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Some Properties of Approximate Solutions of Linear Differential Equations

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Abstract: In this paper, we will consider the Hyers-Ulam stability for the second order inhomogeneous linear differential equation, $u''(x) + \alpha u'(x) + \beta u(x) = r(x)$, with constant coefficients. More precisely, we study the properties of the approximate solutions of the above differential equation in the class of twice continuously differentiable functions with suitable conditions and compare them with the solutions of the homogeneous differential equation $u''(x) + \alpha u'(x) + \beta u(x) = 0$. Several mathematicians have studied the approximate solutions of such differential equation and they obtained good results. In this paper, we use the classical integral method, via the Wronskian, to establish the stability of the second order inhomogeneous linear differential equation with constant coefficients and we will compare our result with previous ones. Specially, for any desired point $c \in \mathbb{R}$ we can have a good approximate solution near $c$ with very small error estimation.

Keywords: linear differential equation; generalized Hyers-Ulam stability; Hyers-Ulam stability; analytic function; approximation

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1. Introduction

Consider the following form of the $n$th order differential equation,

$$G(u^{(n)}, u^{(n-1)}, \ldots, u', u, x) = 0,$$  (1)

where the function $u : I \to \mathbb{R}$ is an $n$ times continuously differentiable function. Given $\epsilon > 0$ and $n$ times continuously differentiable function $u : I \to \mathbb{R}$, we assume that

$$|G(u^{(n)}, u^{(n-1)}, \ldots, u', u, x)| \leq \epsilon \quad \text{(for all } x \in I)$$  (2)

and we want to find a solution $u_0 : I \to \mathbb{R}$ of the differential Equation (1) and a constant $K > 0$ such that

$$|u(x) - u_0(x)| \leq K\epsilon \quad \text{(for any } x \in I).$$  (3)

If the above statement is true for any sufficiently small $\epsilon > 0$ and any $n$ times continuously differentiable function $u : I \to \mathbb{R}$ and if the constant $K$ is independent not only from $\epsilon$ but also from $u$, then the equation in (1) is said to possess the Hyers–Ulam stability property. The reader is referred to [1,2] for more details.
Obłoza has studied the Hyers–Ulam stability of the linear differential equations in [3,4] for the first time. Thereafter, Alsina and Ger [5] studied different types of linear differential equations and their results were further developed by many mathematicians (see [6–11]).

For \( a, b \in \mathbb{R} \), \( I \) denotes the interval \([a, b] \subset \mathbb{R}\) and we consider the second order inhomogeneous linear equation with the following form,

\[
    u''(x) + \alpha u'(x) + \beta u(x) = r(x), \tag{4}
\]

where \( \alpha \) and \( \beta \) are real-valued constants and \( r : I \to \mathbb{R} \) is a continuous function. We note that if we set \( r(x) \equiv 0 \) in (4), then the resulting differential equation,

\[
    u''(x) + \alpha u'(x) + \beta u(x) = 0, \tag{5}
\]

is called the homogeneous differential equation corresponding to (4).

With various additional conditions, the Hyers–Ulam stability of Equation (4) was proved in [12–17]. In this paper, with weaker conditions, we will study the Hyers–Ulam stability of Equation (4) in three different cases with respect to the constant coefficients. In particular, we will try the classical integral method for error estimation. Moreover, using our results, we investigate some properties of approximate solutions of (4) in the class of the twice continuously differentiable functions, and then we compare the approximate solutions with the solutions of the homogeneous differential equation \( u''(x) + \alpha u'(x) + \beta u(x) = 0 \) by estimating the norm of difference between them. Finally, in Section 5 we will conclude this paper.

2. Preliminaries

First, in the following proposition, we introduce the general solution of the second order inhomogeneous linear differential Equation (4).

**Proposition 1** ([18], §2.16). If a general solution \( u_h : I \to \mathbb{R} \) of the homogeneous differential Equation (5) has the form

\[
    u_h(x) = d_1 u_1(x) + d_2 u_2(x) \quad (d_1 \text{ and } d_2 \text{ are arbitrary real constants}),
\]

then a general solution \( u : I \to \mathbb{R} \) of the inhomogeneous linear differential Equation (4) has the form (a variation of constants or Duhamel’s principle)

\[
    u(x) = d_1 u_1(x) + d_2 u_2(x) - u_1(x) \int_c^x \frac{u_2(t)r(t)}{W(u_1, u_2)(t)} \, dt + u_2(x) \int_c^x \frac{u_1(t)r(t)}{W(u_1, u_2)(t)} \, dt,
\]

where \( c \) is an arbitrarily chosen point of \( I \) and

\[
    W(u_1, u_2)(x) := u_1(x)u_2'(x) - u_1'(x)u_2(x)
\]

is the Wronskian of \( u_1 \) and \( u_2 \).

The generalized Hyers–Ulam stability of the second order inhomogeneous linear differential equations with variable coefficients was investigated in ([19], Corollary 3.3) in the class of twice continuously differentiable functions. In the following theorem, we introduce a version of ([19], Corollary 3.3) for the constant coefficients case. The main results in the paper, Theorems 2–4, can be considered as corollaries to ([19], Corollary 3.3). In Theorems 5–7, these results are refined for \( r(x) = 0 \).

Let \( I \) be a (half)-open or closed subinterval of \( \mathbb{R} \) and let \( c \in I \).
**Theorem 1.** Let \( r : \mathbb{R} \to \mathbb{R} \) be a given continuous function, let \( \alpha \) and \( \beta \) be real-valued constants. Suppose a general solution \( u_h : I \to \mathbb{R} \) of the homogeneous differential Equation (5) is of the form \( u_h(x) = d_1 u_1(x) + d_2 u_2(x) \), where \( d_1 \) and \( d_2 \) are arbitrary real-valued constants. Let \( u : I \to \mathbb{R} \) be a twice continuously differentiable function and satisfies the inequality

\[
|u''(x) + \alpha u'(x) + \beta u(x) - r(x)| \leq \sigma(x)
\]

for all \( x \in I \), where the following integral exists with \( \sigma : I \to [0, \infty) \), then there is a solution \( u_0 : I \to \mathbb{R} \) of differential Equation (4) satisfying

\[
|u(x) - u_0(x)| \leq \left| \int_c^x \left[ \frac{u_1(x)u_2(t) - u_1(t)u_2(x)}{W(u_1, u_2)(t)} \right] \sigma(t) dt \right|
\]

for every \( x \in I \), where \( c \) is arbitrarily given in \( I \).

### 3. Differential Equations with Constant Coefficients

Now, we investigate the second order inhomogeneous linear differential Equation (4) with constant coefficients, where \( \alpha \) and \( \beta \) are real-valued constants and \( r : \mathbb{R} \to \mathbb{R} \) is a continuous function.

We denote by \( m_1 \) and \( m_2 \) the roots of the characteristic equation \( m^2 + \alpha m + \beta = 0 \), i.e.,

\[
m_1 = \begin{cases} 
\frac{-\alpha - \sqrt{\alpha^2 - 4\beta}}{2} & \text{(for } \alpha^2 - 4\beta > 0), \\
\frac{-\alpha + i\sqrt{4\beta - \alpha^2}}{2} & \text{(for } \alpha^2 - 4\beta \leq 0) 
\end{cases} \\
m_2 = \begin{cases} 
\frac{-\alpha + \sqrt{\alpha^2 - 4\beta}}{2} & \text{(for } \alpha^2 - 4\beta > 0), \\
\frac{-\alpha - i\sqrt{4\beta - \alpha^2}}{2} & \text{(for } \alpha^2 - 4\beta \leq 0) 
\end{cases}
\]

According to Theorem 1, the general solution of (4) is given as

\[
u(x) = d_1 e^{m_1 x} + d_2 e^{m_2 x} - \frac{e^{m_1 x}}{m_2 - m_1} \int_c^x e^{-m_1 t} r(t) dt + \frac{e^{m_2 x}}{m_2 - m_1} \int_c^x e^{-m_2 t} r(t) dt
\]

for \( \alpha^2 - 4\beta > 0 \); or

\[
u(x) = d_1 e^{\lambda x} + d_2 e^{\lambda x} - e^{\lambda x} \int_c^x t e^{-\lambda t} r(t) dt + x e^{\lambda x} \int_c^x e^{-\lambda t} r(t) dt
\]

for \( \alpha^2 - 4\beta = 0 \); or

\[
u(x) = e^{\lambda x} (d_1 \cos \mu x + d_2 \sin \mu x) - \frac{e^{\lambda x} \cos \mu x}{\mu} \int_c^x e^{-\lambda t} r(t) \sin \mu t dt + \frac{e^{\lambda x} \sin \mu x}{\mu} \int_c^x e^{-\lambda t} r(t) \cos \mu t dt
\]

for \( \alpha^2 - 4\beta < 0 \), where \( c \) is a real number.

In the following theorems, we investigate the Hyers–Ulam stability of the second order inhomogeneous differential Equation (4), with constant coefficients, defined on the whole space \( \mathbb{R} \).

First, we start with the case when \( \alpha^2 - 4\beta > 0 \).
Theorem 2. Let \( r : \mathbb{R} \to \mathbb{R} \) be a given continuous function, let \( a \) and \( \beta \) be given real-valued constants satisfying \( \alpha^2 - 4\beta > 0 \), let \( c \in \mathbb{R} \), and let \( \epsilon > 0 \) be given. If a twice continuously differentiable function \( u : \mathbb{R} \to \mathbb{R} \) satisfies the inequality

\[
|u''(x) + au'(x) + \beta u(x) - r(x)| \leq \epsilon
\]

for every \( x \in \mathbb{R} \), then there exist real-valued constants \( d_1 \) and \( d_2 \) such that

\[
u_c(x) := d_1 e^{m_1 x} + d_2 e^{m_2 x} - \frac{e^{m_1 x}}{m_2 - m_1} \int_c^x e^{-m_1 t} r(t) \, dt + \frac{e^{m_2 x}}{m_2 - m_1} \int_c^x e^{-m_2 t} r(t) \, dt
\]

and

\[
|u(x) - u_c(x)| \leq \begin{cases} 
\frac{\epsilon}{m_2 - m_1} \left( \frac{e^{m_2(x-c)} - 1}{m_2} - \frac{e^{m_1(x-c)} - 1}{m_1} \right) & \text{(for } m_1 \neq 0 \neq m_2) , \\
\frac{\epsilon}{m_2 - m_1} \left( \frac{e^{m_2(x-c)} - 1}{m_2} - (x - c) \right) & \text{(for } m_1 = 0 \), \\
\frac{\epsilon}{m_2 - m_1} \left( x - c \right) & \text{(for } m_2 = 0 \)
\end{cases}
\]

for all \( x \in \mathbb{R} \), where \( m_1 \) and \( m_2 \) are given in (6).

Proof. Since \( \alpha^2 - 4\beta > 0 \), we have \( m_1 < m_2 \), \( u_1(x) = e^{m_1 x} \), \( u_2(x) = e^{m_2 x} \), and \( W(u_1, u_2)(x) = (m_2 - m_1)e^{(m_1 + m_2)x} \). According to Theorem 1 with \( \sigma(x) = \epsilon \) and \( I = \mathbb{R} \), there is a solution \( u_c : \mathbb{R} \to \mathbb{R} \) of (4) satisfying

\[
|u(x) - u_c(x)| \leq \frac{\epsilon}{m_2 - m_1} \left| \int_c^x \left| e^{m_1(x-t)} - e^{m_2(x-t)} \right| \, dt \right|
\]

for all \( x \in \mathbb{R} \).

It further follows from the last inequality that

\[
|u(x) - u_c(x)| \leq \begin{cases} 
\frac{\epsilon}{m_2 - m_1} \int_c^x \left( e^{m_2(x-t)} - e^{m_1(x-t)} \right) \, dt & \text{(for } x \geq c) , \\
\frac{\epsilon}{m_2 - m_1} \int_c^x \left( e^{m_1(x-t)} - e^{m_2(x-t)} \right) \, dt & \text{(for } x < c) 
\end{cases}
\]

\[
= \frac{\epsilon}{m_2 - m_1} \left( e^{m_2(x-c)} - e^{m_1(x-c)} \right)
\]

\[
= \frac{\epsilon}{m_2 - m_1} \left( \lim_{v \to m_2} e^{v(x-c)} - \lim_{w \to m_1} e^{w(x-c)} \right)
\]

\[
= \frac{\epsilon}{m_2 - m_1} \left( \lim_{v \to m_2} e^{v(x-c)} - 1 - \lim_{w \to m_1} e^{w(x-c)} - 1 \right)
\]

for all \( x \in \mathbb{R} \), from which we get the inequality (10) by applying the l'Hospital's rule.

On the other hand, since \( u_c \) is a solution of (4) with \( \alpha^2 - 4\beta > 0 \), it follows from (7) that \( u_c \) has the form given in the statement of our theorem.

We now consider the case when \( \alpha^2 - 4\beta = 0 \), i.e., \( \beta = \frac{\alpha^2}{4} > 0 \) and prove the Hyers–Ulam stability of the inhomogeneous differential Equation (4).
Theorem 3. Let \( r : \mathbb{R} \to \mathbb{R} \) be a continuous function, let \( \alpha \neq 0 \) be a real-valued constant, and let \( c \in \mathbb{R} \) and \( \varepsilon > 0 \) be arbitrary constants. If a twice continuously differentiable function \( u : \mathbb{R} \to \mathbb{R} \) satisfies the inequality
\[
|u''(x) + au'(x) + \frac{\alpha^2}{4}u(x) - r(x)| \leq \varepsilon
\]
for every \( x \in \mathbb{R} \), then there exist real-valued constants \( d_1 \) and \( d_2 \) such that
\[
u_c(x) := d_1 e^{\lambda x} + d_2 x e^{\lambda x} - e^{\lambda x} \int_c^x t e^{-\lambda t} r(t) dt + x e^{\lambda x} \int_c^x e^{-\lambda t} r(t) dt
\]
and
\[
|u(x) - u_c(x)| \leq \varepsilon \left( \lambda(x-c) - 1 \right \frac{e^{\lambda(x-c)}}{\lambda^2} + 1
\]
for every \( x \in \mathbb{R} \), where \( \lambda \) is given in \( (6) \).

Proof. If \( \alpha \neq 0 \) and \( \beta = \frac{\alpha^2}{4} \), then \( \alpha^2 - 4\beta = 0 \), \( m_1 = m_2 = -\frac{\alpha}{2} = \lambda \), \( u_1(x) = e^{\lambda x} \), \( u_2(x) = x e^{\lambda x} \), and \( W(u_1, u_2)(x) = e^{2\lambda x} \). According to Theorem 1 with \( \sigma(x) = \varepsilon \) and \( I = \mathbb{R} \), there is a solution \( u_c : \mathbb{R} \to \mathbb{R} \) of \( (4) \) satisfying
\[
|u(x) - u_c(x)| \leq \varepsilon \int_c^x |x-t| e^{\lambda(x-t)} dt
\]
for all \( x \in \mathbb{R} \).

It further follows from the last inequality that
\[
|u(x) - u_c(x)| \leq \left\{ \begin{array}{ll}
\varepsilon \int_c^x (x-t) e^{\lambda(x-t)} dt & \text{(for } x \geq c), \\
\varepsilon \int_c^x (t-x) e^{\lambda(x-t)} dt & \text{(for } x < c) \\
\varepsilon \int_c^x (x-t) e^{\lambda(x-t)} dt \\
\varepsilon \left( \lambda(x-c) - 1 \right \frac{e^{\lambda(x-c)}}{\lambda^2} + 1
\end{array} \right.
\]
for all \( x \in \mathbb{R} \).

Finally, we know that \( u_c \) is a solution of \( (4) \) with \( \alpha^2 - 4\beta = 0 \). Therefore, in view of \( (8) \), \( u_c \) has the form given in the statement of our theorem.

Now, we are in the position to prove the Hyers–Ulam stability of the inhomogeneous differential Equation \( (4) \) for the case when \( \alpha^2 - 4\beta < 0 \).

Theorem 4. Let \( r : \mathbb{R} \to \mathbb{R} \) be a continuous function, let \( \alpha \) and \( \beta \) be real-valued constants satisfying \( \alpha^2 - 4\beta < 0 \), and let \( c \in \mathbb{R} \) and \( \varepsilon > 0 \) be arbitrary constants. If a twice continuously differentiable function \( u : \mathbb{R} \to \mathbb{R} \) satisfies the inequality
\[
|u''(x) + au'(x) + \beta u(x) - r(x)| \leq \varepsilon
\]
for all \( x \in \mathbb{R} \), then there exist real-valued constants \( d_1 \) and \( d_2 \) such that
\[
u_c(x) := e^{\lambda x} \left( d_1 \cos \mu x + d_2 \sin \mu x \right) \\
- \frac{e^{\lambda x} \cos \mu x}{\mu} \int_c^x e^{-\lambda t} r(t) \sin \mu t dt + \frac{e^{\lambda x} \sin \mu x}{\mu} \int_c^x e^{-\lambda t} r(t) \cos \mu t dt
\]
and

\[ |u(x) - u_\varepsilon(x)| \leq \frac{\varepsilon}{\mu} \int_0^{x-\varepsilon} e^{\lambda t} |\sin \mu t| dt \]  \hspace{1cm} (13)

for all \( x \in \mathbb{R} \), where \( \lambda \) and \( \mu \) are given in (6).

**Proof.** Since \( \alpha^2 - 4\beta < 0 \), we have \( u_1(x) = e^{\lambda x} \cos \mu x \), \( u_2(x) = e^{\lambda x} \sin \mu x \), and \( W(u_1, u_2)(x) = \mu e^{2\lambda x} \).

According to Theorem 1 with \( \sigma(x) = \varepsilon \) and \( I = \mathbb{R} \), there is a solution \( u_\varepsilon : \mathbb{R} \to \mathbb{R} \) of (4) satisfying

\[ |u(x) - u_\varepsilon(x)| \leq \frac{\varepsilon}{\mu} \int_\varepsilon^x e^{\lambda(x-t)} |\sin \mu (x-t)| dt \]

for all \( x \in \mathbb{R} \). If we change the variable with \( \tau = x-t \), then we get

\[ |u(x) - u_\varepsilon(x)| \leq \frac{\varepsilon}{\mu} \int_0^{x-\varepsilon} e^{\lambda \tau} |\sin \mu \tau| d\tau \]

for each \( x \in \mathbb{R} \).

On the other hand, since \( u_\varepsilon \) is a solution of (4) with \( \alpha^2 - 4\beta < 0 \), it follows from (9) that \( u_\varepsilon \) has the form given in the statement of our theorem. □

For any \( x \in \mathbb{R} \), let us denote by \( n(x) = \left[ \frac{\mu x}{\pi} \right] \) the integer satisfying

\[ n(x) \frac{\pi}{\mu} \leq x < (n(x)+1) \frac{\pi}{\mu}. \]  \hspace{1cm} (14)

If \( x \geq 0 \), then we have

\[
\begin{align*}
\left| \int_0^x e^{\lambda t} |\sin \mu t| dt \right| & = \left| \sum_{k=0}^{n(x)-1} \int_{k\pi/\mu}^{(k+1)\pi/\mu} e^{\lambda t} |\sin \mu t| dt \right| + \left| \int_{n(x)\pi/\mu}^x e^{\lambda t} |\sin \mu t| dt \right| \\
& = \sum_{k=0}^{n(x)-1} \left| \int_{k\pi/\mu}^{(k+1)\pi/\mu} e^{\lambda t} |\sin \mu t| dt \right| + \left| \int_{n(x)\pi/\mu}^x e^{\lambda t} |\sin \mu t| dt \right| \\
& = \sum_{k=0}^{n(x)-1} \frac{\mu}{\lambda^2 + \mu^2} \left( 1 + e^{\lambda \frac{\pi}{\mu}} \right) e^{\lambda \frac{k\pi}{\mu}} \\
& \quad + \frac{\lambda}{\lambda^2 + \mu^2} e^{\lambda x} \sin \mu x - \frac{\mu}{\lambda^2 + \mu^2} e^{\lambda x} \cos \mu x + \frac{(-1)^{n(x)} \mu}{\lambda^2 + \mu^2} e^{\lambda n(x) \frac{\pi}{\mu}} \left| \right| \\
& = \frac{\mu}{\lambda^2 + \mu^2} \left( 1 + e^{\lambda \frac{\pi}{\mu}} \right) \frac{1 - e^{-\lambda n(x) \frac{\pi}{\mu}}}{1 - e^{\lambda \frac{\pi}{\mu}}} \\
& \quad + \frac{\lambda}{\lambda^2 + \mu^2} e^{\lambda x} \sin \mu x - \frac{\mu}{\lambda^2 + \mu^2} e^{\lambda x} \cos \mu x + \frac{(-1)^{n(x)} \mu}{\lambda^2 + \mu^2} e^{\lambda n(x) \frac{\pi}{\mu}} \left| .
\end{align*}
\]  \hspace{1cm} (15)
If we replace $\lambda$ by $-\lambda$ in (15), then we get
\[
\left| \int_0^x e^{-\lambda t} \sin \mu t \, dt \right| = \frac{\mu}{\lambda^2 + \mu^2} \left( 1 + e^{-\lambda \frac{\pi}{2}} \right) \frac{1 - e^{-\lambda n(x) \frac{\pi}{2}}}{1 - e^{-\lambda \frac{\pi}{2}}} \tag{16}
\]
\[+ \left| \frac{\lambda}{\lambda^2 + \mu^2} e^{-\lambda x} \sin \mu x - \frac{\mu}{\lambda^2 + \mu^2} e^{-\lambda x} \cos \mu x - \frac{(1/n(x)) \mu}{\lambda^2 + \mu^2} e^{-\lambda n(x) \frac{\pi}{2}} \right|.
\]
For an arbitrary $x < 0$, it follows from (16) that
\[
\left| \int_0^x e^{\lambda t} \sin \mu t \, dt \right| = \left| \int_0^{-x} e^{-\lambda t} \sin \mu t \, dt \right|
\]
\[= \frac{\mu}{\lambda^2 + \mu^2} \left( 1 + e^{-\lambda \frac{\pi}{2}} \right) \frac{1 - e^{-\lambda n(-x) \frac{\pi}{2}}}{1 - e^{-\lambda \frac{\pi}{2}}} \tag{17}
\]
\[+ \left| \frac{\lambda}{\lambda^2 + \mu^2} e^{\lambda x} \sin \mu x - \frac{\mu}{\lambda^2 + \mu^2} e^{\lambda x} \cos \mu x + \frac{(1/n(-x)) \mu}{\lambda^2 + \mu^2} e^{-\lambda n(-x) \frac{\pi}{2}} \right|.
\]

Remark 1. If $x \geq c$ in Theorem 4, then we use (15) to get
\[
\left| u(x) - u_c(x) \right|
\leq \frac{\varepsilon}{\lambda^2 + \mu^2} \left( 1 + e^{\lambda \frac{\pi}{2}} \right) \frac{1 - e^{\lambda n(x-c) \frac{\pi}{2}}}{1 - e^{\lambda \frac{\pi}{2}}} \tag{16}
\]
\[+ \frac{\varepsilon}{\lambda^2 + \mu^2} \left| \frac{\lambda}{\mu} e^{\lambda(x-c)} \sin \mu(x-c) - e^{\lambda(x-c)} \cos \mu(x-c) + (1/n(x-c)) e^{\lambda n(x-c) \frac{\pi}{2}} \right|
\]
and if $x < c$, then we use (17) to get
\[
\left| u(x) - u_c(x) \right|
\leq \frac{\varepsilon}{\lambda^2 + \mu^2} \left( 1 + e^{-\lambda \frac{\pi}{2}} \right) \frac{1 - e^{-\lambda n(c-x) \frac{\pi}{2}}}{1 - e^{-\lambda \frac{\pi}{2}}} \tag{17}
\]
\[+ \frac{\varepsilon}{\lambda^2 + \mu^2} \left| \frac{\lambda}{\mu} e^{\lambda(x-c)} \sin \mu(x-c) - e^{\lambda(x-c)} \cos \mu(x-c) + (1/n(c-x)) e^{-\lambda n(x-c) \frac{\pi}{2}} \right|.
\]

4. Approximation Properties

Let $\alpha$ and $\beta$ be real-valued constants, and let the real vector space $B(\alpha; \beta)$ consist of those $C^2$-functions $u : \mathbb{R} \to \mathbb{R}$ for which there exists a finite constant $M$ (but depending on $u$) such that
\[
\left| u''(x) + au'(x) + \beta u(x) \right| \leq M \tag{18}
\]
for every $x \in \mathbb{R}$. It is clear that $B(\alpha; \beta)$ is a vector space over $\mathbb{R}$.

Let
\[
(v_1 + v_2)(x) := v_1(x) + v_2(x) \quad \text{and} \quad (\gamma v_1)(x) := \gamma v_1(x)
\]
for all $v_1, v_2 \in B(\alpha; \beta)$ and $\gamma \in \mathbb{R}$, then $B(\alpha; \beta)$ is a vector space over $\mathbb{R}$. Therefore, we may remark that $B(\alpha; \beta)$ is a large set enough to be a vector space.

In the next theorems, we study an approximation property of functions from $B(\alpha; \beta)$ by solutions of differential Equation (5).
Theorem 5. Let $\alpha$ and $\beta$ be real-valued constants satisfying $\alpha^2 - 4\beta > 0$ and let $c \in \mathbb{R}$ be given. If $u \in B(\alpha; \beta)$, then there exist real-valued constants $p$ and $q$ such that

$$|u(x) - pe^{\alpha_1 x} - qe^{\alpha_2 x}| = O(|x - c|)$$

as $x \to c$, where $m_1$ and $m_2$ are given in (6).

Proof. Since $\alpha^2 - 4\beta > 0$, we have $m_1 < m_2$, $u_1(x) = e^{\alpha_1 x}$, and $u_2(x) = e^{\alpha_2 x}$. Since $u \in B(\alpha; \beta)$, there is a constant $\varepsilon > 0$ such that (18) is true for every $x \in \mathbb{R}$. On account of Theorem 2 with $r(x) \equiv 0$ and the inequality (11), there exist real-valued constants $p$ and $q$ such that

$$|u(x) - pe^{\alpha_1 x} - qe^{\alpha_2 x}| \leq \frac{\varepsilon}{m_2 - m_1} \left( \lim_{v \to m_2} \frac{e^{v(x-c)} - 1}{v} - \lim_{v \to m_1} \frac{e^{v(x-c)} - 1}{v} \right)$$

for all $x \in \mathbb{R}$. Thus, we have

$$\lim_{x \to c} \frac{|u(x) - pe^{\alpha_1 x} - qe^{\alpha_2 x}|}{x - c} \leq \frac{\varepsilon}{m_2 - m_1} \left( \lim_{v \to m_2} \frac{e^{v(x-c)} - 1}{v} - \lim_{v \to m_1} \frac{e^{v(x-c)} - 1}{v} \right)$$

$$= \frac{\varepsilon}{m_2 - m_1} \left( \lim_{v \to m_2} \frac{e^{v(x-c)} - 1}{v} - \lim_{v \to m_1} \frac{e^{v(x-c)} - 1}{v} \right)$$

which ends our proof. \(\square\)

When $\alpha^2 - 4\beta = 0$, for the proof of the following theorem, we apply Theorem 3.

Theorem 6. Let $\alpha \neq 0$, $\beta = \frac{\alpha^2}{4}$, and $c \in \mathbb{R}$ be real-valued constants. If $y \in B(\alpha; \beta)$, then there exist real-valued constants $p$ and $q$ such that

$$|u(x) - pe^{\lambda x} - qxe^{\lambda x}| = o(|x - c|)$$

as $x \to c$, where $\lambda$ is given in (6).

Proof. Since $\alpha^2 - 4\beta = 0$, we have $m_1 = m_2 = -\frac{\alpha}{2} = \lambda$, $u_1(x) = e^{\lambda x}$, and $u_2(x) = xe^{\lambda x}$. Since $u \in B(\alpha; \beta)$, we have a constant $M < \infty$ such that (18) is true for every $x \in \mathbb{R}$. On account of Theorem 3 with $r(x) \equiv 0$, there exist real-valued constants $p$ and $q$ such that

$$|u(x) - pe^{\lambda x} - qxe^{\lambda x}| \leq \varepsilon \frac{(\lambda(x-c) - 1)e^{\lambda(x-c)} + 1}{\lambda^2}.$$ 

Moreover, we apply the l'Hospital’s rule to get

$$\lim_{x \to c} \frac{|u(x) - pe^{\lambda x} - qxe^{\lambda x}|}{x - c} \leq \lim_{x \to c} \frac{\varepsilon}{\lambda^2} \left( \frac{(\lambda(x-c) - 1)e^{\lambda(x-c)} + 1}{x - c} \right)$$

$$= \lim_{x \to c} \frac{\varepsilon}{\lambda^2} \left( \lambda e^{\lambda(x-c)} + \lambda(\lambda(x-c) - 1)e^{\lambda(x-c)} \right)$$

$$= 0,$$

which completes the proof. \(\square\)
Finally, we investigate an approximation property of the approximate solutions for each function in $B(a; \beta)$ by some solution of the differential Equation (5) when $a^2 - 4\beta < 0$.

**Theorem 7.** Let $\alpha$ and $\beta$ be real-valued constants satisfying $\alpha^2 - 4\beta < 0$ and let $c \in \mathbb{R}$ be a given constant. If $u \in B(a; \beta)$, then there exist real-valued constants $p$ and $q$ such that

$$|u(x) - pe^{\lambda x} \cos \mu x - qe^{\lambda x} \sin \mu x| = o(|x - c|)$$

as $x \to c$, where $\lambda$ and $\mu$ are given in (6).

**Proof.** Since $\alpha^2 - 4\beta < 0$, we have $u_1(x) = e^{\lambda x} \cos \mu x$ and $u_2(x) = e^{\lambda x} \sin \mu x$. By the definition (14) of $n(x)$, we know that

$$c \leq x < c + \frac{\pi}{\mu} \iff n(x - c) = 0 \quad \text{and} \quad c - \frac{\pi}{\mu} < x \leq c \iff n(c - x) = 0.$$ 

Since $u \in B(a; \beta)$, a finite constant $M > 0$ exists such that the inequality in (18) is satisfied for every $x \in \mathbb{R}$. For both cases $c - \frac{\pi}{\mu} < x < c$ and $c \leq x < c + \frac{\pi}{\mu}$, we apply Theorem 4 with $r(x) \equiv 0$ and Remark 1 and we consider the last argument to see that there exist real-valued constants $p$ and $q$ such that

$$|u(x) - pe^{\lambda x} \cos \mu x - qe^{\lambda x} \sin \mu x|$$

$$\leq \frac{M}{\lambda^2 + \mu^2} \left| \frac{\lambda}{\mu} e^{\lambda(x-c)} \sin \mu(x-c) - e^{\lambda(x-c)} \cos \mu(x-c) + 1 \right|$$

for all $c - \frac{\pi}{\mu} < x < c + \frac{\pi}{\mu}$. Furthermore, we apply the l'Hospital's rule to calculate

$$\lim_{x \to c} \frac{|u(x) - pe^{\lambda x} \cos \mu x - qe^{\lambda x} \sin \mu x|}{x - c}$$

$$\leq \frac{M}{\lambda^2 + \mu^2} \left| \frac{\lambda}{\mu} \lim_{x \to c} e^{\lambda(x-c)} \sin \mu(x-c) - \lim_{x \to c} e^{\lambda(x-c)} \cos \mu(x-c) - 1 \right|$$

$$= 0,$$

which completes our proof. \(\square\)

5. Conclusions

In the main theorems, we investigated the Hyers–Ulam stability of second order inhomogeneous linear differential Equation (4), with constant coefficients, defined on the whole space $\mathbb{R}$, while previous papers have investigated the local Hyers–Ulam stability of the second order linear differential equations defined on the bounded intervals like $I = (a, b)$ or $I = [a, b]$ with $-\infty < a < b < +\infty$ under some additional conditions.

Ghaemi et al. [13] recently proved a theorem concerning the Hyers–Ulam stability of the exact second order linear differential equations which yields the following inequality

$$|u(x) - u_c(x)| \leq \frac{\varepsilon}{m^2} \left( m(b - x) + e^{-m(b-a)} - e^{-m(x-a)} \right)$$

instead of (10) under the additional assumptions that $m = \frac{a + \sqrt{a^2 - 4\beta}}{2} > 0$ and the domain of $u$ and $u_c$ is a bounded interval $I = (a, b)$ (see [13], Corollary 2.3). Our results in (10) contain more generalized cases than those in [13]. Moreover, the inequality (19) does not seem to be better than the inequality (10), while we can find an approximate solution with very small error near $c$ for any desired point $c \in \mathbb{R}$. 


Cîmpean and Popa [12] proved Hyers–Ulam stability of the linear differential equations with constant coefficients. From corollaries in [12], one can obtain the following inequalities

\[
|u(x) - u_c(x)| \leq \begin{cases} 
\frac{\epsilon}{|\beta|} & \text{(for either } \alpha^2 - 4\beta > 0 \text{ and } \beta \neq 0), \\
\frac{4\epsilon}{\alpha^2} & \text{(for } \alpha^2 - 4\beta \leq 0 \text{ and } \alpha \neq 0). 
\end{cases}
\]

However, our results in Theorems 2–4 are a better approximation than theirs. Moreover, by our results in Theorems 5–7, for any desired \(c \in \mathbb{R}\), we can obtain an approximation with very small error. Finally, compared to the results in [14–16] we used very classical integral method and have better error estimation than theirs. While we can find an approximate solution generally with very small error near \(c\) for any desired point \(c \in \mathbb{R}\), in their estimates this is only possible in special cases.

**Remark 2.** It would be much nicer if we could provide an estimate of the form

\[
|u(x) - u_c(x)| \leq C \frac{\epsilon}{|\mu|} \left| \int_0^x (x-c) e^{\lambda t} \sin \mu t dt \right|, \quad x \in I
\]

for some constant \(C\), instead of (13) in Theorem 4. We will investigate this problem in next paper.

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