A BOC'S THEORETIC CHARACTERIZATION OF GENDO-SYMMETRIC ALGEBRAS

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Abstract. Gendo-symmetric algebras were recently introduced by Fang and König in [FanKoe]. An algebra is called gendo-symmetric in case it is isomorphic to the endomorphism ring of a generator over a finite dimensional symmetric algebra. We show that a finite dimensional algebra $A$ over a field $K$ is gendo-symmetric if and only if there is a bocs-structure on $(A, D(A))$, where $D = \text{Hom}_K(-, K)$ is the natural duality. Assuming that $A$ is gendo-symmetric, we show that the module category of the bocs $(A, D(A))$ is isomorphic to the module category of the algebra $eAe$, when $e$ is an idempotent such that $eA$ is the unique minimal faithful projective-injective right $A$-module. We also prove some new results about gendo-symmetric algebras using the theory of bocses.

Introduction

A bocs is a generalization of the notion of coalgebra over a field. Bocses are also known under the name coring (see the book [BreWis]). A famous application of bocses has been the proof of the tame and wild dichotomy theorem by Drozd for finite dimensional algebras over an algebraically closed field (see [Dro] and the book [BSZ]). For any given bocs $(A, W)$ over a finite dimensional algebra, one can define a corresponding module category and analyze it. Given a finite dimensional algebra $A$ over a field $K$, it is an interesting question whether for a given $A$-bimodule $W$, there exists a bocs structure on $(A, W)$. The easiest example to consider is the case $W = A$ and in this case the module category one gets is just the module category of the algebra $A$. Every finite dimensional algebra has a duality $D = \text{Hom}_K(-, K)$ and so the next example of an $A$-bimodule to consider is perhaps $W = D(A)$. We will characterize all finite dimensional algebras $A$ such that there is a bocs structure on $(A, D(A))$ and find a surprising connection to a recently introduced class of algebras generalizing symmetric algebras (see [FanKoe2]). Those algebras are called gendo-symmetric and are defined as endomorphism rings of generators of symmetric algebras. Alternatively these are the algebras $A$, where there exists an idempotent $e$ such that $eA$ is a minimal faithful injective-projective module and $D(eA) \cong eA$ as $(eA, eA)$-bimodules. Then $eA$ is the symmetric algebra such that $A \cong \text{End}_{eA}(M)$, for an $eA$-module $M$ that is a generator of $\text{mod-} eA$. Famous examples of non-symmetric gendo-symmetric algebras are Schur algebras $S(n,r)$ with $n \geq r$ and blocks of the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ of a complex semisimple Lie algebra (for a proof of this, using methods close to ours, see [KSX] and for applications see [FanKoe3]). The first section provides the necessary background on bocses and algebras with dominant dimension larger or equal 2. The second section proves our main theorem:

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Theorem A
(Theorem 2.2)
A finite dimensional algebra \( A \) is gendo-symmetric if and only if \( (A, D(A)) \) has a bocs-structure.

We also provide some new structural results about gendo-symmetric algebras in this section. For example we show, using bocs-theoretic methods, that the tensor product over the field \( K \) of two gendo-symmetric algebras is again gendo-symmetric and we proof that \( \text{Hom}_A(D(A), A) \) is isomorphic to the center of \( A \), where \( A^e \) denotes the enveloping algebra of \( A \).

In the final section, we describe the module category \( B \) of the bocs \( (A, D(A)) \) in case \( A \) is gendo-symmetric. The following is our second main result:

Theorem B
(Theorem 3.3)
Let \( A \) be a gendo-symmetric algebra with minimal faithful projective-injective module \( eA \). Then the module category of the bocs \( (A, D(A)) \) is equivalent to \( eAe \)-mod as \( K \)-linear categories.

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1. Preliminaries

We collect here all needed definitions and lemmas to prove the main theorems. Let an algebra always be a finite dimensional algebra over a field \( K \) stated. \( D = \text{Hom}_A(−, K) \) denotes the duality for a given finite dimensional algebra \( A \). \( \text{mod} − A \) denotes the category of finite dimensional right \( A \)-modules and \( \text{proj} (\text{inj}) \) denotes the subcategory of finitely generated projective (injective) \( A \)-modules. We note that we often omit the index in a tensor product, when we calculate with elements. We often identify \( A \otimes_A X \cong X \) for an \( A \)-module \( X \) without explicitly mentioning the natural isomorphism. The Nakayama functor \( ν: \text{mod} − A \rightarrow \text{mod} − A \) is defined as \( D\text{Hom}_A(−, A) \) and is isomorphic to the functor \( (−) \otimes_A D(A) \). The inverse Nakayama functor \( ν^{-1}: \text{mod} − A \rightarrow \text{mod} − A \) is defined as \( D\text{Hom}_A(−, A)D \) and is isomorphic to the functor \( \text{Hom}_A(D(A), −) \) (see \([SkoYam]\) Chapter III section 5 for details). The Nakayama functors play a prominent role in the representation theory of finite dimensional algebras, since \( ν: \text{proj} \rightarrow \text{inj} \) is an equivalence with inverse \( ν^{-1} \). For example they appear in the definition of the Auslander-Reiten translates \( τ \) and \( τ^{-1} \) (see \([SkoYam]\) Chapter III, for the definitions):

1.1. Proposition

Let \( M \) be an \( A \)-module with a minimal injective presentation \( 0 \rightarrow M \rightarrow I_0 \rightarrow I_1 \).

Then the following sequence is exact:

\[
0 \rightarrow ν^{-1}(M) \rightarrow ν^{-1}(I_0) \rightarrow ν^{-1}(I_1) \rightarrow τ^{-1}(M) \rightarrow 0.
\]

Proof. See \([SkoYam]\), Chapter III, Proposition 5.3. (ii). □

The dominant dimension \( \text{domdim}(M) \) of a module \( M \) with a minimal injective resolution \( (I_i): 0 \rightarrow M \rightarrow I_0 \rightarrow I_1 \rightarrow ... \) is defined as:

\[
\text{domdim}(M):=\sup\{n|I_i \text{ is projective for } i=0,1,...,n\}+1, \text{ if } I_0 \text{ is projective, and}
\]

\[
\text{domdim}(M):=0, \text{ if } I_0 \text{ is not projective.}
\]

The dominant dimension of a finite dimensional algebra \( A \) is defined as the dominant dimension of the regular module \( A_A \). It is well-known that an algebra \( A \) has dominant dimension larger than or equal to 1 iff there is an idempotent \( e \) such that \( eA \) is a minimal faithful projective-injective module. The Morita-Tachikawa correspondence (see \([Ta]\) for details) says that the algebras, which are endomorphism rings of
generator-cogenerators are exactly the algebras with dominant dimension at least 2. The full subcategory of modules of dominant dimension at least $i \geq 1$ is denoted by $\text{Dom}_i$. $A$ is called a Morita algebra iff it has dominant dimension larger than or equal to 2 and $D(Ae) \cong eA$ as $A$-right modules. This is equivalent to $A$ being isomorphic to $\text{End}_B(M)$, where $B$ is a selfinjective algebra and $M$ a generator of mod-$B$ (see [KerYam]). $A$ is called a gendo-symmetric algebra iff it has dominant dimension larger than or equal to 2 and $D(eA) \cong eA$ as $(eAe, A)$-bimodules iff it has dominant dimension larger than or equal to 2 and $D(eA) \cong eA$ as $(A, eA)$-bimodules. This is equivalent to $A$ being isomorphic to $\text{End}_B(M)$, where $B$ is a symmetric algebra and $M$ a generator of mod-$B$ and in this case $B = eAe$ (see [FanKoc]).

1.2. Proposition
Let $A$ be a gendo-symmetric algebra and $M$ an $A$-module. Then $M$ has dominant dimension larger or equal to two iff $\nu^{-1}(M) \cong M$.

Proof. See [FanKoc], proposition 3.3. □

The following result gives a formula for the dominant dimension of Morita algebras:

1.3. Proposition
Let $A$ be a Morita algebra with minimal faithful projective-injective module $eA$ and $M$ an $A$-module. Then $\text{domdim}(M) = \inf\{i \geq 0 | \text{Ext}^i(A/AeA, M) \neq 0\}$. Especially, $\text{Hom}_A(A/AeA, A) = 0$ for every Morita algebra, since they always have dominant dimension at least 2.

Proof. This is a special case of [APT], Proposition 2.6. □

The following lemma gives another characterization of gendo-symmetric algebras, which is used in the proof of the main theorem.

1.4. Lemma
Let $A$ be a finite dimensional algebra. Then $A$ is a gendo-symmetric algebra iff $D(A) \otimes_A D(A) \cong D(A)$ as $A$-bimodules. Assume $eA$ is the minimal faithful projective-injective module. In case $A$ is gendo-symmetric, $D(A) \cong eA \otimes_{eAe} eA$ as $A$-bimodules.

Proof. See [FanKoc], Theorem 3.2. and [FanKoc] in the construction of the comultiplication following Definition 2.3. □

1.5. Lemma
An $A$-module $P$ is projective iff there are elements $p_1, p_2, \ldots, p_n \in P$ and elements $\pi_1, \pi_2, \ldots, \pi_n \in \text{Hom}_A(P, A)$ such that the following condition holds:

$$x = \sum_{i=1}^n p_i \pi_i(x) \text{ for every } x \in P.$$  

We then call the $p_1, \ldots, p_n$ a probasis and $\pi_1, \ldots, \pi_n$ a dual probasis of $P$.

Proof. See [Rot], Proposition 3.10. □

1.6. Example
Let $P = eA$, for an idempotent $e$. Then a probasis is given by $p_1 = e$ and the dual probasis is given by $\pi_1 = l_e \in \text{Hom}_A(eA, A)$, which is left multiplication by $e$. $l_e$ can be identified with $e$ under the $(A, eAe)$-bimodule isomorphism $eA \cong \text{Hom}_A(eA, A)$.

1.7. Proposition
1. $\text{Hom}_A(D(A), A)$ is a faithful right $A$-module iff there is an idempotent $e$, such that $eA$ and $eAe$ are faithful and injective.
2. Let $A$ be an algebra with $\text{Hom}_A(D(A), A) \cong A$ as right $A$-modules, then $A$ is a Morita algebra.
Proof. 1. See [KerYam], Theorem 1.
2. See [KerYam], Theorem 3.

1.8. Lemma
Let $Y$ and $Z$ be $A$-bimodules. Then the following is an isomorphism of $A$-bimodules:

$$\text{Hom}_A(Y,D(Z)) \cong D(Y \otimes_A Z).$$

Proof. See [ASS] Appendix 4, Proposition 4.11.

1.9. Definition
Let $A$ be a finite dimensional algebra and $W$ an $A$-bimodule and let $e_r : W \to A \otimes_A W$ and $e_l : W \to W \otimes_A A$ be the canonical isomorphisms. Then the tuple $\mathcal{B} := (A,W)$ is called a bocs (see [Kue]) or the module $W$ is called an $A$-coring (see [BreWis]) if there are $A$-bimodule maps $\mu : W \to W \otimes_A W$ (the comultiplication) and $\epsilon : W \to A$ (the counit) with the following properties:

$$(1_W \otimes_A \epsilon) \mu = e_l, (\epsilon \otimes_A 1_W) \mu = e_r \text{ and } (\mu \otimes_A 1_W) \mu = (1_W \otimes_A \mu) \mu.$$ We often say for short that $W$ is a bocs, if $A$ (and $\mu$ and $\epsilon$) are clear from the context. The category of the finite dimensional bocs modules is defined as follows:

Objects are the finite dimensional right $A$-modules. Homomorphism spaces are $\text{Hom}_B(M,N) := \text{Hom}_A(M, \text{Hom}_A(W,N))$ with the following composition $*$ and units:

Let $g : M \to \text{Hom}_A(W,N) \in \text{Hom}_B(M,N)$ and $f : L \to \text{Hom}_A(W,M) \in \text{Hom}_B(L,M)$. Then $g * f := \text{Hom}_A(\mu, N) \psi \text{Hom}_A(g, W) f$, where $\psi$ is the adjunction isomorphism $\text{Hom}_A(W, \text{Hom}_A(W,N)) \to \text{Hom}_A(W \otimes_A W, N)$. The units $1_M \in \text{Hom}_B(M,M)$ are defined as follows: $1_M := \text{Hom}_A(\epsilon, N) \xi$, where $\xi : M \to \text{Hom}_A(A, M)$ is the canonical isomorphism. Note that the module category of a bocs is $K$-linear. We refer to [Kue] for other equivalent descriptions of the bocs module category and more information.

1.10. Examples
1. $(A, A)$ is always a bocs with the obvious multiplication and comultiplication. The next natural bimodule to look for a bocs-structure is $D(A)$. We will see that $(A, D(A))$ is not a bocs for arbitrary finite dimensional algebras.
2. The next example can be found in 17.6. in [BreWis], to which we refer for more details. Let $P$ be a $(B, A)$-bimodule for two finite dimensional algebras $B$ and $A$ such that $P$ is projective as a right $A$-module and let $P^* := \text{Hom}(P, A)$, which is then a $(A, B)$ bimodule. Let $p_1, p_2, ..., p_n$ be a probasis for $P$ and $\pi_1, \pi_2, ..., \pi_n$ a dual probasis of the projective $A$-module $P$. Denote the $A$-bimodule $P^* \otimes_B P$ by $W$ and define the comultiplication $\mu : W \to W \otimes_A W$ as follows: Let $f \in P^*$ and $p \in P$, then $\mu(f \otimes p) = \sum f(p_i) \otimes (\pi_i \otimes p)$. Define the counit $\epsilon : W \to A$ as follows: $\epsilon(f \otimes p) = f(p)$. Now specialise to $P = e_A$, for an idempotent $e$ and identify $\text{Hom}_A(eA, A) = Ae$. Then $\mu(ae \otimes eb) = (ae \otimes e) \otimes (e \otimes eb)$ and $\epsilon(ae \otimes eb) = aeb$. We will use this special case in the next section to show that $(A, D(A))$ is always a bocs for a gendo-symmetric algebra.
3. Let $(A_1, W_1)$ and $(A_2, W_2)$ be bocses, then $(A_1 \otimes_K A_2, W_1 \otimes_K W_2)$ is again a bocs. See [BreWis] 24.1. for a proof.

2. Characterization of gendo-symmetric algebras

The following lemma, will be important for proving the main theorem.

2.1. Lemma
Assume that $\text{Hom}_A(D(A), A) \cong A \oplus X$ as right $A$-modules for some right $A$-module $X$, then $\text{domdim}(A) \geq 2$ and $X = 0$. 
Proof. By assumption $\text{Hom}_A(D(A), A)$ is faithful and so there is an idempotent $e$ with $eA$ and $Ae$ faithful and injective by [1.7], which implies that $A$ has dominant dimension at least 1. Choose $e$ minimal such that those properties hold. Now look at the minimal injective presentation $0 \to A \to I_0 \to I_1$ of $A$ and note that $I_0 \in \text{add}(eA)$. Using $[1.1]$, there is the following exact sequence: $0 \to \nu^{-1}(A) \to \nu^{-1}(I_0) \to \nu^{-1}(I_1) \to \tau^*(A) \to 0$. But $\nu^{-1}(A) \cong \text{Hom}_A(D(A), A) \cong A \oplus X$ and so there is the embedding: $0 \to A \oplus X \to \nu^{-1}(I_0)$. Note that $\nu^{-1}(I_0) \in \text{add}(eA)$ is the injective hull of $A \oplus X$, since $\nu^{-1} : \text{inj} \to \text{proj}$ is an equivalence and $eA$ is the minimal faithful projective injective module. Thus $\nu^{-1}(I_0)$ has the same number of indecomposable direct summands as $I_0$. Therefore $\text{soc}(X) = 0$ and so $X = 0$, since every indecomposable summand of the socle of the module provides an indecomposable direct summand of the injective hull of that module. Thus $\text{Hom}_A(D(A), A) \cong A$ and $A$ is a Morita algebra by [1.7] 2. and so $A$ has dominant dimension at least 2. $\square$

We now give a bocs-theoretic characterization of gendo-symmetric algebras.

2.2. Theorem
Let $A$ be a finite dimensional algebra. Then the following are equivalent:

1. $A$ is gendo-symmetric.
2. There is a comultiplication and counit such that $B = (A, D(A))$ is a bocs.

Proof. We first show that 1. implies 2.:
Assume that $A$ is gendo-symmetric with minimal faithful projective-injective module $eA$. Set $P := eA$ and apply the second example in [1.10] with $B := eAc$, to see that $B := (A, Ac \otimes_{Ac} eA)$ has the structure of a bocs. Now note that by [1.4] $D(A) \cong Ac \otimes_{Ac} eA$ as $A$-bimodules and one can use this to get a bocs structure for $(A, D(A))$.

Now we show that 2. implies 1.:
Assume that $(A, D(A))$ is a bocs with comultiplication $\mu$ and counit $\epsilon$. Note first that the comultiplication $\mu$ always has to be injective because in the identity $(e \otimes_A 1_W)\mu = c_\varepsilon$ appearing the definition of a bocs, $c_\varepsilon$ is an isomorphism.

So there is a injection $\mu : D(A) \to D(A) \otimes_A D(A)$ which gives a surjection $D(\mu) : D(D(A) \otimes_A D(A)) \to A$. Now using [1.8] we see that $D(D(A) \otimes_A D(A)) \cong \text{Hom}_A(D(A), A)$ as $A$-bimodules.

Since $A$ is projective, $D(\mu)$ is split and $\text{Hom}_A(D(A), A) \cong A \oplus X$ for some $A$-module $X$. By [2.1] this implies $\text{Hom}_A(D(A), A) \cong A$ and comparing dimensions, $D(\mu)$ and thus also $\mu$ have to be isomorphisms. By [1.4] $A$ is gendo-symmetric. $\square$

2.3. Corollary
Let $A$ be a finite dimensional algebra. Then the following two conditions are equivalent:

1. $A$ is gendo-symmetric.
2. $\nu$ is a comonad.

Proof. In [BreWis] 18.28. it is proven that an $A$-bimodule $W$ is a bocs iff the functor $(-) \otimes_A W$ is a comonad. Applying this with $W = D(A)$ and using the previous theorem, the corollary follows. $\square$

2.4. Remark
Theorem 2.2 also shows that the comultiplication of the bocs $(A, D(A))$ is always an $A$-bimodule isomorphism for a gendo-symmetric algebra $A$. In [FanKoe], section 2.2., it is noted that such an isomorphism is unique up to multiples of invertible central elements in $A$. Thus the comultiplication of the bocs is also unique in that sense.
The following proposition gives an application:

2.5. Proposition
Let $A$ and $B$ be gendo-symmetric $K$-algebras. Then $A \otimes_K B$ is again a gendo-symmetric $K$-algebra. In particular, let $F$ be a field extension of $K$ and $A$ a gendo-symmetric $K$-algebra. Then $A \otimes_K F$ is again gendo-symmetric.

Proof. Let $A$ and $B$ be two gendo-symmetric algebras. Then $B_1 = (A, D(A))$ and $B_2 = (B, D(A))$ are bocses. By example 3 of 1.10 also the tensor product of $B_1$ and $B_2$ are bocses, it is the bocs $\mathcal{C} = (A \otimes_K B, D(A) \otimes_K D(B))$. Recall the well known formula $(D(A) \otimes_K D(B)) \cong D(A \otimes_K B)$, which can be found as exercise 12. of chapter II. in [BreWis]. Using this isomorphism one can find a bocs structure on $(A \otimes_K B, D(A \otimes_K B))$ using the bocs structure on $\mathcal{C}$. Thus by our bocs-theoretic characterization of gendo-symmetric algebras, also $A \otimes_K B$ is gendo-symmetric. The second part follows since every field is a symmetric and thus gendo-symmetric algebra.

Let $A^e := A^{op} \otimes_K A$ denote the enveloping algebra of a given algebra $A$. The following proposition can be found in [BreWis], 17.8.

2.6. Proposition
Let $(A, W)$ be a bocs and $c \in W$ with $\mu(c) = \sum_{i=1}^n c_{1,i} \otimes c_{2,i}$.

1. $\text{Hom}_A(W, A)$ has a ring structure with unit $\epsilon$ and product $*^r$, given as follows for $f, g \in \text{Hom}_A(W, A)$:
   $f *^r g = g(f \otimes_A \text{id}_W)\mu$.

   There is a ring anti-morphism $\zeta : A \rightarrow \text{Hom}_A(W, A)$, given by $\zeta(a) = \epsilon(a(-))$.

2. $\text{Hom}_{A^e}(W, A)$ has a ring structure with unit $\epsilon$ and multiplication $*$ given as follows for $f, g \in \text{Hom}_{A^e}(W, A)$:
   $f \ast g(c) = \sum_{i=1}^n f(c_{1,i})g(c_{2,i})$.

   We now describe the ring structures on $\text{Hom}_{A^e}(D(A), A)$ and $\text{Hom}_{A^e}(D(A), A)$.

2.7. Proposition
Let $A$ be gendo-symmetric.

1. $\zeta$, as defined in the previous proposition, is a ring anti-isomorphism $\zeta : A \rightarrow \text{Hom}_{A^e}(D(A), A)$.

2. With the ring structure on $\text{Hom}_{A^e}(D(A), A)$ as defined in the previous proposition,
   $\text{Hom}_{A^e}(D(A), A)$ is isomorphic to the center $Z(A)$ of $A$.

Proof. We use the isomorphism of $A$-bimodule $D(A) \cong A e \otimes_{A e} e A$.

1. Since $A$ and $\text{Hom}_{A^e}(D(A), A)$ have the same $K$-dimension, the only thing left to show is that $\zeta$ is injective. So assume that $\zeta(a) = \epsilon(a(-)) = 0$, for some $a \in A$. This is equivalent to $\epsilon(ax) = 0$ for every $x = e c \otimes d e$ $\in A e \otimes e A$. Now $\epsilon(a c e \otimes d e) = e(a c e \otimes d e)$ is $e A c e A d$. Thus, since $c, d$ were arbitrary, $a A e e A = 0$. This means that $a$ is in the left annihilator $L(A e) e A$ of the two-sided ideal $A e A$. But $L(A e A) = 0$, since $\text{Hom}_{A^e}(A/A e A, A) = 0$, by 1.3 and thus $a = 0$. Therefore $\zeta$ is injective.

2. Define $\psi : \text{Hom}_{A^e}(D(A), A) \rightarrow Z(e A e)$ by $\psi(f) = f(e \otimes e)$, for $f \in \text{Hom}_{A^e}(D(A), A)$. First, we show that this is well-defined, that is $f(e \otimes e)$ is really in the center of $Z(e A e)$. Let $x \in e A e$. Then $xf(e \otimes e) = f(x e \otimes e) = f(e \otimes e x) = f(e \otimes e x e)$ and therefore $f(e \otimes e) \in Z(e A e)$. Clearly, $\psi$ is $K$-linear. Now we show that the map is injective: Assume $\psi(f) = 0$, which is equivalent to $f(e \otimes e) = 0$. Then for any $a, b \in A : f(a e \otimes e b) = 0$, and thus $f = 0$.

Now we show that $\psi$ is surjective. Let $z \in Z(e A e)$ be given. Then define a map
f_z \in \text{Hom}_{\mathcal{A}}(D(A), A) \text{ by } f_z(ac \otimes eb) = zaeb. \text{ Then, since } z \text{ is in the center of } eAe, f \text{ is } A\text{-bilinear and obviously } \psi(f_z) = f_z(c \otimes e) = zce = z. \psi \text{ also preserves the unit and multiplication:} \\
\psi(e) = e(c \otimes e) = e^2 = e \text{ and for two given } f, g \in \text{Hom}_{\mathcal{A}}(D(A), A); \phi(f \ast g) = (f \ast g)(c \otimes e) = (f \ast g)(e \otimes e) = f(e \otimes e)g(e \otimes e), \text{ by the definition of } \ast. \text{ To finish the proof, we use the result from } \text{FanKoe}, \text{ Lemma 2.2., that the map } \phi : Z(A) \to Z(eAe), \phi(z) = zce \text{ is a ring isomorphism in case } A \text{ is gendo-symmetric.} \quad \square

3. Description of the module category of the bocs \((A, D(A))\) for a gendo-symmetric algebra

Let \(A\) be a gendo-symmetric algebra. In this section we describe the module category of the bocs \(\mathcal{B} = (A, D(A))\) as a \(K\)-linear category. We will use the \(A\) bimodule isomorphism \(Ae \otimes_{eAe} eA \cong D(A)\) often without mentioning. Let \(M\) be an arbitrary \(A\)-module. Define for a given \(M\) the map \(I_M : M \to \text{Hom}_A(D(A), M)\) by \(I_M(m) = u_m\) for any \(m \in M\), where \(u_m : D(A) \to M\) is the map \(u_m(ac \otimes eb) = maeb\) for any \(a, b \in A\). Before we get into explicit calculation, let us recall how \(\ast\) is defined in this special case. Let \(f \in \text{Hom}_B(L, M)\) and \(g \in \text{Hom}_B(M, N)\), then for \(l \in L\) and \(a, b \in A : (g \ast f)(l)(ae \otimes eb) = g(f(l)(ae \otimes e))(e \otimes eb)\).

3.1. Proposition

1. \(I_M\) is well defined.
2. \(I_M\) is injective, iff \(M\) has dominant dimension larger or equal 1.
3. \(I_M\) is bijective, iff \(M\) has dominant dimension larger or equal 2.

Proof. 1. We have to show two things: First, \(u_m\) is \(A\)-linear for any \(m \in M\):
\(u_m((a \otimes b)c) = u_m((ae \otimes cbc) = (maeb)c = u_m(ac \otimes eb)c\). Second, \(I_M\) is also \(A\)-linear: \(I_M((mc) \otimes eb) = u_{mc}(ae \otimes eb) = maeb = u_m((ae \otimes eb) = (u_m)eb\). If \(I_M\) is injective iff \((m = 0 \Leftrightarrow u_m = 0)\). Now \(u_m = 0\) is equivalent to \(maeb = 0\) for any \(a, b \in A\). This is equivalent to the condition that the two-sided ideal \(AeA\) annihilates \(m\). Thus there is a nonzero \(m\) with \(u_m = 0\) iff \(\text{Hom}_A(A/AeA, M) \neq 0\) if \(M\) has dominant dimension zero by \([1, 3]\).
3. By \([1, 2]\) \(M\) has dominant dimension larger or equal two iff \(M \cong \nu^{-1}(M)\).

Thus 3. follows by 2. since an injective map between modules of the same dimension is a bijective map. \(\square\)

3.2. Lemma

For any module \(M\), there is an isomorphism
\(\text{Hom}_A(\mu, M) : \text{Hom}_A(D(A), \text{Hom}_A(D(A), M)) \to \text{Hom}(D(A), M)\) and thus \(\nu^{-1}(M) \cong \nu^{-2}(M)\). It follows that every module of the form \(\nu^{-1}(M)\) has dominant dimension at least two.

Proof. The result follows, since \(\psi\) is the canonical isomorphism
\(\psi : \text{Hom}_A(D(A), \text{Hom}_A(D(A), M)) \to \text{Hom}_A(D(A) \otimes_A D(A), M)\) and since \(\mu\) is an isomorphism also \(\text{Hom}_A(AeA, M)\) is an isomorphism. That \(\nu^{-1}(M)\) has dominant dimension at least two, follows now from \([1, 2]\). \(\square\)

We define a functor \(\phi : \text{mod} - A \to \text{mod} - B\) by \(\phi(M) = M \text{ and } \phi(f) = I_Nf\) for an \(A\)-homomorphism \(f : M \to N\). \(\phi\) is obviously \(K\)-linear. The next result shows that it really is a functor and calculates its kernel on objects.

3.3. Theorem

1. \(\phi\) is a \(K\)-linear functor.
2. \(\phi(M) = 0\) iff the two-sided ideal \(AeA\) annihilates \(M\), that is \(M\) is a \(A/AeA\)-module. All modules \(M\) that are annihilated by \(AeA\) have dominant dimension
zero.
3. By restricting φ to Dom2, one gets an equivalence of K-linear categories Dom2 → DomB2, where DomB2 denotes the full subcategory of mod − B having objects all modules of dominant dimension at least 2.
4. Any module A-module M is isomorphic to ν−1(M) in B-mod and thus B-mod is equivalent to Dom2 as K-linear categories, which is equivalent to the module category mod-eAc.

Proof. 1. It was noted above that φ is K-linear. We have to show φ(idM) = \( \text{Hom}(e,M) \zeta \), where \( \zeta : M \to \text{Hom}_A(M,A) \) is the canonical isomorphism, and φ(g ◦ f) = \( I_N(g) \circ I_M(f) \), where f : L → M and g : M → N are A-module homomorphisms. To show the first equality φ(idM) = \( \text{Hom}(e,M) \zeta \), just note that \( \text{Hom}(e,M) \zeta (m)(ae ⊗ eb) = 1_m(e(ae ⊗ eb)) = maeb = I_M(m)(ae ⊗ eb) \), where \( I_m : A \to M \) is left multiplication by m.

Next we show the above equality φ(g ◦ f) = \( I_N(g) \circ I_M(f) \):

Let \( l \in L \) and \( a,b \in A \). First, we calculate φ(g ◦ f)(l)(ae ⊗ eb) = g(f(l))aeb.

Second, \( I_N(g) \circ I_M(f)(l)(ae ⊗ eb) = I_N(g)(I_M(f)(l)(ae ⊗ e))(e ⊗ eb) = I_N(g)(l)(ae ⊗ e)c ⊗ eb = I_N(g)(f(l)(ae))(e ⊗ eb) = g(f(l))aeb. \)

Thus φ(g ◦ f) = \( I_N(g) \circ I_M(f) \) is shown.

2. A module M is zero in the K-category mod-B iff its endomorphism ring \( \text{End}_B(M) \) is zero iff the identity of \( \text{End}_B(M) \) is zero. Thus M is zero in mod-B iff \( I_M(m) = 0 \) for every \( m \in M \). But \( I_M(m) = 0 \) iff mAeA = 0 and so φ(M) = 0 iff M\((A\_{mod}) = 0 \). To see that such an M must have dominant dimension zero, note that \( AeA \) annihilates no element of M iff M has dominant dimension larger or equal 1 by [1,3]

3. Restricting φ to Dom2, φ is obviously still dense by the definition of DomB2.

Now recall that by the previous proposition a module M has dominant dimension at least two iff \( I_M \) is an isomorphism, and then \( h \in \text{Hom}_B(M,N) \) be given with \( M,N \in \text{Dom}_B \). Then φ(\( I_N^{-1}h \)) = \( I_N(\text{Hom}_B(M,N)) \) = h and φ is full. Assume \( h \in \text{Hom}_B(M,N) \), then \( h = 0 \), since \( I_N \) is an isomorphism, and so φ is faithful.

4. Define f ∈ \( \text{Hom}_B(M,\nu^{-1}(M)) \) as \( f = (\text{Hom}_A(\mu,M)\psi)^{-1}I_M \) and \( g \in \text{Hom}_B(\nu^{-1}(M),M) \) as \( g = id_{\nu^{-1}(M)} \). We show that \( f \ast g = I_{\nu^{-1}(M)} \) and \( g \ast f = I_M \), which by 1. are the identities of \( \text{Hom}_B(\nu^{-1}(M),\nu^{-1}(M)) \) and \( \text{Hom}_B(M,M) \). This shows that any module M is isomorphic to \( \nu^{-1}(M) \) in B-mod.

Let \( m \in M \) and \( a,b \in A \).

Then \( (g \ast f)(m)(ae ⊗ eb) = g(f(m)(ae ⊗ e))(e ⊗ eb) = ((\text{Hom}_A(\mu,M)\psi)^{-1}I_M(m))(ae ⊗ e)(e ⊗ eb) = maeb = I_M(m)(ae ⊗ eb) \), where we used that g is the identity on \( \nu^{-1}(M) \). Next we show that \( f \ast g = I_{\nu^{-1}(M)} \): Let \( l \in \nu^{-1}(M) = \text{Hom}_A(D(A),M) \).

First, note that by definition \( I_{\nu^{-1}(M)}(l)(ae ⊗ eb)(a'e ⊗ eb') = l(aeb)(a'e ⊗ eb') = l(aeb(a'e ⊗ eb')) = l(aeb(l)(ae ⊗ eb))(a'e ⊗ eb') = f(l)(ae ⊗ eb)(a'e ⊗ eb') = \( (\text{Hom}_A(\mu,M)\psi)^{-1}I_M(l)(ae ⊗ eb)(a'e ⊗ eb') = l(ae ⊗ eb)(a'e ⊗ eb') \), where we used in the last step that we tensor over \( eAc \).

Now we use [3,2] to show that every module of the form \( \nu^{-1}(M) \) has dominant dimension at least two. Since every module M is isomorphic to \( \nu^{-1}(M) \), B-mod is equivalent to DomB2, which is isomorphic to Dom2 by 3. Now recall that there is an equivalence of categories mod-eAc → Dom2 (this is a special case of [APF] Lemma 3.1.). Combining all those equivalences, we get that B-mod is equivalent to the module category mod-eAc.

3.4. Corollary
In case an \( A \)-module M has dominant dimension larger or equal 2, the map
Let $n \geq 2$ and $A := K[x]/(x^n)$ and $J$ the Jacobson radical of $A$. Let $M := A \oplus \bigoplus_{k=1}^{n-1} J^k$ and $B := \text{End}_A(M)$. Then $B$ is the Auslander algebra of $A$ and $B$ has $n$ simple modules. The idempotent $e$ is in this case primit and corresponds to the unique indecomposable projective-injective module $\text{Hom}_A(M, A)$. By the previous theorem, the kernel of $\phi$ is isomorphic to the module category $\text{mod} - (A/AeA)$. Here $A/AeA$ is isomorphic to the preprojective algebra of type $A_{n-1}$ by [DR] chapter 7.

We describe the bocs module category $\text{B-mod}$ of $(B, D(B))$ for $n = 2$ explicitly. In this case $B$ is isomorphic to the Nakayama algebra with Kupisch series $[2, 3]$. Then $B$ has five indecomposable modules. Let $e_0$ be the primitive idempotent corresponding to the indecomposable projective module with dimension two and $e_1$ the primitive idempotent corresponding to the indecomposable projective module with dimension three. Then $e_1A$ is the unique minimal faithful indecomposable projective-injective module. Let $S_i$ denote the simple $B$-modules. The only indecomposable module annihilated by $Be_1B$ is $S_0$, which is therefore isomorphic to zero in the bocs module category. The two indecomposable projective modules $P_0 = e_0B$ and $P_1 = e_1B$ have dominant dimension at least two and thus are not isomorphic. The only indecomposable module of dominant dimension 1 is $S_1$ and the only indecomposable module of dominant dimension zero, which is not isomorphic to zero in $B$-mod, is $D(Be_0)$. Now let $X = S_1$ or $X = D(Be_0)$, then $\nu^{-1}(X) = \text{Hom}_B(D(B), X) \cong e_0B$. Thus in $B$-mod $S_1 \cong e_0B \cong D(Be_0)$ and $e_1B$ are up to isomorphism the unique indecomposable objects.

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