COLOURED JONES AND ALEXANDER POLYNOMIALS AS TOPOLOGICAL INTERSECTIONS OF CYCLES IN CONFIGURATION SPACES

CRISTINA ANA-MARIA ANGHEL

Abstract. Coloured Jones and Alexander polynomials are sequences of quantum invariants recovering the Jones and Alexander polynomials at the first terms. We show that they can be seen conceptually in the same manner, using topological tools, as intersection pairings in covering spaces between explicit homology classes given by Lagrangian submanifolds.

CONTENTS

1. Introduction 1
2. Representation theory of $U_q(sl(2))$ 4
3. Lawrence representation 8
4. Construction of the generic homology classes 12
5. Topological intersection model for the coloured Jones invariants 15
6. Topological intersection model for the coloured Alexander invariants 17
References 20

1. Introduction

The theory of quantum invariants for knots started with the discovery of the Jones polynomial. After that, Reshetikhin and Turaev developed an algebraic tool which starts with a quantum group and leads to a link invariant. Using this algebraic method, the representation theory of $U_q(sl(2))$ leads to a family of link invariants $\{J_N(L, q) \in \mathbb{Z}[q^{\pm 1}] \}_{N \in \mathbb{N}}$ called coloured Jones polynomials. The first term of this sequence is the original Jones polynomial. On the other hand, the quantum group at roots of unity $U_\xi(sl(2))$, leads to a sequence of invariants, called coloured Alexander polynomials, having the original Alexander invariant as the first term. On the topological side, R. Lawrence introduced a sequence of homological braid group representations based on coverings of configuration spaces and using these, Bigelow and Lawrence gave a homological model for the original Jones polynomial, using the skein nature of the invariant for the proof. Later on, Kohno and Ito ([12], [7], [8]) presented an identification between highest weight quantum representations of the braid group and the homological Lawrence representations.

We are interested in questions concerning topological type models for certain quantum invariants, using homological braid group actions on the homology of configuration spaces. In [2] we presented a topological model for all coloured Jones polynomials, showing that they are graded intersection pairings between two homology classes in a covering of the configuration space in the punctured disc. This result used the formulas from [7]. However, even if the definition of these homology
classes was explicit, it involved difficult functions to deal with from the computational point of view.

Concerning the representation theory of quantum groups at roots of unity, in [8], Ito suggested an identification of highest weight representations at roots of unity with a quotient of the Lawrence representation. Then he concluded a homological model for the coloured Alexander invariants as a sum of traces of these truncated Lawrence representations. Based on Ito’s identification at the root of unity, we showed in [3] a topological model for the coloured Alexander invariants as graded intersection pairings between two homology classes in the truncated Lawrence representation, using a quotient of the homology of the covering of the configuration space in the punctured disc.

Out of these two topological models, we reached with two precise questions: first of all, an explicit formula for the homology classes that occur in these models. On the second direction, we are interested to understand why at the root of unity the truncation occurs at the homological level. In this paper we aim to answer this problem.

In [16], Martel presented a version of Kohno’s identification for the generic quantum group $U_q(sl(2))$ with more explicit bases in the Lawrence representation. In the sequel we will use this identification.

The main result of this paper shows that we can see the coloured Jones polynomials and coloured Alexander polynomials conceptually in the same way, from the Lawrence representation over two variables.

First of all, we start on the algebraic side with quantum representations on weight spaces from the generic Verma module over two variables. We remark that when specialise to one variable with $q$ generic or $q = \xi_N$ root of unity, we can use weight spaces in an $N$-dimensional subspace inside the Verma module. For the generic version this is not surprising, however, for the root of unity case this differs from the usual construction of the coloured Alexander invariants. After we study the precise form of the coefficients of the $R$-matrix after specialisations, we show that we can see both coloured Jones invariants and coloured Alexander invariants from a specific weight space inside the Verma module over two variables. Then, we use identifications with the homological Lawrence representation and we construct certain homology classes. In the last part, we show that the graded intersection pairing between these classes leads by two different specialisation to the two sequences of quantum invariants, namely coloured Jones polynomials and coloured Alexander polynomials.

**Description of the topological tools.** Let $C_{n,m}$ be the unordered configuration space of $m$-points in the $n$-punctured disc. We will use the following tools for our construction:

1. sequence of Lawrence representations $\mathcal{H}_{n,m}$ which are $\mathbb{Z}[x^{\pm 1}, d^{\pm 1}]$-modules (defined from the Borel-Moore homology of a $\mathbb{Z} \oplus \mathbb{Z}$-covering of $C_{n,m}$ - definition 3.2.6)
2. sequence of dual Lawrence representation $\mathcal{H}^\partial_{n,m}$ ( definition 24 )
3. certain topological intersection pairings $\langle, \rangle$ between the Lawrence representations and their dual representations ( definition 3.4 ).

**Notation 1.0.1.** We will use the following specialisations:
1) Generic case ($q$ generic, $\lambda = N − 1 \in \mathbb{N}$)
$$\psi_{q,\lambda} : \mathbb{Z}[x^{\pm}, d^{\pm}] \rightarrow \mathbb{Z}[q^{\pm}]$$
2) Root of unity case ($q = \xi_N = e^{\frac{2\pi i}{N}}, \lambda \in \mathbb{C}$)

$$\psi_{\xi_N, \lambda}: \mathbb{Z}[x^\pm, d^\pm] \to \mathbb{C}.$$ 

given by the formula:

$$\begin{cases} 
\psi_{q, \lambda}(x) = q^{2\lambda} \\
\psi_{q, \lambda}(d) = q^{-2}. 
\end{cases}$$

**Theorem 1.0.2.** (Coloured Jones and Alexander invariants from generic intersection pairings)

There exist two homology classes $F_n^\xi, F_n^\xi_N \in H^2_{n-1}(\mathbb{R}^2 \setminus \mathbb{R}^{N-1})$ which are linear combinations of the basic submanifolds $\mathcal{U}$ and let $G_n^\beta \in H^2_{2n-1}(\mathbb{R}^{n-1})$ be the product of figure-eight configuration spaces from the picture above. Then, if $L = \beta_n$ we have the following models:

$$J_N(L, q) = q^{-(N-1)\omega(\beta_n)} < (\beta_n \otimes I_{n-1}) F_n^\xi, G_n^\beta > |\psi_{\xi_N, N-1}|.$$  

$$\Phi_N(L, \lambda) = \xi_N^{(N-1)\lambda \omega(\beta_n)} < (\beta_n \otimes I_{n-1}) F_n^\xi_N, G_n^\beta > |\psi_{\xi_N, \lambda}|.$$  

**Remark 1.0.3.** This model answers the first question and we see that the classes that give the coloured Alexander and Jones polynomials are given by explicit linear combinations of Lagrangians in the configuration space.

**Remark 1.0.4.** (Recovering Bigelow’s model for the Jones polynomial)

For the case $N = 2$ corresponding to the original Jones polynomial, we see that our classes are similar with Bigelow’s generators, except that we have the point in the middle of the forks pushed towards the boundary.

Now we discuss a bit related to the second question, namely the truncation of the Lawrence representation. The previous models for the coloured Alexander polynomials used this quotient at the homological level. From Theorem 1.0.2 we see that the truncation part at the root of unity is reflected on the homological side directly by the specialisation $\psi_{\xi_N, \lambda}$. In other words, this specialisation is powerful enough to contain the truncation which occurred previously.
Corollary 1.0.5. (Coloured Alexander Invariants from $\mathbb{Z} \oplus \mathbb{Z}_N$ covering spaces)

Let $H^N_{n,m}$ be the Lawrence representation defined using the local system $\mathbb{Z} \oplus \mathbb{Z}_N$ (by projecting the second component of the $\mathbb{Z} \oplus \mathbb{Z}$ local system modulo $N$). We denote the corresponding intersection pairing by $\langle \cdot, \cdot \rangle_N$. Consider the homology classes: $\mathcal{F}^N_n \in H^N_{2n-1,(n-1)(N-1)}$ and $\mathcal{G}^N_n \in H^N_{2n-1,(n-1)(N-1)}$ given by the same geometric submanifolds as above. Then the coloured Alexander invariant is given by the following intersection:

$$\Phi_N(L, \lambda) = \xi_N^{(N-1)\lambda \omega(\beta_n)} < (\beta_n \otimes \mathbb{I}_{n-1}) \mathcal{F}^N_n, \mathcal{G}^N_n > N .$$

Acknowledgements. I would like to thank Professor Christian Blanchet for many beautiful discussions and for asking me these two main questions. Also, I would like to thank Christine Lescop, Jacob Rasmussen, Alexis Virelizier and Emmanuel Wagner for useful conversations.

This paper was prepared at the University of Oxford, and I acknowledge the support of the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement No 674978).

2. REPRESENTATION THEORY OF $U_q(sl(2))$

Definition 2.0.1. Let $q, s$ parameters and consider the ring

$$\mathbb{L}_a := \mathbb{Z}[q^{\pm 1}, s^{\pm 1}].$$

Consider the quantum enveloping algebra $U_q(sl(2))$, to be the algebra over $\mathbb{L}_a$ generated by the elements $\{E, F^{(n)}, K^{\pm 1} | n \in \mathbb{N}^*\}$ with the following relations:

$$KK^{-1} = K^{-1}K = 1; \quad KE = q^2EK; \quad KF^{(n)} = q^{-2n}F^{(n)}K;$$

$$F^{(n)}F^{(m)} = [n+m]^qF^{(n+m)};$$

$$[E, F^{(n+1)}] = F^{(n)}(q^{-n}K - q^nK^{-1}).$$

Then, one has that $U_q(sl(2))$ is a Hopf algebra with the following comultiplication, counit and antipode:

$$\Delta(E) = E \otimes K + 1 \otimes E \quad S(E) = -EK^{-1}$$

$$\Delta(F^{(n)}) = \sum_{j=0}^{n} q^{-(n-j)}K^{j-n}F^{(j)} \otimes F^{(n-j)} \quad S(F^{(n)}) = (-1)^n q^{n(n-1)}K^nF^{(n)}$$

$$\Delta(K) = K \otimes K \quad S(K) = K^{-1}$$

$$\Delta(K^{-1}) = K^{-1} \otimes K^{-1} \quad S(K^{-1}) = K.$$

We will use the following notations:

$$\{x\} := q^x - q^{-x} \quad [x]_q := \frac{q^x - q^{-x}}{q - q^{-1}}$$

$$[n]_q! = [1]_q[2]_q \cdots [n]_q$$

$$\left[ \begin{array}{c} n \\ j \end{array} \right]_q = \frac{[n]_q!}{[n-j]_q[j]_q!}.$$

Definition 2.0.2. (The Verma module)

Consider $V$ be the $\mathbb{L}_a$-module generated by an infinite family of vectors $\{v_0, v_1, \ldots\}$. The following relations define an $U_q(sl(2))$ action on $V$:

$$Kv_i = sq^{-2i}v_i,$$

$$Ev_i = v_{i-1},$$

$$F^{(n)}v_i = [n+i]_q \prod_{k=0}^{n-1} (sq^{-k-i} - s^{-1}q^{k+i})v_{i+n}.$$

In the sequel, we will use certain specialisations of the previous quantum group.
Definition 2.0.3. We consider two types of specialisations of the coefficients, where we specialise the highest weight using $q^\lambda$:

Case a) ($q$ generic, $\lambda = N - 1 \in \mathbb{N}$):

\[\begin{align*}
\eta_{q,\lambda} : \mathbb{Z}[q^\pm, s^\pm] &\rightarrow \mathbb{Z}[q^\pm] \\
\eta_{q,\lambda}(s) &= q^{\lambda}.
\end{align*}\]

(6)

Case b) ($q = \xi_N = e^{2\pi i/N}, \lambda \in \mathbb{C}$ generic):

\[\begin{align*}
\eta_{\xi_N,\lambda} : \mathbb{Z}[q^\pm, s^\pm] &\rightarrow \mathbb{Z}[q^\pm] \\
\eta_{\xi_N,\lambda}(s) &= \xi_N^{\lambda}.
\end{align*}\]

(7)

Using these specialisations, we will consider the corresponding specialised quantum groups and their representation theory. We obtain the following:

| Ring | Quantum Group | Representations | Specialisations |
|------|---------------|----------------|-----------------|
| $L_s = \mathbb{Z}[q^\pm, s^\pm]$ | $U_q(sl(2))$ | $\hat{V}$ | $q, s$ param |
| $L = \mathbb{Z}[q^\pm]$ | $\mathcal{V} = U_q(sl(2)) \otimes_{\eta_{q,N-1} \mathbb{Z}[q^\pm]} \hat{V}_{q,N-1} = \hat{V} \otimes_{\eta_{q,N-1}} \mathbb{Z}[q^\pm]$ | $V_N \subseteq \hat{V}_{q,N-1}$ | $\eta_{q,N-1}$ |
| $L = \mathbb{C}$ | $\mathcal{V}_{\xi_N} = U_q(sl(2)) \otimes_{\eta_{\xi_N,\lambda} \mathbb{C}} \mathbb{C}$ | $\hat{V}_{\xi_N,\lambda} = \hat{V} \otimes_{\eta_{\xi_N,\lambda}} \mathbb{C}$ | $b)$ ($q = \xi_N, \lambda \in \mathbb{C}$) |
| | $U_{\lambda} \subseteq \hat{V}_{\xi_N,\lambda}$ | $\eta_{\xi_N,\lambda}$ |

Lemma 2.0.4. For the cases a) of natural highest weight $\lambda$, $\hat{V}_{q,N-1}$ has an $N$-dimensional $\mathcal{V}$-submodule generated by the first $N$ vectors $\{v_0, ..., v_{N-1}\}$. We denote this by:

$V_N := \langle v_0, ..., v_{N-1} \rangle \subseteq \hat{V}_{q,N-1}$.

Notation 2.0.5. For the root of unity case, we consider the $\mathbb{C}$-vector space generated by the first $N$ vectors in the specialisation of the generic Verma module as follows:

$U_{\lambda} := \langle v_0, ..., v_{N-1} \rangle \subseteq \hat{V}_{\xi_N,\lambda}$.

Definition 2.0.6. ([10],[7]) (Braid group action on the Verma module)

There exist an $R$-matrix for the generic quantum group $R \in U_q(sl(2)) \otimes U_q(sl(2))$ given by the following expression:

\[R = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} E^n \otimes F^{(n)}.
\]

By twisting the two components on which we act with $R$-matrix, we consider the following element:

$R = C \circ R$

$C(v_i \otimes v_j) = s^{j+i} q^{2j} v_j \otimes v_i$. 

Proposition 2.0.7. This element, leads to representations of the braid group on the generic Verma module $\hat{V}$ of $U_q(sl(2))$ as follows

$$\varphi_n^\hat{V} : B_n \rightarrow \text{Aut}_{U_q(sl(2))}(\hat{V}^{\otimes n})$$

$$\sigma^{\pm 1} \rightarrow Id_{V}^{(i-1)} \otimes (\mathcal{R}^{\pm 1} \circ \tau) \otimes Id_{V}^{(n-i-1)}.$$ 

Proposition 2.0.8. 1) The category of finite dimensional representations of $\mathcal{U}$ has the following dualities:

- (1) The category of finite dimensional representations of $U_q(sl(2))$ prescribed by the $K$-action. They will be important in the sequel because they carry homological information.

2) For the root of unity case, we will use the following maps:

$$\text{coev}_{V_N} : \mathbb{L} \rightarrow V_N \otimes V_N^*$$ is given by $1 \mapsto \sum v_j \otimes v_j^*$,

$$\text{ev}_{V_N} : V_N^* \otimes V_N \rightarrow \mathbb{L}$$ is given by $f \otimes w \mapsto f(w),$

$$\text{coev}_{U_\lambda} : \mathbb{C} \rightarrow U_\lambda \otimes U_\lambda^*$$ is given by $1 \mapsto \sum v_j \otimes v_j^*$,

$$\text{ev}_{U_\lambda} : U_\lambda^* \otimes U_\lambda \rightarrow \mathbb{C}$$ is given by $f \otimes w \mapsto f(w),$

$$\text{coev}_{V_N} : V_N \otimes V_N^* \rightarrow \mathbb{L}$$ is given by $v \otimes f \mapsto f(K^{-1}v),$

$$\text{ev}_{V_N} : V_N^* \otimes V_N \rightarrow \mathbb{L}$$ is given by $v \otimes f \mapsto f(K^{-N+1}v),$

for $\{v_j\}$ a basis of $V_N$ and $\{v_j^*\}$ the dual basis of $V_N^*$.

2.1. Weight spaces. In this part, we will consider certain subspaces in tensor powers of $U_q(sl(2))$.

Definition 2.1.1. (Highest weight spaces)

1) Generic case

The $n^{th}$-weight space of the generic Verma module $\hat{V}$ corresponding to the weight $\lambda$ is given by:

$$V_{n,m} := \{v \in \hat{V}^{\otimes n} \mid Kv = s^n q^{-2m} v\}.$$ 

2) The case $q$ generic

The weight space of $\hat{V}_{q,N-1}^{\otimes n}$ corresponding to the weight $m$:

$$V_{n,m}^{N-1} := \{v \in \hat{V}_{q,N-1}^{\otimes n} \mid Kv = q^{n(N-1)-2m} v\}.$$ 

The weight space for the finite dimensional representation $V_N^{\otimes n}$ of weight $m$:

$$V_{n,m}^{N} := \{v \in V_N^{\otimes n} \mid Kv = q^{n(N-1)-2m} v\}.$$ 

Proposition 2.1.2. The representation $\varphi_n^\hat{V}$ induces a well defined action on the generic weight spaces

$$\varphi_{n,m} : B_n \rightarrow \text{Aut}(V_{n,m}).$$

called the generic quantum representation on weight spaces of the Verma module.

We will use the following indexing set:

$$E_{n,m} = \{e = (e_1, \ldots, e_n) \in \mathbb{N}^{n-1} \mid e_1 + \ldots + e_n = m\}.$$

Remark 2.1.3. (Basis for weight spaces) A basis for the generic weight space is given by:

$$\mathcal{B}_{V_{n,m}} = \{v_e := v_{e_1} \otimes \ldots \otimes v_{e_n} \mid e \in E_{n,m}\}.$$
Proposition 2.1.4. Similarly, using the specialisation with generic $q$, we get induced braid group actions as follows.

1) a) $\hat{\varphi}^q_{n,m} : B_n \to \text{Aut}(\hat{V}_{n,m}^q)$

is a well defined action induced by $\hat{\varphi}^q_{n,m}$.

b) $\varphi^q_{n,m} : B_n \to \text{Aut}(V_{n,m}^q)$

is induced by $\varphi^q_n$ called the quantum representation on weight spaces in the finite dimensional module.

Lemma 2.1.5. For the root of unity case, due to the choice of basis, we do not have immediately $U_\lambda$ a submodule over the quantum group. This can be corrected by a change of basis, but for our purpose, we can work with this version. The important part, is that the specialised $R$-matrix at roots of unity, leads to a well defined action onto $U_\lambda \otimes \hat{V}_{n,m}^q$ which commutes with the inclusion:

$B_n \cap U_\lambda \otimes \hat{V}_{n,m}^q \subseteq \hat{V}_{n,m}^q \cap B_n$.

Proof. This can be seen directly from the coefficients of the $R$-matrix and the property that the powers $E^n$ act non-zero to the finite part just if $n \leq N - 1$. In this case, the coefficients of the powers of $F^n$ will contain $\left[ N \right]_\xi \lambda$ which vanish due to the root of unity.

This allows us to define quantum representations for the root of unity case as follows:

Definition 2.1.6. 3) The case with $q = \xi_N$ root of unity and $\lambda \in \mathbb{C}$

The weight space of $\hat{V}_{n,m}^q$ of weight $m$:

$$\hat{V}_{n,m}^q = \{ v \in \hat{V}_{n,m}^q | K v = q^{n\lambda - 2m} v \}.$$  

The weight space of the finite dimensional module $U_\lambda \otimes \hat{V}_{n,m}^q$ corresponding to the weight $m$:

$$V_{n,m}^q = \{ v \in U_\lambda \otimes \hat{V}_{n,m}^q | K v = q^{n\lambda - 2m} v \}.$$  

Proposition 2.1.7. The braid group action from the previous Lemma, induces well defined braid group actions at roots of unity:

2) a) $\hat{\varphi}^q_{n,m} : B_n \to \text{Aut}(\hat{V}_{n,m}^q)$

is a well defined action induced by $\hat{\varphi}^q_{n,m}$.

b) $\varphi^q_{n,m} : B_n \to \text{Aut}(V_{n,m}^q)$

is induced by $\varphi^q_n$ called the quantum representation on weight spaces in the finite dimensional module at root of unity.

2.2. Coloured Jones polynomials as renormalised invariants. The coloured Jones polynomials form a sequence of invariants constructed from the finite dimensional representation $V_N$. In the sequel, we denote by $w : B_n \to \mathbb{Z}$ the map given by the abelianisation.

Proposition 2.2.1. (9) (Coloured Jones polynomial from a braid presentation)

Let us fix $N \in \mathbb{N}$. Consider $L$ be an oriented knot and $\beta \in B_n$ such that $L = \hat{\beta}$ (braid closure) Then, the Reshetikhin-Turaev construction leads to the following formula:

$$J_N(L,q) = \frac{1}{[N]_q} q^{(N-1)w(\beta)} \left( \text{ev}_{V_N} \circ \varphi^q_n \beta_n \otimes \mathbb{I}_n \right) \circ \text{coev}_{V_N} (1).$$
Corollary 2.2.2. Having in mind the normalisation procedure, we can obtain the coloured Jones polynomials by cutting a strand as follows:

\[
J_N(L, q) = q^{-(N-1)w(\beta)} p^\circ \left( (Id \otimes ev_{V_N}) \circ \varphi_{n-1} \cdot \varphi_{n-1}^{-1} \circ (Id \otimes coev_{V_N}) \right)(v_0).
\]

Here \(p : V_N \to \mathbb{Z}[q^\pm]\) is the projection onto the subspace generated by the highest weight vector \(v_0\).

2.3. Coloured Alexander Polynomials.

Proposition 2.3.1. \((8)\) (The ADO invariant from a braid presentation)

Let \(L\) be an oriented knot. Consider \(\beta_n \in B_n\) such that \(L = \hat{\beta}_n\). Then, the ADO invariant of \(L\) can be expressed as follows:

\[
\Phi_N(L, \lambda) = \xi_N^{(N-1)\lambda w(\beta_n)} p^\circ \left( (Id \otimes ev_{U_\lambda}) \circ \varphi_{n-1} \cdot \varphi_{n-1}^{-1} \circ (Id \otimes coev_{U_\lambda}) \right)(v_0).
\]

Here \(p : U_\lambda \to \mathbb{C}\) is the projection onto the subspace generated by the highest weight vector \(v_0\).

3. Lawrence representation

In this part, we will briefly introduce a version of Lawrence representation that will be suitable for our topological models. We will use the definitions from \([16]\). Let us fix two natural numbers \(n\) and \(m\). We consider \(\mathcal{D}_n\) to be the two dimensional disc with boundary, with \(n\)-punctures: \(\mathcal{D}_n = \mathbb{D}^2 \setminus \{1, ..., n\}\). Let us define the unordered configuration space in this punctured disc:

\[
C_{n,m} := \{(x_1, \ldots, x_m) \in (\mathcal{D}_n)^{\times m} \mid x_i \neq x_j, \forall 1 \leq i \leq j \leq m\} / \text{Sym}_m.
\]

For the sequel, we fix \(d_1, \ldots, d_m \in \partial \mathcal{D}_n\) and \(d = (d_1, \ldots, d_m)\) the corresponding base point in the configuration space.

3.1. Homology of the covering space.

Definition 3.1.1. (Local system)

Let \(\rho : \pi_1(C_{n,m}) \to H_1(C_{n,m})\) be the abelianisation map. For \(m \geq 2\), the homology of the unordered configuration space, is known to be:

\[
H_1(C_{n,m}) \simeq \mathbb{Z}^n \oplus \mathbb{Z}
\]

\(\rho(\sigma_i) > \rho(\delta) \quad i \in \{1, ..., n\}\), where \(\sigma_i \in \pi_1(C_{n,m})\) is represented by the loop in \(C_{n,m}\) with \(m - 1\) fixed components and the first one going on a loop in \(D_n\) around the puncture \(p_i\). The generator \(\delta \in \pi_1(C_{n,m})\) is given by a loop in the configuration space with \((m - 2)\) constant points and the first two components which swaps the two initial points. The two generators are presented in the picture below.

\[
\begin{align*}
\text{Let } p : \mathbb{Z}^n \oplus \mathbb{Z} &\to \mathbb{Z} \oplus \mathbb{Z} \text{ be the augmentation map given by:} \\
p(x_1, \ldots, x_m, y) &= (x_1 + \ldots + x_m, y).
\end{align*}
\]
Combining the two morphisms, let us consider the local system:

\[ \phi : \pi_1(C_{n,m}) \to \mathbb{Z} \oplus \mathbb{Z} \]

(19)

\[ \phi = p \circ \rho. \]

**Definition 3.1.2.** (Covering of the configuration space) Let \( \tilde{C}_{n,m} \) be the covering of \( C_{n,m} \) corresponding to the local system \( \phi \). Then, the deck transformations of this covering are given by:

\[
\text{Deck}(\tilde{C}_{n,m}, C_{n,m}) \cong <x><d>.
\]

One of the main tools in this construction is the homology of this covering space. Let us fix a point \( w \in \partial \mathcal{D}_n \).

**Remark 3.1.3.** Let \( H_{m-}^{Hf} (\tilde{C}_{n,m}, \mathbb{Z}) \) be the Borel-Moore Homology and relative to part of the boundary represented by the fiber over the base point \( w \).

**Proposition 3.1.4.** The braid group action from the mapping class group and the Deck transformations action are compatible at the homological level:

\[
B_n \rtimes H_{m-}^{Hf} (\tilde{C}_{n,m}, \mathbb{Z}) \quad \text{as a module over } \mathbb{Z}[x^\pm, d^\pm].
\]

**3.2. 1A) Lawrence representation.** In this part, we recall the definition of the homological Lawrence representations of the braid groups, which we will use as input for the setting of topological traces.

**Definition 3.2.1.** (Multiarcs [16])

Let \( d_1, \ldots, d_m \in \partial \mathcal{D}_n \) points on the boundary which give a base point \( d = \{d_1, \ldots, d_n\} \) in the configuration space. Let us consider a partition \( e \in E_{n,m} \). Then, for each component \( i \in \{1, \ldots, n-1\} \), we consider the space of configurations of \( e_i \) points on the segment in \( D_n \), which starts at the point \( w \) and finishes at the point \( p_i \), as in the figure. Let us denote the projection onto the configuration space by:

\[ \pi_m : D_n^m \setminus \{x = (x_1, \ldots, x_M) | x_i = x_j\} \to C_{n,m} \]

Then, the product of these configuration spaces leads to a submanifold in the configuration space:

\[ F_e := \pi_m(\text{Conf}_{e_1} \times \ldots \times \text{Conf}_{e_{n-1}}) \]

Now, we fix a set of paths to the fixed points: \( \eta_e^k : [0, 1] \to D_n \) as in the picture. The product of these paths gives a path in the configuration space. The set of paths \( \eta_e^k \), going from the segments to \( d \), leads to a path in the configuration space:

\[ \eta_e := \pi_m \circ (\eta_1^e, \ldots, \eta_m^e) : [0, 1] \to C_{n,m}. \]
Let us consider the unique lift of the path $\eta^e$ and denote it by $\tilde{\eta}$ such that
\[
\begin{aligned}
\tilde{\eta} &: [0,1]^m \to \tilde{C}_{n,m} \\
\tilde{\eta}(0) &= d.
\end{aligned}
\]  

3) Multiarcs

Let $\tilde{F}_e$ be the unique lift of the submanifold $F_e$ such that
\[
\begin{aligned}
\tilde{F}_e &: (0,1)^m \to \tilde{C}_{n,m} \\
\tilde{\eta}(1) &= \tilde{F}_e.
\end{aligned}
\] Using this submanifold, we obtain a class in the Borel-Moore homology $[	ilde{F}_e] \in H_{lf, m}^{H_m}(\tilde{C}_{n,m}, \mathbb{Z})$.

This is called the multiarc corresponding to the partition $e \in E_{n,m}$. We denote by $[	ilde{U}_e] \in H_{lf, m}^{H_m}(\tilde{C}_{n,m}, \mathbb{Z})$.

\begin{proposition}
\label{prop:multiarc_basis}
(3.2.2) [16] The set of all multiarcs $\{[\tilde{F}_e] \mid e \in E_{n,m}\}$ is a basis for $H_{lf, m}^{H_m}(\tilde{C}_{n,m}, \mathbb{Z})$.
\end{proposition}

\begin{notation}
(3.2.3) (Normalised multiarc) For $e \in E_{n,m}$, let us consider a normalisation of the multiarc given by:
\[
\mathcal{F}_e := x^{\sum_{i=1}^m i e_i} [\tilde{F}_e] \in H_{lf, m}^{H_m}(\tilde{C}_{n,m}, \mathbb{Z}).
\]
Then $\mathcal{P}_{H_{n,m}} := \{\mathcal{F}_e, e \in E_{n,m}\}$ is a basis for $H_{n,m}$.
\end{notation}

\begin{notation}
(3.2.4) (Code sequence)
\[
\mathcal{U}_e := [\tilde{U}_e] \in H_{lf, m}^{H_m}(\tilde{C}_{n,m}, \mathbb{Z}).
\]
\end{notation}

\begin{proposition}
(3.2.5) (Relation multiarcs and code sequences) Combining the relation between configurations on segments and multi-segments with relations concerning the braking of an arc by a puncture from [16], we have:
\[
\mathcal{U}_e = \prod_{i=1}^n (e_i)! \mathcal{F}_e.
\]
Here, $(i)_d = \frac{1-d^i}{1-d}$. This shows that:
\[
\mathcal{F}_e = \prod_{i=1}^n (e_i)! \mathcal{U}_e.
\]

For our case, we will use a Poincaré-Lefschetz duality, between middle dimensional homologies of the covering space with respect to the different parts of the boundary. For the computational part, we will change slightly the infinity part of the configuration space and denote the following
\[
H_{n,m} := H_{lf, m}^{H_m}(\tilde{C}_{n,m}, \mathbb{Z})
\] the homology relative to the infinity part that encodes the boundary of the configuration space consisting in the multipoints that touch a puncture from the punctured disc or the base point $w$.
\[
H_{n,m}^b := H_{lf, m}^{H_m}(\tilde{C}_{n,m}, \partial \mathbb{Z})
\] the homology relative to the boundary of $\tilde{C}_{n,m}$ which is not in the fiber over $w$ and relative to the Borel-Moore part which corresponds to collisions of points in the configuration space.
We present the detailed definition of this construction in [4]. In the sequel, we use the classes which we discussed before, seen in the modified version of the homology $H_{n,m}$. However, following [4], all relations between the homology classes still hold in this version of the homology.

**Notation 3.2.6. (Lawrence representation)**

We denote the braid group action in the basis given by multiarcs by:

$$l_{n,m} : B_n \rightarrow Aut(H_{n,m}).$$

### 3.3. Identification between weight space representations and homological representations.

The advantage of the basis presented in the above section consists in the fact that this correspond to the basis in the generic weight space given by monomials. More precisely, we have the following:

**Notation 3.3.1.** We will use the following specialisation:

$$\gamma : \mathbb{Z}[x^\pm, d^\pm] \rightarrow \mathbb{Z}[q^\pm, s^\pm]$$

$$\gamma(x) = s^2; \gamma(d) = q^{-2}.$$  

**Theorem 3.3.2. ([16])** The quantum representation on weight spaces is isomorphic to the homological representation of the braid group:

$$B_n \rtimes \hat{V}_{n,m} \simeq H_{n,m} \mid \gamma \rtimes B_{H_{n,m}}(\hat{\phi}_{n,m}, \hat{B}_{\hat{V}_{n,m}}) \mid l_{n,m} \mid \gamma \rtimes B_{H_{n,m}}$$

**(26)**

$$\Theta(v_1 \otimes \ldots \otimes v_{n-1}) = \mathcal{F}_e.$$  

### 3.4. Intersection pairing.

In this part, we present the duality that leads to a topological pairing between the two types of homology of the covering space:

$$<,> : H_{n,m} \otimes H^\partial_{n,m} \rightarrow \mathbb{Z}[x^{\pm 1}, d^{\pm 1}].$$

**Definition 3.4.1. (Intersection form in the covering space) ([5])**

Consider two homology classes $\mathcal{F} \in H_{n,m}, \mathcal{G} \in H^\partial_{n,m}$. Suppose that these classes are represented by two $m$-manifolds $\tilde{M}, \tilde{N} \subseteq \tilde{C}_{n,m}$, which intersect transversely such that:

$$\mathcal{F} = [\tilde{M}]; \mathcal{G} = [\tilde{N}]$$

$$\text{card } |\tilde{M} \cap t\tilde{N}| < \infty, \forall t \in \text{Deck}(\tilde{C}_{n,m}, C_{n,m}).$$

Then, the intersection form is given by:

$$< [\tilde{M}], [\tilde{N}] >= \sum_{(u,v) \in \mathbb{Z}^2} (x^u d^v \tilde{M}, \tilde{N}) x^u d^v.$$

(here, $(\cdot, \cdot)$ is the usual geometric intersection number)

In the following, we will see that if the submanifolds are actually lifts of submanifolds from the configuration space, the intersection pairing in the covering space is encoded in the base space.

Suppose that there exist immersed submanifolds $M, N \subseteq C_{n,m}$ which intersect transversely in a finite number of points such that $\tilde{M}$ is a lift of $M$ through $\tilde{d}$ and $\tilde{N}$ is a lift of $N$ through $d$.

**Proposition 3.4.2. (Computing the intersection pairing from the base space and the local system)**

Let $x \in M \cap N$. We will construct an associated loop $l_x \subseteq C_{n,m}$. Let us denote the geometric intersection number between $M$ and $N$ in $x$ by $\alpha_x$.

a) Construction of $l_x$
Suppose we have two paths $\gamma_M, \delta_N : [0, 1] \to C_{n,m}$ such that:

\[
\begin{align*}
\gamma_M(0) &= d; \gamma_M(1) \in M; \tilde{\gamma}_M(1) \in \tilde{M} \\
\gamma_N(0) &= d; \gamma_N(1) \in N; \tilde{\gamma}_N(1) \in \tilde{N}
\end{align*}
\]

where $\tilde{\gamma}_M, \tilde{\gamma}_N$ are the unique lifts of $\gamma_M, \gamma_N$ through $\tilde{d}$.
Moreover, consider $\bar{\gamma}_M, \bar{\delta}_N : [0, 1] \to C_{n,m}$ such that:

\[
\begin{align*}
\text{Im}(\bar{\gamma}_M) \subseteq M; \bar{\gamma}_M(0) = \gamma_M(1); \bar{\gamma}_M(1) = x \\
\text{Im}(\bar{\delta}_N) \subseteq N; \bar{\delta}_N(0) = \gamma_N(1); \bar{\delta}_N(1) = x.
\end{align*}
\]

Then, consider the loop as follows:

\[l_x := \delta_N \circ \bar{\delta}_N \circ \bar{\gamma}_M^{-1} \circ \gamma_M^{-1}.\]

b) Formula for the intersection form

Then, using this loops and the local system we obtain the intersection form as follows:

\[
< [\tilde{M}], [\tilde{N}] > = \sum_{x \in M \cap N} \alpha_x \cdot \phi(l_x) \in \mathbb{Z}[x^\pm, d^\pm].
\]

4. Construction of the generic homology classes

In this part, we construct certain homology classes in the generic Lawrence representation, which correspond to the evaluation and coevaluation. Since the coevaluation for the root of unity case and generic case differ by a coefficient, we will take this into account in our construction. In the following sections, we will prove that when specialised to the two cases, the generic case and the root of unity case, they lead to the coloured Jones polynomial and coloured Alexander polynomial respectively.

Remark 4.0.1. The main point is that the $R$-matrix for the construction of the coloured Alexander polynomial, whose formula is presented in [2], is obtained from the generic $R$-matrix that we described by the specialisation $\eta_{\kappa, \lambda}$. Thus, at the level of braid group representations, we can use the generic $R$-matrix and then specialise it. The difference occurs in the dualities presented in [4].

Let us start with a knot $K$ that can be presented as a closure of a braid $\beta_n$ with $n$ strands. The Reshetikhin-Turaev construction is obtained from the functor applied to the three main levels of the diagram corresponding to the caps, cups and the braid: As we have seen before, for both coloured Jones case and coloured Alexander case, the braid part comes from the generic action $\hat{\phi}^\vee$. In this part, we will use the definition of the coevaluation and evaluation for the two situations.

1) the evaluation
2) braid level $\beta_n \otimes I_{n-1}$.
3) the coevaluation $\uparrow \otimes \uparrow$

Definition 4.0.2. Let us consider the generic coevaluation up to level $N$ as follows:

\[
\begin{align*}
\text{coev}_N & : \mathbb{Z}[q^{\pm 1}, s^{\pm 1}] \to \hat{V} \otimes \hat{V}^\vee \otimes (\hat{V}^\vee) \otimes I_{n-1} \\
\text{coev}_N(1) &= \sum_{i_1, \ldots, i_{n-1} = 0}^{N-1} v_{i_1} \otimes \ldots \otimes v_{i_{n-1}} \otimes (v_{i_{n-1}})^* \otimes \ldots \otimes (v_{i_1})^*.
\end{align*}
\]
Since coevaluations are morphisms that commute with the $K$ action, we know that:

$$
\begin{cases}
K \left( \Id \otimes \coev_{V_{n-1}} \right) (v_0) = q^{N-1} v_0 = q^{(2n-1)\lambda - 2(n-1)(N-1)} \\
K \left( \Id \otimes \coev_{U_{n-1}} \right) (v_0) = q^{\lambda} v_0 = q^{(2n-1)\lambda - 2(n-1)(N-1)}.
\end{cases}
$$

In other words, the coevaluation from above arrives in the weight space of weight $(n-1)(N-1)$ inside the mixt tensor product $\hat{V}^\otimes n-1 \otimes (\hat{V}^*)^\otimes n-1$.

### 4.1. Construction

In the sequel, we will modify the evaluation and coevaluation by an isomorphism such that it will make the computation more simple on the homological part.

**Notation 4.1.1.** We will use the generic vector space generated by the first $N$ vectors:

$$G_N := \langle v_0, \ldots, v_{N-1} \rangle \subseteq \hat{V}.$$  

For a morphism of vector spaces $f : (\hat{V}^*)^\otimes n-1 \rightarrow \hat{V}^\otimes n-1$ of the type:

$$f \left( v_{n-1}^* \otimes \cdots \otimes v_i^* \right) = v_{N-1-i_{n-1}} \otimes \cdots \otimes v_{N-1-i_1},$$

we denote the deformed evaluation and coevaluation as follows:

$$\begin{align*}
\coev_f : & \mathbb{L} \rightarrow \hat{V}_{2n-2,(n-1)(N-1)} \\
\ev_f : & \hat{V}_{2n-2,(n-1)(N-1)} \rightarrow \mathbb{L}
\end{align*}
$$

$$\begin{align*}
\coev_f & = (\Id_{G_N}^{n-1} \otimes f_n) \circ \coev_{G_N}^{n-1} \\
\ev_f & = (\Id_{G_N}^{n-1} \otimes f_n^{-1})
\end{align*}
$$

**Proposition 4.1.2.** If we act with $\beta_n \otimes I_{n-1}$, we could use the deformed evaluation and coevaluation instead of the usual ones and we have:

$$\begin{align*}
\left((\Id \otimes \coev_{G_N}^{n-1}) \circ \varphi_n^\vee (\beta_n \otimes I_{n-1}) \circ (\Id \otimes \coev_{G_N}^{n-1})\right) (v_0) = \\
\left((\Id \otimes \coev_f^{n-1}) \circ \varphi_n^\vee (\beta_n \otimes I_{n-1}) \circ (\Id \otimes \coev_f^{n-1})\right) (v_0).
\end{align*}
$$

**Proof.** We just have to notice that on the part where $f$ is nontrivial, the braid is trivial, so all what we have to do is to reverse the orientation of the strands and then cancel $f$ with $f^{-1}$. \hfill \Box

**Definition 4.1.3.** *(Choice of normalisation)* We will work at a certain point over a slightly bigger ring where we invert the quantum factorials smaller than $N-1$:

$$\mathbb{L}_N := \mathbb{Z}[(q^\pm, s^\pm)](I_N)^{-1}.$$  

$$I_N := \langle (k)_q | 0 \leq k \leq N - 1 \rangle.$$  

Then we consider $\bar{\gamma}_N : \mathbb{Z}[x^\pm, d^\pm] \rightarrow \mathbb{L}_N$ the extension of $\gamma$ using this new ring.

For our purpose, in order to pass from a multiarc to a code sequence, we will need to invert the coefficients as follows:

$$C_{i_1, \ldots, i_{n-1}} = \frac{1}{\prod_{k=1}^{n-1} (i_k)_q! (N-1-i_k)_q!}.$$  

**Remark 4.1.4.** The specialisations $\eta_{i,N-1}$ and $\eta_{i,N-1,\lambda}$ are well defined on $\mathbb{L}_N$. Here, we use that $\xi_N$ is a root of unity of order $2N$.  

Definition 4.1.5. (Evaluations in the generic case and root of unity case)

If we consider the evaluations for the coloured Jones case and for the coloured Alexander case, they differ by the fact that one uses the $K^{-1}$ action whereas the other has the action of $K^{N-1}$, at the root of unity. This means that:

$$
\overline{ev}_V (w) = \begin{cases} 
  s^{-(n-1)}q^2 \sum_{i=1}^{n} \epsilon_i, & w = v_{i_1} \otimes \ldots \otimes v_{i_{n-1}} \otimes (v_{i_{n-1}})^* \otimes \ldots \otimes (v_{i_1})^* \\
  0, & \text{otherwise}
\end{cases}
$$

by the specialisation $\eta_{\nu,N-1}$.

$$
\overline{ev}_{\nu,\lambda} (w) = \begin{cases} 
  s^{-(n-1)(N-1)}q^{2(N-1)} \sum_{i=1}^{n-1} \epsilon_i, & w = v_{i_1} \otimes \ldots \otimes v_{i_{n-1}} \otimes (v_{i_{n-1}})^* \otimes \ldots \otimes (v_{i_1})^* \\
  0, & \text{otherwise}
\end{cases}
$$

by the specialisation $\eta_{\nu,N,\lambda}$.

Then, we denote the polynomials that encode these evaluation maps as follows:

$$
\begin{align*}
\hat{p}_N^{N-1}(i_1,\ldots,i_{n-1}) &= s^{-(n-1)}q^2 \sum_{i=1}^{n-1} \epsilon_i \\
\hat{p}_N^{N-1}(i_1,\ldots,i_{n-1}) &= s^{-(n-1)(N-1)}q^{2(N-1)} \sum_{i=1}^{n-1} \epsilon_i.
\end{align*}
$$

4.2. First homology class.

Definition 4.2.1. Let us consider $\mathcal{P}_n^N \in H_{2n-1,(n-1)(N-1)}$ given by:

$$
\mathcal{P}_n^N := \sum_{i_1,\ldots,i_{n-1}=0}^{N-1} \mathcal{P}_{0,i_1,\ldots,i_{n-1},N-1-i_{n-1},\ldots,N-1-i_1}.
$$

4.3. Identification using code sequences. For our case, it will be more convenient to use the basis of code sequences instead of multiarcs. In the sequel, we will see that actually for our type of braid actions, we can use this basis instead.

$$(B_n \cup \mathbb{L}_{n-1}) \cap \hat{V}_{2n-1,m} \simeq H_{2n-1,m} \cap (B_n \cup \mathbb{L}_{n-1})$$

(40)

$$
\Theta(v_{e_1} \otimes \ldots \otimes v_{e_{2n-1}}) = \mathcal{F}_e.
$$

Lemma 4.3.1. Let $g : \hat{V}_{2n-1,m} \to \hat{V}_{2n-1,m}$ an isomorphism of vector spaces such that:

$$
g(v_{e_1} \otimes \ldots \otimes v_{e_{2n-1}}) = y(e_n,\ldots,e_{2n-1})v_{e_1} \otimes \ldots \otimes v_{e_{2n-1}}
$$

with $y \in \hat{L}_N$ a rational function which depends just only on the last $n-1$ coordinates.

Then

$$
\Theta \circ g : \hat{V}_{2n-1,m} \mid_{\mathbb{L}} \simeq H_{2n-1,m} \mid_{\hat{L}}
$$

is still equivariant with respect to the $B_n \cup \mathbb{L}_{n-1}$-action and the bases correspond as follows:

$$
\Theta \circ g(v_{e_1} \otimes \ldots \otimes v_{e_{2n-1}}) = \mathcal{F}_e
$$

Remark 4.3.2. This lemma will allow us in the specialised case to use the code sequences instead of the normalised multiarcs. We will have to work over $\hat{L}$, but actually the elements that we are interested in will be defined over $\mathbb{L}$, so overall we will obtain the results over $\mathbb{L}$.

4.4. Second homology class.

Definition 4.4.1. Let us consider the second homology class $\mathcal{Q}_n^N \in H_{n,m}^\partial$ given by the product of $(n-1)$ configuration spaces of $N-1$ points on eights around the symmetric punctures, as in the picture.
(41)

**Remark 4.4.2.** (Figure-eight intersection) Looking at the intersection pairing, we notice that:

\[
\langle \mathcal{U}_{0, i_1, \ldots, i_{n-1}}, N-1-i_{n-1}, \ldots, N-1-i_1, \mathcal{G}_n^N \rangle = x^{-m(i_1, \ldots, i_{n-1})} d^{-m'(i_1, \ldots, i_{n-1})},
\]

where \( m, m' \) are polynomials in \((n-1) - \) variables.

**Proof.** We notice that each configuration of \((N-1)\) points on a fixed figure eight around the point \(i\) and \(2n - 1 - i\) intersects uniquely the collection of red segments with multiplicities \((e_i, N - 1 - e_i)\). Doing this for all pairs indexed by \(i \in \{1, \ldots, n-1\}\), we conclude that \(\mathcal{U}_{0, i_1, \ldots, i_{n-1}}, N-1-i_{n-1}, \ldots, N-1-i_1\) and \(\mathcal{G}_n^N\) intersect in an unique point in the configuration space. Thus, we only need to compute the scalar that corresponds to this point, which will be an element from the Deck transformation group, given by a monomial in \(x\) and \(d\) which we denote by \(x^{-m(i_1, \ldots, i_{n-1})} d^{-m'(i_1, \ldots, i_{n-1})}\). \(\Box\)

**Definition 4.4.3.** (Global classes) Let us consider \(\mathcal{F}_n^N, \mathcal{F}_n^{\xi_n} \in H_{2n-1, (n-1)(N-1)}\) given by:

\[
\mathcal{F}_n^N := \sum_{i_1, \ldots, i_{n-1}=0}^{N-1} x^{-m(i_1, \ldots, i_{n-1})} d^{-m'(i_1, \ldots, i_{n-1})} p_n^N (i_1, \ldots, i_{n-1})
\]

\[
\mathcal{F}_n^{\xi_n} := \sum_{i_1, \ldots, i_{n-1}=0}^{N-1} x^{-m(i_1, \ldots, i_{n-1})} d^{-m'(i_1, \ldots, i_{n-1})} p_n^{\xi_n} (i_1, \ldots, i_{n-1})
\]

5. **Topological intersection model for the coloured Jones invariants**

In this part, we show the model from the main theorem for the Coloured Jones polynomials. We remind the definition from [12]

\[
J_N(L,q) := q^{-(N-1)w(\beta)} \pi \circ \left( (Id \otimes \text{ev}_{V_N}) \circ \varphi_N^{\xi_n} (\beta_n \otimes {I}_{n-1}) \circ (Id \otimes \text{coev}_{V_N} \otimes {n-1}) \right) (v_0).
\]
5.1. **Step I.** First, we will add the extra morphism coming from the normalisation \( f_n \). We notice that we can use this in order to describe the invariant through the particular weight space of weight \((n-1)(N-1)\) as follows:

\[
J_N(L, q) = q^{-(N-1)w(\beta)} \pi \circ (Id \otimes ev_f^{\otimes n-1} |_{\eta_{N-1}})
\]

\[
\circ \varphi_n (\beta_n \otimes \Pi_{n-1}) \circ (Id \otimes ev_f |_{\eta_{N-1}}) \circ (Id \otimes covev_{\Pi_{n-1}} |_{\eta_{N-1}})(v_0).
\]

Using the normalised evaluation and coevaluations, we have the following expression:

\[
J_N(L, q) = q^{-(N-1)w(\beta)} \pi \circ (Id \otimes ev_f^{\otimes n-1} |_{\eta_{N-1}})
\]

\[
\circ \varphi_{2n-1,(n-1)(N-1)} (\beta_n \otimes \Pi_{n-1}) \circ (Id \otimes covev_f^{\otimes n-1} |_{\eta_{N-1}})(v_0).
\]

Then, if we embed the small weight spaces into the generic ones by inclusion \( \iota : V^{N}_{2n-1,(n-1)(N-1)} \hookrightarrow V^{N}_{2n-1,(n-1)(N-1)} \), we have:

\[
J_N(L, q) = q^{-(N-1)w(\beta)} \pi \circ (Id \otimes ev_f^{\otimes n-1} |_{\eta_{N-1}})
\]

\[
\circ \varphi_{2n-1,(n-1)(N-1)} (\beta_n \otimes \Pi_{n-1}) \circ \iota \circ (Id \otimes covev_f^{\otimes n-1} |_{\eta_{N-1}})(v_0).
\]

5.2. **Step II.** Now, we use the property that the quantum representation specialised by \( \psi_{q,N-1} \) preserves small weight spaces inside the weight spaces from generic Verma module. Using this, we obtain:

\[
J_N(L, q) = q^{-(N-1)w(\beta)} \pi \circ (Id \otimes ev_f^{\otimes n-1} |_{\eta_{N-1}})
\]

\[
\circ \varphi_{2n-1,(n-1)(N-1)} (\beta_n \otimes \Pi_{n-1}) \circ \iota \circ (Id \otimes covev_f^{\otimes n-1} |_{\eta_{N-1}})(v_0) =
\]

\[
= q^{-(N-1)w(\beta)} \pi \circ (Id \otimes ev_f^{\otimes n-1} |_{\eta_{N-1}})
\]

\[
\circ \Theta^{-1} \circ \varphi_{2n-1,(n-1)(N-1)} (\beta_n \otimes \Pi_{n-1}) \circ \Theta \circ (Id \otimes covev_f^{\otimes n-1} |_{\eta_{N-1}})(v_0).
\]

Now, we study the identification \( \Theta \) specialised by \( \psi_{q,N-1} \). This preserves the small weight spaces into the ones in the Verma module. Then, we will modify this identification by a function \( g_N \) as in Lemma 4.3.1 by correcting with the inverse of the coefficient between the code sequence and the normalised multiarc from formula 23 on the vectors from the small weight space \( V^{N}_{2n-1,(n-1)(N-1)} \hookrightarrow V^{N}_{2n-1,(n-1)(N-1)} \) of the form:

\[
v_0 \otimes v_1 \otimes \ldots \otimes v_{n-2} \otimes v_{N-1} - i_1 \otimes \ldots \otimes v_{N-1} - i_{n-2}.
\]

Since on the top level, the evaluation sees anyway just this kind of vectors, using the Lemma 4.3.1, we can express the invariant as:

\[
J_N(L, q) = q^{-(N-1)w(\beta)} \pi \circ (Id \otimes ev_f^{\otimes n-1} |_{\eta_{N-1}})
\]

\[
\circ g_N^{-1} \circ \Theta^{-1} \circ \varphi_{2n-1,(n-1)(N-1)} (\beta_n \otimes \Pi_{n-1}) \circ \Theta \circ g_N
\]

\[
\circ \iota \circ (Id \otimes covev_f^{\otimes n-1} |_{\eta_{N-1}})(v_0).
\]

5.3. **Step III.** Using the identification between the quantum and homological representations from Lemma 4.3.1 we have:

\[
J_N(L, q) = q^{-(N-1)w(\beta)} \pi \circ (Id \otimes ev_f^{\otimes n-1} |_{\eta_{N-1}})g_N^{-1} \circ \Theta^{-1}
\]

\[
\circ \iota_{2n-1,(n-1)(N-1)} |_{\psi_{q,N-1}} (\beta_n \otimes \Pi_{n-1}) \Theta^{-1}_N.
\]
5.4. Step IV. In this part, we will discuss which basis we will use in the homological Lawrence representation. In order to pass from the evaluation to the geometric part, we notice that geometrically, we need a submanifold which intersects Ψ_N non-empty if and only if e is a partition symmetric up to the N − 1 reflection with respect to the middle of the disc. In other words, exactly the partitions that occur from the evaluation in the formulas for Ψ_N. Then from the shapes of figure eights that are building blocks for Ψ_N, we see directly by computing the intersection form that they have this property.

We conclude:

\[ J_N(L, q) = \text{Def} \sum_{\text{Def} \sum_{\text{Def} \sum_{\text{Def}}}} \]

We conclude:

\[ J_N(L, q) = D_{e, B} q^{-(N-1)w(\beta)} \pi \circ (\text{Id} \otimes \text{ev}^{\otimes n-1} \mid_{\eta_q, N-1}) g_{N-1} \circ \Theta^{-1} \]

\[ = q^{-(N-1)w(\beta)} \pi \circ \sum_{i_1, \ldots, i_{n-1} = 0}^{N-1} (\text{Id} \otimes \text{ev}^{\otimes n-1} \mid_{\eta_q, N-1}) g_{N-1} \circ \Theta^{-1} \]

This concludes the topological model for the coloured Jones invariants.

6. Topological intersection model for the coloured Alexander invariants

Following equation 52, the coloured Alexander invariant can be expressed as:

\[ \Phi_N(L, \lambda) = \xi_N \langle (N-1)^{\lambda w(\beta)} \rangle \rho \]

\[ = q^{-\langle (N-1)^{\lambda w(\beta)} \rangle} \Phi_n^L(\beta_n \otimes \Pi_{n-1}) \circ (\text{Id} \otimes \text{ev}^{\otimes n-1} \mid_{(n-1)}) (v_0). \]
6.1. **Step I.** (Using the normalised evaluations) For the root of unity case, there is a subtlety related to the version of the quantum group that we are using, coming from the divided powers of the generator $F$. In [3], there is given the definition of the coloured Alexander invariant from the usual quantum group $U_q(sl(2))$. The main point is that if we look at the $R$-matrix, its action on elementary vectors has the same formula as the one that we have, after we specialise at a root of unity. So, at the level of the braid group action, we can use the version of the $R$-matrix presented above. The subtlety occurs at the level of the module. The problem, is that this version $\hat{V}_{\xi,\lambda}$ do not gain an $N$-dimensional submodule as in the case of the quantum group with the usual generators. However, we consider $U_\lambda$ the vector subspace generated by the first $N$-vectors inside the Verma one. The main point now is that we need to deal just with weight spaces and $R$-matrix action. The remark is that even if $U_\lambda$ is not a submodule, its tensor power inside the tensor power of the specialised Verma module is preserved by the braid group action as in Lemma 2.1.5. We conclude the following:

**Proposition 6.1.1.** The inclusion of the weight space corresponding to the finite dimensional part at root of unity into the one inside the corresponding Verma module is preserved by the braid group action:

$$B_n \rhd \overset{\equiv}{\cong} B_n \rhd$$

$$V_{\xi_{n,m}} \hookleftarrow \hat{V}_{\xi_{n,m}}$$

6.2. **Step II.** Secondly, looking at the coevaluation from the root of unity, it has the same formula as the formula from [32]. On the other hand, the evaluation at roots of unity, is exactly the evaluation $ev_{U_\lambda}$ by the specialisation of the coefficients $\eta_{\xi,\lambda}$ following equation 37.

6.3. **Step III.** Putting all together, we conclude that we can obtain the coloured Alexander polynomials through the specialisation $\eta_{\xi,\lambda}$ of the generic weight spaces as follows:

$$\Phi_N(L, \lambda) = \xi_N^{(N-1)\lambda w(\beta_\alpha)} \pi \circ (Id \otimes ev_{U_\lambda}) (Id \otimes f^{-1}) |_{\eta_{\xi,\lambda}}$$

$$\circ \varphi_n^{U_\lambda}(\beta_n \otimes \mathbb{I}_{n-1}) \circ (Id \otimes f |_{\eta_{\xi,\lambda}}) \circ (Id \otimes coev_{U_\lambda}) (v_0) =$$

$$\xi_N^{(N-1)\lambda w(\beta_\alpha)} \pi \circ (Id \otimes ev_{f} |_{\eta_{\xi,\lambda}})$$

$$\circ \varphi_{2n-1,(n-1)(N-1)}(\beta_n \otimes \mathbb{I}_{n-1}) \circ (Id \otimes coev_{f} |_{\xi_{N,\lambda}}) (v_0).$$

(54)

6.4. **Step IV.** Similar to the generic case, using that the inclusion of the weight spaces commutes with the $B_n$ action we conclude:

$$\Phi_N(L, \lambda) = \xi_N^{(N-1)\lambda w(\beta_\alpha)} \pi \circ (Id \otimes ev_{f} |_{\eta_{\xi,\lambda}})$$

$$\circ \varphi_{2n-1,(n-1)(N-1)}(\beta_n \otimes \mathbb{I}_{n-1}) \circ (Id \otimes coev_{f} |_{\xi_{N,\lambda}}) (v_0).$$

(55)

6.5. **Step V.** Passing to the homological part by composing with the isomorphism $\Theta$ we obtain:

$$\Phi_N(L, \lambda) = \xi_N^{(N-1)\lambda w(\beta_\alpha)} \pi \circ (Id \otimes ev_{f} |_{\eta_{\xi,\lambda}})$$

$$\circ \varphi_{2n-1,(n-1)(N-1)}(\beta_n \otimes \mathbb{I}_{n-1}) \circ \Theta^{-1} \circ (Id \otimes coev_{f} |_{\xi_{N,\lambda}}) (v_0).$$

(56)
6.6. **Step VI.** Now, we modify slightly the isomorphism using the function $g_N$ such that we arrive at the basis of code sequences, which is more convenient for us. We would like to emphasise here, that there is a subtle point related to the coefficients that occur in the function $g_N$, which contain non-trivial denominators. However, we use the normalisation just for the vectors from the small weight space, whose corresponding normalisation coefficients have quantum factorials smaller or equal than $N-1$ (since the components of the vectors are all smaller than that) and so do not vanish at the root of unity. Having this in mind, we obtain:

$$\Phi_N(L, \lambda) = \xi_N^{(N-1)\lambda w(\beta_n)} \circ (\text{Id} \otimes \text{ev}_f^\otimes n-1 |_{\eta_{n,N,\lambda}})$$

$$\circ g_N^{-1} \circ \Theta^{-1} \circ \varphi^{2N-1}_{2n-1,(n-1)(N-1)}(\beta_n \otimes I_{n-1}) \circ \Theta \circ g_N$$

$$\circ \iota \circ (\text{Id} \otimes \text{coev}_f^\otimes n-1 |_{\xi_{NI,\lambda}})(v_0).$$

6.7. **Step VII.** Moving the coevaluation on the topological side, we obtain:

$$\Phi_N(L, \lambda) = \xi_N^{(N-1)\lambda w(\beta_n)} \circ (\text{Id} \otimes \text{ev}_f^\otimes n-1 |_{\eta_{n,N,\lambda}})g_N^{-1} \circ \Theta^{-1}$$

$$\circ \iota_{2n-1,(n-1)(N-1)}|_{\psi_{\xi_{NI,\lambda}}(\beta_n \otimes I_{n-1})} \mathcal{P}_n^N.$$

6.8. **Step VIII.**

$$\mathcal{P}_n^N : \mathcal{U}_{0,i_1,...,i_{n-1},N-1-i_{n-1},...,N-1-i_1}$$

$$\mathcal{Q}_n^N : \mathcal{U}_{0,i_1,...,i_{n-1},N-1-i_{n-1},...,N-1-i_1}$$

In the last part, we will move the coevaluation on the homological side, in a similar manner as we did for the generic case, and correct accordingly the classes...
corresponding to the evaluation changing from \( \mathcal{P}_n^N \) to \( \mathcal{P}_n^{\xi N} \):

\[
\Phi_N(L, \lambda) = \partial_{\mathcal{P}_n^{\xi N}} \xi_N^{-1} \lambda (\beta_n) \pi \circ (\text{Id} \otimes \xi^{n-1}_{\varphi, \lambda}) g_N^{-1} \circ \Theta^{-1}
\]

\[
= \xi_N^{-1} \lambda (\beta_n) \pi \sum_{i_1, \ldots, i_{n-1} = 0}^{N-1} \mathcal{U}_{0, i_1, \ldots, i_{n-1}, N-1-i_{n-1}, \ldots, N-1-i_1} (\text{Id} \otimes \xi^{n-1}_{\varphi, \lambda}) g_N^{-1} \circ \Theta^{-1}
\]

\[
= \xi_N^{-1} \lambda (\beta_n) \pi \sum_{i_1, \ldots, i_{n-1} = 0}^{N-1} \mathcal{U}_{0, i_1, \ldots, i_{n-1}, N-1-i_{n-1}, \ldots, N-1-i_1} (\text{Id} \otimes \xi^{n-1}_{\varphi, \lambda}) g_N^{-1} \circ \Theta^{-1}
\]

This concludes the intersection formula for the family of the coloured Alexander invariants.

References

[1] Y. Akustu, T. Deguchi, T. Ohtsuki - Invariants of colored links, J. Knot Theory Ramifications 1 161-184, (1992).
[2] C. Anghel - A topological model for the coloured Alexander invariants, math.GT arXiv:1906.04056, 41 pages, (2019).
[3] C. Anghel - A topological model for the coloured Jones polynomials, math.GT arXiv:1712.04873v2, 50 pages, (2019).
[4] C. Anghel, M. Palmer - Poincaré duality, pairings and bases for different flavours of the Lawrence representations of braid groups
[5] Stephen Bigelow - A homological definition of the Jones polynomial. In Invariants of knots and 3-manifolds (Kyoto, 2001), volume 4 of Geom. Topol. Monogr., pages 29-41. Geom. Topol. Publ., Coventry, (2002).
[6] Stephen Bigelow - Homological representations of the Iwahori-Hecke algebra, Geometry and Topology Monographs, Volume 7: Proceedings of the Casson Fest, Pages 493-507, (2004).
[7] Tetsuya Ito - Reading the dual Garside length of braids from homological and quantum representations. Comm. Math. Phys., 335(1):345-367, (2015).
[8] Tetsuya Ito - A homological representation formula of colored Alexander invariants Adv. Math. 289, 142-160, (2016).
[9] Tetsuya Ito - Topological formula of the loop expansion of the colored Jones polynomials, Trans. Amer. Math. Soc., (2019)
[10] C. Jackson, T. Kerler - The Lawrence-Krammer-Bigelow representations of the braid groups via \( U_q(sl_2) \), Adv. Math. 228, 1689-1717, (2011).
[11] R. Kashaev - The hyperbolic volume of knots from the quantum dilogarithm, Lett. Math. Phys. 39, 269-275, (1997).
[12] Toshitake Kohno - Homological representations of braid groups and KZ connections. J. Singul., 594-108, (2012).
[13] Toshitake Kohno - Quantum and homological representations of braid groups. Configuration Spaces - Geometry, Combinatorics and Topology, Edizioni della Normale, 355-372, (2012).
[14] R. J. Lawrence - Homological representations of the Hecke algebra, Comm. Math. Phys. 135, 141-19, (1990).
[15] R. J. Lawrence - A functorial approach to the one-variable Jones polynomial. J. Differential Geom., 37(3):689-710, (1993).
[16] J. Martel - A homological model for \( U_q(sl(2)) \) Verma-modules and their braid representations, arXiv:2002.05875 (2020).
[17] C. Manolescu - Nilpotent slices, Hilbert schemes, and the Jones polynomial, Duke Mathematical Journal, Vol. 132, 311-369, (2006)
[18] H. Murakami, J. Murakami - The colored Jones polynomials and the simplicial volume of a knot, Acta Math. 186, 85-104, (2001).
[19] P. Seidel, I. Smith- *A link invariant from the symplectic geometry of nilpotent slices*. Duke Math. J. 134:453-514, (2006).

[https://www.maths.ox.ac.uk/people/cristina.palmer-anghel](https://www.maths.ox.ac.uk/people/cristina.palmer-anghel)

Mathematical Institute, University of Oxford, Oxford, United Kingdom

E-mail address: palmeranghel@maths.ox.ac.uk; cristina.anghel@imj-prg.fr