Adaptive Submodular Meta-Learning

Shaojie Tang
Naveen Jindal School of Management, The University of Texas at Dallas
Jing Yuan
Department of Computer Science, The University of Texas at Dallas

Meta-Learning has gained increasing attention in the machine learning and artificial intelligence communities. In this paper, we introduce and study an adaptive submodular meta-learning problem. The input of our problem is a set of items, where each item has a random state which is initially unknown. The only way to observe an item’s state is to select that item. Our objective is to adaptively select a group of items that achieve the best performance over a set of tasks, where each task is represented as an adaptive monotone and submodular function that maps sets of items and their states to a real number. To reduce the computational cost while maintaining a personalized solution for each future task, we first select an initial solution set based on previously observed tasks, then adaptively add the remaining items to the initial set when a new task arrives. As compared to the solution where a brand new solution is computed for each new task, our meta-learning based approach leads to lower computational overhead at test time since the initial solution set is pre-computed in the training stage. To solve this problem, we propose a two-phase greedy policy and show that it achieves a \( \frac{e-1}{2e-1} \) approximation ratio.

1. Introduction

The goal of meta-learning is to leverage a few training examples to improve the performance of the learning algorithm on future tasks (Thrun and Pratt 2012). Among numerous formulations for meta-learning, Model-Agnostic Meta-Learning (Finn et al. 2017) is one of the most popular ones in continuous domain. MAML aims to provide a good initialization of a model’s parameters that can be quickly adapted to a new task using only a small number of gradient steps. Adibi et al. (2020) extend the methodology of MAML to the discrete domain and introduce the submodular meta-learning problem. Under their setting, each task is presented as a monotone and submodular utility function and their goal is to select a group of \( k \) items that achieves the best performance over all tasks. Their submodular meta-learning framework can be done in two parts: They first select a initial set of items based on some observed tasks, then after observing a new task, they add some additional items to that initial set to build a personalized solution for each new task. Their approach
can find a personalized solution for each new task while reducing the computational overhead at test time. This is because the first part, which finds a good initial solution set, is done offline, it does not consume any resource at training time.

In this paper, we extend their study to the adaptive setting. Under our setting, each item has a random state drawn from some known prior distribution. Initially, each item’s state is unknown, we must select an item in order to observe its realized state. Our goal is to adaptively select a group of $k$ items for each incoming task so as to maximize the average expected utility over all tasks. We assume that each task can be represented as an adaptive monotone and submodular function that maps sets of items and their states to a real number. Consider the example of adaptive viral marketing (Golovin and Krause 2011), where we would like to promote a product through a social network. Suppose that we have data on a social network where nodes represent individuals and edges represent social relations, our objective is to choose influential sets of individuals. In this context, items refer to individuals, the state of each item refers to the actual size of influence it generates, and the promotion of a particular product can be considered as a task. As each product may have its own diffusion model that governs the diffusion process of this product, it is reasonable to select different influential sets of individuals for marketing different products. Hence, our objective is to adaptively select some individuals to trigger a large cascade of influence over all products.

Following the framework of meta-learning, our adaptive submodular meta-learning is composed of two stages. In the first stage, we select a initial set of $l$ items non-adaptively based on prior experience. In the second stage, we adaptively add a group of additional $k-l$ items to the initial set after observing the incoming task. Our framework is general enough to capture numerous applications such as machine learning (Dasgupta and Hsu 2008), interactive recommendations (Karbasi et al. 2012), viral marketing (Yuan and Tang 2017a,b), and link prediction (Mitrovic et al. 2019). Note that the two extremes of adaptive submodular meta-learning are non-adaptive setting when $l = k$ (i.e., all $k$ selections are made offline) and fully adaptive setting when $l = 0$ (i.e., all $k$ items are selected in a closed-loop manner after observing the incoming task where each item is selected based on the feedback from previous selections). Clearly, there is trade off between the degree of personalization of our solution and the computational overhead at the test time. In particular, as $l$ increases, we provide a more personalized solution for each incoming task,
however, this also indicates that more selections need to be done adaptively at the test time, which may result in longer response time for a new task. Depending on the requirement of the focal application, the application owner can choose an appropriate $l$ that balances the computational overhead at test time and the degree of personalization of the solution. We leave the selection of an appropriate $l$ to the application owner, while our focus is on finding the best learning policy for a fixed $l$.

Our contributions are threefold:

- We develop a novel framework of adaptive submodular meta-learning where each item has a random state and each task can be represented using an adaptive monotone and adaptive submodular function. Our framework can find numerous applications in machine learning and artificial intelligence.

- We show that the new objective function defined in our framework does not satisfy the property of adaptive submodularity. We propose an effective two-phase greedy policy for the adaptive submodular meta-learning problem. Our policy enjoys the benefit of making selections adaptively while reducing the computational overhead at the test time. We show that our algorithm achieves a $\frac{e-1}{2e-1}$ approximation ratio.

- We conduct extensive experiments to evaluate the performance of our solution. Our results validate our theoretical analysis and show that the proposed solution outperforms all benchmark solutions.

2. Related Work

Meta-learning has been successfully applied to many domains, including reinforcement learning (Duan et al. 2016, Fallah et al. 2020) and one-shot learning (Snell et al. 2017). Model Agnostic Meta-learning (Finn et al. 2017) is one of the most popular forms of meta-learning, it aims at learning a initial model that can easily adapt to the new task from few examples. Most of existing studies, including MAML, consider the case where the feasible parameter space is continuous. Very recently, Adibi et al. (2020) extends this study to the discrete domain, i.e., they consider the case when the parameter space is discrete. Our study follows their work by considering a discrete variant of meta-learning. In Adibi et al. (2020), they assume that each task can be represented using a monotone and submodular utility function. As a result, their objective is to find a good initial solution set that can quickly adapt to a new monotone and submodular function. In this work, we generalize their study
by introducing an adaptive variant of submodular meta-learning. In particular, we assume that each item is associated with a random state whose realization is initially unknown. One must select an item in order to reveal its realized state. The utility function of each task is defined over sets of items as well as their realized states. One natural approach to maximize a utility function under the above setting is to sequentially select a group items, each selection is based on the feedback from previous selections. The previous submodular meta-learning framework falls short in adaptive settings as it requires the decision maker to make selections regardless of the realization of items’ states. To circumvent this issue, we adopt the notation of adaptive submodularity and adaptive monotone (Golovin and Krause 2011), which generalize the classic notations of submodularity and monotonicity from sets to policies. We assume that each task can be represented as an adaptive monotone and adaptive submodular function. Our study is also closely related to batch model active learning (Chen and Krause 2013), where the selection is performed in batches. In each batch, they select multiple items in an offline faction, and receive feedback only after all items from a batch have been selected. They develop a constant factor approximate solution to their problem. Their basic idea is to treat each possible batch as a virtual item, then apply the adaptive greedy algorithm over virtual items to obtain an approximate solution. Our work is different from theirs in two ways: As discussed in Section 3.5, our utility function is not adaptive submodular, thus the standard analysis used in adaptive submodular maximization does not apply to our framework. Moreover, Chen and Krause (2013) assumes that each batch has the same size, while under our setting, we select $l$ items in the first batch (phase) and each of the following batches contains a single item.

3. Preliminaries

We start by introducing some important notations. In the rest of this paper, we use $[m]$ to denote the set $\{1, 2, \cdots, m\}$, and we use $|S|$ to denote the cardinality of a set $S$.

3.1. Items and States

We consider a set $E$ of $n$ items. Each item is in a particular state from $O$. The item states are represented using a function $\phi : E \rightarrow O$, called a realization. Hence, $\phi(e)$ represents the realization of $e$’s state. We use $\Phi = \{\Phi(e) \mid e \in E\}$ to represent the set of random realizations of all items’ states, where $\Phi(e) \in O$ is a random realization of $e$’s state. There is a known prior probability distribution $p = \{\Pr[\Phi = \phi] : \phi \in U\}$ over realizations $U$. The
state $\Phi(e)$ of each item $e \in E$ is initially unknown, and we must select $e$ before observing the value of $\Phi(e)$. After selecting a set of items, we are able to observe a partial realization of those items’ states. For any partial realization $\psi$, we define the domain $\text{dom}(\psi)$ of $\psi$ the set of all items involved in $\psi$. We say a partial realization $\psi$ is consistent with a realization $\phi$, denoted $\phi \sim \psi$, if they are equal everywhere in $\text{dom}(\phi)$. Moreover, we say $\psi$ is a subrealization of $\psi'$, denoted $\psi \subseteq \psi'$, if $\text{dom}(\psi) \subseteq \text{dom}(\psi')$ and they are equal everywhere in the domain $\text{dom}(\psi)$ of $\psi$. Let $p(\phi \mid \psi)$ denote the conditional distribution over realizations conditioned on a partial realization $\psi$: $p(\phi \mid \psi) = \Pr[\Phi = \phi \mid \Phi \sim \psi]$.

3.2. Training and Test Tasks

We consider a family $G$ of tasks, where the size of $G$ could be infinite. Each task $i \in G$ is represented via a utility function $f^i$ from a subset of items and their states to a non-negative real number: $f^i : 2^E \times 2^O \rightarrow \mathbb{R}_{\geq 0}$. We assume that tasks from $G$ arrive randomly according to an underlying probability distribution $\theta$. Although $\theta$ is not always available, we assume that we have observed a group $M$ of $m$ training tasks. Each of the training tasks is sampled independently according to the distribution $\theta$. Our ultimate goal is to maximize the utility function at test time. That is, we aim at achieving the best performance for an incoming task that is sampled independently from the distribution $\theta$.

Consider the task of adaptive viral marketing whose objective is to adaptively select a set of seed users from a social network to help promote some product through the word-of-mouth effect. A social network can be represented as a graph with vertices representing users and edges representing their relations. The information propagation process is governed by some product-specific stochastic cascade model. Notably, Independent Cascade model assigns a product-specific propagation probability to each edge, say $(u,v)$, and it represents the probability that user $u$ successfully promotes a product to user $v$. One approach for solving this problem is to find a fixed group of seed users that can generate the largest average influence over all observed products, and let them promote all other products arriving in the future. As this group of users is pre-computed offline, the aforementioned approach has zero computational overhead at test time, however, it fails to provide a personalized solution for each new task. Another approach is to adaptively select a group of seed users with respect to each new task. Clearly, this approach leads to better performance, but at the cost of spending longer time and computational power on selecting all seed users at the test time.
3.3. Policies
We consider an adaptive optimization problem where we sequentially select a group of items, after each selection, we observe the partial realization of the states of those items which have been previously selected. We define an adaptive policy using a function $\pi$ that maps a set of observations to $E$, specifying which item to select next based on the current task and the partial realization observed so far: $\pi : 2^E \times 2^O \times 2^M \rightarrow E$.

**Definition 1 (Policy Concatenation).** Given two policies $\pi$ and $\pi'$, let $\pi \circ \pi'$ denote a policy that runs $\pi$ first, and then runs $\pi'$, ignoring the observation obtained from running $\pi$.

**Definition 2 (Level-$t$-Truncation of a Policy).** Given a policy $\pi$, we define its level-$t$-truncation $\pi^t$ as a policy that runs $\pi$ until it selects $t$ items.

For each task $i \in M$ and each realization $\phi$, let $E(\pi, \phi, i)$ denote the subset of items selected by $\pi$ under realization $\phi$ for task $i$. The expected utility $f_{\text{avg}}^i(\pi)$ of a policy $\pi$ for task $i$ can be written as

$$f_{\text{avg}}^i(\pi) = \mathbb{E}_{\Phi \sim p} f^i(E(\pi, \Phi, i), \Phi)$$

(1)

3.4. Adaptive Submodularity and Monotonicity
We next introduce several important notations.

**Definition 3 (Conditional Expected Marginal Utility of an Item).** Given a utility function $f^i$, the conditional expected marginal utility $\Delta_i(e \mid \psi)$ of an item $e$ conditioned on $\psi$ is

$$\Delta_i(e \mid \psi) = \mathbb{E}_{\Phi \sim p} [f^i(\text{dom}(\psi) \cup \{e\}, \Phi) - f^i(\text{dom}(\psi, \Phi))] \mid \Phi \sim \psi$$

where the expectation is taken over $\Phi$ with respect to $p(\phi \mid \psi) = \Pr(\Phi = \phi \mid \Phi \sim \psi)$.

**Definition 4 (Conditional Expected Marginal Utility of a Policy).** Given a utility function $f^i$, the conditional expected marginal utility $\Delta_i(\pi \mid \psi)$ of a policy $\pi$ conditioned on a partial realization $\psi$ is

$$\Delta_i(\pi \mid \psi) = \mathbb{E}_{\Phi \sim p} [f^i(\text{dom}(\psi) \cup E(\pi, \Phi, \Phi) - f^i(\text{dom}(\psi), \Phi)] \mid \Phi \sim \psi$$

where the expectation is taken over $\Phi$ with respect to $p(\phi \mid \psi) = \Pr(\Phi = \phi \mid \Phi \sim \psi)$. For any set of items $S \subseteq E$, we define the conditional expected marginal utility $\Delta_i(\pi \mid S)$ of a policy $\pi$ conditioned on $S$ as

$$\Delta_i(\pi \mid S) = \mathbb{E}_{\Phi \sim p} [f^i(S \cup E(\pi, \Phi, \Phi) - f^i(S, \Phi)] \mid \Phi \sim p]$$
We assume that the utility functions of all tasks are adaptive submodularity (Golovin and Krause 2011). That is, for any two partial realizations $\psi$ and $\psi'$ such that $\psi \subseteq \psi'$, the following holds for each $i \in M$ and $e \in E$:

$$\Delta_i(e \mid \psi) \geq \Delta_i(e \mid \psi')$$

Moreover, we assume they are adaptive monotonicity (Golovin and Krause 2011). That is, for any realization $\psi$, the following holds for each $i \in M$ and $e \in E$:

$$\Delta_i(e \mid \psi) \geq 0$$

3.5. Adaptive Submodular Meta-Learning

We next formally introduce the adaptive submodular meta-learning framework. Our framework is done in two stages: In the first (training) stage, we select a set $S \in E$ of size $l$ as the initial solution set, then, in the second (test) stage, we adaptively add the remaining $k - l$ items to $S$ after observing the task at hand. The motivation behind pre-computing an initial solution $S$ is twofold: 1) in some cases it is time-consuming to acquire the partial realization of an item’s state, it is more time-effective to acquire the partial realization of a batch of items’ states at once. As in our case, the selection of $S$ pre-computed regardless of the realization, we can safely select all items from $S$ at once and acquire their partial realization simultaneously, which significantly reduces the response time at the test time compared to full adaptive approach, i.e., $l = 0$. 2) our second motivation inherits from the one behind the non-adaptive submodular meta-learning framework (Adibi et al. 2020). Since computing $S$ does not consume any computational resource such as power in the test stage, our framework helps to reduce the resource consumption in the test stage.

We next consider two extreme cases in terms of $l$:

1. $l = 0$ means that we select all $k$ items adaptively after observing the incoming task. Clearly, this fully adaptive policy, which computes a fully personalized solution for each incoming task, can not perform worse than the case when $l > 0$. However, implementing such a fully adaptive policy is more expensive than implementing an non-adaptive solution where all $k$ items are selected offline at once. For example, if acquiring the partial realization of an item’s state is time-consuming, it is clearly more cost-effective to acquire the partial realization of a batch of items’ states at once.
2. $l = k$ means that we select all $k$ items offline before observing the incoming task. This nonadaptive policy is less computational power- and time-consuming since it requires zero computation at the test time, however, because such an non-adaptive solution can not adapt to the incoming task, its performance could be less satisfactory than the one of an adaptive policy.

One important question that arises from the above discussion is what would be the best $l$. The answer to this question is application-specific, one can balance the computational overhead at test time and the degree of personalization of the solution by tuning the value of $l$. We leave that decision to the application owner, while our focus is on finding the best learning policy for a fixed $l$.

3.6. Problem Statement

We use $\Omega(S,k)$ to denote the set of all policies that 1) pick $S$ as the initial set, i.e., for each $\pi \in \Omega(S,k)$ and each $i \in M$, it holds that $\pi(\emptyset, i) = S$, and 2) $|E(\pi, \phi, i)| \leq k$ for all $\phi$ with $p(\phi) > 0$. Hence, the expected utility $f^i_{avg}(\pi)$ of any $\pi \in \Omega(S,k)$ conditioned on observing task $i$ is

$$f^i_{avg}(\pi) = f^i(S) + \Delta^i_{avg}(\pi|S) \quad (3)$$

The expected utility $\hat{f}_{avg}(\pi)$ of a policy $\pi$ over the distribution $\theta$ of tasks can be written as

$$\hat{f}_{avg}(\pi) = \mathbb{E}_{i \sim \theta}[f^i_{avg}(\pi)] = \mathbb{E}_{i \sim \theta}[f^i(S)] + \mathbb{E}_{i \sim \theta}[\Delta^i_{avg}(\pi|S)] \quad (4)$$

Since the underlying probability distribution $\theta$ of the tasks is not always available, we often collect a group of tasks that are sampled independently according to the distribution $\theta$. As a result, we focus on optimizing the the sample average approximation of $\mathbb{E}_{i \sim p}[f^i_{avg}(\pi)]$ given by

$$f_{avg}(\pi) = \frac{1}{m} \sum_{i \in M} f^i_{avg}(\pi) \quad (5)$$

Thus, our problem can be written as

$$\max_{S \subseteq E, |S| = l} \max_{\pi \in \Omega(S,k)} f_{avg}(\pi) \quad (6)$$

It was worth noting that when $l = 0$, i.e., all $k$ items are selected adaptively after observing the task, our problem is reduced to the adaptive submodular maximization problem.
This is because if the task, say $i$, is known, our objective is reduced to maximizing $f_{\text{avg}}^i(\pi)$ subject to $|E(\pi, \phi, i)| \leq k$ for all $\phi$ with $p(\phi) > 0$. Golovin and Krause (2011) show that the following simple adaptive greedy algorithm achieves a $1 - 1/e$ approximation ratio. It starts with an empty set, and in each round, it selects an item that maximizes the marginal utility on top of the current partial realization. We next show that their result does not apply to the case when $l > 0$. Recall that we must select the first $l$ items before observing the incoming task. Because of the uncertainty associated with the incoming task, our utility function no longer satisfies the adaptive submodularity. Consider a toy example with two items $E = \{1, 2\}$, two tasks $M = \{1, 2\}$, $l = 1$, and one state $O = \{1\}$, i.e., the state of each item is deterministic, and the utility functions are defined as follows: $f_1(\{1\}, 1) = f_1(\{2\}, 1) = f_1(\{1, 2\}, 1) = 0$ and $f_2(\{1\}, 1) = f_2(\{2\}, 1) = 1$, $f_2(\{1, 2\}, 1) = 2$. Hence, the marginal utility of item 1 to an empty set before observing the task is $\frac{1}{2} \times f_1(\{1\}, 1) + \frac{1}{2} \times f_2(\{1\}, 1) = \frac{1}{2}$, and the marginal utility of 1 to an existing set $\{2\}$ after observing the incoming task 2 is $f_2(\{1, 2\}, 1) - f_2(\{2\}, 1) = 1$. This clearly violates the property of adaptive submodularity defined in (2), making the existing results (Chen and Krause 2013) not applicable to our setting.

4. Two-phase Greedy Policy

We next explain the design of our Two-phase Greedy policy $\pi^g$. Our approach is composed of two phases, the first phase is done at the training stage to find a good initial set $S$ of size $l$, the second phase is conducted after observing the incoming task to add the remaining $k - l$ items to $S$ adaptively. A detailed description of $\pi^g$ is listed in Algorithm 1.

- **Phase 1:** Selecting a task-independent initial set $S^g$ of size $l$ according to the following classic non-adaptive greedy algorithm: It starts with an empty set, and then selects a group of $l$ items iteratively. In each round $t \in [1, l]$, it selects an item $e_t$ that maximizes the marginal utility of the average approximation of $m$ utility functions on top of the selected items $S_{t-1}$:

$$e_t \leftarrow \arg \max_{e \in E} \frac{1}{m} \sum_{i \in M} (f_i(S_{t-1} \cup \{e\}) - f_i(S_{t-1}))$$

This process iterates until all $l$ items have been added to $S^g$.

- **Phase 2:** When a task $i \in M$ arrives, $\pi^g$ selects the rest of the $k - l$ items according to the following adaptive greedy algorithm: It starts with $S^g$ and the partial realization
of their states. At each round \( t \in [l + 1, k] \), it selects an item that maximizes the expected marginal utility of \( f_i \) on top of the current partial realization \( \psi^{g_{t-1}} \):

\[
e_t \leftarrow \arg \max_{e \in E} \Delta_i(e | \psi^{g_{t-1}})
\]

After observing the state \( \psi(e_t) \) of \( e_t \), update the current partial realization \( \psi^{g_t} \) using \( \psi^{g_{t-1}} \cup \{ \psi(e_t) \} \). This process iterates until all \( k-l \) items have been selected.

**Algorithm 1 Two-phase Greedy Policy \( \pi_g \)**

1: \( S^g = \emptyset, t = 1. \)
2: \{The first part is executed offline to compute a task-independent set \( S^g \).\}
3: \( \text{while } t \leq l \text{ do} \)
4: \( e_t \leftarrow \arg \max_{e \in E} \frac{1}{m} \sum_{i \in M} (f^i(S_{t-1} \cup \{e\}) - f^i(S_{t-1})); \)
5: \( S_t \leftarrow S_{t-1} \cup \{e_t\}; t \leftarrow t + 1; \)
6: \( S^g \leftarrow S_t; \)
7: \{The second part is executed adaptively upon the arrival of a task \( i \in G. \)\}
8: \( \text{select } S^g \text{ and observe } \psi^{g_l} = \cup_{e \in S^g} \psi(e); t = l + 1; \)
9: \( \text{while } t \leq k \text{ do} \)
10: \( \text{observe } \psi^{g_{t-1}}; \)
11: \( e_t \leftarrow \arg \max_{e \in E} \Delta_i(e | \psi_t); \)
12: \( \text{select } e_t \text{ and observe } \psi(e_t) \)
13: \( \psi^{g_t} = \psi^{g_{t-1}} \cup \{\psi(e_t)\}; t \leftarrow t + 1; \)

The rest of this section is devoted to proving the performance bound of \( \pi_g \). We use \( \pi^o \) to denote an optimal policy and use \( S^o \) to denote the initial set selected by \( \pi^o \). Before presenting the main theorem (Theorem 1) of this paper, we first present three preparatory lemmas.

**Lemma 1.** For all \( i \in M \), we have

\[
f^i_{\text{avg}}(\pi^g @ \pi^o) - f^i_{\text{avg}}(\pi^g @ \pi^o_t) \leq \Delta_i(\pi^g | S^g)
\]

**Proof:** Let \( \psi^{g \oplus \alpha_i} \) denote the partial realization obtained after running \( \pi^g @ \pi^o \), and we use \( \psi^{g \oplus \alpha_i} \subseteq \psi^{g \oplus \alpha_i} \) to denote a subrealization of \( \psi^{g \oplus \alpha_i} \) obtained after running the level-\( l' \)-truncation of \( \pi^g \). Let \( e_t^{opt} \) denote the \( t \)-th item selected by \( \pi^o \), we first bound the expected
marginal utility $\Delta_i(e_t \mid \psi_{gt-1}^{g@o_{t-1}})$ of $e_t$ for each $t \in [l+1,k]$ conditioned on partial realization $\psi_{gt-1}^{g@o_{t-1}}$.

\[
\Delta_i(e_t \mid \psi_{gt-1}^{g@o_{t-1}}) = \max_{e \in E} \Delta_i(e \mid \psi_{gt-1}^{g@o_{t-1}}) \quad (8)
\]

\[
\geq \max_{e \in E} \Delta_i(e \mid \psi_{gt-1}^{g@o_{t-1}}) \quad (9)
\]

\[
\geq \Delta_i(e_i^\text{opt} \mid \psi_{gt-1}^{g@o_{t-1}}) \quad (10)
\]

\[
= f_{\text{avg}}(\pi^g \circ \pi^o_t \mid \psi_{gt-1}^{g@o_{t-1}}) - f_{\text{avg}}(\pi^g \circ \pi^o_{t-1} \mid \psi_{gt-1}^{g@o_{t-1}}) \quad (11)
\]

The first equality is due to $\pi^g$ selects an item that maximizes the conditional expected marginal utility. The first inequality is due to the assumption that $f^i$ is adaptive submodular.

Unfixing $\psi_{gt-1}^{g@o_{t-1}}$, the following inequality holds for all $t \in [l+1,k]$: $\mathbb{E}_{\psi_{gt-1}^{g@o_{t-1}}}[\Delta_i(e_t \mid \psi_{gt-1}^{g@o_{t-1}})] \geq \mathbb{E}_{\psi_{gt-1}^{g@o_{t-1}}} [f_{\text{avg}}(\pi^g \circ \pi^o_t \mid \psi_{gt-1}^{g@o_{t-1}}) - f_{\text{avg}}(\pi^g \circ \pi^o_{t-1} \mid \psi_{gt-1}^{g@o_{t-1}})]$. It follows that

\[
f_{\text{avg}}(\pi^g_t) - f_{\text{avg}}(\pi^g_{t-1}) \geq f_{\text{avg}}(\pi^g_t \circ \pi^o_t) - f_{\text{avg}}(\pi^g_t \circ \pi^o_{t-1}) \quad (12)
\]

We further have

\[
\sum_{t \in [l+1,k]} (f_{\text{avg}}(\pi^g_t) - f_{\text{avg}}(\pi^g_{t-1})) \geq \sum_{t \in [l+1,k]} (f_{\text{avg}}(\pi^g_t \circ \pi^o_t) - f_{\text{avg}}(\pi^g_t \circ \pi^o_{t-1})) \quad (13)
\]

Due to $\Delta_i(\pi^g \mid S^g) = \sum_{t \in [l+1,k]} (f_{\text{avg}}(\pi^g_t) - f_{\text{avg}}(\pi^g_{t-1}))$ and (13), the following inequality holds:

\[
f_i^i(\pi^g \circ \pi^o) - f_i^i(\pi^g \circ \pi^o) \leq \Delta_i(\pi^g \mid S^g) \quad (14)
\]

\[\square\]

**Lemma 2.** For all $i \in M$, we have

\[
\Delta_i(\pi^g \mid S^g) \geq \Delta_i(\pi^g \mid S^g \cup S^o) \quad (15)
\]

**Proof:** We first bound the expected marginal utility $\Delta_i(e_t \mid \psi^{g_{t-1}})$ of $e_t$ conditioned on partial realization $\psi^{g_{t-1}}$.

\[
\Delta_i(e_t \mid \psi^{g_{t-1}}) \geq \mathbb{E}[\Delta_i(e_t \mid \Phi(S^o) \cup \psi^{g_{t-1}}) \mid \Phi \sim \psi^{g_{t-1}}] \quad (16)
\]

The above equality is due to $\psi^{g_{t-1}} \subseteq \Phi(S^o) \cup \psi^{g_{t-1}}$ for any $\Phi$ such that $\Phi \sim \psi^{g_{t-1}}$, thus $\Delta_i(e_t \mid \psi^{g_{t-1}}) \geq \Delta_i(e_t \mid \Phi(S^g \cup S^o))$. 

Then we have
\[ E_{\Psi^{gt-1}}[\Delta_i(e_t | \Psi^{gt-1})] \geq E_{\Psi^{gt-1}}[E[\Delta_i(e_t | \Phi(S^o) \cup \Psi^{gt-1}) | \Phi \sim \Psi^{gt-1}]] \tag{17} \]

The following inequality holds for all \( t > l \):
\[ f_{avg}(\pi^g_t) - f_{avg}(\pi^g_{t-1}) \geq f_{avg}(\pi^g_t \@ \pi^o_t) - f_{avg}(\pi^g_{t-1} \@ \pi^o_t) \tag{18} \]

Then we have
\[ \Delta_i(\pi^g|S^o) = \sum_{t=l+1}^{k} (f_{avg}(\pi^g_t) - f_{avg}(\pi^g_{t-1})) \tag{19} \]
\[ \geq \sum_{t=l+1}^{k} (f_{avg}(\pi^g_t \@ \pi^o_t) - f_{avg}(\pi^g_{t-1} \@ \pi^o_t)) \tag{20} \]
\[ = \Delta_i(\pi^g|S^o \cup S^o) \tag{21} \]

\[ \square \]

**Lemma 3.** \( \sum_{i \in M}(f^i(S^o \cup S^o) - f^i(S^g)) \leq \frac{e}{e-1} \sum_{i \in M} f^i(S^g) \)

**Proof:** Because \( f_i \) is adaptive monotone and adaptive submodular for all \( i \in M \), \( f_i(S) = \mathbb{E}_{\Phi \sim p}[f_i(S, \Phi)] \) is monotone and submodular in terms of \( S \). Hence, \( \sum_{i \in M} f_i(S) \) is also monotone and submodular in terms of \( S \) due the linear combination of monotone submodular functions are still monotone and submodular. It follows that
\[ \sum_{i \in M}(f^i(S^o \cup S^o) - f^i(S^g)) \leq \sum_{i \in M} f^i(S^o) \tag{22} \]

Moreover, since we apply the non-adaptive greedy algorithm (Nemhauser et al. 1978) to obtain \( S^g \), we have \( \sum_{i \in M} f^i(S^g) \geq (1 - 1/e) \sum_{i \in M} f^i(S^*) \) where \( S^* = \arg\max_{S \subseteq E} \sum_{i \in M} f^i(S) \). It follows that
\[ \sum_{i \in M} f^i(S^g) \geq (1 - 1/e) \sum_{i \in M} f^i(S^*) \geq (1 - 1/e) \sum_{i \in M} f^i(S^o) \tag{23} \]

(22) and (23) imply that \( \sum_{i \in M}(f^i(S^o \cup S^o) - f^i(S^g)) \leq \frac{e}{e-1} \sum_{i \in M} f^i(S^g). \square \)

**Theorem 1.** Our two-phase greedy policy \( \pi^g \) achieves a \( \frac{e-1}{2e-1} \) approximation ratio, that is,
\[ f_{avg}(\pi^g) \geq \frac{e-1}{2e-1} f_{avg}(\pi^o) \tag{24} \]
Proof: Recall that $\pi^g@\pi^o$ runs $\pi^g$ first, then runs $\pi^o$ from a fresh start. Hence, the expected utility $f_{avg}^i(\pi^g@\pi^o)$ of $\pi^g@\pi^o$ from task $i$ can be written as:

$$f_{avg}^i(\pi^g@\pi^o) = f^i(S^g) + (f^i(S^g \cup S^o) - f^i(S^g))$$

$$+ \Delta_i(\pi^g|S^g \cup S^o) + (f_{avg}^i(\pi^g@\pi^o) - f_{avg}^i(\pi^g@\pi^o))$$  

It follows that

$$m \times f_{avg}^i(\pi^g@\pi^o) = \sum_{i \in M} f_{avg}^i(\pi^g@\pi^o)$$

$$= \sum_{i \in M} f^i(S^g) + \sum_{i \in M} (f^i(S^g \cup S^o) - f^i(S^g))$$

$$+ \Delta_i(\pi^g|S^g \cup S^o) + \sum_{i \in M} (f_{avg}^i(\pi^g@\pi^o) - f_{avg}^i(\pi^g@\pi^o))$$

$$\leq \frac{2e-1}{e-1} \sum_{i \in M} f^i(S^g) + \sum_{i \in M} \Delta_i(\pi^g|S^g \cup S^o) + \sum_{i \in M} (f_{avg}^i(\pi^g@\pi^o) - f_{avg}^i(\pi^g@\pi^o))$$

$$\leq \frac{2e-1}{e-1} \sum_{i \in M} f^i(S^g) + \sum_{i \in M} \Delta_i(\pi^g|S^g)$$

$$= \frac{2e-1}{e-1} \sum_{i \in M} f^i(S^g) + 2 \sum_{i \in M} \Delta_i(\pi^g|S^g)$$

$$\leq \frac{2e-1}{e-1} \sum_{i \in M} (f^i(S^g) + \Delta_i(\pi^g|S^g)) = \frac{2e-1}{e-1} m \times f_{avg}^i(\pi^g)$$

The first inequality is due to Lemma 3, the second inequality is due to Lemma 1 and Lemma 2. This finishes the proof of the theorem. □

5. Conclusion

In this paper, we develop a novel framework of adaptive submodular meta-learning. Our framework extends the notion of submodular meta-learning to the adaptive setting which allows each item to have a random state. Our goal is to find a initial set of items that can quickly adapt to a new task. We propose a two-phase greedy policy that achieves a $\frac{e-1}{2e-1}$ approximation ratio. We evaluated the performance of our proposed algorithm for the application of adaptive viral marketing.

References

Adibi, Arman, Aryan Mokhtari, Hamed Hassani. 2020. Submodular meta-learning. Advances in Neural Information Processing Systems 33.
Chen, Yuxin, Andreas Krause. 2013. Near-optimal batch mode active learning and adaptive submodular optimization. *ICML (1)* 28 8–1.

Dasgupta, Sanjoy, Daniel Hsu. 2008. Hierarchical sampling for active learning. *Proceedings of the 25th international conference on Machine learning.* 208–215.

Duan, Yan, John Schulman, Xi Chen, Peter L Bartlett, Ilya Sutskever, Pieter Abbeel. 2016. RI^2: Fast reinforcement learning via slow reinforcement learning. *arXiv preprint arXiv:1611.02779*.

Fallah, Alireza, Aryan Mokhtari, Asuman Ozdaglar. 2020. Provably convergent policy gradient methods for model-agnostic meta-reinforcement learning. *arXiv preprint arXiv:2002.05135*.

Finn, Chelsea, Pieter Abbeel, Sergey Levine. 2017. Model-agnostic meta-learning for fast adaptation of deep networks. *arXiv preprint arXiv:1703.03400*.

Golovin, Daniel, Andreas Krause. 2011. Adaptive submodularity: Theory and applications in active learning and stochastic optimization. *Journal of Artificial Intelligence Research* 42 427–486.

Karbasi, Amin, Stratis Ioannidis, et al. 2012. Comparison-based learning with rank nets. *arXiv preprint arXiv:1206.4674*.

Mitrovic, Marko, Ehsan Kazemi, Moran Feldman, Andreas Krause, Amin Karbasi. 2019. Adaptive sequence submodularity. *Advances in Neural Information Processing Systems.* 5352–5363.

Nemhauser, George L, Laurence A Wolsey, Marshall L Fisher. 1978. An analysis of approximations for maximizing submodular set functions-i. *Mathematical programming* 14 265–294.

Snell, Jake, Kevin Swersky, Richard Zemel. 2017. Prototypical networks for few-shot learning. *Advances in neural information processing systems.* 4077–4087.

Thrun, Sebastian, Lorien Pratt. 2012. *Learning to learn.* Springer Science & Business Media.

Yuan, Jing, Shao-Jie Tang. 2017a. Adaptive discount allocation in social networks. *Proceedings of the 18th ACM International Symposium on Mobile Ad Hoc Networking and Computing.* 1–10.

Yuan, Jing, Shaojie Tang. 2017b. No time to observe: adaptive influence maximization with partial feedback. *Proceedings of the 26th International Joint Conference on Artificial Intelligence.* 3908–3914.