GALACTIC FOREGROUNDS: SPATIAL FLUCTUATIONS AND A PROCEDURE FOR REMOVAL

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ABSTRACT

Present-day cosmic microwave background (CMB) studies require more accurate removal of the Galactic foreground emission. This removal becomes even more essential for CMB polarization measurements. In this paper, we consider a method for filtering out the diffuse Galactic fluctuations on the basis of their statistical properties, namely, the power-law spectra of fluctuations. We focus on the statistical properties of two major Galactic foregrounds that arise from magnetized turbulence, namely, the diffuse synchrotron emission and the thermal emission from dust, and describe how their power laws change with the Galactic latitude. We attribute this change to the change in the geometry of the emission region and claim that the universality of the turbulence spectrum provides a new way of removing Galactic foregrounds. For the Galactic synchrotron emission, we mainly focus on the geometry of the synchrotron emitting regions, which will provide useful information for future polarized synchrotron emission studies. Our model calculation suggests that either a one-component extended halo model or a two-component model, an extended halo component (scale height $\gtrsim 1$ kpc) plus a local component, can explain the observed angular spectrum of the synchrotron emission. For thermal emission from Galactic dust, we discuss general properties of a publicly available 94 GHz total dust emission map and explain how we can obtain a polarized dust emission map. Based on a simple model calculation, we obtain the angular spectrum of the polarized dust emission. Our model calculation suggests that $C_\ell \propto l^{-11/3}$ for $l \gtrsim 1000$ and a shallower spectrum for $l \lesssim 1000$. We discuss and demonstrate how we can make use of our findings to remove Galactic foregrounds using a template of spatial fluctuations. In particular, we consider examples of spatial filtering of a foreground at small scales, when the separation into CMB signal and foregrounds is done at larger scales. We demonstrate that the new technique of spatial filtering of foregrounds may be promising for recovering the CMB signal in a situation where foregrounds are known at a scale different from the one being studied. It can also improve filtering by combining measurements obtained at different scales.

Key words: cosmic background radiation -- Galaxy: structure -- ISM: general -- magnetohydrodynamics (MHD) -- turbulence

1. INTRODUCTION

An important problem in the studies of the early universe with the cosmic microwave background (CMB) fluctuations is related to separating them from Galactic foregrounds. The techniques of removing foregrounds are rather elaborate, but in most cases they include using the frequency templates of foreground emission. This requires multi-frequency measurements, which are not always available. Moreover, some foregrounds, e.g., the so-called spinning dust (Draine & Lazarian 1998a, 1998b; Finkbeiner et al. 2002; Lazarian & Finkbeiner 2003), demonstrate rather complex frequency dependence. Due to the utmost importance of obtaining CMB signal free of contamination, it is essential to consider other ways to remove foregrounds. One way to do this is to take into account the known spatial properties of the emission.

If a foreground has well-defined statistics of spatial fluctuations, one can devise techniques for removing the contribution of the foreground to the measured microwave signal (see Section 2). The issue in this case is whether foregrounds have well-defined behavior in terms of their spatial statistics. Determining this with the available data is the first thrust of our present study, which we pursue using the Galactic synchrotron and the Galactic dust emission data. The second thrust is devising possible ways of removing the foregrounds using the self-similarity of Galactic turbulence that gives rise to the foreground fluctuations.

This work continues our brief study in Cho & Lazarian (2002a, henceforth CL02) where we argued that the properties of Galactic foreground radiation can be explained by accepting that the interstellar medium (ISM) that provides the fluctuations is turbulent and therefore the spatial fluctuations of foregrounds inherit the power-law spectra of the underlying magnetohydrodynamic (MHD) turbulence. Since that work, better understanding of the properties of MHD turbulence has been achieved. For instance, it has become clear that the spectrum of density in compressible MHD turbulence can be substantially shallower than the Kolmogorov spectrum, which we assumed in CL02 (see Beresnyak et al. 2005). Moreover, the search for alternative procedures for removing foregrounds became more essential with the attempts to measure the polarization of the CMB radiation, especially the enigmatic $B$-modes. The latter motivates the choice of foregrounds that we deal with in this paper. Synchrotron and dust emissions are the sources of the polarized contamination for the CMB polarization studies.

Diffuse Galactic synchrotron emission is an important polarized foreground source, the understanding of which is essential for CMB studies, especially, in the range of 10–100 GHz. It is known that the observed spectra of synchrotron emission and synchrotron polarization (see de Oliveira-Costa & Tegmark 1999 and references therein) reveal a range of power laws. The polarization of synchrotron emission traces magnetic fields and is perpendicular to the plane-of-sky magnetic field direction. Since the Galactic synchrotron emissivity is roughly proportional to the magnetic energy density, the angular spectrum of synchrotron emission reflects the statistics of magnetic field fluctuations in the Galaxy (see Appendix A.1 for discussions).
Thermal emission from dust is also an important source of polarized foreground emission in the range of frequencies larger than 60 GHz. The emission gets polarized due to grain alignment (see Lazarian 2007 for a review). Therefore, the polarization of dust, similar to synchrotron polarization, traces magnetic field fluctuations.

What is the cause of magnetic field fluctuations? As magnetic field lines are twisted and bent by turbulent motions in the Galaxy, it is natural to think of the turbulence as the origin of magnetic field fluctuations. In fact, several earlier studies addressed the relation between turbulence and the diffuse synchrotron foreground radiation. Tegmark et al. (2000) suggested that the spectra may be relevant to Kolmogorov turbulence. Chepurnov (1999) and CL02 used different approaches, but both showed that the angular spectrum of synchrotron emission reveals Kolmogorov spectrum ($C_l \propto l^{-11/3}$) for large values of multipole $l$. However, they noted that the spectrum can be shallower than the Kolmogorov one for smaller values of multipole $l$, due to density stratification in the halo (Chepurnov 1999) or the Galactic disk geometry (CL02).3

Recent research in compressible magnetized turbulence suggests that the fluctuations may not be necessarily Kolmogorov, to start with (see Beresnyak et al. 2005; Kowal & Lazarian 2007). Nevertheless, both observations and numerical studies confirm the power-law dependence of turbulence, even if the spectrum differs from the Kolmogorov one. This is also supported by theory, which states the self-similarity of turbulence. It is this self-similarity that gives us hope for successful removal of foregrounds arising from turbulent ISMs.

At the same time, it is known that the spatial spectral indexes of foregrounds measured for different Galactic latitudes may differ. If the reason for this is unknown, this may make the weeding out of foregrounds using spatial templates unreliable. CL02 identified these changes with the variations in the geometry of the emission region. Thus, for the same Galactic latitude one should expect the same slope of the spatial spectrum of the fluctuations, which, for instance, allows one to extend the power-law spectrum of the fluctuations measured for low spherical harmonics to higher spherical harmonics. Potentially, if the geometry of the emission region is known, this allows us to predict the expected changes of the index.5 In this paper, we provide more support for the conjecture in CL02.

In this paper, we first present the general properties and structure functions of a publicly available synchrotron foreground emission map. Then we investigate what kinds of Galactic halo structures can produce the observed structure function (and therefore angular spectrum), which will be useful for the study of polarized synchrotron foreground. We also present the properties of a publicly available model dust emission map. Dust emission is one of the most important sources of polarized foreground radiation. Therefore, measurement of the angular power spectrum of such foreground is of great interest. Thus, we provide an estimation of the angular spectrum of polarized emission by foreground dust. This result is of great importance in view of the recent interest in CMB polarization. In Section 2, we explain a method for spatial removal of foreground emission and provide the summary of the expected scaling of foreground fluctuations arising from Galactic turbulence. In Section 3, we present statistical analysis of the Haslam map, which is dominated by diffuse Galactic synchrotron emission. In Section 4, we investigate polarized emission from thermal dust. In Section 5, we discuss how to utilize our knowledge to remove Galactic foregrounds. We provide the discussion of our results in Section 6 and the summary in Section 7. In Appendix A, we review a simple model of the angular spectrum of synchrotron emission arising from MHD turbulence. In Appendix D, we present calculations of high-order structure functions of the synchrotron and the dust maps, and we compare the results with those of turbulence.

2. MOTIVATION: A NEW TECHNIQUE FOR FOREGROUND REMOVAL

2.1. Spatial Removal of Foregrounds

Let us illustrate a possible procedure for the removal of Galactic foregrounds from the CMB signal. The cosmic microwave signals consist of the CMB signal $\tilde{F}^{\text{CMB}}$ and foregrounds $\tilde{F}$. When we correlate the microwave signal at points “1” and “2,” we get

$$\langle I_{1\text{CMB}}^2 + I_{1\text{CMB}} I_{2\text{CMB}} + I_{2\text{CMB}}^2 \rangle = \langle I_{1\text{CMB}}^2 I_{2\text{CMB}}^2 \rangle + \langle I_{1\text{CMB}} I_{2\text{CMB}} \rangle.$$  

where we assume $\tilde{F}^{\text{CMB}}$ and $\tilde{F}$ are uncorrelated. Therefore, the measured angular spectrum $C_{l}^{\text{measured}}$ is just the sum of $C_{l}^{\text{CMB}}$ and $C_{l}^{\text{F}}$, and we have

$$C_{l}^{\text{CMB}} = C_{l}^{\text{measured}} - C_{l}^{\text{F}}.$$  

This means that, if we know the angular spectrum of foregrounds $C_{l}^{\text{F}}$, we can obtain the CMB angular spectrum $C_{l}^{\text{CMB}}$.

How can one obtain $C_{l}^{\text{F}}$? A well-tested way of doing this is to use multi-frequency measurements of the CMB + foreground emission and separate the two components using the frequency templates of the foregrounds. This approach requires many measurements at different frequencies. In addition, for some foregrounds the frequency templates may be difficult to obtain. The so-called spinning dust foreground introduced in Draine & Lazarian (1998a, 1998b) presents an example of such a difficult-to-remove foreground. We also mention that the measurements of foregrounds at different frequencies may have different spatial resolutions and the use of the maps with different resolutions may also present a problem.

In this paper we address a somewhat different problem, which at its extreme6 can be formulated in the following way. Imagine that we separated the foreground and the CMB signals at low resolution $l_{\text{low}}$ using the traditional multi-frequency approach. Is it possible to use this information to remove the foreground contribution from $C_{l}^{\text{measured}}$ for $l > l_{\text{low}}$? For instance, the measurements of the Wilkinson Microwave Anisotropy Probe (WMAP) provide a high accuracy measure of $C_{l}^{\text{F}}$ over a limited range of scales. If we know $C_{l}^{\text{F}}$ as a function of $l_{\text{low}}$ at scales smaller than those measured by WMAP, then one can extrapolate $C_{l}^{\text{F}}$ to $l > l_{\text{low}}$. These values of $C_{l}^{\text{F}}$ can be used to filter the microwave measurements from balloon-borne experiments.

3 See Chepurnov (1999) and CL02.
4 In both approaches larger emissivity toward the disk plane is employed.
5 Inverting arguments in CL02, one can use the changes of the foreground spectra to model the geometry of the emitting volume.
6 Less extreme cases would involve the use of the known spatial properties of $C_{l}^{\text{F}}$ to increase the accuracy of the removal of foregrounds within traditional techniques. We do not discuss these more sophisticated procedures in this paper.
using the procedure given by Equation (2). Note that the balloon-
borne experiments usually have higher spatial resolutions, but
not enough frequency coverage to remove foregrounds using the
frequency templates.

The key question is to what extent we can predict $C_j^F$ over
a range of scales that is different from the range of scales
at which $C_j^F$ was measured. The answer to this question is
trivial if $C_j^F$ is a simple power law. While the actual spectra
of foregrounds are more complex, in this paper we provide both
theoretical arguments and the analysis of the foreground data,
which support the notion that $C_j^F$ can be successfully extended
beyond the range of $l$ that is measured.

We should stress that the filtering above is different from the
accepted techniques of foreground removal using frequency-
dependent templates. The outcome of the latter procedures are
maps of foreground radiation and CMB radiation. The filtering
described above is of statistical nature. Its result is $C_j^{\text{CMB}}$ rather
than emission maps.\footnote{It is easy to see that the phase information of foreground emission is lost in
the process of such a filtering. However, this information is not necessary for
$C_j^{\text{CMB}}$ recovery.}

In view of the above, it is important to determine to what
extent the spatial properties of $C_j^F$ are predictable, in partic-
ular, explore to what extent the power-law approximation is
applicable. This paper provides a study of the spatial statistical
properties of fluctuations of synchrotron and dust emission
and relates those to the properties of the underlying turbulence.
It also provides an example of the filtration procedure that we
advocate.

Another important issue is to determine the reasons for the
change in the power-law behavior with latitude. This is what we
study below for the synchrotron and dust foreground emissions.
The ultimate goal of this research is to obtain models of Galactic
foregrounds that provide a good fit for the foreground spatial
spectrum at arbitrary scales and arbitrary latitudes. This paper
is a step toward constructing such a model.

2.2. Power-law Behavior of Interstellar Turbulence

For the spatial removal procedure to work, we should under-
stand the spatial spectra of foregrounds. Since the spatial
fluctuations of foregrounds inherit the spectra of the underlying
interstellar turbulence, we summarize the spectral behavior of
interstellar turbulence.

It is generally accepted that the ISM is magnetized and
turbulent (see reviews by Elmegreen & Scalo 2004 and McKee
& Ostriker 2007). The so-called Big Power Law in the Sky
corresponding to the Kolmogorov-type turbulence was reported
in Armstrong et al. (1995). Recently, this law, based on the
measurements of the radio scintillations arising from electron
density inhomogeneities, has been extended to larger scales
using the Wisconsin H\alpha Mapper H\alpha fluctuations (Chepurnov
& Lazarian 2010).

Spectra of magnetic turbulence have been studied using
Faraday rotation measurements (see Havercorn et al. 2008) and
starlight polarization (Hildebrand et al. 2009). The interpretation
of the results is more challenging in those cases.

Molecular and atomic spectral lines present a very promising
way of studying turbulence. The interstellar lines are known
to be Doppler-broadened due to turbulent motions. Obtaining
spectra from Doppler-broadened lines is not trivial, however.
A significant portion of the research in this direction based
on the use of the so-called velocity centroids, which were the
main tool for studying velocity fluctuations, has been shown to
produce erroneous results for supersonic turbulence (Lazarian
& Esquivel 2003; Esquivel & Lazarian 2005; Esquivel et al.
2007). At the same time, new techniques based on the theoretical
description of the position–position–velocity (PPV) data cubes,
namely the Velocity Channel Analysis (VCA) and the Velocity
Coordinate Spectrum (VCS), have been developed (Lazarian
& Pogosyan 2000, 2004, 2006, 2008), tested (see Stanimirovic
& Lazarian 2001; Lazarian et al. 2002; Esquivel et al. 2003;
Chepurnov & Lazarian 2009; Padoan et al. 2006, 2009), and
applied to the observational data to obtain the characteristics
of the velocity turbulence (see Lazarian 2009 for a review).

The VCA and VCS techniques reveal that the velocities for
interstellar turbulence may be somewhat steeper than the Kol-
mogorov one, while the density spectra of the fluctuations may
be substantially shallower than the spectrum of Kolmogorov
fluctuations (see Padoan et al. 2006, 2009; Chepurnov et al.
2010). This agrees well with the numerical studies of the mag-
etized supersonic turbulence (Beresnyak et al. 2005; Kowal
et al. 2007; Kowal & Lazarian 2010). For the subsonic tur-
bulence, the spectrum of magnetized media attains values close
to the Kolmogorov index (see Goldreich & Sridhar 1995; Lazarian
& Vishniac 1999; Cho & Vishniac 2000; Müller & Biskamp
2000; Maron & Goldreich 2001; Lithwick & Goldreich 2001;
Cho et al. 2002; Cho & Lazarian 2002b, 2003; Boldyrev 2006;
Beresnyak & Lazarian 2006, 2009). In view of that, we believe
that the Kolmogorov spectrum can be used as a proxy for the
underlying spectra of the velocity and magnetic field, while a
more cautious approach should be adopted for the density in
highly compressible environments, e.g., molecular clouds.

Below we shall show that the spectra of turbulence derived
from the analysis of the foreground are consistent with both the
results of dedicated observations of turbulence as well as the
theoretical expectation for MHD turbulence.

2.3. Data Sets

We use the 408 MHz Haslam all-sky map (Haslam et al. 1982)
and a model 94 GHz dust emission map that are available on
NASA’s LAMBDA Web site.\footnote{http://lambda.gsfc.nasa.gov/}
Both maps were reprocessed for HEALPix (Görski et al. 2005) with nside $= 512 (7^2$ resolution).

The original Haslam data were produced by merging several
different data sets. “The original data were processed in both
the Fourier and spatial domains to mitigate baseline striping
and strong point sources” (see the Web site for details). The
angular resolution of the original Haslam map is $\sim 1^\circ$. Galactic
diffuse synchrotron emission is the dominant source of emission
at 408 MHz.

The 94 GHz dust emission map is based on the work
of Schlegel et al. (1998) and Finkbeiner et al. (1999).
Schlegel et al. (1998) combined 100 $\mu$m maps of the
Infrared Astronomical Satellite (IRAS) and the Diffuse Infrared Back-
ground Experiment (DIRBE) on board the Cosmic Background
Explorer (COBE) satellite and removed the zodiacal foreground
and point sources to construct a full-sky map. Finkbeiner et al.
(1999) extrapolated the 100 $\mu$m emission map and 100/240 $\mu$m
flux ratio maps to submillimeter and microwave wavelengths.
The 94 GHz dust map that we used is identical to the two-
component model 8 of Finkbeiner et al. (1999). The angular
resolution of the 100 $\mu$m map is $\sim 6^\prime$ and that of the
temperature correction derived from the 100/240 $\mu$m ratio

\begin{equation}
\frac{\mu}{240} m \text{ ratio}
map is \( \sim 1^\circ \) (Finkbeiner et al. 1999), which corresponds to \( l \sim 180^\circ /\theta^\circ \sim 180 \).

3. SPATIAL STATISTICS OF DIFFUSE GALACTIC SYNCHROTRON EMISSION

In this section, we analyze the Haslam 408 MHz all-sky map, which is dominated by Galactic diffuse synchrotron emission. Our main goal is to explain the observed synchrotron angular spectrum using simple turbulence models. The results in this section will be useful for the sophisticated modeling of polarized synchrotron emission.

3.1. General Properties of Diffuse Galactic Synchrotron Emission

In this section, we separately study the synchrotron emission from the Galactic halo (i.e., \( b \gtrsim 30^\circ \)) and the Galactic disk (i.e., \( |b| \lesssim 2^\circ \)). Our main goal is to see if the statistics of the synchrotron emission from the halo is consistent with turbulence models. When it comes to synchrotron emission from the Galactic disk, it is not easy to separate diffuse emission and emission from discrete sources. Therefore, we do not try to study turbulence in the Galactic disk. Instead, we will try to estimate which kind of emission is dominant in the Galactic disk.

There exist several models for diffuse Galactic radio emission. Beuermann et al. (1985) showed that a two-component model, a thin disk embedded in a thick disk, can explain the observed synchrotron latitude profile. They claimed that the equivalent width of the disk is about several kiloparsecs and the thin disk has approximately the same equivalent width as the gas disk. They assumed that, in the direction perpendicular to the Galactic plane, the emissivity \( \epsilon \) of each component follows

\[
\epsilon(z) = \epsilon(0) \text{sech}^2(z/z_0),
\]

where \( z \) is the distance from the Galactic plane and \( \epsilon(0), b, \) and \( z_0 \) are constants. The half-equivalent width of the disk, which is proportional to \( z_0 \), at the location of the Sun is \( \sim 2 \) kpc.

Recently, several Galactic synchrotron emission models have been proposed in an effort to separate Galactic components from the WMAP polarization data (see, for example, Page et al. 2007; Sun et al. 2008; Miville-Deschênes et al. 2008; Waelkens et al. 2009). All the models mentioned above assume the existence of a thick disk component with a scale height equal to 1 kpc. Sun et al. (2008) considered an additional local spherical component motivated by the local excess of the synchrotron emission, which might be related to the “local bubble” (see, for example, Fuchs et al. 2009).

Detailed modeling of the Galactic synchrotron emission is beyond the scope of our paper. We will simply assume that there is a thick component with a scale height of \( \sim 1 \) kpc. We will also assume that there could be an additional local spherical component. Then, in the following subsections, we will consider the relation between the spectrum of three-dimensional (3D) turbulence and the observed angular spectrum of the synchrotron emission.

3.2. Structure Function of the 408 MHz Haslam Map

In Appendix A, we discussed the relation between the 3D spatial MHD turbulence spectrum and the observed two-dimensional (2D) angular spectrum of the synchrotron emission (see Equation (A11) for a quick summary). But, in the appendix we assumed that the emission arises from a spherical region filled with homogeneous turbulence. In this section, we will show that the modulation of the synchrotron intensity of the emitting volume can also affect the observed angular spectrum of the synchrotron emission.

Earlier studies showed that the angular spectrum of the 408 MHz Haslam map has a slope close to \( -3; C_l \propto l^{-3} \) (Tegmark & Efstathiou 1996; Bouchet et al. 1996). Recently, La Porta et al. (2008) performed a comprehensive angular power spectrum analysis of all-sky total intensity maps at 408 MHz and 1420 MHz. They found that the slope is close to \( -3 \) for high Galactic latitude regions. Other results also show slopes close to \( -3 \). For example, using Rhodes/HartRAO data at 2326 MHz (Jonas et al. 1998), Giardino et al. (2001b) obtained a slope of \( \sim 2.92 \) for high Galactic latitude regions with \( |b| > 20^\circ \). Giardino et al. (2001a) obtained a slope of \( \sim 3.15 \) for high Galactic latitude regions with \( |b| > 20^\circ \) from the Reich & Reich (1986) survey at 1420 MHz. Bouchet & Gispert (1999) also obtained a slope of \( \sim l^{-3} \) spectrum from the 1420 MHz map.

In general, synchrotron emission from the Galactic disk makes it difficult to measure the angular spectrum of the synchrotron emission from the Galactic halo. In order to avoid the contamination by the Galactic disk, one may mask out the low Galactic latitude regions. This can be done, for example, by setting all synchrotron intensity to zero for regions with \( |b| < b_{\text{cut}} \). However, the angular spectrum obtained with the Galactic mask exhibits spurious oscillations. Moreover, the spectrum obtained with a mask may not be the true one because it is contaminated by the mask. In principle, one may correct such oscillations and estimate the true spectrum using the convolution theorem: the Fourier coefficients (or, in this case, the spherical harmonic coefficients) of the masked data are a convolution of those of the true data and those of the mask. However, the practical implementation of the method is not simple.

Another approach is to first estimate a two-point correlation function and to then extract the angular power spectrum from it (Szapudi et al. 2001; see also Equation (10)). This method is free of artifacts caused by the mask. But the angular spectrum \( C_l \) obtained in this way is, in general, noisy and requires a large number of calculations to accurately measure the slope of the spectrum.

We are interested in the slope of the angular spectrum on small angular scales, which is the same as that of the underlying 3D spatial turbulence spectrum (see Appendix A). Therefore, in this paper, we use yet another approach. We first calculate the second-order angular structure function:

\[
D_2(\theta) = \langle |I(\mathbf{e}_1) - I(\mathbf{e}_2)|^2 \rangle,
\]

where \( I(\mathbf{e}) \) is the intensity of the synchrotron emission, \( \mathbf{e}_1 \) and \( \mathbf{e}_2 \) are unit vectors along the lines of sight, \( \theta \) is the angle between \( \mathbf{e}_1 \) and \( \mathbf{e}_2 \), and the angle brackets denote an average taken over the observed region. Then, we extract the slope of the angular spectrum using the relation

\[
D_2(\theta) \propto \theta^{m-2} \Rightarrow C_l \propto l^{-m} \Rightarrow E_{3D} \propto k^{-m}
\]

for small angular scales (see Appendix A.2).

We note that excessive care is required in the presence of white noise. In this case, the second-order structure function will be

\[
D_2(\theta) = \langle |I(\mathbf{e}_1) + \delta_1 - I(\mathbf{e}_2) - \delta_2|^2 \rangle = \langle |I(\mathbf{e}_1) - I(\mathbf{e}_2)|^2 \rangle + \langle |\delta_1 - \delta_2|^2 \rangle,
\]

where \( \delta_1 \) and \( \delta_2 \) represent the noise (A. Chepurnov 2008, private communication). If the second term on the right
smaller than the values of latitude. From bottom to top, the second-order angular structure functions are obtained for thin stripes ($|\Delta \theta| \leq 2^\circ$) along Galactic latitudes of $30^\circ$, $10^\circ$, and $0^\circ$.

We use geometry is not plane-parallel, but spherical. In what follows, the observer is at the center of a spherical halo. That is, the Galactic plane and $b$ are negligible. The reason is as follows. Our measurements show that $D_2(0;0.15) \sim 0.05$ in the Haslam map. This means that the term is no larger than 0.05, which is sufficiently smaller than the values of $D_2(\theta)$ shown in Figure 1.

In the left panel of Figure 1, we show the second-order structure function for the Galactic halo (i.e., $|b| > 30^\circ$). The slope of the second-order structure function lies between those of two straight lines. The steeper line has a slope of $4/3$, which is shallower than the Kolmogorov spectrum $E_{3D}(k) \propto k^{-11/3}$.

Now the question arises: why is the slope shallower than that of the Kolmogorov spectrum? Chepurnov (1999) provided a model calculation of two straight lines. The steeper line has a slope of $1$. The actual measured slope is $\sim 1.2$. This result implies that the 3D turbulence spectrum is $E_{3D}(k) \propto k^{-3.2}$, which is shallower than the Kolmogorov spectrum $E_{3D}(k) \propto k^{-11/3}$.

We numerically calculate the angular correlation function $w(\theta)$ and the second-order structure $D_2(\theta)$ from

$$ w(\theta) = \int dr_1 \int dr_2 K(|r_1 - r_2|) e(r_1) e(r_2), \quad (6) $$

$$ D_2(\theta) \propto \overline{T} - w(\theta), \quad (7) $$

where $|r_1 - r_2| = r_1^2 + r_2^2 - 2r_1 r_2 \cos \theta$, $e(r)$ is the synchrotron emissivity, $T = \lim_{\theta \to 0} w(\theta)$, and we use the spatial correlation function $K(r)$ obtained from the relation

$$ K(r) \propto \int_0^\infty 4\pi k^2 E_{3D}(k) \frac{\sin kr}{kr} dk, \quad (8) $$

where the spatial spectrum of emissivity $E_{3D}$ has the form

$$ E_{3D}(k) \propto \begin{cases} \text{constant} & \text{if } k \leq k_0, \\ (k/k_0)^{-11/3} & \text{if } k \geq k_0, \end{cases} \quad (9) $$

which is the same as the Kolmogorov spectrum for $k \geq k_0$ ($\sim 1/L$). The reason we use a constant spectrum for $k \leq k_0$ is explained in Appendix B (see also Chepurnov 1999). We obtain the angular spectrum from the relation

$$ C_1 \propto \int P_l(\cos \theta) K(\cos \theta) d(\cos \theta), \quad (10) $$

where $P_l$ is the Legendre polynomial. In Figure 3, we plot the calculation results. The angular correlation function $w(\theta)$ does not change much when $\theta$ is small, and follows $\sim (\pi - \theta)/\sin \theta \sim 1/\theta$ when $\theta$ is large.
The critical angle is a few degrees for the homogeneous model (thick solid curve) and the single-component exponential model (dotted curve). As we discussed earlier, the critical angle for homogeneous turbulence is \(\sim (L/d_{\text{max}})^{\text{ad}} \sim 6^\circ\), where \(d_{\text{max}}\) (=1 kpc in our model) is the distance to the farthest eddy. In Figure 3 (left panel), we clearly see that the slope of \(w(\theta)\) changes near \(\theta \sim 6^\circ\). The second-order structure function \(D_2(\theta)\) also shows a change in the slope near the same critical angle \((\theta \sim 6^\circ)\). In the single-component exponential model (dotted curve), the value of \(d_{\text{max}}\) is not important. Instead, the scale height \(z_0\) is a more important quantity, which is 1 kpc in our model. In the left and middle panels of Figure 3, we observe that the single-component exponential model also shows a change of slope near \(\theta \sim \) a few degrees. Therefore, we can interpret that the critical angle for stratified turbulence is \(\sim L/z_0\), instead of \(\sim L/d_{\text{max}}\).

Now, it is time to answer our earlier question of why the observed slope is shallower than that of Kolmogorov. Let us take a look at the right panel of Figure 3. All three models show that the slope of \(C_l\) is almost the Kolmogorov one for \(l \gtrsim 200 > l_{\text{cr}} \sim \pi d_{\text{max}}/L \sim 30\). However, if we measure the average slope of \(C_l\) between \(l = 10\) and 200, we obtain slopes shallower than the Kolmogorov one. The single-component model and the homogeneous model give similar average slopes around \(-3\). However, the homogeneous model gives a more abrupt change of slope near \(l \sim 30\). In fact, the right panel of Figure 3 shows a noticeable break near \(l \sim 50\). The average slope of the two-component model gives a more or less gradual change of the slope and the observed slope is very close to \(-3\) for a broad range of multipoles \(l\). It is difficult to tell which model is better because the models are highly simplified. Nevertheless, the two-component model looks the most promising, which is not so surprising because the two-component model has more degrees of freedom.

Note that, compared with the two-component model, the single-component model shows a more or less sudden change of slope near \(l \sim l_{\text{cr}} \sim \pi z_0/L \sim 30\). Therefore, if the single-component model is correct, the scale height \(z_0\) cannot be much larger than \(\sim 10\) times the outer scale of turbulence \(L\). If \(z_0\) is much larger than \(\sim 10 L\), \(l_{\text{cr}}\) becomes smaller and we will have an almost Kolmogorov slope for \(l \gtrsim 10\). We also note that it is possible that the 3D spatial turbulence spectrum itself can be shallower than the Kolmogorov one. That is, it is possible that the spectrum of \(B(r)\), hence that of \(B^2(r)\), can be shallower than the Kolmogorov one. For example, some recent studies show that strong MHD turbulence can have a \(\sim k^{-3.5}\) spectrum, rather than \(k^{-11/3}\) (Maron & Goldreich 2001; Boldyrev 2006; Beresnyak & Lazarian 2006). If this is the case, the observed angular spectrum can be slightly shallower than the Kolmogorov for \(l > l_{\text{cr}}\).

To summarize this subsection, our simple model calculations imply that \(C_l\) will be very close to the underlying spectrum of magnetic turbulence for large values of \(l (l > \text{a few times } 100)\). The corresponding spectral slope is expected to be close to the Kolmogorov one (see Section 3). For intermediate values of \(l\) (e.g., \(10 < l < 200\)), the average slope is shallower than the Kolmogorov one. Thus, our modeling shows the consistency of the observational spectra with the expectations. Studies of the observed spectra at higher \(l\) may be useful for better testing of our predictions.

### 3.4. Synchrotron Emission from Galactic Disk

In the right panel of Figure 1, we show how the second-order structure function changes with the Galactic latitude. The lower curve is the second-order angular structure function obtained from pixels in the range of \(28^\circ \leq b \leq 32^\circ\). The middle and upper curves are the second-order angular structure functions obtained from pixels in the range of \(8^\circ \leq b \leq 12^\circ\) and \(-2^\circ \leq b \leq 2^\circ\), respectively. The middle and upper curves clearly show the break in the slopes near \(\theta \sim 3^\circ\) and \(\sim 1^\circ.5\), respectively. When the angular separation is larger than the angle of the break, the structure function becomes almost flat. As we move toward the Galactic plane, the sudden changes in the slopes happen at smaller angles.

There are at least two possible causes for the break in the slope. First, a geometric effect can cause it. As we discuss in Appendix A, the change of slope occurs near \(\theta_{\text{cr}} \sim L/d_{\text{max}}\). As we move toward the Galactic plane, the distance to the farthest eddy, \(d_{\text{max}}\), will increase. As a result, the critical angle \(\theta_{\text{cr}} \sim L/d_{\text{max}}\) will decrease. Therefore, we will have smaller \(\theta_{\text{cr}}\) toward the Galactic plane. This may be what we are observing in the right panel of Figure 1. Second, discrete synchrotron sources can cause flattening of the structure function on angular scales larger than their sizes. Although the map we use was reprocessed to remove strong point sources, there might be unremoved discrete sources. When filamentary discrete sources dominate…

---

9 The reason for the spectrum being shallower in simulations is unclear. It may also be the result of the limited dynamical range in the presence of non-locality of MHD turbulence (Beresnyak & Lazarian 2009).
Synchrotron emission, the second-order structure function will be flat on scales larger than the typical width of the sources. In reality, both effects may work together.

In view of the variations of the spatial spectral slope of the synchrotron emission at low Galactic latitudes, the use of the foreground removal procedure discussed in Section 2 is more challenging at those latitudes. At the same time, it should be possible to reliably remove the high-fluctuations corresponding to high latitudes with the procedure in Section 2 due to the observed regular power-law behavior.

3.5. On the Polarized Synchrotron Emission

Roughly speaking, the shape of the angular spectrum of polarized synchrotron emission will be similar to that of the total intensity at millimeter wavelengths. However, it is expected that at longer wavelengths, Faraday rotation and depolarization effects should cause flattening of the angular spectrum, which has been actually reported (see de Oliveira-Costa et al. 2003 and references therein). On the other hand, La Porta et al. (2006) analyzed the new Dominion Radio Astrophysical Observatory (DRAO) 1.4 GHz polarization survey and obtained angular power spectra with power-law slopes in the range $[-3.0, -2.5]$. More observations on the polarized synchrotron foreground emission can be found in Ponthieu et al. (2005), Giardino et al. (2002), Tucci et al. (2002), and Baccigalupi et al. (2001).

In this paper, we do not discuss the properties of polarized synchrotron emission. Readers may refer to recent models of polarized synchrotron emission (Page et al. 2007; Sun et al. 2008; Miville-Deschênes et al. 2008; Waelkens et al. 2009).

4. POLARIZED EMISSION FROM DUST

Polarized radiation from dust is an important component of Galactic foreground that strongly interferes with the intended CMB polarization measurements (see Lazarian & Prunet 2001). Therefore, the angular spectrum of the polarized radiation from the foreground dust is of great interest. One of the possible ways to estimate the polarized dust radiation at the microwave range is to measure starlight polarization and use the standard formulae (see, for example, Hildebrand et al. 1999) relating polarization at different wavelengths. This approach involves a number of assumptions, the accuracy of which we analyze below. In this section, we describe how we can obtain a map of polarized dust emission using starlight polarization and discuss the angular spectrum of the polarized foreground emission from thermal dust at high Galactic latitude (say, $|b| \gtrsim 20^\circ$).

4.1. Properties of the 94 GHz Dust Emission Map

Let us begin with a model dust emission map created by Finkbeiner et al. (1999), which is available at the NASA LAMBDA Web site. As we will explain later in this section, we can derive a polarized intensity map from this kind of total intensity map.

We note that the difference between the dust emission and synchrotron emission (discussed in the previous section) is expected. The origin of the synchrotron emission is related to cosmic-ray electrons, which are distributed within an extended magnetic halo (Ginzburg & Ptuskin 1976). At the same time, dust is expected to be localized mostly within the Galactic plane. In Figure 4, we present statistical properties of the map. The map shows a rough constancy of the emission for high Galactic latitude region when multiplied by $\sin b$ (left panel of Figure 4). The $\sin b$ factor also appears in the probability density function (PDF); it makes the PDF more symmetric (right panel of Figure 4). Therefore, it is natural to conclude that a disk component dominates the dust map.

As in the Haslam map, we do not try to obtain the angular spectrum of the dust emission map directly. Instead, we use the second-order structure function to reveal the angular spectrum on small angular scales. Indeed, the second-order structure function of the dust map (see Appendix D) shows a slope of $-0.6$, which corresponds to an angular spectrum of $-l^{-2.6}$.

The slope of the angular power spectrum of the model dust emission map is very similar to that of the original FIR data. Schlegel et al. (1998) found a slope of $-2.5$ for the original FIR data. On the other hand, other researchers found slopes close to $-3$ from other observations (see Tegmark et al. 2000 and references therein; see also Masi et al. 2001).

Dust density fluctuations mostly arise from cold dense phases of the ISM (see Draine & Lazarian 1998a for a list of idealized ISM phases). There the turbulence is known to be supersonic, which is vividly revealed, for instance, by Doppler broadening of observed molecular lines from giant molecular clouds (GMCs; see McKee & Ostriker 2007). The shallow spectrum of density fluctuations is consistent with the numerical simulations of supersonic MHD turbulence in Beresnyak et al. (2005). The shallow spectrum was later also reported in supersonic hydro turbulence in Kim & Ryu (2005), which indicates that the effect is not radically changed by the magnetic field. The latter makes the conclusion about the shallow spectrum independent of the degree of ISM magnetization and sub-Alfvénic versus super-Alfvénic character of the turbulence there. Thus, we claim that the observed spectra should be associated with the shallow spectra of underlying density fluctuations in the denser part of the ISM. We predict that the shallow power law arising from density fluctuations extends from the scales of the turbulence energy injection to the dissipation scales. As a result, power-law extension of the observed data and the corresponding filtering using Equation (2) is possible.

4.2. Map of Polarized Dust Emission from Starlight Polarization

In general, it is advantageous to use all possible sources of information about foregrounds in order to improve their removal. In this section, we discuss how the starlight polarization maps can be used to construct the maps of dust polarized emission. In principle, we can construct a polarized dust emission map at millimeter wavelengths ($I_{\text{pol, mm}}(l, b)$) from a total dust emission map ($I_{\text{mm}}(l, b)$) and a degree-of-polarization map ($P_{\text{em, mm}}(l, b)$) at millimeter wavelengths:

$$I_{\text{pol, mm}}(l, b) = P_{\text{em, mm}}(l, b)I_{\text{mm}}(l, b),$$

where $(l, b)$ denotes the Galactic coordinate. However, neither $I_{\text{pol, mm}}(l, b)$ nor $I_{\text{mm}}(l, b)$ is directly available. Therefore, we need indirect methods to get $I_{\text{pol, mm}}(l, b)$ and $I_{\text{mm}}(l, b)$.

Obtaining a total dust emission map ($I_{\text{mm}}(l, b)$) is relatively easy because total dust emission maps at FIR wavelengths are already available from the IRAS and COBE/DIRBE observations. Using the relation

$$I_{\text{mm}}(l, b) = I_{100 \mu m}(l, b)(1 \text{ mm}/100 \mu m)^{-\beta},$$

where $1 \lesssim \beta \lesssim 2$, one can easily obtain an emission map at millimeter wavelengths ($I_{\text{mm}}$) from the maps at 100 $\mu$m or
240 μm. However, more sophisticated model dust emission maps at millimeter wavelengths already exist. For example, Finkbeiner et al. (1999) presented predicted full-sky maps of microwave emission from the diffuse interstellar dust using the FIR emission maps generated by Schlegel et al. (1998). In fact, several numerical codes are publicly available for such calculations (for example, DDSCAT package by Draine & Flatau 1994, Draine & Flatau 2008; ampld.lp.f by Mishchenko 2000). We use ampld.lp.f to calculate the ratio in Equation (15). We assume that the grains are oblate spheroids (grain size is 0.1 μm, \( \lambda_{\text{optical}} = 0.5 \mu m \), and \( \lambda_{\text{mm}} = 1000 \mu m \)). The left panel of Figure 5 shows that the ratio is around 1.5 when the magnetic field is perpendicular to the line of sight. It also shows that the ratio of \( P_{\text{em,mm}}/P_{\text{em,mm}} \) is almost independent of the grain size ratio.

In this subsection, we describe a simple way to obtain a polarized map at millimeter wavelengths. However, actual implementation of the method can be more complicated due to the following reasons. First, we use an assumption that the grains that produce optical absorption also produce microwave emission. But, this is not true in general (see Whittet et al. 2008). Second, the expressions in Equations (15) and (16) are still valid when the magnetic field direction is fixed and perpendicular to the line of sight and all the grains are perfectly aligned with the magnetic field. If this is not the case, Equation (15) will become \( P_{\text{em,mm}} \propto P_{\text{em,mm}} \) with a constant of proportionality that depends on the magnetic field structure and the degree of grain alignment. The effect of partial alignment is expected to be less important.\(^{10} \)

where \( L \) values are defined by

\[
L_1 = \left[ (1 + f^2)/f^2 \right] \left[ 1 - (1/f) \arctan f \right].
\]

\[
L_2 = L_3 = (1 - L_1)/2,
\]

\[
f^2 = \left( a_2/a_1 \right)^2 - 1
\]

(17)

(see, for example, Hildebrand et al. 1999).

However, for optical wavelengths, the condition \( \lambda \gg 2\pi a \) is not always valid and, therefore, the expression in Equation (16) returns only approximate values. For accurate evaluation of the cross sections, one should use numerical methods. Fortunately, several numerical codes are publicly available for such calculations (for example, DDSCAT package by Draine & Flatau 1994, Draine & Flatau 2008; ampld.lp.f by Mishchenko 2000). We use ampld.lp.f to calculate the ratio in Equation (15). We assume that the grains are oblate spheroids (grain size is 0.1 μm, \( \lambda_{\text{optical}} = 0.5 \mu m \), and \( \lambda_{\text{mm}} = 1000 \mu m \)). The left panel of Figure 5 shows that the ratio is around 1.5 when the magnetic field is perpendicular to the line of sight. It also shows that the ratio of \( P_{\text{em,mm}}/P_{\text{em,mm}} \) is almost independent of the grain size ratio.

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The effect of a non-perpendicular magnetic field can be potentially important. We perform a numerical calculation using ampdp.lpf to evaluate this effect. We assume that the grains are oblate spheroids (grain size is 0.1 μm, μoptical = 0.5 μm, and λoptical = 1000 μm). The right panel of Figure 5 shows that the polarization ratio drops from ~1.5 to ~1.1 when the angle (between the magnetic field and the plane of the sky) changes from 90° to ~5°. Therefore, the effect is not very strong and can be potentially corrected for.11

### 4.3. Angular Spectrum of Polarized Emission from Thermal Dust

After we have constructed a map of the polarized emission from thermal dust, we can obtain the angular spectrum. However, if we are interested in only the shape of the angular spectrum, we do not need to construct the polarized thermal dust emission map. We can get the shape of the angular spectrum directly from the starlight polarization map $P_{\text{abs,optical}}(l, b)$.

Equation (11) tells us that $I_{\text{pol-mm}}$ is given by $P_{\text{em-mm}}$ times $I_{\text{mm}}$. From Equations (14) and (15), we have

$$I_{\text{pol-mm}} = P_{\text{em-mm}}I_{\text{mm}} \propto P_{\text{em-opt}}I_{\text{mm}} \approx (P_{\text{abs,opt/τ}})I_{\text{mm}} \propto P_{\text{abs,opt}}.$$  

(18)

Here we use the fact that $τ \propto I_{\text{mm}}$. Note that the constant of proportionality does not affect the shape of the angular spectrum if grain properties do not vary much in the halo. Therefore, as to the power spectrum $C_l$ of $I_{\text{pol-mm}}$, we can use that of $P_{\text{abs,optical}}$:

$$C_l$ of $I_{\text{pol-mm}} \propto C_l$ of $P_{\text{abs,opt}}$.  

(19)

Once we know the angular spectrum of $P_{\text{abs,optical}}$, we can estimate the angular spectrum of $I_{\text{pol-mm}}$.

The angular spectrum of starlight polarization, $P_{\text{abs,opt}}(l, b)$, is already available. Fosalba et al. (2002) obtained $C_l \sim l^{-1.5}$ for starlight polarization. The stars used for the calculation are at different distances from the observer and most of the stars are nearby stars. The sampled stars are mostly in the Galactic disk. CL02 reproduced the observed angular spectrum numerically using a mixture of stars with a realistic distance distribution. CL02 also showed that the slope becomes steeper when only stars with a large fixed distance are used for the calculation. Therefore, it is clear that, if we consider only the nearby stars with a fixed distance, the slope will be steeper. This means that, if we consider stars in the Galactic halo, the slope will be steeper.

The method described above requires measurements of polarization from many distant stars in the Galactic halo. Unfortunately, the number of stars outside the Galactic disk that can be used for this purpose are no more than a few thousands (Heiles 2000; see also discussions in Page et al. 2007 and Dunkley et al. 2009). When more observations are available, accurate estimation of $I_{\text{pol-mm}}(l, b)$ (and $C_l$ of $P_{\text{abs,optical}}$) will be possible.

### 4.4. Model Calculations for Starlight Polarization

We expect the fluctuations of the starlight polarization to arise primarily from the fluctuations of the magnetic fields.12 The latter are expected to have a spectral index close to the Kolmogorov one (see Section 3).

As we discussed in the previous subsection, the angular spectrum, $C_l$ of $P_{\text{abs,optical}}$ for the Galactic halo will be different from the observed $l^{-1.5}$ spectrum for a mixture of stars with different distances in the Galactic disk. However, it is not clear exactly how the former is different from the latter. To deal with this problem we use numerical simulations again. We first generate two sets of magnetic fields on a 2D plane (8192 × 8192 grid points) using Kolmogorov 3D spectra.13 Since we need $P_{\text{abs,optical}}$ for stars well above the Galactic disk, we assume that

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11 We can make use of the right panel of Figure 5 reversely. In the future, when we can accurately measure polarized emission from thermal dust in FIR or millimeter wavelengths, we can obtain the values of [(Cmax - Cmin)/Cmax + Cmin]lum. This result combined with the values of [(Cmax - Cmin)/Cmax + Cmin]optical in optical wavelengths can be used to find the average angle between the magnetic field and the plane of the sky. That is, when we know the ratio $I_{\text{pol-mm}}/I_{\text{pol-optical}}$, we can use the right panel of Figure 5 to find the angle between the magnetic field and the plane of the sky.

12 In addition, fluctuations arising from the variations of the degree of grain alignment (see Lazarian 2007) are expected.

13 Consider a 3D magnetic field with a 3D spectrum $E_{3D}(k_x, k_y, k_z)$ (as $k^{-m}$ with $m = 1.1/3$ for Kolmogorov turbulence). The spectrum of the magnetic field on a 2D sub-plane (e.g., $z = 0$ plane) is $E_{2D}(k_x, k_y) \propto \int_{-\infty}^{\infty} dk_z E_{3D}(k_x, k_y, k_z)$, which we use to generate two sets of magnetic fields on a 2D plane. Note that, although this spectrum does not follow a power law near the outer scale of turbulence, it is close to $k^{-m+1}$ for large values of $k$. 

---

**Figure 5.** Ratio of $P_{\text{em-mm}}/P_{\text{em,optical}}$. Left: the polarization ratio vs. the axis ratio ($a_2/a_1$) of aligned oblate spheroidal grains. The polarization ratio shows only a weak dependence on the axis ratio. We assume that the grains are perfectly aligned with their long axes perpendicular to the magnetic field, the grain size is 0.1 μm, the magnetic field is perpendicular to the line of sight, $P_{\text{optical}} = 0.5 μm$, and $\lambda_{\text{optical}} = 1000 μm$. Right: the polarization ratio vs. the angle between the magnetic field and the plane of the sky. We assume that the grain size is 0.1 μm, the grain axis ratio is 1.5, $P_{\text{optical}} = 0.5 μm$, and $\lambda_{\text{optical}} = 1000 μm$. 

**Table 1.** The Table of Data. 

**Table 2.** The Table of Results. 

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**Figure 6.** The Figure of Data. 

**Figure 7.** The Figure of Results. 

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**Figure 8.** The Figure of Additional Data. 

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**Figure 9.** The Figure of Additional Results. 

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**Figure 10.** The Figure of Further Data. 

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**Figure 11.** The Figure of Further Results. 

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**Figure 12.** The Figure of Additional Further Data. 

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**Figure 13.** The Figure of Additional Further Results. 

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**Figure 14.** The Figure of Further Additional Data. 

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**Figure 15.** The Figure of Further Additional Results. 

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**Figure 16.** The Figure of Further Further Data. 

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**Figure 17.** The Figure of Further Further Results. 

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**Figure 18.** The Figure of Further Further Further Data. 

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**Figure 19.** The Figure of Further Further Further Results. 

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**Figure 20.** The Figure of Further Further Further Further Data. 

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**Figure 21.** The Figure of Further Further Further Further Results. 

---

**Figure 22.** The Figure of Further Further Further Further Further Data. 

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**Figure 23.** The Figure of Further Further Further Further Further Results.
the distance to the stars is fixed in each model. We consider the
following three models.

1. **Case 1: Nearby stars in a homogeneous turbulent medium.**
   We generate three (i.e., $x$, $y$, and $z$) components of the magnetic field on a 2D plane (8192 $\times$ 8192 grid points representing 400 pc $\times$ 400 pc), using the following Kolmogorov 3D spectrum: $E_{\text{3D}}(k) \propto k^{-11/3}$ if $k > k_0$, where $k_0 \sim 1/100$ pc. (The outer scale of turbulence is 100 pc.) We assume that the volume density of the dust is homogeneous. All stars are at a fixed distance of 100 pc from the observer.

2. **Case 2: Distant stars in a homogeneous turbulent medium.**
   We generate three (i.e., $x$, $y$, and $z$) components of the magnetic field on a 2D plane (8192 $\times$ 8192 grid points representing 4 kpc $\times$ 4 kpc), using the following Kolmogorov 3D spectrum: $E_{\text{3D}}(k) \propto k^{-11/3}$ if $k > k_0$, where $k_0 \sim 1/100$ pc. Other setups are the same as those of Case 1, but the distance to the stars is 2 kpc.

3. **Case 3: Stars in a stratified medium.**
   We use the magnetic field generated in Case 1. The volume density of dust shows a sech$^2(z)$ decrease: $\rho(r) = 4\rho_0/[\exp(r/r_0) + \exp(-r/r_0)]^2$. We assume a spherical geometry and $r_0 = 100$ pc. The stars are at $r = 200$ pc from the observer. The outer scale of turbulence is 100 pc.

We assume that the dust grains are oblate spheroids. In the presence of a magnetic field, some grains (especially large grains) are aligned with the magnetic field (see Lazarian 2007 for a review). Therefore, cross sections parallel to and perpendicular to the magnetic field are different. We assume that the parallel cross section is $\sim 30\%$ smaller than the perpendicular one. We use the following equations to follow changes of the Stokes parameters along the path (see Martin 1974 for original equations; see also Dolginov et al. 1996):

\[
I^{-1}dI/ds = -\delta + \Delta \sigma Q/I, \quad (20)
\]

\[
d(Q/I)/ds = \Delta \sigma - \Delta \sigma (Q/I)^2, \quad (21)
\]

\[
d(U/I)/ds = -\Delta \sigma (Q/I)(U/I), \quad (22)
\]

where $\delta = (\sigma_1 + \sigma_2)$, $\Delta \sigma = (\sigma_1 - \sigma_2)$, and

\[
2\sigma_1 = \sigma_\perp, \quad (23)
\]

\[
2\sigma_2 = \sigma_\perp + (\sigma_\perp - \sigma_\parallel) \cos \gamma \quad (24)
\]

(See Draine 1985). Here, $\sigma_\perp$ and $\sigma_\parallel$ are the extinction coefficients and $\gamma$ is the angle between the magnetic field and the plane of the sky. After we get the final values of the Stokes parameters, we calculate the degree of polarization ($\sqrt{Q^2 + U^2}/I$) and, then, the second-order angular structure function of the degree of polarization.

We show the result in Figure 6. When all the stars are at a distance of 100 pc (Case 1), the spectrum is consistent with the Kolmogorov one for small $\theta$. The result for the stratified medium (Case 3) also shows a spectrum compatible with the Kolmogorov one for small $\theta$. When the stars are far away (Case 2), the qualitative behavior is similar. However, if we measure average slope between $\theta = 0.2$ and 20$^\circ$, the result is different: the slope for Case 2 is substantially shallower. Note that $\theta = 0.2$ and 20$^\circ$ correspond to $l = 1000$ and 10, respectively. This means that, when we only have distant stars, the angular spectrum will be shallower than the Kolmogorov one. When we have a mixture of distant and nearby stars, we will have an angular spectrum that is steeper than the case of the distant stars but shallower than the case of the nearby stars, which implies that the spectrum is shallower than the Kolmogorov one. Therefore, it is not surprising that Fosalba et al. (2002) obtained a shallow spectrum of $\sim C_l \propto l^{-1.5}$ for a mixture of nearby and distant stars, mostly in the Galactic disk. Flattening of the spectrum (i.e., $C_l \propto l^{-\alpha}$ with $\alpha \approx 1.3$–1.4) for polarized FIR dust thermal emission is also observed in Prunet et al. (1998; see also Prunet & Lazarian 1999).

Note that the spectrum of the emission polarization for very large values of multipole $l$ is expected to be steeper than the spectrum of starlight polarization measured at small values of $l$. For instance, from our model calculations, we predict that we should see the Kolmogorov spectrum of polarized emission, rather than $\sim C_l \propto l^{-1.5}$. This difference must be taken into account if starlight polarization is used to filter the polarized microwave emission arising from dust. In fact, for filtering one should either make a model of the magnetic field distribution based on the extended samples of stars throughout the Galactic volume or use only distant stars to obtain the spectrum of polarization similar to that expected in the microwave range.

There could be systematic errors in transforming from the starlight polarization spectrum to that of emission. One such error may arise from the variation of the mean magnetic field direction and the line of sight. We expect that, while the starlight polarization spectrum is relatively insensitive to the mean magnetic field direction, the emission spectrum shows a stronger dependence on it. Our preliminary calculations show that such a systematic error is small. We will pursue this possibility in the future.

### 4.5. Comparison with the CMB Polarization

In the right panel of Figure 6, we plot the angular power spectrum of starlight polarization. As we mentioned earlier, the angular spectrum of the degree of starlight polarization should be similar to that of polarized thermal dust emission (see Equations (18) and (19)).

To obtain angular spectra, we use a Gauss–Legendre quadrature integration method as described in Szapudi et al. (2001). To be specific, we first generate magnetic fields from the three models we considered in the previous subsection. Then, we calculate angular correlation functions, $K(\cos \theta)$. Finally, we obtain the angular spectra using Equation (10). Since $C_l$ obtained in this way is very noisy, we plot $C_l$ averaged over the multipole range $(l/1.09, 1.09)$.

We do not show $C_l$ for $l > 1000$ because it is too noisy even with the averaging process. The second-order structure function in the left panel of Figure 6 implies that $l(l + 1)C_l \propto l^{-5/3}$ for $l > 1000$. The straight dashed line for $l > 1000$ reflects this implication.

We normalize the spectra using the condition $\sum_{l=0}^{10}(l + 1)C_l/2\pi = 3(\mu K^2)$ for the case of the stars in the stratified medium. This normalization is based on the values given in Page et al. (2007, their Equation (25)). We assume that the observed band is $W$ band ($\nu = 94$ GHz).

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14 The purpose of this normalization is to match roughly our spectrum and that of Page et al. (2007) on large scales. Equation (25) of Page et al. (2007) reads $(l(l + 1)C_l/2\pi = 1.0(65\text{GHz})^{l-1.8} \sim 31^{-6}(\mu K^2)$ for $\nu = 94$ GHz. When summed from $l = 2$ to $l = 10$, this gives $\sim 2.6(\mu K^2)$. We adopt $\sum_{l=2}^{10}(l + 1)C_l/2\pi = 3(\mu K^2)$ for the case of the stars in a stratified medium.
The plot shows that the slopes for nearby stars (thick solid line) and stars in a stratified medium are shallower than that of the Kolmogorov spectrum for \( l < 1000 \). This result is consistent with that obtained with the angular structure function. Note that the case of the distant stars has a much flatter spectrum for \( l > 1000 \). We believe that our toy model for the stratified medium (Case 3) better represents the actual situation for polarized thermal dust emission from high-latitude Galactic halo. Therefore, we expect that the polarized thermal emission from thermal dust in high-latitude Galactic halo has a spectrum slightly shallower than the Kolmogorov spectrum for \( l < 1000 \). Our calculations do not tell us about the slopes for \( l > 1000 \). However, judging from the behavior of the angular structure function for \( \theta \leq 0.1 \), we expect that in the right panel of Figure 6; see also discussions in Section 6.4).

We also show the polarized CMB “EE” spectrum in the figure (data from CMBFAST online tool at http://lambda.gsfc.nasa.gov/). The figure shows that the EE spectrum dominates polarized thermal dust emission from high-latitude Galactic halo for \( l \geq 100 \). The EE spectrum is expected to be sub-dominant when \( l > 5000 \).

5. SPATIAL FILTERING OF FOREGROUNDS

A unifying theme of this study and that in CL02 is that the principal source of the foreground fluctuations is related to the MHD turbulence in the Galactic ISM. Our analysis of the available observational data in both publications supports this conclusion. In this section, we discuss how this insight into the origin of the foreground fluctuations can be used to remove the foregrounds.

5.1. Statistical Properties of Foregrounds

Removal of Galactic foregrounds has always been a major concern for CMB studies. The challenge is only going to substantially increase now, when CMB polarization studies are attempted.

The knowledge that foregrounds are not an arbitrary noise, but have well-defined statistical properties in terms of their spatial power spectra, is an important additional information that can be utilized to evaluate and eventually eliminate the foreground contribution.

Utilizing the information about underlying turbulence power spectrum is not straightforward, however. Our study shows that the observed power spectrum may depend on the geometry of the emitting volume. Therefore, detailed modeling of the foreground fluctuations should involve accounting for the geometry of the emitting volume.

The latter point stresses the synergy of Galactic foreground and CMB studies. Indeed, our fitting of the power spectra in Figure 3 shows that on the basis of their variations we may distinguish between different models of the emitting turbulent volume. As soon as this is achieved, one can predict, for instance, the level of fluctuations that are expected from the foreground at scales smaller than those studied.

While the previous statements are true in general terms, a number of special cases in which simpler analysis is applicable are available. For instance, a simplification that is expected at higher resolutions that are currently available is that at sufficiently small scales the statistics should become independent of the large-scale distributions of the emitting matter. Moreover, simple power laws are expected and observed (see Figure 1) for the foregrounds at high Galactic latitudes. Therefore, modeling of the distribution of the Galactic emission and the filtering of it using the approach in Section 3 are not necessarily interlocked problems.

5.2. Examples of Filtering

We remind the reader that the approach in Section 3 requires first separating a particular component of a foreground over a range of scales, e.g., using the traditional technique of frequency templates, and then extending the spatial scaling of the foreground’s \( C_l^F \) to higher \( l \). Consider a few examples of utilizing this approach.

\[ \text{If we know the foreground at a single spatial scale, one can use the a priori knowledge of the expected spatial scaling of the foreground spectrum. For instance, if underlying fluctuations are related to a magnetic field, they are likely to have the spectrum close to the Kolmogorov one.} \]
Figure 7. Explanation of our filtering process. We use the Cartesian coordinate system for simplicity and assume that there is only one foreground. Left panels: low angular resolution all-sky map (1024 × 1024 pixels; Δθ ∼ 21'). We generate the low-resolution map by applying a circular top-hat filter on the data shown in the middle panels. The radius of the filter is 4 grid units, which is equivalent to ∼10.5. The size of the resulting data file is 1024 × 1024, the angular resolution of which corresponds to Δθ ∼ 21'. We obtain the spectra in the lower middle panel by direct Fourier transform. Middle panels: all-sky map of the CMB and the foreground signals and their angular spectra. We generate the map in the upper middle panel using the spectra shown in the lower middle panel. The actual resolution of the map we use for the calculation is 8192 × 8192. Right panels: high angular resolution partial-sky map (128 × 128 pixels; Δθ ∼ 5.2). This data file is from the lower right corner of the original 8192 × 8192 data. We obtain this map by skipping every other point in each direction, so that the angular resolution of these data (Δθ ∼ 5.2) is twice worse than the original 8192 × 8192 data, but four times better than the low-resolution data in the middle panels. In the original data file (i.e., in the 8192 × 8192 data), the region spans 256 × 256 grid points. Therefore, the size of the partial-sky map is 128 × 128. The angular size of the map in the upper right panel is 11° × 11°.

For instance, high-resolution measurements of the South Pole Telescope (SPT; see Lueker et al. 2010) and the Atacama Cosmology Telescope (ACT; see Fowler et al. 2010) will provide measurements at high l, but will have limited frequency coverage to remove the foregrounds. Therefore, the spatial extrapolation of C_F obtained with other low-resolution experiments, which, however, provide good frequency coverage required for the C_F identification, may be advantageous.

Prior to the release of the Planck data, the extension of C_F obtained, for instance, with WMAP to higher l may be useful for the foreground filtering for the suborbital missions. After the release of the Planck data, studies of the advocated approach can still be useful (consider, for instance, Figure 6). If Planck measures the spatial spectrum up to l = 2000, then for a higher resolution balloon mission one can evaluate the level of foreground contamination by extrapolating the expected foreground spectrum. Moreover, the study of the B-modes would require new experiments with higher sensitivity and the extrapolation procedure can be useful again.

5.3. Demonstration of Filtering Technique 1

In this subsection, we demonstrate how the filtering process works. For simplicity we use a Cartesian coordinate system. For demonstration purposes, we consider the CMB EE spectrum (see the thin solid line in Figure 6(b)) and Galactic dust foreground for the case of a stratified medium (see the dashed line in Figure 6(b)).

Suppose that low-resolution all-sky maps are already available for many frequency channels. Let us assume that the resolution of the maps is Δθ ∼ 21', which is similar to the WMAP resolution. Since there are many channels, one may use the usual filtering techniques to separate the CMB and the foreground signals. However, since we already know both the CMB and the foreground spectra in advance in this example, we do not follow the usual filtering techniques. Instead, we just assume that we already know the CMB and the foreground spectra for l ≲ 500.

The “all-sky” map in the upper left panel of Figure 7 represents this low-resolution map. The map is defined on a grid of 1024 × 1024 points. The angular resolution of the map is Δθ ∼ 21'. We generate the map from a much higher resolution all-sky map (see upper middle panel), which has a dimension of 8196 × 8196 in this example. When we obtain the low-resolution all-sky map from the much higher resolution all-sky map, we apply a circular top-hat beam pattern with radius ∼10.5 to mimic the actual observation. We calculate the spectra of the CMB and the foreground by direct Fourier transform of the low-resolution map data.

Then, suppose that we perform a high-resolution balloon experiment that can cover only part of the sky. In such
balloon experiments, frequency channels are usually limited and removing foregrounds is a challenging task.

Here we show that, if we know the foreground spectrum, we can easily obtain the CMB spectrum from the balloon data. The upper right panel of Figure 7 represents the high-resolution balloon data. The size of the data is 128 × 128. The angular resolution is ∆θ ∼ 5/2 and the map covers 11° × 11° in the sky. We obtain the map by skipping every other point in each direction in the lower right corner of the original map with 8192 × 8192 points.

Our goal is to obtain the angular spectrum of the CMB signal using the balloon data. We first make the balloon data periodic by proper reflections and translations and obtain the periodic data on a grid of 4096 × 4096 points. Then we multiply the data by a Gaussian profile of width ∼ 11°. The center of the Gaussian profile should locate near the center of the newly constructed 4096 × 4096 pixel data and coincides with the center of the 128 × 128 pixel original data. Then, we perform Fourier transformation of the resulting data of size 4096 × 4096. In this way, we obtain an angular spectrum of the total (i.e., CMB+foreground) fluctuations.

Now, we derive an angular spectrum of the foreground. We first take the angular spectrum from the low-resolution map, which is already available (see the lower left panel of Figure 7). The dashed curve is the foreground spectrum. The foreground spectrum is not defined for l > 500. In principle, we know the foreground spectrum when we know the geometry and turbulence spectrum. Here, we simply assume that the spectrum for l > 500 is Kolmogorov. In this way, we can construct the foreground spectrum for all values of l.

The remaining task is just subtraction: when we subtract the foreground spectrum from that of the total fluctuations, we get the CMB spectrum. The solid curve in Figure 8 is the CMB spectrum obtained in this way. The solid curve shows very good agreement with the original CMB spectrum used for generating the original all-sky data on a grid of 8192 × 8192 points (see the middle panels). The rms relative percentage error (100ΔC_l/C^CMB) for 100 < l < 2000 is 24%, where ΔC_l is the difference between the estimated CMB spectrum (C^estimated_l; solid curve) and the true CMB spectrum (C_l^CMB; dotted curve).

Note that the foreground spectrum depends on the geometry of the emitting regions and the underlying turbulence spectrum. In earlier sections, we discussed the geometry of light-emitting regions. The turbulence spectrum may have some uncertainties (see the Introduction). The Kolmogorov spectrum may be good for a number of cases. However, it is possible that the spectrum deviated from the Kolmogorov one in some regions, such as the molecular clouds. However, such deviation will not affect our filtering process much if the resolution of the balloon data is not far beyond that of the low-resolution all-sky map.

5.4. Demonstration of the Filtering Technique 2

In the previous subsection, we demonstrated a reconstruction of the CMB spectrum. The result is very promising because the dust foreground model we used is not far from a realistic one. Note that we normalized the amplitude of the dust foreground spectrum using the value quoted in an earlier work (Page et al. 2007).

However, there are a couple of issues regarding the previous demonstration. First, in the previous demonstration, we assumed that we know the exact power-law index of the foreground spectrum. In general, we may not know the exact power-law index of the foregrounds. When this is the case, we need to extrapolate the foreground spectrum to larger values of multipoles and figure out to what extent we can safely extrapolate. Second, it is necessary to test how well our method works when the CMB and the foreground spectra are comparable.

In this subsection, we develop a method for deriving the most probable extrapolation of the foreground spectrum. We also test the performance of our technique for the case where the CMB and the foreground spectra are comparable. For this purpose, we scale up the amplitude of the foreground spectrum by a factor of 1000. As in the previous subsection, we assume that multi-channel observations are available for l < 500. Therefore, we have foreground observation for l < 500. However, unlike the previous subsection, we assume that observations are available only for selected values of the multipole l. We mark such values of l in Figure 9. We use the same foreground spectrum as in the previous subsection to generate the four observed points. We also show the 1σ observation error bars for them, which are arbitrarily assigned in this example.

Using the four observed points for foreground, we find the most probable extrapolation for l > 500. It is tempting to use the linear least-squares fit in a log–log plane, in which the x-axis is a logarithm of l and the y-axis is that of l(l + 1)C_l/2π. But, it is not easy to estimate uncertainties in this case.

Therefore, we adopt a slightly different method. We work on the log–log plane. In this subsection, we use the following conventions: x ≡ log l, y ≡ log[l(l + 1)C_l/2π], and (x_i, y_i) (i = 1, . . . , N, where N = 4 in this example) denotes an observed point. For a given x, we want to find the most probable value and the 1σ uncertainty of y. In order to find these, we follow the following procedure. First, we select an arbitrary y and consider all possible lines that pass through (x, y). Let the slope of such a line be a and the y-intercept be b. Then, we calculate the following probability for the chosen y:

\[ P(y) \equiv \exp \left[ -\sum_{i=1}^{N} \frac{(y_i - z_i)^2}{2\sigma_i^2} \right], \quad \text{(25)} \]

16 The motivation for adopting the factor of 1000 is that the CMB BB spectrum by weak gravitational lensing is about 100–1000 times smaller than the CMB EE spectrum for l < 1000 (e.g., Zaldarriaga & Seljak 1998).
where \( z_i = a x_i + b \) and \( \sigma_i \) is the 1σ observation error for \( i \). Second, we repeat similar calculations for different values of \( y \). Third, we find the value of \( y \) where \( P_{\text{tot}}(y) \equiv \sum_{\text{all lines}} P(y) \) has the maximum. Fourth, we find 1σ of the distribution \( P_{\text{tot}}(y) \). Fifth, we repeat similar calculations for different values of \( x \).

We plot the results of this procedure in Figure 9(a). The solid line denotes the most probable values of “\( y \)” and the shaded region represents the 1σ uncertainty. The 1σ uncertainty gets larger as \( l \) gets larger when \( l > 500 \). Therefore, it may not be a good idea to extrapolate for \( l \)'s an order of magnitude larger than 500, which is the maximum \( l \) of the low-resolution multi-channel maps in our current example. However, it seems to be accurate to extrapolate for \( l \)'s a few times larger than 500 in our current example. We can reduce the 1σ uncertainty by reducing the observation error bars, which may enable us to extrapolate further for larger \( l \)’s.

As in the previous subsection, we subtract the most-likely foreground spectrum from the observed (i.e., CMB+foreground) spectrum. We plot the result in Figure 9(b). Figure 9(b) shows that our technique recovers the CMB spectrum quite well for \( l \geq 550 \). Note that Figure 9(a) tells us that the CMB and the foreground signals are almost same at \( l \approx 550 \). This example implies that our technique works when the CMB spectrum is slightly larger than the foreground one.

The spatial filtering that we advocate here may be used as part of a more general filtration procedure that uses both spatial and frequency information. Indeed, it is well recognized that the studies of tensor \( B \)-modes provide an excessively severe challenge to the precision of the removal of foregrounds. In this situation, it is important to reduce the errors in the determination of the foreground signal. The customarily used frequency templates provide filtering that is limited by both systematic and measurement errors. In this situation, any additional information that can help in decreasing the errors is highly valuable. The spatial power spectrum of the fluctuations that we discussed in this paper does provide such information. This approach can be illustrated in Figure 9, which currently only has points for low \( l \). However, it can be seen that the initial error bars of the points can be decreased if we require the points to correspond to a power-law spectrum. It is evident that if we had more points at higher \( l \), the uncertainties could be further reduced. In other words, the constraint that the foreground fluctuations follow a power law allows us to partially remove foregrounds at scales at which no foreground templates based on multi-frequency measurements are available and, at the same time, increase the precision of the foreground removal if the templates are available.

6. DISCUSSION

6.1. Our Approach

The major claim in this paper is that the statistical regularity of the foreground fluctuations enables one to extend their spectra from the scales where observations are available to the scales with no observations. In other words, the spatial spectra of foregrounds are predictable. This, in turn, makes spatial filtering of foregrounds possible.

The predictability of Galactic foreground fluctuations stems from the fact that they are due to ubiquitous Galactic turbulence. MHD turbulence is known to have well-defined statistical properties both in compressible and incompressible limits (see Cho & Lazarian 2005 for a review). Thus, one expects to see a power-law behavior, which in the case of a magnetic field is expected to correspond to the 3D power spectrum close to the Kolmogorov one. In the case of density, the spectrum is Kolmogorov for the subsonic turbulence, but gets shallower than the Kolmogorov one for supersonic turbulence (see Section 4.1). Irrespective of the underlying 3D spectrum being Kolmogorov or not we claim that one can predict the entire spectrum of spatial 2D foreground fluctuations when the measurements over a limited range of scales are available. For this purpose, we need to know the geometric properties of the volume. Our expectations of the underlying spectrum to be, in some instances, e.g., for magnetic fields, close to Kolmogorov only help in increasing the accuracy of our prediction of the 2D foreground spectrum.

With the 2D predicted spectrum, one can then use the procedure of filtering the foreground using Equation (2) (see Section 5.3 for demonstration of the procedure). Alternatively, the good correspondence of the observed spatial spectrum of the foregrounds may serve as an additional proof of the accuracy of the foreground removal procedure. Needless to say, the information of Galactic turbulence and the geometry of the emitting region, which can be a by-product of CMB research, is of astrophysical significance.

A note of warning is due, however. The procedure of statistical filtering that we demonstrated in this paper, as any other procedure of foreground removal, is not ideal. The power-law approximation of \( C \) is definitely not exact. In this paper, we have analyzed the causes for such deviations and provided explanations for the most notable features characterizing the
change. More detailed modeling of Galactic turbulence is required.

6.2. Our Data Analysis

In this paper, we have discussed angular spectra of Galactic foregrounds. We have focused on synchrotron total intensity and polarized thermal dust emission. Our current study, as well as earlier studies (Chepurnov 1999; CL02), predict that $C_l$ will reveal a true 3D turbulence spectrum on small angular scales.

Our model calculations that take into account stratification effects imply that

1. $\theta < \text{a few times } 0.1$ (or $l > \text{a few times 100}$) for synchrotron emission (see Figure 3) and
2. $\theta \lesssim 0.1$ (or $l \gtrsim 1000$) for polarized emission from thermal dust (see Figure 6).

On larger angular scales, spectra are expected to be shallower.

6.3. Results

In this paper, we have analyzed the Haslam 408 MHz map and a model dust emission map, and compared the results with model calculations. We found the following.

1. The Haslam map for the high Galactic latitude ($b > 30^\circ$) can be explained by MHD turbulence in the Galactic halo. The measured second-order angular structure function is proportional to $\theta^{-1.2}$, which corresponds to an angular spectrum of $l^{-3.2}$. The high-order statistics for high Galactic latitude ($b > 30^\circ$) is consistent with that of incompressible MHD turbulence. Our model calculations show that a shallow spectrum of density fluctuations, which characterizes supersonic MHD turbulence, can explain the observed angular spectrum. The one-component model can also explain the observed slope. But, the slope of the spectrum shows a more abrupt change near $l \sim 30$.
2. The model dust emission map may not have anything to do with turbulence on large angular scales. That is, we do not find signatures of turbulence in the map.
3. Both maps show flat high-order structure functions for the Galactic plane. This kind of behavior is expected when discrete structures dominate the map.

We have described how we can obtain the angular spectrum of the polarized emission from thermal dust in high Galactic latitude regions. Our model calculations show that the starlight polarization arising from dust in high Galactic latitude regions will have a Kolmogorov spectrum, $C_l \propto l^{-11/3}$, for $l \gtrsim 1000$ and a shallower spectrum for $l \lesssim 1000$ (Figure 6). We expect that the polarized emission from the same dust also has a similar angular spectrum. That is, we expect that the angular spectrum of the polarized emission from thermal dust is close to a Kolmogorov one for $l \gtrsim 1000$.

We have described a new technique of filtering CMB foregrounds. When we have (1) low angular resolution full-sky measurements (such as WMAP data) and (2) high angular resolution partial-sky measurements with limited frequency channels, we can use this technique to derive the CMB angular spectrum for the high-resolution data. In Sections 5.3 and 5.4, we have demonstrated the technique.

6.4. Comparison with Approaches in the Literature

The existing confusion in the literature includes naive identification of the 2D spectra of foregrounds with the spectra of the underlying fluctuations. Therefore, for instance, from the fact that the spectral slope of $C_l^f$ differs from the Kolmogorov one, the conclusion about the nature of the fluctuations is made.

In this paper, we have shown that the 2D spectra may have a spectral slope different from the underlying spectral slope of the turbulence. We showed that it is essential to take into account the non-trivial geometry of observations with the observer sampling turbulence along the diverging lines of sight and within the volume where the density of emitters change. For the case of the synchrotron fluctuations, we showed that the observed non-Kolmogorov value of the spectral index of $C_l^s$ can be reconciled with the Kolmogorov-type turbulence.

A notable difference between our study and CL02 is that in the latter we tried to necessarily associate the spectra of foregrounds with the underlying Kolmogorov or the Goldreich & Sridhar (1995) turbulence with a spectral slope of $-11/3$. The limitations of this approach are evident in view of the establishment of the shallow spectral index of density fluctuations in supersonic MHD turbulence (Beresnyak et al. 2005). These fluctuations, according to our present study, can explain the observed spectra of Galactic dust emission.

6.5. Limitations and Extensions of our Filtering Procedure

The procedure of spatial statistical filtering discussed in Section 2 is very simplified. It should provide satisfactory results for a simple power-law behavior of $C_l^f$ (see Figure 2). In more complex cases, detailed modeling of the emitting volume and/or the use of the filtering as part of a more sophisticated foreground removal procedure is required. We partially addressed this issue in Section 5. Especially, in Section 5.4, we described a method to estimate uncertainties stemming from a power-law modeling of the foreground spectrum. Nevertheless, more improvement is needed for the new technique. At the same time, we believe that our approach can be applied to the removal of foregrounds not only from the CMB data, but, for instance, from the high-$z$ hydrogen statistics studies.

7. SUMMARY

In this work, we have obtained the following results.

1. We have provided additional evidence that the synchrotron Galactic emissivity is consistent with the halo + disk model. Within this model, we show that the spatial spectrum of the underlying 3D fluctuations is consistent with the Kolmogorov one.
2. Within our model, we related the angular scale for the change of the power spectrum of the synchrotron fluctuations with the ratio of the injection scale to the thickness of the observed region in the direction of observation.
3. We have explained the spectrum of dust foreground emission as arising from the shallow spectrum of density fluctuations, which characterizes supersonic MHD turbulence.
4. We used numerical modeling of grain optical properties to relate the polarization of starlight with the expected submillimeter foreground polarization and have outlined ways for quantitative use of starlight polarization maps to study submillimeter polarized dust foreground. We evaluated the uncertainty in the evaluated submillimeter polarization spectrum arising from the variations of the magnetic field direction with respect to the line of sight.
5. We showed that for a randomly chosen sample of stars, the spectrum of the starlight polarization for the underlying Kolmogorov turbulence depended on how the selected stars
are distributed along the line of sight. For a stratified model of Galactic dust, we predicted a spectrum of spatial fluctuations with the index approaching the Kolmogorov value of $-11/3$ for $l > 1000$.  

6. On the basis of our improved understanding of the self-similar nature of the underlying MHD turbulence, we have proposed and tested a procedure of spatial statistical removal of Galactic foregrounds based upon extending the $C_1^F$ spectrum to higher $l$.

7. We have studied the higher-order correlations of Galactic dust and synchrotron foregrounds and reported substantial difference with the higher-order scalings (see Appendix D). The synchrotron scaling shows intermittencies similar to magnetic fields in incompressible fluids, while dust scaling is closer to the intermittency demonstrated by highly compressible MHD turbulence.

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**APPENDIX A**

**SPECTRUM AND STRUCTURE FUNCTION OF DIFFUSE SYNCHROTRON EMISSION: A MODEL FOR HOMOGENEOUS TURBULENCE REVISITED**

Suppose that 3D MHD turbulence has a 3D spatial power spectrum of the form $E_{3D} \propto k^{-m}$, where $k$ is the wavenumber. Note that in Kolmogorov turbulence, $m = 11/3$. Then what will be the 2D angular spectrum, $C_2$, of the observed synchrotron total intensity? We cannot directly observe the 3D spatial power spectrum. However, we can infer the 3D spectrum from the observed 2D angular spectrum, $C_2$, of the synchrotron total intensity. Then let us find out how the 3D spectrum, $E_{3D}(k)$, and the 2D angular spectrum, $C_2$, are related. We mostly follow discussions in CL02. We also make use of an analytical insight obtained in Lazarian (1992, 1995a, 1995b) and numerical results obtained in CL02. Although we focus on synchrotron emission here, the discussion in this section can be applicable to any kind of emission from an optically thin medium.

**A.1. MHD Turbulence and Synchrotron Emission**

For synchrotron radiation, emissivity at a point $r$ is given by $\epsilon(\mathbf{r}) \propto n(e) |B_1|^2$, where $n(e)$ is the electron number density, $B_1$ is the component of the magnetic field perpendicular to the line of sight. The index $\gamma$ is approximately 2 for radio synchrotron frequencies (see Smoot 1999). If electrons are uniformly distributed over the scales of magnetic field inhomogeneities, the spectrum of synchrotron intensity reflects the statistics of the magnetic field. For small amplitude perturbations $\delta b/B \ll 1$; this is true for scales several times smaller than the outer scale of turbulence if we interpret $B$ as the local mean magnetic field strength and $\delta b$ as the random fluctuating field in the local region), if $\delta b$ has a power-law behavior, the synchrotron emissivity will have the same power-law behavior (see Getmantsev 1959; Lazarian & Shutenkov 1990; Chepurnov 1999). Therefore, we expect that the angular spectrum of synchrotron intensity also reflects the spectrum of 3D MHD turbulence.

When an observer is located inside a turbulent medium, the angular correlation function, and hence the power spectrum, shows two asymptotic behaviors. When the angle is larger than a critical angle, we can show that the angular correlation shows a universal $\theta^{-1}$ scaling. In contrast, when the angular separation is smaller than the critical angle, the angular correlation reflects statistics of turbulence. In this small-angle limit, we can show that the angular power spectrum is very similar to that of turbulence. The critical angle is determined by the geometry. Let the outer scale of turbulence be $L$ and the distance to the farthest eddies be $d_{max}$. Then the critical angle is

$$\theta \sim L/d_{max}. \quad (A1)$$

**A.2. Small-angle Limit in Homogeneous Turbulence**

When the angle between the lines of sight is small (i.e., $\theta < L/d_{max}$), the angular spectrum $C_1$ has the same slope as the 3D energy spectrum of turbulence. Lazarian & Shutenkov (1990) showed that if the 3D spatial spectrum of a variable follows a power law, $E_{3D}(k) \propto k^{-m}$, then the 2D spectrum of the variable projected on the sky also follows the same power law,

$$C_1 \propto l^{-m}, \quad (A2)$$

in the small $\theta$ limit. For Kolmogorov turbulence ($E_{3D} \propto k^{-11/3}$), we expect

$$C_1 \propto l^{-11/3}, \quad \text{if} \ \theta < L/d_{max}. \quad (A3)$$

Note that $l \sim \pi/\theta$.

In some cases, when we have data with incomplete sky coverage, we need to infer $C_1$ from the observation of the angular correlation function

$$w(\theta) = \langle I(\mathbf{e}_1)I(\mathbf{e}_2) \rangle, \quad (A4)$$

where $I(e)$ is the intensity of synchrotron emission, $\mathbf{e}_1$ and $\mathbf{e}_2$ are unit vectors along the lines of sight, $\theta$ is the angle between $\mathbf{e}_1$ and $\mathbf{e}_2$, and the angle brackets denote an average taken over the observed region. As we discuss in Appendix B, when the underlying 3D turbulence spectrum is $\propto k^{-m}$ (e.g., $m = 11/3$ for Kolmogorov turbulence), the angular correlation function $w(\theta)$ is given by

$$w(\theta) \propto \langle I^2 \rangle - \text{const} \theta^{m-2}, \quad \text{if} \ \theta < L/d_{max}. \quad (A5)$$

It is sometimes inconvenient to use the angular correlation function in practice to study turbulence statistics because of the constant $\langle I^2 \rangle$.

A better quantity in small-angle limit would be the second-order angular structure function:

$$D_2(\theta) = \langle |I(\mathbf{e}_1) - I(\mathbf{e}_2)|^2 \rangle \quad (A6)$$

$$= 2\langle I^2 \rangle - 2w(\theta). \quad (A7)$$

Thus, in homogeneous turbulence with 3D spatial spectrum of $E(k) \propto k^{-m}$, we have

$$D_2(\theta) \propto \theta^{m-2}. \quad (A8)$$

When we measure the slope of the angular structure function, we can infer the slope of the 3D spatial power spectrum of turbulence.
which means that the power index $\alpha$ of $C_t$ is $^{17} -1 < \alpha < -11/3$. We expect the following scaling for the second-order angular structure function:

$$D_2(\theta) \propto \begin{cases} \theta^{5/3} & \text{if } \theta < L/d_{\text{max}} \\ \text{constant} & \text{if } \theta > L/d_{\text{max}}. \end{cases}$$ \hspace{1cm} (A12)

The critical angle $\theta_c \sim L/d_{\text{max}}$ depends on the size of the large turbulent eddies and on the length of the line of sight. If we assume that turbulence is homogeneous along the lines of sight and has $L \sim 100$ pc corresponding to the typical size of a supernova remnant, and that $d_{\text{max}} \sim 1$ kpc for a synchrotron halo (see Smoot 1999), we get $\theta_c \sim 6^\circ$.

**APPENDIX B**

**THE SECOND-ORDER ANGULAR STRUCTURE FUNCTION IN THE SMALL-ANGLE LIMIT**

In this appendix, we discuss how underlying 3D statistics and observed 2D statistics are related. This discussion is useful when we infer 3D statistics from observed 2D statistics, or vice versa. Strictly speaking, discussion in this appendix is applicable to homogeneous and isotropic turbulence only.$^{18}$

The angular correlation $w(\theta)$ is given by the integral

$$w(\theta) = \int dl_1 \int dl_2 K(|l_1 - l_2|),$$ \hspace{1cm} (B1)

where $K(r)$ is the 3D spatial correlation function and we change variables: $(r_1, r_2) \rightarrow (r, \psi)$, which is clear from Figure 10. The Jacobian of the transformation is $r/\sin \theta$. We can qualitatively understand the $1/\theta$ behavior as follows. When the angle is large, points along the lines of sight near the observer are still correlated. These points extend from the observer over a distance $\propto 1/\sin(\theta/2)$.

If we assume $L/d_{\text{max}} \ll \theta \ll 1$, we can get the angular power spectrum $C_t$ using Fourier transform:

$$C_t \sim \int \int w(\theta) e^{-i \theta \cdot \mathbf{r}} d\theta_x d\theta_y \sim \int d\theta \theta J_0(\theta) w(\theta) \propto \theta^{-1},$$ \hspace{1cm} (A10)

where $\theta = (\theta_1^2 + \theta_2^2)^{1/2}$, $J_0$ is the Bessel function, and we use $w(\theta) \propto \theta^{-1}$.

**A.4. Expectations for Homogeneous Turbulence**

In summary, for homogeneous Kolmogorov turbulence (i.e., $E(k) \propto k^{-11/3}$), we expect from Equations (A3) and (A10) that

$$C_t \propto \begin{cases} l^{-11/3} & \text{if } l > l_c \\ l^{-1} & \text{if } l < l_c \end{cases},$$ \hspace{1cm} (A11)

$^{17}$ Note that point sources would result in $\alpha \sim 0$.

$^{18}$ When turbulence is inhomogeneous or anisotropic, we may not directly apply the results in this appendix. However, our numerical calculations in Section 3.3 show that the relation between 3D statistics and 2D angular correlation function (or 2D angular structure function) discussed in this appendix is also applicable to inhomogeneous cases.
where \( u = l_1 + l_2 \) and \( w = l_1 - l_2 \). If \( p \leq 1 \), the integration diverges as \( \theta \) goes to zero.\(^{19} \) Therefore, when \( p \leq 1 \), it suffices to perform the integration in the vicinity of \( l_1 = l_2 \) or \( w = 0 \). Then, we have
\[
\frac{u(\theta)}{d\theta} \propto -\int du \int dw [w^2 + u^2 \theta^2 / 4]^{p/2-1} u \theta / 2 \tag{B7}
\]
\[
\approx -\int du (u \theta / 2)^{p-1} u \theta / 2 \propto -\theta^p, \tag{B8}
\]
where we use \( \int_{-\infty}^{\infty} dw / (w^2 + \Lambda^2)^p = A^{1-2n} \int_{-\pi/2}^{\pi/2} d\theta \sec^{2-2n} \theta \). Therefore, for small \( \theta \), we have
\[
w(\theta) \propto C_1 - C_2 \theta^{p+1}, \tag{B9}
\]
where \( C_1 \) and \( C_2 \) are constants. Comparing this equation with
\[
w(\theta) = C_3 - C_4 D_2(\theta), \tag{B10}
\]
we get
\[
D_2(\theta) \propto \theta^{p+1}. \tag{B11}
\]

Analytic expressions for the relation between the angular structure function \( D_2(\theta) \) and the spatial one-dimensional (1D) spectrum \( E(k) \); in the case of Kolmogorov, \( E(k) \propto k^{-5/3} \) can be found in the literature. For example, Lazarian (1995a; see also Lazarian & Shutenkov 1990) derived the following expression:
\[
E(k) = k \int_0^{\ell/R} d\eta \frac{d}{d\eta} (Q(\eta) n_1(k R \eta) + K_5, \tag{B12}
\]
where \( \ell \) can be regarded as the outer scale of turbulence, \( R \) is the size of the system, \( Q(\eta) \sim D_2(\eta) \eta_1, n_1 = \sin \theta, J_1(x) \) is the Bessel function of the first order, and \( K_5 \) is a small correction term.

**APPENDIX C**

**SPATIAL SPECTRUM OF EMISSIVITY**

The synchrotron emissivity is proportional to \( n(e) B^r \propto B^2 \), where \( n(e) \) is the high-energy electron number density. Suppose that the magnetic field is roughly a Gaussian random variable. This may not be exactly true, but should be a good approximation. When a Gaussian random variable \( B(r) \)^{20} follows a Kolmogorov spectrum
\[
E_{B^2,3D} \equiv |\tilde{B}(k)|^2 \propto \begin{cases} 0 & \text{if } k \lesssim k_0, \\ (k/k_0)^{-11/3} & \text{if } k \gtrsim k_0, \end{cases} \tag{C1}
\]
we can show that the 3D spectrum of \( B^2(r) \) follows Equation (9) (see, for example, Chepurnov 1999). The correlation of \( B^2(r) \) and the 3D energy spectrum of \( B^2(r) \) are related by
\[
\langle B^2(x) B^2(x+r) \rangle \propto \int E_{B^2,3D}(k) e^{-ik \cdot r} d^3k, \tag{C2}
\]
where \( \langle \cdots \rangle \) denotes an average over \( x \). A Gaussian random variable satisfies
\[
\langle B^2(x) B^2(x+r) \rangle = \langle B^2(x) \rangle \langle B^2(x+r) \rangle + 2 \langle B(x) B(x+r) \rangle, \tag{C4}
\]
where the first term on the right-hand side is a constant. Therefore, we can ignore this term in what follows. Fourier transform of both sides results in
\[
LHS = E_{B^2,3D}, \tag{C5}
\]
\[
RHS = 2 \int \langle B(x) B(x+r) \rangle e^{-ik \cdot r} d^3r = 2 \int d^3r K_B(r) K_B(r) e^{-ik \cdot r} \tag{C6}
\]
\[
= 2 \int d^3r \int d^3p \int d^3q E_{B,3D}(p) E_{B,3D}(q) e^{(p+q-k) \cdot r} \tag{C8}
\]
\[
= 2 \int d^3p \int d^3q E_{B,3D}(p) E_{B,3D}(q) \delta(p+q-k) \tag{C9}
\]
\[
= 2 \int d^3p E_{B,3D}(p) E_{B,3D}(k-p), \tag{C10}
\]
where \( \delta(k) \) is the Dirac \( \delta \)-function. Therefore, we have
\[
E_{B^2,3D}(k) \approx E_{B^2,3D}(0) \approx 2 \int d^3k |E_{B,3D}(k)|^2 \approx \text{constant} \tag{C11}
\]
for \( k \ll k_0 \).

**APPENDIX D**

**HIGH-ORDER STATISTICS**

While most of the paper is directly related to making use of the knowledge of the underlying spectra and/or two-point correlations in order to remove foregrounds, the part dealing with higher-order statistics is not directly related to the foreground removal. Nevertheless, the correspondence of the intermittencies of foregrounds to those of turbulence provides further support for our understanding of the turbulent origin of foreground fluctuations.

High-order structure functions are used for the study of intermittency, which refers to the non-uniform distribution of structures. Since CMB signals are close to Gaussian, one may think that they do not have strong intermittency. However, it has been reported that the intermittency of CMB signals deviates from Gaussianity (Bershadskii & Screenivasan 2003). If the intermittency of foreground signals is different from that of CMB signals, we can potentially use high-order structure functions to separate the CMB and foreground signals. In addition, we can use high-order structure functions to improve our knowledge of foregrounds.

^{19} \) When \( p = 1 \), the spatial correlation becomes \( K(r) \propto C - r \), where \( C \) is a constant. The corresponding 3D spectrum is \( E(k) \propto k^{-4} \). When the slope of the turbulence spectrum is steeper than \( k^{-4} \), the correlation function has the form \( K(r) \propto k_0 - r^4 \) regardless of the turbulence slope. On the other hand, when the 3D spectrum of turbulence is shallower than \( k^{-4} \), we have \( K(r) \propto k_0 - r^{m-3} \), where \( k_0 \sim L^{m-3} \) is a constant. Therefore, the condition of \( p \leq 1 \) is generally satisfied in a turbulent medium.

^{20} \) For simplicity, we assume that \( B \) is a scalar.
The structure functions of order $p$ for an observable $I$ is defined by

$$S_p(r) = \langle |I(x) - I(x + r)|^p \rangle,$$

(D1)

where the angled brackets denote an average over position $x$. For an observable defined in the plane of the sky, the angular structure function of order $p$ is

$$D_p(\theta) = \langle |I(e_1) - I(e_2)|^p \rangle.$$

(D2)

Traditionally, researchers use high-order structure functions of velocity to probe dissipation structures of turbulence. In fully developed hydrodynamic turbulence, the (longitudinal) velocity structure functions $S_p = \langle \left| (v(x + r) - v(x)) \cdot \hat{r} \right|^p \rangle \equiv \langle \delta v^p \rangle$ are expected to scale as $r^{\zeta_p}$:

$$S_p(r) \propto r^{\zeta_p}.$$

(D3)

One of the key issues in this field is the functional form of the scaling exponents $\zeta_p$. There are several models for $\zeta_p$. Roughly speaking, the dimensionality of the energy dissipation structures plays an important role.

Assuming 1D worm-like dissipation structures, She & Leveque (1994) proposed a scaling relation

$$\zeta^{SL}_p = p/9 + 2[1 - (2/3)^{p/3}]$$

(D4)

for incompressible hydrodynamic turbulence. Note that $\zeta_p$ is the scaling exponent of a structure of order $p$. On the other hand, assuming 2D sheet-like dissipation structures, Müller & Biskamp (2000) proposed the relation

$$\zeta^{MB}_p = p/9 + 1 - (1/3)^{p/3}$$

(D5)

for incompressible MHD turbulence.21

Recently, high-order structure functions of molecular line intensities have also been employed (Padoan et al. 2003; Gustafsson et al. 2006). In the optically thin case, the molecular line intensities are proportional to the column density. Kowal et al. (2007) studied scaling of higher moments of density fluctuations in MHD turbulence. Their numerical results show that, first of all, the scalings of higher moments in 3D can be obtained by studying the distribution of column densities. Then, they showed that the behavior of the scaling exponents for column density depends on sonic Mach number of turbulence.

Bershadskii & Screenivasan (2003) calculated the intermittency of CMB signals. Their result shows that the WMAP data follow the She–Leveque scaling. Therefore, it is interesting to see if foreground signals show different scalings.

We plot the scaling exponents of the Haslam 408 MHz map in Figure 11. The Haslam map shows reasonable agreement with the Müller–Biskamp MHD model. Note that Padoan et al. (2003) also obtained a similar result using $^{13}$CO emission from Perseus and Taurus.

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21 Boldyrev (2002) obtained the same scaling relation for highly supersonic turbulence. However, since it is unlikely that turbulence in the Galactic halo is highly supersonic, we refer the scaling relation to the “Müller–Biskamp” scaling.
Unlike the Haslam map, the 94 GHz dust map does not show agreement with the Müller–Biskamp model. The scaling exponents do not show strong dependence on the order $p$, which makes them similar to the results of the potential part of the supersymmetric flow studied in Kowal & Lazarian (2010) and densities in supersonic super-Alfvénic flows studied in Kowal et al. (2007). One may expect that in the presence of self-gravity this is what is expected for higher-order statistics of density, but further studies are required.

The left and middle panels of Figure 12 show that the slope is around 1 for high-order structure functions. This kind of behavior is expected when discrete structures dominate the map. However, it is not clear what kind of discrete structures dominate. Since the Haslam map and the dust map sample different types of the ISM, it is not surprising that they show different scaling behaviors.

For the Galactic disk, high-order structure functions of both maps show nearly flat structure functions. The structure functions show a nearly flat behavior down to $\theta \sim 1$, which happens to be the actual angular resolution of the maps. Further studies of MHD turbulence in the presence of self-gravity should clarify the origin of such behavior. Alternatively, higher moments of structure functions can be affected by unresolved point sources.

REFERENCES

Armstrong, J., Rickett, B., & Spangler, S. 1995, ApJ, 443, 209
Baccigalupi, C., Burigana, C., Ferrotta, F., De Zotti, G., La Porta, L., Maino, D., Maris, M., & Paladini, R. 2001, A&A, 372, 8
Beresnyak, A., & Lazarian, A. 2006, ApJ, 640, L175
Beresnyak, A., & Lazarian, A. 2009, ApJ, 702, 1190
Beresnyak, A., Lazarian, A., & Cho, J. 2005, ApJ, 624, L93
Bershadskii, A., & Screenivasan, K. R. 2003, Phys. Lett. A, 319, 21
Beuermann, K., Kanbach, G., & Berkhuijen, E. 1985, A&A, 153, 17
Bouchet, F., & Gispert, R. 1999, Astron. Astrophys. Trans., 17, 281
Bouchet, F., Gispert, R., & Pouget, J.-L. 1996, AIP Conf. Proc. 348, The Smoothly Extended Boundaries Can Be the Dominant Structures.

et al. (2007). One may expect that in the presence of self-gravity densities in supersonic super-Alfvénic flows studied in Kowal & Lazarian (2010) and different types of the ISM, it is not surprising that they show different scaling behaviors.

For the Galactic disk, high-order structure functions of both maps show nearly flat structure functions. The structure functions show a nearly flat behavior down to $\theta \sim 1$, which happens to be the actual angular resolution of the maps. Further studies of MHD turbulence in the presence of self-gravity should clarify the origin of such behavior. Alternatively, higher moments of structure functions can be affected by unresolved point sources.

22 However, we can show that thin filamentary structures or point sources with uniform intensity and sharp boundary are not the dominant structures. Consider a circular cloud with a radius $\Delta$ and a uniform intensity $I$ centered at the origin. (For simplicity, let us consider a 2D cloud in the 2D Cartesian coordinate system. Let us assume intensity is defined on the Cartesian grid.) Then the structure function is given by $D_{n}(r) \propto \sum_{i,j} I(|x_{i} - x_{j}|)/N_{pairs}$, where two points $x_{i}$ and $x_{j}$ are separated by the distance $r$ and $N_{pairs}$ is the total number of such pairs. When $r \gg \Delta$, we can show that $\sum_{i,j} I(|x_{i} - x_{j}|) \approx I(0)^n \propto r^{n} \propto \Delta^{n}$ and $N_{pairs} \propto (2\pi r)^{2}$. Therefore, we have $D_{n}(r) \propto r^{n} / \Delta^{n}$ for $r \gg \Delta$. That is, the structure functions show no dependence on $r$. We expect that structure functions for a thin uniform filament also show no dependence on $r$ when $r$ is larger than the width of the filament, because a filament can be viewed as a chain of circular clouds (or a chain of square-like clouds). The high-order structure functions for the dust emission map show $\sim r^n$ power-law scaling for $n \geq 1$ (left panel of Figure 11). Thus, neither thin uniform filaments nor point sources are dominant structures. However, filaments or circular clouds with smoothly extended boundaries can be the dominant structures.
Lazarian, A., & Pogosyan, D. 2008, 
Lazarian, A., Pogosyan, D., & Esquivel, A. 2002, in ASP Conf. Ser. 276, Seeing
Through the Dust: The Detection of H i and the Exploration of the ISM in
Galaxies, ed. A. R. Taylor, T. L. Landecker, & A. G. Willis (San Francisco,
CA: ASP), 182
Lazarian, A., & Prunet, S. 2001, in AIP Conf. Proc. 609, Astrophysical Polarized
Backgrounds, ed. S. Cecchini et al. (Melville, NY: AIP), 32
Lazarian, A., & Shutenkov, V. P. 1990, PAZh, 16, 690 (translated Sov. Astron.
Lett., 16, 297)
Lazarian, A., & Vishniac, E. T. 1999, 
Lee, H., & Draine, B. 1985, 
Lithwick, Y., & Goldreich, P. 2001, 
Lueker, M., et al. 2010, 
Maron, J., & Goldreich, P. 2001, 
Martin, P. 1974, 
Masi, S., et al. 2001, 
Miville-Deschênes, M.-A., Ysard, N., Lavabre, A., Ponthieu, N., Macías-Pérez,
J. F., Aumont, J., & Bernard, J. P. 2008, 
Müller, W.-C., & Biskamp, D. 2000, 
Padoan, P., Boldyrev, S., Langer, W., & Nordlund, A. 2003, 
Padoan, P., Juvela, M., Kritsuk, A., & Norman, M. L. 2006, 
Padoan, P., Juvela, M., Kritsuk, A., & Norman, M. L. 2009, 
Page, L., et al. 2007, 
Ponthieu, N., et al. 2005, 
Prunet, S., & Lazarian, A. 1999, in ASP Conf. Ser. 181, Microwave Foregrounds,
ed. A. de Oliveira-Costa & M. Tegmark (San Francisco, CA: ASP), 113
Prunet, S., Sethi, S. K., Bouchet, F. R., & Miville-Deschênes, M.-A. 1998, 
Reich, P., & Reich, W. 1986, A&AS, 63, 205
Schlegel, D., Finkbeiner, D., & Davis, M. 1998, ApJ, 500, 525
She, Z.-S., & leveque, E. 1994, Phys. Rev. Lett., 72, 336
Smoot, G. F. 1999, in ASP Conf. Ser. 181, Microwave Foregrounds, ed. A. de
Oliveira-Costa & M. Tegmark (San Francisco, CA: ASP), 61
Stanimirovic, S., & Lazarian, A. 2001, 
Sun, X. H., Reich, W., Waelkens, A., & Ensslin, T. A. 2008, 
Szapudi, I., Prunet, S., Pogosyan, D., Szalay, A. S., & Bond, J. R. 2001, 
Tegmark, M., & Efstatthiou, G. 1996, MNRAS, 281, 1297
Tegmark, M., Eisenstein, D. J., Hu, W., & de Oliveire-Costa, A. 2000, 
Tucci, M., Carretti, E., Cecchini, S., Nicastro, L., Fabbri, R., Gaensler, B. M.,
Dickey, J. M., & McClure-Griffiths, N. M. 2002, 
Waëlkens, A., Jaffe, T., Reinecke, M., Kitaura, F., & Ensslin, T. 2009, 
Weingartner, J., & Draine, B. 2001, 
Whittet, D., Hough, J., Lazarian, A., & Hoang, T. 2008, ApJ, 674, 304
Zaldarriaga, M., & Seljak, U. 1998, Phys. Rev. D, 58, 023003

1201