Fuzzy balanced allocation problem with efficiency on servers

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Abstract

This paper deals with the problem of allocation customers to servers with regards to some fuzzy parameters. In this problem each customer is allocated to the nearest server, and assignment of a customer to a server involves the cost to the customer, which is due to the customer's fuzzy distance to the server. Each server has a fuzzy efficiency which is calculated by the data envelopment analysis method with fuzzy parameters. The higher efficiency of the server to which a customer is assigned, cause more profit for the customer. The goal is allocation of all customers to the servers such that the profitability of the least profits for the customers is maximized. In addition, to prevent queuing in some servers, we consider the balancing on allocation customers to the servers. Therefore, the second goal is minimizing the difference between the maximum and minimum number of customers that are assigned to different servers. A fuzzy bi-objective programming model is presented for the problem, then two fuzzy approaches are proposed for solving this model.

Keywords

Allocation; data envelopment analysis (DEA); balancing; fuzzy model

1. Introduction

Data envelopment analysis (DEA) is one of the active areas of decision making. It is a mathematical programming method to measure the efficiency score of a set of homogeneous decision-making units (DMUs) based on observed input and output. The first DEA method is presented by Charnes, Cooper, and Rhodes (CCR) [4] to evaluate the relative efficiency for not-for-profit organizations. So far, many studies have been devoted on various cases of DEA and its applications. On the other side, assignment problem is one of the most used issue in transportation models and location theory. In this problem the goal is assigning customers to servers with minimum cost. The simple model of assignment problem can be solved by Hungarian method which is presented by Kuhn [15]. The problem of finding the location of servers and allocation customer to servers is an extension of assignment problem which called location-allocation problem. In recent years, balanced facility location problems have been gained more attentions. These kinds of facility location models try to find the location of facilities such that the equality in serving to the clients is maximized. Among many researchers in this area, Gavalec and Hudec [10] considered the case of balancing location model such that the maximum difference of the distance between a demand point and its farthest and nearest facility is maximized. Berman et al. [3] proposed some methods for locating $p$ facilities such that the maximum customers that assigned to each facility is minimized. Marin [18] investigated the case of balanced location model that the difference between the number of maximum and minimum customers assigned to different servers is minimized. The balanced 2-median and 2-maxin facility location problems on tree networks, have been considered by Fathali and Zaferanieh [8].

Recently, combinations of DEA and location models has been considered by many researchers. The first serious study on the efficiency of the servers in location models has been done by Thomas et al. [25]. They presented two approaches. In the first approach, they find the optimal location of facilities, then these optimal facilities are used as the input of the DEA model. If the efficiency of DEA model is unity, then the optimal solution is found. Otherwise, the optimal location of new facilities should be found. This method continues until all facilities are

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considered or all DEA scores are efficient. In the second approach, the DEA model and location problem have been considered as a single objective linear programming model to maximize the efficiency of those facilities that are going to be opened. Klimberg & Ratick [14] used the DEA concept for finding the efficient location of facilities. They presented two bi-objective linear programming models for capacitated and un-capacitated facility location models which combined with DEA models. They considered optimization of both spatial interaction between facilities and the customers, and the efficiency of the selected facilities, simultaneously. Smith et al. [20] proposed a bi-criteria efficiency/equity objectives models. Batta et al. [1] presented a location model that deals with the equity and efficiency simultaneously to demonstrate the appropriate use of population, dispersion, and equity criteria. Khodaparast et al. [13] combined the equity servicing and selection efficient facility location for designing the service systems in the public sector. Sayar et al. [19] considered the model of combination of balancing allocation and DEA in which its goal is allocation clients to the servers such that the lowest profit from this allocation is maximized for each customer.

In the real world, it is not always possible to determine the exact parameters of the problem, such as the amount of customer demand and the exact location of customers and facilities, the amount of costs, the creation of facilities, and so on. Models that deal with location-allocation in the space of uncertainty are divided into two categories: probabilistic and fuzzy models. The principle of fuzzy theory was first presented in 1986 [27] by professor Lotfi Asgarzadeh. Fuzzy verb means vague, ambiguous, inaccurate and generally complex. Today, fuzzy theory is widely used in many fields such as management, medicine, operations research and many branches of engineering sciences. Many researches have also been done in fuzzy and probability space on location-allocation issues. Zhou and Liu [28] modeled the location-allocation problem of limited capacity facilities on fuzzy demand for customers. Van and Iwamura [26] proposed an intelligent hybrid algorithm for the problem of location-allocation with fuzzy demand, which uses the simplex algorithm, fuzzy simulation and genetic algorithm. There are many papers on fuzzy location models, see e.g. [5], [6],[21], [22] , [23] and [24]. Guo and Tanaka [11] presented a possible linear programming model in which the coefficients of the definite decision variables and the variables themselves are obtained from fuzzy numbers.

In what follows, a background on fuzzy sets and fuzzy decision making environments are given in Section 2. Section 3, contains the problem definition and mathematical model for balancing allocation problem with fuzzy efficiency on servers and the introduction of a solution to this problem based on the fuzzy interactive approach. In Section 4, we demonstrate the efficiency of this method by providing an example. Finally, Section 5 contains conclusion and conjectures for future researches.

2. Preliminaries

This section contains some basic concepts and definitions of fuzzy set theory, and fuzzy ranking function.

2.1 Fuzzy sets and fuzzy numbers

The following fuzzy concepts are taken from [16].

Definition 1 Let X be the universal set. The set $\tilde{A}$ is called a fuzzy set in X if $\tilde{A}$ is a set of ordered pairs $\tilde{A} = \{(x, \mu_{\tilde{A}}(x))| x \in X\}$; where $\mu_{\tilde{A}}(x)$ is the membership function of $x \in \tilde{A}$. Note that $\mu_{\tilde{A}}(x)$ is a characteristic function for $\tilde{A}$ and it indicates the degree of belonging $x$ to $\tilde{A}$.

Definition 2 The $\alpha$-level set of $\tilde{A}$ is the set $\tilde{A}_\alpha = \{x \in R| \mu_{\tilde{A}}(x) \geq \alpha\}$, where $\alpha \in [0,1]$. The lower and upper bounds of $\alpha$-level set of $\tilde{A}$ are finite numbers represented by $\inf x \in \tilde{A}_\alpha$ and $\sup x \in \tilde{A}_\alpha$, respectively.
**Definition 3** The support of a fuzzy set $\tilde{A}$ is a set of elements in $X$ for which $\mu_{\tilde{A}}(x)$ is positive, that is, $\text{supp} \tilde{A}_x = \{x \in R | \mu_{\tilde{A}}(x) > 0\}$.

**Definition 4** A fuzzy set $\tilde{A}$ is convex if $\mu_{\tilde{A}}(\lambda x + (1 - \lambda)y) \geq \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)\}$ for all $x, y \in X$ and $\lambda \in [0,1]$. 

**Definition 5** A convex fuzzy set $\tilde{A}$ on $R$ is a fuzzy number if the following conditions hold:  
(a) Its membership function is piecewise continuous.  
(b) There exist three intervals $[a, b]$, $[b, c]$ and $[c, d]$ such that $\mu_{\tilde{A}}$ is increasing on $[a, b]$, equal to 1 on $[b, c]$, decreasing on $[c, d]$ and equal to 0 elsewhere.

**Definition 6** The fuzzy number $\tilde{A} = (a^L, a^U, \alpha, \beta)$ is called a trapezoidal fuzzy number, where $[a^L, a^U + \beta]$ is the support of $\tilde{A}$ and $[a^L, a^U]$ is its modal set. We denote the set of all trapezoidal fuzzy numbers by $F(R)$. If $\alpha = \beta$ then a symmetric trapezoidal fuzzy number is obtained. If $\alpha = a^L = a^U$ then a triangular fuzzy number is obtained, which is denoted by $\tilde{A} = (a, \alpha, \beta)$ (see Fig. 1).

For a trapezoidal fuzzy number $\tilde{A} = (a^1, a^2, a^3, a^4)$, the membership function is represented as follow:

$$
\mu_{\tilde{A}}(x) = \begin{cases} 
0 & x < a^1 \\
\frac{x - a^1}{a^2 - a^1} & a^2 - a^1 \leq x \leq a^2 \\
\frac{1}{a^3 - x} & 1a^2 \leq x \leq a^3 \\
\frac{a^4 - x}{a^4 - a^3} & a^3 \leq x \leq a^4 \\
0 & x > a^4
\end{cases}
$$

![Fig. 1. Trapezoidal fuzzy number.](image)

Let $\tilde{A} = (a^1, a^2, a^3, a^4)$ and $\tilde{B} = (b^1, b^2, b^3, b^4)$ be two non-negative trapezoidal fuzzy numbers. The arithmetic on fuzzy numbers are as follow:

$$
x\tilde{A} = (xa^1, xa^2, xa^3, xa^4), \quad \forall x \in R; \quad x \geq 0,
$$

$$
x\tilde{A} = (xa^2, xa^3, -xa^4, -xa^3), \quad \forall x \in R; \quad x < 0,
$$

$$
\tilde{A} \oplus \tilde{B} = (a^1 + b^1, a^2 + b^2, a^3 + b^3, a^4 + b^4),
$$

$$
\tilde{A} \ominus \tilde{B} = (a^1 - b^1, a^2 - b^2, a^3 - b^3, a^4 - b^4),
$$

where

$$
\begin{align*}
& t^1 = \min\{a^1b^1, a^1b^2, a^1b^3, a^1b^4\}, \\
& t^2 = \min\{a^2b^2, a^2b^3, a^2b^3, a^3b^3\}, \\
& t^3 = \max\{a^2b^2, a^2b^3, a^3b^2, a^3b^3\}, \\
& t^4 = \max\{a^1b^1, a^1b^2, a^4b^1, a^4b^4\},
\end{align*}
$$

$$
\tilde{B}^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ b^4 & b^3 & b^2 & b^1 \end{pmatrix}.
$$
\[
\frac{A}{B} = A \otimes B^{-1}.
\]

2.2 Ranking function

Among the various fuzzy ranking methods, the most appropriate method is based on the concept of comparison of fuzzy numbers by using the ranking functions (see e.g. [9,17]). Indeed, an efficient method for ordering the elements of \(F(\mathbb{R})\) is defining a ranking function \(R: F(\mathbb{R}) \to \mathbb{R}\) which maps each fuzzy number into the real numbers.

Let \(\tilde{A}\) and \(\tilde{B}\) be two fuzzy numbers on \(\mathbb{R}\), then the function \(R\) is called a fuzzy ranking function if

\[
\begin{align*}
\tilde{A} &\geq \tilde{B} \iff R(\tilde{A}) \geq R(\tilde{B}), \\
\tilde{A} &> \tilde{B} \iff R(\tilde{A}) > R(\tilde{B}), \\
\tilde{A} &\approx \tilde{B} \iff R(\tilde{A}) = R(\tilde{B}).
\end{align*}
\]

In this paper, the following ranking function for trapezoidal fuzzy number \(\tilde{A} = (a^1, a^2, a^3, a^4)\), is used:

\[
R(\tilde{A}) = \frac{a^1 + a^2 + a^3 + a^4}{4}.
\]

So, for trapezoidal fuzzy numbers \(\tilde{A} = (a^1, a^2, a^3, a^4)\) and \(\tilde{B} = (b^1, b^2, b^3, b^4)\), we have

\[
\tilde{A} \geq \tilde{B} \iff a^1 + a^2 + a^3 + a^4 \geq b^1 + b^2 + b^3 + b^4.
\]

3. Problem definition and proposed model

Let \(X = \{A_1, ..., A_n\}\) be the set of location of \(n\) customers with coordinates \(A_i = (a_i, b_i), i = 1, ..., n\), and \(P = \{p_1, ..., p_K\}\) be the set of location of \(K\) servers on the plane. We suppose that \(K < n\). Also, for each server \(p_k\) has \(m\) trapezoidal fuzzy inputs\((l_{1k}^1, l_{2k}^1, l_{3k}^1, l_{4k}^1), (l_{1k}^2, l_{2k}^2, l_{3k}^2, l_{4k}^2), ..., (l_{1m}^k, l_{2m}^k, l_{3m}^k, l_{4m}^k)\) \) and \(s\) trapezoidal fuzzy outputs \((o_{1k}^1, o_{2k}^1, o_{3k}^1, o_{4k}^1), (o_{1k}^2, o_{2k}^2, o_{3k}^2, o_{4k}^2), ..., (o_{1sk}, o_{2sk}, o_{3sk}, o_{4sk})\). Based on these inputs and outputs data and using data envelopment analysis method, the trapezoidal fuzzy efficiency score \(\tilde{E}_k\) of the server \(p_k\) is calculated as follows:

\[
\tilde{E}_k = (E_{k1}^1, E_{k1}^2, E_{k2}^3, E_{k2}^4) = \frac{\sum_{r=1}^{s} u_r (O_{1k}^r, O_{2k}^r, O_{3k}^r, O_{4k}^r)}{\sum_{i=1}^{m} v_i (l_{1k}^i, l_{2k}^i, l_{3k}^i, l_{4k}^i)}.
\]

Where \(u_r\) and \(v_i\) are the corresponding weights of the \(rth\) output and \(ith\) input, respectively.

The demand of \(ith\) customer, for \(i = 1, ..., n\), is assumed to be a constant value \(a_i\). It is also assumed that each customer is served by one server. For \(i = 1, ..., n and j = 1, ..., K\) suppose that \(\tilde{c}_{ij} = (c_{ij}^1, c_{ij}^2, c_{ij}^3, c_{ij}^4)\) is the trapezoidal fuzzy cost of per unit of the fuzzy distance between customer \(i\) and facility server \(j\). On the other hand, assigning customers to servers with high efficiency, will also be profitable for the customers. The goal is to assign customers to the servers considering to the following two objective functions: 1- maximizing the lowest profit from allocation clients to the servers for each customer, 2- minimizing the difference between the maximum and minimum number of customers allocated to the servers. The second objective function prevents the long queue for taking service which is ultimately benefits for the customers and balances the customer burden assigned to the servers. This problem, in fact is the fuzzy version of the problem that introduced by Sayar et al. [19].

The following notations are used in this paper to model the considered problem:

**Fuzzy Parameters:**

\[
\tilde{d}(A_i, p_j) = (d_{ij}^1, d_{ij}^2, d_{ij}^3, d_{ij}^4): \text{The trapezoidal fuzzy distance between } ith \text{ customer and } jth \text{ server.}
\]
\( \tilde{c}_{ij} = (c_{ij}^1, c_{ij}^2, c_{ij}^3, c_{ij}^4) \): The trapezoidal fuzzy cost of traveling per unit of fuzzy of distance from the \( i \)th customer to the \( j \)th server.

\( \tilde{E}_j = (E_j^1, E_j^2, E_j^3, E_j^4) \): The trapezoidal fuzzy efficiency score of \( j \)th server.

**Crisp parameters:**

\( y \): Profit from each unit of efficiency for customers.

\( cap_j \): Capacity of \( j \)th server.

**Variables:**

\( x_{ij} \): A binary variable which is 1, if \( i \)th customer is assigned to \( j \)th server and zero otherwise.

\( U \): The relative of maximum number of customers that is assigned to a server to the server capacity.

\( L \): The relative of minimum number of customers that is assigned to a server to the server capacity.

The problem can be modeled as the following bi-objective fuzzy programming.

\[
\begin{align*}
\max f_1 &= \min_{i=1,...,n} \sum_{j=1}^{k} \left[ y(E_j^1, E_j^2, E_j^3, E_j^4) - \left( (c_{ij}^1, c_{ij}^2, c_{ij}^3, c_{ij}^4)(d_{ij}^1, d_{ij}^2, d_{ij}^3, d_{ij}^4) \right) \right] x_{ij} \\
\min f_2 &= U - L \\
\text{s.t.} & \sum_{j=1}^{n} x_{ij} = 1 \quad i = 1,2,\ldots,n \\
& \sum_{i=1}^{n} x_{ij} \leq cap_j \quad j = 1,2,\ldots,k \\
& \sum_{i=1}^{n} x_{ij} \leq U cap_j \quad j = 1,2,\ldots,k \\
& \sum_{i=1}^{n} x_{ij} \geq L cap_j \quad j = 1,2,\ldots,k \\
& 0 \leq U, L \leq 1 \\
& x_{ij} \in \{0,1\} \quad i = 1,\ldots,n \quad j = 1,\ldots,k
\end{align*}
\]

Where objective function (1) indicates the maximization of the minimum fuzzy profit from allocation of \( i \)th customer to \( j \)th server. This benefit is derived from the fuzzy profitability of the server for the customer, minus the fuzzy cost of traveling between servers and customers. Objective function (2) indicates the minimization of difference between the maximum and minimum number of customers assigned to the different servers. This objective function refers to the equity problem. Constraint (3) ensures that each client is only assigned to one server. Constraint (4) indicates the capacity of facility \( j \). Constraints (5) and (6) concern with balance conditions. The main form of these conditions are as follows

\[
\frac{\sum_{i=1}^{n} x_{ij}}{cap_j} \leq U \quad j = 1,\ldots,K
\]
They identify the maximum and minimum number of customers that assigned to servers, respectively.

For simplicity, let

\[ \bar{y} = (y^1, y^2, y^3, y^4) \]

\[ = \min_{i=1,...,n} \sum_{j=1}^{k} \left[ y(E_j^1, E_j^2, E_j^3, E_j^4) - \left( (c_{1ij}, c_{2ij}, c_{3ij}, c_{4ij}) (d_{1ij}, d_{2ij}, d_{3ij}, d_{4ij}) \right) \right] x_{ij} \]

Then by adding the following constraints to model (P₁);

\[ \sum_{j=1}^{k} \left[ y(E_j^1, E_j^2, E_j^3, E_j^4) - \left( (c_{1ij}, c_{2ij}, c_{3ij}, c_{4ij}) (d_{1ij}, d_{2ij}, d_{3ij}, d_{4ij}) \right) \right] x_{ij} \geq (y^1, y^2, y^3, y^4) \quad i = 1, \ldots, n \]

the first objective function of model (P₁) can be replaced by

\[ \max \quad f_1 = (y^1, y^2, y^3, y^4). \]

Thus model (P₁) is converted to the following model:

\[ P_2: \]

\[ \begin{align*}
\max f_1 &= (y^1, y^2, y^3, y^4) \\
\min f_2 &= U - L \\
st. & \sum_{j=1}^{k} \left[ y(E_j^1, E_j^2, E_j^3, E_j^4) - \left( (c_{1ij}, c_{2ij}, c_{3ij}, c_{4ij}) (d_{1ij}, d_{2ij}, d_{3ij}, d_{4ij}) \right) \right] x_{ij} \\
& \geq (y^1, y^2, y^3, y^4) \quad i = 1, \ldots, n \\
\end{align*} \]  

Constraints (3) to (8).

4. Fuzzy approaches for solving the problem

In this section, we present two fuzzy approaches to satisfy objective functions. In the first method we use the max-min operator of Belman and Zadeh [3] and the second method is using lexicographic approach for solving fuzzy linear programming models.

4.1. Max-min operator method

As mentioned in [12], in a fuzzy decision problem when fuzzy sets are defined on a set of alternatives, X, fuzzy objective function G and fuzzy constraint C in X may be identified with the given fuzzy sets G and C in X, respectively. For the given fuzzy objective function and fuzzy constraints, a decision making situation in a fuzzy environment can be defined as the intersection of objective function and constraint. Considering the space of decision alternatives X, the fuzzy decision D is defined as a fuzzy set in X given by \( D = G \cap C \) where \( \cap \) is a conjunctive operator in fuzzy environments. In terms of the membership functions, the fuzzy decision can be characterized as

\[ \mu_D(x) = \min (\mu_C(x), \mu_G(x)), \forall x \in X, \]

Where \( \mu_G(x) \) and \( \mu_C(x) \) are the membership functions of the fuzzy objective function and the fuzzy constraint, respectively. In the case that there are \( m \) fuzzy objective functions \( G_i \ (i = 1, \ldots, m) \) and \( n \) fuzzy constraints \( C_j \ (j = 1, \ldots, n) \), then the fuzzy decision is defined as follow,

\[ D = \{G_1 \cap G_2 \cap \ldots \cap G_m\} \cap \{C_1 \cap C_2 \cap \ldots \cap C_n\}. \]

Also, its membership function is formulated as follow

\[ \sum_{i=1}^{n} \frac{x_{ij}}{cap_j} \geq L \quad j = 1, \ldots, K \]
\[
\mu_D(x) = \min \left( \mu_{G_1}(x), \mu_{G_2}(x), \ldots, \mu_{G_m}(x), \mu_{C_1}(x), \mu_{C_2}(x), \ldots, \mu_{C_n}(x) \right), \forall x \in X,
\]

Bellman and Zadeh [3] defined the following non-fuzzy set for maximizing decision \( x^* \),
\[
D = \{ x^* \in X | x^* = \arg \max \{ \mu_D(x) \} = \arg \max \{ \min \{ \mu_G(x), \mu_C(x) \} \} \}.
\]

More generally, in the case that \( m \) fuzzy objective functions and \( n \) fuzzy constraints are given, the optimal decision can be calculated as follows:
\[
D = \{ x^* \in X | x^* = \arg \max \{ \mu_D(x) \} = \arg \max \{ \min \{ \mu_{G_1}(x), \mu_{G_2}(x), \ldots, \mu_{G_m}(x), \mu_{C_1}(x), \mu_{C_2}(x), \ldots, \mu_{C_n}(x) \} \} \}.
\]

In this interactive approach, the Belman and Zadeh [3] max-min operator is used and continues with a number of repetitions that may be necessary to reach a preferred agreement. In this method, first the fuzzy numbers are converted to crisp numbers using the ranking function. Then each of the obtained crisp single-objective function are solved. Finally, by evaluating the answer of both objective functions, the worst (best) lower boundaries and the best (worst) upper boundaries is specified for each of the objectives.

To apply the Belman and Zadeh [3] method in our problem, suppose \( \tilde{y} = (y^1, y^2, y^3, y^4) \) be a trapezoidal fuzzy number, then using considered ranking function we obtain
\[
R(\tilde{y}) = \frac{y^1 + y^2 + y^3 + y^4}{4}.
\]

Then, by solving the following problem the worst lower bound \( f^L_3 \) and the best upper bound \( f^R_3 \) for the objective function are calculated.

\[
P_{21}: \quad \begin{align*}
\max f_3 &= \frac{1}{4} (y^1 + y^2 + y^3 + y^4) \\
\text{s.t.} & \quad \sum_{j=1}^{k} \left[ y(E_j^1 + E_j^2 + E_j^3 + E_j^4) - \frac{1}{4} (c_{ij}^1 + c_{ij}^2 + c_{ij}^3 + c_{ij}^4)(d_{ij}^1 + d_{ij}^2 + d_{ij}^3 + d_{ij}^4) \right] x_{ij} \\
& \quad \geq y^1 + y^2 + y^3 + y^4 \quad i = 1,2,\ldots,n
\end{align*}
\]

Constraints (3) to (8).

Also, the worst lower bound \( f^L_2 \) and the best upper bound \( f^R_2 \) for the objective function of the following problem are calculated.

\[
P_{22}: \quad \begin{align*}
\min f_2 &= U - L \\
\text{s.t.} & \quad \sum_{j=1}^{k} \left[ y(E_j^1 + E_j^2 + E_j^3 + E_j^4) - \frac{1}{4} (c_{ij}^1 + c_{ij}^2 + c_{ij}^3 + c_{ij}^4)(d_{ij}^1 + d_{ij}^2 + d_{ij}^3 + d_{ij}^4) \right] x_{ij} \\
& \quad \geq y^1 + y^2 + y^3 + y^4 \quad i = 1,2,\ldots,n
\end{align*}
\]

Constraints (3) to (8).

Then using these boundaries, the membership function of each of the objective functions in our considered problem are as follows:
\[\mu_{f_3}(\bar{y}) = \begin{cases} 
1 & R(\bar{y}) \geq f_3^R, \\
\frac{R(\bar{y}) - f_3^L}{f_3^R - f_3^L} & f_3^L \leq R(\bar{y}) \leq f_3^R, \\
0 & R(\bar{y}) \leq f_3^L, 
\end{cases}\]

\[\mu_{f_2}(U, L) = \begin{cases} 
1 & U - L \leq f_2^L, \\
\frac{f_2^R - U + L}{f_2^R - f_2^L} & f_2^L \leq U - L \leq f_2^R, \\
0 & U - L \geq f_2^R. 
\end{cases}\]

Therefore, using the max-min Belman and Zadeh operators, the following problem is considered:

\[P_3:\]
\[
\max \lambda \\
\text{s.t.} \\
\lambda \leq \mu_{f_3}(\bar{y}) \\
\lambda \leq \mu_{f_2}(U, L) \\
\sum_{j=1}^{k} \left[ y(E_j^1 + E_j^2 + E_j^3 + E_j^4) - \frac{1}{4} (c_{ij}^1 + c_{ij}^2 + c_{ij}^3 + c_{ij}^4)(d_{ij}^1 + d_{ij}^2 + d_{ij}^3 + d_{ij}^4) \right] x_{ij} \\
\geq y^1 + y^2 + y^3 + y^4 \quad i = 1, 2, \ldots, n
\]

Constraints (3) to (8).

The objective function of problem \(P_3\) has been \(\max \min (\mu_{f_3}(\bar{y}), \mu_{f_2}(U, L))\), which has become \(\max \lambda\) and \(\lambda\) auxiliary variables that indicate the degree of membership.

By solving problem \(P_3\), if the obtained answer is satisfactory, we consider it as the preferred agreement answer, otherwise both objective functions are re-evaluated and the worst boundaries are edited. This approach by reducing the number of repetitions helps us reach a preferred agreement and also gives the most confidence from the obtained answer.

### 4.1. Lexicographic approach

Another method that we refer to in this article is a method for solving fuzzy linear programming models based on the lexicographic approach. This method is presented by Das et al. [7] using some properties of considered ranking function. Lexicographic method is a technique for solving multi-objective linear programming models. In this method several single-objective problems have been solved. In this way, first the objective function with higher priority considering the given constraints is solved. Then a constraint which is the first objective function with right hand side equal to the obtained value is added to the constraints, and solve the problem with the second objective function. After solving this problem, the corresponding constraint to the second objective function and its obtained value is added to the constraints and the third objective function is considered. This method is continued until the value of the last objective function is calculated.

To solve model \((P_2)\) using method of Das et al. [7], based on the considered ranking function, the first fuzzy objective function of this problem (i.e., \(\max f_1 = \bar{y} = (y^1, y^2, y^3, y^4)\)) is transformed into four crisp objective functions as follows:

\[\min f_4 = (y^2 - y^1)\]

\[\max f_5 = y^2\]
\[
\begin{align*}
\max f_6 &= \frac{1}{2}(y^2 + y^3) \\
\max f_7 &= (y^4 - y^3)
\end{align*}
\]

Also, for \( i = 1, \ldots, n \), constraints (10) are written as follows:

\[
\begin{align*}
[yE_j^1 - (c_{ij}^1 d_{ij}^1)]x_{ij} &\geq y^1_i, \\
[yE_j^2 - (c_{ij}^2 d_{ij}^2)]x_{ij} &\geq y^2_i, \\
[yE_j^3 - (c_{ij}^3 d_{ij}^3)]x_{ij} &\geq y^3_i, \\
[yE_j^4 - (c_{ij}^4 d_{ij}^4)]x_{ij} &\geq y^4_i.
\end{align*}
\]

Therefore, model \( P_2 \) is converted to the following five-objective crisp problem.

\[
P_4: \quad \begin{align*}
\min f_4 &= (y^2 - y^1) \\
\max f_5 &= y^2 \\
\max f_6 &= \frac{1}{2}(y^2 + y^3) \\
\max f_7 &= (y^4 - y^3) \\
\min f_2 &= U - L \\
\text{s.t.} \quad & \begin{align*}
[yE_j^1 - (c_{ij}^1 d_{ij}^1)]x_{ij} &\geq y^1_i, & i = 1, \ldots, n \\
[yE_j^2 - (c_{ij}^2 d_{ij}^2)]x_{ij} &\geq y^2_i, & i = 1, \ldots, n \\
[yE_j^3 - (c_{ij}^3 d_{ij}^3)]x_{ij} &\geq y^3_i, & i = 1, \ldots, n \\
[yE_j^4 - (c_{ij}^4 d_{ij}^4)]x_{ij} &\geq y^4_i, & i = 1, \ldots, n \\
y^1 &\geq 0, & y^2 - y^1 &\geq 0, & y^3 - y^2 &\geq 0, & y^4 - y^3 &\geq 0
\end{align*}
\]

Constraints (3) to (8).

Then, this problem is solved using lexicographic approach.

5. Numerical example

To explain presented methods, consider the following example.

**Example 1** Consider a problem with \( n = 2 \) customers and \( m = 3 \) facility servers. Let the values of fuzzy and crisp parameters be given as follows:

\[
\begin{align*}
\tilde{d}_{11} &= (0,1,2,3), \tilde{d}_{12} = (1,2,4,5), \tilde{d}_{13} = (2,3,4,5), \tilde{d}_{21} = (1,2,3,4), \tilde{d}_{22} = (0,2,4,5), \tilde{d}_{23} = (2,4,5,6), \\
\tilde{c}_{11} &= (1,3,4,5), \tilde{c}_{12} = (2,0,3,4), \tilde{c}_{13} = (1,3,4,5), \tilde{c}_{21} = (2,3,4,5), \tilde{c}_{22} = (0,1,2,4), \tilde{c}_{23} = (1,2,4,5), \\
\gamma &= 3, cap_1 = 3, cap_2 = 5, cap_3 = 4.
\end{align*}
\]

Let efficiency values are also obtained using fuzzy input and output variables as follows:

\[
\tilde{E}_1 = (7,8,9,13), \tilde{E}_2 = (7,9,10,14), \tilde{E}_3 = (10,11,12,13).
\]

Thus

\[
\tilde{y} = \min([((21,24,27,39) - (1,3,4,5)(0,1,2,3))x_{11} + \left((21,27,30,42) - (0,2,3,4)(1,2,4,5)\right)x_{12} + \left((30,33,36,39) - (1,3,4,5)(2,3,4,5)\right)x_{13} + \left((21,24,27,39) - (2,3,4,5)(1,2,3,4)\right)x_{21} + \left((21,27,30,42) - (0,1,2,4)(2,3,4,5)\right)x_{22} + \left((30,33,36,39) - (1,2,4,5)(2,4,5,6)\right)x_{23}).
\]

Using Belman and Zadeh [3] method, first the models \( P_{21} \) and \( P_{22} \) should be solved. The optimal solutions of these two single-objective problems are given in Table 1.
Now evaluate the values of both objective functions and obtain the worst and best boundaries for both functions as follows:

\[ 23.125 \leq R(\bar{y}) \leq 23.25 \]
\[ 0.25 \leq U - L \leq 1 \]

Then, the linear membership functions for both goals are obtained as follows:

\[
\mu_{f_3}(\bar{y}) = \begin{cases} 
1 & R(\bar{y}) \geq 23.25 \\
\frac{R(\bar{y}) - 23.125}{23.25 - 23.125} & 23.125 \leq R(\bar{y}) \leq 23.25 \\
0 & R(\bar{y}) \leq 23.125 
\end{cases}
\]

\[
\mu_{f_2}(U, L) = \begin{cases} 
1 & U - L \leq 0.25 \\
\frac{1 - U + L}{0.25 - 0} & 0.25 \leq U - L \leq 1 \\
0 & U - L \geq 1 
\end{cases}
\]

The optimal solution of problem \((P_3)\) using these membership functions is reported in Table 2.

| \(\lambda\) | \(R(\bar{y})\) | \(U\) | \(L\) | \(x_{11}\) | \(x_{12}\) | \(x_{13}\) | \(x_{21}\) | \(x_{22}\) | \(x_{23}\) |
|---|---|---|---|---|---|---|---|---|---|
| 0.8 | 23.225 | 0.4 | 0 | 0 | 1 | 0 | 0 | 1 | 0 |

Table 2. The solution of problem \((P_3)\) in the first iteration.

As one can see, the value of the first objective function \((R(\bar{y}) = 23.225)\) is greater than the worst lower bound. So, we consider it as the new lower bound. Also, the value of the second objective function \((U - L = 0.4)\) is less than the worst upper bound. Thus, we consider it as a new upper bound. Then

\[ 23.225 \leq R(\bar{y}) \leq 23.25 \]
\[ 0.25 \leq U - L \leq 0.4 \]

Then we again compute the linear membership functions for the first and second objective functions as follow:

\[
\mu_{f_3}(\bar{y}) = \begin{cases} 
1 & R(\bar{y}) \geq 23.25 \\
\frac{R(\bar{y}) - 23.225}{23.25 - 23.225} & 23.225 \leq R(\bar{y}) \leq 23.25 \\
0 & R(\bar{y}) \leq 23.225 
\end{cases}
\]

\[
\mu_{f_2}(U, L) = \begin{cases} 
1 & U - L \leq 0.25 \\
\frac{0.4 - U + L}{0.25 - 0} & 0.25 \leq U - L \leq 0.4 \\
0 & U - L \geq 0.4 
\end{cases}
\]
The optimal solution of problem \((P_3)\) in this iteration is reported in Table 3.

| \(\lambda\) | \(R(\hat{y})\) | \(U\) | \(L\) | \(x_{11}\) | \(x_{12}\) | \(x_{13}\) | \(x_{21}\) | \(x_{22}\) | \(x_{23}\) |
|-------------|----------------|------|------|---------|---------|---------|---------|---------|---------|
| 0           | 23.225         | 0.4  | 0    | 0       | 1       | 0       | 0       | 1       | 0       |

Table 3. The final optimal solution of Example 1 using max-min operator.

Since the value of the first objective function is equal to the worst lower bound and the value of the second objective function is equal to the worst upper bound, thus the optimal solution is obtained. Therefore, \(f_1 = 0.4\), \(f_3 = 23.225\).

Now, let we solve Example 1 using lexicographic approach of Das et al. [7]. According to their method the model \((P_4)\) for data in Example 1, is written as follows:

\[
\begin{align*}
\min f_4 &= (y^2 - y^1) \\
\max f_5 &= y^2 \\
\max f_6 &= \frac{1}{2}(y^2 + y^3) \\
\max f_7 &= (y^4 - y^3) \\
\min f_2 &= U - L \\
6x_{11} + x_{12} + 5x_{13} &\geq y^1 \\
16x_{11} + 15x_{12} + 17x_{13} &\geq y^2 \\
24x_{11} + 26x_{12} + 27x_{13} &\geq y^3 \\
39x_{11} + 42x_{12} + 37x_{13} &\geq y^4 \\
x_{21} + x_{22} + 0x_{23} &\geq y^1 \\
12x_{21} + 19x_{22} + 13x_{23} &\geq y^2 \\
21x_{21} + 28x_{22} + 28x_{23} &\geq y^3 \\
37x_{21} + 42x_{22} + 37x_{23} &\geq y^4 \\
\text{Constraints (3) to (8),} \\
y^1 \geq 0, y^2 - y^1 \geq 0, y^3 - y^2 \geq 0, y^4 - y^3 \geq 0.
\end{align*}
\]

By lexicographic method, first the problem with the objective function \(f_4\) is solved. We found that the optimal solution is zero. Then constraint \(y^2 - y^1 = 0\) is added to the previous constraints and the problem with the second objective function is solved. This time the optimal value is equal to one. So, we add the constraint of \(y^2 = 1\) to the previous constraints of the problem and solve it by considering the third objective function. By repeating this procedure, the optimal value of the third and fifth objective functions are \(\frac{1}{2}(y^2 + y^3) = 14\) and \(y^4 - y^3 = 10\), respectively. Finally, the optimal solution is obtained by considering the last objective function. The obtained optimal solution is reported in Table 3.

| \(\hat{y}\) | \(U\) | \(L\) | \(x_{11}\) | \(x_{12}\) | \(x_{13}\) | \(x_{21}\) | \(x_{22}\) | \(x_{23}\) |
|-------------|------|------|---------|---------|---------|---------|---------|---------|
| (1.1,27.37) | 0.25 | 0    | 0       | 1       | 0       | 1       | 1       | 0       |

Table 3. The optimal solution of Example 1 using lexicographic approach.

### 6. Comparing presented methods

To compare two presented method, first consider the obtained results of Example 1. Let \((\hat{y}_B, U_B, L_B)\) and \((\hat{y}_D, U_D, L_D)\) be the obtained solutions of objective function (9), by methods of Belman and Zadeh [3] and Das et al. [7], respectively. By comparing ranking functions of \(\hat{y}_B\) and \(\hat{y}_D\), we found that

\[R(\hat{y}_B) = 23.225 > R(\hat{y}_D) = 16.5.\]

On the other hand, \(U_B - L_B = 0.4 \geq U_D - L_D = 0.25\).
Therefore, for the data in Example 1, the value of the first objective function obtained by the Belman and Zadeh [3] max-min operator is better than those obtained by the Das et al. [7] method. However, the second value of the objective function obtained by Das et al. [7] method is better than max-min operator. We examined some other experiments by both methods, but no significant superiority was found for the two methods.

Second, the lexicographic method of Das et al. [7] is only valid for non-negative fuzzy variables. To investigate this, consider the following example.

**Example 2:** Let the fuzzy values and crisp parameters be given as follows:

\[
\hat{d}_{11} = (2,3,4,4), \hat{d}_{12} = (4,6,7,9), \hat{d}_{13} = (2,4,5,7), \hat{d}_{21} = (3,5,6,10), \hat{d}_{22} = (4,6,8,9), \hat{d}_{23} = (5,7,9,11),
\]

\[
\hat{E}_1 = (4,8,12,14), \hat{E}_2 = (2,10,12,18), \hat{E}_3 = (3,7,11,16).
\]

\[
\tilde{c}_{11} = (1,5,6,8), \tilde{c}_{12} = (1,3,4,6), \tilde{c}_{13} = (5,11,12,16), \tilde{c}_{21} = (4,6,8,10), \tilde{c}_{22} = (3,9,10,12), \tilde{c}_{23} = (3,7,8,11),
\]

\[
\gamma = 2, \text{cap}_1 = 3, \text{cap}_2 = 5, \text{cap}_3 = 4;
\]

By performing fuzzy multiplication and subtraction operations in constraints (10) some negative fuzzy values are obtained. Since in the method of Das et al. [7] non-negative fuzzy variables are considered, solving this example with their method does not have a feasible solution. But this example can be solved using the Belman and Zadeh [3] max-min operator. The optimal solution is reported in Table 4.

| ỹ | u | l | x_{11} | x_{12} | x_{13} | x_{21} | x_{22} | x_{23} |
|---|---|---|-------|-------|-------|-------|-------|-------|
| -36.375 | 0.25 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 |

Table 4. The final optimal solution of Example 2 using max-min operator

**7. Summary and conclusion**

In this paper we considered an optimization fuzzy problem which is a combination of balancing allocation with DEA models. In this problem the goal is balancing allocation clients to the servers such that the lowest profit from this allocation is maximized for each customer. We developed a mathematical model for balancing allocation problem with efficiency on servers. We also used symmetric trapezoidal fuzzy numbers because of the inaccuracy and quality of the data. Then we proposed and compare two fuzzy mathematical methods, namely max-min operator and lexicographic, for solving considered problem. Comparing the presented methods indicates that the max-min operator method is more efficient than lexicographic approach.

**Compliance with ethical standards**

**Conflict of interest** The authors declare that they have no conflict of interest.

**Ethical approval** This article does not contain any studies with human participants performed by any of the authors.

**Informed consent** Informed consent was obtained from all individual participants included in the study.
Authorship contributions

Azam Azodi, Jafar Fathali, Mojtaba Ghiyasi and Tahereh Sayar contributed to the design and implementation of the research, to the analysis of the results and to the writing of the manuscript.

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