General Solutions Of The Spherically Symmetric Vacuum Einstein Field Equations

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Abstract

According to Birkhoff’s theorem the only spherically symmetric solution of the vacuum Einstein field equations is the Schwarzschild solution. Inspite of imposing asymptotically flatness and staticness as initial conditions we obtain that these equations have general solutions with the Schwarzschild metric as merely a special and simplest form of them. It is possible to have perfect and smooth metrics with the same Newtonian and post-Newtonian limits of Schwarzschild by a convenient and correct selection.

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Spherical symmetry requires the dependence of line element $ds^2$ only on the following rotational invariants [1-2],

$$t, dt, \vec{x} \cdot d\vec{x}^2 = r dr, d\vec{x}^2 = dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2)$$

So it can be written

$$ds^2 = B(r, t) dt^2 - A(r, t) dr^2 - 2C(r, t) dtdr - D(r, t)(d\theta^2 + \sin^2 \theta d\varphi^2) \quad (1)$$

There is a common belief that the line element can be transformed to the following standard form by a suitable coordinate transformation. Some texts start from here e.g. [3].

$$ds^2 = B'(r', t') dt'^2 - A'(r', t') dr'^2 - r'^2(d\theta'^2 + \sin^2 \theta' d\varphi'^2) \quad (2)$$

Taking advantage of the mentioned form to compute the field equations it can be shown that it is necessarily static and has a unique Schwarzschild solution [4-5], as required by Birkhoff’s theorem [6-7]. This means that the other solutions are just different forms of this metric which are related by coordinate transformation. What indeed is flawed reasoning, is that the change of the coordinate $r \to \sqrt{D(r, t)}$ with new parameter having the same range of $r$, is not generally true. Because as it turns out as in the Schwarzschild case, the components of the metric may even change their sign in different parts of space-time. So when there is no information about the functional form of $D$ there is no guarantee that $\sqrt{D}$ to be real in the whole range of $r$ and also, its range is exactly the same as $r$ which is between zero and infinity. Accordingly we believe that the steps which follow to arrive at the standard form are not justified. Since it is a hard task to solve the vacuum field equations with the general form of the metric (1) we restrict our investigation to asymptotically flat and static space-time by convention i.e.

$$ds^2 = B(r) dt^2 - A(r) dr^2 - D(r)(d\theta^2 + \sin^2 \theta d\varphi^2) \quad (3)$$

This metric tensor has the nonvanishing components

$$g_{tt} = -B(r), \quad g_{rr} = A(r), \quad g_{\theta\theta} = D(r), \quad g_{\varphi\varphi} = D(r) \sin^2 \theta \quad (4)$$
with functions $A(r)$, $B(r)$ and $D(r)$ that are to be determined by solving the field equations. The nonvanishing contravariant components of the metric are:

$$g^{tt} = -B^{-1}, \quad g^{rr} = A^{-1}, \quad g^{\theta\theta} = D^{-1}, \quad g^{\phi\phi} = D^{-1}\sin^{-2}\theta$$  \hspace{1cm} (5)

The metric connection can be computed by the use of (4) and (5) from the usual definition. Its only nonvanishing components are:

$$\Gamma^r_{rr} = \frac{A'}{2A}, \quad \Gamma^r_{\theta\theta} = -\frac{B'}{2A}, \quad \Gamma^r_{\phi\phi} = -\sin^2\theta \frac{D'}{2A} \quad \Gamma^r_{tt} = \frac{B'}{2A}$$  \hspace{1cm} (6)

where primes stand for differentiation with respect to $r$. With these connections the Ricci tensor can be obtained.

$$R_{rr} = \frac{B''}{2B} - \frac{B'}{4B} \left( \frac{A'}{A} + \frac{B'}{B} \right) - \frac{A'D'}{2AD} + \frac{D''}{D} - \frac{D'^2}{2D^2}$$  \hspace{1cm} (7)

$$R_{\theta\theta} = -1 + \frac{D'}{4A} \left( -\frac{A'}{A} + \frac{B'}{B} \right) + \frac{D''}{2A}$$  \hspace{1cm} (8)

$$R_{\phi\phi} = \sin^2\theta \ R_{\theta\theta}$$

$$R_{tt} = -\frac{B''}{2A} + \frac{B'}{2A} \left( \frac{A'}{A} + \frac{B'}{B} \right) - \frac{B'D'}{2AD}$$  \hspace{1cm} (9)

$$R_{\mu\nu} = 0 \quad \text{for} \ \mu \neq \nu$$

The Einstein field equations for vacuum are $R_{\mu\nu} = 0$. Dividing $R_{rr}$ by $A$ and $R_{tt}$ by $B$ and putting them together we get

$$- \frac{D'}{2AD} \left( \frac{A'}{A} + \frac{B'}{B} \right) + \frac{1}{A} \left( \frac{D''}{D} - \frac{D'^2}{2D^2} \right) = 0$$  \hspace{1cm} (10)
Multiplying (10) by \( \frac{2A'D}{D} \) gives

\[
\frac{A'}{A} + \frac{B'}{B} = \frac{2D''}{D'} - \frac{D'}{D} \tag{11}
\]

Now let integrate (11) with respect to \( r \) and find

\[
AB = C_1 \frac{D'^2}{D} \tag{12}
\]

where \( C_1 \) is a constant of integration which can be fixed by requiring that \( D \) asymptotically approaches to \( r^2 \) and \( A \) and \( B \) to one. This will fix \( C_1 \) to \( \frac{1}{4} \) by (12), thus

\[
AB = \frac{D'^2}{4D} \tag{13}
\]

Now using (9) and dividing the field equation \( R_{tt} = 0 \) by \( \frac{B'}{2A} \) we obtain

\[
-\frac{B''}{B'} + \frac{1}{2} \left( \frac{A'}{A} + \frac{B'}{B} \right) - \frac{D'}{D} = 0 \tag{14}
\]

Substituting (11) in (14) we get

\[
\frac{B''}{B'} + \frac{3D'}{2D} = \frac{D''}{D'} \tag{15}
\]

The next step is to integrate (15) with respect to \( r \) which gives

\[
B' D^2 = C_2 D' \tag{16}
\]

where \( C_2 \) is a constant of integration. Dividing (16) by \( D^2 \) and taking another integration with respect to \( r \) we get

\[
B = -2C_2 D^{-\frac{1}{2}} + C_3 \tag{17}
\]

where \( C_3 \) is another constant of integration. \( C_2 \) and \( C_3 \) can be fixed by considering the Newtonian limit of \( B \) which gives \( C_3 = 1 \) and \( C_2 = M \) \((G = c = 1)\). Thus

\[
B = 1 - 2MD^{-\frac{1}{2}} \tag{18}
\]

\( A \) may be computed by (12) and (18). It is
\[ A = \frac{D'^2}{1 - 2MD^{-\frac{1}{2}}} \]  

(19)

It is a natural expectation that the functional form of \( D \) to be fixed by using the \( \theta \theta \) component of the field equation, that is \( R_{\theta \theta} = 0 \). But using (18), (19) and (11) it turns out that the equation \( R_{\theta \theta} = 0 \) will become an identical relation of zero equal to zero for any arbitrary function of \( D \) which is only restricted to the following constraints

\[ D \to r^2 \quad \text{if} \quad r \to \infty \cup M \to 0 \]  

(20)

This means that \( D \) has the functional form

\[ D(r, M) = r^2 f\left(\frac{M}{r}\right) \]  

(21)

where \( f(0) = 1 \). There is not any obligation that \( D(0, M) \) becomes zero. Since the behavior of \( f \) approaching infinity i.e. \( r \to 0 \) is ambiguous then the transformation \( r \to r' = \sqrt{D(r, M)} = r\sqrt{f\left(\frac{M}{r}\right)} \) does not specify the range of \( r' \) at all. It is clear that the Schwarzschild solution is a special case of the general solution by choosing \( f = 1 \), indeed the simplest case. This does not based on any physical fact. This unjustified choice which is not even stand on any fundamental reasoning, is the source of the trouble of singularity in some parts of space-time and of course can be avoided. The appearance of square root of \( D \) in the final solution reveals that \( D \) cannot adapt negative values otherwise the metric becomes complex which physically is not acceptable. Then we may perform the following coordinate transformation

\[ r \text{ with the range } (0, +\infty) \to r' = D(r, M)^{\frac{1}{2}} \text{ with the range } (D(0, M)^{\frac{1}{2}}, 0) \]  

(22)

where \( D(0, M) \) may be any arbitrarily real non-negative number. Since \( D \) has \([L^2]\) dimension and \( D(0, 0) \) is equal to zero and also \( M \) is the only natural parameter of the system which has length dimension we may conclude that the general form of \( D(0, M) \) is

\[ D(0, M)^{\frac{1}{2}} = \alpha(M)M \]  

(23)

where \( \alpha \) is a dimensionless parameter which for simplicity may take it as a constant independent of \( M \). Now we make another coordinate transformation
The range of new radial coordinate is from zero to infinity. Dropping primes the final form of the metric becomes

\[ ds^2 = (1 - \frac{2M}{r + \alpha M})dt^2 - \frac{dr^2}{1 - \frac{2M}{r + \alpha M}} - (r + \alpha M)^2(d\theta^2 + \sin^2 d\varphi^2) \]  

(25)

Thus \(\alpha\) is an arbitrary constant which its different values define the members of our general solutions family. The only condition on \(\alpha\) is that it should not be much much bigger than one, otherwise it would contradict with Newtonian mechanics predictions. The corresponding metrics related to the values of \(\alpha\) bigger than 2 are all regular in the whole space. The actual metric of the spacetime of course is merely a special member of this class but not necessarily the Schwarzschild metric \((\alpha = 0)\). Specifying this requires further information about the actual properties of spacetime at Schwarzschild scales which at this time there is no access to such data.

**Test for Equivalence**

Someone may be taken in by the apparent form of the general solution that these are just the usual Schwarzschild solution with \(r\) replaced by \(r + \alpha M\) and conclude that they are not new. In the I clarify this point and show that it is not so.

1- In Schwarzschild solution the range of radial coordinate \(r\) is between 0 to \(\infty\) and the center of symmetry is at \(r = 0\). Replacing \(r\) by \(\dot{r} + \alpha M\) does not lead to the new solution because the range of the transformed radial coordinate is between \(-\alpha M\) to \(\infty\). Of course negative values for conventional radial coordinate is meaningless and the center of symmetry is located at \(\dot{r} = -\alpha M\). This is not identical to the new solution because the range of radial coordinate is between 0 to \(\infty\) and the center of symmetry is at \(\dot{r} = 0\). On the other hand in the new metric replacing \(r + \alpha M = \dot{r}\) does not lead to Schwarzschild solution because the range of \(r\) here is between 0 to \(\infty\) and the center of symmetry is at \(r = 0\). While the range of the transformed radial coordinate is between \(\alpha M\) to \(\infty\) and the center of symmetry is at \(\dot{r} = \alpha M\). This evidently is different from Schwarzschild solution. Though the extension of \(\dot{r}\) to values smaller than \(\alpha M\) is physically meaningless because a point by
definition has no internal structure to be extended inside of it, let us consider such a mathematical hypothetical spacetimes. Even these are essentially different from Schwarzschild spacetime because in Schwarzschild the point mass $M$ is at $r = 0$ while in these spacetimes it is located at $r = \alpha M$.

2- The center of spherical symmetry that is the position of point mass $M$ is a common point between the Schwarzschild spacetime and the presented spherically symmetric vacuum spacetimes in this manuscript. As it has been shown the field equations and the given boundary conditions are not sufficient to fix $\alpha$. Thus if we take $\alpha \neq 0$, then these solutions will not be singular at the center of symmetry while the Schwarzschild spacetime possesses an intrinsic singularity at the center of symmetry. If these new metrics were isometric to Schwarzschild metric they should be singular too, because coordinate transformation cannot change the intrinsic properties of spacetime. This clearly shows that the presented metrics are not Kottler solutions of Schwarzschild spacetime.

3- The presented general solution and the Schwarzschild solution have exactly the same space extension. Making use of Cartesian coordinate system as frame of reference will elucidate this fact. It turns out that all components have the same range $(-\infty, +\infty)$.

4- Let us consider hypothetically spacetimes which possesses different lower bound for the surface area of a sphere. Obviously they have different geometrical structures and present different physics.

5- The zone of $r$ of the order of Schwarzschild radius is the domain in which gravitational field is tremendously strong and conventionally we have to give up our common sense and replace the character of $r$ and $t$. So it is not surprising to have a geometry completely different.

We may conclude the discussion with this statement that in contrast with Birkhoff’s theorem the vacuum Einstein field equation spherical solution is not either automatically static nor is uniquely Schwarzschild.
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