ULRICH BUNDLES ON INTERSECTIONS OF TWO 4-DIMENSIONAL QUADRICS

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Abstract. In this paper, we investigate the existence of Ulrich bundles on a smooth complete intersection of two 4-dimensional quadrics in $\mathbb{P}^5$ by two completely different methods. First, we find good ACM curves and use Serre correspondence in order to construct Ulrich bundles, which is analogous to the construction on a cubic threefold by Casanellas-Hartshorne-Geiss-Schreyer. Next, we use Bondal-Orlov’s semiorthogonal decomposition of the derived category of coherent sheaves to analyze Ulrich bundles. Using these methods, we prove that any smooth intersection of two 4-dimensional quadrics in $\mathbb{P}^5$ carries an Ulrich bundle of rank $r$ for every $r \geq 2$. Moreover, we provide a description of the moduli space of stable Ulrich bundles.

1. Introduction

Let $\mathbb{P}^n$ be the $n$-dimensional projective space over the field of complex numbers $\mathbb{C}$. A famous theorem by Horrocks states that a vector bundle $E$ on $\mathbb{P}^n$ splits as the direct sum of line bundles if and only if $E$ has no intermediate cohomology, i.e., $h^i(E(j)) = 0$ for all $0 < i < n$ and $j \in \mathbb{Z}$. It is natural to ask the algebro-geometric meaning of these vanishing conditions for the other varieties. Let $X \subset \mathbb{P}^N$ be an $n$-dimensional smooth projective variety with a fixed polarization $O_X(1) = O_{\mathbb{P}^N}(1)|_X$. We call that a vector bundle $E$ on $X$ is ACM (arithmetically Cohen-Macaulay) if $E$ has no intermediate cohomology with respect to the given polarization $O_X(1)$. Roughly speaking, the presence of nontrivial ACM bundles measures how $X$ is apart from the projective space $\mathbb{P}^n$. Due to their interesting properties, ACM bundles have played a significant role in the study of vector bundles.

In commutative algebra, ACM bundles correspond to MCM (maximal Cohen-Macaulay) modules which are Cohen-Macaulay modules achieving the maximal dimension. A particularly interesting case happens when the minimal free resolution of an MCM module becomes completely linear. Such an MCM module has the maximal possible number of minimal generators which are concentrated on a single degree $37$. Eisenbud and Schreyer made a comprehensive study on the geometric analogue of these linear ACM modules, and named them Ulrich sheaves $[14]$. Thanks to foundational works by Beauville $[5]$ and Eisenbud-Schreyer $[14]$, Ulrich sheaves provide a number of fruitful applications; for example, linear determinantal representations of hypersurfaces, matrix factorizations by linear matrices, the cone of cohomology tables, and Cayley-Chow forms. Eisenbud and Schreyer conjectured that every projective variety carries an Ulrich sheaf $[14]$, and verified it for a few simple cases. The conjecture is still wildly open even for smooth surfaces. In very recent years, there were several progresses on the conjecture for surfaces; for instance, general K3 surfaces $[1]$, abelian surfaces $[6]$, and nonspecial surfaces of $p_g = q = 0$ $[13]$. 

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Much less is known for ACM and Ulrich bundles on threefolds. On a smooth quadric $Q^3 \subset \mathbb{P}^4$, there is only one nontrivial indecomposable ACM bundle, namely, the spinor bundle $[10]$. Arrondo and Costa studied ACM bundles on Fano 3-folds of index 2 of degree $d = 3, 4, 5$ [4]. Madonna studied splitting criteria for rank 2 vector bundles on hypersurfaces in $\mathbb{P}^4$ [29]. He also classified all the possible Chern classes of rank 2 ACM bundles on prime Fano 3-folds and complete intersection Calabi-Yau 3-folds [27]. Their results expected the existence of Ulrich bundles on 3-folds of small degree, however, constructions were not complete except for a very few cases. On the other hand, Beauville showed that a general hypersurface of degree $\leq 5$ in $\mathbb{P}^4$ is linearly Pfaffian. In other words, such a hypersurface carries a rank 2 Ulrich bundle [5]. He also checked that every Fano 3-fold of index 2 carries a rank 2 Ulrich bundle [7]. In particular, a general smooth cubic 3-fold carries Ulrich bundles of rank $r$ for every $r \geq 2$, proved first by Casanellas, Hartshorne, Geiss, and Schreyer [11]. Recently, Lahoz, Macrì, and Stellari extended this result to every smooth cubic 3-fold using the derived category of coherent sheaves and also described the moduli space of stable Ulrich bundles [24].

It is quite natural to ask for the next case, a del Pezzo threefold $X = Q^4_0 \cap Q^4_{\infty}$ of degree four which is the complete intersection of two quadric 4-folds. Indeed, $X$ is very attractive since there are several ways to understand vector bundles on $X$. Since $X$ is a 3-fold, we may construct vector bundles on $X$ by observing curves lying on $X$ via Serre correspondence. On the other hand, it is also well-known that the geometry of the intersection of 2 even dimensional quadrics is closely related to a hyperelliptic curve. Bondal and Orlov showed that the derived category of coherent sheaves on the intersection of 2 even dimensional quadrics has a semiorthogonal decomposition whose components are the derived category of the hyperelliptic curve associated to the 2 given quadrics and the exceptional collection [8]. Recently, there were several attempts to understand vector bundles on a variety using the semiorthogonal decomposition of its derived category. For instance, Kuznetsov studied instanton bundles on some index 2 Fano 3-folds via semiorthogonal decompositions [23]. Lahoz, Macrì, and Stellari studied ACM bundles on cubic 3-folds and 4-folds via semiorthogonal decomposition [24, 25]. Therefore, it is reasonable to apply the semiorthogonal decomposition to understand vector bundles on the intersection of two even dimensional quadrics.

Being motivated by earlier works mentioned above, we investigate the existence and the moduli space of Ulrich bundles on the intersection of two 4-dimensional quadrics by two completely different methods: classical Serre correspondence and Bondal-Orlov theorem. The main result is the following theorem:

**Theorem 1.1** (see Theorem 3.8 and 4.14). The moduli space of stable Ulrich bundles of rank $r \geq 2$ on $X = Q^4_0 \cap Q^4_{\infty}$ is isomorphic to a nonempty open subscheme of $\mathcal{U}_C(r, 2r)$, where $\mathcal{U}_C(r, 2r)$ is the moduli space of stable vector bundles of rank $r$ and degree $2r$ on a curve $C$ of genus 2.

Our approach using Serre correspondence closely follows the works of Arrondo and Costa [4] and of Casanellas, Hartshorne, Geiss, and Schreyer [11], and our approach using derived categories is strongly influenced by the works of Kuznetsov [23] and of Lahoz, Macrì, and Stellari [24, 25]. The structure of this paper is as follows. In Section 2 we recall a few useful facts related to ACM and Ulrich bundles. In Section 3 we construct Ulrich bundles of any rank $r \geq 2$ on a general intersection of two quadric 4-folds $X = Q^4_0 \cap Q^4_{\infty}$ using Serre correspondence and Macaulay2. In Section 4 we
prove the existence of Ulrich bundles of any rank \( r \geq 2 \) on a smooth complete intersection of two quadric 4-folds \( X = Q_4^0 \cap Q_4^\infty \) using Bondal-Orlov theorem. We also analyze the moduli of stable Ulrich bundles of rank \( r \) on \( X \) and provide a description in terms of vector bundles on \( C \).

## 2. Preliminaries on ACM and Ulrich bundles

In this section, we briefly review the definition of ACM and Ulrich bundles and their basic properties.

**Definition 2.1.** Let \( X \subset \mathbb{P}^N \) be an \( n \)-dimensional smooth projective variety embedded by a very ample line bundle \( \mathcal{O}_X(1) \).

1. A coherent sheaf \( \mathcal{E} \) on \( X \) is ACM if \( H^i(\mathcal{E}(j)) = 0 \) for all \( 0 < i < n \) and \( j \in \mathbb{Z} \).
2. An ACM sheaf \( \mathcal{E} \) on \( X \) is Ulrich if \( H^0(\mathcal{E}(-1)) = 0 \) and \( h^0(\mathcal{E}) = \deg(X) \cdot \text{rank}(\mathcal{E}) \).

**Remark 2.2.** Since the underlying space \( X \) is smooth, \( \mathcal{E} \) being ACM implies that \( \mathcal{E} \) is locally free. Hence it is natural to call ACM (Ulrich) bundles for the objects occurring in the above definition.

We recall the following proposition by Eisenbud and Schreyer. We refer to [7, 14] for more details.

**Proposition 2.3** ([7, Theorem 1], [14, Proposition 2.1]). Let \( X \subset \mathbb{P}^N \) and \( \mathcal{E} \) as above. The following are equivalent:

1. \( \mathcal{E} \) is Ulrich;
2. \( H^i(\mathcal{E}(-i)) = 0 \) for all \( i > 0 \) and \( H^j(\mathcal{E}(-j - 1)) = 0 \) for \( j < n \).
3. \( H^i(\mathcal{E}(-j)) = 0 \) for all \( i \) and \( 1 \leq j \leq n \).
4. For some (all) finite linear projections \( \pi : X \to \mathbb{P}^n \), the sheaf \( \pi_* \mathcal{E} \) is isomorphic to the trivial sheaf \( \mathcal{O}_{\mathbb{P}^n}^{\oplus t} \) for some \( t \).
5. The section module \( M := \oplus_j H^0(\mathcal{E}(j)) \) is a linear MCM module, that is, the minimal \( S = \mathbb{C}[x_0, \ldots, x_N] \)-free resolution of \( M \)

\[
F : 0 \to F_{N-n} \to \cdots \to F_1 \to F_0 \to M \to 0
\]

is linear in the sense that \( F_i \) is generated in degree \( i \) for every \( i \).

In particular, by Serre duality, we immediately have the following proposition as a consequence:

**Proposition 2.4.** Let \( X^n \subset \mathbb{P}^N \) be as above, and let \( H := \mathcal{O}_X(1) \) be a very ample line bundle.

1. If \( \mathcal{E} \) is an ACM bundle on \( X \), then \( \mathcal{E}^*(K_X) \) is also an ACM bundle.
2. When \( X \) is subcanonical, that is, \( K_X = \mathcal{O}_X(k) \) for some \( k \in \mathbb{Z} \), \( \mathcal{E} \) is ACM if and only if \( \mathcal{E}^* \) is ACM.
3. If \( \mathcal{E} \) is an Ulrich bundle on \( X \), then \( \mathcal{E}^*(K_X + (n + 1)H) \) is an Ulrich bundle.

The following proposition about the stability is very useful in later sections.

**Proposition 2.5** ([11, Theorem 2.9]). Let \( X \) be a smooth projective variety, and let \( \mathcal{E} \) be an Ulrich bundle on \( X \). Then

1. \( \mathcal{E} \) is semistable and \( \mu \)-semistable.
(2) If $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ is an exact sequence of coherent sheaves with $\mathcal{E}''$ torsion-free, and $\mu(\mathcal{E}') = \mu(\mathcal{E})$, then both $\mathcal{E}'$ and $\mathcal{E}''$ are Ulrich.

(3) If $\mathcal{E}$ is stable, then it is also $\mu$-stable.

3. Geometric approach via Serre correspondence

In this section, we show the existence of Ulrich bundles using Serre correspondence.

3.1. Serre correspondence. We briefly recall Serre correspondence which enables us to construct a vector bundle as an extension from a codimension 2 subscheme. To obtain a vector bundle, such a subscheme has to satisfy certain generating conditions. For instance, it is well-known that a 0-dimensional subscheme on a smooth surface should satisfy Cayley-Bacharach condition to provide a vector bundle as an extension from a codimension 2 subscheme. To obtain a vector bundle, such a local complete intersection and subcanonical [18]. It is clear that not all curves come from vector bundles. When it happens, we cannot construct a vector bundle as an extension. However, still in many cases, it is a powerful tool providing constructions of vector bundles. We refer to [3] for the proof and more details.

**Theorem 3.1** (Serre correspondence). Let $X$ be a smooth variety and let $Y \subset X$ be a local complete intersection subscheme of codimension 2 in $X$. Let $\mathcal{N}$ be the normal bundle of $Y$ in $X$ and let $\mathcal{L}$ be a line bundle on $X$ such that $H^2(\mathcal{L}^*) = 0$. Assume that $(\mathcal{N}^2 \otimes \mathcal{L}^*)|_Y$ has $(r - 1)$ generating global sections $s_1, \ldots, s_{r - 1}$. Then there is a rank $r$ vector bundle $\mathcal{E}$ as an extension

$$0 \to \mathcal{O}_X^{\tau - 1} \xrightarrow{(s_1, \ldots, s_{r - 1})} \mathcal{E} \to \mathcal{I}_{Y/X}(\mathcal{L}) \to 0$$

such that the dependency locus of $(r - 1)$ global sections $s_1, \ldots, s_{r - 1}$ of $\mathcal{E}$ is $Y$ with $\sum_{i=1}^{r-1} s_i \alpha_i|_Y = 0$. Moreover, if $H^1(\mathcal{L}^*) = 0$, such an $\mathcal{E}$ is unique up to isomorphism.

3.2. ACM bundles of rank 2 via Serre correspondence. From now on, let $Q_0, Q_{\infty}$ be two smooth quadric hypersurfaces in $\mathbb{P}^5$ meeting transversally and let $X = Q_0 \cap Q_{\infty} \subset \mathbb{P}^5$ be a smooth Fano 3-fold of degree 4 and index 2, i.e., $\omega_X = \mathcal{O}_X(-2)$.

Let $[H_X], [L_X], [P_X]$ be the class of a hyperplane section, a line, and a point in $X$ respectively. Then,

$$H^2(X, \mathbb{Z}) \simeq \mathbb{Z} \cdot [H_X], \quad H^4(X, \mathbb{Z}) \simeq \mathbb{Z} \cdot [L_X], \quad \text{and} \quad H^6(X, \mathbb{Z}) \simeq \mathbb{Z} \cdot [P_X].$$

The ring structure is given as follows: $H^2_X = 4L_X, H_X \cdot L_X = P_X$. For a vector bundle $\mathcal{F}$ on $X$, we define its slope $\mu$ with respect to $H$ by

$$\mu_H(\mathcal{F}) := \frac{\deg_H \mathcal{F}}{\text{rank } \mathcal{F}}.$$ 

By virtue of (3.1), we fix our convention as follows.

**Notation 3.2.** Via the isomorphisms $\mathbb{Z} \cdot [H_X] \simeq \mathbb{Z}, \mathbb{Z} \cdot [L_X] \simeq \mathbb{Z}$, and $\mathbb{Z} \cdot [P_X] \simeq \mathbb{Z}$, we may regard $c_1(\mathcal{F})$ as an integer, by omitting the cyclic generators of $H^{2i}(X, \mathbb{Z})$. Under this convention, one can easily see that

$$\mu_H(\mathcal{F}) = \frac{c_1(\mathcal{F}) \deg X}{\text{rank } \mathcal{F}} = 4 \cdot \frac{c_1(\mathcal{F})}{\text{rank } \mathcal{F}}.$$
We also omit the redundant coefficient 4 in the formula and redefine the slope of $\mathcal{F}$ as follows:

$$\mu(\mathcal{F}) := \frac{c_1(\mathcal{F})}{\text{rank } \mathcal{F}}.$$ 

The following proposition is useful in later sections.

**Proposition 3.3** ([21 Proposition 1.2.7]). Let $\mathcal{E}$ and $\mathcal{E}'$ be $\mu$-stable bundles with $\mu(\mathcal{E}) > \mu(\mathcal{E}')$. Then $\text{Hom}(\mathcal{E}, \mathcal{E}') = 0$.

Applying Proposition 2.4 to $X = Q_0 \cap Q_\infty$, we get the following:

**Proposition 3.4.** Let $\mathcal{E}$ be an Ulrich bundle of rank $r$ on $X = Q_0 \cap Q_\infty$. Then,

1. $\mu(\mathcal{E}) = 1$, and
2. $\mathcal{E}^*(2)$ is an Ulrich bundle.

In [4], Arrondo and Costa made a comprehensive study of ACM bundles on $X$ extending [36]. They classified the possible Chern classes for ACM bundles under a mild assumption. In particular, they classified all the rank 2 ACM bundles on $X$.

**Theorem 3.5** ([4 Theorem 3.4]). An indecomposable rank 2 ACM vector bundle on $X$ is a twist of one of the following:

1. A line type: a semistable vector bundle $\mathcal{E}_l$ fitting in an exact sequence
   $$0 \to \mathcal{O}_X \to \mathcal{E}_l \to \mathcal{I}_l \to 0$$
   where $l \subset X$ is a line contained in $X$;
2. A conic type: a stable vector bundle $\mathcal{E}_\lambda$ fitting in an exact sequence
   $$0 \to \mathcal{O}_X \to \mathcal{E}_\lambda \to \mathcal{I}_\lambda(1) \to 0$$
   where $\lambda \subset X$ is a conic contained in $X$;
3. An elliptic curve type: a stable vector bundle $\mathcal{E}_e$ fitting in an exact sequence
   $$0 \to \mathcal{O}_X \to \mathcal{E}_e \to \mathcal{I}_e(2) \to 0$$
   where $e \subset X$ is an elliptic curve of degree 6.

It is classically well-known that the Fano scheme $F(X)$ of lines $l \subset X$ is isomorphic to the Jacobian $J(C)$ of the hyperelliptic curve $C$ of genus 2 associated to $X$ (see [28 Theorem 5], [29 Theorem 2] or [34]). Since $\mathcal{E}_l$ has the unique global section up to constants, the space also coincides with the space of line type ACM bundles.

Conic type ACM bundles are also well understood as in the following way. Given a conic $\lambda \subset X$, note that there is only one quadric $Q \in \mathfrak{b} := |Q_0 + tQ_\infty|_{t \in \mathbb{P}^1}$ in a pencil containing the plane $\Lambda = \langle \lambda \rangle$. It is clear that $\Lambda \cap X = \lambda$. Since $Q$ is a 4-dimensional quadric, there is a spinor bundle whose global sections sweep out a family of planes in $Q$ containing $\Lambda$. The bundle $\mathcal{E}_\lambda$ is the restriction of this spinor bundle. Hence, the moduli of conic type ACM bundle can be naturally identified with the space of spinor bundles associated to the pencil $\mathfrak{b}$.

The last case is particularly interesting. When $e \subset X \subset \mathbb{P}^5$ is an elliptic normal curve of degree 6, we have $h^0(\mathcal{I}_e(1)) = 0$ and $h^0(\mathcal{I}_e(2)) = h^0(\mathcal{I}_{e/\mathbb{P}^5}(2)) = 9 - 2 = 7$. Hence $\mathcal{E}_e$ is an
initialized ACM bundle with $h^0(\mathcal{E}_j) = 8 = (\deg X) \cdot (\text{rank} \mathcal{E}_e)$, in other words, it is an Ulrich bundle of rank 2. We refer to [7] for an explicit construction of such curves.

**Proposition 3.6 ([7] Proposition 8).** There exists an Ulrich bundle of rank 2 on $X$.

### 3.3. Ulrich bundles of higher ranks via Serre correspondence and Macaulay2

Similar as the case of cubic 3-folds in $\mathbb{P}^4$, the existence of rank 3 Ulrich bundles on $X$ was expected earlier in [4] Example 4.4. However, as Casanellas and Hartshorne pointed out [11, Remark 5.5], the construction was incorrect not only for cubic 3-folds but also for our $X$. Arrondo and Costa constructed an arithmetically Cohen-Macaulay curve $D$ of degree 15 and genus 12 using a Gorenstein liaison, however, 2 sections of $H^0(\omega_D(-1))$ do not generate the graded module $H^0_D(\omega_D)$. Indeed, in loc. cit., the authors started with a twisted cubic curve $D'$, and then found an arithmetically Gorenstein curve $B'$ of degree 18 containing $D'$ where the residual curve is $D$. Hence we have a short exact sequence

$$0 \to \mathcal{I}_{B'} \to \mathcal{I}_{D'} \to \omega_D(-2) \to 0.$$ 

Since $B'$ is arithmetically Gorenstein, we have a short exact sequence of graded $S = H^0\left(\mathcal{O}_{\mathbb{P}^5}\right)$-modules

$$0 \to H^0(\mathcal{I}_{B'}) \to H^0(\mathcal{I}_{D'}) \to H^0_D(\omega_D(-2)) \to 0. \quad (3.2)$$

Note that $B'$ is the zero locus of a section of $\mathcal{E}_e(1)$, so $\mathcal{I}_{B'}$ fits into the short exact sequence $0 \to \mathcal{O}_X \to \mathcal{E}_e(1) \to \mathcal{I}_{B'}(4) \to 0$. Hence, the first 2 nonzero terms in the sequence (3.2) are

$$H^0(\mathcal{I}_{D'}(1)) \simeq H^0(\omega_D(-1))$$

and

$$H^0(\mathcal{I}_{D'}(2)) \simeq H^0(\omega_D).$$

Via the exact sequence $0 \to \mathcal{I}_{X/P^5} \to \mathcal{I}_{D'/P^5} \to \mathcal{I}_{D'} \to 0$, we may lift the sections in $H^0(\mathcal{I}_{D'}(j))$ as the homogeneous form of degree $j$ in $S$. It is clear that a twisted cubic curve $D' \subset \mathbb{P}^5$ is generated by 2 linear forms and 3 quadratic forms in $S$, and hence $H^0(\omega_D(-1))$ is spanned by these 2 linear forms $l_1$ and $l_2$, namely.

However, sections in the image of $H^0(\omega_D(-1)) \otimes H^0(\mathcal{O}_{\mathbb{P}^5}(1)) \to H^0(\omega_D)$ can only span 11 quadrics, since two sections of $\omega_D(-1)$ admit a linear Koszul relation $l_1l_2 - l_2l_1 = 0$ in $S$. Hence we conclude that $H^0(\omega_D(-1)) \otimes H^0(\mathcal{O}_{\mathbb{P}^5}(1)) \to H^0(\omega_D)$ cannot be surjective.

We need to construct a curve satisfying the generating condition to construct a rank 3 Ulrich bundle $\mathcal{E}$ on $X$. If it exists, then two independent global sections of $\mathcal{E}$ will degenerate along a curve $D$ of degree 15 since $\mathcal{E}$ is globally generated always. It is easy to see that the numerical conditions suggested in [4] Example 4.4 are valid. Hence, we need to construct an ACM curve $D \subset X$ of given invariants such that $H^0(\omega_D(-1))$ has two generating section, that is, the multiplication map

$$H^0(\omega_D(-1)) \otimes H^0(\mathcal{O}_{\mathbb{P}^5}(j)) \to H^0(\omega_D(j - 1))$$

is surjective for each $j \geq 1$.

Since $\omega_D(1 + j)$ is nonspecial for $j \geq 2$, Castelnuovo pencil trick implies that the map is automatically surjective for $j \geq 2$. Hence it is sufficient to check only for the $j = 1$ case. The construction follows from Macaulay2 [17] computations, which is analogous to [11] Appendix or [16]. Although
the proof goes into the same strategy, in particular, the Macaulay2 scripts are almost same, it is worthwhile to write down since the difference between the cubic 3-fold case is not that much straightforward.

**Proposition 3.7** (See also [11] Theorem A.3). The space of pairs \( D \subset X \subset \mathbb{P}^5 \) of smooth ACM curves of degree 15 and genus 12 on a complete intersection of 2 quadrics \( X \) has a component which dominates the Hilbert scheme of intersections of 2 quadrics in \( \mathbb{P}^5 \). Moreover, the module \( H^n_0(\omega_D) \) is generated by its 2 sections in degree \(-1\) as \( S_{2}\mathbb{P}^5 = H^0(\mathcal{O}_{\mathbb{P}^5}) \)-modules for a general pair \( D \subset X \). In particular, a general intersection of two quadrics in \( \mathbb{P}^5 \) carries a desired curve we discussed above.

**Proof.** We prove by constructing a family of such curves as in the following strategy. First, we take a family of smooth curves of genus 12 in \( \mathbb{P}^1 \times \mathbb{P}^2 \). Next, we observe that a general (precisely, a randomly chosen) curve in this family admits an embedding to \( \mathbb{P}^5 \) in a natural way. Finally, we check that such a curve in \( \mathbb{P}^5 \) satisfies the desired properties. Then the whole statement will follow by the deformation theory and the semicontinuity.

Let \( D \) be a smooth projective curve of genus 12 together with line bundles \( L_1 \) and \( L_2 \) with \( |L_1| \) a \( \mathcal{O}_7 \) and \( |L_2| \) a \( \mathcal{O}_0 \). Let \( D' \) be the image of the map

\[
D \frac{|L_1|, |L_2|}{\mathbb{P}^1 \times \mathbb{P}^2}.
\]

Suppose that the maps \( H^0(\mathbb{P}^1 \times \mathbb{P}^2, \mathcal{O}(n, m)) \rightarrow H^0(D, L_1^n \otimes L_2^m) \) are of maximal rank for all \( n, m \geq 1 \). Under this assumption, \( D \) is isomorphic to its image \( D' \) and we may compute the Hilbert series of the truncated ideal

\[
I_{trunc} = \bigoplus_{n, m \geq 3} H^0(\mathcal{I}_D(n, m))
\]

in the Cox ring \( S_{\mathbb{P}^1 \times \mathbb{P}^2} = k[x_0, x_1; y_0, y_1, y_2] \) of \( \mathbb{P}^1 \times \mathbb{P}^2 \), namely,

\[
H_{I_{trunc}}(s, t) = \frac{5s^4t^5 - 11s^4t^4 - 6s^3t^5 + 3s^4t^3 + 10s^3t^4}{(1 - s)^2(1 - t)^3}.
\]

Hence, by reading off the Hilbert series, we may expect that \( I_{trunc} \) admits a bigraded free resolution of type

\[
0 \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow I_{trunc} \rightarrow 0
\]

with modules \( F_0 = S_{\mathbb{P}^1 \times \mathbb{P}^2}(-3, -4)^{10} \oplus S_{\mathbb{P}^1 \times \mathbb{P}^2}(-4, -3)^3 \), \( F_1 = S_{\mathbb{P}^1 \times \mathbb{P}^2}(-3, -5)^{10} \oplus S_{\mathbb{P}^1 \times \mathbb{P}^2}(-4, -4)^{11} \), and \( F_2 = S_{\mathbb{P}^1 \times \mathbb{P}^2}(-4, -5)^5 \).

We will construct a curve \( D' \subset \mathbb{P}^1 \times \mathbb{P}^2 \) in a converse direction. First, we take a free resolution of the above form, and then observe that the module represented by such a resolution is indeed an ideal of a curve \( D' \). Let \( M : F_2 \rightarrow F_1 \) be a general map chosen randomly, and let \( K \) be the cokernel of the dual map \( M^* : F_1^* \rightarrow F_2^* \). The first terms of a minimal free resolution of \( K \) are:

\[
\cdots \rightarrow G \xrightarrow{N'} F_1^* \xrightarrow{M^*} F_2^* \rightarrow K \rightarrow 0
\]

where \( G \) be the module generated by syzygies of \( M^* \). Composing \( N' \) with a general map \( F_0^* \rightarrow G \) and dualizing again, we get a map \( N : F_1 \rightarrow F_0 \). The following script shows that the kernel of \( N^* \) is \( S_{\mathbb{P}^1 \times \mathbb{P}^2} \) so that the entries of the matrix \( S_{\mathbb{P}^1 \times \mathbb{P}^2} \rightarrow F_0^* \) generate an ideal.
i1 : setRandomSeed "RandomCurves";
p=997;
Fp=ZZ/p;
S=Fp[x_0,x_1,y_0..y_2, Degrees=>{2:{1,0},3:{0,1}}]; -- Cox ring
m=ideal basis({1,1},S); -- irrelevant ideal

i2 : randomCurveGenus12Withg17=(S)->(
M:=random(S^{6:{-3,-5},11:{-4,-4}},S^{5:{-4,-5}}); -- random map M
N':=syz transpose M; -- syzygy matrix of the dual of M
N:=transpose(N'*random(source N',S^{3:{4,3},10:{3,4}}));
   ideal syz transpose N) -- the vanishing ideal of the curve

i3 : ID'=saturate(randomCurveGenus12Withg17(S),m); -- ideal of D'

Since the maximal rank assumption is an open condition, the above example provides that there
is a component $H \subset \text{Hilb}(7,10)$, $12$ ($\mathbb{P}^1 \times \mathbb{P}^2$) in the Hilbert scheme of curves of bidegree $(7,10)$ and
genus 12 defined by free resolutions of the above form. Also note that $D' \in H$ admits both $g_1^1$ and
$g_2^{10}$ induced by the natural projections.

We want to verify that a general $D \in H'$ equipped with two natural projections acts like a general
curve $D \in M_{12}$, $L_1$, and $L_2$ in order to show that $H'$ dominates $M_{12}$. Recall from Brill-Noether
theory that for a general curve $D$ of genus $g$, the Brill-Noether locus

$$W^r_d(D) = \{ L \in \text{Pic}(D) | \deg(L) = d, h^0(L) \geq r + 1 \}$$

is nonempty and smooth away from $W^{r+1}_d(D)$ of dimension $\rho$ if and only if

$$\rho = \rho(g,r,d) = g - (r + 1)(g - d + r) \geq 0.$$ 

Also note that the tangent space at $L \in W^r_d(D) \setminus W^{r+1}_d(D)$ is the dual of the cokernel of Petri map

$$H^0(D,L) \otimes H^0(D,\omega_D \otimes L^{-1}) \to H^0(D,\omega_D).$$

We expect that both $L_1$ and $L_2$ are smooth isolated points of dimension $\rho_1 = \rho_2 = 0$, equivalently,
both Petri maps are injective. We refer to [2, Chapter IV] for details on Brill-Noether theory.

Now let $\eta : D \to D'$ be a normalization of a given point $D' \in H$, since we do not know that $D'$
is smooth yet. We check that $L_i$ are smooth points in the associated Brill-Noether loci as follows,
where $L_i$ is a line bundle on $D$ obtained by pulling back natural $g_1^1$ and $g_2^{10}$ on $D'$ for $i = 1, 2$.

We first check $L_2$; we take the plane model $\Gamma \subset \mathbb{P}^2$ of $D'$.

i4 : Sel=Fp[x_0..x_1,y_0..y_2,MonomialOrder=>Eliminate 2];
   R=Fp[y_0..y_2]; -- coordinate ring
   IGammaD=sub(ideal selectInSubring(1,gens gb sub(ID',Sel)),R);
   -- ideal of the plane model

We observe that $\Gamma$ is a curve of desired degree and genus, and its singular locus $\Delta$ consists only of
ordinary double points as follows.
We can also compute the minimal free resolution of $I_{\Delta}$:

\begin{verbatim}
18 : IDelta=saturate IDelta;
   betti res IDelta

   0 1 2
0: 1 . .
1: . . .
2: . .
3: .
4: .
5: 4 .
6: . 3
\end{verbatim}

Thanks to the above Betti table, we immediately check that $\Gamma$ is irreducible since $\Delta$ is not a complete intersection $(4,6)$. Indeed, there is no way to write a degree 10 curve $\Gamma \subset \mathbb{P}^2$ with 24 nodes as a union of 2 curves. In particular, the normalization of $\Gamma$ is isomorphic to a smooth irreducible curve of genus $g = 12$, and thus $D'$ is smooth since $12 = g \leq p_a(D') \leq 12$. Hence from now on, we do not distinguish $D$ and $D'$ since they coincide.

By Riemann-Roch, we have $h^0(D, L_2) = 3$ since $h^1(D, L_2) = h^0(D, \omega_D \otimes L_2^{-1}) = h^0(\mathbb{P}^2, I_{\Delta}(6)) = 4$ by the adjunction formula applied to $D \subset \text{Bl}_{\Delta} \mathbb{P}^2$. Hence $|L_2|$ is complete and the Petri map for $L_2$ is identified with the multiplication

$$H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \otimes H^0(\mathbb{P}^2, I_{\Delta}(6)) \rightarrow H^0(\mathbb{P}^2, I_{\Delta}(7)).$$

Note that the map is injective since there is no linear relation among the 4 sextic generators of $I_{\Delta}$. In fact, the Petri map becomes an isomorphism, and $L_2 \in W^2_10(D)$ is a smooth isolated point of dimension $\rho_2 = 0$. 

To check that $L_1$ is Petri generic, we first compute the embedding $D \to \mathbb{P}^H(\omega_D \otimes L_1^{-1}) = \mathbb{P}^5$ and its minimal free resolution by choosing sections of $H^0(\omega_D) \simeq H^0(\mathbb{P}^2, \mathcal{I}_\Delta(T))$ which vanish on a fiber of $D \to \mathbb{P}^1$ induced by $|L_1|:

\begin{verbatim}
i9 : LK=(mingens IDelta)*random(source mingens IDelta, R^-{12:{-7}}); -- compute a basis
Pt=random(Fp^1,Fp^2); -- a random point in a line
L1=substitute(ID',Pt|vars R); -- fiber over the point
KD=LK*(syz(LK % gens L1))_{0..5}; -- compute a basis for elements in LK vanish in L1
T=Fp[z_0..z_5]; -- coordinate ring
phiKD=map(R,T,KD); -- embedding
ID=preimage_phiKD(IGammaD);
degree ID==15 and genus ID==12
\end{verbatim}

We observe that the curve $D \subset \mathbb{P}^5$ verifies the desired properties. Since the length of the minimal free resolution of $I_D$ equals to the codimension, $D \subset \mathbb{P}^5$ becomes ACM. Note that the dual complex $\text{Hom}^\bullet_S(\mathbb{P}^5, S \mathbb{P}^5(-6))$ gives a resolution of $\oplus_{n\in\mathbb{Z}}H^0(\omega_D(n))$ where $F_D$ is the minimal free resolution of $D$. The Betti table also tells us that this module is generated by its 2 global sections in degree $-1$ and $h^0(L_1) = h^0(\omega_D(-1)) = 2$. Hence, $|L_1|$ is also complete and the Petri map for $L_1$ is identified with

$$H^0(D, \omega_D(-1)) \otimes H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(1)) \to H^0(D, \omega_D).$$

This map is also injective since there is no linear relation between the 2 generators in $H^0(\omega_D(-1))$. Indeed, the Petri map becomes an isomorphism, and $L_1 \in W_1^1(D)$ is a smooth isolated point of dimension $\rho_1 = 0$. As consequences, $\mathcal{H}$ dominates $Z = W_1^1 \times_{\mathcal{M}_{12}} W_2^{10}$ and $\mathcal{M}_{12}$ thanks to Brill-Noether theory.

It remains to check the existence of a dominating family of desired curves in $\mathbb{P}^5$ over the space of intersections of two quadrics in $\mathbb{P}^5$. Since a random curve $D \in \mathcal{H}$ provides an embedding $D \subset \mathbb{P}^5$ given by a Petri generic line bundle $\mathcal{O}_D(1) := \omega_D \otimes L_1^{-1}$, the above construction provides a nonempty component $\mathcal{H}' \subset \text{Hilb}_{15+1-12}(\mathbb{P}^5)$ together with a dominant rational map $\mathcal{H}'//\text{Aut}(\mathbb{P}^5) \to \mathcal{M}_{12}$. Note that choosing an intersection of 2 quadrics $X \subset \mathbb{P}^5$ containing $D$ is equivalent to choosing a
2-dimensional subspace of $H^0(\mathbb{P}^5, \mathcal{I}_{D/\mathbb{P}^5}(2))$. Consider the incidence variety

$$V = \{(D, X) \mid D \in \mathcal{H}' \text{ ACM and } X \in \text{Gr}(2, H^0(\mathbb{P}^5, \mathcal{I}_{D/\mathbb{P}^5}(2))) \text{ smooth}\}.$$ 

Since the graded Betti numbers are upper semicontinuous in a flat family having the same Hilbert function, we observe that $V$ is birational to $\mathcal{H}'$ since $H^0(\mathbb{P}^5, \mathcal{I}_{D/\mathbb{P}^5}(2))$ is spanned by 2 quadrics for a randomly chosen $D$.

We compute the normal sheaf $\mathcal{N}_{D/X}$ for a random pair $(D, X) \in V$ as follows:

```plaintext
i11 : IX=ideal((mingens ID)*random(source mingens ID,T^{2:-2}));
   ID2=saturate(ID^2+IX);
   cNDX=image gens ID / image gens ID2; -- conormal sheaf
   NDX=sheaf Hom(cNDX,T^1/ID); -- normal sheaf
   HH^0 NDX(-1)==0 and HH^1 NDX(-1)==0
o11 = true
i12 : HH^0 NDX==Fp^30 and HH^1 NDX==0
o12 = true
```

In particular, the Hilbert scheme of $X$ is smooth of dimension 30 at $[D \subset X]$, and $h^i(\mathcal{N}_{D/X}(-1)) = 0$ for $i = 0, 1$. We do a similar computation for $\mathcal{N}_{D/\mathbb{P}^5}$:

```plaintext
i13 : cNDP=prune(image (gens ID)/ image gens saturate(ID^2));
   NDP=sheaf Hom(cNDP,T^1/ID);
   HH^0 NDP==Fp^68 and HH^1 NDP==0
o13 = true
```

Hence $\mathcal{H}' \subset Hilb_{15t+1-12}$ is smooth of expected dimension 68 at a general smooth point $[D \subset \mathbb{P}^5] \in \mathcal{H}'$.

Consider the natural projections

$$\xymatrix{ V \ar@{-->}[dr]_{\pi_2} & \mathcal{H}' \ar@{-->}[d]_{\pi_1} \\
& \text{Gr}(2, H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(2))).}$$

We observe that $V$ is irreducible of dimension 68 since the fiber of $\pi_1$ over $D$ is exactly a single point. Also note that the map $\pi_2$ is smooth of dimension $h^0(D, \mathcal{N}_{D/X}) = 30$ at $(D, X)$. Since $\dim \text{Gr}(2, H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(2))) = 38$, we conclude that $\pi_2$ is dominant. In particular, a general $X \in Gr(2, H^0(\mathcal{O}_{\mathbb{P}^5}(2)))$ contains a curve $D \in \mathcal{H}'$. By the semicontinuity, we conclude that a general $(D, X)$ also satisfies the desired properties. □

Existence of such a curve $D$ on $X$ provides a construction of a rank 3 Ulrich bundle on $X$ via Serre correspondence. The idea by Casanellas and Hartshorne also makes sense in our case, and consequently, we have the following theorem:

**Theorem 3.8** (See also [11, Proposition 5.4 and Theorem 5.7]). Let $X \subset \mathbb{P}^5$ be the intersection of 2 general quadrics in $\mathbb{P}^5$. Then $X$ carries an $(r^2+1)$-dimensional family of stable Ulrich bundles of rank for every $r \geq 2$. 

Proof. Since the strategy is almost same as in [11], we only provide a shorter proof here. Note first that there is a rank 2 Ulrich bundle on any smooth complete intersection $X$ [17], namely, an elliptic curve type ACM bundle. Since there is no Ulrich line bundle, any rank 2 Ulrich bundle must be stable by Proposition 2.5. Because of the same reason, if there is a rank 3 Ulrich bundle, then it is also stable.

Proposition 3.7 implies that a general $X$ contains a smooth ACM curve $D$ of degree 15 and genus 12 such that $\omega_D(-1)$ has two sections which generate the graded module $H^1(X, \omega_D)$ as $S^2$-modules. By Serre correspondence, those two generators define a rank 3 vector bundle $E$ as an extension

$$0 \to O_X^2 \to E \to \mathcal{I}_D(3) \to 0.$$ 

Since $D$ is ACM, we immediately check that $H^1(X, \omega_D) = 0$ for every $j \in \mathbb{Z}$. Furthermore, we also have $H^1(X, \omega_D) = H^2(X, -j - 2) = 0$ for every $j \in \mathbb{Z}$ from the dual sequence

$$0 \to O_X(-3) \to E^* \to O_X^2 \to \omega_D(-1) \to 0.$$ 

Hence $E$ is an ACM bundle. Applying Riemann-Roch on $D$, we have $h^0(O_D(2)) = 19 = h^0(O_X(2))$ and thus $h^0(E(-1)) = h^0(O_D(2)) = 0$. Similarly, we have $h^0(O_D(3)) = h^0(O_X(3)) - h^0(O_D(3)) = 10$, and thus $h^0(E) = 12 = (\deg X) \cdot (\rank E)$. Indeed, $E$ is a rank 3 Ulrich bundle. As consequences, we show the existence of Ulrich bundles on $X$ of every rank $r \geq 2$ by taking direct sums of Ulrich bundles of rank 2 and 3.

Suppose first that we have a stable Ulrich bundle $E$ of rank $r$ for every $r \geq 2$. By Riemann-Roch, we have $\chi(E \otimes E^*) = -r^2$. Since the computations in [11] Proposition 5.6 also holds for our $X$, we have $h^2(E \otimes E^*) = h^3(E \otimes E^*) = 0$. Since $E$ is simple, we conclude that $h^0(E \otimes E^*) = 1$ and $h^1(E \otimes E^*) = r^2 + 1$ as desired. Hence the moduli space of stable Ulrich bundles is smooth of expected dimension if it is nonempty.

It only remains to show the existence of stable Ulrich bundles of rank bigger than 3. Let $r \geq 4$, $E'$ and $E'' \neq E'$ be stable Ulrich bundles of rank 2 and $r - 2$, respectively. By Riemann-Roch and [11] Proposition 5.6, we have $h^1(E' \otimes E''^*) = -\chi(E' \otimes E''^*) = 2r - 4 > 0$. Hence the space $\mathbb{P} \Ext^1_X(E'', E')$ is nonempty and each element gives a nonsplit extension

$$0 \to E' \to E \to E'' \to 0$$

where $E$ is a simple and strictly semistable Ulrich bundle of rank $r$. Such extensions form a family of dimension

$$\dim\{E'\} + \dim\{E''\} + \dim \mathbb{P} \Ext^1_X(E'', E') = r^2 - 2r + 5 < r^2 + 1.$$ 

Since all the other extensions by different ranks form smaller families, we conclude that a general Ulrich bundle of rank $r$ is stable. This completes the proof.

\[\square\]

Remark 3.9. We finish this section by a few remarks.

(1) In fact, the proof of Proposition 3.7 implies much stronger results. For instance, one can check that $\mathcal{H}$ is a unirational family which dominates the moduli space $\mathcal{M}_{12}$ of smooth curves of genus 12 as in [11] Appendix.

(2) Because we made a computer-based computation over a finite field, we cannot remove the assumption $X$ being general. It is also mysterious that “how general” $X$ should be.
(3) As we mentioned, the above approach closely follows [11]. In loc. cit., the authors also checked that any smooth cubic 3-fold contains an elliptic normal curve of degree 5. Similarly, any smooth complete intersection of two quadrics in $P^5$ contains an elliptic normal curve of degree 6, as in [7] Proposition 8. It is an interesting task to construct smooth ACM curves of degree 15 and genus 12 on any smooth complete intersection of two 4-dimensional quadrics.

4. Derived categorical approaches

The notion of semiorthogonal decomposition enables us to reduce problems about Ulrich bundles on $X$ to problems about vector bundles on the associated curve $C$. Let us recall some necessary facts about the moduli space of vector bundles on curves and the derived category of coherent sheaves on $X$.

4.1. Stable vector bundles on curves. Let $C$ be a smooth projective curve of genus $g$, $\mathcal{U}_C(r,d)$ be the moduli space of $S$-equivalence classes of rank $r$ semistable vector bundles of degree $d$ on $C$, and $SU_C(r,L)$ be the moduli space of $S$-equivalence classes of rank $r$ semistable vector bundles of determinant $L$ on $C$. We use the superscript $(−)$ to describe the sub-moduli space parametrizing stable objects. It is well-known that $\mathcal{U}_C(r,d)$ and $SU_C(r,L)$ are normal projective varieties (see [28,35]).

The lemma below is one of the well-known results for (semi-)stable bundles on curves.

Lemma 4.1. Let $F$ be a stable vector bundle of rank $\geq 2$ on $C$. Then,

1. $\mu(F) \geq 2g-2$ implies $h^1(F) = 0$, and
2. $\mu(F) \geq 2g-1$ implies that $F$ is globally generated.

If the inequalities on $\mu$ are strict, then the same results are valid for $F$ semistable.

Proof. Assume that $h^1(F) \neq 0$. Then $h^0(F^* \otimes \omega_C) \neq 0$, which is impossible unless $F = \omega_C$ since $F$ is stable and $\deg(F^* \otimes \omega_C) \leq 0$. This proves (1). If $\mu(F) \geq 2g-1$, then $H^1(F(-P)) = 0$ for any $P \in C$, hence $H^0(F) \to F \otimes \kappa(P)$ is surjective. Using Nakayama’s lemma, we conclude that $H^0(F) \otimes O_C \to F$ is surjective.

Lemma 4.2 ([32] Exercise 2.8). Let $F, G$ be vector bundles on $C$ such that $H^p(F \otimes G) = 0$ for $p = 0,1$. Then both $F$ and $G$ are semistable.

Proof. By Riemann-Roch, $\mu(F \otimes G) = g - 1$. Assume that there exists $0 \neq F' \subset F$ such that $\rk F' < \rk F$ and $\mu(F') > \mu(F)$. Then, $\mu(F' \otimes G) > \mu(F \otimes G) = g - 1$. This shows that $\chi(F' \otimes G) > 0$, in particular, $h^0(F' \otimes G) > 0$. This contradicts to $F' \otimes G \subset F \otimes G$ and $h^0(F \otimes G) = 0$. It follows that $F$ is semistable, and the same argument applies to $G$.

Similar as in the case of line bundles, we may define the Brill-Noether locus as follows:

$$W^{k-1}_{r,d}(C) := \{ [F] \in \mathcal{U}_C(r,d) \mid h^0(C,F) \geq k \}$$

which is a subscheme of $\mathcal{U}_C(r,d)$ of expected dimension $\rho^{k-1}_{r,d} = r^2(g-1) + 1 - k(k-d+r(g-1))$.

The following theorem is useful in the future:
Theorem 4.3 ([3] Theorem B]). The locus $W_{r,d}^{k-1}(C)$ is nonempty if and only if 
\[ d > 0, r \leq d + (r - k)g \text{ and } (r, d, k) \neq (r, r, r). \]

4.2. Derived categories of $X$. Let $Q^n \subset \mathbb{P}^{n+1}$ be a smooth quadric hypersurface. We unify all 
the notations which involve spinor bundles in accordance with [8]. Hence, the spinor bundles on the 
quadric $Q^n$ give the semiorthogonal decomposition [22]

\[ D^b(Q^n) = \begin{cases} \langle O(-n + 1), \ldots, O, S \rangle & \text{if } n \text{ is odd} \\ \langle O(-n + 1), \ldots, O, S^+, S^- \rangle & \text{if } n \text{ is even} \end{cases} \]

Especially in the case $n = 4$, $S^\pm$ correspond to the universal quotient bundle and the dual of the 
universal subbundle under the isomorphism $Q^4 \simeq \text{Gr}(2, \mathcal{C}^4)$.

Let $Q_0$, $Q_\infty \subset \mathbb{P}^5$ be two nonsingular 4-dimensional quadrics whose intersection defines $X$. Without 
loss of generalities, we may assume

\[ Q_0 = (x_0^2 + \ldots + x_2^2 = 0) \quad \text{and} \quad Q_\infty = (\lambda_0 x_0^2 + \ldots + \lambda_5 x_2^2 = 0) \]

for some $\lambda_0, \ldots, \lambda_5 \in \mathbb{C}$. We define $X := Q_0 \cap Q_\infty$ a smooth threefold of degree 4. One well 
known approach to $X$ is to associate the quadric pencil $\mathfrak{d} := |Q_0 + tQ_\infty|_{t \in \mathbb{P}^1}$ on $\mathbb{P}^5$. Let us assume 
that the pencil $\mathfrak{d}$ is nonsingular in the sense of [31], namely, each singular quadric $Q_{\mathfrak{d},i}$ $(i = 0, \ldots, 5)$ 
is isomorphic to the cone of a smooth quadric $Q^3 \subset \mathbb{P}^4$ over a point. Note that this condition is 
equivalent to saying that $\lambda_0, \ldots, \lambda_5$ are pairwise distinct. Also note that none of $\lambda_0, \ldots, \lambda_5$ is zero 
since $Q_\infty$ is smooth.

The resolution of indeterminacy of $\varphi_\mathfrak{d} : \mathbb{P}^5 \dashrightarrow \mathbb{P}^1$ gives the relative quadric $Q \to \mathbb{P}^1$. Let $\sigma : C \to \mathbb{P}^1$ 
be the double cover ramified over $[1 : \lambda_0], \ldots, [1 : \lambda_5] \in \mathbb{P}^1$, and let $Q_C := Q \times_{\mathbb{P}^1} C$ be the fiber product. Bondal and Orlov [8] showed that $C$ is the fine moduli space of spinor bundles on the 
quadrics in $\mathfrak{d}$, i.e. there exists a vector bundle $S_{\mathcal{C}}$ on $Q_C$ such that for each $c \in C$, the restriction 
$S_{\mathcal{C}}|_{Q \times \{c\}}$ is one of the spinor bundles on the quadric $Q_{\sigma(c)}$. When $Q_{\sigma(c)}$ is a singular quadric, then 
it is a cone $\mathcal{C}(Q^3)$ of a 3-dimensional quadric over a point $v \in \mathbb{P}^5$. In this case $S_{\sigma(c)}$ is the pullback of the 
unique spinor bundle on $Q^3$ by $\mathcal{C}(Q^3) \setminus \{v\} \to Q^3$. We define the vector bundle $S := S_{\mathcal{C}}|_{X \times C}$.

Theorem 4.4 (Bondal–Orlov [8]). The Fourier–Mukai transform 

\[ \Phi_S : D^b(C) \to D^b(X), \quad F^* \mapsto R\pi_!(Lp_C^*F^* \otimes L) \]

is fully faithful, and induces a semiorthogonal decomposition 

\[ D^b(X) = \langle O_X(-1), O_X, \Phi_S(D^b(C)) \rangle. \]

Furthermore, $X$ can be regarded as the fine moduli space of stable vector bundles of rank 2 with 
fixed determinant of odd degree [29], and $S$ is the universal bundle of this moduli problem. There arises 
an ambiguity of the choice of this fixed determinant (the theorem of Bondal and Orlov is independent 
of the replacement $S \mapsto S \otimes p_C^*L$ for any line bundle $L \in \text{Pic} C$).

Definition 4.5. We choose $\xi$ a line bundle of degree 1, and assume that $S$ is the universal family 
of the fine moduli space $SU_C(2, \xi^*) \simeq X$ which parametrizes the stable vector bundles of rank 2 and 
determinant $\xi^*$. Equivalently, $S$ is determined by imposing the condition $\det S = O_X(1) \boxtimes \xi^*$. 


This choice of $S$ is precisely dual to the same symbol in Section 5 of [23]. We remark that some parts of the next subsection are following the arguments in [23]. This may cause confusions, so we rephrase the details which are necessary for the rest part of the paper.

4.3. Ulrich bundles via derived categories. Let $\text{Coh}(X)$ be the category of coherent sheaves on $X$. There is a natural functor $\text{Coh}(X) \to D^b(X)$ which maps a coherent sheaf $E$ to the complex concentrated at degree zero:

$$\ldots \to 0 \to E \to 0 \to \ldots$$

This identifies $\text{Coh}(X)$ to a full (but not triangulated) subcategory of $D^b(X)$, hence we may regard a coherent sheaf on $X$ as an object in $D^b(X)$. Conversely, we call an object $E^\bullet \in D^b(X)$ a coherent sheaf (resp. a vector bundle) if $E^\bullet$ is isomorphic to an object (resp. a locally free sheaf) in $D^b(X) \cap \text{Coh}(X)$.

We use derived categories to classify Ulrich bundles on $X$. We first assume that there exists an Ulrich bundle $E$ of rank $r \geq 2$ on $X$ (the existence will be proved later). By Proposition 2.3, $H^p(E(-i)) = Hom_{D^b(X)}(E^\bullet(1), O(-i+1)[p]) = 0$ for all $p$ and $i = 1, 2, 3$. Using the semiorthogonal decomposition in Theorem 4.3 one immediately sees that $E^\bullet(1) \in \Phi_S D^b(C)$. Since $D^b(C) \to \Phi_S(D^b(C))$ is an equivalence of categories, the study of Ulrich bundles on $X$ boils down to the study of certain objects in $D^b(C)$. Such objects are obtained by mapping $E^\bullet(1)$ along the projection functor $\Phi^!_S: D^b(X) \to D^b(C)$. Before to proceed, let us note that the projection $\Phi^!_S$ is right adjoint to $\Phi_S$. Therefore, the Ulrich conditions in Proposition 2.3 impose an extra condition on $E^\bullet(1)$ other than $E^\bullet(1) \in \Phi_S D^b(C)$. Indeed, the condition $H^\bullet(E(-3)) = 0$ is not followed by $E^\bullet(1) \in \Phi_S D^b(C)$.

It can be expressed as follows:

$$\text{Hom}_{D^b(X)}(E^\bullet(1), O_X(-2)[p]) = 0$$

$$\Leftrightarrow \text{Hom}_{D^b(X)}(\Phi_S \Phi^!_S(E^\bullet(1)), O_X(-2)[p]) = 0$$

$$\Leftrightarrow \text{Hom}_{D^b(C)}(\Phi_S(E^\bullet(1)), \Phi^!_S(O_X(-2))[p]) = 0.$$  \hfill (4.1)

**Lemma 4.6.** We have $\Phi^!_S(O_X(-2))[2] \simeq R^* \otimes \omega_C^{\otimes 2}$, where $R$ is the second Raynaud bundle which appears in [23] Section 5.4.

**Proof.** By [23], $R \simeq \Phi^!_S O_X[-1] = p_{C*}(S \otimes \phi_X^* O_X) \otimes \omega_C$. Thus,

$$R^* \simeq \text{Hom}_{D^b(C)}(p_{C*} S \otimes \omega_C, O_C)$$

$$\simeq p_{C*} \text{Hom}_{D^b(X)}(S \otimes \phi_X^* \omega_C, \phi_X^* \omega_X[3])$$

$$\simeq p_{C*} (S^* \otimes \phi_X^* O_X(-2)) \otimes \omega_C^{\otimes 3}$$

$$\simeq \Phi^!_S (O_X(-2)) \otimes \omega_C^{\otimes 2}[2],$$

where the second isomorphism is given by Grothendieck-Verdier duality. \hfill \square
Together with the orthogonality condition \(4.1\), we have to understand how the object \(\Phi_S^c(\mathcal{E}^*(1))\) looks like. One standard way is to analyze the restriction to the point \(\Phi_S^c(\mathcal{E}^*(1)) \otimes \kappa(c) \in \mathbb{D}^b(\{c\})\). We fix the notations to avoid confusion as follows.

**Notation 4.7.** For \(x \in X\), we denote by \(\mathcal{S}_x\) the vector bundle over \(C\) determined by the restriction of \(\mathcal{S}\) to \(\{x\} \times C \simeq C\). Similarly, the vector bundle \(\mathcal{S}_c (c \in C)\) over \(X\) is defined to be the restriction of \(\mathcal{S}\) to \(X \times \{c\} \simeq X\).

The proof of the following proposition is essentially due to [23, Theorem 5.10], but we write down the proof to prevent the confusions arising from the choice of a convention.

**Proposition 4.8.** Suppose there exists an Ulrich bundle \(\mathcal{E}\) of rank \(r\) on \(X\). Then, \(F := \Phi_S^c(\mathcal{E}^*(1)) \in \mathbb{D}^b(C)\) is a semistable vector bundle over \(C\) of rank \(r\) and degree \(2r\). Furthermore, \(F\) satisfies

\[
\text{(1) } \text{Ext}^p_c (\mathcal{R}, F^* \otimes \omega_C^{\otimes 2}) = 0 \text{ for } p = 0, 1 \text{ and}
\]

\[
\text{(2) } H^1(F \otimes \mathcal{S}_x) = 0 \text{ for each } x \in X.
\]

Conversely, if \(F\) is a semistable vector bundle over \(C\) of rank \(r\) and degree \(2r\) satisfying the conditions (1) and (2) above, then \(\Phi_S F = \mathcal{E}^*(1)\) for some Ulrich bundle \(\mathcal{E}\) over \(X\).

**Proof.** Let \(c \in C\) be a point. Then \(F \otimes \kappa(c) \in \mathbb{D}^b(\{c\})\) is the complex of \(\mathbb{C}\)-vector spaces whose cohomology sheaves are controlled by

\[
H^{p+1}(X, \mathcal{E}^*(1) \otimes S^*_c) \simeq \text{Ext}_{X}^{p+1}(\mathcal{E}(-1), S^*_c). \quad (4.2)
\]

By [30, p. 310], \(\mu(S^*_c) = -1/2\), regardless whether \(c\) is a ramification point or not. Hence \(\mu(\mathcal{E}(-1)) = 0\), Proposition 3.3 and Serre duality imply that

\[
\text{Ext}_X^0(\mathcal{E}(-1), S^*_c) \simeq \text{Hom}_X(\mathcal{E}(-1), S^*_c) = 0.
\]

Consider the following short exact sequence (cf. [30, Theorem 2.8])

\[
0 \to \mathcal{S}^*_c \to \mathcal{O}^{\mathcal{S}_c}_X \to \mathcal{S}^*_c(1) \to 0 \quad (4.3)
\]

where \(\tau: C \to C\) is the hyperelliptic involution arising from the double cover \(C \to \mathbb{P}^1\). Note that even for the ramification points \(c \in C\), one can compose the sequence \(4.3\) in a natural way. Tensoring \(4.3\) with \(\mathcal{E}^*(j)\) for \(j = -1, 0, 1\), we have \(H^{p+1}(\mathcal{E}^*(1) \otimes S^*_c) \simeq H^{p+2}(\mathcal{E}^* \otimes S^*_c) \simeq H^{p+3}(\mathcal{E}^*(-1) \otimes S^*_c)\) and the latter one vanishes for \(p \geq 1\). This proves that \(4.2\) is zero unless \(p = 0\), in other words, \(F\) is a coherent sheaf concentrated at degree 0. Furthermore, since \(p_X^*(\mathcal{E}^*(1) \otimes S)\) is flat over \(C\), \(c \mapsto \chi(\mathcal{E}^*(1) \otimes S^*_c)\) is a constant function and thus \(F\) is a vector bundle on \(C\).

To compute rank \(F\) and deg \(F\), we use Grothendieck-Riemann-Roch which reads

\[
\text{ch}(\Phi_S F) = \text{ch}(Rp_X^*(p_C^* F \otimes S)) = p_X^* (\text{ch}(p_C^* F) \text{ch}(S) \text{td} (\mathcal{T}_{p_X}))
\]

\[
= (2d - 3s) + \frac{1}{3}(2s - d)P_X - sL_X + (d - 2s)H_X, \quad (4.4)
\]

where \(d = \text{deg} F\) and \(s = \text{rank} F\). The computation method is identical to the one introduced in [23, Lemma 5.2] except that the Fourier-Mukai kernels are dual to each other.
Since $\Phi_S F = \mathcal{E}^*(1)$ is of rank $r$ and of degree zero, we find $2d - 3s = r$ and $d - 2s = 0$. It follows that $s = r$ and $d = 2r$. By (4.1) and Lemma 4.6
\[
\text{Hom}_{\mathcal{D}^b(C)}(\Phi_S^!(\mathcal{E}^*(1)), \Phi_S^!(\mathcal{O}_X(-2))[p]) \simeq \text{Hom}_{\mathcal{D}^b(C)}(F, \mathcal{R}^* \otimes \omega_C^{\otimes 2}[p - 2])
\simeq \text{Ext}^2_{\mathcal{O}_C}(\mathcal{R}, F^* \otimes \omega_C^{\otimes 2}).
\]
Since both $\mathcal{R}$ and $F$ are vector bundles, it suffices to require $\text{Ext}^2_{\mathcal{O}_C}(\mathcal{R}, F^* \otimes \omega_C^{\otimes 2}) = 0$ for $p = 0, 1$ to fulfill (4.1). The semistability of $F$ follows from Lemma 4.2. Finally, $H^1(F \otimes S_x) = 0$ follows from the fact that $\Phi_S F = \mathcal{E}^*(1)$ is a vector bundle on $X$; indeed, $\Phi_S F = Rp_{X*}(p^*_C F \otimes S)$ is the complex concentrated at zero, hence $R^1p_{X*}(p^*_C F \otimes S) = 0$. By the cohomology base change, $H^1(F \otimes S_x) = 0$ for each $x \in X$.

Conversely, assume that $F$ is a semistable vector bundle on $C$ satisfying all the prescribed conditions. The condition (2) implies that $\Phi_S F \in \mathcal{D}^b(X)$ is a vector bundle on $X$. Then $\Phi_S F \in \mathcal{D}^b(C)$ together with (1) can be interpreted as $\text{Ext}^1_X(\Phi_S F, \mathcal{O}_X(-j)) = 0$ for $j = 0, 1, 2$, showing that $\mathcal{E} := (\Phi_S F)^* \otimes \mathcal{O}_X(1)$ is an Ulrich bundle over $X$. □

Using (4.6) and $\Phi_S F = \mathcal{E}^*(1)$, we can immediately check that
\[
(c_i(\mathcal{E}^*)),_i = (1, r, 2r^2 - r, \frac{1}{4}r(r - 2)(2r + 1)).
\]
Proposition 4.8 gives a bijection between the set of Ulrich bundles on $X$ with the set of certain semistable vector bundles on $C$. From now on, we bring our focus into the semistable vector bundles on $C$ satisfying the conditions described in Proposition 4.8. First of all, we prove that a general stable bundle in $\mathcal{U}_C(r, 2r)$ ($r \geq 2$) satisfies the condition (2) of Proposition 4.8.

**Proposition 4.9.** For $r \geq 2$, let $\mathcal{U}_C^s(r, 2r)$ be the moduli space of stable vector bundles on $C$ of rank $r$ and degree $2r$. The subset
\[
\{[F] \in \mathcal{U}_C^s(r, 2r) : h^1(F \otimes S_x) = 0 \text{ for every } x \in X\}
\]
is open and nonempty.

**Proof.** First of all, we claim that the set $\{[F] \in \mathcal{U}_C^s(r, 2r) : h^1(F \otimes S_x) = 0 \text{ for every } x \in X\}$ is open in $\mathcal{U}_C^s(r, 2r)$. Consider the closed subset $Z \subset X \times \mathcal{U}_C^s(r, 2r)$ defined by
\[
\{(x, [F]) : h^1(F \otimes S_x) \geq 1\}.
\]
Since the projection morphism $\text{pr}_2 : X \times \mathcal{U}_C^s(r, 2r) \to \mathcal{U}_C^s(r, 2r)$ is proper, $V := \mathcal{U}_C^s(r, 2r) \setminus \text{pr}_2(Z)$ is open in $\mathcal{U}_C^s(r, 2r)$. Writing down the locus $V$ set-theoretically, we can easily find that
\[
V = \{[F] \in \mathcal{U}_C^s(r, 2r) : h^1(F \otimes S_x) = 0 \text{ for every } x \in X\}.
\]

For $r = 2$, we know that any smooth $X$ carries an Ulrich bundle $\mathcal{E}$ of rank 2 as in Proposition 4.6. Note that its projection image $F := \Phi_S^!(\mathcal{E}^*(1))$ is a rank 2 vector bundle of degree 4 on $C$ satisfying the desired property. Assume that $r \geq 3$. Let $F$ be a stable vector bundle of rank $r$ and degree $2r$, and let $x \in X$. Suppose that $H^1(F \otimes S_x) \simeq \text{Hom}_C(F, S_x^* \otimes \omega_C)^*$ is nonzero. By the stability condition, any nonzero morphism $F \to S_x^* \otimes \omega_C$ must be surjective, so we have a short exact sequence
\[
0 \to F' \to F \to S_x^* \otimes \omega_C \to 0
\]
where $F'$ is a semistable vector bundle of rank $(r - 2)$ and degree $(2r - 5)$. By Riemann-Roch, we have $ext_C^1(S^r_2 \otimes \omega_C, F') = 3r - 4$. Hence, for each $x \in X$, the locus of vector bundles $F$ fit into the above exact sequence has dimension at most $(r - 2)^2 + 1 + (3r - 5) = r^2 - r$. As varying $x \in X$, the bad locus can sweep out a set of dimension at most $r^2 - r + 3 < r^2 + 1$. Hence we conclude that a general $F \in U^s_C(r, 2r)$ does not admit a surjection to $S^r_* \otimes \omega_C$ for any $x \in X$. \hfill \Box

**Remark 4.10.** The formula (4.4) tells us that there is no line bundle $F$ of degree 2 such that $\Phi_S F$ is locally free. Indeed, there is no line bundle $E$ on $X$ such that $\text{ch}(E) = 1 - L_X$. In particular, there is no Ulrich line bundle on $X$.

Our aim is to find a semistable vector bundle $F$ of rank $r$ and degree $2r$ such that $\text{Ext}^p_C(R, F^* \otimes \omega_C^{\otimes 2}) = 0$ for $p = 0, 1$. Since $G := F^* \otimes \omega_C^{\otimes 2}$ is also a semistable vector bundle of rank $r$ and degree $2r$, the following proposition guarantees the existence of Ulrich bundles at least when $r = 3$:

**Proposition 4.11.** $\text{Hom}_C(R, G) = 0$ for a generic stable vector bundle $G$ of rank 3 and degree 6.

**Proof.** Suppose that there is a nontrivial morphism $R \to G$. Note that $R$ is a stable vector bundle [19, Corollary 6.2]. By the stability condition, we observe that the image of $R \to G$ is either a rank 2 vector bundle of degree 3, or a rank 3 vector bundle of degree 4, 5, 6. We show by cases that these conditions are not generic.

1. Suppose that the image of $R \to G$ is a rank 2 vector bundle of degree 3. There are two short exact sequences

$$0 \to G'' \to R \to G' \to 0$$

and

$$0 \to G' \to G \to L \to 0$$

where $G'$ is the image of $R$, $G''$ is a rank 2 vector bundle of degree 1. Note that both $G'$ and $G''$ are stable. Also, $L$ is locally free: indeed, if $\tilde{G}'$ is the kernel of the morphism $G \to L/\text{Tors} L$, then the stability argument forces that $G' = \tilde{G}'$, hence $L = L/\text{Tors} L$ showing that $L$ is locally free. Since $h^0(C, G') > 0$, a nonzero section $s \in H^0(C, G')$ defines the following exact sequence

$$0 \to \mathcal{O}_C(D) \overset{s}{\to} G' \to M \to 0$$

where $D$ is the zero locus $V(s)$ of $s$ and $M = \det G' \otimes \mathcal{O}_C(-D)$ is a line bundle. By the stability, we have either $\deg D = 0$ or 1. Tensoring by $G''^*$, we have

$$0 \to G''^*(D) \to G' \otimes G''^* \to G''^* \otimes M \to 0.$$ 

When $\deg D = 0$, that is, $D = 0$, the stability of $G''$ assures that

$$\dim \text{Hom}_C(G'', G') \leq h^0(C, G''^*) + h^0(C, G''^* \otimes M)$$

$$= 0 + 3 = 3.$$ 

When $\deg D = 1$,

$$\dim \text{Hom}_C(G'', G') \leq h^0(C, G''^*(D)) + h^0(C, G''^* \otimes M)$$

$$= 1 + 2 = 3.$$
since both the Brill-Noether loci $W_{2,1}^1(C)$ and $W_{2,3}^2(C)$ are empty by Theorem 4.3. In any cases, we observe that the Quot scheme $[R \to G'] \in Quot_{2,3}(R)$ has the local dimension at most 3 for any stable quotient $G' \in U_C^2(2, 3)$. The locus of vector bundles $G \in U_C^2(3, 6)$ which is an extension of $L$ by $G'$ has the dimension at most

$$\dim \{G\} \leq \dim Quot_{2,3}(R) + \dim \text{Pic}^3(C) + \dim P \text{Ext}^1_C(L, G)$$

$$\leq 3 + 2 + 4 = 9$$

$$< 10 = \dim U_C^2(3, 6).$$

(2) Suppose that the image of $R$ is a rank 3 vector bundle of degree 4. We have two short exact sequences

$$0 \to L \to R \to G' \to 0$$

and

$$0 \to G' \to G \to T \to 0$$

where $L$ is a line bundle of degree 0, $G'$ is the image of $R$, and $T$ is a torsion sheaf of length 2. Since $\dim \text{Hom}_C(L, R) = 1$ (cf. the proof of [23 Lemma 5.9]), the dimension of the family of stable vector bundles $G' \in U_C(3, 4)$ which fit into the first short exact sequence is at most $\dim \text{Pic}^0(C) = 2$. Hence the dimension of the family of stable vector bundles $G$ which fit into the second short exact sequence is at most $\dim \{T\} + \dim \{G'\} + \dim P \text{Ext}^1_C(T, G') = 9$.

(3) Suppose that the image of $R$ is a rank 3 vector bundle of degree 5. We have two short exact sequences

$$0 \to L \to R \to G' \to 0$$

and

$$0 \to G' \to G \to T \to 0$$

where $L$ is a line bundle of degree $-1$, $G'$ is the image of $R$, and $T$ is a torsion sheaf of length 1. Since $R$ is stable, we have $\dim \text{Ext}^1_C(L, R) = \dim \text{Hom}_C(R, L \otimes \omega_C) = 0$. By Riemann-Roch, we have $\dim \text{Hom}_C(L, R) = 4$, and thus the dimension of the family of stable vector bundles $G' \in U_C(3, 5)$ which fit into the first short exact sequence is at most $\dim \text{Pic}^{-1}(C) + \dim P \text{Hom}_C(L, R) = 5$. Therefore, the dimension of the family of stable vector bundles $G$ which fit into the second short exact sequence is at most $\dim \{T\} + \dim \{G'\} + \dim P \text{Ext}^1_C(T, G') = 8$.

(4) Suppose that the image of $R$ is a rank 3 vector bundle of degree 6, in other words, it coincides with $G$. We have the following short exact sequence

$$0 \to L \to R \to G \to 0$$

where $L$ is a line bundle of degree $-2$. By the stability and Riemann-Roch formula, we have $\dim \text{Hom}_C(L, R) = \chi(L, R) = 8$. Hence the dimension of the family of stable vector bundles $G$ which fits into the above exact sequence is at most $\dim \text{Pic}^{-2}(C) + \dim P \text{Hom}_C(L, R) = 9$.

To sum up, we conclude that a generic stable vector bundle $G \in U_C(3, 6)$ yields $\text{Hom}_C(R, G) = 0$. □

**Corollary 4.12.** For each $r \geq 2$, a generic stable vector bundle $G \in U_C(r, 2r)$ satisfies

$$\text{Ext}^p_C(R, G) = 0, \ p = 0, 1.$$
Proof. Assume that $G_i \in \mathcal{U}_C(r_i, 2r_i)$ ($i = 1, 2$) are stable vector bundles satisfying $\text{Ext}^p_C(\mathcal{R}, G_i) = 0$. Then $G_3 := G_1 \oplus G_2$ is a semistable vector bundle satisfying $\text{Ext}^p_C(\mathcal{R}, G_3) = 0$. By the semicontinuity, we see that $\text{Ext}^p_C(\mathcal{R}, G) = 0$ for a general $G \in \mathcal{U}_C(r_1 + r_2, 2(r_1 + r_2))$. By [7, Proposition 9], Proposition 4.8 and Proposition 4.11, there are vector bundles $G_1 \in \mathcal{U}_C(2, 4)$ and $G_2 \in \mathcal{U}_C(3, 6)$ such that $\text{Ext}^p_C(\mathcal{R}, G_i) = 0$. Since direct sums of $G_1$ and $G_2$ can produce all the ranks $\geq 4$, we get the desired result.

Recall that the projection image $F = \Phi_S^1(\mathcal{E}^*(1))$ is always a semistable vector bundle. It is easy to see that both the stability and the strict semistability are preserved by this Fourier-Mukai projection.

**Proposition 4.13.** Let $\mathcal{E}$ be an Ulrich vector bundle of rank $r \geq 2$, and let $F := \Phi_S^1(\mathcal{E}^*(1))$ be a semistable vector bundle on $C$. If $\mathcal{E}$ is stable (resp. strictly semistable), then so is $F$.

**Proof.** First assume that $\mathcal{E}$ is strictly semistable. There is a destabilizing sequence

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$$

where $\mathcal{E}'$ and $\mathcal{E}''$ are Ulrich bundle of smaller ranks by Proposition 2.5. This gives the following short exact sequence

$$0 \to F'' := \Phi_S^1(\mathcal{E}''^*(1)) \to F = \Phi_S^1(\mathcal{E}^*(1)) \to F' := \Phi_S^1(\mathcal{E}'^*(1)) \to 0.$$ 

Since $\mathcal{E}''$ is Ulrich, we see that $F'' \subset F$ is a vector bundle of slope 2 on $C$, so $F$ cannot be stable. Now assume that $\mathcal{E}$ is stable, but $F$ is strictly semistable. Consider the destabilizing sequence

$$0 \to F'' \to F \to F' \to 0.$$ 

Since $F$ comes from an Ulrich bundle, the conditions in Proposition 4.8 ensures that $h^1(F' \otimes \mathcal{S}_C) = 0$ and $\text{Ext}^p_C(\mathcal{R}, F'^* \otimes \omega_C^2) = 0$. It follows that $\mathcal{E}'$ is Ulrich where $\mathcal{E}'^*(1) := \Phi_S^1(F')$. The existence of the nonzero map $\mathcal{E}'^*(1) \to \mathcal{E}'^*(1)$ leads to a contradiction; indeed, $\mathcal{E}'^*(1)$ is stable of $\mu = 0$ and $\mathcal{E}'^*(1)$ is semistable of $\mu = 0$, thus there is no nonzero map from $\mathcal{E}'^*(1)$ to $\mathcal{E}'^*(1)$. □

To sum up the above discussions, we have the following theorem.

**Theorem 4.14.** Let $\mathcal{M}(r)$ ($r \geq 2$) be the moduli space of $S$-equivalence classes of Ulrich bundles of rank $r$ over $X$. The projection functor $\Phi_S^1 : \text{D}^b(X) \to \text{D}^b(C)$ induces the morphism

$$\varphi : \mathcal{M}(r) \to \mathcal{U}_C(r, 2r), \ [\mathcal{E}] \mapsto \varphi(\mathcal{E}) := [\Phi_S^1(\mathcal{E}^*(1))]$$

of moduli spaces. Moreover, $\varphi$ satisfies the following properties:

1. set-theoretically, $\varphi$ is an injection;
2. $\varphi$ maps stable (resp. semistable) objects to stable (resp. semistable) objects;
3. let $\mathcal{M}'(r)$ be the stable locus. Then $\varphi$ induces an isomorphism of $\mathcal{M}'(r)$ onto

$$\varphi(\mathcal{M}'(r)) = \left\{ [F] \in \mathcal{U}_C^r(r, 2r) : \begin{array}{l} \text{Ext}^p_C(\mathcal{R}, F^* \otimes \omega_C^2) = 0, \ p = 0, 1, \\ h^1(F \otimes \mathcal{S}_x) = 0 \text{ for each } x \in X. \end{array} \right\},$$

which is a nonempty open subscheme of $\mathcal{U}_C(r, 2r)$.
Proposition 4.16. Let $\mathcal{E}$ be a stable Ulrich bundle of rank $r$ over $X$. For a generic line $\ell \subset X$, $\mathcal{E}|_{\ell} \simeq \mathcal{O}_X(1)^{\oplus r}$.  

Proof. First of all, to be well defined, $\varphi$ has to preserve $S$-equivalence classes. Assume that $\mathcal{E}_1$ and $\mathcal{E}_2$ are Ulrich bundles which are $S$-equivalent, i.e. there are Jordan-Hölder filtrations

$$0 = \mathcal{E}^{(0)}_i \subset \mathcal{E}^{(1)}_i \subset \ldots \subset \mathcal{E}^{(m)}_i = \mathcal{E}^*_i(1)$$

such that $\mathcal{E}^{(j)}_i/\mathcal{E}^{(j-1)}_i := \text{gr}_j(\mathcal{E}^*_i(1)) \simeq \text{gr}_j(\mathcal{E}^*_2(1)) := \mathcal{E}^{(j)}_2/\mathcal{E}^{(j-1)}_2$. For each $j$,

$$0 \to \mathcal{E}^{(j-1)}_i \to \mathcal{E}^{(j)}_i \to \text{gr}_j(\mathcal{E}^*_i(1)) \to 0$$

is a short exact sequence of Ulrich bundles by Proposition 2.5. The map $\varphi$ preserves both the stability and the strict semistability by Proposition 4.13, so it immediately follows that

$$0 = \Phi^r_\mathcal{S}(\mathcal{E}^{(0)}_i) \subset \Phi^r_\mathcal{S}(\mathcal{E}^{(1)}_i) \subset \ldots \subset \Phi^r_\mathcal{S}(\mathcal{E}^{(m)}_i) = \varphi(\mathcal{E}_i)$$

is a Jordan-Hölder filtration with $\text{gr}_j(\varphi(\mathcal{E}_i)) \simeq \Phi^r_\mathcal{S}(\text{gr}_j(\mathcal{E}^*_i(1)))$. This shows that $\varphi(\mathcal{E}_1)$ and $\varphi(\mathcal{E}_2)$ are $S$-equivalent.

The statement (1) follows from the fact that $\Phi^r_\mathcal{S} : \mathcal{D}^b(C) \to \Phi^r_\mathcal{S}(\mathcal{D}^b(C))$ is an equivalence of categories, and $\mathcal{E}^*_1(1) \in \Phi^r_\mathcal{S}(\mathcal{D}^b(C))$ for each Ulrich bundle $\mathcal{E}$ over $X$. The statement (2) is already proved in Proposition 4.13 so it only remains to prove (3). For any stable Ulrich bundle $[\mathcal{E}] \in \mathcal{M}^r(r)$, the functor $\Phi^r_\mathcal{S}$ induces

$$T_{[\mathcal{E}]} \mathcal{M}^r(r) \simeq \text{Ext}^1_X(\mathcal{E}, \mathcal{E})$$

$$\simeq \text{Ext}^1_X(\varphi(\mathcal{E}), \varphi(\mathcal{E}))$$

$$\simeq T_{[\varphi(\mathcal{E})]} \mathcal{U}_C^r(r, 2r).$$

Hence together with (1), $\varphi$ is an isomorphism near $[\mathcal{E}]$. Finally, by Proposition 4.13 and Corollary 4.12, $\varphi(\mathcal{M}^r(r))$ is open and nonempty.  

Remark 4.15. It is not true in general that $\varphi(\mathcal{M}^r(r)) = \mathcal{U}_C^r(r, 2r)$. For example, choose a point $P \in C$ and consider a stable bundle $F := R^s \otimes \mathcal{O}_C(-P) \otimes \omega_C^{\otimes 2}$ of rank 4 and degree 8. Then $F^* \otimes \omega_C^{\otimes 2} = \mathcal{R} \otimes \mathcal{O}_C(P)$, hence we see that

$$\text{Hom}_C(\mathcal{R}, F^* \otimes \omega_C^{\otimes 2}) \neq 0.$$ 

This shows that $\varphi(\mathcal{M}^r(4))$ is a proper subset of $\mathcal{U}_C^r(4, 8)$. The relation between jumping lines and instanton bundles has been studied in [23]. For stable Ulrich bundles, we show that a generic line is not jumping. Recall that $\ell \subset X$ is a jumping line for $\mathcal{E}$ if the direct sum decomposition of $\mathcal{E}|_{\ell}$ contains at least two non-isomorphic direct summands.

Proposition 4.16. Let $\mathcal{E}$ be a stable Ulrich bundle of rank $r$ over $X$. For a generic line $\ell \subset X$, $\mathcal{E}|_{\ell} \simeq \mathcal{O}_X(1)^{\oplus r}$.  

Proof. We may assume that $\xi = \mathcal{O}_C(P)$ for a point $P \in C$. Indeed, if we choose a suitable $L \in \text{Pic}^0(C)$ and make a replacement $S' := S \otimes \mathcal{O}(L)$, then all the arguments in this section are still valid for the new Fourier-Mukai transform $\Phi_{S'} : \mathcal{D}^b(C) \to \mathcal{D}^b(X)$ and its right adjoint $\Phi^r_{S'}$. In particular, the Raynaud bundle $\mathcal{R}'$ obtained from $\Phi^r_{S'} \mathcal{O}_X(-2)$ as in Lemma 4.10 satisfies $\mathcal{R}' = \mathcal{R} \otimes L$.  

□
Let $F := \Phi_S^1(\mathcal{E}^*(1))$ and $G := F^* \otimes \omega_C^\otimes 2$. We have $G \otimes \xi^* \neq \mathcal{R}$; otherwise
\[0 = \text{Hom}_C(\mathcal{R}, G) = \text{Hom}_C(\mathcal{R}, \mathcal{R} \otimes \mathcal{O}_C(P)) \neq 0\]
gives a contradiction. Since $G$ is stable and $G \otimes \xi^* \neq \mathcal{R}$, we have $\text{Hom}_C(\mathcal{R}, G \otimes \xi^*) = 0$. By \cite{19} Lemma 2.4 and Theorem 2.5], $H^0(C, L \otimes G \otimes \xi^*) = 0$ for a general $L \in \text{Pic}^0(C)$. On the other hand,
\[
H^p(C, L \otimes G \otimes \xi^*) = \text{Ext}^{1-p}_C(G, L^* \otimes \xi \otimes \omega_C)^*
\]
\[
= \text{Ext}^{1-p}_C(L \otimes \xi^* \otimes \omega_C, F)^*
\]
\[
= \text{Ext}^{1-p}_X(\Phi_S(L \otimes \xi^* \otimes \omega_C), \mathcal{E}^*(1))^*.
\]
We have $\Phi_S(L \otimes \xi^* \otimes \omega_C) = \mathcal{I}_\ell(1)[-1]$ for a line $\ell \subset X$ and its ideal sheaf $\mathcal{I}_\ell$ (cf. \cite{23} Lemma 5.5]). This establishes a bijection between $\text{Pic}^1(C)$ and the Fano variety $F(X)$ of lines in $X$. Thus,
\[
H^p(C, L \otimes G \otimes \xi^*) \simeq \text{Ext}^{2-p}_X(\mathcal{I}_\ell, \mathcal{E}^* )^* \simeq H^{p+1}(\mathcal{E} \otimes \mathcal{I}_\ell(-2)).
\]
In the short exact sequence $0 \rightarrow \mathcal{E} \otimes \mathcal{I}_\ell(-2) \rightarrow \mathcal{E}(-2) \rightarrow \mathcal{E}(-2) \otimes \mathcal{O}_\ell \rightarrow 0$, we easily find that $H^{p+1}(\mathcal{E} \otimes \mathcal{I}_\ell(-2)) \simeq H^p(\mathcal{E}(-2) \otimes \mathcal{O}_\ell)$. In particular, $h^p(\mathcal{E}(-2) \otimes \mathcal{O}_\ell) = 0$ which implies $\mathcal{E}|_\ell \simeq \mathcal{O}_X(1)^{\oplus r}$ for a general $\ell \in F(X)$.

We finish this paper by some important remarks.

Remark 4.17 (Arrondo–Costa revisited).

1. Arrondo–Costa’s classification (Theorem 5.5) also can be interpreted via derived categories of coherent sheaves on $X$. The moduli space of rank 2 ACM bundles of line type is isomorphic to the abelian surface $J(C)$, and the interpretation in terms of categorical language is explained in \cite{23} Lemma 5.5]. The moduli space of rank 2 ACM bundles of conic type is isomorphic to $C$ and this can be explained by the result of \cite{8] because the image of a conic type ACM bundle along the projection functor is a skyscraper sheaf. Finally, rank 2 ACM bundles of elliptic curve type are Ulrich, hence $\mathcal{E} \mapsto \Phi_S^1(\mathcal{E}^*(1))$ shows that the moduli space of ACM bundles of elliptic curve type is isomorphic to an open subset of $U_C(2, 4)$.

2. We observed above that the rank 3 vector bundle $\mathcal{E}$ constructed in \cite{4] Example 4.4] is not Ulrich. Indeed, two global sections of $\omega_D(-1)$ has a nontrivial linear relation, that is,
\[
h^1(\mathcal{E}^*(1)) \simeq \ker[H^0(\omega_D(-1)) \otimes H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)) \rightarrow H^0(\omega_D)] \simeq \mathbb{C}^1.
\]
Hence $h^2(\mathcal{E}(-3)) = h^3(\mathcal{E}(-3)) = 1$. Nevertheless, it is still a very interesting vector bundle as in the following sense. Since $\mathcal{E}(-1)$ and $\mathcal{E}(-2)$ have no cohomology, we see that $\mathcal{E}^*(1)$ is a semistable vector bundle of rank 3 contained in $\Phi_S D^3(C)$. Indeed, the nonzero section of $H^0(\mathcal{E}^*(1)) \simeq H^3(\mathcal{E}(-3))^*$ induces a short exact sequence
\[
0 \rightarrow \mathcal{E} \rightarrow \mathcal{E}(-1) \rightarrow \mathcal{O}_X \rightarrow 0,
\]
where $\mathcal{E}$ is a rank 2 vector bundle so called an “instanton bundle” of charge 3 (see \cite{15] and \cite{23} Definition 1.1 and Theorem 3.10]). Note that rank 2 Ulrich bundles are instanton bundles of charge 2, which are minimal. Arrondo–Costa construction shows the existence of a non-minimal instanton bundle.
Remark 4.18. The second Raynaud bundle \( \mathcal{R} \) has an interesting property. Note that a (semi-)stable vector bundle \( F \) of rank \( r \) and slope \( g - 1 (= 1) \) on \( C \) defines the theta locus

\[
\Theta_F := \{ L \in \text{Pic}^0(C) \mid h^0(C, F \otimes L) \neq 0 \},
\]

which is a natural generalization of the theta divisor. The locus is either a divisor linearly equivalent to \( r\Theta \) where \( \Theta \subset \text{Pic}^0(C) \) is the usual theta divisor, or the whole Picard group \( \text{Pic}^0(C) \). Indeed, the theta map

\[
\theta : \text{SU}_C(r, \det F) \dashrightarrow |r\Theta|
\]
gives a rational map, which is a morphism when \( r \leq 3 \). However, when \( r = 4 \), \( \mathcal{R} \) does not have a theta divisor since \( h^0(\mathcal{R} \otimes L) = 1 \) for every \( L \in \text{Pic}^0(C) \) as treated above (see also [23, Lemma 5.9]). We refer interested readers to [19, 31, 32] for more details on generalized theta divisors and \( \mathcal{R} \).

The strange duality provides a following geometric interpretation in terms of generalized theta divisors. Denote \( L \) by the ample generator of \( \text{Pic} \text{SU}_C(4, \det \mathcal{R}) \), we see that \( \mathcal{R} \) is a base point of \( |L^k| \) if and only if

\[
H^0(C, \mathcal{R} \otimes G) \neq 0, \quad \text{for all } G \in \mathcal{U}_C(k, 0).
\]

By Serre duality, the above condition is equivalent to

\[
\text{Hom}_C(\mathcal{R}, G^* \otimes \omega_C) = \text{Ext}_C^1(\mathcal{R}, G^* \otimes \omega_C) \neq 0.
\]

Note that \( G^* \otimes \omega_C \) is a vector bundle of rank \( k \) and degree \( 2k \).

Corollary 4.12 actually implies that \( \mathcal{R} \) is not a base point of \( |L^k| \) for \( k \geq 2 \). Since Proposition 4.11 holds not only for \( \mathcal{R} \) but for any stable rank 4 vector bundle of degree 4, we conclude that

1. \( \mathcal{R} \notin Bs|L^2| \), i.e., \( Bs|L^2| \) is a proper subset of \( Bs|L| = \{ \text{the set of 16 Raynaud type bundles on } C \} \) which correspond to 16 theta characteristics of \( C \);

2. The linear system \( |L^k| \) is base-point-free for \( k = 3 \).

Since \( |L^k| \) is base-point-free for \( k \geq 4 \) [33, Theorem 8.1], the above statement answers to the question by Popa and Roth for \( g = 2 \) and \( r = 4 \) (cf. [33, Section 8]).

Even though our argument do not assure that a generic vector bundle \( F \in \mathcal{U}_C(2, 4) \) is orthogonal to all the 16 Raynaud type bundles, however, it sounds very promising that \( |L^2| \) is also base-point-free.

Remark 4.19. The strategy in Proposition 4.8 is also useful to classify Ulrich bundles for smooth complete intersection varieties of two even dimensional quadrics of higher dimensions. In higher dimensional cases, we also observe that every Ulrich bundle is a image of Fourier-Mukai transform of a semistable vector bundle on the associated hyperelliptic curve from Bondal-Orlov’s semiorthogonal decomposition. Moreover, the moduli space of stable Ulrich bundles is a smooth Zariski open subset of the moduli space of stable vector bundles on the associated hyperelliptic curve. However, showing the existence becomes more complicated for higher dimensional cases. For instance, there is no Ulrich bundle of rank 2 on such an \( n \)-dimensional del Pezzo variety of degree 4 when \( n \geq 5 \) [12, Theorem 6.3]. By the way, we know the existence of Ulrich bundles of certain rank in these cases from [10]. Therefore, it is interesting to compute all the possible ranks of Ulrich bundles on the higher dimensional smooth complete intersection varieties of two even dimensional quadrics. For example, [33, Theorem 8.1] enables us to make a wild expectation that an Ulrich bundle of rank \( 2^{2g-2} \) might exist.
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