A class of Lie algebras arising from intersection matrices

Li-meng Xia\textsuperscript{1,†}, Naihong Hu\textsuperscript{2}

\textsuperscript{1}Faculty of Science, Jiangsu University, Zhenjiang, 212013, Jiangsu Prov. China,
\textsuperscript{2}Department of Mathematics, East China Normal University, Shanghai, 200241, China

Abstract. In present work, we find a class of Lie algebras, which are defined from the symmetrizable generalized intersection matrices. However, such algebras are different from generalized intersection matrix algebras and intersection matrix algebras. Moreover, such Lie algebras generated by semi-positive definite matrices can be classified by the modified Dynkin diagrams.

Keywords. intersection matrices; extended affine Lie algebras; generators; classification

MSC 17B22, 17B65

1 Introduction

In the early to mid-1980s, Peter Slodowy discovered that matrices like

\[
M = \begin{bmatrix}
2 & -1 & 0 & 1 \\
-1 & 2 & -1 & 1 \\
0 & -2 & 2 & -2 \\
1 & 1 & -1 & 2
\end{bmatrix}
\]

were encoding the intersection form on the second homology group of Milnor fibres for germs of holomorphic maps with an isolated singularity at the origin \cite{19, 20}. These matrices were like the generalized Cartan matrices of Kac-Moody theory in that they had integer entries, 2’s along the diagonal, and \(m_{ij}\) was negative if and only if \(m_{ji}\) was negative. What was new, however, was the presence of positive entries off the diagonal. Slodowy called such matrices generalized intersection matrices:

**Definition 1.** (\cite{19}) An \(n \times n\) integer-valued matrix \(M = (m_{ij})_{1 \leq i,j \leq n}\) is called a generalized intersection matrix (gim) if

\[
m_{ii} = 2,
\]

\[
m_{ij} < 0 \text{ if and only if } m_{ji} < 0, \text{ and}
\]

\[
\]
\( m_{ij} > 0 \) if and only if \( m_{ji} > 0 \) for \( 1 \leq i, j \leq n \) with \( i \neq j \).

Slodowy used these matrices to define a class of Lie algebras that encompassed all the Kac-Moody Lie algebras:

**Definition 2** ([19], [6]). Given an \( n \times n \) generalized intersection matrix \( M = (m_{ij}) \), define a Lie algebra over \( \mathbb{C} \), called a generalized intersection matrix (gim) algebra and denoted by \( \text{gim}(M) \), with:

- **generators:** \( e_1, \ldots, e_n, f_1, \ldots, f_n, h_1, \ldots, h_n \),
- **relations:**
  
  \( (R1) \) for \( 1 \leq i, j \leq n \),
  \[
  [h_i, e_j] = m_{ij} e_j, \quad [h_i, f_j] = -m_{ij} f_j, \quad [e_i, f_i] = h_i,
  \]

  \( (R2) \) for \( m_{ij} \leq 0 \),
  \[
  [e_i, f_j] = 0 = [f_i, e_j], \quad (\text{ad}e_i)^{-m_{ij}+1} e_j = 0 = (\text{ad}f_i)^{-m_{ij}+1} f_j,
  \]

  \( (R3) \) for \( m_{ij} > 0 \), \( i \neq j \),
  \[
  [e_i, e_j] = 0 = [f_i, f_j], \quad (\text{ad}e_i)^{m_{ij}+1} f_j = 0 = (\text{ad}f_i)^{m_{ij}+1} e_j.
  \]

If the \( M \) that we begin with is a generalized Cartan matrix, then the \( 3n \) generators and the first two groups of axioms, \( (R1) \) and \( (R2) \), provide a presentation of the Kac-Moody Lie algebras [7], [9], [14].

Slodowy and, later, Berman showed that the \( \text{gim} \) algebras are also isomorphic to fixed point subalgebras of involutions on larger Kac-Moody algebras [19], [4]. So, in their words, the \( \text{gim}(M) \) algebras lie both "beyond and inside" Kac-Moody algebras.

Further progress came in the 1990s as a byproduct of the work of Berman-Moody, Benkart-Zelmanov, and Neher on the classification of root-graded Lie algebras [6], [3], [16]. Their work revealed that some families of intersection matrix (im) algebras, were universal covering algebras of well understood Lie algebras. An \( \text{im} \) algebra generally is a quotient algebra of a \( \text{gim} \) algebra associated to the ideal generated by homogeneous vectors those have long roots (i.e., \( (\alpha, \alpha) > 2 \)).

A handful of other researchers also began engaging these new algebras. For example, Eswara-Moody-Yokonuma used vertex operator representations to show that \( \text{im} \) algebras were nontrivial [5]. Gao examined compact forms of \( \text{im} \) algebras arising from conjugations over the complex field [10]. Berman-Jurisich-Tan showed that the presentation of \( \text{gim} \) algebras could be put into a broader framework that incorporated Borcherds algebras [5].

Peng found relations between \( \text{im} \) algebras and the representations of tilted algebras via Ringel-Hall algebras [17]. Especially, Peng-Xu studied the root system of GIMs in [18] and defined a new class of Lie algebras in [21]. The Peng-Xu algebra is invariant under the action of braid group, and it can be classified by the root system when GIM is semi-positive definite.
In present paper, for a symmetrizable generalized intersection matrix $M$, a Lie algebra $\text{Pra}(M)$ is defined. Our construction is motivated by the $\text{gim}$ algebras, $\text{im}$ algebras and the extended affine Lie algebras. Such an algebra is named here by partial reflection algebra. For indecomposable symmetrizable generalized intersection matrices, the partial reflection algebras have properties:

- they are quotients of $\text{gim}$ algebras and different from $\text{im}$ algebras;
- they can be classified by modified Dynkin diagrams for semi-positive definite case;
- if $M$ is positive, then $\text{Pra}(M)$ is finite simple;
- if $M$ has co-rank one, then $\text{Pra}(M)$ is an affine Lie algebra.

If there exists a diagonal non-degenerate matrix $S = \text{diag}(s_1, \ldots, s_n)$ such that $SM$ is symmetric, then $M$ is called symmetrizable. In present work, $M$ is always assumed to be a symmetrizable generalized intersection matrix with rank $n - K$.

Lemma 3. For symmetrizable generalized intersection matrix $M$, there exists a $\mathbb{C}$-space $V$ with symmetric non-degenerate bilinear form $(-,-)$ satisfying:

\[ \dim V = n + K, \]

$V$ has a prime root system $\{v_1, \ldots, v_n, u_1, \ldots, u_K\}$ such that

\[ 2(v_i, v_j) = m_{ij}, \]

for $1 \leq i, j \leq n$.

For any non-isotropic element $v$(i.e., $(v, v) \neq 0$), there is a reflection

\[ \rho_v : \lambda \mapsto \lambda - \frac{2(\lambda, v)}{(v, v)} v. \] (1)

Definition 4. Suppose that $M = (m_{ij})_{n \times n}$ is a GIM and $\Pi = \{v_1, \ldots, v_n\} \subset V$ is a prime root system such that $2(v_i, v_j) = m_{ij}$, where $V$ has a non-degenerate quadratic form $(\cdot, \cdot)$ and $\dim V = 2n - \text{rank}(M)$. Let $\Pi' \subset V$. We say that $\Pi$ and $\Pi'$ are braid-equivalent (denoted by $\Pi \sim \Pi'$) if there exists a sequence of transformations of the form:

\[ \Pi \mapsto \cdots \mapsto \Pi_k \mapsto \Pi_{k+1} \cdots \Pi', \]

where

\[ \Pi_k \mapsto \Pi_{k+1} = (\Pi_k \setminus \{\beta\}) \cup \{\rho_\alpha(\beta)\} \]

for some $\alpha, \beta \in \Pi_k$. Particularly, $\Pi'$ is called a braid-equivalent basis of $M$.

Definition 5. Let $\Pi, \Pi'$ are prime root systems of GIMs $M$ and $N$, respectively. We say that $M$ and $N$ are braid-equivalent if $\Pi$ and $\Pi'$ is braid-equivalent.
The referee reminded us to notice Peng-Xu’s previous work. The above definitions are analogues of which appeared in [18] (also see [21]) and we also adopt their terminology braid-equivalent. In fact, this equivalence relation corresponds to the reflections of single lines (not of the whole space). This is why we adopt the name partial reflection algebra. The partial reflection algebra is dependent on the braid-equivalent basis. However, this Lie algebra is different from that defined by Peng-Xu, the difference will be showed partially by Example 3) in below. If the GIM is semi-positive definite, an interesting thing is that both classes of Lie algebras defined by Peng-Xu and us have the same classification (see Theorem 11 below).

2 Lie algebra Pra(M)

Suppose that M is a symmetrizable intersection matrix:

\[ M = (m_{ij})_{i,j \leq n}, \]

and \( \text{corank} M = n - \text{rank} M = K \).

Define lattices \( P^\vee \) as:

\[ P^\vee = \bigoplus_{i=1}^{n} \mathbb{Z} h_i \oplus \bigoplus_{j=1}^{K} \mathbb{Z} d_j, \]

\[ P = \{ \lambda \in H^* \mid \lambda(P^\vee) \subseteq \mathbb{Z} \}, \]

where \( H^* \) is the dual space of \( H = \mathbb{C} \otimes_{\mathbb{Z}} P^\vee \),

and \( H^* \) has a subset

\[ \Pi = \{ \alpha_i \in H^* \mid i = 1, \cdots, n; \alpha_i(h_j) = m_{ji} \} \]

which is linearly independent.

There exists a bilinear form over \( H^* \), such that \( 2(\alpha_i, \alpha_j)/(\alpha_j, \alpha_j) = m_{ij} \).

**Definition 6.** A boundary reflection Lie algebra \( \text{Pra}(M) \) associated with \( (M, P^\vee, P, \Pi) \) is the Lie algebra over complex number field \( \mathbb{C} \) generated by \( e_i, f_i (i = 1, \cdots, n), h \in H \) with defining relations:

\[ [e_i, f_i] = h_i, [h, e_i] = \alpha_i(h)e_i, [h, f_i] = -\alpha_i(h)f_i, \]

associated to the graded decomposition

\[ \text{Pra}(M) = \sum_{\alpha} \text{Pra}(M)_\alpha, \quad \text{Pra}(M)_\alpha = \{ v \mid [h, v] = \alpha(h)v, \forall h \in H \}, \]
for all $\mathcal{B} \sim \Pi$ and all $\alpha, \beta \in \mathcal{B}$, $x \in \text{Pra}(M)_\alpha, y \in \text{Pra}(M)_\beta, z \in \text{Pra}(M)_{-\alpha}$,

$$(\text{ad} x)^{1-\frac{2(\alpha, \beta)}{\langle \alpha, \alpha \rangle}}(y) = 0, [z, y] = 0, (\alpha, \beta) < 0,$$

(7)

$$(\text{ad} z)^{1-\frac{2(\alpha, \beta)}{\langle \alpha, \alpha \rangle}}(y) = 0, [x, y] = 0, (\alpha, \beta) > 0,$$

(8)

$[x, y] = 0, [z, y] = 0, (\alpha, \beta) = 0.$

(9)

**Lemma 7.** If $M$ is a generalized Cartan matrix (GCM) of finite type or of affine type, then $\text{Pra}(M)$ is a generalized Kac-Moody Lie algebra.

**Proof.** (1) If $M$ is of simply-laced type, then the result holds by the properties of root vectors of real roots.

(2) If $M$ is of order 2, then this lemma holds by the definition of Kac-Moody algebra.

(3) If $M$ is of type $B_l$, $B_l^{(1)}$ or $G_2^{(1)}$, then any $\mathcal{B}$ contains one short root, the relations (2.6)-(2.8) hold by the properties of root vectors of real roots.

(4) Assume that $M$ is of type $C_l$. If $(\rho_\alpha(\beta), \gamma) = 0$ and $\rho_\alpha(\beta) \pm \gamma$ are roots, then $\rho_\alpha(\beta) \pm \gamma$ are long roots.

There is a unique long root $\alpha^* \in \mathcal{B}$, then $\mathcal{B} \sim \mathcal{B}_{\omega, \alpha^*}$ for each $\omega \in W$, where $W$ is the Weyl group and

$$\mathcal{B}_{\omega, \alpha^*} = (\mathcal{B} \setminus \{\alpha^*\}) \cup \{\omega(\alpha^*)\}.$$ 

Let $W_0 := W_{S, \mathcal{B}}$ be the subgroup generated by $\{\rho_\mu | \mu \in \mathcal{B}, (\mu, \mu) < (\alpha^*, \alpha^*)\}$. If $\mathcal{B} \cap \{\pm \rho_\alpha(\beta) \pm \gamma\}$ is empty, then $\rho_\alpha(\beta) + \gamma - \alpha^*$ (or $\rho_\alpha(\beta) + \gamma + \alpha^*$) is two times of a combination of short roots in $\mathcal{B}$. Then there exists $\omega \in W_0$ such that $\lambda := \omega(\alpha^*) = \rho(\beta) + \gamma$ (or $\lambda := \omega(\alpha^*) = -(\rho_\alpha(\beta) + \gamma)$) and hence

$$\{\lambda, \rho_\lambda(\gamma), \rho_\alpha(\beta)\} \subset (\mathcal{B}_{\omega, \alpha^*})_{\rho_\lambda, \gamma} \sim (\mathcal{B}_{\omega, \alpha^*}) \sim \mathcal{B}.$$ 

Then $(\mathcal{B}_{\omega, \alpha^*})_{\rho_\lambda, \gamma}$ contains two same roots, and we obtain a contradiction. For the case $\lambda := \omega(\alpha^*) = -(\rho_\alpha(\beta) + \gamma)$, we have

$$\{\lambda, \rho_\lambda(\gamma), -\rho_\alpha(\beta)\} \subset ((\mathcal{B}_{\omega, \alpha^*})_{\rho_\lambda, \gamma})_{\lambda, \rho_\alpha(\beta)} \sim \mathcal{B}_{\omega, \alpha^*} \sim \mathcal{B},$$

and the same contradiction is obtained.

(5) If $M$ is of type $F_4$, let $\alpha_1, \alpha_2$ be the long roots and $\beta_1, \beta_2$ be the short roots in $\mathcal{B}$. Then for each long root $\alpha^*$, there exists

$$\alpha \in \{\pm \alpha_1, \pm \alpha_2, \pm (\alpha_1 + \alpha_2) = \pm \rho_{\alpha_1}(\alpha_2)\}$$

such that $\alpha^*$ belongs to the $W_0$-orbit of $\alpha$, where $W_0 = \langle \rho_{\beta_1}, \rho_{\beta_2} \rangle$. The method of (4) works for this case.

(6) Assume $M$ is of type $C_l^{(1)}, F_4^{(1)}$ or $E_6^{(2)}, A_{2l-1}^{(2)}$. If $\alpha_1, \alpha_2$ are two long roots and $\alpha_1 - \alpha_2$ is isotropic, then $\alpha_1 - \alpha_2$ is an even multiples of the principal imaginary root. Similar to (4) and (5), we also obtain the result.
(7) If $M$ is of type $A_{2l}^{(2)}$, then each $B$ contains a unique longest root and a unique shortest root. So the proof is similar to (4).

(8) If $M$ is of type $D_{2l+1}^{(2)}$, then each $B$ contains two short roots $\beta, \gamma$. If $\langle \beta, \gamma \rangle = 0$, then $\beta \pm \gamma$ are not roots. If $\langle \beta, \gamma \rangle \neq 0$, then $\langle \beta, \gamma \rangle = \pm 2$ and $\beta \mp \gamma$ is an imaginary root. Hence the relations (2.6)-(2.8) hold.

(9) If $M$ is of type $D_{4}^{(3)}$, we only need to get rid of the case that short roots $\beta, \gamma \in B$ such that $\langle \beta, \gamma \rangle > 0$ (respectively, $\langle \beta, \gamma \rangle < 0$) and $\beta + \gamma$ (respectively, $\beta - \gamma$) is still a root. The method is also similar to (4) and (6).

Lemma 8. If $n = 2$, then $\text{Pra}(M)$ is a generalized Kac-Moody algebra.

Proof. First we may assume that $M$ is a GCM, then $m_{1,2} < 0$, $m_{2,1} < 0$. Let $\mathcal{L}$ be the Kac-Moody algebra with structure matrix $M$, $\Pi = \{\alpha_1, \alpha_2\}$ be its prime root system. By Lemma 7, we only need consider det $M < 0$ and we may assume that $L$ is generated by $e_{\alpha_1}, f_{\alpha_1}, h_i (i = 1, 2)$.

Case 1. If $\Pi' = \{\alpha, \beta\}$ satisfies (2.6)-(2.8), then $\Pi'' = \{-\alpha, \beta\}$ satisfies (2.6)-(2.8).

Case 2. If $\Pi' = \{\alpha_1, \rho_{\alpha_1}(\alpha_2)\}$, by case 1, we may assume that $\Pi'' = \{-\alpha_1, \rho_{\alpha_1}(\alpha_2)\}$, so $\Pi'$ is still of Kac-Moody type.

Define a map $\varphi$:

$$
e_{\alpha_2} \mapsto \frac{1}{(m_{1,2})!}(ae_{\alpha_1})^{-m_{1,2}}e_{\alpha_2}, \ f_{\alpha_2} \mapsto \frac{1}{(m_{1,2})!}(ad(-f_{\alpha_1}))^{-m_{1,2}}f_{\alpha_2},$$

$$e_{\alpha_1} \mapsto -f_{\alpha_1}, \ f_{\alpha_1} \mapsto -e_{\alpha_1},$$

then $\varphi$ determines an isomorphism of $\mathcal{L}$. Note that a quantum analogue of this isomorphism is the famous Lusztig symmetry (see [15]). So the Serre relations are preserved. By the definition of braid-equivalent basis, the above two cases are sufficient to show that $\text{Pra}(M) = \mathcal{L}$. Then $\text{Pra}(M)$ is a generalized Kac-Moody algebra.

Examples.

1) If

$$M = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

and $\Pi = \{\alpha_1, \alpha_2, \alpha_3\}$. In [14], it is proved that $\text{gim}(M)$ has an ideal such that the quotient is isomorphic to $sp_6$. In particular, the image of $[e_{\alpha_1}, [e_{\alpha_2}, e_{\alpha_3}]]$ in the quotient is not zero, then in $\text{gim}(M)$, $[e_{\alpha_1}, [e_{\alpha_2}, e_{\alpha_3}]] \neq 0$. However, $[e_{\alpha_1}, [e_{\alpha_2}, e_{\alpha_3}]] = 0$ in $\text{Pra}(M)$, hence $\text{Pra}(M)$ is different from generalized intersection matrix algebra.

Particularly,

$$\Pi \sim \{\alpha_1, \alpha_2, \alpha_3 - \alpha_1\},$$
the associated intersection matrix is

\[
N = \begin{pmatrix}
2 & -1 & -1 \\
-1 & 2 & 0 \\
-1 & 0 & 2 \\
\end{pmatrix},
\]

so \( \text{Pra}(M) \cong \text{Pra}(N) \) is a finite dimensional simple Lie algebra of type \( A_3 \).

2) If

\[
M = \begin{pmatrix}
2 & -1 & 2 \\
-1 & 2 & -1 \\
2 & -1 & 2 \\
\end{pmatrix}
\]

and \( \Pi = \{\alpha_1, \alpha_2, \alpha_3\} \), then

\[
\Pi \sim \{\alpha_1, \alpha_2, \alpha_3 + \alpha_2\} \sim \{\alpha_1, \alpha_2, -\alpha_3 - \alpha_2\},
\]

the associated intersection matrix is

\[
N = \begin{pmatrix}
2 & -1 & -1 \\
-1 & 2 & -1 \\
-1 & -1 & 2 \\
\end{pmatrix},
\]

so \( \text{Pra}(M) \cong \text{Pra}(N) \) is an affine Lie algebra of type \( A_2^{(1)} \).

3) If

\[
M = \begin{pmatrix}
2 & 2 & 2 \\
2 & 2 & 2 \\
2 & 2 & 2 \\
\end{pmatrix}
\]

and \( \Pi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \), then

\[
[e_{\alpha_1}, [e_{\alpha_2}, [e_{\alpha_3}, f_{\alpha_4}]]] \neq 0
\]

in \( \text{Pra}(M) \), but

\[
[e_{\alpha_1}, [e_{\alpha_2}, [e_{\alpha_3}, f_{\alpha_4}]]] = 0
\]

in \( \text{im}(M) \), hence \( \text{Pra}(M) \) is also different from the intersection matrix algebra.

For the relation \([e_{\alpha_1}, [e_{\alpha_2}, [e_{\alpha_3}, f_{\alpha_4}]]] \neq 0 \) in \( \text{Pra}(M) \), we shall give a detailed proof in Appendix.

Example 3) says that the length of roots are not limited by the root length of root vector generators, this is very different from Peng-Xu’s definition. As far as the authors know, this phenomenon inherited from \( \text{gim} \) disappears in other known quotients. This is another reason driving us to study \( \text{Pra}(M) \).

**Theorem 9.** If \( M \) and \( N \) are braid-equivalent, then

\[
\text{Pra}(M) \cong \text{Pra}(N).
\]
**Proof.** Without loss of generality, we can suppose that $\text{Pra}(M)$ and $\text{Pra}(N)$ have the same subspace $H$ and $H^*$ and

$$\Pi_M = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}, \quad \Pi_N = \{\beta_1, \beta_2, \ldots, \beta_n\},$$

also suppose that

1. $\text{Pra}(M)$ is generated by $e_{\alpha_i}, f_{\alpha_i}, h_i, d_j$;
2. $\text{Pra}(N)$ is generated by $x_{\beta_i}, y_{\beta_i}, t_i, s_j$.

**Case 1.** If $\beta_1 = -\alpha_1, \beta_2 = \alpha_2, \ldots, \beta_n = \alpha_n$.

It is obvious that the homomorphism $\varphi$ defined via:

$$e_{\alpha_1} \mapsto y_{\beta_1}, \quad f_{\alpha_1} \mapsto x_{\beta_1}, \quad e_{\alpha_i} \mapsto x_{\beta_i}, \quad f_{\alpha_i} \mapsto x_{\beta_i}, \quad i \neq 1,$$

and $d_j \mapsto s_j$ is an isomorphism of Lie algebras.

**Case 2.** If $\beta_1 = \rho_{\alpha_2}(\alpha_1), \beta_2 = \alpha_2, \ldots, \beta_n = \alpha_n$. Let $L_M$ be generated by $e_{\alpha_1}, f_{\alpha_1}, h_i(i = 1, 2)$ and $L_N$ be generated by $x_{\beta_i}, y_{\beta_i}, t_i(i = 1, 2)$. It suffices to show the Lie algebra isomorphism $L_M \cong L_N$. However, this is a direct consequence of Lemma 8. Then $\text{Pra}(M) \cong \text{Pra}(N)$. \qed

Out of question, $\text{Pra}(M)$ is a generalized Kac-Moody Lie algebra for $n \leq 2$. In the next sections, we always assume that $M$ is indecomposable and $n \geq 3$.

### 3 Classification of $\text{Pra}(M)$ for positive and semi-positive definite $M$

**Theorem 10.** Suppose that $M$ is indecomposable. If there exists a diagonal matrix $S$ with positive components, such that $SM$ is positive definite, then $\text{Pra}(M)$ is a finite dimensional simple Lie algebra.

**Proof.** Since $SM$ is positive definite, then $K = 0$. Suppose that $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ such that

$$\frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} = m_{ij},$$

and let $\Pi_1 = \{\beta_1 = \alpha_1\}, \ M_1 = (2)$. Clearly, $\text{Pra}(M_1)$ is a three dimensional simple Lie algebra. Because $M$ is indecomposable, there is $\alpha_i$, we can assume that $i = 2$ for convenience, then $\beta_2 = \alpha_2$ or $\beta_2 = \rho_{\alpha_1}(\alpha_2)$ such that $(\beta_1, \beta_2) < 0$. Let $\Pi_2 = \{\beta_1, \beta_2\}$ and

$$M_2 = \begin{bmatrix}
2 & \frac{2(\beta_1, \beta_2)}{(\beta_1, \beta_1)} \\
\frac{2(\beta_1, \beta_2)}{(\beta_2, \beta_2)} & 2
\end{bmatrix},$$

then $M_2$ is Cartan matrix and $\text{Pra}(M_2)$ is a finite dimensional simple Lie algebra. Induction on $p$, let $\Pi_p = \{\beta_1, \ldots, \beta_p\}$ be such that

$$M_p = \begin{bmatrix}
2 & \frac{2(\beta_1, \beta_j)}{(\beta_j, \beta_j)} \\
\frac{2(\beta_1, \beta_j)}{(\beta_j, \beta_j)} & 2
\end{bmatrix}_{1 \leq i, j \leq p}$$
is a Cartan matrix and \( \alpha_{p+1} \) be a non-zero weight of \( \text{Pra}(M_p) \), so there is \( \rho \) from \( \text{Pra}(M_p) \)'s Weyl group such that \( (\beta_i, \beta_{p+1}) \leq 0 \) for all \( 1 \leq i \leq p \) (at least one of them is non-zero), where \( \beta_{p+1} = \rho(\alpha_{p+1}) \). Note that the existence of \( \rho \) follows from the positive definite property of \( M_p \). Let \( \Pi_{p+1} = \{ \beta_1, \cdots, \beta_{p+1} \} \) and 
\[
M_{p+1} = \begin{bmatrix}
2(\beta_i, \beta_j) \\
(\beta_j, \beta_j)
\end{bmatrix}_{1 \leq i, j \leq p+1}
\]
is still a Cartan matrix. Then we can get a Cartan matrix \( M_n \) which is braid-equivalent to \( M_p \), so \( \text{Pra}(M_p) \sim \text{Pra}(M_n) \) is a simple Lie algebra of finite type. \( \square \)

**Theorem 11.** Suppose that \( M \) is an indecomposable symmetrizable positive or semi-positive definite generalized intersection matrix, then \( M \) must be braid-equivalent to an intersection matrix determined by one of the modified Dynkin diagrams listed in Figure 1.

\[
\begin{align*}
A_l(r) & : \quad \circ_1 \circ_2 \cdots \circ_{l-1} \circ_l \\
B_l(r, s) & : \quad \circ_1 \circ_2 \cdots \circ_{l-1} \circ_l \\
C_l(r, s) & : \quad \circ_1 \circ_2 \cdots \circ_{l-1} \circ_l \\
D_l(r) & : \quad \circ_1 \circ_2 \cdots \circ_{l-1} \circ_l \\
E_{6,7,8}(r) & : \quad \circ_1 \circ_2 \circ_3 \circ_4 \circ_5 \circ_6 \cdots \circ_l \\
F_4(r, s) & : \quad \circ_1 \circ_2 \circ_3 \circ_4 \\
G_2(r, s) & : \quad \circ_1 \circ_2 \circ_3 \circ_4 \\
A_1(r, s) & : \quad \circ_1 \circ_2 \circ_3 \circ_4 \circ_5 \circ_6 \cdots \circ_l \\
BC_l(r, s, t) & : \quad \circ_1 \circ_2 \circ_3 \circ_4 \circ_5 \circ_6 \cdots \circ_l \circ_{l+1} \\
\end{align*}
\]

**Figure 1**

**Interpretation:**

The circle, the number of lines between circles and the arrows have the same meaning of Dynkin diagrams. The number \( r (or s, t) \) in the circle means the number of copies of the simple root. If any number in circle is 1, the the diagram is just the Dynkin diagram \( (A_1, B_l, C_l, D_l, E_{6,7,8}, F_4, G_2 \text{ and } A_{2l}^{(2)}) \). For example, the modified Dynkin diagram \( B_l(r, s) \) means:
- \( \Pi = \{ \alpha_1, i, \alpha_2, \cdots, \alpha_{r-1}, \alpha_{r+1}, \cdots, \alpha_{r+s-1}, \cdots, \alpha_{l-1}, \alpha_{l+1} \} \) \( 1 \leq i \leq r, 1 \leq j \leq s \);  
- \( (\alpha_1, i, \alpha_2) = -1, (\alpha_{r}, \alpha_{r+1}) = -1, (\alpha_{l-1}, \alpha_{l+1}) = -2, (\alpha_{1,j}, \alpha_{1,k}) = 2, (\alpha_{r}, \alpha_{r}) = 2 \) and \( (\alpha_{l,t}, \alpha_{l,j}) = 1 \). Other pairs of roots are orthogonal. Hence \( \alpha_{l,j} \) is a short root.
- the intersection matrix is of \( (r + s + l - 2) \times (r + s + l - 2) \).
Proof. Let $\Pi = \{\alpha_1, \cdots, \alpha_n\}$, and $\Pi^2 = \{\alpha_1, \cdots, \alpha_{n-K}\}$ be such that

$$M_{n-K} = \begin{bmatrix} 2(\alpha_i, \alpha_j) \\ (\alpha_j, \alpha_j) \end{bmatrix}_{1 \leq i, j \leq n-K}$$

is indecomposable non-degenerate.

Suppose that $W$ is the Weyl group of $\text{Pra}(M_{n-K})$. By Theorem 10, we may assume that $M_{n-K}$ is a Cartan matrix. Restricted on dual space of its Cartan subalgebra, for any $\alpha_j (j > n-K)$, there exists $w_j \in W$ such that

$(w_j(\alpha_j), \alpha_i) \leq 0$ for all $1 \leq i \leq n-K$.

1) If $\Pi$ has only one root length, then $\Pi$ is of type $A_t(r), D_t(r), E_{6,7,8}(r)$.

We prove it in three cases.

(1.a) $\Pi^2$ is of type $E$. Then every $w_j(\alpha_j)$ has to be the minus highest root (up to an imaginary root), otherwise it contradicts to that $M$ is semi-positive definite and $\text{rank}(M) = n-K$. So we may choose $w_j'$ such that $w_j'(\alpha_j) = \alpha_1$ and $\Pi$ is of type $E_{6,7,8}(r)$.

(1.b) $\Pi^2$ is of type $D$. If there exists $w_j(\alpha_j)$ such that $\Pi^2 \cup \{w_j(\alpha_j)\}$ is of type $E_8^{(1)}$, then we may replace $\Pi^2$ by one of type $E_8$ and $\Pi$ is of type $E_8(r)$. For other cases, every $w_j(\alpha_j)$ has to be the minus highest root. So $\Pi$ is of type $D_t(r)$.

(1.c) $\Pi^2$ is of type $A$. If there exists $w_j(\alpha_j)$ such that $\Pi^2 \cup \{w_j(\alpha_j)\}$ is of type $E_7^{(1)}$ or $E_8^{(1)}$, then we may replace $\Pi^2$ by one of type $E_7$ and $\Pi$ is of type $E_7(r)$. For other cases, every $w_j(\alpha_j)$ has to be the minus highest root. So $\Pi$ is of type $D_t(r)$.

2) If $\Pi$ has two different root lengths, but $\Pi^2 = \{\alpha_1\}$, then $\Pi$ is of type $A_1(r, s)$.

Up to imaginary roots, $\Pi \subset \{\pm \alpha_1, \pm \frac{1}{2} \alpha_1\}$ (or $\Pi \subset \{\pm \alpha_1, \pm 2 \alpha_1\}$). Equivalently, $\Pi = \{\alpha_1, -\frac{1}{2} \alpha_1\}$ (or $\Pi = \{\alpha_1, -2 \alpha_1\}$), and $\Pi$ is of type $A_1(r, s)$ which depends the number of $\alpha_1$ and the number of $-\frac{1}{2} \alpha_1$ (or $-2 \alpha_1$).

3) If $\Pi^2$ is of type $G_2$, then $\Pi$ is of type $G_2(r, s)$.

Each $w_j(\alpha_j)$ has to be the minus highest long root or the minus highest short root, this implies the result.

4) If $\Pi$ has two different root lengths, but $\Pi^2 = \{\alpha_1, \alpha_2\}$ has one root length, then $\Pi$ is of type $G_2(r, s)$.

If $w_j(\alpha_j)$ is shorter then the square length of $w_j(\alpha_j)$ has to be $\frac{1}{2}$ of square length of $\alpha_1$. If $w_j(\alpha_j)$ is longer then the square length of $w_j(\alpha_j)$ has to be 3 multiple of square length of $\alpha_1$. Replace $\Pi^2$ by $\Pi^* = \{w_j(\alpha_j), \alpha_2\}$, which is type $G_2$.

5) If $\Pi^2$ is of type $B$ (or $C$) and $\Pi$ has two different root lengths, then $\Pi$ is of type $B_t(r, s)$ (or $C_t(r, s)$). The proof is similar to 3).

In the following cases, the proof is similar and we only state the result.

6) If $\Pi$ has two different root lengths, but $\Pi^2$ is of type $D_t$, then $\Pi$ is of type $B_l(r, s)$ or $C_l(r, s)$.

7) If $\Pi$ has two different root lengths and $\Pi^2$ is of type $B_4$, then $\Pi$ is of type $B_4(r, s)$ or $F_4(r, s)$.
8) If $\Pi$ has two different root lengths and $\Pi^2$ is of type $C_4$, then $\Pi$ is of type $C_4(r, s)$ or $F_4(r, s)$.

9) If $\Pi^2$ is of type $F_4$, then $\Pi$ is of type $F_4(r, s)$.

10) If $\Pi$ has three different root lengths, then $\Pi$ must be of type $BC_l(r, s, t)$.

\[\textbf{Remark 12.} \text{In [13], the same classification to braid-equivalent matrices was given for root systems of GIMs. In this paper, we have provided a different proof.}\]

\[\textbf{Remark 13.} \text{Let } L \text{ be a simple Lie algebra of type } X_1 \text{ with a prime root system } \{\alpha_1, \alpha_2, \cdots, \alpha_l\}, \text{ and let } A \text{ be the Laurent polynomial algebra } C[t_1^{\pm 1}, \cdots, t_l^{\pm 1}] \text{ and } \Omega = AdA/dA. \text{ For convenience, we assume that } \alpha_1 \text{ is a long root. As we know that a toroidal Lie algebra of type } X_1 \text{ and nullity } \nu \text{ can be realized as following:}\]

\[Tor(L) = L \otimes A \otimes \Omega \bigoplus_{i=1}^{\nu} Cd_i.\]

The bracket is given by:

\[\begin{align*}
[x \otimes t^x, y \otimes t^y] &= [x, y] \otimes t^{x+y} + \sum_{i=1}^{\nu} t^i dt^i, \\
[d_j, y \otimes t^y] &= m_j y \otimes t^y, \\
[d_j, t^i dt^i] &= (n_j + m_j)y \otimes t^y, \\
[t^i dt^i, y \otimes t^y] &= 0, \\
[t^i dt^i, t^j dt^j] &= 0, \\
[d_i, d_j] &= 0,
\end{align*}\]

where $t_i = t_1^i \cdots t_l^i$ and $\sum_{i=1}^{\nu} n_i t_i^{\nu} dt_i$.

Let $e_j = x_{\alpha_j}$, $f_j = x_{-\alpha_j}$, $h_j = \alpha_j^L$. It is easy to know that $Tor(L)$ can be generated by elements $\{e_1 \otimes t_i, f_1 \otimes t_i^{-1}, e_j, f_j, h_j, d_i|i = 1, \cdots, \nu, j = 1, \cdots, l\}$.

The roots of $\{e_1 \otimes t_i, e_j|i = 1, \cdots, \nu, j = 1, \cdots, l\}$ forms a set

\[\{\beta_1 = \alpha_1 + \delta_1, \cdots, \beta_{\nu} = \alpha_{1} + \delta_{\nu}, \beta_{\nu+1} = \alpha_1, \cdots, \beta_{\nu+n} = \alpha_n\}.\]

Let

\[M = \begin{bmatrix}
2 & \cdots & 2 a_{11} & a_{12} & \cdots & a_{1n} \\
: & \cdots & : & : & : & : \\
2 & \cdots & 2 a_{11} & a_{12} & \cdots & a_{1n} \\
a_{11} & \cdots & a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & \cdots & a_{21} & a_{22} & \cdots & a_{2n} \\
: & \cdots & : & : & : & : \\
a_{n1} & \cdots & a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}_{(p+n) \times (p+n)} \]
Where \([a_{ij}]_{n \times n}\) is the Cartan matrix of \(L\). It is easy to check that there exists an epimorphism from \(\text{Pra}(M)\) to \(\text{Tor}(L)\).

Appendix: proof of example 3)

Let
\[
A = \begin{pmatrix}
2 & 0 & 0 & 0 & -2 & -2 & -2 \\
0 & 2 & 0 & 0 & -2 & 0 & -2 \\
0 & 0 & 2 & 0 & -2 & -2 & 0 \\
0 & 0 & 0 & 2 & -2 & -2 & -2 \\
0 & -2 & -2 & 2 & 0 & 0 & 0 \\
-2 & 0 & -2 & -2 & 0 & 2 & 0 \\
-2 & -2 & 0 & -2 & 0 & 0 & 2 \\
-2 & -2 & -2 & 0 & 0 & 0 & 2
\end{pmatrix}
\]
and \(\Pi_A = \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7, \beta_8\}\).

Then Kac-Moody Lie algebra \(L_A := \text{Lie}(A)\) has standard generators \(x_i, y_i, \kappa_i (1 \leq i \leq 8)\), where \(\kappa_i\)'s span the Cartan subalgebra, \(x_i\) has root \(\beta_i\) and \(y_i\) has root \(-\beta_i\). By the definition of Kac-Moody algebra,

\([x_1, [x_2, [x_3, x_8]]] \neq 0\).

Define a degree on generators by:
\[
\deg(x_i) = -\deg(y_i) = 1, \deg(\kappa_i) = 0,
\]
then the subalgebra \(L_A^{>0}\) generated by \(\kappa_i, x_i (1 \leq i \leq 8)\) is a graded Lie algebra. Let \(S_0\) be Cartan subalgebra and

\[
S_1 = \bigoplus_{i=1}^{8} \mathbb{C}x_i,
\]
then

\[
L_A^{>0} = \bigoplus_{k \geq 0} S_k,
\]
where \(S_k = \sum_{1 \leq j \leq k-1} [S_{k-j}, S_j]\) for all \(k \geq 2\).

Let \(g_M := \text{gim}(M)\) be the GIM Lie algebra with generators \(e_i, f_i, h_i (1 \leq i \leq 4)\), where \(e_i = e_{\alpha_i}, f_i = f_{\alpha_i}\) and \(h_i = [e_i, f_i]\).

Note that \(\text{Pra}(M)\) naturally is quotient of \(g_M\).
Next we define a degree on \(g_M\) by
\[
\deg(e_i) = \deg(f_i) = 1, \deg(h_i) = 0.
\]
Also define

\[ P_0 = \bigoplus_{i=1}^4 \mathbb{C} h_i, \]
\[ P_1 = P_0 \oplus \bigoplus_{i=1}^4 \mathbb{C} e_i \oplus \bigoplus_{i=1}^4 \mathbb{C} f_i, \]
\[ P_k = P_{k-1} \oplus \sum_{1 \leq j \leq k-1} [P_{k-j}, P_j], \quad \forall k \geq 2, \]

then \( \text{Pra}(M) \) has a filtration

\[ P_{-1} := \{0\} \subset P_0 \subset P_1 \subset P_2 \subset \cdots. \]

Let \( G \) be the graded Lie algebra

\[ G = \bigoplus_{k \geq 0} G_k, \]

where \( G_k = P_k / P_{k-1} \). Let \( E_i, F_i \) denote the image of \( e_i, f_i \) in \( G_1 \), respectively. Define a map \( \phi \) for \( 1 \leq i \leq 4 \):

\[ h_i \mapsto \kappa_i - \kappa_{i+4}, \]
\[ E_i \mapsto x_i, \]
\[ F_i \mapsto x_i + 4. \]

We claim that \( \phi \) induces a Lie algebra injection from \( G \) to \( L_{A}^{>0} \). As a special case, Berman proved that \( g_M \) was a fixed point subalgebra of \( L_{A}(\text{see}[4]) \). In the proof of isomorphism, he constructed such graded algebras. In his proof, our claim holds, then we infer that \([e_1, [e_2, [e_3, f_4]]] \neq 0 \) in \( g_M \).

Now let \( I \) be the ideal of \( g_M \) such that \( \text{Pra}(M) = g_M / I \). Note that

\[ [e_1, [e_2, [e_3, f_4]]] \notin P_3, [e_1, [e_2, [e_3, f_4]]] \notin P_3. \]

Let \( \Gamma = \oplus_{i=1}^4 \mathbb{Z} \alpha_i \). For each \( \mathcal{B} \sim \Pi \) and each \( \alpha \in \mathcal{B} \), we claim that \( \alpha \in \alpha_{i_0} + 2\Gamma \) for some \( i_0 \in \{1, 2, 3, 4\} \). This can be checked by the definitions of reflections and matrix \( M \). So, if \([e_1, [e_2, [e_3, f_4]]] \in I \), we have

\[ \alpha_1 + \alpha_2 + \alpha_3 - \alpha_4 = \alpha \pm \beta \in \alpha_{i_0} + \alpha_{j_0} + 2\Gamma \]

for some \( i_0, j_0 \in \{1, 2, 3, 4\} \), which is a contradiction. Then \([e_1, [e_2, [e_3, f_4]]] \notin I \) and hence \([e_1, [e_2, [e_3, f_4]]] \neq 0 \) in \( \text{Pra}(M) \).

Acknowledgements  The authors are very thankful for the referee’s carefully reading, suggestions and comments. The first author is also thankful to Prof. Peng Liangang for providing
his paper joint with Xu. This work is supported by the NNSF of China (Grant No. 11001110, 11271131), the first author also is supported by Jiangsu Government Scholarship for Overseas Studies.

References

1. B. Allison, G. Benkart, Y. Gao. Lie algebras graded by the root system $BC_r, r \geq 2$. Mem. AMS 2002, 751, x+158
2. S. Bhargava and Y. Gao. Realizations of $BC_r$-graded intersection matrix algebras with grading subalgebras of type $B_r, r \geq 3$. Pacific Journal of Mathematics, 2013, 263(2): 257–281
3. G. Benkart and E. Zelmanov. Lie algebras graded by finite root systems and intersection matrix algebras. Invent. Math., 1996, 126:1–45
4. S. Berman. On generators and relations for certain involutary subalgebras of Kac-Moody Lie Algebras. Comm. Alg. 1989, 17: 3165–3185
5. S. Berman, E. Jurisich and S. Tan. Beyond Borcherds Lie Algebras and Inside, Trans. AMS, 2001, 353: 1183–1219
6. S. Berman, R.V. Moody. Lie algebras graded by finite root systems and the intersection matrix algebras of Slowdowy. Invent. Math., 1992, 108: 323–347
7. R. Carter. Lie Algebras of Finite and Affine Type. Cambridge Univ. Press, Cambridge, 2005
8. S. Eswara Rao, R.V. Moody, T. Yokonuma. Lie algebras and Weyl groups arising from vertex operator representations. Nova J. of Algebra and Geometry, 1992, 1: 15–57
9. O. Gabber, V.G. Kac. On Defining Relations of Certain Infinite-Dimensional Lie Algebras. Bull. AMS (N.S.), 1981, 5: 185–189
10. Y. Gao. Involutive Lie algebras graded by finite root systems and compact forms of IM algebras. Math. Zeitschrift 1996, 223: 651–672
11. Y. Gao, L. Xia. Finite-dimensional representations for a class of generalized intersection matrix algebras. [arXiv:1404.3319v1]
12. J.E. Humphreys. Introduction to Lie Algebras and Representation Theory. Springer, New York, 1972
13. N. Jacobson. Lie Algebras. Inter. science, New York, 1962
14. V.G. Kac. Infinite Dimensional Lie Algebras, 3rd edition. Cambridge Univ. Press, Cambridge, 1990
15. G. Lusztig. Introduction to quantum groups. Birkhauser, Boston, 1993
16. E. Neher. Lie algebras graded by 3-graded root systems and Jordan pairs covered by grids. Amer. J. Math. 1996, 118(2): 439–491
17. L. Peng. Intersection matrix Lie algebras and Ringel-Hall Lie algebras of tilted algebras. Representations of Algebra Vol. I, II, 98–108, Beijing Normal Univ. Press, Beijing, 2002
18. L. Peng, M. Xu. Symmetrizable intersection matrices and their root systems. arXiv: 0912.1024
19. P. Slodowy. Beyond Kac-Moody algebras and inside. Can. Math. Soc. Conf. Proc. 1986, 5: 361–371
20. P. Slodowy. Singularit¨ aten, Kac-Moody Lie-Algebren, assozierte Gruppen und Verallgemeinerungen. Habilitationsschrift, Universität Bonn, March 1984
21. M. Xu, L. Peng. Symmetrizable intersection matrix Lie algebras. Algebra Colloquium 2011, 18(4): 639–646