On Rings whose Simple Singular R-Modules are GP-Injective

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ABSTRACT

In this work we give a characterization of rings whose simple singular right R-modules are Gp-injective. We prove that if R is a quasi-duo ring whose simple singular right R-modules are Gp-injective, then any reduced right ideal of R is a direct summand. We also consider that a zero commutative ring with every simple singular left R-module is Gp-injective

Keywords: Gp-injective, R-modules, Quasi-duo ring, ZC-Ring

حوال الحلقات التي مقاساتها البسيطة المنفردة تكون غامرة من النمط--GP

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الملخص

في هذا البحث ندرس الحلقات التي تكون مقاساتها اليمنى البسيطة المنفردة غامرة من النمط--GP. برهنا أنه في حلقة كواري-ديو التي تكون مقاساتها اليمنى البسيطة المنفردة غامرة من النمط--GP فإن كل مثالياً يمكن مختزل يكون قابل للجمع المباشر، كما بين أن الحلقة شبه ألا بديلية تكون حلقة منتظمة ضعيفة مختزلة إذا كانت مقاساتها اليمنى البسيطة المنفردة غامرة من النمط--GP.

الكلمات المفتاحية: غامرة من النمط--GP، حلقات، مقاسات، حلقات، Quasi-duo

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1. Introduction:
Throughout this paper, \( R \) denotes an associative ring with identity, and all modules are unitary right \( R \)-modules. Recall that: (1) A right \( R \)-module \( M \) is called general right principally injective (briefly right Gp-injective) if for any \( 0 \neq a \in R \) there exists a positive integer \( n \), such that \( a^n \neq 0 \) and any right \( R \)-homomorphism of \( a^nR \) into \( M \) extends to one of \( R \) into \( M \); (2) \( R \) is called reduced if \( R \) has no non-zero nilpotent elements; (3) \( R \) is right (left) quasi-duo ring if every maximal right (left) ideal of \( R \) is an ideal of \( R \); (4) A ring \( R \) is called semi-prime if \( 0 \) is the only nilpotent ideal ; (5) for any element \( a \) in \( R \) we define a right annihilator of \( a \) by \( r(a) = \{ x \in R : ax = 0 \} \) and a left annihilator of \( a \), \( l(a) \) is similarly defined.

2. Rings whose simple singular modules are GP-Injective:
In this section, we study rings whose simple singular right \( R \)-modules are Gp-injective. We begin this section with the following result.

**Proposition 2-1:**
Let \( R \) be a quasi-duo ring, with every simple singular right \( R \)-modules is Gp-injective. Then any reduced right ideal of \( R \) is a direct summand.

**Proof:** Let \( I = aR \) be a reduced principal right ideal of \( R \). We shall show that \( aR + r(a) = R \). if not, there exists a maximal right ideal \( M \) of \( R \) such that \( aR + r(a) \subseteq M \). Now, \( M \) is essential right ideal of \( R \), if not, then there exists a non-zero right ideal \( L \) of \( R \) such that \( M \cap L = 0 \). Then \( aRL \subseteq ML \cap L = 0 \), implies that \( L \subseteq r(a) \subseteq M \), so \( M \cap L = L = 0 \), and this is a contradiction.

So \( M \) must be essential right ideal of \( R \). Therefore \( R/M \) is Gp-injective. Then there exists a positive integer \( n \) such that any \( R \)-homomorphism of \( a^nR \) into \( R/M \) extends to one of \( R \) into \( R/M \). let \( f : a^nR \rightarrow R/M \) be defined by \( f(a^nr) = r + M \). \( f \) is a well-defined \( R \)-homomorphism. Indeed , let \( r_1, r_2 \in R \) such that \( a^nr_1 = a^nr_2 \). Then \( a^nr_1 - a^nr_2 = 0 \), implies that \( a^n(r_1 - r_2) = 0 \), so \( r_1 - r_2 \in r(a^n) \), since \( I \) is reduced. Therefore \( r(a^n) = r(a) \), this implies that \( r_1 - r_2 \in r(a) \subseteq M \). Hence, \( r_1 + M = r_2 + M \). Now \( R/M \) is Gp-injective, so there exists \( c \in R \) such that \( 1 + M = f(a^n) = ca^n + M \). Hence , \( 1 - ca^n \in M \), since \( a^n \in M \) and \( R \) is a quasi-duo ring , then \( ca^n \in M \) and so \( 1 \in M \). This contradicts \( M \neq R \).

Therefore \( aR + r(a) = R \). In particular \( ar + c = 1 \), for some \( r \in R \) and \( c \in r(a) \), whence \( a^2 = a \), if we set \( d = a^2 \in I \), then \( a = a^2d \). Clearly \( (a - ada)^2 = 0 \), since \( I \) is reduced, thus \( a = ada \), and hence \( I = eR \), where \( e = ad \) is an idempotent element. Thus \( I \) is a direct summand.
**Proposition 2-2:**
Let R be a semi-prime ring with every simple singular right R-module is Gp-injective. Then every right ideal of R is an idempotent.

**Proof:** For any right ideal I of R, suppose there exists an element b in I, such that $b \not\in I^2$. Then $bR \neq (bR)^2$. Since R is a semi-prime ring, then $(bR)^2$ is essential in bR. By Zorn's lemma, the set of right ideals J such that $(bR)^2 \subseteq J \subseteq bR$ has a maximal member L. Then $bR/L$ is a simple singular, and therefore is Gp-injective. Now, let $f:bR \rightarrow bR/L$ is the canonical homomorphism defined by $f(br) = br + L$ for all ring R, since $bR/L$ is Gp-injective, so there exists $c \in R$, such that $f(br) = (bc + L)br$. Then $f(b) = (bc + L)b = b + L$, which implies that $b + L = bcb + L$. Hence, $b - bcb \in L$, whence it follows that $b \in L$. Thus $bR \subseteq L$ and this is a contradiction. Therefore $I = I^2$.

**3-Zero Commutative Rings**

In this section we introduce the notion of a zero commutative ring in order to study the connection between rings whose simple singular right R-modules are Gp-injective and other rings.

**Definition 3-1:**
A ring R is called zero commutative (briefly ZC) if for $a, b \in R$, $ab = 0$ if $ba = 0$.

We shall begin this section with the following result.

**Lemma 3-2:**
Let R be a ZC ring. Then $RaR + l(a)$ is an essential left ideal of R.

**Proof:** Given $a \in R$, assume that $[RaR + l(a)] \bigcap I = 0$, where I is a right ideal of R. Then $al \subseteq I \bigcap RaR = 0$, so $I \subseteq (a) \subseteq l(a)$. Hence, $l = 0$; where $RaR + l(a)$ is an essential left ideal of R.

**Lemma 3-3:**
Let R be a ZC ring with every simple singular left R-module is Gp-injective, then R is reduced.

**Proof:** Let $a^2 = 0$. Suppose that $a \neq 0$. By lemma (3-2), $l(a)$ is an essential left ideal of R. Since $a \neq 0$, $l(a) \neq R$. Thus, there exists a maximal essential left ideal M of R containing $l(a)$, therefore $R/M$ is Gp-injective. So any R-homomorphism of $Ra$ into $R/M$ extends to one of R into $R/M$. Let $f: Ra \rightarrow R/M$ be defined by $f(ra) = r + M$. Clearly, f is a well-defined R-
homomorphism. Thus \(1 + M = f(a) = ac + M\). Hence, \(1 - ac \in M\) and so \(1 \in M\), which is a contradiction. Hence \(a = 0\), and so \(R\) is reduced.

**Definition 3-4:**
A ring \(R\) is said to be right weakly regular if for all \(a \in R\), there exists \(b \in RaR\) such that \(a = ab\).

Now, we give the main result.

**Proposition 3-5:**
If \(R\) is ZC and every simple singular left \(R\)-module is Gp-injective, then \(R\) is a reduced weakly regular ring.

**Proof:** By Lemma (3-3), \(R\) is a reduced ring. We shall show that \(RaR + l(a) = R\) for any \(a \in R\). Suppose that there exists \(b \in R\) such that \(RbR + l(b) \neq R\). Then there exists a maximal left ideal \(M\) of \(R\) containing \(RbR + l(b)\). By Lemma (3-2), \(M\) must be essential in \(R\). Therefore \(R/M\) is Gp-injective. So there exists a positive integer \(n\) such that any \(R\)-homomorphism of \(Rb^n\) into \(R/M\) extends to one of \(R\) into \(R/M\). Let \(f: Rb^n \to R/M\) be defined by \(f(rb^n) = r + M\). Since \(R\) is a reduced ring, \(f\) is a well-\(R\)-homomorphism. Now, \(R/M\) is Gp-injective, so there exists \(c \in R\) such that \(1 + M = f(b^n) = b^n c + M\). Hence \(1 - b^n c \in M\) and so \(1 \in M\), which is a contradiction. Therefore \(RaR + l(a) = R\) for any \(a \in R\). Hence \(R\) is a left weakly regular ring. Since \(R\) is reduced, \(RaR + r(a) = R\), implies that \(R\) is a right weakly regular ring. Therefore \(R\) is a weakly regular ring.

**Kim and Nam in [2] proved that.** Rings whose simple right \(R\)-modules are Gp-injective are always semi-prime. But in general rings whose simple singular right \(R\)-modules are Gp-injective need not be semi-prime.

**Proposition 3-6:**
Let \(R\) be a ZC ring, and every simple singular left \(R\)-module is Gp-injective, then \(R\) is a semi-prime ring.

**Proof:** From Lemma (3-3), \(R\) is a reduced ring and then \(R\) is a semi-prime ring.
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