A note on computations of angular momentum and its flux in numerical relativity

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Abstract
The purpose of this note is to point out ambiguities that appear in the calculation of angular momentum and its radiated counterpart when some simple formulae are used to compute them. We illustrate, in two simple different examples, how incorrect results can be obtained with them. Additionally, we discuss the magnitude of possible errors in well-known situations.

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1. Introduction
It is well known that the notion of total angular momentum in general relativity is sensitive to the so-called problem of supertranslation ambiguities. This topic involves subtleties and complications inherent in the infinite-dimensional nature of the asymptotic symmetry group of asymptotically flat spacetimes, namely, the BMS group. On the other hand, there is a clear need to compute these quantities to extract valuable physical information in numerically constructed spacetimes to connect with observations of a variety of sources. A few proposals exist to do this either locally [1–5] or asymptotically [1, 6–12]$^4$. The dependence of any notion of relativistic angular momentum on supertranslations has been the matter of numerous studies. The problem is complex and, unfortunately, most definitions of angular momentum either show supertranslation ambiguities or require additional structure (at the expense of reducing generality in their application).

In spite of all these difficulties, the need for tackling concrete calculations of angular momentum in numerically constructed spacetimes has motivated the use of simple and practical formulae for the computation of angular momentum. These formulae lend themselves to a

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$^4$ We are not presenting an exhaustive list of references, since this is not a review note on angular momentum. We apologize for our arbitrary choice of references.
straightforward computation and, in relatively simple scenarios, have given the expected answers. One option that has been employed in several works has been suggested in [13], where it is proposed to compute the angular momentum from the expression

$$J_{[i]} = \frac{1}{16\pi} \text{Re} \left( \oint_{\Sigma} dS \Phi_{[i]} [2\Psi^0 - 2\sigma^0 \delta^0 - \partial(\sigma^0 \delta^0)] \right),$$

(1)

which is a truncated version of the one defined in [8]; in fact, a supermomentum part is missing. As mentioned above, supertranslation issues can obscure angular momentum definitions (and expressions derived from them) and indeed, as we show in the following sections, equation (1) is susceptible to them as well.

We discuss in the following examples two simple cases where expression (1) gives an incorrect result.

2. Examples

2.1. A supertranslated and boosted Schwarzschild

Let us consider, for simplicity, the case of a spacetime describing two orbiting black holes which after merger give rise to a non-spinning black hole. Furthermore, we consider a case where the final black hole has a kick. This scenario can easily be realized with a non-equal mass binary with individual spins adjusted such that the final spin is zero\(^5\). Therefore, the black hole will generically appear in the grid which is supertranslated from that where the black hole would be at rest.

To simplify this analysis, we further assume that a natural coordinate system can be constructed such that, asymptotically, the system can be surrounded by round spheres; that is, one can construct a Bondi-type coordinate system from it. The delicate issue is that this inertial asymptotic coordinate system results, generically, supertranslated and boosted with respect to the intrinsic asymptotic coordinate system which has the spherical symmetry of the remaining black holes. This, in turn, implies that the asymptotic fields acquire a non-trivial dependence on the supertranslation and boost. To compare what would be obtained in both systems, we denote with primes the asymptotic coordinate system with spherical symmetry and unprimed that naturally constructed in the numerical implementation; then one would have the BMS transformation

$$u = K(\zeta', \bar{\zeta}') (u' - \gamma(\zeta', \bar{\zeta}')) + \frac{u_1(u', \zeta', \bar{\zeta}')}{r'} + O\left( \frac{1}{r^2} \right),$$

(2)

$$r = \frac{r'}{K(\zeta', \bar{\zeta}')} + O(r^0),$$

(3)

$$\zeta = \frac{a\zeta' + b}{c\zeta' + d} + O\left( \frac{1}{r} \right),$$

(4)

where \(K\) is the Lorentz factor associated with the boost, \(\gamma\) determines the supertranslation and \((a, b, c, d)\) are complex constants which determine the boost; for details see [19].

In what follows, it will be convenient to use the notation

$$e^{2\lambda} = \frac{\bar{\zeta}'}{\zeta'} = \left( \frac{c\zeta' + d}{\bar{\zeta}' + d} \right)^2.$$  

(5)

\(^5\) A straightforward calculation can be employed to adjust the physical parameters so that this scenario is realized as discussed by the simple model [14] or the fitted expansions presented in [15, 16]. The resulting kick velocity can be estimated by suitable fits as in [15, 17, 18].
Although the calculation of expression (1) on a spherically symmetric section gives the expected null result, we will show next that it gives a non-zero result when calculated at a supertranslated and boosted section.

As explained elsewhere [20, 21] the above transformation implies a transformation of the null frames, and in particular, the components of the Weyl tensor and the asymptotic shear are affected. More concretely, one has

\[ \Psi_1^0 = \frac{e^{i\theta}}{K^3} \left( \Psi_1^0 - 3 \frac{\delta' u}{K} \Psi_2^0 + 3 \left( \frac{\delta' u}{K} \right)^2 \Psi_3^0 + \left( \frac{\delta' u}{K} \right)^3 \Psi_4^0 \right). \]  

Only \( \Psi_2^0 = -m \) is different from zero in the spherically symmetric frame, therefore one has

\[ \Psi_1^0 = \frac{3m}{K^2} e^{i\theta} \delta' u, \]  

where \( m \) is the mass of the resulting black hole. Similarly, the shear is

\[ \sigma^0 = \frac{-1}{K} e^{i2\theta} \delta^2 \gamma. \]  

Let us now calculate expression (1) in detail, at a \( u = \text{const} \) section, and discuss what is obtained. First, note that the factors \( \Phi_{[1]} \) in the integrand are spin weighted quantities with spin weight \(-1\). Therefore they can be expressed in terms of an \( \text{edth} \) bar operator acting on a spin weight 0 quantity, namely \( \Phi_{[1]} = \bar{\partial} a_{[1]} \), where to pick the angular momentum components \( a_{[1]} \) should be imaginary [11]. Consequently, each \( a_{[1]} \) is a quantity of spin weight 0 satisfying \( \bar{\partial} \partial a_{[1]} = -a_{[1]} \).

We should now express the integration appearing in (1) in terms of the primed, spherically symmetric Bondi system. Let us observe that for all these quantities \( a_{[1]} \), due to the transformation properties of the \( \text{edth} \) operator [22], one has

\[ \bar{\partial} a = \frac{e^{-i\theta}}{K} \bar{\partial}' a', \]  

where \( a' = a'(\xi', \zeta') \) is understood in terms of the primed angular coordinates. Next, one must also take into account that \( dS = K^2 dS' \), then the first term of (1), involving \( \Psi_1^0 \), is

\[ \text{Re} \left\{ \oint_{\Sigma^*} \Phi_{[1]} \Psi_1^0 dS \right\} = \text{Re} \left\{ \oint_{\Sigma^*} e^{-i\theta} \frac{3m}{K^2} \Phi_{[1]} e^{i\theta} \delta'[K(u' - \gamma)] K^2 dS' \right\}. \]  

In order to simplify the discussion we can assume that the calculation is performed at the section \( u = 0 \), which is equivalent to \( u' = \gamma \), therefore, one has \( \delta'[K(u' - \gamma)] = (u' - \gamma) \delta' K - K \delta' \gamma = -K \delta' \gamma \), which implies

\[ \text{Re} \left\{ \oint_{\Sigma^*} \Phi_{[1]} \Psi_1^0 dS \right\} = -\text{Re} \left\{ 3m \oint_{\Sigma^*} \Phi_{[1]} K^{-2} \delta' \gamma dS' \right\}. \]  

Since the factor \( K^{-2} \delta' \gamma \) has spin weight 1, one again can think of a spin-weight 0 potential \( A \) such that \( \delta' A = K^{-2} \delta' \gamma \). Therefore, if the quantity \( A \) were real, this term would not have any contribution, since one would have to evaluate

\[ \text{Re} \left\{ \oint_{\Sigma^*} \Phi_{[1]} \Psi_1^0 dS \right\} = -\text{Re} \left\{ 3m \oint_{\Sigma^*} \delta' a_{[1]}' \delta' A dS' \right\} = \text{Re} \left\{ 3m \oint_{\Sigma^*} \delta' \delta' a_{[1]}' A dS' \right\} = 0; \]  

since \( a_{[1]}' \)'s are imaginary and we are assuming at the moment that \( A \) is real. Unfortunately, it is not difficult to prove that actually \( A \) has an imaginary part, and furthermore, this imaginary
part is only zero when the gradients of $\frac{1}{K^2}$ and $\gamma$ are proportional; something that is not true in our case.

To see this, we next study the equation $\partial A = v^2 \partial \gamma$, where $v = K^{-1}$, and assuming $A$ is a real quantity we will reach a contradiction. Note that for a real spin weight zero field $A$, $\text{Im}[\partial \partial A] = 0$; since $\partial \partial$ is a real operator when acting on a spin weight zero quantity. This last relation can be written in terms of $v$ and $\gamma$ as

$$\partial v^2 \partial \gamma - \partial \gamma \partial v^2 = 0.$$  (13)

Let us denote the gradients on the sphere of the quantities $v^2$ and $\gamma$ by $\nabla_a v^2$ and $\nabla_b \gamma$, respectively. Then the 'cross product' of these vectorial quantities is defined by $\epsilon^{ab} \nabla_a v^2 \nabla_b \gamma$, where $\epsilon^{ab}$ is the surface element of the unit sphere. At this point it is convenient to express the gradients in terms of the edth operator, for example for $v^2$ one has

$$\nabla_a v^2 = -(\partial v^2) \hat{m}_a - (\partial v^2) m_a,$$  (14)

and similarly for $\gamma$. Then the cross product is

$$\epsilon^{ab} \nabla_a v^2 \nabla_b \gamma = (\partial v^2 \partial \gamma - \partial \gamma \partial v^2) \epsilon^{ab} m_a \hat{m}_b,$$  (15)

which should be zero if $A$ were real. But this is true only if the gradients are proportional. Consequently, the term (11) has a non-zero contribution to the quantity $J_{[i]}$.

We now turn our attention to the second term in expression (1) which involves quadratic terms in $\sigma$. One could also, in this case, think of a potential $X$ such that $\sigma^0 = \partial^2 X$. Then, one can prove that if $X$ were real, then the second term of (1) involving the $\sigma^i$’s would not contribute to the integral [23]. However, following a similar line of arguments to above, one can prove that $X$ will have an imaginary part

$$-\frac{1}{K} e^{i\varphi} \neq 1$$  (16)

—recall this is the factor appearing in (8).

Therefore, this second term will also have a non-zero contribution to the quantity $J_{[i]}$, and since it includes independent information not contained in the first term it cannot cancel the first contribution.

Let us discuss further what we have found up to this point. If the adopted coordinate system, for instance the one most naturally constructed from a numerical simulation, results supertranslated and boosted with respect to the final frame adapted to the spherically symmetric final black hole an incorrect result will be obtained for the angular momentum of the system. Note that this conclusion is more subtle than the obvious ‘Newtonian-type’ observation stating that the angular momentum calculated at frames translated with respect to each other is different. In fact, even if the coordinate system is only boosted and supertranslated—without a translation—with respect to that adapted to the spherically symmetric final black hole then $J_{[i]}$ turns out to be different from zero in the Schwarzschild geometry. This is to be contrasted to what would be the case within special relativity paradigm where the absence of a translation does not affect the angular momentum calculation since the Lorentz transformation of a zero tensor is a zero tensor.

### 2.2. A radiating second-order solution

As a second example, let us consider the spacetime given by the metric

$$\text{d}s^2 = \left(-2 \frac{\dot{V}}{V} + K V - 2 \frac{M}{r}\right) \text{d}u^2 + 2 \text{d}u \, \text{d}r - \frac{r^2}{V^2 P_0^2} \, \text{d}\xi \, \text{d}\bar{\xi},$$  (17)

where a dot denotes a derivative with respect to \( u \), \( K_\nu \) is the Gaussian curvature given by
\[
\dot{K}_\nu = 2V^2 \nabla K - 2\delta V \nabla V + V^2.
\]
of the 2-metric
\[
\dot{dS}^2 = \frac{1}{V^2} P_0 \, d\xi \, d\bar{\xi},
\]
and \( P_0(\xi, \bar{\xi}) \) is the conformal factor of the unit sphere. This metric was used by Robinson and Trautman to characterize spacetimes satisfying Einstein equations admitting a congruence of null geodesic without shear and twist [24].

The scalar \( V(u, \zeta, \bar{\zeta}) \) can be expressed in the form \( V = 1 + \Delta(u, \zeta, \bar{\zeta}) \), so that \( \Delta = 0 \) would correspond to the Schwarzschild spacetime. The function \( \Delta \) is required to satisfy the equation
\[
-3M\Delta = (1 + 4\Delta)\vec{\partial}^2 \Delta - \vec{\partial}^2 \Delta \vec{\partial}^2 \Delta.
\]

This spacetime is known as a Robinson–Trautman geometry [25]. By requiring (20) one obtains a solution of vacuum Einstein equations in second order in \( \Delta \). The Robinson–Trautman spacetimes are understood as representing a central object that has been perturbed, and are known to decay in time to the Schwarzschild geometry.

Suppose one has obtained this spacetime and would like to calculate the quantity expressed in (1) at the section \( u = u_0 \). Then, since the asymptotic coordinate system \((u, \zeta, \bar{\zeta})\) is not inertial (not Bondi), one should look for the transformation to a Bondi frame. Let us choose a Bondi coordinate system \((\bar{u}, \zeta, \bar{\zeta})\) such that \( \bar{u} = 0 \) coincide with the section \( u = u_0 \). With this choice one has that on this particular section the Bondi shear is zero; \( \vec{\sigma} = 0 \) [21]. Furthermore, since on this section the null frames are proportional, one also has \( \Psi_0^0 = 0 \). Therefore in this section one obtains \( J_{(1)} \equiv 0 \).

However in any other Bondi section given by \( \bar{u} \neq 0 \) one will obtain \( J_{(1)} \neq 0 \). This can be seen due to two reasons. First, one can note that the shear with respect to the Bondi system is not stationary; in fact, one can prove that \( \vec{\sigma}_{,\bar{u}} = (\vec{\partial}^2 V)V^{-1} \) [25]. One can convince oneself that \( \vec{\sigma} \) cannot be expressed in terms of a real potential \( X \), that is, \( \vec{\sigma} = \vec{\partial}^2 X \) as studied in the previous section. In other words such an \( X \) has an imaginary term. Therefore one deduces that there is always a flux, which in general will include the time variation of \( J_{(1)} \), depending on the arbitrary initial data \( V \). On the other hand one can also infer that \( \Psi_0^0 \) will also be different from zero, at any other Bondi section; due again to the existence of radiation.

Note that \( J_{(1)} \neq 0 \) at any other Bondi section \( \bar{u} \neq 0 \) can also be deduced by taking (Bondi) time derivatives of \( J_{(1)} \). One can see that at \( \bar{u} = 0 \), one has \( J_{(1),\bar{u}} = 0 \), \( J_{(1),\ddot{\bar{u}}} = 0 \) but \( J_{(1),\dddot{\bar{u}}} \neq 0 \).

The weaknesses of formula (1) to calculate the angular momentum also give rise to odd results. Note that the asymptotic value of \( J_{(1)} \), for \( \bar{u} \to \infty \), is zero; since in this regime, \( \bar{u} \) will be supertranslated with respect to the \((u, \zeta, \bar{\zeta})\) frame, but not boosted (see the previous section). So, in this case one would have the curious situation that the initial value of \( J_{(1)} \) is zero, a non-zero flux to the future of the initial section and an ending with a zero value for \( J_{(1)} \) in the asymptotic regime. This alone makes it difficult to identify \( J_{(1)} \) with a physical notion of intrinsic angular momentum.

One instead could use the Bondi system \((u^*, \zeta^*, \bar{\zeta}^*)\) which is defined so as to coincide asymptotically with the R–T system \((u, \zeta, \bar{\zeta})\) in the regime \( u \to \infty \). Then one could calculate the quantity \( J_{(1)} \) for any \( u^* = \text{constant} \). In this case one would obtain that in the limit \( u^* \to \infty \), \( J_{(1)} \) would vanish. However for any finite \( u^* = \text{const} \), the evaluation of \( J_{(1)} \) would give a non-zero value.

This again stresses the difficulties that one encounters in trying to use the expression (1) to obtain intrinsic information of the central object.

5
3. Estimate of the error in spin calculations

In order to provide a different perspective to the quantity \( J_{(i)} \), in the following subsection we recall an unambiguous definition of total intrinsic angular momentum at future null infinity. Actually, since this is the only definition we know of that solves simultaneously the supertranslations ambiguities for center of mass and intrinsic angular momentum [11]—without assuming further structure—we will use it as a reference for comparison.

3.1. An unambiguous definition of intrinsic angular momentum

Arguably one of the most valuable pieces of physical information that can be extracted from a given spacetime is that of intrinsic angular momentum as it relates directly with observable signatures at both gravitational and electromagnetical signals produced in a number of systems. Tied to defining such a quantity is the problem of defining the center of mass of the system with respect to which the intrinsic angular momentum can be defined; consequently, the definition of both concepts must come together. A solution of these difficulties has been formulated in [11], where the expression to calculate the charges associated with BMS generators is given by

\[
Q_{Scm}(w) = \text{Re} \left\{ 8 \int_{Scm} \left( -w_2 \left( \Psi_1^0 + 2\sigma_0 \bar{\sigma}_0 + \bar{\sigma} (\sigma_0 \bar{\sigma}_0) \right) + 2 w_1 \left( \Psi_2^0 + \sigma_0 \bar{\sigma}_0 + \bar{\sigma}^2 \bar{\sigma}_0 \right) \right) dS^2 \right\},
\]

(21)

where \( w_1 \) and \( w_2 \) are components of the two-form \( w \) that is determined by the particular generator of BMS transformations. As with the expression appearing in [8], one can use this type of expression to calculate the total momentum, total supermomentum and total angular momentum. However, the notion of intrinsic angular momentum can only be obtained if one calculates the angular momentum on the center-of-mass sections, here denoted by \( S_{cm} \).

If one calculates the angular momentum in a radiating spacetime at any other section, one would obtain a quantity with angular momentum reminiscence but with unclear intrinsic physical meaning.

3.2. How important in practice are these considerations in the computation of angular momentum?

We would now like to estimate the errors in the value of angular momentum by using an unfortunate choice of frame and section. To do so, we would like to have an estimate of how supertranslated the ‘center-of-mass’ sections are with respect to the sections that are adapted to the coordinates used in numerical computations.

The astrophysical model that we have in mind is similar to the system that was presented in subsection 2.1, i.e, we have some compact objects (for example a binary system) that are undergoing a merge, and where the coordinate grid is in some way following the system until the moment of merger. After such an event, in general, the system will end up with a unique boosted (quasi)-stationary compact object (in the case of black hole collision, we will finish with something similar to a boosted Schwarzschild or Kerr black hole). Then, the new ‘center-of-mass’ sections will be supertranslated and boosted with respect to the originals (those adapted to the numerical grid).

Sections that characterize rest frames are known as ‘nice’ sections [26]. The prescription to find rest frame sections uses a particular notion of supermomentum.
The supermomentum at $S$ is defined \[26\] by
\[
P_{lm}(S) \equiv -\frac{1}{\sqrt{4\pi}} \int_{S} Y_{lm}(\zeta, \bar{\zeta}) \Psi_{1} \, dS^2,
\] (22)
where $\Psi = \Psi_2 + \sigma \dot{\sigma} + \partial^2 \sigma$ and $Y_{lm}$ are spherical harmonics.

Given another section $\tilde{S}$ of future null infinity, one can find another Bondi system $(\tilde{u}, \tilde{\zeta}, \bar{\tilde{\zeta}})$ such that $\tilde{S}$ is determined by $\tilde{u} = 0$. The relation between the new and the original Bondi system is given by a BMS transformation, as given by (2), (3) and (4), where now we use tildes for the new coordinates and un-tilde for the original coordinates. Note that $\tilde{S}$ can also be determined by $u = \gamma(\zeta, \bar{\zeta})$.

The section $\tilde{S}$ is said to be of type nice \[26\] if all the ‘spacelike’ components of the supermomentum, when calculated with respect to the adapted Bondi system, are zero
\[
\tilde{P}_{lm}(\tilde{S}) = 0 \quad \text{for} \quad l \neq 0,
\] (23)
and therefore the only non-vanishing one, $\tilde{P}_{00}(\tilde{S})$, coincides with the total Bondi mass at $\tilde{S}$.

One can prove that $\Psi$ transforms under a BMS transformation as
\[
\tilde{\Psi} = \frac{1}{K^3}(\Psi - \partial^2 \partial^3 \gamma).
\] (24)
Then, for a section to correspond to a ‘nice’ section it must obey $\tilde{\Psi} = \text{constant}$ as otherwise some moments of $P_{lm}$ for $l \neq 0$, will be non-zero. The nice section equation can be understood as a condition for $\gamma(\zeta, \bar{\zeta})$ and $K$.

It was indicated in \[26\] that equation (24) can be expressed by
\[
\partial^2 \partial^3 \gamma = \int_{0}^{\gamma} \Psi(u', \zeta, \bar{\zeta}') \, du' + \Psi(u = 0, \zeta, \bar{\zeta}) + K(\gamma; \zeta, \bar{\zeta})^3 M(\gamma),
\] (25)
where $M(\gamma)$ is the total mass at the section $u = \gamma$ and $K(\gamma; \zeta, \bar{\zeta})$ is the conformal factor of the BMS transformation that aligns its timelike generator (defined as $\tilde{p}_{00} = Y_{00} \frac{\partial}{\partial \tilde{u}}$ \[26\]) with the Bondi momentum at $u = \gamma(\zeta, \bar{\zeta})$.

Note that $\Psi = 4 \partial^2 \partial^3 \gamma$, and then it is proportional to the content of gravitational radiation.

Suppose then that we start by describing our compact bodies system with sections that are close to ‘nice’ sections. Then, after the merger, a new compact object is obtained, and due to the emission of gravitational radiation, the new ‘nice’ sections that ‘follow’ the system will, in general, not only be boosted but also supertranslated by a quantity $\gamma$. Now, let $v$ be the kick velocity of the final compact object, then one can observe that
\[
K^3 \approx 1 + O(v).
\]
On the other hand, by computing the flux of the Bondi momentum expression we can see that
\[
v \approx O\left(\frac{\lambda}{M}\right),
\] (26)
and $\Psi(u = 0, \zeta, \bar{\zeta}) \approx -M$, then
\[
\partial^2 \partial^3 \gamma \approx O(\lambda) + O(vM) \approx O(vM).
\] (27)

From these estimates, and recalling that $\Psi_0^0 \approx O(M)$, and using equation (7), we have that $\Delta \Psi_0^0 \approx O(vM^2)$.

This will be the order of the error that one would have in the computation of angular momentum by not adopting the correct sections. We conclude then that the errors will be, at least, as large as the relativistic velocities of the astrophysical system, i.e., $\Delta J_{ij} \approx O(vM^2)$. Consequently, in scenarios where typical velocities under consideration are low, the error
estimate $\Delta J_{ij}$ is small; thus one can understand why the flux obtained from equation (1) (as calculated in [13]) can produce sensible results (e.g. [27–31]). However, one should be cautious with the use of formulae like (1) in cases where the individual objects move relativistically (as in the studies of high speed collisions [32]), large production of gravitational radiation or producing an object with a small final spin (where the errors described above will be important).

Finally, we stress that all these considerations are true if one assumes that the numerical grid and gauge give rise to an extraction frame/coordinate system which is of Bondi type (which, in general, will not be the case). If it is not of this type, one must consider extra gauge effects such as those reported in [20, 21] in the computation of radiation flux and total momentum, respectively.

4. Final comments

The difficulties shown above are related to the nature of the notion of angular momentum. In special relativity the angular momentum is expressed in terms of an antisymmetric tensor

$$J^{ab} = S^{ab} + P^a R^b - R^a P^b, \quad (28)$$

where $P^a$ is the total momentum, $S^{ab}$ is orthogonal to the momentum and $R^a$ shows the dependence of the angular momentum with respect to some reference point, or origin.

The tensor $S^{ab}$ does not depend on the choice of origin and therefore is the intrinsic angular momentum. At the center-of-mass line one has $J^{ab} = S^{ab}$ and the original expression of the angular momentum gives the intrinsic value.

The expression (1) does not have this translation/supertranslation dependence, and therefore its usage is ambiguous since there is no prescription of what a center of mass is. Instead, we have shown that one could use a definition of angular momentum [11] that does show this translation/supertranslation dependence, and furthermore, it can be used to define center-of-mass sections, in which the calculation of angular momentum gives an unambiguous intrinsic angular momentum. In particular, the application of this intrinsic angular momentum to the two examples presented above gives zero at any center-of-mass section.

Incidentally, note also that for general spacetimes, inferring how much angular momentum is radiated by differentiating with respect to time equation (1) will give results strongly affected by the issues discussed here.

References

[1] Komar A 1959 Covariant conservation laws in general relativity Phys. Rev. 113 934–6
[2] Penrose R 1982 Quasi-local mass and angular momentum in general relativity Proc. R. Soc. Lond. A 381 53–63
[3] Szabados L B 2004 Quasi-local energy–momentum and angular momentum in gr: a review article Living Rev. Rel. 7 4 (http://relativity.livingreviews.org/Articles/lrr-2004-4)
[4] Ashtekar A and Krishnan B 2004 Isolated and dynamical horizons and their holes Living Rev. Rel. 7 10
[5] Hayward S A 2006 Angular momentum conservation for dynamical black holes Phys. Rev. D 74 104013
[6] Bramson B D 1975 Relativistic angular momentum for asymptotically flat Einstein–Maxwell manifolds Proc. R. Soc. Lond. A 341 463–90
[7] Prior C R 1977 Angular momentum in general relativity Proc. R. Soc. Lond. A 354 379
[8] Winicour J 1980 Angular momentum in general relativity General Relativity and Gravitation vol 2 ed A Held (New York: Plenum) pp 71–96
[9] Dray T and Streubel M 1984 Angular momentum at null infinity Class. Quantum Grav. 1 15–26
[10] Rizzi A 2001 Angular momentum in general relativity: the definition at null infinity includes the spatial definition as a special case Phys. Rev. D 63 104002
[11] Moreschi O M 2004 Intrinsic angular momentum and center of mass in general relativity Class. Quantum Grav. 21 5409–25
[12] Szabados L B 2008 Total angular momentum from Dirac eigenspinors Class. Quantum Grav. 25 025007
[13] Lousto C O and Zlochower Y 2007 Practical formula for the radiated angular momentum Phys. Rev. D 76 041502(R)–4
[14] Buonanno A, Kidder L E and Lehner L 2008 Estimating the final spin of a binary black hole coalescence Phys. Rev. D 77 026004
[15] Boyle L, Kesden M and Nisanke S 2008 Binary black hole merger: symmetry and the spin expansion Phys. Rev. Lett. 100 151101
[16] Rezzolla L, Diener P, Dorband E N, Pollney D, Reisswig C, Schnetter E and Seiler J 2008 The final spin from the coalescence of aligned-spin black-hole binaries Astrophys. J. 674 L29
[17] Baker J G, Boggs W D, Centrella J, Kelly B J, McWilliams S T, Miller M C and van Meter J R 2008 Modeling kicks from the merger of generic black-hole binaries Astrophys. J. 682 L29
[18] Campanelli M, Lousto C O, Zlochower Y and Merritt D 2007 Large merger recoils and spin flips from generic black-hole binaries Astrophys. J. 659 L5
[19] Moreschi O M 1986 On angular momentum at future null infinity Class. Quantum Grav. 3 503–25
[20] Lehner L and Moreschi O M 2007 Dealing with delicate issues in waveforms calculations Phys. Rev. D 76 124040–1–12
[21] Gallo E, Lehner L and Moreschi O 2008 Estimating total momentum at finite distances Phys. Rev. D 78 084027
[22] Newman E T and Penrose R 1966 Note on the Bondi–Metzner–Sachs group J. Math. Phys. 7 863–70
[23] Bramson B D 1976 The invariance of spin Proc. R. Soc. Lond. A 364 383–92
[24] Robinson I and Trautman R 1962 Some spherical gravitational waves in general relativity Proc. R. Soc. A 265 463–73
[25] Dain S, Moreschi O M and Gleiser R J 1996 Photon rockets and the Robinson–Trautman geometries Class. Quantum Grav. 13 1155–60
[26] Moreschi O M 1988 Supercenter of mass system at future null infinity Class. Quantum Grav. 5 423–35
[27] Baker J G et al 2008 Mergers of non-spinning black-hole binaries: gravitational radiation characteristics Phys. Rev. D 78 044046
[28] Campanelli M, Lousto C O, Nakano H and Zlochower Y 2008 Comparison of numerical and post-Newtonian waveforms for generic precessing black-hole binaries arXiv:0808.0713 [gr-qc]
[29] Lousto C O and Zlochower Y 2008 Modeling gravitational recoil from precessing highly spinning unequal-mass black-hole binaries arXiv:0805.0159 [gr-qc]
[30] Dain S, Lousto C O and Zlochower Y 2008 Extra-large remnant recoil velocities and spins from near-extremal-Bowen–York-spin black-hole binaries Phys. Rev. D 78 024039
[31] Lousto C O and Zlochower Y 2008 Further insight into gravitational recoil Phys. Rev. D 77 044028
[32] Sperhake U, Cardoso V, Pretorius F, Berti E and Gonzalez J A 2008 The high-energy collision of two black holes Phys. Rev. Lett. 101 161101