Aperiodic Tilings and Entropy
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Aperiodic tilings and entropy

Bruno Durand, Guilhem Gamard, Anaël Grandjean

August 27, 2014
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2 J. Kari & K. Culik’s tileset

3 Aperiodicity

4 Positive entropy

5 Conclusion
Wang tiles

- Finite set of colors
- Alphabet = colored squares
- Adjacent borders have matching colors

\[ c = \{ \text{red, yellow, blue} \} \]
\[ \Sigma = \{ \begin{array}{c} \text{red} \\ \text{yellow} \\ \text{blue} \end{array} \} \]
\[ \Sigma' = \{ \begin{array}{c} \end{array} \} \]
Aperiodic tilesets

**Definition**
A set of tiles is **aperiodic** when:
- it can cover the plane;
- it cannot cover the plane periodically.

Cover such that adjacent borders have matching colors
# The history of small aperiodic tilesets

| Year | Author(s)                                      | Tiles       |
|------|------------------------------------------------|-------------|
| 1964 | R. Berger                                     | > 20,000    |
| 1966 | D. Knuth                                      | 96          |
| 1971 | R. Robinson                                   | 52          |
| 1974 | R. Penrose                                    | 32          |
| 1986 | R. Ammann, B. Grünbaum, G. Shephard          | 16          |

**Self-similar**

| Year | Author(s)                                      | Tiles       |
|------|------------------------------------------------|-------------|
| 1996 | J. Kari                                       | 14          |
| 1996 | K. Culik                                      | 13          |

**Not self-similar**

- R. Berger's 1964 tileset has over 20,000 tiles.
- D. Knuth's 1966 tileset has 96 tiles.
- R. Robinson's 1971 tileset has 52 tiles.
- R. Penrose's 1974 tileset has 32 tiles.
- R. Ammann, B. Grünbaum, G. Shephard's 1986 tileset has 16 tiles.

J. Kari and K. Culik's 1996 tilesets both have 14 and 13 tiles, respectively.
Our result

**Theorem**

The Kari-Culik tileset has positive entropy.

Intuitively:

- Description of a $n \times n$-square takes $\Omega(n^2)$ bits
- It contains dense “random” bits

Note: entropy zero was conjectured.
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A function with aperiodic orbits

Consider this function:

\[ f: \left[ \frac{1}{3}; 2 \right] \rightarrow \left[ \frac{1}{3}; 2 \right] \]

\[ x \mapsto \begin{cases} 
2x & \text{if } x \leq 1 \\
 x/3 & \text{if } x \geq 1 
\end{cases} \]

- Its orbits, i.e. sequences \( u_x = (f^\circ n(x))_{n \in \mathbb{N}} \), are aperiodic
- Its orbits are also dense in \( \left[ \frac{1}{3}; 2 \right] \)
The general idea

Real number representation

\[ f^\circ 3(x) \]

\[ f^\circ 2(x) \]

\[ f(x) \]

\[ x \]

\[ f^\circ -1(x) \]

\[ f^\circ -2(x) \]

\[ f^\circ -3(x) \]
Multiplications done by transducers

(2, 0) stands for “read 2, write 0”
Multiplications done by transducers

(2, 0) stands for “read 2, write 0”
From transducers to tile sets

States of $M_{1/3}$, $M_2$: disjoints colors

One line $=$ one run
An aperiodic set of tiles
An aperiodic set of tiles
An aperiodic set of tiles
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Lines and averages

Definition

The average of a sequence \((u_n)\) is:

\[
\text{avg}(u) = \lim_{n \to \infty} \frac{1}{2n + 1} \sum_{k=-n}^{n} u_k
\]

Tilings: average of a line = average of northern sides
Aperiodicity

Theorem (J. Kari and K. Culik, 1996)

*The Kari-Culik tileset is aperiodic.*

Sketch of proof.

- Suppose there is a periodic tiling. Then each line has an average. The averages are periodic: contradiction.
- To tile the plane, start from …11111111… and run the transducers forever.
Encoding real numbers in bi-infinite sequences

\[ S_x(n) = \lfloor (n + 1)x \rfloor - \lfloor nx \rfloor \]

\[ S_{1/2} = \ldots 01010101010101010101 \ldots \]
\[ S_{7/5} = \ldots 211211121121112112111 \ldots \]
\[ S_{\pi/3} = \ldots 211111111111111111111112111111111111111111111 \ldots \]

- \( S_x \) is on alphabet \( \{ \lfloor x \rfloor, \lceil x \rceil \} \)
- The average of values of \( S_x \) is \( x \)
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Each line has an average

Lemma

In any tiling, each line has an average.

Sketch of proof.

Consider a line without an average.

\[ f^n(x) \quad f^n(y) \]

\[ f(x) \quad f(y) \]

\[ x \quad y \]

Density \[ \Rightarrow \exists n \text{ s.t. } f^n(x) < 1 < f^n(y) \]
Entropy

$C(n) = \text{the number of } n \times n\text{-squares found in any tiling}$

Definition

We call the **entropy** the following quantity:

$$E = \lim_{n \to \infty} \frac{\log C(n)}{n^2}$$

- Classical definition in dynamical systems
- With 13 tiles, if $C(n) \sim 13^{\epsilon n^2}$, then $E = \epsilon$
Substitutive pairs

are pairs of distinct patterns with the same borders.
Substitutive pairs generate entropy

**Lemma**

*If a substitutive square is found in any $n \times n$-square of any tiling, then the entropy of the tiles is positive.*

$n$ possibilities

$2$ possibilities

$2n$ possibilities

$2^4$ possibilities

$3n$ possibilities

$2^9$ possibilities

$\square = $ substitutive square
Substitutive pairs appear often (1/2)

**Lemma**

*Whenever a pattern* $0111^*0$ *occurs on a line of tiles, there is a substitutive square intersecting this pattern.*

**Sketch of proof.**

**Case analysis.**

Middle case
Substitutive pairs appear often (1/2)

**Lemma**

*Whenever a pattern* $0111^\alpha 0$ *occurs on a line of tiles, there is a substitutive square intersecting this pattern.*

**Sketch of proof.**

**Case analysis.**

![Diagram of cases](image)
Substitutive pairs appear often (1/2)

Lemma

*Whenever a pattern* $0111^a0$ *occurs on a line of tiles, there is a substitutive square intersecting this pattern.*

Sketch of proof.

Case analysis.

Rightmost case
Substitutive pairs appear often (2/2)

Lemma

*In any line with an average* \( \in \left[ \frac{3}{4}; \frac{9}{10} \right] \), *a pattern of the form* \( 0111^\alpha 0 \) *appears regularly.*

Sketch of proof.

If there are no “0” regularly, then the average is 1.

If there are no “111” regularly, then the average is \( \leq \frac{3}{4} \).

- All orbits regularly meet the interval \( \left[ \frac{3}{4}; \frac{9}{10} \right] \)
- Hence substitutive squares appear often enough
Introduction

J. Kari & K. Culik’s tileset

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Positive entropy

Conclusion
Many thanks for your attention!

Our result

- The entropy of the Kari-Culik tileset is positive
- The Kari-Culik-tilings are not all self-similar

Open problems

- Characterize the language of words which can appear on K.C.’s lines?
- Is there a tileset working the same way, but with 0 entropy?
- Is there a sub-shift of finite type $A$, with positive entropy, such that any subshift of finite type $\subset A$ also has positive entropy?