Periapsis shift of a quasi-circular orbit in a general static spherically symmetric spacetime

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(Dated: October 17, 2022)

Abstract

We study the periapsis shift of a quasi-circular orbit in a general static spherically symmetric spacetime. We derive two formulae in full order with respect to the gravitational field, one in terms of the gravitational mass $m$ and the other in terms of the orbital angular velocity $\omega_{\phi}$. These formulae reproduce the well-known ones for the prograde shift in the Schwarzschild spacetime. In a general case, the shift deviates from that in the Schwarzschild spacetime due to a particular combination of the components of the Ricci tensor at the radius $r$ of the orbit. The formulae give a retrograde shift due to the extended-mass effect in Newtonian gravity. In the post-Newtonian regime of general relativity near a massive compact object, a retrograde shift implies that the energy density is beyond a critical value $\epsilon_c = 3Gm^2/(2\pi r^4) \sim 3r^2\omega_{\phi}^4/(2\pi G)$, whereas a prograde shift greater than that in the Schwarzschild spacetime implies the violation of the weak energy condition there. Implications to the Galactic Centre are also discussed.

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I. INTRODUCTION

The periapsis shift of Mercury is one of the first classical predictions of general relativity by Einstein (1916) [1], which has been the most successful theory of gravity so far. Recently, this classical test has again attracted attention because a number of very massive and very compact objects that are supposed to be black holes have been observed and accessible by observations of stellar orbits, shadows and gravitational waves. In particular, the general relativistic periapsis shift of a star called S2, whose periapsis distance is \( \sim 10^3 \) times the gravitational radius of the central supermassive compact object at Sagittarius A* (Sgr A*), has been observed by Gravity Collaboration (2020) [2]. Other stars having even closer orbits to the central object have been reported [3].

Although Kerr black holes with almost vacuum surroundings remain a standard assumption for the central objects, a lot of alternative possibilities have been discussed because such a strong field definitely is a new frontier of gravitational physics in this century. As for the nature of the central objects themselves, it has been discussed that they might not be black holes but dense cores [4], boson stars, naked singularities, wormholes or other exotic compact objects. On the other hand, since the black hole no-hair conjecture is broken in certain circumstances (e.g., [5, 6]), it is impossible without detailed observation to exclude such a possibility as the central massive compact object may be a black hole but with a significant hair. Even if the central object is a standard black hole well approximated by a Kerr spacetime, it might be surrounded by a dark matter spike [7] or any other fields. It is also discussed that gravitational theory is significantly modified from Einstein gravity in such a strong field regime, so that the black hole is different from Kerr’s one, which may be realised in some sort of scalar-tensor theories, Chern-Simons gravity, Gauss-Bonnet-dilaton gravity and so on.

It has been found that periapsis shifts in these non-standard scenarios can be very different from those for the Kerr black hole. In particular, the possibility of a retrograde periapsis shift, i.e., a periapsis shift in a direction opposite to the orbital rotation, has been shown near a dense core [8, 9], a boson star [10], a wormhole [11], a naked singularity [12, 13] and also in dark matter distribution around a standard black hole [14, 15] in more or less astrophysically reasonable conditions. In these studies, what can be called the ‘extended-mass effect’ on the periapsis shift should play a key role. This effect has been discovered in
Newtonian gravity by Jiang and Lin (1984) \cite{16}, who showed that the periapsis shift due to this effect is retrograde. This effect has been used to constrain the abundance of dark matter in the solar system \cite{17} and the Galactic Centre \cite{18,19}. The amount of the extended mass inside the orbit of S2 in the Galactic Centre is estimated to be less than 0.5% of the mass of the central compact mass by using the redshift data of S2 \cite{20}.

In this paper, without invoking a weak-field approximation, we derive formulae for the periapsis shift of a quasi-circular orbit in a general static spherically symmetric spacetime, so that we can largely extend the above mentioned Newtonian extended-mass effect to include full orders with respect to the gravitational field and not only the density distribution of extended mass but also any other physical causes which make the Ricci curvature tensor nonvanishing including modified theories of gravity. This paper is organised as follows. In Sec. II we formulate circular orbits in a general static spherically symmetric spacetime. In Sec. III we formulate the periapsis shifts of quasi-circular orbits in the same spacetime and obtain the two formulae that are exact in the sense that we do not approximate the gravitational field. In Sec. IV we discuss several physically interesting limits of the formulae. In Sec. V we discuss the implications of the present result to the Galactic Centre using the periapsis shift of S2. Section VI is devoted to conclusion. In Appendix A we discuss the periapsis shift in the anti-de Sitter spacetime.

II. CIRCULAR ORBITS IN A STATIC SPHERICALLY SYMMETRIC SPACETIME

A. Timelike geodesics

The line element in a static spherically symmetric spacetime can be written in the following form:

\[
ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),
\]

where $\nu$ and $\lambda$ are arbitrary functions of $r$. The gravitational mass $m$ is defined as

\[
e^{\lambda(r)} = \left(1 - \frac{2m(r)}{r}\right)^{-1},
\]

which implies $r > 2m$. 


We model an object orbiting in this spacetime by a test particle, whose trajectory is given by a timelike geodesic. The Lagrangian of the test particle is given by

\[ \mathcal{L} = \frac{1}{2} \left[ -e^\nu \dot{t}^2 + e^\lambda \dot{r}^2 + r^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) \right], \tag{2.3} \]

where the dot denotes the differentiation with respect to the affine parameter. By symmetry, we can assume that the orbit is on the \( \theta = \pi/2 \) plane. There are conserved quantities, energy \( E \) and angular momentum \( L \), associated with the time-translational Killing vector \( \partial_t \) and the rotational Killing vector \( \partial_\phi \), respectively. They are related to \( \dot{t} \) and \( \dot{\phi} \) as follows:

\[ \dot{t} = e^{-\nu} E, \quad \dot{\phi} = \frac{L}{r^2}. \tag{2.4} \]

The Euler-Lagrange equations reduce to

\[ \ddot{r} + V'(r) = 0, \tag{2.5} \]

where the prime denotes the differentiation with respect to \( r \), and the normalisation condition \( g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = -1 \) implies

\[ \frac{1}{2} r^2 + V(r) = 0 \tag{2.6} \]

with \( V(r) \) being the effective potential given by

\[ V(r) = \frac{1}{2} e^{-\lambda} \left[ \left( 1 + \frac{L^2}{r^2} \right) - e^{-\nu} E^2 \right]. \tag{2.7} \]

The affine parameter coincides with the proper time \( \tau \).

### B. Circular orbits

Let us concentrate on circular orbits at \( r = r_0 \). Differentiating \( V(r) \) with respect to \( r \) once and twice, we have

\[ V' = -\lambda V + e^{-\lambda} \left( -\frac{L^2}{r^3} + \frac{\nu'}{2} e^{-\nu} E^2 \right) \tag{2.8} \]

and

\[ V'' = -(\lambda V)' - \lambda e^{-\lambda} (e^\lambda V)' + e^{-\lambda} (e^\lambda V)'' \tag{2.9} \]

respectively.
From Eqs. (2.5) and (2.6), we have \( V = V' = 0 \) at \( r = r_0 \). Together with Eqs. (2.8) and (2.9), this implies
\[
V'' = e^{-\lambda} \left( 3 \frac{L^2}{r^4} + \frac{\nu'' - \nu'^2}{2} e^{-\nu} E^2 \right),
\]
(2.10)
where and hereafter we write \( r \) for \( r_0 \) for brevity unless it is misleading. Equations (2.7) and (2.8) with \( V(r) = V'(r) = 0 \) imply
\[
E^2 = \frac{2e^\nu}{2 - r\nu'}, \quad L^2 = \frac{r^3\nu'}{2 - r\nu'}.
\]
(2.11)
Since \( E^2 > 0 \) and \( L^2 > 0 \), the condition \( 0 < r\nu' < 2 \) must be satisfied for the existence of the circular orbit. Substituting the above into Eq. (2.10), we find
\[
V'' = \frac{3r^{-1}\nu' + \nu'' - (\nu')^2}{2 - r\nu'} e^{-\lambda}.
\]
(2.12)
Since the stability of the circular orbit is determined by the sign of \( V'' \), the circular orbit is stable, unstable and marginally stable if
\[
\nu'' - (\nu')^2 + 3 \frac{\nu'}{r}
\]
(2.13)
is positive, negative and zero, respectively.

### III. EXACT FORMULAE FOR THE PERIAPSIS SHIFT

#### A. Expression in terms of the metric functions

If a stable circular orbit at \( r = r_0 \) with \( V'' > 0 \) is perturbed as \( r = r_0 + \delta r \) with \( \delta r \) being infinitesimally small, it becomes a quasi-circular orbit, where \( \delta r \) obeys a simple harmonic motion as understood in Eq. (2.5). The orbital angular velocity and radial frequency of the particle in terms of the proper time are given by
\[
\omega_\phi = \dot{\phi} = \frac{L}{r^2} = \sqrt{\frac{\nu'}{r(2 - r\nu')}}
\]
(3.1)
and
\[
\omega_r = \sqrt{V''},
\]
(3.2)
respectively, where Eq. (2.11) has been used, and Eq. (3.2) comes from Eq. (2.5) under the perturbation \( r = r_0 + \delta r \). The periapsis shift \( \Delta \phi_p \) is given by
\[
\Delta \phi_p = 2\pi \left( \frac{\omega_\phi}{\omega_r} - 1 \right).
\]
(3.3)
Using Eqs. (2.12), (3.1) and (3.2), this can be calculated to give

\[
\Delta \phi_p = 2\pi \left( \frac{1}{\sqrt{A}} - 1 \right),
\]

where

\[
A := re^{-\lambda} - \nu'' - \left(\frac{\nu'}{r}\right)^2 + \frac{3}{r} \nu'.
\]

The function \( A \) is convenient for further computation. From Eq. (3.4), we can see that the shift is prograde, retrograde and zero, if \( 0 < A < 1 \), \( A > 1 \) and \( A = 1 \), respectively. Note that \( A > 0 \) is guaranteed by the assumption of the stable orbit \( V'' > 0 \).

**B. Periapsis shift in the Schwarzschild spacetime**

In the case of the Schwarzschild spacetime, where

\[
e'' = 1 - \frac{2M}{r}, \quad e' = \left(1 - \frac{2M}{r}\right)^{-1},
\]

Eqs. (2.11) and (3.1) imply

\[
M = \frac{r^3 \omega_p^2}{1 + 3r^2 \omega_p^2}.
\]

Thus, the expression for \( A \) in the Schwarzschild spacetime reduces to

\[
A = 1 - \frac{6M}{r} = \frac{1 - 3r^2 \omega_p^2}{1 + 3r^2 \omega_p^2},
\]

and, hence, we obtain the expressions

\[
\Delta \phi_p = 2\pi \left( \frac{1}{\sqrt{1 - \frac{6M}{r}}} - 1 \right) = 2\pi \left( \sqrt{\frac{1 + 3r^2 \omega_p^2}{1 - 3r^2 \omega_p^2}} - 1 \right).
\]

The above expressions are exact in the sense that we do not assume that the gravitational field is weak. Equation (3.9) reproduces the well-known expressions

\[
\Delta \phi_p \simeq \frac{6\pi M}{r} \simeq 6\pi r^2 \omega_p^2
\]

in the weak-field limit.
C. Expression in terms of the gravitational mass

Let \( \{ \vec{e}_\alpha \}_{\alpha=0,1,2,3} \) be a natural tetrad basis given by

\[
\vec{e}_0 = e^{-\nu/2} \frac{\partial}{\partial t}, \quad \vec{e}_1 = e^{-\lambda/2} \frac{\partial}{\partial r}, \quad \vec{e}_2 = \frac{1}{r} \frac{\partial}{\partial \theta}, \quad \vec{e}_3 = \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi},
\]

(3.11)

The tetrad components of the Einstein tensor, \( G^\alpha_\beta = e^\mu_\alpha e^\nu_\beta G_{\mu\nu} \), where \( G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \) in terms of the Ricci tensor \( R_{\mu\nu} \) and the Ricci scalar \( R = g^{\mu\nu} R_{\mu\nu} \), can be calculated to give

\[
G^0_0 = \frac{1}{r^2} [r(1 - e^{-\lambda})]',
\]

(3.12)

\[
G^1_1 = \frac{e^{-\lambda}}{r} \nu' - \frac{1}{r^2} (1 - e^{-\lambda}),
\]

(3.13)

\[
G^2_2 = G^3_3 = \frac{1}{2} e^{-\lambda} \left( \nu'' + \frac{\nu'^2}{2} - \frac{\nu' \lambda'}{2} + \frac{\nu' - \lambda'}{r} \right),
\]

(3.14)

while all other components of \( G^\alpha_\beta \) vanish. The contracted Bianchi identity \( \nabla_\mu G^\mu_\nu = 0 \) implies

\[
G''_1 + \frac{1}{2} \nu'(G^0_0 + G^1_1) + \frac{2}{r} (G^1_1 - G^2_2) = 0.
\]

(3.15)

Equation (3.14) implies

\[
\nu'' = 2 e^\lambda G^2_2 - \frac{(\nu')^2}{2} + \frac{\nu' \lambda'}{2} - \frac{\nu' - \lambda'}{r},
\]

(3.16)

while Eqs. (3.12) and (3.13) imply

\[
\lambda' = re^\lambda \left[ G^0_0 - \frac{1}{r^2} (1 - e^{-\lambda}) \right],
\]

(3.17)

\[
\nu' = re^\lambda \left[ G^1_1 + \frac{1}{r^2} (1 - e^{-\lambda}) \right],
\]

(3.18)

respectively.

Since Eqs. (2.2) and (3.18) imply

\[
\nu' = r \frac{2m}{1 - \frac{2m}{r}},
\]

(3.19)

the conditions \( E^2 > 0 \) and \( L^2 > 0 \) or \( 0 < r \nu' < 2 \) are satisfied if and only if

\[
-\frac{2m}{r^3} < G^1_1 < \frac{2}{r^2} \left( 1 - \frac{3m}{r} \right)
\]

(3.20)

is satisfied.
From Eqs. (3.5), (3.16), (3.17) and (3.18), we obtain $A = A_{m0} + A_{m1}$, where

$$A_{m0} = 1 - \frac{6mr}{r^2},$$

$$A_{m1} = \left(1 - \frac{2mr}{r^2}\right) \left(\frac{G_{00} + G_{11} + 2G_{22}}{2m} + \frac{G_{00} - 3G_{11}}{r^2}\right) + \frac{1}{2}(G_{00} - 3G_{11})r^2.$$ (3.21)

(3.22)

It is clear that for the locally Ricci-flat spacetime, this expression reproduces the Schwarzschild formula (3.8) or (3.9) with the identification $M = m(r)$. The shift is more and less prograde than that in the Schwarzschild, if and only if $A_{m1}$ is negative and positive, respectively. A spacetime with $A_{m1} = 0$ gives the periapsis shift identical with that in the Schwarzschild spacetime. What we have shown is that under the assumption of the geodesic motion, the periapsis shift is determined by the gravitational mass and the Einstein tensor at the radius of the orbit, irrespective of the gravitational theory.

To demonstrate the vast applicability of the exact formula obtained above, we apply it to maximally symmetric spacetimes. This is delegated to Appendix A.

D. Expression in terms of the orbital angular velocity

Although the expression in terms of the gravitational mass is very useful for theoretical consideration, the gravitational mass at the orbital radius cannot be obtained from observation without the detailed knowledge of the metric functions. In Newtonian gravity, the angular velocity of the circularly orbiting star is used to estimate the mass of the central gravitating object through Kepler’s third law. This is also possible in the Schwarzschild spacetime as is seen in Eq. (3.7). However, in the general case, from Eqs. (2.11), (3.1) and (3.19), we obtain

$$m = \frac{r^3\omega_\phi^2 - \frac{1}{2}(G_{00} + 2G_{22} + G_{11})}{1 + 3r^2\omega_\phi^2}.$$ (3.23)

Therefore, without knowing $G_{11}$, the orbital angular velocity $\omega_\phi$ is not enough to determine $m(r)$ even if we know $r$. In other words, $\omega_\phi$ is more accessible than $m$. Thus, it would be more useful to have the expression for $A$ in terms of $\omega_\phi$ and $r$.

The result is written in the form $A = A_{\omega0} + A_{\omega1}$, where

$$A_{\omega0} = \frac{1 - 3r^2\omega_\phi^2}{1 + 3r^2\omega_\phi^2},$$

$$A_{\omega1} = \frac{1}{\omega_\phi^2} \left[ \frac{1}{2} \left( G_{00} + 2G_{22} + \frac{1 + 7r^2\omega_\phi^2}{1 + 3r^2\omega_\phi^2}G_{11} \right) + r^2\omega_\phi^2(G_{00} + G_{22}) \right].$$ (3.24)

(3.25)
The first term $A_{\omega 0}$ has the form identical to $A$ for the Schwarzschild spacetime, while the second term $A_{\omega 1}$ denotes the deviation from it. If $A_{\omega 1} = 0$, the periapsis shift is identical to that for the Schwarzschild spacetime with the same $\omega_\phi$ and $r$, even though the spacetime may be very different from the Schwarzschild. The shift is more and less prograde if $A_{\omega 1}$ is negative and positive, respectively. It would be interesting that the exact expression (3.25) for $A_{\omega 1}$ is linear with respect to the Ricci tensor, whereas that for $A_{m1}$ is not as seen in Eq. (3.22).

It should be noted that the angular velocity with respect to the proper time, $\omega_\phi = d\phi/d\tau$, is different from the angular velocity measured at infinity, $\Omega_\phi := d\phi/dt$, where $e^{\nu(r)} \to 1$ in the limit $r \to \infty$ is assumed. The latter can be written for the circular orbit as

$$\Omega_\phi = \frac{e^\nu L}{r^2 E} = e^{\nu/2} \sqrt{\frac{\nu'}{2r}},$$

where we have used Eqs. (2.4) and (2.11). The relation between $\omega_\phi$ and $\Omega_\phi$ is written as

$$\omega_\phi = (1 + z)\Omega_\phi,$$

where $1 + z := dt/d\tau$ is the averaged redshift factor, which is given by

$$1 + z = e^{-\nu} E = e^{-\nu/2} \sqrt{\frac{2}{2 - r\nu}},$$

for the circular orbit from Eqs. (2.4) and (2.11). See also [9] for the discussion on the averaged redshift factor. Thus, $\omega_\phi$ can be calculated from the two observables in principle, $\Omega_\phi$ and $1 + z$.

IV. IMPORTANT LIMITS

In this section, for simplicity we use the following notation:

$$G_{00} = 8\pi \epsilon, \quad G_{11} = 8\pi \Sigma, \quad G_{22} = G_{33} = 8\pi \Pi, \quad m = \frac{4\pi}{3} r^3 \bar{\epsilon}.$$  

We also follow the terminology in general relativity, so that $\epsilon$, $\Sigma$, $\Pi$ and $\bar{\epsilon}$ are termed the energy density, radial stress, tangential stress and averaged energy density, respectively. This is purely for convenience and it is trivial how to translate the current result to modified theories of gravity because we do not solve the Einstein equation in any case.
A. Post-Newtonian regime

In the post-Newtonian expansion, assuming \( m/r = O(v^2) \), \( r^2 \omega_\phi^2 = O(v^2) \), \( \epsilon/\bar{\epsilon} = O(1) \), \( \Sigma/\epsilon = O(v^2) \) and \( \Pi/\epsilon = O(v^2) \) with \( v \) being the speed of the orbiting star, we obtain

\[
\Delta \phi_p = -2\pi \left( 1 - \frac{1}{\sqrt{1 + \frac{3\epsilon}{\bar{\epsilon}}}} \right) + 3\pi \left( 1 + \frac{3\epsilon}{\bar{\epsilon}} \right)^{-3/2} \left[ \frac{m}{r} \left( 2 + \frac{\epsilon}{\bar{\epsilon}} \right) - \frac{\Sigma + 2\Pi}{\bar{\epsilon}} + \frac{3\epsilon\Sigma}{\bar{\epsilon}^2} \right] + O(v^4) \] (4.2)

where we have used Eqs. (3.21), (3.22), (3.24) and (3.25), and it turns out that \( m > 0 \) must be satisfied. In the bottom expression, for example, \( \epsilon/\omega^2 \phi \) is understood as the nondimensional quantity \( G\epsilon/(\omega_\phi c)^2 \) if \( c \) and \( G \) are recovered. In both of the above expressions, the first and second terms on the right-hand side stand for the Newtonian and the first post-Newtonian ones, respectively. The first term is negative if \( \epsilon > 0 \) and therefore gives a negative contribution to \( \Delta \phi_p \). This corresponds to the Newtonian extended-mass effect. As we can see, \( \epsilon \), \( \Sigma \) and \( \Pi \) also affect the second term of the post-Newtonian order. On the other hand, if \( \epsilon = \Sigma = \Pi = 0 \), the Newtonian terms vanish, and the post-Newtonian terms give the well-known weak-field formulae (3.10) for the Schwarzschild spacetime.

B. Near a massive central object

Next, we instead assume that \( |\epsilon|, |\Sigma|, |\Pi| \ll |\epsilon| \sim \omega_\phi^2 \). This limit applies if the orbit is near a very massive central gravitating object. The periapsis shift is then calculated to give

\[
\Delta \phi_p = 2\pi \left( \frac{1}{\sqrt{1 - \frac{6m}{r}}} - 1 \right) - 3\pi \left( 1 - \frac{6m}{r} \right)^{-3/2} \left[ \frac{\epsilon + \Sigma + 2\Pi}{\bar{\epsilon}} - \frac{m}{r} \frac{\epsilon + 5\Sigma + 4\Pi}{\bar{\epsilon}} \right] + O(\alpha^2) \] (4.4)

\[
\Delta \phi_p = 2\pi \left( \frac{1 + 3r^2 \omega_\phi^2}{\sqrt{1 - 3r^2 \omega_\phi^2}} - 1 \right) - 4\pi^2 \left( \frac{1 + 3r^2 \omega_\phi^2}{\sqrt{1 - 3r^2 \omega_\phi^2}} \right)^{3/2} \left[ \frac{\epsilon + 2\Pi}{\omega_\phi^2} + \frac{1 + 7r^2 \omega_\phi^2 \Sigma}{1 + 3r^2 \omega_\phi^2 \omega_\phi^2} + \frac{2r^2 \omega_\phi^2 \epsilon + \Pi}{\omega_\phi^2} \right] + O(\alpha^2) \] (4.5)
where we assume that all of $\epsilon/\bar{\epsilon}$, $\Sigma/\bar{\epsilon}$ and $\Pi/\bar{\epsilon}$ are of the order of $\alpha \ll 1$, and it turns out that $m > 0$ must hold. In the above, the first terms on the right-hand side are of $O(\alpha^0)$ and coincide with Eq. (3.9), the Schwarzschild formulae, while the second terms are of $O(\alpha)$.

C. Post-Newtonian regime near a massive central object

If both of the above expansions apply, we can expand $\Delta \phi_p$ as follows:

$$\Delta \phi_p = 3\pi \left(2m \left(\frac{2m}{r} - \frac{\epsilon}{\bar{\epsilon}}\right) + \frac{27\pi m^2}{r^2} - 3\pi \left(\frac{8m \epsilon}{r \bar{\epsilon}} + \frac{\Sigma + 2\Pi}{\bar{\epsilon}}\right) + \frac{27\pi}{4} \left(\frac{\epsilon}{\bar{\epsilon}}\right)^2\right) + O\left(v^6 \alpha^0, v^4 \alpha^1, v^2 \alpha^2, v^0 \alpha^3\right)$$

$$= 3\pi \left(2m r^2 \omega^2 - \frac{4\pi \epsilon}{3 \bar{\omega}^2}\right) + 9\pi r^4 \bar{\omega}^4 - 4\pi^2 \left(11m^2 \omega^2 \frac{\epsilon}{\bar{\omega}^2} + \frac{\Sigma + 2\Pi}{\omega^2}\right)$$

$$+ 12\pi^3 \left(\frac{\epsilon}{\bar{\omega}^2}\right)^2 + O\left(v^6 \alpha^0, v^4 \alpha^1, v^2 \alpha^2, v^0 \alpha^3\right),$$

where the first terms include those of $O(v^2 \alpha^0)$ and $O(v^0 \alpha^1)$, while the second, third and fourth terms are of $O(v^4 \alpha^0)$, $O(v^2 \alpha^1)$ and $O(v^0 \alpha^2)$, respectively. In this regime, the periapsis shift in the lowest orders is prograde (retrograde) if $\frac{2m}{r} > (<) \epsilon/\bar{\epsilon}$, i.e.,

$$\epsilon < (>) \epsilon_c := \frac{2m}{r} \bar{\epsilon} = \frac{3m^2}{2\pi r^4} \simeq 3 \frac{3\pi}{2\pi r^2} \omega^4,$$

where we call $\epsilon_c$ a critical energy density. The shift vanishes in the lowest orders if $\epsilon = \epsilon_c$.

As we can see in Eqs. (4.6) and (4.7), if $\Delta \phi_p > 6\pi m/r \simeq 6\pi r^2 \omega^2$, then, $\epsilon < 0$ must hold, resulting in the violation of the weak energy condition in the sense of general relativity. Note that we can rewrite Eqs. (4.6) and (4.7) in the lowest orders as

$$\Delta \phi_p = \frac{6\pi m}{r} \left(1 - \frac{\epsilon}{\epsilon_c}\right) + O\left(v^4 \alpha^0, v^2 \alpha^1, v^0 \alpha^2\right) = 6\pi r^2 \omega^2 \left(1 - \frac{\epsilon}{\epsilon_c}\right) + O\left(v^4 \alpha^0, v^2 \alpha^1, v^0 \alpha^2\right).$$

D. Vanishing radial stress

In the case of $\Sigma = 0$, we can remove $\Pi$ from Eq. (3.22) using Eq. (3.15) or

$$e^{\lambda m \epsilon} - \left(\frac{2}{r}\right) = 0.$$

The resultant exact expressions are very much simplified as follows:

$$A = 1 - \frac{6m}{r} + 3\frac{\epsilon}{\bar{\epsilon}} = \frac{1 - 3r^2 \omega^2}{1 + 3r^2 \omega^2} + 4\pi \left(1 + 3r^2 \omega^2\right) \frac{\epsilon}{\bar{\omega}^2}.$$
This agrees with the expression for the Einstein cluster. The periapsis shift is prograde (retrograde) if $\epsilon < (>) \epsilon_c$, while it vanishes if the equality holds, where $\epsilon_c$ is defined by Eq. (4.3). The periapsis shift greater than that in the Schwarzschild spacetime implies the violation of the weak energy condition. It is interesting that these statements hold in full order with respect to the gravitational field in this case.

**E. Perfect fluid**

If $\Sigma = \Pi = P$, for which the matter field is called a perfect fluid, we find

$$A = \left(1 - \frac{2m}{r}\right) \frac{\bar{\epsilon} + 3\epsilon + 12P}{\bar{\epsilon} + 3P} - \frac{4m}{r} \frac{4\bar{\epsilon} - 3\epsilon + 9P}{4\epsilon}$$

$$= \frac{1 - 3r^2\omega_\phi^2}{1 + 3r^2\omega_\phi^2} + \frac{4\pi}{\omega_\phi^2} \left[ 1 + 2r^2\omega_\phi^2 \right] \epsilon + \frac{3(1 + 5r^2\omega_\phi^2 + 2r^4\omega_\phi^4)}{1 + 3r^2\omega_\phi^2} P.$$  

(4.12)

The series expansions in different regimes can be obtained by just putting $\Sigma = \Pi = P$ in the corresponding general expressions obtained in subsections IV A, IV B and IV C.

**V. DISCUSSION**

Let us discuss the application of the result obtained so far to S2 star. On applying the result to the elliptical orbit of S2 with the high eccentricity $e \simeq 0.88466...$ and the semi-major axis $a \simeq 970\text{au}$, let us choose the representative radius of the orbit to the semilatus rectum $r = a(1 - e^2) \simeq 210\text{au}$, which is inspired by the weak-field formula for an elliptical orbit, $\Delta \phi_p \simeq 6\pi M/[a(1-e^2)]$ [1,21]. We can estimate the averaged density $\bar{\epsilon}$ and the critical density $\epsilon_c$ there as follows:

$$\bar{\epsilon} \simeq 1.1 \times 10^{-1}M_\odot/\text{au}^3 \left( \frac{m}{4.3 \times 10^6 M_\odot} \right) \left( \frac{r}{210\text{au}} \right)^{-3}$$

(5.1)

$$\epsilon_c \simeq 4.5 \times 10^{-5}M_\odot/\text{au}^3 \left( \frac{m}{4.3 \times 10^6 M_\odot} \right)^2 \left( \frac{r}{210\text{au}} \right)^{-4}.$$  

(5.2)

These values for the mass densities correspond to those for the energy densities, $3.7 \times 10^{16}\text{GeV/cm}^3$ and $1.5 \times 10^{13}\text{GeV/cm}^3$, respectively.

We discuss a possible observational constraint in terms of $\omega_\phi$. If $|A_{\omega 1}| \ll |A_{\omega 0} - 1|$, we can write the deviation of the periapsis shift from the Schwarzschild one in the following
form:
\[
\frac{\Delta \phi_p - \Delta \phi_{p,\text{Sch}}}{\Delta \phi_{p,\text{Sch}}} = -\frac{\pi}{\Delta \phi_{p,\text{Sch}}} \left( 1 + \frac{\Delta \phi_{p,\text{Sch}}}{2\pi} \right)^3 A_{\omega 1} + O(A_{\omega 1}^2),
\] (5.3)

where
\[
\Delta \phi_{p,\text{Sch}} := 2\pi \left( \sqrt{\frac{1 + 3r^2\omega_1^2}{1 - 3r^2\omega_1^2}} - 1 \right). 
\] (5.4)

If we write
\[
\Delta \phi_p = f \Delta \phi_{p,\text{Sch}},
\] (5.5)

we can recast Eq. (5.3) to the following form:
\[
A_{\omega 1} = -\frac{\Delta \phi_{p,\text{Sch}}}{\pi} \left( 1 + \frac{\Delta \phi_{p,\text{Sch}}}{2\pi} \right)^{-3} (f - 1) + O((f - 1)^2). 
\] (5.6)

If we normalise the right-hand side by the observed values \( f = 1.1 \pm 0.19 \) and \( \Delta \phi_{p,\text{Sch}} \approx 12.1' \) for S2 \([2]\), we find that \( A_{\omega 1} \) can be constrained by
\[
|A_{\omega 1}| \lesssim 3.4 \times 10^{-4} \left( \frac{\Delta \phi_{p,\text{Sch}}}{12.1'} \right) \left( \frac{|f - 1|}{0.3} \right). 
\] (5.7)

Let us further assume that the approximation in the post-Newtonian regime near a massive central object is valid for the S2 orbit. This assumption is most plausible. Then, from Eq. (5.5), we can recast Eq. (5.5) to
\[
\epsilon \approx (1 - f)\epsilon_c. 
\] (5.8)

If we adopt the result \( f = 1.1 \pm 0.19 \), we immediately reach the conclusion
\[
\epsilon = (-4.5 \pm 8.5) \times 10^{-6} M_\odot / \text{au}^3 \left( \frac{m}{4.3 \times 10^6 M_\odot} \right)^2 \left( \frac{r}{210 \text{au}} \right)^{-4}
\]
\[
= (-1.5 \pm 2.9) \times 10^{12} \text{GeV/cm}^3 \left( \frac{m}{4.3 \times 10^6 M_\odot} \right)^2 \left( \frac{r}{210 \text{au}} \right)^{-4}. 
\] (5.9)

This is a rather stringent constraint on the matter density in the vicinity of Sgr A*.

In fact, we find that the mass within \( a \approx 970 \text{ au} \) is bounded by \( \approx 1.6 \times 10^4 M_\odot \) by simply multiplying the density by the ball’s volume. Remarkably, this is comparable with the upper bound \( \approx 2 \times 10^4 M_\odot \) obtained by the information of the temporal variation in the redshift of the orbital motion \([20]\). This suggests that the present method for a quasi-circular orbit is at least complementary to the method with the orbital redshift variation, where the latter is clearly more advantageous for elliptical orbits with a higher eccentricity.
It is also intriguing that although the constraint (5.9) is consistent with $\epsilon = 0$, the best-fit value $f = 1.1$ is in the range corresponding to the violation of the weak energy condition. If the weak energy condition must be physically required, we should recall that what is directly constrained is $G_{\hat{0}\hat{0}}$, a particular component of the curvature tensor, rather than the energy density of some physical matter fields and seriously discuss the possibility of the modification of gravity in the vicinity of a massive compact object.

It should, however, be noted that the above constraint should be regarded only as a reference since the orbit of S2 is far from circular. Moreover, the effect of the spin of the central object must be considered. The Kerr black hole gives the deviation of the periapsis shift from that in the Schwarzschild in the 1.5th post-Newtonian order in terms of the orbital velocity of the stars. This deviation can be estimated to be of the order of $10^{-4}$ or less for S2 star in Sgr A*, whereas the post-Newtonian shift in the Schwarzschild spacetime is of the order of $10^{-3}$. Thus, the assumption of spherical symmetry here is justified if the predicted deviation from that in the Schwarzschild spacetime is larger than that due to the spin of the central object.

VI. CONCLUSION

In this paper, we have obtained two expressions in full order with respect to the gravitational field for the periapsis shift of a quasi-circular orbit in a general static spherically symmetric spacetime: one in terms of the gravitational mass and the other the orbital angular velocity. We have found that it is a particular combination of the components of the Einstein tensor on the orbit that makes the periapsis shift of a quasi-circular orbit deviate from that in the Schwarzschild spacetime. A lesson from the present analysis is that by the periapsis shift experiment, we can access the geometrical properties of the spacetime on the orbit of the star but not directly the central gravitating object except for the gravitational mass, although they certainly are valuable. This means that if we want to use the periapsis shift as a probe into the central gravitating object and its surrounding region, we need to invoke some additional experiments, such as light bending and photon sphere observation and/or working assumptions such as modelling dark matter physics and/or choosing gravitational theories. In the post-Newtonian approximation near a massive central object, which is most plausibly valid for S stars near Sgr A*, we have derived the critical energy
density beyond which the periapsis shift becomes retrograde and showed that a prograde shift greater than the Schwarzschild value implies the violation of the weak energy condition in Einstein gravity.

Generalising the present work to include highly non-circular orbits is clearly important for a wider application and more quantitative discussion. To discuss the deviation smaller than that due to the spin of the central object, we should generalise the present formalism to stationary axisymmetric spacetimes (cf. [22]). These interesting generalisations are left for future work.

ACKNOWLEDGMENTS

The authors are grateful to H. Ishihara, T. Kobayashi, Y. Kojima, K. Nakamura, Y. Nakayama, K.-I. Nakao and C.-M. Yoo for their helpful comments. This work was supported by JSPS KAKENHI Grants No. JP19K03876, No. JP19H01895 and No. JP20H05853 (TH); No. JP19K14715 and No. JP22K03611(TI); No. JP19H01900 and No. JP19H00695 (HS, YT).

Appendix A: Periapsis shift in the anti-de Sitter spacetime

Maximally symmetric spacetimes in four dimensions are known to be Minkowski, de Sitter and anti-de Sitter spacetimes. These spacetimes are solutions of the Einstein equations with a cosmological constant \( \Lambda = 0, > 0 \) and \( < 0 \), respectively. The line element in these spacetimes is written in the following form:

\[
ds^2 = - \left(1 - \frac{1}{3} \Lambda r^2\right) dt^2 + \left(1 - \frac{1}{3} \Lambda r^2\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \tag{A1}
\]

and, hence, from Eq. (2.2) we have

\[
m = \frac{1}{6} \Lambda r^3. \tag{A2}
\]

The nonvanishing tetrad components of the Einstein tensor in these spacetimes are given by

\[
G_{00} = \Lambda, \quad G_{11} = G_{22} = G_{33} = -\Lambda. \tag{A3}
\]

Since the condition (3.20) imposes \( \Lambda \) to be negative, only the anti-de Sitter spacetime is admitted for the existence of a circular orbit. From Eqs. (3.21) and (3.22), we obtain \( A = 4 \) and, hence, \( \Delta \phi_p = -\pi \). That is, the periapses repeatedly appear in every half a round.
This can be understood as follows. The effective potential $V(r)$ in this case is given by

$$V(r) = \frac{1 - E^2}{2} - \frac{\Lambda L^2}{6} - \frac{1}{6} \Lambda r^2 + \frac{1}{2} \frac{L^2}{r^2},$$

which has an extremum if and only if $\Lambda < 0$. With $\Lambda < 0$, this agrees with the effective potential of a particle in the isotropic harmonic oscillator potential in Newtonian mechanics up to irrelevant constant terms. In this potential problem, it is well known that the general orbit is an ellipse whose centre is at $r = 0$, where periapses repeatedly appear in every half a round.

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