Automatic Trajectory Synthesis for Real-Time Temporal Logic

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Abstract—Many safety-critical systems must achieve high-level task specifications with guaranteed safety and correctness. Much recent progress towards this goal has been made through controller synthesis from temporal logic specifications. Existing approaches, however, have been limited to relatively short and simple specifications. Furthermore, existing methods either consider some prior discretization of the state-space, deal only with a convex fragment of temporal logic, or are not provably complete. We propose a scalable, provably complete algorithm that synthesizes continuous trajectories to satisfy non-convex Temporal Logic over Reals (RTL) specifications. We separate discrete task planning and continuous motion planning on-the-fly and harness highly efficient boolean satisfiability (SAT) and Linear Programming (LP) solvers to find dynamically feasible trajectories that satisfy non-convex RTL specifications for high dimensional systems. The proposed design algorithms are proven sound and complete, and simulation results demonstrate our approach’s scalability.

I. INTRODUCTION

Autonomous Intelligent Physical Systems (IPSs) must be capable of interpreting and automatically achieving high-level task specifications. Symbolic control proposes to fulfill this need by automatically designing controllers that satisfy formal logic specifications. Temporal logics such as Temporal Logic over Reals (RTL) and Signal Temporal Logic (STL) can express a wide variety of tasks for IPSs \cite{1}. Furthermore, temporal logic formulas are close to natural language and can even be interpreted by verbal commands \cite{2}.

However, today’s large-scale IPSs present unprecedented scalability challenges for symbolic control techniques, and existing symbolic control algorithms cannot solve many real-world problems. This scalability challenge stems from the need to combine logical constraints (from task specifications) with continuous motion restrictions (from physical dynamics).

Early efforts in symbolic control relied on discrete abstractions of continuous dynamical systems. Much work focused on obtaining an equivalent discrete and finite quotient transition system (see \cite{3, 4, 5, 6, 7} and references therein). Given an equivalent transition system, logical constraints can be handled with efficient search techniques in the discrete space. Finding such discrete abstractions is difficult for high-dimensional systems, however, and these approaches are usually limited to systems with less than five continuous state variables \cite{8}.

Recently, a growing body of work has focused on the synthesis of continuous trajectories from high-level logic specifications. One of the significant challenges in this approach is the combination of logical constraints and physical dynamical constraints. Together, these non-convex constraints mean that even determining whether or not a satisfying trajectory exists is a difficult problem. For this reason, existing trajectory synthesis approaches are only provably complete for bounded-time specifications \cite{11}.

Another challenge caused by joint logical/physical constraints is scalability to high-dimensional systems (those with more than ten continuous state variables) \cite{9}. State-of-the-art solvers based on Mixed-Integer Linear Programming (MILP) are exponential in the number of logical predicates, severely limiting scalability to complex systems \cite{12}. Meanwhile, sampling-based and heuristic approaches can achieve impressive results on certain problems, but are not complete and perform poorly on narrow passages \cite{13}.

We directly address this issue for RTL specifications using a two-layer control architecture. By separating the non-convex logical specification (discrete task planning layer) from physical system dynamics (continuous motion planning layer) on-the-fly, we can achieve superior scalability and provable completeness of unbounded specifications.

We focus on RTL in particular because RTL describes specifications over continuous variables and is not time-bounded by definition—the very sort of specification that existing symbolic control techniques struggle to handle. Furthermore, RTL is closely related to commonly used temporal logics like Signal Temporal Logic (STL) and Linear Temporal Logic (LTL). The main difference between RTL and these more commonly used logics is that STL and LTL can include time bounds. In fact, much existing work on symbolic control is restricted to bounded-time subsets of STL and LTL, which enables completeness guarantees \cite{14}. By using RTL, we essentially consider a complementary subset of specifications—those without time bounds. For this reason, extending our results to STL and LTL should be relatively straightforward.

In the discrete planning layer, we use Bounded Satisfiability Checking (BSC) techniques and highly efficient SAT solvers to overcome nonconvexity in the logical specification. Then, in the continuous motion planning layer, finding a corresponding continuous trajectory is as simple as solving a Linear Program (LP). Inspired by the framework of Counterexample-Guided Inductive Synthesis (CEGIS), these two layers work together to ensure completeness: if a continuous trajectory cannot be found for a given discrete plan, a counterexample is generated to guide the discrete planner at the next iteration.

Our main contribution is a trajectory synthesis method that is...
provable sound and complete for unbounded real-time temporal logic specifications. We show that this method is scalable to systems with over 10 state variables and complex logical specifications. Simulation results indicate that our approach is over an order of magnitude faster than the current state-of-the-art.

The rest of the paper is organized as follows. We review related work in Section II. After presenting the necessary preliminaries and a formal problem statement in Section III, we outline our proposed approach in Section IV. Sections V and VI provide a detailed description of the discrete task planning and continuous motion planning algorithms. Section VII details how discrete task planning and continuous motion planning work together, including proofs of soundness and completeness. Finally, Section VIII provides simulation results that illustrate the speed and scalability of our approach, and Section IX concludes the paper.

II. RELATED WORK

Existing approaches for symbolic control based on trajectory synthesis can be roughly divided into three categories: MILP based [9], [14], [15], [16], [17], [18], sampling based [19], [6], and Satisfiability Modulo Theories (SMT) based [10] approaches.

The basic idea of the MILP approach is to rewrite statements with logical expressions into mixed-integer constraints. The addition of auxiliary binary variables to facilitate this rewriting, however, renders the problem intractable for long trajectories. Thus MILP-based approaches such as BluSTL [14] have focused on Model Predictive Control (MPC), which limits the duration (i.e., number of time indices) of the search. LTL OPT [6] proposed an alternative encoding to synthesize controllers from a fragment of Linear Temporal Logic (LTL) and from Metric Temporal Logic (MTL) [16] for longer time horizons. However, this approach faces the same limitations and struggles to efficiently handle nonconvex logical constraints with a long duration (greater than 100 time indices).

When considering only a convex fragment of Signal Temporal Logic (STL), the problem can be efficiently encoded as a Linear Programming (LP) problem [14] instead. Furthermore, the satisfaction of such an STL formula can be measured using robust semantics, which allows for efficient controller synthesis using control barrier functions [20] and prescribed performance control [21]. However, this convex fragment of STL cannot describe many tasks required by real-world IPS. For instance, a quadrotor performing an automated inspection task needs to return to a charging station in a reasonable time infinitely often: this requires nested existential quantifications (always-eventually) and thus cannot be described by a convex fragment of STL. Similarly, a robot operating in a warehouse might have a task that requires logical disjunction: pick up one box OR another box before moving to a goal destination.

Sampling-based approaches such as the Open Motion Planning Library (OMPL) LTL planner [22] propose to combine sampling-based motion planning with discrete search algorithms. Sampling-based motion planning methods are relatively easy to implement and can provide fast solutions to some difficult problems. However, such approaches are suboptimal and are not guaranteed to find a solution if one exists, a property referred to as (in)completeness. Instead, they ensure weaker notions of asymptotical optimality [23] and probabilistic completeness [24], meaning that an optimal solution will be provided, if one exists, given sufficient runtime of the algorithm. These difficulties are exemplified in poor performance on problems with narrow passages [13].

Our proposed approach builds on SMT based symbolic control, which has been used to generate dynamically-feasible trajectories for LTL specifications [10]. Modern SMT solvers can efficiently find satisfying valuations of extensive formulas with complex Boolean structures combining various decidable theories such as lists, arrays, bit vectors, linear integer arithmetic, and linear real arithmetic [25]. SMT based symbolic control from LTL specifications showed encouraging performance for motion planning problems. However, existing approaches are not provably complete. Furthermore, the implementation of real-time specifications is difficult, as existing methods do not offer explicit bounds regarding when the synthesis algorithm will terminate.

To the best of our knowledge, our approach is the first trajectory-based symbolic controller for real-time temporal logic that is provably sound and complete for nonconvex unbounded specifications. Furthermore, we present comparative results showing that our approach scales well to a long duration (greater than 100 time indices) tasks and high-dimensional (greater than 10 continuous state variables) system dynamics.

III. PRELIMINARIES

A. System

Consider the following discrete-time linear control system:

\[ x_{k+1} = Ax_k + Bu_k, \]

where \( x_k \in \mathcal{X} \subseteq \mathbb{R}^n \) are the state variables, \( u_k \in \mathcal{U} \subseteq \mathbb{R}^m \) are the control inputs, \( \mathcal{X} := \{ x_k \in \mathbb{R}^n | A_x x_k \leq b_x \} \) and \( \mathcal{U} := \{ u_k \in \mathbb{R}^m | A_y u_k \leq b_y \} \) are full dimensional polytopes, \( A, B, A_y, A_x \) are matrices and \( b_x, b_y \) are vectors with proper dimensions. We assume that the system is stable (i.e., \( A \) has positive real-valued eigenvalues).

In fact, System (1) can arise from linearization and sampling of a more general continuous system. In this case, we denote the sampling period as \( T_s = t_{k+1} - t_k \).

We will use model checking techniques to verify and control the dynamic behavior of the system. For this purpose, we model this system as a Kripke structure \( M_c \). Kripke structures are a type of transition system model that can represent a large class of systems. They are formally defined as follows:

**Definition 1.** A tuple \( M = (S, Act, T, I, L, \Sigma, F) \) is a Kripke structure where \( S \) is a set of states, \( I \subseteq S \) is a set of initial states, \( Act \) is a set of actions (aka, inputs), \( T \subseteq S \times Act \times S \) is a transition relation, \( L : S \rightarrow 2^{\Sigma} \) is a labeling function over a finite set of symbols \( \Sigma \), and \( F \subseteq S \) is a set of accepted states.
A run of $M$ is a sequence $\xi = s_0 \xrightarrow{a_0} s_1 \xrightarrow{a_1} s_2 \ldots s_k$, where $s_k \in S$, $a_k \in Act$, $s_0 \in I$, $s_k \in F$, and $s_k \xrightarrow{a_k} s_{k+1}$ if and only if $(s_k, a_k, s_{k+1}) \in T$. This run generates a path of $M$ which is defined as a sequence of labels $\sigma = L(s_0)L(s_1)L(s_2)\ldots L(s_k)$. When we need to model infinite behaviors of $M$, we accept a run if it visits the acceptance set $F$ infinitely often. We call such structures that model infinite behavior fair Kripke structures.

We define a continuous structure $M_\epsilon$ for the continuous system $\{1\}$: $M_\epsilon = (S_\epsilon, Act_\epsilon, T_\epsilon, F_\epsilon, \Sigma_\epsilon, \mu_\epsilon)$, where: $S_\epsilon = \mathbb{R}$, $L_\epsilon = \{\emptyset\}$ with $x \in \mathbb{R}$, $Act_\epsilon = \emptyset$, $s_k \xrightarrow{a_k} s_{k+1}$ if and only if $x_{k+1} = Ax_k + Bu_k$, and $F_\epsilon = \mathbb{R}^\ast$. The labels $L_\epsilon$ and symbols $\Sigma$ depend on the logical specification, and are defined in the next subsection. Therefore, a run (trajectory) of $M_\epsilon$ is a sequence $\xi_\epsilon = x_0 \xrightarrow{a_0} x_1 \xrightarrow{a_1} \ldots$.

B. Linear Temporal Logic over Reals

Instead of abstracting the continuous state space directly into finite symbols to provide labels for $M_\epsilon$, we use a formal language that allows us to represent state constraints as half-spaces. As we will see in Section V this allows us to separate logical constraints from continuous dynamics without the expensive computation of a discrete abstraction. Specifically, we consider high-level specifications are given as RTL formulas [26].

**Definition 2.** RTL formulas are defined recursively according to the following syntax:

\[
\phi ::= \pi^\mu|\pi^\mu \phi \quad | \quad \phi_1 \wedge \phi_2 \quad | \quad \phi_1 \vee \phi_2 \quad | \quad \phi_1 \rightarrow \phi_2 \quad | \quad \phi_1 \rightarrow \phi_2 \]  

\[
\varphi ::= \forall \exists \mu \left( x \right) \varphi \quad | \quad \forall \exists \mu \left( x \right) \varphi \quad | \quad \forall \exists \mu \left( x \right) \varphi \quad | \quad \forall \exists \mu \left( x \right) \varphi \]  

where $\varphi$, $\phi_1$, $\phi_2$ are RTL formulas, $\phi_1$ and $\phi_2$ are state formulas, and $\pi^\mu \in \Pi$ is an atomic proposition. Propositions $\pi^\mu : \mathbb{R}^n \rightarrow \{\top, \bot\}$ are defined by a function $\mu$, which we assume is linear affine, i.e., $\mu \left( x \right) = h^T x + a$, $h \in \mathbb{R}^n$ and $a \in \mathbb{R}$.

An RTL formula $\phi$ is defined in terms of state formulas $\varphi$. Note that all state formulas are RTL formulas, but not all RTL formulas are state formulas. We denote a state $x_k \in \mathbb{R}^n$ satisfying a state formula by $x \models \varphi$, and define the notion of satisfaction recursively: $x \models \pi^\mu \varphi$ if and only if $\mu \left( x \right) > 0$, $x \models \neg \pi^\mu \varphi$ if and only if $\mu \left( x \right) < 0$, and $x \models \phi_1 \wedge \phi_2$ if and only if $x \models \phi_1$ and $x \models \phi_2$, and $x \models \phi_1 \vee \phi_2$ if and only if $x \models \phi_1$ or $x \models \phi_2$. With these definitions, we can derive standard Boolean shorthands like negation $\neg$, implication $\rightarrow$, and biconditional $\leftrightarrow$.

These state formulas allow us to define symbols $\Sigma$, and the labeling function $L_\epsilon$ for the transition system is $M_\epsilon$, as follows:

**Definition 3.** For each state formula $\phi$ in the RTL formula $\varphi$, define the symbol $p^\phi$. Then $\Sigma = \{p^\phi\}$ is the set of all such symbols. The labeling function $L_\epsilon : \mathbb{R}^n \rightarrow 2^\Sigma$ maps states to a set of symbols, where $p^\phi \in L_\epsilon \left( x \right)$ for a state $x \in \mathbb{R}^n$ if and only if $x \models \phi$.

In this work, we assume that a valid transition of $M_\epsilon$ changes satisfaction of at most one predicate. In particular, transitions only occur between adjacent labeled state spaces. This assumption is minimally restrictive, since the system $M_\epsilon$ approximate the behavior of a continuous-time system, and regions in state space determine predicates.

**Assumption 1.** We assume that a valid transition $(x_k, u_k, x_{k+1}) \in T_\epsilon$ of $M_\epsilon$ occurs only if there exists at most one predicate $\pi^\mu$ in the specification $\varphi$ such that $x_k \models \pi^\mu$ and $x_{k+1} \models \neg \pi^\mu$.

**Example 1.** Consider an integrator $x_{k+1} = x_k + T_k u_k$, where $u_k$, the input variable, is bounded $|u| \leq 1/m$, and $T_k$ is the sampling time. Consider the formula $(x \leq 1) \wedge (x \geq 2)$. The assumption is satisfied if $Ts < 1$s, since at each timestep, at most one of the predicates $(x \leq 1)$ or $(x \geq 2)$ can change. If the sampling time is too large, however, the state could jump from $(x \leq 1)$ to $(x \geq 2)$ in a single step, violating the assumption.

The meaning (semantics) of an RTL formula is interpreted over a run $\xi$ of $M_\epsilon$. We denote a run $\xi$ satisfying an RTL formula $\varphi$ by $\xi \models \varphi$. We write $\xi \models \varphi$ when the run $x_k \xrightarrow{u_k} x_{k+1} \xrightarrow{u_{k+1}} \ldots$ satisfies the RTL formula $\varphi$.

**Definition 4.** The following semantics define the validity of a formula $\varphi$ with respect to the run $\xi$:

- $\xi \models \varphi$ if and only if $\xi \models \varphi_0$.
- $\xi \models \varphi_1 \wedge \varphi_2$ if and only if $\xi \models \varphi_1$ and $\xi \models \varphi_2$.
- $\xi \models \forall \exists \mu \left( x \right) \varphi_2$ if and only if $\xi \models \varphi_2$ for all $\forall \exists \mu \left( x \right)$.
- $\xi \models \exists \mu \left( x \right) \varphi_2$ if and only if $\exists \mu \left( x \right) \varphi_2$.
- $\xi \models \forall \exists \mu \left( x \right) \varphi_2$, $\forall \exists \mu \left( x \right) \varphi_2$, $\forall \exists \mu \left( x \right) \varphi_2$, $\forall \exists \mu \left( x \right) \varphi_2$.
- $\xi \models \forall \exists \mu \left( x \right) \varphi_2$ if and only if $\forall \exists \mu \left( x \right) \varphi_2$.
- $\xi \models \forall \exists \mu \left( x \right) \varphi_2$, $\forall \exists \mu \left( x \right) \varphi_2$, $\forall \exists \mu \left( x \right) \varphi_2$, $\forall \exists \mu \left( x \right) \varphi_2$.

The operator until $\phi_1 \varphi_2$ means that the sub-formula $\phi_1$ must remain true until $\varphi_2$ becomes true. On the other hand, a specification $\phi_1$ releases $\varphi_2$ ($\varphi_1 \varphi_2$) means that $\varphi_2$ must remain true until $\phi_1$ is true. If $\phi_1$ never true, $\varphi_2$ must remain true forever. Moreover, these definitions allow us to derive the operators “eventually” $\diamond \varphi = \top \varphi$ and “always” $\square \varphi = \bot \varphi$.

**Definition 5.** We define the set of subformulas (closure) $c(\varphi)$ of an RTL formula $\varphi$ as the smallest set satisfying the following conditions: $\varphi \in c(\varphi)$, if $c \varphi_1 \in c(\varphi)$ for $c \in \{\top, \bot\}$ then $\varphi_1 \in c(\varphi)$, and if $\phi_1 \varphi_2 \in c(\varphi)$ for $c \in \{\top, \bot\}$ then $\varphi_1 \varphi_2 \in c(\varphi)$.

**Remark 1.** Note that RTL formulas are time-unbounded by definition. Since dealing with unbounded formulas is particularly difficult for existing symbolic control methods, using RTL rather than related formal logic like STL or MTL allows us to focus on the challenges particular to time-unbounded formulas.

C. Problem formulation

The RTL symbolic control problem is formally defined as follows:

**Problem 1.** Given an RTL formula $\varphi$, and a dynamical system $M_\epsilon$, design a control signal $u = u_0 u_1 \ldots$ such that the corre-
sponding run \( \xi := x_0 \xrightarrow{u_0} x_1 \xrightarrow{u_1} \ldots \) of the system \( M_c \) satisfies \( \phi \).

IV. OVERVIEW

Solving Problem 1 directly in terms of the continuous system \( M_c \) is quite difficult due largely to the nonconvexity introduced by the logical specification \( \phi \). To overcome this nonconvexity, we separate the problem into two parts: discrete task planning and continuous motion planning, as shown in Fig. 1. In the discrete planning phase, we determine a sequence of convex regions in the state space that enforces satisfaction of the task specification \( \phi \). Given a discrete plan, finding a corresponding continuous trajectory (motion planning) can be reduced to a simple linear programming problem.

To find a discrete task plan, we first propose a finite-state abstraction \( M_d \) which is related to \( M_c \) through a simulation relation \( \mathcal{R} \). Unlike early work in symbolic control, this abstraction is built on-the-fly from the predicates of specification \( \phi \). Furthermore, we consider discrete plans to be a fair Kripke structure, which allows us to consider unbounded specifications elegantly. Finally, we propose an encoding that allows us to find a satisfying discrete plan by solving a Boolean satisfiability problem (SAT). While SAT is an NP-complete problem, many fast solvers exist, and SAT/SMT solver performance has been increasing steadily in recent years [12].

Given a discrete plan, we show that finding a corresponding continuous run of a system \( M_c \) can be reduced to solving a linear program (LP). If this LP is infeasible, we treat the corresponding discrete plan as an infeasible counterexample, which is passed back to the discrete planning layer.

A key insight is that the information from previous infeasible discrete plans can be used to generate new plans. Specifically, we show how off-the-shelf incremental SMT solvers like Z3 [27] can use such information from past iterations to improve scalability drastically.

We prove that our approach is sound (any run generated by our approach satisfies the specification \( \phi \)) and complete (if any satisfying run exists, our approach will find a satisfying run). Furthermore, we demonstrate the scalability of our approach in several simulation examples. Unlike approaches that seek to obtain an equivalent discrete and finite abstraction, our approach obtains a simulation abstraction and does not require feedback controllers that guarantee the transitions. Unlike other trajectory-based approaches, our method guarantees soundness, completeness, and for unbounded-time specifications. Furthermore, our approach is scalable to high-dimensional systems and complex specifications.

V. DISCRETE TASK PLANNING

In the discrete task planning layer, we generate a sequence of convex constraints that ensures the satisfaction of the specification \( \phi \). To generate such constraints, we propose a finite discrete transition system \( M_d \), which abstracts the behavior of the continuous system \( M_c \) with respect to the specification. This discrete system \( M_d \), the logical specification \( \phi \), and any counterexamples can be encoded as Boolean formulas and leverage incremental SAT/SMT solvers to rapidly find a discrete plan, which we represent as a Kripke structure \( M_p \).

This process is illustrated in Figure 2. We first generate a discrete abstraction \( M_d \) of the continuous system \( M_c \). We then encode the abstract system \( M_d \), specification \( \phi \), and counterexamples \( M_{ce} \) into a Boolean formula, which is verified with an SAT/SMT solver. If this formula is satisfiable, we decode the satisfying evaluation of the variables to a Kripke structure \( M_p \), which represents behaviors of \( M_d \) that satisfies the specification. Otherwise, we increase the problem bound \( K \) until we determine that no solution exists.

A. Abstraction

The discrete abstraction \( M_d \) is a Kripke structure \( M_d = \langle S_d, \text{Act}_d, T_d, I_d, L_d, \Sigma, F_d \rangle \) over a finite set of states \( S_d = \{s_1, s_2, \ldots, s_{|S_d|}\} \) and an empty set of inputs \( \text{Act}_d = \emptyset \). \( M_d \) is formally related to the continuous system \( M_c \) through the notion of a simulation relation \( \mathcal{R} \).

Definition 6. A relation \( R \subseteq S \times S' \) is a simulation relation from Kripke structure \( M \) to \( M' \) (i.e., \( M' \) simulates \( M \)) if:

1) for every \( s_0 \in I \) there exists \( s'_0 \in I' \) such that \( (s_0, s'_0) \in R \);
2) for every \( s \in F \) there exists \( s' \in F_d \) such that \( (s, s') \in R \);
3) for every \( (s, s') \in R \) we have \( L(s) = L'(s') \);
4) for every \( (s_k, s'_k) \in R \) we have that:
   for every \( a_k \in \text{Act} \) with \( (s_k, a_k, s_{k+1}) \in T \) there exists \( a'_k \in \text{Act}' \) with \( (s'_k, a'_k, s'_{k+1}) \in T' \) satisfying \( (s_{k+1}, s'_{k+1}) \in R \).

We will construct the discrete abstraction \( M_d \) such that \( M_d \) simulates \( M_c \). Intuitively, this means that the discrete model \( M_d \) can express every behavior of \( M_c \) with respect to the specification.

To construct the discrete abstraction, note that an RTL formula \( \phi \) can be used to construct convex polytopic partitions
such that the same predicates hold for all continuous states $x_k \in \mathcal{X}$. An example of such partitions is shown in Figure 3a. We use this property to construct a discrete abstraction $M_d$ which simulates $M_c$, as follows:

1) Construct a finite set of polytopes $P$ representing the state formulas $\phi \in cl(\phi)$, where each polytope $P \in P$ represent a state space such that $L_c(x) = P$ for all $x \in \mathcal{X}$, and $P$ is a set of symbols. Algorithm 1 describes how to construct these polytopes.

2) Each polytope in $P$ corresponds to a state $s \in S_d$. We denote the operation that recovers the polytope of a state $s \in S_d$ by $\mathcal{P}(s) = \mathcal{P}$. The initial state $I_d = \{s_0\}$ is the state $s_0 \in S_d$ such that $\bar{x} \in \mathcal{P}(s_0)$. Moreover, the accepting states $F_d = S_f$. The labeling function $L_d : S_d \rightarrow \Sigma$ is defined such that $p \in L_d(s)$ if and only if $L_c(x) = P$ for all $x \in \mathcal{P}(s)$. Observe that $L_d(s) = P$ of $\mathcal{P}(s) = \mathcal{P}$.

3) Finally, $(\mathcal{P}_k, \mathcal{P}_{k+1}) \in T_d$ if and only if $\mathcal{P}_k$ and $\mathcal{P}_{k+1}$ are adjacent. We call two polytopes adjacent if they are equal or if their intersection is a polytope of dimension $n-1$. For example, a polytope with dimension $n-1$ is a line if $n = 2$ or a plane if $n = 3$.

Algorithm 1 Partition from State Formulas

```
Input: $M_c$

$P \leftarrow \emptyset$;
for $\phi \in cl(\phi)$; do
  for $P_\phi \in toPolytopes(\phi)$; do
    $P_\phi \leftarrow P_\phi \cup P_\phi$;
  end do
  for $P_j \in P_\phi$; do
    $P' \leftarrow P' \cup P_j \cup \mathcal{P}_j$;
  end do
  $P \leftarrow P' ;$
end do

return $P$.
```

This procedure always generates a transition system $M_d$ which simulates $M_c$.

Proposition 1. Given an RTL formula $\phi$ and a transition system $M_c$, there exists a transition system $M_d$ and a simulation relation $R_d$ such that any run $\xi'_d$ of $M_d$ that satisfies the simulation relation $R_d$ for a run $\xi'_d$ that satisfies the formula $\phi$ also satisfies the formula $\phi$, i.e., $\xi'_d \models \phi$ if and only if $\xi'_d \models \phi$ for all $k \in \mathbb{N}$.

Proof. To prove the existence of $M_d$ and $R_d$ by construction. Using the proposed abstraction, the initial state contains the initial state, i.e., $\bar{x} \in \mathcal{P}(s_0)$. Hence, the condition 1 of Definition 6 is satisfied. We also define $F_d = S_f$ such that condition 2 of Definition 6 is satisfied. By construction, the labeling function $L_d : S_d \rightarrow \Sigma^*$ satisfies the condition 3 of Definition 6. Under Assumption 1, the relation $T_d$ satisfies condition 4 of Definition 6. Finally, by condition 3 of Definition 6 $\xi'_d \models \phi$ only if $\xi_d \models \phi$ and $(x_k, s_k) \in R_d$ for all $k \in \mathbb{N}$.

Example 2. As a motivating example, consider a double integrator in $\mathbb{R}^2$ with a sampling time of 1s (i.e., $\bar{x} = u$ where $x$ and $v = \bar{x}$ are state variables $x = [x, v]^T$ and $u$ is the input variable). The system starts at $\bar{x} = [1, -5.5]^T$ and the input is bounded, i.e., $|u| \leq 2$. This problem is inspired by [7] Example 11.5. The system must avoid a forbidden region in state space, visit one of two regions of interest, and reach a target, as illustrated in Figure 3a.

We define 18 atomic propositions which specify predicates representing unsafe states $a$ (blue region), a target $b$ (red region), and areas of interest $c$ (yellow regions). The specification can be written as $\phi = \square((\neg a U b) \land (\neg b U c))$. The first part of this formula $(\neg a U b)$ ensures that for all times before reaching the target $b$, the unsafe state $a$ is avoided. The second part of the formula $(\neg b U c)$ specifies that region $c$ must be visited before region $b$. We choose this example to illustrate our approach because it considers an underactuated system with an unbounded RTL formula. To the best of our knowledge, no existing trajectory synthesis algorithm from RTL specifications can solve this problem with provable soundness and completeness.

Algorithm 1 starts with the workspace and the specification, and generates the discrete abstraction $M_d$. The associated polytopic partitions are shown in Figure 3a while the Kripke structure $M_d$ is illustrated in Figure 3b.

B. RTL equivalent Kripke Structure

Instead of passing a single satisfying run to the continuous planning, we construct a set of RTL language equivalent runs in the form of a Kripke structure. In the continuous planning, this structure essentially defines a sequence of convex constraints, which, if satisfied, guarantees the specification $\phi$.

Definition 7. An RTL equivalent Kripke structure $M'$ from a run $\xi'_d$ is a Kripke structure where every run of this structure satisfies the same RTL formulas that the run $\xi'_d$ satisfies. This means: $\xi'_d \models \phi$ if and only if $\xi'_d \models \phi$ for all runs $\xi'_d$ of $M'$.

We illustrate the process that constructs an RTL equivalent Kripke structure $M'$ from a run $\xi'_d$ in Algorithm 2. If the loop exists, $M'$ is a fair Kripke structure, meaning that it generates infinite runs with a loop. Intuitively, a dynamical system may take more time to pass through the polytopic constraints of a discrete plan. Thus, we construct a Kripke structure that represent these longer runs but preserving the RTL equivalence.

Example 3. Consider the system of Example 2 again. A satisfying run is $\xi'_d = s_1(s_2 s_3 s_4 s_5 s_6 s_7 s_8 s_9 s_{10})$. Fig. 4 shows a graphical representation of $M'$ for this run. First, note that the labelling function $L' : S' \rightarrow S_d$ maps each state to a state from the abstraction $M_d$. At each step of Algorithm 2, we generate a new state $s'$ that has a self-loop and back and forward transitions. When there is a loop, we introduce proxy states $(s_2', s_4', \ldots, s_8')$ in order to visit the accepting state $s_9'$ infinitely often.

We can prove that Algorithm 2 is sound and complete.
Algorithm 2 Construct an RTL equivalent Kripke structure $M'$.

Input: $\xi_d = s_0 \ldots s_{L-1} (s_L \ldots s_K)^\omega, M_d$

Output: $M' = (S', T', I', \ell', S, F')$

1: $i \leftarrow 0$; $S' \leftarrow s_0$; $T' \leftarrow \{(s_i', s_j')\}$; $I_p \leftarrow s_i'$; $L'(s_i') \leftarrow s_0$
2: for $k = 1$ to $K - 1$ do
3: $i \leftarrow i + 1$; $S' \leftarrow s_i'$; $L'(s_i') \leftarrow s_k$
4: $T' \leftarrow \{(s_i', s_j'), (s_{i-1}', s_i'), (s_i', s_{i+1}')\}$
5: if $k = L$ then
6: $i \leftarrow i + 1$; $S' \leftarrow s_i'$; $L'(s_i') \leftarrow s_k$
7: $i \leftarrow i + 1$; $S' \leftarrow s_i'$; $L'(s_i') \leftarrow s_k$
8: if $L < K$ then
9: $i \leftarrow i + 1$; $S' \leftarrow s_i'$; $L'(s_i') \leftarrow s_k$
10: $T' \leftarrow \{(s_i', s_{i-1}'), (s_{i-1}', s_i'), (s_i', s_{i+1}'), (s_{i+1}', s_i')\}$
11: $F' \leftarrow s_i'$;

Fig. 3. Graphical representation of the illustrative example. (a) The workspace after the abstraction and a solution for $x = [1 \ldots 5.5]^T$. (b) A graphical representation of the Kripke structure $M_d$ that abstracts $M_d$.

Fig. 4. Graphical representation of the construction of an RTL equivalent Kripke structure $M'$.

Proposition 2. Algorithm 2 constructs a Kripke structure $M'$ if and only if there exists an RTL Kripke structure $M'$ from the discrete abstraction run $\xi_d$.

Proof. We will prove by structural induction. We define the path constructors that generate paths of $M'$ recursively: $\text{repeat}_\text{at}(i, \xi'_\text{parent})$ and $\text{backward}_\text{at}(i, \xi'_\text{parent})$. Given a run $\xi'_\text{parent} = (s_0')^0 \ldots (s_K')^k$ of $M'$ with length $\sum_{i=0}^{K} \ell'_i$, these operators allow us to construct another run of $\xi'_\text{child} = (s_0')^0 \ldots (s_K')^k$ of $M'$ with length $\sum_{i=0}^{K} \ell'_i$ which is longer, i.e., $\sum_{i=0}^{K} \ell'_i > \sum_{i=0}^{K} \ell'_i$, as follows:

1. $\text{repeat}_\text{at}(i, \xi'_\text{parent}) = \xi'_\text{child} \in (S')^* : \xi'_\text{child} = (s_0')^0 \ldots (s_i')^i \ldots (s_K')^k$;
2. $\text{backward}_\text{at}(i, \xi'_\text{parent}) = \xi'_\text{child} \in (S')^* : \xi'_\text{child} = (s_0')^0 \ldots (s_i')^i \ldots (s_{i-1}')^1 \ldots (s_K')^k$ if $(s_i', s_{i-1}') \in T'$

These operators generate all possible runs of $M'$ because they represent the transitions in $T'$ generated by Algorithm 2. In particular, the operator $\text{repeat}_\text{at}(i, \xi'_\text{parent})$ represents the transitions $(s_i', s_{i+1}') \in T'$, and the operator $\text{backward}_\text{at}(i, \xi'_\text{parent})$ the transitions $(s_i', s_{i-1}') \in T'$ when $i \neq j$. In other words, the combination of these operators over the shortest run of $M'$ generates all possible runs of this structure.

Now, we can start the structural induction proof. First, note that the shortest run $\xi'_\text{child}$ of $M'$ has path equal to the path of the satisfying $\xi_d$ of $M_d$. As a result, $\xi'_\text{child} \models \phi$.

Next, we assume that the run $\xi'_\text{parent}$ satisfies the specification, i.e., $\xi'_\text{parent} \models \phi$. Any run generated by the operators $\text{repeat}_\text{at}(i, \xi'_\text{parent})$ and $\text{backward}_\text{at}(i, \xi'_\text{parent})$ satisfies the specification because the RTL semantics permits repetitions. For example, consider that $\xi'_\text{parent} \models \phi_1 \land \phi_2$ and $\xi'_\text{parent} \models k - 1 \phi_1$. As a result, if we apply the backward operator at instant $k'$, this means that that this formula is satisfied at instant $k \leq k' \leq k' + 2$ because $\xi'_\text{parent} \models k - 1 \phi_1$ and $\xi'_\text{parent} \models k + 2 \phi_2$. This holds for the release operator as well. Therefore, the proposition holds by structural induction.

Remark 2. Intuitively, this is analogous to oscillation behaviors.
exhibited by continuous dynamical systems. The discrete plan indicates regions that the continuous trajectory should evolve through. Sometimes we may need to revisit a region to drive the system trajectory to a goal region, which requires the backward operation.

C. Counterexamples

The discrete abstraction simulates the system; thus, this abstraction has runs that do not render valid runs of the system, which we denote as dynamically infeasible runs. So, we also identify an Irreducibly Inconsistent Set (IIS) [23] for Problem 1. An IIS defines an infeasible subset of constraints such that removing any one constraint renders the subset feasible. We call the constraints in this IIS counter-examples.

Definition 8. Given a feasibility problem with a set of constraints \( \mathcal{C} \), an Irreducibly Inconsistent Set \( \mathcal{I} \) is a subset \( \mathcal{I} \subseteq \mathcal{F} \) such that: (1) the feasibility problem with the constraint set \( \mathcal{I} \) is infeasible; and (2) \( \forall c \in \mathcal{F}, \) the feasibility problem with constraint set \( \mathcal{F} \setminus \{c\} \) is feasible.

Similarly to the discrete plan, we represent these counterexamples as an RTL equivalent Kripke Structure. When we identify an abstraction run that is not feasible, we construct a Kripke structure representing its RTL equivalent runs. In summary, this structure runs are all runs that we can generate using the repeat and backward operators from Proposition 2.

We denote the set of counter-examples found so far as \( M_{\text{cex}} \), where each counter-example \( M_{cex}^i \in M_{\text{cex}} \) is a Kripke structure. We construct this Kripke structure using Algorithm 2 in the same way that we construct discrete plans. Therefore, we can discard all unfeasible runs by the product of the discrete abstraction \( M_d \) and the complement of the counter-examples \( M_{\text{cex}}^i \), i.e., \( M_d \times M_{\text{cex}}^i \).

Example 4. Consider the system of Example 2 but starting at \( \bar{x} = [-4, -8]^\top \) instead. Since the input is bounded, we will not be able to generate a run \( \bar{x}_d \) from \( s_{13} \) to \( s_2 \) without passing through \( s_1 \). Consequently, any run of \( M_d \) with prefix \( s_{13}(s_{13})^*s_2 \), shown in Fig. 5, is dynamically infeasible. We pass the shortest run of the prefix \( s_{13}(s_{13})^*s_2 \) (i.e., \( s_{13}s_2 \)) to Algorithm 2, Fig. 5a and 5b. Then, we add a suffix to this Kripke structure to accept all prefixes, i.e., \( s_{13}(s_{13})^*s_2(s_1 + \cdots + s_{15})^* \), Fig. 5c.

![Fig. 5. Example of a counter-example construction for the discrete plan.](image)

D. Encoding

Given the abstract system \( M_d \), the specification \( \varphi \), and a Kripke structure \( M_{\text{cex}} \) representing counterexamples, we encode the problem of finding a satisfying run of \( M_d \) as a Boolean satisfiability problem using techniques presented in [29], [30]. To do this, we separate the encoding into three components: the abstract system encoding \( \{(M_d)^k\}_k \), the specification encoding \( \{(\varphi)^k\}_k \), and the counter-example encoding \( \{(M_{\text{cex}})^k\}_k \).

These encodings can be combined into one Boolean formula

\[
\{(M_d, \varphi, M_{\text{cex}})^k\} := \{(M_d)^k \land \{(\varphi)^k\} \land \{(M_{\text{cex}})^k\}_k \}
\]

(2)

Particularly, these encodings are indexed by the bound \( K \). The basic idea is to search for short (small \( K \)) solutions first, then incrementally increase this bound until we reach a value, after which the specification \( \varphi \) is unsatisfiable. This iterative structure allows us to harness incremental solvers like Z3 [27] to efficiently find satisfying evaluations for specifications with a high \( K \).

The encodings for the abstract system, specification, and counterexamples are presented below.

Remark 3. The linear nature of this encoding means that the number of constraints increases linearly with \( K \). Additionally, its incremental nature allows incremental SAT/SMT solvers to use learned clauses from previous iterations, drastically improving performance.

1) Abstract System: The abstract system \( M_d \) can be represented symbolically as a Boolean formula that captures finite paths with length \( K \),

\[
\{(M_d)^k\}_k := \bigwedge_{i=0}^{K-1} \bigwedge_{i=0}^{K-1} \bigwedge_{j=0}^{K-1} T(\hat{s}_i, \hat{s}_{i+1}) \land F(\hat{s}_K),
\]

where \( \hat{s}_k \) models states \( s_k \) as bit vectors, \( I(\hat{s}_0) \) and \( F(\hat{s}_K) \) are Boolean formulas ensuring that the state \( \hat{s}_0 \) is the one of the initial states and \( \hat{s}_K \) is one of accepting states, and \( T(\hat{s}_i, \hat{s}_{i+1}) \) encodes the requirements of a transition from \( s_i \) to \( s_{i+1} \). We also define the transformation \( \chi(\hat{s}_k) = s_k \) which returns the state \( s_k \) in \( M_d \) corresponding to an evaluation of \( \hat{s}_k \).

Example 5. Considering the system and specification of Example 2, the states can be abstracted with a vector of four bits, i.e., \( \hat{s}_k \in \{0, 1, \ldots, 15\} \) and \( \chi(\hat{s}_k) = s_k \). The initial conditions are encoded by the formula \( I(\hat{s}_0) = \hat{s}_0 = 12 \). Since, in this example, we do not restrict the final state, the formula that represents the accepting states is trivially true. Finally, the transitions at instant \( k \) are encoded by a formula that is the disjunction of sub-formulas representing each valid transition \( (s_i, s_j) \) as follows:

\[
\hat{s}_k = \chi(\hat{s}_k) \land \hat{s}_{k+1} = \chi^{-1}(s_j).
\]

2) Specification: Following the RTL semantics, we encode the specification \( \varphi \) recursively by considering formula variables \( \{(\varphi')^k\}_k \). For every subformula \( \varphi' \in cl(\varphi) \), a variable \( \{(\varphi')^k\}_k \) is introduced and interpreted as true if and only if \( \bar{x}_d \models \chi(\varphi') \).

The Boolean encoding of propositional operators in RTL formulas for the instants \( 0 \leq k \leq 3 \) is as follows:

- \( \{(p)^k\}_k \) \begin{align*}
&\forall s_t \in (s_k \in S; \forall p \in cl(p)) \quad \hat{s}_k = \chi(s_t), \\
&\{(p_1 \land \varphi_2)^k\}_k \leftrightarrow \{(\varphi_1)^k \land \{(\varphi_2)^k\}_k, \\
&\{(p_1 \lor \varphi_2)^k\}_k \leftrightarrow \{(\varphi_1)^k \lor \{(\varphi_2)^k\}_k, \\
&\{(\lnot p_1)^k\}_k \leftrightarrow \{(\varphi_1)^k \land \{(\varphi_2)^k\}_k.
\end{align*}
For subformulas with temporal operators, we refer to future
formula variables to ensure the temporal behavior. Thus, we
have,
- \(|(φ_k U φ_{k+1})_k \rightarrow ((φ_{k+1})_k \land (φ_k U φ_{k+1})_{k+1})|.
- \(|(φ_k R φ_{k+1})_k \rightarrow ((φ_{k+1})_k \land (φ_k R φ_{k+1})_{k+1})|.

Example 6. Consider the formula \(x \leq 1 U X \leq 1\). The encoding
for a length \(K\) is,
\[
\left[\begin{array}{l}
(x \leq 1 U x \geq 1)_{k} = \\
\left((x \leq 1)_{k} \leftrightarrow p_{1,k} \land (x \geq 1)_{k} \leftrightarrow p_{2,k}\right) \land \\
K \bigwedge_{k=0} \left((x \leq 1 U x \geq 1)_{k} \leftrightarrow \left((x \geq 1)_{k} \land \left((x \leq 1 U x \geq 1)_{k+1}\right)\right)\right),
\end{array}\right]
\]
where \(p_{1,k}\) are Boolean variables encoding \(p_{1} \in I_{d}(\xi_{k})\). ■

We still need to take into account the possible infinite behavior
encoded in the specification. As mentioned above, we
can model infinite behavior as a finite run with a loop. For
this reason, we introduce Boolean variables \(l_{k}\), which are true
only if the loop starts at instant \(k\), \(In_{k}\), which holds only
if the instant \(k\) is within a loop, and \(Exists_{k}\), which holds if
and only if a loop exists. Furthermore, we divide the loop-related
constraints as the base, \(K\)-independent (i.e., \(1 \leq k \leq K\)), and \(K\)-
dependent constraints. We assert the base constraint only once,
at the initialization. At each bound increment, we delete the
old \(K\)-dependent constraint assertions and assert \(K\)-independent
and \(K\)-dependent constraints. This procedure allows us to keep
most of the constraints between steps and harness incremental
solver techniques.

Consequently, the following constraint \(|(Loop)_{K}|\) defines a
loop:
- **Base**: \(l_{0} \leftrightarrow \bot \), and \(In_{0} \leftrightarrow \bot\),
- \(1 \leq k \leq K\): \(l_{k} \leftrightarrow s_{k} \land \neg l_{k} \land In_{k} \land \neg In_{k-1} \land l_{k-1} \rightarrow \neg l_{k}\),
- **\(K\)-dependent**: \(Exists_{K-1} l_{k}\), and \(s_{k} \leftrightarrow s_{k+1}\).

Note that we introduced a proxy variable \(s_{E}\) to separate the \(K\)-
dependent constraints.

Additionally, to compensate for change in the bound \(K\), we
define a set of constraints for the last state \(|(LastState)_{E}|\). For
each subformula \(φ' \in cl(φ)\), we have,
- **Base**: \(\neg Exists_{\rightarrow} (|φ'|_{E} \leftrightarrow \bot)\),
- \(1 \leq k \leq K\): \(l_{k} \leftrightarrow |(φ')_{E}| \leftrightarrow |(φ')_{E}||\),
- **\(K\)-dependent**: \(|(φ')_{E} \leftrightarrow |(φ')_{E}||\) and \(|(φ')_{E} \leftrightarrow |(φ')_{E}||_{k+1}\).

The encoding above allows the case where \(|(φ_1 U φ_{2})_k|\) is true
for all indices of the loop even if \(|(φ_2)|\) is not at any index
of the loop, which violates the RTL semantics. As a result, we
introduce the eventually constraints \(|(EventRTL)_{K}|\) and its
auxiliary formula variables \(|(φ_2)_{E}|\) and \(|(φ_1 U φ_{2})_E|\) such that.

- **Base**: \(|((φ_1 U φ_{2})_E) \rightarrow |(φ_2)_{E}|\), and \(|(φ_2)_{E}|_0 \leftrightarrow \bot\),
- \(1 \leq k \leq K\): \(|(φ_2)_{E} \leftrightarrow |(φ_2)_{E}||_{k-1}|\), or \(|(In_{k} \land |(φ_2)|_{k})|\),
- **\(K\)-dependent**: \(|(φ_2)_{E} \leftrightarrow |(φ_2)_{E}||_K|\).

Similarly, for subformulas \(φ_1 R φ_{2} \in cl(φ)\), we have,
- **Base**: \(|((φ_1 R φ_{2})_E) \rightarrow |(φ_2)_{E}|\), and \(|(φ_2)_{E}|_0 \leftrightarrow \top\),
- \(1 \leq k \leq K\): \(|(φ_2)_{E} \leftrightarrow |(φ_2)_{E}||_{k-1}|\), and \(|(¬In_{k} \lor |(φ_2)|_{k})|\),
- **\(K\)-dependent**: \(|(φ_2)_{E} \leftrightarrow |(φ_2)_{E}||_K|\).

Putting these pieces together, the resulting Boolean formula
that encodes an RTL formula is
\[
|(φ)|_K := |(Loop)|_K \land |(LastState)|_K \land \\
|(EventRTL)|_K \land |(φ)|_0.
\]

Example 7. Consider the specification from Example 2. First,
observe that the temporal operator \(\Box φ_1\) is equivalent to a
release formula \(\Box R φ_1\), where \(φ_1 = ((¬a U b) \land (¬b U c))\). From
the encoding \(|(\Box R φ)|_K\), it follows that \(|(\Box R φ)|_K\) should
always hold true. Then, because of the encoding \(|(LastState)|_K\),
there must always exists a loop. An example of a satisfying
run is \(s_{13}(s_{12}s_{14}s_{5}s_{4}s_{14})^0\). Notice that there is a loop that will
enforce that the states \(s_{12}\) and \(s_4\) (and \(b\) and \(b\)) will always be visited
infinitely often. Note that invalid runs have one of the following characteristics:
(i) they do not have a loop, (ii) they have a loop but do not visit \(s_4\) and one of the states \(s_{12}\) and \(s_{12}\) inside the
loop, (iii) or they visit \(s_1\) or \(s_3\).

3) Counter-Examples: We can encode these counterexamples
in the same way as we encoded the Kripke structure \(M_d\)
of the abstract system: \(|(M_{ce})|_K = \neg \Lambda_{v=1}^{K}|(M_{ce})|_K\), where
\(|(M_{ce})|_K = I_{ce}(s_0) \land \Lambda_{v=0}^{K-1}T_{ce}(s_1, s_{1+v}) \land F_{ce}(s_K)\).

E. Soundness

We establish the correctness of the proposed encoding by the
following proposition:

**Proposition 3.** Given a Kripke structure \(M_d\), an RTL formula
\(φ\), and a set of counterexamples \(M_{ce}\), a run \(ξ\) of \(M_d\) satisfies the
specification (i.e., \(ξ_{E} = φ\)) if there exists \(K \in N\) such that
the encoding \(|(M_d, φ, M_{ce}, K)|\) is satisfiable.

**Proof.** We first prove that if there exists a bound \(K\) such that
\(|(M_d, φ, M_{ce}, K)|\) is satisfiable, then \(ξ_{E} = φ\). Then, the
proposition follows because if the run \(ξ_{E}\) satisfies the specification with
bound \(K\), then it satisfies the specification, i.e., \(ξ_{E} = φ\) implies
that \(ξ_{E} = φ\). The sufficiency of checking runs with only \(K\) steps
follows from the loop structure described in Section [17].

It is easily seen that the constraint \(|(M_d, φ, M_{ce}, K)|\) encodes
all valid finite runs of model \(M_d \setminus M_{ce}\) of length \(K\). Moreover,
the loop constraints \(|(Loop)|_K\) ensure two cases of satisfying runs:
(a) when a loop exists, there will exist an unique index \(1 \leq j \leq K\) such that \(s_j = s_j\) determines when the loop starts, and (b) when there is no loop, the run \(ξ_{E}\) is a
prefix of \(M_d\).

Now, we prove that for any subformula \(φ' \in cl(φ)\), \(ξ_{E} = φ'\)
if \(|(φ')_E|_k\) is true. It is easy to see that the claim holds for the
cases where \(φ\) is an atomic proposition. Moreover, the claim
also trivially holds by induction when \(φ\) is a boolean function
of atomic propositions. We still need to prove the claim for
formulas with temporal operators. In fact, the encoding follows the
one-step identity of temporal operators \(U\) and \(R\). Specifically,
\(|(φ_1 U φ_{2})_k|_k\) is true if either (i) \(∃K' \in \[K..K-1\]\) s.t. \(|(φ_2)|_{K'}\) is true
and $\left\{ (\varphi_i)^l \mid \varphi \right\}$ is true for all $k \leq k'' \leq k'$, or (ii) the proxy variable $\left\{ (\varphi_i U \varphi_j)^l_{k+1} \right\}$ is true and $\left\{ (\varphi_i)^l \right\}$ is true for all $k \leq k'' \leq K$. When no loop exists, $\left\{ (\varphi)^l \right\}_{k+1}$ is false; consequently, the claim holds by induction, when the claim holds for the subformulas. Now, when the loop exists starting at index $1 \leq j \leq K$, the constraint $\left\{ (\text{LastState})^l \right\}$ ensures that the proxy variables $\left\{ (\varphi_j)^l \right\}_{k+1} \equiv (\varphi_j)^l_j$. Moreover, the constraint $\left\{ (\text{EventRTL})^l \right\}$ ensures that there exists $j \leq k \leq K$ s.t. $\left\{ (\varphi_j)^l \right\}$ is true only if there exists $j \leq k' \leq K$ s.t. $\left\{ (\varphi_j)^l \right\}$ is true. Reciprocally, the encoding of the temporal operator $\left\{ (\varphi_i U \varphi_j)^l \right\}$ ensures that $\varphi_i$ holds until $k'$, i.e., $\left\{ (\varphi_i)^l \right\}$ holds true for $0 \leq k' < K$. Thus, the claim holds by induction. The same reasoning applied to the case of the temporal operator $R$. Consequently, any subformula $\varphi' \in c(l(\varphi))$. $\xi_d \models \varphi'$ if $\left\{ (\varphi')^l \right\}_K$ is true. Therefore, there exists a run $\xi$ of $M_d \setminus \{ \varphi \}$ such that $\xi_d \models \varphi'$ if there exists $k$ such that $\left\{ (M_d, \varphi, \varphi_{\text{cex}} K) \right\}$ holds.

F. Completeness

The proposed incremental encoding allows us to use incremental SAT/SMT solvers and determine when to stop increasing the bound $K$. In this regard, our procedure for completeness is based on an inductive procedure proposed in [30].

The main idea is to check if a longer discrete run that satisfies the specification may still exist by removing the $K$-dependent constraints from the encoding. The longest initialized loop-free run, i.e., a run where the initial state of the run is an initial state of the system and all states are distinct, that satisfies the specification is called the recursive diameter and is used as the upper bound for the completeness threshold. We use a straightforward encoding of this loop-free run predicate, whose size is quadratic with the bound (i.e., $O(K^2)$) [30].

First, we define a completeness formula $\left\{ (M_d, \varphi, \varphi_{\text{cex}} K) \right\}$ which consists of exactly the encoding $\left\{ (M_d, \varphi, \varphi_{\text{cex}} K) \right\}$ with all $K$-dependent constraints removed. Intuitively, $\left\{ (M_d, \varphi, \varphi_{\text{cex}} K) \right\}$ is satisfied only if there exists runs $\xi_d$ of $M_d$ that satisfy the specification with length $K$ or longer bounds. Moreover, we propose a simple run formula which is satisfiable for only initialised loop-free runs. Let $\left\{ (\varphi)^l \right\}$ be a bit vector of values of all formula variables $\left\{ (\varphi)^l \right\}_k$, the simple run predicate is defined as follows:

$\left\{ (\text{SR})^l \right\} \triangleq \bigwedge_{0 \leq i \leq j \leq K} \big( \xi_i \neq \xi_j \lor \neg\text{In}_i \lor \neg\text{In}_j \big) \land \left\{ (\varphi)^l \right\}$.

Now, we prove the completeness of this encoding. Note that as an intermediate result, we determine some $K$ above which increasing it does not change the satisfaction.

Proposition 4. Given a Kripke structure $M_d$, an RTL formula $\varphi$, and a set of counterexamples $\varphi_{\text{cex}}$, there is no run $\xi_d$ in $M_d$ by a set of counterexamples that satisfies the specification if for some $K \geq 0$ $\left\{ (M_d, \varphi, \varphi_{\text{cex}} K) \right\}$ is unsatisfiable and either $K = 0$ or $\left\{ (M_d, \varphi, \varphi_{\text{cex}} K - 1) \right\}$ is unsatisfiable.

Proof. First, note that new counterexamples will not change the satisfiability of past checking iterations. Moreover, as mentioned above, $\left\{ (M_d, \varphi, \varphi_{\text{cex}} K') \right\}$ is unsatisfiable implies that $\left\{ (M_d, \varphi, \varphi_{\text{cex}} K) \right\}$ is unsatisfiable for all $K \geq K'$.

Consider that for some $K \geq 0$ $\left\{ (M_d, \varphi, \varphi_{\text{cex}} K) \right\}$ and $\left\{ (\text{SR})^l \right\}$ is unsatisfiable and either $\left\{ (M_d, \varphi, \varphi_{\text{cex}} K - 1) \right\}$ is unsatisfiable or $K = 0$. If $K = 0$, it implies that $\left\{ (M_d, \varphi, \varphi_{\text{cex}} K) \right\}$ is unsatisfiable because $\left\{ (\text{SR})^l \right\}$ is empty; thus, $\left\{ (M_d, \varphi, \varphi_{\text{cex}} K) \right\}$ is unsatisfiable for all $K \geq 0$. Now, notice that if there exist $K$ such that $\left\{ (M_d, \varphi, \varphi_{\text{cex}} K) \right\}$ is satisfiable, then $\left\{ (M_d, \varphi, \varphi_{\text{cex}} K) \right\}$ is satisfiable for $K \leq k$. Thus, if $\left\{ (M_d, \varphi, \varphi_{\text{cex}} K) \right\}$ is unsatisfiable for all $k \geq 0$. Using Proposition [3] we conclude that there is no run $\xi_d$ in $M_d$ without counterexamples such that satisfies the specification.

VI. CONTINUOUS MOTION PLANNING

The continuous planning checks if there exists a dynamically feasible run of the system that satisfies a discrete plan. We decode a satisfying run of $M_d$ from the discrete planning and construct an RTL equivalent Kripke structure $M_p$ from this run.

As described above, the discrete plan has infinite runs. As a result, we need a stop criteria to decide when to decide that the tree has no solution and generate a counter-example. To do so, we harness the simulation relation between the continuous system $M_c$ and the discrete system $M_d$. The basic idea is to take a particular run from the discrete plan and try to find a corresponding run in $M_c$. If we cannot find such a run, we return a counter-example describing runs of $M_d$ to discard in future plans. Unlike counter-examples in traditional model checking, which prove that a system does not always satisfy the specification, this counter-example proves that there exists a prefix of the plan $M_d$ which cannot satisfy the dynamic constraints.

Similarly to the discrete planning, the continuous planning searches for a periodic run of the form $\xi_c^H N = x_0 \overset{u_0}{\longrightarrow} x_1 \overset{u_1}{\longrightarrow} \ldots x_{N-1} \overset{u_{N-1}}{\longrightarrow} x_N \overset{\omega}{\longrightarrow} \ldots$ This structure, commonly used in to address infinite behavior of temporal logic specifications [9], allows us to consider infinite runs with a finite representation. This requirement is necessary when computing a trajectory using LP solvers, and it is frequently used in trajectory synthesis approaches [9].

Now, we formally define the notion of dynamic feasibility.

Definition 9. A run $\xi_c$ of the continuous system $M_c$ is dynamically feasible run of the discrete plan $M_p$, denoted as $\xi_c \in M_p$, if and only if there exists a run $\xi_p = s_{0_p} \ldots s_{N_p} \ldots s_{H_p}$ of the discrete plan $M_p$ with bound $H$ and loop starting at $N$ such that the following problem is feasible:

$$\text{find } \xi_c$$

$$\text{s.t. } x_0 = \bar{x}, x_H = x_{N-1}, \text{ and } \forall k \geq 0 : x_{k+1} \in \text{Post}_g(x_k, u_k), \text{ and } x_{k+1} \in \mathcal{P}(L_p(s_{p,k+1})),$$

where:

- $\text{Post}_g(x_k, u_k) := \{ x \in \mathbb{R}^n : x = Ax_k + Bu_k + \delta, u_k \in \mathcal{U} \}$
- $x_k \in \mathcal{P}(L_p(s_{p,k+1}))$ denotes that the continuous state $x_k$ is contained in the polytope corresponding to the discrete state that is the label of the discrete plan state at instant $k$, i.e., $L_p(s_{p,k+1}) \in S_d$. 

The constraint $x_k \in \mathcal{P}(L_p(s_{p,k+1}))$ enforces that the continuous state resides in a corresponding polytope, which corresponds to a step of a run in the discrete plan $M_p$. We assume that inputs are bounded; thus, $u_k \in \mathcal{U}$ corresponds to the input constraints. Dynamic feasibility is enforced by the constraint $x_{k+1} \in Post_\delta(x_k, u_k)$.

Remark 4. These feasibility constraints are written in terms of $\delta$-completeness \cite{31}. This takes into consideration the fact that some systems exhibit behavior (Zeno behavior) in which improvement towards the satisfaction of given constraints is arbitrarily slow. Setting some positive but arbitrarily small value of $\delta$ allows us to find finite-length satisfying runs for such cases. Note, though, that $\delta$ can be arbitrarily small to correspond to a negligible value.

Example 8. These feasibility constraints are written in terms of $\delta$-completeness \cite{31}. This completeness considers that some systems exhibit behavior (Zeno behavior) in which improvement towards the satisfaction of given constraints is arbitrarily slow. Setting some positive but arbitrarily small value of $\delta$ allows us to find finite-length satisfying runs for such cases. Note, though, that $\delta$ can be arbitrarily small to correspond to a negligible value.

In summary, there are two main challenges to solving the feasibility problem \cite{3}. First, we need to determine when no feasible run $\xi_c \in M_p$ exists. Second, if no such run exists, we need to find an appropriate counterexample $M'_{\text{ex}}$.

A. Dynamical Feasibility

We address these challenges by identifying necessary and sufficient conditions for a feasible run $\xi_c \in M_p$. The basic idea is to check the feasibility of a run by incrementally adding constraints.

We use the discrete plan run in a particular form to define the necessary and sufficient conditions. We can represent a discrete plan in the form of $\xi_p = (s_p^0)^0(s_p^1)^{1}\ldots(s_{p-1})^{l-1}(s_p^P)^L \ldots(s_k)^{0}$, where each two adjacent states $s_i^p$ and $s_{i+1}^p$ are distinct (i.e., $s_i^p \neq s_{i+1}^p$). This form allows us to see the conditions as part of a polytopic tunnel in the system workspace, i.e., $\mathcal{P}(L_p(s_0^0)) \ldots \mathcal{P}(L_p(s_{P-1}^P)) \ldots \mathcal{P}(L_p(s_k^0))$.

Furthermore, we call a segment of this run the sequence $s_{i-1}(s_i)^{l_i}s_{i+1}$ (or $s_i(s_{i+1})^{l_{i+1}}$) if $i = 0$, where $s_i = L_p(s_i^0)$ is the abstraction state $s_i$ that labels the plan state $s_i^p$.

First, there exists a dynamically feasible run $\xi_c \in M_p$ only if all prefixes of the corresponding run $\xi_p$ are also feasible. Formally, we denote a prefix of a run $\xi_p$ of $M_p$ as $\text{Prefix} (\xi_p, P) := \{(s_i^0)^0(s_i^1)^{1}\ldots(s_{P-1})^{l-1}(s_p^P)^L \ldots(s_k)^{0}: l_i > 0 \text{ for } i = 0, \ldots, P - 1\}$. As a result, a prefix is feasible if \cite{3} is feasible for this prefix dropping the loop constraints (i.e., $x_P = x_{P-1}$). Therefore, this is a necessary condition for the existence of a dynamically feasible run and is defined as follows.

Definition 10. A prefix $\text{Prefix}(\xi_p, P)$ is said to be feasible if and only if the solution of the following LP is less than $\delta$:

$$\begin{align*}
\max_{k \in [0, \ell-1]} \min_{u_k \in \mathbb{R}^m} & \quad ||x_{k+1} - Ax_k - Bu_k||_\infty \\
\text{s.t.} & \quad x_0 = \bar{x}, u_k \in \mathcal{U}, x_{k+1} \in \mathcal{P}(L_p(s_{p,k+1})).
\end{align*}$$

Next, if a prefix of the corresponding run $\xi_p$ is feasible, any segment $s_{i-1}(s_{i}^\ell)^s_{i+1}$ (or $s_i(s_{i+1})^\ell_{i+1}$) if $i = 0$ is said to be feasible in $\ell_i$ steps if and only if the solution of the following LP is less than $\delta$:

$$\begin{align*}
\max_{k \in [0, \ell_i-1]} \min_{u_k \in \mathbb{R}^m} & \quad ||x_{k+1} - Ax_k - Bu_k||_\infty \\
\text{s.t.} & \quad x_0 = \bar{x}, u_k \in \mathcal{U}, x_k \in \mathcal{P}(s_{i+1}), \\
& \quad x_{k+1} \in \mathcal{P}(s_i) \text{ if } 0 < k < \ell_i, u_k \in \mathcal{U},
\end{align*}$$

Finally, if a segment $s_{i-1}(s_{i}^\ell)^s_{i+1}$ (or $s_i(s_{i+1})^\ell_{i+1}$) if $i = 0$ is feasible only if it is reachable. Intuitively, the reachability drop the constraint on intermediate system states to satisfies the plan state $s_i$ (i.e., we do not require that $x_i \in \mathcal{P}(s_i)$ for $0 < k < \ell_i$). We formally define the reachability as follows.

Definition 11. A segment $s_{i-1}(s_{i}^\ell)^s_{i+1}$ (or $s_i(s_{i+1})^\ell_{i+1}$) if $i = 0$ is said to be reachable within $\ell_i$ steps if and only if the solution of the following LP is negative:

$$\begin{align*}
\max_{k \in [0, \ell_i-1]} \min_{u_k \in \mathbb{R}^m} & \quad \min_{A_y, b_y} \quad A_y u_k - b_y \\
\text{s.t.} & \quad x_0 \in \mathcal{H}, x_{\ell_i} \in \mathcal{P}(s_{i+1}), \\
& \quad x_{\ell_i} - A_x x_0 = A^{l_i-1}B u_0 + A^{l_i-2}B u_1 + \ldots + B u_{\ell_i},
\end{align*}$$

Remark 5. Note that the constraints of (10), (3) and (6) are linear and max-min problems can be encoded as LP problems using slack variables \cite{32}. Additionally, notice that the objective of this problem is reduce the distance of the variables to the half-spaces of a polytope. As consequence, when the solution is not positive (or greater than $\delta$), all variables in the solution are inside this polytope. Furthermore, we highlight that $\delta$ can be chosen arbitrarily small to correspond to a negligible value.

We now present necessary and sufficient conditions for the existence of a feasible run $\xi_c \in M_p$.

Proposition 5. Given a discrete plan $M_p$ and a dynamical system $M_c$, there exists a feasible run $\xi_c \in M_p$ if and only if there exists a discrete plan run $\xi_p$ such that:

1) for $0 \leq i < K$,
   a) each segment $s_{i-1}(s_i^\ell)^{s}_{i+1}$ is reachable,
   b) each segment $s_i^{l_i}(s_{i+1})^{\ell_{i+1}}$ is feasible,
2) for $0 < P \leq K$, any prefix $\text{Prefix}(\xi_p, P)$ is feasible, and
3) Problem (2) is feasible.

Proof. (⇒) If there exists a feasible run $\xi_c \in M_p$, then, by Definition 9, there exists a run $\xi_p$, such that $x_k \in \mathcal{P}(P(s_p, k))$ for all $k \geq 0$. Since the run $\xi_c$ satisfies the dynamical constraints of $M_c$, it proves any conditions for $\xi_p$. We will prove by contradiction. Assume that there exists a run $\xi_p$ such that all conditions hold but there exists no feasible run $\xi_c \in M_p$. However, all segments and prefixes of the run $\xi_p$ are feasible; thus, there must exist a $\xi_c$, that satisfies the Definition 9, which contradicts the assumption and proves the theorem.

B. Counter-example

The necessary and sufficient conditions for feasibility allow us to identify constraints of the IIS, i.e., the counter-examples. We use Algorithm 3 to extract these constraints from a discrete plan run $\xi_p$.

Algorithm 3 Feasibility Checking

Input: $\xi_p$, $M_c$, $\delta$, $\lambda$, $\gamma$, $P^*$

Output: $\xi_{\text{seg}}$, $\xi_{\text{cex}}$

1) for $i = 2$ to $k$ do
2) $P = i \cdot \text{seg}_{(s_p)} \in \mathcal{P}(s_p)$;
3) $\xi_{\text{seg}} := \text{seg}_{(s_p)}$;
4) $\lambda = \text{LP}(P)$ for $\xi_{\text{seg}}$;
5) if $\lambda$ is infeasible and $\ell'_{i-1} \geq n$ then $\xi_{\text{cex}} := \xi_{\text{seg}}$ return;
6) else if $\lambda$ is feasible then return;
7) else if $\lambda \geq \delta$ and $P^* = P$ then $\xi_{\text{cex}} := \xi_{\text{seg}}$ return;
8) else if $\lambda > \delta$ and $P < P^*$ then $\xi_{\text{cex}} := \xi_{\text{seg}}$ return;
9) else if $\lambda > \delta$ then $\lambda = \lambda$ return;
10) $\lambda = \text{LP}(P)$ for $\xi_{\text{seg}}$;
11) if $\lambda > \delta$ and $P^* = P$ then $\xi_{\text{cex}} := \xi_{\text{seg}}$ return;
12) else if $\lambda > \delta$ and $P < P^*$ then $\xi_{\text{cex}} := \xi_{\text{seg}}$ return;
13) else if $\lambda > \delta$ then $\lambda = \lambda$ return;
14) if $i < k$ then $\lambda = \text{LP}(P)$ for $\text{Prefix}(\xi_{\text{seg}}, P)$;
15) else $\lambda = \text{LP}(P)$ for $\text{Prefix}(\xi_{\text{seg}}, P)$ s.t. $x_{\text{seg}} = x_{\text{seg}+1}$;
16) if $\lambda \geq \delta$ and $P^* = P$ then $\xi_{\text{cex}} := \text{Prefix}(\xi_{\text{seg}}, P)$ return;
17) else if $\lambda > \delta$ and $P < P^*$ then $\xi_{\text{cex}} := \text{Prefix}(\xi_{\text{seg}}, P)$ return;
18) else if $\lambda > \delta$ then $\lambda = \lambda$ return;

We discuss this algorithm in the following proposition.

Proposition 6. Given a discrete plan $M_p$ and a dynamical system $M_c$, Algorithm 3 only returns counter-examples that are constraints of the IIS for a dynamically infeasible discrete plan $M_p$.

Proof. First, from Theorem 5.25, an unconstrained discrete-time linear control system is reachable if and only if $\text{LP}(P)$ for a segment $s_k'((s_k')^a s_k')$ is feasible for $\ell' \leq n$. Moreover, the reachable states of those unconstrained systems does not change by $\ell' \geq n$. Therefore, if increasing the length $\ell'_p$ the solution of $\text{LP}(P)$, the solution does not reduce, the segment $s_k'((s_k')^a s_k')$ is not feasible (lines 5 and 6). For this reason, from Proposition 5 it is a constraints of the IIS. Now, note that $\text{LP}(P)$ for the segment $s_k'((s_k')^a s_k')$ is not feasible if this segment is reachable. Second, if we increase $\ell'_p$, it increases the degree of freedom in the state space. Thus, it must reduce the solution of $\text{LP}(P)$ up to its minimum. Therefore, again, if increasing the length $\ell'_p$ the solution of $\text{LP}(P)$ does not reduce (lines 11 and 12), the segment $s_k'((s_k')^a s_k')$ is not feasible and is a constraint of the IIS. Finally, we have a feasible prefix $\text{Prefix}(\xi_{\text{seg}}, P)$ and add a feasible segment $s_k'((s_k')^a s_k')$ at the end when solving $\text{LP}(P)$, where $b = P - 1$. Thus, increasing $\ell'$ for any $i = 1, \ldots, P$ also increases the degree of freedom in the state space. Hence, any increment in an elements of the parameters $\ell_0, \ell_1, \ldots, \ell_{P-1}$ reduces $\text{LP}(P)$ up to its minimum. Therefore, it does not reduce, the discrete plan $M_p$ is infeasible and the prefix $\text{Prefix}(\xi_{\text{seg}}, P)$ a constraint of the IIS (lines 16 and 17), which concludes our theorem.

Example 9. Consider the running example (2) with initial condition $\hat{x} = [-4, -8]^T$. Let us consider the segment $s_1(s_1) = [1, 2, 3, 4, 5, 6, 7, 8]$. The solution of $\text{LP}(P)$ for $\ell_0 = \{1, 2, 3, 4, 5, 6, 7, 8\}$ is $2, 3, 2.33, 1.5, 0.9, 0.47, 0.18, -0.04$, respectively. Intuitively, after $\ell_0 - 1 > n = 2$, the solution value reduces until it reaches a negative value. In other words, the polytope $\mathcal{P}(s_1)$ is reachable from the initial state. However, if we solve $\text{LP}(P)$ for the same segment, we have $1, 5, \ldots, 5$, respectively, for the same values of $\ell_0$. The reason that the solution does not change the value after $\ell_0 = 3$ is that the bounded input turns the prefix $s_1(s_1) = [5]$ infeasible. Specifically, with input $|u| \leq 2$, the instant $k = 2$ inevitably goes outside $s_1$ and $s_2$. Therefore, we discard any trajectory with the prefix $s_1(s_1) = [5]$.

C. Planning Algorithm

Now, we present how we use Algorithm 3 to search for a dynamically feasible node in the discrete plan $M_p$. In Algorithm 4 we propose a search strategy which uses these necessary and sufficient conditions to find a feasible run $\xi_c \in M_p$. The basic idea is to check that a finite length feasible run exists by checking each segment and prefix. Each time we check reachability or feasibility, we solve a LP problem as defined in Definitions 12, 13, 14 and 10. When such a run cannot be found, this algorithm returns a counterexample representing infeasible prefixes or segments by computing an IIS constraint as per Proposition 5.

Remark 6. The root of the discrete plan $M_p, \text{root}$ is the shortest run that can be generated from this structure.

This algorithm is sound and complete, as shown by the following proposition:

Proposition 7. Given a model $M_c$ and a plan $M_p$, Algorithm 4 returns a run $\xi^*_c$ if and only if this is a feasible run of $M_p$.

Proof. (⇒) It can be easily seen that Algorithm 4 checks the conditions defined in Proposition 5; thus, any run generated from this algorithm is a feasible run $\xi^*_c$ of $M_p$. (⇐) Algorithm 4 considers all possible prefixes of $M_p$. Thus, from Proposition 5 there exists a feasible run $\xi^*_c$ of $M_p$ only if Algorithm 4 returns a run $\xi^*_c$.
Algorithm 4 Continuous Motion Planning

Input: $M_c, M_p, \delta$

Output: $\text{cexSet}, \xi^*_c$

\[ (\lambda, \delta, P, \xi^*_c) \leftarrow \text{Alg. 3}\{M_p, \text{root}, M_c, \delta, \infty, \infty, 0\}; \]
\[ \xi^*_c \leftarrow \emptyset; \text{cexSet} \leftarrow \text{openSet} \leftarrow (M_p, \text{root}, \lambda, \delta, P, \xi^*_c); \]

while openSet $\neq \emptyset$ do

parent $:=$ the lowest mini $\lambda$ with highest $P$ from openSet;

if $0 \leq \max(\lambda, P) \leq \delta$ then break;

for $i = 0$ to $K$ do

$\xi^*_p$ $=$ repeat-at($i, \text{parent}$);

$\langle \lambda, P, \xi^*_c \rangle \leftarrow \text{Alg. 3}\{\xi^*_p, M_c, \delta, \infty, \infty, P\};$

child $:=$ $\langle \lambda, P, \xi^*_c \rangle$;

if child $\text{cexSet} \neq \emptyset$ then $\text{cexSet} \leftarrow \xi^*_c$ else $\text{openSet} \leftarrow \text{child}$;

for $i = 1$ to $K$ do

$\xi^*_p$ $=$ backward-at($i, \text{parent}$);

$\langle \lambda, P, \xi^*_c \rangle \leftarrow \text{Alg. 3}\{\xi^*_p, M_c, \delta, \infty, \infty, P\};$

child $:=$ $\langle \lambda, P, \xi^*_c \rangle$;

if child $\text{cexSet} \neq \emptyset$ then $\text{cexSet} \leftarrow \xi^*_c$ else $\text{openSet} \leftarrow \text{child}$;

if openSet $\neq \emptyset$ then find $\xi^*_c$ solving LP $\square$ for parent s.t. $x_H = x_{N-1}$.

Example 10. Returning to Example 9 the run $\xi^*_d = s_{13}(s_5s_9s_{11}s_5s_6s_{14}s_{13})^0$ labels the shortest run of the discrete plan $M_p$. Executing Algorithm 4 we first obtain the values $\lambda, \delta, P, \xi^*_c$ for the root, which is $\langle \infty, \infty, \infty, 1, \emptyset \rangle$ because the segment $s_5s_6$ is not reachable and $\lambda^0 < n$. Next, inside the while loop, we check all the root children. All of them, except the child $\text{repeat-at}(0, \text{parent})$, will return the same values. This child is different because $\xi^0 = 2 = n$; thus, its values are $\{3, 5, \infty, 1, \emptyset\}$. As a result, this is the next parent and its child generated by $\text{repeat-at}(0, \text{parent})$ will have values $\{2.33, 5, \infty, 1, s_{13}(s_5s_6s_{14}s_{13})^2\}$ because this node is reachable but not feasible. Since $\xi^*_c$ is not empty and all other children have values $\lambda$ and $P$ greater or equal than 2.33 and 5, the prefix $s_{13}(s_5s_6s_{14}s_{13})^2$ is our counter-example.

VII. ITERATIVE DEEPENING TEMPORAL LOGIC OVER REALS

In the previous sections, we presented how to get a discrete plan and check if it is feasible or return a counter-example. Now, we show how to combine both discrete and continuous planning to generate a run of the system $\square$ that ensures the specification. We call our strategy of combining discrete planning and continuous motion planning Iterative Deepening Temporal Logic over Reals (1dRTL). Algorithm 5 describes this strategy. First, we check if there exists a satisfying solution in the discrete task planning layer. If such a solution exists, we check (continuous motion planning) if there is a corresponding feasible run $\xi^*_c$. If so, this is a solution for Problem $\square$.

Otherwise, the continuous planner returns a counter-example and we search for a new discrete plan. This search stops when the formula $\|(M_d, \varphi, M_{cex}, K)\}_c \land |(SR)_K|$ is unsatisfiable. We iteratively increase the length of the discrete plan $K$, which is the reason we call this algorithm 1dRTL. If we reach this stop condition, the algorithm returns no solution.

Algorithm 5 Iterative Deepening Temporal Logic over Reals

Input: $M_c, \varphi, \delta$

\[ K \leftarrow 0; \]

while $\|(M_d, \varphi, M_{cex}, K)\}_c \land |(SR)_K|$ is satisfiable do

if $\|(M_d, \varphi, M_{cex}, K)\}_c \land |(SR)_K|$ is satisfiable then

if continuous motion planning returns a counter-example $M^*_{cex}$ then $M_{cex} \leftarrow M_{cex} \cup M^*_{cex}$

else return $\xi^*_c = \xi^*_p$

else $K \leftarrow K + 1$

return No solution for Problem $\square$

A. Soundness

We show that Algorithm 5 is sound in the sense that any run $\xi^*_c$ generated by Algorithm 3 solves Problem $\square$.

Theorem 1. Given an RTL formula $\varphi$ and a dynamical system $M_c$, any solution $\xi^*_c$ of Algorithm 5 is a solution for Problem $\square$.

Proof. From Proposition 3, if the discrete plan $M_p$ generated in the discrete task planning phase enforces satisfaction of the specification $\varphi$. From Proposition 7, the solution $\xi^*_c$ generated in the continuous motion planning phase is a feasible run of a $M_p$. Therefore, the continuous run $\xi^*_c$ is a solution for Problem $\square$.

B. Completeness

We show that Algorithm 5 is complete in the sense that if no solution is found, then no solution to Problem $\square$ exists.

Theorem 2. Given an RTL formula $\varphi$ and a dynamical system $M_c$, Algorithm 5 returns no solution only if there exists no solution for Problem $\square$.

Proof. As discussed in Section VI, we assume that a solution must be finite or periodic. As a result, from Proposition 7, if Algorithm 4 returns a counter-example for a discrete plan $M_p$, then there exists no feasible run for $M_p$. Since the counter-example is a IIS, next $M_p$ is always different. From Proposition 3 and Proposition 4, if for some $K \geq 0\|(M_d, \varphi, M_{cex}, K)\}_c \land |(SR)_K|$ is unsatisfiable and either $K = 0$ or $\|(M_d, \varphi, M_{cex}, K - 1)\}_c \land |(SR)_K|$ is unsatisfiable, then there exists no discrete plan $M_p$ that satisfies the specification. From Proposition 4 it then follows that there exists no solution for Problem $\square$.

C. Complexity

The complexity of 1dRTL depends on the RTL formula and the number of continuous variables in $M_c$. First, the worst case number of discrete states in the abstraction $M_d$ is exponential in the number of atomic predicates, $O(2^n)$. Our proposed encoding $\|(M_d, \varphi, M_{cex}, K)\}_c \land |(SR)_K|$ depends linearly on the length $K$ and number of subformulas i.e., $O(K|\ell(\varphi)|_1)$, and quadratically on $|\ell(\varphi)|_K$, i.e., $O(K|\ell(\varphi)|^2)$. Finally, the complexity of LP for continuous motion planning depends linearly on the length $H$ of a continuous run $\xi^*_c$ and linearly on the number of continuous variables of $M_c$, i.e., $O(H(n + m) + n)$ for the number of variables and $O(Hn)$ for the number of constraints.
Remark 7. Note that the complexity does not directly depend on the complexity of the atomic predicates. As a result, we can achieve the same performance for arbitrary polytopic constraints as for simpler (rectangular) constraints.

VIII. SIMULATION

In this section, we demonstrate the scalability of our approach and compare with state-of-the-art solvers in three scenarios. First, we show that our approach quickly determines whether a given initial state has a dynamically feasible satisfying trajectory. We contrast these results with LanGuiCS solver [7] which can also determine the inexistence of a solution given an initial state. Next, we evaluate the performance of our approach on a motion planning problem, where the algorithm must be scalable to long trajectories and non-convex constraints. We compare these results with the SatEX solver [10], whose main focus is solvable to high-dimensional systems. We also compare these results with the OMPL solver [22] (sampling-based motion planning), and LTLOpt solver [9] (MILP motion planning). Finally, we validate the scalability of our approach to high-dimensional dynamical systems by considering a quadrotor model with 18 continuous variables. We contrast these results with the SatEX solver [10], whose main focus is scalability to high-dimensional systems.

We implemented our approach using Z3 SMT solver [27], Gurobi LP solver [34], and lrs vertex enumeration solver [35] and is available at https://bitbucket.org/rafael_rodriguesdasilva/idrtl/. All experiments were executed on an Intel Core i7 processor with 32GB RAM.

A. Determining Existence of a Satisfying Trajectory

In this experiment, we use [7, Example 11.5] as a benchmark problem. This problems uses the same system but with different specification, i.e., \( q = (\neg aU b) \land (\neg bU c) \), where \( a \) is represented by the blue regions in Figure 6, \( b \) corresponds to the red region, and \( c \) to the yellow regions. This problem is a particularly challenging problem [5]. The forbidden regions create areas in the workspace where no solution exists. Thus, it is particularly important to decide when the dynamical constraints render a given initial state infeasible.

In this scenario, we selected 10 feasible and 10 infeasible initial states and executed the idRTL and LanGuiCS solvers. Fig. 6(a) illustrates the solutions generated by idRTL for these initial states. Black stars are initial states for which Algorithm 5 returned no solution. The curves are trajectories for initial states where a solution exists. idRTL took 52.6 ± 5.9ms to compute a solution when a solution exists, and 156.9 ± 96.1ms to determine that no solution exists.

Fig. 6(b) shows the same results for LanGuiCS. We can see that both algorithms correctly determined the existence of a solution. LanGuiCS took 1352.6 ± 784.0ms to generate solutions for feasible initial states and 22.3 ± 1.5ms to decide that no solution exists. This solver can achieve a quicker decision for unfeasible initial states because it computes the feasible regions offline. However, LanGuiCS required more than 10min (612.606s) to compute these regions.

These results show that idRTL makes decisions on the existence of a solution in a reasonable time without offline computation. Furthermore, the scalability of our approach is even more evident when a solution exists. The reason for this is that it only takes one valid discrete plan to decide that a solution exists, but it can take several of these plans to rule out the existence of a feasible solution.

![Fig. 6. Comparison between idRTL and LanGuiCS. Note that some continuous trajectories appear to pass through the forbidden blue region, which is an artifact of the time discretization. In practice, we can avoid it by increasing the obstacles’ size or changing the discretization strategy.](Image)

B. Application to Motion Planning

We consider a motion planning problem scenario where a mobile robot must reach a target region while avoiding collisions with obstacles. The main challenge is in considering narrow passages. As the number of passages grows, so does the size of the discrete abstraction.

We model the robot as a two-dimensional double integrator with sampling time \( T_s = 0.5s \), where \( x_k \in \mathcal{X} \subseteq \mathbb{R}^4 \), \( u_k \in \mathcal{U} \subseteq \mathbb{R}^2 \). \( \mathcal{X} = \{ x \in \mathbb{R}^4 : 0 \leq x_1 \leq 30, 0 \leq x_2 \leq 30, -2 \leq x_3 \leq 2, -2 \leq x_4 \leq 2 \} \). \( \mathcal{U} = \{ u \in \mathbb{R}^2 : ||u||_\infty \leq 0.5 \} \). The workspace is illustrated in Fig. 7, where grey regions indicate obstacles and green is the target region.

![Fig. 7. Comparison with sampling-based and MILP-based solvers on a maze problem.](Image)

Table I and Fig. 7 demonstrate the scalability of idRTL in this motion planning application. In Table I we present the computation performance of idRTL and compare with state-of-art sampling-based and MILP-based (LTLOpt [9]) approaches.
This table shows the average run-time of 10 executions for different numbers of passages. The number of passages increases the number of obstacles and the length of a satisfying run, as shown in Fig. 7a and 7b. idRTL is consistently faster than the other approaches. This demonstrates that considering logical and dynamical constraints with a combination of SAT/SMT and optimization solvers is more efficient than solving a MILP problem. The encoding of logical constraints in a MILP problem is especially costly for longer runs. We also compared our approach with RRT [36], a probabilistically complete motion planning algorithm [23]. Even though RRT solves a much less expressive problem, idRTL can still be an order of magnitude faster.

### C. High-dimensional dynamical systems

We demonstrate the scalability of our approach to high-dimensional systems with an example of motion planning for a quadrotor. The quadrotor moves in 3-dimensional Euclidean space and operates with linearized dynamics having 18 continuous variables. We compare these results with SatEx solver, which is also scalable to high-dimensional systems. We consider the collision avoidance problem presented in [10], and increase the complexity of the problem by increasing the length of the x-axis and the number of obstacles.

Fig. 8 shows that our approach can find satisfying solutions for this high-dimensional system even when we increase the problem complexity. idRTL is consistently faster in computing these runs than SatEx. The main reason for this is that our approach searches first for shorter discrete plans to generate the continuous runs. Since the non-convex nature of the problem is generated by the logical constraints, starting with shorter discrete runs reduces the non-convexity in the problem.

### IX. Conclusion

We proposed a fast, scalable, and provably complete symbolic control method for unbounded temporal logic specifications. To address the coupling between nonconvex logical constraints and physical dynamic constraints, we designed a two-layer control architecture which separates discrete task planning and continuous motion planning on-the-fly. By directly addressing this core problem, our approach scales well to high-dimensional systems and complex specifications, as well as offering order-of-magnitude speed improvements over the current state-of-the-art. We hope that this work will provide a step towards safe and provably correct control of complex autonomous systems. Future work will focus on extensions to unknown/dynamic environments and non-linear/hybrid systems.

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