Tautological relations and the \( r \)-spin Witten conjecture

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Abstract

In [23, 24], Y.-P. Lee introduced a notion of universal relation for formal Gromov–Witten potentials. Universal relations are connected to tautological relations in the cohomology ring of the moduli space \( \overline{M}_{g,n} \) of stable curves. Y.-P. Lee conjectured that the two sets of relations coincide and proved the inclusion (tautological relations) \( \subset \) (universal relations) modulo certain results announced by C. Teleman. He also proposed an algorithm that, conjecturally, computes all universal/tautological relations.

Here we give a geometric interpretation of Y.-P. Lee’s algorithm. This leads to a much simpler proof of the fact that every tautological relation gives rise to a universal relation. We also show that Y.-P. Lee’s algorithm computes the tautological relations correctly if and only if the Gorenstein conjecture on the tautological cohomology ring of \( \overline{M}_{g,n} \) is true. These results are first steps in the task of establishing an equivalence between formal and geometric Gromov–Witten theories.

In particular, it implies that in any semi-simple Gromov–Witten theory where arbitrary correlators can be expressed in genus 0 correlators using only tautological relations, the formal and the geometric Gromov–Witten potentials coincide.

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\(2000\) Mathematics Subject Classification: 14H10, 14N35, 53D45, 53D50.

Key words: Quantization of Frobenius manifolds, Gromov–Witten potential, moduli of curves, \( r \)-spin structures, Witten’s conjecture.
As the most important application we show that our results suffice to deduce the statement of a 1991 Witten conjecture on \( r \)-spin structures from the results obtained by Givental for the corresponding formal Gromov–Witten potential.

The conjecture in question states that certain intersection numbers on the moduli space of \( r \)-spin structures can be arranged into a power series that satisfies the \( r \)-KdV (or \( r \)th higher Gelfand-Dikii) hierarchy of partial differential equations.

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1 Introduction

1.1 Tautological relations and universal relations

Tautological cohomology classes of $\overline{M}_{g,n}$ are, roughly, intersections of boundary components with $\psi$-classes and $\kappa$-classes. They are described by so-called dual graphs (Definition 2.11). Sometimes it happens that a nontrivial linear combination $L$ of dual graphs determines a zero cohomology class $[L] = 0$. Such linear combinations are called tautological relations. Until now there is no known way to generate all of them by an algorithm.

Every tautological relation gives rise to a large family of relations in every geometric Gromov–Witten theory. Indeed, let $X_{g,n+n',D} = \overline{M}_{g,n+n'}(X,D)$ be the space of stable maps from genus $g$ curves with $n+n'$ marked points to a target compact Kähler manifold $X$, of degree $D \in H_2(X,\mathbb{Z})$. Denote by $[X_{g,n+n',D}]$ its virtual fundamental class. (Virtual fundamental classes are defined in [2, 26, 35], but we will actually use only certain factorization properties of these classes.) Let $p : X_{g,n+n',D} \to \overline{M}_{g,n}$ be the forgetful map. Further, let $L$ be a tautological relation on $\overline{M}_{g,n}$, and $\beta \in H^*(X_{g,n+n',D})$ a cohomology class of the form $\prod_{i=1}^{n+n'} \psi_i^{d_i} \text{ev}_i^*(\mu_i)$, where $\mu_i$ are cohomology classes of $X$ and $\text{ev}_i$ the evaluation maps. Then the integral

$$\int_{[X_{g,n+n',D}]} p^*[L] \beta$$

(1)

can be expressed as a polynomial in Gromov–Witten invariants of $X$ in a standard way. On the other hand, we know in advance that this integral vanishes because $[L] = 0$.

In [8], A. Givental introduced a notion of formal Gromov–Witten descendant potentials. It is conjectured that formal Gromov–Witten potentials contain all possible geometric Gromov–Witten potentials and more. The relation between formal Gromov–Witten theory and geometry is not established, although C. Teleman announced some important results on this subject\(^1\). Nonetheless, since there exists a universal expression in Gromov–Witten correlators that represents integral (1) for geometric Gromov–Witten potentials, it is natural to conjecture that this expression vanishes not only in the geometric case, but for all formal Gromov–Witten potentials as well.

This led Y.-P. Lee to introduce the notion of a universal relation in [23, 24]. Roughly speaking, it is a linear combination of dual graphs such that the induced relations for Gromov–Witten correlators hold for all

\(^1\)Teleman’s preprint [36] with a classification of semi-simple cohomological field theories appeared a year after the first version of this paper was completed.
formal Gromov–Witten potentials, see Definition 2.15. Y.-P. Lee conjectured that the tautological relations and the universal relations were exactly the same and suggested an algorithm to compute them. To put things clearly, let Taut denote the set of tautological relations, Univ the set of universal relations, and Alg the set of relations computed by Y.-P. Lee’s algorithm.

The inclusion Alg ⊂ Univ is a reformulation of Theorem 4. It was proved by Y.-P. Lee in [24] and we re-explain the proof here in Section 5 with some additional details.

Y.-P. Lee also stated as a theorem the inclusion Taut ⊂ Univ, although his proof is quite complicated and involves a reference to C. Teleman’s results.

In Section 3.2 we prove the following theorem.

Theorem 1 A linear combination of dual graphs $L$ belongs to Alg if and only if the intersection numbers of the class $[L]$ with all tautological classes of complementary dimension vanish.

It follows immediately that Taut ⊂ Alg ⊂ Univ. Moreover, we have Taut = Alg if and only if the so-called Gorenstein conjecture, stating that the Poincaré pairing is nondegenerate on the tautological cohomology ring of $\overline{M}_{g,n}$, is true.

We have no new information about the conjectured equalities $\text{Alg} \cong \text{Univ}$ and $\text{Taut} \cong \text{Univ}$.

Summarizing, we see that if a linear combination $L$ of dual graphs represents a zero cohomology class, then all formal Gromov–Witten potentials “know about it” in the sense that the expressions in correlators that represent intersection numbers with the pull-back of $[L]$ vanish.

This result constitutes a rather powerful tool for establishing equivalences between formal and geometric Gromov–Witten potentials:

In any semi-simple Gromov–Witten theory where any correlator can be expressed in terms of genus 0 correlators using only tautological relations, the formal Gromov–Witten potential coincides with the geometric Gromov–Witten potential.

A more precise formulation is given in Theorem 3.

As the main application, we show that this result suffices to deduce Witten’s conjecture on the space of $r$-spin structures from the results already established by Givental for formal Gromov–Witten potentials.

1.2 Witten’s conjecture

In 1991 E. Witten formulated a conjecture on the intersection theory of the spaces of $r$-spin structures of Riemann surfaces [37]. An $r$-spin structure on
a smooth curve $C$ with $n \geq 1$ marked points $x_1, \ldots, x_n$ is a line bundle $\mathcal{T}$ together with an identification

$$\mathcal{T}^r \simeq K(-\sum a_i x_i),$$

where $K$ is the cotangent line bundle and the integers $a_i \in \{0, \ldots, r - 1\}$ are chosen in such a way that $2g - 2 - \sum a_i$ is divisible by $r$. The space of $r$-spin structures has a natural compactification $\overline{M}_{g, a_1, \ldots, a_n}$ with a forgetful map $p : \overline{M}_{g, a_1, \ldots, a_n}^{1/r} \to \overline{M}_{g, n}$ (see [15, 1, 4]). Using the line bundle $\mathcal{T}$ and the map $p$ one can define on $\overline{M}_{g, n}$ a cohomology class $c_W(a_1, \ldots, a_n)$ of pure dimension, now called Witten’s class. Assuming that $H^0(C, \mathcal{T}) = 0$ for every stable curve $C$, we find that $V = H^1(C, \mathcal{T})$ is a vector bundle over $\overline{M}_{g, a_1, \ldots, a_n}^{1/r}$. Then Witten’s class is defined by

$$c_W(a_1, \ldots, a_n) = \frac{1}{r^g} p_*(c_{\text{top}}(V^\vee)).$$

In other words: take the dual vector bundle of $V$, take its Euler (or top Chern) class, take its push forward to $\overline{M}_{g, n}$, and divide by $r^g$. In the general case, when $H^0(C, \mathcal{T})$ does not vanish identically, Witten’s class has a much more intricate definition that we do not recall here.

Witten’s conjecture states that the intersection numbers of Witten’s class with powers of $\psi$-classes can be arranged into a generating series satisfying the $r$-KdV (or $r$th higher Gelfand-Dikii) hierarchy.

**Theorem 2** Witten’s $r$-spin conjecture is true.

Below we give a summary of our proof of this conjecture. The ideas we use are close to those used by Y.-P. Lee to prove the conjecture in low genus in [21, 22].

1. In his initial paper E. Witten [37] proved that the conjecture was true in genus 0 provided the class $c_W$ satisfied certain postulated properties. The space of $r$-spin structures and the class $c_W$ were later constructed precisely and shown to possess the expected properties by a joint effort of several people [15, 16, 27, 31, 30].

2. A. Givental [8] constructed a transitive group action on all semi-simple formal Gromov–Witten theories. He found a specific group element that takes the Gromov–Witten potential of a point to the string solution of the $r$-KdV hierarchy. (In particular, the genus zero part of this solution coincides with the genus zero part of the generating function for Witten’s correlators.)
3. Y.-P. Lee [23, 24] found an algorithm that allows one to compute universal relations on Gromov–Witten potentials in Givental’s theory, i.e., relations satisfied by all formal Gromov–Witten descendant potentials.

4. In this paper we give a geometric interpretation of Y.-P. Lee’s algorithm and deduce a simple proof of Y.-P. Lee’s claim that any tautological relation in $H^* (\overline{M}_{g,n})$ gives rise to a universal relation in Givental’s theory.

5. The first author together with R. Pandharipande [7] studied the so-called double ramification classes in $H^*(\overline{M}_{g,n})$ and proved that they were tautological. Therefore integrals of Witten’s class and powers of $\psi$-classes on these cycles correspond to some polynomials in correlators of the Gromov–Witten potential in Givental’s setting.

6. Finally, the last two authors [34] showed that tautological relations on double ramification classes were sufficient to express Witten’s correlator in terms of genus zero correlators.

The logic of our proof is the following. Item 6 gives an algorithm that expresses any given correlator in terms of genus 0 correlators. The steps of this algorithm can be performed in Givental’s theory by Item 4. Thus the generating series for the correlators coincides with the Gromov–Witten potential given by Givental. But the latter is known to give a solution of the $r$-KdV hierarchy by Item 2.

Useful references on Witten’s $r$-spin conjecture also include [20, 1, 3, 17, 18, 19, 4, 5, 33].

2 Formal Gromov–Witten potentials and tautological classes

In this section we give a short overview of the aspects of Givental’s theory of formal Gromov–Witten potentials that we will need.

2.1 Formal Gromov–Witten descendant potentials

Definition 2.1 The genus $g$ Gromov–Witten descendant potential of a point is the formal power series

$$F_g^{pt}(t_0, t_1, \ldots) = \sum_{n \geq 0} \sum_{d_1, \ldots, d_n} \langle \tau_{d_1} \ldots \tau_{d_n} \rangle \frac{t_{d_1} \ldots t_{d_n}}{n!},$$
where
\[ \langle \tau_{d_1} \ldots \tau_{d_n} \rangle = \int_{\overline{M}_{g,n}} \psi_1^{d_1} \ldots \psi_n^{d_n}, \]
g being determined by the condition \( \dim \overline{M}_{g,n} = 3g - 3 + n = \sum d_i \). The total Gromov–Witten descendant potential of a point is \( F_{\text{pt}} = \sum F_{\text{pt}}^g h^{g-1} \) and its exponential \( Z_{\text{pt}} = \exp F_{\text{pt}} \) is called the Gromov–Witten partition function of a point.

A formal Gromov–Witten potential is defined to model certain properties of \( F_{\text{pt}} \) and those of Gromov–Witten potentials of more general target Kähler manifolds \( X \). We restrict our considerations to the even part of the cohomology of \( X \). In our description we explain in brackets the geometric aspects that motivate the axiomatic definitions.

Let \( V \) be a complex vector space [the space \( H_{\text{even}}(X, \mathbb{C}) \)] in which we choose a basis \( A \). The space \( V \) possesses a distinguished element 1 [cohomology class 1], which we usually assume to be the first vector of the basis. \( V \) also carries a nondegenerate symmetric bilinear form \( \eta \) [Poincaré pairing]. The coefficients of \( \eta \) in the basis will be denoted by \( \eta_{\mu \nu} \) and the coefficients of the inverse matrix by \( \eta^{\mu \nu} \).

**Definition 2.2** Let \( M \) be a neighborhood of the origin in \( V \). Let \( F_0 \) be a power series in variables \( t_\mu^{d}, d = 1, 2, 3, \ldots, \mu \in A \), whose coefficients are analytic functions on \( M \) in variables \( t_\mu^{0} \). The coefficients of \( F_0 \) are denoted by
\[ F_0 = \sum_{n \geq 0} \sum_{d_1, \ldots, d_n \geq 0, \mu_1, \ldots, \mu_n \in A} \langle \tau_{d_1,\mu_1} \ldots \tau_{d_n,\mu_n} \rangle \frac{t_{d_1}^{\mu_1} \ldots t_{d_n}^{\mu_n}}{n!}. \]

\( F_0 \) is called a formal genus 0 Gromov–Witten potential if it satisfies the string equation, the dilaton equation, and the topological recursion relation (see, for instance, [8], [10] or [11]). The open set \( M \) is called a Frobenius manifold\(^2\).

**Remark 2.3** If \( X \) is a Kähler manifold and \( E \subset H_2(X, \mathbb{Z}) \) its semi-group of effective 2-cycles, one usually considers Gromov–Witten potentials with coefficients not in \( \mathbb{C} \), but in the Novikov ring of power series of the form
\[ \sum_{D \in E} c_D Q^D, \quad c_D \in \mathbb{C}. \]

\(^2\)Sometimes the definition of a Frobenius manifold also includes an Euler field and \( F_0 \) is required to satisfy certain homogeneity conditions with respect to this field. In other sources Frobenius manifolds with an Euler field are called conformal.
We will mostly work with Gromov–Witten potentials over \( \mathbb{C} \), since we do not need Novikov rings for our main application, namely the Witten conjecture. However, we indicate in remarks the modifications that must be made in the general case. A detailed introduction to formal Gromov–Witten potentials, including a discussion of Novikov rings, can be found in [25].

Given a formal genus 0 Gromov–Witten potential \( F_0(t^\mu_0) \), let \( f_0(t^\mu_0) \) be the series obtained from \( F_0 \) by setting \( t^\mu_d = 0 \) for \( d \geq 1 \). On every tangent space to \( M \) one defines an algebra via the structural constants

\[
C_{\mu_1, \mu_2}^{\mu_3}(t^\mu_0) = \sum_{\nu} \frac{\partial^3 f_0}{\partial t^\mu_1_0 \partial t^\mu_2_0 \partial t^\mu_0} \eta^{\nu, \mu_3}
\]

depending on \( t^\mu_0 \in M \). Together with the bilinear form \( \eta \), one gets the structure of a Frobenius algebra in every tangent space to \( M \).

**Definition 2.4** \( F_0 \) is called semi-simple if the algebra structure is semi-simple for generic \( t^\mu_0 \in M \). The rank of \( F_0 \) is the dimension of \( V \).

**Remark 2.5** A (genus 0) Gromov–Witten potential defined over a Novikov ring \( R \) is called semi-simple if the algebra structure at a generic point is semi-simple over the algebraic closure of the field of fractions of \( R \).

**Example 2.6** Let \( F \) be the string solution of the 3-KdV hierarchy (the Gromov–Witten potential that appears in Witten’s conjecture for \( r = 3 \), see Section 4.1). Denote \( x = t^0_1, y = t^1_0 \). Then

\[
f_0(x, y) = \frac{x^2y}{2} + \frac{y^4}{72}.
\]

Let \( F \) be the Gromov–Witten potential of \( \mathbb{C}P^1 \). Denote \( x = t^0_1, y = t^\omega_0 \), where \( 1 \in H^0(\mathbb{C}P^1) \) and \( \omega \in H^2(\mathbb{C}P^1) \) form the natural basis of \( H^*(\mathbb{C}P^1) \). Then

\[
f_0(x, y) = \frac{x^2y}{2} + Qe^y.
\]

In [10], Givental constructs an action of the so-called twisted loop group on Gromov–Witten potentials of rank \( k \). More precisely, the group itself is ill-defined, but there is a well-defined action of its upper triangular and lower triangular parts.

This action is almost transitive on semi-simple potentials of rank \( k \). Denote by

\[
F^{pt, \alpha} = \sum \hbar^{g-1} \alpha^{2g-\sum d_i} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle \frac{t_{d_1} \cdots t_{d_n}}{n!}
\]

(2)
a rescaled Gromov–Witten potential of the point.

Then, whenever $F_0$ is semi-simple and of rank $k$, there exists an element $S$ of the lower-triangular group and an element $R$ of the upper-triangular group such that $SR$ takes $F^\text{pt.}^\alpha_1 \oplus \cdots \oplus F^\text{pt.}^\alpha_k$ to $F_0$, where $\alpha_1, \ldots, \alpha_k$ are appropriately chosen constants.

A quantized version of the same group (described in [8]), that we will call Givental’s group, acts on Gromov–Witten descendant partition functions. The quantization $\hat{S}(h) \hat{R}(h)$ of $SR$ acts on power series in $h$ and $t_d^\mu$. It takes $Z^\text{pt.}^\alpha_1 \times \cdots \times Z^\text{pt.}^\alpha_k$ to some power series $Z = \exp \sum h^{g-1} F_g$. More precisely, $Z$ is in general a power series in variables $t_d^\mu$ for $d \geq 1$, whose coefficients are functions on $M$ in variables $t_d^0$. These functions are analytic on $M$ outside of the discriminant, i.e., the subvariety where the tangent Frobenius algebra is not semi-simple.

In practice it often happens that for mysterious reasons the coefficient functions of $Z$ turn out to be regular on the discriminant of $M$. In particular, this is the case for the $A_r$ singularity, which is the main example of interest for us, and for Gromov–Witten potentials of Kähler manifolds. In the sequel we will only consider the cases where $Z$ is regular at the origin and thus can be decomposed into a power series. However, the examples of [6], Section 6, show that singularities on the discriminant do appear in many cases.

Taking the lowest degree term in $h$ (that is, the coefficient of $h^{-1}$) in the logarithm of $Z$, we recover the action of $SR$.

In Section 5 we will give a precise description of the upper and lower triangular groups and their actions.

**Definition 2.7** The series $F = \ln Z$ is called a *genus expansion* of the genus zero potential $F_0$ or a *formal Gromov–Witten descendant potential*.

In terminology coming from physics, the Taylor coefficients $\langle \tau_{d_1,\mu_1} \cdots \tau_{d_n,\mu_n} \rangle_{g,D}$ of the Gromov–Witten potential $F$ are called *correlators*, while the elements of the basis $A$ are called *primary fields*.

The genus expansion of a formal genus 0 potential is not unique, because the action of the twisted loop group is not free. Indeed, the upper-triangular subgroup of the rank 1 twisted loop group acts trivially on $F^\text{pt.}_0$. The direct product of $k$ copies of the rank 1 upper-triangular subgroup forms a subgroup of the rank $k$ upper triangular subgroup. This subgroup acts trivially on $F^\text{pt.}^\alpha_1 \oplus \cdots \oplus F^\text{pt.}^\alpha_k$. But the quantizations of the elements of this subgroup do not act trivially on $Z^\text{pt.}^\alpha_1 \times \cdots \times Z^\text{pt.}^\alpha_k$, and thus we obtain several different genus expansions.

However, the ambiguity of the genus expansion can be fixed using an additional property of *homogeneity*. 
To illustrate this property, let us first assume that \( Z = \exp \sum \hbar^{g-1} F_g \) is the descendant Gromov–Witten partition function of a target Kähler manifold \( X \). Choose a homogeneous basis \( A \) of \( H^*(X) \). To each variable \( t^\mu_d \) we assign its weight \( w(t^\mu_d) = d + \deg(\mu) - 1 \), where \( \deg(\mu) \) is the algebraic degree of \( \mu \in A \). Further, introduce a weight function on the Novikov ring: \( w(Q^D) = -\langle D, c_1(TX) \rangle \) for an effective divisor \( D \). Finally, denote by \( \dim X \) the dimension of \( X \).

The expected dimension of \( X_{g,n,D} \) is \( n + (1 - g)(\dim X - 3) + \langle D, c_1(TX) \rangle \). Therefore the correlator \( \langle \tau_{d_1,\mu_1} \ldots \tau_{d_n,\mu_n} \rangle_{g,D} \) vanishes unless

\[
\sum d_i + \sum \deg(\mu_i) = n + (1 - g)(\dim X - 3) + \langle D, c_1(TX) \rangle
\]

\[
\iff \sum w(t^\mu_d) + w(Q^D) = (1 - g)(\dim X - 3).
\]

In other words, \( F_g \) is quasihomogeneous of total weight \( (1 - g)(\dim X - 3) \).

This property is formalized in the following definition.

**Definition 2.8** Introduce a weight map \( w : A \to \mathbb{Q} \) such that \( w(1) = -1 \), let \( w(t^\mu_d) = d + w(\mu) \). Also introduce a weight function on the Novikov ring. Let \( \dim \in \mathbb{Q} \) be a constant.

A formal Gromov–Witten potential \( F = \sum \hbar^{g-1} F_g \) is called *homogeneous* with respect to \( w \) and \( \dim \) if every \( F_g \) is quasihomogeneous of total weight \( (1 - g)(\dim X - 3) \).

A homogeneous formal genus 0 Gromov–Witten potential has a unique homogeneous genus expansion ([8] Proposition 6.7 c and Remark 6.9 a).

Givental conjectured that if \( F_0 \) is the geometric genus 0 potential of a target space \( X \), then the total geometric Gromov–Witten potential of \( X \) coincides with the homogeneous genus expansion of its genus 0 part.

### 2.2 Tautological classes and dual graphs

#### 2.2.1 The \( \kappa \)-classes

**Definition 2.9** The cohomology class \( \kappa_{k_1,\ldots,k_m} \) on \( \overline{M}_{g,n} \) is defined by

\[
\kappa_{k_1,\ldots,k_m} = \pi_* (\psi_{n+1}^{k_1+1} \ldots \psi_{n+m}^{k_m+1}),
\]

where

\[
\pi : \overline{M}_{g,n+m} \to \overline{M}_{g,n}
\]

is the forgetful map.
This definition is compatible with the usual definition of $\kappa$-classes $\kappa_k$ (for $m = 1$). The classes $\kappa_{k_1, \ldots, k_m}$ and the monomials $\kappa_{k_1} \ldots \kappa_{k_m}$ form two bases of the same vector space and the matrix of basis change is triangular. Indeed, we have

$$\kappa_{k_1, \ldots, k_m} = \sum_{\sigma \in S_m} \prod_{c = \text{cycle of } \sigma} \kappa_{k(c)},$$

where $k(c) = \sum_{i \in c} k_i$.

For instance,

$$\kappa_{k_1, k_2} = \kappa_{k_1} \kappa_{k_2} + \kappa_{k_1 + k_2},$$

$$\kappa_{k_1, k_2, k_3} = \kappa_{k_1} \kappa_{k_2} \kappa_{k_3} + \kappa_{k_1 + k_2} \kappa_{k_3} + \kappa_{k_1 + k_3} \kappa_{k_2} + \kappa_{k_2 + k_3} \kappa_{k_1} + 2 \kappa_{k_1 + k_2 + k_3}.$$

We prefer to work with the classes $\kappa_{k_1, \ldots, k_m}$ because they are easier to express in terms of Gromov–Witten correlators.

Let $p : \overline{M}_{g,n+1} \to \overline{M}_{g,n}$ be the forgetful map. Let $r : \overline{M}_{g-1,n+2} \to \overline{M}_{g,n}$ and $q : \overline{M}_{g_1,n_1+1} \times \overline{M}_{g_2,n_2+1} \to \overline{M}_{g,n}$ be the usual “gluing” mappings to the boundary components of $\overline{M}_{g,n}$.

**Lemma 2.10** We have

$$p^*(\kappa_{k_1, \ldots, k_m}) = \kappa_{k_1, \ldots, k_m} - \sum_{i=1}^{m} \psi_{n+1}^{k_i} \kappa_{\hat{k}_i, \ldots, k_m},$$

$$q^*(\kappa_{k_1, \ldots, k_m}) = \sum_{I \cup J = \{1, \ldots, m\}} \kappa_{k_I} \times \kappa_{k_J},$$

$$r^*(\kappa_{k_1, \ldots, k_m}) = \kappa_{k_1, \ldots, k_m},$$

where $k_I = \{k_i\}_{i \in I}$, $k_J = \{k_i\}_{i \in J}$, and $\hat{k}_i$ means that the index is omitted.

**Proof.** Only the first equality is nontrivial. Consider the forgetful map

$$\tilde{p} : \overline{M}_{g,n+1+m} \to \overline{M}_{g,n+m}.$$

To avoid confusion in indices, suppose the $n + m$ marked points are numbered from 1 to $n + m$, while the forgotten point is labeled with $\alpha$. In $H^2(\overline{M}_{g,n+m+1})$ we have $\tilde{p}^*(\psi_{n+i}) = \psi_{n+i} - D_{n+i,\alpha}$, where $D_{n+i,\alpha}$ is the divisor of curves on which the points $n + i$ and $\alpha$ lie on a separate sphere with no other marked points. From the relations

$$\psi_{n+i} D_{n+i,\alpha} = 0 \quad \text{for} \quad 1 \leq i \leq m, \quad D_{n+i,\alpha} D_{n+j,\alpha} = 0 \quad \text{for} \quad i \neq j,$$

we obtain that

$$\tilde{p}^* \left( \prod_{i=1}^{m} \psi_{n+i}^{k_i+1} \right) = \prod_{i=1}^{m} \psi_{n+i}^{k_i+1} + \sum_{i=1}^{m} \psi_{n+1}^{k_1+1} \ldots (D_{n+i,\alpha})^{k_i+1} \ldots \psi_{n+m}^{k_m+1}.$$
Taking the push-forward of this class to $\overline{\mathcal{M}}_{g,n+1}$ we obtain the right-hand side of the first equality of the lemma.

The second equality comes from the fact that each of the $m$ points forgotten by the map $\pi : \overline{\mathcal{M}}_{g,n+m} \to \overline{\mathcal{M}}_{g,n}$ can find itself on either of the two components of the boundary curves. The third equality follows directly from the definition.

\[ \diamond \]

### 2.2.2 The dual graphs

Consider a stable curve $C$ of genus $g$ with $n$ marked points in a one-to-one correspondence with a set of markings $S$. The topological type of $C$ can be described by a graph $G$ obtained by replacing every irreducible component of the curve by a vertex and every node of the curve by an edge. Every marked point is replaced by a tail (an edge that does not lead to any vertex) retaining the same marking as the marked point. Each vertex $v$ is labeled by an integer $g_v$: the geometric genus of the corresponding component. The $g_v$'s and the first Betti number of $G$ add up to $g$.

In order to avoid problems with automorphisms, we will label all the half-edges of $G$. To $G$ we assign the space

$$\overline{\mathcal{M}}_G = \prod_v \overline{\mathcal{M}}_{g_v,n_v},$$

where the product goes over the set of vertices of $G$, $g_v$ is the genus of the vertex $v$, and $n_v$ its valency (the number of half-edges and tails adjacent to it). The space $\overline{\mathcal{M}}_G$ comes with a natural map $p : \overline{\mathcal{M}}_G \to \overline{\mathcal{M}}_{g,n}$ whose image is the closure of the set of stable curves homeomorphic to $C$. Note that $p_*[\overline{\mathcal{M}}_G] = |\text{Aut}(G)| \cdot [p(\overline{\mathcal{M}}_G)]$.

We can define a cohomology class on $\overline{\mathcal{M}}_G$ (and hence on $\overline{\mathcal{M}}_{g,n}$ taking the push-forward by $p$) by assigning a class $\kappa_{k_1,\ldots,k_m}$ to each vertex of $G$ and a power $\psi^d$ of the $\psi$-class to each half-edge and each tail of $G$.

**Definition 2.11** A graph $G$ with labeled half-edges and tails, describing the topological type of a stable curve, with an additional label $\kappa_{k_1,\ldots,k_m}(v)$ assigned to each vertex $v$ and a nonnegative integer $d_e$ assigned to each half-edge and tail $e$ is called a **stable dual graph**. The corresponding cohomology class of $\overline{\mathcal{M}}_{g,n}$ is called the **tautological class** assigned to $G$ and denoted by $[G]$. The **genus** of a dual graph is the genus of the corresponding stable curves, its **degree** is the algebraic degree of the corresponding cohomology class, and its **dimension** is $\dim \overline{\mathcal{M}}_{g,n} - \text{degree} = 3g - 3 + n - \text{degree}$.

This definition allows several modifications.
First of all, we can consider not necessarily connected dual graphs. They represent tautological classes on direct products of several moduli spaces.

Second, we can consider dual graphs that describe the topology of a semi-stable curve that is not necessarily stable. In this case we will also label each tail of the graph with a primary field \( \mu \in A \). If we work over a Novikov ring, we also assign an effective 2-cycle \( D \in E \) to the whole graph. Such a graph will be called a semi-stable dual graph with primary fields. It describes a tautological cohomology class in the space of stable maps \( \overline{M}_{g,n+n'}(X,D) \).

A “dual graph” with no other specifications will mean “stable connected dual graph”.

### 2.3 Universal relations

Let \( F = \sum \hbar^{g-1} F_g \) be the geometric Gromov–Witten potential of some target Kähler manifold \( X \), \( A \) a basis of \( H^*(X) \) and \( \eta \) the Poincaré pairing. Let \( L = \sum c_i G_i \) be a linear combination of dual graphs representing a class \( [L] \in H^*(\overline{M}_{g,n}) \). As in the introduction, let \( X_{g,n+n',D} = \overline{M}_{g,n+n'}(X,D) \), where \( D \in E \) is an effective 2-cycle, let \( [X_{g,n+n',D}] \) be its virtual fundamental class, and let \( p : X_{g,n+n',D} \to \overline{M}_{g,n} \) be the forgetful map. We are going to describe a way to express the integrals of the form

\[
\sum_{D \in E} Q^D \int_{[X_{g,n+n',D}]} p^*([L]) \prod_{i=1}^{n+n'} \psi_{d_i}^{d_i} \operatorname{ev}_i^*(\mu_i)
\]

via \( \eta^{\mu\nu} \) and the coefficients of the series \( F_0, \ldots, F_g \).

We start with assigning a polynomial in correlators and coefficients \( \eta^{\mu\nu} \) to any stable dual graph or any semi-stable dual graph with primary fields that does not contain \( \kappa \)-classes.

**Definition 2.12** Let \( G \) be a stable dual graph or a semi-stable dual graph with primary fields such that no \( \kappa \)-classes are assigned to its vertices. We define the polynomial \( P_G \) by the following procedure. (i) Assign a primary field \( \mu \in A \) to every half-edge of \( G \). In the case of a stable dual graph, assign, moreover, the distinguished primary field \( 1 \in A \) to the tails. (ii) To every vertex \( v \) assign the correlator \( \langle \tau_{d_1,\mu_1} \cdots \tau_{d_{n_v},\mu_{n_v}} \rangle_{g_v} \), where \( g_v \) is the genus of \( v \), \( n_v \) its valency, and \( d_i, \mu_i \) the labels on the half-edges and tails adjacent to \( v \). (iii) To every edge assign the coefficient \( \eta^{\mu\nu} \), where \( \mu \) and \( \nu \) are the primary fields corresponding to its half-edges. (iv) Take the product of all the correlators and the coefficients \( \eta^{\mu\nu} \) thus obtained. (v) Sum over all the ways to attribute primary fields to the half-edges. If we work over a Novikov
ring, we must also sum over all the ways to assign effective 2-cycles $D_v$ to the vertices in such a way that $\sum D_v = D$.

Note that every edge of $G$ introduces a contraction of indices via the bilinear form $\eta$. This comes from the fact that the class of the diagonal in $X \times X$ equals $\sum_{\mu,\nu} \eta^\mu\nu \mu \times \nu$.

Note also that the definition works perfectly well for not necessarily connected dual graphs.

Now we go back to our problem of constructing a pull-back in $X_{g,n+n',D}$ of the tautological class $[L]$. Let $G$ be a stable dual graph participating in the linear combination $L$.

**Step 1: eliminating the $\kappa$-classes.** If a vertex $v$ of $G$ is labeled with $\kappa_{k_1},\ldots,k_m$, erase this label and replace it by $m$ new tails issuing from $v$ with labels $\psi^{k_1+1},\ldots,\psi^{k_m+1}$ on them. Thus we obtain a new dual graph $G_1$. The newly added tails will be called $\kappa$-tails.

This rule is justified by the following remark: if $\pi : \overline{M}_{g,n+m} \to \overline{M}_{g,n}$ is the forgetful map, we have

$$\psi_1^{d_1} \ldots \psi_n^{d_n} \kappa_{k_1},\ldots,k_m = \pi_*(\psi_1^{d_1} \ldots \psi_n^{d_n} \psi_1^{k_1+1} \ldots \psi_n^{k_m+1})$$

in $H^*(\overline{M}_{g,n})$.

**Example 2.13** Let

$$G = \begin{array}{ccc}
\psi^0 & \kappa_1 & \psi^0 \\
g = 2 & & g = 1 \\
\end{array}$$

(here $g = 3$, $n = 1$). Then we have

$$P_{G_1} = \sum_{\mu,\nu \in A} \langle \tau_{0,1} \tau_{2,1} \tau_{0,\nu} \rangle_2 \eta^\nu\mu \langle \tau_{1,\mu} \rangle_1.$$ 

**Step 2: recomputing the $\psi$-classes.** There is a difference between the $\psi$-classes on $\overline{M}_{g,n}$ and on $X_{g,n+n',D}$, because of the presence of additional marked points and because of the appearance of semi-stable source curves. To take this into account, we modify $G_1$ according to the following rule: replace every half-edge and every tail with label $\psi^d$ by the linear combination

$$\bullet \psi^d$$
\[
\sum_{p=0}^{d} (-1)^p \sum_{d_0 + \ldots + d_p = d-p} \psi^{d_p} \psi^0 0 \psi^{d_p-1} \ldots \psi^0 0 \psi^{d_1} \psi^0 0 \psi^{d_0}
\]

Thus we obtain a linear combination of dual graphs \( G_2 \). When we perform Steps 1 and 2 with all graphs of the linear combination \( L \), we obtain a new linear combination \( L_2 \) of semi-stable dual graphs (without primary fields). It represents the class \( p^*(L) \) and will therefore be denoted by \( L_2 = p^*(L) \).

In terms of polynomials \( P_{G_1} \) assigned to the dual graphs, replacing \( G_1 \) by the linear combination of dual graphs \( G_2 \) is equivalent to making the following substitutions in \( P_{G_1} \):

\[
\tau_{1,\mu} \mapsto \tau_{1,\mu} - \sum_{\mu_1,\mu_1} \langle \tau_{0,\mu_0,\mu_1} \rangle_0 \eta^{\mu_1\mu_1} \tau_{0,\mu_1},
\]

\[
\tau_{2,\mu} \mapsto \tau_{2,\mu} - \sum_{\mu_1,\mu_1} \langle \tau_{1,\mu_0,\mu_1} \rangle_0 \eta^{\mu_1\mu_1} \tau_{0,\mu_1} - \sum_{\mu_1,\mu_2} \langle \tau_{0,\mu_0,\mu_2} \rangle_0 \eta^{\mu_1\mu_1} \tau_{1,\mu_1}
\]

\[
+ \sum_{\mu_1,\mu_1,\mu_2,\mu_2} \langle \tau_{0,\mu_0,\mu_1} \rangle_0 \eta^{\mu_1\mu_1} \langle \tau_{0,\mu_1,\mu_2} \rangle_0 \eta^{\mu_2\mu_2} \tau_{0,\mu_2},
\]

and so on. In general, every insertion of \( \tau_{d,\mu} \) must be replaced by

\[
\tau_{d,\mu} + \sum_{p=1}^{d} (-1)^p \sum_{d_0 + \ldots + d_p = d-p} \langle \tau_{d_0,\mu_0,\mu_1} \rangle_0 \eta^{\mu_1\mu_1} \langle \tau_{d_1,\mu_1,\mu_2} \rangle_0 \eta^{\mu_2\mu_2} \ldots \eta^{\mu_p\mu_p} \tau_{d_p,\mu_p}, \tag{3}
\]

This formula is used in the following way: the symbol \( \tau_{d,\mu} \) was part of some correlator \( \langle x \rangle \); now we put \( \tau_{d_p,\mu_p} \) in its place, while the other factors of the formula become factors in front of \( \langle x \rangle \).

It is important to note that the procedure of Step 2 for expressing ancestor \( \psi \)-classes in terms of descendant \( \psi \)-classes is universal. This means that the expression remains valid for any number of additional marked points \( n' \) and for any target manifold \( X \).

**Step 3:** multiplying by \( \prod_{i=1}^{n+n'} \psi_i^{d_i} \text{ev}^* \psi_i^*(\mu) \). Let \( G_2 \) be one of the dual graphs involved in the linear combination \( L_2 \). It has two kinds of tails: the \( \kappa \)-tails and the ordinary tails numbered from 1 to \( n \). Add \( n' \) more tails numbered from \( n+1 \) to \( n+n' \) by attaching them to the vertices of \( G_2 \) in all possible ways.

Now, the \( \kappa \)-tails already bear labels \( \psi^k \). We also label them with the distinguished primary field 1.

The tails from 1 to \( n \) also bear labels \( \psi^{d_i}, 1 \leq i \leq n \). We replace \( \psi^{d_i} \) on the \( i \)-th tail by \( \psi^{d_i+d_i} \) and also label the \( i \)-th tail with the primary field \( \mu_i \).
The tails from $n + 1$ to $n + n'$ have no labels. We label them with $\psi^{d_i}$ and $\mu_i$, $n + 1 \leq i \leq n + n'$.

Thus we obtain a linear combination $L_3$ of semi-stable dual graphs with primary fields. This linear combination will be denoted by $L_3 = p^*(L) \prod_{i=1}^{n+n'} \psi_i^{d_i} \text{ev}_i^*(\mu_i)$.

In terms of polynomials $P_G$, replacing $P_{L_2}$ by $P_{L_3}$ is equivalent to (i) replacing the symbols $\tau_{d_i,1}$ corresponding to the tails from 1 to $n$ by symbols $\tau_{d_i+d_i,\mu_i}$ and (ii) inserting new symbols $\tau_{d_i,\mu_i}$ for $n + 1 \leq i \leq n + n'$ in the existing correlators in all possible ways.

The corresponding polynomial in correlators will be denoted by $P_{L_3} = \left\langle p^*(L) \prod_{i=1}^{n+n'} \psi_i^{d_i} \text{ev}_i^*(\mu_i) \right\rangle_g$.

If $F$ is the geometric Gromov–Witten potential of some target space $X$, we have

$$\left\langle p^*(L) \prod_{i=1}^{n+n'} \psi_i^{d_i} \text{ev}_i^*(\mu_i) \right\rangle_g = \sum_{D \in H_2(X)} \int_{[X,g,n+n',D]} p^*([L]) \prod_{i=1}^{n+n'} \psi_i^{d_i} \text{ev}_i^*(\mu_i).$$

However it will be important for us that the expression $\left\langle p^*(L) \prod_{i=1}^{n+n'} \psi_i^{d_i} \text{ev}_i^*(\mu_i) \right\rangle_g$ makes sense for any power series $F = \sum \hbar^{g-1} F_g$ in variables $t^g_{\mu_i}$, in particular for formal Gromov–Witten potentials. Note that if two different linear combinations $L$ and $L'$ of dual graphs represent the same cohomology class, it is, for the time being, not at all obvious that $\left\langle p^*(L) \prod_{i=1}^{n+n'} \psi_i^{d_i} \text{ev}_i^*(\mu_i) \right\rangle_g$ and $\left\langle p^*(L') \prod_{i=1}^{n+n'} \psi_i^{d_i} \text{ev}_i^*(\mu_i) \right\rangle_g$ coincide for every formal Gromov–Witten potential.

**Definition 2.14** Let $L$ be a linear combination of dual graphs. Let $F = \sum \hbar^{g-1} F_g$ be a power series in variables $t^g_{\mu_i}$. The infinite vector of the values of $\left\langle p^*(L) \prod_{i=1}^{n+n'} \psi_i^{d_i} \text{ev}_i^*(\mu_i) \right\rangle_g$ for all $n' \geq 0$ and for all $d_i, \mu_i$, is called the *induced vector* of $L$ and is denoted by $F_L$.

**Definition 2.15** A linear combination $L$ of dual graphs is called a *universal relation* if the vector $F_L$ is equal to 0 for any semi-simple formal Gromov–Witten descendant potential $F$, as defined by Givental.

**Proposition 2.16** The vanishing of $F_L$ can be expressed as an infinite family of partial differential equations on $F_0$, $F_1$, . . . .
This is a standard fact so we do not prove it here but instead illustrate it with an example.

**Example 2.17** The tautological relation

\[
\begin{bmatrix}
g = 1 \\
\psi
\end{bmatrix} - \frac{1}{12} \begin{bmatrix}
g = 0
\end{bmatrix} = 0
\]

gives rise to the following family of partial differential equations indexed by \(d \geq 0\) and \(\rho \in A\):

\[
\frac{\partial F_1}{\partial t_{d+1}} - \sum_{\mu,\nu} \frac{\partial F_1}{\partial t_0}\eta^{\mu\nu} \frac{\partial^2 F_0}{\partial t_0^\rho dt_d^\nu} - \frac{1}{24} \sum_{\mu,\nu} \frac{\partial^3 F_0}{\partial t_0^\mu \partial t_0^\nu dt_d^\rho \eta^{\mu\nu}} = 0.
\]

The notion of universal relation is naturally generalized to not necessarily connected stable dual graphs, but this should be done carefully. The simplest way to define a universal relation in this case is the following.

Consider any graph as a product of its connected components. Then a linear combination \(L\) of not necessarily connected stable dual graphs is called a *universal relation* if it can be represented as a sum of several products such that every term of the sum contains a universal relation for connected graphs as a factor.

In other words, if we consider the product \(\overline{\mathcal{M}}_{g_1,n_1} \times \cdots \times \overline{\mathcal{M}}_{g_k,n_k}\), a universal relation in one of the factors times any class in the product of the remaining factors is a universal relation, and a linear combination of universal relations is still a universal relation.

**Remark 2.18** The method we used to define the induced vector \(F_L\) works for a linear combination \(L\) of not necessarily connected stable dual graphs. However, in this case a more natural notion is that of an *extended induced vector* \(\hat{F}_L\). Its definition is similar to that of \(F_L\), with the difference that when we add new marked points to the curve, we are allowed to prescribe the connected component to which every point should go. \(L\) is a universal relation if and only if \(\hat{F}_L\) vanishes. We do not know whether the conditions \(\hat{F}_L = 0\) and \(F_L = 0\) are equivalent.

Now we can give a precise formulation of the theorem announced in the introduction.

Let \(F = \sum \h^{g-1} F_g\) be a power series in \(\h\) and \(t_d^\mu\), where \(F_0\) is a formal genus 0 Gromov–Witten potential. We introduce five properties of \(F\).

1. **Homogeneity.** \(F\) is homogeneous (in the sense of Definition 2.8) for some \(w\) and \(\text{dim.}\)
2. **Geometricity.** $F_L$ vanishes for every tautological relation $L$.

3. **Semi-simplicity.** $F_0$ is semi-simple at some point $t$.

4. **Reducibility to genus 0.** Every correlator of $F$ can be expressed in terms of genus 0 correlators using only properties 1 and 2.

5. **Analyticity.** The homogeneous genus expansion of $F_0$ is regular at the origin.

All geometric Gromov–Witten potentials of target Kähler manifolds satisfy conditions 1 and 2. Conditions 3, 4, and 5, on the other hand, must be checked in every particular case.

In Section 4 we will see an example of a potential that satisfies all five conditions without being the geometric potential of a target manifold.

**Theorem 3** A power series $F$ satisfying conditions (1-5) coincides with the homogeneous genus expansion of $F_0$.

**Proof.** Let $\hat{F}$ be the homogeneous genus expansion of $F_0$. The expressions $F_L$ and $\hat{F}_L$ vanish for all tautological relations $L$ (the vanishing of $\hat{F}_L$ follows from Theorems 1 and 4). Similarly, the nonhomogeneous correlators vanish both for $F$ and for $\hat{F}$. According to condition 4, these vanishing conditions are enough to express every correlator in terms of genus 0 correlators. But the genus 0 correlators coincide, since both are given by $F_0$. Thus $F = \hat{F}$. ♦

In [24], Y.-P. Lee studied the action of Givental’s group on the induced vectors $F_L$ and constructed an algorithm that computes certain, conjecturally all, universal relations. We are now going to state his results.

### 2.4 Y.-P. Lee’s algorithm

#### 2.4.1 The operators $\tau_k$

We are going to define linear operators $\tau_k$ acting on the space of linear combinations of dual graphs. Here $k$ is an arbitrary positive integer.

If $G$ is a connected dual graph whose tails are labeled by a set $S$, then $\tau_k(G)$ is a linear combination of not necessarily connected dual graphs with labeling set $S \cup \{\alpha, \beta\}$. The graphs of this linear combination are obtained from $G$ by the following operations.
1. Cut an edge of $G$ into two tails. Change their labels to $\alpha$ and $\beta$ in both possible ways. If the labels on the half-edges were $\psi^a$ (on $\alpha$) and $\psi^b$ (on $\beta$) before the cutting, we now label them first with $\psi^{a+k}$ and $\psi^b$ and then with $\psi^a$ and $\psi^{b+k}$. The first stable graph thus obtained is taken with coefficient 1 while the second is taken with coefficient $(-1)^{k-1}$.

\[
\begin{bmatrix}
\psi^a & \psi^b
\end{bmatrix} \longrightarrow \begin{bmatrix}
\psi^{a+k} & \psi^b
\end{bmatrix} + (-1)^{k-1} \begin{bmatrix}
\psi^a & \psi^{b+k}
\end{bmatrix} + \text{two more terms.}
\]

2. Split a vertex $v$ of $G$ in two, and add a new tail on each of them, one marked by $\alpha$ and the other one by $\beta$. If the genus of $v$ was $g$, assign to the new vertices genera $g_1$ and $g_2$ such that $g_1 + g_2 = g$ in all possible ways. Distribute the edges that were going out of $v$ between the two vertices in all possible ways. If $v$ carried the label $\kappa_{k_1, \ldots, k_m}$, split the set $\{k_1, \ldots, k_m\}$ in two disjoint subsets $I$ and $J$ in all possible ways and assign to the new vertices the labels $\kappa_I$ and $\kappa_J$. Label tail $\alpha$ with $\psi^i$ and tail $\beta$ with $\psi^j$, in all possible ways with the condition $i + j = k - 1$. The dual graph thus obtained is taken with coefficient $(-1)^{j+1}$. Keep only stable graphs and sum over all the possibilities described above.

\[
\begin{bmatrix}
g
\end{bmatrix} \kappa_{k_1, \ldots, k_m} \longrightarrow \sum (-1)^{j+1} \begin{bmatrix}
g_1 & \psi^i & \psi^j & g_2
\end{bmatrix} + \sum (-1)^{j+1} \begin{bmatrix}
g_1 & \psi^i & \psi^j & g_2
\end{bmatrix} + \ldots
\]

3. Choose a vertex of $G$, decrease its genus by 1 and add two new tails on it, one marked by $\alpha$ and the other one by $\beta$. Label tail $\alpha$ with $\psi^i$ and tail $\beta$ with $\psi^j$, in all possible ways with the condition $i + j = k - 1$. The dual graph thus obtained is taken with coefficient $(-1)^{j+1}$. Sum over all possible $i$ and $j$.

\[
\begin{bmatrix}
g
\end{bmatrix} \longrightarrow \sum (-1)^{j+1} \begin{bmatrix}
\psi^j
\end{bmatrix} + \begin{bmatrix}
\psi^i
\end{bmatrix} + \ldots
\]
The operations $\tau_k$ are extended to linear combinations of dual graphs by linearity.

### 2.4.2 The algorithm

**Theorem 4 (Y.-P. Lee)** A linear combination $L$ of dual graphs is a universal relation whenever (i) $F^\text{pt}_L = 0$ and (ii) $\tau_k(L)$ is a universal relation for all $k \geq 1$.

This theorem is actually an algorithm for computing universal relations for Gromov–Witten potentials. Indeed, the vector $F^\text{pt}_L$ has an infinite number of entries, but they can be expressed via a finite number of entries using the string and dilaton equations. Therefore it is enough to check Condition (i) for a finite number of entries. Now, the dimension (see Definition 2.11) of $\tau_k(L)$ is smaller than that of $L$, therefore we can proceed by induction on the dimension of the relations.

Y.-P. Lee conjectures that this algorithm finds all universal relations on formal Gromov–Witten potentials and that these universal relations arise from tautological relations in the tautological ring of $\overline{M}_{g,n}$. However, neither of these claims is proved.

Y.-P. Lee has also observed that in many cases Condition (i) is unnecessary and conjectured that checking Condition (ii) is enough for $\dim L \geq 1$. However, it follows from our geometric interpretation that this conjecture is wrong. The first example where it fails is in $\overline{M}_{2,1}$ in dimension 1.

Y.-P. Lee’s proof of Theorem 4 is summarized in Section 5.

### 3 Geometrical interpretation

In this section we give a geometric interpretation of the operators $\tau_k$ as intersections with the boundary. (This interpretation was discovered independently by Y.-P. Lee and R. Pandharipande - private communication.) Using it, we prove that a linear combination $L$ of dual graphs is obtained as a relation in Y.-P. Lee’s algorithm if and only if the intersection of $[L]$ with all tautological classes of complementary dimension vanishes.

#### 3.1 Operators $\tau_i$ and boundary classes

In the universal curve $\overline{C}_{g,n}$, consider the codimension 2 subvariety $\Delta$ of nodes in the singular fibers of $\overline{C}_{g,n}$. Each point of $\Delta$ is a node of a stable curve and we will label by $\alpha$ and $\beta$ the two marked points of its normalization identified at this node. This can be done in two ways, hence we obtain a double covering.
of $\Delta$ that we will call $D$. The space $D = D_{g,n}$ has one connected component isomorphic to $\overline{M}_{g-1,n+2}$ (unless $g = 0$) and $(g+1)2^n - 2(n+1)$ other connected components, each of which is isomorphic to $\overline{M}_{g_i,n_i+1} \times \overline{M}_{g_2,n_2+1}$ for suitable $g_i$ and $n_i$ with $g_1 + g_2 = g$ and $n_1 + n_2 = n$. It comes with a natural map $p : D \to \overline{M}_{g,n}$, whose image is the boundary $\partial \overline{M}_{g,n} = \overline{M}_{g,n} \setminus M_{g,n}$. Note that all tautological classes supported on $\partial \overline{M}_{g,n}$ are defined as push-forwards of classes on $D$ under $p$.

On $D$ we define the following cohomology classes:

$$
\begin{align*}
\rho_1 &= 1 \\
\rho_2 &= \psi_\alpha - \psi_\beta \\
\rho_3 &= \psi_\alpha^2 - \psi_\alpha \psi_\beta + \psi_\beta^2 \\
\rho_4 &= \psi_\alpha^3 - \psi_\alpha^2 \psi_\beta + \psi_\alpha \psi_\beta^2 - \psi_\beta^3
\end{align*}
$$

and so on.

**Proposition 3.1** Let $G$ be a dual graph of genus $g$ with $n$ tails. Then we have

$$[	au_k(G)] = -\rho_k p^*[G]$$

in the cohomology ring of $D$.

**Corollary 3.2** Let $L$ be a linear combination of dual graphs and suppose that the class $[	au_1(L)]$ vanishes (respectively, has zero intersection with all tautological classes of complementary dimension). Then the class $[\tau_k(L)]$ vanishes (respectively, has zero intersection with all tautological classes of complementary dimension) for all $k$.

**Proof.** The class $[\tau_k(L)]$ is obtained from $[\tau_1(L)]$ by a multiplication by $\rho_k$.  

This corollary confirms Y.-P. Lee’s experimental observation that requiring $\tau_k(L)$ to be a universal relation for all $k \geq 1$ is equivalent to requiring just $\tau_1(L)$ to be a universal relation ([23], Section 2.2, Remark (iii)).

**Proof of Proposition 3.1.** The main idea of the proof is very simple. On every boundary component of $\overline{M}_{g,n}$ we can define the classes $\psi_\alpha$ and $\psi_\beta$ corresponding to the marked points $\alpha$ and $\beta$ identified at the node. It is well-known that the first Chern class of the normal line bundle to the boundary component in $\overline{M}_{g,n}$ equals $-(\psi_\alpha + \psi_\beta)$.

Now, when we intersect a tautological class with a boundary component of $\overline{M}_{g,n}$ two cases can occur: either the class is entirely contained in the
component, or it intersects it transversally. In the first case we must multiply our class by the first Chern class of the normal line bundle $-(\psi^{\alpha} + \psi^{\beta})$ and then by $-\rho_k$. Their product is equal to $\psi^{k}\psi^{\alpha} + (-1)^{k-1}\psi^{\beta}$. In the second case, we must add a new node, that either separates a component of the curve in two or is a nonseparating node. In both cases we multiply the class thus obtained by $-\rho_k = \sum_{i+j=k-1}(-1)^{i+1}\psi^{i}\psi^{j}$. These three possibilities correspond to the three operations in the definition of $\tau_k$.

Now we present the proof with all necessary details.

To begin with, consider the case where $G$ is a dual graph of genus $g$ with $n$ tails with trivial labels assigned to all vertices, tails, and half-edges. Thus $[G]$ is the cohomology class $p_G^*\overline{\mathcal{M}}_G$ on $\overline{\mathcal{M}}_{g,n}$.

The computation of $p^*[G]$ fits in the framework of [12], A.4. The contributing graphs are of two kinds: either they have one more edge than $G$ or they have the same number of edges as $G$.

Consider a contributing graph with one extra edge. Contracting the extra edge, we obtain the graph $\Gamma$ and the contracted edge determines a unique vertex $v$ of $G$; let $g_v$ be the geometric genus of the corresponding component and let $n_v$ be the valence of $G$ at $v$ including both half-edges and tails. Unless $g_v = 0$, there is exactly one contributing graph corresponding to $v$ whose extra edge is a self-edge at a vertex with geometric genus $g_v - 1$. Moreover, there are exactly $(g_v+1)2^{n_v} - 2(n_v+1)$ other contributing graphs corresponding to $v$ whose extra edge connects two distinct vertices. In other words, the contributing graphs whose extra edge contracts to $v$ are in one-to-one correspondence with the connected components of the double covering $\mathcal{D}_{g_v,n_v}$.

Let $\Gamma$ be a contributing graph with extra edge $e$. Let $h_\alpha$ and $h_\beta$ be the labeled half-edges constituting $e$ and let $\alpha$ and $\beta$ be the corresponding tails arising upon cutting $e$. If cutting $e$ disconnects the graph, put

$$\overline{\mathcal{M}}_e := \overline{\mathcal{M}}_{g_\alpha,n_\alpha} \times \overline{\mathcal{M}}_{g_\beta,n_\beta}$$

where $g_\alpha$ resp. $g_\beta$ and $n_\alpha$ resp. $n_\beta$ are the genus and the number of tails of the connected component containing $\alpha$ resp. $\beta$. If $e$ is non-disconnecting, put

$$\overline{\mathcal{M}}_e := \overline{\mathcal{M}}_{g-1,n+2}.$$ 

Further, let

$$p_{\Gamma,e} : \overline{\mathcal{M}}_{\Gamma} \rightarrow \overline{\mathcal{M}}_e \subset \mathcal{D}_{g,n}$$

be the map obtained via cutting $e$ and contracting all other edges. Then by [12], A.4 the contribution of $(\Gamma, e)$ to $p^*[G]$ equals

$$p_{\Gamma,e}[\overline{\mathcal{M}}_{\Gamma}].$$
Next, consider a contributing graph with the same number of edges as $G$. The graph may then be identified with $G$ and exactly one edge $e$ of $G$ has been selected: it is identified by the excess line bundle. Define $\overline{\mathcal{M}}_e$ as above and let

$$p_{G,e}: \overline{\mathcal{M}}_G \to \overline{\mathcal{M}}_e \subset \mathcal{D}_{g,n}$$

be the map obtained via cutting $e$ and contracting all other edges. Then by [12], A.4 the contribution of $(G,e)$ to $p^*[G]$ equals

$$p_{G,e*}(-\psi_\alpha - \psi_\beta) = (-\psi_\alpha - \psi_\beta)p_{G,e*}[\overline{\mathcal{M}}_G]$$

(with a slight abuse of notation).

We find the following formula for $p^*[G]$:

$$p^*[G] = \sum_{(\Gamma,e): \Gamma/e=G} p_{\Gamma,e*}[\Gamma] + \sum_{e=(h_\alpha,h_\beta) \in E(G)} (-\psi_\alpha - \psi_\beta)p_{G,e*}[G].$$

If $v_0(G)$ denotes the number of vertices of $G$ of geometric genus 0, the number of summands in the first sum equals

$$\sum_{v \in V(G)} (g_v + 1)2^{n_v} - 2n_v - 2 + |V(G)| - v_0(G),$$

while the second sum runs over the half-edges of $G$.

Having dealt with the case of a dual graph with trivial labels, we consider the case where $G$ is a decorated dual graph: each vertex $v$ of $G$ has been assigned a class $\kappa_{K_v}$, where $K_v$ is a collection of $m_v$ nonnegative integers, and each half-edge or tail $h$ has been assigned a power $\psi_h^{d_h}$ of its $\psi$-class $\psi_h$.

Each half-edge or tail on $G$ determines a unique such on $\Gamma$. Taking this into account, the formula above for $p^*[G]$ continues to hold in the case of decorated half-edges or tails.

Let $v$ be a vertex of $G$. In case the extra edge $e$ on $\Gamma$ doesn’t contract to $v$, a unique vertex $w$ of $\Gamma$ corresponds to $v$ and it is decorated with the corresponding class $\kappa_{K_v}$. We proceed analogously in case $e$ is a self-edge at a vertex $w$ (with genus $g_w = g_v - 1$) contracting to $v$. Finally, if $e$ contracts to $v$ and connects two distinct vertices, the decoration $\kappa_{K_v}$ has to be divided up in all $2^{m_v}$ possible ways over the two vertices (cf. Lemma 2.10). The formula above for $p^*[G]$ continues to hold for an arbitrary decorated dual graph; note that the number of summands in the first sum equals

$$\sum_{v \in V(G)} 2^{m_v}((g_v + 1)2^{n_v} - 2n_v - 2) + |V(G)| - v_0(G).$$

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These considerations prove the statement for \( \tau_1 \). The statement for \( \tau_k \) follows from the trivial remark that
\[
\rho_k(\psi_\alpha + \psi_\beta) = \psi_\alpha^k + (-1)^{k-1}\psi_\beta^k.
\]

\[\Box\]

3.2 The algorithm and the Gorenstein conjecture

The Gorenstein conjecture [29] states that the tautological ring of \( \overline{M}_{g,n} \) has the form of the cohomology ring of a smooth manifold, that is, it has a unique top degree class and a non-degenerate Poincaré duality. The conjecture is made for the \( \mathbb{Q} \)-subalgebra \( R^*(\overline{M}_{g,n}) \) of the rational Chow ring generated by the tautological classes. In this case the first claim holds (see [13, 7]), but little is known about the second claim. In this paper we consider the Gorenstein conjecture for the tautological cohomology ring \( RH^*(\overline{M}_{g,n}) \), the image in cohomology of \( R^*(\overline{M}_{g,n}) \). The first claim is then obvious: the top degree class is the class of a point and it can be represented by any dual graph of maximal degeneration without \( \psi \)- and \( \kappa \)-classes. The second claim is again not known.

Definition 3.3 We say that a tautological cohomology class on \( \overline{M}_{g,n} \) is Gorenstein vanishing if its intersection numbers with all tautological classes of complementary dimension vanish.

Here we are going to prove Theorem 1 stating that a linear combination \( L \) of dual graphs appears as a relation in Y.-P. Lee’s algorithm if and only if the class \([L]\) is Gorenstein vanishing.

Proof of Theorem 1. Denote by \( p : \overline{M}_{g,n+n'} \to \overline{M}_{g,n} \) the forgetful map and by \( q : \mathcal{D} \to \overline{M}_{g,n} \) the natural projection.

Taking into account the form of Y.-P. Lee’s algorithm (Section 2.4.2), and the geometric interpretation of the operators \( \tau_k \), the theorem can be reformulated as follows:

A tautological class \( \alpha \in H^*(\overline{M}_{g,n}) \) is Gorenstein vanishing if and only if

(i) all the intersection numbers

\[
\int_{\overline{M}_{g,n+n'}} p^*(\alpha) \prod_{i=1}^{n+n'} \psi_i^{d_i},
\]

vanish; and
(ii) the classes $\rho_k q^*(\alpha)$ are Gorenstein vanishing for all $k$.

Since $\rho_1 = 1$, the second condition can, of course, be replaced by the condition $q^*(\alpha)$ is Gorenstein vanishing.

**The only if part.** Suppose $\alpha$ is Gorenstein vanishing. Then
\[
\int_{\mathcal{M}_{g,n+n'}} p^*(\alpha) \prod_{i=1}^{n+n'} \psi_i^{d_i} = \int_{\mathcal{M}_{g,n}} \alpha p^* \left( \prod_{i=1}^{n+n'} \psi_i^{d_i} \right) = 0,
\]
because the class $p^* \left( \prod_{i=1}^{n+n'} \psi_i^{d_i} \right)$ is tautological.

Similarly, if $\beta$ is a tautological class on $\mathcal{D}$, then
\[
\int_{\mathcal{D}} q^*(\alpha) \beta = \int_{\mathcal{M}_{g,n}} \alpha q^* (\beta) = 0,
\]
because $q^* (\beta)$ is tautological.

**The if part.** Suppose $\alpha$ satisfies Conditions (i) and (ii), and let $G$ be a dual graph of genus $g$ with $n$ tails, of complementary dimension to $\alpha$. We wish to prove that $\alpha \cap [G] = 0$.

First consider the case when $G$ has no edges. This means that $[G]$ is a polynomial in $\psi$- and $\kappa$-classes. A class like that can be represented as a linear combination of several classes $p^* \left( \prod_{i=1}^{n+n'} \psi_i^{d_i} \right)$. But, by the same equality that we used in the “if” part, we have
\[
\int_{\mathcal{M}_{g,n}} \alpha p^* \left( \prod_{i=1}^{n+n'} \psi_i^{d_i} \right) = \int_{\mathcal{M}_{g,n+n'}} p^* (\alpha) \prod_{i=1}^{n+n'} \psi_i^{d_i} = 0
\]
by Condition (i).

Now suppose $G$ has at least one edge. Then the class $[G]$ is the push-forward of some tautological class $\beta$ on $\mathcal{D}$, in other words, $[G] = q^* (\beta)$. Thus
\[
\alpha \cap [G] = \int_{\mathcal{M}_{g,n}} \alpha q^* (\beta) = \int_{\mathcal{D}} q^* (\alpha) \beta = 0,
\]
because $q^* (\alpha)$ is Gorenstein vanishing by Condition (ii).

\[\Diamond\]

4 A proof of Witten’s conjecture

In this section we explain in more detail the plan of the proof of Witten’s conjecture outlined in the introduction.
4.1 Witten’s conjecture and Gromov–Witten theories

The generating series studied in Witten’s conjecture is

\[ F^r = \sum_{g \geq 0} \sum_{n \geq 1} \int_{\overline{M}_{g,n}} c_W(a_1, \ldots, a_n) \psi_1^{d_1} \ldots \psi_n^{d_n} \cdot \frac{t_1^{a_1} \ldots t_n^{a_n}}{n!}, \]

where \( c_W(a_1, \ldots, a_n) \), also depending on \( r \) and \( g \), is the Witten class briefly introduced in the introduction.

Although this series is not the Gromov–Witten potential of any target space, it is part of the framework of Gromov–Witten theories, because its genus 0 part satisfies the string, the dilaton, and the TRR equations (see [16]).

The set of primary fields \( A \), its distinguished element, and the bilinear form \( \eta \) (that participates in the TRR equation) are determined by the following properties of Witten’s class, proved in [30].

1. If \( a_i = r - 1 \) for some \( i \), then \( c_W(a_1, \ldots, a_n) = 0 \). Thus the set \( A \) equals \( \{0, \ldots, r - 2\} \).
2. Let \( p : \overline{M}_{g,n+1} \to \overline{M}_{g,n} \) be the forgetful map. Then

\[ p^*(c_W(a_1, \ldots, a_n)) = c_W(a_1, \ldots, a_n, 0). \]

It follows that the distinguished primary field is 0. (There is an unfortunate clash of notation with the Givental theory, where the distinguished primary field is usually denoted by 1, a convention that we followed in our paper except in the applications to Witten’s conjecture.)

3. Let \( r : \overline{M}_{g-1,n+2} \to \overline{M}_{g,n} \) and \( q : \overline{M}_{g_1,n_1+1} \times \overline{M}_{g_2,n_2+1} \to \overline{M}_{g,n} \) be the gluing mappings to the boundary components of \( \overline{M}_{g,n} \). In the second case we assume, for simplicity, that the \( n_1 \) marked points on the component of genus \( g_1 \) have numbers \( 1, \ldots, n_1 \), while the \( n_2 \) marked points on the component of genus \( g_2 \) have numbers \( n_1 + 1, \ldots, n \). Then we have

\[ q^*(c_W(a_1, \ldots, a_n)) = \sum_{a' + a'' = r - 2} c_W(a_1, \ldots, a_{n_1}, a') \times c_W(a'', a_{n_1+1}, \ldots, a_n), \]

(where at most one term of the sum is actually nonzero, because of the condition \( 2g_1 - 2 - \sum_{i=1}^{n_1} a_i - a' \equiv 2g_2 - 2 - \sum_{i=n_1+1}^{n} a_i - a'' \equiv 0 \mod r \)), and

\[ r^*(c_W(a_1, \ldots, a_n)) = \sum_{a' + a'' = r - 2} c_W(a_1, \ldots, a_n, a', a''). \]

It follows that the bilinear form is given by

\[ \eta^{ab} = \delta^{a+b,r-2}. \]
The string, the dilaton, and the TRR equations allow us to express all genus zero correlators of the series $F^r$ using only the correlators for $g = 0$, $n = 3, 4$, which were computed in Witten’s original paper.

In [9], Givental found a specific element of Givental’s group that sends the series

$$Z^{pt} \times \cdots \times Z^{pt} \quad (r - 1 \text{ factors})$$

to the series $Z^r = \exp \sum F_g^{(r)} h^{g-1}$ such that $Z^r|_{h=1}$ is the $\tau$-function of the string solution of the $r$-KdV hierarchy, while $F^r = \sum F_g^{(r)}$ is the string solution itself. The genus 0 part of $F^r$ coincides with the genus 0 part of $F^r$. Indeed, both satisfy the string, the dilaton, and the TRR equations and it is easy to check that they have the same coefficients for $n = 3, 4$.

Thus what remains to be proved is that the geometric series $F^r$ and the formal Gromov–Witten potential $F^r$ coincide also in higher genus. To do that, we will use Theorem 3, so let us check the conditions (1-5) involved in its formulation.

**Homogeneity.** The correlator

$$\langle \tau_{d_1,a_1} \cdots \tau_{d_n,a_n} \rangle_g$$

vanishes unless

$$\sum d_i + \sum \frac{a_i}{r} = n + \left( \frac{r-2}{r} - 3 \right) (1 - g).$$

Indeed, from the definitions of Section 1.2, we get $c_1(T) = (2g - 2 - \sum a_i)/r$, hence the rank of $V$ and the degree of $c_W$ equal

$$\deg c_W = \frac{(r - 2)(g - 1) + \sum a_i}{r}.$$

Thus the series $F^r$ satisfies the homogeneity property for the weight function $w(i) = \frac{i}{r} - 1$, and the “dimension” $\dim = \frac{r-2}{r}$ (There is no need for a weight function on a Novikov ring, because we are working over $\mathbb{C}$).

**Geometricity.** It is obvious that the Gromov–Witten potential $F^r$ respects all tautological relations because of its geometric origin.

**Semi-simplicity.** There exists a bi-polynomial isomorphism

$$\mathbb{C}^{r-2} \to \mathbb{C}^{r-2} : (t_{0,1}, \ldots, t_{0,r-2}) \mapsto (s_1, \ldots, s_{r-2})$$

such that the algebra described in Definition 2.4 is naturally identified with the algebra

$$\mathbb{C}[X]/(X^{r-1} + s_1 X^{r-3} + \cdots + s_{r-2}).$$

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Thus it is semi-simple whenever the polynomial has \( r - 1 \) distinct roots.

For instance, for \( r = 5 \), the algebra of Definition 2.4 is isomorphic to \( \mathbb{C}[X]/P' \), where

\[
P(X) = X^5 - t_{0,3}X^3 - t_{0,2}X^2 + (t_{0,3}^2/5 - t_{0,1})X
\]

and \( P' \) is its derivative.

For more details, see [9].

**Reducibility to genus 0.** This property will be established in Theorem 5.

**Analyticity.** In [9] the genus expansion of \( F_0^{(r)} \) is given in the form of a power series at the origin.

### 4.2 Admissible covers and double ramification cycles

The spaces of admissible covers and the double ramification cycles were first introduced by Ionel [14] and proved very useful in the study of moduli spaces. Let us briefly recall their definitions.

Consider a map \( \varphi \) from a smooth curve \( C \) with \( n \) marked points to the sphere \( S = \mathbb{C}P^1 \). On \( S \) we mark all branch points of \( \varphi \) and the images of the \( n \) marked points of \( C \). On \( C \) we then mark all the preimages of the points that are marked on \( S \). Now choose several disjoint simple loops on \( S \), that do not pass through the marked points. Suppose that if we contract these loops we obtain a stable genus 0 curve \( S' \). Now contract also all the preimages of the loops in \( C \) to obtain a nodal curve \( C' \) that turns out to be automatically stable. We have obtained a map \( \varphi' \) from a stable curve \( C' \) of genus \( g \) to a stable curve \( S' \) of genus 0. It has the same degree over every component of \( S' \). Moreover, at each node of \( C' \), the projection \( \varphi' \) has the same local multiplicity on both components meeting at the node.

**Definition 4.1** A map from a stable curve of genus \( g \) to a stable curve of genus 0 topologically equivalent to a map described above is called an admissible covering.

We will be particularly interested in the space of admissible coverings with multiple ramifications over only 2 points labeled with 0 and \( \infty \), the other ramification points being simple.

**Definition 4.2** Consider the space of admissible coverings of some given genus \( g \) with prescribed ramification types over two points labeled 0 and \( \infty \), and with simple ramifications elsewhere. The normalization of this space is called a double ramification space or a DR-space.
Definition 4.3 Let $k_1, \ldots, k_{n+p}$ be a list of integers such that $\sum k_i = 0$ and $k_i \neq 0$ for $n + 1 \leq i \leq n + p$. Consider the set of smooth curves $(C, x_1, \ldots, x_n) \in \mathcal{M}_{g,n}$ such that there exist $p$ more marked points $x_{n+1}, \ldots, x_{n+p}$ and a meromorphic function on $C$ with no zeroes or poles outside of $x_1, \ldots, x_{n+p}$, the orders of zeroes or poles being prescribed by the list $k_1, \ldots, k_{n+p}$ ($k_i > 0$ for the zeroes, $k_i < 0$ for the poles, and $k_i = 0$ for the marked points that are neither zeroes nor poles). The closure of this set in $\overline{\mathcal{M}}_{g,n}$ is called the double ramification cycle or a DR-cycle.

Here are some basic facts about the double ramification cycles that make them so useful.

1. The codimension of a DR-cycle is equal to $g - p$ whenever there is at least one positive and one negative number among $k_1, \ldots, k_n$ (see [28, 14]). Assuming that this condition is satisfied we see that for $p = g$ the DR-cycle coincides with the moduli space $\mathcal{M}_{g,n}$.

2. The cohomology class Poincaré dual to any DR-cycle belongs to the tautological ring of $\overline{\mathcal{M}}_{g,n}$ (proved in [7]). This makes the results of this paper applicable to DR-cycles.

3. Each DR-cycle is the image of the corresponding DR-space under the forgetful map $\pi$ that forgets the covering and all the marked points except $x_1, \ldots, x_n$, but retains and stabilizes the source curve with the $n$ remaining marked points. The map $\pi$ sends the fundamental homology class of the DR-space to a multiple of the fundamental homology class of the DR-cycle.

4. Every class $\pi^*(\psi_i)$ on a DR-space can be expressed as a linear combination of boundary divisors $[14, 32]$.

This is a very important property that can be used to compute integrals involving $\psi$-classes. Indeed, it allows us to get rid of the $\psi$-classes one by one by reducing the integral to simpler integrals over smaller spaces. Using this procedure, the following result was established in [34], Theorem 1.

Theorem 5 [34] Every correlator in the $r$-spin Witten conjecture can be expressed in genus zero correlators using only tautological relations.

The last result shows that Theorem 3 is applicable to the $r$-spin Witten conjecture and suffices to prove it.
The proof of Theorem 5 in [34] goes as follows. As explained in Item 1, the fundamental class of $\overline{M}_{g,n}$ can be represented as a DR-cycle with $p = g$. If this is done in an intelligent way, then the elimination of $\psi$-classes according to Item 4, leads us to boundary divisors that can themselves be expressed as DR-cycles on the boundary. We end up with the integral of Witten’s class $c_W$ (without $\psi$-classes) over a DR-cycle. A dimension count shows that an integral like that may be nonzero only if $g = 0$ or 1 and the codimension of the DR-cycle is equal to the genus (0 for $g = 0$ and 1 for $g = 1$). In the latter case we must do some more work: putting one $\psi$-class back into the integral and expressing it as a linear combination of boundary divisors in different ways we obtain certain relations between genus 1 and genus 0 integrals of $c_W$. It turns out that these relations suffice to reduce all genus 1 integrals to genus 0 integrals.

However, now we can give a simpler, although less constructive, proof.

**Proof of Theorem 5.** In [14], Ionel proved the following assertion:

Let $M$ be a monomial in $\psi$- and $\kappa$-classes on $\overline{M}_{g,n}$, of degree at least $g$ for $g \geq 1$ or at least 1 for $g = 0$. Then the class $M$ can be represented as a linear combination of classes of the form

$$q_* [(\text{DR-cycle})_1 \times \cdots \times (\text{DR-cycle})_k].$$

Here $k \geq 1$ is an integer that can be different for different terms of the sum, $q : \overline{M}_{g_1,n_1} \times \cdots \times \overline{M}_{g_k,n_k} \to \overline{M}_{g,n}$ is the gluing map from a product of smaller moduli spaces to a boundary stratum of $\overline{M}_{g,n}$ and the cycles $(\text{DR-cycle})_j$ are DR-cycles on the smaller moduli spaces.

It was established in [7] that every DR-cycle is tautological and that Ionel’s theorem can therefore be improved in the following way:

Let $M$ be a monomial in $\psi$- and $\kappa$-classes on $\overline{M}_{g,n}$, of degree at least $g$ for $g \geq 1$ or at least 1 for $g = 0$. Then the class $M$ can be represented by a linear combination of dual graphs each of which has at least one edge. We will call this property the $g$-reduction.

Now, a simple dimension count shows that the integral

$$\int_{\overline{M}_{g,n}} \beta \cdot c_W(a_1, \ldots, a_n)$$

vanishes unless the class $\beta$ has complex degree at least $g$. Indeed, the degree of Witten’s class equals

$$\deg c_W = \frac{(r - 2)(g - 1) + \sum a_i}{r} \leq \frac{(r - 2)(n + g - 1)}{r},$$

\footnote{The degree of a $\psi$-class equals 1, while the degree of $\kappa_k$ equals $k$.}
while the dimension of $\overline{M}_{g,n}$ is $3g - 3 + n$ (use the exact expression for $g = 1$ and the upper bound for $g \geq 2$).

Recall that the pull-back of Witten’s class to the boundary components is given by the factorization property (3) of Section 4.1. This makes it easy to apply the $g$-reduction to integrals involving Witten’s class.

The rest of the proof is simple. Suppose we wish to compute the integral

$$\int_{\overline{M}_{g,n}} c_W(a_1, \ldots, a_n) \psi_1^{d_1} \cdots \psi_n^{d_n}.$$  

Apply the $g$-reduction in iteration as many times as possible, starting with the class $\psi_1^{d_1} \cdots \psi_n^{d_n}$. In the end we will obtain an expression of $\prod_{i=1}^{n} \psi_i^{d_i}$ as a linear combination of dual graphs $G$ satisfying the following condition. Suppose a vertex $v$ of $G$ is labeled with genus $g_v > 0$ and with a class $\kappa_{k_1, \ldots, k_m}$, and suppose the half-edges and tails issuing from $v$ are labeled with $\psi_1^{d_1}, \ldots, \psi_n^{d_n}$. Then $\sum k_i + \sum d_i < g_v$. (Indeed, if $\sum k_i + \sum d_i \geq g_v$ for at least one vertex, we can apply the $g$-reduction to this vertex.) But, as we have already explained, the integral of Witten’s class over the class $[G]$ represented by a dual graph like that vanishes whenever there is at least one vertex of nonzero genus. Thus the only contribution comes from graphs with only genus zero vertices with no $\psi$- or $\kappa$-classes. So we have reduced any given correlator involved in Witten’s conjecture to a linear combination of products of genus 0 correlators. More precisely, the only remaining correlators are integrals of Witten’s class with no $\psi$-classes over genus zero moduli spaces.

As we have already explained, this implies that the formal Gromov–Witten potential $F^{(r)}$ coincides with the geometric Gromov–Witten potential $F^{[r]}$ and proves the Witten conjecture.

5 More on Givental’s quantization and Y.-P. Lee’s theorem

A proof of Theorem 4 is contained in [24]. However there are some missing details that we would like to fill in here. First, we would like to explain precisely why the operators $\tau_k$ act on the $\kappa$-classes in the way described in Section 2.4.1 (this is done in Proposition 5.5). Second, we explain more precisely how Givental’s quantization is applied to prove the theorem.

“Givental’s group” is really not a group, but a collection of two groups: the so-called “lower triangular” and “upper triangular” groups. This is analogous to the Birkhoff decomposition in the finite-dimensional case. However,
because both groups are infinite-dimensional, it is in general not possible to multiply their elements, like it is impossible to multiply a power series in $z$ and a power series in $z^{-1}$. On the other hand, it turns out that, under some conditions, one can apply first an element of the upper triangular group and then an element of the lower triangular group to a Gromov–Witten potential.

Both the lower triangular and the upper triangular group possess Lie algebras. An element of the lower triangular Lie algebra is a series $s(z^{-1}) = \sum_{l\geq 1} s_l z^{-l}$ of linear operators on the vector space $V$. The operators $s_l$ are self-adjoint for $l$ odd and skew-self-adjoint for $l$ even with respect to the quadratic form $\eta$.

Similarly, an element of the upper triangular Lie algebra is a series $r(z) = \sum_{l\geq 1} r_l z^l$ of linear operators on $V$. The operators $r_l$ are self-adjoint for $l$ odd and skew-self-adjoint for $l$ even with respect to the quadratic form $\eta$.

The coefficients of $s_l$ and $r_l$ in the basis $A$ will be denoted by $(s_l)^\nu_\mu$ and $(r_l)^\nu_\mu$. We will also need the bivectors and the bilinear forms given by

$$(s_l)^\mu_\nu = \sum_\rho \eta^\rho_\mu (s_l)^\rho_\nu; \quad (s_l)^\mu_\nu = \sum_\rho \eta^\rho_\mu (s_l)^\rho_\nu.$$  

$$ (r_l)^\mu_\nu = \sum_\rho \eta^\rho_\mu (r_l)^\rho_\nu; \quad (r_l)^\mu_\nu = \sum_\rho \eta^\rho_\mu (r_l)^\rho_\nu. $$

The matrices $(s_l)^\mu_\nu$, $(r_l)^\mu_\nu$, $(s_l)^\mu_\nu$, and $(r_l)^\mu_\nu$ are symmetric for odd $l$ and skew-symmetric for even $l$.

Y.-P. Lee [24] writes down explicit formulas for the action of $s$ and $r$ on any given correlator of a Gromov–Witten potential $F$ (see below). Once this is done, the main problem is to understand what happens when we apply these formulas to the induced vector of a tautological class: indeed, both Y.-P. Lee’s formulas for the derivatives of an individual correlator and the expression of the induced vector in terms of correlators (described in Section 2.3) are fairly complicated.

Below we sum up the argument of [24] and give more detailed statements of certain results.

If $s(z^{-1}) = \sum_{l\geq 1} s_l z^{-l}$ is an element of the lower triangular Lie algebra and $r(z) = \sum_{l\geq 1} r_l z^l$ an element of the upper triangular Lie algebra, denote, for shortness

$$s_l(\tau_{d,\mu}) = \sum_\nu (s_l)^\nu_\mu \tau_{d-l,\nu}; \quad r_l(\tau_{d,\mu}) = \sum_\nu (r_l)^\nu_\mu \tau_{d+l,\nu}.$$  

Now we are going to follow the path from the Gromov–Witten potential of a point to the general formal semi-simple Gromov–Witten potential.
5.1 The upper triangular group

Let \( t_0^\mu \) be flat coordinates on a semi-simple Frobenius manifold \( M \) of dimension \( k \). Let \( f(t_0^\mu) \) be the corresponding genus 0 potential. We assume that \( f \) is an analytic function. At a semi-simple point, the tangent Frobenius algebra \( T^*_M \) to the Frobenius manifold \( M \) possesses a basis of primitive idempotents. Denote by \( \alpha_1, \ldots, \alpha_k \) their scalar squares in the metric \( \eta \). Givental constructs an element \( R \) of the upper triangular group whose action transforms the constant Frobenius structure on \( T^*_M \) into the Frobenius structure of \( M \) at the neighborhood of the semi-simple point.

In other words, the first step of Givental’s quantization is to apply the quantized action of \( R \) to the partition function \( Z_{pt, \alpha_1} \times \cdots \times Z_{pt, \alpha_k} \), see Eq. (2).

The potential \( F_{pt, \alpha_1} \oplus \cdots \oplus F_{pt, \alpha_k} \) obviously possesses the two following crucial properties.

**Definition 5.1** A Gromov–Witten potential is *tame* if \( \langle \tau_{d_1 \cdots d_n}^\mu_1 \cdots \tau_{d_1 \cdots d_n}^\mu_n \rangle \) vanishes whenever \( \sum d_i > 3g - 3 + n \).

**Definition 5.2** A Gromov–Witten potential is an *ancestor* potential if its correlators with \( 2 - 2g - n \geq 0 \) vanish.

Let \( r = \ln R \) be an element of the upper triangular Lie algebra. In general, the action of \( R \) on a power series is not well defined. However, it is easy to check (cf. Proposition 5.3) that the action of \( r_l \) increases the grading \( \sum d_i - n - 3g + 3 \) by \( l \). Therefore the action of \( R \) on a tame series is well-defined and is equal to the exponential of the action of \( r \).

According to Givental’s formulas, \( r = \sum_{l \geq 1} r_l z^l \) acts on the partition function \( Z \) via the second order differential operator

\[
\hat{r} = -\sum_{l \geq 1} (r_l^\mu)_{l+1} \frac{\partial}{\partial t_{l+1}} + \sum_{d_0, d \geq 1} (r_l^\mu)_{d+l} \frac{\partial}{\partial t_{d+l}} + \frac{\hbar}{2} \sum_{d_1, d_2 \geq 0} (r_{d_1, d_2+1})^{\mu_1, \mu_2} \frac{\partial^2}{\partial t_{d_1}^{\mu_1} \partial t_{d_2}^{\mu_2}}.
\]

The next proposition gives the action of \( r \) on individual correlators.
Proposition 5.3 ([24], Equation (23))

\[ r. \langle \tau_{d_1, \mu_1} \cdots \tau_{d_n, \mu_n} \rangle_g = \]

\[ - \sum_{l=1}^{\infty} \langle r_l(\tau_{1,1}) \tau_{d_1, \mu_1} \cdots \tau_{d_n, \mu_n} \rangle_g \]

\[ + \sum_{l=1}^{\infty} \sum_{i=1}^{n} \langle \tau_{d_1, \mu_1} \cdots r_l(\tau_{d_i, \mu_i}) \cdots \tau_{d_n, \mu_n} \rangle_g \]

\[ + \frac{1}{2} \sum_{l=1}^{\infty} \sum_{m+m'=l-1} (-1)^{m+1} \sum_{\mu, \nu} (r_l)^{\mu \nu} \langle \tau_{m, \mu} \tau_{m', \nu} \tau_{d_1, \mu_1} \cdots \tau_{d_n, \mu_n} \rangle_{g-1} \]

\[ + \frac{1}{2} \sum_{l=1}^{\infty} \sum_{m+m'=l-1} (-1)^{m+1} \sum_{\mu, \nu} \sum_{g_1+g_2=g} \sum_{l\cup J=\{1, \ldots, n\}} \langle \tau_{m, \mu} \prod_{i \in I} \tau_{d_i, \mu_i} \rangle_{g_1} \langle \tau_{m', \nu} \prod_{i \in J} \tau_{d_i, \mu_i} \rangle_{g_2}. \]

Remark 5.4 One can easily deduce from these formulas that the action of the upper triangular group preserves the tameness property and the property of being an ancestor potential.

Proposition 5.5 (based on [24], Section 6) Let \( L \) be a linear combination of dual graphs and \( r \) an element of the upper triangular Lie subalgebra of Givental’s Lie algebra. Let \( F \) be a formal tame ancestor potential. Then

\[ r. \left( p^*(L) \prod_{i=1}^{n+n'} \psi_i^{d_i} \ev_i^*(\mu_i) \right)_g \]

is a linear combination of polynomials in correlators of the form

(i) \( \left( p^*(L) \prod_{i=1}^{n+n'+1} \psi_i^{d_i} \ev_i^*(\mu_i) \right)_g \),

where \( d_{n+n'+1} \geq 2 \) and \( \mu_{n+n'+1} \) is a primary field;

(ii) \( \left( p^*(L) \prod_{i=1}^{n+n'} \psi_i^{d_i} \ev_i^*(\mu'_i) \right)_g \),

where \( d'_i = d_i, \mu'_i = \mu_i \) for all \( i \) except one, and \( d'_i > d_i \) for one \( i \);

(iii) \( \left( p^*(\tau(L)) \prod_{i=1}^{n+n'} \psi_i^{d_i} \ev_i^*(\mu_i) \cdot \ev^*_\alpha(\mu_\alpha) \ev^*_\beta(\mu_\beta) \right)_g \),
where \( l \geq 1 \) and \( \mu_\alpha, \mu_\beta \) are primary fields;

\[
\langle \tau_{d_\alpha, \mu_\alpha} \prod_{i \in I} \tau_{d_i, \mu_i} \rangle_0 \left< p^*(L) \psi^{d_\beta}_\beta \text{ev}^*_\beta(\mu_\beta) \prod_{i \in J} \psi^{d_i}_i \text{ev}^*_i(\mu_i) \right>_g,
\]

where \( I \subset \{ n + 1, \ldots, n + n' \} \), \( I \sqcup J = \{ 1, \ldots, n + n' \} \), \( d_\alpha \) and \( d_\beta \) are nonnegative integers, and \( \mu_\alpha, \mu_\beta \) are primary fields;

\[
\langle \tau_{d_\alpha, \mu_\alpha} \tau_{d_j, \mu_j} \prod_{i \in I} \tau_{d_i, \mu_i} \rangle_0 \left< p^*(L) \psi^{d_\beta}_\beta \text{ev}^*_\beta(\mu_\beta) \prod_{i \in J} \psi^{d_i}_i \text{ev}^*_i(\mu_i) \right>_g,
\]

where \( I \subset \{ n + 1, \ldots, n + n' \} \), \( j \in \{ 1, \ldots, n \} \), \( I \sqcup J \sqcup \{ j \} = \{ 1, \ldots, n + n' \} \), \( d_\alpha \) and \( d_\beta \) are nonnegative integers, and \( \mu_\alpha, \mu_\beta \) are primary fields.

**Remark 5.6** If we work over a Novikov ring, we must introduce a summation over \( D_1 + D_2 = D \) in the last term of the equality in Proposition 5.3 as well as in terms (iv) and (v) of Proposition 5.5.

**Corollary 5.7** Let \( L \) be a linear combination of genus \( g \) dual graphs with \( n \) tails such that \( \tau_k(L) \) is a universal relation for all \( k \). Assume that \( F^\text{pt}_L \) vanishes. Then \( F^\text{pt}_L \) also vanishes for every \( F \) that can be obtained from \( F^\text{pt}_0, \alpha_1 \oplus \cdots \oplus F^\text{pt}_0, \alpha_k \) by the upper triangular group action.

**Proof.** The polynomials (i) to (v) are either entries of \( F^\text{pt}_L \) of \( F^\text{pt}_\tau(L) \). In the latter case they vanish by assumption. Therefore the vector \( F^\text{pt}_L \) on the orbit \( e^{tr}F \) is the solution of a linear differential equation with vanishing initial conditions. Thus \( F^\text{pt}_L = 0 \) on the whole orbit. \( \Box \)

**Proof of Proposition 5.5.**

1. We first prove the proposition in the particular case \( n' = 0 \), \( d_1 = \cdots = d_n = 0 \). In Section 2.3 we gave a three steps algorithm to determine the linear combination of graphs \( p^*(L) \prod_{i=1}^{n+n'} \psi^d_i \text{ev}^*_i(\mu_i) \). In our particular case, the steps are greatly simplified: Step 1 (replacing \( \kappa \)-classes by additional tails) remains unchanged; Step 2 (expressing ancestor \( \psi \)-classes in terms of descendant \( \psi \)-classes) can be skipped, since all genus zero 2-point correlators vanish for an ancestor potential and since there are no additional marked points; Step 3 (adding tails from \( n+1 \) to \( n+n' \) and multiplying by \( \psi^d_i \text{ev}^*_i(\mu_i) \)) reduces to assigning the markings \( \mu_1, \ldots, \mu_n \) to tails 1 to \( n \).

As for the claim of our proposition, if \( n' = 0 \), then terms (iv) and (v) disappear, because \( I \subset \{ n + 1, \ldots, n + n' \} \) is then empty, and an ancestor potential does not have genus 0 correlators with fewer than 3 entries. Thus we
must prove that \( r. \langle p^*(L) \prod_{i=1}^n \text{ev}_i^*(\mu_i) \rangle_g \) is a linear combination of terms (i), (ii), and (iii).

Let \( G \) be a graph in \( L \).

1a. First suppose that \( G \) has only one vertex (of genus \( g \) and valency \( n \)) with a class \( \kappa_{k_1}, \ldots, k_m \) assigned to the vertex and classes \( \psi_{d_i} \) assigned to the tails. Applying Steps 1 and 3 we obtain

\[
\left\langle p^*(G) \prod_{i=1}^n \text{ev}_i^*(\mu_i) \right\rangle = \left\langle \prod_{i=1}^m \tau_{k_i+1,1} \prod_{i=1}^n \tau_{d_i,\mu_i} \right\rangle.
\]

To determine the action of \( r \) on the correlator in the right-hand side we apply Equation (4) to it.

Applying the third and fourth terms in Equation (4) gives us term (iii) involving \( \tau_l(G) \). Note that when we apply the fourth term of Equation (4), the indices \( k_1, \ldots, k_m \) are distributed among the two correlators in all possible ways, according to the description of \( \tau_l \).

Now apply the first two terms of Equation (4) to our correlator. We regroup them in the following way:

\[
\sum_{l=1}^\infty \sum_{\mu} (r_l)_\mu^1 \left[ \sum_{j=1}^m \left\langle \tau_{k_j+t+1,\mu} \prod_{i \neq j} \tau_{k_i+1,1} \prod_{i=1}^n \tau_{d_i,\mu_i} \right\rangle - \left\langle \tau_{l+1,\mu} \prod_{i=1}^m \tau_{k_i+1,1} \prod_{i=1}^n \tau_{d_i,\mu_i} \right\rangle \right] + \sum_{j=1}^n \left\langle r_l(\tau_{d_j,\mu_j}) \prod_{i=1}^m \tau_{k_i+1,1} \prod_{i \neq j} \tau_{d_i,\mu_i} \right\rangle. \tag{5}
\]

Consider the forgetful map \( \pi : \mathcal{M}_{g,n+1} \to \mathcal{M}_{g,n} \). The pull-back \( \pi^*(\kappa_{k_1}, \ldots, k_m) \) given by Lemma 2.10 imitates the expression in square brackets of (5). In addition, we have \( \pi^*(\psi_{d_i}^j) \psi_{n+1}^{l+1} = \psi_{d_i}^l \psi_{n+1}^{l+1} \). It follows that the terms in square brackets add up to give term (i).

The last term of Equation (5) replaces one of the symbols \( \tau_{d_j,\mu_j} \) by \( \tau_{d_j',\mu_j'} \) with \( d_j' > d_j \), which gives us term (ii).

1b. Now let \( G \) be an arbitrary dual graph. Then the terms in square brackets of Equation (5) will appear for each vertex of the graph and their sum will still represent the class \( \pi^*(G) \cdot \psi_{n+1}^{l+1} \text{ev}_{n+1}^*(\mu) \).

The last term of Equation (5) replaces as before, one of the symbols \( \tau_{d_j,\mu_j} \) by \( \tau_{d_j',\mu_j'} \). But now this symbol can either correspond to a tail or to a half-edge of \( G \). If the symbol we replace corresponds to a tail, it gives rise to term (ii) in the proposition. If it corresponds to a half-edge of \( G \), the corresponding term \( \left\langle r_l(\tau_{d_j,\mu_j}) \prod_{i=1}^m \tau_{k_i+1,1} \prod_{i \neq j} \tau_{d_i,\mu_i} \right\rangle \) contributes to term (iii) of
the proposition. Finally, the third and fourth terms in Equation (4) complete the expression for term (iii).

2. Now we return to the general case, when \( n' \) and \( d_1, \ldots, d_{n+n'} \) are arbitrary. Consider the forgetful maps

\[
\overline{\mathcal{M}}_{g,n+n'+1} \xrightarrow{\pi_1} \overline{\mathcal{M}}_{g,n+n'} \xrightarrow{\pi_2} \overline{\mathcal{M}}_{g,n}
\]

and the composition \( \pi = \pi_2 \circ \pi_1 \).

If \( L \) is our initial linear combination of dual graphs, then \( L' = \pi_2^*(L) \prod_{i=1}^{n+n'} \psi_i^{d_i} \) is a well-defined linear combination of dual graphs obtained by the usual three steps algorithm, but without attaching the markings \( \mu_i \) to the tails.

Now we can apply the particular case that we have just proved to the linear combination \( L' \). All we have to do is reinterpret the answer in terms of \( L \). We claim the following.

Term (i) for \( L' \) gives term (i) for \( L \). Indeed, the classes on the moduli spaces of curves involved in this term are

\[
\pi_1^*(L') \psi_{n+n'+1}^{d_{n+n'+1}} = \pi_1^*(L) \prod_{i=1}^{n+n'} \psi_i^{d_i} \psi_{n+n'+1}^{d_{n+n'+1}}.
\]

But since \( d_{n+n'+1} \geq 2 \), we have \( \pi_1^*(\psi_i^{d_i}) \psi_{n+n'+1}^{d_{n+n'+1}} = \psi_i^{d_i} \psi_{n+n'+1}^{d_{n+n'+1}} \) on \( \overline{\mathcal{M}}_{g,n+n'+1} \).

Term (ii) for \( L' \) gives term (ii) for \( L \). This is obvious.

Term (iii) for \( L' \) gives terms (iii), (iv), and (v) for \( L \). Indeed, the space \( \mathcal{D}_{g,n+n'} \) has two kinds of irreducible components: those that project to a component of \( \mathcal{D}_{g,n} \) under \( \pi_2 \) and those that project onto \( \overline{\mathcal{M}}_{g,n} \) under \( \pi_2 \). In the first case we obtain term (iii) of the proposition. As to the second case, it occurs when the generic curve of the component of \( \mathcal{D}_{g,n+n'} \) has one component of genus \( g \) and one component of genus 0 that is contracted by \( \pi_2 \). The class \( \pi_2^*(L) \) is supported on the genus \( g \) component. The contracted component of genus 0 contains either 0 or 1 point with markings \( 1, \ldots, n \) (those that are not forgotten by \( \pi_2 \)). According to these two cases we obtain either term (iv) or term (v) from the proposition.

5.2 The lower triangular group

By the action of the upper triangular group we have obtained a Gromov–Witten potential that describes the semi-simple Frobenius manifold \( M \) that we started with. Recall that this Gromov–Witten potential is a power series in variables \( t_0, t_1, t_2, \ldots \) whose coefficients are functions in variables \( t_0 \) analytic outside the discriminant of \( M \).
Now the action of the lower triangular group re-expands the Gromov–Witten potential at a different (possibly non semi-simple) point of $M$ and simultaneously incorporates 1- and 2-point genus 0 correlators.

The element $s = \sum_{l \geq 1} s_l z^{-l}$ acts on the partition function $Z$ via the first order differential operator

$$\hat{s} = -\sum_{\mu} (s_1)_{\mu}^1 \frac{\partial}{\partial t_0^\mu} + \frac{1}{\hbar} \sum_{d, \mu} (s_{d+2})_{1, \mu} t_d^\mu + \sum_{d, \mu} (s_l)_{\mu}^\nu t_{d+1}^{\mu, \nu} \frac{\partial}{\partial t_d^\mu} + \frac{1}{2\hbar} \sum_{d_1, d_2, \mu_1, \mu_2} (-1)^{d_1} (s_{d_1+d_2+1})_{\mu_1, \mu_2} t_{d_1}^{\mu_1} t_{d_2}^{\mu_2}.$$

In this expression we have omitted the term $-\frac{1}{2\hbar} (s_3)_{1,1}$. Indeed, it commutes with all other terms and the action of its exponential only adds the constant $(s_3)_{1,1}$ to $F_0$. Similarly, in the sequel we will consider $F_0$ to be defined up to an additive constant and omit those terms in differential operators that do nothing more than changing this constant.

The following proposition makes the action of $s$ on individual correlators explicit.

**Proposition 5.8** ([24], Equations (19),(20)) For $2 - 2g - n < 0$, we have

$$s.\langle \tau_{d_1, \mu_1} \cdots \tau_{d_n, \mu_n} \rangle_g =$$

$$= \frac{1}{2\hbar} \sum_{d, \mu} (s_{d+2})_{1, \mu} t_d^\mu + \sum_{d, \mu} (s_l)_{\mu}^\nu t_{d+1}^{\mu, \nu} \frac{\partial}{\partial t_d^\mu} + \frac{1}{2\hbar} \sum_{d_1, d_2, \mu_1, \mu_2} (-1)^{d_1} (s_{d_1+d_2+1})_{\mu_1, \mu_2} t_{d_1}^{\mu_1} t_{d_2}^{\mu_2}.$$

If $g = 0$ and $n = 2$, we have

$$s.\langle \tau_{d_1, \mu_1} \tau_{d_2, \mu_2} \rangle_0 =$$

$$= -\langle s_1(\tau_{1,1}) \tau_{d_1, \mu_1} \tau_{d_2, \mu_2} \rangle_0 + \sum_{l=1}^\infty \sum_{i=1}^n \langle \tau_{d_1, \mu_1} \cdots s_l(\tau_{d_1, \mu_i}) \cdots \tau_{d_n, \mu_n} \rangle_0.$$

If $g = 0$ and $n = 1$, we have

$$s.\langle \tau_{d, \mu} \rangle_0 = -\langle s_1(\tau_{1,1}) \tau_{d, \mu} \rangle_0 + \sum_{l=1}^\infty \langle s_l(\tau_{d, \mu}) \rangle_0 + (s_{d+2})_{1, \mu}.$$

Looking at Equations (6), (7), (8) we see that $s_l$ decreases the grading $\sum d_i$ by $l$, except for the first term involving $s_1(\tau_{1,1})$. This annoying term
leads to a problem that we have to discuss in more detail. The corresponding term

\[ \hat{u} = \sum_{\mu} (s_1)^{\mu} \frac{\partial}{\partial \mu_0} \]

in the differential operator \( \hat{s} \) is simply a partial derivative in the direction \( s_1(1) \). The exponential \( \exp(\hat{u}) \) is then a shift of the coordinates \( t^\mu_0 \) in this direction. A shift of coordinates is not a well-defined operation for formal power series. In our case, we consider power series in variables \( t^\mu_1, t^\mu_2, \ldots \), whose coefficients are analytic in \( t^\mu_0 \) on the Frobenius manifold except perhaps its discriminant. Thus the shift is a well-defined operation, but it can, in some cases, lead us out of the realm of power series. In this case the genus expansion of the genus 0 Gromov–Witten potential will be a power series in \( t^\mu_1, t^\mu_2, \ldots \), whose coefficients are analytic functions in \( t^\mu_0 \) with a singularity at the origin. (See, for instance, [6], Section 6, where \( F \) and \( G \) are the Gromov–Witten potentials without descendants for genus 0 and 1 respectively.)

We also need to give a proper definition of the action of \( S = \exp(s) \). We have, up to omitted constant terms,

\[
[s, \hat{u}] = \frac{1}{\hbar} \sum_{d,\mu} (s_1 s_{d+1})_{1,\mu} t^\mu_d,
\]

\[
[s, [s, \hat{u}]] = \frac{1}{\hbar} \sum_{d,\mu} (s_1 s_2 s_{d+1})_{1,\mu} t^\mu_{d+1},
\]

\[
[s, [s, [s, \hat{u}]]) = \frac{1}{\hbar} \sum_{d_1, d_2, \mu} (s_1 s_2 s_3 s_{d+1})_{1,\mu} t^\mu_{d_1+d_2+1},
\]

dec. On the other hand, \([\hat{u}, [\hat{u}, \hat{s}]] = 0 \) (again, up to constant term). Thus, according to the Baker–Campbell–Hausdorff formula, we have \( e^s = e^{-\hat{u}} e^{\hat{s}} \), where

\[
\hat{v} = \hat{u} + s - \frac{1}{2} [s, \hat{u}] + \frac{1}{12} [s, [s, \hat{u}]] - \frac{1}{720} [s, [s, [s, \hat{u}]]) + \ldots \quad (9)
\]

\[
= \sum_{d, l, \mu, \nu} (s_l)^{\mu} t^\nu_{d+l} \frac{\partial}{\partial t^\mu_d} + \frac{1}{2\hbar} \sum_{d_1, d_2, \mu_1, \mu_2} (-1)^{d_1} (s_{d_1+d_2+1})_{\mu_1, \mu_2} t^{\mu_1}_{d_1} t^{\mu_2}_{d_2} + \sum C_{d, \mu} t^\mu_d,
\]

where every \( C_{d, \mu} \) is a finite polynomial in the matrix elements \( (s_l)^{\mu} \).

**Proposition 5.9** For \( 2 - 2g - n < 0 \), we have

\[
\hat{v}. \langle \tau_{d_1, \mu_1} \ldots \tau_{d_n, \mu_n} \rangle_g = \sum_{l=1}^{\infty} \sum_{i=1}^{n} \langle \tau_{d_1, \mu_1} \ldots \tau_{d_i, \mu_i} \rangle \langle \tau_{d_{i+1}, \mu_i} \ldots \tau_{d_n, \mu_n} \rangle_g. \quad (10)
\]
If \( g = 0 \) and \( n = 2 \), we have

\[
\hat{v}. \left\langle \tau_{d_1, \mu_1} \tau_{d_2, \mu_2} \right\rangle_0 = \sum_{l=1}^{\infty} \left[ \left\langle s_l(\tau_{d_1, \mu_1}) \tau_{d_2, \mu_2} \right\rangle_0 + \left\langle \tau_{d_1, \mu_1} s_l(\tau_{d_2, \mu_2}) \right\rangle_0 \right] + (-1)^{d_1} (s_{d_1+d_2+1})_{\mu_1 \mu_2}.
\]

**Remark 5.10** The action of \( \hat{v} \) on genus 0 one point correlators involves the coefficients \( C^\mu_d \) and is much more complicated. However it is easy to see that such correlators never appear in the induced vectors of tautological relations. Therefore the action of \( \hat{v} \) on them is immaterial to us.

**Lemma 5.11** The action of \( e^\hat{v} \) is well-defined on power series; in other words, every coefficient in the series \( e^\hat{v} Z \) is a finite polynomial in coefficients of \( Z \).

**Proof.** The action of \( \hat{v} \) on a correlator is a polynomial involving correlators with strictly smaller \( \sum d_i \) and matrix elements of the matrices \( s_l \).

**Definition 5.12** The action of \( e^\hat{v} \) on \( Z \) is defined as the action of \( e^\hat{v} \) followed by a translation of coordinates \( t_{0}^{\mu} \) by the vector \(-s_0\).

To sum up: the same element \( s \) of the lower triangular Lie subalgebra of Givental’s Lie algebra determines three differential operators: \( \hat{s} \), \( \hat{u} \), and \( \hat{v} \). They have the following properties: the action of \( e^\hat{s} \) is well-defined on power series; the action of \( e^\hat{u} \) is a translation of coordinates; we have \( e^\hat{s} = e^{-\hat{u}} e^\hat{v} \).

Now let \( L \) be a linear combination of dual graphs of genus \( g \) with \( n \) tails and \( \left\langle p^*(L) \prod_{i=1}^{n+n'} \psi_i^{d_i} \psi_i^{*}(\mu_i) \right\rangle \) an entry of its induced vector. Let \( s \) be an element of the lower triangular Lie subalgebra of Givental’s Lie algebra and \( \hat{v} \) the operator defined by Equation (9).

**Proposition 5.13** The result of the action

\[
\hat{v}. \left\langle p^*(L) \prod_{i=1}^{n+n'} \psi_i^{d_i} \psi_i^{*}(\mu_i) \right\rangle_g
\]

is a linear combination of polynomials in correlators of the form

\[
\left\langle p^*(L) \prod_{i=1}^{n+n'} \psi_i^{d'_i} \psi_i^{*}(\mu'_i) \right\rangle_g
\]

with \( d'_i = d_i \), \( \mu'_i = \mu_i \) for all \( i \) except one, while \( d'_i < d_i \) for one \( i \).
Corollary 5.14 If the induced vector $F_L$ vanishes for one formal Gromov–Witten potential $F$ then it also vanishes for all potentials that can be obtained from $F$ by the action of the lower triangular subgroup of Givental’s group.

Proof. The polynomials (12) are themselves entries of $F_L$. Therefore the vector $F_L$ on the orbit $e^{i\mathfrak{g}}F$ is the solution of a linear differential equation with vanishing initial conditions. Thus $F_L = 0$ on the whole orbit.

We know that the vanishing of $F_L$ can be expressed as a family of partial differential equations with constant coefficients. These equations are preserved by translations of $t_0^\mu$. It follows that the condition $F_L = 0$ is preserved by the action of the lower triangular group. ♦

Proof of Proposition 5.13. We must compute

$$\hat{v} : \left< p^*(L) \prod_{i=1}^{n+n'} \psi^d_i \text{ev}_i^*(\mu_i) \right>_g.$$

The main element of the proof is the following observation: every contribution of the second term in Equation (11) cancels with some contribution of the first term in Equations (10) and (11).

Indeed, recall that in the definition of the induced vector $F_L$ every half-edge and every tail of each graph in $L$ was replaced by a “stick” of several edges (Step 2), and then new tails were added in all possible ways to the new graphs (Step 3). Consider the linear combination $L_3$ of graphs obtained after Step 3 of the procedure. In $L_3$ take two graphs $\Gamma_1$ and $\Gamma_2$ differing only in one fragment:

$$\Gamma_1 = \quad \ldots \quad I \quad \ldots \quad \psi^{d_0+d_b+1} \quad \ldots \quad$$

$$\Gamma_2 = \quad \ldots \quad I \quad \ldots \quad \psi^{d_b} \quad \psi^{d_a} \quad \ldots \quad$$

Here $I \subset \{n + 1, \ldots, n + n'\}$ is a set of labels of the tails added in Step 3.

The polynomial $P_{\Gamma_1}$ contains the factor

$$Q_{\Gamma_1} = \left< \tau^{d_a+d_b+1} \nu^0 \nu^0 \prod_{i \in I} \tau_{d_i} \mu_i \right>_0.$$
The polynomial $P_{\Gamma_2}$ contains the factor

$$Q_{\Gamma_2} = \sum_{\mu_a, \nu_a} \langle \tau_{d_a, \mu_a} \tau_{0, \nu_a} \rangle_0 \eta^{\nu_a \mu_b} \left( \tau_{d_b, \mu_b} \tau_{0, \nu_b} \prod_{i \in I} \tau_{d_i, \mu_i} \right)_0. $$

Apply the operator $\hat{v}$ to the correlator $\langle \tau_{d_a, \mu_a} \tau_{0, \nu_a} \rangle_0$ in the expression $Q_{\Gamma_2}$, and take the contribution of the second term in Equation (11). We obtain

$$(-1)^d_a \sum_{\mu_a, \nu_a} (s_{d_a+1})_{\mu_a \nu_a} \eta^{\nu_a \mu_b} \left( \tau_{d_b, \mu_b} \tau_{0, \nu_b} \prod_{i \in I} \tau_{d_i, \mu_i} \right)_0 = \left( s_{d_a+1} (\tau_{d_a+d_b+1, \mu_b}) \tau_{0, \nu_b} \prod_{i \in I} \tau_{d_i, \mu_i} \right)_0. $$

But this term is part of the action of $\hat{v}$ on $P_{\Gamma_1}$. Indeed, it is part of the sum in the first term of Equations (10) and (11) applied to

$$\tau_{d_a+d_b+1, \mu_b} \tau_{0, \nu_b} \prod_{i \in I} \tau_{d_i, \mu_i} \rangle_0.$$ 

Since $\Gamma_2$ has one edge more than $\Gamma_1$, these terms appear with opposite signs in Substitution (1) and hence cancel.

Let us look at the contributions of the first term of Equations (10) and (11) that survived the cancellation. They are exactly those where $s_l$ is applied to the symbols $\tau_{d, \mu}$ corresponding to the $n + n'$ marked points and such that $l$ is smaller than or equal to the corresponding $d_i$. These contributions combine into the expression given in the proposition.

**Proof of Theorem 4.** The result follows from Corollaries 5.7 and 5.14.

**Remark 5.15** Unfortunately, for the time being we cannot prove the implication ($L$ is a universal relation) $\Rightarrow$ ($\tau_1(L)$ is a universal relation). There are two reasons for that. First, as explained in Remark 2.18, for not necessarily connected graphs checking that $Z_{\tau_1(L)} = 0$ does not imply that $\tau_1(L)$ is a universal relation. Second, term (iii) of Proposition 5.5 does not allow us to multiply $\tau_1(L)$ by arbitrary powers of $\psi_\alpha$ and $\psi_\beta$, but only by some of their combinations, namely, the classes $\rho_k$ of Section 3.1.

**Acknowledgements**

The authors are deeply grateful to M. Kazarian for his explanations on Frobenius manifolds and Givental’s quantization. We thank B. Dubrovin for pointing out the necessity of analyticity conditions on Gromov–Witten potentials.
The third author benefited greatly from a discussion with Tom Coates. We also wish to thank the participants of the Moduli Spaces program at the Mittag-Leffler Institute (Djursholm, Sweden) for the stimulating atmosphere and lots of interesting discussions. We thank the Institute for hospitality and support.

C.F. is supported by the grants 622-2003-1123 from the Swedish Research Council and DMS-0600803 from the National Science Foundation and by the Göran Gustafsson Foundation for Research in Natural Sciences and Medicine. D.Z. is partly supported by the ANR project “Geometry and Integrability in Mathematical Physics” ANR-05-BLAN-0029-01.

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