Palatini formulation of the $R^{-1}$-modified gravity with an additional squared scalar curvature term

Xinhe Meng$^{1,2,3}$ and Peng Wang$^2$

$^1$ CCAST (World Laboratory), PO Box 8730, Beijing 100080, People’s Republic of China
$^2$ Department of Physics, Nankai University, Tianjin 300071, People’s Republic of China
$^3$ Institute of Theoretical Physics, CAS, Beijing 100080, People’s Republic of China

E-mail: xhmeng@phys.nankai.edu.cn and pewang@eyou.com

Received 22 March 2004, in final form 11 October 2004
Published 3 December 2004
Online at stacks.iop.org/CQG/22/23

Abstract

In this paper by deriving the modified Friedmann equation in the Palatini formulation of $R^2$ gravity, first we discuss the problem of whether in Palatini formulation an additional $R^2$ term in Einstein’s general relativity action can drive an inflation. We show that the Palatini formulation of $R^2$ gravity cannot lead to gravity-driven inflation as in the metric formalism. If considering no zero radiation and matter energy densities, we obtain that only under rather restrictive assumptions about the radiation and matter energy densities will there be a mild power-law inflation $a(t) \sim t^2$, which is obviously different from the original vacuum energy-like driven inflation. Then we demonstrate that in the Palatini formulation of a more generally modified gravity, i.e., the $1/R + R^2$ model that intends to explain both the current cosmic acceleration and early time inflation, accelerating cosmic expansion is achieved at late universe evolution times under model parameters satisfying $\alpha \ll \beta$.

PACS numbers: 98.80.Bp, 98.65.Dx, 98.80.Es

1. Introduction

Although the fact that the expansion of our universe is currently in an accelerating phase now seems well established [1], so far the mechanism responsible for this is not yet very clear. Many authors introduce a mysterious cosmic fluid called dark energy in the general relativity (GR) and Roberson–Walker metric framework to explain this. See [2] for a review and [3] for some recent models.

On the other hand, some authors have suggested that maybe such a mysterious dark energy does not exist, but the observed cosmic acceleration is a signal of our first real lack of understanding of gravitational physics [4, 6]. An example is the braneworld theory of Dvali et al [5]. Recently, some authors proposed to add a $1/R$ term in the Einstein–Hilbert
action to modify the general relativity (GR) [6, 7]. It is interesting that such a term may be predicted by string/M-theory [8]. In the metric formulation, this additional term will give fourth-order field equations. It was shown in their works that this additional term can give accelerating expansion solutions for the field equations without dark energy. In this framework, Dick [9] considered the problem of weak field approximation, and Soussa and Woodard [10] considered the gravitational response to a diffuse source.

Based on this modified action, Vollick [11] has used the Palatini variational principle to derive the field equations. In the Palatini formulation, instead of varying the action only with respect to the metric, one views the metric and connection as independent field variables and varies the action with respect to them independently. This would give second-order field equations. For the original Einstein–Hilbert action, this approach gives the same field equations as the metric variation. For a more general action, those two formulations are inequivalent, they will lead to different field equations and thus describe different physics [12]. Flanagan [13] derived the equivalent scalar–tensor description of the Palatini formulation. In [14], Dolgov and Kawasaki argued that the fourth-order field equations in metric formulation suffer a serious instability problem. Indeed if this is the case, the Palatini formulation appears even more appealing, because the second-order field equations in Palatini formulation are free of this sort of instability [15]. Furthermore, Chiba [16] argued that the theory derived using metric variation is in conflict with the solar system experiments. However, the most convincing motivation to take the Palatini formalism seriously is that the modified Friedmann (MF) equation following from it fits the SN Ia data at an acceptable level [15].

At the other end of cosmic time, the very early stage, it is now generally believed that the universe also undergoes an acceleration phase called inflation. The mechanism driving inflation is also presently unclear. The most popular explanation is that inflation is driven by some inflaton field [17]. Also, some authors suggest that modified gravity could be responsible for inflation [18, 19]. Revealing the mechanisms for current acceleration and early inflation are two of the most important objects of modern cosmology.

As originally proposed by Carroll et al [6] and later implemented by Nojiri and Odintsov [19], adding a correction term $R^m$ with $m > 0$ in addition to the $1/R$ term may explain both the early time inflation and current acceleration without introducing inflaton and dark energy. Furthermore, Nojiri and Odintsov [19] showed that adding a $R^m$ term can avoid the above-mentioned instability when considering the theory in metric formulation. In this paper, we will show that in the Palatini formulation, the $R^2$ term contribution is not the same as the conclusion reached when considering the theory in metric formulation [18]. And the $1/R + R^2$ model will have some theoretical inconsistencies as well as conflict with particle experiments that might invalidate this model.

Besides, there are many activities in the study of quantum versions of $R^2$ gravity which seems to be a multiplicatively renormalizable theory (for a review, see [22]). However, such a theory has had a serious problem: possible non-unitarity due to the presence of higher derivative terms. It is very promising that in the Palatini formalism higher derivative terms do not play such a role as in metric formalism, such that the unitarity problem of $R^2$ gravity may be resolved in the Palatini formalism. Also, it is interesting to explore the $R^2$ correction to the chaotic inflation scenario [25] in the Palatini formulation. When written in the Einstein frame, in metric formulation, this will correspond to two-scalar field inflation; in the Palatini formulation, the model will correspond to a type of $k$-inflation [26]. More detailed investigations of this idea can be found in our recently published work [30].

This paper is arranged as follows: in section 2 we review the framework of deriving field equations and modified Friedmann (MF) equations in the Palatini formulation; in section 3 we discuss $R^2$ gravity in the Palatini formulation and show the cosmology implications in
section 4 we discuss the combined effects of both a $1/R$ term and an $R^2$ term and section 5 is devoted to conclusions and discussions.

2. Deriving the modified Friedmann equation in the Palatini formulation

Firstly, we briefly review deriving field equations from a generalized Einstein–Hilbert action by using the Palatini variational principle. See [11, 12, 15] for details.

The field equations follow from the variation in the Palatini approach of the generalized Einstein–Hilbert action

$$ S = - \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} L(R) + S_M $$

where $\kappa^2 = 8\pi G, L$ is a function of the scalar curvature $R$ and $S_M$ is the matter action.

Varying with respect to $g_{\mu\nu}$ gives

$$ L'(R) R_{\mu\nu} - \frac{1}{2} L(R) g_{\mu\nu} = \kappa^2 T_{\mu\nu} \tag{2} $$

where a prime denotes differentiation with respect to $R$ and $T_{\mu\nu}$ is the energy–momentum tensor given by

$$ T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}} \tag{3} $$

We assume that the universe contains dust and radiation, thus $T_{\mu}^{\nu} = \{-\rho_m - \rho_r, p_r, p_r, p_r\}$ where $\rho_m$ and $\rho_r$ are the energy densities for dust and radiation, respectively, $p_r$ is the pressure of the radiation. Note that $T = g^{\mu\nu} T_{\mu\nu} = -\rho_m$ because of the relation $p_r = \rho_r/3$.

In the Palatini formulation, the connection is not associated with $g_{\mu\nu}$, but with $h_{\mu\nu} \equiv L'(R) g_{\mu\nu}$, which is known from varying the action with respect to $\Gamma^\lambda_{\mu\nu}$. Thus, the Christoffel symbol with respect to $h_{\mu\nu}$ is given by

$$ \Gamma^\lambda_{\mu\nu} = \left\{ \lambda_{\mu\nu} \right\} - \frac{1}{2L'} \left[ 2\delta^\lambda_{(\mu} \partial_{\nu)} L' - g_{\mu\nu} g^{\sigma\sigma} \partial_\sigma L' \right] \tag{4} $$

where the subscript $g$ signifies that this is the Christoffel symbol with respect to the metric $g_{\mu\nu}$.

The Ricci curvature tensor is given by

$$ R_{\mu\nu} = R_{\mu\nu}(g) + \frac{1}{2} (L')^{-2} \nabla_\mu L' \nabla_\nu L' - (L')^{-1} \nabla_\mu \nabla_\nu L' - \frac{1}{2} (L')^{-1} g_{\mu\nu} \nabla^\sigma \nabla_\sigma L' \tag{5} $$

and

$$ R = R(g) - 3 (L')^{-1} \nabla_\mu \nabla^{\mu} L' + \frac{1}{2} (L')^{-2} \nabla_\mu L' \nabla^{\mu} L' \tag{6} $$

where $R_{\mu\nu}(g)$ is the Ricci tensor with respect to $g_{\mu\nu}$ and $R = g^{\mu\nu} R_{\mu\nu}$. Note by contracting (2), we get

$$ L'(R) R - 2 L(R) = \kappa^2 T \tag{7} $$

Assume that we can solve $R$ as a function of $T$ from (7). Thus, (5), (6) define the Ricci tensor with respect to $h_{\mu\nu}$.

Then we review the general framework for deriving the modified Friedmann equation in the Palatini formalism [15]. Let us work with the Robertson–Walker metric describing the cosmological evolution,

$$ ds^2 = -dt^2 + a(t)^2 (dx^2 + dy^2 + dz^2). \tag{8} $$

Note that we only consider a flat metric, which is favoured by present observations [1].
From (8), (5) we can get the non-vanishing components of the Ricci tensor:

\[
R_{00} = -\frac{3}{a} \ddot{a} + \frac{3}{2} (L')^{-2} (\partial_0 L')^2 - \frac{3}{2} (L')^{-1} \nabla_0 \nabla_0 L' \tag{9}
\]

\[
R_{ij} = \left[ a \ddot{a} + 2 \dot{a}^2 + (L')^{-1} \begin{pmatrix} 0 \\ i \\ j \\ g \end{pmatrix} \partial_0 L' + \frac{a^2}{2} (L')^{-1} \nabla_0 \nabla_0 L' \right] \delta_{ij}. \tag{10}
\]

Substituting equations (9) and (10) into the field equations (2), we can get

\[
6H^2 + 3H (L')^{-1} \partial_0 L' + \frac{3}{2} (L')^{-2} (\partial_0 L')^2 = \frac{\kappa^2 (\rho + 3p)}{L'} + \frac{L}{L'} \tag{11}
\]

where \( H \equiv \dot{a}/a \) is the Hubble parameter, \( \rho \) and \( p \) are the total energy density and total pressure, respectively. Assume that we can solve \( R \) in terms of \( T \) from equation (7), substitute it into the expressions for \( L' \) and \( \partial_0 L' \), we can get the MF equation.

In this paper, we will consider the Palatini formulation of the following model suggested by Carroll et al [6] and implemented in the metric formulation by Nojiri and Odintsov [19]:

\[
L = R - \frac{\alpha^2}{3R} + \frac{R^2}{3\beta} \tag{12}
\]

where \( \alpha \) and \( \beta \) are parameters both with dimensions \((\text{eV})^2\).

Since in the early universe, the \( R^2 \) term dominates, in order to find how this term functions, we first consider the Palatini formulation of the modified action only with an \( R^2 \) term, that is

\[
L = R + \frac{R^2}{3\beta}. \tag{13}
\]

This action has been studied by Starobinsky in metric formulation [18] and it has been shown that a gravity-driven inflation can be achieved.

### 3. Palatini formulation of \( R^2 \) gravity

The field equations follow by substituting equation (13) into equation (2)

\[
\left( 1 + \frac{2R}{3\beta} \right) R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \left( R + \frac{R^2}{3\beta} \right) = \kappa^2 T_{\mu\nu}. \tag{14}
\]

Contracting indices gives

\[
R = -\kappa^2 T = \kappa^2 \rho_m. \tag{15}
\]

The second equality follows because the radiation has vanishing trace of momentum-energy tensor. This equation is quite remarkable, since it is formally the same as the one given by GR, with only one difference: \( R_{\mu\nu} \) is associated with the conformal transformed metric \( h_{\mu\nu} = L' (R) g_{\mu\nu} \) and \( R = g^{\mu\nu} R_{\mu\nu} \).

From the conservation equation \( \dot{\rho}_m + 3H \rho_m = 0 \) and equation (15), we can find that

\[
\partial_0 L' = -2 \frac{\kappa^2 \rho_m}{\beta} H. \tag{16}
\]

Substituting this into equation (11) we can get the modified Friedmann equation for the \( R^2 \) gravity:

\[
H^2 = \frac{2\kappa^2 (\rho_m + \rho_r) + (\kappa^2 \rho_m)^2}{\left( 1 + \frac{2\kappa^2 \rho_m}{3\beta} \right) \left[ 6 + 3 F_0 \left( \frac{\kappa^2 \rho_m}{\beta} \right) \left( 1 + \frac{1}{2} F_0 \left( \frac{\kappa^2 \rho_m}{\beta} \right) \right) \right]} \tag{17}
\]
where the function $F_0$ is given by
\[ F_0(x) = -\frac{2x}{1 + \frac{3}{2}x}. \] (18)

It is interesting to see from equation (17) that all the effects of the $R^2$ term are determined by $\rho_m$. If $\rho_m = 0$, equation (17) simply reduces to the standard Friedmann equation.

Now let us come to the discussion of inflation. To begin with, note that in the metric formulation of the $R^2$ gravity, inflation is driven by the vacuum gravitational field, i.e., we assume that the radiation and matter energy densities are zero during inflation, thus called ‘gravity-driven’ inflation. However, in the Palatini formulation, when the radiation and matter energy densities are zero, it can be seen directly from equation (17) that the expansion rate will be zero and thus no inflation will occur. Thus, in the Palatini formulation of $R^2$ gravity, we cannot have gravity-driven inflation. So the only hope that the $R^2$ term can drive an inflation without an inflaton field is that the relationship between the expansion rate and the energy density of radiation and matter will be changed which can lead to inflation (thus what we are talking of now is similar to the ‘Cardassian’ scenario of Freese and Lewis [27]: the current accelerated expansion of the universe is driven by the changed relationship between the expansion rate and matter energy density). We will see that naturally there will be no inflation and power-law inflation can occur only under a specific assumption on $\rho_m$ and $\rho_r$.

First, in a typical model of $R^2$ inflation, $\beta$ is often taken to be the order of the Planck scale [18]. This is also the most natural value of $\beta$ from an effective field point of view. Thus, we naturally have $\kappa^2 \rho_m / \beta \ll 1$. Under this condition, it can be seen that from equation (18), the MF equation (17) reduces to the standard Friedmann equation:
\[ H^2 = \frac{\kappa^2}{3} (\rho_m + \rho_r). \] (19)

Thus, it is obvious that in this case there will be no inflation. Also note that from the BBN constraints on the Friedmann equation [20], $\beta$ should be sufficiently large so that the condition $\kappa^2 \rho_m / \beta \ll 1$ is satisfied at least in the era of BBN. Thus we conclude that in the most natural case, the Palatini formulation of $R^2$ gravity cannot lead to inflation.

Second, let us assume that in the very early universe, we have $\kappa^2 \rho_m / \beta \gg 1$, which is the interesting possibility of deviating GR. In this case, from equation (18), the MF equation (17) will reduce to
\[ H^2 = \frac{\kappa^2 \rho_m}{21} + \frac{2\beta \rho_r}{7 \rho_m} + \frac{2\beta}{7}. \] (20)

Then we can see that if the $\beta$ term could dominate over the other two terms, it would drive an exponential expansion by the effective cosmological constant $\beta$. But note that this equation is derived under the assumption that $\beta \ll \kappa^2 \rho_m$. Thus, inflation cannot be driven by the $\beta$ term. On the other hand, if we further assume that $\rho_r \gg \kappa^2 \rho_m / \beta$, i.e., the second term dominates in the MF equation (20) over some time interval if $\rho_{r0} \gg \kappa^2 \rho_{m0} / \beta$, and then if the matter and radiation evolve independently so, from the relation $\rho_r \propto a^{-4}$ and $\rho_m \propto a^{-3}$, the MF equation (20) can be solved to give $a(t) \propto t^2$, neglecting numerical factors. Thus, only in this case can we get a mild power-law inflation that quite differs from the original exponent inflation with enough e-folding for solving the hot big-bang cosmology puzzles: lack of defects, flatness, horizon and homogeneous problems. This kind of mild inflation will occur at a time smaller than the time scale associated with $\beta^{-1/2}$ (in this MF equation for a possible inflation there is a particular time scale associated with the parameter $\beta$ being $\beta^{-1/2}$ by dimension analysis), which may be unrealistic if the time scale is of the order of the Planck time. However, current constraint with cosmic background radiation anisotropies (power
spectrum) analysis on the rate of power-law inflation reads \( p > 21 \) where \( a \propto t^p \) (see, e.g., [29]). So this case is not a viable model of inflation.

Besides, we can see it in another aspect with the large enough e-folding number \( N \) required for solving the original cosmology problems.

\[
N = \ln \left[ \frac{a_f}{a_i} \right] = 2 \ln (t_f/t_i).
\]  

(21)

If we assume the e-folding number \( N > 60 \) then we have \( t_f > e^{30} t_i \), i.e., this kind of power-law inflation lasts not less than \( 10^{13} \) times the initial time which is against the primordial inflation required. As additional comments we should mention that there is some low energy-scale inflation possibly taking place, especially if one considers alternative ways to generate density perturbations, such as the curvaton mechanism or a modulated inflaton decay constant, but these scenarios are not relevant to what we focus on above.

At late cosmological times when \( \kappa^2 \rho_m/\beta \ll 1 \), \( F_0 \sim 0 \), the MF equation (17) reduces to the standard Friedmann equation, which implies that early universe dynamics is dominated by a larger curvature term and enlightens us to describe late times cosmologies by possibly including the sort of \( R^n \) term with \( n \) as negative integer when the small curvature term dominates. Moreover, as a reduction, this type of inflation also occurs in a more complicated model with an additional \( R^{-1} \) term, when taking this term’s coupling constant going to zero directly, as we will discuss in the section following.

In summary, in the Palatini formulation, the modified gravity theory with a \( R^2 \) correction term would not lead to an early time gravity-driven inflation, contrary to the famous conclusion when considering the theory in the metric formulation. The difference of those two formulations is now quite obvious. Now, we still cannot tell which one is physical. But this makes these results more interesting. It is conceivable that the quantum effects of the \( R^2 \) theory in the Palatini formulation would also be different from the metric formulation (see [22] for a review). Such higher derivative terms similar to the \( R^2 \) term may be induced by the quantum effects, e.g., trace anomaly [22, 23]. It has recently been shown [23] that phantom cosmology implemented by trace anomaly induced terms also admits both early time inflation and late time cosmic acceleration. It follows from our consideration that the \( R^2 \) term in the Palatini formulation does not support inflation, then we expect that also in phantom cosmology with quantum effects considered in the Palatini formulation, inflation does not occur either.

4. Palatini formulation of a \( 1/R + R^2 \) gravity

In this section we will consider mainly the cosmological consequences of an \( R^{-1} + R^2 \) gravity theory, when this is analysed in the Palatini formulation. The \( R^{-1} \) phenomenological theory has gained some interest since it seems to be able to account for cosmological observations in Supernovae Ia, as an alternative model to dark energy; this topic nowadays is a hot subject under discussion [28]. Now the additional \( R^{-1} \) term is coupled to an \( R^2 \) term to investigate possibly more interesting cosmological features, such as if the above-discussed kind of mild power law with power 2 (early universe) inflation could also occur. By qualitative reasoning, the large curvature terms dominate cosmic global evolution at early times where the inverse curvature terms effect can be neglected for the early universe evolution stage, which is also reflected in inflation theory. In this case the discussions in the above section apply here.

Now let us turn to discussions of the \( 1/R + R^2 \) gravity more mathematically, especially the relative strength of the two additional curvature terms.
The field equations follow by substituting equation (12) into (2)
\[
\left(1 + \frac{\alpha^2}{3R^2} + \frac{2R}{3\beta}\right) R_{\mu\nu} - \frac{1}{2g_{\mu\nu}} \left( R - \frac{\alpha^2}{3R} + \frac{R^2}{3\beta} \right) = \kappa^2 T_{\mu\nu}.
\] (22)

Contracting indices gives an explicit expression for the scalar curvature
\[
R = \frac{1}{2\alpha} \left[ -\frac{\kappa^2 T}{\alpha} + 2\sqrt{1 + \frac{1}{4} \left( \frac{\kappa^2 T}{\alpha} \right)^2} \right] = \frac{1}{2\alpha} \left[ \frac{\kappa^2 \rho_m}{\alpha} + 2\sqrt{1 + \frac{1}{4} \left( \frac{\kappa^2 \rho_m}{\alpha} \right)^2} \right]
\] (23)

where we take the plus sign in the two root solutions if the matter density is positive, and as in section 3, we assume that the universe contains dust and radiation. It is interesting to note that equation (23) is the same as the one in 1/R gravity [15].

From the conservation equation \(\dot{\rho}_m + 3H\rho_m = 0\) and equation (23), we can find that
\[
\partial_0 L' = \frac{\left(\frac{x}{2}\right)^2 - \frac{x}{2}}{\sqrt{1 + \frac{1}{4} \left( \frac{x^2}{2}\right)^2}} \frac{\kappa^2 \rho_m}{\alpha} H.
\] (24)

Substituting this into equation (11) we can get the MF equation:
\[
H^2 = \frac{\kappa^2 \rho_m + 2\kappa^2 \rho_r + \alpha \left[ G\left(\frac{\kappa^2 \rho_m}{\alpha}\right) = \frac{1}{3G\left(\frac{\kappa^2 \rho_m}{\alpha}\right)} + \frac{2n}{3\beta} G\left(\frac{\kappa^2 \rho_m}{\alpha}\right)^2 \right]}{\left[1 + \frac{1}{3G\left(\frac{\kappa^2 \rho_m}{\alpha}\right)} + \frac{2n}{3\beta} G\left(\frac{\kappa^2 \rho_m}{\alpha}\right) \right] \left[ 6 + 3F\left(\frac{\kappa^2 \rho_m}{\alpha}\right)(1 + \frac{1}{2} F\left(\frac{\kappa^2 \rho_m}{\alpha}\right)) \right]}
\] (25)

where the two functions \(G\) and \(F\) are given by
\[
G(x) = \frac{1}{2} \left[ x + 2\sqrt{1 + \frac{1}{4} x^2} \right]
\] (26)
\[
F(x) = \frac{(1 - \frac{x}{2} G(x))^3 x}{(G(x)^2 + \frac{2n}{3\beta} G(x)^3 + \frac{1}{4}) \sqrt{1 + \frac{1}{4} x^2}^3}
\] (27)

In order to be consistent with observations, we should have \(\alpha \ll \beta\). We can see this in two different ways.

Firstly, when \(\kappa^2 \rho_m \gg \alpha\), from equation (26), \(G \sim \kappa^2 \rho_m/\alpha\). From the BBN constraints, we know the MF equation should reduce to the standard one in the BBN era [20]. This can be achieved only when \(F \sim 0\) [15] and from equation (27), this can be achieved only when \(\alpha \ll \beta\) and \(1 \ll \kappa^2 \rho_m/\alpha \ll (\beta/\alpha)^{1/3}\).

Secondly, when \(\kappa^2 \rho_m \ll \alpha\) (this is just the case we are interested in for considering the deviation from GR), we can expand the rhs of equation (25) to the first order in \(\kappa^2 \rho_m/\alpha\):
\[
H^2 = \frac{11+4n/\beta}{11+4n/\beta} \kappa^2 \rho_m + \frac{3}{11+4n/\beta} \kappa^2 \rho_r + \frac{1}{2} \alpha \sqrt{1 + \frac{1}{2} \left( \frac{\kappa^2 \rho_m}{\alpha} \right)^2}
\] (28)

When \(\alpha \ll \beta\), this will reduces exactly to the first-order MF equation in the 1/R theory [15]. Since we have shown there that the MF equation in 1/R theory can fit the SN Ia data at an acceptable level, the above MF equation cannot greatly deviate from it, thus the condition below should be satisfied consistently,
\[
\alpha \ll \beta.
\] (29)

The coupling constant \(\alpha\) is very small and the inverse curvature term functions only at larger cosmic scale as expected, which can also be seen from the recent analysis by Carroll et al [6, 28].
Following [13], we have an equivalent scalar–tensor description of the Palatini formulation of modified gravity. When considering the $1/R + R^2$ gravity, the potential is given by [13, 16]

$$V(\Phi) = \frac{2\alpha^2}{3^3} \exp \left(-2\sqrt{\frac{2\kappa^2}{3}} \Phi\right)$$

where $\phi$ is determined from $\Phi$ by

$$\frac{2}{3} \left(\frac{\phi}{\beta}\right)^3 - \left(\exp\left(\sqrt{\frac{2\kappa^2}{3}} \Phi\right) - 1\right) \left(\frac{\phi}{\beta}\right)^2 + \frac{1}{3} \left(\frac{\alpha}{\beta}\right)^2 = 0.$$  (31)

Now we can see a problem of $1/R + R^2$ gravity. The determinant of equation (31) is $\Delta = \frac{22}{27} \left(\frac{\alpha}{\beta}\right)^2 \left[\left(\exp\left(\sqrt{\frac{2\kappa^2}{3}} \Phi\right) - 1\right)^3 - \left(\frac{\alpha}{\beta}\right)^3\right]$. When it is positive, i.e., $\sqrt{\kappa^2/\Phi} > \sqrt{3/2} \ln(1+(\alpha/\beta)^{2/3})$, equation (31) has three distinct real solutions. Thus, the correspondence to scalar–tensor theory is not one-to-one now. According to [21], this is a strong indication that the $1/R + R^2$ theory is not a consistent theory. Furthermore, as also pointed out to us by Flanagan [24], this implies that when $R$ exceeds the critical value $R_0 = (\alpha^2/\beta)^{1/3}$ which satisfies $L''(R_0) = 0$, the $1/R + R^2$ theory does not have a well-behaved initial-value formulation.

In [13], Flanagan also showed that, if we assume the $1/R$ model is also applicable at small scales, there will be severe conflict with electron–electron scattering experiments. This conflict is due to the smallness of the energy scale of the potential near the extremal point, which is $\alpha/\kappa^2 \sim 10^{-12}$ (eV)^2 in the $1/R$ case. In the $1/R + R^2$ case, we can find that the extremal value $V'(\Phi_0) = 0$ is given by $\sqrt{\kappa^2/\Phi_0} = \sqrt{3/2} \ln(4/3 + 2\alpha/3\beta)$. This corresponds to $\phi = \alpha$. Substituting this into equation (30) and using the fact $\alpha \ll \beta$ found above we can find that near the extremal point $\Phi_0$ the potential is of the order $\alpha/\kappa^2$, i.e., the same as the $1/R$ case. Thus, the conflict will still appear.

However, the above conflict is due to the fact that we assume that the $1/R$ corrected action is applicable in very small scales in addition to the astrophysical scales where it was originally suggested to be effective to explain the cosmic acceleration. Thus, this $1/R$ gravity theory cannot be a fundamental theory and if we can find some way to guarantee that it is only effective on a large scale, we can still use it to discuss cosmological issues. A concrete way to achieve this is still under investigation.

5. Conclusions and discussions

In this paper we have shown that in the Palatini formulation, an $R^2$ term cannot lead to an early time inflation, contrary to the conclusion when considering the theory in the metric variation. Furthermore, in the more general $1/R + R^2$ model that intends to explain both the current cosmic acceleration and the early time inflation, we have demonstrated that accelerating cosmology at late times can be obtained without dark energy introduced under the model coupling constants consistently satisfying the condition $\alpha \ll \beta$.

Intuitively speaking, the cosmic global evolution at early times is dominated by a large curvature term like $R^n (n > 0)$ and at later times by a kind of $R^n (n < 0)$ term when the small curvature term dominates the global evolution. The current ‘standard theory’ of gravitation, Einstein’s general relativity (GR), has passed many tests within the solar system. To reconcile the successful GR predictions within the solar system, the extended gravity theories may be required to be scale sensitive. It could be challenging and profound to locate the additional curvature terms in our above discussions about what form of scale dependence may exist.
Acknowledgments

The insightful comments by three referees that solidify this work considerably are highly appreciated. We would especially like to thank Professor Sergei Odintsov too for his careful reading of this manuscript and for much helpful advice, which have improved this paper a lot. Specially, he informed us of the non-unitarity problem in $R^2$ gravity and suggested reconsidering this problem in the Palatini formalism. We would also like to thank Professors Sean Carroll, Eanna Flanagan, Nadeem Haque, Shin’ichi Nojiri, Mark Trodden and A A Starobinsky for helpful discussions and Professors Mauro Francaviglia and Igor Volovich for helping us find their earlier works. This work is partly supported by China NSF, Doctoral Foundation of National Education Ministry and ICSC-World Laboratory Scholarship.

References

[1] Perlmutter S et al 2000 Nature 404 955
   Perlmutter S et al 1999 Astrophys. J. 517 565
   Riess A et al 1998 Astrophys. J. 116 1009
   Riess A et al 2001 Astrophys. J. 560 49
   Wang Y 2000 Astrophys. J. 536 531

[2] Carroll S M 2001 Living Rev. Rel. 4 1
   Carroll S M 2003 Preprint astro-ph/03010324 (Preprint astro-ph/0004075)
   Padmanabhan T 2003 Phys. Rep. 380 235 (Preprint hep-th/0212290)

[3] Carroll S M, Hoffman M and Trodden M 2003 Preprint astro-ph/0301273
   Nojiri S and Odintsov S D 2003 Preprint hep-th/0303117
   Caldwell R R, Dave R and Steinhardt P J 1998 Phys. Rev. Lett. 80 1582
   Kamenshchik A, Moschella U and Pasquier V 2001 Phys. Lett. B 511 265
   Frolov A, Kofman L and Starobinsky A 2002 Preprint hep-th/0204187

[4] Lue A, Scoccimarro R and Starkman G 2003 Preprint astro-ph/0307034

[5] Dvali G, Gabadadze G and Porrati M 2000 Phys. Lett. B 485 208

[6] Carroll S M, Duvvuri V, Trodden M and Turner M 2004 Phys. Rev. D 70 043528 (Preprint astro-ph/0306438)

[7] Capozziello S, Carloni S and Troisi A 2003 Preprint astro-ph/0303041

[8] Nojiri S and Odintsov S D 2003 Phys. Lett. B 576 5 (Preprint hep-th/0307071)

[9] Dick R 2003 Preprint gr-qc/0307052

[10] Soussa M E and Woodard R P 2003 Preprint astro-ph/0308114

[11] Vollick D N 2003 Phys. Rev. D 68 063510 (Preprint astro-ph/0306630)

[12] Ferraris M, Francaviglia M and Volovich I 1993 Nuovo Cimento B 108 1313 (Preprint gr-qc/9303007)
   Ferraris M, Francaviglia M and Volovich I 1994 Class. Quantum Grav. 11 1505

[13] Flanagan E E 2004 Phys. Rev. Lett. 92 071101 (Preprint astro-ph/0308111)

[14] Dolgov A D and Kawasaki M 2003 Preprint astro-ph/0307285

[15] Meng X H and Wang P 2004 Class. Quantum Grav. 21 951 (Preprint astro-ph/0308031)
   Meng X H and Wang P 2003 Class. Quantum Grav. 20 4949 (Preprint astro-ph/0307354)

[16] Chiba T 2003 Preprint astro-ph/0307338

[17] Liddle A R and Lyth D H 2000 Cosmological Inflation and Large Scale Structure (Cambridge: Cambridge University Press)

[18] Starobinsky A A 1980 Phys. Lett. B 91 99

[19] Nojiri S and Odintsov S D 2003 Phys. Rev. D 68 123512 (Preprint hep-th/0307288)

[20] Carroll S M and Kaplinghat M 2001 Preprint astro-ph/0108002
   Olive K A, Steigman G and Walker T P 2000 Phys. Rep. 333–334 389 (Preprint astro-ph/9905320)
[21] Magnano G and Sokolowski L M 1994 Phys. Rev. D 50 5039 (Preprint gr-qc/9312008)

[22] Buchbinder I L, Odintsov S D and Shapiro I L 1992 Effective Action in Quantum Gravity (Bristol: Institute of Physics Publishing)

[23] Nojiri S and Odintsov S D 2003 Preprint hep-th/0308176

[24] Flanagan É 2003 private communications

[25] Gottlober S, Muller V and Starobinsky A 1991 Phys. Rev. D 43 2510

Cardenas V H, del Campo S and Herrera R 2003 Preprint gr-qc/0308040

[26] Armendariz-Picon C, Damour T and Mukhanov V 1999 Phys. Lett. B 458 209

[27] Freese K and Lewis M 2002 Phys. Lett. B 540 1 (Preprint astro-ph/0201229)

[28] Abdalla M, Nojiri S and Odintsov S 2004 Preprint hep-th/0409177

Carroll S et al 2004 Preprint astro-ph/0410031

[29] Armendariz-Picon C and Lim E A 2003 J. Cosmol. Astropart. Phys. JCAP12(2003)006 (Preprint hep-th/0303103)

[30] Meng X H and Wang P 2004 Class. Quantum Grav. 21 2029 (Preprint qr-qc/0402011)