Water waves problem with surface tension in a corner domain I: A priori estimates with constrained contact angle

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Abstract

We study the two dimensional water waves problem with surface tension in the case when there is a non-zero contact angle between the free surface and the bottom. In the presence of surface tension, dissipations take place at the contact point. Moreover, when the contact angle is less than \(\frac{\pi}{6}\), no singularity appears in our settings. Using elliptic estimates in corner domains and a geometric approach, we prove an a priori estimate for the water waves problem.

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1 Introduction

In this paper, we consider the two dimensional water waves problem on an unbounded corner domain $\Omega_t$ with an upper free surface $\Gamma_t$ and a fixed flat bottom $\Gamma_b$. Particularly, if we parametrize our domain by denoting $\Gamma_t = \{(x, z) | z = \eta(t, x)\}$ and $\Gamma_b = \{(x, z) | z = b(x)\}$, we can write at time $t$

$$\Omega_t = \{(x, z) | b(x) < z < \eta(t, x)\}.$$ 

To avoid technical complexity, we assume that $\Gamma_t$ and $\Gamma_b$ only have one intersection point $X_c$ (the contact point) at the left end (See figure 1). The contact angle between the free surface and the bottom is denoted by $\omega(t)$ (or sometimes simply $\omega$).

Without loss of generality, we pose the contact point at $t = 0$ at the origin. Moreover, we assume that the bottom $\Gamma_b$ is a line segment near the origin, and $\Gamma_b$ becomes a horizontal line away from the origin. Our domain $\Omega_t$ has a finite depth, which means that there exists a constant $h > 0$ such that the distance between $\Gamma_t$, $\Gamma_b$ is always less that $h$. Our domain corresponds to the scene of sea waves moving near a beach, and the contact point denotes the intersection point between the sea and the rigid bottom in two dimensional case.

The water waves problem investigates an ideal fluid with a free surface, which is supposed to be inviscid and incompressible. We assume that the fluid is under the influence from the gravity in the domain $\Omega_t$ and the surface tension on the free surface $\Gamma_t$. Moreover, the problem is also assumed to be irrotational.

The water waves problem in our case involves surface tension on the free surface. Compared to the case without surface tension, it is important to find a proper condition at the contact point. In fact, T. Young [40] had proved a long time ago that, in the stationary case, the (stationary) contact angle $\omega_s$ is a physical constant depending only on the materials of the bottom and the fluid:

$$\cos \omega_s = \frac{[\gamma]}{\sigma}$$

where $\sigma$ is the surface tension coefficient along the free surface and $[\gamma] = \gamma_1 - \gamma_2$ with $\gamma_1, \gamma_2$ are measures of the free-energy per unit length associated to the solid-vapor and solid-fluid interaction respectively. Based on this theory, when the fluid is moving, a similar condition
from W. Ren and W. E \cite{34} is adopted in our paper, which takes the fluid speed at the contact point into account:

$$\beta_c v_c = \sigma (\cos \omega_s - \cos \omega),$$  \hfill (1.1)

where

$$v_c = -v \cdot \tau_b$$

denotes the fluid speed at the contact point and $\beta_c$ is the effective friction coefficient determined by interfacial widths, interactions between the fluid and the bottom, and the normal stress contributions.

Based on the condition above and letting $v$ be the fluid velocity and $P$ be the total pressure, we consider the following water waves problem on the domain $\Omega$ at time $t > 0$:

\begin{equation}
(WW) \begin{cases}
\partial_t v + v \cdot \nabla v = -\nabla P - g, \\
\text{div} v = 0, \quad \text{curl} v = 0, \quad \text{on} \quad \Omega_t \\
P|_{\Gamma_t} = \sigma \kappa, \\
\partial_t + v \cdot \nabla \text{ is tangent to } \Gamma_t, \\
v \cdot n_b|_{\Gamma_b} = 0, \\
\beta_c v_c = \sigma (\cos \omega_s - \cos \omega), \quad \text{at} \quad X_c
\end{cases}
\end{equation}

with $\kappa$ the mean curvature of the free surface and $g$ the gravity vector.

Before stating our result, firstly we would like to recall some previous works on the well-posedness of classical water waves problems. Here 'classical' means water waves in smooth domains, where smooth domains refer to domains with smooth boundaries. Compared to smooth domains, when we say non-smooth domains, we always refer to domains with Lipschitz boundaries. For example, our domain $\Omega$ with a corner is a non-smooth domain.

In the case when surface tension is ignored, some early works such as V.I. Nalimov \cite{30}, H. Yosihara \cite{38}, and W. Craig \cite{13} established the local well-posedness with small data in 2 dimensional case. The local well-posedness of general initial data for 2 and 3 dimensional are solved by S. Wu \cite{41,42} in the case of infinite depth. D. Lannes \cite{25} considered the water waves problem in the case of finite depth under Eulerian setting. Moreover, H. Lindblad \cite{26} proved the existence of solutions for the general problem of a liquid body in vacuum and P. Zhang and Z. Zhang \cite{46} showed the local well-posedness for the rotational problem.

Concerning the problem with surface tension, H. Yosihara \cite{39} proved an early result on the local well-posedness with small data in infinite-depth case. T. Iguchi \cite{21} and D. Ambrose \cite{6} studied the local well-posedness of the irrotational problem in 2 dimensional case. B. Schweizer \cite{35} showed the existence for the general 3 dimensional problem. J. Shatah and C. Zeng \cite{36,37} proved a priori estimates and the local well-posedness (even when the fluid domains are not simply connected), and they solved the problem using a geometric approach and the fluid is rotational and with surface tension. A similar geometric approach had also been used by K. Beyer and M. Günther\cite{9,10} to study the irrotational problem for star-shaped domains. Moreover, Moreover, D. Coutand and S. Shkoller \cite{12} proved local well-posedness for the rotational problem under Lagrangian coordinates.

For the global well-posedness for the water waves, the first result is given by S. Wu \cite{44} who proved the almost global existence for the gravity problem in two dimensions. Later, P. Germain, N. Masmoudi and J. Shatah \cite{14} and S. Wu \cite{45} proved the global existence of gravity waves in three dimensions respectively. T. Alazard and J.M. Delort \cite{3} and A.D.
Ionescu and F. Pusateri [23] gave the proof of the global regularity for the gravitational water
waves system in two dimensions independently. Recently, J. Hunter, M. Ifrim and D. Tataru
[18, 19, 20] used the conform mapping method to give another proof of the global existence
for the gravitational problem in two dimensions.

There are more works on water waves, and we only mention some here: Ambrose and
Masmoudi [7, 8], M. Ogawa and A. Tani [31, 32], B. Alvarez-Samaniego and D. Lannes [5],
T. Alazard, Burq and Zuily [2], M. Ming and Z. Zhang [28], M. Ming, P. Zhang and Z. Zhang
[29] e.t.c..

Secondly, we recall some recent results on water waves in non-smooth domains. In fact,
theoretical research on the non-smooth domain only started several years ago and remains a
lot of open problems. There was a work by T. Alazard, N. Burq and C. Zuily [3] for some right
angle with vertical walls when there is no surface tension, where they used symmetrizing and
periodizing to turn this problem into a classical water waves problem. Later, R.H. Kinsey
and S. Wu [24] and S. Wu [43] proved a priori estimates and local well-posedness for the two
dimensional water waves with angled crests, where a conformal mapping is used to convert
the boundary singularities. T. de Poyferré [33] gave a priori estimates for the rotational water
waves problem in a compact domain (corresponds to a beach type) in general n dimensions,
where the contact angle is smaller than a dimensional constant and therefore no singularity
appears. Moreover, this work is done in absence of surface tension.

In our paper, we consider a beach-type domain \( \Omega_t \), which is two dimensional with only
one contact point. Different from T. de Poyferré [33], we take surface tension on the free
surface into account, which leads to a big difference in the whole energy formulation and will
be discussed very soon.

To study the water wave problem (WW), the first main difficult comes from the Dirichlet-
Neumann operator or equivalently the related elliptic systems on corner domains. As already
explained in details in our previous work [27], non-smooth domain generates singularities from
related elliptic systems. Moreover, Remark 5.20 [27] tells us directly that smaller contact angle
leads to higher regularities for elliptic systems. Motived from this point, we want to study the
water waves problem on corner domains under a proper formulation firstly with no singularity.
In fact, we choose the geometric approach from J. Shatah and C. Zeng [36, 37], which turns
out to be a good choice for corner domains. Meanwhile, to avoid singularity, we work under
a comparatively low regularity (will be explained in the following main theorem) as the first
step for our project. In this paper, we prove an a priori estimate for the water waves problem
(WW), which is performed under a comparatively low regularity and no singularity appears.

Compared to the classical water waves, our problem is also variational, but the variation
formulation is very different. In fact, one can see in Section 3 that, the Lagrangian Action
contains a potential at the contact point, and meanwhile there is a dissipation relating to
the contact point in the variation equation (4.1). In a word, our variation formulation is new
compared to the classical water waves.

Now we stress the role of surface tension. As mentioned above, the energy formulations
with and without surface tension are completely different. When there is no surface tension,
the friction at the contact point is ignored as in [33], so the energy is conserved. On the other
hand, when surface tension is taken into considerations especially at the contact point, the
boundary condition (1.1) appears, and the water waves problem (WW) generates a different
basic energy in Eulerian coordinates

\[ E_0 = \frac{1}{2} \int_{\Omega_t} |v|^2 dX + g \int_{\Omega_t} X \cdot e_z dX + \sigma S(u) + [\gamma]X \cdot \tau_b|_{X_c} \]  

(1.3)

where the last term denotes the friction potential at the contact point. Moreover, one can find in Section 4 that our problem (WW) satisfies the following dissipation equation:

\[ \frac{d}{dt} E_0 + \beta_c |v_c|^2|_{X_c} = 0 \]  

(1.4)

with \( \beta_c > 0 \) the friction coefficient and \( v_c \) the speed at the contact point along the bottom. This implies that some energy dissipation takes place at the contact point. By the way, our dissipation formulation is indeed similar as that in Y. Guo and I. Tice [17], which works on the contact line problem for the Stokes equation.

As a result, the dissipation formulation leads to a big difference in the a priori estimate: The terms at the contact point need to be treated carefully, which is a completely new part in water waves problem. Similarly as in [36], our energy estimate is proved firstly for the equation of the main part \( J \) from \( \nabla P \), and then we go back to (WW). Compared to [36], another difference in the estimates relates to the lower boundary \( \Gamma_b \), which also needs much care since our estimates are performed very often in variational sense, and some special Sobolev spaces from P. Grisvard [15] such as \( \tilde{H}^{\frac{7}{2}}(\Gamma_b) \), \( \tilde{H}^{-\frac{7}{2}}(\Gamma_b) \) are applied in our paper.

Now it’s the time to state the main theorem in our paper. To begin with, we introduce the energy functional

\[ E(t) = \| \nabla_{\tau_t} J^\perp \|^2_{L^2(\Gamma_t)} + \| \mathcal{D}_t J \|^2_{L^2(\Omega_t)} + \| \Gamma_t \|^2_{\tilde{H}^{\frac{7}{2}}} + \| v \|^2_{L^2(\Omega_t)}, \]

and the dissipation at the contact point

\[ F(t) = | \sin \omega \nabla_{\tau_t} J^\perp |_{X_c} |^2. \]

So we can present the main theorem as below:

**Theorem 1.1** Assume that the initial data \((\Gamma_0, v_0) \in H^4 \times H^3(\Omega_0)\) and the initial contact angle \( \omega_0 \in (0, \pi/6) \). Let \((\Gamma_t, v) \in H^4 \times H^3(\Omega_t)\) be a strong solution of (WW), then there exists a constant \( T_0 \) depending on the initial data such that the following a priori estimate holds

\[ \sup_{0 \leq t \leq T_0} E(t) + \int_0^{T_0} F(t) dt \leq E(0) + \int_0^{T_0} P(E(t)) dt, \]

where \( P(\cdot) \) is a polynomial with positive constant coefficients depending on \( \sigma, \beta_c, \Gamma_b \).

**Remark 1.1** We consider the irrotational case in this paper, but our formulation may also work for the rotational case.

**Remark 1.2** In this paper, we need at most \( H^4 \) estimates for the elliptic systems in \( \Omega_t \). One can see directly from Remark 5.20 [27] that, to avoid the singularity, one needs the contact angle \( \omega \in (0, \pi/6) \). The energy estimates for the water waves problem remains an open problem for a more general angle.
Remark 1.3 Notice that the dissipation $F(t)$ contains $\sin \omega$, which means smaller contact angle $\omega$ leads to smaller dissipation as long as $\nabla_{\tau_t} J^\perp|_{X_c}$ remains the same. On the other hand, we have $J \in H^2(\Omega_t)$ in our settings, so the dissipation term $\int_0^{T_0} F(t) dt$ on the left side of the a priori estimate is some kind of smoothing estimates such that $\nabla_{\tau_t} J^\perp$ makes sense at the contact point $X_c$ for $t \in [0, T_0]$.

Organization of this paper. Section 2 introduces some notations used in this paper. In Section 3, we explain the water waves in a geometric approach and prove that our problem is variational. In Section 4, a dissipation equation is deduced from the water waves problem. In Section 5, we recall some trace theorems and elliptic estimates from our previous paper. Section 6 deals with some commutators and derive the equation for $J$. In the end, we prove our a priori estimates in Section 7.

2 Notations

- $\Omega_0$ is the initial domain at time $t = 0$, and $\Omega_t$ is the domain at time $t$.
- We denote by $Y$ a point in $\Omega_0$, and by $X$ a point in $\Omega_t$.
- $X_c$ is the coordinate of the contact point at time $t$, which corresponds to $Y_c \in \Omega_0$ satisfying $X_c = u(t, Y_c)$.
- $v_c = -v \cdot \tau_b|_{X_c}$ is the speed of the contact point $X_c$ along the bottom $\Gamma_b$.
- We denote by $\tilde{n}_b, \tilde{\tau}_b$ unit orthogonal extensions onto $\Omega_t$ for $n_b, \tau_b$ on $\Gamma_b$.
- $D_t = \partial_t + v \cdot \nabla$ is the material derivative.
- $M^*$ denotes the transport of a matrix $M$.
- $A \cdot B$ denotes the inner product of two vectors or two matrices $A, B$.
- $w^\perp$ on $\Gamma_t$: $w \cdot n_t$ for a vector $w \in T_X \Gamma_t$.
- $w^\top$ on $\Gamma_t$: $(w \cdot \tau_t) \tau_t$. Sometimes we also use $w^\top$ on $\Gamma_b$ with a similar definition.
- $\Pi$ is the second fundamental form where $\Pi(w) = \nabla w n_t \in T_X \Gamma_t$ for $w \in T_X \Gamma_t$.
- $\Pi(v, w)$ denotes $\Pi(v) \cdot w$. Moreover, $\Pi$ is symmetric: $\Pi(v, w) = \Pi(w, v)$.
- $||\Pi||^2 = tr(\Pi \Pi^*)$.
- $\kappa = tr \Pi = \nabla_{\tau_t} n_t \cdot \tau_t$ is the mean curvature.
- $(D \cdot \Pi)(w) = (D_{\tau_t} \Pi)(w) \cdot \tau_t = D_{\tau_t}(\Pi(w)) - \Pi(D_{\tau_t} w)$.
- $\Pi \cdot D^\top: \Pi(\tau_t) \cdot \nabla_{\tau_t}$.
- $\Delta_{\tau_t}$ is the Beltrami-Laplace operator on $\Gamma_t$:
  \[ \Delta_{\tau_t} f = D^2 f(\tau_t, \tau_t) = D \cdot (\nabla^\top f) = \nabla_{\tau_t} \nabla_{\tau_t} f - \nabla_{D_{\tau_t} \tau_t} f. \]

- $D^2 f(\tau_t, \tau_t') = D^2 f(\tau_t, \tau_t') - (\Pi(\tau_t) \cdot \tau_t') \nabla_{n_t} f$.
- $H(f)$ or $f_H$ is the harmonic extension for some function $f$ on $\Gamma_t$, which is defined by the elliptic system
  \[
  \begin{cases}
  \Delta H(f) = 0, & \text{on } \Omega_t, \\
  H(f)|_{\Gamma_t} = f, \quad \nabla_{n_b} H(f)|_{\Gamma_b} = 0.
  \end{cases}
  \]

- $\Delta^{-1}(h, g)$ denotes the solution of the system
  \[
  \begin{cases}
  \Delta u = h & \text{on } \Omega \\
  u|_{\Gamma_t} = 0, \quad \nabla_{n_b} u|_{\Gamma_b} = g.
  \end{cases}
  \]
- $[\gamma] = \gamma_1 - \gamma_2$, where $\gamma_1$, $\gamma_2$ are the surface tension coefficients denoting the solid-air and solid-fluid interactions respectively.
- $\beta_c$ is the effective friction coefficient determined by interfacial widths, interactions between the fluid and the bottom, and the normal stress contributions.
- $g$ denotes the constant gravity vector or the gravity coefficient.
- $P(E(t))$: Some polynomial for the energy $E(t)$.
- The Sobolev norm $H^s$ on $\Omega_t$ can be defined by restrictions

$$\|u\|_{H^s(\Omega)} = \inf\{\|U\|_{H^s(\mathbb{R}^2)}, U|_{\Omega} = u\},$$

and similar definitions also work on $\Gamma_t$, $\Gamma_b$.
- $\tilde{H}^{\frac{1}{2}}(\Gamma_b)$, a subspace of $H^{\frac{1}{2}}(\Gamma_b)$ related to corner domains is defined as

$$\tilde{H}^{\frac{1}{2}}(\Gamma_b) = \left\{ u \in \tilde{H}^{\frac{1}{2}}(\Gamma_b) \mid \rho^{-\frac{1}{2}}u \in L^2(\Gamma_b) \right\}$$

where $\tilde{H}^{\frac{1}{2}}(\Gamma_b)$ is the corresponding homogeneous space, and $\rho = \rho(X)$ is the distance (arc length) between the point $X \in \Gamma_b$ and the left end $X_c$. The norm is defined as

$$\|u\|^2_{\tilde{H}^{\frac{1}{2}}} = \|u\|^2_{H^{\frac{1}{2}}} + \int_{\Gamma_b} \rho^{-1}|u|^2dX.$$

Moreover, we use $\tilde{H}^{-\frac{1}{2}}(\Gamma_b)$ to denote the dual space of $\tilde{H}^{\frac{1}{2}}(\Gamma_b)$. For more details, see [15].

3 Geometry and variation

In this section, we introduce the geometry behind the water waves problem following the notations from [36]. And one can see that our problem on the corner domain is also variational in nature, while the new point here is about the dissipation at the contact point. In the end, we will compute the second variation of the basic energy $E_0$ to find out the leading-order term in the linearization of our problem (WW), which turns out to be the same as [36].

Let $X = u(t, Y)$ for any $Y \in \Omega_0$ be the Lagrangian coordinates map solving

$$\frac{dX}{dt} = v(t, X), \quad X(0) = Y$$

and one can also denote the velocity as $v = u_t \circ u^{-1}$.

Since $v$ is divergence free, the trajectory map $u$ is volume-preserving. We define the manifold

$$\Gamma = \{ \Phi : \Omega_0 \to \mathbb{R}^2 \mid \Phi \text{ is a volume-preserving homeomorphism} \},$$

and consequently the tangent space of $\Gamma$ is given by

$$T_0\Gamma = \{ \tilde{w} : \Omega_0 \to \mathbb{R}^2 \mid w = \tilde{w} \circ \Phi^{-1} \text{ satisfying } \nabla \cdot w = 0 \text{ on } \Phi(\Omega_0) \text{ and } w \cdot n_b|_{\Gamma_b} = 0 \}$$

where $\Gamma_b$, $n_b$ denote also the bottom and the unit outward normal vector of $\Phi(\Omega_0)$ respectively.
Based on the tangent space, we need to consider \((T_{\Phi})^\perp\) and the Hodge decomposition. In fact, for any vector field \(w : \Phi(\Omega_0) \to \mathbb{R}^2\), we look for the Hodge decomposition for \(w\) in the form
\[
w = w_1 - \nabla q
\]
where \(\tilde{w}_1 = w_1 \circ \Phi \in T_{\Phi}\Gamma\) and \(q\) is decided by \(\Phi, w\). A direct computation shows that
\[
(T_{\Phi})^\perp = \{-(\nabla q) \circ \Phi \mid \Delta q = -\nabla \cdot w, \text{ with } q|_{\Gamma_t} = 0, \nabla_n q|_{\Gamma_b} = -w \cdot n_b\}
\]
with \(\Gamma_t\) the upper surface of \(\Phi(\Omega_0)\).

Now we are ready to apply the Hodge decomposition. For a path \(u(t, \cdot) \in \Gamma\) with \(\tilde{v} = u_t\), and any vector \(\tilde{w}(t, \cdot) \in T_{u(t)}\Gamma\), we decompose \(\tilde{w}_t\) and find the covariant derivative \(\tilde{D}_t\tilde{w}\) and the second fundamental form \(II_{u(t)}(\tilde{w}, \tilde{v})\) satisfying
\[
\tilde{w}_t = \tilde{D}_t\tilde{w} + II_{u(t)}(\tilde{w}, \tilde{v}),
\]
where
\[
\tilde{D}_t\tilde{w} = T_{u(t)}\Gamma, \quad II_{u(t)}(\tilde{w}, \tilde{v}) \in (T_{u(t)}\Gamma)^\perp.
\]
From the Hodge decomposition we know that
\[
II_{u(t)}(\tilde{w}, \tilde{v}) = -(\nabla P_{w,v}) \circ u
\]
with \(P_{w,v}\) solving the system
\[
\begin{cases}
-\Delta P_{w,v} = tr(\nabla w \nabla v), & \text{on } \Omega \\
P_{w,v}|_{\Gamma_t} = 0, \quad \nabla_n P_{w,v}|_{\Gamma_b} = w \cdot \nabla n_b|_{\Gamma_b}.
\end{cases}
\]
As a result, denoting \(D_tw = (\tilde{D}_t\tilde{w}) \circ u^{-1}\), we arrive at the decomposition of \(D_tw\):
\[
D_tw = D_tw + \nabla P_{w,v}.
\]

3.1 Lagrangian variation

Our problem on the corner domain \(\Omega\) has a Lagrangian formulation with dissipations at the corner. In fact, we introduce the Lagrangian Action on a time interval \([0, T]\):
\[
I(u) = \int_0^T \left( \int_{\Omega_t} \frac{1}{2} |u|^2 dX - g \int_{\Omega_t} zdX - \sigma S(u) - [\gamma] X \cdot \tau_0|_{X_c} \right) dt
\]
where \(\tau_0\) near \(X_c\) is a constant vector due to the definition of the bottom, and the surface potential is
\[
S(u) = \lim_{A \to +\infty} \left( \int_{\Gamma_t \cap \{x \leq A\}} ds - \int_{\Gamma_s \cap \{x \leq A\}} ds \right)
\]
with \(\Gamma_s\) the reference surface.

On the other hand, we also define the dissipation at the contact point as
\[
F(u, q) = \frac{1}{2} \beta_c (\psi \circ u^{-1}) \cdot \tau_0|_{X_c}
\]
with \(q = u_t\).
Considering any path $u(s,t,\cdot) \in \Gamma$ with $\bar{w} = u_s|_{s=0}$ and $w = \bar{w} \circ u^{-1}|_{s=0}$ satisfying
\[ \bar{w}|_{t=0} = \bar{w}|_{t=T} = 0, \]
we will show that the Euler equation and the condition at the contact point from system (WW) can be deduced from the variation formulation
\[ \langle I'(u), \bar{w} \rangle = \int_0^T \langle F_q(u, u_t), \bar{w} \rangle dt. \]
To prove this, we firstly start with the left side and compute on $\Omega_0$ to find:
\[ \langle I'(u), \bar{w} \rangle = \int_0^T \left( \int_{\Omega_0} u_t \cdot \bar{w}_t dY - g \int_{\Omega_0} e_z \cdot \bar{w} dY - \sigma \langle S'(u), \bar{w} \rangle - [\gamma] \bar{w} \cdot \tau_b \right) dt. \]
A direct calculation as in [36] leads to
\[ \langle S'(u), \bar{w} \rangle = \int_{\Gamma_t} (\kappa w^\perp + D \cdot w^\top) ds, \]
where in our 2 dimensional case, we can write in particular that
\[ D \cdot w^\top = \nabla_{\tau_t} (w \cdot \tau_t) = -\frac{d}{ds} (w \cdot \tau_t) \]
with $s$ the arclength parameter on $\Gamma_t$ starting from $X_c$. Consequently, we find
\[ \int_{\Gamma_t} D \cdot w^\top ds = -\int_0^\infty \frac{d}{ds} (w \cdot \tau_t) ds = w \cdot \tau_t|_{X_c}, \]
which infers that
\[ \langle S'(u), \bar{w} \rangle = \int_{\Gamma_t} \kappa w^\perp ds + w \cdot \tau_t|_{X_c} = \int_{\Omega_t} \nabla \kappa H \cdot wdX + w \cdot \tau_t|_{X_c}. \]
Notice that compared to the classical case, there is an extra term concerning the contact point. As a result, we arrive at
\[ \langle I'(u), \bar{w} \rangle = \int_0^T \left( -\int_{\Omega} (D_t v + ge_z + \sigma \nabla \kappa H) \cdot wdX - (\sigma w \cdot \tau_t + [\gamma] w \cdot \tau_b)|_{X_c} \right) dt. \]
On the other hand, the right side of the variation formulation turns out to be
\[ \int_0^T \langle F_q(u, u_t), \bar{w} \rangle dt = \int_0^T \beta_c(v \cdot \tau_b)(w \cdot \tau_b)|_{X_c} dt, \]
which together with the left side implies the following equality:
\[ \int_0^T \left( -\int_{\Omega} (D_t v + ge_z + \sigma \nabla \kappa H) \cdot wdX - (\sigma w \cdot \tau_t + [\gamma] w \cdot \tau_b)|_{X_c} \right) dt = \int_0^T \beta_c(v \cdot \tau_b)(w \cdot \tau_b)|_{X_c} dt. \]
Consequently, we retrieve the Euler equation
\[ D_t v = -\nabla P_{v,v} - \sigma \nabla \kappa H - ge_z \quad \text{on} \quad \Omega \]
where the total pressure \( P \) in (WW) is decomposed into two parts

\[
P = P_{v,v} + \sigma \kappa_\mathcal{H}
\]

with \( P_{v,v} \) defined by the Hodge decomposition and the second part from the mean curvature \( \kappa \).

Moreover, we also find the equality at the corner:

\[
- \int_0^T (\sigma w \cdot \tau_t + [\gamma] w \cdot \tau_b)|_{X_c} dt = \int_0^T \beta_c (v \cdot \tau_b) (w \cdot \tau_b)|_{X_c} dt.
\]

Remembering the notations

\[
v \cdot \tau_b = -v_c, \quad w \cdot \tau_b = -w_c,
\]

one can have

\[
w \cdot \tau_t = w_c \cos \omega.
\]

Substituting these computations into the equality above, we derive the condition at the contact point:

\[
\beta_c v_c = [\gamma] - \sigma \cos \omega,
\]

which can also be written as

\[
\beta_c v_c = \sigma (\cos \omega_s - \cos \omega)
\]

while recalling that \( \cos \omega_s = [\gamma]/\sigma \) stands for the cosine of the static contact angle \( \omega_s \).

### 3.2 Second variation of the basic energy

Recall that the basic energy \( E_0 \) in Eulerian coordinates takes the form

\[
E_0 = \frac{1}{2} \int_{\Omega_t} |v|^2 dX + g \int_{\Omega_t} X \cdot e_z dX + \sigma S(u) + [\gamma] X \cdot \tau_b|_{X_c}
\]

where \( S(u) \) is the surface potential defined in (3.1), and \( [\gamma] u \cdot \tau_b|_{X_c} \) is the interaction energy at the corner.

We try to analyze the water wave problem (WW) by a linearization. The basic energy provides us a good way for linearization and finding out the leading-order operator. Since it turns out that the velocity part and the gravity part in \( E_0 \) are lower-order terms (see [36, 37], and also verified in [33]), we focus on the last two terms in (3.2).

In fact, we denote

\[
E_c = \sigma S(u) + [\gamma] X \cdot \tau_b|_{X_c}
\]

as the part of energy related to the leading-order operator as well as the contact point. We will find out variations of \( E_c \).

Firstly, consider a path \( u(s, t, \cdot) \in \Gamma \) and \( \bar{w} = u_s|_{s=0} \in T_u \Gamma \) with \( w = \bar{w} \circ u^{-1}|_{s=0} \), we compute the first variation as

\[
\langle E'_c, \bar{w} \rangle = \sigma \langle S'(u), \bar{w} \rangle + [\gamma] [w \cdot \tau_b]|_{X_c}
\]
where we already know from last subsection that

\[
(S'(u), \bar{w}) = \int_{\Gamma_t} \kappa w^\perp ds + w \cdot \tau_t|_{X_c}.
\]

Secondly, we consider the second variation, and we will start with the surface potential \( S(u) \). Let \( h(s, \cdot) \) be a geodesic on \( \Gamma \) with \( h(0) = u \) and \( \bar{w} = h_{s|s=0} \), which means

\[
\bar{D}_s \bar{w} = (D_s w + \nabla P_{w, w}) \circ h = 0.
\]

To begin with, the second variation for \( S(u) \) can be written as

\[
\bar{D}^2 S(u)(\bar{w}, \bar{w}) = \frac{d}{ds} \int_{\Gamma_t} \kappa w^\perp ds|_{s=0} + \frac{d}{ds} w \cdot \tau_t|_{X_c}|_{s=0} := A + B.
\]

Similarly as in [36] a direct computation leads to

\[
A = \int_{\Gamma_t} (D_s \kappa w \cdot n_t + \kappa D_s w \cdot n_t + \kappa w \cdot D_s n_t) ds + \kappa w \cdot n_t D_s ds|_{s=0}
\]

\[
= \int_{\Gamma_t} \left( (\Delta_{\Gamma_t} w^\perp - w^\perp |\Pi|^2 + \mathcal{D} \cdot \Pi (w^\top)) w^\perp - \kappa \nabla P_{w, w} \cdot n_t - \kappa \nabla w^\top w \cdot n_t
\]

\[
+ \kappa w^\perp (\kappa w^\perp + \mathcal{D} \cdot w^\top) \right) ds
\]

where one applied equation (6.6) for \( D_s \kappa \) and

\[
D_s w = -\nabla P_{w, w}, \quad D_s n_t = -((\nabla w)^* n_t)^\top, \quad D_s ds = (\kappa w^\perp + \mathcal{D} \cdot w^\top) ds.
\]

As a result, one can tell that the leading-order term is the first one, so one writes

\[
A = \int_{\Gamma_t} (-\Delta_{\Gamma_t} w^\perp) w^\perp ds + \text{lower-order terms}.
\]

Moreover, in 2 dimensional case we can have in particular that

\[
\int_{\Gamma_t} (-\Delta_{\Gamma_t} w^\perp) w^\perp ds = -\int_{\Gamma_t} \mathcal{D} \cdot (\nabla^\top w^\perp) w^\perp ds
\]

\[
= \int_0^{+\infty} \frac{d}{ds} (\nabla_{\tau_t} w^\perp) w^\perp ds
\]

\[
= -w^\perp \nabla_{\tau_t} w^\perp|_{X_c} + \int_{\Gamma_t} |\nabla_{\tau_t} w^\perp|^2 ds
\]

which implies that

\[
A = \int_{\Gamma_t} |\nabla_{\tau_t} w^\perp|^2 ds - w^\perp \nabla_{\tau_t} w^\perp|_{X_c} + \text{lower-order terms}.
\]

Next, we turn to deal with term \( B \) to find

\[
B = D_s w \cdot \tau_t + w \cdot D_s \tau_t|_{X_c} = -\nabla P_{w, w} \cdot \tau_t + w \cdot (\nabla_{\tau_t} w \cdot n_t) n_t|_{X_c}
\]

\[
= w^\perp \nabla_{\tau_t} w^\perp - w^\perp (w \cdot \nabla_{\tau_t} n_t) - \nabla P_{w, w} \cdot \tau_t|_{X_c},
\]
which infers that
\[ B = w^\bot \nabla_{\tau} w^\bot|_{X_c} + \text{lower-order terms} \]
by noticing \( \nabla P_{w,w} \) is also a lower-order part from Proposition 7.2.

As a result, combining the analysis on \( A, B \) we can conclude that
\[
\bar{D}^2 S(u)(\bar{w}, \bar{w}) = \int_{\Gamma_t} |\nabla_{\tau} w^\bot|^2 ds + \text{lower-order terms.}
\]

On the other hand, we consider the second variation for the second term in \( E_c \) to find that
\[
\frac{d}{ds} \left( [\gamma] w \cdot \tau_b|_{X_c} \right)_{s=0} = [\gamma] \left( D_s w \cdot \tau_b + w \cdot D_s \tau_b \right)|_{X_c} = -[\gamma] \nabla P_{w,w} \cdot \tau_b|_{X_c},
\]
where \( D_s \tau_b = 0 \) since \( \tau_b \) is constant near the contact point. Consequently, one can see that this term also turns out to be a lower-order term.

Summing up the analysis above, we finally arrive at the second variation for \( E_c \) as
\[
\bar{D}^2 E_c(u)(\bar{w}, \bar{w}) = \int_{\Gamma_t} |\nabla_{\tau} w^\bot|^2 ds + \text{lower-order terms.}
\]

Based on this expression, we conclude that, in the presence of the contact point, the leading-order part in our basic energy \( E_0 \) remains the same as the related classical case (see [36]), and the terms related to the corner are only lower-order ones.

Therefore, this computation tells us that in the linearization of the water waves problem, there is no difference for the leading-order operator compared with the classical case, which will be verified in the following text. In fact, one can see in our paper that, higher-order terms related to the corner appear only in the dissipation part.

## 4 Dissipation equation

In this section, we will show that the water waves problem (WW) satisfies the following energy-dissipation equality
\[
\frac{d}{dt} E_0 + \beta_c |v_c|^2 = 0,
\]
where the proof is similar as the proof for the variation formulation. Compared to classical water waves problems, our energy has a dissipation related to the contact point, which is a completely new case. Moreover, one can also find dissipations from the contact point in our higher-order energy in the following sections.

To prove the dissipation equality, recalling again the definition (1.3) for the basic energy \( E_0 \), we firstly compute that
\[
\frac{d}{dt} E_0 = \int_{\Omega_t} v \cdot D_t v dX + g \int_{\Omega_t} D_t z dX + \sigma \partial_t S(u) + [\gamma] v \cdot \tau_b|_{X_c},
\]
where
\[
\partial_t S(u) = \int_{\Omega_t} \nabla \kappa_{\mathcal{H}} \cdot v dX + v \cdot \tau_t|_{X_c}.
\]
Noticing that
\[
v_c = -v \cdot \tau_b|_{X_c}, \quad v \cdot \tau_t|_{X_c} = v_c \cos \omega,
\]

...
we arrive at
\[
\frac{d}{dt} E_0 = \int_{\Omega_t} v \cdot (D_t v + g e_z + \sigma \nabla \kappa_H) dX + (\sigma v_c \cos \omega - [\gamma] v_c) \bigg|_{X_c}.
\]
Plugging in the Euler equation and the condition at the corner from (WW), we find that
\[
\frac{d}{dt} E_0 = -\int_{\Omega_t} v \cdot \nabla P_{v,v} dX + \sigma v_c (\cos \omega - [\gamma] \sigma) \bigg|_{X_c} = -\beta_c |v_c|^2 |_{X_c}
\]
and the proof ends.

5 Elliptic estimates and trace theorems in the corner domain

In order to perform the energy estimates, some technical preparations are needed. This section provides us some useful elliptic estimates and trace theorems on the corner domain, which are adjusted from [27, 15].

To begin with, the following mixed-boundary elliptic system are used very often in our paper:
\[
\begin{cases}
\Delta u = h, & \text{on } \Omega_t, \\
u|_{\Gamma_t} = f, & \nabla_{n_b} u + b_0 \nabla_{\tau_b} u |_{\Gamma_b} = g,
\end{cases}
\tag{5.1}
\]
where \(b_0\) is a constant coefficient. The elliptic estimate related to this system is given below.

**Theorem 5.1** If the contact angle \(\omega(t) < \pi/4\), we have the following estimate for system (5.1):
\[
\|u\|_{H^s(\Omega_t)} \leq C\left(\|\Gamma_t\|_{H^{\frac{7}{2}}(\Omega_t)}, \|\Gamma_b\|_{H^{\frac{7}{2}}(\Gamma_b)}, \|h\|_{H^{s-2}(\Omega_t)} + \|f\|_{H^{s-\frac{1}{2}}(\Gamma_t)} + \|g\|_{H^{s-\frac{3}{2}}(\Gamma_b)}\right),
\]
where \(2 \leq s \leq 3\). For \(s = 4\) and \(\omega(t) < \pi/6\), we have
\[
\|u\|_{H^s(\Omega_t)} \leq C\left(\|\Gamma_t\|_{H^{\frac{7}{2}}(\Omega_t)}, \|\Gamma_b\|_{H^{\frac{7}{2}}(\Gamma_b)}, \|h\|_{H^2(\Omega_t)} + \|f\|_{H^{\frac{7}{2}}(\Gamma_t)} + \|g\|_{H^{\frac{7}{2}}(\Gamma_b)}\right).
\]

**Proof.** This theorem comes directly from Proposition 5.19 [27] when \(s = 2, 3, 4\). For \(2 < s < 3\), we use the complex interpolation theorem to get it.

On the other hand, we give the a priori estimate for (5.1) when \(b_0\) is a function with a bounded support on \(\Gamma_b\), which will be used in Proposition 7.1.

**Lemma 5.2** Let the contact angle \(\omega(t) < \pi/4\) and \(u \in H^3(\Omega_t)\) be a solution of system (5.1) with \(h \in H^1(\Omega_t), f \in H^{\frac{7}{2}}(\Gamma_t)\) and \(g \in H^{\frac{7}{2}}(\Gamma_b)\). Moreover, assume that the coefficient \(b_0 \in H^{\frac{7}{2}}(\Gamma_b)\) is a function and vanishes away from the contact point. Then one has the following elliptic estimate
\[
\|u\|_{H^3(\Omega_t)} \leq C\left(\|\Gamma_t\|_{H^{\frac{7}{2}}(\Omega_t)} , \|\Gamma_b\|_{H^{\frac{7}{2}}(\Gamma_b)} , \|b_0\|_{H^1(\Gamma_b)} , \|h\|_{H^1(\Omega_t)} , \|f\|_{H^{\frac{7}{2}}(\Gamma_t)} , \|g\|_{H^{\frac{7}{2}}(\Gamma_b)}\right).
\]
Proof. The proof is based on Theorem 5.1 and we will use a unit decomposition to localize system (5.1).

To begin with, since \( b_0 \) has a bounded support, we choose a unit decomposition \( \sum_{i=1}^{k} \chi_i = 1 \) with some \( k \in \mathbb{N} \) such that \( \chi_1 \) is supported near \( X_c \) and \( b_0 \) vanishes on the support of \( \chi_k \). Moreover, the horizontal size of the supports for \( \chi_1, \ldots, \chi_{k-1} \) is a small constant \( \delta > 0 \) to be fixed later. As a result, we have the decomposition

\[
u = \sum_{i=1}^{k} \chi_i u := \sum_{i=1}^{k} u_i
\]

and we can prove the desired estimate for each \( u_i \) to close the proof. In fact, we only need to focus on the estimate for \( u_1 \), and the remaining parts will be similar and classical.

A direct computation leads to the system for \( u_1 \):

\[
\begin{cases}
\Delta u_1 = h + [\Delta, \chi_1]u, & \text{on } \Omega_t, \\
 u_1|_{\Gamma_t} = \chi_1 f, \\
 \nabla_{n_b} u_1 + b_0(X_c)\nabla_{n_b} u_1|_{\Gamma_b} = g_1
\end{cases}
\]

where

\[
g_1 = \chi_1 g + u\nabla_{n_b} \chi_1 + b_0 u\nabla_{n_b} \chi_1 + (b_0(X_c) - b_0)\nabla_{n_b} u_1|_{\Gamma_b}.
\]

In order to prove the estimate for \( u_1 \), we firstly deal with \( g_1 \) on \( \Gamma_b \) to arrive at

\[
\|g_1\|_{H^{3/2}(\Gamma_b)} \leq C\|g\|_{H^{3/2}(\Gamma_b)} + C(1 + \|b_0\|_{H^{3/2}(\Gamma_b)})\|u\|_{H^{5/2}(\Omega_t)} + \|b_0(X_c) - b_0\|_{L^\infty(\text{supp} \chi_1)}\|u_1\|_{H^3(\Omega_t}).
\]

Applying Theorem 5.1, we have

\[
\|u_1\|_{H^3(\Omega_t)} \leq C(\|\Gamma_t\|_{H^{3/2}}, \|\Gamma_b\|_{H^{3/2}}, \|b_0\|_{H^{3/2}(\Gamma_b)})(\|h\|_{H^1(\Omega_t)} + \|u\|_{H^{5/2}(\Omega_t)} + \|f\|_{H^{3/2}(\Gamma_t)} + \|g\|_{H^{3/2}(\Gamma_b)})
\]

\[+ \|b_0(X_c) - b_0\|_{L^\infty(\text{supp} \chi_1)}\|u_1\|_{H^3(\Omega_t)}.
\]

Consequently, when the horizontal size \( \delta \) of \( \text{supp} \chi_1 \) is small enough, we can find a small constant \( C_\delta \) such that

\[
\|b_0(X_c) - b_0\|_{L^\infty(\text{supp} \chi_1)} \leq C_\delta,
\]

which leads to the estimate for \( u_1 \):

\[
\|u_1\|_{H^3(\Omega_t)} \leq C(\|\Gamma_t\|_{H^{3/2}}, \|\Gamma_b\|_{H^{3/2}}, \|b_0\|_{H^{3/2}(\Gamma_b)})
\]

\[+ \|h\|_{H^1(\Omega_t)} + \|u\|_{H^{5/2}(\Omega_t)} + \|f\|_{H^{3/2}(\Gamma_t)} + \|g\|_{H^{3/2}(\Gamma_b)}.
\]

Moreover, \( \|u\|_{H^{3/2}(\Omega_t)} \) in the right side of the above inequality can be handled by an interpolation, Theorem 5.3, and the Poincaré inequality.

As mentioned above, the estimates for the remaining parts are similar, therefore our proof is finished. 

Except for the elliptic estimates above, we present here some useful trace theorems in the corner domain.
Theorem 5.3 (Traces on $\Gamma_l$ or $\Gamma_b$) Let $l < s - \frac{1}{2}$ with some $s > \frac{1}{2}$, we define the map:

$$u \to \{u, \partial_{n_j} u, \ldots \partial_{n_j}^l u\}|_{\Gamma_j},$$

for $u \in \mathcal{D}(\bar{\Omega}_b)$ and $n_j$ is the unit outward normal vector on $\Gamma_j$ where $\Gamma_j$ takes $\Gamma_b$ or $\Gamma_l$. Then, the map has a unique continuous extension as an operator from

$$H^s(\Omega_t) \text{ onto } H^{s-l-\frac{1}{2}}(\Gamma_l) \times H^{s-l-\frac{1}{2}}(\Gamma_b), \quad l \leq s - 1.$$ 

Moreover, one has the estimate

$$\|u\|_{H^{s-rac{1}{2}}(\Gamma_l)} + \|u\|_{H^{s-rac{1}{2}}(\Gamma_b)} \leq C(\|\Gamma_l\|_{H^{s-rac{1}{2}}})\|u\|_{H^s(\Omega_t)}.$$ 

Proof. This result is adjusted from Theorem 1.5.2.1 [15] and Remark 4.2 [27] by a cut-off function argument and interpolations.

Theorem 5.4 (Trace theorem with mixed boundary conditions) Let $2 \leq m \leq 4$ be an integer and functions $f \in H^{m-\frac{1}{2}}(\Gamma_l)$, $g \in H^{m-\frac{1}{2}}(\Gamma_b)$ be given. Then there exists a function $u \in H^m(\Omega_t)$ satisfying the following mixed boundary conditions

$$u|_{\Gamma_l} = f, \quad \partial_{n_0} u + b_0 \partial_{n_0} u|_{\Gamma_b} = g$$

where $b_0$ is a constant coefficient and the two vectos $n_b + b_0 \tau_b \parallel \tau_l$. Moreover, one has the estimate

$$\|u\|_{H^m(\Omega_t)} \leq C(\delta, \gamma, \|\Gamma_l\|_{H^{m-\frac{1}{2}}}(f) + \|g\|_{H^{m-\frac{1}{2}}(\Gamma_b)}),$$

and

$$\|u\|_{H^{m-rac{1}{2}}(\Gamma_b)} \leq C(\delta, \gamma, \|\Gamma_l\|_{H^{m-\frac{1}{2}}}(f) + \|g\|_{H^{m-1}(\Gamma_b)}).$$

Proof. The estimate (5.2) can be found directly from Theorem 4.6 [27]. For the estimate (5.3), it can be proved by a complex interpolation theorem. Besides, the condition $n_b + b_0 \tau_b \parallel \tau_l$ can always be satisfied in this paper.

Next, some special trace theorems involving $\vec{H}^{\frac{1}{2}}(\Gamma_b)$ and $\vec{H}^{-\frac{1}{2}}(\Gamma_l)$ are also needed in our paper.

Lemma 5.5 Assume that $u|_{\Gamma_l} = 0$ and let $f = u|_{\Gamma_b}$ for $u \in H^1(\Omega_t)$. Then the mapping $w \mapsto f$ is linear continuous from $H^1(\Omega_t)$ onto the subspace of $\vec{H}^{\frac{1}{2}}(\Gamma_b)$:

$$\|f\|_{\vec{H}^{\frac{1}{2}}(\Gamma_b)} \leq C(\|\Gamma_l\|_{H^{\frac{1}{2}}})\|u\|_{H^1(\Omega_t)}.$$ 

Proof. This is an adaption from Theorem 8.1 [27]. In fact, using a cut-off function $\chi$ near the contact point, one can prove the estimate for $\chi u$ by using Theorem 8.1 [27]:

$$\|\chi u\|_{\vec{H}^{\frac{1}{2}}(\Gamma_b)} \leq C(\|\Gamma_l\|_{H^{\frac{1}{2}}})\|\chi u\|_{H^1(\Omega_t)}.$$ 

The remainder part $(1 - \chi)u$ can be dealt with a classical trace theorem to have

$$\| (1 - \chi) f \|_{\vec{H}^{\frac{1}{2}}(\Gamma_b)} \leq C(\|\Gamma_l\|_{H^{\frac{1}{2}}})\|(1 - \chi)u\|_{H^1(\Omega_t)}.$$ 

Combining these two estimates, the proof is finished.
Lemma 5.6 If $u$ belongs to $H^{\frac{1}{2}}(\Gamma_b)$, then $\nabla_\tau b u$ belongs to $\tilde{H}^{-\frac{1}{2}}(\Gamma_b)$ and satisfies the estimate
\[ \| \nabla_\tau b u \|_{\tilde{H}^{-\frac{1}{2}}(\Gamma_b)} \leq C (\| \Gamma_t \|_{H^{\frac{5}{2}}} \| u \|_{H^{\frac{1}{2}}(\Gamma_b)}). \]

Proof. This lemma is an application of Theorem 8.2 [27] and the proof of Theorem 1.4.4.6 [15]. ■

Lemma 5.7 Let $u \in E(\Delta; L^2(\Omega_t)) = \{ u \in H^1(\Omega_t) | \Delta u \in L^2(\Omega_t) \}$. Then the mapping $u \mapsto \nabla_n b u |_{\Gamma_b}$ is continuous from $E(\Delta; L^2(\Omega_t))$ into $\tilde{H}^{-\frac{1}{2}}(\Gamma_b)$:
\[ \| \nabla_n b u \|_{\tilde{H}^{-\frac{1}{2}}(\Gamma_b)} \leq C (\| \Gamma_t \|_{H^{\frac{5}{2}}} (\| u \|_{H^1(\Omega_t)} + \| \Delta u \|_{L^2(\Omega_t)}). \]

Proof. Similarly as the previous lemmas, it can be proved directly from Theorem 1.5.3.10 [15]. ■

It turns out that the Sobolev’s embedding theorem also works on the corner domain, and we only pick up three cases here.

Lemma 5.8 We have the following embeddings:
\[ \| u \|_{L^4(\Omega_t)} \leq C (\| \Gamma_t \|_{H^{\frac{5}{2}}} \| u \|_{H^{\frac{1}{2}}(\Omega_t)} \]
for $u \in H^{\frac{1}{2}}(\Omega_t)$, and
\[ \| u \|_{L^\infty(\Omega_t)} \leq C (\| \Gamma_t \|_{H^{\frac{5}{2}}} \| u \|_{H^{s_1}(\Omega_t)} \]
for $u \in H^{s_1}(\Omega_t)$ with $s_1 = 1 + \epsilon$ ($\epsilon > 0$ is a small constant). Moreover, for $f \in H^{s_2}(\Gamma_b)$ with $s_2 = \frac{1}{2} + \epsilon$, the embedding holds:
\[ \| f \|_{L^\infty(\Gamma_b)} \leq C (\| \Gamma_t \|_{H^{\frac{5}{2}}} \| f \|_{H^{s_2}(\Gamma_b)}. \]

Proof. Using an extension theorem (for example Theorem 1.4.3.1 [15] and the classical Sobolev’s embedding theorem, this lemma can be proved. ■

Moreover, except for the trace theorems and embedding results above, we will meet with $H^1$-type elliptic estimates frequently in the energy estimates. The following estimate is the $H^1$ estimate related to the harmonic extension operator $\mathcal{H}$.

Lemma 5.9 Let $\mathcal{H}(f)$ be the harmonic extension of $f \in H^{\frac{1}{2}}(\Gamma_t)$:
\[ \begin{cases} \Delta \mathcal{H}(f) = 0, & \text{on } \Omega_t, \\ \mathcal{H}(f)|_{\Gamma_t} = f, & \nabla_n b \mathcal{H}(f)|_{\Gamma_b} = 0. \end{cases} \]

Then one has $\mathcal{H}(f) \in H^1(\Omega_t)$ and the following estimate
\[ \| \mathcal{H}(f) \|_{H^1(\Omega_t)} \leq C (\| \Gamma_t \|_{H^{\frac{5}{2}}} \| f \|_{H^{\frac{1}{2}}(\Gamma_t)}). \]
Proof. Step 1: Find \( u_1 \in H^1(\Omega_t) \) satisfying
\[
\begin{cases}
\Delta u_1 = 0, & \text{on } \Omega_t, \\
 u_1|_{\Gamma_t} = f.
\end{cases}
\]
In fact, the only boundary condition here is on \( \Gamma_t \), so one can extend \( \Gamma_t \) and \( f \) such that \( f^{ex} \) is defined on a horizontally infinite curve \( \Gamma_t^{ex} \). Moreover, one can also extend the domain \( \Omega_t \) to a horizontally infinite strip \( \Omega_t^{ex} \) as in [25] and pose zero Dirichlet boundary condition on the bottom if necessary. Therefore, the system for \( u_1 \) can be extended to a system for \( u_1^{ex} \) on \( \Omega_t^{ex} \). Applying Lemma 2.12 [25] leads to the variational estimate
\[
\|u_1^{ex}\|_{H^1(\Omega_t^{ex})} \leq C(\|\Gamma_t\|_{H^{3/2}_2(\Gamma_t^{ex})}) \|f^{ex}\|_{H^{3/2}_2(\Gamma_t^{ex})},
\]
which infers that \( u_1 = u_1^{ex}|_{\Omega_t} \) solves the system above and
\[
\|u_1\|_{H^1(\Omega_t)} \leq C(\|\Gamma_t\|_{H^{3/2}_2(\Gamma)}) \|f\|_{H^{3/2}_2(\Gamma_t)}. \]

Step 2: Define \( u_2 \) by the system
\[
\begin{cases}
\Delta u_2 = 0, & \text{on } \Omega_t, \\
u_2|_{\Gamma_t} = 0, & \nabla n_b u_2|_{\Gamma_b} = -\nabla n_b u_1|_{\Gamma_b}.
\end{cases}
\]
According to Lemma 5.7, we know that \( \nabla n_B u_1|_{\Gamma_b} \in \tilde{H}^{-1/2}(\Gamma_b) \), so the boundary conditions make sense. The variation equation for \( u_2 \) is
\[
\int_{\Omega_t} \nabla u_2 \cdot \nabla \phi dX = -\int_{\Gamma_b} \nabla n_B u_1 \phi ds
\]
with any \( \phi \in V = \{ \phi \in H^1(\Omega_t) \mid \phi|_{\Gamma_t} = 0 \} \). From Lemma 5.5, we know that \( \phi \in \tilde{H}^{3/2}(\Gamma_b) \), so the right-hand side of the variation equation admits the estimate
\[
\|\int_{\Gamma_b} \nabla n_B u_1 \phi ds\| \leq \|\nabla n_B u_1\|_{\tilde{H}^{-1/2}(\Gamma_b)} \|\phi\|_{\tilde{H}^{3/2}(\Gamma_b)} \leq C(\|\Gamma_t\|_{H^{3/2}_2(\Gamma)}) \|u_1\|_{H^1(\Omega_t)} \|\phi\|_{H^1(\Omega_t)}.
\]
As a result, there exists a unique solution \( u_2 \) solving the variational system with the estimate
\[
\|u_2\|_{H^1(\Omega_t)} \leq C(\|\Gamma_t\|_{H^{3/2}_2(\Gamma)}) \|u_1\|_{H^1(\Omega_t)} \leq C(\|\Gamma_t\|_{H^{3/2}_2(\Gamma)}) \|f\|_{H^{3/2}_2(\Gamma_t)},
\]
where the Poincaré inequality is applied.

Step 3: Summing the first two steps together and let
\[
\mathcal{H}(f) = u_1 + u_2,
\]
we conclude that \( \mathcal{H}(f) \in H^1(\Omega) \) solves system (5.4) and the desired estimate holds. \( \blacksquare \)

In the end of this section, we consider elliptic estimates for a Neumann-type system
\[
\begin{cases}
\Delta u = h, & \text{on } \Omega_t, \\
\nabla n_B u|_{\Gamma_b} = f, & \partial_B u|_{\Gamma_b} = g.
\end{cases}
\]
satisfying the condition
\[ \int_{\Omega_t} h = \int_{\Gamma_t} f + \int_{\Gamma_b} g. \]

One can check from [27] with the Neumann conditions above and conclude the following result for system (5.5). The proof is similar as that for Proposition 5.19 [27] with no singularity and hence is omitted.

**Theorem 5.10** Let the contact angle \( \omega(t) < \pi/4 \) and \( u \in H^{\frac{5}{2}}(\Omega_t) \) be a solution for system (5.5). Then we have the estimate
\[ \| u \|_{H^{\frac{5}{2}}(\Omega_t)} \leq C(\| \Gamma_t \|_{H^{\frac{5}{2}}}) (\| f \|_{H^1(\Gamma_t)} + \| g \|_{H^1(\Gamma_b)}). \]

6 New formulation for \( J \)

In this section, we will derive the equation of a good unknown \( J = \nabla\kappa_H \) which is introduced in [36]. This quantity is indeed the main part in the pressure term \( \nabla P \) from the Euler equation by recalling that
\[ \nabla P = J + \nabla P_{v,v}. \]
with \( P_{v,v} \) the lower-order term. To find out the equation for \( J \), we need some expansions of some commutators. The computations here follow the formulation of Shatah-Zeng, see [36] (some expressions are quoted from there directly). In order to be self-contained, we will also recall some details from their work.

Before the commutators, we recall the following computation directly from [36]:
\[ D_t n_t = -((\nabla v)^* n_t)^\top \quad \text{on} \quad \Gamma_t. \]
Moreover, since our domain is two dimensional in this paper, we know directly that \( \tau_t \) is the parallel-transporting tangent basis satisfies
\[ D_t \tau_t = (\nabla \tau_t \cdot n_t)n_t \quad \text{and} \quad D_{\tau_t} \tau_t = 0 \quad \text{on} \quad \Gamma_t. \]
These expressions will be used repeatedly in the following sections.

6.1 Commutators and computations about \( \kappa \)

1. \([D_t, H]\). For a given function \( f \) on \( \Gamma_t \), recall that the harmonic extension \( Hf \) or equivalently \( f_H \) on \( \Omega_t \) is defined by the elliptic system
\[
\begin{cases}
\Delta f_H = 0 & \text{on } \Omega_t \\
 f_H|_{\Gamma_t} = f, \quad \nabla n_b f_H|_{\Gamma_b} = 0.
\end{cases}
\]
Recall also the definition of \( \Delta^{-1}(h, g) \) as the solution of the system
\[
\begin{cases}
\Delta u = h & \text{on } \Omega \\
u|_{\Gamma_t} = 0, \quad \nabla n_b u|_{\Gamma_b} = g.
\end{cases} \quad (6.1)
\]
In order to analyze the commutator, we start with the elliptic system of $D_tf_H$. In fact, it satisfies the equation

$$\Delta D_tf_H = [\Delta, D_t]f_H$$

and the boundary condition on $\Gamma_b$:

$$\nabla_{n_b}D_tf_H|_{\Gamma_b} = [\nabla_{n_b}, D_t]f_H|_{\Gamma_b}.$$ 

In a word, direct computations shows that $D_tf_H$ solves the system

$$\begin{cases}
\Delta D_tf_H = 2\nabla v \cdot \nabla^2 f_H + \Delta v \cdot \nabla f_H & \text{on } \Omega \\
D_tf_H|_{\Gamma_t} = D_t f, & \nabla_{n_b}D_tf_H|_{\Gamma_b} = (\nabla_{n_b}v - \nabla vn_b) \cdot \nabla f_H|_{\Gamma_b}.
\end{cases}$$

As a result, we arrive at

$$D_tf_H = \mathcal{H}(D_tf) + \Delta^{-1}(2\nabla v \cdot \nabla^2 f_H + \Delta v \cdot \nabla f_H, (\nabla_{n_b}v - \nabla vn_b) \cdot \nabla f_H) \quad (6.2)$$

2. $[D_t, \Delta^{-1}]$. Denoting $u = \Delta^{-1}(h, g)$, we know that $u$ solves system $[6.1]$. To compute this commutator, we investigate the system for $D_t u$.

In fact, direct computations shows that $D_t u$ satisfies

$$\begin{cases}
\Delta D_t u = D_t h + 2\nabla v \cdot \nabla^2 u + \Delta v \cdot \nabla u & \text{on } \Omega \\
D_t u|_{\Gamma_t} = 0, & \nabla_{n_b}D_t u|_{\Gamma_b} = D_t g + (\nabla_{n_b}v - \nabla vn_b) \cdot \nabla u|_{\Gamma_b}.
\end{cases}$$

Consequently, by using a simple decomposition we conclude that

$$D_t \Delta^{-1}(h, g) = \Delta^{-1}(D_th, D_t g) + \Delta^{-1}(2\nabla v \cdot \nabla^2 \Delta^{-1}(h, g) + \Delta v \cdot \nabla \Delta^{-1}(h, g), (\nabla_{n_b}v - \nabla vn_b) \cdot \nabla \Delta^{-1}(h, g)). \quad (6.3)$$

3. $[D_t, \mathcal{N}]$. Recalling the definition of the Dirichlet-Neumann operator that

$$\mathcal{N} f := \nabla_{n_t} f_H \quad \text{on } \Gamma_t,$$

for a given function $f$ on $\Gamma_t$. We know directly from Theorem 1.2 [27] that

$$[D_t, \mathcal{N}] f = \nabla_{n_t} \Delta^{-1}(2\nabla v \cdot \nabla^2 f_H + \Delta v \cdot \nabla f_H, (\nabla_{n_b}v - \nabla vn_b) \cdot \nabla f_H)$$

$$- \nabla_{n_t}v \cdot \nabla f_H - \nabla(\nabla f_H)\nabla v \cdot n_t, \quad \text{on } \Gamma_t. \quad (6.4)$$

4. $[D_t, \Delta_{\Gamma_t}]$. One can have by a direct computation that

$$D_t \Delta_{\Gamma_t} f = \Delta_{\Gamma_t} D_t f + 2\nabla^2 f(\tau_t, (\nabla_{\tau_t} v)\nabla) - \nabla f \cdot \Delta_{\Gamma_t} v + \kappa \nabla(\nabla f)\nabla v \cdot n_t \quad \text{on } \Gamma_t. \quad (6.5)$$

5. $D_t \kappa$. Direct computations lead to the following expression

$$D_t \kappa = -\Delta_{\Gamma_t} v^\perp - v^\perp|\Pi|^2 + \mathcal{D} \cdot \Pi(v^\perp) \quad (6.6)$$

or equivalently

$$D_t \kappa = -\Delta_{\Gamma_t} v \cdot n_t - 2\Pi(\tau_t) \cdot \nabla_{\tau_t} v \quad \text{on } \Gamma_t. \quad (6.7)$$
6. $D_t^2 \kappa$. We have

$$D_t^2 \kappa = - n_t \cdot \Delta \Gamma_t D_t v + 2 \sigma \Pi \cdot (D^T J) + R_1 \quad \text{on} \quad \Gamma_t$$

(6.8)

with

$$R_1 = 2 \left[ D ((\nabla v)^* n_t)^\top + \Pi ((D^T v)^\top) \right] \cdot D^T v + \Delta \Gamma_t v \cdot ((\nabla v)^* n_t)^\top + 2 \Pi \cdot ((\nabla v)^2)^\top
- 2 (D^T v \cdot n_t) (\Pi(\cdot) \cdot \nabla n_t v + \Pi(\cdot) \cdot (\nabla v)^* n_t) - 2 n_t \cdot D^2 v(\tau_t, (\nabla \tau_t v)^\top)
- 2 (\nabla n_t v \cdot n_t) (\Pi \cdot D^T v) + n_t \cdot \nabla v ((\Delta \Gamma_t v)^\top) - \kappa |((\nabla v)^* n_t)^\top|^2 + 2 \Pi \cdot (D^T \nabla P_{e,v})$$

So one can see that the leading-order terms in $R_1$ are like $\nabla^2 v, \nabla n_t$.

6.2 Equations for $J$

With the preparations above, we are ready to derive the equations for $J$. In the following analysis, the idea is trying to express $D_t J, D_t^2 J$ using $v, D_t v, J$ and $D_t J$ instead of $\kappa, D_t \kappa$ or $D_t^2 \kappa$.

To get started, one recalls the definition $J = \nabla \kappa H$ and applies $D_t$ to find

$$D_t J = D_t \nabla \kappa H = \nabla D_t \kappa H - (\nabla v)^* J
= \nabla H(D_t \kappa) + [D_t, \mathcal{H}] \kappa - (\nabla v)^* J$$

where we used the commutator

$$[D_t, \nabla] = -(\nabla v)^* \nabla.$$

Plugging (6.7) and (6.2) into the equation above, we can find the equation for $D_t J$:

$$D_t J = \nabla H(D_t \kappa) + \nabla \Delta^{-1} (2 \nabla v \cdot \nabla J + \Delta v \cdot J, (\nabla n_b v - \nabla v n_b) \cdot J) - (\nabla v)^* J
= \nabla H(D_t \kappa) + \nabla \Delta^{-1} (2 \nabla v \cdot \nabla J + \Delta v \cdot J, (\nabla n_b v - \nabla v n_b) \cdot J)
- (\nabla v)^* J$$

(6.9)

Secondly, to derive the equation for $D_t^2 J$, we start with

$$D_t^2 J = D_t^2 \nabla \kappa H = D_t (\nabla D_t \kappa H + [D_t, \nabla] \kappa H)
= \nabla D_t^2 \kappa H - (\nabla v)^* \nabla D_t \kappa H - (D_t \nabla v)^* J - (\nabla v)^* D_t J
= \nabla D_t^2 \kappa H - 2 (\nabla v)^* D_t J - (\nabla D_t v)^* J - ((\nabla v)^2)^* J + (\nabla v)^* \nabla v J$$

where

$$\nabla D_t \kappa H = D_t J + (\nabla v)^* J$$

and

$$(D_t \nabla v)^* J = (\nabla D_t v)^* J - ((\nabla v)^2)^* J.$$
where we applied (6.2) repeatedly. Plugging this expression above into the equation for $D_t^2 J$, we arrive at

$$D_t^2 J = \nabla \mathcal{H}(D_t^2 \kappa) + A_1 + A_2 + A_3$$

(6.10)

where

$$A_1 = \nabla[D_t, \mathcal{H}]D_t \kappa, \quad A_2 = \nabla D_t \Delta^{-1} (2\nabla v \cdot \nabla J + \Delta v \cdot J, (\nabla_{n_b} v - \nabla_{v n_b}) \cdot J)$$

and

$$A_3 = -2(\nabla v)^* D_t J - (\nabla D_t v)^* J - ((\nabla v)^2)^* J + (\nabla v)^* \nabla v J.$$

Applying (6.2) and (6.9) we find that

$$w = \Delta^{-1} (2\nabla v \cdot \nabla \mathcal{H}(D_t \kappa) + \Delta v \cdot \nabla \mathcal{H}(D_t \kappa), (\nabla_{n_b} v - \nabla_{v n_b}) \cdot \nabla \mathcal{H}(D_t \kappa)).$$

(6.11)

On the other hand, applying (6.3) to $A_2$ we have

$$A_2 = \nabla \Delta^{-1} (2\nabla v \cdot \nabla^2 w_{A_2} + \Delta v \cdot \nabla w_{A_2}, (\nabla_{n_b} v - \nabla_{v n_b}) \cdot \nabla w_{A_2}) + \nabla \Delta^{-1} (h_{A_2}, g_{A_2})$$

(6.12)

where

$$w_{A_2} = \Delta^{-1} (2\nabla v \cdot \nabla J + \Delta v \cdot J, (\nabla_{n_b} v - \nabla_{v n_b}) \cdot J),$$

$$h_{A_2} = 2\nabla v \cdot (\nabla D_t J - (\nabla v)^* J) + 2(\nabla D_t v - (\nabla v)^* \nabla v \cdot J + D_t J \cdot \Delta v$$

$$+ J \cdot (\Delta D_t v - \Delta v \cdot \nabla v - 2\nabla v \cdot \nabla^2 v),$$

$$g_{A_2} = (\nabla_{n_b} v - \nabla_{v n_b}) \cdot \nabla v \cdot J + \nabla_{n_b} D_t \cdot J - (D_t v - \nabla v) \cdot \nabla_{n_b} \cdot J$$

$$+ \nabla v ((\nabla v)^* n_b) ^\top \cdot J + (\nabla_{n_b} v - \nabla_{v n_b}) \cdot D_t J.$$

We can see the leading-order terms in $A_1, A_2$ are like $J, D_t J, \nabla v, \nabla D_t v$.

It remains to rewrite $\nabla \mathcal{H}(D_t^2 \kappa)$ in (6.10). In fact, substituting (6.8) into this term, one has

$$\nabla \mathcal{H}(D_t^2 \kappa) = \nabla \mathcal{H}(\sigma n_t \cdot \Delta \Gamma, D_t v + 2\sigma \Pi \cdot (D^\top J) + R_1)$$

$$= \nabla \mathcal{H}(\sigma n_t \cdot \Delta \Gamma, J + n_t \cdot \Delta \Gamma, \nabla P_{v,v} + 2\sigma \Pi \cdot (D^\top J) + R_1)$$

where the Euler equation is applied. Moreover, direct computations show that

$$2\sigma \Pi \cdot D^\top J = 2\sigma \nabla_{\tau t} n_t \cdot \nabla_{\tau t} J = \sigma \Delta \Gamma, J_{\perp} - \sigma \Delta \Gamma, n_t \cdot J - \sigma n_t \cdot \Delta \Gamma, J,$$

which leads to

$$\nabla \mathcal{H}(D_t^2 \kappa) = \sigma \nabla \mathcal{H}(\Delta \Gamma, J_{\perp}) - \sigma \nabla \mathcal{H}(\Delta \Gamma, n_t \cdot J) + \nabla \mathcal{H}(n_t \cdot \Delta \Gamma, \nabla P_{v,v}) + \nabla \mathcal{H}(R_1).$$
Therefore, plugging the expression above into (6.10), we finally conclude that

\[ D_t^2 J = \sigma \nabla H(\Delta_{\Gamma_t} J^\perp) + R_0. \]  

(6.13)

with

\[ R_0 = -\sigma \nabla H(J \cdot \Delta_{\Gamma_t} n_t) + \nabla H(n_t \cdot \Delta_{\Gamma_t} \nabla P_{J,v}) + \nabla H(R_1) + A_1 + A_2 + A_3. \]

Based on the equations for \(D_t J\) and \(D_t^2 J\), we will also find the equation for \(D_t J\) for the energy estimate.

In fact, we know from the Hodge decomposition that

\[ D_t J = D_t J + \nabla P_{J,v} \]

where \(P_{J,v}\) satisfies the system

\[
\begin{cases}
\Delta P_{J,v} = -tr(\nabla J \nabla v), & \text{on } \Omega_t \\
P_{J,v}|_{\Gamma_t} = 0, & \nabla n_b P_{J,v}|_{\Gamma_b} = \nabla v n_b \cdot J.
\end{cases}
\]

Similarly, apply the Hodge decomposition on \(D_t^2 J\) to find that

\[ D_t D_t J = D_t(D_t J + \nabla P_{J,v}) = D_t^2 J + D_t \nabla P_{J,v}. \]

Substituting (6.13) into the equation above, we conclude the equation for \(D_t J\):

\[ D_t D_t J + \sigma A J = R \]  

(6.14)

where the operator \(A\) is defined by

\[ A(w) = \nabla H(-\Delta_{\Gamma_t}(w|_{\Gamma_t})^\perp) \]

and the remainder term \(R\) is

\[ R = R_0 + D_t \nabla P_{J,v}. \]

To close this section, we consider about \(D_t \nabla P_{J,v}\) in \(R\). Indeed, direct computations and (6.3) lead to the expression for \(D_t \nabla P_{J,v}\), and the details are omitted here.

## 7 A priori estimates

We are going to prove Theorem 1.1 in this section. The idea here is firstly to define an energy together with a dissipation for \(J\), and after we finish the energy estimate for \(J\), we will go back to the estimate for \(v\) and \(\Gamma_t\) using Proposition 7.1.

Besides, one can see that the pressure \(P\) is decomposed into

\[ P = P_{J,v} + \kappa_{\mathcal{H}}, \]

therefore, the estimate for the pressure is determined by the estimates for \(v, \Gamma_t\).

To get started, we recall the energy functional

\[ E(t) = \|\nabla_{\tau_t} J^\perp\|^2_{L^2(\Gamma_t)} + \|D_t J\|^2_{L^2(\Omega_t)} + \|\Gamma_t\|^2_H/2 + \|v\|^2_{L^2(\Omega_t)}, \]

and the dissipation

\[ F(t) = |\sin \omega \nabla_{\tau_t} J^\perp|_{X_e}|^2. \]

Since the higher-order part from the energy \(E(t)\) and the dissipation \(F(t)\) above relate with \(J\), we need to consider the relationship between \(v, \Gamma_t\) and \(E(t), F(t)\). The following proposition shows that \(v, n_t\) can be controlled by \(E(t)\).
Proposition 7.1 Let $\Gamma_t \in H^1$, $v \in H^3(\Omega_t)$ and $\omega < \frac{c}{b}$, then we have
\[
\|v\|_{H^3(\Omega_t)} \leq P(E(t)),
\] (7.1)
and
\[
\|n_t\|_{H^3(\Gamma_t)} \leq P(E(t)).
\] (7.2)

Proof. Step 1: estimates for $n_t$. To estimate $n_t$, we will start with $\kappa_{H}$. Recalling that
\[
k = tr \Pi = \nabla_{\tau_t} n_t \cdot \tau_t \quad \text{with} \quad \nabla_{\tau_t} n_t \parallel \tau_t \quad \text{on} \quad \Gamma_t,
\]
one can see that the higher-order estimate of $D$ we can also derive the estimate for $n$. Moreover, since $\delta$
and $\kappa$
Plugging (7.4) into (7.3) and choosing a suitable $\delta$
which together with (7.5) implies that
As a result, we arrive at by Theorem 5.3
Moreover, we derive from Lemma 5.9 that
which together with (7.5) implies that
As a result, we arrive at by Theorem 5.3
Moreover, since
\[
D_t J = D_t J + \nabla P_{J,v},
\]
we can also derive the estimate for $D_t J$ from (7.6):
\[
\|D_t J\|_{L^2(\Omega_t)} \leq \|D_t J\|_{L^2(\Omega_t)} + \|P_{J,v}\|_{H^2(\Omega_t)} \leq P(E(t))(1 + \|v\|_{H^2(\Omega_t)}),
\] (7.7)
where we applied Theorem 5.1 on $\|P_J,v\|_{H^2(\Omega_t)}$.

Step 2: estimates for $v$. Recall from (WW) that the velocity $v$ satisfies

$$\begin{cases} \\
\Delta v = 0 & \text{on } \Omega_t, \\
v \cdot n_b|_{\Gamma_b} = 0 \quad (7.8)
\end{cases}$$

As a result, in order to have estimates for $v$, a natural way is to find some condition on $\Gamma_t$ for $v$. We choose to consider Neumann condition on the free surface. As long as we establish the estimate for this condition, we can finish the estimate for $v$ using the elliptic system above.

To begin with, we denote the outward normal derivative of $v$ on $\Gamma_t$ by $\nu = \nabla n_t v$.

In order to estimate $\nu$, we plan to deal with its two components $\nu^\top$ and $\nu^\perp$ respectively.

(i) The estimate for $\nu^\top$. This part follows the proof of Proposition 4.3 [36] and hence some details are omitted. By a direct calculation, we have

$$\Delta_{\Gamma_t} \nu^\top = \nabla n_t (D \cdot \nu^\top) n_t,$$

which results in the following estimate

$$\|\nu^\top\|_{H^3(\Gamma_t)} \leq C(\|\Gamma_t\|_{H^5(\Gamma_t)} \|D \cdot \nu^\top\|_{H^\frac{5}{2}(\Gamma_t)}). \quad (7.9)$$

Next, we give the estimates of $D \cdot \nu^\top$. From the definition of $\nu$, we can write

$$D \cdot \nu^\top = \Delta_{\Gamma_t} v \cdot n_t + \Pi \cdot D^\top v, \quad \text{on } \Gamma_t$$

where the second term of right hand can be controlled by

$$\|\Pi \cdot D^\top v\|_{H^\frac{5}{2}(\Gamma_t)} \leq C(\|\Gamma_t\|_{H^5(\Gamma_t)} \|n_t\|_{H^\frac{5}{2}(\Gamma_t)} \|v\|_{H^\frac{5}{2}(\Omega_t)}) \leq C(\|\Gamma_t\|_{H^5(\Omega_t)} \|v\|_{H^\frac{5}{2}(\Omega_t)}). \quad (7.10)$$

On the other hand, for the first term $\Delta_{\Gamma_t} v \cdot n_t$, recalling equation (6.9) for $D_t J$ and using (7.6) to find that

$$\|D_t J + \nabla \mathcal{H}(\Delta_{\Gamma_t} v \cdot n_t)\|_{L^2(\Omega_t)} \leq P(E(t)) \|v\|_{H^\frac{5}{2}(\Omega_t)},$$

and moving $D_t J$ to the right side and applying (7.7) and the Poincaré inequality lead to

$$\|\mathcal{H}(\Delta_{\Gamma_t} v \cdot n_t)\|_{H^1(\Omega_t)} \leq P(E(t))(1 + \|v\|_{H^\frac{5}{2}(\Omega_t)}). \quad (7.11)$$

As a result, combining (7.10) and (7.11) and Theorem 5.3 to get

$$\|D \cdot \nu^\top\|_{H^\frac{5}{2}(\Gamma_t)} \leq P(E(t))(1 + \|v\|_{H^\frac{5}{2}(\Omega_t)}),$$

and furthermore plugging the above estimates into (7.9), we finally obtain

$$\|\nu^\top\|_{H^\frac{5}{2}(\Gamma_t)} \leq P(E(t))(1 + \|v\|_{H^\frac{5}{2}(\Omega_t)}). \quad (7.12)$$

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(ii) The estimate for $\nu^\perp$. Since

$$\nu^\perp = \nabla n_t v \cdot n_t = \nabla n_t (v \cdot \tilde{n}_t) - v \cdot \nabla n_t \tilde{n}_t \quad \text{on} \quad \Gamma_t$$  \hspace{1cm} (7.13)

for some extension $\tilde{n}_t$ of $n_t$ on $\Omega_t$, the idea in this part is to deal with $\nu^\perp := v \cdot \tilde{n}_t$ first and then go back to $\nu^\perp$ (the extension $\tilde{n}_t$ will be defined later). Again, the estimate for $\nu^\perp$ relies on an elliptic system with boundary conditions.

Firstly, we look for the estimate for $\nu^\perp$ on $\Gamma_t$. In fact, recalling equation (6.6) for $\kappa$ to get that

$$\|D_t \kappa\|_{L^2(\Gamma_t)} \leq C(\|\Gamma_t\|_{H^\frac{5}{2}(\Omega_t)}) \|v\|_{H^\frac{5}{2}(\Omega_t)}$$  \hspace{1cm} (7.14)

and moreover

$$\|\Delta \Gamma_t \nu^\perp\|_{H^\frac{1}{2}(\Gamma_t)} \leq C(\|\Gamma_t\|_{H^\frac{5}{2}(\Gamma_t)})(\|D_t \kappa\|_{H^\frac{5}{2}(\Gamma_t)} + \|v\|_{H^\frac{5}{2}(\Omega_t)})$$

which implies that

$$\|\nu^\perp\|_{H^\frac{5}{2}(\Gamma_t)} \leq C(\|\Gamma_t\|_{H^\frac{5}{2}(\Gamma_t)})(\|D_t \kappa\|_{H^\frac{5}{2}(\Gamma_t)} + \|v\|_{H^\frac{5}{2}(\Omega_t)}).$$  \hspace{1cm} (7.15)

To close the estimate above, we need to find the estimate for $\|D_t \kappa\|_{H^\frac{5}{2}(\Gamma_t)}$. In fact, we notice that

$$D_t J = D_t \nabla \kappa_H = \nabla D_t \kappa_H - (\nabla v)^* \nabla \kappa_H$$

which leads to

$$\|\nabla D_t \kappa_H\|_{L^2(\Omega_t)} \leq C(\|\nabla v\|_{L^2(\Omega_t)}) \|D_t \kappa\|_{L^2(\Gamma_t)} + \|D_t J\|_{L^2(\Gamma_t)}.$$

On the other hand, for any two points $X_1 \in \Omega_t$ and $X_2 \in \Gamma_t$ where $X_1$, $X_2$ stay in a vertical line, we can write

$$D_t \kappa_H(X_1) = D_t \kappa(X_2) + \int_{X_1}^{X_2} dD_t \kappa_H,$$

so combining the estimate above for $\|\nabla D_t \kappa_H\|_{L^2(\Omega_t)}$ and remembering that our domain $\Omega_t$ has finite depth, we arrive at

$$\|D_t \kappa_H\|_{H^1(\Omega_t)} \leq C(\|\nabla v\|_{L^2(\Omega_t)}) \|D_t \kappa\|_{L^2(\Gamma_t)} + \|D_t J\|_{L^2(\Gamma_t)} + \|D_t \kappa\|_{L^2(\Gamma_t)}).$$  \hspace{1cm} (7.16)

Moreover, using Theorem 5.3, (7.17) and the estimate for $\kappa_H$ in Step 1 to derive from (7.16) that

$$\|D_t \kappa\|_{H^\frac{5}{2}(\Gamma_t)} \leq P(E(t))(1 + \|v\|_{H^\frac{5}{2}(\Omega_t)}) + \|D_t \kappa\|_{L^2(\Gamma_t)}).$$

Consequently, plugging the estimate above and (7.14) into (7.15) to obtain the estimate for $\nu^\perp$ on $\Gamma_t$:

$$\|\nu^\perp\|_{H^\frac{5}{2}(\Gamma_t)} \leq P(E(t))(1 + \|v\|_{H^\frac{5}{2}(\Omega_t)}).$$  \hspace{1cm} (7.17)
Next, to prove the estimate from the elliptic equation for $v^\perp$, we need also to find the condition for $v^\perp$ on the bottom, $\Gamma_b$. To begin with, we define the extension of $n_t$. In fact, one can define Dirichlet boundaries as

$$\tilde{n}_t|_{\Gamma_c} = n_t, \quad \text{and} \quad \tilde{n}_t = -\tau_b, \quad \text{away from } X_c \text{ on } \Gamma_t,$$

and apply a Dirichlet-type trace theorem (for example Theorem 4.7 [27]) to obtain $\tilde{n}_t$ defined on $\Omega_t$ satisfying

$$\|\tilde{n}_t\|_{H^3(\Omega_t)} \leq C\left(\|\Gamma_t\|_{H^{3/2}(\Gamma_t)} + \|\tau_b\|_{H^{3/2}(\Gamma_b)}\right).$$

Now we can use the following condition

$$\nabla \cdot v = 0, \quad \text{and} \quad \nabla \times v = 0,$$

to find directly that

$$\nabla_{n_b} v^\perp = \nabla_{n_b} \left((\tilde{n}_t \cdot \tilde{\tau}_b)v \cdot \tilde{\tau}_b + (\tilde{n}_t \cdot \tilde{n}_b)v \cdot \tilde{n}_b\right)$$

$$= - (\tilde{n}_t \cdot n_b) \nabla_{\tau_b} (v \cdot \tau_b) + r_\perp, \quad \text{on } \Gamma_b,$$

with

$$r_\perp = - (\tilde{n}_t \cdot \tilde{\tau}_b)v \cdot (\nabla_{\tau_b} n_b - \nabla_{n_b} \tilde{\tau}_b) + (\tilde{n}_t \cdot n_b)v \cdot (\nabla_{\tau_b} \tilde{\tau}_b + \nabla_{n_b} \tilde{n}_b)$$

$$+ \nabla_{n_b} (\tilde{n}_t \cdot \tilde{\tau}_b)v \cdot \tau_b + \nabla_{n_b} (\tilde{n}_t \cdot \tilde{n}_b)v \cdot n_b, \quad \text{on } \Gamma_b$$

where $\tilde{n}_b$, $\tilde{\tau}_b$ are unit orthogonal extensions for $n_b$, $\tau_b$. On the other hand, since $v \cdot n_b|_{\Gamma_b} = 0$, we can arrive at the following oblique condition

$$\nabla_{n_b} v^\perp + b_\perp \nabla_{\tau_b} v^\perp = R_\perp, \quad \text{on } \Gamma_b,$$

with

$$b_\perp = \frac{\tilde{n}_t \cdot n_b}{\tilde{n}_t \cdot \tau_b}, \quad R_\perp = r_\perp - b_\perp \nabla_{\tau_b} (\tilde{n}_t \cdot \tau_b)v \cdot \tau_b.$$

Summing up the boundary conditions above, we finally conclude the following elliptic system for $v^\perp$:

$$\begin{cases}
\Delta v^\perp = 2\nabla v \cdot \nabla \tilde{n}_t + v \cdot \Delta \tilde{n}_t, & \text{on } \Omega_t, \\
v^\perp|_{\Gamma_t} \in H^{3/2}(\Gamma_t), \quad \nabla_{n_b} v^\perp + b_\perp \nabla_{\tau_b} v^\perp|_{\Gamma_b} = R_\perp|_{\Gamma_b}.
\end{cases}$$

As a result, based on the elliptic system for $v^\perp$ above, we are ready to find the estimate for $v^\perp$. Since the coefficient $b_\perp$ is a function with bounded support near the contact point, Lemma 5.2 is applied here to have

$$\|v^\perp\|_{H^3(\Omega_t)} \leq P(E(t))(1 + \|v\|_{H^{3/2}(\Omega_t)}),$$

which together with (7.13) implies that

$$\|v^\perp\|_{H^{3/2}(\Gamma_t)} \leq P(E(t))(1 + \|v\|_{H^{3/2}(\Omega_t)}).$$

(7.18)

(iii) The estimate for $v$. Firstly, combing the estimates (7.12) and (7.18) one can conclude that

$$\|\nabla_{n_t} v\|_{H^{3/2}(\Gamma_t)} = \|v\|_{H^{3/2}(\Omega_t)} \leq P(E(t))(1 + \|v\|_{H^{3/2}(\Omega_t)}).$$

(7.19)
In order to prove the estimate for \( v \), we need to check again on system (7.8). We can see that the condition on \( \Gamma_b \) is \( v \cdot n_b = 0 \), which is the condition for \( v \cdot n_b \) instead of \( v \). Therefore, we will derive estimates for \( v \cdot \tilde{n}_b \) and then \( v \cdot \tilde{\tau}_b \) to close the estimate.

To start with, we write down directly the system for \( v \cdot \tilde{n}_b \) as

\[
\begin{aligned}
\Delta(v \cdot \tilde{n}_b) &= [\Delta, \tilde{n}_b]v, & \text{on } \Omega_t, \\
\nabla_{n_b}(v \cdot \tilde{n}_b)|_{\Gamma_t} &\in H^{\frac{3}{2}}(\Gamma_t), & v \cdot n_b|_{\Gamma_b} = 0.
\end{aligned}
\]

where the Neumann condition on the free surface is obtained easily from (7.19). Besides, one can see that this system is again a mixed-boundary problem, although it is slightly different from system (5.1) in Section 5 by switching the two boundary conditions. The elliptic estimate for this system turns out to be similar as in Theorem 5.1. As a result, we arrive at

\[
\|v \cdot \tilde{n}_b\|_{H^3(\Omega_t)} \leq C(\|\Gamma_t\|_{H^{\frac{5}{2}}})(\|\Delta, \tilde{n}_b\|_{H^1(\Omega_t)} + \|\nabla_{n_b}(v \cdot \tilde{n}_b)\|_{H^{\frac{3}{2}}(\Gamma_t)})
\]

\[
\leq P(E(t))(1 + \|v\|_{H^{\frac{3}{2}}(\Omega_t)}),
\]

where we used (7.19).

Next, we will retrieve \( v \cdot \tilde{\tau}_b \) from \( v \cdot \tilde{n}_b \). By the divergence free and the curl free conditions for \( v \), it is straightforward to get that

\[
\begin{aligned}
\nabla_{\tilde{\tau}_b}(v \cdot \tilde{n}_b) &= -\nabla_{\tilde{n}_b}(v \cdot \tilde{n}_b) + v \cdot (\nabla_{\tilde{n}_b} \tilde{\tau}_b + \nabla_{\tilde{\tau}_b} \tilde{n}_b), & \text{and} \\
\nabla_{\tilde{n}_b}(v \cdot \tilde{\tau}_b) &= \nabla_{\tilde{n}_b}(v \cdot \tilde{n}_b) - v \cdot (\nabla_{\tilde{n}_b} \tilde{n}_b - \nabla_{\tilde{n}_b} \tilde{\tau}_b),
\end{aligned}
\]

Therefore we can have

\[
\|v \cdot \tilde{\tau}_b\|_{H^3(\Omega_t)} \leq P(E(t))(1 + \|v\|_{H^{\frac{3}{2}}(\Omega_t)}),
\]

which implies that

\[
\|v\|_{H^3(\Omega_t)} \leq P(E(t))(1 + \|v\|_{H^{\frac{3}{2}}(\Omega_t)}).
\]

Finally, we can prove the desired estimate by an interpolation.

\[\blacksquare\]

### 7.1 Estimates for \( P_{w,v} \) terms

Before considering the energy estimates, some more preparations are still needed. We present the estimates related to the pressure term \( P_{w,v} \) in this part, which will be used frequently later.

In fact, for some vector field \( w \) on \( \Omega_t \), we firstly recall the system for \( P_{w,v} \):

\[
\begin{aligned}
\Delta P_{w,v} &= -tr(\nabla w \nabla v), & \text{on } \Omega_t, \\
P_{w,v}|_{\Gamma_t} &= 0, & \partial_{n_b} P_{w,v}|_{\Gamma_b} = w \cdot \nabla v n_b.
\end{aligned}
\]

The following proposition gives the estimate for \( P_{w,v} \).

**Proposition 7.2** Let \( \omega < \frac{2}{5} \) and \( 1 \leq s \leq 4 \). For a vector field \( w \in H^{s-1}(\Omega_t) \) the estimate below holds:

\[
\|P_{w,v}\|_{H^s(\Omega_t)} \leq P(E(t))\|w\|_{H^{s-1}(\Omega_t)}.
\]

Moreover, if \( w = D_t J \), we have

\[
\|P_{D_t J,v}\|_{H^1(\Omega_t)} \leq P(E(t)).
\]
Proof. The first result comes directly from Theorem 5.1, so we focus on the second one, which is a variational estimate.

In fact, for all \( \phi \in \mathcal{V} = \{ \phi \in H^1(\Omega_t) \mid \phi|_{\Gamma_t} = 0 \} \), we have the variation equation

\[
\int_{\Omega_t} \nabla P_{D_tJ,v} \cdot \nabla \phi dX = \int_{\Omega_t} \text{tr}(\nabla D_t J \nabla v) \phi dX + \int_{\Gamma_b} (D_t J \cdot \nabla v n_b) \phi ds.
\]

Firstly, we need to deal with the right side in the equation above. We have for the first term that

\[
\int_{\Omega_t} \text{tr}(\nabla D_t J \nabla v) \cdot \phi dx \leq \| \nabla D_t J \|_{H^{-1}(\Omega_t)} \| \nabla v \phi \|_{H^1(\Omega_t)}
\]

\[
\leq C \| D_t J \|_{L^2(\Omega_t)} \| \nabla v \|_{H^2(\Omega_t)} \| \phi \|_{H^1(\Omega_t)}
\]

\[
\leq P(E(t)) \| \phi \|_{H^1(\Omega_t)}
\]

where (7.7) and Proposition 7.1 are applied here.

For the second term on the right side, we can write

\[ D_t J \cdot \nabla v n_b = g \nabla \tau_b D_t \kappa_H - \nabla v n_b \cdot (\nabla v)^* J \]

where we notice that \( \nabla v n_b = (\nabla v n_b \cdot \tau_b) \tau_b \) and denote \( g = \nabla v n_b \cdot \tau_b \). Therefore we obtain

\[
\int_{\Gamma_b} (D_t J \cdot \nabla v n_b) \phi ds = \int_{\Gamma_b} (g \nabla \tau_b D_t \kappa_H - \nabla v n_b \cdot (\nabla v)^* J) \phi ds
\]

\[
\leq C \| \nabla \tau_b D_t \kappa_H \|_{H^{-1}(\Gamma_b)} \| g \phi \|_{H^2(\Gamma_b)} + \| \nabla v n_b \cdot (\nabla v)^* J \|_{L^2(\Gamma_b)} \| \phi \|_{L^2(\Gamma_b)}
\]

\[
\leq P(E(t)) \| \phi \|_{H^1(\Omega_t)}
\]

where we used Lemma 5.5 for \( g \phi \in \tilde{H}^2(\Gamma_b) \), and moreover, Lemma 5.6, Theorem 5.3, Proposition 7.1, and (7.16) for \( D_t \kappa_H \). Besides, we note that the estimate for \( J \) here is given in (7.6).

Combining all the estimates above, we have for the left side of the variation equation that

\[
\int_{\Omega_t} \nabla P_{D_tJ,v} \cdot \nabla \phi dX \leq P(E(t)) \| w \|_{H^1(\Omega_t)}
\]

which leads to the unique existence of \( P_{D_tJ,v} \in \mathcal{V} \) and the desired estimate. \( \blacksquare \)

Based on the estimate for \( P_{w,v} \), we will consider here some more related terms which will be used later. In fact, although the estimates for \( P_{J,v} \) and \( D_t J \) are already mentioned before, we still write them down here again.

1. \( P_{J,v} \) and \( P_{v,v} \). Applying Proposition 7.2 together with Proposition 7.1 and 7.6 we have that

\[ \| P_{J,v} \|_{H^2(\Omega_t)} + \| P_{v,v} \|_{H^4(\Omega_t)} \leq P(E(t)). \]  

(7.20)

2. Higher-order estimates for \( P_{J,v} \). Recalling the definition of \( P_{J,v} \) and applying Theorem 5.1 to obtain

\[ \| P_{J,v} \|_{H^2(\Omega_t)} \leq P(E(t)) \| J \|_{H^1(\Omega_t)} \]
and
\[ \|P_{J,v}\|_{H^3(\Omega_t)} \leq P(E(t)) \|J\|_{H^2(\Omega_t)} \]
assuming that we have \( J \in H^2(\Omega_t) \). Thus, by the complex interpolation theory, we derive the estimate
\[ \|P_{J,v}\|_{H^2(\Omega_t)} \leq P(E(t)) \|J\|_{H^2(\Omega_t)}. \]
As a result, combining (7.6) we conclude that
\[ \|P_{J,v}\|_{H^5(\Omega_t)} \leq P(E(t)) \|J\|_{H^2(\Omega_t)}. \]

3. \( D_tJ \). Combining (7.7) and Proposition 7.1, we have
\[ \|D_tJ\|_{L^2(\Omega_t)} \leq P(E(t)). \]

4. \( D_tP_{v,v} \) and \( D_tP_{J,v} \). Recalling that \( P_{v,v} = \Delta - \frac{1}{\beta} (\text{tr} \nabla v)^2 \), direct computations using (6.3) lead to
\[
D_tP_{v,v} = \Delta^{-1} \left( -\text{tr}(\nabla D_t v - (\nabla v)^* \nabla v) \nabla v + \nabla v (\nabla D_t v - (\nabla v)^* \nabla v) \right),
\]
\[
\nabla v \nabla v - (\nabla v)^* \nabla v - (\nabla v)^* \nabla v - \nabla (\nabla v)^* \nabla v + \Delta^{-1} \left( 2 \nabla v \cdot \nabla^2 v + \Delta v \cdot \nabla P_{v,v} + (\nabla v \cdot \nabla P_{v,v}) \right)
\]
Consequently, applying the Euler equation from (WW), Theorem 5.1, (7.20) and a complex interpolation, we can have
\[ \|D_tP_{v,v}\|_{H^2(\Omega_t)} \leq P(E(t)). \]
Similarly, we can also show by a variational argument as in the proof of Proposition 7.2 that
\[ \|D_tP_{J,v}\|_{H^1(\Omega_t)} \leq P(E(t)). \]

7.2 The dissipation at the contact point

Before our a priori estimates, we want to deal with the dissipation term of \( J \) on the contact point, which will appear later in the estimates and plays a key role. Besides, one can see from this lemma that a lower bound for the contact angle \( \omega \) is needed.

Lemma 7.1 We have the following equation at the contact point
\[ (D_tJ)^\perp (\nabla_{\tau_c}J)^\perp |_{X_c} = -\frac{\sigma^2}{\beta_c} F(t) + r_c, \]
where
\[ r_c = \left( -r + \cot \omega (J \cdot D_t n_b) \right) \sin \omega \nabla_{\tau_c} J \]

at \( X_c \)
with
\[
\frac{\partial}{\partial t} \tau = -\sigma \sin \omega (\nabla_{\tau_t} v \cdot \nabla_{\tau_t} n_t) + \sigma \tau_t \cdot D_t n_t (\nabla_{\tau_t} v \cdot n_t)
+ \sigma \sin \omega (\nabla_{\tau_t} v P_{v,v} - [D_t, \nabla_{\tau_t}] v) \cdot n_t - \beta_c D_t \nabla P_{v,v} \cdot \tau_b.
\]

Moreover if the contact angle \(\omega \in (0, \pi/6)\) and \(\sin \omega \geq c_0\) for some small constant \(c_0 > 0\), we have the following estimate
\[
|r_c| \leq P(E(t)) F(t)^{1/2}.
\] (7.25)

**Proof.** To begin with, we recall the boundary condition on \(X_c\) from (WW):
\[
[\gamma] - \sigma \cos \omega = \beta_c v_c \quad \text{at} \quad X_c,
\]
and we also have at \(X_c\) that
\[
\cos \omega = -\tau_t \cdot \tau_b, \quad v_c = -v \cdot \tau_b,
\]
which implies
\[
[\gamma] + \sigma \tau_t \cdot \tau_b = -\beta_c v \cdot \tau_b \quad \text{at} \quad X_c.
\] (7.26)

On the other hand, we recall the computations
\[
D_t n_t = -((\nabla v)^* n_t)^\top \quad \text{and} \quad D_t \tau_t = (\nabla_{\tau_t} v \cdot n_t)n_t \quad \text{on} \quad \Gamma_t,
\]
which will be used in the following lines.

Taking \(\partial_t\) on both sides of (7.26) to obtain
\[
\sigma (D_t \tau_t) \cdot \tau_b = -\beta_c (D_t v) \cdot \tau_b \quad \text{at} \quad X_c,
\]
where we notice that \(\tau_b, n_b\) are constant vectors near \(X_c\). Substituting the Euler equation from (WW) into the equation above to get
\[
\sigma (D_t \tau_t) \cdot \tau_b = \beta_c (\sigma J + \nabla P_{v,v} + g) \cdot \tau_b \quad \text{at} \quad X_c.
\]

Now we take \(\partial_t\) again on both sides of the equation above to have at \(X_c\) that
\[
\beta_c \sigma D_t J \cdot \tau_b + \beta_c D_t \nabla P_{v,v} \cdot \tau_b = \sigma D_t^2 \tau_t \cdot \tau_b = \sigma \tau_b \cdot D_t ((\nabla_{\tau_t} v \cdot n_t)n_t)
+ \sigma \tau_b \cdot (D_t \nabla_{\tau_t} v \cdot n_t)n_t + \sigma \tau_b \cdot (\nabla_{\tau_t} v \cdot D_t n_t)n_t
+ \sigma \tau_b \cdot (\nabla_{\tau_t} v \cdot n_t) D_t n_t
- \sigma \sin \omega (D_t \nabla_{\tau_t} v \cdot n_t + \nabla_{\tau_t} v \cdot D_t n_t)
+ \sigma \tau_b \cdot D_t n_t (\nabla_{\tau_t} v \cdot n_t),
\] (7.27)

where we used that \(n_t \cdot \tau_b = -\sin \omega\) at \(X_c\).

On the other hand, applying \(\nabla_{\tau_t}\) on the Euler equation in (WW) (constrained on \(\Gamma_t\)) to arrive at
\[
D_t \nabla_{\tau_t} v \cdot n_t = (\nabla_{\tau_t} D_t v) \cdot n_t + [D_t, \nabla_{\tau_t}] v \cdot n_t = -\nabla_{\tau_t} (\sigma J + \nabla P_{v,v}) \cdot n_t + [D_t, \nabla_{\tau_t}] v \cdot n_t,
\]
which can be substituted into (7.27) to obtain
\[
\beta_c \sigma D_t J \cdot n_b = \sigma^2 \sin \omega \nabla_{\tau_b} J \cdot n_t + r \quad \text{at } X_c, \quad (7.28)
\]
where \(r\) is defined by
\[
\begin{align*}
  r &= -\sigma \sin \omega (\nabla_{\tau_b} v \cdot D_t n_t) + \sigma \tau_b \cdot D_t n_t (\nabla_{\tau_b} v \cdot n_t) \\
  & \quad + \sigma \sin \omega (\nabla_{\tau_b} \nabla P_{v,v} - [D_t, \nabla_{\tau_b}]v) \cdot n_t - \beta_b D_t \nabla P_{v,v} \cdot \tau_b.
\end{align*}
\]

Next, we plan to compute \(D_t J \cdot n_b\) to retrieve \(D_t J\). Firstly, we recall the definition of \(J\) to find
\[
J \cdot n_b = 0 \quad \text{on } \Gamma_b.
\]
Taking \(D_t\) on both sides of the condition above to get
\[
D_t J \cdot n_b = -J \cdot D_t n_b \quad \text{on } \Gamma_b. \quad (7.29)
\]
Combining (7.28) and (7.29) to obtain that
\[
D_t J = \frac{\sigma^2}{\beta_c} \sin \omega (\nabla_{\tau_b} J) \cdot \tau_b + (r \tau_b - (J \cdot D_t n_b) n_b) \quad \text{on } \Gamma_b. \quad (7.30)
\]
As a result, we have
\[
(D_t J)^\perp (\nabla_{\tau_b} J)^\perp |_{X_c} = -\frac{\sigma^2}{\beta_c} F(t) + r_c.
\]
with
\[
r_c = (r \tau_b - (J \cdot D_t n_b) n_b) \cdot n_t \nabla_{\tau_b} J^\perp.
\]
In the end, we prove the estimate (7.25). In fact, applying Lemma 5.8, Proposition 7.1, Proposition 7.2 and (7.23) it’s straightforward to prove that
\[
|r|_{X_c} + |J \cdot D_t n_b|_{X_c} \leq P(E(t))
\]
with \(D_t n_b = -((\nabla v)^* n_b)\). Therefore, the proof is finished.

\[\square\]

7.3 Energy estimates.

Now we are in a position to prove the a priori estimate for \(E(t)\). For the sake of simplicity, we perform the energy estimates directly on \(J\) here. In fact, to be more strict, the following estimates should be performed firstly on a sequence of smooth functions which converges to \(J\), and then we show that the final energy estimate also holds when the limit is taken.

To begin with, taking the inner product with \(D_t J\) for both sides of (6.14) and integrating on \(\Omega_t\) to obtain that
\[
\int_{\Omega_t} D_t D_t J \cdot D_t Jdx + \sigma \int_{\Omega_t} AJ : D_t Jdx = \int_{\Omega_t} R \cdot D_t Jdx \quad (7.31)
\]
where \( R = R_0 + D_t \nabla P_{J,v} \).

For the first term on the left side of (7.31), we simply have
\[
\int_{\Omega_t} D_t D_t J \cdot D_t J dX = \frac{1}{2} \partial_t \int_{\Omega_t} |D_t J|^2 dX.
\]

Applying (7.24) on \( D_t \nabla P_{J,v} \) term from the right side, we find
\[
\int_{\Omega_t} D_t \nabla P_{J,v} \cdot D_t J dX \leq \|D_t \nabla P_{J,v}\|_{L^2(\Omega_t)} \|D_t J\|_{L^2(\Omega_t)} \leq P(E(t)).
\]

Next, we focus on the second term on the left side of (7.31) to derive that
\[
\int_{\Omega_t} A J \cdot D_t J dX = \int_{\Omega_t} \nabla \mathcal{H}(-\Delta_{\Gamma_t} J^\perp) \cdot D_t J dX
\]
\[
= -\int_{\Gamma_t} \Delta_{\Gamma_t} J^\perp (D_t J \cdot n_t) dX
\]
where one recalls that
\[ D_t J = D_t J + \nabla P_{J,v} \quad \text{and} \quad D_t J \circ u \in T_{u(t)} \Gamma \]
from the Hodge decomposition in Section 3. Consequently, we can have by integrating by parts
\[
\int_{\Omega_t} A J \cdot D_t J dX = -\int_{\Gamma_t} \Delta_{\Gamma_t} J^\perp (D_t J \cdot n_t) ds - \int_{\Gamma_t} \Delta_{\Gamma_t} J^\perp (\nabla P_{J,v} \cdot n_t) ds
\]
\[
= \int_{\Gamma_t} \nabla_{\tau_t} J^\perp \nabla_{\tau_t} (D_t J \cdot n_t) ds - \left( (D_t J)^\perp \nabla_{\tau_t} J^\perp \right) |_{X_c} - \int_{\Gamma_t} \Delta_{\Gamma_t} J^\perp (\nabla P_{J,v} \cdot n_t) ds. \tag{7.32}
\]

For the first term on the right side of (7.32), one deduces that
\[
\int_{\Gamma_t} \nabla_{\tau_t} J^\perp \nabla_{\tau_t} (D_t J \cdot n_t) ds
\]
\[
= \int_{\Gamma_t} \nabla_{\tau_t} J^\perp D_t (\nabla_{\tau_t} J^\perp) ds - \int_{\Gamma_t} \nabla_{\tau_t} J^\perp \nabla_{\tau_t} (J \cdot D_t n_t) ds - \int_{\Gamma_t} \nabla_{\tau_t} J^\perp [D_t, \nabla_{\tau_t}] J^\perp ds
\]
\[
= \frac{1}{2} \partial_t \int_{\Gamma_t} |\nabla_{\tau_t} J^\perp|^2 ds - \int_{\Gamma_t} \nabla_{\tau_t} J^\perp \nabla_{\tau_t} (J \cdot D_t n_t) ds - \int_{\Gamma_t} \nabla_{\tau_t} J^\perp [D_t, \nabla_{\tau_t}] J^\perp ds.
\]

For the second term on the right side of (7.32), applying Lemma 7.4 one can have
\[
-(D_t J)^\perp \nabla_{\tau_t} J^\perp |_{X_c} = \frac{\sigma^2}{\beta_c} F(t) - r_c.
\]

with
\[
|r_c| \leq P(E(t)) F(t)^{\frac{1}{2}}
\]
as long as
\[
\sin \omega \geq c_0 > 0 \quad \text{for some constant } c_0,
\]

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which will be checked in the end of this paper.

Combining all the estimates above, we have
\[
\partial_t \left( \frac{1}{2} \int_{\Omega_t} |D_t J|^2 dX + \frac{\sigma}{2} \int_{\Omega_t} |\nabla_{\tau_t} J^\perp|^2 ds \right) + \frac{\sigma^3}{2 \beta_c} F(t) \\
\leq P(E(t)) + \sigma \int_{\Gamma_t} \Delta_{\tau_t} J^\perp \nabla P_{J,v} \cdot n_t ds + \sigma \int_{\Gamma_t} \nabla_{\tau_t} J^\perp \nabla_{\tau_t} (J \cdot D_t n_t) ds \\
+ \sigma \int_{\Gamma_t} \nabla_{\tau_t} J^\perp [D_t, \nabla_{\tau_t}] J^\perp ds + \int_{\Omega_t} R_0 \cdot D_t J dX.
\]

So now it remains to deal with the right side of the energy estimate above. In fact, it is straightforward to show from Theorem 5.3, Proposition 7.1 and (7.6) that
\[
\int_{\Gamma_t} \nabla_{\tau_t} J^\perp \nabla_{\tau_t} (J \cdot D_t n_t) ds + \int_{\Gamma_t} \nabla_{\tau_t} J^\perp [D_t, \nabla_{\tau_t}] J^\perp ds \leq P(E(t)),
\]
and moreover, a direct computation and applying (7.21) one can also get
\[
\int_{\Gamma_t} \Delta_{\tau_t} J^\perp \nabla P_{J,v} \cdot n_t ds = - \int_{\Gamma_t} \nabla_{\tau_t} J^\perp \nabla_{\tau_t} \nabla P_{J,v} \cdot n_t ds + (\nabla_{\tau_t} J^\perp \nabla P_{J,v} \cdot n_t) \big|_{X_c} \\
\leq \|\nabla_{\tau_t} J^\perp\|_{L^2(\Gamma_t)} \|\nabla_{\tau_t} \nabla P_{J,v} \cdot n_t\|_{L^2(\Gamma_t)} + F(t) \frac{1}{2} \left\| \frac{1}{\sin \omega} \nabla P_{J,v} \cdot n_t \right\|_{X_c} \\
\leq \frac{1}{4} \frac{\sigma^2}{\beta_c} F(t) + P(E(t))
\]
as long as we have \(\sin \omega \geq c_0 > 0\).

As a result, we arrive at the following estimate
\[
\partial_t \left( \int_{\Omega_t} |D_t J|^2 dX + \int_{\Gamma_t} |\nabla_{\tau_t} J^\perp|^2 ds \right) + F(t) \leq P(E(t)) + \|R_0\|_{L^2(\Omega_t)}^2,
\]
which tells us that, to close the energy estimates, the only thing left is to prove the estimate for the reminder term \(R_0\).

In fact, one can see that the boundary conditions on \(\Gamma_b\) play an important role in the variational estimates, which are handled differently compared to the smooth-domain case. Therefore, a lemma is presented here focusing on a typical type of boundary conditions needed in the estimate for \(R_0\).

**Lemma 7.2** Let \(w \in H^1(\Omega_t)\) and \(\Delta w \in L^2(\Omega_t)\). Then the boundary condition
\[
(\nabla_{n_b} v - \nabla_v n_b) \cdot \nabla w \big|_{\Gamma_b},
\]
makes sense in the variation formulation: For any \(\phi \in V = \{\phi \in H^1(\Omega_t) \big| \phi|_{\Gamma_t} = 0\}\), one has
\[
\int_{\Gamma_b} (\nabla_{n_b} v - \nabla_v n_b) \cdot \nabla w \phi ds \leq P(E(t)) \left( \|w\|_{H^1(\Omega_t)} + \|\Delta w\|_{L^2(\Omega_t)} \right) \|\phi\|_{H^1(\Omega_t)}.
\]

**Proof.** The proof lies in clarifying the space for the boundary condition.
Firstly, we need to rewrite $\nabla_{n_b}v - \nabla_vn_b$. Since $n_b$ is a unit normal vector, we know that $\nabla_vn_b = (\nabla_vn_b \cdot \tau_b)\tau_b$. Besides, we decompose $\nabla_{n_b}v$ with respect to $\tau_b$ and $n_b$ to have

$$\nabla_{n_b}v - \nabla_vn_b = (\nabla_{n_b}v \cdot \tau_b - \nabla_vn_b \cdot \tau_b)\tau_b + (\nabla_{n_b}v \cdot n_b)n_b.$$

Consequently, the boundary condition can be written as

$$(\nabla_{n_b}v - \nabla_vn_b) \cdot \nabla w|_{\Gamma_b} = (\nabla_{n_b}v \cdot \tau_b - \nabla_vn_b \cdot \tau_b)\nabla \tau_b w + (\nabla_{n_b}v \cdot n_b)\nabla n_b w|_{\Gamma_b}.$$ 

Now we are ready to show that this condition makes sense. In fact, since $w \in H^\frac{1}{2}(\Gamma_b)$, we have $w \in H^\frac{1}{2}(\Gamma_b)$ by Theorem 5.3. Applying Lemma 5.6 we can see also that $\nabla \tau_b w \in H^\frac{1}{2}(\Gamma_b)$ with the estimate

$$\|\nabla \tau_b w\|_{H^\frac{1}{2}(\Gamma_b)} \leq P(E(t))\|w\|_{H^\frac{1}{2}(\Gamma_b)} \leq P(E(t))\|w\|_{H^1(\Omega_t)}.$$ 

On the other hand, using Lemma 5.7 to have $\nabla_{n_b}w|_{\Gamma_b} \in H^{-\frac{1}{2}}(\Gamma_b)$ with

$$\|\nabla_{n_b}w\|_{H^{-\frac{1}{2}}(\Gamma_b)} \leq P(E(t))(\|w\|_{H^1(\Omega_t)} + \|\Delta w\|_{L^2(\Omega_t)}).$$

Finally, summing the estimates above up leads to

$$\int_{\Gamma_b} (\nabla_{n_b}v - \nabla_vn_b) \cdot \nabla w \phi ds$$

$$\leq \|\nabla \tau_b w\|_{H^\frac{1}{2}(\Gamma_b)}(\nabla_{n_b}v \cdot \tau_b - \nabla_vn_b \cdot \tau_b)\phi|_{\Gamma_b} + \|\nabla_{n_b}w\|_{H^{-\frac{1}{2}}(\Gamma_b)}(\nabla_{n_b}v \cdot n_b)\phi|_{\Gamma_b}$$

$$\leq P(E(t))(\|w\|_{H^1(\Omega_t)} + \|\Delta w\|_{L^2(\Omega_t)}\|\phi\|_{H^1(\Omega_t)},$$

where Theorem 5.3, Lemma 5.5, Lemma 5.8 and Proposition 7.1 are applied. 

Now, it’s the time to present the estimate for $R_0$.

**Proposition 7.3** We have the following estimate for the remainder term $R_0$ defined in (6.13):

$$\|R_0\|_{L^2(\Omega_t)} \leq P(E(t)).$$

**Proof.** Recall from (6.13) that

$$R_0 = -\sigma\nabla \mathcal{H}(J \cdot \Delta_{\Gamma_t}n_t) + \nabla \mathcal{H}(n_t \cdot \Delta_{\Gamma_t}\nabla P_{v,u}) + \nabla \mathcal{H}(R_1) + A_1 + A_2 + A_3$$

where $R_1$ and $A_1, A_2, A_3$ are defined in (6.8) and (6.11), (6.12), (6.10) respectively, so the estimate for $R_0$ lies in the estimates for all the terms above.

- Estimate for $\sigma\nabla \mathcal{H}(J \cdot \Delta_{\Gamma_t}n_t)$. Applying Theorem 5.3, Lemma 5.9 and 7.6 one finds that

$$\|\sigma\nabla \mathcal{H}(J \cdot \Delta_{\Gamma_t}n_t)\|_{L^2(\Omega_t)} \leq P(E(t))\|J \cdot \Delta_{\Gamma_t}n_t\|_{H^\frac{1}{2}(\Gamma_t)} \leq P(E(t)).$$

- Estimate for $\nabla \mathcal{H}(n_t \cdot \Delta_{\Gamma_t}\nabla P_{v,u})$. Similarly as the previous term above one has

$$\|\nabla \mathcal{H}(n_t \cdot \Delta_{\Gamma_t}\nabla P_{v,u})\|_{L^2(\Omega_t)} \leq P(E(t)).$$
- Estimate for $\nabla \mathcal{H}(R_1)$. Examining the expression for $R_1$ from (6.8) carefully, one can see that the leading-order terms in $R_1$ are $\partial^2 v$, $\partial n_t$, $\kappa$ and $\partial^2 P_{v,v}$, so all the terms in $R_1$ can be dealt directly and the details are omitted here. As a result, we find
\[
\|\nabla \mathcal{H}(R_1)\|_{L^2(\Omega_t)} \leq P(E(t)).
\]

- Estimate for $A_1$. Recall from (6.11) that
\[
A_1 = \nabla w
\]
with the notation
\[
w = \Delta^{-1}(2\nabla v \cdot \nabla^2 \mathcal{H}(D_t\kappa) + \Delta v \cdot \nabla \mathcal{H}(D_t\kappa), (\nabla_{n_b} v - \nabla_v n_b) \cdot \nabla \mathcal{H}(D_t\kappa)),
\]
so one needs to deal with the estimate for $w$. In fact, by the notation $\Delta^{-1}$ we know that $w$ satisfies the system
\[
\begin{cases}
\Delta w = 2\nabla v \cdot \nabla^2 \mathcal{H}(D_t\kappa) + \Delta v \cdot \nabla \mathcal{H}(D_t\kappa), & \text{on } \Omega_t, \\
|w|_{\Gamma_t} = 0, \quad \nabla_{n_b} w|_{\Gamma_b} = (\nabla_{n_b} v - \nabla_v n_b) \cdot \nabla \mathcal{H}(D_t\kappa)|_{\Gamma_b}
\end{cases}
\]
which is defined by the variation equation
\[
\int_{\Omega_t} \nabla w \cdot \nabla \phi dX = - \int_{\Omega_t} \left(2\nabla v \cdot \nabla^2 \mathcal{H}(D_t\kappa) + \Delta v \cdot \nabla \mathcal{H}(D_t\kappa)\right) \phi dX + \int_{\Gamma_b} (\nabla_{n_b} v - \nabla_v n_b) \cdot \nabla \mathcal{H}(D_t\kappa) \phi ds
\]
for any $\phi \in \mathcal{V} = \{\phi \in H^1(\Omega_t) \mid \phi|_{\Gamma_t} = 0\}$.

One needs to prove the estimate for $w$ from the variation equation. First of all, one has from the proof of Proposition 7.1 that
\[
\|D_t\kappa\|_{H^1(\Gamma_t)} \leq P(E(t)),
\]
which together with Lemma 5.9 leads to
\[
\|\mathcal{H}(D_t\kappa)\|_{H^1(\Omega_t)} \leq C \|D_t\kappa\|_{H^1(\Gamma_t)} \leq P(E(t)).
\]
Therefore, we have the estimate for the first term in the right side of the variation equation:
\[
\left| \int_{\Omega_t} \left(2\nabla v \cdot \nabla^2 \mathcal{H}(D_t\kappa) + \Delta v \cdot \nabla \mathcal{H}(D_t\kappa)\right) \phi dX \right|
\leq \|\nabla^2 \mathcal{H}(D_t\kappa)\|_{H^{-1}(\Omega_t)} \|2\nabla v \phi\|_{H^1(\Omega_t)} + \|\nabla \mathcal{H}(D_t\kappa)\|_{L^2(\Omega_t)} \|\Delta v \phi\|_{L^2(\Omega_t)}
\leq P(E(t)) \|\nabla \mathcal{H}(D_t\kappa)\|_{L^2(\Omega_t)} \left(\|2\nabla v \phi\|_{H^1(\Omega_t)} + \|\Delta v \phi\|_{L^2(\Omega_t)}\right)
\leq P(E(t)) \|\phi\|_{H^1(\Omega_t)}
\]
where Lemma 5.8 and Proposition 7.1 are used.

Secondly, we need to deal with the boundary term in the variation equation. Indeed, one can check directly that $\mathcal{H}(D_t\kappa) \in H^1(\Omega_t)$ satisfies the condition in Lemma 7.2 so applying this lemma leads to
\[
\left| \int_{\Gamma_b} (\nabla_{n_b} v - \nabla_v n_b) \cdot \nabla \mathcal{H}(D_t\kappa) \phi ds \right| \leq P(E(t)) \|\phi\|_{H^1(\Omega_t)}.
\]
Summing up these estimates above we finally conclude that the variation equation for \( w \) admits a unique solution \( w \in H^1(\Omega_t) \) with the estimate

\[
\|w\|_{H^1(\Omega_t)} \leq P(E(t)),
\]

which implies that

\[
\|A_1\|_{L^2(\Omega_t)} \leq P(E(t)).
\]

- Estimate for \( A_2 \). The estimate is similar as that for \( A_1 \). Recall from (6.12) that

\[
A_2 = \nabla \Delta^{-1}(2\nabla v \cdot \nabla^2 w_{A2} + \Delta v \cdot \nabla w_{A2}, (\nabla_{n_b} v - \nabla_v n_b) \cdot \nabla w_{A2}) + \nabla \Delta^{-1}(h_{A2}, g_{A2})
\]

\[
:= A_{21} + A_{22}
\]

where

\[
w_{A2} = \Delta^{-1}(2\nabla v \cdot \nabla J + \Delta v \cdot J, (\nabla_{n_b} v - \nabla_v n_b) \cdot J),
\]

\[
h_{A2} = 2\nabla v \cdot (\nabla D_t J - (\nabla v)^* J) + 2(\nabla D_t v - (\nabla v)^* v) \cdot \nabla J + D_t J \cdot \Delta v
\]

\[
+ J \cdot (\Delta D_t v - \Delta v \cdot \nabla v - 2\nabla v \cdot \nabla^2 v),
\]

\[
g_{A2} = (\nabla_{n_b} v - \nabla_v n_b) \cdot \Delta v \cdot J + \nabla_{n_b} D_t v \cdot J - (D_t v - \nabla_v v) \cdot \nabla n_b \cdot J
\]

\[
+ \nabla v ((\nabla v)^* n_b) \nabla J + (\nabla_{n_b} v - \nabla_v n_b) \cdot D_t J.
\]

Firstly, in order to deal with \( A_{21} \), one needs to handle \( w_{A2} \). In fact, \( w_{A2} \) satisfies the system

\[
\begin{cases}
\Delta w_{A2} = 2\nabla v \cdot \nabla J + \Delta v \cdot J, & \text{on } \Omega_t \\
w_{A2}|_{\Gamma_t} = 0, & \nabla_{n_b} w_{A2}|_{\Gamma_b} = (\nabla_{n_b} v - \nabla_v n_b) \cdot J.
\end{cases}
\]

Noticing that \( J = \nabla \kappa_H \in H^{\frac{3}{4}}(\Omega_t) \), the variational estimate for \( w_{A2} \in H^1(\Omega_t) \) can be done similarly as the \( w \) system in the estimate for \( A_1 \), which turns out to be

\[
\|w_{A2}\|_{H^1(\Omega_t)} \leq C \left( \|2\nabla v \cdot \nabla J + \Delta v \cdot J\|_{L^2(\Omega_t)} + P(E(t))\|\kappa_H\|_{H^1(\Omega_t)} \right)
\]

\[
\leq P(E(t)).
\]

Moreover, since

\[
\Delta w_{A2} = 2\nabla v \cdot \nabla J + \Delta v \cdot J \in L^2(\Omega_t),
\]

we also have that \( w_{A2} \in E(\Delta; L^2(\Omega_t)) \).

Now we can close the estimate for \( A_{21} \). In fact, the variational estimate for \( A_{21} \) is almost the same as the estimate for \( w \) in \( A_1 \) part, so we omit the details to write directly that

\[
\|A_{21}\|_{L^2(\Omega_t)} \leq P(E(t)),
\]

where Lemma 7.2 is applied to the boundary condition on \( \Gamma_b \) since \( w_{A2} \in E(\Delta; L^2(\Omega_t)) \).

Secondly, we consider the estimate for \( A_{22} \). Letting \( u_{A2} = \Delta^{-1}(h_{A2}, g_{A2}) \), we have

\[
A_{22} = \nabla u_{A2}.
\]

Similarly as before, we will deal with the variational estimate for the \( u_{A2} \) system again. In fact, \( u_{A2} \) is defined by the variation equation

\[
\int_{\Omega_t} \nabla u_{A2} \cdot \nabla \phi dX = -\int_{\Omega_t} h_{A2} \phi dX + \int_{\Gamma_b} g_{A2} \phi ds \tag{7.33}
\]
where \( \phi \in \mathcal{V} = \{ \phi \in H^1(\Omega_t) \mid \phi|_{\Gamma_t} = 0 \} \). So the variational estimate lies in the estimates for the two integrals on the right side.

For the term of \( h_{A2} \), since the estimate here is similar as before, the details are omitted and we directly write down the estimate

\[
\left| \int_{\Omega_t} h_{A2} \phi dX \right| \leq P(E(t)) \| \phi \|_{H^1(\Omega_t)}. \tag{7.34}
\]

On the other hand, we consider the estimate for \( \int_{\Gamma_b} g_{A2} \phi ds \) from (7.33). Plugging in the expression for \( g_{A2} \) to arrive at

\[
\int_{\Gamma_b} g_{A2} \phi ds = \int_{\Gamma_b} (\nabla v n_b - \nabla_v v) \cdot \nabla_J \phi ds + \int_{\Gamma_b} \nabla_n b D_J v \cdot J \phi ds - \int_{\Gamma_b} (D_J v - \nabla v) \cdot \nabla n_b \cdot J \phi ds
\]

\[
+ \int_{\Gamma_b} \nabla_v ((\nabla v)^* n_b) \cdot J \phi ds + \int_{\Gamma_b} (\nabla_n v - \nabla v n_b) \cdot D_t J \phi ds
\]

\[
:= B_1 + B_2 + \cdots + B_5.
\]

and we will check the terms one by one.

In fact, similar estimates as before lead to

\[
|B_1 + \cdots + B_4| \leq P(E(t)) \| w \|_{H^1(\Omega_t)};
\]

and it remains to deal with the last term \( B_5 \). Since

\[
D_t J = D_t \nabla \kappa_H = \nabla D_t \kappa_H - (\nabla v)^* J
\]

where we know that \( D_t \kappa_H \in H^1(\Omega_t) \) and

\[
\Delta D_t \kappa_H = 2 \nabla v \cdot J + \Delta v \cdot J \in L^2(\Omega_t).
\]

As a result, one has that

\[
|B_5| \leq \left| \int_{\Gamma_b} (\nabla_n v - \nabla_v n_b) \cdot \nabla D_t \kappa_H \phi ds \right| + \left| \int_{\Gamma_b} (\nabla_n v - \nabla_v n_b) \cdot (\nabla v)^* J \phi ds \right|
\]

\[
\leq P(E(t)) \left( \| D_t \kappa_H \|_{H^1(\Omega_t)} + \| \Delta D_t \kappa_H \|_{L^2(\Omega_t)} \right) \| \phi \|_{H^1(\Omega_t)}
\]

\[
+ \| (\nabla v + \nabla n_b) \cdot (\nabla v)^* J \|_{L^2(\Gamma_b)} \| \phi \|_{L^2(\Gamma_b)}
\]

\[
\leq P(E(t)) \| \phi \|_{H^1(\Omega_t)}
\]

where we applied Lemma 7.2, Lemma 5.8, Proposition 7.1 and (7.16).

Summing up the estimates from \( B_1 \) to \( B_5 \), we have

\[
\left| \int_{\Gamma_b} g_{A2} \phi ds \right| \leq P(E(t)) \| \phi \|_{H^1(\Omega_t)}. \tag{7.35}
\]

Combining (7.34) and (7.35) above, we finally conclude that the variation equation (7.33) admits an unique solution \( u_{A2} \in \mathcal{V} \) and derive the estimate for \( A_{22} \):

\[
\| A_{22} \|_{L^2(\Omega_t)} = \| \nabla u_{A2} \|_{L^2(\Omega_t)} \leq P(E(t)).
\]

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As a result, we are able to finish the estimate for $A_2$.

- Estimate for $A_3$. This term can be handled directly to arrive at the estimate

$$\|A_3\|_{L^2(\Omega_t)} \leq P(E(t)).$$

In the end, combining all the estimates above, the proof is finished.

Now, we can finally conclude that

$$\partial_t \left( \int_{\Omega_t} |\nabla J|^2 \, dX + \int_{\Gamma_t} |\nabla J^1|^2 \, ds \right) + F(t) \leq P(E(t)). \quad (7.36)$$

To close the energy, we still need to give the estimates of $\|\Gamma_t\|_{H^{5/2}}$ and $\|v\|_{L^2(\Omega_t)}$. Firstly, applying the Euler equation from (WW) with (7.20) and (7.6), we can prove directly that

$$\partial_t \|v\|_{L^2(\Omega_t)}^2 \leq P(E(t)). \quad (7.37)$$

Secondly, in order to deal with $\|\Gamma_t\|_{H^{5/2}}$, we can parametrize $\Gamma_t$ under Eulerian coordinates $(x, z)$ by

$$\Gamma_t = \{(x, z) \mid z = \eta(t, x), \ t > 0, \ x \geq c(t)\}$$

where $c(t)$ is the $x$ coordinate for the contact point $X_c$. Therefore one can write

$$\|\Gamma_t\|_{H^{5/2}} = \|\eta\|_{H^{5/2}(c(t), \infty)}$$

and the estimate for $\Gamma_t$ means the estimate for $\eta$.

On the other hand, the water waves problem (WW) contains the kinematic condition on $\Gamma_t$, which can be written in form of $\eta, v$ as

$$\partial_t \eta + v_1 \partial_x \eta = v_2, \quad \text{on} \quad [c(t), \infty),$$

where $v = (v_1, v_2)^t$. Moreover, a direct computation shows that the material derivative $D_t$ under the parametrization is simply

$$D_t = \partial_t + v_1 \partial_x,$$

so the equation above for $\eta$ can be rewritten as

$$D_t \eta = v_2.$$

As a result, it is straightforward to derive the estimate

$$\partial_t \|\Gamma_t\|_{H^{5/2}}^2 = \partial_t \|\eta\|_{H^{5/2}(c(t), \infty)}^2 \leq P(E(t)) \quad (7.38)$$

Now plugging (7.38) and (7.37) into (7.36), we finally arrive at the energy estimate

$$\partial_t E(t) + F(t) \leq P(E(t)),$$

and integrating on both sides on a time interval $[0, T_0]$ (to be fixed later) leads to

$$\sup_{t \in [0, T_0]} E(t) + \int_0^{T_0} F(t) \leq E(0) + \int_0^{T_0} P(E(t)).$$
In the end, we need to consider about the evolution of the contact angle \( \omega(t) \) and check on the condition
\[
\sin \omega(t) \geq c_0 > 0 \quad \text{on } [0, T_0],
\]
for some constant \( c_0 \).

In fact, we have at the initial time \( t = 0 \) that
\[
\omega(0) \in (0, \frac{\pi}{6}) \quad \text{and} \quad \sin \omega(0) = -n_t(0) \cdot \tau_b(0)|_{X_c(0)} > 0,
\]
where \( \tau_b(0) \) is a constant vector since we set in the beginning that \( \Gamma_b \) becomes a straight line near the contact point. So we choose \( T_0 \) small enough such that for any \( t \in [0, T_0] \), \( \tau_b(t) \) stays the same as \( \tau_b(0) \) to prove that
\[
|\sin \omega(0) - \sin \omega(t)| = |n_t(t)|_{X_c(t)} - n_t(0)|_{X_c(0)}| \tau_b(0)|
\leq T_0 \sup_{[0, T_0]} |\partial_t n_t|_{X_c(t)}| \tau_b(0)|
\leq T_0 \sup_{[0, T_0]} P(E(t))
\]
Consequently, when \( T_0 \) is small enough, one finds a small constant \( \delta > 0 \) such that
\[
|\sin \omega(0) - \sin \omega(t)| < \delta \quad \text{for any } t \in [0, T_0],
\]
which infers that
\[
\sin \omega(t) \geq c_0 \quad \text{for any } t \in [0, T_0]
\]
for some constant \( c_0 > 0 \). Moreover, we can also have from the arguments above that
\[
\omega(t) \in (0, \frac{\pi}{6}) \quad \text{for any } t \in [0, T_0].
\]
As a result, our main theorem is proved.

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