IRREDUCIBLE REPRESENTATIONS OF DEFORMED OSCILLATOR ALGEBRA AND \( q \)-SPECIAL FUNCTIONS

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Abstract

Different generators of a deformed oscillator algebra give rise to one-parameter families of \( q \)-exponential functions and \( q \)-Hermite polynomials related by generating functions. Connections of the Stieltjes and Hamburger classical moment problems with the corresponding resolution of unity for the \( q \)-coherent states and with 'coordinate' operators - Jacobi matrices, are also pointed out.

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1. Suggestions to change the canonical commutation relations for improving some properties of quantum field theory have already appeared in the works of the founders of quantum mechanics (see [1, 2] and Refs therein).

In the early fifties, E. P. Wigner [1] posed the following question: “what kind of functions, \( f(H) \), appearing in the right-hand side of the commutator \([p, x] = \imath \hbar f(H)\) are compatible with the given expression of the Hamiltonian, \( H \), and the standard equations of motions”? He found that, in the case of the harmonic oscillator Hamiltonian the usual expression \( f(H) = 1 \) was not unique. These investigations were continued in [3] where, for a special case, now called “\( q \)-root of unity”, it was found that, together with the fermionic (two-dimensional) and bosonic (infinite dimensional) cases, there exist cases with dimensions equal to \( m \) which are related with the parastatistics not connected to the Green’s Ansatz.

Twenty years later, the generalization of the Veneziano amplitude, by substitution of the \( q \)-\( \Gamma \)-function instead of the standard \( \Gamma \)-function [4], gave rise in the operator formalism to the \( q \)-oscillator commutation relation [5],

\[
aa - qa^\dagger = 1, \quad 0 < q < 1. \tag{1}
\]

A rejuvenation of the problem of oscillator deformations in the end of the 80’s was associated with the growing interest in quantum groups. Such popularity appeared after the works [5, 7], in which a deformation based on the relation \( AA^\dagger - q^{1/2}A^\dagger A = q^{-N/2} \), was considered in connection with Schwinger’s realization of the quantum algebra \( su_q(2) \) [8] (for a \( q \)-boson description of the \( su_q(1, 1) \), see [9]). Further interest in the \( q \)-oscillator problem was stimulated by research in the multimode case [10], supersymmetries [11], and relations to the \( q \)-analysis [12] (for more details and Refs, see [13]).

Different generators of the deformed oscillator algebra give rise to one-parameter families of \( q \)-exponential functions [13, 14, 15, 16], \( q \)-Hermite polynomials and other \( q \)-special functions [17, 18, 19, 20]. Consideration of the resolution of unity (completeness of the system of \( q \)-coherent states) for the \( q \)-Bargmann - Fock realization of irreducible representations of deformed oscillator algebra, and the spectral properties [18] of the ‘coordinate’ operator (which is represented as a Jacobi matrix), pointed out deep connections with the classical Stieltjes and Hamburger moment problems [21].

Recently, the \( q \)-oscillator was applied to the study of the phonon spectrum in \(^4\text{He} \) [22], a specific case of the one-dimensional Schrödinger equation [23], different quantum mechanical models [24], and the trapped atom problem.

2. The deformed oscillator algebra, \( \mathcal{A}(q) \), is generated by three elements \( a, a^\dagger, N \) with defining relations

\[
aa - qa^\dagger = 1, \quad [N, a] = -a, \quad [N, a^\dagger] = a^\dagger. \tag{2}
\]

The generator \( N \) is considered as an independent element, and we restrict ourselves to the
case of positive real $q \in (0, \infty)$. The algebra $\mathcal{A}(q)$ has a central element $\zeta$,

$$
\zeta = q^{-N}([N; q] - a^\dagger a); \quad [N; q] := (1 - q^N)/(1 - q)
$$

(for a more general three-generator algebra $\mathcal{A}(q)$ with $[a, a^\dagger] = F(N)$, see [26, 27]).

In the original papers, the irreducible representation of $\mathcal{A}(q)$ with the vacuum state $|0\rangle$ ($a|0\rangle = 0$) was considered. The oscillator-type representation space $\mathcal{H}_0$, in the basis of eigenvectors of the operator $N$, is

$$
\mathcal{H}_0 = \{ |n\rangle; \quad n = 0, 1, 2, \ldots; \quad a|0\rangle = 0, \quad |n\rangle = ([n; q]!)^{-1/2}(a^\dagger)^n|0\rangle \}.
$$

Due to the existence of a non-trivial central element, $\zeta$, in addition to $\mathcal{H}_0$, the algebra $\mathcal{A}(q)$ has a set of inequivalent irreducible representations ($0 < q < 1$) in the spaces $\mathcal{H}_\gamma$ ($\gamma \geq \gamma_c = (1 - q)^{-1}$) parameterised by the value of the central element $\zeta = -\gamma$ [25], with the spectrum of $N$, the set of all integers $\mathbb{Z}$. The matrix $a^\dagger$ in the number operator basis is

$$
(a^\dagger)_{nk} = c_n\delta_{n,k+1}, \quad a^\dagger|n - 1\rangle = c_n|n\rangle, \quad (c_n)^2 = \gamma q^n + [n; q].
$$

These irreducible representations are connected with different symplectic leaves of Poisson brackets in $R^3$, which correspond to the quasiclassical limit of the $q$-oscillator commutation relation [4].

Considering $\mathcal{A}(q)$ as an associative algebra, any invertible transformation of the generators is admissible; in particular, there are some natural sets of the generators:

$$
AA^\dagger - q^{1/2}A^\dagger A = q^{-N/2}, \quad [N, A] = -A, \quad [N, A^\dagger] = A^\dagger,
$$

related to the quantum algebra $sl_q(2)$ via the Schwinger realization [6, 7], and the following set related to the $sl_q(2)$ algebra by a contraction procedure with fixed $q$ [25],

$$
[a, a^\dagger] = q^{-N}, \quad [N, a] = -a, \quad [N, a^\dagger] = a^\dagger.
$$

The equivalence of these generators is given by the equalities $a = q^{N/2}\alpha = q^{N/4}A$ [8, 13], with an obvious one-parameter generalization, namely,

$$
a(\lambda) = q^{-\frac{1}{4}\lambda N}a, \quad a^\dagger(\lambda) = a^\dagger q^{-\frac{1}{4}\lambda N}.
$$

This leads to the commutation relations (still one degree of freedom)

$$
a(\lambda)a^\dagger(\lambda) - q^{1-\lambda}a^\dagger(\lambda)a(\lambda) = q^{-\lambda N}.
$$

Sometimes, these generators and relation [8] are called a two-parameter deformed oscillator [28]: $p \leftrightarrow q^{1-\lambda}$ and $r \leftrightarrow q^{-\lambda}$, $aa^\dagger - pa^\dagger a = rN$. However, they define the same algebra $\mathcal{A}(q)$ in the case of general $q = p/r$.  

3
One more formal parameter $\nu \in R$ can be added by a shift $N \rightarrow N + \nu$. The corresponding set of $\mathcal{A}(q)$ generators is denoted by $W_{p,\nu}(q)$. As a consequence of (9), namely,

$$a(\lambda)(a(\lambda))^m = (pa(\lambda))^m a(\lambda) + (pa(\lambda))^{m-1} r^N [m; \frac{r}{p}],$$

the normalized basis vectors of $\mathcal{H}_0$ in terms of $a(\lambda)$ are given by

$$|n\rangle = ([n; q, \lambda] |z\rangle)^{-1/2} (a(\lambda))^n |0\rangle$$

with the factorials defined as

$$[n; q, \lambda]! = \prod_{k=1}^{n} [k; q, \lambda], \quad [m; q, \lambda] = q^{\lambda(1-m)} [m; q].$$

3. In the theory of Lie groups and quantum mechanics, special functions appear as particular matrix elements (overlap coefficients) of appropriate operators in corresponding representations (realizations): examples are exponential functions, as coherent states in the Bargmann-Fock representation of $\mathcal{H}_0$, of the usual boson oscillator $[b, b^\dagger] = 1$,

$$\exp(\sqrt{z}) = \langle w|z\rangle, \quad |z\rangle = e^{z b^\dagger}|0\rangle, \quad b|z\rangle = z|z\rangle,$$

and Hermite polynomials, as eigenvectors of the operator $N$, in the coordinate representation,

$$H_n(x) \sim \langle n|x\rangle, \quad (b + b^\dagger)|x\rangle = 2x|x\rangle.$$

The simple action of the annihilation and creation operators in the coherent state representation leads to the generating function of the Hermite polynomials

$$\omega(z; x) = \langle z|x\rangle = \exp(2xz - \frac{1}{2}z^2).$$

The coherent states of the annihilation operator $a$ of the $q$-oscillator in $\mathcal{H}_0$ were introduced in [5]:

$$a|z\rangle = z|z\rangle, \quad |z\rangle = e_q(z a^\dagger)|0\rangle,$$

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n; q]!}.$$

In the $q$-Bargman-Fock space, related to the $q$-coherent states, the creation operator $a^\dagger$ is the operator of multiplication by $\bar{z}$,

$$|f\rangle \rightarrow f(z) = \langle z|f\rangle,$$

$$\langle z|a^\dagger |f\rangle = (a|z\rangle)^\dagger |f\rangle = \bar{z} \langle z|f\rangle = \bar{z} f(z),$$

4
the annihilation operator, $a$, is a $q$-difference operator $D_q$, and the $q$-coherent state is given by the $q$-exponent (15),

$$af(z) = D_q f(z) = f(z) - f(qz) = \frac{f(z) - f(qz)}{z(1 - q)}, \quad \langle z|\zeta \rangle = e_q(\bar{z}\zeta).$$

The $q$-exponent above (see (13)) is well known in $q$-analysis [14]. The scalar product in the $q$-Bargman-Fock realization of $\mathcal{H}_0$ is given by [3]

$$\langle \phi | f \rangle = \frac{1}{2\pi} \int \bar{\phi}(z) f(z) d\mu(z),$$

where the measure is defined by the resolution of unity,

$$\frac{1}{2\pi} \int_0^{1/(1-q)} \int_0^{2\pi} |z\rangle\langle z| (e_q(q|z|^2))^{-1} d\phi \, dq |z|^2 = \sum_{n=0}^{\infty} |n\rangle\langle n| = I,$$

this completeness relation was proved in [3] using the product representation of the $q$-exponent (15), namely,

$$e_q(x) = \left( \prod_{k=0}^{\infty} (1 - (1 - q)q^k x) \right)^{-1} = \frac{1}{((1 - q)x; q)_\infty},$$

and the Jackson $q$-integral, $\int_0^b f(x) \, dq \, x = (1 - q) \sum_{m=0}^{\infty} q^m b f(q^m b)$ [14].

Following the same pattern, other choices for the generators of the $q$-oscillator algebra $\mathcal{A}(q)$ give rise to different $q$-exponential functions [9, 13],

$$\alpha|z\rangle_\alpha = z|z\rangle_\alpha, \quad |z\rangle_\alpha = e_{1/q}(z\alpha^\dagger)|0\rangle,$$

$$A|z\rangle_A = z|z\rangle_A, \quad |z\rangle_A = E_q(zA^\dagger)|0\rangle,$$

where the symmetric $q$-exponent is

$$E_q(x) = \sum_{m=0}^{\infty} \frac{x^m}{[m]_q!}, \quad [m]_q = \frac{q^m - q^{-m}}{q - q^{-1}},$$

The one-parameter $q$-exponential function $\exp(z; q, \lambda)$ is connected with the annihilation operator $a(\lambda)$ (8), (9)

$$a(\lambda)|z; \lambda\rangle = z|z; \lambda\rangle, \quad |z; \lambda\rangle = \exp(za(\lambda)^\dagger; q, \lambda)|0\rangle;$$

$$\exp(z; q, \lambda) = \sum_{m=0}^{\infty} q^{\lambda n (n-1)/2} \frac{x^n}{[n, q]!}. $
The properties of these \( q \)-exponents \( \exp(z; q, \lambda) \) are quite different \([15, 16]\); for example, for \( 0 < q < 1 \) and \( \lambda < 0 \), the \( q \)-exponent \( \exp(z; q, \lambda) \) \([25]\) has zero radius of convergence. It would be interesting to relate different \( q \)-exponential functions and their properties with particular physical systems.

The corresponding resolution of unity in the \((q, \lambda)\)-Bargman-Fock realization of \( \mathcal{H}_0 \), where the annihilation operator \( (8) \) acts as a difference operator \( D^{(\lambda)}_q \) \([15]\),

\[
\frac{1}{2\pi} \int_{0}^{2\pi} |z; \lambda\rangle \langle z; \lambda| \, d\phi \, d_q \sigma(|z|^2) = \sum_{n=0}^{\infty} |n\rangle \langle n| = I,
\]

results in the classical moment problem (MP) \([21]\) for the measure \( d_q \sigma(|z|^2) \),

\[
\int_{0}^{\infty} x^n \, d_q \sigma(|z|^2) = s_n(q; \lambda), \quad s_n(q; \lambda) = [n; q, \lambda]!.
\]

Depending on the behaviour of the moments \( s_n(q; \lambda) \) as \( n \to \infty \), the MP can be determinate (a unique solution, if any: this is the case of the \( q \)-oscillator \([2]\)), or indeterminate (many solutions: these cases are realized for the \( q \)-oscillators \([3]\) or \([4]\)). The completeness (the system is overcomplete) was proved for \( s_n(q; \lambda) = [n; q]! \) \([30]\), and \( s_n(q; \lambda) = [n; q^{-1}]! \) \([31]\). Complete subsystems of \( q \)-coherent states \([14]\) (or \([24]\) for \( \lambda = 0 \)) are discussed in \([32]\). The classical MP refers also to \( q \)-Hermite polynomials: the latter are nothing but polynomials of the first kind \([21]\) for a Jacobi matrix \( J \) which is constructed as a “generalized coordinate” from the \( q \)-oscillator creation and annihilation operators \([17]\),

\[
J(\lambda) = a(\lambda) + a^\dagger(\lambda), \quad J(\lambda) \, |x\rangle = 2x \, |x\rangle, \quad (28)
\]

\[
|x\rangle = \sum_{n=0}^{\infty} H_n(x; q, \lambda) |n\rangle. (29)
\]

Due to \([28]\), these \( q \)-Hermite polynomials satisfy the following three-term recurrence relation:

\[
c_n(\lambda) H_{n-1}(x; q, \lambda) + c_{n+1}(\lambda) H_{n+1}(x; q, \lambda) = x \, H_n(x; q, \lambda). (30)
\]

The corresponding generating function can be introduced as in the oscillator case \([13]\),

\[
\omega(z, x; \lambda) = \langle \bar{z}; \lambda | x \rangle, \quad \text{however its form will depend on the chosen generators of } A(q) \quad [17]
\]

\[
\langle \bar{z}; \lambda | (a(\lambda) + a^\dagger(\lambda)) | x \rangle = (D^{(\lambda)}_q + z) \, \omega(z, x; \lambda) = 2x \, \omega(z, x; \lambda).
\]

This difference equation for \( \omega(z, x; \lambda) \) will include two points for \( \lambda = 0, 1 \), so its solution will be given by the ‘standard’ \( q \)-exponent \([15]\) and three points: \( z, q^{-\lambda} z, q^{1-\lambda} z \) for the general \( \lambda \). The measure entering into the \( q \)-Hermite polynomials \( H_n(x; q, \lambda) \) orthogonality relations
is connected with the solution of the Hamburger MP: this measure is known explicitly for some cases (see e.g. [18]). This connection of the MP with Jacobi matrices gives rise to a generalized deformation of the oscillator identifying the matrix \( c_k \delta_{n+1,k}^+, c_k > 0 \) with an annihilation operator \( a \). Then one gets the Wigner commutation relation \([a, a^\dagger] = F(N)\) with \( F(n) = c_{n+1}^2 - c_n^2 \) and its central element \( \zeta = (c^2(N) - a^\dagger a) + \text{const} \) (see also [2, 26, 27]). The \( q \)-special functions related to the other irreducible representations \( \mathcal{H}_\gamma \) of \( A(q) \) are discussed in [31]. In particular, for the generators (2) the normalized \( q \)-coherent states exist in \( \mathcal{H}_\gamma \) for the creation operator \( a^\dagger \) and \( z > \gamma_c = (1 - q)^{-1} \).

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