Uncertainties in the heliosheath ion temperatures

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Abstract. The Voyager plasma observations show that the physics of the heliosheath is rather complex, and especially that the temperature derived from observation differs from expectations. To explain this fact the temperature in the heliosheath should be based on $\kappa$ distributions instead of Maxwellians because the former allows for much higher temperature. Here we show an easy way to calculate the $\kappa$ temperatures when those estimated from the data are given as Maxwellian temperatures. We use the moments of the Maxwellian and $\kappa$ distributions to estimate the $\kappa$ temperature. Moreover, we show that the pressure (temperature) given by a truncated $\kappa$ distribution is similar to that given by a Maxwellian and only starts to increase for higher truncation velocities. We deduce a simple formula to convert the Maxwellian to $\kappa$ pressure or temperature. We apply this result to the Voyager-2 observations in the heliosheath.

1 Introduction

Knowledge about the temperature of an astrophysical plasma is of significance for the correct hydrodynamical treatment of various plasma processes like, e.g., heat conduction, wave propagation, compression or charge exchange. The temperature of a space plasma is, of course, not directly measurable but must be derived indirectly. If the velocity distribution function of a plasma constituent is known, the temperature can be computed as a second velocity moment. If not, assumptions have to be made about this distribution function. Consequently, an uncertainty regarding the latter will translate into an uncertainty regarding the temperature. For simplicity, unmeasured velocity distributions are often assumed to be Maxwellians. One must be aware that the derived ‘Maxwellian temperature’ is an assumption and might be rather different from the actual thermodynamically relevant temperature of the considered plasma (e.g., Fahr and Siewert [2013], Nicolaou and Livadiotis [2016]).

One such example is the plasma in the inner heliosheath, i.e. the region of the heliosphere between the solar wind termination shock and the heliopause. The temperatures for the inner heliosheath can, in difference to those determined for the upstream solar wind (Bridge et al. [1977], only be derived from Voyager 2 measurements under the Maxwellian assumption (e.g., Richardson and Wang [2012]). From the modeling of fluxes of energetic neutral atoms (ENAs) it is known, however, that the distribution function of protons in this region is probably not a Maxwellian but a so-called $\kappa$-distribution with the parameter $\kappa < 2$ (e.g., Heerikhuisen et al. [2008]). Consequently, the temperature of the proton population should be computed from these non-Maxwellian distributions. In the present paper we quantify these alternative $\kappa$-temperatures and discuss their significant difference to the Maxwellian ones. Before both temperatures are discussed in detail, we briefly review the $\kappa$-distributions and the physical reason for their importance downstream of the termination shock.

2 $\kappa$-distributions

The $\kappa$-distributions are an often used tool for the quantitative treatment of non-Maxwellian plasmas as is described in the reviews by Pierrard and Lazar (2010) and Livadiotis and McComas (2013). In the following, we assume that the actual distribution function is isotropic in the bulk velocity frame, because the protons undergo efficient pitch angle scattering at Alfvén waves (MHD wave turbulences) leading to this form of isotropy. Furthermore, without the capabilities to measure 2D or 3D distributions and implicitly
their anisotropy, the observations always integrated over all pitch-angles, and thus isotropic. Thus, the definition of the isotropic $\kappa$-distribution reads:

$$f_\kappa(v) = n(r) \frac{\Gamma(\kappa + 1)}{\left(\sqrt{\pi} \sqrt{\kappa \Theta}\right)^{3} \Gamma \left(\kappa - \frac{1}{2}\right)} \left(1 + \frac{v^2}{\kappa \Theta^2}\right)^{-\left(\kappa + 1\right)}$$  \hspace{1cm} (1)

where $\Theta$ is a fitting parameter, which normalises the speed, often it is identified with a thermal speed. In the limit $\kappa \rightarrow \infty$ these distributions converge to the Maxwellian distribution:

$$f_m(v) = \frac{n_0(r)}{2\pi k T} \sqrt{\frac{m}{\pi \kappa \Theta}} \exp\left(-\frac{mv^2}{2kT}\right)$$  \hspace{1cm} (2)

$$= \frac{n_0(r)}{\left(\pi v_p^2\right)^{3/2}} \exp\left(-\frac{v^2}{v_p^2}\right)$$

where the fitting parameter is usually the ‘thermal speed’ $v_p \equiv \sqrt{2kT/m}$, with the temperature $T$, the proton mass $m$ and the Boltzmann constant $k$.

Both distribution functions are normalized to the number density $n(r)$ so that

$$\int f_{m,\kappa} d^3 v = 4\pi \int_0^\infty v^2 f_{m,\kappa} dv = n(r)$$  \hspace{1cm} (3)

Furthermore, for the following, we normalize all speeds $(v, \Theta)$ to the solar wind speed $u_{sw} = 436 \text{ km/s}$ which corresponds to a proton energy of 1 keV.

In order to fit the thermal core of the Maxwellian, it is required that the speed $\Theta$ is equal to the thermal speed $\Theta = v_p$, as shown in appendix A. The latter relation corresponds to the preferable of two alternatives to interpret $\kappa$-temperatures, as is discussed in detail in Lazar et al. (2015) and Lazar et al. (2016).

### 2.1 Pick-up proton induced $\kappa$-distributions

Due to the different velocity-space processes acting upon solar wind protons when moving out from 1 AU to great radial solar distances, the resulting distribution function is far from being an equilibrium distribution in the form of a shifted Maxwellian. This is theoretically evident mainly from the fact that the solar wind proton plasma is permanently loaded with newly injected pick-up protons. This, in connection with wave-driven diffusion processes, keeps the resulting ion distribution function off a Maxwellian equilibrium shape. The transport of such pick-up ions was a subject of many investigations in the past, see Fahr (1973), Holzer and Leer (1973), and Vasyliunas and Siscoe (1976) with an ever increasing sophistication in treating the exact form of this pick-up ion incorporation process (Isenberg, 1995; Fichtner et al., 1996; Schwadron et al., 1996; Chalov and Fahr, 1998; Pogorelov et al., 2016). Also from in-situ observations it had been clearly recognized that pick-up protons show a typical core-distribution below the velocity injection border and an extended power-law tail above (Gloeckler, 2003; Fisk et al., 2010; Hill et al., 2009; McComas et al., 2015b). This then justified the theoretical endeavor of Fahr et al. (2014) to treat the evolving joint solar wind ion distribution as a $\kappa$-distribution with a $\kappa$-parameter evolving with radial distance. These authors could show with the help of an adequate pressure-moment transport equation that the resulting ion kappa-distribution evolves into a highly non-thermal distribution with $\kappa \leq 2.0$ by the time the solar wind plasma arrives at the termination shock.

From this finding it became evident that the passage of such a non-equilibrium proton distribution over the solar wind termination shock would generate a distribution downstream of the shock with strong non-equilibrium signatures. There are good reasons given by the Liouville theorem, see Siewert and Fahr (2007) that this downstream distribution function will also be as a kappa function obeying the relation

$$f_2(v) = s \cdot f_1\left(\frac{v}{\sqrt{s}}\right)$$  \hspace{1cm} (4)

where the indices ”1” and ”2” indicate upstream and downstream quantities, respectively, and where $s$ denotes the shock compression ratio. This then shows that downstream of the shock the following kappa function has to be expected

$$f_2(v) = As \left(1 + \frac{v^2}{\kappa_1 s \Theta_1^2}\right)^{-\left(\kappa_1 + 1\right)}$$  \hspace{1cm} (5)

This expresses the fact that downstream one must expect a kappa distribution with $\kappa$ identical to its upstream value $\kappa_1$, but with an increased core width $\Theta_2 = s \Theta_1$.

The above finding raises the question how in-situ low-energy ion measurements, e.g. by Voyager-2, should be interpreted in terms of velocity-moments of the distribution function. Solar wind ion data downstream of the shock are, in general, not displayed as spectral flux ion data, but as plasma parameters ($n$, flow speed, and $T$) derived from a fit of the measured fluxes to a convected isotropic Maxwellian distribution, see e.g. Richardson and Wang (2012). The question arises on the basis of whether the underlying distribution function is a Maxwellian and if not how the actual plasma parameters differ from those derived using a Maxwellian fit. A recently published paper by Nicolaou and Livadiotis (2016) demonstrated, how different values of the velocity moments can be, if based on Maxwellian or on $\kappa$-distributions. In the following part of the paper we shall thus focus on the specific question, how Voyager-2 heliosheath plasma parameters displayed in the literature change values when interpreted on the basis of non-thermal ion kappa distributions.

### 3 Maxwellian temperatures along the Voyager-2 trajectory

Richardson and Wang (2012) have shown that during the years 2008 to 2011 (denoted by $t_8$ and $t_{11}$ in the following
ions, originating from the cold LISM H-atoms (where \(U_0\) is the bulk speed of the proton fluid in the heliosheath). This equation evidently can be simplified to

\[
\frac{1}{T} \frac{d}{dt} (T) = -n_H \sigma_{ex} v_{rel} \tag{7}
\]

and with \(v_{rel} \approx \sqrt{8\pi kT/m}\) leads to

\[
\frac{1}{T^{3/2}} \frac{d}{dt} (T) = -n_H \sigma_{ex} \sqrt{8\pi k/m} \tag{8}
\]

yielding with \(\sigma_{ex} = const\)

\[
\frac{1}{\sqrt{T}} - \frac{1}{\sqrt{T_8}} = n_H \sigma_{ex} \sqrt{\frac{2\pi k}{m}} (\tau - \tau_8) \tag{9}
\]

where \(\tau_8 = t_8\) = Jan. 2008, the date when Voyager-2 started moving within the heliosheath. Adopting a time period \((\tau_{11} - \tau_8) \approx (t_{11} - t_8) = 9.6 \cdot 10^7 s\) we find

\[
\sqrt{T_{11}/T_8} = \frac{1}{1 + n_H \sigma_{ex} \sqrt{2\pi kT/m} (t_{11} - t_8)} = 0.119 \tag{10}
\]

assuming \(\sqrt{2\pi kT/m} = 100\) km/s and \(n_H \sigma \approx 10^{-16} cm^{-1}\). This yields \(T_{11} = T_8/1.119^2 = 988630\) K. This estimate has to be compared with temperatures of about 40000 K were measured by Voyager-2 in 2011 and indicates that an essential part of the ion cooling may be ascribed to ongoing charge exchange reactions removing high energy ions and replacing them by low energetic ones.

Because the heliosheath plasma is convected along streamlines which originate at different points on the termination shock, it must be kept in mind, that Voyager-2 crosses different streamlines (see Fig. 2), when moving through the inner heliosheath. On these different streamlines the solar wind evolves differently, as has been quantified in [Fahr et al. 2016]. Furthermore, in that paper we assumed that the locally prevailing temperature is a \(\kappa\)-temperature \(T_\kappa\) resulting from an incompressible plasma flow

\[
T_\kappa(s) = \frac{m}{2K} \Theta^2(s) \frac{\kappa(s)}{\kappa(s) - 3/2} \tag{11}
\]

In the following, after discussing the variation of \(\kappa\) and \(\kappa\)-temperature along the Voyager-2 trajectory, we derive how the Maxwellian temperatures can be translated in \(\kappa\)-temperatures.

4 Heliosheath plasma along the Voyager-2 trajectory

In Fahr et al. [2016] we have developed a numerical procedure to calculate the ion pressure evolution along arbitrarily selected flow lines of the heliosheath plasma flow on the basis of a pressure transport equation in which it was assumed that at each location in the plasma flow the underlying distribution function can be approximated by a \(\kappa\)-distribution.

Figure 1. Daily (points) and 11 day running averages (lines) of the radial speed, density, and temperature observed at V2. Taken from Richardson and Wang (2012).

In this volume a proper time \(\tau\) is counted (i.e. the time of the comoving clock), and we want to describe the temperature change within the box as function of \(\tau\). For an adequate estimate we use a simple thermodynamic transport equation describing the proton temperature as the result of charge-exchange-related energy exchanges between the plasma and the neutral gas in the following form:

\[
\frac{d}{d\tau} (n_p kT) = -n_p n_H \sigma_{ex} v_{rel} \cdot (kT - kT_0) \tag{6}
\]

with \(n_p \approx const\) (see the assumption of incompressible heliosheath plasma flow as made in Fahr et al. [2016]), \(n_H \approx const\), \(kT \gg kT_0\), and \(T(\tau) \sim O(10^3 K)\) (see Richardson and Wang, 2012) denoting the actual, local ion temperature, \(T_0\) denoting the thermalized heliosheath pick-up ion temperature downstream of the shock with \(T_0 \approx (3/2 K) mU_0^2 = O(10^3 K)\) denoting the temperature of the newly injected
Some streamlines originating at different points at the termination shock are shown with different colors. The Voyager-2 trajectory is the black straight line, crossing the various streamlines on which the solar wind has evolved differently [Fahr et al. 2016].

$k$-evolution for three different momentum diffusion coefficients: $D_0 = 10^{-10}, 10^{-9}, 10^{-8}$ (red green and blue curve respectively) along the Voyager-2 trajectory. See Fahr et al. (2016) for details.

The $k$-temperature derived from Eq. (9) of Fahr et al. (2016), using the results shown in Fig. 3 as shown in Fig. 4.

Sketch of a realistic situation for fitting the data (histogram, i.e. currents in the instrument cups). The black line shows a shifted Maxwellian and three shifted $k$-distribution with $k = 1.6$ (red), $k = 3$ (blue), and $k = 10$ (green). The number on the x-axis are the channel numbers which can be translated to energy or velocity. The y-axis is an arbitrary units. The distribution functions are multiplied with the velocity, which is approximately a representation of the currents, see Bridge et al. (1977).

$f(s,v) = f_k(s,v)$. This method allowed us to estimate the evolution of the $k$-parameter along each streamline.

In Fig. 3 we show, how the $k$-parameter varies, in turn, along the Voyager-2 trajectory (displayed as heliocentric distance in AU). Depending on the magnitude of the velocity diffusion coefficient $D_0$ (Fahr et al. 2016), one can see along the Voyager-2 trajectory different $k$-values (see green, red, or blue curves). The more efficient the velocity diffusion process is, the lower are the resulting $k$-values at Voyager-2. Directly connected with these $k$-values are the associated $k$-temperatures $T_k$ along the Voyager-2 trajectory (see Eq. (11)) as shown in Fig. 4.

$k$-temperatures along the Voyager-2 trajectory

We use the standard coordinate system for distribution functions, namely the space coordinates in the solar frame and the velocity coordinates in the rest frame (bulk frame) of the fluid. We assume isotropic distributions as defined in section 1.1 above in each case.

The data fitting procedure

A problem now arises unavoidably when using observations, like those from the plasma instrument on-board of Voyager-2 [Richardson 2008] where one has only a few data points in a limited energy range to fit a distribution function.

The physical parameters, like density, temperature and velocity are obtained from the currents of the different cups in the PLS-instrument on-board of Voyager-2. These currents correspond to a kinetic energy (momentum) from which
The truncated moments $M_{\kappa,x}^0$ (upper panel) and $M_{\kappa,x}^2$ (lower panel) for different $\kappa$'s. The logarithmic x-axis displays the speed normalized to 100 km/s and the y-axis is in arbitrary units. See text for details. The dotted vertical line indicates a speed of approximately 200 km/s, which is about the solar wind speed in the heliosheath. While a speed about 30 units corresponds to that of protons as measured in the lowest energy channel of the LECP instrument on-board of Voyager.

5.2 Truncated distribution functions

Based on the above discussion we have only available truncated distribution functions which have to deliver the fit. In the case of a Maxwellian this is not a problem, but for the $\kappa$-distribution, because the core for different $\kappa$ values can be fitted quite well, while the very different tails, not taken into account into the fit, imply, that particles far off the core contribute remarkably to the temperature moments. It can be seen in Fig. 5 that the fits to the $\kappa$-distribution can easily produce substantial differences in the expected values for different $\kappa$. This is shown in Fig. 6, where we have plotted the truncated zeroth and second order moments normalized to the number density. That is:

\[
M_{\kappa,x}^0 = 4\pi \int_0^x v^2 f_\kappa dv \quad M_{\kappa,x}^2 = 4\pi \int_0^x v^4 f_\kappa dv
\]

It can be seen in the upper panel of Fig. 6 that for different sets of $(\kappa, \Theta)$ for small normalized speed values the different $\kappa$-distribution zeroth order moments are approximately the same, but strongly differ in the second order moments for small $\kappa \lesssim 2$ above a truncation speed (see lower panel), which is usually higher than the range covered by observations. These incomplete distribution functions are the true problem when interpreting observational data, while the problem faced by Nicolaou and Livadiotis (2016) considering complete (idealized) distribution functions appears more academic. Measured distributions are constrained by the instrument limitations, for instance, given by the energy range detected by the instrument, where the highest energy channel may define a truncation limit of the distribution function and its moments.

We have extended the range of the x-axis to match the lowest channel of the LECP instrument on-board of Voyager-2, which measures energetic protons in the energy range of ${\approx}40$ keV corresponding to roughly 3000 km/s. It becomes obvious from Fig. 6 that these particles do not essentially contribute to the number density for different $\kappa$'s, but they heavily influence the second order moment (the pressure). Because the contribution of the high energy particles to the number density is so small, it is not an easy task to determine the $\kappa$-value. It will require more advanced models, which go far beyond the scope of this paper to determine the $\kappa$-distribution from data of different instruments.

5.3 Analytic estimate of the $\kappa$-distribution temperature

From the second moments we can find a relation between the pressures of a $\kappa$-distribution $P_\kappa$ and that of a Maxwellian $P_m$ (see Eq. (13))

\[
P_\kappa = \frac{2\kappa}{2\kappa - 3} \frac{\Theta^2}{v_p^2} P_m
\]

Now we can either calculate the Maxwellian pressure from the observed Maxwellian temperature $T_m$ by

\[
P_m = nk_BT_m
\]
and compare the “temperatures”. The result is the same, except for absolute numbers. With this definition we find:

\[ T_\kappa = \frac{2\kappa}{2\kappa - 3} \frac{\Theta^2}{v_p^2} T_m \]  

(16)

which is identical to Eq. (11).

Furthermore, because for \( \kappa \to \infty \) the \( \kappa \)-temperature must be equal to that of the Maxwellian, we find that \( v_p = \Theta \) and finally, since \( \Theta \) is independent of \( \kappa \),

\[ T_\kappa = \frac{2\kappa}{2\kappa - 3} T_m \]  

(17)

depends only on \( \kappa \) and can be easily calculated from the Maxwellian temperature knowing the corresponding \( \kappa \). Thus one can use the temperatures derived from a Maxwellian distribution and calculate from that the \( \kappa \)-temperatures, where \( \kappa \) and \( \Theta \) can be estimated elsewhere, for instance by solving a kinetic transport equation as done in Fahr et al. (2016).

That means also that the presented temperatures derived with a Maxwellian as available from the Omniweb (http://omniweb.gsfc.nasa.gov/coho/form/voyager2.html) must be taken with care, because they should translate into \( \kappa \)-distribution temperatures along the Voyager-2 trajectory through the inner heliosheath. With formula (17) this is an easy task knowing \( \kappa \).

In Fig. 7 we have plotted the Voyager-2 temperature data between 84 to 112 AU, together with those obtained from the \( \kappa \)-parameter of Fahr et al. (2016) using Eq. (17). In the upper panel of Fig. 7 the \( \kappa \)-parameter reaches the limiting value of 1.5 and, to avoid numerical problems it was set to \( \kappa = 1.5001 \) beyond \( r \approx 98 \) AU. Thus beyond 98 AU the model by Fahr et al. (2016) is not applicable for such high values of the diffusion coefficient (\( D_0 \approx 10^{-8} \) cm\(^2\)/s), while it can nicely explain a non-adiabatic behavior of the temperature between \( \approx 88 \) and \( \approx 98 \) AU.

This shows, that the temperature highly depends on the underlying distribution function, and thus temperature data have to be taken with care, because they can easily lead to erroneous interpretations of the data. This holds true in general.

6 Conclusion

The above study shows how Maxwellian temperatures derived from Voyager-2 measurements can easily be translated into corresponding \( \kappa \)-temperatures. We have also shown (like Nicolaou and Livadiotis 2016) that the temperature of a plasma highly depends on the underlying distribution function and differs from that obtained by a Maxwellian. This implies the need of the knowledge of the underlying distribution function, because, otherwise, the temperature is not well defined.

The recently presented study by Nicolaou and Livadiotis (2016) is not helpful in the “data-relevant” aspects discussed...
here, because the authors compare moments calculated on the basis of \( \kappa \)- or Maxwell-distributions, however, taken from an infinite velocity range, while in reality data only support moments in a very limited velocity range. This is the important aspect that has to be taken serious in these matters.

Nevertheless, after deriving a \( \kappa \)-distribution from theoretical considerations it is an easy task to determine the corresponding \( \kappa \)-temperature when a Maxwellian temperature is given. From IBEX observations (McComas et al., 2015; McComas et al., 2015) it might be possible to obtain the \( \kappa \)-value from which the temperature can be estimated. Note, that the previous attempts to do so (e.g. Livadiotis et al., 2011; Zimstein and McComas, 2015) use a \( \kappa \)-independent temperature, which is a concept under debate (Lazar et al., 2016, 2017).

The procedure discussed in the present paper, may not only be applied to spacecraft observations but to all observations in which the temperature is derived with the assumption that the underlying distribution function is a Maxwellian.

**Appendix A: Moments of the distribution function**

From the moments of a distribution function the macroscopically observable quantities can be derived (for example Goedbloed and Poedts, 2004 and many others). To calculate the moments one has to integrate the distribution function times some power \( \alpha \) of the speed. For the Maxwell distribution the integrals are well known and can be found elsewhere (e.g. Gradshteyn and Ryzhik, 2007). For the \( \kappa \)-distribution they can also be found in a more general form in Gradshteyn and Ryzhik (2007), Nr 3.251.2:

\[
I = \frac{\Gamma(\nu)}{\Gamma(\mu)} \int_0^\infty v^\mu \left(1 + \frac{\beta v^2}{\kappa\Theta^2}\right)^{-\nu} dv
\]

where \( \Gamma \) is the Gamma function. With

\[
\mu = 3 + \alpha, \quad \beta = \frac{1}{\kappa\Theta^2}, \quad \nu = \kappa + 1, \quad p = 2
\]

we have the integral (Fahr et al., 2014)

\[
I_\alpha = \int_0^\infty v^{\alpha+2} \left(1 + \frac{v^2}{\kappa\Theta^2}\right)^{-\kappa-1} dv
\]

\[
= \frac{1}{2} \sqrt{\frac{\kappa\Theta^2}{\pi}} \frac{\Gamma(\frac{3+\alpha}{2})}{\Gamma(\kappa+1)} \frac{\Gamma(\frac{3}{2} + \frac{\alpha}{2})}{\Gamma(\frac{3}{2})} \Gamma(\kappa - \frac{1+\alpha}{2})
\]

and thus

\[
I_0 = g(\kappa, \Theta) = \frac{n(r)}{\sqrt{\pi\Theta}} \frac{\Gamma(\kappa+1)}{\Gamma(\kappa)} \frac{\Gamma(\kappa + 1)}{\Gamma(\kappa - 1/2)}
\]

is the normalisation factor of the \( \kappa \)-distribution defined in Eq. [I].

With the above integral we thus can easily calculate the following moments of the \( \kappa \)-distribution

\[
M^0 = \int f_\kappa(r, v, t) d^3v \quad \text{number density}
\]

\[
M^1 = \frac{1}{n(r, t)} \int v f_\kappa(r, v, t) d^3v = 0 \quad \text{velocity}
\]

\[
M^2 = \int v \otimes v f_\kappa(r, v, t) d^3v \quad \text{stress tensor per unit mass}
\]

of \( f_\kappa \) and compare them to those for a Maxwell distribution \( f_m \). In the comoving reference frame the first moment \( M^1 \) vanishes. Furthermore, we assume that the stress tensor will be described by an isotropic pressure and thus the dyadic \( v \otimes v \) can be contracted to a scalar \( M^2 \) as \( v \otimes v \rightarrow v^2 \). In addition we calculate the most probable speed by

\[
M^1 = \frac{1}{n(r, t)} \int v f(r, v, t) d^3v = v_p
\]

We find for the different moments:

\[
M^0_m = M^0_\kappa = n(r, t)
\]

\[
M^1_m = \frac{2}{\sqrt{\pi}} \sqrt{\frac{2kT}{m}} \equiv \frac{2}{\sqrt{\pi}} v_p
\]

\[
M^2_m = \frac{3}{2} v_p^2 n(r, t)
\]

\[
M^1_\kappa = \frac{2}{\sqrt{\pi}} \frac{1}{(\kappa - 1) \sqrt{\kappa}} \frac{\Gamma(\kappa + 1)}{\Gamma(\frac{3}{2})}
\]

\[
M^2_\kappa = \frac{3kT^2}{2\kappa - 3} n(r, t)
\]

The pressure ratio is:

\[
P_\kappa = \frac{2\kappa}{2\kappa - 3} \frac{\Theta^2}{v_p^2} P_m
\]

Finally, replacing the velocity in the fluid rest frame by that in the observer frame, that is: \( v = u - w \), where \( u \) is the fluid bulk velocity and \( w \) the thermal velocity, one easily finds that the zeroth moment remains, the first moment gives the bulk speed and in the second moment the ram pressure appears as an additional term.

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