Information theoretic axioms for Quantum Theory

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In this paper we derive the complex Hilbert space formalism of quantum theory from four simple information theoretic axioms. It is shown that quantum theory is the only non classical probabilistic theory satisfying the following axioms: distinguishability, conservation, reversibility, composition. The new results of this reconstruction compared to other reconstructions by other authors are: (i) we get rid of axiom “subspace” in favor of axiom conservation eliminating mathematical requirements contained in previous axiomatics; (ii) we are able to classify all the probabilistic theories that are consistent requiring (a) only the first two axioms (b) only the first three axioms; this could be useful in experimental tests of quantum theory since it gives the possibility to understand whether or not other mathematical models could be consistent with such tests; (iii) we provide a connection between two different approaches to quantum foundations, quantum logic and the one based on information theoretic primitives showing that any theory satisfying the first two axioms given above either is classical or is a theory in which physical systems are described by a projective geometry.

One of the most curious facts about quantum theory is that it explains almost every physical phenomenon except from gravity and it is still not clear why it works so well. The first attempts to give rigorous foundations for the rules of quantum theory initiated a subject called quantum logic [1-8]. The starting point in quantum logic is that propositions related to measurements performable on a quantum system can be associated to sentences of a propositional calculus. When the system is classical the propositions related to classical measurements form a Boolean algebra and Boolean algebras are the algebraic models of the calculus of classical logic. The main question that quantum logic addresses is: when a Boolean algebra is relaxed into an orthomodular nondistributive lattice (i.e. a generalization of the lattice of subspaces of a hilbert space), which logic is it the model of? “Quantum Logic” is the name that designates the answer but there are several views about the content of this name and its physical significance [1]. Other notable and conceptually simpler ways to look for a physical explanation of the rules of quantum theory is to consider hidden variables models [10, 12]. The starting point to formulate these models is that the state of a quantum system describes an ontic property of the system and the randomness in quantum experiments is simply a consequence of mediating the result on many repetitions of the experiment. These models are appealing because they give simple explanations to questions regarding the ontology of the state of a quantum system or the nature of the measurement process. In such models a physical system in fact always possesses a definite value of a physical quantity and a measurement is just a read of this value. One of the main drawback to consider this as the physical explanation of quantum phenomena is that hidden variables in these models cannot be Lorentz invariant and, at the same time, be used in a model equivalent to quantum theory from the predictive point of view [13]. The advent of quantum information theory opened a new direction in the research of foundations of quantum theory. Quantum information showed that the controversial physical phenomena arising from the mathematics of quantum theory, can be exploited for information theoretic tasks that are impossible in a world governed by classical physics [14, 15]. It is then natural to ask whether it is possible to put as foundations of the mathematical rules of quantum theory a set of information theoretic principles. The first work that partially answered to this question in a positive way was [20]. Almost ten years after the appearance of that paper, finding informational principles for quantum theory is becoming a quite active area of research [15, 21, 24, 26]. Based on the reconstruction of [20], in [23] it is given an argument to eliminate the use of one of the axioms in the first reconstruction, Simplicity. In [22], the argument developed in [23] is used to derive a new set of requirements that, if imposed to a probabilistic theory, are equivalent to the mathematical formalism of quantum theory. In [23] it is considered purification as a foundational principle at the base of all the new information theoretic features of quantum information. It is shown that all theories satisfying purification (namely all states of a system A are in one to one correspondence with pure states defined on a larger composite system AB) and local discriminability (that is shown to be equivalent to local tomography) are very similar to quantum theory from an informational point of view. Starting from this work the same authors gave in [21] a reconstruction of quantum theory from six informational axioms, two of which are purification and local discriminability. The author of [20], ten years after his first reconstruction, invented a formulation of quantum theory based on a new mathematical object called Duotensor and gave a new set of operational/informational principles that are proved to be equivalent to this formulation of quantum theory [20].

In this paper we give a new reconstruction of quantum theory based on a new set of informational principles. The main result is the following: the only non classical probabilistic theory satisfying a list of four axioms - Dis-
I. PROBABILISTIC THEORIES

A generic probabilistic theory is a mathematical framework in which it is possible to model any experimental set up and to calculate probabilities for all the possible configurations of a set up. In such a framework preparation, transformation and measurement devices are represented by collections of outcomes, e.g.:

$$\rho \in \{\rho_i\}_{i \in X} \quad M = \{a_j\}_{j \in Y} \quad T = \{\mathcal{T}_k\}_{k \in Z}$$

where \(\rho, M, T\) are respectively a preparation, transformation and measurement while \(\phi_i, \mathcal{T}_k, a_j\) represent the corresponding outcomes in some outcome sets \(X, Z, Y\).

An outcome set of a physical device, in general, is not something that have a well defined probability distribution on its own. The probability distribution of the outcome set of a measurement device in an experimental setup, depends on the settings of the preparation device. This is clear since if we perform a measurement on a system prepared in some way we have a given probability distribution of the measurement outcomes, while, if we perform the same measurement on a system prepared in some different way we obtain, in general, a different probability distribution.

Given a preparation and a measurement, \(\rho = \{\phi_i\}_{i \in X}, M = \{a_j\}_{j \in Y}\) we define their composition as:

$$\circ: (a_j, \rho_i) \rightarrow a_j \circ \rho_i \quad \forall (i,j) \in X \times Y$$

(2)

Since to a preparation followed by a measurement outcome is always associated the probability of the measurement outcome given that preparation we can associate to \(a_j \circ \rho_i\) the probability of seeing measurement outcome \(a_j\) when performing measurement \(\{a_j\}_{j \in Y}\) on a system prepared in state \(\rho_i\), namely:

$$p: a_j \circ \rho_i \rightarrow p(a_j | \rho_i)$$

(3)

To use a notation that resembles the bra-ket notation of standard quantum theory we will also define:

$$\langle a_j | \rho_i \rangle := p(a_j | \rho_i)$$

(4)

Composing the two maps \(\circ\) and \(p\) we can define a new map \(M\) turning any pair \((a_i, \rho_j)\) into a probability:

$$M: (a_j, \rho_i) \rightarrow (a_j | \rho_i)$$

(5)

The probabilistic structure defined by (3) turns every \(\rho_i\) into a function from measurement outcomes to probabilities, given by \(M[(i, \rho_i)]\). If \(\rho_i, \rho_i'\) induce the same function, then it is impossible to distinguish between them from the statistics of an experiment. This means that the two outcomes of the preparation device are equivalent: accordingly, we will take equivalence classes with respect to this equivalence relation. We will thus identify the outcomes \(\rho_i, \rho_i'\) with the corresponding function \(\rho_i\) and will call it state. Accordingly, we will refer to preparation devices as collections of states \(\{\rho_i\}_{i \in X}\). The
same construction holds for measurements: every measurement outcome \( a_j \) induces a function from preparations to probabilities, given by \( M[a_j] \). If two outcomes \( a_j, a'_j \) induce the same function, then it is impossible to distinguish between them from the statistics of an experiment. This means that the two outcomes are equivalent: accordingly, we will take equivalence classes with respect to this equivalence relation. We will thus identify the outcome \( a_j \) with the corresponding function and we will call it \textit{effect}. Accordingly, we will refer to measurement devices as collection of effects \( \{a_j\}_{j \in Y} \).

The state of a system provides the information regarding the probabilities of all the possible outcomes in all the possible measurements that can be performed on a system prepared in a given configuration. Thus the state of a system can be represented by a list of probabilities that, in principle, could contain an infinite number of entries. Hence for a state \( \rho \) of a system we can write:

\[
\rho = \begin{pmatrix}
\vdots \\
p_{a_j} \\
\vdots
\end{pmatrix}
\]  

(6)

where \( p_{a_j} \) is the probability of the effect labeled by \( a_j \) given the state \( \rho \). \( \rho \) is thus represented by the vector \( \{p_{a_j}\} \) and two different state vectors of a given system will differ at least for one of their entries. In principle the number of entries for a state expressed as in (6) is infinite since, at least for one of their entries. In principle the number of such entries is 4, i.e. the number of real parameters that are necessary in order to specify a hermitian matrix acting on \( \mathbb{C}^2 \). In \cite{2}, it is argued this kind of compression is a general feature of all physical theories and it is taken as a starting point to formulate a framework for quantum gravity. In this paper we will assume that the number of entries in the list of probabilities representing a state of a physical system as in (6) is finite.

From the above considerations we have that states can be represented as elements of a real finite dimensional vector space while effects can be represented in the vector space constituting the dual of the state space. As a consequence transformations of a system \( A \) can be represented as operators defined on the vector space in which are represented the states of \( A \).

**Definition 1** The dimension of the real vector space where states of a system are represented as vectors of probabilities is the dimension of the system.

For a quantum system seen as an object of a probabilistic theory the dimension of the hilbert space associated to a system does not coincide with the dimension of the system. The dimension of a quantum system in this framework is the dimension of the real vector space in which are defined density matrices representing states. This is the real vector space generated by hermitian operators acting on the hilbert space associated to the system. The dimension of such real vector space is \( d = n^2 \) where \( \mathbb{C}^n \) is the hilbert space associated to the system. This notion of dimension of a system is introduced in \cite{20}.

The probability distribution of the effects \( A = \{a_j\}_{j \in Y} \) in measurement \( A \) always depends on what state is prepared. It then must hold that the probability of happening of state \( \rho_i \) must be independent of the measurement that is performed. If it were not so then the probability distribution of the effects in a measurement would depend on the state of the physical system and, at the same time, would be something on which the state of the system depends. If this was the case then states and effects would be related in a non linear way and the function associating pairs of state and effect to a probability would be non linear. This would prevent the possibility to form mixture of states (see below) and of representing states in a real vector space. Since the probability of happening of state \( \rho_i \) must be independent of the measurement performed we must have:

\[
(\sum_{j \in Y} a_j |\rho_i)) = (\sum_{l \in S} b_l |\rho_i) \quad \forall \{a_j\}_{j \in Y}, \{b_l\}_{l \in S} \quad \quad (7)
\]

where \( \sum_{j \in Y} a_j, \sum_{l \in S} b_l \) represents respectively a singleton (i.e. a single outcome measurement) constituted by the coarse graining of all the outcomes in measurements \( A = \{a_j\}_{j \in Y} \) and \( B = \{b_l\}_{l \in S} \). Condition (7) is stated in \cite{23} and is associated to \textit{causality}. We now give the following:

**Definition 2** Given any measurement \( M = \{m_j\}_{j \in Y} \), the coarse graining of the effects in \( M, \cup_{j \in Y} m_j \), is called deterministic effect.

We now state the following characterization of (7):

**Proposition 1** \cite{7} is equivalent to require that the deterministic effect is unique for all measurements.

This proposition is proved in \cite{23}.

Given (7) we must have that the probability distribution of the states of a preparation device \( \rho = \{\rho_i\}_{i \in X} \) must be independent of the settings and outcomes on other devices in any experiment. This implies that to every preparation \( \rho = \{\rho_i\}_{i \in X} \) must be associated a probability distribution \( \{\rho_i\}_{i \in X} \) for the corresponding states. Hence every state \( \rho \) can be represented as a convex combination of other states, namely:

\[
\rho = \sum_{i \in X} p_i \rho_i 
\]

(8)

From (6) and the fact that states can be represented as in (7) we have that the state space of any physical system is a compact convex set. This leads us immediately to the following definition:
**Definition 3** A state is mixed if it can be represented as a convex combination of at least two other states. A state is pure if the convex combination representing it contains only one state.

Since we have that \( \mathcal{T}_p \rho = \sum_{i \in X} p_i \mathcal{T}_p \rho_i \) for every transformation \( \mathcal{T}_p \) and every state \( \rho \) we must have that transformations are linear operators. Note however that the convex decomposition of a state \( \rho \) is not required to be unique. There can be many ways to express the same state \( \rho \) as a convex combination of other states.

**Definition 4** The refinement of state \( \rho \) is defined to be the set of states that can appear in a convex decomposition representing \( \rho \).

We will denote the refinement set of a state \( \rho \) as \( F_\rho \). The notion of refinement of a state will play a crucial role in this derivation. It is introduced in the framework of probabilistic theories in [21]. From the mathematical point of view, the refinement of a state coincides with the notion of face of a convex set. A face \( F \) of a convex set, \( S \), is defined to be a convex subset of \( S \) such that given two points, \( x_1, x_2 \in S \) if \( \lambda x_1 + (1 - \lambda)x_2 \in F \) then \( x_1, x_2 \in F \). From definition 4, we see that the refinement of a state \( \rho \) is a face of the (convex) state space of the system of which \( \rho \) represents a preparation.

**Definition 5** A state \( \rho \) is completely mixed relatively to a set of states \( S \), if all the states in \( S \) can appear in the convex decomposition of \( \rho \).

A physical system can be used to store information. Storage of information into a physical system is possible if it can be defined a protocol that can be used to read that information. To define such a protocol we give the following:

**Definition 6** The set of states \( \{\rho_i\}_{i=1}^N \) is perfectly distinguishable if there exists a measurement \( A = \{a_j\}_{j=1}^N \) such that \( (a_j|\rho_i) = \delta_{ij} \). The set of effects \( \{a_j\}_{j=1}^N \) is a set of perfectly discriminating effects.

Given the above definition we see that if the set of states of a system contains at least two perfectly distinguishable states, the system can be used to store information. Referring to the above definition, if one prepares a state belonging to a set of perfectly distinguishable states, \( \{\rho_i\}_{i=1}^N \), say \( \rho_{00} \), then performing the measurement \( A = \{a_j\}_{j=1}^N \) one can say with certainty that the state prepared was \( \rho_{00} \) upon seeing the effect \( a_{00} \). This in turn defines a protocol to read the information that can be stored in the physical system. A list of \( L \) symbols chosen among the set \( \{i\}_{i=1}^N \) can be stored in \( L \) copies of the system with preparations chosen in the set of states \( \{\rho_i\}_{i=1}^N \).

In Quantum theory a system having states defined on a Hilbert space of dimension \( n \), \( \mathbb{C}^n \) has \( n \) perfectly distinguishable states.

We conclude this section with the following two definitions

**Definition 7** The set of perfectly distinguishable states \( \{\rho_i\}_{i=1}^N \in S \), where \( S \) is a generic set of states, is maximal in \( S \) if there does not exist a state \( \sigma \in S \) such that \( \{\rho_i\}_{i=1}^N \cup \sigma \) is a perfectly distinguishable set of states.

**Definition 8** The information capacity of a set of states is the cardinality of the largest maximal set of perfectly distinguishable states in that set.

II. AXIOMS FOR QUANTUM THEORY

In what follows we will impose a set of four axioms that are very natural for a generic probabilistic theory in which systems can be used to store information. It will be shown in the following sections that the only non classical theory satisfying all these axioms is quantum theory. The axioms are called Distinguishability, Conservation, Reversibility, Composition. The starting point to formulate them is that every physical system can be used to store information. To understand the meaning of this consider a system as simple as a die. On every face of a die there is a certain number of dots from one to six. Suppose that one has to store and retrieve a text written using an alphabet of six elements \( \{a, b, ..., f\} \). He can encode every letter of the alphabet into a number from one to six. The text is an ensemble of letters that have a certain probability of appearing \( \{p_a, p_b, ..., p_f\} \). This text can thus be stored into an ensemble of dice that are prepared according to the probability distribution \( \{p_a, p_b, ..., p_f\} \).

The preparation of the ensemble for storage of the text can be performed leaning each die in the ensemble on a surface in such a way that the face corresponding to the stored letter is hided. One can retrieve the text simply looking at which is the hided face for every die. The above protocol to store a text into an ensemble of systems can be accomplished with a quantum system with hilbert space dimension six as well. One can choose a basis for the system, \( \{|i\}_{i=1}^6 \) and encode every letter of the alphabet into one of these states. Storage of the text into an ensemble of physical systems can be performed preparing each quantum system in the ensemble in one of these states. The text can thus be retrieved measuring each element of the ensemble of quantum systems in the above basis.

We are now going to state and explain the axioms.

**Axiom 1 - Distinguishability** Every state that is not completely mixed relatively to a given refinement set is perfectly distinguishable from another state in that set.

This axiom is a stronger version of axiom distinguishability used in [21]. Recall that the notion of refinement set of a state coincides with the notion of face of a convex set. This axiom tells that if \( \rho \) is not completely mixed in a given set of states of a system \( F \) constituting a face, then there exist another state \( \phi \) in that face and a measurement \( A = \{a_\rho, a_\phi\} \) that can be performed on system
states if regarded as states of a system with information capacity \( n \) is a simplex in \( \mathbb{R}^{n-1} \). The refinement of a mixed state \( \rho \) that is not completely mixed, constitutes a simplex in \( \mathbb{R}^m \) with \( m < n - 1 \). Any pure state \( \phi \) not belonging to \( F_\rho \) is represented by a vector in \( \mathbb{R}^{n-1} \) orthogonal to the subspace in which is embedded \( F_\rho \). Hence we have that \( \phi \) is perfectly distinguishable from all states in \( F_\rho \) hence also from \( \rho \) itself. The axiom holds also in quantum theory. The refinement set of a mixed state \( \rho \) of a system with information capacity \( n \), is the convex hull of a \( C^D \) where \( F_\rho \) has information capacity \( m \leq n \). Any pure state \( \phi \) not belonging to \( F_\rho \) is represented by a complex vector orthogonal to the subspace representing \( F_\rho \). Hence \( \phi \) is perfectly distinguishable from all the pure states in \( F_\rho \) and thus from \( \rho \) itself.

**Axiom 2 - Conservation** The information capacity of the refinement of any mixture of two states is less than or equal to the sum of the information capacities of the refinements of the two states composing the mixture, with equality if the states are perfectly distinguishable.

Let \( \eta = \rho \sigma + (1 - \sigma)\rho \). The axiom tells that there cannot exist sets of states that are not perfectly distinguishable in \( F_\sigma \) and sets of states that are not perfectly distinguishable in \( F_\rho \) that become sets of perfectly distinguishable states if regarded as states of \( F_\eta \). Moreover if \( \rho \) and \( \sigma \) are perfectly distinguishable states, then the information capacity of \( F_\eta \) is the sum of the information capacities of \( F_\rho \) and \( F_\sigma \). Hence the number of pure states usable to store information in \( F_\rho \) is the sum of the number of pure states usable to store information in \( F_\rho \) and the number of pure states usable to store information in \( F_\sigma \). To have an example consider two mixtures of different faces of a die, say a mixture of "one" and "six" \( \{p_{\text{one}}, p_{\text{six}} = (1 - p_{\text{one}})\} \) and a mixture of "two" and "four" \( \{p_{\text{two}}, p_{\text{four}} = (1 - p_{\text{two}})\} \). The axiom states that if we prepare the following mixture \( \{q_{\text{one}}, q_{\text{six}}, q_{\text{two}}, q_{\text{four}} = (1 - q_{\text{two}})\} \) with \( q = 1 - t \), then the number of values that can be used to store information in this mixture is not greater than four that is the number of values used in preparing the above two mixtures, namely, "one," "six," "two," "four." This axiom clearly holds in classical theory. \( \sigma \) and \( \rho \) are such that \( F_\sigma \) and \( F_\rho \) are two simplexes with \( s \) and \( r \) vertices respectively and live in \( \mathbb{R}^{s-1} \) and \( \mathbb{R}^{r-1} \). The refinement set of \( \eta = \rho \sigma + (1 - \sigma)\rho \) has dimension \( s + r - t - 1 \) where we assumed the intersection of \( F_\rho \) and \( F_\sigma \) to be a simplex in \( \mathbb{R}^{t-1} \). \( F_\eta \) is a simplex in \( \mathbb{R}^{s+r-t-1} \) and has information capacity \( r + s - t \) where \( t \geq 0 \) with equality iff \( \rho \) and \( \sigma \) are perfectly distinguishable. The axiom holds in quantum theory as well. Take a convex combination of two density matrices \( \rho \) and \( \sigma \). \( F_\rho \) and \( F_\sigma \) are the convex hull of the space of rays in \( \mathbb{C}^r \) and \( \mathbb{C}^s \) respectively. \( F_\eta \) is the convex hull of the rays belonging to the smallest complex vector space containing both \( \mathbb{C}^r \) and \( \mathbb{C}^s \), i.e. \( \mathbb{C}^{r+s-t} \) where \( \mathbb{C}^d \) is the intersection of \( \mathbb{C}^r \) and \( \mathbb{C}^s \). The information capacity of \( F_\eta \) is thus \( r + s - t \) with \( t \geq 0 \) with equality iff \( \rho \) and \( \sigma \) perfectly distinguishable.

**Axiom 3 - Reversibility** For any two pure states of a system, \( \phi, \psi \), there exists a reversible transformation \( \mathcal{D} \) such that \( \phi = \mathcal{D} \psi \).

The significance of the above axiom is simply that one can transform any pure state into any other with a reversible transformation. This axiom clearly holds classically since one can transform any pure state of a simplex in \( \mathbb{R}^{n-1} \) into any other applying a transformation in the symmetric group \( S_n \) (i.e. the group whose elements are the permutations of \( n \) objects with composition rule being sequential composition of permutations). This axiom also holds in quantum theory since the set of pure states of a system with hilbert space \( \mathbb{C}^n \) is such that any pure state can be transformed into any other with a transformation belonging to the group \( SU(n) \).

**Axiom 4 - Composition** If \( d_A \) and \( c_A \) are the dimension and the information capacity of system \( A \) then

\[
d_{ABC..} = d_A d_B d_C \cdots
\]

and

\[
c_{ABC..} = c_A c_B c_C \cdots
\]

where \( ABC.. \) is the system composed of systems \( A, B, C.. \).

The first part of the above axiom is called local tomography. Every state of a system in a probabilistic theory can be represented as a list of a certain number of probabilities obtained performing an equal number of different measurements on the system in that state (see [3]). The number of measurements that are sufficient in order to uniquely determine a state as a vector of probabilities is the number of degrees of freedom of the system that is equal to its dimension by definition [2] (see also [4]). The first part of axiom composition requires that the number of degrees of freedom of a composite system be the product of the number of degrees of freedom of the components. An important consequence of this axiom is that the real vector space in which it can be represented
the state space of a composite system $S_{AB}$ is the vector space tensor product of the vector spaces in which are represented the state spaces of the component systems, $S_A$, $S_B$. The second part states simply that for a system composed of a certain number of component systems, the information capacity of the composite system is the product of the information capacities of the components. To see that the above axiom holds classically note that the state space of a composite system is a simplex in the real vector space tensor product of the vector spaces where live the components. Hence the first part of the axiom holds. Since the information capacity of a classical system is equal to the number of degrees of freedom of the system we have that also the second part holds classically. The axiom holds in quantum theory as well. Indeed a density matrix of a composite system holds classically. The axiom holds in quantum theory as part of the axiom holds. Since the information capacity of a system composed of a certain number of component systems, the state space of a composite system $S_{AB}$ is the vector space tensor product of the vector spaces in which are represented the state spaces of the component systems $S_A$ and $S_B$, respectively. Hence the first part of the axiom holds. The second part holds as well since the information capacity of the state space of a quantum system coincides with the dimension of the complex vector space on which are defined density matrices describing states of the systems. $C^{nA}$ and $C^{nB}$ are Hilbert spaces of component systems with information capacity $n_A$ and $n_B$ respectively, and $C^{nA} \otimes C^{nB}$ is the Hilbert space for the composite system with information capacity $n_{A+B}$.

**III. PROOF OF THE MAIN RESULT**

### A. Many quantum theories

In this section we show that a probabilistic theory satisfying axioms 1,2 either is classical, or is a probabilistic theory in which pure states are points of a projective space. The fact that classical probability theory satisfies axioms 1,2 is already proved in the previous section. We thus are going to prove the remaining alternative, namely, pure states of a system with information capacity $n+1$ of a probabilistic theory satisfying axioms 1,2 are points in a projective space of dimension $n$. From this fact it will follow that this class of theories is such that pure states are elements in a vector space defined over a generic field of numbers (more precisely, a generic field of numbers is called division ring). This implies that for theories in this class it holds the superposition principle. Pure states of such theories can be represented by elements of a vector space in which all states in a subspace $A$ are perfectly distinguishable from all states in a subspace $B$ disjoint from $A$ and linear combinations of elements in disjoint subspaces represent allowed pure states.

A projective space of generic dimension $n$ is defined in the following:

**Definition 9** Projective space of dimension $n$

An arbitrary set $S$, together with a family of subsets, $F_j$, that are called subspaces of dimension $j$, is a projective space of dimension $n$ if the following holds:

(i) The only $-1$-dimensional subspace is the empty set.

(ii) 0-dimensional subspaces of $F^n$ are the point subsets of $S$.

(iii) There is a unique subspace of dimension $n$, $F^n$.

(iv) If $F^r$ and $F^s$ are two subspaces of $F^n$, and $F^r$ is contained in $F^s$ then $F^r \subseteq F^s$ if and only if $r = s$.

(v) Given two subspaces $F^r$ and $F^s$ of $F^n$, if $F^r$ is the intersection of $F^s$ and $F^t$ then $F^s$ is a subspace of $F^n$.

(vi) Given two subspaces $F^r$ and $F^s$ of $F^n$, if $F^{n+1} = F^r \cup F^s$ and $F^t$ is the intersection of $F^r$ and $F^s$, then:

$s + r = t + m$.

In the following we will show that for pure states and for a family of subsets of these states of a system with information capacity $n+1$ described by a probabilistic theory satisfying axioms 1,2, definition 9 holds. To prove that definition 9 holds for pure states of a system of a probabilistic theory we have to define the notion of subspace in this context. In what follows we will denote $S$ the set of states of a generic system.

**Definition 10** Given the set of states of a generic system $S$ with information capacity $n+1$ and any state $\rho \in S$ such that the cardinality of the largest maximal set of perfectly distinguishable states in $F^r$, is $j+1 \leq n+1$, the set $F^r_{\rho}$ is called subspace of $S$ with dimension $j$ and denoted $F^r_{\rho}$.

**Definition 11** The empty set $\emptyset$ is defined to be a subspace of the set of states of a system $S$ with dimension -1 and is denoted $F^{-1}$.

**Definition 12** Pure states in $S$ are defined to be subspaces of dimension 0 and denoted $F^0$. In the following we will suppose that $S$ is the set of states of a generic system with information capacity $n+1$ of a generic probabilistic theory satisfying axioms 1,2. Lemmas 1-4 and theorem 5 are needed to prove theorem 6 containing the most important part of this section.

**Lemma 1** The only subspace of dimension $n$ in $S$ is $S$ itself.

**Proof:** Let $F^0_\theta$ be a subspace of dimension $n$ in $S$ generated by the refinement of a mixed state $\theta$. We now prove the thesis showing that $\theta$ is completely mixed in $S$. To see this suppose it is not so. Then $\theta$ is not completely mixed. From axiom distinguishability $\theta$ is perfectly distinguishable from some state $\phi$. The state $p\theta + (1-p)\phi$ is in $S$ and its refinement has information capacity greater than or equal to $n+2$ from axiom conservation. This contradicts the hypothesis that $S$ has information capacity $n+1$ and proves the thesis. ■

**Lemma 2** If $F^r_\rho$ and $F^s_\sigma$ are both subspaces of $S$ and $F^r_\rho \subseteq F^s_\sigma$ then $r \leq s$. $r = s$ if and only if $F^r_\rho = F^s_\sigma$.
Proof: If $F_\rho \subseteq F_\sigma$ then the information capacity of $F_\rho$ cannot exceed that of $F_\sigma$. If $F_\rho$ and $F_\sigma$ are the same subspace then clearly $r = s$. To prove the converse note that by hypothesis we have that for all $\phi \in F_\rho^r$ we must have $\phi \in F_\sigma^r$. Now suppose that $r = s$ and $F_\rho^r \subseteq F_\sigma^r$. This means that $\rho$ is not completely in $F_\sigma$. From axiom distinguishability $\rho$ is perfectly distinguishable from some state $\phi$ in $F_\sigma^r$. The refinement of the state $\omega = p\rho + (1-p)\phi$ must have information capacity equal to $s + 2$ by hypothesis and from axiom conservation. But this is absurd since $\omega \in F_\sigma^r$ and $F_\sigma^r$ must have information capacity $s + 1$ by definition. Hence if $F_\rho^r \subseteq F_\sigma^r$ and $r = s$ then we must have $F_\rho^r = F_\sigma^r$. ||

Lemma 3 The refinement of a mixture of any two pure states of a system has information capacity two.

Proof: From axiom conservation the refinement of a convex combination of any two pure states of a system cannot have information capacity greater than two. Since any of the two pure states is not completely mixed by definition, from axiom distinguishability we have the thesis. ■

Definition 13 Given two subspaces of a system $S$, $F_\rho^r$ and $F_\sigma^s$, we denote their intersection as $F_\rho^r \wedge F_\sigma^s$.

Lemma 4 Given two subspaces of $S$, $F_\rho^r$ and $F_\sigma^s$, $F_\rho^r \wedge F_\sigma^s$ is a subspace of $S$.

Proof: The intersection of two convex sets is a convex set. Hence there exists a state $\tau$ that is completely mixed in the set $F_\tau = F_\rho^r \wedge F_\sigma^s$. The set of states of $S$ in $F_\tau$ constitutes a subspace of $S$ with information capacity greater than or equal to two from axiom distinguishability. ■

Definition 14 Given two subspaces, $F_\rho^r$ and $F_\sigma^s$ we define their span and denote it as $F_\rho^r \vee F_\sigma^s$ the set of states in the refinement of a convex combination of $\rho$ and $\sigma$.

Theorem 1 Given two distinct subspaces of $S$, $F_\rho^r$ and $F_\sigma^s$, if $F_{\eta}^m = F_\rho^r \vee F_\sigma^s$ and $F_\tau^r = F_\rho^r \wedge F_\sigma^s$ with $\eta = p\rho + (1-p)\sigma + (1-p)\rho$ and $\tau$ completely mixed state in $F_\rho^r \wedge F_\sigma^s$, then $r + s = t + m$.

Proof: In the case both $\rho$ and $\sigma$ are pure states the thesis holds from lemma and definition.

Suppose that only one of them, say $\sigma$, is mixed. Then either $\rho$ is in $F_\sigma$ and then $\tau = \rho$ or $\rho$ is not in $F_\sigma$ and $\tau$ is the empty set. In both these cases the thesis trivially holds.

Suppose now that both $\rho$ and $\sigma$ are mixed states. By hypothesis and axiom distinguishability $\tau$ is perfectly distinguishable from some state $\phi_1$ in $F_\rho^r$ that we choose w.l.o.g. pure. Let $\tau_1 = p\tau + (1-p)\phi_1$, $0 < p < 1$. From axiom conservation the information capacity of $F_{\tau_1}$ is $t + 2$. Thus we have constructed the subspace $F_{\tau_1}^{r+1}$. The state $\sigma_1 = p\rho + (1-p)\phi_1$, $0 < p < 1$, is such that $F_{\sigma_1}$ has information capacity less than or equal to $s + 1$ from axiom conservation. Moreover, from axiom distinguishability, there exists a pure state $\psi_1 \in F_{\sigma_1}$ that is perfectly distinguishable from $\sigma$ and such that the state $\sigma_1' = p\sigma + (1-p)\psi_1$ has refinement $F_{\sigma_1'}$ with information capacity equal to $s + 1$. Since by hypothesis $\sigma_1' \in F_{\sigma_1}$ we must have that the information capacity of $F_{\sigma_1}$ is $s + 1$. In this way we have constructed the subspace $F_{\sigma_1}^{r+1}$. Now we can have either $\tau_1$ completely mixed in $F_{\rho}^r$ or not. If the latter is the case then, in the same way we constructed $F_{\sigma_1}^{r+1}$ with $\sigma_1$ and $\phi_1$ we construct $F_{\eta}^{m+1}$ with $F_{\sigma_1}^{r+1}$ and a pure state $\phi_2$ in $F_{\rho}^r$. In the same way as before we can also construct $F_{\sigma_1'}^{r+1}$ with $F_{\sigma_1'}^{r+1}$ and $\phi_2$. Iterating this procedure we will end for some finite $k$ to a state $\tau_k = p\tau_{k-1} + (1-p)\phi_k$, $0 < p < 1$ that is completely mixed in $F_{\rho}^r$. This happens when there are no more states in $F_{\rho}^r$ outside $F_{\tau_k}^{r+1}$. $k$ is finite since, if it were not so, we would have infinite perfectly distinguishable states in $F_{\rho}^r$ contradicting the hypothesis ($r$ is finite since $n$ is). Iterating the above construction we also obtain the subspace $F_{\sigma_1+k}^{m+k}$ where $\sigma_k = p\sigma_{k-1} + (1-p)\phi_k$, $0 < p < 1$. By construction we have $F_{\sigma_k} \subset F_{\sigma_1}$. Moreover $F_{\sigma_1+k}$ contains every state in $F_{\rho}^r$ and there cannot exist states in $F_{\rho}^r$ outside $F_{\sigma_1+k}$ since this would contradict the facts that $F_{\tau_k}^{r+k} = F_{\rho}^r \subset F_{\sigma_1+k}^{m+k}$. But this in turn means that every state in $F_{\rho}^r$ must pertain to $F_{\sigma_1+k}$ and thus $\sigma_k$ is completely mixed in $F_{\rho}^r$. All this implies that $F_{\sigma_1+k} = F_{\rho}^r$ and $F_{\tau_k}^{r+k} = F_{\sigma_1}^{m+k}$ thus $k = r-t$ and $m = s+k$. Thus we find $m = s + r - t$ and prove the thesis. ■

Theorem 2 If the states of a system with information capacity $n + 1$ are described by a probabilistic theory satisfying axioms 1,2 then pure states of that system are points of a projective space of dimension $n$.

Proof: For a system $S$ with information capacity $n + 1$ satisfying axioms 1,2 there exists a family of subspaces $F_1^i$ of dimension $j$ such that:

(1)-(2) (i)-(ii) in definition hold by definitions 1, 12.

(3) (iii) in definition holds by lemma 1.

(4) (iv) in definition holds by lemma 1.

(5) (v) in definition holds by lemma 1.

(6) (vi) in definition holds by theorem 1.

Projective spaces can be very wild geometrical objects. All spaces in which points are one dimensional subspaces of a vector space defined over some field of numbers constitute projective spaces but not all projective spaces can be defined in this way. Examples of spaces in these class are the so called non-Desarguesian planes. Fortunately, the following theorem characterizes projective spaces of dimension greater than two, and, as a consequence, the set of pure states of systems described by theories obeying axioms 1,2 with information capacity greater than three.

Theorem 3 Veblen and Young theorem

If the dimension of a projective space is $n \geq 3$ then it is isomorphic to a $\mathbb{K}P^n$, i.e. a projective space of dimension $n$ over some division ring $\mathbb{K}$. 
The proof of this mathematical result is not in the scope of this paper. A projective space over a division ring $K$ is defined as a space whose points are one dimensional subspaces of a vector space defined over the division ring $K$. A division ring (also called skew field) is a mathematical generalization of the concept of field of numbers (e.g. reals or complex numbers), in which multiplication is not needed to be commutative. The most popular example of skew field in which multiplication is not commutative are the quaternions. In the rest of the paper we will refer to a skew filed of numbers simply as a “field” for brevity.

Theorem 3 characterizes the state space of systems with information capacity greater than or equal to four described by a generic probabilistic theory satisfying axioms 1,2. We have thus proved that in a theory satisfying axioms 1,2, systems with information capacity greater than three either are classical or are such that pure states in a superposition of these states are points of a $K^P^n$. These can be seen as elements of a vector space defined over the division ring $K$. In this landscape a set of pure perfectly distinguishable states $\{\phi_0, \phi_1, ..., \phi_n\}$ of a system with information capacity $n+1$ is a set of $n+1$ disjoint subspaces of the vector space in which are represented pure states of the system. If a system can be in one of the states $\{\phi_0, \phi_1, ..., \phi_n\}$ then it can also be found in a superposition of these states $\phi_s = \sum_{i=0}^n k_i \phi_i$ with $\{k_i\} \in K$ since this linear combination represents an element of the vector space and thus an allowed pure state. Hence in a probabilistic theory satisfying axioms 1,2, systems with information capacity greater than three are either classical (since classical theory satisfies them) or a generalization of quantum systems in which superposition principle holds with coefficients not necessarily complex but to a generic (skew) field of numbers $K$. In section III C it will be proved that this characterization also holds for system with information capacity three or two.

B. State space of an elementary system

An elementary system is a system with information capacity two. We will denote the set of states of an elementary system $S_2$. We now show that all theories satisfying axioms 1-3 are such that the set of pure states in the refinement of a state with information capacity two, constitutes a sphere embedded in euclidean space of some dimension $d_2$. If $\rho$ is a completely mixed state in $S_2$ this result implies that the number of entries of a real vector representing a normalized state of an elementary system is $d_2+1$ where the normalization degree of freedom is explicitly taken into account. In quantum theory $d_2 = 3$.

We are now going to show that, given a state $\rho$, if $F_\rho$ has information capacity two, then states in $F_\rho$ constitute a sphere.

**Lemma 5** Given a state $\rho$, if $F_\rho$ has information capacity two, then the set of pure states in $F_\rho$ is a sphere.

**Proof:** From axiom reversibility, for any two pure states in $F_\rho$, $\psi$, $\phi$, there exists a transformation $G$ such that $\phi = G\psi$. The set of transformations mapping pure states in $F_\rho$ into pure states in $F_\rho$ forms a group. This group must be compact since vectors representing pure states in $F_\rho$ constitute a real representation of it and have bounded entries. From this fact and the fact that any compact group admits a representation by means of orthogonal $d \times d$ matrices, we know that we can represent pure states in $F_\rho$ as points of a $d$-dimensional sphere. We now prove that every point of the sphere represents a pure state. In order to prove it suppose it is not so. Then it exists a mixed state, $\sigma$, on the border of the convex set $F_\rho$ that is not completely mixed. From axiom distinguishability, $\sigma$ must be perfectly distinguishable from some other pure state, $\phi$, not in $F_\sigma$. $F_\sigma$ contains at least two perfectly distinguishable states from axiom distinguishability since $\sigma$ is by hypothesis mixed, and there must be a pure state in $F_\sigma$ that is perfectly distinguishable from some other state in $F_\sigma$. The state $\omega = p\phi + (1-p)\sigma$, $0 < p < 1$, is in $F_\rho$ thus by hypothesis $F_\omega$ cannot have information capacity larger than two. Indeed from axiom conservation, $F_\omega$ has information capacity equal to the sum of one plus the information capacity of $F_\sigma$ thus having information capacity greater than two. Hence we find a contradiction proving that the set of pure states in $F_\rho$ constitutes a sphere.

The above lemma permits to conclude that an elementary system satisfying axioms 1-3 is either classical or is a generalized Bloch sphere in dimension $d_2$. In the following lemma we will explicit the Bloch representation for the generalized elementary system.

**Lemma 6** A point $\psi$ of the sphere constituting the state space of an elementary system has the following form:

$$\psi = \begin{pmatrix} 2p(x_1) - 1 \\ 2p(x_2) - 1 \\ \vdots \\ 1 \end{pmatrix}$$

where $\{x_i\}_{i=1}^{d_2}$ is a set of fiducial effects for $S_2$.

**Proof:** In the representation where $\psi$ is a vector on the unit sphere in $\mathbb{R}^{d_2}$, the probability of seeing an effect $\phi$ in a measurement given that it is prepared state $\psi \in S_2$ is:

$$E_\phi(\psi) = 1/2(1 + \phi^T\psi)$$

with $\phi$ representing some other unit vector on the sphere. The orthonormal basis for $\mathbb{R}^{d_2}$ is:

$$x_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad x_{22} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

It represents a set of fiducial effects for $S_2$ since all effects can be represented as unit vectors on the sphere. For
any state $\psi \in S_2$ we have that the probability for the $i$-th fiducial effect is $p(x_i) = E_i(\phi) = (1 + \phi^i)/2$, hence $\phi^i = 2p(x_i) - 1$. This proves the thesis. ■

C. Quantum theory

At this point of our derivation we have not yet considered composite systems. In this section we are going to show that the only non classical theory satisfying axioms 1-3 and axiom composition is quantum theory.

First we will use axiom composition to prove what in [20, 22, 23] is called “axiom subspace” and what in [21] is derived from “axiom compression” (a slightly different restatement of axiom subspace). In section 1 it will be discussed the significance of that axiom and will be pointed out that it may be regarded as a mathematical requirement on the state space of a physical system rather than a natural informational or operational constraint.

The strategy to prove “axiom subspace” is to show that the field of numbers $\mathbb{K}$ on which is defined a projective space of a physical system satisfying axioms 1-4 does not depend on the system considered but is a property of the theory. This will imply that the object describing a system with information capacity $m + 1$, a $\mathbb{K}P^m$, is the same object describing a subspace with information capacity $m + 1$ of a larger system described by a $\mathbb{K}P^n$ with $n > m$. From this fact, any representation of the state space of the system with information capacity $m + 1$ can be equivalently considered a representation of a subspace with information capacity $m + 1$ of a larger system and this will prove axiom subspace in our derivation.

We will now consider two projective spaces representing two physical systems of a theory satisfying axioms 1-4 and the composite system obtained composing them. Considering any of the subsystems of the composite system, a subspace of the composite system, we will show that all projective spaces describing systems in a theory satisfying axioms 1-4 must be defined over the same field of numbers.

Definition 15 Two projective spaces $P_1$ and $P_2$ are isomorphic iff there exists a bijective map between $P_1$ and $P_2$ such that the points in a subspace of dimension $n$ of $P_1$ are mapped into points of a subspace of dimension $n$ of $P_2$ and conversely.

In the following lemma we will consider two different projective spaces $\mathbb{K}P^m$ and $\mathbb{L}P^m$ describing two systems of information capacity $n + 1$ and $m + 1$ respectively and the projective space describing the system obtained composing the two systems $\mathbb{J}P^l$ that must have information capacity $l + 1 = (m + 1)(n + 1)$. We will then construct an isomorphism between a subspace of dimension $n$ ($m$) of $\mathbb{J}P^l$ and $\mathbb{K}P^n$ ($\mathbb{L}P^m$).

Lemma 7 For any probabilistic theory satisfying axioms 1-4, given two systems of information capacity $m + 1$ and $n + 1$, and the composite system obtained composing them, there exists an isomorphism between any of the subsystems and a subspace of the composite system with the same information capacity.

Proof: Suppose to have two systems $K$ and $L$ satisfying axioms 1-4 of information capacity $n + 1$ and $m + 1$ respectively. According to theorem 3 their state spaces constitute two projective spaces of dimension $n$ and $m$ respectively defined over a skew field. Let $K$ be represented by $\mathbb{K}P^n$ and $L$ by $\mathbb{L}P^m$. Composing the two systems we obtain, from axioms 1-4, another projective space that can be represented in $(dn + 1)(dm + 1) - 1$ (excluding the normalization degree of freedom) real euclidean space and that we denote as $J = \mathbb{J}P^l$ with $l + 1 = (m + 1)(n + 1)$. Consider any state $\rho \in K$ and a fixed pure state $\phi \in L$. The map obtained as:

$$\phi : \rho \to \rho \otimes \phi$$

(11)

Is a map between states in $K$ and states in $J$. We now show that $\phi$ is an isomorphism between $K$ and a subspace of dimension $n$ of $J$ showing that $\mathbb{K}P^n$ is isomorphic to a subspace of dimension $n$ of $\mathbb{J}P^l$. To see this let $\rho'$ be a completely mixed state in $K$. $F_{\rho'} \otimes \phi$ is, by definition, a subspace of dimension $n$ of $J$. The map $\phi$ is injective from $K$ to $F_{\rho'} \otimes \phi$ and, since $\phi$ is a fixed pure state, also surjective. This means that we have a bijective map between pure states in $K$ and pure states in $F_{\rho'} \otimes \phi$. Moreover, mapping any state in a subspace of dimension $h < n$ of $K$ results a state in a subspace of dimension $h$ of $F_{\rho'} \otimes \phi$. Thus we have an isomorphism between a subspace of $J$ with dimension $n$, $\mathbb{J}P^n$, and the $\mathbb{K}P^n$ describing $K$. Reversing the roles of $L$ and $K$ in the above argument we have the thesis. ■

Theorem 4 The field of numbers on which are defined projective spaces representing pure states of systems in a theory satisfying axioms 1-4 is a property of the theory and does not depend on the system considered.

Proof: Composing a system $K$ described by $\mathbb{K}P^n$ with a system $L$ described by $\mathbb{L}P^m$, it results a composite system $J$ described by $\mathbb{J}P^l$ with $l + 1 = (m + 1)(n + 1)$. From lemma 4 it exists an isomorphism between $\mathbb{L}P^m$ and a subspace of dimension $m$ of $J$, $\mathbb{J}P^m$ and also an isomorphism between $\mathbb{K}P^n$ and a subspace of dimension $n$ of $J$, $\mathbb{J}P^n$. From standard projective geometry, [30], this is possible only if $J = K = L$. Since this holds for systems of arbitrary finite information capacity we have the thesis. ■

We now turn to the last part of our derivation where we will show that in a theory satisfying axioms 1-4, the field of numbers $\mathbb{K} = \mathbb{C}$. From axiom composition we know that the system composed of two elementary systems has information capacity four and we denote the set of states of the composed system $S_1$. We also know that if $d_2 + 1$ is the number of parameters required to specify a state of an elementary system, $(d_2 + 1)^2$ parameters will suffice to...
specify a state of the system composed of two elementary systems.

In [21, it is shown that if a theory satisfies axiom composition then the dimension of an elementary system must be odd. Since the group of transformations of an elementary system must be transitive on a sphere in \( \mathbb{R}^{d_2} \), it must coincide with one of the Lie groups whose action is transitive on a sphere in \( \mathbb{R}^{d_2} \) with odd \( d_2 \). The possibilities are: \( SO(d_2) \); the smallest exceptional Lie group usually denoted as \( G_2 \), whose fundamental representation is a subgroup of \( SO(7) \). This observation can be found in [22]. We now rule out the last possibility in that list with the following:

**Lemma 8** \( G_2 \) cannot be the group of transformations of an elementary system in a theory satisfying axioms 1-4.

**Proof:** \( G_2 \) is the group of automorphisms of the algebra of octonions. If this were the group of transformations of an elementary system, then the state space of such a system would be a projective space of dimension 1 over octonions. Consider a system composed of two elementary systems of this kind. The state space of the composite system must form a projective space of dimension 3 from axiom composition and axioms 1,2. Considering one of the two subsystems as a subspace of the composite system, the 3-dimensional projective space should contain a one dimensional subspace which is an octonion dimensional projective space. But this would mean that the 3-dimensional projective space we are dealing with is an octonionic projective space and this contradicts theorem 3 since octonions are not a skew filed (they lack associativity of multiplication). Hence we reach a contradiction and we prove the thesis.

The following proposition will be used in the proof of the subsequent lemma.

**Proposition 2** A separable state \( \psi \) is pure iff the marginal states on both subsystems are pure.

**Proof:** Let \( \phi_1 \) and \( \phi_2 \) be the marginal states of \( \psi \) on system A and B respectively. Suppose \( \phi_1, \phi_2 \) pure and \( \psi \) mixed. Then \( \psi = p_\alpha \phi_1^\alpha \otimes \phi_2^\beta + p_\gamma \phi_1^\gamma \otimes \phi_2^\delta \) and the marginal state on any of the subsystems would be mixed. This proves one implication. Suppose by converse that \( \psi \) is pure and that \( \phi_1 \) is mixed. Then we would have \( \psi = (p_\alpha \phi_1^\alpha + p_\beta \phi_1^\beta) \otimes \phi_2 = p_\alpha \phi_1^\alpha \otimes \phi_2 + p_\beta \phi_1^\beta \otimes \phi_2 \) and \( \psi \) would be mixed. This proves the thesis.

From axiom reversibility we know that the set of transformations of \( S_4 \) forms a compact group and any such group can be represented by means of orthogonal matrices. In the following lemma we will find such representation. The argument used in the proof of the following lemma is invented in [23].

**Lemma 9** The representation of pure states of \( S_4 \) in which the group of transformations is represented by means of orthogonal matrices is such that a pure normalized state \( \psi \) is represented as follows:

\[
\psi = \begin{pmatrix}
\alpha_i = 2p(x_i) - 1 \\
\vdots \\
\beta_j = 2p(y_j) - 1 \\
\gamma_{ij} = 4p(x_i, y_j) - 2p(x_i) - 2p(y_j) + 1 \\
1
\end{pmatrix}
\]  

where \( \{x_i\}_{i=1}^{d_2}, \{y_j\}_{j=1}^{d_2} \) are sets of fiducial effects for \( S_2 \), \( \{\alpha_i\}, \{\beta_j\} \) represent the marginal states of \( \psi \), \( \{\gamma_{ij}\}_{i,j=1}^{d_2} \) the correlation matrix and the last entry represents normalization.

**Proof:** A normalized state \( \psi \in S_4 \) by axiom Local Tomography can be represented as in [12]. By axiom Reversibility we know that every pure state in \( S_4 \) can be expressed as \( A\phi_1 \otimes \phi_2 \) with \( A \) matrix (not necessarily orthogonal) representing an element in the group of reversible transformations of \( S_4 \) and \( \phi_1 \otimes \phi_2 \) pure state represented as in [12]. In the case \( A \) is product we already know that \( A \) is an orthogonal matrix by lemma 8. Since the group of reversible transformations in \( S_4 \) is compact we know that it exists a matrix \( S \) such that \( S^{-1}AS = O \) with \( O \) orthogonal and \( S \in \mathbb{R}^{(d_2^2 + 2d_2)} \times \mathbb{R}^{(d_2^2 + 2d_2)} \) for every reversible transformation in \( S_4 \) (not only the product ones). The matrix \( S \) is non singular since it must perform a change of basis from the representation [12] to the representation in which \( A \) is an orthogonal matrix. We will show that \( S \) is proportional to the identity, namely, those two representations are the same. If \( A \) is product then it is orthogonal; from this we have that if \( A \) is product then \( A \equiv O_1^i \otimes O_2^j \) and \( SA = AS \). Product transformations form a subgroup of the group of transformations of \( S_4 \). The following three subspaces: i) real span of the vectors \((\alpha_1, ..., \alpha_{d_2})^T\); ii) real span of vectors \((\beta_1, ..., \beta_{d_2})^T\); iii) real span of matrices \(\{\gamma_{ij}\}_{i,j=1}^{d_2} \) are three invariant subspaces for the subgroup of product transformations. By Shur’s lemma we thus have:

\[
S = \begin{pmatrix}
aI_{d_2} & 0 & 0 \\
0 & bI_{d_2} & 0 \\
0 & 0 & sI_{d_2}
\end{pmatrix}
\]  

for some \( a, b, s > 0 \) where \( I_d \) is the identity \( d \times d \) matrix. Now define \( \phi^0 = \theta \) and \( \phi^1 = -\theta \) with \( \theta \in \mathbb{R}^{d_2} \) and \( \phi^{0,1} \) pure perfectly distinguishable states in \( S_2 \). Since product of two pure states is pure, from proposition 8 we have \( \phi^a \otimes \phi^b \) is pure \( \forall a, b \in \{0, 1\} \). By axiom Reversibility there exist transformations \( G_{sw} \) and \( G_{cnot} \) such that \( G_{sw}\phi^a \otimes \phi^b = \phi^b \otimes \phi^a \) and \( G_{cnot}\phi^a \otimes \phi^b = \phi^a \otimes \phi^{a \oplus b} \) where \( \oplus \) denotes sum modulo 2. This implies that \( G_{sw}(\theta, 0, 0) = (0, \theta, 0) \) while \( G_{cnot}(0, 0, \theta^T) = (0, \theta, 0) \) where the first and second entries in these vectors represent vectors \( \alpha \) and \( \beta \), i.e. the marginal states of the two
component systems, while the third entry represents matrix $\gamma$ containing the information regarding correlations of the component systems (see (12)). Rewriting these two expressions in the representation where transformations are orthogonal matrices we have:

$$SG_{sw}S^{-1}(a\theta, 0, 0) = (0, b\theta, 0)$$

and

$$SG_{cna}S^{-1}(0, 0, s\theta T) = (0, b\theta, 0)$$

From lemma 8 we have $||\theta||^2 = 1$ and $||\gamma_{\phi_0}\phi_0||^2 = \text{Tr}[\gamma_{\phi_0}\phi_0\gamma_{\phi_0}\phi_0] = 1$ where $\gamma_{\phi_0}\phi_0 = \theta \theta^T$. Since $SGS^{-1}$ is orthogonal for all $G$ we have that $(a\theta, 0, 0)$, $(0, 0, s\theta T)$ and $(0, b\theta, 0)$ have the same modulus hence $a = s = b$. This implies that $S = aI$ and proves the thesis. ■

Lemma 10 For every pure state $\psi \in S_4$ there exists an effect that gives probability one on it.

Proof: The thesis holds for pure product states of $S_4$ from the results obtained in subsection III-B and the fact that probabilities for product states and product effects factorize. Consider performing a transformation $O$ on the composed system in a product state $\psi_{prod}$, then performing its inverse $O^{-1}$ and, after that, making a measurement containing the effects giving probability 1 on $\psi_{prod}$. $O$ acts after state $\psi_{prod}$ while $O^{-1}$ acts before the effect giving probability 1 on $\psi_{prod}$ thus transforming this effect. Expliciting this using the notation of (3), (4), we have:

$$(\psi_{prod}| O \psi_{prod}) = (\psi_{prod}O^{-1}| O \psi_{prod}) = 1 \quad (14)$$

Where $(\psi_{prod}|$ represents the effect giving probability 1 on state $\psi_{prod}$. By axiom reversibility, every state in $S_4$ can be written as $O\psi_{prod}$ for some reversible transformation matrix $O$. This implies the thesis. ■

Definition 16 Given a state $\psi \in S_4$, the effect giving probability one on $\psi$ involved in lemma 11 will be called the effect corresponding to $\psi$.

Lemma 11 Any state vector $\psi \in S_4$ in the representation of lemma 8 is such that $||\psi||^2 = 4$.

Proof: The thesis holds by lemma 4 for pure product states. By axiom Reversibility and the fact that in the representation of lemma 8 transformations are represented as orthogonal matrices the thesis follows also for every $\psi \in S_4$. ■

Lemma 12 In the representation of lemma 8, given any state $\psi$, the effect corresponding to $\psi$ is represented by a vector proportional to that representing $\psi$.

Proof: For product states the thesis holds with the corresponding effect being a product effect. This comes from lemma 8 and the fact that probabilities for product states and product effects factorize. In the representation of lemma 8, since transformations are represented as orthogonal matrices, we have: $(\psi_{prod}|O^{-1}| = 1/4\psi_{prod}^T O$ (the factor 1/4 comes from normalization) while $|O\psi_{prod}| = O\psi_{prod}$ and the probability is simply the scalar product of these two vectors. ■

We already know that a two dimensional subspace of the state space of $S_4$ is a representation of the state space of an elementary system. In the following lemma we will show that the set of effects corresponding to the pure states in a two dimensional subspace of $S_4$ are the set of effects of an elementary system. We will thus prove the following:

Lemma 13 The set of effects corresponding (in the sense of definition 14) to the pure states in a two dimensional subspace of $S_4$ forms the same manifold as the effects associated to an elementary system.

Proof: From lemma 8 any two dimensional subspace $F^2_\rho$ of $S_4$ constitutes a sphere. From lemma 8 this is the same manifold formed by the state space of an elementary system $S_2$, namely, a $d_2$-dimensional sphere with transitivity group $SO(d_2)$ with $d_2$ odd. This is the case since both $F^2_\rho$ and $S_2$ are a projective space of dimension one over a given field of numbers $\mathbb{K}$. Let $T$ be the subset of the set of transformations of $S_1$ that transforms pure states in $F^2_\rho$ into pure states in $F^2_\rho$. Since the manifold representing states in $F^2_\rho$ is the same representing states in $S_2$, the action of $T$ on these state vectors represents the group of transformations of an elementary system. Since $T$ represents such a group, we have that also the effects corresponding to the pure states in $F^2_\rho$, by definition 14, are represented by a $d_2$-dimensional sphere. This implies that states in $F^2_\rho$ and the effects corresponding to them are an equivalent representation of the states and effects of an elementary system. ■

In the following lemmas we will use the following notation:

- $(e)$ represents the deterministic effect
- $s_2(e|\psi)_{S_4}$ indicates the marginal state of $\psi$, namely $\alpha_\psi$ or $\beta_\psi$.
- $s_2(a|\psi)_{S_4}$ indicates the not normalized state of $S_2$ obtained measuring an effect $(a)$ on one component system of the two bits system in state $\psi \in S_4$.
- $s_2(b) \otimes s_2(a|\psi)_{S_4} = (ab|\psi)$ where $(a)$, $(b)$ are two effects in $S_2$.

Lemma 14 Given $(0), (1)$ two perfectly discriminating effects in $S_2$, the deterministic effect for the system composed of two bits is:

$$s_4(e) = (00) + (11) + (10) + (01) \quad (15)$$

where $(ij) = (i) \otimes (j)$, $i, j = \{0, 1\}$
Proof: The effect in \( [15] \) represents the coarse graining of the outcomes obtained in a measurement for the composite system. ■

**Lemma 15** Given \( \{0,1\} \) two perfectly discriminating effects in \( S_2 \), there exists a state \( \psi_{\text{ent}} \in S_4 \) such that:

\[
(00|\psi_{\text{ent}}) = (11|\psi_{\text{ent}}) = 1/2
\]

where \((ii) = (i) \otimes (i), i = 0, 1\)

Proof: If \( \{0,1\} \) are two perfectly discriminating effects there will be two pure perfectly distinguishable states \( \{0,1\} \in S_2 \) that are perfectly discriminated by them. Let \( \rho = 1/2(00 + 11) \) be a state of \( S_4 \). \( F_\rho \) is a two dimensional subspace of \( S_4 \). From lemma \([4]\) the set of states in \( F_\rho \) and the set of effects corresponding to them are represented by the same manifolds of the corresponding sets of an elementary system. If we represent these two sets in real euclidean space, then they represent one the dual of the other as in the case of an elementary system. It then must exist a state \( \psi_{\text{ent}} \in F_\rho \) with the claimed property. This is represented by a state in the equator of the sphere representing pure states in \( F_\rho \). ■

**Lemma 16** Using the representation of lemma \([3]\) for \( \psi_{\text{ent}} \) we have \( \alpha_{\psi_{\text{ent}}} = \beta_{\psi_{\text{ent}}} = 0 \) and

\[
\gamma_{\psi_{\text{ent}}} = \begin{pmatrix} 1 & 0 \\ 0 & C \end{pmatrix}
\]

where \( C \) is a \( d_2 - 1 \) dimensional real matrix.

Proof: \( \alpha_{\psi_{\text{ent}}} \) and \( \beta_{\psi_{\text{ent}}} \) are representations of the two marginal states \( s_0(e|\psi_{\text{ent}})_s s_1 \) on the two component systems of \( S_4 \). \( \{0,1\} \) are the two perfectly distinguishable states of lemma \([3]\). In the representation of lemma \([3]\) these are represented by two antipodal vectors \( \phi^0, \phi^1 \) on the unit sphere in \( \mathbb{R}^{d_2} \). \( \{0,1\} \) are the two corresponding discriminating effects. We know that:

\[
s_2(e|\psi_{\text{ent}})_s s_4 = s_2(0|\psi_{\text{ent}}) s_1 + s_2(1|\psi_{\text{ent}}) s_1 = p(x) \pm (1-p) |x'\rangle
\]

where \( p(x), (1-p) |x'\rangle \) are two not normalized states in \( S_2 \). The state \( \rho = 1/2(00 + 11) \) is perfectly distinguishable from \( \{1\}0 \) with the measurement \( \{00 + 11\} \) of \( 00 + 11 \) represents the coarse graining of the corresponding outcomes. The state \( \psi_{\text{ent}} \in F_\rho \) perfectly distinguishable from \( \{1\}0 \) with the same measurement. This implies that \( (1|0|\psi_{\text{ent}}) = 0 \).

This in turns means that \( p(1|x) = 0 \) that is true iff \( (1|x) = 0 \). \( |x\rangle \) is a pure normalized state in \( S_2 \) that gives probability 0 on the effect \( \{1\} \) and the unique normalized state in \( S_2 \) with this property is \( |0\rangle \). This implies \( |x\rangle = |0\rangle \).

With the same argument, we can conclude that \( (0|x') = 0 \) and thus \( |x'\rangle = |1\rangle \). Since by hypothesis \((00|\psi_{\text{ent}}) = (11|\psi_{\text{ent}}) = 1/2 \) we must have \( p = 1/2 \). Hence we have \( s_2(e|\psi_{\text{ent}}) s_4 = 1/2(00 + 11) \). This implies that \( \alpha_{\psi_{\text{ent}}} = \beta_{\psi_{\text{ent}}} = 0 \) since \( 00 \) and \( 11 \) are antipodal vectors in the representation we are working with. Without loss of generality we can choose the vector representing \( |0\rangle \) equal to \( x_1 \) where \( x_1 \) is defined in \([10]\) and \( |1\rangle \) is the antipode of \( x_1 \). This is due to the arbitrariness of the choice of \( \{0,1\} \) in lemma \([5]\). Since from this lemma \( \psi_{\text{ent}} \) is such that:

\[
(00|\psi_{\text{ent}}) + (11|\psi_{\text{ent}}) = 1
\]

we conclude that \( \gamma^i_{\psi_{\text{ent}}} = 1 \) where \( \gamma^i_{\psi_{\text{ent}}} \) is the \( i \)-th entry of the matrix \( \gamma_{\psi_{\text{ent}}} \). From the fact that \( \alpha_{\psi_{\text{ent}}} = \beta_{\psi_{\text{ent}}} = 0 \) the action of an element of the group of product transformations on \( \psi_{\text{ent}} \) is:

\[
O_1 \otimes O_2 \psi_{\text{ent}} = O_1 \gamma_{\psi_{\text{ent}}^1} O_2^T
\]

with \( O_1 \) orthogonal matrix in \( SO(d_2) \) and \( \gamma_{\psi_{\text{ent}}} \) \( d_2 \) dimensional real matrix. Suppose now that the vector \( \{\gamma^i_{\psi_{\text{ent}}^1}\}_{i=1}^{d_2} \) is such that it exists \( \gamma^i_{\psi_{\text{ent}}} \neq 0 \) for some \( i \neq 1 \).

Then, since the group of transformations of \( S_2 \) is transitive on the sphere, it would exist an \( O_1 \) such that the vector \( \{\gamma^i_{\psi_{\text{ent}}} \}_{i=1}^{d_2} \) would have an entry with modulus greater than 1. Such entry would pertain to a state vector representing a pure state in \( S_4 \). The representation we are working with, is by hypothesis the one found in lemma \([5]\) it is clear from \([5]\) that there cannot exist entries of state vectors with modulus greater than one in this representation. This implies that \( \gamma^i_{\psi_{\text{ent}}} = 0 \) for all \( i \neq 1 \).

We now have to show that the dimensionality of the state space of an elementary system in a theory satisfying axioms 1-4 is three. In order to do so we will use a strategy inspired to that invented in \([23]\).

We introduce the following state:

\[
\psi^i_{\text{ent}} := R^i \otimes I \psi_{\text{ent}}
\]

\( I \) is the identity in \( S_2 \) while \( R^i \) is the matrix having entries:

\[
R^i_{kl} = 0 \text{ if } k \neq l, \quad R^i_{kk} = 1 \text{ if } k \neq i, \quad R^i_{kk} = -1 \text{ if } k = i.
\]

We know that \( \psi^i_{\text{ent}} \) is a state in \( S_4 \) for all \( i \) since, for all \( i \), \( R^i \) is in \( SO(d_2) \) that coincides with the group of reversible transformations in \( S_2 \).

**Lemma 17** \( \psi^i_{\text{ent}} \) and \( \psi_{\text{ent}} \) are such that \( (\psi^i_{\text{ent}}|\psi_{\text{ent}}) = 0 \)

Proof: From lemma \([4]\) we know that:

\[
(00) + (11) + (01) + (10) = (e)
\]

where \( 00 = x_1 \), \( x_1 \) is defined in \([10]\) and \( 11 \) is represented by the vector antipodal to \( 00 \) in the \( d_2 \)-dimensional sphere representing effects of \( S_2 \). This choice is the same done in lemma \([10]\).

From lemma \([3]\) we also have that:

\[
(e|\psi_{\text{ent}}) = (00 + 11) (\psi_{\text{ent}}) = 1
\]

that implies \( (10) (\psi_{\text{ent}}) = (01) (\psi_{\text{ent}}) = 0 \). Now also note that:

\[
(00 + 11)(R^{i-1} \otimes I)(R^i \otimes I) \psi_{\text{ent}} = (00 + 11)|\psi_{\text{ent}}\rangle = 1
\]
and since:

\[(00 + 11)(R_i^{-1} \otimes I)(R_i \otimes I)\psi_{\text{ent}}) = (10 + 01)\psi_{\text{ent}} = 1\]

it follows that \((00)\psi_{\text{ent}} = (11)\psi_{\text{ent}} = 0\). This implies that the test \(\{00, 11\}\) is perfectly distinguishing for \(\psi_{\text{ent}}, \psi_{\text{ent}}\) hence the two states are perfectly distinguishable. From lemma 13 the set of states and the set of effects of \(F_{\rho}\) have the same representation as geometrical object in real euclidean space as the corresponding sets of an elementary system. This means that \(\psi_{\text{ent}}, \psi_{\text{ent}}\) are two antipodal points of the sphere thus \((\psi_{\text{ent}}^i, \psi_{\text{ent}}) = 0\). ■

Theorem 5 The dimension of the state space of an elementary system is \(d_2 = 3\).

Proof: Let \(\rho = 1/2(\psi_{\text{ent}}^i + \psi_{\text{ent}})\). \(F_{\rho}\) has information capacity two. From lemma 17 and lemma 13 we know that

\[(\psi_{\text{ent}}^i, \psi_{\text{ent}}) = 0 \quad (23)\]

We will now explicit (23) using the representation of lemma 1. First we change notation for the matrix \(\gamma_{\psi_{\text{ent}}}^x\):

\[
\gamma_{\psi_{\text{ent}}}^x = (\gamma_{\psi_{\text{ent}}}^x_1, \gamma_{\psi_{\text{ent}}}^x_2, \cdots, \gamma_{\psi_{\text{ent}}}^x_{d_2})
\]

where \(\gamma_{\psi_{\text{ent}}}^x_i\) is the \(i\)-th column vector of the matrix \(\gamma_{\psi_{\text{ent}}}^x\)

Note that by lemmas 1 and 11 we must have

\[||\gamma_{\psi_{\text{ent}}}^x_i||^2 = 1 \quad (25)\]

and

\[
\sum_{\theta=1}^{d_2} ||\gamma_{\psi_{\text{ent}}}^x_\theta||^2 = 3. \quad (26)
\]

(24) can be written as

\[0 = 1/4(1 + \psi^T \psi^i) = 1/4(1 + \text{Tr}[\gamma_{\psi_{\text{ent}}}^T \gamma_{\psi_{\text{ent}}}]) =
\]

\[= 1/4(1 + \sum_{\theta=1}^{d_2} ||\gamma_{\psi_{\text{ent}}}^x_\theta||^2 - 2 ||\gamma_{\psi_{\text{ent}}}^x||^2 - 2 ||\gamma_{\psi_{\text{ent}}}^x_1||^2) \quad (27)
\]

This, (23), (26) imply that:

\[||\gamma_{\psi_{\text{ent}}}^x||^2 = ||\gamma_{\psi_{\text{ent}}}^x_1||^2 = 1 \quad (28)
\]

Since by definition \(i = 2, \cdots d_2\) we must have that \(||\gamma_{\psi_{\text{ent}}}^x_i||^2 = 1\) for all \(i\). From (24) the thesis follows. ■

Corollary 2 Pure states of a system with information capacity \(n+1\) described by a probabilistic theory satisfying axioms 1-4 are points of a \(\mathbb{CP}^n\).

Proof: From theorem 6 the state space of a system with information capacity \(n + 1\) must be a projective space of dimension \(n\) over some field \(\mathbb{K}\). From lemma 7 and corollary 1, every one dimensional subspace of this projective space must be a \(\mathbb{CP}^1\). This implies the thesis. ■

IV. DISCUSSION

A. The subspace axiom

In [20], [22], [23], it was assumed what we here generically call “subspace axiom”. In [21] it was assumed axiom compression stating more or less the same thing as axiom subspace in a way consistent with the graphical formalism invented by the same authors. The requirement in these axioms is that it must exist a linear mapping between a subspace of a system and the state space of a different system with information capacity equal to that subspace such that the two spaces (i.e. the state space of one system and the subspace of the other) can be regarded as different representations of the same mathematical object. The intuitive justification for this axiom is that the mathematical object used to represent a set of states depends only on the information capacity of that set. This is clearly a mathematical requirement on the state space of a physical system and does not deal with information theory. We see from theorem 6 that this property is derived by the projective space structure of quantum theory. This structure, in turn, is derived from axiom distinguishability and axiom conservation that deal with storage and retrieving of information into a physical system (see section III) and do not refer to any mathematical property that the state space of a physical system must have. In [20] it has been formulated a new axiom \textit{sturdiness} to prove axiom subspace in the duotensor framework. Despite its “operational” formulation, \textit{sturdiness} strongly relies on the notion of filter as used in quantum mechanics. This axiom can thus be hardly regarded as an information theoretic constraint on a probabilistic theory and in the context of the present reconstruction it would sound strange.

Requiring axioms like the subspace one in a derivation of quantum theory from informational/operational requirements means to require a priori much of the mathematical structure of the theory without giving a justification in terms of more basic principles. In the present reconstruction we are able to get rid of axiom subspace in favor of two simpler axioms related to storing and retrieving of information into physical systems. It is remarkable that deriving an important piece of mathematical structure of quantum theory from more basic principles permits us to classify probabilistic models different from quantum theory that share with quantum theory only
some of the natural requirements we imposed in section 1 but not others.

B. Complex amplitudes and composite Systems

It is shown in section II A that if a probabilistic theory satisfies axioms 1-5 then either is classical or it constitutes a generalization of quantum theory in which the superposition principle holds with amplitudes not necessarily complex but belonging to a generic field of numbers. Among these theories we find quantum theory but also quantum theory over reals and quantum theory over quaternions corresponding respectively to amplitudes in superpositions being reals and quaternions. The result in corollary 2 implies that among the theories in this class, using complex amplitudes in superpositions is equivalent to require axiom composition for the description of composite systems.

It is shown in [29] that quantum theory over reals satisfies 2-Local Tomography for the composite system.

Definition 17 A theory satisfies 2-Local Tomography if the state of a composite system (ABC...) can be determined from the statistics of measurements on the single components (A, B, C,...) and the statistics of bipartite measurements on two components at one time (AB, AC, BC, ...).

2-local tomography differs from local tomography because in the latter the statistics of measurements on the single components suffice to determine the state. The result of [28] combined with the present derivation of quantum theory is very interesting. We have in fact proved that quantum theory over reals is, with quantum theory, in the class of probabilistic theories satisfying axioms 1-5. On the other hand quantum theory is the only theory in this class satisfying axiom composition with local tomography. In [28] it is shown that quantum theory over reals satisfies 2-local tomography in place of local tomography. This implies that quantum theory over reals satisfies the same informational axioms of quantum theory except from the one concerning composite systems in which local tomography must be substituted with 2-local tomography. We are now in a position to state a conjecture similar to the one stated in [28]: quantum theory over reals is the only probabilistic theory satisfying the following list of axioms: system, distinguishability, capacity, conservation, reversibility and composition with 2-local tomography.

It is very natural that our description of the physical world involves local tomography in spite of 2-local tomography for describing state of a composite system.

Indeed if the latter were used then the mere fact that two different physical systems are described at once as a single system would imply for the composite system to have more degrees of freedom than the cartesian product of the degrees of freedom of the components.

V. CONCLUSION

The mathematical rules governing quantum theory can be understood in terms of information theoretic principles. A physical system is thought as something in which can be stored information. The maximal amount of information that can be stored in a given system has a definite value. Probabilistic theories are a very general framework in which physical systems can be described in terms of operational primitives such as preparations, transformations and measurement outcomes. Five natural axioms for physical systems in the probabilistic theories framework are Distinguishability, Conservation, Reversibility, Composition. Theories satisfying the first two axioms are either classical or such that pure states of a physical system are points of a projective space defined over a generic field of numbers. This latter class of theories constitutes a generalization of quantum theory in which superposition principle holds with amplitudes not necessarily complex but belonging to a generic field of numbers. Examples of theories in this class are quantum theory, real quantum theory, quaternionic quantum theory. This result establishes a connection between two different approaches to quantum foundations, quantum logic and the one based on information theoretic primitives. Such connection is provided by theorem 4. More importantly, this result permits to get rid of the subspace axiom in the reconstruction. Axiom subspace was always more or less explicitly used in previous reconstructions and contains a mathematical constraint on the state space of a physical system. They are also classified all the possible probabilistic theories that are consistent with three subsets of the set of axioms given and this could be useful in experimental tests of quantum mechanics to check whether the results of these tests could be consistent with other mathematical models.

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