Generalizations of The Chung-Feller Theorem II

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Abstract

The classical Chung-Feller theorem [2] tells us that the number of Dyck paths of length \( n \) with \( m \) flaws is the \( n \)-th Catalan number and independent on \( m \). L. Shapiro [9] found the Chung-Feller properties for the Motzkin paths. Mohanty’s book [5] devotes an entire section to exploring Chung-Feller theorem. Many Chung-Feller theorems are consequences of the results in [5]. In this paper, we consider the \((n, m)\)-lattice paths. We study two parameters for an \((n, m)\)-lattice path: the non-positive length and the rightmost minimum length. We obtain the Chung-Feller theorems of the \((n, m)\)-lattice path on these two parameters by bijection methods. We are more interested in the pointed \((n, m)\)-lattice paths. We investigate two parameters for an pointed \((n, m)\)-lattice path: the pointed non-positive length and the pointed rightmost minimum length. We generalize the results in [5]. Using the main results in this paper, we may find the Chung-Feller theorems of many different lattice paths.

Keywords: Chung-Feller Theorem; Dyck path; Motzkin path

1 Introduction

Let \( Z \) denote the set of the integers and \([n] := \{1, 2, \ldots, n\}\). We consider \( n \)-Dyck paths in the plane \( Z \times Z \) using \textit{up} \((1, 1)\) and \textit{down} \((1, -1)\) steps that go from the origin to the point \((2n, 0)\). We say \( n \) the \textit{semilength} because there are \(2n\) steps. An \textit{n-flawed} path is an \( n \)-Dyck path that contains some steps under the \( x \)-axis. The number of \( n \)-Dyck path that never pass below the \( x \)-axis is the \( n \)-th Catalan number \( c_n = \frac{1}{n+1}\binom{2n}{n} \). Such paths are called the \textit{Catalan paths of length} \( n \). A Dyck path is called a \textit{(n, r)-flawed} path if it contains \( r \) up steps under the \( x \)-axis and its semilength is \( n \). Clearly, \( 0 \leq r \leq n \). The classical Chung-Feller theorem [2] says that the number of the \((n, r)\)-flawed paths is equal to \( c_n \) and independent on \( r \).

The classical Chung-Feller Theorem were proved by MacMahon [7]. Chung and Feller reproved this theorem by using analytic method in [2]. T.V.Narayana [8] showed the Chung-Feller Theorem

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by combinatorial methods. S. P. Eu et al. [3] proved the Chung-Feller Theorem by using the Taylor expansions of generating functions and gave a refinement of this theorem. In [4], they gave a strengthening of the Chung-Feller Theorem and a weighted version for Schröder paths. Y.M. Chen [1] revisited the Chung-Feller Theorem by establishing a bijection.

Mohanty’s book [5] devotes an entire section to exploring Chung-Feller theorem. We state the result from [5] as the following lemma.

**Lemma 1.1** [5] Given a positive integer \( n \), let \( Y = (y_1, \ldots, y_{n+1}) \) be a sequence of integers with \( 1 - n \leq y_i \leq 1 \) for all \( i \in [n+1] \) such that \( \sum_{i=1}^{n+1} y_i = 1 \). Furthermore, let \( E(Y) = |\{ i \mid \sum_{j=1}^{i} y_j \leq 0 \}| \).

Let \( Y_i \) be the \( i \)-th cyclic permutation of \( Y \) (i.e., \( Y_i = (y_i, y_{i+1}, \ldots, y_{n+i+1}) \) with \( y_{n+r+1} = y_r \)). Then there exists a permutation \( i_1, \ldots, i_{n+1} \) on the set \([n+1]\) such that \( E(Y_{i_1}) > E(Y_{i_2}) > \cdots > E(Y_{i_{n+1}}) \).

Many Chung-Feller theorems are consequences of lemma 1.1. First, let \( \phi \) be a mapping from \( \mathbb{Z} \) to \( \mathbb{P} \), where \( \mathbb{P} \) is a set of all the positive integers. Let the sequence \( Y = (y_1, \ldots, y_{n+1}) \) satisfy the conditions in Lemma 1.1. Using \( (\phi(y_i), y_i) \) steps, we can obtain a lattice path \( P(Y) = (\phi(y_1), y_1)(\phi(y_2), y_2) \cdots (\phi(y_{n+1}), y_{n+1}) \) in the plane \( \mathbb{Z} \times \mathbb{Z} \) that go from the origin to the point \( \sum_{i=1}^{n+1}(\phi(y_i), 1) \). Using Lemma 1.1, we will derive the classical Chung-Feller theorem for Dyck paths if we let \( y_i \in \{1, -1\} \) and set \( \phi(y) = 1 \) for all \( y \in \mathbb{Z} \); we will derive the Chung-Feller theorem for Schröder paths if we let \( y_i \in \{1, 0, -1\} \) and set \( \phi(0) = 2 \) and \( \phi(y) = 1 \) for \( y \neq 0 \); we will derive the Chung-Feller theorem for Motzkin paths if we let \( y_i \in \{1, 0, -1\} \) and set \( \phi(0) = 1 \) and \( \phi(y) = 1 \) for \( y \neq 0 \) and so on.

How to derive the Chung-Feller theorem for lattice paths in the plane \( \mathbb{Z} \times \mathbb{Z} \) using \((1, -1), (1, 1), (1, 0), (2, 0)\) steps? For answering this problem, the authors of this paper [6] proved the Chung-Feller theorems for three classes of lattice paths by using the method of the generating functions. It is interesting that these Chung-Feller theorems can’t be derivable as a special case from lemma 1.1. This implies that we may generalize the results of Lemma 1.1.

In this paper, first we give the definition of the \((n, m)\)-lattice paths. We consider two parameters for an \((n, m)\)-lattice path: the non-positive length and the rightmost minimum length. Using bijection methods, we obtain the Chung-Feller theorems of the \((n, m)\)-lattice path on these two parameters. Then we study the pointed \((n, m)\)-lattice paths. We investigate two parameters for an pointed \((n, m)\)-lattice path: the pointed non-positive length and the pointed rightmost minimum length. We give generalizations of the results in [5] and prove the Chung-Feller theorems of the pointed \((n, m)\)-lattice path on these two parameters. Finally, using the main theorems of this paper, we may find the Chung-Feller theorems of many different \((n, m)\)-lattice paths.

This paper is organized as follows. In Section 2, we focus on the \((n, m)\)-lattice paths. Using bijection methods, we obtain the Chung-Feller theorems of the \((n, m)\)-lattice path. In Section 3,
we study the pointed \((n, m)\)-lattice paths and give generalizations of the results in [5]. In Section 4, using the main theorems of this paper, we find the Chung-Feller theorems of many different \((n, m)\)-lattice paths.

## 2 The \((n, m)\)-lattice paths

Throughout the paper, we always let \(n\) and \(m\) be two positive integers with \(m \geq n + 1\). In this section, we will consider the \((n, m)\)-lattice paths. We will define two parameters for an \((n, m)\)-lattice path: the non-positive length and the rightmost minimum length. Using bijection methods, we will obtain the Chung-Feller theorems of the \((n, m)\)-lattice path on these two parameters. First, we give the definition of the \((n, m)\)-lattice paths as follows.

**Definition 2.1** An \((n, m)\)-lattice paths \(P\) is a sequence of the vectors \((x_1, y_1)(x_2, y_2)\ldots(x_{n+1}, y_{n+1})\) in \(\mathbb{Z}^2\) such that:

1. \(1 - n \leq y_i \leq 1\) and \(\sum_{i=1}^{n+1} y_i = 1\)
2. \(1 \leq x_i \leq m - 1\) and \(\sum_{i=1}^{n+1} x_i = m\).

\((x_i, y_i)\) is called the steps of \(P\) for any \(i \in [n + 1]\). Since \(P\) can be viewed as a path from the origin to \((m, 1)\) in the plane \(\mathbb{Z} \times \mathbb{Z}\) and has \(n + 1\) steps, we say that \(P\) is of order \(n + 1\) and length \(m\).

### 2.1 The non-positive length of an \((n, m)\)-lattice paths

Given an \((n, m)\)-lattice path \(P = (x_1, y_1)(x_2, y_2)\ldots(x_{n+1}, y_{n+1})\), we let \(NP(P) = \{i \mid \sum_{j=1}^{i} y_j \leq 0\}\) and \(NPL(P) = \sum_{i \in NP(P)} x_i\). Clearly, \(0 \leq NPL(P) \leq m - x_{n+1} \leq m - 1\) since \(n + 1 \neq NP(P)\).

We say that \(NPL(P)\) is the non-positive length of the \((n, m)\)-lattice path \(P\). Moreover, we define a linear order \(<_P\) on the set \([n + 1]\) by the following rules:

- for any \(i, j \in [n + 1]\), \(i <_P j\) if either (1) \(\sum_{k=1}^{i} y_k < \sum_{k=1}^{j} y_k\) or (2) \(\sum_{k=1}^{i} y_k = \sum_{k=1}^{j} y_k\) and \(i > j\).

The sequence formed by writing \([n + 1]\) in the increasing order with respect to \(<_P\) is denoted by \(\pi_P = (\pi_P(1), \pi_P(2), \ldots, \pi_P(n + 1))\).

**Example 2.2** Let \(n = 8\) and \(m = 11\). We draw an \((8, 11)\)-lattice path

\[P = (1, 1)(1, -2)(2, 1)(1, 1)(1, -1)(1, -1)(1, 1)(1, 1)(2, 0)\]

as follows.
Then $NP(P) = \{2, 3, 5, 6, 7\}$, $NPL(P) = 6$ and $\pi_P = (6, 2, 7, 5, 3, 9, 8, 4, 1)$.

We use $\mathcal{L}_{n,m,r}$ to denote the set of all the $(n,m)$-lattice paths $P$ such that $NPL(P) = r$. In particular, we use $\tilde{\mathcal{L}}_{n,m,0}$ to denote the set of all the lattice paths $P = (x_1, y_1)(x_2, y_2)\ldots(x_{n+1}, y_{n+1})$ in the set $\mathcal{L}_{n,m,0}$ such that $x_{n+1} = 1$. Clearly, $\tilde{\mathcal{L}}_{n,m,0} \subset \mathcal{L}_{n,m,0}$.

**Lemma 2.3**

1. The number of the $(n,m)$-lattice paths $P$ such that $NP(P) = 0$ is equal to $\binom{m-1}{n}c_n$;
2. The number of the $(n,m)$-lattice paths $P = (x_1, y_1)(x_2, y_2)\ldots(x_{n+1}, y_{n+1})$ such that $NPL(P) = 0$ and $x_{n+1} = 1$ is equal to $\binom{m-2}{n-1}c_n$.

**Proof.** (1) It is well known that the number of the solutions of the equation $\sum_{i=1}^{n+1} y_i = 1$ such that $1 - n \leq y_i \leq 1$ and $NP(P) = \emptyset$ is $c_n$ and the number of the solutions of the equation $\sum_{i=1}^{n+1} x_i = m$ in positive integers is $\binom{m-1}{n}$. Hence, The number of the $(n,m)$-lattice paths $P$ such that $NPL(P) = 0$ is equal to $\binom{m-1}{n}c_n$.

(2) Note that the number of the solutions of the equation $\sum_{i=1}^{n} x_i = m - 1$ in positive integers is $\binom{m-2}{n-1}$. We immediately obtain that the number of the $(n,m)$-lattice paths $P$ such that $NPL(P) = 0$ and $x_{n+1} = 1$ is equal to $\binom{m-2}{n-1}c_n$. 

**Lemma 2.4** There is a bijection $\Phi$ from $\mathcal{L}_{n,m,r}$ to $\mathcal{L}_{n,m,r+1}$ for any $1 \leq r \leq m - 2$.

**Proof.** Let $P = (x_1, y_1)(x_2, y_2)\ldots(x_{n+1}, y_{n+1}) \in \mathcal{L}_{n,m,r}$. Consider the sequence $\pi_P$. Suppose $\pi_P(k) = n + 1$ for some $k$. Since $r \geq 1$, we have $k \geq 2$. We discuss the following two cases.

**Case I.** $k \leq n$

If $x_{n+1} = 1$, then let $i = \pi_P(k + 1)$ and

$$\Phi(P) = (x_{i+1}, y_{i+1})\ldots(x_{n+1}, y_{n+1})(x_1, y_1)\ldots(x_i, y_i).$$

If $x_{n+1} \geq 2$, then let $i = \pi_P(k - 1)$ and

$$\Phi(P) = (x_1, y_1)\ldots(x_i + 1, y_i)\ldots(x_{n+1} - 1, y_{n+1}).$$

**Case II.** $k = n + 1$
Note that \( x_{n+1} \geq 2 \) since \( r \leq m - 2 \). We let \( i = \pi_p(n) \) and

\[
\Phi(P) = (x_1, y_1) \ldots (x_i + 1, y_i) \ldots (x_{n+1} - 1, y_{n+1}).
\]

It is easy to see that \( \Phi(P) \in L_{n,m} \) for Cases I and II.

For proving that \( \Phi \) is a bijection, we describe the inverse of \( \Phi \) as follows.

Let \( P = (x_1, y_1)(x_2, y_2) \ldots (x_{n+1}, y_{n+1}) \in L_{n,m} \), where \( 1 \leq r \leq m - 2 \). Suppose \( \pi_p(k) = n+1 \) for some \( k \). Let \( i = \pi_p(k-1) \). If \( x_i = 1 \), then let

\[
\Phi^{-1}(P) = (x_{i+1}, y_{i+1}) \ldots (x_{n+2}, y_{n+2}) (x_1, y_1) \ldots (x_i, y_i);
\]

otherwise, let

\[
\Phi^{-1}(P) = (x_1, y_1) \ldots (x_i - 1, y_i) \ldots (x_{n+1} + 1, y_{n+1}).
\]

This complete the proof.

**Example 2.5** Let \( n = 3 \) and \( m = 5 \). We draw \((3,5)\)-lattice paths

\[
P_1 = (1,1)(1,1)(1,-2)(2,1) \quad P_2 = (1,1)(1,1)(2,-2)(1,1)
\]

\[
P_3 = (1,1)(2,-2)(1,1)(1,1) \quad P_4 = (2,-2)(1,1)(1,1)(1,1)
\]

as follows.

![Lattice Paths](image)

We have \( \Phi(P_i) = P_{i+1} \) and \( NPL(P_i) = i \).

**Lemma 2.6** There is a bijection from \( \tilde{L}_{n,m,0} \) to \( L_{n,m,1} \).

**Proof.** Let \( P = (x_1, y_1)(x_2, y_2) \ldots (x_{n+1}, y_{n+1}) \in \tilde{L}_{n,m,0} \). Consider the sequence \( \pi_p \). Note that \( \pi_p(1) = n+1 \) for any \( P \in \tilde{L}_{n,m,0} \). So, let \( i = \pi_p(2) \). Let the mapping \( \Phi \) be defined as that in Lemma 2.4, i.e., \( \Phi(P) = (x_{i+1}, y_{i+1}) \ldots (x_{n+1}, y_{n+1})(x_1, y_1) \ldots (x_i, y_i) \). Then \( \Phi(P) \in L_{n,m,1} \). Conversely, for any \( P = (x_1, y_1)(x_2, y_2) \ldots (x_{n+1}, y_{n+1}) \in L_{n,m,1} \), we have \( \pi_p(2) = n+1 \). Suppose \( \pi_p(1) = i \), then \( x_i = 1 \). This tells us that \( \Phi \) is a bijection from \( \tilde{L}_{n,m,0} \) to \( L_{n,m,1} \).

**Theorem 2.7** For any \( 1 \leq r \leq m - 1 \), the number of the \((n,m)\)-lattice paths \( P \) such that \( NPL(P) = r \) is equal to the number of the \((n,m)\)-lattice paths \( P = (x_1, y_1)(x_2, y_2) \ldots (x_{n+1}, y_{n+1}) \) such that \( NPL(P) = 0 \) and \( x_{n+1} = 1 \) and independent on \( r \).

**Proof.** Combining Lemmas 2.4 and 2.6, we immediately obtain the results as desired.
2.2 The rightmost minimum length of an \((n,m)\)-lattice paths

Given a \((n,m)\)-lattice path

\[ P = (x_1, y_1)(x_2, y_2) \ldots (x_{n+1}, y_{n+1}) \]

we let

\[ a_0 = 0, \quad b_0 = 0, \quad a_i = \sum_{j=1}^{i} y_j \]

and

\[ b_i = \sum_{j=1}^{i} x_j \]

for \(i \geq 1\). Then the \((n,m)\)-lattice path \(P\) can be viewed as a sequence of the points in the plane \(\mathbb{Z} \times \mathbb{Z}\)

\[ (b_0, a_0), (b_1, a_1), \ldots, (b_{n+1}, a_{n+1}). \]

A minimum point of the path \(P\) is a point \((b_i, a_i)\) such that \(a_i \leq a_j\) for all \(j \neq i\). A rightmost minimum point is a minimum point \((b_i, a_i)\) such that the point is the rightmost one among all the minimum points. If \((b_i, a_i)\) is the minimum point of the path \(P\), we call \(b_i\) the rightmost minimum length of the \((n,m)\)-lattice paths \(P\), denoted by \(\text{RML}(P)\).

**Example 2.8** We consider the path \(P\) in Example 2.2. The point \((7, -1)\) is the rightmost minimum point and \(\text{RML}(P) = 7\).

We use \(\mathcal{M}_{n,m,r}\) to denote the set of all the \((n,m)\)-lattice paths \(P\) such that \(\text{RML}(P) = r\).

**Lemma 2.9** There is a bijection \(\Psi\) from \(\mathcal{M}_{n,m,r}\) to \(\mathcal{M}_{n,m,r+1}\) for any \(1 \leq r \leq m - 2\).

**Proof.** Let \(P = (x_1, y_1)(x_2, y_2) \ldots (x_{n+1}, y_{n+1}) \in \mathcal{M}_{n,m,r}\). If \(x_{n+1} = 1\), we let

\[ \Psi(P) = (x_{n+1}, y_{n+1})(x_1, y_1) \ldots (x_n, y_n); \]

otherwise let

\[ \Psi(P) = (x_1 + 1, y_1)(x_2, y_2) \ldots (x_n, y_n)(x_{n+1} - 1, y_{n+1}). \]

It is easy to see that \(\Psi(P) \in \mathcal{M}_{n,m,r+1}\).

For proving that \(\Phi\) is a bijection, we describe the inverse of \(\Phi\) as follows.

If \(x_1 = 1\), we let

\[ \Psi(P) = (x_2, y_2)(x_3, y_3) \ldots (x_{n+1}, y_{n+1})(x_1, y_1); \]

otherwise let

\[ \Psi(P) = (x_1 - 1, y_1)(x_2, y_2) \ldots (x_n, y_n)(x_{n+1} + 1, y_{n+1}). \]

This complete the proof.

**Example 2.10** Let \(n = 3\) and \(m = 5\). We draw \((3,5)\)-lattice paths

\[
\begin{align*}
P_1 & = (1, -2)(2, 1)(1, 1)(1, 1) \quad P_2 = (1, 1)(1, -2)(2, 1)(1, 1) \\
P_3 & = (1, 1)(1, 1)(1, -2)(2, 1) \quad P_4 = (2, 1)(1, 1)(1, -2)(1, 1)
\end{align*}
\]

as follows.
We have $\Psi(P) = P_{i+1}$ and $RML(P) = i$.

Note that $NPL(P) = 0$ if and only if $RML(P) = 0$ for any $(n, m)$-lattice path. Recall that $\tilde{L}_{n,m,0}$ is the set of all the lattice paths $P = (x_1, y_1)(x_2, y_2)\ldots(x_{n+1}, y_{n+1})$ in the set $L_{n,m,0}$ such that $x_{n+1} = 1$. Hence, also $\tilde{L}_{n,m,0}$ is the set of all the lattice paths $P = (x_1, y_1)(x_2, y_2)\ldots(x_{n+1}, y_{n+1})$ in the set $M_{n,m,0}$ such that $x_{n+1} = 1$.

**Lemma 2.11** There is a bijection from $\tilde{L}_{n,m,0}$ to $M_{n,m,1}$.

**Proof.** Let $P = (x_1, y_1)(x_2, y_2)\ldots(x_{n+1}, y_{n+1}) \in \tilde{L}_{n,m,0}$. Then $x_{n+1} = 1$ and $y_{n+1} \leq 0$. We let

$$
\Psi(P) = (x_{n+1}, y_{n+1})(x_1, y_1)\ldots(x_n, y_n).
$$

Clearly, $\Psi(P) \in M_{n,m,1}$.

Conversely, let $P = (x_1, y_1)(x_2, y_2)\ldots(x_{n+1}, y_{n+1}) \in \tilde{L}_{n,m,1}$. Then $x_1 = 1$ and $y_1 \leq 0$. We let

$$
\Psi(P) = (x_2, y_2)(x_3, y_3)\ldots(x_{n+1}, y_{n+1})(x_1, y_1).
$$

This complete the proof.

**Theorem 2.12** For any $1 \leq r \leq m - 1$, the number of the $(n, m)$-lattice paths $P$ such that $RML(P) = r$ is equal to the number of the $(n, m)$-lattice paths $P = (x_1, y_1)(x_2, y_2)\ldots(x_{n+1}, y_{n+1})$ such that $RML(P) = 0$ and $x_{n+1} = 1$ and independent on $r$.

**Proof.** Combining Lemmas 2.9 and 2.11, we immediately obtain the results as desired.

### 3 The pointed $(n, m)$-lattice path

In this section, we will consider the pointed $(n, m)$-lattice paths. We will define two parameters for an pointed $(n, m)$-lattice path: the pointed non-positive length and the pointed rightmost minimum length. We will give generalizations of the results in [5]. We will prove the Chung-Feller theorems of the pointed $(n, m)$-lattice path on these two parameters. First, we give the definition of the pointed $(n, m)$-lattice paths as follows.
Definition 3.1 A pointed \((n, m)\)-lattice paths \(\hat{P}\) is a pair \([P; j]\) such that:

1. \(P = (x_1, y_1)(x_2, y_2) \ldots (x_{n+1}, y_{n+1})\) is an \((n, m)\)-lattice paths;
2. \(0 \leq j \leq x_{n+1} - 1\).

We call the point \((m - j, 0)\) the root of \(P\). We use \(\mathcal{L}_{n,m}\) to denote the set of the pointed \((n, m)\)-lattice paths.

Lemma 3.2 The number of the pointed \((n, m)\)-lattice paths is \(\binom{2n}{n}\binom{m}{n+1}\).

Proof. Note that the number of the solutions of the equation \(\sum_{i=1}^{n+1} y_i = 1\) such that \(1 - n \leq y_i \leq 1\) is \(\binom{2n}{n}\). On the other hand, we let \(z_i = x_i\) for all \(i \in [n]\), \(z_{n+1} = x_{n+1} - j\) and \(z_{n+2} = j\). Since \(\sum_{i=1}^{n+2} x_i = m, x_i \geq 1\) and \(0 \leq j \leq x_{n+1} - 1\), we have \(\sum_{i=1}^{n+2} z_i = m\), \(z_i \geq 1\) for all \(i \in [n+1]\) and \(z_{n+2} \geq 0\). It is easy to see that the number of the solutions of the equation \(\sum_{i=1}^{n+2} z_i = m\) such that \(z_i \geq 1\) for all \(i \in [n+1]\) and \(z_{n+2} \geq 0\) is \(\binom{m}{n+1}\). Hence, the number of the pointed \((n, m)\)-lattice paths is \(\binom{2n}{n}\binom{m}{n+1}\).

3.1 The pointed non-positive length of an pointed \((n, m)\)-lattice paths

Given a pointed \((n, m)\)-lattice path \(\hat{P} = [P; j]\), where \(P = (x_1, y_1)(x_2, y_2) \ldots (x_{n+1}, y_{n+1})\) and \(0 \leq j \leq x_{n+1} - 1\), we let \(PNPL(\hat{P}) = NPL(P) + j\). Clearly, \(0 \leq PNPL(\hat{P}) \leq m - 1\). We say that \(PNPL(\hat{P})\) is the pointed non-positive length of the path \(\hat{P}\).

By Lemma 2.3 (1), we have the following lemma.

Lemma 3.3 The number of the pointed \((n, m)\)-lattice paths with pointed non-positive length 0 is \(\binom{m-1}{n}\).

Given an \((n, m)\)-lattice path \(P = (x_1, y_1)(x_2, y_2) \ldots (x_{n+1}, y_{n+1})\), we let

\[
P_i = (x_{i+1}, y_{i+1}) \ldots (x_{n+1}, y_{n+1})(x_1, y_1) \ldots (x_i, y_i).
\]

\(P_i\) is called the \(i\)th cyclic permutation of \(P\). Furthermore, setting the point \((m - j, 0)\) to be the root of \(P_i\), where \(0 \leq j \leq x_i - 1\), we get a pointed \((n, m)\)-lattice paths \([P_i; j]\), denoted by \(\hat{P}(i; j)\). Finally, we define a set \(\mathcal{PL}(P)\) as follows:

\[
\mathcal{PL}(P) = \{\hat{P}(i; j) \mid i \in [n + 1] \text{ and } 0 \leq j \leq x_i - 1\}.
\]

Clearly, we have the following lemma.

Lemma 3.4 \(|\mathcal{PL}(P)| = m\).
Recall that $<_P$ is the linear order on the set $[n+1]$. We define a linear order $<_P$ on the set $\mathcal{PL}(P)$ by the following rules:

for any $\hat{P}(i_1; j_1), \hat{P}(i_2; j_2) \in \mathcal{PL}(P)$, $\hat{P}(i_1; j_1) <_P \hat{P}(i_2; j_2)$ if either (1) $i_1 <_P i_2$ or (2) $i_1 = i_2$ and $j_1 < j_2$.

The sequence, which is formed by the elements in the set $\mathcal{PL}(P)$ in the increasing order with respect to $<_P$, reduce a bijection from the sets $[m]$ to $\mathcal{PL}(P)$, denoted by $\Theta = \Theta_P$.

**Example 3.5** Let $n = 3$ and $m = 5$. Let $P = (1,1)(1,-2)(1,1)(2,1)$. We draw the pointed $(3,5)$-lattice path $\hat{P} = [P; 1]$ as follows.

![Diagram of a pointed lattice path]

where the root is the point $(4,0)$ denoted by $\bullet$. Then $PNPL(\hat{P}) = 3$. We write the bijection $\Theta_P$ as the following $2 \times 5$ matrix.

$$
\Theta_P = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 \\
\hat{P}(2; 0) & \hat{P}(3; 0) & \hat{P}(4; 0) & \hat{P}(4; 1) & \hat{P}(1; 0)
\end{pmatrix}
$$

**Theorem 3.6** Let $P$ be an $(n,m)$-lattice path, $\mathcal{PL}(P)$ and $\Theta_P$ defined as above. Then

$$
PNPL(\Theta(r)) = r - 1
$$

for any $r \in [m]$.

**Proof.** Note that $0 \leq PNPL(\Theta(r)) \leq m - 1$ for any $r \in [m]$. It is sufficient to prove that $PNPL(\Theta(r + 1)) = PNPL(\Theta(r)) + 1$ for any $r \in [m-2]$. Suppose

$$
P = (x_1, y_1)(x_2, y_2) \ldots (x_{n+1}, y_{n+1})
$$

and $\Theta(r) = \hat{P}(s; t) \in \mathcal{PL}(P)$. Let $\pi_P$ be the sequence formed by writing $[n+1]$ in the increasing order with respect to $<_P$ and $\pi_P^{-1}(s) = k$. Then $PNPL(\Theta(r)) = \sum_{j=1}^{k-1} x_{\pi_P(j)} + t$. Now, suppose $\Theta(r + 1) = \hat{P}(\tilde{s}; \tilde{t})$. We discuss the following two cases:

**Case I.** $s = \tilde{s}$

Then $\tilde{t} = t + 1$. This implies $PNPL(\Theta(r + 1)) = PNPL(\Theta(r)) + 1$.

**Case II.** $s <_P \tilde{s}$

Then $\pi_P(k + 1) = \tilde{s}$, $t = x_s - 1$ and $\tilde{t} = 0$. Thus,

$$
PNPL(\Theta(r + 1)) = \sum_{j=1}^{k} x_{\pi_P(j)} = \sum_{j=1}^{k-1} x_{\pi_P(j)} + x_s = PNPL(\Theta(r)) + 1.
$$

This complete the proof. □
Example 3.7 We consider the path $P$ in Example 3.5. We draw the pointed lattice path $\Theta(r)$ as follows:

![Lattice paths](image)

Remark 3.8 Let $\hat{P} = [P; j]$ be a pointed $(n, m)$-lattice path, where $P = (x_1, y_1) \ldots (x_{n+1}, y_{n+1})$ and $0 \leq j \leq x_{n+1} - 1$. Setting $m = n + 1$, we have $x_i = 1$ for all $i$ and $j = 0$. Let $Y = (y_1, \ldots, y_{n+1})$. Then $E(Y) = \text{PNPL}(\hat{P})$. This tells us that Lemma 1.1 can be viewed as a corollary of Theorem 3.6.

We use $\mathcal{L}_{n,m,r}$ to denote the set of the pointed $(n, m)$-lattice paths with pointed non-positive length $r$. Clearly, $\mathcal{L}_{n,m} = \bigcup_{r=0}^{m-1} \mathcal{L}_{n,m,r}$. Let $l_{n,m,r} = |\mathcal{L}_{n,m,r}|$.

Corollary 3.9 For any $0 \leq r \leq m - 1$, the number of the pointed $(n, m)$-lattice paths with pointed non-positive length $r$ is equal to the number of the pointed $(n, m)$-lattice paths with pointed non-positive length 0 and independent on $r$, i.e., $l_{n,m,r} = \frac{1}{m} \binom{2n}{n} \binom{m}{n+1}$.

Proof. First, we define an equivalent relation on the set $\mathcal{L}_{n,m}$. Let $\hat{P} = [P; i]$ and $\hat{Q} = [Q; j]$ be two pointed $(n, m)$-lattice paths. Suppose $P = (x_1, y_1) \ldots (x_{n+1}, y_{n+1})$. Recall $P_k$ denote the $k$th cyclic permutation of $P$, i.e., $P_k = (x_{k+1}, y_{k+1}) \ldots (x_{n+1}, y_{n+1})(x_1, y_1) \ldots (x_k, y_k)$. We say $\hat{Q}$ and $\hat{P}$ is equivalent, denoted by $\hat{Q} \sim \hat{P}$, if $Q = P_k$ for some $k \in [n+1]$. Hence, given a pointed lattice path $\hat{P} \in \mathcal{L}_{n,m}$, we define a set $EQ(\hat{P})$ as $EQ(\hat{P}) = \{ \hat{Q} \in \mathcal{L}_{n,m} \mid \hat{Q} \sim \hat{P} \}$. We say that the set $EQ(\hat{P})$ is an equivalent class of the set $\mathcal{L}_{n,m}$. Clearly, $|EQ(\hat{P})| = m$. Now, we may suppose that the set $\mathcal{L}_{n,m}$ has $t$ equivalent class. Then $t = \frac{1}{m} \binom{2n}{n} \binom{m}{n+1}$. For any $0 \leq r \leq m - 1$, from Theorem 3.6, every equivalent class contains exactly one element with pointed non-positive length $r$. Hence, $l_{n,m,r} = t = \frac{1}{m} \binom{2n}{n} \binom{m}{n+1}$.
3.2 The pointed rightmost minimum length of an pointed \((n, m)\)-lattice paths

Let \(\hat{P} = [P; j]\) be a pointed \((n, m)\)-lattice path, where \(P = (x_1, y_1)(x_2, y_2)\ldots(x_{n+1}, y_{n+1})\) is a \((n, m)\)-lattice path and \(0 \leq j \leq x_{n+1} - 1\). Recall that \(RML(P)\) is the rightmost minimum length of \(P\). We let \(PRML(\hat{P}) = RML(P) + j\) and call \(PRML(\hat{P})\) the pointed rightmost minimum length of \(\hat{P}\).

Note that \(PNPL(P) = 0\) if and only if \(PRML(\hat{P}) = 0\) for any pointed \((n, m)\)-lattice path. We immediately obtain the following lemma.

**Lemma 3.10** The number of the pointed \((n, m)\)-lattice paths with pointed rightmost minimum length 0 is \(\binom{m-1}{n}\).

First, given a \((n, m)\)-lattice path \(P\), we recall that \(\pi_P\) is the sequence formed by writing \([n+1]\) in the increasing order with respect to \(<_P\). Suppose \(\pi_P(1) = i\). Let \(\sigma_P = (\sigma_P(1), \sigma_P(2), \ldots, \sigma_P(n+1)) = (i, i-1, \ldots, 1, n+1, n, \ldots, i+1)\).

Using \(\sigma_P\), we define a new linear order \(<^*_P\) on the set \(\mathcal{P}L(P) = \{\hat{P}(i; j)\mid i \in [n+1] \text{ and } 0 \leq j \leq x_i - 1\}\) by the following rules:

- for any \(\hat{P}(i_1; j_1), \hat{P}(i_2; j_2) \in \mathcal{P}L(P)\), \(\hat{P}(i_1; j_1) <^*_P \hat{P}(i_2; j_2)\) if either (1) \(\sigma_P^{-1}(i_1) < \sigma_P^{-1}(i_2)\) or (2) \(i_1 = i_2\) and \(j_1 < j_2\).

The sequence, which is formed by the elements in the set \(\mathcal{P}L(P)\) in the increasing order with respect to \(<^*_P\), reduce a bijection from the sets \([m]\) to \(\mathcal{P}L(P)\), denoted by \(\Gamma = \Gamma_P\).

**Example 3.11** Consider the path \(P\) and the pointed path \(\hat{P}\) in Example 3.5. we have \(PRML(\hat{P}) = 3\). It is easy to see \(\sigma_P = (2, 1, 4, 3)\). We write the bijection \(\Gamma_P\) as the following \(2 \times 5\) matrix.

\[
\Gamma_P = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 \\
\hat{P}(2; 0) & \hat{P}(1; 0) & \hat{P}(4; 0) & \hat{P}(4; 1) & \hat{P}(3; 0)
\end{pmatrix}
\]

**Theorem 3.12** Let \(P\) be an \((n, m)\)-lattice path and \(\Gamma\) defined as above. Then

\[PRML(\Gamma(r)) = r - 1\]

for any \(r \in [m]\).

**Proof.** It is sufficient to prove that \(PRML(\Gamma(r+1)) = PRML(\Gamma(r)) + 1\). Suppose \(\Gamma(r) = \hat{P}(i_1; j_1)\) and \(\Gamma(r+1) = \hat{P}(i_2; j_2)\). If \(i_1 = i_2\), then \(j_1 + 1 = j_2\). Clearly, \(PRML(\Gamma(r+1)) = PRML(\Gamma(r)) + 1\).

We consider the case with \(\sigma_P^{-1}(i_1) < \sigma_P^{-1}(i_2)\). Let \(k = \sigma_P^{-1}(i_1)\). Then \(\sigma_P^{-1}(i_2) = k + 1, j_1 = x_{i_1} - 1\) and \(j_2 = 0\). We have \(PRML(\hat{P}(i_2; j_2)) = \sum_{j=1}^{k} x_{\sigma_P(j)} = \sum_{j=1}^{k-1} x_{\sigma_P(j)} + x_{i_1} = PRML(\hat{P}(i_1; j_1)) + 1\). \(\blacksquare\)
Example 3.13 We consider the path $P$ in Example 3.5. We draw the pointed lattice path $\Gamma(r)$ as follows:

We use $\mathcal{M}_{n,m,r}$ to denote the set of the pointed $(n,m)$-lattice paths with pointed rightmost minimum length $r$. Clearly, $\mathcal{L}_{n,m} = \bigcup_{r=0}^{m-1} \mathcal{M}_{n,m,r}$. Let $d_{n,m,r} = |\mathcal{M}_{n,m,r}|$.

Corollary 3.14 For any $0 \leq r \leq m - 1$, the number of the pointed $(n,m)$-lattice paths with pointed rightmost minimum length $r$ is equal to the number of the pointed $(n,m)$-lattice paths with pointed rightmost minimum length 0 and independent on $r$, i.e., $d_{n,m,r} = \frac{1}{m} \binom{2n}{n} \binom{m}{n+1}$.

Proof. Similar to the proof of Corollary 3.9, we can obtain the results as desired.

4 The application of the main theorem

In fact, by Theorems 3.6 and 3.12, we may find the Chung-Feller theorems of many different $(n,m)$-lattice paths on the parameter: the pointed non-positive length and the pointed rightmost minimum length. For example, we let $A$ and $B$ be two finite subsets of the set $\mathbb{P}$. Let $\mathcal{S} = \mathcal{S}_A \cup \mathcal{S}_B \cup \{(1,1)\}$, where $\mathcal{S}_A = \{(2i-1,-1) \mid i \in A\}$ and $\mathcal{S}_B = \{(2i,0) \mid i \in B\}$. In [6], we have proved the following corollary by the generating function methods. Using Theorems 3.6 and 3.12, we can reobtain the corollary.

Corollary 4.1 Let $\mathcal{P}_{n,m}$ be the set of the pointed lattice paths in the plane $\mathbb{Z} \times \mathbb{Z}$ which (1) only use steps in the set $\mathcal{S}$; (2) have $n+1$ steps; (3) go from the origin to the point $(m,1)$. Then in $\mathcal{P}_{n,m}$,
(1) the number of the pointed lattice paths with pointed non-positive length \( r \) is equal to the number of the pointed lattice paths with pointed non-positive length 0 and independent on \( r \);
(2) the number of the pointed lattice paths with pointed rightmost minimum length \( r \) is equal to the number of the pointed lattice paths with pointed rightmost minimum length 0 and independent on \( r \).

**Proof.** (1) It is easy to see that a pointed lattice path \( P \) in \( \mathcal{P}_{n,m} \) can be viewed as a pointed \((n,m)\)-lattice path \((x_1,y_1) \ldots (x_{n+1},y_{n+1})\) such that \((x_i,y_i) \in \mathcal{S}\) for all \( i \in [n+1] \). By Theorem 3.6, using a similar method as Corollary 3.9, we get the results as desired.
(2) The proof is omitted.

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