Adaptive Differentially Private Empirical Risk Minimization

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Abstract

We propose an adaptive (stochastic) gradient perturbation method for differentially private empirical risk minimization. At each iteration, the random noise added to the gradient is optimally adapted to the stepsize; we name this process adaptive differentially private (ADP) learning. Given the same privacy budget, we prove that the ADP method considerably improves the utility guarantee compared to the standard differentially private method in which vanilla random noise is added. Our method is particularly useful for gradient-based algorithms with time-varying learning rates, including variants of AdaGrad (Duchi et al., 2011). We provide extensive numerical experiments to demonstrate the effectiveness of the proposed adaptive differentially private algorithm.

1 Introduction

Publishing deep neural networks such as ResNets [He et al., 2016] and Transformers [Vaswani et al., 2017] (with billions of parameters) trained on private datasets has become a major concern in the machine learning community; these models can memorize the private training data and can thus leak personal information, such as social security numbers [Carlini et al., 2020]. Moreover, these models are vulnerable to privacy attacks, such as membership inference [Shokri et al., 2017, Gupta et al., 2021] and reconstruction [Fredrikson et al., 2015, Nakamura et al., 2020]. Therefore, over the past few years, a considerable number of methods have been proposed to address the privacy concerns described above. One main approach to preserving data privacy is to apply differentially private (DP) algorithms [Dwork et al., 2006a, 2014, Abadi et al., 2016, Jayaraman et al., 2020] to train these models on private datasets. Differentially private stochastic gradient descent (DP-SGD) is a common privacy-preserving algorithm used for training a model via gradient-based optimization; DP-SGD adds random noise to the gradients during the optimization process [Bassily et al., 2014, Song et al., 2013, Bassily et al., 2020].

To be concrete, consider the empirical risk minimization (ERM) on a dataset \( \mathcal{D} = \{x_i\}_{i=1}^n \), where each data point \( x_i \in \mathcal{X} \). We aim to obtain a private high dimensional parameter \( \theta \in \mathbb{R}^d \) by solving

\[
\min_{\theta \in \mathbb{R}^d} F(\theta) := \frac{1}{n} \sum_{i=1}^n f_i(\theta), \quad \text{with } f_i(\theta) = f(\theta; x_i) \tag{1}
\]

where the loss function \( f(\cdot) : \mathbb{R}^d \times \mathcal{X} \to \mathbb{R} \) is non-convex and smooth at each data point. To measure the performance of gradient-based algorithms for ERM, which enjoys privacy guarantees, we define the utility by using the expected \( \ell_2 \)-norm of gradient, i.e., \( \mathbb{E}[\|\nabla F(\theta)\|] \), where the expectation is taken over the randomness of the algorithm [Wang et al., 2017, Zhang et al., 2017, Wang et al., 2019, Zhou et al., 2020a].\(^1\) The DP-SGD

\(^1\)We examine convergence through the lens of utility guarantees; one may interchangeably use the two words “utility” or “convergence”.
Figure 1: **Comparison between $\alpha_t = 1$ and $\alpha_t = 1/\sqrt{\eta_t}$ in (3).** Set the stepsize $\eta_t = 1/\sqrt{20 + t}$ and the same privacy budget at final iteration. The green curves in the left and right plots are the same; they correspond to the left vertical y-axis illustrating the actual Gaussian noise (i.e., $\eta_t \alpha_t Z$) added to the parameter $\theta_t$ for $\alpha_t = 1$ (dash line) and $\alpha_t = 1/\sqrt{\eta_t}$ (solid line). The blue (black) curves on the left (right) plot corresponding to the right vertical y-axis show the overall privacy (validation accuracy) for $\alpha_t = 1$ (DP-SGD), represented by the dashed line, and $\alpha_t = 1/\sqrt{\eta_t}$ (ADP-SGD), represented by the solid line. The variance of the perturbation using our proposed ADP-SGD decreases more slowly than that using DP-SGD, and so spreads across the entire optimization process more evenly than DP-SGD. Note that the privacy value $\bar{\varepsilon} = \varepsilon/16 = 3.2$ is based on the theoretical upper bound. The validation accuracy (black curves) is for CIFAR10 dataset with the gradient clipping $C_G = 2.5$ comparable to $G$ (see detailed explanation in Section 6).

with a Gaussian mechanism solves ERM in (1) by performing the following update with the released gradient $g_t$ at the $t$-th iteration:

$$\text{DP-SGD: } \theta_{t+1} = \theta_t - \eta_t g_t; g_t = \nabla f_{\xi_t}(\theta_t) + Z,$$

(2)

where $Z \sim N(0, \sigma^2 I)$, $\xi_t \sim \text{Uniform} \{1, 2, \ldots, n\}$, and $\eta_t > 0$ is the stepsize or learning rate. Choosing the appropriate stepsize $\eta_t$ is challenging in practice, as $\eta_t$ depends on the unknown Lipschitz parameter of the gradient $\nabla f(\theta; x_i)$ Ghadimi and Lan [2013]. Recent popular techniques for tuning $\eta_t$ include adaptive gradient methods Duchi et al. [2011] and decaying stepsize schedules Goyal et al. [2017]. When applying non-constant stepsizes, most of the existing differentially private algorithms directly follow the standard DP-SGD strategy by adding a simple perturbation (i.e., $Z \sim N(0, \sigma^2 I)$) to each gradient over the entire sequence of iterations [Zhou et al., 2020a]. This results in a uniformly-distributed privacy budget for each iteration [Bassily et al., 2014].

Several theoretical, as well as experimental results, corroborate the validity of the DP-SGD method with a uniformly-distributed privacy budget [Bu et al., 2020, Zhou et al., 2020b, a]. Indeed, using a constant perturbation intuitively makes sense after noticing that the update in (2) is equivalent to $\theta_{t+1} = \theta_t - \eta_t \nabla f(\theta_t; x_{\xi_t}) - \eta_t Z$. This implies that the size of the true perturbation (i.e., $\eta_t Z$) added to the updated parameters is controlled by $\eta_t$. The decaying learning rate thus diminishes the true perturbation added to $\theta_t$. Although the DP-SGD method with decaying noise $\eta_t Z$ is reasonable, prior to this paper it was unknown whether this is the optimal strategy using the utility measure.

To study the above question, we propose adding a hyperparameter $\alpha_t > 0$ to the private mechanism:

$$\text{ADP-SGD: } \theta_{t+1} = \theta_t - \eta_t g_t; g_t = \nabla f_{\xi_t}(\theta_t) + \eta_t \alpha_t Z.$$

(3)

The role of the hyperparameter $\alpha_t$ is to adjust the variance of the added random noise given the stepsize $\eta_t$. It is thus natural to add “adaptive” in front of the name DP-SGD and call our proposed algorithm ADP-SGD. To establish the privacy and utility guarantees of this new method, we first extend the advanced composition
theorem [Dwork et al., 2014] so that it treats the case of a non-uniformly distributed privacy budget. We then show that our method achieves an improved utility guarantee when choosing $\alpha_t = 1/\sqrt{t}$, compared to the standard method using uniformly-distributed privacy budget, which corresponds to $\alpha_t = 1$. This relationship between $\alpha_t$ and $\eta_t$ is surprising. Given the same privacy budget and the decaying stepsize $\eta_t < 1$, the best choice $-\alpha_t = 1/\sqrt{t}$ results in $\theta_{t+1} = \theta_t - \eta_t \nabla f(\theta_t; x_{t}) - \sqrt{\eta_t} Z$. This implies that the actual Gaussian noise $\sqrt{\eta_t} Z$ of ADP-SGD decreases more slowly than that of the conventional DP-SGD (i.e., $\eta_t Z$). To some extent, this is counter-intuitive in terms of convergence: one may anticipate that a more accurate gradient or smaller perturbation will be necessary as the parameter $\theta_t$ reaches a stationary point (i.e., as $\|\nabla F(\theta_t)\| \rightarrow 0$) Lee and Kifer [2018]. See Figure 1 for an illustration. We will explain how this interesting finding is derived in Section 4.

**Contribution.** Our contributions include:

- We propose an adaptive (stochastic) gradient perturbation method — “Adaptive Differentially Private Stochastic Gradient Descent” (ADP-SGD) (Algorithm 1 or (3)) — and show how it can be used to perform differentially private empirical risk minimization. We show that ADP-SGD provides a solution to the core question of this paper: *given the same overall privacy budget and iteration complexity, how should we select the gradient perturbation adaptively - across the entire SGD optimization process - to achieve better utility guarantees?* To answer this, we establish the privacy guarantee of ADP-SGD (Theorem 4.1) and find that the best choice of $\alpha_t$ follows an interesting dynamic: $\alpha_t = 1/\sqrt{t}$ (Theorem 4.2). Compared to the conventional DP-SGD, ADP-SGD with $\alpha_t = 1/\sqrt{t}$ results in a better utility given the same privacy budget $\varepsilon$ and complexity $T$.

- As the ADP-SGD method can be applied using any generic $\eta_t$, we discuss the two widely-used stepsize schedules: (1) the polynomially decaying stepsize of the form $\eta_t = 1/\sqrt{t + 1}$, and (2) $\eta_t$ updated by the gradients Duchi et al. [2011]. When using $\eta_t = 1/\sqrt{t + 1}$, given the same privacy budgets $\varepsilon$, we obtain a stochastic sequence $\{\theta_t^{ADP}\}$ for ADP-SGD with $\alpha_t = 1/\sqrt{t}$, and $\{\theta_t^{DP}\}$ for standard DP-SGD. We have the utility guarantees of the two methods, respectively$^2$

$$
\mathbb{E}[\|\nabla F(\theta_t^{ADP})\|^2] = \tilde{O}\left(\frac{\log(T)}{\sqrt{T}} + \frac{d \sqrt{T}}{n^2 \varepsilon^2}\right); \quad \mathbb{E}[\|\nabla F(\theta_t^{DP})\|^2] = \tilde{O}\left(\frac{\log(T)}{\sqrt{T}} + \frac{d \log(T) \sqrt{T}}{n^2 \varepsilon^2}\right)
$$

where $\tau := \arg\min_{k \in [T-1]} \mathbb{E}[\|\nabla F(\theta_k)\|^2]$. Compared to the standard DP-SGD, ADP-SGD with $\alpha_t = 1/\sqrt{t}$ improves the bound by a factor of $O(\log(T))$ when $T$ and $d$ are large (i.e. high-dimensional settings). When $\eta_t$ is updated by the gradients Duchi et al. [2011], the same result holds. See Section 5 for the detailed discussion.

- Finally, we perform numerical experiments to systematically compare the two algorithms: ADP-SGD ($\alpha_t = 1/\sqrt{t}$) and DP-SGD. In particular, we verify that ADP-SGD with $\alpha_t = 1/\sqrt{t}$ consistently outperforms DP-SGD when $d$ and $T$ are large. Based on these theoretical bounds and supporting numerical evidence, we believe ADP-SGD has important advantages over past work on differentially private empirical risk minimization.

**Notation.** In the paper, $[N] := \{0, 1, 2, \ldots, N\}$ and $\{\cdot\} := \{\cdot\}_{i=1}^T$. We write $\|\cdot\|$ for the $\ell_2$-norm. $F^*$ is a global minimum of $F$ assuming $F^* > 0$. We use $D_F := F(\theta_0) - F^*$ and set stepsize $\eta_t = \eta_t/b_{t+1}$.

### 2 Preliminaries

We first make the following assumptions for the objective loss function in (1).

**Assumption 2.1.** Each component function $f(\cdot)$ in (1) has $L$-Lipschitz gradient, i.e.,

$$
\|\nabla f(x) - \nabla f(y)\| \leq L \|x - y\|, \quad \forall x, y \in \mathbb{R}^d.
$$

**Assumption 2.2.** Each component function $f(\cdot)$ in (1) has bounded gradient, i.e.,

$$
\|\nabla f(x)\| \leq G, \quad \forall x \in \mathbb{R}^d.
$$

$^2$This is an informal statement of Proposition 5.1: the order $\tilde{O}$ hides $\log(1/\delta)$, $LG^2$ and $F(\theta_0) - F^*$ terms. We keep the iteration number $T$ in our results since the theoretical best value of $T$ depends on some unknown parameters such as the Lipschitz parameter of the gradient, which we try to tackle using non-constant stepsizes.
The bounded gradient assumption is a common assumption for the analysis of DP-SGD algorithms [Wang et al., 2017, Zhou et al., 2020a,b] and also frequently used in general adaptive gradient methods such as Adam [Reddi et al., 2021, Chen et al., 2018, Reddi et al., 2018]. One recent popular approach to relax this assumption is using the gradient clipping method [Chen et al., 2020, Andrew et al., 2019, Pichapati et al., 2019], which we will discuss more in Section 6 as well as in Appendix A. Nonetheless, this assumption would serve as a good starting point to analyze our proposed method. Next, we introduce differential privacy [Dwork et al., 2006b].

**Definition 2.1 ((ε, δ)-DP).** A randomized mechanism \( M : D \rightarrow R \) with domain \( D \) and range \( R \) is \((ε, δ)\)-differentially private if for any two adjacent datasets \( D, D' \) differing in one sample, and for any subset of outputs \( S \subseteq R \), we have

\[
\Pr[M(D) \in S] \leq e^ε \Pr[M(D') \in S] + δ.
\]

**Lemma 2.1 (Gaussian Mechanism).** For a given function \( h : D \rightarrow R^d \), the Gaussian mechanism \( M(D) = h(D) + Z \) with \( Z \sim N(0, \sigma^2 I_d) \) satisfies \((\sqrt{2\log(1.25/δ)}\Delta/σ, δ)-DP\), where \( \Delta = \sup_{D,D'} \|h(D) - h(D')\| \). \( D, D' \) are two adjacent datasets, and \( ε, δ > 0 \).

To achieve differential privacy, we can use the above Gaussian mechanism [Dwork et al., 2014]. In our paper, we consider iterative differentially private algorithms, which prompts us to use privacy composition results to establish the algorithms’ privacy guarantees after the completion of the final iteration. To this end, we extend the advanced composition theorem [Dwork et al., 2014] to the case in which each mechanism \( M_i \) has its own specific \( ε_i \) and \( δ_i \) parameters.

**Lemma 2.2 (Extended Advanced Composition).** Consider two sequences \( \{ε_i\}_{i=1}^k, \{δ_i\}_{i=1}^k \) of positive numbers satisfying \( ε_i \in (0, 1) \) and \( δ_i \in (0, 1) \). Let \( M_i \) be \((ε_i, δ_i)\)-differentially private for all \( i \in \{1, 2, \ldots, k\} \). Then \( M = (M_1, \ldots, M_k) \) is \((\tilde{ε}, \tilde{δ})\)-differentially private for \( \tilde{ε} \in (0, 1) \) and

\[
\tilde{ε} = \sqrt{\sum_{i=1}^k 2ε_i^2 \log \left( \frac{1}{\tilde{δ}} \right)} + \sum_{i=1}^k \frac{ε_i(ε_i - 1)}{ε_i + 1}, \quad \tilde{δ} = 1 - (1 - δ_1)(1 - δ_2)\ldots(1 - δ_k) + δ'.
\]

When \( ε_i = ε_0 \) and \( δ_i = δ_0 \) for all \( i \), Lemma 2.2 reduces to the classical advanced composition theorem [Dwork et al., 2014] restated in Lemma A.2 in the Appendix.

### 3 The ADP-SGD algorithm

In this section, we present our proposed algorithm: adaptive differentially private stochastic gradient descent (ADP-SGD, Algorithm 1). The “adaptive” part of the algorithm is tightly connected with the choice of the hyper-parameter \( α_t \) (see line 5 of Algorithm 1). For \( α_t = 1 \), ADP-SGD reduces to DP-SGD. As mentioned before, we aim to investigate whether an uneven allocation of the privacy budget for each iteration (via ADP-SGD) will provide a better utility guarantee than the default DP-SGD given the same privacy budget. To achieve this, our proposed ADP-SGD with hyper-parameter \( α_t \) adjusts the privacy budget consumed at the \( t \)-th iteration according to the current learning rate \( η/b_{t+1} \) (see line 6 of Algorithm 1). Moreover, we will update \( α_t \) dynamically (see line 5 of Algorithm 1) and show how to choose \( α_t \) in Section 4. Before proceeding to analyze Algorithm 1, we state Definition 3.1 to clearly explain the adaptive privacy mechanism for the algorithm.

**Definition 3.1 (Adaptive Gaussian Mechanism).** At iteration \( t \) in Algorithm 1, the privacy mechanism \( M_t : R^d \rightarrow R^d \) is:

\[
M_t(X) = \nabla f(θ_t; x_t) + α_{t+1}c_t.
\]

The hyper-parameter \( α_{t+1} \) is adaptive to the DP-SGD algorithm specifically to the stepsize \( η_t := η/b_t \).

Algorithm 1 is a general framework that can cover many variants of stepsize update schedules, including the adaptive gradient algorithms [Duchi et al., 2011, Kingma and Ba, 2014]. Rewriting \( η_t = η/b_{t+1} \) in Algorithm 1 is equivalent to (3). In particular, we use functions \( φ_1 : \mathbb{R}^2 \rightarrow \mathbb{R} \) and \( φ_2 : \mathbb{R}^2 \rightarrow \mathbb{R} \) to denote the updating rules for parameters \( b \) and \( α_t \), respectively. For example, when \( φ_1 = 1/\sqrt{a + ct} \) and \( φ_2 \) is the constant 1 for all \( t \) and \( a,c > 0 \), ADP-SGD reduces to DP-SGD with polynomial decaying stepsizes [Bassily et al.,
are independent of each other and also independent of \( \theta \).

Assumption 3.1. \( \nabla^2 \) 2018, Ward et al., 2019], we will use Assumption 3.1 in addition to Assumption 2.2 and Assumption 2.1. 

reduces to the vanilla SGD.

\[
(\text{Theorem 4.1})
\]

In this section, we provide the main results for our method – the privacy and utility guarantees.

4 Theoretical results for ADP-SGD

In this section, we provide the main results for our method – the privacy and utility guarantees.

Theorem 4.1 (Privacy Guarantee). Suppose the sequence \( \{\alpha_t\}^{T}_{t=1} \) is known in advance and that Assumption 2.2 holds. Algorithm 1 satisfies \((\varepsilon, \delta)\)-DP if the random noise \( c_t \) has variance

\[
\sigma^2 = \frac{(16G)^2 B_3}{n^2 \varepsilon^2} \sum_{t=0}^{T-1} \frac{1}{\alpha_{t+1}} \quad \text{with} \quad B_3 = \log \left( \frac{16T}{n\delta} \right) \log \left( \frac{1.25}{\delta} \right). \tag{6}
\]

The theorem is proved by using Lemma 2.2 and Definition 3.1 (see Appendix C.1 for details). Note that the term \( B_3 \) could be improved by using the moments accountant method [Mironov et al., 2019], to \( O(\log(1.25/\delta)) \) independent of \( T \) but with some additional constraints [Abadi et al., 2016]. We keep this format of \( B_3 \) as in (6) in order to compare directly with [Bassily et al., 2014].

Theorem 4.1 shows that \( \sigma^2 \) must scale with \( \sum_{t=1}^{T} 1/\alpha_t^2 \). When the complexity \( T \) increases, the variance \( \sigma^2 \), regarded as a function of \( T \), could be either large or small, depending on the sequence \( \{\alpha_t\} \).

If \( \alpha_t^2 \propto t^p, p \in [0, 1] \), then \( \sigma^2 \propto \begin{cases} T^{1-p} & 0 < p < 1 \\ \log(T) & p = 1 \end{cases} \)

and \( p = 0 \) is the default DP-SGD. From a convergence view, \( \theta_{t+1} = \theta_t - \eta_t \nabla f(\theta_t; x_{\xi_t}) - \eta_t \alpha_t Z \) implies that the actual Gaussian noise added to the updated parameter \( \theta_t \) has variance \( \eta^2 \alpha_t^2 \sigma^2 \). Therefore, it is subtle to determine what \( p \) would be the best choice for ensuring convergence. In Theorem 4.2, we will see that the optimal choice of the sequence \( \{\alpha_t\}^{T}_{t=1} \) is closely related to the stepsize.

Theorem 4.2 (Convergence for ADP-SGD). Suppose we choose \( \sigma^2 \) - the variance of the random noise in Algorithm 1 - according to (6) in Theorem 4.1 and that Assumption 2.1, 2.2 and 3.1 hold. Furthermore, suppose \( \alpha_t, b_t \) are deterministic. The utility guarantee of Algorithm 1 with \( \tau \triangleq \arg\min_{k \in [T-1]} \mathbb{E}[\|\nabla F(\theta_k)\|^2] \) and \( B_3 = \log(16T/(n\delta)) \log(1.25/\delta) \) is

\[
\mathbb{E}[\|\nabla F(\theta_\tau)\|^2] \leq \frac{1}{\sum_{t=0}^{T-1} b_{t+1}} \left( W_{opt} + \frac{d(16G)^2 B_3}{2n^2 \varepsilon^2} M(\{\alpha_t\}, \{b_t\}) \right) \tag{7}
\]
where \( W_{\text{opt}} := \frac{D_f}{n} + \frac{n\epsilon}{2} \sum_{t=0}^{T-1} \mathbb{E} \left[ \frac{1}{b_t} \right] \) and \( M(\{\alpha_t\}, \{b_t\}) \triangleq \sum_{t=0}^{T-1} (\alpha_t/b_t)^2 \sum_{t=1}^{T} 1/\alpha_t^2 \).

Although the theorem assumes independence between \( b_{t+1} \) and the stochastic gradient \( \nabla f(\theta_t; x_{t}) \), we shall see in Section 5.2 that a similar bound holds for correlated \( b_t \) and \( \nabla f(\theta_t; x_{t}) \).

**Remark 4.1 (An optimal relationship between \( \alpha_t \) and \( b_t \)).** According to (7), the utility guarantee of Algorithm 1 consists of two terms. The first term \( W_{\text{opt}} \) corresponds to the optimization error and the last term \( \frac{\alpha_t}{2} M(\{\alpha_t\}, \{b_t\}) \) is introduced by the privacy mechanism, which is also the dominating term. Note that if we fix \( \{b_t\} \) and minimize \( M \) with respect to \( \{\alpha_t\} \), the minimal value denoted by \( M_{\text{adp}} \) expresses as

\[
\min_{\{\alpha_t\}} M(\{\alpha_t\}, \{b_t\}) = M_{\text{adp}} \triangleq \left( \sum_{t=0}^{T-1} 1/b_{t+1} \right)^2. \tag{8}
\]

Furthermore, \( M(\{\alpha_t\}, \{b_t\}) = M_{\text{adp}} \) if \( \alpha_t^2 = b_t \). Therefore, if we choose \( \alpha_t, b_t \) such that the relationship of \( \alpha_t^2 = b_t \) holds, we can achieve the minimum utility guarantee for Algorithm 1.

Based on the utility bound in Theorem 4.2, we now compare the strategies between using the arbitrary setting of \( \{\alpha_t\} \) and the optimal setting \( \alpha_t^2 = b_t \) by examining the ratio \( M(\{\alpha_t\}, \{b_t\})/M_{\text{adp}} \): a large value of this ratio implies a significant reduction in the utility bound is achieved by using Algorithm 1 with \( \alpha_t = \sqrt{b_t} \).

For example, for the standard DP-SGD method, the function \( M \) reduces to \( M_{\text{dp}} \triangleq T \sum_{t=0}^{T-1} 1/b_{t+1}^2 \). Our proposed method - involving \( \alpha_t = \sqrt{b_t} \) - admits a bound improved by a factor of

\[
M_{\text{dp}}/M_{\text{adp}} = T \left( \frac{\sum_{t=0}^{T-1} 1/b_{t+1}^2}{\sum_{t=0}^{T-1} 1/b_{t+1}} \right)^2 \geq 1,
\]

where (a) is due to the Cauchy-Schwarz inequality; thus, ADP-SGD is not worse than DP-SGD for any choice of \( \{b_t\} \). In the following section, we will analyze this factor of \( M_{\text{dp}}/M_{\text{adp}} \) for two widely-used stepsize schedules: (a) the polynomially decaying stepsize given by \( \eta_t = 1/(\sqrt{t+1}) \); and (b) a variant of adaptive gradient methods [Duchi et al., 2011].

Note that, in addition to \( \alpha_t^2 = b_t \), there are other relationships between the sequences \( \{(\alpha_t/b_t)^2\} \) and \( \{\alpha_t^2\} \) that could lead to the same \( M_{\text{adp}} \). For instance, setting \( \alpha_t^2 = b_{t-1} \) is another possibility. Nevertheless, in this paper, we will focus on the \( \alpha_t^2 = b_t \) relation, and leave the investigation of other appropriate choices to future work. We emphasize that the bound in Theorem 4.2 only assumes \( f \) to have Lipschitz smooth gradients and be bounded. Thus, the theorem applies to both convex or non-convex functions. Since our focus is on the improvement factor \( M_{\text{dp}}/M_{\text{adp}} \), we will assume our functions are non-convex, but the results will also hold for convex functions.

## 5 Examples for ADP-SGD

We now analyze the convergence bound given in Theorem 4.2 and obtain an explicit form for \( M \) in terms of \( T \) by setting the stepsize to be \( 1/b_{t+1} \propto 1/\sqrt{T} \), which is closely related to the polynomially decreasing rate of adaptive gradient methods [Duchi et al., 2011, Ward et al., 2019] studied in Section 5.2.

**Constant stepsize v.s. time-varying stepsize.** If the constant step size is used, then there is no need to use the adaptive DP mechanism proposed in this paper as we verify that constant perturbation to the gradient is optimal in terms of convergence. However, as we explained in the introduction, to ease the difficulty of stepsize tuning, time-varying stepsize is widely used in many practical applications of deep learning. We will discuss two examples below. In these cases, the standard DP mechanism (i.e., constant perturbation to the gradient) is not the most suitable technique, and our proposed adaptive DP mechanism can give better utility results.

**Achieving \( \log T \) improvement.** We present Proposition 5.1 and Proposition 5.2 to show that our method achieves \( \log(T) \) improvement over the vanilla DP-SGD. Although this \( \log(T) \) improvement can also be achieved by using the moments accountant method (MAM) [Mironov et al., 2019], we emphasize that our proposed method is orthogonal and complementary to MAM. This is because the \( \log(T) \) improvement using MAM is over \( B_3 \) (see discussion after Theorem 4.1), while ours is during the optimization process depending
on stepsizes. Nevertheless, since the two techniques are complementary to each other, we can apply them simultaneously and achieve a $\log^2(T)$ improvement over DP-SGD using the advanced composition for $O(1/\sqrt{T})$ stepsizes, compared to a $\log(T)$ improvement using either of them. Thus, an adaptive DP mechanism for algorithms with time-varying stepsizes is advantageous.

### 5.1 Example 1: ADP-SGD with polynomially decaying stepsizes

The first case we consider is the stochastic gradient descent with polynomially decaying stepsizes. More specifically, we let $b_t = (a + ct)^{1/2}$, $a > 0$, $c > 0$.

**Proposition 5.1 (ADP-SGD v.s. DP-SGD for a polynomially decaying stepsize schedule).** Under the conditions of Theorem 4.2 on $f$ and $\sigma^2$, let $b_t = (a + ct)^{1/2}$ with $a > 0, c > 0$ in Algorithm 1. Denote $\tau = \arg \min_{t \in [T-1]} \mathbb{E}[\|\nabla F(\theta_t)\|^2]$, and $B_\delta = \log(16T/(n\delta)) \log(1.25/\delta)$. If we choose $T \geq 5 + 4a/c$, we have the following utility guarantee for ADP-SGD ($\alpha_t^2 = b_t$) and DP-SGD ($\alpha_t^2 = 1$) respectively,

\[
\begin{align*}
(ADP-SGD) & \quad \mathbb{E}[\|\nabla F(\theta^*_{\text{ADP}})\|^2] \leq \frac{W^\text{decay}_{\text{opt}}}{\sqrt{T-1}} + \frac{\eta dL(16G)^2 B_\delta \sqrt{T}}{2n^2 \epsilon^2 \sqrt{c}}; \\
(DP-SGD) & \quad \mathbb{E}[\|\nabla F(\theta^*_{\text{DP}})\|^2] \leq \frac{W^\text{decay}_{\text{opt}}}{\sqrt{T-1}} + \frac{\eta dL(16G)^2 B_\delta \sqrt{T} \log \left(1 + \frac{T}{c^2}\right)}{n^2 \epsilon^2 \sqrt{c}}.
\end{align*}
\]

where $W^\text{decay}_{\text{opt}} = \sqrt{\mathbb{E} \left( \frac{D^2}{n^2} + \frac{n^2 L B^2}{2e} \right)}$.

The proof of Proposition 5.1 is given in Appendix D.1 and Appendix D.2. Proposition 5.1 implies $M_{\text{dp}}/M_{\text{adp}} = \mathcal{O}(\log T)$—that is, ADP-SGD has an improved utility guarantee compared to DP-SGD. Such an improvement can be significant when $d$ is large and $\log^2$ is large.

### 5.2 Example 2: ADP-SGD with adaptive stepsizes

We now examine another choice of the term $b_t$, which relies on a variant of adaptive gradient methods [Duchi et al., 2011]. To be precise, we assume $b_t$ is updated according to the norm of the gradient, i.e.,

\[
b_{t+1}^2 = b_t^2 + \max\{\|\nabla f(\theta_t; x_{\xi_t})\|^2, \nu\},
\]

where $\nu > 0$ is a small value to prevent the extreme case in which $1/b_{t+1}$ goes to infinity (when $b_t^2 = \|\nabla f(\theta_t; x_{\xi_t})\|^2 \to 0$, then $\eta/b_t \to \infty$). We choose this precise equation formula because it is simple, and it also represents the core of adaptive gradient methods - updating the stepsize on-the-fly by the gradients [Levy et al., 2018, Ward et al., 2019]. The conclusions for this variant may transfer to other versions of adaptive stepsizes, and we defer this to future work.

Observe that $b_t \propto t^{1/2}$ since $b_t^2 \in [b_0^2 + tv, b_0^2 + tG]$, which at a first glance indicates that the bound for this adaptive stepsize could be derived via a straightforward application of Proposition 5.1. However, since $b_t$ is now a random variable correlated to the stochastic gradient $\nabla f(\theta_t; x_{\xi_t})$, we cannot directly apply Theorem 4.2 to study $b_t$. To tackle this, we adapt the proof technique from [Ward et al., 2019] and obtain Theorem D.1, which we defer to Appendix D.3.

As we see, $b_t$ is updated on the fly during the optimization process. Applying our proposed method with $\alpha_t^2 = b_t$ for this adaptive stepsize is not possible since $\alpha_t$ has to be set beforehand according to Equation (6) in Theorem 4.1. To address this, we note $b_t^2 \in [b_0^2 + tv, b_0^2 + tG]$. Thus, we propose to set $\alpha_t^2 = \sqrt{b_t^2 + tC}$ for some $C \in [\nu, G^2]$ and obtain Proposition 5.2 based on Theorem D.1.

**Proposition 5.2 (ADP v.s. DP with an adaptive stepsize schedule).** Under the same conditions of Theorem D.1 on $f$, $\sigma^2$, and $b_t$, if $\alpha_t = (b_0^2 + tC)^{1/4}$ for some $C \in [\nu, G^2]$, then

\[
\begin{align*}
(ADP-SGD) & \quad \mathbb{E}[\|\nabla F(\theta^*_{\text{ADP}})\|^2] \leq \frac{W^\text{adapt}_{\text{opt}}}{\sqrt{T-1}} + \frac{128G^3 \eta dL B_\delta \sqrt{T}}{n^2 \epsilon^2 \nu} \\
(DP-SGD) & \quad \mathbb{E}[\|\nabla F(\theta^*_{\text{DP}})\|^2] \leq \frac{W^\text{adapt}_{\text{opt}}}{\sqrt{T-1}} + \frac{32G^3 \eta dL B_\delta \sqrt{T} \log \left(1 + \frac{T}{c^2}\right)}{n^2 \epsilon^2 \nu}.
\end{align*}
\]

where $W^\text{adapt}_{\text{opt}} = 2G \left(2G + \frac{\eta L}{T}\right) \left(1 + \log \left(\frac{T(G^2 + \nu^2)}{b_0^2} + 1\right)\right) + \frac{2GD\epsilon}{\eta}$. 


See the proof in Appendix D.4. Similar to the comparison in Proposition 5.1, the key difference between two bounds in Proposition 5.2 is the last term; using ADP-SGD gives us a tighter utility guarantee than the one provided by DP-SGD by a factor of $O(\log(T))$. This improvement is significant when the dimension $d$ is very high, or when either $L$, $G$, or $T$ are sufficiently large. Note that the bound in Proposition 5.2 does not reflect the effect of the different choice of $C$, as the bound corresponds to the worst case scenarios. We will perform experiments testing a wide range of $C$ values and this will allow us to thoroughly examine the properties of ADP-SGD for adaptive stepsizes.

6 Experiments

In this section, we present numerical results to support the theoretical findings of our proposed methods. We perform two sets of experiments: (1) when $\eta_t$ is polynomially decaying, we compare ADP-SGD ($\alpha_t^2 = b_t$) with DP-SGD (setting $\alpha_t = 1$ in Algorithm 1); and (2) when $b_t$ is updated by the norm of the gradients, we compare ADP-SGD ($\alpha_t^2 = \sqrt{b_0^2 + tC}$) with DP-SGD. The first set of experiments is designed to examine the case when the learning rate is precisely set in advanced (i.e., Proposition 5.1), while the second concerns when the learning rate is not known ahead (i.e., Proposition 5.2). In addition to the experiments above, in the supplementary material (Appendix F.3), we present strong empirical evidence in support of the claim that using a decaying stepsize schedule yields better results than simply employing a constant stepsize schedule. See Section G for our code demonstration.

Assumption 2.2 and the gradient clipping method. One limitation for the above proposition is the bounded gradient (Assumption 2.2). However, as discussed in Section 2, this is common. But $G$ could be very large, particularly for all $\theta \in \mathbb{R}^d$ in highly over-parameterized models (such as neural networks). To make the algorithm work in practice, we use gradient clipping [Chen et al., 2020, Andrew et al., 2019, Pichapati et al., 2019]. That is, given the current gradient $\nabla f_{\xi_t}(\theta_t)$, we apply a function $h(\cdot; C_G) : \mathbb{R}^d \to \mathbb{R}^d$, which depends on the the positive constant $C_G > 0$ such that $\|h(\nabla f_{\xi_t}(\theta_t))\| \leq C_G$. Thus, the implementation of our algorithms (sample codes are shown in Figure 8) are

\begin{align}
\text{ADP-SGD:} \quad \theta_{t+1} &= \theta_t - \eta_t g_t; \quad g_t = h(\nabla f_{\xi_t}(\theta_t), C_G) + \eta_t \alpha_t Z. \\
\text{DP-SGD:} \quad \theta_{t+1} &= \theta_t - \eta_t g_t; \quad g_t = h(\nabla f_{\xi_t}(\theta_t), C_G) + Z. \tag{11}
\end{align}

Regarding the convergence result of using the gradient clipping method instead of the bounded gradient assumption (Assumption 2.2), [Chen et al., 2020] show that if the gradient distribution is “approximately” symmetric, then the gradient norm goes to zero (Corollary 1). Furthermore, [Chen et al., 2020] showed (in Theorem 5) that the convergence of DP-SGD with clipping (without bounded gradient assumption) is $O(\sqrt{d}/(\alpha \varepsilon)) +$ clipping bias with the specified constant learning rate $O(1/\sqrt{T})$.

There is a straightforward way to apply our adaptive perturbation to the above clipping result (e.g., Theorem 5 in [Chen et al., 2020]) using the time-varying learning rate. The bounds for ADP-SGD and DP-SGD respectively are (9)+clipping bias and (10)+clipping bias where the constant $G$ in (9) and (10) is now replaced with $C_G$. Thus, there is still a $\log(T)$ factor gain if clipping bias is not larger than the bounds in (9) and (10). In our experiments, we will be using various gradient clipping values $C_G \in \{0.5, 1.0, 2.5, 5.0\}$ to understand how it affects our utility.

Datasets and models. We perform our experiments on CIFAR-10 [Krizhevsky et al., 2009], using a convolution neural network (CNN). See our CNN design in the appendix. Notably, following previous work [Abadi et al., 2016], the CNN model is pre-trained on CIFAR-100 and fine-tuned on CIFAR-10. The mini-batch size is 256, and each independent experiment runs on one GPU. We set $\eta = 1$ in Algorithm 1 (line 6) and use the gradient clipping with $C_G \in \{0.5, 1.0, 2.5, 5.0\}$ [Chen et al., 2020, Andrew et al., 2019, Pichapati et al., 2019]. Note that one might need to think about $C_G$ as being approximately closer to the bounded gradient parameter $G$. We provide a more detailed discussion in Appendix 6.1. The privacy budget is set to be $\varepsilon = \varepsilon/C_G \in \{0.8, 1.2, 1.6, 3.2, 6.4\}$ and we choose $\delta = 10^{-5}$.\(^3\) Given these privacy budgets, we calculate the corresponding variance by Theorem 4.1 (See Appendix G for the code to obtain $\sigma$). We acknowledge that our empirical investigation is limited by the computing budget and this limitation forced us to choose

\(^3\)The constant $C_G = 16$ in (6). Although $\varepsilon = 16\varepsilon$ is large for $\varepsilon \in \{0.8, 1.2, 1.6, 3.2, 6.4\}$, they match the numerical privacy $\{0.29, 0.43, 0.57, 1.23, 3.24\}$ calculated by the moments accountant with the noise determined by $T = 11700$ (60 epochs) and the gradient clipping $C_G = 1.0$. The code is based on https://github.com/tensorflow/privacy
between diversity in the choice of iteration complexity $T$ and type of stepsizes as opposed to diversity in the model architectures and datasets.

**Performance measurement.** There are 50000 images used for training and 10000 images for validation, respectively. We repeat the experiments five times. For the $i$-th independent experiment, we calculate the validation accuracy at every 20 iterations and select the best validation accuracy $\text{acc}_{i}^{\text{best}}$ during the optimization process (over the entire iteration $\{t\}_{t=1}^{T}$). In Table 1 and Table 2, we report the average and standard deviation of $\{\text{acc}_{i}^{\text{best}}\}_{i=1}^{5}$, which is closely related to $\min_{t \in [T]} \mathbb{E}[\|\nabla F(\theta_t^{\text{ADP}})\|^2]$ and $\min_{t \in [T]} \mathbb{E}[\|\nabla F(\theta_t^{\text{DP}})\|^2]$, the convergence metric for our theoretical analysis in Theorem 4.2. Additionally, to further understand the method’s final performance, we report in Table 4 and 5 the average and standard deviation of the accuracy $\{\text{acc}_{i}^{\text{last}}\}_{i=1}^{5}$ where $\text{acc}_{i}^{\text{last}}$ represents the validation accuracy at iteration $T$ for the $i$-th experiment.

6.1 ADP-SGD v.s. DP-SGD for polynomially decaying stepsizes

We focus on understanding the optimality of the theoretical guarantees of Theorem 4.2 and Proposition 5.1; the experiments help us further understand how this optimality reflects in generalization. We consider training with $T = 11760, 23520, 39200$ iterations corresponding to 60, 120, 200 training epochs (196 iterations/epoch), which represents the practical scenarios of limited, standard and large computational time budgets. We use two kinds of monotone learning rate schedules: i) $\eta_t = 0.1 - \alpha_T \sqrt{t}$ (see orange curves in Figure 2) and report the results in Table 1; ii) $\eta_t = \eta_0 / (\bar{C}_t + 1) = 1 / \sqrt{20 + t}$ (see blue curves in Figure 2), in which the results are given in Table 2. The former learning rate schedule is designed to make sure the learning rate reaches close to zero at $T$, while the latter one is to match precisely Proposition 5.1.

![Figure 2: Two schedules of decaying learning rates.](image)

**Observation from Table 1 and Table 2.** The results from the two tables show that the overall performances of our method (ADP-SGD) are mostly better than DP-SGD given a fixed privacy budget and the same complexity $T$, which matches our theoretical analysis. Particularly, the increasing $T$ tends to enlarge the gap between ADP-SGD and DP-SGD, especially for smaller privacy: for $\bar{\varepsilon} = 0.8$ with $C_G = 1$ in Table 2, we have improvements of 0.8% at epoch 60, 1.48% at epoch 120, and 7.03% at epoch 200. This result is reasonable since, as explained in Proposition 5.1, ADP-SGD improves over DP-SGD by a factor $\log(T)$.

Furthermore, our method is more robust to the predefined complexity $T$ and thus provides an advantage when using longer iterations. For example, for $\bar{\varepsilon} = 3.2$ with $C_G = 2.5$ in Table 2, our method increases from 65.34% to 66.41% accuracy when the iteration complexity of 60 epochs is doubled; it maintains the accuracy 65.74% at the longer epoch 200. In contrast, under the same privacy budget and gradient clipping, DP-SGD suffers the degradation from 66.08% (epoch 60) to 65.17% (epoch 200).

**Discussion on gradient clipping.** Both results in Table 1 and Table 2 indicate that: (1) Smaller gradient clipping $C_G$ could help achieve better accuracy when the privacy requirement is strict (i.e., small privacy $\bar{\varepsilon} \in \{0.8, 1.2\}$). For a large privacy requirement, i.e., $\bar{\varepsilon} \in \{3.2, 6.4\}$, a bigger gradient clipping value is more advantageous; (2) As the gradient clipping $C_G$ increases, the gap between DP and ADP tends to be more significant. This matches our theoretical analysis (e.g. Proposition 5.1) that the improvement of ADP-SGD
over DP-SGD is by a magnitude $O(dL G^2 \log(T)\sqrt{T}/n^2)$ where $G_C$ can replace $G$ as we discussed in paragraph “Assumption 2.3” and the gradient clipping method”.

Table 1: Mean accuracy of ADP-SGD/DP-SGD with polynomially decaying step sizes $\eta_t = 0.1 - \alpha T$ where $\alpha_T$ is the ratio depending on the final epochs/iterations $T$ such that the learning rate at $T$ is $\eta_T = 10^{-10}$ (see the orange curves in Figure 2). This table reports accuracy for CIFAR10 with the mean and the corresponding standard deviation over $\{acc_{best}^i\}_{i=1}^5$. Here, $acc_{best}^i$ is the best validation accuracy over the entire iteration process for the $i$-th independent experiment. Each set $\{acc_{best}^i\}_{i=1}^5$ corresponds to a pair of $(\bar{\varepsilon}, C_G, T, Alg)$. The difference (“Gap”) between DP and ADP is provided for visualization purpose. The results suggest that the more iterations or epochs we use, the more improvements ADP-SGD can potentially gain over DP-SGD. The results are reported in percentage (%). The bolded number is the best accuracy in a row among epoch 60, 120 and 200 for the same $C_G$. See paragraph Data sets and models and Performance measurement for detailed information.

| $\varepsilon$ | Alg      | Gradient clipping $C_G = 0.5$ |             |             |
|---------------|----------|-------------------------------|-------------|-------------|
|               |          | epoch=60                      | epoch=120   | epoch=200   |
| 0.8           | ADP-SGD  | 57.06 ± 0.56                  | 51.24 ± 0.64 | 44.93 ± 0.59 |
|               | DP-SGD   | 56.12 ± 0.63                  | 44.16 ± 0.140 | 31.79 ± 1.131 |
|               | Gap      | 0.93                         | 7.98        | 14.14       |
| 1.2           | ADP-SGD  | 58.84 ± 0.245                 | 50.61 ± 0.833 | 45.78 ± 0.432 |
|               | DP-SGD   | 56.12 ± 0.63                  | 44.16 ± 0.140 | 31.79 ± 1.131 |
|               | Gap      | 0.93                         | 7.98        | 14.14       |

Table 2: Mean accuracy of ADP-SGD/DP-SGD with polynomially decaying step sizes $\eta_t = \eta_0/(1 + \sqrt{20} + t)$ (see the blue curve in Figure 2). This table reports accuracy for CIFAR10 with the mean and the corresponding standard deviation over $\{acc_{best}^i\}_{i=1}^5$. Here, $acc_{best}^i$ is the best validation accuracy over the entire iteration process for the $i$-th independent experiment. See Table 1 for reading instruction.
Figure 3: Validation accuracy with respect to iteration $t$ using ADP-SGD/DP-SGD with polynomially decaying stepsizes $\eta_t = \eta_0^{-1} - 1 \sqrt{20 + t}$ (see the blue curve in Figure 2). The black lines corresponding to the right y-axis are measured by the validation accuracy on CIFAR10 for a CNN model using ADP-SGD (solid line) and DP-SGD (dash line). The shaded region is the one standard deviation. Each plot corresponds to a privacy budget $\bar{\varepsilon} = 3.2$ with a fixed gradient clipping value $C_G$ (see title) and a fixed $T$ (see x-axis) where the top (bottom) row is for 120 (200) training epochs. Same as Figure 1, the monotone green curves, which correspond to the left vertical y-axis, show the actual noise for $\alpha_t = 1/19$ (ADP-SGD, the solid line) and $\alpha_t = 1$ (DP-SGD, the dashed line). The top middle plot is the same as the right plot in Figure 1.

To further understand DP and ADP with respect to different privacy parameters and the gradient clipping values, we present the detailed performance in Figure 3 for $T = 23520$ (120 epochs) where each plot corresponds to a pair of $(\bar{\varepsilon} = 3.2, C_G)$. We see that from Figure 3 that the gap between ADP-SGD and DP-SGD becomes more significant as $C_G$ increases from 1 to 5; ADP achieves the highest accuracy at $C_G = 2.5$ with the best mean accuracy 66.41%. The intuition is that the noise added to the gradient with a large gradient clipping $C_G$ is much higher than that with a small gradient clipping value. In this situation, our method proves to be helpful by spreading out the total noise across the entire optimization process more evenly than DP-SGD (see the green curves in Figure 3). On the other hand, using a large gradient clipping $C_G = 5$, our method suffers an over-fitting issue while DP-SGD performs considerably poorer. Thus one should be cautious when selecting the gradient clipping values $C_G$.

6.2 ADP-SGD v.s. DP-SGD for adaptive stepsizes

In this section, we focus on understanding the optimality of the theoretical guarantees of Proposition 5.2; we study the numerical performance of ADP-SGD with stepsizes updated by the gradients. We notice that, at the beginning of the training, the gradient norm in our model lies between 0.0001 and 0.001 when $C_G = 1.0$. To remedy this small gradient issue, we let $b_t$ follow a more general form: $b_{t+1} = b_t^2 + \max \{\beta_t \|\nabla f(\theta_t; x_\xi_t)\|^2, 10^{-5}\}$ with $\beta_t > 1$. Specifically, we set $\beta_t = \max \{\beta/(t \mod 195)+1\}, 1\}$ with $\beta$ searching in a set $\{1, 512, 1024, 2048, 4096, 8192\}$. See Appendix F.2 for a detailed description. As mentioned in Section 5.2, we set $\alpha_t^2 = \sqrt{b_0^2 + tC}$ in advance with $b_0^2 = 20$, and choose $C \in \{10^{-5}, 10^{-4}, 0.001, 0.01, 0.1, 1\}$.\footnote{This set for $\beta$ is due to the values of gradient norm as mentioned in the main text. These elements cover a wide range of values that the best test errors are doing as good as or better than the ones given in Table 2.}
ε can be updated on the fly in line with AdaGrad and Adam while maintaining a predefined privacy budget. The magnitude of the privacy budget available, our approach with a decaying stepsize schedule shows an improvement in convergence by a factor with \( \log(T/n) \) relative to DP-SGD. The results indicate that when using adaptive stepsizes, ADP-SGD with various \( C \) performs better than DP-SGD.

| \( C_\epsilon \) | Alg | \( \bar{\varepsilon} = 0.8 \) | \( \bar{\varepsilon} = 1.0 \) | \( \bar{\varepsilon} = 1.2 \) | \( \bar{\varepsilon} = 1.4 \) |
|---|---|---|---|---|---|
| 1.0 | ADP | 56.58 ± 0.436 (57.65) | 62.09 ± 0.346 (62.57) | 64.33 ± 0.329 (65.03) | 67.34 ± 0.191 (67.99) |
| | DP | 56.24 ± 0.353 (57.02) | 62.02 ± 0.264 (62.33) | 64.33 ± 0.329 (65.03) | 67.42 ± 0.141 (67.7) |
| 2.5 | ADP | 56.27 ± 0.174 (56.46) | 62.38 ± 0.428 (62.86) | 64.29 ± 0.408 (64.85) | 67.55 ± 0.156 (67.77) |
| | DP | 55.65 ± 0.448 (55.98) | 62.23 ± 0.238 (62.62) | 64.26 ± 0.140 (64.39) | 66.23 ± 0.367 (66.62) |

Table 3: Errors of ADP-SGD vs. DP-SGD with adaptive stepsizes. This table reports accuracy with the mean and the corresponding standard deviation over five independent runs. The value inside the bracket is the highest accuracy over the five runs. Each entry is the best value over 36 pairs of \((\beta, C)\) for ADP-SGD and 6 values of \( \beta \) for DP-SGD. See the corresponding \((\beta, C)\) in Table 6. The results indicate that when using adaptive stepsizes, ADP-SGD with various \( C \) performs better than DP-SGD.

We consider the number of iterations to be \( T = 11700 \) with the gradient clipping 1.0 and 2.5. Table 3 summarizes the results of DP-SGD and ADP-SGD with the best hyper-parameters.

7 Related work

Differentially private empirical risk minimization. Differentially Private Empirical Risk Minimization (DP-ERM) has been widely studied over the past decade. Many algorithms have been proposed to solve DP-ERM including objective perturbation [Chaudhuri et al., 2011, Kifer et al., 2012, Iyengar et al., 2019], output perturbation [Wu et al., 2017, Zhang et al., 2017], and gradient perturbation [Bassily et al., 2014, Wang et al., 2017, Jayaraman and Wang, 2018]. While most of them focus on convex functions, we study DP-ERM with nonconvex loss functions. As most existing algorithms achieving differential privacy in ERM are based on the gradient perturbation [Bassily et al., 2014, Wang et al., 2017, 2019, Zhou et al., 2020a], we thus study gradient perturbation.

Non-constant stepsizes for SGD and DP-SGD. To ease the difficulty of stepsize tuning, we could apply polynomially decaying stepsize schedules [Ge et al., 2019] or adaptive gradient methods that update the stepsize using the gradient information [Duchi et al., 2011, McMahan and Streeter, 2010]. We called them adaptive stepsizes to distinguish our adaptive deferentially private methods. These non-private algorithms update the stepsize according to the noisy gradients, and achieve favorable convergence behavior [Levy et al., 2018, Li and Orabona, 2019, Ward et al., 2019, Reddi et al., 2021].

Empirical evidence suggests that differential privacy with adaptive stepsizes could perform almost as well as – and sometimes better than – DP-SGD with well-tuned stepsizes. This results in a significant reduction in stepsize tuning efforts and also avoids the extra privacy cost [Bu et al., 2020, Zhou et al., 2020a,b]. Several works [Lee and Kifer, 2018, Koskela and Honkela, 2020] also studied the nonuniform allocation of the privacy budget for each iteration. However, Lee and Kifer [2018] only proposes a heuristic method and the purpose of Koskela and Honkela [2020] is to avoid the need for a validation set used to tune stepsizes. In this work, we emphasize the optimal relationship between the stepsize and the variance of the random noise, and aim to improve the utility guarantee of our proposed method.

8 Conclusion and future work

In this paper, we proposed an adaptive differentially private stochastic gradient descent method in which the privacy mechanisms can be optimally adapted to the choice of stepsizes at each round, and thus obtain improved utility guarantees over prior work. Our proposed method has not only strong theoretical guarantees but also superior empirical performance. Given high-dimensional settings with only a fixed privacy budget available, our approach with a decaying stepsize schedule shows an improvement in convergence by a factor of \( d \log(T)/\sqrt{n^2} \) or a factor with \( \log(T) \) relative to DP-SGD.

Note that the sequence \( \{\alpha_t\} \) has to be fixed before the optimization process begins, as our method require that the variance \( \sigma^2 \) for some privacy budget \( \varepsilon \) depends on the \( \{\alpha_t\} \) (Theorem 4.1). However, our theorem suggests that the optimal choice of \( \alpha_t \) depends on the stepsize (Theorem 4.2), meaning that we have to know the stepsize a priori; this is not possible for those stepsizes updated on the fly, such as AdaGrad [Duchi et al., 2011] and Adam [Kingma and Ba, 2014]. Thus, one potential avenue of future work is to see whether \( \{\alpha_t\} \) can be updated on the fly in line with AdaGrad and Adam while maintaining a predefined privacy budget \( \varepsilon \).
text, the relation $\alpha_t^2 = b_t$ is not the unique setting to achieve the improved utility guarantees. A thorough investigation on $\alpha_t$ and $b_t$ with various gradient clipping values would therefore be an interesting extension. Finally, our adaptive differential privacy is applied only to a simple first-order optimization; generalizing to variance-reduced or momentum methods is another potential direction.

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A Privacy guarantees and convergence of DP-SGD

With the preliminaries given in Section 2, we will briefly summarize the analysis of privacy guarantees for the standard differentially private stochastic gradient descent (DP-SGD) described in Algorithm 1 with $\alpha_t = 1, \forall t \in [T]$. To make our algorithm more general and suitable to the practice where we select $m < n$ samples instead of selecting a single sample for each iteration Goyal et al. [2017], Wang et al. [2020], we restate the DP-SGD algorithm with $m$ random samples in Algorithm 2. This $m$ is called size of mini-batch. Denote $B_i = \{x_{i1}, \ldots, x_{im}\}$ for the $i$-th mini-batch where $B_i \cap B_j = \emptyset$ and each element in $\{i_k\}$ is chosen uniformly in $[n]$ without replacement. In our experiments, $m = 256$ and $n = 50000$ for CIFAR10 (see Section 6 for details). For the rest of this section, we focus on analysis of Algorithm 2.

Algorithm 2 DP-SGD with mini-batch size $m$

1. Input $\theta_0, b_0$ and $\eta, m < n/2$.
2. for $t = 1, \ldots, T$ do
3. prepare mini-batches $B_1, B_2, \ldots, B_{\lceil n/m \rceil}$ such that $B_i \cap B_j = \emptyset$ for $i \neq j$ and $|B_i| = m$
4. get $x_t \sim \text{Uniform}(1, ..., \lceil n/m \rceil)$ and $c_t \sim N(0, \sigma I)$
5. update $b_{t+1} = \phi_1 \left( b_t, 1/|B_{t}| \sum_{i \in B_{t}} \nabla f(\theta_t; x_i) \right)$
6. release gradient $g_t^b = \eta b_{t+1} \left( 1/|B_{t}| \sum_{i \in B_{t}} \nabla f(\theta_t; x_i) + c_t \right)$
7. update $\theta_{t+1} = \theta_t - g_t^b$
8. end for

Theorem A.1 presented below has been well studied in prior work Bassily et al. [2014], Song et al. [2013], Wang et al. [2020]. We stated here for the completeness of the paper and to clarify the constant in the expression of $\sigma^2$.

Lemma A.1 (Privacy Amplification via Sampling Kasiviswanathan et al. [2011]). Let the mechanism $M : D \rightarrow R$ be $(\epsilon, \delta)$-DP. Consider $M_q$ follows the two steps (1) sample a random $q$ fraction of $D$ (2) run $M$ on the sample. Then the mechanism $M_q$ is $(\epsilon(\frac{q}{1-q}), \delta')$-DP.

Lemma A.2 (Advanced Composition [Dwork et al., 2006a]). For all $\epsilon_0, \delta_0, \delta' > 0$, let $M = (M_1, \ldots, M_k)$ be a sequence of $(\epsilon_0, \delta_0)$-differentially private algorithms. Then, $M$ is $(\epsilon, \delta)$-differentially private, where $\epsilon = \epsilon_0 \sqrt{2k \log(1/\delta')} + k \epsilon_0 \frac{e^\delta \epsilon_0 - 1}{e^\delta 0 - 1}$ and $\delta = 1 - (1 - \delta_0)^k + \delta'$. \hspace{1cm} 6

Theorem A.1 (Privacy Guarantee for DP-SGD). Suppose the sequence $\{\alpha_t\}_{t=1}^T$ is all constant 1 and that Assumption 2.2 holds. Algorithm 2 satisfies $(\epsilon, \delta)$-DP if the random noise $c_t$ has variance

$$\sigma^2 = \frac{(16G^2)B_\delta T}{n^2 \epsilon^2} \text{ with } B_\delta = \log(16T m/\epsilon n^2) \log(1.25/\delta),$$

where $T \geq n^2 \epsilon^2 B_\delta / (8m^2 \log(1.25/\delta_0))$.

Proof. The proof can be summarized into three steps Bassily et al. [2014]:
- **Step One.** By Assumption 2.2, the gradient sensitivity of the loss function is

$$\Delta = \frac{1}{m} \sup_{x \in D, x' \in D^c} \|\nabla f(w; x) - \nabla f(w; x')\| \leq 2G/m.$$

Given the privacy budget $\epsilon$, we apply Gaussian privacy mechanism with $\sigma$ define in (13). By Lemma 2.1, the Gaussian privacy mechanism given in Definition 3.1 with $\alpha_t = 1$ satisfies that $\epsilon_0 = 2G/m \sqrt{2 \log(1.25/\delta_0)} / \sigma$ for some $\delta_0 \leq 10^{-5}$ such that $\delta_0 T m / n \ll 0.1$ Kamath [2020]. As we have $T \geq n^2 \epsilon^2 B_\delta / (32m^2 \log(1.25/\delta_0))$, the privacy

$$\epsilon^2 = 8G^2 \log(1.25/\delta_0) n^2 \epsilon^2 / (16G^2 m^2 B_\delta T) = n^2 \epsilon^2 \log(1.25/\delta_0) / (32m^2 B_\delta T) \leq 1.$$

- **Step Two.** Applying amplification by sub-sampling (i.e. Lemma A.1). We have $(\epsilon_p, \delta_p) - DP$ for each step in DP-SGD with $\epsilon_p = 2\epsilon_0 m / n \geq (e^\delta - 1)m/n$ and $\delta_p = \delta_0 m / n$, since $p = m/n < 0.5$ and $\epsilon_0 \leq 1$.

\hspace{1cm} 5The amplification by subsampling, a standard tool for SGD analysis Bassily et al. [2014], is first appear in Kasiviswanathan et al. [2011]. The proof can be also found here http://www.ccs.neu.edu/home/jullman/cs7880s17/HW1sol.pdf

\hspace{1cm} 6In Theorem III.3 Dwork et al. [2010], $\delta = k \delta_0 + \delta'$, a further simplification of $\delta = 1 - (1 - \delta_0)^k + \delta'$
• Step Three. After Step One and Two, we apply advanced (strong) composition stated in Lemma A.2 (Theorem III.3 Dwork et al. [2010] or Theorem 3.20 in Dwork et al. [2014]) for the \(T\) iterations. Then, Algorithm 1 follows (\(\varepsilon_{\text{dpsgd}}, \delta_{\text{dpsgd}}\))-DP satisfying for some \(\delta'\) such that \(\delta_0 T m/(0.25n) \leq \delta' \leq \delta_0 T m/(0.1n) \ll 1\).

\[
\delta_{\text{dpsgd}} = 1 - (1 - \delta_0 m/n) + \delta' \quad \text{(a)}
\]
\[
\varepsilon_{\text{dpsgd}} = \varepsilon_p \sqrt{2T \log(1/\delta')} + T \varepsilon_p e^{\varepsilon_p} - 1
\]
\[
\varepsilon_{\text{dpsgd}} = A_n \varepsilon_p \sqrt{2T \log(1/\delta')} + T \varepsilon_p e^{\varepsilon_p} - 1
\]

where (a) follows from the fact that \(1 - (1 - \delta_0 m/n) \leq \delta_0 m/n\). We now simplify (15)

\[
\varepsilon_{\text{dpsgd}} = \varepsilon_p \sqrt{2T \log(1/\delta')} + T \varepsilon_p e^{\varepsilon_p} - 1
\]
\[
\leq \varepsilon_p (\sqrt{2T \log(1/\delta')} + T \varepsilon_p)
\]
\[
\leq 2(2\varepsilon_p m/n) \sqrt{2T \log(1/\delta')}
\]
\[
\leq 8mG/m \sqrt{2\log(1/\delta'_{0})} / \sigma \leq 2T \log(1/\delta'),
\]

where the last inequality is from \(\varepsilon^2 \leq 16G \sqrt{T \log(1/\delta'_{0})}/B_{\sigma}^2\); (c) follows by substituting the \(\varepsilon_0 = 2G \sqrt{2T \log(1/\delta'_{0})}/\sigma\) given in Step One; (d) follows from the (14).

Now we let \(\delta_{\text{dpsgd}} = \delta\) and compare the relationship between \(\varepsilon_{\text{dpsgd}}\) and \(\varepsilon\):

\[
\varepsilon^2_{\text{dpsgd}} \leq \frac{(16G)^2 T \log(16T m/(n \delta_{\text{dpsgd}})) \log(1.25/\delta_{\text{dpsgd}})}{n^3 \sigma^2}
\]
\[
= \frac{(16G)^2 T \log(16T m/(n \delta_{\text{dpsgd}})) \log(1.25/\delta_{\text{dpsgd}})}{n^2 \varepsilon^2} \leq \frac{\varepsilon^2 \log(16T m/(n \delta_{\text{dpsgd}})) \log(1.25/\delta_{\text{dpsgd}})}{\log(16T m/(n \delta)) \log(1.25/\delta)}
\]
\[
\leq \varepsilon^2.
\]

Thus, setting the \(\sigma\) in (13) is sufficient to obtain an \((\varepsilon, \delta)\)-DP algorithm.

Proposition A.1. (DP-SGD with constant stepsizes)\(^7\) Under the conditions of Theorem 4.2 on \(f\). Set \(\sigma^2\) satisfying (13) in Theorem A.1. Let \(b_t = 1\) in Algorithm 2 and denote \(\tau = \arg \min_{i \in [T-1]} E[\|\nabla F(\theta_i)\|^2]\) and \(B_{\delta} = \log(16T m/n\delta)) \log(1.25/\delta)\). Then the gradients follow

\[
E[\|\nabla F(\theta)\|^2] \leq \frac{2D_F}{nT} + \eta L G^2 \left(1 + \frac{16^2 B_{\delta} T}{n^2 \varepsilon^2}\right).
\]

We omit the proof of the proposition as it can be found in Wang et al. [2020]. In fact, the proof is straightforward by applying Theorem C.2 and noticing that

\[
\| \frac{1}{m} \sum_{i \in B} \nabla f(w; x_i) \|^2 \leq \frac{m}{m^2} \sum_{i \in B} \|\nabla f(w; x_i)\|^2 \leq G^2.
\]

Set \(\eta = \sqrt{1/T}\), (16) becomes

\[
E[\|\nabla F(\theta)\|^2] \leq \frac{2D_F + LG^2}{\sqrt{T}} + \frac{L(16G^2) B_{\delta} \sqrt{T}}{n^2 \varepsilon^2}.
\]

\(^7\) Setting \(\sigma = 0\) for Theorem 7 in Wang et al. [2020] reduces to our bound.
Let us compare DP-SGDs between the constant stepsize $\eta/b_t = \sqrt{1/T}$ and the decaying stepsize $\eta/b_t = 1/\sqrt{a + ct}$. Suppose the second term introduced by the privacy mechanism dominates the bound. We see that the ratio of second term in the bound using the decaying stepsize (i.e., \(26\) in Proposition 5.1) to that using the constant stepsize (i.e., \(17\)) is $O(\log(T) / \sqrt{c})$. Thus if we set $\sqrt{c} = \log(T)$, the second term in both \(17\) and \(26\) have the same order.

Let us now compare between DP-SGD with the constant stepsize $\eta/b$ and the decaying stepsize $\eta/b_t = \sqrt{1/T}$ for a mini-batch described in Algorithm 3.

Let us restate Theorem 4.1 in Theorem C.1 for a mini-batch described in Algorithm 3.

### B Proof for Extended Advanced Composition Theorem

We restate Lemma 2.2 as follows.

**Lemma B.1 (Extended Advanced Composition).** Consider two sequences $\{\varepsilon_i\}_{i=1}^k, \{\delta_i\}_{i=1}^k$ of positive numbers satisfying $\varepsilon_i \in (0, 1)$ and $\delta_i \in (0, 1)$. Let $\mathcal{M}_i$ be $(\varepsilon_i, \delta_i)$-differentially private for all $i \in \{1, 2, \ldots, k\}$. Then $\mathcal{M} = (\mathcal{M}_1, \ldots, \mathcal{M}_k)$ is $(\bar{\varepsilon}, \bar{\delta})$-differentially private for $\delta' \in (0, 1)$ and

$$\bar{\varepsilon} = \sqrt{\sum_{i=1}^k 2\varepsilon_i^2 \log \left( \frac{1}{\delta_i} \right)} + \sum_{i=1}^k \varepsilon_i (e^{\varepsilon_i} - 1) \left( e^{\varepsilon_i} + 1 \right), \quad \bar{\delta} = 1 - (1 - \delta_1)(1 - \delta_2) \ldots (1 - \delta_k) + \delta'. $$

The result follows immediately from Theorem 3.5 of Kairouz et al. [2017]. Alternative proof would be using Renyi DP. This result immediately follows by invoking Lemmas 2.6 and 2.7 of Feldman et al. [2020]. Particularly, Lemma 2.7 in Feldman et al. [2020]) gives a composition rule for Renyi differential privacy, which can then be used to obtain our version of composition for $(\varepsilon, \delta)$-differential privacy. Lemma 2.6 in Feldman et al. [2020] allows translating Renyi differential privacy to $(\varepsilon, \delta)$-differential privacy.

### C Proof for Section 4

As explain in Appendix A, we will select $m \leq n/2$ samples instead of selecting a single sample for each iteration. We restate Algorithm 1 in Algorithm 3 with mini-batches $m$ variable.

**Algorithm 3 ADP-SGD with mini-batch size $m$**

1: Input: $\theta_0, b_0, \alpha_0, \eta > 0$ and $m \leq n/2$
2: for $t = 0, 1, \ldots, T - 1$ do
3: prepare mini-batches $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_{[n/m]}$ such that $\mathcal{B}_i \cap \mathcal{B}_j = \emptyset$ for $i \neq j$ and $|\mathcal{B}_i| = m$
4: get $\xi_t \sim \text{Uniform}(1, \ldots, [n/m])$ and $c_j \sim \mathcal{N}(0, \sigma I)$
5: update $b_{t+1} = \phi_1(b_t, \frac{1}{|\mathcal{B}_i|} \sum_{i \in \mathcal{B}_i} \nabla f(\theta_t; x_i))$
6: update $\alpha_{t+1} = \phi_2(\alpha_t, b_{t+1})$
7: release $g_t^b = \frac{1}{|\mathcal{B}_i|} \sum_{i \in \mathcal{B}_i} \nabla f(\theta_t; x_i) + \alpha_{t+1} c_j$
8: update $\theta_{t+1} = \theta_t - g_t^b$
9: end for

#### C.1 Proof for Theorem 4.1

Let us restate Theorem 4.1 in Theorem C.1 for a mini-batch described in Algorithm 3.
Theorem C.1 (Privacy Guarantee). Suppose the sequence \(\{\alpha_t\}_{t=1}^T\) is known in advance and that Assumption 2.2 holds. Denote \(B_\delta = \log(16Tm/\delta_0)) \log(1.25/\delta)\) as in Theorem A.1. Algorithm 3 with \(m\) satisfies \((\varepsilon, \delta)\)-DP if the random noise \(c_j\) has variance
\[
\sigma^2 = \frac{(16G)^2B_\delta}{n^2\varepsilon^2} \sum_{t=0}^{T-1} \frac{1}{\alpha_{t+1}^2},
\]  
where \(T\) is required to satisfy \(\alpha_t^2 \sum_{t=1}^T 1/\alpha_t^2 \geq n^2\varepsilon^2 B_\delta/(32m^2 \log(1.25/\delta_0))\).

Proof. Similar to the proof in Theorem A.1, we will follow three steps.

• **Step One.** At \(t\) iteration, the Gaussian privacy mechanism given in Definition 3.1 with any \(\alpha_t\) satisfies that \((\varepsilon_t, \delta_0)\)-DP where \(\varepsilon_t = 2G/m \sqrt{2 \log(1.25/\delta_0)/\alpha_t}\) for some \(\delta_0 \ll 10^{-5}\) such that \(\delta_0 T/n \ll 0.1\). Note that the privacy
\[
(\varepsilon_t)^2 = \frac{8G^2 \log(1.25/\delta_0)}{\alpha_t^2 m^2} \frac{n^2\varepsilon^2}{(16G)^2 B_\delta \sum_{t=1}^T 1/\alpha_t^2} = n^2\varepsilon^2 \log(1.25/\delta_0)/(32m^2 B_\delta \sum_{t=1}^T 1/\alpha_t^2) \leq 1,
\]
where the last inequality is due to the fact that \(\alpha_t^2 \sum_{t=1}^T 1/\alpha_t^2 \geq n^2\varepsilon^2 B_\delta/(32m^2 \log(1.25/\delta_0))\).

• **Step Two.** Applying amplification by sub-sampling (i.e. Lemma A.1), we have \((\varepsilon_p, \delta_p)\)-DP for each step \(p\) in DP-SGD with \(\varepsilon_p = 2\varepsilon_t m/n \geq (\varepsilon_t - 1)m/n\) and \(\delta_p = \delta_0 m/n\), since \(p = m/n\) and \(\varepsilon_t \leq 1\).

• **Step Three.** Using Lemma 2.2 or Lemma B.1, we have Algorithm 1 satisfying \((\varepsilon_{adpsgd}, \delta_{adpsgd})\)-DP for some \(\delta'\) such that \(\delta_0 T \log(0.25n) \leq \delta \leq \delta_0 T \log(0.1n) < 1\).
\[
\delta_{adpsgd} = 1 - (1 - \delta_0 m/n)^T = 1 - 1.25\delta' \leq 12.5 \delta_0 T \log(0.25n)/n, \tag{21}
\]
\[
\varepsilon_{adpsgd} = \sqrt{T \sum_{t=1}^T 2(\varepsilon_p^t)^2 \log \left( \frac{1}{\delta'} \right) + \sum_{t=1}^T \varepsilon_p^t (\varepsilon_p'^t - 1)/\alpha_t} \leq 2 \sqrt{2 \sum_{t=1}^T (\varepsilon_p^t)^2 \log \left( \frac{1}{\delta'} \right)} \tag{22}
\]
where \((a)\) is due to the fact that \(\sum_{t=1}^T \varepsilon_p^t (\varepsilon_p'^t - 1)/(\varepsilon_p'^t + 1)\) is considerably smaller than \(\sqrt{T \sum_{t=1}^T (\varepsilon_p^t)^2 \log \left( \frac{1}{\delta'} \right)}\). Indeed, \[
\sum_{t=1}^T \varepsilon_p^t (\varepsilon_p'^t - 1)/(\varepsilon_p'^t + 1) \leq \sum_{t=1}^T (\varepsilon_p^t)^2 = \frac{4m^2}{n^2} \sum_{t=1}^T \varepsilon_p^t = \frac{32G^2 \log(1.25/\delta_0)}{n^2\varepsilon^2} \sum_{t=1}^T \frac{1}{\alpha_t^2} = \frac{\log(1.25/\delta_0)\varepsilon^2}{8B_\delta},
\]
\[
\sqrt{T \sum_{t=1}^T (\varepsilon_p^t)^2 \log \left( \frac{1}{\delta'} \right)} = \sqrt{\frac{\log(1.25/\delta_0)\varepsilon^2}{8B_\delta} \log \left( \frac{1}{\delta'} \right) \frac{(a)}{\log(1/\delta')/B_\delta} \geq \sum_{t=1}^T \frac{\varepsilon_p^t (\varepsilon_p'^t - 1)}{(\varepsilon_p'^t + 1)},
\]
where \((a)\) is due to the fact that \(\log(1.25/\delta_0)\varepsilon^2/\log(1/\delta')/B_\delta < 1\). Let \(\delta_{adpsgd} = \delta\). We now further simply \(\varepsilon_{adpsgd}\) as follows
\[
\varepsilon_{adpsgd}^2 \leq 8 \log(1/\delta') \sum_{t=1}^T (\varepsilon_p^t)^2 = \frac{\log(1/\delta') \log(1.25/\delta_0)\varepsilon^2}{B_\delta} \leq \varepsilon^2,
\]
where the last step \((a)\) is due to (21).
\[\square\]

C.2 Proof for Theorem 4.2

We restate Theorem C.2 with the following theorem for a mini-batch described in Algorithm 3.

**Theorem C.2 (Convergence for ADP-SGD).** Suppose we choose \(\sigma^2\) - the variance of the random noise in Algorithm 3 - according to (18) in Theorem C.1. Suppose Assumption 2.1, 2.2 and 3.1 hold. Furthermore, suppose \(\alpha_t, b_t\) are deterministic. The utility guarantee of Algorithm 3 with \(T \equiv \arg\min_{k \in [T-1]} E[\|\nabla F(\theta_k)\|^2]\) and \(B_\delta = (\log(16Tm/\delta_0)) \log(1.25/\delta)\) is
\[
E[\|\nabla F(\theta_T)\|^2] \leq \frac{1}{\sum_{t=0}^{T-1} \frac{1}{b_{t+1}}} \left( \frac{D_F}{\eta} + \frac{\eta L}{2} \sum_{t=0}^{T-1} \mathbb{E} \left[ \frac{\|g_{\xi_t}\|^2}{b_{t+1}} \right] + \frac{d(16G)^2B_\delta}{2n^2\varepsilon^2} M \right),
\]  
(23)
where
\[ g_t = \frac{1}{|B_t|} \sum_{i \in B_t} \nabla f(\theta_i; x_i) \] and \( M(\{\alpha_t\}, \{b_t\}) \triangleq \sum_{t=0}^{T-1} (\alpha_{t+1}/b_{t+1})^2 \sum_{t=1}^{T-1} 1/\alpha_t^2. \)

**Proof.** Recall the update in Algorithm 1:
\[ \theta_{t+1} = \theta_t - \frac{\eta}{b_{t+1}} g_{\xi_t} - \frac{\alpha_{t+1}}{b_{t+1}} c_j. \]

By Lipschitz gradient smoothness (c.f. Lemma E.1):
\[ F(\theta_{j+1}) \leq F(\theta_j) - \eta \langle \nabla F(\theta_j), \frac{1}{b_{j+1}} g_{\xi_j} - \frac{\alpha_{j+1}}{b_{j+1}} c_j \rangle + \frac{\eta^2 L}{2} \| g_{\xi_j} \| \| \frac{\alpha_{j+1}}{b_{j+1}} c_j \|, \]
\[ \mathbb{E}[F(\theta_{j+1})] \leq \mathbb{E}[F(\theta_j)] + \mathbb{E} \left[ \frac{-\eta}{b_{j+1}} \langle \nabla F(\theta_j), g_{\xi_j} \rangle + \frac{\eta^2 L}{2b_{j+1}^2} \| g_{\xi_j} \|^2 + \frac{\eta^2 dL^2}{2} \frac{\alpha_{j+1}^2}{b_{j+1}^2} \right]. \]

So we have by telescoping from \( j = 0 \) to \( j = T - 1 \)
\[ \mathbb{E}[F(\theta_{T-1})] \leq \mathbb{E}[F(\theta_0)] + \sum_{j=0}^{T-1} \mathbb{E} \left[ \frac{-\eta}{b_{j+1}} \langle \nabla F(\theta_j), g_{\xi_j} \rangle + \frac{\eta^2 L}{2b_{j+1}^2} \| g_{\xi_j} \|^2 + \frac{\eta^2 dL^2}{2} \frac{\alpha_{j+1}^2}{b_{j+1}^2} \right]. \]

Moving the term \( \sum_{j=0}^{T-1} \mathbb{E} \left[ \frac{-\eta}{b_{j+1}} \langle \nabla F(\theta_j), g_{\xi_j} \rangle \right] \) to the left hand side gives
\[ \sum_{j=0}^{T-1} \mathbb{E} \left[ \frac{\langle \nabla F(\theta_j), g_{\xi_j} \rangle}{b_{j+1}} \right] \leq \frac{\alpha_{T+1}}{b_{T+1}} \sum_{j=0}^{T-1} \mathbb{E} \left[ \| \nabla F(\theta_j) \|/b_{j+1} \right] + \frac{\eta dL(16G)^2 B_{d, M}}{2n^2 \xi^2}, \]
where (a) follows by substituting \( \sigma \) with (18) and denoting \( M = \sum_{t=0}^{T-1} \frac{\alpha_{t+1}^2}{b_{t+1}^2} \sum_{t=0}^{T-1} \frac{1}{\alpha_t^2}. \) We finish the proof by simplifying the left hand side in above inequality as follows
\[ \sum_{j=0}^{T-1} \mathbb{E} \left[ \frac{\langle \nabla F(\theta_j), g_{\xi_j} \rangle}{b_{j+1}} \right] \geq \sum_{j=0}^{T-1} \mathbb{E} \left[ \| \nabla F(\theta_j) \| \right] \sum_{j=0}^{T-1} \frac{1}{b_{j+1}}. \tag{24} \]

Now let us take a look at the Remark 4.1. For \( M \), note that
\[ M = \sum_{t=0}^{T-1} \frac{\alpha_{t+1}^2}{b_{t+1}^2} \sum_{t=0}^{T-1} \frac{1}{\alpha_t^2} \geq \left( \sum_{t=0}^{T-1} \frac{1}{\alpha_{t+1}} \right)^2 = \left( \sum_{t=0}^{T-1} \frac{1}{\alpha_{t+1}} \right)^2 = M_{\text{adp}}, \]
where we apply the fact that \( \| u \|^2 \| v \|^2 \geq |\langle u, v \rangle|^2 \) and the equality holds when \( \frac{\alpha_{t+1}}{b_{t+1}} = \frac{1}{\alpha_{t+1}}. \) So the optimal relationship for \( t = 1, 2, \ldots, T \) is \( \alpha_t^2 = b_t. \) On the other hand, observe that
\[ M = \sum_{t=0}^{T-1} \frac{\alpha_{t+1}^2}{b_{t+1}^2} \sum_{t=0}^{T-1} \frac{1}{\alpha_t^2 - t} \geq \left( \sum_{t=0}^{T-1} \frac{1}{\alpha_{t+1}} \right)^2 = \left( \sum_{t=0}^{T-1} \frac{1}{b_{t+1}} \right)^2 = M_{\text{adp}}, \]
where the equality in (a) holds if \( \alpha_t \alpha_{T-(t-1)} = b_t. \)
D Proofs for Section 5

For this section, we restate the Proposition 5.1 with Proposition D.1 for a more general setup – Algorithm 3 with a mini-batch of size \( m \).

Proposition D.1 (ADP-SGD v.s. DP-SGD with a polynomially decaying stepsize schedule). Under the conditions of Theorem C.2 on \( f \) and \( a^2 \), let \( b_t = (a + ct)^{1/2} \) in Algorithm 3, where \( a > 0, c > 0 \). Denote \( \tau = \arg \min_{t \in [T - 1]} \| \nabla F(\theta_t) \|_2^2 \), \( B_\delta = \log(16Tm/\delta^2) \log(1/\delta) \) and \( B_T = \log(1 + Tc/a) \). If we choose \( T \geq 5 + 4a/c \), and \( \alpha_t^2 = b_t \), we have the following utility guarantee for ADP-SGD

\[
(ADP-SGD) \quad \mathbb{E}[\| \nabla F(\theta_{\tau}) \|^2] \leq \frac{\sqrt{c} (D_F + \eta G^2 L \tau) + \eta dL(16G)^2 B_\delta \sqrt{T}}{\sqrt{T-1} \tau} + \frac{\eta dL(16G)^2 B_T}{\eta c^2 T^{1/2}}. \tag{25}
\]

In addition, if we choose \( T \geq 5 + 4a/c \) and \( \alpha_t = 1 \), we have the utility guarantee for DP-SGD:

\[
(DP-SGD) \quad \mathbb{E}[\| \nabla F(\theta_{\tau}) \|^2] \leq \frac{\sqrt{c} (D_F + \eta G^2 L \tau) + \eta dL(16G)^2 B_\delta \sqrt{T}}{\sqrt{T-1} \tau} + \frac{\eta dL(16G)^2 B_T (\sqrt{T-1} + 1)}{2n^2 \eta c^2}. \tag{26}
\]

Proof. The proof will be divided into two parts: Appendix D.1 is for ADP-SGD and Appendix D.2 is for DP-SGD.

D.1 Proof for Proposition D.1 – ADP-SGD with \( b_t = \sqrt{a + ct} \)

Note from Lemma E.2 we have

\[
\frac{2}{\sqrt{c}} \left( \sqrt{\frac{a}{c} + T} - \sqrt{\frac{a}{c} + 1} \right) \leq \sum_{j=0}^{T-1} \frac{1}{b_{j+1}^2} = \sum_{j=1}^{T} \frac{1}{\sqrt{a + ct} + c} \leq \frac{2}{\sqrt{c}} \left( \sqrt{\frac{a}{c} + T} - \sqrt{\frac{a}{c}} \right). \tag{27}
\]

Set \( \tilde{B}_1 = \mathbb{E} \left[ \sum_{t=0}^{T-1} \frac{\| g_t \|^2}{b_{t+1}^2} \right] \). Continue with the bound (23) of Theorem C.2 with \( \alpha_t^2 = b_t \)

\[
\mathbb{E}[\| \nabla F(\theta_{\tau}) \|^2] \leq \frac{L}{b_{T+1}} \left( \frac{D_F}{\eta L} + \frac{\eta T - 1}{2} \sum_{t=0}^{T-1} \mathbb{E}[\| g_t \|^2] \right) + \frac{\eta dL(16G)^2 B_\delta}{2n^2 \eta c^2} \left( \sum_{t=0}^{T-1} \frac{1}{b_{t+1}^2} \right) \leq \frac{D_F}{\eta L} + \frac{\eta L \tilde{B}_1 / 2}{2(T - 1)} + \frac{\eta dL(16G)^2 B_\delta}{2n^2 \eta c^2} \left( \sqrt{\frac{a}{c} + T} - \sqrt{\frac{a}{c}} \right),
\]

where (a) is by replacing \( \alpha_t^2 = b_t \) in \( M \); (b) follows by the fact that

\[
\frac{2}{\sqrt{c}} \left( \sqrt{\frac{a}{c} + T} - \sqrt{\frac{a}{c} + 1} \right) = \frac{2(T - 1)}{\sqrt{a + ct} + \sqrt{a + c}} \quad \text{and} \quad \sqrt{\frac{a}{c} + T} - \sqrt{\frac{a}{c}} \leq \sqrt{T},
\]

(c) is true due to the fact that \( \sqrt{a + Tc + \frac{a^2}{c} + \frac{a^2}{c^2}} \leq 2\sqrt{\frac{a^2}{c^2} + \frac{a}{c}} \leq 2\sqrt{T}^{1/2} \) as \( T \geq 5 + 4a/c \); (6) with \( B_T = \log(1 + Tc/a) \) is due to

\[
\tilde{B}_1 = \mathbb{E} \left[ \sum_{t=0}^{T-1} \frac{\| g_t \|^2}{a + c(t+1)} \right] \leq \sum_{t=0}^{T-1} \frac{G^2}{a + c(t+1)} \leq \frac{G^2}{c} \log \left( 1 + \frac{cT}{a} \right) = G^2 B_T/c,
\]

where we use Lemma E.2 with \( p = 1 \).
D.2 Proof for Proposition D.1 – DP-SGD with $b_t = \sqrt{a + ct}$

Applying the fact in (24), (27), and that $\|g_\xi\|^2 \leq G^2$, the bound (23) of Theorem C.2 with $\alpha_t^2 = 1$ reduces to

$$
\min_{k \in \{T-1\}} \mathbb{E}[\|\nabla F(\theta_k)\|^2] \leq \left(\frac{\sqrt{a + ct} + \sqrt{a + c}}{2(T - 1)} \left(\frac{D_F}{\eta} + \frac{\eta LG^2 B_T}{2c} + \frac{\eta dL(16G)^2T}{2cn^2\varepsilon^2} B_B B_T \right) \right)
\leq \left(\frac{\sqrt{a + ct} + \sqrt{a + c}}{T - 1} \left(\frac{D_F}{\eta} + \frac{\eta LG^2 B_T}{2c} \right) + \frac{\eta dL(16G)^2}{2n^2\varepsilon^2 \sqrt{c}} B_B B_T \frac{T}{\sqrt{T - 1}},
\right)
$$

where (a) is by Lemma E.2 (see (24)); (b) follows by substituting $\sigma$ and setting $B_T = \log (1 + T \frac{\varepsilon}{a})$; (c) is true due to $\frac{\sqrt{a + ct} + \sqrt{a + c}}{T - 1} \leq 2\frac{\sqrt{a + c}}{T - 1} + \frac{\sqrt{a + c}}{\sqrt{T - 1}} \leq \frac{2\sqrt{a + c}}{\sqrt{T - 1}}$ as $T \geq 5 + 4 \varepsilon$. \hfill \square

D.3 Convergence for an adaptive stepsize schedule

**Theorem D.1 (Convergence for an adaptive stepsize schedule).** Under the conditions of Theorem C.2 on $f$ and $\sigma^2$, let $b_{t+1}^2 = b_t^2 + \max \left\{ \frac{1}{(\delta L)^2} \sum_{j \in E_j} \|\nabla f(\theta; x_i)\|^2, \nu \right\}, \nu \in (0, G]$ in Algorithm 3. Denote $B_\delta = \log(16 T m / \nu d) \log(1.25 / \delta)$ and $\tau = \arg \min_{\xi \in \{T-1\}} \mathbb{E}[\|\nabla F(\theta_\xi)\|^2]$. If $T \geq 5 + 4b_0^2/G^2$, then the utility guarantee follows

$$
\mathbb{E}[\|\nabla F(\theta_\tau)\|^2] \leq \frac{2G}{\sqrt{T - 1}} \left( B_{\text{sgd}} + \frac{\eta dL(16G)^2 B_B \mathbb{E}[M]}{2n^2\varepsilon^2} \right),
$$

where $B_{\text{sgd}} = \frac{D_F^2}{\eta} + \left(2G + \frac{\eta L}{2}\right) \left(1 + \log \left(\frac{T(G^2 + \nu^2)}{b_0^2} + 1\right)\right)$ and $M = \sum_{t=0}^{T-1} \frac{\alpha_{t+1}}{b_{t+1}^2} + \sum_{t=1}^{T-1} \frac{1}{\alpha_{t+1}}$.

When $\alpha_t = 0$, Proposition 5.1 (result in (25) with $M = 0$) and Theorem D.1 (with $M = 0$) corresponds to the standard SGD algorithms with decaying and adaptive stepsizes, respectively. In particular, if we set $a = b_0^2$, $c = G^2$ and $\alpha_t = 0$ in Proposition 5.1, then the result in Proposition 5.1 becomes $G (D_F/\eta + \eta G^2 B_T/2c) / \sqrt{T - 1} \leq Q_{\text{decay}}$, while the result in Theorem D.1 is $2G B_{\text{sgd}} / \sqrt{T - 1} \leq Q_{\text{adapt}}$. We see that for a sufficiently large $G > L$, the advantage of using this variant of adaptive stepizes is that $Q_{\text{adapt}} = O(G^3 \log(T)) / \sqrt{T}$ is smaller than $Q_{\text{decay}} = O(G^3 \log(T)) / \sqrt{T}$ by an order of $G$. Note that the proof follows closely with Ward et al. [2019].

**Proof.** Write $F_j = F(\theta_j)$ and $g_{\xi_j} = \frac{1}{|E_j|} \sum_{x_i \in E_j} \nabla f(\theta; x_i)$ . In addition, we write $\mathbb{E}_{\cdot}[]$ means taking expectation with respect to the randomness of $\xi_j$ and $c_j$ conditional on $\{\xi_i\}_{t=0}^{t-1}$ and $\{c_i\}_{t=0}^{t-1}$; $\mathbb{E}_{c_j}[]$ means taking expectation with respect to the randomness of $c_j$ conditional on $\{\xi_i\}_{t=0}^{t-1}$ and $\{c_i\}_{t=0}^{t-1}$; $\mathbb{E}_{\xi_j}[]$ means taking expectation with respect to the randomness of $\xi_j$ conditional on $\{\xi_i\}_{t=0}^{t-1}$ and $\{c_i\}_{t=0}^{t-1}$. Note that since $c_j$ and $\xi_j$ is independent, thus we have $\mathbb{E}_{\cdot}[] = \mathbb{E}_{c_j}[] \mathbb{E}_{\xi_j}[]$ .

By Decent Lemma E.1,

$$
F_{j+1} \leq F_j - \frac{\eta}{b_{j+1}} (\nabla F_j, g_{\xi_j} + \alpha_{j+1} c_j) + \frac{\eta^2 L}{2b_{j+1}^2} \|g_{\xi_j} + \alpha_{j+1} c_j\|^2
= F_j - \frac{\eta}{b_{j+1}^2} \|\nabla F_j\|^2 + \frac{\eta}{b_{j+1}} (\nabla F_j, \nabla F_j - g_{\xi_j}) - \frac{\eta \alpha_{j+1}}{b_{j+1}} (\nabla F_j, c_j)
+ \frac{\eta^2 L \alpha_{j+1}}{b_{j+1}^2} (g_{\xi_j}, c_j) + \frac{\eta^2 L \alpha_{j+1}^2}{2b_{j+1}^2} \|c_j\|^2 + \frac{\eta^2 L}{2b_{j+1}^2} \|g_{\xi_j}\|^2.
$$

Observe that taking expectation with respect to $c_j$, conditional on $\xi_1, \ldots, \xi_{j-1}, \xi_j$ gives

$$
\mathbb{E}_{c_j} [\|\nabla g_{\xi_j}, c_j\|] = 0 \quad \mathbb{E}_{c_j} [\langle \nabla F_j, c_j \rangle] = 0 \text{ and } \mathbb{E}_{c_j} [\|c_j\|^2] = \delta \alpha^2.
$$

Thus, we have

$$
\mathbb{E}_{c_j} [F_{j+1}] \leq F_j - \frac{\eta}{b_{j+1}} \|\nabla F_j\|^2 + \frac{\eta}{b_{j+1}} (\nabla F_j, \nabla F_j - g_{\xi_j}) + \frac{\eta^2 L}{2b_{j+1}^2} (\|g_{\xi_j}\|^2 + \alpha_{j+1}^2 \delta \alpha^2).
$$

23
Note that taking expectation of $\frac{1}{b_j+G}\langle \nabla F_j, \nabla F_j - g_{\xi_j}\rangle$ with respect to $\xi_j$ conditional on $\xi_1, c_1, \ldots, \xi_{j-1}, c_{j-1}$ gives

$$E_{\xi_j}\left[ \frac{1}{b_j+G}\langle \nabla F_j, \nabla F_j - g_{\xi_j}\rangle \right] = \frac{1}{b_j+G} E_{\xi_j}\left[ \langle \nabla F_j, \nabla F_j - g_{\xi_j}\rangle \right] = 0. \quad (31)$$

Applying above inequalities back to the inequality (30) becomes

$$E_j\left[ \frac{F_{j+1}}{\eta} \right] \leq \frac{F_j}{\eta} - \frac{\| \nabla F_j \|^2}{b_j+G} + \eta L \left[ \frac{\| g_{\xi_j} \|^2 + \alpha \sigma^2}{b_{j+1}} \right]. \quad (32)$$

Observe the identity

$$\frac{1}{b_j+G} - \frac{1}{b_{j+1}} = \frac{\max \{ \| g_{\xi_j} \|^2, \nu \}}{b_{j+1}(b_j+G)(b_j+b_{j+1})} - \frac{G}{b_{j+1}(b_j+G)};$$

thus, applying Cauchy-Schwarz,

$$\left( \frac{1}{b_j+G} - \frac{1}{b_{j+1}} \right) \langle \nabla F_j, g_{\xi_j} \rangle = \left( \frac{\max \{ \| g_{\xi_j} \|^2, \nu \}}{b_{j+1}(b_j+G)(b_j+b_{j+1})} \right) \langle \nabla F_j, g_{\xi_j} \rangle \leq \max \{ \| g_{\xi_j} \|^2, \nu \} \| \nabla F_j \| \leq \left( \frac{\max \{ \| g_{\xi_j} \|^2, \nu \}}{b_{j+1}(b_j+G)(b_j+b_{j+1})} \right) \langle \nabla F_j, g_{\xi_j} \rangle \leq \frac{\| \nabla F_j \|^2}{b_{j+1}^2} \leq \frac{G^2}{b_j+G} \frac{\max \{ \| g_{\xi_j} \|^2, \nu \}}{b_{j+1}^2} + \frac{\nu \| \nabla F_j \|^2}{4(b_j+G)^2}. \quad (33)$$

By applying the inequality $\sqrt{ab} \leq \frac{a^2}{2} + \frac{1}{2a} b^2$ with $\lambda = \frac{\nu}{b_j+G}$, $a = \frac{\| g_{\xi_j} \|^2}{b_{j+1}^2}$, and $b = \frac{\| \nabla F_j \|^2}{(b_j+G)^2}$, the first term in (33) can be bounded as

$$E_{\xi_j} \frac{\| \nabla F_j \| \| g_{\xi_j} \|}{b_{j+1}(b_j+G)} \leq G \frac{\| \nabla F_j \| \| g_{\xi_j} \|}{b_{j+1}(b_j+G)} \leq G \frac{\| \nabla F_j \| \| g_{\xi_j} \|}{b_{j+1}(b_j+G)}.$$

Similarly, applying the inequality $\sqrt{ab} \leq \frac{a^2}{2} + \frac{1}{2a} b^2$ with $\lambda = \frac{\nu}{b_j+G}$, $a = \frac{\| g_{\xi_j} \|^2}{b_{j+1}^2}$, and $b = \frac{\| \nabla F_j \|^2}{(b_j+G)^2}$, the second term of the right hand side in equation (33) is bounded by

$$E_{\xi_j} \frac{\| \nabla F_j \| \| g_{\xi_j} \|}{b_{j+1}^2} \leq G \frac{\| \nabla F_j \| \| g_{\xi_j} \|}{b_{j+1}^2} \leq G \frac{\| \nabla F_j \| \| g_{\xi_j} \|}{b_{j+1}^2}.$$

Thus, we have

$$E_{\xi_j} \left[ \left( \frac{1}{b_j} - \frac{1}{b_{j+1} + G} \right) \langle \nabla F_j, g_{\xi_j} \rangle \right] \leq 2\frac{\| \nabla F_j \| \| g_{\xi_j} \|}{b_{j+1}^2} \leq 2G \frac{\| \nabla F_j \| \| g_{\xi_j} \|}{b_{j+1}^2}.$$

and, therefore, back to (32),

$$E_{\xi_j} [F_{j+1}] \leq F_j - \frac{\eta \| \nabla F_j \|^2}{b_j + G} + 2\eta G E_{\xi_j} \left[ \frac{\| g_{\xi_j} \|^2, \nu}{b_{j+1}^2} \right] \leq F_j - \frac{\eta \| \nabla F_j \|^2}{b_j + G} + 2\eta G E_{\xi_j} \left[ \frac{\| g_{\xi_j} \|^2, \nu}{b_{j+1}^2} \right].$$
We divided above inequality by $\eta$ and then move the term $\frac{\|\nabla F_j\|^2}{2(b_j + G)}$ to the left hand side:

$$\frac{\|\nabla F_j\|^2}{2(b_j + G)} \leq \frac{F_j - E_{\xi_j} F_{j+1}}{\eta} + (2G + \frac{\eta L}{2})E_{\xi_j} \left[ \max \left\{ \frac{\|g_{\xi_j}\|^2}{b_{j+1}^2}, \nu \right\} \right] + E_{\xi_j} \left[ \frac{\eta d L \alpha_{j+1}^2 \sigma^2}{b_{j+1}^2} \right].$$

Applying the law of total expectation, we take the expectation of each side with respect to $z_{j-1}, \xi_j-1, z_j-2, \xi_j-2, \ldots$, and arrive at the recursion

$$E \left[ \frac{\|\nabla F_j\|^2}{2(b_j + G)} \right] \leq \frac{E[F_j] - E[F_{j+1}]}{\eta} + (2G + \frac{\eta L}{2})E \sum_{k=0}^{T-1} \left[ \max \left\{ \frac{\|\nabla f_{\xi_k}\|^2, \nu \right\} \right] b_{k+1}^2 + \eta d L E \sum_{k=0}^{T-1} \frac{\alpha_{k+1}^2 \sigma^2}{b_{k+1}^2}. \quad (36)$$

For the second term of right hand side in inequality 36, we apply Lemma E.3 and then Jensen’s inequality to bound the final summation:

$$E \sum_{k=0}^{T-1} \left[ \max \left\{ \frac{\|\nabla f_{\xi_k}\|^2, \nu \right\} \right] b_{k+1}^2 \leq E \left[ 1 + \log \left( 1 + \sum_{k=0}^{T-1} \max \left\{ \frac{\|\nabla f_{\xi_k}\|^2, \nu \right\} / b_0^2 \right) \right] \leq 1 + \log \left( \frac{T(G^2 + \nu)}{b_0^2} + 1 \right) \leq D_1 \quad (37)$$

As for term of left hand side in equation (36), we obtain

$$E \left[ \frac{\|\nabla F_k\|^2}{2(b_k + G)} \right] \geq \frac{E\|\nabla F_k\|^2}{2 \sqrt{b_0^2 + (k+1)G^2}} \quad (38)$$

since we have $b_k = \sqrt{b_0^2 + \sum_{t=0}^{k-1} \max \{\|\nabla f_{\xi_t}\|^2, \nu \}} \leq \sqrt{b_0^2 + kG^2}$ since $\nu \leq G^2$.

Thus (36) arrives at the inequality

$$\min_{0 \leq k \leq T-1} E[\|\nabla F_k\|^2] \leq \sum_{k=1}^{T} \frac{1}{2 \sqrt{b_0^2 + kG^2} \sqrt{b_0^2 + (k+1)G^2}} \leq \frac{F_0 - F^*}{\eta} + (2G + \frac{\eta L}{2})D_1 + \eta d L \sum_{k=0}^{T-1} \frac{\alpha_{k+1}^2 \sigma^2}{2b_{k+1}^2}. \quad (39)$$

Divided by $\sum_{k=1}^{T} \frac{1}{2 \sqrt{b_0^2 + kG^2}}$ and replaced $\sigma$ with

$$\sigma^2 = \frac{(16G^2 \beta_3 \sqrt{2} n^2 \varepsilon^2)}{\sum_{t=0}^{T-1} \alpha_{t+1}^2 \sigma_{t+1}^2}. \quad (40)$$

the above inequality (39), for $T \geq T_0 = \frac{4b_0^2}{G^2}$, results in

$$\min_{E} E\|\nabla f_{\xi_t}\|^2 \leq \sum_{k=1}^{T} \frac{1}{2 \sqrt{b_0^2 + kG^2}} \left( B_{sgd} + \frac{\eta d L (16G^2 \beta_3 \sqrt{2} n^2 \varepsilon^2)}{2n^2 \varepsilon^2} \sum_{j=0}^{T-1} \frac{\alpha_{j+1}^2 \sum_{t=0}^{T-1} \frac{1}{\alpha_{t+1}^2}}{b_{j+1}^2 \sum_{t=0}^{T-1} \frac{1}{\alpha_{t+1}^2}} \right). \quad (41)$$

Observe that

$$\sum_{k=1}^{T} \frac{1}{2 \sqrt{b_0^2 + kG^2}} \geq \frac{1}{G} \left( \sqrt{T + (b_0/G)^2} - \sqrt{1 + (b_0/G)^2} \right) = \frac{T - 1}{\sqrt{TG^2 + b_0^2} \sqrt{b_0^2 + G^2}} \quad (42)$$

and the fact that

$$\sqrt{T G^2 + b_0^2} + \sqrt{G^2 + b_0^2} \leq \frac{2 \sqrt{G^2 + b_0^2} + G}{T - 1} \leq \frac{2G}{\sqrt{T - 1}} \leq \frac{2G}{\sqrt{T - 4}} \quad \text{for } T \geq 5 + \frac{4b_0^2}{G^2}. \quad (43)$$

Thus, applying (42) and (43) to (41) finishes the proof. \qed
D.4 Proof for Proposition D.1

We restate Proposition 5.2 with Proposition D.2 for a more general algorithm – Algorithm 3.

**Proposition D.2 (ADP v.s. DP with an adaptive stepsize schedule).** Under the same conditions of Theorem D.1 on \( f, \sigma^2, \) and \( b_t, \) if \( \alpha_t = (b_0^2 + tC)^{1/4} \) for some \( C \in [\nu, G^2], \) then

\[
(EPSILON) \quad E\|\nabla F(\theta^D_p)\|^2 \leq \frac{2GB_{gsp} + 4G(16G)^2\eta dL\log^2(1.25/\delta)(\sqrt{T}-1+1)}{n^2\varepsilon^2\nu}.
\]

In addition, if \( \alpha_t = 1, \) then

\[
(DP-SGD) \quad E\|\nabla F(\theta^D_p)\|^2 \leq \frac{2GB_{gsp} + G(16G)^2\eta dL\log^2(1.25/\delta)(\sqrt{T}-1+1)\log (1 + T\frac{\varepsilon}{\nu})}{n^2\varepsilon^2\nu}.
\]

**Proof.** Starting with \( M \) in Theorem D.1 with \( \alpha_t^2 = \sqrt{b_0^2 + tC}, \) we have

\[
T \sum_{t=0}^{T-1} \frac{1}{\alpha_{t+1}^2} \leq T \sum_{t=0}^{T-1} \frac{1}{\sqrt{b_0^2 + jC}} \leq \frac{2\sqrt{T}}{\nu} \sum_{j=1}^{\nu T} \frac{1}{\sqrt{b_0^2 + j
u}} \leq \frac{4T \log \left( \frac{T}{\nu} \right)}{\nu}.
\]

Thus, the bound in Theorem D.1 reduces to

\[
\min_{t \in [T-1]} E\|\nabla F_t\|^2 \leq \frac{2G}{\sqrt{T}-1} \left( B_{gsp} + \frac{2(16G)^2\eta dLb_0^2T}{n^2\varepsilon^2\nu} \right).
\]

As for \( \alpha_t^2 = 1, \) we have

\[
M = \sum_{j=0}^{T-1} \frac{1}{b_{j+1}^2} \leq T \sum_{j=0}^{T-1} \frac{1}{b_{j+1}^2} \leq T \sum_{k=0}^{T-1} \frac{1}{b_0^2 + k\nu} \leq T \left( \frac{1}{\nu} \log \left( 1 + T\nu/b_0^2 \right) \right).
\]

Applying the above inequality for \( M \) reduces to the bound for \( \alpha_t^2 = 1. \)

\[ \square \]

E Technical Lemma

**Lemma E.1 (Descent Lemma).** Let \( F \in C^1_{\nu}. \) Then,

\[ F(x) \leq F(y) + \langle \nabla F(y), x - y \rangle + \frac{L}{2} \|x - y\|^2. \]

**Lemma E.2 (Summation with power \( p \)).** For any positive number \( a_1 \) and \( a_2 \)

\[
\sum_{t=1}^{T} \frac{1}{(a_1 + a_2 t)^p} \leq \begin{cases} \frac{1}{(1-p)a_2^p} \left( (a_1/a_2 + T)^{1-p} - (a_1/a_2)^{1-p} \right) & p < 1 \\ \frac{1}{a_2} \log(1 + T a_2/a_1) & p = 1 \end{cases}
\]

\[
\sum_{t=1}^{T} \frac{1}{(a_1 + a_2 t)^p} \geq \begin{cases} \frac{1}{(1-p)a_2^p} \left( (a_1/a_2 + 1 + T)^{1-p} - (a_1/a_2 + 1)^{1-p} \right) & p < 1 \\ \frac{1}{a_2} \log(1 + T/(a_1/a_2 + 1)) & p = 1 \end{cases}
\]

**Lemma E.3.** For any non-negative \( a_1, \cdots, a_T, \) such that \( a_1 > 1, \)

\[
\frac{T}{\sum_{t=1}^{T} a_t} \leq \log \left( \sum_{t=1}^{T} a_t \right) + 1.
\]
F Additional Experiments

Table 4: Mean accuracy of ASD-GP/DP-SGD with polynomially decaying stepsizes \( \eta_t = 0.1 + \alpha_T t \) where \( \alpha_T \) is the ratio depending on the final epochs/iterations \( T \) such that the learning rate at \( T \) is \( \eta_T = 10^{-10} \) (see the orange curves in Figure 2). This table reports accuracy for CIFAR10 with the mean and the corresponding standard deviation over \( \{acc_{last}^{i}\}_{i=1}^{5} \). Here, \( acc_{last}^{i} \) is the accuracy at the final iteration for the \( i \)-th independent experiment. Each set \( \{acc_{last}^{i}\}_{i=1}^{5} \) corresponds to a pair of \( (\varepsilon, C_G, T, Alg) \). The difference ("Gap") between DP and ADP is provided for visualization purpose. However, we ignore those differences ("Gap") between DP and ADP when one has an accuracy of less than 15\%. The results suggest that the more iterations or epochs we use, the more improvements ASD-GP can potentially gain over DP-SGD. The results are reported in percentage (%). The bolded number is the best accuracy in a row among epoch 60, 120 and 200 for the same gradient clipping \( C_G \). See paragraph Datasets and models and Performance Measurement for details.

| \( \varepsilon \) | Alg     | Gradient clipping \( C_G = 0.5 \) | Gradient clipping \( C_G = 1 \) | Gradient clipping \( C_G = 2.5 \) |
|--------------|--------|-----------------|-----------------|-----------------|
| \( \varepsilon \) | | epoch=60 | epoch=120 | epoch=200 | epoch=60 | epoch=120 | epoch=200 | epoch=60 | epoch=120 | epoch=200 |
| 0.8 | ADSGD | 56.44 ± 0.397 | 47.46 ± 0.431 | 26.25 ± 1.850 | 47.34 ± 0.887 | 21.34 ± 2.404 | 11.48 ± 2.767 | 13.22 ± 1.922 | 10.08 ± 0.249 | 10.15 ± 0.113 |
|     | DPSGD | 55.99 ± 0.647 | 44.05 ± 0.194 | 22.56 ± 3.488 | 43.99 ± 1.345 | 10.04 ± 0.021 | 10.04 ± 0.047 | 10.11 ± 0.054 | 10.47 ± 0.940 | 10.37 ± 0.704 |
|     | Gap   | 0.45   | 3.41   | 3.09   | 3.35   | 11.3   | N/A   | N/A   | N/A   | N/A   |
| 1.2 | ADSGD | 59.52 ± 0.390 | 58.23 ± 1.086 | 48.21 ± 0.765 | 57.85 ± 1.180 | 41.38 ± 0.920 | 22.63 ± 2.249 | 32.46 ± 1.140 | 10.14 ± 0.196 | 10.06 ± 0.095 |
|     | DPSGD | 59.64 ± 0.671 | 56.87 ± 0.526 | 45.03 ± 1.312 | 57.05 ± 0.322 | 39.1 ± 1.047 | 16.77 ± 0.753 | 30.63 ± 2.282 | 11.56 ± 3.047 | 10.0 ± 0.030 |
|     | Gap   | 0.12   | 3.36   | 3.18   | 0.8    | 2.47   | 5.86   | 1.83   | N/A   | N/A   |
| 1.6 | ADSGD | 61.09 ± 0.284 | 61.45 ± 0.291 | 57.40 ± 0.414 | 61.75 ± 0.455 | 52.29 ± 0.525 | 36.4 ± 1.243 | 48.08 ± 0.886 | 18.89 ± 0.589 | 10.68 ± 1.055 |
|     | DPSGD | 61.26 ± 0.216 | 55.22 ± 0.981 | 55.66 ± 0.748 | 61.64 ± 0.674 | 52.58 ± 0.469 | 35.7 ± 2.453 | 45.44 ± 0.672 | 15.73 ± 6.957 | 9.98 ± 0.046 |
|     | Gap   | 0.35   | 0.19   | 2.23   | 0.35   | 1.71   | 0.7    | 2.95   | 2.66   | N/A   |
| 3.2 | ADSGD | 61.71 ± 0.252 | 65.57 ± 0.371 | 60.23 ± 0.587 | 60.99 ± 0.345 | 65.6 ± 0.134 | 62.77 ± 0.491 | 65.54 ± 0.384 | 55.44 ± 0.359 | 38.65 ± 1.554 |
|     | DPSGD | 61.61 ± 0.290 | 65.36 ± 0.126 | 60.7 ± 0.168 | 60.95 ± 0.233 | 65.71 ± 0.343 | 61.56 ± 0.475 | 64.56 ± 0.467 | 53.32 ± 0.939 | 32.94 ± 5.487 |
|     | Gap   | 0.09   | 0.21   | 0.14   | 0.06   | –0.11  | 1.21   | 0.98   | 2.12   | 5.71   |
| 6.4 | ADSGD | 62.09 ± 0.622 | 66.13 ± 0.202 | 68.29 ± 0.141 | 65.97 ± 0.103 | 68.87 ± 0.220 | 69.6 ± 0.189 | 69.3 ± 0.180 | 68.89 ± 0.369 | 64.66 ± 0.413 |
|     | DPSGD | 61.86 ± 0.441 | 68.18 ± 0.281 | 68.23 ± 0.238 | 66.29 ± 0.255 | 68.05 ± 0.185 | 69.15 ± 0.161 | 69.29 ± 0.080 | 68.66 ± 0.251 | 65.33 ± 0.277 |
|     | Gap   | 0.21   | –0.05  | 0.06   | –0.32  | 0.22   | 0.45   | 0.01   | 0.23   | 1.01   |

Table 5: Mean accuracy of ADP-SGD/DP-SGD with polynomially decaying stepsizes \( \eta_t = \eta_0/(1 + t) = 1/\sqrt{20 + t} \) (see the blue curve in Figure 2). This table reports accuracy for CIFAR10 with the mean and the corresponding standard deviation over \( \{acc_{last}^{i}\}_{i=1}^{5} \). Here, \( acc_{last}^{i} \) is the accuracy at the final iteration for the \( i \)-th independent experiment. Each set \( \{acc_{last}^{i}\}_{i=1}^{5} \) corresponds to a pair of \( (\varepsilon, C_G, T, Alg) \). See Table 4 for reading instruction.
F.1 DP-SGD v.s. ADP-SGD with decaying stepsizes

See Table 4 and Table 5 for the statistics at the last iteration. Comparing Table 4 and Table 1 (Table 5 and 2), it appears that both DP-SGD and ADP-SGD do better at earlier iterations. It appears that the difference (the row “Gap” in the tables) between the best iteration and iteration $T$ is pretty minimal for large $\varepsilon$ and $T$. Thus, our observation in the main text for Table 1 and 2 holds also for Table 4 and 5.

F.2 DP-SGD v.s. ADP-SGD with adaptive stepsizes

For this set of experiments, we first tune the hyper-parameters, namely $\beta$ for DP-SGD and a pair of $(\beta, C)$ for ADP-SGD, whose optimal values are shown in Table 6. Based on these hyper-parameters, we repeat the experiments five times and report the results in Table 3.

| Gradient Clipping | Algorithms       | $\varepsilon = 0.8$ | $\varepsilon = 1.6$ | $\varepsilon = 3.2$ | $\varepsilon = 6.4$ |
|-------------------|------------------|---------------------|---------------------|---------------------|---------------------|
| $C_G = 1.0$       | DP-SGD with $\beta$ | 1024                | 512                 | 1                   | 1                   |
|                   | ADP-SGD with $(\beta, C)$ | (1024, $10^{-5}$)  | (512, $10^{-4}$)    | (512, $10^{-4}$)    | (1, $10^{-4}$)      |
| $C_G = 2.5$       | DP-SGD with $\beta$ | 1024                | 512                 | 1                   | 1                   |
|                   | ADP-SGD with $(\beta, C)$ | (1024, $10^{-5}$)  | (512, $10^{-5}$)    | (512, $10^{-5}$)    | (1, $10^{-2}$)      |

Table 6: ADP-SGD v.s. DP-SGD with adaptive stepsizes. The corresponding $(\beta, C)$ for Table 3.

F.3 Constant stepsizes v.s. decaying stepsizes for DP-SGD

In this section, we justify why using a decaying stepsie in Algorithm 1 $\eta_t = \eta_0 b_{t+1} = 1/\sqrt{2 + ct}, c > 0$ is better than a constant one for DP-SGD $\eta_t = 1/\sqrt{2}$. We use a convolutional neural network (with the network parameters randomly initialized, see Figure 6 for the architecture design) applied to the MNIST dataset. We analyze the accuracy of the classification results under several noise regimes characterized by $\sigma \in \{1.6, 3.2, 6.4, 12.8\}$ in Algorithm 1. We vary $c$ in $\{0, 10^{-5}, 10^{-3}, 10^{-2}, 10^{-1}, 5 \cdot 10^{-1}\}$. We note that $c = 0$ and $a = 2$ correspond to a constant learning rate of $1/\sqrt{2}$.

We plot the test error (not accuracy) with respect to epoch, in order to better understand how the test error varies over time (see Figure 4). On the same plot, we will also represent the privacy budget $\varepsilon$, obtained at each epoch, computed by using both the available code\footnote{https://github.com/tensorflow/privacy/blob/master/tensorflow_privacy/privacy/analysis/rdp_accountant.py} and the theoretical bound. From Figure 4, we see that in all cases $c > 0$ (corresponding to a non-constant learning rate) consistently performs better than constant learning rate $c = 0$. 

8https://github.com/tensorflow/privacy/blob/master/tensorflow_privacy/privacy/analysis/rdp_accountant.py
Figure 4: **Constant stepsize v.s. decaying stepsize for DP-SGD.** We plot the test error (not accuracy), corresponding to the left y-axis, with respect to the epoch. Each plot corresponds to a fixed noise $\sigma$. Different color corresponds to a learning rate schedule $\eta_t = \eta_0/bt = 1/\sqrt{2 + ct}$ with $c$ described in the legend. In addition, we plot the numerical $\varepsilon$ (dash line) and theoretical $\bar{\varepsilon}$ (dot plot), corresponding to the right y-axis. We see that the constant learning rate ($c = 0$) is not as good as the decaying ones ($c > 0$).

Figure 5: **Convolutional Neural Network for CIFAR10.**

```python
class Net_CIFAR10(nn.Module):
    def __init__(self, num_classes=10):
        super(Net_CIFAR10, self).__init__()
        self.conv1 = nn.Conv2d(3, 32, 3)
        self.conv2 = nn.Conv2d(32, 64, 3)
        self.conv3 = nn.Conv2d(64, 128, 3)
        self.pool = nn.MaxPool2d(2, 2)
        self.fc1 = nn.Linear(128 * 4 * 4, 128)
        self.fc2 = nn.Linear(128, num_classes)
        def forward(self, x):
            x = self.pool(F.relu(self.conv1(x)))
            x = self.pool(F.relu(self.conv2(x)))
            x = self.pool(F.relu(self.conv3(x)))
            x = x.view(-1, 32 * 4 * 4)
            x = F.relu(self.fc1(x))
            x = F.relu(self.fc2(x))
            return x
```

Figure 6: **Convolutional Neural Network for MNIST.**

```python
class Net_MNIST(nn.Module):
    def __init__(self, num_classes=10):
        super(Net_MNIST, self).__init__()
        self.conv1 = nn.Conv2d(1, 16, 3, padding=1)
        self.conv2 = nn.Conv2d(16, 32, 3)
        self.fc1 = nn.Linear(32 * 4 * 4, 32)
        self.fc2 = nn.Linear(32, num_classes)
        def forward(self, x):
            x = F.relu(self.conv1(x))  # -> [B, 16, 14, 14]
            x = F.relu(self.conv2(x))  # -> [B, 32, 5, 5]
            x = F.max_pool2d(x, 2, 1)  # -> [B, 32, 4, 4]
            x = x.view(-1, 32 * 4 * 4)
            x = F.relu(self.fc1(x))
            x = self.fc2(x)
            return x
```

F.4 Model architectures

In Figure 5 and 6, we present the CNN models in our experiments written in Python code based on PyTorch.\(^9\)

\(^9\)https://pytorch.org/
```python
import numpy as np

# Primary arguments
epochs = 200  # Suppose we run for 120 epochs
iter_per_epoch = 196  # This match CIFAR10 dataset (size 50000) when using a mini-batch size 256 i.e., 50000//256+1
T = (epochs*iter_per_epoch)  # Total iteration complexity T
log_delta = np.log(100000)  # Set delta = 10^-5
epsilon_G = 3.2  # Set the number epsilon_G = epsilon/(16)^2 where epsilon is our theoretical budget and (16)^2
a,c,p = 20,1,0.5  # Learning rate setting eta_t = 1/(a+c+t)^0.5
gradient_clip = 2.5  # We use gradient clipping for our experiments

# Obtain the variance of gradient perturbation for DP-SGD and ADP-SGD
n = iter_per_epoch  # This is due to using mini-batch SGD. If we use one-sample SGD, n=50000

# Calculate the variance of the Gaussian Mechanism for DP-SGD
sigma_l = log_delta/(n*epsilon_G)*np.sqrt(T)  # Note that this is a simplified version

# Calculate the variance of the Gaussian Mechanism for ADP-SGD
G_t = nablal_l + alpha * t

# ADP-SGD-epsilon = log(1/delta)/(n*sigma_l) (sum_tT^1/alpha_t^2)^0.5

# DP-SGD-epsilon = log(1/delta)/(n*sigma_l) (sum_tT^1)^0.5 (here, alpha_t=1)

# So the condition of DP-SGD-epsilon = ADP-SGD-epsilon implies that sigma_l^2 = sigma_l^2 * (sum_tT^1/alpha_t^2)/T <= epsilon

# Determine sigma_l based on above equation (1)

area = 0
lrs = []

for epoch in range(epochs):
    for k in range(iter_per_epoch):
        area += 1.0/(a+c*t(k*iter_per_epoch+epoch))**0.5
        lrs.append(1.0/(a+c*t(k*iter_per_epoch+epoch))**0.5)

# Gathering gradient perturbation with respect to each iteration t

privacy = []
sigmas = []
squared_ep1, squared_ep2 = 0, 0

for epoch in range(epochs):
    for k in range(iter_per_epoch):
        ep1 = log_delta/(iter_per_epoch*sigma_l)*squared_ep1**0.5  # Theoretical Privacy for DP-SGD
        squared_ep1 += 1.0/(a+c*t(k*iter_per_epoch+epoch))**0.5
        ep2 = log_delta/(iter_per_epoch*sigma_l)*squared_ep2**0.5  # Theoretical Privacy for ADP-SGD
        squared_ep2 += 1.0/(a+c*t(k*iter_per_epoch+epoch))**0.5
        privacy.append(ep1, ep2)

        signal_square = (gradient_clip*sigma_l)**2/(a+c*t(k*iter_per_epoch+epoch))  # Variance of the actual perturbation for DP-SGD
        signal_square = (gradient_clip*sigma_l)**2/(a+c*t(k*iter_per_epoch+epoch))**0.5  # Variance of the actual perturbation for ADP-SGD
        sigmas.append(signal_square, signal_square)
```

Figure 7: Code to obtain Figure 1
Figure 8: Sample code (one epoch) based on PyTorch for training a CNN model over CIFAR10 data, whose results are shown in Table 5 and 4.