Classical and quantum motion on the orbifold limit of the Eguchi-Hanson metric

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Abstract. – We investigate the behaviour of a particle moving on the orbifold limit of the EH metric as the two centers approach each other. In the classical region of the configuration space we specify the physically acceptable solutions and observe a tendency of the radial wave function to concentrate around the conical singularity for small values of its argument. In the quantum case, using Schr"{o}dinger’s equation, we determine the energy spectra and the radial eigenfunctions for a class of potentials.

Introduction. –

Hyper-K"{a}hler manifolds in four dimensions have been studied extensively in connection with the theory of gravitational instantons, their algebraic generalizations through Penrose’s nonlinear graviton theory and the heavenly equations over the past twenty years.

Motivated by their presence in supersymmetric field theories we extract the EH metric (which is a member of the complete regular SO(3)-invariant hyper-K"{a}hler family in four dimensions) from the K"{a}hler potential of the most general $N = 2$ nonlinear $\sigma$-models action. Using well known transformations we bring EH metric in a more familiar form appropriate for performing our calculations. When the two centers of the EH metric coincide the manifold exhibits a singular behaviour at $r = 0$ and is recognized to be the orbifold $M = \mathbb{C}^2/\mathbb{Z}_2$. The generalization of this statement to all members of the $A_e$-series translates into the appearance of a self-dual, positive definite solution of Einstein’s equations in vacuum (a self-dual gravitational instanton).

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1 A Riemannian manifold $(M, g)$ is hyper-K"{a}hler if it is equipped with three automorphisms $J_i; i = 1, 2, 3$ of the tangent bundle which satisfy the quaternion algebra: $J_i J_j = -\delta_{ij} + \epsilon_{ijk} J_k$, $[J_i, J_j] = 2\epsilon_{ijk} J_k$ and are covariant constant with respect to the Levi-Civita connection: $\nabla J_i = 0$ for $i = 1, 2, 3$.

2 A complete hyper-K"{a}hler manifold is a self-dual, positive definite solution of Einstein’s equations in vacuum (a self-dual gravitational instanton).
of the manifold \( M = C^2/Z_{k+1} \). Taking advantage of the Kähler structure of the manifold we specify an invariant quantity, quadratic w.r.t. the Weyl tensor, for which after integration over \( SO(3) \) produces the correct Hirzebruch signature. The Euler characteristic and the Hirzebruch signature are also evaluated for our case study.

Starting from the classical wave equation describing the motion of a massive particle we show that there exist radial polynomial solutions (generalized Bessel’s functions) which strongly depend on the dimension of our manifold. Increasing values of \( l \) for \( \rho \ll 1 \) force the radial function to accumulate around the point of highest symmetry (the fixed point \( r = 0 \)). The angular part is expressed in terms of the Jacobi polynomials which are closely related to the associated Legendre functions of the first kind upon angle coordinate reduction of the original four dimensional manifold down to the three dimensional space. The solutions are suitably normalized in such a way that one can recover known orthonormality results in the reduced space.

The quantum behaviour on the manifold is exploited by a detailed study of some finite dimensional quantum models. This is achieved with the help of Schrödinger’s equation and switching on different potentials. For simplicity we consider two types of potentials (the harmonic and the Coulomb potential) which lead to discretized energy spectra. The energy levels of the harmonic potential in one dimension and in the absence of angular momenta are identical to the expected ones. In the Coulomb case the radial eigenfunctions turn out to be Laguerre’s polynomials which reduce to the predicted ones in three dimensions.

Rederivation of the Eguchi-Hanson metric. –

The N=2 action we consider is \( \int d^8z \left( \sum_{i=1}^n |\Phi_i^+|^2 e^V + \sum_{i=1}^n |\Phi_i^-|^2 e^{-V} - cV \right) + \int d^6z \left[ \sum_{i=1}^n \Phi_i^- \Phi_i^+ - b \right] S + h.c \).

where \( d^n z = dx d\theta d\bar{\theta} \), \( V \) is a real (\( N = 1 \) four-dimensional) superfield, \( \Phi_{\pm} \)'s are 2n independent complex chiral (right-, left-handed) superfields satisfying \( D_{\alpha} \Phi_+ = D_{\bar{\alpha}} \Phi_- = 0 \), with \( D_{\alpha}, \bar{D}_{\alpha} \) the usual two component spinor covariant derivatives. \( \Phi_{\pm} \)'s can be chosen to be in a representation of \( SU(n) \) such that \( \Phi_{\pm} \) is \( SU(n) \) invariant. \( S \) is a Lagrange multiplier chiral superfield satisfying \( \bar{D}_{\alpha} S = D_{\bar{\alpha}} S = 0 \), the bar and the symbol h.c. denote complex and Hermitian conjugation. The constants c and b are real and complex respectively and parametrize the particular linear combination of the coordinate systems we are using to describe the manifold. The term \( c \int d^8z V(x, \theta, \bar{\theta}) \) represents the Fayet-Iliopoulos term which is gauge invariant and supersymmetric. The equation of motion for the auxiliary fields \( V \) and \( S \) is:

\[
\sum_{i=1}^n |\Phi_i^+|^2 e^V - \sum_{i=1}^n |\Phi_i^-|^2 e^{-V} = c
\]

\[
\sum_{i=1}^n \Phi_i^- \Phi_i^+ = b.
\]

We restrict our attention to the case where \( b = 0 \) and there are only two chiral complex superfields \( n = 2 \). Choosing the gauge:
we solve the equation of motion for $V$ ($\sinh V = \frac{c}{2r}$). The result is:

$$V = \ln \left[ \frac{c}{2r} + \sqrt{1 + \frac{c^2}{4r^2}} \right]$$

where $r = |Z_1|^2 + |Z_2|^2 = Z_i \bar{Z}_i$ and we keep only the positive root of the quadratic equation for $V$. The Kähler potential is found to be:

$$K(Z_i, \bar{Z}_i) = 2r \cosh V - cV = 2r \sqrt{1 + \frac{c^2}{4r^2}} - c \ln \left( \frac{c}{2r} + \sqrt{1 + \frac{c^2}{4r^2}} \right)$$

and the action now takes the form:

$$I = \int d^4x d^2\theta d^2\bar{\theta} K(Z_i, \bar{Z}_i).$$

The Kähler metric is given as usual by:

$$g_{ij} = \frac{\partial^2 K(Z, \bar{Z})}{\partial Z^i \partial \bar{Z}^j} = 2\delta_{ij} \sqrt{1 + \frac{c^2}{4r^2}} - Z_i Z_j \frac{c^2}{2r^3 \sqrt{1 + \frac{c^2}{4r^2}}}$$

with inverse:

$$g^{ij} = (g_{ij})^{-1} = \frac{1}{2} \delta^{ij} \sqrt{1 + \frac{c^2}{4r^2}} - \frac{c^2}{8r^3} \epsilon^{ik} \epsilon^{jm} Z^k \bar{Z}^m \sqrt{1 + \frac{c^2}{4r^2}}. \quad (9)$$

We recognize this metric to be the Calabi metric which in four real dimensions is the Eguchi-Hanson metric, i.e., the simplest Asymptotically Locally Euclidean (ALE) gravitational instanton \[^3\]. To prove this we change to Cartesian coordinates \[^3\] using the transformations (the Jacobian of the transformation is $(r \sin \theta)/16$):

\[^3\) ALE spaces describe a Riemannian 4-manifold geodesically complete and such that:

1. the curvature 2-form is anti(self)-
dual
2. the Riemannian metric is required to approximate a Euclidean metric up to $O(r^{-4})$, $g^{ij} = \delta^{ij} + a^{ij} = \delta^{ij} + O(r^{-4})$ with appropriate decay in the derivatives of $g^{ij}$. In other words, $
abla^2 a^{ij} = O(r^{-4-p})$, $p \geq 0$ where $r^2 = \sum x_i^2$ and $\nabla$ denotes differentiation w.r.t. the coordinates $x_i$.

This would agree with the intuitive picture of instantons as being localized in finite regions of space-time. The above picture is only verified modulo an additional subtlety: the basic manifold at infinity resembles a quotient $R^4/T$, $T$ being a finite group of identifications.

\[^4\) From now on we treat the components $(Z_i)$ of the chiral superfields $(\Phi^\pm_i)$ as ordinary coordinates on the four-dimensional space.
\[ Z_1 = \sqrt{r} \cos\left(\frac{\theta}{2}\right) e^{i(\psi + \phi)}, \quad Z_2 = \sqrt{r} \sin\left(\frac{\theta}{2}\right) e^{i(\psi - \phi)} \] (10)

and setting \( \alpha^2 = \frac{a^2}{4} \) one finds for the line element:

\[ ds^2_{EH} = g_{ij} dZ^i dZ^j = \frac{1}{4} \frac{dr^2}{\sqrt{1 + (\frac{\alpha}{r})^2}} + \frac{r}{\sqrt{1 + (\frac{\alpha}{r})^2}} \sigma_z^2 + r \sqrt{1 + (\frac{\alpha}{r})^2 (\sigma^2_x + \sigma^2_y)} \] (11)

where \( \sigma_x, \sigma_y \) and \( \sigma_z \) are the left-invariant one-forms of the \( SU(2) \) group (see Appendix A for their definition). The variables \( r, \theta, \psi, \phi \) are constrained by \( \alpha \leq r \leq \infty, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi, \quad 0 \leq \psi \leq 2\pi \). The restricted range of \( \psi \) reflects the \( Z_2 \) identification of antipodal points. Using the coordinate transformation \( r^2 = \rho^4 \) and that \( \alpha^2 = a^4 \) we end up with the familiar form of the Eguchi-Hanson metric [6]:

\[ ds^2_{EH} = 1 \frac{1}{\sqrt{1 + (\frac{a}{\rho})^4}} \left( d\rho^2 + \rho^2 \sigma_z^2 \right) + \rho^2 \sqrt{1 + (\frac{a}{\rho})^4 (\sigma_x^2 + \sigma_y^2)} \] (12)

The determinant of the metric in this case is given by

\[ g = 1 \frac{1}{64} \rho^6 \left[ 1 + (\frac{a}{\rho})^4 \right] \left[ 1 + (\frac{a}{\rho})^4 \sin^2 \theta \right]. \]

We can also write the line element in the equivalent form:

\[ ds^2 = V^{-1} (d\phi + A_i dx^i)^2 + V \gamma_{ij} dx^i dx^j, \] (13)

where \( V, A_i, \gamma_{ij} \) being all independent of \( \phi \), and \( V \) satisfies:

\[ V(\vec{x}) = \epsilon + 2m \sum_{i=1}^{k+1} \frac{1}{|\vec{x} - \vec{x}_i|} \] (14)

\[ \vec{\nabla} V = -\vec{\nabla} \times \vec{A} \] (15)

\[ \vec{\nabla} \cdot \vec{\nabla} V = 2m \sum_{i=1}^{k+1} \delta^{(3)}(\vec{x} - \vec{x}_i). \] (16)

The choice \( \epsilon = 0, \quad m = \frac{1}{2} \) corresponds to an admissible metric on the \( k^{th} \) representative of the \( A \)-series space. For the EH instanton \((k = 1)\) after some straightforward calculations we find that: \([5]\)

\[ V^{-1} = \frac{1}{4} (c_1 \sin^2 \theta \cos^2 \psi + c_2 \sin^2 \theta \sin^2 \psi + c_3 \cos^2 \theta) \] (17)

\[ \vec{A} = \frac{1}{4} V(0, (c_2 - c_1) \sin \psi \cos \psi \sin \theta, c_3 \cos \theta) \] (18)

\[ \gamma_{rr} = \frac{c_0}{V}, \quad \gamma_{\psi \psi} = \frac{1}{16} (c_3 \sin^2 \theta (c_1 \cos^2 \psi + c_2 \sin^2 \psi)), \]

\[ \gamma_{\theta \theta} = \frac{1}{16} (c_1 c_2 \sin^2 \theta + c_3 \cos^2 \theta (c_1 \sin^2 \psi + c_2 \cos^2 \psi)), \]

\([5]\) In [5] the coefficient \( \frac{1}{4} \) of equation (32) should be replaced by \( \frac{V}{4} \).
\[ \gamma_{\theta\psi} = -\frac{1}{16} [c_3 (c_2 - c_1) \sin \psi \cos \psi \sin \theta \cos \theta] \] (19)

where \( c_0 = 1/\sqrt{r^2 + \alpha^2} \), \( c_1 = c_2 = \sqrt{r^2 + \alpha^2} \), and \( c_3 = r^2/\sqrt{r^2 + \alpha^2} \). The explicit expressions for \( V \), \( \vec{A} \) and the components of the \( \gamma \) metric are:

\[ V = 4\sqrt{r^2 + \alpha^2} r^2 + \alpha^2 \sin^2 \theta \] (20)

\[ \vec{A} = \left( 0, 0, \frac{r^2 \cos \theta}{r^2 + \alpha^2 \sin^2 \theta} \right) \] (21)

\[ \gamma_{rr} = \frac{r^2 + \alpha^2 \sin^2 \theta}{16(r^2 + \alpha^2)}, \gamma_{\psi\psi} = \frac{r^2}{16} \sin^2 \theta, \gamma_{\theta\theta} = \frac{1}{16}(r^2 + \alpha^2 \sin^2 \theta), \gamma_{\theta\psi} = 0 \] (22)

The space is flat because the Ricci tensor vanishes:

\[ R_{ij} = g^{kl} R_{klij} = R'_{ij} = -\frac{\partial^2}{\partial Z_i \partial Z_j} \ln(\det(g_{ij})) = 0. \] (23)

since \( \det(g_{ij}) = 4 \). A manifold equipped with a Ricci flat metric, by Yau’s proof of the Calabi conjecture, implies that the first Chern class must vanish: \( c_1^R(K) = 0 \). As a result the dimensions of the Dolbeault cohomology groups \( H^{(2,0)}(M) \) and \( H^{(0,2)}(M) \) are equal to one.

**Manifold structure and topological invariants.** –

The global topology of the EH manifold is \( S^2 \times \mathbb{R} \)

1. close to \( r = c \), the manifold is homotopic to \( S^2 \) \( (M^{EH} \approx R^2 \times S^2) \) and has the same Euler characteristic as \( S^2 \) i.e. \( \chi = 2 \),

2. when \( r \to \infty \) the metric approaches a flat metric and the constant-\( r \) hypersurfaces are distorted three-spheres with opposite points identified with respect to the origin. The group manifold is then \( M_{EH} \approx \mathbb{R} \times \mathbb{R} \mathbb{P}^3 \) where \( \mathbb{R} \mathbb{P}^3 \) is the real projective space \( \mathbb{R} \mathbb{P}^3 = SO(3) = S^3/Z_2 \) for which \( S^3 = SU(2) \) is the double covering.

The Gauss-Bonnet theorem states that it is possible to obtain the Euler characteristic of a closed Riemannian manifold of even dimension \( \dim M = 2l \) from the volume integral of the 2l-form \( \Omega \):

\[ \chi_{M^{2l}} = \int_{M^{2l}} \Omega = \frac{(-1)^l}{4\pi^l} \int_{M^{2l}} \epsilon_{a_1 \cdots a_{2l}} R^{a_1 a_2} \cdots \wedge R^{a_{2l-1} a_{2l}}. \] (24)

In four dimensions it takes the familiar form:

\[ \chi_{M^4} = \frac{1}{32\pi^2} \int_{M^4} \epsilon_{abcd} R^{abcd} \wedge R^{cd} \] (25)

where the curvature two-form \( R^a_{\bar{b}} \) is defined by the spin connection one-forms \( \omega_a^b \) as:

\(^{(k)}\) The first Chern class is represented by the differential form \( h = R_{i\bar{j}} dZ^i dZ^\bar{j} \) and is exact, i.e. that \( h = \partial \beta \) for some \( \beta \).
It has also been proved that $\Omega$ can be expressed as the exterior derivative of a $(2l - 1)$-form in $M^{4l-1}$ constructed by the unit tangent vectors of $M^{2l}$:

$$\Omega = -dD. \quad (27)$$

In this way the original integral of $\Omega$ over $M^{2l}$ can be performed over a submanifold $U^{2l}$ obtained as the image in $M^{4l-1}$ of a continuous unit tangent vector field over $M^{2l}$ with some isolated singular points. Applying Stoke’s theorem one thus get:

$$\chi_{M^{2l}} = \int_{M^{2l}} \Omega = \int_{U^{2l}} \Omega = \int_{\partial U^{2l}} D. \quad (28)$$

For manifolds with a boundary, the above formula can be generalized to include boundary corrections:

$$\chi_{M^{2l}} = \int_{M^{2l}} \Omega - \int_{\partial M^{2l}} D = \int_{\partial U^{2l}} D - \int_{\partial M^{2l}} D. \quad (29)$$

The 3-form $D$ in four dimensions is given by \[6, 8\]:

$$D = -\frac{1}{16\pi^2} \epsilon_{abcd}(\theta^{ab} \wedge R^{cd} - \frac{2}{3} \theta^{ab} \wedge \theta^c \wedge \theta^{cd}) \quad (30)$$

where $\theta^a_b$ is the second fundamental form of the Lorentz group i.e. the difference between the spin connection of the original metric, computed on the boundary, and the spin connection obtained from the boundary if the metric were locally a product near the boundary:

$$\theta^a_b = \omega^a_b - (\omega_0)_b^a. \quad (31)$$

For the EH-manifold the boundary $\partial M_{EH}$ will be represented by a slice at $r = r_0$ which in the end we will have to send to infinity. Since the EH metric factorizes on $\partial M_{EH}$ one easily calculates the components of the second fundamental form to be:

$$\theta_{12} = \theta_{23} = \theta_{31} = 0$$

$$\theta_{01} = -\left(1 - \left(\frac{\alpha}{r_0}\right)^4\right)^{\frac{1}{2}} \sigma_x$$

$$\theta_{02} = -\left(1 - \left(\frac{\alpha}{r_0}\right)^4\right)^{\frac{1}{2}} \sigma_y \quad (32)$$

$$\theta_{03} = -\left(1 + \left(\frac{\alpha}{r_0}\right)^4\right) \sigma_z$$

From (29), (30) and (32) it follows that:

$$\chi_{EH} = \frac{3}{2} + \frac{1}{2} = 2. \quad (33)$$
The Hirzebruch signature $\tau$ receives contribution only from the “bulk” and is given by:

$$
\tau_{EH} = \frac{1}{24\pi^2} \int_{M_{EH}} Tr(R \wedge R) = \frac{1}{12\pi^2} \int_{M_{EH}} R_{ab}R_{ba} = 1.
$$

(34)

An alternative calculation for $\tau_{EH}$ is presented in the Appendix B. In general for the multi-Taub NUT metrics the Euler characteristic and the Hirzebruch signature are connected through the relation $\chi = \tau + 1$ where $\tau = n - 1$, $n$ being the number of centers.

The orbifold limit of the EH-metric.

In the limit when $\vec{x}_i \to 0$, or equivalently $\vec{x} \to \infty$ the two-centered metric degenerates to the orbifold metric on $M = C^2/\Gamma$ [7]. $\Gamma$ in general will be a Kleinian subgroup of $SU(2)$ (for multi-centered metrics is $\mathbb{Z}_{k+1}$) but in the Eguchi-Hanson case (two-centered or $k = 1$) will be identified to be $\mathbb{Z}_2$. The limit $\vec{x}_i \to 0$ is also equivalent to taking $a \to 0$ in the line element i.e.

$$
ds^2_{\bar{C}^2/Z_2} = \lim_{a \to 0} ds^2_{EH} = \frac{1}{4r} dr^2 + r (\sigma_x^2 + \sigma_y^2 + \sigma_z^2)
$$

$$
= \frac{1}{4r} dr^2 + \frac{r}{4} (d\phi^2 + d\theta^2 + d\psi^2 + 2 \cos^2 \theta d\psi d\phi).
$$

(35)

When we approach the origin we encounter a conical singularity which corresponds to a point of higher symmetry [8]. The self-dual components of the spin connection are easily computed to be:

$$
\omega_0^1 = \omega_3^2 = \sigma_x = \frac{e^1}{r},
$$

$$
\omega_0^2 = \omega_1^3 = \sigma_y = \frac{e^2}{r},
$$

$$
\omega_0^3 = \omega_2^1 = \sigma_z = \frac{e^3}{r}
$$

(36)

where the orthonormal vierbein basis is $e^a = (e^0, e^1, e^2, e^3) = (dr, r\sigma_x, r\sigma_y, r\sigma_z)$. A factor of $4r$ has been absorbed in the line element i.e.

$$
ds^2 = dr^2 + r^2 (\sigma_x^2 + \sigma_y^2 + \sigma_z^2).
$$

(37)

From the definition of the curvature two-form we find that its components vanish so this metric is flat everywhere apart from the fixed point.

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(7) The ALE spaces can also be described as the complex affine variety [9] in $C^3$ characterized by the vanishing locus $W(x, y, z) = 0$, where $x, y, z$ are the coordinates on $C^3$. For the A-series the corresponding discrete subgroups of $SU(2)$ are the binary cyclic groups $C_n = Z_{2n}$ whose action on $x, y$ is generated by $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} e^{i\frac{\pi}{n}} & 0 \\ 0 & e^{-i\frac{\pi}{n}} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. The smallest set which generates all polynomials invariant under $C_n$ is clearly $X = x^{2n}$, $Y = y^{2n}$, $Z = xy$ and these generators obey the single relation $XY = Z^{2n}$. In the vicinity of the Kleinian singularity of $C^2/Z_2$ the real varieties have a double cone structure (the tips of the cones face each other).

(8) In this case both vector fields $\frac{\partial}{\partial \sigma_x}$ and $\frac{\partial}{\partial \sigma_y}$ generate translational isometries and since flat space is self-dual as well as anti-self-dual the rotational character of $\frac{\partial}{\partial \sigma_x}$ or $\frac{\partial}{\partial \sigma_y}$ disappears.
In terms of our original expression (eq. (8)) for the Kähler metric the orbifold limit corresponds to taking both $c, r$ to zero. We can express $V, \vec{A}$ and $\gamma_{ij}$ of the line element in terms of the $Z^1, Z^2$ coordinates as follows (see Appendix B):

\[
V_{\text{orb}} = \frac{4}{|Z^1|^2 + |Z^2|^2}, \quad A_{\text{orb}} = \pm \frac{|Z^1|^2 - |Z^2|^2}{|Z^1|^2 + |Z^2|^2}, \\
\gamma_{rr} = \frac{1}{16}, \quad \gamma_{\psi\psi} = \frac{1}{4} |Z^1|^2 |Z^2|^2, \quad \gamma_{\theta\theta} = \frac{1}{16} (|Z^1|^2 + |Z^2|^2)^2, \quad \gamma_{\theta\psi} = 0.
\]  

(38)

The “bulk” contributions to the Euler characteristic and the Hirzebruch signature vanish since $R_{ab} = 0$ so we end up with:

\[
\chi_{C^2/Z_2} = 1, \quad \tau_{C^2/Z_2} = 0.
\]  

(39)

These results are also confirmed by the prementioned formula of the Euler characteristic expressed by $\chi = \tau + 1 = n$.

**Classical motion on the $C^2/Z_2$ orbifold.** –

Consider the classical wave equation of a particle with mass $\mu$ on the quotient manifold $M = C^2/Z_2$ described by:

\[
\partial_{\mu} (\sqrt{g} g^{\mu\nu} \partial_{\nu} F) - \sqrt{g} \mu^2 F = 0
\]  

(40)

where $F = F(r, \theta, \phi, \psi) = R(r)Q(\theta, \phi, \psi)$ and $g = \det g_{ij} = r^6 \sin^2 \theta$. The radial equation takes the form:

\[
\frac{d^2 R}{dr^2} + \frac{(q-1)}{r} \frac{dR}{dr} - (\mu^2 + \frac{\lambda^2}{r^2}) R = 0
\]  

(41)

with $\lambda^2 = l(l + 1)$ a positive constant determining the opening angle and $q = \dim M$ is the dimension of the manifold. Changing variable to $\rho = r \mu$ we obtain the generalized Bessel’s equation [10]:

\[
\frac{d^2 J}{d\rho^2} + \frac{(q-1)}{\rho} \frac{dJ}{d\rho} - (1 + \frac{\lambda^2}{\rho^2}) J = 0.
\]  

(42)

Solving (42) in the standard way we find the solution for the $(s^+ - s^-) \notin Z^+$ case to be (see Appendix C for the other solutions):

\[
J(\rho) = \begin{cases} 
AJ_{s^+}(\rho) + BJ_{s^-}(\rho) & \\
= A \left(\frac{\rho}{2}\right)^{s^+} \sum_{j=0}^{\infty} \left(\frac{\rho}{2}\right)^{2j} \frac{1}{j! \Gamma(1 + j + d)} + B \left(\frac{\rho}{2}\right)^{s^-} \sum_{j=0}^{\infty} \left(\frac{\rho}{2}\right)^{2j} \frac{1}{j! \Gamma(1 + j + d)} & 
\end{cases}
\]  

(43)

where $s^\pm = s^\pm_{(l,q)} = -\frac{(q-2)}{2} \pm d = -\frac{(q-2)}{2} \pm \sqrt{\lambda^2 + \frac{(q-2)^2}{4}}$, $i = 1, 2$. The limiting forms of the Bessel’s functions for small and large values of their arguments are given by the leading terms:
\[
\rho \ll 1 , \quad J_{s}^{\pm} (\rho) \rightarrow \frac{1}{\Gamma \left( s^{\pm} + \frac{1}{2} \right)^{2} } \left( \frac{\rho^{\pm}}{2} \right)^{s^{\pm}} 
\]
\[
\rho \gg 1 , \quad J_{s}^{\pm} (\rho) \rightarrow \frac{2^{s^{\pm}}}{\sqrt{\pi \rho^{q-1}}} \cos \left( \rho - \frac{s^{\pm} \pi}{2} - \frac{(q-1) \pi}{4} \right). 
\]

Notice that for \( \rho \ll 1 \) the solutions are always square integrable around the singular point \( r = 0 \) and there is a tendency of the wave function to concentrate around \( r = 0 \) when \( l \) decreases.

The azimuthal equation has solutions the eigenfunctions of the operators \( L_{2} \), \( L_{3} \), \( L_{2}^{\prime} \) and \( L_{3}^{\prime} \) i.e.

\[
L_{2}^{\prime} |l, m, n \rangle = l(l+1) |l, m, n \rangle \\
L_{3}^{\prime} |l, m, n \rangle = n |l, m, n \rangle \\
L_{2}^{\prime} |l, m, n \rangle = l(l+1) |l, m, n \rangle \\
L_{3}^{\prime} |l, m, n \rangle = m |l, m, n \rangle 
\]

where \( |l, m, n \rangle = Q_{l, m, n} (\theta, \phi, \psi) = \Phi_{l, m}(\phi) \Psi_{n}(\psi) W_{l, m, n}(\theta) \) and \( |m| \leq l, |n| \leq l \). The generators \( L_{i} \) obey the \( SU(2) \) algebra \([L_{i}, L_{j}] = i \epsilon_{ijk} L_{k}\), \([L_{2}^{\prime}, L_{3}^{\prime}] = 0\), while their dual partners the commutation relations \([\tilde{L}_{i}, \tilde{L}_{j}] = -i \epsilon_{ijk} \tilde{L}_{k}\), \([\tilde{L}_{2}^{\prime}, \tilde{L}_{3}^{\prime}] = 0\). The first equation of (46) can be written equivalently as:

\[
\frac{1}{\sin \theta} \frac{d}{d \theta} \left( \sin \theta \frac{d W}{d \theta} \right) + \left[ l(l+1) - \frac{m^{2} + n^{2}}{\sin^{2} \theta} + 2 \frac{\cos \theta}{\sin^{2} \theta} mn \right] W = 0
\]

and substituting \( u = \cos \theta \) for \( \theta \) as well as putting \( W(\theta) = P(u) \), (47) becomes:

\[
\frac{d}{du} \left[ (1-u^{2}) \frac{dP}{du} \right] + \left[ l(l+1) - \frac{m^{2} + n^{2}}{1-u^{2}} + 2\frac{u}{1-u^{2}} mn \right] P = 0.
\]

The solutions of the differential eq. (48) have the following symmetries:

1. they are invariant under the interchange of \( m, n \):

\[
P_{m,n}^{l}(u) = P_{n,m}^{l}(u) = P_{-m,-n}^{l}(u)
\]

2. we recover the familiar result of the associated Legendre functions of the first kind when \( m = 0 \) or \( n = 0 \):

\[
P_{m,0}^{l}(u) = P_{m}^{l}(u) = P_{0,-n}^{l}(u) = P_{n}^{l}(u).
\]

The singular regular points of eq. (48) are \( u = \pm 1 \). Near the points \( u = \mp 1 \) the dominant behaviour is \((u-1)^{a/2}\) and \((1+u)^{b/2}\) respectively, where \( a = |m-n| \) and \( b = |m+n| \). Consider now the expansion of the form:
with \( N_{m,n}^l \) a normalization factor to be determined later on and \( U_{m,n}^l(u) \) satisfying the differential equation:

\[
(1 - u^2) \frac{d^2 U}{du^2} + [b - a - (2 + a + b)] \frac{dU}{du} + \left[ l(l + 1) - \frac{a + b}{2} (\frac{a + b}{2} - 1) \right] U.
\]

We recognize (52) to be a hypergeometric equation with solutions the Jacobi polynomials [1], provided that \( l = \tilde{\rho} + \frac{a+b}{2} \) and \( \tilde{\rho} \in \mathbb{Z} \). One can prove that equation:

\[
\frac{d}{du} \left[ (1 - u)^{a+1} (1 + u)^{b+1} \frac{dU}{du} \right] + \tilde{\rho}(\tilde{\rho} + a + b + 1)U = 0
\]

has solutions given by the Rodrigue’s formula:

\[
T_{k}^{(a,b)}(u) = \frac{(-1)^k}{2^k k!} (1 - u)^{-a} (1 + u)^{-b} \frac{d^k}{du^k} [(1 - u)^{k+a} (1 + u)^{k+b}]
\]

\[
= \sum_{\nu=0}^{k} \binom{k + a}{k - \nu} \binom{k + b}{\nu} \left( \frac{u - 1}{2} \right)^\nu \left( \frac{1 + u}{2} \right)^{k-\nu}
\]

and obeying the important identity:

\[
T_{k}^{(a,b)}(u) = (-1)^k T_{k}^{(b,a)}(-u).
\]

In our case the solutions \( P_{m,n}^l \) are given explicitly by:

\[
P_{m,n}^l(u) = e^{i\phi}\mathbf{Y}_{m,n}^l(\theta, \phi)
\]

\[
= \left[ \frac{2l + 1}{4\pi} \frac{(l-m)!(l+m)!}{(l-n)!(l+n)!} \right]^{\frac{1}{2}} e^{im\phi} \zeta_{m,n}^{\psi} (1 - u^2)^{\frac{1}{2}} (1 + u)^{\frac{1}{2}}
\]

and form a complete set of orthonormal polynomials \( i.e.\):

\[
\int_0^{2\pi} d\theta \int_0^{2\pi} d\phi \int_{-1}^{+1} dz Q_{m,n}^l(u, \phi, \psi) Q_{m',n'}^{l'}(u, \phi, \psi) = \frac{8\pi^2}{2l + 1} \delta_{l,l'} \delta_{m,m'} \delta_{n,n'}
\]

\[
\sum_{l,m,n} Q_{m,n}^l(\theta, \phi, \psi) Q_{m,n}^l(\theta', \phi', \psi') = \frac{8\pi^2}{2l + 1} \delta(\cos \theta - \cos \theta') \delta(\phi - \phi') \delta(\psi - \psi').
\]

(\textsuperscript{3}) It was shown in [12] that in complex stereographic coordinates \((\zeta, \bar{\zeta})\), defined by \( \zeta = e^{i\phi} \cot \frac{\theta}{2} \), the spin-s spherical harmonics take the form \( Y_{m,s}^l = \left[ \frac{(2l + 1)(l-m)!(l+m)!}{4\pi(l+s)!(l+s)!} \right]^{\frac{1}{2}} (-1)^{l-m} \times (1 + \zeta \bar{\zeta})^{-1} \sum_{p} \binom{l - s}{p} \binom{l + s}{p + s - m} \zeta^p (-\bar{\zeta})^{p+s-m}. \) In Appendix D we prove that these solutions are related to the Jacobi polynomials up to a symmetry.
Schrödinger’s equation on $C^2/Z_2$.

In the presence of the harmonic potential $V(r) = \frac{1}{2}Kr^2$ Schrödinger’s equation becomes:

$$\frac{\hbar^2}{2\mu}\left[\frac{d^2R}{dr^2} + \frac{(q-1)}{r}\frac{dR}{dr}\right] - \left[2(V - E) + \frac{2\mu l(l+1)}{\hbar^2}r^2\right]R = 0$$

which by changing variable to $\xi = \left(\frac{2\mu K}{\hbar^2}\right)^{\frac{1}{4}}r$ and introducing the quantities $a^2 = \frac{2\mu K}{\hbar^2}$ and $\lambda = \frac{2\bar{\hbar}E}{\hbar^2}$ it can be rewritten in the following dimensionless form as:

$$\frac{d^2R}{d\xi^2} + \frac{(q-1)}{\xi}\frac{dR}{d\xi} + \left[\lambda - \xi^2 - \frac{l(l+1)}{\xi^2}\right]R = 0.$$  

The solution of eq. (60) is facilitated on one hand by examining the dominant behaviour of $R(\xi)$ in the asymptotic region $\xi \to +\infty$ and on the other hand by demanding finiteness at $\xi = 0$. Then the desired solution is:

$$R(\xi) = e^{-\frac{1}{2}\xi^2}\xi^{s^+}L(\xi)$$

where $L(\xi)$ satisfies the equation:

$$\xi L'' + (2s^+ + q - 1 - 2\xi^2)L' + 2\nu\xi L = 0.$$  

Energy levels are determined in the usual way and are found to be:

$$E_{l,\nu,q} = \frac{\hbar\omega_c}{\sqrt{2}}\left[\nu + 1 + \sqrt{(q - 2)^2 + 4l(l+1)}\right]$$

with $\nu$ being an integer ($\nu \geq 0$) and $\omega_c = \sqrt{\frac{K}{\mu}}$ the angular frequency of the corresponding classical harmonic oscillator. In one dimension and when $l = 0$ we obtain the expected result.

The solutions (61) fall into two classes depending on whether $\nu$ is even or odd integer.

1. $\nu = 2n$:

$$R_{even}(\xi) = e^{-\frac{1}{2}\xi^2}\xi^{s^+}\left[\frac{\Gamma(-n)}{\Gamma(d)} + \sum_{k=1}^{2(n-1)} \xi^{2k}\frac{\Gamma(k-n)}{\Gamma(k)\Gamma(1+k+d)}\right]$$

2. $\nu = 2n + 1$:

$$R_{odd}(\xi) = e^{-\frac{1}{2}\xi^2}\xi^{s^+}\left[\frac{\Gamma(-\frac{1}{2}-n)}{\Gamma(d)} + \sum_{k=1}^{2n-1} \xi^{2k}\frac{\Gamma(k-n-\frac{1}{2})}{\Gamma(k)\Gamma(1+k+d)}\right].$$
where $\Gamma(-z) = -\frac{\pi}{\sin\pi z} z$.

For the Coulomb potential $V(r) = -\frac{a}{r}$ inserting the quantity $\lambda = \frac{4}{\lambda'} \left( \frac{E'}{E'} \right)^{\frac{1}{2}}$ (where $E' = |E|$) and the new variable $\xi = \frac{4a\mu}{\hbar^2} \frac{1}{\nu + \frac{1}{2} + \sqrt{(q-2)^2/4 + l(l+1)}} r$ into Schrödinger’s equation we get:

$$\frac{d^2 R}{d\xi^2} + \frac{(q-1)}{\xi} \frac{dR}{d\xi} + \left[ \frac{\lambda}{\xi} - \frac{1}{4} - \frac{l(l+1)}{\xi^2} \right] R = 0.$$

(66)

Again the energy spectrum is specified by the asymptotic behaviour of the power series $L(\xi)$ and is:

$$E_{l,\nu,q} = -\frac{\alpha^2 \mu}{\hbar^2} \frac{1}{\nu + \frac{1}{2} + \sqrt{(q-2)^2/4 + l(l+1)}}$$

(67)

The solutions satisfying the same boundary conditions as in the case of the harmonic potential are:

$$R_{\nu,l,q}(\xi) = e^{-\frac{1}{2\xi} \xi^{s^+} L_{\nu}^{2s^+ + q-2}(\xi)}$$

(68)

where $L_{\nu}^{2s^+ + q-2}(\xi)$ are Laguerre’s polynomials defined by the series:

$$L_{\nu}^{2s^+ + q-2}(\xi) = \sum_{k=0}^{\nu} \frac{\Gamma(\nu + 2s^+ + q - 1)}{\Gamma(k + 2s^+ + q - 1) k!(\nu - k)!} (-\xi)^k$$

(69)

with $2s^+ + q - 2 = \nu + \frac{1}{2} + \sqrt{(q-2)^2/4 + l(l+1)} \in \mathbb{R}$. We can easily check from (69) that in three dimensions we obtain the correct results.

Conclusions. –

We have presented a simple way to derive the Eguchi-Hanson metric from the $N=2$ action of nonlinear $\sigma$-models. The Hirzebruch signature of the manifold has been calculated by using the Kähler structure that accompanies it. A simple way has been proposed to construct the orbifold $M = C^2/Z_2$ from the EH metric as well as the study of its topological invariant features have been explored. The classical and the quantum behaviour of a massive particle on the singular manifold has been investigated on one hand by studying the classical wave equation and on the other hand by Schrödinger’s equation and switching on some potentials.

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Appendix A. –

The left-invariant one forms $\sigma_i$ on the manifold of the group $SU(2)_L = S^3$, have the explicit expression:

$$\sigma_x = \frac{1}{2}(\sin\psi d\theta - \cos\psi \sin\theta d\phi)$$

$$\sigma_y = \frac{1}{2}(\sin\psi \sin\theta d\phi - \cos\psi d\theta)$$
\[ \sigma_z = \frac{1}{2} (d\psi + \cos \theta d\phi). \] 

(A.1)

and obey the relation:

\[ d\sigma_i = \epsilon_{ijk} \sigma_j \wedge \sigma_k. \] 

(A.2)

In an analogous way one can define the dual one-forms \( \tilde{\sigma} \) invariant under the \( SU(2)_R \) supersymmetry and in terms of Euler angles they read:

\[ \tilde{\sigma}_x = \frac{1}{2} (\sin \phi d\theta - \cos \phi \sin \theta d\psi) \]

\[ \tilde{\sigma}_y = \frac{1}{2} (\sin \phi \sin \theta d\psi - \cos \phi d\theta) \]

\[ \tilde{\sigma}_z = \frac{1}{2} (d\phi + \cos \theta d\psi). \] 

(A.3)

The generators of \( SU(2) \) are expressed in terms of the killing vectors \( \xi^{(j)} \) of the metric by the formula \( L_j = -i \xi^{(j)} \partial \). In spherical coordinates they read:

\[ L_1 = -i \left( \sin \psi \frac{\partial}{\partial \theta} - \frac{\cos \psi}{\sin \theta} \frac{\partial}{\partial \phi} + \frac{\cos \theta \cos \psi}{\sin \theta} \frac{\partial}{\partial \psi} \right) \]

\[ L_2 = i \left( \cos \psi \frac{\partial}{\partial \theta} + \frac{\sin \psi}{\sin \theta} \frac{\partial}{\partial \phi} - \frac{\cos \theta \sin \psi}{\sin \theta} \frac{\partial}{\partial \psi} \right) \]

\[ L_3 = -i \frac{\partial}{\partial \psi} \] 

(A.4)

where \( L_3 \) is associated with a diagonal \( U(1) \). Similar expressions hold for the dual generators provided we interchange the roles of \( \phi \) and \( \psi \).

Appendix B.

Let \( M \) be a complex manifold with local coordinates \((Z^1, Z^2, \ldots, Z^n)\). A Hermitian metric on \( M \) is given by an expression of the form:

\[ g_{ab}dz^a \otimes d\bar{Z}^b. \] 

(B.1)

That \( g_{ab} \) is Hermitian means:

\[ g_{ab} = g_{ba}. \] 

(B.2)

We also require:

\[ g_{ab} = g_{ba}, g_{ab} = 0 = g_{\bar{a}b}. \] 

(B.3)

The affine connection and the Riemann tensor greatly simplify on a Kähler manifold as one can realise. The Kähler 2-form \( K = g_{ab}dz^a \wedge d\bar{z}^b \) is closed \( (dK = 0) \) and this implies that:

\[ \partial_z g_{ab} = \partial_a g_{zb}, \partial_z g_{ab} = \partial_b g_{az}. \] 

(B.4)
The affine connection is given as usual by $\Gamma_{ij,k} = (\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})/2$ and taking into account (B.1) one finds that the only nonzero components are:

$$
\Gamma_{ab,\bar{c}} = \partial_b g_{a\bar{c}}, \quad \Gamma_{\bar{a}b,c} = \partial_b g_{a\bar{c}}.
$$

For the covariant Riemann curvature the definition:

$$
R_{ijkl} = \frac{1}{2} \left( \frac{\partial^2 g_{ik}}{\partial \xi^j \partial \xi^l} - \frac{\partial^2 g_{il}}{\partial \xi^j \partial \xi^k} - \frac{\partial^2 g_{jk}}{\partial \xi^i \partial \xi^l} + \frac{\partial^2 g_{jl}}{\partial \xi^i \partial \xi^k} \right) + g_{np}(\Gamma^n_{ik} \Gamma^p_{jl} - \Gamma^n_{il} \Gamma^p_{jk})
$$

becomes:

$$
R_{abcd} = -\frac{1}{2}(\partial_b \Gamma_{ac,d} + \partial_d \Gamma_{ac,b}) - g^{\bar{e}p}(\Gamma_{ac,r} \Gamma_{bd,p} - \Gamma_{ad,r} \Gamma_{bc,p}).
$$

The above expression provides the nonzero components which turn out to be:

$$
R_{abcd}, R_{ab\bar{c}d}, R_{a\bar{b}c\bar{d}}, R_{\bar{a}b\bar{c}d}.
$$

The first Bianchi identity:

$$
R_{ijkl} + R_{kijl} + R_{jkil} = 0
$$

simplifies to the statement that:

$$
R_{abcd,\bar{e}} = R_{\bar{c}dab}
$$

while the second Bianchi identity becomes:

$$
R_{abcd,f} + R_{b\bar{f}cd,a} + R_{facd,\bar{b}} = 0.
$$

The nonzero components of the affine connection for the metric (8) are:

$$
\begin{align*}
\Gamma_{11,1} &= \frac{\partial_1 g_{11}}{4r^6(1 + \frac{c^2}{4r^2})^{3/2}} = \left( \frac{2}{|Z_1|^6 - 6 |Z_1|^2 |Z_2|^2 + |Z_1|^2 c^2 - 4 |Z_2|^2} \right) Z_1 c^2 \\
\Gamma_{12,1} &= \frac{\partial_1 g_{12}}{4r^6(1 + \frac{c^2}{4r^2})^{3/2}} = \left( \frac{6 |Z_1|^6 + 12 |Z_1|^2 |Z_2|^2 + |Z_1|^2 c^2 - 4 |Z_2|^2 - 4 |Z_1|^2} \right) Z_2 c^2 \\
\Gamma_{22,2} &= \frac{\partial_2 g_{22}}{4r^6(1 + \frac{c^2}{4r^2})^{3/2}} = \left( \frac{2 |Z_2|^6 - 6 |Z_1|^2 |Z_2|^2 - 4 |Z_1|^4} \right) Z_2 c^2 \\
\Gamma_{21,2} &= \frac{\partial_2 g_{21}}{4r^6(1 + \frac{c^2}{4r^2})^{3/2}} = \left( \frac{6 |Z_1|^2 + 12 |Z_1|^2 |Z_2|^2 + |Z_1|^2 c^2} \right) Z_2 c^2 \\
\Gamma_{11,2} &= \frac{\partial_1 g_{11}}{4r^6(1 + \frac{c^2}{4r^2})^{3/2}} = \left( \frac{6 |Z_1|^4 + 12 |Z_1|^2 |Z_2|^2 + 6 |Z_2|^4 + c^2} \right) Z_2 c^2 \\
\Gamma_{22,1} &= \frac{\partial_2 g_{22}}{4r^6(1 + \frac{c^2}{4r^2})^{3/2}} = \left( \frac{6 |Z_1|^4 + 12 |Z_1|^2 |Z_2|^2 + 6 |Z_2|^4 + c^2} \right) Z_2 c^2.
\end{align*}
$$
The Weyl tensor in our case equals the Riemann curvature and the invariant we consider is the quadratic expression:

\[ W = R_{ijkl} \bar{R}_{ijkl}. \]  

(B.18)

The nonzero components of the Weyl tensor are (excluding components connected through obvious symmetries):

\[ R_{1111}, R_{2222}, R_{1112}, R_{1121}, R_{2212}, R_{1122}, R_{1212}, R_{1221}, R_{2121} \]  

(B.19)

and the quadratic invariant thus takes the form:

\[
W = 4R_{1111} \bar{R}_{1111} + 4R_{2222} \bar{R}_{2222} + 8R_{1112} \bar{R}_{1112} + 8R_{1121} \bar{R}_{1121} + 8R_{1212} \bar{R}_{1212} + 8R_{1221} \bar{R}_{1221} + 8R_{2121} \bar{R}_{2121}.
\]  

(B.20)

For the metric (8) the result reads:

\[
W = \frac{384c^4}{(4c^2 + c^2)^3}
\]  

(B.21)

which vanishes when \( c = 0 \). Integration over the whole \( SO(3) \) sphere gives:

\[
\frac{1}{24\pi^2} \int Tr(R \wedge R) = 1.
\]  

(B.22)

The inverse transformations read:

\[
\begin{align*}
 r &= |Z^1|^2 + |Z^2|^2, \quad \cos \frac{\theta}{2} = \frac{|Z^1|}{\sqrt{|Z^1|^2 + |Z^2|^2}}, \quad \sin \frac{\theta}{2} = \frac{|Z^2|}{\sqrt{|Z^1|^2 + |Z^2|^2}}, \\
 \psi &= -i \ln \left( \frac{Z^1 Z^2}{|Z^1||Z^2|} \right), \quad \phi = i \ln \left( \frac{Z^2 |Z^1|}{|Z^1||Z^2|} \right)
\end{align*}
\]  

(B.23)

bearing in mind that \( 0 \leq \theta \leq \pi, \ 0 \leq \phi \leq 2\pi, \ 0 \leq \psi \leq 2\pi \).

**Appendix C.**

The other two solutions of Bessel’s equation are:

1. \((s^+ - s^-) \in Z^+\):

\[
J(\rho) = AJ_s^+(\rho) + B[J_s^-(\rho) + CJ_s^+(\rho) \ln |\rho|] \tag{C.1}
\]

2. \(s^+ - s^- = 0\):

\[
J(\rho) = AJ_s^+(\rho) + B[\rho^{s^+ + 1}\sum_{j=0}^{\infty} a_j \rho^{2j} + J_s^+(\rho) \ln |\rho|]. \tag{C.2}
\]
The roots of the Bessel’s function are given by the asymptotic formula of \( J_{s}(\rho) \):

\[
\rho_{\lambda,q,n}^{\pm} = n\pi + \frac{\pi}{2} \left[ \frac{1}{2} \pm \sqrt{\lambda^2 + \left(\frac{q-2}{2}\right)^2} \right].
\]

The normalization of Bessel’s function is:

\[
\int_{0}^{\alpha} \rho J_{s}(\rho) J_{s}(x_{s+}, q, n') d\rho = \frac{\alpha^2}{2} J_{s+1}^{2}(x_{s+}, q, n\delta_{n,n'}). \tag{C.3}
\]

**Appendix D.**

The generating function of the Jacobi polynomials is:

\[
\sum_{k=0}^{\infty} T_{k}^{(a,b)}(u) w^{k} = 2^{s+b}(1 - 2uw + w^2)^{-\frac{1}{2}}[1 - w + (1 - 2uw + w^2)^{\frac{1}{2}}]^{-a} -\frac{1}{w}[1 + w + (1 - 2uw + w^2)^{\frac{1}{2}}]^{-b}
\]

where the expressions \([\cdot \cdot \cdot]^{-a}\) and \([\cdot \cdot \cdot]^{-b}\) must be taken positive for \( w = 0 \).

The recurrence relations are:

\[
u T_{k}^{(a,b)}(u) = \frac{2(k+a)(k+b)}{(2k+a+b)(2k+a+b+1)} T_{k-1}^{(a,b)}(u) - \frac{(a^2 - b^2)}{(2k+a+b)(2k+a+b+2)} T_{k}^{(a,b)}(u) + \frac{2(k+1)(k+a+b+1)}{(2k+a+b+1)(2k+a+b+2)} T_{k+1}^{(a,b)}(u)
\]

and:

\[
(1 - u^2) \frac{dT_{k}^{(a,b)}(u)}{du} = \frac{2(k+a)(k+b)(k+a+b+1)}{(2k+a+b)(2k+a+b+1)} T_{k-1}^{(a,b)}(u) + \frac{2k(a-b)(k+a+b+1)}{(2k+a+b)(2k+a+b+2)} T_{k}^{(a,b)}(u) - \frac{2k(k+1)(k+a+b+1)}{(2k+a+b+1)(2k+a+b+2)} T_{k+1}^{(a,b)}(u)
\]

where \( k = 2, 3, \cdots, T_{0}^{(a,b)}(u) = 1 \) and \( T_{1}^{(a,b)}(u) = \frac{1}{2}(a + b + 2)u + \frac{1}{2}(a - b) \).

The equivalence between the spin-s spherical harmonics and Jacobi polynomials goes as follows:

\[
P_{m,n}^{l}(u) = N_{m,n}^{l} \left( \frac{1-u}{2} \right)^{\frac{1}{2}} \left( \frac{1+u}{2} \right)^{\frac{1}{2}} T_{l-m}^{(a,b)}(u)
\]

\[
= N_{m,n}^{l} \left( \frac{1-u}{2} \right)^{\frac{1}{2}} \left( \frac{1+u}{2} \right)^{\frac{1}{2}} (-1)^{k} T_{l+m-1}^{(a,b)}(-u)
\]
Thus

\[\frac{(-1)^{l-m}}{2^l}(1-u)^\frac{1}{2}\frac{1}{(1-u)}(1-u)^{l-m}\]

\[\times \left(\frac{1+u}{1-u}\right) \left(1-u\right)^{l-m}\]

\[= N_{m,n}^l \left(\frac{1-u}{2}\right)^l\]

\[\times \sum_{\nu=0}^{l-m} \binom{l+n}{\nu+m+n} \binom{l-n}{\nu} (-1)^{l-m+\nu} \left(\frac{1+u}{1-u}\right)^{\nu-(m+n)}\]

\[m,n \geq 0.\]

Setting in (D.4) \(m = -m\) we obtain the expression:

\[P_{-m,m}(u) = N_{-m,m}^l \left\{\frac{1-u}{2}\right\}^l\]

\[\times \sum_{\nu=0}^{l+m} \binom{l+n}{\nu-m+n} \binom{l-n}{\nu} (-1)^{l+m+\nu} \left(\frac{1+u}{1-u}\right)^{\nu-(m+n)}\]

\[N_{-m,m}^l = N_{m,n}^l (\sin \frac{\theta}{2})^{2l}\]

\[\times \sum_{\nu=0}^{l+m} \binom{l+n}{\nu-m+n} \binom{l-n}{\nu} (-1)^{l-m-\nu} (\cot \frac{\theta}{2})^{2\nu-m+n}\]

\[m,n \geq 0.\]

which is the one found in [12]. The normalization factor \(N_{m,n}^l\) can be constructed by exploring initially the case when \(a = b = |m|\). Then one finds, making use of the orthogonality of the associated Legendre functions \(P_{l,m}(u)|^2\) \(du = \frac{2}{2l+1} \binom{l+m}{|m|}^2\), that:

\[\frac{(l!)^2}{[(l+m)!]^2} \int_0^{2\pi} \frac{d\phi}{\int_{-1}^{1} du P_{m,n}^l(u)P_{m,n}^l(u) = \frac{4\pi}{2l+1} \frac{(l!)^2}{(l+m)!/(l-m)!}.}\]

Thus \(N_{m,n=0}^l = \left[\frac{2l+1}{4\pi} \frac{(l-m)!/(l+m)!}{(l!)^2}\right]^{\frac{1}{2}}\).

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