Generalized Weierstrass Relation for a Submanifold $S^k$ in $\mathbb{E}^n$

Coming from Submanifold Dirac Operator

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§1. Introduction

Using the submanifold quantum mechanical scheme [dC, JK], the restricted Dirac operator in a $k$-spin submanifold immersed in euclidean space $\mathbb{E}^n$ ($0 < k < n$) is defined [BJ, Mat1-10]. We call it submanifold Dirac operator. Then it is shown that the zero modes of the Dirac operator express the local properties of the submanifold, such as the Frenet-Serret and generalized Weierstrass relations. In other words this article gives a representation of a further generalized Weierstrass relation for the submanifold.

Before we start to explain the idea of the submanifold quantum mechanics, we will recall three facts.

(1) Let us consider a left-differential ring $P$ and its element $Q$ over a manifold $M$. In the book of Björk [Remark 1.2.16 in Bj], it is stated that to treat its adjoint right operator is difficult. Its essential is as follows. Assume that $M$ is Riemannian. For smooth functions $f_1$ and $f_2$ whose support is compact, we consider the following integral as a bilinear form of $f_1$ and $f_2$ formally

$$\int_M dvol(f_1Qf_2).$$

What is a natural adjoint of $Q$? One might regard an action to $f_1$ obtained by partial integral as its adjoint. However there exists an obstacle because the measure depends upon the local coordinate. Hence concept of adjoint operator is very subtle.

(2) In a quantum mechanical problem, we sometimes encounter the situation that for an eigen function (and thus its zero mode) $\psi$ of a differential operator $P$,

$$P\psi = E\psi,$$ \hspace{1cm} (1-2)

is a vector of a representation space of a group $G$. In the case, if one finds a solution of $P\psi = 0$, he obtains a representation of the group $G$.

Further suppose that $P$ is decomposed by $P = P_1 + P_2$. Let us consider a kernel of $P_2$, $\text{Ker}P_2$, in a certain function space. If an element $\psi_1 \in \text{Ker}P_2$ satisfies

$$P_1\psi_1 = E\psi_1,$$ \hspace{1cm} (1-3)
we also obtain a representation of the group $G$.

(3) In the quantum mechanical problem over a manifold $M$, there are typical two pairings for the function space over $M$ in general, i.e., 1) global pairing $\langle,\rangle$ induced from $L^2$-norm, such as (1-1) and 2) point-wise pairing $\cdot$ which is connected with the probability density. (Of course, the point-wise pairing can be regarded as $\langle \circ, \delta(p)\times \rangle$ in terms of the Dirac distribution $\delta()$ at $p$ in $M$.)

Even though the concept of adjoint operator is subtle, we could define a natural adjoint operator if the measure is fixed by some reason, e.g., Haar measure. We note that the ordinary Lebesgue measure in a euclidean space $\mathbb{E}^n$ is a typical Haar measure of the translation group. The quantum mechanics in $\mathbb{E}^n$ is based on the measure and concept of the adjoint operator plays essential roles. In such a case, by fixing a typical measure and $L^2$-type paring $\langle,\rangle: \Omega^* \times \Omega \to \mathbb{C}$, for an operator $Q \in P$ whose domain is $\Omega$, we can define a right-adjoint operator, $Ad(Q)$, with the domain $\Omega^*$ by

$$\langle f, Qg \rangle = \langle fAd(Q), g \rangle, \quad \text{for } (f, g) \in \Omega^* \times \Omega.$$  \hspace{1cm} (1-4)

Assume that there is an isomorphism $\varphi$ between domains $\Omega$ and $\Omega^*$ as a vector space, $\varphi: \Omega \to \Omega^*$. Then we can define the left-adjoint operator $Q^*$ as $Q^*f := \varphi^{-1}(\varphi(f)Ad(Q))$.

Triplet $(\Omega^* \times \Omega, \langle,\rangle, \varphi)$ becomes a preHilbert space $\mathcal{H}$ by letting the inner product $(,)_\varphi: \Omega \times \Omega \to \mathbb{C}$ be $(f, g)_\varphi := \langle \varphi(f), g \rangle$. (Then $(P^*f, g)_\varphi = (f, Py)_\varphi$.)

Suppose that $Q \in P$ is self-adjoint, i.e., the domains of $Q^*$ and $Q$ coincide and $Q^* = Q$ over there. Then we have the following properties:

1. The kernel of $Q$, Ker$Q$, is isomorphic to the Ker$(Ad(Q))$ i.e.,

   $$(\text{Ker}(Q))^* := \varphi(\text{Ker}(Q)) = \text{Ker}(Ad(Q)).$$  \hspace{1cm} (1-5)

2. $((\text{Ker}Q)^* \times \text{Ker}Q, \langle,\rangle, \varphi)$ becomes a preHilbert space.

The projection $\pi$ from $\Omega^* \times \Omega$ to $(\text{Ker}Q)^* \times \text{Ker}Q$ is commutative with $\varphi$, i.e.

$$\varphi\pi|_{\Omega} = \pi|_{\Omega^*}\varphi, \quad (\varphi(\pi|_{\Omega}f) = \pi|_{\Omega^*}\varphi(f) \equiv \varphi(f)Ad(\pi|_{\Omega})).$$  \hspace{1cm} (1-6)

For $\pi$ satisfying (1-6), we will say that $\pi$ consists with the inner product. In fact (1-6) means that $\pi|_{\Omega^*} = \pi|_{\Omega}$ due to the relation $\pi|_{\Omega^*}f \equiv \varphi^{-1}(\varphi(f)Ad(\pi|_{\Omega})) = \pi|_{\Omega}f$.

Next assume that $P_2$ is not self-adjoint in the preHilbert space $\mathcal{H} \equiv (\Omega^* \times \Omega, \langle,\rangle, \varphi)$. Even for the case, if $P_2$ is a certain operator, we could construct a transformation $\eta_{sa}$ of the preHilbert space and its operators, and find a preHilbert space $\mathcal{H}'$ satisfying the following conditions,

1. There exists an isomorphism $\eta_{sa}: \Omega^* \times \Omega \to \tilde{\Omega}^* \times \tilde{\Omega}$ as a vector space.

2. By defining a pairing $\langle,\rangle P_2 := \eta_{sa}|_{\Omega^*\circ}, \eta_{sa}|_{\Omega^*\times}$, and $\tilde{\varphi} := \eta_{sa}|_{\Omega^*}\varphi\eta_{sa}^{-1}|_{\tilde{\Omega}}$, $\mathcal{H}' \equiv (\tilde{\Omega}^* \times \tilde{\Omega}, (\cdot, \cdot), \tilde{\varphi})$.

3. An operator $P$ for $\Omega$ is transformed as $\eta_{sa}|_{\Omega^*}P\eta_{sa}^{-1}|_{\Omega}$

4. $P_2$ itself is a self-adjoint in $\mathcal{H}'$. 

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We call $\eta_{sa}$ self-adjointization: $\eta_{sa} : \mathcal{H} \to \mathcal{H}'$. Of course, the self-adjointization is not a unitary operation for $P_2$.

As mentioned above, $\text{Ker}(P_2) \subset \tilde{\Omega}$ also becomes a preHilbert space denoted by $\mathcal{H}'' := ((\text{Ker}(P_2))^* \times \text{Ker}(P_2), (,), \varphi)$. Letting the projection of $\mathcal{H}' \to \mathcal{H}''$ be denoted by $\pi_{P_2}$, we have a sequence,

$$\mathcal{H} \xrightarrow{\eta_{sa}} \mathcal{H}' \xrightarrow{\pi_{P_2}} \mathcal{H}''.$$ \hspace{1cm} (1-7)

This sequence is a key of the submanifold quantum mechanics. Instead of considering $P\psi = E\psi$ in $\mathcal{H}$, we might search a solution of $f$ in $\eta_{sa}(P)\psi_1 = E\psi_1$ in $\mathcal{H}''$.

Let us explain the idea of the submanifold quantum mechanics. We note that for a smooth $k$-submanifold $S$ embedded in the $n$-euclidean space $\mathbb{E}^n$ ($0 < k < n$), we can find a natural adjoint operator for a differential operator defined over $S$ by fixing the induced metric of $S$ from $\mathbb{E}^n$, even though $S$ is a curved space. For a Schrödinger equation in $\mathbb{E}^n$ with the $L^2$-type Hilbert space $\mathcal{H}$,

$$-\Delta \psi = E\psi,$$ \hspace{1cm} (1-8)

we should regard the Laplace operator $\Delta$ as a Casimir operator for the Lie group with respect to the translation. By considering $\Delta$ over a tubular neighborhood of $S$, $\Delta$ includes the normal differential operator $\partial_\perp$. We regard the normal differential $\partial_\perp$ as above $P_2$. As $\partial_\perp$ is not self-adjoint in general, we step the above sequence. In the self-adjointization $\eta_{sa}$, we obtain an extra potential in the differential equation. By considering kernel of purely normal component of the differential $\partial_\perp$ and restricting the its definition region of $\eta_{sa}(\Delta)$ at $S$, we define a differential operator,

$$\Delta_{S \to \mathbb{E}^n} := \eta_{sa}(\Delta)|_{\text{Ker} \partial_\perp}|_S.$$ \hspace{1cm} (1-9)

Then it turns out to be

$$\Delta_{S \to \mathbb{E}^n} = \Delta_S + U(\kappa_i),$$ \hspace{1cm} (1-10)

where $\Delta_S$ is the Beltrami-Laplace operator on $S$ which exhibits the intrinsic properties of $S$ and $U(\kappa_i)$ is an invariant functional of principal curvature $\kappa$’s of $S$ in $\mathbb{E}^n$.

Due to the self-adjointness of $\partial_\perp$ in $\mathcal{H}''$, we are allowed to consider the Hilbert space $\mathcal{H}''$ for $\Delta_{S \to \mathbb{E}^n}$ naturally. The point-wise product in $\text{Ker}(\partial_\perp)|_S$ also has meaning; the probability density is well-defined there. Thus we can consider the submanifold Schrödinger equation,

$$-\Delta_{S \to \mathbb{E}^n} \psi = E\psi,$$

as a quantum mechanical problem and a representation of translational group.

For the case a smooth surface $S$ embedded in $\mathbb{E}^3$, by letting $K$ and $H$ denote the Gauss and mean curvature, we obtain

$$\Delta_{S \to \mathbb{E}^3} = \Delta_S + H^2 - K.$$ \hspace{1cm} (1-11)
Hence it is expected that $\Delta_{S \hookrightarrow \mathbb{E}^n}$ and its zero mode exhibit the extrinsic properties, e.g., umbilical points, of the submanifold.

The submanifold quantum mechanics was opened by Jensen and Koppe about thirty years ago and rediscovered by da Costa [JK, dC]; even though they did not mention essentials of the submanifold quantum mechanics as described above, they obtained $\Delta_{S \hookrightarrow \mathbb{E}^n}$. We should note that as the above operation is local, our consideration can be extended to an immersed submanifold $S$ in $\mathbb{E}^n$.

As I have been considering the Dirac operator version of above quantum system for this decade [Mat1-10], which is our main subject in this paper. In the investigation of the Dirac operator, we should recall the fact that solutions $\{\Psi\}$ of the Dirac equation,

$$D_{\mathbb{E}^n} \Psi = 0,$$

locally represents the spin group. By letting denote point-wise pairing, $(\varphi_{pt}(\{\Psi\}) \times \{\Psi\}, \cdot, \varphi_{pt})$ for a certain map $\varphi_{pt}$ becomes a preHilbert space and for an appropriate matrix and solution $\Psi$, $\varphi_{pt}(\Psi) \gamma \Psi$ exhibits a section of the $\text{SO}(n)$ principal bundle $\text{SO}_{\mathbb{E}^n}(T\mathbb{E}^n)$ over $\mathbb{E}^n$.

Thus we apply the submanifold quantum mechanical scheme to the Dirac operator over a $k$-spin submanifold $S$ immersed in $\mathbb{E}^n$. Then we have a representation of $\text{SO}_{\mathbb{E}^n}(T\mathbb{E}^n)|_S$ at $S$ immersed in $\mathbb{E}^n$, which is the generalized Weierstrass relation. Our main theorem is Theorem 3.15.

The organization of this article is as follows. Section 2 devotes the preliminary on the geometrical setting [E] and conventions of the Clifford module and its related objects [BGV, Tas]. In §3, we give an construction algorithm of the submanifold Dirac equation and investigate its properties. Section 4 gives its example.

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§2. Preliminary

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As we consider a restriction of the differential operator and the function space over a manifold, we should deal with it in the framework of the sheaf theory and thus we employ the following notations.

2.1 Conventions. [Mal] For a fiber bundle $A$ over a differential manifold $M$ and an open set $U \subset M$, let $\Gamma(U,A)$ denote a set of smooth sections of the fiber bundle $A$ over $U$. Further for a point $p$ in $M$, let $\Gamma(p,A)$ denote a stalk at $p$ of a set of smooth sections of the fiber bundle $A$.

Further we use Einstein convention and let $\mathbb{C}$ ($\mathbb{R}$) denote the complex (real) field. For brevity, we use the notations $\partial_{u^\mu} := \partial/\partial u^\mu$ for a certain parameter $u^\mu$. For a real number $x \in [n, n+1)$ and an integer $n$, $[x]$ denotes $n$. Let $\mathbb{C}_M$ ($\mathbb{R}_M$) denote a complex (real) line bundle over a manifold $M$.

In order to define the submanifold Dirac operator, let us consider a smooth spin $k$-submanifold $S$ immersed in $n$ euclidean space $\mathbb{E}^n$. As mentioned in §3, the Dirac operator can be constructed locally. Thus in order to simplify the argument, we assume that $S$ satisfies several properties as follows.

2.2 Assumptions/Notations on the submanifold $S$ and $T_S$.

1) Let $S$ be diffeomorphic to $\mathbb{R}^k$ as a chart $(s^1, \cdots, s^k) \in \mathbb{R}^k$.
2) Let a tubular neighborhood of $S$ be denoted by $T_S$, $\pi_{T_S} : T_S \to S$.
3) Let $T_S$ have a foliation structure by $(dq^{\dot{\alpha}} = 0)_{\dot{\alpha}=k+1,\cdots,n}$, $T_S \approx \mathbb{R}^k \times \mathbb{R}^{n-k} \ni (s^1, \cdots, s^k, q^{k+1}, \cdots, q^n) \equiv (u^1, \cdots, u^n)$. (Let beginning of Greek indices $\alpha, \beta$ run from 1 to $k$ and those with dot $\dot{\alpha}, \dot{\beta}$ run from $k+1$ to $n$. Let the middle of Greek indices $\mu, \nu$ run from 1 to $n$.)
4) Let each leaf of the foliation parameterized by $q = (q^{k+1}, \cdots q^n)$ be denoted by $S_q$: $S_{q=0} \equiv S$.
5) Let $g_{T_S}$, $g_{S_q}$ and $g_S$ denote the Riemannian metric of $T_S$, $S_q$ and $S$ induced from that of $\mathbb{E}^n$ respectively.
6) Let $g_{T_S}^{i,j} := g_{T_S}(\partial_{x^i}, \partial_{x^j})$, $g_{T_S}^{\mu,\nu} := g_{T_S}(\partial_{u^\mu}, \partial_{u^\nu})$, $g_{T_S} := det g_{T_S}^{\mu,\nu}$ and so on for $T_S$, $S$ and $\mathbb{E}^n$.
7) Let $(dq^{\dot{\alpha}} = 0)_{\dot{\alpha}=k+1,\cdots,n}$ be an orthonormal base $g_{S_q}(\partial_{q^{\alpha}}, \partial_{q^{\beta}}) = \delta_{\alpha,\beta}$ and satisfy $g_{T_S}(\partial_{q^{\alpha}}, \partial_{q^{\alpha}}) = 0$ for $\alpha = 1, \cdots, k$ and $\dot{\alpha} = k+1, \cdots, n$. In other words, we have

$$g_{T_S} = g_{S_q} \oplus \delta_{\dot{\alpha},\dot{\beta}} dq^{\dot{\alpha}} \otimes dq^{\dot{\beta}}.$$ (2-1)

8) At a point $p$ in $S_q$, let an orthonormal frame of the cotangent space $T^*S_q$ be denoted by $d\xi := (d\xi^\mu) := (d\zeta^\alpha, dq^{\dot{\alpha}})$.

We call this parameterization $q$ satisfying (2)-(7) canonical parameterization.
2.3 Notation. For a point \( p \) of \( S \), let Weingarten map be denoted by \( -\gamma_\beta : T_p S \rightarrow T_p \mathbb{E}^n \); for bases \( e_\alpha \) of \( TS \) and \( \tilde{e}_\beta \in TS^\perp \) \((T_p \mathbb{E}^n = T_p S \oplus T_p S^\perp)\),

\[
\gamma_\beta(e_\alpha) := \partial_\alpha \tilde{e}_\beta = \gamma^\alpha_{\beta \alpha} e_\alpha + \beta_{\beta \alpha} e_\beta.
\] (2-2)

It is not trivial whether a submanifold in \( \mathbb{E}^n \) has the canonical parameterization \( q \) or not in general. However it is not so difficult to prove that a local chart of the submanifold has canonical parameterization. In the proof, 2.2 (7) requires some arguments on the Weingarten map but by tuning the frame \( \tilde{e}_\beta \) in (2-2) using \( \text{SO}(n-k) \) action, we can construct it due to the following Proposition [Mat10], which guarantees the existence of \( S \) satisfying the assumptions in 2.2.

2.4 Proposition. For a base \( e_\alpha \) of \( TS \), there is an orthonormal frame \( e_\alpha = \delta_{\alpha \beta} dq^\beta \in TS^\perp \) satisfying

\[
\partial_\alpha e_\beta = \gamma_{\beta \alpha} e_\beta.
\] (2-3)

In terms of the properties, we have the moving frame and the metric as follows.

2.5 Lemma. For the moving frame \( e_\alpha = \delta_{\alpha \beta} dq^\beta \) in Proposition 2.4, the moving frame \( E^i = E^i_\mu \partial_\mu \), \( E^i_\mu = \partial_\mu x^i \) in \( S_q \) is expressed by

\[
E^i_\alpha = e^i_\alpha + q^{\alpha} \gamma_{\alpha \beta} e^i_\beta, \quad E^i_{\dot{\alpha}} = e^i_{\dot{\alpha}}.
\] (2-4)

Proof. As a point \( x \equiv (x^i) \) in \( S_q \) is expressed by \( x = y + e_\alpha q^{\dot{\alpha}} \) using \( y := \pi_{TS} x \), we obtain them. \( \square \)

2.6 Corollary. Let \( g_{TS} := \det_{n \times n}(g_{TS,\mu,\nu}) \), \( g_{S_q} := \det_{k \times k}(g_{S_q,\alpha,\beta}) \) and \( g_S := g_{S_q} |_{q=0} \).

1. \( g_{TS} \equiv g_{S_q} \).

2. \( g_{S_q} \) is expressed as \( g_{S_q} = g_S + g_S^{(1)} q^{\dot{\alpha}} + g_S^{(2)} (q^{\dot{\alpha}})^2 \), and locally

\[
g_{TS}(\partial_\alpha, \partial_\beta) = g_{S\alpha \beta} + [\gamma^\gamma_{\alpha \alpha} g_{S\gamma \beta} + g_{S\alpha \gamma} \gamma^\gamma_{\beta \beta}] q^{\dot{\alpha}} + [\gamma^\gamma_{\alpha \alpha} g_{S\delta \gamma} \gamma^\gamma_{\beta \beta}] q^{\dot{\beta}}.
\] (2-5)

3. When we factorize \( g_{S_q} \) as \( g_{S_q} = g_S \cdot \rho_{S_q} \), the factor \( \rho_{S_q} \) is given by

\[
\rho_{S_q} = 1 + 2 \text{tr}_{k \times k}(\gamma^{\dot{\alpha} \dot{\beta}}) q^{\dot{\alpha}} + \left[ 2 \text{tr}_{k \times k}(\gamma^{\dot{\alpha} \dot{\beta}}) \text{tr}_{k \times k}(\gamma^{\alpha \beta}) - \text{tr}_{k \times k}(\gamma^{\dot{\alpha} \dot{\beta}}) \gamma^{\alpha \beta} \right] q^{\dot{\alpha}} q^{\dot{\beta}} + \mathcal{O}(q^{\dot{\alpha}} q^{\dot{\beta}}),
\] (2-6)

where \( \mathcal{O} \) is Landau symbol.

Further we will recall properties of Clifford algebra and its related quantities, and show our conventions of them.
2.7 Clifford Algebra and Spinor Representations. [BGV, Tas]

(1) Let $\text{CLIFF}(R^n)$ denote the Clifford algebra for the vector space $R^n$ and let $\text{CLIFF}^C(R^n) := \text{CLIFF}(R^n) \otimes C$. Let $\text{CLIFF}^{\text{even}}(R^n)$ denote the subaglebra of $\text{CLIFF}^C(R^n)$ consisting of even degrees of generators in $\text{CLIFF}^C(R^n)$. Further let $\text{SPIN}(R^n)$ denote the spin group for $R^n$.

(2) For the exterior algebra $\wedge R^n = \bigoplus_{j=0}^{n} \wedge^j R^n$, there is an isomorphism as a vector space, called symbol map $\text{CLIFF}(R^n) \rightarrow \wedge R^n$. Let its inverse be denoted by $\gamma$, $\gamma : \wedge R^n \rightarrow \text{CLIFF}(R^n)$, (2.7)

which is called gamma-matrix.

(3) For even $n$ case, let $\text{Cliff}(R^n)$ denote a $\text{CLIFF}^C(R^n)$-module whose endomorphism is isomorphic to $\text{CLIFF}^C(R^n)$, which is a $2^{n/2}$ dimensional $C$-vector space. For odd $n$ case, let $\text{Cliff}(R^n)$ denote a $\text{CLIFF}^C(R^n)$-module whose endomorphism is homomorphic to $\text{CLIFF}^C(R^n)$ as a $2^{n/2}$ dimensional $C$-vector space representation and elements are invariant for the action of $\gamma(e_1) \gamma(e_2) \cdots \gamma(e_n)$. Here $e_1, e_2, \cdots, e_n$ are orthonormal base of $R^n$.

In order to simplify the argument, we will fix the expressions of the $\gamma$-matrices and so on as follows.

2.8 Conventions.

(1) We recall the fact $\text{CLIFF}^C(R^{n+2}) \approx \text{END}(C^2) \otimes \text{CLIFF}^C(R^n)$, where $\text{END}(C^2)$ is the endomorphism of $C^2$ and $\text{END}(C^2)$ can be generated by the Pauli matrices:

$$
\sigma_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (2.8)
$$

(2) For even $n$ case, noting $\text{CLIFF}^C(R^n) \approx \text{END}(C^{[n/2]})$, we use the conventions,

\[ \gamma(e_1) = \sigma_1 \otimes \sigma_1 \otimes \cdots \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1, \]
\[ \gamma(e_2) = \sigma_1 \otimes \sigma_1 \otimes \cdots \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_2, \]
\[ \gamma(e_3) = \sigma_1 \otimes \sigma_1 \otimes \cdots \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_3, \]
\[ \gamma(e_4) = \sigma_1 \otimes \sigma_1 \otimes \cdots \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_0 \]
\[ \gamma(e_5) = \sigma_1 \otimes \sigma_1 \otimes \cdots \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_3 \otimes \sigma_0 \]
\[ \cdots \]
\[ \gamma(e_{n-1}) = \sigma_1 \otimes \sigma_3 \otimes \cdots \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_0, \]
\[ \gamma(e_n) = \sigma_2 \otimes \sigma_0 \otimes \cdots \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_0. \quad (2.9) \]
(3) For odd $n$ case, noting $\text{CLIFF}^C(\mathbb{R}^n) \approx \text{END}(\mathbb{C}^{(n-1)/2}) \oplus \text{END}(\mathbb{C}^{(n-1)/2})$, we use the conventions,

\begin{align*}
\gamma(e_1) &= \sigma_1 \otimes \sigma_1 \otimes \cdots \otimes \sigma_1 \otimes \epsilon, \\
\gamma(e_2) &= \sigma_1 \otimes \sigma_1 \otimes \cdots \otimes \sigma_1 \otimes \sigma_2 \otimes 1, \\
\gamma(e_3) &= \sigma_1 \otimes \sigma_1 \otimes \cdots \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_3 \otimes 1, \\
\cdots \\
\gamma(e_{n-2}) &= \sigma_1 \otimes \sigma_3 \otimes \cdots \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_0 \otimes 1, \\
\gamma(e_{n-1}) &= \sigma_2 \otimes \sigma_0 \otimes \cdots \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_0 \otimes 1, \\
\gamma(e_n) &= \sigma_3 \otimes \sigma_0 \otimes \cdots \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_0 \otimes 1, \\
\end{align*}

where $\epsilon$ is defined as the generator of $\text{CLIFF}^C(\mathbb{R}) = \mathbb{R}[[\epsilon]]/(1 - \epsilon^2)$.

(4) By introducing $b_+ := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $b_- := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\text{Cliff}(\mathbb{R}^n)$ is spanned by the bases

$$
\Xi_\epsilon = b_{\epsilon_1} \otimes b_{\epsilon_2} \otimes \cdots \otimes b_{\epsilon_{[n/2]-1}} \otimes b_{\epsilon_{[n/2]}},
$$

where $\epsilon = (\epsilon_1, \epsilon_2, \cdots, \epsilon_{[n/2]})$ and $\epsilon_a = \pm (a = 1, \cdots, [n/2])$. Similarly, $\text{Cliff}(\mathbb{R}^n)^*$ is spanned by

$$
\Xi_\epsilon = \overline{b}_{\epsilon_1} \otimes \overline{b}_{\epsilon_2} \otimes \cdots \otimes \overline{b}_{\epsilon_{[n/2]-1}} \otimes \overline{b}_{\epsilon_{[n/2]}},
$$

where $\overline{b}_+ := (1, 0)$ and $\overline{b}_- := (0, 1)$. By appropriately numbering, let $\epsilon^{[c]} (c = 1, \cdots, 2^{[n/2]})$ denotes each $\epsilon$. The isomorphic map $\varphi : \text{Cliff}(\mathbb{R}^n) \rightarrow \text{Cliff}(\mathbb{R}^n)^*$ is given by

$$
\varphi\left( \sum_{c=1}^{2^{[n/2]}} a_c \Xi_\epsilon^{[c]} \right) = \sum_{c=1}^{2^{[n/2]}} \overline{a}_c \Xi_{\epsilon^{[c]}},
$$

for $a_c \in \mathbb{C}$ and its complex conjugate $\overline{a}_c$.

(5) Defining $b_1 := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ b_2 := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \ b_3 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \overline{b}_1 := \frac{1}{\sqrt{2}} (1, 1), \ \overline{b}_2 := \frac{1}{\sqrt{2}} (1, -1)$ and $\overline{b}_3 := (1, 0)$,

$$
\overline{b}_a \sigma_b b_a = \delta_{a,b}, \quad (a, b = 1, 2, 3), \quad \text{(not summed over } a). \quad (2-14)
$$

(6) For $j = 1, \cdots, [n/2]$, define

$$
\Psi^{(2j-1)} := b_1 \otimes b_1 \otimes \cdots \otimes b_1 \otimes b_3 \otimes b_1 \otimes \cdots \otimes b_1,
$$

$$
\Psi^{(2j)} := b_1 \otimes b_1 \otimes \cdots \otimes b_1 \otimes b_2 \otimes b_1 \otimes \cdots \otimes b_1.
$$

(2-15)
Further for odd $n$ case, we define,

$$
\Psi^{(n)} := b_3 \otimes b_1 \otimes \cdots \otimes b_1 \otimes b_1 \otimes \cdots \otimes b_1,
$$

(2-16)

and assume that $e\Psi^{(i)} = \Psi^{(i)}$. Further define $\overline{\Psi}^{(k)}$ as in (2-13). Then $\Psi^{(a)}$ is an element of $\text{Cliff}(\mathbb{R}^n)$ $(a = 1, \cdots, n)$ satisfying

$$
\overline{\Psi}^{(a)}_{\{dx\}} \gamma(e^b) \Psi^{(a)}_{\{dx\}} = \delta^a_b, \text{ not summed over } a.
$$

(2-17)

As we wish to consider a spin principal subbundle over $S$ induced from spin principal bundle over $E^n$, we recall the facts:

2.9 Lemma.

(1) For $k < n$, $\text{CLIFF}^C(\mathbb{R}^k)$ is a natural sub-vector space of $\text{CLIFF}^C(\mathbb{R}^n)$ by the inclusion of generators such that $\text{CLIFF}^C_{\text{even}}(\mathbb{R}^k)$ is a subring of $\text{CLIFF}^C_{\text{even}}(\mathbb{R}^n)$.

(2) For $k < n$, $\text{SPIN}(\mathbb{R}^k)$ is a natural subgroup of $\text{SPIN}(\mathbb{R}^n)$.

Proof. The statements are proved by considering five cases. 1) $k = 2l$, $n = 2l + 1$ case, 2) $k = 2l$, $n = 2l + 2$ case, 3) $k = 2l + 1$, $n = 2l + 2$ cases, 4) $k = 2l + 1$, $n = 2l + 3$ case, and 5) otherwise. The fifth case can be proved by combinations of the other cases. The first case is trivial. The second case is a key because the third and forth cases are similarly proved as the second case. Therefore we concentrate our attention only on the $k = 2l$, $n = 2l + 2$ case. Recalling the facts in 2.8, we have a natural inclusion as a set by the generators for the bases $e_i$'s of $\mathbb{R}^k$ and $E_i$'s of $\mathbb{R}^n$,

$$
\tau_{k,n} : \gamma(e) :\mapsto \gamma(E_i) := \sigma_1 \otimes \gamma(e_i), \quad i = 2, 3, \cdots, k,
$$

(2-18)

and define $\tau_{k,n}(\gamma(e_i)\gamma(e_j)) := \tau_{k,n}(\gamma(e_i))\tau_{k,n}(\gamma(e_j))$, $\tau_{k,n}(\gamma(e_i)\gamma(e_j)\gamma(e_k)) := \tau_{k,n}(\gamma(e_i))\tau_{k,n}(\gamma(e_j))\tau_{k,n}(\gamma(e_k))$ and so on. On the other hand, for $c^i \in \text{CLIFF}(\mathbb{R}^k)$, we have a natural inclusion as an algebra,

$$
t_{k,n} : \text{CLIFF}(\mathbb{R}^k) \hookrightarrow \text{CLIFF}(\mathbb{R}^n), \quad C_i = \sigma_0 \otimes c_i,
$$

(2-19)

and then we have homomorphism, $t_{k,n}(c_i c_j) = t_{k,n}(c_i) t_{k,n}(c_j)$. We note that for $1 \leq i, j \leq k$, $\gamma(E_i)\gamma(E_j) = t_{k,n}(\gamma(e_i)\gamma(e_j))$ and thus in even subring $\text{CLIFF}^C_{\text{even}}(\mathbb{R}^n)$, image $\tau$ and $\iota$ agree. Thus (1) is proved. Accordingly $\exp(\tau_{k,n}(\gamma(e_j)\gamma(e_i)))$ can be regarded as an elements of $\text{SPIN}(\mathbb{R}^n)$ and (2) is proved. □

§3. Construction of Submanifold Dirac Operator

An algorithm to construct submanifold Dirac operator is the following six steps.
Step 1: Set the Dirac equation \( D_{E^n} \Psi_{E^n} = 0 \) in a euclidean space \( E^n \) and embed the \( k \)-smooth Spin submanifold \( S \) into \( E^n \) \((0 < k < n)\).

Here we will give our notations in order to express the Dirac equation \( D_{E^n} \Psi_{E^n} = 0 \).

3.1 Clifford module etc..

1. Let \( \text{CLIFF}^{C}_{E^n}(T^*E^n) \) denote the Clifford bundle over \( E^n \) which has the Clifford ring \( \text{CLIFF}^{C}(\mathbb{R}^n) \) structure associated with the cotangent bundle \( T^*E^n \) and the set of differential forms \( \Omega(E^n) := \sum_{\alpha=1}^{n} \Omega^{\alpha}(E^n) \).

2. Let \( \text{Cliff}_{E^n}(T^*E^n) \) denote the Clifford module over \( E^n \) associated with the cotangent space \( T^*E^n \), which is modeled by \( \text{Cliff}(\mathbb{R}^n) \).

3. Let \( \varphi_{pt} \) denote the natural bijection between the spaces as a \( \text{CLIFF}^{C}_{E^n}(T^*E^n) \)-module modeled by (2-13), i.e.,

\[
\varphi_{pt} : \Gamma(p, \text{Cliff}_{E^n}(T^*E^n)) \rightarrow \Gamma(p, \text{Cliff}_{E^n}(T^*E^n)^*),
\]

for a point \( p \) in \( E^n \).

4. For an orthonormal frame \( e^i \in T^*E^n \) at \( p \in E^n \), let \( \gamma_{\{e\}} \) denote the \( \gamma \)-matrix as the map from \( \Omega(E^n) \) to \( \text{CLIFF}^{C}_{E^n}(T^*E^n) \) as (2-7),

\[
\gamma_{\{e\}} : e^i \mapsto \gamma_{\{e\}}(e^i) \in \text{CLIFF}^{C}_{E^n}(T^*E^n)|_p.
\]

For later convenience, we also employ a notation for a one-from \( du^\mu = E^\mu_i e^i \in \Gamma(p, \Omega(1)(E^n)) \),

\[
\gamma_{\{e\}}(du^\mu) = E^\mu_i \gamma_{\{e\}}(e^i).
\]

For simplicity, for a Cartesian coordinate system \( (x^i) = (x^1, \cdots, x^n) \) in \( E^n \) we fix \( e^i = dx^i \) and in \( T_S \), let \( e^\mu = d\xi^\mu \) in 2.2 (8).

5. Let \( \text{SPIN}_{E^n}(T^*E^n) \) denote a spin principal bundle over \( E^n \). Let a natural bundle map from \( \text{SPIN}_{E^n}(T^*E^n) \) to \( \text{SO}(n) \)-principal bundle \( \text{SO}_{E^n}(T^*E^n) \) be denoted by \( \tau_{E^n} \); for \( p \in E^n \), the orthonormal frame \( e \in T_p E^n \) and \( e^\Omega \in \text{SPIN}_p(T^*E^n) \), the action of \( \text{SO}(n) \) is defined by

\[
\tau_{E^n}(e^\Omega)e^i := \gamma_{\{e\}}^{-1}(e^\Omega \gamma_{\{e\}}(e^i)e^{-\Omega}).
\]

6. The Dirac operator \( D_{E^n} \) in the euclidean space \( E^n \), as an endomorphism between germs of Clifford module \( \text{Cliff}_{E^n}(T^*E^n) \) over \( E^n \),

\[
D_{E^n} : \Gamma(p, \text{Cliff}_{E^n}(T^*E^n)) \rightarrow \Gamma(p, \text{Cliff}_{E^n}(T^*E^n)),
\]

whose representation element is given by

\[
D_{x,\{dx\}} = \gamma_{\{dx\}}(dx^i) \partial_{x^i}.
\]

It is obvious that the following Proposition holds.
3.2 Proposition. For a point $p \in \mathbb{E}^n$ and a doublet $(\mathbf{Ψ}_{\{dx\}}, \mathbf{Ψ}_{\{dx\}})$ in $\Gamma(p, \text{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n)^*) \times \Gamma(p, \text{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n))$, the followings holds:

1. $\delta_{ij} \mathbf{Ψ}_{\{dx\}}(x) \gamma_{\{dx\}}(dx^i) \mathbf{Ψ}_{\{dx\}}(x) dx^j$ is an element of $\Gamma(p, \Omega^1(T^*\mathbb{E}^n))$.
2. $\mathbf{Ψ}_{\{dx\}}(x) \mathbf{Ψ}_{\{dx\}}(x)$ is an element of $\Gamma(p, \mathbb{C}_{\mathbb{E}^n})$.

Using the conventions (2-11), the following proposition is easily obtained.

3.3 Proposition. For a point $p$ in $\mathbb{E}^n$, $\{\mathbf{Ψ}^a_{\{dx\}} := \Xi_{\{x\}}^a\}_{a=1, \ldots, 2^{\lfloor n/2 \rfloor}}$ satisfy the Dirac equation,

$$ D_{x,\{dx\}} \mathbf{Ψ}_{\mathbb{E}^n}(x) = 0, $$

and are bases of $\Gamma(p, \text{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n))$ as a $\Gamma(p, \mathbb{C}_{\mathbb{E}^n})$-vector space. It means that they are also the bases of the fiber $\text{Cliff}_p(T^*\mathbb{E}^n)$ of $\text{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n)$ at $p$.

Further noting (2-17), we have the following important proposition.

3.4 Proposition. There exist $\mathbf{Ψ}^{(i)}_{\{dx\}} (i = 1, \ldots, n)$ and $\mathbf{Ψ}^{(i)}_{\{dx\}} := \phi_{\mu t}(\mathbf{Ψ}^{(i)}_{\{dx\}})$ satisfying

$$ \sum_{j,k=1}^n \mathbf{Ψ}^{(i)}_{\{dx\}} \gamma_{\{dx\}}(dx^j) \mathbf{Ψ}^{(i)}_{\{dx\}} dx^k = dx^i. $$

We note that since $\mathcal{W}_p := \Gamma(p, \text{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n)^*) \times \Gamma(p, \text{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n))$ has a point-wise-pairing $\cdot$ as mentioned in Proposition 3.2 (2), for a point $p \in \mathbb{E}^n$, $\mathcal{H}_p := \langle \mathcal{W}_p, \cdot, \phi_{\mu t} \rangle$ becomes the preHilbert space and due to Proposition 3.2 (1), $\mathcal{W}_p$ gives stalk $\Gamma(p, \text{SO}_{\mathbb{E}^n}(T^*\mathbb{E}^n))$.

Step 2: By setting the canonical parameterization in the tubular neighborhood $T_S$ of $S$, let the Dirac operator, the Clifford module and so on be expressed by the parameterization.

Let $\Gamma_{vc}(T_S, \text{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n))$ denote a set of sections of the Clifford module $\text{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n)$ whose support is in $T_S$. We go on to express its stalk at $p \in T_S$ by $\Gamma_{vc}(p, \text{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n))$. Let $\psi_{\{dx\}}$ denote a germ of $\Gamma(p, \text{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n))$ for a point $p \in T_S$. Recalling (3-3), $dx^i = E^i_{\mu} du^\mu$ and decomposing $E^i_{\mu} = G^i_{\mu} \Lambda^i_{\nu}$ as $dx^i = \Lambda^i_{\mu} d\xi^\mu$, we can find $e^\Omega \in \Gamma(p, \text{SO}_{\mathbb{E}^n}(T^*\mathbb{E}^n))$ satisfying the following relations, called gauge transformation,

$$ \psi_{\{dx\}}(u) = e^{-\Omega} \psi_{\{dx\}}(x), \quad \psi_{\{dx\}}(u) = \psi_{\{dx\}}(x) e^\Omega, $$

$$ e^{-\Omega} \psi_{\{dx\}}(dx^i)e^\Omega = \gamma_{\{dx\}}(\Lambda^i_{\mu} d\xi^\mu) $$

$$ = \Lambda^i_{\rho} G^i_{\mu} \gamma_{\{dx\}}(d\xi^\nu) $$

$$ = E^i_{\mu} \gamma_{\{dx\}}(du^\mu). \quad (3-6) $$
As $\Psi_{\{d\xi\}} \equiv \varphi_{pt}(\Psi_{\{d\xi\}})$, $\varphi_{pt}$ does not depend upon the orthonormal frame. In terms of these expressions, we have an assertion of Proposition of 3.2,

$$
\delta_{ij} \Psi_{\{dx\}}(x) \gamma_{\{dx\}}(dx^i) \Psi_{\{dx\}}(x) dx^j = g_{T_S, \mu\nu} \Psi_{\{d\xi\}}(u) \gamma_{\{d\xi\}}(du^\mu) \Psi_{\{d\xi\}}(u) du^\nu,
$$

$$
\Psi_{\{dx\}}(x) \Psi_{\{dx\}}(x) = \Psi_{\{d\xi\}}(u) \Psi_{\{d\xi\}}(u).
$$

Further the representation of the Dirac operator $D_{x,\{dx\}}$ is transformed to $D_{u,\{d\xi\}}$ by means of the gauge transformation,

$$
D_{u,\{d\xi\}} = e^{-\Omega} D_{x,\{dx\}} e^\Omega = \gamma_{\{d\xi\}}(du^\mu) e^{-\Omega} \partial_\mu e^\Omega
$$

$$
= \gamma_{\{d\xi\}}(du^\mu)(\partial_\mu + \partial_\mu \Omega).
$$

Then we have the following lemma:

**3.5 Lemma.** *We can regard $D_{u,\{d\xi\}}$ as a representation of a map $D_{T_S}$, *

$$
D_{T_S} : \Gamma_{vc}(p, \text{Cliff}_{E^n}(T^*E^n)) \to \Gamma_{vc}(p, \text{Cliff}_{E^n}(T^*E^n)),
$$

*for a point $p$ in $T_S$.*

Noting Lemma 3.5 and the fact that zero-section is in $\text{Cliff}_{E^n}(T^*E^n)$, it is not difficult to prove that the solution space of $D_{u,\{d\xi\}} \Psi = 0$ in $\Gamma(p, \mathbb{C}_{2^{[n/2]}}^{2^{[n/2]}})$ belongs to $\Gamma(p, \text{Cliff}_{E^n}(T^*E^n))$. Since $D_{u,\{d\xi\}}$ is a $2^{[n/2]} \times 2^{[n/2]}$-matrix type first order differential operator of rank $2^{[n/2]}$, the solution space gives a $2^{[n/2]}$-dimensional orthonormal frame in $\Gamma(p, \text{Cliff}_{E^n}(T^*E^n))$ as a $\Gamma(p, \mathbb{C}_{2^{[n/2]}})$-vector space. In fact, due to Proposition 3.2, we can find one as $e^{-\Omega} \Psi_{\{dx\}}$ by the gauge transformation.

Noting the Propositions 3.2-3.5 and (3-6)-(3-8), we have a key proposition:

**3.6 Proposition.**

1. *For a point $p \in T_S$, there exist $(\Psi_{\{d\xi\}}^{[a]}_{\{dx\}})_{a=1,\ldots,2^{[n/2]}}$ in $\Gamma(p, \mathbb{C}_{2^{[n/2]}}^{2^{[n/2]}})$ such that they are orthonormal frame in $\Gamma(p, \text{Cliff}_{E^n}(T^*E^n))$ and satisfy the Dirac equation

$$
D_{u,\{d\xi\}} \Psi_{\{d\xi\}}(u) = 0,
$$

i.e.,

$$
\Psi_{\{d\xi\}}^{[a]} \Psi_{\{d\xi\}}^{[b]} = \delta_{a,b},
$$

for $\Psi_{\{d\xi\}}^{[a]} := \varphi_{pt}(\Psi_{\{d\xi\}}^{[a]}).$

2. *For $\Psi_{\{dx\}}^{[a]} \in \Gamma(p, \text{Cliff}_{E^n}(T^*E^n))$ in (1) and $\Psi_{\{dx\}}^{[a]} \in \Gamma(p, \text{Cliff}_{E^n}(T^*E^n))$ in Proposition 3.3, there exists a germ $e\Omega \in \Gamma(p, \text{SPIN}_{E^n}(T^*E^n))$ as a $(2^{[n/2]} \times 2^{[n/2]})$-matrix satisfying

$$
\Psi_{\{dx\}}^{[a]} = e^{\Omega} \Psi_{\{dx\}}^{[a]}, \quad (a = 1, \ldots, 2^{[n/2]}).$

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(3) For $e^\Omega \in \Gamma(p, \text{SPIN}_n(T^*\mathbb{E}^n))$ in (2), by letting $\Psi^{(i)}_{\{\xi\}} := e^{-\Omega}\Psi^{(i)}_{\{\xi\}}$ for $(i = 1, \ldots, n)$ in Proposition 3.4, we have the relation,

$$g_{T_\Sigma, \mu \nu} \overline{\Psi}^{(i)}_{\{\xi\}}(u) \gamma_{\{\mu\}}(d\mu) \Psi^{(i)}_{\{\xi\}}(u) du^\nu = dx^i, \quad (i = 1, \ldots, n). \quad (3-11)$$

Proof. The proof of this proposition is done by the gauge transformation except well-definedness of (3-11). We should check the dependence of the orthonormal frame $\Psi^{[a]}_{\{\xi\}}$. Let us take another one $\Psi'_{\{\xi\}}^{[a]}$. By letting $e^{\Omega'}$ defined as $\Psi^{[a]}_{\{\xi\}} = e^{\Omega'}\Psi'_{\{\xi\}}^{[a]}$, we have $\Psi_{\{\xi\}}^{[a]} = e^{-\Omega}e^{\Omega'}\Psi_{\{\xi\}}'[a]$. When one rewrites (3-11) in terms of the frame $\Psi_{\{\xi\}}^{[a]}$, there appears $e^{-\Omega'}e^{-\Omega}\gamma_{\{\xi\}}(d\mu)e^{-\Omega}e^{\Omega'}$. However as both are solutions of the Dirac equation (3-9), the gauge transformation for $e^{-\Omega}e^{\Omega'}$ must leave the Dirac operator invariant. Hence the $\partial_\mu$ component gives that $e^{-\Omega'}e^{-\Omega}\gamma_{\{\xi\}}(d\mu)e^{-\Omega}e^{\Omega'} = \gamma_{\{\xi\}}(d\mu)$. (3-11) does not depend on the choice of the orthonormal frame. □

Here in order to investigate the domain of the new Dirac operator, we wish to apply the facts in Lemma 2.9 to the fiber bundles. Let $\text{CLIFF}^C_S(T^*S)$ (or $\text{Cliff}_S(T^*S)$) be restricted its base space to the submanifold $S$ by $\text{CLIFF}^C_{\mathbb{E}^n}(T^*\mathbb{E}^n)|_S$ (or $\text{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n)|_S$) as a vector bundle over $S$. As the fiber of $\text{SPIN}_p(T^*S)$ is a subgroup of $\text{SPIN}_p(T^*\mathbb{E}^n)$, we also express $\text{SPIN}_S(T^*\mathbb{E}^n) := \text{SPIN}_{\mathbb{E}^n}(T^*\mathbb{E}^n)|_S$. Here the reader should note the difference between $\Gamma(p, \text{Cliff}_S(T^*\mathbb{E}^n))$ and $\Gamma(p, \text{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n))$; the former is $\{\psi(s)\}$ and the later is $\{\psi(u)\}$.

3.7 Definition.

(1) Let us denote the inclusion as a set due to the correspondence of generators modeled by Lemma 2.9,

$$\tau_{S, \mathbb{E}^n} : \text{CLIFF}^C_S(T^*S) \rightarrow \text{CLIFF}^C_S(T^*\mathbb{E}^n), \quad (\tau_{S, \mathbb{E}^n} : \gamma_{S, \{\xi\}}(d\xi) \mapsto \gamma_{\{\xi\}}(d\xi)), \quad so \ that \ this \ map \ induces \ a \ ring \ homomorphism \ from \ \text{CLIFF}^{C_{even}}_S(T^*S) \rightarrow \text{CLIFF}^{C_{even}}_S(T^*\mathbb{E}^n). \ Here \ \gamma_{S, \{\xi\}} \ denotes \ the \ \gamma-matrix \ over \ S \ associated \ with \ T^*S \ and \ its \ orthonormal \ frame \ \{d\xi\}.$$

(2) Let us denote the inclusion as a group modeled by Lemma 2.9,

$$\iota_{S, \mathbb{E}^n} : \text{SPIN}_S(T^*S) \rightarrow \text{SPIN}_S(T^*\mathbb{E}^n).$$

Step 3: In order to construct a self-adjointization of the normal differential operator, let us define a pairing in $T_S$.

As $q$ is a natural coordinate of $T_S$ and we are now considering the affine geometry in category of differential geometry, let us introduce the Haar measure with respect to the
local affine transformation along the normal direction \( \{ q \} \), which is an invariant measure for the rotation and translations of \( q \), i.e.,
\[
e^{\frac{1}{2}x^*q^*} g_S^{-\frac{1}{2}} d^k s^0 d^{n-k} q = g_S^{-\frac{1}{2}} d^k s^0 d^{n-k} q.
\]
Here \( g_S^{-\frac{1}{2}} \) is not a mistype of \( g_T S^{-\frac{1}{2}} \). We call this measure normal affine invariance measure. Further we prepare a generic normal differential operator \( \partial_\perp := b^{\hat{\alpha}} \partial_{q^{\hat{\alpha}}} \) for generic real constant number \( b^{\hat{\alpha}} \)'s. As mentioned in Introduction, we apply the scheme in the submanifold quantum mechanics to this operator.

3.8 Definition. For a point \( p \in S \) and \( (\Psi_1,\{d\xi\}), \Psi_2,\{d\xi\}) \in \Gamma_v c(\pi_{-1}^{-1}(p), \text{Cliff}_{E^n}(T^*E^n)^*) \times \Gamma_v c(\pi_{-1}^{-1}(p), \text{Cliff}_{E^n}(T^*E^n)) \), we introduce \( L^2 \)-type pairing \( < \cdot | \cdot > \) in \( T_S \) as a fiber integral,
\[
< \Psi_1,\{d\xi\}| D_{\Psi_1,\{d\xi\}} \Psi_2,\{d\xi\} > = \int_{\pi_{-1}^{-1}(p)} g_{T_S}^{-\frac{1}{2}} d^{n-k} q \Psi_1,\{d\xi\}(u) D_{\Psi_1,\{d\xi\}} \Psi_2,\{d\xi\}(u) \in \Gamma(p, C_S),
\]
and a transformation \( \eta_{sa} \) so that its measure is the normal affine invariance measure,
\[
< \Psi_1,\{d\xi\}| D_{\Psi_1,\{d\xi\}} \Psi_2,\{d\xi\} > = (\Phi_1,\{d\xi\}| D_{\Phi_1,\{d\xi\}} \Phi_2,\{d\xi\})
= \int_{\pi_{-1}^{-1}(p)} g_{T_S}^{-\frac{1}{2}} d^{n-k} q \Phi_1,\{d\xi\}(u) D_{\Phi_1,\{d\xi\}} \Phi_2,\{d\xi\}(u) \in \Gamma(p, C_S),
\]
where
\[
\Phi_1,\{d\xi\} := \eta_{sa}(\Psi_1,\{d\xi\}) \equiv \rho s_q^{-\frac{1}{4}} \Psi_1,\{d\xi\}, \quad \Phi_2,\{d\xi\} := \eta_{sa}(\Psi_2,\{d\xi\}) \equiv \rho s_q^{-\frac{1}{4}} \Psi_2,\{d\xi\}, \quad D_{\Phi_1,\{d\xi\}} := \eta_{sa}(D_{\Psi_1,\{d\xi\}}) \equiv \rho s_q^{-\frac{1}{4}} D_{\Psi_1,\{d\xi\}} \rho s_q^{-\frac{1}{4}}.
\]
Further let \( \tilde{\varphi}_{pt} \) denote \( \eta_{sa} \varphi_{pt} \eta_{sa}^{-1} \).

We should note that \( \eta_{sa} \) can be defined for more general differential operators but for simplicity, we only define it for the Dirac operator. Further as the metric \( g_{T_S} \) is not singular, \( \eta_{sa} \) gives diffeomorphism. The following Lemma is naturally obtained.

3.9 Lemma. For points \( p \in S_q \) and \( p' \in S \), the triplets
\[
\mathcal{H}_p' := (\eta_{sa}(\Gamma_v c(p, \text{Cliff}_{E^n}(T^*E^n)^*)) \times \Gamma_v c(p, \text{Cliff}_{E^n}(T^*E^n)), \cdot, \tilde{\varphi}_{pt}),
\]
and
\[
\mathcal{H}' := (\eta_{sa}(\Gamma_v c(\pi_{-1}^{-1} p', \text{Cliff}_{E^n}(T^*E^n)^*)) \times \Gamma_v c(\pi_{-1}^{-1} p', \text{Cliff}_{E^n}(T^*E^n)), (|), \tilde{\varphi}_{pt}),
\]
become preHilbert spaces. In \( \mathcal{H}' \), The operator \( \partial_\perp \) is anti-hermite, i.e., \( \partial_\perp^* = -\partial_\perp \) for each \( b^{\hat{\alpha}} \).
Step 4: Decompose $\mathcal{D} T_s$ to normal part and tangential part.
Noting the orthonormal frame $d\xi$ in $T_S$ consisting of $(d\xi^1, \cdots, d\xi^k, dq^{k+1}, \cdots, dq^n)$, let us decompose $\mathcal{D} T_s$ to
\[
\mathcal{D}_{u,\{d\xi\}} = \mathcal{D}_{u,\{d\xi\}}^\parallel + \mathcal{D}_{u,\{d\xi\}}^\perp,
\]
where $\mathcal{D}_{u,\{d\xi\}}^\perp := \gamma(d\xi^\alpha)(dq^\alpha)\partial_{q^\alpha}$. Then the following lemma is not difficult to proved.

3.10 Lemma.

(1) $\mathcal{D}_{u,\{d\xi\}}^\parallel$ does not contain the vertical differential operator $\partial_{q^\alpha}$.

(2) $\mathcal{D}_{u,\{d\xi\}}^\perp * = -\mathcal{D}_{u,\{d\xi\}}^\perp$ in $\mathcal{H}'$.

(3) For a point $p \in T_S$, $\mathcal{D}_{u,\{d\xi\}}^\perp$ is a homomorphism of Clifford module,
\[
\mathcal{D}_{u,\{d\xi\}}^\perp : \eta_{sa}(\Gamma_{vc}(p, \text{Cliff}(T^* T_S))) \rightarrow \eta_{sa}(\Gamma_{vc}(p, \text{Cliff}(T^* T_S))).
\]

For a point $p \in T_S$, let us define a projection $\pi$,
\[
\pi : \eta_{sa}((\Gamma_{vc}(p, \text{Cliff}(T^* T_S)) \ast \Gamma_{vc}(p, \text{Cliff}(T^* T_S)))) \rightarrow (\text{Ker}_p(Ad(\partial_\perp)) \ast \text{Ker}_p(\partial_\perp)),
\]
where $\text{Ker}_p(P)$ denotes the set of germs of the kernel of $P$ at the point $p$ and $Ad(P)$ denotes the right-adjoint of $P$. Here we note that $\text{Ker}_p(\partial_\perp)(\subset \Gamma_{vc}(p, \text{Cliff}(T^* T_S))$ is given as the intersection of $\text{Ker}_p(\partial_\alpha)$ for every $\alpha = k + 1, \cdots, n$.

Then noting (2-6) and the fact that $\rho_{S_q} = 1$ at $S$, it is obvious that the following relations hold.

3.11 Lemma.

(1) $\varphi_{pt}|_S = \varphi_{pt}|_S$ and $\eta_{sa}(\Psi_{\{d\xi\}})|_S \equiv \Psi_{\{d\xi\}}|_S$ for $p \in S$ and $\Psi_{\{d\xi\}} \in \Gamma_{vc}(p, \text{Cliff}_{\mathbb{E}^n}(T^* \mathbb{E}^n))$.

(2) For a point $p \in T_S$, $\varphi_{pt}|_{\text{Ker}_p(\partial_\perp)}$ is a bijection:
\[
(\text{Ker}_p(\partial_\perp))^* := \varphi_{pt}(\text{Ker}_p(\partial_\perp)) \approx \text{Ker}_p(Ad(\partial_\perp)).
\]

(3) $\pi$ consists with the inner product, $\pi^* = \pi$,
\[
\pi \varphi_{pt} = \varphi_{pt} \pi, \quad \text{and} \quad \pi \varphi_{pt} = \varphi_{pt} \pi \text{ at } S.
\]

(4) For a point $p \in T_S$, the triplet $\mathcal{H}_p^\nu := (\text{Ker}_p(\partial_\perp))^* \times \text{Ker}_p(\partial_\perp), \varphi_{pt}$.成为 a preHilbert space. Let $\mathcal{H}_p(S \hookrightarrow \mathbb{E}^n) := \mathcal{H}_p^\nu$ for $p \in S$.

(5) Any element in $\text{Ker}_p(\partial_\perp)$ belongs to kernel of $\mathcal{D}_{u,\{d\xi\}}^\perp$, i.e.,
\[
\text{Ker}_p(\partial_\perp) \subset \text{Ker}_p(\mathcal{D}_{u,\{d\xi\}}^\perp).
\]
Step 5: Define submanifold Dirac operator \( \mathcal{D}_{S \hookrightarrow \mathbb{E}^n} \) at \( S \hookrightarrow \mathbb{E}^n \) by restricting its domain \( \text{Ker}_S(\partial_\perp) \).

Let us define \( \mathcal{D}_{S \hookrightarrow \mathbb{E}^n} \) by restricting the its domain as \( \bigcup_{p \in S} \text{Ker}_p(\partial_\perp) \), i.e.,

\[
\mathcal{D}_{S \hookrightarrow \mathbb{E}^n} := \mathbb{D}_T S \big|_{\bigcup_{p \in S} \text{Ker}_p(\partial_\perp)},
\]

with a local expression \( \mathcal{D}_{s,\{d\xi\}} := \mathbb{D}_{u,\{d\xi\}}|_{q=0,\partial q = \mathcal{D}}, \) whose explicit form is given by the following proposition.

3.12 Proposition. For abbreviation of \( \tau_{S,\mathbb{E}^n}(\gamma_{s,\partial}(ds^\alpha)) \) by its image \( \gamma_{\{d\xi\}}(ds^\alpha) \), the explicit form of \( \mathcal{D}_{s,\{d\xi\}} \) is given as

\[
\mathcal{D}_{s,\{d\xi\}} = \tau_{S,\mathbb{E}^n}(\mathcal{D}_{s,s,\{d\zeta\}}) + \gamma_{\{d\xi\}}(dq^\alpha)tr_{k \times k}(\gamma^\alpha_{\alpha\beta}),
\]

where \( \mathcal{D}_{s,s,\{d\zeta\}} \) is a proper (or intrinsic) Dirac operator of \( S \),

\[
\mathcal{D}_{s,s,\{d\zeta\}} := \gamma_{\{d\xi\}}(ds^\alpha)(\partial s^\alpha + \partial s^\alpha \Omega|_{q=0}),
\]

by fixing the coordinate \( s \) and the orthonormal frame \( \{d\zeta\} \).

Proof. [BJ, Mat1-10, MT] From (2-6), \( \rho_{s_q}^{1/4} = 1 + \frac{1}{2} tr_{k \times k}(\gamma^\alpha_{\alpha\beta})q^\beta + \mathcal{O}(q^\alpha q^\beta). \) Hence,

\[
\rho_{s_q}^{1/4} \partial_\alpha \rho_{s_q}^{-1/4} = \partial_\alpha - \frac{1}{2} tr_{k \times k}(\gamma^\alpha_{\alpha\beta}) + \mathcal{O}(q^\alpha),
\]

\[
\rho_{s_q}^{1/4} \partial_\alpha \rho_{s_q}^{-1/4} = \partial_\alpha + \mathcal{O}(q^\alpha).
\]

The second term in (3-15) is obtained. As \( \tau_{S,\mathbb{E}^n} \) induces the group inclusion \( \iota_{S,\mathbb{E}^n} \), we proved them. \( \square \)

Following proposition is important but is not difficult to be proved.

3.13 Proposition. For a point \( S \), the stalks of \( \Gamma(p,\text{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n)) \) and \( \Gamma_{vc}(p,\text{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n)) \) are bijective.

As we are dealing with a germ at a point \( p \) in \( S \) hereafter, let us neglect the difference between \( \Gamma(p,\text{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n)) \) and \( \Gamma_{vc}(p,\text{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n)) \).

Let \( p \) denote a point in \( S \). We are considering the kernel of \( \mathbb{D}_{u,\{d\xi\}} \) at \( p \) as a domain of the Dirac operator \( \mathcal{D}_{S \hookrightarrow \mathbb{E}^n} \). For given \( \psi_0(s) \in \Gamma(p,\text{Cliff}_S(T^*\mathbb{E}^n)) \), we assume that \( \Phi(u) \in \Gamma(p,\text{Cliff}_{\mathbb{E}^n}(T^*\mathbb{E}^n)) \) satisfies \( \mathbb{D}_{u,\{d\xi\}} \Phi = 0 \) with a boundary condition \( \Phi(s,0) = \psi_0(s) \) at \( S \). The existence of \( \Phi \) is obvious because we can find its solution as \( \Phi(u) = \psi_0(s) \) due to \( \partial q^\alpha \psi_0 \equiv 0 \). Therefore we regard that \( \Gamma(p,\text{Cliff}_S(T^*\mathbb{E}^n)) \) is a subset of \( \text{Ker}_p(\partial_\perp) \). Further we can evaluate \( \Phi(u) \in \text{Ker}_p(\partial_\perp) \) around a point \( p \) in \( S \) as

\[
\Phi(u) = \psi_0(s) + \sum \psi_\alpha(s)q^\alpha + \sum \psi_\alpha\beta(s)q^\alpha q^\beta + \cdots, \quad \psi_\alpha(s) \equiv 0
\]

where \( \psi_0(s) \in \Gamma(p,\text{Cliff}_S(T^*\mathbb{E}^n)) \), \( \psi_\beta(s), \psi_\alpha\beta(s) \in \Gamma(p,\mathbb{C}_{\mathbb{E}^n}) \) and so on. Accordingly we can identify \( \text{Ker}_p(\partial_\perp)|_{q=0} \) with \( \Gamma(p,\text{Cliff}_S(T^*\mathbb{E}^n)) \) for a point \( p \in S \).

Thus noting the fact that \( \mathcal{D}_{s,\{d\xi\}} \) is expressed by coordinate-free expressions and does not include \( \partial q^\alpha \), Definition 3.7, Lemma 2.9, 3.11 and Proposition 3.12 we have the following proposition:
3.14 Proposition. The Dirac operator \( \mathcal{D}_{S \to \mathbb{E}^n} \) is an endomorphism in germs of Clifford module \( \text{Cliff}_S(T^*\mathbb{E}^n) \) at a point \( p \) in \( S \),
\[
\mathcal{D}_{S \to \mathbb{E}^n} : \Gamma(p, \text{Cliff}_S(T^*\mathbb{E}^n)) \to \Gamma(p, \text{Cliff}_S(T^*\mathbb{E}^n)),
\]
whose representation element is given by \( \mathcal{D}_s,\{d\xi\} \).

**Step 6:** Consider solution of the submanifold Dirac equation \( \mathcal{D}_{S \to \mathbb{E}^n} \psi = 0 \) of \( S \to \mathbb{E}^n \).

Let us consider the solution of \( \mathcal{D}_{S \to \mathbb{E}^n} \psi = 0 \) at a point \( p \in S \). For non-vanishing \( \psi \in \Gamma(p, \mathbb{C}^{2^{[n/2]}}_{\mathbb{E}^n}) \) such that
\[
\mathcal{D}_s,\{d\xi\} \psi(s) = 0,
\]
\( \psi(s) \) can be regarded as an element of \( \Gamma(S, \text{Cliff}_S(T^*\mathbb{E}^n)) \). By solving a boundary problem for \( \Phi(u) \in \Gamma(S, \mathbb{C}^{2^{[n/2]}}_{\mathbb{E}^n}) \),
\[
\mathcal{D}_u,\{d\xi\} \Phi(u) = 0,
\]
with boundary condition
\[
\Phi|_S = \psi, \text{ at } S,
\]
we have \( \Phi(u) \) satisfying \( \mathcal{D}_{u,\{d\xi\}} \Phi(u) = 0 \) and \( \mathcal{D}_{u,\{d\xi\}} \eta_{sa}^{-1}(\Phi(u)) = 0 \) at \( p \in S \).

Noting that \( \mathcal{D}_{u,\{d\xi\}}|_{q=0} = \mathcal{D}_{s,\{d\xi\}} + \mathcal{D}_u,\{d\xi\} \), and \( \mathcal{D}_s,\{d\xi\} \) does not include the parameter \( q \), we can apply the separation of variables to this system. Thus it is expected that there exists an orthonormal frame \( (\eta_{sa}^{-1} \Phi[a]_{\{d\xi\}})_{a=1,\ldots,2^{[n/2]}} \in \Gamma(p, \text{Cliff}_S(T^*\mathbb{E}^n)) \) such that each \( \Phi[a]_{\{d\xi\}} \) belongs to \( \text{Ker}_p(\partial_\perp) \) and satisfies the Dirac equation (3-20); each \( (\eta_{sa}^{-1} \Phi[a]_{\{d\xi\}}) \) is a solution of the Dirac equation (3-9). Noting Lemma 3.11 (1), we regard \( (\Phi[a]_{\{d\xi\}}|_{q=0})_{a=1,\ldots,2^{[n/2]}} \) as the orthonormal frame of \( \Gamma(p, \text{Cliff}_S(T^*\mathbb{E}^n)) \) and solutions of the submanifold Dirac equation (3-19).

Inversely, as the Dirac operator \( \mathcal{D}_{s,\{d\xi\}} \) is also a \( 2^{[n/2]} \times 2^{[n/2]} \)-matrix type first order differential operator, let us assume that we find an orthonormal frame of \( \Gamma(p, \text{Cliff}_S(\mathbb{E}^n)) \) belonging to the solution space of the submanifold Dirac equation (3-19). Let it be denoted by \( \psi[a]_{\{d\xi\}} (a = 1, \ldots, 2^{[n/2]}), \) i.e., \( \varphi_{pt}(\psi[a]_{\{d\xi\}}) \psi[b]_{\{d\xi\}} = \delta_{a,b} \). For each \( \psi[a]_{\{d\xi\}}(s) \), we find an element \( \Psi[a]_{\{d\xi\}}(u) \) in the solution space of \( \mathcal{D}_{u,\{d\xi\}} \eta_{sa}(\Psi(u)) = 0 \) or \( \mathcal{D}_{u,\{d\xi\}} \Psi(u) = 0 \) with the boundary condition \( \Psi[a]_{\{d\xi\}}(u)|_{q=0} = \psi[a]_{\{d\xi\}}(s) \) at \( p \).

Then due to Proposition 3.6 (2), we have an element of \( e^\Omega \in \Gamma(p, \text{SPIN}_{\mathbb{E}^n}(\mathbb{E}^n)) \) such that,
\[
\Psi[a]_{\{d\xi\}} = e^\Omega \tilde{\Psi[a]_{\{d\xi\}}}, \quad (a = 1, \ldots, 2^{[n/2]}).
\]
Using the element, we define \( \Psi^{(i)}_{\{d\xi\}} := e^{-\Omega} \Psi^{(i)}_{\{d\xi\}}, (i = 1, \ldots, n) \). Then Proposition 3.6 (3) gives the relation,
\[
g_{TS,\mu\nu} \Psi^{(i)}_{\{d\xi\}}(d\omega^\mu) \Psi^{(i)}_{\{d\xi\}} d\omega^\nu = dx^i, \quad (i = 1, \ldots, n).
\]
Therefore by defining $\psi_{\{d\xi\}}^{(i)}(s) := \Psi_{\{d\xi\}}^{(i)}(u)|_{q=0}$ and extracting the $ds^\alpha$ component, we have

$$g_{S,\alpha\beta}\psi_{\{d\xi\}}^{(i)}(ds^\alpha)e^{\Omega_i}\psi_{\{d\xi\}}^{(i)} = \frac{\partial x^i}{\partial s^\alpha}, \quad (i = 1, \ldots, n, \alpha = 1, \ldots, k). \quad (3-23)$$

This is a representation of the tangential space of $S$ or the generalized Weierstrass relation. Here we note that $\psi_{\{d\xi\}}^{(i)}$ is also a solution of the submanifold Dirac equation (3-19).

As mentioned above, our theory is locally constructed and obtained objects do not depend on the local parameterization. Accordingly we can easily extended it to that for a general smooth spin $k$-submanifold in immersed $\mathbb{E}^n$. In other words, we regard the assumptions in 2.2 as those for a local chart of the submanifold $S$. Since $\mathbb{E}^n$ is a spin manifold, we can define Cliff$_{\mathbb{E}^n}(T^*\mathbb{E}^n)$, SPIN$_{\mathbb{E}^n}(T^*\mathbb{E}^n)$ and their global sections. By restricting it on $S$, we have Cliff$_S(T^*\mathbb{E}^n)$ and SPIN$_S(T^*\mathbb{E}^n)$ structure over $S$. On the hand, $S$ has its own Cliff$_S(T^*S)$ and SPIN$_S(T^*S)$ structure. Due to Proposition 3.12, $S$ has the submanifold Dirac operator $\mathcal{D}_{S \hookrightarrow \mathbb{E}^n}$ with the local representation.

**3.15 Theorem.** A smooth spin $k$-submanifold $S$ immersed in $\mathbb{E}^n$ has a Clifford module Cliff$_S(T^*\mathbb{E}^n)$ by locally constructing in terms of the solution space of the submanifold Dirac equation $\mathcal{D}_{S \hookrightarrow \mathbb{E}^n}\psi = 0$ at each chart and then the solutions give data of tangential vector in $T\mathbb{E}^n$ at each point as follows:

Let $p \in S$ be expressed by an affine coordinate $(x^i)$. In a set $\{\psi\}$ of non-vanishing germs of $\Gamma(p, \mathbb{C}^{2^{[n/2]}})$ satisfying $\mathcal{D}_{S \hookrightarrow \mathbb{E}^n}\psi = 0$, there exists an orthonormal frame $\{\psi_{\{d\xi\}}^{[a]}\}_{a=1,\ldots,2^{[n/2]}}$ (⊂ $\{\psi\}$) of $\Gamma(p, \text{Cliff}_S(T^*\mathbb{E}^n))$ as a $\Gamma(p, \mathbb{C}_{\mathbb{E}^n})$-vector space. For these bases, there exists a germ $e^{\Omega} \in \Gamma(p, \text{Spin}_{\mathbb{E}^n}(T^*\mathbb{E}^n))$ such that they are transformed to $\{\psi_{\{d\xi\}}^{[a]}|_{S}\}_{a=1,\ldots,2^{[n/2]}}$ by $\psi_{\{d\xi\}}^{[a]} = e^{-\Omega}\psi_{\{d\xi\}}^{[a]}|_{S}$, $(a = 1, \ldots, 2^{[n/2]}).$ We define $\psi^{(i)} := e^{-\Omega}\psi_{\{d\xi\}}^{(i)}|_{S}$ and $\bar{\psi}^{(i)} := \overline{\psi_{\{d\xi\}}^{(i)}e^{-\Omega}}|_{S}$ $(i = 1, \ldots, n)$. Then the following relation holds:

$$g_{S,\alpha\beta}\bar{\psi}^{(i)}[\tau_{S,\mathbb{E}^n}(\gamma_S\{d\xi\})(ds^\beta)]\psi^{(i)} = \partial_{s^\alpha}x^i, \quad (i = 1, \ldots, n, s = 1, \ldots, k). \quad (3-24)$$

This is the generalized Weierstrass relation for a spin submanifold immersed in $\mathbb{E}^n$. Since the equation is given by the local argument, the proof of the theorem does not require a global argument at all.

We note that even though we fix the coordinate system, for a point $p$ in $S$, $\left(\frac{\partial x^i}{\partial s^\alpha}\right)_{\alpha=1,\ldots,k;i=1,\ldots,n}$ gives a data of embedding $T_pS \cong \mathbb{R}^k$ in $T_p\mathbb{E}^n \cong \mathbb{R}^n$ up to action of $\text{GL}(\mathbb{R}^k)$ and $\text{GL}(\mathbb{R}^{n-k})$. In other words, the submanifold Dirac operator $\mathcal{D}_{S \hookrightarrow \mathbb{E}^n}$ exhibits data of Grassmannian bundle over $S$. 

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As we have assumed the canonical parameterization on $T_S$ in the definition of the submanifold Dirac operator, it should be extended to one without the assumption. Then we can argue the global behavior of the submanifold Dirac operator $\mathcal{D}_{S\rightarrow \mathbb{E}^n}$ and its zero modes over $S$. It is expected that the zero modes bring us a global data of the orthonormal frame of $\text{Cliff}_S(T^*\mathbb{E}^n)$, e.g., topological defect and so on. Of course, for the low codimension cases, we can find the global form of $\mathcal{D}_{S\rightarrow \mathbb{E}^n}$ easily and then we can also argue it. Inversely, if a $k$-spin manifold $M$ and a differential operator over $M$ satisfy appropriate conditions, we also obtain the data of immersion of $M$ in $\mathbb{E}^n$ following the philosophy of the Weierstrass relation.

**3.16 Remark.** The generalized Weierstrass relation for a conformal surface in $\mathbb{E}^3$ was discovered by Kenmotsu [Ke] as a generalization of the Weierstrass relation for a minimal surface in $\mathbb{E}^3$. The Dirac type relation was found by Konopelchenko in 1995 [Ko1] and I showed that the submanifold Dirac equation is identified with the relation [Mat8]. I also computed the submanifold Dirac operators for the generalized Weierstrass relation for a conformal surface in $\mathbb{E}^4$, which was also discovered by Konopelchenko [KL, Ko1-2] and Pinkall and Pedit [PP]. Recently there are so many studies on the relations between submanifolds and Clifford bundles [Bo, Ko1-2, KL, KT, Fr, Tai1-2, Tr]. However no article mentioned the transformation $\eta_{bs}$ as long as I know. For this decade, I have applied the transformation to the relation between the Dirac operators and submanifolds, especially curves immersed in $\mathbb{E}^n$. For the curve case, the submanifold Dirac equation is identified with Frenet-Serret relation [Mat1-7]. However I could not reach the direct connection between a more general submanifold and zero modes of the Dirac operator because the net meaning of the transformation remained as a question in my mind. After the conference, I noticed the essentials of the submanifold quantum mechanics as mentioned in the Introduction, which naturally leads me to the further generalized Weierstrass relation or Theorem 3.15.

Though we did not mention in this article, it is known, from the physical point of view, that the obtained Dirac operator has the following properties:

1. The index of the Dirac operator is related to the topological index for the case of curves [Mat2, 5].
2. The operator determinants are associated with the energy functionals, such as Euler-Bernoulli functional and Willmore functional for the cases of space curves and of immersed conformal surfaces respectively [Mat6, 10].
3. The deformations preserving all eigenvalues of the submanifold Dirac operators become the soliton equations, such as MKdV equation [Mat2, 6, 7, MT], complex MKdV equation, Nonlinear Schrödinger equation [Mat1, 3], modified Novikov-Veselov equations [Ko1, Ko2, KL, KT, Tai1] depending on submanifolds.

As we showed in this article, we modeled the theory of Thom class [BT, BGV] on construction of this theory of the submanifold Dirac operator. Both stand upon vary similar
geometrical situation. As the Dirac operators are generally related to some characteristic class of fiber bundle, it is expected that the submanifold Dirac operator might be related to the Thom class and/or generalization of Riemann-Roch [PP].

Further our scheme does not need global properties and thus might be extended to a subgroup manifold immersed in more general group manifold with a Casimir operator and a Haar measure as a continuous group version of induced representation for finite group.

§4. Dirac Operator on a conformal surface in $E^4$

As an example, we will consider the case of a conformal surface immersed in $E^4$. In [Mat10], we gave an explicit local form (4-1) of the submanifold Dirac operator in this case and a conjecture that the submanifold Dirac operator represents the surface. As the conjecture was actually proved by Konopelchenko [KL, Ko2] and Pedit and Pinkall [PP], we will give another proof of the conjecture by means of the submanifold Dirac system method. This means that my conjecture in [Mat10] was based upon physical assurance.

First we will give the properties of the Dirac operators in a conformal surface $S$ in $E^n$, we can set the metric given by

$$g_{S\alpha\beta} = \rho^{-1} \delta_{\alpha\beta},$$

and the orthonormal frame $\{d\zeta\}$ given by $d\zeta^\alpha := \rho^{-1/2} ds^\alpha$. Let complex parameterization $dz := ds^1 \pm \sqrt{-1} ds^2$. We introduce another transformation $\eta_{\text{conf}}$ as follows. For a point $p \in S$ and $\psi_{\{d\xi\}} \in \Gamma(p, \text{Cliff}_S(T^*S^{\mathbb{E}^n}))$, let $\varphi_{\{d\xi\}} := \eta_{\text{conf}}(\psi_{\{d\xi\}}) \equiv \rho^{1/2} \psi_{\{d\xi\}}$ and $\bar{\varphi}_{\{d\xi\}} := \eta_{\text{conf}}(\bar{\psi}_{\{d\xi\}}) \equiv \varphi_{\text{pt}}(\psi_{\{d\xi\}})$. Then we have the following properties.

4.1 Lemma.

1. The proper Dirac operator of $S$ is given by

$$D_{S,\{d\zeta\}} = \rho^{-1} \sigma^\alpha \partial_\alpha \rho^{1/2},$$

by letting $\gamma_{S,\{d\zeta\}}(d\zeta^\alpha) = \sigma^\alpha$.

2. For a doublet $(\varphi_{\{d\xi\}}, \bar{\varphi}_{\{d\xi\}})$ in $\eta_{\text{conf}} \mathcal{H}_p(S \hookrightarrow \mathbb{E}^n)$, $\varphi_{\{d\xi\}} \tau_{S,\mathbb{E}^n} [\gamma_{\{d\xi\}}(d\xi^\alpha)] \varphi_{\{d\xi\}} ds^\alpha$ is invariant form for choice of local parameterization of $S$.

Proof. (1) is obvious [P, Mat8-9]. By setting $\gamma_{\{d\xi\}}(ds^\alpha) = \rho^{-1/2} \gamma_{\{d\xi\}}(d\xi^\alpha)$, the relation in Proposition 3.2 (1) becomes

$$g_{S\alpha\beta} \psi_{\{d\xi\}} \tau_{S,\mathbb{E}^n} [\gamma_{\{d\xi\}}(d\xi^\alpha)] \psi_{\{d\xi\}} ds^\alpha = \bar{\varphi}_{\{d\xi\}} \tau_{S,\mathbb{E}^n} [\gamma_{\{d\xi\}}(d\xi^\alpha)] \varphi_{\{d\xi\}} ds^\alpha,$$

and thus (2) is proved □
4.2 Proposition.

(1) For the case of a conformal surface in $\mathbb{E}^4$, the submanifold Dirac operator is given by

$$\mathcal{D}_{s,(d\xi)} = 2 \left( \begin{array}{cc} |p_c| & \partial \\ \frac{\partial}{\partial s} & -p_c \end{array} \right),$$

(4-1)

where $\partial := (\partial s^1 - \sqrt{-1}\partial s^2)/2$, $\bar{\partial} := (\partial s^1 + \sqrt{-1}\partial s^2)/2$ and $p_c$ is

$$p_c := -\frac{1}{2} \rho^{1/2} \text{tr}_{2 \times 2} (\gamma_{3\beta}^\alpha + \sqrt{-1}\gamma_{4\beta}^\alpha).$$

(2) Let $\mathbb{E}^4 \approx \mathbb{C} \times \mathbb{C} \in (Z^1, Z^2) \equiv (x^1 + \sqrt{-1}x^2, x^3 + \sqrt{-1}x^4)$. For the affine coordinate $(Z^1, Z^2)$ of the surface, the relations,

$$dZ^1 = fmdz - gndz, \quad dZ^2 = fmdz + gmdz, \quad d\bar{Z}^1 = \bar{d}Z^1, \quad d\bar{Z}^2 = \bar{d}Z^2,$$

hold if $\varphi_1 := \begin{pmatrix} f \\ g \\ 0 \\ 0 \end{pmatrix}$, $\varphi_2 := \begin{pmatrix} 0 \\ 0 \\ m \\ n \end{pmatrix}$ are solutions of

$$\mathcal{D}_{s,(d\xi)} \varphi_a = 0,$$

and $(|f|^2 + |g|^2)(|m|^2 + |n|^2) = \rho^{1/2}$.

Direct computation leads the next lemma.

4.3 Lemma. For the four dimensional euclidean space $\mathbb{E}^4$, we can fix the representation of the gamma matrices of $\{dx^i\}$ system as $\gamma_{\{dx\}}(dx^i) = \sigma^i \otimes \sigma^i$ for $i = 1, 2, 3$ and $\gamma_{\{dx\}}(dx^4) = \sigma^2 \otimes 1$. By letting

$$\Psi_1 := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \Psi_2 := \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad \Psi_3 := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \Psi_4 := \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix},$$

$\Psi_1 = (0, 1, 0, 1), \quad \Psi_2 = (1, 0, 1, 0), \quad \Psi_3 = (0, -1, 1, 0), \quad \text{and} \quad \Psi_4 = (1, 0, 0, -1)$, we have the following relations:

$$\sum_i \Psi_1 \gamma_{\{dx\}}(dx^i) \Psi_1 dx^i = 2dZ_1, \quad \sum_i \Psi_2 \gamma_{\{dx\}}(dx^i) \Psi_2 dx^i = 2d\bar{Z}_1,$$

$$\sum_i \Psi_3 \gamma_{\{dx\}}(dx^i) \Psi_3 dx^i = 2dZ_2, \quad \sum_i \Psi_4 \gamma_{\{dx\}}(dx^i) \Psi_4 dx^i = 2d\bar{Z}_2.$$
Proof of Proposition 4.2. (1) is proved by Theorem 3.15. We consider (2). For \( \varphi_a \)'s, their independent partner solutions are given by

\[
\varphi_3 := \begin{pmatrix} -\vec{g} \\ f \\ 0 \\ 0 \end{pmatrix}, \quad \varphi_4 := \begin{pmatrix} 0 \\ 0 \\ -\pi \\ m \end{pmatrix}.
\]

Since we have fixed \( \gamma_{S,\{d\xi\}}(d\zeta^\beta) = \sigma^\beta \), the convention in 2.8 gives

\[
\tau_{S,\bar{\zeta}^2}(\gamma_{S,\{d\xi\}}(d\zeta^\beta)) = \sigma^1 \otimes \sigma^\beta.
\]

We define \( \bar{\varphi}_1 := \varphi_1 + \varphi_2, \bar{\varphi}_2 := \varphi_3 + \varphi_4, \bar{\varphi}_3 := \varphi_1 + \varphi_4, \text{ and } \bar{\varphi}_4 := \varphi_3 + \varphi_2 \). Let us assume that \( \bar{\varphi}_a = \rho^{1/2}e^\Omega \Psi_a|_{q^a = 0} \) \((a = 1, 2, 3, 4)\) of Lemma 4.3. Then we find the spin matrix as

\[
\rho^{1/2}e^\Omega = \begin{pmatrix} f & -\vec{g} & 0 & 0 \\ g & \vec{f} & 0 & 0 \\ 0 & 0 & m & -\pi \\ 0 & 0 & n & m \end{pmatrix}.
\]

We have these dual bases, \( \bar{\varphi}_a = \bar{\Psi}_a e^{-\Omega}|_{q^a = 0} \), and obtain the relation,

\[
2d\bar{Z}_1 = \bar{\varphi}_1 \sigma_1 \otimes \sigma^\alpha \bar{\varphi}_1 ds^\alpha, \quad 2d\bar{Z}_1 = \bar{\varphi}_2 \sigma_1 \otimes \sigma^\alpha \bar{\varphi}_2 ds^\alpha,
\]

\[
2d\bar{Z}_2 = \bar{\varphi}_3 \sigma_1 \otimes \sigma^\alpha \bar{\varphi}_3 ds^\alpha, \quad 2d\bar{Z}_2 = \bar{\varphi}_4 \sigma_1 \otimes \sigma^\alpha \bar{\varphi}_4 ds^\alpha.
\]

Explicit representation of them proves (2). \( \Box \)

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