Self-similar solutions to the Hesse flow

Shun Maeta

Received: 26 January 2021 / Revised: 27 May 2021 / Accepted: 9 August 2021 / Published online: 17 August 2021
© The Author(s), under exclusive licence to Springer Nature Singapore Pte Ltd. 2021

Abstract
We define a Hesse soliton, that is, a self-similar solution to the Hesse flow on Hessian manifolds. On information geometry, the $e$-connection is important, which does not coincide with the Levi–Civita one. Therefore, it is interesting to consider a Hessian manifold with a flat connection which does not coincide with the Levi–Civita one. We call it a proper Hessian manifold. In this paper, we show that any compact proper Hesse soliton is expanding and any non-trivial compact gradient Hesse soliton is proper. Furthermore, we show that the dual space of a Hesse–Einstein manifold can be understood as a Hesse soliton.

Keywords Hesse flow · Hesse solitons · Hessian manifolds · Information geometry

Mathematics Subject Classification Primary 53B12 · 53E99 · 35C06; Secondary 62B11

1 Introduction

Information geometry is a method of exploring the world of information by means of modern geometry, which was introduced by S. Amari. On information geometry, exponential families of probability distributions are important. It is well known that many important smooth families of probability distributions, for example normal distributions, are included in exponential families of probability distributions. Interestingly, they carry Hessian structure.

Information geometry is built on the basis of differential geometry. The geometric flows are one of the most powerful tools in the theory of differential geometry. In fact,
as is well known, Poincare conjecture was solved by the Ricci flow. The Ricci flow is defined as follows by using the Ricci tensor $\text{Ric}(g(t))$ (cf. [7]):

$$\frac{\partial}{\partial t} g(t) = -2\text{Ric}(g(t)),$$

where $g(t)$ is the time dependent Riemannian metric on a Riemannian manifold $(M, g(t))$. Therefore, Hessian manifolds can be deeply understood by considering the geometric flow for the second Koszul form $\beta$. In fact, $\beta$ plays a similar role to that of the Ricci tensor in a Hessian manifold (i.e., a manifold with a Hessian structure). The flow is called the Hesse flow (or the Hesse–Koszul flow) defined by Mirghafouri and Malek (cf. [9], see also [11]):

$$\frac{\partial}{\partial t} g(t) = 2\beta(g(t)).$$

They proved the short-time existence, the global existence and the uniqueness of it on compact Hessian manifolds. Puechmorel and Tô [11] gave convergence theorems of it on compact Hessian manifolds. In particular, Puechmorel and Tô showed that the Hesse flow converges to the unique Hesse–Einstein metric when the first affine Chern class is negative.

**Remark 1.1** (1) Interestingly, the Hesse flow of a Hessian manifold corresponds to the Kähler Ricci flow of its tangent bundle (see for example [5]). The history and comprehensive reference on the Kähler Ricci flow can be seen in [2].

(2) As in [9,11], the Hesse flow deforms the Hessian potential $u$ via the parabolic Monge–Ampère operator $\frac{\partial}{\partial t} u = 2 \log(\det(D^2 u))$. The equation and its generalization were studied (cf. [8,14]).

A self-similar solution i.e., a soliton equation plays an important and fundamental role in the study of a geometric flow. In fact, the Ricci soliton which is the self-similar solution to the Ricci flow plays an important role in solving Poincare conjecture and the geometrization conjecture.

Therefore, we define the self-similar solution to the Hesse flow. We call it a Hesse soliton. In this paper, we give some existence and non-existence results of Hesse solitons. In particular, we show that nontrivial compact Hesse solitons are expanding. Furthermore, we show that one can understand the dual space of a Hesse–Einstein manifold as a Hesse soliton.

**2 Preliminary**

In this section, we set up terminology and define some notions which are related to Hessian manifolds and information geometry.
2.1 Riemannian geometry

Let \((M, g)\) be an \(n\)-dimensional Riemannian manifold. As is well known, the Levi–Civita connection \(\nabla : TM \times C^\infty(TM) \to C^\infty(TM)\) is the unique connection on \(TM\), which is compatible with the metric and is torsion free:

\[
\begin{align*}
X g(Y, Z) &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z), \\
\nabla_X Y - \nabla_Y X &= [X, Y].
\end{align*}
\]

The Riemannian curvature tensor is defined by

\[
Rm(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,
\]

for any vector field \(X, Y, Z \in \mathfrak{X}(M)\). We use the notations \(R^i_{jkl}\) as

\[
R^i_{jkl} = g^{ip} R_{p jkl},
\]

and \(R_{ijkl} = g_{ip} R^p_{jkl}\). The Ricci and scalar curvatures are defined by \(R_{ij} = R_{ijik}\) and \(R = R_{ii}\).

2.2 Hessian manifolds

Let \(M\) be an \(n\)-dimensional smooth manifold. A connection \(D\) is said to be flat if \(D\) is torsion free and curvature free, that is,

\[
D_X Y - D_Y X - [X, Y] = 0,
\]

and

\[
D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z = 0.
\]

**Definition 2.1** A Riemannian metric \(g\) on a flat manifold \((M, D)\) is called a Hessian metric if \(g\) can be locally expressed by

\[
g = Dd\varphi,
\]

for some smooth function \(\varphi\), that is,

\[
g_{ij} = \frac{\partial^2 \varphi}{\partial x_i \partial x_j},
\]

for an affine coordinate system. \((D, g)\) is called a Hessian structure. \((M, D, g)\) is called a Hessian manifold.
Let \( v_g \) be the volume form of \( g \) and \( X \) be a vector field on \( M \). The first and second Koszul forms \( \alpha \) and \( \beta \) of \((M, D)\) are defined by

\[
\begin{align*}
D_X v_g &= \alpha(X)v_g, \\
\beta &= D\alpha.
\end{align*}
\]

We denote by \( \gamma \) the difference tensor of \( \nabla \) and \( D \):

\[
\gamma_X Y = \nabla_X Y - D_X Y.
\]

Here we remark that since \( D_{\partial_i} \partial_j = 0 \), the components \( \gamma^i_{jk} \) of \( \gamma \) with respect to affine coordinate systems coincide with the Christoffel symbols \( \Gamma^i_{jk} \) of \( \nabla \), where \( \partial_i = \frac{\partial}{\partial x_i} \).

A tensor field \( H \) of type \((1, 3)\) defined by the covariant differential

\[
H = D\gamma
\]

of \( \gamma \) is said to be the Hessian curvature tensor for \((D, g)\). The components \( H^i_{jkl} \) of \( H \) is given by

\[
H^i_{jkl} = \frac{\partial \gamma^i_{jl}}{\partial x_k}.
\]

Let \( D' = 2\nabla - D \), then \( D' \) is also a flat connection and \((D', g)\) is a Hessian structure. \( D' \) is called the dual connection and \((D', g)\) is called the dual Hessian structure of \((D, g)\).

The flat connection \( D \) and the dual one \( D' \) satisfy that

\[
X g(Y, Z) = g(D_X Y, Z) + g(Y, D'_X Z).
\]

**Proposition 2.2** (Proposition 3.4 in [13]) *On a Hessian manifold, the following holds:*

1. \( \alpha(X) = \text{Trace} \gamma_X \).
2. \( \alpha_i = \gamma^r_{ri} \).
3. \( \beta_{ij} = H^r_{rij} = H_{ijr}^r \).

The second Koszul form \( \beta \) plays a similar role to that of the Ricci tensor and one can define Hesse–Einstein manifolds:

**Definition 2.3** Let \((M, D, g)\) be a Hessian manifold. If the following holds

\[
\beta = \lambda g,
\]

then \((M, D, g)\) is called a Hesse–Einstein manifold.

**Proposition 2.4** (Proposition 2.3 in [13]) *The curvature tensor of a Hessian manifold \((M, D)\) is given as follows:*

\[
R^i_{jkl} = \gamma^r_{lr} \gamma^r_{jk} - \gamma^i_{kr} \gamma^r_{jl}.
\]

\( \Box \) Springer
This implies the following:

\[ R_{jk} = R^s_{jks} = \gamma^s_{kr} \gamma^r_{js} - \alpha_r \gamma^r_{jk}, \]  

(2.4)

and

\[ R = |\gamma|^2 - |\alpha|^2. \]  

(2.5)

We can use the following notations without confusion: for example, \( \gamma^{ijk}_{\gamma^{ist}} = \gamma^{i}_{jk} \gamma^{st}_{ist}, H_{rrij} = H^r_{rij}, \alpha_r \gamma_{rij} = \alpha^r \gamma_{rij}, \) etc.

### 2.3 The connection between information geometry and Hessian manifolds

Hessian manifolds play an important role in information geometry: For \( \Omega_n = \{1, 2, \ldots, n\}, \) let

\[ S_{n-1} := \left\{ p : \Omega_n \to \mathbb{R}_+; \sum_{\omega \in \Omega_n} p(\omega) = 1 \right\} \]

be a set of all probability distribution on \( \Omega_n, \) where \( \mathbb{R}_+ = \{x \in \mathbb{R}; x > 0\}. \) As is well known, one can regard it as a manifold (see for example [6]). A metric \( g^F \) on \( S_{n-1} \) such as

\[ g^F_p (X, Y) = \sum_{\omega=1}^n p(\omega)(X \log p(\omega))(Y \log p(\omega)) \]

is called a Fisher information metric. For each \( \alpha \in \mathbb{R}, \) \( \nabla^{(\alpha)} \) is determined by

\[ g^F_p (\nabla^{(\alpha)} Y, Z) = g^F_p (\nabla_X Y, Z) - \frac{\alpha}{2} \sum_{\omega=1}^n p(\omega)(X \log p(\omega))(Y \log p(\omega))(Z \log p(\omega)), \]

where \( \nabla \) is the Levi–Civita connection compatible with \( g^F. \) \( \nabla^{(\alpha)} \) is called the \( \alpha \)-connection.

Chentsov (cf. [4]) shows that an extremely natural invariance requirement of \( S_{n-1} \) determines a metric and a connection of \( S_{n-1}, \) that is, the metric is the Fisher information metric and the connection is the \( \alpha \)-connection on \( S_{n-1}. \) This means that within the frame of information geometry, the \( \alpha \)-connection is the most natural connection.

The \( \alpha \)-connection \( \nabla^{(\alpha)} \) satisfies that

\[ X g^F (Y, Z) = g^F (\nabla^{(\alpha)}_X Y, Z) + g^F (Y, \nabla^{(-\alpha)}_X Z). \]

The most important case is \( \alpha = 1. \) It is known that for \( (g^F, \nabla^{(1)}, \nabla^{(-1)}), S_{n-1} \) is the dual flat manifold and \( g^F \) can be written \( g^F_{ij} = \partial_i \partial_j \varphi \) for an affine coordinate system (cf. [1]). Hence, \( (S_{n-1}, \nabla^{(1)}, g^F) \) is a Hessian manifold with \( \nabla^{(1)} \neq \nabla. \)
**Example** [13] Let $\mathcal{X}$ be a discrete set (countable set) or $\mathbb{R}^m$. A family of probability distributions

$$
\mathcal{P} = \{ p(x; \theta) | \theta \in \Theta \}
$$

is said to be an exponential family if there exist functions $c(x), f_1(x), \ldots, f_n(x)$ on $\mathcal{X}$, and a function $\varphi(\theta)$ on $\Theta$, such that

$$
p(x; \theta) = \exp \left\{ c(x) + \sum_{i=1}^{n} f_i(x) \theta^i - \varphi(\theta) \right\}.
$$

Then, $\left( \nabla^{(1)}, g = \left( \frac{\partial^2 \varphi}{\partial \theta^i \partial \theta^j} \right) \right)$ is a Hessian structure on $\Theta$.

From the above arguments, to apply Hessian geometry to information geometry, it is important to consider a Hessian manifold $(M, D, g)$ with $D \neq \nabla$. From this, we define the following:

**Definition 2.5** Let $(M, D)$ be a Hessian manifold. If $D \neq \nabla$, $M$ is called a proper Hessian manifold.

### 3 Hesse solitons

Let $(M, D, g)$ be a Hessian manifold. In this section, we consider self-similar solutions to the Hesse flow $\partial_t g = 2 \beta$:

$$
g(t) = \sigma(t) \psi^*(t) g(0),
$$

where $g(0)$ is the Hessian metric $g$, $\sigma(t) : \mathbb{R} \to \mathbb{R}_+$ is a smooth function and $\psi(t) : M \to M$ is a 1-parameter family of diffeomorphisms. By differentiating, we have

$$
2 \beta(g(t)) = \left( \frac{d}{dt} \sigma(t) \right) \psi^*(t) g(0) + \sigma(t) \psi^*(t) (L_X g(0)),
$$

where $L_X$ denotes the Lie derivative by the time dependent vector field $X$ such that $X(\psi(t)(p)) = \frac{d}{dt}(\psi(t)(p))$ for any $p \in M$.

**Proposition 3.1** $\beta(c g) = \beta(g)$ for any positive constant $c \in \mathbb{R}$.

**Proof** Let $v_g$ and $v_{cg}$ be the volume forms of $g$ and $c g$, respectively. Assume that $\alpha^c$ is the first Koszul form for $c g$. By Definition 2.1, we have

$$
D_X v_{cg} = \alpha^c(X) v_{cg}.
$$

From this and the definition of the volume form, we obtain

$$
D_X v_g = \alpha^c(X) v_g.
$$

$\copyright$ Springer
Therefore, we have
\[ \alpha^c = \alpha. \]

From this and the definition of the second Koszul form (2.2),
\[ \beta(cg) = D\alpha^c = D\alpha = \beta(g). \]

\[ \square \]

By (3.1) and Proposition 3.1, one has
\[ 2\beta(g(t)) = \frac{d}{dt}\sigma(t)g(t) + \mathcal{L}_Y g(t), \]
where \( Y(t) = \sigma(t)X(t) \). Therefore, we define self-similar solutions to the Hesse flow as follows:

**Definition 3.2** Let \((M, D, g = Dd\varphi)\) be a Hessian manifold. If there exist a vector field \( X \) and \( \lambda \in \mathbb{R} \), such that
\[ \beta - \frac{1}{2}\mathcal{L}_X g = \lambda g, \]
then, \( M \) is called a Hesse soliton. If \( \lambda > 0 \), \( \lambda = 0 \), \( \lambda < 0 \), then the Hesse soliton is called expanding, steady or shrinking, respectively. If there exists a smooth function \( f \) on \( M \) such that \( X = \text{grad} f \), that is,
\[ \beta = \nabla\nabla f = \lambda g, \]
then the Hesse soliton \((M, D, g, f)\) is called a gradient Hesse soliton, where \( \nabla\nabla f \) is the Hessian of \( f \). \( f \) is called a potential function.

Hesse–Einstein manifolds are trivial solutions of Hesse solitons. Therefore, if a Hesse soliton is Hesse–Einstein, it is called trivial. If a Hesse soliton is a proper Hessian manifold, it is called a proper Hesse soliton.

## 4 Existence and non-existence theorems for Hesse solitons

In this section, we show some existence and non-existence theorems for Hesse solitons.

**Theorem 4.1**
(1) There exist no compact shrinking Hesse solitons.
(2) Any compact steady Hesse soliton is non proper and trivial.

Unlike in the case (1), in the case (2), we remark that there exist non proper and trivial steady Hesse solitons. We will consider it later.

We first show the following lemma.
Lemma 4.2  On any Hessian manifold, the following formula holds:

\[
\frac{1}{2} \Delta R = \nabla_i \nabla_j \alpha_k \gamma_{ijk} - \nabla_r \nabla_r \alpha_i + |\nabla \gamma|^2 + |\text{Rm}|^2 + |\text{Ric}|^2 \tag{4.1}
\]

\[+ R_{ij} \beta_{ij} - R_{ij} \nabla_i \alpha_j. \]

Proof Since

\[\gamma_{ijk} = \frac{1}{2} \frac{\partial g_{ij}}{\partial x_k} \quad \text{and} \quad g_{ij} = \frac{\partial i}{\partial j} \phi, \]

we have

\[\nabla_i \gamma_{jkl} = \frac{1}{2} \frac{\partial i}{\partial j} \frac{\partial k}{\partial l} \phi - (\gamma_{rij} \gamma_{rkl} + \gamma_{rik} \gamma_{rjl} + \gamma_{ril} \gamma_{rjk}).\]

This means that \(\nabla_i \gamma_{jkl}\) is symmetry with respect to \(i, j, k, l\).

A direct computation shows that

\[\nabla_r \nabla_r \gamma_{ijk} = \nabla_r \nabla_i \gamma_{rjk}\]

\[= \nabla_r \nabla_r \gamma_{rjk} + (\gamma_{rpt} \gamma_{tit} - \alpha_t \gamma_{tip}) \gamma_{pjk}\]

\[+ (\gamma_{rpt} \gamma_{tij} - \gamma_{rjt} \gamma_{tip}) \gamma_{rpk} + (\gamma_{rpt} \gamma_{tik} - \gamma_{rkt} \gamma_{tip}) \gamma_{rjp}\]

\[= \nabla_i \nabla_j \alpha_k + \gamma_{rpt} (\gamma_{tir} \gamma_{pjk} + \gamma_{tij} \gamma_{rpk} + \gamma_{tik} \gamma_{rjp})\]

\[- \alpha_t \gamma_{tip} \gamma_{pjk} - \gamma_{rjt} \gamma_{tip} \gamma_{rpk} - \gamma_{rkt} \gamma_{tip} \gamma_{rjp},\]

where the first and third equalities follow from the symmetric property of \(\nabla_i \gamma_{jkl}\) with respect to \(i, j, k, l\), and the second one follows from the Ricci identity. From this, one has

\[\frac{1}{2} \Delta |\gamma|^2 = \nabla_r \nabla_r \gamma_{ijk} \gamma_{ijk} + |\nabla \gamma|^2 \tag{4.2}\]

\[= \left\{ \nabla_i \nabla_j \alpha_k + \gamma_{rpt} (\gamma_{tir} \gamma_{pjk} + \gamma_{tij} \gamma_{rpk} + \gamma_{tik} \gamma_{rjp})\right.\]

\[- \alpha_t \gamma_{tip} \gamma_{pjk} - \gamma_{rjt} \gamma_{tip} \gamma_{rpk} - \gamma_{rkt} \gamma_{tip} \gamma_{rjp}\}

\[\gamma_{ijk} + |\nabla \gamma|^2.\]

Substituting

\[|\text{Rm}|^2 = R_{ijkl} R_{ijkl}\]

\[= \gamma_{rpt} \gamma_{tir} \gamma_{pjk} \gamma_{tij} + \gamma_{rpt} \gamma_{tij} \gamma_{rpk} \gamma_{tijk} - \gamma_{rjt} \gamma_{tip} \gamma_{rpk} \gamma_{tijk} - \gamma_{rkt} \gamma_{tip} \gamma_{rjp} \gamma_{tijk},\]

into (4.2), we have

\[\frac{1}{2} \Delta |\gamma|^2 = \nabla_i \nabla_j \alpha_k \gamma_{ijk} + |\nabla \gamma|^2 + |\text{Rm}|^2 + \gamma_{rpk} \gamma_{tik} \gamma_{rjp} \gamma_{tijk}\]

\[- \alpha_t \gamma_{tip} \gamma_{pjk} \gamma_{tijk}.\]
From this, (2.4),

\[ |\text{Ric}|^2 = \gamma_{skr} \gamma_{rjs} \gamma_{kip} \gamma_{ijp} - \gamma_{skr} \gamma_{rjs} \alpha_i \gamma_{ijk} - \alpha_r \gamma_{rjk} \gamma_{kip} \gamma_{pji} + \alpha_r \alpha_i \gamma_{rjk} \gamma_{ijk}, \]

and

\[ \nabla_i \alpha_j = \beta_{ij} - \gamma_{rrij} \alpha_r, \]

one has

\[ \frac{1}{2} \Delta |\gamma|^2 = \nabla_i \nabla_j \alpha_k \gamma_{ijk} + |\nabla \gamma|^2 + |\text{Rm}|^2 + |\text{Ric}|^2 + \gamma_{ijkl} \gamma_{jkl} \alpha_p \gamma_{ipr} - \alpha_i \alpha_j \gamma_{ijk} \gamma_{rjk} \]

\[ = \nabla_i \nabla_j \alpha_k \gamma_{ijk} + |\nabla \gamma|^2 + |\text{Rm}|^2 + |\text{Ric}|^2 + R_{ij} \beta_{ij} - R_{ij} \nabla_i \alpha_j. \]

Therefore, we have

\[ \frac{1}{2} \Delta R = \nabla_i \nabla_j \alpha_k \gamma_{ijk} - \nabla_r \nabla_r \alpha_i \alpha_i + |\nabla \gamma|^2 + |\text{Rm}|^2 + |\text{Ric}|^2 \]

\[ + R_{ij} \beta_{ij} - R_{ij} \nabla_i \alpha_j. \]

\[ \square \]

By using the above lemma, one can show Theorem 4.1.

**Proof of Theorem 4.1** By taking the trace of (3.3), we have

\[ \beta_{ii} - \text{div} \, X = \lambda n. \]

From this, one has

\[ \lambda n \, \text{Vol}(M, g) = \int_M \beta_{ii} - \text{div} \, X \, v_g \]

\[ = \int_M \nabla_i \alpha_i + |\alpha|^2 \, v_g \]

\[ = \int_M |\alpha|^2 \, v_g \geq 0, \]

where the last equality follows from Stokes’ theorem. Hence, one has \( \lambda \geq 0 \). Therefore, there exist no compact shrinking Hesse solitons.

If \( \lambda = 0 \), then one has \( \alpha = 0 \). Furthermore, by Lemma 4.2, we have

\[ \frac{1}{2} \Delta R = |\nabla \gamma|^2 + |\text{Rm}|^2 + |\text{Ric}|^2. \]

By Green’s formula, one has

\[ \int_M |\text{Ric}|^2 + |\text{Rm}|^2 + |\nabla \gamma|^2 \, v_g = 0. \]
Therefore, $M$ is flat, in particular $R = 0$. From this and (2.5), we have $\gamma = 0$, that is, $D = \nabla$. Furthermore, we also have $\beta = 0$. Therefore, it is also trivial.

As mentioned above, we consider the properness of steady Hesse solitons. The equation of steady Hesse solitons is

$$\beta - \frac{1}{2} \mathcal{L}_X g = 0.$$ 

From the proof of (2) of Theorem 4.1, any compact steady Hesse soliton is flat and non proper. Hence, any compact steady Hesse soliton is

$$\mathcal{L}_X g = 0.$$ 

One can construct many examples of steady Hesse solitons by taking the vector field $X$ to be a Killing vector field:

**Proposition 4.3** Let $(M, g, X)$ be a non proper Hessian manifold with a Killing vector field $X$ (and the Levi–Civita connection $\nabla$). Then, $\beta$ and $\mathcal{L}_X g$ vanishes, and therefore, $(M, g, X)$ is a steady Hesse soliton.

We remark that if the first Koszul form coincides with the dual one, that is, $\alpha = \alpha'$, then $D = \nabla$ on compact Hessian manifolds. In fact, since the volume form is parallel, $\alpha'(X) = D'Xv_g = (2\nabla - D)v_g = -D_Xv_g = -\alpha(X)$, that is, $\alpha = -\alpha'$, thus one has $\alpha = \alpha' = 0$. By Lemma 4.2 and Green’s formula, one has

$$\int_M |\text{Rm}|^2 + |\text{Ric}|^2 + |\nabla \gamma|^2 v_g = 0.$$ 

From this and (2.5), we have $\gamma = 0$, that is, $D = \nabla$. On a complete Hessian manifold, it is well known that E. Calabi obtained the same conclusion (cf. [3]), that is, any complete Hessian manifold with $\alpha = 0$ satisfies $D = \nabla$.

From the above arguments, we consider a more general problem. Obviously, if a Hessian manifold $M$ is non proper, then $\beta = \beta'(= 0)$. In particular, $\nabla \alpha = 0$. In fact, since $\alpha = -\alpha'$, one has $\beta' = D'\alpha' = -D'\alpha = -(2\nabla - D)\alpha = \beta - 2\nabla \alpha$. Therefore, $\beta' - \beta = 2\nabla \alpha$.

However, the converse is not true, that is, even if $\beta = \beta'$, $M$ might not satisfies that $D = \nabla$. In fact, the following example satisfies $\beta = \beta' = \frac{n}{2} g$, but $D \neq \nabla$. This means that there exist proper Hesse–Einstein manifolds.

**Example** Let

$$\Omega = \left\{ x \in \mathbb{R}^n; x_n > \sqrt{\sum_{i=1}^{n-1} (x_i)^2} \right\} \text{ and } \varphi = -\log \left( x_n^2 - \left( \sum_{i=1}^{n-1} (x_i)^2 \right) \right).$$

Then, $(\Omega, D, g = Dd\varphi)$ is a Hessian structure on $\Omega$.  
From the above argument, it is interesting to consider the problem: “Does there exist non trivial Hesse soliton with $\beta = \beta'$ (that is, $\nabla\alpha = 0$)?”

We first consider a complete Einstein Hessian manifold with a non-negative Einstein constant $\lambda$, that is, a Hessian manifold with $\text{Ric} = \lambda g$ with $\lambda \geq 0$.

**Proposition 4.4** Any complete Einstein Hessian manifold with a non-negative Einstein constant $\lambda$ and $\beta = \beta'$ is flat and $\nabla\gamma = 0$.

**Proof** By the assumption,

$$0 = \nabla_i\alpha_j = \beta_{ij} - \gamma^r_{ij}\alpha_r = \beta_{ij} - g^{rs}\gamma_{rij}\alpha_s.$$ 

From this, (4.1) and the assumption,

$$\frac{1}{2} \Delta R = |\nabla\gamma|^2 + |\text{Rm}|^2 + |\text{Ric}|^2 + \lambda\beta_{ii} = |\nabla\gamma|^2 + |\text{Rm}|^2 + |\text{Ric}|^2 + \lambda|\alpha|^2 \\
\geq \frac{1}{n} R^2,$$

where the last inequality follows from the Schwarz inequality. Since the Ricci curvature is non-negative, by the Omori–Yau maximum principle (cf. [10,15]), $R = 0$. Hence $M$ is flat and $\nabla\gamma = 0$. \(\square\)

**Lemma 4.5** Any Hesse soliton with $\beta = \beta'$ has constant $\text{div} X$.

**Proof** Since

$$0 = \nabla_i\alpha_j = \beta_{ij} - \gamma^r_{ij}\alpha_r = \beta_{ij} - g^{rs}\gamma_{rij}\alpha_s,$$

we have

$$\beta_{ii} = |\alpha|^2.$$

Hence, one has

$$\nabla_k\beta_{ii} = 0.$$

By the equation of Hesse solitons (3.3),

$$0 = \nabla_k\beta_{ii} = \nabla_k(\text{div} X + n\lambda) = \nabla_k\text{div} X,$$

which implies that $\text{div} X$ is constant. \(\square\)

By Lemma 4.5, one can show the following:

**Proposition 4.6** If compact Hesse solitons satisfy $\beta = \beta'$, then $\text{div} X = 0$. 
**Proof** By Lemma 4.5, \( \text{div } X \) is constant, say \( C \). By Stokes’ theorem,

\[
0 = \int_M \text{div } X v_g = C \text{Vol}(M).
\]

Thus, \( C = 0 \), that is, \( \text{div } X = 0 \).

In particular, if \( M \) is gradient, by the standard maximum principle, one can obtain the following.

**Corollary 4.7** Any compact gradient Hesse soliton with \( \beta = \beta' \) is trivial.

A similar result for complete Hesse solitons can be obtained.

**Proposition 4.8** Any complete gradient Hesse soliton with \( \beta = \beta' \) and non-negative Ricci curvature is trivial.

**Proof** By Lemma 4.5,

\[
\Delta |\nabla f|^2 = 2|\nabla \nabla f|^2 + 2\text{Ric}(\nabla f, \nabla f) + 2g(\nabla f, \nabla \Delta f) \\
= 2|\nabla \nabla f|^2 + 2\text{Ric}(\nabla f, \nabla f) \geq 0.
\]

Hence, Omori–Yau maximum principle shows that \( |\nabla f|^2 \) is constant, say \( C \). Assume that \( C > 0 \). Since \( \nabla \nabla f = 0 \), \( \Delta f = 0 \). From this,

\[
\Delta e^f = |\nabla f|^2 e^f + \Delta f e^f = |\nabla f|^2 e^f > 0.
\]

By Omori–Yau maximum principle again, \( e^f \) is constant, that is, \( f \) is constant, which is a contradiction.

By Corollary 4.7, it is interesting to consider non trivial gradient Hesse solitons from the point of view of information geometry.

**Corollary 4.9** Any compact non trivial gradient Hesse soliton is proper.

**Proof** By Corollary 4.7, \( \nabla \alpha \neq 0 \) at some point \( p \in M \), i.e., on some open set \( \Omega \ni p \). Hence, \( \nabla \gamma \neq 0 \) on \( \Omega \). In fact, if \( \nabla \gamma = 0 \) at \( q \in \Omega \), then we have \( \nabla \alpha = 0 \) at \( q \), which is a contradiction.

Therefore, \( \gamma \neq 0 \) on some set \( \tilde{\Omega} \) of \( M \), which means that the soliton is proper. □

By the same argument, one can show the following.

**Corollary 4.10** Any complete non trivial gradient Hesse soliton with non-negative Ricci curvature is proper.

**Remark 4.11** Interestingly Puechmorel and Tô showed that

**Theorem 4.12** [11] Any compact Hessian manifold whose first affine Chern class is negative converges to a Hesse-Einstein metric under normalized Hesse flow.

One of the referees observed that this result suggests that the universal affine cover of a compact Hesse soliton must contain a line [12].
5 Dual Hessian structure

In this section, we consider the dual space of a Hessian manifold \((M, D, g)\) and show that one can understand the dual space of a Hesse-Einstein manifold as a Hesse soliton.

**Theorem 5.1** Let \((M, D, g)\) be a Hesse soliton,

\[
\beta - \frac{1}{2} \mathcal{L}_X g = \lambda g,
\]

then the dual space \((M, D', g)\) is also a Hesse soliton which satisfies that

\[
\beta' - \frac{1}{2} \mathcal{L}_{(X - 2\alpha\sharp)} g = \lambda g,
\]

where \(\sharp\) is the musical isomorphism \(\sharp: TM^* \rightarrow TM\),

\[
g(\alpha\sharp, X) = \alpha(X),
\]

for any vector field \(X\) on \(M\).

**Proof** By the definition of the musical isomorphism \(\sharp\),

\[
(\nabla\alpha)(Y, Z) = (\nabla_Y \alpha)(Z)
= Y\alpha(Z) - \alpha(\nabla_Y Z)
= Yg(\alpha\sharp, Z) - g(\alpha\sharp, \nabla_Y Z)
= g(\nabla_Y \alpha\sharp, Z) + g(\alpha\sharp, \nabla_Y Z) - g(\alpha\sharp, \nabla_Y Z)
= g(\nabla_Y \alpha\sharp, Z).
\]

Since \(\beta\) and \(\beta'\) are symmetric 2 forms, \(\nabla\alpha = \frac{1}{2}(\beta' - \beta)\) is also a symmetric 2 form. Thus,

\[
2(\nabla\alpha)(Y, Z) = (\nabla\alpha)(Y, Z) + (\nabla\alpha)(Z, Y)
= g(\nabla_Y \alpha\sharp, Z) + g(\nabla_Z \alpha\sharp, Y)
= \mathcal{L}_{\alpha\sharp} g(Y, Z).
\]

Since \(2\nabla\alpha = \beta - \beta'\) and \((M, D, g)\) is a Hesse soliton

\[
\beta - \frac{1}{2} \mathcal{L}_X(Y, Z) = \lambda g,
\]

\(\odot\) Springer
one has

\[
\beta'(Y, Z) = \beta(Y, Z) - 2(\nabla\alpha)(Y, Z)
= \frac{1}{2} \mathcal{L}_X g(Y, Z) + \lambda g(Y, Z) - \mathcal{L}_{\alpha^\sharp} g(Y, Z)
= \frac{1}{2} \mathcal{L}_{(X - 2\alpha^\sharp)} g(Y, Z) + \lambda g(Y, Z).
\]

We consider gradient Hesse solitons. If the first Koszul form \( \alpha \) is exact, that is, \( \alpha = dF \) for some smooth function \( F \) on \( M \), then the Hesse soliton of the dual space is also gradient.

**Corollary 5.2** Let \( (M, D, g, f) \) be a gradient Hesse soliton, such that the first Koszul form is exact, that is, \( \alpha = dF \) for some smooth function \( F \) on \( M \). Then the dual space \( (M, D', g) \) is also a gradient Hesse soliton with the potential function \( f - 2F \).

**Proof** Since \( \alpha = dF \), we have

\[
g(\alpha^\sharp, Y) = \alpha(Y) = dF(Y) = XF = g(\nabla F, Y),
\]

for any vector field \( Y \) on \( M \). Thus, we have

\[
\alpha^\sharp = \nabla F.
\]

By Theorem 5.1, the proof is complete. \( \square \)

One can understand the dual space of Hesse–Einstein manifolds as Hesse solitons.

**Corollary 5.3** Let \( (M, D, g) \) be a Hesse–Einstein manifold,

\[
\beta = \lambda g,
\]

then the dual space is a Hesse soliton \( (M, D', g, -2\alpha^\sharp) \), that is, it satisfies

\[
\beta' - \frac{1}{2} \mathcal{L}_{(-2\alpha^\sharp)} g = \lambda g.
\]

**Acknowledgements** The author would like to express his gratitude to Professor Shun-ichi Amari for his encouragement. He also would like to thank Professor Tat Dat Tô and the referees for their useful comments and suggestions.

**References**

1. Amari, S.: Information Geometry and Its Applications, Applied Mathematical Sciences, vol. 194. Springer, Tokyo (2016)
2. Boucksom, S., Eyssidieux, P., Guedj, V.: An Introduction to the Kähler–Ricci Flow, Lecture Notes in Mathematics, vol. 2086. Springer, Cham (2013)
3. Calabi, E.: Improper affine hyperspheres of convex type and a generalization of a theorem by K. Jörgens. Mich. Math. J. 5, 105–126 (1958)
4. Chentsov, N.N.: Statistical Decision Rules and Optimal Inference. American Mathematical Society, Providence (1982)
5. Dombrowski, P.: On the geometry of the tangent bundle. J. Reine Angew. Math. 210, 73–88 (1962)
6. Fujiwara, A.: Foundations of Information Geometry. Makinoshoten (2015)
7. Hamilton, R.: Three-manifolds with positive Ricci curvature. J. Differ. Geom. 17, 255–306 (1982)
8. Kitagawa, J.: A parabolic flow toward solutions of the optimal transportation problem on domains with boundary. J. Reine Angew. Math. 672, 127–160 (2012)
9. Mirghafouri, M., Malek, F.: Long-time existence of a geometric flow on closed Hessian manifolds. J. Geom. Phys. 119, 54–65 (2017)
10. Omori, H.: Isometric immersions of Riemannian manifolds. J. Math. Soc. Japan 19, 205–214 (1967)
11. Puechmorel, S., Tô, T.D.: Convergence of the Hesse–Koszul flow on compact Hessian manifolds. arXiv:2001.02940 [math.DG]
12. Referee’s report.: Irreducible compact Hessian manifolds admit Hesse–Einstein metrics
13. Shima, H.: The Geometry of Hessian Structures. World Scientific Publishing Co. Pte. Ltd., Hackensack (2007)
14. Schnürer, O. C., Smoczyk, K.: Neumann and second boundary value problems for Hessian and Gauss curvature flows. Ann. Inst. H. Poincarè Anal. Non Linèaire 20(6), 1043–1073 (2003)
15. Yau, S.-T.: Harmonic functions on complete Riemannian manifolds. Commun. Pure Appl. Math. 28, 201–228 (1975)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.