On bi-Hamiltonian deformations of exact pencils of hydrodynamic type

Alessandro Arsie\textsuperscript{1} and Paolo Lorenzoni\textsuperscript{2}

\textsuperscript{1} Department of Mathematics, University of Toledo, 2801 W. Bancroft Street, Toledo, OH 43606, USA
\textsuperscript{2} Dipartimento di Matematica, Università di Milano—Bicocca, Via Roberto Cozzi 53, I-20125 Milano, Italy
E-mail: alessandro.arsie@utoledo.edu and paolo.lorenzoni@unimib.it

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Abstract
In this paper, we are interested in nontrivial bi-Hamiltonian deformations of the Poisson pencil $\omega_\lambda = \omega_2 + \lambda \omega_1 = u\delta'(x - y) + \frac{1}{2} u_x \delta(x - y) + \lambda \delta'(x - y)$.
Deformations are generated by a sequence of vector fields \{\(X_2, X_3, X_4, \ldots\)\}, where each \(X_k\) is homogeneous of degree \(k\) with respect to a grading induced by rescaling. Constructing recursively the vector fields \(X_k\), one obtains two types of relations involving their unknown coefficients: one set of linear relations and an other one which involves quadratic relations. We prove that the set of linear relations has a geometric meaning: using Miura-quasitriviality, the set of linear relations expresses the tangency of the vector fields \(X_k\) to the symplectic leaves of \(\omega_1\) and this tangency condition is equivalent to the exactness of the pencil \(\omega_\lambda\). Moreover, extending the results of Lorenzoni P (2002 J. Geom. Phys. \textbf{44} 331–75), we construct the nontrivial deformations of the Poisson pencil \(\omega_\lambda\), up to the eighth order in the deformation parameter, showing therefore that deformations are unobstructed and that both Poisson structures are polynomial in the derivatives of \(u\) up to that order.

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1. Introduction
One of the most relevant problems in the modern theory of integrable systems is the classification of systems of PDEs of evolutionary type of the following form:

\[ u_t^i = F^i(u, u_x, u_{xx}, \ldots, u_{(n)}, \ldots), \]  

(1.1)

where \(u^i\) are functions of variables \((x, t)\), \(u_x^i\) denotes the first derivative of \(u^i\) with respect to \(x\) and \(u_{(n)}^i\) is the \(n\)th derivative with respect to \(x\) and \(i = 1, \ldots, N\). In particular, the functions
are required to satisfy the usual boundary conditions of the formal calculus of variations, i.e. either \( u^i \in C^\infty(S^1, \mathbb{R}) \) or \( u^i \in C^\infty(\mathbb{R}, \mathbb{R}) \) with vanishing conditions at infinity. To fix the ideas, in this work we will restrict our attention to the periodic case. Moreover, we assume that the functions \( F^i \) are polynomials in the derivatives with respect to \( x \) of the functions \( u^j \).

Systems of the form (1.1) can be thought as dynamical systems on the space of infinite jets \( J^\infty(S^1, \mathbb{R}^n) \) considering the complete set of differential consequences of (1.1):

\[
\frac{\partial}{\partial x} F^j(u, u_x, u_{xx}, \ldots, u^{(n)}, \ldots)
\]

where \( \partial_x \) is the total derivative with respect to \( x \) acting on \( F^j \) as follows:

\[
\frac{\partial}{\partial x} F^j(u, u_x, u_{xx}, \ldots, u^{(n)}, \ldots) = \sum_{k=0}^{N} \sum_{i=1}^{N} \frac{\partial F^j}{\partial u^i} u^{(k+i)}
\]

and \( u^{(l)} \) stands for \( u^l \).

Rescaling the independent variables \( t \mapsto \epsilon t \), \( x \mapsto \epsilon x \), the system (1.1) transforms into

\[
\frac{\partial}{\partial t} u^i = \frac{1}{\epsilon} F^i_0(u) + \sum_{k=1}^{\infty} \epsilon^k F^i_k(u, u_x, u_{xx}, \ldots, u^{(n)}, \ldots),
\]

where \( F^i_0 \) do not depend on the derivatives of the functions \( u \) with respect to \( x \), and \( F^i_k \) are homogeneous polynomials of degree \( k \) in the derivatives of \( u \), assigning degree 0 to functions of \( u \) and \( \deg(u^{(l)}) = l \). Even though it is not strictly necessary, the rescaling of the independent variables and the ensuing presence of the parameter \( \epsilon \) have the effect to clearly separate the various homogeneous components: this is a key aspect of the perturbative approach to the classification problem.

In particular, we will be concerned in this work with the classification of systems of PDEs of type (1.3) having a well-defined dispersionless limit as \( \epsilon \) goes to zero. In order to have a well-defined system in this case, it is necessary to restrict our analysis to the family of PDEs in which \( F^i_0(u) \) is identically zero. Therefore, from now on we restrict our attention to systems of the following form:

\[
\frac{\partial}{\partial t} u^i = V^i_j(u) u^j_x + \sum_{k=1}^{\infty} \epsilon^k F^i_k(u, u_x, u_{xx}, \ldots, u^{(n)}, \ldots).
\]

The main idea of the perturbative approach to classification is to deal with the terms \( F^i_k, k \geq 1 \), as perturbations of the dispersionless limit \( u^i = V^i_j(u) u^j_x \) and to reconstruct the integrability of the full system (1.4) starting from the integrability [22] of the quasilinear system

\[
\frac{\partial}{\partial t} u^i = V^i_j(u) u^j_x.
\]

A trivial way to produce integrable perturbations starting from a quasilinear system (1.5) consists in a change of dependent variables of the form

\[
\tilde{u}^i = F^i_0(u) + \sum_{k=1}^{\infty} \epsilon^k F^i_k(u, u_x, u_{xx}, \ldots),
\]

where \( F^i_k \) are differential polynomials (in the derivatives of the \( u^i \))s of degree \( k \) and \( \det \frac{\partial F^i_k}{\partial u_x} \neq 0 \). Such transformations are called Miura transformations. We will not assume the convergence of the series on the right-hand side of (1.6). In this formal setting, two systems of the form (1.4), which are related by a Miura transformation, will be considered equivalent.

From the perturbative point of view to the classification, different approaches are possible (and have been explored).
• Fix a local Hamiltonian structure and extend to all orders the conservation laws of the unperturbed system in a recursive way [9].
• Extend to all orders the symmetries of the hydrodynamic limit [21].
• One approach is based on the additional assumptions that the systems (1.1) one is dealing with are reductions of a \((2 + 1)\) integrable PDE [10].
• Another approach is the approach proposed by Dubrovin and Zhang in [7]. It can be applied to a quasilinear system possessing a local bi-Hamiltonian structure. In this case, instead of studying the deformations of the system it is more convenient to study and classify the deformations of its bi-Hamiltonian structure.

The aim of this paper is to apply the approach of Dubrovin and Zhang to the simplest possible case: the local bi-Hamiltonian structure

\[
\{u(x), u(y)\}_\lambda = 2u\delta'(x - y) + u_x\delta(x - y) - \lambda\delta'(x - y)
\]

of the Hopf equation

\[
u_t = uu_x.
\]

The deformations of this structure has been classified, up to the fourth order, in [17]. They depend on a certain number of parameters. All these parameters, except one, are irrelevant and correspond to Miura equivalent deformations. The remaining one parametrizes the space of nonequivalent deformations.

The starting observation of this paper was that, using the freedom in the choice of the irrelevant parameters, one can write the fourth order deformations of the pencil (1.7) in the simple form:

\[
2u(x)\delta'(x - y) + u_x\delta(x - y) - \lambda\delta'(x - y) + \sum_k \epsilon^k \left[ c_k(u)\delta^{(2k)}(x - y) + \partial_x^{2k}[c_k(u)\delta'(x - y)] \right]
\]

where \(c_2\) is the relevant functional parameter, \(c_3\) is a free parameter, and \(c_4 = -\frac{1}{u_a}(c_2)^2\). This observation suggested us to look for higher-order deformations of the same form:

\[
2u(x)\delta'(x - y) + u_x\delta(x - y) - \lambda\delta'(x - y) + \sum_k \epsilon^k \left[ (-1)^k c_k(u)\delta^{(2k)}(x - y) + \partial_x^{2k}[c_k(u)\delta'(x - y)] \right]
\]

For a suitable choice of the functions \(c_k\) one obtains the two known cases corresponding to KdV and Camassa–Holm hierarchy. In the KdV case the function \(c_2\) is constant and all the remaining functions vanish, while in the Camassa–Holm case we have \(c_k = u\) for all \(k \geq 2, k\) even, while \(c_k = -u\) for \(k \geq 3, k\) odd. So, motivated by this preliminary observation, we started to study higher-order deformations of (1.7). We realized very soon (at order 6) that our optimistic conjecture about the form of the deformations was wrong. However, we also realized that part of the constraints imposed by the Jacobi identity has a simple geometrical interpretation: the deformed Poisson pencil can always be written in the form

\[
\text{Lie}_{X}\delta'(x - y) + O(\epsilon^8) - \lambda\delta'(x - y),
\]

where the vector field \(X\) is always tangent to the symplectic leaves of \(\delta'(x - y)\). Using some important results due to Dubrovin, Liu and Zhang, we have been able to prove this result to all orders: indeed we prove that the tangency of \(X\) to the symplectic leaves of \(\delta'(x - y)\) is valid at any order in the deformation parameter and depends crucially on the exactness of the pencil (1.7) (indeed it is an equivalent condition).
This paper is organized as follows. In section 2, we fix the notations and we review the basic results about the subject. This section does not contain new material, except that we provide a complete proof of proposition 4, which was instead sketched in [15]. Due to the amount of results accumulated in the last years we believe that it might be helpful for the reader to have a short review of them. Moreover, part of these results will be used in the remaining sections. In section 3, we recall a powerful formalism developed by several authors which is very convenient to make computations and we state two computational lemmas that are proved in a final appendix. Although proofs are not difficult, we have not been able to locate them in the literature. In section 4, we use such a formalism to show that the form of the deformations suggested by lower-order deformations is unfortunately too optimistic. In section 5, we prove that the vector field generating the deformation \( X_\epsilon \) is tangent to the symplectic leaves of \( \delta'(x - y) \) at any order (provided the deformation exists) if and only if the undeformed Poisson pencil is exact. In section 4, we find the general form for a scalar pencil to be exact. These results will help us to simplify the computations (performed with the help of Maple) of the subsequent section 6. The corrective terms are computed in section 6 where we extend the results of [17] up to the eighth order in the deformation parameter, proving that the deformation is unobstructed up to that order. Section 7 provides some remarks and final comments.

2. The Dubrovin–Zhang bi-Hamiltonian approach

A very important class of integrable systems is given by bi-Hamiltonian systems introduced for the first time in [18]. A system is bi-Hamiltonian if it can be written as a Hamiltonian system with respect to two compatible Poisson structures \( \omega_1 \) and \( \omega_2 \), where compatibility conditions entail the fact that \( \omega_1 + \lambda \omega_2 \) is a Poisson structure for any value of \( \lambda \in \mathbb{R} \). For this class of systems, the presence of a bi-Hamiltonian structure captures all the integrability properties.

In the case of systems of hydrodynamic type, the class of Hamiltonian structures to be considered was introduced by Dubrovin and Novikov. Let us briefly outline the key points in their construction. Consider the functionals

\[
\mathcal{F}[u] := \int_{S^1} f(u, u_x, u_{xx}, \ldots) \, dx,
\]

and

\[
\mathcal{G}[u] := \int_{S^1} g(u, u_x, u_{xx}, \ldots) \, dx
\]

and define a bracket between them as follows:

\[
\{ F, G \}[u] := \iint_{S^1 \times S^1} \frac{\delta F}{\delta u'(x)} \frac{\delta G}{\delta u'(y)} \, dx \, dy,
\]

(2.1)

where \( \frac{\delta}{\delta u'} \) denotes the variational derivative with respect to \( u' \) and the bivector \( P^{ij} \) has the following form:

\[
P^{ij} = g^{ij} \delta'(x - y) + \Gamma^i_{jk} u^j_x \delta(x - y).
\]

A deep result of Dubrovin and Novikov characterizes under which conditions the bracket (2.1) is Poisson.

**Theorem 1** [5]. If \( \det(g^{ij}) \neq 0 \), then the bracket (2.1) is Poisson if and only if the metric \( g^{ij} \) is flat and the functions \( \Gamma^i_{jk} \) are related to the Christoffel symbols of \( g_{ij} \) (the inverse of \( g^{ij} \)) by the formula \( \Gamma^i_{jk} = -g^{ij} \Gamma^j_{ik} \).
In this work, when referring to brackets of Dubrovin–Novikov in the previous theorem, we will call them brackets of hydrodynamic type, discarding the case of non-local brackets of hydrodynamic type arising from non-flat metrics. Let us remark that in the flat coordinates \( \{ f_1, \ldots, f_N \} \), the brackets of the form (2.1) reduce to
\[
\eta^{ij} \delta'(x - y),
\]
where \( \eta^{ij} \) is a constant matrix. Also observe that such a bracket is clearly degenerate and its Casimirs are the integrals of the flat coordinates:
\[
C_j = \int_{S^1} f_j \, dx.
\]

In this set-up, we have the following definition:

**Definition 2.** A bi-Hamiltonian structure of hydrodynamic type is given by a pair of Poisson bivectors \( P^{ij}_1, P^{ij}_2 \) satisfying separately the conditions of theorem 1 (for two different contravariant metrics \( g^{ij}_1, g^{ij}_2 \)) and satisfying an additional compatibility condition requiring \( Q^{ij}_\lambda := P^{ij}_1 + \lambda P^{ij}_2 \) to be Poisson for any \( \lambda \). Dubrovin proved that from a differential-geometric point of view, the compatibility of \( P_1 \) and \( P_2 \) is equivalent to the fact that \( g_\lambda := g^{ij}_1 + \lambda g^{ij}_2 \) is a flat pencil of metrics which means [6]

(i) the Riemann tensor \( R_\lambda \) of the pencil \( g_\lambda \) vanishes for any value of \( \lambda \);
(ii) the Christoffel symbols \( (\Gamma_1)_k^{ij} \) of the pencil are given by \( (\Gamma_1)_k^{ij} + \lambda(\Gamma_2)_k^{ij} \).

Since the bi-Hamiltonian structure encodes all the characteristics of an integrable system, Dubrovin and Zhang proposed to study the integrable perturbations of systems of PDEs of the type described above, by studying perturbations of their associated bi-Hamiltonian structure and classifying them modulo Miura transformations. They also conjectured that, under suitable additional assumptions coming from the Gromov–Witten theory, the perturbed bi-Hamiltonian hierarchies exist and are uniquely determined by their dispersionless limit. They also made important steps toward the proof of the conjecture. One of the very important missing gaps, concerning the polynomiality of the dispersive corrections, was recently filled in the remarkable preprint [2], where it is also proved that one of the Hamiltonian structure is polynomial. However, the polynomiality of the second Hamiltonian structure to all orders in the deformation parameter \( \epsilon \) is still an open problem.

2.1. Poisson structures on the space of loops

On the space of smooth loops \( \mathcal{L}(\mathbb{R}^n) := \{ h : S^1 \to \mathbb{R}^n, h \in C^\infty \} \) we consider the ring \( \mathcal{A} \) of differential polynomials:
\[
f(x, u, u_1, \ldots, u_N) := \sum_{i_1, i_2, \ldots, i_n} f_{i_1, i_2, \ldots, i_n} (x, u) u_{(i_1)}^{i_1} \cdots u_{(i_n)}^{i_n},
\]
where \( u = (u_1, \ldots, u_N) \), \( u_{(i)} = (u_{(i)}^{1}, \ldots, u_{(i)}^{N}) \) with \( u_{(i)}^{j} := \frac{d}{dx} u_{(i)}^{j} \). Moreover, the coefficients \( f_{i_1, i_2, \ldots, i_n} (x, u) \) of these differential polynomial are required to be smooth functions on \( S^1 \times \mathbb{R}^n \).

Denote by \( \mathcal{A}_0 := \mathcal{A}/\mathbb{R} \) the space of differential polynomials modulo constants, and \( \mathcal{A}_1 := \mathcal{A}_0 \, dx \). Then one has a well-defined map \( d : \mathcal{A}_0 \to \mathcal{A}_1 \):
\[
f \mapsto df := \left( \frac{df}{dx} + \sum_{l,s} \frac{df}{du_{(s)}^{i(s)}} u_{(l+1)}^{i(l+1)} \right) \, dx.
\]

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The quotient $A := \mathcal{A}_1 / d\mathcal{A}_0$ is called the space of local functionals on $\mathcal{L}(\mathbb{R}^n)$. Elements of $A$ (local functionals) are expressed as integrals over $S^1$ of a representative differential polynomial:

$$A \ni \lambda = \int_{S^1} f(x, u, u_x, \ldots, u_{(n)}) \, dx.$$  \hspace{1cm} (2.6)

Observe that by the fact that we are dealing with suitable boundary conditions, two elements $\lambda_1$ and $\lambda_2$ are identified by a differential polynomial up to a total derivative.

In order to study Poisson bi-vectors on the loop space, we need to introduce the notion of local multi-vectors (k-vectors). A local k-vector $\alpha$ on the loop space $\mathcal{L}(\mathbb{R}^n)$ is defined to be a formal possibly infinite sum

$$\alpha = \sum_{p_1, \ldots, p_k \geq 0}^{1} \frac{1}{k!} \partial_{s_1} x_{i_1} \ldots \partial_{s_k} x_{i_k} A^{i_1 \ldots i_k}_{[p_1 \ldots p_k]} \frac{\partial}{\partial u_{(i_1)}(x_1)} \wedge \ldots \wedge \frac{\partial}{\partial u_{(i_k)}(x_k)},$$  \hspace{1cm} (2.7)

with coefficients

$$A^{i_1 \ldots i_k}_{[p_1 \ldots p_k]} = \sum_{u(x_1), u_x(x_1), \ldots} B^{i_1 \ldots i_k}_{p_1 \ldots p_k}(u(x_1), u_x(x_1), \ldots) \delta^{(p_1)}(x_1 - x_2), \ldots, \delta^{(p_k)}(x_1 - x_k).$$  \hspace{1cm} (2.8)

The coefficients $B^{i_1 \ldots i_k}_{p_1 \ldots p_k}(u(x_1), u_x(x_1), \ldots)$ belong to $\mathcal{A}$, the ring of differential polynomials. The coefficients $A^{i_1 \ldots i_k}$ are skew-symmetric with respect to the simultaneous exchange $(i_r, x_r)$ with $(i_s, x_s)$ and they are called the components of the $\alpha$ k-vector.

The space of local k-vectors is denoted as $\Lambda^{loc}_k$. Specializing to the case of 1-vectors, we obtain the class of local vector fields on the loop space $\mathcal{L}(\mathbb{R}^n)$. They are expressed by the following formula:

$$\xi = \sum_{i=1}^{N} \sum_{s \geq 0} \partial_{s} \xi^i(\gamma(x), u_{(i)}(x)), \frac{\partial}{\partial u_{(i)}(x)}.$$  \hspace{1cm} (2.9)

Their components do not depend explicitly on $x$ and thus they are called translation invariant evolutionary vector fields.

The subspace $\Lambda^{0}_{loc}$ of $\Lambda_{loc}$ is identified with the space of local functionals of the form

$$F := \int_{S^1} f(u(x), u_x(x), \ldots) \, dx, \quad f(u(x), u_{(i)}(x), \ldots) \in \mathcal{A}_0.$$  \hspace{1cm} (2.10)

Since we will focus our attention to local bivectors, we provide their general expression. A local bivector $\omega$ has the form

$$\omega = \frac{1}{2} \sum_{i,j} \partial_{i}^x \partial_{j}^y \omega_{ij} \frac{\partial}{\partial u_{(i)}(x)} \wedge \frac{\partial}{\partial u_{(j)}(y)},$$  \hspace{1cm} (2.11)

where

$$\omega_{ij} = A^{ij}(x - y; u(x), u_x(x), \ldots) = \sum_{r=0}^{\infty} A^{ij}_r(u(x), u_x(x), \ldots) \delta^{(r)}(x - y).$$  \hspace{1cm} (2.12)

In order to characterize which local bivector corresponds to a Poisson bivector, it is necessary to introduce on the space of local multi-vectors with its natural gradation

$$\Lambda^{+}_{loc} := \Lambda^{0}_{loc} \oplus \Lambda^{1}_{loc} \oplus \Lambda^{2}_{loc} \oplus \cdots$$

a bilinear operation:

$$[\cdot, \cdot] : \Lambda^{s}_{loc} \times \Lambda^{r}_{loc} \rightarrow \Lambda^{r+s-1}_{loc}, \quad r, s \geq 0,$$  \hspace{1cm} (2.13)

called the Schouten–Nijenhuis bracket. Let us describe how the Schouten–Nijenhuis bracket operates on certain pairs of local multi-vectors. We have that for any $F, G \in \Lambda^{0}_{loc}$ $[F, G] = 0$. 6
identically, while if $\xi \in \Lambda^1_{\text{loc}}$ is of the form (2.9) and $F$ is a local functional of the form (2.10), then

$$\[\xi, F\] = \int_{S} \sum_{t \geq 0} \sum_{i=1}^{N} \left( \partial_{x}^{t} \xi_{i} \right) \frac{\partial f}{\partial u^{t}(x)} \, dx = \int_{S} \sum_{i=1}^{N} \xi_{i} \frac{\delta F}{\delta u^{i}(x)} \, dx \tag{2.14}$$

where

$$\frac{\delta F}{\delta u^{i}(x)} = \sum_{t \geq 0} (-1)^{t} \partial_{x}^{t} \left( \frac{\partial f}{\partial u^{i}(t)} \right)$$

is the variational derivative of the local functional $F$. Observe that $[\xi, F]$ is indeed an element of $\Lambda^0_{\text{loc}}$. The Schouten–Nijenhuis bracket of two local vector fields $\xi, \eta$ is again a vector field $\mu$ described by the following formula:

$$\mu = [\xi, \eta] = \sum_{s,i} \partial_{s}^{t} \mu_{i} \frac{\partial}{\partial u^{s}(x)} = \sum_{s,j,t} \partial_{s}^{t} \left( \xi_{j}(t) \frac{\partial \eta_{i}}{\partial u^{j}(t)} - \eta_{j}(t) \frac{\partial \xi_{i}}{\partial u^{j}(t)} \right) \frac{\partial}{\partial u^{s}(x)} \tag{2.15}$$

where the components $\mu_{i}$ of $\mu$ are given as

$$\mu_{i} = \sum_{s,j} \partial_{s}^{t} \xi_{j}(t) \frac{\partial \eta_{i}}{\partial u^{j}(t)} - \eta_{j}(t) \frac{\partial \xi_{i}}{\partial u^{j}(t)}.$$

The Schouten–Nijenhuis bracket of a local bivector $\omega$ of the form (2.11) and a local functional $F$ gives rise to a local vector field whose components are

$$[\omega, F]^{i} = \sum_{j,k} A^{ij}_{k} \frac{\partial}{\partial u^{j}(x)} \frac{\delta F}{\delta u^{k}(x)} \tag{2.16}.$$

Analogously the Schouten–Nijenhuis bracket of a local bivector $\omega$ and local vector field $\xi$ is again a local bivector whose components are given by

$$[\omega, \xi]^{i} = \sum_{k,l} \partial_{k}^{t} \xi^{l} \left(u(x), \ldots \right) \frac{\partial A^{ij}}{\partial u^{k}(x)} - \frac{\partial \xi^{l}(u(y), \ldots)}{\partial u^{k}(y)} A^{ij} \tag{2.17}.$$

Finally, if $P$ and $Q$ are two translation invariant bivectors, then their Schouten–Nijenhuis bracket $[P, Q]$ is a translation invariant trivector, whose complicated formula can be found in [7] together with many other details.

**Remark 1.** An alternative efficient way to compute Schouten brackets is based on the idea of substituting multivectors with superfunctional and Schouten bracket with Poisson brackets between superfunctionals [13], see also [1]. We recall this formalism in section 3.

The Schouten–Nijenhuis bracket also satisfies the following properties (a graded Jacobi identity and a graded skew-symmetry):

$$(-1)^{li}[[a, b], c] + (-1)^{lj}[[b, c], a] + (-1)^{lj}[[c, a], b] = 0, \quad \text{graded Jacobi identity} \tag{2.18}$$

$$[a, b] = (-1)^{li}[b, a], \quad a \in \Lambda^k_{\text{loc}}, \quad b \in \Lambda^l_{\text{loc}}, \quad c \in \Lambda^i_{\text{loc}}. \tag{2.19}$$

The importance of the Schouten–Nijenhuis bracket stems from the fact that a local Poisson structure can be characterized in the following way.

**Definition 3.** A local bivector $\omega \in \Lambda^2_{\text{loc}}$ of the form (2.11) is a local Poisson structure on $\mathcal{L}(\mathbb{R}^n)$ if $[\omega, \omega] = 0$. 

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A local Poisson structure gives rise to a Poisson bracket on the space of local functionals in the following way:

$$\{F, G\} := \int_{S^1} \sum_{k \geq 0} \frac{\delta F}{\delta u^i(x)} A^i_k(u, u_x, \ldots) \frac{\delta G}{\delta u^i(x)} \, dx.$$  \hspace{1cm} (2.20)

Using a special choice of local functionals $F = \int u^i(w) \delta(w - x) \, dw$, $G = \int u^j(w) \delta(w - y) \, dw$, we recover the usual representation of a Poisson structure as

$$[u^i(x), u^j(x)] = \sum_{k \geq 0} A^i_k(u(x), u_x(x), \ldots) \delta^{(k)}(x - y).$$

As we have seen in the introduction (see equation (1.3)), the scaling $\psi_{\epsilon} : x \mapsto x\epsilon$ decomposes the evolutionary vector field into homogeneous components. Analogously, this same scaling induces a natural gradation on the space $\Lambda^k_{\text{loc}}$. Now we detail how the various ingredients rescale under $\psi_{\epsilon}$. First of all we define $(\psi_{\epsilon} u^i)(x) = u^i(\epsilon x)$.

Moreover, to see how the $\delta$ distribution and its derivatives rescale, consider

$$\int f(x) \delta^{(s)}(\epsilon x) \, dx = \int f \left( \frac{z}{\epsilon} \right) \delta^{(s)}(z) \frac{dz}{\epsilon} = \left( -1 \right)^s \int \frac{d^i f \left( \frac{z}{\epsilon} \right)}{dz^i} \delta(z) \frac{dz}{\epsilon} = \left( -1 \right)^s f^{(i)}(0) \frac{1}{\epsilon^{s+1}} \int f(x) \delta^{(s)}(x) \frac{1}{\epsilon^{s+1}} \, dx,$$

from which

$$\delta^{(s)}(\epsilon x) = \frac{1}{\epsilon^{s+1}} \left( -1 \right)^s f^{(i)}(0) \delta^{(s)}(x).$$

or

$$\delta^{(s)}(x) = \epsilon^{s+1} (\psi_{\epsilon} (\delta^{(s)}))(x).$$

With this information, we can show that the rescaling $\psi_{\epsilon}$ induces a decomposition on $\Lambda^k_{\text{loc}}$ into monomials of different degrees. For simplicity we focus on the cases of $\Lambda^0_{\text{loc}}$ and $\Lambda^2_{\text{loc}}$. Any local vector field has the form

$$\xi = \sum_{i=1}^N \sum_{s \geq 0} \partial^s_x \xi^i(u(x), u_x(x), \ldots) \frac{\partial}{\partial u^i_{(s)}},$$

where its components $\xi^i$ are elements of the ring $A$ of differential polynomials. Since the rescalings induced in $\partial^s_x$ and $\frac{\partial}{\partial u^i_{(s)}}$ are one the reciprocal of the other, the splitting of the $\xi$ into homogeneous monomials depends only on its components $\xi^i$. In general the components $\xi^i$ split under rescaling into homogeneous monomials as follows:

$$\xi^i := a^i(u) + \epsilon \sum_{j=1}^N b^i_j(u) u^j_{(s)} + \epsilon^2 \left( \sum_{j=1}^N c^i_j(u) u^j_{(s)^2} + \sum_{j,l=1}^N c^i_{jl}(u) u^j_{(s)} u^l_{(s)} \right) + \ldots.$$  \hspace{1cm} (2.22)
and this gives rise to an analogous decomposition as
\[ \Lambda^1_{\text{loc}} = \bigoplus_{k=0}^{\infty} \Lambda^1_{k,\text{loc}}, \]
where \( \Lambda^1_{k,\text{loc}} \) is the space of local vector fields \( \xi \) whose components \( \xi^i \) are homogeneous differential polynomials of degree \( k \). Let us apply the same analysis to the case of elements of \( \Lambda^2_{\text{loc}} \). Recall that a local bivector \( \omega \) is given by
\[ \omega = \frac{1}{2} \sum_{i<j} \partial_i \partial_j \omega^{ij} \left( \frac{\partial}{\partial u^i_r(x)} \wedge \frac{\partial}{\partial u^j_s(y)} \right), \]
where \( \omega^{ij} = A_{ij}(x-y; u(x), u_s(x), \ldots) = \sum_{t \geq 0} A_{ij}^t (u(x), u_s(x), \ldots) \delta(t)(x-y). \)
Since the terms \( \partial_i \partial_j \) and \( \frac{\partial}{\partial u^i_r(x)} \wedge \frac{\partial}{\partial u^j_s(y)} \) have reciprocal scaling factors, the decomposition of \( \omega \) in homogeneous monomials under rescaling is completely controlled by the way in which its components \( \omega^{ij} \) decompose. Rewrite \( \omega^{ij} \) as follows:
\[ \sum_{t \geq 0} \sum_{l \geq 0} (A_{ij}^t)_l \epsilon^{l+t} \delta(t)(x-y), \]
where \( (A_{ij}^t)_l \) is the homogeneous components of degree \( l \) of the differential polynomial \( A_{ij}^t \). Rescaling under \( \psi_\epsilon \) gives rise to
\[ \sum_{t \geq 0} \sum_{l \geq 0} (A_{ij}^t)_l \epsilon^{l+t+1} \delta(t)(x-y), \]
which can be rewritten setting \( k = l + t + 1 \) as
\[ \sum_{k=1}^{\infty} \epsilon^k \sum_{l=0}^{k-1} (A_{ij}^l)_{k-1-l} \delta(l)(x-y). \]
In this way the components of \( \omega^{ij} \) of a bivector decompose into homogeneous terms \([\omega^{ij}]_k\) of the form
\[ [\omega^{ij}]_k := \sum_{l=0}^{k-1} (A_{ij}^l)_{k-1-l} \delta(l)(x-y). \]
Therefore, we have an induced decomposition of \( \Lambda^2_{\text{loc}} \) as follows:
\[ \Lambda^2_{\text{loc}} = \bigoplus_{k \geq 1} \Lambda^2_{k,\text{loc}}, \]
where \( \omega \) is in \( \Lambda^2_{k,\text{loc}} \) exactly when its components \( \omega^{ij} \) are of the form of the addends in (2.24).
Note that the bivectors of hydrodynamic type \( \omega \) introduced by Dubrovin and Novikov (2.2) are elements of \( \Lambda^2_{2,\text{loc}} \), and indeed any element of \( \Lambda^2_{2,\text{loc}} \) is a bivector of hydrodynamic type. Following [5], we call any Poisson structure of the form
\[ (\omega + P) \in \Lambda^2_{\text{loc}}, \]
where \( P = \sum_{k \geq 1} P_k \), \( P_k \in \Lambda_{k+2,\text{loc}} \) a deformation of \( \omega \). Note that due to the rescaling \( \psi_\epsilon \), the deformation (2.26) transforms into \( \epsilon^2 \left( \omega + \sum_{k \geq 1} \epsilon^k P_k \right) \), so we can rewrite the deformation as
\[ \omega + \sum_{k \geq 1} \epsilon^k P_k. \]
As we did for bivectors, the space of \( j \)-multivectors \( \Lambda^j_\text{loc} \) can be decomposed in terms that are homogeneous under rescaling:

\[
\Lambda^j_\text{loc} = \bigoplus_k \Lambda^j_{k,\text{loc}}.
\]

Any element \( P \in \Lambda^j_{k,\text{loc}} \) is transformed into \( \epsilon^k P \) under rescaling.

### 2.2. Poisson cohomology

Fix \( \omega \in \Lambda^2_\text{loc} \) and assume that \( \omega \) is Poisson. Consider the map

\[
d_\omega : \Lambda^j_\text{loc} \rightarrow \Lambda^{j+1}_\text{loc}, \quad d_\omega(a) = [\omega, a].
\]

In particular, it is immediate to see that the map \( d_\omega \) maps \( \Lambda^j_{k,\text{loc}} \) to \( \Lambda^{j+1}_{k+2,\text{loc}} \). This map has the property that \( d_\omega \) maps \( \Lambda^j_{k,\text{loc}} \) to \( \Lambda^{j+1}_{k,\text{loc}} \). Due to the graded Jacobi identity satisfied by the Schouten–Nijenhuis bracket and the fact that \( \omega \) is Poisson, this enables one to define cohomology groups, known as Poisson cohomology groups in the following way:

\[
H^j (\mathcal{L}(\mathbb{R}^n), \omega) := \frac{\ker \left\{ d_\omega : \Lambda^j_\text{loc} \rightarrow \Lambda^{j+1}_\text{loc} \right\}}{\operatorname{im} \left\{ d_\omega : \Lambda^{j-1}_\text{loc} \rightarrow \Lambda^j_\text{loc} \right\}}.
\] (2.27)

These cohomology groups are completely analogous to those defined in the case of finite-dimensional Poisson manifolds by Lichnerowicz [14].

Due to the fact that the space of \( j \)-multivectors \( \Lambda^j_j \) has a natural decomposition in terms of components homogeneous under rescaling and the fact that \( d_\omega \) preserves this homogeneous decomposition, each cohomology group inherits a natural decomposition in homogeneous parts. Indeed, we can introduce

\[
H^j_k (\mathcal{L}(\mathbb{R}^n), \omega) := \frac{\ker \left\{ d_\omega : \Lambda^j_{k,\text{loc}} \rightarrow \Lambda^{j+1}_{k+2,\text{loc}} \right\}}{\operatorname{im} \left\{ d_\omega : \Lambda^{j-1}_{k-2,\text{loc}} \rightarrow \Lambda^j_{k,\text{loc}} \right\}},
\] (2.28)

where a class \([\alpha]\) is in \( H^j_k (\mathcal{L}(\mathbb{R}^n), \omega) \) exactly when any of its representatives can be chosen in \( \Lambda^j_{k,\text{loc}} \). Naturally, one has

\[
H^j (\mathcal{L}(\mathbb{R}^n), \omega) = \bigoplus_k H^j_k (\mathcal{L}(\mathbb{R}^n), \omega).
\] (2.29)

Let us remark that a decomposition like (2.29) is typical of the infinite-dimensional situation, and it does not have an analogous correspondence in the finite-dimensional case.

For Poisson structures of hydrodynamic type such as (2.2), it has been proved in [13] (see also [4] for an independent proof of the cases \( n = 1, 2 \)) that \( H^k (\mathcal{L}(\mathbb{R}^n), \omega) = 0 \) for \( k = 1, 2, \ldots \).

The vanishing of these cohomology groups implies that any deformation of \( \omega \) of the form

\[
P^k = \omega + \sum_{n=1}^{\infty} \epsilon^P P_n,
\]

where \( P_n \in \Lambda^2_{k+2,\text{loc}} \) can be obtained from \( \omega \) by performing a Miura transformation. Indeed, from the Poisson condition

\[
[P^k, P^l] = 0
\]

it follows that \( P_1 \) is a cocycle of \( \omega \) and therefore a coboundary

\[
P_1 = \text{Lie}_X \omega
\]
for a suitable vector field $X_1$. This means that, performing the Miura transformation generated by the vector field $-X_1$, we can eliminate the term $\epsilon$ and obtain a local Poisson bivector of the form

$$\tilde{P} = \omega + \sum_{n=2}^{\infty} \epsilon^n \tilde{P}_n.$$ 

Using the same arguments we can show that

$$\tilde{P}_2 = \operatorname{Lie}_{X_2} \omega$$

and therefore it can be eliminated by the Miura transformation generated by the vector field $-X_2$. In this way, step by step, we reduce $P$ to $\omega$. The reducing Miura transformation is the composition of the infinite sequence of Miura transformations generated by $-X_1$, $-X_2$, $\ldots$.

Totally different is the case in which we deform a pencil of local bivectors. Without loss of generality we can assume such a pencil of the form

$$P_\lambda = \omega_2 - \lambda \omega_1 + \sum_{n=1}^{\infty} \epsilon^n P_n,$$

where $\omega_1$ and $\omega_2$ are a pair of compatible bivectors of hydrodynamic type:

$$\omega_a = g^{ij}_a \delta'(x - y) + \Gamma^{ij}_k(u^k_1 \delta(x - y), \quad a = 1, 2. \quad (2.31)$$

Indeed, due to the triviality of $H^2(\mathcal{L}(\mathbb{R}^n), \omega)$, the $\epsilon$-corrections to $\omega_1$ can be eliminated by a Miura transformation. However, the requirement that $P_\lambda$ is a Poisson bivector, namely

$$\left[ P_\lambda^\epsilon, P_\lambda^\epsilon \right] = 0$$

for any $\lambda \in \mathbb{R}$ imposes some restrictions on the bivector $P_\lambda^\epsilon = \omega_2 + \sum_{n=1}^{\infty} \epsilon^n P_n$.

First it must be compatible with $\omega_1$, that is to say, indicating with $d_1(\cdot) := [\omega_1, \cdot]$

$$d_1 P_n = 0,$$ \quad (2.32)

which means that all the terms $P_n$ must be the coboundary of $\omega_1$.

Second it must be a Poisson bivector. This is equivalent to the system of conditions

$$d_2 P_n = -\frac{1}{2} n \sum_{k=1}^{n-1} [P_k, P_{n-k} - k], \quad n = 1, 2, \ldots.$$ \quad (2.33)

**Proposition 4.** The system (2.33) is compatible.

**Proof.** Indeed due to the vanishing of $H^3(\mathcal{L}(\mathbb{R}^n), \omega_2)$ compatibility is equivalent to the requirement that the right-hand sides are cocycles of $\omega_2$:

$$d_2 \left( \sum_{k=1}^{n-1} [P_k, P_{n-k}] \right) = 0, \quad n = 1, 2, \ldots.$$ 

This can be proved by induction using the graded Jacobi identity

$$d_2 \left( \sum_{k=1}^{n-1} [P_k, P_{n-k}] \right) = \sum_{k=1}^{n-1} \left[ \omega_2, [P_k, P_{n-k}] \right]$$

$$= - \sum_{k=1}^{n-1} [P_k, [\omega_2, P_{n-k}]] - \sum_{k=1}^{n-1} [P_{n-k}, [P_k, \omega_2]]$$

$$= -2 \sum_{k=1}^{n-1} [P_k, [\omega_2, P_{n-k}]].$$
This means that we can solve recursively equations (2.33) and the compatibility is proved. □

At each step the solution of (2.33) is defined up to a coboundary of $\omega_2$. The problem is to prove that it is always possible to choose these coboundaries in such a way that the resulting Poisson bivector is compatible with $\omega_1$. In other words the problem is to prove that any solution $P_n$ of (2.33) has the form

$$P_n = d_1 X_n + d_2 Y_n$$

if $P_1, \ldots, P_{n-2}, P_{n-1}$ are coboundaries of $\omega_1$. This is a nontrivial open problem. Note that if $P_1, \ldots, P_{n-2}, P_{n-1}$ are coboundaries of $\omega_1$, then the bivectors $P_n$ defined by (2.33) satisfy the condition

$$d_1 d_2 P_n = 0.$$  

(2.35)

Indeed

$$d_1 \left( \sum_{k=1}^{2n-1} [P_{2k}, P_{2n-2k}] \right) = \sum_{k=1}^{2n-1} [\omega_1, [P_{2k}, P_{2n-2k}]] = \sum_{k=1}^{2n-1} [P_{2k}, [\omega_1, P_{2n-2k}]] - \sum_{k=1}^{2n-1} [P_{2n-2k}, [P_{2k}, \omega_1]] = 0.$$ 

Unfortunately, this is not sufficient to conclude that $P_n$ has the form (2.34). The possible obstruction lives in the bi-Hamiltonian cohomology group

$$H^3(L(\mathbb{R}^n), \omega_1, \omega_2) = \frac{\text{Ker}(d_1 d_2|_{\Lambda^1_{loc}})}{\text{Im}(d_1|_{\Lambda^1_{loc}}) \oplus \text{Im}(d_1|_{\Lambda^1_{loc}})}.$$ 

To the best of our knowledge, bi-Hamiltonian cohomology groups were initially introduced and studied in [12]. However, bi-Hamiltonian cohomology groups in the framework of integrable PDEs were first used in [7].

Since the deformations $P_\epsilon = \sum_{k=1}^{\infty} \epsilon^k P_k$ are coboundaries, namely $P_\epsilon = \text{Lie}_{X_\epsilon} \omega_1$, in order to explicitly construct the components $P_k$ it is convenient to solve the equations for the vector fields $X_\epsilon$ generating the deformation, instead of solving the corresponding equations for the bivectors $P_\epsilon$ that, in general, are more involved.

Let us consider, for instance, first-order deformations, that is, $P^\epsilon_1 := \omega_2 + \epsilon \text{Lie}_{X_1} \omega_1 - \lambda \omega_1$. Since we want $P^\epsilon_1$ to be Poisson up to the order $\epsilon$ included, we require $[P^\epsilon_1, P^\epsilon_1] = o(\epsilon)$. This implies that

$$[\omega_2, P_1] = d_1 P_1 = d_2 d_1 X_1 = -d_1 d_2 X_1 = 0,$$

(2.36)

where we have used the fact that $(d_1 + d_2)^2 = 0$.

Among all vector fields that satisfy the equation $d_1 d_2 X_1 = 0$ we have to single out those defining trivial deformations, that is, those generating deformations $P_1$ that can be obtained by infinitesimal change of coordinates:

$$\text{Lie}_{X_1} \omega_1 = 0$$  

(2.37)

$$\text{Lie}_{X_1} \omega_2 = P_1.$$  

(2.38)

Note that the vector field $\tilde{X}$ does not coincide with the vector field $X_1$ defining the deformation.
Theorem 5. Nontrivial first-order deformations are the elements of the group

\[ H_2^1(\mathcal{L}(\mathbb{R}^n), \omega_1, \omega_2) = \frac{\text{Ker}(d_1 d_2|_{\Lambda^1_{1,\text{loc}}})}{\text{Im}(d_1|_{\Lambda^0_{1,\text{loc}}})} \oplus \text{Im}(d_1|_{\Lambda^0_{1,\text{loc}}}). \]

Proof: Suppose that \( X_1 = d_1 a + d_2 b \) then

\[ P_1 = d_1 d_2 b = -d_2 d_1 b. \]

This means \( P_1 = \text{Lie}_\tilde{X} \omega_2 \) with \( \tilde{X} = -d_1 b \). Moreover,

\[ \text{Lie}_\tilde{X} \omega_1 = 0 \]

and therefore the deformation is trivial.

Assume now that the deformation is trivial. Then, by definition, we have

\[ \text{Lie}_\tilde{X} \omega_1 = 0, \quad \text{Lie}_\tilde{X} \omega_2 = P_1 \]

which implies, due to the vanishing of the first cohomology group,

\[ \tilde{X} = d_1 b. \]

The above condition entails

\[ -d_1 d_2 b = d_2 d_1 b = \text{Lie}_\tilde{X} \omega_2 = P_1 = d_1 Y_1 \]

with \( Y_1 = -d_2 b + d_1 a \).

Higher-order deformations can be treated in a similar way. It turns out that nontrivial deformations are related to the cohomology groups

\[ H_2^k(\mathcal{L}(\mathbb{R}^n), \omega_1, \omega_2) = \frac{\text{Ker}(d_1 d_2|_{\Lambda^1_{1,\text{loc}}})}{\text{Im}(d_1|_{\Lambda^0_{1,\text{loc}}})} \oplus \text{Im}(d_1|_{\Lambda^0_{1,\text{loc}}}). \]

The study of such cohomology groups has been done by Liu and Zhang in [15] in the semisimple case, that is, assuming that the eigenvalues \( u^1, \ldots, u^n \) of the matrix \( g_1^{-1} g_2 \) define a set of local coordinates, called canonical coordinates (here \( g_1 \) and \( g_2 \) are the contravariant metrics defining the two undeformed Poisson structures of hydrodynamic type \( \omega_1 \) and \( \omega_2 \)). Under this additional assumption they showed that

\[ H_2^k(\mathcal{L}(\mathbb{R}^n), \omega_1, \omega_2) = 0 \quad \forall \ k \neq 3 \]

and that the elements of

\[ H_2^3(\mathcal{L}(\mathbb{R}^n), \omega_1, \omega_2) \]

are vector fields of the form

\[ d_2 \left( \sum_{i=1}^n \int c^i(u') u'_i \log u'_i \, dx \right) - d_1 \left( \sum_{i=1}^n \int u' c^i(u') u'_i \log u'_i \, dx \right), \tag{2.39} \]

where \( c^i(u') \) are arbitrary functions of a single variable. Note that the functionals in brackets in formula (2.39) do not belong to \( \Lambda^1_{1,\text{loc}} \) due to the non-polynomial dependence on the \( x \)-derivatives of the \( u \)'s. If we allow such a dependence all the elements in \( H_2^3(\mathcal{L}(\mathbb{R}^n), \omega_1, \omega_2) \) will become ‘trivial’. This remark justifies the following definitions [7].

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Definition 6. The transformations of the form

\[ u^i \rightarrow v^i = u^i + \sum_{k=1}^{\infty} \epsilon^k F_k^i (u; u_x, ..., u^{(nk)}), \quad i = 1, ..., n, \]

where the coefficients \( F_k^i \) are quasihomogeneous of the degree \( k \) rational functions in the derivatives \( u_x, ..., u^{(nk)} \), are called quasi-Miura transformations.

Definition 7. A deformation of a Poisson pencil of hydrodynamic type is called quasitrivial if there exists a quasi-Miura transformation reducing the pencil to its leading term.

Clearly the transformation generated by the vector field

\[ d_1 \left( -\sum_{i=1}^{n} \int c^i (u^i) u_x^i \log u_x^i \, dx \right) \]

is quasitrivial. In other words, in the semisimple case, all second-order deformations are quasitrivial. In [16] Liu and Zhang proved that, in the scalar case, the deformations (if they exist) are quasitrivial at any order (an alternative independent proof was given later in [1]). We will use this fact in a subsequent section. The quasitriviality of deformations of semisimple bi-Hamiltonian structures of hydrodynamic type was instead proved in [8].

3. An alternative formalism

We already pointed out that it is possible to compute the Schouten bracket of two bivectors using a different formalism, initially introduced by Dorfman, Gelfand [11], Olver [19] and further developed by Getzler [13] and Barakat [1]. One of the key advantages of this formalism is that it turns difficult and time-consuming computations into extremely fast and straightforward calculations. Consider the graded algebra \( A := \bigoplus_{k \in \mathbb{Z}} A_k \) over the ring \( C^\infty(\mathbb{R}^n) \), where \( A = C^\infty(\mathbb{R}^n)[u_1, u_2, \ldots] \) is just the polynomial algebra with coefficients in \( C^\infty(\mathbb{R}^n) \) generated by countable generators \( \{u_1, \ldots, u_k, \ldots\} \) with the grading induced by assigning \( \text{deg}(u_k) = k \). On \( A \) it is defined as a total derivative with respect to \( x \):

\[ \partial_x := \sum_{i=1}^{n} \sum_{k=0}^{\infty} u_{(k+1)}^i \frac{\partial}{\partial u_{(k)}^i}, \quad (3.1) \]

and a variational derivative with respect to \( u^i \)

\[ \frac{\delta}{\delta u^i} := \sum_{k \geq 0} (-\partial_x)^k \frac{\partial}{\partial u_{(k)}^i}. \quad (3.2) \]

Now instead of dealing with \( \delta\)-Dirac distributions and their derivatives, one introduces a polynomial algebra over anticommuting variables \( \theta_i^j, \quad i = 1, \ldots, n, \quad k \in \mathbb{Z}_{\geq 0} \), that satisfies the following relations:

\[ \theta_i^j \wedge \theta_k^l = -\theta_k^l \wedge \theta_i^j, \quad (3.3) \]

\[ \partial_i \theta_k^j = \theta_k^{j+1}, \quad (3.4) \]

where \( \partial_i \) behaves like a derivation:

\[ \partial_i (\theta_i^j \wedge \theta_i^k) = \theta_k^{j+1} \wedge \theta_i^j + \theta_i^k \wedge \theta_k^j. \]

Formally, it is possible to express \( \partial_i \) as a combination of partial derivatives with respect to \( \theta_i^j \):

\[ \partial_i = \sum_{j=1}^{n} \sum_{k=0}^{\infty} \theta_{k+1}^{j+1} \frac{\partial}{\partial \theta_k^j}. \quad (3.5) \]
To apply correctly formula (3.5) obtaining results consistent with \[ \partial_x (\theta_i^k \wedge \theta_j^l) = \theta_i^{k+1} \wedge \theta_j^l + \theta_i^k \wedge \theta_j^{l+1}, \]

we need to bring the term \( \theta_i^k \) in front using the anti-commutation rule \( \theta_j^l \wedge \theta_i^k = -\theta_i^k \wedge \theta_j^l \) and only after that apply the operator \( \frac{\partial}{\partial \theta_i^k} \). For instance, suppose that in the scalar case we want to compute \( \partial (\theta_1^1 \wedge \theta_3^3) = \theta_2^1 \wedge \theta_3^3 + \theta_1^1 \wedge \theta_4^4 \), using equation (3.5). We have

\[
\sum_{k=0}^{\infty} \theta_{k+1} \frac{\partial}{\partial \theta_k^k} (\theta_1 \wedge \theta_3) = \theta_2 \frac{\partial}{\partial \theta_1} (\theta_1 \wedge \theta_3) + \theta_4 \frac{\partial}{\partial \theta_3} (\theta_1 \wedge \theta_3)
\]

\[
= \theta_2 \wedge \theta_3 - \theta_3 \frac{\partial}{\partial \theta_3} (\theta_1 \wedge \theta_3)
\]

\[
= \theta_2 \wedge \theta_3 - \theta_2 \wedge \theta_1 = \theta_2 \wedge \theta_3 + \theta_1 \wedge \theta_4.
\]

From now on, when the operator \( \frac{\partial}{\partial \theta_i^k} \) appears, we will assume that this procedure has been enforced. In this way the total derivative \( \partial_x \) is extended to \( A(\Theta) := A[\theta_i^0, i = 1, \ldots, n, k \geq 0] \):

\[
\partial_x = \sum_{i=1}^{n} \sum_{k=0}^{\infty} \left[ \theta_{i(k+1)} \frac{\partial}{\partial \theta_i^k} + \theta_{i(k+1)} \frac{\partial}{\partial \theta_i^k} \right].
\]

(3.6)

Analogously, we can consider the variational derivative with respect to \( \theta^i \) as given by the following formula:

\[
\frac{\delta}{\delta \theta^i} := \sum_{k=0}^{\infty} (-\partial)^k \frac{\partial}{\partial \theta^i_k}. \quad (3.7)
\]

For instance,

\[
\frac{\delta}{\delta \theta^i} (\theta^i \wedge \theta^j) = \frac{\partial}{\partial \theta^i} (\theta^i \wedge \theta^j) - \partial_x \left( \frac{\partial}{\partial \theta^i} (\theta^i \wedge \theta^j) \right)
\]

\[
= \theta_j^i + \partial_x \left( \frac{\partial}{\partial \theta^i} (\theta^i \wedge \theta^j) \right) = 2 \theta_j^i.
\]

As in the classical case, we have the following important lemma.

**Lemma 8.** The following identities hold:

\[
\frac{\delta}{\delta \theta^i} \partial_x = 0, \quad \frac{\delta}{\delta \theta^i} \partial_x = 0. \quad (3.8)
\]

This lemma is a generalization of the well-known fact that to a Lagrangian function it is possible to add a closed form or a total derivative without affecting the equations of motion (Euler–Lagrange equations).

**Proof.** The same proof of [20], theorem 4.7, applies to this case. \( \Box \)

It is also possible to introduce higher-order variational derivatives with respect to \( \theta^i \) and \( u_{(k)} \):

\[
\frac{\delta}{\delta \theta^i_k} := \sum_{l=0}^{\infty} (-1)^l \begin{pmatrix} k+l \cr k \end{pmatrix} \frac{\partial l}{\partial \theta^i_{k+l}}, \quad (3.9)
\]

\[
\frac{\delta}{\delta u_{(k)}} := \sum_{l=0}^{\infty} (-1)^l \begin{pmatrix} k+l \cr k \end{pmatrix} \frac{\partial l}{\partial u_{(k+l)}}. \quad (3.10)
\]
The higher-order variational derivatives are related to the ordinary partial derivatives through the following lemma.

**Lemma 9.** The following identities hold true:

\[ \frac{\partial}{\partial \theta^i_k} = \sum_{j=k}^{\infty} \binom{j}{k} \frac{\partial^{j-k}}{\partial \theta^j} \delta \frac{\partial^j}{\partial \theta^j}, \]

(3.11)

where \( \frac{\delta}{\delta \theta^i} \) are given by (3.9) and

\[ \frac{\partial}{\partial u'_j} = \sum_{j=k}^{\infty} \binom{j}{k} \frac{\partial^{j-k}}{\partial u'_j} \delta \frac{\partial^j}{\partial u'_j}, \]

(3.12)

where \( \frac{\delta}{\delta u'_j} \) are given by (3.10).

**Proof.** See the appendix. \( \square \)

An important identity relating higher variational derivatives is the following one.

**Lemma 10.** For any differential polynomials \( f, g \in A \), the following identity holds:

\[ \sum_{j \geq 0} \partial^j_x (f) \frac{\partial g}{\partial u(j)} = \sum_{j \geq 0} \partial^j_x \left( f \frac{\delta g}{\delta u(j)} \right). \]

(3.13)

This lemma is actually the definition of higher variational derivatives as given in [20]. This lemma holds true in a more general situation, where \( f, g \) are not required to depend polynomially on the derivatives of \( u \).

As we will see, this formalism has several advantages. First, we need to recall how it is related to the construction of \( k \)-multivectors and evolutionary vector fields introduced before. Given a \( k \)-multivector \( P \) written in the Dubrovin–Zhang formalism, to rewrite it in this formalism it is sufficient to substitute each occurrence of \( \delta^{(k)} \) with \( \theta_k \) and finally multiply by \( \theta \) on the left. For instance, \( \omega_1 = \delta'(x - y) \) is written as \( \omega_1 = \theta \wedge \theta_1 \), while \( \omega_2 = u \delta'(x - y) + \frac{1}{2} u \delta(x - y) \) is written as \( \omega_2 = \theta \wedge (u \theta_1 + \frac{1}{2} u \theta) = u \theta \wedge \theta_1 \). The same procedure applies in particular to evolutionary vector fields. Since an evolutionary vector field is written as \( X = f \frac{\partial}{\partial u} + \frac{\partial f}{\partial u'_1} + \cdots, f \in A \) in the Dubrovin–Zhang formalism, in this formalism the same vector field appears as \( X = f \theta \). In the case of systems, if we are given a Poisson tensor of hydrodynamic type as \( P_{ij} = g_{ij} \delta'(x - y) + \Gamma^1_{ij} u_k \delta(x - y) \), we can write it in terms of the anti-commuting variables \( \theta^i \) as \( P^i = g^i j \theta^j + \Gamma^i^j u_k \theta^j \), since the sum over \( i, j \) is assumed and \( \Gamma^{ij} \) is symmetric, while \( \theta^i \theta^j \) is skew.

Thus, using this formalism, a \( k \)-multivector \( P \) is represented as a sum of terms of the form \( f \theta^i_1 \wedge \cdots \wedge \theta^i_k \), where \( f \in A \). In particular, the Schouten bracket between a \( k \)-multivector \( P \) and a \( k' \)-multivector \( Q \) is a \((k + k' - 1)\)-multivector given by the following expression:

\[ [P, Q] = \sum_{i=1}^{N} \frac{\delta P}{\delta \theta^i} \frac{\delta Q}{\delta u'} - (-1)^{k+1} \frac{\delta P}{\delta u'} \frac{\delta Q}{\delta \theta^i}. \]

(3.14)

From (3.14) and (3.8) we immediately get the following lemma:

**Lemma 11.** Let \( P \) and \( Q \) be a \( k \)-multivector and a \( k' \)-multivector, respectively. Then

\[ [P, \partial Q] = 0. \]
Using this formalism, it might be useful to have a way to express $\frac{\delta}{\delta u^i}(f \theta^j_p)$ using a formula in which the $\theta$-variables do not appear under an operator sign. This is provided by the following.

**Lemma 12.** Let $f \in \mathcal{A}$ be homogeneous of degree $k$. Then the following formula holds true:

$$\frac{\delta}{\delta u^i}(f \theta^j_p) = \sum_{l=0}^{k} (-1)^l \frac{\delta}{\delta u^i}^{(l)}(f) \theta^{j+p} \delta^l,$$

where moreover

$$\frac{\delta}{\delta u^i}^{(l)}(f) = \sum_{h=0}^{k-l} (-1)^h \left( \frac{l}{h} \right) \partial_h \left( \frac{\partial f}{\partial u} \delta^l \right).$$

**Proof.** See the appendix. □

Entirely similar formulas hold for more complicated expressions.

Once we have an expression written using the anti-commutative variables $\theta^i$ and their derivatives, in order to revert to the Dubrovin–Zhang formalism it is necessary to apply a normalization operator $\mathcal{N} := \sum_i \theta^i \frac{\delta}{\delta \theta^i}$ (introduced in [1]) before deleting all the instances of $\theta^i$ appearing on the left and making the substitution $\theta_i \mapsto \theta^{(i)}$. Let us work out in more detail a simple example assuming that the target space is one dimensional. Consider a vector field $X = \partial n x f_n$, where $f_n(u)$ is an arbitrary function of $u$. We want to compute the Lie derivative of $\omega_1 = \delta^{(1)}$ with respect to $X$. First we transform $\omega_1$ in the formalism with $\theta$'s, where it appears as $\omega_1 = \theta \theta_1$, while $X = \partial n x \delta^{(n)} f_n$. Then we recall that the Poisson cohomology operator $d_1$ associated with $\omega_1$ is equal to $d_1 = \delta^1 \delta$.$\quad$So we have

$$\text{Lie}_X(\omega_1) = d_1(X) = 2 \theta_1 \delta \left( \frac{\partial f_n}{\partial u} \theta_1 \right) = 2 \theta_1 \delta^{(n)}((-1)^n f_n),$$

where the last equality holds integrating by part inside the variational derivative and recalling that the variational derivative of a total derivative is identically zero. Thus, we obtain

$$\text{Lie}_X(\omega_1) = (-1)^n \frac{\partial f_n}{\partial u} \delta(n)(x - y).$$

At this point, applying the normalization operator $\mathcal{N}$ we obtain

$$\theta \frac{\delta}{\delta \theta^i} \left( (-1)^n \frac{\partial f_n}{\partial u} \theta_1 \theta_n \right) = (-1)^n \theta \left( -\theta_1 \right) \frac{\partial}{\partial \theta^i} \left( 2 \frac{\partial f_n}{\partial u} \theta_1 \theta_n \right)$$

$$+ (-1)^n \theta \frac{\partial}{\partial \theta^i} \left( 2 \frac{\partial f_n}{\partial u} \theta_1 \theta_n \right)$$

$$= -(-1)^n \theta \frac{\partial}{\partial \theta^i} \left( 2 \frac{\partial f_n}{\partial u} \theta_1 \theta_n \right) - \theta \frac{\partial}{\partial \theta^i} \left( \frac{\partial f_n}{\partial u} \theta_1 \theta_n \right)$$

$$= -(-1)^n \theta \frac{\partial}{\partial \theta^i} \left( 2 \frac{\partial f_n}{\partial u} \theta_1 \theta_n \right) - \theta \frac{\partial}{\partial \theta^i} \left( 2 \frac{\partial f_n}{\partial u} \theta_1 \theta_n \right).$$

Now we can cancel the $\theta$ appearing on the left and substitute $\theta_1$ with $\delta^{(1)}(x - y)$ and $\theta_n$ with $\delta^{(n)}(x - y)$, obtaining

$$-(1)^n \theta \left( 2 \frac{\partial f_n}{\partial u} \delta^{(n)}(x - y) \right) - \theta \left( 2 \frac{\partial f_n}{\partial u} \delta^{(1)}(x - y) \right).$$

This is exactly the formula for $d_1 X_n$ appearing in theorem 13.
4. The scalar case

In the scalar case we have
\[
    \omega_1 = f(u) \delta'(x - y) + \frac{1}{2} f_x \delta(x - y)
\]
and
\[
    \omega_2 = g(u) \delta'(x - y) + \frac{1}{2} g_x \delta(x - y).
\]

Without loss of generality we can assume \( f(u) = 1 \). For simplicity we will consider the special case \( g(u) = 2u \). In this case the pencil
\[
    \omega_2 - \lambda \omega_1
\]
is exact. This means that there exists a vector field \((X = \frac{1}{2})\) such that
\[
    \text{Lie}_X \omega_2 = \omega_1, \quad \text{Lie}_X \omega_1 = 0.
\]

For more about exact Poisson pencils and their deformations, see the next section. Only two examples of deformations of \( P_\lambda = \omega_2 - \lambda \omega_1 \) are known. One is the Poisson pencil of KdV which is
\[
    P_\lambda = \omega_2 - \lambda \omega_1 + c \delta''''(x - y),
\]
the second is the Poisson pencil of Camassa–Holm equation, that can be written in the form
\[
    \omega_2 - \lambda \omega_1 + P_\epsilon
\]
with \( P_\epsilon \) given by
\[
    \sum_{n=1}^{2n} \epsilon^{2n} \left[ \partial_k (u \delta^{(2n)}(x - y)) + \partial_{2n} (u \delta(x - y)) \right]
    + \sum_{n=1}^{2n+1} \epsilon^{2n+1} \left[ \partial_k (u \delta^{(2n+1)}(x - y)) - \partial_{2n+1} (u \delta(x - y)) \right].
\]

**Theorem 13.** Up to the fifth order, all deformations of the pencil
\[
    P_\lambda = \omega_2 - \lambda \omega_1 = 2u(x) \delta'(x - y) + u_x \delta(x - y) - \lambda \delta'(x - y)
\]
can be reduced, by the action of Miura group, to the following form:
\[
    P_\epsilon^\lambda = \omega_2 - \lambda \omega_1 - \sum_{k=1}^{5} \epsilon^k d_k X_n + \mathcal{O}(\epsilon^6)
\]
where
\[
    X_n = (\partial^n_k f_n) \frac{\partial}{\partial u} + \sum_{k,n} \epsilon^k \left( \frac{\partial^n_k f_n}{\partial u^n} \right) \frac{\partial}{\partial u^n}, \quad n = 1, \ldots, 5
\]
\[
    d_k X_n = -(-1)^n \partial_k \left[ \frac{\partial f}{\partial u} \delta^{(n)}(x - y) \right] - \partial^n_k \left[ \frac{\partial f}{\partial u} \delta(x - y) \right]
\]
\[
    \frac{\partial f_k}{\partial u} = -\frac{\partial}{\partial u} \left( \frac{\partial f_2}{\partial u} \right)^2
\]
\[
    \frac{\partial^2 f_5}{\partial u^2} = -2 \frac{\partial^2 f_3}{\partial u^2} \frac{\partial^2 f_2}{\partial u^2}
\]
and \( f_2, f_3 \) are arbitrary. The deformations are trivial if and only if \( f_2 = 0 \).
This theorem has been proved in [17]. For the convenience of the reader, and as an example of the alternative formalism we outlined in the previous section, we will prove that the bivector defined above is Poisson up to terms of order $O(\epsilon^6)$.

**Proof.** First of all let us check that

$$[P\lambda^\epsilon, P\lambda^\epsilon] = O(\epsilon^6)$$

or, using the alternative formalism that

$$\{\hat{P}\lambda^\epsilon, \hat{P}\lambda^\epsilon\} = O(\epsilon^6)$$

(we denote with $\hat{P}$ the corresponding quantity in the alternative formalism introduced in section 3). This is equivalent to

$$2\{\omega_2, d_1X_n\} + \sum_{k=1}^{n-1} [d_1X_{2k}, d_1X_{2n-2k}], \quad n = 1, 2.$$

Using the identities

$$\frac{\partial}{\partial u_1} \hat{a}_n^\epsilon = \sum_{l=0}^{n} \binom{n}{l} \frac{\partial}{\partial u_{n-l}}$$

$$\sum_{s=0}^{n} (-1)^s \binom{n}{s} \binom{n}{s} = (-1)^n \delta_{0,k}$$

it is easy to prove that (from now on we will omit the symbol of the wedge product among the anti-commuting variables $\theta$)

$$\frac{\delta d_1X_n}{\delta u} = 2(-1)^n \frac{\partial^2 f_n}{\partial u^2} \theta_1 \theta_n \quad (4.1)$$

$$\frac{\delta d_1X_n}{\delta \theta} = -2(-1)^n \theta_1 \left( \frac{\partial f_n}{\partial u} \theta_1 \right) - 2\theta_1 \left( \frac{\partial f_n}{\partial u} \theta_1 \right). \quad (4.2)$$

Using the formulas above, we obtain

$$2\{\omega_2, d_1X_{2n}\} + \sum_{k=1}^{n-1} [\hat{P}_k, \hat{P}_{n-k}] = 4 \frac{\partial f_n}{\partial u} \theta_1 \theta_{n+1} + 4\theta_1 \left( \frac{\partial f_n}{\partial u} \theta_1 \right) \frac{\partial f_n}{\partial u} \theta_1$$

$$+ 8 \sum_{k=1}^{n-1} \frac{\partial f_k}{\partial u} \frac{\partial^2 f_{n-k}}{\partial u^2} \theta_1 \theta_{n-k} \theta_{k+1} + 8 \sum_{k=1}^{n-1} (-1)^k \theta_1 \left[ \frac{\partial f_k}{\partial u} \theta_1 \right] \left[ \frac{\partial^2 f_{n-k}}{\partial u^2} \theta_1 \theta_{n-k} \right].$$

Taking into account that

$$\partial_n^\epsilon (\theta_1 \theta_1) = \theta_1 \theta_{1+1} + \sum_{k=1}^{n} \left[ \binom{n}{k} - \binom{n}{k-1} \right] \theta_k \theta_{n+1-k}$$

(where the square bracket denotes the integer part of the fraction) and dividing by 4, we obtain

$$\frac{\partial f_n}{\partial u} \theta_1 \theta_{n+1} + \frac{\partial f_n}{\partial u} \theta_1 \theta_{n+1} + \sum_{k=1}^{n} \left[ \binom{n}{k} - \binom{n}{k-1} \right] \frac{\partial f_n}{\partial u} \theta_k \theta_{n+1-k}$$

$$+ 2 \sum_{k=1}^{n-1} \frac{\partial f_{n-k}}{\partial u} \frac{\partial^2 f_k}{\partial u^2} - \frac{\partial f_{n-1-k}}{\partial u^2} \frac{\partial^2 f_{n+1-k}}{\partial u^2} \theta_1 \theta_{n+1-k}$$

$$+ 2 \sum_{k=1}^{n-1} (-1)^k \theta_1 \theta_{n-k} \left[ \frac{\partial f_{n-k}}{\partial u} \theta_1 \right] \left[ \frac{\partial^2 f_k}{\partial u^2} \theta_1 \theta_k \right].$$
that implies
\[
\sum_{k=2}^{\lfloor \frac{n}{2} \rfloor} \left\{ \binom{n}{k} - \binom{n}{k-1} \right\} \frac{\partial f_n}{\partial u} + 2 \frac{\partial f_{n-k}}{\partial u} \frac{\partial^2 f_k}{\partial u^2} - 2 \frac{\partial f_{k-1}}{\partial u} \frac{\partial^2 f_{n-k}}{\partial u^2} \left( \frac{\partial^2 f_k}{\partial u^2} \right) \theta_1 \theta_k \theta_{n+1-k} \\
+ 2 \sum_{k=2}^{n-2} (-1)^{n-k} \frac{\partial f_{n-k}}{\partial u} \left[ \frac{\partial^2 f_k}{\partial u^2} \right] \theta_1 \theta_k = 0.
\]

In the case \( n = 2 \) the equation above is clearly satisfied for arbitrary \( f_2 \) and \( f_3 \). For \( n = 4 \) we obtain
\[
\left\{ \binom{4}{2} - \binom{4}{1} \right\} \frac{\partial f_4}{\partial u} + 2 \frac{\partial f_2}{\partial u} \frac{\partial^2 f_2}{\partial u^2} \theta_1 \theta_2 \theta_3 + 2 \theta_1 \left[ \frac{\partial^2 f_2}{\partial u^2} \right] \left[ \frac{\partial^2 f_2}{\partial u^2} \right] \theta_1 \theta_2 \theta_3 = 0
\]

that implies
\[
\frac{\partial f_4}{\partial u} = - \frac{\partial}{\partial u} \left( \frac{\partial^2 f_2}{\partial u^2} \right)^2.
\]

In the case \( n = 5 \) we obtain
\[
\left\{ \binom{5}{2} - \binom{5}{1} \right\} \frac{\partial f_5}{\partial u} + 2 \frac{\partial f_3}{\partial u} \frac{\partial^2 f_3}{\partial u^2} \theta_1 \theta_2 \theta_3 + 2 \sum_{k=2}^{3} (-1)^{5-k} \frac{\partial f_{5-k}}{\partial u} \left[ \frac{\partial^2 f_k}{\partial u^2} \right] \theta_1 \theta_k = 0
\]

that implies
\[
\frac{\partial^2 f_5}{\partial u^2} = -2 \frac{\partial^2 f_3}{\partial u^2} \frac{\partial^2 f_3}{\partial u^2}.
\]

Unfortunately, as we mentioned in the introduction, it is not possible to extend the previous formulas to the case \( n = 6 \). Indeed in this case we obtain
\[
\sum_{k=2}^{\lfloor \frac{6}{2} \rfloor} \left\{ \binom{6}{k} - \binom{6}{k-1} \right\} \frac{\partial f_6}{\partial u} + 2 \frac{\partial f_{6-k}}{\partial u} \frac{\partial^2 f_k}{\partial u^2} - 2 \frac{\partial f_{k-1}}{\partial u} \frac{\partial^2 f_{6-k}}{\partial u^2} \left( \frac{\partial^2 f_k}{\partial u^2} \right) \theta_1 \theta_k \theta_{7-k} \\
+ 2 \sum_{k=2}^{4} (-1)^{6-k} \frac{\partial f_{6-k}}{\partial u} \left[ \frac{\partial^2 f_k}{\partial u^2} \right] \theta_1 \theta_k = 0
\]

that implies
\[
\frac{\partial f_6}{\partial u} = - \frac{\partial}{\partial u} \left( \frac{\partial^2 f_6}{\partial u^2} \right)
\]

plus two additional conditions relating \( f_2, f_3, f_4 \) which are compatible only if \( f_2(u) \) is a polynomial of degree 2.

As we will see the higher-order deformations are much more complicated.
5. Deformations of exact pencils in the scalar case

In general, if \( \omega_1 \) is a Poisson structure, then \( \omega_2 := \text{Lie}_X (\omega_1) \) is compatible with \( \omega_1 \) (in the sense that \( \{ \omega_1, \text{Lie}_X (\omega_1) \} = 0 \)) but it might fail to be Poisson itself. A simple sufficient condition ensuring that \( \omega_2 \) is Poisson is the notion of the exact pencil, which was introduced in [3].

**Definition 14.** Given a Poisson structure \( \omega_1 \) and a vector field \( X \) such that \( \omega_2 := \text{Lie}_X (\omega_1) \neq 0 \) and \( \text{Lie}_X (\omega_2) = 0 \), we say that the pencil \( \omega_2 := \omega_1 - \lambda \omega_2 \) is an exact pencil.

Note that the condition \( \text{Lie}_X (\omega_2) = 0 \) guarantees that \( \text{Lie}_X (\omega_1) \) is indeed a Poisson structure. In fact \( 0 = \text{Lie}_X ([\omega_2, \omega_1] = [\text{Lie}_X (\omega_2), \omega_1] + [\omega_2, \text{Lie}_X (\omega_1)] \), but \( \text{Lie}_X (\omega_2) = 0 \) by definition, so \( \{\omega_2, \text{Lie}_X (\omega_1)\} = \{\omega_2, \omega_1\} = 0 \).

The following lemma classifies exact Poisson pencils of hydrodynamic type in the scalar case.

**Lemma 15.** In the scalar case, all exact Poisson pencils have the form
\[
\omega_2 = \omega_1 - \lambda \omega_2 = (au + b) \delta(x - y) + \frac{1}{2} au \delta(x - y) - \lambda \delta(x - y) = (au + b) \theta \theta_1 - \lambda \theta \theta_1
\]
for arbitrary constants \( a, b \).

**Proof.** By triviality of the Poisson cohomology, we can always assume that one of the Poisson structures is \( \delta_1 \). Therefore, without loss of generality we can take \( \omega_1 := \delta_1 = \theta \theta_1 \). We look for a vector field \( X = f(u) \theta \), such that \( \text{Lie}_X (\omega_1) = d_1 (X) = 0 \). Since \( d_1 = 2 \frac{\partial}{\partial u} \), we have that \( d_1 (X) = 0 \) is equivalent to requiring \( f(u) \) to be a constant, call it \( c \), so that \( X = c \theta \). Given \( \omega_2 := g(u) \theta \theta_1 \), we search under which conditions on \( g(u) \) \( d_1 (X) = \text{Lie}_X (\omega_2) \) is equal to \( \omega_1 \). We have
\[
d_2 (X) = \left( 2 g(u) \theta \theta_1 + \frac{\partial g}{\partial u} u \theta \right) \frac{\delta}{\delta u} (c \theta) + \frac{\partial g}{\partial u} \theta \theta_1 \frac{\delta}{\delta \theta} (c \theta) = c \frac{\partial g}{\partial u} \theta \theta_1.
\]
Therefore, \( d_1 (X) = \omega_1 \) (up to the action of the normalization operator \( S \), which in this case acts as multiplication by 2) if and only if \( g(u) \) is at most affine in \( u \): \( g(u) = au + b \). Moreover, since \( c \) is an arbitrary constant different from zero, we can choose \( c = \frac{1}{\lambda} \), so that \( d_2 (X) \) is indeed \( \omega_1 \). Finally it is immediate to check that if \( \omega_2 \) has the form \( (au + b) \theta \theta_1 \), then \( d_1 (X) = 0 \) and \( \omega_2 \) is indeed Poisson.

Now we consider the deformation of a general pencil \( \omega_2 - \lambda \omega_1 \) as follows:
\[
\omega_2 - \lambda \omega_1 - d_1 X_e = 2 g(u) \delta(x - y) + \frac{\partial g}{\partial u} u \delta(x - y) - \lambda \delta(x - y) - d_1 X_e,
\]
where \( X_e = \sum_{k=1}^{\infty} \epsilon^k X_k \) and \( \text{deg}(X_k) = k \). (Note that we have exchanged the names of \( \omega_1 \) and \( \omega_2 \), since we want to emphasize the role of \( \delta(x - y) \)). By Miura quasitriviality, the deformation vector field \( X_e \) always exists, coming from Hamiltonian functionals which are possibly not polynomial in the derivatives of \( u \).

It is not restrictive to assume that the odd powers in \( \epsilon \) are missing, since this can be always achieved by performing a suitable Miura transformation. For instance, the third- and fifth-order deformations obtained in the previous section can be eliminated just by putting \( f_3 = 0 \), when \( g(u) = u \). For deformations of the form (5.1) we have the following result.

**Theorem 16.** In the scalar case, the vector field \( X_e \) is tangent to the symplectic leaves of \( \omega_1 \) if and only if the undeformed pencil \( \omega_2 - \lambda \omega_1 \) is exact.
Proof. The tangency of $X_\epsilon$ to the symplectic leaves of $\omega_1$ is equivalent to impose the following condition
\[
\int_{S^1} X_\epsilon \frac{\delta C}{\delta u} \, dx = 0
\]
for all the Casimirs $C$ of $\omega_1$. On the other hand, the Casimirs of a Poisson bracket of hydrodynamic type are the integrals of the flat coordinates of the metric defining the bracket. In the case of $\omega_1$ we have only one Casimir given by
\[
C = \int_{S^1} u \, dx
\]
and the tangency condition reads
\[
\int_{S^1} X_\epsilon \, dx = 0
\]
which is equivalent to
\[
X_\epsilon = \partial_x F_\epsilon
\]
for a suitable differential polynomial $F_\epsilon$. Without loss of generality we can assume
\[
X_\epsilon = \sum_{k=1}^{\infty} \epsilon^{2k} X_2^{(2k)}.
\]

By the quasitriviality of deformations, we have that the deformed pencil:
\[
P_2 - \lambda \omega_2 = \omega_2 + \text{Lie}_{X_\epsilon} \omega_1 = \omega_2 + \sum_{k=1}^{\infty} \epsilon^{2k} P_2^{(2k)} = \omega_2 + \sum_{k=1}^{\infty} \epsilon^{2k} \text{Lie}_{X_2^{(2k)}} \omega_1
\]
(5.2)
can be reduced to its dispersionless limit by iterating quasi Miura transformations (here $\omega_2 = \omega_2 - \lambda \omega_1$). Following [16] we show how to construct such transformations. This will give us a crucial piece of information on the vector fields $X_2^{(2k)}$.

By hypothesis the vector field $X_2^{(2)}$ satisfies the condition
\[
d_1 d_2 X_2^{(2)} = 0.
\]
Moreover we have seen that it can be written as
\[
X_2^{(2)} = d_1 H_2^{(2)} - d_2 K_2^{(2)}
\]
for two suitable functionals $H_2^{(2)}$ and $K_2^{(2)}$. Let us consider the quasi Miura transformation generated by the vector field
\[
\epsilon^2 Z_2^{(2)} = -\epsilon^2 d_1 K_2^{(2)}.
\]
Since
\[
\text{Lie}_{Z_2^{(2)}} \omega_1 = 0
\]
\[
\text{Lie}_{Z_2^{(2)}} \omega_2 = -P_2^{(2)} = -\text{Lie}_{X_2^{(2)}} \omega_1.
\]
this transformation does not modify $\omega_1$ while
\[
P_2 \rightarrow \tilde{P}_2 = P_2 + \sum_{k=1}^{\infty} \frac{\epsilon^{2k}}{k!} \text{Lie}_{Z_2^{(2)}}^{k} P_2 = \omega_2 + \epsilon^4 \left( P_2^{(4)} + \text{Lie}_{Z_2^{(2)}} P_2^{(2)} + \frac{1}{2} \text{Lie}_{Z_2^{(2)}}^3 \omega_1 \right)
\]
\[
+ \epsilon^6 \left( P_2^{(6)} + \text{Lie}_{Z_2^{(2)}} P_2^{(4)} + \frac{1}{2} \text{Lie}_{Z_2^{(2)}}^3 P_2^{(2)} + \frac{1}{6} \text{Lie}_{Z_2^{(2)}}^5 \omega_2 \right) + O(\epsilon^8)
\]
\[
= \omega_2 + \epsilon^4 \left( P_2^{(4)} - \frac{1}{2} \text{Lie}_{Z_2^{(2)}}^2 \omega_1 \right) + \epsilon^6 \left( P_2^{(6)} + \text{Lie}_{Z_2^{(2)}} P_2^{(4)} - \frac{1}{3} \text{Lie}_{Z_2^{(2)}}^3 \omega_2 \right) + O(\epsilon^8)
\]

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Notice that, using twice graded Jacobi identity, the second term of order $O(\epsilon^4)$ can be written as

$$\Lie_{Z_2^{(2)}} \omega_2 = [d_1 K_2^{(2)}, [d_1 K_2^{(2)}, \omega_2]] = -[d_1 K_2^{(2)}, [d_2 K_2^{(2)}, \omega_1]] = \Lie_{x_2^{(4)} d_1 K_2^{(2)}, d_2 K_2^{(2)}} \omega_1$$

and therefore we can write

$$\tilde{P}_2 = \omega_2 + \Lie_{x_2^{(4)}} \omega_1 + O(\epsilon^6)$$

with

$$\tilde{X}_2^{(4)} = X_2^{(4)} - \frac{1}{2} [d_1 K_2^{(2)}, d_2 K_2^{(2)}]$$

(5.4)

Moreover

$$d_1 d_2 \tilde{X}_2^{(4)} = 0.$$

According to the main result of [16] (extended to the non scalar case in [8]) this implies that

$$\tilde{X}_2^{(4)} = d_1 H_2^{(4)} - d_2 K_2^{(4)}.$$

Let us consider now the quasi Miura transformation generated by the vector field

$$\epsilon^4 Z_2^{(4)} = -\epsilon^4 d_1 K_2^{(4)}.$$

Since

$$\Lie_{Z_2^{(2)}} \omega_1 = 0$$

(5.5)

$$\Lie_{Z_2^{(2)}} \omega_2 = -\left( P_2^{(4)} - \frac{1}{2} \Lie_{Z_2^{(2)}} \omega_2 \right) = -\Lie_{x_2^{(4)}} \omega_1,$$

(5.6)

this transformation does not modify $\omega_1$ while

$$\tilde{P}_2 \rightarrow \tilde{P}_2 = \omega_2 + \epsilon^6 \left( P_2^{(6)} + \Lie_{x_2^{(4)}} P_2^{(2)} + \Lie_{Z_2^{(2)}} P_2^{(4)} - \frac{1}{2} \Lie_{Z_2^{(2)}} \omega_2 \right) + \ldots$$

Notice that, using twice graded Jacobi identity, the second and the third term in the term of order $O(\epsilon^8)$ can be written as

$$\Lie_{Z_2^{(4)}} \Lie_{Z_2^{(2)}} \omega_2 = \left[ d_1 K_2^{(4)}, [d_1 K_2^{(2)}, \omega_2] \right] = -\left[ d_1 K_2^{(4)}, [d_2 K_2^{(2)}, \omega_1] \right] = \Lie_{\{d_1 K_2^{(4)}, d_2 K_2^{(2)}\}} \omega_1$$

$$\Lie_{Z_2^{(4)}} \Lie_{Z_2^{(2)}} \omega_2 = \left[ [d_1 K_2^{(4)}, d_1 K_2^{(2)}], [d_1 K_2^{(2)}, \omega_2] \right] = -\left[ [d_1 K_2^{(4)}, d_2 K_2^{(2)}], [d_2 K_2^{(2)}, \omega_1] \right] = \Lie_{\{d_1 K_2^{(4)}, d_2 K_2^{(2)}\}} \omega_1$$

and therefore we can write

$$\tilde{P}_2 = \omega_2 + \Lie_{x_2^{(6)}} \omega_1 + O(\epsilon^8)$$

with

$$\tilde{X}_2^{(6)} = X_2^{(6)} + [d_1 K_2^{(2)}, d_2 K_2^{(2)}] + [d_1 K_2^{(4)}, d_2 K_2^{(2)}] - \frac{1}{2} [[d_1 K_2^{(2)}, d_1 K_2^{(2)}], d_2 K_2^{(2)}]]$$

Moreover

$$d_1 d_2 \tilde{X}_2^{(6)} = 0.$$

Again this implies that

$$\tilde{X}_2^{(6)} = d_1 H_2^{(6)} - d_2 K_2^{(6)}$$

The quasi Miura transformation generated by the vector field

$$\epsilon^6 Z_2^{(6)} = -\epsilon^6 d_1 K_2^{(6)}$$
reduces the pencil to the form
\[ \tilde{P}_\lambda \to \tilde{\tilde{P}}_\lambda = \omega_\lambda + O(\epsilon^8). \]
The higher terms can be treated in a similar way. This is the procedure to construct the quasi-Miura transformation reducing the pencil to its dispersionless limit presented in [8, 16]. What is important for our purposes is that the vector fields \( X_{2k}, k = 4, 5, \ldots \) can always be written as linear combination of Hamiltonian vector fields (w.r.t. \( \omega_1 \) or \( \omega_2 \)) and commutators (or iterated commutators) of Hamiltonian vector fields. For instance
\[ X_{2}^{(4)} = d_1 H_2^{(4)} - d_2 K_2^{(4)} + \frac{1}{2}[d_1 K_2^{(4)}, d_2 K_2^{(4)}] \]
and
\[ X_{2}^{(6)} = d_1 H_2^{(6)} - d_2 K_2^{(6)} - [d_1 K_2^{(2)}, d_2 K_2^{(4)}] - [d_1 K_2^{(4)}, d_2 K_2^{(2)}] + \frac{1}{2}[[d_1 K_2^{(2)}, [d_1 K_2^{(2)}, d_2 K_2^{(2)}]]. \]

To conclude the first part of the proof we have to show that
- If two translation invariant vector fields are tangent to the symplectic leaves of \( \omega_1 \) the same is true for their commutator.
- If \( g(u) = au + b \) then the vectors fields \( d_1 H \) and \( d_2 K \) are tangent to the symplectic leaves of \( \omega_1 \) for any choice of the functionals \( H \) and \( K \).

Concerning the first point it follows immediately by the formula of commutator of two translation invariant vector fields
\[ X = X_0 \frac{\partial}{\partial u} + \sum_{k=1}^{\infty} \partial_x^k X_0 \frac{\partial}{\partial u(k)} \]
and
\[ Y = Y_0 \frac{\partial}{\partial u} + \sum_{k=1}^{\infty} \partial_x^k Y_0 \frac{\partial}{\partial u(k)} . \]
Indeed, integrating by parts we have
\[ \int_{S^1} [X, Y]_0 dx = \int_{S^1} \sum_{k=0}^{\infty} \left[ \partial_x^k X_0 \frac{\partial Y_0}{\partial u(k)} - \partial_x^k Y_0 \frac{\partial X_0}{\partial u(k)} \right] dx \]
\[ = \int_{S^1} \left[ X_0 \sum_{k=0}^{\infty} (-1)^k \partial_x^k \frac{\partial Y_0}{\partial u(k)} - Y_0 \sum_{k=0}^{\infty} (-1)^k \partial_x^k \frac{\partial X_0}{\partial u(k)} \right] dx \]
\[ = \int_{S^1} \left[ X_0 \frac{\delta Y_0}{\delta u} - Y_0 \frac{\delta X_0}{\delta u} \right] dx \]
that vanishes since, by hypothesis
\[ \frac{\delta Y_0}{\delta u} = \frac{\delta X_0}{\delta u} = 0. \]

Concerning the second point we observe that the tangency of \( d_1 H \) is trivial while the tangency of \( d_2 K \) can be easily checked by straightforward computation:
\[ \int_{S^1} \left( 2g(u) \frac{\partial g}{\partial u} + \frac{\partial g}{\partial u} \right) \frac{\delta K}{\delta u} dx = \int_{S^1} \left[ \partial_x \left( 2g(u) \frac{\delta K}{\delta u} \right) - \frac{\partial g}{\partial u} \frac{\partial K}{\partial u} \right] dx \]
\[ = - \int_{S^1} \sum_{k=0}^{\infty} (-1)^k \partial_x^k \left( \frac{\delta K}{\delta u(k)} \right) u_x \frac{\partial g}{\partial u} dx . \]
Indeed if \( g(u) = au + b \), using repeated integration by parts, we have

\[
\int_{S^1} \sum_{k=0}^{\infty} (-1)^k \frac{\partial K}{\partial u^{(k)}} \, u_k \frac{\partial g}{\partial u} \, dx = \int_{S^1} \sum_{k=0}^{\infty} (-1)^k \frac{\partial K}{\partial u^{(k)}} \, u_k a \, dx
\]

\[
= a \int_{S^1} \sum_{k=0}^{\infty} \frac{\partial K}{\partial u^{(k)}} \, u^{(k+1)} \, dx = a \int S^1 \partial_x(K) \, dx = 0.
\]

To conclude the proof we show that if \( g(u) \neq au + b \) the vector field \( X_2^{(2)} \) is no longer tangent to the symplectic leaves of \( \omega_1 \). Indeed, the deformations of the pencil

\[
2g(u)\delta'(x - y) + g' \, u_x \delta(x - y) - \lambda \delta'(x - y),
\]

up to the second order, are given by the following formula (see [9] where also fourth order deformations are computed)

\[
2g(u)\delta'(x - y) + g' \, u_x \delta(x - y) - \lambda \delta'(x - y)
\]

\[
+ \epsilon^2 \left\{ \left( \frac{c_g}{4} \right) \delta''(x - y) + \frac{3}{8}(c_{g'})u_x \delta''(x - y)
\]

\[
+ \left[ \left( \frac{c''_g}{8} + \frac{c'_g}{3} + \frac{5c_{g''}}{24} \right) u_x^2 + \left( \frac{c''_g}{8} + \frac{7c_{g''}}{24} \right) u_{xx} \right] \delta'(x - y)
\]

\[
+ \left[ \left( \frac{c''_g}{24} + \frac{c'_g}{12} + \frac{c_{g''}}{24} \right) u_x^3 + \frac{1}{6}(c'_{g''} + c_{g''})u_x u_{xx} + \frac{c_{g''}}{12} u_{xxx} \right] \delta(x - y) \right\}
\]

The term in the bracket \( \{ \ldots \} \) should be equal to

\[
\text{Lie}_{(Y(u)ux + Z(u)x^2)} \delta'(x - y)
\]

for a suitable choice of \( Y \) and \( Z \). Taking into account the tangency condition which implies \( Z = \frac{\partial Y}{\partial u} \), we would obtain

\[
\{ \ldots \} = 2Y \delta''(x - y) + 3Y_x \delta''(x - y) + Y_{xx} \delta'(x - y).
\]

But this is impossible due to the presence, in the left hand side, of the term

\[
\left[ \left( \frac{c''_g}{24} + \frac{c'_g}{12} + \frac{c_{g''}}{24} \right) u_x^3 + \frac{1}{6}(c'_{g''} + c_{g''})u_x u_{xx} + \frac{c_{g''}}{12} u_{xxx} \right] \delta(x - y)
\]

vanishing only if \( c'' = 0 \). □

6. Deformations up to the eighth order

In this section we compute deformations up to the eighth order and we show that there are no obstructions to the existence of a polynomial deformation up to that order. Such a result has been obtained taking full advantage of the computational capabilities of Maple and of the statement of theorem 16.

**Theorem 17.** Up to the eighth order, the deformations of the pencil

\[
2u \delta'(x - y) + u_x \delta(x - y) - \lambda \delta'(x - y)
\]

are unobstructed and can be reduced to the following form:

\[
P^\epsilon_x = \omega_2 - \lambda \omega_1 - \sum_{k=1}^{4} \epsilon^{2k} d_i(F_{2k}) + O(\epsilon^{10}),
\]

25
where the vector field $X_e = \sum_{k=1}^{4} e^{2k} (F_{2k})$ generating the deformation has homogeneous components given by

$$
F_2 = \partial_x^2 f_2 \\
F_4 = \partial_x^4 f_4 \\
F_6 = \partial_x \left( r_0 u_{xxxxx} + l_1 u_{xxxx} u_x + l_2 u_{xxxx} u_x^2 + l_3 u_{xxxx} u_x^3 + l_4 u_x^4 \right) \\
F_8 = \partial_x \left( r_0 u_{xxxxx} + r_1 u_{xxxxx} u_x + r_2 u_{xxxxx} u_x^2 + r_3 u_{xxxxx} u_x^3 + r_4 u_{xxxxx} u_x^4 \right)
$$

where

$$
\frac{\partial f_4}{\partial u} = -\frac{\partial}{\partial u} \left( \frac{\partial f_2}{\partial u} \right)^2,
$$

$$
t_0 := -\frac{1}{2} \left( \frac{\partial f_2}{\partial u} \right)^2 \frac{\partial^3 f_2}{\partial u^3} - \left( \frac{\partial^2 f_2}{\partial u^2} \right)^2 \frac{\partial f_2}{\partial u},
$$

$$
t_1 := \frac{1}{2} t_2 - \frac{1}{4} \left( \frac{\partial^2 f_2}{\partial u^2} \right)^3 - \frac{3}{8} \left( \frac{\partial f_2}{\partial u} \right)^2 \frac{\partial^4 f_2}{\partial u^4} - \frac{19}{12} \frac{\partial f_2}{\partial u} \frac{\partial^3 f_2}{\partial u^3} \frac{\partial f_2}{\partial u},
$$

$$
t_5 := \frac{5}{6} t_4 - \frac{1}{6} \frac{\partial t_1}{\partial u} - \frac{1}{6} \frac{\partial t_2}{\partial u} - 4 \left( \frac{\partial f_2}{\partial u} \right)^2 \frac{\partial^3 f_2}{\partial u^3} - \frac{23}{8} \frac{\partial f_2}{\partial u} \frac{\partial^3 f_2}{\partial u^3} - \frac{9}{16} \left( \frac{\partial f_2}{\partial u} \right)^2 \frac{\partial^5 f_2}{\partial u^5} - \frac{7}{2} \frac{\partial f_2}{\partial u} \frac{\partial^4 f_2}{\partial u^4} \frac{\partial f_2}{\partial u},
$$

$$
r_0 := -\frac{1}{6} \frac{\partial f_2}{\partial u} \frac{\partial^3 f_2}{\partial u^3} - \frac{\partial^2 f_2}{\partial u^2} \frac{\partial^3 f_2}{\partial u^3} - \frac{3}{2} \frac{\partial f_2}{\partial u} \left( \frac{\partial f_2}{\partial u} \right)^2 \frac{\partial^3 f_2}{\partial u^3} - \frac{7}{2} \frac{\partial f_2}{\partial u} \frac{\partial^4 f_2}{\partial u^4} \frac{\partial f_2}{\partial u},
$$

$$
r_1 := \frac{205}{468} \frac{\partial^2 f_2}{\partial u^2} \left( \frac{\partial f_2}{\partial u} \right)^2 \frac{\partial^4 f_2}{\partial u^4} - \frac{1}{12} \left( \frac{\partial f_2}{\partial u} \right)^3 \frac{\partial^5 f_2}{\partial u^5} - \frac{83}{234} \left( \frac{\partial^2 f_2}{\partial u^2} \right)^2 \frac{\partial f_2}{\partial u} \frac{\partial^3 f_2}{\partial u^3} + \frac{1}{26} \left( \frac{\partial^2 f_2}{\partial u^2} \right)^4 \left( \frac{\partial f_2}{\partial u} \right)^2 - \frac{107}{234} \frac{\partial^3 f_2}{\partial u^3} \frac{\partial f_2}{\partial u} + \frac{7}{13} \frac{r_2}{r_4} - \frac{2}{39} \frac{\partial t_1}{\partial u} - \frac{2}{39} \frac{\partial t_2}{\partial u} - \frac{2}{39} \frac{\partial t_3}{\partial u} + \frac{2}{39} \frac{t_4}{u},
$$

$$
r_5 := \frac{7}{9} r_5 - \frac{773}{81} \frac{\partial^3 f_2}{\partial u^3} \left( \frac{\partial f_2}{\partial u} \right)^2 \frac{\partial^4 f_2}{\partial u^4} - \frac{15715}{1944} \frac{\partial^2 f_2}{\partial u^2} \frac{\partial^3 f_2}{\partial u^3} - \frac{7}{9} \frac{t_2}{27} - \frac{8}{81} \frac{\partial t_1}{\partial u} + \frac{35}{81} \frac{\partial^3 f_2}{\partial u^3} + \frac{35}{81} \frac{\partial^2 f_2}{\partial u^2} \frac{\partial^2 f_2}{\partial u^2} - \frac{7}{81} \frac{\partial t_2}{\partial u} + \frac{35}{81} \frac{\partial^2 f_2}{\partial u^2} + \frac{7}{81} \frac{\partial t_3}{\partial u} - \frac{7}{81} \frac{\partial t_4}{\partial u},
$$

$$
r_7 := \frac{1003}{2592} \frac{\partial^6 f_2}{\partial u^6} - \frac{7}{9} \frac{\partial^3 f_2}{\partial u^3} + \frac{1}{35} \frac{\partial^2 f_2}{\partial u^2} + \frac{7}{7776} \frac{\partial^2 f_2}{\partial u^2} \frac{\partial^2 f_2}{\partial u^2} - \frac{7}{9} \frac{\partial^2 f_2}{\partial u^2} + \frac{35}{81} \frac{\partial^2 f_2}{\partial u^2} + \frac{35}{81} \frac{\partial^2 f_2}{\partial u^2} \frac{\partial^2 f_2}{\partial u^2} - \frac{7}{81} \frac{\partial t_1}{\partial u} - \frac{7}{81} \frac{\partial t_2}{\partial u} + \frac{20249}{81} \frac{\partial^3 f_2}{\partial u^3} \frac{\partial f_2}{\partial u} + \frac{2972}{81} \frac{\partial^2 f_2}{\partial u^2} \frac{\partial f_2}{\partial u} \left( \frac{\partial^3 f_2}{\partial u^3} \right)^2 - \frac{97}{8} \frac{\partial^3 f_2}{\partial u^3} \frac{\partial f_2}{\partial u} \frac{\partial^4 f_2}{\partial u^4}.\]
\[ r_4 := \frac{341}{108} \left( \frac{\partial^3 f_2}{\partial u^2} \right)^2 \left( \frac{\partial f_2}{\partial u} \right)^2 + \frac{5}{3} r_2 + \frac{128}{27} \frac{\partial^2 f_2}{\partial u^2} \left( \frac{\partial f_2}{\partial u} \right)^2 \frac{\partial^4 f_2}{\partial u^4} + \frac{65}{144} \left( \frac{\partial f_2}{\partial u} \right)^3 \frac{\partial^3 f_2}{\partial u^3} + \frac{1175}{108} \left( \frac{\partial^2 f_2}{\partial u^2} \right)^2 \frac{\partial f_2}{\partial u} \frac{\partial^3 f_2}{\partial u^3} + \left( \frac{\partial^2 f_2}{\partial u^2} \right)^4 + \frac{1}{9} \frac{\partial t_1}{\partial u} + \frac{1}{9} \frac{\partial t_2}{\partial u} + \frac{5}{9} \frac{\partial f_2}{\partial u} + \frac{20}{9} \frac{\partial f_2}{\partial u}. \]

\[ r_5 := \frac{568}{45} \left( \frac{\partial f_2}{\partial u} \right)^2 \frac{\partial^4 f_2}{\partial u^4} + \frac{5107}{540} \left( \frac{\partial^2 f_2}{\partial u^2} \right)^3 \frac{\partial^3 f_2}{\partial u^3} + \frac{991}{216} \left( \frac{\partial^2 f_2}{\partial u^2} \right)^2 \frac{\partial^2 f_2}{\partial u^2} \left( \frac{\partial f_2}{\partial u} \right)^2 + \frac{11}{5} r_7 - \frac{3}{5} r_8 + \frac{2}{3} \frac{\partial r_2}{\partial u} + \frac{101}{90} \frac{\partial^2 f_2}{\partial u^2} + \frac{2}{45} \frac{\partial^2 t_2}{\partial u^2} + \frac{2}{45} \frac{\partial^2 t_2}{\partial u^2} + \frac{29}{1080} \frac{\partial^2 f_2}{\partial u^2} \left( \frac{\partial f_2}{\partial u} \right)^2 + \frac{247}{20} \left( \frac{\partial^2 f_2}{\partial u^2} \right)^2 \frac{\partial f_2}{\partial u} \frac{\partial^4 f_2}{\partial u^4} + \frac{1673}{3600} \left( \frac{\partial f_2}{\partial u} \right)^6. \]

\[ r_6 := \frac{10801}{972} \left( \frac{\partial^4 f_2}{\partial u^4} \right)^2 \left( \frac{\partial f_2}{\partial u} \right)^2 + \frac{55271}{5832} \left( \frac{\partial^2 f_2}{\partial u^2} \right)^3 \frac{\partial^4 f_2}{\partial u^4} + \frac{286849}{23328} \left( \frac{\partial^2 f_2}{\partial u^2} \right)^2 \frac{\partial f_2}{\partial u} \frac{\partial^2 f_2}{\partial u^2} + \frac{698971}{4656} \left( \frac{\partial f_2}{\partial u} \right)^2 \frac{\partial^2 f_2}{\partial u^2} + \frac{578641}{11664} \left( \frac{\partial^2 f_2}{\partial u^2} \right)^2 \frac{\partial^3 f_2}{\partial u^3} + \frac{39635}{1458} \left( \frac{\partial f_2}{\partial u} \right)^3 \frac{\partial f_2}{\partial u} + \frac{r_9}{\partial u} + \frac{3}{5} \frac{\partial f_2}{\partial u} + \frac{20}{9} \frac{\partial f_2}{\partial u} + \frac{70}{81} \frac{\partial f_2}{\partial u} + \frac{2}{3} \frac{\partial f_2}{\partial u} + \frac{383}{486} \frac{\partial^2 f_2}{\partial u^2} - \frac{383}{486} \frac{\partial^2 f_2}{\partial u^2} - \frac{679}{486} \frac{\partial^2 f_2}{\partial u^2} + \frac{14}{486} \frac{\partial^2 f_2}{\partial u^2} - \frac{14}{486} \frac{\partial^2 f_2}{\partial u^2} + \frac{243}{486} \frac{\partial^2 f_2}{\partial u^2} - \frac{222}{486} \frac{\partial^2 f_2}{\partial u^2} + \frac{502015}{5832} \frac{\partial^2 f_2}{\partial u^2} + \frac{5983}{1552} \frac{\partial f_2}{\partial u} + \frac{3 \partial^3 f_2}{\partial u^3}. \]

\[ r_{10} := \frac{20}{39} \frac{\partial^2 f_2}{\partial u^2} + \frac{20}{39} \frac{\partial^2 f_2}{\partial u^2} + \frac{173507}{2808} \frac{\partial^3 f_2}{\partial u^3} \frac{\partial f_2}{\partial u} \frac{\partial^2 f_2}{\partial u^2} + \frac{5}{3} \frac{\partial^2 f_2}{\partial u^2} - \frac{20}{39} \frac{\partial^4 f_2}{\partial u^4} - \frac{209}{1560} \left( \frac{\partial f_2}{\partial u} \right)^3 \frac{\partial^3 f_2}{\partial u^3} - \frac{12811}{312} \left( \frac{\partial^3 f_2}{\partial u^3} \right)^2 \frac{\partial^2 f_2}{\partial u^2} + \frac{5}{9} \frac{\partial^2 f_2}{\partial u^2} - \frac{5}{9} \frac{\partial^2 f_2}{\partial u^2} - \frac{9907}{468} \left( \frac{\partial^2 f_2}{\partial u^2} \right)^3 \frac{\partial f_2}{\partial u} + \frac{35}{39} \frac{\partial^2 f_2}{\partial u^2} \frac{\partial f_2}{\partial u} \frac{\partial^2 f_2}{\partial u^2}. \]
\[ \begin{align*}
- \frac{25}{3} \frac{\partial^2 f_2}{\partial u^2} t_6 - \frac{5}{78} \frac{\partial t_4}{\partial u} \frac{\partial^3 f_2}{\partial u^3} - 66479 \ \frac{\partial^3 f_2}{\partial u^3} \left( \frac{\partial^2 f_2}{\partial u^2} \right) \frac{\partial^4 f_2}{\partial u^4} \\
- \frac{32501}{2340} \left( \frac{\partial^2 f_2}{\partial u^2} \right)^2 \frac{\partial f_2}{\partial u} \frac{\partial^6 f_2}{\partial u^6} - \frac{1323}{520} \frac{\partial^2 f_2}{\partial u^2} \left( \frac{\partial f_2}{\partial u} \right)^2 \frac{\partial^7 f_2}{\partial u^7} \\
+ \frac{5}{78} \frac{\partial^2 t_1}{\partial u^2} \frac{\partial^3 f_2}{\partial u^3} + \frac{5}{78} \frac{\partial^2 t_2}{\partial u^2} \frac{\partial^3 f_2}{\partial u^3} - \frac{7353}{1040} \frac{\partial^3 f_2}{\partial u^3} \left( \frac{\partial f_2}{\partial u} \right)^2 \frac{\partial^6 f_2}{\partial u^6} \\
- \frac{557}{52} \frac{\partial^4 f_2}{\partial u^4} \left( \frac{\partial f_2}{\partial u} \right)^2 \frac{\partial f_2}{\partial u} - \frac{12094}{351} \left( \frac{\partial^4 f_2}{\partial u^4} \right) ^2 \frac{\partial f_2}{\partial u} \frac{\partial f_2}{\partial u} \\
- \frac{35393}{702} \left( \frac{\partial^3 f_2}{\partial u^3} \right)^2 \frac{\partial f_2}{\partial u} \frac{\partial^4 f_2}{\partial u^4} + \frac{17}{26} \frac{\partial t_9}{\partial u} - \frac{55}{39} \frac{\partial t_{11}}{\partial u} + \frac{5}{13} \frac{\partial^7 f_2}{\partial u^7} \\
- \frac{35}{117} \frac{\partial^4 t_4}{\partial u^4} \frac{\partial f_2}{\partial u} + \frac{14}{13} \frac{\partial^3 r_8}{\partial u^3} - \frac{20}{13} \frac{\partial^2 r_7}{\partial u^2} - \frac{17}{39} \frac{\partial^6 t_6}{\partial u^6} \frac{\partial f_2}{\partial u} - \frac{39}{39} \frac{\partial^6 t_6}{\partial u^6} \frac{\partial f_2}{\partial u}.
\end{align*} \]

All the remaining parameters, namely \( t_2, t_4, t_5, t_6, r_2, r_7, r_8, r_9, t_{11}, r_{12}, r_{13}, r_{14} \) are free and can be chosen arbitrarily.

Using the freedom in the choice of the parameters we can simplify the previous expression of \( X_\lambda \) and of the corresponding deformation. For instance, up to the sixth order in \( \epsilon \) we have the following result, where, in order to have a simpler expression, the free parameter \( f_3 \) appearing in theorem 13 has been fixed to 0.

**Theorem 18.** Up to Miura transformations, the deformations of the pencil \( P_\lambda = u \delta^1 + \frac{1}{2} \epsilon u (1) \delta - \lambda \delta^{(1)} \) can be reduced to the following form:

\[ P_\lambda^* = P_\lambda - \epsilon^2 \left\{ c_2^3 (c_2 \delta^{(1)}(x - y)) + c_2 \delta^{(3)}(x - y) + (\partial_x c_2) \delta^{(2)}(x - y) \right\} \\
- \epsilon^4 \left\{ c_4^4 (c_4 \delta^{(1)}(x - y)) + c_4 \delta^{(5)}(x - y) + (\partial_x c_4) \delta^{(4)}(x - y) \right\} \\
- \epsilon^6 \left\{ c_6^6 (c_6 \delta^{(1)}(x - y)) + c_6 \delta^{(7)}(x - y) + (\partial_x c_6) \delta^{(6)}(x - y) \right\} \\
+ \epsilon^6 \left\{ h^8 \delta^{(3)}(x - y) + (\partial_x h) \delta^{(2)}(x - y) + \partial^2 (h \delta^{(1)}(x - y)) \right\} \\
+ \epsilon^6 \left\{ \partial_x^2 (h^2 \delta^{(2)}(x - y)) + \partial_x (\partial_x h) \delta^{(3)}(x - y) \right\} \\
+ \left\{ \partial_x^2 g \delta^{(5)}(x - y) + (\partial_x^2 g) \delta^{(4)}(x - y) \right\}, \tag{6.1} \]

where

\[ c_2 = \frac{\partial f_2}{\partial u}, \]

\( c_4 \) and \( c_6 \) are related to \( c_2 \) via the following equations:

\[ c_4 = \frac{\partial f_4}{\partial u} = -\frac{\partial}{\partial u} (c_2)^2, \tag{6.2} \]

\[ c_6 = -\frac{1}{2} \frac{\partial}{\partial u} \left( c_2^2 \frac{\partial c_2}{\partial u} \right), \tag{6.3} \]

while \( g \) is given by

\[ g = \frac{\partial}{\partial u} \left\{ \frac{3}{2} \frac{\partial^2 c_2}{\partial u^2} + \left( \frac{\partial c_2}{\partial u} \right)^2 + \frac{19}{3} \frac{\partial^2 c_2}{\partial u^2} \frac{\partial c_2}{\partial u} \right\} \tag{6.4} \]

and \( h := h_1 + h_2 + h_3 + h_4 \), and the \( h_i \)’s have the following expression:

\[ h_1 = u_x^2 \left( \frac{97}{60} c_2^2 \left( \frac{\partial c_2}{\partial u} \right)^2 + \frac{8}{3} \left( \frac{\partial c_2}{\partial u} \right)^2 \frac{\partial^2 c_2}{\partial u^2} + \frac{21}{40} c_2^2 \frac{\partial^4 c_2}{\partial u^4} + \frac{49}{15} c_2 \left( \frac{\partial^3 c_2}{\partial u^3} \right) \frac{\partial c_2}{\partial u} \right). \tag{6.5} \]
\[ h_2 = u_i^4 \left( \frac{254}{3} \left( \frac{\partial c_2}{\partial u} \right)^2 + \frac{17}{5} \left( \frac{\partial^4 c_2}{\partial u^4} \right) \right) + \frac{176}{3} \left( \frac{\partial^3 c_2}{\partial u^3} \right)^2 \]
\[ + 14 \frac{1512}{45} \left( \frac{\partial c_2}{\partial u} \right) \left( \frac{\partial^3 c_2}{\partial u^3} \right) \frac{\partial^2 c_2}{\partial u^2} \]
\[ + \frac{176}{45} \left( \frac{\partial^2 c_2}{\partial u^2} \right)^2 \]
\[ + \frac{176}{45} \left( \frac{\partial c_2}{\partial u} \right)^2 \left( \frac{\partial^2 c_2}{\partial u^2} \right) \left( \frac{\partial^3 c_2}{\partial u^3} \right) \]
\[ + \frac{176}{45} \left( \frac{\partial c_2}{\partial u} \right)^2 \left( \frac{\partial^3 c_2}{\partial u^3} \right)^2 \]
\[ + \frac{176}{45} \left( \frac{\partial^2 c_2}{\partial u^2} \right)^2 \left( \frac{\partial^3 c_2}{\partial u^3} \right) \left( \frac{\partial^4 c_2}{\partial u^4} \right) \]
\[ + \frac{176}{45} \left( \frac{\partial^3 c_2}{\partial u^3} \right)^2 \left( \frac{\partial^4 c_2}{\partial u^4} \right) \left( \frac{\partial^5 c_2}{\partial u^5} \right) \] \[ \cdot (6.6) \]

\[ h_3 = u_{xxx} u_i \left( \frac{3}{10} \left( \frac{\partial^4 c_2}{\partial u^4} \right) + \frac{2}{3} \left( \frac{\partial c_2}{\partial u} \right)^2 \frac{\partial^2 c_2}{\partial u^2} + \frac{1}{15} \left( \frac{\partial c_2}{\partial u} \right)^2 \left( \frac{\partial^3 c_2}{\partial u^3} \right)^2 \right) \] \[ + 28 \frac{15}{11} \left( \frac{\partial c_2}{\partial u} \right)^2 \left( \frac{\partial^3 c_2}{\partial u^3} \right) \left( \frac{\partial^4 c_2}{\partial u^4} \right) \]
\[ + 28 \frac{15}{11} \left( \frac{\partial c_2}{\partial u} \right)^2 \left( \frac{\partial^5 c_2}{\partial u^5} \right) \] \[ \cdot (6.7) \]

\[ h_4 = u_{xxx} u_i^2 \left( \frac{139}{10} \left( \frac{\partial c_2}{\partial u} \right)^2 \frac{\partial^2 c_2}{\partial u^2} + \frac{178}{15} \left( \frac{\partial c_2}{\partial u} \right)^2 \left( \frac{\partial^3 c_2}{\partial u^3} \right)^2 + \frac{21}{20} \left( \frac{\partial^5 c_2}{\partial u^5} \right)^2 \] \[ + \frac{259}{30} \left( \frac{\partial^2 c_2}{\partial u^2} \right) \frac{\partial^2 c_2}{\partial u^2} + \frac{13}{20} \left( \frac{\partial c_2}{\partial u} \right)^2 \left( \frac{\partial^3 c_2}{\partial u^3} \right) \left( \frac{\partial^4 c_2}{\partial u^4} \right) \] \[ + \frac{13}{20} \left( \frac{\partial c_2}{\partial u} \right)^2 \left( \frac{\partial^3 c_2}{\partial u^3} \right)^2 \] \[ + \frac{13}{20} \left( \frac{\partial^2 c_2}{\partial u^2} \right)^2 \left( \frac{\partial^3 c_2}{\partial u^3} \right) \left( \frac{\partial^4 c_2}{\partial u^4} \right) \] \[ + \frac{13}{20} \left( \frac{\partial^3 c_2}{\partial u^3} \right)^2 \left( \frac{\partial^4 c_2}{\partial u^4} \right) \left( \frac{\partial^5 c_2}{\partial u^5} \right) \] \[ \cdot (6.8) \]

Let us observe then that although deformations up to the fourth order in \( \epsilon \) appear to have a quite regular structure, the pattern is already broken at the sixth order.

7. Conclusions

In this paper, we proved that the linear relations on the unknown coefficients of the vector fields \( X_{2k} \) generating the deformations, obtained by solving recursively (2.33), have a geometric meaning in the case of the pencil \( \omega_1 \) we considered. The linear relations entail the tangency of \( X_{2k} \) to the symplectic leaves of \( \omega_1 \) and this geometric property is equivalent to the exactness of the pencil \( \omega_1 \). How much of this extends to the case of systems is not clear. Although an essential ingredient in our proof, namely the Miura-quasitriviality, has been established also for systems of hydrodynamic type, preliminary computations show that this is not enough to relate the exactness of the pencil to the tangency of the vector fields generating the deformation. This will be explored elsewhere.

Our computation about deformations up to the eighth order shows, unfortunately, that it is not feasible to guess a general formula for either the vector fields \( X_{2k} \) or the Poisson structures, even though deformations up to the fourth order might have suggested otherwise. On the other hand, the formulas we have found might turn out to be useful in guessing general formulas not for arbitrary central invariants, but for specific ones. Indeed for a specific choice of central invariant, the formulas simply drastically (even to the eighth order) and some patterns start to emerge. It seems therefore reasonable that using these formulas, using a specific central invariant, one might be able to guess a general structure (for that specific central invariant) and construct the associated new integrable PDE.

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Appendix

Proof of lemma 9. Since the proof is based on combinatorial identities, we focus on the proof of (3.11), since the proof of (3.12) is entirely similar. Using direct substitution of (3.9) in (3.11) we have

\[ \frac{\partial}{\partial \theta_i^j} \frac{\partial}{\partial \theta_i^{j+l}} \sum_{l=0}^{\infty} \binom{j+l}{j} \sum_{k=0}^{\infty} \binom{j}{j-l} \frac{\partial}{\partial \theta_i^{j+l}} = \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{l} \binom{j}{j-l} \binom{j+l}{j} \frac{\partial}{\partial \theta_i^{j+l}}. \]

Setting \( q = l + j \) we get that the previous expression is equal to

\[ \sum_{q=0}^{\infty} (-1)^{q-j} \binom{j}{j-l} \frac{\partial}{\partial \theta_i^{j+l}} = \sum_{q=0}^{\infty} \sum_{j=0}^{\infty} (-1)^{q-j} \binom{j}{j-l} \frac{\partial}{\partial \theta_i^{j+l}}. \]

where this last equality comes from exchanging vertical with horizontal summation in a lattice. Finally, using the identity

\[ \sum_{j=0}^{\infty} (-1)^{j} \binom{j}{j-l} \frac{\partial}{\partial \theta_i^{j+l}} = (-1)^{q} \delta_q, \]

where \( \delta_q \) is the Kronecker delta, we obtain that the last expression is equal to

\[ (-1)^{q} \sum_{q=0}^{\infty} (-1)^{q-j} \delta_q \frac{\partial}{\partial \theta_i^{j+l}} = \frac{\partial}{\partial \theta_i^{j+l}}, \]

thus proving the identity. The proof of formula (3.12) is completely analogous. \( \square \)

Proof of lemma 12. By definition we have

\[ \frac{\delta}{\delta u^j}(f \theta_j^p) = \sum_{r=0}^{\infty} (-1)^{r} \frac{\partial}{\partial \theta_j^p}(f \theta_j^p) \]

\[ = \sum_{r=0}^{\infty} (-1)^{r} \frac{\partial f}{\partial \theta_j^p} \theta_j^p = \sum_{r=0}^{\infty} (-1)^{r} \frac{\partial f}{\partial u^r} \theta_j^p. \]

where the last equality is due to the fact that \( f \) is homogeneous of degree \( k \). Now we have

\[ \sum_{r=0}^{k} (-1)^{r} \frac{\partial f}{\partial u^r} \theta_j^p = \sum_{r=0}^{k} (-1)^{r} \sum_{h=0}^{r} \binom{r}{h} \frac{\partial f}{\partial u^h} \theta_j^p \theta_j^{p-r-h}. \]

Setting \( r - h = l \) and remembering that \( f \) is homogeneous of degree \( k \), the previous sum can be rewritten as

\[ \sum_{l=0}^{k} \sum_{h=0}^{l} (-1)^{l+h} \binom{l+h}{h} \frac{\partial f}{\partial u^{l+h}} \theta_j^p = \sum_{l=0}^{k} (-1)^{l} \sum_{h=0}^{l} (-1)^{h} \binom{l+h}{h} \frac{\partial f}{\partial u^{l+h}} \theta_j^p \theta_j^{l+h}. \]

where the last equality holds by definition of higher-order variational derivative (see (3.10)) and the fact that \( f \) is homogeneous of degree \( k \). \( \square \)
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