HARDY-SOBOLEV INEQUALITY WITH HIGHER DIMENSIONAL
SINGULARITY

EL HADJI ABDOUNAYE THIAM

Abstract. For $N \geq 4$, we let $\Omega$ to be a smooth bounded domain of $\mathbb{R}^N$, $\Gamma$ a smooth
closed submanifold of $\Omega$ of dimension $k$ with $1 \leq k \leq N-2$ and $h$ a continuous
function defined on $\Omega$. We denote by $\rho_{\Gamma}(\cdot) := \text{dist}_{\text{g}}(\cdot, \Gamma)$ the distance function
to $\Gamma$. For $\sigma \in (0, 2)$, we study existence of positive solutions $u \in H^1_0(\Omega)$ to the
nonlinear equation
$$-\Delta u + hu = \rho_{\Gamma}^{-\sigma} \frac{u^{2^*(\sigma) - 1}}{2^*(\sigma)}$$
in $\Omega$,
where $2^*(\sigma) := \frac{2(N-\sigma)}{N-2}$ is the critical Hardy-Sobolev exponent. In particular, we
provide existence of solution under the influence of the local geometry of $\Gamma$ and the
potential $h$.

1. Introduction

We consider the following Hardy-Sobolev inequality with cylindrical weight: for $N \geq 3$, $0 \leq k \leq N-1$ and $0 \leq \sigma \leq 2$, we have
\begin{equation}
\int_{\mathbb{R}^N} |\nabla v|^2 \, dx \geq C \left( \int_{\mathbb{R}^N} |z|^{-\sigma} |v|^{2^*_\sigma} \, dx \right)^{2/2^*_\sigma} \quad \text{for all } v \in D^{1,2}(\mathbb{R}^N),
\end{equation}
where $x = (t, z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$, $C$ is a positive constant depending only on $N$, $k$ and $\sigma$, $2^*_\sigma := \frac{2(N-\sigma)}{N-2}$ is the critical Hardy-Sobolev exponent and $D^{1,2}(\mathbb{R}^N)$ is the completion of $C^\infty_c(\mathbb{R}^N)$ with respect to the norm
$$v \mapsto \left( \int_{\mathbb{R}^N} |\nabla v|^2 \, dx \right)^{1/2}.$$ 
For $\sigma = 0$, inequality (1.1) corresponds to the following classical Sobolev inequality:
\begin{equation}
\int_{\mathbb{R}^N} |\nabla v|^2 \, dx \geq C \left( \int_{\mathbb{R}^N} |v|^{2^*_0} \, dx \right)^{2/2^*_0} \quad \text{for all } v \in D^{1,2}(\mathbb{R}^N).
\end{equation}
In this case, the best constant (denoted $S_{N,0}$) is achieved by the function $w(x) = c \left( 1 + |x|^2 \right)^{-N/2}$
and hence the value $S_{N,0} = N(N-2) [\Gamma(N/2) / \Gamma(N)]$ explicitly (see Aubin [1], Lieb [28]
and Talenti [32]). Here $\Gamma$ is the classical Euler function.

For $\sigma = 2$ and $k \neq N-2$, inequality (1.1) corresponds to the following classical Hardy inequality:
\begin{equation}
\int_{\mathbb{R}^N} |\nabla v|^2 \, dx \geq \left( \frac{N-k-2}{2} \right)^2 \int_{\mathbb{R}^N} |z|^{-2} |v|^2 \, dx \quad \text{for all } v \in D^{1,2}(\mathbb{R}^N).
\end{equation}
The constant $\left( \frac{N-k-2}{2} \right)^2$ is optimal but it is never achieved. This fact suggests that
it is possible to improve this inequality, see Brezis-Vasquez [5] and references therein.
For improved Hardy inequality on compact Riemannian manifolds, see the paper of the author [11].
For $\sigma \in (0, 2)$, the best constant in (1.1) is given by

\[ S_{N,\sigma} := \inf \left\{ \int_{\mathbb{R}^N} |\nabla u|^2 \, dx, \quad u \in D^{1,2}(\mathbb{R}^N) \quad \text{and} \quad \int_{\mathbb{R}^N} |z|^{-\sigma} |u|^2 \, dx = 1 \right\} \]

and it is attained, see Badiale-Tarantello [2]. Moreover extremal functions are cylindrical symmetric, see Fabbri-Mancini-Sandeep [12]. However few of them are known explicitily.

Indeed, when $k = 0$, they are given up to scaling by $w(x) = \left((N - \sigma)(N - 2)\right)^{\frac{N - \sigma}{2(N - 2)}} \left(1 + |x|^{2 - \sigma}\right)^{\frac{2 - N}{2σ}}$. Thus the best constant is

\[ S_{N,\sigma} := (N - 2)(N - \sigma) \left[ \frac{w_{N-1} \Gamma^2(N - \sigma)}{2 - \sigma} \frac{2 - \sigma}{\Gamma(\frac{2(N - \sigma)}{2 - \sigma})} \right] \frac{2 - \sigma}{N - \sigma}, \]

see Lieb [28]. When $\sigma = 1$, the authors in [12] showed that the minimizers are given by

\[ w(x) = \left((N - k)(k - 1)\right)^{\frac{N - 2k}{2(N - 2)}} \left(1 + |x|^2\right)^{\frac{2 - N}{2}} \]

up to scaling in the full variable and translations in the $t$-direction.

In this paper, we consider a Hardy-Sobolev inequality in a bounded domain of the Euclidean space with singularity a closed submanifold of higher dimensional singularity. In particular, we let $\Omega$ be a bounded domain of $\mathbb{R}^N$, $N \geq 3$, and $h$ a continuous function defined on $\Omega$. Let $\Gamma \subset \Omega$ be a smooth closed submanifold in $\Omega$ of dimension $k$, with $1 \leq k \leq N - 2$. We are concerned with the existence of minimizers for the following Hardy-Sobolev best constant:

\[ \mu_{h,\sigma}(\Omega, \Gamma) := \inf_{u \in H^1_0(\Omega) \setminus \{0\}} \left\{ \int_{\Omega} |\nabla u|^2 \, dx + \int_{\Omega} hu^2 \, dx \right\} \left( \int_{\Omega} \rho_v^{-\sigma} |u|^2 \, dx \right)^{\frac{1}{2}}, \]

where $\sigma \in (0, 2)$, $2^*_v := \frac{2(N - \sigma)}{N - 2}$ and $\rho_v(x) := \text{dist}(x, \Gamma)$ is the distance function to $\Gamma$. Here and in the following, we assume that $-\Delta + h$ defines a coercive bilinear form on $H^1_0(\Omega)$. We are interested with the effect of the local geometry of the submanifold $\Gamma$ on the existence of minimizer for $\mu_{h,\sigma}(\Omega, \Gamma)$.

When $k = 1$ (i.e. $\Gamma$ is a curve), we have the following result due to the author and Fall [14].

**Theorem 1.1.** Let $N \geq 3$, $\sigma \in (0, 2)$ and $\Omega$ be a bounded domain of $\mathbb{R}^N$. Consider $\Gamma$ a smooth closed curve contained in $\Omega$. Let $h$ be a continuous function such that the linear operator $-\Delta + h$ is coercive. Then there exists a positive constant $C_{N,\sigma}$, only depending on $N$ and $\sigma$ with the property that if there exists $y_0 \in \Gamma$ such that

\[
\begin{aligned}
&h(y_0) + C_{N,\sigma} |\kappa(y_0)|^2 < 0 & \quad \text{for } N \geq 4 \\
&m(y_0) > 0 & \quad \text{for } N = 3
\end{aligned}
\]

then $\mu_{h,\sigma}(\Omega, \Gamma) < S_{N,\sigma}$, and $\mu_{h,\sigma}(\Omega, \Gamma)$ is achieved by a positive function. Here $\kappa : \Gamma \to \mathbb{R}^N$ is the curvature vector of $\Gamma$ and $m : \Omega \to \mathbb{R}$ is the mass-the trace of the regular part of the Green function of the operator $-\Delta + h$ with zero Dirichlet data.

This result shows the dichotomy between the case $N \geq 4$ and the case $N = 3$ as in Brezis-Nirenberg [3], Druet [8], Jaber [25] et references therein.

Our main result deals with the case $2 \leq k \leq N - 2$ and $N \geq 4$. Then we have
Theorem 1.2. Let \( N \geq 4, \sigma \in (0, 2) \) and \( \Omega \) be a bounded domain of \( \mathbb{R}^N \). Consider \( \Gamma \) a smooth closed submanifold contained in \( \Omega \) of dimension \( k \) with \( 2 \leq k \leq N - 2 \). Let \( h \) be a continuous function such that the linear operator \( -\Delta + h \) is coercive. Then there exists positive constants \( C_{N,\sigma}^1 \) and \( C_{N,\sigma}^2 \), only depending on \( N \) and \( \sigma \) with the property that if there exists \( y_0 \in \Gamma \) such that
\[
C_{N,\sigma}^1 H^4(y_0) + C_{N,\sigma}^2 R_\sigma(y_0) + h(y_0) < 0
\]
then \( \mu_{h,\sigma}(\Omega, \Gamma) < S_{N,\sigma} \), and \( \mu_{h,\sigma}(\Omega, \Gamma) \) is achieved by a positive function. Here \( H^2 \) and \( R_\sigma \) are respectively the norms of the mean curvature and the scalar curvature of \( \Gamma \).

The explicit values of \( C_{N,\sigma}^1 \) and \( C_{N,\sigma}^2 \) appearing in (1.7) are given by weighted integrals involving partial derivatives of \( w \), a minimizer for \( S_{N,\sigma} \), see Proposition 4.3 below. When \( k = 1 \) then \( R_\sigma(x_0) = 0 \). Hence \( H = \kappa \), so that we recover Theorem 1.1.

In the litterature several authors studied Hardy-Sobolev inequalities in domains of the Euclidean space and in Riemannian manifolds, see \([6, 7, 15, 21, 25, 27]\) and references therein. For instance, we let \( \Omega \) to be a smooth bounded domain of \( \mathbb{R}^N \) with \( 0 \in \Omega \) and consider the following Hardy-Sobolev constant
\[
\mu_\sigma(\Omega) := \inf \left\{ \int_\Omega |\nabla u|^2 \, dx, \ u \in H_0^1(\Omega) \text{ and } \int_\Omega |x|^{-\sigma}|u|^2 \, dx = 1 \right\},
\]
with \( \sigma \in [0, 2) \). It is well known that the value of \( \mu_\sigma(\Omega) \) is independent of \( \Omega \) thanks to scaling invariant. Moreover \( \mu_\sigma(\Omega) = S_{N,\sigma} \) given by (1.4) and it is not attained for all bounded domains, see Ghoussoub-Yuan \([21]\) and Struwe \([30]\). However the situation changes when we add a little perturbation. For example, let \( h \) be a continuous function on \( \Omega \). Consider the following Hardy-Sobolev best constant
\[
\mu_{h,\sigma}(\Omega) := \inf \left\{ \int_\Omega |\nabla u|^2 \, dx + \int_\Omega h u^2 \, dx, \ u \in H_0^1(\Omega) \text{ and } \int_\Omega |x|^{-\sigma}|u|^2 \, dx = 1 \right\}.
\]
When \( \sigma = 0 \), (1.9) corresponds to the famous Brezis-Nirenberg problem (see \([4]\)) and when \( \sigma = 2 \), this kind of problem was study by the author on compact Riemannian manifolds, see \([33]\). In the non-singular case (\( \sigma = 0 \)), authors in \([4]\) showed that, for \( N \geq 4 \) it is enough that \( h(y_0) < 0 \) to get minimizer for some \( y_0 \in \Omega \). While for \( N = 3 \), the problem is no more local and existence of minimizers is guaranted by the positiveness of a certain mass-the trace of the regular part of the Green function of the operator \( -\Delta + h \) with zero Dirichlet data, see \([5, 10]\). Related references for this Brezis-Nirenberg type problem are Druet \([9]\), Hebey-Vaugon \([23, 24]\), Egnell \([11]\) and references therein.

When \( \sigma = 2 \) and \( h \equiv \lambda \) is a real parameter and \( \Omega \) is replaced by a compact Riemannian manifold, then the author in \([32]\) proved the existence of a threshold \( \lambda^*(\Omega) \) such that the best constant in (1.9) has a solution if and only if \( \lambda < \lambda^* \). See also \([34]\).

A very interesting case in the litterature is when \( 0 \in \partial \Omega \). The result of the attainability for the Hardy-Sobolev best constant \( \mu_\sigma(\Omega) \) defined in (1.5) is quite different from that in the situation where \( 0 \in \Omega \). The fact that things may be different when \( 0 \in \partial \Omega \) first emerged in the paper of Egnell \([11]\) where he considers open cones of the form \( C = \{ x \in \mathbb{R}^N; x = r\theta, \theta \in D \text{ and } r > 0 \} \) where the base \( D \) is a connected domain of the unit sphere \( S^{N-1} \) of \( \mathbb{R}^N \). Egnell showed that \( \mu_\sigma(C) \) is then attained for \( 0 < s < 2 \) even when \( C \neq \mathbb{R}^N \). Later Ghoussoub and Kang in \([21]\) showed that if all the principal curvatures of \( \partial \Omega \) at 0 are negative then \( \mu_\sigma(\Omega) < \mu_\sigma(\mathbb{R}^N) \) and it is achieved. Demyanov and Nazarov in \([17]\) proved that the extremals for \( \mu_\sigma(\Omega) \) exist when \( \Omega \) is average concave in a neighborhood of the origin. Later Ghoussoub and Robert in \([18]\) proved the existence of extremals when the boundary is smooth and the mean curvature at 0 is negative. For
more results in this direction and generalizations, we refer to Ghoussoub-Robert [15–17], Chern-Lin [8], Lin-Li [27], Lin-Walade [29, 31], the Fall, Minlend and the author [13] and references therein.

The proof of Theorem 1.2 rely on test function methods. Namely to build appropriate test functions allowing to compare \( \mu_{h,\sigma}(\Omega, \Gamma) \) and \( S_{N,\sigma} \). While it always holds that \( \mu_{h,\sigma}(\Omega, \Gamma) \leq S_{N,\sigma} \), our main task is to find a function for which \( \mu_{h,\sigma}(\Omega, \Gamma) < S_{N,\sigma} \). This then allows to recover compactness and thus every minimizing sequence for \( \mu_{h,\sigma}(\Omega, \Gamma) \) has a subsequence which converges to a minimizer. Building these approximates solutions requires to have sharp decay estimates of a minimizer \( w \) for \( S_{N,\sigma} \), see Lemma 2.3 below. In Section 3 we prove existence result when \( \mu_{h,\sigma}(\Omega, \Gamma) < S_{N,\sigma} \). In Section 4 we build continuous family of test functions \((u_\varepsilon)_{\varepsilon>0}\) concentrating at a point \( y_0 \in \Gamma \) which yields \( \mu_{h,\sigma}(\Omega, \Gamma) < S_{N,\sigma} \), as \( \varepsilon \to 0 \), provided (1.7) holds.

\[ \text{Lemma 2.3} \]

Let \( \Gamma \subset \mathbb{R}^N \) be a smooth closed submanifold of dimension \( k \) with \( 2 \leq k \leq N - 2 \). For \( y_0 \in \Gamma \), we let \( (E_1; \ldots ; E_k) \) be an orthonormal basis of \( T_{y_0}\Gamma \), the tangent space of \( \Gamma \) at \( y_0 \). For \( r > 0 \) small, a neighborhood of \( y_0 \in \Gamma \) can be parametrized by the mapping \( f : B_{\varepsilon}((0, r)) \rightarrow \Gamma \) defined by

\[ f(t) := \exp_{y_0}^\Gamma \left( \sum_{a=1}^{k} t_a E_a \right), \]

where \( \exp_{y_0}^\Gamma \) is the exponential map of \( \Gamma \) at \( y_0 \) and \( B_{\varepsilon}((0, r)) \) is the ball of \( \mathbb{R}^k \) centered at 0 and of radius \( r \). We choose a smooth orthonormal frame field \( (E_{k+1}(f(t)); \ldots ; E_N(f(t))) \) on the normal bundle of \( \Gamma \) such that \( (E_1(f(t)); \ldots ; E_N(f(t))) \) is an oriented basis of \( \mathbb{R}^N \) for every \( t \in B_{\varepsilon}^k \), with \( E_i(f(0)) = E_i \). We fix the following notation, that will be used a lot in the paper,

\[ Q_r := B_{\varepsilon}^k(0, r) \times B_{\varepsilon}^{N-k}(0, r), \]

where \( B_{\varepsilon}^m(0, r) \) denotes the ball in \( \mathbb{R}^m \) with radius \( r \) centered at the origin. Provided \( r > 0 \) small, the map \( F_{y_0} : Q_r \rightarrow \Omega \), given by

\[ (t, z) \mapsto F_{y_0}(t, z) := f(t) + \sum_{i=2}^{N} z_i E_i(f(t)), \]

is smooth and parameterizes a neighborhood of \( y_0 = F_{y_0}(0, 0) \). We consider \( \rho_{F} : \Omega \rightarrow \mathbb{R} \) the distance function to the submanifold given by

\[ \rho_{F}(t) = \min_{\Gamma \in \Gamma} |t - \Gamma|. \]

In the above coordinates, we have

\[ \rho_{F}(F_{y_0}(x)) = |z| \quad \text{for every } x = (t, z) \in Q_r. \]

Since the basis \( \{ E_i \} \) is orthonormal, then for every \( t \in B_{\varepsilon}^k(0, r) ; a, b = 1, \ldots , k \) and \( i, j = k + 1, \ldots , N \), there exists real numbers \( \Gamma_{ab}^i(f(t)) \) and \( \beta_{ij}^a(f(t)) \) such that we can write

\[ dE_i \circ \frac{\partial f}{\partial t_a} = - \sum_{k=1}^{k} \Gamma_{ab}^i \frac{\partial f}{\partial t_b} + \sum_{i

The quantity \( \Gamma_{ab}^i(f(t)) \) and \( \beta_{ij}^a(f(t)) \) are the second fundamental form and the "torsion" of \( \Gamma \). The norms of the second fundamental form and the mean curvature are then given
respectively by
\[
\Gamma := \left( \sum_{i=0}^{k} \sum_{a=1}^{N} \left( \Gamma^i_{ab} \right)^2 \right)^{1/2}
\quad \text{and} \quad
H := \left( \sum_{i=k+1}^{N} \sum_{a=1}^{k} \left( \Gamma^i_{aa} \right)^2 \right)^{1/2}.
\]

We note that provided \( r > 0 \) small, \( \Gamma^i_{ab} \) and \( \beta^i_{ja} \) are smooth functions. Moreover, it is easy to see that
\[
\beta^i_{ja}(f(t)) = -\beta^i_{ja}(f(t)) \quad \text{for } i, j = 2, \ldots, N \text{ and } a = 1, \ldots, k.
\]

Next, we derive the expansion of the metric induced by the parameterization \( F_{y_0} \) defined above. For \( x = (t, z) \in Q_r \), we define
\[
\tilde{g}_{ab}(x) := \partial_{t_a} F_{y_0}(x) \cdot \partial_{t_b} F_{y_0}(x), \quad g_{ai}(x) := \partial_{t_a} F_{y_0}(x) \cdot \partial_{z_i} F_{y_0}(x)
\]
and
\[
g_{ij}(x) := \partial_{z_i} F_{y_0}(x) \cdot \partial_{z_j} F_{y_0}(x).
\]

Then we have the following

**Lemma 2.1.** For any \( a, b = 1, \ldots, k \) and for any \( i, j = k+1, \ldots, N \), we have
\[
g_{ab}(x) = \delta_{ab} - 2 \sum_{i=k+1}^{N} z_i \Gamma^i_{ab} + \sum_{i=k+1}^{N} \sum_{j=k+1}^{N} \sum_{c=1}^{k} z_i z_j \Gamma^i_{ac} \Gamma^j_{cb} + \frac{1}{3} \sum_{c,d=1}^{k} R_{abcd}(x_0) \partial_{z_c} \partial_{z_d} + O(|x|^3)
\]
and
\[
g_{ai}(x) = \sum_{j=k+1}^{N} z_j \beta^j_{ia} \quad \text{and} \quad g_{ij}(x) = \delta_{ij},
\]
where the curvature terms \( \Gamma^i_{ab} \) and \( \beta^i_{ja} \) are computed at the point \( f(t) \).

**Proof.** We use the expression in (2.1) to get
\[
\frac{\partial F}{\partial t_a} = \frac{\partial f}{\partial t_a} + \sum_{i=k+1}^{N} z_i dE_i \circ \frac{\partial f}{\partial t_a}, \quad \text{and} \quad \frac{\partial F}{\partial z_i} = E_i.
\]

Then using (2.5) and the fact that \( \frac{\partial f}{\partial t_a} \in T_{f(t)} \Gamma \), we easily get
\[
g_{ai}(x) = \sum_{j=k+1}^{N} z_j \beta^j_{ia} \quad \text{and} \quad g_{ij}(x) = \delta_{ij}.
\]

We have also that
\[
g_{ab}(x) = \left( \frac{\partial f}{\partial t_a} \cdot \frac{\partial f}{\partial t_b} \right) + \sum_{i=k+1}^{N} z_i (dE_i \circ \frac{\partial f}{\partial t_a} \cdot \frac{\partial f}{\partial t_b}) + \sum_{i=k+1}^{N} z_i z_j (dE_i \circ \frac{\partial f}{\partial t_a} dE_j \circ \frac{\partial f}{\partial t_b}).
\]

The expansion of the induced metric \( \tilde{g}_{ab} = \left( \frac{\partial f}{\partial t_a} \cdot \frac{\partial f}{\partial t_b} \right) \) in the local chart of the exponential map is given by
\[
\tilde{g}_{ab}(x) = \delta_{ab} - \frac{1}{3} \sum_{c,d=1}^{k} R_{abcd}(x_0) \partial_{z_c} \partial_{z_d} + O(|x|^3),
\]
where the \( R_{abcd} \) are the components of the tensor curvature of \( \Gamma \), see [22]. We then plug (2.7) in (2.6) to get

\[
g_{ab}(x) = \delta_{ab} - 2 \sum_{i=k+1}^{N} z_i \Gamma^i_{ab} + \sum_{ij=k+1}^{N} \sum_{c=1}^{k} z_i z_j \Gamma^i_{ac} \Gamma^j_{bc} + \sum_{ij=k+1}^{N} \sum_{\substack{c=1,\cdots,k \atop j \neq j}}^{k} z_i z_j \beta^i_{la} \beta^j_{lb} - \frac{1}{3} \sum_{cd=1}^{k} R_{abcd}(x_0) t_c t_d + O(|x|^3).
\]

This ends the proof. \( \square \)

We will need the following result deduced from Lemma 2.1.

**Lemma 2.2.** In a small neighborhood of the point \( y_0 \in \Omega \) the expansion of the square root of the determinant of the metric is given by

\[
\sqrt{|g|}(x) = 1 - \sum_{i=k+1}^{N} z_i H^i - \frac{1}{2} \sum_{ij=k+1}^{N} \sum_{ab=1}^{k} z_i z_j \Gamma^i_{ab} \Gamma^j_{ab} + \frac{1}{3} \sum_{ij=k+1}^{N} z_i z_j H^i H^j
\]

(2.8)

Moreover the components of the inverse of the metric are

\[
g^{ab}(x) = \delta_{ab} + 2 \sum_{i=k+1}^{N} z_i \Gamma^i_{ab} + 3 \sum_{ij=k+1}^{N} \sum_{c=1}^{k} z_i z_j \Gamma^i_{ac} \Gamma^j_{bc} + \frac{1}{3} \sum_{ij=k+1}^{N} \sum_{\substack{c=1,\cdots,k \atop j \neq j}}^{k} z_i z_j H^i H^j + O(|x|^3)
\]

(2.9)

\[
g^{ai}(x) = - \sum_{j=k+1}^{N} z_j \beta^i_{la} - 2 \sum_{c=1}^{k} \sum_{lm=k+1}^{N} z_l z_m \Gamma^i_{ac} \Gamma^m_{ae} + O(|x|^3)
\]

\[
g^{ij}(x) = \delta_{ij} + \sum_{c=1}^{k} \sum_{lm=k+1}^{N} z_l z_m \beta^i_{lc} \beta^j_{mc} + O(|x|^3).
\]

**Proof.** We can write \( g(x) = I + A \). Then we have the classical expansion

\[
\sqrt{\det(I + A)}(x) = 1 + \frac{\text{tr} A}{2} + \frac{\left(\text{tr} A\right)^2}{4} - \frac{\text{tr} (A^2)}{4} + O(|A|^3),
\]

where we have

\[
\frac{\text{tr} A}{2} = - \sum_{i=k+1}^{N} z_i H^i + \sum_{ij=k+1}^{N} \sum_{ab=1}^{k} z_i z_j \Gamma^i_{ab} \Gamma^j_{ab} + \sum_{ij=k+1}^{N} \sum_{\substack{c=1,\cdots,k \atop j \neq j}}^{k} z_i z_j \beta^i_{la} \beta^j_{lb} - \frac{1}{6} \sum_{cd=1}^{k} R_{cd}(x_0) t_c t_d + O(|x|^3)
\]

and

\[
\frac{\left(\text{tr} A\right)^2}{4} = \sum_{ij=k+1}^{N} z_i z_j H^i H^j + O(|x|^3),
\]

where for \( i = k + 1, \ldots, N \) we have

\[
H^i = \sum_{a=1}^{k} \Gamma^i_{aa}
\]
are the components of the mean curvature of $\Gamma$. Moreover using the fact that the matrix $A$ is symmetric, we get
\[
trA^2 = \sum_{\alpha=1}^{N} (A^2)_{\alpha\alpha} = \sum_{\alpha=1}^{N} \left( \sum_{\beta=1}^{N} A_{\alpha\beta}(x)A_{\alpha\beta}(x) \right).
\]
Then
\[
-\frac{trA^2}{4} = -\frac{1}{4} \left( \sum_{\alpha=1}^{N} A_{\alpha\beta}(x) + 2 \sum_{a=1}^{N} \sum_{i=k+1}^{N} A_{ai}(x) + \sum_{i=j=k+1}^{N} A_{ij}^2(x) \right).
\]
Therefore
\[
(2.13)
-\frac{trA^2}{4} = -\sum_{ij=k+1}^{N} \sum_{ab=1}^{k} z_i z_j \Gamma_{ab}^i \Gamma_{ab}^j - \frac{1}{2} \sum_{ij=k+1}^{N} \sum_{ab=1}^{k} z_i z_j \beta_\alpha^i \beta_\alpha^j.
\]
By (2.10), (2.11), (2.12) and (2.13), we finally obtain
\[
\sqrt{|g|}(x) = 1 - \sum_{i=k+1}^{N} z_i H^i - \frac{1}{2} \sum_{ij=k+1}^{N} \sum_{ab=1}^{k} z_i z_j \Gamma_{ab}^i \Gamma_{ab}^j
+ \sum_{ij=k+1}^{N} z_i z_j H^i H^j - \frac{1}{6} \sum_{cd=1}^{k} Ric_{cd}(x_0) t_c t_d + O (|x|^3).
\]
We write
\[
g(x) = I + A(x) + B(x) + O (|x|^3),
\]
where $A$ and $B$ are symmetric matrix given by
\[
A_{ab}(x) = -2 \sum_{i=k+1}^{N} z_i \Gamma_{ab}^i; \quad A_{ai}(x) = \sum_{j=k+1}^{N} z_j \beta_\alpha^j \text{ and } A_{ij}(x) = 0
\]
and
\[
B_{ab}(x) = \sum_{ij=k+1}^{N} \sum_{c=1}^{k} z_i z_j \Gamma_{ac}^i \Gamma_{bc}^j + \sum_{ij=k+1}^{N} \sum_{ab=1}^{k} z_i z_j \beta_\alpha^i \beta_\alpha^j - \frac{3}{2} \sum_{cd=1}^{k} R_{abcd}(x_0) t_c t_d
\]
and
\[
B_{ai}(x) = B_{ij}(x) = 0.
\]
It’s clear that the inverse of the metric $g^{-1}$ is given by
\[
g^{-1}(x) = I - A(x) - B(x) + A^2(x) + O (|x|^3).
\]
This yields
\[
g^{ab}(x) = \delta_{ab} - A_{ab}(x) - B_{ab}(x) + \sum_{c=1}^{k} A_{ac}(x)A_{bc}(x) + \sum_{i=k+1}^{N} A_{ai}(x)A_{bi}(x) + O (|x|^3)
\]
\[
g^{ai}(x) = -A_{ai}(x) - B_{ai}(x) + \sum_{c=1}^{k} A_{ac}(x)A_{ic}(x) + \sum_{j=k+1}^{N} A_{aj}(x)A_{ij}(x) + O (|x|^3)
\]
\[
g^{ij}(x) = \delta_{ij} - A_{ij}(x) - B_{ij}(x) + \sum_{c=1}^{k} A_{ac}(x)A_{jc}(x) + \sum_{l=k+1}^{N} A_{ul}(x)A_{jl}(x) + O (|x|^3).
\]
Hence we obtain that
\[(2.14)\]
\[g^{ab}(x) = \delta_{ab} + 2 \sum_{i=k+1}^{N} z_i \Gamma_{ai}^{i} + 3 \sum_{ij=k+1}^{N} \sum_{c=1}^{k} z_i z_j \Gamma_{ac}^{i} \Gamma_{bc}^{j} + \frac{1}{3} \sum_{cld=1}^{k} R_{acbd}(x) t_c t_d + O \left( |x|^3 \right) \]
\[g^{ai}(x) = - \sum_{j=k+1}^{N} z_j \beta_{ia}^{j} - 2 \sum_{c=1}^{k} \sum_{lm=k+1}^{N} z_l z_c \Gamma_{ac}^{i} \Gamma_{lc}^{m} + O \left( |x|^3 \right) \]
\[g^{ij}(x) = \delta_{ij} + \sum_{c=1}^{k} \sum_{lm=k+1}^{N} z_l z_m \beta_{ic}^{i} \beta_{jc}^{m} + O \left( |x|^3 \right) . \]

This ends the proof of the lemma. \( \square \)

We consider the best constant for the cylindrical Hardy-Sobolev inequality
\[S_{N,\sigma} = \min \left\{ \int_{\mathbb{R}^N} |\nabla w|^2 \, dx : w \in D^{1,2}(\mathbb{R}^N), \int_{\mathbb{R}^N} |z|^{-\sigma} |w|^{2^*} \, dx = 1 \right\} . \]

As mentioned in the first section, it is attained by a positive function \( w \in D^{1,2}(\mathbb{R}^N) \), satisfying
\[(2.15)\]
\[- \Delta w = S_{N,\sigma} |z|^{-\sigma} w^{2^*-1} \quad \text{in} \ \mathbb{R}^N, \]
see e.g. [2]. Moreover from [12], we have
\[(2.16)\]
\[w(x) = w(t, z) = \theta(|t|, |z|) \quad \text{for a function} \quad \theta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+. \]

We will need the following preliminary result in the sequel.

**Lemma 2.3.** Let \( w \) be a ground state for \( S_{N,\sigma} \) then there exist positive constants \( C_1, C_2 \), only depending on \( N \) and \( \sigma \), such that

(i) For every \( x \in \mathbb{R}^N \)
\[(2.17)\]
\[\frac{C_1}{1 + |x|^{N-2}} \leq w(x) \leq \frac{C_2}{1 + |x|^{N-2}}. \]

(ii) For \( |x| = |(t, z)| \leq 1 \)
\[|\nabla w(x)| + |x| |D^2 w(x)| \leq C_2 |z|^{1-\sigma} \]

(iii) For \( |x| = |(t, z)| \geq 1 \)
\[|\nabla w(x)| + |x| |D^2 w(x)| \leq C_2 \max(1, |z|^{-\sigma}) |x|^{1-N} . \]

Fabbri, Mancini and Sandeep proved (i) in [12]. The proof of (ii) and (iii) are done by the Fall and the author in [14].

3. Existence Result

Let \( \Omega \) be a bounded domain of \( \mathbb{R}^N \), \( N \geq 3 \), and \( h \) a continuous function on \( \Omega \). Let \( \Gamma \) be a smooth closed submanifold contained in \( \Omega \). We consider
\[(3.1)\]
\[\mu_{h, \sigma}(\Omega, \Gamma) := \inf_{u \in H_0^1(\Omega)} \left( \int_{\Omega} |\nabla u|^2 \, dx + \int_{\Omega} hu^2 \, dy \right)^{1/2} \left( \int_{\Omega} \rho_t^{-\sigma} |u|^{2^*} \, dy \right)^{1/2} . \]

We also recall that
\[(3.2)\]
\[S_{N,\sigma} = \inf_{w \in D^{1,2}(\mathbb{R}^N)} \left( \int_{\mathbb{R}^N} |\nabla w|^2 \, dx \right)^{1/2} \left( \int_{\mathbb{R}^N} |z|^{-\sigma} |w|^{2^*} \, dx \right)^{1/2}. \]
with $x = (t, z) \in \mathbb{R} \times \mathbb{R}^{N-1}$. Our aim in this section is to show that if $\mu_{h, \sigma}(\Omega, \Gamma) < S_{N, \sigma}$ then the best constant $\mu_{h, \sigma}(\Omega, \Gamma)$ is achieved. The argument of proof is standard. However, for sake of completeness, we add the proof. We start with the following

**Lemma 3.1.** Let $\Omega$ be an open subset of $\mathbb{R}^N$, with $N \geq 3$, and let $\Gamma \subset \Omega$ be a smooth closed submanifold. Then for every $r > 0$, there exist positive constants $c_r > 0$, only depending on $\Omega, \Gamma, N, \sigma$ and $r$, such that for every $u \in H^1_0(\Omega)$

$$S_{N, \sigma} \left( \int_\Omega \rho_{T}^{-\sigma}|u|^{2^*_s} \, dy \right)^{2/2^*_s} \leq (1 + r) \int_\Omega |\nabla u|^2 \, dy + c_r \left( \int_\Omega |u|^{2^*_s} \, dy \right)^{2/2^*_s},$$

where $2^*_s = \frac{2(N-\sigma)}{N-2}$ and $\sigma \in (0, 2)$.

**Proof.** We let $r > 0$ small. We can cover a tubular neighborhood of $\Gamma$ by a finite number of sets $(T_r^i)_{1 \leq i \leq m}$ given by

$$T_r^i := F_{y_i}(Q_r), \quad \text{with } y_i \in \Gamma.$$

We refer to Section 22 for the parameterization $F_{y_i} : Q_r \to \Omega$. See e.g. [1, Section 2.27], there exists $(\varphi_i)_{1 \leq i \leq m}$ a partition of unity subordinated to this covering such that

$$\sum_i \varphi_i = 1 \quad \text{and} \quad |\nabla \varphi_i| \leq K \quad \text{in } U := \bigcup_{i=1}^m T_r^i,$$

for some constant $K > 0$. We define

$$\psi_i(y) := \varphi_i^{1/2^*_s}(y) u(y) \quad \text{and} \quad \tilde{\psi}_i(x) = \psi_i(F_{y_i}(x)).$$

Recall that $\rho_{T}^i > C > 0$ on $\Omega \setminus U$, for some positive constant $C > 0$. Therefore, since $\frac{1}{2^*_s} < 1$, by (3.3) we get

$$\left( \int_{T_r^i} \rho_{T}^{-\sigma} |\tilde{\psi}_i|^{2^*_s} \, dy \right)^{2/2^*_s} \leq \left( \int_U \rho_{T}^{-\sigma} |u|^{2^*_s} \, dy \right)^{2/2^*_s} + \left( \int_{\Omega \setminus U} |u|^{2^*_s} \, dy \right)^{2/2^*_s} \leq \sum_i \left( \int_{T_r^i} \rho_{T}^{-\sigma} |\tilde{\psi}_i|^{2^*_s} \, dy \right)^{2/2^*_s} + c_r \left( \int_\Omega |u|^{2^*_s} \, dy \right)^{2/2^*_s} \leq \sum_i \left( \int_{T_r^i} \rho_{T}^{-\sigma} |\tilde{\psi}_i|^{2^*_s} \, dy \right)^{2/2^*_s} + c_r \left( \int_\Omega |u|^{2^*_s} \, dy \right)^{2/2^*_s}.$$

By change of variables and Lemma 222 we have

$$\left( \int_{T_r^i} \rho_{T}^{-\sigma} |\tilde{\psi}_i|^{2^*_s} \, dy \right)^{2/2^*_s} = \left( \int_{Q_r} \left| z \right|^{-\sigma} |\tilde{\psi}_i|^{2^*_s} \sqrt{|g|(x)} \, dx \right)^{2/2^*_s} \leq (1 + cr) \left( \int_{Q_r} \left| z \right|^{-\sigma} |\tilde{\psi}_i|^{2^*_s} \, dx \right)^{2/2^*_s}.$$ 

In addition by the Hardy-Sobolev best constant (3.2), we have

$$S_{N, \sigma} \left( \int_{Q_r} \left| z \right|^{-\sigma} \left| \tilde{\psi}_i \right|^{2^*_s} \, dx \right)^{2/2^*_s} \leq \left( \int_{Q_r} \left| \nabla \tilde{\psi}_i \right|^2 \, dx \right)^{2/2^*_s}. $$

Therefore by change of variables and Lemma 222 we get

$$S_{N, \sigma} \left( \int_{T_r^i} \rho_{T}^{-\sigma} |\tilde{\psi}_i|^{2^*_s} \, dy \right)^{2/2^*_s} \leq (1 + cr) \int_{Q_r} \left| \nabla \tilde{\psi}_i \right|^2 \, dx \leq (1 + c'r) \int_{T_r^i} \left| \nabla (\varphi_i^{1/2^*_s} u) \right|^2 \, dy = (1 + c'r) \int_{T_r^i} |\varphi_i^{1/2^*_s} \nabla u + u \nabla \varphi_i^{1/2^*_s}|^2 \, dy + c_r \int_\Omega |u|^2 \, dy.$$
Applying Young’s inequality using (3.3) and (3.4), we find that
\[
S_{N,\sigma} \left( \int \rho_{\tau}^{-\sigma} \psi_i^2 \, dy \right)^{2/\sigma} \leq (1 + c') (1 + \varepsilon) \int \varphi_i^2 \left( \frac{|\nabla u|^2}{m} \right) dy + c_r(\varepsilon) \int |u|^2 \, dy \\
\leq (1 + c') (1 + \varepsilon) \int |\nabla u|^2 \, dy + c_r(\varepsilon) \int |u|^2 \, dy.
\]
Summing for \( i \) equal 1 to \( m \), we get
\[
S_{N,\sigma} \sum_{i=1}^{m} \left( \int \rho_{\tau}^{-\sigma} \psi_i^2 \, dy \right)^{2/\sigma} \leq (1 + c') (1 + \varepsilon) \left( \int |\nabla u|^2 \, dy \right)^{1/2} + c_r(\varepsilon) \left( \int |u|^2 \, dy \right)^{1/2}.
\]
This together with (3.5) give
\[
S_{N,\sigma} \left( \int \rho_{\tau}^{-\sigma} |u|^2 \, dy \right)^{2/\sigma} \leq (1 + c') (1 + \varepsilon) \int |\nabla u|^2 \, dy + c_r(\varepsilon) \int |u|^2 \, dy + c_r \left( \int |u|^2 \, dy \right)^{2/\sigma}.
\]
Since \( \varepsilon \) and \( r \) can be chosen arbitrarily small, we get the desired result. \( \square \)

We can now prove the following existence result.

**Proposition 3.2.** Consider \( \mu_{h,\sigma}(\Omega, \Gamma) \) and \( S_{N,\sigma} \) given by (3.1) and (3.2) respectively. Suppose that
\[
(3.6) \quad \mu_{h,\sigma}(\Omega, \Gamma) < S_{N,\sigma}.
\]
Then \( \mu_{h,\sigma}(\Omega, \Gamma) \) is achieved by a positive function.

**Proof.** Let \( (u_n)_{n \in \mathbb{N}} \) be a minimizing sequence for \( \mu_{h,\sigma}(\Omega, \Gamma) \) normalized so that
\[
(3.7) \quad \int \rho_{\tau}^{-\sigma} |u|^2 \, dx = 1 \quad \text{and} \quad \mu_{h,\sigma}(\Omega, \Gamma) = \int \nabla u_n|^2 \, dx + \int u_n^2 \, dx + o(1).
\]
By coercivity of \( -\Delta + h \), the sequence \( (u_n)_{n \in \mathbb{N}} \) is bounded in \( H_0^1(\Omega) \) and thus, up to a subsequence,
\[
u_n \rightharpoonup u \quad \text{weakly in } H_0^1(\Omega),
\]
and
\[
(3.8) \quad u_n \to u \quad \text{strongly in } L^p(\Omega) \quad \text{for } 1 \leq p < 2^*_0 := \frac{2N}{N-2}.
\]
The weak convergence in \( H_0^1(\Omega) \) implies that
\[
(3.9) \quad \int |\nabla u_n|^2 \, dx = \int |\nabla (u_n - u)|^2 \, dx + \int |\nabla u|^2 \, dx + o(1).
\]
By Brezis-Lieb lemma \( 3 \) and the strong convergence in the Lebesgue spaces \( L^p(\Omega) \), we have
\[
(3.10) \quad 1 = \int \rho_{\tau}^{-\sigma} |u|^2 \, dx = \int \rho_{\tau}^{-\sigma} |u - u_n|^2 \, dx + \int \rho_{\tau}^{-\sigma} |u_n|^2 \, dx + o(1).
\]
By Lemma 3.1 and 3.3 — note that \( 2^*_0 < 2^*_0 \), we then deduce that
\[
(3.11) \quad S_{N,\sigma} \left( \int \rho_{\tau}^{-\sigma} |u - u_n|^2 \, dx \right)^{2/\sigma} \leq (1 + r) \int |\nabla (u - u_n)|^2 \, dx + o(1).
\]
Using (3.9), (3.10) and (3.11), we have

\[
S_{N, \sigma} \left( 1 - \int_{\Omega} \rho_{\Gamma}^{-\sigma} |u|^{2^*_\sigma} \, dx \right)^{2^{*}_\sigma} \leq (1 + r) \left( \int_{\Omega} |\nabla u_h|^2 \, dx - \int_{\Omega} |\nabla u|^2 \, dx \right) + o(1)
\]

\[
= (1 + r) \left( \mu_{h, \sigma}(\Omega, \Gamma) - \int_{\Omega} h u_h^2 \, dx - \int_{\Omega} |\nabla u|^2 \, dx \right) + o(1)
\]

\[
= (1 + r) \left( \mu_{h, \sigma}(\Omega, \Gamma) - \int_{\Omega} h u^2 \, dx - \int_{\Omega} |\nabla u|^2 \, dx \right) + o(1)
\]

(3.12)

By the concavity of the map \( t \mapsto t^{2^{*}_\sigma} \) on \([0, 1]\), we have

\[
1 \leq \left( 1 - \int_{\Omega} \rho_{\Gamma}^{-\sigma} |u|^{2^*_\sigma} \, dx \right)^{2^{*}_\sigma} + \left( \int_{\Omega} \rho_{\Gamma}^{-\sigma} |u|^{2^*_\sigma} \, dx \right)^{2^{*}_\sigma}.
\]

From this, then taking the limits respectively as \( n \to +\infty \) and as \( r \to 0 \) in (3.12), we find that

\[
[S_{N, \sigma} - \mu_{h, \sigma}(\Omega, \Gamma)] \left( 1 - \left( \int_{\Omega} \rho_{\Gamma}^{-\sigma} |u|^{2^*_\sigma} \, dx \right)^{2^{*}_\sigma} \right) \leq 0.
\]

Thanks to (3.6), we then get

\[
1 \leq \int_{\Omega} \rho_{\Gamma}^{-\sigma} |u|^{2^*_\sigma} \, dx.
\]

Since by (3.7) and Fatou’s lemma,

\[
1 = \int_{\Omega} \rho_{\Gamma}^{-\sigma} |u_h|^{2^*_\sigma} \, dx \geq \int_{\Omega} \rho_{\Gamma}^{-\sigma} |u|^{2^*_\sigma} \, dx,
\]

we conclude that

\[
\int_{\Omega} \rho_{\Gamma}^{-\sigma} |u|^{2^*_\sigma} \, dx = 1.
\]

It then follows from (3.7) that \( u_h \to u \) in \( L^{2^*_\sigma}(\Omega; \rho_{\Gamma}^{-\sigma}) \) and thus \( u_h \to u \) in \( H_0^1(\Omega) \). Therefore \( u \) is a minimizer for \( \mu_{h, \sigma}(\Omega, \Gamma) \). Since \( |u| \) is also a minimizer for \( \mu_{h, \sigma}(\Omega, \Gamma) \), we may assume that \( u \geq 0 \). Therefore \( u > 0 \) by the maximum principle. \( \square \)

4. Comparing \( S_{N, \sigma} \) and \( \mu_{h, \sigma}(\Omega, \Gamma) \)

**Lemma 4.1.** Let \( v \in D^{1,2}(\mathbb{R}^N), N \geq 3 \), satisfy \( v(t, z) = \Theta(|t|, |z|) \), for some some function \( \Theta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R} \). Then for \( 0 < r < R \), we have

\[
\int_{Q_R \setminus Q_r} |\nabla v|^2 \, \sqrt{g} \, dx = \int_{Q_R \setminus Q_r} |\nabla v|^2 \, dx + \frac{3 \Gamma^2 - 2 H^2}{k(N - k)} \int_{Q_R \setminus Q_r} |z|^2 |\nabla v|^2 \, dx
\]

\[
+ \frac{R_g(x_0)}{3k^2} \int_{Q_R \setminus Q_r} |t|^2 |\nabla v|^2 \, dx + \frac{H^2 - \Gamma^2}{N - k} \int_{Q_R \setminus Q_r} |z|^2 |\nabla v|^2 \, dx
\]

\[
- \frac{R_g(x_0)}{6k} \int_{Q_R \setminus Q_r} |t|^2 |\nabla v|^2 \, dx + O \left( \int_{Q_R \setminus Q_r} |x|^3 |\nabla v|^2 \, dx \right).
\]
Proof. It is easy to see that

\begin{equation}
\int_{Q_R \setminus Q_r} |\nabla v|^2 \sqrt{|g|} dx = \int_{Q_R \setminus Q_r} |\nabla v|^2 dx + \int_{Q_R \setminus Q_r} (|\nabla v|^2 - |\nabla v|^2) \sqrt{|g|} dx
\end{equation}

(4.2)

\begin{equation}
\sum_{ij=k+1}^N \beta_{ij}^a \beta_{ij}^b \frac{z_2^2 + z_3^2}{|z|^2} |\nabla v|^2 dx
\end{equation}

(4.3)

\begin{equation}
\sum_{a=1}^k \sum_{i=2}^N g^{ia} (\partial_{ia} \nabla v \cdot \nabla v) z_i t_a \sqrt{|g|} dx
\end{equation}

(4.6)

We recall that

\begin{equation}
|\nabla v|^2(x) - |\nabla v|^2 = \sum_{\alpha \beta=1}^N \left[ g^{\alpha \beta} (x) - \delta_{\alpha \beta} \right] \partial_{\alpha x} v(x) \partial_{\beta x} v(x).
\end{equation}

It then follows that

\begin{equation}
\int_{Q_R \setminus Q_r} (|\nabla v|^2 - |\nabla v|^2) \sqrt{|g|} dx = \sum_{ij=k+1}^N \int_{Q_R \setminus Q_r} \left[ g^{ij} - \delta_{ij} \right] \partial_{zi} v \partial_{zj} v \sqrt{|g|} dx
\end{equation}

(4.4)

We first use Lemma 2.1, Lemma 2.2 and (2.4), to get

\begin{equation}
\sum_{ij=k+1}^N \int_{Q_R \setminus Q_r} \left[ g^{ij} - \delta_{ij} \right] \partial_{zi} v \partial_{zj} v \sqrt{|g|} dx
\end{equation}

(4.5)

Using again Lemma 2.1 and Lemma 2.2 it easy follows that

\begin{equation}
\sum_{a=1}^k \sum_{i=2}^N \int_{Q_R \setminus Q_r} g^{ia} (\partial_{ia} v \partial_{zj} v) \sqrt{|g|} dx = \sum_{a=1}^k \sum_{i=2}^N \int_{Q_R \setminus Q_r} g^{ia} (\nabla v \cdot \nabla v) z_i t_a \sqrt{|g|} dx
\end{equation}

(4.6)
By Lemma 2.1 and Lemma 2.2 we then have

\begin{align*}
\sum_{ab=1}^{k} \int_{Q_r \cap Q_x} [g^{ab} - \delta_{ab}] (\partial_{x_a} v \partial_{x_b} v) \sqrt{|g|} dx &= \sum_{ab=1}^{k} \int_{Q_r \cap Q_x} \left[ -2 \sum_{ij=k+1}^{N} z_{i} z_{j} H^{i} \Gamma_{ab}^{i} + 3 \sum_{ij=k+1}^{N} z_{i} z_{j} \Gamma_{a l i} \Gamma_{b c j} + \frac{1}{3} \sum_{c d=1}^{k} R_{a c b d}(x_0) t_{c} t_{d} + O \left( |x|^3 \right) \right] \frac{t_{a} t_{b}}{|t|^2} |\nabla_{x} v|^2 dx.
\end{align*}

Therefore

\begin{align*}
\sum_{ab=1}^{k} \int_{Q_r \cap Q_x} [g^{ab} - \delta_{ab}] (\partial_{x_a} v \partial_{x_b} v) \sqrt{|g|} dx &= \frac{3 \Gamma^2 - 2 H^2}{k(N-k)} \int_{Q_r \cap Q_x} |z|^2 |\nabla_{x} v|^2 dx + O \left( \int_{Q_r \cap Q_x} |x|^3 |\nabla_{x} v|^2 dx \right)
\end{align*}

(4.7)

By Lemma 2.2 we have

\begin{align*}
\int_{Q_r \cap Q_x} |\nabla v|^2 (\sqrt{|g|} - 1) dx &= \frac{H^2 - \Gamma^2/2}{N-k} \int_{Q_r \cap Q_x} |z|^2 |\nabla_{x} v|^2 dx
\end{align*}

(4.8)

\begin{align*}
- \frac{R_{a}(x_0)}{6 k} \int_{Q_r \cap Q_x} |t|^2 |\nabla_{x} v|^2 dx + O \left( \int_{Q_r \cap Q_x} |x|^3 |\nabla_{x} v|^2 dx \right)
\end{align*}

(4.9)

The result follows from (4.1), (4.4), (4.5), (4.6), (4.7) and (4.8). This then ends the proof. \(\square\)

We consider \(\Omega\) a bounded domain of \(\mathbb{R}^N\), \(N \geq 3\), and \(\Gamma \subset \Omega\) be a smooth closed submanifold of dimension \(k\) with \(2 \leq k \leq N - 2\). For \(u \in H^1_0(\Omega) \setminus \{0\}\), we define the ratio

\begin{align*}
J(u) := \frac{\int_{\Omega} |\nabla u|^2 dy + \int_{\Omega} h u^2 dy}{\left( \int_{\Omega} \rho_{\Gamma}^{-\sigma} |u|^{2\sigma} dy \right)^{2/2\sigma}}.
\end{align*}

(4.10)

We let \(\eta \in C^\infty_c(Q_{2r})\) be such that

\begin{align*}
0 \leq \eta \leq 1 \quad \text{and} \quad \eta \equiv 1 \quad \text{in} \ Q_r.
\end{align*}

For \(\varepsilon > 0\), we consider \(u_\varepsilon : \Omega \to \mathbb{R} \) given by

\begin{align*}
u_\varepsilon(y) := \varepsilon^{-\frac{2-N}{2}} \eta(F_{y_0}^{-1}(y)) w \left( \varepsilon^{-1} F_{y_0}^{-1}(y) \right).
\end{align*}

(4.11)

In particular, for every \(x = (t, z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}\), we have

\begin{align*}
u_\varepsilon(F_{y_0}(x)) := \varepsilon^{\frac{2-N}{2}} \eta(x) \theta(|t|/\varepsilon, |z|/\varepsilon).
\end{align*}

(4.12)

It is clear that \(u_\varepsilon \in H^1_0(\Omega)\). Then we have the following expansion.
Lemma 4.2. For $J$ given by (4.10) and $u_\varepsilon$ given by (4.11), as $\varepsilon \to 0$, we have

$$J(u_\varepsilon) = S_{N,\sigma} + \varepsilon^2 \frac{H^2 - 3R_0(x_0)}{k(N - k)} \int_{Q_{\varepsilon}} |z|^2 |\nabla \varepsilon w|^2 dx + \varepsilon^2 \frac{R_0(x_0)}{3k^2} \int_{Q_{\varepsilon}} |t|^2 |\nabla \varepsilon w|^2 dx$$

(4.13)

$$+ \varepsilon^2 \frac{H^2 + R_0(x_0)}{2(\varepsilon^2 - k)} \int_{Q_{\varepsilon}} |z|^2 |\nabla w|^2 dx - \varepsilon^2 \frac{R_0(x_0)}{6k} \int_{Q_{\varepsilon}} |t|^2 |\nabla w|^2 dx$$

(4.14)

$$+ \varepsilon^2 \frac{H^2 + R_0(x_0)}{2(\varepsilon^2 - k)} S_{N,\sigma} \int_{Q_{\varepsilon}} |z|^2 |\nabla w|^2 dx - \varepsilon^2 \frac{R_0(x_0)}{2(\varepsilon^2 - k)} S_{N,\sigma} \int_{Q_{\varepsilon}} |t|^2 |\nabla w|^2 dx$$

(4.15)

$$+ \varepsilon^2 h(y_0) \int_{Q_{\varepsilon}} \omega^2 dx + O \left( \varepsilon^2 \int_{Q_{\varepsilon}} |h(F_{y_0}(\varepsilon x) - h(y_0)||\nabla w|^2 dx \right) + O \left( \varepsilon^{N-2} \right).$$

Proof. To simplify the notations, we will write $F$ in the place of $F_{y_0}$. Recalling (4.11), we write

$$u_\varepsilon(y) = \frac{2-N}{\varepsilon} \eta(F^{-1}(y)) W_\varepsilon(y),$$

where $W_\varepsilon(y) = W \left( \frac{y}{\varepsilon} \right).$ Then $|\nabla u_\varepsilon|^2 = \varepsilon^{2-N} \left( \eta^2 |\nabla W_\varepsilon|^2 + \eta^2 |\nabla W_\varepsilon|^2 + \frac{1}{2} \nabla W_\varepsilon \cdot \nabla \eta^2 \right).$

Integrating by parts, we have

$$\int |\nabla u_\varepsilon|^2 dy = \varepsilon^{2-N} \int_{F(Q_{2\varepsilon})} \eta^2 |\nabla W_\varepsilon|^2 dy + \varepsilon^{2-N} \int_{F(Q_{2\varepsilon}) \setminus F(Q_\varepsilon)} W_\varepsilon^2 \left( |\nabla \eta|^2 - \frac{1}{2} \Delta \eta^2 \right) dy$$

$$= \varepsilon^{2-N} \int_{F(Q_{2\varepsilon})} \eta^2 |\nabla W_\varepsilon|^2 dy - \varepsilon^{2-N} \int_{F(Q_{2\varepsilon}) \setminus F(Q_\varepsilon)} W_\varepsilon^2 \eta \Delta \eta dy$$

(4.16)

$$= \varepsilon^{2-N} \int_{F(Q_{2\varepsilon})} \eta^2 |\nabla W_\varepsilon|^2 dy + O \left( \varepsilon^{2-N} \int_{F(Q_{2\varepsilon}) \setminus F(Q_\varepsilon)} W_\varepsilon^2 \eta dy \right).$$

By the change of variable $y = \frac{F(z)}{\varepsilon}$ and (4.12), we can apply Lemma 4.1 to get

$$\int |\nabla u_\varepsilon|^2 dy = \int_{Q_{\varepsilon}} |\nabla w|_{Q_{2\varepsilon}} |\nabla w|_{Q_{2\varepsilon},Q_{\varepsilon}} \int_{Q_{\varepsilon}} |\nabla w|^2 dx$$

$$= \int_{Q_{\varepsilon}} |\nabla w|^2 dx + \varepsilon^2 \frac{3\Gamma^2 - 2H^2}{k(N - k)} \int_{Q_{\varepsilon}} |z|^2 |\nabla \varepsilon w|^2 dx + \varepsilon^2 \frac{R_0(x_0)}{3k^2} \int_{Q_{\varepsilon}} |t|^2 |\nabla \varepsilon w|^2 dx$$

$$+ \varepsilon^2 \frac{H^2 - \Gamma^2}{N - k} \int_{Q_{\varepsilon}} |z|^2 |\nabla w|^2 dx - \varepsilon^2 \frac{R_0(x_0)}{6k} \int_{Q_{\varepsilon}} |t|^2 |\nabla w|^2 dx + O(\rho(\varepsilon))$$

$$= S_{N,\sigma} + \varepsilon^2 \frac{3\Gamma^2 - 2H^2}{k(N - k)} \int_{Q_{\varepsilon}} |z|^2 |\nabla w|^2 dx + \varepsilon^2 \frac{R_0(x_0)}{3k^2} \int_{Q_{\varepsilon}} |t|^2 |\nabla w|^2 dx$$

$$+ \varepsilon^2 \frac{H^2 - \Gamma^2}{N - k} \int_{Q_{\varepsilon}} |z|^2 |\nabla w|^2 dx - \varepsilon^2 \frac{R_0(x_0)}{6k} \int_{Q_{\varepsilon}} |t|^2 |\nabla w|^2 dx + O(\rho(\varepsilon)),$$

where

$$\rho(\varepsilon) = \varepsilon^3 \int_{Q_{\varepsilon}} |z|^3 |\nabla w|^2 dx + \varepsilon^3 \int_{Q_{2\varepsilon}} |w|^2 dx + \int_{R^N \setminus Q_{\varepsilon}} |\nabla w|^2 dx + \varepsilon^2 \int_{Q_{2\varepsilon} \setminus Q_{\varepsilon}} |z|^2 |\nabla w|^2 dx.$$
Using Lemma 2.3, we find that $\rho(\varepsilon) = O(\varepsilon^{N-2})$. Therefore

$$
\int_{\Omega} |\nabla u_\varepsilon|^2 \, dy = S_{N,\sigma} + \varepsilon^2 \frac{3\Gamma^2 - 2H^2}{k(N-k)} \int_{Q_{r/\varepsilon}} |z|^2 |\nabla w|^2 \, dx + \varepsilon^2 \frac{R_\varepsilon(x_0)}{3k^2} \int_{Q_{r/\varepsilon}} |t|^2 |\nabla w|^2 \, dx
$$

(4.17)

$$
+ \varepsilon^2 \frac{H^2 - \Gamma^2/2}{N-k} \int_{Q_{r/\varepsilon}} |z|^2 |\nabla w|^2 \, dx - \varepsilon^2 \frac{R_\varepsilon(x_0)}{6k} \int_{Q_{r/\varepsilon}} |t|^2 |\nabla w|^2 \, dx + O(\varepsilon^{N-2}).
$$

(4.18)

By the change of variable $y = \frac{F(x)}{\varepsilon}$, and using the fact that $\rho(F(x)) = |z|$, we get

$$\int_{Q_{r/\varepsilon}} |\nabla u_\varepsilon|^2 \, dy = \int_{Q_{r/\varepsilon}} |z|^{-\sigma} w^2\sqrt{|\gamma_k|} \, dx + O \left( \int_{Q_{2r/\varepsilon}\setminus Q_{r/\varepsilon}} |z|^{-\sigma} (\eta(\varepsilon x) w) \right)$$

$$+ O \left( \int_{Q_{2r/\varepsilon}\setminus Q_{r/\varepsilon}} |z|^{-\sigma} w^2 \, dx \right).$$

Using (4.17), we have

$$\varepsilon^3 \int_{Q_{r/\varepsilon}} |z|^3 |z|^{-\sigma} w^2 \, dx + \int_{\Omega\setminus Q_{r/\varepsilon}} |z|^{-\sigma} w^2 \, dx + \int_{Q_{2r/\varepsilon}\setminus Q_{r/\varepsilon}} |z|^{-\sigma} w^2 \, dx = O(\varepsilon^{N-\sigma}).$$

Hence by Taylor expanding, we get

$$\left( \int_{\Omega} \rho^{-\sigma} |u_\varepsilon|^{2^{*}} \, dx \right)^{2/2^*} = 1 + \varepsilon^2 \frac{2H^2 - \Gamma^2}{2N-k} \int_{Q_{r/\varepsilon}} |z|^{-\sigma} w^2 \, dx - \varepsilon^2 \frac{R_\varepsilon(x_0)}{2^{*}(3k)} \int_{Q_{r/\varepsilon}} |t|^2 |z|^{-\sigma} w^2 \, dx + O(\varepsilon^{N-\sigma}).$$

Finally, by (4.10), we conclude that

$$J(u_\varepsilon) = S_{N,\sigma} + \frac{3\Gamma^2 - 2H^2}{k(N-k)} \int_{Q_{r/\varepsilon}} |z|^2 |\nabla w|^2 \, dx + \frac{3R_\varepsilon(x_0)}{3k^2} \int_{Q_{r/\varepsilon}} |t|^2 |\nabla w|^2 \, dx$$

$$+ \varepsilon^2 \frac{H^2 - \Gamma^2/2}{N-k} \int_{Q_{r/\varepsilon}} |z|^2 |\nabla w|^2 \, dx - \varepsilon^2 \frac{R_\varepsilon(x_0)}{6k} \int_{Q_{r/\varepsilon}} |t|^2 |\nabla w|^2 \, dx$$

$$+ \varepsilon^2 \frac{2H^2 - \Gamma^2}{2N-k} \int_{Q_{r/\varepsilon}} |z|^{-\sigma} w^2 \, dx - \varepsilon^2 \frac{R_\varepsilon(x_0)}{2^{*}(3k)} \int_{Q_{r/\varepsilon}} |t|^2 |z|^{-\sigma} w^2 \, dx$$

$$+ \varepsilon^2 h(y_0) \int_{Q_{r/\varepsilon}} w^2 \, dx + O \left( \varepsilon^2 \int_{Q_{r/\varepsilon}} |h(F_\varepsilon(x)) - h(y_0)| w^2 \, dx \right) + O(\varepsilon^{N-2}).$$

We thus get the desired result by using the Gauss equation $\Gamma^2 = H^2 - R_\varepsilon(x_0)$, see [22]. Chapter 4.
Proposition 4.3. For $N \geq 5$, we define

\[ A_{N, \sigma} = \frac{1}{k(N-k)} \int_{Q_{r/\varepsilon}} |z|^2 |\nabla w|^2 \, dx + \frac{1}{2(N-k)} \int_{Q_{r/\varepsilon}} |z|^2 |\nabla w|^2 \, dx + \frac{1}{2k(N-k)} S_{N, \sigma} \int_{Q_{r/\varepsilon}} |z|^{2-\sigma} w^{2\sigma} \, dx, \]

\[ B_{N, \sigma} = -\frac{3}{k(N-k)} \int_{Q_{r/\varepsilon}} |z|^2 |\nabla w|^2 \, dx + \frac{3}{2k} \int_{Q_{r/\varepsilon}} |t|^2 |\nabla w|^2 \, dx + \frac{1}{2(N-k)} \int_{Q_{r/\varepsilon}} |z|^2 |\nabla w|^2 \, dx \]

\[-\frac{1}{6k} \int_{Q_{r/\varepsilon}} |t|^2 |\nabla w|^2 \, dx + \frac{S_{N, \sigma}}{2k(N-k)} \int_{Q_{r/\varepsilon}} |z|^{2-\sigma} w^{2\sigma} \, dx - \frac{S_{N, \sigma}}{2k(3k)} \int_{Q_{r/\varepsilon}} |t|^2 |z|^{-\sigma} w^{2\sigma} \, dx \]

and

\[ C_{N, \sigma} = \int_{\mathbb{R}^N} |\nabla w|^2 \, dx. \]

Assume that, for some $y_0 \in \Gamma$, we have

\[
\left\{ \begin{array}{l}
A_{N, \sigma} H^2 + B_{N, \sigma} R_0(y_0) \quad \text{for } N \geq 5 \\
A_4 H^2(y_0) + B_4 R_0(y_0) + h(y_0) \quad \text{for } N = 4.
\end{array} \right.
\]

Then

\[ \mu_{h, \sigma} (\Omega, \Gamma) < S_{N, \sigma}. \]

Proof. We claim that

\[ S_{N, \sigma} \int_{Q_{r/\varepsilon}} |z|^{2-\sigma} w^{2\sigma} \, dx = \int_{Q_{r/\varepsilon}} |z|^2 |\nabla w|^2 \, dx - (N-k) \int_{Q_{r/\varepsilon}} w^2 \, dx + O(\varepsilon^{N-2}) \]

(4.19)

\[ S_{N, \sigma} \int_{Q_{r/\varepsilon}} |t|^2 |z|^{-\sigma} w^{2\sigma} \, dx = \int_{Q_{r/\varepsilon}} |t|^2 |\nabla w|^2 \, dx - k \int_{Q_{r/\varepsilon}} w^2 \, dx + O(\varepsilon^{N-2}) \]

(4.20)

To prove this claim, we let \( \eta_k(x) = \eta(\varepsilon x) \). We multiply \( \eta_k \) by \( |z|^2 \eta_k w \) and integrate by parts to get

\[ S_{N, \sigma} \int_{Q_{2r/\varepsilon}} \eta_k |z|^{2-\sigma} w^{2\sigma} \, dx = \int_{Q_{2r/\varepsilon}} \nabla w \cdot \nabla (\eta_k |z|^2 w) \, dx \]

\[ = \int_{Q_{2r/\varepsilon}} \eta_k |z|^2 |\nabla w|^2 \, dx + \frac{1}{2} \int_{Q_{2r/\varepsilon}} \nabla w^2 \cdot \nabla (|z|^2 \eta_k) \, dx \int_{Q_{2r/\varepsilon}} \eta_k |z|^2 |\nabla w|^2 \, dx - \frac{1}{2} \int_{Q_{2r/\varepsilon}} w^2 \Delta (|z|^2 \eta_k) \, dx \]

\[ = \int_{Q_{2r/\varepsilon}} \eta_k |z|^2 |\nabla w|^2 \, dx - (N-1) \int_{Q_{2r/\varepsilon}} w^2 \eta_k \, dx = -\frac{1}{2} \int_{Q_{2r/\varepsilon}} w^2 (|z|^2 \Delta \eta_k + 4 \nabla \eta_k \cdot z) \, dx. \]

We then deduce that

\[ S_{N, \sigma} \int_{Q_{r/\varepsilon}} |z|^{2-\sigma} w^{2\sigma} \, dx = \int_{Q_{r/\varepsilon}} |z|^2 |\nabla w|^2 \, dx - (N-1) \int_{Q_{r/\varepsilon}} w^2 \, dx \]

\[ + O \left( \int_{Q_{2r/\varepsilon} \setminus Q_{r/\varepsilon}} |z|^{2-\sigma} w^{2\sigma} \, dx + \int_{Q_{2r/\varepsilon} \setminus Q_{r/\varepsilon}} |z|^2 |\nabla w|^2 \, dx + \int_{Q_{2r/\varepsilon} \setminus Q_{r/\varepsilon}} w^2 \, dx \right) \]

\[ + O \left( \varepsilon \int_{Q_{2r/\varepsilon} \setminus Q_{r/\varepsilon}} |z| |\nabla w| \, dx + \varepsilon^2 \int_{Q_{2r/\varepsilon} \setminus Q_{r/\varepsilon}} |z|^2 w^2 \, dx \right). \]

Thanks to Lemma 2.3 we get the first equation of (4.19) as claimed. For the second one we multiply (2.15) by \( |t|^2 \eta_k w \) and integrate by parts as in the first one.

Next, by the continuity of \( h \), for \( \delta > 0 \), we can find \( r_3 > 0 \) such that

\[ |h(y) - h(y_0)| < \delta \quad \text{for ever } y \in F(Q_{r_3}). \]
Case $N \geq 5$.

Using (4.19) and (4.21) in (4.22), we obtain, for every $r \in (0, r_4)$

\begin{equation}
J(u_\varepsilon) = S_{N, \sigma} + \varepsilon^2 \frac{H^2 - 3R_\sigma(x_0)}{k(N-k)} \int_{\mathbb{R}^N} |z|^2 |\nabla w|^2 dx + \varepsilon^2 \frac{R_\sigma(x_0)}{3k^2} \int_{\mathbb{R}^N} |t|^2 |\nabla w|^2 dx
\end{equation}

(4.22)

\begin{equation}
+ \varepsilon^2 \frac{H^2 + R_\sigma(x_0)}{2(N-k)} \int_{\mathbb{R}^N} |z|^2 |\nabla w|^2 dx - \varepsilon^2 \frac{R_\sigma(x_0)}{6k} \int_{Q_{r/\varepsilon}} |t|^2 |\nabla w|^2 dx
\end{equation}

(4.23)

\begin{equation}
+ \varepsilon^2 \frac{H^2 + R_\sigma(x_0)}{2\varepsilon(N-k)} S_{N, \sigma} \int_{\mathbb{R}^N} |z|^{2-\sigma} w^{2\sigma} dx - \varepsilon^2 \frac{R_\sigma(x_0)}{2\varepsilon^2(3k)} S_{N, \sigma} \int_{\mathbb{R}^N} |t|^{2} |z|^{-\sigma} w^{2\sigma} dx
\end{equation}

(4.24)

\begin{equation}
+ \varepsilon^2 h(y_0) \int_{\mathbb{R}^N} w^2 dx + O \left( \varepsilon^2 \delta \int_{\mathbb{R}^N} w^2 dx \right) + O \left( \varepsilon^{N-2} \right),
\end{equation}

(4.25)

where we have used Lemma 2.3 to get the estimates

\[ \int_{\mathbb{R}^N \setminus Q_{r/\varepsilon}} |z|^2 |\nabla w|^2 dx + \int_{\mathbb{R}^N \setminus Q_{r/\varepsilon}} w^2 dx = O(\varepsilon). \]

It follows that, for every $r \in (0, r_4)$,

\[ J(u_\varepsilon) = S_{N, \sigma} + \varepsilon^2 \left\{ A_{N, \sigma} H^2(y_0) + B_{N, \sigma} R_\sigma(y_0) + C_{N, \sigma} h(y_0) \right\} + O(\delta \varepsilon^2 B_{N, \sigma}) + O(\varepsilon^3). \]

Suppose now that

\[ A_{N, \sigma} H^2(y_0) + B_{N, \sigma} R_\sigma(y_0) + C_{N, \sigma} h(y_0) < 0 \]

We can thus choose respectively $\delta > 0$ small and $\varepsilon > 0$ small so that $J(u_\varepsilon) < S_{N, \sigma}$. Hence we get

\[ \mu_{h, \sigma}(\Omega, \Gamma) < S_{N, \sigma}. \]

Case $N = 4$.

From (4.22) and (4.21), we estimate, for every $r \in (0, r_4)$

\begin{equation}
J(u_\varepsilon) \leq S_{N, \sigma} + \varepsilon^2 \frac{|H - 3R_\sigma(x_0)|}{k(N-k)} \int_{Q_{r/\varepsilon}} |z|^2 |\nabla w|^2 dx + \varepsilon^2 \frac{|R_\sigma(x_0)|}{3k^2} \int_{Q_{r/\varepsilon}} |t|^2 |\nabla w|^2 dx
\end{equation}

(4.22)

\begin{equation}
+ \varepsilon^2 \frac{H^2 + R_\sigma(x_0)}{2(N-k)} \int_{Q_{r/\varepsilon}} |z|^2 |\nabla w|^2 dx - \varepsilon^2 \frac{R_\sigma(x_0)}{6k} \int_{Q_{r/\varepsilon}} |t|^2 |\nabla w|^2 dx
\end{equation}

(4.23)

\begin{equation}
+ \varepsilon^2 \frac{H^2 + R_\sigma(x_0)}{2\varepsilon(N-k)} S_{N, \sigma} \int_{Q_{r/\varepsilon}} |z|^{2-\sigma} w^{2\sigma} dx - \varepsilon^2 \frac{R_\sigma(x_0)}{2\varepsilon^2(3k)} S_{N, \sigma} \int_{Q_{r/\varepsilon}} |t|^{2} |z|^{-\sigma} w^{2\sigma} dx
\end{equation}

(4.24)

\begin{equation}
+ \varepsilon^2 h(y_0) \int_{Q_{r/\varepsilon}} w^2 dx + O \left( \varepsilon^2 \delta \int_{Q_{r/\varepsilon}} w^2 dx \right) + O \left( \varepsilon^{N-2} \right).
\end{equation}

Further since, by (4.17),

\[ \int_{Q_{r/\varepsilon}} |z|^{2-\sigma} w^{2\sigma} dx = O(1), \]
then by (4.19), we get
\[ J(u_c) \leq S_{N, \sigma} + \varepsilon^2 \left[ \frac{H^2 - 3R_0(x_0)}{k} \int_{Q_{r/\varepsilon}} w^2 \, dx + \varepsilon^2 \frac{|R_0(x_0)|}{3k} \int_{Q_{r/\varepsilon}} w^2 \, dx \right. \\
+ \varepsilon^2 \frac{|H^2 + R_0(x_0)|}{2} \int_{Q_{r/\varepsilon}} w^2 \, dx - \varepsilon^2 \frac{|R_0(x_0)|}{6} \int_{Q_{r/\varepsilon}} w^2 \, dx \\
+ \varepsilon^2 h(y_0) \int_{Q_{r/\varepsilon}} w^2 \, dx + O \left( \varepsilon^2 \delta \int_{Q_{r/\varepsilon}} w^2 \, dx \right) + O \left( \varepsilon^{N-2} \right). \]

Therefore
\[ J(u_c) \leq S_{N, \sigma} + \varepsilon^2 \left[ \frac{|H^2(y_0) - 3R_0(y_0)|}{k} + \frac{|R_0(y_0)|}{3k} + \frac{H^2(y_0)}{2} + \frac{R_0(y_0)}{3} + h(y_0) \right] \int_{Q_{r/\varepsilon}} w^2 \, dx \]
\[ + O \left( \varepsilon^2 \delta \int_{Q_{r/\varepsilon}} w^2 \, dx \right) + O \left( \varepsilon^{N-2} \right). \]
Thus
\[ J(u_c) \leq S_{N, \sigma} + \varepsilon^2 \left[ \frac{|H^2(y_0) - 3R_0(y_0)|}{k} + \frac{|R_0(y_0)|}{3k} + \frac{H^2(y_0)}{2} + \frac{R_0(y_0)}{3} + h(y_0) \right] \int_{Q_{r/\varepsilon}} w^2 \, dx \]
\[ + O \left( \varepsilon^2 \delta \int_{Q_{r/\varepsilon}} w^2 \, dx \right) + C\varepsilon^2, \]
for some positive constant $C$ independent on $\varepsilon$. By (2.44), we have that
\[ \int_{Q_{r/\varepsilon}} \frac{C^2}{1 + |x|^2} \, dx \leq \int_{Q_{r/\varepsilon}} w^2 \, dx \leq \int_{Q_{r/\varepsilon}} \frac{C^2}{1 + |x|^2} \, dx, \]
so that
\[ (4.26) \int_{B_{\frac{1}{2} (0, r/\varepsilon)}} \frac{C^2}{1 + |x|^2} \, dx \leq \int_{Q_{r/\varepsilon}} w^2 \, dx \leq \int_{B_{\frac{1}{2} (0, r/\varepsilon)}} \frac{C^2}{1 + |x|^2} \, dx. \]
Using polar coordinates and a change of variable, for $R > 0$, we have
\[ \int_{B_{\frac{1}{2} (0, R)}} \frac{dx}{(1 + |x|^2)^2} \, dx = \left| S^3 \right| \int_0^R \frac{t^3}{(1 + t^2)^2} \, dt \]
\[ = \left| S^3 \right| \int_0^{\sqrt{R}} \frac{s^2}{2(1 + s)^2} \, ds \]
\[ = \frac{\left| S^3 \right|}{2} \left( \log \left( 1 + \sqrt{R} \right) - \frac{\sqrt{R}}{1 + \sqrt{R}} \right). \]
Therefore, there exist numerical constants $c, \overline{c} > 0$ such that for every $\varepsilon > 0$ small, we have
\[ (4.27) c \log \varepsilon \leq \int_{Q_{r/\varepsilon}} w^2 \, dx \leq \overline{c} \log \varepsilon. \]

Now we assume that
\[ \frac{|H^2(y_0) - 3R_0(y_0)|}{k} + \frac{|R_0(y_0)|}{3k} + \frac{H^2(y_0)}{2} + \frac{R_0(y_0)}{3} + h(y_0) < 0. \]
Therefore by Lemma 122 and 127, we get
\[ J(u_c) \leq S_{N, \sigma + \overline{c}} \left[ \frac{|H^2(y_0) - 3R_0(y_0)|}{k} + \frac{|R_0(y_0)|}{3k} + \frac{H^2(y_0)}{2} + \frac{R_0(y_0)}{3} + h(y_0) \right] \varepsilon^2 \log \varepsilon + \overline{c} \delta \varepsilon^2 \| \log \varepsilon \| + C\varepsilon^2. \]
Then choosing $\delta > 0$ small and $\varepsilon$ small, respectively, we deduce that $\mu_{h,\sigma}(\Omega, \Gamma) < J(u_\varepsilon) < S_{4,\sigma}$. This ends the proof of the proposition.

\textbf{Proof of Theorem 1.2 (completed).} We know that when $\mu_{h,\sigma}(\Omega, \Gamma) < S_{N,\sigma}$ then $\mu_{h,\sigma}(\Omega, \Gamma)$ is achieved by a positive function $u$, see Proposition 3.2 above. Therefore by Proposition 4.3, we get the result with $C_{1,\sigma}^{4} = A_{N,\sigma}^{4} C_{N,\sigma}$ and $C_{2,\sigma}^{4} = B_{N,\sigma}^{4} C_{N,\sigma}$ for $N \geq 5$. When $N = 4$, we get $C_{1,\sigma}^{4}$ and $C_{2,\sigma}^{4}$ depending on the signs of $H^2(y_0) - 3R_\gamma(y_0)$ and $R_\gamma(y_0)$. They are given by:

$$C_{1,\sigma}^{4} = 1 \quad \text{and} \quad C_{2,\sigma}^{4} = \frac{10}{3k} \quad \text{when} \quad H^2 \leq R_\gamma(y_0) \leq 0.$$  

$$C_{1,\sigma}^{4} = \frac{1}{2} + \frac{1}{k} \quad \text{and} \quad C_{2,\sigma}^{4} = \frac{1}{2} - \frac{8}{3k} \quad \text{when} \quad H^2 \geq 3R_\gamma(y_0) \geq 0.$$  

(4.28)

$$C_{1,\sigma}^{4} = \frac{1}{2} + \frac{1}{k} \quad \text{and} \quad C_{2,\sigma}^{4} = \frac{1}{2} - \frac{10}{3k} \quad \text{when} \quad R_\gamma(y_0) \leq 0.$$  

$$C_{1,\sigma}^{4} = \frac{1}{2} + \frac{1}{k} \quad \text{and} \quad C_{2,\sigma}^{4} = \frac{1}{2} - \frac{10}{3k} \quad \text{when} \quad H^2 - 3R_\gamma(y_0) \leq 0 \quad \text{and} \quad R_\gamma(y_0) \geq 0.$$  

\textbf{Acknowledgement:} I wish to thanks my supervisor Mouhamed Moustapha Fall for useful discussions and remarks. This work is supported by the German Academic Exchange Service (DAAD).

\textbf{References}

[1] T. Aubin, \textit{Problèmes isopérimétriques de Sobolev}, J. Differential Geom. 11(1976) 573-598.

[2] M. Badiale, G. Tarantello, \textit{A Sobolev-Hardy inequality with applications to a nonlinear elliptic equation arising in astrophysics}, Arch. Rational Mech. Anal. 163(4)(2002) 259-293.

[3] Brezis H., Lieb E., \textit{A relation between pointwise convergence of functions and convergence of functionals}, Proc. Amer. Math. Soc. 88(1983), no.3, 486-490.

[4] H. Brezis and L. Nirenberg, \textit{Positive solutions of nonlinear elliptic equations involving critical exponents}, Comm. Pure Appl. Math 36 (1983), 437-477.

[5] Brezis H., Vasquez J.L., \textit{Blow-up solutions of some nonlinear elliptic problems}, Rev. Mat. Univ. Complut. Madr. 10(1997), 443-469.

[6] Chern J-L. and Lin C-S., \textit{Minimizers of Caffarelli-Kohn-Nirenberg Inequalities with the singularity on the boundary}, Archive for rational mechanics and Analysis, Volume 197, No. 2 (2010), 401-432.

[7] Demyanov A.V., Nazarov A.I., \textit{On the solvability of the Dirichlet problem for the semilinear Schrödinger equation with a singular potential}, (Russian) Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklor, (POMI) 336 (2006). Kravch. Zadachi Mat. Fiz. i Smezh. Vopr. Teor. Funkts. 37, 25–45, 274; translation in J. Math. Sci. (N.Y.) 143 (2007), no. 2, 2857-2868.

[8] O. Druet, \textit{Elliptic equations with critical Sobolev exponents in dimension 3}, Ann. Inst. H. Poincaré Anal. Non Linéaire 19(2002), no.2, 125-142(English, with English and French summaries).

[9] O. Druet, \textit{The best constants problem in Sobolev inequalities}. Math. Ann., 314, 1999, 327-346.

[10] O. Druet, \textit{Optimal Sobolev inequalities and extremals functions. The three-dimensional case}, Indiana Univ. Math. J. 51(2002), no.1, 69-88.

[11] H. Egide, \textit{Positive solutions of semilinear equations in cones}, Tran. Amer. Math. Soc 11(1992), 191-201.

[12] Fabbrri I., Mancini G., Sandeep K., \textit{Classification of solutions of a critical Hardy-Sobolev operator}, J. Differential equations 224 (2006), 258-276.

[13] Fall M. M., Minlend I., Thiama E. H. A, \textit{The role of the mean curvature in a Hardy-Sobolev inequality}, NoDEA Nonlinear Differential Equations Appli. 22 (2015), no. 5, 1047-1066.
[14] M. M. Fall and E. H. A. Thiam, *Hardy-Sobolev inequality with singularity a curve*, DOI: 10.12775/TMNA.2017.045.

[15] N. Ghoussoub and F. Robert, *On the Hardy-Schrödinger operator with a boundary singularity*, Preprint 2014. https://arxiv.org/abs/1410.1913.

[16] N. Ghoussoub and F. Robert, *Sobolev inequalities for the Hardy-Schrödinger operator: extremals and critical dimensions*, Bull. Math. Sci. 6 (2016), no. 1, 89-144.

[17] N. Ghoussoub and F. Robert, *Elliptic Equations with Critical Growth and a Large Set of Boundary Singularities*, Trans. Amer. Math. Soc., Vol. 361, No. 9 (Sep., 2009), pp. 4843-4870.

[18] N. Ghoussoub and F. Robert, *Concentration estimates for Emden-Fowler equations with boundary singularities and critical growth*, IMRP Int. Math. Res. Pap. (2006), 21867, 1-85.

[19] N. Ghoussoub and F. Robert, *The effect of curvature on the best constant in the Hardy Sobolev inequalities*, Geom. Funct. Anal. 16(6), 1201-1245(2006).

[20] Ghoussoub N., Kang X. S., *Hardy-Sobolev critical elliptic equations with boundary singularities*, Ann. Inst. H. Poincaré Anal. Non Linéaire 21 (2004), no. 6, 767-793.

[21] N. Ghoussoub, C. Yuan, *Multiple solutions for quasi-linear PDEs involving the critical Sobolev and Hardy exponents*, Trans. Amer. Math. Soc. 12 (2000), 5703-5743.

[22] A. Gray, *Tubes*, second edition, Springer Science and Business Media, 2004.

[23] E. Hebey and M. Vaugon, *The best constant problem in the Sobolev embedding theorem for complete Riemannian manifolds*, Duke Mathematical Journal, 1995, 79(1), 235-279.

[24] E. Hebey and M. Vaugon, *Meilleures constantes dans le théorème d’inclusion de Sobolev*, In Annales de l’Institut Henri Poincaré (C) Non Linear Analysis, 2016, 13(1), 57-93.

[25] Jaber H., *Hardy-Sobolev equations on compact Riemannian manifolds*, Nonlinear Anal. 421 (2015) 1869-1888.

[26] Jaber H., *Optimal Hardy-Sobolev equations on compact Riemannian manifolds*, J. Math. Appl. 421 (2015) 1869-1888.

[27] Li Y., Lin C., *A Nonlinear Elliptic PDE with Two Sobolev-Hardy Critical Exponents*, published online september 28, 2011-© Springer-Verlag (2011).

[28] Lieb E.H., *Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities*, Ann. of Mathematics 118 (1983), 349-374.

[29] C-S. Lin, H. Wadade, *Minimizing problems for the Hardy-Sobolev type inequality with the singularity on the boundary*, Tohoku Math. J. (2) 64 (2012), no. 1, 79-103.

[30] M. Struwe, *Variational Methods: Applications to nonlinear Partial Differential Equations and Hamiltonian Systems*, Springer Science and Business Media, 2008, Vol. 34.

[31] C-S. Lin, H. Wadade, *On the attainability for the best constant of the Sobolev-Hardy type inequality*, RIMS Kôyûroku 1740 (2011), 141-157.

[32] Talenti G., *Best constant in Sobolev inequality*, Ann. di Matem. Pura ed. Appli. 110 (1976), 353-372.

[33] E. H. A. Thiam, *Hardy and Hardy-Sobolev Inequalities on Riemannian manifolds*, Imhotep Mathematical Journal, Vol. 2, No. 1, (2017), pp 14-35.

[34] E. H. A. Thiam, *Weighted Hardy Inequality On Riemannian manifolds*, Communications in Contemporary Mathematics, 2016, 18(6), 1550072, 25pp.

E. H. A. T.: AFRICAN INSTITUTE FOR MATHEMATICAL SCIENCES IN SENEGAL, KM 2, ROUTE DE JOAL, B.P. 14 18. Mbour, SENEGAL.
E-mail address: elhadji@aims-senegal.org