A ZARISKI THEOREM FOR MONODROMY OF A-HYPERGEOMETRIC SYSTEMS

JENS FORSGÅRD AND LAURA FELICIA MATUSEVICH

Abstract. We give conditions under which the monodromy group of an A-hypergeometric system is invariant under modifications of the collection of characters A. The key ingredient is a Zariski–Lefschetz type theorem for principal A-determinants.

1. Introduction

This article concerns the study of A-hypergeometric monodromy. A-hypergeometric systems were introduced by Gel’fand, Graev, Kapranov and Zelevinsky in the late twentieth century [14, 16, 17] in order to provide a uniform theory for multivariate hypergeometric functions (see also [2]), as well as a bridge to toric geometry. Through this connection, important concepts such as canonical series solutions [24] and holonomic rank [1, 3, 23] can be described in combinatorial terms.

The combinatorial tractability of A-hypergeometric systems, combined with existing deep results on monodromy of classical hypergeometric functions (see, e.g., [7, 9]), gives hope of a correspondingly rich theory of A-hypergeometric monodromy. However, the literature in this direction is sparse (but, see [6, 26]). To understand why this is the case, we recall that, by definition, the monodromy group of a system of differential equations is a representation of the fundamental group of the complement of its singular locus. The singular locus of an A-hypergeometric system is the zero set of a polynomial called the principal A-determinant or full discriminant. Geometrically, this is a union of discriminantal hypersurfaces. Computing the fundamental group of the complement of an algebraic hypersurface is a deep and important question, which is challenging in the case of discriminantal hypersurfaces [11, 22]. For principal A-determinants, we are not aware of any general results in this direction, and this has obstructed progress on A-hypergeometric monodromy. Consequently, our first major goal is to prove a Zariski–Lefschetz-type theorem for principal A-determinants.

To make this more precise, recall that an A-hypergeometric system \( H_A(\beta) \) is defined by a finite collection \( A \) of algebraic characters of the torus \( (\mathbb{C}^*)^{1+n} \), and a parameter vector \( \beta \in \mathbb{C}^{1+n} \). (See §5 for a precise definition.) Denote by \( V_A \subset \mathbb{C}^A \) the singular locus of \( H_A(\beta) \), which is independent of \( \beta \). Here, \( \mathbb{C}^A \) denotes the complex affine space of dimension \( k = |A| \).

Removing a character from \( A \) corresponds to restricting, in the space \( \mathbb{C}^A \), to a coordinate hyperplane. To study homotopy groups of complements of embedded algebraic varieties through intersections with linear spaces is classical [18, 21, 27], and numerous theorems exist in the literature; in most instances with some smoothness assumption.
However, the variety $V_A$ is highly singular. Typically, its Whitney stratification has reducible nonempty strata in each codimension. The combination of a highly singular variety and a coordinate hyperplane implies that standard Lefschetz and Zariski type theorems do not apply. As a first main result, we provide a combinatorial condition that allows us to add or remove characters and still control the effect on homotopy.

Let $N$ denote the Newton polytope (i.e., convex hull) of $A$. We say that $A$ has an interior point if at least one element of $A$ is interior to $N$. For each face $\Gamma$ of $N$ the face lattice of $A$ relative $\Gamma$ is the affine lattice spanned by the elements of $A \cap \Gamma$. We define the face saturation $A^s$ of $A$ to be the largest subset of $N \times \mathbb{Z}^{1+n}$ such that the face lattices of $A^s$ coincide with the face lattices of $A$. (See Definition 4.1 for a precise statement). Generalizing a Zariski-style theorem from [4], we conclude the following.

**Theorem 1.1.** Assume that $A$ has an interior point. Then, the inclusion $C_A \to C_A^s$ given by appending zeros to $x$ for the coordinates corresponding to the characters $A^s \setminus A$, induces a surjective morphism

\[
\eta: \pi_1(C_A \setminus V_A, x) \to \pi_1(C_A^s \setminus V_A^s, (x, 0)).
\]

Throughout this paper, we say that a morphism between fundamental groups is canonical if it is induced by an inclusion of topological spaces, as in Theorem 1.1. An interesting question, which we do not address in detail, is whether the canonical morphism in Theorem 1.1 is an isomorphism. This can be deduced in special cases, as in §.

With this result in hand, we return to $A$-hypergeometric monodromy. The article [6] describes an algorithm to compute the monodromy group $\text{Mon}_x(A, \beta)$ of an $A$-hypergeometric system with parameter vector $\beta$ at the base point $x \in C_A$. However, this method is only applicable if the collection $A$ satisfies some assumptions. The milder of these assumptions, that $A$ admits a so-called Mellin–Barnes basis, fails already for relatively small collections $A$. One can deduce from [12] [13] that the set of all collections $A$ which admit a Mellin–Barnes basis is a semi-ideal (or, downward closed set) in the poset lattice of all collections with Newton polytope $N$. In other words, a suitable subcollection of $A$ can admit a Mellin–Barnes basis, even if $A$ does not. Our main result supplements this semi-ideal property by describing conditions under which the monodromy group is invariant under the actions of deleting (or adding) characters from $A$.

**Theorem 1.2.** Assume that $A$ has an interior point. If the parameter $\beta$ is sufficiently generic (i.e., nonresonant; see Definition 5.3), then

\[
\text{Mon}_x(A, \beta) \simeq \text{Mon}_{(x, 0)}(A^s, \beta).
\]

The genericity assumption on the parameter $\beta$ cannot be removed, since it is known to characterize $A$-hypergeometric systems with irreducible monodromy representations [5] [25]. Without the genericity assumption, it might even be that the dimensions of the solution spaces differ [8] [24].

Finally, we remark that we prove stronger versions of Theorems 1.1 and 1.2, which apply also in the situation when $A$ has no interior points. The necessary definitions are, naturally, more technical, and we have saved the details for Definition 4.1 and Theorems 4.4 and 5.8.
Outline. Section 2 sets notation and reviews necessary background. Section 3 begins our study of fundamental groups. Section 4 links these results to the combinatorics of the collection $A$, and contains the proof of Theorem 1.1. We turn to $A$-hypergeometric monodromy and prove Theorem 1.2 in Section 5. Finally, we apply these results and Beuker’s method to compute the monodromy groups for $A$-hypergeometric systems associated to monomial curves in Section 6.

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2. Preliminaries

We use, with slight adjustments, the notation of [15, Chapter 10].

Throughout this article, $A = \{\alpha_1, \ldots, \alpha_k\} \subset \mathbb{Z}^{1+n}$ denotes a quasi-homogeneous collection of algebraic characters of the torus $(\mathbb{C}^*)^{1+n}$, and $\beta \in \mathbb{C}^{1+n}$ is a parameter vector. We often write $A$ as a matrix whose columns are the characters $\alpha_i$. We denote by $X_A$ the toric variety associated to $A$, with $Z_A$ its character lattice, and we let $N = \text{conv}(A) \subset \mathbb{R} \otimes Z_A$ denote the corresponding Newton polytope. The collection $A$ is said to be saturated if $A = N \cap Z_A$. However, we impose no such assumption. Let $N_A = N[A]$ denote the monoid generated by $A$ and the origin, so that $Z_A$ is the group completion of $N_A$. Given an affine lattice $L$, the lattice volume defined by $L$ is the unique translation invariant measure on $\mathbb{R} \otimes L$ such that a minimal simplex in $L$ has volume one.

Let $F$ denote the face poset lattice of $N$, and let $F_{\text{int}} \subset F$ denote the semi-ideal generated by all faces $\Gamma \leq N$ that contain a relative interior point in $A$. Given $\Gamma \in F$, let $\Gamma_R$ denote the linear span of $\Gamma$ and the origin. That is, $\Gamma_R$ is the linear span of the cone generated by the face $\Gamma \leq N$. (Some authors prefer to consider the cone over $N$ rather than $N$; the face lattices of the two coincide except for the apex of the cone.)

Define the index

$$i(A, \Gamma) = \left[ Z_A \cap \Gamma_R : Z_{A \cap \Gamma} \right].$$

Set $Z_A/\Gamma = Z_A/(Z_A \cap \Gamma_R)$, and consider the admissible semigroup

$$(2.1) \quad N_A/\Gamma = (Z_A \cap \Gamma_R)/(Z_A \cap \Gamma_R) \subset Z_A/\Gamma.$$ 

Let $v(A, \Gamma)$ denote the subdiagram volume of $N_A/\Gamma$. That is, $v(A, \Gamma)$ denotes the lattice volume of the set difference between the convex hulls of $N_A/\Gamma$ and $(N_A/\Gamma)^* = (N_A/\Gamma) \setminus \{0\}$ in $\mathbb{R} \otimes Z_A/\Gamma$. By convention, the subdiagram volume of the trivial semigroup is one.

Let $z = (z_0, \ldots, z_n)$ be coordinates on the torus $(\mathbb{C}^*)^{1+n}$. Given $\alpha_i \in \mathbb{Z}^{1+n}$, the associated character is the monomial $z \mapsto z^{\alpha_i}$. We use $C^A$ to denote the space of polynomials

$$C^A = \{ y_1 z^{\alpha_1} + \cdots + y_k z^{\alpha_k} \mid y_1, \ldots, y_k \in C \}. $$

That is, $C^A$ is complex vector space of dimension $k$ with coordinates $y = (y_1, \ldots, y_k)$.

The collection $A$ defines a projective toric variety $X_A \subset \mathbf{P}(C^A)$. Let $\mathbf{P}(C^A)$ denote the dual space of $\mathbf{P}(C^k)$. The $A$-discriminantal variety is the projectively dual

$$(2.2) \quad \tilde{X}_A \subset \mathbf{P}(C^A).$$
The collection $A$ is said to be nondefective if the $A$-discriminant is a hypersurface, in which case we denote by $D_A$ its defining homogeneous polynomial (unique up to sign, if the coefficients are required to be relatively prime integers). If $A$ is defective, then $D_A = 1$. The affine cone $\tilde{X}_A \subset \mathbb{C}^A$ is the closure of the rational locus of all polynomials $f \in (\mathbb{C}^*)^A$ which has a singularity in $(\mathbb{C}^*)^{1+n}$.

Following [15], the principal $A$-determinant is defined as the toric resultant

$$E_A(f) = R_A\left(\frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n}\right).$$

We make use of the formula [15, Ch. 10, Thm. 1.2], up to a nonzero constant,

$$E_A(f) = \prod_{\Gamma \in \mathcal{F}} D_{A \cap \Gamma}(x)^{m(A, \Gamma)},$$

where the multiplicities $m(A, \Gamma)$ are given by

$$m(A, \Gamma) = i(A, \Gamma) v(A, \Gamma).$$

Notice that $m(A, N) = 1$ and that $m(A, \Gamma) \geq 1$ for all $\Gamma \in \mathcal{F}$. As fundamental groups are topological rather than algebraic, we often replace the principal $A$-determinant with the reduced polynomial

$$\widehat{E}_A(x) = \prod_{\Gamma \in \mathcal{F}} D_{A \cap \Gamma}(x).$$

**Lemma 2.1.** Assume that $\alpha_i \in A$ is not a vertex of $N$. Set $Y_i = \{ x \in \mathbb{C}^A \mid x_i = 0 \}$ and $A_i = A \setminus \{\alpha_i\}$. Then, each irreducible component of $V_A \cap Y_i$ is contained in $V_{A_i}$.

**Proof.** Since $\alpha_i$ is not a vertex of $N$, the hyperplane $Y_i$ is not contained in $V_A$. It follows that the restriction of $E_A$ to $Y_i$ is nontrivial. The statement then follows from (2.3). \qed

In particular, there is an identity of sets $V_A \cap Y_i = V_{A_i}$. However, the irreducible components of $V_A \cap Y_i$ need not appear with the same multiplicity in $V_{A_i}$. In practice, tracing how the multiplicities of irreducible components in $V_A$ change when we restrict to coordinate hyperplanes is a central part of our investigation.

Consider a (regular) triangulation $T$ of the Newton polytope $N$, with vertices in $A$. We express $T$ as the set of full-dimensional cells $\sigma$. Consider the characteristic function $\varphi_T : A \to \mathbb{Z}$ defined by

$$\varphi_T(\alpha) = \sum_{\sigma \in T \mid \alpha \in \text{vert}(\sigma)} \text{vol}_{\mathbb{Z}^A}(\sigma).$$

That is, $\varphi_T(\alpha)$ is the sum of the lattice volumes of all simplices in $T$ containing $\alpha$ as a vertex. The secondary polytope $\Sigma_A$ is defined as the convex hull of the vectors

$$\varphi_T(A) = (\varphi_T(\alpha_1), \ldots, \varphi_T(\alpha_k)) \in \mathbb{Z}^k$$

as $T$ ranges over all (regular) triangulations of $A$ [15, Ch. 7]. The secondary polytope coincides with the Newton polytope of the principal $A$-determinant $E_A$ [15, Ch. 10, Thm. 1.4]. We make the following remark, where we use coordinates $u$ for $\mathbb{Z}^k$. 


Lemma 2.2. Assume that $\alpha_i \in A$ is not a vertex of $N$. If $Z_A = Z_{A_i}$, then the secondary polytope $\Sigma_A$ coincides with the facet of $\Sigma_A$ contained in the hyperplane $u_i = 0$.

Proof. Since $Z_A = Z_{A_i}$, they induce the same lattice volume. Hence, it suffices to note that a regular triangulation $T$ of $N$, with vertices in $A$, is such that $u = \varphi_T(A)$ has $u_i = 0$ if and only if $\alpha_i$ is not a vertex of any simplex in $T$. \flushright{□}

3. Fundamental Groups

Throughout this section, let us consider a general polynomial $P \in \mathbb{C}[y_1, \ldots, y_k]$. By slight abuse of notation, we denote by $V \subset \mathbb{C}^k$ both the vanishing locus of $P$ and the set of irreducible components of $V$. An irreducible hypersurface in $V$ will be denoted by a capital letter, and points in $V$ will be denoted by lowercase letters. We use $y$ as coordinates, and denote the base point of the fundamental group by $x$.

Let $Z \in V$ be an irreducible component, and choose a smooth point $y \in Z$. For a generic line $\ell$ passing through $Z$, and a sufficiently small open neighborhood $U$ of $y$, the complement $(U \cap \ell) \setminus Z$ is a punctured disc. Choose an auxiliary point $\hat{x} \in (U \cap \ell) \setminus Z$, choose a generator $\hat{\gamma}$ of $\pi_1((U \cap \ell) \setminus Z, \hat{x}) \approx \mathbb{Z}$, and choose a path $\rho$ from $x$ to $\hat{x}$ in $\mathbb{C}^k \setminus V$. Then, the generator-of-the-monodromy (gom) of $\pi_1(\mathbb{C}^k \setminus V, x)$, around $Z$ and determined by the above choices, is the path

$$\gamma = \rho^{-1} \circ \hat{\gamma} \circ \rho.$$ 

The nomenclature is self-explanatory; it is well known that the set of goms around all irreducible components $Z \in V$ generates the fundamental group $\pi_1(\mathbb{C}^k \setminus V, x)$. The following lemma is also classical.

Lemma 3.1 (See, e.g., [4, Lem. 2.1]). Let $V_1$ and $V_2$ be two disjoint families of irreducible hypersurfaces in $\mathbb{C}^k$, and choose $x \in \mathbb{C}^k \setminus (V_1 \cup V_2)$. Then, the canonical homomorphism

$$\eta: \pi_1(\mathbb{C}^k \setminus (V_1 \cup V_2), x) \to \pi_1(\mathbb{C}^k \setminus V_1, x)$$

is surjective and, more precisely:

1. Each gom of $\pi_1(\mathbb{C}^k \setminus V_1, x)$ lifts to a gom of $\pi_1(\mathbb{C}^k \setminus (V_1 \cup V_2), x)$.

2. The kernel of $\eta$ is generated by the goms around components of $V_2$. \flushright{□}

Let $P \in \mathbb{C}[y_1, \ldots, y_k]$ with vanishing locus $V$. We are interested in the intersection $V \cap Y_i$ where $Y_i = \{ y \in \mathbb{C}^k | y_i = 0 \}$. To simplify the presentation, we assume that $i = k$. In the following proposition, we work against an auxiliary variable, which we can assume to be $y_1$. We use $y = (y_1, \bar{y}, y_k)$, where $\bar{y} = (y_2, \ldots, y_{k-1})$, as coordinates of $\mathbb{C}^k$. Let

$$\text{Disc}_j(P) = \text{Res}_j \left( P, \frac{\partial P}{\partial y_j} \right)$$

denote the discriminant of $P$ with respect to $y_j$. That is, $\text{Disc}_j(P)$ is the resultant of $P$ the derivative $\partial P/\partial y_j$ with respect to the variable $y_j$. Notice that $\text{Disc}_j(P)$ does not depend on $y_j$. 


**Proposition 3.2.** Let $V$ be a hypersurface defined by a polynomial $P \in \mathbb{C}[y_1, \bar{y}, y_k]$. Let $K(\bar{y}, y_k) = \text{Disc}(P)(\bar{y}, y_k)$ denote the discriminant of $P$ with respect to the auxiliary variable $y_1$. If

1. all common factors of $K$ and $P$ belong to $\mathbb{C}[\bar{y}]$, and
2. $K$ restricted to $Y_k$ is nontrivial

then, for a base point $x = (x_1, \bar{x}, 0) \in Y_k \setminus V$, the canonical morphism

$$\eta : \pi_1(Y_k \setminus (Y_k \cap V), x) \to \pi_1(\mathbb{C}^k \setminus V, x)$$

is surjective.

**Proof.** Let $W$ be the set of irreducible components of the hypersurface $K(\bar{y}, y) = 0$ in $\mathbb{C}^k$. By the assumptions (1) and (2) we can choose a base point $x \in (Y_1 \cap Y_k) \setminus (V \cup W)$. Write $V = V_K \cup V_{\bar{K}}$, where $V_K = V \setminus W$ and $V_{\bar{K}} = V \setminus W$. (Here, we view $V$, $V_K$, and $V_{\bar{K}}$ as sets of irreducible hypersurfaces in $\mathbb{C}^k$.)

By the assumption (1), each element of $V_K$ is the vanishing locus of a polynomial from $\mathbb{C}[\bar{y}] \subset \mathbb{C}[y_1, \bar{y}, y_k]$ and, hence, $\mathbb{C}^k \setminus V_K$ is a trivial bundle over $Y_k \setminus (Y_k \cap V_K)$ with fibers isomorphic to $\mathbb{C}$. It follows that the morphism $\eta'$ in the commutative diagram

$$
\begin{array}{ccc}
\pi_1(Y_k \setminus (Y_k \cap V), x) & \xrightarrow{\eta} & \pi_1(\mathbb{C}^k \setminus V, x) \\
\downarrow & & \downarrow \\
\pi_1(Y_k \setminus (Y_k \cap V_K), x) & \xrightarrow{\eta'} & \pi_1(\mathbb{C}^k \setminus V_K, x)
\end{array}
$$

is an isomorphism. Let $\gamma \in \pi_1(\mathbb{C}^k \setminus V, x)$. Since $\eta'$ is an isomorphism, there is an element $\gamma' \in \pi_1(Y_k \setminus (Y_k \cap V), x)$ such that $\theta(\gamma)\eta'(\gamma') = 0$, and it follows from Lemma 3.1 part (1) that $\gamma'$ lifts to $\pi_1(Y_k \setminus (Y_k \cap V), x)$. Hence, it suffices to show that each $\gamma \in \pi_1(\mathbb{C}^k \setminus V, x)$ with $\theta(\gamma) = 0$ belongs to the image of $\eta$.

Assume that $\gamma \in \pi_1(\mathbb{C}^k \setminus V, x)$ belongs to the kernel of the morphism $\theta$. It follows from Lemma 3.1 part (2) that $\gamma$ belongs to the subgroup generated by the goms around $V_{\bar{K}}$. Hence, there is no loss of generality in assuming that $\gamma$ is a gom around $V_{\bar{K}}$.

The final step is analogous to the argument of [4, Thm. 2.5]. Let $\delta$ be the degree of $P$ in the variable $y_1$. We obtain a trivial fiber bundle

$$\mathbb{C}^k \setminus (V \cup W) \to Y_1 \setminus W,$$

whose fibers are complex lines with $\delta$ points removed, and we obtain the long exact sequence of homotopy groups

$$\ldots \longrightarrow \pi_1(L \setminus (L \cap V), x) \xrightarrow{\tau} \pi_1(\mathbb{C}^k \setminus (V \cup W), x) \xrightarrow{\tau'} \pi_1(Y_1 \setminus W, x) \longrightarrow 0.$$ 

Any gom $\gamma$ of $V_{\bar{K}}$ in $\pi_1(\mathbb{C}^k \setminus V, x)$ lifts by Lemma 3.1 part (1) to a gom of $V_{\bar{K}}$ in $\pi_1(\mathbb{C}^k \setminus (V \cup W), x)$, which belongs to the kernel of the morphism $\tau'$. Hence, $\gamma$ lies in the image of the morphism $\tau$. But (2) implies that we can choose the fiber $L$ inside the plane $Y_k$ and, hence, $\tau$ is simply the morphism $\eta$ restricted to $\pi_1(L \setminus (L \cap V), x)$. The result follows. \qed
We now translate the geometric conditions of Proposition 3.2 in terms of combinatorial conditions on the collection \( A \). Recall that restricting to the hyperplane \( Y_k \) corresponds to deleting the point \( \alpha_k \in A \). The auxiliary variable \( y_1 \) corresponds to an auxiliary point \( \alpha_1 \in A \). Before stating this result, we need a combinatorial definition.

**Definition 3.3.** Let \( \alpha_i \in A \) and set \( A_i = A \setminus \{ \alpha_i \} \). We say that \( \alpha_i \) is **lattice redundant** if all face lattices of \( A \) and \( A_i \) coincide. That is, if for each face \( \Gamma \leq N \) we have that \( Z_{A \cap \Gamma} = Z_{A_i \cap \Gamma} \).

**Proposition 3.4.** Let \( \alpha_k \in A \) be lattice redundant. Let \( \alpha_1 \) be an auxiliary point, contained in a minimal face \( \Gamma_1 \leq N \). Assume, in addition, that

1. \( \alpha_k \) is contained in the closure of \( \Gamma_1 \), and
2. if \( \alpha_1 \in \Gamma_2 \), then either \( A \cap \Gamma_2 \) is defective or \( m(A, \Gamma_2) = m(A_k, \Gamma_2) \).

Then, the canonical morphism

\[
\eta: \pi_1(C^{A_k \setminus E_{A_k}, (x_1, \bar{x})}) \to \pi_1(C^{A \setminus E_A, (x_1, \bar{x}, 0)})
\]

is surjective.

**Proof.** Recall that \( D_{A \cap \Gamma} \) denotes the \( A \cap \Gamma \)-discriminant, which appear as a factor of the principal \( A \)-determinant \( E_A(f) \) with multiplicity \( m(A, \Gamma) \), see \([2,2]\) and \([2,4]\).

We first claim that for each face \( \Gamma \leq N \) there is a polynomial \( Q_\Gamma \in \mathbb{C}[\bar{y}] \) such that

\[
D_{A \setminus \Gamma}(y_1, \bar{y}, 0) = Q_\Gamma(\bar{y})D_{A_k \setminus \Gamma}(y_1, \bar{y}).
\]

A priori, we know, by Lemma \([2,1]\) that \( Q_\Gamma \in \mathbb{C}[y_1, \bar{y}] \) is a product of \( A \cap F \)-discriminants of faces \( F \leq \Gamma \) (possibly including the face \( F = \Gamma \)). Let \( d(F) \) denote the multiplicity of the \( A \cap F \)-discriminant in the product of all coefficients \( Q_\Gamma \) as \( \Gamma \) ranges over all faces of \( N \). It follows that, up to a constant,

\[
E_A(y_1, \bar{y}, 0) = \prod_{\Gamma \leq N} D_{A_k \setminus \Gamma}^{m(A, \Gamma) + d(\Gamma)}(y_1, \bar{y}).
\]

Since \( \alpha_k \) is lattice redundant, the Newton polytope of \( E_A(y_1, \bar{y}, 0) \in \mathbb{C}[y_1, \bar{y}] \) coincides with the secondary polytope \( \Sigma_{A_k} \) by Lemma \([2,2]\). Hence, the degree in \( y_1 \) of \( E_A(y_1, \bar{y}, 0) \) and \( E_{A_k}(y_1, \bar{y}) \) coincide. Counting said degree, we obtain

\[
\sum_{\Gamma \leq N} (m(A, \Gamma) + d(\Gamma)) \deg_1(D_{A \setminus \Gamma}) = \sum_{\Gamma \leq N} m(A_k, \Gamma) \deg_1(D_{A_k \setminus \Gamma}).
\]

If \( A \cap \Gamma \) is defective or if \( \alpha_1 \notin \Gamma \), then \( \deg_1(D_{A_k \setminus \Gamma}) = 0 \). For the remaining faces, we are under the assumption that \( m(A, \Gamma) = m(A_k, \Gamma) \). It follows that

\[
\sum_{\Gamma \leq N} d(\Gamma) \deg_1(D_{A_k \setminus \Gamma}) = 0,
\]

where the sum is taken over all faces \( \Gamma \) containing \( y_1 \) such that \( A \cap \Gamma \) is nondefective. As the degree \( \deg_1(D_{A_k \setminus \Gamma}) \) is positive for such faces, we conclude that \( d(\Gamma) = 0 \) for each such face. It follows that no coefficient \( Q_\Gamma \) contains a factor which depends on \( y_1 \) and, hence, \( Q_\Gamma \in \mathbb{C}[\bar{y}] \) as claimed.
That $Q_\Gamma(y)$ does not depend on $y_1$ implies that
\[
\left.\operatorname{Disc}_1(\widehat{E}_A)(\bar{y}, y_k)\right|_{y_k} = \operatorname{Disc}_1(\widehat{E}_A)(\bar{y}, 0) = \left(\operatorname{Disc}_1(\widehat{E}_{A_k})(\bar{y})\right) \prod_{\Gamma \in \mathcal{N}} Q_\Gamma(y),
\]
where $\widehat{E}_A$ and $\widehat{E}_{A_k}$ denotes the reduced principal $A$-determinants from [2.6]. Since $\widehat{E}_{A_k}$ is reduced, the first factor in the right hand side is nontrivial. It follows that
\[
K(\bar{y}, y_k) = \operatorname{Disc}_1(\widehat{E}_A)(\bar{y}, y_k)
\]
is nontrivial when restricted to $Y_k$. The common factors of $K(\bar{y}, y_k)$ and $\widehat{E}_A(y_1, \bar{y}, y_k)$ are the $(A \cap \Gamma)$-discriminants for faces $\Gamma$ not containing $y_1$. By assumption, any face containing $y_k$ also contains $y_1$ and, hence, said common factors belongs to $C[\bar{y}]$. That is, all assumptions of Proposition 3.2 are fulfilled. $\Box$

4. Saturations

Recall that $A$ is saturated if $A = N \cap Z_A$. We now give the precise definition of the face saturations alluded to in the introduction. Recall that $\mathcal{F}$ denotes the face poset lattice of $N$, and that $\mathcal{F}_{\text{int}} \subset \mathcal{F}$ denote the semi-ideal of $\mathcal{F}$ generated by all faces $\Gamma \leq N$ that contain a relative interior point in $A$.

**Definition 4.1.** The face saturation $A^s$ and the partial face saturation $A^p$ of the collection $A$ are given by
\[
A^s = \bigcup_{\Gamma \in \mathcal{F}} \Gamma^\circ \cap Z_{A \cap \Gamma} \text{ and } A^p = \bigcup_{\Gamma \in \mathcal{F}_{\text{int}}} \Gamma^\circ \cap Z_{A \cap \Gamma},
\]
where $\Gamma^\circ$ denotes the relative interior of the face $\Gamma$.

That is, $A^s$ is obtained from $A$ by adjoining, for each face $\Gamma \leq N$, all points of the lattice $Z_{A \cap \Gamma}$ which are contained in the relative interior of $\Gamma$. We say that we saturate each face of $N$. The partial face saturation differs from the face saturation in that we only saturate the faces which belong to the closure of a face that already contains a relative interior point.

**Example 4.2.** Consider the two dimensional collection
\[
A = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
0 & 3 & 0 & 1 & 0 \\
0 & 0 & 3 & 0 & 2
\end{bmatrix}.
\]
Notice that $Z_A = Z^3$, see Figure 1. The relative face saturation $A^p$ is obtained by saturating the two edges of $N$ which contain a relative interior point. That face saturation
$A^*$ is obtained from $A^p$ by also saturating the full-dimensional face relative $Z_A$. Note that we only add the interior point $(1, 1, 1)$ at this step. The saturation $N \cap Z_A$ consists of all integer points in the Newton polytope.

**Proposition 4.3.** Let $\alpha_k \in A$ be lattice redundant, and assume that there is a face $\Gamma \in F_{\text{int}}$, containing $\alpha_k$, which has a relative interior point $\alpha_1$ distinct from $\alpha_k$. Then, the canonical morphism

$$\eta: \pi_1(C^{A^\ell} \setminus E_{A_k}, (x_1, \bar{x})) \to \pi_1(C^{A\ell} \setminus E_A, (x_1, \bar{x}, 0))$$

is surjective.

**Proof.** We only need to check that the conditions of Proposition 3.4 are fulfilled, with $\alpha_1$ as the auxiliary point. Recall the expression for the multiplicities $m(A, \Gamma)$ from (2.5). Since $\alpha_k$ is lattice redundant, we have that $i(A, \Gamma) = i(A_k, \Gamma)$ for all faces $\Gamma \in F$. It remains to be checked that the subdiagram volume is invariant for any face $\Gamma$ containing $\alpha_1$. This follows a fortiori from the equality of semigroups $N_{A_k}/\Gamma = N_A/\Gamma$ (cf. (2.1)). Indeed, that $Z_A \cap \Gamma_R = Z_{A_k} \cap \Gamma_R$ holds because $\alpha_k$ is lattice redundant. The only generator of $N_A$ not contained in $N_{A_k}$ is $\alpha_k$, but $\alpha_k \in Z_{A_k} \cap \Gamma = Z_{A_k} \cap \Gamma_R$, where the last equality holds because of lattice reducancy. Consequently, $Z_{A_k} \cap \Gamma + N_A = Z_{A_k} \cap \Gamma + N_{A_k}$, from which we deduce that $N_{A_k}/\Gamma = N_A/\Gamma$. Since the semigroups coincide, so do their subdiagram volumes. □

We are now ready to prove the stronger version of Theorem 1.1.

**Theorem 4.4.** Assume that $A$ has an interior point. Then, the inclusion $C^A \to C^{A^p}$ given by appending zeros to $x$ for the coordinates corresponding to the characters $A^p \setminus A$, induces a surjective morphism

$$\eta: \pi_1(C^{A^\ell} \setminus V_A, x) \to \pi_1(C^{A\ell} \setminus V_A^p, (x, 0)).$$

**Proof.** The proof is a simple induction using Proposition 4.3. □

**Proof of Theorem 1.1** If $A$ has a relative interior point, then $A^p = A^*$. Hence, the statement follows from Theorem 4.4. □

For the remainder of this section, we discuss the question of whether, in general, the partial face saturation $A^p$ can be replaced by the face saturation $A^*$ in Theorem 4.4. The main remark is that Proposition 3.4 is much stronger than what is needed to prove Proposition 4.3.

**Proposition 4.5.** If the collection $A$ has dimension at most two, then Theorem 1.1 still holds if the partial face saturation $A^p$ is replaced by the face saturation $A^*$.

**Proof.** If $A$ has dimension one, then $A^p = A^*$. Suppose that $A$ has dimension two. There is no loss of generality in assuming that $A = A^p$ and that $A$ has no relative interior points. Let $\alpha_1$ be a vertex of $N$ and, for $i = 2, 3$, let $\Gamma_i = \text{conv}\{\alpha_1, \alpha_i\}$ denote the edges of $N$ incident to $\alpha_1$. Let $\ell_i$ denote the lattice length of $\Gamma_i$ relative $Z_{A \cap \Gamma_i}$, so that $\alpha_1 + (\alpha_i - \alpha_1)/\ell_i$ is the closest point in to $\alpha_1$ in $Z_{A \cap \Gamma_i}$. Our first claim is that either $Z_A$ has no points in $N^\circ$, or we can find a vertex $\alpha_1$ of $N$ such that

$$\alpha' = \alpha_1 + (\alpha_2 - \alpha_1)/\ell_2 + (\alpha_3 - \alpha_1)/\ell_3 \in N^\circ.$$
Figure 2. The image of $A^s$ under the projections $\pi_1$ and $\pi_2$, with the images of $\Gamma_1 \cap A$ in blue, the images of $\Gamma_2 \cap A$ in black, and the images of $\alpha_8$ in red. Notice that $p_i(\alpha_8)$ is a vertex of $\text{conv}(p_i(A^s \setminus \Gamma_i))$ for $i = 1, 2$.

The proof is elementary planar geometry, and is left to the reader.

If $Z_A$ has no points in $N^o$, then $A^s = A^p$ and there is nothing to prove. If $Z_A$ has a point in $N^o$, then $A' = A \cup \{\alpha'\}$. By construction, $\alpha'$ is lattice redundant in $Z_A$. We claim that $\alpha_1$ can act as an auxiliary point, fulfilling the conditions of Proposition 3.4.

Proposition 4.5 implies that Theorem 1.1 is suboptimal in the sense that we can possibly, in the statement of the theorem, replace $A^p$ with a larger collection and still obtain a surjective morphism $\eta$. That said, Proposition 3.4 is not sufficiently strong to conclude that we can replace $A^p$ by $A^s$ in general, as shown in the following example.

Example 4.6. Consider the collection $A$, and its face saturation,

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 2 & 1 & 2 & 1 & 0 & 1 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 & 3 & 4 \ \end{bmatrix} \quad \text{and} \quad A^s = A \cup \begin{bmatrix} 1 & 1 & 1 \ \end{bmatrix}.$$

There are two faces of $N$ with relative interior points in $A$, given by $\Gamma_1 \cap A = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ and $\Gamma_2 \cap A = \{\alpha_5, \alpha_6, \alpha_7\}$. All strict faces of $A$ are saturated relative to their induced lattices. We have that $Z_A = Z^4$, and there is a unique point $\alpha_8 \in A^s \setminus A$. We claim that there is no point in $A$ which can act as the auxiliary point in Proposition 3.4 for $\alpha_8$. It suffices to show that we have strict inequalities of subdiagram volumes $v(A, \Gamma_i) > v(A^s, \Gamma_i)$ for $i = 1, 2$. Let $p_i$ denote the projection whose kernel is the linear span of $\Gamma_i$. Then, the strict inequalities of subdiagram volumes is a consequence of that $p_i(\alpha_8)$ is a vertex of the polytope $\text{conv}(p_i(A^s \setminus \Gamma_i))$, see Figure 2.

5. Monodromy

In this section, we interchangeably think of $A$ as a subset of $Z_1^{1+n}$ and as a $(1+n) \times k$ integer matrix whose entries are denoted by $\alpha_{ij}$. Let $D$ denote the Weyl algebra over $C^A$, that is, the ring of linear partial differential operators in $k$ variables with polynomial
coefficients over \( C \). This ring is generated as a \( C \)-algebra by \( y_1, \ldots, y_k, \partial_1, \ldots, \partial_k \) subject to the relations imposed by the Leibniz rule for derivatives. Here \( \partial_j \) stands for the partial derivative operator \( \partial_j/\partial y_j \).

We observe that the polynomial ring \( C[\partial_1, \ldots, \partial_k] = C[\partial] \) is a commutative subring of the noncommutative ring \( D \). The following \( C[\partial] \)-ideal is known as the toric ideal associated to \( A \).

\[
I_A = \langle \partial^{u_+} - \partial^{u_-} \mid u \in \mathbb{Z}^k, A \cdot u = 0 \rangle \subset C[\partial],
\]

where for \( u \in \mathbb{Z}^k \), \( (u_+)_i = \max(u_i, 0) \) and \( (u_-)_i = \max(-u_i, 0) \) for \( i = 1, \ldots, k \), so in particular, \( u_+, u_- \in \mathbb{N}^k \) and \( u = u_+ - u_- \). Note that the monomials \( \partial^{u_+} \) and \( \partial^{u_-} \) do not have any variables in common by construction.

We remark that the homogeneity assumption on \( A \) means that the ideal \( I_A \) is homogeneous with respect to the standard (total degree) \( \mathbb{N} \)-grading on \( C[\partial] \).

We can use \( A \) to define more differential operators. Set

\[
E_i = \sum_{j=1}^k \alpha_{ij} y_j \partial_j, \quad i = 1, \ldots, 1+n.
\]

For \( \beta \in \mathbb{C}^{1+n} \) we denote \( E - \beta = \{ E_1 - \beta_1, \ldots, E_{1+n} - \beta_{1+n} \} \). These are called Euler operators. Note that \( F(y) \) is annihilated by the operators \( E - \beta \) if and only if

\[
F(z^{\alpha_1}y_1, \ldots, z^{\alpha_k}y_k) = z^\beta F(y_1, \ldots, y_k),
\]

for \( z \) in a nonempty open subset of \( (\mathbb{C}^*)^{1+n} \) (with respect to the Euclidean topology on \( \mathbb{C}^* \)). If \( F(y) = \sum \lambda_u y^u \) is a formal power series, then \( F \) is annihilated by \( E - \beta \) if and only if \( A \cdot u = \beta \) whenever \( \lambda_u \neq 0 \).

**Definition 5.1.** Let \( A \) be an \( (1+n) \times k \) integer matrix and let \( \beta \in \mathbb{C}^{1+n} \). The \( A \)-hypergeometric system with parameter \( \beta \) is the left \( D_k \)-ideal

\[
H_A(\beta) = D_k \cdot (I_A + \langle E - \beta \rangle).
\]

If \( x \in \mathbb{C}^A \) is a nonsingular point of the system of partial differential operators \( H_A(\beta) \), then the space of germs of complex holomorphic solutions of \( H_A(\beta) \) at \( x \) is denoted \( \text{Sol}_x(A, \beta) \). The dimension of the \( C \)-vector space \( \text{Sol}_x(A, \beta) \) is the holonomic rank of \( H_A(\beta) \), denoted by \( \text{rank}(H_A(\beta)) \).

**Remark 5.2.** Since \( I_A \) is homogeneous, \( D/H_A(\beta) \) is a regular holonomic \( D \)-module for all \( \beta \).

In order to state the main result in this section, we need one more definition.

**Definition 5.3.** A vector \( \beta \in \mathbb{C}^{1+n} \) is said to be a resonant parameter of \( A \) (or simply \( A \)-resonant) if there is \( \gamma \in \mathbb{Z}^{1+n} \) such that \( \beta - \gamma \) lies in the linear span of a codimension one face of \( \text{conv}(A) \). Parameters that are not resonant are called nonresonant (or \( A \)-nonresonant).

It is well known \([1, 17, 23]\) that if \( \beta \) is nonresonant then \( \text{rank}(H_A(\beta)) = \text{vol}(A) \). Note that the set of resonant parameters of \( A \) is the union of an infinite but locally finite collection of hyperplanes in \( \mathbb{C}^{1+n} \), so that nonresonant parameters are very generic.
Theorem 5.4. Assume that $A$ and $A_k$ span the same lattice, and $\text{conv}(A) = \text{conv}(A_k)$ (so in particular $\text{vol}(A) = \text{vol}(A_k)$, and $A$-nonresonance coincides with $A_k$-nonresonance). Let $\beta$ be nonresonant. If $x$ is a generic nonsingular point of $H_{A_k}(\beta)$, then $(x, 0)$ is a nonsingular point of $H_A(\beta)$ and the morphism

$$\Psi: \text{Sol}_{(x,0)}(A, \beta) \to \text{Sol}_x(A_k, \beta),$$

given by $\Psi(F)(y_1, \ldots, y_k) = F(y_1, \ldots, y_{k-1}, 0)$, is an isomorphism.

Note that since $\text{vol}(A) = \text{vol}(A_k)$ and $\beta$ is nonresonant (for both $A$ and $A_k$), we have $\text{rank}(H_A(\beta)) = \text{rank}(H_{A_k}(\beta))$, so that the solution spaces we are interested in have the same vector space dimension. The content of Theorem 5.4 is not that the two vector spaces are isomorphic (which is obvious from comparing dimensions), but that evaluating $y_k \mapsto 0$ gives an isomorphism. A proof of the following auxiliary result can be found in [10] Lemma 7.11.

Lemma 5.5. If $\beta$ is $A$-nonresonant then for any $\gamma \in \mathbb{N}^k$, right multiplication by $\partial^{-\gamma}$ induces a $\mathcal{D}$-module isomorphism $\mathcal{D}/H_A(\beta - A \cdot \gamma) \to \mathcal{D}/H_A(\beta)$. The induced linear transformation of solution spaces $\text{Sol}_x(A, \beta) \to \text{Sol}_x(A, \beta - A \cdot \gamma)$ is given by differentiation. We denote the inverse of this linear transformation by $\partial^{-\gamma}$.

□

Remark 5.6. In the previous statement, the precise form of the inverse linear transformation between solution spaces depends on $\beta$, and denoting it by $\partial^{-\gamma}$ is an abuse of notation. This is justified because whenever we write $\partial^{-\gamma}F$, the parameter of the hypergeometric function $F$ will be understood. Since $\partial_i$ and $\partial_j$ commute for $i \neq j$, $\partial_i$ and $\partial^{-1}_i$ also commute. Identities such as $\partial^{-1}_i \partial_i^{-1} = \partial^{-2}_i$ hold as well, since both are inverses of $\partial^2_i$. It follows that, as long as $\beta$ is nonresonant, $\partial^u$ is well-defined for $u \in \mathbb{Z}^k$.

If $A \cdot u = 0$ and $\beta$ is nonresonant, $\partial^u$ acts as the identity on $\text{Sol}_x(A, \beta)$: if $F$ is a nonzero solution of $H_A(\beta)$, then $\partial^u(\partial^u F - F) = \partial^u F - \partial^u F = 0$. Therefore, $\partial^u F - F$ is in the kernel of the linear isomorphism $\partial^u$, whence $\partial^u F = F$.

Proof of Theorem 5.4. Since $\alpha_k$ is not a vertex of $\text{conv}(A)$, the singular locus of $H_A(\beta)$ does not contain the hyperplane $y_k = 0$. Thus, if $x \in \mathbb{C}^{k-1}$ is a sufficiently generic nonsingular point of $H_{A_k}(\beta)$, then $(x, 0)$ is a nonsingular point of $H_A(\beta)$.

For $u = (u_1, \ldots, u_k) \in \mathbb{Z}^k$, denote $\bar{u} = (u_1, \ldots, u_{k-1}) \in \mathbb{Z}^{k-1}$.

Let $\psi \in \text{Sol}_x(A, \beta)$ applied to $A_k$ (and not $A$). Let $u \in \mathbb{Z}^k$ such that $A \cdot u = 0$ and $u_k = \ell \geq 0$. We set $\psi_{\ell} = \partial^{-\bar{u}} \psi$. Note that if $v \in \mathbb{Z}^k$ with $A \cdot v = 0$ and $v_k = \ell$, then $A_k \cdot (\bar{u} - \bar{v}) = 0$. It follows that $\partial^{-\bar{u}} \psi = \psi$, or equivalently $\partial^{-\bar{u}} \psi = \partial^{-\bar{v}} \psi$. In other words, $\psi_{\ell}$ does not depend on the choice of $u$. Note that $\psi_0 = \psi$.

Given $\ell > 0$, if there is no $u \in \mathbb{Z}^k$ with $A \cdot u = 0$ and $u_k = \ell$, we set $\psi_{\ell} = 0$. By construction, $\psi_{\ell} \in \text{Sol}_x(A_k, \beta - A_k \cdot \bar{u}) = \text{Sol}_x(A_k, \beta - \ell \alpha_k)$. We define

$$F = F(y_1, \ldots, y_k) = \sum_{\ell=0}^{\infty} \frac{y_{\ell}^k}{\ell!} \psi_{\ell}(y_1, \ldots, y_{k-1}).$$
We claim that \( F \) is a formal solution of \( H_A(\beta) \). By construction, each summand of \( F \) is annihilated by \( E - \beta \), so consider \( v \in \mathbb{Z}^k \) and the operator \( \partial^{v_+} - \partial^{v_-} \). If \( v_k = 0 \), then this operator belongs to \( I_A \) and, therefore, annihilates every summand of \( F \).

Assume now that \( v_k \neq 0 \), say \( v_k > 0 \). Let \( u \in \mathbb{Z}^k \) such that \( A \cdot u = 0 \) and \( u_k = \ell \geq v_k \). Then \( (u - v)_k = \ell - v_k \) and, hence,

\[
\psi_{\ell-v_k} = \partial^{-(u-v)}\psi = \partial^{-\ell+v_k} = \partial^{-\ell} \partial^{v_k+} \psi = \partial^{-\ell} \partial^{v_k} \psi = \partial^{-\ell-v} \partial^v \psi_{\ell}.
\]

It follows that \( \partial^{v_+} \psi_{\ell} = \partial^{v_-} \psi_{\ell-v_k} \). Now,

\[
\partial^{v_+} F = \sum_{\ell \geq v_k} \frac{y_k^{\ell-v_k}}{(\ell - v_k)!} \partial^{v_+} \psi_{\ell} = \sum_{\ell \geq v_k} \frac{y_k^{\ell-v_k}}{(\ell - v_k)!} \partial^{v_-} \psi_{\ell-v_k} = \partial^{v_-} \sum_{\ell \geq v_k} \frac{y_k^{\ell-v_k}}{(\ell - v_k)!} \psi_{\ell-v_k} = \partial^{v_-} F.
\]

We have verified that \( F \) is a formal solution of \( H_A(\beta) \). By construction, \( F(x,0) = \psi(x) \), so that, in particular, \( F \) is defined at \((x,0)\).

If \( \psi \) is given as a Nilsson series converging in a neighborhood of \( x \in \mathbb{C}^A \), then \( F \) is by construction a formal Nilsson series. Since \( D/H_A(\beta) \) is a regular holonomic \( D \)-module, and \( F(x,0) \) is defined, we may assume that \( F \) converges in an open neighborhood of \((x,0)\) (after perturbing \( x \) if necessary). Thus, if we apply this construction to the elements a basis of \( \text{Sol}_x(A_k,\beta) \), then we obtain a set of solutions of \( H_A(\beta) \) with a common domain of convergence in a neighborhood of \((x,0)\), whose images under \( y_k \mapsto 0 \) are the elements of the basis of \( \text{Sol}_x(A_k,\beta) \). It follows that the linear transformation \( \Psi : \text{Sol}_{(x,0)}(A,\beta) \to \text{Sol}_x(A_k,\beta) \) is surjective. Since both vector spaces involved have dimension \( \text{vol}(A) \), we conclude that \( \Psi \) is an isomorphism.

**Corollary 5.7.** Let \( \alpha_k \in A \) be lattice redundant, and assume that there is a face \( \Gamma \in F_{\text{int}} \), containing \( \alpha_k \), which has a relative interior point \( \alpha_1 \) distinct form \( \alpha_k \). Let \( \beta \) be nonresonant. If \( x \) is a generic nonsingular point of \( H_{A_k}(\beta) \), so that \((x,0)\) is a nonsingular point of \( H_A(\beta) \), then

\[
\text{Mon}_{(x,0)}(A,\beta) = \text{Mon}_x(A_k,\beta).
\]

**Proof.** Let \( r = \text{vol}(A) \) denote the lattice volume of \( A \). Since \( \beta \) is nonresonant, the solution space \( \text{Sol}_{(x,0)}(A,\beta) \) has dimension \( r \). By Theorem 5.4, we can choose a basis \( F_1, \ldots, F_r \) of \( \text{Sol}_{(x,0)}(A,\beta) \) such that \( F_1(\bar{y},0), \ldots, F_r(\bar{y},0) \) is a basis for the solution space of \( \text{Sol}_x(A_k,\beta) \). By Proposition 4.3, we find a set \( G \subset \pi_1(\mathbb{C}^A \setminus V_{A_k},x) \) which generates both \( \pi_1(\mathbb{C}^A \setminus V_{A_k},x) \) and \( \pi_1(\mathbb{C}^A \setminus V_A,(x,0)) \). It follows that both \( \text{Mon}_{(x,0)}(A,\beta) \) and \( \text{Mon}_x(A_k,\beta) \) are generated by monodromy matrices obtain by analytic continuation of the functions \( F_1, \ldots, F_r \) along the paths \( \gamma \in G \). Hence, the two monodromy groups have the same set of generators.

**Theorem 5.8.** If \( \beta \) is a nonresonant parameter, then

\[
\text{Mon}_x(A,\beta) \cong \text{Mon}_{(x,0)}(A^*,\beta).
\]

**Proof.** The proof is a simple induction using Corollary 5.7. □

**Proof of Theorem 1.2.** If \( A \) has a relative interior point, then \( A^p = A^* \). Hence, the statement follows from Theorem 5.8. □
6. Monomial Curves

Let us exemplify our main theorems by considering a one-dimensional collection

\[ A = \begin{bmatrix} 1 & 1 & \ldots & 1 & 1 \\ 0 & \alpha_1 & \ldots & \alpha_m & \delta \end{bmatrix}, \]

where \( 0 < \alpha_1 < \cdots < \alpha_m < \delta \). The toric variety associated to such a one-dimensional collection is called a monomial curve. The integer \( \delta \) is the toric degree of \( A \). There is no loss of generality in assuming that \( \gcd(\alpha_1, \ldots, \alpha_m, \delta) = 1 \), so that \( \mathbb{Z}_A = \mathbb{Z}^2 \). In this case, the partial face saturation and the face saturation of \( A \) both coincide with the saturation \( N \cap \mathbb{Z}_A \). The space \( (\mathbb{C}^*)^A \) is identified with the space of all univariate polynomials

\[ f(z) = y_0 + y_1 z^{\alpha_1} + \cdots + y_m z^{\alpha_m} + y_{m+1} z^\delta. \]

The reduced principal \( A \)-determinant is \( \tilde{E}_A(y) = y_0 y_{m+1} D_A(y) \), where \( D_A(y) \) is the \( A \)-discriminant. In particular, there is a map

\[ V : \mathbb{C}^A \setminus E_A \to C_\delta(\mathbb{C}^*), \]

where \( C_\delta(\mathbb{C}^*) \) denotes the configuration space of \( \delta \) distinct points in \( \mathbb{C}^* \). Taking fundamental groups, we obtain the braid map

\[ \text{br} : \pi_1(\mathbb{C}^A \setminus E_A, x) \to \mathcal{C}B_\delta, \]

where \( \mathcal{C}B_\delta \) denotes the cyclic braid group on \( \delta \) strands.

**Corollary 6.1.** The braid map \( \text{br} \) is surjective.

**Proof.** If \( A \) is saturated, then there is a map \( C_\delta(\mathbb{C}^*) \to \mathbb{C}^A \setminus E_A \) which takes a set of \( \delta \) points to the monic polynomial of degree \( \delta \) vanishing on said set. This fact, and Theorem 1.1, imply that the braid map is surjective also in the general case. \( \square \)

The braid map is not injective. This is primarily an effect of working with the principal \( A \)-determinant rather then the \( A \)-discriminant. For example, the cycle

\[ t \mapsto e^{2\pi it} y \]

is nontrivial (and acts diagonally, yet nontrivially, in monodromy) in \( \mathbb{C}^A \setminus E_A \). We conclude that there is a surjective map

\[ \text{br} : \pi_1(\mathbb{C}^A \setminus E_A, x) \to \mathbb{Z} \times \mathcal{C}B_\delta. \]

There are simple arguments, using for example the order map of the coamoeba [13], which imply that the extended braid map \( \text{br} \) is injective whenever \( A \) consists of three elements that generate \( \mathbb{Z} \). It follows that the braid map is injective whenever there exists an \( i \in \{1, \ldots, m\} \) such that \( \gcd(a_i, \delta) = 1 \). While we expect that the extended braid map is always injective, no such simple argument is available in the general case.

In terms of computing the monodromy group of the \( A \)-hypergeometric system, it is beneficial to choose \( A \) with as few points as possible. The monodromy group \( \text{Mon}_x(A, \beta) \)
only depends on $\beta$ and the toric degree $\delta$. Indeed, the collection $A$ has the same saturation as the triple

$$A' = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & \delta \end{bmatrix},$$

which is of codimension one and, hence, admits a Mellin–Barnes basis. Using the torus action, we find that $\mathbb{C}^{A'\setminus E_{A'}}$ is a trivial bundle, with fibers $(\mathbb{C}^*)^2$, and base $\mathbb{C}\setminus D_{A'}$, where $D_{A'}$ is the dehomogenized discriminant defined as the intersection of $D_{A'}$ with the linear subspace $x_1 = x_2 = 1$. We get a long exact sequence

$$\ldots \rightarrow \mathbb{Z}^2 \rightarrow \pi_1(\mathbb{C}^{A'\setminus E_{A'}}, x) \rightarrow \pi_1(\mathbb{C}\setminus D_{A'}, x) \rightarrow 0.$$

The dehomogenized discriminant $D_{A'}$ is a rational zero-dimensional variety and, hence, consists of a single point [20]. It follows that $\pi_1(\mathbb{C}^{A'\setminus E_{A'}}, x)$ has three generators, in this case. The corresponding monodromy matrices can be computed using Beukers’ method [6]. For example, if $\delta = 3$, then the monodromy group is generated by

$$\begin{bmatrix} e^{2\pi i \beta_1} & 0 & 0 \\ 0 & e^{2\pi i \beta_1} & 0 \\ 0 & 0 & e^{2\pi i \beta_1} \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & e^{2\pi i (\beta_2 - \beta_1)} \\ e^{2\pi i \beta_1} & 0 & 0 \end{bmatrix}, \text{ and } \begin{bmatrix} -1 & 1 & 1 \\ 0 & e^{2\pi i \beta_1} & 0 \\ 0 & 0 & e^{2\pi i \beta_1} \end{bmatrix}.$$

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Jens Forsgård, Universiteit Utrecht, Mathematisch Instituut, Postbus 80010, 3508 TA Utrecht, The Netherlands

E-mail address: j.b.forsgaard@uu.nl

Laura Felicia Matusevich, Department of Mathematics, Texas A&M University, College Station, TX 77843.

E-mail address: laura@math.tamu.edu