Controlling an Alien Predator Population by Regional Controls

Sebastian Anița∗ Vincenzo Capasso† Gabriel Dimitriu‡

Abstract

We investigate the problem of minimizing the total cost of the damages produced by an alien predator population and of the regional control paid to reduce this population. The dynamics of the predators is described by a prey-predator system with either local or nonlocal reaction terms. A sufficient condition for the zero-stabilizability (eradicability) of predators is given in terms of the sign of the principal eigenvalue of an appropriate operator that is not self-adjoint, and a stabilizing feedback control with a very simple structure is indicated. The minimization related to such a feedback control is treated for a closely related minimization problem viewed as a regional control problem. The level set method is a key ingredient. An iterative algorithm to decrease the total cost is obtained and numerical results show the effectiveness of the theoretical results.

A spatially structured SIR problem may be described by the same system; in this case the above mentioned minimization problem is related to the problem of eradication of an epidemic by regional controls.

Keywords: Zero-stabilization; regional control; prey-predator system; SIR system.

1 Setting of the problem

Consider the following reaction–diffusion system which describes the dynamics of two interacting populations: prey and predator that are free to move in the habitat $\Omega \subset \mathbb{R}^2$ and are subject to a control acting in a subset $\omega \subset \Omega$ on the predators.

$$
\begin{align*}
\frac{\partial}{\partial t} h(x, t) - d_1 \Delta h(x, t) &= r(x) h(x, t) - \rho(x) h(x, t)^2 - h(x, t) (Bp(\cdot, t))(x), \quad (x, t) \in Q, \\
\frac{\partial}{\partial t} p(x, t) - d_2 \Delta p(x, t) &= -a(x) p(x, t) + c_0 h(x, t) (Bp(\cdot, t))(x) + \chi_\omega(x) u(x, t), \quad (x, t) \in Q, \\
\frac{\partial}{\partial \nu} h(x, t) &= \frac{\partial}{\partial \nu} p(x, t) = 0, \quad (x, t) \in \Sigma, \\
h(x, 0) &= h_0(x), \quad p(x, 0) = p_0(x),
\end{align*}
$$

(1.1)

∗Corresponding author. Faculty of Mathematics, “Alexandru Ioan Cuza” University of Iași, and “Octav Mayer” Institute of Mathematics of the Romanian Academy, Iași 700506, Romania. Email: sanita@uaic.ro.
†ADAMSS (Centre for Advanced Applied Mathematical and Statistical Sciences), Università degli Studi di Milano, 20133 Milano, Italy. Email: vincenzo.capasso@unimi.it.
‡Department of Medical Informatics and Biostatistics, University of Medicine and Pharmacy “Grigore T. Popa”, Iași 700115, Romania. Email: gabriel.dimitriu@umfiasi.ro.
Here $\Omega$ is a bounded domain (open and connected) with a sufficiently smooth boundary $\partial \Omega$, $\omega$ is an open subset, $Q = \Omega \times (0, +\infty)$, $\Sigma = \partial \Omega \times (0, +\infty)$. $h(x, t)$ and $p(x, t)$ are the spatial densities at time $t$ of the prey, respectively predator populations. The diffusion coefficients $d_1$ and $d_2$ are positive constants, $r(x)$ is the growth rate and $\rho(x)h(x, t)^2$ is a local logistic term of preys. $a(x)$ is the decreasing rate of the predator population. The quantity $h(x, t)(Bp(\cdot, t))(x)$ gives the density of captured prey population at position $x$, which is transformed into biomass via a conversion rate $c_0 \in (0, +\infty)$. Possible choices of the operator $B \in L(L^2(\Omega))$ will be discussed later.

The homogeneous Neumann boundary conditions describe the no flux of populations across the boundary of the habitat. $h_0(x)$ and $p_0(x)$ denote the initial prey and predator population densities at position $x$, respectively. The control $u$ acts on the predators only, in the subregion $\omega$; $\chi_\omega$ is the characteristic function of $\omega$.

We assume that

(A1) $r, \rho, a, h_0, p_0 \in L^\infty(\Omega)$, \( r(x) \geq r_0, \rho(x) \geq \rho_0 \) a.e. $x \in \Omega$ ($r_0, \rho_0$ are positive constants);

\[ h_0(x) \geq 0, \quad p_0(x) \geq 0 \quad \text{a.e.} \quad x \in \Omega, \]

and $h_0$ and $p_0$ are not identically zero.

(A2) $B \in L(L^2(\Omega)) \cap L(L^\infty(\Omega))$, $(B)y(x) \geq 0$ a.e. $x \in \Omega$, for any $y \in L^2(\Omega)$ such that $y(x) \geq 0$ a.e. $x \in \Omega$.

Two cases are of particular interest to us:

CASE 1. If $(B)y(x) = c(x)y(x)$ for $y \in L^2(\Omega)$, where $c \in L^\infty(\Omega)$, $c(x) \geq 0$ a.e. $x \in \Omega$, then the functional response to predation is of the usual Lotka-Volterra type.

CASE 2. If $(B)y(x) = \int_\Omega \kappa(x, x')y(x') dx'$ for $y \in L^2(\Omega)$, where $\kappa \in L^\infty(\Omega \times \Omega)$, $\kappa(x, x') \geq 0$ a.e. $(x, x') \in \Omega \times \Omega$, then the functional response to predation is such that predators, however coming from any position $x'$, upon predation at position $x$ will stay and produce offsprings at this new position (the predators follow the prey). For other prey-predator systems with nonlocal terms see [29].

We may notice that system (1.1) may also model a spatially structured SIR epidemic system, in which case $h(x, t)$ and $p(x, t)$ represent the spatial density of the susceptible and infective population, respectively. With $c_0 = 1$, CASE 1 presented above corresponds to a local infection rate, while CASE 2 corresponds to a nonlocal infection rate as proposed by D.G. Kendall [34] (see also [14], and [20]).

For the SIR system $r(x) = b(x) - \mu(x)$, where $b(x)$ is the birth rate and $\mu(x)$ is the natural death rate at position $x$; $a(x) = \mu(x) + \tilde{\mu}(x)$, where $\tilde{\mu}(x)$ is the additional removal rate due to the extra death rate caused by the disease and the possible natural recovery rate. The infective population
do not have offsprings and the recovered individuals acquire immunity. The control $u$ may describe the additional removal of infectives (because of either recovery by treatment or isolation) due to a planned regional intervention by the relevant public health system.

If we view $p$ as an alien pest population density or as an infective population density, then it is of great interest to know if there exists a control $u$ such that, for the solution $(h^u, p^u)$ to (1.1),

$$\lim_{t \to +\infty} p^u(\cdot, t) = 0 \text{ in an appropriate functional space.}$$

**Definition.** The predator population is zero-stabilizable (eradicable), if for any $h_0, p_0$ satisfying (A1), there exists $u \in L^\infty_{\text{loc}}(\Omega \times [0, +\infty))$ such that

$$h^u(x, t), \ p^u(x, t) \geq 0 \text{ a.e. } (x, t) \in Q, \quad (1.2)$$

and

$$\lim_{t \to +\infty} p^u(\cdot, t) = 0 \text{ in } L^\infty(\Omega). \quad (1.3)$$

We are dealing with zero-stabilizability (eradicability) with state constraints.

We will see that the problem of eradicability is deeply related to the sign of the principal eigenvalue $\lambda^\omega_{1, \gamma}$ for

$$\begin{cases}
-d_2 \Delta \Psi(x) + a(x) \Psi(x) + \gamma \chi_\omega(x) \Psi(x) - c_0 K(x)(B \Psi)(x) = \lambda \Psi(x), & x \in \Omega, \\
\partial_\nu \Psi(x) = 0, & x \in \partial \Omega,
\end{cases} \quad (1.4)$$

where $\gamma \in [0, +\infty)$, and $K$ is the unique maximal nonnegative solution to

$$\begin{cases}
-d_1 \Delta K(x) = r(x) K(x) - \rho(x) K(x)^2, & x \in \Omega, \\
\partial_\nu K(x) = 0, & x \in \partial \Omega.
\end{cases} \quad (1.5)$$

(Actually (1.5) has two nonnegative solutions, the trivial one and $K$; see [10]).

Notice that the elliptic operator in (1.4) is not self-adjoint, so we cannot use the same arguments as in [9] to derive the basic properties of $\lambda^\omega_{1, \gamma}$ and of the corresponding eigenspace. We may however apply the Krein-Rutman Theorem (for details see the Appendix).

Our strategy will be to diminish the pest (resp. infective) population using a bilinear control with a very simple structure $u := -\gamma p$. Here $\gamma \in [0, +\infty)$ represents a constant affordable predator elimination (resp. treatment or isolation) rate. If we consider the feedback control $u := -\gamma p$ ($\gamma \in [0, +\infty)$), then Banach’s fixed point theorem implies that (1.1) has a unique solution $(h, p)$ which has nonnegative components.

The following zero-stabilizability (eradicability) shall be proved in the next section and extends a result in [10].

**Theorem 1.1** If $\lambda^\omega_{1, \gamma} > 0$, where $\gamma \in [0, +\infty)$, then the feedback control $u := -\gamma p$ realizes (1.2) and (1.3), for any $h_0, p_0$ satisfying (A1).
We shall see that moreover,

\[
\lim_{t \to +\infty} p(\cdot, t) = 0 \quad \text{in} \quad L^\infty(\Omega),
\]

at the rate of \(e^{-\lambda t}t\).

From the point of view of Geometric Measure Theory, in 2D, the geometry of \(\omega\) can be described by its three Minkowski functionals. Actually in this work we shall consider only the area and the perimeter of \(\omega\) (denoted by \(\text{length}(\partial(\omega))\)). We assume that the cost to be paid in order to act in \(\omega\) is

\[
\alpha \cdot \text{area}(\omega) + \beta \cdot \text{length}(\partial\omega),
\]

where \(\alpha, \beta > 0\), and is paid once for all for installing the harvesting devices (resp. for treatment or isolation units) (see also [9]).

After all our goal is to find a subregion \(\omega\) which minimizes the total cost of the damages produced by the pest population (resp. the cost of the treatment for the infective population), and of the costs associated with the intervention in \(\omega\), namely

\[
\text{(P)} \quad \text{Minimize} \left\{ \theta \int_0^\infty \int_\Omega h(x, t)(Bp(\cdot, t))(x) \, dx \, dt + \alpha \cdot \text{area}(\omega) + \beta \cdot \text{length}(\partial\omega) \right\},
\]

subject to \(\omega\), where \(\theta\) is a positive constant.

We shall see that usually \(h(x, t) \leq K(x)\) a.e. in \(Q\), and \(p(x, t) \leq y(x, t)\) a.e. in \(Q\), where \(y\) is the solution to

\[
\begin{aligned}
\frac{\partial y(x, t)}{\partial t} - d_2 \Delta y(x, t) &= -a(x) y(x, t) + c_0 K(x) (By(\cdot, t))(x) \\
-\gamma \chi_\omega(x) y(x, t) &= 0, \\
y(x, 0) &= p_0(x),
\end{aligned}
\]

\((1.6)\)

This implies that

\[
\theta \int_0^\infty \int_\Omega h(x, t)(Bp(\cdot, t))(x) \, dx \, dt \leq \theta \int_0^\infty \int_\Omega K(x)(By(\cdot, t))(x) \, dx \, dt,
\]

and consequently, if for a certain \(\omega\) we get a small value of the cost functional in

\[
\text{(P)} \quad \text{Minimize} \left\{ \theta \int_0^\infty \int_\Omega K(x)(By(\cdot, t))(x) \, dx \, dt + \alpha \cdot \text{area}(\omega) + \beta \cdot \text{length}(\partial\omega) \right\},
\]

(subject to \(\omega\)), then for the same \(\omega\) we get even a smaller value for the cost functional in \((\check{P})\).

We shall treat problem \((P)\) as a shape optimization problem, using the level set method (see [39] and references therein). We shall use the implicit interface according to which \(\partial\omega\) is the zero-isocountour of a certain function \(\varphi : \overline{\Omega} \to \mathbb{R}\) and \(\omega = \{x \in \Omega; \varphi(x) > 0\}\), while \(\partial\omega = \{x \in \overline{\Omega}; \varphi(x) = 0\}\). If \(\varphi\) is an implicit function of \(\omega\), then

\[
\text{area}(\omega) = \int_\Omega H(\varphi(x)) \, dx \quad \text{and} \quad \text{length}(\partial\omega) = \int_\Omega \delta(\varphi(x))|\nabla \varphi(x)| \, dx.
\]
Here $H$ is the Heaviside function and $\delta$ is its Dirac Delta generalized derivative. Hence, we may rewrite problem (P) as

\[
\begin{align*}
\text{Minimize} & \quad \theta \int_0^\infty \int_\Omega K(x)(By(\cdot,t))(x) \, dx \, dt + \alpha \int_\Omega H(\varphi(x)) \, dx + \beta \int_\Omega \delta(\varphi(x))|\nabla \varphi(x)| \, dx \\
\text{subject to} & \quad \varphi, \quad \text{where } y \text{ is the solution to}
\end{align*}
\]

\[
\begin{cases}
\partial_t y(x,t) - d_2 \Delta y(x,t) = -a(x)y(x,t) + c_0 K(x)(By(\cdot,t))(x) - \gamma H(\varphi(x))y(x,t), & (x,t) \in Q, \\
\partial_\nu y(x,t) = 0, & (x,t) \in \Sigma, \\
y(x,0) = p_0(x), & x \in \Omega.
\end{cases}
\]

For stabilization problems related to reaction-diffusion systems in biology we refer to [3]-[7]. Some optimal control problems in mathematical biology have been treated in [1], [2], [11], [12], [16], [17], [23], [24], [27], [30], [33], [35]-[38], [43]. We have also to mention some recent results concerning the regional control in population dynamics [8]. For basic notions and methods in shape optimization theory see [19], [21], [25], [31], [32], [39], [41], [42].

The present paper contains a continuation of the investigations started in [9], in the more complicated situation of a non-selfadjoint operator $B$. We shall use here most of the notations adopted in [9].

Here is the plan of the paper. Section 2 concerns the proof of Theorem 1.1. Section 3 is devoted to an “approximation” of problem (P). An iterative algorithm to decrease the total cost by changing $\omega$ is derived in Section 4. Some numerical results for a nonlocal interaction are given. Final remarks are presented in the next section. Basic properties of $\lambda^\omega_{1,\gamma}$ and of the corresponding eigenfunctions are proved in the Appendix.

## 2 Proof of Theorem 1.1

Let $\gamma \in [0, +\infty)$ such that $\lambda^\omega_{1,\gamma} > 0$. For any sufficiently small $\varepsilon > 0$ we have that $\lambda^\omega_{1,\gamma}(\varepsilon) > 0$, where $\lambda^\omega_{1,\gamma}(\varepsilon)$ is the principal eigenfunction to (2.7) (we have used that $\lim_{\varepsilon \to 0} \lambda^\omega_{1,\gamma}(\varepsilon) = \lambda^\omega_{1,\gamma}$; see the Appendix).

\[
\begin{cases}
-d_2 \Delta \Psi(x) + a(x)\Psi(x) + \gamma \chi_\omega(x)\Psi(x) - c_0(K(x) + \varepsilon)(B\Psi)(x) = \lambda \Psi(x), & x \in \Omega, \\
\partial_\nu \Psi(x) = 0, & x \in \partial \Omega.
\end{cases}
\] (2.7)

Consider $\Psi_1$ an eigenfunction to (2.7) corresponding to the eigenvalue $\lambda^\omega_{1,\gamma}(\varepsilon)$ and satisfying

\[
\Psi_1(x) > 0, \quad \forall x \in \overline{\Omega}
\]

(notice that $\Psi_1 \in C(\overline{\Omega})$). It follows that there exists $\zeta \in (0, +\infty)$ such that

\[
\Psi_1(x) \geq \zeta, \quad \forall x \in \overline{\Omega}.
\]
Since the solution \((h,p)\) to (1.1), corresponding to \(u := -\gamma p\), satisfies
\[
0 \leq h(x,t) \leq \tilde{h}(x,t) \quad \text{a.e. in } Q,
\]
where \(\tilde{h}\) is the unique solution to
\[
\begin{aligned}
\partial_t h(x,t) - d_1 \Delta h(x,t) &= r(x)h(x,t) - \rho(x)h(x,t)^2, \quad (x,t) \in Q, \\
\partial_x h(x,t) &= 0, \quad (x,t) \in \Sigma, \\
h(x,0) &= h_0(x), \quad x \in \Omega.
\end{aligned}
\]
(this follows in a standard manner using the fact that \(0 \leq h(x,t)(B\psi(\cdot,t))(x)\) a.e. in \(Q\); for comparison results for parabolic equations see [15], [28], [40]), and using that
\[
\lim_{t \to \infty} \tilde{h}(\cdot,t) = K \quad \text{in } L^\infty(\Omega),
\]
we may conclude that for any sufficiently small \(\varepsilon > 0\), there exists \(T(\varepsilon) \in (0, +\infty)\) such that
\[
h(x,t) \leq K(x) + \varepsilon \quad \text{a.e. in } \Omega \times (T(\varepsilon), +\infty),
\]
and consequently
\[
0 \leq c_0 h(x,t)(B\psi)(x) \leq c_0(K(x) + \varepsilon)(B\psi)(x) \quad \text{a.e. in } \Omega \times (T(\varepsilon), +\infty),
\]
for any \(\psi \in L^\infty(\Omega), \psi(x) \geq 0 \quad \text{a.e.} \ x \in \Omega\). Since \(p(\cdot, T(\varepsilon)) \in L^\infty(\Omega)\), it follows that there exists \(\tau \in (0, +\infty)\) such that
\[
0 \leq p(x, T(\varepsilon)) \leq \tau \Psi_1(x) \quad \text{a.e.} \ x \in \Omega.
\]

On the other hand, the function \(z(x,t) = \tau \Psi_1(x)e^{-\lambda_\tau(\varepsilon)t}\) is the solution to
\[
\begin{aligned}
\partial_t z(x,t) - d_2 \Delta z(x,t) &= -a(x)z(x,t) + c_0(K(x) + \varepsilon)(Bz(\cdot,t))(x) \\
-\gamma \chi_\omega(x)z(x,t), \quad (x,t) \in \Omega \times (T(\varepsilon), +\infty), \\
\partial_x z(x,t) &= 0, \quad (x,t) \in \partial \Omega \times (T(\varepsilon), +\infty), \\
z(x, T(\varepsilon)) &= \tau \Psi_1(x), \quad x \in \Omega,
\end{aligned}
\]
and in a standard manner we may conclude the following comparison result:
\[
0 \leq p(x,t) \leq \tau \Psi_1(x)e^{-\lambda_\tau(\varepsilon)t} \quad \text{a.e. in } \Omega \times (T(\varepsilon), +\infty).
\]
This implies that \(\lim_{t \to +\infty} p(\cdot,t) = 0\) in \(L^\infty(\Omega)\) at the rate of \(e^{-\lambda_\tau(\varepsilon)t}\).

**Remark.** Usually, even without the presence of predators the initial density \(h_0(x)\) is less or equal than \(K(x)\) for \(x \in \Omega\). In this case we get that \(h(x,t) \leq K(x)\) a.e. in \(Q\), and consequently we may infer (this follows in a standard way; see [28], [40])
\[
0 \leq p(x,t) \leq \bar{p}(x,t) \quad \text{a.e. in } Q,
\]
where \( \bar{p} \) is the solution to
\[
\begin{aligned}
\partial_t \bar{p}(x, t) - d_2 \Delta \bar{p}(x, t) &= -a(x)\bar{p}(x, t) + c_0 K(x)(B\bar{p}(\cdot, t))(x) - \gamma \chi_\omega(x)\bar{p}(x, t), \quad (x, t) \in Q, \\
\partial_p \bar{p}(x, t) &= 0, \\
\bar{p}(x, 0) &= \tau_0 \Psi_1(x),
\end{aligned}
\]
where \( \tau_0 \in (0, +\infty) \) and \( \Psi_1 \) is an eigenfunction to \( L_1 \) corresponding to the eigenvalue \( \lambda_1 \), satisfying
\[
\Psi_1(x) \geq \zeta_0, \quad \forall x \in \Omega.
\]
Here \( \zeta_0 \in (0, +\infty) \) (actually, \( \Psi_1 \in C(\Omega) \)), and \( p_0(x) \leq \tau_0 \zeta_0 \) a.e. \( x \in \Omega \). We may infer that
\[
0 \leq p(x, t) \leq \tau_0 \Psi_1(x)e^{-\lambda_1 t} \quad \text{a.e. in } Q
\]
(because \( \bar{p}(x, t) = \tau_0 \Psi_1(x)e^{-\lambda_1 t}, \quad (x, t) \in \Omega \)) and that \( \lim_{t \to +\infty} p(\cdot, t) = 0 \) in \( L^\infty(\Omega) \) at the rate of \( e^{-\lambda_1 t} \).

3 Regional control of the pest population

In the sequel we shall denote by \( T > 0 \) a large number, \( Q_T = \Omega \times (0, T), \Sigma_T = \partial \Omega \times (0, T), d = d_2, y_0 = p_0 \). \( H_\sigma(s) = \frac{1}{2} (1 + \frac{2}{\pi} \arctan \frac{s}{\sigma}) \) is a mollified version of \( H(s) \) and its derivative \( \delta_\sigma(s) = \frac{\sigma}{\pi(\sigma^2 + s^2)} \) is a mollified version of \( \delta(s) \).

Without loss of generality, we may assume that \( \theta = 1 \). Assume that \( h_0(x) \leq K(x) \) a.e. \( x \in \Omega \). Problem \( \hat{P} \) may be “approximated” by the following regional control problem:

\[
\text{(RC)} \quad \text{Minimize } J(\varphi),
\]
where \( \varphi : \overline{\Omega} \to \mathbb{R} \) is a smooth function,
\[
J(\varphi) = J_{\text{damage}}(\varphi) + \alpha J_{\text{area}}(\varphi) + \beta J_{\text{perimeter}}(\varphi).
\]
Here
\[
J_{\text{damage}}(\varphi) = \int_0^T \int_\Omega K(x)(B\varphi(\cdot, t))(x) \, dx \, dt
\]
is the cost of the damages produced by the predators,
\[
J_{\text{area}}(\varphi) = \int_\Omega H_\sigma(\varphi(x)) \, dx
\]
is an approximation of the area of \( \omega = \{ x \in \Omega; \varphi(x) > 0 \} \), and
\[
J_{\text{perimeter}}(\varphi) = \int_\Omega \delta_\sigma(\varphi(x))|\nabla \varphi(x)| \, dx,
\]
is an approximation of the perimeter of \( \omega \) (length of \( \partial \omega \)); \( y^\sigma \) is the solution of
\[
\begin{aligned}
\partial_t y(x, t) - d\Delta y(x, t) &= -a(x)y(x, t) + c_0 K(x)(B\varphi(\cdot, t))(x) - \gamma H_\sigma(\varphi(x))y(x, t), \quad (x, t) \in Q_T, \\
\partial_p y(x, t) &= 0, \\
y(x, 0) &= y_0(x),
\end{aligned}
\]
\( (x, t) \in \Sigma_T \),
\( x \in \Omega \).
Theorem 3.1 For any smooth functions \( \varphi, \psi : \overline{\Omega} \to \mathbb{R} \) we have that

\[
d J(\varphi)(\psi) = \int_\Omega \delta_\sigma(\varphi(x))\psi(x) \left[ \gamma \int_0^T r^\varphi(x,t)y^\varphi(x,t) \, dt + \alpha \right. \\
- \beta \nabla \cdot \left( \frac{\nabla \varphi(x)}{|\nabla \varphi(x)|} \right) dx + \beta \int_{\partial \Omega} \delta_\sigma(\varphi(x)) \partial_\nu \varphi(x) \psi(x) \, dl,
\]

where \( r^\varphi \) is the solution to

\[
\begin{aligned}
\partial_t r(x,t) + d\Delta r(x,t) &= a(x)r(x,t) - c_0(\mathcal{B}^*(K(\cdot)r(\cdot,t))) (x) \\
&+ \gamma H_\sigma(\varphi(x))r(x,t) + (\mathcal{B}^* \mathcal{K})(x), \\
\partial_\nu r(x,t) &= 0, \\
r(x,T) &= 0,
\end{aligned}
\tag{3.10}
\]

Here \( \mathcal{B}^* \) is the adjoint of \( \mathcal{B} \in L(L^2(\Omega)) \).

Sketch of the proof. As in the proof of Lemma 3 in [8], it is possible to prove that, for any smooth functions \( \varphi, \psi : \overline{\Omega} \to \mathbb{R} \), we have

\[
\lim_{s \to 0} \frac{1}{s} [y^{\varphi+s\psi} - y^{\varphi}] = z \quad \text{in } C([0,T]; L^\infty(\Omega)),
\]

where \( z \) is the solution to the problem

\[
\begin{aligned}
\partial_t z(x,t) - d\Delta z(x,t) &= -a(x)z(x,t) + c_0 K(x)(Bz(\cdot,t))(x) \\
&- \gamma H_\sigma(\varphi(x))z(x,t) - \gamma \delta_\sigma(\varphi(x))y^{\varphi}(x,t)\psi(x), \\
\partial_\nu z(x,t) &= 0, \\
z(x,T) &= 0,
\end{aligned}
\tag{3.11}
\]

If \( \varphi, \psi : \overline{\Omega} \to \mathbb{R} \) are arbitrary and smooth functions, then

\[
\lim_{s \to 0} \frac{1}{s} [J(\varphi + s\psi) - J(\varphi)] = \int_0^T \int_\Omega K(x)(Bz(\cdot,t))(x) dx \, dt + \alpha \int_\Omega \delta_\sigma(\varphi(x)) \psi(x) dx \\
+ \beta \int_\Omega \delta'_\sigma(\varphi(x)) \psi(x) |\nabla \varphi(x)| dx + \beta \int_\Omega \delta_\sigma(\varphi(x)) \frac{\nabla \varphi(x) \cdot \nabla \psi(x)}{|\nabla \varphi(x)|} dx.
\]

After some calculations, we get as in [8] that

\[
d J(\varphi)(\psi) = \int_0^T \int_\Omega K(x)(Bz^\varphi(\cdot,t))(x) dx \, dt + \alpha \int_\Omega \delta_\sigma(\varphi(x)) \psi(x) dx \\
- \beta \int_\Omega \delta_\sigma(\varphi(x)) \nabla \cdot \left( \frac{\nabla \varphi(x)}{|\nabla \varphi(x)|} \right) \psi(x) dx + \beta \int_{\partial \Omega} \delta_\sigma(\varphi(x)) \partial_\nu \varphi(x) \psi(x) \, dl. \tag{3.12}
\]

If we multiply the first equation in (3.10) by \( z \) and integrate over \( Q_T \), we obtain after an easy calculation, and using (3.10) and (3.11), that

\[
\int_0^T \int_\Omega K(x)(Bz(\cdot,t))(x) dx \, dt = \gamma \int_0^T \int_\Omega \delta_\sigma(\varphi(x)) r^{\varphi}(x,t)y^{\varphi}(x,t) dx \, dt. \tag{3.13}
\]
By (3.12) and (3.13) we get the conclusion.

**Remark.** By (3.9) we obtain the gradient descent with respect to $\varphi$ (see [31])

$$\partial_s \varphi(x, s) = \delta_\sigma(\varphi(x, s)) \left[-\gamma \int_0^T r^\varphi(x, t)y^\varphi(x, t) \, dt \right.$$

$$\left. - \alpha + \beta \text{ div} \left(\frac{\nabla \varphi(x, s)}{|\nabla \varphi(x, s)|}\right)\right], \quad x \in \Omega, \ s > 0,$$

$$\delta_\sigma(\varphi(x, s)) \frac{\partial \varphi(x, s)}{|\nabla \varphi(x, s)|} \partial_\nu \varphi(x, s) = 0, \quad x \in \partial \Omega, \ s > 0. \quad (3.14)$$

**Remark.** If $B$ is the operator in CASE 1, then $r^\varphi$ is the solution to

$$\left\{ \begin{array}{l}
\partial_t r(x, t) + d \Delta r(x, t) = a(x)r(x, t) - c_0c(x)K(x)r(x, t) + c(x)K(x) + \gamma H(\varphi(x))r(x, t), \ (x, t) \in Q_T, \\
\delta_\sigma(\varphi(x, s)) \frac{\partial \varphi(x, s)}{|\nabla \varphi(x, s)|} \partial_\nu \varphi(x, s) = 0, \\
r(x, T) = 0, \quad x \in \Omega.
\end{array} \right.$$ \quad (x, t) \in \Sigma_T, \quad x \in \partial \Omega.

**Remark.** If $B$ is the operator in CASE 2, then $r^\varphi$ is the solution to

$$\left\{ \begin{array}{l}
\partial_t r(x, t) + d \Delta r(x, t) = a(x)r(x, t) - c_0c(x)K(x)r(x, t) + c(x)K(x) + \gamma H(\varphi(x))r(x, t), \ (x, t) \in Q_T, \\
\delta_\sigma(\varphi(x, s)) \frac{\partial \varphi(x, s)}{|\nabla \varphi(x, s)|} \partial_\nu \varphi(x, s) = 0, \\
r(x, T) = 0, \quad x \in \Omega.
\end{array} \right.$$ \quad (x, t) \in \Sigma_T, \quad x \in \partial \Omega.$$

4 **Computational issues**

In this section we approach numerically the optimal control problem for the model (3.8) on a two-dimensional domain defining an isolated habitat $\Omega$. The systems resulted by discretization were solved iteratively using Matlab’s built in function gmres, which implements the generalized minimal residual method. For our numerical simulations, we have used gmres algorithm without “restarts” of the iterative method, and found satisfactory to use no preconditioners, and a tolerance for the relative error of $10^{-3}$. Gmres algorithm was also applied to an optimal control problem for a two-prey and one-predator model with diffusion in [12].

The set of the model parameters was asserted to have the following values:

- parameters defining the discretization process for space variable $x = (x_1, x_2)$ in the domain $\Omega = [0, 1] \times [0, 1]$ and time: space steps $\Delta x_1 = \Delta x_2 = 2.78e-2$, time step $\Delta t = 2.78e-2$ (36 discretization points on both space and time axes);
- final time $T$, and maximum number of iterations, maxiter: $T = 1$, maxiter = 50;
- diffusion parameter: $d = 1.e-2$;
- parameters defining birth and mortality rates: $a \equiv 1$, $c_0 = 1$, $K \equiv 1$, $\gamma = 1$;
– prescribed convergence parameters for \( J \) and shape function \( \varphi \), given by \( \varepsilon_1 \) and \( \varepsilon_2 \) respectively, and \( \sigma \), a parameter for mollified version of the Heaviside function: \( \varepsilon_1 = 1.e^{-4}, \varepsilon_2 = 1.e^{-5}, \) and \( \sigma = 1.e^{-2}; \)

– parameters representing the weights in the cost functional \( J: \theta = 1 \), and \( \alpha \) and \( \beta \) vary successively in the set \( \mathcal{W} = \{ 0, 0.00001, 0.0001, 0.001, 0.1, 1, 50, 75, 100 \} \).

Theorem 3.1 allows us to construct an iterative procedure to update the shape function \( \varphi \) defining the subregion where the control acts. The following three stop criteria have been used in the Algorithm detailed below:

\[
\text{iter} > \text{maxiter}, \quad \left| J^{(\text{iter}+1)} - J^{(\text{iter})} \right| < \varepsilon_1, \quad \text{and} \quad \| \varphi^{(\text{iter}+1)} - \varphi^{(\text{iter})} \|_{L^2(\Omega)/\text{area}(\Omega)} < \varepsilon_2.
\]

**Algorithm 1**: Iterative scheme to update the shape function \( \varphi \) (the subregion where the control acts).

1. Set \( \text{iter} := 0 \); Choose the positive constants \( T, d, e_0, \gamma, \text{maxiter}, \sigma \) (parameter for mollified Heaviside function), \( \varepsilon_1 \) and \( \varepsilon_2 \).
2. Define the operator \( B \), as well as the functions \( a \) and \( K \).
3. Choose a large value for \( J^{(0)} \) and a small constant for \( s_0 > 0 \) (artificial time).
4. Initialize shape function: \( \varphi^{(0)} := \varphi^{(0)}(x,0) \).
5. Compute \( y^{(\text{iter}+1)} \) the solution of \( \text{Lin} \) corresponding to \( \varphi^{(\text{iter})} := \varphi^{(\text{iter})}(x,0) \).
6. Compute \( J^{(\text{iter}+1)} := \int_0^T \int_\Omega K(x)(By^{(\text{iter}+1)})(x,t) \, dx \, dt + \alpha \int_\Omega H_\sigma(\varphi^{(\text{iter})}(x,0)) \, dx + \beta \int_\Omega \delta_\sigma(\varphi^{(\text{iter})}(x,0)) |\nabla \varphi^{(\text{iter})}(x,0)| \, dx. \)
7. IF \( |J^{(\text{iter}+1)} - J^{(\text{iter})}| < \varepsilon_1 \) THEN STOP
   ELSE GO TO 8:
8. Compute \( r^{(\text{iter}+1)} \) the solution of problem \( \text{Lin} \) corresponding to \( \varphi^{(\text{iter})}(-,0) \) and \( y^{(\text{iter}+1)} \).
9. Compute \( \varphi^{(\text{iter}+1)} \) using \( \text{Lin} \) and the initial condition \( \varphi^{(\text{iter}+1)}(x,0) := \varphi^{(\text{iter})}(x,s_0) \) using a semi-implicit timestep scheme.
10. IF \( \| \varphi^{(\text{iter}+1)} - \varphi^{(\text{iter})} \|_{L^2(\Omega)/\text{area}(\Omega)} < \varepsilon_2 \) THEN STOP
    ELSE IF \( \text{iter} > \text{maxiter} \) THEN STOP
        \( \text{iter} := \text{iter} + 1 \)
    GO TO 5:

The tolerances \( \varepsilon_1 > 0 \) in Step 7 and \( \varepsilon_2 > 0 \) in Step 10 are prescribed convergence parameters. For details about the gradient methods, see [13].

In what follows, we present results of several numerical simulations corresponding to CASE 2 (Section 1, p. 2), when \( (By)(x) = \int_\Omega \kappa(x,x')y(x') \, dx' \) for \( y \in L^2(\Omega) \), where \( \kappa \in L^\infty(\Omega \times \Omega) \), \( \kappa(x,x') \geq 0 \) a.e. \( (x,x') \in \Omega \times \Omega \). In this case, the numerical response to predation shows that the predators from position \( x' \) that captured preys at position \( x \) will stay and produce offsprings at this new position (the predators follow the prey).
Experiment 1. We have used the following initial levels of the state \( y \) and function \( \varphi \) for our numerical simulations: \( y_0(x_1, x_2) = 1 \), and
\[
\varphi_0(x_1, x_2) = \exp(-3(x_1 - 0.5)^2 - 3(x_2 - 0.5)^2) + \sin(3\pi x_1)\sin(5\pi x_2) - 0.75.
\]
The function \( \kappa(x, x') \) is defined by
\[
\kappa(x, x') \equiv |\kappa_1(x_1, x_2)\kappa_2(x_1', x_2')|,
\]
where
\[
\kappa_1(x_1, x_2) = x_1^2 \sin(\pi x_1) + x_2^2 \sin(\pi x_2), \quad \text{and} \quad \kappa_2(x_1', x_2') = 100(x_1'^2 \cos(\pi x_1') + x_2'^2 \cos(\pi x_2')).
\]

Figure [1] depicts the shape of the subdomain \( \omega \) (plotted with light color) for \( \alpha = 100 \) and \( \beta = 0.1 \) at different iterations. Variations of the functionals \( J_{\text{damage}} \), \( J_{\text{area}} \), \( J_{\text{perimeter}} \) and \( J \) for the weights \( \alpha = 100 \) and \( \beta = 0.1 \) are presented in Figure [2]. We note that \( J_{\text{damage}} \) has an oscillatory behavior in the first 10 iterations of the optimization, and then tends to a stabilizing value. At the same time, the functional \( J_{\text{area}} \) is continuously decreasing, whereas \( J_{\text{perimeter}} \) although presents a general decreasing tendency has spurious fluctuations during the whole iterative process. These fluctuations being of small magnitude, they do not affect the decreasing evolution of the global functional \( J \).

Analogous plots are obtained in Figures [3][4] for \( \alpha = 0 \), and \( \beta = 100 \). In this case (with the new values for the weights, \( \alpha \) and \( \beta \)) the algorithm is not convergent. The both functionals \( J_{\text{area}} \) and \( J_{\text{perimeter}} \) indicate an increasing tendency (even a strictly increasing for \( J_{\text{perimeter}} \)) that induce a similar behaviour for \( J \). All these functionals tend to stabilize to a certain value.

Experiment 2. In this experiment, we have used the same initial condition, \( y_0(x_1, x_2) \), and the same initial shape function \( \phi_0(x_1, x_2) \) as in Experiment 1, but we have chosen an asymmetric form for the function \( \kappa(x, x') \) with
\[
\kappa_1(x_1, x_2) = 500\sin(3\pi x_1)\cos(5\pi x_2)\exp(-(x_1 - x_2 - 0.2)^2 - 3(x_1 - x_2 - 0.8)^2), \quad \text{and}
\]
\[
\kappa_2(x_1', x_2') = 500\sin(3\pi x_1')\cos(5\pi x_2')\exp(-5(x_1' - 0.2)^2 - (x_2' - 0.8)^2).
\]

The shape of the subdomain \( \omega \) (where the control acts, area marked with light color) for \( \alpha = 100 \) and \( \beta = 0.1 \) at different iterations is illustrated in Figure [5]. The variation of the functionals \( J_{\text{damage}} \), \( J_{\text{area}} \), \( J_{\text{perimeter}} \) and \( J \) for \( \alpha = 100 \) and \( \beta = 0.1 \) is presented in Figure [6]. In this case the algorithm is convergent in 39 iterations. Zoomed areas in Figures [2] and [6] (bottom–right) show more clear the decreasing of the global functional \( J \).

The last two figures (Figure [7] and Figure [8]) give an idea about the robustness of the minimization algorithm, when the weight parameters in the global functional \( J \), \( \alpha \) and \( \beta \), vary in a certain range of values. Thus, Figure [7] shows the variations of the functionals \( J_{\text{damage}} \), \( J_{\text{area}} \), \( J_{\text{perimeter}} \) and \( J \) during iterative process, when \( \alpha = 50 \) and \( \beta \) takes successively values in the set \( W = \{0.000001, 0.0001, 0.001, 0.01, 0.1, 50, 75, 100\} \). Analogously, Figure [8] presents the variations of the functionals \( J_{\text{damage}} \), \( J_{\text{area}} \), \( J_{\text{perimeter}} \) and \( J \) during iterative method, when \( \beta = 75 \) and \( \alpha \) takes values in the set \( W \).
5 Final remarks

This work has proposed a novel regional control strategy of minimizing the total cost of the damages produced by an alien predator population and to reduce this population. The dynamics of the predators is described by a prey-predator system with local or nonlocal reaction terms. A sufficient condition for the zero-stabilizability (eradicability) of predators is given in terms of the sign of the principal eigenvalue of an appropriate operator that is not self-adjoint, and a stabilizing feedback control with a very simple structure is indicated. The minimization related to such a feedback control is treated for a closely related minimization problem viewed as a regional control problem. The level set method has been adopted for handling the Minkowski functionals of the relevant subregion.

An iterative algorithm to decrease the total cost was obtained and numerical results showed the effectiveness of the theoretical results. Several numerical simulations have been carried out for the prey-predator system with nonlocal reaction terms. In this case, the numerical response to predation reflects the interactions among individuals in an actual habitat, when the predators from position $x'$ that captured preys at position $x$, will stay and produce offsprings at this new position (the predators follow the prey). As a conclusion regarding the numerical realization, the proposed algorithm is strongly affected by the values of the model parameters, and not in the least, the results also depend on the resolution of the discretisation (both space and time steps). Its convergence is attained when an appropriate selection of the weight parameters, $\alpha$ and $\beta$, is done. These selections should maintain a balance of the functionals $J_{\text{damage}}$, $J_{\text{area}}$ and $J_{\text{perimeter}}$ with respect to their order of magnitude.

Finally, one may remark that a spatially structured SIR problem may be described by the same system, and the above mentioned minimization problem may be viewed as the problem of minimizing the effects of an epidemics by regional controls.

6 Appendix

We establish here some auxiliary results. Consider the following eigenvalue problem

$$
\begin{cases}
-d\Delta \psi(x) + \eta(x)\psi(x) - K(x)(B\psi)(x) = \lambda \psi(x), & x \in \Omega, \\
\partial_{\nu}\psi(x) = 0, & x \in \partial\Omega.
\end{cases}
$$

(6.15)

Here $d \in (0, +\infty)$, $\eta, K \in L^\infty(\Omega)$, $K(x) \geq 0$ a.e. in $\Omega$, and $B$ satisfies (A2).

**Lemma 1.** Problem (6.15) has a simple eigenvalue $\lambda_1 \in \mathbb{R}$, which corresponds to a positive eigenfunction. None of the other eigenvalues corresponds to a positive eigenfunction.

$\lambda_1$ is called the principal eigenvalue for (6.15).
Remark. There exists $\psi_1$ an eigenfunction for (6.15), corresponding to $\lambda_1$, such that

$$\psi_1(x) \geq \zeta > 0, \quad \forall x \in \Omega$$

(actually, $\psi_1 \in C(\overline{\Omega})$).

Proof of Lemma. Let us use the Krein-Rutman theorem. We may assume without loss of generality, that there exists $\eta_0$ such that

$$\eta(x) \geq \eta_0 > \|K\|\|B\|$$

a.e. $x \in \Omega$ (where $\|K\| = \|K\|_{L^\infty(\Omega)}$, $\|B\| = \|B\|_{L^2(\Omega)}$). If this hypothesis is not satisfied, then we reduce our problem to this situation by translating $\lambda$.

Let $X = L^\infty(\Omega), C = \{w \in L^\infty(\Omega) ; w(x) \geq 0$ a.e. $x \in \Omega\}$, and $T : X \rightarrow X$ given by

$$Tf = \psi,$$

where $f \in L^\infty(\Omega)$ and $\psi$ is the unique solution to

$$\begin{cases}
-d\Delta \psi(x) + \eta(x)\psi(x) - K(x)(B\psi)(x) = f(x), & x \in \Omega, \\
\partial_\nu \psi(x) = 0, & x \in \partial\Omega.
\end{cases} \quad (6.16)$$

We have that $X$ is a Banach space and $C$ is a solid cone ($C$ is a closed convex cone with nonempty interior).

Let us prove that $T$ is a compact linear operator which is strictly positive (i.e., if $f \in C$ and $f \neq 0_X$, then $Tf \in \text{Int} C$). It is obvious that the hypotheses in Lax-Milgram lemma are satisfied if we view $f$ as an element of $L^2(\Omega)$. Hence, (6.16) has a unique weak solution $\psi \in W^{2,2}(\Omega)$.

Actually, by Theorem IX.26 in [18] we get that $\psi \in W^{2,2}(\Omega)$, and there exists a positive constant $\tilde{c}_2$ such that

$$\|\psi\|_{W^{2,2}(\Omega)} \leq \tilde{c}_2 \|f\|_{L^2(\Omega)}$$

($\tilde{c}_2$ is independent of $f$).

Finally, we get that there exists $\tilde{c} > 0$ such that

$$\|Tf\|_{W^{2,2}(\Omega)} \leq \tilde{c} \|f\|_{L^\infty(\Omega)}.$$

Since $W^{2,2}(\Omega) \subset C(\overline{\Omega})$ continuously, we obtain that $Tf \in C(\overline{\Omega}) \subset L^\infty(\Omega)$, and $T$ is linear and bounded. On the other hand, since the embedding $W^{2,2}(\Omega) \subset C(\Omega)$ is compact (see [18]), we may infer that $T$ is a compact linear operator. Let us prove that $T$ is strictly positive. Let $f \in L^\infty(\Omega)$, $f(x) \geq 0$ a.e. $x \in \Omega$, and $f \neq 0_X$. Let us prove that $\psi$, the weak solution to (6.16) satisfies $\psi(x) \geq 0$ a.e. $x \in \Omega$. Indeed, if we consider $\psi^- \in W^{1,2}(\Omega)$ then

$$d \int_\Omega \nabla \psi(x) \cdot \nabla \psi^-(x) dx + \int_\Omega \eta(x)\psi(x)\psi^-(x) dx - \int_\Omega K(x)(B\psi)(x)\psi^-(x) dx = \int_\Omega f(x)\psi^-(x) dx \geq 0.$$
This implies that
\[-d \int_{\Omega} |\nabla \psi^- (x)|^2 \, dx - \int_{\Omega} \eta(x)|\psi^- (x)|^2 \, dx + \int_{\Omega} K(x)(B\psi^-)(x)\psi^- (x) \, dx \geq 0,\]
and consequently,
\[\|K\|_{\infty} \cdot \|B\| \cdot \|\psi^-\|_{L^2(\Omega)}^2 \geq \int_{\Omega} \eta(x)|\psi^- (x)|^2 \, dx \geq \eta_0 \|\psi^-\|_{L^2(\Omega)}^2.\]

Since \(\eta_0 > \|K\|_{\infty} \cdot \|B\|\), it follows that \(\psi^- = 0_{L^2(\Omega)}\), and so \(\psi(x) \geq 0\) a.e. \(x \in \Omega\).

Let us prove that \(\psi(x) > 0\), for any \(x \in \overline{\Omega}\) (recall that \(\psi \in C(\overline{\Omega})\)). Indeed, since \(K(x)(B\psi(\cdot))(x) \geq 0\) a.e. \(x \in \Omega\), we conclude that
\[\begin{cases}
-d \Delta \psi(x) + \|\eta\|_{\infty} \psi(x) \geq 0, \quad x \in \Omega, \\
\partial_{\nu} \psi(x) = 0, \quad x \in \partial \Omega.
\end{cases}
\]

It follows that for any \(q \in [1, +\infty)\), there exists \(c_q > 0\) such that
\[\|\psi\|_{L^q(\Omega)} \leq c_q \cdot \inf_{x \in \Omega} \psi(x)\]
(see Lemma 2.1 in [22]).

If \(\inf_{x \in \Omega} \psi(x) = \min_{x \in \Omega^-} \psi(x) > 0\), then we get that \(Tf > 0\).

Indeed, if we assume, by contradiction, that \(\inf_{x \in \Omega} \psi(x) = 0\), then it follows that \(\|\psi\|_{L^q(\Omega)} = 0\), for any \(q \in [1, +\infty)\). We get that \(\psi(x) = 0\) a.e. \(x \in \Omega\), and so \(\psi(x) = 0\), \(\forall x \in \Omega\). This implies that \(f = 0_X\) which is a contradiction. Hence, \(T\) is strictly positive.

By Theorem 1.2 in [26] we get that the spectral radius of \(T\) satisfies \(r(T) > 0\), and \(r(T)\) is a simple eigenvalue with an eigenvector \(\psi \in \text{Int} C\); there is no other eigenvalue with positive eigenvector. The conclusion of Lemma 1 is now obvious.

The second result concerns the principal eigenvalues for
\[\begin{cases}
-d \Delta \psi(x) + \eta(x)\psi(x) - (K(x) + \varepsilon)(B\psi)(x) = \lambda \psi(x), \quad x \in \Omega, \\
\partial_{\nu} \psi(x) = 0, \quad x \in \partial \Omega,
\end{cases}\]
where \(\varepsilon > 0\). We denote by \(\lambda_1(\varepsilon)\) the principal eigenvalue to (6.17).

**Lemma 2.**
\[\lim_{\varepsilon \to 0} \lambda_1(\varepsilon) = \lambda_1.\]

**Proof.** Let \(\psi_\varepsilon\) be the positive eigenfunction corresponding to \(\lambda_1(\varepsilon)\), and satisfying
\[\|\psi_\varepsilon\|_{L^2(\Omega)} = 1.\]

Let us prove that if \(0 \leq \varepsilon_1 < \varepsilon_2\), then
\[\lambda_1(\varepsilon_1) \geq \lambda_2(\varepsilon_2).\]
Consider \( \tilde{\lambda}_1(\varepsilon_1) \) the principal eigenvalue for

\[
\begin{cases}
-d\Delta \psi(x) + \eta(x)\psi(x) + (B^*((K(\cdot) + \varepsilon_1)\psi(\cdot)))(x) = \lambda\psi(x), & x \in \Omega, \\
\partial_n \psi(x) = 0, & x \in \partial\Omega.
\end{cases}
\] (6.18)

The existence and basic properties related to it follow as for (6.15). In fact \( \lambda_1(\varepsilon) \in \mathbb{R} \), and let \( \tilde{\psi}_{\varepsilon_1} \) be a corresponding and positive eigenfunction.

Using (6.17) and (6.18) we get that

\[-(\varepsilon_2 - \varepsilon_1) \int_\Omega (B\psi_{\varepsilon_2})(x) \cdot \tilde{\psi}_{\varepsilon_1}(x) \, dx = (\lambda_1(\varepsilon_2) - \tilde{\lambda}_1(\varepsilon_1)) \int_\Omega \psi_{\varepsilon_2}(x) \tilde{\psi}_{\varepsilon_1}(x) \, dx,
\] (6.19)

and

\[0 = (\lambda_1(\varepsilon_1) - \tilde{\lambda}_1(\varepsilon_1)) \int_\Omega \psi_{\varepsilon_1}(x) \tilde{\psi}_{\varepsilon_1}(x) \, dx.
\]

Since \( \int_\Omega \psi_{\varepsilon_1}(x) \tilde{\psi}_{\varepsilon_1}(x) \, dx > 0 \), we may conclude that \( \tilde{\lambda}_1(\varepsilon_1) = \lambda_1(\varepsilon_1) \).

On the other hand, since \( \int_\Omega (B\psi_{\varepsilon_2})(x) \tilde{\psi}_{\varepsilon_1}(x) \, dx \geq 0 \), and \( \int_\Omega \psi_{\varepsilon_2}(x) \tilde{\psi}_{\varepsilon_1}(x) \, dx > 0 \), we get by (6.19) that \( \lambda_1(\varepsilon_2) \leq \lambda_1(\varepsilon_1) \).

We also conclude that \( \lambda_1(\varepsilon) \leq \lambda_1 \), for any \( \varepsilon > 0 \). We may infer that there exists \( \lim_{\varepsilon \to 0} \lambda_1(\varepsilon) = \tilde{\lambda}_1 \leq \lambda_1 \).

Let us prove that actually we have equality. If \( 0 < \varepsilon < 1 \), then by (6.17) we get that

\[d \int_\Omega |\nabla \psi_\varepsilon|^2 \, dx + \int_\Omega \eta(x)|\psi_\varepsilon|^2 \, dx \leq \lambda_1(\varepsilon) + (\|K\|_\infty + \varepsilon) \cdot \|B\|,
\]

and consequently, \( \psi_\varepsilon \) is bounded in \( W^{1,2}(\Omega) \). Therefore, there exists a sequence \( (\psi_{\varepsilon_n}) \, (\varepsilon_n \to 0) \), such that

\[\psi_{\varepsilon_n} \to \psi_0 \quad \text{in} \quad W^{1,2}(\Omega), \quad \text{and} \quad \psi_{\varepsilon_n} \to \psi_0 \quad \text{in} \quad L^2(\Omega),
\]

which implies that \( \|\psi_0\|_{L^2(\Omega)} = 1 \), and \( \psi_0(x) \geq 0 \) a.e. \( x \in \Omega \). Since \( \psi_{\varepsilon_n} \) satisfies

\[d \int_\Omega \nabla \psi_{\varepsilon_n}(x) \cdot \nabla \psi(x) \, dx + \int_\Omega \eta(x)\psi_{\varepsilon_n}(x)\psi(x) \, dx
\]

\[-\int_\Omega (K(x) + \varepsilon_n)(B\psi_{\varepsilon_n})(x)\psi(x) \, dx = \lambda_1(\varepsilon) \int_\Omega \psi_{\varepsilon_n}(x)\psi(x) \, dx,
\]

for any \( \psi \in W^{1,2}(\Omega) \), we may pass to the limit, and obtain that \( \psi_0 \) is a weak solution to (6.15) corresponding to \( \tilde{\lambda}_1 \), i.e. \( \psi_0 \) is a nonnegative eigenfunction for (6.15), corresponding to \( \lambda := \tilde{\lambda}_1 \), and so \( \tilde{\lambda}_1 \) is an eigenvalue. By Lemma 1 we conclude that \( \tilde{\lambda}_1 = \lambda_1 \).

Using the same method we have used for Lemma 2 may prove that

**Lemma 3.** The mapping \( \gamma \mapsto \lambda_{1,\gamma}^\nu \) is strictly increasing.
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Figure 1: Variation of the subdomain $\omega$ (marked with light color) for $\alpha = 100$ and $\beta = 0.1$ at iterations 0, 20, 34 and 42 (Experiment 1).

Figure 2: Variation of the functionals $J_{\text{damage}}$, $J_{\text{area}}$, $J_{\text{perimeter}}$ and $J$ for $\alpha = 100$ and $\beta = 0.1$ (Experiment 1).
Figure 3: Variation of the subdomain \( \omega \) (marked with light color) for \( \alpha = 0 \) and \( \beta = 100 \) at iterations 0, 20, 38 and 50 (Experiment 1).

Figure 4: Variation of the functionals \( J_{\text{damage}} \), \( J_{\text{area}} \), \( J_{\text{perimeter}} \) and \( J \) for \( \alpha = 0 \) and \( \beta = 100 \) (Experiment 1).
Figure 5: Variation of the subdomain $\omega$ (marked with light color) for $\alpha = 100$ and $\beta = 0.1$ at iterations 0, 2, 7 and 39 (Experiment 2).

Figure 6: Variation of the functionals $J_{\text{damage}}$, $J_{\text{area}}$, $J_{\text{perimeter}}$ and $J$ for $\alpha = 100$ and $\beta = 0.1$ (Experiment 2).
Figure 7: Variation of the functionals $J_\text{damage}$, $J_\text{area}$, $J_\text{perimeter}$ and $J$ for $\alpha = 50$ and several values of $\beta$ (Experiment 2).
Figure 8: Variation of the functionals $J_{\text{damage}}$, $J_{\text{area}}$, $J_{\text{length}}$, and $J$ for $\beta = 75$ and several values of $\alpha$ (Experiment 2).