Kostka-Foulkes polynomials cyclage graphs and charge statistic for the root system $C_n$

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Abstract

We establish a Morris type recurrence formula for the root system $C_n$. Next we introduce cyclage graphs for the corresponding Kashiwara-Nakashima’s tableaux and use them to define a charge statistic. Finally we conjecture that this charge may be used to compute the Kostka-Foulkes polynomials for type $C_n$.

1 Introduction

There exists $q$-analogues of the multiplicities of weights in the irreducible representations of the classical Lie algebras. They are obtained by substituting the ordinary Kostant’s partition function $P$ by the $q$-analogue $P_q$ defined from the equality

$$
\prod_{\alpha \text{ positive root}} \frac{1}{(1 - qx^{\alpha})} = \sum_{\beta} P_q(\beta) x^{\beta}.
$$

Given $\lambda$ and $\mu$ two partitions

$$
K_{\lambda,\mu}(q) = \sum_{\sigma \in W} (-1)^{l(\sigma)} P_q(\sigma(\lambda + \rho) - (\mu + \rho)).
$$

As shown by Lusztig [17] $K_{\lambda,\mu}(q)$ is a polynomial in $q$ with non negative integer coefficients. These polynomials naturally appear in the classical theory of Hall-Littlewood polynomials. They coincide with the Kostka-Foulkes polynomials that is, with the entries of the transition matrix between the basis of Hall-Littelwood polynomials and the basis of Schur functions [18]. Many interpretations of the Kostka-Foulkes polynomials exist. For example, they appear in the filtrations of weight spaces by the kernels of powers of a regular nilpotent element, and degree in harmonic polynomials [1], [2]. We recover them in the expansion of the Hall Littlewood polynomials in terms of the affine Hecke algebra (see [19]).

For type $A_{n-1}$ the positivity of the Kostka-Foulkes Polynomials can also be proved by a purely combinatorial method. Recall that for any partitions $\lambda$ and $\mu$ with $n$ parts the number of semi-standard tableaux of shape $\lambda$ and weight $\mu$ is equal to the multiplicity of the weight $\mu$ in the finite dimensional irreducible module of $U_q(sl_n)$ with highest weight $\lambda$. In [14] Lascoux and Schützenberger have introduced a beautiful statistic $\text{ch}_{A}$ on standard tableaux called the charge and, by using Morris recurrence formula, have proved the equality

$$
K_{\lambda,\mu}(q) = \sum_{T \in ST(\mu)} q^{\text{ch}_{A}(T)}
$$

where $ST(\mu)_{\lambda}$ is the set of semi-standard tableaux of shape $\lambda$ and weight $\mu$. Set $A_n = \{1 < \cdots < n\}$. The charge may be defined by endowing $ST(\mu)$ the set of semi-standard tableaux of weight $\mu$ with a structure of graph defined from Lascoux-Schützenberger’s plactic monoid. Recall that the plactic monoid is the quotient set of $A^* \times \mathbb{N}$ by the Knuth relations

$$
abx \equiv \begin{cases}
  bax & \text{if } a < x \leq b \\
  axb & \text{if } x \leq a < b
\end{cases}
$$

For any tableau $T$ we denote by $w(T)$ the column reading of $T$ that is, the word obtained by reading the columns of $T$ from right to left and from top to bottom. The cyclage graph structure on $ST(\mu)$ can be defined as follows.

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We draw an arrow $T \to T'$ between the two tableaux $T$ and $T'$ of $ST(\mu)$, if and only if there exists $u$ in $A_n^*$ and $x \neq 1 \in A_n$ such that $w(T) = xu$ and $w(T') = ux$. Then we say that $T'$ is a cocycle of $T$. The essential tool to define this graph structure is the insertion algorithm for the semi-standard tableaux. The cyclage graph $ST(\mu)$ contains a unique row tableau $L_\mu$ which can not be obtained as the cocycle of another tableau of $ST(\mu)$. Let $T_\mu$ be the unique semi-standard tableau of shape $\mu$ belonging to $ST(\mu)$. Then there is no cocycle of $T_\mu$. For any $T \in ST(\mu)$ all the paths joining $L_\mu$ to $T$ have the same length. This length is called the cocharge of $T$ and denoted $coch_A(T)$. Similarly, all the paths joining $T$ to $T_\mu$ have the same length called the charge of $T$. The maximal value of $coch_A$ is $||\mu|| = coh_A(L_\lambda) = \sum_i (i-1)\mu_i$. Moreover the charge and the cocharge satisfy the equality $coh_A(T) = ||\mu|| - coh_A(T)$ for any $T \in ST(\mu)$.

The initial cocycle of the tableau $T$ of reading $w(T) = xu$ with $x \neq 1$ is obtained by inserting $x$ in the sub-tableau of $T$ of reading $u$. Every tableau $T \in ST(\mu)$ can be related to $T_\mu$ by a sequence of initial cocycles. So it is enough to consider initial cocycles to define $ coh_A$.

The charge of $T$ can also be defined directly from $w(T)$ when $\mu$ is a dominant weight. Moreover it can be characterized in terms of the geometry of the crystal graph $B(\lambda)$ associated to $V(\lambda)$ [12].

In this article we restrict ourselves to the root system $C_n$. Our aim is two folds. First we establish Morris type recurrence formula for type $C_n$ and use it to give explicit formulae for Kostka-Foulkes polynomials when $\lambda$ is a row partition or a column partition of height 2. Next we introduce a cyclage graph structure and a notion of charge for type $C_n$. For any dominant weight $\lambda$, let $V(\lambda)$ be the finite dimensional irreducible $U_q(sp_{2n})$-module with highest weight $\lambda$. In [11] Kashiwara and Nakashima have given a combinatorial description of $B(\lambda)$ the crystal graph of $V(\lambda)$ in terms of symplectic tableaux analogous to the semi-standard tableaux for type $C_n$. From the plactic monoid and the insertion algorithm described in [15] it is natural to try to obtain cyclage graphs for symplectic tableaux. Nevertheless the situation is more complex than for type $A_{n-1}$. First we have to restrict the possible cyclage operations to the initial cyclage to avoid loops in our cyclage graphs. Moreover if we use the complete insertion algorithm for type $C_n$, the number of boxes of the cyclage of a tableau $T$ may be strictly less than that of $T$ due to the contraction relation in the plactic monoid. The cyclage graphs obtained by this mean seem to be not relevant to define a charge related to the Kostka-Foulkes polynomials. To overcome this problem we will execute the insertion algorithm without this contraction relation and consider that the symplectic tableaux are filled by letters of the totally ordered alphabet

$$C_n = \{\pi \cdots \pi \ < \ 1 \cdots < n\}$$

which can be naturally embedded in the infinite alphabet

$$C_\infty = \{\cdots < \pi \cdots < \pi \ < 1 \cdots < n \cdots\}.$$ 

Our convention for the alphabet $C_n$ is not identical to that of [11] to dispose of a natural infinite extension of $C_n$. Denote by $ST(n)$ the set of symplectic tableaux defined on $C_n$. If $T \in ST(n)$ the initial cocycle (without contraction) of $T$ does not belong to $ST(n)$ in general but belongs to $ST(n+1)$. So it is natural to consider the cyclage graph structure of $ST = \bigcup_{n \geq 1} ST(n)$.

Let $\mu$ be a dominant weight for the root system $C_n$. With our convention $\mu$ may be identified with the partition $(\mu_1, \ldots, \mu_\ell)$. We are going to endow $ST(\mu)$ the subset of $ST$ containing the symplectic tableaux $T$ such that for any $m \geq 1$, the number of letters $\bar{k}$ in $T$ minus the number of letters $k$ is $\mu_1$ if $k \leq n$, 0 otherwise with a structure of cyclage graph. This structure is more complex than for type $A_n$. In particular $ST(\mu)$ decomposes into connected components. These components can be isomorphic and do not necessarily contain a row tableau. We define on $ST(\mu)$ a charge statistic $ch_n$. Many computations allows us to conjecture that an analogue to [11] exists for type $C_n$ with the charge $ch_n$. However it seems to be impossible to derive it from our Morris type recurrence formula.

In Section 1 we recall the Background on Kostka-Foulkes polynomials and crystal basis theory that we need in the sequel. We also summarize the basic properties of the insertion algorithm introduced in [15]. Section 2 is devoted to the Morris type recurrence formula for type $C_n$ and its applications. In Section 3 we define the cyclage graph structure on $ST(\mu)$ and give some of its properties. Finally we introduced $ch_n$ in Section 4 and conjecture that it permits to compute the Kostka-Foulkes polynomials for type $C_n$. 

2
2 Background

2.1 Kostka-Foulkes polynomials for type $C_n$

We choose to label the Dynkin diagram of $sp_{2n}$ by

$$\begin{array}{ccccccc}
0 & \Rightarrow & 1 & - & 2 & - & 3 & - & \cdots & - & n & - & 1.
\end{array}$$

(2)

The weight lattice $P_n$ of $C_n$ can be identified with $\mathbb{Z}^n$ equipped with the orthonormal basis $\varepsilon_i$, $i = 1, \ldots, n$. We take for the simple roots

$$\alpha_0 = 2\varepsilon_T$$

and

$$\alpha_i = \varepsilon_{i+1} - \varepsilon_i, \quad i = 1, \ldots, n - 1.$$  

Then the set of positive roots of $sp_{2n}$ is

$$R^+_n = \{\varepsilon_T - \varepsilon_i, \varepsilon_i + \varepsilon_T \mid 1 \leq j < i \leq n\} \cup \{2\varepsilon_T \mid 1 \leq i \leq n\}.$$

Denote by $P^+_n$ the set of dominant weights of $sp_{2n}$. Write $\Lambda_0, \ldots, \Lambda_{n-1}$ for the fundamentals weights. Then we have $\Lambda_i = \varepsilon_i + \cdots + \varepsilon_{i+n-1}$, $0 \leq i \leq n - 1$. Consider $\lambda \in P^+_n$ and set $\lambda = \sum_{i=0}^{n-1} \lambda_i \Lambda_i$ with $\lambda_i \in \mathbb{N}$. The dominant weight $\lambda$ is characterized by the partition $(\lambda_0, \ldots, \lambda_T)$ where $\lambda_i = \lambda_{i-1} + \ldots + \lambda_{i-T}$, $i = 1, \ldots, n$. In the sequel we will identify $\lambda$ and $(\lambda_0, \ldots, \lambda_T)$ by setting $\lambda = (\lambda_0, \ldots, \lambda_T \lambda')$. Then $\lambda = \lambda_0 \varepsilon + \cdots + \lambda_T \varepsilon_T$ that is, the $\lambda_i$'s are the coordinates of $\lambda$ on the basis $(\varepsilon_0, \ldots, \varepsilon_T)$. Let $\rho$ be the half sum of positive roots. We have $\rho = (n, n-1, \ldots, 1)$. For any $\lambda \in P^+_n$, set $|\lambda| = \lambda_T + \cdots + \lambda_T$.

The Weyl group $W_n$ of $sp_{2n}$ can be regarded as the sub group of the permutation group of $C_n = \{0, \ldots, 2, 1, 2, \ldots, n\}$ generated by $s_i = (i, i+1)(i, i+1)$, $i = 1, \ldots, n-1$ and $s_0 = (1, 2)$ where for $a, b \in C_n$ we have $a \neq b$, $(a, b)$ is the simple transposition which switches $a$ and $b$. Note that any $\sigma \in W_n$ verifies $\sigma(i) = \sigma(i)$ for $i \in \{1, \ldots, n\}$. We denote by $l$ the length function corresponding to the set of generators $s_i$, $i = 0, \ldots, n-1$.

The action of $\sigma \in W_n$ on $\beta = (\beta_0, \ldots, \beta_T) \in P_n$ is given by

$$\sigma \cdot (\beta_0, \ldots, \beta_T) = (\beta_0', \ldots, \beta_T'),$$

where $\beta_0' = \beta_{\sigma(i)}$ if $\sigma(i) \in \{1, \ldots, n\}$ and $\beta_0' = -\beta_{\sigma(i)}$ otherwise.

Let $Q^+_n$ be the set of nonnegative integral linear combinations of positive roots. For any $\beta = (\beta_0, \ldots, \beta_T) \in P_n$ we set $x^\beta = x^\beta_0 \cdots x^\beta_T$ where $x_1, \ldots, x_n$ are fixed indeterminates. The $q$-analogue of the Kostant function partition $P_q$ is defined by

$$\prod_{\beta \in Q^+_n} \frac{1}{1-qx^\beta} = \sum_{\beta \in Q^+_n} P_q(\beta)x^\beta \quad \text{and} \quad P_q(\beta) = 0 \quad \text{if} \quad \beta \notin Q^+_n.$$

**Definition 2.1.1** Let $\lambda, \mu \in P^+_n$. The Kostka-Foulkes polynomial $K_{\lambda, \mu}(q)$ is defined by

$$K_{\lambda, \mu}(q) = \sum_{\sigma \in W_n} (-1)^{l(\sigma)} P_q(\sigma(\lambda + \mu) - (\mu + \mu)).$$

**Lemma 2.1.2** Consider $\lambda, \mu \in P^+_n$ such that $\lambda_T = \mu_T$ and write $\lambda' = (\lambda_{n-T}, \ldots, \lambda_T)$, $\mu' = (\mu_{n-T}, \ldots, \mu_T)$. Then

$$K_{\lambda, \mu}(q) = K_{\lambda', \mu'}(q).$$

**Proof.** Note first that $K_{\lambda', \mu'}(q)$ is a Kostka-Foulkes polynomial for type $C_{n-1}$. We identify $W_{n-1}$ with the sub-group of $W_n$ generated by $s_i$, $i = 1, \ldots, n - 2$ and $s_0$. Consider $\sigma \in W_n$ such that $\sigma \notin W_{n-1}$ and set $\beta = \sigma(\lambda + \mu) - (\mu + \mu) \in P_n$. We must have $\sigma(T) \neq T$ since $\sigma \notin W_{n-1}$. This implies that $\beta = (\beta_0, \ldots, \beta_T)$ with $\beta_0 < 0$ because $\lambda_T = \mu_T$. Thus $\beta \notin Q^+_n$ and $P_q(\beta) = 0$. This means that

$$K_{\lambda, \mu}(q) = \sum_{\sigma \in W_{n-1}} (-1)^{l(\sigma)} P_q(\sigma(\lambda' + \mu') - (\mu' + \mu')).$$

with $\rho' = (n-1, \ldots, 1)$ the half sum of the positive roots of the roots system $C_{n-1}$. Hence $K_{\lambda, \mu}(q) = K_{\lambda', \mu'}(q)$. □
Let $\beta \in P_n$. We set

$$a_\beta = \sum_{\sigma \in W_n} (-1)^{|\sigma|} (\sigma \cdot x^\beta)$$

where $\sigma \cdot x^\mu = x^{\sigma(\mu)}$. The Schur function $s_\beta$ is defined by

$$s_\beta = \frac{a_\beta + \rho}{a_\rho}.$$

When $\lambda \in P_n^+$, $s_\lambda$ is the Weyl character of $V(\lambda)$ the finite dimensional irreducible $U_q(s_{2n})$-module with highest weight $\lambda$. For any $\sigma \in W_n$, the dot action of $\sigma$ on $\beta \in P_n$ is defined by $\sigma \cdot \beta = \sigma \cdot (\beta + \rho) - \rho$. We have the following straightening law for the Schur functions. For any $\beta \in P_n$ there exists a unique $\lambda \in P_n^+$ such that $s_\beta = (-1)^{|\lambda|} s_\lambda$ with $\sigma \in W_n$ and $\lambda = \sigma \cdot \beta$. Set $K = \mathbb{Z}[q, q^{-1}]$ and write $K[P_n]$ for the $K$ module generated by $x^\beta$, $\beta \in P_n$. Set $K[P_n]^{W_n} = \{ f \in K[P_n], \sigma \cdot f = f \text{ for any } \sigma \in W_n \}$. Then $\{ s_\lambda \}$ is a basis of $K[P_n]^{W_n}$.

To each positive root $\alpha$, we associate the raising operator $R_\alpha : P_n \to P_n$ defined by

$$R_\alpha(\beta) = \alpha + \beta.$$ 

Given $\alpha_1, ..., \alpha_p$ positives roots and $\beta \in P_n$, we set $(R_{\alpha_1} \cdot \cdots \cdot R_{\alpha_p}) s_\beta = s_{R_{\alpha_p} \cdots R_{\alpha_1} (\beta)}$. Composing the action of raising operators on Schur function should be avoided in general. For example $(R_{\alpha_1} R_{\alpha_2})(s_\beta)$ is not necessarily equal to $(R_{\alpha_1})(R_{\alpha_2} s_\beta)$ (see example p 360 in [19]). For all $\beta \in P_n$, we define the Hall-Littelwood polynomial $Q_\beta$ by

$$Q_\beta = \left( \prod_{\alpha \in R_n^+} \frac{1}{1 - q R_\alpha} \right) s_\beta$$

where $\frac{1}{1 - q R_\alpha} = \sum_{k=0}^{+\infty} q^k R_\alpha^k$.

**Theorem 2.1.3** [18] For any $\lambda, \mu \in P_n^+$, $K_{\lambda, \mu}(q)$ is the coefficient of $s_\lambda$ in $Q_\mu$ that is,

$$Q_\mu = \sum_{\lambda \in P_n^+} K_{\lambda, \mu}(q) s_\lambda.$$

When $n = 1$, the root system $C_1$ can be regarded as the root system $A_1$ and the Kostka-Foulkes polynomial $K_{\lambda, \mu}(q)$ where $\lambda$ and $\mu$ are partitions of length 1 satisfies

$$K_{\lambda, \mu}(q) = q^{(|\lambda| - |\mu|)/2}.$$ (3)

### 2.2 Crystal graphs for type $C_n$

Recall that crystal graphs for the $U_q(s_{2n})$-modules are oriented colored graphs with colors $i \in \{0, ..., n-1\}$. An arrow $a \rightarrow b$ means that $\bar{f}_i(a) = b$ and $\bar{e}_i(b) = a$ where $\bar{e}_i$ and $\bar{f}_i$ are the crystal graph operators (for a review of crystal bases and crystal graphs see [10]). A vertex $v^0 \in B$ satisfying $\bar{e}_i(v^0) = 0$ for any $i \in \{0, ..., n-1\}$ is called a highest weight vertex. The decomposition of $V$ into its irreducible components is reflected into the decomposition of $B$ into its connected components. Each connected component of $B$ contains a unique highest weight vertex. The crystals graphs of two isomorphic irreducible components are isomorphic as oriented colored graphs. The action of $\bar{e}_i$ and $\bar{f}_i$ on $B \otimes B' = \{ b \otimes b'; b \in B, b' \in B' \}$ is given by:

$$\bar{f}_i(u \otimes v) = \begin{cases} \bar{f}_i(u) \otimes v & \text{if } \varphi_i(u) > \varepsilon_i(v) \\ u \otimes \bar{f}_i(v) & \text{if } \varphi_i(u) \leq \varepsilon_i(v) \end{cases}$$

and

$$\bar{e}_i(u \otimes v) = \begin{cases} u \otimes \bar{e}_i(v) & \text{if } \varphi_i(u) < \varepsilon_i(v) \\ \bar{e}_i(u) \otimes v & \text{if } \varphi_i(u) \geq \varepsilon_i(v) \end{cases}$$

where $u \otimes v$ is the tensor product of $u$ and $v$. An oriented colored graph is isomorphic to an oriented colored graph if there exists a bijection between the vertex sets preserving the orientation and the colors.
where $\varepsilon_i(u) = \max\{k; \tilde{e}_i^k(u) \neq 0\}$ and $\varphi_i(u) = \max\{k; \tilde{f}_i^k(u) \neq 0\}$. The weight of the vertex $u$ is defined by $\text{wt}(u) = \sum_{i=0}^{n-1} (\varphi_i(u) - \varepsilon_i(u))\Lambda_i$.

The following lemma is a straightforward consequence of (4) and (5).

**Lemma 2.2.1** Let $u \otimes v \in B \otimes B'$ $u \otimes v$ is a highest weight vertex of $B \otimes B'$ if and only if for any $i \in \{0, \ldots, n-1\}$ $\tilde{e}_i(u) = 0$ (i.e. $u$ is of highest weight) and $\varepsilon_i(v) \leq \varphi_i(u)$.

The Weyl group $W_n$ acts on $B$ by:

$$s_i(u) = (\tilde{f}_i)^{\varphi_i(u) - \varepsilon_i(u)}(u) \text{ if } \varphi_i(u) - \varepsilon_i(u) \geq 0,$$

$$s_i(u) = (\tilde{e}_i)^{\varepsilon_i(u) - \varphi_i(u)}(u) \text{ if } \varphi_i(u) - \varepsilon_i(u) < 0.$$

We have the equality $\text{wt}(\sigma(u)) = \sigma(\text{wt}(u))$ for any $\sigma \in W_n$ and $u \in B$. For any $\lambda \in P_n^+$, we denote by $B(\lambda)$ the crystal graph of $V(\lambda)$.

According to (2) we have

$$B(\Lambda_{n-1}) : \pi \rightarrow \sum_{i=0}^{n-2} \rightarrow \cdots \rightarrow \sum_{i=0}^1 \rightarrow \sum_{i=1}^0 \rightarrow 1 \rightarrow 2 \rightarrow \cdots \rightarrow n \rightarrow n-1 \rightarrow \cdots \rightarrow 1 \rightarrow \pi.$$

Kashiwara-Nakashima’s combinatorial description of the crystal graphs $B(\lambda)$ is based on the notion of symplectic tableaux analogous for type $C_n$ to semi-standard tableaux. In the sequel we use De Concini’s version of these tableaux which is equivalent to Kashiwara-Nakashima’s one.

We defined a total order on $C_n$ by setting $C_n = \{\pi < \cdots < \pi \leq 1 \leq \cdots < \pi \}$.

For any letter $x \in C_n$ we set $\pi = x$. Note that our convention for labelling the crystal graph of the vector representation are not those used by Kashiwara and Nakashima. To obtain the original description of $B(\lambda)$ from that used in the sequel it suffices to change each letter $k \in \{1, \ldots, n\}$ of $C_n$ into $n-k+1$ and each letter $\bar{k} \in \{\bar{1}, \ldots, \bar{\pi}\}$ into $n-k+1$. The interest of this change of convention will appear in Sections 4 and 5.

We identify the vertices of the crystal graph $G_n = \bigoplus_{\ell} B(\Lambda_1)^{\otimes \ell}$ with the words on $C_n$. For any $w \in G_n$ we have $\text{wt}(w) = d_{\pi} \pi + d_{\pi-\bar{1}} \pi - d_{\bar{1}} \bar{1} + \cdots + d_{\bar{\pi}} \bar{\pi}$ where for any $i = 1, \ldots, n$ $d_i$ is the number of letters $\bar{i}$ of $w$ minus the number of its letters $i$. Using Formulas (4) and (5) we obtain a simple rule to compute the action of $\tilde{e}_i$, $\tilde{f}_i$ or $s_i$ on $w \in G_n$ that we will use in Section 4. Consider the subword $w_i$ of $w$ containing only the letters $i+1$, $\bar{i}$, $i+1$. Then encode in $w_i$ each letter $\bar{i+1}$ or $i$ by the symbol $+$ and each letter $\bar{i}$ or $i+1$ by the symbol $-$. Because $\tilde{e}_i(+) = \tilde{f}_i(-) = 0$ in $B(\Lambda_{n-1}) \oplus B(\Lambda_{n-1})$ the factors of type $+$ may be ignored in $w_i$. So we obtain a subword $w_i(1)$ in which we can ignore all the factors $+$ to construct a new subword $w_i(2)$ etc. Finally we obtain a subword $\rho(w)$ of $w$ of type $\rho(w) = -r + s$.

Then we have the

**Note 2.2.2**

- If $r > 0$, $\tilde{e}_i(w)$ is obtained by changing the rightmost symbol $-$ of $\rho(w)$ into its corresponding symbol $+$ (i.e. $i+1$ into $i$ and $\bar{i}$ into $i+1$) the others letters of $w$ being unchanged. If $r = 0$, $\tilde{e}_i(w) = 0$.

- If $s > 0$, $\tilde{f}_i(w)$ is obtained by changing the leftmost symbol $+$ of $\rho(w)$ into its corresponding symbols $-$ (i.e. $i$ into $i+1$ and $\bar{i+1}$ into $\bar{i}$) the others letters of $w$ being unchanged. If $s = 0$, $\tilde{f}_i(w) = 0$.

- If $r > s$, $s_i(w)$ is obtained by changing the $r-s$ rightmost symbols $-$ of $\rho(w)$ into its corresponding symbol $+$, otherwise $s_i(w)$ is obtained by changing the $s-r$ leftmost symbols $+$ of $\rho(w)$ into its corresponding symbols $-$.
A column on $C_n$ is a Young diagram $C$ of column shape filled from top to bottom by increasing letters of $C_n$. The height $h(C)$ of a column $C$ is the number of its letters. Set $C(n, h)$ for the set of columns of height $h$ on $C_n$ i.e. with letters in $C_n$. The reading of the column $C \in C(n, h)$ is the word $w(C)$ of $C_n$ obtained by reading the letters of $C$ from top to bottom. We will say that a column $C$ contains the pair $(z, \bar{z})$ when $C$ contains the unbarred letter $z \geq 1$ and the barred letter $\bar{z} \leq \bar{1}$. Let $C_1$ and $C_2$ be two columns. We will write $C_1 \leq C_2$ when $h(C_1) \geq h(C_2)$ and the rows of the tableau $C_1C_2$ weakly increase.

**Definition 2.2.3** Let $C$ be a column on $C_n$ and $I_C = \{z_1 < \cdots < z_r\}$ the set of unbarred letters $z$ such that the pair $(z, \bar{z})$ occurs in $C$. The column $C$ is $n$-admissible when there exists a set of unbarred letters $J_C = \{t_1 < \cdots < t_s\} \subset C_n$ such that:

- $t_1$ is the lowest letter of $C_n$ satisfying: $t_1 > z_1, t_1 \notin C$ and $\bar{t_1} \notin C$,
- for $i = 2, \ldots, r$, $t_i$ is the lowest letter of $C_n$ satisfying: $t_i > \max(t_{i-1}, z_i)$, $t_i \notin C$ and $\bar{t_i} \notin C$.

In this case we write:

- $rC$ for the column obtained from $C$ by changing $z_i$ into $t_i$ for each letter $z_i \in I_C$,
- $lC$ for the column obtained from $C$ by changing $\bar{z}_i$ into $\bar{t}_i$ for each letter $z_i \in I_C$.

Consider $C = \begin{array}{c} 3 \\ 2 \\ 2 \\ 3 \end{array}$. Then $C$ is not 4-admissible but is 5-admissible with $rC = \begin{array}{c} 3 \\ 2 \\ 4 \\ 5 \\ 3 \end{array}$ and $lC = \begin{array}{c} 3 \\ 2 \\ 4 \\ 5 \\ 3 \end{array}$. As usually, we associate to each partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ the Young diagram $Y(\lambda)$ whose $i$-th row has length $\lambda_{i+1}$. By definition, a $n$-symplectic tableau $T$ of shape $\lambda$ is a filling of $Y(\lambda)$ by letters of $C_n$ satisfying the following conditions:

- the columns $C_i$ of $T = C_1 \cdots C_s$ are $n$-admissible,
- for $i = 1, \ldots, s - 1 : rC_i \leq lC_{i+1}$.

The set of $n$-symplectic tableaux will be denoted $\text{ST}(n)$. If $T = C_1C_2 \cdots C_r \in \text{ST}(n)$, the reading of $T$ is the word $w(T) = w(C_r) \cdots w(C_2)w(C_1)$. From $\text{III}$ we deduce the

**Theorem 2.2.4**

(i): The vertices of $B(\Lambda_p)$ $p = 0, \ldots, n - 1$ are in one-to-one correspondence with the readings of $n$-admissible columns of height $n - p$.

(ii): The vertices of $B(\Lambda)$ are in one-to-one correspondence with the readings of the $n$-symplectic tableaux of shape $\lambda$.

More precisely Kashiwara and Nakashima realize $B(\lambda)$ into a tensor power $B(\Lambda_{n-1}) \otimes 1$. Given $p = 0, \ldots, n - 1$, $B(\Lambda_p)$ can then be identified with the connected component of $G_n$ whose highest weight vertex is $b_\lambda = \pi(n-1) \cdots p + 1$. In this identification, the vertices of $B(\Lambda_p)$ are the readings of the admissible columns of height $n - p$. If $\lambda = \sum_{p=0}^{n-1} \lambda_p \Lambda_p$, $B(\lambda)$ is identified with the connected component whose highest weight vertex is $b_\lambda = b_\lambda \otimes \lambda_{n-1} \cdots \otimes \lambda_1 b_0 \otimes \lambda_0$.

By identifying $U_q(\mathfrak{sp}_{2(n-1)})$ with the sub-algebra of $U_q(\mathfrak{sp}_{2n})$ generated by the Chevalley’s generators $e_i, f_i$ and $t_i, i = 0, \ldots, n - 1$, we endow $B(\lambda)$ with a structure of crystal graph for type $C_{n-1}$. The decomposition of $B(\lambda)$ into its $U_q(\mathfrak{sp}_{2(n-1)})$-connected components is obtained by erasing all the arrows of color $n - 1$. 

6
2.3 Insertion scheme for symplectic tableaux

In [15] we have introduced an insertion scheme for symplectic tableaux analogous for type $C_n$ to the bumping algorithm on Young tableaux. Now we are going to summarize the properties of this scheme that we shall need in Section 4.

Consider first a letter $x$ and a column $C$. The insertion of the letter $x$ in the $n$-admissible column $C$ is denoted $x \rightarrow C$. If $x$ is strictly greater to the greatest letter of $C$ then $x \rightarrow C$ is the column obtained by adding a box containing $x$ on bottom of $C$, that is, $x \rightarrow C = C \begin{array}{c} x \end{array}$. Now suppose that $x$ is less than the greatest letter of $C$. Then $x \rightarrow C$ is a symplectic tableau of two columns defined recursively as follows:

1. if $C = \begin{array}{c} a \end{array}$ contains only one column then $x \rightarrow C = \begin{array}{c} x \end{array} a$,
2. if $C = \begin{array}{c} a \end{array} b$ contains two letters, $x \rightarrow C = \begin{array}{c} a \end{array} b$ if $a < x \leq b$ and $b \neq x$,
3. $x \rightarrow C = \begin{array}{c} b + 1 \end{array} x b + 1$ if $a = b$ and $b \leq x \leq b$,
4. $\overline{b} \rightarrow C = \begin{array}{c} b - 1 \end{array} a$ if $x = b$ and $b < a < b$.

Consider a $n$-admissible column $C$ of height $k \geq 3$ and suppose we have defined our insertion for the $n$-admissible columns of height $< k$. Set $w(C) = a_1 \cdots a_{k-1}a_k$ and $x \rightarrow \begin{array}{c} a_{k-1} \end{array} a_k = \begin{array}{c} \delta_{k-1} \end{array} y$. Then we have $y > a_{k-2}$ and the column $C'$ of reading $a_1 \cdots a_{k-2}y$ is $n$-admissible. Write $\delta_{k-1} \rightarrow C' = \begin{array}{c} \overline{d_1} \overline{z} \end{array}$. We set $x \rightarrow C = \begin{array}{c} \overline{d_1} \overline{z} \end{array}$. This can be pictured by

\[
\begin{array}{c} a_1 \end{array} = \begin{array}{c} a_1 \end{array} = \begin{array}{c} a_1 \end{array} \quad \begin{array}{c} \overline{d_1} \overline{z} \end{array} = \begin{array}{c} \overline{d_1} \overline{z} \end{array}.
\]

During each step we apply one of the transformations 1 to 4 below. We have proved in [15] that $x \rightarrow C$ is then a $n$-symplectic tableau with two columns respectively of height $h(C)$ and 1.

**Example 2.3.1** Suppose $n = 5$.

\[
\begin{array}{c} 4 \end{array} = \begin{array}{c} 4 \end{array},
\begin{array}{c} 2 \end{array} = \begin{array}{c} 2 \end{array},
\begin{array}{c} 2 \end{array} = \begin{array}{c} 2 \end{array},
\begin{array}{c} 3 \end{array} = \begin{array}{c} 3 \end{array},
\begin{array}{c} 4 \end{array} = \begin{array}{c} 4 \end{array},
\begin{array}{c} 5 \end{array} = \begin{array}{c} 5 \end{array}.
\]
Example 2.3.2 Suppose $n = 3$. Then $2 \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 \\ 3 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 \\ 2 & 3 \end{pmatrix}$.

Remarks:
(i) The insertion scheme described below do not suffice to define a complete insertion algorithm for the $n$-symplectic tableaux since the first column $C'$ of $x \rightarrow T$ may not be $n$-admissible when $x$ is greater than the greatest letter of $C_1$. To obtain a complete insertion algorithm we have to apply relation 8 to $C'$. This give a column $D$ of reading $x_1 \cdots x_p$. Finally we compute successively the insertions $x_p(\rightarrow x_{p-1} \cdots (x_1 \rightarrow C_2 \cdots C_r))$. In the sequel we only use insertion algorithm without the contraction relation 8.
(ii) To each $w = x_1 \cdots x_r \in C_n^r$ of length $r$ we can associate recursively a symplectic tableau $P(w)$ by setting $P(w) = [x_1]$ if $r = 1$ and $P(w) = x_r \rightarrow P(x_1 \cdots x_{r-1})$ otherwise. If $P(w)$ belongs to $\text{ST}(m)$ with $m \geq n$, results of 15 implies that $P(w) \equiv_n w$. Moreover $P(w)$ is the unique $m$-symplectic tableau with this property. Denote by $\sim_m$ the equivalence relation defining on the vertices of $G_m$ by $w_1 \sim_m w_2$ if and only if $w_2$ and $w_3$ belong to the same connected component of $G_m$. Given two words $w_1$ and $w_2$ such that $P(w_1)$ and $P(w_2)$ belong $\text{ST}(m)$ we have the equivalences

\[ w_1 \equiv_m w_2 \iff P(w_1) = P(w_2) \iff P(w_1) \sim_m P(w_2). \]
Moreover we have for any \( \sigma \in W_m \)
\[
P(\sigma(w)) = \sigma(P(w)).
\]

(iii) : The insertion algorithm is reversible in the sense that if we know the tableau \( T' \) such that \( x \to T = T' \) and the shape of \( T \) we can recover the tableau \( T \) and the letter \( x \). This follows from the fact that the transformations 1 to 4 are reversible. More precisely, \( T' \) has one box more than \( T \). Let \( y \) be the letter belonging to that box. Then if we apply transformations 1 to 4 from right to left starting from \( y \), we recover \( T \) and \( x \).

(iv) : In Section 3.1 Pieri rule for type \( r \).

Example 2.3.3 Suppose \( n = 3 \) and \( T' = \begin{array}{ccc}
2 & 1 & 2 \\
1 & 2 & \\
3 & & 
\end{array} \). Then by applying reverse insertion algorithm to each outside corners of \( T' \) we obtain the pairs \( \begin{pmatrix} 3, & 2, & 1, & 2 \end{pmatrix}, \ \begin{pmatrix} 1, & 1, & 2, & 1 \end{pmatrix}, \ \begin{pmatrix} 1, & 1, & 2 \end{pmatrix} \).

3 Morris type recurrence formula

In this section we introduce a recurrence formula for computing Kostka polynomials analogous for type \( C_n \) to Morris recurrence formula. It allows to explain the Kostka polynomials for type \( C_n \) as combinations of Kostka polynomials for type \( C_{n-1} \). We embed type \( C_{n-1} \) in type \( C_n \) by identifying \( U_q(s_{2n-1}) \) with the sub-algebra of \( U_q(s_{2n}) \) generated by the Chevalley operators \( e_i, f_i \) and \( t_i, i = 0, \ldots, n-2 \). The weight lattice \( P_{n-1} \) of \( U_q(s_{2n}) \) is the \( \mathbb{Z} \)-lattice generated by \( \gamma_i, i = 1, \ldots, n-1 \) and \( P_{n-1} = P_n \cap P_{n-1} \) is the set of dominant weights. The Weyl group \( W_{n-1} \) is the sub-group of \( W_n \) generated by the \( s_i, i = 0, \ldots, n-2 \) and we have \( R_n \cap R_n \).

Given any positive integer \( r \), write \((r)_n \) for the row partition \((p, 0, \ldots, 0)\) of length \( n \). To obtain our recurrence formula we need to describe the decomposition \( B(\gamma) = B((r)_n) \) with \( \gamma \in P_n^+ \) and \( r > 0 \) an integer into its irreducible components. This is analogous for type \( C_n \) to Pieri rule.

3.1 Pieri rule for type \( C_n \)

Let \( \gamma = (\gamma_1, \ldots, \gamma_n) \in P_n^+ \). By Theorem 2.2.1 the vertices of \( B((r)_n) \) are the words
\[
L = (n)^{k_n} \cdots (2)^{k_2}(1)^{k_1}(\overline{1})^{k_1}(\overline{2})^{k_2} \cdots (\overline{n})^{k_n}
\]
where \( k_1, k_2, \ldots \) are positive integers, \((x)^k \) means that the letter \( x \) is repeated \( k \) times in \( L \) and \( k_1 + \cdots + k_n = r \). Let \( b_\gamma \) be the highest weight vertex of \( B(\gamma) \).

Lemma 3.1.1 \( b_\gamma \otimes L \) is a highest weight vertex of \( B((\gamma)_n) \) if and only if the following conditions holds:
(i) : \( \gamma_i - k_i \geq \gamma_{i+1} \) for \( i = 2, \ldots, n \) and \( \gamma_1 - k_1 \geq 0 \).
(ii) : \( \gamma_i + k_i \leq \gamma_{i+1} + k_{i+1} \) for \( i = 1, \ldots, n-1 \).

Proof. By Lemma 2.2.1 \( b_\gamma \otimes L \) is a highest weight vertex if and only if for any \( m = 1, \ldots, r \), each vertex \( b_\gamma \otimes L_m \) (where \( L_m \) is the word obtained by reading the \( m \) leftmost letters of \( L \)) is a highest weight vertex. It means that \((\gamma_1 - k_1, \ldots, \gamma_s - k_s, \gamma_{s+1}, \ldots, \gamma_t) \) and \((\gamma_1 - k_1, \ldots, \gamma_{s+1} + k_{i+1}, \gamma_i - k_i + k_T, \gamma_T + k_T - k_T) \) are partitions respectively for \( s = n, \ldots, 1 \) and \( t = 1, \ldots, n-1 \). This is equivalent to the conditions
\[
\begin{cases}
\gamma_i - k_i \geq \gamma_{i+1} - k_{i+1} & \text{for } s = n, n-2 \text{ and } t \leq n-1 \\
\gamma_i - k_i + k_T \leq \gamma_{i+1} - k_{i+1} & \text{for } t = 1, \ldots, n-1.
\end{cases}
\]

Corollary 3.1.2 \( B(\gamma) \otimes B((r)_n) = \bigoplus_{\lambda \in P_n^+} B(\lambda)^{\oplus n_\lambda} \) where \( n_\lambda \) is the number of vertices \( L \in B((r)_n) \) such that \( \lambda \in P_n^+ \)
(i) : \( k - k_i = \lambda_i - \gamma_i \) for \( i = 1, \ldots, n \).
(ii) : \( \lambda_i \geq \lambda_{i+1} - k_{i+1} \) for \( i = 1, \ldots, n-1 \).
(iii) : \( \lambda_i - k_i \geq \lambda_{i+1} + k_{i+1} - k_{i+1} \) for \( i = 1, \ldots, n \).

\[9\]
Proof. The multiplicity $n_\lambda$ is equal to the number of highest weight vertices $b_\gamma \otimes L \in B(\gamma) \otimes B((r)_n)$ of weight $\lambda$. The condition $\text{wt}(b_\gamma \otimes L) = \lambda$ is equivalent to

$$\gamma_i - k_i + k_\gamma = \lambda_i$$

which gives (i). The assertions (ii) and (iii) are respectively obtained by replacing for any $i$, $\gamma_i$ by $\lambda_i + k_i - k_\gamma$ in assertions (ii) and (i) of the previous lemma. ■

Note that $B(\gamma) \otimes B((r)_n)$ is not multiplicity free in general.

3.2 Recurrence formula

Theorem 3.2.1 Let $\mu \in P^+_{n}$ with $n \geq 2$ and write $\mu = (\mu_\tau, \mu')$ where $\mu_\tau$ is the first part of $\mu$ and $\mu' = (\mu_{\gamma-1}, ..., \mu_\gamma) \in P^+_{n-1}$. Then

$$Q_\mu = \sum_{r \geq 0} \sum_{m \geq 0} q^{m+r} \sum_{\lambda \in B(\gamma) \otimes B((r)_n)} K_{\lambda, \mu'}(q)s_{(\mu_\tau, \lambda)}$$

(12)

Proof. We start from $Q_\mu = \left( \prod_{\alpha \in R^+_n} \frac{1}{1 - qR_\alpha} \right) s_\mu$. By Proposition 3.5 of [19] we write

$$Q_\mu = \left( \prod_{\alpha \in R^+_n} \frac{1}{1 - qR_\alpha} \right) \left( \sum_{\lambda \in P^+_{n-1}} K_{\lambda, \mu'}(q)s_{(\mu_\tau, \lambda)} \right).$$

Then by applying Theorem 2.1.3 we obtain

$$Q_\mu = \left( \prod_{\alpha \in R^+_n} \frac{1}{1 - qR_\alpha} \right) \left( \sum_{\lambda \in P^+_{n-1}} K_{\lambda, \mu'}(q)s_{(\mu_\tau, \lambda)} \right).$$

(13)

Set $R_\tau = R_{\gamma - k_\gamma}$ for $i = 1, ..., n - 1$ and $R_i = R_{\gamma + k_\gamma}$ for $i = 1, ..., n$. Recall that for any $\beta \in P_{n-1}$, $R_\tau(\beta) = \beta + \varepsilon_\tau - \varepsilon_\gamma$ and $R_i(\beta) = \beta + \varepsilon_\tau + \varepsilon_\gamma$. Then (13) implies

$$Q_\mu = \sum_{\lambda \in P^+_{n-1}} K_{\lambda, \mu'}(q) \times$$

$$\left( \sum_{r \geq 0} \sum_{m \geq 0} q^{m+r} (R_\mu)^m (R_1)^{k_1} (R_\gamma)^{k_\gamma} \cdots (R_{n-1})^{k_{n-1}} (R_{n-1})^{k_{n-1}} (R_{n-1})^{k_{n-1}} s_{(\mu_\tau, \lambda)} \right),$$

$$Q_\mu = \sum_{r \geq 0} \sum_{m \geq 0} q^{m+r} \sum_{\lambda \in P^+_{n-1}} K_{\lambda, \mu'}(q) \sum_{k_1 + \cdots + k_n = r} s_{(\mu_\tau + m, \lambda + k_1, ..., \lambda + k_n)}.$$

Fix $\lambda, m$ and $r$ and consider

$$S = \sum_{k_1 + \cdots + k_n = r} s_{(\mu_\tau + m, \lambda + k_1, ..., \lambda + k_n)}.$$

Set $\gamma = (\lambda_{n-1} + k_{n-1} - k_{n-1} - k_\gamma, ..., \lambda_1 + k_1 - k_\gamma)$.

1: Suppose first that there exits $i \in \{1, ..., n - 2\}$ such that $\gamma_i > \lambda_i - k_{n-1}$. Set $\tilde{\gamma} = s_i \circ \gamma$ that is

$\tilde{\gamma} = s_i(\gamma_{n-1} + n - 1, ..., \gamma_{i+1} + n - i + 1, \gamma_i + n - i, ..., \gamma_1 + 1) - (n - 1, ..., 1).$
Then $\gamma_{\pi} = \tilde{\gamma}_{\pi}$ for $s \neq i + 1, i$, $\tilde{\gamma}_{\pi + i} = \gamma_{\pi} - 1$ and $\tilde{\gamma}_{\pi} = \gamma_{\pi + i} + 1$ that is

$$
\begin{align*}
\tilde{\gamma}_{\pi + i} &= \gamma_{\pi} + k_i - k_{\pi} - 1 \\
\tilde{\gamma}_{\pi} &= \gamma_{\pi + i} + k_{i+1} - k_{\pi + i} + 1
\end{align*}
$$

Write $\tilde{k}_{i+1} = k_i$, $\tilde{k}_i = k_{i+1}$, $\tilde{k}_{i+1} = \gamma_{\pi + i} - \gamma_{\pi} + k_i + 1$ and $\tilde{k}_{\pi} = \gamma_{\pi} - \gamma_{\pi + i} + k_{i+1} - 1$. To make our notation homogeneous set $\tilde{k}_i = k_i$ for any $t \neq i, i+1, i, i+1$. Then $\tilde{\gamma}_{\pi} > \gamma_{\pi + i} - k_{\pi + i}$. We have $\tilde{k}_{i+1} > 0$ and $\tilde{k}_{\pi} = \gamma_{\pi} - \gamma_{\pi + i} + k_{i+1} - 1 \geq 0$ since $\gamma_{\pi} > \gamma_{\pi + i} - k_{\pi + i}$. Moreover $\tilde{k}_{i} + \cdots + k_{\pi} + k_{i} + \cdots + k_{n} = r$ and for any $s \in \{1, \ldots, n-2\}$

$$
\tilde{\gamma}_{\pi} = \gamma_{\pi} + \tilde{k}_{s} - \tilde{k}_{\pi}.
$$

2 : Suppose that $\gamma_{\pi} \leq \gamma_{\pi + i} - k_{\pi + i}$ for any $i = 1, \ldots, n-2$ and $\gamma_{\pi} - k_{\pi} < 0$. Set $\tilde{\gamma} = s_0 \circ \gamma$. Then $\gamma_{\pi} = \tilde{\gamma}_{\pi}$ for $s \neq 1$ and $\tilde{\gamma}_{\pi} = -\gamma_{\pi} - k_{\pi} + k_{\pi} - 2$. Write $k_i = k_i$, $\tilde{k}_i = k_{\pi}$ for any $i = 2, \ldots, n-1$ and set $\tilde{k}_1 = k_{\pi} - \gamma_{\pi} - 1$, $\tilde{k}_{1+1} = \gamma_{\pi} + 1$. We have $\tilde{k}_i \geq 0$, $\gamma_{\pi} \leq \gamma_{\pi + i} - k_{\pi + i}$ for any $i = 1, \ldots, n-2$ and $\gamma_{\pi} - \tilde{k}_{\pi} < 0$. Moreover $\tilde{k}_{i} + \cdots + k_{\pi} + k_{i} + \cdots + k_{n} = r$.

3 : Now suppose that $\gamma_{\pi} \leq \gamma_{\pi + i} - k_{\pi + i}$ for any $s \in \{1, \ldots, n-2\}$, $\gamma_{\pi} - k_{\pi} \geq 0$ and there exists $i \in \{1, \ldots, n-2\}$ such that $\gamma_{\pi + i} - k_{\pi + i} < \gamma_{\pi} + k_{i} - k_{\pi}$. Define $\tilde{\gamma} = s_0 \circ \gamma$ as above. Set $\tilde{k}_{i+1} = k_{i+1}$, $\tilde{k}_i = k_{i}$, $\tilde{k}_{i+1} = \gamma_{\pi} - \gamma_{\pi + i} - k_{i} + k_{i+1} - 1$ and $\gamma_{\pi} = (\lambda, \nu) = (\lambda_{\pi} - k_{\pi} - k_{i} + k_{i+1} + 1) + k_{i} + k_{i+1} + 1$. Write $k_i = k_i$ for any $t \neq i, i+1, i, i+1$. We obtain $\tilde{k}_i \geq 0$ and $\tilde{k}_{i+1} \geq 0$ since $\gamma_{\pi} \leq \gamma_{\pi + i} - k_{\pi + i}$ and $\gamma_{\pi} - \gamma_{\pi + i} - k_{i} + k_{i+1} < \gamma_{\pi} + k_{i} - k_{\pi}$. Since $\tilde{k}_i = k_{i}$ for any $s = 1, \ldots, n-1$, we have $\gamma_{\pi} \leq \gamma_{\pi + i} - k_{i} - k_{i+1} < \gamma_{\pi} + k_{i} - k_{\pi}$ holds since it is equivalent to $0 < k_{i+1} + 1$. Finally $\tilde{k}_{i} + \cdots + k_{\pi} + k_{i} + \cdots + k_{n} = r$ and for any $s \in \{1, \ldots, n-2\}$

$$
\tilde{\gamma}_{\pi} = \gamma_{\pi} + \tilde{k}_{s} - \tilde{k}_{\pi}.
$$

Denote by $E_1$, $E_2$ and $E_3$ the sets of multi-indices $(k_1, \ldots, k_{\pi}, k_1, \ldots, k_{n})$ such that $k_{\pi} + \cdots + k_{\pi} + k_{i} + \cdots + k_{n} = r$ and satisfying respectively the assertions 1, 2, 3. Let $\chi$ be the map defined on $E_1 \cup E_2 \cup E_3$ by

$$
\chi(\gamma) = \tilde{\gamma}.
$$

Then by the above arguments $\chi$ is a bijection which verifies $\chi(E_i) = E_i$ for $i = 1, 2, 3$. Now the pairing $\gamma \mapsto \tilde{\gamma}$ provides the cancellation of all the $s$, with $\gamma = (\lambda_{\pi} - k_{n-1} - k_{\pi} - 1, \lambda_{\pi} + k_{1} - k_{\pi})$ such that $(\lambda_{\pi}, \lambda_{\pi}, k_{1}, \ldots, k_{n}) \in E_1 \cup E_2 \cup E_3$ appearing in $S$. Indeed $s_{\langle \mu_{\pi} + r + 2m, \gamma \rangle} = -s_{\langle \mu_{\pi} + r + 2m, \tilde{\gamma} \rangle}$. By Corollary 3.2.2, it means that

$$
S = \sum_{\gamma \in \Gamma_{\pi(n-1)}^{+}} s_{\langle \mu_{\pi} + r + 2m, \gamma \rangle}
$$

and the theorem is proved.

Note that the theorem is also true for $n = 2$. In this case $R_{n-1}$ is the set of positive roots of the root system $A_1$.

**Corollary 3.2.2** Let $\nu, \mu \in P_{n}^{+}$ such that $\mu_{\pi} \geq \nu_{\pi(n-1)}$. Set $l = \nu_{\pi} - \mu_{\pi} \geq 0$ and $\nu' = (\nu_{n-1}, \ldots, \nu_{1})$. Then

$$
K_{\nu, \mu}(q) = \sum_{r+2m=l} q^{r+2m} \sum_{\lambda \in \lambda_{B(\nu) \cap B(\nu_{n-1})}} K_{\lambda, \nu'}(q).
$$

**Proof.** Let $m, r$ be two integers such that $(\mu_{\pi} + r + 2m, \nu') = \nu$. Then $l = r + 2m$. Consider $s_{\langle \mu_{\pi} + r + 2m, \nu \rangle}$ and $s_{\langle \mu_{\pi} + r + 2m, \nu' \rangle}$ appearing in $S$. Suppose that there exists $\sigma \in W_n$ such that $(\mu_{\pi} + r + 2m + \nu, \gamma) = \sigma \circ (\mu_{\pi} + r + 2m + \nu, \gamma - n - 1, \nu) = (n, \ldots, 1)$. We cannot have $\sigma(\nu) = \pi$ with $p \in \{1, \ldots, n\}$ otherwise $\mu_{\pi} + r + 2m + \nu < 0$. Set $\sigma(\pi) = \pi$ with $p \in \{1, \ldots, n\}$. If $p < n$ we must have

$$
\nu_{\pi} + p = \mu_{\pi} + r + 2m + n.
$$

Thus $\nu_{\pi} = \mu_{\pi} + r + 2m + n - p > \mu_{\pi}$ which contradicts the hypothesis $\mu_{\pi} \geq \nu_{\pi(n-1)}$. Hence $\sigma(\pi) = \pi$ and $r + 2m = l$ that is $\sigma \in W_{n-1}$. Moreover we have $\gamma = \nu'$ since $w \circ \nu' = \gamma$ and $\nu, \gamma \in P_{n-1}$. This proves that
\( s_{(\mu_{\tau}+r+2m,\nu')} \) can not be obtained by applying the straightening law for Schur functions on \( s_{(\mu_{\tau}+r+2m,\gamma)} \) with \( (\mu_{\tau}+r+2m,\gamma) \neq (\mu_{\tau}+r+2m,\nu') \). Then the corollary directly follows from [12] and Theorem 2.1.3. \( \square \)

Now suppose that \( \nu = (p)_n \in \mathbb{P}^n \). Then for any \( \mu \in \mathbb{P}^n \) we must have

\[
K_{(p)_n,\mu}(q) = \sum_{r+2m=l} q^{r+m} K_{(r)_{n-1},\mu}(q)
\]

with \( l = p - \mu_{\tau} \). This implies that \( K_{(p)_n,\mu}(q) \) may be computed recursively. We are going to give an explicit formula for \( K_{(p)_n,\mu}(q) \). The vertices of \( B((p)_n)_\mu = \{ L \in B(p\Lambda_{n-1}), \text{wt}(b) = \mu \} \) are the words

\[
L = (n)^{k_n} \cdots (2)^{k_2}(1)^{k_1}(\overline{1})^{k_1}(\overline{2})^{k_2} \cdots (\overline{n})^{k_n}
\]

with \( \mu_{\tau} = k_1 - \nu_i \) for \( i = 1, \ldots, n \) and \( k_n + \cdots + k_{\tau} + k_1 + \cdots + k_n = p \).

**Proposition 3.2.3** Let \( p \geq 1 \) be an integer. For any \( \mu \in \mathbb{P}^n \) we have

\[
K_{(p)_n,\mu}(q) = q^{f_\mu(n)} \sum_{L \in B((p)_n)_\mu} q^{\theta_n(L)}
\]

where \( f_\mu(n) = \sum_{i=1}^{n} (n-i)\mu_{\tau} \) and \( \theta_n(L) = \sum_{i=1}^{n} (2 \times (n-i)+1)(k_1-\mu_{\tau}) \).

**Proof.** We proceed by induction on \( n \). Suppose \( n = 1 \) we have \( f_1(\mu) = 0 \). We can write \( L = (1)^{p-k_1}(\overline{1})^{k_1} \) and \( \mu_{\tau} = 2k_1 - p \). Thus \( \theta_1(L) = k_1 - \mu_{\tau} = (p - \mu_{\tau})/2 \) and the proposition is true by [13].

Now suppose the proposition true for \( n - 1 \). First note that \( f_\mu(n) = f_{n-1}(\mu') + \sum_{i=1}^{n-1} \mu_{\tau} \). The set of vertices obtained by erasing the letters \( n \) and \( \overline{n} \) in \( B((p)_n)_\mu \) is the disjoint union of the \( B((r)_{n-1})_{\mu'} \) with \( r \in \{0, \ldots, l = p - \mu_{\tau} \} \) since the number of letters \( n \) or \( \overline{n} \) belonging to a vertex \( L \in B((p)_n)_\mu \) is a least equal to \( \mu_{\tau} \). Its reflects the decomposition of \( B((p)_n)_\mu \) into its \( Uq(sp_{2(n-1)}) \)-connected components. Consider \( L \in B((p)_n)_\mu \) and denote by \( L' \) the vertex obtained by erasing all the letters \( n \) and \( \overline{n} \) in \( L \). Let \( r \) be such that \( L' \in B((r)_{n-1})_{\mu'} \). Then \( r \in \{0, \ldots, l \} \) and \( l - r \) is even since it is equal to the number of pairs \( (n, \overline{n}) \) erased in \( L \). We set \( l - r = 2m \). Then \( k_1 = \mu_{\tau} + m \).

We have

\[
\theta_n(L) = \theta_{n-1}(L') + 2 \sum_{i=1}^{n-1} (k_i - \mu_{\tau}) + (k_1 - \mu_{\tau}).
\]

From the equality \( r = \sum_{1 \leq i \leq n-1} \mu_{\tau} + 2 \sum_{1 \leq i \leq n-1} (k_i - \mu_{\tau}) \) we deduce

\[
\theta_n(L) = \theta_{n-1}(L') + r - \sum_{i=1}^{n-1} \mu_{\tau} + m.
\]

Set

\[
K = \sum_{L \in B((p)_n)_\mu} q^{\theta_n(L)}.
\]

Then by the above arguments

\[
K = \sum_{r+2m=l} \sum_{L' \in B((r)_{n-1})_{\mu'}} q^{\theta_{n-1}(L') + r - \sum_{i=1}^{n-1} \mu_{\tau} + m + f_{n-1}(\mu') + \sum_{i=1}^{n-1} \mu_{\tau}} = \sum_{r+2m=l} q^{r+m} \times \sum_{L' \in B((r)_{n-1})_{\mu'}} q^{\theta_{n-1}(L') + f_{n-1}(\mu')}.
\]

Thus we obtain by the induction hypothesis

\[
K = \sum_{r+2m=l} q^{r+m} K_{(r)_{n-1},\mu'}(q).
\]

Finally \( K = K_{(p)_n,\mu}(q) \) by [14]. \( \square \)
Corollary 3.2.4 Write \((1^2)_n\) for the partition of length \(n\) equal to \((1,1,0,...,0)\). Then
\[
K_{(1^2)_n,0}(q) = \sum_{i=1}^{n-1} q^{2i}.
\]
To prove this corollary we need the more general lemma above

Lemma 3.2.5 Write \((1^p)_n\) for the partition of length \(n\) \((1,...,1,0,...,0)\) with \(p \geq 2\) parts equal to 1. Then
\[
K_{(1^p)_n,0}(q) = (q-1)K_{1^p,0}(q) + qK_{(1^p)_{n-1},0}(q) + qK_{(1^p-2)_{n-1},0}(q)
\]
where \(1^p = (2,1,...,1,0,...,0)\) \(\in \mathcal{P}_{n-1}\) contains \(p - 2\) parts equal to 1.

Proof. With \(\mu = 0\) formula \((12)\) becomes
\[
Q_0 = \sum_{\gamma \in \mathcal{P}_{n-1}^+} \sum_{r=0}^{+\infty} \sum_{m=0}^{+\infty} q^{m+r} \prod_{\lambda \in \mathcal{B}(\gamma) \otimes \mathcal{B}(1_{n-1}^\gamma)} K_{\lambda,0}(q) s(r+2m,\gamma).
\]
By using the straightening law for Schur functions and Theorem \(2.1.3\) we have to find all the \((r+2m,\gamma)\) such that there exists \(\pi \in \mathcal{W}_{n-1}\) satisfying \(\pi \circ (r+2m,\gamma) = (1^p)_n\) that is
\[
\sigma(r+2m+n,\gamma_{n-1}^\pi) = n-1, \ldots, \gamma_1^\pi + 1 = n+1, \ldots, n-p+2, n-p, \ldots, 1.
\]
We have \(r+2m+n \geq n\) hence \(\pi(\sigma) \in \{\pi,\pi^{-1}\}\).

(i) : If \(\pi(\sigma) = \pi\) then \(r = 1\) and \(m = 0\). For \(k \notin \{1,n\}\), \(\gamma_k + k > 1\) thus \(\sigma(\pi) = \pi\). By a straightforward induction we obtain \(\gamma_k = k\) \(\forall k \in \{1,\ldots,n-p\}\). Moreover we have \(\gamma_k + k \leq n\) for \(k < n\). This implies that \(\gamma_{n-1} = 0,1\) since \(\gamma_{n-1} + n - 1 \geq n - 1\). We can not have \(\gamma_{n-1}^\pi = 0\) otherwise \(\gamma_{n-1}^\pi = 0\) for any \(k < n\) and the value \(n\) in the left hand side of \((15)\) is not attained. Hence \(\gamma_{n-1} = 1\) and \(\sigma(\pi) = \pi^{-1}\). By induction we can prove that
\[
\gamma_{n-1} = \cdots = \gamma_{n-p+1} = 1 \text{ and } \sigma(\pi) = \pi^{-1} \text{ for } k \in \{n-1,\ldots,n-p+1\}.
\]
It means that \(\sigma = id, R = 1, m = 0\) and \(\gamma = (1^{p-1})_{n-1}\).

(ii) : If \(\pi(\sigma) = \pi^{-1}\) then \(R = m = 0\). By using similar arguments than above we obtain \(\gamma_{n-1} = 2, \gamma_{n-2} = \cdots = \gamma_{n-p+2} = 1\) and \(\gamma_{n-p} = \cdots = \gamma_0 = 0\). It means that \(\sigma = s_n\) and \(\gamma = 1^p\). Note that \(s(0,1^p) = -s((1^p)_n)\) since \(s_n \circ (0,1^p_n) = (1^p)_n\) and \(l(s_n) = 1\).

Finally by Theorem \(2.1.3\) we must have
\[
K_{(1^p)_n,0}(q) = q \times \sum_{\lambda \in \mathcal{B}(1^{p-1}_{n-1}) \otimes \mathcal{B}(1_{n-1})} K_{\lambda,0}(q) - K_{\gamma^p,0}(q) = (q-1)K_{1^p,0}(q) + qK_{(1^p)_{n-1},0} + qK_{(1^p-2)_{n-1},0}.
\]

Proof. (of Corollary 3.2.4) We proceed by induction on \(n\). For \(n = 2\), \(K_{(1^2),0}(q) = q^2\). Suppose the corollary true for \(k < n\). Then by applying Lemma 3.2.5 we obtain
\[
K_{(1^2)_n,0}(q) = (q-1)K_{(1^2)_{n-1},0}(q) + qK_{(1^2)_{n-1},0}(q) + q.
\]
It follows from Proposition 3.2.3 that
\[
K_{(1^2)_{n-1},0}(q) = \sum_{i=1}^{n-1} q^{2i-1}.
\]
Thus
\[
K_{(1^2)_n,0}(q) = (q-1) \sum_{i=1}^{n-1} q^{2i-1} + q \sum_{i=1}^{n-2} q^{2i} = \sum_{i=1}^{n-1} q^{2i} - \sum_{i=1}^{n-1} q^{2i-1} + \sum_{i=1}^{n-2} q^{2i+1} + q = \sum_{i=1}^{n} q^{2i}.
\]

Note that we can not deduce an explicit formula for \(K_{(1^p)_n,0}(q)\) with \(p > 2\) from the recurrence formula of Lemma 3.2.5 as we have do in Proposition 3.2.3 since we have no explicit formula for \(K_{\gamma^p,0}(q)\) as soon as \(p > 2\). Nevertheless we will give a conjectural general formula for \(K_{(1^p)_n,0}(q)\) in Section 5.
4 Cyclage graphs for symplectic tableaux

Given a symplectic tableau \( T \in ST(n) \), we can factorize \( w(T) \) in a unique way by setting \( w(T) = xu \) where \( u \) is a word and \( x \) is a letter. It is easy to verify that \( u \) is also the reading of a symplectic tableau, say \( T_0 \in ST(n) \). The initial cyclage operation on \( T \) consists in the insertion \( x \to T_0 \). We are going to see that all the initial cocyclage operations are not relevant for defining a charge.

It follows from Paragraph 2.3 that the tableau obtained by cocycling a tableau \( T \in ST(n) \) does not belong to \( ST(n) \) in general but belongs to \( ST(n+1) \). To overcome this problem we are going to define our cyclage operation directly on the complete symplectic tableaux set \( ST = \bigcup_{n \geq 1} ST(n) \).

4.1 Cocyclage operation

Set \( C_\infty = \bigcup_{n \geq 1} C_n \). Then \( C_\infty \) is totally ordered by \( \leq \). Given any \( T \in ST \) there exists an integer \( m \geq 1 \) such that \( T \in ST(m) \). Recall that \( d_T \) is the number of letters \( i \) of \( T \) minus the number of letters \( i \). For any weight \( \mu \in P_n \), we will say that \( T \in ST \) is a tableau of weight \( \mu \) if \( T \in ST(m) \) with \( m \geq n \), \( d_T = 0 \) for \( i > n \) and \( d_T = \mu_T \) for \( k = 1, \ldots, n \). For any \( \mu \in P_n \), the set of tableaux of weight \( \mu \) is denoted \( ST(\mu) \). If \( T \in ST(\mu) \), the number of letters \( k \) with \( k > n \) that belong to \( T \) is equal to the number of letters \( \overline{k} \).

Let \( w \in C_\infty \) and write \( w = xu \) with \( x \) a letter and \( u \in C^*_\infty \). The cocyclage shift \( \xi \) is the map defined on \( C_\infty \) by \( \xi(w) = ux \).

Lemma 4.1.1 For any \( n \geq 1 \), \( \sigma \in W_n \), and \( w \in C^*_\infty \), \( \xi(\sigma(w)) = \sigma(\xi(w)) \).

Proof. The proof is analogous to that of Proposition 5.6.1 of [10].

Consider a symplectic tableau \( T = C_1 \cdots C_r \in ST(m) \) with \( r \geq 1 \). We will say that the cocyclage operation is authorized for \( T \) if there is no letter \( y \in C_m \) such that \( y \in C_i \) for any \( i = 1, \ldots, r \) and \( \overline{\gamma} \notin T \). It means that the cocyclage operation is not authorized for \( T \) when there exits an integer \( p \in \{1, \ldots, m\} \) such that \( |d_{\overline{\gamma}}| \) is equal to \( r \). If the cocyclage operation is authorized for \( T \), we write \( w(T) = xw(T_0) \) where \( T_0 \in ST(m) \) and \( x \in C_m \), and we set \( U(T) = x \to T_0 \).

Remarks:
(i): \( U(T) \) belongs to \( ST \) and \( wt(U(T)) = wt(T) \). More precisely when \( T \in ST(m) \), \( U(T) \in ST(m) \) if the heights of the first columns of \( T \) and \( U(T) \) are equal, \( U(T) \in ST(m+1) \) otherwise.
(ii): If \( wt(T) = 0 \) then the cocyclage operation is always authorized.
(iii): There is no cocyclage operation on the columns.

Example 4.1.2 Consider the tableaux \( T_1 = \begin{bmatrix} 4 & 3 & 2 \\ 2 & 2 & 1 \\ 2 \end{bmatrix}, T_2 = \begin{bmatrix} 4 & 3 & 4 \\ 2 & 2 \end{bmatrix} \) and \( T_3 = \begin{bmatrix} 4 & 3 & 2 \\ 2 & 2 & 1 \end{bmatrix} \). Then the cocyclage operation is authorized for \( T_1 \) and \( T_2 \) but not in \( T_3 \). We obtain \( U(T_1) = \begin{bmatrix} 4 & 3 & 1 \\ 3 & 2 \end{bmatrix}, U(T_2) = \begin{bmatrix} 4 & 2 \\ 2 & 3 \\ 2 \end{bmatrix} \).

Lemma 4.1.3 Suppose \( T \in ST(m) \) and consider \( \sigma \in W_m \). Then the cocyclage operation is authorized for \( T \) if and only if it is authorized for \( \sigma(T) \). In this case we have \( U(\sigma(T)) = \sigma(U(T)) \).

Proof. The lemma directly follows from Lemma 4.1.1 and (11).

4.2 Cyclage graphs

We endow the set \( ST \) with a structure of graph by drawing an array \( T \to T' \) if and only if the cocyclage operation is authorized on \( T \) and \( U(T) = T' \). Write \( \Gamma(T) \) for the connected component containing \( T \). Let \( t \) be the translation operation on letters of \( C_\infty \) defined by \( t(k) = k+1 \) and \( t(\overline{k}) = \overline{k+1} \) for \( k \geq 1 \). We write \( t(w) \) (resp. \( t(T) \)) for the word (resp. the tableau) obtained by applying \( t \) to each letter of \( w \in C^*_\infty \) (resp. to each letter of \( T \in ST \)).
Lemma 4.2.1

(i) Suppose $T \in \text{ST}(m)$. Then $\Gamma(T)$ and $\Gamma(\sigma(T))$ are isomorphic for any $\sigma \in W_m$.

(ii) The cycle graphs $\Gamma(T)$ and $t(\Gamma(T))$ are isomorphic.

(iii) Suppose that $T_1 \neq T_2 \in \Gamma(T)$ are such that $U(T_1) = U(T_2) = T$. Then $T_1$ and $T_2$ have different shapes.

Proof. Assertion (i) follows immediately from Lemma 4.1.3.

Let $w_1$ and $w_2$ be two words of $C_m$. Then $w_1 \equiv_m w_2$ if and only if $t(w_1) \equiv_{m+1} t(w_2)$. This implies that $P(t(w)) = t(P(w))$ for any word $w \in C_m$. Hence $t$ commutes with $U$. Since $t$ is a bijection, it is also an isomorphism between $\Gamma(T)$ and $t(\Gamma(T))$ which proves (ii).

Suppose that $T_1, T_2 \in \text{ST}(m)$. Write $w(T_1) = w_1 w(R)$ and $w(T_2) = w_2 w(S)$ with $w_1, w_2$ two letters and $R, S$ two symplectic tableaux. Then $w(R)x \equiv_{m+1} w(S)y$ since $P(w(R)x) = P(w(S)y) = T$. Suppose that $T_1$ and $T_2$ have the same shape. Then $R$ and $S$ have the same shape $Y$. The highest weight vertices of the connected components of $G_{m+1}$ containing $w(R)x$ and $w(S)y$ may be respectively written $w(Y_0)x_0$ and $w(Y_0)y_0$ where $Y_0 \in \text{ST}(m + 1)$ is the highest weight tableau of shape $Y$. The congruence $w(R)x \equiv_{m+1} w(S)y$ implies the congruence $w(Y_0)x_0 \equiv_{m+1} w(Y_0)y_0$. Thus we must have $wt(w(Y_0)x_0) = wt(w(Y_0)x_0)$. It means that $x_0 = y_0$. Hence $w(R)x$ and $w(S)y$ are congruent and belong to the same connected component. This implies that $w(R)x = w(S)y$, thus $xw(R) = yw(S)$ and $T_1 = T_2$. So (iii) is proved.

Assertion (i) of the above lemma permits to restrict to the cycle graphs $\Gamma(T)$ with $T \in \text{ST}(\mu)$ and $\mu \in P^+_n$.

Suppose first that $\mu = 0$. We have seen that the cocycle operation is always authorized on symplectic tableaux of weight 0 with at least two columns. So we can define from $T$ a sequence $(T_n)$ of symplectic tableaux by setting $T_0 = T$ and $T_{k+1} = U(T_k)$ while $T$ is not a column.

Proposition 4.2.2 The sequence $(T_n)$ is finite without repetition and there exists an integer $e$ such that $T_e$ is a column of weight 0.

To prove this proposition we need two technical lemmas. Given two words $w_1, w_2 \in C^*_\infty$, write $w_1 \prec w_2$ if $w_1$ and $w_2$ can respectively be written $w_1 = u_1 x_1 v$ and $w_2 = u_2 x_2 v$ where $u_1, u_2, v \in C^*_\infty$ and $x_1, x_2 \in C_\infty$ verify $x_1 < x_2$. It means that $\prec$ is the inverse lexicographic order on words of $C^*_\infty$. For any symplectic tableau $T$ with $r > 1$ columns, we denote by $N_r(T)$ the number of boxes belonging to the $r - 1$ rightmost columns of $T$.

Lemma 4.2.3 Consider $\mu \in P^+_n$ and $\tau \in \text{ST}(\mu)$ a tableau with $r > 1$ columns. Let $T, T'$ two tableaux of $\Gamma(\mu)$ such that $T = U^{(i)}(\tau)$ and $T' = U^{(i+1)}(\tau)$ with $i \geq 0$ an integer. Then the following assertions hold.

1. $T$ and $T'$ contains at most $r + 1$ columns.

2. Suppose that $T$ contains $r$ or $r + 1$ columns and $N_r(T) = N_r(T')$. Then only one of the following situations can happen:

   (i) : $T$ and $T'$ contain $r$ columns and their $r$-th columns have the same height.

   (ii) : $T$ contains $r$ columns and $T'$ contains $r + 1$ columns.

   (iii) : $T$ and $T'$ contains $r + 1$ columns.

   (iv) : $T$ contains $r + 1$ columns, $T$ contains $r$ columns and the height of the last column $C'_r$ of $T'$ is equal to $h(C_r) + 1$.

   Moreover in each case we can write $w(T) = x_* w(T_*)$ and $w(T') = x'_* w(T'_*)$ with $w(T_*) \prec w(T_*)$.

3. Suppose that $T$ contains $r$ or $r + 1$ columns and $N_r(T) \neq N_r(T')$. Then $N_r(T) > N_r(T')$.

Proof. 1 : Let $j$ be an integer such that $U^{(i)}(\tau)$ is defined and contains $r + 1$ columns. Then the height of its last column is equal to 1. The assertion follows immediately.

2 : In case (i) the box which is added to $T_*$ during the insertion $x_* \rightarrow T_*$ appears on the bottom of the last column $C_{r,*}$. This insertion can be written $x_* \rightarrow T_* = (x_* \rightarrow C_1 \cdots C_{r-1})C_{r,*} = (C'_1 \cdots C'_{r-1}) (y \rightarrow C_{r,*})$ that is, $x_*$ is first inserted in the sub-tableau composed of the $r - 1$ leftmost columns of $T_*$ which gives a new tableau $C'_1 \cdots C'_{r-1}$ and a letter $y$. This letter is then inserted on the bottom of $C_{r,*}$. Suppose that there exists an integer $i \in \{1, ..., r - 1\}$ such that $C'_i \neq C_i$. Then if we choose $i$ minimal we have $w(C'_i) \prec w(C_i)$ by (ii) and finally $w(T'_*) \prec w(T_*)$. Now if $C'_i = C_i$ for $i = 1, ..., r - 1$ we have $y = x_*$ and $x_* \in C_i$, $x_* \notin C_i$ for any $i = 1, ..., r - 1$. So the letter $x_*$ belongs to all the columns of $T$. Then $x_*$ is a barred letter since
\( \mu \in P_n^+ \) and \( r > 1 \). Moreover \( \overline{x}_s \not\in C_r^* \) for \( x_s \to C_r^* \) is a column. Thus \( \overline{x}_s \not\in T \). This contradicts the fact that the cocyclage operation is authorized for \( T \).

In case (ii) a new column of height 1 is added to the shape of \( T \). The insertion can be written \( x_s \to T_s = (x_s \to C_1 \cdots C_{r-1})C_{r,s} = ((C'_1 \cdots C'_{r-1})(y \to C_{r,s}) = C'_1 \cdots C'_{r-1}C_{r,s} x_s \) that is \( x_s \) is inserted in the sub-tableau composed of the \( r-1 \) leftmost columns of \( T \) which gives a new tableau \( C'_1 \cdots C'_{r-1} \) and a letter \( y \). This letter is then inserted in \( C_{r,s} \) which gives the column \( C_{r,s}' \) and the letter \( x_s' \). If \( y \neq x_s \), we terminate as in case (i). Otherwise we have \( C'_i = C_i \) for any \( i = 1, \ldots, r-1 \). We can not have \( x_s' = x_s \), since it would imply that \( x_s \in C_{r,s} \) which is impossible since \( C_r \) can not contain two letters \( x_s \). Thus \( x_s' > x_s \), \( w(C_{r,s}') < w(C_{r,s}) \) and finally \( w(T'_s) < w(T_s) \).

Case (iii) is similar to case (i) with \( h(C_r) = 1 \).

In case (iv) \( C_{r+1} \) contains only the letter \( x_s \) and a new box appears on the bottom of the column \( C_r \) of \( T \) during the insertion \( x_s \to T_s \). The insertion can be written \( x_s \to T_s = x_s \to (C_1 \cdots C_r) = (C'_1 \cdots C'_{r-1})(y \to C_{r,s}) = C'_1 \cdots C'_{r-1}C_{r,s} \), that is \( x_s \) is inserted in the sub-tableau composed of the \( r-1 \) leftmost columns of \( T \) which gives the tableau \((C'_1 \cdots C'_{r-1})\) and the letter \( y \). This letter is then inserted on the bottom of \( C_{r,s} \) which gives the column \( C_{r,s}' \). Suppose that \( y = x_s \). We must have \( x_s \in C_i \) and \( \overline{x}_s \not\in C_i \) for any \( i = 1, \ldots, r-1 \). Then \( x_s = \overline{x} \) with \( q \geq 1 \) that is, is a barred letter as in (i). Moreover \( \overline{x}_s \not\in C_i \) because \( x_s \to C_r \) is a column. Thus \( d_{\overline{x}}(w(T)) = r \). Now \( T \) contains \( r+1 \) columns, hence \( T \not\equiv \tau \). Let \( j \) minimal such that \( R = U^{(r-j)}(\tau) \) contains \( r \) columns. Then \( d_{\overline{x}}(w(R)) = d_{\overline{x}}(w(T)) = r \). Thus the cocyclage operation in not authorized in \( R \) and we obtain a contradiction. It means that \( y \neq x_s \). So we can terminate as in case (i).

3 : It is clear from the definition of the cocyclage operation.

Lemma 4.2.4 Let \( T = C_1 \cdots C_p \in ST(0) \) with \( r > 1 \) columns. Then there exists an integer \( k \) such that \( T_k \) has at most \( r-1 \) columns. Moreover if \( k \) is minimal the sequence \( T_0, \ldots, T_k \) is without repetition.

Proof. Let \( m \geq 1 \) be an integer such that \( T_0, \ldots, T_m \) have \( r \) or \( r+1 \) columns and \( N_r(T_i) = N_r(T) \) for any \( i = 1, \ldots, m \). Then by Lemma 4.2.3 we can write \( w(T_i) = x_i w(T_{i,s}), i = 0, \ldots, m \) with \( w(T_{m,s}) < \cdots < w(T_{1,s}) < w(T_{0,s}) \). (16)

This implies that the sequence \( T_0, \ldots, T_m \) is without repetition. Now suppose that \( T \in ST(n) \). Then \( T_i \in ST(n) \) for any \( i = 0, \ldots, m \). Indeed the height of the first column of any tableau \( T_i, i = 0, \ldots, m \) is always equal to that of \( T_0 \) since \( N_r(T_i) = N_r(T) \) (see Remark (i) after Lemma 1.11). Denote by \( p \) the number of boxes in \( T_0 \). Since the number of symplectic tableaux with \( p \) boxes belonging to \( ST(n) \) is finite there exits an integer \( s_1 \) minimal such that \( N_r(T_{s_1}) = N_r(T) + 1 \) or \( T_{s_1} \) has at most \( r-1 \) columns. Then the sequence \( T_0, \ldots, T_{s_1} \) is without repetition. If \( T_{s_1} \) has at most \( r-1 \) columns we take \( k = s_1 \). Otherwise \( T_{s_1} \) has \( r \) or \( r+1 \) columns and we can obtain similarly starting from \( T_{s_1} \) an integer \( s_2 \) minimal such that \( N_r(T_{s_2}) = N_r(T_{s_1}) + 1 \) or \( T_{s_2} \) has at most \( r-1 \) columns. The sequence \( T_{s_1}, \ldots, T_{s_2} \) is without repetition. Then the sequence \( T_0, \ldots, T_{s_2} \) is also without repetition. Indeed a tableau \( T_i \) with \( i \in \{0, \ldots, s_2 - 1 \} \) can not be equal to a tableau \( T_j \) with \( j \in \{s_1, \ldots, s_2 - 1 \} \) since \( N_r(T_i) \neq N_r(T_j) \). By induction we can construct \( T_{s_2+1} \) from \( T_{s_2} \) while \( s_2 + 1 \) columns such that \( N_r(T_{s_2+1}) = N_r(T_{s_2}) + 1 \) and the sequence \( T_0, \ldots, T_{s_2+1} \) is without repetition. The procedure terminates since the number of boxes belonging to the columns \( r \) and \( r+1 \) decreases by 1 to each step. So the lemma is proved.

Proof. (of Proposition 4.2.2) Let \( r > 1 \) be the number of columns of \( T \). By Lemma 4.2.3 we can obtain from \( T = T_0 \) a tableau \( T_k \) with at most \( r-1 \) columns and such that the sequence \( T_0, \ldots, T_k \) is without repetition. If \( r-1 > 1 \) we can obtain a tableau \( T_k \) at \( T_k \), with at most \( r-2 \) columns and such that the sequence \( T_0, \ldots, T_k \) is without repetition. We can define \( T_{k+1} \) from \( T_k \) while \( r-s > 1 \) such that the sequence \( T_0, \ldots, T_{k+1} \) is without repetition. It is clear that the procedure terminates when \( T_k = T_r \) is a column of weight 0.

It follows from Proposition 4.2.2 that \( wt(T_1) = wt(T_2) \iff \Gamma(T_1) = \Gamma(T_2) \) in general. For example all the columns of weight 0 occur in different connected components.

We give below \( \Gamma \left( \begin{array}{cccc} 1 & 1 & 1 & 1 \end{array} \right), \Gamma \left( \begin{array}{cccc} 3 & 3 & 2 & 1 & 1 \end{array} \right), \Gamma \left( \begin{array}{ccc} 3 & 1 & \ \ 2 & 2 & \ 1 & 3 \end{array} \right) \) and \( \Gamma \left( \begin{array}{cc} 2 & 1 \ 1 & 2 \end{array} \right) \).
Remarks:
(i): Given \( T' \in \text{ST} \), it is possible to find the tableaux \( T \) (if there is any) such that \( U(T) = T' \). To do this we find all the pairs \((x, T_*)\) obtained by applying the reverse insertion algorithm on the outside corners of \( T' \). By definition of \( U \), the tableaux \( T \) are precisely those which verify \( w(T) = xw(T_*) \) for a pair \((x, T_*)\). They are determined by the pairs \((x, T_*)\) for which \( xw(T_*) \) is the reading of a symplectic tableau. For example \( T' = \begin{array}{c}
2 \\
1 \\
3 \\
\end{array} \) has only one outside corner which gives \( x = \bar{1} \) and \( T_* = \begin{array}{c}
3 \\
1 \\
1 \\
\end{array} \). There is no tableau \( T \) such that \( U(T) = T' \) since \( \bar{1}(T1) \) is not the reading of a symplectic tableau.

(ii): In the definition of \( U(T) \) we have restricted to the initial cocyclages. For type \( A \) the cyclages graphs take also into account non initial cyclages. When \( T \) is of dominant evaluation, they are obtained by considering all the factorizations \( w(T) = yw(Y) \) in the plactic monoid with \( y \neq 1 \) a letter and \( Y \) a semi-standard tableau. The use of non initial cocyclages with symplectic tableaux is problematic because it can make appear loops in the cyclages graphs. For example consider \( Z = \begin{array}{c}
3 \\
1 \\
\end{array} \). Then \( w(Z) = (3\bar{3}(T)1) \equiv (2\bar{T}1) \equiv \bar{T}21 \). Now if we compute \( P(\xi(\bar{T}21)) \) we obtain \( Z' = P(T21\bar{T}) = \begin{array}{c}
2 \\
1 \\
\end{array} \). So we have a loop since \( U(3)(Z') = Z \) (see the cyclage graph \( \Gamma(\begin{array}{c}
1 \\
1 \\
1 \\
1 \\
\end{array}) \) above).

(iii): For type \( A \), every semi-standard tableau belongs to the cyclage graph containing a row tableau. By considering \( \Gamma(\begin{array}{c}
2 \\
1 \\
1 \\
2 \\
\end{array}) \) we see that such a property is false with the symplectic tableaux even if we consider non initial cyclages. This explains why we have to consider the cocyclage operation and not the cyclage one.

4.3 Reduction operations

Consider \( T \in \text{ST}(\mu) \) with \( \mu \in P^+_n \). If the cocyclage operation is not authorized for \( T \), then \( T \) do not contain any letters \( n \). Indeed there exists \( p \in \{1, ..., n\} \) such that \( \mu_\bar{p} = \mu_p \) is equal to the number of columns of \( T \). Thus \( p_\bar{p} = \mu_\bar{p} \), \( p_\bar{p} \geq \mu_P \), so each column of \( T \) contains a letter \( \bar{p} \) and no letter \( n \). Let \( T_\bar{p} \) be the tableau obtained first by erasing the letters \( \bar{p} \) in \( T \) next by applying \( t \) to the letters \( x \in T \) such that \( \bar{p} = x < n \). It is easy to verify that \( T_\bar{p} \in \text{ST}(\mu') \) with \( \mu' = (\mu_{\bar{n}-1}, ..., \mu_{\bar{p}}, 0) \in P_n^+ \). Now if the cocyclage operation is not authorized on \( T_\bar{p} \), we can compute \( (T_\bar{p})_\bar{p} \) and so on until obtain a symplectic tableau \( \bar{T} \) which is either a column (eventually empty) or a symplectic tableau for which the cocyclage operation is authorized. We will say that \( \bar{T} \) is obtained by reduction operations from \( T \). By convention we set \( \bar{T} = T \) if the cocyclage operation is already defined for the symplectic tableau \( T \).

Remark: When a reduction operation is done in \( T \in \text{ST}(\mu) \) with \( \mu \in P^+_n \), \( \mu_\bar{p} = \mu_{\bar{p}} \) is equal to the first part of the shape of \( T \). To define a charge statistic on symplectic tableaux related to Kostka-Foulkes polynomials, it seems natural by Lemma 2.1.2 to impose that the charges associated to \( T \) and \( \bar{T} \) should be equal as we will do in Section 5.

From \( T \in \text{ST}(\mu) \) we can compute a sequence of symplectic tableaux by setting \( T_0 = T \) and

\[ T_{k+1} = U(\bar{T}_k) \]

while \( \bar{T}_k \) is not a column.

Proposition 4.3.1 The sequence \( (T_n) \) is finite without repetition and the last symplectic tableau obtained is a column of weight 0 (eventually empty).

Proof. Suppose first that there is a loop in the sequence \( (T_n) \) that is, there exits two integers \( k \) and \( s \) such that \( T_k = T_{k+s} \). Then \( T_i = \bar{T}_i \) for any \( i = k, ..., k + s - 1 \). Choose \( p \in \{k, ..., k + s - 1\} \) such that the number of columns of \( T_p \) is minimal among all the tableaux \( T_i \), \( i = k, ..., k + s - 1 \). Denote by \( r \) the number of columns of \( T_p \). Then by assertion 1 of Lemma 1.2.4 \( T_{k+s} \) contain \( r \) or \( r+1 \) columns since every \( T_i \), \( i = k, ..., k+s-1 \) can be obtained by cocyclage operations from \( T_p \). We must have \( N_r(T_k) \leq \cdots \leq N_r(T_{k+s}) \). This
implies that $N_r(T_k) = \cdots = N_r(T_{k+s-1})$ for $T_{k+s} = T_k$. Then by assertion 2 of Lemma \ref{lem3.7} we can write

$$w(T_i) = x_i w(T_{i+1}), i = k, \ldots, k + s$$

with

$$w(T_{k+s}) < \cdots < w(T_s).$$

We obtain a contradiction since $w(T_{k+s}) = w(T_s)$. It means that there is no loop in the sequence $(T_n)$. Hence this sequence is without repetition.

Now suppose that this sequence is infinite. Then there exits an integer $a$ such that the sequence $(T_{n+a})_{n \geq 0}$ is infinite without reduction operation. In the proof of Lemma \ref{lem3.7} the hypothesis $\mu = 0$ is only used to assure that the sequence of the cocycled tableaux is defined. It means that this lemma is yet true for the sequence $(T_{n+a})_{n \geq 0}$. Thus we can define by induction as in proof of Proposition \ref{lem4.2.2} an infinite sequence of tableaux $(T_v)_{\mu \geq 0}$ such that $T_{v_0} = T_{n+a}$ and for any $j, T_{v_{j+1}}$ has one column less than $T_{v_j}$. We derive a contradiction since the number of columns of $T_a$ is finite. It means that the sequence $(T_n)$ is finite.

Finally $T_a$ the last tableau of this sequence is necessarily a column such that $T_a = T_u$ that is, $T_u$ is a column of weight 0. ■

**Example 4.3.2** The cocycle operation is not authorized for $T = \begin{array}{c}
3 \\
2 \\
1 \\
\end{array}$. We have $\hat{T} = \begin{array}{c}
3 \\
2 \\
1 \\
\end{array}$. Then $T_1 = \begin{array}{c}
3 \\
2 \\
\end{array}$ and $T_2 = \begin{array}{c}
3 \\
\end{array}$.

### 4.4 Embedding of cycle graphs

Each connected component $\Gamma(T)$ contains a tableau $Y$ which admits no cocycle. This tableau is necessarily unique. Suppose that $Y \in \mathbf{ST}(m)$. Then $\Gamma(T) \subset \mathbf{ST}(m)$ thus is finite. Moreover for any $Z \in \Gamma(T)$ there exits an integer $k$ such that $U(k)(Z) = T$. This means that $\Gamma(T)$ has a tree structure.

**Proposition 4.4.1** Let $\mu \in P_n$ and consider $T_{\mu} \in \mathbf{ST}(\mu)$. Suppose that there exists $j \leq i \leq n$ such that $\mu' > \mu'_j \geq 0$. Set $\nu \in P_n$ defined by $\nu_k = \mu'_k$ for $k \neq i,j$, $\nu_i = \mu'_i - 1$ and $\nu_j = \mu'_j + 1$. Then there exists a tableau $T_{\nu} \in \mathbf{ST}(\nu)$ and a unique embedding from $\Gamma(T_{\mu})$ to $\Gamma(T_{\nu})$ which commutes with $U$ and preserves the shape of the tableaux.

**Proof.** We have seen that $\Gamma(T_{\mu})$ has a finite number of vertices. Write $m \geq n$ for the lowest integer such that $\Gamma(T_{\mu})$ is contained in $\mathbf{ST}(m)$. By abuse of notation we also denote $\mu$ and $\nu$ the weights of $P_m$ defined by $\mu = (0, \ldots, 0, \mu_{\overline{m}}, \ldots, \mu_{\overline{1}})$ and $\nu = (0, \ldots, 0, \nu_{\overline{m}}, \ldots, \nu_{\overline{1}})$. Let $\sigma \in W_m$ such that $\sigma(\overline{m}) = \overline{m}$, $\sigma(j) = \overline{m} - 1$ and $\sigma(k) = \overline{k}$ for $k \neq i,j$. Set $\sigma(\mu) = \mu'$ and $\sigma(\nu) = \nu'$. Write $T_{\mu'} = \sigma(T_{\mu})$. We have $\Gamma(T_{\mu'}) \subset \mathbf{ST}(m)$. Then for any $T \in \Gamma(T_{\mu'})$, $\tilde{f}_{m-1}(w(T)) \neq 0$. Indeed the crystal graph $G^{(m-1)}_{\nu_{\overline{m}}}$ obtained by erasing in $G_{\nu_{\overline{m}}}$ all the arrows of color $i \neq m - 1$ and all the letters $x \notin \{ \overline{m}, \overline{m-1}, m-1, m \}$ is a $U_q(sl_2)_{m-1}$-crystal where $U_q(sl_2)_{m-1}$ is the sub-algebra of $U_q(sl_2)_{\overline{m}}$ isomorphic to $U_q(sl_2)_{m-1}$ generated by $e_{m-1}, f_{m-1}$ and $t_{m-1}$. The vertex $w(T)_{m-1}$ of $G^{(m-1)}_{\nu_{\overline{m}}}$ obtained from $w(T)$ is of weight $(\mu_{\overline{m}}', \mu_{\overline{m}}') \neq 0$. Since $\mu'_j > \mu'_j \geq 0$ we have $\mu_{\overline{m}}' > \mu_{\overline{m}}' \geq 0$. Thus $w(T)_{m-1}$ is a highest weight vertex and there is an arrow of color $m - 1$ and length $\mu_{\overline{m}}' - \mu_{\overline{m}}'$ which starts from $w(T)$.

Now consider $T \in \Gamma(T_{\mu'})$ such that $U(T) = T'$ is defined. Write $w(T) = x_s w(T_s)$. We must have $x_s = \overline{m}$ since the cocycle operation is authorized for $T$. Moreover $x_s \neq m$. Otherwise the first column of $T'$ would contain the letters $m$ and $\overline{m}$ (because $\mu_{\overline{m}}'$ > 0) and $T' \notin \mathbf{ST}(m)$. We are going to prove that

$$\tilde{f}_{m-1}(x_s w(T_s)) = x_s \tilde{f}_{m-1}(w(T_s)).$$

It suffices to establish (18) for $x_s = m - 1$. When $x_s = m - 1$ there is no letter $\overline{m} - 1$ in the second row of $T$ since $T' \in \mathbf{ST}(m)$. Thus all the letters $\overline{m} - 1$ of $T$ belong to its first row. Denote by $T_1$ the sub-tableau of $T$ containing all the columns whose the lowest letter is $\overline{m}$. The tableau $T_1$ can be regarded as the juxtaposition $T_1 T_2$ of the tableaux $T_1$ and $T_2$ where $T_2$ is the sub-tableau obtained by considering the columns of $T$ which do not occur in $T_1$. Then $T_1$ do not contain any letter $m$ or $\overline{m}$ and $T_2$ do not contain any letter $\overline{m}$. Suppose that $\tilde{f}_{m-1}(x_s w(T_s)) = \tilde{f}_{m-1}(x_s) w(T_s)$. Then with the notation of Note \ref{lem4.2.2} we can write $\rho(w(T_2)) = (+)^s$ since
\[ x_s = m - 1 = + \text{ is not ignored during the encoding procedure. The pairs } (+-) \text{ ignored are pairs } (m-1, m) \text{ or } (m-1, m) \text{ for } m \text{ do not belong to } w(T_2). \text{ Since } \rho(w(T_2)) \text{ contains only symbols } +, \text{ all the letters } m-1 \text{ can be paired with letters } m-1. \text{ Thus the number of letters } m-1 \text{ in } w(T_2) \text{ is strictly greater than that of letters } m-1. \text{ It is also true for } w(T) \text{ because } w(T_1) \text{ does not contain any letter } m-1. \text{ This contradicts the inequality } \mu_{m-1}' \geq 0. \text{ Thus } [13] \text{ is true.}

Denote by \( V \) the symplectic tableau of reading \( \tilde{f}_{m-1}(w(T'_n)) \). \text{ We are going to prove that } \Psi : \Gamma(T'_n) \to \Gamma(V) \text{ defined by } \Psi(T) = S \text{ if and only if } w(S) = \tilde{f}_{m-1}(w(T)) \text{ is an embedding which commutes with } U \text{ and preserves the shape of the tableaux.} \text{ We have } \Psi(U(T)) = P \left( \tilde{f}_{m-1}(w(T_1)x_s) \right). \text{ Suppose that there exists } p \in \{1, \ldots, m\} \text{ such that } \nu'_p \text{ is equal to } r \text{ the number of columns of } S. \text{ Then since } U(T) \text{ is defined and } \nu'_p = \mu'_k \text{ for any } k \neq m, m-1 \text{ we must have } p \in \{m, m-1\}. \text{ If } \nu'_p = r \text{ then we obtain } \mu'_p = r+1 \text{ which is impossible for } T \text{ contains only } r \text{ columns. If } \nu'_m = r \text{ then we obtain } \mu'_m \geq r \text{ since } r \text{ and } \mu'_m > \mu'_{m-1}. \text{ This contradicts the fact that the cocyclage operation is authorized for } T. \text{ Hence the cocyclage operation is authorized for } S \text{ and we can write}

\[ U(S) = U(\Psi(T)) = P \left( \xi(\tilde{f}_{m-1}(x_s w(T_s))) \right) = P \left( \xi(x_s(\tilde{f}_{m-1}(w(T_s)))) \right) = P \left( \tilde{f}_{m-1}(w(T_s))x_s \right). \]

Thus we have to show that

\[ \tilde{f}_{m-1}(w(T_s))x_s = \tilde{f}_{m-1}(w(T_s))x_s. \]

By [11] it is equivalent to

\[ \varphi_{m-1}(w(T_s)) > \varepsilon_{m-1}(x_s) \tag{19} \]

We have seen that \( x_s \neq m \) and for \( x_s \neq m-1 \) [19] is true since \( \varepsilon_{m-1}(x_s) = 0 \) and \( \varphi_{m-1}(w(T_s)) \geq 1 \). Suppose that \( x_s = m-1 \). Then the vertex of \( G^{(m-1)}_{\Gamma(T_s)} \) obtained from \( w(T_s) \) as above is of weight \( (\mu_{m-1}, \mu_{m-1} - 1) \). Thus \( \varphi_{m-1}(w(T_s)) \geq \mu_{m-1} - \mu_{m-1} + 1 \geq 2 \) for \( \mu_{m-1} > \mu_{m-1}. \) So [19] is satisfied. It is clear that \( T \) and \( \Psi(T) \) have the same shape. Moreover by (iii) of Lemma 4.2.1 \( \Psi \) is the unique map from \( \Gamma(T'_n) \) to \( \Gamma(V) \) which commutes with \( U \) and preserves the shape of the tableaux. Finally, using \( \sigma^{-1} \) we obtain from \( \Psi \) a unique embedding \( \Psi_\sigma \) satisfying \( \Psi_\sigma(T) = \sigma^{-1}\Psi(T) \) from \( \Gamma(T'_n) \) to \( \Gamma(T'_n) \) with \( T'_n = \sigma^{-1}\Psi(T'_n). \)

Note that \( T'_n \) is not unique in general since \( \Gamma(T'_n) \) may contain fewer connected components isomorphic to \( \Gamma(T'_n). \)

**Corollary 4.4.2** Let \( \mu \in P_n^+ \) and \( T_\mu \in ST(\mu) \). Write respectively \( m \) and \( p \) for the sum of the non zero parts and the number of zero parts in \( \mu \). Define \( \kappa \in P_n^{m+p} \) by \( \kappa_i = 1 \) for \( p+1 \leq i \leq m+p \) and \( \kappa_i = 0 \) otherwise. Then there exists a tableau \( T_\kappa \in ST(\kappa) \) and a unique embedding of \( \Gamma(T_\mu) \) into \( \Gamma(T_\kappa) \) which commutes with \( U \) and preserves the shape of the tableaux.

**Proof.** The corollary directly follows by composing embeddings obtained in the previous Proposition. ■

**Example 4.4.3** If \( n = 3 \) and \( \mu = (2, 1, 0) \) then \( \kappa = (1, 1, 1, 0) \). The cyclage graph \( \Gamma(\begin{array}{cccc} 3 & 3 & 2 & 1 \\ 3 & 2 & 1 & 1 \end{array}) \) of [17] may be uniquely embedded in \( \Gamma(\begin{array}{cccc} 4 & 3 & 2 & 1 \\ 3 & 2 & 1 & 1 \end{array}) \).
5 A charge for symplectic tableaux

5.1 Definition of $\text{ch}_n$

**Definition 5.1.1** Let $C$ be a column of weight 0. Write $E_C = \{ i \geq 1, i \in C, i + 1 \notin C \}$. The charge $\text{ch}_n(C)$ of the column $C$ is

$$\text{ch}_n(C) = 2 \sum_{i \in E_C} (n - i).$$

Note that for any column $T$ of weight 0,

$$\text{ch}_{n+1}(t(C)) = \text{ch}_n(C) + 2\text{card}(E_C)$$

and $\text{ch}_{n+1}(t(C)) = \text{ch}_n(C)$.

Moreover for any $i \geq 1$, we have $\varepsilon_i(w(C)) = \begin{cases} 1 \text{ if } i \in E_C \\ 0 \text{ otherwise} \end{cases}$. Thus for any $n$-admissible column $C$ of weight 0

$$\text{ch}_n(C) = 2 \sum_{i=1}^{n-1} (n - i)\varepsilon_i(w(C)). \quad (20)$$

Now consider $T \in \text{ST}(\mu)$ with $\mu \in P^+_n$. Let $\{T_0, ..., T_p\}$ with $T_p = C_T$ a column of weight 0 be the above sequence defined from $T$.

**Definition 5.1.2** The charge $\text{ch}_n(T)$ is

$$\text{ch}_n(T) = \text{ch}_n(C_T) + p.$$  

For any tableau $T$ we have by Lemma 12.1

$$\text{ch}_{n+1}(t(T)) = \text{ch}_n(T).$$

5.2 Conjectures

**Conjecture 5.2.1** Let $\Lambda^A_k$ and $\Lambda^A_{n-k}$ be respectively the $k$-th and $(n-k)$-th fundamentals weights of $U_q(sl_n)$. Set $\lambda_k = \Lambda^A_k + \Lambda^A_{n-k}$. Then we have the equality:

$$K_{\Lambda^A_{2k},0}(q) = K_{\Lambda^A_{k},0}(q^2)$$

where $K_{\Lambda^A_{k},0}(q^2)$ is the Kostka-Foulkes polynomial for the root system $A_{n-1}$ corresponding to $\mu = 0$ evaluated in $q^2$.

Write $B(2k)_{0} = \{ b \in B(\Lambda_{2k}), wt(b) = 0 \}$. By identifying $U_q(sl_n)$ with the subalgebra of $U_q(sp_{2n})$ generated by the Chevalley’s generators $e_i, f_i$ and $t_i$, $i = 1,...,n-1$, $B(\Lambda_{2k})$ has a structure of crystal graph for $U_q(sl_n)$ obtained by erasing all the arrows of color 0 that we denote $B^A(\Lambda_{2k})$. This graph decomposes into non isomorphic connected components and in this decomposition $B_0(2k)$ is exactly the set of vertices of weight 0 of the connected component isomorphic to $B^A(\lambda_k)$.

By Theorem 5.1 of [12] we can write

$$K_{\Lambda^A_{k},0}(q^2) = \sum_{w(C) \in B(2k)_{0}} q^{2d'(w(C))}$$

with $d'(w(C)) = \sum_{i=1}^{n-1} (n - i)\varepsilon_i(w(C))$. Then it follows from (20) that Conjecture 5.2.1 is equivalent to the equality

$$K_{\Lambda^A_{2k},0}(q) = \sum_{w(C) \in B(2k)_{0}} q^{\text{ch}_n(C)}.$$  

So by Corollary 5.2.1 this conjecture is true for $k = 1$.

More generally many computations suggest that $\text{ch}_n$ is an analogue for the root system $C_n$ of Lascoux-Schützenberger’s charge on semi-standard tableaux.
**Conjecture 5.2.2** Consider \( \lambda, \mu \in P_n^+ \). Then

\[
K_{\lambda, \mu}(q) = \sum_{w(T) \in B(\lambda)_{\mu}} q^{\text{ch}_n(T)}
\]

where \( B(\lambda)_{\mu} = \{ T \in B(\lambda), \text{wt}(T) = \mu \} \).

**Example 5.2.3**

1. Suppose \( n = 4 \), \( \lambda^{(1)} = (2, 1, 1, 1) \) and \( \mu^{(1)} = (1, 1, 1, 0) \). There are 4 tableaux in \( \text{ST}(4) \) of shape \( \lambda^{(1)} \) and weight \( \mu^{(1)} \). They appear in the cyclage graph of Example 4.4.3. Set \( C = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \\ 1 \end{pmatrix} \). We have \( \text{ch}_4(C) = 0 \) since \( \hat{C} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \). Hence the charges of these 4 tableaux are 1, 2, 3 and 4. This gives the Kostka-Foulkes polynomial \( K_{\lambda^{(1)}, \mu^{(1)}}(q) = q + q^2 + q^3 + q^4 \).

2. Suppose \( n = 3 \), \( \lambda^{(2)} = (2, 2, 0) \) and \( \mu^{(2)} = (0, 0, 0) \). There are 6 tableaux in \( \text{ST}(3) \) of shape \( \lambda^{(2)} \) and weight \( \mu^{(2)} \):

\[
T_1 = \begin{array}{c} 2 \\ 1 \\ 2 \end{array} T_2 = \begin{array}{c} 2 \\ 1 \\ 2 \end{array} T_3 = \begin{array}{c} 3 \\ 1 \\ 3 \end{array} T_4 = \begin{array}{c} 3 \\ 1 \\ 3 \end{array} T_5 = \begin{array}{c} 3 \\ 2 \\ 3 \end{array} \text{ and } T_6 = \begin{array}{c} 3 \\ 2 \\ 3 \end{array}.
\]

Note that \( T_2, T_3 \in \Gamma(\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array}) \) and \( \Gamma(T_1) \) is given in (17). We obtain \( \text{ch}_3(T_1) = 2 + \text{ch}_3(\begin{array}{c} 2 \\ 1 \\ 2 \\ 1 \\ 1 \end{array}) \).

\[
4, \text{ch}_3(T_2) = 4 + \text{ch}_3(\begin{array}{c} 3 \\ 1 \\ 3 \\ 1 \end{array}) = 8 \text{ and } \text{ch}_3(T_3) = 6. \text{ Moreover } T_5 = t(T_1) \text{ and } T_6 = t(T_2). \text{ Thus } \text{ch}_3(T_5) = 4 - 2 = 2 \text{ and } \text{ch}_3(T_6) = 8 - 2 \times 2 = 4. \text{ By an easy computation we obtain } U^{(4)}(T_4) = \begin{pmatrix} 4 \\ 1 \\ 1 \\ 4 \end{pmatrix}. \text{ Hence } \text{ch}_3(T_4) = 4 + 2(2 - 1) = 6. \text{ This gives the Kostka-Foulkes polynomial } K_{\lambda^{(2)}, \mu^{(2)}}(q) = q^2 + 2q^3 + 2q^6 + q^8.
\]

**Remark:**

(i) : Once \( \text{ch}_n \) defined on \( \text{ST} \), it is possible to define \( \text{ch}_n \) for any words of \( C^* \) by setting

\[
\text{ch}_n(w) = \text{ch}_n(P(w)).
\]

Then given \( w_1, w_2 \in C^* \), the congruence \( w_1 \equiv_n w_2 \) implies that \( \text{ch}_n(w_1) = \text{ch}_n(w_2) \), that is \( \text{ch}_n \) is a plactic invariant. We recover a property of the Lascoux-Schützenberger’s charge \( \text{ch}_A \) for type \( \text{A} \) [11] [20]. Nevertheless, it seems difficult to define \( \text{ch}_n \) directly on words as it possible for \( \text{ch}_A \). In [12], the statistic \( \text{ch}_A \) is characterized in terms of the combinatorics of crystal graphs. We have not found such a characterization for the symplectic charge \( \text{ch}_n \).

(ii) : It seems to be impossible to define a simple charge statistic on \( \text{ST}(n) \) by using a cocyclage operation taking into account the contraction relation (8) and relevant for computing Kostka-Foulkes polynomials. Consider for example \( T = \begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array} \) for \( n = 3 \). If we apply cocyclages operations based on the complete insertion scheme (with the contraction relations) we obtain the symplectic tableaux of \( \text{ST}(3) \), \( \begin{array}{c} 1 \\ 1 \\ 1 \\ 2 \\ 1 \\ 2 \end{array} \).
that a charge for $T$ must necessarily be odd and by Corollary 3.2.4 a charge for $\begin{array}{c} 1 \\ 1 \end{array}$ must be even. So we can not deduce the charge of $T$ from that of $\begin{array}{c} 1 \\ 1 \end{array}$ by simply counting the number of cocyclage operations.

(iii) : Lascoux-Schützenberger’s proof of the equality

$$K_{\lambda,\mu}(q) = \sum_{w(T) \in B(\lambda)\mu} q^{ch(T)}$$

for type $A$ is based on the Morris recurrence formula. We have seen that Theorem 3.2.1 can be regarded as an analogue of this formula for type $C_n$. It permits to decompose a Kostka-Foulkes polynomial for type $C_n$ in terms of Kostka-Foulkes polynomials for type $C_{n-1}$. Unfortunately a charge statistic must take into account the contraction relations to be compatible with the decomposition obtained in this way since the partitions $\lambda$ such that $B(\lambda)$ appears in a decomposition of type $B(\gamma) \otimes B(\psi)_{n-1}$ may be such that $|\lambda| < |\mu|$. This is a reason why we are not able to deduce Conjecture 5.2.2 from Theorem 3.2.1.

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