K-theory of some C*-algebras and buildings.

Alina Vdovina

Abstract

We compute an exact formula for the order of the class of the identity in the $K_0$ group of an infinite class of two-dimensional Kuntz-Crieger algebras.

Introduction

The class of the identity $[1]$ in $K_0$ of different classes of crossed product C*-algebras was broadly investigated in the literature, see [9], [14], [13], [1], [17], [10].

We will concentrate on a case associated to two-dimensional Euclidean buildings. Let group $G$ acts simply transitively on the vertices of a $\tilde{A}_2$ building $\Delta$. Then there is an induced action on the boundary $\Omega$ of $\Delta$, the crossed product algebra $C(\Omega) \rtimes \Gamma$ depends only on $G$ and is classified by its $K$-groups together with the class $[1]$ in $K_0$ of the identity element $1$ of $C(\Omega) \rtimes \Gamma$. It is interesting therefore to identify this class.

We will consider special class of $\tilde{A}_2$ groups $\Gamma_{T_0}$ described in [7], which embed as arithmetic subgroups of $PGL(3, F_q(X))$. For this class of groups we prove the following result, which was conjectured for all $\tilde{A}_2$ groups in [17].

**Theorem.** The order of the class $[1]$ of the identity element $1$ of $C(\Omega) \rtimes \Gamma$ in $K_0(C(\Omega) \rtimes \Gamma)$ is $q - 1$, where $\Gamma$ is a $\Gamma_{T_0}$ group and $q \equiv 1 (mod 3)$.

Polygonal presentation and construction of polyhedra.

A polyhedron is a two-dimensional complex which is obtained from several oriented $p$-gons by identification of corresponding sides. Let’s take a sphere of a small radius at a point of the polyhedron. The intersection of the sphere with the polyhedron is a graph, which is called the link at this point. We consider a case, when all sides of a polyhedron are regular euclidean triangles and links at all vertices are incidence graphs of finite projective planes. The universal covering of such a polyhedron is an euclidean $\tilde{A}_2$ building [2], [5] and with the metric introduced in [3] p. 165] it is a complete metric space of non-positive curvature in the sense of Alexandrov and Busemann. It follows from [4], that the fundamental groups of our polyhedra satisfy the property (T) of Kazhdan. (Another relevant reference is [21].)
Definition. Let $\mathcal{P}$ be a tesselation of the Euclidean plane by regular triangles, with angles $\pi/3$ in each vertex. A Euclidean $A_2$ building is a polygonal complex $X$, which can be expressed as the union of subcomplexes called apartments such that:

1. Every apartment is isomorphic to $\mathcal{P}$.
2. For any two polygons of $X$, there is an apartment containing both of them.
3. For any two apartments $A_1, A_2 \in X$ containing the same polygon, there exists an isomorphism $A_1 \rightarrow A_2$ fixing $A_1 \cap A_2$.

Recall that a generalized $m$-gon is a connected, bipartite graph of diameter $m$ and girth $2m$, in which each vertex lies on at least two edges. A graph is bipartite if its set of vertices can be partitioned into two disjoint subsets such that no two vertices in the same subset lie on a common edge. The vertices of one subset we will call black vertices and the vertices of the other subset the white ones. The diameter is the maximum distance between two vertices and the girth is the length of a shortest circuit. Incidence graphs of finite projective planes are exactly generalized triangles.

We recall a definition of polygonal presentation introduced in [20].

Definition. Suppose we have $n$ disjoint connected bipartite graphs $G_1, G_2, \ldots, G_n$. Let $P_i$ and $Q_i$ be the sets of black and white vertices respectively in $G_i$, $i = 1, \ldots, n$; let $P = \bigcup P_i$, $Q = \bigcup Q_i$, $P_i \cap P_j = \emptyset$, $Q_i \cap Q_j = \emptyset$ for $i \neq j$ and let $\lambda$ be a bijection $\lambda : P \rightarrow Q$.

A set $\mathcal{K}$ of $k$-tuples $(x_1, x_2, \ldots, x_k), x_i \in P$, will be called a polygonal presentation over $P$ compatible with $\lambda$ if

1. $(x_1, x_2, x_3, \ldots, x_k) \in \mathcal{K}$ implies that $(x_2, x_3, \ldots, x_k, x_1) \in \mathcal{K}$;
2. given $x_1, x_2 \in P$, then $(x_1, x_2, x_3, \ldots, x_k) \in \mathcal{K}$ for some $x_3, \ldots, x_k$ if and only if $x_2$ and $\lambda(x_1)$ are incident in some $G_i$;
3. given $x_1, x_2 \in P$, then $(x_1, x_2, x_3, \ldots, x_k) \in \mathcal{K}$ for at most one $x_3 \in P$.

If there exists such $\mathcal{K}$, we will call $\lambda$ a basic bijection.

We can associate a polyhedron $K$ on $n$ vertices with each polygonal presentation $\mathcal{K}$ as follows: for every cyclic $k$-tuple $(x_1, x_2, x_3, \ldots, x_k)$ we take an oriented $k$-gon on the boundary of which the word $x_1, x_2, x_3, \ldots, x_k$ is written. To obtain the polyhedron we identify the corresponding sides of our polygons, respecting orientation.

Lemma [20] A polyhedron $K$ which corresponds to a polygonal presentation $\mathcal{K}$ has graphs $G_1, G_2, \ldots, G_n$ as the links.

1 Balanced polygonal presentation

We will use a particular case of polygonal presentation, so-called triangle presentation, described in [7]. We repeat now the construction from [7] for completeness.

Consider the Desarguesian projective plane $(P, L) = \text{PG}(2, q)$ of prime power order $q$, in which the points and lines are 1- and 2-dimensional subspaces, respectively, of a 3-dimensional vector space $V$ over $\mathbb{F}_q$, with incidence being inclusion. We may take $V \cong \mathbb{F}_q^3$. Consider a regular quadratic form on $\mathbb{F}_q^3 (x_0, y_0) \rightarrow$
triples is a triangle presentation $Tr(x_0,y_0)$, where $Tr$ is the trace of the field extension $F_\beta/F_\gamma$. For $x \in P$, set

$$\lambda_0 = \{y \in P : Tr(x,y) = 0\}.$$ 

This defines a point-line correspondence $\lambda_0 : P \to L$. The following set of triples is a triangle presentation $T_0$ compatible with $\lambda_0$:

$$T_0 = \{(x,x\xi, x\xi^{l+1}) | x, \xi \in P, Tr(\xi) = 0\}.$$ 

It is convenient to denote elements of $P$ with letters of an alphabet $X = \{x_1, \ldots, x_{q^2+q+1}\}$.

We describe now a new polygonal presentation $T_1$. Take an alphabet $Y = \mathcal{A}, \mathcal{B}, \mathcal{C}$, were every subalphabet $\mathcal{A}, \mathcal{B}, \mathcal{C}$ contains $q^2 + q + 1$ elements. $\mathcal{A} = \{a_1\}, \mathcal{B} = \{b_1\}, \mathcal{C} = \{c_1\}$, $i = 1, \ldots, q^2 + q + 1$.

Define $T_1$ as the following set of triples:

$$\{(a_i, b_i, c_m), (b_j, c_i, a_m), (c_k, a_i, b_m) \in T_1 \iff (x_k, x_i, x_m) \in T_0\}$$

Define bijections $\lambda_1, \lambda_2, \lambda_3$ in the following way $\lambda_1(x_i) = a_i$, $\lambda_2(x_i) = b_i$, $\lambda_3(x_i) = c_i$.

**Lemma 1.** There exists a subset of $T_1$, such that every element of $Y$ occurs exactly once. We will call such a subset $S$ basic subset. A polygonal presentation containing a basic subset will be called balanced presentation.

**Proof.** Let’s consider such an element $x \in P$, that is a generator of $P$ as a cyclic group. Fix $\xi \in P$ such that $Tr(\xi) = 0$. Now, consider the following set of triples $(a_i, b_i, c_i), i = 1, \ldots, q^2 + q + 1$, where $a_i = \lambda_1(x_i), b_i = \lambda_2(x_i), c_i = \lambda_3(x_i) = c_i$.

### 2 Subshift of a balanced polygonal presentation

Let $T$ be a polygonal presentation with $n = 3$, $k = 3$, where all there graphs $G_1, G_2$ and $G_3$ are incidence graphs of finite projective planes of order $q$. The polyhedron, which corresponds to $T$, has triangular faces and three vertices. We will consider polyhedra such that all three vertices of each triangle have different graphs as links. In this case we can give a Euclidean metric to every face. In this metric all sides of the triangles are geodesics of the same length. The universal covering of the polyhedron is an Euclidean building $\Delta$, see [2, 3]. Each element of $T$ may be identified with an oriented basepointed triangle in $\Delta$. We now construct a 2-dimensional shift system associated with $T$. The transition matrices $M_1, M_2$ in the way, defined as in [17, p.828]: if $\alpha = (x_1, x_2, x_3), \beta = (y_1, y_2, y_3) \in T$ say that $M_1(\beta, \alpha) = 1$ if and only if there exists $\psi = (x_3, z, y_1)$ and $M_1(\beta, \alpha) = 0$ otherwise (Figure 2). In a similar way, $M_2(\gamma, \alpha) = 1$ for $\alpha = (x_1, x_2, x_3), \gamma = (y_1, y_2, y_3) \in T$ if and only if there exists $\psi = (x_2, y_1, z)$ and $M_2(\gamma, \alpha) = 0$ otherwise.

The matrices $M_1, M_2$ of order $3(q+1)(q^2+q+1)$ are nonzero $[0, 1]$ matrices. We will use $T$ as an alphabet and $M_1, M_2$ as transition matrices to build up 2-dimensional words as in [18]. Let $[m, n]$ denote $[m, m+1, \ldots, n]$, where $m \leq n$ are integers. If $m, n \in \mathbb{Z}^2$, say that $m \leq n$ if $m_j \leq n_j$ for $j=1,2$, and when $m \leq n$ let $[m, n] = [m_1, n_1] \times [m_2, n_2]$. In $\mathbb{Z}^2$, let $0$ denote the zero vector and let $e_j$ denote the $j$-th standard unit basis vector. If $m \in \mathbb{Z}^2 = [m \in \mathbb{Z}^2; m \geq 0]$, let
In order to apply the theory from [18] we need the matrices $M_1, M_2$ to satisfy the following conditions:

(H0) Each $M_i$ is a nonzero $\{0, 1\}$-matrix.

(H1a) $M_1 M_2 = M_2 M_1$.

(H2) The directed graph with vertices $\alpha \in T$ and directed edges $(\alpha, \beta)$ whenever $M_i(\alpha, \beta) = 1$ for some $i$ is irreducible.

(H3) For any nonzero $p \in \mathbb{Z}^2$, there exists a word $w \in W$ which is not $p$ - periodic, i.e., there exists $l$ so that $w(l)$ and $w(l+p)$ are both defined but not equal.

In [18] some $C^*$-algebra is defined by partial isometries of the system of words $W_m$, where $m \in \mathbb{Z}^2$. It is proved there, that if the matrices $M_1, M_2$ satisfy the conditions (H0),(H1a,b),(H2),(H3), then this algebra is simple, purely infinite and nuclear.

Now we prove the conditions (H0), (H1a,b), (H2), (H3) for our two-dimensional shift. By definition of matrices $M_1, M_2$ they are nonzero $\{0, 1\}$ matrices, so (H0) holds. If we have $\alpha, \beta, \psi$, such that $M_1(\alpha, \beta) = 1$, $M_2(\beta, \psi) = 1$, then $\gamma$ such that $M_2(\alpha, \gamma) = 1, M_1(\gamma, \psi) = 1$, is uniquely defined because of properties of finite projective planes. Conditions (H1a,b) follow. To prove (H2) we need to color sides of triangles in three different colors. This is possible since there are three vertices in the polyhedron with different graphs as links. So, all triangles from $T$ have one of three possible colorings. We need to show, that for any $\alpha, \beta \in T$ we can choose $r > 0$ such that $M_r(\alpha, \beta) > 0$, where $j = 1, 2$. Geometrically it means that any $\alpha, \beta \in T$ can be realized so that $\beta$ lies in some sector with base $\alpha$ (for more details see [13]). Without loss of generality we can assume, that $j = 1$. We will say, that $\beta \in T$ is reachable from $\alpha \in T$ in $r$ steps, if there is $r > 0$ such that $M_r(\alpha, \beta) > 0$. It is easy to see, that every triangle is reachable from some triangle of other color in one or two steps. So, to prove (H2) we need to show, that any triangle is reachable from another one of the same color. Now we can use the proof of the Theorem
1.3 from [18], since at each step of this proof it is only used, that the link at each vertex of the building is an incidence graph of a finite projective plane, which is true in our case too. The proof of (H3) is identical to the proof of (H3) in the case of the subshift considered in [19].

Now, as a set of triangles we consider all elements of $T_1$, every cyclic word $(a_i, b_j, c_k) \in T_1$ brings three basepointed triangles (Figure 2).

**Lemma 2.** The set $M(S^s)$ consists of $q - 1$ copies of every element of $T_1^b$ and one copy of $S^b$.

Denote $S^s$ tiles of $S$ starting with $a_i, i = 1, \ldots, g^2 + q + 1$ and analyse, which elements of $T_1$, and how many of them can be obtained by one left shift from $S^s$. In general, from each tile in $A_1$ case, one can obtain $q^2$ tiles by one left(right) shift as a consequence of properties of finite projective planes (see [RS] for details). Now, each tile $\gamma \in T_1^b$ can be obtained from some tile $\alpha \in S^s$ by $q^2$ times. So, the total number of tiles which can be obtained from $S^s$ is $q^2(q^2 + q + 1)$. Since every $c_i$ appears exactly once, every $\gamma \in T_1^b$ will appear in $M(S^s)$ exactly $q$ times if $\gamma \in S^b$ and $q - 1$ otherwise. So, the set $M(S^s)$ consists of $q - 1$ copies of every element of $T_1^b$ and one copy of $S^b$.

Two subsequent copies of this lemma one gets by substitution $a$ by $b$, $b$ by $c$ and $a$ by $c, b$ by $a$.

### 3 The class of the identity in $K$-theory.

**Proof of the Theorem.**

It was shown in [17], that the $K$-theory of the $C^*$-algebra can be found as the abelian group with generators, which are the elements of the alphabet $T_1^b$ with the following relations:

$$ t = \sum_{s \in T_1} M_1(s, t) $$

It follows from [8], [17], that the identity function in $C(\Omega)$ can be expressed as the sum of all tiles of the alphabet $T_1$,

$$ \sum_{s \in T_1} t. $$

So, we will use the system of relations

$$ t = \sum_{s \in T_1} M_1(s, t) $$
to express \( \sum_{t \in T} t \).

It follows from Lemma 2, that
\[
\sum_{t \in S} a_t = (q-1) \sum_{t \in T} b_t + \sum_{t \in S} b_t
\]
\[
\sum_{t \in S} b_t = (q-1) \sum_{t \in T} c_t + \sum_{t \in S} c_t
\]

By addition of these three equalities we get
\[
(q-1) \sum_{t \in T} t = 0,
\]
so \((q-1)1 = 1\). But it was shown in [17], that the order of 1 is at least \(q-1\) in the case when \(q \equiv 1 (\text{mod} \ 3)\), what completes the proof.

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