Entangling and squeezing atoms by weak measurement

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(Dated: January 1, 2019)

Creating multipartite entangled spin states is of general interest to quantum metrology [1–3] and quantum information science [4–6]. In particular, spin squeezed states (SSS) — a special category of entangled spin states [7] whose total-spin fluctuations along a direction are smaller than the sum of the individual-spin fluctuations, can enable atomic clocks [8, 9] and magnetometers [10, 11] with sensitivities surpassing the standard quantum limit. To date, several mechanisms have been investigated to produce SSS, including nonlinear interactions among the individual atoms either directly or mediated by light [12–20], quantum-state transfer from squeezed light to atomic ensemble [21–28], and QND projective measurements of collective spin [29–35]. Among them, QND measurement [36] is most general as it is experimentally realizable in many atomic systems, ranging from cold atoms [34, 37] to room temperature vapors [38]. It usually involves off-resonant atom-light interaction and produces entanglement between light and the atomic spin. A projection measurement of light then maps the atoms into a spin-spin entangled state, yielding a conditional quantum-noise reduction \( \propto 1/(1 + \kappa^2) \), where \( \kappa \) is the coupling strength between light and atoms which can be increased by either applying a cavity or using an elongated cooled atomic ensemble. However, many precision measurement systems in free space, in particular those with large atom numbers making the most sensitive atomic sensors [39], have inherently low coupling strength \( \kappa \), for instance, \( \kappa \approx 1 \) in hot atomic vapors [40] and \( \kappa \approx 0.6 \) in cold atomic vapors [35], where large squeezing is unattainable with currently available QND techniques.

In this paper, we propose a novel scheme to entangle and squeeze atoms using weak measurement (WM) [41]. The scheme gives enhanced squeezing compared to the normal QND method especially in the weak coupling regime. The WM, a quantum measurement protocol first introduced by Aharonov et al. [41], investigates a situation where the coupling between the system and the probe is weak. Appropriately pre- and post-selected system states, i.e., nearly orthogonal, yield a counterintuitive result — the distribution of measured values can be dramatically amplified and thus lie significantly outside the range of eigenvalues of the observable operator. The ability of WM to amplify tiny physical effects has led to numerous applications, including the measurements of small frequency shifts [42], the amplification of small phase [43] and optical nonlinearities [44]. We here show that the WM can be used to produce and strengthen the entanglement among atomic spins. Unlike traditional conditional QND squeezing, the proposed WM-based protocol is unconditional, i.e., the resulted quantum spin distribution always has zero mean and is squeezed along the same direction, even independent of the coupling strength. Furthermore, by properly choosing the post-selection parameter and using a series of optical probe pulses, the QND-type squeezing can be converted into one-axis twisting (OAT) and two-axis twisting (TAT) squeezing, leading to a quantum-noise reduction that scales exponentially with the coupling strength.

\[ H = \hbar \chi S_z J_z, \]

\[ J = (S_x, S_y, S_z) \]

Squeezing generation by weak measurement. We consider an atomic ensemble consisting of \( N_A \) four-level atoms in the ground states \( |\uparrow\rangle, |\downarrow\rangle \) (with \( x \) as the quantization axis) interacting off-resonantly with a light beam propagating along the \( z \)-axis, as shown in Fig. 1. If the light is tuned far from resonance, one may adiabatically eliminate the exited states and obtain the well-known Faraday rotation (FR) Hamiltonian [4]. \( H = \hbar \chi S_z J_z \), where \( \chi \) is the coupling constant, \( S = (S_x, S_y, S_z) \) denotes the Stokes vector, and \( J = (J_x, J_y, J_z) \) stands for the collective spin operators for the ground states of atoms. They obey the angular momentum commu-

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The atom ensemble is projected into a state $|\Phi\rangle$ with $V \in \{S, J\}$ and $j,k,l \in \{x,y,z\}$. In particular, $S_j = \frac{1}{\sqrt{2}}(a_x^j a_y^j - a_y^j a_x^j)$ and $J_z = \frac{1}{N} \sum_{i=1}^{N_A} (|j_i\rangle \langle j_i| - |j_i\rangle \langle j_i|)$, where $a_x a_y$ is the annihilation operator of $x(y)$-polarized light. We assume the light field contains a strong $x$-polarized component and a weak $y$-polarized component, such that one can treat the $x$-mode operators $a_x, a_y$ as classical numbers and the $y$-mode field as the quantum field which is relevant here. Under this condition, the time evolution operator can be written as $U = \exp(-i\kappa \hat{P}_L \hat{J}_z)$, where we have defined a new constant $\kappa = \chi \sqrt{N_{ph}/2}$ with $N_{ph}$ the photon number of the light mode and the canonical operators $X_L = (a_x + a_y)/\sqrt{2}$, $P_L = -i(a_x - a_y)/\sqrt{2}$ for $y$-polarized light, satisfying $[X_L, P_L] = i$.

We begin by revisiting the traditional QND-based spin squeezing. The atoms are initially prepared in the coherent spin state (CSS) $|\Psi_A\rangle_{in} = |\uparrow\rangle^\otimes N_A$ along $x$, and the relevant $y$-polarization component of the input $x$-polarized light can be written as a vacuum state $|\varphi_L\rangle = |0\rangle_L$. After the QND interaction, a homodyne detection of the $X_L$ component is performed, leading to a random measurement result $X_L = x_L$, for which $\langle x_L|P_L|0\rangle_L = \langle x_L|0\rangle_L u + \langle x_L|0\rangle_L (1-u)$. For $\chi \ll 1$, the atom ensemble is projected into a state $|\Psi_A\rangle_{out} = \langle x_L|U|\varphi_L\rangle |\Psi_A\rangle_{in} \approx \langle x_L|0\rangle_L |\uparrow\rangle^\otimes N_A - i \kappa \sqrt{N_{ph}/2} \langle x_L|0\rangle_L |\downarrow\rangle^\otimes N_A + \frac{\kappa^2}{2 N_A} (1 - 2 \chi^2) \langle x_L|0\rangle_L |\downarrow\rangle^\otimes N_A$, where we have kept only up to the second order in $U$, and defined the single-spin-flipped state $|\downarrow\rangle^{SSF} = 2iJ_z |\Psi_A\rangle_{in} = \sum_{j=1}^{N_A} |j\rangle |\uparrow\rangle^\otimes (N_A - 1)$ as well as the pairwise-spin-flipped state $|\downarrow\rangle_{PSF} = (N_A - 4J_z^2) |\Psi_A\rangle_{in} = \sum_{j,k=1}^{N_A} |j\rangle |k\rangle |\uparrow\rangle^\otimes (N_A - 2)$, and also introduced $k^2 = N_{ph}/2 = \eta \sigma$, with the on-resonance optical depth $\alpha_0 = N_{ph}/\sigma$ and the scattered photon number per atom $\eta = N_{ph}/\sqrt{N_{ph}/2}$, where $\sigma$ is the resonant light scattering cross section of a single atom, $A$ is the optical beam cross section, $\Gamma$ is the spontaneous decay rate, and $\Delta$ the detuning from resonance. We note that the second term $|\rangle_{SSF}^{\downarrow}$ contributes an average mean value (proportional to $x_L$) of $J_z$, which is often different for each detection and makes the protocol conditional; the third term $|\downarrow\rangle_{PSF}^{\downarrow}$ is the essence of spin squeezing [45], but has small weight for weak coupling.

Here, we propose to use the WM approach to enhance the spin-spin entanglement and thus the collective spin squeezing. As shown in Fig. 1, a $y$-polarized single photon is sent through a beam splitter (BS1), with reflectivity $r$ and transmissivity $t = \sqrt{1 - r^2}$. As a result, the spatial modes $a$ and $b$ of the two output ports of BS1 are prepared in the entangled state $|\varphi_L\rangle = r |0\rangle_a |b\rangle + t |a\rangle_b$. Next, the spatial mode $b$ and a strong $x$-polarized beam are combined by a PBS to produce a single output beam with independent $x$ and $y$ polarizations. The output beam is then injected into the atomic ensemble to experience the FR interaction, and after the interaction the probe light is guided into another PBS to separate the two polarization modes. The $y$-polarized photons in $a$ and $b$ are then recombined at BS2, with reflectivity $r'$ and transmissivity $t' = \sqrt{1 - r'^2}$. The recombined photons are finally detected at upper detector PD1 and lower detector PD2. If PD1 detects ‘no photon’ and PD2 detects ‘one photon’, which corresponds to post-selecting the optical fields in a state, $|\varphi_L\rangle = r' |0\rangle_a |b\rangle - t' |a\rangle_b$, collapsing the collective spin into the state $|\Psi \rangle_{out} = \langle \varphi_L', U | \varphi_L \rangle |\Psi_A\rangle_{in} \approx \langle \varphi_L', \varphi_L \rangle \langle \uparrow \rangle^\otimes N_A - \frac{A_w \kappa^2}{4 N_A} |\downarrow\rangle_{PSF}$. (1)

We note that (i) if the postselected state $|\varphi_L'\rangle$ is near orthogonal to the input state $|\varphi_L\rangle$, $|\langle \varphi_L', \varphi_L \rangle|^2$ is very small, leading to a large $A_w$, which therefore enhances the coupling strength $\kappa$ effectively by a large factor $\sqrt{A_w}$, as seen from Eq.(1). (ii) the cancellation of the single-spin-flipped term $|\downarrow\rangle_{SSF}$ ensures that the created squeezed state has a deterministic mean value of zero, indicating that the squeezing process is unconditional. The probability of getting such a state is given by $P = \langle |\varphi_L', \varphi_L \rangle |^2 \asymp |\langle \varphi_L', \varphi_L \rangle |^2 = (rr' - tt')^2$, showing that the price for the enhancement in squeezing is a reduction in the success probability.

Now we quantitatively show the amount of squeezing created by this WM approach. We employ the Holstein-Primakoff approximation [36], and assume that the atomic ensemble is strongly polarized with large $N_A$ so that the small deviations from perfect polarization direction during interaction can be neglected. One can then define the canonical variables for atoms $X_A = J_y/\sqrt{N_A/2}$ and $P_A = J_z/\sqrt{N_A/2}$, satisfying $[X_A, P_A] = i$ and $X_A^2/2 + P_A^2/2 = n_A + 1/2$, where
n_A denotes the atomic number operator [37]. With this notation we can rewrite the unitary evolution as 
U = \exp(-i\kappa P_{0}P_{A}) and the CSS in terms of p_A eigenstates as \(|\Psi_{A}\rangle_{in} = \int dp_{A} \exp[-p_{A}^2/2]\phi_{L}(|p_{A}\rangle)\). (hereafter we omit normalization constants for simplicity), and, after taking into account effects of the higher-order terms of U, we have

\[ |\Psi_{A}\rangle_{out} = \left(1 + \sum_{n=1}^{\infty} \left(\frac{i\kappa}{2}\right)^{n}\frac{p_{A}^{2n}}{(2n)!} + \frac{1}{(2n+1)!} \xi^{2n} p_{A}^{2n}\right) |\Psi_{A}\rangle_{in} = \int dp_{A} \left(1 - \tilde{A}_{w} \kappa^{2} p_{A}^{2}\right) e^{-\frac{p_{A}^{2}}{2\xi^{2}}} |p_{A}\rangle \] (3)

where \(\tilde{A}_{w} = A_{w}/2 - 1/4\) and \(\xi^{2} = 1/(1+\kappa^{2}/2)\). For this state, one can calculate the atomic variance \((\Delta P_{A})^{2} = \xi^{4}\) and thus the squeezing parameter \(\xi^{2} = (\Delta P_{A})^{2}/(\Delta P_{A}^{2})\). Optimizing \(\xi^{2}\) with respect to \(A_{w}\), we get \(\xi_{opt}^{2} = (3 - \sqrt{5})\xi_{w}^{2}\) for \(A_{w} = 4(3 - \sqrt{5})/(3\kappa^{2}) + (15 - 4\sqrt{5})/6\), compared to the traditional QND whose squeezing parameter is \(1/(1+\kappa^{2})\), this schemes can enhance the amount of squeezing especially in the weak coupling regime. For instance, as \(\kappa \to 0\), QND gives nearly no squeezing, and the squeezing enhancement by our WM method reaches the order of about 2.6 dB (given by the coefficient 0.55). In Fig. 2(a), we plot the squeezing created by this WM protocol and the QND protocol, which verifies that the enhancement is more significant in the weaker coupling regime and decreases for larger \(\kappa\).

An interesting aspect of this scheme is that a non-Gaussian entangled state can be created (at large weak values \(A_{w}\)) which also has applications in quantum metrology [38], even though such entangled state does not favor squeezing and is precisely the reason for the above 2.6 dB limit in squeezing enhancement. The physics can be better understood if we express the state of Eq. (5) in terms of the atomic number state \(|n_{A}\rangle\): 

\[ |\Psi_{A}\rangle_{out} = (1 - \kappa^{2} A_{w} \xi_{w}^{2}/2)|0_{A}\rangle_{\xi} + \kappa^{2} A_{w} \xi_{w}^{2}/2|2_{A}\rangle_{\xi}, \]

where \(|n_{A}\rangle_{\xi} = S(\zeta)|n_{A}\rangle\) denotes the squeezed atomic number state, with the squeezing operator \(S(\zeta) = \exp[-\zeta(a_{A}^{\dagger} - a_{A}^{\dagger})]/2\), where \(\zeta = \ln \xi_{w}^{2}\) and \(a_{A}\) is the atomic annihilation operator. (i) For small \(\zeta, \zeta\) is nearly zero, and the squeezing \(S(\zeta)\) has negligible contribution to squeezing, i.e., \(|n_{A}\rangle_{\xi} \approx |n_{A}\rangle\). In this regime, by choosing suitable \(A_{w}\) in post-selection, one can collapse the atomic state from the initial vacuum state \(|0_{A}\rangle\) to a superposition state \(|\phi_{2}\rangle = (1/2 + 1/\sqrt{3})/2|0_{A}\rangle + (1/2 - 1/\sqrt{3})/2|2_{A}\rangle\), and even to an equal superposition state (for larger \(A_{w}\) \(|\phi_{2}\rangle = (|0_{A}\rangle - |2_{A}\rangle)/\sqrt{2}\). The state \(|\phi_{2}\rangle\) has the narrowest probability distributions in the \(P_{A}\) basis (shown as the dashed line in the inset of Fig. 2(a)], corresponding to the scenario of best enhancement in squeezing compared to QND. For much larger \(A_{w}\), \(|\phi_{2}\rangle\) approaches \(|\phi_{2}\rangle\), which deviates from the normal Gaussian state (solid line in the inset of Fig. 2(a)] and reaches its maximum value \(P = 0.58\) at \(\kappa = 1.1\). Such characteristics are attractive since the relatively large success probability results in near-deterministic unconditional squeezing, and more importantly, many free-space atomic systems, such as warm vapor cell [49] and cold atoms [39], work well for precision measurement with this small coupling-strength.

Multi-detection scheme and one(two)-axis-twisting squeezing. As discussed above, the enhancement in squeezing over QND is moderate and determined by the state \(|\phi_{2}\rangle\). However, one can obtain much higher enhancement by employing the multi-detection scheme and constructing higher-dimensional superposition state \(|\phi_{2n}\rangle = \sum_{j=0}^{n} c_{j}(|2_{A}\rangle_{j})\) with \(n > 1\) and \(c_{j}\) the normalization constant, which is realized by breaking the strong x-polarized light pulse into \(n\) subpulses. Each subpulse...
co-propagates with a y-polarized superposition state described above, and then together experience the FR interaction and subsequent single photon detections. If all the detections succeed, the spin state is collapsed into:

$$\langle \Psi_A \rangle_{out} = \int dp_A \prod_{j=1}^{\tilde{n}} \left( 1 - \tilde{A}_w \vartheta_j^2 \kappa^2 \frac{P_A}{P_A} \right) e^{-\frac{p_A^2}{2\omega}} |p_A\rangle,$$

where $\vartheta_j < 1$ relates to weight of each subpulse in photon number and satisfies $\sum_{j=1}^{\tilde{n}} \vartheta_j^2 = 1$. Corresponding to this state, in Fig. 2(a) we also plot the optimal squeezing of the protocol (optimized with respect to $\vartheta_j$) for varying $\kappa$ under different detection times $n = 2, 3$, which indicates that the improvement in the enhancement can be significant. Figure 2(b) shows the enhancement as a function of detection times $n$, for $\kappa \to 0$ and $\infty$, indicating that, when $n > 1$, the QND squeezing are completely surpassed by the WM squeezing, and an enhancement of $9\text{ dB}$ is obtainable at $n = 9$.

Although of solely theoretical value, interesting results appear when we set $\vartheta_j^2 = 1/n$ (which corresponds to the case of $n$ equal subpulse) with $n \to \infty$ and further manipulate the weak value $A_w$. We have

$$\langle \Psi_A \rangle_{out} \approx \int dp_A e^{-\frac{A_w \vartheta_j^2 \kappa^2}{2} \omega} \frac{p_A^2}{2\omega} |p_A\rangle.$$

For this state, if $A_w = 1$, the output state is exactly the same as the state created by QND scheme 47. Therefore, any $A_w > 1$ will improve the performance of squeezing. Note that $A_w$ can be a positive, negative and even complex number 50. Consider the particular case when $A_w$ is imaginary 51, i.e., $A_w \rightarrow i\tilde{A}_w$, then, according to Eq. (5), the net effect of the multi-detection is equivalent to a unitary transformation $U_{QND} = e^{-iA_w \vartheta_j^2 \kappa^2 \omega^2}$, which is exactly the well-known OAT transformation 27, yielding a squeezing $\xi^2_{OAT} \approx 1/A_w^2 \kappa^4$ that is a quadratic improvement of the QND protocol. Further improvement is possible if one can transform OAT into TAT 27. To do so, the subpulses described above are sent through the sample alternately along the $x$ and $y$ directions. The subsequent (successful) post-selection measurements result in the effective transformation $U_{QND} \approx 1 + iA_w \kappa^2 X_{\tilde{A}}^2 / n \kappa^2 \omega^2 \exp[-iA_w \vartheta_j^2 \kappa^2 (P_A^2 - X_{\tilde{A}}^2)]$, where the post-selection processes are chosen such that the $y$-direction yields $A_w \rightarrow -i\tilde{A}_w$ while the $z$-direction generates $A_w \rightarrow i\tilde{A}_w$. The transformation $U_{TAT}$ creates a squeezing $\xi^2_{TAT} = e^{-A_w \vartheta_j^2 \kappa^2}$, indicating that the spin fluctuations are now shrunk monotonically.

**General considerations.** A major challenge in implementing the multi-detection protocol is that its success odds decreases dramatically with $n$, as shown in the inset of Fig 2(b). To increase the odds, we suggest utilizing the non-maximally entangled NooN state $|\varphi_L\rangle = r|0\rangle_m |m_i\rangle + t|m_o\rangle |0\rangle$ as the input state and, correspondingly, the state $|\varphi_D\rangle = r'|0\rangle_m m_i - t'|m_o\rangle |0\rangle$ as the post-selection state, which, for weak coupling, leads to the same state as Eq. (1) and yields the $m$-photon weak value $A_w,m = 2m\tilde{A}_w$ for large $m$. Since the success odds of the multi-detection protocol is $P_n \propto 1/(2A_w)^{2n}$, the use of multi-photon NooN state can thus greatly increase the odds, that is $P_n \rightarrow m^{2n}P_{nw}$.

We next show that the proposed scheme can also be extended to the case of coherent-state input $|\alpha\rangle$, which is easier to implement than the single-photon Fock state 53. Then, the initial state after BS1 is $|\varphi_L\rangle = |\alpha\rangle_a |\alpha\rangle_b$. If the light state is successfully post-selected in the state $|\varphi_D\rangle = \sqrt{2}r't'(0_m) |0_o\rangle + |0_o\rangle |0\rangle + (t'^2 - r'^2)|1_o\rangle |1\rangle$, the spin state collapses into $|\Psi_A\rangle_{out} \approx |\varphi_D\rangle |\varphi_L\rangle (1 - i\kappa A_0 P_A - (i\kappa^2 A_{\text{w},a} P_A^2) |\Psi_A\rangle$ with $A_{\text{w},a} = (1 - i\alpha A_0 - 2r_0'/2\alpha^2)/2$, where we have assumed $t = 1/\sqrt{2} A_0 = i(r_0' - \alpha^2/2 + 1)/\alpha$ and $r_0' = 2r't'/2(r'^2 - r'^2)$. If $2(r'^2 + 1)$, we get $A_0 = 0$ and $A_{\text{w},a} = 1/(2(r_0'^2 + 1))$. The spin state obtained is exactly the same as the state in Eq. (1), but with $A_w$ replaced by $A_{\text{w},a}$. Large values of $A_{\text{w},a}$ are obtainable when $r_0'$ approaches $-1$.

So far, we have ignored the decoherence processes where the transverse $J_y$ and $J_z$ components decay with coefficient $\eta$ and the $y$-polarized quantum field is absorbed with a ratio $\epsilon = \eta N_A / N_{ph} \approx \eta$ for $N_A \sim N_{ph}$ 54. For the case of weak coupling, such losses can be neglected since $\eta$ is small. Another inevitable imperfection is the inefficiency of the photodetectors. The single-photon detectors usually have a detection efficiency $\xi_D < 1$, which can be modeled as a sequence of (virtual) beam splitters with reflectivity $\eta_D = 1 - \xi_D$ followed by an idealized detector 55. Figure 2(c) shows how our protocols perform for different $\eta_D$, from which we can see that, for a feasible value $\eta_D = 5%$ 56 57, the squeezing produced is still well above the QND bound on the spin squeezing, while higher-efficiency detectors are required to make the multi-detection scheme work well.

**Conclusion.** We proposed a novel scheme to prepare atoms in an entangled or squeezed state, by using weak measurement. The scheme is probabilistic but unconditional, and can have significant enhancement in squeezing over traditional QND scheme, especially in the weak coupling regime. The QND-type interaction can be even turned into one-axis-twisting and two-axis-twisting type, simply by controlling the post-selection parameters. This scheme can be used as a general state-preparation method and should be useful in the context of entanglement study and quantum metrology.

We thank Vladan Vuletić for enlightening discussions and acknowledge support from National Key Research Program of China under Grant No. 2016YFA0302000 and NNSFC under Grants No. 11504273 (M.Wang), 91636107, 61675047.
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