Geometric Phases of the Uhlmann and Sjöqvist et al Types for $O(3)$-Orbits of $n$-Level Gibbsian Density Matrices

Paul B. Slater
ISBER, University of California, Santa Barbara, CA 93106-2150
e-mail: slater@itp.ucsb.edu, FAX: (805) 893-7995
(November 5, 2018)

We accept the implicit challenge of A. Uhlmann in his 1994 paper, “Parallel Lifts and Holonomy along Density Operators: Computable Examples Using $O(3)$-Orbits,” by, in fact, computing the holonomy invariants for rotations of certain $n$-level Gibbsian density matrices ($n = 2, \ldots, 11$). From these we derive, by the tracing operation, the associated geometric phases and visibilities, which we analyze and display. We then proceed analogously, implementing the alternative methodology presented by E. Sjöqvist et al in their letter, “Geometric Phases for Mixed States in Interferometry” (Phys. Rev. Lett. 85, 2845 [2000]). For the Uhlmann case, we are able to also compute several higher-order holonomy invariants. We compare the various geometric phases and visibilities for different values of $n$, and also directly compare the two forms of analysis. By setting one parameter ($a$) in the Uhlmann analysis to zero, we find that the so-reduced form of the first-order holonomy invariant is simply equal to $(-1)^{n+1}$ times the holonomy invariant in the Sjöqvist et al method. Additional phenomena of interest are reported too.

PACS Numbers 03.65.Vf, 05.30.-d, 05.70.-a

Contents

I. Introduction

II. Uhlmann geometric phases for Gibbsian $n$-level systems ($n = 2, \ldots, 11$)

III. Uhlmann visibilities for Gibbsian $n$-level systems ($n = 2, \ldots, 11$)

IV. Comparisons across Gibbsian $n$-level systems of Uhlmann geometric phases and visibilities

V. Higher-order Uhlmann holonomy invariants

A. Traces of powers of Uhlmann holonomy invariants

B. Cross-comparisons of Uhlmann holonomy invariants

C. Eigenvalues of Uhlmann holonomy invariant for $n = 2, 3, 5$

VI. Sjöqvist et al geometric phases for Gibbsian $n$-level systems ($n = 2, \ldots, 11$)

VII. Sjöqvist et al visibilities for Gibbsian $n$-level systems ($n = 2, \ldots, 11$)

VIII. Comparisons across Gibbsian $n$-level systems of Sjöqvist et al geometric phases and visibilities

IX. Direct comparisons of Uhlmann and Sjöqvist et al results

X. Summary

I. INTRODUCTION

In a recent paper, Sjöqvist et al [1] provided “a new formalism of the geometric phase for mixed states in the experimental context of quantum interferometry”. Only in passing, did these authors note that “Uhlmann was probably the first to address the issue of mixed state holonomy, but as a purely mathematical problem”. Interested in possible (previously uninvestigated) relationships between the work of Sjöqvist et al [1] and that of Uhlmann [2], the present author compared the two approaches in terms of two spin-$\frac{1}{2}$ scenarios [4]. In the first of these, the spin-$\frac{1}{2}$ systems undergo unitary evolution along geodesic triangles [2], while in the second the unitary
evolution takes place along circular paths [3]. In [3], Uhlmann had also proposed an additional scenario, which was (initially) left unanalyzed in [4]. It involves “the Gibbsian states of the form”,

\[
\rho = \frac{e^\alpha \bar{\mathbf{J}}}{\text{trace} e^\alpha \bar{\mathbf{J}}},
\]

which fill for a given value of \(\alpha\) a 2-sphere called \(S^2_\alpha\) if \(\bar{\mathbf{J}}\) runs through all directions in 3-space. Now, for the starting point of the unitary evolution, one sets \(\bar{\mathbf{n}}\) = (0, 0, 1), while \(\bar{\mathbf{n}}\) = (0, \(\sin \theta\), \(\cos \theta\)) is chosen as rotational axis. The curve of state evolution is given by

\[
\phi \rightarrow U(\phi)\rho_0 U(-\phi), \quad U(\phi) = e^{-i\phi(\sin \theta J_y + \cos \theta J_z)},
\]

and the associated parallel lift of this curve with initial value \(\rho^{1/2}\) is

\[
\phi \rightarrow U(\phi)\rho_0^{1/2} V(\phi), \quad V(\phi) = e^{i\phi \tilde{H}},
\]

where

\[
\tilde{H} = \cos \theta J_z + a \sin \theta J_y, \quad \text{and} \quad a = \frac{1}{\cosh \alpha^2}.
\]

It is possible to regard \(V(\phi)\) as a rotation with angle

\[
\tilde{\phi} = \kappa \phi, \quad \kappa = \sqrt{\cos^2 \theta + a^2 \sin^2 \theta} \leq 1
\]

and rotation axis

\[
\tilde{\xi} = \left(0, \frac{\sin \theta}{\kappa}, \frac{a \cos \theta}{\kappa}\right).
\]

The holonomy invariant can then be written as

\[
(-1)^{2j} \rho_0^{1/2} e^{2\pi i \tilde{H}} \rho_0^{1/2} = (-1)^{2j} \rho_0^{1/2} e^{2\pi i \kappa \tilde{\xi} \cdot \bar{\mathbf{J}}} \rho_0^{1/2}.
\]

(All the preceding equations are directly adopted from [3].)

In this paper, to begin, we compute the trace of this invariant [3] for all \(j = \frac{1}{2}, 1, \frac{3}{2}, \ldots, \frac{9}{2}\). We plot the arguments of these traces, that is the corresponding geometric phases \(\gamma_j\), in Figs. I - II (sec. I), and their absolute values, that is the visibilies \(\nu_j\), in Figs. III - II (sec. II). For \(j = \frac{1}{2}\), the results are equivalent to the first (non-Gibbsian) set of models in [3], which was compared with the analyses of Sjöqvist et al [1] in [3]. Let us note that the value \(\alpha = 0\) corresponds to the fully mixed (classical) state, while \(\alpha = \pm \infty\) correspond to pure states.

In sec. III we compare certain aspects of these results across the ten distinct values of \(n\). All the various results up to this point can be considered as first-order in nature. In sec. IV, on the other hand, we compute certain higher-order invariants. In secs. V, VI, and VII we perform analyses precisely analogous to those in secs. I, II, but now in terms not of the approach of Uhlmann to mixed state holonomy, but that of Sjöqvist et al [1]. In our final analytical section (sec. VIII), before the summary (sec. IX), we directly compare results obtained by the two different procedures.

One of our main findings is that if one sets the implicit parameter \(a\) in the Uhlmann holonomy invariant [3] to zero, then its resultant trace is simply equal to \((-1)^{n+1}\) times the holonomy invariant [3, eq. (15)] yielded by the methodology of Sjöqvist et al.
II. UHLMANN GEOMETRIC PHASES FOR GIBBSIAN $N$-LEVEL SYSTEMS ($N = 2, \ldots, 11$)

FIG. 1. Uhlmann geometric phase for Gibbsian spin-$\frac{1}{2}$ systems
FIG. 2. Uhlmann geometric phase for Gibbsian spin-1 systems

FIG. 3. Uhlmann geometric phase for Gibbsian spin-$\frac{3}{2}$ systems
FIG. 4. Uhlmann geometric phase for Gibbsian spin-2 systems

FIG. 5. Uhlmann geometric phase for Gibbsian spin-$\frac{3}{2}$ systems
FIG. 6. Uhlmann geometric phase for Gibbsian spin-3 systems

FIG. 7. Uhlmann geometric phase for Gibbsian spin-$\frac{7}{2}$ systems
FIG. 8. Uhlmann geometric phase for Gibbsian spin-4 systems

FIG. 9. Uhlmann geometric phase for Gibbsian spin-$\frac{9}{2}$ systems
III. UHLMANN VISIBILITIES FOR GIBBSIAN $N$-LEVEL SYSTEMS ($N = 2, \ldots, 11$)
FIG. 12. Uhlmann visibility for Gibbsian spin-1 systems

FIG. 13. Uhlmann visibility for Gibbsian spin-$\frac{3}{2}$ systems
FIG. 14. Uhlmann visibility for Gibbsian spin-2 systems

FIG. 15. Uhlmann visibility for Gibbsian spin-\(\frac{5}{2}\) systems
FIG. 16. Uhlmann visibility for Gibbsian spin-3 systems

FIG. 17. Uhlmann visibility for Gibbsian spin-\(\frac{7}{2}\) systems
FIG. 18. Uhlmann visibility for Gibbsian spin-4 systems

FIG. 19. Uhlmann visibility for Gibbsian spin-$\frac{9}{2}$ systems
IV. COMPARISONS ACROSS GIBBSIAN $N$-LEVEL SYSTEMS OF UHLMANN GEOMETRIC PHASES AND VISIBILITIES

In the spin-$\frac{1}{2}$ case (Figs. 1 and 11), we have that the geometric phase $\gamma_{\frac{1}{2}}$ is the arctangent of $\frac{y}{x}$, taking into account which quadrant the point $(x, y)$ lies in, where

$$x = -\cosh i\pi\kappa, \quad y = -\frac{\cos \theta \sin \pi \kappa \tanh \frac{\alpha}{2}}{\kappa}$$

(8)

Also, the visibility in this spin-$\frac{1}{2}$ case is given by

$$\nu_{\frac{1}{2}} = \sqrt{1 + \frac{4 \sinh^2 i\pi\kappa}{\zeta}}$$

(9)

where

$$\zeta = 3 - \cos 2\theta + 2 \cos^2 \theta \cosh \alpha,$$

(10)

and $\kappa$ is as defined in (3). The comparable results in the spin-1 case (Figs. 2 and 12) are similarly based on

$$x = \frac{e^{-2i\pi\kappa} \left(2(e^\alpha + 2e^{2i\pi\kappa} + e^{\alpha+4i\pi\kappa}) \cos^2 \theta + (2 + 2e^{4i\pi\kappa} + e^\alpha(1 + e^{2i\pi\kappa})^2) \text{sech}^2 \frac{\alpha}{2} \sin^2 \theta \right)}{4\kappa^2(1 + 2 \cosh \alpha)},$$

(11)

and

$$y = \frac{e^\alpha \cos \theta \sin 2\pi\kappa}{\kappa + 2\kappa \cos \alpha}.$$  

(12)

One feature distinguishing the (strikingly similar) visibility plots (Figs. 11 and 20) from one another is that as $j$ increases, $\nu_j$ decreases at the (boundary) point $\alpha = 5, \theta = \frac{\pi}{2}$. For instance this value is .871618 in Fig. 11 and .498776 in Fig. 15.
In Fig. 21, we plot $\gamma_j$ ($j = \frac{1}{2}, \ldots$) versus $\theta$, holding $\alpha$ fixed at 1. Curves for lower-dimensional Gibbsian systems here strictly dominate those for higher-dimensional systems. (As $\alpha$ increases, however, above certain thresholds for $\theta$, this simple monotonic behavior vanishes [cf. Fig. 24].)

FIG. 21. Uhlmann geometric phases for $n$-level Gibbsian systems ($n = 2, \ldots, 11$) holding $\alpha = 1$. The curve for $n = 2$ dominates that for $n = 3$, which dominates that for $n = 4, \ldots$

In Fig. 22, we “reverse” this scenario, now holding $\theta$ fixed at $\frac{\pi}{10}$ and letting $\alpha$ vary over $[0, 5]$. The monotonicity of the ten curves is completely analogous to that in Fig. 21, with curves for lower $j$ dominating those for higher $j$.

FIG. 22. Uhlmann geometric phases for $n$-level Gibbsian systems ($n = 2, \ldots, 11$) holding $\theta$ fixed at $\frac{\pi}{10}$. The curve for $n = 2$ dominates that for $n = 3$, which dominates that for $n = 4, \ldots$

In Fig. 23, we hold $\alpha$ at 2 and plot the visibilities for the ten spin scenarios. Again, as in Figs. 21 and 22, curves for lower values of $j$ dominate those for higher values.

FIG. 23. Uhlmann geometric phases for $n$-level Gibbsian systems ($n = 2, \ldots, 11$) holding $\alpha = 2$. The curve for $n = 2$ dominates that for $n = 3$, which dominates that for $n = 4, \ldots$
FIG. 23. Uhlmann visibilities for \( n \)-level Gibbsian systems (\( n = 2, \ldots, 11 \)) holding \( \alpha \) fixed at 2. The curve for \( n = 2 \) dominates that for \( n = 3 \), which dominates that for \( n = 4, \ldots \)

While in Fig. 21 the inverse temperature parameter \( \alpha \) was fixed at 1, in Fig. 24, it is held at 2. Now the same simple monotonicity with \( j \) observed in the preceding three figures, holds below \( \theta \approx 0.65 \). However, it is lost at higher values of \( \theta \), with the curves for the higher spin states oscillating from \(-\frac{\pi}{2}\) to \(\frac{\pi}{2}\) at lower values of \( \theta \) than do the curves for some of the lower spin states.

FIG. 24. Uhlmann geometric phases for \( n \)-level Gibbsian systems (\( n = 2, \ldots, 11 \)) holding \( \alpha = 2 \). Only below \( \theta \approx 0.65 \) does the curve for \( n = 2 \) dominate that for \( n = 3 \), which dominates that for \( n = 4, \ldots \)

In Fig. 25, in which \( \theta \) is fixed at \( \frac{\pi}{5} \), similar behavior to that observed in Fig. 24 takes place. Below \( \alpha \approx 1.43 \), the simple monotonicity with \( j \) holds, then the curves begin to jump from \(-\frac{\pi}{2}\) to \(\frac{\pi}{2}\) with the curves for higher \( j \) jumping first.
FIG. 25. Uhlmann geometric phases for \( n \)-level Gibbsian systems \((n = 2, \ldots, 11)\) holding \( \theta = \frac{\pi}{5} \). Only below \( \alpha \approx 2 \) does the curve for \( n = 2 \) dominate that for \( n = 3 \), which dominates that for \( n = 4, \ldots \).

In Fig. 26, holding \( \theta = \frac{\pi}{2} \), we plot the various visibilities over the range \( \alpha \in [1, 4] \). At \( \alpha = 2.2 \), the values of \( \nu_j \) monotonically decline from \( j = \frac{1}{2} \) to \( j = \frac{9}{2} \).

FIG. 26. Uhlmann visibilities for \( n \)-level Gibbsian systems \((n = 2, \ldots, 11)\) fixing \( \theta = \frac{\pi}{2} \). At \( \alpha = 2.1 \), the values of \( \nu \) monotonically decline as \( n \) increases. The highest peak belongs to \( n = 2 \).

V. HIGHER-ORDER UHLMANN HOLOMONY INVARIANTS

A. Traces of powers of Uhlmann holonomy invariants

All the analyses above have been conducted on the basis of the trace of the holonomy invariant \([\text{eq. } (7)]\). In addition, however, the traces of the higher powers of \([\text{eq. } (7)]\) are also invariants \([\text{eq. } (102)]\) (all reducing, however, for pure states, to simple powers of the “Berry phase” \([\text{eq. } (112.5)]\) (cf. \([\text{eq. } (7)]\)). In Fig. 27, we show the argument of the second power of the holonomy invariant for the spin-\( \frac{1}{2} \) case \((n = 2)\), which we express in the form of the arctangent of \( \frac{x}{y} \), taking into account which quadrant the point \((x, y)\) lies in, where

\[
x = \frac{1}{2 \zeta} \left( \zeta (-1 + (1 + \cosh \alpha) \cosh 2i\pi \kappa) \text{sech}^2 \frac{\alpha}{2} - 8 \sinh^2 i\pi \kappa \right), \quad y = -\frac{2i \kappa \cos \theta \sinh \alpha \sinh 2i\pi \kappa}{\zeta}.
\]  

The corresponding absolute value (Fig. 28) is the square root of

\[
x^2 + y^2 = 1 - \frac{2 \cosh 2i\pi \kappa}{1 + \cosh \alpha} + \frac{1}{4} \text{sech}^2 \frac{\alpha}{2} + \frac{4(2 + \cosh \alpha) \text{sech}^2 \frac{\alpha}{2} \sinh^2 i\pi \kappa}{\zeta} + \frac{16 \sinh^4 i\pi \kappa}{\zeta^2}.
\]  

16
(In some of the later [computationally-intensive] figures in this section, it proved essentially necessary to omit the [otherwise obvious] axes labels, due to some quite distinct peculiarities of the graphics facilities for the local installation of MATHEMATICA on the more powerful workstations.)

FIG. 27. Argument of the trace of the second power of the holonomy invariant (7) for the two-level Gibbsian systems ($n = 2$)

FIG. 28. Absolute value of the trace of the second power of the holonomy invariant (7) for the two-level Gibbsian systems ($n = 2$)

In Figs. 29 and 30 are shown the analogous quantities for Gibbsian spin-1 systems.
FIG. 29. Argument of the trace of the second power of the holonomy invariant \( \Theta \) for the three-level Gibbsian systems \( (n = 3) \).

FIG. 30. Absolute value of the trace of the second power of the holonomy invariant \( \Theta \) for the three-level Gibbsian systems \( (n = 3) \).
FIG. 31. Argument of the trace of the third power of the holonomy invariant (7) for the three-level Gibbsian systems (n = 3)

FIG. 32. Absolute value of the trace of the third power of the holonomy invariant (7) for the three-level Gibbsian systems (n = 3)
FIG. 33. Argument of the trace of the second power of the holonomy invariant (7) for the four-level Gibbsian systems \((n = 4)\)

FIG. 34. Absolute value of the trace of the second power of the holonomy invariant (7) for the four-level Gibbsian systems \((n = 4)\)
FIG. 35. Argument of the trace of the third power of the holonomy invariant (7) for the four-level Gibbsian systems ($n = 4$)

FIG. 36. Absolute value of the trace of the third power of the holonomy invariant (7) for the four-level Gibbsian systems ($n = 4$)
FIG. 37. Argument of the trace of the fourth power of the holonomy invariant (7) for the four-level Gibbsian systems (n = 4)

FIG. 38. Absolute value of the trace of the fourth power of the holonomy invariant (7) for the four-level Gibbsian systems (n = 4)
FIG. 39. Argument of the trace of the second power of the holonomy invariant for the five-level Gibbsian systems ($n = 5$)

FIG. 40. Absolute value of the trace of the second power of the holonomy invariant for the five-level Gibbsian systems ($n = 5$)
FIG. 41. Argument of the trace of the third power of the holonomy invariant (7) for the five-level Gibbsian systems ($n = 5$)

FIG. 42. Absolute value of the trace of the third power of the holonomy invariant (7) for the five-level Gibbsian systems ($n = 5$)
FIG. 43. Argument of the trace of the fourth power of the holonomy invariant (7) for the five-level Gibbsian systems \( n = 5 \)

FIG. 44. Absolute value of the trace of the fourth power of the holonomy invariant (7) for the five-level Gibbsian systems \( n = 5 \)
FIG. 45. Argument of the trace of the fifth power of the holonomy invariant \( \mathcal{I} \) for the five-level Gibbsian systems \( (n = 5) \)

FIG. 46. Absolute value of the trace of the fifth power of the holonomy invariant \( \mathcal{I} \) for the five-level Gibbsian systems \( (n = 5) \)
FIG. 47. Argument of the trace of the second power of the holonomy invariant \( (\mathcal{H}) \) for the six-level Gibbsian systems \( (n = 6) \)

FIG. 48. Absolute value of the trace of the second power of the holonomy invariant \( (\mathcal{H}) \) for the six-level Gibbsian systems \( (n = 6) \)
FIG. 49. Argument of the trace of the third power of the holonomy invariant (7) for the six-level Gibbsian systems ($n = 6$)

FIG. 50. Absolute value of the trace of the third power of the holonomy invariant (7) for the six-level Gibbsian systems ($n = 6$)
FIG. 51. Argument of the trace of the fourth power of the holonomy invariant (7) for the six-level Gibbsian systems ($n = 6$).

FIG. 52. Absolute value of the trace of the fourth power of the holonomy invariant (7) for the six-level Gibbsian systems ($n = 6$).
FIG. 53. Argument of the trace of the fifth power of the holonomy invariant $\mathcal{H}$ for the six-level Gibbsian systems ($n = 6$)

FIG. 54. Absolute value of the trace of the fifth power of the holonomy invariant $\mathcal{H}$ for the six-level Gibbsian systems ($n = 6$)
B. Cross-comparisons of Uhlmann holonomy invariants

FIG. 55. Ratio of the argument of the trace of the second power of the holonomy invariant \( \xi \) to the argument of the trace of the first power for the eight-level Gibbsian systems.

FIG. 56. Ratio of the absolute value of the trace of the second power of the holonomy invariant \( \xi \) to the absolute value of the trace of the first power for the eight-level Gibbsian systems.
C. Eigenvalues of Uhlmann holonomy invariant for $n = 2, 3, 5$

To supplement our consideration above of the *traces* of various powers of the Uhlmann holonomy invariant (6), here we examine the eigenvalues themselves. (Both these traces and eigenvalues are themselves invariants, of course.) In Figs. 59 and 60 are shown the arguments of the dominant and subordinate eigenvalues of the holonomy invariant (7) for the case $n = 2$, and in Figs. 61 and 62 the corresponding absolute values.
FIG. 59. Argument of the dominant eigenvalue of the Uhlmann holonomy invariant \( \mathcal{H} \) for the two-level Gibbsian density matrices.

FIG. 60. Argument of the subordinate eigenvalue of the Uhlmann holonomy invariant \( \mathcal{H} \) for the two-level Gibbsian density matrices.
In Figs. 63 and 64, we plot the arguments of the dominant and subordinate eigenvalues of the holonomy invariant (7) for the case $n = 3$ and in Figs. 65 and 67, the absolute values of the two corresponding eigenvalues. The argument of the intermediate (in absolute value) one of the three eigenvalues appears to always be identically zero, that is, this intermediate eigenvalue is real (Fig. 66). It also appears numerically, on the basis of further analyses we have conducted, that this result is generalizable to the proposition that for odd $n$, the central/most intermediate one (that is, the $\left\lceil \frac{n}{2} \right\rceil$-th) of the $n$ eigenvalues is real. (Also, the centrally located diagonal entry of the holonomy invariant for odd $n$ appears to be real, in general.)
FIG. 63. Argument of the dominant eigenvalue of the Uhlmann holonomy invariant $\lambda$ for the three-level Gibbsian density matrices.

FIG. 64. Argument of the subordinate eigenvalue of the Uhlmann holonomy invariant $\lambda$ for the three-level Gibbsian density matrices.
FIG. 65. Absolute value of the dominant eigenvalue of the Uhlmann holonomy invariant (7) for the three-level Gibbsian density matrices

FIG. 66. Intermediate eigenvalue of the Uhlmann holonomy invariant (7) for the three-level Gibbsian density matrices
FIG. 67. Absolute value of the subordinate eigenvalue of the Uhlmann holonomy invariant \( \lambda \) for the three-level Gibbsian density matrices.

In Figs. 68-71, we show the arguments of the eigenvalues of the Uhlmann holonomy invariant \( \lambda \) in the case \( n = 5 \). (The case \( n = 4 \) proved more intractable in nature.) The third eigenvalue is simply real. In Figs. 72, 73, 75, 76 are shown the corresponding absolute values. In Fig. 74 is shown the value of the (real) intermediate eigenvalue.

FIG. 68. Argument of the leading eigenvalue of the Uhlmann holonomy invariant \( \lambda \) for the five-level Gibbsian density matrices.
FIG. 69. Argument of the second leading eigenvalue of the Uhlmann holonomy invariant (7) for the five-level Gibbsian density matrices.

FIG. 70. Argument of the second smallest eigenvalue of the Uhlmann holonomy invariant (7) for the five-level Gibbsian density matrices.
FIG. 71. Argument of the last eigenvalue of the Uhlmann holonomy invariant (6) for the five-level Gibbsian density matrices

FIG. 72. Absolute value of the leading eigenvalue of the Uhlmann holonomy invariant (6) for the five-level Gibbsian density matrices
FIG. 73. Absolute value of the second leading eigenvalue of the Uhlmann holonomy invariant (7) for the five-level Gibbsian density matrices

FIG. 74. Central eigenvalue of the Uhlmann holonomy invariant (7) for the five-level Gibbsian density matrices
FIG. 75. Absolute value of the second smallest eigenvalue of the Uhlmann holonomy invariant (7) for the five-level Gibbsian density matrices

FIG. 76. Absolute value of the last eigenvalue of the Uhlmann holonomy invariant (7) for the five-level Gibbsian density matrices

VI. SJÖQVIST ET AL GEOMETRIC PHASES FOR GIBBSIAN N-LEVEL SYSTEMS ($N = 2, \ldots, 11$)

In the approach of Sjöqvist et al[1] every pure state that diagonalizes the initial density matrix is parallel transported separately. By weighting the associated holonomy of each such pure state by the corresponding eigenvalues, we obtain the measure proposed in [1] “in the experimental context of quantum interferometry”. Since it is essentially trivial to obtain the pure states that diagonalize the initial (diagonal) $n$-level Gibbsian density matrix and the associated eigenvalues, and since Uhlmann has presented the holonomy invariant (the trace of which is the Berry phase)

$$W_{2\pi} W_0^* = (-1)^{2j} \rho_0^{1/2} \rho_0^{1/2} e^{2\pi i \cos \theta J_z} = e^{-2\pi i m(1 - \cos \theta)} |m > m|, \tag{15}$$
for the circular evolution of pure states, we can directly compute the Sjöqvist et al geometric phase ($\gamma$) and visibility ($\nu$) for the $n$-level Gibbsian density matrices. These are presented in Figs. 77 to 86 and Figs. 87 to 96. The trace of the weighted (by the eigenvalues) sum of the holonomy invariants (15) for $n = 2$ is

$$\cos (\pi \cos \theta) + i \sin (\pi \cos \theta) \tanh \frac{\alpha}{2},$$

(16)

while for $n = 3$, it is

$$\frac{1 + 2 \cos (2\pi \cos \theta) \cosh \alpha + 2i \sin (2\pi \cos \theta) \sinh \alpha}{1 + 2 \cosh \alpha},$$

(17)

for $n = 4$,

$$\frac{e^{-3i\pi \cos \theta}(1 + e^{\alpha + 2i\pi \cos \theta})(1 + e^{2\alpha + 4i\pi \cos \theta})}{(1 + e^{\alpha})(1 + e^{2\alpha})}.$$

(18)

and for $n = 5$,

$$\frac{1 + 2 \cos (2\pi \cos \theta) \cosh \alpha + 2 \cos (4\pi \cos \theta) \cosh 2\alpha + 2i \sin (2\pi \cos \theta) \sinh \alpha + 2i \sin (4\pi \cos \theta) \sinh 2\alpha}{1 + 2 \cosh \alpha + 2 \cosh 2\alpha}.$$

(19)

FIG. 77. Sjöqvist et al geometric phase for Gibbsian spin-$\frac{1}{2}$ systems
FIG. 78. Sjöqvist et al geometric phase for Gibbsian spin-1 systems

FIG. 79. Sjöqvist et al geometric phase for Gibbsian spin-$\frac{3}{2}$ systems
FIG. 80. Sjöqvist et al geometric phase for Gibbsian spin-2 systems

FIG. 81. Sjöqvist et al geometric phase for Gibbsian spin-$\frac{5}{2}$ systems
FIG. 82. Sjöqvist et al geometric phase for Gibbsian spin-3 systems

FIG. 83. Sjöqvist et al geometric phase for Gibbsian spin-$\frac{7}{2}$ systems
FIG. 84. Sjöqvist et al geometric phase for Gibbsian spin-4 systems

FIG. 85. Sjöqvist et al geometric phase for Gibbsian spin-$\frac{9}{2}$ systems
FIG. 86. Sjöqvist et al geometric phase for Gibbsian spin-5 systems
VII. SJÖQVIST ET AL VISIBILITIES FOR GIBBSIAN \(N\)-LEVEL SYSTEMS \((N = 2, \ldots, 11)\)

FIG. 87. Sjöqvist et al visibility for Gibbsian spin-\(\frac{1}{2}\) systems
FIG. 88. Sjöqvist et al visibility for Gibbsian spin-1 systems

FIG. 89. Sjöqvist et al visibility for Gibbsian spin-$\frac{3}{2}$ systems
FIG. 90. Sjöqvist et al visibility for Gibbsian spin-2 systems

FIG. 91. Sjöqvist et al visibility for Gibbsian spin-\( \frac{5}{2} \) systems
FIG. 92. Sjöqvist et al visibility for Gibbsian spin-3 systems

FIG. 93. Sjöqvist et al visibility for Gibbsian spin-7 systems
FIG. 94. Sjöqvist et al visibility for Gibbsian spin-4 systems

FIG. 95. Sjöqvist et al visibility for Gibbsian spin-$\frac{3}{2}$ systems
VIII. COMPARISONS ACROSS GIBBSIAN N-LEVEL SYSTEMS OF SJÖQVIST ET AL GEOMETRIC PHASES AND VISIBILITIES

In Figs. 97-101, we display the direct counterparts of Figs. 21-25, but now based on the methodology \[1\] of Sjöqvist et al rather than Uhlmann. (We omit the counterpart of Fig. 26 because it appears to consist of essentially noise.)
FIG. 98. Sjöqvist et al geometric phases for $n$-level Gibbsian systems ($n = 2, \ldots, 11$) holding $\theta$ fixed at $\frac{\pi}{10}$. In monotonically decreasing order are the curves for (the even) $n = 2, 4, 6, 8, 10$, followed by those for (the odd) $n = 3, 5, 7, 9, 11$, cf. Fig. 22.

FIG. 99. Sjöqvist et al visibilities for $n$-level Gibbsian systems ($n = 2, \ldots, 11$) holding $\alpha$ fixed at 2. The curve for $n = 2$ stands out from the other nine, cf. Fig. 23.

FIG. 100. Sjöqvist et al geometric phases for $n$-level Gibbsian systems ($n = 2, \ldots, 11$) holding $\alpha = 2$. In order of decreasing dominance at $\theta = .25$ are the curves for (the even) $n = 2, 4, 6, 8, 10$, followed by those for (the odd) $n = 3, 5, 7, 9, 11$, cf. Fig. 24.
FIG. 101. Sjöqvist et al. geometric phases for $n$-level Gibbsian systems ($n = 2, \ldots, 11$) holding $\theta = \frac{\pi}{5}$. At $\alpha = 2$, in order of decreasing dominance are the curves for $n = 7, 2, 9, 4, 11, 6, 8, 3, 10, 5$, cf. Fig. 25.

IX. DIRECT COMPARISONS OF UHLMANN AND SJÖQVIST ET AL RESULTS

In Fig. 102, we plot for $n = 6$ the ratio of the Sjöqvist et al. geometric phase (Fig. 81) to the Uhlmann geometric phase (Fig. 5), and in Fig. 103, the corresponding ratio for $n = 11$. In the next two figures (Figs. 104 and 105), we show the ratios of the associated visibililities.

FIG. 102. Ratio of the Sjöqvist et al geometric phase (Fig. 81) to the Uhlmann geometric phase (Fig. 5) for six-level Gibbsian systems.
FIG. 103. Ratio of the Sjöqvist et al geometric phase (Fig. 86) to the Uhlmann geometric phase (Fig. 10) for the eleven-level Gibbsian systems

FIG. 104. Ratio of the Sjöqvist et al visibility (Fig. 91) to the Uhlmann visibility (Fig. 15) for the six-level Gibbsian systems
FIG. 105. Ratio of the Sjöqvist et al visibility (Fig. 96) to the Uhlmann visibility (Fig. 20) for the eleven-level Gibbsian systems

X. SUMMARY

We have reported an in-depth study here of two different methodologies for determining geometric phases for mixed states [1,2]. The analysis has been framed in terms of rotations (O(3)-orbits) of n-level Gibbsian systems, as originally proposed by Uhlmann [3].

There are, of course, many features in these results. One of these is that the Sjöqvist et al geometric phases (sec. VI) appear to be less sensitive to the inverse temperature parameter (α) than do the Uhlmann geometric phases (sec. II).

In particular, we have found (sec. [VI]) that as the number of levels of the Gibbsian density matrices increases (in the sample we have studied) from n = 2 to 11, the Uhlmann methodology often yields simple monotonic behavior. On the other hand, this seems to be largely absent using the Sjöqvist et al methodology (sec. VIII), which appears among other things to be sensitive to the bosonic (odd n) or fermionic (even n) character of the system under consideration.

It is certainly important to note that if one sets the parameter a — given in (4) — equal to zero in the approach of Uhlmann (sec. [I]), the resulting holonomy invariant (7) is simply equal to (−1)^{n+1} times the holonomy invariant of Sjöqvist et al (eq. (15)). This would appear to help to explain the added complexity of the plots of the Uhlmann geometric phases (sec. [II]) vis-à-vis those of the Sjöqvist et al plots (sec. [VI]).

We have found compelling numerical evidence that for odd n the central (\left\lceil \frac{n}{2} \right\rceil,\left\lceil \frac{n}{2} \right\rceil)-entry of the Uhlmann holonomy invariant (7) is always a real number (sec. [V C]). (By the fundamental theorem of algebra, for odd n, one of the n eigenvalues must of course be real — since complex roots come in conjugate pairs — but clearly not necessarily the centrally located eigenvalue.) Also, the (\left\lceil \frac{n}{2} \right\rceil,\left\lceil \frac{n}{2} \right\rceil)-entry of the Uhlmann holonomy invariant (7) is itself real, for odd n. In terms of absolute values, the diagonal entries of the invariant (7) appear to be always monotonically decreasing from the (upper left) (1,1)-entry to the (lower right) (n,n)-entry. (We note that the angular momentum operator J_z — entering in our equations (2), (4) and (15) — itself has declining [real] diagonal entries, that is, n − 1/over2, . . ., −\frac{n−1}{2}.)
ACKNOWLEDGMENTS

I would like to express appreciation to the Institute for Theoretical Physics for computational support in this research.

[1] E. Sjöqvist, A. K. Pati, A. Ekert, J. S. Anandan, M. Ericsson, D. K. L. Oi and V. Vedral, Phys. Rev. Lett. 85, 2845 (2000).
[2] A. Uhlmann, in Nonlinear, Dissipative, Irreversible Quantum Systems, edited by H.-D. Doebner, V. K. Dobrev and P. Nattermann, (Clausthal, 1994).
[3] A. Uhlmann, in Symmetries in Science VI, edited by B. Gruber, (Plenum, New York, 1993), p. 741.
[4] P. B. Slater, e-print, math-ph/0111014.
[5] J. Dittmann and A. Uhlmann, J. Math. Phys. 40, 3246-67 (1999).
[6] R. Jackiw, Intl. J. Mod. Phys. A 3, 285-297 (1989).

List of Figures

1 Uhlmann geometric phase for Gibbsian spin-3/2 systems ................................................. 3
2 Uhlmann geometric phase for Gibbsian spin-1 systems ......................................................... 4
3 Uhlmann geometric phase for Gibbsian spin-3 systems .................................................................. 4
4 Uhlmann geometric phase for Gibbsian spin-2 systems ................................................................. 5
5 Uhlmann geometric phase for Gibbsian spin-7/2 systems ............................................................. 5
6 Uhlmann geometric phase for Gibbsian spin-3 systems ............................................................. 6
7 Uhlmann geometric phase for Gibbsian spin-7/2 systems ............................................................. 6
8 Uhlmann geometric phase for Gibbsian spin-4 systems ............................................................. 7
9 Uhlmann geometric phase for Gibbsian spin-7/2 systems ............................................................. 7
10 Uhlmann geometric phase for Gibbsian spin-5 systems .............................................................. 8
11 Uhlmann visibility for Gibbsian spin-1 systems ........................................................................... 8
12 Uhlmann visibility for Gibbsian spin-1 systems ........................................................................... 9
13 Uhlmann visibility for Gibbsian spin-3 systems ........................................................................... 9
14 Uhlmann visibility for Gibbsian spin-2 systems .......................................................................... 10
15 Uhlmann visibility for Gibbsian spin-7/2 systems ....................................................................... 10
16 Uhlmann visibility for Gibbsian spin-3 systems ........................................................................... 11
17 Uhlmann visibility for Gibbsian spin-7/2 systems ....................................................................... 11
18 Uhlmann visibility for Gibbsian spin-4 systems ........................................................................... 12
19 Uhlmann visibility for Gibbsian spin-7/2 systems ....................................................................... 12
20 Uhlmann visibility for Gibbsian spin-5 systems ........................................................................... 13
21 Uhlmann geometric phases for n-level Gibbsian systems \(n = 2, \ldots, 11\) holding \(\alpha = 1\). The curve for \(n = 2\) dominates that for \(n = 3\), which dominates that for \(n = 4, \ldots\) .................................................. 14
22 Uhlmann geometric phases for n-level Gibbsian systems \(n = 2, \ldots, 11\) holding \(\theta\) fixed at \(\frac{\pi}{2}\). The curve for \(n = 2\) dominates that for \(n = 3\), which dominates that for \(n = 4, \ldots\) .................................................. 14
23 Uhlmann visibilities for n-level Gibbsian systems \(n = 2, \ldots, 11\) holding \(\alpha\) fixed at 2. The curve for \(n = 2\) dominates that for \(n = 3\), which dominates that for \(n = 4, \ldots\) .................................................. 15
24 Uhlmann geometric phases for n-level Gibbsian systems \(n = 2, \ldots, 11\) holding \(\alpha = 2\). Only below \(\theta \approx 0.65\) does the curve for \(n = 2\) dominate that for \(n = 3\), which dominates that for \(n = 4, \ldots\) .................................................. 15
25 Uhlmann geometric phases for n-level Gibbsian systems \(n = 2, \ldots, 11\) holding \(\theta = \frac{\pi}{2}\). Only below \(\alpha \approx 2\) does the curve for \(n = 2\) dominate that for \(n = 3\), which dominates that for \(n = 4, \ldots\) .................................................. 16
26 Uhlmann visibilities for n-level Gibbsian systems \(n = 2, \ldots, 11\) fixing \(\theta = \frac{\pi}{2}\). At \(\alpha = 2.1\), the values of \(\nu\) monotonically decline as \(n\) increases. The highest peak belongs to \(n = 2\) .................................................. 16
27 Argument of the trace of the second power of the holonomy invariant \((7)\) for the two-level Gibbsian systems \((n = 2)\) ................................................................. 17
28 Absolute value of the trace of the second power of the holonomy invariant \((7)\) for the two-level Gibbsian systems \((n = 2)\) ................................................................. 17
29 Argument of the trace of the second power of the holonomy invariant (7) for the three-level Gibbsian systems
30 Absolute value of the trace of the second power of the holonomy invariant (7) for the three-level Gibbsian systems
31 Argument of the trace of the third power of the holonomy invariant (7) for the three-level Gibbsian systems
32 Absolute value of the trace of the third power of the holonomy invariant (7) for the three-level Gibbsian systems
33 Argument of the trace of the second power of the holonomy invariant (7) for the four-level Gibbsian systems
34 Absolute value of the trace of the second power of the holonomy invariant (7) for the four-level Gibbsian systems
35 Argument of the trace of the third power of the holonomy invariant (7) for the four-level Gibbsian systems
36 Absolute value of the trace of the third power of the holonomy invariant (7) for the four-level Gibbsian systems
37 Argument of the trace of the fourth power of the holonomy invariant (7) for the four-level Gibbsian systems
38 Absolute value of the trace of the fourth power of the holonomy invariant (7) for the four-level Gibbsian systems
39 Argument of the trace of the second power of the holonomy invariant (7) for the five-level Gibbsian systems
40 Absolute value of the trace of the second power of the holonomy invariant (7) for the five-level Gibbsian systems
41 Argument of the trace of the third power of the holonomy invariant (7) for the five-level Gibbsian systems
42 Absolute value of the trace of the third power of the holonomy invariant (7) for the five-level Gibbsian systems
43 Argument of the trace of the fourth power of the holonomy invariant (7) for the five-level Gibbsian systems
44 Absolute value of the trace of the fourth power of the holonomy invariant (7) for the five-level Gibbsian systems
45 Argument of the trace of the fifth power of the holonomy invariant (7) for the five-level Gibbsian systems
46 Absolute value of the trace of the fifth power of the holonomy invariant (7) for the five-level Gibbsian systems
47 Argument of the trace of the second power of the holonomy invariant (7) for the six-level Gibbsian systems
48 Absolute value of the trace of the second power of the holonomy invariant (7) for the six-level Gibbsian systems
49 Argument of the trace of the third power of the holonomy invariant (7) for the six-level Gibbsian systems
50 Absolute value of the trace of the third power of the holonomy invariant (7) for the six-level Gibbsian systems
51 Argument of the trace of the fourth power of the holonomy invariant (7) for the six-level Gibbsian systems
52 Absolute value of the trace of the fourth power of the holonomy invariant (7) for the six-level Gibbsian systems
53 Argument of the trace of the fifth power of the holonomy invariant (7) for the six-level Gibbsian systems
54 Absolute value of the trace of the fifth power of the holonomy invariant (7) for the six-level Gibbsian systems
55 Ratio of the argument of the trace of the second power of the holonomy invariant (7) to the argument of the trace of the first power for the eight-level Gibbsian systems
56 Ratio of the absolute value of the trace of the second power of the holonomy invariant (7) to the absolute value of the trace of the first power for the eight-level Gibbsian systems
61 Argument of the dominant eigenvalue of the Uhlmann holonomy invariant (7) for the two-level Gibbsian density matrices

62 Absolute value of the subordinate eigenvalue of the Uhlmann holonomy invariant (7) for the two-level Gibbsian density matrices

63 Argument of the dominant eigenvalue of the Uhlmann holonomy invariant (7) for the three-level Gibbsian density matrices

64 Argument of the subordinate eigenvalue of the Uhlmann holonomy invariant (7) for the three-level Gibbsian density matrices

65 Absolute value of the dominant eigenvalue of the Uhlmann holonomy invariant (7) for the three-level Gibbsian density matrices

66 Intermediate eigenvalue of the Uhlmann holonomy invariant (7) for the three-level Gibbsian density matrices

67 Absolute value of the subordinate eigenvalue of the Uhlmann holonomy invariant (7) for the three-level Gibbsian density matrices

68 Argument of the leading eigenvalue of the Uhlmann holonomy invariant (7) for the five-level Gibbsian density matrices

69 Argument of the second leading eigenvalue of the Uhlmann holonomy invariant (7) for the five-level Gibbsian density matrices

70 Argument of the second smallest eigenvalue of the Uhlmann holonomy invariant (7) for the five-level Gibbsian density matrices

71 Argument of the last eigenvalue of the Uhlmann holonomy invariant (7) for the five-level Gibbsian density matrices

72 Absolute value of the leading eigenvalue of the Uhlmann holonomy invariant (7) for the five-level Gibbsian density matrices

73 Absolute value of the second leading eigenvalue of the Uhlmann holonomy invariant (7) for the five-level Gibbsian density matrices

74 Central eigenvalue of the Uhlmann holonomy invariant (7) for the five-level Gibbsian density matrices

75 Absolute value of the second smallest eigenvalue of the Uhlmann holonomy invariant (7) for the five-level Gibbsian density matrices

76 Absolute value of the last eigenvalue of the Uhlmann holonomy invariant (7) for the five-level Gibbsian density matrices

77 Sjöqvist et al geometric phase for Gibbsian spin-$\frac{1}{5}$ systems

78 Sjöqvist et al geometric phase for Gibbsian spin-$\frac{1}{4}$ systems

79 Sjöqvist et al geometric phase for Gibbsian spin-$\frac{1}{3}$ systems

80 Sjöqvist et al geometric phase for Gibbsian spin-$\frac{1}{2}$ systems

81 Sjöqvist et al geometric phase for Gibbsian spin-$\frac{5}{6}$ systems

82 Sjöqvist et al geometric phase for Gibbsian spin-$\frac{3}{5}$ systems

83 Sjöqvist et al geometric phase for Gibbsian spin-$\frac{5}{3}$ systems

84 Sjöqvist et al geometric phase for Gibbsian spin-$\frac{4}{5}$ systems

85 Sjöqvist et al geometric phase for Gibbsian spin-$\frac{4}{3}$ systems

86 Sjöqvist et al geometric phase for Gibbsian spin-$\frac{3}{4}$ systems

87 Sjöqvist et al visibility for Gibbsian spin-$\frac{1}{2}$ systems

88 Sjöqvist et al visibility for Gibbsian spin-$\frac{3}{5}$ systems

89 Sjöqvist et al visibility for Gibbsian spin-$\frac{3}{4}$ systems

90 Sjöqvist et al visibility for Gibbsian spin-$\frac{5}{6}$ systems

91 Sjöqvist et al visibility for Gibbsian spin-$\frac{4}{3}$ systems

92 Sjöqvist et al visibility for Gibbsian spin-$\frac{5}{3}$ systems

93 Sjöqvist et al visibility for Gibbsian spin-$\frac{4}{5}$ systems
Sjöqvist et al geometric phases for $n$-level Gibbsian systems ($n = 2, \ldots, 11$) holding $\alpha = 1$. At $\theta = .25$, the curve for $n = 2$ is dominant, followed in order by those for $n = 4, 6, 8, 10$, (all having positive values at $\theta = .25$) and $n = 3, 5, 7, 9, 11$ (all having negative values at $\theta = .25$), cf. Fig. 21.

In monotonically decreasing order are the curves for (the even) $n = 2, 4, 6, 8, 10$, followed by those for (the odd) $n = 3, 5, 7, 9, 11$, cf. Fig. 22.

At $\alpha = 2$, in order of decreasing dominance at $\theta = .25$ are the curves for $n = 7, 2, 9, 4, 11, 6, 8, 3, 10, 5$, cf. Fig. 25.

Ratio of the Sjöqvist et al geometric phase (Fig. 81) to the Uhlmann geometric phase (Fig. 3) for six-level Gibbsian systems.

Ratio of the Sjöqvist et al geometric phase (Fig. 86) to the Uhlmann geometric phase (Fig. 10) for the eleven-level Gibbsian systems.

Ratio of the Sjöqvist et al visibility (Fig. 91) to the Uhlmann visibility (Fig. 15) for the six-level Gibbsian systems.

Ratio of the Sjöqvist et al visibility (Fig. 96) to the Uhlmann visibility (Fig. 20) for the eleven-level Gibbsian systems.