Expansion properties of metric spaces not admitting a coarse embedding into a Hilbert space

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Abstract. The main purpose of the paper is to find some expansion properties of locally finite metric spaces which do not embed coarsely into a Hilbert space. The obtained result is used to show that infinite locally finite graphs excluding a minor embed coarsely into a Hilbert space. In an appendix a direct proof of the latter result is given.

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A metric space \((M, d_M)\) is called locally finite if all balls in it have finitely many elements. We say that \((M, d_M)\) has bounded geometry if for each \(r > 0\) there is \(U(r) < \infty\) such that each ball of radius \(r\) in \(M\) has at most \(U(r)\) elements. Let \(A\) and \(B\) be metric spaces. A mapping \(f: A \to B\) is called a coarse embedding if there exist non-decreasing functions \(\rho_1, \rho_2: [0, \infty) \to [0, \infty)\) such that (1) \(\forall x, y \in A \rho_1(d_A(x, y)) \leq d_B(f(x), f(y)) \leq \rho_2(d_A(x, y));\) (2) \(\lim_{r \to \infty} \rho_1(r) = \infty\).

We are interested in conditions under which a locally finite metric space \(M\) embeds coarsely into a Hilbert space. See [Gro93], [Roe03], and [Yu06] for motivation and background for this problem. Since, as it is well-known (see e.g. [Ost09, Section 4]), coarse embeddability into a Hilbert space is equivalent to coarse embeddability into \(L_1\), we consider coarse embeddability into \(L_1\).

Locally finite metric space which are not coarsely embeddable into \(L_1\) were characterized in [Ost09] and [Tes09]. We reproduce the characterization as it is stated in [Ost09].

Theorem 1 ([Ost09, Theorem 2.4]) Let \((M, d_M)\) be a locally finite metric space which is not coarsely embeddable into \(L_1\). Then there exists a constant \(D\), depending on \(M\) only, such that for each \(n \in \mathbb{N}\) there exists a finite set \(M_n \subset M\) and a probability measure \(\mu_n\) on \(M_n \times M_n\) such that

\(\bullet\) \(d_M(u, v) \geq n\) for each \((u, v) \in \text{supp}\mu_n\).
• For each Lipschitz function \( f : M \to L_1 \) we have

\[
\int_{M_n \times M_n} ||f(u) - f(v)||_{L_1} d\mu_n(u, v) \leq D\text{Lip}(f).
\]

Our first purpose is to find some expansion properties of sets \( M_n \).

Let \( s \) be a positive integer. We consider graphs \( G(n, s) = (M_n, E(M_n, s)) \), where the edge set \( E(M_n, s) \) is obtained by joining those pairs of vertices of \( M_n \) which are at distance \( \leq s \). The graphs \( \{G(n, s)\}_{n=1}^\infty \) have uniformly bounded degrees if the metric space \( M \) has bounded geometry.

**Observation:** Each vertex cut of \( G(n, s) \) separates it into pieces with \( d_M \)-distance between them at least \( s \).

If we would prove in the bounded geometry case that the condition

(*) For some \( s \in \mathbb{N} \) there is a number \( h_s > 0 \) and subgraphs \( H_n \) of \( G(n, s) \) of indefinitely growing sizes (as \( n \to \infty \)) such that the expansion constants of \( \{H_n\} \) are uniformly bounded from below by \( h_s \)

is satisfied, it would solve the well-known problem (see [GK04], [Ost09], [Tes09]): whether each metric space with bounded geometry which does not embed coarsely into a Hilbert space contains weak expanders? For spaces with bounded geometry weak expanders are defined as Lipschitz images \( f_m(X_m) \) of (vertex sets) of a family of expanders with uniformly bounded Lipschitz constants of \( \{f_m\}_{m=1}^\infty \) and without dominating pre-images in the sense that \( \lim_{m \to \infty} \max_{z \in f_m(X_m)} (|f_m^{-1}(z)|/|X_m|) = 0 \).

**Remark.** When we consider a connected graph as a metric space, we identify the graph with its vertex set endowed with the standard graph distance.

The well-known proof of non-embeddability of expanders (see [Gro00], [Mat97], [Roe03, Section 11.3]) shows that a metric space with bounded geometry containing weak expanders does not embed coarsely into a Hilbert space.)

In this paper we prove only the following weaker expansion property of the graphs \( G(n, s) \). We introduce the measure \( \nu_n \) on \( M_n \) by \( \nu_n(A) = \mu_n(A \times M_n) \). Let \( F \) be an induced subgraph of \( G(n, s) \). We denote the vertex boundary of a set \( A \) of vertices in \( F \) by \( \delta_FA \).

**Theorem 2** Let \( s \) and \( n \) be such that \( 2n > s > 8D \). Let \( \varphi(D, s) = \frac{s}{4D} - 2 \). Then \( G(n, s) \) contains an induced subgraph \( F \) with \( d_M \)-diameter \( \geq n - \frac{s}{2} \), such that each subset \( A \subset F \) with \( d_M \)-diameter \( < n - \frac{s}{2} \) satisfies the condition: \( \nu_n(\delta_FA) > \varphi(D, s)\nu_n(A) \).

**Proof of Theorem 2** Suppose that for some \( n, s \in \mathbb{N} \) satisfying \( 2n > s > 8D \) there is no such subgraph in \( G(n, s) \). Then for each induced subgraph \( F \) in \( G(n, s) \) of \( d_M \)-diameter \( \geq n - \frac{s}{2} \) we can find a subset \( A \subset F \) of \( d_M \)-diameter \( < n - \frac{s}{2} \) such
that \( \nu_n(\delta_F A) \leq \varphi(D, s)\nu_n(A) \). We start with \( F_1 = G(n, s) \) (the definitions of \( M_n \) and \( \mu_n \) imply that the \( d_M \)-diameter of \( M_n \) is \( \geq n \)), find a subset \( A_1 \subset F_1 \) of \( d_M \)-diameter \( < n - \frac{s}{2} \) such that \( \nu_n(\delta_F A_1) \leq \varphi(D, s)\nu_n(A_1) \), and remove \( A_1 \cup \delta_F A_1 \) from \( G(n, s) \). If the obtained graph \( F_2 \) still has \( d_M \)-diameter \( \geq n - \frac{s}{2} \), we find a subset \( A_2 \) in it such that \( \nu_n(\delta_F A_2) \leq \varphi(D, s)\nu_n(A_2) \). We remove the subset \( A_2 \cup \delta_F A_2 \) from \( F_2 \). We continue in an obvious way till we get a set of \( d_M \)-diameter \( < n - \frac{s}{2} \) (this should eventually happen since \( M_n \) is finite). We denote this set \( A_p \), where \( p \) is the number of steps in the process.

**Remark.** This exhaustion process is similar to the one used in [LS93].

Observe that each of the sets \( A_i \) has diameter \( < n - \frac{s}{2} \), and that the \( d_M \)-distance between any \( A_i \) and \( A_j \) \((i \neq j)\) is at least \( s \) (see the observation above).

We introduce a family of 1-Lipschitz functions \( f_\theta \) on \( M \), where \( \theta = \{\theta_i\}_{i=1}^p \in \Theta = \{-1, 1\}^p \) by the formula:

\[
f_\theta(x) = \begin{cases} 
\theta_j \left( \frac{s}{2} - \text{dist}(x, A_j) \right) & \text{if dist}(x, A_j) < \frac{s}{2} \\
0 & \text{if dist}(x, \cup_{i=1}^p A_i) \geq \frac{s}{2}.
\end{cases}
\]

The function is well-defined since the inequality \( \text{dist}(x, A_j) < \frac{s}{2} \) cannot be satisfied for more than one value of \( j \). Straightforward verification shows that this function is 1-Lipschitz.

We endow \( \Theta = \{-1, 1\}^p \) with the natural probability measure \( P \) and introduce for each \( x \in M \) a function \( F_x \in L_1(\Theta, P) \) given by \( F_x(\theta) = f_\theta(x) \). It is clear that the mapping \( x \mapsto F_x \) is 1-Lipschitz.

Applying inequality (1) to this mapping we get

\[
D \geq \int_{M_n \times M_n} |F_x(\theta) - F_y(\theta)|d\mu_n(x, y) \geq \int_{M_n \times M_n} \int_\Theta |f_\theta(x) - f_\theta(y)|dP(\theta)d\mu_n(x, y) \\
\geq \int_{M_n \times M_n} \int_{\Psi(x, y)} |f_\theta(x)|dP(\theta)d\mu_n(x, y),
\]

where \( \Psi(x, y) \) is the subset of \( \Theta \) for which \( f_\theta(x) \) and \( f_\theta(y) \) have different signs (we mean that signs have values in \( \{-1, 0, 1\} \)). Observe that the value of \( |f_\theta(x)| \) does not depend on \( \theta \). We get

\[
\int_{M_n \times M_n} \int_{\Psi(x, y)} |f_\theta(x)|dP(\theta)d\mu_n(x, y) \geq \int_{(\cup_{i=1}^p A_i) \times M_n} |f_\theta(x)| \int_{\Psi(x, y)} dP(\theta)d\mu_n(x, y).
\]

Now we observe that for \( x \in A_j \) and \( y \) satisfying \( (x, y) \in \text{supp}\mu_n \) we have \( d_M(x, y) \geq n \) and therefore \( d_M(y, A_j) \geq \frac{s}{2} \) (recall that the diameter of \( A_j \) is \( < n - \frac{s}{2} \)). Hence \( P(\Psi(x, y)) \geq \frac{1}{2} \) for each pair \( (x, y) \) from \( \text{supp}\mu_n \). We get
\[
\int_{(\cup_{i=1}^{p}A_i) \times M_n} |f_{\theta}(x)| \int_{\Psi(x,y)} d\mathcal{P}(\theta) d\mu_n(x,y) \geq \int_{(\cup_{i=1}^{p}A_i) \times M_n} \frac{s}{2} \cdot \frac{1}{2} d\mu_n(x,y) = \frac{s}{4} \nu_n(\cup_i A_i).
\]

Remark. The idea of “random” signing of functions in a similar situation was used in [Rao99].

Recalling the beginning of this chain of inequalities, we get

\[
D \geq \frac{s}{4} \nu_n(\cup_i A_i). \tag{2}
\]

Observe that \(\nu_n(\cup_i A_i) + \nu_n(\cup_i \delta F_i A_i) = 1\) and \(\nu_n(\cup_i \delta F_i A_i) \leq \varphi(D, s) \nu_n(\cup_i A_i)\). Therefore

\[
(1 + \varphi(D, s)) \nu_n(\cup_i A_i) \geq 1 \tag{3}
\]

Combining (2) and (3) we get

\[
D \geq \frac{s}{4(1 + \varphi(D, s))},
\]

or \(\varphi(D, s) \geq \frac{s}{4D} - 1\), a contradiction. ■

Now we combine Theorem [2] with some results and technique from [KPR93] (some of the estimates from [KPR93] were improved in [FT03] but we do not use this improvement).

**Theorem 3** Let \(r \in \mathbb{N}\) and \(G\) be a locally finite connected graph which does not have \(K_r\)-minors, let \(d_G\) be the graph distance on \(G\). Then \((G, d_G)\) embeds coarsely into \(L_1\).

**Proof.** Assume the contrary. We apply Theorem [1] to \(G\) and denote by \(D, M_n, \) and \(\mu_n\) the corresponding constant (depending only on \(G\), finite sets, and probability measures. Let \(\nu_n\) be measures introduced in Theorem [2]. According to Theorem [2] for each \(2n > s > 8D\) there is an induced subgraph \(F = F(n, s)\) in \(G(n, s)\) such that the condition of Theorem [2] is satisfied. The condition \(\nu_n(\delta_F A) > \varphi(D, s) \nu_n(A)\) implies that \(\nu_n(F) > 0\).

Now we use a modified construction from [KPR93, Section 4]. Let \(t, s \in \mathbb{N}\) (we shall specify our choice of these numbers later). Let \(\Delta = t + 2s\). We pick a vertex \(x_1 \in G\), \(\alpha \in \{0, 1, 2, \ldots, \Delta - 1\}\), and let

\[
D_1 = \{v \in G : (d_G(v, x_1) - \alpha) \ (\text{mod} \ \Delta) \in \{1, 2, \ldots, 2s\}\}
\]

(that is, \(D_1\) consists of infinitely many ‘annuluses’ of width \(2s\) each, with distances \(t\) between them). We choose \(\alpha\) in such a way that \(\nu_n(D_1 \cap F)\) is the minimal possible. Using averaging argument we get that \(\alpha\) can be chosen in such a way that \(\nu_n(D_1 \cap F) \leq \left(\frac{2s}{2s+t}\right) \nu_n(F)\).
We delete $D_1$ from $G$. The second round of deletions is: we repeat the same procedure for each of the components of the obtained graph endowed with its own graph distance. Each time we choose the corresponding $\alpha$ (the level of cut) in such a way $\nu_n(D \cap F) \leq \left(\frac{2s}{2s+t}\right) \nu_n(F \cap X)$, where $X$ is the component under consideration and $D$ is the set of vertices deleted this time.

We do $r$ rounds of deletions. Let $\{G_i\}$ be the components of the remaining graph. The argument of [KPR93, Theorem 4.2] shows that the $d_G$-diameter of each of $G_i$ does not exceed $(r - 1)(4(r + 1)t + 1)$ (where $r$ is from the statement of the theorem). It is also easy to see that

$$\nu_n(F \cap (\cup_i G_i)) \geq \left(\frac{t}{2s + t}\right)^r \nu_n(F).$$

Now we impose additional conditions on $s$, $t$, and $n$ (the condition $2n > s > 8D$ was imposed in Theorem). The conditions are

$$(\varphi(D, s) + 1) \left(\frac{t}{2s + t}\right)^r > 1$$

and

$$(r - 1)(4(r + 1)t + 1) < n - \frac{s}{2}.$$  

These conditions can be satisfied. In fact, we choose $s > 8D$ first. Then we choose $t$ such that (5) is satisfied, and then $n$ such that (6) is satisfied.

Let $R_i = F \cap G_i$. Our choice of parameters implies that the $d_G$-diameter of $R_i$ is $< n - \frac{s}{2}$. Therefore $\nu_n(\delta_F R_i) > \varphi(D, s) \nu_n(R_i)$. Since $\{\delta_F R_i\}$ are disjoint (this was the reason why we deleted ‘annuluses’ of width $2s$), we get

$$\nu_n(F) \geq \nu_n(\cup_i \delta_F R_i) + \nu_n(\cup_i R_i) > (\varphi(D, s) + 1) \nu_n(\cup_i R_i)$$

$$\geq (\varphi(D, s) + 1) \left(\frac{t}{2s + t}\right)^r \nu_n(F).$$

We get a contradiction with (5). □

**Appendix: Coarse embeddability of graphs with excluded minors. Second proof**

The purpose of this appendix is to show that coarse embeddability of graphs excluding $K_r$ as a minor can be proved using the techniques from [KPR93] and [Rao99] (see also [FT03]), without using Theorems 1 and 2.

**SECOND PROOF OF THEOREM 3.** For $\Delta \in \mathbb{N}$ by $[\Delta]$ we denote the set $\{1, \ldots, \Delta\}$. For each $\Delta \in \mathbb{N}$ we consider the probability space

$$\Omega_\Delta = \Lambda_\Delta \times \Theta,$$

where

$$\Lambda_\Delta = [\Delta]^r \text{ and } \Theta = \{-1, 1\}^\mathbb{N}.$$

For each point $\omega \in \Omega_\Delta$ we define a function $f_{\Delta, \omega} : X \to \mathbb{R}$ in the following way.
We assume that elements of $X$ are enumerated, so $X = \{x_k : k \in \mathbb{N}\}$. Let 
$$(\{r_j\}_{j=1}^r, \{\theta_j\}_{j=1}^\infty) \in \Omega_\Delta$$

We denote by $D_1$ the set of all vertices $v$ in $X$ with $d(v, x_1) = r_1$ (mod $\Delta$).

We delete the set $D_1$ from $X$. We label connected components of the obtained graph by the numbers of the least subscripts of vertices contained in them. For the component where $x_j$ is the vertex with the least subscript, we do the same procedure as above (with the respect to the graph distance defined by the subgraph) with $d(v, x_j) = r_2$ (mod $\Delta$). So the number $r_2$ is used for all of the components of this level.

We denote the set of all obtained vertices by $D_2$ and delete it from the graph. We repeat the procedure $r$ times. Let $\{X_i\}_{i=1}^\infty$ be components of the obtained graph.

We define the function $f_{\Delta, \omega}(u)$ corresponding to $\omega = (\{r_j\}_{j=1}^r, \{\theta_j\}_{j=1}^\infty)$ by

$$f_{\Delta, \omega}(u) = \theta_k \text{dist} (u, \bigcup_{i=1}^n D_i),$$

where $k$ is the least subscript of a point $x_k$ belonging to the same component of $X \setminus (\bigcup_{i=1}^r D_i)$ as $u$. An obvious and very important property of $f_{\Delta, \omega}$ is that it is a real-valued 1-Lipschitz function.

One of the main results of [KPR93] (Theorem 4.2) (see also [FT03]) implies that the diameters of the components $X_i$ are $< (r - 1)(4(r + 1)\Delta + 1) =: d_{\Delta, r}$.

Now, for each vertex $u$ in $X$ we introduce a function $F_{\Delta, u}(\omega)$ in $L_1(\Omega_\Delta)$ given by

$$F_{\Delta, u}(\omega) = f_{\Delta, \omega}(u)$$

It is easy to see that $|F_{\Delta, u}(\omega)| \leq \Delta/2$ for all $u$ and $\omega$. The function $F_{\Delta, u}(\omega)$ is measurable because all subsets of $\Omega_\Delta$ are measurable. (It is worth mentioning that for each $u$ the value of the function at $\omega$ depends only on finitely many values of $\theta_i$. In fact, for a fixed $u$ the value of $f_{\Delta, \omega}(u)$ can depend only on those $\theta_k$ for which $x_k$ is in the same component $X_i$ as $u$. But for such $x_k$ we have $d(u, x_k) \leq (r - 1)(4(r + 1)\Delta + 1)$. Since $X$ is locally finite, there are only finitely many $x_k$ satisfying this condition.)

The following inequality is a very important property of the functions $F_{\Delta, u}$:

$$\int_{\Omega_\Delta} |F_{\Delta, u}(\omega)| \, d\omega \geq \varepsilon_r \Delta, \quad (8)$$

where $\varepsilon_r$ depends on $r$ only (see [Rao99] Lemma 3), the dependence obtained in this way is of the form $\delta^r$, where $0 < \delta < 1$). Furthermore, if we write $\omega = (\lambda, \theta)$ according to (7), we have

$$\int_{\Omega_\Delta} |F_{\Delta, \omega}(\lambda, \theta)| \, d\lambda \geq \varepsilon_r \Delta \, \forall \theta \in \Theta. \quad (9)$$

If $d(u, v) \geq d_{\Delta, r}$, then $u$ and $v$ are in different pieces of the decomposition no matter how $\lambda = \{r_j\}_{j=1}^r$ is chosen. Therefore, with probability $\frac{1}{2}$, the signs of $f_{\Delta, \omega}(u)$ and $f_{\Delta, \omega}(v)$
are different, Let \( \Psi(\lambda) \subset \Theta \) be the subset for which the signs \( f_{\Delta,\lambda,\theta}(u) \) and \( f_{\Delta,\lambda,\theta}(v) \) are different. Then

\[
\|F_{\Delta,u} - F_{\Delta,v}\|_{L_1(\Omega_\Delta)} = \int_{\Lambda_\Delta} \int_{\Theta} \|F_{\Delta,u}(\lambda, \theta) - F_{\Delta,v}(\lambda, \theta)\| d\theta d\lambda
\]

\[
\geq \int_{\Lambda_\Delta} \int_{\Psi(\lambda)} \|F_{\Delta,u}(\lambda, \theta) - F_{\Delta,v}(\lambda, \theta)\| d\theta d\lambda
\]

\[
= \int_{\Lambda_\Delta} \int_{\Psi(\lambda)} (|F_{\Delta,u}(\lambda, \theta)| + |F_{\Delta,v}(\lambda, \theta)|) d\theta d\lambda
\]

\[
(\text{observe that the integrand does not depend on } \theta)
\]

\[
= \frac{1}{2} \int_{\Lambda_\Delta} (|F_{\Delta,u}(\lambda, \theta)| + |F_{\Delta,v}(\lambda, \theta)|) d\lambda
\]

\[
\geq \varepsilon, \Delta.
\]

(10)

We apply this construction with \( \Delta = 2, 4, \ldots, 2^i, \ldots \). Let \( \Omega = \bigcup_{i=1}^{\infty} \Omega_{2^i} \) be the disjoint union of the measure spaces \( \Omega_{2^i} \). Let \( O \) be one of the vertices of \( X \). We introduce an embedding \( \varphi : X \to L_1(\Omega) \) by

\[
\varphi(v)|_{\Omega_{2^i}} = \left(\frac{2}{3}\right)^i (F_{2^i,v}(\omega) - F_{2^i,O}(\omega)).
\]

To complete the proof of the theorem it remains to show that \( \varphi \) is a well-defined mapping and that it is a coarse embedding.

Since \( f_{\Delta,\omega}(u) \) are 1-Lipschitz (as functions of \( u \)) real-valued functions, the mappings \( \varphi_i(v) := F_{2^i,v} \in L_1(\Omega_{2^i}) \) are also 1-Lipschitz. Therefore \( \|\varphi_i(v) - \varphi_i(O)\|_{L_1(\Omega_{2^i})} \leq d(O, v) \) and \( \varphi(v) \in L_1(\Omega) \).

To show that \( \varphi \) is a coarse embedding it suffices to establish the following two inequalities:

\[
\|\varphi(u) - \varphi(v)\|_{L_1(\Omega)} \leq 3d(u, v),
\]

(11)

\[
d(u, v) \geq d_{2^i,n} \Rightarrow \|\varphi(u) - \varphi(v)\|_{L_1(\Omega)} \geq \left(\frac{4}{3}\right)^i \varepsilon.\]

(12)

The inequality (11) is an immediate consequence of the fact that \( \varphi_i \) are 1-Lipschitz:

\[
\|\varphi(u) - \varphi(v)\|_{L_1(\Omega)} = \sum_{i=0}^{\infty} \left(\frac{2}{3}\right)^i \|\varphi_i(u) - \varphi_i(v)\|_{L_1(\Omega_{2^i})} \leq d(u, v) \sum_{i=0}^{\infty} \left(\frac{2}{3}\right)^i = 3d(u, v).
\]

If \( d(u, v) \geq d_{2^i,n} \), we apply the inequality (10) and get

\[
\|\varphi(u) - \varphi(v)\|_{L_1(\Omega)} \geq \left(\frac{2}{3}\right)^i \|\varphi_i(u) - \varphi_i(v)\|_{L_1(\Omega_{2^i})} \geq \left(\frac{2}{3}\right)^i \varepsilon = \left(\frac{4}{3}\right)^i \varepsilon.\]

\[\blacksquare\]
References

[FT03] J. Fakcharoenphol and K. Talwar, An improved decomposition theorem for graphs excluding a fixed minor, in: Approximation, randomization, and combinatorial optimization, 36–46, Lecture Notes in Comput. Sci., 2764, Springer, Berlin, 2003.

[Gro00] M. Gromov, Spaces and questions, GAFA 2000 (Tel Aviv, 1999), Geom. Funct. Anal. 2000, Special Volume, Part I, 118–161.

[Gro93] M. Gromov, Asymptotic invariants of infinite groups, in: A. Niblo, M. Roller (Eds.) Geometric group theory, London Math. Soc. Lecture Notes, 182, 1–295, Cambridge University Press, 1993.

[GK04] E. Guentner, J. Kaminker, Geometric and analytic properties of groups, in: Noncommutative geometry, Ed. by S. Doplicher and R. Longo, Lecture Notes Math., 1831 (2004), 253–262.

[KPR93] P. Klein, S. Plotkin, and S. Rao, Excluded minors, network decomposition, and multicommodity flow. In: Proc. 25th Annual ACM Symposium on the Theory of Computing, pp. 682–690, 1993.

[LS93] N. Linial and M. Saks, Low diameter graph decompositions, Combinatorica, 13 (1993), 441–454.

[Mat97] J. Matoušek, On embedding expanders into ℓ_p spaces, Israel J. Math., 102 (1997), 189–197.

[Ost09] M. I. Ostrovskii, Coarse embeddability into Banach spaces, Topology Proceedings, 33 (2009) pp. 163–183; arXiv:0802.3666

[Rao99] S. Rao, Small distortion and volume preserving embeddings for planar and Euclidean metrics, in: Proceedings of the Fifteenth Annual Symposium on Computational Geometry (Miami Beach, FL, 1999), 300–306, ACM, New York, 1999.

[Roe03] J. Roe, Lectures on coarse geometry, University Lecture Series, 31, American Mathematical Society, Providence, R.I., 2003.

[Tes09] R. Tessera, Coarse embeddings into a Hilbert space, Haagerup property and Poincaré inequalities, Journal of Topology and Analysis, 1 (2009); arXiv:0802.2531

[Yu06] G. Yu, Higher index theory of elliptic operators and geometry of groups, in: International Congress of Mathematicians, Vol. II, 1623–1639, Eur. Math. Soc., Zürich, 2006.