Kepler’s Area Law in the Principia: Filling in some details in Newton’s proof of Prop. 1.

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Abstract

During the past 25 years there has been a controversy regarding the adequacy of Newton’s proof of Prop. 1 in Book 1 of the Principia. This proposition is of central importance because its proof of Kepler’s area law allowed Newton to introduce a geometric measure for time to solve problems in orbital dynamics in the Principia. It is shown here that the critics of Prop. 1 have misunderstood Newton’s fundamental limit argument by neglecting to consider the justification for this limit which he gave in Lemma 3. We clarify the proof of Prop. 1 by filling in some details left out by Newton which show that his proof of this proposition was adequate and well grounded.

*Rigor merely sanctions the conquests of sound intuition*, Jacques Hadamard

Key Words: Kepler’s area law, Newton’s Principia

Introduction

In Prop. 1 of the Principia Newton gave a proof that Kepler’s empirical area law for planetary orbits and the confinement of these orbits to a plane are consequences of his laws of motion for central forces. In his words,

“The areas which bodies made to move in orbits described by radii drawn to an unmoving center of force lie in unmoving planes and are proportional to the times”
This proposition is justifiably regarded as a cornerstone of the Principia, because the proportionality between the area swept out by the radius vector of the orbit and the elapsed time which this law entails enabled Newton to solve dynamical problems by purely geometrical methods supplemented by continuum limit arguments which he had developed. Although the validity of Newton’s proof was not questioned by his contemporaries, alternative analytic proofs of the area law were given by Jacob Hermann and John Keill based on the analytic form of the calculus which had been developed by Newton and by Leibniz (Guicciardini 1999). During the past years, however, two influential historians of science, D.T. Whiteside and A.J. Aiton have criticized Newton’s proof, claiming that it was inadequate and applied only to an infinitesimal arc of the orbit (MP 6:35-36, footnote 19) (Aiton 1989). Referring to this claim, Whiteside remarked that there were underlying subtleties in the proof that Newton did not fully appreciated, and that Newton continued “to believe in its superficial simplicities” although, Whiteside admitted, not even “Johann Bernoulli, his arch critic ... saw fit to impugn the adequacy of Newton’s demonstration”. The basis for the Aiton-Whiteside criticism is that Newton had treated incorrectly the continuum limit of a discrete polygonal orbit due to a sequence of central force impulses. Subsequently, this criticism has been accepted by many Newtonian scholars although some arguments have been presented that it is not valid (Erlichson 1992) (Nauenberg 1998a). For example, in his new translation and guide to Newton’s Principia, I.B. Cohen warmly endorsed Whiteside’s analysis (Cohen 1999) while N. Guicciardini in his new book Reading the Principia questioned whether Newton’s limit arguments in Prop. 1 are well grounded, acknowledging, however, dissenting views (Guicciardini 1999). Other recent authors discussing the Principia either neglected to examine the validity of Newton’s limit arguments (Brackenridge 1995) (Chandrasekhar 1995), or failed to understand them (Densmore 1995).

In this paper we fill in details left out by Newton in his presentation of Prop. 1 which we hope will clarify the content of this proposition, and help resolve the current controversy over Newton’s proof. Specifically, we discuss his mathematical procedure to obtain a continuous orbit as the limit of a discrete polygonal orbit, and we show also that in this limit a sequence of discrete impulses leads to a continuous force with a measure which is proportional to the measure of force which Newton gave in Prop. 6. In this related proposition, Newton started directly with the assumption that the
orbit is a continuous curve which satisfies the area law, and then obtained a measure for the central force by a somewhat different limit argument from the one which he presented in Prop. 1. In Prop. 6 this measure is equal to the acceleration in units of time proportional to the area swept by the radius vector. In the following discussion the reader should consult Props. 1 and 6 which we do not reproduce here except for some quotations and the diagrams shown in Figs. 1 and 4. All of these quotations come from a new English translation of the original Latin version of the *Principia* by I.B. Cohen and Ann Whitman (Cohen 1999).

The current controversy with Prop. 1 arises partly because Newton’s statement of his limit argument is succinct:

“Now let the number of triangles be increased and their widths decreased indefinitely, and their ultimate perimeter $ADF$ will (by lemma. 3, corol. 4) be a curved line ...”

Newton did not give any details how this *indefinite* increase in the number of triangles and the corresponding reduction of their widths would have to be tailored to lead to a well defined and unique continuum limit. It is clear that to understand Newton’s procedure one has to consult Lemma 3 which is given by him as justification for his limit arguments. Remarkably, however, neither Whiteside (MP 6: footnote 19) (Whiteside 1991) nor Aiton (Aiton 1989) commented on this important Lemma, which was also neglected by one of their critics (Erlichson 1992), and by other recent commentators of the *Principia* (Brackenridge 1995) (Chandrasekhar 1995) (Cohen 1999) (Densmore 1995). In Lemmas 2 and 3 and its corrolaries, Newton described how the area bounded by a given curve and a line and the length of the curve can be approximated by a sequence of parallelograms. In Lemma 2, he proved rigorously the existence of a limit for this area by obtaining lower and upper bounds given by the area of the inscribed and circumscribed rectangles of equal width, showing that for a concave curve the difference between these two bounds is the area of the first rectangle. Consequently, as the number of these rectangles increases indefinitely while their widths approach zero this difference vanishes, and the sum of the area of these rectangles approach the same limiting value. By definition, this limit is the area under the curve. Indeed, modern calculus books reproduce Newton’s proof for the area under a curve, but attribute it to either Cauchy or Riemman. In Lemma 3 Newton extended his proof in Lemma 2 to the case of rectangles of unequal width,
Figure 1: This diagram corresponds to Newton's diagram in Prop. 1, but with the lines $Sc, Sd, Se$ and $Sf$ deleted, and with an additional curve through points $ABCDEF$. 
“The same ultimate ratios are also ratios of equality when the widths AB, BC, CD, ... of the parallelograms are unequal and are all diminished indefinitely”.

It is possible that this extension was included in the *Principia* mainly for its application to Prop. 1, because the rectangles which can be associated with the vertices in the corresponding diagram, by taking the initial radial position AS as the horizontal axis, see Fig. 1, would also have unequal widths. In Cor. 2 of this Lemma Newton asserted that

“...the rectilinear figure that is comprehended by the chords of the vanishing arcs...coincides ultimately with the curvilinear figure”

Finally, in Cor. 4 of Lemma 3 he concluded,

“And therefore these ultimate figures (with respect to their perimeter acE) are not rectilinear, but curvilinear limits of rectilinear figures.”

Hence, Newton’s reference to Lemma 3 is conclusive evidence that in Prop. 1 Newton envisioned that the vertices of the polygon in his diagram, Fig. 1, are located on a given geometrical curve which remains fixed as the number of vertices in this polygon increased indefinitely. Newton’s construction of this polygon requires that these vertices all lie on a plane, and consequently this curve must also be planar. Unfortunately, this limit curve was not shown in the diagram associated with Prop. 1, and this apparently has confused readers who did not consult Lemma 3. If any doubt remains for our interpretation, it should be dispelled by Corollaries 2 and 3 of Prop. 1, which were added to later editions of the *Principia*, where Newton called the sides AB and BC of the polygon

“chords of two arcs successively described by the same body in equal times...”.

Referring to Fig. 1, it can be seen that with this interpretation Newton’s entire polygonal construction is determined by fixing the length of the first chord AB. This construction proceeds as follows: the extension Bc of this chord is set equal in length to AB, and the first deflection Cc is determined
by the condition that it is a line parallel to $BS$ starting at $c$ which intersects the given curve at the point $C$. This procedure is iterated with the next chord $BC$ which is now determined, by setting its extension $Cd$ equal in magnitude to $BC$ and the deflection $Dd$ parallel to $CS$, starting at $d$, and ending at the intersection $D$ with the given curve. This iterative process continues until the last point $F$ on the curve is reached. The extension $Bc$ of the chord $AB$ and the deflection $Cc$ must lie on the plane of the initial triangle $SAB$ and therefore the vertex $C$ is also on the same plane. Similarly, this property holds also for all subsequent vertices of this polygon, or as Newton stated in his proposition,

“...making the body ...describe the individual lines $CD, DE, EF, ...$, all these lines will lie in the same plane”

Note, however, that for these lines to intersect a given curve, this curve must also be planar and lie in this plane. The orientation of this plane in space is determined by the initial radial position $AS$ and by the initial chord $AB$ which is in the direction of the initial velocity for the polygonal orbit. Furthermore, in Newton’s continuum limit in which the length of the chord $AB$ vanishes, the plane’s orientation remains unchanged, as the direction of $AB$ approaches that of the initial velocity which is directed along the tangent of the curve at the initial position at $A$.

In Prop.1 Newton did not specify directly how the magnitude of the deflections $Cc, Dd, ...$ are obtained, nor how the magnitude must vary with the number $n$ of vertices. We have seen, however, that these deflections can be determined by the assumption that the polygon vertices are attached to a fixed planar curve. An alternative possibility, which in fact was considered by Hooke, (Nauenberg 1994b, 1998b) is that these deflections depend on the radial distance of the vertices of the polygon. Then to obtain the continuum limit a rule has to be given for how these deflections scale with the number of vertices. In this case Newton would have had to invoke his curvature lemma, Lemma 11, which implies that the deflections scale as the square of the length of the adjacent chords. But instead, in Prop. 1 he referred to Lemma 3 which is relevant to the continuum limit when a fixed curve is given.

Newton’s discrete construction in Prop. 1 refers to a sequence of triangles rather than rectangles as in Lemma 3, but it is easy to see that this lemma remains also valid in this case. There is a detail, however, regarding this application which needs some clarification. As will be shown below, given
a curve of finite length Newton’s polygonal construction does not in general
cover the entire curve for a finite number of triangles. This problem, however,
disappears in the limit that the number of triangles increases indefinitely, and
the proof is given in Appendix A.

Newton’s description of the continuum limit quoted above continues as follows:

“...and thus the centripetal force by which the body is continually
drawn back from the tangent of this curve will act uninterrupt-
edly...”

In Corols. 3 and 4 to Prop. 1 Newton stated that the ratio of forces at
two distinct point on the curve where given by the limit of the ratio of the
displacements caused by central force impulses at these points, but he did
not show that in the continuum limit the measure of this force is proportional
to the measure for force which he gave in Prop. 6. This is another detail
that will be discussed after the next section.

**The historical context of Prop. 1**

Before we begin our discussion of Prop. 1 it is necessary to understand
what Newton’s meant by the term *orbit* when he formulated Prop. 1, In the
Principia the term *orbit* is not defined explicitly, but it has been generally
understood to mean a geometrical curve which describes the position of a
moving body in space. Mathematically an orbit is a continuous curve which
is parameterized by the time variable. In Prop. 1, however, Newton had
to have a more restrictive definition, because he was dealing there with the
special case of the motion of a body under the action of a central force, or in
his words,

“... bodies made to move in orbits.. by...an unmoving center of
force”.

To elucidate this point we turn now to the historical circumstances which led
Newton to discover this crucial proposition.

One of Newton’s first ideas about orbital motion was to consider the ac-
tion of a continuous force as the limiting case of a sequence of force impulses.
As can be seen from his earliest surviving drafts on orbital motion in the
Figure 2: This diagram in Newton’s *Waste Book* shows an octagon inscribed in a circle. Here the deflections $gc, hd...$ are shown from the respective tangents $bg, ch...$ rather than from the extensions of the corresponding chords $ab, bc...$ as shown in Fig. 1.
Waste Book (Herivel 1965) (Whiteside 1991), Newton approximated circular motion by a regular polygon with its vertices located on a circle, see Fig. 2. He also obtained an expression for the continuous force as the limit of force impulses (Brackenridge 1995). Apparently, however, he did not generalize this idea to non-circular motion until shortly after his correspondence in 1679 with Hooke (Nauenberg 1994b) who had suggested in a letter to Newton a somewhat similar conceptual scheme to understand the orbital motion of planets moving around the sun. On Nov. 24, 1679 Hooke had written to Newton:

“But particularly if you will let me know your thoughts of that compounding the celestial motions of the planetts of a direct motion by the tangent and an attractive motion towards the central body...”

Indeed, years later Newton recalled that

“In the year 1679 in answer to a letter from Dr. Hook ... I found now [my italics] that whatsoever was the law of the forces which kept the Planets in their Orbs, the area described by the Radius drawn from them to the Sun would be proportional to the times in which they were described...”

In fact, Prop. 1 appeared for the first time as Theorem 1 in a short manuscript, De Motu, which Newton had sent in 1684 to Halley containing the beginning draft of what became later his Principia. In this manuscript Newton describes the continuum limit in similar words,

“Now let these triangles be infinite in number and infinitely small, so that each individual triangle corresponds to the individual moment of time, the centripetal force acting without diminishing and the proposition will be established”

Apart from the mathematical language, which we would regard today as far less precise than the language in Prop.1, quoted previously, it is noteworthy that Newton gave no reference here to any lemmas which would justify his limit argument. Indeed, he did not give even a hint on how to construct a limiting procedure in De Motu. Nevertheless, Hooke, who was one of the first members of the Royal Society to see this manuscript (Nauenberg
1994b, 1998b) recognized that for a finite number of impulses Newton’s polygonal construction gave an approximate solution for orbital motion along the lines which he had suggested to Newton in his Nov. 24, 1679 letter quoted above. The best evidence for this supposition is that shortly after the appearance of De Motu, Hooke implemented Newton’s construction as an \textit{algorithm} to construct the orbit when the magnitude of the force impulses is proportional to the distance from the center. In Sept. 1685, almost two years before the Principia was published, Hooke obtained by this procedure a remarkably accurate graphical drawing of an elliptical orbit (Nauenberg 1994b, 1998b) with its center located at the center of force by setting the deflections proportional to the distance from the center. An enlarged version of the upper part of his diagram, excluding some auxiliary lines, is shown in Fig. 3.

This reveals quite clearly the relation of Hooke’s diagram to Newton’s diagram in Prop. 1, which is shown in Fig. 1. The main difference is that in Hooke’s figure the initial position is located above the center of force and the initial velocity points to the right, leading to clockwise motion, while in Newton’s figure the initial position is to the right of the center of force and the initial velocity is mainly upwards leading to counter-clockwise motion. Indeed, Hooke had also conjectured that the force of gravity consisted of discrete pulses. In one of his Cutlerian lectures entitled \textit{A Discourse on the Nature of Comets}, read at a meeting of the Royal Society soon after Michaelmas 1682, but published only after his death, Hooke speculated that bodies emitted periodic gravitational pulse in analogy with his theory of sound and light, and deduced that the intensity decreased with the inverse square distance from the source:

“This propagated Pulse I take to be the Cause of the Descent of Bodies towards the Earth... Suppose for Instance there should be a 1000 of these pulses in a Second of Time, then must the Grave body receive all those thousand impressions within the space of that Second, and a thousand more the next...(Hooke 1705)"

But the important question of how the magnitude of the deflection caused by the force impulses scales with the size of the triangles, which is essential to establish the existence of a continuum limit in this application of Newton’s polygonal construction, was not - and undoubtedly could not be - raised by Hooke. Only later did Newton show, in his Lemma 11 on curvature which appeared in section 1 of the Principia, how these deflections scale
Figure 3: This diagram is a blowup of the upper part of Hooke’s Sept. 1685 diagram (see reference [12] with some auxiliary lines deleted to show more clearly its correspondence with Newton’s diagram in Prop. 1, see Fig. 1.
with the size of the adjacent chord or arc length, namely as the square of these quantities.

From the foregoing it is therefore reasonable to conclude that in Prop. 1 Newton had in mind that any orbit under the action of central forces is the continuum limit of a polygonal orbit, which is caused by a sequence of force impulses. In this case the body moves along straight lines between impulses, or by “direct motion” as envisioned by Hooke, where this motion is along the sides of a polygon, while the force impulse give rise to a linear deflection or an “attractive motion” towards the center. Since all the impulses are directed to this common center the resulting polygonal orbit must be in a plane as Newton demonstrated in Prop. 1. The orientation of this plane is determined by the direction of the velocity and the position of the body relative to the center of force at some initial time. Newton was well aware of the important role of initial conditions to fix the orbit, and in Prop.17 he discussed these conditions for the case of elliptical motion under the action of inverse square forces,

“Suppose that the centripetal force is inversely proportional to the square of the distance of places from the center ... it is required to find the line which a body describes when going forth from a given place with a given velocity along a given straight line.”

Setting the sequence of central force impulses at equal time intervals, Newton gave a proof in Prop. 1 that the areas of the triangles associated with the resulting polygonal orbit are equal. Since planarity as well as the this area law are properties of any polygonal orbit due to central force impulses, it is reasonable to expect that these properties remain also valid in a properly defined continuum limit, but Newton did not give any details about how this limit is obtained apart from the brief sentence quoted in our Introduction, and his reference to Cor. 4 Lemma 3. In the next section we will attempt to fill in some of the details left out in Newton’s discussion. We should keep in mind that for any finite polygon the time associated with the vertices of the polygon is different from the corresponding time associated with the continuous orbit. Newton was not always careful about this distinction. For example, in Cor. 2 of Prop.1 quoted above, Newton considered “two arcs successively described in equal times”, when he was referring to the time
intervals defined by his polygonal construction, but for a finite polygon these time intervals differ. The reason is that the area of a finite triangle differs by a small amount from the area of the corresponding “pie” which is bounded by an arc instead of a chord of the orbital curve, and these areas are proportional to time intervals.

**Filling in some details of Newton’s proof of Prop. 1**

Referring to the diagram in Prop. 1, see Fig.1, we assume that the vertices $A, B, C, D, E$ and $F$ of Newton’s polygon are located on a given curve. Since this polygon is planar we expect that this curve should also lie on the same plane. We will show that in the continuum limit, Newton’s polygonal construction determines a parameterization of this curve as a function of time, describing orbital motion under the action of a central force centered at the point $S$, and that in this limit the magnitude of the central force obtained from Prop. 1 is equivalent to that defined in Prop. 6. There are some restrictions on the possible planar curves which can support Newton’s polygonal construction. For example, the radius vector $\vec{r}$ with origin at $S$ cannot become tangential to the curve, because in the neighborhood of any such point Newton’s polygonal approximation cannot be constructed. This construction also fails when the curve crosses this origin, which corresponds to orbital motion when the central force diverges as $1/r^3$ or faster, and when the curvature approaches infinity. Therefore, our discussion will be confined to regions of space where the central force and the curvature of the orbit remain finite.

As shown in the Introduction, given the length of the initial chord $AB$ it is evident that Newton’s polygonal construction is uniquely specified by the condition that the vertices of the polygon lie on a fixed planar curve. That Newton had such a given curve in mind, although it did not appear in the diagram in Prop. 1, is clear from his reference to Corol. 4, Lemma 3 for the continuum limit, as we argued in detail previously. While in this lemma the approximation to a continuous curve is discussed for a subdivision in rectangles, the extension to triangles is quite straightforward, but there is a detail which needs to be worked out: if $A$ is the initial point of the curve then unless the length of the initial chord $AB$ is suitably chosen the last point $F$ of the polygon will in general not lie at the endpoint of the given curve. But in the continuum limit this is not a problem. Suppose that the last vertex
$F$ occurs before the endpoint of the curve. Apply Newton’s construction by extending chord $EF$ to $g$, and draw a line from $g$ parallel to $FS$. Then either 
a) this line intersects the curve at a new vertex $G$ or b) it does not intersect
the curve at all. In case a) repeat Newton’s construction until case b) is
reached. When $F$ is the last vertex, and deflections due to the central force
impulse are small, it is expected that the distance of $F$ from the endpoint
of the curve decreases as the length of the initial chord $AB$ is decreased. Then
in the continuum limit all the cord lengths becomes vanishingly small, and
the last point $F$ reaches the end point of the curve. A rigorous proof for this
assertion is given in Appendix A which is based on a suggestion by Bruce
Pourciau (private communication). These considerations are valid provided
that the curvature of the orbit is finite, in which case the difference between
the chord and the arc length is second order in the chord length.

In Prop. 1, the deflection at each vertex due to the central force impulse
is similar to the deflection that Newton described in Prop. 6. The main
difference is that in Prop. 1 the extension at the end of the adjacent chord
replaces the tangent to the curve in Prop. 6. For example, referring to
Figs. 1 and 4, at vertex $B$ the extension $Bc$ of the chord $AB$ in Prop. 1
corresponds to the tangent line $PR$ in Prop. 6, with $P$ equivalent to $B$,
and the deflection $Cc$ which is parallel to $BS$ in Prop. 1 corresponding to
the deflection $RQ$ which is parallel to $PS$ in Prop. 6. In the limit that the
chord length approaches zero, the difference between the tangent and the
chord becomes vanishingly small, and consequently these two constructions
become similar, except that the magnitude of the deflection at a vertex in
Prop. 1 is twice as large as that in Prop. 6. Hence, in the continuum
limit Newton would have obtained a measure for central force in Prop. 1
equivalent to that in Prop. 6, by dividing the deflection at a vertex, which
depends quadratically on the adjacent chord length, by the square of twice
the area of the triangles. For example, the measure of the force at $B$ which
is the continuum limit of $Cc/({\Delta SBC})^2$, where $\Delta SBC$ is twice the area of
the triangle $SBC$, is twice the measure of the force at $P$ given in Prop. 6,
which is the limit of $QR/({\Delta SPR})^2$, where $\Delta SPR = SP \times QT$ is twice the area of the triangle $SPR$. 
Figure 4: Newton’s diagram for Prop.6
Conclusion

We have shown that apart from some details which we have discussed here, Newton’s polygonal construction in Prop. 1, Fig. 1, has a well defined continuum limit which is justified by Lemmas 2 and 3 in section 1 of the *Princípia*. This limit parameterizes a continuous planar curve as a function of time which describes orbital motion under the action of central forces with origin at $S$. This orbit satisfies a generalization of Kepler’s area law which states that the time interval between any two points on the orbit is proportional to the area swept out by the corresponding radius vector. This definition of an orbit for central forces is the one which Newton introduced in Prop. 6 to evaluate the magnitude of the central force. The planarity property of the orbit is a straightforward consequence of the requirement that an orbital curve is the limit of a Newtonian polygonal construction, because the vertices of this polygon all lie in the same plane when the force impulses are directed to a common center. The orientation of this plane is determined entirely by some initial conditions, i.e., the position and velocity of the moving body at some given time. We also have shown that in the continuum limit these impulses lead to a continuum central force which is proportional to the force measure described in Prop. 6.

Historically, Hooke played an important role in prompting Newton in 1679 to take a new approach to orbital dynamics which led him to prove the area law for central forces (Nauenberg 1994b) which Kepler had found empirically (by fitting the orbit of the planet Mars to the observations of Tycho Brahe). There is evidence that by 1679 Newton had been pursuing a different approach to orbital dynamics based on his development of curvature (Nauenberg 1994a, Brackenridge and Nauenberg 2001). This led him to a local description of central forces in which the area law was not apparent. We conclude that apart from some details left out by Newton, which have been discussed here, Prop. 1 is well grounded and provides a valid proof for central force of the generalization of Kepler’s area law and the property that the orbits lie in a plane.

Appendix A

Following a suggestion by Bruce Pourciau (private communication), we give here a rigorous proof that for an orbit with finite curvature all the chord
lengths in Newton’s polygonal construction in Prop. 1 vanish in a mathematically well defined continuum limit, and that in this limit this construction covers a given length of this orbit. This condition is necessary for the application of Lemma 3 to Prop. 1.

Let \( s(j) \) be the cord length associated with the \( j \)th and \( (j+1) \)th vertices, and \( e(j) = s(j + 1) - s(j) \) the difference in length between adjacent chords. Then \( e(j) \approx d(j)\cos(\theta(j)) \) where \( d(j) \) is the magnitude of the deflection at the \( j \)th vertex and \( \theta(j) \) is the angle between this deflection and the adjacent chord. For an orbit with finite curvature \( d(j) \approx c(j)s(j)^2/\sin(\theta(j)) \), where the constants \( c(j) \) are proportional to the curvature at the \( j \)-th vertex as \( s(j) \) approaches zero. Given an orbital curve, the length of the first chord \( s(1) \) then determines the length of all the other cords in Newton’s polygonal construction, where

\[
s(j) = s(1) + e(1) + e(2) + \ldots + e(j - 1) \tag{1}
\]

for \( j = 2, 3, \ldots n \). Let \( s1 = L/n \) where \( L \) is some fixed length. Then

\[
e(1) = c'(1) \ast (L/n)^2, \tag{2}
\]

and for \( j = 2, 3, \ldots n \)

\[
e(j) = c'(j) \ast (L/n)^2 + o(l/n)^3 \tag{3}
\]

where \( c'(j) = c(j) \ast \cot(\theta(j)) \) and \( o(L/n)^3 \) refer to all terms proportional to \( (L/n)^3 \) and higher powers of \( (L/n) \). Hence, the sum

\[
e(1) + e(2) + \ldots + e(j) = (L/n)^2 \ast (c(1) + c(2) + \ldots + c(j)) + o(L/n)^3 \tag{4}
\]

Now suppose that \( c \) is the maximum curvature of the orbit. Then

\[
e(1) + e(2) + \ldots + e(j) < (L/n)^2 \ast (j - 1) \ast c + o(L/n)^3, \tag{5}
\]

and

\[
e(1) + e(2) + \ldots + e(n) < (L/n)^2 \ast (n - 1) \ast c + o(L/n)^2 \tag{6}
\]

As \( n \) goes to infinity both of these two sums go to zero, and therefore, according to Eq. 4 all the chords \( s(j) \) vanish in this limit. Therefore in this limit Newton’s polygonal construction in Prop. 1 covers any chosen segment of the orbit with a length given by the limit of the sum \( s(1) + s(2) + s(3) + \ldots + s(n) \). This length depends on the parameter \( L \), but it is equal to \( L \) only in the special case of a circular orbit. In this case the angle \( \theta(j) \) is equal \( 90^\circ \), and the \( e(j) \)'s vanishes as \( (L/n)^3 \)
Appendix B

We express the content of Prop. 1 in modern vector notation, which may be helpful in clarifying some of the main points which we emphasized previously. Assuming that there are \( n \) vertices in the polygon in Fig. 1 of Prop. 1, let \( \vec{r}_j \) be the position vector and \( \vec{v}_j \) the velocity vector at the \( j \)-th vertex where \( j = 1, 2, \ldots n \). Then Newton’s construction take the the form

\[
\vec{r}_{j+1} = \vec{r}_j + \vec{v}_j \Delta t,
\]

and

\[
\vec{v}_{j+1} = \vec{v}_j + \Delta \vec{v}_{j+1},
\]

where \( \Delta t \) is the equal time interval between impulses introduced by Newton, and \( \Delta \vec{v}_j \) is instantaneous velocity change \( \Delta \vec{v}_j = \vec{d}_j / \Delta t \) where \( \vec{d}_j \) is the deflection at the \( j \)-th vertex. Then the crossed product of Eqs. 7 and 8

\[
\vec{r}_{j+1} \times \vec{v}_{j+1} = \vec{r}_j \times \vec{v}_j + \vec{r}_{j+1} \times \Delta \vec{v}_{j+1}
\]

where \((1/2)|\vec{r}_j \times \vec{v}_j|\) is the area of the triangle associated with the \( j \)-the vertex. Since the deflection \( \vec{d}_j \) and corresponding velocity change \( \Delta \vec{v}_j \) are parallel to \( \vec{r}_j \), which is Newton’s definition of central force impulses in Prop. 1, then the last term in Eq. 9 vanishes, and consequently a) the area of the triangles are equal, and b) the vertices of the polygon lie on a plane. This constitutes Newton’s proof of Prop. 1 in the language of vector calculus.

The problem of the continuum limit is to describe how the time interval \( \Delta t \) and the deflections \( \vec{d}_j \) should vary as \( n \) approaches infinity. Newton’s statement

“Let the time be divided in equal times...”

corresponds to setting \( \Delta t = T/n \), where \( T \) is some finite time interval, and his reference to Cor. 4 Lemma 3 implies that the deflections \( \vec{d}_j \) are determined by the condition that the vertices of the polygon are located on a given orbital curve. This curve can be described by a vector \( \vec{R}(u) \) where \( u \) is a scalar parameter which can be chosen arbitrarily. For example, \( u \) can be the area swept by this radius vector with origin at the center of force in which case it will turn out to be proportional to time for the continuum orbit. But it can also be the arc length, or the angular variable in polar coordinates which
are non-linear functions of time. Then the condition that the j-th vertex is located on this curve is given by

$$\vec{r}_j = \vec{R}(u_j), \quad (10)$$

where \( u_j \) is the value of \( u \) at the position of this vertex. It can be readily seen that in this case

$$\vec{d}_j = \vec{R}(u_{j+1}) + \vec{R}(u_{j-1}) - 2\vec{R}_j, \quad (11)$$

and as \( \Delta t \) vanishes the ratio

$$\frac{d_j}{(v_j \Delta t)^2} \quad (12)$$

where \( d_j = |\vec{d}_j| \) and \( v_j = |\vec{v}_j| \), is proportional to the curvature of the orbit at time \( t \). Hence this ratio as a well defined limit in the case that this curvature is finite. Likewise,

$$\frac{\Delta v_j}{\Delta t} = \frac{\vec{d}_j}{\Delta t^2}. \quad (13)$$

has a continuum limit corresponding to the standard calculus definition of the acceleration or force/unit mass \( \vec{a} \)

$$\vec{a} = \lim_{n \to \infty} \frac{\Delta v_j}{\Delta t} = \frac{d^2 \vec{R}}{dt^2} \quad (14)$$

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On p. 99 Densmore claims that “the bases of triangles in the proposition (Prop. 1) ... do not circumscribe and are not inscribed in, this ultimate curve, nor do they connect to it or follow it in any other way as a kind of ‘ghost curve’ ”, and she claims that “Newton has not offered an argument that the limiting procedure in the proposition is a unique curve...”. But Densmore ignores the fact that Newton invoked cor. 4, Lemma 3 to justify his limiting procedure.

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In this paper, some of the criticisms of Whiteside’s analysis of Prop. 1 are wrong, because Whiteside correctly identified the deflections due to the force impulses as “second order infinitesimal magnitudes...”.

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