Naked Singularities in the Charged Vaidya-deSitter Spacetime

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Abstract

We study the occurrence of naked singularities in the spherically symmetric collapse of a charged null fluid in an expanding deSitter background - a piece of charged Vaidya-deSitter spacetime. The necessary conditions for the formation of a naked singularity are found. The results for the uncharged solutions can be recovered from our analysis.

KEY WORDS: Gravitational collapse, naked singularity, cosmic censorship

1 Introduction

In recent years there has been much interest in the problem of gravitational collapse in general relativity (see [1] for recent reviews). The end state of gravitational collapse of a sufficiently massive body is a gravitational singularity. The conjecture that such a singularity from a regular initial surface must always be hidden behind an event horizon is called the cosmic censorship hypothesis (CCH) and was proposed by Penrose [2]. The weak CCH allows for the occurrence of locally naked singularities but not globally naked ones, whereas the strong CCH does not allow either. This conjecture remains as the central open issue in general relativity. Hence, examples showing the existence of naked singularities (which appear to violate the conjecture) remain important and may be valuable when one attempts to formulate the notion of the CCH in concrete mathematical terms.

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In recent years significant attention has been given to self-similar spacetimes in the context of naked singularities \[3\]. However, self similarity is a strong geometric condition on the spacetime and thus gives rise to the possibility that the naked singularity could be the result of a geometric condition rather than the matter content of the spacetime. So, it is useful to construct examples which are not self similar and develop naked singularities. The Vaidya solution \[4\] is most commonly used as a testing ground for various violations of the CCH, e.g., the Vaidya solution representing null dust collapse exhibits naked singularities \[5\]. It was recently shown \[6\] by means of an example that whether the spacetime is asymptotically flat or not does not make any difference to the occurrence of a locally naked singularity.

The ingoing charged Vaidya solution \[7\] represents a radial infall of a stream of charged photons. Lake and Zannias \[8\], under the assumption of homothecity, found that naked singularities can be formed. The usefulness of the model is that rich structure is exhibited. In this context, it is worthwhile to examine gravitational collapse of charged radiation shells in an expanding deSitter background with reference to the occurrence of naked singularities and the CCH. The metric for this purpose is already known \[9\]. The organisation of the paper is as follows: In Section 2 we introduce the charged Vaidya deSitter spacetime and note that the weak energy condition is satisfied. Section 3 is devoted in finding the analytical solution and condition for the existence of a naked singularity. The paper ends with the discussion in Section 4.

## 2 Charged Vaidya-deSitter Spacetime

The charged Vaidya-deSitter metric in \((v, r, \theta, \phi)\) coordinates is \[9\]

\[
\begin{align*}
  ds^2 &= -(1 - \frac{2m(v)}{r} + \frac{e^2(v)}{r^2} - \frac{\Lambda r^2}{3})dv^2 + 2dvdr + r^2d\Omega^2 \\
\end{align*}
\]

where \(d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2\), \(v\) represents advanced Eddington time, in which \(r\) is decreasing towards the future along a ray \(v = \text{const.}\), the two arbitrary functions \(m(v)\) and \(e(v)\) (which are restricted only by the energy conditions), represent, respectively, the mass and electric charge at advanced time \(v\), and \(\Lambda\) is the cosmological constant. This metric \[9\] represents a solution to the Einstein equations for a collapsing charged null fluid in an expanding deSitter background.

The energy momentum tensor can be written in the form

\[
T_{ab} = T_{ab}^{(n)} + T_{ab}^{(m)}
\]

\[2\]
where
\[ T_{ab}^{(n)} = \frac{1}{4\pi r^3} \left[ r\dot{m}(v) - e(v)e'(v) \right] k_ak_b \] (3)

with the null vector \( k_a \) satisfying \( k_a = -\delta_a^v \) and \( k_ak^a = 0 \), \( T_{ab}^{(m)} \) is related to the electromagnetic tensor \( F_{ab} \):
\[ T_{ab}^{(m)} = \frac{1}{4\pi} \left( F_{ac}F_b^c - \frac{1}{4}g_{ab}F_{cd}F^{cd} \right) \] (4)

which satisfies Maxwell’s field equations
\[ F_{[abc]} = 0 \text{ and } F_{ab;c}g^{bc} = 4\pi J_a \] (5)

where \( J_a \) is the four-current vector.

Clearly, for the weak energy condition to be satisfied we require the bracketed quantity in eq. (3) to be non-negative. We note that the stress tensor in general may not obey the weak energy condition. In particular, if \( dm/de > 0 \) then there always exists a critical radius \( r_c = e\dot{e}/\dot{m} \) such that when \( r < r_c \) the weak energy condition is always violated. However, in realistic situations, the particle cannot get into the region \( r < r_c \) because of the Lorentz force and so the energy condition is still preserved [9, 10].

The Kretschmann scalar for the metric (1) reduces to
\[ K = \frac{48}{r^6} \left[ m^2(v) - \frac{2}{r} e^2(v)m(v) + \frac{7}{6} \frac{e^4(v)}{r^2} \right] + \frac{8}{3} \Lambda^2 \] (6)

So the Kretschmann scalar diverges along \( r = 0 \), establishing that metric (1) is scalar polynomial singular.

### 3 The Existence and Nature of Naked Singularities

The physical situation is that of a radial influx of charged null fluid in an initially empty region of the deSitter universe. The first shell arrives at \( r = 0 \) at time \( v = 0 \) and the final at \( v = T \). A central singularity of growing mass is developed at \( r = 0 \). For \( v < 0 \) we have \( m(v) = e(v) = 0 \), i.e., an empty deSitter metric, and for \( v > T \), \( \dot{m}(v) = \dot{e}(v) = 0 \), \( m(v) \) and \( e^2(v) \) are positive definite. The metric for \( v = 0 \) to \( v = T \) is charged Vaidya-deSitter, and for \( v > T \) we have the Reissner Nordström deSitter solution.
In order to get an analytical solution, we choose $m(v) \propto v$ and $e^2(v) \propto v^2$. To be specific we take

$$m(v) = \begin{cases} 
0, & v < 0, \\
\lambda v (\lambda > 0) & 0 \leq v \leq T, \\
m_0 (> 0) & v > T.
\end{cases} \tag{7}$$

and

$$e^2(v) = \begin{cases} 
0, & v < 0, \\
\mu^2 v^2 (\mu^2 > 0) & 0 \leq v \leq T, \\
e_0^2 (> 0) & v > T.
\end{cases} \tag{8}$$

When $\Lambda = 0$, the spacetime is self similar, admitting a homothetic Killing vector. However if $\Lambda \neq 0$, the basic requirement of self similarity \cite{12} is not met. So, the line element (1) does not admit any proper conformal Killing symmetries.

Radial ($\theta$ and $\phi = \text{const.}$) null geodesics of the metric (1) must satisfy the null condition

$$\frac{dr}{dv} = \frac{1}{2} \left[ 1 - \frac{2m(v)}{r} + \frac{e^2(v)}{r^2} - \frac{\Lambda r^2}{3} \right] \tag{9}$$

which, upon using eqs. (7) and (8) turns out to be

$$\frac{dr}{dv} = \frac{1}{2} \left[ 1 - 2\lambda X + \mu^2 X^2 - \frac{\Lambda r^2}{3} \right] \tag{10}$$

where $X \equiv v/r$ is the tangent to a possible outgoing geodesic. Clearly, the above differential equation has a singularity at $r = 0, v = 0$. The nature (a naked singularity or a black hole) of the collapsing solutions can be characterised by the existence of radial null geodesics coming out from the singularity.

In order to determine the nature of the limiting value of $X$ at $r = 0, v = 0$ on a singular geodesic, we let

$$X_0 = \lim_{r \to 0} \frac{v}{r} = \lim_{v \to 0} \frac{v}{r} \tag{11}$$

Using (10) and L’Hôpital’s rule we get

$$X_0 = \lim_{r \to 0} \frac{v}{r} = \lim_{v \to 0} \frac{dv}{dr} = \frac{2}{1 - 2\lambda X_0 + \mu^2 X_0^2} \tag{12}$$
which implies,
\[ \mu^2 X_0^3 - 2\lambda X_0^2 + X_0 - 2 = 0 \] (13)

This algebraic equation governs the behaviour of the tangent vector near the singular point. Thus by studying the solution of this algebraic equation, the nature of the singularity can be determined. The central shell focusing singularity is at least locally naked (for brevity we have addressed it as naked throughout this paper), if eq. (13) admits one or more positive real roots. However, the locally naked singularities should not be treated less seriously [13]. When there are no positive real roots to eq. (13), the central singularity is not naked because in that case there are no outgoing future directed null geodesics from the singularity. Hence in the absence of positive real roots, the collapse will always lead to a black hole. The condition under which this locally naked singularity could be globally naked is well discussed [3] and we shall not discuss it here. Thus, the occurrence of positive real roots implies that the strong CCH is violated, though not necessarily the weak CCH. We now examine the condition for the occurrence of a naked singularity.

We know that a cubic equation admits at least one real root. Eq. (13) will have three real roots if \( \lambda^2 + 18\lambda\mu^2 \geq 16\lambda^3 + \mu^2 + 27\mu^4 \). However, since \( \lambda > 0 \) and \( \mu^2 > 0 \), eq. (13) cannot have any negative roots. Hence eq. (13) has at least one positive real root. It follows that the gravitational collapse of a charged null fluid must lead to a naked singularity.

The Kretschmann scalar with the help of eqs. (7) and (8), takes the form
\[ K = \frac{48}{r^4} \left( \lambda^2 X^2 - 2\lambda\mu^2 X^3 + \frac{7}{6}\mu^4 X^4 \right) + \frac{8}{3}\Lambda^2 \] (14)

which diverges at the naked singularity and hence the singularity is a scalar polynomial singularity.

Having seen that the naked singularity in our model is a scalar polynomial singularity, we now turn our attention to the Weyl tensor. It is well known that the Weyl tensor vanishes at all points for any conformally flat space time. The divergence of the Weyl tensor at the singularity could imply that such singularities are not associated with the matter distribution and hence are to be taken seriously (see [14], for more details).

The surviving components of the Weyl tensor are
\[ C_{0101} = \frac{2}{r^2} \left[ \frac{-m(v)}{r} + \frac{e^2(v)}{r^2} \right] \]
\[ C_{0202} = \frac{m(v)}{r} \left[ 1 - \frac{2m(v)}{r} + \frac{3e^2(v)}{r^2} - \frac{\Lambda}{3}r^2 \right] - \frac{e^2(v)}{r^2} \left[ 1 + \frac{e^2(v)}{r^2} - \frac{\Lambda}{3}r^2 \right] \]

\[ C_{0212} = \left[ \frac{-m(v)}{r} + \frac{e^2(v)}{r^2} \right] \]

\[ C_{0303} = \frac{m(v)}{r} \left[ 1 - \frac{2m(v)}{r} + \frac{3e^2(v)}{r^2} - \frac{\Lambda}{3}r^2 \right] \sin^2\theta - \frac{e^2(v)}{r^2} \left[ 1 + \frac{e^2(v)}{r^2} - \frac{\Lambda}{3}r^2 \right] \sin^2\theta \]

\[ C_{0313} = \frac{2}{r^2} \left[ \frac{-m(v)}{r} + \frac{e^2(v)}{r^2} \right] \sin^2\theta \]

\[ C_{2323} = -2 \left[ -rm(v) + e^2(v) \right] \sin^2\theta \]

The Weyl scalar \( (C = C_{abcd}C^{abcd}, \ C_{abcd} \text{ is the Weyl tensor}) \), upon inserting eqs. (7) and (8), reads

\[ C(v, r) = \frac{48}{r^6} \left[ m^2(v) - \frac{2m(v)e^2(v)}{r} + \frac{e^4(v)}{r^2} \right] \]

\[ = \frac{48}{r^6} \left[ \lambda^2 v^2 - \frac{2\lambda\mu^2 v^3}{r} + \frac{\mu^4 v^4}{r^2} \right] \] (15)

It can be noted that the Weyl scalar is zero in the deSitter region \( (v < 0) \). It is also zero at \( r \neq 0, v = 0 \). Eq. (13) can be written as

\[ C(X, r) = \frac{48}{r^4} \left[ \lambda^2 X^2 - 2\lambda\mu^2 X^3 + \mu^4 X^4 \right] \] (16)

Thus the Weyl scalar diverges at the naked singularity. The effect of the energy momentum tensor on the geometry can be found by evaluating the Ricci scalar \( (R = R_{ab}^{ab}, R_{ab} \text{ the Ricci tensor}) \) and comparing with the Weyl scalar [14]. The Ricci scalar for the metric (1) is

\[ R = 4 \left( \Lambda^2 + \frac{e^4(v)}{r^8} \right) = 4 \left( \Lambda^2 + \frac{\mu^4 X^4}{r^4} \right) \] (17)

In contrast to the uncharged Vaidya space time [14], in this case, the Ricci scalar diverges at the same rate as the Weyl scalar. So the Weyl scalar does not dominate in our case.
4 Discussion

We have shown the development of naked curvature singularities in the charged Vaidya-deSitter spacetime. In the limit $\mu \to 0$, our results reduce to the those obtained previously [6] for the uncharged case. Lake and Zannias [8] had shown that naked singularities occur in the charged Vaidya spacetime. We have shown that the asymptotic flatness of the spacetime does not alter the result, i.e., naked singularities also occur if $\Lambda \neq 0$. This agrees with the conclusion of Wagh and Maharaj [6], i.e., the occurrence of a naked singularity does not depend upon whether the asymptotic spacetime is expanding or not.

Wagh and Maharaj [6] had shown that in the case of the uncharged Vaidya-deSitter spacetime, a naked singularity occurs only if $\lambda \leq 1/8$. Most earlier work [3] on naked singularities had shown that they occur only for specific values of the parameters. In contrast, we have shown that for the charged case, a naked singularity always occurs, irrespective of the value of $\lambda$.

We found that the Kretschmann scalar diverges in the limiting approach to the singularity along radial null geodesics. So, the singularity can be considered as a physically significant curvature singularity, and hence cannot be ignored.

The behaviour of the Ricci scalar is greatly affected by the presence of charge. It diverges at the same rate as the Weyl scalar. Therefore, one can say that the role played by initial conditions in the deciding metric is equally important to that of the energy momentum tensor whenever a naked singularity occurs. Further, the Weyl tensor is that part of the curvature of spacetime that is not locally determined by matter, and hence the divergence of the Weyl scalar at the naked singularity implies that the singularity is not associated with the local matter distribution and hence should be taken seriously.

In conclusion, we have obtained another example showing that the formation of a naked singularity is not restricted to self similar spacetimes and to an asymptotically flat setting. However, it remains to be shown whether or not the singularity satisfies the strong curvature condition.

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[3] see, for example, Joshi (Ref. 1)

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[11] $m \propto v$ was introduced by A. Papapetrou (see Ref 5 above) and subsequently used by many authors, $e^2 \propto v^2$ has been examined by Lake and Zannias (see Ref 7 above).

[12] A spherical symmetric space-time is self similar if $g_{tt}(ct, cr) = g_{tt}(t, r)$ and $g_{rr}(ct, cr) = g_{rr}(t, r)$ for every $c > 0$. A self similar spacetime is characterised by the existence of a homothetic Killing vector.

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