Relative Density and Exact Recovery in Heterogeneous Stochastic Block Models

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Abstract

The Stochastic Block Model (SBM) is a widely used random graph model for networks with communities. Despite the recent burst of interest in recovering communities in the SBM from statistical and computational points of view, there are still gaps in understanding the fundamental information theoretic and computational limits of recovery. In this paper, we consider the SBM in its full generality, where there is no restriction on the number and sizes of communities or how they grow with the number of nodes, as well as on the connection probabilities inside or across communities. This generality allows us to move past the artifacts of homogenous SBM, and understand the right parameters (such as the relative densities of communities) that define the various recovery thresholds. We outline the implications of our generalizations via a set of illustrative examples. For instance, \( \log n \) is considered to be the standard lower bound on the cluster size for exact recovery via convex methods, for homogenous SBM. We show that it is possible, in the right circumstances (when sizes are spread and the smaller the cluster, the denser), to recover very small clusters (up to \( \sqrt{\log n} \) size), if there are just a few of them (at most polylogarithmic in \( n \)).

1 Introduction

A fundamental problem in network science and machine learning is to discover structures in large, complex real-world networks (e.g., biological, social, or information networks). Communities are one of the most basic structures to look for, and are useful in many ways including simplifying network analysis. Community or cluster detection also arises in machine learning and underlies many decision tasks, as a basic step that uses pairwise relations between data points in order to understand more global structures in the data. Applications of community detection are numerous, and include recommendation systems [XWZ+14], image segmentation [SM00, MS01], learning gene network structures in bioinformatics, e.g., in protein detection [CY06] and population genetics [JTZ04].

In spite of a long history of heuristic algorithms (see, e.g., [LLM10] for an empirical overview), as well as strong research interest in recent years on the theoretical side as reviewed in the next section, there are still gaps in understanding the fundamental information theoretic limits of recoverability (i.e., if there is enough information to reveal the communities) and computational tractability (if there are efficient algorithms to recover them). This is particularly true in the case of sparse graphs (that test the limits of recoverability), graphs with heterogeneous communities (communities varying greatly in size and connectivity), graphs with a number of communities that grows with the number of nodes, and partially observed graphs (with various observation models).

In this paper, we study recovery regimes and algorithms for community detection in sparse graphs generated under a heterogeneous stochastic block model, where there is no restriction on the number and sizes of communities or how they grow with the number of nodes, as well as the connection probabilities inside or across communities. We propose key network descriptors, called relative densities (defined in (1.3)), that govern the exact recoverability of the communities, and determine ranges of these parameters that lead to various regimes of difficulty of recovery. The implications of our generalizations are outlined in Section 1.3 where illustrative examples provide insight into our results in Section 2.
1.1 The Heterogenous Stochastic Block Model and Exact Recovery

The stochastic block model (SBM), first introduced and studied in mathematical sociology by Holland, Laskey and Leinhardt in 1983 [HLL83], can be described as follows. Start with \( n \) vertices and partition the vertex set \( \{1, 2, \ldots, n\} \) into \( r \) groups \( V_1, V_2, \ldots, V_r \), of sizes \( n_1, n_2, \ldots, n_r \) respectively. Then, we draw an edge between two nodes with a probability depending on which communities they belong to; i.e., the probability of an edge between vertices \( i \) and \( j \) (denoted by \( i \sim j \)) is given by

\[
\mathbb{P}(i \sim j) = \begin{cases} 
    p_k & \text{if there is a } k \in \{1, 2, \ldots, r\} \text{ such that } i, j \in V_k \\
    q & \text{otherwise}
\end{cases}
\]  

(1.1)

where we assume \( q < \min_k p_k \) in order for the idea of communities to make sense. Such inter-cluster edges are also known as “ambient” edges. Notice that each of the \( V_k \)'s is endowed with an Erdős-Rényi graph structure \( G(n_k, p_k) \) (within each community \( V_k \), the probability of an edge is given by the local probability \( p_k \)). This defines a distribution over random graphs known as the stochastic block model. To contrast our study of this general setting with previous works where homogenous SBM is considered (where the sizes and probabilities associated to the communities are equal, e.g., in [CX14]), or other special cases of SBMs are studied (e.g., when the number of communities is fixed or grows slowly with the number of nodes such as in [AS15a]), we sometimes refer to the above model as the heterogenous stochastic block model.

The community detection problem studied in this paper is then stated simply as: given the adjacency matrix of a graph generated by the heterogenous stochastic block model, can we recover the labels of all vertices, with high probability, using an algorithm that has been proved to do so, whether in polynomial time or not. Note that recovery with high probability is the best one can hope for, as–with tiny probability–the model can generate graphs where the partition is unrecoverable, e.g., the complete graph. Whether this problem is solvable depends on the parameters involved, and our results characterize parts of the model space for which such recovery is possible. Moreover, based on the computational complexity of the proposed algorithm, we can be in different subregimes, hard (recovery is possible theoretically, but not necessarily efficiently), easy (recovery can be done efficiently; i.e., there is a polynomial-time algorithm), and simple (recovery can be done by simple counting and thresholding procedures), as explained in the next section.

In the next subsection, we mention other natural questions in community detection and review existing results in the literature. We summarize our new results in section 1.3.

1.2 Related Work

What we can infer about the community structure from a single draw of the random graph varies based on the regime of model parameters. Often, the following scenarios are considered.

1. **Exact Recovery (Strong Consistency).** In this regime it is possible to recover all labels, with high probability. That is, an algorithm has been proved to do so, whether in polynomial time or not. Notice that we need the nodes in all communities to be connected for the exact recovery to be possible.

2. **Almost Exact Recovery (Weak Consistency).** A total of \( n - o(n) \) labels are recoverable, but no more. For example, consider the case where the graph has multiple components, all but one of which are tiny; the tiny components cannot be classified correctly.

3. **Partial Recovery or Approximation Regime.** Only a fraction of vertices, i.e. \( (1 - \epsilon)n \) for some \( \epsilon > 0 \), can be guaranteed to be recovered correctly. For example, in the case of two symmetric communities, this fraction should be greater than 1/2 (which one can obtain just by random guessing).

4. **Detectability.** One may construct a partition of the graph which is correlated with the true partition, but one cannot guarantee any kind of quantitative improvement over random guessing. This happens in very sparse regimes when some \( p_k \)'s and \( q \) are of the same, small, order; e.g. see [MNS14c].

It may appear at first that the differences between exact recovery with strong and weak consistencies (the first two regimes above) are small; to illustrate the differences, consider the situation when one has a

\footnote{In this context, this means doing better than guessing.}
very large (sized $n$) social network with a particular set of nodes of interest, which may also be large but $o(n)$. An exact recovery algorithm with strong consistency guarantees that, with high probability, all of the nodes of interest will be correctly labeled. An exact recovery algorithm with weak consistency can guarantee that any of the nodes will be correctly labeled with high probability, but may yield absolutely no guarantees about the entire set (in fact, depending on the set size, the probability that some nodes will be mislabeled may be $O(1)$). In other words, in such setting, while the probability of correct recovery for a fixed set of $n - o(n)$ vertices may be zero, the probability of correct recovery for some set of $n - o(n)$ vertices is close to one.

Thresholds. Recently, there has been significant interest in determining sharp thresholds (or phase transitions) for the various parameter regimes. Currently, the best understood case is the SBM with only two communities of equal size (which we refer to as binary SBM hereafter) for which all of the four regimes above have been identified and characterized in a series of recent papers [CO10, MNS14c, MNS13, Mas14, MNS14a, MNS14b, ABH14, HWX14]. Moreover, tractable algorithms have been proposed and they work down to the information-theoretical thresholds; i.e., information-theoretical and computational thresholds coincide for the case of binary SBM.

Aside from this case, Abbe and Sandon [AS15a] proved the existence of an information-theoretic threshold for exact recovery in the case when the number $r$ of communities is fixed and all community sizes are $O(n)$ (while the connectivity probabilities $p_k, q$ are $O(\log n/n)$). In particular, in [AS15a], they provided an almost linear-time algorithm using the knowledge of model parameters that works down to this information-theoretic threshold. Such knowledge is shown to be unnecessary in a fully agnostic algorithm developed in [AS15b].

Outside of the settings described above, results tend to be inconclusive where not all the regimes are well understood and the bounds incorporate large or unknown constants. Although we do not aim to give an exhaustive review of the existing literature, we will mention the main state-of-the-art results for the regimes identified above.

1. **Exact Recovery (Strong Consistency).** Many partial results are available for general SBM, yielding upper bounds on the thresholds for efficient regimes, or lower bounds for exact recoverability; for example Chen and Xu [CX14] which served as an inspiration for this paper. The results in [CX14] cover the regime when all clusters are equivalent, that is, all $p_k = p$ and there are $r$ clusters, each of size $K := n/r$; $r$ and $p$ are allowed to vary with $n$. Depending on $K$, $p$, $q$, and $n$, they characterize the conditions under which 1) exact recovery is impossible, 2) exact recovery is possible *theoretically*, but not necessarily *efficiently*, e.g., by the Maximum Likelihood Estimator, 3) exact recovery can be done efficiently, e.g., by a semidefinite programming relaxation of the ML estimator, 4) exact recovery can be done by a simple counting and thresholding procedure.

The bounds for these regimes in [CX14] are not shown to be sharp thresholds, but they work down to the limit of cluster connectivity for $p$ and $K$, which with $K = O(n^2)$ for some constant $0 \leq \beta \leq 1$, results in $p = O(\log n/K)$ (further lowering of $p$ will result in a disconnected graph, and as such strong recovery becomes impossible.) The downside of [CX14] lies in the very strong assumption of equivalent clusters. The difficulty of such assumption in heterogeneous SBM will be discussed in detail in Section 1.4.

2. **Almost Exact Recovery (Weak Consistency).** This case has not been extensively treated in the literature. Yun and Proutiere [YP14] studied the case when there is a finite number of clusters, all of size $O(n)$, and such that all intra-cluster probabilities $p_k$ are equal to $p$. They find a characterizing condition for weakly consistent recovery in terms of $p$, $q$, and $n$; this condition was rediscovered in the case of the binary SBM by Mossel, Neeman and Sly [MNS14b]; for this latter case it can be stated as

$$n \frac{(p - q)^2}{p + q} \to \infty .$$  

[YP14] is the first to give a lower bound on the threshold. In their studied case this lower bound coincides with the upper bound, which they show by providing a spectral algorithm (based on an algorithm by Coja-Oghlan [CO10]) with a simpler analysis.

Previous to their results, there have been other methods/algorithms to show the possibility of weakly consistent recovery; although the algorithms used may be even simpler (e.g., Rohe, Chatterjee, Yu [RCY11], which is spectral), they generally do not come close to the threshold.
Previously, weakly consistent recovery has been studied by Rohe, Chatterjee, Yu [RCY11] using a spectral algorithm (based on an algorithm by Coja-Oghlan [CO10] with a simpler analysis), but the results do not come close to the threshold where \( p, q \) is required to be almost \( O(1) \) (up to logarithmic factors).

Recently, Zhang and Zhou [ZZ15] obtained similar results as (1.2) under approximately same-sized communities, with the smallest inter-cluster connectivity parameter \( p \) and the highest intra-cluster connectivity parameter \( q \), by adopting a minimax approach. They show that weak recovery is possible if

\[
\frac{n(p-q)^2}{pK \log K} \rightarrow \infty,
\]

and is impossible if

\[
\frac{n(p-q)^2}{pK} = O(1)
\]

where \( K \) is the number of clusters which is allowed to grow. Later, [GMZZ15] proposed a computationally feasible algorithm that provably achieves the optimal misclassification proportion given above.

3. Partial Recovery. Coja-Oghlan [CO10] extended the asymptotic analysis of SBM to bounded degree regimes and was the first to give partial recovery results. For the binary SBM case, his conditions amount roughly to the following: for \( p = a/n \) and \( q = b/n \) for some constants \( a, b \), there exists some large constant \( C \) such that, if \( (a-b)^2 \geq C(a+b) \log(a+b) \), then partial recovery is possible, and the fraction of recovered vertices is upper bounded by a function of \( C \). Following [CO10], a series of works by [DKMZ11, MNS14c, Mas14, MNS13] extended the asymptotic analysis of SBM to bounded degree regimes and was the first to give partial recovery results. For the binary SBM case, his conditions amount roughly to the following: for \( p = a/n \) and \( q = b/n \) for some constants \( a, b \), there exists some large constant \( C \) such that, if \( (a-b)^2 \geq C(a+b) \log(a+b) \), then partial recovery is possible, and the fraction of recovered vertices is upper bounded by a function of \( C \). Following [CO10], a series of works by [DKMZ11, MNS14c, Mas14, MNS13] established a sharp threshold for detection in binary SBM. Decelle et al [DKMZ11] conjectured a sharp threshold at \( (a-b)^2 = 2(a+b) \), based on non-rigorous ideas from statistical physics. Later, [MNS14c] showed that below this threshold it is impossible to cluster, or even to estimate the model parameters from the graph. Finally, [Mas14, MNS13] provided an algorithm which efficiently outputs a labeling that is correlated with the true community assignment when \( (a-b)^2 > 2(a+b) \). Mossel, Neeman and Sly [MNS14a] proposed an algorithm using a variant of belief propagation that is optimal in the sense that if \( (a-b)^2 > C(a+b) \) for some constant \( C \), then the algorithm achieves the optimal fraction of nodes labelled correctly.

For the general SBM in the bounded average degree regime, recently, Guedon and Vershynin [GV14] analyzed a convex optimization based approach, and Le, Levina, and Vershynin [LLV15] analyzed a simple spectral algorithm, achieving similar upper bounds on the threshold of partial recovery. The proofs make use of the Grothendieck inequality. [GV14] offers a convex optimization approach for obtaining a correct labeling of a \((1-\epsilon)\) fraction of the vertices with arbitrarily small \( \epsilon \). The particular formulation of the convex problem is not crucial and can be changed without significant change to the bound itself. However, it is unclear how their results evolve when the networks have unbounded average degrees.

Le, Levina, and Vershynin [LLV15] proposed a spectral method with degree correction when the average degree regime of the network is bounded. As a result of the degree correction, the graph Laplacian concentrates (which otherwise does not, in the bounded average degree regime) and hence the leading eigenvectors of the Laplacian can be used to approximately recover the labels. A similar degree correction trick was adopted in [QR13]. It should be noted that in [RCY11], the authors used the fact that although the Laplacian does not concentrate, the square of the Laplacian does, and obtained good partial solutions in a much denser regime (smallest degree being \( \Omega(n) \)).

4. Detectability/Impossibility. As mentioned above, for the binary SBM with \( p = a/n \) and \( q = b/n \), Decelle et al [DKMZ11] conjectured that if \( (a-b)^2 < 2(a+b) \) one cannot infer the community assignments with better than 50% accuracy which can be achieved by random guessing. The conjecture was later verified by [MNS14c] as pointed out above. For the symmetric SBM with \( r \) equivalent communities (of the same size and connection probabilities), the strongly empirically-supported conjecture of Decelle et al [DKMZ11] states that when \( (a-b)^2 < c(r)(a+(r-1)b) \) for some \( c(r) \leq r \), the model is indistinguishable from a general Erdős-Rényi model; e.g. see Conjecture 7.2 in [MNS14c] for details.

As mentioned in the beginning of this section, it has been proven that there is no gap between the information-theoretic and computational thresholds for binary SBM. On the other hand, while the information-theoretic threshold for partial recovery of more than 2 communities is still unknown, [MNS14c] conjectured
a gap exists for partial recovery for more than 4 communities. Similarly, sharp thresholds for exact recovery of multiple communities are still unknown (see [AS15a]).

In addition to the papers mentioned above, the interested reader will find good surveys of current literature in [CX14, AS15a, AL14, MNS14a, MNS14b].

1.3 This paper

In this paper we study the general setup presented in Section 1.1, where the communities are not constrained to have the same size and connection probabilities, and where \( r \) is allowed to grow with \( n \). Our work is concerned with exact recovery and is based on [CX14]. We provide the following:

- An information-theoretic lower bound, describing an impossibility regime (Theorem 4),
- An upper bound, describing a potentially "hard" regime in which recovery is always possible, though not necessarily in an efficient way (Theorem 3). Here we assume the sizes of the communities \( n_k \), for \( k = 1, \ldots, r \), are known.
- An upper bound for efficient recovery via a convex optimization algorithm similar to the one in [CX14], describing an “easy” regime (Theorems 1 and 2). Here we assume the quantity \( \sum_k n_k^2 \) is known.
- A bound characterizing a very simple and efficiently solvable thresholding algorithm, if model parameters \( p_k, q \) are known (Theorem 12).
- Extensions of the above bounds to the case of partial observations, i.e., when each entry of the matrix is observed uniformly with some probability \( \gamma \) and the results recorded.

Our setup is general and allows for any mix of clusters of all magnitudes and densities. We illustrate the importance of considering such a model, as opposed to using summary statistics such as \( n_{\text{min}} \) and \( p_{\text{min}} \), by some examples later in this section. This setup allowed us to identify the crucial quantities

\[
\rho_k = n_k(p_k - q), \quad \tilde{D}(p_k, q) = \frac{(p_k - q)^2}{q(1-q)}, \quad \tilde{D}(q, p_k) = \frac{(p_k - q)^2}{p_k(1-p_k)},
\]

(1.3)

where \( \rho_k \) is called the relative cluster density for a cluster \( k \), and \( \tilde{D} \) represents the Chi-square divergence between two Bernoulli random variables with the given probabilities. We elaborate on these quantities in the beginning of Section 2. The bounds resulting from our inequalities bear resemblance to, and appear to be generalizations of McSherry’s [McS01], allowing for the different \( n_k \)'s and \( p_k \)'s. It is worth mentioning that we have explored the possibility of allowing for a whole matrix of inter- and intra-cluster connectivity probabilities (in other words, we looked at the case when instead of a uniform probability \( q \) of inter-cluster connection, we have different connectivity probabilities \( q_{kl} \) for each pair of clusters \( (k,l) \), for \( k \neq l \)). The calculations can be followed through but at the cost of added notation complexity, with no clear shortcut, which we decided not to pursue.

Our results cover a wider set of cases than present in the existing literature. We give illustrative examples in Section 1.4 to show that the setup we consider and the results we obtain represent a clear improvement over previous work. The examples emphasize how Theorems 1, 2 and 3 (given in Section 2 with proofs and more details given in Appendices A, B), complement each other, and how they compare and contrast with existing literature. More details and justification for the claims made in the examples are given in Appendix D.

1.4 Examples

In the following, a configuration is a list of cluster sizes \( n_k \), their connection probabilities \( p_k \), and the inter-cluster connection probability \( q \). A triple \( (m, p, k) \) indicates \( k \) clusters of size \( m \) each, with connectivity parameter \( p \). We do not worry about whether \( m \) and \( k \) are always integers; if they are not, one can always round up or down as needed so that the total number of vertices is \( n \), without changing the asymptotics. Moreover, when the \( O(\) \) notation is used, we mean that appropriate constants can be determined.
1.4.1 Counter-examples for the \((p_{\min}, n_{\min})\) heuristic

In a heterogenous setup, one might think one can plug in \((p_{\min}, n_{\min})\) in the results for homogenous SBM to identify recoverability regimes. While this simplistic approach will indeed yield some upper bounds on some of the “positive” thresholds (i.e., if you can solve it for the simplistic case, you can also solve it for the more complex one, it can completely fail to correctly identify solvable subregimes. The first two examples show why such a heuristic used for generalization attempts in the literature is not useful enough.

Example 1 Suppose we have two clusters of sizes \(n_1 = n - \sqrt{n}, n_2 = \sqrt{n}\), with \(p_1 = n^{-2/3}\) and \(p_2 = 1/\log n\) while \(q = n^{-2/3-0.01}\). As we will see, the bounds we obtain here in Theorem 3 make it clear that this case is theoretically solvable (in the hard regime). By contrast, Theorem 3.1 in [CL14] (specialized for the case of no outliers), requiring

\[
\min(p_{\min} - q)^2 \gtrsim (\sqrt{p_{\min}n_{\min}} + \sqrt{q})^2 \log n, \tag{1.4}
\]

would fail and provide no guarantee for recoverability.

Example 2 Consider a configuration as

\[
(n - n^{2/3}, n^{-1/3+\epsilon}, 1), (\sqrt{n}, O(1/\log n), n^{1/6}), \quad q = n^{-2/3+3\epsilon},
\]

where \(\epsilon\) is some small quantity, e.g., \(\epsilon = 0.1\). Either of Theorems 1 and 2 verify that this case is in the easy regime, and the partition can be recovered efficiently by solving a convex program, with high probability. By contrast, using the \(p_{\min} = n^{-1/3+\epsilon}\) and \(n_{\min} = \sqrt{n}\) heuristic, neither the condition of Theorem 3.1 in [CL14] (given in (1.4)) nor the condition of Theorem 2.5 in [CX14] is fulfilled, and thus we have no means of reaching the same conclusion based on the \((p_{\min}, n_{\min})\) heuristic.

1.4.2 Cluster sizes: small, large, and in-between

The next three examples attempt to provide an idea of how wide the spread of cluster sizes can be, as characterized by our results. Most algorithms for clustering the SBM run into the problem of small clusters [CSX12, Bop87, McS01], often because the models employed do not allow for enough parameter variation to identify the key quantities involved. The bounds we obtain in this paper indicate that the “correct” parameters are not the pairs \((p_k, n_k)\), but rather the relative cluster densities \(\rho_k = (p_k - q)n_k\) (which are related to the “effective densities” appearing in [VOH14]). This allows us to significantly vary the sizes of the clusters, and still be able to obtain exact recovery, as long as the relative densities are large enough.

Example 3 (smallest cluster size for convex recovery) Consider a configuration as

\[
(\sqrt{\log n}, O(1), m), (n_2, O(\log n), \sqrt{n}), \quad q = O(\log n/n),
\]

where \(n_2 = \sqrt{n} - m\sqrt{\log n}/n\) to ensure a total of \(n\) vertices. Here, we assume \(m \leq n/(2\sqrt{\log n})\) which implies \(n_2 \geq \sqrt{n}/2\). It is straightforward to verify the conditions of Theorem 1. Notice that, in verifying...
the first condition for the second group of clusters (with \( p_2 = O(\frac{\log n}{\sqrt{n}}) \), we need \( p_2 n_2 \gtrsim \log n_2 \), which is satisfied when \( m \) is a constant.

There are two important things to note in this example. First, to our knowledge, this is the first example in the literature for which SDP-based recovery works and allows the recovery of (a few) clusters of size smaller than \( \log n \). Previously, \( \log n \) was considered to be the standard bound on the cluster size for exact recovery, as illustrated by Theorem 2.5 of [CX14] in the case of equivalent clusters. We have thus shown that it is possible, in the right circumstances (when sizes are spread and the smaller the cluster, the denser), to recover very small clusters (up to \( \sqrt{\log n} \) size), if there are just a few of them (at most polylogarithmic in \( n \)).

For more details, see Section A.1.

Secondly, the condition of Theorem 3 is not satisfied. This is not an inconsistency (as Theorem 3 gives only an upper bound for the threshold), but indicates the limitation of this theorem in characterizing all recoverable cases.

### Spreading the sizes

The previous example allows us to go lower than the standard \( \log n \) bound on the cluster size for exact recovery; however, we can solve only if the number of very small clusters is finite. On the other hand, Theorem 2 provides us with the option of having many small clusters but requires the smallest cluster to be of size \( O(\log n) \). Since the maximum cluster size is \( O(n) \), one may ask what kind of a spread can be achieved with the help of Theorem 2. In Example 4, we assume a cluster of size \( O(n) \) and examine how small \( n_{\text{min}} \) can be for Theorem 2 to guarantee exact recovery by the convex program. Similarly, in Example 5, we fix \( n_{\text{min}} = \log n \) and examine how large \( n_{\text{max}} \) can be.

#### Example 4

Consider a configuration where small clusters are dense and we have a big cluster,

\[
\left( \frac{1}{2} n', O(1), n^{1-\epsilon} \right), \quad \left( \frac{1}{4} n, n^{-\alpha} \log n, 1 \right), \quad q = O(n^{-\beta} \log n),
\]

with \( 0 < \epsilon < 1 \) and \( 0 < \alpha < \beta < 1 \). Then the conditions of Theorems 1 and 2 both require that

\[
\frac{1}{2}(1 - \alpha) < \epsilon < 2(1 - \alpha), \quad \epsilon > 2\alpha - \beta
\]

and are depicted in Figure 1. Since we have not specified the constants in our results, we only consider strict inequalities.

![Figure 1](image-url)

**Figure 1:** The space of parameters in Equation 1.5. The face defined by \( \beta = \alpha \) is shown with dotted edges. The three gray faces correspond to \( \beta = 1, \alpha = 0 \) and \( \epsilon = 1 \). The green plane (corresponding to the last condition in (1.5)) comes from controlling the intra-cluster interactions uniformly (see (A.7) and (A.8)) which might be only an artifact of our proof and can be possibly improved.
Notice that the small clusters are as dense as can be, but the large one is not necessarily very dense. By picking \( \epsilon \) to be just over 1/4, we can make \( \alpha \) just shy of 1/2, and \( \beta \) very close to 1. As far as we can tell, there are no results in the literature surveyed that cover such a case, although the clever “peeling” strategy introduced in [ACH13] would recover the largest cluster. The strongest result in [ACH13] that seems applicable here is Corollary 4 (which works for non-constant probabilities). The [ACH13] algorithm works to recover a large cluster (larger than \( O(\sqrt{n \log^2 n}) \)), subject to existence of a gap in the cluster sizes (roughly, there should be no cluster sizes between \( O(\sqrt{n}) \) and \( O(\sqrt{n \log^2 n}) \)). Therefore, in this example, after a single iteration, the algorithm will stop, despite the continued existence of a gap, as there is no cluster with size above the gap. Hence the “peeling” strategy on this example would fail to recover all the clusters.

Example 5 Consider a configuration with many small dense clusters. We are interested to see how large the spread of cluster sizes can be for the convex recovery approach to work. As required by Theorems 1 and 2 and to control \( \sigma_{\text{max}} \) (defined in (2.2)) the larger a cluster, the smaller its connectivity probability should be; therefore we choose the largest cluster at the threshold of connectivity (required for recovery). Consider the following cluster sizes and probabilities:

\[
\left( \log n, O(1), \frac{n}{\log n} - m\sqrt{\frac{n}{\log n}}, \sqrt{n \log n}, O(\sqrt{\log n} / n), m \right), \quad q = O\left(\frac{\log n}{n}\right),
\]

where \( m \) is a constant. Again, we round up or down where necessary to make sure the sizes are integers and the total number of vertices is \( n \). All the conditions of Theorem 2 are satisfied, hence we conclude that exact convex recovery is possible in this case.

Note that the last condition of Theorem 1 is not satisfied since there are too many small clusters. Also note that alternate methods proposed in the literature surveyed would not be applicable; in particular, the gap condition in [ACH13] is not satisfied for this case from the start.

1.4.3 Closeness of \( p_{\text{min}} \) and \( q \)

Finally, the following examples illustrate how small \( p_{\text{min}} - q \) can be in order for the recovery, respectively, the convex recovery algorithms to still be guaranteed to work. Note that the difference in \( p_{\text{min}} - q \) for the two types of recovery is noticeable, indicating that there is a significant difference between what we know to be recoverable and what we can recover efficiently by our convex method. We consider both dense graphs (where \( p_{\text{min}} \) is \( O(1) \)) and sparse ones.

Example 6 Consider a configuration where all of the probabilities are of \( O(1) \) and

\[
(n_1, p_{\text{min}}, 1), \quad (n_2, p_2, 1), \quad (n_3, p_3, \frac{n - n_1 - p_{\text{min}}}{n_3}), \quad q = O(1),
\]

where \( p_2 - q \) and \( p_3 - q \) are \( O(1) \). On the other hand, we assume \( p_{\text{min}} - q = f(n) \) is small. For recoverability by Theorem 3, we need \( f(n) \gtrsim (\log n) / n_{\text{min}} \) and \( f^2(n) \gtrsim (\log n) / n_1 \). Notice that, since \( n \gtrsim n_1 \gtrsim n_{\text{min}} \), we should have \( f(n) \gtrsim \sqrt{\log n / n} \). For the convex program to recover this configuration (by Theorem 1 or 2), we need \( n_{\text{min}} \gtrsim \sqrt{n} \) and \( f^2(n) \gtrsim \max\{n / n_1^c, n / n_{\text{min}}\} \), while all the probabilities are \( O(1) \).

For a similar configuration to Example 6, where the probabilities are not \( O(1) \), recoverability by Theorem 3 requires \( f(n) \gtrsim \max\{\sqrt{p_{\text{min}}(\log n) / n}, n^{-c}\} \) for some appropriate \( c > 0 \).

Note that if all the probabilities, as well as \( p_{\text{min}} - q \), are \( O(1) \), then by Theorem 3 all clusters down to a logarithmic size should be recoverable. However, the success of convex recovery is guaranteed by Theorems 1 and 2 when \( n_{\text{min}} \gtrsim \sqrt{n} \).

2 Main Results

In this paper, we consider the heterogenous stochastic block model described in Section 1.1. Consider a partition of the \( n \) nodes into \( V_0, V_1, \ldots, V_r \), where \( |V_k| = n_k, k = 0, 1, \ldots, r \). Consider \( \bar{n} = \sum_{k=1}^r n_k \) and denote the number of isolated nodes by \( n_0 \); hence, \( n_0 + \bar{n} = n \). Ignoring \( n_0 \), we further define \( n_{\text{min}} = \min\{n_k : k = 1, \ldots, r\} \) and \( n_{\text{max}} = \max\{n_k : k = 1, \ldots, r\} \). The nodes in \( V_0 \) are isolated and the nodes in \( V_k \) form the community \( C_k = V_k \times V_k \), for \( k = 1, \ldots, r \). The union of communities is denoted by \( \mathcal{C} = \cup_{k=1}^r C_k \),
and \( \mathcal{C}^c \) denotes the complement; i.e. \( \mathcal{C}^c = \{(i,j) : (i,j) \not\in \mathcal{C}_k \text{ for any } k = 1,\ldots,r, \text{ and } i,j = 1,\ldots,n\} \). Denote by \( \mathcal{Y} \) the set of admissible adjacency matrices according to a community assignment as above, i.e.

\[
\mathcal{Y} := \{Y \in \{0,1\}^{n \times n} : Y \text{ is a valid clustering matrix over the partition } V_0, V_1, \ldots, V_r \text{ where } |V_k| = n_k \}.
\]

We will denote by \( \log \) the natural logarithm (base \( e \)), and the notation \( \theta \geq 1 \) is equivalent to \( \theta \geq O(1) \). A Bernoulli random variable with parameter \( p \) is denoted by \( \text{Ber}(p) \), and a Binomial random variable with parameters \( n \) and \( p \) is denoted by \( \text{Bin}(n,p) \).

Consider a distribution over random graphs with \( V \) as their node set as defined in (1.1). Each subset \( V_k \) is endowed with an Erdős-Rényi graph structure \( \mathcal{G}(n_k,p_k) \) for \( k = 1,\ldots,r \), and an edge is drawn between two nodes in different communities, independent of other edges, with probability \( q \). We assume that \( p_k \geq q \) for \( k = 1,\ldots,r \). The goal is to recover the underlying clustering matrix \( Y^* \) exactly given a single graph drawn from this distribution. We will need the following definitions:

- **Define the relative density of a community as**

  \[
  \rho_k = (p_k - q)n_k
  \]

  which gives \( \sum_{k=1}^r \rho_k = \sum_{k=1}^r p_k n_k - qn \).

- **The Neyman Chi-square divergence** (e.g., see [CR84]) between two discrete random variables \( \mu \) and \( \pi \) (on the same support set of size \( t \)) is defined as

  \[
  D_{\chi^2}^2(\mu, \pi) = \sum_{i=1}^t \frac{(\mu_i - \pi_i)^2}{\pi_i}
  \]

  and is always bounded below by the KL divergence; due to \( \log x \leq x - 1 \). In the case of two Bernoulli random variables \( \text{Ber}(p) \) and \( \text{Ber}(q) \), the Neyman Chi-square divergence is given by

  \[
  \tilde{D}(p, q) := \frac{(p-q)^2}{q(1-q)}
  \]

  and we have \( \tilde{D}(p, q) \geq D_{\text{KL}}(p, q) := D_{\text{KL}}(\text{Ber}(p), \text{Ber}(q)) \); see (B.16). Moreover, for \( q < p \), when both \( p \) and \( q/p \) are bounded away from \( 1 \), we have

  \[
  \tilde{D}(q, p) = p \left(1 - \frac{q/p}{p}\right)^2 \approx p.
  \]

Chi-square divergence is an instance of a more general family of divergence functions called \( f \)-divergences or Ali-Silvey distances [AS66]. This family also has KL-divergence, total variation distance, Hellinger distance and Chernoff distance as special cases. Moreover, the divergence used in [AS15a] is an \( f \)-divergence.

- **Define the total variance** \( \sigma_k^2 = n_k p_k (1 - p_k) \) over the \( k \)th community, and let \( \sigma^2_0 = nq(1-q) \). Also, define

  \[
  \sigma_{\max}^2 = \max_{k=1,\ldots,r} \sigma_k^2 = \max_{k=1,\ldots,r} n_k p_k (1 - p_k).
  \]

### 2.1 Convex Recovery

We consider a convex optimization program for recovering the underlying clustering matrix \( Y^* = \sum_{k=1}^r \mathbf{1}_{c_k} \) and characterize the models that are exactly recoverable using this program. In the following, \( \| \cdot \| \) denotes the matrix nuclear norm or trace norm, i.e., the sum of singular values of the matrix. The dual to the nuclear norm is the spectral norm, denoted by \( \| \cdot \|_* \).
Convex Recovery

input: $\sum_{k=1}^r n_k^2$

output:

$$\hat{Y} = \arg \max_Y \sum A_{ij} Y_{ij}$$
subject to $||Y||_* \leq ||Y^*||_* = n$
$$\sum_{ij} Y_{ij} = \sum_k n_k^2$$
$$0 \leq Y_{ij} \leq 1$$

(2.3)

We prove two theorems giving conditions under which the above convex program outputs the true clustering matrix with high probability. While the theorems are similar in terms of the methodology used, they differ in terms of the conditions we must impose. As we will see, Theorem 1 allows us to describe a regime in which tiny communities of size $O(\sqrt{\log n})$ are recoverable (provided that they are very dense and that only few tiny or small clusters exist; see Example 3), while Theorem 2 covers a less restrictive regime in terms of cluster sizes, but allows us to recover clusters only down to $O(\log n)$; see Example 5. The proofs for both theorems along with auxiliary lemmas are given in Appendix A.

**Theorem 1** Under the heterogenous stochastic block model, the output of convex recovery program in (2.3) coincides with $Y^*$ with high probability, provided that

$$\rho_k^2 \geq \sigma_k^2 \log n_k \ , \ \bar{D}(p_{\min}, q) \geq \frac{\log n_{\min}}{n_k} \ , \ \rho_{\min}^2 \geq \max\{\sigma_{\max}^2 \ , \ n_k(1-q) \ , \ \log n\} \ , \ \sum_{k=1}^r n_k^{-\alpha} = o(1)$$

for some $\alpha > 0$, where $\sigma_k^2 = n_k p_k (1-p_k)$.

The assumption $\sum_{i=1}^r n_i^{-\alpha} = o(1)$ above is tantamount to saying that the number of small or tiny communities (where by tiny we mean communities of size $O(\sqrt{\log n})$) cannot be too large (e.g., the number of polylogarithmic-size communities cannot be a power of $n$). In other words, one needs to have mostly large communities (growing like $n^\epsilon$, for some $\epsilon > 0$) for this assumption to be satisfied. Note, however, that the condition does not restrict the number of clusters of size $n^\epsilon$ for any fixed $\epsilon > 0$. The second theorem imposes more stringent conditions on the relative density, but relaxes the condition that only a very small number of nodes can be in small clusters.

**Theorem 2** Under the heterogenous stochastic block model, the output of convex recovery program in (2.3) coincides with $Y^*$, with high probability, provided that

$$\rho_k^2 \geq \sigma_k^2 \log n \ , \ \bar{D}(p_{\min}, q) \geq \frac{\log n}{n_k} \ , \ \rho_{\min}^2 \geq \max\{\sigma_{\max}^2 \ , \ n_k(1-q)\}$$

**Remark 1** For exact recovery to be possible, we need all communities (but at most one) to be connected. Therefore, in each subgraph, which is generated by $G(n_k, p_k)$, we need $p_k n_k \geq \log n_k,$ for $k = 1, \ldots, r$. Observe that this connectivity requirement is implicit in the first condition of Theorems 1, 2 which can be seen from (2.1).

Note that any convex optimization problem that involves the nuclear norm $||Y||_*$ (or equivalently, $\text{tr}(Y)$ for $Y \succeq 0$) in its objective function or constraints, will have a bottleneck similar to the specific convex problem we analyzed here. Namely, for any such program to succeed we need a subgradient of the nuclear norm at $Y^*$ which has a component $Z$ with spectral norm bounded by 1 (see the proof of Theorem 1 in Appendix A). For example, when all $p_k$ and $q$ are $O(1)$, this requires the minimum cluster size to be at least $O(\sqrt{n})$; also see Example 6.

It is worth mentioning that for some community configurations, a simple counting argument can provide us with the exact underlying community structure; hence no need to solve a semidefinite program as above. We present one such algorithm in Appendix C and characterize exact recovery guarantees.

In the following, we attempt to provide a better picture of the model space in terms of recoverability. Section 2.2 considers a modified maximum likelihood estimator to identify bigger parts of the model space that can be recovered exactly. Section 2.3 provides an information-theoretic argument to exclude part of the model space that are impossible to recover exactly.
2.2 Exactly Recoverable Models

Next, we consider an estimator, inspired by maximum likelihood estimation, and characterize a subset of the model space which is exactly recoverable via this simple estimation method. The proposed estimation approach is not computationally tractable and is only used to examine the conditions for which exact recovery is possible. For a fixed $Y \in \mathcal{Y}$ and an observed matrix $A$, the likelihood function is given by

$$
\mathbb{P}_Y(A) = \prod_{i<j} p_{A_{ij}}^{A_{ij}} (1 - p_{\tau(i,j)})^{(1 - A_{ij})} q^{A_{ij}} (1 - q)^{1 - A_{ij}},
$$

where $\tau: \{1, \ldots, n\}^2 \to \{1, \ldots, r\}$ and $\tau(i,j) = k$ if and only if $(i,j) \in C_k$, and arbitrary in $\{1, \ldots, r\}$ otherwise. The log-likelihood function is given by

$$
\log \mathbb{P}_Y(A) = \sum_{i<j} \log \left( \frac{1 - q}{q(1 - p_{\tau(i,j)})} \right) A_{ij} + \sum_{i<j} \log \frac{1 - p_{\tau(i,j)}}{1 - q} Y_{ij} + \text{terms not involving } \{Y_{ij}\}.
$$

Maximizing the log-likelihood involves maximizing a weighted sum of $\{Y_{ij}\}$'s where the weights depend on the (usually unknown) values of $q, p_1, \ldots, p_r$. To be able to work with less information, we will use the following modification of maximum likelihood estimation, which only uses the knowledge of $n_0, n_1, \ldots, n_r$.

\[ \begin{align*}
\text{Non-convex Recovery} & \\
\text{input: } & \{n_k\} \\
\text{output: } & \hat{Y} = \arg \max_{Y} \left\{ \sum_{i,j} A_{ij} Y_{ij} : Y \in \mathcal{Y} \right\} \tag{2.4}
\end{align*} \]

**Theorem 3** Suppose $n_{\min} \geq 2$ and $n \geq 8$. Under the heterogenous stochastic block model, provided that

$$
\rho_{\min} \geq 4(17 + \eta) \left( \frac{1}{3} + \frac{p_{\min}(1 - p_{\min}) + q(1 - q)}{p_{\min} - q} \right) \log n,
$$

for some choice of $\eta > 0$, the optimal solution $\hat{Y}$ of the non-convex recovery program in (2.4) coincides with $Y^*$, with a probability not less than $1 - \frac{5p_{\max} - 2q}{p_{\min} - q} n^{2-\eta}$.

Notice that $\rho_{\min} = \min_{k=1,\ldots,r} n_k(p_k - q)$ and $p_{\min} = \min_{k=1,\ldots,r} p_k$ do not necessarily correspond to the same community.

2.3 When is Exact Recovery Impossible?

**Theorem 4** If any of the following conditions holds,

1. $2 \leq n_k \leq n/e$, and

$$
4 \sum_{i=1}^{r} n_k^2 \tilde{D}(p_k, q) \leq \frac{1}{2} \sum_{k} n_k \log \frac{n}{n_k} - r - 2 \tag{2.5}
$$

2. $2 \leq n_k \leq n/e$, and

$$
\frac{1}{r} + \log \frac{1}{r} + \frac{1}{r} - \frac{p_{\min}}{p_{\max}} + 1 + \sum_{k} n_k^2 p_k \leq \left( \frac{1}{4} - \sum_{k} n_k^2 p_k \right) \log n + \sum_{k} (n_k p_k - \frac{1}{r}) n_k \log n_k \tag{2.6}
$$

3. $n \geq 128$, $r \geq 2$, and

$$
\max_{k} n_k \left( \tilde{D}(p_k, q) + \tilde{D}(q, p_k) \right) \leq \frac{1}{12} \log(n - n_{\min}) \tag{2.7}
$$

then

$$
\inf_{\hat{Y}} \sup_{Y \neq Y^*} \mathbb{P} [\hat{Y} \neq Y^*] \geq \frac{1}{2}
$$

where the infimum is taken over all measurable estimators $\hat{Y}$ based on the realization $A$ generated according to the heterogenous stochastic block model (HSBM).
2.4 Partial Observations

In the general stochastic block model, we assume that the entries of a symmetric adjacency matrix $A \in \{0, 1\}^{n \times n}$ have been generated according to a combination of Erdős-Rényi models with parameters that depend on the true clustering matrix. In the case of partial observations, we assume that the entries of $A$ has been observed independently with probability $\gamma$. In fact, every entry of the input matrix falls into one of these categories: observed as one denoted by $\Omega_1$, observed as zero denoted by $\Omega_0$, and unobserved which corresponds to $\Omega^c$ where $\Omega = \Omega_0 \cup \Omega_1$. If an estimator only takes the observed part of the matrix as the input, one can revise the underlying probabilistic model to incorporate both the stochastic block model and the observation model; i.e. a revised distribution for entries of $A$ as

$$A_{ij} = \begin{cases} \text{Ber}(\gamma p_k) & (i, j) \in C_k \text{ for some } k \\ \text{Ber}(\gamma q) & i \in C_k \text{ and } j \in C_l \text{ for } k \neq l. \end{cases}$$

yields the same output from an estimator that only takes in the observed values. Therefore, the algorithms in (2.3) and (2.4), as well as the results of Theorems 1, 2, 3, can be easily adapted to the case of partially observed graphs.

3 Future Directions

We have provided a series of extensions to prior work (especially [CX14, AS15a]), however there are still interesting problems that remain open. Future directions for research on this topic include the following.

Models for Partial Observation. We considered the case where a subset of the edges in the underlying graph were observed uniformly at random. In practice, however, the observed edges are often not uniformly sampled, and care will be needed to model the effect of nonuniform sampling. Also, in many practical problems, the observed edges may be chosen by the algorithm based on some prior information (non-adaptive), or based on observations made so far (adaptive); e.g., see Yun and Proutiere [YP14]. It will be interesting to examine what the algorithms can achieve in these scenarios.

Overlapping Communities. SBMs with overlapping communities represent a more realistic model than the non-overlapping case; it has been shown that the large social and information network community structure is quite complex and that very large communities tend to have significant overlap. Only a few references in the literature have considered this problem (e.g., [AS15a]), and there are many open questions on recovery regimes and algorithms. It would be interesting to develop a convex optimization-based algorithm for recovery of models generated by SBM with overlapping communities.

Outlier Nodes. A practically important extension to the SBM is to allow for adversarial outlier nodes. Cai and Li in [CL14] proposed a semidefinite program that can recover the clusters in an SBM in the presence of outlier nodes connected to other nodes in an arbitrary way, provided that the number of outliers is small enough. Their result is comparable to the best known results in the case of balanced clusters and equal probabilities. However, their complexity results are still parametrized by $p_{\text{min}}$ and $n_{\text{min}}$, which excludes useful examples, as discussed in Section 1.3. Extending our results to the setting of [CL14] is a direction for future work.

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A Proofs for Convex Recovery

In the following, we present the proofs of Theorems 1 and 2.

A.1 Proof of Theorem 1

We are going to prove that under the HSBM, with high probability, the output of the convex recovery program in (2.3) coincides with the underlying clustering matrix $Y^*$ provided that

$$ \rho_k^2 \geq n_k p_k (1 - p_k) \log n_k $$

$$ (p_{\min} - q)^2 \geq q(1 - q) \frac{\log n_{\min}}{n_{\min}} $$

as well as $\sum_{k=1}^{r} n_k^{-\alpha} = o(1)$ for some $\alpha > 0$. Notice that $p_k (1 - p_k) n_k \geq \log n_k$, for all $k = 1, \ldots, r$, is implied by the first condition, as mentioned in Remark 1.

Before proving Theorem 1, we state a crucial result from random matrix theory that allows us to bound the spectral radius of the matrix $A - E(A)$ where $A$ is an instance of adjacency matrix under HSBM. This result appears, for example, as Theorem 3.4 in [Cha12]. Although Lemma 2 from [TM10] appears to state a weaker version of this result, the proof presented there actually supports the version we give below in Lemma 5. Finally, Lemma 8 from [Vu14] states the same result and presents a very brief sketch of the proof idea, along the lines of the proof presented fully in [TM10].

**Lemma 5** Let $A = \{a_{ij}\}$ be a $n \times n$ symmetric random matrix such that each $a_{ij}$ represents an independent random Bernoulli variable with $E(a_{ij}) = p_{ij}$. Assume that there exists a constant $C_0$ such that $\sigma^2 = \max_{i,j} p_{ij} (1 - p_{ij}) \geq C_0 \log n / n$. Then for each constant $C_1 > 0$ there exists $C_2 > 0$ such that

$$ \mathbb{P} \left( \|A - E(A)\| \geq C_2 \sigma \sqrt{n} \right) \leq n^{-C_1}.$$

As an immediate consequence of this, we have the following corollary.

**Corollary 6** Let $A = \{a_{ij}\}$ be a $n \times n$ symmetric random matrix such that each $a_{ij}$ represents an independent random Bernoulli variable with $E(a_{ij}) = p_{ij}$. Assume that there exists a constant $C_0$ such that $\sigma^2 = \max_{i,j} p_{ij} (1 - p_{ij}) \leq C_0 \log n / n$. Then for each constant $C_1 > 0$ there exists $C_3 > 0$ such that such that

$$ \mathbb{P} \left( \|A - E(A)\| \geq C_3 \sqrt{\log n} \right) \leq n^{-C_1}.$$

**Proof.** The corollary follows from Lemma 5, by replacing the $(1,1)$ entry of $A$ with a Bernoulli variable of probability $p_{11} = C_0 \log n / n$. Given that the old $(1,1)$ entry and the new $(1,1)$ entry are both Bernoulli variables, this can change $\|A - E(A)\|$ by at most 1. The new maximal variance is equal to $\max_{i,j} p_{ij} (1 - p_{ij}) = C_0 \log n / n$. Therefore Lemma 5 is applicable to the new matrix and the conclusion holds.

We use Lemma 5 to prove the following result.

**Lemma 7** Let $A$ be generated according to the heterogenous stochastic block model (HSBM). Suppose

1. $p_k (1 - p_k) n_k \geq \log n_k$, for $k = 1, \ldots, r$, and
2. there exists an $\alpha > 0$ such that $\sum_{k=1}^{r} n_k^{-\alpha} = o(1)$.

Then with probability at least $1 - o(1)$ we have

$$ \|A - E(A)\| \leq \max_{i} \sqrt{p_i (1 - p_i) n_i} + \sqrt{\max \{q(1 - q)n, \log n\}}.$$

---

2As a more general result about the norms of rectangular matrices, but with the slightly stronger growth condition $\sigma^2 \geq \log^{\alpha + \epsilon} n / n$. 

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Proof. We split the matrix $A$ into two matrices, $B_1$ and $B_2$. $B_1$ consists of the block-diagonal projection onto the clusters, and $B_2$ is the rest. Denote the blocks on the diagonal of $B_1$ by $C_1, C_2, \ldots, C_r$, where $C_i$ corresponds to the $i$th cluster. Then $\|B_1 - E(B_1)\| = \max_i \|C_i - E(C_i)\|$, and for each $i$, $\|C_i - E(C_i)\| \geq \sqrt{p_i(1 - p_i)n_i}$ with probability at most $n_i^{-a}$, by Lemma 5. By assumptions (1) and (2) of Lemma 7 and applying a union bound, we conclude that

$$\|B_1 - E(B_1)\| \preceq \max_i \sqrt{p_i(1 - p_i)n_i}$$

with probability at least $1 - \sum_{i=1}^r n_i^{-a} = 1 - o(1)$. We shall now turn our attention to $B_2$. Let $\sigma^2 = \max\{q(1 - q), \log n/n\}$. By Corollary 6, $\|B_2 - E(B_2)\| \preceq \max\{\sqrt{q(1 - q)n}, \sqrt{\log n}\}$, with high probability. Putting the two norm estimates together, the conclusion of Lemma 7 follows. \qed

We are now in the position to prove Theorem 1.

Proof. [of Theorem 1] We need to show that for any feasible $Y \neq Y^*$, we have $\Delta(Y) := \langle A, Y^* - Y \rangle > 0$. Rewrite $\Delta(Y)$ as

$$\Delta(Y) = \langle A, Y^* - Y \rangle = \langle E[A], Y^* - Y \rangle + \langle A - E[A], Y^* - Y \rangle.$$ 

Note that $\sum_{i,j} Y^*_{ij} = \sum_{i,j} Y_{ij} = \sum_{k=1}^r n_k^2$, thus $\sum_{i,j} (Y^*_{ij} - Y_{ij}) = 0$. Express this as

$$\sum_{k=1}^r \sum_{i,j \in V_k} (Y^* - Y)_{ij} = - \sum_{k' \neq k''} \sum_{i \in V_{k'}, j \in V_{k''}} (Y^* - Y)_{ij}.$$

Then we have

$$\langle E[A], Y^* - Y \rangle = \sum_{k=1}^r \sum_{i,j \in V_k^*} p_k (Y^* - Y)_{ij} + \sum_{k' \neq k''} \sum_{i \in V_{k'}, j \in V_{k''}} q(Y^* - Y)_{ij} = \sum_{k=1}^r \sum_{i,j \in V_k} (p_k - q)(Y^* - Y)_{ij}.$$ 

Finally, since $0 \leq (Y^* - Y)_{ij} \leq 1$ for $i, j \in V_k$, we can write

$$\langle E[A], Y^* - Y \rangle = \sum_{k=1}^r \sum_{i,j \in V_k} (p_k - q)\|(Y^* - Y)_{c_k}\|_1.$$ 

(A.2)

Next, recall that the subdifferential (i.e., the set of all subgradients) of $\| \cdot \|_*$ at $Y^*$ is given by

$$\partial \|Y^*\|_* = \{UU^T + Z | U^T Z = ZU = 0, \|Z\| \leq 1\}$$

where $Y^* = UKU^T$ is the singular value decomposition for $Y^*$ with $U \in \mathbb{R}^{n \times r}$, $K = \text{diag}(n_1, \ldots, n_r)$, and $U_{ik} = 1/\sqrt{n_k}$ if node $i$ is in cluster $C_k$ and $U_{ik} = 0$ otherwise.

Let $M := A - E[A]$. Since conditions (1) and (2) of Lemma 7 are verified, there exists $C_1 > 0$ such that $\|M\| \leq \lambda$, with probability $1 - o(1)$, where

$$\lambda := C_1 \left( \max_i \sqrt{p_i(1 - p_i)n_i} + \sqrt{\max\{q(1 - q)n, \log n\}} \right).$$

(A.3)

Furthermore, let the projection operator onto a subspace $T$ be defined by

$$\mathcal{P}_T(M) := UU^T M + MUU^T - UU^T MUU^T,$$

and also $\mathcal{P}_{T^\perp} = I - \mathcal{P}_T$, where $I$ is the identity map. Since $\|\mathcal{P}_{T^\perp}(M)\| \leq \|M\| \leq \lambda$ with high probability, $UU^T + \frac{1}{\lambda} \mathcal{P}_{T^\perp}(M) \in \partial \|Y^*\|_*$ with high probability. Now, by the constraints of the convex program, we have

$$0 \geq \|Y\|_* - \|Y^*\|_* \geq \langle UU^T + \frac{1}{\lambda} \mathcal{P}_{T^\perp}(M), Y - Y^* \rangle = \langle UU^T - \frac{1}{\lambda} \mathcal{P}_T(M), Y - Y^* \rangle + \frac{1}{\lambda} \langle M, Y - Y^* \rangle,$$

(A.4)
which implies \( \langle M, Y^*-Y \rangle \geq \langle \mathcal{P}_T(M) - \lambda UU^T, Y^*-Y \rangle \). Combining (A.1) and (A.2) we get,

\[
\Delta(Y) \geq \sum_{k=1}^{r} (p_k - q_k) \| (Y^*-Y)_{C_k} \|_1 + \langle \mathcal{P}_T(M) - \lambda UU^T, Y^*-Y \rangle \\
\geq \sum_{k=1}^{r} (p_k - q_k) \| (Y^*-Y)_{C_k} \|_1 \\
- \sum_{k=1}^{r} \| (\mathcal{P}_T(M) - \lambda UU^T)_{C_k} \|_\infty \| (Y^*-Y)_{C_k} \|_1 \\
- \sum_{k' \neq k''} \| (\mathcal{P}_T(M) - \lambda UU^T)_{V_{k'} \times V_{k''}} \|_\infty \| (Y^*-Y)_{V_{k'} \times V_{k''}} \|_1
\]

(A.5)

where we have made use of the fact that an inner product can be bounded by a product of dual norms. We now derive bounds for the quantities \( \mu_{kk} \) and \( \mu_{k'k''} \) marked above. Note that the former indicates sums over the clusters, while the latter indicates sums outside the clusters.

For \( \mu_{kk} \), if \((i,j) \in C_k \) then

\[
(\mathcal{P}_T(M) - \lambda UU^T)_{ij} = (UU^T M + M UU^T - UU^T M UU^T - \lambda UU^T)_{ij} \\
= \frac{1}{n_k} \sum_{l \in C_k} M_{lj} + \frac{1}{n_k} \sum_{l \in C_k} M_{il} - \frac{1}{n_k^2} \sum_{l,l' \in C_k} M_{ll'} - \frac{\lambda}{n_k}.
\]

Recall Bernstein’s inequality (e.g. see Theorem 1.6.1 in [Tro15]):

**Proposition 8** (Bernstein Inequality) Let \( S_1, S_2, \ldots, S_n \) be independent, centered, real random variables, and assume that each one is uniformly bounded:

\[
\mathbb{E}[S_k] = 0 \quad \text{and} \quad |S_k| \leq L \quad \text{for each} \quad k = 1, \ldots, n.
\]

Introduce the sum \( Z = \sum_{k=1}^{n} S_k \), and let \( \nu(Z) \) denote the variance of the sum:

\[
\nu(Z) = \mathbb{E}[Z^2] = \sum_{k=1}^{n} \mathbb{E}[S_k^2].
\]

Then

\[
\mathbb{P}[ |Z| \geq t ] \leq 2 \exp \left( -\frac{t^2/2}{\nu(Z) + Lt/3} \right) \quad \text{for all} \quad t \geq 0.
\]

We will apply it to bound the three sums in \( \mu_{kk} \), using the fact that each of the sums contains only centered, independent, and bounded variables, and that the variance of each entry in the sum is \( p_k(1-p_k) \).

For the first two sums, we can use \( t \sim \sqrt{n_k p_k (1-p_k) \log n_k} \) to obtain a combined failure probability (over the entire cluster) of \( O(n_k^{-\alpha}) \). Finally, for the third sum, we may choose \( t \sim n_k \sqrt{p_k (1-p_k) \log n_k} \), again for a combined failure probability over the whole cluster of no more than \( O(n_k^{-\alpha}) \).

We have thusly

\[
\mu_{kk} \leq \frac{1}{n_k} \sum_{l \in C_k} M_{lj} + \frac{1}{n_k} \sum_{l \in C_k} M_{il} + \frac{\lambda}{n_k} \sum_{l,l' \in C_k} M_{ll'} + \frac{\lambda}{n_k} \leq \sqrt{\frac{p_k (1-p_k)}{n_k} \log n_k} + \sqrt{\frac{p_k (1-p_k)}{n_k} \log n_k} + \frac{\lambda}{n_k},
\]

for all \( i,j \in C_k \), with probability \( 1 - O(n_k^{-\alpha}) \). Note that in the inequality above, the second term is much smaller in magnitude than the first, so we can disregard it; using (A.3), we obtain

\[
\mu_{kk} \lesssim \frac{1}{n_k} \left( \sqrt{n_k p_k (1-p_k) \log n_k} + \max_i \sqrt{p_i (1-p_i) n_i} + \sqrt{\max \{ q(1-q) n, \log n \}} \right), \quad (A.6)
\]
and by taking a union bound over $k$ we can conclude that the probability that any of these bounds fail is $o(1)$. Similarly, for $\mu_{k',k''}$, for $k' \neq k''$, we can calculate that

$$
\mu_{k',k''} \leq \frac{1}{n_{k'}} \sum_{l \in c_{k'}} M_{ij} + \frac{1}{n_{k'}} \sum_{l \in c_{k'}} M_{il} + \frac{1}{n_{k'}n_{k''}} \sum_{l' \in c_{k'},l'' \in c_{k''}} M_{l',l''} \lesssim \sqrt{q(1-q) \log n_{k'} n_{k''} + \sqrt{q(1-q) \log(n_{k'} n_{k''}) / n_{k'} n_{k''}}}
$$

with failure probability over all $i \in c_{k'}$, $j \in c_{k''}$ of no more than $O(n_{k'}^{-\alpha} n_{k''}^{-\alpha})$. We do this by taking $t \sim \sqrt{n_{k'} q(1-q) \log(n_{k'} n_{k''})}$, respectively $t \sim \sqrt{n_{k'} q(1-q) \log(n_{k'} n_{k''})}$ in the first two sums. For the third, we just take $t \sim \sqrt{n_{k'} n_{k''} q(1-q) \log(n_{k'} n_{k''})}$. As before, note that the second term is much smaller in magnitude than the first, and hence we can disregard it to obtain

$$
\mu_{k',k''} \lesssim \max_k \sqrt{q(1-q) \log n_k n_k} = \sqrt{q(1-q) \log n_{\min}} := \mu_{off},
$$

as the function $\log x/x$ is strictly increasing if $x \geq 3$, with the probability that all of the above are simultaneously true being $1 - o(1)$. Since the bound on $\mu_{k',k''}$ is independent of $k'$ and $k''$ we can rewrite (A.5) as

$$
\Delta(Y) \geq \sum_{k=1}^r (p_k - q) \| (Y^* - Y) c_k \|_1 - \sum_{k=1}^r \mu_{kk} \| (Y^* - Y) c_k \|_1 - \mu_{k',k''} \| (Y^* - Y) c_{k'} \|_1
$$

$$
\geq \sum_{k=1}^r (p_k - q - \mu_{kk} - \mu_{off}) \| (Y^* - Y) c_k \|_1
$$

where we use the fact that $\sum_{k' \neq k''} \| (Y^* - Y) c_{k'} \|_1 = \sum_{k=1}^r \| (Y^* - Y) c_k \|_1$. Finally, the conditions of theorem guarantee the nonnegativity of the right hand side, hence the optimality of $Y^*$ as the solution to the convex recovery program in (2.3).

### A.2 Proof of Theorem 2

We use a different result than Lemma 7, which we state below.

**Lemma 9 (Corollary 3.12 in [BvH14])** Let $X$ be an $n \times n$ symmetric matrix whose entries $X_{ij}$ are independent symmetric random variables. Then there exists for any $0 < \epsilon \leq \frac{1}{2}$ a universal constant $c_\epsilon$ such that for every $t \geq 0$

$$
\| X \| \leq 2(1 + \epsilon) \hat{\sigma} + t,
$$

with probability at least $1 - n \exp(-\frac{\epsilon^2}{\hat{\sigma}^2 \sigma^2})$, where

$$
\hat{\sigma} = \max_i \sqrt{\sum_j \mathbb{E}[X_{ij}^2]} , \quad \sigma_* = \max_{i,j} \| X_{ij} \|_\infty.
$$

We specialize Lemma 9 to HSBM to get the following result.

**Lemma 10** Let $A$ be generated according to the heterogeneous stochastic block model (HSBM). Then there exists for any $0 < \epsilon \leq \frac{1}{2}$ a universal constant $c_\epsilon$ such that

$$
\| A - \mathbb{E}(A) \| \leq 4(1 + \epsilon) \max \{ \sigma_{\max}, \sigma_0 \} + \sqrt{2c_\epsilon \log n}
$$

with probability at least $1 - n^{-1}$.

We can now present the proof for Theorem 2.

**Proof.** The proof follows the same lines as the proof of Theorem 1. Given the similarities between the proofs, we will only describe here the differences between the tools employed, and how they affect the conditions in
Theorem 2. The proof proceeds identically as before, up to the definition of $\lambda$, which—since we use Lemma 10 rather than 7—becomes

$$\lambda := C_2 \max \{\sigma_{\text{max}}, \sigma_0, \sqrt{\log n} \},$$

(A.9)

where $C_2$ was chosen as a good upper bounding constant for Lemma 10.

The other two small changes come from the fact that we will need to make sure that the failure probabilities for the quantities $\mu_{kk}$ and $\mu_{kk'}$ are polynomial in $1/n$, which leads to the replacement of $\log n_k$ in either of them by a $\log n$. The rest of the proof proceeds exactly in the same way.

B Proofs for Recoverability and Non-recoverability

B.1 Proofs for Recoverability

Proof. [of Theorem 3] For $\Delta(Y) := \langle A, Y^* - Y \rangle$, we have to show that for any feasible $Y \neq Y^*$, we have $\Delta(Y) > 0$. For simplicity we assume $Y_{ii} = Y^*_{ii} = 0$ for all $i \in \{1, \ldots, n\}$. Consider an splitting as

$$\Delta(Y) = \langle A, Y^* - Y \rangle = \langle E(A), Y^* - Y \rangle + \langle A - E(A), Y^* - Y \rangle.$$  

(B.1)

Notice that $Y^* = \sum_{k=1}^r 1_{c_k}$ and $E(A) = q11^T + \sum_{k=1}^r (p_k - q)1_{c_k}$. Considering $d_k(Y) = \langle Y^*_{c_k}, Y^* - Y \rangle$, the number of entries on $C_k$ on which $Y$ and $Y^*$ do not match, we get

$$\langle E(A), Y^* - Y \rangle = \sum_{k=1}^r (p_k - q)d_k(Y)$$

(B.2)

where we used the fact that $Y, Y^* \in Y$ and have the same number of ones and zeros, hence $\sum_{i,j} Y_{ij} = \sum_{i,j} Y^*_{ij}$. On the other hand, the second term in (B.1) can be represented as

$$T(Y) := \langle A - E(A), Y^* - Y \rangle = \sum_{Y_{ij}=1, Y_{ij}=0} (A - E(A))_{ij} + \sum_{Y^*_{ij}=0, Y_{ij}=1} (E(A) - A)_{ij}$$

where each term is a centered Bernoulli random variable bounded by 1. Observe that the total variance for all the summands in the above is given by

$$\sigma^2 = \sum_{k=1}^r d_k(Y)p_k(1 - p_k) + q(1 - q)\sum_{k=1}^r d_k(Y).$$

Then, combining (B.1) and (B.2), and applying the Bernstein inequality yields

$$\mathbb{P}(\Delta(Y) \leq 0) = \mathbb{P} \left( T(Y) \leq \frac{r^2}{2\sigma^2 + 2t/3} \right) \leq \exp \left( -\frac{t^2}{2\sigma^2 + 2t/3} \right) \leq \exp \left( -\sum_{k=1}^r (p_k - q)d_k(Y) \right)$$

where $t = \sum_k (p_k - q)d_k(Y)$ and

$$\nu(Y) = \frac{\sigma^2}{t} = \sum_{k=1}^r \frac{(p_k(1-p_k) + q(1-q))d_k(Y)}{\sum_{k}(p_k-q)d_k(Y)} \leq \max_k \frac{p_k(1-p_k) + q(1-q)}{p_k-q} = \frac{p_{\min}(1-p_{\min}) + q(1-q)}{p_{\min}-q} := \nu_0.$$

Considering $\tilde{\nu} := 2\nu_0 + 2/3$ and $\theta_k := \frac{p_k - q}{p_{\min} - q}$, we get

$$\mathbb{P}(\Delta(Y) \leq 0) \leq \exp \left( -\frac{1}{\tilde{\nu}} \sum_k (p_k - q)d_k(Y) \right) \leq \exp \left( -\frac{1}{\tilde{\nu}} (p_{\min} - q)\sum_k \theta_k d_k(Y) \right)$$

(B.3)

which can be bounded using the next lemma which is a direct extension of Lemma 4 in [CX14].
Lemma 11: Given the values of $\theta_k$ and $n_k$, for $k = 1, \ldots, r$, and for each integer value $\xi \in [\min \theta_k (2n_k - 1), \sum_k \theta_k n_k^2]$, we have

$$|\{Y \in \mathcal{Y} : \sum_{k=1}^r \theta_k d_k(Y) = \xi\}| \leq \left(\frac{4\xi}{\tau}\right)^2 n^{16\xi/\tau}$$

(B.4)

where $\tau := \min_k \theta_k n_k$, and $\mathcal{Y} = \{Y' \in \mathcal{Y} : Y_{ij}^* Y_{ij}^* = Y_{ij}^* Y_{ij}^*\}$.

Now plugging in the result of Lemma 11 into (B.3) yields,

$$\mathbb{P}\left(\exists Y \in \mathcal{Y} : Y \neq Y^*, \Delta(Y) \leq 0\right) \leq \sum_\xi \mathbb{P}\left(\exists Y \in \mathcal{Y} : \sum_k \theta_k d_k(Y) = \xi, \Delta(Y) \leq 0\right) \leq 2 \sum_\xi \left(\frac{4\xi}{\tau}\right)^2 n^{16\xi/\tau} \exp\left(-\frac{1}{\bar{\nu}}(p_{\min} - q)\xi\right)$$

$$= 32 \sum_\xi \left(\frac{\xi}{\tau}\right)^2 \exp\left((16 \log n - \frac{1}{\bar{\nu}}(p_{\min} - q)\tau)\frac{\xi}{\tau}\right)$$

$$\leq 32 \sum_\xi \left(\frac{\xi}{\tau}\right)^2 \exp\left((16 \log n - \frac{1}{\bar{\nu}}p_{\min})\frac{\xi}{\tau}\right)$$

(B.5)

In order to have a meaningful bound for the above probability, we need the exponential term in (B.5) to be decreasing. Hence, we require $p_{\min} \geq 64\bar{\nu} \log n$. Moreover, the function in (B.5) is a decreasing function of $\xi/\tau$ for $\tau \geq 4\bar{\nu} p_{\min}$, requiring the following condition (for some $\eta > 0$ which will be determined later),

$$p_{\min} \geq 2(16 + \eta)\bar{\nu} \log n + 4\bar{\nu}$$

(B.7)

implies

$$\frac{\xi}{\tau} \geq 1 \geq \frac{4}{4 + 2\eta \log n} \geq \frac{4\bar{\nu}}{p_{\min} - 32\bar{\nu} \log n}$$

and allows us to bound the summation in (B.5) with the largest term (corresponding to the smallest value of $\xi/\tau$, or an even smaller value, namely 1) times the number of summands; i.e.,

$$(B.5) \leq 32 \left(\sum_k \theta_k n_k^2\right) \exp\left(16 \log n - \frac{1}{\bar{\nu}}p_{\min}\right)$$

(B.8)

$$\leq 32 \sum_k \theta_k n_k^2 \exp(-2 - \eta \log n)$$

(B.9)

$$\leq 5 \theta_{\max} n^2 - \eta$$

(B.10)

$$\leq 5 \frac{p_{\max} - q}{p_{\min} - q} n^2 - \eta$$

(B.11)

or, similarly,

$$(B.5) \leq 32 \sum k \theta_k n_k^2 \exp(-2 - \eta \log n) \leq 5 \frac{\sum k=1^r \rho_k n^1 - \eta}{p_{\min} - q}$$

(B.12)

Hence, if the condition in (B.7) holds we get the optimality of $Y^*$ with a probability at least equal to the above. Finally, $n \geq 8$ implies log $n \geq 2$ and (B.7) can be insured by

$$p_{\min} \geq 4(17 + \eta) \left(\frac{1}{3} + \frac{p_{\min}(1 - p_{\min}) + q(1 - q)}{p_{\min} - q}\right) \log n.$$
Proof. [of Lemma 11] We extend the proof of Lemma 4 in [CX14] to our case. Fix a $Y \in \mathcal{Y}$ with $\sum_{k=1}^{r} \theta_k d_k(Y) = \xi$ and consider the corresponding $r$ clusters as well as the set of isolated nodes. Notice that for any $Y' \in [Y]$ we also have $\sum_{k=1}^{r} \theta_k d_k(Y') = \xi$. In the following, we will construct an ordering for the clusters of $Y$ according to $Y^*$. Denote the clusters of $Y^*$ by $V_1^*, \ldots, V_r^*$, and $V_{r+1}^*$.

Consider the set of values of cluster sizes $\{n_1, \ldots, n_r\} = \{\eta_1, \ldots, \eta_s\}$ where $\eta_1, \ldots, \eta_s$ are distinct, and define $\mathcal{I}_\ell = \{k : n_k = \eta_\ell\} \subseteq \{1, \ldots, r\}$ for $\ell = 1, \ldots, s$. For any $\eta_\ell$ of multiplicity 1 (i.e., $|\mathcal{I}_\ell| = 1$), the cluster in $Y \in \mathcal{Y}$ of size $\eta_\ell$ can be uniquely assigned to a cluster among $V_1^*, \ldots, V_r^*$ of similar size. We now define an ordering for the remaining clusters. Consider a $\eta_\ell$ of multiplicity larger than 1, and restrict the attention to clusters $V$ of size $\eta_\ell$ and clusters $V_k^*$ for $k \in \mathcal{I}_\ell$ (all clusters in $Y^*$ of size $\eta_\ell$). This is similar to the case in [CX14] where all sizes are equal: For each new cluster $V$ of size $\eta_\ell$, if there exists a $k \in \mathcal{I}_\ell$ such that $|V \cap V_k^*| > \frac{1}{2} \eta_\ell$ then we label this cluster as $V_k$; this label is unique. The remaining unlabeled clusters are labeled arbitrarily by a number in $\mathcal{I}_\ell$.

Hence, we labeled all the clusters of $Y$ according to the clusters of $Y^*$. For each $(k, k') \in \{1, \ldots, r\} \times \{1, \ldots, r + 1\}$, we use $\alpha_{kk'} := |V_k^* \cap V_{k'}^*|$ to denote the sizes of intersections of the true and new clusters. We observe that the new clusters $(V_1, \ldots, V_{r+1})$ have the following properties:

(A1) $(V_1, \ldots, V_{r+1})$ is a partition of $\{1, \ldots, n\}$ with $|V_k| = n_k$ for all $k = 1, \ldots, r$; since $Y \in \mathcal{Y}$.

(A2) For $\ell \in \{1, \ldots, s\}$ with $|\mathcal{I}_\ell| = 1$, we have $\alpha_{kk} = n_k$ for the index $k \in \mathcal{I}_\ell$.

(A3) For $\ell \in \{1, \ldots, s\}$ with $|\mathcal{I}_\ell| > 1$, consider any $k \in \mathcal{I}_\ell$. Then, exactly one of the following is true: (1) $\alpha_{kk} > \frac{1}{2} n_k$; (2) $\alpha_{kk'} \leq \frac{1}{2} n_k$ for all $k' \in \mathcal{I}_\ell$.

(A4) For $d_k(Y) = (Y_k^*, Y^* - Y)$, where $k = 1, \ldots, r$, we have

$$d_k(Y) = |\{(i, j) : (i, j) \in C_k^*, V_{ij} = 0\}|$$
$$= |\{(i, j) : (i, j) \in C_k^*, (i, j) \in C^r+1\}| + \sum_{k' \neq k''} |\{(i, j) : (i, j) \in C_k^*, (i, j) \in V_{k'} \times V_{k''}\}|$$
$$= \alpha_{k(r+1)}^2 + \sum_{k' \neq k''} \alpha_{kk'} \alpha_{kk''},$$

which implies

$$\xi = \sum_{k=1}^{r} \theta_k d_k(Y) = \sum_{k=1}^{r} \theta_k \alpha_{k(r+1)}^2 + \sum_{k=1}^{r} \sum_{k' \neq k''} \theta_k \alpha_{kk'} \alpha_{kk''}. $$

Unless specified otherwise, all the summations involving $k'$ or $k''$ are over the range $1, \ldots, r + 1$.

We showed that the ordered partition for a $Y \in \mathcal{Y}$ with $\sum_{k=1}^{r} \theta_k d_k(Y) = \xi$ satisfies the above properties. Therefore,

$$\left| \{Y \in \mathcal{Y} : \sum_{k=1}^{r} \theta_k d_k(Y) = \xi \} \right| \leq \left| \{(V_1, \ldots, V_{r+1}) satisfying the above conditions} \right|. $$

Next, we upper bound the right hand side of the above.

Fix an ordered clustering $(V_1, \ldots, V_{r+1})$ which satisfies the above conditions. Define,

$$m_1 := \sum_{k' \neq 1} \alpha_{1k'},$$

as the number of nodes in $V_1^*$ that are misclassified by $Y$; hence $m_1 + \alpha_{11} = n_1$. Consider the following two cases:

- if $\alpha_{11} > n_1/4$ we have

$$\sum_{k' \neq k''} \alpha_{1k'} \alpha_{kk''} \geq \alpha_{11} \sum_{k' \neq k''} \alpha_{1k''} > \frac{1}{4} n_1 m_1$$

- otherwise, we have

$$\sum_{k' \neq k''} \alpha_{1k'} \alpha_{kk''} \leq \frac{1}{4} n_1 m_1.$$
Theorem 4. Let $\mathbb{P}(Y^*, A)$ be the joint distribution of $Y^*$ and $A$, where $Y^*$ is sampled uniformly from $Y$ and $A$ is generated according to the heterogenous stochastic block model conditioning on $Y^*$. Note that

$$\inf_{\hat{Y}} \sup_{Y^* \in Y} \mathbb{P}[\hat{Y} \neq Y^*] \geq \inf_{\hat{Y}} \mathbb{P}(Y^*, A)[\hat{Y} \neq Y^*].$$

By Fano’s inequality we have,

$$\mathbb{P}(Y^*, A)[\hat{Y} \neq Y^*] \geq 1 - \frac{I(Y^*; A) + 1}{\log |\mathcal{Y}|},$$

(B.13)

where $I(X; Z)$ is the mutual information, and $H(X)$ is the Shannon entropy for $X$. By counting argument we find that $|\mathcal{Y}| = \binom{\tilde{n}}{n_1 \ldots n_r}$. Using $\sqrt{n(n/e)^n} \leq n! \leq e\sqrt{n(n/e)^n}$ and $\binom{\tilde{n}}{n_1 \ldots n_r} \geq (n/\tilde{n})^n$, it follows that

$$|\mathcal{Y}| \geq \frac{n^\tilde{n} \sqrt{n}}{e^r \sqrt{n_1 \ldots n_r n_1^\ldots n_r}}.$$

which gives

$$\log |\mathcal{Y}| \geq \sum_{i=1}^r n_i (\log \frac{n}{n_i} - \log \frac{n_i}{2n_i}) - r \geq \frac{1}{2} \sum_{i=1}^r n_i \log \frac{n}{n_i} - r.$$
On the other hand, note that $H(A) \leq \binom{n}{2} H(A_{12})$ by chain rule, the fact that $H(X|Y) \leq H(X)$, and the symmetry among identically distributed $A_{ij}$’s. Furthermore $A_{ij}$’s are conditionally independent and hence $H(A|Y^*) = \binom{n}{2} H(A_{12}|Y^*)$. Now it follows that

$$I(Y^*; A) = H(A) - H(A|Y^*) \leq \binom{n}{2} I(Y^*; A_{12}).$$

Observe that

$$\mathbb{P}(Y^*_{12} = 1, (1, 2) \in C_i) = \frac{\binom{n-2}{n_i} \binom{n-n_i}{n_r-n_i}}{\binom{n}{n_i}} \frac{n_i(n_i-1)}{n(n-1)} := \alpha_i.$$  

Using the properties of KL-divergence, we have $\mathbb{P}(A_{12} = 1) = \sum_{i=1}^{r} \alpha_i p_i + (1 - \sum_{i} \alpha_i) q := \beta$. Therefore,

$$I(Y^*_1, A_{12}) = \sum_{i=1}^{r} \alpha_i D_{KL}(p_i, \beta) + (1 - \sum_{i} \alpha_i) D_{KL}(q, \beta) = H(\beta) - \sum_{i} \alpha_i H(p_i) - (1 - \sum_{i} \alpha_i) H(q) \quad (B.14)$$

Since $I(Y^*; A) \leq \binom{n}{2} I(Y^*_1; A_{12})$, plugging in the following condition in Fano’s inequality (B.13),

$$\left( \frac{n}{2} \sum_{i} n_i \log \frac{n_i}{n_i} - r \right) \geq 2 + 2 \binom{n}{2} I(Y^*_1; A_{12}), \quad (B.15)$$

guarantees $\mathbb{P}(Y^*; A)(Y^* \neq Y^*) \geq \frac{1}{4}$. In the following, we bound $I(Y^*_1; A_{12})$ in two different ways to derive conditions 1 and 2 of Theorem 4. Throughout the proof we use the following inequality from [CX14] for the Kullback-Leibler divergence of Bernoulli variables,

$$D_{KL}(p, q) := D_{KL}(\text{Ber}(p), \text{Ber}(q)) = p \log \frac{p}{q} + (1-p) \log \frac{1-p}{1-q} \leq \frac{(p-q)^2}{q(1-q)}, \quad (B.16)$$

where the inequality is established by $\log x \leq x - 1$, for any $x \geq 0$.

- From (B.14), we have

$$I(Y^*_1, A_{12}) \leq \sum_{i=1}^{r} 4 \alpha_i (p_i - q)^2 q(1-q) \leq 4 \sum_{i=1}^{r} n_i^2 (p_i - q)^2 q(1-q) \quad (B.17)$$

where we assumed $\sum n_i^2 \leq \frac{1}{2} n^2$. Now, the right hand side of B.15 can be bounded as

$$2 \binom{n}{2} I(Y^*_1; A_{12}) \leq 4 \sum_{i=1}^{r} n_i^2 (p_i - q)^2 q(1-q) = 4 \sum_{i=1}^{r} n_i^2 D(p_i, q)$$

and gives the sufficient condition 1 of Theorem 4.

- Again from (B.14), we have

$$I(Y^*_1; A_{12}) = \sum_{i} \alpha_i \left( p_i \log \frac{p_i}{\beta} + (1-p_i) \log \frac{1-p_i}{1-\beta} \right) + (1 - \sum_{i} \alpha_i) D_{KL}(q, \beta) \leq \sum_{i} \alpha_i p_i \log \frac{1}{\alpha_i} + \log c + (1 - \sum_{i} \alpha_i) \frac{(q-\beta)^2}{\beta(1-\beta)}$$

where the first term is bounded via $\beta \geq \sum_{i} \alpha_i p_i \geq \alpha_i p_i$, the second term is bounded via $\beta \leq p_{\text{max}}$ and $c = (1-p_{\text{min}})/(1-p_{\text{max}})$, and we used (B.16) for the last term. Since $1 - \beta = 1 - q - \sum_{i} \alpha_i (p_i - q) \geq (1 - \sum_{i} \alpha_i)(1 - q)$, the last term can be bounded as

$$\left( 1 - \sum_{i} \alpha_i \right) \frac{(q-\beta)^2}{\beta(1-\beta)} \leq \left( 1 - \sum_{i} \alpha_i \right) \frac{(\sum_{i} \alpha_i (p_i - q))^2}{(\sum_{i} \alpha_i p_i)(1-\sum_{i} \alpha_i)(1-q)} \leq \sum_{i} \alpha_i (p_i - q) \leq \sum_{i} \alpha_i p_i.$$
This implies
\[
I(Y_{12}^*; A_{12}) \leq \sum_i \alpha_ip_i \log \frac{1}{\alpha_i} + \sum_i \alpha_ip_i + \log c \leq \sum_i \alpha_ip_i \log \frac{1}{\alpha_i} + \log c.
\] (B.18)

Since \(n_i \geq 2\), \(\alpha_i = \frac{n_i(n_i - 1)}{n(n - 1)} \geq \frac{1}{en}.\) Hence
\[
2 \binom{n}{2} I(Y_{12}^*; A_{12}) \leq n(n - 1) \sum_i \frac{n_in_i(n_i - 1)}{n(n - 1)} p_i \log \frac{e^2n^2}{n_i^2} + 2 \log c \leq 2 \sum_i n_i^2 p_i \log \frac{en}{n_i} + 2 \log c
\]

which gives the sufficient condition 2 of Theorem 4.

\[\]

Proof. [of case 3 in Theorem 4] Without loss of generality assume \(n_1 \leq n_2 \leq \ldots \leq n_r\). Let \(M := \bar{n} - n_{\text{min}} = \bar{n} - n_1\), and \(\bar{Y} := \{Y_0, Y_1, \ldots, Y_M\}\). \(Y_0\) is the clustering matrix with clusters \(\{C_i\}_{i=1}^{\ell}\) that correspond to \(V_1 = \{1, \ldots, n_1\}\), \(V_{\ell} = \{\sum_{i=1}^{\ell-1} n_i + 1, \ldots, \sum_{i=1}^{\ell} n_i\}\) for \(\ell = 2, \ldots, r\). Other members of \(\bar{Y}\) are given by swapping an element of \(U_{12}^*\) with an element of \(V_1\). Let \(P_i\) be the distributional law of the graph \(A\) conditioned on \(Y^* = Y_i\). Since \(P_i\) is product of \(\frac{1}{2n(n - 1)}\) Bernoulli random variables, we have
\[
I(Y^*; A) = E_Y [D_{KL}(P(A|Y), P(A))]
\]
\[
= \frac{1}{M+1} \sum_{i=0}^{M} D_{KL}(P_i, \sum_{j=0}^{M} P_j)
\]
\[
\leq \frac{1}{(M+1)^2} \sum_{i,j=0}^{M} D_{KL}(P_i, P_j)
\]
\[
\leq \max_{i,j=0,\ldots,M} D_{KL}(P_i, P_j)
\]
\[
\leq \max_{i,j,k=1,\ldots,r} \sum_{j=1}^{3} \left( \frac{n_i(p_i - q)^2}{q(1 - q)} + \frac{n_j(p_j - q)^2}{p_j(1 - p_j)} \right)
\]
\[
\leq 3 \max_{i=1,\ldots,r} \left( \frac{n_i(p_i - q)^2}{q(1 - q)} + \frac{n_i(p_i - q)^2}{p_i(1 - p_i)} \right)
\]
where the third line follows from the convexity of KL-divergence, and the line before the last follows from the construction of \(\bar{Y}\) and (B.16). Now if the condition of the theorem holds, then \(I(Y^*; A) \leq \frac{1}{4} \log(n - n_{\text{min}}) = \frac{1}{4} \log |\bar{Y}|\). Note that for \(n \geq 128\) we get \(\log |\bar{Y}| = \log(n - n_{\text{min}}) \geq \log(n/2) \geq 4\). The conclusion follows by Fano’s inequality in (B.13) restricting the supremum to be taken over \(\bar{Y}\).

\[\]

C Recovery by a Simple Counting Algorithm

In Section 2.1, we considered a tractable approach for exact recovery of (partially) observed models generated according to the heterogenous stochastic block model. However, in the interest of computational effort, one can further characterize a subset of models that are recoverable via a much simpler method than the convex program. The following algorithm is a proposal to do so. Moreover, the next theorem provides a characterization for models for which this simple thresholding algorithm is effective for exact recovery. Here, we allow for isolated nodes as described in Section 2.
Algorithm 1 Simple Thresholding Algorithm

1: (Find isolated notes) For each node \( v \), compute its degree \( d_v \). Declare \( v \) as isolated if
\[
d_v < \min_k \left( \frac{(n_k - 1)(p_k - q)}{2} \right) + (n - 1)q.
\]

2: (Find all communities) For every pair of nodes \((v, u)\), compute the number of common neighbors \( S_{vu} := \sum_{w \neq v, u} A_{vw}A_{uw} \). Declare \( v, u \) as in the same community if
\[
S_{vu} > nq^2 + \frac{1}{2} \left( \min_k \left( (n_k - 2)p_k^2 - n_kq^2 \right) + q \cdot \max_{i \neq j} (\rho_k - p_k + \rho_l - p_l) \right)
\]
where \( \rho_k = n_k(p_k - q) \).

Theorem 12 Under the stochastic block model, with probability at least \( 1 - 2n^{-1} \), the simple counting algorithm 1 find the isolated nodes provided
\[
\min_k (n_k - 1)^2(p_k - q)^2 \geq 19(1 - q) \left( \max_k n_kp_k + nq \right) \log n.
\]
(C.1)

Furthermore the algorithm finds the cluster if
\[
\left[ \min_k \left\{ (n_k - 2)p_k^2 + (n - n_k)q^2 \right\} - q \max_{k \neq l} \left\{ (n_k - 1)p_k + (n_l - 1)p_l + (n - n_k - n_l)q \right\} \right]^2 \geq 26(1 - q^2) \left( \max_k n_kp_k^2 + nq^2 \right) \log n,
\]
(C.2)

while the term inside the bracket (which is squared) is assumed to be non-negative.

We remark that the following is a slightly more restrictive condition than (C.2)
\[
\left[ \min_k n_k(p_k^2 - q^2) - 2q\max_k \right]^2 \geq 26(1 - q^2) \left[ nq^2 + \max_k n_kp_k^2 \right] \log n.
\]
(C.3)

with better interpretability.

Proof. [of Theorem 12] For node \( v \), let \( d_v \) denote its degree. Let \( \bar{V} = \bigcup_{i=1}^r V_i \) denote the set of nodes which belong to one of the clusters, and \( V_0 \) be isolated nodes. If \( v \in V_i \) for some \( i = 1, \ldots, r \), then \( d_v \) is distributed as a sum of independent binomial random variables \( \text{Bin}(n_i - 1, p_i) \) and \( \text{Bin}(n - n_i, q) \). If \( v \in V_0 \), then \( d_v \) is distributed as \( \text{Bin}(n - 1, q) \). Hence we have,
\[
\mathbb{E}[d_v] = \begin{cases} 
(n_i - 1)p_i + (n - n_i)q & v \in V_i \subset \bar{V} \\
(n - 1)q & v \in V_0,
\end{cases}
\]
and
\[
\text{Var}[d_v] = \begin{cases} 
(n_i - 1)p_i(1 - p_i) + (n - n_i)q(1 - q) & v \in V_i \subset \bar{V} \\
(n - 1)q(1 - q) & v \in V_0.
\end{cases}
\]

Let \( \kappa_0^2 := \max_i n_ip_i(1 - q) + nq(1 - q) \), and \( t = \min_i \frac{(n_i - 1)(p_i - q)}{2} \leq \kappa_0^2 / 2 \). Then \( \text{Var}[d_v] \leq \kappa_0^2 \) for any \( v \in V_0 \cup \bar{V} \).

By Bernstein’s inequality we get
\[
\Pr \left[ \left| d_v - \mathbb{E}[d_v] \right| > t \right] \leq 2 \exp \left( -\frac{t^2}{2\kappa_0^2 + 2t/3} \right) \leq 2 \exp \left( -\frac{3 \min_i (n_i - 1)^2(p_i - q)^2}{28\kappa_0^2} \right) \leq 2n^{-2},
\]
(C.4)

where the last inequality follows from the condition (C.1). Now by union bound over all nodes, with probability at least \( 1 - 2n^{-1} \), for node \( v \in V_i \subset \bar{V} \) we have,
\[
d_v \geq (n_i - 1)p_i + (n - n_i)q - t > \min_i \frac{(n_i - 1)(p_i - q)}{2} + (n - 1)q,
\]
(C.5)
and for node $v \in V_0$,
\[
d_v \leq (n - 1)q(1 - q) + t < \min_i \frac{(n_i - 1)(p_i - q)}{2} + (n - 1)q. \tag{C.6}
\]
This proves the first statement of the theorem, and all the isolated nodes are correctly identified. For the second statement, let $S_{vu}$ denote the common neighbor for nodes $v, u \in \bar{V}$. Then
\[
S_{vu} \sim_d \begin{cases} 
  \text{Bin}(n_i - 2, p_i^2) + \text{Bin}(n - n_i, q^2) & (v, u) \in V_i \times V_i \\
  \text{Bin}(n_i - 1, p_i q) + \text{Bin}(n_j - 1, p_j q) + \text{Bin}(n - n_i - n_j, q^2) & (v, u) \in V_i \times V_j, \ i \neq j
\end{cases}
\]
where $\sim_d$ denotes equality in distribution and $+$ denotes the summation of independent random variables. Hence
\[
E[S_{vu}] = \begin{cases} 
  (n_i - 2)p_i^2 + (n - n_i)q^2 & (v, u) \in V_i \times V_i \\
  (n_i - 1)p_i q + (n_j - 1)p_j q + (n - n_i - n_j)q^2 & (v, u) \in V_i \times V_j, \ i \neq j
\end{cases}
\]
and
\[
\text{Var}[S_{vu}] = \begin{cases} 
  (n_i - 2)p_i^2(1 - p_i^2) + (n - n_i)q^2(1 - q^2) & (v, u) \in V_i \times V_i \\
  (n_i - 1)p_i q(1 - p_i q) + (n_j - 1)p_j q(1 - p_j q) + (n - n_i - n_j)q^2(1 - q^2) & (v, u) \in V_i \times V_j, \ i \neq j
\end{cases}
\]
Let
\[
\Delta = \min_i \left( (n_i - 2)p_i^2 + (n - n_i)q^2 \right) - \max_j \left( 2(n_j - 1)p_j q + (n - 2n_j)q^2 \right)
\]
\[
= \min_i \left( (n_i - 2)p_i^2 - n_i q^2 \right) - \max_j \left( 2(n_j - 1)p_j q - 2n_j q^2 \right),
\]
Let $\kappa_i^2 := 2 \max_i n_i p_i^2(1 - p_i^2) + n_i q^2(1 - q^2)$. Then $\text{Var}[S_{vu}] \leq \kappa_i^2$ for all $v, u$. Then $\Delta \leq \kappa_i^2/2$. Bernstein’s inequality with $t = \Delta/2$ yields
\[
\mathbb{P}[|S_{vu} - E[S_{vu}]| > t] \leq 2 \exp \left( -\frac{t^2}{2\kappa_i^2 + 2t/3} \right) \leq 2 \exp \left( -\frac{3\Delta^2}{26\kappa_i^2} \right) \leq 2n^{-3}, \tag{C.7}
\]
where the last line follows from assumption (C.2). By union bound over all pair of nodes $(v, u)$, we get with probability at least $1 - 2n^{-1}$, $S_{vu} > \Gamma$ for all $v, u$ in the same cluster and $S_{vu} < \Gamma$ otherwise. Here
\[
\Gamma := \frac{1}{2} \left( \min_i \left( (n_i - 2)p_i^2 + (n - n_i)q^2 \right) + \max_{i \neq j} \left( (n_i - 1)p_i q + (n_j - 1)p_j q + (n - n_i - n_j)q^2 \right) \right).
\]

\section{Detailed Computations for Examples in Section 1.3}

In the following, we present the detailed computations for the examples in Section 1.3 and summarized in Table 1. When there is no impact on the final result, quantities are approximated as denoted by \( \approx \).

First, we repeat the conditions of Theorems 1 and 2. The conditions of Theorem 1 can be equivalently stated as
\begin{itemize}
  \item $p_k^2 \geq n_k p_k (1 - p_k) \log n_k = \sigma_k^2 \log n_k$
  \item $(p_{\text{min}} - q)^2 \geq q(1 - q) \frac{\log n_{\text{min}}}{n_{\text{min}}}$
  \item $p_{\text{min}}^2 \geq \max \{ \log n, nq(1 - q), \max_k n_k p_k (1 - p_k) \}$
  \item $\sum_{k=1}^r n_k^{-\alpha} = O(1)$ for some $\alpha > 0$.
\end{itemize}
Notice that $n_k p_k (1 - p_k) \geq \log n_k$, for $k = 1, \ldots, r$, is implied by the first condition, as mentioned in Remark 1. The conditions of Theorem 2 can be equivalently stated as
\begin{itemize}
\item \( \rho_k^2 \gtrsim n_k p_k (1 - p_k) \log n \)
\item \((p_{\min} - q)^2 \gtrsim q(1 - q)^{\frac{\log n}{p_{\min}}} \)
\item \( \rho_{\min}^2 \gtrsim \max \{ nq(1 - q), \max_k \ n_k p_k (1 - p_k) \} \).
\end{itemize}

**Remark 2** Provided that both \( p_k \) and \( q/p_k \) are bounded away from 1, we have
\[
\tilde{D}(q, p_k) = p_k \frac{(1 - q/p_k)^2}{1 - p_k} \approx p_k \quad \text{and} \quad \frac{p_k^2}{\sigma_k^2} = \frac{(1 - q/p_k)^2}{1 - p_k} n_k p_k \approx n_k p_k.
\]

This simplifies the first condition of Theorem 1 to a simple connectivity requirement. Hence, we can rewrite the conditions of Theorems 1, 2 as

1: \( n_k p_k \gtrsim \log n \), \( \tilde{D}(p_{\min}, q) \gtrsim \frac{\log n_{\min}}{n_{\min}} \), \( \rho_{\min}^2 \gtrsim \max \{ \sigma_{\max}^2, nq(1 - q), \log n \} \), \( \sum_{k=1}^{r} n_k^{-\alpha} = o(1) \) for some \( \alpha > 0 \)

2: \( n_k p_k \gtrsim \log n \), \( \tilde{D}(p_{\min}, q) \gtrsim \frac{\log n}{n_{\min}} \), \( \rho_{\min}^2 \gtrsim \max \{ \sigma_{\max}^2, nq(1 - q) \} \).

**Example 1:** In a configuration with two communities \( (n - \sqrt{n}, n^{-2/3}, 1) \) and \( (\sqrt{n}, \frac{1}{\log n}, 1) \) with \( q = n^{-2/3-0.01} \), we have \( n_{\min} = \sqrt{n} \) and \( p_{\min} = n^{-2/3} \). We have,
\[
\tilde{D}(p_{\min}, q) \approx n^{-2/3+0.01}
\]
which does not exceed either \( \frac{\log n_{\min}}{n_{\min}} \approx \log n \) or \( \frac{\log n}{n_{\min}} \approx \sqrt{n} \), and we get no recovery guarantee from Theorems 1 and 2 respectively. However, as \( p_{\min} - q \) is not much smaller than \( q \), while \( \rho_{\min} \approx n^{1/3} \) grows much faster than \( \log n \), the condition of Theorem 3 trivially holds.

Here are the related quantities for this configuration:
\[
\rho_1 = n_1 (p_1 - q) = (n - \sqrt{n}) (n^{-2/3} - n^{-2/3-0.01}) \approx n^{1/3}, \quad \rho_2 = n_2 (p_2 - q) = \sqrt{n} (\frac{1}{\log n} - n^{-2/3-0.01}) \approx \frac{\sqrt{n}}{\log n}
\]
which gives \( \rho_{\min} \approx n^{1/3} \). Furthermore,
\[
\sigma_1^2 = n_1 p_1 (1 - p_1) \approx n^{1/3}, \quad \sigma_2^2 = n_2 p_2 (1 - p_2) = \frac{\sqrt{n}}{\log n},
\]
which gives \( \sigma_{\max} = \frac{\sqrt{n}}{\log n} \). On the other hand \( nq(1 - q) \approx n^{1/3-0.01} \) which is smaller than \( \sigma_{\max}^2 \).

**Example 2:** Consider a configuration with \( (n - n^{2/3}, n^{-1/3+\epsilon}, 1) \) and \( (\sqrt{n}, \frac{c}{\log n}, n^{1/6}) \) and \( q = n^{-2/3+3\epsilon} \). Since all \( p_k \)'s and \( q/p_k \)'s are much less than 1, the first condition of both Theorems 1 and 2 can be verified by Remark 2. Moreover, \( n_{\min} = \sqrt{n} \) and \( p_{\min} = n^{-1/3+\epsilon} \) which gives
\[
\tilde{D}(p_{\min}, q) = n^{-\epsilon}
\]
and verifies \( \tilde{D}(p_{\min}, q) \gtrsim \frac{\log n_{\min}}{n_{\min}} \) for 1, as well as \( \tilde{D}(p_{\min}, q) \gtrsim \frac{\log n}{n_{\min}} \) for 2. Moreover, \( \rho_1 \approx n^{2/3+\epsilon} \) and \( \rho_2 \approx \frac{\sqrt{n}}{\log n} \) which gives \( \rho_{\min} \approx \frac{\sqrt{n}}{\log n} \gtrsim \sqrt{\log n} \). On the other hand, \( \sigma_1^2 \approx n^{2/3+\epsilon} \) and \( \sigma_2^2 \approx \frac{\sqrt{n}}{\log n} \) which gives
\[
\max \{ \sigma_{\max}^2, nq(1 - q) \} \approx n^{2/3+\epsilon}.
\]
Thus all conditions of Theorems 1 and 2 are satisfied. Moreover, as \( p_{\min} - q \) is not much smaller than \( q \), while \( \rho_{\min} \approx \frac{\sqrt{n}}{\log n} \) is growing much faster than \( \log n \), the condition of Theorem 3 trivially holds.
Example 3: Consider a configuration with \((\sqrt{\log n}, O(1), m)\) and \((n_2, O(\log n), \sqrt{n})\) and \(q = O(\log n/n)\), where \(n_2 = \sqrt{n} - m\sqrt{\log n/n}\). Here, we assume \(m \leq n/(2\sqrt{\log n})\) which implies \(n_2 \geq \sqrt{n}/2\). Since all \(p_k\)'s and \(q/p_k\)'s are much less than 1, we can use Remark 2: the first condition of Theorem 1 holds as \(n_1p_1 \approx \sqrt{\log n} \geq \log n_1 \approx \log\log n \geq \log n_2\). However, \(n_1p_1 \approx \sqrt{\log n} \geq \log n\) and Theorem 2 does not offer a guarantee for this configuration.

Moreover, \(n_{min} = \sqrt{\log n}\) and \(p_{min} = O(\frac{\log n}{\sqrt{n}})\) which gives

\[
\tilde{D}(p_{min}, q) = \log n
\]

and verifies \(\tilde{D}(p_{min}, q) \geq \frac{\log n}{n_{min}} \approx \frac{\log n}{\sqrt{\log n}}\) for 1, as well as \(\tilde{D}(p_{min}, q) \geq \frac{\log n}{n_{min}} = \sqrt{\log n}\) for 2. Moreover, \(\sigma_1^2 = \log n\) (also \(\rho_1\)) and \(\sigma_2^2 = \log n\) (also \(\rho_2\)) which gives

\[
\max\{\sigma_{max}^2, nq(1-q)\} \approx \log n
\]

and \(\rho_{min}^2 \approx \log n\). For the last condition of Theorem 1 we need

\[
m(\log n)^{-\alpha/2} + \sqrt{n}(\sqrt{n} - k\sqrt{\frac{\log n}{n}})^{-\alpha} = o(1)
\]

for some \(\alpha > 0\) which can be guaranteed provided that \(m\) grows at most polylogarithmically in \(n\). All in all, we verified the conditions of Theorem 1 while the first condition of 2 fails. Observe that \(\rho_{min}\) fails the condition of Theorem 3.

Alternatively, consider a configuration with \((\sqrt{\log n}, O(1), m)\) and \((\sqrt{n}, O(\frac{\log n}{\sqrt{n}}), m')\) and \(q = O(\frac{\log n}{n})\), where \(m' = \sqrt{n} - m\sqrt{\log n/n}\) to ensure a total of \(n\) vertices. Here, we assume \(m = n/(2\sqrt{\log n})\) which implies \(m' \geq \sqrt{n}/2\). Similarly, all conditions of Theorem 1 can be verified provided that \(m\) grows at most polylogarithmically in \(n\). Moreover, the conditions of Theorems 2 and 3 fail to satisfy.

Example 4: Consider a configuration with \((\frac{1}{2}n^e, O(1), n^{-1})\) and \((\frac{1}{2}n, n^{-\alpha} \log n, 1)\) and \(q = n^{-\beta} \log n\), where \(0 < \alpha < \beta < 1\) and \(0 < \epsilon < 1\).

We have \(\rho_1 \approx n^\epsilon\) and \(\rho_2 \approx n^{1-\alpha} \log n\). Since \(\rho_{min}^2 \geq \log n\), the last condition of Theorem 1 holds, and \(\log n_{min} \approx \log n\), we need to check for similar conditions to be able to use Theorems 1 and 2. Using Remark 2, the first condition of both Theorems holds because of \(n_1p_1 \approx n^\epsilon \geq \log n\) and \(n_2p_2 \approx n^{1-\alpha} \log n \geq \log n\). Moreover, the condition

\[
\tilde{D}(p_{min}, q) \approx n^{\beta-2\alpha} \log n \geq \frac{\log n}{n_{min}} \approx \frac{\log n}{n^2}
\]

is equivalent to \(\beta + \epsilon > 2\alpha\). Furthermore, \(\sigma_1^2 = n^\epsilon\) and \(\sigma_2^2 = n^{1-\alpha} \log n\), and for the last condition we need

\[
\min\{n^{2\epsilon}, n^{2-2\alpha} \log^2 n\} \geq \max\{n^\epsilon, n^{1-\alpha} \log n, n^{1-\beta} \log n\}
\]

which is equivalent to \(2\epsilon + \alpha > 1\) and \(\epsilon + 2\alpha < 2\). Notice that \(\beta + 1 > 2\alpha\) is automatically satisfied when we have \(\beta + \epsilon > 2\alpha\) from the previous part.

Example 5: Consider a configuration with \((\log n, O(1), \frac{n}{\log n} - m\sqrt{\frac{n}{\log n}})\) and \((\sqrt{n} \log n, O(\sqrt{\frac{\log n}{n}}), m)\) and \(q = O(\frac{\log n}{n})\). All of \(\rho_1, \rho_2, \sigma_1^2, \sigma_2^2,\) and \(nq(1-q)\), are approximately equal to \(\log n\). Thus, the first and third conditions of Theorems 1 and 2 are satisfied. Moreover,

\[
\tilde{D}(p_{min}, q) \approx 1 \geq \frac{\log n_{min}}{n_{min}} \approx \frac{\log n}{\log n}
\]

which establishes the conditions of Theorem 2. On the other hand, the last condition of Theorem 1 is not satisfied as one cannot find a constant value \(\alpha > 0\) for which

\[
\sum_{k=1}^{r} n_k^\alpha = \left(\frac{n}{\log n} - m\sqrt{\frac{n}{\log n}}\right) \log^{-\alpha} n + m(n \log n)^{-\alpha/2}
\]

is \(o(1)\) while \(n\) grows.
Example 6: For the first configuration, Theorem 1 requires \( f^2(n) \gtrsim \max\{ \frac{\log n}{n_1}, \frac{\log n_{\text{min}}}{n_{\text{min}}}, \frac{n}{n_1} \} \) while Theorem 2 requires \( f^2(n) \gtrsim \max\{ \frac{\log n_1}{n_{\text{min}}}, \frac{\log n}{n_{\text{min}}}, \frac{n}{n_1} \} \) and both require \( n_{\text{min}} \gtrsim \sqrt{n} \). Therefore, both set of requirements can be written as

\[
f^2(n) \gtrsim \max\{ \frac{\log n}{n_{\text{min}}}, \frac{n}{n_1} \} , \quad n_{\text{min}} \gtrsim \sqrt{n} .
\]