A SAT Encoding for the \( n \)-Fractions Problem

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Abstract. This note describes a SAT encoding for the \( n \)-fractions puzzle which is problem 041 of the CSPLib. Using a SAT solver we obtain a solution for two of the six remaining open instances of this problem.

1 Introduction

The \( n \)-fractions puzzle \([1]\) is problem 041 of the CSPLib. The original puzzle is specified as follows: find nine distinct non-zero digits, \{\(A, B, C, D, E, F, G, H, I\}\), that satisfy

\[
\frac{A}{BC} + \frac{D}{EF} + \frac{G}{HI} = 1
\]

where \(BC\) is shorthand for \(10B + C\), \(EF\) for \(10E + F\), and \(H\) for \(10H + I\). A simple generalization is as follows: find \(3n\) nonzero digits, \(x_i, y_i, z_i\) \((1 \leq i \leq n)\), satisfying

\[
\sum_{i=1}^{n} \frac{x_i}{y_i z_i} = 1 \tag{1}
\]

where \(y_i z_i\) is shorthand for \(10y_i + z_i\) and the number of occurrences of each digit in \{1, \ldots, 9\} is between 1 and \([n/3]\). An interesting problem is to find the greatest \(n\) such that at least one solution exists. Since each fraction is at least \(1/99\), this family of problems has solutions for at most \(n \leq 99\). Malapert and Provillard prove in a recent paper \([2]\) that the puzzle has no solution for \(n \geq 45\).

Two models are described in the literature (see \([2]\)) to solve the \( n \)-fractions puzzle. The division model handles Equation \((1)\) with floating point arithmetic. This approach returns invalid solutions because of rounding errors. The product model only needs integer arithmetic because Equation \((1)\) is reformulated as follows:

\[
\sum_{i=1}^{n} \left( x_i \prod_{k \neq i} y_k z_k \right) = \prod_{i=1}^{n} y_k z_k \tag{2}
\]

The main problem with the product model is that the number of bits required to represent the products grows exponentially with the size of \(n\). For example, the multiplication term on the right side of Equation \((2)\) overflows a 32-bit integer for \(n = 6\).
Malapert and Provillard [2] propose an integer factorization model and demonstrate that applying this model they can find solutions for all of the instances with \( n < 45 \) except for six: where \( n \in \{36, 39, 41, 42, 43, 44\} \). Their approach comprises two basic ideas: The first idea is to solve the following constraint instead of that expressed as Equation (1):

\[
\sum_{i=1}^{n} x_i \times \frac{L}{y_i z_i} = L
\]

where \( L \) is the lowest common multiple of the integers \( \{ y_i z_i \mid 1 \leq i \leq n \} \). In this formalization, each of the terms, \( \frac{L}{y_i z_i} \) on the left side of Equation (3) is an integer. In theory, the products in Equation (3) still grow exponentially.

In practice, based on this formulation, it is possible to solve large \( n \)-fractions puzzles. The second idea is to represent the integer variables in Equation (3) in terms of their prime factorizations.

In this note we describe a simple LCM model for the \( n \)-fractions problem. The approach is based on Equation (3). We encode the constraints of this model to SAT using a standard binary representation for integers. Our approach is able to solve two of the instances left open in the paper by Malapert and Provillard [2]. These are the 36-fraction puzzle and the 39-fraction puzzle.

## 2 The LCM Constraint Model

In this section we describe a simple LCM model for the \( n \)-fractions problem in terms of finite integer constraints. These are then compiled to CNF using the finite-domain constraint compiler \texttt{BEE} [3] which compiles constraints to CNF. The (conjunctions of) constraints in our model (in \texttt{BEE} syntax) are detailed below as framed text.

### 2.1 Domain and Counting Constraints

For \( 1 \leq i \leq n \), the variables \( x_i, y_i, z_i \) take integer values in the domain \( \{1, \ldots, 9\} \). The number of occurrences of each digit is constrained to be between 1 and \( \lceil n/3 \rceil \).

The variables \( y_i z_i = 10 \times y_i + z_i \) take integer values in the domain \( \{11, \ldots, 99\} \).

In \texttt{BEE} an integer variable \( x \) is declared to be in unary or binary representation, \texttt{new\_int}(\( x, \text{lb}, \text{ub} \)) or \texttt{new\_binary}(\( x, \text{lb}, \text{ub} \)), where \text{lb} and \text{ub} are lower and upper bounds.

The variables \( x_i, y_i, z_i \) and \( y_i z_i \) are represented in unary representation. The variables \( x_i \) and \( y_i z_i \) are represented also through channelling to their binary representation. This is because the counting constraints (on the digits) are best encoded to CNF using the unary representation while the arithmetic constraints described in Sections 2.3 and 2.4 are best encoded to CNF using the binary representation. In the constraint model, detailed as Figure 1, we denote the digits \( [x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_n] \) by \([\text{dig}_1, \ldots, \text{dig}_{3n}]\) and then the (Boolean) variables \( \text{dig}_{i,j} \) denote that \( \text{dig}_i \) takes value \( j \) and the (integer) variables \( s_j \)
denote the number of occurrences of the value $j$ among $[dig_1, \ldots, dig_{3n}]$ (for $1 \leq i \leq 3n, 1 \leq j \leq 9$).

### 2.2 Symmetry Breaking and Redundant Constraints

We add the symmetry breaking constraints and a redundant constraint proposed by Frisch \cite{4}.

\[
(y_i, z_i, x_i) \leq_{lex} (y_{i+1}, z_{i+1}, x_{i+1}) \quad 1 \leq i < n \tag{4}
\]

\[
\min_{1 \leq i \leq n} y_i z_i \leq \sum_{i=1}^{n} x_i \leq \max_{1 \leq i \leq n} y_i z_i \tag{5}
\]

For the BEE syntax see Figure 2.

### 2.3 LCM Constraints

The least common multiple, $L$ of a set of positive integers $S$ is the smallest positive integer that is divisible by each of the integers in $S$. In the context of Equation (3), it is sufficient if $L$ is any common multiple.

We introduce integer variables $L$ and $\{d_1, \ldots, d_n\}$. The variable $L$ takes values in the domain $\{1, \ldots, maxL\}$ where $maxL$ is a parameter of the encoding.
The variables $d_i$ take values in the domain $[\text{maxL}/11]$. The following constraint states that $L$ is divided by each of the numbers $y_i z_i$. This constraint also “determines” the variables $d_i$, or more precisely, the relation between the variables $y_i z_i$, $L$ and $d_i$.

$$\bigwedge_{i=1}^{n} y_i z_i \times d_i = L \quad (6)$$

For an optimization, we observe that often many of the values in the sequence $y_1 z_1, \ldots, y_n z_n$ are repeated (see Table 2). Moreover, because of the specific symmetry break of Equation (4), repeated values $y_i z_i$ occur consecutively in this sequence. Instead of encoding the LCM constraints using Equation (6), we encode them with the following constraints

$$\bigwedge_{i=1}^{n-1} \text{if } (y_i z_i = y_{i+1} z_{i+1}) \text{ then } (d_i = d_{i+1}) \text{ else } (y_i z_i \times d_i = L) \quad (7)$$

In Figure 3, the variables $[\ell_1, \ldots, \ell_n]$ are such that $y_i z_i \times d_i = \ell_i$. If we constrain all of the $\ell_i$ to equal $\ell_1$ then $\ell_1$ is a common multiplier of the divisors $(y_i, z_i)$. Instead we only constrain $\ell_i = \ell_1$ where the divisor $y_i z_i$ occurs first (not repeated) in the sequence of divisors.

2.4 The Puzzle Constraint

Equation (3) is modeled by the following constraint expressed in terms of the variables $d_i$ introduced in the model as described in Section 2.3. We encode Equation (11) as

$$\sum_{i=1}^{n} x_i \times d_i = L \quad (8)$$
For the BEE syntax see Figure 4

3 Experimental Results

The computations described in this note are performed using the finite-domain constraint compiler BEE [3] which compiles constraints to a CNF, and solves it applying an underlying SAT solver. We use Glucose 4.0 [5]. All computations were performed on an Intel E8400 core, clocked at 2 GHz, able to run a total of 12 parallel threads. Each of the cores in the cluster has computational power comparable to a core on a standard desktop computer. Each SAT instance is run on a single thread, and all running times reported in this paper are CPU times.

Table 1 describes the experimental evaluation. The first two columns describe the instance: \( n \) and the maximum value of a common multiple in the solution. The column titled “BEE” is the compile time (seconds) from constraints to CNF. The next two columns specify the CNF size in number of clauses and variables. The right most column specifies the SAT solving time in seconds (except where marked as hours).

In the experiments we search for suitable values of \( \text{maxL} \). Basically, for smaller values of \( n \), we start from 100 and increment by 100 until a solution is found. For larger values of \( n \), we start from 1000 and increment by 500, and then refine the value from the largest multiple of 1000 that has a solution incrementing by 100.

Table 2 details the solutions found using our encoding. The first column details the number \( n \) of fractions. The second column details the common multiplier (the value of \( L \)) in the solution found. The third column details the solution found. Note that for \( n < 3 \) there is no solution as the constraint that states that the number of occurrences of each digit in \( \{1, \ldots, 9\} \) is between 1 and \( \left\lceil \frac{n}{3} \right\rceil \) is trivially violated.

References

1. Frisch, A., Jefferson, C., Miguel, I., Walsh, T.: CSPLib problem 041: The n-fractions puzzle. \url{http://www.csplib.org/Problems/prob041}
2. Malapert, A., Provillard, J.: Puzzlesolving the n-fractions puzzle as a constraint programming problem. INFORMS Transactions on Education 0(0) (0) null
3. Metodi, A., Codish, M., Stuckey, P.J.: Boolean equi-propagation for concise and efficient SAT encodings of combinatorial problems. J. Artif. Intell. Res. (JAIR) 46 (2013) 303–341
4. Frisch, A.M., Jefferson, C., Miguel, I.: Symmetry breaking as a prelude to implied constraints: A constraint modelling pattern. In: ECAI. Volume 16. (2004) 171
5. Audemard, G., Simon, L.: Glucose 4.0 SAT Solver. \url{http://www.labri.fr/perso/lsimon/glucose/}
Table 1. Solving $n$-fractions with BEE

| $n$ | maxL | BEE | # cl | # var | sat     |
|-----|------|-----|------|------|---------|
| 3   | 300  | 0.05| 10954| 1663 | 0.11    |
| 4   | 100  | 0.07| 14171| 2054 | 0.03    |
| 5   | 100  | 0.09| 18231| 2596 | 0.04    |
| 6   | 100  | 0.16| 22370| 3122 | 0.09    |
| 7   | 100  | 0.15| 27330| 3788 | 0.13    |
| 8   | 100  | 0.15| 31937| 4341 | 0.21    |
| 9   | 100  | 0.28| 36661| 4915 | 0.10    |
| 10  | 100  | 0.27| 42207| 5526 | 0.24    |
| 11  | 100  | 0.29| 47414| 6143 | 0.28    |
| 12  | 100  | 0.33| 52313| 6644 | 0.57    |
| 13  | 100  | 0.25| 58444| 7373 | 0.30    |
| 14  | 100  | 0.49| 63490| 7914 | 0.79    |
| 15  | 120  | 0.35| 71762| 9466 | 6.95    |
| 16  | 100  | 0.52| 74966| 9161 | 2.15    |
| 17  | 100  | 0.45| 79991| 9759 | 1.79    |
| 18  | 300  | 0.47| 90836| 11952| 7.03    |
| 19  | 100  | 0.54| 91356| 10988| 6.03    |
| 20  | 300  | 0.71| 102790|13372| 16.61   |
| 21  | 300  | 0.80| 108010|14060| 28.08   |
| 22  | 300  | 0.83| 115090|14793| 202.55  |
| 23  | 300  | 0.57| 120344|15506| 257.02  |
| 24  | 300  | 0.91| 125787|16131| 14.05   |
| 25  | 300  | 1.06| 132945|16977| 374.90  |
| 26  | 300  | 0.77| 138824|17604| 382.66  |
| 27  | 400  | 1.12| 147838|19311| 16.70   |
| 28  | 300  | 1.14| 151870|19077| 769.62  |
| 29  | 400  | 1.23| 161155|20856| 951.97  |
| 30  | 500  | 1.16| 166856|21755| 162.36  |
| 31  | 500  | 1.31| 174467|22638| 253.78  |
| 32  | 500  | 0.87| 179809|23317| 983.09  |
| 33  | 1900 | 1.70| 206067|28633| 8427.08 |
| 34  | 500  | 1.53| 192702|24875| 4690.07 |
| 35  | 2400 | 1.91| 217837|31579| 6.11 hr |
| 36  | 2400 | 1.16| 223404|32427| 37.99 hr|
| 37  | 2400 | 1.04| 185947|31670| 67.77 hr|
| 38  | 2400 | 1.99| 237793|34322| 66.58 hr|
| 39  | 8400 | 4.86| 326435|72219| 102.20 hr|
Table 2. Solutions