On a Parametric Spectral Estimation Problem

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Abstract: We consider an open question posed in Zhu and Baggio (2017) on the uniqueness of the solution to a parametric spectral estimation problem.

Keywords: Spectral estimation, generalized moment problem, global inverse function theorem, spectral factorization.

1. INTRODUCTION

In this paper, we consider a spectral estimation problem subjected to a generalized moment constraint, a framework pioneered by Byrnes, Georgiou, and Lindquist in Byrnes et al. (2000); Georgiou and Lindquist (2003). The formulation of the problem can be seen as a generalization of earlier work on rational covariance extension, cf. e.g., Kalman (1982); Georgiou (1983); Byrnes et al. (1995, 1998, 2001b), and Nevanlinna–Pick interpolation, cf. Georgiou (1987); Byrnes et al. (2001a) and references therein.

A standard setup of the problem is as follows. Suppose we have a zero-mean wide-sense stationary vector signal $y(t)$ with an unknown spectral density matrix $\Phi(z)$. In order to estimate the spectrum, we perform the following steps.

Step 1. Feed the signal $y(t)$ into a filter bank with a transfer function

$$G(z) = (zI - A)^{-1}B$$

(1)

to get an output $x(t)$. The corresponding time domain representation is just

$$x(t+1) = Ax(t) + By(t).$$

(2)

We have some extra specifications on the system matrices, which include

- $A \in \mathbb{C}^{n \times n}$ is Schur stable, i.e., all its eigenvalues have moduli less than 1;
- $B \in \mathbb{C}^{n \times m}$ is of full column rank with $n \geq m$;
- The pair $(A, B)$ is reachable.

Step 2. Compute an estimate of the steady-state covariance matrix $\Sigma := \mathbb{E}x(t)x(t)^*$ of the state vector $x(t)$; cf. e.g., Ferrante et al. (2012) for such structured covariance estimation problem. Hence we have

$$\int G\Phi G^* = \Sigma,$$

(3)

where the function is integrated on the unit circle $\mathbb{T}$ against the normalized Lebesgue measure, i.e.,

$$\int F := \int_{-\pi}^{\pi} F(e^{i\theta}) \frac{d\theta}{2\pi}.$$ 

This simplified notation will be adopted throughout.

Step 3. Given the estimated $\Sigma > 0$, find a spectral density $\Phi$ such that the generalized moment constraint (3) is satisfied.

We must point out that existence of a bounded and coercive $\Phi$ satisfying (3) is not trivial in general. Such feasibility problem was addressed in Georgiou (2002), see also e.g., Ferrante et al. (2010, 2012). In this paper, we shall always assume the feasibility in the sense explained next. Let $C(\mathbb{T}; \mathcal{S}_m)$ denote the space of $m \times m$ Hermitian matrix-valued continuous functions on the unit circle and let $\mathcal{S}_m$ be the vector space of $n \times n$ Hermitian matrices.

Define the linear operator

$$\Gamma : C(\mathbb{T}; \mathcal{S}_m) \rightarrow \mathcal{S}_n,$$

$$\Phi \mapsto \int G\Phi G^*.$$ 

(4)

We shall denote the image/range of this map by $\text{im} \Gamma$ for short. Then we assume that the covariance matrix $\Sigma \in \text{im} \Gamma$. According to (Ferrante et al., 2012, Proposition 3.1), $\text{im} \Gamma$ is a linear space with real dimension $m(2n - m)$.

Given a positive definite $\Sigma \in \text{im} \Gamma$, there are in general infinitely many spectral densities that would solve (3). The mainstream approach today to remedy such ill-posedness is to first introduce a prior matrix spectral density $\Psi$, which represents our guess of the “true” density $\Phi$. Then one tries to define an entropy-like distance index $d(\Phi, \Psi)$ between two spectral densities, and to find the “best” $\Phi$ by solving the constrained optimization problem

$$\text{minimize } d(\Phi, \Psi) \text{ subject to } (3),$$

$\Phi \in \mathcal{S}_m$.

where $\mathcal{S}_m$ is the family of $m \times m$ bounded and coercive spectral densities. Due to the page limit, we refer the readers to e.g., Zhu and Baggio (2017) for a brief review of the literature in this direction.

In this work however, we attempt to attack the problem in a direction different from optimization, as a continuation of the work in Ferrante et al. (2010), where a parametric family of spectral densities was introduced, and a certain map from the parameter space to the space of generalized moments was studied. The question whether a solution to the parametric spectral estimation problem in fact exists
was essentially left open in Ferrante et al. (2010) until recently, such existence result has been worked out in Zhu and Baggio (2017). In this paper, we try to approach the question of uniqueness of the solution and even well-posedness of the problem. The main tool here is the global inverse function theorem of Hadamard that is reported e.g., in Gordon (1972). However, we do not claim to have answered such questions to a satisfactory level. Instead, we only provide a possible way to the answer.

The outline of this paper is as follows. In Section 2, we review the problem formulation and characterize the solution in a special case. A spectral factorization problem is discussed in Section 3, whose result will be useful for the development in Section 4, where we present our main results.

2. A PARAMETRIC FORMULATION AND THE SOLUTION IN A SPECIAL CASE

Let us first define the set
\[ \mathcal{L}_+ := \{ \Lambda \in \mathcal{S}_n : G^*(z)AG(z) > 0, \forall z \in \mathbb{T} \}, \] (5)
which obviously contains all the Hermitian positive definite matrices, since \( G(z) \) is of full column rank for any \( z \in \mathbb{T} \) which readily follows from the problem setup. By the continuous dependence of eigenvalues on the matrix entries, one can verify that \( \mathcal{L}_+ \) is an open subset of \( \mathcal{S}_n \).

For \( \Lambda \in \mathcal{L}_+ \), take \( W_\Lambda \) as the unique stable and minimum phase (right) spectral factor of \( G^* \Lambda G \) (Ferrante et al., 2010, Lemma 11.4.1), i.e.,
\[ G^* \Lambda G = W_\Lambda^* W_\Lambda. \] (6)
The spectral factor \( W_\Lambda \) can be written as
\[ W_\Lambda(z) = L^{-1} B^* PA(zI - A)^{-1} B + L, \] (7)
where \( P \) is the unique stabilizing solution of the Discrete-time Algebraic Riccati Equation (DARE)
\[ \Pi = A^* \Pi A - A^* \Pi B (B^* \Pi B)^{-1} B^* \Pi A + \Lambda, \] (8)
and \( L \) is the right Cholesky factor of the positive matrix \( B^* PB \), i.e.,
\[ B^* PB = L^* L \] (9)
with \( L \) being lower triangular having real and positive diagonal entries. It is worth pointing out that the DARE (8) above is not a standard one, as \( \Lambda \in \mathcal{L}_+ \) can be indefinite. A formal proof for the existence of a stabilizing solution can be found in the appendix of (Avventi, 2011, Paper A).

To avoid any redundancy in the parameterization, we have to define the set \( \mathcal{L}_+^\perp := \mathcal{L}_+ \cap \text{im} \Gamma \). This is due to a simple geometric reason. More precisely, the adjoint map of \( \Gamma \) in (4) is given by (cf. Ferrante et al. (2010))
\[ \Gamma^* : \mathcal{S}_n \to C(\mathbb{T}; \mathcal{S}_m) \]
\[ X \mapsto G^* X G, \] (10)
and we have the relation
\[ (\text{im} \Gamma)^\perp = \ker \Gamma^* = \{ X \in \mathcal{S}_n : G^*(z)XG(z) = 0, \forall z \in \mathbb{T} \}. \] (11)
Hence for any \( \Lambda \in \mathcal{L}_+ \), we have the orthogonal decomposition
\[ \Lambda = \Lambda^\Gamma + \Lambda^\perp \]
with \( \Lambda^\Gamma \in \text{im} \Gamma \) and \( \Lambda^\perp \) in the orthogonal complement. In view of (11), the part \( \Lambda^\perp \) does not contribute to the function value of \( G^* \Lambda G \) on the unit circle, and we simply have
\[ \mathcal{L}_+^\perp = \Pi_{\text{im} \Gamma} \mathcal{L}_+, \]
where \( \Pi_{\text{im} \Gamma} \) denotes the orthogonal projection operator onto the linear space \text{im} \Gamma.

From this point on, we shall take the prior \( \Psi \in \mathcal{S}_m \) to be continuous on \( \mathbb{T} \), which would facilitate reasoning. We can now define a parametric family of spectral densities
\[ \mathcal{F} := \{ \Phi_\Lambda = W_\Lambda^{-1} \Psi W_\Lambda^* : \Lambda \in \mathcal{L}_+^\perp \}. \] (12)
We have the map
\[ \Lambda \mapsto W_\Lambda \mapsto W_\Lambda^{-1} \Psi W_\Lambda^* \]
from the parameter \( \Lambda \in \mathcal{L}_+^\perp \) to the density function \( \Phi_\Lambda \).

Remark 1. In the scalar case, the form of spectral densities in the family (12) reduces to
\[ \Phi_\Lambda = \frac{\Psi}{G^* \Lambda G}, \]
which is precisely the solution (4.3) in Georgiou and Lindquist (2003) of a constrained optimization problem in terms of the Lagrange multiplier \( \Lambda \). An alternative matricial parametrization has been proposed and studied in Georgiou (2006).

Our problem is formulated as follows.

Problem 2. Given the filter bank \( G(z) \) in (1), the prior \( \Psi \in \mathcal{S}_m \) continuous, and a positive definite matrix \( \Sigma \in \text{im} \Gamma \), find a spectral density in the parametric family \( \mathcal{F} \) defined in (12) such that
\[ \int G \Phi_\Lambda G^* = \Sigma. \] (13)
The above problem has an equivalent formulation. Define \( \text{im} \Gamma := \text{im} \Gamma \cap \mathcal{S}_{m,n} \) where \( \mathcal{S}_{m,n} \) is the open set of \( n \times n \) Hermitian positive definite matrices. Consider the map
\[ \omega : \mathcal{L}_+^\perp \to \text{im} \Gamma \]
\[ \Lambda \mapsto \int \Phi_\Lambda G^*. \] (14)
Then Problem 2 is asking: what is the preimage of \( \Sigma \in \text{im} \Gamma \) under the map \( \omega \)? As shown in Zhu and Baggio (2017), this is a continuous surjective map between open subsets of the linear space \text{im} \Gamma, and thus a solution to Problem 2 always exists. The question now is whether the solution is unique. We show next that uniqueness is indeed true if the prior \( \Psi \) has a special structure.

2.1 Well-posedness given a scalar prior

In the case of a scalar prior, in which we take \( \Psi(z) = \psi(z)I_m \), where the scalar-valued function \( \psi(z) \in \mathcal{F}_1 \) is continuous, the map \( \omega \) would reduce to
\[ \tilde{\omega} : \mathcal{L}_+^\perp \to \text{im} \Gamma \]
\[ \Lambda \mapsto \int \psi G(G^* \Lambda G)^{-1} G^*, \] (15)
and the family of spectral densities becomes
\[ \tilde{\mathcal{F}} := \{ \Phi_\Lambda = \psi(G^* \Lambda G)^{-1} G^* : \Lambda \in \mathcal{L}_+^\perp \}. \] (16)
According to Ferrante et al. (2010), solution to Problem 2 under a scalar prior exists and is unique. We shall next show that given a continuous prior \( \psi \), the map \( \tilde{\omega} \) is a \( C^1 \)
diffeomorphism\footnote{The word “diffeomorphism” in the sequel should always be understood in the $C^1$ sense. Hence the attributive $C^1$ will be omitted.} between $\mathcal{L}^+$ and im $\Gamma$, which in particular, means that the solution $\Lambda$ depends continuously on the covariance data $\Sigma$, and thus the problem is well-posed in the sense of Hadamard. The proof is an application of the global inverse function theorem of Hadamard that appears e.g., in Gordon (1972).

**Theorem 3.** (Hadamard). Let $M_1$ and $M_2$ be connected, oriented, boundary-less $n$-dimensional manifolds of class $C^1$, and suppose that $M_2$ is simply connected. Then a $C^1$ map $f : M_1 \rightarrow M_2$ is a diffeomorphism if and only if $f$ is proper and the Jacobian determinant of $f$ never vanishes.

Conditions on the domain and codomain of $\tilde{\omega}$ will be verified easily. In fact, the set $\mathcal{L}^+$ is easily seen to be open and path-connected since both $\mathcal{L}$ and im $\Gamma$ are such. The simple connectedness of im $\Gamma$ has been reported in (Zhu and Baggio, 2017, Proposition 1). The fact that $\tilde{\omega}$ is of class $C^1$ can be seen along the proof of (Zhu and Baggio, 2017, Lemma 1). Moreover, properness of the more general map $\omega$ has been proven in (Ferrante et al., 2010, Theorem 11.4.1). Therefore, it is only left to check the Jacobian of $\tilde{\omega}$. The next result can be viewed as an interpretation of (Ferrante et al., 2010, Theorem 11.4.2). Here and in the sequel, we shall introduce the notation $\Phi(z; \Lambda)$ to denote a spectral density function that depends on the parameter $\Lambda$, and use it interchangeably with $\Phi_\Lambda(z)$.

**Proposition 4.** The Jacobian determinant of $\tilde{\omega}$ never vanishes in $\mathcal{L}^+$, and hence the map $\tilde{\omega}$ is a diffeomorphism.

**Proof.** From (Zhu and Baggio, 2017, Lemma 1), the differential of $\tilde{\omega}$ at $\Lambda \in \mathcal{L}^+$ is

$$\delta \tilde{\omega}(\Lambda; \delta \Lambda) = -\int \psi G(G^* \Lambda G)^{-1}(G^* \delta \Lambda G)(G^* \Lambda G)^{-1} G^*$$

such that $\delta \Lambda \in \text{im $\Gamma$}$. Our target is to show that

$$\delta \omega(\Lambda; \delta \Lambda) = 0 \implies \delta \Lambda = 0.$$  

To this end, first notice that the middle part of the integrand in (17) is just the differential of the spectral density $\Phi_\Lambda = \psi(G^* \Lambda G)^{-1}$ w.r.t. $\Lambda$:

$$\delta \Phi(z; \Lambda; \delta \Lambda) := -\psi(G^* \Lambda G)^{-1}(G^* \delta \Lambda G)(G^* \Lambda G)^{-1}.$$  

Then the condition $\delta \tilde{\omega}(\Lambda; \delta \Lambda) = 0$ means that

$$\delta \Phi(z; \Lambda; \delta \Lambda) \in \ker \Gamma = (\text{im $\Gamma$})^\perp,$$  

which in view of (10), reads

$$\langle G^* X G, \delta \Phi(z; \Lambda; \delta \Lambda) \rangle = \text{tr} \int G^* X G \delta \Phi(z; \Lambda; \delta \Lambda) = 0,$$  

\quad $\forall X \in \mathfrak{h}_n$.

In particular, following (Ferrante et al., 2010, Eqns. 11.44–11.45), choosing $X = \delta \Lambda$ would lead to

$$G^* \delta \Lambda G \equiv 0,$$  

which by (11), implies that $\delta \Lambda \in (\text{im $\Gamma$})^\perp$. Since at the same time $\delta \Lambda \in \text{im $\Gamma$}$, it is necessary that $\delta \Lambda = 0$. The rest is just an application of Theorem 3.

**Remark 5.** The unique solution in $\mathcal{X}$ to the spectral estimation problem has an interesting characterization in terms of an optimization problem; cf. (Avventi, 2011, Paper A) for details.

A difficulty arises when one tries to extend the analysis in the previous proposition to the more general map $\omega$, as it would entail the differentiation of the spectral factor $W_\Lambda$ in (6) w.r.t. the parameter $\Lambda$. Such a difficulty can be bypassed by introducing a spectral factorization as will be discussed next.

### 3. A DIFFEOMORPHIC SPECTRAL FACTORIZATION

Following the lines of Avventi (2011), given the stabilizing solution $P$ of the DARE (8), let us introduce a change of variables by setting

$$C := L^{-*} B^* P.$$  

Then it is not difficult to recover the relation $L = CB$ for the Cholesky factor in (9). In this way, the spectral factor (7) can be rewritten as

$$W_\Lambda(z) = CA(zI - A)^{-1}B + CB = zCG,$$  

where the second equality holds because of the identity $A(zI - A)^{-1}B = z(zI - A)^{-1}$. In view of this, the factorization (6) can then be rewritten as

$$G^* \Lambda G = G^* C^* CG, \quad \forall z \in \mathbb{T}.$$  

This relation has also been expressed in (Ferrante et al., 2010, Equation 11.29). In the sequel, we shall also call the $m \times n$ matrix $C$ a “spectral factor”.

As reported in (Avventi, 2011, Section A.5.5), it is possible to build a homeomorphic factorization by carefully choosing the set where the factor $C$ lives. More precisely, let the set $\mathcal{C}_+ \subset \mathbb{C}^{m \times n}$ contain those matrices $C$ that satisfy the following two conditions

- $CB$ is lower triangular with real and positive diagonal entries,
- $A - B(CB)^{-1}CA$ has eigenvalues strictly inside the unit circle.

Define the map

$$h : \mathcal{L}^+ \rightarrow \mathcal{C}_+,$$  

$$\Lambda \mapsto C \text{ via (18)}.$$

Then according to (Avventi, 2011, Theorem A.5.5), the map $h$ of spectral factorization is a homeomorphism. We shall next strengthen this result by showing that the map $h$ is in fact a diffeomorphism using Theorem 3.

#### 3.1 Characterization of diffeomorphism

We are going to apply Theorem 3 to the inverse of $h$

$$h^{-1} : \mathcal{C}_+ \rightarrow \mathcal{L}^+,$$  

$$C \mapsto \Lambda := \Pi_{\text{im $\Gamma$}}(C^* C).$$  

Those technical requirements on the domain and codomain of $h^{-1}$ can be verified without difficulty. The set $\mathcal{C}_+$ is an open subset of the linear space

$$\mathcal{C} := \{ C \in \mathbb{C}^{m \times n} : CB \text{ is lower triangular with real diagonal entries} \},$$  

whose real dimension coincides with $\text{im $\Gamma$}$ (cf. Avventi (2011)). The fact that $\mathcal{C}_+$ is also path-connected is a consequence of $h$ being a homeomorphism. Furthermore,
the proof of $\mathcal{L}_F^\top$ being simply connected can be adapted easily from (Zhu and Baggio, 2017, Proposition 1).

The map $h^{-1}$ is actually smooth (hence of course $C^1$) because it is a composition of the quadratic map $C \mapsto C^*C$ and the projection $\Pi_m$, both of which are smooth. The fact that $h^{-1}$ is proper has also been reported in Avventi (2011). Therefore, it remains to investigate the Jacobian of $h^{-1}$. In order to carry out explicit computation, it is necessary to choose bases for the two linear spaces $\mathcal{E}$ and im $\Gamma$.

Let $M := m(2n - m)$, and let $\{A_1, A_2, \ldots, A_M\}$ and $\{C_1, \ldots, C_M\}$ be orthonormal bases of im $\Gamma$ and $\mathcal{E}$, respectively. Then one can parameterize $\Lambda \in \mathcal{L}_F^\top$ and $C \in \mathcal{E}_+$ as

$$
\Lambda(x) = x_1A_1 + x_2A_2 + \cdots + x_mA_M,
C(y) = y_1C_1 + y_2C_2 + \cdots + y_mC_M,
$$

for some $x_j, y_j \in \mathbb{R}$, $j = 1, \ldots, M$. The map $h^{-1}$ can then be expressed coordinate-wise as

$$
x_j = \langle \Lambda_j, C^k(y)^*C(y) \rangle.
$$

Then the partial derivatives can be computed as

$$
\frac{\partial x_j}{\partial y_k} = \langle \Lambda_j, C_k^*C(y) + C^*(y)C_k \rangle,
$$

which is the $(j, k)$ element of the Jacobian matrix denoted as $J_{h^{-1}}(y)$. We need some ancillary results in order to show that $h^{-1}$ has everywhere nonvanishing Jacobian.

**Proposition 6.** If $v \in \mathbb{C}^n$ is such that $v^*G(z) = 0$ for all $z \in \mathbb{T}$, then $v = 0$.

**Proof.** The condition that $v^*G(z) = 0$ for all $z \in \mathbb{T}$ implies that $v^* \int GG^*v = 0$.

Under our problem setting stated in Section 1, we have $\int GG^* > 0$ and thus the assertion of the proposition follows. To see the fact of positive definiteness, note first that the following expansion holds

$$
G(z) = (zI - A)^{-1}B
= z^{-1} \sum_{k=0}^{\infty} z^{-k}A^kB^k, \quad \text{for } |z| \geq 1,
$$

since $A$ is stable. Then by the Parseval identity, we have

$$
\int GG^* = \sum_{k=0}^{\infty} A^kB^*A^k = RR^*,
$$

where $R = [B, AB, \ldots, A^kB^k, \ldots]$. The above is the unique solution of the discrete-time Lyapunov equation

$$
X - AXA^* = BB^*.
$$

Since $(A, B)$ is by assumption reachable, $R$ is of full row rank, and therefore $\int GG^* > 0$.

**Proposition 7.** Given $C \in \mathcal{E}_+$, the rational matrix equation in the unknown $V \in \mathbb{C}^{m \times n}$

$$
G^*(C^*V + V^*C)G = 0, \quad \forall z \in \mathbb{T}
$$

has the general solution

$$
V = QC
$$

where $Q \in \mathbb{C}^{m \times m}$ is an arbitrary constant skew-Hermitian matrix. If one further requires $V \in \mathcal{E}$, then (28) has only the trivial solution $V = 0$.

**Proof.** The equation (28) is equivalent to

$$
z^*G^*(C^*V + V^*C)Gz = 0, \quad \forall z \in \mathbb{T}.
$$

Let

$$
zCG(z) = zC(zI - A)^{-1}B = \frac{P_C(z)}{z^n \det(zI - A)},
$$

where $P_C(z) := z^{-n+1}C \text{adj}(zI - A)B$ and $\text{adj}(\cdot)$ denotes the adjugate matrix. Obviously, $P_C(z)$ is a matrix polynomial in the indeterminate $z^{-1}$, which is intended to conform to the engineering convention. From (26), we have

$$
\lim_{z \to \infty} zCG = CB = \lim_{z \to \infty} P_C(z),
$$

where the second equality holds since $\lim_{z \to \infty} z^{-n} \det(zI - A) = 1$. Moreover, the scalar polynomial $\det P_C(z)$ has all its roots inside $\mathbb{D}$, which can be seen from (19) as $zCG$ is minimum phase, i.e., admits a stable inverse.

Define similarly $P_V(z) := z^{-n+1}V \text{adj}(zI - A)B$. Then one can reduce (30) to the matrix polynomial equation

$$
P_C(z)P_V(z) + P_V(z)P_C(z) = 0, \quad \forall z \in \mathbb{T}.
$$

in which we have

$$
P_C^*(0) = \left[ \lim_{z \to \infty} P_C(z) \right]^* = (CB)^*\n$$

nonsingular because $C \in \mathcal{E}_+$. By the identity theorem for holomorphic functions, if the above equation holds on $\mathbb{T}$, then it holds for any $z \in \mathbb{C}$ except for 0 (and $\infty$). Hence the restriction $z \in \mathbb{T}$ can be removed here. Since $P_C^*$ is anti-stable and $P_C(0)$ nonsingular, according to (a variant of) (Ježek, 1986, Theorem MP1), the general solution of (31) is

$$
P_V = QC,
$$

where $Q \in \mathbb{C}^{m \times m}$ is an arbitrary constant skew-Hermitian matrix. This in turn implies that

$$
VG(z) = QCG(z), \quad \forall z \in \mathbb{T},
$$

which in view of Proposition 6, further implies that $V = QC$.

To prove the remaining part of the claim, just apply the power series expansion (26) to (32), and notice that all the Fourier coefficients on the two sides of (32) must coincide. This in particular means that

$$
VB = QC\B.
$$

Since we have $C \in \mathcal{E}_+$ and $V \in \mathcal{E}$ in addition, both $VB$ and $CB$ are lower triangular and the latter is invertible. Therefore $Q$ turns out to be also lower triangular and at the same time skew-Hermitian, which necessarily means that $Q$ is equal to 0 and so is $V$.

**Theorem 8.** The Jacobian determinant of $h^{-1}$ never vanishes in $\mathcal{E}_+$, and hence the map $h$ in (21) is a diffeomorphism.

**Proof.** Suppose $v \in \mathbb{R}^M$ is such that $J_{h^{-1}}(y)v = 0$. We need to show that $v = 0$. To this end, notice from (25) that equivalently we have for $j = 1, 2, \ldots, M$,

$$
0 = \sum_{k=1}^{M} v_k \langle A_j, C^k(y) + C^*(y)C_k \rangle
= \langle A_j, C^*(y)C(y) + C^*(y)C(v) \rangle,
$$

which implies that

$$
C^*(y)C(y) + C^*(y)C(v) \in \text{im } \Gamma.
$$
In view of (11), this in turn means
\[ G^*(z) [C^*(v)C(g) + C^*(y)C(v)] G(z) = 0, \quad \forall z \in T. \]
By Proposition 7, the only solution is \( v = 0 \). Thus Theorem 3 is applicable and this completes the proof.

4. THE GENERAL MAP \( \omega \)

Let us return to the map \( \omega \) defined in (14). We shall use the result obtained in the previous section to attack the uniqueness conjecture posed in Zhu and Baggio (2017). Given the relation (19), the spectral density \( \Phi_\Lambda \) can be reparameterized in \( C \) as
\[ \Phi_\Lambda \equiv \Phi_C := (CG)^{-1} \Psi(CG)^{-*}. \] (33)
In this way, the map \( \omega \) can be expressed as a composition
\[ \omega = \tau \circ h : \omega(\Lambda) = \tau(h(\Lambda)), \] (34)
with \( h \) in (21) and
\[ \tau : \mathcal{C}_+ \to \text{im}_+ \Gamma. \] (35)
Since \( h \) has been proved to be a diffeomorphism, we can restrict our attention to the map \( \tau \) due to the next simple result.

**Proposition 9.** Let \( X, Y, Z \) be open subsets of \( \mathbb{R}^n \). Suppose we have functions \( f : X \to Y, \ g : Y \to Z \) and \( f \) is a diffeomorphism between \( X \) and \( Y \). Define the composite function
\[ h = g \circ f : X \to Z. \] (36)
Then \( h \) is a diffeomorphism between \( X \) and \( Y \) if and only if \( g \) is a diffeomorphism between \( Y \) and \( Z \).

**Proof.** The “if” part is trivial since a composition of two diffeomorphisms is again a diffeomorphism. To see the converse, for \( y \in Y \), let \( x = f^{-1}(y) \in X \) and put it into (36) as an argument of \( h \). Then one gets
\[ g = h \circ f^{-1}, \]
which is again a composition of two diffeomorphisms.

Since properness of the map \( \omega \) has already been proven, it remains to show that \( \omega \) is continuously differentiable and has everywhere nonvanishing Jacobian. In view of the relation (34) and the previous proposition, it would be sufficient and necessary that the map \( \tau \) possesses such two properties. We need the next lemma before proving the continuous differentiability.

**Lemma 10.** Let a sequence \( \{ \Lambda_k \}_{k \geq 1} \subset \mathcal{L}_+ \) converge to some \( \Lambda \in \mathcal{L}_+ \). Then there exists a real number \( \mu > 0 \) such that
\[ G^*(e^{i\theta})\Lambda_k G(e^{i\theta}) \geq \mu I, \quad \forall k, \theta. \]

**Proof.** The claim of the lemma follows from the continuity of the function \( G^*(e^{i\theta})\Lambda_k G(e^{i\theta}) \) in \( \Lambda \) and \( \theta \), and the uniform convergence of the sequence of functions \( \{ G^{*\theta}\Lambda_k G \}_{k \geq 1} \) to \( G^{*\theta}\Lambda G \).

**Proposition 11.** The map \( \tau \) in (35) is of class \( C^1 \).

**Proof.** We can proceed by mimicking the proof of (Zhu and Baggio, 2017, Lemma 1), although the argument here is slightly more general. First compute the differential of \( \Phi(z; C) \) w.r.t. \( C \in \mathcal{C}_+ \) as
\[ \delta\Phi(z; C; \delta C) = -(CG)^{-1}\delta CG\Phi_C - \Phi_C G^* \delta C^*(CG)^{-*}. \] (37)
which is easily seen to be continuous in \( C \) and \( \theta \in [-\pi, \pi] \) for a fixed \( \delta C \in \mathcal{C} \). This means that we can take the differential of the map \( \tau \) inside the integral in (35)
\[ \delta\tau(C; \delta C) = \int G\delta\Phi(e^{i\theta}; C; \delta C) G^*. \] (38)
Next we show that the above differential is continuous in \( C \) for a fixed \( \delta C \). To this end, suppose we have a sequence \( \{ C_k \}_{k \geq 1} \subset \mathcal{C}_+ \) that converges to some \( C \in \mathcal{C}_+ \) as \( k \to \infty \). Due to the relation (20), we have for each \( k \)
\[ G^*\Lambda_k G = G^*\bar{C} \Gamma_k G, \quad \forall \in T, \] (39)
where \( \Lambda_k = h^{-1}(C_k) \in \mathcal{L}_+ \). Since \( h \) is a diffeomorphism by Theorem 8, we have
\[ \lim_{k \to \infty} \Lambda_k = \bar{\Lambda} := \bar{h}^{-1}(\bar{C}). \]
Let \( \lambda_{\min,k}(\theta) \) be the smallest eigenvalue of \( G^*(e^{i\theta})\Lambda_k G(e^{i\theta}) \), and \( \sigma_{\min,k}(\theta) \) be the smallest singular value of \( G_k G(e^{i\theta}) \). In view of (39), we have
\[ \lambda_{\min,k}(\theta) = \sigma^2_{\min,k}(\theta) \]
By Lemma 10, there exist a real number \( \mu > 0 \) such that
\[ \lambda_{\min,k}(\theta) \geq \mu \Rightarrow \sigma_{\min,k}(\theta) \geq \sqrt{\mu}, \quad \forall k, \theta. \]
Then we have
\[ \| G\delta\Phi(e^{i\theta}; C_k; \delta C) G^* \|_F \leq 2 \| (G^{*\theta})^{-1} \delta CG\Phi_C \|_2 \leq 2 \| (G^{*\theta})^{-1} \|_2 \| \delta CG\|_2 \| \Phi_C \|_2 \leq \frac{2}{\sigma_{\min,k}(\theta)} \| CG\|_F \| \Phi \|_F \leq K, \]
where the constant
\[ K = \frac{2}{\mu^{3/2}} \max_{\theta} \| CG(e^{i\theta}) \|_F \max_{\theta} \| \Phi(e^{i\theta}) \|_F. \]
We can now bound the integrand in (38). For any \( \theta \in [-\pi, \pi] \) and \( k \geq 1 \), we have
\[ \int |G\delta\Phi(e^{i\theta}; C_k; \delta C) G^*| \leq K \| CG(e^{i\theta}) \|_F \leq \kappa \| CG(e^{i\theta}) \|_F \leq \kappa K \| G \|_F^2 \leq \kappa K \| G \|_{\text{max}}^2 \]
where \( K_{\text{max}} := \max_{\theta \in [-\pi, \pi]} \text{tr} \left\{ G^*(e^{i\theta}) G^* \right\} \)
and \( \kappa \) is a constant for norm equivalence. The last step is an application of Lebesgue’s dominated convergence theorem to conclude
\[ \lim_{k \to \infty} \delta\tau(C_k; \delta C) = \delta\tau(C; \delta C), \]
which completes the proof.

We are now left with the task of investigating whether the Jacobian of \( \tau \) vanishes nowhere, which can be approached via the differential (38). However, the trick of orthogonality in the proof of Proposition 4 does not apply in a straightforward manner to the general map \( \omega \). The desired result can be obtained if an additional constraint is imposed on the prior \( \Psi \), and this is reported in the next proposition.

**Proposition 12.** If the prior \( \Psi \) is such that the equality
\[ \text{tr} \int F^* \Psi F = \text{tr} \int F \Psi F^* \] (41)
holds for any \( C \in \mathcal{C}_+ \) and any \( V \in \mathcal{C} \), where the matrix function \( F = VG(CG)^{-1} \), then the Jacobian determinant of \( \tau \) vanishes nowhere in \( \mathcal{C}_+ \), and hence the map \( \omega \) is a diffeomorphism.
Proof. Fix $C \in \mathcal{C}_+$ and let $\delta r(C; V) = 0$ for some $V \in \mathcal{C}$. In view of (38), this would imply that

$$\delta \Phi(z; C; V) \in \ker \Gamma = (\text{im } \Gamma^*)^\perp,$$

which in view of (10), means

$$\langle G^* XG, \delta \Phi(z; C; V) \rangle = \int G^* XG \delta \Phi(z; C; V) = 0, \quad \forall X \in \mathcal{S}_n.$$  \hspace{1cm} (42)

Choosing $X = C^* V + V^* C$ in (42) would lead to the relation

$$\int 2F \Psi F^* + \Psi F^* F^* + FF \Psi = 0$$

after some manipulations of the variables using (37). The left-hand side in the above equation is different from

$$\int (F + F^*) \Psi (F + F^*)$$ \hspace{1cm} (43)

in only one term. If the equality (41) holds for any $C \in \mathcal{C}_+$ and $V \in \mathcal{C}$, then we would have the expression (43) equal to zero, which, by the same reasoning as in Proposition 4, implies

$$F + F^* \equiv 0, \quad \forall z \in \mathcal{T},$$

which is equivalent to (28). In view of Proposition 7, this in turn implies $V = 0$.

The above proposition does not improve much over Proposition 4 for the scalar case, since the requirement on the prior seems very artificial and a matrix-valued $\Psi$ in general does not satisfy it, as illustrated in the next example.

Example 13. Consider a static case in which $n = m$, $B = I$, and the matrix $A$, that is, the transfer function (1) reduces to $G = z^{-1} I$ and the output of the linear system is identical to the 1-step delayed input. Let us fix $\Psi \equiv \text{diag}(1, 2)$ and $C = I$, and then (41) would reduce to

$$\int V^* \Psi V = \int V \Psi V^*.$$  

The only requirement on $V$ is being lower-triangular with real diagonal entries. Hence we can take, e.g.,

$$V = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix},$$

and it is straightforward to check that the above equality does not hold. However, in this overly simplified example, the solution to Problem 2 is still unique. Indeed, given $\Sigma \in \mathbb{S}_m$ and $\Psi \in \mathcal{S}_m$, one is looking for a parameter $C \in \mathcal{C}_+$ such that

$$\int C^{-1} \Psi C^{-1} = \Sigma.$$  

Clearly, this implies

$$C^{-1} L_R = L_{2\Sigma} U,$$

where the notation $L_A$ denotes the usual Cholesky factor of $A > 0$, $U$ is a unitary matrix, and $R := \int \Psi > 0$. It then follows that $U$ is lower triangular with real and positive diagonal entries, since such are all $C$, $L_R$, and $L_\Sigma$. Hence $U$ is necessarily equal to identity, and $C = L_R L_{2\Sigma}^{-1}$. This means that the condition on the prior in Proposition 12 is not necessary for the uniqueness of the solution.

5. CONCLUSION

We have shown that a parametric spectral estimation problem is well-posed if the chosen prior is special. It would be interesting to investigate whether the claim would still hold when the prior is arbitrarily matrix-valued, and this is left for future work.

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