Production of soft photons from the quark-gluon plasma in hot QCD
— Screening of mass singularities

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Abstract

It has been reported that, within the hard-thermal-loop resummation scheme, the production rate of soft real photons from a hot quark-gluon plasma exhibits unscreened mass singularities. We show that still higher-order resummations screen the mass singularities and obtain the finite soft-photon production rate to leading order at logarithmic accuracy $\mathcal{O}(\alpha s \ln^2 \alpha_s)$.  

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1 Introduction

It has been established by Braaten and Pisarski [1] that, in perturbative thermal QCD, the resummations of the leading-order terms, called hard thermal loops, are necessary. In thermal massless QCD, we encounter the problem of infrared and mass or collinear singularities. The hard-thermal-loop (HTL) resummed propagators soften or screen the infrared singularities, and render otherwise divergent physical quantities finite, if they are not sensitive to a further resummation of the corrections of $\mathcal{O}(g^2 T)$.

[For a compact review of infrared and mass singularities in thermal field theory, we refer to [2].]

The calculation of the production rate of soft real photons ($E = \mathcal{O}(g T)$) from a hot quark-gluon plasma, to leading order $\mathcal{O}(\alpha_s \ln \alpha_s)$, within the HTL resummation scheme has recently been reported [3, 4] with the conclusion that the result is divergent, owing to mass singularities.

The purpose of this paper is to show that resummations of formally still higher-order contributions screen the above-mentioned mass singularities and the production rate of $\mathcal{O}(\alpha_s \ln^2 \alpha_s)$ results. In evaluating the soft-photon production rate, the imaginary-time formalism has been used in [3], while, in [4], use has been made of the retarded/advanced formalism of real-time thermal field theory. For the purpose of this paper, the ordinary real-time formalism is convenient, since, in this formalism, identifications of the physical processes that lead to mass singularities mentioned above are straightforward [3].

In Sect. 2, we compute the singular part of the production rate of soft photons to leading order in HTL-resummation scheme of real-time thermal field theory. Although the result is known [3, 4], we repeat the calculation in such a manner that the procedure fits our purpose of further resummation. In Sect. 3, we carry out further resummations of (formally) higher-order contributions and show that the mass singularities are screened. Then, the production rate is evaluated at leading logarithmic accuracy, i.e., the constant $c$ in $E dW/d^3 p = c\alpha_s \ln^2 \alpha_s$ is computed. Sect. 4 is devoted to discussions and conclusions. Appendix collects some formulae used in the text.
2 Leading-order calculation

After summing over the polarizations of the photon, the production rate is given by

\[ E \frac{dW}{d^3p} = \frac{i}{2(2\pi)^3} g_{\mu\nu} \Pi_{12}^{\mu\nu}(E, p). \]  

(1)

In (1), \( \Pi_{12}^{\mu\nu} \) is the (1,2) component of the photon polarization tensor in the real-time formalism based on the time path \( C_1 \oplus C_2 \oplus C_3 \) in the complex time plane; \( C_1 = -\infty \rightarrow +\infty, \ C_2 = +\infty \rightarrow -\infty, \ C_3 = -\infty \rightarrow -\infty - i/T \). [The time-path segment \( C_3 \) does not play any explicit role in the present context.] The fields whose time arguments are lying on \( C_1 \) and on \( C_2 \) are referred, respectively, to as the type-1 and type-2 fields. A vertex of type-1 (type-2) fields is called a type-1 (type-2) vertex. Then \( \Pi_{12}^{\mu\nu} \) in (1) is the “thermal vacuum polarization between the type-2 photon and the type-1 photon”. [It is worth mentioning that the “Feynman rules” of the above-mentioned real-time formalism is equivalent to the circled diagram rules introduced by Kobes and Semenoff \( [4] \), provided that the type-1 (type-2) field is identified with the field of “uncircled” (“circled”) type.]

To leading order, Fig. 1 is the only diagram \( [3, 4] \) that contributes to \( E dW/d^3p \). In Fig. 1, \( p_0 = p = E \) and 2 and 1 at the effective photon-quark vertices stand for the type of vertices (cf. \( [4] \) below). Fig. 1 gives

\[ \Pi_{12}^{\mu\nu}(P) = -i e_q e^2 N_c \int \frac{d^4K}{(2\pi)^4} \sum_{i_1, \ldots, i_4=1}^2 tr \left[ *S_{i_1 i_4}(K) \cdot (\Gamma^\nu(K, K'))^2 *S_{i_3 i_2}(K') (\Gamma^\mu(K', K))_{i_2 i_1} \right], \]

(2)

where \( i_1, \ldots, i_4 \) are the thermal indices that specify the field types. In (2), all the momenta \( P, K \) and \( K' \) are soft (\( \sim gT \)), so that both photon-quark vertices, \( \Gamma^\nu \) and \( \Gamma^\mu \), as well as both quark propagators, \( *S_{i_1 i_4} \) and \( *S_{i_3 i_2} \), are HTL-resummed effective ones (cf. \( [4] \) and \( [5.1] - [5.6] \)). [Throughout this paper, a capital letter like \( P \) denotes the four momentum, \( P = (p_0, p) \), and a lower-case letter like \( p \) denotes the length of the three vector, \( p = |p| \). The unit three vector along the direction of, say, \( p \) is denoted as \( \hat{p} \equiv p/p \). The null four vector like \( \hat{P} \) is defined as \( \hat{P} = (1, \hat{p}) \).]

For our purpose, it is convenient to decompose \( g_{\mu\nu} \) in (1) into two parts as

\[ g_{\mu\nu} = g_{\mu\nu}^{(0)} + g_{\mu\nu}^{(\hat{e})} \]

(3)

\[ g_{\mu\nu}^{(\hat{e})} = g_{\mu 0} \hat{P}_\nu + g_{\nu 0} \hat{P}_\mu - \hat{P}_\mu \hat{P}_\nu. \]

(4)
Substituting (3) for $g_{\mu\nu}$ in (4), we have, with an obvious notation,

$$\frac{E\,dW}{d^3p} = E\frac{dW^{(t)}}{d^3p} + E\frac{dW^{(\ell)}}{d^3p}. \tag{5}$$

Now we observe that $(^{*}\Gamma^\mu)_{ji}^{1}$ in (2) is written as

$$(^{*}\Gamma^\mu)_{ji}^{1} = \gamma_\mu \delta_{1j} \delta_{1i} + (^{*}\tilde{\Gamma}^\mu)_{ji}^{1},$$

$$(^{*}\tilde{\Gamma}^\mu)_{ji}^{1} = \int d\Omega \hat{Q}^\mu_i \hat{Q}_j f_{ji}^{1}(\hat{Q}_1, K', K), \quad (\ell = 1, 2), \tag{6}$$

where $(^{*}\tilde{\Gamma}^\mu)_{ji}^{1}$ is the HTL correction, in terms of an angular integral, and $Q^\mu_i = q_1 \hat{Q}^\mu_i$ is the hard momentum circulating along the HTL. Here, the explicit form of the function $f_{ji}^{1}$ in (3) is not necessary. The expression for $(^{*}\Gamma^\nu)_{ji}^{2}$ is given by (4) with $\gamma_\nu \delta_{1j} \delta_{1i} \rightarrow -\gamma_\nu \delta_{2j} \delta_{2i}$, $\hat{Q}^\mu_i \rightarrow \hat{Q}^\nu_i$, $\hat{Q}_1 \rightarrow \hat{Q}_2$, and $f_{ji}^{1}(\hat{Q}_1, K', K) \rightarrow f_{ji}^{2}(\hat{Q}_2, K, K')$. It turns out that the production rate $E\,dW/d^3p$ diverges (3, 4) due to the mass singularities. As in (3, 4), we are only interested in the divergent parts neglecting all finite contributions.

**Computation of $E\,dW^{(t)}/d^3p$**

As is seen below, the mass singularities arise (3, 4) from the factors $(1 - \sigma \hat{p} \cdot \hat{q}_1)^{-1} (\sigma = \pm)$ and/or $(1 - \sigma' \hat{p} \cdot \hat{q}_2)^{-1} (\sigma' = \pm)$, which come from $(^{*}\tilde{\Gamma}^\mu)_{i_2i_1}^{1}$ and/or $(^{*}\tilde{\Gamma}^\nu)_{i_3i_2}^{2}$ in (2) (cf. (3)). When integrating over the directions of $\hat{q}_1$ and/or $\hat{q}_2$, these factors lead to divergences at $\hat{p} \cdot \hat{q}_1 = \sigma$ and/or at $\hat{p} \cdot \hat{q}_2 = \sigma'$. Let us see the numerator factors in the integrand of $E\,dW^{(t)}/d^3p$, which are obtained after taking the trace of Dirac matrices under the HTL approximation (cf. (2) - (3)).

- When one of the photon-quark vertices in Fig. 1 is the HTL correction and the other is the bare vertex, it can be shown that the numerator is a linear combination of the terms, each of which contains the combination $(\hat{q}_j \cdot a) - (\hat{q}_j \cdot \hat{p})(\hat{p} \cdot a)$, $j = 1$ or 2, with $a$ being $\hat{q}_j$ or $\hat{k}$ or $\hat{k}'$. Then, the numerator vanishes at $\pm|\hat{q}_j| || \hat{p}$, meaning that there is no singularity in the integrand.

- When both photon-quark vertices in Fig. 1 are the HTL corrections, there emerges the numerator factor $\hat{q}_1 \cdot \hat{q}_2 - (\hat{q}_1 \cdot \hat{p})(\hat{p} \cdot \hat{q}_2)$. For $\hat{q}_1 = \sigma \hat{p}$ (\sigma =
\( \pm \) and \( \hat{q}_2 \neq \sigma' \hat{p} \) (\( \sigma' = \pm \)), or for \( \hat{q}_2 = \sigma' \hat{p} \) and \( \hat{q}_1 \neq \sigma \hat{p} \), this numerator factor vanishes. For \( \hat{q}_1 = \sigma \hat{p} \) and \( \hat{q}_2 = \sigma' \hat{p} \), the numerator factor gives \( (1 - \sigma \hat{p} \cdot \hat{q}_1)^{1/2}(1 - \sigma' \hat{p} \cdot \hat{q}_2)^{1/2} \), and the integrations over the directions of \( \hat{q}_1 \) and \( \hat{q}_2 \) converge.

Thus, \( E dW^{(t)}/d^3p \) is free from singularity.

**Computation of** \( E dW^{(t)}/d^3p \)**

Let us compute \( ig^{(t)}_{\mu\nu} \Pi_{12}^{\mu\nu} \) with \( g^{(t)}_{\mu\nu} \) as given in (4). It is not difficult to see that the term with \( \hat{P}_\mu \hat{P}_\nu \) in (4) does not yield mass-singular contribution (see below). Thus we obtain, for the singular contributions,

\[
ig^{(t)}_{\mu\nu} \Pi_{12}^{\mu\nu}(P) \approx e_q^2 e^2 N_c \left[ g_{\mu 0} \hat{P}_\nu + \hat{P}_\mu g_{\nu 0} \right] \int \frac{d^4 K}{(2\pi)^4} \sum_{i_1, \ldots, i_4 = 1}^{2} \text{tr} \left[ S_{i_1 i_4}(K) \cdot \left( \tilde{T}^{\mu}(K', K') \right)^2_{i_4 i_3} S_{i_3 i_2}(K') \left( \tilde{T}^{\mu}(K', K) \right)_{i_2 i_1}^{1} \right],
\]

where the symbol “\( \approx \)” is used to denote an approximation that is valid for keeping the singular contributions. Using the Ward-Takahashi relations, (11,12), we can simplify (4) as

\[
ig^{(t)}_{\mu\nu} \Pi_{12}^{\mu\nu}(P) \approx e_q^2 e^2 N_c \int \frac{d^4 K}{(2\pi)^4} \sum_{i = 1}^{2} \left\{ \text{tr} \left[ \tilde{T}^{\mu}(K', K) \left( \tilde{T}^{\mu}(K', K) \right)_{i_2}^{1} - \left( \tilde{T}^{\mu}(K', K') \right)_{i_2}^{1} S_{i_2}(K) \right] \right. \\
+ \text{tr} \left[ \left( \tilde{T}^{\mu}(K', K') \right)_{i_1}^{1} S_{i_1}(K') - \tilde{T}^{\mu}(K', K) \left( \tilde{T}^{\mu}(K', K') \right)_{i_1}^{2} \right] \left. \right\}.
\]

We may easily see that the two terms in each set of square brackets give the same contributions, and the mass-singular contributions arises from the HTL-corrected parts (eq. (3)) \( \tilde{T}^{\mu} \)'s of \( \tilde{T}^{\mu} \)'s in (8). It is also not difficult to see that the mass singularities arise [3,4] from the terms with \( \left( \tilde{T}^{\mu} \right)_{i_j}^{\xi} \) with \( i \neq j \).

For the sake of convenience of the calculation in the next section, we do not use the known simple expressions for \( \tilde{T}^{\mu} \)'s, but rather use their defining expressions (cf. Fig. 2):

\[
\left( \tilde{T}^{\mu}(K', K) \right)_{i_2}^{1} = - ig^2 N_c^2 \left[ \frac{1}{2 N_c} \int \frac{d^4 Q}{(2\pi)^4} \gamma_\xi S_{21}(Q) \gamma_\mu S_{11}(Q + P) \gamma_\zeta \Delta_{12}^{\zeta}(Q - K') \right],
\]
\[
\left(\tilde{\Gamma}^{\nu}(K, K')\right)_{21}^2 = i g^2 \frac{N_c^2 - 1}{2 N_c} \int \frac{d^4 Q}{(2\pi)^4} \gamma_\xi S_{22}(Q + P) \gamma^{\nu} S_{21}(Q) \gamma_\zeta \Delta^{E}_{12}(Q - K') ,
\]

where \( S_{ij} \) and \( \Delta^{E}_{12} \) are the bare thermal propagators of a quark and a gluon, respectively (cf. Appendix). In [9], \( Q \) is hard while \( P, K \) and \( K' \) are soft. Inserting (9) with \( \mu = \nu = 0 \) into (8), we obtain

\[
i g^{(t)}_{\mu \nu} \Pi^{\mu \nu}_{12}(P) \simeq 4 i g^2 e_q^2 e^2 N_c \frac{N_c^2 - 1}{2 N_c} \frac{1}{E} \int \frac{d^4 K}{(2\pi)^4} \int \frac{d^4 Q}{(2\pi)^4} \Delta_{12}(Q - K') \tilde{S}_{21}(Q) \delta (Q^2) \delta ((Q - K')^2) \sum_{\sigma, \tau = \pm} [\left( \hat{K}_\sigma \cdot Q \right) + q^0 (\hat{K}_\sigma \cdot \hat{Q}_\tau) - (Q \cdot \hat{Q}_\tau)]
\]

\[
\cdot \rho_{\sigma}(K) \left[ \frac{1}{q_0 (1 + i\epsilon) + p_0 - \tau |q + p|} + \text{c.c.} \right] ,
\]

where \( \hat{Q}_\tau = (1, \tau \hat{q}) \) and the HTL approximation \( \epsilon(q_0 - k'_0) n_B(q_0 - k'_0) \simeq \epsilon(q_0) n_B(q_0) \) has been made. In (11), \( \rho_{\pm} \) (eq. (A.3)) is the absorptive part of the effective thermal quark propagator. Due to the factor \( \delta(Q^2) \) in (11), we have two contributions, the one with \( q_0 = q \) and the other with \( q_0 = -q \). The former (latter) contribution is singular when \( \tau = + (-) \);

\[
g_0 + p_0 - \tau |q + p| \simeq E (1 - \tau \hat{p} \cdot \hat{q}) = E (1 - \epsilon(q_0) \hat{p} \cdot \hat{q}) .
\]

When \( \hat{P}_\mu \hat{P}_\mu \) term in \( g^{(t)}_{\mu \nu} \) (eq. (9)) is substituted for \( g_{\mu \nu} \) in \( i g_{\mu \nu} \Pi^{\mu \nu}_{12} \), we obtain the Dirac trace

\[
\text{tr} \left\{ \hat{K}_\sigma Q \hat{P} \hat{Q} \right\} = 4 \left( 2(K_\sigma \cdot Q)(\hat{P} \cdot Q) - (K_\sigma \cdot \hat{P}) Q^2 \right) .
\]

The \( Q^2 \) term gives vanishing contribution due to \( \delta(Q^2) \) (cf. (11)), and \( \hat{P} \cdot Q = q_0 - \hat{p} \cdot q = q_0 (1 - \epsilon(q_0) \hat{p} \cdot \hat{q}) \) eliminates the singularity (cf. (12)) of the factors in the square brackets in (11) and no mass singularity arises.
Carrying out the integration over $q_0$, we obtain

\[ i g^{(0)}_{\mu\nu} \Pi_{12}^{\mu\nu}(P) \simeq g^2 e_q^2 e^2 N_c \frac{N_c^2 - 1}{2 N_c} \frac{1}{E^2} \frac{1}{\pi} \int \frac{d^4 K}{(2\pi)^3} \sum_{\sigma = \pm} \rho_{\sigma}(K) \]

\[ \cdot \int q^2 dq \int d(\hat{p} \cdot \hat{q}) \sum_{\tau = \pm} \tau \left( 1 - \sigma \tau \hat{q} \cdot \hat{k} \right) \left\{ \frac{1}{1 - \tau \hat{p} \cdot \hat{q} + i\epsilon} + c.c. \right\} \]

\[ \cdot \left[ \{ \theta(-\tau) + n_B(q) \} \{ \theta(\tau) - n_F(q) \} \right. \]

\[ \left. \cdot \delta \left( (\tau q + p_0 - k_0)^2 - (q + p - k)^2 \right) \right], \]

(13)

where we have set $n_F(k_0) = \frac{1}{2}$. This is because, in the HTL approximations, $e^{k_0/T} = 1 + \mathcal{O}(g)$.

In (13), the integration over $\hat{p} \cdot \hat{q}$ diverges at $\hat{q} = \hat{p}$, which is nothing but the mass singularity. Using the HTL approximation $p << q$ in (13), and carrying out the integrations over $q$ and over $\hat{p} \cdot \hat{q}$ with the small quark mass ($\mu$) regulator, we have

\[ E \frac{dW}{d^3 p} \simeq \frac{1}{2\pi} e_q^2 \alpha N_c \frac{m_f^2}{E} \ln \left( \frac{ET}{\mu^2} \right) \int \frac{d^4 K}{(2\pi)^3} \delta(P \cdot K) \sum_{\sigma = \pm} (1 - \sigma \hat{p} \cdot \hat{k}) \rho_{\sigma}(K), \]

(14)

where $m_f$ (eq. (A.6)) is the thermal mass of a quark. The expression (14) is valid as far as the singular contribution is concerned.

We encounter the same integral as in (14) in the hard-photon production case [3]. Here we recall that $K$ is soft $\sim \mathcal{O}(gT)$. Then, the upper limit $k^*$ of the integration over $k$ is in the range, $gT << k^* << T$. (In [3], $k^*$ is chosen as $k^* = T$.) Referring to [4], we have

\[ E \frac{dW}{d^3 p} \simeq \frac{e_q^2 \alpha \alpha_s}{4\pi^2} T^2 \left( \frac{m_f}{E} \right)^2 \ln \frac{ET}{\mu^2} \left[ \ln \left( \frac{k^*}{m_f} \right)^2 - 1.31 \right]. \]

(15)

Thus, we have reproduced the result reported in [3, 4].

3 Further resummation

We study still higher order corrections. In the course of calculations in the previous section, we need not take care of the gauge invariance, since HTL corrections are gauge invariant [5]. Let us consider adding additional gluon lines to Fig. 1. In a covariant
gauge, the gauge-parameter dependent part of a gluon propagator \( \Delta_{ji}^{\mu\nu}(K) \) carries a factor \( K_\mu K_\nu \). Then, we see [10] that, upon summation over all possible positions on the quark lines, where the gluon can be inserted, the gauge-parameter dependent parts cancel, thanks to the trivial identity \((Q + K)^{-1}K(Q)^{-1} = (Q)^{-1} - (Q + K)^{-1}\). Thus, we can use the Feynman gauge.

Here is an observation of crucial importance: The insertions of gluon lines to the “parts” of Fig. 1, which do not show up mass singularities, yield higher-order contributions, i.e., the contributions which are at least \( g \) times the contributions under consideration. Then, the only part that should be considered to the order in consideration, \( \mathcal{O}(\alpha_s) \) up to logarithmic factors, is [10]. The same reasoning as above tells us that only the gluon-line insertions to the quark propagators with momentum \( Q + P \) are the only corrections that we should take into account, the quark propagators which originate the mass singularity in (15). This amounts to replacing \( S_{11} - S_{22} \) (cf. [10] with the one-loop self-energy-part resummed one (see Appendix),

\[
\begin{align}
\delta S_{11}(R) - \delta S_{22}(R) &= \sum_{\tau = \pm} \hat{R}_\tau \left\{ \delta \tilde{S}_{11}^{(\tau)}(R) - \delta \tilde{S}_{22}^{(\tau)}(R) \right\}, \quad R = Q + P, \quad (16) \\
\delta \tilde{S}_{11}^{(\tau)}(R) - \delta \tilde{S}_{22}^{(\tau)}(R) &= -\frac{1}{2} \left[ \frac{1}{D_\tau(R) + (r_0 - \tau r)a + R^2 b} + \text{c.c.} \right], \quad (17)
\end{align}
\]

where use has been made of (A.2) with (A.18). In (17), \( D_\tau \) is as in (A.4) and \( r_0 (= q_0 + p_0) \) should read \( r_0 (1 + i\epsilon) \). Explicit expressions for \( \tilde{a} = \tilde{a}(r_0, r) \) and \( \tilde{b} = \tilde{b}(r_0, r) \) are given in (A.14) - (A.17). We first note that the delta-function contributions of quasiparticle modes coming from two terms in (17) cancel, since they are pure imaginary. From (A.14) - (A.17), we see that \( \tilde{a}(r_0, r) \) and \( \tilde{b}(r_0, r) \) have imaginary parts in the whole range of \( r_0 \). We also see from (12) that \( r_0 - \tau r = q_0 + p_0 - \tau |q + p| \) in (17) is non negative. Then, using (A.4), we obtain

\[
\delta \tilde{S}_{11}^{(\tau)}(R) - \delta \tilde{S}_{22}^{(\tau)}(R) = \frac{(r_0 - \tau r)(1 + \mathcal{F}_\tau) - \tau \frac{m^2}{2r^2}}{(r_0 - \tau r)(1 + \mathcal{F}_\tau) - \tau \frac{m^2}{r^2}} + (r_0 - \tau r)^2 \mathcal{G}_\tau^2, \quad (18)
\]

where

\[
\begin{align}
\mathcal{F}_\tau &= \tau \frac{m^2}{2r^2} \ln \frac{r_0 + r}{r_0 - r} - Re \tilde{a} - (r_0 + \tau r)Re \tilde{b}, \\
\mathcal{G}_\tau &= Im \tilde{a} + (r_0 + \tau r)Im \tilde{b}. \quad (19)
\end{align}
\]
The factor $\tilde{S}_{11}(Q + P) - \tilde{S}_{22}(Q + P)$ in (10) is replaced by (18), which means that the quantity in the square brackets in (11) is replaced by two times (18). As discussed above after (11), the mass-singular contributions have arisen from the region $r_0 - \tau r \simeq E(1 - \tau \hat{q} \cdot \hat{p}) \simeq 0$ (cf. (12)). Then, to obtain the leading contribution from (11), with the above-mentioned substitution being made, it is sufficient to carry out the $\hat{p} \cdot \hat{q}$ integration over the small region $\hat{p} \cdot \hat{q} \simeq \tau$. Setting $\hat{p} \cdot \hat{q} = \tau$ in all the factors except in the factor (18), we obtain

$$\int_{\hat{p} \cdot \hat{q} \simeq 0} d(\hat{p} \cdot \hat{q}) \simeq \frac{1}{E} \ln \left( \frac{qE}{m_f^2} \right) \simeq \frac{1}{2E} \ln \left( \frac{1}{\alpha_s} \right),$$

(20)

where use has been made of the fact that $F_\tau$ and $G_\tau$ are of $\mathcal{O}(g^2)$ (cf. (19), (A.14)-(A.17)), and $F_\tau$ and $G_\tau$ are ignored on the r.h.s. of (20). The factor $m_f^2$ in (20) or in (18) comes from the hard loop momentum region in the (momentum-)integral representation of $\tilde{\Sigma}_F$ in (A.13).

Thus, we finally obtain

$$E \frac{dW}{d^3p} \simeq \frac{e^2}{8\pi^2} \frac{\alpha \alpha_s}{T^2} \left( \frac{m_f}{E} \right)^2 \ln \left( \frac{1}{\alpha_s} \right) \ln \left( \frac{c'}{\alpha_s} \left( \frac{k^*}{T} \right)^2 \right),$$

(21)

$$c' \simeq \frac{4N_c}{\pi(N_c^2 - 1)} e^{-1.31} \simeq 0.129, \quad (N_c = 3).$$

As is obvious from our derivation, the result (21) is valid at logarithmic accuracy. In order to obtain a full contribution of $\mathcal{O}(\alpha \alpha_s)$, we need to evaluate all the contributions, which we ignored in various stages.

### 4 Discussions and conclusions

Within the HTL resummation scheme, the soft-photon production rate (15) diverges [3, 4]. This divergence arises from mass singularities.

For any given diagram in real-time formalism, general rules of identifying the physical processes are available [5]. According to the rules, we can identify the physical processes leading to mass singularities: For $q_0 > 0$ (cf. Fig. 2), an example of such diagrams is depicted in Fig. 3 with the final-state cut line $C_1$. In Fig. 3, the left part
of the final-state cut line represents the $S$-matrix element in vacuum theory, while the right part represents the $S^*$-matrix element. The group of particles on top of Fig. 3 stands for spectator particles which are constituents of the quark-gluon plasma. When $q \parallel p$, the exchanged quark, $q(Q + P)$, is on the mass shell, which causes the mass singularity.

Here it is worth mentioning the following issue. In vacuum theory, it has long been known [11] that, when mass singularities arise in a rate of some reaction, the states that degenerate with the final state participate in the intermediate states. For such a case, the general theorem [11] of cancelling mass singularities states that, when we sum up a set of reaction rates, whose initial and final states as a whole include all the degenerate sets, the cancellations of mass singularities occur and finite reaction rate results. Physical interpretation of the theorem is fully discussed in [11]. It has been proved [12] that this theorem also applies to the case of thermal field theory.

We see from Fig. 3 with the final-state cut line $C_1$ that, when $q \parallel p$, the single-quark state $q(Q + P)$ in the intermediate state degenerate with the two-particle state, $q(Q) + \gamma(P)$, in the final state. Following the theorem stated above, we consider the processes whose final state is $q(Q + P)$. Fig. 3 with the final-state cut line $C_2$ is an example of such processes. We can see [3] that such processes are represented by

$$\int_\Delta \frac{d^4P}{(2\pi)^4} \sum_{\ell=1}^2 \left[ \Delta^{\mu\nu}_{\ell\ell} \left\{ \Pi_{\ell\ell}(P) \right\}_{\mu\nu} \right],$$

where $\Delta$ is the small region where $P^2 \simeq 0$. Although, presenting the elaborate formulae is not the purpose of this paper, we can confirm the above-mentioned theorem in the case of the soft-photon production rate: Within the HTL resummation scheme, the cancellation of mass singularities occurs between $E dW/d^3p$ in (15) and the rate extracted from (22), and their sum is free from singularity. In the case of, e.g., $q_0 > 0$ (Fig. 3), the former represents the production rate of the two-particle state $q(Q) + \gamma(P)$, while the latter represents the production of $q(Q + P)$.

In conclusion, an explicit computation in this paper of the soft-photon production rate demonstrates that mass singularities due to the exchanges of (massless) hard quarks are shielded by formally higher-order contributions, in a hot quark-gluon plasma. This screening mechanism allows us to compute the soft-photon production rate at logarithmic accuracy (cf. (21)): The appearance of the second logarithm
\[ \ln \left\{ \frac{e^2}{\alpha_s} \left( \frac{k^*}{T} \right)^2 \right\} \] reflects the HTL resummed soft quark propagators \( \tilde{S}_{i\bar{i}}(K) \) and \( \tilde{S}_{i\bar{i}2}(K') \) in (21), while the first logarithm \[ \ln \left( \frac{1}{\alpha_s} \right) \] arises from the resummed hard quark propagators, \( \tilde{S}_{11}(Q + P) \) and \( \tilde{S}_{22}(Q + P) \) (cf. (16)).

Finally, the following observation is in order. For shielding quark mass singularities, only the quark propagator with momentum \( Q + P \) (cf. Figs. 2 and 3) and the two onshell lines with momenta \( P \) and \( Q \) participate, and “other parts” of the diagram are not directly relevant. Then, the mechanism of shielding quark mass singularities works universally, being independent of reactions.

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Appendix

Here we display various expressions and useful formulae.

- Effective thermal propagator of a quark with soft momentum

\[ \tilde{S}_{ji}(K) = \sum_{\sigma = \pm} \hat{K}_\sigma \tilde{S}_{ji}^{(\sigma)}(K), \quad (j, i = 1, 2), \quad (A.1) \]

where

\[ \hat{K}_\sigma = (1, \sigma \hat{k}), \]

\[ \tilde{S}_{11}^{(\sigma)}(K) = - \left( \tilde{S}_{22}^{(\sigma)}(K) \right)^*, \]

\[ = - \frac{1}{2D_\sigma(K)} + i\pi \epsilon(k_0)n_F(|k_0|)\rho_\sigma(K), \quad (A.2) \]

\[ \tilde{S}_{12(21)}^{(\sigma)}(K) = \pm i\pi n_F(\pm k_0)\rho_\sigma(K), \quad (A.3) \]

with

\[ D_\sigma(K) = -k_0 + \sigma k + \frac{m^2_f}{2k} \left[ \left( 1 - \sigma \frac{k_0}{k} \right) \ln \frac{k_0 + k}{k_0 - k} + 2\sigma \right], \quad (A.4) \]

\[ \rho_\sigma(K) = \epsilon(k_0) \frac{1}{2\pi i} \left[ \frac{1}{D_\sigma(k_0(1 + i0^+), k)} - \frac{1}{D_\sigma(k_0(1 - i0^+), k)} \right], \quad (A.5) \]
\[ n_F(x) = \frac{1}{e^{x/T} - 1}, \]
\[ m^2_f = \frac{\pi \alpha_s N_c^2 - 1}{2N_c} T^2. \] (A.6)

In (A.4), \( k_0 \) should read \( k_0(1 + i\epsilon) \). \( D_{\sigma}(K) \) in (A.4) is first calculated in [13].

- **Bare thermal propagator of a quark**

The bare thermal propagators \( S_{ji}(K) \) \((j, i = 1, 2)\) are obtained from (A.1) - (A.6) with \( m_f = 0 \) and \( \rho_{\sigma}(K) = \delta(k_0 - \sigma k) \): In the text, there appears the combination,

\[ \tilde{S}_{11}^{(\sigma)}(K) - \tilde{S}_{22}^{(\sigma)}(K) = \frac{1}{2} \left[ \frac{1}{k_0(1 + i\epsilon) - \sigma k} + \text{c.c.} \right]. \] (A.7)

For the purpose of practical use, it is convenient to use the “one-term” forms;

\[ S_{j\ell}(K) = \mathcal{K} \tilde{S}_{j\ell}(K), \] (A.8)
\[ \tilde{S}_{11}(K) = -\left\{ \tilde{S}_{22}(K) \right\}^*, \]
\[ = \frac{1}{K^2 + i\epsilon} + 2\pi i n_F(|k_0|) \delta(K^2), \]
\[ \tilde{S}_{12(21)}(K) = 2\pi i \epsilon(\pm k_0) n_F(\pm k_0) \delta(K^2). \] (A.9)

- **Bare thermal gluon propagator**

\[ \Delta_{12}^{\mu\nu}(Q) = -g^{\mu\nu} \Delta_{12}(Q), \] (A.10)
\[ \Delta_{12}(Q) = -2\pi i \epsilon(q_0) n_B(q_0) \delta(Q^2), \] (A.11)
\[ n_B(x) = \frac{1}{e^{x/T} - 1}. \]

- **HTL-corrected Ward-Takahashi relations**

\[ (K - K')_\mu \sum_{i_2, j_2 = 1}^2 \left( \Gamma^{\mu}(K', K) \right)_{i_2 j_1} \left( S_{j_1 i_1}^{(\sigma)}(K') \right) \delta_{i_1 i} \delta_{j_1 j} \tilde{S}_{ji}(K) = \delta_{i_1 i} \delta_{j_1 j} \tilde{S}_{ji}(K) - \delta_{i_2 i} \delta_{j_2 j} \tilde{S}_{ji}(K), \] (A.12)

where summations are not taken over the repeated indices on the r.h.s.
• Thermal self-energy part of a quark and resummed quark propagator

The quasiparticle or diagonalized self-energy part \( \tilde{\Sigma}_F \) is related \([7]\) to the so-called analytic self-energy part \( \Sigma \) through \( \tilde{\Sigma}_F(r_0, r) = \Sigma(r_0(1 + i0^+), r) \). In real-time thermal field theory, \( \tilde{\Sigma}_F \) is evaluated \([7]\) through 
\[
\tilde{\Sigma}_F(r_0, r) = \Sigma_1(r_0(1 + i0^+), r) + \Sigma_2(r_0(1 + i0^+), r).
\]
Decomposing \( \tilde{\Sigma}_F \) as 
\[
\tilde{\Sigma}_F(r_0, r) = a(r_0, r) \hat{R} + b(r_0, r) \gamma^0,
\]
we obtain, to one-loop order,
\[
a(r_0, r) = -\frac{m_f^2}{r^2} \left( 1 - \frac{r_0}{2r} \ln \frac{r_0 + r}{r_0 - r} \right) + \tilde{a}(r_0, r),
\]
\[
b(r_0, r) = m_f^2 \frac{r_0}{r^2} \left( 1 - \frac{R^2}{2r_0^2} \ln \frac{r_0 + r}{r_0 - r} \right) + R^2 \tilde{b}(r_0, r),
\]
\[
\tilde{a}(r_0, r) = g^2 \frac{N_c^2 - 1}{2N_c} \frac{1}{r^2} \left[ (r_0^2 + r^2) I_B + R^2 I_F - 2r_0 (J_B + J_F) \right],
\]
\[
\tilde{b}(r_0, r) = -g^2 \frac{N_c^2 - 1}{2N_c} \frac{1}{r^2} \left[ r_0 (I_B + I_F) - 2(J_B + J_F) \right],
\]
with
\[
I_B = \frac{1}{16\pi^2} \frac{1}{r} \int_0^\infty dk n_B(k) \ln \left\{ \prod_{\xi = \pm} \frac{r_0 + \xi(r + 2k)}{r_0 - \xi(r + 2k)} \right\},
\]
\[
J_B = \frac{1}{16\pi^2} \frac{1}{r} \int_0^\infty dk n_B(k) \ln \left\{ \prod_{\xi = \pm} \frac{r_0 + r - 2\xi k}{r_0 - r - 2\xi k} \right\}.
\]

\( I_F \) and \( J_F \) in \((A.14)\) and \((A.13)\) are given, respectively, by \( I_B \) and \( J_B \) with \( n_B(k) \to n_F(k) \). It is to be noted that, when \( R \) is soft, \( I_B, J_B, I_F \) and \( J_F \) may be neglected in the HTL approximation, and we can reproduce \((A.1) - (A.6)\).

The self-energy-part corrected propagator of a quark \( ^8 S_{j\alpha}(R) \) is written \([7]\) as in \((A.1) - (A.6)\), provided \( D_\sigma \) in \((A.4)\) is replaced by
\[
D_\sigma(R) \to (-r_0 + \sigma r) \left\{ 1 - a(r_0, r) \right\} + b(r_0, r)
\]
\[
= D_\sigma(R) + (r_0 - \sigma r) \tilde{a} + R^2 \tilde{b}.
\]
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Figure captions

Fig. 1. Effective one-loop diagram for the production of a real soft photon. The blobs indicate the effective quark propagators and the effective quark-photon vertices. 1 and 2 at the effective photon-quark vertices designate the type of vertex.

Fig. 2. Structure of the HTL correction, \( \tilde{\Gamma}_{21}^{\mu}(K', K) \), to the photon-quark vertex.

Fig. 3. An example of the physical processes taking place in a quark-gluon plasma, which is responsible for the singular contribution (15) to \( E dW/d^3p \). The left side of the final-state cut line, \( C_1 \) or \( C_2 \), represents the \( S \)-matrix element, while the right side represents the \( S^* \)-matrix element. For the diagram with the final-state cut line \( C_1 \), when the effective quark propagator \( \tilde{S}_{11}(Q + P) \) (cf. (16)) is used, the mass singularity is screened. As to the photons and gluons in the diagram, in addition to the transverse components, the longitudinal and scalar components are included, because of the factor \( g_{\mu\nu} \) in (I) and of the usage of the Feynman gauge. If we use the Coulomb gauge and \( g^{(t)}_{\mu\nu} \) (eq. (3)) in place of \( g_{\mu\nu} \) in (I), only transverse components take place. In this case, the diagrams which are topologically different from Fig. 1 (and then also this diagram) contribute to the singular contributions.
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