A Multi-Bennett 8R Mechanism Obtained From Factorization of Bivariate Motion Polynomials

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We present a closed-loop 8R mechanism with two degrees of freedom whose motion exhibits curious properties. In any point of a two-dimensional component of its configuration variety it is possible to fix every second joint while retaining one degree of freedom. This shows that the even and the odd axes, respectively, always form a Bennett mechanism. In this mechanism, opposite distances and angles are equal and all offsets are zero. The 8R mechanism has four “totally aligned” configurations in which the common normals of any pair of consecutive axes coincide.

1 Introduction

Overconstrained linkages is a long-lasting but still highly active topic of research in mechanism science. For several decades, researchers focused on overconstrained mechanisms consisting of a single loop of \( n \leq 6 \) revolute joints (R), prismatic joints (P), or, sometimes, helical joints (H). New linkages of that type are continuously being discovered, often by craftily combining known linkages [2, 3, 28], sometimes via novel concepts for their construction. One of these concepts is the factorization of motion polynomials [8]. It gave rise to the construction of the only class of overconstrained 6R linkages with still unknown relations between its Denavit-Hartenberg parameters. In [6, 9, 17, 20], motion polynomial factorization was exploited for the synthesis of linkages.

In spite of some attempts, a complete classification of overconstrained single-loop linkages is currently out of reach. It is thus natural that research efforts shifted towards the investigation of single-loop linkages consisting of \( n \geq 7 \) links with, generically, \( n-6 \geq 1 \) degrees of freedom. (The classification of single-loop linkages with \( n \geq 7 \) links and more
than \(n - 6\) degrees of freedom has recently been completed in [4].) A guiding principle for their construction is existence of “interesting” properties of the mechanism’s motion or its configuration variety. One example is [13], where 7R linkages whose configuration variety contains irreducible components of different dimensions – a property that has been named kinematotropic in [29] – are constructed. [24] combines mobile 4R linkages (Bennett linkages) or RPRP linkages to loops of 7R/P joints whose configuration variety is reducible. The motion of the original 4R/P linkages is obtained by locking of joints in certain configurations. Analogically [23] restricted a specially designed single-loop 8R mechanism to its possible sub-motions. [20] and [19] pursue similar aims but use motion polynomial factorization techniques. Joint locking is also used in [12] for restricting a mechanism to a certain subvariety of its total configuration space, although for a class of parallel mechanisms.

Our contribution in this article is of similar spirit as the works cited above but also differs in several aspects. We present an 8R linkage with two degrees of freedom that has the weird property that it retains one degree of freedom when simultaneously locking every second joint in any configuration of a two-dimensional subvariety of its total configuration space. This unique property immediately implies that the quadruples of “even” or “odd” axes form respective Bennett mechanisms in any configuration because Bennett mechanisms constitute the only class of mobile spatial closed-loop linkages with four revolute axes [11]. We therefore refer to this mechanism by the name “multi-Bennett 8R mechanism”. Our aim in this paper is to prove these properties by an algebraic construction and use this to derive some geometric and kinematic characteristics of the thus obtained mechanism.

While combination of Bennett linkages is a common technique in this area [1, 7, 13, 24, 27], our example seems to be novel. It is not geometrically motivated – at least in the current state of our understanding – but rather based on an algebraic construction. As suggested by examples in [16], there exist bivariate motion polynomials that admit, in a non-trivial way, two factorizations into products of linear univariate factors with alternating indeterminates. These two times four factors give rise to the revolute axes of the 8R mechanism and describe their relative motions. The underlying bivariate factorization theory is currently being explored [15, 16] and is considerably harder than in the univariate case. This is witnessed by our proof of existence in Theorem 3.3.

In spite of its algebraic construction, the 8R linkage is subject to severe geometric constraints. We demonstrate this by computing simple necessary relations between its Denavit-Hartenberg parameters in Theorem 4.2. In Theorems 4.1 and 4.3 we describe remarkable properties of several discrete configurations. Our algebraic approach is efficient for proving existence of multi-Bennett 8R mechanisms and some aspects of its geometry. A complete geometric characterization, which exists for the vast majority of comparable mechanisms, can not be obtained in this way and is probably rather difficult.

We feel that its numerous special properties (simple Denavit-Hartenberg parameters, two degrees of freedom that are easy to control via low-degree rational parametrization, existence of special configurations) make our mechanism a promising candidate for yet to be explored applications.

We continue this text by recalling some basic facts about motion polynomials and
their relation to mechanism science in Section 2. In Section 3 we provide a proof for existence of quaternion polynomials with two non-trivial univariate factorizations. The proof is constructive and provides a good method to directly compute the underlying 8R linkage. Nonetheless, we found the procedure insufficient for obtaining results that are suitable for further processing and in particular for the computation of Denavit-Hartenberg parameters. Thus, our further investigation of the multi-Bennett 8R mechanism in Section 4 is based on carefully selected coordinate frames and configurations. This simplification results in formulas that are tractable by means of computer algebra and, ultimately, provides the desired necessary relations among the Denavit-Hartenberg parameters (Theorem 4.2).

This paper is a continuation of [14], a conference paper which verifies most of the claims made in this article at hand of a concrete numeric example. Strict mathematical proofs of the claimed facts are presented herein for the first time.

### 2 Preliminaries

Our construction of the multi-Bennett mechanism is based on certain factorizations of bivariate quaternion polynomials. In this section we provide a brief introduction to some fundamental concepts that will be used later in this text and we settle our notation.

Denote by $\mathbb{H}$ the four-dimensional associative real algebra of (real) quaternions. It is generated by basis elements $i, j,$ and $k$ via the relations

$$i^2 = j^2 = k^2 = ijk = -1.$$  

A quaternion $h \in \mathbb{H}$ can be written as $h = h_0 + h_1i + h_2j + h_3k$ with real numbers $h_0, h_1, h_2, h_3$. Extending real scalars to dual numbers $a + b\varepsilon$ with $a, b \in \mathbb{R}$ and $\varepsilon^2 = 0$ yields the algebra $\mathbb{DH}$ of dual quaternions

$$\mathbb{DH} = \{h = hp + \varepsilon hd \mid hp, hd \in \mathbb{H}\}.$$  

The quaternions $hp$ and $hd$ are called **primal part** and **dual part** of $h$. Any quaternion can be viewed as a dual quaternion with vanishing dual part. We will therefore sometimes use the same symbol $h$ for real ($h \in \mathbb{H}$) and dual ($h \in \mathbb{DH}$) quaternions.

The conjugate dual quaternion $h^*$ is obtained by replacing $i, j,$ and $k$ with $-i, -j,$ and $-k$, respectively, the $\varepsilon$-conjugate $h_\varepsilon$ of a dual quaternion is obtained by replacing $\varepsilon$ with $-\varepsilon$.

Given a dual quaternion $h = hp + \varepsilon hd$, where $hp = h_0 + h_1i + h_2j + h_3k$ and $hd = h_4 + h_5i + h_6j + h_7k$, the value $\text{Scal}(h) := \frac{1}{2}(h + h^*) = h_0 + \varepsilon h_4$ is called the **scalar part** and $\text{Vect}(h) := \frac{1}{2}(h - h^*) = h_1i + h_2j + h_3k + \varepsilon(h_5i + h_6j + h_7k)$ the **vector part** of $h$.

The dual quaternion norm of $h$ is $hh^*$. It is the dual number

$$hh^* = hp hp^* + \varepsilon(hp hd^* + hd hp^*) = h_0^2 + h_1^2 + h_2^2 + h_3^2 + 2\varepsilon(h_0 h_4 + h_1 h_5 + h_2 h_6 + h_3 h_7) \in \mathbb{D}.$$  

(1)
Dual quaternions satisfying $hh^* = 1$ are said to be normalized or unit. In this case, the dual part in (1) vanishes, that is

$$h_0h_4 + h_1h_5 + h_2h_6 + h_3h_7 = 0.$$  \hspace{1cm} (2)

This is well-known under the name Study condition.

The dual quaternion $h = h_p + \varepsilon h_d$ is invertible if and only if $h_p \neq 0$. In this case, we have

$$h^{-1} = h_p^{-1}(1 - \varepsilon h_d h_p^{-1}), \quad \text{where} \quad h_p^{-1} = \frac{h_p^*}{h_p h_p^*}.$$  \hspace{1cm} (4)

If $h$ is unit, then $h^{-1} = h^*$. The multiplicative sub-group $\mathbb{DH}^\times := \{ h \in \mathbb{DH} \mid hh^* \in \mathbb{R} \setminus \{0\} \}$ modulo the real multiplicative group $\mathbb{R}^\times$ is isomorphic to $\text{SE}(3)$, the group of rigid body displacements.

Using homogeneous coordinates in the projective space $\mathbb{P}^3(\mathbb{R}) = \mathbb{P}(\langle 1, \varepsilon i, \varepsilon j, \varepsilon k \rangle)$, the action of $h \in \mathbb{DH}^\times$ on $x = x_0 + \varepsilon(x_1i + x_2j + x_3k) \in \mathbb{P}^3(\mathbb{R})$ is given by

$$x \mapsto hxh^*.$$  \hspace{1cm} (3)

2.1 Dual Quaternions and Line Geometry

In this paper, rotations around a fixed axis but with variable rotation angle will play an important role. We therefore have a closer look at the representation of straight lines (revolute axes) and rotations within the framework of dual quaternions. Identifying the oriented revolute axis with normalized Plücker coordinates $(p_1, p_2, p_3, q_1, q_2, q_3)$ in the sense of [25, Section 2] with the unit dual quaternion $r = p_1i + p_2j + p_3k + \varepsilon(q_1i + q_2j + q_3k)$, the rotation with angle $\varphi$ around $r$ is given by the unit dual quaternion

$$h := \cos(\varphi) + \sin(\varphi)r, \quad \varphi \in [0, 2\pi),$$  \hspace{1cm} (4)

or, because we use homogeneous coordinates, by any of its non-zero real multiples. Note that the dual quaternion $h$ of (4) satisfies the Study condition (2) because $r$ satisfies the Plücker condition $rr^* \in \mathbb{R} \setminus \{0\}$.

The action (3) on points can be used to transform straight lines by transforming points on them. Points on a straight line given by its Plücker coordinates can be found, for example, by [25, Equation (2.4)]. A straightforward calculation also provides us with a direct formula for displacing a straight line $\ell$ whose Plücker coordinates are given as vectorial dual quaternions:

$$\ell \mapsto (h\ell h^*)_\ell.$$  \hspace{1cm} (5)

2.2 Dual Quaternion Polynomials

The representation (4) of a rotation around an oriented general axis is only unique up to multiplication with a real scalar. Assuming, for the time being, $\varphi \neq 0$, we can divide (4) by $\sin(\varphi)$, substitute $\cot(\varphi)$ with $-t$ and multiply the result with $-1$ to see that the linear dual quaternion polynomial

$$C = t - r, \quad t \in \mathbb{R}$$  \hspace{1cm} (6)
parametrizes all rotations with non-vanishing rotation angle around $r$ as well. In order
to also account for $\varphi = 0$, we should extend the parameter range in (6) to $\mathbb{R} \cup \{\infty\}$. With
the natural understanding that $C(\infty) := \lim_{t \to \infty} \frac{1}{t} C(t) = 1$, the parameter value $t = \infty$
indeed corresponds to the rotation angle $\varphi = 0$, that is, the identity transformation.

We would like to emphasize that the dual quaternion in (4) represents a rotation with
a fixed rotation axis $r$ and rotation angle $\varphi$. The polynomial in (6) parametrizes all
rotations with fixed rotation axis $r$. The rotation angle is dependent on the parameter
$t$ (we have $\varphi = 2 \arccot(-t)$).

More generally, we can consider arbitrary polynomials $C = \sum_{i=0}^d c_i t^i$ with coefficients $c_i \in \mathbb{DH}$. Since the indeterminate $t$ typically serves as a real parameter in our context,
multiplication, conjugation and evaluation at real values of polynomials are defined by
the conventions that $t$ commutes with all coefficients and $t^* = t$. The thus obtained ring
of polynomials is denoted by $\mathbb{DH}[t]$. Similarly, we can also consider the ring of bivariate
dual quaternion polynomials $\mathbb{DH}[s, t]$ in $s$ and $t$. Its multiplication, conjugation and
evaluation at real values is defined by similar conventions and the assumption that $s$ and $t$ commute with all coefficients and with each other.

The linear polynomial $t - r$ from Equation (6) satisfies $(t - r)(t - r)^* \in \mathbb{R}[t]$ and also
$(t - r)(t - r)^* \neq 0$ (note that $r$ satisfies the Plücker condition). A generalization of this
property leads to

Definition 1. A polynomial $C \in \mathbb{DH}[t]$ or in $\mathbb{DH}[s, t]$ is called a motion polynomial if
$CC^* \in \mathbb{R}[t]$ or in $\mathbb{R}[s, t]$, respectively, and $CC^* \neq 0$.

The name “motion polynomial” is justified by the observation that the action (3) on
points allows the parametric version

$$x \mapsto y(s, t) = C_\varepsilon(s, t) x C^*(s, t), \quad s, t \in (\mathbb{R} \cup \{\infty\}) \times (\mathbb{R} \cup \{\infty\}).$$

Equation (7) is a polynomial map in homogeneous coordinates. The Cartesian coordinates of $y(s, t)$ are rational functions so that (7) describes a rigid body motion with rational surfaces as trajectories.

Univariate motion polynomials have been originally defined in [8]. There, it was implicitly assumed that motion polynomials are monic. We rather replace this assumption
by the condition $CC^* \neq 0$ which, together with a proper evaluation at $s = \infty$ or $t = \infty$
and the possibility of rational re-parametrizations, suffices for our purpose.

Definition 2. The value of the motion polynomial $C \in \mathbb{DH}[t]$ at $t = \infty$ is defined as
$C(\infty) := \lim_{t \to \infty} t^{-\deg C} C(t)$. It is the leading coefficient of $C$. The value of $C \in \mathbb{DH}[s, t]$ at $(s, t)$ where $s = \infty$ or (not exclusively) $t = \infty$ is defined by similar limits. It is the leading coefficient in $s$ or $t$ (or in both), respectively.

Re-parametrizations that preserve polynomiality and degree of univariate motion polynomials are maps of the form

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}, \quad a, b, c, d \in \mathbb{R}, \quad ad - bc \neq 0,$$
combined with multiplying away denominators. With \( C = \sum_{i=0}^{d} c_i t^i \) we have
\[
C(\tau) = \sum_{i=0}^{d} c_i (a\tau + b)^i (c\tau + d)^{d-i}.
\]

It is noteworthy that \( (8) \) naturally is a map from \( \mathbb{R} \cup \{\infty\} \) to \( \mathbb{R} \cup \{\infty\} \). Assuming \( c \neq 0 \), we have
\[
\infty \mapsto \frac{a}{c} \quad \text{and} \quad -\frac{d}{c} \mapsto \infty.
\]

If \( c = 0 \), then \( \infty \) is a fix point of \( (8) \). Re-parametrizations of type \( (8) \) do not change the property of being a motion polynomial.

An extension of \( (8) \) to bivariate polynomials is straightforward. Let us illustrate some definitions and concepts so far for linear motion polynomials, which constitute an important special example.

**Example 2.1.** The linear polynomial \( C = t - h \) with \( h = h_p + \varepsilon h_d \), where \( h_p = h_0 + h_1 i + h_2 j + h_3 k \) and \( h_d = h_4 + h_5 i + h_6 j + h_7 k \), is a motion polynomial by Definition 1 if
\[
CC^* = (t - h)(t - h)^* = t^2 - (h + h^*) t + hh^*
\]
is a real polynomial. This is equivalent to \( h + h^* = 2(h_0 + \varepsilon h_4) \) and \( hh^* = h_0^2 + h_1^2 + h_2^2 + h_3^2 + 2\varepsilon(h_0 h_4 + h_1 h_5 + h_2 h_6 + h_3 h_7) \) both being real whence \( h_d + h_d^* = 0 \) and \( h_p h_d^* + h_d h_p^* = 0 \) or, equivalently,
\[
h_4 = 0 \quad \text{and} \quad h_1 h_5 + h_2 h_6 + h_3 h_7 = 0. \tag{9}
\]

In this case the motion polynomial \( C \) describes a rotation around the straight line with Plücker coordinates \( \text{Vect}(h) = h_1 i + h_2 j + h_3 k + \varepsilon(h_5 i + h_6 j + h_7 k) \), a fact which should not be surprising. We already demonstrated the relation between linear motion polynomials and rotations. The second equation in \( (9) \) is just the Plücker condition. Note that the rotation angle depends on both, \( t \) and \( h \). By Definition 2 the value \( C(\infty) \) equals \( \lim_{t \to \infty} t^{-1} C(t) = 1 \) which is the identity displacement. Obviously, \( C(0) = -h \). The re-parametrization \( \tau \mapsto \frac{1}{\tau} \) is of type \( (8) \) with \( a = d = 0 \) and \( b = c = 1 \). It interchanges 0 and \( \infty \). Indeed, \( C(\tau) = 1 - \tau h \) and
\[
C(\tau)|_{\tau=0} = 1, \quad C(\tau)|_{\tau=\infty} = \lim_{\tau \to \infty} \tau^{-1} C(\tau) = -h,
\]
as expected.

In the next section we will study bivariate motion polynomials which can be written as products of linear motion polynomials.

### 3 Alternating Factorizations of Bivariate Quaternion Polynomials

Given a bivariate dual quaternion polynomial \( C \in \mathbb{DH}[s,t] \), we denote its bi-degree by \( \text{bdeg}(C) \). We wish to find a motion polynomial \( C \in \mathbb{DH}[s,t] \) with \( \text{bdeg}(C) = (2,2) \) that
admits two different factorizations with alternating univariate linear factors, i.e.,

\[ C = (t - h)(s - \ell)(t - m)(s - n) = (s - n')(t - m')(s - \ell')(t - h'), \quad (10) \]

where \( h, \ell, m, n, n', m', \ell', h' \in \mathbb{DH} \setminus \{0\} \) and \((t - r)(t - r')^* = (t - r')(t - r)^* \in \mathbb{R}[t], (s - u)(s - u')^* = (s - u')(s - u)^* \in \mathbb{R}[s] \) for \( r \in \{h, m\} \) and \( u \in \{\ell, n\} \), a requirement that is seen to be necessary by taking norms on both sides of \([10]\). We call these factorizations alternating since the \( s \)- and \( t \)-factors appear in alternating order. By the considerations in Section 2, the linear factors will represent rotations around fixed axes.

Motion polynomials of shape \([10]\) immediately lead to closed-loop 8R mechanisms with the properties mentioned in the introduction:

- Each factorization gives rise to a two-parametric motion of an open 4R chain. Since the factorizations agree, the two distal links can be rigidly connected to form a closed-loop 8R linkage with the same two degrees of freedom.

- In this two-dimensional motion component (we conjecture that other components exist as well), the motion of any axis is determined by either \( s \) or \( t \). Locking one axis, that is, fixing \( s \) or \( t \), automatically locks every second axis while the axes parametrized by the other parameter still move. In terms of linear motion polynomials, the expression “locking an axis” can be read as follows: Each linear polynomial in \([10]\) represents a rotation around a fixed axis (c. f. Example 2.1). As outlined in the previous section, the rotation angle is dependent on the dual quaternion as well as on \( t \). We may now choose a fixed real number \( t_0 \in \mathbb{R} \) and consider the expressions \( t_0 - h, t_0 - m, t_0 - m' \) and \( t_0 - h' \). All of them represent rotations with a fixed rotation angle and we refer to the respective rotation axes as “locked axes”. By locking all axes parametrized by \( t \), we obtain a “sub-mechanism” with four moving axes (the axes parametrized by \( s \)) which, by a naive counting of degrees of freedom, should be rigid. However, the two factorizations in Equation \([10]\) guarantee that the sub-mechanism still moves. Such a closed-loop mobile 4R mechanism is called a Bennett linkage and we use the term “s-Bennett linkage”. Interchanging \( s \) and \( t \) leads to another sub-mechanism, the “t-Bennett linkage”.

Up to now, only isolated examples of this kind of polynomials have been known (c. f. [14, 16]). We will present a systematic construction of these polynomials, and thus of multi-Bennett 8R mechanisms, and start with a simple yet crucial lemma:

**Lemma 3.1.** Let \( C \in \mathbb{DH}[s,t] \) be a dual quaternion polynomial that admits two alternating factorizations:

\[ C = (t - h)(s - \ell)(t - m)(s - n) = (s - n')(t - m')(s - \ell')(t - h'), \quad (11) \]

Then

\[ (s - \ell)(s - n) = (s - n')(s - \ell') \quad \text{and} \quad (t - h)(t - m) = (t - m')(t - h'). \]
where \((s - \ell)(s - n) = (s - n')(s - \ell')\). The second statement follows by interchanging the roles of \(s\) and \(t\).

**Remark 3.2.** If the polynomial \(C = (t - h)(s - \ell)(t - m)(s - n)\) admits a second alternating factorization as in (11), it can always be computed by so-called Bennett flips \([18, Definition 4]\). The name is motivated by the observation that the revolute axes to \(\ell, n, \ell', \) and \(n'\) (and also to \(h, m, h'\) and \(m'\)) form, in that order, a Bennett linkage. More precisely, the quaternions \(n', \ell', m', h' \in \mathbb{D}\mathbb{H}\) can be computed by replacing the univariate polynomials \((s - \ell)(s - n)\) and \((t - h)(t - m)\) by their second factorization with linear factors. In \([18, Definition 4]\), it is shown that the second factorization of a univariate polynomial \((u - h_1)(u - h_2) \in \mathbb{D}\mathbb{H}[u]\) is obtained via the formulas

\[
k_2 = -(h_1^* - h_2)^{-1}(h_1h_2 - h_1h_1^*) \quad \text{and} \quad k_1 = h_1 + h_2 - k_2,
\]

where \((u - h_1)(u - h_2) = (u - k_1)(u - k_2)\) and \((u - h_1)(u - h_1)^* = (u - k_2)(u - k_2)^*,\)

\((u - h_2)(u - h_2)^* = (u - k_1)(u - k_1)^*\).

In the remainder of this section, we will provide a systematic procedure for the construction of motion polynomials with two alternating factorizations. We would like to emphasize that we are not aware of any factorization results for bivariate dual quaternion polynomials in existing literature. Our construction is based on the following idea: In Section 3.1, we construct bivariate real quaternion polynomials with two alternating factorizations. In Section 3.2, we extend our results to dual quaternion polynomials.

### 3.1 Quaternion polynomials with two alternating factorizations

The following theorem is the centerpiece of the present section. It presents a method that can be used to construct quaternion polynomials of bi-degree \((2, 2)\) that admit two different factorizations with linear factors.

**Theorem 3.3.** Let \(h, m, n \in \mathbb{H}\) be quaternions. Moreover, assume that either \(\text{Scal}(h) \neq \text{Scal}(m)\) or, if \(\text{Scal}(h) = \text{Scal}(m)\), that \(hh^* \neq mm^*\). Then there exists a suitable quaternion \(\ell \in \mathbb{H}\) such that the polynomial

\[
C := (t - h)(s - \ell)(t - m)(s - n) \in \mathbb{H}[s, t]
\]

admits a second factorization with univariate linear factors.

**Proof.** We briefly explain the main idea of the proof: According to (13), the polynomial \(C\) has a left factor of the form \(t - h\). By choosing the quaternion \(\ell\) in a special way, we force the polynomial \(C\) to admit another factorization with a right factor \(t - h'\) of the same norm, that is

\[
(t - h)(s - \ell)(t - m)(s - n) = A(t - h') \quad \text{with} \quad A \in \mathbb{H}[s, t], \; \text{bdeg}(A) = (1, 2),
\]
and \((t-h)(t-h)^* = (t-h')(t-h')^*\). In [14, 26], it is shown that polynomials of degree one in \(t\) admit factorizations with univariate linear factors as long as the corresponding norm polynomial splits into a product of real univariate polynomials[1]. This is indeed the case for the polynomial \(A\) in [14] since \(AA^* = PR\) with \(P = (t-m)(t-m)^* \in \mathbb{R}[t]\) and \(R = (s-\ell)(s-\ell^*)(s-n)(s-n)^* \in \mathbb{R}[s]\). Therefore,

\[
A = (u_1 - h_1)(u_2 - h_2)(u_3 - h_3) \quad \text{with} \quad u_i \in \{s, t\} \text{ and } h_i \in \mathbb{H} \text{ for } i = 1, 2, 3 \quad (15)
\]

and \(C\) admits a second factorization with univariate linear factors. All possible combinations of linear \(s\)- and \(t\)-factors will be discussed in the proof of Corollary 3.4.

In order to show (14), we define \(M := (t-h)(t-h)^* \in \mathbb{R}[t]\), view \(C\) and \(M\) as univariate polynomials with coefficients in \(\mathbb{H}[s]\) and apply division with remainder of \(C\) by \(M\):

\[
C = (t-h)(s-\ell)(t-m)(s-n) = TM + R, \quad (16)
\]

where \(T, R \in \mathbb{H}[s, t]\) and \(\mathrm{bdeg}(R) = (d_t, d_s)\) with \(d_t \leq 1\) and \(d_s \leq 2\). We compare the coefficients of \(t^2\) on the left-hand and right-hand side of equation (16) and conclude \(T = (s-\ell)(s-n)\) since \(R\) is at most linear in \(t\). Therefore, the linear factor \(s-n\) is a right factor of both \(C\) and \(T\). Representation (16) then shows that it is also a right factor of \(R\) (we used the fact \(TM = MT\) since the polynomial \(M \in \mathbb{R}[t]\) is real and commutes with other polynomials). Similarly, the linear factor \(t-h\) is a left factor of both \(C\) and \(M = (t-h)(t-h)^*\) and hence also a left factor of \(R\). We can write \(R = (t-h)R' = R't-hR' \in \mathbb{H}[s, t]\) with \(R' \in \mathbb{H}[s]\). Since \(s-n\) divides \(R\) from the right it needs to divide each coefficient of \(R\) when viewed as polynomials in \(t\) with coefficients in \(\mathbb{H}[s]\). We conclude that \(s-n\) is a right factor of \(R\). Hence \(R\) is necessarily of the form

\[
R = (t-h)(r_1 s + r_0)(s-n) \quad (17)
\]

with \(r_1, r_0 \in \mathbb{H}\). The quaternions \(r_1\) and \(r_0\) are obtained by comparing appropriate coefficients in (16): Up to now, we always considered \(C\) and \(R\) as univariate polynomials with coefficients in \(\mathbb{H}[s]\). We now view them as bivariate polynomials, which allows us to compare the coefficients of \(ts^2\) and \(t\): Comparing coefficients of \(ts^2\) in (16) yields

\[
-h-m = -h-h^* + r_1
\]

and hence \(r_1 = h^* - m\). Comparing coefficients of \(t\) leads to

\[
-\ell mn - h\ell n = -\ell nh - \ell nh^* - r_0 n
\]

and hence

\[
r_0 = [(\ell m + h\ell n) - \ell n(h + h^*)]n^{-1} = \ell m + h\ell - \ell(h + h^*) = h\ell - \ell(r_1 + h).
\]

\[\bigstar\]

The original reference is [26, Lemma 2.9], but in [16, p. 9] we provide an algorithm that can be used to compute a factorization of the desired form.
In (*) we used the fact \( h + h^* \in \mathbb{R} \). Let us recall the main idea of the proof: We need to force \( C \) to admit a factorization with the right factor \( t - h' \), where \( h' \) is yet to be determined. Alternatively, we can force \( R \) to have the right factor \( t - h' \). By (16), it is then also a right factor of \( C \) (note that we require \( M = (t - h')^*(t - h') \)). We write

\[
R = r_1(t - r_1^{-1}hr_1)(s + r_1^{-1}r_0)(s - n) = r_1(t - h')(s + r_1^{-1}r_0)(s - n),
\]

where \( h' := r_1^{-1}hr_1 \). The polynomial \( t - h' \) indeed satisfies the required condition \( (t - h')(t - h')^* = M \). If the polynomial \( S := (s + r_1^{-1}r_0)(s - n) \in \mathbb{H}[s] \) was a real polynomial, the factor \( t - h' \) would commute with \( S \) and hence be a right factor of \( R \). In case \(-r_1^{-1}r_0 = n^* \) we obtain \( S = (s - n^*)(s - n) \in \mathbb{R}[s] \). Therefore, we need to find a quaternion \( \ell \in \mathbb{H} \) such that \(-r_0 = r_1n^* \), that is

\[
\ell(r_1 + h) - h\ell = r_1n^*.
\]

Above equation is a linear equation in the quaternion unknown \( \ell \in \mathbb{H} \). By (10) Theorem 2.3 it is uniquely solvable if and only if \( \text{Scal}(A) \neq -\text{Scal}(B) \) or \( \text{Scal}(A) = -\text{Scal}(B) \) and \( AA^* \neq BB^* \), where \( A := -h \) and \( B := h + r_1 \). This is equivalent to our theorem’s assumption \( \text{Scal}(h) \neq \text{Scal}(m) \) or \( \text{Scal}(h) = \text{Scal}(m) \) and \( hh^* \neq mm^* \). In the referenced Theorem 2.3 of (10), an explicit formula for the solution \( \ell \in \mathbb{H} \) is provided:

\[
\ell = (2(2 \text{Scal}(h) - \text{Scal}(m)) - h - (h + r_1)(h + r_1)^*h^{-1})^{-1}(r_1n^* - h^{-1}r_1n^*(h + r_1)^*).
\]

This proves the claim.

In order to construct mechanisms, we need to guarantee that the second factorization in Theorem 3.3 is alternating as well. This is ensured by some additional assumptions stated in the (rather technical) Corollary 3.4.

**Corollary 3.4.** Let \( h, m, n \in \mathbb{H} \) be quaternions and \( h', m' \in \mathbb{H} \) be such that \((t - h)(t - m) = (t - m')(t - h')\) (c.f. Remark 3.2). If \( mn \neq nm \) and \( h'n \neq nh' \), the second factorization in Theorem 3.3 is alternating as well, that is

\[
C = (t - h)(s - \ell)(t - m)(s - n) = (s - n')(t - m')(s - \ell')(t - h'),
\]

where \((t - r)(t - r)^* = (t - r')(t - r')^* \), \((s - u)(s - u)^* = (s - u')(s - u')^* \) for \( r \in \{h, m\} \) and \( u \in \{\ell, n\} \).

**Proof.** We have \( 3! = 6 \) possibilities for factorizations of \( A \) with univariate linear factors, where \( A \) is defined in (15). We use representation (14) and obtain

\[
\begin{align*}
C &= (t - h)(s - \ell)(t - m)(s - n) = (s - \ell)(t - m')(s - \ell)(t - h'), \quad (18) \\
C &= (t - h)(s - \ell)(t - m)(s - n) = (t - m')(s - \ell)(s - n)(t - h'), \quad (19) \\
C &= (t - h)(s - \ell)(t - m)(s - n) = (s - \ell)(s - n)(t - m')(t - h'), \quad (20) \\
C &= (t - h)(s - \ell)(t - m)(s - n) = (t - m')(s - n')(s - \ell')(t - h'), \quad (21) \\
C &= (t - h)(s - \ell)(t - m)(s - n) = (s - n')(s - \ell')(t - m')(t - h'), \quad (22) \\
C &= (t - h)(s - \ell)(t - m)(s - n) = (s - n')(t - m')(s - \ell')(t - h'). \quad (23)
\end{align*}
\]
We highlighted the different possibilities for factorizations of $A$ by using bold letters. Note that the linear $s$-factors on the left have the same norms as the linear $s$-factors on the right but the $s$-factors possibly appear in a different order. If the order is different, the $s$-factors must correspond in Bennett flips by arguments as in the proof of Lemma 3.1 and are denoted by a prime, i.e. $n', \ell'$. If the order is the same, the $s$-factors are equal by arguments similar to Lemma 3.1 and [8] Lemma 3. The same arguments apply to linear $t$-factors.

The two factorizations in (18) are $t$-equivalent in the sense of [16] Definition 4.3. By [16] Proposition 4.6, we conclude $h'n = nh'$ (and also $h\ell = \ell h$), a case which is excluded by assumption. The same can be said for the two factorizations in (19). The second factorization in (20) can be rewritten as $(s-\ell)(s-n)(t-h)(t-m)$ and therefore turns out to be $s$-equivalent to the first factorization in (20). We again use [16] Proposition 4.6 and conclude $mn = nm$, which is also excluded by assumption. The second factorizations in (21) and also in (22) are coincident with the second factorizations in (19) and (20) after applying Bennett flips of $(s-n')(s-\ell')$. Therefore, $C$ needs to admit two different factorizations of the form (23).

**Remark 3.5.** Under the weak assumptions of Corollary 3.4, Theorem 3.3 guarantees existence of a quaternion $\ell \in \mathbb{H}$ such that $C = (t-h)(s-\ell)(t-m)(s-n)$ admits a second alternating factorization. While our proofs are constructive, the actual computation of the second factorization can be simplified a lot with the help of Remark 3.2. At first, we compute quaternions $h', \ell', m', n' \in \mathbb{H}$ via Bennett flips (12) of the univariate polynomials $(t-h)(t-m)$ and $(s-\ell)(s-n)$, respectively. The second factorization is then given by $C = (s-n')(t-m')(s-\ell')(t-h')$. Pseudocode for this approach is given in Algorithm 1.

**Algorithm 1** Polynomials with two alternating factorizations

**Require:** Quaternions $h$, $m$, $n \in \mathbb{H}$ satisfying the assumptions of Theorem 3.3 and Corollary 3.4

**Output:** Two tuples $(t-h, s-\ell, t-m, s-n)$ and $(s-n', t-m', s-\ell', t-h')$ such that $(t-h)(s-\ell)(t-m)(s-n) = (s-n')(t-m')(s-\ell')(t-h')$.

1. $r_1 \leftarrow h^* - m$
2. $\ell \leftarrow (2(2\text{Scal}(h) - \text{Scal}(m))) - h - (h + r_1)(h + r_1)^*h^{-1}(r_1n^* - h^{-1}r_1n^*(h + r_1)^*)$
3. $h' \leftarrow -(h^* - m)^{-1}(hm - hh^*)$, $m' \leftarrow h + m - h'$
4. $\ell' \leftarrow -(\ell^* - n)^{-1}(\ell n - \ell\ell^*)$, $n' \leftarrow \ell + n - \ell'$
5. return $(t-h, s-\ell, t-m, s-n)$, $(s-n', t-m', s-\ell', t-h')$

**Example 3.6.** Setting

$h = 2i - j - 3k, \quad m = -6 - 2i + 3j - 3k, \quad n = -j$

*In [16] Definition 4.3, two different factorizations of bivariate quaternion polynomials with linear factors are called $t$-equivalent, if the linear $s$-factors appear in the same order. This is the case in [18] and [19] since $s-\ell$ is the first and $s-n$ the second $s$-factor in both factorizations. Such factorizations are special since they can be transferred into each other by applying Bennett flips and letting appropriate $s$- and $t$-factors commute with each other (c. f. [16] Proposition 4.6).*
and applying Algorithm 1 yields the two alternating factorizations

\[(t - 2i + j + 3k)(s + i - j)(t + 6 + 2i - 3j + 3k)(s + j) = (s - j)(t - 2i - 3j + 3k + 6)(s + i + j)(t + 2i + j + 3k).\]

### 3.2 An extension to dual quaternion polynomials

When it comes to applications in space kinematics, it is necessary to formulate our statements for dual quaternion polynomials. The extension of two different alternating factorizations over the quaternions to dual quaternions is straightforward by using the following approach:

**Step 1:** We start with two dual quaternions \( h = h_p + \varepsilon h_d \) and \( m = m_p + \varepsilon m_d \) such that \( t - h \) and \( t - m \) satisfy the motion polynomial condition of Definition 1 and we compute Bennett flips of \( h \) and \( m \) to obtain dual quaternions \( h' = h_p' + \varepsilon h_d' \) and \( m' = m_p' + \varepsilon m_d' \), respectively.

**Step 2:** We choose a quaternion \( n_p \in \mathbb{H} \) such that \( h_p, m_p, \) and \( n_p \) satisfy the conditions of Corollary 3.4 and apply Theorem 3.3 to the quaternions \( h_p, m_p, \) and \( n_p \). This gives a quaternion polynomial \( C_p \in \mathbb{H}[s,t] \) that admits the two different alternating factorizations

\[ C_p = (t - h_p)(s - \ell_p)(t - m_p)(s - n_p) = (s - n_p')(t - m_p')(t - \ell_p')(t - h_p'). \]  \quad (24)

**Step 3:** Finally, we have to determine the respective dual parts \( \ell_d, n_d, \ell'_d, n'_d \in \mathbb{H} \) of the dual quaternions \( \ell = \ell_p + \varepsilon\ell_d, n = n_p + \varepsilon n_d, \ell' = \ell_p' + \varepsilon\ell_d', \) and \( n' = n_p' + \varepsilon n_d' \), to allow for two factorizations of

\[ C := (t - h_p - \varepsilon h_d)(s - \ell_p - \varepsilon\ell_d)(t - m_p - \varepsilon m_d)(s - n_p - \varepsilon n_d) = (s - n_p' - \varepsilon n_d')(t - m_p' - \varepsilon m_d')(s - \ell_p' - \varepsilon\ell_d')(t - h_p' - \varepsilon h_d'). \]  \quad (25)

The yet unknown quaternions are highlighted in bold letters. Comparing coefficients in \( t \) and \( s \) for all quaternion coefficients on the left-hand and right-hand side of equation (25) yields a system of 32 equations in 16 unknowns. (Note that the primal parts are equal by construction.) Additionally, we have to impose the motion polynomial conditions of Definition 1 on the linear \( s \)-polynomials, leading to eight further linear equations in 16 unknowns (c. f. Example 2.1):

\[ \ell_p\ell_d^* + \ell_d\ell_p^* = 0, \quad \ell_d + \ell_d^* = 0, \]
\[ \ell_p'\ell_d'^* + \ell_d'\ell_p'^* = 0, \quad \ell_d' + \ell_d'^* = 0, \]
\[ n_p n_d^* + n_d n_p^* = 0, \quad n_d + n_d^* = 0, \]
\[ n_p' n_d'^* + n_d' n_p'^* = 0, \quad n_d' + n_d'^* = 0. \]

In total, we have to solve a system of 40 linear equations in 16 unknowns. The respective linear system of equations seems to be highly overconstrained. Quite surprisingly, it turns out to always admit a solution. This will be proved by a straightforward computation in Section 4.1 so that we have:
Theorem 3.7. The construction outlined in above Steps 1 to 3 generically yields a motion polynomial $C$ satisfying

$$C = (t - h)(s - \ell)(t - m)(s - n) = (s - n')(t - m')(s - \ell')(t - h')$$

with linear motion polynomials $t - h$, $s - \ell$, $t - m$, $s - n$, $t - m'$, $s - \ell'$, and $t - h'$.  

Example 3.8. We build on Example 3.6 and additionally choose quaternions

$$h_d := 23i - 74j + 40k \quad \text{and} \quad m_d := -45i - 66j - 36k.$$  

The polynomials $t - h_p - \varepsilon h_d$ and $t - m_p - \varepsilon m_d$ are motion polynomials:

$$(t - h_p - \varepsilon h_d)(t - h_p - \varepsilon h_d)^* = t^2 + 14 \in \mathbb{R}[t] \setminus \{0\},$$

$$(t - m_p - \varepsilon m_d)(t - m_p - \varepsilon m_d)^* = t^2 + 12t + 58 \in \mathbb{R}[t] \setminus \{0\}.$$  

We compute Bennett flips of $h := h_p + \varepsilon h_d$ and $m := m_p + \varepsilon m_d$ and obtain

$$m' = -6 + 2i + 3j - 3k - \varepsilon(21i + 22j + 36k),$$

$$h' = -2i - j - 3k - \varepsilon(i + 118j - 40k).$$

The unknowns $\ell_d$, $n_d$, $\ell'_d$, $n'_d$ are obtained by solving the respective system of linear equations:

$$\ell_d = -11i - 11j + 2k, \quad n_d = -3i - 2k,$$

$$\ell'_d = 11i - 11j - 2k, \quad n'_d = -25i + 2k.$$  

Finally, we get a motion polynomial in $\mathbb{D}H[s, t]$ with two alternating factorizations:

$$(t - 2i + j + 3k + \varepsilon(-23i + 74j - 40k))(s + i - j + \varepsilon(11i + 11j - 2k))$$

$$= (s + i + j - \varepsilon(25i - 2k))(t + 6 - 2i - 3j + 3k + \varepsilon(21i + 22j + 36k))$$

$$= (s + i + j - \varepsilon(11i - 11j - 2k))(t + 2i + j + 3k + \varepsilon(i + 118j - 40k)).$$

4 The Multi-Bennett 8R Mechanism

In the preceding section we proved existence of bivariate quaternion polynomials $C \in \mathbb{H}[s, t]$ that admit two factorizations with linear quaternion polynomials and we hinted at the possibility to extend this to motion polynomials of the shape

$$C = (t - h)(s - \ell)(t - m)(s - n) = (s - n')(t - m')(s - \ell')(t - h')$$

with dual quaternions $h$, $\ell$, $m$, $n$, $n'$, $m'$, $\ell'$, and $h'$. By construction, each linear factor in $t$ or in $s$ parametrizes a rotation around a straight line in space so that each of the two factorizations gives rise to an open 4R chain whose end-effectors share the two-parametric rational motion parametrized by $C$. Thus, this motion is contained in the
configuration variety of the closed-loop 8R linkage formed by the two open 4R chains. Investigation of properties of this 8R linkage is the topic of this section. In doing so, we only consider the generic case, i.e., we assume that no special algebraic relations between the input parameters are fulfilled. At present, a comprehensive discussion of all special cases seems of little value.

The linkage’s zero configuration is given by \( t = s = \infty \) because then there is zero rotation in all joints (c.f. Definition 2). The axes’ Plücker coordinates in this zero configuration are simply the respective vector parts of the linear factors. The positions of these axes in the configuration determined by a general parameter pair \((s,t)\) can be computed via (5) as

\[
\begin{align*}
H(s,t) &= \text{Vect}(h), \\
L(s,t) &= ((t-h)\text{Vect}(\ell)(t-h^*))_\varepsilon, \\
M(s,t) &= ((t-h)(s-\ell)\text{Vect}(m)(s-\ell^*)(t-h^*))_\varepsilon, \\
N(s,t) &= ((t-h)(s-\ell)(t-m)\text{Vect}(n)(t-m^*)(s-\ell^*)(t-h^*))_\varepsilon, \\
N'(s,t) &= \text{Vect}(n'), \\
M'(s,t) &= ((s-n')\text{Vect}(m')(s-n'^*))_\varepsilon, \\
L'(s,t) &= ((s-n')(t-m')\text{Vect}(\ell')(t-m'^*)(s-n'^*))_\varepsilon, \\
H'(s,t) &= ((s-n')(t-m')(s-\ell')\text{Vect}(h')(s-\ell'^*)(t-m'^*)(s-n'^*))_\varepsilon.
\end{align*}
\]

Note that \( H(s,t) \) and \( N'(s,t) \) are independent of \( t \) and \( s \), \( L(s,t) \) depends only on \( t \) and \( M'(s,t) \) depends only on \( s \).

For fixed \( t = t_0 \), the axes \( L(s,t_0), N(s,t_0), L'(s,t_0), \) and \( N'(s,t_0) \) form, in that order, a Bennett linkage whose motion is parametrized by \( s \). We call it the \textit{s-Bennett linkage at} \( t_0 \). Similarly, for fixed \( s = s_0 \) we obtain a \textit{t-Bennett linkage at} \( s_0 \), formed by \( H(s_0, t), M(s_0, t), H'(s_0, t), \) and \( M'(s_0, t) \).

It is well-known (and follows from Bennett’s original description of his mechanism as isogram, c.f. [22 Section 10.3]) that for given \( t_0 \) there exist two values for \( s \in \mathbb{R} \cup \{ \infty \} \) at which the \( t_0 \)-Bennett mechanism is in a configuration where its four axes have the same common perpendicular. We call this an \textit{aligned configuration}. A similar statement holds true for every \( s_0 \)-Bennett mechanism.

The aligned configurations will play a crucial role in our computation of the 8R-linkage’s Denavit-Hartenberg parameters in the next section. The 8R-linkage itself exhibits an interesting aligning behavior as well that will be investigated in more detail in the forthcoming Section 4.2.

### 4.1 Denavit-Hartenberg Parameters

The aim of this section is the proof of simple relations among the multi-Bennett’s Denavit-Hartenberg parameters. In order to do so, we will compute parametrizations of
its moving axes with respect to special coordinates. None of these assumptions is a loss of generality so that the resulting statements are of general validity and are suitable for proving the missing piece in Theorem 3.7.

Our computation of the 8R-linkage’s Denavit-Hartenberg parameters will profit a lot from the geometry of its t- and s-Bennett linkages. According to [22, Section 10.3], the axes of any Bennett linkage can be computed by

- picking two arbitrary points $F_h, F_m$ and a straight line $z$,
- rotating $F_h$ and $F_m$, respectively, around $z$ by a rotation angle of $180^\circ$ to obtain a spatial quadrilateral $F_h, F_m, F_h', F_m'$ with equal opposite sides, and
- selecting the axes $H, M, H', M'$ as the perpendiculars to the quadrilateral’s sides at $F_h, F_m, F_m'$, and $F_h'$, respectively (Figure 1).

In order to compute the linkage’s Denavit-Hartenberg parameters, we assume that the t-Bennett linkage at $s_0 = \infty$ is aligned for $t_0 = \infty$. This is no loss of generality as it can be achieved via re-parametrizations of type (8). Moreover, we assume that the axes in this configuration intersect the first coordinate axis perpendicularly and that the axis of half-turn symmetry is the third coordinate axis. This entails a slight alteration of the construction from above. We assign coordinates

$$F_h = (h_1, 0, 0), \quad F_m = (m_1, 0, 0), \quad F_{h'} = (-h_1, 0, 0), \quad F_{m'} = (-m_1, 0, 0)$$

to the common normal feet and

$$D_h = (0, h_2, h_3), \quad D_m = (0, m_2, m_3), \quad D_{h'} = (0, -h_2, h_3), \quad D_{m'} = (0, -m_2, m_3).$$
to the corresponding axis directions. By this choice, we ensure equal opposite distance and angles but not equality of Bennett ratios. A straightforward computation yields that this can be satisfied by

\[ m_1 = \frac{h_1 h_3 m_2}{h_2 m_3} \quad \text{or} \quad m_1 = \frac{h_1 h_2 m_3}{h_3 m_2}. \]

Since both expressions are equal up to interchanging \( h_2 \) with \( h_3 \) and \( m_2 \) with \( m_3 \) we can use either of them. The following computations use \( m_1 = \frac{h_1 h_3 m_2}{(h_2 m_3)} \).

Now, we compute the Plücker coordinates, viewed as dual quaternions, of the axes in the zero configuration as

\[
\begin{align*}
\text{Vect}(h) &= D_h + \varepsilon(F_h \times D_h) = h_2 j + h_3 k - h_1 \varepsilon(h_3 j - h_2 k), \\
\text{Vect}(m) &= D_m + \varepsilon(F_m \times D_m) = m_2 j + m_3 k - \frac{h_1 h_3 m_2}{h_2 m_3} \varepsilon(m_3 j - m_2 k), \\
\text{Vect}(h') &= D_{h'} + \varepsilon(F_{h'} \times D_{h'}) = -h_2 j + h_3 k + h_1 \varepsilon(h_3 j + h_2 k), \\
\text{Vect}(m') &= D_{m'} + \varepsilon(F_{m'} \times D_{m'}) = -m_2 j + m_3 k + \frac{h_1 h_3 m_2}{h_2 m_3} \varepsilon(m_3 j + m_2 k).
\end{align*}
\]

Here, we identified in the usual way vectors in \( \mathbb{R}^3 \) with vectorial quaternions. The coefficients \( h, m, h', \) and \( m' \) in the factors \( t-h, t-m, t-h', t-m' \) of the sought motion polynomial are linear combinations of 1 and \( \text{Vect}(h), \text{Vect}(m), \text{Vect}(h'), \) and \( \text{Vect}(m') \), respectively. The coefficients cannot be chosen arbitrarily but are subject to the closure condition \( (t-h)(t-m) = (t-m')(t-h') \). This is ensured by having

\[
\begin{align*}
h &= \mu - \nu m_2 (j + \frac{h_3}{h_2} k) + \frac{\nu h_1 m_2}{h_2} \varepsilon(h_3 j - h_2 k), \\
m &= \mu + \nu (m_2 j + m_3 k) - \frac{\nu h_1 h_3 m_2}{h_2 m_3} \varepsilon(m_3 j - m_2 k), \\
h' &= \mu + \nu m_2 (j - \frac{h_3}{h_2} k) - \frac{\nu h_1 m_2}{h_2} \varepsilon(h_3 j + h_2 k), \\
m' &= \mu - \nu (m_2 j - m_3 k) + \frac{\nu h_1 h_3 m_2}{h_2 m_3} \varepsilon(m_3 j + m_2 k)
\end{align*}
\]

with parameters \( \nu, \mu \in \mathbb{R} \).

So far, we have followed Step 1 of Section 3 and computed, in full generality but at a special configuration, the axes and corresponding dual quaternions that move with parameter \( t \). For Steps 2 and 3 we make the general ansatz

\[
\ell = \ell_p + \varepsilon \ell_d, \quad n = n_p + \varepsilon n_d, \quad \ell' = \ell'_p + \varepsilon \ell'_d, \quad n' = n'_p + \varepsilon n'_d
\]

\[3\]An important characteristic of a Bennett mechanism is its Bennett ratio, the ratio between sine of angle and distance of two consecutive axes, which is independent of the chosen pair of consecutive axes [21] Equation (11.69)].
with $\ell_p', \ell_d, n_p, n_d, \ell'_p, \ell'_d, n'_p, n'_d \in \mathbb{H}$. Step 2 gives the primal parts $\ell_p$, $\ell'_p$, and $n'_p$ in terms of the indetermined coefficients of $n_p = n_0 + n_1 i + n_2 j + n_3 k$:

$$
\ell_p = \frac{1}{h_2 m_3 + h_3 m_2} \left( h_2 (m_3 n_0 - 2m_2 n_1) + h_3 m_2 n_0 + n_1 (h_2 m_3 - h_3 m_2) i 
+ (h_2 (m_3 n_2 - 2m_2 n_3) - h_3 m_2 n_2) j - n_3 (h_2 m_3 + h_3 m_2) k \right),
$$

$$
\ell'_p = \frac{1}{h_2 m_3 + h_3 m_2} \left( h_2 (m_3 n_0 - 2m_2 n_1) + h_3 m_2 n_0 + n_1 (h_2 m_3 - h_3 m_2) i 
+ (h_2 (m_3 n_2 - 2m_2 n_3) - h_3 m_2 n_2) j + n_3 (h_2 m_3 + h_3 m_2) k \right),
$$

$$
n'_p = n_0 + n_1 i + n_2 j - n_3 k.
$$

This ensures that the primal parts on both sides of

$$(t - h)(s - \ell)(t - m)(s - n) = (s - n')(t - m')(s - \ell')(t - h')$$

agree. Equality of the respective dual parts together with the motion polynomial condition boils down to a system of linear equations (Step 3) for the real coefficients of $\ell_d, \ell'_d, n_d, \text{and } n'_d$ which we solve with a computer algebra system. There is, indeed, a unique solution whence we have provided the missing piece in the proof of Theorem 3.7.

The solutions are just a bit too long to be displayed here. Therefore, and also having in mind forthcoming computations, we strive for further simplifications. By a rational re-parametrization we can achieve that the revolute axis $\text{Vect}(\ell)$ is perpendicular to the first coordinate axis in the zero configuration, at $s_0 = \infty$. This having done, we see that necessarily $n_1 = 0$. With this admissible simplification, the solutions for the dual parts are

$$
\ell_d = \frac{1}{\Delta} \left( n_3 (h_2 m_3 + h_3 m_2)(h_2 m_2 n_3 - h_2 m_3 n_2 + h_3 m_2 n_2 + h_3 m_3 n_3) j 
- (h_2 m_2 n_3 - h_2 m_3 n_2 + h_3 m_2 n_2 + h_3 m_3 n_3)(2h_2 m_2 n_3 - h_2 m_3 n_2 + h_3 m_2 n_2) k \right),
$$

$$
n_d = \frac{1}{\Delta} \left( -n_3 (2h_2 m_2 n_3 - h_2 m_3 n_2 + h_3 m_2 n_2 + h_3 m_3 n_3) j 
+ n_2 (2h_2 m_2 n_3 - h_2 m_3 n_2 + h_3 m_2 n_2)(h_2 m_3 n_3 - h_2 m_3 n_2 + h_3 m_2 n_2 + h_3 m_3 n_3) k \right),
$$

$$
\ell'_d = \frac{1}{\Delta} \left( n_3 (h_2 m_3 + h_3 m_2)(h_2 m_2 n_3 - h_2 m_3 n_2 + h_3 m_2 n_2 + h_3 m_3 n_3) j 
+ (h_2 m_2 n_3 - h_2 m_3 n_2 + h_3 m_2 n_2 + h_3 m_3 n_3)(2h_2 m_2 n_3 - h_2 m_3 n_2 + h_3 m_2 n_2) k \right)
$$

$$
n'_d = \frac{1}{\Delta} \left( -n_3 (2h_2 m_2 n_3 - h_2 m_3 n_2 + h_3 m_2 n_2 + h_3 m_3 n_3) j 
- n_2 (2h_2 m_2 n_3 - h_2 m_3 n_2 + h_3 m_2 n_2)(h_2 m_3 n_3 - h_2 m_3 n_2 + h_3 m_2 n_2 + h_3 m_3 n_3) k \right)
$$

where

$$
\Delta = h_2 m_3 (h_3 m_2 - h_2 m_3)(n_2^2 - n_3^2) + 2(h_2 m_2 + h_3 m_3) n_2 n_3.
$$

But having $n_1 = 0$ has further consequences:
A glance at (28) immediately confirms that all coefficients of $i$ vanish for all revolute axes in the zero configuration. Therefore, all revolute axes in the zero configuration are perpendicular to the first coordinate axis.

It can readily be verified that the intersection conditions

$$\text{Vect}(\ell)i - i\text{Vect}(\ell) = \text{Vect}(\ell')i - i\text{Vect}(\ell') = 0.$$ 

between the first coordinate axis (with Plücker coordinates $i$) and all mechanism axes that move with parameter $s$ in the zero configuration are satisfied.

This means that in the zero configuration all revolute axes intersect the first coordinate axis perpendicularly. We infer that not only the $t$-Bennett mechanism but also the $s$-Bennett mechanism aligns and both share the common perpendicular of their axis. Since each Bennett mechanism has two aligned configurations and there is nothing special about our zero configuration, we can say:

**Theorem 4.1.** The multi-Bennett 8R mechanism has four aligned configurations in which all eight revolute axes share a common perpendicular line.

The four aligned configurations of an example can be seen in the corners of Figure 2. From the representations (27) and (28) of the axes’ Plücker coordinates, it is straightforward to compute the mechanism’s Denavit-Hartenberg parameters. The information given in [25, Section 2.1.2] is sufficient for that purpose but more explicit formulas are also available, for example in [5]. Using computer algebra, it is easy to verify

**Theorem 4.2.** The offsets of a multi-Bennett 8R mechanism are all zero. Opposite distances as well as opposite angles are equal.

Remarkably, the four distances are rational expressions in the input parameters, no square roots appear:

$$d_1 = \frac{1}{\Phi} (h_1(m_2 n_3 - m_3 n_2)(2h_2^2 m_2 n_3 - h_2^2 m_3 n_2 + h_2 h_3 m_2 n_2 + h_2 h_3 m_3 n_3 + h_3^2 m_2 n_3)),$$

$$d_2 = \frac{1}{\Phi} (h_1(m_2 n_3 - m_3 n_2)(h_2 n_2 - h_3 n_3)(h_2 m_3 - h_3 m_2)),$$

$$d_3 = \frac{1}{\Phi} (h_1(m_2 n_2 + m_3 n_3)(h_2 n_3 + h_3 n_2)(h_2 m_3 - h_3 m_2)),$$

$$d_4 = \frac{1}{\Phi} (h_1(h_2 n_3 + h_3 n_2)(2h_2 m_2^2 n_3 - h_2 m_2 m_3 n_2 + h_2 m_3^2 n_3 + h_3 m_2^2 n_2 + h_3 m_2 m_3 n_3)),$$

where

$$\Phi = h_2 m_3 ((h_3 m_2 - h_2 m_3)(n_2^2 - n_3^2) + 2n_2 n_3(h_2 m_2 + h_3 m_3)).$$
The squared cosines of the corresponding angles are

\[
\begin{align*}
\cos^2 \alpha_1 &= \frac{(m_2n_2 + m_3n_3)^2}{(n_2^2 + n_3^2)(m_2^2 + m_3^2)}, \\
\cos^2 \alpha_2 &= \frac{(2h_2m_2n_3 - h_2m_2n_2 + h_2m_3n_3 + h_3m_2n_3)^2}{(m_2^2 + m_3^2)^2}, \\
\cos^2 \alpha_3 &= \frac{(2h_2m_2n_3 - h_2m_3n_2 + h_2m_3n_3 + h_3m_3n_3)^2}{(h_2^2 + h_3^2)^2}, \\
\cos^2 \alpha_4 &= \frac{(h_2n_2 - h_3n_3)^2}{(h_2^2 + h_3^2)(n_2^2 + n_3^2)}
\end{align*}
\]

where

\[
\Psi = 4h_2^2m_2n_3(m_2n_3 - m_3n_2) + (h_2^2m_2^2 + h_3^2m_3^2)(n_2^2 + n_3^2) + 2h_2h_3m_2(2m_2n_3 - m_3(n_2^2 - n_3^2)).
\]

We conjecture that the necessary conditions of Theorem 4.2 on the mechanism’s Denavit-Hartenberg parameters are not sufficient to characterize a multi-Bennett 8R mechanism.

### 4.2 Bennett Sub-Mechanisms

We have already mentioned that for fixed \( s = s_0 \) the axes \( H(s, t), M(s, t), H'(s, t), \) and \( M'(s, t) \) to the respective factors \( t - h, t - m, t - h', \) and \( t - m' \) form a Bennett mechanism. The same is true for fixed \( t = t_0 \) and the axes \( L(s, t), N(s, t), L'(s, t), \) \( N'(s, t) \) to the respective factors \( s - \ell, s - n, s - \ell', s - n' \). We refer to the respective Bennett mechanisms as \( t \)-Bennett mechanism at \( s_0 \) and as \( s \)-Bennett mechanism at \( t_0 \). The \( t \)-Bennett mechanism aligns for precisely two parameter values \( t', t'' \). By means of (26) it can readily be verified that aligning of \( t \)-Bennett linkage happens at

\[
t' = \infty, \quad t'' = \mu
\]

while an \( s \)-Bennett linkage aligns at

\[
s' = \infty, \quad s'' = n_0.
\]

The most remarkable thing about Equations (29) and (30) is that that \( t' \) and \( t'' \) do not depend on \( s \) and \( s' \), \( s'' \) do not depend on \( t \). Abstracting from our special geometric description to the general case, we can thus state:

**Theorem 4.3.** In a multi-Bennett 8R mechanism, the \( t \)-Bennett sub-mechanisms align precisely for two fixed parameter values \( t', t'' \) and the \( s \)-Bennett sub-mechanisms align precisely for two fixed parameter values \( s', s'' \). The points \( (t', s'), (t', s''), (t'', s'), \) and \( (t'', s'') \) in the configuration space correspond to the four aligned states of the complete mechanism, c.f. Theorem 4.1.
Theorem 4.3 is illustrated in Figure 2. There, the eight links are visualized by cylinders around the common normals of consecutive joint axes. This is clearly visible in the four totally aligned configurations in the corners. The motions between neighbouring corners have \( t = t', t = t'', s = s' \), or \( s = s'' \). Figure 2 also illustrates the multi-Bennett’s configuration space, a torus, and the four curves, meridian and lateral circles on the torus, along which Bennett sub-mechanisms align.

As expected, the Bennett ratio is not constant within the family of \( t \)-Bennett mechanism but depends on \( s \) (and vice versa for \( s \)-Bennett mechanisms). However, a noteworthy property is:

**Theorem 4.4.** The Bennett ratio within the family of \( t \)-Bennett linkages is a rational function of degree four in \( s \) and vice versa for the \( s \)-Bennett linkages.

**Proof.** A direct computation using computer algebra yields the value

\[
\tau = \frac{h_2m_2((h_2m_3 + h_3m_2)^2(s^4 - 4n_0s^3) + c_2s^2 + c_1s + c_0)}{h_1\sqrt{h_2^2 + h_3^2\sqrt{m_2^2 + m_3^2}(s^2 - 2n_0s + n_0^2 + n_2^2 + n_3^2)((h_3^2m_2^2 - h_2^2m_3^2)s(s - 2n_0) + d_0)}
\]

for the \( t \)-Bennett ratio where

\[
c_2 = 6(h_2m_3 + h_3m_2)^2n_0^2 + (2h_2^2m_3^2 + 2h_3^2m_2^2 + 4h_2^2m_3^2)n_2^2 + 2(2h_2^2m_2^2 + h_2^2m_3^2 + h_3^2m_2^2)n_3^2 - 4n_2n_3(h_2m_2 - h_3m_3)(h_2m_3 - h_3m_2),
\]

\[
c_1 = -4n_0^2(h_2m_3 + h_3m_2)^2 - 4n_0n_2^2(h_2^2m_3^2 + h_3^2m_2^2 + 2h_2^2m_3^2) + 8n_0n_2n_3(h_2m_2 - h_3m_3)(h_2m_3 - h_3m_2) - 4n_0n_3^2(2h_2^2m_2^2 + h_2^2m_3^2 + h_3^2m_2^2),
\]

\[
c_0 = n_0^4(h_2m_3 + h_3m_2)^2 + 2n_0^2n_2^2(h_2^2m_3^2 + h_3^2m_2^2 + 2h_2^2m_3^2)
+ 4n_0^2n_2n_3(h_2m_2 - h_3m_3)(h_3m_2 - h_2m_3) + 2n_0^2n_3^2(2h_2^2m_2^2 + h_2^2m_3^2 + h_3^2m_2^2) + n_1^2(h_2m_3 - h_3m_2)^2
+ 4n_2n_3^2(h_2m_2 + h_3m_3)(h_3m_2 - h_2m_3) + 2n_2n_3^2(2h_2^2m_2^2 - h_2^2m_3^2 + 6h_2h_3m_2m_3 - h_2^3m_2^2 + 2h_3^2m_2^2)
+ 4n_2n_3^2(h_2m_2 + h_3m_3)(h_2m_3 - h_3m_2) + n_3^4(h_2m_3 - h_3m_2)^2,
\]

and

\[
d_0 = -4h_2n_2n_3(2h_2m_2m_3 - h_3m_2^2 + h_2m_3^2) + n_0^2(h_2^3m_2^2 - h_2^2m_3^2)
+ n_2^2(3h_2m_3 - h_3m_2)(h_2m_3 - h_3m_2) + n_3^2(4h_2^2m_2^2 - h_2^2m_3^2 + 4h_2h_3m_2m_3 + h_3^2m_2^2).
\]

A similar formula can be derived for the \( s \)-Bennett ratio.

\[\square\]

5 Conclusion and Future Research

We presented the first example of a mechanism constructed from the factorization of *bivariate* motion polynomials and described some of its fundamental properties. Of course, open questions remain.
Figure 2: Configuration space of multi-Bennett 8R mechanism and animated transitions between four totally aligned configurations in the corners; points correspond to depicted configurations.
The simple conditions on the mechanism’s DH parameters which we describe in Theorem 4.2 are necessary but, so we believe, not sufficient. It would be desirable to augment them with further conditions to obtain a set of sufficient conditions.

We further believe that the configuration space parametrized by the underlying motion polynomial $C(s,t)$ is only a part of the mechanism’s complete configuration space. Obtaining a clearer picture of possible assembly modes or bifurcations of the motion is certainly a worthy topic of future research.

The configuration space component described by $C(s,t)$ has many attractive features for potential applications: It has a rational parametrization with low degree parameter lines. The motion along a parameter line is the well-understood coupler motion of a Bennett mechanism. Simple parametrization but also the unusual separation into joints that only move with parameter $t$ and joints that only move with parameter $s$ is expected to be beneficial for the control of a multi-Bennett 8R mechanism. It can also be viewed as an adjustable Bennett mechanism where rotation in one group of joints (say those parametrized by $s$) changes the geometry of the $t$-Bennett mechanism.

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