Current-induced spin polarization of a magnetized two-dimensional electron gas with Rashba spin-orbit interaction

A. Dyrdal\textsuperscript{1}, J. Barnaś\textsuperscript{1,2} and V. K. Dugaev\textsuperscript{3}

\textsuperscript{1}Faculty of Physics, Adam Mickiewicz University, ul. Umultowska 85, 61-614 Poznań, Poland
\textsuperscript{2}Institute of Molecular Physics, Polish Academy of Sciences, ul. M. Smoluchowskiego 17, 60-179 Poznań, Poland
\textsuperscript{3}Department of Physics and Medical Engineering, Rzeszów University of Technology, ul. Powstańców Warszawy 6, 35-959 Rzeszów, Poland

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Nowadays the current-induced spin polarization can also arise in a magnetic system, when it includes spin-orbit coupling. This effect was predicted theoretically in the ‘70s\textsuperscript{12–26}. These observations initiated a wide interest in the field- and thermally-induced spin-orbit torques and new ways of magnetization switching, that could be alternative to the switching induced by spin transfer torques\textsuperscript{30,32}. In this paper we present theoretical results on the current-induced spin polarization of a magnetic 2DEG with Rashba spin-orbit interaction. Such a system is a basic model of various magnetic semiconductor heterostructures. The system consists of a 2DEG deposited on a magnetic substrate and interacting with the substrate via exchange interaction (see also Fig.1). To calculate the current-induced spin polarization we use the Matsubara Green function formalism which enables description of the temperature variation of the induced spin polarization. We derive some general formulas for the polarization and also present numerical results. The induced spin polarization is shown to include generally a term due to Berry curvature of the corresponding electron bands. Similar terms also appear in the spin-orbit torques following from exchange interaction of the electrons and magnetic underlayer.

The paper is organized as follows. In section 2 we describe the model system and also present the theoretical formalism and derive general formulas for the current-induced spin polarization. In Section 3 we present analytical and numerical results in some specific situations; first, we consider the nonequilibrium spin polarization in the absence of exchange field (Section 3 A), then we present results for exchange field oriented perpendicularly to the plane of 2DEG (Section 3 B) and for exchange field oriented in plane of 2DEG and collinear (perpendicular) to the electric current, Section 3C (Section 3 D). In Section 4 we discuss the spin polarization in a general

I. INTRODUCTION

Spin-orbit interaction leads, in general, to a number of interesting transport phenomena, that enable generation and control of spin currents in a pure electrical manner. Two of the most prominent examples are the spin Hall and spin Nernst effects. The former (latter) effect consists in generation of pure spin current flowing perpendicularly to an external electric field (temperature gradient) applied to the system. These effects play currently an essential role in the processes of electrical generation and detection of spin currents\textsuperscript{13,14}. For instance, the spin current can be used as origin of spin torque exerted on magnetic moments of a ferromagnet in a bilayer system consisting of a magnetic layer attached to a nonmagnetic one with strong spin-orbit coupling. This torque, in turn, may induce magnetic dynamics and even can reverse magnetic moment of the magnetic layer when the spin current exceeds some critical value.

Another consequence of the spin-orbit interaction in a system with mobile electrons is the current-induced nonequilibrium spin polarization of conduction electrons. This effect was predicted theoretically in the ‘70s\textsuperscript{12,24} for a two-dimensional electron gas (2DEG) with Rashba spin-orbit interaction, and then it was studied in various systems exhibiting spin-orbit interaction\textsuperscript{12,25,26,28,29}. The current-induced spin polarization was also observed experimentally\textsuperscript{30–32}, and currently it is attracting attention of many researchers\textsuperscript{15,16,28}.

The current-induced spin polarization can also arise in a magnetic system, when it includes spin-orbit coupling. In such a case the induced non-equilibrium spin polarization interacts with the local magnetization via exchange coupling and creates a torque exerted on the magnetic moment\textsuperscript{25,27}. Moreover, it has been also shown that not only external electric field, but also a temperature gradient may lead to spin-orbit driven spin polarization\textsuperscript{3,20,26,29}.
FIG. 1. Current-induced spin polarization in a magnetized 2DEG. Schematic of the device (top), and coordinates used in theoretical description. Electric field is oriented along the axis $x$.

case of arbitrarily oriented exchange field. In Section 5, in turn, we consider relation of the nonequilibrium spin polarization with the Berry curvature of the corresponding electronic bands. The induced spin-orbit torque is briefly discussed in Section 6, while summary and final conclusions are in Section 7.

II. THEORETICAL OUTLINE

We consider a magnetized 2DEG with Rashba spin-orbit interaction, as shown schematically in Fig. 1. The 2DEG is assumed to be deposited on a ferromagnetic substrate which creates an effective exchange field acting on the electron gas.

A. Model

The single-particle Hamiltonian describing such a system can be written in the following form:

$$
\hat{H} = \frac{\hbar^2 k^2}{2m} \sigma_0 + \alpha (k_x \sigma_z - k_y \sigma_y) + \mathbf{H} \cdot \mathbf{\sigma},
$$

(1)

where $\sigma_0$ and $\sigma_n$ (for $n = \{x, y, z\}$) are the unit and Pauli matrices defined in the spin space, the parameter $\alpha$ in the second term of the Hamiltonian describes strength of the Rashba spin-orbit interaction, while $k_x$ and $k_y$ are the in-plane wavevector components. The third term of the above Hamiltonian describes the effect of exchange field due to a magnetic substrate. This exchange field can be written as $\mathbf{H} = J \mathbf{M}$, with $J$ standing for the exchange parameter ($J > 0$ for a ferromagnetic coupling between the 2DEG and magnetic substrate). Note, the exchange field $\mathbf{H}$ is measured here in energy units. In spherical coordinates (see Fig. 1), components of the exchange field, $\mathbf{H} = (H_x, H_y, H_z)$, can be written as

$$
H_x = JM_x = JM \sin(\theta) \cos(\xi),
$$

(2a)

$$
H_y = JM_y = JM \sin(\theta) \sin(\xi),
$$

(2b)

$$
H_z = JM_c = JM \cos(\theta),
$$

(2c)

where $M = |\mathbf{M}|$, while $\theta$ and $\xi$ are the polar and azimuthal angles, as defined in Fig. 1. In general, we take into account the temperature dependence of the magnetization $M(T)$, and assume it obeys the Bloch’s law $M = M(T) = M_0 \left[1 - (T/T_c)^3\right]$, where $T_c$ is the Curie temperature of the magnetic substrate, and $M_0$ is the corresponding zero-temperature magnetization.

Eigenvalues of the Hamiltonian $\hat{H}$ take the form

$$
E_{\pm} = \varepsilon_k \pm \lambda_k,
$$

(3)

where $\varepsilon_k = \hbar^2 k^2 / 2m$ (with $k^2 = k_x^2 + k_y^2$), while $\lambda_k = [H^2 + \alpha^2 k^2 - 2\alpha (H_y k_x - H_x k_y)]^{1/2}$.

Below we present the theoretical method based on the Matsubara-Green function formalism, and also derive a general formula for the nonequilibrium spin polarization induced by an external electric field.

B. Method and general solution for current-induced spin polarization

To describe spin polarization induced by an external electric field we introduce a time-dependent external electromagnetic field of frequency $\omega / \hbar$ (note, here $\omega$ is energy) described by the vector potential $\mathbf{A}(t) = \mathbf{A}(\omega) \exp(-i \omega t / \hbar)$. The electric field is related to $\mathbf{A}$ via the formula $\mathbf{E}(\omega) = (\hbar / i \omega) \mathbf{A}(\omega)$. Hamiltonian $H^E_A$ describing interaction of the system with the external field (treated as a perturbation) takes the form

$$
\hat{H}^E_A(t) = -\mathbf{j}^{el} \cdot \mathbf{A}(t).
$$

(4)

Here, the operator of the electric current density is defined as $\mathbf{j}^{el} = e \mathbf{\hat{v}}$; with $e$ being the charge of electron ($e < 0$), and $\mathbf{\hat{v}} = (1/\hbar) \partial \hat{H} / \partial \mathbf{k}$ being the electron velocity operator. The $x$ and $y$ components of the velocity operator have the following explicit form:

$$
\hat{v}_x = \frac{\hbar k}{m} \cos(\phi) \sigma_0 - \frac{\alpha}{\hbar} \sigma_y,
$$

(5)

$$
\hat{v}_y = \frac{\hbar k}{m} \sin(\phi) \sigma_0 + \frac{\alpha}{\hbar} \sigma_x,
$$

(6)

where $\phi$ is the angle between the wavevector $\mathbf{k}$ and the axis $x$, i.e. $k_x = k \cos(\phi)$ and $k_y = k \sin(\phi)$, while the last terms in Eq. (5) and Eq. (6) represent components of the anomalous velocity that originates from the Rashba spin-orbit interaction.

Without loss of generality, we assume in this paper that the external electric field is oriented along the $x$-axis. Thus, the $\alpha$-th ($\alpha = x, y, z$) component of the quantum-mechanical average value of spin polarization induced by
the external electric field can be found in the Matsubara-Green functions formalism from the following formula:

\[ S_\alpha(i\omega_m) = \frac{1}{\beta} \sum_{k,n} \text{Tr} \left\{ \delta_\alpha G_k(i\varepsilon_n + i\omega_m) \hat{H}_{A}^{\text{eff}}(i\omega_m) G_k(i\varepsilon_n) \right\} , \]

where \( \delta_\alpha = \hbar \sigma_\alpha / 2 \) is the operator of the \( \alpha \)'s spin component, \( \beta = 1/k_B T \) (with \( T \) and \( k_B \) denoting the temperature and Boltzmann constant, respectively), \( \varepsilon_n = (2n + 1)i\pi \hbar k_B T \) and \( \omega_m = 2m\pi \hbar k_B T \) are the Matsubara energies, while \( G_k(i\varepsilon_n) \) are the Matsubara Green functions (in the \( 2 \times 2 \) matrix form). Note, the perturbation term takes now the form \( \hat{H}_{A}^{\text{eff}}(i\omega_m) = -ev_x A_x(i\omega_m), \) with the amplitude of the vector potential \( A_x(i\omega_m) \) determined by the amplitude \( E_x(i\omega_m) \) of electric field through the relation \( A_x(i\omega_m) = \frac{E_x(i\omega_m)}{i\omega_m} \).

Taking into account the explicit form of \( \hat{H}_{A}^{\text{eff}}(i\omega_m) \), one can rewrite Eq. (7) in the form

\[ S_\alpha(i\omega_m) = -\frac{1}{\beta} \frac{eE_x(i\omega_m)\hbar}{i(\omega_m)} \times \sum_{k,n} \text{Tr} \left\{ \delta_\alpha G_k(i\varepsilon_n + i\omega_m) \hat{v}_x G_k(i\varepsilon_n) \right\} . \]

The sum over Matsubara energies in the above expression can be calculated by the method of contour integration \(^{33,34}\).

\[ \frac{1}{\beta} \sum_n \delta_\alpha G_k(i\varepsilon_n + i\omega_m) \hat{v}_x G_k(i\varepsilon_n) = -\int_{\mathcal{C}} \frac{dz}{2\pi i} f(z) \delta_\alpha G_k(z + i\omega_m) \hat{v}_x G_k(z), \]

where \( \mathcal{C} \) denotes the appropriate contour of integration and \( f(z) \) is a meromorphic function of the form \((e^{iz} + 1)^{-1}, \) that has simple poles at the odd integers \( n, z = \varepsilon_n \) (for details see Refs. \(^{33,34}\)).

Upon analytical continuation one obtains

\[ S_\alpha(\omega) = -\frac{e\hbar}{\omega} E_x \text{Tr} \sum_k \int \frac{d\varepsilon}{2\pi} f(\varepsilon) \delta_\alpha \left( G_k^{R}(\varepsilon + \omega) \hat{v}_x [G_k^{R}(\varepsilon) - G_k^{A}(\varepsilon)] \right. \\
\left. + [G_k^{R}(\varepsilon) - G_k^{A}(\varepsilon)] \hat{v}_x G_k^{A}(\varepsilon - \omega) \right) . \]

Here, \( f(\varepsilon) \) is the Fermi-Dirac distribution function and \( G_k^{R/A}(\varepsilon) \) is the impurity-averaged retarded/advanced Green function corresponding to the Hamiltonian (1). The Green functions take the following explicit form:

\[ G_k^{R/A}(\varepsilon) = G_k^{R/A}(\varepsilon) \sigma_0 + G_k^{R/A}(\varepsilon) \sigma_x + G_k^{R/A}(\varepsilon) \sigma_y + G_k^{R/A}(\varepsilon) \sigma_z , \]

where

\[ G_k^{R/A}(\varepsilon) = \frac{1}{2} [G_+(\varepsilon) + G_- (\varepsilon)], \]

\[ G_k^{R/A}(\varepsilon) = \frac{1}{2\lambda_k} (-\alpha k_y + H_z)[G_+(\varepsilon) - G_- (\varepsilon)], \]

\[ G_k^{R/A}(\varepsilon) = \frac{1}{2\lambda_k} (-\alpha k_x + H_y)[G_+(\varepsilon) - G_- (\varepsilon)], \]

\[ G_k^{R/A}(\varepsilon) = \frac{1}{2\lambda_k} H_z [G_+(\varepsilon) - G_- (\varepsilon)], \]

with \( G_k^{R}(\varepsilon) = [\varepsilon + \mu - E_k + i\Gamma]^{-1} \) and \( G_k^{A}(\varepsilon) = [\varepsilon + \mu - E_k - i\Gamma]^{-1} \). Note, we assumed \( \Gamma = \hbar / 2\tau, \) with equal effective relaxation time \( \tau \) in the two subbands.

Using equation (10) as a starting point and performing integration over \( \varepsilon \) we get finally the following formula for the three components of the current-induced spin polarization:

\[ S_x = -eE_x \hbar \times \int \frac{d^2 k}{(2\pi)^2} \left\{ \frac{1}{2\Gamma} \frac{\hbar^2 k_x}{2m\lambda_k} (\alpha k_y + H_z) [f'(E_+) - f'(E_-)] + \frac{\alpha (\alpha k_y + H_z)(\alpha k_x - H_y)}{(2\lambda k)^2 + (2\Gamma)^2} [f'(E_+) + f'(E_-)] - \frac{\alpha H_z}{2\lambda k} \frac{f(E_-) - f(E_+)}{(E_+ - E_-)^2 + (2\Gamma)^2} \right\} , \]

\[ S_y = eE_y \hbar \times \int \frac{d^2 k}{(2\pi)^2} \left\{ \frac{1}{4\Gamma} \frac{\hbar^2}{\lambda_k} (\alpha k_x - H_y)^2 [f'(E_+) + f'(E_-)] + \frac{\hbar^2 k_x}{m\lambda_k} \frac{1}{4\Gamma} [f'(E_+) - f'(E_-)] + \frac{\alpha H_z}{\lambda_k} \frac{1}{(2\lambda k)^2 + (2\Gamma)^2} \frac{f'(E_+) + f'(E_-)}{(E_+ - E_-)^2 + (2\Gamma)^2} \right\} , \]

\[ S_z = -eE_z \hbar \times \int \frac{d^2 k}{(2\pi)^2} \left\{ \frac{\alpha H_z}{\Gamma} \frac{\hbar^2 k_x}{2(2\lambda k)^2 + (2\Gamma)^2} [f'(E_+) + f'(E_-)] + \frac{\hbar^2 k_z}{2m\lambda_k} \frac{1}{2\Gamma} [f'(E_+) - f'(E_-)] - \frac{\alpha (2\Gamma)^2 (H_z + \alpha k_y)}{4\lambda_k} \frac{f'(E_+) + f'(E_-)}{(E_+ - E_-)^2 + (2\Gamma)^2} - \frac{\alpha (2\Gamma)^2 (H_z + \alpha k_y)}{4\lambda_k} \frac{f(E_-) - f(E_+)}{(E_+ - E_-)^2 + (2\Gamma)^2} \right\} . \]

Details on the derivation of the above equations are presented in the Appendix A. Before presenting results on the current-induced spin polarization for an arbitrary orientation of the exchange field, we consider first some special cases.
III. SPECIAL CASES

A. Zero exchange field

First, we reconsider the limit of zero exchange field, i.e. the limit of a nonmagnetized 2DEG, when only the $y$ component of spin polarization survives. The general expression for $S_y$ takes then the following form:

$$S_y = eE_y \int \frac{dk}{(2\pi)^2} \left\{ \frac{x}{4\Gamma} \left[ f(E_+ - \mu) + f(E_- - \mu) \right] ight. 
+ \alpha \pi \left\{ \frac{\Gamma}{2} \right. - \frac{\Gamma}{2} \left. \right\} 
+ \frac{\hbar^2 k}{4m\Gamma} \left\{ f(E_+ - \mu) - f(E_- - \mu) \right\}, \quad (16)$$

where $f' = \partial f/\partial E$. Note that in this limit the eigenvalues have the form $E_{\pm} = \varepsilon_k \pm \alpha k$.

In the low-temperature regime, the above integrals can be evaluated analytically and one arrives at

$$S_y = -eE_y \int \frac{dk}{(2\pi)^2} \left\{ \frac{x}{4\Gamma} \left\{ \delta(E_+ - \mu) + \delta(E_- - \mu) \right\} 
+ \alpha \pi \left( \frac{\Gamma}{2} + (2\alpha k/\hbar) \right) 
+ \frac{\hbar^2 k}{4m\Gamma} \left\{ \delta(E_+ - \mu) - \delta(E_- - \mu) \right\} \right\}, \quad (17)$$

When both subbands are occupied (which corresponds to $\mu > 0$), the Dirac delta functions in the above equation can be written in the form

$$\delta(E_{\pm} - \mu) = \frac{m}{\sqrt{2m\hbar^2} + m^2\alpha^2} \delta(k - k_{\pm}), \quad (18)$$

and finally one obtains

$$S_y = \frac{1}{2} eE_y \frac{m\alpha}{2\pi \hbar^2} \tau 
- \frac{eE_y}{16\pi \Gamma \sqrt{2m\hbar^2 + m^2\alpha^2}} \left[ \frac{k_+}{1 + (\alpha k_+/\Gamma)^2} + \frac{k_-}{1 + (\alpha k_-/\Gamma)^2} \right], \quad (19)$$

with $k_{\pm} = \mp \frac{m\alpha}{\hbar^2} + \frac{1}{\hbar^2} \sqrt{m^2\alpha^2 + 2m\hbar^2}$. The first term of Eq.(19) corresponds to the Edelstein expression for the current-induced spin polarization in the so-called bubble approximation,

$$S^0_{y} = \frac{1}{2} eE_y \frac{m\alpha}{2\pi \hbar^2} \tau. \quad (20)$$

Note, the impurity vertex correction is neglected in our considerations. Such a correction leads to some renormalization of the spin polarization (for details see e.g. Ref. 3 and 32). The second term in Eq.(19) is a correction which originates from the imaginary term in the nominator of the Green function and products of two retarded or two advanced Green’s functions (omitted in Ref. 3). Note, the second term in Eq.(19) vanishes in the quasi-ballistic limit (low impurities concentration), when $\Gamma \to 0$.

In the general case, i.e. for arbitrary $T$ and arbitrary chemical potential $\mu$, one should use the general formula (16). However, one point requires some comment. It is known, that for impurities with short-range ($\delta$-like) potential and $\mu > 0$, the parameter $\Gamma$ is constant, while for negative $\mu$ it increases and diverges when $\mu$ approaches the bottom of the lower energy band. Thus, at a certain value of $\mu$, $\mu = \mu_{loc}$, the Ioffe-Regel localization condition is obeyed, and the states become localized below $\mu_{loc}$. Accordingly, the results are valid beyond the localization regime, i.e. for $\mu > \mu_{loc}$.

Now, we present some numerical results. In Fig.2(a) we show the temperature dependence of spin polarization for four different values of chemical potential $\mu$. Here, we should mention that the chemical potential also depends on temperature, thus a fixed value of chemical potential means that the carrier concentration varies. If however the system is gated one can keep chemical potential constant in the temperature range shown in Fig.2(a), and is roughly equal to the corresponding value of $S^0_{y}$. For still lower values

![FIG. 2. Temperature dependence of the spin polarization for different values of $\mu$ (a) and for different values of $\Gamma$ (b). Spin polarization as a function of chemical potential for different values of temperature $T$ (c), and spin polarization as a function of $\Gamma$ for different values of chemical potential $\mu$ (d). The spin polarization is normalized to the Edelstein term $S^0_y$ [see Eq.(20)]. In numerical calculations it was assumed: $m = 0.07m_0$ ($m_0$ being the electron rest mass), $\alpha = 2 \cdot 10^{-11}$ eV m, and $eE_x = -5 \cdot 10^{-2}$ eV/m. Other parameters as indicated.](image-url)
of \( \mu \), the temperature dependence is nonmonotonous - it first decreases and then slightly increases with temperature. To understand this behaviour we plot in Fig.2(c) the spin polarization \( S_y \) as a function of chemical potential for several values of temperature and the same \( \Gamma \) as in Fig.2(a). This figure clearly shows that spin polarization tends to \( S_y^0 \) with increasing \( \mu \). Such a behavior is reasonable as the second term in Eq.(19) decreases with increasing \( \mu \) (the effective role of finite \( \Gamma \) decreases with increasing \( E_+ - E_- \)). For small values of \( \mu \), however, the second term in Eq.(19) plays a role and the spin polarization is reduced. The temperature dependence appears when \( E_+ - E_- \) at the Fermi level is of the order or smaller than \( kT \), which takes place in the region of small values of \( \mu \). Moreover, this figure also shows that \( S_y \) decreases with increasing \( T \), except a narrow region of small values of \( \mu \), where the temperature dependence is nonmonotonous, exactly like in Fig.2(a). The temperature dependence is also shown in Fig.2(b) for several values of \( \Gamma \). This figure also shows that the correction due to the second term in Eq.(19) increases with increasing \( \Gamma \). The latter behavior is shown explicitly in Fig.2(d) for indicated values of \( \mu \). The decrease of spin polarization with increasing \( \Gamma \) is physically clear as the effective separation of the two Rashba bands becomes reduced with increasing \( \Gamma \). The second term in Eq.(19) plays then an important role and leads to reduction of spin polarization.

B. Exchange field perpendicular to plane of 2DEG

Consider now a magnetized 2DEG and let us begin with the situation when the exchange field (or equivalently substrate magnetization) is perpendicular to the plane of 2DEG, \( H = (0, 0, H_z) \). The eigenvalues of Hamiltonian (1) reduce now to the form \( E_{\pm} = \varepsilon_\pm + \zeta \), with \( \zeta = \sqrt{H_z^2 + \alpha^2 k^2} \).

The general expressions describing the two nonzero components of spin polarization take the forms

\[
S_x = eE_x \hbar \int \frac{dk}{2\pi} \frac{\alpha H_z}{(2\zeta)^2} \left\{ \frac{\Gamma^2}{\zeta^2 + \Gamma^2} [f(E_+) - f(E_-)] - \frac{1}{\zeta} [f(E_+) - f(E_-)] \right\}, \quad (21)
\]

\[
S_y = eE_x \hbar \int \frac{dk}{2\pi} \frac{\alpha^2 k^2}{4\zeta^2} \left\{ \frac{h^2 k^2}{4m\zeta} [f(E_+) + f(E_-)] + \Gamma \left( 2 - \frac{\alpha^2 k^2}{\zeta^2} \right) \right\}, \quad (22)
\]

while \( S_z = 0 \). Accordingly, the electric field generates now spin polarization with both in-plane components nonzero, while the component normal to the plane of 2DEG (along the exchange field) vanishes exactly. Thus, the exchange field generates spin polarization along the electric field and also modifies the spin polarization along the axis \( y \).

In the low temperature limit equations (21) and (22) lead to the following analytical expressions:

\[
S_x = -\hbar e \frac{e}{8\pi} E_x \frac{H_z}{\alpha} \left\{ \frac{\zeta_+ - \zeta_-}{\zeta_+ + \zeta_-} \right\} + \alpha^2 \left( \frac{\zeta_+}{\zeta_+ + (\zeta_+ + \zeta_-)\Gamma^2} - \frac{\zeta_-}{\zeta_- + (\zeta_+ + \zeta_-)\Gamma^2} \right) \quad (23)
\]

\[
S_y = -\hbar e \frac{e}{16\pi} E_x \frac{H_z}{\alpha} \left[ \frac{k^2}{\zeta_+} - \frac{k^2}{\zeta_-} - \hbar e E_x \left( \frac{1 - H_z}{\zeta_+} \right) \frac{\zeta_+}{1 + (\zeta_+ + \zeta_-)\Gamma^2} \right] + \alpha^2 \left( \frac{\zeta_+}{\zeta_+ + (\zeta_+ + \zeta_-)\Gamma^2} - \frac{\zeta_-}{\zeta_- + (\zeta_+ + \zeta_-)\Gamma^2} \right), \quad (24)
\]

where \( \nu_{\pm} = \frac{m}{\hbar^2}(1 \pm \frac{\alpha^2 k^2}{\hbar^2 \zeta^2})^{-1} \) represent the density of states corresponding to the \( E_\pm \) subbands, respectively, \( \zeta_\pm = \zeta(k = k_\perp) \), and \( k_\perp \) is \( \sqrt{m \alpha^2 + \hbar^2 \nu_{\pm}^2 \zeta^2} \). The Fermi wavevectors corresponding to the two subbands.

In the ballistic limit (extremely long relaxation time), Eq. (21) takes the form

\[
S_x = -eE_x \hbar \int \frac{d^2k}{4\pi^2} \left[ f(E_+) - f(E_-) \right],
\]

which after integration over \( k \) leads to

\[
S_x = -\hbar e \frac{e}{8\pi} E_x \frac{H_z}{\alpha} \frac{\zeta_+ - \zeta_-}{\zeta_+ + \zeta_-}, \quad (26)
\]

i.e. to the first term in Eq. (23). The above expression does not depend on the relaxation time and due to the mathematical form of Eq. (25) it may be identified as the Berry phase related contribution to the spin polarization, that in turn may be responsible for anti-damping spin-torque. For details see Section V.

Numerical results on the current-induced spin polarization of the magnetized 2DEG with the exchange field oriented perpendicularly to the plane are shown in Fig.3. The dependence of \( S_x \) and \( S_z \) on the exchange field \( \nu M_0 \) is presented in Figs.3(a) and 3(b), respectively. Figure 3(a) clearly shows that \( S_x \) vanishes in the limit of zero exchange field and then its magnitude grows rather fast with increasing \( \nu M_0 \). Then, it decreases to zero for large exchange fields. Magnitude of \( S_y \), in turn, is nozero for zero exchange field, and increases with increasing \( \nu M_0 \). It reaches a maximum at some value of \( \nu M_0 \), and then decreases with a further increase in \( \nu M_0 \). Such a behavior with \( \nu M_0 \) can be understood since the relative role of Rashba coupling decreases with increasing \( \nu M_0 \). Note, that the \( S_z \) component is antisymmetrical with respect to sign reversal of \( \nu M_0 \), while the \( S_y \) component is then symmetrical. Figures 3(c) and 3(d) present the \( x \) and \( y \) components of spin polarization as a function of temperature for fixed values of chemical potential and \( \nu M_0 \). In
numerical calculations we have assumed $T_c = 150$ K and therefore the $S_y$ component vanishes for $T \geq 150$ K. In turn, the $S_y$ component is remarkably enhanced below $T_c$ and drops to a weakly temperature dependent value (for fixed chemical potential and the parameter $\Gamma$) when $T \geq 150$ K.

Variation of spin polarization with the chemical potential is presented in Figs (3e) and (3f) for different magnitudes of the exchange field $JM_0$. Magnitudes of both components increase monotonously with $\mu$ when $\mu$ is inside the energy region between the bottom edges of the two subbands. For $\mu$ in the vicinity of the bottom of higher energy band, these components reach maximum values and for larger $\mu$ they decrease with increasing $\mu$. Note, the component $S_x$ is roughly three orders of magnitude smaller than the $S_y$ component.

In Fig. (3g) and (3h) we show the $x$ and $y$ components of the spin polarization as a function of the Rashba coupling constant. These figures clearly show that the absolute values of both components increase roughly linearly with $\alpha$. However, some deviations from this linear dependence appear above certain values of $\alpha$, where the increase is smaller.

C. Exchange field in plane of 2DEG and perpendicular to electric field

In this section we consider the current-induced spin polarization for the magnetization vector (exchange field) oriented along the $y$ axis, i.e. when the exchange field is in the plane of two-dimensional electron gas and perpendicular to the current. In such a case the $x$ and $z$ components of the current-induced spin polarization vanish exactly, and the only nonzero component is $S_y$ - like in the case of zero exchange field. This component, however, is modified by the exchange field.

Numerical results for $S_y$ are shown in Fig.4 where variation of $S_y$ with the exchange field $JM_0$, Fig.4(a), clearly shows that the spin polarization decreases relatively fast with increasing absolute value of $JM_0$ and is suppressed.
when the Zeeman-like term (due to exchange coupling to the substrate) dominates over the Rashba term. The suppression to zero of spin polarization at large \( JM_0 \) appears due to strong modification of electronic states by the Zeeman like term, and takes place for all values of chemical potential.

Temperature dependence of \( S_y \) is shown in Fig.4b) for two values of chemical potential and two values of \( JM_0 \). For the larger value of \( JM_0 \), the spin polarization \( S_y \) vanishes in a broad temperature region and then increases when \( T \) approaches the Curie temperature, reaching the magnitude of \( S_y \) limit in the limit of a nonmagnetized 2DEG. This behavior is consistent with that in Fig.4a.

In Fig.4(c) we show \( S_y \) as a function of chemical potential. As follows from this figure, \( S_y \) increases monotonously with the chemical potential increasing from the minimum of the lower subband, and then becomes saturated for large values of \( \mu \). The rate of this increase as well as the chemical potential at which the saturation appears depend on \( JM_0 \). Spin polarization as a function of the Rashba parameter \( \alpha \) is shown in Fig.4(d) for indicated values of the exchange field. In general, the \( y \) component of spin polarization increases now nonlinearly with the Rashba constant.

D. Exchange field in plane of 2DEG and collinear with electric field

When the exchange field is oriented along the \( x \) axis, i.e. it is collinear with the external electric field, the \( x \) component of spin polarization vanishes, whereas the \( y \) and \( z \) components are non-zero. In general, the \( S_z \) component of spin polarization is roughly three orders of magnitude smaller than the \( S_y \) component. Variation of both components with the exchange field \( JM_0 \), temperature, chemical potential, and Rashba constant is presented in Figs 5 (a-d).

Behavior of the \( S_y \) and \( S_z \) components with \( JM_0 \), \( T \), \( \mu \) and \( \alpha \) is qualitatively similar to the corresponding behavior of the components \( S_x \) and \( S_y \) in the case with the exchange field normal to the plane of 2DEG, see Fig.3. There are some differences of rather quantitative character, which follow from different electronic bands in these two situations. For instance, the \( S_z \) component varies with the chemical potential in a slightly different manner than the \( S_x \) component in Fig.3. Weak difference also appear in the variation of the \( S_y \) component with temperature for \( T \) below the Curie temperature \( T_c \). Similarly as in Fig.3, both components behave almost linearly with the Rashba parameter \( \alpha \).

IV. NUMERICAL RESULTS FOR ARBITRARILY ORIENTED EXCHANGE FIELD

Up to now we have discussed only some specific situations, when the exchange field is oriented along the three main directions: (i) along the electric field, (ii) normal to the electric field and to the plane of 2DEG, and (iii) normal to the electric field and oriented in the plane of 2DEG. Now let us consider a general case, when the exchange field is oriented arbitrarily. This orientation is described by the polar \( \theta \) and azimuthal \( \xi \) angles, as shown in Fig.1. Generally, all three components of spin polarization (i.e. \( S_x \), \( S_y \) and \( S_z \)) can be nonzero. In Fig.5 we present these components as a function of both \( \theta \) and \( \xi \) angles, see left panel in this figure. The right panel, in turn, presents several vertical cross-sections of the corresponding density plots from the left panel. In the specific configurations considered in the preceding section, the results shown in Fig. 6 reduce to the corresponding ones discussed in Sec.3. This figure shows the regions in the \((\theta, \xi)\) plane, where particular components of the spin polarization are large and where are small or suppressed to zero.

The results in a general case, like those presented in Fig. 5, are required when considering magnetic dynamics induced by spin torque due to spin polarization. Magnetic moment (and thus also exchange field) precesses then in space, and this time evolution is associated with time evolution of the spin polarization. In this paper, however, we do not consider dynamical properties and focus rather on evaluating spin polarization in static situations.

V. RELATION WITH THE BERRY CURVATURE

Recently H. Kurebayashi et al.\(^{39} \) based on experimental results, have proposed the anti-damping spin-orbit torque mediated by the Berry phase. In other words, they showed that the Berry curvature gives rise to the spin-orbit torque in systems with broken inversion symmetry. Our results given by Eqs. (13)-(15) show that when the exchange field is nonzero, the inversion symmetry is broken and the general expressions for the \( x \) and \( z \) components of the spin polarization contain terms that do not depend on relaxation rate, but are functions of the Fermi-Dirac distribution function instead of its derivative. Thus, taking into account the notation well known in the context of the anomalous Hall effect, we can rewrite Eqs (13)-(15) as follows:

\[
S_\alpha = S_\alpha^I + S_\alpha^{II}
\]

(27)

where the first term depends on the states in a close vicinity of the Fermi level: \( S_\alpha^I = S_\alpha[f(E_\pm)] \), while the second term contains information from all electronic states: \( S_\alpha^{II} = S_\alpha[f(E_\pm)] \). Now we show that the terms \( S_\alpha^{II} \) are related to the Berry curvature.

To do this let us rewrite the Hamiltonian (1) in the following form:

\[
H = \varepsilon_0 + \mathbf{n} \cdot \mathbf{\sigma}
\]

(28)
FIG. 5. Current-induced spin polarization in case when the exchange field is parallel to the electric field for indicated parameters. Variation of the spin polarization components with the exchange field for indicated values of chemical potential (a,b); with temperature for $J M_0 = 10$ meV (red curves), $J M_0 = 1$ meV (blue curves), and for indicated chemical potential $\mu$ (c,d); with the chemical potential for indicated values of $J M_0$ (e,f); and with the Rashba coupling parameter $\alpha$ for indicated values of chemical potential (f,g). Note, the parameters used for the top panel are the same as the corresponding ones in the bottom panel. Other parameters as in Fig. 3

where $n = (\alpha k_x - H_x, -\alpha k_x - H_y, -H_z)$, and $\varepsilon_k = \hbar^2(k_x^2 + k_y^2)/2m$. The eigenvectors corresponding to the eigenvalues $E_\pm$ can be written as

$$|\Psi_+\rangle = \left(\sqrt{a_{k_x} + n_z}, a_{k_x} + n_z, \sqrt{2a_{k_x}(a_{k_x} + n_z)}\right),$$

$$|\Psi_-\rangle = \left(-\sqrt{2a_{k_x}(a_{k_x} + n_z)}, a_{k_x} + n_z, \sqrt{a_{k_x} + n_z}\right).$$

The Berry curvature of the $n$-th ($n = 1, 2$) band, $B^z_{n}(\mathbf{k})$, is defined as the rotation of the Berry connection $A_n(\mathbf{k}) = i \langle \Psi_n | \nabla_k | \Psi_n \rangle$ (for details see Refs [11,13]). Thus one can write

$$B^z_{n}(\mathbf{k}) = i \left[ \frac{\partial}{\partial k_x} \langle \Psi_n | \frac{\partial}{\partial k_y} | \Psi_n \rangle - \frac{\partial}{\partial k_y} \langle \Psi_n | \frac{\partial}{\partial k_x} | \Psi_n \rangle \right].$$

Combining Eqs. (28) to (30) we find for the Berry curvature,

$$B^z_{\pm}(\mathbf{k}) = \mp \frac{\alpha^2 H_z}{2a_{k_x}^3}.$$  

Taking the expression above into account, the Berry phase related terms in the electrically generated spin polarization can be written as

$$S_{x}^{II} = \frac{1}{2} e E_x h \sum_n \int \frac{d^2 k}{(2\pi)^2} f(E_n) B^z_{n}(\mathbf{k})$$

$$S_{y}^{II} = 0$$

$$S_{z}^{II} = -\frac{1}{2} e E_z h \sum_n \int \frac{d^2 k}{(2\pi)^2} f(E_n) \frac{\alpha k_y + H_y}{\alpha H_z} B^z_{n}(\mathbf{k})$$

Note, these terms disappear in the absence of exchange field.

VI. SPIN-ORBIT TORQUE

Due to exchange interaction, the current induced spin polarization exerts a torque $\tau$ on the magnetic moment $\mathbf{M}$. This torque enters the Landau-Lifshitz-Gilbert equation for magnetic dynamics,

$$\frac{\partial \mathbf{m}}{\partial t} = -\gamma \mathbf{m} \times \mathbf{h}_{\text{eff}} + \alpha_g \mathbf{m} \times \frac{\partial \mathbf{m}}{\partial t} + \tau,$$

where $\mathbf{m} = \mathbf{M}/M$ is a unit vector along magnetic moment $\mathbf{M}$, $\mathbf{h}_{\text{eff}}$ is the effective magnetic field which includes external magnetic field, dipolar field, and anisotropy field, $\alpha_g$ is the Gilbert damping factor, and $\gamma$ is the gyromagnetic factor.

To find the torque $\tau$ we write the coupling energy of the magnetic moment and induced spin polarization as $$(2J/\hbar)S \cdot \mathbf{M} = -\mathbf{M} \cdot \mathbf{h}_{\text{so}},$$ where $\mathbf{h}_{\text{so}}$ is defined as

$$\mathbf{h}_{\text{so}} = -\frac{2J}{\hbar} \mathbf{S}.$$ 

Taking the above into account, one can write the torque $\tau$ as a sum of a field-like torque $\tau_f$ and damping-like torque $\tau_d$,

$$\tau = \tau_f + \tau_d.$$ 

These components can be written in terms of the spin-orbit field $\mathbf{h}_{\text{so}}$ as

$$\tau_f = -\gamma \mathbf{m} \times \mathbf{h}_{\text{so}}$$

and the damping-like term, and

$$\tau_d = -\alpha_g \gamma \mathbf{m} \times (\mathbf{m} \times \mathbf{h}_{\text{so}}).$$
FIG. 6. Spin polarization induced by an external electric field in the presence of arbitrarily oriented exchange field. Left panel presents the three components of spin polarization as a function of $\theta$ and $\xi$. The right panel shows the corresponding vertical cross-sections (for a few values of the angle $\xi$). The parameters assumed in numerical calculations are: $\mu = 0.02$ eV, $\alpha = 2 \cdot 10^{-11}$ eVm, $JM_0 = 0.01$ eV, $\Gamma = 0.5 \cdot 10^{-6}$ eV and $T = 50$ K. The other parameters as in Fig.3.

for the damping-like term. Since the spin polarization includes terms related to the Berry curvature, the resulting spin-orbit torques include terms related to the Berry curvature as well.

VII. SUMMARY AND CONCLUSIONS

Using the Matsubara Green function method we have calculated current-induced spin polarization in a magnetized two-dimensional electron gas with the Rashba spin-orbit interaction. The exchange field is shown to have a significant impact on the spin polarization. First, for some orientations of the exchange field, the component of spin polarization that appears in the absence of exchange field can be enhanced by the exchange field, while for other orientations this component can be suppressed. Second, exchange field also generates the components of spin polarization which are absent in the limit of vanishing exchange field. We also note, that the states at the band edges may become localized due to disorder and the results may be not valid in the localization regime.

Analytical and/or numerical results have been presented in some special cases, when exchange field is oriented along current or perpendicular to current (in-plane and perpendicular to the plane of 2DEG in the latter case). Numerical results have been also presented in a general case of arbitrary orientation of exchange field. We have found that the exchange field leads to terms in the spin polarization that can be related to the Berry curvature of the corresponding electron bands. Since the calculated spin polarization generates a torque which may induce dynamics of the magnetic moment, this torque includes terms related to the Berry curvature as well.

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Appendix A: Derivation of Eqs. (13), (14), (15)

The current induced spin polarization is evaluated starting from the equation (10) that we rewrite in the following form:

$$S_\alpha = -eE_x \frac{\hbar}{\omega} \int \frac{d^2k}{(2\pi)^2} \left( T^{(1)}_{S_\alpha} + T^{(2)}_{S_\alpha} \right),$$

(A1)

where:

$$T^{(1)}_{S_\alpha} = \int \frac{d\epsilon}{2\pi} f(\epsilon) T^{(1)}_{S_\alpha} (\epsilon + \omega, \epsilon),$$

(A2)

$$T^{(2)}_{S_\alpha} = \int \frac{d\epsilon}{2\pi} f(\epsilon) T^{(2)}_{S_\alpha} (\epsilon, \epsilon - \omega),$$

(A3)
and the following notation has been introduced:

\[ T^{(1)}_{S_\alpha}(\varepsilon + \omega, \varepsilon) = \text{Tr} \left\{ \hat{s}_\alpha G^{R}_{k}(\varepsilon + \omega) \hat{\epsilon}_x [G^{R}_{k}(\varepsilon) - G^{A}_{k}(\varepsilon)] \right\}, \]  \hspace{1cm} (A4)

\[ T^{(2)}_{S_\alpha}(\varepsilon, \varepsilon - \omega) = \text{Tr} \left\{ \hat{s}_\alpha [G^{A}_{k}(\varepsilon) - G^{A}_{k}(\varepsilon)] \hat{\epsilon}_x G^{A}_{k}(\varepsilon - \omega) \right\}, \]  \hspace{1cm} (A5)

According to the above notation the \( S_x \) component of spin polarization is described by the following expressions:

\[ T^{(1)}_{S_x}(\varepsilon + \omega, \varepsilon) = \frac{\hbar^2 k_x}{2m\lambda_k} (a \k_y - H_x) \left[ G^{R}_{k-}(\varepsilon + \omega) G^{A}_{k-}(\varepsilon) - G^{R}_{k-}(\varepsilon + \omega) G^{R}_{k+}(\varepsilon) + G^{R}_{k+}(\varepsilon + \omega) G^{R}_{k+}(\varepsilon) - G^{R}_{k-}(\varepsilon + \omega) G^{A}_{k+}(\varepsilon) \right] - \frac{\alpha}{2\lambda_k} (a \k_y - H_x) (a \k_x + H_y) \left[ G^{A}_{k-}(\varepsilon + \omega) G^{A}_{k-}(\varepsilon) - G^{R}_{k-}(\varepsilon + \omega) G^{A}_{k+}(\varepsilon) - G^{A}_{k+}(\varepsilon + \omega) G^{R}_{k-}(\varepsilon) + G^{R}_{k+}(\varepsilon + \omega) G^{R}_{k+}(\varepsilon) \right] - \frac{\alpha}{2\lambda_k} (a \k_y - H_x) (a \k_x + H_y) \left[ G^{R}_{k-}(\varepsilon + \omega) G^{A}_{k-}(\varepsilon) - G^{R}_{k-}(\varepsilon + \omega) G^{A}_{k+}(\varepsilon) + G^{R}_{k+}(\varepsilon + \omega) G^{R}_{k-}(\varepsilon) + G^{R}_{k+}(\varepsilon + \omega) G^{R}_{k+}(\varepsilon) \right] - \frac{i}{2\lambda_k} H_z \left[ G^{R}_{k-}(\varepsilon + \omega) G^{A}_{k+}(\varepsilon) - G^{A}_{k+}(\varepsilon + \omega) G^{A}_{k+}(\varepsilon) + G^{R}_{k+}(\varepsilon + \omega) G^{A}_{k+}(\varepsilon) - G^{A}_{k+}(\varepsilon + \omega) G^{R}_{k+}(\varepsilon) \right] \]  \hspace{1cm} (A6)

\[ T^{(2)}_{S_x}(\varepsilon, \varepsilon - \omega) = \frac{\hbar^2 k_x}{2m\lambda_k} (a \k_y - H_x) \left[ G^{A}_{k-}(\varepsilon) G^{A}_{k-}(\varepsilon) - G^{A}_{k+}(\varepsilon) G^{A}_{k+}(\varepsilon) - G^{A}_{k+}(\varepsilon) G^{A}_{k-}(\varepsilon) + G^{A}_{k+}(\varepsilon) G^{A}_{k+}(\varepsilon) \right] - \frac{\alpha}{2\lambda_k} (a \k_x + H_y) (a \k_y - H_x) \left[ G^{A}_{k-}(\varepsilon - \omega) G^{A}_{k-}(\varepsilon) - G^{A}_{k+}(\varepsilon) G^{A}_{k+}(\varepsilon) - G^{A}_{k+}(\varepsilon) G^{A}_{k-}(\varepsilon) + G^{A}_{k+}(\varepsilon) G^{A}_{k+}(\varepsilon) \right] - \frac{\alpha}{2\lambda_k} (a \k_x + H_y) (a \k_y - H_x) \left[ G^{A}_{k-}(\varepsilon - \omega) G^{A}_{k-}(\varepsilon) - G^{A}_{k+}(\varepsilon) G^{A}_{k+}(\varepsilon) + G^{A}_{k+}(\varepsilon) G^{A}_{k+}(\varepsilon) - G^{A}_{k+}(\varepsilon) G^{A}_{k+}(\varepsilon) \right] - \frac{i}{\lambda_k} H_z \left[ G^{A}_{k-}(\varepsilon - \omega) G^{A}_{k+}(\varepsilon) - G^{A}_{k+}(\varepsilon - \omega) G^{A}_{k+}(\varepsilon) + G^{A}_{k+}(\varepsilon - \omega) G^{A}_{k+}(\varepsilon) - G^{A}_{k+}(\varepsilon - \omega) G^{A}_{k+}(\varepsilon) \right] \]  \hspace{1cm} (A7)

Inserting Eqs. (A6), (A7) into Eqs. (A2) and (A3) respectively we get:

\[ \text{Re} \left[ T^{(1)}_{S_x} + T^{(2)}_{S_x} \right] = \frac{\hbar^2 k_x}{2m\lambda_k} (a \k_y - H_x) \left[ \frac{2\Gamma}{\omega^2 + (2\Gamma)^2} f'(E+) - f'(E-) \right] + \frac{\alpha}{2\lambda_k} (a \k_x + H_y) (a \k_y - H_x) \left[ \frac{2\Gamma}{\omega^2 + (2\Gamma)^2} f'(E+) - f'(E-) \right] - \frac{\omega\Gamma}{\lambda_k} (a \k_x + H_y) (a \k_y - H_x) \left[ \frac{f'(E_+)}{(E_+ - E_+ - \omega)^2 + (2\Gamma)^2} + \frac{f'(E_-)}{(E_+ - E_- - \omega)^2 + (2\Gamma)^2} \right] - \frac{\omega\Gamma}{\lambda_k} H_z \left[ \frac{f'(E_+)}{(E_+ - E_+ - \omega)^2 + (2\Gamma)^2} - \frac{f'(E_-)}{(E_+ - E_- - \omega)^2 + (2\Gamma)^2} \right] + \frac{\omega}{2\lambda_k} H_z \left[ \frac{E_+ - E_-}{(E_+ - E_- - \omega)^2 - \omega^2} f'(E_+) - f'(E_-) \right] - \omega \frac{\lambda_k}{\lambda_k} \left[ \frac{E_+ - E_-}{(E_+ - E_- - \omega)^2 - \omega^2} f'(E_+) - f'(E_-) \right] + \omega \frac{\lambda_k}{\lambda_k} H_z \left[ \frac{f'(E_+)}{E_+ - E_- - \omega} - \frac{f'(E_-)}{E_+ - E_- - \omega} \right] \]  \hspace{1cm} (A8)

In the limit of \( \omega \to 0 \) we find \( x \) component of current-induced spin polarization given by Eq. (13).

In turn, the \( S_y \) component of spin polarization is expressed by the following functions:

\[ T^{(1)}_{S_y}(\varepsilon + \omega, \varepsilon) = \frac{\alpha}{2} \left[ G^{R}_{k-}(\varepsilon + \omega) G^{A}_{k-}(\varepsilon) + G^{R}_{k+}(\varepsilon + \omega) G^{A}_{k+}(\varepsilon) - G^{R}_{k-}(\varepsilon + \omega) G^{A}_{k+}(\varepsilon) - G^{R}_{k+}(\varepsilon + \omega) G^{A}_{k-}(\varepsilon) \right] - \frac{\hbar^2 k_x}{2m\lambda_k} (a \k_y + H_y) \left[ G^{R}_{k-}(\varepsilon + \omega) G^{A}_{k-}(\varepsilon) - G^{R}_{k-}(\varepsilon + \omega) G^{R}_{k+}(\varepsilon) + G^{R}_{k+}(\varepsilon + \omega) G^{R}_{k+}(\varepsilon) - G^{R}_{k-}(\varepsilon + \omega) G^{A}_{k+}(\varepsilon) \right] - \frac{\alpha}{2\lambda_k} (a \k_y + H_y) \left[ (a \k_y - H_x)^2 + H_y^2 \right] \left[ G^{R}_{k-}(\varepsilon + \omega) G^{A}_{k-}(\varepsilon) - G^{R}_{k-}(\varepsilon + \omega) G^{A}_{k+}(\varepsilon) + G^{R}_{k-}(\varepsilon + \omega) G^{R}_{k+}(\varepsilon) + G^{R}_{k+}(\varepsilon + \omega) G^{R}_{k+}(\varepsilon) \right] + G^{R}_{k-}(\varepsilon + \omega) G^{R}_{k+}(\varepsilon) - G^{R}_{k-}(\varepsilon + \omega) G^{R}_{k+}(\varepsilon) + G^{R}_{k+}(\varepsilon + \omega) G^{R}_{k+}(\varepsilon) \]  \hspace{1cm} (A9)
\[ I_{S_y}^{(2)}(\varepsilon, \varepsilon - \omega) = \frac{\alpha}{2} \left[ G^{R}_{k-}(\varepsilon)G^{A}_{k-}(\varepsilon - \omega) - G^{R}_{k-}(\varepsilon)G^{A}_{k-}(\varepsilon - \omega) + G^{A}_{k+}(\varepsilon)G^{A}_{k+}(\varepsilon - \omega) - G^{R}_{k+}(\varepsilon)G^{A}_{k+}(\varepsilon - \omega) \right] \]

\[ - \frac{\hbar^2 k_x}{2m \lambda_k} (\alpha k_x + H_y) \left[ G^{A}_{k-}(\varepsilon)G^{A}_{k-}(\varepsilon - \omega) - G^{A}_{k-}(\varepsilon)G^{A}_{k-}(\varepsilon - \omega) - G^{R}_{k+}(\varepsilon)G^{A}_{k+}(\varepsilon - \omega) + G^{R}_{k+}(\varepsilon)G^{A}_{k+}(\varepsilon - \omega) \right] \]

\[ - \frac{\alpha}{2 \lambda_k^2} \left[ (\alpha k_y - H_x)^2 + H_y^2 \right] \left[ G^{A}_{k-}(\varepsilon)G^{A}_{k-}(\varepsilon - \omega) - G^{A}_{k+}(\varepsilon)G^{A}_{k+}(\varepsilon - \omega) - G^{R}_{k+}(\varepsilon)G^{A}_{k+}(\varepsilon - \omega) + G^{R}_{k+}(\varepsilon)G^{A}_{k+}(\varepsilon - \omega) \right] \]

(A10)

After integration over \( \varepsilon \) in Eqs. (A2) and (A3) with integrands given by (A9), (A10) we obtain the following expression:

\[ 9 \text{Re} \left[ I_{S_y}^{(1)} + I_{S_y}^{(2)} \right] = \frac{\hbar^2 k_x}{2m \lambda_k} (\alpha k_x + H_y) \frac{2 \Gamma_\omega}{\omega^2 + (2 \Gamma)^2} \left[ f'(E_+) - f'(E_-) \right] - \frac{\alpha}{\lambda_k^2} (\alpha k_x + H_y) \frac{2 \Gamma_\omega}{\omega^2 + (2 \Gamma)^2} \left[ f'(E_+) - f'(E_-) \right] \]

\[ - \frac{\alpha}{\lambda_k^2} \Gamma_\omega \left[ (\alpha k_y - H_x)^2 + H_y^2 \right] \left[ \frac{f'(E_-)}{(E_+ - E_- - \omega)^2 + (2 \Gamma)^2} + \frac{f'(E_+)}{(E_+ - E_- + \omega)^2 + (2 \Gamma)^2} \right] \]

(A11)

In the limit \( \omega \to 0 \) we obtain the formula describing \( y \) component of current-induced spin polarization given by Eq. (13).

Finally, the \( S_z \) component of the nonequilibrium spin polarization is described by following traces:

\[ I_{S_z}^{(1)}(\varepsilon, \varepsilon, \varepsilon) = \frac{\alpha^2 k_x}{2 \lambda_k^2} H_z \left[ G^{R}_{k-}(\varepsilon + \omega)G^{A}_{k-}(\varepsilon - \omega) - G^{R}_{k-}(\varepsilon + \omega)G^{A}_{k-}(\varepsilon - \omega) - G^{R}_{k+}(\varepsilon + \omega)G^{R}_{k+}(\varepsilon) \right] \]

\[ - G^{R}_{k+}(\varepsilon + \omega)G^{A}_{k-}(\varepsilon + \omega)G^{A}_{k+}(\varepsilon) + G^{R}_{k+}(\varepsilon + \omega)G^{R}_{k-}(\varepsilon) - G^{R}_{k+}(\varepsilon + \omega)G^{R}_{k+}(\varepsilon) \]

\[ + \frac{\hbar^2 k_x}{2m \lambda_k} H_z \left[ G^{R}_{k-}(\varepsilon + \omega)G^{R}_{k-}(\varepsilon + \omega)G^{A}_{k-}(\varepsilon) + G^{R}_{k+}(\varepsilon + \omega)G^{A}_{k+}(\varepsilon) - G^{R}_{k+}(\varepsilon + \omega)G^{A}_{k+}(\varepsilon) \right] \]

\[ + \frac{\alpha}{\lambda_k^2} (H_x - \alpha k_y) \left[ G^{R}_{k-}(\varepsilon + \omega)G^{A}_{k+}(\varepsilon) + G^{R}_{k-}(\varepsilon + \omega)G^{R}_{k+}(\varepsilon) - G^{R}_{k+}(\varepsilon + \omega)G^{R}_{k+}(\varepsilon) \right] \]

(A12)

\[ I_{S_z}^{(2)}(\varepsilon, \varepsilon - \omega) = \frac{\alpha^2 k_x}{2 \lambda_k^2} H_z \left[ G^{A}_{k-}(\varepsilon)G^{A}_{k-}(\varepsilon - \omega) - G^{A}_{k+}(\varepsilon)G^{A}_{k-}(\varepsilon - \omega) - G^{R}_{k-}(\varepsilon)G^{A}_{k-}(\varepsilon - \omega) + G^{R}_{k+}(\varepsilon)G^{A}_{k-}(\varepsilon - \omega) \right] \]

\[ - G^{A}_{k-}(\varepsilon)G^{A}_{k+}(\varepsilon - \omega) + G^{A}_{k+}(\varepsilon)G^{A}_{k+}(\varepsilon - \omega) - G^{R}_{k+}(\varepsilon)G^{A}_{k+}(\varepsilon - \omega) - G^{R}_{k+}(\varepsilon)G^{A}_{k+}(\varepsilon - \omega) \]

\[ + \frac{\hbar^2 k_x}{2m \lambda_k} H_z \left[ G^{A}_{k+}(\varepsilon)G^{A}_{k+}(\varepsilon - \omega) - G^{A}_{k-}(\varepsilon)G^{A}_{k-}(\varepsilon - \omega) - G^{R}_{k+}(\varepsilon)G^{A}_{k+}(\varepsilon - \omega) - G^{R}_{k+}(\varepsilon)G^{A}_{k+}(\varepsilon - \omega) \right] \]

\[ + \frac{\alpha}{\lambda_k^2} (H_x - \alpha k_y) \left[ G^{A}_{k-}(\varepsilon)G^{A}_{k+}(\varepsilon - \omega) - G^{A}_{k+}(\varepsilon)G^{A}_{k-}(\varepsilon - \omega) - G^{R}_{k+}(\varepsilon)G^{A}_{k+}(\varepsilon - \omega) + G^{R}_{k+}(\varepsilon)G^{A}_{k+}(\varepsilon - \omega) \right] \]

(A13)
These two equations combining with Eqs. 2 and 3 lead to the following expression:

\[
\Re \left[ T_{S_k}^{II} + T_{S_k}^{III} \right] = -\omega \frac{2\Gamma}{\omega^2 + (2\Gamma)^2} \frac{\hbar^2 k_x}{2m\lambda_k} \left[ f'(E_+) - f'(E_-) \right] - \omega \frac{\alpha}{2\lambda_k} H_z (ak_x + H_y) \frac{2\Gamma}{\omega^2 + (2\Gamma)^2} \left[ f'(E_+) + f'(E_-) \right] + \omega \frac{\alpha}{\lambda_k} (ak_x + H_y) H_z \left[ \frac{f'(E_-)}{(E_+ - E_- - \omega)^2 + (2\Gamma)^2} + \frac{f'(E_+)}{(E_+ - E_- + \omega)^2 + (2\Gamma)^2} \right] + \omega \frac{\alpha}{\lambda_k} (H_x - ak_y) \left[ \frac{E_+ - E_- - \omega}{(E_+ - E_- - \omega)^2 + (2\Gamma)^2} f'(E_-) + \frac{E_+ - E_- + \omega}{(E_+ - E_- + \omega)^2 + (2\Gamma)^2} f'(E_+) \right] - \omega \alpha (H_x - ak_y) \frac{f'(E_+) + f'(E_-)}{(E_+ - E_-)^2 - \omega^2} - \omega \frac{\alpha}{\lambda_k} (H_x - ak_y) \frac{f(E_-) - f(E_+)}{(E_+ - E_-)^2 - \omega^2} - \omega^2 \frac{\alpha}{2\lambda_k} (H_x - ak_y) \frac{f'(E_-) - f'(E_+)}{(E_+ - E_-)^2 - \omega^2}
\]

(A14)

In the dc-limit we get Eq. 15.
Jungwirth, Nature Nanotech. 9, 211217 (2014).
40 M. V. Berry, Proc. R. Soc. Lond. Ser. A 392, 45 (1984).
41 G. E. Volovik, Zh. Eksp. Teor. Fiz. 94, 123 (1988).
42 Di Xiao, Ming-Che Chang, and Qian Niu, Rev. Mod. Phys. 82, 1959 (2010).
43 Naoto Nagaosa, Jairo Sinova, Shigeki Onoda, A. H. MacDonald, and N. P. Ong, Rev. Mod. Phys. 82, 1539 (2010).