SET REPRESENTATIONS OF LINEGRAPHS

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Abstract. Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. A family $\mathcal{S}$ of nonempty sets $\{S_1, \ldots, S_n\}$ is a set representation of $G$ if there exists a one-to-one correspondence between the vertices $v_1, \ldots, v_n$ in $V(G)$ and the sets in $\mathcal{S}$ such that $v_i v_j \in E(G)$ if and only if $S_i \cap S_j \neq \emptyset$. A set representation $\mathcal{S}$ is a distinct (respectively, antichain, uniform and simple) set representation if any two sets $S_i$ and $S_j$ in $\mathcal{S}$ have the property $S_i \neq S_j$ (respectively, $S_i \nsubseteq S_j$, $|S_i| = |S_j|$ and $|S_i \cap S_j| \leq 1$). Let $\mathcal{U}(\mathcal{S}) = \bigcup_{i=1}^{n} S_i$. Two set representations $\mathcal{S}$ and $\mathcal{S}'$ are isomorphic if $\mathcal{S}'$ can be obtained from $\mathcal{S}$ by a bijection from $\mathcal{U}(\mathcal{S})$ to $\mathcal{U}(\mathcal{S}')$. Let $\mathcal{F}$ denote a class of set representations of a graph $G$. The type of $\mathcal{F}$ is the number of equivalence classes under the isomorphism relation. In this paper, we investigate types of set representations for linegraphs. We determine the types for the following categories of set representations: simple-distinct, simple-antichain, simple-uniform and simple-distinct-uniform.

Keywords: Set representation; Uniquely intersectable; Clique partition; Line graph.

1 Introduction

Let $G$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The degree of $v$ is denoted by $d_G(v)$. For brevity, $V(G)$, $E(G)$ and $d_G(v)$ are simply written as $V$, $E$ and $d(v)$ when the context is clear.

\textbf{Definition 1.} Let $\mathcal{S}$ be a multiset of nonempty sets $\{S_1, \ldots, S_p\}$, that is, $S_1, \ldots, S_p$ might not be distinct. We write

$$\mathcal{U}(\mathcal{S}) = \bigcup_{i=1}^{p} S_i,$$

and we call $\mathcal{U}$ the universe of $\mathcal{S}$.

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The intersection graph of $S$ is the graph $G(S) = (V, E)$ with

$$ V = \{ S_1, \ldots, S_p \} \quad \text{and} \quad E = \{ (S_i, S_j) : i \neq j \quad \text{and} \quad S_i \cap S_j \neq \emptyset \}. $$

We say that $S$ is a set representation of the intersection graph $G$. We write $F(G)$ for the set of all set representations of $G$. The smallest cardinality of a universe, i.e., $|U(S)|$, for which $G$ has a set representation is called the intersection number of $G$ and it is denoted by $\theta(G)$.

We distinguish the following categories of set representations $S = \{ S_1, \ldots, S_p \}$.

- **Distinct** if no two sets $S_i$ and $S_j$ in $S$ are the same.
- **Antichain** if no set $S_i$ is a subset of another set $S_j$.
- **Uniform** if all subsets $S_i$ have the same cardinality.
- **Simple** if any two subsets have at most one element in common.

Below follows some notation and terminology.

1. The intersection numbers of distinct, antichain, uniform and simple set representations $S$ of $G$ are denoted by $\theta_d(G), \theta_a(G), \theta_u(G),$ and $\theta_s(G)$, respectively.

2. Let $F_d(G), F_a(G), F_u(G)$ and $F_s(G)$ be the sets of all minimum distinct, antichain, uniform, and simple set representations $S$ of $G$. That is,

$$ |U(S)| = \theta_d(G) \quad \text{if} \quad S \in F_d(G), $$

and similarly for the minimum universes of the other types.

3. We write
   (a) $F_{sd}(G) = F_s(G) \cap F_d(G)$,
   (b) $F_{sa}(G) = F_s(G) \cap F_a(G)$,
   (c) $F_{su}(G) = F_s(G) \cap F_u(G)$ and
   (d) $F_{sdu}(G) = F_{sd}(G) \cap F_{su}(G)$.

4. A set representation $S$ in $F_{sd}(G)$ is called an sd-set representation. The $sa, su$ and $sdu$-representations are defined similarly. The minimal cardinalities of a universe $U(S)$ with $S \in F_{sd}(G)$, is denoted by $\theta_{sd}(G)$. The parameters $\theta_{sa}(G)$, $\theta_{su}(G)$ and $\theta_{sdu}(G)$ are defined similarly.

It seems that Szpiłrajn-Marczewski [15] first came up with the idea of an intersection graph and a set representation although, often it is attributed to Erdős, Goodman and Pósa [6]. In [10], Kou, Stockmeyer and Wong proved that the computation of $\theta(G)$ is an NP-complete problem. Poljak, Rödl and Turzik proved the NP-completeness of $\theta_d(G)$ and $\theta_u(G)$ [14]. We refer to [18] for the NP-completeness of $\theta_a(G)$ and $\theta_u(G)$. Kong and Wu investigated bounds and relations between the various categories of set representations [9]. We remark that the sets $F_d(G), F_a(G), F_u(G)$ and $F_s(G)$ are not empty (see, eg, [8] Theorem 2.5).
Definition 2. Two set representations $S$ and $S'$ of $F(G)$ are isomorphic if there is a bijection $U(S) \rightarrow U(S')$ which maps each set of $S$ to a unique set of $S'$.

Definition 3. A graph $G$ is uniquely intersectable if all elements of $F(G)$ are isomorphic.

Alter and Wang [1] studied uniquely intersectable graphs. Unique simple, distinct, and antichain intersectability was subsequently studied in [5,12,16].

We parameterize intersectability as follows.

Definition 4. Let $F$ be some category of set representations. We say that $F$ is of type $\ell$ if its members are partitioned into $\ell$ equivalence classes by the isomorphism relation. We call $\ell$ the type of $F$ and we denote it by $\tau(F)$.

When $F$ is the collection of all set representations of a certain category $x$, then we also write $\tau_x(G)$ instead of $\tau(F)$. Thus, by definition, $
\tau(F_s(G)) = \tau_s(G) = 1, \quad \tau(F_d(G)) = \tau_d(G) = 1$ \quad and \quad $\tau(F_a(G)) = \tau_a(G) = 1$.

The linegraph of a graph $G = (V, E)$ is the graph $G^*$ with $V(G^*) = E$ and $E(G^*) = \{ef : \{e, f\} \subseteq E \text{ and } e \cap f \neq \emptyset\}$.

Bylka and Komar [5] and Li and Chang [11] investigated the characterization of graphs $G$ with $\tau(F_s(G^*)) = 1$.

We summarize our results in this paper as follows.

(1) If $G$ is not one of the following graphs: $K_4$, $W_t$, $3K_2 \lor K_1$, a star, or a 1t-peacock, then

$$\theta_{sd}(G^*) = |V_i| + \sum_{i=1}^{k} m_i.$$  

The set $V_i$ is the set of ‘inland vertices,’ which we define in Definition [12 on page 14]. The sum is the total number of vertices of degree one that are adjacent to some inland vertex. We prove this formula in Theorem [8].

(2) If $G$ is not one of the following graphs: $K_3$, $K_4$, $W_t$, a star, or a tailed peacock, then

$$\theta_{sa}(G^*) = |V_i| + \sum_{i=1}^{k} (m_i + 1).$$  

We prove this formula in Theorem [11 on page 28].

(3) 

$$\tau_{sd}(G^*) = \begin{cases} 
2 & \text{if } G \text{ is } K_4, W_t, \text{ or a TP}_1, \\
3 & \text{if } G \text{ is } 3K_2 \lor K_1, \\
2 + N_{PP}(d(v), r) & \text{if } G \text{ is a v-star with } d(v) \geq 3, \\
2^{V_{5w}^l} & \text{otherwise.}
\end{cases}$$  

3
all distinct

two kinds of nontrivial edge-clique covers

clique contains only one vertex. A 3-clique is also called a triangle.

in graph

a bijection introduced by Erdős, Goodman and Pósa, one obtains at least three kinds of nonisomorphic simple set representations \( S \) of \( K_n \) with \( \theta_S(K_n) = n \).

\section{Preliminaries}

For \( n \in \mathbb{N} \), let \([n] = \{1, \ldots, n\} \). If every pair of vertices in a graph is adjacent then the graph is called a clique. A clique with \( n \) vertices is denoted as \( K_n \). A \( k \)-clique in graph \( G \) is an induced subgraph of \( G \) which is a clique with \( k \) vertices. trivial clique contains only one vertex. A 3-clique is also called a triangle.

\textbf{Definition 5.} A set \( \Omega = \{Q_1, \ldots, Q_p\} \) of cliques in \( G \) is an edge-clique cover of \( G \) if

\[ V(G) = \bigcup_{i=1}^{p} V(Q_i) \quad \text{and} \quad E(G) = \bigcup_{i=1}^{p} E(Q_i). \]

An edge-clique cover \( \Omega \) is called an edge-clique partition if \( E(Q_i) \cap E(Q_j) = \emptyset \) for all distinct \( i, j \in [p] \).

In Section 2.1, we recall that, by the Erdős – De Bruijn theorem, there exist two kinds of nontrivial edge-clique covers \( \Omega \) of \( K_n \) with \( |\Omega| = n \). We introduce a third, trivial one for ease of future arguments. In Section 2.2 we show that, via a bijection introduced by Erdős, Goodman and Pósa, one obtains at least three kinds of nonisomorphic simple set representations \( S \) of \( K_n \) with \( \theta_S(K_n) = n \).
2.1 Edge clique-covers $\Omega$ of $K_n$ with $|\Omega| = n$

**Definition 6.** A finite linear space $\Gamma$, abbreviated FLS, is a pair $\Gamma = (P, L)$, where $P$ is a set of $n$ points and $L$ is a set of lines satisfying the following conditions:

(L1) A line is a set of at least two and at most $n - 1$ points.
(L2) Any two points are on exactly one line.

**Definition 7.** An FLS is a finite projective plane, abbreviated PP, if the following two conditions are satisfied:

(P1) Any two distinct lines intersect in exactly one point.
(P2) There are four points of which no three are on a line.

The following theorem is well-known, see, e.g., [2].

**Theorem 1.** If $\Pi$ is a PP with $n$ points and $l$ lines, then there exists an integer $r$, called the order of the plane, such that

$$n = l = r^2 + r + 1.$$

Furthermore, each point lies on $r + 1$ lines and each line contains $r + 1$ points.

The PP of order 2 is the well-known Fano Plane (See Figure 1).

![Fano Plane](image)

**Fig. 1.** Fano Plane.

It is well-known that there are unique PP's of orders 2, 3, 4, 5, 7 and 8 and none of orders 6 or 10. Moreover, there are exactly four non-isomorphic PP's of order 9, namely the Desarguesian Plane, the Left Nearfield Plane, the Right Nearfield Plane and the Hughes Plane. In this paper, we use $N_{PP}(n, r)$ to denote the number of non-isomorphic PP's with $n$ points and order $r$.

Let $n \in \mathbb{N}$ and $m \in \mathbb{N}$ and assume $m > 1$. Consider a set of $n$ elements, say $\mathcal{U} = [n]$. Let $A_1, \ldots, A_m$ be subsets of $\mathcal{U}$ such that every pair of elements of $\mathcal{U}$ is contained in exactly one of them. De Bruijn and Erdős proved the following theorem [4].

**Theorem 2.** We have that $m \geq n$. Equality occurs only in one of the two following cases.
(a) One set, say $A_1$, contains $n - 1$ elements, say $A_1 = [n - 1]$. The other sets are then, without loss of generality,

$$A_2 = \{1, n\}, A_3 = \{2, n\}, \ldots, A_n = \{n - 1, n\}.$$  

(b) For some $r \in \mathbb{N}$, $n = r^2 + r + 1$. All subsets have $r + 1$ elements and each element of $U$ is in exactly $r + 1$ subsets.

**Remark 1.** Notice that we may assume that each subset $A_i$ has at least two elements, otherwise we could simply remove some of them.

**Corollary 1.** Let $n \geq 3$ and let $\Omega$ be an edge-clique partition of $K_n$ with $|\Omega| > 1$. If $\Omega$ contains no trivial clique, then $|\Omega| \geq n$ and equality holds only in one of the two following cases.

(a) $\Omega$ consists of one clique with $n - 1$ vertices and $n - 1$ copies of $K_2$ or,

(b) the FLS implied by $\Omega$ is a PP.

The FLS’s corresponding to edge-clique partitions as in Condition (3) of Corollary 1 are conventionally referred to as near-pencils, abbreviated N-P. Let N-P($K_n$) and PP($K_n$) denote the edge-clique partitions of $K_n$ obtained from its corresponding N-P and PP. Table 1 illustrates N-P($K_7$) and PP($K_7$). Note that clique $Q_i$ of PP($K_7$) is corresponding to $\ell_i$ for $1 \leq i \leq 7$ of Fano plane as shown in Figure 1.

| $\Omega$ | N-P($K_7$) | PP($K_7$) | SP($K_7$) |
|----------|-------------|------------|------------|
| $Q_1$    | $\{v_1, v_6\}$ | $\{v_1, v_2, v_3\}$ | $\{v_1, \ldots, v_7\}$ |
| $Q_2$    | $\{v_1, v_7\}$ | $\{v_1, v_4, v_5\}$ | $\{v_2\}$ |
| $Q_3$    | $\{v_2, v_7\}$ | $\{v_1, v_6, v_7\}$ | $\{v_3\}$ |
| $Q_4$    | $\{v_3, v_7\}$ | $\{v_2, v_4, v_6\}$ | $\{v_4\}$ |
| $Q_5$    | $\{v_4, v_7\}$ | $\{v_2, v_5, v_7\}$ | $\{v_5\}$ |
| $Q_6$    | $\{v_5, v_7\}$ | $\{v_3, v_4, v_6\}$ | $\{v_6\}$ |
| $Q_7$    | $\{v_6, v_7\}$ | $\{v_3, v_5, v_6\}$ | $\{v_7\}$ |

A third edge-clique cover $\Omega$ of $K_n$ with $|\Omega| = n$ is defined as follows. Let $\Omega = \{Q_1, \ldots, Q_n\}$ with $Q_1 = \{v_1, \ldots, v_n\}$ and $Q_i = \{v_i\}$ for $i \in [n] \setminus \{1\}$. It is a trivial partition since all edges are in $Q_1$ and the other $Q_i$ contain no edges. We call $\Omega$ the silly partition and denote it by SP($K_n$). We introduce it because it eases some of the arguments.

### 2.2 A bijection between set representations and edge-clique covers

In [6], Erdős, Goodman and Pósa found a bijection, called EGP-bijection, between set representations and edge-clique covers of a graph $G$, as described below.
Let \( \Omega = \{Q_1, \ldots, Q_p\} \) be an edge-clique cover of graph \( G \). For every \( v_i \in V \) with \( i \in [n] \), let
\[
S_i = \{Q_j : j \in [p] \text{ and } v_i \in Q_j\}. \tag{1}
\]

Obviously, \( v_i v_j \in E \) if and only if \( S_i \cap S_j \neq \emptyset \). Thus \( \mathcal{S} = \{S_i \mid v_i \in V\} \in F(G) \). Conversely, let \( \mathcal{S} \in F(G) \). We obtain an edge-clique cover for \( G \) as follows. Let
\[
\mathcal{S} = \{S_i : v_i \in V\} \text{ and } \cup(\mathcal{S}) = \{s_1, s_2, \ldots, s_p\}.
\]
Define \( Q_j = \{v_i : j \in [p] \text{ and } s_j \in S_i\} \). Now for any edge \( pq \in E \),
\[
\{p, q\} \subseteq Q_j \text{ if and only if } s_j \in S_p \cap S_q. \tag{2}
\]
Thus \( \Omega = \{Q_1, \ldots, Q_p\} \) covers the edges of \( G \).

Hereafter, we call the set representation which is the EGP-image of an edge-clique cover an EGP-set. Likewise, we call the clique cover which is the EGP-image of a set representation an EGP-cover. We use EGP\((\mathcal{S})\) and EGP\((\Omega)\) to denote the EGP-cover and EGP-set, respectively. When the EGP-image of an edge-clique cover \( \Omega \) is simple we denote it by EGP\(_s(\Omega)\).

When \( \Omega \) is an edge-clique partition then, by Equation (1), any two sets \( S_x \) and \( S_y \) in EGP\(_s(\Omega)\) intersect in at most one element. That is, EGP\(_s(\Omega) \in F_s(G)\). Conversely, let \( \mathcal{S} \in F_s(G) \) and let \( \Omega \) be the EGP-image of \( \mathcal{S} \). Let \( xy \in E(G) \). By Equation (2), there is exactly one \( s_j \in S_x \cap S_y \) and so \( (x, y) \) is in exactly one \( Q_j \in \Omega \). That is, \( \Omega \) is an edge-clique partition. Table 2 illustrates the EGP-sets from the edge-clique covers in Table 1.

| \( \mathcal{S} \) | EGP\(_s(\text{N-P}(K_7))\) | EGP\(_s(\text{PP}(K_7))\) | EGP\(_s(\text{SP}(K_7))\) |
|---|---|---|---|
| \( \mathcal{S}_1 \) | \{Q_1, Q_2\} | \{Q_1, Q_2, Q_3\} | \{Q_1\} |
| \( \mathcal{S}_2 \) | \{Q_1, Q_3\} | \{Q_1, Q_4, Q_5\} | \{Q_1, Q_2\} |
| \( \mathcal{S}_3 \) | \{Q_1, Q_4\} | \{Q_1, Q_6, Q_7\} | \{Q_1, Q_3\} |
| \( \mathcal{S}_4 \) | \{Q_1, Q_5\} | \{Q_2, Q_4, Q_5\} | \{Q_1, Q_4\} |
| \( \mathcal{S}_5 \) | \{Q_1, Q_6\} | \{Q_2, Q_5, Q_7\} | \{Q_1, Q_5\} |
| \( \mathcal{S}_6 \) | \{Q_1, Q_7\} | \{Q_3, Q_4, Q_7\} | \{Q_1, Q_6\} |
| \( \mathcal{S}_7 \) | \{Q_2, \ldots, Q_7\} | \{Q_3, Q_5, Q_6\} | \{Q_1, Q_7\} |

**Lemma 1.** For \( n \geq 3 \), the set representations EGP\((\text{N-P}(K_n))\), EGP\((\text{PP}(K_n))\) and EGP\((\text{SP}(K_n))\) are all in \( F_{sd}(K_n) \).

**Proof.** First consider EGP\((\text{SP}(K_n))\). It is easy to verify that, for all \( i \neq j \),
\[
S_i \neq S_j \quad \text{and} \quad |S_i \cap S_j| = 1.
\]
Thus $\text{EGP}(\text{SP}(K_n)) \in F_{sd}(K_n)$.

Since $\text{N-P}(K_n)$ and $\text{PP}(K_n)$ are edge-clique partitions of $K_n$, the set representations $\text{EGP}(\text{N-P}(K_n))$ and $\text{EGP}(\text{PP}(K_n))$ are in $F_s(K_n)$. The sets of $\text{EGP}(\text{N-P}(K_n))$, constructed by Equation (1) from the cliques in Theorem 2 on page 5 (a), are

$$S_1 = \{Q_1, Q_2\}, S_2 = \{Q_3, Q_4\}, \ldots, S_{n-1} = \{Q_{2n-1}, Q_{2n}\}, S_n = \{Q_{2n+1}, \ldots, Q_n\}.$$ 

No two sets are the same and every pair intersect in one element. This proves $\text{EGP}(\text{N-P}(K_n)) \in F_{sd}(K_n)$.

Now consider the edge-clique partition $\text{PP}(K_n)$. For any vertex $x$, all the lines that contain $x$ intersect only in $x$. By Equation (1) and the fact that $n \geq 3$, this implies that $|S_x \cap S_y| = 1$ and $S_x \neq S_y$ whenever $x \neq y$. This proves $\text{EGP}(\text{PP}(K_n)) \in F_{sd}(K_n)$. This completes the proof. □

**Definition 8.** An $h$-punctured PP is a PP with $h$ points deleted.

We denote an edge-clique partition derived from an $h$-punctured PP by $\text{PP}_h(K_n)$. The EGP-image of $\text{PP}_h(K_n)$ is denoted by $\text{EGP}_s(\text{PP}_h(K_n-h))$.

**Example 1.** Consider the 2-punctured PP of the Fano plane in Figure 1 with $v_6$ and $v_7$ removed. It contains the following lines.

$$\ell_1 = \{v_1, v_2, v_3\} \quad \ell_2 = \{v_1, v_4, v_5\} \quad \ell_3 = \{v_2, v_4\}$$
$$\ell_4 = \{v_2, v_5\} \quad \ell_5 = \{v_3, v_4\} \quad \ell_6 = \{v_3, v_5\}.$$

For the edge-clique cover we define $Q_i = \ell_i$ for $1 \leq i \leq 6$. Accordingly, the EGP-bijection gives $\text{EGP}_s(\text{PP}_2(K_5)) = \{S_1, \ldots, S_5\}$ (see Figure 2), where

$$S_1 = \{Q_1, Q_2\} \quad S_2 = \{Q_1, Q_3, Q_4\} \quad S_3 = \{Q_1, Q_3\}$$
$$S_4 = \{Q_2, Q_4\} \quad S_5 = \{Q_3, Q_4\}.$$

**Fig. 2.** $\text{EGP}_s(\text{PP}_2(K_5))$.

For a proof of the following theorem we refer to [3].

**Theorem 3.** Let $\Gamma = (P, L)$ be an FLS with $n$ points and $\ell$ lines. The equality $\ell = n + 1$ holds if and only if $\Gamma$ is either a 1-punctured PP or the 2-punctured Fano Plane.
3 The types $\tau_{sd}(K_n)$, $\tau_{sa}(K_n)$ and $\tau_{sdu}(K_n)$

In this section, we investigate the types of $F_\text{sd}(K_n)$, $F_\text{sa}(K_n)$ and $F_\text{sdu}(K_n)$. In [7], Guo, Wang and Wang proved the following theorem.

**Theorem 4.** For $n \geq 1$, $\theta_{sd}(K_n) = n$ and for $n \geq 3$, $\tau_{sd}(K_n) = 2 + N_{PP}(n, r)$.

**Remark 2.** In Theorem 4, the equation $\tau_{sd}(K_n) = 2 + N_{PP}(n, r)$ is due to the fact that every member in $F_{sd}(K_n)$ for $n \geq 3$ is either $\text{EGP}_s(N-P(K_n))$ or $\text{EGP}_s(PP(K_n))$ or $\text{EGP}_s(SP(K_n))$.

In Theorems 5 and 6, we derive the types $\tau_{sa}(K_n)$ and $\tau_{sdu}(K_n)$.

**Theorem 5.** For $n \geq 3$, $\theta_{sa}(K_n) = n$ and $\tau_{sa}(K_n) = 1 + N_{PP}(n, r)$.

**Proof.** Assume that $n \geq 3$. By Theorem 4, $\theta_{sd}(K_n) = n$. Notice that $F_{sa}(K_n) \subset F_{sd}(K_n)$ implies that $\theta_{sa}(K_n) \geq \theta_{sd}(K_n) = n$. Clearly, no set representation of $\text{EGP}_s(N-P(K_n))$ is contained in another one. Thus $\text{EGP}_s(N-P(K_n)) \subseteq F_{sa}(K_n)$. This implies that $\theta_{sa}(K_n) \leq n$. As a consequence, $\theta_{sa}(K_n) = n$.

Since $\theta_{sa}(K_n) = n$, by Remark 2, there are only three kinds of set representations in $F_{sa}(K_n)$. It is quickly verified that $\text{EGP}_s(N-P(K_n))$ and $\text{EGP}_s(PP(K_n))$ are in $F_{sa}(K_n)$ but $\text{EGP}_s(SP(K_n))$ is not. Thus the theorem follows. \qed

**Definition 9.** Let $S \in F(G)$. An element of $U(S)$ is a monopolist if it appears in exactly one set of $S$.

**Theorem 6.**

$$\theta_{sdu}(K_n) = \begin{cases} n & \text{if } n = 3, \\ n & \text{if } n \geq 4, n = r^2 + r + 1 \text{ and } N_{PP}(n, r) \neq 0, \\ n + 1 & \text{otherwise.} \end{cases}$$

$$\tau_{sdu}(K_n) = \begin{cases} 1 & \text{if } n = 3, \\ N_{PP}(n, r) & \text{if } n \geq 4 \text{ and } \theta_{sdu}(K_n) = n, \\ 1 + N_{PP}(n + 1, r) & \text{if } n \geq 4 \text{ and } \theta_{sdu}(K_n) = n + 1. \end{cases}$$

**Proof.** We claim that $n \leq \theta_{sdu}(K_n) \leq n + 1$. By Theorem 4 since $F_{sdu}(K_n) \subset F_{sd}(K_n)$, we have $\theta_{sdu}(K_n) \geq n$. To prove that $\theta_{sdu}(K_n) \leq n + 1$, consider the following set representation. Let $S = \{S_1, \ldots, S_n\}$ where $S_i = \{s_i, s_{i+1}\}$ for $1 \leq i \leq n$. Clearly, $S$ is simple, distinct and uniform. Also, $|U(S)| = n + 1$. This implies that $\theta_{sdu}(K_n) \leq n + 1$. This proves the claim.

We first analyze the cases where $\theta_{sdu}(K_n) = n$ for some $n \geq 3$. Notice that $\text{EGP}_s(SP(K_n))$ is not uniform for $n \geq 3$. We also have that $\text{EGP}_s(N-P(K_n))$ is in $F_{sdu}(K_n)$ only for $n = 3$. In a PP of order $r$, each point PP is on exactly $r + 1$ lines. This implies that the set representation of the projective plane is in $F_{sdu}(K_n)$. Thus $F_{sdu}(K_n)$ contains $\text{EGP}_s(N-P(K_3))$ (for $n = 3$) and $\text{EGP}_s(PP(K_n))$ for $n =
$r^2 + r + 1$ (n ≥ 7) and $N_{pp}(n, r) \neq 0$. This shows that, if n ≥ 4 and there does not exist a PP of order r with $n = r^2 + r + 1$, then $\theta_{sd_u}(K_n) = n + 1$. Conversely, if n = 3 or $n = r^2 + r + 1$ and $N_{pp}(n, r) \neq 0$, then

$$\theta_{sd_u}(K_n) = n \quad \text{and} \quad \tau_{sd_u}(K_n) = \begin{cases} 1 & \text{if } n = 3 \\ N_{pp}(n, r) & \text{otherwise.} \end{cases}$$

Let $S \in F_{sd_u}(K_n)$, n ≥ 4, and assume that $|U(S)| = n + 1$. We claim that $S$ is either $\text{EGP}_s(\text{PP}_1(K_n))$ or $S = \{S_1, S_2, \ldots, S_n\}$ with $S_i = \{s_1, s_{i+1}\}$ for $i \in [n]$. Delete all monopolists from the sets of $S$ and let $S'$ be the result. Clearly, $S' \in F_s(K_n)$. Furthermore, $\text{EGP}(S')$ is an edge-clique partition of $K_n$ which contains at most $n + 1$ cliques and no trivial ones. By Corollary 1 and Theorem 3, either $|\text{EGP}(S')| = 1$ or $|\text{EGP}(S')|$ is an N-P, a PP, a 1-punctured PP, or a 2-punctured Fano Plane. If $|\text{EGP}(S')| = 1$ or $|\text{EGP}(S')|$ is an N-P or PP, then $S$ is either an $\text{EGP}_s(N-P(K_n))$ or an $\text{EGP}_s(\text{PP}(K_n))$ and then $U(S) \neq n + 1$. If $|\text{EGP}(S')|$ is an $\text{EGP}_s(\text{PP}_2(K_n))$ (see Figure 2 for an illustration), then $S' = S$ and $S$ is not uniform. Thus either $|\text{EGP}(S)| = 1$ or $|\text{EGP}(S)|$ is an $\text{EGP}_s(\text{PP}_1(K_n))$. This concludes the proof of this theorem.

4 The type $\tau_{sd}(G^*)$

In the rest of this paper, let $G$ be a connected graph and $\mathcal{P}$ an edge-clique partition of $G^*$. Our results on $\tau_{sd}(G^*), \tau_{sa}(G^*)$ and $\tau_{sd_u}(G^*)$ are based on the results in [13]. Hence, we follow most of the terms used in [13]. However, for ease of readability, we repeat some of them as follows.

If $uv \in E(G)$, then we use $uv$ to denote the corresponding vertex in $G^*$. An edge in $G^*$ with endpoints $vu$ and $vw$ is denoted by $(vu, vw)$ and a k-clique in $G^*$ containing vertices $vu_1, vu_2, \ldots, vu_k$ is denoted by $\{vu_1, vu_2, \ldots, vu_k\}$. In this case, we also say that the k-clique in $G^*$ is induced by the edges $vu_1, vu_2, \ldots, vu_k$ in $G$. We also use $uvw$ to denote a triangle when $u, v$ and $w$ are the vertices in the $K_3$.

Definition 10. A v-star in $G$ is a subgraph of $G$ consisting of a set of edges incident with a common vertex $v$ (see Figure 3(a)). Denote by $S_v$ a star consisting of $k$ edges incident with $v$. A v-star $S_v$ is saturated (respectively, a trivial) if $i = d(v)$ (respectively, $i = 1$). A v-wing in $G$ is a $K_3$ in which only $d_G(v) > 2$, and we call $v$ the stalk vertex of the wing (see Figure 3(b)). A 3-wing is a v-wing with $d(v) = 3$. A semiwing is a $K_3$ with exactly one vertex of degree 2 in $G$ (see Figure 3(c)). In a semiwing, the stalk vertices are the vertices with degree greater than 2. Let $v_n(\mathcal{P})$ denote the number of 3-cliques in $\mathcal{P}$ which are induced by $v$-wings. Let $V_{3w}$ be the set of stalk vertices of the 3-wings in $G$.

The join of graphs $G$ and $H$, denoted by $G \vee H$, is the graph $G \vee H = (V, E)$ where $V(G \vee H) = V(G) \cup V(H)$ and $E(G \vee H) = E(G) \cup E(H) \cup \{gh : g \in V(G), h \in V(H)\}$. For $t \geq 2$, let $W_t = tK_1 \vee K_2$ (see Figure 3(d)). The base edge of $W_t$ is the edge of which both endpoints are of degree greater than 2.
Definition 11. A plume in \( G \) is a vertex of degree 1. A graph is a one-tail peacock graph if it is composed of a \( K_3 \) in which exactly one vertex is adjacent to \( t \) plumes and the other two vertices are of degree 2, where \( t \) is a positive integer (see Figure 4(a)). A graph is a two-tail peacock graph if it composes of a \( K_3 \) in which exactly two vertices have plumes as neighbors (see Figure 4(b)). A graph is a diamond-back one-tail peacock graph (respectively, diamond-back two-tail peacock graph) if it composes of a \( W_t \) in which exactly one endpoint (respectively, both endpoints) of the base edge has plumes as neighbors (see Figures 4(c) and 4(d)).

For brevity, one-tail peacocks, two-tail peacocks, diamond-back one-tail peacocks and diamond-back two-tail peacocks are abbreviated as 1t-peacocks, 2t-peacocks, d1t-peacocks and d2t-peacocks, respectively, and are denoted by TP\(_1\), TP\(_2\), TP\(_{d1}\) and TP\(_{d2}\), respectively. We use ‘peacock’ as a generic term for either one of the peacock species mentioned above and denote it by TP.

Recall Whitney’s theorem \[19\].

**Theorem 7.** Any clique in \( G^* \) is induced either by a star or a \( K_3 \) in \( G \).

**Lemma 2 (Lemma 2.13 in \[13\]).** If \( w_v(P) > 0 \) for some \( v \in V(G) \), then \( P \) contains at least \( w_v(P) + 1 \) cliques induced by \( v \)-stars in \( G \), with equality occurring only when \( w_v(P) = 1 \) and \( d(v) = 3 \), or \( w_v(P) = 3 \) and \( G = 3K_2 \lor K_1 \).
McGuinness and Rees [13] define a surjection \( f : \mathcal{P} \to V_2(G) \) for \( G \neq K_3 \), where \( V_2(G) = \{ v \in V(G) : d(v) \geq 2 \} \). We describe the surjection as follows (and we make some modifications). Assume that \( G \neq K_3 \). The set \( V_2(G) \) can be partitioned into the following three subsets with respect to \( \mathcal{P} \):

\[ \begin{align*}
(1) & \quad R_\mathcal{P} = \{ v \in V_2(G) : d(v) \geq 3 \text{ and no clique in } \mathcal{P} \text{ is induced by } v \text{-stars} \}, \\
(2) & \quad \text{NW}_\mathcal{P} = \{ v \in V_2(G) \mid \text{v does not lies in a wing of } G \}, \text{ and} \\
(3) & \quad W = \{ v \in V_2(G) : v \text{ lies in a wing of } G \}.
\end{align*} \]

By definition, \( \text{NW}_\mathcal{P} \) is disjoint with \( R_\mathcal{P} \) and \( W \). Whitney’s theorem and Lemma 2 imply that \( R_\mathcal{P} \) and \( W \) are disjoint.

**Example 2.** Consider the graph \( G \) in Figure 5(a). It has 9 vertices and 13 edges. The vertex set \( V \) is \( \{ a, b, \ldots, i \} \) and the edges are labeled by \( 1, \ldots, 13 \). Figure 5(b) shows the linegraph \( G^* \) of \( G \). Figure 5(c) is an edge-clique partition \( \mathcal{P} \) of \( G^* \), where \( \mathcal{P} = \{ Q_1, \ldots, Q_{17} \} \) and \( Q_1 = \{ 1, 2 \}, Q_2 = \{ 1, 5, 7, 8, 9, 10 \}, \ldots \), as shown in Figure 5(c). Clearly, \( W = \{ c, d, e, f, g \} \) since \( cde \) and \( cfg \) are wings. Cliques \( Q_{16} \) and \( Q_{17} \) are induced by an \( h \)-star and an \( i \)-star, respectively. Thus \( \text{NW}_\mathcal{P} = \{ h, i \} \). Note that \( d_{G}(b) = 3 \) and no clique in \( \mathcal{P} \) is induced by \( b \)-star. Thus \( R_\mathcal{P} = \{ b \} \). The partition of \( V_2(G) \) is shown as in Figure 5(d).
Proposition 1. Let $P$ be an edge-clique partition of $G^*$ and $v$ a stalk vertex in $W$. If there are $t$ $v$-wings, then there are at least $2t + 1$ cliques in $P$.

Proof. For the stalk vertex $v \in W$, let

$$\{vx_{2i-1}x_{2i} : i \in [t]\}$$

be the collection of $v$-wings in $G$. We bound the cliques of $P$ from below by partitioning them into the following types. First, by definition, there are $w_v(P)$ $v$-wings that are 3-cliques in $P$. Secondly, by Whitney’s theorem and Lemma 2, there are $w_v(P) + 1$ cliques in $P$ induced by $v$-stars. Thirdly, for each $v$-wing $vx_{2i-1}x_{2i}$ for $w_v(P) + 1 \leq i \leq t$, there exist two cliques in $P$ induced by $S^2_{x_{2i-1}}$ and $S^2_{x_{2i}}$.

Adding them up, we find that each stalk vertex $v \in W$ gives rise to at least

$$w_v(P) + (w_v(P) + 1) + 2(t - w_v(P)) = 2t + 1$$

cliques in $P$. This completes the proof. \qed

Partition the nontrivial cliques $C \in P$ in the following subsets.

I. $P_r = \{C : C$ is induced by a $K_3$ in $G$ that is neither a wing nor a semiwing\},

II. $P_{nw} = \{C : C$ is induced by a semiwing or a $v$-star with $v \in NW_P\}$ and

III. $P_w = \{C : C$ is induced by a $v$-star or a $v$-wing for $v \in W\}$.

Notice that the sets $P_r, P_{nw}$ and $P_w$ are disjoint.

Example 3. We use Figure 5 to illustrate the sets $P_r, P_{nw}$ and $P_w$. In Figure 5(a), we can find that triangles $bch$, $bhi$, $bci$ and $chi$ are neither wings nor semiwings. Cliques $Q_4, Q_5$ and $Q_9$ in $P$ are induced by triangles $bch$, $bct$ and $bhi$, respectively, in $G$. Thus $P_r = \{Q_4, Q_5, Q_9\}$ (see Figure 5(d)).

It is easy to verify that

$$P_w = \{Q_1, Q_2, Q_3, Q_6, Q_7, Q_8, Q_10, \ldots, Q_{15}\}$$

in which

(i) $Q_{10}$ is induced by a $c$-wing,

(ii) $Q_{14}$ is induced by an $f$-star,

(iii) $Q_{15}$ is induced by a $g$-star, and

(iv) all other cliques are induced by $c$-stars.

The remaining collection is $P_{nw} = \{Q_{16}, Q_{17}\}$. Note that $Q_{16}$ is an $h$-star and $Q_{17}$ is an $i$-star.

The following lemma appears in [13, Lemma 2.8].
Lemma 3. \(|P_2| \geq |R_2|\), with equality occurring only if either \(R_2 = \emptyset\) or \(G = K_4\).

For a proof of the following lemma see [13, Lemma 2.9]

Lemma 4. If \(G \neq W_1\) and \(\mathcal{P}\) contains a clique induced by a semiwing \(wuv\) in \(G\) with non-stalk vertex \(w\), then \(\mathcal{P}\) contains at least two cliques induced by \(x\)-stars for some \(x \in \{u, v\}\).

McGuinness and Rees define three functions in [13], which we describe shortly.

1. A surjection \(f_1 : P_r \to R_p\),
2. a bijection \(f_2 : P_{nw} \to NW_p\) and
3. another surjection \(f_3 : P_w \to W\).

Since \(|P_r| \geq |R_p|\) by Lemma [13] \(f_1\) can be designed to be surjective. For each stalk vertex \(v \in W\), by Proposition [11] there exist at least \(2t + 1\) cliques in \(P_w\) which can be assigned to the \(2t + 1\) vertices lying in \(v\)-wings by \(f_3\). Hence, \(f_3\) is surjective.

We impose some additional rules on the construction of \(f_3\). That is, for each \(v\)-wing \(vxy\) in \(G\), if \(vxy\) induces a \(K_2\) in \(P_w\), then we always assign the \(K_2\) to \(x\) or \(y\); otherwise, assign the two \(2\)-cliques in \(P_w\) induced by \(S_x^2\) and \(S_y^2\) to \(x\) and \(y\), respectively. This completes the description of \(f_3\).

A bijection \(f_2 : P_{nw} \to NW_p\) is described as follows. Let \(u \in NW_p\). If \(d(u) = 2\) and \(u\) lies in no triangle in \(G\), then \(S_u^2\) induces a \(K_2\) in \(P_{nw}\) which is assigned to \(u\) in the bijection. If \(d(u) = 2\) and \(u\) lies in a triangle in \(G\), then \(u\) is the non-stalk vertex of a semiwing (assuming \(G \neq K_4\)). Thus \(P_{nw}\) contains either the \(K_2\) induced by \(S_{u}^2\) or the \(K_3\) induced by the semiwing and either of them is assigned to \(u\). If \(d(u) \geq 3\), then \(\mathcal{P}\) contains cliques induced by \(u\)-stars, which are in \(P_{nw}\) by definition, and these are assigned to \(u\).

To ensure that \(f_2\) is well-defined, recall that the non-stalk vertex of a semiwing \(wuv\), say \(w\), must be in \(NW_p\). Therefore, if \(wuv\) induces a clique \(C\) in \(P_{nw}\), then \(f_2(C) = w\). Thus \(f_2\) is a bijection function.

We obtain a surjection \(f : \mathcal{P} \to V_2(G)\) as follows. When restricted to one of the domains \(P_2, P_{nw}\) or \(P_w\), the function is locally defined by \(f_1, f_2\) or \(f_3\). Finally, we assigning remaining trivial cliques in \(\mathcal{P}\) arbitrarily to vertices in \(V_2(G)\). For \(v \in V_2(G)\), let

\[ C_v = \{ C : C \in \mathcal{P} \text{ and } f(C) = v \}. \]

Figure 5(e) is an example of \(C_v\).

Before investigating \(F_{sd}(G^*)\), we define some terms. In the rest of this section we assume that \(S \in F_{sd}(G^*), \mathcal{P} = EGP(S)\) and \(S(uv)\) denotes the sets in \(S\) assigned to \(uv \in V(G^*)\). Note that \(\mathcal{P}\) might contain some trivial cliques.

Definition 12. A vertex \(v \in V_2(G)\) is critical if \(v\) has a neighbor with degree 1; otherwise, it is an inland vertex. Let \(V_c\) and \(V_i\) denote the sets of critical and inland vertices.
Assume that $V_c = \{v_1, \ldots, v_k\}$. For $i \in [k]$, let $v^1_i, \ldots, v^{m_i}_i$ be the neighbors of $v_i$ of degree one. Also, let $v^{m_i+1}_i, \ldots, v^{d(v_i)}_i$ be the remaining neighbors of $v_i$.

We introduce some more notations.

$$V_{cw} = V_c \cap W$$

$$V_{cnw} = V_c \cap NW_P$$

$$\gamma = |V_i| + \sum_{i=1}^{k} m_i$$

$$P_c = \{C \in \mathcal{P} : C \text{ is induced by a } v\text{-star with } v \in V_c\} \quad \text{and}$$

$$P_i = \mathcal{P} \setminus P_c.$$  \hspace{1cm} (3)

For $i \in [k]$, let

$$S_i = \{S(v_i v_1^1), S(v_i v_2^1), \ldots, S(v_i v^{m_i}_i)\}. \quad \text{ (8)}$$

Obviously, $S_i \in F_{sd}(K_{m_i})$. By Theorem $[4]$, $\|\text{EGP}(S_i)\| \geq m_i$ for $i \in [k]$.

**Example 4.** We use Figure 5 to illustrate the terminology that we introduced above. In Figure 5(a), the set of critical vertices is $V_c = \{c\}$ since $c$ is the only vertex which has a neighbor that is of degree one. The set of inland vertices is $V_i = V_2(G) \setminus V_c = V \setminus \{a, c\}$.

Thus as defined in Equations (3) up to (7) and (8):

$$V_{cw} = \{c\}$$

$$V_{cnw} = \emptyset$$

$$\gamma = |V_i| + \sum_{i=1}^{k} m_i = 7 + 1 = 8$$

$$P_c = \{Q_1, Q_2, Q_3, Q_6, Q_7, Q_8, Q_{11}, Q_{12}, Q_{13}\}$$

$$P_i = \{Q_4, Q_5, Q_9, Q_{10}, Q_{14}, \ldots, Q_{17}\} \quad \text{and}$$

$$S_1 = \{S(ca)\}.$$ 

**Proposition 2.** All cliques in $\text{EGP}(S_i), \ i \in [k]$, are sub-cliques of some cliques in $\mathcal{P}$ induced by $v_i$-stars. Furthermore, $V_c \cap R_P = \emptyset$ and $\mathcal{P}$ contains at least $m_i$ cliques induced by $v_i$-stars for $i \in [k]$.

**Corollary 2.** If $|U(S)| \leq \gamma$, then $|P_i| \leq |V_i|$.
Proof. By Proposition 2 and $|P| = |P_i| + |P_c|$, we obtain that

$$|P| \geq |P_i| + \sum_{i=1}^{k} |\text{EGP}(S_i)|.$$  

If $|P_i| > |V_i|$, then

$$|U(S)| = |P| > |V_i| + \sum_{i=1}^{k} |\text{EGP}(S_i)| \geq |V_i| + \sum_{i=1}^{k} m_i = \gamma.$$  

This is a contradiction. \hfill \qed

Lemma 5. If $G \neq K_3$, then $|P_i| \geq |V_i| - \sum_{v \in V_{cw}} w_v(P)$.  

Proof. Let $V'_2(G) = \{v \in V_2(G) : C'_v \subseteq P_c\}$, where $C'_v$ is the set obtained by removing all trivial cliques from $C_v$. For all $v \in V_2(G) \setminus V'_2(G)$, it follows that $C_v \cap P_i \neq \emptyset$ implies that $|P_i| \geq |V_2(G)| - |V'_2(G)|$.

By the construction of $f_1$, $V'_2(G) \cap R_P = \emptyset$. We claim that $v \in NW_P \cap V'_2(G)$ implies that $v \in V_{cw}$. The reason is the following.

If $v \in NW_P \cap V'_2(G)$, then all nontrivial cliques in $C_v$ are also in $P_c$. By the construction of $f_2$, all those cliques are induced by $v$-stars. Hence $v \in V_c$. This proves the claim.

By the construction of $f_3$, for each $v \in V_{cw}$, there are exactly $1 + w_v(P)$ vertices in $V'_2(G)$. Moreover, for each stalk vertex $v \in W \setminus V_c$, no vertex in a $v$-wing is in $V'_2(G)$. This proves the correctness of the following derivation:

$$|P_i| \geq |V_2(G)| - |V'_2(G)| = |V_2(G)| - (|V_{cnw}| + \sum_{v \in V_{cw}} (1 + w_v(P))) = |V_2(G)| - (|V_{cnw}| + |V_{cw}| + \sum_{v \in V_{cw}} w_v(P)) = |V_2(G)| - (|V_c| + \sum_{v \in V_{cw}} w_v(P)) \quad \text{(by Proposition 2)}$$

$$= |V_i| - \sum_{v \in V_{cw}} w_v(P).$$

This completes the proof. \hfill \qed
Lemma 6. If $G$ is not a TP$_1$ and $|U(S)| \leq \gamma$, then $w_v(P) = 0$ for every $v \in V_{cw}$.

Proof. Suppose to the contrary there exists a $v \in V_{cw}$ with $w_v(P) \neq 0$. Let $u_1, \ldots, u_m$ be $v$'s neighbors that have degree one. For $1 \leq i \leq w_v(P)$, let $vx_{2i-1}x_{2i}$ be the $v$-wings which induce 3-cliques in $P$.

Let $G'$ be the subgraph of $G^*$ with

$$V(G') = \{vu_1, vu_2, \ldots, vu_m, vx_1, vx_2, \ldots, vx_{2w_v(P)}\}$$

$$E(G') = E(G[V']) \setminus \{(vx_{2i-1}, vx_{2i}) : 1 \leq i \leq w_v(P)\}.$$  

By Corollary 1, any edge-clique partition of $G'$ has at least $m + w_v(P)$ cliques since there exists a clique of size $m + w_v(P)$ in $G'$. Moreover, any clique in $P$ which contains edges of $G'$ is clearly induced by a $v$-star. Thus $P$ has at least $m + w_v(P)$ cliques induced by $v$-stars.

The case where $P$ has exactly $m + w_v(P)$ cliques induced by $v$-stars occurs only when $w_v(P) = 1$. The reason is the following.

When $w_v(P) = 1$, the edges of $G' \cup (vx_1, vx_2)$ are partitioned by an $N$-$P$ consisting of:

1. a clique of size $m + 2w_v(P) - 1$ induced by the set of vertices $V(G') \setminus \{vx_1\}$ and,
2. $m + 2w_v(P) - 1$ $K_2$'s intersecting in vertex $vx_1$.

Note that the 2-clique $\{vx_1, vx_2\}$ is contained in the $K_3$ in $P$ induced by $vx_1x_2$. Thus $P$ has exactly $m + w_v(P)$ cliques induced by $v$-stars only when $w_v(P) = 1$.

Consider the case where $w_v(P) = 1$. Since $G$ is not a TP$_1$, there exists another $v$-wing. Thus $v$ has a neighbor $y \notin \{u_1, u_2, \ldots, u_m, x_1, x_2\}$.

Add the vertex $vy$ to $G'$ together with all those edges in $E(G^*)$ that have endpoint $vy$ and the other endpoint in $V(G')$. Let $G^+$ be this subgraph of $G^*$. Since $y$ is not adjacent to any vertex in $\{u_1, u_2, \ldots, u_m, x_1, x_2\}$ of $G$, any clique in $P$ which contains edges of $G^+$ is still induced by a $v$-star. Therefore, by Corollary 1, $P$ has at least $m + 2$ cliques induced by $v$-stars. That is, if $w_v(P) \neq 0$, then $P$ contains more than $m + w_v(P)$ cliques induced by $v$-stars.

On the other hand, by Proposition 2 for each $v_i \in V_c \setminus W \cup V_{cw}$ with $w_{v_i}(P) = 0$, $P$ contains at least $m_i$ cliques induced by $v_i$-stars. It follows that

$$|P_c| > \sum_{i=1}^{k} m_i + \sum_{v \in V_{cw}} w_v(P).$$
Thus by Lemma 5,

\[ |U(S)| = |\mathcal{P}| = |\mathcal{P}_i| + |\mathcal{P}_c| > |V_i| - \sum_{v \in V_{cw}} w_v(\mathcal{P}) + \sum_{i=1}^{k} m_i + \sum_{v \in V_{cw}} w_v(\mathcal{P}) = \gamma. \]

This contradiction concludes our proof. \( \square \)

**Corollary 3.** If \( G \) is not a TP and \( |U(S)| \leq \gamma \), then \( C_v \cap \mathcal{P}_i \neq \emptyset \) for all \( v \in V_i \).

**Proof.** Let \( v \in V_i \). First assume that \( v \in R_\mathcal{P} \cup NW_\mathcal{P} \). By the construction of \( f_1 \) and \( f_2 \), we can obtain that \( C_v \) has a clique induced by a triangle or a nontrivial \( v \)-star in \( G \). Such a clique is in \( \mathcal{P}_i \) and therefore the corollary holds in this case.

Now assume that \( v \in V_{cw} \). Since \( w_v(\mathcal{P}) = 0 \) by Lemma 6, every non-stalk vertex \( x \) of the \( v \)-wings (which is in \( V_i \)) is assigned by \( f_3 \) to the clique induced by \( S_2^x \) (which is in \( \mathcal{P}_i \)). For a stalk vertex \( v \in W \setminus V_c \), all cliques induced by \( v \)-wings or \( v \)-stars are in \( \mathcal{P}_i \). Therefore, by the construction of \( f_3 \), for all vertices \( x \) in \( v \)-wings, \( C_x \cap \mathcal{P}_i \neq \emptyset \). This completes the proof. \( \square \)

**Lemma 7.** If \( G \) is not a TP and \( |U(S)| \leq \gamma \), then the following statements hold.

1. \( |\mathcal{P}_i| = |V_i| \)
2. for \( i \in [k] \) and \( v_i \in V_c \), \( \mathcal{P} \) contains exactly \( m_i \) cliques induced by \( v_i \)-stars,
3. for every \( v \in V_i \setminus V_{3w} \), \( \mathcal{P} \) contains at most one nontrivial clique induced by \( v \)-stars unless \( G = 3K_2 \cup K_1 \).

**Proof.** By Lemmas 5 and 6 \( |\mathcal{P}_i| \geq |V_i| \). By Corollary 2 on page 15 the first statement holds.

If, for some \( i \in [k] \), \( \mathcal{P} \) contains more than \( m_i \) cliques induced by \( v_i \)-stars, then, by the first statement and Proposition 2 on page 15,

\[ |U(S)| = |\mathcal{P}| = |\mathcal{P}_i| + |\mathcal{P}_c| > \gamma. \]

This is a contradiction. Therefore the second statement holds.

Consider the last statement. Suppose that \( G \neq 3K_2 \cup K_1 \) and suppose that there are two cliques induced by \( v \)-stars in \( \mathcal{P} \), for some vertex \( v \in V_i \setminus V_{3w} \). By the construction of \( f_i \), the set \( C_v \) contains these two cliques unless \( v \) is a stalk vertex in \( W \).

First assume that \( v \) is a stalk vertex in \( W \) and that \( w_v(\mathcal{P}) > 0 \). By Lemma 2 on page 11 the set \( \mathcal{P} \) contains more than \( w_v(\mathcal{P}) + 1 \) cliques induced by \( v \)-stars.

\^ See Definition 10 on page 10 to recall the definition of \( V_{3w} \).
Now assume that \( w_v(\mathcal{P}) = 0 \). The set \( \mathcal{P} \) still contains more than one clique induced by a \( v \)-star, since \( w_v(\mathcal{P}) + 1 = 1 \). We conclude that the number of cliques induced by \( v \)-wings and \( v \)-stars is greater than the number of vertices in the \( v \)-wings.

Since \( v \in V_i \), all cliques induced by \( v \)-wings and \( v \)-stars are in \( \mathcal{P}_i \). Consequently, by the construction of \( f_3 \), \(|C_x \cap \mathcal{P}_i| \geq 2\) for some vertex \( x \) in a \( v \)-wing. However, in that case, \(|\mathcal{P}_i| > |V_i|\) by Corollary 3 on the preceding page. This contradicts the first statement.

Assume that \( v \) is not a stalk vertex in \( W \). The set \( C_v \) contains two cliques induced by \( v \)-stars, that is, \(|C_v \cap \mathcal{P}_i| \geq 2\). This is also a contradiction. This completes the proof. \( \square \)

**Lemma 8.** If \( G \) is neither \( K_4 \) nor a \( TP_1 \) and \(|U(S)| \leq \gamma\), then \( \mathcal{P}_r = \emptyset \).

**Proof.** Suppose that \( \mathcal{P}_r \neq \emptyset \). First assume that \( R_{\mathcal{P}} \neq \emptyset \). By Lemma 3 on page 14, \(|\mathcal{P}_r| > |R_{\mathcal{P}}|\). By the construction of \( f_1 \), this implies that \(|C_v \cap \mathcal{P}_r| \geq 2\) for some \( v \in R_{\mathcal{P}} \).

Now assume that \( R_{\mathcal{P}} = \emptyset \), namely, there is no \( f_1 \). However, there is at least one clique in \( \mathcal{P}_r \). Since \( \mathcal{P}_r \subset \mathcal{P}_i \), both cases imply \(|\mathcal{P}_i| > |V_i|\) by Corollary 3. This contradicts the first statement in Lemma 7. This proves the lemma. \( \square \)

**Lemma 9.** If \( G \) is neither a \( W_4 \) nor a \( TP_1 \) and if \(|U(S)| \leq \gamma\) then \( \mathcal{P} \) contains no \( K_3 \) induced by a semiwing in \( G \).

**Proof.** Assume that there is a \( K_3 \) in \( \mathcal{P} \) which is induced by a semiwing \( wuv \), with \( d(w) = 2 \). By Lemma 4 on page 14, \( \mathcal{P} \) contains two cliques induced by \( u \)-stars. Clearly, \( u \notin V_{3w} \) and \( G \neq 3K_2 \cup K_1 \). Thus by the third statement in Lemma 7, \( u \in V_c \).

Let \( u_1, \ldots, u_m \) be \( u \)'s neighbors that have degree one. By Corollary 1, \( \mathcal{P} \) has at least \( m + 1 \) \( u \)-stars partitioning the edges of

\[
G^*[[uu_1, \ldots, uu_m, uv, uw]]
\]

besides the edge \( (uv, uw) \). This contradicts the second statement of Lemma 7. This proves the lemma. \( \square \)

**Lemma 10.** If \( G \) is neither \( 3K_2 \cup K_1 \) nor a \( TP_1 \) and if \(|U(S)| \leq \gamma\), then \( \mathcal{P} \) contains no \( K_3 \) that is induced by a \( v \)-wing for \( v \notin V_{3w} \).

\(^5\) Recall the definition of \( \mathcal{P}_r \), it's Item II on Page 13.
Proof. Suppose that some \( v \)-wing, for some \( v \notin V_{3w} \) induces a \( K_3 \) in \( \mathcal{P} \). By Lemma \( 6 \) \( v \notin V_c \). By Lemma \( 2 \) \( \mathcal{P} \) contains at least two cliques induced by \( v \)-stars. This contradicts the third statement in Lemma \( 7 \). This proves the lemma.

\[ \square \]

Corollary 4. If \( G \notin \{ K_4, W_4, 3K_2 \lor K_1, TP_1 \} \) and \( |U(S)| \leq \gamma \), then any clique in \( \mathcal{P} \) is induced by either a star or 3-wing in \( G \).

Theorem 8. If \( G \notin \{ K_4, W_4, 3K_2 \lor K_1, a \text{ star}, TP_1 \} \), then

\[ \theta_{sd}(G^*) = |V| + \sum_{i=1}^{k} m_i. \]

Proof. Recall that \( |V| + \sum_{i=1}^{k} m_i = \gamma \). First consider a vertex \( v \in V_i \) which is not in a 3-wing. By the third statement in Lemma \( 7 \), the set \( \mathcal{P} \) contains at most one clique induced by \( v \)-star. Thus if \( \mathcal{P} \) does not contain the clique induced by the saturated \( v \)-star, then it contains a clique induced by a triangle that contains \( v \). By Corollary \( 4 \), \( v \) is in a 3-wing, a contradiction. Thus for each \( v \in V_i \) which is not in a 3-wing, \( \mathcal{P} \) contains the clique induced by the saturated \( v \)-star.

Now consider a vertex \( v_i \in V_c \) for \( i \in [k] \). Since \( G \) is not a star, \( v_i \) has a neighbor, say \( z \), with \( d(z) \geq 2 \). By the second statement in Lemma \( 7 \) and Proposition \( 2 \), \( |E_{GP}(S_i)| = m_i \). Therefore, by Remark \( 2 \), \( E_{GP}(S_i) \) is an N-P, a PP, or an edge-clique cover containing a \( K_{m_i} \) and \( m_i - 1 \) trivial cliques. For the first two cases, since \( v_i z \) is adjacent to all of \( v_i v_i^1, \ldots, v_i v_i^{m_i} \) in \( G^* \), \( S(v_i z) \) contains at least two elements, say \( a \) and \( b \), in \( U(S_i) \). However, by definition of PP, any two cliques in an N-P or PP intersect at exactly one vertex. Suppose the two cliques in \( E_{GP}(S_i) \) corresponding to \( a \) and \( b \) in the EGP bijection intersect at a vertex \( v_i v_i^j \) in \( G^* \). This means that \( a, b \in S(v_i v_i^j) \) and \( |S(v_i z) \cap S(v_i v_i^j)| \geq 2 \), a contradiction. Thus we conclude that, for each \( v_i \in V_c \), \( \mathcal{P} \) contains the clique induced by the saturated \( v_i \)-star and \( m_i - 1 \) trivial cliques induced by \( \{v_i v_i^1, \ldots, v_i v_i^{m_i}\} \).

We conclude that, if \( |U(S)| \leq \gamma \), then \( \mathcal{P} \) consists of \( \sum_{i=1}^{k} (m_i - 1) \) trivial cliques and \( |V_2(G)| \) cliques induced by saturated \( v \)-stars for all \( v \in V_2(G) \) except that, for each 3-wing, say \( vxy \), with \( d(v) = 3 \) in \( G \), \( \mathcal{P} \) may contain either a \( K_3 \) induced by \( vxy \) and two \( K_2 \)'s induced by \( v \)-stars or three cliques induced by saturated \( v \)-, \( x \)- and \( y \)-stars, respectively.

Accordingly, consider an \( S \) with its corresponding \( \mathcal{P} \) as described above. Every \( S(uv) \in S \) contains two elements corresponding to two cliques in \( \mathcal{P} \) where both \( d(u) \) and \( d(v) \) are greater than or equal to 2 and \( uv \in E(G) \) does not connect the two non-stalk vertices of a wing. Clearly, \( S(uv) \) is the unique set in \( S \) containing these two elements. Due to the elements in \( U(S) \) corresponding to trivial cliques
in \( P \), \( S(vu_1) \neq S(vu_2) \) for any pair of vertices \( vu_1 \) and \( vu_2 \) in \( G^* \) with \( d(u_1) = d(u_2) = 1 \). Thus \( S \in F_{sd}(G^*) \). Hence

\[
\theta_{sd}(G^*) = |V_2(G)| + \sum_{i=1}^{k} (m_i - 1) = |V_i| + |V_c| + \sum_{i=1}^{k} (m_i - 1) = \gamma.
\]

This completes the proof. \( \square \)

**Theorem 9.** The type of \( F_{sd}(G^*) \) is

\[
\tau_{sd}(G^*) = \begin{cases} 
2 & \text{if } G \text{ is } K_4, W_t, \text{ or a } TP_1, \\
3 & \text{if } G \text{ is } 3K_2 \lor K_1, \\
2 + N_{pp}(d(v), r) & \text{if } G \text{ is a } v\text{-star with } d(v) \geq 3, \\
2 & \text{otherwise.}
\end{cases}
\]

**Proof.** It is easy to verify that

\[
\tau_{sd}(G^*) = \begin{cases} 
2 & \text{if } G \text{ is } K_4 \text{ or } W_t, \\
3 & \text{if } G = 3K_2 \lor K_1.
\end{cases}
\]

By Theorem 4 on page 9, \( \tau_{sd}(G^*) = 2 + N_{pp}(d(v), r) \) when \( G \) is a \( v\)-star with \( d(v) \geq 3 \). The ‘otherwise’-statement follows directly from the proof of Theorem 8.

It remains to prove that \( \tau_{sd}(G^*) = 2 \) when \( G \) is a \( TP_1 \). Let \( G \) be a \( TP_1 \) that is not \( K_3 \). There are two classes of \( sd \)-set representations for \( G \), depending on whether \( P \) contains a \( K_3 \) induced by a wing or not.

First, we consider the case where \( P \) contains a \( K_3 \) induced by a wing. Let \( vxy \) be such a \( v \)-wing and let \( G' \) be the subgraph of \( G^* \) induced by the saturated \( v \)-star but without the edge \((vx, vy)\). By Corollary 1 on page 6, \( P \) contains at least \( m + 1 \) cliques in \( G' \) which are induced by \( v \)-stars, where

\[
m = |\{u : u \in V(G) \text{ and } d(u) = 1\}|.
\]

The case where \( P \) contains exactly \( m + 1 \) cliques induced by \( v \)-stars occurs only when the \( m + 1 \) cliques together with \{\( vx, vy \)\}, form an \( N-P \). Thus, in this case, \( \tau_{sd}(G^*) = 2 \).

Consider the case where \( P \) does not contain a \( K_3 \) induced by a wing. By using an argument similar to the one in Theorem 8 we also easily derive \( \tau_{sd}(G^*) = 2 \).

This completes the proof. \( \square \)
5 The types $\tau_{sa}(G^*)$ and $\tau_{sdu}(G^*)$

In this section, we assume that $S \in F_{sa}(G^*)$. The set $S(uv)$ denotes the set in the set representation which is assigned to $uv \in V(G^*)$ with respect to $S$. Recall that $S_i = \{S(v_1v_1^i), S(v_1v_2^i), \ldots, S(v_1v_m^i)\}$, where $v_i \in V_c$. Let $P = \text{EGP}(S)$ and let $\gamma' = |V_i| + \sum_{i=1}^k (m_i + 1)$.

**Lemma 11.** If $G$ is not a star and $|U(S)| \leq \gamma'$, then $|P| \leq |V_i|$.

**Proof.** Assume $|\text{EGP}(S_i)| = m_i \geq 3$ for some $i \in [k]$. By Theorem 5, $\text{EGP}(S_i)$ is an N-P or PP. An argument similar as in Theorem 8 gives a contradiction. Thus we only need to consider $m_i \in \{1, 2\}$.

If $m_i = 1$ for some $i \in [k]$, then $|\text{EGP}(S_i)| \geq 2$; otherwise we would have that, for any $z$ adjacent to $v_i$ with $d(z) \geq 2$, $S(v_1v_1^i) \subset S(v_1z)$, which is a contradiction.

Assume $m_i = 2$ for some $i \in [k]$. We can obtain that $|\text{EGP}(S_i)| \geq 3$ since $\text{EGP}(K_2) = 3$. As a consequence, $|\text{EGP}(S_i)| \geq m_i + 1$ for $i \in [k]$. By Proposition 2 on page 15 and the fact that $|U(S)| = |P| + |P_i|$, we obtain $|P_i| > |V_i|$. This implies that $|U(S)| > \gamma'$ which is a contradiction. This completes the proof. \[\Box\]

Assume $G \neq K_3$. In the following, we construct another surjection

$$f' : P \to V_2(G).$$

The construction goes by the same rules as in Section 4, except that the construction of $f_3$ is slightly modified.

If $P$ contains a $K_3$ induced by a wing $vx_0y$, say with stalk vertex $v$ in $G$, then $S(x_0y)$ must contain a monopolist. In this case, in constructing $f_3$, assign the trivial clique in $P$ which corresponds to the monopolist to one of $x$ and $y$, and the $K_3$ in $P$ induced by $vx_0y$ to the other of $x$ and $y$. Let $f_3'$ be the modified $f_3$. Based on the adjustment, we have the following lemma.

**Lemma 12.** If $G \neq K_3$, then, for all $v \in V_i$, $C_v \cap P_i \neq \emptyset$ and $|P_i| \geq |V_i|$.

**Proof.** For every $v \in V_i \cap R_P$, $C_v$ is assigned a clique from $P_i$ by $f_1$, since $f_1$ is surjective. Similarly, for every $v \in V_i \cap NW_P$, $C_v$ is assigned a clique from $P_i$ by $f_2$ and for every $v \in V_i \cap W$, $C_v$ is assigned a clique from $P_i$ by $f_3'$.

That proves the lemma. \[\Box\]

**Lemma 13.** If $G$ is neither a $K_3$ nor a star and if $|U(S)| \leq \gamma'$, then the following statements hold.

1. $|P_i| = |V_i|$.

---

6 A monopolist was defined in Definition 9 on page 9.
Lemma 4, at least two $u$-cliques in $P$.

**Proof.** Lemma 14 implies that $|EGP(S_i)| \geq m_i + 1$ for $i \in [k]$. By Proposition 1, $P$ contains at least $m_i + 1$ cliques induced by $v_i$-stars for $i \in [k]$. If either $|P_i| \neq |V|$ or $P$ contains more than $m_i + 1$ cliques induced by $v_i$-stars for some $i \in [k]$, then, by Lemma 12

$$|U(S)| = |P| = |P_1| + |P_2| > |V| + \sum_{i=1}^{k} (m_i + 1) = \gamma'.$$

This contradicts the assumption that $|U(S)| \leq \gamma'$. Thus $|P| = |V|$ and $P$ contains precisely $m_i + 1$ cliques induced by $v_i$-stars for $i \in [k]$. Consequently, for all $i \in [k]$, $|EGP(S_i)| = m_i + 1$. This proves the first two statements.

It remains to prove the third statement. By Lemma 14 for all $v \in V_i$, it follows that $C_v \cap P_i \neq \emptyset$ and $|P_i| \geq |V_i|$. Suppose that $P$ contains two cliques induced by $v$-stars for some $v \in V_i$. By the construction of $f'$, it follows that $C_v$ contains those two cliques unless $v$ is a stalk vertex in $W$. However, if $v$ is a stalk vertex in $W$, then $C_v$, for some $v' \in V_i$, contains two cliques in $P_i$ by the construction of $f'$. Thus, in any case, there exists a $v \in V_i$ with $|C_v \cap P_i| \geq 2$. This results in $|P| \geq |V|$ which contradicts the first statement. This completes the proof.

**Lemma 14.** If $G \notin \{K_3, K_4, a \text{ star}\}$ and $|U(S)| \leq \gamma'$, then $P_r = \emptyset$.

**Proof.** This lemma follows by replacing Corollary 3 and Lemma 7 by Lemmas 12 and 13 respectively, in the proof of Lemma 8.

**Lemma 15.** Assume that $G \notin \{K_3, W_1, a \text{ star}\}$ and assume that $|U(S)| \leq \gamma'$. If a semiwing in $G$ induces a $K_3$ in $P$, then at least one stalk vertex of this semiwing is in $V_c$.

**Proof.** Let $wuv$ with $d(w) = 2$ be a semiwing which induces a $K_3$ in $P$. By Lemma 14 at least two $u$-stars induce nontrivial cliques in $P$. By the third statement in Lemma 13 $u \in V_c$. This completes the proof.

**Lemma 16.** Assume that $G \notin \{K_3, a \text{ star}\}$ and assume that $|U(S)| \leq \gamma'$. If there is a $v$-wing in $G$ inducing a $K_3$ in $P$, then $v \in V_c$.

**Proof.** By Lemma 2 $P$ contains two cliques induced by $v$-stars. By the third statement in Lemma 13 $v \in V_c$. This completes the proof.
By Corollary 11 and Theorem 3 if there exists an EGP\( (K_{m_i}) \) such that \( 1 < |\text{EGP}| \leq m_i + 1 \) and there is no trivial clique in it, then it can only be an N-P, PP, 1-punctured PP, or 2-punctured Fano Plane. Thus, by the second statement in Lemma 13, for each \( i \in \{k\} \), EGP\( (S_i) \) can only be one of the following five edge-clique covers: N-P plus one trivial clique, PP plus one trivial clique, 1-punctured PP, 2-punctured Fano Plane, or \( K_{m_i} \) plus \( m_i \) distinct trivial cliques. However, both of PP plus one trivial clique and 2-punctured Fano Plane are impossible as we will show later. We distinguish the following types of edge-clique covers for EGP\( (S_i) \).

**Type I:** EGP\( (S_i) \) is an N-P plus one trivial clique.

In this case, the unique way to configure the set representation is as follows.

The clique \( K_{m_i-1} \) in N-P and the trivial clique \( K_1 \) are disjoint in EGP\( (S_i) \). Furthermore, every \( S(v_i v_i') \) for \( m_i + 1 \leq j \leq d(v_i) \) contains the two elements in \( U(S_i) \) corresponding to \( K_{m_i-1} \) and \( K_1 \). Notice that \( d(v_i) > m_i + 1 \) implies \( |S(v_i v_i') \cap S(v_i v_i'')| \geq 2 \). Thus this case occurs only when \( d(v_i) = m_i + 1 \).

**Type II:** EGP\( (S_i) \) is a 1-punctured PP.

Let \( r \) be the order of the PP and let \( x \) be the deleted point. Thus \( m_i = r^2 + r \). By Theorem 11, the PP has \( r + 1 \) lines that go through \( x \). Therefore, EGP\( (S_i) \) has exactly \( r + 1 \) pairwise disjoint cliques of cardinality \( r \). The total number of these vertices is \( r^2 + r = m_i \). Of course, every other pair of lines in EGP\( (S_i) \) intersect in one point. Thus \( S(v_i v_i') \) contains the \( r + 1 \) elements in \( U(S_i) \) corresponding to these \( r + 1 \) cliques. This case occurs when \( d(v_i) = m_i + 1 \); since otherwise \( |S(v_i v_i') \cap S(v_i v_i'')| \geq 2 \) which is a contradiction.

**Type III:** EGP\( (S_i) \) consists of \( K_{m_i} \) plus \( m_i \) trivial cliques.

In this case, \( S(v_i v_i') \) for each \( j \) with \( m_i + 1 \leq j \leq d(v_i) \) contains either

(a) the element in \( U(S_i) \) that corresponds to the \( K_{m_i} \), or,

(b) the elements in \( U(S_i) \) that correspond to the \( m_i \) trivial cliques.

**Type IV:** EGP\( (S_i) \) is a PP plus one trivial clique.

Since a PP of \( K_{m_i} \) does not have a clique \( K_{m_i-1} \), the configuration for Type I cover cannot be applied in this case. Thus this case is impossible.

**Type V:** EGP\( (S_i) \) is a 2-punctured Fano Plane.

In a 2-punctured Fano Plane (see the EGP-set in Figure 2 as an example), \( \{Q_3, Q_6\} \) and \( \{Q_4, Q_5\} \) are the only sets of pairwise disjoint cliques. However, if \( S(v_i v_i') \) contains the two elements in \( U(S_i) \) corresponding to \( Q_3 \) and \( Q_6 \), then it contains another element corresponding to \( Q_1 \) or \( Q_2 \). Consequently, \( |S(v_i v_i') \cap S(v_i v_i'')| \geq 2 \) for some \( 1 \leq j \leq m_i \), a contradiction. A similar reasoning applied to \( Q_4 \) and \( Q_5 \) also leads to a contradiction. Thus this case is also impossible.

**Lemma 17.** If \( G \) is neither \( K_3 \) nor a star and \( |U(S)| \leq \gamma' \), then the following two statements are true.
Now consider the case that \( EGP(d(v_i)) \) consists of a \( K_{m_i} \) and \( m_i \) distinct trivial cliques. Furthermore, each \( S(v_i^j) \cap U(S_i) \) for \( m_i + 1 \leq j \leq d(v_i) \), contains either the element corresponding to the \( K_{m_i} \) or the elements corresponding to the \( m_i \) trivial cliques.

**Proof.** By inspection, Types I and II covers occur only when \( d(v_i) = m_i + 1 \). Type III covers might have \( d(v_i) > m_i + 1 \). Thus we have to consider these three types of covers when proving the first statement and we only need to consider Type III when proving the second statement.

By the definition of a Type III cover, the set \( EGP(S_i) \) is a \( K_{m_i} \) plus \( m_i \) distinct trivial cliques. This implies that \( S(v_i^j) \cap U(S_i) \) for \( m_i + 1 \leq j \leq d(v_i) \) contains either the element corresponding to the \( K_{m_i} \) or the elements corresponding to the \( m_i \) trivial cliques. This proves the second statement.

Next, we prove the first statement. Consider the case that \( EGP(S_i) \) is a Type I cover. The trivial clique in \( EGP(S_i) \) contains the intersection of all the \( K_2 \)'s in the N-P. Furthermore, \( S(v_i^{m_i+1}) \cap U(S_i) \) contains the two elements corresponding to the \( K_{m_i-1} \) and \( K_1 \) in \( EGP(S_i) \). Therefore, \( v_iv_i^{m_i+1} \) together with the vertices in the \( K_{m_i-1} \) induce a clique in \( \mathcal{P} \). Vertex \( v_iv_i^{m_i+1} \) together with the vertex in \( K_1 \) induce another clique in \( \mathcal{P} \). As a consequence, the cliques in \( \mathcal{P} \) induced by \( v_i \)-stars constitute an N-P of \( K_{d(v_i)} \).

Now consider the case that \( EGP(S_i) \) is a Type II cover. That is, \( EGP(S_i) \) is a PP with a vertex, say \( x \), deleted. Furthermore, \( S(v_i^{m_i+1}) \cap U(S_i) \) contains the cliques in \( EGP(S_i) \) that, originally, contained the vertex \( x \). Therefore, \( v_iv_i^{m_i+1} \) together with the vertices in each of those cliques induce a clique in \( \mathcal{P} \). Accordingly, the cliques in \( \mathcal{P} \) induced by \( v_i \)-stars constitute a PP of \( K_{d(v_i)} \).

Finally, we consider the case that \( EGP(S_i) \) is a Type III cover. In this case, if \( d(v_i) = m_i + 1 \) with \( m_i \geq 2 \) and \( S(v_i^{m_i+1}) \) contains the elements corresponding to the \( m_i \) trivial cliques, then clique \( \{v_iv_i^{m_i+1}, v_i^{m_i+1}\} \in \mathcal{P} \) for \( 1 \leq j \leq m_i \). This means that the cliques in \( \mathcal{P} \) induced by \( v_i \)-stars constitute an N-P of \( K_{d(v_i)} \).

If \( d(v_i) = m_i + 1 \) and \( S(v_i^{m_i+1}) \) contains the element corresponding to the \( K_{m_i} \), then \( S_{v_i}^{d(v_i)} \) induces a clique in \( \mathcal{P} \). In this way, the cliques in \( \mathcal{P} \) induced by \( v_i \)-stars are a \( K_{d(v_i)} \) and \( m_i \) trivial cliques. This completes the proof. \( \square \)

**Theorem 10.** For tailed peacocks \( G \),

\[
\tau_{sa}(G^*) = \begin{cases} 
5 & \text{if } G \text{ is TP}_2 \text{ with } m_1, m_2 \geq 2 \text{ and } m_1 \neq m_2, \\
4 & \text{if } G \text{ is TP}_2 \text{ with } m_1 = m_2 \geq 2, \\
3 & \text{if } G \text{ is TP}_2 \text{ with } m_1 \geq 2 \text{ and } m_2 = 1, \\
2 & \text{otherwise.}
\end{cases}
\]
Proof. Let $G$ be a TP. We count $\tau_{sa}(G^*)$ by analyzing all possible set representations $S$ of $G^*$ with $|\cup(S)| \leq \gamma'$. We consider two cases.

Case 1. $G$ is a TP$_1$ or a TP$_2$.

If $G$ is a TP$_1$ or a TP$_2$, then $G$ has a stalk vertex $v_1$ or two stalk vertices $v_1$ and $v_2$. By Lemma [17] for each stalk vertex $v_i$ with $i \in [k]$, the set $\text{EGP}(S_i)$ consists of a $K_{m_i}$ and $m_i$ distinct trivial cliques, and $S(v_i v_i^{m_i+1})$ and $S(v_i v_i^{m_i+2})$ contain either the $K_{m_i}$ or the $m_i$ trivial cliques. Note that, in this case, $k \in \{1, 2\}$.

When $m_i \geq 2$, the sets $S(v_i v_i^{m_i+1})$ and $S(v_i v_i^{m_i+2})$ cannot contain the $m_i$ trivial cliques at the same time. If $k = 1$, then $\tau_{sa}(G^*) = 2$, that is, one for $S(v_1 v_1^{m_1+1}) = S(v_1 v_1^{m_1+2})$ and the other for $S(v_1 v_1^{m_1+1}) \neq S(v_1 v_1^{m_1+2})$.

For the case where $k = 2$, since the triangle in $G$ induces a $K_3$ in $P$, this results in $S(v_1 v_1^{m_1+1}) \neq S(v_1 v_1^{m_1+2})$ which further implies $S(v_2 v_2^{m_2+1}) \neq S(v_2 v_2^{m_2+2})$ and vice versa. Thus if $k = 2$ and $m_1, m_2 \geq 2$, then $\tau_{sa}(G^*) = 5$, unless $m_1 = m_2$. If $k = 2$ and $m_1 = m_2 \geq 2$, then there are two isomorphic $S$'s. Here, for $i \in \{1, 2\}$,

$$S(v_i v_i^{m_i+1}) \neq S(v_i v_i^{m_i+2})$$

and $S(v_1 v_2) \cap \cup(S_i)$ contains the $K_{m_i}$ for one of $i \in \{1, 2\}$ and the $m_i$ trivial cliques for the other $i \in \{1, 2\}$. In this case, $\tau_{sa}(G^*) = 4$.

If $m_2 = 1$ and $m_1 \geq 2$ (or $m_1 = 1$ and $m_2 \geq 2$), then there are two possibilities for $S(v_1 v_1^{m_1+1}) \cap \cup(S_i)$ for $j \in \{1, 2\}$. We obtain $\tau_{sa}(G^*) = 3$. If $m_1 = m_2 = 1$, then it is easy to see that $\tau_{sa}(G^*) = 2$.

Case 2. $G$ is a TP$_{d_1}$ or a TP$_{d_2}$.

In this case, for each $v_i \in V_c$, all the vertices $v_i v_i^j$, with $m_i + 1 \leq j \leq d(v_i)$ and with $d(v_i^j) = 2$, are nonadjacent. Therefore, all $S(v_i v_i^j) \cap \cup(S_i)$ for $m_i + 1 \leq j \leq d(v_i)$ with $d(v_i^j) = 2$ contain the same element. If the set $S(v_i v_i^j)$, with $d(v_i^j) \geq 3$, contains the same element as above, this yields $\tau_{sa}(G^*) = 2$. If the set $S(v_i v_i^j)$ with $d(v_i^j) \geq 3$ does not contain the same element, then all triangles in $G$ induce $K_3$'s in $P$ and, therefore, $S(v_i^j x)$, for all $x$ adjacent to $v_i$ with $d(x) = 2$, contains a common element which is not in $S(v_i v_i^j)$. This completes the proof. □

Lemma 18. If $G \notin \{K_3, K_4, W_t, \text{ a star, TP}\}$ and $|\cup(S)| \leq \gamma'$, then, for each $v_i \in V_c$ with $d(v_i) > m_i + 1$,

$$\big| \bigcap_{m_i + 1 \leq j \leq d(v_i)} S(v_i v_i^j) \cap \cup(S_i) \big| = 1,$$

and the common element is the element corresponding to the clique $K_{m_i}$ in $\text{EGP}(S_i)$ when $m_i \geq 2$. 26
A similar argument as in the previous case leads to the conclusion that Case 2.

By the second statement in Lemma 13, either $y$ is, either $S$ induced by the triangle $v_i v_i^{m+1}$ and therefore $v_i^1$ is adjacent to $v_i^{m+1}$. Thus, by Lemma 14 all $v_i^j$, for $m_i + 2 \leq j \leq d(v_i)$, are of degree two if $d(v_i) > m_i + 2$.

Thus, for $m_i + 2 \leq j \leq d(v_i)$, every $S(v_i v_i^1)$ contains the clique $K_{m_i}$ in $EGP(S_i)$, for otherwise $|S(v_i v_i^1) \cap S(v_i v_i^{m+1})| > 1$. By the second statement in Lemma 14 for $m_i + 2 \leq j \leq d(v_i)$, the element in $S(v_i v_i^1) \cap S(v_i v_i^{m+1})$ corresponds to a $K_3$ in $P$ induced by the triangle $v_i v_i^{m+1}$ and therefore $v_i^1$ is adjacent to $v_i^{m+1}$. Thus, the edges $v_i v_i^{m+1}$ and $v_i v_i^{m+1}$ induce a triangle in $P$. Therefore, $v_i^{m+1} \in V_c$ by the third statement in Lemma 13.

Thus, the edges $(v_i^{m+1}, v_i^{m+1}, v_i)$ and $(v_i^{m+1}, v_i^{m+1}, v_i^{m+2})$ in $G^*$ are covered by cliques in $P$ that are induced by distinct $v_i^{m+1}$-stars, since $y$ is not adjacent to $v_i^{m+2}$ and $v_i v_i^{m+1} v_i^{m+2}$ induces a triangle in $P$. Therefore, $v_i^{m+1} \in V_c$ by the third statement in Lemma 13.

For brevity, let $v_z$ denote the critical vertex $v_i^{m+1}$ for some $z \in [k]$. By Lemma 17 the set $EGP(S_z)$ consists of a $K_{m_z}$ and $m_z$ trivial cliques. Due to the $K_3$ in $P$ induced by $v_z v_z v_z^s$, we can obtain that $S(v_z v_z v_z^s) = S(v_z v_z v_z) \cap S(v_z v_z)$. That is, either $S(v_z y) \cap S(v_z z) = S(v_z y) \cap S(v_z z)$ or $S(v_z y) \cap S(v_z z) = S(v_z v_z v_z^s) \cap S(v_z v_z v_z^s)$. By the second statement in Lemma 13 either $y$ is adjacent to $v_z^s$ or $y$ is adjacent to $v_z$. The first option contradicts $d(v_z^s) = 2$ and the second contradicts that $y$ is not adjacent to $v_z$. The conclusion is that all neighbors of $v_i^{m+1}$ are plumes, except $v_i$ and $v_i^1$, for $m_i + 2 \leq x \leq d(v_i)$. Thus $G$ is a TP$_{d2}$, a contradiction.

**Case 2.** $d(v_i) = m_i + 2$ and $d(v_i^{m+2}) = 2$.

A similar argument as in the previous case leads to the conclusion that $G$ is a TP$_2$; again a contradiction.

**Case 3.** $d(v_i) = m_i + 2$ and $d(v_i^{m+1}) = 2$.

We prove that $v_i^{m+2}$ has no neighbor with degree at least 2, except $v_i$ and $v_i^{m+1}$.

A plume was defined in Definition 11.
By Lemmas 14-16, 

\[ P \]

This completes the proof.

Finally, consider the case where \( m_1 = 1 \). Suppose that \( S(\{v_i v_i^{m_i+1}\}) \) and \( S(\{v_i v_i^{m_i+2}\}) \) contain the two elements in \( U(S_i) \), where \( v_i \in V_c \) with \( d(v_i) > m_i + 1 \). By a similar argument as above, we obtain that

i. \( v_i^{m_i+1} \) is adjacent to \( v_i^{m_i+2} \),
ii. the triangle \( v_i v_i^{m_i+1} v_i^{m_i+2} \) induces a \( K_3 \) in \( \mathcal{P} \), and
iii. either \( v_i^{m_i+1} \) or \( v_i^{m_i+2} \) has degree two.

Without loss of generality, assume that \( d(v_i^{m_i+1}) = 2 \). By using a similar argument as above, we obtain that every neighbor \( y \neq v_i \) of \( v_i^{m_i+2} \), with \( d(y) \geq 2 \) is adjacent to \( v_i \). If there exists such a vertex \( y \), which is not \( v_i^{m_i+1} \), then either \( S(v_i y) \cup U(S_i) \neq S(\{v_i v_i^{m_i+1}\}) \cup U(S_i) \) or \( S(v_i y) \cup U(S_i) \neq S(\{v_i v_i^{m_i+2}\}) \cup U(S_i) \) since \( v_i v_i^{m_i+1} v_i^{m_i+2} \) induces a \( K_3 \) in \( \mathcal{P} \).

In the first case, \( y \) is adjacent to \( v_i^{m_i+1} \), which contradicts \( d(v_i^{m_i+1}) = 2 \). In the second case, \( v_i y v_i^{m_i+2} \) induces a triangle in \( \mathcal{P} \). Thus \( d(y) = 2 \). Moreover, by a similar argument we obtain that all neighbors \( x \) of \( v_i \), with \( d(x) \geq 2 \) and \( x \neq v_i^{m_i+2} \), are adjacent to \( v_i^{m_i+2} \). This yields \( d(x) = 2 \). Thus \( G \) is a TP, a contradiction. This completes the proof.

\[ \square \]

**Theorem 11.** If \( G \notin \{K_3, K_4, W_4, a\ star, TP\} \), then \( \theta_{sa}(G^*) = |V_i| + \sum_{i=1}^{k} (m_i + 1) \).

**Proof.** First, we show that there exists an \( S \in \mathcal{F}_{sa}(G^*) \) with \( |U(S)| \leq \gamma' \). By Lemma 13 the set \( \mathcal{P} \) contains the clique induced by the saturated \( v_i \)-star for each \( i \in [k] \) with \( d(v_i) > m_i + 1 \). Therefore, \( \mathcal{P} \) contains no clique induced by a \( K_3 \) in \( G \) that contains some \( v_i \in V_c \).

By Lemmas 14-16, \( \mathcal{P} \) contains no clique induced by a \( K_3 \) in \( G \) and therefore, by the third statement in Lemma 13, \( \mathcal{P} \) contains the clique induced by the saturated \( v \)-star for every \( v \in V_i \). By Lemma 17, there exists an \( S \) with \( |U(S)| \leq \gamma' \).

On the other hand, by using a similar argument as in Lemma 8 an \( S \) with its corresponding \( \mathcal{P} \) as described above is in \( \mathcal{F}_{sa}(G^*) \). Hence

\[ \theta_{sa}(G^*) = \gamma' = |V_i| + \sum_{i=1}^{k} (m_i + 1) \]

This completes the proof. 

\[ \square \]
Theorem 12. For a graph $G$ which is not a TP,

$$\tau_{sa}(G^*) = \begin{cases} 
1 & \text{if } G = K_2, \\
2 & \text{if } G = K_4 \text{ or } W_4, \\
1 + N_{pp}(d(v), r) & \text{if } G \text{ is a } v\text{-star with } d(v) \geq 3, \\
2^x y^z & \text{otherwise}, 
\end{cases}$$

where $x = |\{v_i : m_i = 2 \text{ and } d(v_i) = 3\}|$, $y = 3 + N_{pp}(m_i + 1, r)$ and $z = |\{v_i : m_i \geq 3 \text{ and } d(v_i) = m_i + 1\}|$.

Proof. Clearly, $\tau_{sa}(G^*) = 3$ when $G = K_3$ and $\tau_{sa}(G^*) = \tau_{sd}(G^*) = 2$ when $G \in \{K_4, W_4\}$. By Theorem 13, $\tau_{sa}(G^*) = 1 + N_{pp}(d(v), r)$ when $G$ is a $v$-star with $d(v) \geq 3$. It remains to prove that

$$\tau_{sa}(G^*) = 2^x y^z$$

for the remaining cases, where $x = |\{v_i : m_i = 2 \text{ and } d(v_i) = 3\}|$, $y = 3 + N_{pp}(m_i + 1, r)$ and $z = |\{v_i : m_i \geq 3 \text{ and } d(v_i) = m_i + 1\}|$. Recall that for each $i \in [k]$ with $d(v_i) = m_i$, the cliques in $\mathcal{P}$ induced by $v_i$-stars are either

i. a $K_{d(v_i)}$ with $m_i$ trivial cliques $\{v_i v_i^1\}, \ldots, \{v_i v_i^{m_i}\}$, or
ii. an N-P or a PP of $K_{d(v_i)}$.

Furthermore, if $d(v_i) = m_i + 1$ and the cliques in $\mathcal{P}$ induced by $v_i$-stars constitute an N-P, then there are two non-isomorphic $S$ which are determined according to the intersection of the $K_2$’s in the N-P is $v_i v_i^{m_i+1}$ or not unless $m_i = 2$. This proves the last equality in the theorem. \qed

Theorem 13. For a graph $G$,

$$\tau_{sdu}(G^*) = \begin{cases} 
2 & \text{if } G \in \{K_4, W_2, TP_1\} \text{ with } m_1 = 1, \\
N_{pp}(d(v), r) & \text{if } G \text{ is } H \text{ with } \theta_{sdu}(K_{d(v)}) = d(v), \\
1 + N_{pp}(d(v) + 1, r) & \text{if } G \text{ is } H \text{ with } \theta_{sdu}(K_{d(v)}) = d(v) + 1, \\
1 & \text{otherwise}, 
\end{cases}$$

where $H$ is a $v$-star with $d(v) \geq 4$

Proof. If $G$ is neither $K_3$ nor a star, then $G^*$ is not a complete graph. Therefore, any $S \in F_s(G^*)$ has $|S(e)| \geq 2$ for some $e \in E(G)$. This further implies that any $S \in F_{sv}(G^*)$ has $|S(e)| \geq 2$ for all $e \in E(G)$.

Any pair of edges $e$ and $e'$ with $e \neq e'$ have $S(e) \nsubseteq S(e')$ since $|S(e) \cap S(e')| \leq 1$. Thus those members, if any, of $F_{sa}(G^*)$ in which all sets have the same cardinality, constitute $F_{sv}(G^*)$ and $F_{sdu}(G^*)$. For the case where $G$ is a $v$-star, we can obtain $\tau_{sdu}(G^*)$ directly from Theorem 6. This completes the proof. \qed
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