Discrete soliton equations and convergence acceleration algorithms

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Abstract

Some of the well-known convergence acceleration algorithms, when viewed as two-variable difference equations, are equivalent to discrete soliton equations. It is shown that the $\eta$—algorithm is nothing but the discrete KdV equation. In addition, one generalized version of the $\rho$—algorithm is considered to be integrable discretization of the cylindrical KdV equation.

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1 Introduction

Recently, it has been claimed that good algorithms in the field of numerical analysis play important roles in the theory of nonlinear integrable systems. In 1982, Symes [1] pointed out that one step in the QR algorithm, which is the most popular method to solve matrix eigenvalue problems, is equivalent to time evolution of the finite nonperiodic Toda equation. In 1992, Hirota, Tsujimoto, and Imai [2] showed that the LR algorithm, which is another successful tool to find eigenvalues of a given matrix, is no other than the time-discrete Toda equation. In 1993, Papageorgiou, Grammaticos, and Ramani [3] showed that a well-known convergence acceleration scheme, the $\varepsilon$-algorithm, is nothing but the discrete potential KdV equation.

Our main interest in this paper is on the convergence acceleration algorithms. Let $\{S_m\}$ be a sequence of numbers which converges to $S_\infty$. In order to find $S_\infty$ by direct calculation, we often need a large amount of data. In such cases we transform the original sequence $\{S_m\}$ into another sequence $\{T_m\}$ instead of calculating directly. If $\{T_m\}$ converges to $S_\infty$ faster than $\{S_m\}$, that is

$$\lim_{m \to \infty} \frac{T_m - S_\infty}{S_m - S_\infty} = 0,$$

we say that the transformation $T : \{S_m\} \to \{T_m\}$ accelerates the convergence of the sequence $\{S_m\}$. We now have many convergence acceleration algorithms. Among them, we here focus our attention on the $\eta$, $\varepsilon$, and $\rho$-algorithms [4-6]. We clarify that there is a strong tie between these algorithms and discrete soliton equations.

In §2, we show that Bauer’s $\eta$-algorithm is considered to be the discrete KdV equation in ref. [7]. We also look over the result by Papageorgiou et al., the equivalence between Wynn’s $\varepsilon$-algorithm and the discrete potential KdV equation. In §3, we introduce a different type of algorithm, Wynn’s $\rho$-algorithm. In spite of its similarity with the $\varepsilon$-algorithm, it possesses noticeably different characteristics not only as a convergence accelerator but also as a discrete soliton equation. We show that the $\rho$-algorithm relates with the cylindrical KdV equation [8],

$$u_t + 6uu_x + u_{xxx} + \frac{1}{2t}u = 0.$$

Concluding remarks are given in §4.
2 The $\eta$–algorithm and the $\varepsilon$–algorithm

In this section we show that Bauer’s $\eta$–algorithm [4], which is one of the famous convergence acceleration algorithms, is equivalent to the discrete KdV equation. Let initial values $\eta_0^{(m)}$ and $\eta_1^{(m)}$ be

$$\eta_0^{(m)} = \infty, \quad \eta_1^{(m)} = c_m \equiv \Delta S_{m-1}, \quad (m = 0, 1, 2, \ldots), \quad S_{-1} \equiv 0,$$  \hspace{1cm} (3)

where $\Delta$ is the forward difference operator given by $\Delta \equiv \Delta^n \equiv a_{n+1} - a_n$. Then all the other elements are calculated from the following recurrence relations called the $\eta$–algorithm;

$$\begin{cases}
\frac{\eta_{2n+1}^{(m)}}{\eta_{2n+2}^{(m)}} + \frac{\eta_{2n}^{(m)}}{\eta_{2n+1}^{(m)}} = \frac{\eta_{2n+1}^{(m+1)}}{\eta_{2n+2}^{(m+1)}} + \frac{\eta_{2n}^{(m+1)}}{\eta_{2n+1}^{(m+1)}} \\
\frac{1}{\eta_{2n+1}^{(m)}} + \frac{1}{\eta_{2n+2}^{(m)}} = \frac{1}{\eta_{2n+1}^{(m+1)}} + \frac{1}{\eta_{2n+2}^{(m+1)}}
\end{cases} \quad \text{(rhombus rules)} \hspace{1cm} (4)

This defines a transformation of a given series $c_m = \eta_1^{(m)}$, $m = 0, 1, 2, \ldots$ to a new series $c_n' = \eta_0^{(0)}$, $n = 1, 2, \ldots$ such that $\sum_{n=1}^{\infty} c_n'$ converges more rapidly to the same limit $S_\infty$. The quantities $\eta_n^{(m)}$ are given by the following ratios of Hankel determinants;

$$\eta_{2n+1}^{(m)} = \frac{\begin{vmatrix}
c_m & \cdots & c_{m+n} \\
\vdots & \ddots & \vdots \\
c_{m+n} & \cdots & c_{m+2n}
\end{vmatrix}}{\begin{vmatrix}
\Delta c_m & \cdots & \Delta c_{m+n-1} \\
\vdots & \ddots & \vdots \\
\Delta c_{m+n-1} & \cdots & \Delta c_{m+2n-2}
\end{vmatrix}}, \hspace{1cm} (5)

$$\eta_{2n+2}^{(m)} = \frac{\begin{vmatrix}
c_m & \cdots & c_{m+n} \\
\vdots & \ddots & \vdots \\
c_{m+n} & \cdots & c_{m+2n}
\end{vmatrix}}{\begin{vmatrix}
\Delta c_m & \cdots & \Delta c_{m+n} \\
\vdots & \ddots & \vdots \\
\Delta c_{m+n} & \cdots & \Delta c_{m+2n}
\end{vmatrix}}. \hspace{1cm} (6)

If we introduce dependent variable transformations,

$$X_{2n}^{(m)} = \frac{1}{\eta_{2n}^{(m)}}, \quad X_{2n-1}^{(m)} = \eta_{2n-1}^{(m)}, \hspace{1cm} (7)$$
the $\eta$-algorithm is rewritten as
\begin{align}
X_{n+1}^{(m)} - X_{n-1}^{(m+1)} &= \frac{1}{X_n^{(m+1)}} - \frac{1}{X_n^{(m)}},
\end{align}
which is the discrete KdV equation. Let us replace variables $n$ and $m$ by
\begin{align}
n &= \frac{t}{\epsilon^3}, \quad m - \frac{1}{2} = \frac{x - ct}{\epsilon^3},
\end{align}
respectively and rewrite $X_n^{(m)}$ as
\begin{align}
\epsilon^2 u(x - \frac{\epsilon}{2}, t) = \frac{1}{p + \epsilon^2 u(x + \frac{\epsilon}{2}, t)} - \frac{1}{p + \epsilon^2 u(x - \frac{\epsilon}{2}, t)}.
\end{align}
If we take the small limit of $\epsilon$, eq. (11) yields the KdV equation
\begin{align}
u_t - \frac{1}{p^3} u u_x + \frac{1}{48p^2}(1 - \frac{1}{p^3}) u_{xxx} = 0
\end{align}
at the order of $\epsilon^5$.

The discrete KdV eq. (8) is rewritten as
\begin{align}
\{\tau(n + 2, m - 1)\tau(n - 1, m) + \tau(n + 1, m)\tau(n, m - 1)\}\tau(n - 1, m + 1)
= \{\tau(n - 2, m + 1)\tau(n + 1, m) + \tau(n - 1, m)\tau(n, m + 1)\}\tau(n + 1, m - 1),
\end{align}
through a dependent variable transformation,
\begin{align}
X_n^{(m)} &= \frac{\tau(n + 1, m - 1)\tau(n - 1, m)}{\tau(n, m - 1)\tau(n, m)}.
\end{align}
Subtracting $\tau(n - 1, m + 1)\tau(n + 1, m - 1)\tau(n, m)$ from both sides of eq. (13), we obtain the bilinear form of the discrete KdV equation,
\begin{align}
\tau(n + 2, m - 1)\tau(n - 1, m) + \tau(n + 1, m)\tau(n, m - 1) - \tau(n + 1, m - 1)\tau(n, m) = 0.
\end{align}
It is noted that the solutions (5) and (6) are recovered by putting
\begin{align}
\tau(2n, m) &\equiv \begin{bmatrix} c_{m+1} & \cdots & c_{m+n} \\ \vdots & \vdots & \vdots \\ c_{m+n} & \cdots & c_{m+2n-1} \end{bmatrix}, \\
\tau(2n + 1, m) &\equiv \begin{bmatrix} \Delta c_{m+1} & \cdots & \Delta c_{m+n} \\ \vdots & \vdots & \vdots \\ \Delta c_{m+n} & \cdots & \Delta c_{m+2n-1} \end{bmatrix}
\end{align}
in eq. (14).

Next, following the result by Papageorgiou et al., we briefly review the equivalence between the $\varepsilon-$algorithm and the discrete potential KdV equation. The $\varepsilon-$algorithm originates with Shanks [9] and Wynn [5]. Define $\varepsilon_0^{(m)}$ and $\varepsilon_1^{(m)}$ by

$$
\varepsilon_0^{(m)} = 0, \quad \varepsilon_1^{(m)} = S_m \ (m = 0, 1, 2, \ldots).
$$

(18)

Then all the other quantities obey the following rhombus rule called the $\varepsilon-$algorithm;

$$(\varepsilon_n^{(m)} - \varepsilon_{n-1}^{(m)}) (\varepsilon_n^{(m+1)} - \varepsilon_n^{(m)}) = 1.
$$

(19)

According as $n$ becomes large, $\varepsilon_{2n+1}^{(m)}$ converges more rapidly to $S_\infty$ as $m \to \infty$. On the other hand, $\varepsilon_{2n}^{(m)}$ diverges as $m \to \infty$. This fact reminds us of the idea of the singularity confinement [10], since a singularity at $(2n, m)$ is confined and $\varepsilon_{2n+1}^{(m)}$ converges to the same limit as the original sequence $\varepsilon_1^{(m)}$.

It has been shown that the $\varepsilon-$algorithm (19) is regarded as the discrete potential KdV equation. The quantities $\varepsilon_n^{(m)}$ are also given by the following ratios of Hankel determinants;

$$
\varepsilon_{2n+1}^{(m)} = \frac{\Delta^2 S_m \Delta^2 S_{m+1} \cdots \Delta^2 S_{m+n-1}}{\Delta^2 S_{m+1} \Delta^2 S_{m+2} \cdots \Delta^2 S_{m+n}} \cdots \frac{\Delta^2 S_{m+n} \Delta^2 S_{m+n+1} \cdots \Delta^2 S_{m+2n-2}}{
\varepsilon_{2n+2}^{(m)} = \frac{\Delta^3 S_m \Delta^3 S_{m+1} \cdots \Delta^3 S_{m+n-1}}{\Delta^3 S_{m+1} \Delta^3 S_{m+2} \cdots \Delta^3 S_{m+n}} \cdots \frac{\Delta^3 S_{m+n-1} \Delta^3 S_{m+n} \cdots \Delta^3 S_{m+2n-2}}{
\varepsilon_n^{(m)} = 0, \quad \varepsilon_1^{(m)} = S_m \ (m = 0, 1, 2, \ldots).
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\varepsilon_{2n+2}^{(m)} = \frac{\Delta^3 S_m \Delta^3 S_{m+1} \cdots \Delta^3 S_{m+n-1}}{\Delta^3 S_{m+1} \Delta^3 S_{m+2} \cdots \Delta^3 S_{m+n}} \cdots \frac{\Delta^3 S_{m+n-1} \Delta^3 S_{m+n} \cdots \Delta^3 S_{m+2n-2}}{
\varepsilon_n^{(m)} = 0, \quad \varepsilon_1^{(m)} = S_m \ (m = 0, 1, 2, \ldots).
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Equation (20) is called the Shanks transformation [9]. Substitution of \( n = 1 \) in eq. (20) gives the well-known Aitken acceleration algorithm.

We have so far seen that the \( \eta \)- and the \( \varepsilon \)-algorithms are interpreted as the discrete KdV and the discrete potential KdV equations, respectively. Therefore, these two algorithms are the same in their performance as convergence acceleration algorithms. This equivalence can also be understood from the fact [4] that the quantities \( \eta_{2n}^{(m)} \) and \( \varepsilon_{2n+1}^{(m)} \) are related by

\[
\eta_{2n}^{(m)} = \varepsilon_{2n+1}^{(m)} - \varepsilon_{2n-1}^{(m)}, \quad \eta_{2n+1}^{(m)} = \varepsilon_{2n+1}^{(m)} - \varepsilon_{2n+1}^{(m-1)}.
\]

(22)

3 The \( \rho \)-algorithm

The \( \rho \)-algorithm is traced back to Thiele’s rational interpolation [11]. It was first used as a convergence accelerator by Wynn [5]. The initial values of the algorithm are given by

\[
\rho_0^{(m)} = 0, \quad \rho_1^{(m)} = S_m \quad (m = 0, 1, 2, \ldots),
\]

(23)

and all the other elements fulfill the following rhombus rule;

\[
(\rho_{n+1}^{(m)} - \rho_{n-1}^{(m+1)})(\rho_n^{(m+1)} - \rho_n^{(m)}) = n.
\]

(24)

The \( \rho \)-algorithm is almost the same as the \( \varepsilon \)-algorithm except that “1” in the right hand side of eq. (19) is replaced by “\( n \)” in eq. (24). This slight change, however, yields considerable differences in various aspects between these two algorithms.

The first difference is in their performance. As one can find in ref. [12], the \( \varepsilon \)-algorithm accelerates exponentially or alternatively decaying sequences, while the \( \rho \)-algorithm does rationally decaying sequences.

The second difference is in their determinant expressions. The quantities \( \varepsilon_{2n}^{(m)} \) are given by ratios of Hankel determinants, while the quantities \( \rho_n^{(m)} \) are given by

\[
\rho_n^{(m)} = (-1)^{[\frac{m-1}{2}]} \tilde{\tau}_n^{(m)},
\]

(25)

where \([x]\) stands for the greatest integer less than or equal to \( x \). Moreover, the functions \( \tau_n^{(m)} \) and \( \tilde{\tau}_n^{(m)} \) are expressed as the following double Casorati determinants;

\[
\tau_n^{(m)} = \begin{cases} 
  u^{(m)}(k; k) & n = 2k, \\
  u^{(m)}(k + 1; k) & n = 2k + 1,
\end{cases}
\]

(26)
\[
\tau^{(m)}_n = \begin{cases} 
  u^{(m)}(k+1; k-1) & n = 2k, \\
  u^{(m)}(k; k+1) & n = 2k+1,
\end{cases}
\] (27)

where

\[
u^{(m)}(p; q) = \det \begin{bmatrix} 
  1 & m & \cdots & m^{p-1} \\
  1 & m+1 & \cdots & (m+1)^{p-1} \\
  \vdots & \vdots & \ddots & \vdots \\
  1 & m+p+q-1 & \cdots & (m+p+q-1)^{p-1}
\end{bmatrix}
\] (28)

The third difference is in the relation with discrete soliton equations. We have seen in the previous section that the \( \varepsilon \)–algorithm is regarded as the discrete potential KdV equation. Before discussing the relation of the \( \rho \)–algorithm with soliton equations, let us survey the result by Papageorgiou et al. again. They considered the most general form of the algorithm given by

\[
(x^{(m)}_{n+1} - x^{(m+1)}_{n-1})(x^{(m+1)}_n - x^{(m)}_n) = z^{(m)}_n,
\] (29)

and applied the singularity confinement test to eq. (29). As a result, when \( z^{(m)}_n \) is of the form,

\[
z^{(m)}_n = f(n+m) + g(m),
\] (30)

eq. (29) passes the test and is expected to be integrable. If we put

\[
f(x) = x, \quad g(x) = -x,
\] (31)

eq. (29) gives the \( \rho \)–algorithm (24), which indicates that there is a chance for eq. (24) to be some discrete analogue of soliton equations. Instead of the \( \rho \)–algorithm (24) itself, we consider the algorithm of the following form;

\[
(\rho^{(m)}_{n+1} - \rho^{(m+1)}_{n-1})(\rho^{(m+1)}_n - \rho^{(m)}_n) = an + b(m+1),
\] (32)

where \( a \) and \( b \) are constant. Employing a dependent variable transformation,

\[
Y^{(m)}_n = \rho^{(m)}_n - \rho^{(m-1)}_n,
\] (33)
we obtain

\[ Y_{n+1}^{(m)} - Y_{n}^{(m+1)} = \frac{an + bm + b}{Y_n^{(m+1)}} - \frac{an + bm}{Y_n^{(m)}} \]  \hspace{1cm} (34)

from eq. (32). Equation (34) possesses a form similar to the discrete KdV eq. (8). However, the nonautonomous property of eq. (34) yields an essential difference in its continuous limit. Let us introduce new variables \( t, x \) defined by

\[ t = \frac{an + bm}{\epsilon^3}, \quad x = \frac{cn + m - 1}{2}, \]  \hspace{1cm} (35)

and rewrite \( Y_n^{(m)} \) as \( \epsilon^{-3/2} \sqrt{t} \{ p + \epsilon^2 u(x - \epsilon/2, t) \} \), where \( \epsilon \) is a small parameter and \( p, c \) are finite constants satisfying

\[ p^2 = \frac{b}{2a - b}, \quad c = \frac{1}{2} - \frac{1}{2p^2} = \frac{b - a}{b}. \]  \hspace{1cm} (36)

Then eq. (34) becomes

\[
\frac{\epsilon^2}{\epsilon^2} u(x - \frac{\epsilon}{2} + ce, t + a\epsilon^3) - \frac{\epsilon^2}{\epsilon^2} u(x + \frac{\epsilon}{2} - ce, t + (b - a)\epsilon^3) \\
- \frac{1}{p + \epsilon^2 u(x + \frac{\epsilon}{2}, t + \epsilon^3)} + \frac{1}{p + \epsilon^2 u(x - \frac{\epsilon}{2}, t)} \\
+ \frac{\epsilon^3}{2t} \left[ a \left\{ p + \epsilon^2 u(x - \frac{\epsilon}{2} + ce, t + a\epsilon^3) \right\} \\
+ (a - b) \left\{ p + \epsilon^2 u(x + \frac{\epsilon}{2} - ce, t + (b - a)\epsilon^3) \right\} \\
- \frac{b}{p + \epsilon^2 u(x + \frac{\epsilon}{2}, t + \epsilon^3)} \right] = 0.
\]  \hspace{1cm} (37)

Taking the small limit of \( \epsilon \), we have

\[ (2a - b)u_t - \frac{1}{p^3} uu_x + \frac{1}{48p^2} (1 - \frac{1}{p^3}) u_{xxx} + \frac{(2a - b)}{2t} u = 0 \]  \hspace{1cm} (38)

at the order of \( \epsilon^5 \) from eq. (37). Since the coefficient of \( u_t \) is always twice as large as that of \( u/t \), eq. (32) is considered as one integrable discretization of the cylindrical KdV equation. It is interesting to note that the \( \rho \)-algorithm (24) is not exactly discretization of the cylindrical KdV equation. This is because we have \( p = 0 \) in eq. (36) and coefficients of \( uu_x \) and \( u_{xxx} \) become infinite in the case of the \( \rho \)-algorithm.

The third difference can be understood clearly from a viewpoint of the Hirota’s formalism. Employing the same dependent variable transformation as eq. (33), we obtain

\[ Y_{n+1}^{(m)} - Y_{n-1}^{(m+1)} = \frac{n}{Y_{n}^{(m+1)}} - \frac{n}{Y_{n}^{(m)}} \]  \hspace{1cm} (39)
from eq. (24). Moreover, through the same dependent variable transformation as eq. (14),
\[ Y_n^{(m)} = \frac{F(n+1, m-1)F(n-1, m)}{F(n, m-1)F(n, m)} \quad (40) \]
eq. (39) is rewritten as the following trilinear form;
\[
\begin{vmatrix}
-F(n+2, m-1) & F(n+1, m-1) & nF(n, m-1) \\
F(n+1, m) & 0 & F(n-1, m) \\
-nF(n, m+1) & F(n-1, m+1) & F(n-2, m+1)
\end{vmatrix} = 0. \quad (41)
\]
The functions \( F(n, m) \) and \( \tau_n^{(m)} \) in eq. (26) are related by
\[ F(n, m) = (-1)^{a(n)}\tau_n^{(m)}, \quad (42) \]
where \( a(n) \) satisfies
\[ a(n) \equiv a(n-2) + \left[ \frac{n-2}{2} \right] \quad (\text{mod 2}), \quad a(0) = a(1) = 0. \quad (43) \]
Because of nonautonomous property of eq. (24), there is no way to derive a bilinear form
with a single variable \( F(n, m) \) from the trilinear eq. (41). This fact reminds us of the
similarity constraint of the discrete KdV equation \[13\]. It should be noted, however, that
a pair of functions \( \tau_n^{(m)} \) and \( \tilde{\tau}_n^{(m)} \) given by eqs. (26) and (27)
satisfy bilinear equations,
\[
\begin{align*}
\tau_n^{(m)}\tau_{n+1}^{(m+1)} - \tau_{n+1}^{(m)}\tau_n^{(m+1)} + \tau_n^{(m+1)}\tilde{\tau}_n^{(m)} & = 0, \quad (44) \\
\tau_{n-1}^{(m+1)}\tilde{\tau}_n^{(m)} + \tau_{n+1}^{(m)}\tilde{\tau}_{n-1}^{(m+1)} - n\tau_n^{(m)}\tilde{\tau}_n^{(m+1)} & = 0, \quad (45)
\end{align*}
\]
which are considered to be the Jacobi and the Plücker identities for determinants, re-
spectively.

4 Concluding Remarks

We have seen that the \( \eta \)–algorithm and the \( \varepsilon \)–algorithm are equivalent to the discrete
KdV and the discrete potential KdV equations, respectively and that their performance
as convergence acceleration algorithms is completely the same. We have also shown that
the one generalization of the \( \rho \)–algorithm is considered as integrable discretization of
the cylindrical KdV equation. The \( \varepsilon \)– and the \( \rho \)– algorithms, despite their apparent
similarity, possess different properties both as convergence accelerators and as discrete
soliton equations. The difference in performance of these two algorithms depends on their different determinant expressions.

When we apply $\varepsilon$-- and $\rho$--algorithms to a convergent sequence, odd terms converge to the same limit as the original sequence though even terms diverge. This fact agrees with the idea of the singularity confinement. It is a future problem to clarify how two different notions, acceleration and integrability, are associated with each other. In other words, we should consider whether we can construct new convergence acceleration algorithms from the other discrete soliton equations\(^1\) and what kind of equations the other algorithms correspond to. The solution of these problems will shed a new light on the study of discrete integrable systems and numerical analysis.

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