Generating functions for tensor products.

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Abstract: This is the first of two articles devoted to a comprehensive exposition of the generating-function method for computing fusion rules in affine Lie algebras. The present paper is entirely devoted to the study of the tensor-product (infinite-level) limit of fusions rules. We consider thus in detail the problem of constructing tensor-product generating functions in finite Lie algebras. From the beginning, the problem is recast in terms of the concept of a model, which is an algebra whose Poincaré series is the generating function under study. We start by reviewing Sharp’s character method. Simple examples are worked out in detail, illustrating thereby its intrinsic limitations. An alternative approach to the construction of tensor-product generating function is then presented which overcomes most of the technical difficulties associated to the character method. It is based on the reformulation of the problem of calculating tensor products in terms of the solution of a set of linear and homogeneous Diophantine equations whose elementary solutions represent “elementary couplings”. Grobner bases provide a tool for generating the complete set of relations between elementary couplings and, most importantly, as an algorithm for specifying a complete, compatible set of “forbidden couplings”. This machinery is then applied to the construction of various tensor-product generating functions.

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1. Introduction

1.1. Orientation

Fusion rules yield the number of independent couplings between three given primary fields in conformal field theories. We are interested in fusion rules in unitary conformal field theories that have a Lie group symmetry, that is, those whose generating spectrum algebra is an affine Lie algebra at integer level. These are the Wess-Zumino-Witten models [1,2]. Primary fields in these cases are in 1-1 correspondence with the integrable representations of the appropriate affine Lie algebra at level \( k \). Denote this set by \( P_{+}^{(k)} \) and a primary field by the corresponding affine weight \( \hat{\lambda} \). Fusion coefficients \( N^{(k)}_{\hat{\lambda}\hat{\mu}\hat{\nu}} \) are defined by the product

\[
\hat{\lambda} \times \hat{\mu} = \sum_{\nu \in P_{+}^{(k)}} N^{(k)}_{\hat{\lambda}\hat{\mu}} \hat{\nu} \hat{\nu}
\]  

(1.1)

In the infinite-level limit and for fields with finite conformal dimensions (i.e. finite values of the finite (non-affine) Dynkin labels – see below), the purely affine condition on weight integrability is relaxed and the primary fields are solely characterized by their finite part, required to be an integrable weight of the corresponding finite Lie algebra. Recall that a weight \( \lambda \) is characterized by its expansion coefficients in terms of the fundamental weights \( \omega_i \)

\[
\lambda = \sum_{i=1}^{r} \lambda_i \omega_i = (\lambda_1, \ldots, \lambda_r)
\]

(1.2)

where \( r \) is the rank of the algebra. The numbers \( \lambda_i \)'s are the Dynkin labels. The set of weights with nonnegative Dynkin labels (the integrable weights) is denoted by \( P_{+} \).

Similarly, in the infinite-level limit, the fusion coefficients reduce to tensor-product coefficients:

\[
\lim_{k \to \infty} N^{(k)}_{\hat{\lambda}\hat{\mu}} \hat{\nu} = N_{\lambda\mu} \nu.
\]

(1.3)

where \( N_{\lambda\mu} \nu \) is defined by

\[
\lambda \otimes \mu = \sum_{\nu \in P_{+}} N_{\lambda\mu} \nu \nu
\]

(1.4)

By abuse of notation, we use the same symbol for the highest weight and the highest-weight representation. Notice that

\[
N_{\lambda\mu} \nu = N_{\lambda\mu\nu^*}
\]

(1.5)
where $\nu^*$ denotes the highest weight of the representation conjugate to that of $\nu$. Equivalently, $N_{\lambda \mu \nu^*}$ gives the multiplicity of the scalar representation in the triple product $\lambda \otimes \mu \otimes \nu^*$.

This paper is devoted to a detailed analysis of tensor products from the point of view of generating-function methods, preparing the ground for an analysis of fusion rules along the same lines. This will be the subject of a following article.

1.2. Tensor-product generating functions

A tensor-product generating function codes the information for all the tensor products of a given algebra in a single function. This function is first defined in terms of an infinite series as follows. We introduce the dummy variables $L_i, M_i, N_i$ with $i = 1, \ldots, r$ and multiply the tensor product coefficient $N_{\lambda \mu \nu}$ by $L^\lambda M^\mu N^\nu$ where

$$L^\lambda = L_{\lambda^1}^{\lambda^1} \cdots L_{\lambda^r}^{\lambda^r} \quad (1.6)$$

and sum over all integrable values of $\lambda, \mu, \nu$:

$$G(L, M, N) = \sum_{\lambda, \mu, \nu \in P_+} N_{\lambda \mu \nu} L^\lambda M^\mu N^\nu \quad (1.7)$$

$G$ can generally be expressed in terms of a rather simple closed function of its variables. For instance, for $su(2)$, it reads

$$G(L, M, N) = \frac{1}{(1 - LM)(1 - LN)(1 - MN)} \quad (1.8)$$

The above generating function contains all the global information concerning the tensor products in $su(2)$. An example of global information that can be extracted from such an expression is the number of couplings having a particular property. Suppose that we are interested in the total number of triple products with $\lambda_1 + \mu_1 + \nu_1 = 2n$ for a given positive integer value of $n$. We can rescale the three dummy variables by a factor $s$, giving

$$G(L, M, N; s) = \frac{1}{(1 - s^2 LM)(1 - s^2 LN)(1 - s^2 MN)}$$

$$= \sum_{n=0}^{\infty} s^{2n} \sum_{\lambda_1, \mu_1, \nu_1 \geq 0} N_{\lambda_1 \mu_1 \nu_1} L^{\lambda_1} M^{\mu_1} N^{\nu_1} \quad (1.9)$$
We then set $L = M = N = 1$ and read the desired information from the coefficient of $s^{2n}$ in the expansion:

$$G(s) = \frac{1}{(1 - s^2)^3} = \sum_{n=0}^{\infty} s^{2n} \sum_{\lambda_1+\mu_1+\nu_1=2n} N_{\lambda_1\mu_1}^{\nu_1}$$

that is

$$\sum_{\lambda_1+\mu_1+\nu_1=2n} N_{\lambda_1\mu_1}^{\nu_1} = \frac{(n+2)(n+1)}{2}$$

Furthermore, the generating function contains local information. A simple example of useful local information is the value of an explicit tensor-product coefficient. Given a closed expression for the generating function, explicit tensor-product coefficients can be read from its Taylor series expansion.

This in turn shows that another example of basic global information that can be deduced from a generating function is the integrality as well as the positivity of the tensor-product coefficients. In fact, tensor-product generating functions $G$ can always be written in a manifestly positive form.

More importantly, from our point of view, is that in the context of fusion rules, the construction of the simplest generating functions led to the discovery of the notion of threshold levels [3]. Moreover, as shown in the sequel paper, setting up a fusion generating function is a way to obtain explicit expressions for these threshold levels. Our new approach to fusion-rule generating functions, which originates from the generalization of techniques developed in the present paper on tensor products, leads to a further new concept, that of fusion basis.

1.3. Overview of the paper

The present article is organized as follows. We start by explaining in detail the construction of tensor-product generating functions for finite Lie algebras. The first construction which is presented is the character method developed by Sharp and his collaborators (section 2). Here the starting point is the construction of the generating function for the characters of all the irreducible representations which serves as the building block for the construction of the tensor-product generating functions. Simple examples of generating functions are worked out in details ($su(2)$, $su(3)$, $sp(4)$ and $osp(1,2)$). We also introduce
and exemplify the concept of a \textit{model} for a generating function. This is an algebra whose Poincaré series is the generating function under study. Models allow us to make use of powerful techniques from computational algebra to find generating functions as described below.

Although it is conceptually very simple, the character method is strongly limited by its inherent computational difficulties: the disproportion between the simplicity of the resulting form of the generating function and the intermediate calculations is enormous. This motivates our alternative approach to the construction of tensor-product generating function. It is based on the reformulation of the problem of calculating tensor products in terms of the solution of a set of linear and homogeneous Diophantine equations (cf. section 3). The Diophantine-equation reformulation of the problem is equivalent to its expression in terms of counting the number of integer solutions in a convex polytope and the polytope equations appropriate to any classical Lie algebra have been described by Berenstein and Zelevinsky. For \textsc{su}(N), these inequalities are equivalent to the Littlewood-Richardson algorithm, presented in section 4. The elementary solutions of these Diophantine equations represent “elementary couplings”. For \textsc{sp}(4), the use of the Berenstein-Zelevinsky inequalities to obtain the elementary couplings and their relations (cf. the analysis of section 6) is new.

The essential difficulty in constructing a generating function, given a set of inequalities is not finding the elementary solutions. For examples of the size we consider efficient algorithms exist to solve this problem. The key difficulty is related to the numerous relations – just alluded to – that exist in general between the elementary solutions. From the Diophantine-equation point of view, the decomposition of a solution may not be unique because different sums of elementary solutions could yield the same result.

These redundancies must then be eliminated by forbidding one of the two sums of elementary solutions occurring in each relation. However, the procedure for doing this when there are more that one relation is somewhat tricky: we cannot select at random a sum to be forbidden from each relation and be sure that no further relations will arise. This is the problem of finding a compatible set of forbidden couplings. To solve this problem we first “exponentiate” it: given a solution \(\alpha = (\alpha_1, \ldots, \alpha_k)\) to our system of linear Diophantine equations, we introduce formal variables \(X_1, \ldots, X_k\) and consider the monomial \(X_1^{\alpha_1} \cdots X_k^{\alpha_k}\). The linear span, \(R\), of all such monomials is a model for the generating function for the solutions to the original set of linear Diophantine equations.
(see section 5). To find the Poincaré series of \( R \) we use the theory of Grobner bases. Given a set of elementary couplings, or in this case generating monomials, Grobner basis techniques provide a method for generating a complete set of relations. In addition, and more importantly, they provide an algorithm for specifying a compatible set of “forbidden couplings”. In other words one can calculate the Poincaré series of \( R \) and hence the required generating function, directly from the set of relations provided by the Grobner basis method. It is then shown how this machinery applies to the construction of tensor-product generating functions. This is the subject of section 5.

A trivial modification of the description of tensor products in terms of inequalities follows by introducing extra positive labels that transforms the inequalities into equalities. For \( su(N) \), this gives rise to a remarkable graphical construction, the famous Berenstein-Zelevinsky triangles. These are introduced in section 7. We also discuss the analogous construction for \( sp(4) \), whose diagrammatic representation is new. But the main interest of these reformulations is that it yields a simple and systematic way of obtaining the elementary couplings from the construction of a vector basis. A projection operation yields the elementary couplings. In the process, we get a new way of constructing the generating functions. This vector-basis approach to the construction of tensor-product generating functions is illustrated with \( su(3) \) and \( sp(4) \).

1.4. What is new?

In part, this article is a comprehensive review of existing techniques for constructing tensor-product generating functions. In addition, the efficiency and the power of the different methods of construction are compared. We also, from a practical point of view, discuss the connections between commutative algebra and the computation and properties of generating functions. On the mathematical side, these links have been developed by Stanley [4]. In this article, emphasis is placed on the concept of a model of a generating function, which is an algebra whose Poincaré series is equal to the given generating function. This allows us to make use of powerful techniques in computational commutative algebra including Grobner bases, which were first used in this context in [5].

In this way we arrive at new derivations of previously known results. But those pertaining to \( sp(4) \), and in particular the use of the Berenstein-Zelevinsky inequalities as a way to fix unambiguously the linear relations between the “elementary couplings”, are new and should be stressed. However, the main achievement is a sound reformulation of the problem that is well-adapted to its extension to the fusion case.
2. Generating-function for tensor products: the character method

2.1. The character method for the construction of the tensor-product generating function: the su(2) case

The method developed by Sharp and collaborators for constructing generating functions for tensor products is based on manipulations of the character generating functions [6]. Although simple in principle, these manipulations become rather cumbersome as the rank of the algebra is increased. To illustrate the method, we will work in complete detail the simplest example, the su(2) case.

The first step is the derivation of the character generating function. The Weyl character formula for a general algebra of rank \( r \) and a highest-weight representation \( \lambda \) is

\[
\chi_\lambda = \frac{\xi_{\lambda + \rho}}{\xi_\rho}
\]  

(2.1)

where \( \rho \) is the finite Weyl vector, \( \rho = \sum_{i=1}^{r} \omega_i \), and where the characteristic function \( \xi \) is defined as

\[
\xi_{\lambda + \rho} = \sum_{w \in W} \epsilon(w)e^{w(\lambda + \rho)}
\]  

(2.2)

where \( \epsilon(w) \) is the signature of the Weyl reflection \( w \) and \( W \) stands for the Weyl group.

For \( su(2) \), the Weyl group contains two elements: 1, \( s_1 \). With

\[
x = e^{\omega_1}
\]  

(2.3)

the su(2) characteristic function \( \xi \) for the representation of highest weight \( m\omega_1 \equiv (m) \) is

\[
\xi_{m+1} = x^{m+1} - x^{-m-1}
\]  

(2.4)

The character reads then

\[
\chi_m = \frac{x^{m+1} - x^{-m-1}}{x - x^{-1}} = \frac{x^m - x^{-m-2}}{1 - x^{-2}} = x^m + x^{m-2} + \cdots + x^{-m}
\]  

(2.5)

The character generating function \( \chi_L \) is obtained by multiplying the above expression by \( L^m \) where \( L \) is a dummy variable, and summing over all positive values of \( m \):

\[
\chi_L(x) = \sum_{0}^{\infty} L^m \chi_m = \frac{1}{x - x^{-1}} \sum_{0}^{\infty} L^m (x^{m+1} - x^{-m-1})
\]

\[
= \frac{1}{1 - x^{-2}} \left( \frac{1}{1 - Lx} - \frac{x^{-2}}{1 - Lx^{-1}} \right) = \frac{1}{(1 - Lx)(1 - Lx^{-1})}
\]  

(2.6)
By construction, the character of the highest weight \((m)\) can be recovered from the power expansion of \(\chi_L\) as the coefficient of the term \(L^m\). The characteristic generating function \(\xi_L\) is defined by

\[
\chi_L(x) = \frac{\xi_L}{\xi_0}
\]  

(2.7)

and it reads

\[
\xi_L(x) = \frac{x - x^{-1}}{(1 - Lx)(1 - Lx^{-1})} = \frac{x}{1 - Lx} - \frac{x^{-1}}{1 - Lx^{-1}}
\]  

(2.8)

the last form being the one that results directly from (2.5).

The tensor product of two highest-weight representations can be obtained from the product of the corresponding characters:

\[
\chi_m \chi_n = \sum_{\ell} N_{mn}^\ell \chi_\ell
\]  

(2.9)

This information can be extracted from the product of the corresponding generating functions. We are thus led to consider the product \(\chi_L(x)\chi_M(x)\). To simplify the analysis of the resulting expression, notice that the information concerning the representations occurring in the tensor product is coded in the leading term of the character, i.e., the term \(x^{m+1}\). The rest of the representation is easily reconstructed by the application of the Weyl group and the action of the ladder operator. Actually, to insure that every positive power of \(x\) singles out a highest-weight representation, we can multiply both sides by \(\xi_0\): To read off these terms, we can focus on the terms with strictly positive powers of \(x\) in the product \(\chi_L(x)\chi_M(x)\xi_0(x)\). If we want to read off directly the Dynkin label of the representations (and not their shifted value), it is more convenient to divide by \(x\) before doing the projection, now restricted to the nonnegative powers of \(x\). The truncation of an expression by its negative powers of \(x\) will be denoted by the MacMahon symbol \([7]\ \Omega\), defined by

\[
\Omega \geq -\sum_{n \geq 0} c_n x^n = \sum_{n \geq 0} c_n x^n
\]  

(2.10)

When there is no ambiguity concerning the variable in terms of which the projection is defined, it is omitted from the \(\Omega\) symbol.

We are thus interested in the projection of the following expression

\[
\chi_L(x)\chi_M(x)\xi_0(x)x^{-1} = \chi_L(x)\xi_M(x)x^{-1}
\]

\[
= \frac{1}{(1 - Lx)(1 - Lx^{-1})} \left( \frac{1}{1 - Mx} - \frac{x^{-2}}{1 - Mx^{-1}} \right)
\]  

(2.11)
For these manipulations, we use systematically the following simple identities:

\[
\frac{1}{(1 - Ax)(1 - Bx^{-1})} = \frac{1}{(1 - AB)} \left( \frac{1}{1 - Ax} + \frac{Bx^{-1}}{1 - Bx^{-1}} \right)
\]

\[
= \frac{1}{(1 - AB)} \left( \frac{Ax}{1 - Ax} + \frac{1}{1 - Bx^{-1}} \right)
\]

\[
= \frac{1}{(1 - AB)} \left( \frac{1}{1 - Ax} + \frac{1}{1 - Bx^{-1}} - 1 \right)
\]

Which one is used is dictated by the context (a good choice often simplifies the subsequent analysis, the last one being somewhat more algorithmic).

We thus have two terms to analyze. The first one is

\[
\Omega \geq \frac{1}{(1 - Lx)(1 - Lx^{-1})(1 - Mx)} = \frac{1}{(1 - Lx)(1 - LM)} \left( \frac{1}{1 - Mx} + \frac{Lx^{-1}}{1 - Lx^{-1}} \right)
\]

(2.13)

The first part is not affected by the projection. The second one can be written as

\[
\frac{Lx^{-1}}{(1 - Lx)(1 - LM)(1 - Lx^{-1})} = \frac{Lx^{-1}}{(1 - LM)(1 - L^2)} \left( \frac{Lx}{1 - Lx} + \frac{1}{1 - Lx^{-1}} \right)
\]

(2.14)

The second part of the above expression contains only negative powers of \(x\) and can thus be ignored. The first part is unaffected by the projection. We have thus, for the first term of (2.11)

\[
\Omega \geq \frac{1}{(1 - Lx)(1 - Lx^{-1})(1 - Mx)} = \frac{1}{(1 - Lx)(1 - LM)} \left( \frac{1}{1 - Mx} + \frac{L^2}{1 - L^2} \right)
\]

(2.15)

The other term that needs to be considered is

\[
\Omega \geq \frac{x^{-2}}{(1 - Lx)(1 - Lx^{-1})(1 - Mx^{-1})}
\]

\[
= \frac{x^{-2}}{(1 - Lx^{-1})(1 - LM)(1 - Lx)}
\]

\[
= \frac{x^{-2}}{(1 - LM)(1 - L^2)} \left( \frac{Lx}{1 - Lx} + \frac{1}{1 - Lx^{-1}} \right)
\]

\[
= \frac{Lx^{-1}}{(1 - LM)(1 - L^2)(1 - Lx)}
\]

\[
= \frac{L^2}{(1 - LM)(1 - L^2)(1 - Lx)}
\]

(2.16)
Subtracting (2.16) from (2.15), we find that

\[ \Omega \chi_L(x) \xi_M(x) x^{-1} = \frac{1}{(1 - LM)(1 - Lx)(1 - Mx)} \] (2.17)

Replacing \( x \) by \( N \), we thus get

\[ G^{su(2)}(L, M, N) = \frac{1}{(1 - LM)(1 - LN)(1 - MN)} \] (2.18)

as the generating function for \( su(2) \) tensor products. To read off the representations \( (n) \) that occur in the tensor product \( (\ell) \otimes (m) \), we expand the generating function (as a Taylor series) and collect all terms \( N^n \) that are multiplied by \( L^\ell M^m \). All such values of \( n \) are the representations we are looking for. This is simply a restatement of the following relation between the tensor-product coefficients \( N_{\ell m}^n \) and the generating function:

\[ G(L, M, N) = \sum_{\ell, m, n \geq 0} N_{\ell m}^n L^\ell M^m N^n \] (2.19)

2.2. The abstract setting: Poincaré series, elementary couplings and relations; defining a model

As we shall see it is frequently useful have a model, \( R \), for a generating function \( G(X_1, \ldots, X_k) \) such as (2.18). By this we mean a commutative \( \mathbb{Q} \)-algebra with an identity, graded by \( \mathbb{N}^k \),

\[ R = \bigoplus_{\alpha \in \mathbb{N}^k} R_\alpha, \quad R_\alpha R_\beta \subseteq R_{\alpha + \beta} \] (2.20)

and such that its Poincaré series \( F(R) = \sum_{\alpha \in \mathbb{N}^k} \dim_{\mathbb{Q}}(R_\alpha)X^\alpha \)

satisfies

\[ F(R) = G(X_1, \ldots, X_k). \] (2.21)

For example, for (2.18), with \( X_1 = L, X_2 = M, X_3 = N \), we can take \( R = \mathbb{Q}[E_1, E_2, E_3] \) (in fact all our examples \( R \) is either a subring or quotient of a polynomial) with the

\[ \text{\footnotesize Such series are also called Hilbert series.} \]
grading of \( E_1, E_2 \) and \( E_3 \) being \((1, 1, 0), (1, 0, 1) \) and \((0, 1, 1)\). The homogeneous subspaces are spanned by \( E_1^a E_2^b E_3^c \), \( a, b, c \in \mathbb{N} \) with grade \((a + b, a + c, b + c)\) and so

\[
F(R) = \sum_{(a, b, c) \in \mathbb{N}^3} X_1^{a+b} X_2^{a+c} X_3^{b+c} = G^{su(2)}(X_1, X_2, X_3) \tag{2.22}
\]
as required.

If \( R \) is generated by elements \( E_1, \ldots, E_s \) and is a model for a generating function \( G \) for tensor products (or fusion products) then we call \( E_1, \ldots, E_s \) a set of “elementary couplings” for \( G \).

It should perhaps be stressed that \textit{a priori} the variables \( X_1, \ldots, X_k \) and \( E_1, \ldots, E_s \) are unrelated. We shall refer to the \( E \)’s as \textit{model variables} and the \( X \)’s as \textit{grading variables}. If the grading vector of \( E_i \) is \( \alpha^i \), \( i = 1, \ldots, s \) then there is an associated monomial in the grading variables: \( X^{\alpha^i} \), for which we will use the notation \( g(E_i) \). For example in the above example we have \( g(E_1) = X_1^1 X_2^1 X_3^0 = LM \). However, to avoid tedious repetition when writing down generating functions we shall often write, for example, \( 1/(1 - E_1) \) rather than \( 1/(1 - g(E_1)) \). In \textit{all} such cases where model variables appear in a generating function they should be replaced by the corresponding monomial in the grading variables. The utility of this convention can be seen by examining some of the examples of generating functions contained in the following sections and comparing them with the much more unwieldy expressions obtained by everywhere replacing the model variables by grading variables.

In the case of tensor products we use the notation “\( E : g(E) : \text{product} \)” to denote a set of elementary couplings with their “exponentiated” grading and the corresponding term in the tensor product. So in the example above we would write:

\[
E_1 : LM : (1) \otimes (1) \supset (0), \\
E_2 : LN : (1) \otimes (0) \supset (1), \\
E_3 : MN : (0) \otimes (1) \supset (1) \tag{2.23}
\]

Having made the distinction between grading and model variables, it should be noted that there are cases where we can identify the model as a ring generated by monomials in the grading variables. So in the above example we could define \( E_1 = LM, E_2 = LN \) and \( E_3 = MN \) and take the model for our generating function to be the subring of \( Q[L, M, N] \) generated by \( E_1, E_2 \) and \( E_3 \). However, it is not always desirable, or even possible, to make this identification.
As we shall see, identifying a generating function with the Poincaré series of a model will allow us to link the generating function more explicitly with combinatorial rules for calculating tensor product (and other) multiplicities. It will also allow us to make use of powerful algebraic techniques for computing Poincaré series.

In the example above we are given a generating function from which it is easy to find a model. Of course in general we will start with a model from which we construct the required generating function. We close this section with two examples of how models for the $su(2)$ character generating function can be constructed.

The first method, which has been exploited by Sharp et al (see [6]) to construct character generating functions, amounts to finding an algebra $R$ which is a module for the Lie algebra $su(2)$ and such that as an $su(2)$ module $R$ is isomorphic to $\oplus_{i \geq 1} V_i$ where $V_i$ is the irreducible $su(2)$ module of dimension $i$.

In this case we can take $R = \mathbb{Q}[p, q]$ with the generators of $su(2)$ being given by differential operators:

$$h = p \frac{\partial}{\partial p} - q \frac{\partial}{\partial q}, \quad x_+ = q \frac{\partial}{\partial p}, \quad x_- = p \frac{\partial}{\partial q}$$

The $su(2)$ highest-weight vectors are $p^i$, $i \geq 0$ and a basis of the irreducible submodule of dimension $i$ is just given by the monomials of degree $i$ in $p$ and $q$. We can give $R$ an $\mathbb{N}^3$ grading by taking the degree of $p$ to be $(1, 1, 0)$ and of $q$ to be $(1, 0, 1)$. Here the first grading index specifies the representation while the other two refer to a particular weight. As $R = \mathbb{Q}[p, q]$ the Poincaré function for $R$ is,

$$\frac{1}{(1 - p)(1 - q)}$$

with the understanding, as explained above, that $p$ and $q$ should be replaced by the corresponding expression in terms of the grading variables. Let us denote these grading variables here by $L$ (which exponentiates the representation index) and $x, y$ (exponentially related to the weights). The Poincaré function reads then

$$\frac{1}{(1 - Lx)(1 - Ly)}$$

Of course in this case the distinction involved and the usefulness of our convention is not immediately apparent.
Note that in this example we could take a $\mathbb{Z}^2$ grading with the grades of $p$ and $q$ being $(1, 1)$ and $(1, -1)$. However in general allowing negative weights means we have to be much more careful when taking Laurent series expansions and so we avoid this complication when possible. This grading is related to a single variable description of the weights $x^i, \cdots, x^{-i}$. Setting $y = x^{-1}$ gives the corresponding form of the generating function.

Another way of constructing a model for the weight generating function, which makes more natural the $\mathbb{N}^3$ grading, is to observe that the complete set $SU(2)$ weight vectors of finite dimensional irreducible $su(2)$ modules are in 1-1 correspondence with one-rowed Young tableaux. If the Young tableau has $c$ boxes filled with $a$ 1’s and $b$ 2’s then there is a constraint

$$a + b - c = 0, \quad a, b, c \geq 0 \quad (2.26)$$

and so the solutions to this linear Diophantine equation are in 1-1 correspondence with the complete set of $SU(2)$ weight vectors. Thus to find a model for the weight generating function it is sufficient to find a model for the solutions to (2.26). In this example it is not difficult to see that every solution to this equation is a linear combination (with non-negative coefficients) of the two fundamental solutions: $(a, b, c) = (1, 0, 1)$ and $(a, b, c) = (0, 1, 1)$. Let $R$ be the subring of $\mathbb{Q}[A, B, C]$ generated by the monomials $E_1 = AC$, $E_2 = BC$. Considering the exponents of the monomials $E_1$ and $E_2$, we see that the monomials in $R$ correspond to the solutions of (2.26) and hence taking the natural grading on $R$ ensures that the Poincaré series of $R$ is the generating function for the solutions to (2.26) and hence is the required generating function. In this example there are no relations between $E_1$ and $E_2$ and so $R$ is isomorphic to the polynomial ring in two variables (as expected) and so the Poincaré function is once again (with $A \to x, B \to y, C \to L$):

$$\frac{1}{(1 - Lx)(1 - Ly)} \quad (2.27)$$

In all these simple examples we have found a model which is a polynomial ring. In general this will not be the case. This is illustrated in the next section.

2.3. Multiple $su(2)$ tensor products

In order to illustrate in a rather simple context the occurrence of relations between elementary couplings, we will consider a slight generalisation of the previous problem.
namely, finding the multiplicity of a given representation $\zeta$ in the triple product $\lambda \otimes \mu \otimes \nu$. In terms of character generating functions, this amounts to considering the product $\chi_L(x)\chi_M(x)\chi_N(x) \supset \chi_P(x)$, or equivalently, $\chi_L(x)\chi_M(x)\xi_N(x)x^{-1} \supset \xi_P(x)x^{-1}$. The left side is then projected onto positive powers of $x$. We are thus led to consider

$$\Omega \geq \left(1 - Lx\right)\left(1 - Lx^{-1}\right)\left(1 - Mx\right)\left(1 - Mx^{-1}\right) \left(\frac{1}{1 - Nx} - \frac{x^{-2}}{1 - Nx^{-1}}\right)$$

(2.28)

The projection of each term is worked out as previously and the resulting expression is found to be, with $x$ replaced by $P$:

$$G(L, M, N, P) = \frac{1 - LMNP}{(1 - LP)(1 - MP)(1 - NP)(1 - LM)(1 - LN)(1 - MN)}$$

(2.29)

This is the sought for generating function. Here we would like to have a model with 6 elementary couplings corresponding to the terms in the denominator of the generating function:

$$E_1 : LM : (1) \otimes (1) \otimes (0) \supset (0)$$

$$E_2 : LN : (1) \otimes (0) \otimes (1) \supset (0)$$

$$E_3 : LP : (1) \otimes (0) \otimes (0) \supset (1)$$

$$E_4 : MN : (0) \otimes (1) \otimes (1) \supset (0)$$

$$E_5 : MP : (0) \otimes (1) \otimes (0) \supset (1)$$

$$E_6 : NP : (0) \otimes (0) \otimes (1) \supset (1)$$

(2.30)

and there must be a linear relation between the following products (signalled by a term in the numerator) which have grading $LMNP$:

$$E_1E_6, \quad E_2E_5, \quad E_3E_4$$

(2.31)

It is not difficult to see that a model is given by $\mathbb{Q}[e_1, e_2, e_3, e_4, e_5, e_6]/I$ where $E_i = e_i + I, i = 1, \ldots, 6$ and $I = (ae_1e_6 + be_2e_5 + ce_3e_4)$ is the ideal generated by the polynomial $ae_1e_6 + be_2e_5 + ce_3e_4$ with $a, b, c \in \mathbb{Q}$ not all zero. In particular, the model is far from unique. As we shall see later, particular methods of construction will select one particular model.

---

4 Relations between products of elementary couplings are often called syzygies in the physics literature (see in particular [6] and related works). However, the proper mathematical meaning of a syzygy is somewhat different and for this reason we stick to the more correct terminology of relation.
Before leaving this example, we would like to rework it from a different point of view, as an illustration of the ‘composition’ technique of generating functions. Let \( G(L, M, R) \) describe the tensor product corresponding to \( \chi_L \chi_M \supset \chi_R \) and similarly let \( G(Q, N, P) \) correspond to \( \chi_Q \chi_N \supset \chi_P \). We are interested the product \( \chi_L(x) \chi_M(x) \chi_N(x) \supset \chi_P(x) \) but treated from the product of the two generating functions \( G \). We thus want to enforce the constraint \( R = Q \) in the product \( G(L, M, R)G(Q, N, P) \). The idea is to multiply this product by \( (1 - Q^{-1}R^{-1})^{-1} \) and, in the expansion in powers of \( R \) and \( Q \), keep only terms of order zero in both variables: with an obvious notation we have

\[
\frac{RQ}{\Omega} G(L, M, R)G(Q, N, P) \frac{1}{1 - Q^{-1}R^{-1}} = \frac{RQ}{\Omega} \sum_n A_n(L, M) R^n \sum_m B_m(N, P) Q^m \sum_\ell R^{-\ell}Q^{-\ell}
\]

which is manifestly equivalent to considering

\[
\frac{x}{\Omega} G(L, M, x)G(x^{-1}, N, P)
\]

With the explicit expressions for the generating functions, we have thus

\[
\frac{x}{\Omega} = \frac{1}{(1 - Lx)(1 - Mx)(1 - LM)} \frac{1}{(1 - P^{-1}x^{-1})(1 - N^{-1}x^{-1})(1 - NP)}
\]

A brief and by now standard analysis yields directly the generating function (2.29).

2.4. The osp(1,2) case

The simplest example after \( su(2) \) is that of the superalgebra \( \text{osp}(1,2) \). Very little information is needed about superalgebras for the study of the \( \text{osp}(1,2) \) representations. We only need the fact that the highest weight \( \text{osp}(1,2) \) representations \( \{m\} \) decompose into a direct sum of two \( su(2) \) representations; \( (m) \oplus (m - 1/2) \) [8]. The character is thus

\[
\chi\{m\} = \frac{x^{m+1} + x^m - x^{-m} - x^{-m-1}}{x - x^{-1}}
\]

---

5 This trick is used in different references in [6], mainly in relation with the construction of generating functions for branching functions.
It is a simple exercise to check that the generating character function is

\[ \chi_L = \frac{1 + L}{(1 - Lx)(1 - Lx^{-1})} \]  \hspace{1cm} (2.36)

The tensor-product generating function is found to be [9]

\[ G^{osp(1,2)}(L, M, N) = \frac{1 - (LMN)^2}{(1 - LM)(1 - LN)(1 - MN)(1 - LMN)} \]
\[ = \frac{1 + LMN}{(1 - LM)(1 - LN)(1 - MN)} \]  \hspace{1cm} (2.37)

An underlying model would thus have four elementary couplings:

\begin{align*}
E_1 : LM : & \{1\} \otimes \{1\} \supset \{0\} \\
E_2 : LN : & \{1\} \otimes \{0\} \supset \{1\} \\
E_3 : MN : & \{0\} \otimes \{1\} \supset \{1\} \\
E_4 : LMN : & \{1\} \otimes \{1\} \supset \{1\} \\
\end{align*}

and the numerator indicates a linear relation between products of elementary couplings of degree \((LMN)^2\) namely:

\[ E_1E_2E_3 \quad \text{and} \quad E_4^2 \]  \hspace{1cm} (2.38)

A model is given by \( \mathbb{Q}[e_1, e_2, e_3, e_4]/I \) where \( E_i = e_i + I, i = 1, \ldots, 4 \) and \( I = (ae_1e_2e_3 + be_4^2) \) is the ideal generated by the polynomial \( ae_1e_2e_3 + be_4^2 \) with \( a, b \in \mathbb{Q} \), but not both zero.

2.5. The su(3) case

The next example in complexity is \( su(3) \). With

\[ x_i = e^{\omega_i} \quad i = 1, 2 \]  \hspace{1cm} (2.40)

the characteristic function for a representation of highest weight \( (m, n) = m\omega_1 + n\omega_2 \) is

\[ \xi_{(m,n)} = x_1^{m+1}x_2^{n+1} - x_1^{-m-1}x_2^{m+n+2} - x_1^{n+m+2}x_2^{-n-1} \\
+ x_1^{n+1}x_2^{-m-n-2} + x_1^{-m-n-2}x_2^{m+1} - x_1^{-n-1}x_2^{-m-1} \]  \hspace{1cm} (2.41)
The characteristic generating function is obtained by multiplying this result by $L_1^m L_2^n$ and summing over all positives values of $m, n$. The result is

$$\xi_{L_1L_2} = \frac{1 - L_1L_2}{(1 - L_1x_1)(1 - L_1x_1^{-1}x_2)(1 - L_2x_2)(1 - L_2x_2^{-1})(1 - L_2x_2^{-1})}$$

(2.42)

The construction of the tensor-product generating function proceeds exactly as for $\mathfrak{su}(2)$, but here it is much more complicated from the simple fact that there are two variables. The result is [6]

$$G^{\mathfrak{su}(3)}(L_1, L_2, M_1, M_2, N_1, N_2) = (1 - L_1L_2M_1M_2N_1N_2)$$

$$\times [(1 - L_1N_2)(1 - L_1M_2)(1 - L_2M_1)(1 - L_2N_1)]^{-1}$$

$$\times [(1 - M_2N_1)(1 - M_1N_2)(1 - L_1M_1N_1)(1 - L_2M_2N_2)]^{-1}$$

(2.43)

From the denominator we see that it is natural to seek a model with eight elementary couplings:

$$E_1 : L_1M_2, \quad E_2 : L_1N_2, \quad E_3 : M_1N_2$$

$$E_4 : L_2M_1, \quad E_5 : L_2N_1, \quad E_6 : M_2N_1$$

$$E_7 : L_1M_1N_1, \quad E_8 : L_2M_2N_2$$

(2.44)

The numerator indicates a relation between these elementary couplings that need to be taken into account to avoid over-counting multiplicities: there must be a linear relation between the following three products:

$$E_1E_3E_5, \quad E_2E_4E_6 \quad \text{and} \quad E_7E_8$$

(2.45)

which are the three terms with grading $L_1L_2M_1M_2N_1N_2$. This is the only relation required and a model for this generating function is given by $R = \mathbb{Q}[e_1, \ldots, e_8]/I$, with $E_i = e_i + I, i = 1, \ldots, 8$ and $I = (ae_1e_3e_5 + be_2e_4e_6 + ce_7e_8)$ with $ae_1e_3e_5 + be_2e_4e_6 + ce_7e_8 \neq 0$.

The elements of $R$ have the form $m + I$ with $m \in \mathbb{Q}[e_1, \ldots, e_8]$. However there is no canonical way of choosing the representatives $m$. Take for example the case $a = b = c = 1$. (Usually we will construct a model for our generating function as explained above and this construction will fix the values of $a, b$ and $c$). In $R$ we have $E_1E_3E_5 = -(E_2E_4E_6 + E_7E_8)$ and so we can take as a basis for $R$ the set of (equivalences classes of ) monomials which do not contain the product $E_1E_3E_5$. In this case we say that we have chosen to make $E_1E_3E_5$ a ‘forbidden product’. Similarly we can forbid the products $E_2E_4E_6$ or $E_7E_8$. As we shall see later, the choice of forbidden products corresponds to a choice of term ordering.
We can write the generating function in three different ways, each making manifest the fact that one of the above products never appears. Note once again we use the convention that $E_i$ should be replaced by $g(E_i)$ in these expressions:

\[
G = \left( \prod_{i \neq 1, 3, 5} (1 - E_i)^{-1} \right) \left( 1 + \frac{E_1}{(1 - E_1)(1 - E_5)} \right) \\
+ \frac{E_3}{(1 - E_3)(1 - E_1)} + \frac{E_5}{(1 - E_5)(1 - E_3)} \\
= \left( \prod_{i \neq 2, 4, 6} (1 - E_i)^{-1} \right) \left( 1 + \frac{E_2}{(1 - E_2)(1 - E_4)} \right) \\
+ \frac{E_4}{(1 - E_4)(1 - E_6)} + \frac{E_6}{(1 - E_6)(1 - E_2)} \\
= \left( \prod_{i \neq 7, 8} (1 - E_i)^{-1} \right) \left( 1 + \frac{E_7}{(1 - E_7)} + \frac{E_8}{(1 - E_8)} \right)
\]

(2.46)

It is clear that in expanding the first form, we will never encounter a term corresponding to a product of the three factors $E_1 E_3 E_5$. Similarly no product $E_2 E_4 E_6$ corresponds to a term in the second form, while the last expression amounts to forbidding all factors containing $E_7 E_8$. Therefore, although $G$ is unique, its expression in terms of the $E_i$’s is not, because the forbidden couplings may be chosen in different ways.

Before leaving this example, let us mention another way of constructing the generating function using the idea of ‘composition’ of generating functions described previously. This uses the Giambelli formula that expresses a general representation in terms of a difference of products of representations with a single non-zero Dynkin label, i.e.,

\[
(\mu_1, \mu_2) = (\mu_1 + \mu_2, 0) \otimes (\mu_2, 0) - (\mu_1 + \mu_2 + 1, 0) \otimes (\mu_2 - 1, 0)
\]

(2.47)

First consider the generating function:

\[
G(L_1, L_2, M_1, N_1, N_2)
\]

(2.48)

which is the generating function for products of the form: $(\lambda_1, \lambda_2) \otimes (\mu_1, 0)$. From this generating function we form:

\[
H(L_1, L_2, M_1, M_2, R_1, R_2) = \frac{N}{Q} G(L_1, L_2, M_1, N_1, N_2) G(N^{-1}_1, N^{-1}_2, M_2, R_1, R_2)
\]

(2.49)
which is the generating function for products of the form

\[(\lambda_1, \lambda_2) \otimes (\mu_1, 0) \otimes (\mu_2, 0)\] (2.50)

Note that the generating function for products

\[(\lambda_1, \lambda_2) \otimes (\mu_1 + 1, 0) \otimes (\mu_2 - 1, 0)\] (2.51)

is \(HM_2M_1^{-1}\) and so, by (2.47), the generating function for products \((\lambda_1, \lambda_2) \otimes (\mu_1, \mu_2)\) is:

\[\frac{M}{\Omega} [H - HM_2M_1^{-1}]\] (2.52)

The coefficient of \(M_1^{\mu_1}M_2^{\mu_2}\) is the multiplicity of the representation with Dynkin labels \((\mu_1 - \mu_2, \mu_2)\) in the product

\[\xi_{(\mu_1, 0) \otimes (\mu_2, 0) - (\mu_1 + 1, 0) \otimes (\mu_2 - 1, 0)}\] (2.53)

To change to variables which carry the Dynkin labels we make the substitution \(M_2 \mapsto M_2M_1^{-1}\), so that \(M_1\) now carries the first Dynkin label. This introduces negative powers of \(M_1\), corresponding to products

\[(\lambda_1, \lambda_2) \otimes [(\mu_1, 0) \otimes (\mu_2, 0) - (\mu_1 + 1, 0) \otimes (\mu_2 - 1, 0)]\] (2.54)

with \(\mu_1 < \mu_2\), which are not required. So we must keep only positive degree terms in \(M_1\) to obtain the final generating function.

2.6. The \(sp(4)\) case

As a final example, consider the \(sp(4)\) case. With the \(x_i\) defined as in (2.40), the characteristic function is found to be

\[\xi_{(m,n)} = x_1^{m+1}x_2^{n+1} - x_1^{-m-1}x_2^{m+n+2} - x_1^{n+m+5}x_2^{-n-1} + x_1^{m+2n+3}x_2^{-m-n-2} + x_1^{-m-2n-3}x_2^{m+n+2} - x_1^{m+1}x_2^{-m-n-} - x_1^{-m-2n-3}x_2^n + x_1^{-m-1}x_2^{-n-1}\] (2.55)

and the characteristic generating function is

\[\xi_{L_1,L_2} = \frac{1}{(1 - L_1x_1)(1 - L_1x_1x_2^{-1})(1 - L_2x_2^{-1})(1 - L_2x_2x_2^{-1})} (1 + \frac{L_2}{(1 - L_2x_1x_2^{-1})(1 - L_2x_2^{-1})} + \frac{(1 + L_2)L_1x_1}{(1 - L_1x_1)(1 - L_2x_1^{-1}x_2)} + \frac{L_1x_1^{-1}x_2}{(1 - L_1x_1)(1 - L_1x_1^{-1}x_2)})\] (2.56)

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From this characteristic generating function, we construct the character generating function and then we can proceed to the tensor-product generating function. This is again extremely cumbersome. The result is [10]

\[
G_{sp}^{(4)}(L_1, L_2, M_1, M_2, N_1, N_2) = \frac{1}{[(1 - M_1 N_1)(1 - L_1 N_1)(1 - L_1 M_1)(1 - M_2 N_2)(1 - L_2 N_2)(1 - L_2 M_2)]^{-1}}
\]

\[
= \left( \frac{1}{(1 - L_2 M_1 N_1)(1 - L_2 M_1^2 N_2)} + \frac{L_2 M_2 N_1^2}{(1 - L_2 M_1 N_1)(1 - L_2 M_2 N_1^2)} \right)
\]

\[
+ \frac{L_1 M_2 N_1 N_2}{(1 - L_1 M_2 N_1)(1 - L_1 M_2^2 N_2)} + \frac{L_1 M_2 N_1}{(1 - L_1 M_2 N_1)(1 - L_2 M_2 N_1^2)}
\]

\[
+ \frac{L_2 M_2 N_2}{(1 - L_1 M_1 N_2)(1 - L_1 M_2 N_2)} + \frac{L_1 M_1 N_2}{(1 - L_1 M_1 N_2)(1 - L_2 M_1^2 N_2)}
\]

From this expression, we read off the following list of elementary couplings (recall that the first variable is a model variable and then we write the corresponding monomial in the grading variables):

\[
A_1 : M_1 N_1, \quad A_2 : L_1 N_1, \quad A_3 : L_1 M_1 \\
B_1 : M_2 N_2, \quad B_2 : L_2 N_2, \quad B_3 : L_2 M_2 \\
C_1 : L_2 M_1 N_1, \quad C_2 : L_1 M_2 N_1, \quad C_3 : L_1 M_1 N_2 \\
D_1 : L_1^2 M_2 N_2, \quad D_2 : L_2 M_1^2 N_2, \quad D_3 : L_2 M_2 N_1^2.
\]

However, not all the products of the model variables can be linearly independent: there are linear relations between:

\[
C_i C_j, \quad A_k D_k, \quad \text{and} \quad A_i A_j B_k \\
D_i D_j, \quad A_k^2 B_i B_j, \quad \text{and} \quad B_k C_k^2 \\
C_i D_i, \quad A_j B_k C_k, \quad \text{and} \quad A_k B_j C_j
\]

for \(i, j, k\) a cyclic permutation of 1, 2, 3 (and repeated indices are not summed). (It is plain that the three sets of products found to be linearly related must have the same Dynkin labels.) A specific form of the generating functions amounts to a specific choice of a set of forbidden couplings among those that are related by a linear relation.

2.7. Technical remarks on the character method

The character method provides a first principle approach to the construction of tensor-product generating functions. This is certainly its great virtue. However, the last two
examples indicate the essential limitation of this method for constructing generating functions for tensor products: the calculations are extremely complicated. Further progress requires the search for more powerful techniques.

It should also be clear from the last example that in general the problem of finding a model and computing its Poincaré series is not trivial even when we know the form of the generating function. Of course we shall be interested in the inverse procedure, that is, in constructing models in order to compute the associated generating function. This requires a systematic procedure for computing Poincaré series and this will be provided by the theory of Grobner bases which we shall discuss shortly.

3. Tensor-product descriptions

3.1. The need for a tensor-product description

As already mentioned, the fundamental limitation of the character method for constructing tensor-product generating functions lies in the complication of the intermediate calculations associated to the projection to positive powers of the $x_i$'s variables of the different terms of the generating character products. The complication of these intermediate steps should be contrasted with the relative simplicity of the resulting generating functions. This state of affair strongly suggests that there are much more efficient ways of obtaining these generating functions.

It is clear that one major technical complication of the character method is that it starts at too fundamental a level, namely the character of the separate representations. As a result, we need to take care of the action of the Weyl group: this generates many terms and the $\Omega$ projection of each term is rather complicated. But the fact that their final sum conspires to produce a rather simple result suggests that bypassing the use of the Weyl group would induce substantial simplifications.

One natural way to proceed is to start from a combinatorial description of the tensor-product rules. Such a description already takes into account the action of the Weyl group and encodes the various subtractions of the singular vectors.

---

6 This continues to be the case when affine Weyl reflections are included for the fusion-rule generating functions, with the additional complication that extra projections are required to ensure that the representation labels are in the fundamental region of the affine Weyl group.
But how do we make the connection with the generating-function approach? The key is to find a combinatorial description which can be expressed as a set of linear Diophantine inequalities.

Given this set of inequalities, there is an algorithm, again due to MacMahon, for constructing a generating function. In this context, the generating-function method appears as a general approach to the solutions of a system of inequalities. This is particularly well illustrated in MacMahon’s book [7]. This provides then a direct route from the Diophantine inequalities to the generating function. This method is conceptually similar to the character method, except that the starting point is substantially closer to the end result. Let us illustrate MacMahon’s approach with a simple example.

3.2. MacMahon’s theory for the solution of Diophantine inequalities

We look for all the positive integer solutions of the inequalities (see e.g. [7] vol. 2 no 356 p. 109)

\[ x_1 \geq x_2 \quad x_1 \geq x_3 \]  

(3.1)

To impose these constraints on a free series of the form

\[ \sum X_1^{x_1} X_2^{x_2} X_3^{x_3} \]  

(3.2)

we introduce extra parameters \( t \) and \( s \) as follows. To take into account the first inequality, we replace \( X_1 \) by \( tX_1 \) and \( X_2 \) by \( t^{-1}X_2 \) and project onto positive powers of \( t \). Similarly, the second inequality is taken care by the replacements \( X_1 \to sX_1 \) and \( X_3 \to s^{-1}X_3 \) and projecting onto positive powers of \( s \). This leads to

\[
\frac{t \cdot s}{\Omega \cdot \Omega} \geq \frac{1}{(1-t sX_1)(1-t^{-1}X_2)(1-s^{-1}X_3)} \]  

(3.3)

whose projections read

\[
\frac{1-stX_1^2X_2X_3}{(1-t sX_1)(1-sX_1X_2)(1-tX_1X_3)(1-X_1X_2X_3)} \]  

(3.4)

7 This is an adaptation of a method developed by Elliot [11] for the analysis of linear Diophantine equalities and for this reason the algorithm is often referred to as the Elliot-MacMahon method.

8 See in particular vol. 2 section VIII.
We then set the auxiliary variables $s, t$ equal to 1 and obtain:

$$\frac{1 - X_1^2X_2X_3}{(1 - X_1)(1 - X_1X_2)(1 - X_1X_3)(1 - X_1X_2X_3)}$$  \hspace{1cm} (3.5)

From this we read the four elementary solutions to the Diophantine inequalities under study:

$$\alpha_1 = (1, 0, 0), \quad \alpha_2 = (1, 1, 0)$$
$$\alpha_3 = (1, 0, 1), \quad \alpha_4 = (1, 1, 1)$$  \hspace{1cm} \text{(3.6)}

with the ordering $(x_1, x_2, x_3)$ and the linear relation

$$\alpha_2 + \alpha_3 = \alpha_1 + \alpha_4$$  \hspace{1cm} (3.7)

3.3. Seeking a road from elementary couplings to generating functions

Although the description of tensor products via linear Diophantine equations is a more efficient route to finding the generating function than the character one, complications associated to the Ω projections remain a source of technical difficulty that severely limits the practical applicability of the method.

A more powerful approach to our problem is to use the techniques of computational algebra. We start with a description of the tensor-product multiplicities as solutions to linear Diophantine inequalities. From these we find directly a model for the generating function.

In this approach, we thus work with a particular model of our generating function. But the advantage of this is that we can use Grobner-basis techniques, described below, to find the Poincaré series of the model and hence the required generating function.

4. The LR rule ($su(N)$)

For $su(N)$ tensor products, there is a particularly convenient description, which is that of the Littlewood-Richardson tableaux, supplemented by the stretched-product operation (defined below) [5].

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9 This is roughly the inverse of the MacMahon’s method which was originally conceived as a technique to generate the elementary couplings and their linear relations through the construction of the generating function. Here, the elementary couplings and their relations are first obtained and used as the input for the construction of the generating function.
Integrable weights in $su(N)$ can be represented by tableaux: the weight $(\lambda_1, \lambda_2, \cdots, \lambda_{N-1})$ is associated to a left justified tableau of $N-1$ rows with $\lambda_1 + \lambda_2 + \cdots + \lambda_{N-1}$ boxes in the first row, $\lambda_1 + \lambda_2 + \cdots + \lambda_{N-2}$ boxes in the second row, etc. Equivalently, the tableau has $\lambda_1$ columns of 1 box, $\lambda_2$ columns of 2 boxes, etc. The scalar representation has no boxes, or equivalently, any number of columns of $N$ boxes.

The Littlewood-Richardson rule is a simple combinatorial description of the tensor product of two $su(N)$ representations $\lambda \otimes \mu$. The second tableau ($\mu$) is filled with numbers as follows: the first row with 1’s, the second row with 2’s, etc. All the boxes with a 1 are then added to the first tableau according to following restrictions:

1) the resulting tableau must be regular: the number of boxes in a given row must be smaller or equal to the number of boxes in the row just above;

2) the resulting tableau must not contain two boxes marked by 1 in the same column.

All the boxes marked by a 2 are the added to the resulting tableaux according to the above two rules (with 1 is replaced by 2) and the further restriction:

3) in counting from right to left and top to bottom, the number of 1’s must always be greater or equal to the number of 2’s.

The process is repeated with the boxes marked by a 3, 4, $\cdots$, with the additional rule that the number of $i$’s must always be greater or equal to the number of $i+1$’s when counted from right to left and top to bottom. The resulting Littlewood-Richardson (LR) tableaux are the Young tableaux of the irreducible representations occurring in the decomposition.

These rules can be rephrased in an algebraic way as follows. Define $n_{ij}$ to be the number of boxes $i$ that appear in the LR tableau in the row $j$. The LR conditions read

$$\lambda_{j-1} + \sum_{i=1}^{k-1} n_{i,j-1} - \sum_{i=1}^{\min(k,N-1)} n_{ij} \geq 0 \quad 1 \leq k \leq j \leq N \quad j \neq 1 \quad (4.1)$$

and

$$\sum_{j=i}^{k} n_{i-1,j-1} - \sum_{j=i}^{k} n_{ij} \geq 0 \quad 2 \leq i \leq k \leq N \quad \text{et} \quad i \leq N-1. \quad (4.2)$$
The weight $\mu$ of the second tableau and the weight $\nu$ of the resulting LR tableau are respectively given by

$$\begin{align*}
\sum_{j=i}^{N} n_{ij} &= \sum_{j=i}^{N-1} \mu_j \quad i = 1, 2, \ldots, N - 1 , \\
\nu_j - \lambda_j + \sum_{i=1}^{N-1} n_{i,j+1} &= \sum_{i=1}^{\min(j,N-1)} n_{ij} \quad j = 1, 2, \ldots, N - 1 .
\end{align*}$$

Hence, given three weights $\lambda, \mu$ and $\nu$, the number of positive integers solutions $\{n_{ij}\}$ satisfying the above conditions gives the multiplicity $N_{\lambda\mu}^{\nu}$ of $\nu$ in the tensor product $\lambda \otimes \mu$.

The combined equations (4.1) and (4.2) constitute a set of linear and homogeneous inequalities. As described in [4], the Hilbert basis theorem guarantees that every solution can be expanded in terms of the elementary solutions of these inequalities.

As explained in section 3.2 we can construct a model for the solutions of the equations (4.1) and (4.2) by introducing new formal variables $A_i$, $1 \leq i \leq t$ where $t$ is the total number of variables in (4.1) and (4.2). Then the subring of $\mathbb{Q}[A_i; 1 \leq i \leq t]$ generated by the monomials $A^\alpha$ with $\alpha$ a solution of (4.1) and (4.2) provides the required model. This ring $R$ will be generated by a finite set of monomials $E_j$ $1 \leq j \leq s$ which we call elementary couplings corresponding to the elementary solutions of (4.1) and (4.2). Thus $R$ is isomorphic to $\mathbb{Q}[e_1, \ldots, e_s]/I$ under the mapping $\phi : e_i \rightarrow E_i$ where $I$ is an ideal. Each element of $I$ corresponds, via the map $\phi$, to a relation between the elementary couplings. As we shall see shortly, calculating a generating set of elements of $I$ (or more particularly a Grobner basis) is the key step in our calculations.

In the case of LR tableaux, there is a nice pictorial representation of the model $R$. Consider the set of formal linear combinations of LR tableaux with rational coefficients. It is given a ring structure by defining the stretched product of two LR tableaux (denoted by $\cdot$) to be the tableau obtained by fusing the two tableaux and reorder the numbers in each row in increasing order. More algebraically, if we denote the void boxes of a LR tableau by a 0, so that

$$n_{0j} = \sum_{i=j}^{N-1} \lambda_i \quad j = 1, 2, \ldots, N - 1$$

we can characterize completely a tableau by the data $\{n_{ij}\}$ with now $i \geq 0$. It is clear the set of numbers $\{n_{ij}\}$ with $i \geq 0$, or equivalently, $\{\lambda_i, n_{ij}\}$ with $i \geq 1$, is a complete
set of variables for the description of the tensor products. Then, the tableau obtained by
the stretched product of the tableaux \( \{n_{ij}\} \) and \( \{n'_{ij}\} \) is simply described by the numbers
\( \{n_{ij} + n'_{ij}\} \). Here is a simple example:

\[
\begin{array}{ccc}
1 & 2 & 1 \\
2 & 3 & 2 \\
4 & & \\
\end{array}
\cdot
\begin{array}{ccc}
1 & & 1 \\
1 & 2 & 2 \\
2 & 2 & 3 \\
\end{array}
= 
\begin{array}{ccc}
1 & 1 & 1 \\
1 & 2 & 2 \\
2 & 2 & 3 \\
4 & & \\
\end{array}
\]

(4.5)

This ring of tableaux, then, is isomorphic to the model \( R \) constructed above and we
do not distinguish between them. Thus we specify a set of elementary couplings (i.e. a set
of generators of \( R \)) as a set of elementary LR Tableaux.

4.1. Example: the \( su(2) \) case

The complete set of inequalities for \( su(2) \) variables \( \{\lambda_1, n_{11}, n_{12}\} \) is simply

\[
\lambda_1 \geq n_{12} \quad n_{11} \geq 0 \quad n_{12} \geq 0
\]

(4.6)
The other weights are fixed by

\[
\mu_1 = n_{11} + n_{12} \quad \nu_1 = \lambda_1 + n_{11} - n_{12}
\]

(4.7)
By inspection, the elementary solutions of this set of inequalities are

\[
(\lambda_1, n_{11}, n_{12}) = (1, 0, 1), \quad (1, 0, 0), \quad (0, 1, 0)
\]

(4.8)
which correspond respectively to \( E_1, E_2, E_3 \) in (2.23). These correspond to the following
LR tableaux:

\[
E_1 : \begin{array}{c}
1 \\
\end{array}, \quad E_2 : \begin{array}{c}
\end{array}, \quad E_3 : \begin{array}{c}
1 \\
\end{array}
\]

(4.9)
It is also manifest that there are no linear relations between these couplings. The generating
function is thus simply:

\[
G^{su(2)} = \frac{1}{(1 - E_1)(1 - E_2)(1 - E_3)}
\]

(4.10)
(Recall once again our convention concerning grading and model variables: the LR tableaux
in this generating function must be replaced by the corresponding monomials in the grading
variables).
4.2. Example: multiple tensor products in the $su(2)$ case

Before we turn to more complicated algebras, it is interesting to reconsider the problem of multiple tensor products treated previously from the character method. Let us look for the multiplicity of the representation $\zeta$ in the triple product $\lambda \otimes \mu \otimes \nu \supset \zeta$. In the first step, the LR rule applies as before: with $n_{11} + n_{12} = \mu_1$, we have $\lambda_1 \geq n_{12}$. After the first product, we re-apply the LR rule with now $\lambda_1$ replaced by $\lambda_1 + n_{11} - n_{12}$ and $n_{ij}$ replaced by $m_{ij}$ with $m_{11} + m_{12} = \nu_1$. The LR gives $\lambda_1 + n_{11} - n_{12} \geq m_{12}$. The two inequalities that defines the $su(2)$ LR basis for the quadruple product are then

$$
\lambda_1 \geq n_{12} \quad \lambda_1 + n_{11} - n_{12} \geq m_{12} \quad n_{ij} \geq 0 \quad m_{ij} \geq 0 \quad (4.11)
$$

The elementary solutions are then, in the order: name of the coupling, corresponding Dynkin labels and the 5-vector $(\lambda_1, n_{11}, n_{12}, m_{11}, m_{12})$:.

$$
\begin{align*}
E_1 : (1) \otimes (1) \otimes (0) & \supset (0) \\
E_2 : (1) \otimes (0) \otimes (1) & \supset (0) \\
E_3 : (1) \otimes (0) \otimes (0) & \supset (1) \\
E_4 : (0) \otimes (1) \otimes (1) & \supset (0) \\
E_5 : (0) \otimes (1) \otimes (0) & \supset (1) \\
E_6 : (0) \otimes (0) \otimes (1) & \supset (1)
\end{align*}
$$

The linear relation, whose existence was signalled by the character method, is

$$
E_3E_4 = E_2E_5 : (1, 1, 0, 0, 1), \quad \neq E_1E_6 : (1, 0, 1, 1, 0) \quad (4.13)
$$

Choosing to forbid the product $E_3E_4$, the generating function can be written in the form

$$
G = \frac{1 - E_3E_4}{(1 - E_1)(1 - E_2)(1 - E_3)(1 - E_4)(1 - E_5)(1 - E_6)}
$$

$$
= \left( \prod_{i=1,2,5,6} \frac{1}{1 - E_i} \right) \left( \frac{1}{1 - E_3} + \frac{E_4}{1 - E_4} \right) \quad (4.14)
$$

The latter form makes manifest the absence of $E_3E_4$.

We could represent the elementary couplings in terms of tableaux, where the boxes with 1 refers to the $\mu$ tableau and those with 2 originates from the $\nu$ tableau. (Warning: the
resulting tableaux describing the four-products are not necessarily LR tableaux.) Hence, \( n_{1j} \) gives the number of 1 in row \( j \) of the composed tableau while \( m_{1k} \) gives the number of 2 in row \( k \). The elementary tableaux are

\[
E_1 : \begin{array}{c}
1 \\
\end{array}, \quad E_2 : \begin{array}{c}
2 \\
1 \\
\end{array}, \quad E_3 : \begin{array}{c}
\end{array}
\]

\[
E_4 : \begin{array}{c}
1 \\
2 \\
\end{array}, \quad E_5 : \begin{array}{c}
1 \\
\end{array}, \quad E_6 : \begin{array}{c}
2 \\
\end{array}
\]

(4.15)

From this representation, the relation reads

\[
E_3E_4 = E_2E_5 : \begin{array}{c}
1 \\
2 \\
\end{array}, \quad \neq E_1E_6 : \begin{array}{c}
2 \\
1 \\
\end{array}
\]

(4.16)

It is worth pointing out here the relation between the multiplicities of the \( su(2) \) triple tensor products and \( su(3) \) weight multiplicities in \( su(3) \) irreducible representations. To make this relation explicit, we first consider void boxes to be filled with 0 and then reshuffle all filling numbers by 1, i.e.,

\[
\begin{array}{c}
2 \\
1 \\
\end{array} \rightarrow \begin{array}{c}
1 \\
3 \\
2 \\
\end{array}
\]

(4.17)

The multiplicity of the product \( \lambda \otimes \mu \otimes \nu \supset \zeta \) is then seen to be the number of tableaux filled by \( \lambda_1 \)'s, \( \mu_1 \) 2's, \( \nu_1 \) 3's (the tableau has thus a total number of \( \lambda_1 + \mu_1 + \nu_1 \) boxes) with the restriction that there should be \( \zeta_1 \) columns of 1 box. Since by construction, the numbers are strictly increasing in each column from top to bottom and non-decreasing in each row from left to right, these tableaux are nothing but \( su(3) \) semistandard tableaux. The resulting tableau has \( su(3) \) weight

\[
\lambda_1 \omega_1 + \mu_1 (-\omega_1 + \omega_2) - \nu_1 \omega_2
\]

(4.18)

and row lengths: \( (\lambda_1 + \mu_1 + \nu_1 + \zeta_1)/2 \) and \( (\lambda_1 + \mu_1 + \nu_1 - \zeta_1)/2 \). In terms of Dynkin labels, the number of such semistandard tableaux is exactly the multiplicity of the \( su(3) \) weight \( (\lambda_1 - \mu_1, \mu_1 - \nu_1) \) in the representation of highest weight \( (\zeta_1, (\lambda_1 + \mu_1 + \nu_1 - \zeta_1)/2) \).

For instance, the \( su(2) \) product \((1) \otimes (1) \otimes (1) \supset (1)\) has multiplicity 2, corresponding to the products \( E_3E_4 \) (forbidding \( E_2E_5 \)) and \( E_1E_6 \), or equivalently, to the two tableaux

\[
\begin{array}{c}
1 \\
2 \\
\end{array}, \quad \begin{array}{c}
2 \\
1 \\
\end{array}
\]

(4.19)

\footnote{In a \( su(N) \) semistandard tableau, a box filled by \( i \) has weight \(-\omega_{i-1} + \omega_i\), with \( \omega_0 = \omega_N = 0 \).}
and this is exactly the multiplicity of the \( su(3) \) weight \((0,0)\) in the highest-weight representation \((1,1)\), whose two semistandard tableaux are

\[
\begin{array}{|c|c|}
\hline
1 & 2 \\
\hline
3 \\
\hline
\end{array}, \quad \begin{array}{|c|}
\hline
1 \\
\hline
3 \\
\hline
2 \\
\hline
\end{array}
\]

(4.20)

Equivalently, since the multiplicity of the \( su(2) \) product \( \lambda \otimes \mu \otimes \nu \supset \zeta \) is the same as \( \lambda \otimes \mu \otimes \nu \otimes \zeta \supset (0) \), by filling the boxes of \( \zeta \) with 4’s, one can relate the resulting multiplicity to a \( su(4) \) weight multiplicity in an irreducible representation of two equal rows. More precisely, the resulting tableaux have \( \lambda_1 1’s, \mu_1 2’s, \nu_1 3’s \) and \( \zeta_1 4’s \), which corresponds to a weight

\[
\lambda_1 \omega_1 + \mu_1 (-\omega_1 + \omega_2) + \nu_1 (-\omega_2 + \omega_3) - \zeta_1 \omega_3
\]

(4.21)

and the corresponding irreducible representation has weight \((0,n,0,0)\) with

\[
2n = \lambda_1 + \mu_1 + \nu_1 + \zeta_1
\]

(4.22)

More generally, the multiplicity of the multiple \( N \geq 3 \) \( su(2) \) product

\[
\bigotimes_{i=1}^{N} \lambda^{(i)} \supset (0)
\]

(4.23)

is equal to the multiplicity of the \( su(N) \) weight \( \Lambda' \) in the highest-weight representation \( \Lambda \), with

\[
\Lambda' = \sum_{i=1}^{N} \lambda^{(i)} (-\omega_{i-1} + \omega_i) \quad \Lambda = (0, \frac{1}{2} \sum_{i=1}^{N} \lambda^{(i)}_{i}, 0, \ldots, 0)
\]

(4.24)

(with \( \omega_0 = \omega_N = 0 \)).

4.3. Example: the \( su(3) \) case

Let us return to standard triple tensor products and turn to \( su(3) \). The LR conditions for \( su(3) \) are

\[
\lambda_1 \geq n_{12} \\
\lambda_2 \geq n_{13} \\
\lambda_2 + n_{12} \geq n_{13} + n_{23} \\
n_{11} \geq n_{22} \\
n_{11} + n_{12} \geq n_{22} + n_{23}
\]

(4.25)
The other weights are given by

\[
\begin{align*}
\mu_1 &= n_{11} + n_{12} + n_{13} - n_{22} - n_{23} \\
\mu_2 &= n_{22} + n_{23} \\
\nu_1 &= \lambda_1 + n_{11} - n_{12} - n_{22} \\
\nu_2 &= \lambda_2 + n_{12} + n_{22} - n_{13} - n_{23}
\end{align*}
\]

The elementary solutions of the set of inequalities (4.25) are again easily found by inspection and they are given by the following set of numbers: \((\lambda_1, \lambda_2, n_{11}, n_{12}, n_{13}, n_{22}, n_{23})\):

\[
\begin{align*}
E_1 &: (1, 0) \otimes (0, 1) \supset (0, 0) : (1, 0, 0, 1, 0, 0, 1) \\
E_2 &: (1, 0) \otimes (0, 0) \supset (1, 0) : (1, 0, 0, 0, 0, 0, 0) \\
E_3 &: (0, 0) \otimes (1, 0) \supset (1, 0) : (0, 0, 1, 0, 0, 0, 0) \\
E_4 &: (0, 1) \otimes (1, 0) \supset (0, 0) : (0, 1, 0, 0, 1, 0, 0) \\
E_5 &: (0, 1) \otimes (0, 0) \supset (0, 1) : (0, 1, 0, 0, 0, 0, 0) \\
E_6 &: (0, 0) \otimes (0, 1) \supset (0, 1) : (0, 0, 1, 0, 0, 1, 0) \\
E_7 &: (1, 0) \otimes (1, 0) \supset (0, 1) : (1, 0, 0, 1, 0, 0, 0) \\
E_8 &: (0, 1) \otimes (0, 1) \supset (1, 0) : (0, 1, 1, 0, 0, 0, 1)
\end{align*}
\]

The corresponding tableaux are

\[
\begin{align*}
E_1 : \begin{array}{c}
\begin{array}{c}
1 \\
2
\end{array}, \quad E_2 : & \begin{array}{c}
\begin{array}{c}
\end{array}, \quad E_3 : & \begin{array}{c}
\begin{array}{c}
1
\end{array}, \quad E_4 : & \begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
\end{array}
\end{align*}
\]

\[
\begin{align*}
E_5 : & \begin{array}{c}
\begin{array}{c}
\end{array}, \quad E_6 : & \begin{array}{c}
\begin{array}{c}
1
\end{array}, \quad E_7 : & \begin{array}{c}
\begin{array}{c}
\end{array}, \quad E_8 : & \begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
\end{array}
\end{align*}
\]

If we use the 7-component vector as a description of an elementary coupling, we see that there is one relation

\[
E_1E_3E_5 = E_7E_8 : (1, 1, 1, 1, 0, 0, 1)
\]
This is confirmed by the construction of the corresponding LR tableaux:

\[ E_1 E_3 E_5 : \begin{array}{|c|c|} \hline 1 & 2 \\
\hline \end{array} \cdot \begin{array}{|c|} \hline 1 \\
\hline \end{array} \cdot \begin{array}{|c|} \hline 1 \\
\hline \end{array} = \begin{array}{|c|c|c|} \hline 1 & 2 & 1 \\
\hline \end{array} \] (4.30)

and

\[ E_7 E_8 : \begin{array}{|c|} \hline 1 \\
\hline \end{array} \cdot \begin{array}{|c|} \hline 1 \\
\hline \end{array} = \begin{array}{|c|} \hline 1 \\
\hline \end{array} \] (4.31)

The generating function can thus be constructed by either forbidding \( E_1 E_3 E_5 \) or \( E_7 E_8 \). That yields directly the first or the third expression of (2.46) respectively. \[ \square \]

4.4. Example: the su(4) case

To the su(4) LR known conditions are

\[ \lambda_1 \geq n_{12} \]
\[ \lambda_2 \geq n_{13} \]
\[ \lambda_2 + n_{12} \geq n_{13} + n_{23} \]
\[ \lambda_3 \geq n_{14} \]
\[ \lambda_3 + n_{13} \geq n_{14} + n_{24} \]
\[ \lambda_3 + n_{13} + n_{23} \geq n_{14} + n_{24} + n_{34} \] (4.32)
\[ n_{11} \geq n_{22} \]
\[ n_{11} + n_{12} \geq n_{22} + n_{23} \]
\[ n_{11} + n_{12} + n_{13} \geq n_{22} + n_{23} + n_{24} \]
\[ n_{22} \geq n_{33} \]
\[ n_{22} + n_{23} \geq n_{33} + n_{34} \]

The tensor-product elementary couplings, that is, the elementary solutions to these inequalities are best written directly in terms of LR tableaux:

\[ A_1 : \begin{array}{|c|} \hline 1 \\
\hline \end{array}, \quad A_2 : \begin{array}{|c|} \hline 2 \\
\hline \end{array}, \quad A_3 : \quad, \quad B_1 : \begin{array}{|c|} \hline 1 \\
\hline \end{array}, \quad B_2 : \begin{array}{|c|} \hline 1 \\
\hline \end{array}, \quad B_3 : \quad, \quad C_1 : \quad, \quad C_2 : \begin{array}{|c|} \hline 1 \\
\hline \end{array}, \quad C_3 : \quad, \quad D'_1 : \begin{array}{|c|} \hline 1 \\
\hline \end{array}, \quad D'_2 : \quad, \quad D'_3 : \begin{array}{|c|} \hline 1 \\
\hline \end{array}, \quad D_1 : \quad, \quad D_2 : \quad, \quad D_3 : \begin{array}{|c|} \hline 1 \\
\hline \end{array}, \quad \] (4.33)

\[ ^{12} \text{In this basis, } E_2 E_4 E_6 \text{ is an independent product.} \]
together with

\[
\begin{align*}
D_1: & \begin{array}{c} \ \ \ \ \ \ \ \ \ \ \ \ \ \ 1 \\
& \begin{array}{c} \ 2 \\
& \begin{array}{c} \ 3 \end{array}
\end{array}
\end{array} , \\
D_2: & \begin{array}{c} \ \ \ \ \ \ \ \ \ \ \ \ \ \ 1 \\
& \begin{array}{c} \ 2 \\
& \begin{array}{c} \ 3 \end{array}
\end{array}
\end{array} , \\
D_3: & \begin{array}{c} \ \ \ \ \ \ \ \ \ \ \ \ \ \ 1 \\
& \begin{array}{c} \ 2 \\
& \begin{array}{c} \ 3 \end{array}
\end{array}
\end{array} ,
\end{align*}
\]

(4.34)

The Dynkin-label transcription of the elementary couplings reads

\[
\begin{align*}
A_1 : (0,0,0) & \otimes (0,0,1) \supset (0,0,1) & D'_1 : (0,1,0) & \otimes (1,0,0) \supset (0,0,1) \\
A_2 : (0,0,1) & \otimes (1,0,0) \supset (0,0,0) & D'_2 : (1,0,0) & \otimes (1,0,0) \supset (0,1,0) \\
A_3 : (1,0,0) & \otimes (0,0,0) \supset (1,0,0) & D'_3 : (1,0,0) & \otimes (0,1,0) \supset (0,0,1) \\
B_1 : (0,0,0) & \otimes (0,1,0) \supset (0,1,0) & D_1 : (0,1,0) & \otimes (0,0,1) \supset (1,0,0) \\
B_2 : (0,1,0) & \otimes (0,1,0) \supset (0,0,0) & D_2 : (0,0,1) & \otimes (0,1,0) \supset (0,1,0) \\
B_3 : (0,1,0) & \otimes (0,0,0) \supset (0,1,0) & D_3 : (0,0,1) & \otimes (1,0,0) \supset (1,0,0) \\
C_1 : (0,0,0) & \otimes (1,0,0) \supset (1,0,0) & E_1 : (1,0,1) & \otimes (0,1,0) \supset (0,1,0) \\
C_2 : (1,0,0) & \otimes (0,0,1) \supset (0,0,0) & E_2 : (0,1,0) & \otimes (0,1,0) \supset (1,0,1) \\
C_3 : (0,0,1) & \otimes (0,0,0) \supset (0,0,1) & E_3 : (0,1,0) & \otimes (1,0,1) \supset (0,1,0)
\end{align*}
\]

(4.35)

For \(su(4)\), there is a large number of linear relations: in fact there are 15 relations [12,5] :

\[
\begin{align*}
D'_jD_k & = C_iE_i & D_jD'_k & = B_jC_jC_k & E_iE_j & = B_kD_kD'_i \\
D_iE_i & = C_jB_kD_k & D'_iE_i & = B_jD'_jC_k
\end{align*}
\]

(4.36)

with \(i,j,k\) a cyclic permutation of 1,2,3.

To construct the generating function, we need to select forbidden couplings. It turns out that when there are more that one relation, complications may arise. We must ensure that the selected forbidden couplings are complete, which means that no further (usually higher-order) relations are required for a unique decomposition of a given coupling. How do we select a set of complete compatible forbidden couplings? A technique that is tailor-made for dealing with problems of that type is that of Grobner bases. This will be introduced in the next section. At this point, we simply indicate a complete choice of forbidden couplings, namely \(\{E_iE_j, D'_iE_i, D_iE_i, D_jD'_i, D'_jD_i\}\). This yields then a model for the generating
4.5. A remark on the reduction of grading variables and higher multiplicities

In this section, we would like to stress the fact that a generating function built from a complete basis (or a complete description of tensor products) has ‘multiplicity coefficients’ all either 0 or 1: the ‘coupling’ either exists or not. Higher multiplicities can be generated only after the number of grading variables has been reduced (that is, some grading variables have been set equal to 1). To be explicit, suppose that we consider the generating function for the $su(N)$ tensor products starting from a LR description. This is a system of inequalities for the variables $\{\lambda_i, n_{jk}\}$ with $i, j = 1, \ldots, N - 1$ and $j \leq k \leq N$. Introducing the grading variables $\{L_i, N_{jk}\}$, the generating function reads:

$$G = \sum C_{\{\lambda_i, n_{jk}\}} L_i^{\lambda_i} N_{jk}^{n_{jk}}$$

and the coefficient $C_{\{\lambda_i, n_{jk}\}}$ is either 1, if the solution exits, or 0, if the solution does not exist. In other words, when expressed in terms of a complete set of variables, the multiplicity is 0 or 1. Larger values for the multiplicity can only result from the reduction in the number of grading variables. Typically, we consider the reduction from the set $\{\lambda_i, n_{jk}\}$ to the set $\{\lambda_i, \mu_i, \nu_i\}$. The multiplicity, which now has the interpretation of a tensor-product coefficient, is then no longer trivially 0 or 1 (except for $su(2)$ where this changes of variables does not induce a reduction).
5. Diophantine inequalities: elementary couplings, relations and Grobner bases

We will introduce the idea of the Grobner basis via a simple example. Suppose we know that $R = Q[x, y, z, t]/I$ where $I = (xy - t, zy - t)$, with an $\mathbb{N}^2$ grading given by $(1, 0), (0, 1), (1, 0)$ and $(1, 1)$ for $x, y, z$ and $t$, is a model for a generating function. Writing $\bar{x} = x + I$ and similarly for the other variables, we have in $R$ that $\bar{x}\bar{y} = \bar{t}$ and $\bar{z}\bar{y} = \bar{t}$. These two expressions give two re-write rules: $xy \mapsto t$ and $zy \mapsto t$. These rules can be used to simplify any monomial. The aim is to find a re-write rule which, when iterated, produces unique representatives for the classes of $I$. If this is the case, then a vector space basis of $R$ would consist of terms of the form $m + I$ with $m$ a monomial which is not divisible by any of the left-hand sides of the rewrite rules.

In the example above, if we had ‘good’ rewrite rules then a basis for $R$ would be represented by monomials not containing $xy$ or $zy$, i.e. monomials of the form either $y^a t^b$ or $x^a z^b t^c$. The generating function which counts there monomials, taking into account the grading and potential over-counting, is:

$$
\frac{1}{(1 - AB)} \left( \frac{B}{1 - B} + \frac{1}{(1 - A)^2} \right),
$$

(5.1)

The exponent of $A$ carries the first grading index and $B$ the second.

However this generating function is not correct. It contains the term $2A^2 B$ corresponding to the 2 monomials $xt$ and $zt$. But the polynomial $z(xy - t) - x(zy - t) = xt - zt$ is also in $I$ and hence in $R$ we have $\bar{x}\bar{t} = \bar{z}\bar{t}$ and so the space of grade $(2, 1)$ has dimension 1 rather than 2. This problem can also be seen as a problem with the re-write rules. If we start with $xyz$ then we can use the first re-write rule: $xyz \mapsto tz$ or the second: $xyz \mapsto xt$. We cannot apply any further re-write rules and so this set of re-write rules does not produce a unique representative. The solution is to include the rule $xt \mapsto zt$. This gives a set of 3 rules: $xy \mapsto t$, $zy \mapsto t$ and $xt \mapsto zt$. It turns out that this is a ‘good’ set and so a basis

---

13 For an elementary introduction to Grobner bases, see for instance [13].

14 In more details: with the specified grading, $y^a t^b$ corresponds to the term $B^a (AB)^b$ which is generated by $[(1 - AB)(1 - B)]^{-1}$ while $x^a z^b t^c$ corresponds to $A^a A^b (AB)^c$ which is generated by $[(1 - AB)(1 - A)^2]^{-1}$. Since the constant term would be counted twice if we simply add these two pieces, we multiply the first one by $B$. 

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for $R$ is given by (the classes of) monomials of the form $y^at^b$, $x^az^b$ and $z^at^b$ which gives the generating function:

$$\frac{1}{(1-AB)(1-B)} + \frac{A}{(1-A)^2} + \frac{A}{(1-A)(1-AB)}$$

(5.2)

The set of ‘good’ generators, $xy-t$, $zy-t$, $xt-zt$ we have found for $I$ is known as a Grobner basis [14].

The general procedure for constructing a Grobner basis given a set of generating polynomials is as follows. First choose a term ordering, which is an ordering on monomials with the property that any chain $m_1 > m_2 > \ldots$ has finite length. For example we can order the variables by $x > y > z > t$ and then order all monomials by the corresponding lexicographic (dictionary) order, for example: $x^2y > xyz > y^3$. For each generator of our ideal $I$, select the monomial which is highest with respect to the given term ordering. This is then the term which appears on the left of the re-write rule. The lexicographic ordering gives the first two re-write rules of our example: $xy \mapsto t$ and $zy \mapsto t$. Next, for each pair of leading terms find the lowest common multiple and simplify it in the two possible ways. In this case there is only one pair of leading terms and the lowest common multiple is $xyz$ which simplifies to $xt$ and $yt$. Continue to apply the re-write rules until the terms do not simplify further. If the resulting pair of terms are the same, then proceed to the next pair of leading terms, otherwise add a new re-write rule. In this case we add $xt \mapsto yt$. Proceed until no pair of leading terms gives a new rule. This is the case for the rules we now have. For example the two rules $xy \mapsto t$ and $xt \mapsto zt$ appears to give a new rule by simplifying $xyt$ to both $t^2$ and $yzt$. However the second term can be further reduced to $t^2$ and so no new rule is required. Improvements on this basic algorithm, known as the Buchberger [14] algorithm, mean that it is now feasible to find Grobner bases for quite large sets of generating polynomials.\[15\]

Although it is not clear from this example, the technique of Grobner bases is a very versatile tool for performing explicit calculations. We end this section with an illustrative example relevant to our discussion of tensor-product generating functions.

Consider a set of linear Diophantine equations:

$$M\alpha = 0, \quad \alpha \geq 0$$

(5.3)

\[15\] The web pages of the computer-algebra information network at [http://cand.cand.nl/CAIN](http://cand.cand.nl/CAIN) contain information about many of the programs currently available.
with $M$ an integer matrix and $\alpha$ a vector of non-negative integers. We would like to construct a generating function for the solutions to this set of equations:

$$\sum_{\alpha} x^{\alpha}.$$  \hspace{1cm} (5.4)

A non-trivial example is given by the Diophantine equations that describe a $3 \times 3$ magic square:

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$  \hspace{1cm} (5.5)

with non-negative entries and equal row and column sums. The magic square condition (the sum of each row and each column is the same, say equal to $t$) gives the following set of equations:

$$\begin{align*}
a + b + c &= t \\
d + e + f &= t \\
g + h + i &= t \\
a + d + g &= t \\
b + e + h &= t \\
c + f + i &= t
\end{align*}$$  \hspace{1cm} (5.6)

With $\alpha$ standing for the column vector with entries $(a, b, c, d, e, f, g, h, i, t)$, the matrix $M$ reads

$$M = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -1 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & -1
\end{pmatrix}$$  \hspace{1cm} (5.7)

It can be show than any solution of $M\alpha = 0$ is a linear combination (with non-negative coefficients!) of a finite number of basic solutions $\alpha_1, \cdots, \alpha_s$ (see for example [4]). Moreover there is a straightforward algorithm for finding this basic set of solutions [15]. In this case
we have the following set:
\[ \alpha_1 = (0, 0, 1, 0, 1, 0, 1, 0, 0, 1) \]
\[ \alpha_2 = (0, 1, 0, 0, 1, 1, 0, 0, 1, 0) \]
\[ \alpha_3 = (0, 0, 1, 1, 0, 0, 1, 0, 1, 0) \]
\[ \alpha_4 = (1, 0, 0, 0, 1, 0, 1, 0, 1, 0) \]
\[ \alpha_5 = (0, 1, 0, 1, 0, 0, 0, 1, 1, 0) \]
\[ \alpha_6 = (1, 0, 0, 0, 1, 0, 0, 0, 1, 1) \] (5.8)

We shall use \( A, B, \ldots, T \) to denote the “grading variables” of this example so that the exponent of \( A \) carries the value of \( a \) and so on. A model for the generating function is given by the subring \( S \) of \( \mathbb{Q}[A, B, C, D, E, F, G, H, I, T] \) generated by monomials corresponding to the 6 elementary solutions,
\[ E_1 = CEGT, \quad E_2 = BFGT, \quad E_3 = CDHT, \]
\[ E_4 = AFHT, \quad E_5 = BDIT, \quad E_6 = AEIT \] (5.9)
The monomials in \( S \) correspond to magic squares. For example \( E_2^2E_4E_6 = A^2C^2E^3FG^2HT^4 \in S \) corresponds to a square with row and column sums equal to 4:
\[
\begin{pmatrix}
2 & 0 & 2 \\
0 & 3 & 1 \\
2 & 1 & 1
\end{pmatrix}
\] (5.10)

Note that in this example it is convenient to construct our model as a subring of the ring of grading variables. Thus each “elementary coupling” \( E_i \) is actually equal to the corresponding monomial in the grading variables.

However, there are relations between these generators and so it is not immediately clear how to construct the Poincaré series for \( S \). What we require is an isomorphism of \( S \) with \( R = \mathbb{Q}[e_1, \ldots, e_6]/I \) such that \( e_i \mapsto E_i, \ i = 1, \ldots, 6 \) and such that we have a Grobner basis of the ideal \( I \) (the ‘ideal of relations’).

Fortunately, such an isomorphism is easily constructed using Grobner-basis methods. Introduce the ring \( \mathbb{Q}[A, B, C, D, E, F, G, H, I, T, E_1, \ldots, E_6] \) with the lexicographic ordering
\[ A > B > C > D > E > F > G > H > I > T > e_1 > \ldots > e_6 \] (5.11)

Let \( J \) be the ideal generated by \( E_1 - e_1, \ldots, E_6 - e_6 \). This is not necessarily a Grobner basis with respect to this term ordering. Let \( G \) be the Grobner basis for \( J \) with the given
ordering. Then it can be shown [14] that $G \cap \mathbb{Q}[e_1, \ldots, e_6]$ is a Grobner basis for the ideal of relations $I$ which we require. In this case $G$ is quite large, but its intersection with $\mathbb{Q}[e_1, \ldots, e_6]$ is $e_1 e_4 e_5 - e_2 e_3 e_6$. The corresponding relation in $R$ is $E_1 E_4 E_5 - E_2 E_3 E_6$ and it is straightforward to verify that these two terms do indeed give the same magic square, so that indeed we have found a relation between the generators of $R$. The Poincaré series for $\mathbb{Q}[e_1, \ldots, e_6]/I$ is easily computed:

$$\frac{1}{(1 - E_2)(1 - E_3)(1 - E_6)} \left( \frac{1}{(1 - E_1)(1 - E_4)} + \frac{E_5}{(1 - E_1)(1 - E_5)} + \frac{E_4 E_5}{(1 - E_4)(1 - E_5)} \right)$$

(5.12)

If we do not need to keep track of the row and column sums we can set $T = 1$ in this generating function. Similarly, if we simply want a generating function for the number of magic squares with given row and column sums, we can set all variables except $T$ equal to 1, which yields:

$$\frac{1 + T + T^2}{(1 - T)^5}$$

(5.13)

It is perhaps worth pointing out that the model we constructed above is still a model for (5.13) if we change the grading by keeping just the last component of the grading vector. Now, however, the grading variable is simply $T$ and so we cannot identify the grading and model variables. Thus to obtain (5.13) from (5.12) we once again must use our convention that model variables are replaced by the corresponding grading variables. In this case this corresponds precisely to setting all the variables except $T$ equal to 1.

We have made use of the techniques described in this section to verify the results we have quoted previously.

6. The Berenstein-Zelevinsky basis

6.1. Generalities

The presentation of the $su(2, 3, 4)$ cases has made clear the usefulness of a basis, or more precisely, a re-expression of the tensor-product calculation in terms of Diophantine equations. The Littlewood-Richardson algorithm yields a set of such equations only for $su(N)$. Fortunately, Berenstein and Zelevinsky [16] have expressed the solution of the
multiplicity of a given tensor product as a counting problem for the number of integral points in a convex polytope. For a given algebra, the polytope is formulated in terms of a characteristic set of inequalities. These inequalities can be interpreted as defining a tensor-product basis. For \( su(N) \), this basis reduces to the LR set of inequalities. For the other classical algebras but \( sp(4) \), the proposed basis is a conjecture. Hence, in order to analyse our next example, \( sp(4) \), we first need to present the corresponding Berenstein-Zelevinsky inequalities.

6.2. The BZ \( sp(4) \) basis

The combinatorial description of tensor products for \( sp(4) \) is not as simple as in the \( su(N) \) case: a standard LR product must be supplemented by a division operation and modification rules [17]. Various natural trials for the search of a linear basis failed (and for instance, a stretched product making manifest the relations between the elementary couplings cannot be defined).

Given the BZ set of inequalities, the natural way to proceed, as just mentioned, is to interpret these as the appropriate basis for the description of the tensor products. These inequalities are as follows:

\[
\begin{align*}
\lambda_1 & \geq p_{12} \\
\lambda_2 & \geq r_1/2 \\
\lambda_2 & \geq r_1/2 + q_{12} - p_{12} \\
\lambda_2 & \geq r_2/2 + q_{12} - p_{12} \\
\mu_1 & \geq q_{12} \\
\mu_1 & \geq q_{12} + r_1 - r_2 \\
\mu_1 & \geq p_{12} + r_1 - r_2 \\
\mu_2 & \geq r_2/2 \\
\nu_1 & = r_2 - r_1 - 2p_{12} + \lambda_1 + \mu_1 \\
\nu_2 & = p_{12} - q_{12} - r_2 + \lambda_2 + \mu_2
\end{align*}
\]

\( (6.1) \)

\(^{16}\) Our notation is different from that used in [16]. The relation is \( r_1 = m_1, r_2 = m_2, p_{12} = m_{12}, q_{12} = m_{12}^\dagger \).
The $sp(4)$ tensor product coefficient $N_{\lambda\mu\nu}$ is thus given by the number of solutions of the above system with $r_1, r_2 \in 2 \mathbb{Z}_+$ et $p_{12}, q_{12} \in \mathbb{Z}_+$.

A proper set of variables for a complete description of a particular tensor-product coupling is thus \{\lambda_1, \lambda_2, \mu_1, \mu_2, r_1, r_2, p_{12}, q_{12}\}. We give the list of elementary couplings, adding to each coupling the corresponding four-vector $[r_1, r_2, p_{12}, q_{12}]$:

- $A_1 : (0, 0) \otimes (1, 0) \supset (1, 0) [0, 0, 0, 0]$
- $A_2 : (1, 0) \otimes (0, 0) \supset (1, 0) [0, 0, 0, 0]$
- $A_3 : (1, 0) \otimes (1, 0) \supset (0, 0) [0, 0, 1, 1]$
- $B_1 : (0, 0) \otimes (0, 1) \supset (0, 1) [0, 0, 0, 0]$
- $B_2 : (0, 1) \otimes (0, 0) \supset (0, 1) [0, 0, 0, 0]$
- $B_3 : (0, 1) \otimes (0, 1) \supset (0, 0) [2, 2, 0, 0]$
- $C_1 : (0, 1) \otimes (1, 0) \supset (1, 0) [0, 0, 0, 1]$
- $C_2 : (1, 0) \otimes (0, 1) \supset (1, 0) [0, 2, 1, 0]$
- $C_3 : (1, 0) \otimes (1, 0) \supset (0, 1) [0, 0, 1, 0]$
- $D_1 : (2, 0) \otimes (0, 1) \supset (0, 1) [0, 2, 2, 0]$
- $D_2 : (0, 1) \otimes (2, 0) \supset (0, 1) [2, 0, 0, 0]$
- $D_3 : (0, 1) \otimes (0, 1) \supset (2, 0) [0, 2, 0, 0]$

The unspecified linear relations mentioned in (2.59) can now be obtained. To find those products that are really equal in the present basis, we only need to compare their corresponding sets of four-vectors $[r_1, r_2, p_{12}, q_{12}]$ (which are additive in products of couplings). We thus find for instance that $C_1 C_2 = A_3 D_3 : [0, 2, 1, 1] \neq A_1 A_2 B_3 : [2, 2, 0, 0]$ (6.3)

Proceeding in this way for the other cases, we find the following complete list of relations:

- $C_1 C_2 = A_3 D_3$, \quad $C_2 C_3 = A_1 D_1$ \quad $C_3 C_1 = A_1 A_3 B_2$
- $D_1 D_2 = B_3 C_3^2$, \quad $D_2 D_3 = A_1^2 B_2 B_3$ \quad $D_1 D_3 = B_2 C_2^2$ (6.4)
- $C_1 D_1 = A_3 B_2 C_2$ \quad $C_2 D_2 = A_1 B_3 C_3$ \quad $C_3 D_3 = A_1 B_2 C_2$

The use of the BZ basis to find the elementary couplings and their relations seems to be novel.\footnote{Our construction provides an indirect proof of the validity of the BZ basis in that from it we recover the result of [10] derived from the character method.}

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A possible choice of forbidden products is the one given in [10]:

\{C_i C_j, D_i D_j, C_i D_i\} \tag{6.5}

with \(i, j = 1, 2, 3\) and \(i \neq j\). It leads to the generating function:

\[
G^{sp(4)} = \left( \prod_{i=1}^{3} \tilde{A}_i \tilde{B}_i \right) \left( \tilde{C}_1 \tilde{D}_2 + D_3 \tilde{C}_1 \tilde{D}_3 + C_2 D_1 \tilde{C}_2 \tilde{D}_1 + C_2 \tilde{C}_2 \tilde{D}_3 + D_1 \tilde{C}_3 \tilde{D}_1 + C_3 \tilde{C}_3 \tilde{D}_2 \right) \tag{6.6}
\]

Of course, by modifying the ordering in the Grobner basis, we can get other choices of forbidden couplings. Here is another set of forbidden couplings that can be obtained:

\(\{D_i D_j, C_i D_i, A_1 D_1, A_3 D_3, A_1 A_3 B_2\}\). The corresponding generating function reads

\[
G^{sp(4)} = \tilde{B}_1 \tilde{B}_2 \tilde{B}_3 \left( \prod_{i=1}^{3} \tilde{A}_i \right) \tilde{C}_i (1 - A_1 A_3 B_2) + D_3 \tilde{D}_3 \tilde{A}_1 \tilde{A}_2 \tilde{C}_1 \tilde{C}_2 + D_1 \tilde{D}_1 \tilde{A}_2 \tilde{A}_3 \tilde{C}_2 \tilde{C}_3 + D_2 \tilde{D}_2 \tilde{A}_1 \tilde{A}_2 \tilde{A}_3 \tilde{C}_1 \tilde{C}_3 (1 - A_1 A_3 B_2) \tag{6.7}\]

These two generating functions are equivalent when rewritten in terms of the grading variables, that is, in terms of Dynkin labels. However, they originate from two distinct models. The second one turns out to be well adapted to the fusion extension.

7. A vector-basis approach to the construction of generating functions

In this section, we present a simple and systematic way of generating by hand all the elementary solutions of a set of linear homogeneous inequalities starting from the well-known construction of a vector basis. The first step amounts to reformulate the system of inequalities in terms of equalities. We then look for the elementary independent solutions by relaxing the positivity requirement. In other words, we construct the vector basis. In a final step, we find the minimal linear combinations of these vector-basis elements that yield positive solutions. This will also provide a further illustration of the MacMahon’s projection technique. But actually, the outcome of this projection is the desired tensor-product generating function. Hence, this approach turns out to be a new way of constructing the
generating functions. This method will play a key role in our subsequent description of fusion-rule generating functions.

In the first subsection, we present the equality version of the LR system of inequalities which holds for $su(N)$. This happens to lead to a remarkable graphical description of $su(N)$ tensor products that has been found by Berenstein and Zelevinsky [18]. In the following section, we illustrate the procedure by working out the vector-basis argument for $su(3)$. A more general discussion of this technique is presented in the next section. We finish with an analysis of the $sp(4)$ case, for which we provide a novel graphical representation of tensor products.

7.1. $su(N)$ tensor-product basis: from inequalities to equalities: graphical representations as BZ triangles for $su(N)$

Consider the direct transformation of the LR inequalities to equalities by introducing an appropriate number of new non-negative integer variables. Consider first the $su(2)$ case, for which there is a single inequality: $\lambda_1 \geq n_{12}$. We transform this into an equality by introducing the positive integer $a$ defined by

$$\lambda_1 = n_{12} + a \quad (7.1)$$

The expression for $\nu_1$ becomes then $\nu_1 = \lambda_1 + n_{11} - n_{12} = a + n_{11}$. Since $\mu_1 = n_{11} + n_{12}$, we are led naturally to a triangle representation of the tensor product:

$$\lambda \otimes \mu \supset \nu \leftrightarrow \begin{array}{c} a \\ n_{12} \\ n_{11} \end{array} \quad (7.2)$$

We read off the Dynkin label of the $\lambda$ representation from the sum of the two integers that form the left side of the triangle, that of the $\mu$ representation from the bottom of the triangle and the $\nu_1$ label is the sum of the two integers that form the right side. A more uniform notation amounts to setting $a = m_{12}$ and $n_{11} = l_{12}$, in terms of which the triangle looks quite symmetrical:

$$\begin{array}{c} m_{12} \\ n_{12} \\ l_{12} \end{array} \quad (7.3)$$

18 This method, referred to as being novel, is probably well-known but we could not trace it precisely in the literature. There are implicit remarks in that direction in the first reference of [4].
with
\[ \lambda_1 = m_{12} + n_{12} \quad \mu_1 = n_{12} + l_{12} \quad \nu_1 = m_{12} + l_{12} \quad (7.4) \]

These numbers \( m_{12} \) and \( l_{12} \) play the role of \( n_{12} \) in the permuted versions of the tensor product. The triangle combinatorial reformulation of the tensor product problem is as follows: the number of triangles that can be formed from nonnegative integers \( n_{12}, m_{12} \) and \( l_{12} \) that add up to the Dynkin labels of the representations under study according to the above relations gives the multiplicity of the triple coupling \( \lambda \otimes \mu \supset \nu \), or equivalently, the multiplicity of the scalar representation in the product \( \lambda \otimes \mu \otimes \nu \supset (0) \) (since for \( su(2) \), \( \nu^* = \nu \)).

For \( su(3) \) the situation is somewhat more complicated. The transformation of the inequalities (4.25) into equalities takes the form
\[
\begin{align*}
\lambda_1 &= n_{12} + a \\
\lambda_2 &= n_{13} + b \\
\lambda_2 + n_{12} &= n_{13} + n_{23} + c \\
n_{11} &= n_{22} + d \\
n_{11} + n_{12} &= n_{22} + n_{23} + e
\end{align*}
\quad (7.5)
\]

The expression for the other weights becomes
\[
\begin{align*}
\mu_1 &= n_{13} + e \\
\mu_2 &= n_{22} + n_{23} \\
\nu_1 &= a + d \\
\nu_2 &= n_{22} + c
\end{align*}
\quad (7.6)
\]

Since there are two expressions for both \( n_{11} \) and \( \lambda_2 \), there follows the compatibility relations:
\[
\begin{align*}
n_{12} + d &= n_{23} + e \\
n_{23} + c &= b + n_{12}
\end{align*}
\quad (7.7)
\]

By adding these two relations, we find:
\[
c + d = b + e
\quad (7.8)
\]

Again we are led naturally to a triangle representation: with \( \zeta = \nu^* \) this reads

\[
\begin{array}{cccc}
\text{a} & n_{12} & d \\
\text{b} & e & n_{23} & n_{22}
\end{array}
\quad (7.9)
\]
We read the Dynkin labels from the sides of the triangles, from $\lambda_1$ to $\zeta_2$ in an anti-clockwise rotation starting from the top of the triangle, exactly as for $su(2)$, except that here there are two labels on each sides. Notice that the compatibility conditions amounts to the equality of the sums of the extremal points of the three pairs of opposite sides of the hexagon obtained by dropping the three corners of the triangle.

Again a more symmetrical notation calls for itself:

$$\lambda_1 = m_{13} + n_{12}, \quad \lambda_2 = m_{23} + n_{13},$$  
$$\mu_1 = n_{13} + l_{12}, \quad \mu_2 = n_{23} + l_{13},$$  
$$\zeta_1 = l_{13} + m_{12}, \quad \zeta_2 = l_{23} + m_{13}$$

The hexagon conditions read:

$$n_{12} + m_{23} = n_{23} + m_{12},$$  
$$l_{12} + m_{23} = l_{23} + m_{12},$$  
$$l_{12} + n_{23} = l_{23} + n_{12}.$$  

In terms of triangles, the problem of finding the multiplicity of the $su(3)$ tensor product $\lambda \otimes \mu \otimes \zeta \supset 0$ boils down to enumerating the number of triangles made with nonnegative integers that form a bipartition of the Dynkin labels and that satisfy the above three hexagon relations.

For $su(4)$ the BZ triangle is obtained in a similar way. One first transforms the
inequalities (4.32) into equalities by introducing positive integers $a, b, \cdots, \kappa$:

\[
\begin{align*}
\lambda_1 &= n_{12} + a \\
\lambda_2 &= n_{13} + b \\
\lambda_2 + n_{12} &= n_{13} + n_{23} + c \\
\lambda_3 &= n_{14} + d \\
\lambda_3 + n_{13} &= n_{14} + n_{24} + e \\
\lambda_3 + n_{13} + n_{23} &= n_{14} + n_{24} + n_{34} + f \\
n_{11} &= n_{22} + g \\
n_{11} + n_{12} &= n_{22} + n_{23} + h \\
n_{11} + n_{12} + n_{13} &= n_{22} + n_{23} + n_{24} + i \\
n_{22} &= n_{33} + j \\
n_{22} + n_{23} &= n_{33} + n_{34} + \kappa
\end{align*}
\]

The Dynkin labels of $\mu$ and $\zeta = \nu^*$ becomes

\[
\begin{align*}
\mu_1 &= n_{14} + i, & \mu_2 &= n_{24} + \kappa, & \mu_3 &= n_{33} + n_{34} \\
\zeta_1 &= n_{33} + f, & \zeta_2 &= c + j, & \zeta_3 &= a + g
\end{align*}
\]

Since every Dynkin label is the sum of two positive integers, a triangle representation is again natural:

\[
\begin{array}{cccccccc}
a & n_{12} & g \\
x & y & u \\
d & i & s & t & n_{34} & n_{33}
\end{array}
\]

However the position of the integers specifying the labels 2 is ambiguous at this point: $x$ can be either $b$ or $n_{13}$ ($y$ being the other one), similarly $(u, v)$ is related to the doublet $(j, c)$ and $(s, t)$ to $(n_{24}, \kappa)$. Moreover not all the needed integers appear in this triangle: $e, h, n_{23}$ are missing. In order to take into account the various compatibility relations, it is natural to insert the three remaining points in the center of the big triangle forming then three hexagons:
Quite remarkably, the different conditions are simply the equality of the opposite sides of every hexagon, exactly as for \( su(3) \), and these completely fix the position of every integers in the triangle. For instance, replacing \( \lambda_2 = n_{13} + b \) into \( \lambda_2 + n_{12} = n_{13} + n_{23} + c \) yields

\[
n_{12} + b = n_{23} + c
\]

(7.18)

For this to be an hexagon relation, \( b \) and \( c \) must belong to the same hexagon and since they are on opposite sides of the triangles, they must belong to the first hexagon: hence \( x = b \) and \( u = c \). That also fixes the position of \( n_{23} \) inside the big triangle. Proceeding in this way with the other constraints, we end up with the following representation:

\[
\begin{array}{cccccc}
  & a & \\
  n_{12} & g & c \\
  b & n_{13} & h & n_{23} & j \\
  & d & e & f \\
  n_{14} & i & n_{24} & \kappa & n_{34} & n_{33}
\end{array}
\]

(7.19)

Finally, a more symmetrical form is obtained by redefining the name of the positive integers \( a, \cdots, \kappa \) and \( n_{33} \) as follows:

\[
\begin{array}{cccccccc}
  & m_{14} & \\
  n_{12} & l_{34} & m_{13} \\
  n_{13} & l_{23} & n_{23} & l_{24} & m_{24} \\
  n_{14} & l_{12} & n_{24} & l_{13} & n_{34} & l_{14} & m_{34}
\end{array}
\]

(7.20)

They are related to the Dynkin labels by

\[
\begin{align*}
\lambda_1 &= m_{14} + n_{12} & \lambda_2 &= m_{24} + n_{13} & \lambda_3 &= m_{34} + n_{14} \\
\mu_1 &= n_{14} + l_{12} & \mu_2 &= n_{24} + l_{13} & \mu_3 &= n_{34} + l_{14} \\
\zeta_1 &= l_{14} + m_{12} & \zeta_2 &= l_{24} + m_{13} & \zeta_3 &= l_{34} + m_{14}
\end{align*}
\]

(7.21)

and the hexagon relations read:

\[
\begin{align*}
n_{12} + m_{24} &= m_{13} + n_{23} & n_{12} + l_{34} &= l_{23} + n_{23} & m_{24} + l_{23} &= l_{34} + m_{13} \\
n_{13} + l_{23} &= l_{12} + n_{24} & n_{13} + m_{34} &= n_{24} + m_{23} & m_{34} + l_{12} &= l_{23} + m_{23} \\
l_{24} + n_{23} &= l_{13} + n_{34} & n_{23} + m_{23} &= m_{12} + n_{34} & l_{13} + m_{23} &= l_{24} + m_{12}
\end{align*}
\]

(7.22)

The \( su(N) \) generalisation is obvious; the triangle is built out of \( (N - 1)(N - 2)/2 \) hexagons and three corner points.
Here is the rationale for the labelling $n_{ij}, m_{ij}, l_{ij}$ from the triangle point of view [18]. If $e_i$ are orthonormal vectors in $\mathbb{R}^N$, then the positive roots of $su(N)$ can be represented in the form $e_i - e_j$, $1 \leq i < j \leq N$. The triangle encodes three sums of positive roots:

$$\begin{align*}
\mu + \zeta - \lambda^* & = \sum_{i<j} l_{ij}(e_i - e_j), \\
\zeta + \lambda - \mu^* & = \sum_{i<j} m_{ij}(e_i - e_j), \quad (7.23) \\
\lambda + \mu - \zeta^* & = \sum_{i<j} n_{ij}(e_i - e_j),
\end{align*}$$

The hexagon relations are simply the consistency conditions for these three expansions. Clearly, the variables $n_{ij}$ that appear in the above relations are exactly the $n_{ij}$ that appear in the LR tableaux for the product $\lambda \otimes \mu \supset \zeta^* = \nu$.

### 7.2. From a vector basis to the generating function: the $su(3)$ case

Given the transcription of inequalities into equalities, we can easily extract the corresponding basis vectors. This is the starting point of a new method for constructing the tensor-product generating functions. To keep things concrete, we focus on the $su(3)$ case. The goal is to first get a vector basis and then to project it to get the elementary couplings. The generating function is a direct result of this procedure.

The equality version of the LR inequalities have already been presented in the previous subsection: these are (7.12) and (7.13); they underlie the construction of the BZ triangle (7.11). The last hexagon condition of (7.13) is the difference of the previous two so it is not an independent relations. We thus have a total of 15 variables: $\lambda_1, \ldots, \nu_2, l_{12}, \ldots, n_{23}$ and 8 equations. The number of independent variables is thus 7. These will be chosen to be $m_{13}, m_{23}, l_{13}, l_{23}, n_{12}, n_{13}, n_{23}$. The dependent variables are fixed as follows:

$$\begin{align*}
\lambda_1 & = m_{13} + n_{12} \\
\lambda_2 & = m_{23} + n_{13} \\
\mu_1 & = n_{13} + n_{12} + l_{23} - n_{23} \\
\mu_2 & = n_{23} + l_{13} \\
\zeta_1 & = n_{12} + m_{23} + l_{13} - n_{23} \\
\zeta_2 & = l_{23} + m_{13} \\
l_{12} & = n_{12} + l_{23} - n_{23} \\
m_{12} & = n_{12} + m_{23} - n_{23}
\end{align*}$$

(7.24)
We now look for the elementary solutions of this system (without invoking the constraint that all the above dependent variables should be necessarily positive). The sought basis vectors are obtained by setting one of the variable $m_{13}, \ldots, n_{23}$ to 1 and all other set equal to zero. This produces (in order) the triangles $E_2, E_5, E_6, E_3, E_7, E_4$ and $Z_1$ displayed below:

$$
E_2 : (1, 0)(0, 0)(0, 1) \quad E_3 : (0, 0)(1, 0)(0, 1)
$$

$$
\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
$$

$$
E_4 : (0, 1)(1, 0)(0, 0) \quad E_5 : (0, 1)(0, 0)(1, 0) \quad E_6 : (0, 0)(0, 1)(1, 0)
$$

$$
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
\end{array}
$$

$$
E_7 : (1, 0)(1, 0)(1, 0) \quad Z_1 : (0, 0)(-1, 1)(-1, 0)
$$

$$
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 \\
\end{array}
$$

These are all genuine BZ triangles except for $Z_1$ which has some negative entries. However, at this level, there are no relations between these elementary solutions (the basis vectors are independent), hence the decomposition of any solution in terms of these 7 basic ones is unique. All solutions are then freely generated from the following function:

$$
G = \frac{1}{(1 - E_2)(1 - E_3)(1 - E_4)(1 - E_5)(1 - E_6)(1 - E_7)(1 - Z_1)} \quad (7.26)
$$

To recover the generating function for all tensor products from the above expression, we need to project out terms that lead to triangles with negative entries. To achieve this, we introduce the grading variables associated to the above couplings (compare the above triangles with the general form given in (7.11)):

$$
E_2 : M_{13}, \quad E_3 : L_{12}L_{23} \quad E_4 : N_{13} \quad E_5 : M_{12}M_{23} \quad E_6 : L_{13} \quad E_7 : L_{12}M_{12}N_{12} \quad Z_1 : L_{12}^{-1}M_{12}^{-1}N_{23} \quad (7.27)
$$

Our generating function follows from the projection of the above function $G$, re-expressed in terms of the grading variables, to positive powers of $L_{12}$ and $M_{12}$. Equivalently, one can
rescale $L_{12}$ by $x$ and $M_{12}$ by $y$ and project to positive powers of $x$ and $y$ and set $x = y = 1$ in the result. This is equivalent to the rescaling

$$E_3 \rightarrow xE_3 \quad E_5 \rightarrow yE_5 \quad E_7 \rightarrow xyE_7 \quad Z_1 \rightarrow x^{-1}y^{-1}Z_1$$  \hspace{1cm} (7.28)

We are thus led to consider

$$\Omega \geq \Omega G(E_2, xE_3, \cdots, x^{-1}y^{-1}Z_1)$$  \hspace{1cm} (7.29)

Keeping only those terms which depend explicitly upon $x$ or $y$, we have then

$$\frac{x \cdot y}{\Omega \geq \Omega} \frac{1}{(1-xE_3)(1-yE_5)(1-xyE_7)(1-x^{-1}y^{-1}Z_1)} = \frac{1}{(1-xE_3)(1-yE_5)(1-E_7Z_1)} \left( \frac{1}{1-xyE_7} + \frac{x^{-1}y^{-1}Z_1}{1-x^{-1}y^{-1}Z_1} \right)$$  \hspace{1cm} (7.30)

No more work is need for the first term. For the second one, we have

$$\frac{x \cdot y}{\Omega \geq \Omega} \frac{x^{-1}y^{-1}Z_1}{(1-xE_3)(1-E_7Z_1)(1-x^{-1}Z_1E_5)} \left( \frac{yE_5}{1-yE_5} + \frac{1}{1-x^{-1}y^{-1}Z_1} \right)
\begin{align*}
= \frac{x}{\Omega \geq (1-E_5)(1-E_7Z_1)(1-xE_3)(1-x^{-1}Z_1E_5)} \\
= \frac{x}{\Omega \geq (1-E_5)(1-E_7Z_1)(1-E_3E_5Z_1)(1-E_3)} \\
= \frac{x^{-1}E_5Z_1}{E_3E_5Z_1} \left( \frac{xE_3}{1-xE_3} + \frac{1}{1-x^{-1}Z_1E_5} \right)  \hspace{1cm} (7.31)
\end{align*}

We then introduce the following two new elementary couplings

$$E_1 = E_7Z_1 \quad E_8 = E_3E_5Z_1$$  \hspace{1cm} (7.32)

Collecting the two terms resulting from the projection, we end up with

$$G^{su(3)} = \prod_{i=1}^{8} \tilde{E}_i (1 - E_7E_8)$$  \hspace{1cm} (7.33)

which is indeed the $su(3)$ tensor-product generating function.

As a side remark, we indicate how the various triangles associated to a given triple product are related to each others. For this question, we consider the three weights to be fixed. The system of equations is now nonhomogeneous and solutions are given by a linear
combination of a particular solution and the sum of all homogeneous solutions, that is, solutions of the system with all Dynkin labels set equal to zero:

\[
\begin{align*}
0 &= m_{13} + n_{12} \\
0 &= m_{23} + n_{13} \\
0 &= n_{13} + n_{12} + l_{23} - n_{23} \\
0 &= n_{23} + l_{13} \\
0 &= n_{12} + m_{23} + l_{13} - n_{23} \\
0 &= l_{23} + m_{13} \\
0 &= n_{12} + l_{23} - n_{23} - l_{12} \\
0 &= n_{12} + m_{23} - n_{23} - m_{12}
\end{align*}
\] (7.34)

These solutions are given by \( n \Delta \) with \( n \in \mathbb{Z} \) and

\[
\Delta = \begin{pmatrix} 1 & -1 & -1 \\ -1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}
\] (7.35)

Hence, given an allowed triangle, all other triangles related to the same tensor product can be obtained by adding or subtracting a number of times \( \Delta \) while ensuring that all triangle entries are positive [19].

The great advantage of the vector-basis procedure is that the basis vectors are very easily obtained and their number grows slowly with \( N \) for \( su(N) \) (their number being \((N - 1)(N + 4)/2\)) as compared to the number of tensor-product elementary couplings:

\[
\begin{array}{ccc}
\text{algebra} & \# \text{ elem. coupl.} & \text{dim. vect. space} \\
\text{su}(2) & 3 & 3 \\
\text{su}(3) & 8 & 7 \\
\text{su}(4) & 18 & 12 \\
\text{su}(5) & 45 & 18 \\
\text{su}(6) & 138 & 25 \\
\text{su}(7) & 526 & 33 \\
\end{array}
\] (7.36)
7.3. General aspects of the vector-basis construction

In general, of course, the fundamental solutions to the linear system may have non-integral values of the variables. However the corresponding terms in the generating function can be eliminated by rationalising all the denominator terms and then keeping only those terms in the numerator that have integral exponents. This suggests the following modification of MacMahon’s algorithm.

Consider the system of equations

\[ Mx = 0, \quad x \in \mathbb{N}_k \]  

where \( M \) is a matrix of rank \( s \). We thus have \( k \) variables and \( s \) relations between them. The dimension of the vector basis is thus \( k - s \). We will denote the independent (free) variables as \( x_i, \; i = 1, \cdots, k - s \) and the remaining ones as \( \tilde{x}_j, \; j = 1, \cdots, s \). To find a generating function for the solutions of this system:

1. First construct a basis in \( \mathbb{Q}^k \) for the solutions of \( Mx = 0 \) by setting \( x_i = 1 \) with all other \( x_j \) zero \( (j = 1, \cdots, k - s, \; j \neq i) \). Denote by \( \tilde{x}_j^{(1)} \) the value of the dependent variable \( \tilde{x}_j \) evaluated at \( x_1 = 1 \) with all other \( x_i \) zero. The basis then reads

\[
\epsilon_1 = (1, 0, 0 \ldots, 0; \{\tilde{x}_j^{(1)}\}),
\epsilon_2 = (0, 1, 0 \ldots, 0; \{\tilde{x}_j^{(2)}\}),
\ldots
\epsilon_{k-s} = (0, 0, 0 \ldots, 1; \{\tilde{x}_j^{(k-s)}\})
\]

By construction, the \( \epsilon_i \)'s are linearly independent. However notice that in general the \( \tilde{x}_j^{(i)} \) might be rational.

2. From the form of the \( \epsilon_i \)'s, it follows that any solution to (7.37) can be written as \( \sum_i c_i \epsilon_i \) with \( c_i \) non-negative integers. In particular this means that every solution to (7.37) corresponds to a term in the generating function:

\[
G(X) = \frac{1}{(1 - X^{\epsilon_1})(1 - X^{\epsilon_2}) \ldots (1 - X^{\epsilon_{s}})}
\]

where \( X_1, \ldots, X_k \) are grading variables.

3. \( G(x) \) may contain negative or fractional exponents due to the occurrence of \( \tilde{x}_j^{(i)} \) in the exponents. These are eliminated by first using MacMahon’s algorithm to eliminate
any negative exponents and then rationalizing denominators and keeping only terms with integral exponents in the numerators.

The result is the generating function for the solutions to (7.37). This algorithm, however, does not seem to be optimal in all case. Consider the following example:

\[
\begin{align*}
3x - 2y + z - 3t &= 0 \\
2x + y - 2z - t &= 0
\end{align*}
\]  

with \(x, y, z, t \in \mathbb{N}\). Using linear algebra, we find two basic solutions: \((\frac{3}{7}, \frac{8}{7}, 1, 0)\) and \((\frac{5}{7}, -\frac{3}{7}, 0, 1)\). This means that the initial generating function is

\[
\frac{1}{(1 - X^{3/7}Y^{8/7}Z)(1 - X^{5/7}Y^{-3/7}T)}
\]

from which we must eliminate negative and fractional exponents. This is somewhat lengthy. However, the calculation can be shortened by some observations. First note that we can take as the fundamental solutions: \((\frac{3}{56}, \frac{1}{7}, \frac{1}{8}, 0)\) and \((\frac{5}{21}, -\frac{1}{7}, 0, \frac{1}{3})\) since once again any solution to (7.40) is a non-negative linear combination of these two solutions. This give as the initial generating function:

\[
\frac{1}{(1 - X^{3/56}Y^{1/7}Z^{1/8})(1 - X^{5/21}Y^{-1/7}T^{1/3})}
\]  

Keeping the positive exponents in \(Y\) yields:

\[
\frac{1}{(1 - X^{7/24}Z^{1/8}T^{1/3})(1 - X^{3/56}Y^{1/7}Z^{1/8})}
\]

and keeping integral exponents in \(T\) and \(Y\) gives:

\[
\frac{1}{(1 - X^{7/8}Z^{3/8}T)(1 - X^{3/8}YZ^{7/8})}
\]

Rationalizing these denominators and keeping integral exponents in the numerator yields the final generating function:

\[
(1 + X^2Z^3TY^3 + X^4Z^6T^2Y^6 + X^3Z^2T^3Y + X^5Z^5T^4Y^4 + X^7Z^8T^5Y^7 + X^6Z^4T^6Y^2 + X^8Z^7T^7Y^5)(1 - X^7Z^3T^8)^{-1}(1 - X^3Y^8Z^7)^{-1}
\]

Notice however that in applications to tensor products, it seems we never encounter rational exponents but simply negative ones. The procedure is thus usually simpler.
7.4. Multiple $su(2)$ products from the vector-basis construction

A simple and different application of the formalism just developed is furnished by the analysis of $su(2)$ quadruple tensor products. This application is different in that it does not rely on the triangle description and as such, its formulation is less direct.\footnote{This does not mean however that there are no diagrammatic representations for the quadruple product. In fact, having a set of inequalities, we can transform them into equalities, as it is done below, and from them set up a diagrammatic representation. In the present case, it could correspond to two adjacent $su(2)$ triangles, one upside down, with their adjacent sides forced to be equal. Here we simply mean that the analysis will not rely on such a description.} It will serve as a preparation the somewhat more complicated $sp(4)$ example treated in the following section.

The Diophantine description of this problem has been presented in section 4.2. It is based on the two inequalities (4.11) which are readily transformed into equalities by the introduction of two positive integers $a_1, a_2$:

\begin{align}
\lambda_1 &= n_{12} + a_1 \\
\lambda_1 + n_{11} - n_{12} &= m_{12} + a_2
\end{align}

However this system does not contain any reference to the variable $m_{11}$ and for this reason we introduce the further constraint $m_{11} \geq 0$ which calls for a new integer variable:

\begin{equation}
{m_{11} = a_3}
\end{equation}

We have thus a total of 8 variables: $\{\lambda_1, n_{11}, n_{12}, m_{11}, m_{12}, a_1, a_2, a_3\}$ and 3 equations. There are thus 5 independent variables, chosen to be $\{a_1, a_2, a_3, n_{12}, m_{12}\}$. The basis vectors, with components ordered as follows

\begin{equation}
(a_1, a_2, a_3, n_{12}, m_{12}; \lambda_1, n_{11}, m_{11})
\end{equation}

are obtained by successively setting equal to 1 one of $\{a_1, a_2, a_3, n_{12}, m_{12}\}$ and the others equal to 0. These basis vectors together with their exponentiated version written in terms of appropriate grading variables read:

\begin{align}
(1, 0, 0, 0, 0; 1, -1, 0) & : L_1 N_{11}^{-1} A_1 \\
(0, 1, 0, 0, 0; 0, 1, 0) & : N_{11} A_2 \\
(0, 0, 1, 0, 0; 0, 0, 1) & : M_{11} A_3 \\
(0, 0, 0, 1, 0; 1, 0, 0) & : L_1 N_{12} \\
(0, 0, 0, 0, 1; 0, 1, 0) & : N_{11} M_{12}
\end{align}
The desired generating function is obtained from the projection to positive powers of $N_{11}$ of the function

$$\frac{1}{(1 - L_1 N_{11}^{-1} A_1)(1 - N_{11} A_2)(1 - L_1 N_{12})(1 - N_{11} M_{12})(1 - M_{11} A_3)}$$  \hspace{1cm} (7.49)

The projection operation is done by the familiar method and the result, after setting all $A_i = 1$ is

$$G = \frac{1 - L_1 N_{11} M_{12}}{(1 - L_1 N_{12})(1 - L_1 M_{12})(1 - L_1)(1 - N_{11} M_{12})(1 - N_{11})(1 - M_{11})}$$  \hspace{1cm} (7.50)

from which we read of the 6 elementary couplings $E_1, \ldots, E_6$ (in the order where they appear in the denominator) given in (4.12) and the relation $E_3 E_4 = E_2 E_5$. The above function is exactly the one derived in section 4.2.

7.5. $sp(4)$ diamonds and the vector-basis derivation of the generating function

The system of inequalities (6.1) pertaining to $sp(4)$ can be transformed into a system of equations in the standard way: by setting $r_1/2 = R_1$ and $r_2/2 = R_2$ and introducing the integers $a_i$, we get:

$$\begin{align*}
\lambda_1 &= p_{12} + a_1 \\
\lambda_2 &= R_1 + a_2 \\
\mu_1 &= q_{12} + a_5 \\
\mu_2 &= R_2 + a_8 \\
\nu_1 &= a_1 + a_7 \\
\nu_2 &= a_4 + a_8 \\
a_2 + p_{12} &= a_3 + q_{12} \\
a_3 + R_1 &= a_4 + R_2 \\
a_5 + 2R_2 &= a_6 + 2R_1 \\
a_6 + q_{12} &= a_7 + p_{12}
\end{align*}$$  \hspace{1cm} (7.51)

This leads to a diamond-type graphical representation of the tensor product that has the advantage over the one presented in [18] of being linear (the sum of two diamonds is also a diamond):

---

20 The original idea of looking for a diagrammatic representation of $sp(4)$ tensor products along these lines is due to M. Walton.
In this picture, all data pertaining to the first (second) Dynkin label appear at the left (right). Dotted lines relate those two points that compose the label indicated beside it. Opposite continuous lines are constrained to be equal, with the length of a line being defined as the sum of its extremal points except for the lines delimited by the points \((a_6, R_1)\) and \((a_5, R_2)\) where the point \(R_i\) is counted twice (the little bar besides \(R_1\) and \(R_2\) being a reminder of this particularity). Explicitly, for those lines, we have thus the constraint \(a_6 + 2R_1 = a_5 + 2R_2\). Given a triple \(sp(4)\) product, the number of such diamonds that can be drawn with non-negative entries yields the multiplicity of the product. For instance, the two diamonds that describe the triple coupling \((1, 1) \otimes (1, 1) \otimes (2, 0)\) are:
The dimension of the vector basis is 8 (18 variables and 10 equations, the last four equations above being linearly independent). As our free variables, we choose the set
\{\mathcal{R}_1, \mathcal{R}_2, p_{12}, q_{12}, a_1, a_3, a_6, a_8\}. The 8 basis vectors in terms of grading variables are

\begin{align*}
E_1 & : L_2 M_2 N_2 A_4 A_5 R_1 & E_2 & : M_1^{-2} M_2 N_2^{-1} A_4^{-1} A_5^{-2} R_2 \\
E_3 & : L_1 L_2^{-1} N_1^{-1} A_2^{-1} A_7^{-1} P_{12} & E_4 & : L_2 M_1 N_1 A_2 A_7 Q_{12} \\
E_5 & : L_1 N_1 A_1 & E_6 & : L_2 N_2 A_2 A_3 A_4 \\
E_7 & : M_1 N_1 A_5 A_6 A_7 & E_8 & : M_2 N_2 A_8
\end{align*}

The generating function is obtained by first projecting of the function \(\prod (1 - E_i)^{-1}\) to positive powers for each grading variables and then by setting all grading variables equal to 1 except for \(L_i, M_i, N_i\)’s. The \(sp(4)\) elementary couplings are simple products of the \(E_i\)’s (the following \(A_{1,2,3}\) should not be confused with the above grading variables):

\begin{align*}
A_1 &= E_7 & A_2 &= E_5 & A_3 &= E_3 E_4 \\
B_1 &= E_8 & B_2 &= E_6 & B_3 &= E_1 E_2 \\
C_1 &= E_4 & C_2 &= E_2 E_3 E_6 E_7 & C_3 &= E_1 E_3 E_7 \\
D_1 &= E_2 E_3^2 E_6^2 E_7^2 & D_2 &= E_1 & D_3 &= E_2 E_6 E_7^2
\end{align*}

The complete list of \(sp(4)\) elementary couplings (6.2) are thus recovered.

8. Summary and conclusion

8.1. Listing the methods considered

Here is the list of methods that have been reviewed here for construction tensor-product generating functions:

1- The character method

This has been extensively exemplified in section 2. It starts from first principles (the multiplication of two characters or more precisely, two character generating functions) but it is computationally complicated.

2- Composition of generating function

This technique uses other, simpler generating functions to construct the required generating function. The method has been illustrated in sections 2.3 and 2.5. One example of this method is the use of the Giambelli formula.
In this approach, we reformulate the problem of computing tensor products in terms of a set of Diophantine inequalities (section 4). We then look for the elementary solutions and their relations and from these, we construct the generating function. We can distinguish two ways of obtaining the elementary solutions and their relations:

i) MacMahon’s method which yields the generating function as an output of his method for finding the elementary solutions and their relations (here these are the syzygies of first, second and possibly higher orders). This approach is plagued by the technical difficulties of the intermediate projection operations. It is discussed in section 3.

ii) Apply the Huet’s algorithm to find the elementary couplings and then apply Grobner basis techniques to calculate a complete set of forbidden couplings. This is certainly powerful and it appears to be the most effective approach. Grobner bases are presented in section 5 and this method is illustrated in section 4 and 6.

4- Diophantine equalities – The vector basis

In this approach, we first re-express the inequalities in terms of equalities and then write the vector basis, relaxing the positivity constraint inherent to the elementary couplings. Since the basis vectors are independent the initial generating function is simply a product of terms of the form \( 1/(1 - E) \) for suitable monomials \( E \). The elementary couplings are then recovered from the projection to positive solutions (and a second projection onto terms with integral exponents may also be necessary in general) and the result of the projection(s) is the desired generating function. This method and various examples are worked out in section 7.

8.2. Conclusion

As it has already been stressed in the introduction, the main purpose of this work is to prepare the ground for the analysis of fusion rules, which is the subject of a sequel paper. In the present work, we have reviewed the existing techniques for computing tensor-product generating functions and presented a comparative critical assessment of their respective virtues and limitations. In addition, we have focused on a model formulation linking generating functions to Poincaré series, an idea that has first been introduced in [4] and further extended in [5]. Our contribution in that respect has been to rephrase this program
more explicitly, clarify some issues and to exemplify the procedure with many examples, some of which are new.

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