Well-posedness of the three-dimensional Lagrangian averaged Navier-Stokes equations

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Abstract

In this dissertation we study the well-posedness of the three-dimensional Lagrangian averaged Navier-Stokes (LANS-α) equations. The LANS-α equations are a system of PDEs designed to capture the large scale dynamics of the incompressible Navier-Stokes equations. In the Lagrangian averaging approach, the motion at spatial scales smaller than a chosen parameter $\alpha > 0$ is filtered without the use of artificial viscosity. There are two types of LANS-α equations: the anisotropic version in which the fluctuation tensor is a dynamic variable that is coupled with the evolution equations for the mean velocity, and the isotropic version in which the covariance tensor is assumed to be a constant multiple of the identity matrix.

We prove the global-in-time existence and uniqueness of weak solutions to the isotropic LANS-α equations for the case of no-slip boundary conditions, generalizing the known periodic box result [14]. Our proof makes use of a formulation of the equations on bounded domains provided by Marsden and Shkoller [21]. In the anisotropic model, there are two choices for the divergence-free projection of the viscosity term. One choice is the classic Leray projector. In this case, Marsden and Shkoller [22] have shown the local-in-time well-posedness of the anisotropic equations in the periodic box. We extend their result by considering the second choice of projector, the generalized Stokes projector. The local-in-time well-posedness of the anisotropic LANS-α with this viscosity term is proven by using quasi-linear PDE-type methods.

We numerically compute strong solutions to the anisotropic equations in the laminar channel and pipe by considering steady fluid flow with no-slip boundary conditions. In particular, given a steady velocity vector that solves the Navier-Stokes equations, we numerically calculate the covariance tensor such that the pair solves the anisotropic LANS-α equations. Our solutions are in good agreement with the results contained in [11]. Namely, we confirm the logarithmic degeneracy rate of $v$.
the covariance tensor near the boundary and show that in elementary domains the sup-norm of the covariance tensor is unbounded in time near the wall. We conclude the dissertation by showing the existence of shear flow solutions to the anisotropic LANS-\( \alpha \) equations.
1.1 The Navier-Stokes equations.

The Navier-Stokes equations for an incompressible fluid are a system of partial differential equations that model the velocity vector $u = u(t, x)$ and pressure function $p = p(t, x)$ of a fluid whose velocity is divergence-free. We assume the fluid is contained in a fixed domain $\Omega \subset \mathbb{R}^n$ with boundary $\delta \Omega$ (possibly empty) and has a constant density $\rho$ with value $\rho = 1$. In this case, the Navier-Stokes equations are given as

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\nabla p + \nu \Delta u + f,$$

$$\text{div } u = 0, \ u(0, x) = u_0(x), \ u = 0 \text{ on } \partial \Omega,$$  \hspace{1cm} (1.1.1)

where $\nu > 0$ is the kinematic viscosity and the vector $f$ represents external forces. The kinematic viscosity is defined as the coefficient of viscosity divided by the density of the fluid. The pressure function is determined (modulo a constant) from the incompressibility constraint $\text{div } u = 0$ by solving the Neumann problem $-\Delta p = \text{div}(u \cdot \nabla)u$ with boundary condition $\nabla p \cdot n = 0$, where $n$ denotes the outward normal vector to the boundary. In the limiting case of viscosity $\nu \to 0$, the Navier-
Stokes equations are reduce to the idealized setting of the Euler equations.

The ratio of forces from the convective nonlinearity \((u \cdot \nabla)u\) term and the linear diffusion \(\nu \Delta u\) term is called the Reynolds number. As the Reynolds number increases, the contribution to the motion of the fluid from the viscous term decreases and turbulence is introduced. In turbulent regimes, the nonlinear effects send energy from the large spatial scales to smaller and smaller scales until the energy reaches the Kolmogorov dissipation scale, at which it is abolished by the linear dispersive mechanism. To resolve a numerical simulation of the Navier-Stokes equations (1.1.1), enough grid points or Fourier modes must be used so that the approximation captures the energy cascade in all scales down to the Kolmogorov scale. For turbulent flows such resolution requirements are not yet achievable, making the problem of turbulence an important unsolved problem in physics. The numerical inability in resolving small spatial scales motivates the study of the averaged motion of an incompressible fluid.

The averaged fluid methodology, discussed in the next section, captures the dynamics of the large scale motion while averaging the computationally unresolvable scales of the Navier-Stokes equations.

### 1.2 Averaged fluid motion.

An approach to modeling the averaged motion of an incompressible fluid is to suppose that the velocity of the fluid is a random variable represented by the decomposition

\[
    u(t, x) = U(t, x) + u'(t, x),
\]

where \(U\) denotes the mean velocity field and \(u'\) is a random variable with mean zero. In a statistical theory for turbulence, the evolution of the fluid at large spatial scales is the primary focus. The process of substituting equation (1.2.2) into the Navier-Stokes equations and averaging results in the Reynolds averaged Navier-Stokes equations.
(RANS) equations \[15, 30\]. The RANS equations are written as
\[
\frac{\partial U}{\partial t} + (U \cdot \nabla) U + \text{div}(u' \otimes u') = -\nabla p + \nu \Delta U + f
\]
\[
\text{div} U = 0, \quad U(0, x) = u_0(x) \quad (1.2.3)
\]
\[
U = 0 \text{ on } \partial \Omega.
\]
The dynamics of the mean velocity \(U\) is well-defined when the Reynolds stress tensor \((u' \otimes u')\) is expressed in terms of \(U\). The effect the motion at small length scales has on the evolution of the averaged velocity \(U\) is called the turbulence closure problem. Classically, it is assumed that the Reynolds stress term is of the form \(\text{div}(u' \otimes u') = \nu_E(t, x, \text{Def} U) \cdot \text{Def} U\), where \(\nu_E\) is the eddy viscosity and \(\text{Def} U\) is the rate of deformation tensor defined as
\[
\text{Def} U = \frac{1}{2} \left[ \nabla U + (\nabla U)^T \right]. \quad (1.2.4)
\]
Under this assumption, viscosity, that is not naturally present in the physical model, is added into the system. This artificial viscosity augments the inherent dissipative mechanism and assists in the removal of the energy contained in the small scales at which \(u'\) resides. Since it is still necessary to guess the form of \(\nu_E\), an improvement to the procedure of modeling the averaged motion of a fluid is needed.

The Lagrangian averaged Navier-Stokes (LANS-\(\alpha\)) equations are a system of partial differential equations designed to capture the large scale dynamics of the incompressible Navier-Stokes equations without the use of artificial viscosity or dissipation. In the Lagrangian averaging approach, the motion at spatial scales smaller than a chosen parameter \(\alpha > 0\) are filtered. The inviscid form of the LANS-\(\alpha\) equations, the Lagrangian averaged Euler (LAE-\(\alpha\)) equations, first appeared in Holm, Marsden, and Ratiu \[17, 16\] as a \(n\)-dimensional generalization of the one-dimensional Camassa-Holm equation. The authors expressed the LAE-\(\alpha\) equations as the Euler-Poincaré equations corresponding to a Lagrangian given by an \(H^1\)-equivalent norm. In the next section, we discuss their results in greater detail.
Marsden and Shkoller provide a complete derivation of the LANS-$\alpha$ equation in [21, 22]. Rather than averaging at the level of the Navier-Stokes equations, the authors average at the level of the action functional. For the LAE-$\alpha$ equations, the action functional is an $\alpha^2$-modification to the kinetic energy action functional of the classical Euler equations. Unlike the Reynolds averaging procedure described above, averaging at the level of the action functional preserves the variational structure of the incompressible fluid. In particular, the solutions to the LAE-$\alpha$ equation are the extrema of an $H^1$-equivalent energy functional just as the solutions to the Euler equations are minimizers of the total kinetic energy. To highlight additional features of the Lagrangian averaging approach, we outline Marsden and Shkoller’s recent derivation [22]. For a chosen parameter $\alpha > 0$, the authors define the initial data $u^\varepsilon_0 = u_0 + \varepsilon \omega$, $\omega \in S^2$, in a ball of radius $0 < \varepsilon < \alpha$ centered at given initial data $u_0$. Solving the Euler equations with initial data $u_0$ and $u^\varepsilon_0$ results in a velocity field $u$ and a perturbed velocity field $u^\varepsilon$. Define $\eta$ and $\eta^\varepsilon$ as the Lagrangian trajectories associated with the velocity solutions $u$ and $u^\varepsilon$. The vector $\eta$ solves the first order initial value problem

$$\dot{\eta}(t, x) = u(t, \eta(t, x)), \quad \eta(0, x) = x,$$

where the dot represents the partial derivative with respect to time. The vector $\eta^\varepsilon$ solves equation (1.2.5) with $u$ replaced with the solution $u^\varepsilon$. We can regard the differential equation (1.2.5) as a map from the space of $u$’s (spatial or Eulerian description) to the space of $(\eta, \dot{\eta})$ (material or Lagrangian description). The Lagrangian fluctuation $\xi^\varepsilon := \eta^\varepsilon \circ \eta^{-1}$ is a volume preserving diffeomorphism that plays the role of the Reynolds decomposition (1.2.2). Since the decomposition is made in the Lagrangian reference frame, the interplay between the Euler and Lagrangian frameworks is of fundamental importance. This interplay is insignificant in the Reynolds averaging approach where the decomposition and the averaging only occur in the Eulerian ref-
ence frame. After asymptotically expanding $u^\varepsilon$ in terms of $\varepsilon$, Marsden and Shkoller take the ensemble average of the total kinetic energy

$$ S = \frac{1}{2} \int_0^T \int_\Omega |u^\varepsilon|^2 \, dx \, dt $$

over all possible solutions $u^\varepsilon$. At this point, the turbulence closure problem is confronted. In the Lagrangian averaging approach, the turbulence closure problem amounts to specifying the fluctuations $\xi'$ and $\xi''$ (′ denotes the derivative with respect to $\varepsilon$ evaluated at $\varepsilon = 0$) as functions of the mean velocity $u$. The generalized Taylor “frozen turbulence” hypothesis provides the solution and assumes the fluctuation $\xi'$ is frozen or Lie advected in the mean flow as a divergence-free vector field. In addition, the fluctuation $\xi''$ satisfies

$$ \frac{D}{Dt} \langle \xi'' \rangle \perp u \text{ in } L^2(\Omega), $$

where $\langle \cdot \rangle$ is the ensemble average over the solutions $u^\varepsilon$ and the operator $\frac{D}{Dt}$ is the usual total derivative. Marsden and Shkoller’s derivation results in an averaged action functional which includes all terms up to order $\alpha^2$. To conclude their calculation, the authors apply Hamilton’s principle to the averaged action functional therefore yielding the LAE-α equations.

The derivation outlined above has recently been generalized by Bhat et al. [4] in their computation of the LAE-α equations for a compressible fluid. Unlike the previous derivation, the authors averaged over a tube of trajectories centered around a given Lagrangian flow. The tube is constructed by specifying the Lagrangian fluctuation $\xi^\varepsilon$ at $t = 0$ and deciding on a “flow rule” which evolves $\xi^\varepsilon$ to later time. For example, the flow rule for an incompressible fluid is the frozen Taylor hypothesis. In the model of compressible fluid motion, the flow rule is the answer to the closure problem and is chosen to be one of two different physical properties. One physical property is appropriate for isotropic fluid conditions and the other property yields the anisotropic model for bounded fluid containers. The remainder of the author’s
construction follows the outline presented in the previous paragraph.

1.3 Geometric structure of the Euler and LAE-α equations.

The Euler-Poincaré equations, developed originally by Poincaré [26] in his study of Euler-type equations, are determined once a Lagrangian map \( L : g \rightarrow \mathbb{R} \) is specified in a Lie algebra \( g \). For any point \( \zeta \in g \), the evolution of the variable \( \zeta \) is determined by the Euler-Poincaré equations

\[
\frac{d}{dt} \frac{\delta L}{\delta \zeta} = \text{ad}_\zeta^* \frac{\delta L}{\delta \zeta}.
\] (1.3.6)

The map \( \text{ad}_\zeta : g \rightarrow g \), the adjoint representation of the Lie algebra, is the linear map \( \eta \mapsto [\zeta, \eta] \), where \([\zeta, \eta]\) denotes the Lie bracket of \( \zeta \) and \( \eta \). The map \( \text{ad}_\zeta^* \) is the dual linear map associated with \( \text{ad}_\zeta \). The LAE-α equations and the classical Euler equations are the Euler-Poincaré equations posed on the same Lie algebra using different Lagrangian maps. The configuration space that yields the appropriate Lie algebra for incompressible fluid motion was unknown until Arnold [2] and Ebin and Marsden [12].

Ebin and Marsden [12] define the configuration space for incompressible fluid motion as

\[
\mathcal{D}^s_{\mu}(\Omega) := \{ \eta(t, \cdot) : \Omega \rightarrow \Omega \mid \eta \in H^s(\Omega), \eta \text{ is a bijection}, \\
\eta^{-1}(t, \cdot) : \Omega \rightarrow \Omega \text{ is in } H^s(\Omega), \text{ and } \det D\eta = 1 \},
\]

the group of \( H^s \)-class volume preserving diffeomorphisms on \( \Omega \). The group \( \mathcal{D}^s_{\mu}(\Omega) \) (under composition) is a \( C^\infty \) differentiable manifold but is not a Lie group [12] (right multiplication is smooth, but left multiplication is not). It does however have an exponential map and associated Lie algebra in the usual sense of Lie groups. The tangent space to \( \mathcal{D}^s_{\mu}(\Omega) \) at the identity is identified with the space \( \mathcal{X}^s_{\text{div}}(\Omega) \), the space of \( H^s \) divergence-free vector fields on \( \Omega \) that are tangent to the boundary \( \partial \Omega \).
The Euler equations can be written as the Euler-Poincaré equations posed on the Lie algebra \( \mathcal{X}_{\text{div}}^s(\Omega) \) with the Lagrangian defined as the total kinetic energy. This connection was first made by Arnold [2]. The Euler equations arise from an application of Hamilton’s principle of least action to the \( L^2 \) Lagrangian

\[
\ell(u) = \frac{1}{2} \int_{\Omega} |u(t, x)|^2 \, dx.
\]  

(1.3.7)

Euler-Poincaré reduction techniques (see Marsden and Ratiu [20]) show that Hamilton’s principle reduces to the following variational principle with respect to Eulerian velocities:

\[
\delta \int_{a}^{b} \ell(u) \, dt = 0,
\]

(1.3.8)

which should hold for all variations \( \delta u \) of the form

\[
\delta u = \dot{w} + (u \cdot \nabla)w - (w \cdot \nabla)u,
\]

where \( w \) is a time dependent vector field representing the infinitesimal particle displacement vanishing at temporal endpoints. The Hamiltonian structure, along with references to the literature, can be found in Marsden and Weinstein [23], Arnold and Khesin [3], and Marsden and Ratiu [20].

Arnold [2] discovered that solutions to the Euler equation correspond to geodesics of \( \mathcal{D}^s_{\mu}(\Omega) \) with respect a \( L^2 \) right invariant metric. The \( L^2 \) metric is defined to be the weak Riemannian metric on \( \mathcal{D}^s_{\mu}(\Omega) \) whose value at the identity is

\[
\langle u(t, x), w(t, x) \rangle_{L^2} = \int_{\Omega} u(t, x) \cdot w(t, x) \, dx, \quad u, w \in \mathcal{X}_{\text{div}}^s(\Omega).
\]

(1.3.9)

The word weak is used here because this metric need not define the topology on the tangent space but may define a weaker topology. The bridge between the group \( \mathcal{D}^s_{\mu}(\Omega) \) and hydrodynamics is the following. If \( \eta(t, x) \in \mathcal{D}^s_{\mu}(\Omega) \) is a geodesic with respect to the \( L^2 \) right-invariant metric (1.3.9) and the velocity of the fluid is defined as \( u(t, x) = \dot{\eta}(t, \eta^{-1}(t, x)) \), then velocity vector \( u \) is a solution to the classical Euler
equations
\[
\frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\text{grad } p + f, \\
d\text{iv}_u = 0, \quad u(0, x) = u(0),
\]
(1.3.10)
with the boundary condition that \(u\) is tangent to \(\partial \Omega\). The geodesic \(\eta\), which satisfies the first order differential equation (1.2.5), is the particle path associated with the vector velocity solution to the Euler equations. By showing the existence of a smooth geodesic on the group \(D^s_\mu(\Omega)\) for \(s > (n/2) + 1\), Ebin and Marsden [12] obtain short-time well-posedness of the Euler equations on a smooth \(n\)-dimensional Riemannian manifold.

We summarize the geometric results of the Euler equations on \(\Omega\) with the following theorem.

**Theorem 1.3.1** (see [16]) The following statements are equivalent:

(i) The velocity vector \(u \in X^s_{\text{div}}(\Omega)\) solves the Euler equations (1.3.10).

(ii) The particle path \(\eta \in D^s_\mu(\Omega)\), given by (1.2.5), is a geodesic on \(D^s_\mu(\Omega)\) with respect to the \(L^2\) right invariant weak metric (1.3.9).

(iii) Hamilton’s principle of least action (1.3.8) holds for variations of the form \(\delta u = \dot{w} + (u \cdot \nabla)w - (w \cdot \nabla)u\).

(iv) The Euler-Poincaré equations
\[
\frac{d}{dt} \frac{\delta \ell}{\delta u} = \text{ad}^*_u \frac{\delta \ell}{\delta u}
\]
holds for the Lagrangian \(\ell\) defined by (1.3.7) on the Lie algebra \(X^s_{\text{div}}(\Omega)\).

The Lagrangian averaged Euler equations contain a rich geometric structure similar to the geometric framework of the Euler equations. The LAE-\(\alpha\) equations were first written by Holm, Marsden, and Ratiu [10] as an \(n\)-dimensional generalization of the one-dimensional Camassa-Holm equation. For a given parameter \(\alpha > 0\), the
authors define the Lagrangian for the mean fluid velocity $u \in \mathcal{X}_{\text{div}}^s(\Omega)$ as the $H^1$-equivalent norm

$$L(u) = \frac{1}{2} \int_{\Omega} u \cdot u + 2\alpha^2 \text{Def } u : \text{Def } u \, dx,$$

(1.3.11)

where the differential operator $\text{Def}$ is the deformation tensor defined by (1.2.4) and $: \,$ is the contraction on two indices given by $a : b = a_{ij}b_{ij}$. This $H^1$ Lagrangian defines a right-invariant weak metric on the Lie algebra of divergence-free vector spaces $\mathcal{X}_{\text{div}}^s(\Omega)$. Holm, Marsden, and Ratiu [16] calculate the Euler-Poincaré equations with the Lagrangian defined by equation (1.3.11). The result is the Lagrangian averaged Euler equations in Euclidean space

$$\frac{\partial v}{\partial t} + (u \cdot \nabla)v + [\nabla u]^T \cdot v = -\text{grad } p,$$

$$v = (1 - \alpha^2 \Delta)u,$$

(1.3.12)

$$\text{div } v = 0, \quad v(0,x) = v_0,$$

with the boundary condition that $v$ is tangent to $\partial \Omega$. The authors conclude that the solutions to the LAE-$\alpha$ equations correspond to the geodesics of $\mathcal{D}^s_\mu(\Omega)$ with respect to the $H^1$ right-invariant metric defined by (1.3.11). In [27] and [28], Shkoller generalizes equation (1.3.12) to a smooth $n$-dimensional Riemannian manifold. By showing the existence of smooth geodesic flow on $\mathcal{D}^s_\mu(\Omega)$ with respect to the $H^1$ metric defined by (1.3.11), Shkoller proves the smooth-in-time local (global when $n = 2$) existence and uniqueness of strong solutions with $H^s$ initial data, $s > (n/2) + 1$.

### 1.4 Organization of the dissertation.

In this dissertation we study the well-posedness of the Lagrangian averaged Navier-Stokes equation on fluid containers located in $\mathbb{R}^3$. There are two types of LANS-$\alpha$ equations: the anisotropic version in which the (fluctuation) covariance tensor is a dynamic variable that is coupled with the evolution equations for the mean velocity, and the isotropic version in which the covariance tensor is assumed to be a constant
multiple of the identity matrix. A brief history of the LANS-α equations is presented in Chapter 2.

In Chapter 2, we consider the isotropic LANS-α equations on bounded domains. We prove the global-in-time existence and uniqueness of weak solutions for the case of no-slip boundary data, extending the periodic box result of Foias, Holm, and Titi [14]. We make use of a formulation of the LANS-α equations on bounded domains given by Shkoller [28] and Marsden and Shkoller [21] which reveals the correct boundary conditions. We begin the proof of existence with a sequence of strong solutions to the LANS-α equations. Known to exist from [21], these vector fields trivially solve the weak formulation of the equations. We use standard interpolating inequalities to establish estimates independent of the sequence index. Classical compactness arguments enable us to conclude that the sequence converges in the correct space with the limit satisfying the weak formulation of the LANS-α equations. At the end of the chapter we prove the existence of a nonempty, compact, convex, and connected global attractor.

Chapters 3 and 4 focus on the anisotropic equations. In bounded domains, the covariance tensor plays a prominent role in the mechanics of fluid motion. The anisotropic LANS-α are therefore the correct model to capture the large scale motion of the fluid. There are two choices for the divergence-free projection of the viscosity term. One choice is the classical $L^2$-orthogonal Leray projector. In this case, Marsden and Shkoller [22] show that strong solutions exist and are unique in the three-dimensional periodic box for a finite time interval. In Chapter 3, we extend this result by considering the second choice of projector, the generalized Stokes projector. The generalized Stokes projector, defined in detail in Chapter 3, assigns to the divergence-free vector field the no-slip boundary condition. Without fluid motion on the boundary, the fluctuation tensor is also zero on the boundary, making the Stokes projector the appropriate projector for the anisotropic equations. The inclu-
sion of the generalized Stokes projector viscosity term in the anisotropic equations on a periodic box is an essential step towards understanding averaged flow on bounded domains. We prove the local-in-time existence and uniqueness of classical solutions to the anisotropic LANS-\(\alpha\) on the periodic box with this viscosity term by using quasi-linear partial differential equation type methods. We begin the proof by obtaining an approximate solution using the Galerkin projection of the anisotropic equations onto a finite dimensional vector space. The generalized Stokes projector term forces us to repose the problem in terms of the momentum rather than the velocity. After proving an elliptic regularity-type result, we show that our approximations remain in the correct vector spaces independent of the projection. We use classical compactness arguments to establish the existence of solutions for a short period of time.

In Chapter 4, we examine the anisotropic LANS-\(\alpha\) equations in the channel and pipe geometry. Recently, Coutand and Shkoller [11] have proposed a turbulent channel flow theory founded on the anisotropic equations. By posing the problem in the correct functional framework, Coutand and Shkoller [11] proved that weak solutions of the anisotropic LANS-\(\alpha\) equations exist and are unique throughout the entire channel for all time. In Chapter 4, we numerically compute strong solutions to the anisotropic equations by considering steady fluid flow with no-slip boundary conditions. In particular, given a steady velocity vector that solves the Navier-Stokes equations, we numerically calculate the covariance tensor such that the pair solves the anisotropic LANS-\(\alpha\) equations. Our solutions are in good agreement with analytic results contained in [11]. Namely, we confirm the logarithmic degeneracy rate of the covariance tensor near the boundary and show that in these elementary domains the sup-norm of the covariance tensor is unbounded in time near the wall. We conclude the chapter by showing the existence of a shear flow velocity solution to the anisotropic LANS-\(\alpha\) equations.

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Chapter 2

Weak solutions of the isotropic LANS-\(\alpha\) equations

2.1 Introduction.

The isotropic Lagrangian averaged Navier-Stokes (LANS-\(\alpha\)) equations for an incompressible viscous fluid moving in a bounded fluid container \(\Omega \subset \mathbb{R}^n\) with smooth (at least \(C^3\)) boundary \(\partial \Omega\) may be written as the following system of partial differential equations:

\[
\begin{align*}
\partial_t u + \nabla u \cdot u + U^\alpha(u) &= -(1 - \alpha^2 \Delta)^{-1} \text{grad} p - \nu Au + F, \\
\text{div} u &= 0, \\
\text{div} u &= 0 \text{ on } \partial \Omega, \\
u(0, x) &= u_0(x),
\end{align*}
\]

where

\[
U^\alpha(u) = \alpha^2 (1 - \alpha^2 \Delta)^{-1} \text{Div}(\nabla u \cdot \nabla u^T + \nabla u \cdot \nabla u - \nabla u^T \cdot \nabla u),
\]

\(\text{Div} \) denotes the divergence operator.
and $\partial_t$ represents the partial derivative with respect to $t$. We use $u(t,x)$ to denote the large-scale (or averaged) velocity field of the fluid, assumed to have constant density. The pressure function $p(t,x)$ is determined (modulo constants) from the incompressibility constraint (2.1.1b). The constant $\nu$ denotes the kinematic viscosity of the fluid, and $\alpha > 0$ is the spatial scale at which fluid motion is filtered, i.e. spatial scales smaller than $\alpha$ are averaged out. The additional term $F(x) \in H^2 \cap H^1_0$ represents the external force acting on the system and is assumed, for simplicity, to be time-independent. We let $A := -P\Delta$ denote the Stokes operator, with $P$ the Leray projector onto divergence-free vector fields.

It has been a longstanding problem in fluid dynamics to derive a model for the large scale motion of a fluid that averages or course-grains the small, computationally unresolvable, scales of the Navier-Stokes equations. The LANS-$\alpha$ equations provide one such averaged model, and have been studied rather extensively from both the analytical, as well numerical, points of view. The numerical simulations of both forced and decaying isotropic turbulence given by Chen et al. [5, 6, 7, 8] and Mohseni et al. [24] are briefly reviewed in Chapter 4.

The LANS-$\alpha$ equations can be written in an isotropic and anisotropic version. The isotropic equations, given by equations (2.1.1) are the focus of this chapter. As discussed in Chapter 1, the inviscid ($\nu = 0$) version of equations (2.1.1), known as the Lagrangian averaged Euler (LAE-$\alpha$) or Euler-$\alpha$ equations, was first given by Holm, Marsden, and Ratiu [16] in the case that $\Omega = \mathbb{R}^n$ as

$$\partial_t v + \nabla_u v + [\nabla u]^T \cdot v = -\text{grad} p,$$

$$\text{div} u = 0,$$

where the variable

$$v = (1 - \alpha^2 \Delta) u$$

may be thought of as the momentum. Foias, Holm, and Titi [14] first added viscous dissipation to (2.1.3); they argued on physical grounds that the momentum $v$ rather
than the velocity $u$, need be diffused. By assuming periodic boundary conditions, they obtained the following form of the LANS-$\alpha$ equations:

$$\begin{align*}
\partial_t v + \nabla u v - \alpha^2 [\nabla u]^T \cdot \Delta u &= -\text{grad } p + \nu \Delta v + g, \\
\text{div } u &= 0, \\
u(0, x) &= u_0(x),
\end{align*}$$

(2.1.4)

with $g$ taken in $L^2$. While yielding the correct equations on a periodic box, the question of how to appropriately prescribe boundary data in the no-slip $u = 0$ case remained open. Specifically, inversion of the dissipative term $\nu \Delta v = \nu (1 - \alpha^2 \Delta) u$, a fourth-order operator, requires further constraints than simply $u = 0$ on $\partial \Omega$.

Shkoller [28] and Marsden and Shkoller [21] supplied the additional boundary condition by reformulating (2.1.4) as the system of equations (2.1.1). In the formulation (2.1.1), it is clear that if $u = 0$ on $\partial \Omega$, then

$$Au = 0 \quad \text{on} \quad \partial \Omega$$

as well. This follows since each term in the inviscid equations identically vanishes on the boundary, thanks to the inversion of $(1 - \alpha^2 \Delta)$ with Dirichlet boundary conditions. The viscous term in the formulation (2.1.1) was obtained by treating the Lagrangian trajectory as a stochastic process, and replacing deterministic time derivatives with backward-in-time mean stochastic derivatives, exactly following the usual procedure for obtaining the viscous dissipation term in the Navier-Stokes equations as done by Chorin [9] and Peskin [25].

The term $U^\alpha(u)$, given in equation (2.1.2), provides a regularization to the Navier-Stokes equations which is dispersive, rather than dissipative, in character. This regularization is geometric in nature, and arises as the geodesic flow of an $H^1$ right-invariant Riemannian metric on the Hilbert group of volume-preserving diffeomorphisms of the fluid container (discussed in Chapter 1); as such, this regularizer yields an a priori $L^\infty - H^1$ estimate in three-dimensions (in general, $n$ dimensions for $n \geq 2$),
and one is thus tempted to ask whether the LANS-\(\alpha\) system is globally well-posed (even though, as is clear from (2.1.1a), no additional artificial viscosity is being added to the Navier-Stokes equations).

In the case of the periodic box, Foias, Holm, and Titi \[14\] proved the global well-posedness of \(H^1\) weak solutions in dimension three, but as we noted, their formulation (2.1.4) did not provide the obvious extension to bounded domains. Using the equations (2.1.1), Marsden and Shkoller \[21\] proved the global well-posedness of classical solutions in dimension three in the case of no-slip boundary data. In this chapter, we give an extension of that result to the \(H^1\) weak solutions of Foias, Holm, and Titi \[14\]. In particular, we prove the global-in-time existence, uniqueness, and regularity of weak solutions to the LANS-\(\alpha\) equations for initial data in the class \(\{u \in H^1_0 | \text{div } u = 0\}\). The analogous two-dimensional result follows trivially (as it is already known for the original Navier-Stokes equations).

The well-posedness result leads to the existence of a nonempty, compact, convex, and connected global \(H^1\) attractor in both two- and three-dimensions. In three-dimensions, the global attractor has the identical bound as that obtained by Foias, Holm, and Titi \[14\] for periodic boundary conditions. This upper bound depends on \(1/\alpha\) and consequently tends to infinity as \(\alpha\) tends to zero. Due to the difference in the Lieb-Thirring inequality in two- and three-dimensions, the two-dimensional bound for the global attractor is \(\alpha\)-independent. The global attractor of the two-dimensional LANS-\(\alpha\) equations is similar to the bound for the global attractor of the two-dimensional Navier-Stokes equations. This is shown in detail in Coutand \textit{et. al.} \[10\].

The remainder of this chapter is organized into two sections. In Section 2.2 we establish the global well-posedness result, while Section 2.3 is devoted to showing the existence of a global \(H^1\) attractor.
2.2 Global well-posedness.

In this section, we establish the global existence of unique $H^1$ weak solutions to the LANS-\(\alpha\) equations on bounded domains $\Omega \in \mathbb{R}^3$. Rather than using the standard Galerkin method, we instead take a sequence of classical solutions, via the existence result in [21], and prove that this sequence converges in $C([0,T],H^1)$ to a $H^1$ weak solution of equations (2.1.1) for all $T > 0$.

2.2.1 Notation and some classical inequalities.

We work in the following Hilbert spaces: $V^s = H^s \cap H_0^1$ and $V^s_\mu = \{ u \in V^s \mid \text{div } u = 0 \}$ for $s \geq 1$, and $\tilde{V}^s_\mu = \{ u \in V^s_\mu \mid Au = 0 \text{ on } \partial \Omega \}$ for $s \geq 3$. We endow $V^1_\mu$ with the following scalar product:

$$\langle f, g \rangle_{1,\alpha} = \int_{\Omega} f \cdot g + \alpha^2 \text{Def } f : \text{Def } g \, dx,$$

where $\text{Def } u$ is the rate of deformation tensor defined by equation (1.2.4), $\cdot$ denotes the usual dot product, and $: \text{ is the contraction of two indices, e.g. } a : b = a_{ij}b_{ij}$.

Furthermore for any integer $s \geq 0$, we set

$$D^s u = \{ D^\beta u : |\beta| = s \}, \quad ||D^s u||_{L^p} = \sum_{|\beta| = s} ||D^\beta u||_{L^p},$$

where $\beta = (\beta_1, \ldots, \beta_n)$ denotes a multi-index, and

$$|\beta| = \beta_1 + \cdots + \beta_n, \quad D^\beta = \partial^\beta_1 \cdots \partial^\beta_n,$$

where

$$\partial_k := \frac{\partial}{\partial x_k}.$$

Throughout the dissertation, we let $C > 0$ denote a generic constant. For simplicity in notation, we write $u(t) = u(t, \cdot)$. The following standard inequalities are used frequently and we include their definitions for completeness.
Gagliardo-Nirenberg inequalities \([1]\). Suppose

\[
\frac{1}{p} = \frac{i}{n} + a \left( \frac{1}{r} - \frac{m}{n} \right) + (1 - a) \frac{1}{q}
\]

where \(1/m \leq a \leq 1\) (if \(m - n - n/r\) is an integer \(\geq 1\), only \(a < 1\) is allowed). Then for \(f : \Omega \to \mathbb{R}^n\),

\[
\| D^i f \|_{L^p} \leq C \| D^m f \|_{L^r}^a \cdot \| f \|_{L^q}^{1-a}.
\] (2.2.6)

Two specific cases of (2.2.6) in dimension three are

\[
\| v \|_{L^4} \leq 4 \| Dv \|_{L^2}^{3/4} \| v \|_{L^2}^{1/4},
\] (2.2.7)

\[
\| D^i v \|_{L^2} \leq C \| v \|_{L^2}^{1-i/m} \| D^m v \|_{L^2}^{i/m}.
\] (2.2.8)

**Sobolev Embedding Theorem.** If \(u \in H^s(\mathbb{R}^n)\) for \(s > \frac{n}{2} + k\), then \(u \in C^k(\mathbb{R}^n)\). In particular for \(s > \frac{n}{2}\), \(H^s(\mathbb{R}^n) \subset L^{\infty}(\mathbb{R}^n)\) with the bound

\[
\| u \|_{L^{\infty}(\mathbb{R}^n)} \leq C \| u \|_{H^s(\mathbb{R}^n)}
\] (2.2.9)

where the constant \(C\) only depends on \(s\) and \(n\).

**Young’s Inequality.** Let \(1 < p, q < \infty\), \(\frac{1}{p} + \frac{1}{q} = 1\). Then

\[
ab \leq \varepsilon a^p + C(\varepsilon)b^p \quad (a, b > 0, \varepsilon > 0)
\] (2.2.10)

for \(C(\varepsilon) = (\varepsilon p)^{-q/p}q^{-1}\).

**Gronwall’s Inequality.** Let \(\eta(\cdot)\) be a nonnegative, absolutely continuous function on \([0, T]\) which satisfies for a.e. \(t\) the differential inequality

\[
\eta'(t) \leq \phi(t)\eta(t) + \psi(t),
\]

where \(\phi(t)\) and \(\psi(t)\) are nonnegative, summable functions on \([0, T]\). Then

\[
\eta(t) \leq e^{\int_0^t \phi(s) \, ds} \left[ \eta(0) + \int_0^t \psi(s) \, ds \right]
\] (2.2.11)

for all \(0 \leq t \leq T\).
2.2.2 Three equivalent forms of the LANS-α equations.

Three equivalent forms of the LANS-α equations will be useful to us.

**LANS-1:**

\[
\partial_t u + \nabla u u + U^{\alpha}(u) = -\nu A u - (1 - \alpha^2 \Delta)^{-1} \text{grad} p + (1 - \alpha^2 \Delta)^{-1} f
\]

\[
\text{div } u(t, x) = 0,
\]

\[
u(1 - \alpha^2 \Delta) Au - \text{grad } p + f,
\]

(2.2.12)

where \( f \in L^2 \). (It is convenient to replace \( F \) in equation (2.1.1a) with \((1 - \alpha^2 \Delta)^{-1} f\); there is no loss in generality as \((1 - \alpha^2 \Delta)\) with domain \( H^2 \cap H^1_0 \) is an isomorphism.)

The Stokes operator \( Au = -P \Delta u \), is the Leray projection of \(-\Delta u\) onto divergence-free vector fields and has domain \( D(A) = H^2 \cap H^1_0 \). As we noted above, when \( u = 0 \) on \( \partial \Omega \), then \( Au \) must also equal zero on \( \partial \Omega \).

**LANS-2:** This form is equivalent to LANS-1 in view of our remark that LANS-1 implies \( Au = 0 \) on \( \partial \Omega \):

\[
\partial_t (1 - \alpha^2 \Delta) u + \nabla u [(1 - \alpha^2 \Delta) u] - \alpha^2 \nabla u^T \cdot \Delta u = \nu(1 - \alpha^2 \Delta) Au - \text{grad } p + f,
\]

(2.2.13)

together with the constraint \( \text{div } u(t, x) = 0 \) and boundary data \( u = Au = 0 \) on \( \partial \Omega \).

Note that when the domain \( \Omega \) is the period box \( \mathbb{T}^3 \), the Stokes operator is given by \(-\Delta\), and formulation (2.2.13) reduces to equations (2.1.4), the LANS-α equations used by Foias, Holm, and Titi [14].

**LANS-3:** This form is the analog of the Navier-Stokes equations written in terms of the Helmholtz-Hodge projection:

\[
\partial_t u + \nu Au + \mathcal{P}^{\alpha} \left[ \nabla u u + U^{\alpha}(u) - (1 - \alpha^2 \Delta)^{-1} f \right] = 0,
\]

(2.2.14)

where for \( s \geq 1 \), \( \mathcal{P}^{\alpha} \) is the Stokes projector defined below.
CHAPTER 2.  Weak solutions of the LANS-$\alpha$ equations on bounded domains

**Definition 2.2.1** For $s \geq 1$, we let $P^\alpha: \mathcal{V}^s \to \mathcal{V}_\mu^s$ denote the Stokes projector, a continuous $\langle \cdot, \cdot \rangle_{1,\alpha}$-orthogonal idempotent operator (see Proposition 1 of [28]). It is defined as

$$P^\alpha(w) = w - (1 - \alpha^2 \Delta)^{-1} \text{grad} p = v,$$

where $(v, p)$ solve the Stokes problem: given $w \in \mathcal{V}^s$, there is a unique vector field $v \in \mathcal{V}_\mu^s$ and a function $p$ (unique up to an additive constant) such that

$$(1 - \alpha^2 \Delta)v + \text{grad} p = (1 - \alpha^2 \Delta)w,$$

$$\text{div} v = 0,$$

$$v = 0 \text{ on } \partial \Omega.$$  

**2.2.3 Results.**

We begin with two elementary lemmas.

**Lemma 2.2.1** $\dot{\mathcal{V}}^4_{\mu}$ is dense in $\mathcal{V}_\mu^1$.

**Proof:** Let $v \in \mathcal{V}_\mu^1$. We find a sequence $v_m \in \dot{\mathcal{V}}^4_{\mu}$ which converges to $v$ in $H^1$. The proof makes use of the Lax-Milgram Theorem to provide a compact operator from $\mathcal{V}_\mu^1$ to $\dot{\mathcal{V}}^4_{\mu}$. Define the bilinear form $E: \dot{\mathcal{V}}^4_{\mu} \times \dot{\mathcal{V}}^4_{\mu} \to \mathbb{R}$ by

$$E[u, v] = \langle u, v \rangle_{H^4}.$$ 

The function space $\dot{\mathcal{V}}^4_{\mu}$ is a closed subspace of $H^4$ and therefore a Hilbert space endowed with the usual $H^4$ topology. Since $E$ is defined as the usual inner product on $H^4$, it is coercive and continuous. We define, using the Riesz-Representation Theorem, the bounded linear function $\tilde{f} \in \left( \dot{\mathcal{V}}^4_{\mu} \right)^*$ by

$$\tilde{f}(v) = (f, v)_{L^2} \quad \forall v \in \dot{\mathcal{V}}^4_{\mu},$$

where $f \in \dot{\mathcal{V}}^4_{\mu}$. By the Lax-Milgram Theorem, there exists a unique $u(f) \in \dot{\mathcal{V}}^4_{\mu}$ such that

$$E[u, v] = \langle u, v \rangle_{H^4} = (f, v)_{L^2}, \quad \forall v \in \dot{\mathcal{V}}^4_{\mu}.$$
This is the weak formulation of an elliptic problem \(Lu = f\) where \(L\) is an eighth order elliptic differential operator. The map \(L^{-1} : f \mapsto u\) is a continuous map from \(\dot{\mathcal{V}}_{\mu}^4 \to \dot{\mathcal{V}}_{\mu}^4\). Since \(H^4\) is compactly embedded in \(H^1\), \(L^{-1}\) is a compact operator from \(\mathcal{V}_\mu^4 \to \dot{\mathcal{V}}_{\mu}^4\), and consequently has a discrete spectrum of eigenfunctions \(\{e_i\}_{i \in \mathbb{N}}\). The eigenfunctions \(\{e_i\}_{i \in \mathbb{N}}\) form a Hilbert basis of \(\mathcal{V}_\mu^1\) and therefore we write \(v = \sum_{i=1}^{\infty} c_i e_i\).

The ellipticity of \(L^{-1}\) implies that \(\{e_i\}_{i \in \mathbb{N}}\) are also eigenfunctions of \(L\), allowing us to conclude that \(\{e_i\}_{i \in \mathbb{N}} \in \dot{\mathcal{V}}_{\mu}^4\). Consequently, the sequence \(v_m = \sum_{i=1}^{m} c_i e_i \in \dot{\mathcal{V}}_{\mu}^4\) converges in \(H^1\) to \(v\) as \(m \to \infty\).

Lemma 2.2.2 Let \(\lambda_1 = \inf_{v \in \mathcal{V}_\mu^2} \frac{||\nabla v||^2_{L^2}}{||v||^2_{L^2}}\) be the smallest eigenvalue of the Stokes operator. Then for all \(v \in \mathcal{V}_{\mu}^2\),

\[
||\nabla v||^2_{L^2} + \alpha^2||Av||^2_{L^2} \geq \lambda_1 \left\{ ||v||^2_{L^2} + \alpha^2||\nabla v||^2_{L^2} \right\}
\]

Proof: We just need to show that \(||Av||^2_{L^2} \geq \lambda_1||\nabla v||^2_{L^2}\) (since \(||\nabla v||^2_{L^2} \geq \lambda_1||v||^2_{L^2}\) by definition). Let \(\{h_i\}_{i \in \mathbb{N}}\) be eigenfunctions associated to the Stokes operator \(A\) with corresponding eigenvalues \(\lambda_i\). For all \(v \in \mathcal{V}_{\mu}^2\), we write \(v = \sum_{i=1}^{\infty} c_i h_i\). Then \(Av = \sum_{i=1}^{\infty} c_i \lambda_i h_i\). The orthogonality property of \(h_i\) implies \((Av, v) = \sum_{i=1}^{\infty} c_i^2 \lambda_i = ||\nabla v||^2_{L^2}\) and consequently,

\[
||Av||^2_{L^2} = \sum_{i=1}^{\infty} c_i^2 \lambda_i^2 \geq \left( \sum_{i=1}^{\infty} c_i^2 \lambda_i \right) \lambda_1 \geq \lambda_1 (Av, v) = \lambda_1 ||\nabla v||^2_{L^2}
\]

which completes the proof.

Weak solutions to the isotropic LANS-\(\alpha\) equations (2.1.1) are defined below.

Definition 2.2.2 Let \(f \in L^2(\Omega)\) and \(u_0 \in \mathcal{V}_{\mu}^1\). For any \(T > 0\), a function

\[u \in C([0, T]; \mathcal{V}_{\mu}^1) \cap L^2([0, T]; D(A))\]

with \(\frac{du}{dt} \in L^2([0, T]; L^2) \cap L^\infty((0, T); \mathcal{V}_{\mu}^1)\) and \(u(0) = u_0\) is said to be a weak solution to the LANS-\(\alpha\) equations with initial data \(u_0\) in the interval \([0, T]\) provided
\[
\left\langle \frac{d}{dt}(1-\alpha^2\Delta)u, w \right\rangle_{D(A)} + \nu \left\langle (1-\alpha^2\Delta)Au, w \right\rangle_{D(A)} + \left\langle B(u, (1-\alpha^2\Delta)u), w \right\rangle_{D(A)} = (f, w) \tag{2.2.15}
\]

for every \(w = w(x) \in D(A)\), the domain of the Stokes operator, and for almost every \(t \in [0, T]\) with

\[
B(u, v) = \nabla u \cdot \nabla v + \nabla u^T \cdot v;
\]

moreover, \(u(0) = u_0 \in \mathcal{V}^1_\mu\). Here, the equation (2.2.15) is understood in the following sense: for every \(t_0, t \in [0, T]\),

\[
\left\langle u(t), (1-\alpha^2\Delta)w \right\rangle_{D(A)} + \int_{t_0}^{t} \left\langle B(u(s), (1-\alpha^2\Delta)u(s)), w \right\rangle_{D(A)} \, ds = -\int_{t_0}^{t} \nu \left\langle Au(s), (1-\alpha^2\Delta)w \right\rangle_{D(A)} \, ds + \int_{t_0}^{t} (f, w) \, ds.
\]

For our proof, we shall make use of the following results from Marsden and Shkoller [21]

**Lemma 2.2.3 (Theorem 5.2 of [21])** For \(u_0 \in \mathcal{V}^s_\mu\), \(s \in [3, 5]\), and \(f \in L^2(\Omega)\), there exists a unique solution \(u\) to equation (2.2.12) in \(C([0, \infty), \mathcal{V}^s_\mu)\).

and

**Lemma 2.2.4 (Lemma 5.1 of [21])** For \(s \geq 3\), \(U^\alpha : \mathcal{V}^s \to \mathcal{V}^s\) and \(U^\alpha : \mathcal{V}^2 \to H^{1+\sigma}\) for \(\sigma \in (0, \frac{1}{2})\).

We can now state our main result.

**Theorem 2.2.1** For \(f \in L^2(\Omega)\) and \(u_0 \in \mathcal{V}^1_\mu\), there exists a unique weak solution

\[
u \in C([0, \infty), \mathcal{V}^1_\mu) \cap L^\infty((s, \infty), \mathcal{V}^3_\mu), \quad \forall s > 0
\]

to equation (2.2.12). The solution depends continuously on the initial data \(u_0\).
Proof: Consider a sequence of initial velocity \( \{u_\epsilon^0\}_{\epsilon=1}^{\infty} \in \hat{V}_\mu^4 \) and force \( \{f^\epsilon\}_{\epsilon=1}^{\infty} \in H^1 \) such that \( u_\epsilon^0 \to u_0 \) (by Lemma 2.2.1) and \( f^\epsilon \to f \). By Lemma 2.2.3 for each \( \epsilon \in \mathbb{N} \) there exists a unique solution \( u^\epsilon \) of the LANS-\( \alpha \) equations in \( C([0, \infty), \hat{V}_\mu^4) \). This solution has sufficient regularity to satisfy the weak formulation:

\[
\langle u^\epsilon(t), (1 - \alpha^2 \Delta)w \rangle_{D(A)} + \int_{t_0}^{t} \langle B(u^\epsilon(s), (1 - \alpha^2 \Delta)u^\epsilon(s)), w \rangle_{D(A)} \, ds \\
= - \int_{t_0}^{t} \nu \langle Au^\epsilon(s), (1 - \alpha^2 \Delta)w \rangle_{D(A)} \, ds + \int_{t_0}^{t} (f^\epsilon, w) \, ds,
\]

for all \( w \in D(A) \). In order to show the limit as \( \epsilon \to \infty \) is also a weak solution, we need to develop the appropriate energy estimates.

**An \( H^1 \) Estimate.** Define \( a^\epsilon = \nabla u^\epsilon + \mathcal{U}^\alpha(u^\epsilon) - (1 - \alpha^2 \Delta)^{-1} f^\epsilon \). Since \( u^\epsilon \in H^1_0 \) we may use it as a test function. Taking the \( \langle \cdot, \cdot \rangle_{1,\alpha} \) inner product, defined by (2.2.5), of LANS-3 together with \( u^\epsilon \),

\[
\left\langle \frac{d}{dt} u^\epsilon, u^\epsilon \right\rangle_{1,\alpha} + \nu \langle Au^\epsilon, u^\epsilon \rangle_{1,\alpha} + \langle \mathcal{P}^\alpha(a^\epsilon), u^\epsilon \rangle_{1,\alpha} = 0.
\]

By the definition of \( \langle \cdot, \cdot \rangle_{1,\alpha} \),

\[
\left\langle \frac{d}{dt} u^\epsilon, u^\epsilon \right\rangle_{1,\alpha} + \nu \langle Au^\epsilon, u^\epsilon \rangle_{1,\alpha} = \frac{1}{2} \frac{d}{dt} \left[ \|u^\epsilon\|^2_{L^2} + \alpha^2 \|\nabla u^\epsilon\|^2_{L^2} \right] + \nu \left[ \|\nabla u^\epsilon\|^2_{L^2} + \alpha^2 \|Au^\epsilon\|^2_{L^2} \right].
\]

Notice that the previous integration by parts used the additional boundary condition 
\( Au^\epsilon = 0 \). Using the properties of \( \mathcal{P}^\alpha \), we find

\[
\langle \mathcal{P}^\alpha(a^\epsilon), u^\epsilon \rangle_{1,\alpha} = -(f^\epsilon, u^\epsilon).
\]

Therefore,

\[
\frac{1}{2} \frac{d}{dt} \left[ \|u^\epsilon\|^2_{L^2} + \alpha^2 \|\nabla u^\epsilon\|^2_{L^2} \right] + \nu \left[ \|\nabla u^\epsilon\|^2_{L^2} + \alpha^2 \|Au^\epsilon\|^2_{L^2} \right] = (f^\epsilon, u^\epsilon). \tag{2.2.16}
\]

By Poincaré’s inequality and Young’s inequality (2.2.10),

\[
\frac{d}{dt} \left[ \|u^\epsilon\|^2_{L^2} + \alpha^2 \|\nabla u^\epsilon\|^2_{L^2} \right] + \nu \left[ \|\nabla u^\epsilon\|^2_{L^2} + \alpha^2 \|Au^\epsilon\|^2_{L^2} \right] \leq K_1 \tag{2.2.17}
\]
where $K_1 = \frac{||f'\epsilon||_{L^2}^2}{\nu \lambda_1}$. By Lemma 2.2.2,
\[
\frac{d}{dt} \left[ ||u'\epsilon||_{L^2}^2 + \alpha^2 ||\nabla u'\epsilon||_{L^2}^2 \right] + \nu \lambda_1 \left[ ||u'\epsilon||_{L^2}^2 + \alpha^2 ||\nabla u'\epsilon||_{L^2}^2 \right] \leq K_1.
\]

Using Gronwall’s inequality (2.2.11), we get the $H^1$ estimate
\[
||u'\epsilon(t)||_{L^2}^2 + \alpha^2 ||\nabla u'\epsilon(t)||_{L^2}^2 \leq e^{-\nu \lambda_1 t} \left( ||u'\epsilon(0)||_{L^2}^2 + \alpha_2 ||\nabla u'\epsilon(0)||_{L^2}^2 \right) + \frac{K_1}{\nu \lambda_1} (1 - e^{-\nu \lambda_1 t})
\]
\[
\leq k_1 := ||u_0'\epsilon||_{L^2}^2 + \alpha^2 ||\nabla u'\epsilon(0)||_{L^2}^2 + \frac{K_1}{\nu \lambda_1} \quad (2.2.18)
\]
where $\lambda_1$ is defined in Lemma 2.2.2. Therefore we have proved that $u'\epsilon \in L^\infty([0, \infty), \mathcal{V}_\mu^1)$ independently of $\epsilon$.

**An $H^2$ Estimate.** Since $Au'\epsilon$ is divergence-free, we take the $L^2$ inner product of LANS-2 together with $Au'\epsilon$ to get
\[
\frac{1}{2} \frac{d}{dt} \left[ ||\nabla u'\epsilon||_{L^2}^2 + \alpha^2 ||Au'\epsilon||_{L^2}^2 \right] + \nu \left[ ||Au'\epsilon||_{L^2}^2 + \alpha^2 ||\nabla Au'\epsilon||_{L^2}^2 \right] + (B(u'\epsilon, (1 - \alpha^2 \Delta)u'\epsilon), Au'\epsilon) = (f'\epsilon, Au'\epsilon). \quad (2.2.19)
\]

By estimate (2.2.18), we use the Sobolev embedding result $W^{1,\infty} \subset L^4$ to bound the nonlinear term of equation (2.2.19)
\[
||\nabla u'\epsilon, Au'\epsilon|| \leq ||\nabla u'\epsilon||_{L^2} ||u'\epsilon||_{L^4} ||Au'\epsilon||_{L^4} \leq C k_1 ||u'\epsilon||_{H^1} ||Au'\epsilon||_{L^4} \leq C k_1^2 ||\nabla Au'\epsilon||_{L^2} \leq \frac{C k_1^2}{4} ||\nabla Au'\epsilon||_{L^2}^3 + \nu ||Au'\epsilon||_{L^2}^2 + C (||u'\epsilon||_{H^1}), \quad (2.2.20)
\]
may be estimated by the Gagliardo-Nirenberg inequality (2.2.6) and a repeated use of Young’s inequality. The boundary conditions, the incompressibility constraint, and one integration by parts lead to
\[
||\nabla u'\epsilon \nabla u'\epsilon, Au'\epsilon|| = 0. \quad (2.2.21)
\]
For the third term, using (2.2.6), (2.2.7), and (2.2.18) we have the estimate,

\[
\left| \left( (\nabla u^T) \cdot \Delta u^\epsilon, Au^\epsilon \right) \right| \leq \| \nabla \Delta u^\epsilon \|_{L^2} \| u^\epsilon \|_{L^4} \| Au^\epsilon \|_{L^4} \leq C \| u^\epsilon \|_{H^1} \| \nabla Au^\epsilon \|_{L^2}^{7/4} \| Au^\epsilon \|_{L^2}^{1/4}
\]

\[
\leq C k_1 \| \nabla Au^\epsilon \|_{L^2}^{23/12} \| u^\epsilon \|_{L^2}^{1/12} \leq C k_1^{13/12} \| \nabla Au^\epsilon \|_{L^2}^{23/12}
\]

\[
\leq \frac{\nu}{4} \| \nabla Au^\epsilon \|_{L^2}^2 + C(\| u^\epsilon \|_{H^1}). \tag{2.2.22}
\]

Again Young’s inequality implies

\[
\| (f^\epsilon, Au^\epsilon) \| \leq \| f^\epsilon \|_{L^2} \| Au^\epsilon \|_{L^2} \leq \frac{1}{\nu} \| f^\epsilon \|_{L^2}^2 + \frac{\nu}{4} \| Au^\epsilon \|_{L^2}^2.
\]

Therefore by (2.2.20) - (2.2.22), the equation (2.2.19) becomes

\[
\frac{1}{2} \frac{d}{dt} \left[ \| \nabla u^\epsilon \|_{L^2}^2 + \alpha^2 \| Au^\epsilon \|_{L^2}^2 \right] + \nu \left[ \| Au^\epsilon \|_{L^2}^2 + \alpha^2 \| \nabla Au^\epsilon \|_{L^2}^2 \right]
\]

\[
\leq \| (B(u^\epsilon, (1 - \alpha^2 \Delta) u^\epsilon), Au^\epsilon) \| + \| (f^\epsilon, Au^\epsilon) \|
\]

\[
\leq \frac{\nu \alpha^2}{2} \| \nabla Au^\epsilon \|_{L^2}^2 + \frac{\nu}{2} \| Au^\epsilon \|_{L^2}^2 + \frac{\nu}{4} \| f^\epsilon \|_{L^2}^2 + C.
\]

Hence by Lemma 2.2.2 we have

\[
\frac{d}{dt} \left[ \| \nabla u^\epsilon \|_{L^2}^2 + \alpha^2 \| Au^\epsilon \|_{L^2}^2 \right] + \nu \lambda_1 \left[ \| \nabla u^\epsilon \|_{L^2}^2 + \alpha^2 \| Au^\epsilon \|_{L^2}^2 \right] \leq K_2, \tag{2.2.23}
\]

\[
\frac{d}{dt} \left[ \| \nabla u(t) \|_{L^2}^2 + \alpha^2 \| Au(t) \|_{L^2}^2 \right] e^{\nu \lambda_1 t} \leq K_2 e^{\nu \lambda_1 t}.
\]

where \( K_2 = \frac{2}{\nu} \| f^\epsilon \|_{L^2}^2 + C \). Since \( \| Au \|_{L^2}^2 \) is not necessarily bounded at \( t = 0 \), for every \( s_1 > 0 \) we integrate both sides from \([s_1, t],[s_1, t] \),

\[
\| \nabla u(t) \|_{L^2}^2 + \alpha^2 \| Au(t) \|_{L^2}^2 \leq \| \nabla u(s_1) \|_{L^2}^2 + \alpha^2 \| Au(s_1) \|_{L^2}^2 \right) e^{\nu \lambda_1 (s_1 - t)} + \nu \lambda_1 K_2. \tag{2.2.24}
\]

Thus, independently of \( \epsilon \),

\[
u^\epsilon \in \mathcal{L}^\infty((s_1, \infty), \mathcal{V}_s^2(\Omega)),
\]

for all \( s_1 > 0 \)

**An \( H^3 \) Estimate.** Since \( A^2 u^\epsilon \) is not necessarily equal to zero on \( \partial \Omega \), we do not use it to derive an \( H^3 \)-estimate. To achieve this estimate we make use of the Ladyzhenskaya
Using (2.2.7) and Young’s inequality, the second term becomes,

\[ \partial_t (1 - \alpha^2 \Delta) u_\epsilon^t + \nabla u_\epsilon^t (1 - \alpha^2 \Delta) u_\epsilon^t + \nabla u_\epsilon^t (1 - \alpha^2 \Delta) u_\epsilon^t - \alpha^2 (\nabla u_\epsilon^t)^T \cdot \Delta u_\epsilon^t - \alpha^2 (\nabla u_\epsilon^t)^T \cdot \Delta u_\epsilon^t = -\text{grad} p_t - \nu (1 - \alpha^2 \Delta) Au_\epsilon^t. \]

Noting that \( u_\epsilon^t \in D(A) \), we take the \( L^2 \) inner product with \( u_t \) to get

\[
\frac{1}{2} \frac{d}{dt} \left[ ||u_\epsilon^t||_{L^2}^2 + \alpha^2 ||\nabla u_\epsilon^t||_{L^2}^2 \right] + \nu \left[ ||\nabla u_\epsilon^t||_{L^2}^2 + \alpha^2 ||Au_\epsilon^t||_{L^2}^2 \right] \leq |(u_\epsilon^t, \nabla u_\epsilon^t)| + \alpha^2 |(\Delta u_\epsilon^t, \nabla u_\epsilon^t + \nabla u_\epsilon^t)|. \tag{2.2.25}
\]

Before estimating the right hand side, we would like a bound for \( ||u_\epsilon^t||_{L^2}^2 \) which will be used in the later computations. For each \( \epsilon \), \( u_\epsilon \) satisfies the equation LANS-3 and consequently,

\[
||u_\epsilon^t||_{L^2}^2 \leq \nu |(Au_\epsilon^t, u_\epsilon^t)| + |(\nabla u_\epsilon^t, u_\epsilon^t)| + |(U^\alpha(u_\epsilon^t), u_\epsilon^t)| + |((1 - \alpha^2 \Delta)^{-1} f_\epsilon, u_\epsilon^t)|. \tag{2.2.26}
\]

The first term is simply \( \nu |(Au_\epsilon^t, u_\epsilon^t)| \leq \frac{1}{8} ||u_\epsilon^t||_{L^2}^2 + 2\nu^2 ||Au_\epsilon^t||_{L^2}^2 \) by Young’s inequality. Using (2.2.7) and Young’s inequality, the second term becomes,

\[
|(|(\nabla u_\epsilon^t, u_\epsilon^t)| \leq 16 ||\nabla u_\epsilon^t||_{L^2} ||u_\epsilon^t||^{1/4}_{L^2} ||u_\epsilon^t||^{3/4}_{L^2} ||Au_\epsilon^t||^{3/4}_{L^2}
\leq \nu^2 ||Au_\epsilon^t||_{L^2}^2 + C \|| \nabla u_\epsilon^t ||_{L^2}^{5/2} ||u_\epsilon^t ||_{L^2}^{2/5} ||u_\epsilon^t ||_{L^2}^{8/5}
\leq \nu^2 ||Au_\epsilon^t||_{L^2}^2 + \frac{1}{8} ||u_\epsilon^t||_{L^2}^2 + C k_1^{10},
\]

where \( k_1 \) is the time independent \( H^1 \) bound (2.2.18). Let \( v_\epsilon = (1 - \alpha^2 \Delta)^{-1} u_\epsilon^t \). Then, by integration by parts, the third term of (2.2.26) is

\[
|(U^\alpha(u_\epsilon^t), u_\epsilon^t)| \leq \alpha^2 |(\Delta u_\epsilon^t \cdot \nabla u_\epsilon^t, v_\epsilon)| \leq \alpha^2 ||u_\epsilon^t||_{H^2}^2 ||v_\epsilon||_{H^1}
\leq \frac{\alpha^2}{4} ||v_\epsilon||_{H^1}^2 + \alpha^2 ||v_\epsilon||_{H^2}^4.
\]
By definition, \( v^\epsilon - \alpha^2 \Delta v^\epsilon = u_t^\epsilon \). Taking the \( L^2 \) inner product of both sides with \( v^\epsilon \) we have an \( H^1 \) estimate
\[
\|v^\epsilon\|^2_{L^2} + \alpha^2 \|\nabla v^\epsilon\|^2_{L^2} = (u_t^\epsilon, v^\epsilon) \leq \frac{1}{2} \|u_t^\epsilon\|^2_{L^2} + \frac{1}{2} \|v^\epsilon\|^2_{L^2}.
\]
Hence \( \|v^\epsilon\|^2_{H^1} \leq \frac{1}{2\alpha^2} \|u_t^\epsilon\|^2_{L^2} \) and therefore
\[
|(U^\alpha(u^\epsilon), u_t^\epsilon)| \leq \frac{1}{8} \|u_t^\epsilon\|^2_{L^2} + \alpha^2 \|u^\epsilon\|^4_{H^2}.
\]
The last term of (2.2.26) can be estimated in a similar way
\[
|((1 - \alpha^2 \Delta)^{-1} f^\epsilon, u_t^\epsilon)| = |(f^\epsilon, v^\epsilon)| \leq \frac{1}{8} \|v^\epsilon\|^2_{L^2} + 2 \|f^\epsilon\|^2_{L^2} \leq \frac{1}{8} \|u_t^\epsilon\|^2_{L^2} + 2 \|f^\epsilon\|^2_{L^2}.
\]
Combining the above estimates, we obtain the bound
\[
\|u_t^\epsilon\|^2_{L^2} \leq 6\nu^2 \|Au^\epsilon\|^2_{L^2} + 2\alpha^2 \|u^\epsilon\|^4_{H^2} + 4 \|f^\epsilon\|^2_{L^2} + Ck_1^{10}, \quad (2.2.27)
\]
From the \( H^2 \) estimate (2.2.24) we conclude that \( u_t^\epsilon \in L^\infty((s_1, \infty), L^2) \) for all \( s_1 > 0 \).
Using this result, we now estimate each of the terms on the right hand side of (2.2.25).
Using (2.2.6)-(2.2.8) and Young’s inequality, we get
\[
|(u_t^\epsilon, \nabla u_t^\epsilon)| + \alpha^2 |(\Delta u_t^\epsilon, \nabla u_t^\epsilon + \nabla u_t^\epsilon)|
\leq C \left[ \| \nabla u_t^\epsilon \|_{L^2}^{\frac{3}{4}} \| u_t^\epsilon \|_{L^2}^{\frac{5}{4}} \| u^\epsilon \|_{H^2} + \alpha^2 \|Au_t^\epsilon\|_{L^2}^{15/8} \| u_t^\epsilon \|_{L^2}^{1/8} \| u^\epsilon \|_{H^1} \right]
+ \alpha^2 |Au_t^\epsilon|_{L^2} \| \nabla u_t^\epsilon \|_{L^2}^{\frac{3}{4}} \| u^\epsilon \|_{H^1}^{1/4} \| u^\epsilon \|_{H^1}
\leq \frac{\nu}{2} \| \nabla u_t^\epsilon \|^2_{L^2} + \frac{\nu \alpha^2}{2} \|Au_t^\epsilon\|^2_{L^2} + C_{s_1},
\]
where \( C_{s_1} \) is a constant multiple of the \( H^2 \) bound depending only on the lower time bound \( s_1 > 0 \). Therefore (2.2.25) becomes
\[
\frac{d}{dt} \left[ \|u_t^\epsilon\|^2_{L^2} + \alpha^2 \|\nabla u_t^\epsilon\|^2_{L^2} \right] + \nu \left[ \| \nabla u_t^\epsilon \|^2_{L^2} + \alpha^2 \|Au_t^\epsilon\|^2_{L^2} \right] \leq C_{s_1}.
\]
By Lemma 2.2.2,
\[
\frac{d}{dt} \left[ \|u_t^\epsilon\|^2_{L^2} + \alpha^2 \|\nabla u_t^\epsilon\|^2_{L^2} \right] + \nu \lambda_1 \left[ \|u_t^\epsilon\|^2_{L^2} + \alpha^2 \|\nabla u_t^\epsilon\|^2_{L^2} \right] \leq C_{s_1}.
\]
By Lemma 2.2.2,
\[
\frac{d}{dt} \left[ \|u_t^\epsilon\|^2_{L^2} + \alpha^2 \|\nabla u_t^\epsilon\|^2_{L^2} \right] + \nu \lambda_1 \left[ \|u_t^\epsilon\|^2_{L^2} + \alpha^2 \|\nabla u_t^\epsilon\|^2_{L^2} \right] e^{\nu \lambda_1 t} \leq C_{s_1} e^{\nu \lambda_1 t}.
For every \( s_2 > 0 \), we integrate from \([s_2, t]\) and obtain the \( H^1 \) bound of \( u^\epsilon_t \),

\[
||u^\epsilon_t||_{L^2}^2 + \alpha^2 ||\nabla u^\epsilon_t||_{L^2}^2 \leq (||u^\epsilon_t(s_2)||_{L^2}^2 + \alpha^2 ||\nabla u^\epsilon_t(s_2)||_{L^2}^2) e^{\nu \lambda_1 (s_2 - t)} + \nu \lambda_1 C_{s_1}
\]

Hence, independently of \( \epsilon \),

\[
 u^\epsilon_t \in L^\infty((s_2, \infty), V^1_\mu),
\]

for all \( s_2 > 0 \).

Let \( s = \min\{s_1, s_2\} \). By standard Sobolev inequalities, we also have \( \nabla u^\epsilon \) belongs \( L^\infty((s, \infty), H^1) \). Therefore,

\[
||P^\alpha (\nabla u^\epsilon)||_{V^1_\mu} \leq ||\nabla u^\epsilon||_{H^1},
\]

implies that for all \( s > 0 \), \( P^\alpha (\nabla u^\epsilon) \) belongs to \( L^\infty((s, \infty), V^1_\mu) \), independently of \( \epsilon \). From Lemma 2.2.1 and our \( H^2 \) estimate (2.2.21), we infer that \( U^\alpha(u^\epsilon) \) is contained in \( L^\infty((s, \infty), H^{1+\sigma}) \) for \( 0 < \sigma < 1/3 \) and consequently \( P^\alpha(U^\alpha(u^\epsilon)) \) belongs \( L^\infty((s, \infty), V^{1+\sigma}_\mu) \), independently of \( \epsilon \).

Thus using the equation LANS-3 and the results for \( u^\epsilon_t \), we have that for any \( s > 0 \),

\[
\nu A u^\epsilon = -P^\alpha [\nabla u^\epsilon u^\epsilon + U^\alpha(u^\epsilon) - (1 - \alpha^2 \Delta)^{-1} f^\epsilon] + u^\epsilon_t
\]

belongs to \( L^\infty((s, \infty), V^1_\mu) \), independently of \( \epsilon \). By the elliptic regularity of the Stokes operator \( A \), we conclude that

\[
u Au^\epsilon \in L^\infty((s, \infty), V^3_\mu).
\]

Define \( H = \{u \in L^2 \mid \text{div } u = 0\} \). For all \( T > 0 \), (2.2.27) implies \( u^\epsilon_t \in L^2([0, T], H) \) (independently of \( \epsilon \)) and therefore the classical compactness theorem (see for instance [13, 19]) enables us to conclude that there is a subsequence \( u^\epsilon' \) such that for all \( T > 0 \),

\[
 u^\epsilon' \to u \quad \text{weakly in} \quad L^2([0, T], D(A)),
\]

\[
 u^\epsilon' \to u \quad \text{strongly in} \quad L^2([0, T], V^1_\mu),
\]

\[
 u^\epsilon' \to u \quad \text{in} \quad C([0, T], H).
\]
It is straightforward to verify that $u$ satisfies the weak formulation associated with $f$. Indeed, from an identical argument provided in [14], we may conclude that $u \in C([0, \infty), \mathcal{V}_\mu^1)$. Furthermore, we infer that $u(0) = u_0$. Hence we are lead to the existence of weak solutions for the LANS-\(\alpha\) equation. The uniqueness and continuous dependence on initial data of weak solutions can be proved in the same classical way as was done in [14] on Page 14 for the periodic case.

**Remark 2.2.1** A standard contraction mapping argument can be used to show the existence of LANS-\(\alpha\) solutions $u$ in $C([0,T], \mathcal{V}_\mu^s)$ for $s > 1$. In the absence of forcing or when $f$ is $C^\infty$, the solution $u$ of the LANS-\(\alpha\) equations is instantly regularized so that $u \in C^\infty((0, \infty) \times \Omega)$.

### 2.3 Estimating the dimension of the global attractor.

Theorem 2.2.1 is sufficient to define the semi-group $S(t)$ by

$$S(t) : u_0 \in \mathcal{V}_\mu^1 \mapsto u(t) \in \mathcal{V}_\mu^1.$$ 

We now show the uniform compactness property of the operator $S$ and the existence of an $H^2$ absorbing set. By Theorem I.1.1 in [29], this implies the existence of a maximal compact global $H^1$ attractor.

#### 2.3.1 Absorbing sets and attractor.

In this section we prove the existence of an absorbing set in $\mathcal{V}_\mu^2$. Usually proving the existence of absorbing sets amounts to proving a priori estimates; for the LANS-\(\alpha\) equations, these estimates are established in Section 2.3. Thus estimate (2.2.24) implies

$$\limsup_{t \to +\infty} c\alpha^2 \|u\|^2_{H^2} \leq \rho_2^2 := \nu\lambda_1 K_2 = 2\lambda_1 \|f\|_{L^2}^2 + C(\|u\|_{H^1}).$$
CHAPTER 2. Weak solutions of the LANS-\(\alpha\) equations on bounded domains

We conclude that the ball \(B_{V^2}(0, \rho_2)\) of \(V^2\), denoted \(B_2\), is an absorbing set in \(V^2\) for the semi-group \(S(t)\). We choose \(\rho_2' > \rho_2\) and denote \(B_0\) the ball \(B_{V^2}(0, \rho_2')\). If \(B\) is any bounded set of \(V^1\), then from (2.2.24), \(S(t)B \subseteq B_0\) for \(t \geq t_0(B, \rho_2')\), where

\[
t_0 = s_1 + \frac{1}{\nu \lambda_1} \log \frac{|\nabla u(s_1)|^2_{L^2} + \alpha^2 |Au(s_1)|^2_{L^2}}{\rho_2'^2 - \rho_2^2}.
\]

At the same time, this result proves the uniform compactness of \(S(t)\); any bounded set \(B\) of \(V^1\) is included in such a ball, and for \(u_0 \in V^1\) and \(t \geq t_0\), \(u(t)\) belongs to \(B_2\) which is bounded in \(H^2\) and relatively compact in \(H^1\). Consequently, by Theorem I.1.1 in [29], the LANS-\(\alpha\) equations have a nonempty compact, convex, and connected global attractor in \(H^1\),

\[A^\alpha = \cap_{s > 0} (\cup_{t \geq s} S(t)B_2).\]

2.3.2 Dimension of the attractor.

The dimension of the three-dimensional attractor for the LANS-\(\alpha\) equations follows the arguments of Theorem 6 in [14] (stated below), and leads to the same bound as in the periodic case. It is important to note that the bound tends to infinity as \(\alpha \to 0\).

**Theorem 2.3.1 (Theorem 6 of [14])** The Hausdorff and fractal dimensions of the global attractor of the LANS-\(\alpha\) equations, \(d_H(A^\alpha)\) and \(d_F(A^\alpha)\), respectively, satisfy:

\[
d_H(A^\alpha) \leq d_F(A^\alpha) \leq c' \max \left\{ G^{4/3} \left( \frac{1}{\alpha^4 \lambda_1^2} \right)^{2/3}, G^{3/2} \left( \frac{1}{\alpha^6 \lambda_1^3} \right)^{3/8} \right\},
\]

where \(G = \frac{|f|_{L^2}}{\nu^2 \lambda_1^2}\) is the Grashoff number, and \(c' > 0\) is a constant depending only of the shape of \(\Omega\).

In two dimensions, [10] proved that the dimension of the attractor can be bounded independently of \(\alpha\) by the Grashoff number. Hence, in two-dimensions, we find a similar bound as for the Navier-Stokes equations.
**Theorem 2.3.2 (Theorem 3 of [10])** The Hausdorff and fractal dimensions of the global attractor of the LANS-\(\alpha\) equations, \(d_H(A)\) and \(d_F(A)\), respectively, satisfy:

\[
d_H(A) \leq d_F(A) \leq C' G,
\]

where \(G = \frac{\|f\|_{L^2}^2}{\nu^2 \lambda_1}\) is the Grashoff number, and \(C' > 0\) is a constant depending only on the shape of \(\Omega\).
Chapter 3

Local well-posedness of the anisotropic LANS-$\alpha$ equations on $\mathbb{T}^3$

3.1 The anisotropic LANS-$\alpha$ equations.

The anisotropic Lagrangian averaged Navier-Stokes (LANS-$\alpha$) equations are given on $\Omega \times (0, T)$ by

\begin{align*}
(1 - \alpha^2 C) \left( \partial_t u + \text{div } u \otimes u - \nu \mathbb{P} Cu \right) &= -\text{grad } p, \quad (3.1.1a) \\
\text{div } u(t, x) &= 0, \quad (3.1.1b) \\
\partial_t F + \nabla F \cdot u &= \nabla u \cdot F + [\nabla u \cdot F]^T, \quad (3.1.1c) \\
u(0, x) &= u_0(x), \quad F(0, x) = F_0(x) \geq 0, \quad (3.1.1d)
\end{align*}

where $u(t, x)$ denotes the divergence-free mean velocity vector, $p(t, x)$ represents the scalar pressure field, $\nu$ is the kinematic viscosity, $F(t, x)$ denotes the covariance tensor (a $3 \times 3$ matrix), and the notation $F_0 \geq 0$ means the initial covariance matrix is assumed positive semi-definite. In [22], the authors define the covariance tensor $F$ as
the ensemble average of the tensor product of the Lagrangian fluctuation vector $\xi'$ given explicitly as

$$F(t, x) = \langle \xi'(t, x) \otimes \xi'(t, x) \rangle.$$ 

Furthermore, $Cu := \text{div}[\nabla u \cdot F]$ and the operator $\mathbb{P}$ is a projection onto divergence-free vector fields. When $F$ is assumed to be a constant multiple of the identity matrix, the anisotropic equations reduce to the isotropic equations of Chapter 2 for only the averaged fluid velocity. A brief history of the isotropic equations can be found in Chapter 2.

The operator $\mathbb{P}$ can be chosen to be either the Leray projector $P : L^2(\Omega) \to \{v \in L^2(\Omega) \mid \text{div} v = 0, v \cdot n = 0 \text{ on } \partial \Omega\}$, or the generalized Stokes projector $P^\alpha_F := [P(1 - \alpha^2 C)]^{-1}P(1 - \alpha^2 C)$ defined in detail by the following definition.

**Definition 3.1.1** For any $v$ in the domain of $(1 - \alpha^2 C)$ and $F > 0$, we set the generalized Stokes projector $P^\alpha_F(v) = w$, where $w$ is the solution of the generalized Stokes problem

$$(1 - \alpha^2 C)w + \text{grad} p = (1 - \alpha^2 C)v$$

$$\text{div} w = 0 \quad w = 0 \text{ on } \partial \Omega;$$

thus

$$P^\alpha_F(v) = v - (1 - \alpha^2 C)^{-1} \text{grad} p.$$ 

**Remark 3.1.1** The Stokes projector defined in Definition 2.2.1 is a special case of Definition 3.1.1 with the condition $F = \text{Id}$.

The generalized Stokes projector $P^\alpha_F$ maps onto the space of $H^1$ divergence-free vector fields that vanish on $\partial \Omega$. Since the Lagrangian fluctuations are necessarily zero along $\partial \Omega$, we know $F(t, x) = 0$ on $\partial \Omega$ for all time $t \geq 0$. Unlike the Leray projector which only assigns the normal component of $u$ to vanish on the boundary, the gener-
alized Stokes projector provides the correct boundary condition for $F$. Therefore on bounded domains, it is important to use this projector.

Modeling incompressible flow on bounded domains is the purpose of the anisotropic model. The first step towards this goal is proving well-posedness of the equations on unbounded domains. In this chapter, we shall restrict our attention to the three-dimensional periodic box $\Omega = \mathbb{T}^3$. We define for $s \geq 0$,

$$H^s_{\text{per}} = \{ u \in [H^s(\mathbb{T}^3)]^3 | \int_{\mathbb{T}^3} u(x) \, dx = 0 \}$$

and

$$H^s_{\text{div}} = \{ u \in H^s_{\text{per}} | \text{div } u = 0 \}.$$

Marsden and Shkoller [22] prove the local well-posedness of the anisotropic LANS-$\alpha$ equations with the projector $\mathbb{P} = P$ for initial velocity fields in the class $H^s_{\text{div}}$ and positive $F_0 \in [H^s_{\text{per}}(\mathbb{T}^3)]^{3 \times 3}$ for $s > 3.5$. They use the classical Galerkin projection to obtain smooth finite-dimensional approximations $u_m$ and $F_m$. After establishing a priori bounds independent of $m$, the authors appeal to compactness results to extract a subsequence of $u_m$ which converges in $C^0([0,T]; H^s_{\text{div}})$ uniformly in $m$. An important tool in obtaining the proper estimates is the control of the regularity of the covariance tensor $F$ by the one derivative higher regularity of the mean velocity $u$. This allows the authors to establish estimates only dependent on the initial covariance tensor.

The next step towards applying the anisotropic equations to bounded domains is the study of solutions to the equations with the generalized Stokes projector viscosity term. As discussed above, the generalized Stokes projector preserved the zero boundary condition for the covariance tensor. Although we known the correct projection for bounded domains, the appropriate additional boundary condition necessary to solve the forth order equation (3.1.1) is unknown. In this chapter, we show the well-posedness of the anisotropic LANS-$\alpha$ equations on $\mathbb{T}^3$ with the projector $\mathbb{P} = \mathbb{P}_F^\alpha$ by a method similar to the one used by Marsden and Shkoller [22]. However, in ob-
taining a priori $H^s$-type bounds, it is important that we use the differential operator $D^s P(1 - \alpha^2 C)$, $|\sigma| \leq s - 2$, rather than the standard $D^s$, where $D^s$ denotes all partials derivatives of order $s$. This particular approach is different than what is done in [22]. It allows us to estimate the viscosity term without commuting the operator $D^s$ with the inverse operator $[P(1 - \alpha^2 C)]^{-1}$ and leads us to bounds involving the Sobolev norms of $P(1 - \alpha^2 C)u$ and $u$. We then appeal to the elliptic regularity estimates established in Proposition 3.2.1 to write all bounds in terms of $P(1 - \alpha^2 C)u$. The same compactness argument leads us to the existence of solutions to the anisotropic LANS-$\alpha$ equations (3.1.1).

3.2 Local well-posedness with Stokes projector.

Before proving the existence and uniqueness of classical solutions to the anisotropic LANS-$\alpha$ equations stated above, we establish an elliptic regularity estimate for the operator $P(1 - \alpha^2 C)$ which holds assuming the least regularity of $F$. The proof relies on a standard commutator estimate which we include below for completeness. For the remainder of the chapter, $C$ represents an arbitrary positive constant unless, by the context, it is clear that $C$ is the operator defined above.

**Lemma 3.2.1** For $f, g \in H^s(\Omega)$,

$$||[D^s, f]g||_{L^2} \leq C (||f||_{H^s} ||g||_{L^\infty} + ||\nabla f||_{L^\infty} ||g||_{H^{s-1}}) \quad (3.2.2)$$

**Proposition 3.2.1** For $F > 0$ and $F \in \left[H^{k-1}_{\text{per}}(T^3)\right]^{3 \times 3}$, suppose $u \in H^k(T^3)$ solves $P(1 - \alpha^2 C)u = w$ for $k \geq 3$. If $w \in H^1(T^3)$, then

$$||u||^2_{H^3} \leq C(1 + C ||F||_{H^2}^4) ||w||^2_{H^1}, \quad (3.2.3)$$

and for $w \in H^{k-2}(T^3)$,

$$||u||^2_{H^k} \leq C ||w||^2_{H^{k-2}} + C ||F||^2_{H^{k-1}} ||u||^2_{H^{k-1}}, \quad k \geq 4. \quad (3.2.4)$$
Moreover, if $F$ solves (3.1.1) then for some $T > 0$,

$$\|u\|_{H^3}^2 \leq C(t, \|F_0\|_{H^2}) \|w\|_{H^1}^2 + C(\|F_0\|_{H^2}) \|w\|_{H^1}^4,$$  \hspace{1cm} (3.2.5)

$$\|u\|_{H^4}^2 \leq C(t, \|F_0\|_{H^2}) \|w\|_{H^2}^2 + C(\|F_0\|_{H^3}) \|w\|_{H^1}^4,$$  \hspace{1cm} (3.2.6)

$$\|u\|_{H^k}^2 \leq C \|w\|_{H^{k-2}}^2 + C(t) \|F_0\|_{H^{k-1}}^2 \|u\|_{H^{k-1}}^2 + C(\|F_0\|_{H^{k-1}}^4 \|w\|_{H^{k-1}}^4,$$  \hspace{1cm} (3.2.7)

for $k \geq 4$ and all $t < T$.

**Remark 3.2.1** In a classical elliptic regularity statement one assumes $u \in H^1$ solves the elliptic second order differential equation and attempts to prove $u$ has two derivatives more regularity than the inhomogeneous term using difference quotients. Since we are assuming that $\|F\|_{L^\infty}$ is not necessarily bounded, this regularity is not guaranteed and we add the assumption $u \in H^k$. Later in the chapter, the proposition is used when $u \in C^\infty([0,T] \times \mathbb{T}^3)$ and therefore the regularity of $u$ is not delicate.

**Remark 3.2.2** If $F \in [C^\infty(\mathbb{T}^3)]^{3 \times 3}$, then equations (3.2.3) and (3.2.4) reduce to the standard elliptic regularity result

$$\|u\|_{H^k}^2 \leq C(\|F\|_{W^{k-1, \infty}}) \|w\|_{H^{k-1}}^2.$$  \hspace{1cm} (3.2.8)

**Proof.** We first write $P(1 - \alpha^2C)u = w$ in the energy form

$$(u, v)_{L^2} + \alpha^2(F \cdot \nabla u, \nabla v)_{L^2} = (w, v)_{L^2}, \quad v \in H^k_{\text{per}}(\mathbb{T}^3) \quad k \geq 1.$$  \hspace{1cm} (3.2.9)

We begin by showing the special case of $k = 3$, namely inequality (3.2.3). When $v = u$, uniform ellipticity of F implies $(F \cdot \nabla u, \nabla u)_{L^2} \geq \lambda \|u\|_{H^1}^2$ from which we find

$$\|u\|_{H^1}^2 \leq C \|w\|_{H^1}^2.$$  \hspace{1cm} (3.2.10)

Now suppose $v = D^4u$ and input this into inequality (3.2.3). Integrating by parts gives

$$\alpha^2(D^2(F \cdot \nabla u), D^2 \nabla u)_{L^2} \leq (w, D^4u)_{L^2}.$$
Expanding the left hand side,

$$
\alpha^2 (D^2 (F \cdot \nabla u), D^2 \nabla u)_L^2 = \alpha^2 (F \cdot D^2 \nabla u, D^2 \nabla u)_L^2 + 2 \alpha^2 (\nabla F \cdot D \nabla u, D^2 \nabla u)_L^2
$$

$$
+ \alpha^2 (D^2 F \cdot \nabla u, D^2 \nabla u)_L^2 =: I + II + III,
$$

where $I \geq \alpha^2 \lambda ||u||_{H^3}^2$ by uniform ellipticity of $F$ and

$$
II \leq C ||\nabla F||_{L^4} ||D^2 u||_{L^4} ||D^3 u||_{L^2} \leq C ||F||_{H^2} ||D^2 u||_{L^4} ||u||_{H^3},
$$

$$
III \leq C ||D^2 F||_{L^2} ||\nabla u||_{L^\infty} ||D^3 u||_{L^2} \leq C ||F||_{H^2} ||u||_{W^{2,4}} ||u||_{H^3}
$$

by standard Sobolev inequalities. Combining the estimates above, we get

$$
||u||_{H^3}^2 \leq C (w, D^4 u) + C ||F||_{H^2} ||D^2 u||_{L^4} ||u||_{H^3}
$$

$$
\leq C ||w||_{H^1} ||D^4 u||_{H^{-1}} + C ||F||_{H^2} ||u||_{H^1}^{1/8} ||u||_{H^3}^{7/8}
$$

$$
\leq \varepsilon ||u||_{H^3}^2 + C ||w||_{H^1}^2 + C ||F||_{H^2} ||u||_{H^1}^2,
$$

where we have used the Gagliardo-Nirenberg inequality (2.2.6) for the second inequality and Young’s inequality (2.2.10) for the last. Using estimate (3.2.10) and choosing $\varepsilon > 0$ sufficiently small, we can absorb the $||u||_{H^3}^2$-term into the left side, obtaining

$$
||u||_{H^3}^2 \leq C (1 + C ||F||_{H^2}^1 ||w||_{H^1}^2)
$$

For $k \geq 4$, we substitute $v = (-1)^{k-1} D^{2k-2} u$ into (3.2.9). After integrating by parts,

$$
\alpha^2 (D^{k-1} (F \cdot \nabla u), D^{k-1} \nabla u)_L^2 \leq (D^{k-1} w, D^{k-1} u)_L^2.
$$

Expanding the left hand side we see

$$
\alpha^2 (D^{k-1} (F \cdot \nabla u), D^{k-1} \nabla u)_L^2 = \alpha^2 (F \cdot D^{k-1} \nabla u, D^{k-1} \nabla u)_L^2
$$

$$
+ \alpha^2 ([D^{k-1}, F] \nabla u, D^{k-1} \nabla u)_L^2 =: I + II,
$$
where \( I \geq \alpha^2 \lambda \| u \|_{H^k} \). Unlike the case when \( k = 3, \ k \geq 4 > \frac{7}{2} \) implies \( H^{k-1} \subset W^{1,\infty} \) and therefore by a standard commutator estimate (3.2.2),

\[
II \leq C(\| F \|_{H^{k-1}} \| \nabla u \|_{L^\infty} + \| \nabla F \|_{L^\infty} \| \nabla u \|_{H^{k-2}}) \| u \|_{H^k} \leq C \| F \|_{H^{k-1}} \| u \|_{H^{k-1}} \| u \|_{H^k}.
\]

Hence

\[
\| u \|_{H^k}^2 \leq C \| D^{k-1} w \|_{H^{k-1}} \| D^{k-1} u \|_{H^1} + C \| F \|_{H^{k-1}} \| u \|_{H^{k-1}} \| u \|_{H^k}
\]

\[ \leq \varepsilon \| u \|_{H^k}^2 + C \| w \|_{H^{k-2}}^2 + C \| F \|_{H^{k-1}}^2 \| u \|_{H^{k-1}}^2. \]

By choosing \( \varepsilon > 0 \) sufficiently small we achieve inequality (3.2.4).

To show inequality (3.2.5), recall from [22], equation (33), that if \( F \) solves (3.1.1c) for a given \( u \), we may estimate \( \| F \|_{H^s} \) by one higher order derivative of \( u \) in the following way,

\[
\| F \|_{H^s} \leq C \| F_0 \|_{H^s} \exp \int_0^t \| u(s) \|_{H^{s+1}} \, ds \leq C \| F_0 \|_{H^s} \left( 1 + t \| u \|_{H^{s+1}} + O(t^2) \right).
\]

(3.2.11)

For \( k = 3 \), it follows that inequality (3.2.3) can be written as

\[
\| u \|_{H^3}^2 \leq C \| w \|_{H^1}^2 + C \| F_0 \|_{H^2}^{16} \left( 1 + 16t \| u \|_{H^3} + O(t^2) \right) \| w \|_{H^1}^2,
\]

\[ \leq C \| w \|_{H^1}^2 + C \| F_0 \|_{H^2}^{16} \left( 1 + O(t^2) \right) \| w \|_{H^1}^2 + tC \| F_0 \|_{H^2}^{16} \| w \|_{H^1} \| u \|_{H^3}, \]

\[ \leq C \| w \|_{H^1}^2 + C(t) \| F_0 \|_{H^2}^{16} \| w \|_{H^1}^2 + C \| F_0 \|_{H^2}^{32} \| w \|_{H^1}^2 + \varepsilon t^2 \| u \|_{H^3}^2, \]

For \( t < T \), we choose \( \varepsilon > 0 \) sufficiently small to conclude inequality (3.2.5). The estimate for \( k \geq 4 \) follows from the same procedure.

**Theorem 3.2.1** For \( s > 7/2 \), and \( u_0 \in H^s_{div}(\mathbb{T}^3) \), \( F_0 \in [H^s_{per}(\mathbb{T}^3)]^{3x3} \), with \( F_0 > 0 \), there exists a unique solution \((u, F)\) with \( u \in C^0([0,T];H^s_{div}) \cap L^2(0,T;H^{s+1}_{div}) \) and \( F \in C^0([0,T];[H^s_{per}]^{3x3}) \) to the anisotropic LANS-\( \alpha \) equations, where \( T \) depends on the initial data.
Proof. The evolution equation for the mean velocity (3.1.1a) may be written as
\[ \partial_t u + P^\alpha_F \text{div}(u \otimes u) = \nu P^\alpha_F Cu \]
\[ u(0, x) = u_0(x). \]

Approximate solutions. Let \( v_i = v_i(\cdot, x) (i = 1, 2, \ldots) \) denote the smooth periodic orthogonal basis of \( H^1_{\text{per}} \) given by the eigenfunctions of the Stokes operator. Define
\[ P_m : H^1_{\text{per}} \to V^m := \text{span}\{v_1, \ldots, v_m\} \text{ and } u_m = P_m u. \]
Consider the Galerkin projection of (3.1.1) given by
\[ \partial_t u_m + P_m P^\alpha_F \text{div}(u_m \otimes u_m) = \nu P_m P^\alpha_F C_m u_m, \]
\[ u_m(0, x) = P_m u_0, \]
with \( C_m u_m := \text{div}(F_m \cdot \nabla u_m) \) where \( F_m \) is a solution to
\[ \partial_t F_m + \nabla F_m \cdot u_m = \nabla u_m \cdot F_m + [\nabla u_m \cdot F_m]^T, \]
\[ F_m(0, x) = F_0(t, x) > 0. \]

For each \( m \in \mathbb{N} \), there is a smooth solution; in order to pass to the limit as \( m \to \infty \) to produce a solution to (3.1.1), it suffices to obtain a priori estimates independent of \( m \).

\( H^s \) estimate. Since each term on the right-hand side of equation (3.2.13) involves the generalized Stokes projector \( P^\alpha_F := [P(1 - \alpha^2 C_m)]^{-1} P(1 - \alpha^2 C_m) \), we use the operator \( P(1 - \alpha^2 C_m) \) before the standard differential operator in the following \( H^s \)-energy estimate. By letting \( \sigma \) denote a multi-index with \(|\sigma| \leq s - 2\), we apply \( D^\sigma P(1 - \alpha^2 C_m) \) to both sides of equation (3.2.13) and take the \( L^2 \)-inner product with \( D^\sigma P(1 - \alpha^2 C_m) u_m \). The left-hand side reduces to
\[ (D^\sigma P(1 - \alpha^2 C_m) \partial_t u_m, D^\sigma w_m)_{L^2} = (D^\sigma \partial_t w_m, D^\sigma w_m)_{L^2} - \alpha^2 (D^\sigma P[C_m, \partial_t] u_m, D^\sigma w_m)_{L^2} \]
\[ = (\partial_t D^\sigma w_m, D^\sigma w_m)_{L^2} + \alpha^2 (D^\sigma P \text{div}(\partial_t F_m \cdot \nabla u_m), D^\sigma w_m)_{L^2} \]
\[ = \frac{1}{2} \frac{d}{dt} ||D^\sigma w_m||^2_{L^2} - \alpha^2 (D^\sigma P(\partial_t F_m \cdot \nabla u_m), \nabla D^\sigma w_m)_{L^2}, \]
where we define $w_m := P(1 - \alpha^2 C_m)u_m$. In addition
\[
\frac{d}{dt} \| D^\sigma w_m \|_{L^2}^2 \leq 2\alpha^2(D^\sigma P(\partial_t F_m \cdot \nabla u_m), \nabla D^\sigma w_m)_{L^2}
- 2(D^\sigma P(1 - \alpha^2 C_m) \text{div}(u_m \otimes u_m), D^\sigma w_m)_{L^2}
+ 2\nu(D^\sigma P(1 - \alpha^2 C_m)C_mu_m, D^\sigma w_m)_{L^2}
= I + II + III,
\] (3.2.15)
where we have used the definition of $\mathcal{P}_{F_m}^\alpha := [P(1 - \alpha^2 C_m)]^{-1}P(1 - \alpha^2 C_m)$ for the $II$ and $III$ terms.

**Estimate for I.** By the continuity of the Leray projector $P$ and the fact that $H^r$ is an multiplicative algebra for $r > \frac{3}{2}$, we use equation (3.2.14) to write the first term as
\[
|I| \leq C(\alpha^2)||P(\partial_t F_m \cdot \nabla u_m)||_{H^{s-2}}||w_m||_{H^{s-1}} \leq C(\alpha^2)||\partial_t F_m||_{H^{s-2}}||u_m||_{H^{s-1}}||w_m||_{H^{s-1}}
\leq C(\alpha^2)(||\nabla F_m \cdot u_m||_{H^{s-2}} + ||\nabla u_m \cdot F_m||_{H^{s-2}})||u_m||_{H^{s-1}}||w_m||_{H^{s-1}}
\leq C(\alpha^2)||F_m||_{H^{s-1}}||u_m||_{H^{s-1}}^2||w_m||_{H^{s-1}}
\leq \varepsilon_1||w_m||_{H^{s-1}}^2 + C(\varepsilon_1, \alpha^4)||F_m||_{H^{s-1}}^2||u_m||_{H^{s-1}}^4,
\]
where we have used Young’s inequality for the last estimate.

**Estimate for II.** Since $\text{div}D^\sigma w_m = 0$,
\[
|II| \leq |(D^{s-2}(1 - \alpha^2 C_m) \text{div}(u_m \otimes u_m), D^{s-2}w_m)|
\leq C||\text{div}(u_m \otimes u_m)||_{H^{s-2}}||w_m||_{H^{s-2}}
+C(\alpha^2)||D^{s-2}(F_m \cdot \nabla \text{div}(u_m \otimes u_m), \nabla D^{s-2}w_m)||
\leq C||\nabla u_m \cdot u_m||_{H^{s-2}}||w_m||_{H^{s-2}} + C(\alpha^2)||F_m \cdot \nabla \text{div}(u_m \otimes u_m)||_{H^{s-2}}||w_m||_{H^{s-1}}.
\]
By Young’s inequality we conclude
\[
|II| \leq \varepsilon_2||w_m||_{H^{s-1}}^2 + C||u_m||_{H^{s-2}}||u_m||_{H^{s-1}}||w_m||_{H^{s-2}} + C(\alpha^4)||F_m||_{H^{s-2}}^2||u_m||_{H^{s-1}}^2||u_m||_{H^{s-1}}^2.
\]
**Estimate for III.** Using the coercivity of $F_m$, the estimate for expression $III$ includes a negative coefficient in front of the $||w_m||_{H^{s-1}}$-term. The negative coefficient allows
us to control the $\|w_m\|_{H^{s-1}}$-terms from estimates for expressions I and II.

\[
III = 2\nu(D^\sigma P(1 - \alpha^2 C_m)C_m u_m, D^\sigma w_m)_{L^2} = 2\nu(D^\sigma PC_m(1 - \alpha^2 C_m)u_m, D^\sigma w_m)_{L^2} \\
= 2\nu(D^\sigma C_m P(1 - \alpha^2 C_m)u_m, D^\sigma w_m)_{L^2} + 2\nu(D^\sigma [P, C_m](1 - \alpha^2 C_m)u_m, D^\sigma w_m)_{L^2} \\
= -2\nu(D^\sigma (F_m \cdot \nabla w_m), \nabla D^\sigma w_m)_{L^2} + 2\nu(D^\sigma [P, C_m]u_m, D^\sigma w_m)_{L^2} \\
- 2\alpha^2 \nu(D^\sigma [P, C_m]C_m u_m, D^\sigma w)_{L^2}.
\] (3.2.16)

Since $\text{div } u_m = 0$, the definition of the Leray projector $P$ implies $[P, C_m]u_m = PC_m u_m - C_m Pu_m = -\text{grad } p_1$ where $\Delta p_1 = \text{div } C_m u_m$. Therefore the second term on the right hand side of equation (3.2.16) vanishes. Also

\[
[P, C_m]C_m u_m = PC_m^2 u_m - C_m PC_m u_m = C_m u_m - \text{grad } p_2 - C_m^2 u_m + C_m \text{grad } p_1 \\
= -\text{grad } p_2 + \text{grad } C_m p_1 + [C_m, \nabla] p_1 =: \text{grad } P - \text{div } (\nabla F_m \cdot \nabla p_1),
\]

where $\Delta p_2 = \text{div } C_m^2 u_m$. Using Leibniz’ formula on the first term of (3.2.16) we find

\[
III = -2\nu(F_m \cdot \nabla D^\sigma w_m, \nabla D^\sigma w_m)_{L^2} + C \sum_{0 < |\beta| \leq |\sigma|} (D^\beta F_m D^{\sigma-\beta} \nabla w_m, \nabla D^\sigma w_m)_{L^2} \\
- 2\alpha^2 \nu(D^\sigma (\nabla F_m \cdot \nabla p_1), \nabla D^\sigma w_m)_{L^2}.
\] (3.2.17)

Rather than introducing the commutator in equation (3.2.17), we use the Leibniz’ formula to obtain the correct power on the term $\|w_m\|_{H^3}$ for the critical case of $\sigma = 4$. With a standard commutator estimate the best we can obtain is a bound by $\|F_m\|_{H^3} \|w_m\|_{H^3}^2$, which doesn’t allow the correct isolation of the $\|w_m\|_{H^3}$-term.

By the uniform ellipticity of $F_m$, the first term of (3.2.17)

\[-2\nu(F_m \cdot \nabla D^\sigma w_m, \nabla D^\sigma w_m)_{L^2} \leq -\lambda \|w_m\|_{H^{s-1}}^2.\]
The second term of inequality (3.2.17)

\[ \sum_{0 < |\beta| \leq |\sigma|} (D^\beta F_m D^{\sigma-\beta} \nabla w_m, \nabla D^\sigma w_m)_{L^2} \leq C \sum_{0 < |\beta| \leq |\sigma|} \|D^\beta F_m\|_{L^4} \|D^{\sigma-\beta+1} w_m\|_{L^4} \|\nabla D^\sigma w_m\|_{L^2} \]

\[ \leq C \|D^{s-2} F_m\|_{H^1} \|D^{s-2} w_m\|_{L^4} \|w_m\|_{H^{s-1}} \]

\[ \leq C \|F_m\|_{H^{s-1}} \|D^{s-1} w_m\|_{L^4} \|D^{s-2} w_m\|_{L^4} \|w_m\|_{H^{s-1}} \]

\[ \leq C \|F_m\|_{H^{s-1}} \|w_m\|_{H^{\frac{4}{5}}_{s-2}} \|w_m\|_{H^{\frac{4}{5}}_{s-1}}, \]

where we have used the Gagliardo-Nirenberg inequality for the third inequality. Finally, the estimates for \( I \)

\[ -2\alpha^2 \nu (D^\sigma (\nabla F_m \cdot \nabla p_1), \nabla D^\sigma w_m)_{L^2} \leq C(\alpha^2) \|F_m\|_{H^{s-1}} \|\nabla p_1\|_{H^{s-2}} \|w_m\|_{H^{s-1}} \]

\[ \leq C(\alpha^2) \|F_m\|_{H^{s-1}} \|C_m u_m\|_{H^{s-2}} \|w_m\|_{H^{s-1}} \]

\[ \leq C(\alpha^2) \|F_m\|^2_{H^{s-1}} \|u_m\|_{H^2} \|w_m\|_{H^{s-1}} \]

Hence by Young’s inequality we combine the above estimates to write inequality (3.2.17) as

\[ |III| \leq (\lambda - \epsilon_3) \|w_m\|^2_{H^{s-1}} + C \|F_m\|^8_{H^{s-1}} \|w_m\|^2_{H^{s-2}} + C(\alpha^2) \|F_m\|^4_{H^{s-1}} \|u_m\|^2_{H^2} \]

Finally, the estimates for \( I, II, \) and \( III \) allow us to conclude

\[ \frac{d}{dt} \|w_m\|^2_{H^{s-2}} \leq (\lambda - \epsilon) \|w_m\|^2_{H^{s-1}} + C \|F_m\|^8_{H^{s-1}} \|w_m\|^2_{H^{s-2}} + C \|u_m\|_{H^{s-1}} \|w_m\|_{H^{s-2}} + C \|F_m\|^2_{H^{s-2}} \|u_m\|^2_{H^{s-1}} \|u_m\|^2_{H^s} + C \|F_m\|^4_{H^{s-1}} \|u_m\|^2_{H^{s-1}} + C \|F_m\|^4_{H^{s-1}} \|u_m\|^2_{H^{s-1}} \]

Using inequality (3.2.11) and the elliptic regularity estimates of Proposition 3.2.1, we can bound the right hand side of (3.2.18) by only \( t, \|F_0\|_{H^k}, \) and \( \|w_m\|_{H^k} \) for \( k \leq s - 1 \). This is precisely the bound we need to show the existence of classical solutions using Proposition 3.2.1. As an example of this computation, we include the case when \( s = 4 \).
Example $s = 4$. We now use Proposition 3.2.1 and inequality (3.2.11) to write the estimate above for $s = 4$, in terms of $w_m$ and $F_0$ alone. We begin with the second term,

$$||F_m||_{H^3}^2 w_m||^2_{H^2} \leq C||F_0||_{H^3}^8 (1 + 8t||u_m||_{H^4} + O(t^2))||w_m||^2_{H^2}$$

$$\leq C||F_0||_{H^3}^8 (1 + O(t^2))||w_m||^2_{H^2} + tC||F_0||_{H^3}^8 ||u_m||_{H^4}||w_m||^2_{H^2}$$

$$\leq C(t, ||F_0||_{H^3}) ||w_m||^2_{H^2} + C(||F_0||_{H^3}) ||w_m||^4_{H^2} + C(t, ||F_0||_{H^3}) ||u_m||^2_{H^4}$$

for the last. The third term in inequality (3.2.18) can be estimated by a repeated use of Young’s inequality,

$$||u_m||_{H^2} ||u_m||_{H^3} ||w_m||_{H^2} \leq C||w_m||^2_{H^2} + C||u_m||^4_{H^3}$$

$$\leq C||w_m||^2_{H^2} + C(t, ||F_0||_{H^3}) ||w_m||^4_{H^2} + C(t, ||F_0||_{H^3}) ||w_m||^8_{H^4}$$

The fourth term of inequality (3.2.18) can be estimated by

$$||F_m||_{H^2}^2 ||u_m||_{H^3}^2 ||u_m||_{H^4}^2 \leq C(t)||F_0||_{H^2}^2 ||u_m||^4_{H^4} + tC||F_0||_{H^2}^2 ||u_m||^3_{H^5} ||u_m||^2_{H^4}$$

$$\leq C(t, ||F_0||_{H^3}) ||u_m||^4_{H^4} + C(t, ||F_0||_{H^2}) ||u_m||^6_{H^3}$$

$$\leq C(t, ||F_0||_{H^2}) ||w_m||^4_{H^2} + C(t, ||F||_{H^2}) ||w_m||^8_{H^3}$$

$$+ C(t, ||F_0||_{H^3}) ||w_m||^4_{H^2}.$$
by inequalities (3.2.5), (3.2.6), and Young’s inequality. The second to last term of inequality (3.2.18)

\[ |F_m||u_m| \leq C(\|F_0||u_0| + tC\|F_0||u_m|^2 + tC\|F_0||u_m|^2 + C(t, \|F_0||H^1)) \|u_m||^4_{H^4} \]

where we have used Young’s inequality to write \|u_m||^2_{H^4} \leq C\|u_m||^2_{H^4} + C\|u_m||^4_{H^4}.

The final term of inequality (3.2.18) can be estimated by inequality (3.2.3) as

\[ |F_m||u_m| \leq C\|F_m||u_0|| + C\|F_m||^2_{H^2}|w_m| \]

Finally, combining the estimates above we achieve the a priori estimate

\[
\frac{d}{dt}|w_m|^2_{H^2} \leq (-\lambda + \varepsilon)|w_m|^2_{H^2} + C|w_m|^2_{H^4} + C|w_m|^2_{H^4} + C|w_m|^8_{H^1},
\]

(3.2.19)

where \( C = C(t, \alpha, \|F_0||H^1) > 0 \).

**Convergence to a strong solution to equation (3.1.1).**

From inequality (3.2.19) and equation (3.2.13), we may conclude that for some

\( T > 0 \), \( w_m \) is bounded in \( L^\infty(0, T; H^2_{div}) \cap W^{1,\infty}(0, T; L^2_{div}) \cap L^2(0, T; H^3_{div}) \) uniformly in \( m \), and hence by inequality (3.2.6),

\( u_m \) is bounded in \( L^\infty(0, T; H^4_{div}) \cap W^{1,\infty}(0, T; H^2_{div}) \cap L^2(0, T; H^3_{div}) \),

in the strong sense.
uniformly in $m$. By the weak compactness theorem, there exists a subsequence $u_{m_k}$ such that for all $T > 0$,

$$u_{m_k} \rightharpoonup u \text{ in } L^\infty(0, T; H^4_{\text{div}}) \cap W^{1, \infty}(0, T; H^2_{\text{div}}) \cap L^2(0, T; H^5_{\text{div}}).$$

Since $W^{1, \infty}(0, T; H^2_{\text{div}}) \subset C^0([0, T]; H^2_{\text{div}})$, Arzela-Ascoli compactness criterion implies

$$u_{m_k} \rightarrow u \text{ in } C^0([0, T]; H^2_{\text{div}}).$$

Furthermore by interpolation, $u_m \in W^{0, \infty}(0, T; H^4_{\text{div}}) \cap W^{1, \infty}(0, T; H^2_{\text{div}})$ implies

$$u_{m_k} \rightarrow u \text{ in } C^\delta([0, T]; H^{4-\delta}_{\text{div}}),$$

and therefore by Sobolev’s embedding theorem (2.2.9),

$$u_{m_k} \rightarrow u \text{ in } C^0([0, T]; C^2_{\text{div}})$$

for $\delta$ taken sufficiently small. Thus all the terms on the right-hand side of equation (3.2.13) converge strongly and $\partial_t u_m$ converges weakly. To see that $u \in C^0([0, T]; H^4_{\text{div}})$, it suffices to show that $||u(t, \cdot)||_{H^4}$ is continuous on $[0, T]$, but this follows from the inequalities (3.2.6) and (3.2.19).

The uniqueness of classical solutions to (3.1.1) follows the same arguments made in [22]. The only deviation being the evolution of the difference of two solutions of (3.2.12) is given as

$$\partial_t(u_1 - u_2) + (P^\alpha_1 - P^\alpha_2) \nabla_1 u_1 \cdot u_1 + P^\alpha_2 \nabla u_1 \cdot (u_1 - u_2) + P^\alpha_3 \nabla (u_1 - u_2) \cdot u_2$$

$$= \nu P^\alpha_1 \text{ div} ((F_1 - F_2) \cdot \nabla u_1) + \nu P^\alpha_1 \text{ div} (F_2 \cdot \nabla u_1) + \nu P^\alpha_2 \text{ div} (F_2 \cdot \nabla u_2)$$

$$= \nu P^\alpha_1 \text{ div} ((F_1 - F_2) \cdot \nabla u_1) + \nu P^\alpha_1 \text{ div} (F_2 \cdot \nabla (u_1 - u_2))$$

$$+ \nu (P^\alpha_1 - P^\alpha_2) \text{ div} (F_2 \cdot \nabla u_1),$$

where $F_i$ solves equation (3.1.1e) with $u = u_i$ for $i = 1, 2$. That is, if

$$y(t) := ||u_1(t) - u_2(t)||_{H^1}^2 + ||F_1(t) - F_2(t)||_{H^1}^2$$
we use the fact that \( u \in L^\infty(0,T;H^s_{\text{div}}) \cap L^2(0,T;H^{s+1}_{\text{div}}) \) and computations similar to that used for existence to obtain the differential inequality
\[
\frac{d}{dt} y(t) \leq C(t) y(t), \quad C(t) = C(||u_i||_{H^s}, ||F_i||_{H^s}), \quad i = 1, 2,
\]
from which uniqueness follows.

**Remark 3.2.3** As \( \alpha \to 0 \), the anisotropic LANS-\( \alpha \) equations should reduce to the Navier-Stokes equations. Once it is shown that \( F \to \text{Id} \) as \( \alpha \to 0 \), the anisotropic equations reduce to the isotropic equations. Marsden and Shkoller [21] have proven for \( s \geq 3 \), solutions to the isotropic LANS-\( \alpha \) equations converge in \( H^s \) for short time on intervals which are governed by the existence theory for the Navier-Stokes equations.
Chapter 4

Numerical solutions to the anisotropic LANS-α equations

4.1 Introduction.

Examining the behavior of fluid flow in elementary domains is more than a common numerical test for a new fluid model, it is also a means of highlighting interesting phenomena inherent in the model. The direct numerical simulation (DNS) of turbulent flow at small to moderate Reynolds number has become an important computational tool in understanding large scale turbulence motion. Unfortunately, DNS are still computationally expensive in turbulent regimes. Alternate approaches to brute-force DNS are the Reynolds averaged Navier-Stokes (RANS) simulations and large eddy simulations (LES). The RANS equations (1.2.3) are briefly discussed in Chapter 1. In LES, a spatial averaging operator (filter) is applied to the Navier-Stokes equations to obtain a new set of equations for the averaged (filtered) variables. Due to the computational limit of DNS for large Reynolds number, LES have become one of the standard methods in solving for fluid flow.

The behavior of small spatial scales in turbulent flow is often characterized by
statistical isotropy and homogeneity away from the boundary of the fluid container. Therefore, the isotropic LANS-α equations appear to be an appropriate model in isotropic regimes and have been studied recently from the numerical point of view. As a proposed model for large scale turbulence, Mohseni et. al. [24] compare the isotropic LANS-α equations to known results for DNS and LES methods. The authors demonstrate the utility of the isotropic LANS-α as a sub-grid stress model for three-dimensional isotropic forced and decaying turbulence. They perform two sets of forced isotropic turbulence simulations and compare their results to DNS and LES results where appropriate. In the LANS-α simulations, Mohseni et. al. conclude that selecting an appropriate α is a delicate comprise between the accuracy of the model at the the large scales and the minimum resolution requirements. In particular, the LANS-α equations accurately mimic the behavior of the Navier-Stokes equations at large spatial scales as long as a minimum resolution is observed. The accuracy improves as $\alpha \to 0$, but the computation requires higher resolution. The higher resolution comes with a price: an unresolved LANS-α computation could result in the contamination of large scales and a loss of accuracy.

In 1998, Chen et. al. [5] studied the mean velocity of turbulent channel and pipe flows. They proposed using stationary solutions to isotropic LANS-α equations as a closure approximation for the Reynolds-averaged equations. Since in the near-wall region the fluctuations are highly anisotropic, their results were only in good agreement with experimental data away from the viscous boundary layer.

By restricting the anisotropic LANS-α equations to the channel, Coutand and Shkoller [11] propose a turbulent channel theory that models the large scale fluid motion throughout the entire domain. Unlike the isotropic equations, the solutions to the anisotropic equations consist of both a mean velocity field $u$ and a $3 \times 3$ covariance tensor $F$, defined, in detail, in the next section. An important property of $F$ is its degeneracy to zero at the wall. Coutand and Shkoller [11] show that near
the wall, the degeneracy scales like $d \sqrt{|\log d|}$, where $d$ is the normalized distance function to the wall. The authors compensate for the degeneracy in the boundary layer by working in weighted Sobolev spaces. In this functional framework, they prove the global-in-time existence and uniqueness of weak solutions to the anisotropic LANS-$\alpha$ equations. In particular, they restrict the anisotropic equations to the channel and make the assumption that the initial covariance tensor is given in the form $F_0 = F(0, x, y, z) = \rho(z) \text{Id}$. By assuming the fluid is moving in one direction, the anisotropic LANS-$\alpha$ equations reduce to a one-dimensional partial differential equation for the mean velocity. The authors use the Galerkin method to obtain the existence of weak solutions to the anisotropic equations.

In this chapter, we use the anisotropic LANS-$\alpha$ equations as a numerical model for two classical examples of laminar velocity profiles. Unlike other approaches to the modeling of turbulence, the Lagrangian averaging approach allows us to capture the large scale motion of a fluid in laminar regimes. The first example that we consider is the Poiseuille flow in the channel and pipe domains. Laminar Poiseuille flow occurs when an incompressible fluid with no-slip boundary conditions is driven by a constant upstream pressure gradient, yielding a symmetric parabolic stream-wise profile. We assume the velocity $u$ satisfies the steady Navier-Stokes equations and we numerically solve the anisotropic equations for the initial covariance tensor $F_0$. In particular, we calculate the matrix $F$ such that pair $(u, F)$ is a solution to the anisotropic LANS-$\alpha$ equations. The degeneracy rate of our numerical solution $F$ near the wall of the channel is in good agreement with the logarithmic decay rate given by Coutand and Shkoller [11]. In addition, we show that in the boundary layer, $||F(t, \cdot)||_{L^\infty(\Omega)}$ does not remain bounded as $t \to \infty$, answering, at least numerically, a question stated in [11].

In the last section of the chapter, we study shear flow solutions to the anisotropic LANS-$\alpha$ equations. Shear flow occurs in the channel when one side of the boundary
is moving while the other side remains fixed. The derivation by Marsden and Shkoller [22] relied upon no-slip boundary conditions for the mean velocity field $u$. In order to find a solution to the inhomogeneous problem, we use the classical technique of introducing a new vector field that vanishes on the boundary and that solves the anisotropic equations with additional forcing terms. For the domains and the initial data that submit a unique solution to the new formulation, we are able to solve for the unknown mean velocity. In Section 4.3, we show that shear flow velocity solutions to the anisotropic LANS-α equations exist if the initial covariance tensor is not required to be positive. This turns out to be a natural assumption and we compute the shear flow solutions directly.

4.1.1 The covariance tensor $F(t, x)$.

The matrix $F$ is defined by Marsden and Shkoller [22] as the ensemble average of the tensor product of the Lagrangian fluctuation vector. Specifically, if $\eta$ and $\eta^\varepsilon$ are the particle trajectories of the fluid velocity $u$ and averaged fluid velocity $u^\varepsilon$, respectively, then the Lagrangian fluctuation vector $\xi^\varepsilon := \eta^\varepsilon \circ \eta^{-1}$ (see Figure 4.1) and

$$F := \left\langle \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \xi^\varepsilon \otimes \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \xi^\varepsilon \right\rangle$$

where $< \cdot >$ denotes the average over all possible solutions $u^\varepsilon$ (see [22]). Because these fluctuations are necessarily zero along the boundary,

$$F(t, x) = 0 \text{ for } t \geq 0$$

on the boundary of the domain. As noted in [11], it is unknown a priori whether $\|F(t, \cdot)\|_{L^\infty}$ remains bounded for all time. In next section, we show numerically that $\|F(t, \cdot)\|_{L^\infty}$ is strictly increasing in time in the viscous boundary layer of the channel and pipe under a steady mean flow.
4.2 Homogeneous boundary conditions.

Laminar Poiseuille flow occurs when an incompressible fluid in a straight channel, or pipe, is driven by a constant upstream pressure gradient, yielding a symmetric parabolic stream-wise velocity profile. In this section, we begin with the steady Poiseuille flow associated with the Navier-Stokes equations. We construct an initial covariance tensor such that the steady flow solution and the covariance tensor solve the anisotropic equations in the channel. First, we reduce the anisotropic LANS-$\alpha$ equations to the channel and pipe under the assumption that the initial covariance tensor is a multiple of the identity matrix.

4.2.1 Given the initial covariance matrix.

**Channel.** The three-dimensional channel is given by $\Omega = \mathbb{R}^2 \times [-h, h]$ with coordinates $x = (x, y, z)$. We shall assume that the velocity vector is of the form

$$u(t, x, y, z) = (u(t, z), 0, 0),$$

and that the initial covariance matrix is written as

$$F_0 = F(0, x, y, z) = \rho(z)\text{Id},$$
such that $\rho(\pm h) = 0$.

Suppose $\rho(z)$ is given. As discussed in [11], when no-slip boundary conditions are prescribed for the mean velocity, the nonlinear term vanishes and equation (3.1.1) reduces to the following system

$$
\partial_t (u - \alpha^2 (\rho u')') - \nu ((\rho u')' - \alpha^2 (\rho (\rho u')''))' = -c,
$$

$$
u(t, \pm h) = 0, \quad u(0, z) = u_0(z),
$$

where $c := \partial_x p$ is constant since $\partial_y p = \partial_z p = 0$. To address the degeneracy at the boundary, Coutand and Shkoller define a weighted Sobolev to set as the functional framework. This space is based on the following function: For $\epsilon < \delta << 1$ define $\rho_{CS}(z) \in C^\infty[-h, h]$ to be the positive function

$$
\rho_{CS}(z) = \begin{cases}
d \sqrt{|\log d|}, & 0 \leq d(z) \leq \epsilon, \\
1, & d(z) \geq \delta,
\end{cases}
$$

$d = \text{normalized distance function to the boundary}.

Coutand and Shkoller prove the existence of a unique global weak solution to equation (4.2.1) in a weighted Sobolev space. In addition, they show that the covariance tensor $F$ must degenerate like $\rho_{CS}$ in the viscous boundary layer. In the next section, we verify that our numerical solution $\rho$ matches $\rho_{CS}$ near the walls of the channel.

**Pipe.** The three-dimensional pipe is given by $\Omega = \mathbb{R}^3$ with coordinates $\mathbf{x} = (x, y, z)$ such that $0 \leq \sqrt{x^2 + y^2} \leq a$. We shall assume that the velocity vector is radially symmetric of the form

$$
\mathbf{u}(t, x, y, z) = (0, 0, u(t, r)),
$$

where $r := \sqrt{x^2 + y^2}$ and that the initial covariance matrix

$$
F_0 = F(0, x, y, z) = \rho(r) \text{Id}.
$$

We let $\eta$ denote the Lagrangian flow of $\mathbf{u}$ satisfying the initial value problem (4.2.5). Then definition (4.2.3) implies that

$$
\eta(t, \mathbf{x}) = \left( x, y, z + \int_0^t u(s, r) \, ds \right).
$$
Therefore
\[ F(t, x) := D\eta(t, x) \cdot F_0(x) \cdot D\eta(t, x)^T = \rho(r) \left[ \begin{array}{ccc} 1 & 0 & \frac{\nu U}{r} \\ 0 & 1 & \frac{\nu U}{r} \\ \frac{\nu U}{r} & \frac{\nu U}{r} & 1 + \nu^2 \end{array} \right], \] (4.2.4)

where \( U(t, x) := \int_0^t \partial_r u(s, r) \, ds \). Together with definition (4.2.3), \( Cu := \text{div}(\nabla u \cdot F) \) reduces to
\[ Cu = (0, 0, \frac{1}{r} (r \rho u')') \]
with \( u'(t, r) := \partial_r u(t, r) \). The nonlinear term in the anisotropic LANS-\( \alpha \) vanishes and the equations reduce to the following system
\[ \partial_t (u - \alpha^2 \frac{1}{r} (r \rho u')') - \frac{\nu}{r} \left( (r \rho u')' - \alpha^2 (r \rho (\frac{1}{r} (r \rho u')'))' \right) = -c, \]
(4.2.5)
\[ u(t, a) = 0, \quad u(0, r) = u_0(r), \]
where \( c := \partial_z p \) is constant since \( \partial_x p = \partial_y p = 0 \).

### 4.2.2 Given the mean velocity profile.

In this section, we provide the numerical results for steady channel and pipe flow assuming no-slip boundary conditions. We start with the steady solution \( u \) to the Navier-Stokes equations and proceed to find a \( \rho \), and therefore a \( F \), such that \((u, F)\) solve the anisotropic LANS-\( \alpha \) equations.

**Channel.** We begin by assuming the velocity field is the steady flow given as \( u(t, x, y, z) = (u(z), 0, 0) \) that satisfies the no-slip boundary condition \( u(\pm h) = 0 \). For the Navier-Stokes equations, the classical Poiseuille flow is given by
\[ u(z) = \beta (h^2 - z^2) \]
(4.2.6)
\[ p(x) = p_0 - 2\nu \beta x, \]
where \( \beta, p_0 > 0 \). In order for \((u, \rho)\) to solve equation (4.2.1) with the velocity \( u(z) \)
given by equation (4.2.6), the function $\rho(z)$ must solve the ordinary differential equation
\begin{align*}
(z\rho)' - \alpha^2(\rho(z)'' \rho)' &= 1, \\
\rho(\pm h) &= 0.
\end{align*}
(4.2.7)
Solving equation (4.2.7) numerically leads us to the following Proposition.

**Proposition 4.2.1** Let the function $u$ be the steady solution (4.2.6) to the incompressible Navier-Stokes equations in the smooth three-dimensional channel. The pair $(u, \rho)$ solves the channel anisotropic LANS-$\alpha$ equation (4.2.1) when the function $\rho(z)$ solves equation (4.2.7). A numerical solution is given in Figure 4.2(a) when $\alpha = 0.1$ and $h = \beta = 1$. In addition, the covariance tensor $F$ is defined as
\[
F(t, z) := \rho(z) D\eta(t, x) D\eta(t, x)^T = \rho(z) \begin{bmatrix}
1 + 4\beta^2 t^2 z^2 & 0 & -2\beta t z \\
0 & 1 & 0 \\
-2\beta t z & 0 & 1
\end{bmatrix}.
\]

**Remark 4.2.1** We assume that $\rho(z)$ is positive and shares the same symmetry across the channel as the velocity $u(z)$. In addition, we assign in the center of the channel the values $\rho(0) = A$ and $\rho'(0) = 0$. By a numerical shooting method, we find a value for $A$ that forces the zero boundary condition $\rho(1) = 0$. This solution is plotted in Figure 4.2(a).

**Remark 4.2.2** The numerical solution $\rho$ decays at the same rate as the function $\rho_{CS}$ near the boundary. In Figure 4.2(b), we plot both functions near the wall of the channel.

To illustrate the dynamics of $F$ in the channel, we calculate its eigenvalues and conclude the following Proposition.

**Proposition 4.2.2** The eigenvalues of $F(t, z)$ are
\begin{align*}
\lambda_1(t, z) &= \rho(z), \\
\lambda_{2,3}(t, z) &= \frac{1}{2} \rho(z) \left( 2 + u^2 \pm |u|\sqrt{u^2 + 4} \right) = \rho(z) \left( 1 + 2\beta^2 t^2 z^2 \pm 2\beta t |z| \sqrt{1 + \beta^2 t^2 z^2} \right).
\end{align*}
The eigenvalues are plotted in Figure 4.2(c) for the case of $\alpha = 0.1$, $\beta = 1$, and the function $\rho$ given as the solution to equation (4.2.7). Furthermore, $\|F(t, \cdot)\|_{L^\infty(\Omega)} = O(t^2)$ and although our velocity is steady throughout the channel, the covariance tensor

$$\|F(t, \cdot)\|_{L^\infty(\Omega)} \to \infty \quad \text{as} \quad t \to \infty$$

near the boundary in the direction of the flow. This is illustrated in Figure 4.2(d).
(a) The solution $\rho(z)$ of equation (4.2.7).

(b) The solution $\rho(z)$ and the function $\rho(z)$ used in [22] near the boundary.

(c) The eigenvalues of $F(t, z)$ at $t = 2$.

(d) The evolution of $||F||_{L^\infty}$.

Figure 4.2: For $h = \beta = 1$ and $\alpha = 0.1$. Because of symmetry, the graphs (a), (c), and (d) are on $[0, 1]$ only.
Proof. The eigenvalues are straightforward to compute. Since the largest eigenvalue \( \lambda_2(\cdot, x) = O(t^2) \) and \( \lambda_2 \leq ||F(t, z)||_{L^\infty} \), then \( ||F(t, \cdot)||_{L^\infty} = O(t^2) \). The vector

\[
v(t, z) = C \begin{bmatrix} tz + \sqrt{1 + t^2 z^2} \\
0 \\
-1
\end{bmatrix}
\]

solves the eigenvalue equation \( Fv = \lambda_2v \). Therefore, as \( t \to \infty \), the norm of \( F \) increases to infinity in the direction of the flow (x-direction).

Pipe. The format and conclusions of this subsection follow closely the results in the channel. We begin by assuming the velocity field is a steady velocity field given as \( u(t, x) = (0, 0, u(r)) \) that satisfies the no-slip boundary condition \( u(\pm a) = 0 \). For the Navier-Stokes equations, the classical pipe flow is given by

\[
\begin{align*}
  u(r) &= \beta(a^2 - r^2) \\
p(r) &= p_0 - 4\nu\beta r,
\end{align*}
\]

where \( \beta, p_0 > 0 \). In order for \((u, \rho)\) to solve equation (4.2.5) with the velocity \( u(r) \) given by equation (4.2.8), the function \( \rho(r) \) must solve the ordinary differential equation

\[
r(rp' + 2\rho) - \alpha^2(rp(rp'' + 3\rho'))' = 2r \\
\rho(a) = 0.
\]

As in the channel domains, we arrive at the following Proposition.

**Proposition 4.2.3** Let the function \( u \) be the steady solution (4.2.8) to the incompressible Navier-Stokes equations in the smooth three-dimensional pipe. The pair \((u, \rho)\) solves the pipe anisotropic LANS-\(\alpha\) equation (4.2.5) when the function \( \rho(r) \) solves equation (4.2.9). A numerical solution is given in Figure 4.3 when \( \alpha = 0.1 \), \( \beta = 1 \), and the radius of the pipe is \( a = 1 \). In addition, the covariance tensor \( F \) is given by (4.2.4) and has the same eigenvalues as the channel covariance tensor.
CHAPTER 4. Numerical solutions to the anisotropic LANS-α equations

4.2.3 Conclusions.

As expected, the anisotropy of the fluid is of fundamental importance in bounded domains. We have demonstrated that the dynamics of $F$ are just as important in laminar regimes as they are in turbulent regimes. Supposing that the covariance tensor is a constant multiple of the identity matrix, while accurate for very short time and in the center of the channel, becomes an inaccurate assumption in the viscous boundary layer. In both the channel and the pipe geometry, the evolution of $F$, illustrated by Figure 4.2(c), is non-decreasing in time and achieves its greatest value in this boundary layer. As time increases, the location of the maximum value of $\|F\|_{\infty}$ in the channel approaches the limit $z = 0.89$. Therefore an accurate model for fluid motion in the entire channel or pipe should be founded on the anisotropic model, rather than the isotropic version, in this boundary region.

The logarithmic degeneracy rate of the covariance tensor in the channel agrees well with the decay rate of the function computed in [11]. Since the dynamics of fluid motion in the channel and pipe are similar, it is not surprising that the covariance tensor for the channel and pipe are the same. The next step in studying the numerical
properties of the anisotropic model is to solve for the covariance tensor when the velocity is given as a time-dependent solution to the Navier-Stokes problem. This is a necessary step in understanding the dynamics of the covariance tensor $F$.

4.3 Inhomogeneous boundary conditions.

In many physical models, ranging from the study of blood flow to the modeling of earthquakes, at least one of the boundary components of the fluid container are in motion. In these cases, the mean velocity of the fluid at the wall will no longer be zero. In this section, we study the anisotropic LANS-$\alpha$ equations with inhomogeneous boundary data. As a specific example, we consider the mean fluid motion in a channel when one of the boundary walls is not fixed. This motion is called shear flow.

4.3.1 The LANS-$\alpha$ equations with inhomogeneous boundary data.

Let $\Omega$ be a bounded fluid container in $\mathbb{R}^n$ with boundary $\partial \Omega$ and suppose that the mean velocity field $u = g \neq 0$ on $\partial \Omega$. To find a solution pair $(u, F)$ that solves the anisotropic LANS-$\alpha$ equations with inhomogeneous boundary conditions, we choose a divergence-free vector field $v(t, \cdot) \in C^\infty(\Omega)$ such that $v = g$ on the boundary. With $v$ given, we define a new vector field $w = u - v$. The vector field $w$ is zero on the boundary and solves the anisotropic LANS-$\alpha$ equations with $u$ replaced with $w + v$. Namely, we now search for a solution to the following system of partial differential equations

\[
\begin{align*}
(1 - \alpha^2 C) \left( \partial_t w - \nu \mathbb{P} C w + \text{div}(v \otimes w + w \otimes v) \right) &= -\text{grad} p - (1 - \alpha^2 C) \left( \partial_t v - \nu \mathbb{P} C v \right), \\
\text{div} w(t, x) &= 0, \quad w = 0 \text{ on } \partial \Omega, \\
\partial_t F + \nabla F \cdot w - \left( \nabla w \cdot F + [\nabla w \cdot F]^T \right) &= \nabla F \cdot v + \nabla v \cdot F + [\nabla v \cdot F]^T, \\
w(0, x) &= u_0(x) - v(0, x), F(0, x) = F_0(x).
\end{align*}
\]

(4.3.10)
In the three-dimensional torus $\mathbb{T}^3$, the trivial existence and uniqueness of a solution to (4.3.10) follows from the smoothness of $v$, Theorem 1 in [22], and Theorem 3.2.1 in Chapter 3. This result is stated as the following proposition.

**Proposition 4.3.1** For $s > 7/2$, and $w_0 \in H^{s}_{\text{div}}(\mathbb{T}^3)$, $F_0 \in [H^s_{\text{per}}(\mathbb{T}^3)]^{3 \times 3}$, with $F_0 > 0$, there exists a unique solution $(w, F)$ with $w \in C^0([0, T]; H^{s}_{\text{div}}) \cap L^2(0, T; H^{s+1}_{\text{div}})$ and $F \in C^0([0, T]; [H^s_{\text{per}}]^{3 \times 3})$ to equations (4.3.10), where $T$ depends on the initial data.

In arbitrary bounded domains, it is unknown whether solutions exist to the anisotropic equations. As we demonstrate in the next section, under certain limiting conditions we can find a solution to equation (4.3.10).

**4.3.2 Given the initial covariance matrix.**

To study shear flow velocity solutions to the anisotropic LANS-$\alpha$ equations, we need to restrict the full equations to the three-dimensional channel by making a number of limiting assumptions. We assume that the initial covariance tensor $F_0 = F(0, x, y, z) = \rho(z)\text{Id}$ is given and the mean velocity has the form $u = (u(t, z), 0, 0)$ for $z \in [-h, h]$, where $u \neq 0$ on the boundary.

As was done in the general case above, we choose a vector field $v(t, \cdot) \in C^\infty[-h, h]$ such that $v(t, \pm h) = u(t, \pm h)$ for all $t \geq 0$. The vector field $w = u - v$ then solves the following one-dimensional problem

$$L^\alpha w = -c + f,$$

$$w(\pm h) = 0, w(0, z) = u_0(z) - v(0, z)$$

(4.3.11)

where $L^\alpha$ is the linear operator defined as

$$L^\alpha u := \partial_t(u - \alpha^2(\rho u')') - \nu((\rho u')' - \alpha^2(\rho(\rho u'')))',$$

and $f := -L^\alpha v$. Then $u(t, x, y, z) := (w(t, z) + v(t, z), 0, 0)$ solves equation (4.2.1) with inhomogeneous boundary conditions.
4.3.3 Given the mean velocity field.

Suppose that we are given the steady velocity vector field \( u = (u(z), 0, 0) \) with \( u(z) \neq 0 \) on the boundary. To find a solution to the anisotropic model, we need to find a function \( \rho(z) \) such that

\[
\nu((\rho u')' - \alpha^2(\rho u'')') = -c.
\]

But since the vector field \( u \) doesn’t satisfy the no-slip boundary conditions, we do not know the correct boundary conditions for \( \rho \). Rather, we choose \( v \in C^\infty[-h, h] \) with \( v = u \) on boundary. Then \( w = u - v \) and the pair \((w, \rho)\) solve

\[
\mathcal{L}^\alpha w = -c + f, \tag{4.3.12}
\]

where \( f := -\mathcal{L}^\alpha v \). Since \( w = 0 \) on the boundary we also have the boundary condition \( \rho(\pm h) = 0 \). Since \( w \) is given we may find \( \rho \) satisfying equation (4.3.12) with zero boundary conditions. However, unlike the case when \( u \) satisfies no-slip boundary conditions, the forcing in equation (4.3.12) removes any a priori statement about the positivity of \( \rho \).

**Shear flow solution.** As an example, we show the existence of a shear flow velocity solution to the anisotropic LANS-\( \alpha \) equations. The steady shear flow velocity solution to the Navier-Stokes equations with the boundary conditions \( u(-h) = 0 \) and \( u(h) = U > 0 \) is

\[
u(z) = \frac{U}{2h}(z + h) \tag{4.3.13}
\]

with the pressure function \( p \equiv 0 \). The function

\[
w(z) = \frac{U}{4h^2}(h - z)(h + z)
\]

is zero on the boundary and solves (4.3.12) with \( f = -\mathcal{L}^\alpha \left( \frac{U^2}{4h^2}(z + h)^2 \right) \) when \( \rho(z) \)
is a solution of

\[ \rho' - \alpha^2 (\rho \rho'')' = 0, \]

\[ \rho(\pm h) = 0. \]

Namely,

\[ \rho(z) = \begin{cases} 
\frac{1}{2\alpha^2} (z - a)(z - b) & \text{for } -h \leq a \leq z \leq b \leq h \\
0 & \text{otherwise}
\end{cases}. \]

If we want \( \rho \in C^1 [-h, h] \) then \( a = -h, b = h \). We conclude that the pair \((u, \rho)\) with \( u \) define by equation \((4.3.13)\) and \( \rho(z) = \frac{1}{2\alpha^2} (z - h)(z + h) \) is a shear flow solution which solves equation \((4.2.1)\) with inhomogeneous boundary conditions.
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