**Research Article**

**Product Antimagic Labeling of Caterpillars**

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Received 22 July 2021; Revised 29 September 2021; Accepted 4 October 2021; Published 16 October 2021

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Let $G$ be a graph with $m$ edges. A product antimagic labeling of $G$ is a bijection from the edge set $E(G)$ to $\{1, 2, \ldots, m\}$ such that the vertex-products are pairwise distinct, where the vertex-product of a vertex $v$ is the product of labels on the incident edges of $v$. A graph is called product antimagic if it admits a product antimagic labeling. In this paper, we will show that caterpillars with at least three edges are product antimagic by an $O(m \log m)$ algorithm.

1. Introduction

Let $b > a$ be two integers. We use $[a, b]$ to denote the set \{a, a + 1, \ldots, b\} and simply write $[1, a]$ as $[a]$. All graphs considered in this paper are simple and finite. Let $G$ be a graph with $m$ edges. The vertex set and edge set of $G$ are denoted by $V(G)$ and $E(G)$, respectively. For two vertex sets $X, Y \subseteq V(G)$, the set of edges with one end in $X$ and the other end in $Y$ is denoted by $E(X, Y)$. An antimagic labeling of $G$ is a bijection $\tau$ from $E(G)$ to $[m]$ such that for any two distinct vertices $u$ and $v$ in $G$, the sum of labels on the edges incident with $u$ differs from that of $v$. A graph is said to be antimagic if it admits an antimagic labeling. The concept of antimagic labeling was proposed by Hartsfield and Ringel in 1990 [1]. In the same paper, they conjectured that every connected graph other than $K_2$ is antimagic. This topic was investigated by many researchers; for instance, see [2–6]. Recently, Lozano et al. [7] proved that caterpillars are antimagic, where a caterpillar is a tree with at least three vertices such that the removal of its leaves produces a path.

In 2000, Figueroa-Centeno et al. [8] introduced multiplicative variation of antimagic labeling. A product antimagic labeling of a graph $G$ with $m$ edges is a bijection $\varphi$ from $E(G)$ to $[m]$ such that the vertex-products are pairwise distinct, where the vertex-product $p(v)$ of a vertex $v$ is the product of labels on the incident edges of $v$. A graph $G$ is called product antimagic if there is a product antimagic labeling of $G$. In [8], the authors proved that paths with at least four vertices and 2-regular graphs are product antimagic. Furthermore, they proposed the following conjecture.

**Conjecture 1.** A connected graph with at least three edges is product antimagic.

Kaplan et al. [9] proved that the following graphs are product antimagic: the disjoint union of cycles and paths where each path has at least three edges; connected graphs with $n$ vertices and $m$ edges where $m \geq 4nlnn$; graphs $G$ where each component has at least two edges and the minimum degree of $G$ is at least $8\sqrt{\ln|E|\ln(\ln|E|)}$; and all complete $k$-partite graphs except $K_2$ and $K_{1,2}$. In [10], Pikhurko characterizes all large graphs that are product antimagic. More precisely, it is shown that there is an integer $n_0$ such that a graph with $n \geq n_0$ vertices is product antimagic if and only if it does not belong to any of the following four classes: graphs that have at least one isolated edge; graphs that have at least two isolated vertices; unions of vertex-disjoint copies of $K_{1,2}$; graphs consisting of one isolated vertex; and graphs obtained by subdividing some edges of the star $K_{1,k+1}$.

In this paper, we prove that Conjecture 1 is affirmative for caterpillars with at least three edges.

**Theorem 1.** Every caterpillar with at least three edges is product antimagic.
There are some other variations of antimagic labeling, such as antimagic orientation; see [11–14] for some results of trees. For more types of labelings, refer to the survey of Joseph [15]. The remainder of this paper is organized as below. In the next section, we prove Theorem 1. In Section 3, we write the labeling procedure of Theorem 1 as an algorithm.

2. Proof of Theorem 1

Let $T$ be a caterpillar. A leaf of $T$ is a vertex of degree one in $T$, a spine of $T$ is a longest path of $T$, and a leg of $T$ is an edge that does not belong to the spine of $T$.

Proof. of Theorem 1. Let $T$ be a caterpillar with $m(\geq 3)$ edges. Since it is proved that paths with at least three edges are product antimagic, we may assume that $T$ is not a path. Let $P = v_0v_1, \ldots, v_r$ be a spine of $T$. Let $U = \{v_h, v_{h+1}, \ldots, v_k\} \subseteq V(P)$ be the set of vertices of degree at least three in $T$, where $h_1 < h_2 < \cdots < h_i$. Define

$$X = \{x \in V(T) \mid V(P) \mid x \text{ is adjacent to a vertex in } U\}. \quad (1)$$

In fact, a vertex in $X$ is a leaf incident with a leg in $T$. Note that $V(P) \cup X = V(T)$. Let $M$ be a maximum matching in $E(U, X)$. Therefore, $M$ is a matching of size $|U|$ that saturates all vertices in $U$. We define a product antimagic labeling of $T$ in three steps in the following, and see Figure 1 for example.

Step 1. Label the edges in $E(P)$. If $\ell$ is odd, starting from the edge $v_0v_1$, we label the edges in $E(P)$ consecutively with $2, m - (\ell - 1)/2 + 1, 3, m - (\ell - 1)/2 + 2, \ldots, \ell - 1/2 + 1, m, (\ell - 1)/2 + 2$, respectively.

If $\ell$ is even, starting from the edge $v_0v_1$, we label the edges in $E(P)$ consecutively with $m - \ell/2 + 1, 2, m - \ell/2 + 2, 3, \ldots, m - 1, \ell/2, m, \ell/2 + 1$, respectively.

The current labeling is denoted by $\varphi_1$, and the vertex-product of a vertex $x$ in $V(T)$ is denoted by $p_1(x)$. It can be seen that

$$\varphi_1(v_i, v_{i+1}) = \begin{cases} \frac{i}{2} + 2, & i \equiv 0 \pmod{2}, \\ m - \frac{\ell - 1}{2} + \frac{i + 1}{2}, & i \equiv 1 \pmod{2}, \end{cases} \quad (2)$$

where $i \in [0, \ell - 1]$, in which $\ell$ is odd.

And

$$\varphi_1(v_i, v_{i+1}) = \begin{cases} m - \frac{\ell}{2} + \frac{i + 1}{2}, & i \equiv 0 \pmod{2}, \\ \frac{i + 1}{2} + 1, & i \equiv 1 \pmod{2}, \end{cases} \quad (3)$$

where $i \in [0, \ell - 1]$, in which $\ell$ is even.

Claim 1. For any two distinct vertices, $v_i, v_j \in V(P)$ and $p_1(v_i) \neq p_1(v_j)$.

Proof. By the definition of vertex-product,

$$p_1(v_i) = \begin{cases} \varphi_1(v_0v_i), & i = 0, \\ \varphi_1(v_{i-1}v_i) \cdot \varphi_1(v_iv_{i+1}), & i \in [1, \ell - 1], \\ \varphi_1(v_{\ell-1}v_\ell), & i = \ell. \end{cases} \quad (4)$$

By equations (2) and (3), it is easy to verify that

$$p_1(v_1) < p_1(v_2) < \cdots < p_1(v_{\ell-1}). \quad (5)$$

Furthermore, $p_1(v_0) < p_1(v_1), p_1(v_2) < p_1(v_1)$, and $p_1(v_0) \neq p_1(v_1)$. So, for any two distinct vertices, $v_i, v_j \in V(P)$ and $p_1(v_i) \neq p_1(v_j)$.

Step 2. Label edges in $E(U, X) \cup E(M)$ if it is not an empty set.

If $\ell$ is odd, label the edges in $E(U, X) \cup E(M)$ using numbers in $[1] \cup [(\ell - 1)/2 + 3, m - (\ell - 1)/2 - t]$ one by one arbitrarily (we reserve numbers in $[m - (\ell - 1)/2 - t + 1, m - (\ell - 1)/2]$ for edges in $M$).

If $\ell$ is even, label the edges in $E(U, X) \cup E(M)$ using numbers in $[1] \cup [\ell/2 + 1, m - \ell/2 - t]$ one by one arbitrarily (we reserve numbers in $[m - \ell/2 - t + 1, m - \ell/2]$ for edges in $M$).

Denote the current labeling by $\varphi_2$ and the partial vertex-product of a vertex $x \in V(T)$ by $p_2(x)$.

Step 3. Label edges in $M$.

Case 1. $E(U, X) \cup E(M) \neq \emptyset$.

Assume that $p_2(v_i) \leq p_2(v_{i+1}) \leq \cdots \leq p_2(v_{i+1})$, where $\{i_1, i_2, \ldots, i_t\} = \{h_1, h_2, \ldots, h_t\}$. Label edges in $M$ incident with $v_{i_1}, v_{i_2}, \ldots, v_{i_t}$ with $m - \ell/2 - t + 1, \ldots, m - \ell/2$, respectively, in this order. The resulting labeling is denoted by $\varphi_3$, and the vertex-product of a vertex $x \in V(T)$ is denoted by $p_3(x)$.

Claim 2. $\varphi_3$ is a product antimagic labeling of $T$.

Proof. By the way of assigning labels to edges in $M$, we have

$$p_3(v_i) < p_3(v_{i+1}) < \cdots < p_3(v_{i+t}). \quad (6)$$

It is easy to see that $\varphi_3$ is a bijection from $E(T)$ to $[m]$. We show that $\varphi_3$ is a product antimagic labeling of $T$ in the following. Let $V_1$ and $V_2$ be the sets of vertices of degree 1 and 2, respectively. Then, $V(T) = V_1 \cup V_2 \cup U$. For each vertex $x \in V_1 \cup V(P)$, we know that $p_3(x) \leq m - \ell/2$ by Step 3. Combining with Claim 1 and the labeling steps, it follows that no vertex in $V_1$ receives the same vertex-product as other vertices in $T$. Also, by Claim 1 and equation (5), it suffices to prove that for any vertex $u \in V_2$ and any vertex $v \in U$, $p_3(u) \neq p_3(v)$.

If $\ell$ is odd, then

$$p_3(u) \leq m \cdot \left(\frac{\ell - 1}{2} + 2\right), \quad (7)$$

$$p_3(v) \geq 2 \cdot \left(m - \frac{\ell - 1}{2} + 1\right) \cdot \left(\frac{\ell - 1}{2} + 3\right).$$
Proof. By equation (5) in the proof of Claim 1 and the assumption $\ell$ is even, then $\phi(\ell) = m + \ell/2$. If $\ell$ is even, then $\phi_3(u) < \phi_3(v)$. If $\ell$ is odd, then $\phi_3(u) \geq \phi_3(v)$. Therefore, $\phi_3(u) < \phi_3(v)$. Since $\ell/2 + 1 < \ell/2 + 2$ and $m < \ell/2 + 1$, it follows that $\phi_3(u) < \phi_3(v)$. By the above discussion, $\phi_3$ is a product antimagic labeling of $T$.

Case 2. $E(U, X) = E(M)$.

In this case, Step 2 does not exist. We assign the remaining labels $1, [\ell/2] + 2, [\ell/2] + 3, \ldots, m - [\ell/2]$ to the edges in $E(M)$ incident with $v_1, v_2, \ldots, v_{\ell/2}$ one by one, respectively. The labeling of $T$ is denoted by $\phi'$, and the vertex-product of a vertex $x \in V(T)$ is denoted by $p'(x)$.

Claim 3. $\phi'$ is a product antimagic labeling of $T$. 

Proof. By equation (5) in the proof of Claim 1 and the fact that $\phi_1(\ell) = m \cdot ([\ell/2] + 1)$. By the labeling of edges in $E(M)$, we know that $p'(v_1) = m \cdot ([\ell/2] + 1)$. Therefore, the cost of labeling $T$ is $O(m \log m)$. Since $T$ is a tree, the vertex number $n = m + 1$, so the cost can be also expressed as $O(n \log n)$.

3. An Algorithm

In this section, we will write the steps of labeling a caterpillar as Algorithm 1. The notation follows from the last section. Finally, we show that Algorithm 1 runs in time $O(m \log m)$. Assignments in Step 1 of the algorithm can be done in constant time. Therefore, the cost of Step 1 is $O(m \log m)$.

Since each vertex in $U$ is of degree three, Step 2 does not run. Otherwise, Step 2 visits at most $m$ edges and assigns a random label to each of them, but labels can be chosen increasingly from the unused labels in $\{1, 2, \ldots, m - [\ell/2]\}$, thus giving a cost $O(m \log m)$. Step 3 requires time $O(m \log m)$ due to the fact that the partial vertex-products must be sorted (line 10). The total cost of the algorithm is, then, $O(m \log m)$. Since $T$ is a tree, the vertex number $n = m + 1$, so the cost can be also expressed as $O(n \log n)$.

Algorithm 1: Product antimagic labeling of a caterpillar.

Input: A caterpillar $T$ with $m$ edges
Output: A product antimagic labeling of $T$

Step 1: Label the edges in path $P$
(1) $2 \rightarrow \phi(v_0v_1)$
(2) $m - [\ell/2] + 1 \rightarrow \phi(v_1v_2)$
if $\ell$ is even then
(3) Exchange $\phi(v_0v_1)$ and $\phi(v_1v_2)$
(4) for $i = 2$ to $\ell - 1$ do
(5) $\phi(v_{i-1}v_i) + 1 \rightarrow \phi(v_{i-1}v_i)$
Step 2: Label the edges in the legs except one incident with each vertex in $U$
(6) for all $v \in U$ do
(7) for all legs $e$ incident with $v$ except one do
(8) a random label from $(1) \cup \{m - [\ell/2] + 2\}) \phi(E(T)) \rightarrow \phi(e)$
Step 3: Label the remaining edges
(9) Sort the vertices in $U$ as $u_1, u_2, \ldots, u_t$ s.t. $p(u_i) \leq p(u_{i+1})$ for all $i \in [1, t - 1]$
(10) Sort the labels in $(1) \cup \{m - [\ell/2] + 2\}) \phi(E(T))$ as $l_1, \ldots, l_t$ in increasing order
(11) for $i = 1$ to $t$ do
(12) $l_i \rightarrow \phi(e_i^*)$, where $\phi(e_i^*)$ is the unlabeled leg incident with $u_i$
Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The idea of this paper follows from the works in [7, 11]. So, the authors are grateful to them. This work was supported by NSFC (Nos. 11901263, 61802158, and 12071194) and NSFC of Gansu Province (No. 20JR5RA229).

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