A positive answer to the Riemann hypothesis: 
A new result predicting the location of zeros

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Abstract
In this paper, a positive answer to the Riemann hypothesis is given 
by using a new result predicting the exact location of zeros of the 
alternating zeta function on the critical strip.

Keywords: Riemann hypothesis, alternating zeta function, location of 
zeros.

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““The greatest problem for mathematicians now is probably the Riemann hypothesis. But it’s not a problem that can be simply stated.”” –Andrew J. Wiles.

“If I were to awaken after having slept for five hundred years, my first question would be: Has the Riemann hypothesis been proven?” –David Hilbert.

“If you could be the Devil and offer a mathematician to sell his soul for the proof of one theorem - what theorem would most mathematicians ask for? I think it would be the Riemann hypothesis.” –Hugh L. Montgomery.

“It would be very discouraging if somewhere down the line you could ask a computer if the Riemann hypothesis is correct and it said, “Yes, it is true, but you won’t be able to understand the proof.” –Ronald L. Graham.
“It will be another million years, at least, before we understand the primes.”
– Paul Erdős.

1 Introduction

Let $s = \alpha + i\beta$ be a complex number. The complex Riemann zeta function is defined in the half-plane $\alpha > 1$ by the absolutely convergent series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

and in the whole complex plane $\mathbb{C}$ by analytic continuation. As shown by B. Riemann, the zeta function (1) extends to $\mathbb{C}$ as a meromorphic function with only a simple pole at $s = 1$ with residue 1. In [1] Riemann obtained an analytic formula for the number of primes up to a preassigned limit in terms of the zeros of the zeta function (1). This principal result implies that natural primes are distributed as regularly as possible if the Riemann hypothesis is true.

Riemann Hypothesis. The nontrivial zeros of $\zeta(s)$ have real part equal to $\alpha = \frac{1}{2}$.

The Riemann hypothesis is probably the most important open problem in pure mathematics today [2]. The unsolved Riemann hypothesis is part of the Hilbert’s eighth problem, along with the Goldbach conjecture. It is also one of the Clay mathematics institute millennium prize problems. This hypothesis has been checked to be true for the first 1500000000 solutions. However, a mathematical proof is not formulated since its formulation in 1859.

It is well known that the Riemann hypothesis is equivalent to the statement that all the zeros of the Dirichlet eta function (the alternating zeta function)

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}$$

falling in the critical strip $D = \{ s = \alpha + i\beta \in \mathbb{C} \text{ with } 0 < \alpha < 1 \}$ lie on the critical line $\alpha = \frac{1}{2}$. See [3-4-8, pp. 49]. We have $\eta(s) = (1 - 2^{1-s}) \zeta(s)$ and the series $\eta(s)$ given by (2) converges only for $s = \alpha + i\beta \in \mathbb{C}$ with $\alpha > 0$. The function $\eta(s)$ is a non-constant analytic for all $s = \alpha + i\beta \in \mathbb{C}$ with $\alpha > 0$. 

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2 Evaluating the possible values of $\alpha$

The functional equation for $\eta(s)$ restricted to the critical strip $D$ is given by

\[
\begin{align*}
\eta(s) &= \varphi(s) \eta(1-s) \\
\varphi(s) &= 2^{\frac{1-2^{-1+s}}{1-2^s}} \pi^{s-1} \sin \left( \frac{\pi s}{2} \right) \Gamma(1-s)
\end{align*}
\]

Let

\[
\begin{align*}
\eta(s) &= x(s) + iy(s) \\
\eta(1-s) &= u(s) + iv(s)
\end{align*}
\]

be the algebraic forms of $\eta(s)$ and $\eta(1-s)$ where

\[
\begin{align*}
x(s) &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^\alpha} \cos(\beta \ln n) \\
y(s) &= -\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^\alpha} \sin(\beta \ln n) \\
u(s) &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{1-\alpha}} \cos(\beta \ln n) \\
v(s) &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{1-\alpha}} \sin(\beta \ln n)
\end{align*}
\]

Also, the function $\varphi(s)$ can be written as $\varphi(s) = \varphi_1(s) + i\varphi_2(s)$ and therefore it is given uniquely by the polar form

$$\varphi(s) = \rho(s) \exp(i\theta(s))$$

where, $\rho(s) = \sqrt{\varphi_1^2(s) + \varphi_2^2(s)} > 0$ (since $s \in D$) is the magnitude and $\theta(s) = \arg \varphi(s) \in \mathbb{R}$, is the argument of $\varphi(s)$. Here, we have $\varphi_1(s) = \rho(s) \cos \theta(s)$ and $\varphi_2(s) = \rho(s) \sin \theta(s)$ since such complex numbers are entirely determined by their modulus and angle. If $\varphi(s) = 0$, i.e., equivalently $\rho(s) = 0$, then the polar form is undefined since the argument of the complex number 0 is undefined. This result show that Theorem 1 below is not valid for negative even integers roots of $\eta(s) = 0$. The main reason here is that these roots are exactly the roots of the equation $\rho(s) = 0$. This remark has no relation with the Riemann hypothesis, but it is very interesting to see clearly the behavior of the roots of $\eta(s) = 0$.

The fact that $\varphi(s)$ is analytic implies that $\theta(s)$ and $\rho(s)$ cannot be constant everywhere in any open subset of $\mathbb{C}$. Also, we note that $\rho(s)$ and
θ(s) are real-valued, resp. $\mathbb{S}^1$-valued; therefore they cannot be analytic. This fact will be used later in the proof of the main Theorem 1 below.

Note that all the identities concerning arguments in this paper holds only modulo factors of $2\pi$ if the argument is being restricted to $(-\pi, \pi]$. To compute these values we can use function $\arctan 2$ defined as follow

$$\arctan 2(a, b) = \begin{cases} 
\arctan \left( \frac{b}{a} \right), & \text{if } a > 0 \\
\arctan \left( \frac{b}{a} \right) \pm \pi, & \text{if } a < 0 \\
\pm \frac{\pi}{2}, & \text{if } a = 0 \\
\text{undefined if } b = 0, a = 0
\end{cases}$$

with principal values in the range $(-\pi, \pi]$. Here, $(a, b) = a + ib \in \mathbb{C}$ and the sign "−" corresponding to the case $b < 0$. The main reason here is that the statement concerning the argument of a product used between Equations (12) and (13) below assumes the argument as an equivalence class of numbers. If we need the argument as a single number one needs to fix a half-open interval of length $2\pi$ to which the argument belongs by definition, i.e, we choose $(-\pi, \pi]$ for the purpose of calculations in this paper. The angle associated with the product of two complex numbers $z_1z_2$ is $\theta_1 + \theta_2$. Hence, all the equations involving arguments from and below equation (12) are calculated using addition modulo $2\pi$. We simply write $\arg (z_1z_2) = (\theta_1 + \theta_2) \pmod{2\pi}$ and take only the principal value of any angle.

Let us define the following set

$$\Omega = \{ s = \alpha + i\beta \in D : x(s) = u(s), y(s) = -v(s) \}$$

We know that there are infinitely many zeros of the equation $\eta(s) = 0$. Hence, the set $\Omega$ is not empty since if $s$ is any zero then we obtain $x(s) = u(s) = 0$ and $y(s) = -v(s) = 0$.

The main result for the proof of the Riemann hypothesis is given as follow:

**Theorem 1** The complex number $s = \alpha + i\beta \in D$ is a solution of $\eta(s) = 0$ and $\alpha = \frac{1}{2}$ if and only if $\theta(s) \neq 0 \pmod{2\pi}$ and $s \in \Omega$.

**Proof.** We note that the assumptions of Theorem 1 are valid only for single $s \in D$ since all zeros of a non-constant analytic functions are isolated. Hence, in all what follow we deals only with a single value $s \in D$. 4
(1) Proving that if \( s = \alpha + i\beta \in D \) satisfying \( \theta (s) \neq 0 \) (mod \( 2\pi \)) and \( s \in \Omega \), then \( s \) is a solution of \( \eta (s) = 0 \) with \( \alpha = \frac{1}{2} \).

For this purpose, substituting the algebraic forms of \( \eta (s) \) and \( \eta (1 - s) \) into equation (3) to obtain

\[
x(s) = u(s) \varphi_1(s) - v(s) \varphi_2(s)
y(s) = u(s) \varphi_2(s) + v(s) \varphi_1(s)
\]

or

\[
\begin{pmatrix} x(s) \\ y(s) \end{pmatrix} = A(s) \begin{pmatrix} u(s) \\ v(s) \end{pmatrix}
\]

\[
A(s) = \rho(s) \begin{pmatrix} \cos \theta(s) & -\sin \theta(s) \\ \sin \theta(s) & \cos \theta(s) \end{pmatrix}
\]

and the inverse transformation is given by

\[
\begin{align*}
  u(s) &= \frac{x(s)}{\rho^2(s)} \varphi_1(s) + \frac{y(s)}{\rho^2(s)} \varphi_2(s) \\
  v(s) &= -\frac{x(s)}{\rho^2(s)} \varphi_2(s) + \frac{y(s)}{\rho^2(s)} \varphi_1(s)
\end{align*}
\]

By setting \( z(s) = \frac{x(s)}{\rho(s)} \) and \( w(s) = \frac{y(s)}{\rho(s)} \), we obtain a counterclockwise rotation transformation of the form

\[
\begin{pmatrix} z(s) \\ w(s) \end{pmatrix} = B(s) \begin{pmatrix} u(s) \\ v(s) \end{pmatrix}
\]

\[
B(s) = \begin{pmatrix} \cos \theta(s) & -\sin \theta(s) \\ \sin \theta(s) & \cos \theta(s) \end{pmatrix}
\]

The matrix \( B(s) \) in (7) is invertible for all \( s \in D \) since its determinant is 1. It is well known that a non trivial rotation must have a unique fixed point, its rotocenter. The rotation in (5) is non trivial if \( \varphi(s) \neq 1 \) (here we assumed that \( \theta(s) \neq 0 \) (mod \( 2\pi \)) in the second part of Theorem 1). The reason is that the trivial rotation corresponding to the identity matrix, in which no rotation takes place. The fixed point of the rotation in (7) must satisfies

\[
(I_2 - B(s)) \begin{pmatrix} u(s) \\ v(s) \end{pmatrix} = 0
\]
where $I_2$ is the $2 \times 2$ unit matrix. The determinant of the matrix $(I_2 - B(s))$ is $-2(\cos \theta(s) - 1)$ and it is not zero since $\theta(s) \neq 0 \pmod{2\pi}$, this means that $s$ is a solution of $\eta(1 - s) = 0$ and by using (3) we conclude that $s$ is a root of $\eta(s) = 0$.

Now, for $s \in D$, the assumptions $x(s) = u(s)$ and $y(s) = -v(s)$ in the set $\Omega$ can be reformulated by using (5) and (6) as follow:

$$\begin{cases} 
    x(s) = u(s) \varphi_1(s) - v(s) \varphi_2(s) = u(s) = x(s) \varphi_1(s) + y(s) \varphi_2(s) \\
    y(s) = u(s) \varphi_2(s) + v(s) \varphi_1(s) = -v(s) = -(y(s) \varphi_1(s) - x(s) \varphi_2(s))
\end{cases}$$

that is,

$$\begin{cases} 
    u(s) \varphi_1(s) - v(s) \varphi_2(s) = x(s) \varphi_1(s) + y(s) \varphi_2(s) \\
    u(s) \varphi_2(s) + v(s) \varphi_1(s) = -(x(s) \varphi_1(s) - y(s) \varphi_2(s))
\end{cases}$$

by replacing the values of $\varphi_1(s) = \rho(s) \cos \theta(s)$, $\varphi_2(s) = \rho(s) \sin \theta(s)$, and the values of $x(s), y(s), u(s)$ and $v(s)$ from (4), we obtain the following equation

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^\alpha} (1 - n^{2\alpha-1} \rho^2(\alpha + i\beta)) \exp(i\beta \ln n) = 0 \quad (8)$$

Originally, the set $\Omega$ is expressed in term of $x(s), u(s), y(s)$ and $v(s)$. From (*) we can write $\Omega$ in term of $\alpha$ and $\beta$ as follow

$$\Omega_{\alpha,\beta} = \{ (\alpha, \beta) \in \mathbb{R}^2 : \text{Equation (8) is verified with } 0 < \alpha < 1 \}$$

Hence, in the sequel we replace $\Omega$ by $\Omega_{\alpha,\beta}$.

At this stage assuming $\theta(s) \neq 0 \pmod{2\pi}$ and $s \in \Omega_{\alpha,\beta}$, then $s$ is a root of $\eta(s) = 0$. It is well known that all roots of a non-zero analytic function are isolated. By using this fact and by a simple remark, we conclude that when $\alpha = \frac{1}{2}$, the factor $(1 - n^{2\alpha-1} \rho^2(s))$ in (8) is zero because it is easy to check that $\rho(s) = 1$ by using direct calculations. See [9] for more details. Since we are speaking about a single and isolated root $s$ of the function $\eta(s)$, the only solution $s = \alpha + i\beta$ of (8) is obtained when

$$\alpha = \frac{1}{2} \quad (9)$$
for some $\beta \in \mathbb{R}$. In this case $s = \frac{1}{2} + i\beta$ is a root of equation $\eta(s) = 0$.

(2) Proving that if $s = \alpha + i\beta \in D$ is a solution of $\eta(s) = 0$ with $\alpha = \frac{1}{2}$ then $\theta(s) \neq 0 \pmod{2\pi}$ and $s \in \Omega$:

For this purpose, assume that $s \in D$ is a solution of $\eta(s) = 0$ with $\alpha = \frac{1}{2}$, then, trivially, we have $x(s) = u(s) = 0$ and $y(s) = -v(s) = 0$, hence $s \in \Omega_{\alpha, \beta}$. Thus we only need to prove that $\theta(s) \neq 0 \pmod{2\pi}$. From the second equation of (3) we have

$$\varphi \left( \frac{1}{2} + i\beta \right) = \sqrt{2} \pi^{i\beta - \frac{1}{2}} \left( \frac{2i\beta - \frac{1}{2} - 1}{2i\beta + \frac{1}{2} - 1} \right) \left( \cosh \frac{1}{2} \pi \beta + i \sinh \frac{1}{2} \pi \beta \right) \Gamma \left( \frac{1}{2} - i\beta \right)$$

and by direct calculations, we get

$$\left\{ \begin{array}{l}
\pi^{i\beta - \frac{1}{2}} = \frac{1}{\sqrt{\pi}} \left( \cos \left( \beta \ln \pi \right) + i \sin \left( \beta \ln \pi \right) \right) \\
\frac{2i\beta - \frac{1}{2} - 1}{2i\beta + \frac{1}{2} - 1} = \frac{3 \cos(\beta \ln 2) - \frac{3}{2} \sqrt{2}}{4 \cos(\beta \ln 2) - \frac{1}{2} \sqrt{2}} - \frac{1}{4} i \frac{\sin(\beta \ln 2)}{\cos(\beta \ln 2) - \frac{1}{2} \sqrt{2}} \\
\Gamma \left( \frac{1}{2} - i\beta \right) = \frac{\sqrt{\pi} \exp \left( -i \left( 2\vartheta(\beta) + \beta \ln 2 + \arctan \left( \tanh \left( \frac{\pi}{2} \right) \right) \right) \right)}{\sqrt{\cosh \pi \beta}}
\end{array} \right. (11)$$

where $\vartheta(t)$ is the Riemann Siegel function [10] given by

$$\vartheta(t) = \arg \left( \Gamma \left( \frac{2it + 1}{4} \right) \right) - \frac{\log \pi}{2} t, t \in \mathbb{R}.$$  

The argument is chosen such that a continuous function is obtained and $\vartheta(0) = 0$ holds, i.e., in the same way that the principal branch of the log Gamma function is defined. Also, the argument of $\Gamma(s)$ is well defined and harmonic on the plane with a cut along the negative real axis.

On one hand, the last formula of equation (12) can be proved easily for $\xi = \mu + it, \mu, t \in \mathbb{R}$ by using the following equations

$$\left\{ \begin{array}{l}
\cos \left( \frac{\pi \xi}{2} \right) = \frac{1}{\sqrt{\pi}} \sqrt{\cosh(\pi t)} \exp \left( -i \arctan \left( \tanh \left( \frac{\pi t}{2} \right) \right) \right) \\
\Gamma \left( \frac{1}{2} + i\frac{t}{2} \right) = \left| \Gamma \left( \frac{1}{4} + i\frac{t}{2} \right) \right| \exp \left( i \left( \vartheta(t) + \frac{t}{2} \ln \pi \right) \right) \\
\Gamma(\xi) \Gamma \left( \xi + \frac{1}{2} \right) = 2^{1-2\xi} \sqrt{\pi} \Gamma(2\xi) \\
\Gamma(\xi) \Gamma \left( 1 - \xi \right) = \frac{\pi}{\sin \pi \xi}
\end{array} \right.$$
and then we get the formula by setting $\xi = \frac{1}{4} + it$ and replacing $t$ by $-\beta$. Indeed, we have
\[
\Gamma (2\xi) = 2^{2\xi-1}\pi^{-\frac{1}{2}}\Gamma (\xi) \frac{\pi}{\cos (\pi\xi)} \frac{1}{\Gamma \left(\frac{1}{2} - \xi\right)}
\]
For $\xi = \frac{1}{4} + it, t \in \mathbb{R}$, we have
\[
\Gamma \left(\frac{1}{2} + it\right) = \frac{2^{-\frac{1}{2}+it\pi\frac{1}{2}} \Gamma \left(\frac{1}{4} + it\right)}{\cos (\pi (\frac{1}{4} + it)) \Gamma \left(\frac{1}{4} - it\right)}
\]
Hence, we have
\[
\Gamma \left(\frac{1}{2} + it\right) = \frac{2^{it\pi\frac{1}{2}}\pi^{it}}{\cosh \pi t \exp \left(-i \arctan (\tanh \left(\frac{\pi x}{2}\right))\right)}
\]
Thus, we obtain
\[
\Gamma \left(\frac{1}{2} + it\right) = \frac{\sqrt{\pi} \exp \left(i \left(2\vartheta (t) + t \ln (2\pi) + \arctan (\tanh \left(\frac{\pi t}{2}\right))\right)\right)}{\sqrt{\cosh (\pi t)}}
\]
For $t = -\beta$, we have
\[
\Gamma \left(\frac{1}{2} - i\beta\right) = \frac{\sqrt{\pi} \exp \left(-i \left(2\vartheta (\beta) + \beta \ln (2\pi) + \arctan (\tanh \left(\frac{\pi \beta}{2}\right))\right)\right)}{\sqrt{\cosh (\pi \beta)}}
\]
because the function $t \to \arctan (\tanh \left(\frac{\pi t}{2}\right))$ is odd and since the function $\vartheta (t)$ is also odd as shown in [10]. We note that other proof of the last equation of (12) uses the functional equation of the zeta function.

On the other hand, we can calculate the arguments of the quantities in
(12) as follow

\[
\begin{align*}
\arg (\sqrt{2}) &= 0 \pmod{2\pi} \\
\arg \left( \pi^{\imath\beta - \frac{1}{2}} \right) &= (\beta \ln \pi) \pmod{2\pi} \\
\arg \left( \frac{2^{\beta - \frac{1}{2}} - 1}{2^{\beta + \frac{1}{2}} - 1} \right) &= \varpi(\beta) \pmod{2\pi} \\
\arg \left( \cosh \frac{\pi \beta}{2} + i \sinh \frac{\pi \beta}{2} \right) &= \phi(\beta) \pmod{2\pi} \\
\arg \left( \Gamma \left( \frac{1}{2} - \imath \beta \right) \right) &= \psi(\beta) \pmod{2\pi}
\end{align*}
\]

The arguments: \( t \ln \pi, \varpi(t), \phi(t) \) and \( \psi(t) \) in (12) are well defined since the functions \( \pi^{it - \frac{1}{2}}, \cosh \left( \frac{1}{2} \pi t \right) + i \sinh \left( \frac{1}{2} \pi t \right), \frac{2^{it - \frac{1}{2}} - 1}{2^{it + \frac{1}{2}} - 1} \) and \( \Gamma \left( \frac{1}{2} - it \right) \) are not zero for all \( t \in \mathbb{R} \). In particular, this is true if \( t = \beta = \text{Im}(s) \).

Since the complex argument of a product of two numbers is equal to the sum of their arguments, then from (12) and the remark following Equation (4) we have

\[
\theta \left( \frac{1}{2} + \imath \beta \right) = (0 + \beta \ln \pi + \varpi + \phi + \psi) \pmod{2\pi} \quad (13)
\]

Hence, we have

\[
\begin{align*}
\theta \left( \frac{1}{2} + \imath \beta \right) &= 2 \left( g(\beta) - \vartheta(\beta) \right) \pmod{2\pi} \\
g(\beta) &= \frac{1}{2} \left( \beta \ln 2 + \arctan \left( \frac{\sin(\beta \ln 2)}{3 \cos(\beta \ln 2) - \frac{1}{3} \sqrt{2}} \right) \right) \\
\vartheta(\beta) &= \left[ g(\beta) - \frac{1}{2} \theta \left( \frac{1}{2} + \imath \beta \right) \right] \pmod{2\pi}
\end{align*}
\]

Thus, from the first equation of (14) we get

\[
\vartheta(\beta) = \left[ g(\beta) - \frac{1}{2} \theta \left( \frac{1}{2} + \imath \beta \right) \right] \pmod{2\pi} \quad (15)
\]
The only possible value of equations \( \theta \left( \frac{1}{2} + i\beta \right) = 2k\pi \) is when \( k = 0 \). Indeed, assume that \( \theta \left( \frac{1}{2} + i\beta \right) = 2k\pi \). We know from [10] that \( \vartheta (\beta) \) is an odd function and \( g (\beta) \) is also an odd function by direct calculations. Hence, from (15) we have \( \vartheta (-\beta) = -g (\beta) - k\pi = -g (\beta) + k\pi \), that is, \( k = 0 \). As mentioned above, we must prove that \( \theta \left( \frac{1}{2} + i\beta \right) \neq 0 \) (mod\( 2\pi \)) if \( \frac{1}{2} + i\beta \) is a root of \( \eta (s) = 0 \). We know that the function \( \vartheta (\beta) \) has only three roots 0 and \( \pm 17.8455995405 \ldots \). Hence, the equation \( g (\beta) = 0 \) has only the same three roots, that is the equation \( \theta \left( \frac{1}{2} + i\beta \right) = 2g (\beta) \) has only the same three roots. The only possible case to get \( \theta \left( \frac{1}{2} + i\beta \right) = 0 \) (mod\( 2\pi \)) is when \( \beta = 0 \) since we have \( g (0) = 0 \), \( g (17.8455995405) = -4.8774 \ldots \) and \( g (-17.8455995405) = 4.8774 \ldots \). However, the real \( s = \frac{1}{2} + i0 \) is not a root of \( \eta (s) = 0 \) (we have \( \eta \left( \frac{1}{2} \right) \approx 0.60440 \)). Hence, we have proved that if \( s \in D \) is a solution of \( \eta (s) = 0 \) with \( \alpha = \frac{1}{2} \), then \( \theta (s) \neq 0 \) (mod\( 2\pi \)).

Theorem 1 proves that the Riemann hypothesis is true for the function \( \eta (s) \) if and only if \( \theta (s) \neq 0 \) (mod\( 2\pi \)) and \( s \in \Omega_{\alpha,\beta} \) for every root \( s \in D \) of the equation \( \eta (s) = 0 \). Hence, to prove that the Riemann hypothesis is true for the function \( \eta (s) \), we need only a proof of the second part in Theorem 1 since we have an equivalence between the two cases. Clearly, if \( s \in D \) is a root of \( \eta (s) = 0 \), then \( s \in \Omega = \Omega_{\alpha,\beta} \). Hence, we need only to prove that if \( s \) is a root of \( \eta (s) = 0 \), then \( \theta (s) \neq 0 \) (mod\( 2\pi \)).

In fact we can prove the following equivalence:

\[
\text{s} \in D \text{ and } \theta (s) \neq 0 \text{ (mod\( 2\pi \)) } \iff \alpha = \frac{1}{2}
\]

Indeed, the first implication: \( \alpha = \frac{1}{2} \implies \theta (s) \neq 0 \text{ (mod\( 2\pi \)) } \) holds true as shown in the second part of the proof of Theorem 1. For the second implication: \( \theta (s) \neq 0 \text{ (mod\( 2\pi \)) } \implies \alpha = \frac{1}{2} \), we can remark that if \( s \in D \) and \( \theta (s) \neq 0 \text{ (mod\( 2\pi \)) } \), then \( s \) is a root of \( \eta (s) = 0 \), hence \( s \in \Omega = \Omega_{\alpha,\beta} \) which give equation (8). Again, since \( s \) is an isolated root, then we must have \( \alpha = \frac{1}{2} \).

Finally, we can concludes that

**Theorem 2** The Riemann hypothesis is true, i.e., all the nontrivial zeros of \( \zeta (s) \) have real part equal to \( \alpha = \frac{1}{2} \).

**Proof.** The fact that the Dirichlet eta function (2) have the same zeros as the zeta function (1) in the critical strip \( D \), implies that all nontrivial zeros of \( \zeta (s) \) have real part equal to \( \alpha = \frac{1}{2} \).
3 Conclusion

At this end, the hope that primes are distributed as regularly as possible is now become a truth by this unique solution of the Riemann hypothesis. Also, all propositions which are known to be equivalent to or true under the Riemann hypothesis are now correct. Examples includes, growth of arithmetic functions, Lindelöf hypothesis and growth of the zeta function, large prime gap conjecture...etc.

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