Poincaré constant on manifolds with ends

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Abstract
We obtain optimal estimates of the Poincaré constant of central balls on manifolds with finitely many ends. Surprisingly enough, the Poincaré constant is determined by the second largest end. The proof is based on the argument by Kusuoka–Stroock where the heat kernel estimates on the central balls play an essential role. For this purpose, we extend earlier heat kernel estimates obtained by the authors to a larger class of parabolic manifolds with ends.

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1 | INTRODUCTION

Let \( M \) be a Riemannian manifold. Denote by \( \mu \) the Riemannian measure on \( M \) and by \( \nabla \) the gradient. For a precompact connected open set \( U \subset M \), define the Poincaré constant \( \Lambda(U) \) as the smallest number such that the following inequality holds for all \( f \in C^1(\overline{U}) \):

\[
\int_U |f - f_U|^2 d\mu \leq \Lambda(U) \int_U |\nabla f|^2 d\mu,
\]

where \( f_U := \frac{1}{\mu(U)} \int_U f d\mu \). Equivalently, we have

\[
\Lambda(U) = \frac{1}{\lambda(U)},
\]

where \( \lambda(U) \) is the smallest positive eigenvalue of \(-\Delta \) in \( U \) with the Neumann condition on \( \partial U \). Here \( \Delta \) is the Laplace–Beltrami operator on \( M \). Estimating the Poincaré constant has many applications; see \([1, 2–4, 6, 22]\) for examples and references therein.

Denote by \( d(x, y) \) the geodesic distance on \( M \) and by \( B(x, r) \) — open geodesic balls on \( M \). In this paper we are concerned with estimating the Poincaré constant \( \Lambda(B(x, r)) \). It is well known that in \( \mathbb{R}^n \)

\[
\Lambda(B(x, r)) = C_n r^2.
\]

It is also known by \([19]\) that on complete non-compact manifolds with non-negative Ricci curvature

\[
\Lambda(B(x, r)) \simeq r^2.
\] (1.1)

There are other classes of manifolds satisfying (1.1), for example, Lie groups of polynomial volume growth (see \([7, 8]\)), the tube manifold around the square lattice \( \mathbb{Z}^d \) (or jungle gym).

However, there are natural examples of manifolds where (1.1) does not hold, for example, the hyperbolic spaces where \( \Lambda(B(x, r)) \) grows exponentially in \( r \). Another example that is more relevant for this paper is the connected sum \( \mathbb{R}^n \# \mathbb{R}^n \), where \( n \geq 2 \). By \( \mathbb{R}^n \# \mathbb{R}^n \) we denote any manifold that is obtained by gluing together two copies of \( \mathbb{R}^n \) over a compact tube (see Figure 1).

It follows from the results of this paper (Theorems 2.9 and 2.12) that on such a manifold

\[
\Lambda(B(o, r)) \simeq \begin{cases} 
  r^n & \text{if } n > 2, \\
  r^2 \log r & \text{if } n = 2,
\end{cases}
\]
\[ \lambda(B(o, r)) \simeq \begin{cases} \frac{1}{r^n} & \text{if } n > 2, \\ \frac{1}{r^2 \log r} & \text{if } n = 2 \end{cases} \]

for all large \( r \) and for a central reference point \( o \in \mathbb{R}^n \# \mathbb{R}^n \) (see Section 3). In these simple cases, these results are well known to the specialists of the subject.

Consider now a connected sum \( M = M_1 \# \cdots \# M_k \) of \( k \) model manifolds \( M_1, \ldots, M_k \) with a same dimension \( N \) (see Section 3 for details of this construction). For example, \( M_i \) can be a surface of revolution (see Figure 2). Assume that the volume growth function \( V_i(r) \) of \( M_i \) satisfies for some
\[ \alpha_i > 0 \text{ and } \beta_i \in \mathbb{R} \]

\[ V_i(r) \simeq r^{\alpha_i}(\log r)^{\beta_i}, \quad (i = 1, \ldots, k, \ r \gg 1). \]

We assume that

\[ (N, 0) \succeq (\alpha_1, \beta_1) \succeq (\alpha_2, \beta_2) \succeq \cdots \succeq (\alpha_k, \beta_k) \]

in the sense of the lexicographical order which implies that

\[ V_1(r) \succeq V_2(r) \succeq \cdots \succeq V_k(r) \]

(see Figure 2). It follows from the main results of this paper (Theorems 2.9 and 2.12) that the Poincaré constant \( \Lambda(B(o, r)) \) on \( M \) is determined, quite surprisingly, solely by the second largest end \( M_2 \):

\[
\Lambda(B(o, r)) \simeq \begin{cases} 
  r^{\alpha_2}(\log r)^{\beta_2} & \text{if } (\alpha_2, \beta_2) > (2, 1), \\
  r^2(\log r)(\log \log r) & \text{if } (\alpha_2, \beta_2) = (2, 1), \\
  r^2 \log r & \text{if } \alpha_2 = 2, \beta_2 < 1, \\
  r^2 & \text{if } \alpha_2 < 2
\end{cases}
\]

for all large \( r \) (see Example 2.14 for details).

Let us mention for comparison that if \( V_i(r) \simeq r^{\alpha_i} \) then the heat kernel long-time behavior is determined by the end \( M_i \) having \( \alpha_i \) nearest to 2 (see Example 2.22).

In the next section, we describe a more general class of manifolds with ends where we obtain two-sided estimates of the Poincaré constant. The main results are stated in Theorems 2.9 and 2.12. Proofs are given in Sections 5 and A and use crucially heat kernel estimates.

**Notation.** The notation \( f \simeq g \) for two non-negative functions \( f, g \) means that there are two positive constants \( c_1, c_2 \) such that \( c_1 g \leq f \leq c_2 g \) for the specified range of the arguments of \( f \) and \( g \). Similarly, the notation \( f \gtrsim g \) (resp., \( f \lesssim g \)) means that there is a positive constant \( c \) such that \( cf \gtrsim g \) (resp., \( f \lesssim cg \)). Throughout this article, the letters \( c, C, b, \ldots \) denote positive constants whose values may be different at different instances. When the value of a constant is significant, it will be explicitly stated.

## 2 | MAIN RESULTS

### 2.1 | Heat kernels

Let us recall some known results about the heat kernel on manifolds. Let \( M \) be a Riemannian manifold. Denote by \( \text{vol} \) the Riemannian measure on \( M \). Given a smooth positive function \( \sigma \) on \( M \), define a measure \( \mu \) on \( M \) by \( d\mu = \sigma d\text{vol} \). The pair \((M, \mu)\) is called a weighted manifold. Any Riemannian manifold can be considered as a weighted manifold with \( \sigma = 1 \). The Laplace operator \( \Delta \) of the weighted manifold \((M, \mu)\) is defined by

\[
\Delta = \frac{1}{\sigma} \text{div}(\sigma \nabla),
\]
where \( \text{div} \) and \( \nabla \) are the divergence and the gradient of the Riemannian metric of \( M \). The operator \( \Delta \) is known to be symmetric with respect to the measure \( \mu \) (see [10]).

Set \( V(x, r) = \mu(B(x, r)) \).

**Definition 2.1.** We say that a weighted manifold \( M \) satisfies the *volume doubling* (VD) condition if there exists a constant \( C \) such that for all \( x \in M \) and \( r > 0 \),

\[
V(x, 2r) \leq CV(x, r).
\]

**Definition 2.2.** We say that a weighted manifold \( M \) admits the *scale invariant Poincaré inequality* (PI) if there exist constants \( C > 0 \) and \( \kappa \geq 1 \) such that for all \( x \in M \) and \( r > 0 \), and for all \( f \in C^1(B(x, \kappa r)) \),

\[
\int_{B(x, r)} |f - f_{B(x, r)}|^2 d\mu \leq Cr^2 \int_{B(x, \kappa r)} |\nabla f|^2 d\mu.
\]

Denote by \( p(t, x, y) \) the heat kernel of \( M \), that is, the minimal positive fundamental solution of the heat equation \( \partial_t u = \Delta u \). The following theorem is a combined result of [8, 21] based on previous contributions of Moser [20] and Kusuoka–Stroock [18].

**Theorem 2.3.** On a geodesically complete, non-compact weighted manifold \( M \), the following conditions are equivalent:

(i) (PI) and (VD).

(ii) *The Li–Yau type heat kernel estimates:*

\[
p(t, x, y) \asymp \frac{C}{V(x, \sqrt{t})} \exp \left( -b \frac{d^2(x, y)}{t} \right),
\]

where the sign \( \asymp \) means that both \( \leq \) and \( \geq \) hold but with different values of the positive constants \( C \) and \( b \).

(iii) *The uniform parabolic Harnack inequality (for the definition see [15, Section 2.1]).*

The estimate (2.1) was proved for the first time by Li and Yau [19] on manifolds of non-negative Ricci curvature.

**Definition 2.4.** We say that a manifold \( M \) is *nice* if \( M \) is a geodesically complete, non-compact weighted manifold which admits the PI and the VD condition, so that \( M \) satisfies each of the conditions of Theorem 2.3.

### 2.2 Manifold with ends

Fix a natural number \( k \geq 2 \). Let \( M_1, \ldots, M_k \) be a sequence of geodesically complete, non-compact weighted manifolds of the same dimension.
Definition 2.5. Let $M$ be a weighted manifold. We say that $M$ is a manifold with $k$ ends $M_1, \ldots, M_k$ and write

$$M = M_1 \# \cdots \# M_k$$

(2.2)

if there is a compact set $K \subset M$ so that $M \setminus K$ consists of $k$ connected components $E_1, E_2, \ldots, E_k$ such that each $E_i$ is isometric (as a weighted manifold) to $M_i \setminus K_i$ for some compact set $K_i \subset M_i$ (see Figure 3). Each $E_i$ will be referred to as an end of $M$.

For the manifold (2.2) we will also use the notation $M = \#_{i \in I} M_i$ where

$$I = \{1, 2, \ldots, k\}.$$

It will be important for us to distinguish between parabolic and non-parabolic manifolds.

Definition 2.6. We say that a weighted manifold $N$ is parabolic if the weighted Laplace operator $\Delta$ has no positive Green function. Equivalently $N$ is parabolic if and only if

$$\int_{\infty}^{\infty} p(t, x, y)dt = \infty$$

(2.3)

for all/some $x, y \in N$.

If the integral in (2.3) converges then it determines the minimal positive Green function

$$g(x, y) = \int_{0}^{\infty} p(t, x, y)dt.$$

If $N$ satisfies (VD) and (PI), then $N$ is parabolic if and only if

$$\int_{\infty}^{\infty} \frac{rdr}{V(x, r)} = \infty$$

for all/some $x \in N$. 

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**Figure 3** Manifold with ends.
For a manifold with ends \((2.2)\), we say that an end \(E_i\) is parabolic (or non-parabolic) if \(M_i\) is parabolic (resp., non-parabolic). It is easy to verify that \(M\) is parabolic if and only if all the ends \(E_i\) are parabolic.

In the sequel, we always assume that each end \(M_i\) is nice (see Definition 2.4). Besides, if \(M_i\) is parabolic then we assume in addition that \(M_i\) satisfies the following condition (relatively connected annuli, RCA).

**Definition 2.7 (RCA).** We say that a Riemannian manifold \(N\) has RCA with respect to a reference point \(o \in N\) if there exists a constant \(A > 1\) such that, for any \(r > A^2\), any two points \(x, y \in N\) satisfying \(d(x, o) = d(y, o) = r\), are connected by a continuous path in \(B(o, Ar) \setminus B(o, A^{-1}r)\).

For each \(i = 1, \ldots, k\), let \(\mu_i\) be the reference measure on \(M_i\) and \(d_i\) be the geodesic distance on \(M_i\). Denote by \(B_i(x, r)\) geodesic balls on \(M_i\) and set \(V_i(x, r) = \mu_i(B_i(x, r))\). Fix a reference point \(o_i \in K_i\). Set
\[
V_i(r) = V_i(o_i, r) \quad (2.4)
\]
and refer to \(V_i\) as the (pointed) volume function of \(M_i\). Note that for small \(r\), we have \(V_i(r) \approx V_j(r) \approx r^n\) where \(n = \dim M\).

Clearly, \(M_i\) is parabolic if and only if
\[
\int_{r}^\infty \frac{r dr}{V_i(r)} = \infty.
\]

Define a function \(h_i(r)\) for all \(r > 0\) by
\[
h_i(r) = 1 + \left(\int_{1}^{r} \frac{s ds}{V_i(s)}\right)_{+}. \quad (2.5)
\]

Then \(M_i\) is parabolic if \(h_i(r) \to \infty\) as \(r \to \infty\) and non-parabolic if \(h_i \approx 1\).

**Definition 2.8.** We say that \(M_i\) is subcritical if for all \(r \gg 1\)
\[
h_i(r) \approx \frac{r^2}{V_i(r)}. \quad (2.6)
\]

Note that the subcriticality condition (2.6) is equivalent to
\[
h_i(r) \lesssim \frac{r^2}{V_i(r)},
\]

since the opposite inequality
\[
h_i(r) \gtrsim \frac{r^2}{V_i(r)} \quad (2.7)
\]
follows trivially from (2.5) and the doubling property of \(V_i(r)\).
For example, if \( V_i(r) \approx r^\alpha \) for large \( r \) then

\[
h_i(r) \approx \begin{cases} 
  r^{2-\alpha} & \text{if } \alpha < 2, \\
  \log r & \text{if } \alpha = 2, \\
  1 & \text{if } \alpha > 2,
\end{cases}
\]

for large \( r \). In this case \( M_i \) is parabolic if and only if \( \alpha \leq 2 \) and subcritical if and only if \( \alpha < 2 \).

If \( V_i(r) \approx r^2 (\log r)^\beta \) for large \( r \) then

\[
h_i(r) \approx \begin{cases} 
  (\log r)^{1-\beta} & \text{if } \beta < 1, \\
  \log \log r & \text{if } \beta = 1, \\
  1 & \text{if } \beta > 1,
\end{cases}
\]

for large \( r \). In this case \( M_i \) is non-subcritical for all \( \beta \in \mathbb{R} \) and parabolic if and only if \( \beta \leq 1 \).

In the next section we state the main results of this paper about the Poincaré constant on manifolds with ends. For all Poincaré type estimates obtained in this article, the range of interest is \( r \gg 1 \). Similarly, for all heat kernel estimates, the range of interest is \( t \gg 1 \).

### 2.3 Poincaré constant

First let us state a general definition of Poincaré constant. Let \((M, \mu)\) be a weighted manifold. For precompact connected open sets \( U \subset U' \subset M \), define the Poincaré constant \( \Lambda(U, U') \) of the couple \((U, U')\) as the smallest number such that the following inequality holds for all \( f \in C^1(U') \):

\[
\int_{U} |f - f_U|^2 d\mu \leq \Lambda(U, U') \int_{U'} |\nabla f|^2 d\mu,
\]

where \( f_U := \frac{1}{\mu(U)} \int_{U} f d\mu \). Equivalently, we have

\[
\Lambda(U, U') = \sup_{f \in C^1(U')} \inf_{\xi \in \mathbb{R}} \frac{\int_{U} |f - \xi|^2 d\mu}{\int_{U'} |\nabla f|^2 d\mu}. \tag{2.8}
\]

We note that the PI is equivalent to hold for all \( x \in M \), and \( r > 0 \),

\[
\Lambda(B(x, r), B(x, xr)) \lesssim r^2.
\]

The function \( (U, U') \to \Lambda(U, U') \) is clearly monotone in the following sense: if \( W \subset U \subset U' \subset W' \) then \( \Lambda(W, W') \leq \Lambda(U, U') \). If \( U = U' \), then we use a shorter notation

\[
\Lambda(U) := \Lambda(U, U).
\]

In this case the number

\[
\lambda(U) := \frac{1}{\Lambda(U)} = \inf_{f \in C^1(U), f \not= \text{const}} \frac{\int_{U} |\nabla f|^2 d\mu}{\int_{U} |f - f_U|^2 d\mu} \tag{2.9}
\]
is the spectral gap of $-\Delta$ on $U$, that is, the smallest positive eigenvalue of $-\Delta$ in $U$ with the Neumann condition on $\partial U$.

Let $M = \#_{i \in I} M_i$ be a manifold with nice ends as described above. For any $r > 0$ define natural numbers $m = m(r)$ and $n = n(r)$ so that

$$V_m(r) = \max_{i \in I} V_i(r)$$

and

$$V_n(r) = \max_{i \in I \setminus \{m\}} V_i(r).$$

That is, $V_m(r)$ is the largest volume function at scale $r$, and $V_n(r)$ is the second largest volume function at scale $r$.

Fix a central reference point $o \in K$. We first state our result in the case when all $M_i$ are non-parabolic.

**Theorem 2.9** (All ends are non-parabolic). Let $M = \#_{i \in I} M_i$ be a manifold with nice ends. Assume that all $M_i$ are non-parabolic. Then, for all $r \gg 1$,

$$\Lambda(B(o,r)) \lesssim V_n(r),$$

where $n$ is defined in (2.11). If, in addition each $V_i$ satisfies for all $r \gg 1$

$$rV'_i(r) \lesssim V_i(r),$$

then, for all $r \gg 1$,

$$\Lambda(B(o,r)) \simeq V_n(r).$$

As this theorem shows, the Poincaré constant is determined by the second largest volume function at scale $r$ via the simple formula $\Lambda(B(o,r)) \simeq V_n(r)$.

The case when some ends are parabolic is more challenging and our results are less complete. In this case, we will need the following definition which requires to subdivide the set of ends into three subsets $I_{\text{super}}$, $I_{\text{middle}}$ and $I_{\text{sub}}$. The subset $I_{\text{super}}$ is made of ‘large ends’. The set of $I_{\text{sub}}$ is made of ‘small ends’. Here, large and small are defined by two fixed positive parameter $\epsilon$ and $\delta$. Restrictive hypothesis are made on the ends in $I_{\text{middle}}$.

**Definition 2.10** (Critically ordered ends, COE). Let $M = \#_{i \in I} M_i$ be a manifold with nice ends. We say that $M$ has COE if there exist $\epsilon, \delta, \gamma_1, \gamma_2 > 0$ such that

$$\gamma_1 < \epsilon, \ \gamma_1 + \gamma_2 < \delta < 2, \ 2\gamma_1 + \gamma_2 < 2,$$

and a decomposition

$$I = I_{\text{super}} \cup I_{\text{middle}} \cup I_{\text{sub}}$$

such that the following conditions are satisfied.
(a) For each $i \in I_{\text{super}}$ and all $r \geq 1$,
\[ V_i(r) \gtrsim r^{2+\varepsilon}. \]

(b) For each $i \in I_{\text{sub}}$, $V_i$ is subcritical (see Definition 2.8) and
\[ V_i(r) \lesssim r^{2-\delta}. \tag{2.15} \]

(c) For each $i \in I_{\text{middle}}$,
\[ \left( \frac{R}{r} \right)^{2-\gamma_2} \lesssim \frac{V_i(R)}{V_i(r)} \lesssim \left( \frac{R}{r} \right)^{2+\gamma_1} \quad \text{for all } 1 \leq r \leq R. \tag{2.16} \]

For any pair $i, j \in I_{\text{middle}}$, $V_i \gtrsim V_j$ or $V_j \gtrsim V_i$ (that is, the ends in $I_{\text{middle}}$ can be ordered according to their volume growth uniformly over $r \in [1, \infty)$) and $V_i \gtrsim V_j$ implies that $V_i h_i \gtrsim V_j h_j$. Finally, if $M$ is parabolic (that is, all ends are parabolic) then we assume that $V_i \gtrsim V_j$ also implies $V_i h_i^2 \lesssim V_j h_j^2$.

Note that (2.6) implies (2.15) with some $\delta > 0$ (see Section 4.1), but here we need $\delta$ also to satisfy (2.13).

**Example 2.11.** Let all functions $V_i(r)$ have for $r \gg 1$ the form
\[ V_i(r) = r^\alpha_i (\log r)^\beta_i \tag{2.17} \]
for some $\alpha_i > 0$ and $\beta_i \in \mathbb{R}$. Let us show that $M$ satisfies (COE). Define $I_{\text{super}}$ to consist of all indices $i$ such that $\alpha_i > 2$, $I_{\text{sub}}$ to consist of all $i$ such that $\alpha_i < 2$, and $I_{\text{middle}}$ to consist of all $i$ with $\alpha_i = 2$. Clearly, (a) and (b) are satisfied. Let us verify (c). Indeed, in the case of $\alpha_i = 2$, we have
\[ h_i(r) \approx \begin{cases} 
1, & \text{if } \beta_i > 1 \\
\log \log r, & \text{if } \beta_i = 1 \\
(\log r)^{1-\beta_i}, & \text{if } \beta_i < 1
\end{cases} \]
and, hence,
\[ V_i(r)h_i(r) \approx \begin{cases} 
1, & \text{if } \beta_i > 1 \\
\log \log r, & \text{if } \beta_i = 1 \\
(\log r)^{1-\beta_i}, & \text{if } \beta_i < 1
\end{cases} \]

We see that
\[ V_i \gtrsim V_j \Leftrightarrow \beta_i \geq \beta_j \Leftrightarrow V_i h_i \gtrsim V_j h_j. \]

Moreover, if $M$ is parabolic then all $\beta_i \leq 1$ and in this case
\[ V_i(r)h_i^2(r) \approx \begin{cases} 
1, & \text{if } \beta_i = 1 \\
(\log r)^2, & \text{if } \beta_i < 1
\end{cases} \]
so that

\[ V_i \gtrsim V_j \iff \beta_i \gtrsim \beta_j \iff V_i h_i^2 \gtrsim V_j h_j^2. \]

Hence, \( M \) satisfies (COE). Further examples of such manifolds can be found in Section 3.

Now we can state our main result in the case when the ends may be parabolic.

**Theorem 2.12** (At least one end is parabolic). *Let \( M = \#_{i \in I} M_i \) be a manifold with nice ends. Assume that each parabolic end \( M_i \) satisfies also (RCA) with a reference point \( o_i \). Assume further that \( M \) satisfies (COE). Then, for all \( r \gg 1 \), we have*

\[
\Lambda(B(o,r)) \lesssim V_n(r) h_n(r),
\]

*(2.18)*

*where \( n \) is defined in (2.11). If, in addition each \( V_i \) satisfies for all \( r \gg 1 \)*

\[
r V_i'(r) \lesssim V_i(r),
\]

*(2.19)*

*then, for all \( r \gg 1 \),*

\[
\Lambda(B(o,r)) \simeq V_n(r) h_n(r).
\]

Hence, in this case the Poincaré constant \( \Lambda(B(o,r)) \) is again determined by the second largest volume function \( V_n \) but the formula also involves the associated function \( h_n \).

**Remark 2.13.** To obtain the upper bound of \( \Lambda(B(o,r)) \) in (2.12) and (2.18), we first prove the upper bound of \( \Lambda(B(o,r), B(o, \kappa r)) \) for some \( \kappa > 1 \) (see Section 5.3). To reduce \( \kappa > 1 \) to \( \kappa = 1 \), we need an additional argument presented in Section A.

**Example 2.14.** Assume that, under the hypotheses of Theorem 2.12, all \( M_i \) have volume functions (2.17) as in Example 2.11. Define on the set of all pair \( (\alpha_i, \beta_i) \) the lexicographical order \( \geq \), that is,

\[
(\alpha_i, \beta_i) \geq (\alpha_j, \beta_j)
\]

if \( \alpha_i > \alpha_j \) or \( \alpha_i = \alpha_j \) and \( \beta_i \geq \beta_j \). Then we have

\[
V_i \gtrsim V_j \iff (\alpha_i, \beta_i) \geq (\alpha_j, \beta_j).
\]

Clearly, each of the function (2.17) satisfies (2.19). By Theorem 2.12, we conclude that the Poincaré constant on \( M \) is determined by the second largest pair \( (\alpha_n, \beta_n) \), that is,

\[
\Lambda(B(o,r)) \simeq V_n(r) h_n(r) \simeq \begin{cases}
  r^{\alpha_n} (\log r)^{\beta_n} & \text{if } (\alpha_n, \beta_n) > (2, 1), \\
  r^2 \log r \log \log r & \text{if } (\alpha_n, \beta_n) = (2, 1), \\
  r^2 \log r & \text{if } \alpha_n = 2, \beta_n < 1, \\
  r^2 & \text{if } \alpha_n < 2.
\end{cases}
\]
Corollary 2.15. Under the hypotheses of Theorem 2.12, if
\[ |I_{\text{super}}| + |I_{\text{middle}}| \leq 1, \tag{2.20} \]
then
\[ \Lambda(B(o, r)) \lesssim r^2, \]
that is,
\[ \lambda(B(o, r)) \gtrsim \frac{1}{r^2} \]
for all large \( r \). Consequently, \( M \) satisfies (PI).

Proof. Indeed, under the hypothesis (2.20) the second largest end \( M_n \) belongs to \( I_{\text{sub}} \). By Definition 2.10, for a subcritical end we have
\[ h_n(r) \simeq r^2 / V_n(r) \]
when
\[ \Lambda(B(o, r)) \lesssim V_n(r) h_n(r) \simeq r^2. \]
Note also that if \( B(x, \kappa r) \subset E_i \) then, by (PI) on \( M_i \), we have
\[ \Lambda(B(x, r), B(x, \kappa r)) \lesssim C r^2. \tag{2.21} \]
Hence, using the terminology of [15, Section 4], (PI) holds for anchored and remote balls in \( M \). By [15, Proposition 4.2], (PI) holds for all balls in \( M \).

For some applications it is desirable to get rid of the hypothesis (COE). We state the target result here as a conjecture.

Conjecture 2.16. Let \( M = \#_{i \in I} M_i \) be a manifold with nice ends. Assume also that each parabolic end satisfies (RCA). For any \( r \gg 1 \), define \( m = m(r) \) and \( n = n(r) \) so that
\[ V_m(r) h_m(r) \gtrsim V_n(r) h_n(r) \gtrsim V_i(r) h_i(r) \text{ for all } i \neq n, m. \]
Then
\[ \Lambda(B(o, r)) \lesssim V_n(r) h_n(r). \]

Obviously, Theorems 2.9 and 2.12 support this conjecture. A typical case that is not covered by these theorems is when \( M = M_1 \# M_2 \) where both volume functions \( V_1, V_2 \) are close to the critical case \( V(r) = r^2 \) but not ordered in the sense of \( \geq \); see [12] for such examples.

2.4 Heat kernel estimates on manifolds with nice ends

Our strategy of the proof of Theorems 2.9 and 2.12 is inspired by the argument of Kusuoka–Stroock [18] (see also [22]), where (PI) was deduced from the heat kernel estimates (2.1).
Let $M = \#_{i \in I} M_i$ be a manifold with nice ends as above. We obtain the estimates of the Poincaré constant on $M$ by using heat kernel bounds on $M$. In the case when $M$ is non-parabolic, matching upper and lower estimates of the heat kernel on $M$ were obtained in [16]. To state this result, let us introduce the following notation:
\[
\bar{V}_i := V_i h_i^2,
\]
where $V_i(r)$ and $h_i(r)$ were defined in (2.4) and (2.5), respectively.

**Theorem 2.17** [16, Corollary 6.8][17]. Let $M = \#_{i \in I} M_i$ be a manifold with nice ends where each parabolic $M_i$ satisfies (RCA). If $M$ is non-parabolic then, for all $t > 0$, we have
\[
p(t, o, o) \sim \frac{1}{\min_{i \in I} \bar{V}_i(\sqrt{t})}.
\]
In particular, when all $M_i$ are non-parabolic then $\bar{V}_i \sim V_i$ and we obtain
\[
p(t, o, o) \sim \frac{1}{\min_{i \in I} V_i(\sqrt{t})},
\]
that is, $p(t, o, o)$ is determined by the end with the smallest volume function.

In the case when $M$ is parabolic, similar estimates were obtained in [11], however, with certain restriction on the volume functions $V_i(r)$ near the critical case $V_i(r) \sim r^2$. For example, the result of [11] includes $V_i(r) \sim r^2$ with $\alpha \leq 2$ but does not include $V_i(r) \sim r^2 \log r$ or $r^2 / \log r$.

In this paper, we prove matching upper and lower heat kernel bounds on parabolic manifolds with ends under weaker restrictions than in [11].

**Definition 2.18.** A parabolic end $M_i$ is called regular if it satisfies (2.16) with positive exponents $\gamma_1$ and $\gamma_2$ such that $2\gamma_1 + \gamma_2 < 2$, that is,
\[
c \left( \frac{R}{r} \right)^{2-\gamma_2} \leq \frac{V_i(R)}{V_i(r)} \leq C \left( \frac{R}{r} \right)^{2+\gamma_1} \quad \text{for all } 1 \leq r \leq R.
\]
(2.22)

For example, if $M$ satisfies (COE) (see Definition 2.10) then any $M_i$ with $i \in I_{\text{middle}}$ is regular.

**Definition 2.19** (Dominates in the order, DOE). We say that $M = \#_{i \in I} M_i$ has an end that DOE if there exists $l \in I$ such that, for all $i \in I$
\[
V_l \geq V_i \quad \text{and} \quad \bar{V}_l \leq \bar{V}_i.
\]
(2.23)

**Remark 2.20.** If a manifold $M = \#_{i \in I} M_i$ satisfies (COE) with $I_{\text{super}} = \emptyset$ and $I_{\text{middle}} \neq \emptyset$, then $M$ admits (DOE) with $l \in I_{\text{middle}}$ and for all $r \gg 1$:
\[
V_l(r) \sim V_m(r),
\]
where $m = m(r)$ is the index of the largest end defined in (2.10); see Lemma 5.2 for details.
The next theorem is our main result regarding heat kernel estimates.

**Theorem 2.21.** Let \( M = \#_{i \in I} M_i \) be a manifold with parabolic nice ends, where each end satisfies (RCA) and is regular or subcritical. If there exists at least one non-subcritical regular end, assume also that \( M \) admits (DOE). Then, for all \( t > 0 \),

\[
p(t, o, o) \simeq \frac{1}{V_m(\sqrt{t})},
\]

(2.24)

where \( m \) is defined in (2.10).

Hence, in the situation covered by this theorem, the long-time behavior of the heat kernel is determined by the largest volume function, in contrast to the case of non-parabolic ends where, as we have seen above, \( p(t, o, o) \) is determined by the smallest volume function. The estimate (2.24) implies sharp matching upper and lower bounds for \( p(t, x, y) \) for all \( x, y \in M \) by means of the gluing techniques of [16] (see also Theorem 4.3).

**Example 2.22.** Let \( M = \#_{i \in I} M_i \) be a manifold with nice ends, where each parabolic \( M_i \) satisfies (RCA). Assume that the volume growth function \( V_i(r) \) of \( M_i \) satisfies for some \( \alpha_i > 0 \)

\[
V_i(r) \simeq r^{\alpha_i}, \quad (i \in I, r \gg 1).
\]

Then Theorems 2.17 and 2.21 imply that for all \( t > 0 \)

\[
p(t, o, o) \simeq \frac{1}{t^{\alpha/2}},
\]

where \( \alpha = 2 + \min_{i \in I} |\alpha_i - 2| \) so that \( p(t, o, o) \) is determined by \( \alpha_i \) nearest to 2, unlike the Poincaré constant \( \Lambda(B(o, r)) \).

**Example 2.23.** Let \( M = \#_{i \in I} M_i \) be as above. Assume that \( V_i(r) \) satisfies for some \( \beta_i \in \mathbb{R} \)

\[
V_i(r) \simeq r^{2(log r)^{\beta_i}}, \quad (i \in I, r \gg 1).
\]

Then Theorems 2.17 and 2.21 imply that for all \( t > 0 \)

\[
p(t, o, o) \simeq \begin{cases} 
\frac{1}{t(log t)^{\beta}} & \text{if } \beta_i \neq 1 \text{ for all } i \in I, \\
\frac{1}{t(log t)(log log t)^{\beta}} & \text{if } \beta_i = 1 \text{ for some } i \in I,
\end{cases}
\]

where \( \beta = 1 + \min_{i \in I} |\beta_i - 1| \) so that \( p(t, o, o) \) is determined by \( \beta_i \) nearest to 1.

It seems that the condition (DOE) in Theorem 2.21 is technical and we conjecture, that in general the following is true:
Conjecture 2.24. Let $M = \#_{i \in I} M_i$ be a manifold with nice ends, and each parabolic end also satisfies (RCA). Then, for all $t > 0$, we have

$$p(t, o, o) \simeq \frac{\min_{i \in I} h_i^2(\sqrt{t})}{\min_{i \in I} V_i(\sqrt{t})}. \quad (2.25)$$

Remark 2.25. Regarding condition (DOE), observe that the first condition $V_i \geq V_i$ in (2.23) does not imply in general the second condition $\tilde{V}_i \leq \tilde{V}_i$. However, if all ends are regular and there exists $\eta \geq 2\gamma_1$ such that for all $i$

$$\frac{V_i(r)}{\tilde{V}_i(r)} \geq C r^\eta \quad \text{for all } r \geq 1, \quad (2.26)$$

then $\tilde{V}_i \leq \tilde{V}_i$ is satisfied. Indeed, by the regularity of $V_i$ and (2.5), we obtain

$$h_i(r) = 1 + \frac{1}{V_i(r)} \int_1^r \frac{V_i(r)}{V_i(s)} ds \leq 1 + \frac{1}{V_i(r)} \int_1^r \left( \frac{r}{s} \right)^{2+\gamma_1} ds \leq \frac{r^{2+\gamma_1}}{V_i(r)},$$

which implies

$$\tilde{V}_i(r) = V_i(r) h_i^2(r) \leq \frac{r^{4+2\gamma_1}}{V_i(r)}.$$

Since $h_i(r) \geq \frac{r^2}{V_i(r)}$, we obtain

$$\tilde{V}_i(r) = V_i(r) h_i^2(r) \geq \frac{r^4}{V_i(r)}.$$

Hence, (2.26) implies that

$$\frac{\tilde{V}_i(r)}{\tilde{V}_i(r)} \leq \frac{V_i(r)}{\tilde{V}_i(r)} \leq r^{2\gamma_1 - \eta} \leq 1,$$

as was claimed.

### 2.5 Structure of the paper

In Section 3 we give examples of application of Theorems 2.9 and 2.12 in the case when all ends are model manifolds.

In Section 4, we first survey the previous results of [11, 16] on heat kernel estimates on manifolds with ends and then prove Theorem 2.21.

In Section 5 we give the proofs of Theorems 2.9 and 2.12. For this, we first obtain in Section 5.2 a lower bound for the Dirichlet heat kernel in balls (Lemma 5.4) using the two-sided off-diagonal bounds of the heat kernel $p(t, x, y)$ on $M$ that follow from Theorem 2.21. By means of this estimate, we obtain in Section 5.3 an upper bound for $\Lambda(B(o, r), B(o, kx))$ for some large enough $k > 1$. We
use an additional argument to reduce $\kappa$ to 1 stated in Section A. The lower bound for $\Lambda(B(o, r))$ is proved in Section 5.4.

In Section A, we obtain a general upper bound of the Poincaré constant $\Lambda(B(o, r))$ by using a collection of $\Lambda(B(x, s), B(x, \kappa s))$, where $B(x, s) \subset B(o, r)$ and $\kappa \geq 1$.

3 | MODEL MANIFOLD AND EXAMPLES

Let $\psi$ be a smooth positive function on $(0, +\infty)$ such that such $\psi(r) = r$ for $r < 1$. Fix an integer $N \geq 2$ and consider in $\mathbb{R}^N$ a Riemannian metric $g_\psi$ given in the polar coordinates $(r, \vartheta) \in (0, \infty) \times \mathbb{S}^{N-1}$ by

$$g_\psi = dr^2 + \psi(r)^2 d\vartheta^2.$$ 

In a punctured neighborhood of the origin $o \in \mathbb{R}^N$, this metric coincides with the Euclidean one and, hence, extends to the full neighborhood of $o$, so that $g_\psi$ is defined on the entire $\mathbb{R}^N$. The Riemannian manifold $(\mathbb{R}^N, g_\psi)$ is called a model manifold. For example, if $\psi(r) = r$ then $g_\psi$ is the canonical Euclidean metric.

Let us assume in addition that, for some $C > 0$ and all $r \gg 1$,

$$\begin{align*}
\sup_{[r, 2r]} \psi & \leq C \inf_{[r, 2r]} \psi, \\
\psi(r) & \leq Cr, \\
\int_0^r \psi^{N-1}(s) ds & \leq Cr\psi^{N-1}(r).
\end{align*} \tag{3.1}$$

Since for any bounded range $r \in (0, r_0)$ the condition (3.1) is trivially satisfied, we obtain by [15, Proposition 4.10] that $(\mathbb{R}^N, g_\psi)$ satisfies (PI) and (VD). It is also obvious that $(\mathbb{R}^N, g_\psi)$ satisfies (RCA) with respect to the origin $o$ because the geodesic balls $B(o, r)$ coincide with the Euclidean balls.

Let us reformulate conditions (3.1) in terms of the volume $V(r)$ of a ball $B(o, r)$ on $(\mathbb{R}^N, g_\psi)$:

$$V(r) = V(o, r) = \omega_N \int_0^r \psi^{N-1}(s) ds. \tag{3.2}$$

Lemma 3.1. The conditions (3.1) are equivalent to the following conditions to be satisfied for all $r \gg 1$:

$$\begin{align*}
V(r) & \leq Cr^N \\
V(r) & \simeq rV'(r). \tag{3.3}
\end{align*}$$

Proof. The inequality $V(r) \leq Cr^N$ follows from $\psi(r) \leq r$ and (3.2). The third condition in (3.1) is equivalent to

$$V(r) \leq CrV'(r)$$
that is the upper bound in the second condition in (3.3). To obtain a similar lower bound, observe that the first condition in (3.1) is equivalent to

\[ V'(s) \simeq V'(r) \text{ for all } s \in \left[ \frac{1}{2} r, r \right], \]  

which implies

\[ V(r) \geq \int_{r/2}^{r} V'(s) ds \simeq rV'(r). \]

Let us now prove that (3.3) implies (3.1). Clearly, the third condition in (3.1) follows from

\[ V(r) \simeq rV'(r). \]  

(3.5)

The conditions (3.3) imply the second condition in (3.1) as follows:

\[ \psi(r)^{N-1} = \omega_N^{-1} V'(r) \simeq \frac{V(r)}{r} \leq C r^{N-1}. \]  

(3.6)

Finally, let us prove the first condition in (3.1), or, equivalently, (3.4). In the view of (3.5) and (3.6) it suffices to prove that \( V \) satisfies the doubling property, that is,

\[ V(2r) \leq CV(r). \]  

(3.7)

Indeed, applying again (3.5), we obtain

\[ \ln V(2r) - \ln V(r) = \int_{r}^{2r} \frac{V'(t)}{V(t)} dt \simeq \int_{r}^{2r} \frac{dt}{t} = \ln 2, \]

when (3.7) follows.

In view of Lemma 3.1, it will be more convenient to describe a model manifold in terms of the volume function \( V(r) \) rather than \( \psi(r) \). Hence, we denote the model manifold \((\mathbb{R}^N, g_{\psi})\) shortly by \( \mathcal{M}_V \).

Given \( k \) model manifolds \( \mathcal{M}_{V_1}, \ldots, \mathcal{M}_{V_k} \) satisfying (3.3) (and, hence, (VD) and (PI)), consider their connected sum

\[ M = \mathcal{M}_{V_1} \# \cdots \# \mathcal{M}_{V_k}. \]

Assume that the standing assumptions of either Theorem 2.9 or Theorem 2.12 are satisfied, that is, either all ends are non-parabolic or all ends are (COE). Then, Theorems 2.9 and 2.12 yield that

\[ \Lambda(B(o, r)) \simeq V_n(r) h_n(r), \]

that is,

\[ \lambda(B(o, r)) \simeq \frac{1}{V_n(r) h_n(r)}, \]

where \( n \) is the index of the second largest end.
FIGURE 4  Model manifolds $\mathcal{M}_V$ of dimension $N = 2$ with $V(r) = r^\alpha$, $r \gg 1$.

**Example 3.2.** Assume that, for all $r \gg 1$,

$$V(r) = r^\alpha \prod_{j=1}^{J} (\log_{[j]} r)^{\beta(j)},$$

(3.8)

where $\alpha > 0$, $\beta(1), \ldots, \beta(J) \in \mathbb{R}$ and $\log_{[j]} r$ is the $j$-times iterated logarithm. Clearly, in this case $\mathcal{M}_V$ is parabolic if and only if

$$(\alpha, \beta(1), \ldots, \beta(J)) \preceq (2, 1, \ldots, 1),$$

(3.9)

where $\preceq$ denotes the lexicographical order. Also, it is easy to see that $V(r)$ satisfies (3.3) and hence the parabolic Harnack inequality if and only if

$$(\alpha, \beta(1), \ldots, \beta(J)) \preceq (N, 0, \ldots, 0).$$

(3.10)

Note for comparison that the function $V(r) = (\log r)^{\beta}$ with $\beta > 0$ does not satisfy the second condition in (3.3). Some examples of model manifolds with $V(r) = r^\alpha$ are shown on Figure 4.

Assume now that for each $i$ and all $r \gg 1$

$$V_i(r) = r^{\alpha_i} \prod_{j=1}^{J} (\log_{[j]} r)^{\beta_i(j)},$$

(3.11)

where each $(J + 1)$-tuple $(\alpha_i, \beta_i(1), \ldots, \beta_i(J))$ satisfies (3.10). The condition (COE) holds in this case with the following decomposition of the index set $I$:

$$i \in \begin{cases} 
I_{\text{super}} & \text{if } \alpha_i > 2, \\
I_{\text{sub}} & \text{if } \alpha_i < 2, \\
I_{\text{middle}} & \text{if } \alpha_i = 2.
\end{cases}$$
Let us compute $V_i h_i$. If $V_i$ is non-parabolic, that is, if

$$\alpha_i, \beta_i(1), \ldots, \beta_i(J) > (2, 1, \ldots, 1),$$

then by (2.5) $h_i \simeq 1$ and, hence, $V_i h_i \simeq V_i$.

Let now $V_i$ be parabolic, that is,

$$\alpha_i, \beta_i(1), \ldots, \beta_i(J) \leq (2, 1, \ldots, 1). \quad (3.12)$$

If $\alpha_i < 2$ then $V_i$ is subcritical and, hence,

$$V_i h_i(r) \simeq r^2. \quad (3.13)$$

In the case $\alpha_i = 2$, we denote by $J_i$ the smallest index $j = 1, \ldots, J$ such that $\beta_i(j) < 1$; if such $j$ does not exist then take $J_i = J + 1$. The parabolicity of $V_i$ implies that

$$\beta_i(j) = 1 \text{ for all } j < J_i.$$

With this notation we obtain, for all $r \gg 1$,

$$h_i(r) \simeq \begin{cases} 
  r^{2-\alpha_i} \prod_{j=1}^{J_i} (\log_{[j]} r)^{-\beta_i(j)} & \text{if } \alpha_i < 2 \\
  \log_{[J_i]} r \prod_{j=J_i}^{J} (\log_{[j]} r)^{-\beta_i(j)} & \text{if } \alpha_i = 2.
\end{cases}$$

Combining this, (3.13) and (3.11), we obtain

$$V_i(r) h_i(r) \simeq \begin{cases} 
  r^2 & \text{if } \alpha_i < 2, \\
  r^2 \prod_{j=1}^{J_i} \log_{[j]} r & \text{if } \alpha_i = 2
\end{cases} \quad (3.14)$$

and

$$\bar{V}_i(r) \simeq \begin{cases} 
  r^{4-\alpha_i} \prod_{j=1}^{J_i} (\log_{[j]} r)^{-\beta_i(j)} & \text{if } \alpha_i < 2 \\
  r^2 \prod_{j=1}^{J_i-1} (\log_{[j]} r)(\log_{[J_i]} r)^2 \prod_{j=J_i}^{J} (\log_{[j]} r)^{-\beta_i(j)} & \text{if } \alpha_i = 2.
\end{cases} \quad (3.15)$$

Theorems 2.9, 2.12, 2.21 and 2.17 yield the following. Let $m$ be the index for the largest volume function and let $n$ be the index for the second largest volume function, that is, for all $i \neq m, n$,

$$(\alpha_m, \beta_m(1), \ldots, \beta_m(J)) \geq (\alpha_n, \beta_n(1), \ldots, \beta_n(J)) \geq (\alpha_i, \beta_i(1), \ldots, \beta_i(J)).$$

Case I. Let $M$ be non-parabolic. Then

$$p(t, o, o) \simeq \frac{1}{\min_{i \in I} \bar{V}_i(\sqrt{t})}, \quad (3.16)$$
that is, \( p(t, o, o) \) is determined by the smallest function \( \tilde{V}_i(\sqrt{t}) \) (see further examples and comments in [16]). Note that it is not easy to give an explicit expression for \( \min_{i \in I} \tilde{V}_i \) because the form of the result varies depending on the exact nature of \((\alpha_i, \beta_i(1), \ldots, \beta_i(J)), 1 \leq i \leq k\). Of course, in any particular example, one can compute \( \min_{i \in I} \tilde{V}_i \) explicitly.

**Case 2.** Let \( M \) be parabolic. Then

\[
p(t, o, o) \approx \frac{1}{V_m(\sqrt{t})} \approx \frac{1}{t^{\alpha_m/2} \prod_{j=1}^{J} (\log_{[j]} t)^{\beta_m(j)}}.
\]

(3.17)

If \( V_i(r) \approx r^{\alpha_i} \), then the above estimates in (3.16) and (3.17) imply that

\[
p(t, o, o) \approx \frac{1}{t^{\alpha/2}},
\]

where

\[
\alpha = 2 + \min_{i \in I} |\alpha_i - 2|
\]

so that \( p(t, o, o) \) is determined by the volume growth function nearest to \( r^2 \).

In both cases 1 and 2, the Poincaré constant \( \Lambda(B(o, r)) \) is determined by the second largest volume function \( V_n(r) \), namely, for all \( r \gg 1 \)

\[
\Lambda(B(o, r)) \approx V_n(r)h_n(r).
\]

If \( V_n \) is non-parabolic, then \( V_nh_n \approx V_n \), when we obtain

\[
\Lambda(B(o, r)) \approx r^{\alpha_n} \prod_{j=1}^{J} (\log_{[j]} r)^{\beta_n(j)},
\]

that is,

\[
\lambda(B(o, r)) \approx r^{-\alpha_n} \prod_{j=1}^{J} (\log_{[j]} r)^{-\beta_n(j)}.
\]

If \( V_n \) is parabolic, then by (3.14) we obtain

\[
\Lambda(B(o, r)) \approx \begin{cases} 
  r^2 & \text{if } \alpha_n < 2, \\
  r^2 \prod_{j=1}^{J} \log_{[j]} r & \text{if } \alpha_n = 2,
\end{cases}
\]

(3.18)

that is,

\[
\lambda(B(o, r)) \approx \begin{cases} 
  \frac{1}{r^2} & \text{if } \alpha_n < 2, \\
  \frac{1}{r^2 \prod_{j=1}^{J} \log_{[j]} r} & \text{if } \alpha_n = 2.
\end{cases}
\]

(3.19)
For example, (3.18) holds provided at most one of the ends of $M$ is non-parabolic, because in this case the second largest $V_n$ is parabolic. We see from (3.18) that in this case the estimate of the Poincaré constant exhibits a certain rigidity — it does not depend on the exponents $\alpha_i$ and $\beta_i(j)$, although it does depend on $J_n$.

4  |  HEAT KERNEL ESTIMATES (CENTRAL ESTIMATES)

This section is devoted to obtain heat kernel estimates on manifolds with nice ends, which play a key role in the proof of Theorem 2.12. The main result here is Theorem 2.21 which provides two-sided matching estimates of $p(t, o, o)$ on parabolic manifolds with nice ends. This result is of interest by itself independently of the application to bounding the Poincaré constant. Theorem 2.21 implies the off-diagonal estimates of $p(t, x, y)$ as in Theorem 4.3.

4.1  |  Known results

Let us first recall some known results. As before, consider a manifold with ends $M = M_1 \# \cdots \# M_k$ where each $M_i$ is nice, namely $M_i$ is a geodesically complete, weighted manifold that satisfies (PI) and (VD) (see Definition 2.4). Besides, if $M_i$ is parabolic then we assume that it satisfies (RCA). Fix a reference point $o_i \in M_i$, set $V_i(r) = V_i(o_i, r)$ and define $h_i(r)$ by (2.5).

In the case when $M$ is non-parabolic the matching upper and lower bounds of $p(t, o, o)$ are provided by Theorem 2.17. Let us discuss the case when $M$ is parabolic.

Recall that an end $M_i$ is called subcritical if, for all $r \gg 1$,

$$h_i(r) \simeq \frac{r^2}{V_i(r)} \quad \text{(4.1)}$$

(see Definition 2.8). For example, if $V_i(r) \simeq r^\alpha$ then $M_i$ is subcritical if and only if $\alpha < 2$. Conversely, if $M_i$ is subcritical, then there exists $\delta > 0$ such that

$$V_i(r) \lesssim r^{2-\delta}. \quad \text{(4.2)}$$

Indeed, (2.8) implies that, for some $\delta > 0$,

$$(\log h_i(r))' \geq \frac{\delta}{r},$$

which implies upon integration that $h_i(r) \gtrsim r^\delta$, which together with (2.8) yields (4.2).

**Definition 4.1.** An end $M_i$ is called **critical** if

$$V_i(r) \simeq r^2.$$ 

Note that all critical and subcritical ends are parabolic.
Theorem 4.2 [11, Theorem 2.1]. Let \( M = M_1 \# \cdots \# M_k \) where each end \( M_i \) is either critical or subcritical. Then, for all \( t > 0 \), we have

\[
p(t, o, o) \approx \frac{1}{V_m(\sqrt{t})},
\]

where \( m = m(r) \) is the index of the largest volume function at scale \( r \).

Our main goal in this section is to prove Theorem 2.21 stated in Introduction that provides the same estimates of \( p(t, o, o) \) but for a more general class of parabolic manifolds with ends (see Section 3).

Let us first explain how the estimates of \( p(t, o, o) \) lead to the estimates of \( p(t, x, y) \) for all \( x, y \in M \). For \( r, t > 0 \), set

\[
H_i(r, t) = \frac{r^2}{V_i(r)h_i(r)} + \frac{1}{h_i(\sqrt{t})} \left( \int_r^{\sqrt{t}} \frac{sd s}{V_i(s)} \right)_+ = \frac{r^2}{V_i(r)h_i(r)} + \frac{(h_i(\sqrt{t}) - h_i(r))_+}{h_i(\sqrt{t})}
\]

and

\[
D_i(r, t) = \frac{h_i(r)}{h_i(r) + h_i(\sqrt{t})}.
\]

The following result follows from [11, (5.1), (5.6), (5.7); 13, Thms 3.3, 4.9; 14, Thms 4.4, 4.6; 16, Theorems 3.5] and the local Harnack inequality near \( o \in K \).

Theorem 4.3. Let \( M = M_1 \# \cdots \# M_k \) be a manifold with nice ends, where each parabolic end satisfies (RCA). Then, for all \( x \in E_i, y \in E_j, t \gg 1 \) we have

\[
p(t, x, y) \approx \delta_{ij} \frac{1}{V_i(x, \sqrt{t})} D_i(|x|, t) D_j(|y|, t) e^{-b \frac{d^2(x, y)}{t}} + p(t, o, o) H_i(|x|, t) H_j(|y|, t) e^{-b \frac{|x|^2 + |y|^2}{t}}
\]

\[
+ \int_1^t p(s, o, o) ds \left[ \frac{D_i(|x|, t) H_j(|y|, t)}{V_i(\sqrt{t}) h_i(\sqrt{t})} + \frac{D_j(|y|, t) H_i(|x|, t)}{V_j(\sqrt{t}) h_j(\sqrt{t})} \right] e^{-b \frac{|x|^2 + |y|^2}{t}}.
\]

This is a very general estimate. In the view of this result, the on-diagonal value \( p(t, o, o) \) of the heat kernel plays a key role in estimating of \( p(t, x, y) \). One of the aim of this paper is to improve upon the existing results from [11]; see Theorem 2.21.

4.2 Preliminary estimates

A classical method of obtaining on-diagonal heat kernel bounds is to use a Nash-type functional inequality, which gives a uniform upper bound for \( p(t, x, x) \) for all \( x \in M \). Indeed, such
an inequality can be proved on manifold $M$ in the setting of Theorem 4.3 (see [17]) but in the case when $M$ is parabolic, this upper bound is not optimal. In order to obtain an optimal upper bound, we have developed in [11, Section 3] a different method based on the integrated resolvent kernel $\gamma_\lambda(x)$ and its derivative $\dot{\gamma}_\lambda(x)$. However, in [11, Section 3] we could handle only critical and subcritical ends (cf. Theorem 4.2). For the proof of Theorem 2.21, we apply the method of [11, Section 3], but with significant improvements that allow us to handle much more general ends.

Let $(M, \mu)$ be a geodesically complete, non-compact weighted manifold. For any $\lambda > 0$, we define the resolvent operator $G_\lambda$ acting on non-negative measurable functions $f$ on $M$ by

$$G_\lambda f(x) = \int_M \int_0^\infty e^{-\lambda t} p(t, x, z) f(z) dz dt.$$ 

We remark that $u = G_\lambda f$ is the minimal non-negative weak solution of the equation

$$\Delta u - \lambda u = -f \quad (4.3)$$

(see [10]). Fix a compact set $K \subset M$. We define the functions $\gamma_\lambda$ and $\dot{\gamma}_\lambda$ on $M$ by

$$\gamma_\lambda(x) := G_\lambda 1_K(x) = \int_K \int_0^\infty e^{-\lambda t} p(t, x, z) dz dt$$

and

$$\dot{\gamma}_\lambda(x) := G_\lambda \gamma_\lambda = -\frac{\partial}{\partial \lambda} \gamma_\lambda(x) = \int_K \int_0^\infty t e^{-\lambda t} p(t, x, z) dz dt.$$ 

Fix a reference point $o \in K$. The following lemma follows from the estimates in [11, (3.10), (3.11)].

**Lemma 4.4.** There exist constants $C > 0$ and $t_0 > 0$ depending on $K$ and such that, for all $x \in K$ and $t \geq t_0$,

$$p(t, o, o) \leq \frac{C}{t} \gamma_\lambda(x), \quad (4.4)$$

$$p(t, o, o) \leq \frac{C}{t^2} \gamma_\lambda(x). \quad (4.5)$$

As in [11, Section 4], in order to estimate $\gamma_\lambda$ and $\dot{\gamma}_\lambda$, we will use the functions $\Phi_\lambda^\Omega$ and $\Psi_\lambda^\Omega$ defined below. Denote by $p_\Omega^\lambda(t, x, y)$ the Dirichlet heat kernel in an open set $\Omega \subset M$. For any $\lambda > 0$, define the resolvent operator $G_\lambda^\Omega$ on non-negative measurable functions $f$ in $\Omega$ by

$$G_\lambda^\Omega f(x) = \int_\Omega \int_0^\infty e^{-\lambda t} p_\Omega^\lambda(t, x, z) f(z) dz dt,$$

so that $G_\lambda^\Omega f$ is the minimal non-negative weak solution of the equation $\Delta u - \lambda u = -f$ in $\Omega$.

Now we define the functions $\Phi_\lambda^\Omega$ and $\Psi_\lambda^\Omega$ in $\Omega$ by

$$\Phi_\lambda^\Omega = \lambda G_\lambda^\Omega 1,$$

$$\Psi_\lambda^\Omega = G_\lambda^\Omega (1 - \Phi_\lambda^\Omega).$$
These functions have certain probabilistic meaning. Indeed, denote by \( \{X_t\}_{t \geq 0}, \{P_x\}_{x \in M} \) Brownian motion on \( M \). For any open set \( \Omega \subset M \), let \( \tau_\Omega \) be the first exit time of \( X_t \) from \( \Omega \), that is,

\[
\tau_\Omega = \inf\{t > 0 : X_t \notin \Omega\}.
\]

Then, by [11, (3.17), (3.25)], for any \( x \in \Omega \),

\[
\Phi^\Omega_\lambda(x) = \int_0^\infty \lambda e^{-\lambda t} P_x(\tau_\Omega > t) dt, \tag{4.6}
\]

\[
\Psi^\Omega_\lambda(x) = \int_0^\infty t e^{-\lambda t} \partial_t P_x(\tau_\Omega \leq t) dt. \tag{4.7}
\]

Now let us assume that the manifold \( M \) satisfies (PI), (VD) and (RCA). Let \( A \) be a precompact open set containing

\[
K_\varepsilon := \{ z \in M : d(K, z) < \varepsilon \}
\]

for some \( \varepsilon > 0 \). Define function \( h(r) \) for \( r > 0 \) by

\[
h(r) = 1 + \left( \int_1^r \frac{s ds}{V(s)} \right)_+, \tag{4.8}
\]

where \( V(r) = V(o, r) \). In the next lemmas, we obtain some new estimates of \( \Phi^K_\lambda \) and \( \Psi^K_\lambda \).

**Lemma 4.5.** There exists \( c, \lambda_0 > 0 \) depending on \( K \) and \( A \), and such that, for all \( \lambda \in (0, \lambda_0) \),

\[
\inf_{\partial A} \Phi^K_\lambda \geq c \frac{h(\frac{1}{\sqrt{\lambda}})}{h(\frac{1}{\sqrt{\lambda}})}. \tag{4.9}
\]

**Proof.** By [14, (4.47)], we have, for all \( x \in \partial A \) and \( t \geq t_1 \),

\[
P_x(\tau_K > t) \approx \frac{h(|x|)}{h(\sqrt{t})}
\]

where \( t_1 > 0 \) depends on \( A \). Substituting this into (4.6) and using the fact that \( \inf_{\partial A} h(|x|) > 0 \), we obtain, for all \( \lambda < 1/(2t_1) \) and \( x \in \partial A \),

\[
\Phi^K_\lambda(x) \geq \int_{1/(2\lambda)}^{1/\lambda} \lambda e^{-\lambda t} \frac{c}{h(\sqrt{t})} dt \geq \frac{c}{h(\frac{1}{\sqrt{\lambda}})} \int_{1/(2\lambda)}^{1/\lambda} \lambda e^{-\lambda t} dt \geq \frac{c}{h(\frac{1}{\sqrt{\lambda}})}. \tag{4.9}
\]

**Remark 4.6.** If \( M \) is critical, then the estimate in (4.9) implies that

\[
\inf_{\partial A} \Phi^K_\lambda \geq \frac{c}{\log \frac{1}{\sqrt{\lambda}}},
\]

which coincides with [11, (3.34)]. If \( M \) is subcritical, the estimate in (4.9) implies that

\[
\inf_{\partial A} \Phi^K_\lambda \geq c\lambda V(\frac{1}{\sqrt{\lambda}}),
\]

which is identical to [11, (3.33)].
Recall that $V(r)$ is regular if it satisfies (2.22), that is,
\[ c \left( \frac{R}{r} \right)^{2-\gamma_2} \leq \frac{V(R)}{V(r)} \leq C \left( \frac{R}{r} \right)^{2+\gamma_1} \quad \text{for all } 1 \leq r \leq R \]
for some positive $\gamma_1, \gamma_2$ such that $2\gamma_1 + \gamma_2 < 2$.

Set $\tilde{V}(r) = V(r)h^2(r)$. Then we obtain the following upper estimate of $\tilde{V}(r)$ on manifold with regular volume function.

**Lemma 4.7.** Let $M$ be regular, that is, the volume function $V$ is regular. Then, for all $1 \leq r \leq R$,
\[ \tilde{V}(r) \leq r^{2+\gamma_1+\gamma_2} \leq r^4, \quad (4.10) \]

\[ \frac{\tilde{V}(R)}{\tilde{V}(r)} \leq \left( \frac{R}{r} \right)^{2+2\gamma_1+\gamma_2}. \quad (4.11) \]

**Proof.** By (4.8) and regularity, we have
\[ V(r)h(r) = V(r) + \int_1^r \frac{V(r)}{V(s)} ds \leq C \frac{V(r)}{V(1)} + C \int_1^r \left( \frac{r}{s} \right)^{2+\gamma_1} ds \leq C'r^{2+\gamma_1}. \quad (4.12) \]

Similarly, we have
\[ h(r) = 1 + \int_1^r \frac{ds}{V(s)} \leq 1 + \frac{1}{V(1)} \int_1^r \frac{V(1)}{V(s)} ds \leq 1 + C \int_1^r \frac{ds}{s^{2-\gamma_2}} \leq C'r^{\gamma_2}, \]
when (4.10) follows.

Now we prove (4.11). By the regularity of $V$ and
\[ h(r) \geq c \frac{r^2}{V(r)}, \]
we obtain
\[ \frac{\tilde{V}(R)}{\tilde{V}(r)} = \frac{V(R)h^2(R)}{V(r)h^2(r)} = \frac{V(R)(h(r) + \int_r^R \frac{V(s)}{V(s)} ds)^2}{V(r)h^2(r)} = \frac{V(R)}{V(r)} \left( 1 + \frac{\int_r^R \frac{V(s)}{V(s)} ds}{h(r)} \right)^2 \]
\[ \leq \frac{V(R)}{V(r)} \left( 1 + C \frac{V(r)}{r^2} \int_r^R \frac{V(s)}{V(s)} ds \right)^2 \]
\[ \leq \frac{V(R)}{V(r)} \left( 1 + C \frac{V(r)}{r^2} \int_r^\infty \left( \frac{R}{s} \right)^{2+\gamma_1} ds \right)^2 \]
\[ \leq C \frac{V(R)}{V(r)} \left( 1 + \frac{V(r)}{V(R)} \left( \frac{R}{r} \right)^{2+\gamma_1} \right)^2 \]
\[ \leq C \left( \frac{R}{r} \right)^{2+\gamma_1} + C \left( \frac{R}{r} \right)^{2+2\gamma_1+\gamma_2} \leq C \left( \frac{R}{r} \right)^{2+2\gamma_1+\gamma_2}, \]
which proves (4.11).
Lemma 4.8. Assume that the manifold $M$ is either subcritical or regular. Then there exist $C, \lambda_0 > 0$ depending on $K$ and $A$, such that, for all $\lambda \in (0, \lambda_0)$,

$$\sup_{\partial A} \Psi_{\lambda}^{Kc} \leq \frac{C}{\lambda^2 \bar{V}\left(\frac{1}{\sqrt{\lambda}}\right)}.$$  \hspace{1cm} (4.13)

Proof. Set $\Omega = K^c$. By [14, (4.48)] we have, for all $x \in \partial A$ and $t \geq t_0 > 0$

$$\partial_t P_x (\tau_{\Omega} \leq t) \leq \frac{C}{\bar{V}\left(\sqrt{t}\right)}.$$  \hspace{1cm} (4.14)

Taking $\lambda_0 \leq \frac{1}{t_0}$, we obtain by (4.7) for any $0 < \lambda < \lambda_0$

$$\Psi_{\lambda}^{\Omega}(x) = \int_0^{t_0} t e^{-\lambda t} \partial_t P_x (\tau_{\Omega} \leq t) dt + \int_{t_0}^{\infty} t e^{-\lambda t} \partial_t P_x (\tau_{\Omega} \leq t) dt \leq t_0 + C \int_{t_0}^{\infty} e^{-\lambda t} \frac{t}{\bar{V}\left(\sqrt{t}\right)} dt.$$  \hspace{1cm} (4.15)

Observe that if $M$ is regular then by (4.10) we have for all $r \geq 1$

$$\bar{V}(r) \leq Cr^4.$$  \hspace{1cm} (4.16)

If $M$ is subcritical, we have

$$\bar{V}(r) = V(r) h^2(r) \approx V(r) \left(\frac{r^2}{\bar{V}(r)}\right)^2 = \frac{r^4}{\bar{V}(r)},$$  \hspace{1cm} (4.17)

which again implies (4.15).

In particular, it follows from (4.15) for $r = \frac{1}{\sqrt{\lambda}}$ that

$$\frac{1}{\lambda^2 \bar{V}\left(\frac{1}{\sqrt{\lambda}}\right)} \geq c.$$  \hspace{1cm} (4.18)

so that the constant term $t_0$ in (4.14) can be estimated from above as follows:

$$t_0 \leq \frac{1}{\lambda^2 \bar{V}\left(\frac{1}{\sqrt{\lambda}}\right)}.$$  \hspace{1cm} (4.19)

Next, let us decompose the integral in (4.14) into two intervals: $[t_0, 1/\lambda]$ and $[1/\lambda, \infty)$. For the second interval, we obtain

$$\int_{1/\lambda}^{\infty} e^{-\lambda t} \frac{t}{\bar{V}\left(\sqrt{t}\right)} dt \leq \frac{1}{\bar{V}\left(\frac{1}{\sqrt{\lambda}}\right)} \int_{1/\lambda}^{\infty} te^{-\lambda t} dt \approx \frac{1}{\lambda^2 \bar{V}\left(\frac{1}{\sqrt{\lambda}}\right)}.$$  \hspace{1cm} (4.19)
Let us estimate the first integral. If $M$ is regular then, by using (4.11), we obtain

$$\int_{t_0}^{1/\lambda} e^{-\lambda t} \frac{t}{\widetilde{V}(\sqrt{t})} dt \leq \frac{1}{\widetilde{V}(\frac{1}{\sqrt{\lambda}})} \int_{t_0}^{1/\lambda} \frac{1}{\sqrt{t}} dt$$

$$\leq \frac{1}{\lambda^2 \widetilde{V}(\frac{1}{\sqrt{\lambda}})} \int_{t_0}^{1/\lambda} \left(\frac{1}{\lambda t}\right)^{1+\gamma_1 + \gamma_2/2} (\lambda t) d(\lambda t)$$

$$= \frac{1}{\lambda^2 \widetilde{V}(\frac{1}{\sqrt{\lambda}})} \int_{t_0}^{1} s^{-(\gamma_1 + \gamma_2/2)} ds$$

$$\approx \frac{1}{\lambda^2 \widetilde{V}(\frac{1}{\sqrt{\lambda}})} \int_{t_0}^{1/\lambda} t^{1+\gamma_1 + \gamma_2/2} dt$$

$$= \frac{1}{\lambda^2 \widetilde{V}(\frac{1}{\sqrt{\lambda}})} \int_{t_0}^{1/\lambda} \frac{1}{\sqrt{t}} dt$$

because $\gamma_1 + \gamma_2/2 < 1$.

If $M$ is subcritical, then we use the fact that $(VD)$ implies the reverse $VD$ (see, for instance, [22, Lemma 5.2.8]), that is, for some $\theta > 0$ and for all $R \geq r \geq 1$,

$$\frac{V(r)}{V(R)} \approx \left(\frac{r}{R}\right)^\theta.$$  \hspace{1cm} (4.21)

Then we obtain by (4.16) and (4.21) that

$$\int_{t_0}^{1/\lambda} e^{-\lambda t} \frac{t}{\widetilde{V}(\sqrt{t})} dt \leq \int_{t_0}^{1/\lambda} \frac{t}{\sqrt{t}} dt$$

$$\approx \int_{t_0}^{1/\lambda} t^{1/2} / \sqrt{t} dt$$

$$= V(\frac{1}{\sqrt{\lambda}}) \int_{t_0}^{\frac{1}{\sqrt{\lambda}}} V(\sqrt{t}) \frac{1}{t} dt$$

$$\approx V(\frac{1}{\sqrt{\lambda}}) \int_{t_0}^{\frac{1}{\sqrt{\lambda}}} (\lambda t)^{\theta/2} \frac{1}{\lambda t} d(\lambda t)$$

$$\approx V(\frac{1}{\sqrt{\lambda}}) \frac{1}{\lambda^2 \widetilde{V}(\frac{1}{\sqrt{\lambda}})}.$$  \hspace{1cm} (4.22)

Combining (4.18), (4.19), (4.20) and (4.22), we obtain (4.13). \hfill \square

**Remark 4.9.** If $M$ is critical, that is, $V(r) \approx r^2$, then Lemma 4.8 implies that

$$\sup_{\theta A} \psi_{\nu}^{Kc} \leq \frac{C}{\lambda \log^2 \frac{1}{\lambda}},$$
which coincides with [11, (3.39)]. If $M$ is subcritical, then Lemma 4.8 implies that
\[
\sup_{\partial A} \psi^{k_{c}}_{\lambda} \leq CV\left(\frac{1}{\sqrt{\lambda}}\right).
\]
Using further (4.2), we obtain
\[
\sup_{\partial A} \psi^{k_{c}}_{\lambda} \leq C\left(\frac{1}{\lambda}\right)^{1-\delta/2},
\]
which is a significant improvement of [11, (3.39)].

4.3 Proof of Theorem 2.21

Here we prove Theorem 2.21. Recall that we consider a manifold with nice ends $M = \#_{i\in I} M_{i}$ where each end is parabolic satisfying (RCA) (see Figure 3). Besides, we assume that each end satisfies either regular or subcritical, and also, (DOE) if there exists at least one non-subcritical regular end (see Definition 2.19). Our aim is to prove the estimate (2.24) that is,
\[
p(t, o, o) \approx \frac{1}{V_{m}(\sqrt{t})},
\]
where $V_{m}(r)$ is the largest volume function at scale $r$.

As before, let $K$ be the central part of $M$. Let $A$ be a connected, precompact open subset of $M$ with smooth boundary and such that $K_{\varepsilon} \subset A$ for some large enough $\varepsilon > 0$. Set
\[
\partial A_{i} := (\partial A) \cap E_{i}, \quad i = 1, \ldots, k.
\]

First we prove the following heat kernel upper bound.

**Proposition 4.10.** Let $M = \#_{i\in I} M_{i}$ be a manifold with parabolic ends, where each $M_{i}$ is either regular (Definition 2.18) or subcritical (Definition 2.10). Then for any $t \gg 1$
\[
p(t, o, o) \lesssim \min_{i} h_{i}^{2}(\sqrt{t}) \min_{i} \tilde{V}_{i}(\sqrt{t}).
\]

We use here the following two Lemmas from [11] that were proved for arbitrary manifolds with ends.

**Lemma 4.11** [11, Lemma 4.1]. There exists a constant $C = C(A, K) > 0$ such that, for all $\lambda > 0$,
\[
(\sup_{\partial K} \gamma_{\lambda}) \sum_{i=1}^{k} \inf_{\partial A_{i}} \Phi^{E_{i}}_{\lambda} \leq C.
\]
The estimate (4.25) combined with the estimates in (4.4) and (4.9) implies that
\[
p(t, o, o) \leq \frac{1}{t} \min_{i} h_{i}(\sqrt{t}).
\]
As was pointed out in [11, Remark 3.2], this estimate gives the optimal upper bound of \( p(t, o, o) \) when all \( M_i \) are subcritical. However, if there exists at least one critical end, this estimate yields
\[
p(t, o, o) \leq C \frac{\log t}{t},
\]
instead of the desired bound
\[
p(t, o, o) \leq \frac{C}{t}.
\]
In order to obtain an optimal bound of \( p(t, o, o) \) on parabolic manifolds with non-subcritical ends, we will use the following result.

**Lemma 4.12** [11, Lemma 4.2]. There exists a constant \( C > 0 \) depending on \( A, K \) such that for any \( \lambda > 0 \)
\[
(\sup_{\partial K} \gamma_{\lambda}) \sum_{i=1}^{k} \inf_{\partial A_i} \Phi_{\lambda}^{E_i} \leq C + (\sup_{\partial K} \gamma_{\lambda}) \left( C + \sum_{i=1}^{k} \sup_{\partial A_i} \Psi_{\lambda}^{E_i} \right). \tag{4.26}
\]

**Proof of Proposition 4.10.** Substituting (4.9) into (4.25), we obtain
\[
\sup_{\partial K} \gamma_{\lambda} \lesssim \frac{1}{\sum_{i=1}^{k} \inf_{\partial A_i} \Phi_{\lambda}^{E_i}} \lesssim \frac{1}{\sum_{i=1}^{k} \frac{1}{h_i(\frac{1}{\sqrt{\lambda}})}} \lesssim \min_{i \in I} h_i(\frac{1}{\sqrt{\lambda}}) = w(\lambda). \tag{4.27}
\]
Applying (4.26) with (4.27) and (4.13), we obtain, for small \( \lambda > 0 \),
\[
\sup_{\partial K} \gamma_{\lambda} \lesssim w(\lambda) \left( 1 + w(\lambda) \left( 1 + \sum_{i=1}^{k} \sup_{\partial A_i} \Psi_{\lambda}^{E_i} \right) \right) \lesssim \frac{w^2(\lambda)}{\lambda^2 \min_i \bar{V}_i(\frac{1}{\sqrt{\lambda}})},
\]
where we have also used that \( w \geq 1 \) and, by (4.15),
\[
\lambda^2 \min_i \bar{V}_i(\frac{1}{\sqrt{\lambda}}) \leq 1.
\]
Hence, we have proved that
\[
\sup_{\partial K} \gamma_{\lambda} \lesssim \frac{\min_i h_i^2(\frac{1}{\sqrt{\lambda}})}{\lambda^2 \min_i \bar{V}_i(\frac{1}{\sqrt{\lambda}})}.
\]
Taking here \( \lambda = t^{-1} \) and substituting this estimate of \( \gamma_{\lambda} \) into (4.5), we conclude Proposition 4.10.

**Proof of Theorem 2.21 (estimate (4.23)).** First, if all ends are subcritical, then we have by definition
\[
h_i(r) \simeq \frac{r^2}{V_i(r)}, \quad \bar{V}_i(r) \simeq r^2.
\]
Then the estimate in (4.24) implies that
\[
p(t, o, o) \leq \frac{\min_i \frac{t}{V_i(\sqrt{t})}}{\frac{t}{V_m(\sqrt{t})}} \approx \frac{1}{V(\sqrt{t})},
\]
(4.28)

where \(V(o, r)\) is the measure of the geodesic ball in \(M\) with radius \(r\) centered at \(o\).

If there exists at least one non-subcritical regular end, then by the assumption of (DOE), we have
\[
\min_i h_i^2(r) \simeq h_i^2(r), \quad \min_i \bar{V}_i(r) \simeq \bar{V}_i(r).
\]
Substituting this into the estimate in (4.24), it follows that for all \(t \gg 1\)
\[
p(t, o, o) \leq \frac{h_i^2(\sqrt{t})}{\bar{V}_i(\sqrt{t})} = \frac{1}{\bar{V}_i(\sqrt{t})},
\]
(4.29)

Because \(V(o, r) \simeq V_m(r) \simeq V_i(r)\) (see Lemma 5.2), we obtain, for all \(t \gg 1\)
\[
p(t, o, o) \leq \frac{1}{V(o, \sqrt{t})},
\]
which gives the same upper estimate as in (4.28).

For a bounded range of \(t\), this estimate is trivially satisfied, so that it holds for all \(t > 0\). Since \(V_m(r)\) is doubling, the volume function \(V(o, r)\) on \(M\) is also doubling. By [5], we conclude that \(p(t, o, o)\) satisfies also a matching lower bound
\[
p(t, o, o) \geq \frac{1}{V(o, \sqrt{t})} \approx \frac{1}{V_m(\sqrt{t})}.
\]
Hence, we obtain the estimate in (4.23), which concludes Theorem 2.21.

\[\square\]

Remark 4.13. If all ends \(M_i\) are either critical (that is, \(V(r) \simeq r^2\)) or subcritical, then (2.24) was already obtained in our previous paper [11, Theorem 2.1]. When all ends are subcritical, \(m\) may depend on \(r\) (see Theorem 4.2).

If (DOE) is not satisfied then the upper estimate in (4.29) might not be optimal. We expect to obtain such an example in the case \(M = M_1 \# M_2\) with \(V_i(r) = r^2 \varphi_i(r)\), where each \(\varphi_i\) is a slowly varying function and \(V_1, V_2\) are not properly ordered in any of the above sense; see [12] for an example of volume functions \(V_1, V_2\) behaving that way.

5 | ESTIMATE OF THE POINCARÉ CONSTANT

5.1 | Remarks on the conditions (COE), (DOE) and regularity

In the next two lemmas we collect some already known properties.
Lemma 5.1. Let $V$ be regular with parameters $\gamma_1, \gamma_2 > 0$ satisfying $2\gamma_1 + \gamma_2 < 2$. Then for all $r \geq 1$ we have

\begin{align}
r^2 - \gamma_2 &\lesssim V(r) \lesssim r^{2+\gamma_1}, \\
r^2 &\lesssim V(r) h(r) \lesssim r^{2+\gamma_1}, \\
v(r) &\lesssim \overline{v}(r) \lesssim r^{2+\gamma_1 + \gamma_2}.
\end{align}

Proof. The estimates (5.1) follow immediately from (2.22). The lower bound in (5.2) is equivalent to (2.7). The upper bound in (5.2) is equivalent to (4.12).

The lower bound in (5.3) is trivial because $h \geq 1$. The upper bound in (5.3) coincides with (4.10). □

Lemma 5.2. Let $M = \#_{i \in I} M_i$ be a manifold with nice ends. Assume that $M$ satisfies (COE) with parameters $\varepsilon, \delta, \gamma_1, \gamma_2$ (see Definition 2.10).

- In general, we have
  
  (a) if $i \in I_{\text{super}}$, then $M_i$ is non-parabolic and $h_i(r) \simeq 1$;
  
  (b) if $i \in I_{\text{super}}$, $j \in I_{\text{middle}}$ and $k \in I_{\text{sub}}$, then, for all $r \gg 1$,

  \[ V_i(r) \gtrsim V_j(r) \gtrsim V_k(r), \quad \text{and} \quad V_i(r) \gtrsim V_j(r) h_j(r) \gtrsim V_k(r) h_k(r) \simeq r^2 \tag{5.4} \]

  and

  \[ \overline{V}_j(r) \lesssim \overline{V}_k(r); \tag{5.5} \]

- if all ends are parabolic and $I_{\text{middle}} \neq \emptyset$ then the following properties hold:
  
  (c) $I_{\text{super}} = \emptyset$ and $M$ admits (DOE), introduced in Definition 2.19, that is, there exists a dominating volume function $V_l$ with $l \in I_{\text{middle}}$, which is a volume function such that for all $i \in I$

  \[ V_l \gtrsim V_i, \quad \overline{V}_l \lesssim \overline{V}_i; \tag{5.6} \]

  (d) for all $r \gg 1$,

  \[ V_l(r) \simeq V_m(r) \quad \text{and} \quad \min_i h_i(r) \simeq h_l(r) \simeq h_m(r), \]

  where $m = m(r)$ is the index of the largest end defined in (2.10);

  (e) for all $r \gg 1$,

  \[ \max_{j \neq l} \{ V_j(r) h_j(r) \} \simeq V_n(r) h_n(r), \]

  where $n = n(r)$ is the index of the second largest end defined in (2.11).

Proof. (a) For any $i \in I_{\text{super}}$ we have by definition $V_i(r) \gtrsim r^{2+\varepsilon}$ and, hence,

\[ 1 \leq h_i(r) \lesssim 1 + \int_1^r \frac{1}{s^{2+\varepsilon}} ds \lesssim 1. \]

This means non-parabolicity of $M_i$ because of (2.1).
(b) By (2.13) \( \varepsilon > \gamma_1 \) and \( \gamma_2 < \delta \), we obtain, by using Definition 2.10 that

\[
V_i(r) \gtrsim r^{2+\varepsilon} \gtrsim r^{2+\gamma_1} \gtrsim V_j(r) \gtrsim r^{2-\gamma_2} \gtrsim r^{2-\delta} \gtrsim V_k(r).
\]

By (5.2), subcriticality of \( V_k \) and \( \varepsilon > \gamma_1 \) we obtain

\[
V_k(r)h_k(r) \simeq r^2 \lesssim V_j(r)h_j(r) \lesssim r^{2+\gamma_1} \lesssim V_i(r),
\]

which proved (5.4).

Since \( V_k \) is subcritical, we have

\[
\mathcal{V}_k(r) \simeq \frac{r^4}{V_k(r)} \gtrsim \varepsilon r^{2+\delta}.
\]

Because \( \gamma_1 + \gamma_2 < \delta \), we obtain by (5.3)

\[
\mathcal{V}_j(r) \lesssim r^{2+\gamma_1+\gamma_2} \lesssim r^{2+\delta} \lesssim \mathcal{V}_k(r).
\]

Now we assume that all ends are parabolic and \( I_{\text{middle}} \neq \emptyset \).

(c) Subpart (a) implies \( I_{\text{super}} = \emptyset \) immediately. By the definition of \( I_{\text{middle}} \) and the estimates in (5.4), there exists an end \( i \in I_{\text{middle}} \) such that for all \( i \in I \)

\[
V_i \gtrsim V_i. \tag{5.7}
\]

By the estimates in (5.5), we obtain

\[
\mathcal{V}_i \lesssim \mathcal{V}_i,
\]

which concludes (DOE).

(d) \( V_i \gtrsim V_m \) and \( V_m(r) \gtrsim V_i(r) \) imply \( V_i(r) \simeq V_m(r) \) immediately. By the estimates in (5.7), we obtain

\[
\min_{i \in I} h_i \simeq h_i.
\]

Next, \( V_i \gtrsim V_m \) implies that \( h_i \lesssim h_m \) and, by the assumption of (COE)

\[
V_i h_i \gtrsim V_i h_i.
\]

Since \( V_i(r) \simeq V_m(r) \) at \( r \), we obtain

\[
h_i(r) \gtrsim h_m(r),
\]

which concludes \( h_i(r) \simeq h_m(r) \).

(e) If \( |I_{\text{middle}}| = 1 \), then for all \( r \gg 1 \), \( m(r) = l \) and the second largest end is subcritical. Hence we can prove (e) easily (see also Lemma 5.2).

If \( |I_{\text{middle}}| \geq 2 \), then for all \( r \gg 1 \), \( m \) and \( n \) are in \( I_{\text{middle}} \). Assume first that \( m = m(r) = l \). Let \( j \in I_{\text{middle}} \) be the index so that \( V_j(r)h_j(r) = \max_{i \in I \setminus \{l\}} \{V_i(r)h_i(r)\} \). By the condition of (COE),

Either \( V_n \gtrsim V_j \) or \( V_j \gtrsim V_n \) holds. If \( V_n \gtrsim V_j \), then \( V_n h_n \gtrsim V_j h_j \), which implies (e). If \( V_j \gtrsim V_n \), we note that \( V_n(r) \gtrsim V_j(r) \) by \( j \neq m = l \). Then same argument as in (a) concludes \( V_j(r) \simeq V_n(r) \) and \( h_j(r) \simeq h_n(r) \) which concludes (e).
Next, assume that \( m \neq l \). In this case, we have

\[
V_l \gtrsim V_m \quad \text{and} \quad V_l \gtrsim V_n,
\]

\[
V_m(r) \gtrsim V_n(r) \gtrsim V_l(r).
\]

Then we obtain \( V_l(r) \simeq V_m(r) \simeq V_n(r) \) and, by the same argument as in (d), we obtain

\[
h_l(r) \simeq h_m(r) \simeq h_n(r),
\]

when we conclude (e).

\[\square\]

5.2  Lower bound on Dirichlet heat kernel

Recall that \( M \) is a manifold with nice ends \( M_1, \ldots, M_k \). Moreover, assume (RCA) on each parabolic end. First, we prove the following technical lemmas.

**Lemma 5.3.** Let \( M \) be as above. Assume also that for all \( t > 0 \)

\[
p(t, o, o) \simeq \min_i h_i^2(\sqrt{t}) \min_i \tilde{V}_i(\sqrt{t}).
\]

(5.8)

Then we have for all \( r \gg 1 \)

\[
\int_1^r p(s, o, o)ds \simeq \min_i h_i(r).
\]

(5.9)

**Proof.** If \( M \) is non-parabolic, although the estimates in (5.8) holds (see Theorem 2.17), we can prove the estimate (5.9) without using (5.8). Indeed, by the definition of the non-parabolicity we have for \( r \gg 1 \)

\[
\int_1^r p(s, o, o)ds \simeq 1.
\]

Because \( h_i \simeq 1 \) for each non-parabolic end, we have

\[
\min_i h_i \simeq 1.
\]

Then we conclude (5.9).

Next, suppose that \( M \) is parabolic. By the assumption (5.8),

\[
\int_1^r p(s, o, o)ds \simeq \int_1^r \frac{\min_i h_i^2(s)}{\min_i \tilde{V}_i(s)} ds.
\]

Observe that

\[
h_i(r) = \frac{1}{\int_r^\infty \frac{sds}{\tilde{V}_i(s)}},
\]
which is obtained by integration of the identity
\[
\left( \frac{1}{h(r)} \right)' = - \frac{h'(r)}{h^2(r)} = - \frac{r/V(r)}{h^2(r)} = - \frac{r}{\tilde{V}_i(r)}
\]
and
\[
\frac{1}{h(r)} \to 0 \text{ as } r \to \infty
\]
by the parabolicity. Then we have
\[
\min_i h_i(r) = \frac{1}{\max_i \int_r^\infty \frac{ds}{V_i(s)}} \approx \frac{1}{\sum_i \int_r^\infty \frac{ds}{V_i(s)}} = \frac{1}{\int_r^\infty \frac{ds}{W(s)}} =: h(r),
\]
where
\[
W(r) = \min_i \tilde{V}_i(r).
\]
Because
\[
- \frac{h'(r)}{h^2(r)} = \left( \frac{1}{h(r)} \right)' = \frac{-r}{W(r)}
\]
and \(h\) is a doubling function, we conclude that for all \(r \geq 2\)
\[
\int_1^r \frac{h^2(s)}{h_i(s)} ds \approx \int_1^r \frac{h^2(s)}{W(s)} ds = \int_1^r h'(s) ds = h(r) - h(1) \approx h(r).
\]
\(\square\)

For \(i = 1, \ldots, k\), set
\[
A_i(r) = (B(o, r) \setminus B(o, r/2)) \cap E_i.
\]
(5.10)

For \(\kappa > 0\), \(\kappa B\) means \(B(o, \kappa r)\). Recall that \(p^D_{\kappa B}(t, x, y)\) is the Dirichlet heat kernel on \(\kappa B\).

The following estimates of \(p^D_{\kappa B}(t, x, y)\) has a key role for the estimate of the Poincaré constant:

**Lemma 5.4.** Assume that a connected sum \(M = M_1 \# \cdots \# M_k\) satisfies all the hypothesis of Lemma 5.3. Then there exists \(\kappa \geq 1\) such that for all \(r \gg 1\), for any \(x \in A_i(r), y \in B(o, r) \cap E_j\),
\[
p^D_{\kappa B}(r^2, x, y) \approx \frac{1}{V_i(r) h_i(r)} \left[ \delta_{ij} h_j(|y|) + \min_j \frac{h_j(r)}{h_i(r)} \left( \int_1^r \frac{ds}{V_j(s)} + \frac{r^2 h_j(|y|)}{V_j(r) h_j(r)} \right) \right].
\]
Moreover, this implies that
\[
\inf_{x \in A_i(r), y \in B} p^D_{\kappa B}(r^2, x, y) \approx \frac{r^2 \min_j h_j(r)}{V_i(r) h_i(r) \max_j \{ V_j(r) h_j(r) \}}.
\]
(5.11)
FIGURE 5 \((\kappa B)^c \cap E_\xi)^c\) (shadow area).

Proof. We prove the estimate in (5.11) only from below (upper bound follows from the trivial bound \(p^D_{\kappa B}(r^2, x, y) \leq p(r^2, x, y)\) and the estimate in (5.14)).

Recall that \(\tau_{\kappa B}\) is the first exit time from \(\kappa B\). By the definition of the Dirichlet heat kernel and the strong Markov property of the Brownian motion on \(M\) (see also [16]), we obtain

\[
p^D_{\kappa B}(r^2, x, y) = p(r^2, x, y) - \mathbb{E}_y(1_{\{\tau_{\kappa B} \leq r^2\}}p(r^2 - \tau_{\kappa B}, X_{\tau_{\kappa B}}, x)),
\]

where \(\kappa \geq 1\) will be chosen later. By the structure of the connected sum, we observe that

\[
\{\tau_{\kappa B} \leq r^2\} = \bigcup_{\xi=1}^k \{\tau_{\kappa B} \leq r^2, X_{\tau_{\kappa B}} \in E_\xi\} \subset \bigcup_{\xi=1}^k \{\tau((\kappa B)^c \cap E_\xi)^c \leq r^2\}
\]

(see Figure 5).

We note that

\[
\partial((\kappa B)^c \cap E_\xi)^c = (\partial \kappa B) \cap E_\xi.
\]

Then we obtain

\[
p^D_{\kappa B}(r^2, x, y) = p(r^2, x, y) - \sum_{\xi=1}^k \mathbb{E}_y\left(1_{\{\tau_{\kappa B} \leq r^2, X_{\tau_{\kappa B}} \in E_\xi\}}p(r^2 - \tau_{\kappa B}, X_{\tau_{\kappa B}}, x)\right)
\]

\[
\geq p(r^2, x, y) - \sum_{\xi=1}^k \mathbb{P}_y\left(\tau((\kappa B)^c \cap E_\xi)^c \leq r^2\right) \sup_{0<s\leq r^2, z \in (\partial \kappa B) \cap E_\xi} p(s, z, x).
\]  

(5.13)

We estimate \(p(r^2, x, y), \mathbb{P}_y(\tau((\kappa B)^c \cap E_\xi)^c \leq r^2)\) and \(\sup_{0<s\leq r^2, z \in (\partial \kappa B) \cap E_\xi} p(s, z, x)\) separately.
Step 1: Estimate of $p(r^2, x, y)$.
We will prove that for any $x \in A_i(r)$, $y \in B(o, r) \cap E_j$

$$p(r^2, x, y) \simeq \frac{1}{V_i(r)h_j(r)} \left[ \delta_{ij}h_j(|y|) + \frac{\min_{\eta} h_{\eta}(r)}{h_j(r)} \left( \int_{|y|}^{r} \frac{sds}{V_j(s)} + \frac{r^2h_j(|y|)}{V_j(r)h_j(r)} \right) \right],$$

which is nothing but the same estimate as in (5.11) for the full heat kernel $p(r^2, x, y)$ instead of $p^{D}_{xB}(r^2, x, y)$. Because $p^{D}_{xB}(r^2, x, y) \leq p(r^2, x, y)$ is always true, this estimate gives the upper bound in (5.11).

To prove (5.14), we use Theorem 4.3 and the location of $x, y$ to obtain

$$p(r^2, x, y) \simeq \delta_{ij} \frac{1}{V_i(r)h_j(r)} h_j(|y|) + p(r^2, o, o) \frac{1}{V_i(r)h_i(r)} \int_{|y|}^{r} \frac{sds}{V_j(s)} + \frac{r^2}{V_i(r)h_i(r)} \int_{|y|}^{r} \frac{sds}{V_j(s)} h_j(|y|) h_j(r) + \frac{\min_{\eta} h_{\eta}(r)}{h_j(r)} \int_{|y|}^{r} \frac{sds}{V_j(s)} h_j(|y|).$$

(5.15)

Because $p(t, o, o)$ is monotone decreasing, it is easy to see that

$$r^2p(r^2, o, o) \leq C \int_{1}^{r^2} p(s, o, o) ds,$$

which makes the third term of the right hand side in (5.15) dominate the second term. By using Lemma 5.3, we obtain (5.14).

Step 2: Estimate of $P_y(\tau_{((B)\cap \Omega)^c} \leq r^2)$.
We will prove that for any $y \in B(o, r) \cap E_j$

$$P_y(\tau_{((B)\cap \Omega)^c} \leq r^2) \simeq C \left[ \frac{1}{V_j(\sqrt{s})h_j(|y|)} + \frac{\min_{\eta} h_{\eta}(r)}{h_j(r)} \int_{|y|}^{r} \frac{sds}{V_j(s)} + \frac{r^2}{V_j(r)h_j(r)} \int_{|y|}^{r} \frac{sds}{V_j(s)} h_j(|y|) \right] e^{-b\frac{r^2}{s}}.$$

(5.17)

Since $y \in B(o, r) \cap E_j$, applying [14, Theorem 3.5], we obtain

$$P_y(\tau_{((B)\cap \Omega)^c} \leq r^2) \leq P_y(\tau_{(F^c)} \leq r^2) \leq 2\text{cap}(F, \Omega) \int_{1}^{r^2} \sup_{\omega \in \Omega \setminus F} p(s, \omega, y) ds,$$

where we chose $\kappa > 2$ and

$$F = \left\{ \omega \in E_\xi : \frac{3\kappa r}{4} \leq |\omega| \leq \frac{5\kappa r}{4} \right\}, \quad \Omega = \left\{ \omega \in E_\xi : \frac{\kappa r}{2} < |\omega| < \frac{3\kappa r}{2} \right\}.$$

Here we remark that $(\partial x B) \cap E_\xi \subset F \subset \Omega$ and $y \not\in \Omega$ because of $\kappa > 2$.

By the estimate in Theorem 4.3,

$$\sup_{\omega \in \Omega \setminus F} p(s, \omega, y) \simeq C \left[ \frac{1}{V_j(\sqrt{s})h_j(|y|)} + \frac{p(s, o, o)}{V_j(r)h_j(r)} P_y(\tau_K \leq s) + \frac{r^2}{V_j(r)h_j(r)} \int_{|y|}^{r} \frac{sds}{V_j(s)} h_j(|y|) \right] e^{-b\frac{r^2}{s}}.$$
Then we obtain
\[
\int_1^{r^2} \sup_{\omega \in \Omega \setminus F} p(s, \omega, y) ds 
\leq C \left[ \delta_{ij} \frac{r^2}{V_j(r)} h_j(r) + \int_1^{r^2} p(s, o, o) ds \frac{r^2}{V_j(r) h_j(r)} \mathbb{P}_j(\tau_K \leq r^2) \right] e^{-b \xi^2}.
\]

Since
\[
\text{cap}(F, \Omega) = \text{cap} \left( B_\xi \left( o, \frac{5r}{4} \right), B_\xi \left( o, \frac{3r}{4} \right) \right) + \text{cap} \left( B_\xi \left( o, \frac{5r}{4} \right), B_\xi \left( o, \frac{3r}{2} \right) \right),
\]
where \( B_\xi(x, r) \) is a geodesic open ball in \( M_\xi \). By using the general estimate for any \( 0 < r < R \)
\[
\text{cap}(B(o, r), B(o, R)) \leq 2 \int_r^R \frac{s ds}{\mu(B(o, s))}
\]
(see [9, Theorem 7.1; 13, (4.5)] for the detail), we obtain
\[
\text{cap}(F, \Omega) \leq C \frac{V_\xi(r)}{r^2}.
\]

By [14, Theorem 4.6],
\[
\mathbb{P}_j(\tau_K \leq r^2) \leq \frac{C}{h_j(r)} \int_{|y|}^r \frac{s ds}{V_j(s)}.
\]
Combining these estimate and Lemma 5.3, we conclude (5.17).

**Step 3: Estimate of \( \sup_{0 < s \leq r^2, z \in (\partial \Omega) \cap E_\xi} p(s, z, x) \).**

We will prove that for any \( x \in A_i(r) \)
\[
\sup_{0 < s \leq r^2, z \in (\partial \Omega) \cap E_\xi} p(s, z, x) \leq \frac{C}{\kappa^{2\alpha}} \left[ \delta_{ij} \frac{1}{V_i(r)} + \min_{\eta} h_\eta(r) \frac{r^2}{V_i(r) h_\eta(r) V_\xi(r) h_\xi(r)} \right].
\]
(5.18)

By the estimates in Theorem 4.3, for all \( 0 < s \leq r^2 \) and \( z \in (\partial \Omega) \cap E_\xi \)
\[
p(s, x, z) \propto C e^{b(4r^2)/s} \left[ \delta_{ij} \frac{1}{V_i(\sqrt{s})} + p(s, o, o) \frac{r^2}{V_i(r) h_i(r) V_\xi(r) h_\xi(r)} \right.
\]
\[
+ \int_1^s p(u, o, o) du \left( \frac{1}{V_i(\sqrt{s}) h_i(\sqrt{s})} \frac{r^2}{V_\xi(r) h_\xi(r)} + \frac{r^2}{V_i(r) h_i(r) V_\xi(\sqrt{s}) h_\xi(\sqrt{s})} \right) \right].
\]
Since \( V_i \) and \( \tilde{V}_i \) are doubling, there exist \( C, \alpha > 0 \) such that
\[
\frac{1}{V_i(\sqrt{s})} \leq C \left( \frac{r^2}{s} \right)^\alpha,
\]
\[ p(s, o, o) \simeq p(r^2, o, o) \]
\[ \leq p(r^2, o, o) \frac{\min_i \tilde{V}_i(r)}{\min_i \tilde{V}_i(\sqrt{s})} \leq C \left( \frac{r^2}{s} \right)^{\alpha}, \]
\[ \frac{1}{V_i(\sqrt{s})h_i(\sqrt{s})} = \frac{h_i(\sqrt{s})}{\tilde{V}_i(\sqrt{s})} \leq \frac{C}{V_i(r)h_i(r)} \left( \frac{r^2}{s} \right)^{\alpha}. \]

Because the inequality
\[ r^\alpha e^{-t} \leq \alpha^\alpha e^{-\alpha} \]
is always true for all \( t > 0 \), we have
\[ \left( \frac{r^2}{s} \right)^{\alpha} e^{-b^2 \frac{r^2}{s}} \leq \frac{1}{b^\alpha \alpha^{2\alpha}} \alpha^\alpha e^{-\alpha}. \]

Then we obtain
\[ \sup_{0 < s \leq r^2} p(s, z, x) \leq \frac{C}{\kappa^{2\alpha}} \left[ \delta_{i \xi} \frac{1}{V_i(r)} + p(r^2, o, o) \frac{r^2}{V_i(r)h_i(r)} \frac{r^2}{V_\xi(r)h_\xi(r)} \right. \]
\[ \left. + \int_1^{r^2} p(u, o, o)du \left( \frac{1}{V_i(r)h_i(r)} \frac{r^2}{V_\xi(r)h_\xi(r)} + \frac{r^2}{V_i(r)h_i(r)} \frac{1}{V_\xi(r)h_\xi(r)} \right) \right]. \]

Using (5.16) and Lemma 5.3 again, we conclude (5.18).

**Step 4: Proof of (5.11)**.
Combining (5.17) and (5.18), we obtain
\[ \sum_{\xi = 1}^k p_j(\tau_{(x_{\xi})} \cap E_\xi) \leq r^2 \sup_{0 < s \leq r^2} p(s, z, x) \]
\[ \leq Ce^{-b^2 \frac{r^2}{s}} \sum_{\xi = 1}^k \left( \delta_{i \xi} \frac{h_j(|y|)}{h_j(r)} + \min_{\eta} h_{\eta}(r) \frac{1}{h_j(r)} \int_{|y|}^{r} sds \right) \]
\[ \times \left( \delta_{\xi \beta} \frac{1}{V_i(r)} + \min_{\eta} h_{\eta}(r) \frac{r^2}{V_i(r)h_i(r)} \frac{1}{V_\xi(r)h_\xi(r)} \right) \]
\[ = Ce^{-b^2 \frac{r^2}{s}} \frac{1}{V_i(r)h_j(r)} \sum_{\xi = 1}^k \left( \delta_{i \xi} h_j(|y|) + \min_{\eta} h_{\eta}(r) \frac{r^2}{V_i(r)h_i(r)} \frac{1}{V_\xi(r)h_\xi(r)} \right) \]
\[ \times \left( \delta_{\xi \beta} \frac{1}{h_j(r)} \frac{r^2}{V_\xi(r)h_\xi(r)} \right). \]
\[
\frac{C e^{-b \kappa^2}}{\kappa^{2\alpha}} \frac{1}{V_i(r)h_j(r)} \left[ \delta_{ij} h_j(|y|) \right.
+ \left. \min_{\eta} \frac{h_{\eta}(r)}{h_i(r)} \left( \int_{|y|}^{r} \frac{sd\sigma}{V_j(s)} \left( 1 + \frac{r^2 \min_{\eta} h_{\eta}(r)}{\min_{\eta} \overline{V_j}(r)} \right) + \frac{r^2}{V_j(r)h_j(r)} h_j(|y|) \right) \right].
\]

By using Lemma 5.3 and a trivial lower bound \( V_{\eta}(r)h_{\eta}(r) \gtrsim r^2 \), we obtain

\[
r^2 \min_{\eta} h_{\eta}(r) \lesssim \min_{\eta} \overline{V}_{\eta}(r)
\]

which implies that

\[
1 + \frac{r^2 \min_{\eta} h_{\eta}(r)}{\min_{\eta} \overline{V}_{\eta}(r)} \approx 1.
\]

Then we have

\[
\sum_{\xi=1}^{k} P_y(\tau(xB) \cap E_\xi \subseteq r^2) \sup_{0 < s \leq r^2} p(s, z, x) \leq \frac{C e^{-b \kappa^2}}{\kappa^{2\alpha}} \frac{1}{V_i(r)h_j(r)} \left[ \delta_{ij} h_j(|y|) \right.
+ \left. \min_{\eta} \frac{h_{\eta}(r)}{h_i(r)} \left( \int_{|y|}^{r} \frac{sd\sigma}{V_j(s)} + \frac{r^2}{V_j(r)h_j(r)} h_j(|y|) \right) \right], \tag{5.19}
\]

which is same as (5.14) up to \( C e^{-b \kappa^2} / \kappa^{2\alpha} \). Choosing \( \kappa > 2 \) large enough and substituting (5.14) and (5.19) into (5.13), we conclude (5.11).

**Step 5: Proof of (5.12).**

If \( i = j \), then we obtain by the estimate in (5.11) and the fact that \( h_j(|y|) + \int_{|y|}^{r} \frac{sd\sigma}{V_j(s)} = h_j(r) \)

\[
p_{xB}(r^2, x, y) \geq c \frac{\min_{\eta} h_{\eta}(r)}{V_i(r)h_i(r)} h_i(r) = c \frac{\min_{\eta} h_{\eta}(r)}{V_i(r)h_i(r)}.
\tag{5.20}
\]

If \( i \neq j \), then the function

\[
F(t) = \int_{t}^{r} \frac{sd\sigma}{V_j(s)} + \frac{Cr^2}{V_j(r)h_j(r)} h_j(t)
\]

is monotone decreasing for some constant \( C > 0 \) and for all \( r \gg 1, t \approx r \) gives the lower bound of \( F \), namely we obtain

\[
F(r) \geq c \frac{r^2}{V_j(r)}.
\]

Then we obtain by the estimate in (5.11)

\[
p_{xB}(r^2, x, y) \geq c \frac{\min_{\eta} h_{\eta}(r)}{V_i(r)h_i(r)} \frac{r^2}{V_j(r)}
\]

\[
= c \frac{\min_{\eta} h_{\eta}(r)}{V_i(r)h_i(r)} \frac{r^2}{V_j(r)h_j(r)}.
\tag{5.21}
\]
Because $V_j(r)h_j(r) \geq cr^2$ is always true, combining estimates in (5.20) and (5.21), we conclude the estimate in (5.12).

5.3 | Upper bound on Poincaré constant

We prove here the upper bounds of the Poincaré constant: (2.12) of Theorem 2.9 and (2.18) of Theorem 2.12 with some large $\kappa \geq 1$. The following lemma inspired by Kusuoka–Stroock [18, Theorem 5.10] (see also [22, 5.5.1]) plays a key role to obtain an upper bound of the Poincaré constant from a lower bound of the Dirichlet heat kernel.

**Lemma 5.5.** Let $(M, \mu)$ be a weighted manifold. Let $U \subset U' \subset M$ be precompact connected open sets. Then for any $t > 0$ and any open set $A \subset U'$,

$$\Lambda(U, U') \leq \frac{2t}{\mu(A) \inf_{x \in A, y \in U} p_{U'}^D(t, x, y)}.$$ 

**Proof.** Let $\{P_t^{N,U'}\}$ be the Neumann heat semigroup on $U'$ and $p_t^{N,U'}(t, x, y)$ its kernel function (Neumann heat kernel). By a well-known fact for Dirichlet forms, for any $f \in C^1(U')$, and for any $t > 0$,

$$\int_{U'} |\nabla f|^2 d\mu = \mathcal{E}(f, f)_{U'} \geq \left( f - \frac{P_{2t}^{N,U'} f}{2t}, f \right)_{L^2(U')}.$$ 

Then we obtain for any open set $A \subset U'$

$$2t \int_{U'} |\nabla f|^2 d\mu \geq \left( f - \frac{P_{2t}^{N,U'} f}{2t}, f \right)_{L^2(U')} \geq \int_{A} P_t^{N,U'} \left[ f - P_t^{N,U'} f(x) \right]^2 (x) d\mu(x) \geq \mu(A) \inf_{x \in A} P_t^{N,U'} \left[ f - P_t^{N,U'} f(x) \right]^2 (x).$$ (5.22)

Because the Neumann heat kernel is always larger than the Dirichlet heat kernel, we obtain for all $x \in A$

$$p_t^{N,U'} \left[ f - P_t^{N,U'} f(x) \right]^2 (x) = \int_{U'} P_t^{N,U'}(t, x, y) \left[ f(y) - P_t^{N,U'} f(x) \right]^2 d\mu(y) \geq \int_U P_t^{D}(t, x, y) \left[ f(y) - P_t^{N,U'} f(x) \right]^2 d\mu(y) \geq \inf_{y \in U} P_t^{D}(t, x, y) \int_U |f(y) - P_t^{N,U'} f(x)|^2 d\mu(y) \geq \inf_{y \in U} P_t^{D}(t, x, y) \int_U |f(y) - f(x)|^2 d\mu(y).$$ (5.23)
Here the final line follows from the fact that $f_U$ minimizes the function $\xi \mapsto \int_U |f - \xi|^2\,d\mu$. Combining the estimates in (5.22) and (5.23), we complete the proof.

Proof of Theorems 2.9 and 2.12. Now we start to prove the upper bound of the Poincaré constant in the main theorems. For the lower bound, see section 5.4.

Recall that $M$ is the manifold with nice ends $M_1, \ldots, M_k$, where each parabolic end satisfies (RCA). If there is at least one parabolic end, we assume also that $M$ admits (COE) defined in Definition 2.10. By the results in Theorems 2.17 and 2.21, $M$ satisfies the estimate in (5.8) which allows us to use Lemma 5.4. Let $\kappa > 1$ be the constant from (5.11). Apply Lemma 5.5 to the case of $U = B = B(o, r), U' = \kappa B = B(o, \kappa r), t = r^2$ and $A = A_i(r)$ given in (5.10). Then we obtain

$$\Lambda(B, \kappa B) \leq \frac{2r^2}{\mu(A_i(r)) \inf_{x \in A_i(r), y \in B} P^D_{xB}(r^2, x, y)}.$$ 

Since $\mu(A_i(r)) \simeq V_i(r)$ by (VD) on $M_i$, substituting the estimates in (5.12) to the above, we obtain for all $i = 1, \ldots, k$

$$\Lambda(B, \kappa B) \lesssim \frac{h_i(r)}{\min \eta \, h_\eta(r)} \max_{j \neq i} \{V_j(r)h_j(r)\}. \quad (5.24)$$

If all ends are non-parabolic, then $h_i(r) \simeq 1$ for all $i = 1, \ldots, k$. By taking $i = m = m(r)$, the index of the largest end at $r$, we obtain

$$\Lambda(B, \kappa B) \lesssim \max_{j \neq m} V_j(r) \simeq V_n(r).$$

Hence, the estimate in (2.12) holds for some $\kappa \geq 1$.

Next, let us assume that $M = \#_{i \in I} M_i$ is a manifold with (COE). Assume first that $I_{\text{super}} \neq \emptyset$. Then the index of the largest end $m$ is in $I_{\text{super}}$ and $\min \eta \, h_\eta(r) \simeq 1$. Taking $i = m$, we obtain

$$\Lambda(B, \kappa B) \lesssim \max_{j \neq m} \{V_j(r)h_j(r)\}.$$ 

By the second condition in (5.4) and the definition of (COE), we conclude

$$\max_{j \neq m} \{V_j(r)h_j(r)\} \simeq V_n(r)h_n(r),$$

which gives the estimate in (2.18) for some $\kappa \geq 1$.

If $I_{\text{super}} = \emptyset$ and $I_{\text{middle}} \neq \emptyset$ then, by Lemma 5.2 there exists a dominating volume function $V_l$ satisfying (5.6) with $l \in I_{\text{middle}}$ and $\min \eta \, h_\eta(r) \simeq h_l(r)$. Estimate (5.24) with $i = l$ implies that

$$\Lambda(B, \kappa B) \lesssim \max_{j \neq l} \{V_j(r)h_j(r)\} \simeq V_n(r)h_n(r),$$

which gives the estimate in (2.18) for some $\kappa \geq 1$.

Finally, assume that $I_{\text{super}} = I_{\text{middle}} = \emptyset$, that is, all ends are subcritical. By the definition of subcriticality given in Definition 2.10(b),

$$\min \eta \, h_\eta(r) \lesssim \min \eta \frac{r^2}{V_\eta(r)} = \frac{r^2}{V_m(r)}.$$
Taking $i = m$ in the estimate in (5.24), we obtain
\[
\Lambda(B, xB) \lesssim \max_{j \neq m} \{V_j(r)h_j(r)\} \simeq r^2,
\]
which completes the proof of the estimate in (2.18) with $\kappa \geq 1$.

By using an additional argument in Section A, we will show that we can reduce $\kappa > 1$ to $\kappa = 1$.

\[\square\]

5.4 Lower bound on the Poincaré constant

In this section we prove the matching lower bound for the Poincaré constant under a different hypothesis.

**Theorem 5.6.** Let $M = M_1 \# \cdots \# M_k$ be a manifolds with ends such that each end $M_i$ satisfies (VD) and
\[
CV_i(r) \geq rV'_i(r) \quad \text{for all } r \geq r_0.
\]

Then for all large $r \gg 1$
\[
\Lambda(B(o, r)) \geq c \max_{i \neq m} \{V_i(r)h_i(r)\},
\]
where $m = m(r)$ is the index of the largest end at scale $r$, namely for all $i \in I$
\[
V_m(r) \geq V_i(r).
\]

Let $n = n(r)$ be the index of the second largest end. Then the estimate in (5.26) implies that
\[
\Lambda(B(o, r)) \geq cV_n(r)h_n(r).
\]

In view of the upper estimate of the Poincaré constant in Theorems 2.9 and 2.12, the lower bound in (5.28) is optimal if either all ends are non-parabolic, or $M$ is a manifold with (COE).

**Proof.** Set $B = B(o, r)$. First we show that for any $\varepsilon \in (0, 1)$,
\[
\varepsilon \sup_{f \in C^1(B)} \frac{\int_B |f|^2 d\mu}{\mu(B | \{f = 0\} \geq \varepsilon \mu(B)} \leq \Lambda(B, B).
\]

Indeed, for any $f \in C^1(B)$ with $\mu(B \cap \{f = 0\}) \geq \varepsilon \mu(B)$,
\[
\int_B |f - f_B|^2 d\mu = \int_B |f|^2 d\mu - \mu(B)f_B^2
= \int_B |f|^2 d\mu - \frac{1}{\mu(B)} \left( \int_{B \cap \{f \neq 0\}} f d\mu \right)^2.
\]
Cauchy–Schwarz inequality implies that
\[
\int_B |f - f_B|^2 d\mu \geq \int_B |f|^2 d\mu - \frac{\mu(B \cap \{f \neq 0\})}{\mu(B)} \int_B |f|^2 d\mu
\]
\[
= \frac{\mu(B \cap \{f = 0\})}{\mu(B)} \int_B |f|^2 d\mu \geq \varepsilon \int_B |f|^2 d\mu.
\]
By the expression of the Poincaré constant (2.8), we obtain the estimate in (5.29).

Now we prove the estimate in (5.26) by choosing a test function. For \(i \neq m\), take a \(C^1\)-function \(f_i\) defined by
\[
f_i(x) = \begin{cases} 
    h_i(|x|) & x \in B \cap E_i, \\
    0 & x \in B \cap E_j, j \neq i.
\end{cases}
\]
We note that the assumption (5.27) implies that
\[
\mu(B \cap E_m) \geq \frac{1}{2} \mu(B \cap E_j)
\]
for large \(r \gg 1\) and all \(j \in \{1, \ldots, k\}\). Because \(f_i = 0\) in \(B \cap E_m\), we observe that
\[
\mu(B \cap \{f_i = 0\}) \geq \mu(B \cap E_m)
\]
\[
= \frac{1}{2k} (2k \mu(B \cap E_m) + \mu(K)) - \frac{1}{2k} \mu(K)
\]
\[
\geq \frac{1}{2k} (\mu(B) - \mu(K)).
\]
Then for large \(r \gg 1\) so that \(\mu(B) \geq 2\mu(K)\), we obtain
\[
\mu(B \cap \{f = 0\}) \geq \frac{1}{4k} \mu(B).
\]
Hence, (5.29) implies that
\[
\frac{1}{4k} \int_B |f_i|^2 d\mu \leq \Lambda(B, B).
\]
By using the assumptions (VD) and (5.25), we obtain for large \(r \gg 1\)
\[
\int_B |f_i|^2 d\mu \geq \int_{(B \setminus \frac{1}{2}B) \cap E_i} |f_i|^2 d\mu
\]
\[
\geq \left( \mu(B \cap E_i) - \mu\left(\frac{1}{2}B \cap E_i\right) \right) \inf_{(B \setminus \frac{1}{2}B) \cap E_i} |f_i|^2
\]
\[
\geq cV_i(r)h_i^2(r),
\]
\[
\int_B |\nabla f_i|^2 d\mu = \int_{B(0,r_0)} |\nabla f_i|^2 d\mu + \int_{r_0}^r |\partial_s h_i(s)|^2 V'(s)ds
\]
\[
= C + \int_{r_0}^r \frac{sV'(s)}{V(s)} ds \leq C + Ch_i(r)
\]
\[
\leq C'h_i(r).
\]
Then we obtain for any \( i \neq m \),

\[
eq cV_i(r)h_i(r) \leq \Lambda(B)
\]

which concludes (5.26).

**APPENDIX: SPECTRAL GAP ON CENTRAL BALLS**

Recall that \( M \) is a connected sum of manifolds \( M_1, \ldots, M_k \) with a central part \( K \). The purpose of this section is to obtain a general upper bound of the Poincaré constant \( \Lambda(B(o, r)) = \Lambda(B(o, r), B(o, r)) \) at the central reference point \( o \in K \) for any large \( r > 0 \) by using a collection of \( \Lambda(B(x, s), B(x, ks)) \), where \( B(x, s) \subset B(o, r) \) and \( k \geq 1 \). This is an important procedure to obtain a lower bound of the spectral gap \( \lambda(B(o, r)) \) of \(-\Delta\) on \( B(o, r) \) defined in (2.9).

First of all, we only assume the volume doubling (VD) condition on each \( M_i \).

The main tool to obtain a desired bound is a Whitney covering \( \mathcal{W} \) of \( B(o, r) \) which is a collection of balls defined as the following.

**Definition A.1.** For any \( 0 < \eta < 1 \), a collection of balls \( \mathcal{W} = \mathcal{W}(\eta) \) in \( B(o, r) \) is called a Whitney covering of \( B(o, r) \) with parameter \( \eta \) if

1. (W1) all \( F \in \mathcal{W} \) are disjoint;
2. (W2) \( \bigcup_{F \in \mathcal{W}} 3F = B(o, r) \);
3. (W3) \( r(F) = \eta d(F, B(o, r)^c) \); here \( r(F) \) is the radius of \( F \) and \( B(o, r)^c \) is the complement of \( B(o, r) \); and
4. (W4) for any \( \alpha \geq 1 \), there exists \( N = N(\eta, \alpha) \) independent of \( r \) such that for any \( x \in B(o, r) \)

\[
\#\{F \in \mathcal{W} : x \in \alpha F\} \leq N.
\]

It is a well-known fact that there exists such a covering. For \( F \in \mathcal{W} \), we denote by \( \gamma_F \) a distance-minimizing curve joining the center of \( F \) and \( o \). Let \( F(F) = (F_0, F_1, \ldots, F_{l(F)}) \) be a string of \( F \), that is, a sequence of balls in \( \mathcal{W} \) so that \( 3F_j \cap 3F_{j+1} \neq \emptyset \) \( (j = 0, 1, \ldots, l(F) - 1), o \in 3F_0 \) and \( F_{l(F)} = F \). It is known that there exists such a string \( F(F) \) for any \( F \in \mathcal{W} \); see [22, Section 5.3.3] for details.

**Proposition A.2** (c.f. [22, Section 5.3.3]). Let \( \mathcal{W} \) be a Whitney covering of \( B(o, r) \) with parameter \( 0 < \eta < 1/4 \). Then \( \mathcal{W} \) satisfies the following.

1. (PW1) \([22, \text{Lemma 5.3.6}]\): For any \( F, F' \in \mathcal{W} \) so that \( 3F' \cap \gamma_F \neq \emptyset \),

\[
r(F') \geq \frac{1}{4\eta + 1}r(F).
\]

2. (PW2) \([22, \text{Lemma 5.3.7}]\): For \( F \in \mathcal{W} \), let \( F(F) = (F_0, F_1, \ldots, F_{l(F)}) \) be a string of \( F \). Then for any \( j = 0, 1, \ldots, l(F) - 1 \),

\[
(\eta^{-1} - 4)r(F_j) \leq (4 + \eta^{-1})r(F_{j+1})
\]

\[
(\eta^{-1} - 4)r(F_{j+1}) \leq (4 + \eta^{-1})r(F_j),
\]
and

\[ 3F_{j+1} \subset \left( \frac{6^{\eta^{-1}} + 4}{\eta^{-1} - 4} + 3 \right) F_j. \] (A.1)

We note that \( \frac{6^{\eta^{-1}} + 4}{\eta^{-1} - 4} + 3 < 12 \) if \( \eta < 1/20 \). Thus, we always assume that \( \eta < 1/20 \) in the sequel.

**Lemma A.3.** For any \( x \geq 1 \) and any \( 0 < \eta < 1/20 \), let \( B_0 = B(o, 12x\eta r) \). Then for any \( r \) large enough so that \( \text{diam} K \leq \eta r \) and for any \( F \in \mathcal{W} \) so that

\[ F \cap B_0 = \emptyset, \] (A.2)

we have

\[ 12xF \cap K = \emptyset. \]

**Proof.** Since \( r(F) \leq \eta r \) by (W3), the condition (A.2) implies that

\[
d(o, o(F)) \geq r(F) + r(B_0) = r(F) + 12x\eta r
= r(F) + (12x - 1)\eta r + \eta
\geq 12xr(F) + \text{diam} K,
\]

which concludes the lemma.

In the sequel, we always assume that \( r \) is large enough so that

\[ \text{diam} K \leq \eta r. \] (A.3)

**Lemma A.4.** For \( F \in \mathcal{W} \), if \( 3F \not\subset 3B_0 \), then \( F \cap B_0 = \emptyset \).

**Proof.** Suppose that \( F \cap B_0 \neq \emptyset \). Then for any \( x \in 3F \) and \( z \in F \cap B_0 \),

\[
d(o, x) \leq d(o, z) + d(z, x)
\leq r(B_0) + 4r(F)
\leq 12x\eta r + 4\eta r \leq 3r(B_0).
\]

This implies that \( 3F \subset 3B_0 \). By contraposition, we conclude the lemma.

**Lemma A.5.** For \( F \in \mathcal{W} \), if \( 3F \cap 3B_0 \neq \emptyset \), then

\[ 3F \subset B_1 = B(o, (36x + 6)\eta r). \]

**Proof.** For \( x \in 3F \) and \( z \in 3F \cap 3B_0 \), we obtain

\[
d(o, x) \leq d(o, z) + d(z, x)
\leq 36x\eta r + 6r(F)
\leq (36x + 6)\eta r,
\]

which concludes the lemma.
Now we modify the Whitney covering $\mathcal{W} = \mathcal{W}(\eta)$ to fit the manifolds with ends in concern. Set

$$\mathcal{W}' = \{ F \in \mathcal{W} : 3F \not\subset 3B_0 \}. $$

Here we note that

$$B(o, r) = \cup_{F \in \mathcal{W}'} 3F \cup 3B_0. \quad (A.4)$$

Moreover, for $i = 1, \ldots, k$, set $\mathcal{W}_i' = \{ F \in \mathcal{W}' : F \subset E_i \}$. We can easily see that they are disjoint each other by the definition, and, Lemmas A.3 and A.4 imply that

$$\mathcal{W}' = \cup_{i=1}^k \mathcal{W}_i'. $$

For $F \in \mathcal{W}_i'$, we retake a string $P(F) = (F_0, F_1, \ldots, F_{\ell'(F)})$ given in (PW2) so that $F_j \subset E_i$ ($j \neq 0$) and $F_0 = B_0$.

The following inclusion conditions play a key role in the proof of Theorem A.7.

**Lemma A.6.** For a constant $\kappa \geq 1$, let $\eta > 0$ be a parameter satisfying

$$\eta < \frac{1}{\kappa(36\kappa + 6)}. \quad (A.5)$$

For such $\eta > 0$, let $r > 0$ be a number so that (A.3) holds. Then we have

$$3\kappa B_0 \subset \kappa B_1 \subset B(o, r), \quad (A.6)$$

$$12\kappa F \subset B(o, r) \cap E_i \quad (F \in \mathcal{W}_i'). \quad (A.7)$$

The main result of this section is the following.

**Theorem A.7.** Let $M$ be a connected sum of manifolds $M_1, \ldots, M_k$ with a compact central part $K$, where each $M_i$ admits (VD). Let $o \in K$ be a central reference point. Fix $\kappa \geq 1$. Fix $\eta > 0$ satisfying (A.5). For any $r > 0$ satisfying $\text{diam}K \leq \eta r$, there exists a constant $C > 0$ such that

$$\Lambda(B(o, r)) \leq C \left( \Lambda(3B_0, 3\kappa B_0) + \Lambda(B_1, \kappa B_1) + \max_{F \in \mathcal{W}'(\eta)} \{ \Lambda(3F, 3\kappa F), \Lambda(12F, 12\kappa F) \} \right), $$

where $B_0 = B(o, 12\kappa \eta r)$ and $B_1 = B(o, (36\kappa + 6)\eta r)$.

We note that by (A.7), if $F \in \mathcal{W}_i'$, then $12\kappa F \subset E_i$. Hence, the Poincaré constants $\Lambda(3F, 3\kappa F)$ and $\Lambda(12F, 12\kappa F)$ for $F \in \mathcal{W}_i'$ can be computed by using a local information of $E_i$.

**Proof.** By using (A.4), we have

$$\int_{B(o, r)} |f - f_{B(o, r)}|^2 d\mu \leq \int_{B(o, r)} |f - f_{3B_0}|^2 d\mu$$

$$\leq \int_{3B_0} |f - f_{3B_0}|^2 d\mu + \sum_{F \in \mathcal{W}} \int_{3F} |f - f_{3B_0}|^2 d\mu.$$
\[ \leq \int_{3B_0} |f - f_{3B_0}|^2 d\mu + 4 \sum_{F \in \mathcal{W}'} \int_{3F} |f - f_{3F}|^2 d\mu \]
\[ + 4 \sum_{i=1}^k \sum_{F \in \mathcal{W}_i'} \int_{3F} |f_{3F} - f_{3F}|^2 d\mu + 4 \sum_{i=1}^k \sum_{F \in \mathcal{W}_i'} \int_{3F} |f_{3F} - f_{3B_0}|^2 d\mu \]
\[ = I + 4II + 4 \sum_{i=1}^k III_i + 4 \sum_{i=1}^k IV_i, \quad (A.8) \]

where \( F_1 \) is an element of the string \( F(F) \). We estimate terms \( I, II, III_i \) and \( IV_i \) in \( (A.8) \) separately.

**Estimate of \( I \)**

By using \((A.6)\), we obtain
\[ I = \int_{3B_0} |f - f_{3B_0}|^2 d\mu \leq \Lambda(3B_0, 3\kappa B_0) \int_{3\kappa B_0} |\nabla f|^2 d\mu \]
\[ \leq \Lambda(3B_0, 3\kappa B_0) \int_{B(o,r)} |\nabla f|^2 d\mu, \]
which gives a desired bound.

**Estimate of \( II \)**

By using \((W2)\) and \((W4)\), we obtain
\[ II = \sum_{F \in \mathcal{W}'} \int_{3F} |f - f_{3F}|^2 d\mu \leq \sum_{F \in \mathcal{W}'} \Lambda(3F, 3\kappa F) \int_{3\kappa F} |\nabla f|^2 d\mu \]
\[ \leq \max_{F \in \mathcal{W}'} \Lambda(3F, 3\kappa F) \sum_{F \in \mathcal{W}'} \int_{3\kappa F} |\nabla f|^2 d\mu \]
\[ \leq N(\eta, 3\kappa) \max_{F \in \mathcal{W}'} \Lambda(3F, 3\kappa F) \int_{B(o,r)} |\nabla f|^2 d\mu, \]
which gives a desired bound.

**Estimate of \( III_i \)**

For \( F \in \mathcal{W}_i' \), let \( F_j, F_{j+1} \in F(F) \) \((j \neq 0)\). Then the Cauchy–Schwartz inequality implies that
\[ |f_{3F_{j+1}} - f_{3F_j}| \leq \frac{1}{\mu(3F_{j+1})^{1/2}} \left( \int_{3F_{j+1} \times 3F_j} |f(x) - f(y)|^2 d\mu(x)d\mu(y) \right)^{1/2}. \]
We note that for any open set \( D \subset M \),
\[ \int_{D \times D} |f(x) - f(y)|^2 d\mu(x)d\mu(y) = 2\mu(D) \int_D |f - f_D|^2 d\mu. \quad (A.9) \]
Moreover, by using \((A.1)\), \((A.9)\), and the VD property of \( M_i \), we have
\[ |f_{3F_{j+1}} - f_{3F_j}| \leq \left( \frac{C}{\mu(F_j)} \int_{12F_j} |f - f_{12F_j}|^2 d\mu \right)^{1/2}. \]
Then we obtain

\[
|f_{3F} - f_{3F_1}| \leq \sum_{j=1}^{l'(F)-1} |f_{3F_{j+1}} - f_{3F_j}|
\]

\[
\leq \sum_{j=1}^{l'(F)-1} \left( \frac{C}{\mu(F_j)} \int_{12F_j} |f - f_{12F_j}|^2 d\mu \right)^{1/2}
\]

\[
\leq \sum_{j=1}^{l'(F)-1} \left( \frac{C\Lambda(12F_j, 12xF_j)}{\mu(F_j)} \int_{12xF_j} |\nabla f|^2 d\mu \right)^{1/2}.
\]

By using [22, Lemma 5.3.8], for any \(F_j \in \mathcal{P}(F)\) with \(j \neq 0\),

\[
F \subset AF_j \cap E_i,
\]

where \(A = 8 + 4\eta + \eta^{-1}\). Then we obtain

\[
III_i = \sum_{F \in \mathcal{W}'_i} \int_{3F} |f_{3F} - f_{3F_1}|^2 = \int_{B \cap E_i} \sum_{F \in \mathcal{W}'_i} \chi_{3F}(x) |f_{3F} - f_{3F_1}|^2 d\mu(x)
\]

\[
\leq C \int_{B \cap E_i} \sum_{F \in \mathcal{W}'_i} \chi_{3F}(x) \left( \sum_{j=1}^{l'(F)-1} \left( \frac{\Lambda(12F_j, 12xF_j)}{\mu(F_j)} \int_{12xF_j} |\nabla f|^2 d\mu \right)^{1/2} \right)^2 d\mu(x)
\]

\[
= C \int_{B \cap E_i} \sum_{F \in \mathcal{W}'_i} \chi_{3F}(x) \left( \sum_{j=1}^{l'(F)-1} \left( \frac{\Lambda(12F_j, 12xF_j)}{\mu(F_j)} \int_{12xF_j} |\nabla f|^2 d\mu \right)^{1/2} \chi_{AF_j \cap E_i}(x) \right)^2 d\mu(x)
\]

\[
\leq C \int_{B \cap E_i} \sum_{F \in \mathcal{W}'_i} \chi_{3F}(x) \left( \sum_{G \in \mathcal{W}'_i} \left( \frac{\Lambda(12G, 12xG)}{\mu(G)} \int_{12xG} |\nabla f|^2 d\mu \right)^{1/2} \chi_{AG \cap E_i}(x) \right)^2 d\mu(x)
\]

\[
\leq CN(\eta, 3) \max_{G \in \mathcal{W}'_i} \Lambda(12G, 12xG) \int_{B \cap E_i} \left( \sum_{G \in \mathcal{W}'_i} \left( \frac{1}{\mu(G)} \int_{12xG} |\nabla f|^2 d\mu \right)^{1/2} \chi_{AG \cap E_i}(x) \right)^2 d\mu(x)
\]

\[
= CN(\eta, 3) \max_{G \in \mathcal{W}'_i} \Lambda(12G, 12xG) \left\| \sum_{G \in \mathcal{W}'_i} a_G \chi_{AG \cap E_i} \right\|^2_{L^2(E_i)},
\]

where \(\chi_D\) is the characteristic function of a set \(D\) and

\[
a_G = \left( \frac{1}{\mu(G)} \int_{12xG} |\nabla f|^2 d\mu \right)^{1/2}.
\]
Since $E_i$ satisfies (VD), applying Lemma [22, 5.3.12], there exists a constant $C = C(A) > 0$ such that

$$
\left\| \sum_{G \in \mathcal{W}'_i} a_G \chi_{G \cap E_i} \right\|_{L^2(E_i)} \leq C \left\| \sum_{G \in \mathcal{W}'_i} a_G \chi_G \right\|_{L^2(E_i)}.
$$

By using (W1), (W4) and (A.7), we obtain

$$
III_i \leq C' N(\eta, 3) \max_{F \in \mathcal{W}'_i} \Lambda(12F, 12xF) \int_{B \cap E_i} \left( \sum_{F \in \mathcal{W}'_i} a_F \chi_F(x) \right)^2 d\mu(x)
$$

$$
= C' N(\eta, 3) \max_{F \in \mathcal{W}'_i} \Lambda(12F, 12xF) \int_{B \cap E_i} \frac{1}{\mu(F)} \int_{12xF} |\nabla f|^2 d\mu \chi_F(x) d\mu(x)
$$

$$
= C' N(\eta, 3) \max_{F \in \mathcal{W}'_i} \Lambda(12F, 12xF) \sum_{F \in \mathcal{W}'_i} \int_{12xF} |\nabla f|^2 d\mu
$$

$$
\leq C' N(\eta, 3) N(\eta, 12x) \max_{F \in \mathcal{W}'_i} \Lambda(12F, 12xF) \int_B |\nabla f|^2 d\mu,
$$

which gives a desired bound.

**Estimate of $IV_i$**

Because $3F_1, 3B_0 \subset B_1$, by using the same argument as in the estimate of $III_i$, we obtain

$$
IV_i = \sum_{F \in \mathcal{W}'_i} \int_{3F} |f_{3F_1} - f_{3B_0}|^2 d\mu
$$

$$
\leq \sum_{F \in \mathcal{W}'_i} \frac{\mu(3F)}{\mu(3F_1) \mu(3B_0)} \int_{B_1 \times B_1} |f(x) - f(y)|^2 d\mu(x) d\mu(y)
$$

$$
= 2 \left( \sum_{F \in \mathcal{W}'_i} \frac{\mu(3F)}{\mu(3F_1)} \right) \frac{\mu(B_1)}{\mu(3B_0)} \int_{B_1} |f - f_{B_1}|^2 d\mu.
$$

Here we recall that the central ball satisfies (VD). Indeed,

$$
\mu(B(o, s)) \simeq \mu(B_1(o_1, s)) + \mu(B_2(o_2, s)) + \ldots + \mu(B_k(o_k, s))
$$

$$
\simeq \max_i \mu(B_i(o_i, s)), \quad (A.10)
$$

where $B_i(o_i, s)$ is the geodesic ball in $M_i$ centered at $o_i \in K_i$. Since each $B_i(o_i, s)$ satisfies (VD), so does $B(o, s)$ by (A.10). Hence $\mu(B_1)/\mu(3B_0)$ is bounded in $r$. Now we estimate $\mu(3F_1)$ from below. By using (W3), Lemma A.5 and (A.6), for any $F \in \mathcal{W}'_i$

$$
r(F_1) = \eta d(F_1, B(o, r)^c)
$$

$$
\geq \eta d(B_1, B(o, r)^c) = \eta(1 - (36x + 6)\eta)r,
$$
which implies that

\[ A' r(F_1) \geq 2r, \]

where \( A' = \frac{2}{\eta(1-(36\alpha+6)\eta)} \). Then we obtain

\[ A' F_1 \supset B(o,r). \]

Since each end \( E_i \) is doubling, we obtain

\[ \mu(3F_1) \geq \mu(A'F_1 \cap E_i) \geq \mu(B(o,r) \cap E_i). \]

By using (W2), we obtain for all \( i = 1, \ldots, k \)

\[ \sum_{F \in W'_i} \frac{\mu(3F)}{\mu(3F_1)} \leq \frac{C}{\mu(B(o,r) \cap E_i)} \sum_{F \in W'_i} \mu(3F) \leq C. \]

Combining the above estimates, we obtain

\[ IV_i \leq C' \int_{B_1} |f - f_{B_1}|^2 d\mu \]

\[ \leq C' \Lambda(B_1, \kappa B_1) \int_{\kappa B_1} |\nabla f|^2 d\mu. \]

Since \( \kappa B_1 \subset B(o,r) \) by (A.6), we conclude a desired estimate of \( IV_i \).

We use the following corollary to show the upper bound of the Poincaré constant (that is, lower bound of the spectral gap) of central balls in Theorems 2.9 and 2.12.

**Corollary A.8.** Let \( M \) be a manifold with nice ends \( M_1, \ldots, M_k \), where each parabolic end satisfies (RCA). If there is at least one parabolic end, assume that \( M \) satisfies (COE). Then for any large \( r > 0 \)

\[ \Lambda(B(o,r)) \lesssim V_n(r) h_n(r), \]

that is,

\[ \lambda(B(o,r)) \gtrsim \frac{1}{V_n(r) h_n(r)}, \]

where \( n \) is the index of the second largest end defined in (2.11).

**Proof.** Because \( 3xF, 12xF \subset E_i \) for all \( F \in W'_i \) by (A.7), the assumption of (VD) and (PI) on each \( M_i \) implies that for all \( F \in W' \)

\[ \Lambda(3F, 3xF), \Lambda(12F, 12xF) \lesssim r^2. \]

Using the estimates in section 5.3, we obtain

\[ \Lambda(3B_0, 3xB_0) + \Lambda(B_1, \kappa B_1) \lesssim V_n(r) h_n(r). \]
Since the inequality $V_i(r)h_i(r) \gtrsim r^2$ is always true by (VD), we conclude the corollary from Theorem A.7 immediately. \qed

For the estimates of Poincaré constants of central balls on some examples of manifolds with ends, we refer to Section 3.

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