THE FU-YAU EQUATION ON COMPACT ASTHENO-KÄHLER MANIFOLDS

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Abstract. In this paper, we study the Fu-Yau equation on compact Hermitian manifolds and prove the existence of solutions of equation on astheno-Kähler manifolds. We also prove the uniqueness of solutions of Fu-Yau equation when the slope parameter \( \alpha \) is negative.

1. Introduction

Let \((M, \omega)\) be an \(n\)-dimensional compact Kähler manifold. As a reduced generalized Strominger system in higher dimensions, Fu and Yau introduced the following fully nonlinear equation for \( \varphi \) \[\text{[8]}, \]

\[
\sqrt{-1} \partial \bar{\partial} (e^\varphi \omega - \alpha e^{-\varphi} \rho) \wedge \omega^{n-2} 
+ n \alpha \sqrt{-1} \partial \bar{\partial} \varphi \wedge \sqrt{-1} \partial \bar{\partial} \varphi \wedge \omega^{n-2} + \mu \frac{\omega^n}{n!} = 0, \tag{1.1}
\]

where \( \alpha \) is a non-zero constant called the slope parameter, \( \rho \) is a real smooth \((1,1)\) form, \( \mu \) is a smooth function. For \( \varphi \), we impose the elliptic condition

\[
\tilde{\omega} = e^\varphi \omega + \alpha e^{-\varphi} \rho + 2n \alpha \sqrt{-1} \partial \bar{\partial} \varphi \in \Gamma_2(M) \tag{1.2}
\]

and the normalization condition

\[
\| e^{-\varphi} \|_{L^1} = A, \tag{1.3}
\]

where

\[
\Gamma_2(M) = \{ \alpha \in A^{1,1}(M) \mid \frac{\alpha^1 \wedge \omega^{n-1}}{\omega^n} > 0, \frac{\alpha^2 \wedge \omega^{n-2}}{\omega^n} > 0 \}
\]

and \( A^{1,1}(M) \) is the space of smooth real \((1,1)\) forms on \( M \).

When \( n = 2 \), (1.1) is equivalent to the Strominger system on a toric fibration over a \( K3 \) surface constructed by Goldstein and Prokushki \[10\], which was solved by Fu and Yau for \( \alpha > 0 \) and \( \alpha < 0 \) in \[8\] and \[7\], respectively. (1.1) is usually called Fu-Yau equation (cf. \[22\], \[9\]).

In case of \( \alpha < 0 \), Phong, Picard and Zhang \[22\] recently proved the existence of solutions of (1.1) with the condition (1.3) is replaced by

\[
\| e^\varphi \|_{L^1} = \frac{1}{A} \gg 1.
\]

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In [2], we prove that there exists constant $A_0$ depending only on $\alpha$, $\rho$, $\mu$ and $(M, \omega)$ such that for any $A \leq A_0$ and any $\alpha \neq 0$, (1.1) has a solution satisfying (1.2) and (1.3). Our result is new and different from that of [22] which deals with only the case that $\alpha < 0$.

Since the Strominger system comes from non-Kähler geometry [25], it is natural to consider (1.1) on Hermitian manifolds. In the Kähler case, all the proofs in [8, 7, 22, 2] relied heavily on the Kähler condition $d\omega = 0$. It seems to be very difficult to solve (1.1) on general Hermitian manifolds. In this paper, we focus on a class of Hermitian manifolds which satisfies the astheno-Kähler condition

$$\sqrt{-1} \partial \bar{\partial} \omega^{n-2} = 0.$$  

Astheno-Kähler manifold was first introduced in the paper of Jost-Yau [13], where they extended Siu’s rigidity results in Kähler manifolds to astheno-Kähler manifolds [24]. Such manifolds have many naturally properties as Kähler manifolds. For example, every holomorphic 1-form on a compact astheno-Kähler manifold is closed [13, Lemma 6].

There are many examples of astheno-Kähler manifolds, see [16, 17, 19, 18, 6, 5, 15]. For example, the product of a complex curve with a Kähler metric and a complex surface with a non-Kähler Gauduchon metric satisfies (1.4).

The purpose of this paper is to generalize the main result in [2] to astheno-Kähler manifolds. Namely, we prove

**Theorem 1.1.** Let $(M, \omega)$ be an $n$-dimensional compact astheno-Kähler manifold. Then there exist constants $A_0$, $C_0$, $\delta_0$, $M_0$ and $D_0$ depending only on $\alpha$, $\rho$, $\mu$ and $(M, \omega)$ such that for any $A \leq A_0$, there exists a unique solution $\varphi$ of (1.1) satisfying (1.2), (1.3) and

$$e^{-\varphi} \leq \delta_0, \ |\partial \varphi| g \leq D, \ D_0 \leq D \text{ and } A \leq \frac{1}{C_0 M_0 D}.$$  

Since in the non-Kähler case the constant is not a trivial solution in the continuity method when $t = 0$ in [2], we introduce a new continuous path to solve (1.1) as follows ($t \in [0, 1]$),

$$\sqrt{-1} \partial \bar{\partial} (e^{\varphi} \omega - te^{-\varphi} \rho) \wedge \omega^{n-2} + n\alpha \sqrt{-1} \partial \bar{\partial} \varphi \wedge \sqrt{-1} \partial \bar{\partial} \rho \wedge \omega^{n-2}$$

$$+ n\alpha (t - 1) \sqrt{-1} \partial \bar{\partial} h \wedge \sqrt{-1} \partial \bar{\partial} h \wedge \omega^{n-2} + t\mu \omega^n = 0,$$

where $h$ is a smooth function. Clearly, (1.6) is equivalent to (1.1) when $t = 1$. We will show that there is $h$ such that (1.6) can be solved when $t = 0$ (cf. Lemma 4.1).

In the proof of openness for the solvable set of $t$, the astheno-Kähler condition (1.4) will play an important role. (1.4) guarantees that the adjoint

\footnote{Phong, Pacard and Zhang posted a paper [23] with a similar result after we posted the paper in arXiv.}
of linearized operator $L$ has no zero order terms (cf. (4.4)), then the strong maximum principle can be applied. 

(1.4) will also be used for the $C^0$-estimate (cf. Lemma 2.1). In fact, instead of $L^1$-integral of $\varphi$ in [2], we first estimate a $L^{k_0}$-integral for some $k_0 \ll 1$, then apply the Moser iteration to derive the $C^0$-estimate. The $C^1, C^2$-estimates for solutions of (1.6) can be obtained by the argument in [2]. For the reader’s convenience, we give a sketch of the proofs in Section 3. Actually, the argument there are valid for solutions of (1.1) on any Hermitian manifolds $(M, \omega)$.

In the next part of this paper, we improve Theorem 1.1 without the restriction condition (1.5) in case of $\alpha < 0$. In fact, we prove the following uniqueness of Fu-Yau equation.

**Theorem 1.2.** Let $\alpha < 0$ and $(M, \omega)$ be an $n$-dimensional compact astheno-Kähler manifold. There exists a constant $A_0$ depending only on $\alpha$, $\rho$, $\mu$ and $(M, \omega)$ such that for any $A \leq A_0$, (1.7) has a unique smooth solution satisfying (1.2) and the $L^n$-normalization condition

$$
\|e^{-\varphi}\|_{L^n} = A.
$$

Furthermore, investigating the structure of the Fu-Yau equation, we obtain the monotonicity property of solutions.

**Theorem 1.3.** Let $\alpha < 0$ and $(M, \omega)$ be an $n$-dimensional compact Kähler manifold. Suppose that $\varphi$ and $\tilde{\varphi}$ are solutions of (1.1) satisfying (1.2). If $\|e^{-\varphi}\|_{L^n} = A$, $\|e^{-\tilde{\varphi}}\|_{L^n} = \tilde{A}$ and $A < \tilde{A} \leq A_0$, then we have $\varphi > \tilde{\varphi}$ on $M$, where $A_0$ is a constant depending only on $\alpha$, $\rho$, $\mu$ and $(M, \omega)$.

**Remark 1.4.** In addition, if $\text{tr} \omega \rho \geq 0$, both Theorem 1.2 and 1.3 are still true when $L^n$-normalization condition (1.7) is replaced by a weaker condition $\|e^{-\varphi}\|_{L^1} = A$ (see Remark 8.2). In particular, Theorem 1.2 is an improvement of main results in [7, 22] in Kähler case.

Roughly speaking, Theorem 1.2 and 1.3 are consequences of a priori estimates for $\varphi$. Compared to the proof of Theorem 1.1, we need to derive a strong $C^0, C^1, C^2$ estimates without (1.5). In order to use the blow-up argument for $C^1, C^2$ estimates, we establish an estimate

$$
\sup_M |\partial^3 \varphi|_g \leq C_A (1 + \sup_M |\partial \varphi|_g^2),
$$

where $C_A$ is a constant depending only on $A$, $\alpha$, $\rho$, $\mu$ and $(M, \omega)$. Such a kind of estimate (1.8) was widely studied in Monge-Ampère equations and $\sigma_k$ Hessian equations (cf. 30, 12, 4, 29, 22, 26, 27). In our case, we adopt an auxiliary function involving the largest eigenvalue $\lambda_1$ of $\omega$ with respect to $\omega$. This advantage gives us enough good third order terms to deal with the bad terms when we use the maximal principle as in [1]. Also the sign of $\alpha$ plays a crucial role. We note that (1.1) is not degenerate when $\alpha < 0$ (cf. (6.1)).
As we know, (1.1) can be rewritten as a $\sigma_2$-type equation on a Hermitian manifold with function $F$ at the right hand including the gradient term of solution (cf. (5.1), (6.1)). In [1], we generalized $\sigma_2$-equation to an almost Hermitian manifold and obtained a $C^2$-estimate for the solutions, which depends only on the gradient of solutions and background data. It is interesting to studying the $C^2$-estimate for solutions of $\sigma_k$-type equation in space of $\Gamma_k$ ($k \geq 2$) of $k$-convex functions (cf. [11, 26, 20, 21], etc.). But it seems nontrivial to generalize the method for $\sigma_2$-equation to $\sigma_k$-equation even on Kähler manifolds if $F$ involves the gradient term of solution.

The organization of paper is as follows. In Section 2 and Section 3, we give the $C^0$-estimate, and $C^1$, $C^2$-estimates for solutions of (1.1) under the condition in Theorem 1.1, respectively. Theorem 1.1 is proved in Section 4. In Section 5, we improve the $C^0$-estimate in Section 2 in case of $\alpha < 0$. In Section 6, we give another method to get strong $C^1$, $C^2$-estimates in case of $\alpha < 0$. Theorem 1.2 and Theorem 1.3 will be proved in Section 7, 8, respectively.

2. Zero order estimate (1)

In this section, we use the Moser iteration to do $C^0$-estimate for solutions $\varphi$ of (1.1). First, we prove a lemma for $L^2$-estimate of gradient $\partial \varphi$.

Lemma 2.1. Let $\varphi$ be a smooth solution of (1.1) satisfying (1.2). Let $f(t)$ be a smooth function in $\mathbb{R}$ such that $f' \geq 0$. Then we have

$$\int_M f'(\varphi)\sqrt{-1}\partial \varphi \wedge \overline{\partial} \varphi \wedge (e^\varphi \omega + \alpha e^{-\varphi} \rho) \wedge \omega^{n-2} \leq -2 \int_M f'(\varphi)\sqrt{-1}\partial \varphi \wedge (e^\varphi \omega - \alpha e^{-\varphi} \rho) \wedge \omega^{n-2} - 2 \int_M f(\varphi)\mu \omega^n. \tag{2.1}$$

Proof. Since $\tilde{\omega} \in \Gamma_2(M)$ and $f'(\varphi) \geq 0$, it is clear that

$$\int_M f'(\varphi)\sqrt{-1}\partial \varphi \wedge \overline{\partial} \varphi \wedge \tilde{\omega} \wedge \omega^{n-2} \geq 0.$$

By the Stokes’ formula , it follows

$$\int_M f'(\varphi)\sqrt{-1}\partial \varphi \wedge \overline{\partial} \varphi \wedge (e^\varphi \omega + \alpha e^{-\varphi} \rho) \wedge \omega^{n-2} \geq -2n\alpha \int_M \sqrt{-1}\partial f(\varphi) \wedge \overline{\partial} \varphi \wedge \sqrt{-1}\partial \overline{\partial} \varphi \wedge \omega^{n-2} = -2n\alpha \int_M f(\varphi)\overline{\partial} \varphi \wedge \sqrt{-1}\partial \overline{\partial} \varphi \wedge \sqrt{-1}\partial \omega^{n-2} + 2n\alpha \int_M f(\varphi)\mu \omega^n.$$
On the other hand, by (1.1), we have

\[
2n\alpha \int_M \bar{f}(\varphi) \sqrt{-1} \partial \bar{\partial} \varphi \wedge \sqrt{-1} \partial \bar{\partial} \varphi \wedge \omega^{n-2} = -2 \int_M \bar{f}(\varphi) \sqrt{-1} \partial \bar{\partial} (e^{\varphi} \omega - \alpha e^{-\varphi} \rho) \wedge \omega^{n-2} - 2 \int_M \bar{f}(\varphi) \frac{\omega^n}{n!}.
\]

Thus

\[
\int_M \bar{f}'(\varphi) \sqrt{-1} \partial \varphi \wedge \sqrt{-1} \partial \varphi \wedge (e^{\varphi} \omega + \alpha e^{-\varphi} \rho) \wedge \omega^{n-2} - 2 \int_M \bar{f}(\varphi) \sqrt{-1} \partial \varphi \wedge \sqrt{-1} \partial \varphi \wedge \omega^n - 2 \int_M \bar{f}(\varphi) \frac{\omega^n}{n!}.
\] (2.2)

For the first term on the right hand of the inequality (2.2), we have

\[
-2n\alpha \int_M \bar{f}(\varphi) \delta \varphi \wedge \sqrt{-1} \partial \varphi \wedge \sqrt{-1} \partial \omega^{n-2} = 2n\alpha \int_M \bar{\partial}(\bar{f}(\varphi) \delta \varphi) \wedge \sqrt{-1} \partial \varphi \wedge \sqrt{-1} \partial \omega^{n-2} + 2n\alpha \int_M \bar{f}(\varphi) \delta \varphi \wedge \sqrt{-1} \partial \varphi \wedge \sqrt{-1} \partial \omega^{n-2} = 2n\alpha \int_M \bar{f}(\varphi) \delta \varphi \wedge \sqrt{-1} \partial \varphi \wedge \sqrt{-1} \partial \omega^{n-2}.
\] (2.3)

For the second term of (2.2), we compute

\[
-2 \int_M \bar{f}(\varphi) \sqrt{-1} \partial \bar{\partial} (e^{\varphi} \omega - \alpha e^{-\varphi} \rho) \wedge \omega^{n-2} = 2 \int_M \sqrt{-1} \partial f(\varphi) \wedge \sqrt{-1} \partial \bar{\partial}(e^{\varphi} \omega - \alpha e^{-\varphi} \rho) \wedge \omega^{n-2} - 2 \int_M \bar{f}(\varphi) \delta (e^{\varphi} \omega - \alpha e^{-\varphi} \rho) \wedge \sqrt{-1} \partial \omega^{n-2}.
\]

and then,

\[
-2 \int_M \bar{f}(\varphi) \sqrt{-1} \partial \bar{\partial} (e^{\varphi} \omega - \alpha e^{-\varphi} \rho) \wedge \omega^{n-2} = 2 \int_M \bar{f}'(\varphi) \sqrt{-1} \partial \varphi \wedge \sqrt{-1} \partial \bar{\partial} \varphi \wedge (e^{\varphi} \omega + \alpha e^{-\varphi} \rho) \wedge \omega^{n-2} + 2 \int_M \bar{\partial} f(\varphi) \wedge (e^{\varphi} \omega - \alpha e^{-\varphi} \rho) \wedge \sqrt{-1} \partial \omega^{n-2} + 2 \int_M \bar{f}(\varphi)(e^{\varphi} \omega - \alpha e^{-\varphi} \rho) \wedge \sqrt{-1} \partial \bar{\partial} \omega^{n-2}.
\] (2.4)
Thus substituting (2.3) and (2.4) into (2.2), we see that
\[
\int_M f'(\varphi) \sqrt{-1} \partial \varphi \wedge \overline{\partial} \varphi \wedge (e^\varphi \omega + \alpha e^{-\varphi} \rho) \wedge \omega^{n-2}
\]
\[
\leq -2 \int_M f'(\varphi) \sqrt{-1} \partial \varphi \wedge (e^\varphi \overline{\partial} \omega - \alpha e^{-\varphi} \overline{\partial} \rho) \wedge \omega^{n-2}
\]
\[
-2 \int_M f'(\varphi) \overline{\partial} \varphi \wedge (e^\varphi \omega - \alpha e^{-\varphi} \rho) \wedge \sqrt{-1} \partial \omega^{n-2}
\]
\[
+ 2 \int_M f(\varphi) (e^\varphi \omega - \alpha e^{-\varphi} \rho) \wedge \sqrt{-1} \partial \overline{\partial} \omega^{n-2}
\]
\[
- 2na \int_M f(\varphi) \sqrt{-1} \partial \varphi \wedge \overline{\partial} \varphi \wedge \sqrt{-1} \partial \overline{\partial} \omega^{n-2} + 2 \int_M f(\varphi) \mu \omega^n / n!.
\]
Note that \(\sqrt{-1} \partial \overline{\partial} \omega^{n-2} = 0\). Hence, we get (2.1).

By Lemma 2.1, we prove the following \(C^0\)-estimate.

**Proposition 2.2.** Let \(\varphi\) be a smooth solution of (1.1) satisfying (1.2) and (1.3). There exist constants \(\delta_0\), \(A_0\) and \(M_0\) depending only on \(\alpha\), \(\rho\), \(\mu\) and \((M, \omega)\) such that if

\[
e^{-\varphi} \leq \delta_0 \quad \text{and} \quad \|e^{-\varphi}\|_{L^1} = A \leq A_0,
\]

then

\[
\frac{A}{M_0} \leq e^{-\varphi} \leq M_0 A.
\]

**Proof.** First, we estimate the positive infimum of \(e^\varphi\). At the expense of decreasing \(\delta_0\), we assume that

\[
e^\varphi \omega + \alpha e^{-\varphi} \rho \geq \frac{1}{2} e^\varphi \omega.
\]

Then by taking \(f(\varphi) = -e^{-(k+1)\varphi}\) \((k \geq 1)\) in Lemma 2.1, we have

\[
\int_M e^{-k\varphi} |\partial \varphi|^2 \omega^n \leq C \int_M e^{-k\varphi} |\partial \varphi| g \omega^n + C \int_M e^{-(k+1)\varphi} \omega^n.
\]

By the Cauchy-Schwarz inequality, it follows that

\[
\int_M e^{-k\varphi} |\partial \varphi|^2 \omega^n \leq C \int_M e^{-k\varphi} \omega^n.
\]

Hence, by the above relation together with the Sobolev inequality, one can use the Moser iteration to derive

\[
\|e^{-\varphi}\|_{L^\infty} \leq C \|e^{-\varphi}\|_{L^1} = CA.
\]

Next we estimate the supremum of \(e^\varphi\). As in the proof of (2.7), by taking

\[
f(\varphi) = \frac{1}{k-1} e^{(k-1)\varphi}
\]

in Lemma 2.1, we can also get

\[
\int_M e^{k\varphi} |\partial \varphi|^2 \omega^n \leq C \left(1 + \frac{1}{|k-1|}\right) \int_M e^{k\varphi} \omega^n.
\]
**Claim 1.** There exists a positive constant $k_0 \ll 1$ depending only on $\alpha$, $\rho$, $\mu$ and $(M, \omega)$ such that

\begin{equation}
\|e^\varphi\|_{L^{k_0}} \leq \frac{C}{A}.
\end{equation}

By (2.9), we use the Moser iteration to obtain

\[ \|e^\varphi\|_{L^\infty} \leq C_0 \|e^\varphi\|_{L^2}. \]

By (2.10), it follows

\[ \|e^\varphi\|_{L^\infty} \leq \frac{C_0^{k_0}}{k_0} \|e^\varphi\|_{L^{k_0}} \leq \frac{C}{A}. \]

Thus, the proof of Proposition 2.2 is complete.

It remains to prove Claim 1. Without loss of generality, we assume that $\Vol(M, \omega) = 1$. We define

\[ U = \{ x \in M \mid e^{-\varphi(x)} \geq \frac{A}{2} \}. \]

Then by (2.8), we have

\[ A = \int_M e^{-\varphi} \omega^n = \int_U e^{-\varphi} \omega^n + \int_{M \setminus U} e^{-\varphi} \omega^n \]

\[ \leq e^{-\inf_M \varphi} \Vol(U) + \frac{A}{2} (1 - \Vol(U)) \]

\[ \leq \left( C - \frac{1}{2} \right) A \Vol(U) + \frac{A}{2}, \]

which implies

\begin{equation}
\Vol(U) \geq \frac{1}{C_0}. 
\end{equation}

On the other hand, by the Poincaré inequality, we have

\[ \int_M e^{k_0 \varphi} \omega^n - \left( \int_M e^{k_0 \varphi} \omega^n \right)^2 \leq C \int_M |\partial e^{k_0 \varphi}|^2 \omega^n \leq C k_0^2 \int_M e^{k_0 \varphi} \omega^n. \]

It then follows that

\begin{equation}
\int_M e^{k_0 \varphi} \omega^n \leq \frac{1}{1 - C_0 k_0^2} \left( \int_M e^{k_0 \varphi} \omega^n \right)^2.
\end{equation}
Combining this with the Cauchy-Schwarz inequality, we obtain
\[
\left( \int_M e^{k_0 \varphi} \omega^n \right)^2 \leq (1 + \mathcal{C}_0) \left( \int_U e^{k_0 \varphi} \omega^n \right)^2 + \left( 1 + \frac{1}{\mathcal{C}_0} \right) \left( \int_{M \setminus U} e^{k_0 \varphi} \omega^n \right)^2.
\]
\[
\leq \frac{(1 + \mathcal{C}_0)2k_0}{\mathcal{A}k_0} (\text{Vol}(U))^2 + \left( 1 + \frac{1}{\mathcal{C}_0} \right) \left( 1 - \text{Vol}(U) \right)^2 \int_M e^{k_0 \varphi} \omega^n
\]
\[
\leq \frac{(1 + \mathcal{C}_0)2k_0}{\mathcal{A}k_0} + \left( 1 - \frac{1}{\mathcal{C}_0^2} \right) \frac{1}{1 - \mathcal{C}_0 k_0} \left( \int_M e^{k_0 \varphi} \omega^n \right)^2.
\]
By choosing \( k_0 \ll 1 \), we see that
\[
\left( \int_M e^{k_0 \varphi} \omega^n \right)^2 \leq \frac{\mathcal{C}}{\mathcal{A}k_0}.
\]
Thus, we get from (2.12),
\[
\int_M e^{k_0 \varphi} \omega^n \leq \frac{\mathcal{C}}{\mathcal{A}k_0}.
\]
Claim 1 is proved.

3. First and second order estimates (I)

In this section, we give a sketch of proofs of \( C^1, C^2 \) estimates of \( \varphi \). As in [2], the basic idea is to rewrite (1.1) as a \( \sigma_2 \)-type equation,
\[
\sigma_2(\tilde{\omega}) = \frac{n(n-1)}{2} \left( e^{2\varphi} - 4\alpha e^{-\varphi} |\partial\varphi|^2 + n(n-1) \right) f,
\]
where
\[
f \omega^n = 2\alpha \rho \wedge \omega^{n-1} + \alpha^2 e^{-2\varphi} \rho^2 \wedge \omega^{n-2} - 4n\alpha \mu \frac{\varphi^n}{n!}
\]
\[
+ 4\alpha^2 e^{-\varphi} \sqrt{-1} \left( \partial\varphi \wedge \overline{\partial}\varphi \wedge \rho - \partial\varphi \wedge \overline{\partial}\rho - \partial\varphi \wedge \overline{\partial} + \overline{\partial}\varphi + \partial\overline{\partial}\rho \right) \wedge \omega^{n-2}
\]
\[
+ 4nae^{\varphi} \sqrt{-1} \left( \partial\omega \wedge \overline{\partial}\varphi + \partial\varphi \wedge \overline{\partial}\omega + \partial\overline{\partial}\omega \right) \wedge \omega^{n-2}.
\]
As in [2], we define \( \tilde{\omega} = e^{-\varphi} \omega \). Then (3.1) becomes
\[
\sigma_2(\tilde{\omega}) = \frac{n(n-1)}{2} \left( 1 - 4\alpha e^{-\varphi} |\partial\varphi|^2 \right) + \frac{n(n-1)}{2} e^{2\varphi} f.
\]
Since \( \omega \) is not Kähler, the function \( f \) is more complicated than one in [2]. Precisely, more terms involving \( e^{\varphi} \) and \( \partial\varphi \) appears. However, for the right hand side of (3.3), the leading term is still \(-2n(n-1)\alpha e^{-\varphi} |\partial\varphi|^2 \). Thus we will obtain a similar inequality as in Kähler case when we differentiate (3.3). This is why we can prove an analogy of [2] Proposition 3.1, 4.1 as follows.
Proposition 3.1. Let $\varphi$ be a smooth solution of (1.1) satisfying (1.2) and $\frac{1}{M_0 A} \leq e^{-\varphi} \leq M_0 A$ for some uniform constant $M_0$. There exist uniform constants $D_0$ and $C_0$ such that if

$$|\partial \varphi|_g \leq D, \quad D_0 \leq D \text{ and } A \leq A_D := \frac{1}{C_0 M_0 D},$$

then

$$|\partial \varphi|_g^2 \leq M_1 \text{ and } |\partial \varphi|_g \leq \frac{D}{2}.$$

Proof. i). $C^1$-estimate. As in [2], we consider the following auxiliary function,

$$Q = \log |\partial \varphi|_g^2 + \frac{\varphi}{B},$$

where $B$ is a uniform constant to be determined. Let $x_0$ be the maximum point of $Q$ and $\{e_i\}_{i=1}^n$ be a local unitary frame in a neighbourhood of $x_0$ such that, at $x_0$,

$$\tilde{g}_\sigma = \delta_i^j \tilde{g}_\sigma = \delta_i^j (e^\varphi + \alpha e^{-\varphi} \rho_{ik} + 2n\alpha \varphi_k).$$

We use the following notations

$$F_i^j = \frac{\partial \sigma_2(\hat{\omega})}{\partial \tilde{g}_{ij}} \text{ and } F_i^j,k = \frac{\partial^2 \sigma_2(\hat{\omega})}{\partial \tilde{g}_{ij} \partial \tilde{g}_{kl}},$$

where $\hat{\omega} = e^{-\varphi}\hat{\omega}$. By (3.4), we know that

$$(3.6) \quad \left| F_i^j - (n-1) \right| \leq \frac{1}{100}.$$

Then by a direct calculation, we have

$$F_i^j e_i e_j (|\partial \varphi|_g^2) \geq \frac{4}{5} \sum_{i,j}(|e_i e_j(\varphi)|^2 + |e_i e_j(\varphi)|^2) - C|\partial \varphi|_g^2$$

$$\quad \quad \quad + \sum_k F_i^j (e_k(\varphi_{ik}) + e_k(\varphi_{ik})\varphi_k). \quad (3.7)$$

Next, we deal with the terms involving three derivatives of $\varphi$ in (3.7). Differentiating (3.3) along $e_k$ at $x_0$, we have

$$2n\alpha F_i^j e_k (\varphi_{ik}) = 2\alpha e^{-\varphi} \varphi_k F_i^j \rho_{ik} - \alpha e^{-\varphi} F_i^j e_k(\rho_{ik}) + 2n\alpha \varphi_k F_i^j \varphi_{ik}$$

$$\quad - 2\alpha(n-1)(|\partial \varphi|_g^2)_k + 2\alpha n(n-1)|\partial \varphi|_g^2 \varphi_k$$

$$\quad \quad \quad - n(n-1)e^{-\varphi} f \varphi_k + \frac{n(n-1)}{2} e^{-\varphi} f_k. \quad (3.8)$$
which implies

\[
\sum_k F^\varphi_k (e_k(\varphi^\varphi)\varphi_k + \bar{e}_k(\varphi^\bar{\varphi})\varphi_k) \\
\geq - C e^{-\varphi} |\partial \varphi|_g^2 - C e^{-\varphi} |\partial \varphi|_g + 2|\partial \varphi|_g^2 F^\varphi \varphi_k \\
+ 2(n-1)|\partial \varphi|_g^4 - 2(n-1)\text{Re} \left( \sum_k (|\partial \varphi|_g^2)_{k,\varphi_k} \right) \\
- \frac{n-1}{\alpha} e^{-\varphi} |\partial \varphi|_g^2 f + \frac{n-1}{2\alpha} e^{-\varphi} \text{Re} \left( \sum_k f_k \varphi_k \right). 
\]

(3.9)

For the third and fourth term of (3.9), by the argument of [2, (3.14)], we obtain

\[
2|\partial \varphi|_g^2 F^\varphi \varphi_k + 2(n-1)|\partial \varphi|_g^4 \\
\geq - \frac{1}{10} \sum_{i,j} |e_i \bar{e}_j(\varphi)|^2 - \left( C e^{-2\varphi} + \frac{1}{B} \right) |\partial \varphi|_g^4 - C|\partial \varphi|_g^2. 
\]

(3.10)

For the last two terms of (3.9). By the similar calculation of [2, (3.10)] and the expression of \( f \) (3.2), at \( x_0 \), we get

\[
- \frac{n-1}{\alpha} e^{-\varphi} |\partial \varphi|_g^2 f + \frac{n-1}{2\alpha} e^{-\varphi} \text{Re} \left( \sum_k f_k \varphi_k \right) \\
\geq - C \left( e^{-2\varphi} |\partial \varphi|_g + e^{-2\varphi} + |\partial \varphi|_g \right) \sum_{i,j} (|e_i \bar{e}_j(\varphi)| + |e_i e_j(\varphi)|) \\
- C e^{-\varphi} |\partial \varphi|_g^4 - C e^{-\varphi} |\partial \varphi|_g^2 - C e^{-\varphi} |\partial \varphi|_g \\
- C|\partial \varphi|_g^3 - C|\partial \varphi|_g^2 \\
\geq - \frac{1}{10} \sum_{i,j} (|e_i \bar{e}_j(\varphi)|^2 + |e_i e_j(\varphi)|^2) - \left( C e^{-\varphi} + \frac{1}{2B} \right) |\partial \varphi|_g^4 \\
- CB^3, 
\]

(3.11)

where we used the Cauchy-Schwarz inequality in the last inequality. Substituting (3.10) and (3.11) into (3.9), we derive

\[
\sum_k F^\varphi_k (e_k(\varphi^\varphi)\varphi_k + \bar{e}_k(\varphi^\bar{\varphi})\varphi_k) \\
\geq - \frac{1}{5} \sum_{i,j} (|e_i \bar{e}_j(\varphi)|^2 + |e_i e_j(\varphi)|^2) - 2(n-1)\text{Re} \left( \sum_k (|\partial \varphi|_g^2)_{k,\varphi_k} \right) \\
- \left( C e^{-\varphi} + \frac{3}{2B} \right) |\partial \varphi|_g^4 - C|\partial \varphi|_g^2 - CB^3. 
\]

(3.12)
Hence, substituting this into (3.7), we see that
\[
F_{i j} \overline{e}_i e_j (|\partial \varphi|_g^2) \\
\geq \frac{3}{5} \sum_{i, j} (|e_i \overline{e}_j(\varphi)|^2 + |e_i e_j(\varphi)|^2) - 2(n - 1) \text{Re} \left( \sum_k (|\partial \varphi|_g^2) k \varphi \right) \\
- \left( Ce^{-\varphi} + \frac{3}{2B} \right) |\partial \varphi|_g^4 - C |\partial \varphi|_g^2 - CB^3.
\]
By the maximum principle, at \( x_0 \), we obtain
\[
0 \geq F_{i j} e_i \overline{e}_j (Q) \\
\geq \frac{1}{2|\partial \varphi|_g^2} \sum_{i, j} (|e_i \overline{e}_j(\varphi)|^2 + |e_i e_j(\varphi)|^2) - \frac{2(n - 1) \text{Re} \left( \sum_k (|\partial \varphi|_g^2) k \varphi \right)}{|\partial \varphi|_g^2} \\
- \frac{F_{i j}^\ast |e_i (|\partial \varphi|_g^2)|^2}{|\partial \varphi|_g^2} - \left( Ce^{-\varphi} + \frac{1}{B} \right) |\partial \varphi|_g^2 - C + \frac{1}{B} F_{i j}^\ast e_i \overline{e}_j (\varphi).
\]
On the other hand, by the fact \( dQ = 0 \) and the Cauchy-Schwarz inequality, we get (cf. [2, (3.17)-(3.19)])
\[
- \frac{2(n - 1) \text{Re} \left( \sum_k (|\partial \varphi|_g^2) k \varphi \right)}{|\partial \varphi|_g^2} + \frac{F_{i j}^\ast |e_i (|\partial \varphi|_g^2)|^2}{|\partial \varphi|_g^2} + \frac{1}{B} F_{i j}^\ast e_i \overline{e}_j (\varphi) \\
\geq - \frac{1}{4|\partial \varphi|_g^2} \sum_{i, j} |e_i \overline{e}_j(\varphi)|^2 + \frac{2(n - 1)}{B} |\partial \varphi|_g^2 - \frac{C}{B^2} |\partial \varphi|_g^2.
\]
Substituting this into (3.14), we see that
\[
0 \geq \frac{1}{4|\partial \varphi|_g^2} \sum_{i, j} (|e_i \overline{e}_j(\varphi)|^2 + |e_i e_j(\varphi)|^2) - C_0 B^3 \\
+ \left( \frac{4n - 7}{2B} - \frac{C_0}{B^2} - C_0 e^{-\varphi} \right) |\partial \varphi|_g^2.
\]
Since \( A \ll 1 \), we may assume
\[
C_0 e^{-\varphi} \leq \frac{1}{32C_0}.
\]
By choosing \( B = 4C_0 \) in (3.15), we see that
\[
|\partial \varphi|_g^2 (x_0) \leq 2^{11} C_0^5.
\]
Note that \( \frac{1}{M_0 A} \leq e^{-\varphi} \leq M_0 A \). Hence, we obtain
\[
\max_M |\partial \varphi|_g^2 \leq e^{\frac{1}{2}(\sup_M \varphi - \inf_M \varphi)} |\partial \varphi|_g^2 (x_0) \leq C.
\]
ii). \( C^2 \)-estimate. The proof is almost as same as [2 Proposition 4.1]. We consider the following auxiliary function,
\[
Q = |\overline{\partial} \varphi|_g^2 + B |\partial \varphi|_g^2.
\]
where $B$ is a uniform constant to be determined. Let $x_0$ be the maximum point of $Q$ and $\{e_i\}_{i=1}^n$ be the local unitary frame such that $\tilde{g}(x_0)$ is diagonal. By direct calculation, we have

$$F^{\tilde{e}_i \tilde{e}_i}(\xi, \tilde{\varphi}^2) = 2 \sum_{k,l} F^{\tilde{e}_i \tilde{e}_i}(\varphi_{kl}) \varphi_{kl} + 2 \sum_{k,l} F^{\tilde{e}_i \tilde{e}_i}(\varphi_{kl}) \varphi_{kl}$$

$$= -2 |\partial\varphi| g \sum_{k,l} |F^{\tilde{e}_i \tilde{e}_i}(\varphi_{kl})| + \sum_{i,j,p} |e_p e_j e_i(\varphi)|^2$$

$$- C \sum_{i,j} (|e_i e_j(\varphi)|^2 + |e_i e_j(\varphi)|^2) - C.$$

To deal with the fourth order terms $\sum_{k,l} F^{\tilde{e}_i \tilde{e}_i}(\varphi_{kl}) \varphi_{kl}$ in (3.17), we differentiate (3.13) twice along $e_k$ and $\tilde{e}_i$, we get

$$F^{\tilde{e}_i \tilde{e}_i}(e^{-\varphi} \tilde{g}^{-\varphi} \tilde{g}^{-\varphi} \tilde{g}^{-\varphi}) + F^{\tilde{e}_i \tilde{e}_i}(e^{-\varphi} \tilde{g}^{-\varphi} \tilde{g}^{-\varphi} \tilde{g}^{-\varphi})$$

$$= -2n(n-1)\alpha e_k \tilde{e}_i (e^{-\varphi} |\partial\varphi|^2) + \frac{n(n-1)}{2} e_k \tilde{e}_i (e^{-2\varphi} f).$$

By the similar argument of [2] Lemma 4.2 and the expression of $f$ (3.2), we obtain

$$|F^{\tilde{e}_i \tilde{e}_i}(\varphi_{kl})| \leq 8n |\alpha| e^{-\varphi} \sum_{i,j,p} |e_p e_j e_i(\varphi)|^2 + C \sum_{i,j,p} |e_p e_j e_i(\varphi)|^2 + C \sum_{i,j} (|e_i e_j(\varphi)|^2 + |e_i e_j(\varphi)|^2) + C.$$

Substituting this into (3.17) and using (3.1), we obtain

$$F^{\tilde{e}_i \tilde{e}_i}(\xi, \tilde{\varphi}^2) \geq -C_0 (D + 1) \sum_{i,j} (|e_i e_j(\varphi)|^2 + |e_i e_j(\varphi)|^2) - C_0.$$

On the other hand, by (3.13) and $C^1$-estimate, we have

$$F^{\tilde{e}_i \tilde{e}_i}(\xi, \tilde{\varphi}^2) \geq \frac{1}{2} \sum_{i,j} (|e_i e_j(\varphi)|^2 + |e_i e_j(\varphi)|^2) - C_1.$$

Hence, by the maximum principle, at $x_0$, we get

$$0 \geq F^{\tilde{e}_i \tilde{e}_i}(Q)$$

$$= F^{\tilde{e}_i \tilde{e}_i}(\xi, \tilde{\varphi}^2) + B F^{\tilde{e}_i \tilde{e}_i}(\xi, \tilde{\varphi}^2)$$

$$\geq \left( \frac{B}{2} - C_0 D - C_0 \right) \sum_{i,j} (|e_i e_j(\varphi)|^2 + |e_i e_j(\varphi)|^2) - C_0 - C_1 B.$$

Choose $B = 8C_0 D + 8C_0$. It follows that

$$|\partial\varphi|^2(x_0) \leq C.$$
Therefore, by (3.16), at the expense of increasing $D_0$, we obtain
\[
\max_M |\partial \bar{\partial} \varphi|^2 \leq |\partial \bar{\partial} \varphi|^2(x_0) + BC \leq CD \leq \frac{D^2}{4}.
\]
□

4. Proof of Theorem 1.1

In this section, we solve (1.6) when any $t \in [0,1]$. The following lemma shows the existence of solutions when $t = 0$.

Lemma 4.1. Let $(M, \omega)$ be an $n$-dimensional compact astheno-Kähler manifold. Then there exists a function $h \in C^\infty(M)$, unique up to addition of a constant, such that
\[
(4.1) \quad \sqrt{-1} \partial \bar{\partial} (e^h \omega) \wedge \omega^{n-2} = 0.
\]

Proof. First we prove the existence. We define an elliptic operator $\tilde{L}$ by
\[
(\tilde{L}u)\omega^n = \sqrt{-1} \partial \bar{\partial} (u \omega) \wedge \omega^{n-2}.
\]
Let $\tilde{L}^*$ be a $L^2$-adjoint operator of $\tilde{L}$. Then by Stokes’ formula and the condition $\sqrt{-1} \partial \bar{\partial} \omega^{n-2} = 0$, we see that
\[
(\tilde{L}^* v)\omega^n = \sqrt{-1} \partial \bar{\partial} v \wedge \omega^{n-1} + \sqrt{-1} \partial v \wedge \bar{\partial} \omega^{n-2} - \bar{\partial} v \wedge \partial \omega^{n-2}.
\]
It follows that $\text{Ker} \tilde{L}^* = \{\text{constants}\}$ and $\text{Ind}(\tilde{L}) = \text{Ind}(\tilde{L}^*) = 0$. Thus
\[
\dim(\text{Ker} \tilde{L}) = 1.
\]
Denote the generator of $\text{Ker} \tilde{L}$ by $v_0$. Then $v_0$ does not change the sign, and we may assume that $v_0 \geq 0$. By the strong maximum principle, we know that $v_0 > 0$. Hence, $h = \log v_0$ satisfies (4.1).

For the uniqueness, by $\dim(\text{Ker} \tilde{L}) = 1$, we see that the only solution of (4.1) is $h + c$, where $c$ is a constant. □

Choose the function $h$ in (1.6) as a solution of (4.1). We consider solution $\varphi = \varphi_t$ of (1.6) which satisfies the elliptic condition
\[
(4.2) \quad e^{\varphi} \omega + tae^{-\varphi} + 2n\alpha \sqrt{-1} \partial \bar{\partial} \varphi \in \Gamma_2(M)
\]
and the normalization condition
\[
(4.3) \quad ||e^{-\varphi}||_{L^1} = A.
\]
By taking $\varphi_0 = h + \log ||e^{-h}||_{L^1} - \log A$, it is easy to see that $||e^{-\varphi_0}||_{L^1} = A$ and $\varphi_0$ is a solution of (1.6) when $t = 0$. Moreover, since $A \ll 1$,
\[
e^{\varphi_0} \omega + 2n\alpha \sqrt{-1} \partial \bar{\partial} \varphi_0 = \frac{||e^{-h}||_{L^1}}{A} e^h \omega + 2n\alpha \sqrt{-1} \partial \bar{\partial} h > 0.
\]
Thus $\varphi_0$ satisfies (4.2) and (4.3).
For a fixed $\beta \in (0, 1)$, we define
\[ B = \{ \varphi \in C^{2,\beta}(M) \mid \|e^{-\varphi}\|_{L^1} = A \}, \]
\[ B_1 = \{ (\varphi, t) \in B \times [0, 1] \mid \varphi \text{ satisfies (4.2)} \}, \]
\[ B_2 = \{ u \in C^\beta(M) \mid \int_M u\omega^n = 0 \}. \]

Then $B_1$ is an open subset of $B \times [0, 1]$. Since $\int_M \mu\omega^n = 0$, we introduce a map $\Phi : B_1 \to B_2$,
\[ \Phi(\varphi, t) = \sqrt{-1} \partial\overline{\partial}(e^\varphi\omega - t\alpha e^{-\varphi}\rho) \wedge \omega^{n-2} \]
\[ + n\alpha \sqrt{-1} \partial\overline{\partial} \varphi \wedge \sqrt{-1} \partial\overline{\partial} \varphi \wedge \omega^{n-2} \]
\[ + 2n\alpha (t - 1) \sqrt{-1} \partial\overline{\partial} h \wedge \sqrt{-1} \partial\overline{\partial} h \wedge \omega^{n-2} + t\mu \frac{\omega^n}{n!}. \]

Set
\[ I = \{ t \in [0, 1] \mid \text{there exists } (\varphi, t) \in B_1 \text{ such that } \Phi(\varphi, t) = 0 \}. \]

Then the existence of solutions of (1.1) is reduced to proving that $I$ is both open and closed.

**Proof of Theorem 1.1. Openness.** Suppose that $\hat{t} \in I$ and there exists $(\hat{\varphi}, \hat{t}) \in B_1$ such that $\Phi(\hat{\varphi}, \hat{t}) = 0$. Let
\[ L : \{ u \in C^{2,\beta}(M) \mid \int_M u e^{-\hat{\varphi}}\omega^n = 0 \} \to \{ v \in C^\beta(M) \mid \int_M v\omega^n = 0 \} \]
be a linearized operator of $\Phi$ at $\hat{\varphi}$. Then
\[ (Lu)\omega^n = \sqrt{-1} \partial\overline{\partial}(ue^{-\hat{\varphi}}\omega + t\alpha e^{-\hat{\varphi}}\rho) \wedge \omega^{n-2} \]
\[ + 2n\alpha (t - 1) \sqrt{-1} \partial\overline{\partial} h \wedge \omega^{n-2} + t\mu \frac{\omega^n}{n!}. \]

By the implicit function theorem, it suffices to prove that $L$ is injective and surjective.

Let $L^*$ be a $L^2$-adjoint operator of $L$. By the fact $\sqrt{-1} \partial\overline{\partial} \omega^{n-2} = 0$ and Stokes’ formula, it follows that
\[ L^*(v)\omega^n = \sqrt{-1} \partial\overline{\partial} v \wedge \left( e^\varphi\omega + t\alpha e^{-\varphi}\rho + 2n\alpha \sqrt{-1} \partial\overline{\partial} \varphi \right) \wedge \omega^{n-2} \]
\[ + \sqrt{-1} \partial\overline{\partial} v \wedge \left( e^\varphi\omega + t\alpha e^{-\varphi}\rho + 2n\alpha \sqrt{-1} \partial\overline{\partial} \varphi \right) \wedge \overline{\partial} \omega^{n-2} \]
\[ - \overline{\partial} v \wedge \left( e^\varphi\omega + t\alpha e^{-\varphi}\rho + 2n\alpha \sqrt{-1} \partial\overline{\partial} \varphi \right) \wedge \sqrt{-1} \partial\overline{\partial} \omega^{n-2}. \]

Thus $L^*$ has no zero order terms. By the strong maximum principle, we see that
\[ \text{Ker} L^* \subset \{ \text{constants} \}. \]
As a consequence, \( \text{Ker} L \subset \{ cu_0 \mid c \in \mathbb{R} \} \) for some smooth function \( u_0 \) by \( \text{Ind}(L) = 0 \). On the other hand, again by the strong maximum principle, we may assume that \( u_0 > 0 \). Thus

\[
u_0 \notin \{ u \in C^{2,\beta}(M) \mid \int_M u e^{-\tilde{\varphi}} \omega^n = 0 \},
\]

which implies \( \text{Ker} L = 0 \), and so \( L \) is injective.

Next, for any \( w \in \{ v \in C^{\beta}(M) \mid \int_M v \omega^n = 0 \} \), by the Fredholm alternative and regularity theory of elliptic equations, there exists a function \( \tilde{u} \in C^{2,\beta}(M) \) such that

\[
L(\tilde{u} + c_0 u_0) = w \text{ and } \tilde{u} + c_0 u_0 \in \{ u \in C^{2,\beta}(M) \mid \int_M u e^{-\tilde{\varphi}} \omega^n = 0 \},
\]

where

\[
c_0 = -\frac{\int_M \tilde{u} e^{-\tilde{\varphi}} \omega^n}{\int_M u_0 e^{-\varphi_0} \omega^n}.
\]

This implies that \( L \) is surjective.

**Closeness.** First we prove the \( C^0 \)-estimate along (1.6). Recalling \( \varphi_0 = h + \log \| e^{-h} \|_{L^1} - \log A \) and \( A \ll 1 \), we have \( \sup_M e^{-\varphi_0} \leq M_0 A \). We claim

**Claim 2.**

\[
\sup_M e^{-\varphi_t} \leq 2M_0 A, \quad \forall t \in [0,1].
\]

If the claim is false, there exists \( \tilde{t} \in (0,1) \) such that

\[
(4.6) \quad \sup_M e^{-\varphi_{\tilde{t}}} = 2M_0 A.
\]

Since \( A \ll 1 \), we assume that \( 2M_0 A \leq \delta_0 \), where \( \delta_0 \) is the constant in Proposition 2.2. Then, applying Proposition 2.2 while \( \rho \) and \( \mu \) are replaced by \( t\rho \) and \( n\alpha(t-1)n! \sqrt{-1} \partial \bar{\partial} \omega^n \wedge \omega^{n-2} + t\mu \), we obtain \( \sup_M e^{-\varphi_t} \leq M_0 A \), which contradicts with (4.6). Thus the claim is true.

By Claim 2, we see that Proposition 2.2 and Proposition 3.1 hold for \( \varphi_\tilde{t} \). As a consequence, we get the \( C^2 \)-estimate for \( \varphi_\tilde{t} \) along (1.6). Then combining the \( C^{2,\alpha} \)-estimate (cf. [28, Theorem 1.1]) and the bootstrapping argument, we complete the proof of closeness (for more details, we refer the reader to [2, Section 5.2]).

**Uniqueness.** The uniqueness of solutions of (1.1) can be proved by a similar argument of [2, Section 5.3] (also see the proof of Theorem 1.2 in Section 7 below). It suffices to prove that \( \varphi_0 \) of (1.6) is unique when \( t = 0 \) by the estimates in Proposition 2.2 and Proposition 3.1. But the latter is guaranteed by Remark 7.2 in Section 7. \( \square \)
In this section, we improve Proposition 2.2 in the case of $\alpha < 0$. The key point is to drop the condition $e^{-\phi} \leq \delta_0$. We begin with the following lemma.

**Lemma 5.1.** Let $\alpha < 0$ and $\phi$ be a smooth solution of (1.1) satisfying (1.2) and (1.7). There exists constant $A_0$ depending only on $\alpha$, $\rho$, $\mu$ and $(M, \omega)$ such that if any $A \leq A_0$, then

$$e^{-\phi} \leq C_0.$$ \hfill (5.1)

**Proof.** The elliptic condition $\tilde{\omega} \in \Gamma_2(M)$ implies that

$$\int_M e^{-k\varphi} \tilde{\omega} \wedge \omega^{n-1} \geq 0.$$ 

Namely,

$$0 \leq \int_M e^{-k\varphi} (e^{\varphi} \omega + \alpha e^{-\varphi} \rho + 2n\alpha \sqrt{-1} \partial \overline{\partial} \varphi) \wedge \omega^{n-1}.$$ 

By the Stokes’ formula, for $k > 1$, it follows that

$$0 \leq \int_M e^{-(k-1)\varphi} \omega^n - |\alpha| \int_M e^{-(k+1)\varphi} \rho \wedge \omega^{n-1}$$

$$+ 2n\alpha \int_M e^{-k\varphi} \sqrt{-1} \partial \omega^{n-1}$$

$$- 2n|\alpha|k \int_M e^{-k\varphi} \sqrt{-1} \partial \varphi \wedge \overline{\partial} \varphi \wedge \omega^{n-1}$$

$$\leq C \int_M \left( e^{-(k-1)\varphi} + e^{-(k+1)\varphi} \right) \omega^n$$

$$- 2n\alpha k \int_M e^{-k\varphi} \sqrt{-1} \partial \overline{\partial} \omega^{n-1}$$

$$- 2n|\alpha|k \int_M e^{-k\varphi} \sqrt{-1} \partial \varphi \wedge \overline{\partial} \varphi \wedge \omega^{n-1}.$$ \hfill (5.2)

Thus by the Sobolev inequality, we obtain

$$\left( \int_M e^{-k\beta \varphi} \omega^n \right)^{\frac{1}{n}} \leq C_1 k \int_M e^{-(k-1)\varphi} \omega^n + C_1 \int_M e^{-(k+1)\varphi} \omega^n + C \int_M e^{-k\varphi} \omega^n,$$ \hfill (5.3)

where $\beta = \frac{n}{n-1}$.

Next, we prove that $\|e^{-\varphi}\|_{L^{n+1}} \leq C$ by (5.3). In fact, by taking $k = n$ in (5.3) and the Hölder inequality, we see that

$$\|e^{-\varphi}\|_{L^{n, \beta}} \leq C_1 n \|e^{-\varphi}\|_{L^{n-1}} + C_1 n \|e^{-\varphi}\|_{L^n} \|e^{-\varphi}\|_{L^{n, \beta}} + C \|e^{-\varphi}\|_{L^n}$$

$$\leq CA^{n-1} + C_1 n A \|e^{-\varphi}\|_{L^{n, \beta}} + CA.$$
Note that $C_1 n A \leq \frac{1}{2}$. Then

$$\|e^{-\varphi}\|_{L^{n, \beta}} \leq C.$$ 

Thus we get

(5.4)

$$\|e^{-\varphi}\|_{L^{n+1}} \leq \|e^{-\varphi}\|_{L^{n, \beta}} \leq C.$$ 

Finally, we use the iteration to obtain (5.1). Let $H = e^{-\varphi} + 1$. It suffices to prove that $\|H\|_{L^\infty} \leq C$. By (5.3), it is easy to see that

$$\|H\|_{L^k} \leq (Ck)^{\frac{k}{k+1}} \|H\|_{L^{k+1}}^{\frac{k+1}{k}}.$$ 

For $j \geq 0$, we define

$$p_j = \beta^j + \frac{\beta}{\beta - 1}, \quad a_j = (Cp_j - p_j)^{\frac{1}{p_j - 1}} \quad \text{and} \quad b_j = \frac{p_j}{p_j - 1}.$$ 

It then follows that

$$\|H\|_{L^{p_j+1}} \leq a_j \|H\|_{L^{p_j}}^{b_j},$$

which implies

(5.5)

$$\|H\|_{L^{p_{j+1}}} \leq a_j a_{j-1} \cdots a_0 b_j \cdots b_1 \|H\|_{L^{p_0}}^{b_j \cdots b_1}.$$ 

Note that $\prod_{i=1}^{\infty} a_i < \infty$ and $\prod_{i=1}^{\infty} b_i < \infty$. Thus as $j \to \infty$, we obtain from (5.5),

$$\|H\|_{L^\infty} \leq C \|H\|_{L^{n+1}}^{C} \leq C'.$$

Here we used (5.4). \qed

Now, we apply Lemma 5.1 and Lemma 2.1 to improve Proposition 2.2 as follows.

**Proposition 5.2.** Let $\alpha < 0$ and $\varphi$ be a smooth solution of (1.1) satisfying (1.2) and (1.7). There exist constants $A_0$ and $M_0$ depending only on $\alpha$, $\rho$, $\mu$ and $(M, \omega)$ such that if $A \leq A_0$, then

$$\frac{A}{M_0} \leq e^{-\varphi} \leq M_0 A.$$ 

**Proof.** By taking $f(\varphi) = -e^{-k\varphi}$ ($k \geq 2$) in Lemma 2.1 we have

$$k \int_M e^{-k\varphi} (e^{\varphi} \omega + \alpha e^{-\varphi} \rho) \wedge \sqrt{-1} \partial \varphi \wedge \overline{\partial} \varphi \wedge \omega^{n-2} \leq -2k \int_M e^{-k\varphi} \sqrt{-1} \partial \varphi \wedge (e^{\varphi} \overline{\partial} \omega - \alpha e^{-\varphi} \overline{\partial} \rho) \wedge \omega^{n-2}$$

$$-2k \int_M e^{-k\varphi} \overline{\partial} \varphi \wedge (e^{\varphi} \omega - \alpha e^{-\varphi} \rho) \wedge \sqrt{-1} \partial \omega^{n-2} - 2 \int_M e^{-k\varphi} \mu \frac{\omega^n}{n!}.$$ 

It follows that

$$\int_M e^{-(k-1)\varphi} |\partial \varphi|^2 \omega^n \leq C \int_M \left( e^{-(k+1)\varphi} |\partial \varphi|_g + e^{-k\varphi} \right) \omega^n.$$
Combining this with Lemma 5.1 and the Cauchy-Shwarz inequality, we get
\[
\int_M e^{-(k-1)\varphi} |\partial \varphi|_{\omega}^2 \omega^n \leq C \int_M \left( e^{-(k+3)\varphi} + e^{-k\varphi} \right) \omega^n \\
\leq C (C_0^4 + C_0) \int_M e^{-(k-1)\varphi} \omega^n.
\]

Thus by the Moser iteration, we derive
\[
\|e^{-\varphi}\|_{L^\infty} \leq C \|e^{-\varphi}\|_{L^1} \leq CA.
\]

Note that \(A \ll 1\). Hence \(e^{-\varphi} \ll 1\). Now we can apply Proposition 2.2 to obtain
\[
\frac{A}{M_0} \leq e^{-\varphi} \leq M_0 A.
\]

6. First and Second Order Estimates (II)

In this section, we provide another proof to derive a prior \(C^1, C^2\) estimates for \(\varphi\) of (1.1) in case of \(\alpha < 0\), but without the restriction condition (1.5).

For convenience, we say a constant \(C\) is uniform if it depends only on \(\alpha, \rho, \mu\) and \((M, \omega)\), and we use \(C_A\) to denote a uniform constant depending on \(A\). The main goal in this section is to prove the following proposition.

**Proposition 6.1.** Let \(\alpha < 0\) and \(\varphi\) be a smooth solution of (1.1) on a Hermitian manifold \((M, \omega)\), which satisfies (1.2) and (1.7). Then
\[
\sup_M |\partial \varphi|_{\omega} \leq C_A \sup_M |\partial \varphi|_{\omega}^2 + C_A,
\]
where \(C_A\) is a uniform constant depending on \(A\).

For simplicity, we write (1.6) as
\[
(6.1) \quad \sigma_2(\tilde{\omega}) = F,
\]
where
\[
F = \frac{n(n-1)}{2} (e^{2\varphi} + 4|\alpha|e^{\varphi}|\partial \varphi|_{\omega}^2) + \frac{n(n-1)}{2} f
\]
and \(f\) is defined by (3.2). Let \(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\) be the eigenvalues of \(\tilde{\omega}\) with respect to \(\omega\). Since \(\tilde{\omega} \in \Gamma_2(M)\), by Proposition 5.2, it is clear that
\[
|\partial \tilde{\omega} \varphi|_{\omega} \leq C_A \lambda_1 + C_A.
\]

In Hermitian case, more troublesome terms will appear when we commute the covariant derivatives (cf. (6.4) below). To deal with these bad terms, we consider the following auxiliary function as in [1],
\[
Q = \log \lambda_1 + h(|\partial \varphi|_{\omega}^2) + e^{B\varphi},
\]
where \(B\) is a constant to be determined later,
\[
h(t) = -\frac{1}{2} \log(2K - t) \quad \text{and} \quad K = \sup_M |\partial \varphi|_{\omega}^2 + 1.
\]
By directly calculation, we have

\[
\frac{1}{4K} \leq h' \leq \frac{1}{2K} \quad \text{and} \quad h'' = 2(h')^2.
\]

Let \( x_0 \) be the maximum point of \( Q \). Around \( x_0 \), we choose holomorphic coordinate \((z^1, z^2, \ldots, z^n)\) such that at \( x_0 \),

\[
g_{\overline{ij}} = \delta_{ij} \quad \text{and} \quad \tilde{g}_{\overline{ij}} = \delta_{ij} \lambda_i.
\]

To prove Proposition 6.1, by (6.2), it suffices to prove \( \lambda_1 \leq C_A K \). Without loss of generality, we assume that \( \lambda_1 \gg C_A K \). Moreover, we may suppose that \( Q \) is smooth at \( x_0 \). Otherwise, we just need to apply a perturbation argument (cf. \([26, 27, 3]\)).

In the following calculation, we use the covariant derivatives with respect to the Chern connection \((\nabla, T^CM)\) induced by \( \omega \). Let us recall the commutation formulas for covariant derivatives:

\[
\varphi_{\overline{jk}} = \varphi_{\overline{kj}} - T_{ki}^p \varphi_{\overline{pj}} - R_{\overline{ikj}}^p \varphi_p,
\]

\[
\varphi_{\overline{jk}} = \varphi_{\overline{kj}} - T_{ki}^p \varphi_{\overline{pj}} - \tilde{T}_{ij}^p \varphi_{\overline{kj}} + \varphi_{\overline{ikj}} R_{\overline{pj}} - \varphi_{\overline{ikj}} R_{\overline{pj}} - T_{ki}^p \tilde{T}_{ij}^q \varphi_{\overline{pq}},
\]

where \( T_{ij}^k \) and \( R_{\overline{ikj}}^k \) are components of torsion tensor and curvature tensor induced by the Chern connection. Let

\[
G_{\overline{j}} = \frac{\partial \sigma_{\frac{1}{2}}(\tilde{\omega})}{\partial \tilde{g}_{\overline{j}}}, \quad G_{\overline{j}k} = \frac{\partial^2 \sigma_{\frac{1}{2}}(\tilde{\omega})}{\partial \tilde{g}_{\overline{j}} \partial \tilde{g}_{\overline{k}}}.
\]

The following lemmas are devoted to deriving lower bounds of \( G_{\overline{k}k} \varphi_{\overline{k}k} \), \( G_{\overline{k}k}(|\varphi|^2)_{\overline{k}k} \) and \( G_{\overline{k}k}(\lambda_1)_{\overline{k}k} \).

**Lemma 6.2.** At \( x_0 \), we have

\[
G_{\overline{k}k} \varphi_{\overline{k}k} \geq \sum_k G_{\overline{k}k} - CF_{\frac{1}{2}}
\]

and

\[
G_{\overline{k}k}(|\varphi|^2)_{\overline{k}k} \geq \frac{1}{2} \sum_i G_{\overline{k}k} (|\varphi_{ik}|^2 + |\varphi_{ik}|^2) - C_A K_{\frac{3}{2}}
\]

\[
- C_A K_{\frac{1}{2}} \sum_{i,k} (|\varphi_{ik}| + |\varphi_{ik}|) - C \sum_k G_{\overline{k}k}.
\]

**Proof.** By the definition of \( \tilde{\omega} \), we have

\[
2n \alpha G_{\overline{k}k} \varphi_{\overline{k}k} = G_{\overline{k}k} (\tilde{g}_{\overline{k}k} - e^\varphi g_{\overline{k}k} - \alpha e^{-\varphi} \rho_{\overline{k}k})
\]

\[
= \sigma_{\frac{1}{2}}(\tilde{\omega}) - e^\varphi \sum_k G_{\overline{k}k} - \alpha e^{-\varphi} G_{\overline{k}k} \rho_{\overline{k}k}
\]

\[
= F_{\frac{1}{2}} - e^\varphi \sum_k G_{\overline{k}k} - \alpha e^{-\varphi} G_{\overline{k}k} \rho_{\overline{k}k}.
\]
Note that $A \ll 1$ and $\alpha < 0$. Then by Proposition 5.2, we get

$$G^{k\overline{k}} \varphi_{k\overline{k}} \geq \sum_k G^{k\overline{k}} - C F^{1\overline{1}}.$$  

To prove the second inequality in the lemma, we compute

\begin{align*}
G^{k\overline{k}} (|\partial \varphi|_g^2)_{k\overline{k}} &= \sum_i G^{k\overline{k}} (|\varphi_{ik}|^2 + |\varphi_{\overline{i}k}|^2) + 2 \text{Re} \left( \sum_i G^{k\overline{k}} \varphi_i \varphi_{k\overline{k}} \right) \\
&\geq \sum_i G^{k\overline{k}} (|\varphi_{ik}|^2 + |\varphi_{\overline{i}k}|^2) + 2 \text{Re} \left( \sum_i G^{k\overline{k}} \varphi_i \varphi_{k\overline{k}} \right) - C \sum_i G^{k\overline{k}} (|\varphi_{ik}| + |\varphi_{\overline{i}k}|) \\
&\geq \frac{1}{2} \sum_i G^{k\overline{k}} (|\varphi_{ik}|^2 + |\varphi_{\overline{i}k}|^2) + 2 \text{Re} \left( \sum_i G^{k\overline{k}} \varphi_i \varphi_{k\overline{k}} \right) - C \sum_k G^{k\overline{k}}.
\end{align*}

On the other hand, by differentiating (6.1) along $\nabla_{\overline{i}}$, we see that

$$2 \text{Re} \left( \sum_i G^{k\overline{k}} \varphi_i \varphi_{k\overline{k}} \right) = 2 \text{Re} \left( \sum_i \varphi_i (F^{\overline{1}}_{\overline{1}})_{\overline{i}} \right) = \frac{1}{F^{\overline{1}}_{\overline{1}}} \text{Re} \left( \sum_i \varphi_i F^{\overline{1}}_{\overline{i}} \right).$$

Since

$$F \geq \frac{1}{C} (e^{2\varphi} + e^{\varphi} |\partial \varphi|_g^2)$$

and

$$|F^{\overline{1}}_{\overline{1}}| \leq C e^{2\varphi} |\partial \varphi|_g + C e^{\varphi} |\partial \varphi|_g^3 + C e^{\varphi} (|\partial \varphi|_g + 1) \sum_{i,k} (|\varphi_{ik}| + |\varphi_{\overline{i}k}|) + C,$$

we get

\begin{align*}
2 \text{Re} \left( \sum_i G^{k\overline{k}} \varphi_i \varphi_{k\overline{k}} \right) &\geq - C e^{\frac{3}{2}\varphi} |\partial \varphi|_g - C e^{\frac{1}{2}\varphi} |\partial \varphi|_g^3 \\
&\quad + C e^{\frac{1}{2}\varphi} (|\partial \varphi|_g + 1) \sum_{i,k} (|\varphi_{ik}| + |\varphi_{\overline{i}k}|) - C.
\end{align*}
Hence, substituting this into (6.5), we obtain
\[
G^{k\bar{k}}(\partial \varphi^2)_{k\bar{k}} \geq \frac{1}{2} \sum_i G^{k\bar{k}} (|\varphi_{ik}|^2 + |\varphi_{k\bar{k}}|^2) - C e^{\frac{1}{2} \varphi} |\partial \varphi|_g - C e^{\frac{1}{2} \varphi} |\partial \varphi|_g^3
\]
\[
- C e^{\frac{1}{2} \varphi} (|\partial \varphi|_g + 1) \sum_{i,k} (|\varphi_{ik}| + |\varphi_{k\bar{k}}|) - C \sum_k G^{k\bar{k}} - C
\]
\[
\geq \frac{1}{2} \sum_i G^{k\bar{k}} (|\varphi_{ik}|^2 + |\varphi_{k\bar{k}}|^2) - C_A K^{\frac{3}{2}}
\]
\[
- C_A K^{\frac{3}{2}} \sum_{i,k} (|\varphi_{ik}| + |\varphi_{k\bar{k}}|) - C \sum_k G^{k\bar{k}}.
\]

\[\square\]

Lemma 6.3. At \(x_0\), we have
\[
G^{k\bar{k}}(\lambda_1)_{k\bar{k}} \geq -G^{ij,k\bar{l}} \nabla_1 \tilde{g}_{ij} \nabla \tilde{g}_{k\bar{l}} - C \sum_k G^{k\bar{k}} |\varphi_{1\bar{k}}| - C_A \sum_k |\varphi_{1k}| - C_A (\lambda_1 + K) \sum_i G^{i\bar{i}}
\]
\[
- C_A (\lambda_1 + K) \sum_{i,k} G^{i\bar{i}} - C_A K^{\frac{3}{2}} \sum_{i,k} |\varphi_{i\bar{k}}|^2 - C_A K^{\frac{3}{2}}
\]

Proof. By the formulas for the derivatives of \(\lambda_1\) (cf. \([26, 27, 3]\)), we have (6.6)
\[
G^{k\bar{k}}(\lambda_1)_{k\bar{k}} = G^{k\bar{k}} \nabla_k \nabla_{1\bar{k}} + 2 \sum_{j>1} G^{k\bar{k}} \frac{\nabla_k \tilde{g}_{ij}}{\lambda_1 - \lambda_j}
\]
\[
\geq 2n \alpha G^{k\bar{k}} |\varphi_{1\bar{k}}| + \frac{2}{n \lambda_1} \sum_{j>i} G^{k\bar{k}} |\nabla_k \tilde{g}_{ij}|^2 - C e^\varphi (\lambda_1 + K) \sum_i G_{i\bar{i}}
\]
\[
\geq 2n \alpha G^{k\bar{k}} |\varphi_{1\bar{k}}| + \frac{2}{n \lambda_1} \sum_{j>i} G^{k\bar{k}} |\varphi_{i\bar{k}}|^2 - C e^\varphi (\lambda_1 + K) \sum_i G_{i\bar{i}}
\]

On the other hand, by differentiating (6.1) along \(\nabla_T \nabla_1\), we have
\[
G^{k\bar{k}} \nabla_T \nabla_1 (e^\varphi \tilde{g}_{k\bar{k}} + \alpha e^{-\varphi} \tilde{g}_{kk} + 2n \alpha \varphi_{k\bar{k}}) = (F^{\frac{1}{2}})_{1\bar{1}} - G^{ij,k\bar{l}} \nabla_1 \tilde{g}_{ij} \nabla_T \tilde{g}_{k\bar{l}}
\]

Then by the commutation formula (6.4), we see that
\[
2n \alpha G^{k\bar{k}} |\varphi_{1\bar{k}}| \geq -C \sum_{i,k} G^{k\bar{k}} |\varphi_{i\bar{k}}| - C e^\varphi (\lambda_1 + K) \sum_i G_{i\bar{i}}
\]
\[
+ (F^{\frac{1}{2}})_{1\bar{1}} - G^{ij,k\bar{l}} \nabla_1 \tilde{g}_{ij} \nabla_T \tilde{g}_{k\bar{l}}
\]

Hence, substituting this into (6.6) and using the Cauchy-Schwarz inequality, we obtain
\[
G^{k\bar{k}}(\lambda_1)_{k\bar{k}} \geq -C \sum_k G^{k\bar{k}} |\varphi_{1\bar{k}}| - C e^\varphi (\lambda_1 + K) \sum_i G_{i\bar{i}}
\]
\[
+ (F^{\frac{1}{2}})_{1\bar{1}} - G^{ij,k\bar{l}} \nabla_1 \tilde{g}_{ij} \nabla_T \tilde{g}_{k\bar{l}}
\]
Next, we deal with the term \((F^2)_{1T}\). Clearly,

\[(6.8) \quad (F^2)_{1T} = \frac{1}{2F^2} \left( F_{1T} - \frac{|F_1|^2}{2F} \right). \]

For the first term of \((6.8)\), we compute

\[F_{1T} = 2n(n-1)|\alpha|(e^\varphi|\partial \varphi_g|^2)_{1T} + \frac{n(n-1)}{2}(e^{2\varphi})_{1T} + \frac{n(n-1)}{2}f_{1T}\]

\[\geq 2n(n-1)|\alpha|e^\varphi \sum_k (|\varphi_{k1}|^2 + |\varphi_{kT}|^2) - Ce^\varphi|\partial \varphi_g| \sum_k |\varphi_{k1}| \]

\[- Ce^\varphi|\partial \varphi_g|^2|\varphi_{k1}| - Ce^\varphi|\partial \varphi_g|^4 - Ce^\varphi|\partial \varphi_g|^2 \sum_k (|\varphi_{k1}| + |\varphi_{kT}|) \]

\[- Ce^\varphi|\varphi_{k1}| + \frac{n(n-1)}{2}f_{1T}\]

\[\geq \frac{3}{2} n(n-1)|\alpha|e^\varphi \sum_k (|\varphi_{k1}|^2 + |\varphi_{kT}|^2) - Ce^\varphi(|\partial \varphi_g| + 1) \sum_k |\varphi_{k1}| \]

\[- Ce^\varphi|\partial \varphi_g|^4 - Ce^{3\varphi}. \]

For the second term of \((6.8)\), we have

\[F_1 = 2n(n-1)|\alpha|e^\varphi \nabla_1 |\partial \varphi_g|^2 + 2n(n-1)|\alpha|e^\varphi |\partial \varphi_g|^2 \varphi_1 \]

\[+ n(n-1)e^{2\varphi} \varphi_1 + \frac{n(n-1)}{2}f_1, \]

which implies

\[|F_1|^2 \leq \left( 4 + \frac{1}{200} \right) n^2(n-1)^2 \alpha^2 e^{2\varphi}|\partial \varphi_g|^2 \sum_k |\varphi_{k1}|^2 \]

\[+ Ce^{2\varphi}|\partial \varphi_g|^2 \sum_k |\varphi_{kT}|^2 + Ce^{2\varphi}|\partial \varphi_g|^6 + Ce^{4\varphi}|\partial \varphi_g|^2 + C|f_1|^2 \]

\[\leq \left( 4 + \frac{1}{100} \right) n^2(n-1)^2 \alpha^2 e^{2\varphi}|\partial \varphi_g|^2 \sum_k |\varphi_{k1}|^2 + Ce^{2\varphi} \sum_k |\varphi_{k1}|^2 \]

\[+ Ce^{2\varphi}(|\partial \varphi_g|^2 + 1) \sum_k |\varphi_{kT}|^2 + Ce^{2\varphi}|\partial \varphi_g|^6 + Ce^{4\varphi}|\partial \varphi_g|^2 + C. \]

On the other hand, by the definition of \(F\), we have

\[2F \geq \frac{99}{100} n(n-1) \left( e^{2\varphi} + 4|\alpha|e^\varphi |\partial \varphi_g|^2 \right). \]
Thus, substituting these estimates into (6.8), we get

\[
(\tilde{F}_1^2 )_1 \geq - C \frac{e^\phi |\partial \phi|_g}{F^\frac{1}{4}} \left( e^\phi (|\partial \phi|_g + 1) \sum_k |\varphi_{kT}| + e^\phi \sum_{k,l} |\varphi_{kT}|^2 + e^\phi |\varphi_{kT}|_g^4 + e^{2\phi} \right) \\
- C e^\frac{1}{4} \sum_k |\varphi_{kT}| - C e^\phi F^{-\frac{1}{4}} \sum_{k,l} |\varphi_{kT}|^2 - C e^\phi |\varphi_{kT}|_g^2 - C e^{2\phi}.
\]

We note that a similar estimate of (6.9) was also appeared in [22]. Combining (6.7) and (6.9), we finally prove that

\[
G^{\overline{k}\overline{k}} (\lambda_1)_{\overline{k}k} \geq G^{\overline{j}\overline{k}} \nabla_{\overline{1}} \tilde{g}^{\overline{j}} \nabla_{\overline{T}} \tilde{g}_{\overline{\eta} \overline{\eta}} - C \sum_{k} G^{\overline{k}\overline{k}} |\varphi_{1T_k}| - C e^\frac{1}{4} \sum_k |\varphi_{kT}| \\
- C e^\phi (\lambda_1 + K) \sum_{i} G^{\overline{i}\overline{i}} - C e^\phi F^{-\frac{1}{4}} \sum_{i,k} |\varphi_{ik}|^2 - C e^\phi |\varphi_{ik}|_g^2 - C e^{2\phi} \\
- G^{\overline{j}\overline{k}} \nabla_{\overline{1}} \tilde{g}^{\overline{j}} \nabla_{\overline{T}} \tilde{g}_{\overline{\eta} \overline{\eta}} - C \sum_{k} G^{\overline{k}\overline{k}} |\varphi_{1T_k}| - C A \sum_{k} |\varphi_{kT}| \\
- C A (\lambda_1 + K) \sum_{i} G^{\overline{i}\overline{i}} - C A K^{-\frac{1}{4}} \sum_{i,k} |\varphi_{ik}|^2 - C A K^{\frac{1}{2}}.
\]

Using the above lemmas, we prove the following lower bound of \( G^{\overline{k}\overline{k}} Q_{k\overline{k}} \) at \( x_0 \).

**Lemma 6.4.** At \( x_0 \), for any \( \varepsilon \in (0,1) \), we have

\[
0 \geq G^{\overline{k}\overline{k}} Q_{k\overline{k}} \geq - G^{\overline{j}\overline{k}} \nabla_{\overline{1}} \tilde{g}^{\overline{j}} \nabla_{\overline{T}} \tilde{g}_{\overline{\eta} \overline{\eta}} - (1 + \varepsilon) \frac{G^{\overline{k}\overline{k}} |\nabla_k \tilde{g}_{1T_k}|^2}{\lambda_1^2} \\
+ \frac{h'}{4} \sum_{i} G^{\overline{k}\overline{k}} (|\varphi_{ik}|^2 + |\varphi_{ik}|_g^2) + h'' G^{\overline{k}\overline{k}} |\nabla_k |\varphi|_g|^2 \\
+ B^2 e^{\phi} G^{\overline{k}\overline{k}} |\varphi_k|^2 + \left( \frac{B}{\varepsilon} - C A \right) \sum_{k} G^{\overline{k}\overline{k}}.
\]
Proof. By (6.3), Lemma 6.2 and Lemma 6.3 we see that

\[
G^{\ell \ell} Q_{k\ell}(x_0) \geq - \frac{G_{\ell \ell}^{1 \ell} \nabla_{i \ell} g_{1 \ell} \nabla_{\ell i} g_{1 \ell}}{\lambda_1} - \frac{G_{\ell \ell}^{k \ell} |\nabla_k g_{1 \ell}|^2}{\lambda_1^2} - C \sum_k G_k^{k \ell} |\varphi_{1 \ell k}| 
\]

(6.11)

\[+ \frac{C_A}{\lambda_1} \sum_k G_k^{k \ell} |\varphi_{1 \ell k}| + \frac{h'}{2} \sum_i G_{\ell i}^{k \ell} (|\varphi_{ik}|^2 + |\varphi_{ik}|^2) - C_A K^{\frac{3}{2}} h' \]

\[+ h'' G_{\ell \ell}^{k \ell} |\nabla_k |\partial \varphi|^2| \|
\]

\[+ C_A K^{\frac{3}{2}} h' \sum_{i,k} (|\varphi_{ik}| + |\varphi_{ik}|) - C_A K^{\frac{3}{2}} 
\]

\[+ B^2 e^{2B} G_{\ell \ell}^{k \ell} |\varphi_{k}|^2 + (B e^{2B} - C_A) \sum_k G_k^{k \ell} - C B e^{2B} F^{\frac{3}{2}}. \]

On the other hand, by the definition of \( \tilde{\omega} \), we have

(6.12)

\[2n \alpha \varphi_{1 \ell k} = \nabla_k \tilde{g}_{1 \ell} - e^\varphi \varphi_k - \alpha (e^{-\varphi} \rho_{1 \ell})_k, \]

which implies

\[- \frac{C \sum_k G_k^{k \ell} |\varphi_{1 \ell k}|}{\lambda_1} \geq - \frac{C}{\lambda_1} \sum_k G_k^{k \ell} (|\nabla_k \tilde{g}_{1 \ell}| + e^\varphi K) \]

(6.13)

\[\geq - \frac{h'' G_{\ell \ell}^{k \ell} |\nabla_k |\partial \varphi|^2|}{\lambda_1^2} - \frac{C_A}{\varepsilon} \sum_k G_k^{k \ell}, \]

where \( \varepsilon \in (0, 1) \). Note that \( \nabla_k Q(x_0) = 0 \). Then

(6.14)

\[\frac{\nabla_k \tilde{g}_{1 \ell}}{\lambda_1} = h' \nabla_k |\partial \varphi|^2 - B e^{2B} \varphi_k. \]

Thus combining this with (6.12), it follows that

\[- \frac{C_A \sum_k |\varphi_{1 \ell k}|}{\lambda_1} \geq - C_A K^{\frac{3}{2}} h' \sum_{i,k} (|\varphi_{ik}| + |\varphi_{ik}|) - C_A K^{\frac{3}{2}} B e^{2B} \varphi. \]

Hence, substituting the above estimates into (6.11), we get

\[0 \geq - \frac{G_{\ell \ell}^{1 \ell} \nabla_{i \ell} g_{1 \ell} \nabla_{\ell i} g_{1 \ell}}{\lambda_1} - (1 + \varepsilon) \frac{G_{\ell \ell}^{k \ell} |\nabla_k g_{1 \ell}|^2}{\lambda_1^2} + h'' G_{\ell \ell}^{k \ell} |\nabla_k |\partial \varphi|^2| \]

(6.15)

\[+ \frac{h'}{2} \sum_i G_{\ell i}^{k \ell} (|\varphi_{ik}|^2 + |\varphi_{ik}|^2) - C_A K^{\frac{3}{2}} h' \sum_{i,k} (|\varphi_{ik}| + |\varphi_{ik}|) \]

\[+ B^2 e^{2B} G_{\ell \ell}^{k \ell} |\varphi_k|^2 + \left( B e^{2B} - \frac{C_A}{\varepsilon} \right) \sum_k G_k^{k \ell} \]

\[- C_A K^{\frac{3}{2}} h' - C_A K^{\frac{3}{2}} - C B e^{2B} F^{\frac{3}{2}} - C_A K^{\frac{3}{2}} B e^{2B}. \]
By the definitions of $G^{k\bar{k}}$, it is clear that

$$
\sum_k G^{k\bar{k}} = \frac{n-1}{2F^2} \lambda_1 + (n-1)G^{1\bar{1}} \geq \frac{n-1}{2F^2} \lambda_1.
$$

Combining this with (6.3) and $G^{k\bar{k}} \geq \frac{F^2}{n\lambda_1}$, it follows that

$$
C_A K^{1/2} h' \sum_{i,k} (|\varphi_{ik}| + |\varphi_{\bar{k}i}|) \leq \frac{h'}{10} \frac{F^2}{n\lambda_1} \sum_{i,k} (|\varphi_{ik}|^2 + |\varphi_{\bar{k}i}|^2) + C_A K^{1/2} \frac{\lambda_1}{F^2}
$$

$$
\leq \frac{h'}{10} \sum_i G^{k\bar{k}} (|\varphi_{ik}|^2 + |\varphi_{\bar{k}i}|^2) + C_A \sum_k G^{k\bar{k}}.
$$

By the definition of $F$ and $\lambda_1 \gg C_A K$, we have

$$
C_A K^{1/2} \lambda' + C_A K^{1/2} + C B e^{B\varphi} F^{1/2} + C_A K^{1/2} B e^{B\varphi} \leq C_A K^{1/2} B e^{B\varphi}
$$

and

$$
C_A K^{1/2} B e^{B\varphi} \leq \frac{n-1}{4F^2} \lambda_1 B e^{B\varphi} \leq B \frac{2}{2} e^{B\varphi} \sum_k G^{k\bar{k}}.
$$

Substituting these estimates into (6.15), we obtain (6.10) immediately. □

**Proof of Proposition 6.1.** By (6.14) and (6.3), we have

$$
(1 + \varepsilon) G^{k\bar{k}} |\nabla_k \tilde{g}_{1\bar{1}}|^2 = (1 + \varepsilon) G^{k\bar{k}} |\nabla_k |\partial \varphi|_g|^2 + B e^{B\varphi} \varphi_k |^2
$$

$$
\leq 2(h')^2 G^{k\bar{k}} |\nabla_k |\partial \varphi|_g|^2 + C B^2 e^{2B\varphi} G^{k\bar{k}} |\varphi_k|^2
$$

$$
\leq \frac{h''}{16K} G^{k\bar{k}} |\nabla_k |\partial \varphi|_g|^2 + C B^2 e^{2B\varphi} G^{k\bar{k}} |\varphi_k|^2.
$$

Since $G^{i\bar{j},k\bar{l}} \nabla_1 \tilde{g}_{\bar{j}} \nabla_1 \tilde{g}_{\bar{l}} \varphi_i \leq 0$, by (6.10) (taking $\varepsilon = \frac{1}{2}$), we get

$$
0 \geq \frac{h'}{8} \sum_i G^{k\bar{k}} (|\varphi_{ik}|^2 + |\varphi_{\bar{k}i}|^2) - C_A K B^2 e^{2B\varphi} \sum_k G^{k\bar{k}}
$$

$$
\geq \frac{h'}{16K} \sum_i G^{k\bar{k}} (|\varphi_{ik}|^2 - C_A B^2 e^{2B\varphi} K \sum_k G^{k\bar{k}}).
$$

On other hand hand, for the $\sigma_2$ function, we have (cf. [14 Theorem 1]),

$$
G^{i\bar{i}} \geq \frac{1}{C} \sum_k G^{k\bar{k}}, \forall i \geq 2.
$$

Substituting this into (6.17), it follows that

$$
0 \geq \frac{1}{16K} G^{i\bar{i}} |\varphi_i|^2 - C_A B^2 e^{2B\varphi} G^{i\bar{i}}.
$$

Hence, we obtain

$$
\lambda_i \leq C_A |\varphi_i| + C_A \leq C_A B K, \forall i \geq 2,
$$

where $C_{A,B}$ is a uniform constant depending on $A$ and $B$. 


By (6.10) and (6.16) for \( k = 1 \), we have

\[
0 \geq - \frac{G_{\bar{k}i}}{\lambda_1} \frac{\nabla_{k} \tilde{g}}{\lambda_1^2} - \left(1 + \varepsilon\right) \sum_{k \geq 2} G_{k}^{\bar{k}} \frac{\nabla_{k} \tilde{g}}{\lambda_1^2}^2
\]

\[+ \frac{h'}{8} \sum_{i} G_{k}^{\bar{k}} \left(|\varphi_{ik}|^2 + |\varphi_{ik}|^2\right) + \sum_{k \geq 2} h'' G_{k}^{\bar{k}} \frac{\nabla_{k} \tilde{g}}{\lambda_1^2}^2\]

\[+ B^2 e^{B_\varphi} G_{k}^{\bar{k}} |\varphi_k|^2 + \left(\frac{B}{2} e^{B_\varphi} - \frac{C_A}{\varepsilon}\right) \sum_{k} G_{k}^{\bar{k}} - CK B^2 e^{2B_\varphi} G_{1}^{\bar{1}}.\]

We need to deal with bad third order term

\[
(1 + \varepsilon) \sum_{k \geq 2} G_{k}^{\bar{k}} \frac{\nabla_{k} \tilde{g}}{\lambda_1^2}^2
\]

\[(6.20)\]

\[= (1 - 2\varepsilon) \sum_{k \geq 2} G_{k}^{\bar{k}} \frac{\nabla_{k} \tilde{g}}{\lambda_1^2}^2 + 3\varepsilon \sum_{k \geq 2} G_{k}^{\bar{k}} \frac{\nabla_{k} \tilde{g}}{\lambda_1^2}^2 \]

For the first term of (6.20), we use (6.18) to see that

\[
(1 - 2\varepsilon) \sum_{k \geq 2} G_{k}^{\bar{k}} \frac{\nabla_{k} \tilde{g}}{\lambda_1^2}^2 \]

\[\leq (1 - \varepsilon) \sum_{k \geq 2} G_{k}^{\bar{k}} \frac{\nabla_{k} \tilde{g}}{\lambda_1^2}^2 + \frac{C}{\varepsilon} \sum_{k} G_{k}^{\bar{k}} \]

\[(6.21)\]

\[\leq (1 - \varepsilon) \sum_{k \geq 2} \lambda_1 + C \frac{A,B,K}{\varepsilon} |\nabla_{1} \tilde{g}|^2 + \frac{C}{\varepsilon} \sum_{k} G_{k}^{\bar{k}} \]

\[\leq - \sum_{k \geq 2} G_{1,k}^{\bar{1}} \frac{\nabla_{1} \tilde{g}}{\lambda_1}^2 + \frac{C}{\varepsilon} \sum_{k} G_{k}^{\bar{k}},\]

as long as \( \lambda_1 \geq \frac{C \frac{A,B,K}{\varepsilon}}{\varepsilon} \). For the second term of (6.20), we use (6.14) to get

\[
3\varepsilon \sum_{k \geq 2} G_{k}^{\bar{k}} \frac{\nabla_{k} \tilde{g}}{\lambda_1^2}^2 \]

\[(6.22)\]

\[\leq 6\varepsilon (h')^2 \sum_{k \geq 2} G_{k}^{\bar{k}} |\nabla_{k} \tilde{g}|^2 + 6\varepsilon B^2 e^{2B_\varphi} \sum_{k \geq 2} G_{k}^{\bar{k}} |\varphi_k|^2.\]

Thus substituting (6.21), (6.22) and (6.20) into (6.19), we obtain

\[
0 \geq \frac{h'}{8} \sum_{i} G_{k}^{\bar{k}} \left(|\varphi_{ik}|^2 + |\varphi_{ik}|^2\right) + \left(\frac{B}{2} e^{B_\varphi} - \frac{C_A}{\varepsilon}\right) \sum_{k} G_{k}^{\bar{k}} - CK B^2 e^{2B_\varphi} G_{1}^{\bar{1}}.\]
Choose $B = 12C_A + 1$ and $\varepsilon = e^{-B\varphi(x_0)}$, so that
\begin{align*}
B^2 e^{B\varphi} - 6\varepsilon B^2 e^{2B\varphi} = 0 \quad \text{and} \quad \left(\frac{B}{2} e^{B\varphi} - \frac{C_A}{\varepsilon}\right) \sum_k G^k \geq 0.
\end{align*}
Hence,
\begin{align*}
\frac{h'}{8} G^{1\Gamma} |\varphi_{1\Gamma}|^2 \leq C_A K B^2 e^{2B\varphi} G^{1\Gamma},
\end{align*}
which implies $\lambda_1 \leq C_A K$. We complete the proof.

As a corollary of Proposition 6.1, we obtain the following estimate.

**Theorem 6.5.** Let $\alpha < 0$ and $\varphi$ be a smooth solution of (1.6) satisfying (4.2) and (1.7). Then there exists a uniform constant $A_0$ such that if $A \leq A_0$, then we have the following estimate
\begin{align*}
\|\varphi\|_{C^{k,\alpha}} \leq C_{A,k},
\end{align*}
where $C_{A,k}$ depends only on $A$, $k$, $\alpha$, $\rho$, $\mu$ and $(M, \omega)$.

**Proof.** By Proposition 6.1 while $\rho$ and $\mu$ are replaced by
\begin{align*}
\frac{n\alpha(t - 1)!n\sqrt{-1}\partial\bar{\partial} h}{} \sqrt{-1}\partial\bar{\partial} h + n\alpha(\sqrt{-1}\partial\bar{\partial} h) + \omega^{n-2} + 2n\alpha \sqrt{-1}\partial\bar{\partial} \varphi + \sqrt{-1}\partial\bar{\partial} h + \omega^{n-2} = 0.
\end{align*}
we use the blow-up argument to derive (cf. [4, 20]),
\begin{align*}
\sup_M |\partial\bar{\partial} \varphi| \leq C_A.
\end{align*}
By the $C^{2,\alpha}$-estimate (cf. [28, Theorem 1.1]), it follows
\begin{align*}
\|\varphi\|_{C^{2,\alpha}} \leq C'_{A}.
\end{align*}
Hence, by the bootstrapping argument, we complete the proof.

### 7. Proof of Theorem 1.2

In this section, we give the proof of Theorem 1.2. First we prove the uniqueness of solutions of (1.6) in case of $\alpha < 0$ when $t = 0$.

**Lemma 7.1.** When $t = 0$, (1.6) has a unique solution satisfying (4.2) and (1.7)
\begin{align*}
\varphi_0 = h + \ln \|e^{-h}\|_{L^n} - \ln A,
\end{align*}
where $h$ is the function in Lemma 4.1.

**Proof.** It suffices to prove that $\varphi - h$ is constant. For convenience, we define
\begin{align*}
\tilde{\varphi} = \varphi - h \quad \text{and} \quad \omega_h = e^h \omega.
\end{align*}
Then, when $t = 0$, (1.6) and (1.2) can be expressed as
\begin{align*}
\sqrt{-1}\partial\bar{\partial}(e^{\tilde{\varphi}} \omega_h) \wedge \omega^{n-2} + n\alpha \sqrt{-1}\partial\bar{\partial} \tilde{\varphi} \wedge \sqrt{-1}\partial\bar{\partial} \tilde{\varphi} \wedge \omega^{n-2}
\end{align*}
and
\begin{align*}
e^{\tilde{\varphi}} \omega_h + 2n\alpha \sqrt{-1}\partial\bar{\partial}(\tilde{\varphi} + h) \in \Gamma_2(M). 
\end{align*}
By the similar calculation of Lemma 2.1, we have

\[
\int_M e^{2\tilde{\phi}} \sqrt{-1} \partial \tilde{\phi} \wedge \overline{\partial} \tilde{\phi} \wedge \omega_h \wedge \omega^{n-2} \geq -2n\alpha \int_M \sqrt{-1} \partial e^{\tilde{\phi}} \wedge \overline{\partial} e^{\tilde{\phi}} \wedge \sqrt{-1} \partial \overline{\partial} (\tilde{\phi} + h) \wedge \omega^{n-2} \\
= -2n\alpha \int_M e^{\tilde{\phi}} \wedge \overline{\partial} e^{\tilde{\phi}} \wedge \sqrt{-1} \partial \overline{\partial} (\tilde{\phi} + h) \wedge \sqrt{-1} \partial \omega^{n-2} \\
+ 2n\alpha \int_M e^{\tilde{\phi}} \wedge \sqrt{-1} \partial \overline{\partial} (e^{\tilde{\phi}} \omega_h) \wedge \omega^{n-2} \\
= -2n\alpha \int_M e^{\tilde{\phi}} \wedge \sqrt{-1} \partial \overline{\partial} (e^{\tilde{\phi}} \omega_h) \wedge \omega^{n-2} \\
- 2 \int_M e^{2\tilde{\phi}} \sqrt{-1} \partial \overline{\partial} (e^{\tilde{\phi}} \omega_h) \wedge \omega^{n-2} \\
- 2n\alpha \int_M e^{\tilde{\phi}} \wedge \sqrt{-1} \partial \overline{\partial} (\tilde{\phi} + h) \wedge \sqrt{-1} \partial \omega^{n-2}.
\]

(7.1)

Since \(\sqrt{-1} \partial \overline{\partial} \omega^{n-2} = 0\), the first term of (7.1) vanishes. For the second term, we compute

\[
-2 \int_M e^{\tilde{\phi}} \sqrt{-1} \partial \overline{\partial} (e^{\tilde{\phi}} \omega_h) \wedge \omega^{n-2} \\
= 2 \int_M e^{\tilde{\phi}} \sqrt{-1} \partial \tilde{\phi} \wedge \overline{\partial} (e^{\tilde{\phi}} \omega_h) \wedge \omega^{n-2} - 2 \int_M e^{\tilde{\phi}} \overline{\partial} (e^{\tilde{\phi}} \omega_h) \wedge \sqrt{-1} \partial \omega^{n-2} \\
= 2 \int_M (e^{2\tilde{\phi}} \sqrt{-1} \partial \tilde{\phi} \wedge \overline{\partial} \tilde{\phi} \wedge \omega_h + \frac{1}{2} \sqrt{-1} \partial e^{2\tilde{\phi}} \wedge \overline{\partial} \omega_h) \wedge \omega^{n-2} \\
- \int_M \overline{\partial} e^{2\tilde{\phi}} \wedge \omega_h \wedge \sqrt{-1} \partial \omega^{n-2} - 2 \int_M e^{2\tilde{\phi}} \overline{\partial} \omega_h \wedge \sqrt{-1} \partial \omega^{n-2} \\
= \int_M e^{2\tilde{\phi}} \left( -\partial (\overline{\partial} \omega_h \wedge \omega^{n-1}) + \overline{\partial} (\omega_h \wedge \partial \omega^{n-1}) - 2\overline{\partial} \omega_h \wedge \partial \omega^{n-1} \right) \\
+ 2 \int_M e^{2\tilde{\phi}} \sqrt{-1} \partial \tilde{\phi} \wedge \overline{\partial} \tilde{\phi} \wedge \omega_h \\
= 2 \int_M e^{2\tilde{\phi}} \sqrt{-1} \partial \tilde{\phi} \wedge \overline{\partial} \tilde{\phi} \wedge \omega_h \wedge \omega^{n-2},
\]

where we used the relations in the last equality,

\(\sqrt{-1} \partial \overline{\partial} \omega^{n-2} = 0\) and \(\sqrt{-1} \partial \overline{\partial} (e^h \omega) \wedge \omega^{n-2} = 0\).
For the third term of (7.1), we see that
\[
-2n\alpha \int_M e^{\tilde{\varphi}} \sqrt{-1} \partial \bar{\partial} \tilde{\varphi} \wedge \sqrt{-1} \partial \bar{\partial} h \wedge \omega^{n-2}
= 2n\alpha \int_M e^{\tilde{\varphi}} \sqrt{-1} \partial \bar{\partial} \tilde{\varphi} \wedge \sqrt{-1} \partial \bar{\partial} h \wedge \omega^{n-2}
-2n\alpha \int_M \bar{\partial} e^{\tilde{\varphi}} \wedge \sqrt{-1} \partial \bar{\partial} h \wedge \sqrt{-1} \omega^{n-2}
= 2n\alpha \int_M e^{\tilde{\varphi}} \sqrt{-1} \partial \bar{\partial} \tilde{\varphi} \wedge \sqrt{-1} \partial \bar{\partial} h \wedge \omega^{n-2}.
\]
Substituting the above estimates into (7.1), we get the inequality
\[
\int_M e^{2\tilde{\varphi}} \sqrt{-1} \partial \bar{\partial} \tilde{\varphi} \wedge \omega_h \wedge \omega^{n-2}
\leq -2n\alpha \int_M e^{\tilde{\varphi}} \sqrt{-1} \partial \bar{\partial} \tilde{\varphi} \wedge \sqrt{-1} \partial \bar{\partial} h \wedge \omega^{n-2}
\leq C \int_M e^{\tilde{\varphi}} \sqrt{-1} \partial \bar{\partial} \tilde{\varphi} \wedge \omega_h \wedge \omega^{n-2}.
\]
(7.2)

On the other hand, by Proposition 5.2, we have
\[
e^{\tilde{\varphi}} = e^{\varphi - h} \geq \frac{1}{CA}.
\]
(7.3)
Combining this with (7.2) and \(A \ll 1\), we prove
\[
\int_M e^{2\tilde{\varphi}} \sqrt{-1} \partial \bar{\partial} \tilde{\varphi} \wedge \omega_h \wedge \omega^{n-2} = 0,
\]
which implies \(\partial \tilde{\varphi} = 0\). Therefore, \(\tilde{\varphi}\) is constant. \(\square\)

**Remark 7.2.** Lemma 7.1 is also true from the above proof if (1.7) is replaced by the condition (2.5), and the solution of (1.6) when \(t = 0\) is given by \(\varphi_0 = h + \ln \|e^{-h}\|_{L^1} - \ln A\). The reason is that (7.3) holds by Proposition 2.2. In this case, the lemma holds for any \(\alpha \neq 0\).

Now we are in a position to prove Theorem 1.2.

**Proof of Theorem 1.2.** The existence is proved in Theorem 1.1. It suffices to prove the uniqueness. Assume that we have two solutions \(\varphi\) and \(\varphi'\). Then as in [2] Section 5.3, we use the continuity method to solve (1.6) from \(t = 1\) to \(0\). By Theorem 6.5 there are two families of solutions \(\{\varphi_t\}\) and \(\{\varphi'_t\}\) of (1.6) satisfying (1.2) and \(L^n\)-normalization conditions. We also have \(\varphi_1 = \varphi\) and \(\varphi'_1 = \varphi'\). By Lemma 7.1 we see that
\[
\varphi_0 = \varphi'_0 = -\ln A.
\]
Let
\[
J = \{t \in [0, 1] \mid \varphi_t = \varphi'_t\}.
\]
Clearly, \( J \) is closed. Applying the implicit function theorem, we see that \( J \) is open. Then \( J = [0, 1] \) and so
\[
\varphi = \varphi_1 = \varphi_1' = \varphi'.
\]
This completes the proof of uniqueness. \( \square \)

8. Proof of Theorem 1.3

In this section, we give the proof of Theorem 1.3. When \( A \) is sufficiently small, Theorem 1.2 implies that there exists a unique solution of (1.6) satisfying (1.2) and (1.7). For convenience, we denote it by \( \phi_{t,A} \).

Lemma 8.1. \( \phi_{t,A} \) is smooth with respect to \( t \) and \( A \).

Proof. For \( \beta \in (0, 1) \), we define the sets
\[
\tilde{B} = C^{2,\beta}(M) \times [0, 1] \times [0, 1],
\]
\[
\tilde{B}_1 = \{(\varphi, t, A) \in \tilde{B} \mid \varphi \text{ satisfies (1.2)}\},
\]
\[
\tilde{B}_2 = \mathbb{R} \times \{u \in C^{\beta}(M) \mid \int_M u \omega^n = 0\},
\]
and map \( \Phi : \tilde{B}_1 \to \tilde{B}_2 \)
\[
\Phi(\varphi, t, A) = (\Phi_1(\varphi, A), \Phi_2(\varphi, t)) ,
\]
where
\[
\Phi_1(\varphi, A) = \int_M e^{-n\varphi} \omega^n - A^n
\]
and
\[
\Phi_2(\varphi, t) = \sqrt{-1} \partial \bar{\partial}(e^{\varphi} \omega - t \alpha e^{-\varphi} \rho) \land \omega^{n-2}
\]
\[
+ n \alpha \sqrt{-1} \partial \bar{\partial} \varphi \land \sqrt{-1} \partial \bar{\partial} \varphi \land \omega^{n-2}
\]
\[
+ n \alpha(t - 1) \sqrt{-1} \partial \bar{\partial} \rho \land \sqrt{-1} \partial \bar{\partial} \rho \land \omega^{n-2} + t \mu \omega^n / n!.
\]

By the same argument of Section 4, we have
\[
(D_{\varphi} \Phi)_{(\hat{\varphi}, \hat{t}, \hat{A})}(u) = \left( -\int_M n e^{-n\hat{\varphi}} u \omega^n, Lu \right),
\]
where
\[
(Lu) \omega^n = \sqrt{-1} \partial \bar{\partial}(ue^{\hat{\varphi}} \omega + \hat{t} \alpha u e^{-\hat{\varphi}} \rho) \land \omega^{n-2}
\]
\[
+ 2n \alpha \sqrt{-1} \partial \bar{\partial} \hat{\varphi} \land \sqrt{-1} \partial \bar{\partial} u \land \omega^{n-2}.
\]

and
\[
(8.1) \quad \text{Ker} L = \{cu_0 \mid c \in \mathbb{R}\},
\]
where \( u_0 \) is a positive function. Similarly, we see that \( (D_{\varphi} \Phi)_{(\hat{\varphi}, \hat{t}, \hat{A})} \) is invertible. Using the implicit function theorem, near \((\hat{\varphi}, \hat{t}, \hat{A})\), there exists
a smooth map $F(t, A)$ such that $\Phi(F(t, A), t, A) = (0, 0)$, where $\varphi_{t, A}$ (for convenience, we denote it by $\hat{\varphi}$) satisfies
$$\Phi(\hat{\varphi}, \hat{t}, \hat{A}) = (0, 0).$$
This implies that $F(t, A)$ is the solution of (1.6) satisfying the elliptic and $L^n$-normalization conditions. Thanks to Theorem 1.2, we have $\varphi_{t, A} = F(t, A)$. Hence, $\varphi_{t, A}$ is smooth at $(\hat{t}, \hat{A})$. Since $(\hat{t}, \hat{A})$ is arbitrary, we complete the proof.

**Proof of Theorem 1.3.** By the definition of $\varphi_{t, A}$, it suffices that prove $\varphi_{1, A} > \varphi_{1, \hat{A}}$. Define $\varphi(s) = \varphi_{1, A^s}$. Then it follows that
$$\sqrt{-1} \partial \overline{\partial} (e^{\varphi(s)} \omega - tae^{-\varphi(s)} \rho) \wedge \omega^{n-2} + n\alpha \sqrt{-1} \partial \overline{\partial} \varphi(s) \wedge \sqrt{-1} \partial \overline{\partial} \varphi(s) \wedge \omega^{n-2} + t\mu \omega^n = 0,$$
and
$$\int_M e^{-n\varphi(s)} \omega^n = A^{ns} \hat{A}^{n(1-s)}.$$
By Lemma [8.1], we can differentiate the above two equations with respect to $s$, respectively, and we obtain
$$\frac{\partial \varphi(s)}{\partial s} \in \text{Ker} L \text{ and } \int_M e^{-n\varphi(s)} \frac{\partial \varphi(s)}{\partial s} \omega^n > 0.$$
Recalling (8.1), we see that $\frac{\partial \varphi(s)}{\partial s} > 0$, which implies
$$\varphi_{1, A} - \varphi_{1, \hat{A}} = \int_0^1 \frac{\partial \varphi(s)}{\partial s} ds > 0.$$
Theorem 1.3 is proved. $\Box$

**Remark 8.2.** When $tr\omega \rho \geq 0$, using (5.2) and Sobolev inequality, we obtain
$$\left( \int_M e^{-k\beta \varphi} \omega^n \right)^{\frac{1}{\beta}} \leq Ck \int_M e^{-(k-1)\varphi} \omega^n,$$
which implies $\|e^{-\varphi}\|_{L^\infty} \leq C$. By the similar argument in Section 5, we get the analogous estimate of Theorem 6.5 under the normalization $\|e^{-\varphi}\|_{L^1} = A$. This estimate is enough for the proofs of Theorem 1.2 and 1.3.

**References**

[1] J. Chu, L. Huang and X. Zhu, The 2-nd Hessian type equation on almost Hermitian manifolds, preprint, arXiv: 1707.04072.
[2] J. Chu, L. Huang and X. Zhu, The Fu-Yau equation in higher dimensions, preprint, arXiv: 1801.09351.
[3] J. Chu, V. Tosatti and B. Weinkove, The Monge-Ampère equation for non-integrable almost complex structures, to appear in J. Eur. Math. Soc. (J EMS)
[4] S. Dinew and S. Kołodziej, Liouville and Calabi-Yau type theorems for complex Hessian equations, Amer. J. Math. 139 (2017), no. 2, 403–415.
A. Fino, G. Grantcharov and L. Vezzoni, Astheno-Kähler and balanced structures on fibrations, preprint, arXiv: 1608.06743.

A. Fino and A. Tomassini, On astheno-Kähler metrics, J. London Math. Soc. 83 (2011), 290–308.

J.-X. Fu and S.-T. Yau, A Monge-Ampère-type equation motivated by string theory, Comm. Anal. Geom. 15 (2007), no. 1, 29–75.

J.-X. Fu and S.-T. Yau, The theory of superstring with flux on non-Kähler manifolds and the complex Monge-Ampère equation, J. Differential Geom. 78 (2008), no. 3, 369–428.

M. Garcia-Fernandez, Lectures on the Strominger system, preprint, arXiv: 1609.02615.

E. Goldstein and S. Prokushkin, Geometric model for complex non-Kähler manifolds with SU(3) structure, Comm. Math. Phys. 251 (2004), no. 1, 65–78.

P. Guan, C. Ren and Z. Wang, Global $C^2$-estimates for convex solutions of curvature equations, Comm. Pure Appl. Math. 68 (2015), no. 8, 1287–1325.

Z. Hou, X.-N. Ma and D. Wu, A second order estimate for complex Hessian equations on a compact Kähler manifold, Math. Res. Lett. 17 (2010), no. 3, 547–561.

J. Jost and S.-T. Yau, A nonlinear elliptic system for maps from Hermitian to Riemannian manifolds and rigidity theorems in Hermitian geometry, Acta Math. 170 (1993), no. 2, 221–254; Correction, Acta Math. 173 (1994), no. 2, 307.

M. Lin and N. S. Trudinger, On some inequalities for elementary symmetric functions, Bull. Aust. Math. Soc. 50 (1994), 317–326.

A. Latorre and L. Ugarte, On non-Kähler compact complex manifolds with balanced and astheno-Kähler metrics, Com. Ren. Acad. Sci. Math. 355 (2017), 90–93.

J. Li and S.-T. Yau, Hermitian-Yang-Mills connection on non-Kähler manifolds, Mathematical aspects of string theory (San Diego, Calif., 1986), 560–573, Adv. Ser. Math. Phys., 1, World Sci. Publishing, Singapore, 1987.

J. Li, S.-T. Yau and F. Zheng, On projectively flat Hermitian manifolds, Comm. Anal. Geom. 2 (1994), 103–109.

K. Matsuo, Astheno-Kähler structures on Calabi-Eckmann manifolds, Colloq. Math. 115 (2009), no. 1, 33–39.

K. Matsuo and T. Takahashi, On compact astheno-Kähler manifolds, Colloq. Math. 89 (2001), no. 2, 213–221.

D. H. Phong, S. Picard and X. Zhang, On estimates for the Fu-Yau generalization of a Strominger system, to appear in J. Reine Angew. Math.

D. H. Phong, S. Picard and X. Zhang, A second order estimate for general complex Hessian equations, Anal. PDE 9 (2016), no. 7, 1693–1709.

D. H. Phong, S. Picard and X. Zhang, The Fu-Yau equation with negative slope parameter, Invent. Math. 209 (2017), no. 2, 541–576.

D. H. Phong, S. Picard and X. Zhang, Fu-Yau Hessian equations, preprint, arXiv: 1801.09842.

Y. T. Siu, The complex analyticity of harmonic maps and the strong rigidity of compact Kähler manifolds. Ann. of Math. 112 (1980), 73–111.

A. Strominger, Superstrings with torsion, Nuclear Phys. B 274 (1986), no. 2, 253–284.

G. Székelyhidi, Fully non-linear elliptic equations on compact Hermitian manifolds, to appear in J. Differential Geom.

G. Székelyhidi, V. Tosatti and B. Weinkove, Gauduchon metrics with prescribed volume form, Acta Math. 219 (2017), no. 1, 181–211.

V. Tosatti, Y. Wang, B. Weinkove and X. Yang, $C^{2,\alpha}$ estimate for nonlinear elliptic equations in complex and almost complex geometry, Calc. Var. Partial Differential Equations 54 (2015), no. 1, 431–453.
[29] V. Tosatti and B. Weinkove The Monge-Ampre equation for \((n-1)\)-plurisubharmonic functions on a compact Kähler manifold, J. Amer. Math. Soc. 30 (2017), no. 2, 311–346.

[30] M. Warren and Y. Yuan, Hessian estimates for the sigma-2 equation in dimension 3, Comm. Pure Appl. Math. 62 (2009), no. 3, 305–321.

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