WARPED PRODUCT KÄHLER MANIFOLDS AND BOCHNER-KÄHLER METRICS

G. GANCHEV AND V. MIHOVA

Abstract. Using as an underlying manifold an alpha-Sasakian manifold we introduce warped product Kaehler manifolds. We prove that if the underlying manifold is an alpha-Sasakian space form, then the corresponding Kaehler manifold is of quasi-constant holomorphic sectional curvatures with special distribution. Conversely, we prove that any Kaehler manifold of quasi-constant holomorphic sectional curvatures with special distribution locally has the structure of a warped product Kaehler manifold whose base is an alpha-Sasakian space form. Considering the scalar distribution generated by the scalar curvature of a Kaehler manifold, we give a new approach to the local theory of Bochner-Kaehler manifolds. We study the class of Bochner-Kaehler manifolds whose scalar distribution is of special type. Taking into account that any manifold of this class locally is a warped product Kaehler manifold, we describe all warped product Bochner-Kaehler metrics. We find four families of complete metrics of this type.

1. Introduction

In [6] we considered Kähler manifolds \((M, g, J, D)\) endowed with a \(J\)-invariant distribution \(D\) of real codimension 2. If \(D^\perp\) is the 2-dimensional distribution, orthogonal to \(D\), then every holomorphic section \(E(m), m \in M\), determines an angle \(\vartheta = \angle(E(m), D^\perp(m))\).

A Kähler manifold \((M, g, J, D)\) is said to be of quasi-constant holomorphic sectional curvatures if its holomorphic sectional curvatures only depend on the point \(m\) and the angle \(\vartheta\). The distribution \(D\) of such manifolds is of pointwise constant holomorphic sectional curvatures \(a(m)\).

In [6] we studied the case of a Kähler manifold of quasi-constant holomorphic sectional curvatures when the distribution \(D\) is non-involutive. This implies \(da \neq 0\) and the 1-form \(\eta = \frac{da}{\|a\|}\) generates an involutive distribution \(\Delta\) (determined by the nullity spaces of \(\eta\)). Then \(D\) is a \(B_0\)-distribution with a geometric function \(k \neq 0\). We proved that the integral submanifolds of \(\Delta\) are \(\alpha\)-Sasakian manifolds of constant \(\varphi\)-holomorphic sectional curvatures of type determined by \(\text{sign}(a + k^2)\). This result establishes a relation between the Kähler manifolds of quasi-constant holomorphic sectional curvatures and the \(\alpha\)-Sasakian space forms.

In this paper we show that by using \(\alpha_0\)-Sasakian space forms we can construct locally all Kähler manifolds of quasi-constant holomorphic sectional curvatures with \(B_0\)-distribution.

In Section 3 we consider warped product manifolds using as a base an \(\alpha_0\)-Sasakian manifold and introduce geometrically determined complex structure \(J\). We prove that the warped

1991 Mathematics Subject Classification. Primary 53B35, Secondary 53C25.

Key words and phrases. \(\alpha\)-Sasakian manifolds, warped product Kähler manifolds, Bochner-Kähler manifolds with special scalar distribution, warped product Bochner-Kähler metrics.
product manifold with the complex structure $J$ becomes a locally conformal Kähler manifold. Further we define a Kähler metric on these manifolds.

In Proposition 3.5 we prove that any warped product Kähler manifold is of quasi-constant holomorphic sectional curvatures if and only if the underlying manifold is an $\alpha_0$-Sasakian space form.

The basic result in Section 3 (Theorem 3.7) states as follows:

Any Kähler manifold of quasi-constant holomorphic sectional curvatures with $B_0$-distribution, satisfying one of the conditions

$$a + k^2 > 0, \quad a + k^2 = 0, \quad a + k^2 < 0,$$

locally has the structure of a warped product Kähler manifold from the Examples (E).

In Section 4 we study the local theory of Bochner-Kähler manifolds with respect to the $J$-invariant scalar distribution generated by the scalar curvature $\tau$ of the manifold. We prove that the $(1,0)$-part of the vector field $\text{grad} \tau$ is a holomorphic vector field and the function

$$\mathfrak{B} = \|\rho\|^2 - \frac{\tau^2}{2(n + 1)} + \frac{\Delta \tau}{n + 1}$$

is a constant, which we call the Bochner constant of the manifold.

In Section 5 we investigate Bochner-Kähler manifolds whose scalar distribution is a $B_0$-distribution. In Proposition 5.1 we give the following curvature characterization of these manifolds:

Let $(M, g, J)$ $(\dim M = 2n \geq 6)$ be a Bochner-Kähler manifold with $d\tau \neq 0$. If the scalar distribution $D_\tau$ is a $B_0$-distribution, then the manifold is of quasi-constant holomorphic sectional curvatures and

$$R = a\pi + b\Phi, \quad b \neq 0.$$

On any of the above manifolds the function $b_0 = \frac{2a - b}{2}$ is a constant. The constants $\mathfrak{B}$ and $b_0$ determine uniquely the constant $\mathfrak{R}$.

Theorem 3.7 shows that the study of Bochner-Kähler manifolds whose scalar distribution is a $B_0$-distribution is equivalent to the study of warped product Bochner-Kähler manifolds.

In Section 6 we describe completely all families of warped product Bochner-Kähler metrics in terms of the underlying $\alpha_0$-Sasakian space form $Q_0$ and the constants $\mathfrak{R}, b_0$.

Finally we show that four of these families consist of complete metrics.

2. Preliminaries

Let $(M, g, J, D)$ be a $2n$-dimensional Kähler manifold with metric $g$, complex structure $J$ and $J$-invariant distribution $D$ of codimension 2. The Lee algebra of all $C^\infty$ vector fields on $M$ will be denoted by $\mathfrak{X}M$ and $T_mM$ will stand for the tangent space to $M$ at an arbitrary point $m \in M$. Any tangent space to $M$ has the structure $T_mM = D(m) \oplus D^\perp(m)$, where $D^\perp(m)$ is the 2-dimensional $J$-invariant orthogonal complement to the space $D(m)$, $m \in M$. Then the structural group of these manifolds is the subgroup $U(n - 1) \times U(1)$ of $U(n)$.

Locally we can always choose a unit vector field $\xi$ such that $D^\perp = \text{span}\{\xi, J\xi\}$. The 1-forms corresponding to the vector fields $\xi$ and $J\xi$ with respect to the metric $g$ are given by:

$$\eta(X) = g(X, \xi), \quad \bar{\eta}(X) = g(X, J\xi) = -\eta(JX), \quad X \in \mathfrak{X}M.$$
The Kähler form $\Omega$ of the Kähler structure $(g, J)$ is given by $\Omega(X, Y) = g(JX, Y)$, $X, Y \in \mathfrak{X} M$.

Let $\nabla$ be the Levi-Civita connection of the metric $g$. The Riemannian curvature tensor $R$, the Ricci tensor $\rho$ and the scalar curvature $\tau$ of $\nabla$ are given as follows:

$$
R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z,
$$

$$
R(X, Y, Z, U) = g(R(X, Y)Z, U), \quad X, Y, Z, U \in \mathfrak{X} M,
$$

$$
\rho(Y, Z) = \sum_{i=1}^{2n} R(e_i, Y, Z, e_i), \quad Y, Z \in T_m M,
$$

$$
\tau = \sum_{i=1}^{2n} \rho(e_i, e_i),
$$

where $\{e_i\}$, $i = 1, \ldots, 2n$ is an orthonormal basis for $T_m M$, $m \in M$. The structure $(g, J, D)$ also gives rise to the following functions

$$
\varkappa := R(\xi, J\xi, J\xi, \xi), \quad \sigma := \rho(\xi, \xi) = \rho(J\xi, J\xi)
$$

and tensors

$$
4\pi(X, Y)Z := g(Y, Z)X - g(X, Z)Y - 2g(JX, Y)JZ + g(JY, Z)JX - g(JX, Z)JY,
$$

$$
8\Phi(X, Y)Z := g(Y, Z)(\eta(X)\xi + \bar{\eta}(X)J\xi) - g(X, Z)(\eta(Y)\xi + \bar{\eta}(Y)J\xi) + g(JY, Z)(\eta(X)J\xi - \bar{\eta}(X)J\xi) - g(JX, Z)(\eta(Y)J\xi - \bar{\eta}(Y)J\xi) - 2g(JX, Y)(\eta(Z)J\xi - \bar{\eta}(Z)J\xi)
$$

$$
+ (\eta(Y)\eta(Z) + \bar{\eta}(Y)\bar{\eta}(Z))X - (\eta(X)\eta(Z) + \bar{\eta}(X)\bar{\eta}(Z))Y + (\eta(Y)\bar{\eta}(Z) - \eta(Z)\bar{\eta}(Y))JX - (\eta(X)\bar{\eta}(Z) - \eta(Z)\bar{\eta}(X))JY - 2(\eta(X)\bar{\eta}(Y) - \bar{\eta}(X)\eta(Y))JZ,
$$

$$
\Psi(X, Y)Z := g(Y, Z)(\bar{\eta}(X)\xi - \eta(X)\bar{\eta}(Y)J\xi) + g(X, \bar{\eta}(Y)\bar{\eta}(Z)J\xi - \eta(Y)\bar{\eta}(X)\bar{\eta}(Z)J\xi
$$

$$
= (\eta \wedge \bar{\eta})(X, Y)(\bar{\eta}(Z)\xi - \eta(Z)J\xi), \quad X, Y, Z \in \mathfrak{X} M.
$$

These basic tensors are invariant under the action of the structural group $U(n - 1) \times U(1)$ in the sense of [17].

A Kähler manifold $(M, g, J, D)$ (dim $M = 2n \geq 4$) with $J$-invariant distribution $D$ is of quasi-constant holomorphic sectional curvatures (briefly a QCH-manifold) if and only if

$$
R = a\pi + b\Phi + c\Psi,
$$

where $\pi, \Phi, \Psi$ are the tensors (2.1) and

$$
a = \frac{\tau - 4\sigma + 2\varkappa}{n(n - 1)}, \quad b = \frac{-2\tau + 4(n + 2)\sigma - 4(n + 1)\varkappa}{n(n - 1)},
$$

$$
c = \frac{\tau - 4(n + 1)\sigma + (n + 1)(n + 2)\varkappa}{n(n - 1)}.
$$
Let \((M, g)\) be a Riemannian manifold endowed with a unit vector field \(\xi\), and \(\eta\) be the 1-form corresponding to \(\xi\) with respect to the metric \(g\). We assume that the distribution \(\Delta\), determined by the nullity spaces of \(\eta\), is involutive. An involutive distribution \(\Delta\) is characterized by the condition 5
\[
d\eta = -\theta \wedge \eta,
\]
where \(\theta(X) = g(\nabla_\xi \xi, X), \quad X \in \mathfrak{X}M\).

A \(C^\infty\) function \(u\) on \(M\) is said to be a proper function for the involutive distribution \(\Delta\) if
\[
du = \xi(u) \eta = \|du\| \eta.
\]

Now, let \((M, g, J)\) be a Kähler manifold endowed with a unit vector field \(\xi\) and \(\eta, \tilde{\eta}\) be the 1-forms corresponding to the vector fields \(\xi, J\xi\), respectively. Then the vector field \(\xi\) generates the distributions:
\[
\Delta(m) := \{x \in T_mM \mid \eta(x) = 0\}, \quad m \in M,
\]
\[
D(m) = \{x_0 \in T_mM \mid \eta(x_0) = \tilde{\eta}(x_0) = 0\}, \quad D^\perp(m) = \text{span}\{\xi, J\xi\}, \quad m \in M.
\]

As a rule we shall use the following denotations for the different kinds of vectors (vector fields) on \(M\):
\[
X \in T_mM (\mathfrak{X}M), \quad x \in \Delta (\mathfrak{X}\Delta), \quad x_0 \in D(m) (\mathfrak{X}D), \quad m \in M.
\]

Any vector \(X \in T_mM\) can be decomposed in a unique way in the form
\[
X = x_0 + \tilde{\eta}(X)J\xi + \eta(X)\xi,
\]
where \(x_0\) is the projection of \(X\) into \(D(m)\).

The following notion is essential in our considerations:

**Definition 2.1.** \(\mathbf{[5]}\) A \(J\)-invariant distribution \(D\) generated by the unit vector field \(\xi\) is said to be a \(B_0\)-distribution if it satisfies the following conditions:
\[
i) \quad \nabla_{x_0} \xi = \frac{k}{2} x_0, \quad k \neq 0, \quad x_0 \in D,
\]
\[
ii) \quad \nabla_{J\xi} \xi = -p^*J\xi,
\]
\[
iii) \quad \nabla_\xi \xi = 0.
\]

The conditions (2.4) in Definition 2.1 are equivalent to the equality
\[
(2.5) \quad \nabla_X \xi = \frac{k}{2} \{X - \eta(X)\xi - \tilde{\eta}(X)J\xi\} - p^*\tilde{\eta}(X)J\xi, \quad X \in \mathfrak{X}M.
\]

The last formula implies that
\[
(2.6) \quad dk = \xi(k) \eta, \quad p^* = -\frac{\xi(k) + k^2}{k}.
\]

Introducing the relative divergences \(\text{div}_0 \xi\) and \(\text{div}_0 J\xi\) (relative codifferentials \(\delta_0 \eta\) and \(\delta_0 \tilde{\eta}\)) of the vector fields \(\xi\) and \(J\xi\) (1-forms \(\eta\) and \(\tilde{\eta}\)) with respect to the distribution \(D\) by
\[
\text{div}_0 \xi = -\delta_0 \eta = \sum_{i=1}^{2(n-1)} (\nabla_{e_i} \eta) e_i, \quad \text{div}_0 J\xi = -\delta_0 \tilde{\eta} = \sum_{i=1}^{2(n-1)} (\nabla_{e_i} \tilde{\eta}) e_i,
\]
\( \{ e_1, \ldots, e_{2(n-1)} \} \) being an orthonormal basis of \( D(m) \), \( m \in M \), from (2.5) we obtain:

\[
k = \frac{\text{div}_0 \xi}{n-1}, \quad \text{div}_0 J \xi = 0.
\]

Let \((M, g, J, D)\) \((\dim M = 2n \geq 6)\) be a Kähler QCH-manifold with non-involutive distribution \( D \) \((D^\perp = \text{span}\{\xi, J\xi\}\)\) From (2.5) it follows that \( D \) is a \( B_0 \)-distribution with geometric function \( k \neq 0 \) and the equality (2.2) implies that \( D \) is of pointwise constant holomorphic sectional curvatures \( a \). In [7] we have shown that the function \( a + k^2 \) plays an important role in the study of the class of the Kähler QCH-manifolds with \( B_0 \)-distribution.

We divided this class into three subclasses with respect to \( \text{sign} (a + k^2) \):\[
a + k^2 > 0, \quad a + k^2 = 0, \quad a + k^2 < 0.
\]

In [7] we have proved that the integral submanifolds of the distribution \( \Delta \) are \( \alpha \)-Sasakian space forms.

Next we give some basic notions concerning the class of \( \alpha \)-Sasakian manifolds.

Let \((Q_0, g_0, \varphi, \tilde{\xi}, \tilde{\eta}_0)\) \((\dim Q_0 = 2n - 1 \geq 5)\) be an almost contact Riemannian manifold. The structures of the manifold satisfy the following conditions:

\[
\begin{align*}
\tilde{\eta}_0(x) &= g_0(\tilde{\xi}_0, x), \quad \varphi \tilde{\xi}_0 = 0, \quad \varphi^2 x = -x + \tilde{\eta}_0(x)\tilde{\xi}_0, \quad x \in \mathfrak{X}Q_0, \\
g_0(\varphi x, \varphi y) &= g_0(x, y) - \tilde{\eta}_0(x)\tilde{\eta}_0(y), \quad x, y \in \mathfrak{X}Q_0.
\end{align*}
\]

Let \( \mathcal{D}^0 \) be the Levi-Civita connection of the metric \( g_0 \). The manifold \((Q_0, g_0, \varphi, \tilde{\xi}_0, \tilde{\eta}_0)\) is called an \( \alpha_0 \)-Sasakian manifold [8] if

\[
\mathcal{D}^0_x \tilde{\xi}_0 = \alpha_0 \varphi x, \quad x \in \mathfrak{X}Q_0^{2n-1}, \quad \alpha_0 = \text{const};
\]

\[
(\mathcal{D}^0 \varphi)y = \alpha_0 \{ \tilde{\eta}_0(y)x - g_0(x, y)\tilde{\xi}_0 \}, \quad x, y \in \mathfrak{X}Q_0.
\]

From (2.8) it follows that

\[
d \tilde{\eta}_0(x, y) = 2\alpha_0 g_0(\varphi x, y), \quad x, y \in \mathfrak{X}Q_0.
\]

If \( K^0 \) is the Riemannian curvature tensor of \( g_0 \), then (2.7), (2.8) and (2.9) imply that

\[
K^0(x, y)\tilde{\xi}_0 = \alpha_0^2 \{ \tilde{\eta}_0(y)x - \tilde{\eta}_0(x)y \}, \quad x, y \in \mathfrak{X}Q_0.
\]

The last equality gives the geometric meaning of the constant \( \alpha_0 \):

\[
K^0(x_0, \tilde{\xi}_0, x_0) = \alpha_0^2, \quad x_0 \perp \tilde{\xi}_0, \quad g_0(x_0, x_0) = 1.
\]

The change of the direction of \( \tilde{\xi}_0 \) has as a consequence the change of \( \text{sign} \alpha_0 \). Thus we can assume that \( \alpha_0 > 0 \).

The class of the Sasakian manifolds is the subclass of \( \alpha_0 \)-Sasakian manifolds with \( \alpha_0 = 1 \).

The distribution orthogonal to \( \tilde{\xi}_0 \) is denoted by \( D \):

\[
D(m) := \{ x_0 \in T_mQ_0 \mid \tilde{\eta}_0(x_0) = 0 \}, \quad m \in Q_0.
\]

Then the vector space \((D, \varphi)\) is a Hermitian vector space at every point of \( Q_0 \).

An \( \alpha_0 \)-Sasakian manifold is said to be an \( \alpha_0 \)-Sasakian space form [11] if it is of constant \( \varphi \)-holomorphic sectional curvatures \( H \), i.e.

\[
K^0(x_0, \varphi x_0, \varphi x_0, x_0) = H_0, \quad x_0 \in D, \quad g_0(x_0, x_0) = 1.
\]
Any $\alpha_0$-Sasakian space form is characterized by the following curvature identity [11, 8]

$$K^0 = \frac{H_0 + 3\alpha_0^2}{4} \pi_{01} + \frac{H_0 - \alpha_0^2}{4} (\pi_{02} - \pi_{03}),$$

(2.11)

where $\pi_{01}, \pi_{02}, \pi_{03}$ are the basic invariant tensors with respect to the structure $(g_0, \tilde{\xi}_0, \tilde{\eta}_0)$:

$$\pi_{01}(x, y, z, u) = g_0(y, z)g_0(x, u) - g_0(x, z)g_0(y, u),$$

$$\pi_{02}(x, y, z, u) = g_0(\varphi y, z)g_0(\varphi x, u) - g_0(\varphi x, z)g_0(\varphi y, u) - 2g_0(\varphi x, y)g_0(\varphi z, u),$$

$$\pi_{03}(x, y, z, u) = g_0(y, z)\tilde{\eta}_0(x)\tilde{\eta}_0(u) - g_0(x, z)\tilde{\eta}_0(y)\tilde{\eta}_0(u) + \tilde{\eta}_0(y)\tilde{\eta}_0(z)g_0(x, u) - \tilde{\eta}_0(x)\tilde{\eta}_0(z)g_0(y, u),$$

(2.12)

for any $x, y, z, u \in \mathcal{X}Q_0$.

Keeping in mind the projections $g_0 - \tilde{\eta}_0 \otimes \tilde{\eta}_0$ and $\tilde{\eta}_0 \otimes \tilde{\eta}_0$ of the metric $g_0$ onto the distribution $D$ and span $\{\tilde{\xi}_0\}$, we consider the following bihomothetical transformation of the structure $(g_0, \tilde{\xi}_0, \tilde{\eta}_0)$:

$$g = p^2\{(g_0 - \tilde{\eta}_0 \otimes \tilde{\eta}_0) + q^2 \tilde{\eta}_0 \otimes \tilde{\eta}_0\}$$

$$\tilde{\xi} = \frac{1}{pq} \tilde{\xi}_0, \quad \tilde{\eta} = pq \tilde{\eta}_0,$$

(2.13)

where $p$ and $q$ are positive constants. In the case $p = q$, the transformation (2.13) is called a D-homothetic deformation of the structure [13].

If $D$ is the Levi-Civita connection of the new metric $g$, then

$$D_x \tilde{\xi} = \alpha_0 \frac{q}{p} \varphi x,$$

$$\langle D_x \varphi \rangle y = \alpha_0 \frac{q}{p} \{\tilde{\eta}(y)x - g(x, y)\tilde{\xi}\},$$

(2.14)

where $x, y \in \mathcal{X}Q_0$.

From (2.14) it follows that $(Q_0, g, \varphi, \tilde{\xi}, \tilde{\eta})$ is an $\alpha$-Sasakian manifold with $\alpha = \frac{q}{p} \alpha_0$.

If $\pi_1, \pi_2, \pi_3$ denote the basic invariant tensors (2.12) with respect to the structure $(g, \tilde{\xi}, \tilde{\eta})$ given by (2.13), then the relations between $\pi_1, \pi_2, \pi_3$ and $\pi_{01}, \pi_{02}, \pi_{03}$ are the following:

$$\pi_1 = p^4\{\pi_{01} + (q^2 - 1)\pi_{03}\},$$

$$\pi_2 = p^4 \pi_{02},$$

$$\pi_3 = p^4q^2 \pi_{03}.$$

(2.15)

The Levi-Civita connection $D$ of $g$, as a consequence of (2.13), is expressed by the equality

$$D_x y = D^0_x y + \alpha_0 (q^2 - 1)\{\tilde{\eta}_0(y) \varphi x + \tilde{\eta}_0(x) \varphi y\}, \quad x, y \in \mathcal{X}Q_0.$$

Then the curvature tensor $K$ of the metric $g$ has the form:

$$K = \frac{H + 3\alpha_0^2}{4} \pi_1 + \frac{H - \alpha_0^2}{4} (\pi_2 - \pi_3),$$

(2.16)

where

$$H = \frac{H_0 + 3\alpha_0^2}{p^2} - \frac{3q^2 \alpha_0^2}{p^2}, \quad \alpha = \frac{q}{p} \alpha_0.$$

(2.17)
Hence

\[(2.18)\quad H + 3\alpha^2 = \frac{H_0 + 3\alpha_0^2}{\rho^2},\]

which shows that \(\text{sign}(H_0 + 3\alpha_0^2)\) is invariant under the bihomothetical changes \((2.13)\) of the structure \((g_0, \xi_0, \bar{\eta}_0)\).

There are three types of \(\alpha_0\)-Sasakian space forms corresponding to the \(\text{sign}\) of the constant \(H_0 + 3\alpha_0^2\):

\[H_0 + 3\alpha_0^2 > 0, \quad H_0 + 3\alpha_0^2 = 0, \quad H_0 + 3\alpha_0^2 < 0.\]

The formula \((2.18)\) shows that the bihomothetical transformations \((2.13)\) of the structure \((g_0, \xi_0, \bar{\eta}_0)\) do not change the type of the space form.

3. Warped Product Kähler Manifolds

Now let \((Q_0, g_0, \varphi, \xi_0, \bar{\eta}_0)\) \((\dim Q_0 = 2n - 1 \geq 5)\) be an \(\alpha_0\)-Sasakian manifold with constant \(\alpha_0 > 0\). Further, let \(\mathbb{R}\) be equipped with a coordinate system \(O\varepsilon\) and the standard inner product determined by \(\varepsilon^2 = 1\). For an arbitrary point \(t \in \mathbb{R}\) we denote \(\xi := \frac{d}{dt}\) and \(\eta := dt\).

The manifolds \(Q_0\) and \(\mathbb{R}\) generate the product manifold \(N = Q_0 \times \mathbb{R}\) with the standard product metric \(g_0 + \eta \otimes \eta\).

Let \(p(t),\; t \in I \subset \mathbb{R}\) with \(p(t) > 0,\; p(t_0) = 1\) be a real function. On any leaf \(Q(t)\) \((\dim Q = 2n - 1 \geq 5)\) \((t \text{-fixed})\) with the \(\alpha_0\)-Sasakian structure \((g_0, \varphi, \xi_0, \bar{\eta}_0)\), we define the structure \((\tilde{g}, \varphi, \tilde{\xi}, \tilde{\eta})\):

\[(3.1)\]

\[
\tilde{g}(t) := p^2(t)g_0, \quad \tilde{\xi} := \frac{1}{p(t)}\xi_0, \quad \tilde{\eta} := p(t)\bar{\eta}_0.
\]

This structure is homothetical to \((g_0, \varphi, \xi_0, \bar{\eta}_0)\) with constant \(p(t)\) \((t \text{-fixed})\). Then the structure \((\tilde{g}, \varphi, \tilde{\xi}, \tilde{\eta})\) defined by \((3.1)\) \((t\text{-fixed})\) is an \(\tilde{\alpha}\)-Sasakian with

\[
\tilde{\alpha}(t) = \frac{1}{p(t)}\alpha_0.
\]

We furnish the manifold \(N = Q_0 \times \mathbb{R}\) with the warped product metric \(\Pi\):

\[(3.2)\]

\[
G := p^2(t)g_0 + \eta \otimes \eta.
\]

Then \(\eta\) and \(\bar{\eta}\) are the 1-forms corresponding to \(\xi\) and \(\bar{\xi}\) with respect to the metric \(G\):

\[
\eta(X) = G(\xi, X), \quad \bar{\eta}(X) = G(\bar{\xi}, X), \quad X \in \mathfrak{X}N.
\]

Let \(T_mN\) be the tangent space to \(N\) at an arbitrary point \(m \in N\). The structure \((G, \tilde{\xi}, \tilde{\xi})\) gives rise to the following distributions on \(N\):

\[
D(m) := \{x_0 \in T_mN \mid \tilde{\eta}(x_0) = \eta(x_0) = 0\}, \quad D^\perp(m) := \text{span} \{\tilde{\xi}, \tilde{\xi}\},
\]

\[
\Delta(m) := \{x \in T_mN \mid \bar{\eta}(x) = 0\}, \quad m \in N.
\]

Then any vector \((\text{vector field})\) \(X\) in \(T_mN\) \((\mathfrak{X}N)\) can be decomposed uniquely as follows:

\[
X = x_0 + \tilde{\eta}(X)\tilde{\xi} + \bar{\eta}(X)\bar{\xi},
\]

where \(x_0\) is the projection of \(X\) into \(D\).
Any vector (vector field) $x$ in $\Delta$ ($\mathfrak{X}\Delta$) can be decomposed in a unique way as follows:

$$x = x_0 + \tilde{\eta}(x)\tilde{\xi},$$

where $x_0$ is the projection of $x$ into $D$.

Denoting by $\nabla$ and $\bar{\nabla}$ the Levi-Civita connections of the manifold $(N, G)$ and the submanifold $(Q(t), \bar{g}(t))$, respectively, we have

**Lemma 3.1.** The Gauss and Weingarten formulas for the submanifold $(Q(t), \bar{g}(t))$ of $N$ with normal vector field $\xi$ are:

(3.3) $\nabla_x y = \bar{\nabla}_x y - \xi(ln p) \bar{g}(x, y) \xi$, $x, y \in \mathfrak{X}\Delta$,

(3.4) $\bar{\nabla}_x \xi = \xi(ln p) x$, $x \in \mathfrak{X}\Delta$.

**Proof.** Since $N = Q_0 \times \mathbb{R}$, then

$$\nabla_x \xi = 0$$

and $[\xi, \xi_0] = [\xi, x_0] = 0$ for every $x_0 \in \mathfrak{X}D$. From (3.1) it follows

$$[\xi, \xi] = \nabla_\xi \tilde{\xi} - \nabla_{\tilde{\xi}} \xi = \frac{1}{p} [\xi, \xi_0] - \xi(ln p) \tilde{\xi} = -\xi(ln p) \tilde{\xi}.$$

Using the Koszul formula for the Levi-Civita connection $\nabla$ of the metric $G$

$$2G(\nabla_X Y, Z) = XG(Y, Z) +YG(Z, X) - ZG(X, Y) +G([X, Y], Z) - G([Y, Z], X) + G([Z, X], Y), \quad X, Y, Z \in \mathfrak{X}N,$$

we calculate the components of $\nabla X Y$, $X, Y \in \mathfrak{X}N$:

$$\nabla_{x_0} \xi = \bar{\nabla}_{x_0} \xi = \bar{\alpha} \varphi x_0, \quad \bar{\nabla}_{\xi} x_0 = \bar{\alpha} \varphi x_0, \quad x_0 \in \mathfrak{X}D,$$

(3.5)

$$\nabla_{\xi} \xi = 0, \quad \bar{\nabla}_{\xi} \xi = -\xi(ln p) \xi, \quad \bar{\nabla}_{\xi} \xi = \xi(ln p) \xi, \quad \nabla_{\xi} \xi = 0,$$

$$\bar{\nabla}_{x_0} \xi = \bar{\nabla}_{x_0} \xi = \xi(ln p) x_0, \quad \bar{\nabla}(\nabla_{x_0} y_0, \xi) = -\xi(ln p) \bar{g}(x_0, y_0), \quad x_0, y_0 \in \mathfrak{X}D.$$

The above equalities imply the assertion of the lemma. QED

Next we define a complex structure $J$ on the warped product manifold $(N, G)$ by the help of the geometric structures $(\varphi, \tilde{\xi}, \tilde{\xi})$ in the following way:

(3.6) $J|_D := \varphi, \quad J\xi := \tilde{\xi}, \quad J\xi := -\tilde{\xi}.$

Thus $(N, G, J)$ becomes an almost Hermitian manifold.

More precisely we have

**Lemma 3.2.** The covariant derivative of the structure $J$ on $(N, G)$, given by (3.6), satisfies the following identity:

(3.7) $(\bar{\nabla}_X J)Y = \{\xi(ln p) - \bar{\alpha}\} \{G(X, Y)\xi + G(JX, Y)\xi - \bar{\eta}(Y)X - \bar{\eta}(Y)JX\}, \quad X, Y \in \mathfrak{X}N.$

**Proof.** Let $X, Y \in \mathfrak{X}N$, $x, y$ be the projections of $X, Y$ into $\mathfrak{X}\Delta$ and $x_0, y_0$ - the projections of $X, Y$ into $\mathfrak{X}D$. Then we have

(3.8) $X = x + \tilde{\eta}(X)\xi = x_0 + \bar{\eta}(X)\xi + \bar{\eta}(X)\xi,$

(3.9) $Y = y + \bar{\eta}(Y)\xi = y_0 + \bar{\eta}(Y)\xi + \bar{\eta}(Y)\xi.$
Using (3.3), (3.6) and (3.9) we calculate \( \nabla_x J y - J \nabla_x y \):

\[
(3.10) \quad (\nabla_x J)y = \{\xi(\ln p) - \tilde{\alpha}\} \{\bar{g}(x, y)\bar{\xi} + \bar{g}(\varphi x, y)\bar{\xi} - \bar{\eta}(y) x\}, \quad x, y \in \mathfrak{X}\Delta.
\]

Using (3.3), (3.4), (3.6) and (3.8) we compute \( \nabla_x J \bar{\xi} - J \nabla_x \bar{\xi} \):

\[
(3.11) \quad (\nabla_x J)\bar{\xi} = -\{\xi(\ln p) - \bar{\alpha}\} \varphi x, \quad x \in \mathfrak{X}\Delta.
\]

Finally, by using (3.5) and (3.6), we find

\[
(\nabla_{\bar{\xi}} J)\bar{\xi} = 0, \quad (\nabla_{\bar{\xi}} J)\bar{\xi} = 0, \quad (\nabla_{\bar{\xi}} J)x_0 = 0, \quad x_0 \in \mathfrak{X}D,
\]

which imply

\[
(3.12) \quad (\nabla_{\bar{\xi}} J)X = 0, \quad X \in \mathfrak{X}N.
\]

Now (3.10), (3.11) and (3.12) give (3.7).

The identity (3.7) implies that the almost complex structure \( J \), defined by (3.6) is integrable, i.e. \((N, G, J)\) is a Hermitian manifold. More precisely, this manifold is in the class \( W_4 \), according to the classification of almost Hermitian manifolds given in [4], with Lee form \( \{\bar{\alpha} - \xi(\ln p)\} \bar{\eta} \). Since \( \bar{\alpha} - \xi(\ln p) \) is a function of \( t \) and \( \bar{\eta} = dt \), then the Lee form is closed in all dimensions \( 2n \geq 4 \) and \((N, G, J)\) is a locally conformal Kähler manifold.

Let \( q(t) > 0, t \in I \subset \mathbb{R} \) be a real \( C^\infty \)-function satisfying the condition \( q(t_0) = 1 \). We consider the following complex dilatational transformation of the structure \((G, \xi)\) (cf [3]):

\[
(3.13) \quad g = G + (q^2(t) - 1)(\bar{\eta} \otimes \bar{\eta} + \bar{\eta} \otimes \bar{\eta}), \quad \xi = \frac{1}{q} \bar{\xi}, \quad \eta = q \bar{\eta}; \quad \bar{\xi} = \frac{1}{q} \bar{\xi}, \quad \bar{\eta} = q \bar{\eta}.
\]

**Proposition 3.3.** The Hermitian manifold \((N, g, J, \xi, \bar{\xi})\) with structures given by (3.6) and (3.13) is Kählerian if and only if the positive functions \( p(t) \) and \( q(t) \) are related as follows

\[
q^2(t) = \frac{1}{\alpha_0} \bar{\xi}(p).
\]

**Proof.** From (3.13) we obtain that the Kähler form \( g(JX, Y), X, Y \in \mathfrak{X}N \), is closed if and only if \( \bar{\xi}(\ln p) - \bar{\alpha}q^2 = 0 \). Since \( \bar{\alpha}(t) = \frac{1}{p(t)} \alpha_0 \), then \( q^2 = \frac{\bar{\xi}(p)}{\alpha_0} \). \( \quad \Box \)

**Remark 3.4.** In view of (3.13) the 1-form \( \eta \) is closed, i.e. \( \eta = ds \) locally. Taking into account the equality \( \eta = q \bar{\eta} \) and choosing \( s(t_0) = 0 \) we obtain \( s(t) = \int_{t_0}^{t} q(t) dt \). Considering the functions \( p(t) \) and \( q(t) \) as functions of \( s \) we have

\[
\eta = ds, \quad \xi = \frac{d}{ds}, \quad q = \sqrt{\frac{1}{\alpha_0} \frac{dp}{dt}} = \frac{1}{\alpha_0} \xi(p) = \frac{1}{\alpha_0} \frac{dp}{ds} = \frac{p'}{\alpha_0}.
\]

From now on the derivatives with respect to the parameter \( s \) will be denoted as usual by \( ()', ()'', ... \)

Let \( Q_0(g_0, \varphi, \bar{\xi}_0, \bar{\eta}_0) \) be an \( \alpha_0 \)-Sasakian manifold and \( p(s), s \in I_0 \subset \mathbb{R} \), be a \( C^\infty \)-real function, satisfying the conditions

\[
(3.14) \quad p(s) > 0, \quad p'(s) > 0, \quad p(0) = 1, \quad p'(0) = \alpha_0.
\]
We consider the following Kähler manifolds \((N, g, J, \xi)\) \((\dim N = 2n \geq 6)\):

\[
N = Q_0 \times \mathbb{R},
\]

\[
g = \bar{g} - \bar{\eta} \otimes \bar{\eta} + \frac{p^2}{\alpha_0^2} (\bar{\eta} \otimes \bar{\eta} + \bar{\bar{\eta}} \otimes \bar{\bar{\eta}})
= p^2(s) \left\{ g_0 + \left( \frac{p''(s)}{\alpha_0^2} - 1 \right) \bar{\eta}_0 \otimes \bar{\eta}_0 \right\} + \frac{p^2(s)}{\alpha_0^2} \bar{\eta} \otimes \bar{\eta},
\]

\[
\tilde{\xi} = \frac{1}{p} \xi_0, \quad \tilde{\eta} = p \bar{\eta}_0, \quad \tilde{\xi} = \frac{\alpha_0}{p'} \tilde{\xi}, \quad \tilde{\eta} = \frac{p'}{\alpha_0} \tilde{\eta}_0,
\]

\[
\tilde{\xi} = \frac{dt}{dt}, \quad \tilde{\eta} = dt, \quad \xi = \frac{\alpha_0}{p'} \tilde{\xi}, \quad \eta = \frac{p'}{\alpha_0} \tilde{\eta},
\]

\[
J_\tilde{\xi} := \varphi, \quad J_\tilde{\eta} := \tilde{\xi}, \quad J_\tilde{\xi} := -\xi,
\]

where the function \(p(s), s \in I_0 \subset \mathbb{R}\), satisfies the conditions (3.14) and \(\alpha_0 = \text{const} > 0\).

In what follows we call the manifolds \((N, g, J, \xi)\) given by (3.15) warped product Kähler manifolds.

If \(\nabla\) is the Levi-Civita connection of the Kähler metric \(g\), then the equalities (3.13), (3.4) and \(\nabla_\xi \xi = 0\) imply that

\[
\nabla_X \xi = \frac{p'}{p} \{ X - \bar{\eta}(X) \tilde{\xi} - \eta(X) \xi \} + \frac{p^2 + pp''}{pp'} \eta(X) \tilde{\xi}, \quad X \in \mathfrak{X} N.
\]

Since \(\xi = \frac{\alpha_0}{p'} \tilde{\xi}\), then the unit vector field \(\xi\) generates the same distributions \(\Delta, D\) and \(D^\perp\).

Comparing the equality (3.16) with (2.5) we conclude that:

- The distribution \(D\) of the Kähler manifold (3.15) is a \(B_0\)-distribution with functions

\[
\frac{k}{2} = \frac{p'}{p}, \quad p^* = -\frac{p^2 + pp''}{pp'}.
\]

Let \(R, \rho, \tau\) be the curvature tensor, the Ricci tensor and the scalar curvature of \(g\), respectively. From the equality (3.16) we compute:

\[
R(X, Y)\xi = \frac{p''}{p} \{ \eta(X)Y - \eta(Y)X - \bar{\eta}(X)JY + \bar{\eta}(Y)JX + 2g(JX, Y)J\xi \}
+ \frac{pp'' - p'p''}{pp'} \{ \eta(X)\bar{\eta}(Y) - \eta(Y)\bar{\eta}(X) \} J\xi, \quad X, Y \in \mathfrak{X} N.
\]

Taking a trace in (3.18) we find

\[
\rho(X, \xi) = -\left\{ \frac{pp'' - p'p''}{pp'} + 2(n + 1) \frac{p''}{p} \right\} \eta(X), \quad X \in \mathfrak{X} N.
\]

Then the functions \(\kappa\) and \(\sigma\) are

\[
\kappa = R(\xi, J\xi, J\xi, \xi) = -\left\{ \frac{pp'' - p'p''}{pp'} + 4 \frac{p''}{p} \right\},
\]

\[
\sigma = \rho(\xi, \xi) = \rho(J\xi, J\xi) = -\left\{ \frac{pp'' - p'p''}{pp'} + 2(n + 1) \frac{p''}{p} \right\}.
\]
Proposition 3.5. Any warped product Kähler manifold \((N, g, J, \xi)\) given by (3.15) is of quasi-constant holomorphic sectional curvatures if and only if the base manifold \(Q_0\) is an \(\alpha_0\)-Sasakian space form.

Proof. Let \((N, g, J, \xi)\) be a warped product Kähler manifold defined by (3.15). Then any leaf \(Q(s)\) with the restriction of the metric \(g\) is a submanifold of \(N\) with unit normal vector field \(\xi\). Denoting by \(\mathcal{D}\) the Levi-Civita connection of the restriction of \(g\) onto \(Q(s)\) we find the Gauss formula:

\[
\nabla_x y = \mathcal{D}_x y - \left( \frac{p'}{p} g(x, y) + \frac{p''}{p'} \tilde{\eta}(x) \tilde{\eta}(y) \right) \xi, \quad x, y \in \mathfrak{X}\Delta. 
\]

The submanifold \(Q(s)\) carries the standard induced structure \((g, \varphi, \tilde{\xi}, \tilde{\eta})\) \([15, 16]\), where

\[
\varphi x = Jx + \tilde{\eta}(x) \xi, \quad x \in \mathfrak{X}\Delta.
\]

Then the Gauss equation is

\[
R|\Delta = K - \frac{p'^2}{p^2} \pi_1|\Delta - \frac{p''}{p} \pi_3|\Delta,
\]

where \(K\) is the curvature tensor of \(Q(s)\) and \(\pi_1, \pi_3\) are the tensors (2.12) with respect to the structure \((g, \varphi, \tilde{\xi}, \tilde{\eta})\) on \(Q(s)\).

The equalities (3.22) and (3.16) imply that

\[
\mathcal{D}_x \tilde{\xi} = \frac{p'}{p} \varphi x, \quad x \in \mathfrak{X}\Delta,
\]

\[
(\mathcal{D}_x \varphi)y = \frac{p'}{p} \{\tilde{\eta}(y)x - g(x, y)\tilde{\xi}\}, \quad x, y \in \mathfrak{X}\Delta.
\]

Hence \(Q(s)\) with the induced almost contact Riemannian structure is an \(\alpha\)-Sasakian manifold with

\[
\alpha(s) = \frac{p'(s)}{p(s)} = \text{const} > 0, \quad s - \text{fixed}.
\]

Now let the base manifold \((Q_0, g_0, \varphi, \tilde{\xi}_0, \tilde{\eta}_0)\) be an \(\alpha_0\)-Sasakian space form with constant \(\varphi\)-holomorphic sectional curvatures \(H_0\).

The equalities (3.15) imply that the restriction of the Kähler metric \(g\) onto the leaf \(Q(s), s \in I_0 \subset \mathbb{R}\) (s-fixed), is

\[
g(s) = p^2(s) \left\{ g_0 + \left( \frac{p'^2(s)}{\alpha_0^2} - 1 \right) \tilde{\eta}_0 \otimes \tilde{\eta}_0 \right\},
\]

which means that the metrics \(g(s)\) and \(g_0\) are in the bihomothetical relation (2.13) with \(q(s) = \frac{p'(s)}{\alpha_0}\).

Hence \(Q(s) \subset N\) is an \(\alpha\)-Sasakian space form with curvature tensor \(K\) satisfying (2.16), where

\[
\alpha = \frac{p'}{p} > 0, \quad H = \frac{1}{p^2} \{H_0 + 3\alpha_0^2 - 3p'^2\}.
\]

Replacing the tensor \(K\) into (3.24) we find

\[
R|\Delta = \frac{1}{4} \left( H - \frac{p'^2}{p^2} \right) \pi_1|\Delta + \frac{1}{4} \left( H - \frac{p'^2}{p^2} + 4 \frac{p''}{p} \right) \pi_3|\Delta.
\]
where $\pi_1, \pi_2$ and $\pi_3$ are the tensors (2.12) with respect to the structure $(g, \varphi, \tilde{\xi}, \tilde{\eta})$.

Taking into account (3.18), (3.27) and (3.23) we obtain

$$R = a\pi + b\Phi + c\Psi,$$

where

$$a = H - \frac{p'^2}{p^2}, \quad b = -2 \left( H - \frac{p'^2}{p^2} + 4\frac{p''}{p} \right), \quad c = H - \frac{p'^2}{p^2} + 5\frac{p''}{p} - \frac{p'''}{p}.$$

According to [6] $(N, g, J)$ is a Kähler manifold of quasi-constant holomorphic sectional curvatures.

For the inverse, let the warped product Kähler manifold (3.15) be of quasi-constant holomorphic sectional curvatures. Then any integral submanifold $Q(s)$ of the distribution $\Delta$ is an $\alpha$-Sasakian space form [7]. Since $Q(s)$ and $Q_0$ are in bihomothetical correspondence (2.13), then the base manifold $Q_0$ is an $\alpha_0$-Sasakian space form. QED

Thus we constructed the following

**Examples** of Kähler manifolds of quasi-constant holomorphic sectional curvatures with $B_0$-distribution:

All warped product Kähler manifolds $(N = Q_0 \times \mathbb{R}, g, J, D)$, whose underlying base manifold $Q_0$ is an $\alpha_0$-Sasakian space form.

**Remark 3.6.** From (3.17), (3.25), (3.28) and (2.18) it follows that

$$a + k^2 = H + 3a'^2 = \frac{H_0 + 3\alpha_0^2}{p^2}.$$

Hence, the type of the underlying $\alpha_0$-Sasakian space form $Q_0$ (i.e. $H_0 + 3\alpha_0^2 \geq 0$) determines uniquely the type of the corresponding warped product Kähler manifold of quasi-constant holomorphic sectional curvatures (i.e. $a + k^2 \geq 0$).

Now we shall prove the main theorem in this section.

**Theorem 3.7.** Any Kähler manifold of quasi-constant holomorphic sectional curvatures with $B_0$-distribution, satisfying one of the conditions

$$a + k^2 > 0, \quad a + k^2 = 0, \quad a + k^2 < 0,$$

locally has the structure of a warped product Kähler manifold from the Examples (E).

**Proof.** Let $(M, g, J, D)$ (dim $M = 2n \geq 6$) be a Kähler $QCH$-manifold with $B_0$-distribution $D$ and functions $k, p^*, a$, which satisfy one of the inequalities in the theorem and $m_0$ be an arbitrary point in $M$. The Levi-Civita connection of the metric $g$ is denoted as usual by $\nabla$.

Since $\eta$ is closed, we can find a coordinate neighborhood $U$ about $m_0$ with coordinate functions $(w^1, ..., w^{2n-1}, s)$, where $(w^1, ..., w^{2n-1})$ are in a domain $W \subseteq \mathbb{R}^{2n-1}$, $s \in I' = (-\varepsilon, \varepsilon)$, $\varepsilon > 0$ and $\eta = ds$. The integral submanifolds $Q(s)$ of $\Delta$ in $U$ are determined by $Q(s) : s = \text{const} \in I'$. Especially the submanifold $Q_0$ of $\Delta$ through $m_0$ in $U$ is given by $Q_0 : s = 0$. We have shown that the integral submanifold $Q_0$ of $\Delta$ carries an $\alpha_0$-Sasakian structure $(g_0, \varphi, \tilde{\xi}_0, \tilde{\eta}_0)$ with $\alpha_0 = \frac{k(0)}{2}$ and constant $\varphi$-holomorphic sectional curvatures $H_0 = a(0) + \frac{k^2(0)}{4}$. With the help of the local base $\left\{ \frac{\partial}{\partial w^i} \right\}$, $i = 1, ..., 2n - 1$, of $Q_0$
we construct a frame field \( \{ e_\beta, \varphi e_\beta, \tilde{\xi}_0 \} \), \( \beta = 1, ..., n - 1 \), which is a \( \varphi \)-base for \( T_mQ_0 \) at every point \( m \in Q_0 \). We note that this frame field is a \( \varphi \)-base on every \( Q(s) \), \( s \in I' \).

We need the following lemma.

**Lemma 3.8.** Let \( \{ e_\beta, \varphi e_\beta, \tilde{\xi}_0 \} \), \( \beta = 1, ..., n - 1 \) be a frame field on \( Q_0 \), which is a \( \varphi \)-base at every point of \( Q_0 \). There exist unique positive functions \( \lambda(s), \mu(s), s \in I' \), such that \( \lambda(0) = \mu(0) = 1 \) and the vector fields \( \tilde{e}_\beta = \lambda e_\beta, \varphi \tilde{e}_\beta = \lambda \varphi e_\beta, \tilde{\xi} = \mu \tilde{\xi}_0, \beta = 1, ..., n - 1 \), in \( \mathfrak{X}U \) are parallel along the integral curves of \( \xi \).

**Proof.** The conditions
\[
\frac{\partial}{\partial w}, \frac{\partial}{\partial s} = 0
\]
imply that \( \lambda(e_\beta, \xi) = [\varphi e_\beta, \xi] = [\tilde{\xi}_0, \xi] = 0, \beta = 1, ..., n - 1 \), or equivalently
\[
\nabla_\xi e_\beta = \frac{k}{2} e_\beta, \quad \nabla_\xi (\varphi e_\beta) = \frac{k}{2} (\varphi e_\beta), \quad \nabla_\xi \tilde{\xi}_0 = -p^* \tilde{\xi}_0.
\]
The last equalities determine uniquely the functions
\[
\lambda(s) = e^{-\int_0^s \frac{k}{2} ds}, \quad \mu(s) = e^{\int_0^s p^*(s) ds}, \quad s \in I',
\]
such that
\[
\nabla_\xi (\lambda e_\beta) = \nabla_\xi \tilde{e}_\beta = 0, \quad \nabla_\xi \lambda (\varphi e_\beta) = \nabla_\xi (\varphi \tilde{e}_\beta) = 0, \quad \nabla_\xi (\mu \tilde{\xi}_0) = \nabla_\xi \tilde{\xi} = 0,
\]
which proves the lemma.

We continue the proof of the theorem.

Because of (2.6) and (3.29) the functions \( \lambda(s), \mu(s) \) are related as follows:
\[
(3.30) \quad \frac{1}{\mu} = \frac{1}{\alpha_0} \frac{1}{\lambda} \left( \frac{1}{\lambda} \right)'.
\]

Since span \( \{ \frac{\partial}{\partial w}, \frac{\partial}{\partial s} \} = T_mQ(s) \) at any point \( m \in Q(s), s \) fixed in \( I' \), we can consider the metric \( g_0 \) acting on \( T_mQ(s) \) which is identified with the metric \( g_0 \) of the basic leaf \( Q_0 \).

We give the relation between the metric \( g(s) \) and the metric \( g_0 \) in \( U \).

Let \( \tilde{e}_\beta, \varphi \tilde{e}_\beta, \tilde{\xi}, \beta = 1, ..., n - 1 \), be as in Lemma 3.8. Since \( \tilde{e}_\beta, \varphi \tilde{e}_\beta, \beta = 1, ..., n - 1 \), are parallel along the integral curves of \( \xi \), we have
\[
g(s)(\tilde{e}_\beta, \tilde{e}_\beta) = \lambda^2(s) g(e_\beta, e_\beta) = 1 = g_0(e_\beta, e_\beta).
\]
Hence
\[
g(s)(e_\beta, e_\beta) = \frac{1}{\lambda^2} g_0(e_\beta, e_\beta).
\]
Similarly we have
\[
g(s)(\varphi e_\beta, \varphi e_\beta) = \frac{1}{\lambda^2} g_0(\varphi e_\beta, \varphi e_\beta),
\]
\[
g(s)(\tilde{\xi}_0, \tilde{\xi}_0) = \frac{1}{\mu^2} g_0(\tilde{\xi}_0, \tilde{\xi}_0).
\]

Thus we obtain
\[
(3.31) \quad g(s) = \frac{1}{\lambda^2} (g_0 - \tilde{\eta}_0 \otimes \tilde{\eta}_0) + \frac{1}{\mu^2} \tilde{\eta}_0 \otimes \tilde{\eta}_0.
\]
Putting \( p(s) = \frac{1}{\lambda(s)} \), \( s \in I' \), in view of (3.30) and (3.31), we get
\[
g(s) = p^2(g_0 - \tilde{\eta}_0 \otimes \tilde{\eta}_0) + \left(\frac{pp'}{\alpha_0}\right)^2 \tilde{\eta}_0 \otimes \tilde{\eta}_0.
\] (3.32)

Further the proof of the theorem ends in the following scheme.

Let \( Q_0 \times I' \) be the standard product manifold, defined in the coordinate set \( W \times I' \). In this case we denote \( \xi' = \frac{\partial}{\partial s}, \eta' = ds, \Delta' = \text{span} \left\{ \frac{\partial}{\partial w^1}, ..., \frac{\partial}{\partial w^{2n-1}} \right\} \). Then every leaf of \( Q_0 \times I' \) carries the \( \alpha_0 \)-Sasakian structure \( (g_0, \varphi, \tilde{\xi}_0, \tilde{\eta}_0) \) of the base manifold \( Q_0 \). Using the function \( p(s) = \frac{1}{\lambda(s)} \), where \( \lambda(s) \) is given by (3.29), we endow the manifold \( U' = Q_0 \times I' \) with the warped product Kähler structure \( (g', J', \xi', \eta') \) as in (3.15).

We shall show that the natural coordinate diffeomorphism
\[ F : U \rightarrow U' \]
is an equivalence, i.e. \( F \) preserves the structures \( g, J, \xi \) (consequently \( \eta, \Delta, D \)).

By definition \( F \) preserves the vector field \( \xi = \frac{\partial}{\partial s} \), the 1-form \( \eta = ds \) and the distribution \( \Delta \).

From the condition that \( F \) preserves the vector fields \( \left\{ \frac{\partial}{\partial w^i} \right\}, i = 1, ..., 2n-1 \), it follows that \( F \) preserves the \( \varphi \)-base \( \{ e_\beta, \varphi e_\beta, \tilde{\xi}_0 \}, \beta = 1, ..., n-1 \) in \( \mathfrak{X}U \). This means that \( F \) preserves the structure \( \varphi \) on every leaf \( Q(s), s \in I' \). Taking into account (3.9), we conclude that \( F \) preserves the complex structure \( J \).

Finally, the metric \( g(s) \) on \( Q(s) \subset U, s \in I' \) has the form (3.32). On the other hand, according to (3.15), the restriction of the metric \( g' \) of \( U' \) onto the leaves \( Q'(s) \) is given by the same formula (3.32), written with respect to the metric \( g_0 \) of the base \( Q_0 \). Hence \( F \) is also an isometry.

**QED**

### 4. On the local theory of Bochner-Kähler manifolds

Let \( (M, g, J) \) (\( \dim M = 2n \geq 4 \)) be a Kähler manifold. In the next calculations we shall use the complexification \( T_m^C M, m \in M \) and its standard splitting
\[ T_m^C M = T_m^{1,0} M \oplus T_m^{0,1} M. \]

Any complex basis of \( T_m^{1,0} M \) will be denoted by \( \{ Z_\alpha \}, \alpha = 1, ..., n \), and the conjugate basis \( \{ \bar{Z}_\alpha = \overline{Z_\alpha} \}, \bar{\alpha} = \bar{I}, ..., \bar{n} \) will span \( T_m^{0,1} M \). Unless otherwise stated, the Greek indices \( \alpha, \beta, \gamma, \delta, \varepsilon \) will run through \( 1, ..., n \).

We recall that the Bochner curvature operator \( B \) acts on the curvature tensor \( R \) of \( (M, g, J) \) with respect to a complex base as follows:
\[
(B(R))_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - \frac{1}{n+2} \frac{1}{\tau} (g_{\alpha\beta} \rho_{\gamma\delta} + g_{\gamma\beta} \rho_{\alpha\delta} + g_{\gamma\delta} \rho_{\alpha\beta} + g_{\alpha\delta} \rho_{\gamma\beta}) + \frac{2(n+1)(n+2)}{\tau} (g_{\alpha\beta} g_{\gamma\delta} + g_{\gamma\beta} g_{\alpha\delta}).
\]
The manifold \((M, g, J)\) is said to be Bochner-Kähler if its Bochner curvature tensor \(B(R)\) vanishes identically, which is equivalent to the curvature identity:

\[
R_{\alpha\beta\gamma\delta} = \frac{1}{n+2} \left( g_{\alpha\delta\beta} \rho_{\gamma\delta} + g_{\gamma\delta\beta} \rho_{\alpha\delta} + g_{\gamma\delta\alpha} \rho_{\beta\delta} + g_{\beta\delta\alpha} \rho_{\gamma\delta} \right) - \frac{1}{2(n+1)(n+2)} (g_{\alpha\beta} g_{\gamma\delta} + g_{\gamma\delta} g_{\alpha\beta}).
\]

(4.1)

The curvature tensor of any Kähler manifold as a consequence of the second Bianchi identity satisfies the equalities:

\[
\nabla_{\alpha} \rho_{\gamma\beta} = \nabla_{\gamma} \rho_{\alpha\beta} = \nabla_{\beta} \rho_{\gamma\alpha} = \nabla_{\gamma} \nabla_{\beta} \rho_{\alpha\gamma}.
\]

The other meaning of the identity \(\nabla_{\alpha} \rho_{\gamma\beta} = \nabla_{\gamma} \rho_{\alpha\beta}\) is that the Ricci form \(\rho(JX, Y)\), \(X, Y \in \mathfrak{X}M\) is closed, i.e.

\[(\nabla X \rho)(JY, Z) + (\nabla Y \rho)(JZ, X) + (\nabla Z \rho)(JX, Y) = 0, \quad X, Y, Z \in \mathfrak{X}M.\]

(4.2)

\[(\nabla X \rho)(JY, Z) + (\nabla Y \rho)(JZ, X) + (\nabla Z \rho)(JX, Y) = 0, \quad X, Y, Z \in \mathfrak{X}M.\]

On the other hand (4.1) implies

\[
\nabla_{\gamma} \rho_{\alpha\beta} = \frac{1}{2(n+1)} (\tau_{\alpha\rho_{\gamma\beta}} + \tau_{\gamma\rho_{\alpha\beta}}).
\]

(4.3)

**Remark 4.1.** The identity (4.3) in view of (4.2) is equivalent to

\[
(\nabla X \rho)(Y, Z) = -\frac{1}{4(n+1)} \{ 2d\tau(X)g(Y, Z) + d\tau(Y)g(X, Z) + d\tau(Z)g(X, Y) + d\tau(JY)g(X, JZ) + d\tau(JZ)g(X, JY) \}, \quad X, Y, Z \in \mathfrak{X}M.
\]

From (4.1) and (4.3) it follows that the covariant derivative of the curvature tensor \(R\) of any Bochner-Kähler manifold satisfies the following identity:

\[
\nabla_{\alpha} R_{\beta\gamma\delta} = \frac{1}{(n+1)(n+2)} (\tau_{\alpha\beta\gamma\delta} + \tau_{\beta\gamma\delta\alpha} + \tau_{\gamma\delta\alpha\beta}).
\]

(4.4)

Studying the integrability conditions for (4.3) we obtain the basic formulas connecting derivatives of the scalar curvature \(\tau\) with curvature properties of the given manifold.

**Proposition 4.2.** Let \((M, g, J)\) be a Bochner-Kähler manifold and \(T = \text{grad} \, \tau\). Then

(i) the vector field \(\frac{T - iJT}{2}\) is holomorphic;

(ii) \((n+2)\nabla_{\alpha} \tau_{\beta} + 2(n+1)\rho_{\alpha\beta}^2 - \tau \rho_{\alpha\beta} = \frac{(n+2)\Delta \tau + 2(n+1)\|\rho\|^2 - \tau^2}{2n} g_{\alpha\beta},\)

where \(\rho_{\alpha\beta}^2 = \rho_{\alpha\beta} \rho_{\beta\alpha};\)

(iii) \(2\rho(X, T) = -X(\Delta \tau), \quad X \in \mathfrak{X}M.\)

**Proof.** Applying the standard Ricci formula in Riemannian geometry

\[
\nabla_{\alpha} \nabla_{\beta} \rho_{\gamma\delta} - \nabla_{\beta} \nabla_{\alpha} \rho_{\gamma\delta} = \frac{1}{2(n+1)} (g_{\beta\delta} \nabla_{\alpha} \tau_{\gamma} - g_{\alpha\delta} \nabla_{\beta} \tau_{\gamma}) = 0,
\]

(4.5)
\[\nabla_\alpha \nabla_\beta \rho_{\gamma\delta} - \nabla_\beta \nabla_\alpha \rho_{\gamma\delta} = -\frac{1}{n+2} \left( g_{\gamma\beta} \rho_{\alpha\delta}^2 - g_{\alpha\delta} \rho_{\gamma\beta}^2 \right) \]
\[+ \frac{1}{2(n+1)(n+2)} \left( g_{\gamma\beta} \rho_{\alpha\delta} - g_{\alpha\delta} \rho_{\gamma\beta} \right) \]
\[= \frac{1}{2(n+1)} \left( g_{\gamma\beta} \nabla_\alpha \tau_\delta - g_{\alpha\delta} \nabla_\gamma \tau_\beta \right). \]

After taking a trace in (4.5) we find
\[\nabla_\beta \tau_\gamma = 0. \]

This equality implies that the \((1,0)\)-part \(T - iJT\) of the vector field \(T = \text{grad} \tau\) is holomorphic, which proves (i).

In a similar way (4.6) implies
\[\nabla_\gamma \tau_\beta + 2(n+1)\rho_{\gamma\beta}^2 - \tau \rho_{\gamma\beta} = \frac{(n+2)\Delta \tau + 2(n+1)\|\rho\|^2 - \tau^2}{2n} g_{\gamma\beta}, \]
which is (ii).

To prove (iii) we consider the identity
\[\nabla_\alpha \nabla_\beta \tau_\gamma - \nabla_\beta \nabla_\alpha \tau_\gamma = -R^\xi_{\alpha\beta\gamma} \tau_\xi. \]

Keeping in mind (4.7) we take a trace in the last equality and get
\[-\frac{1}{2}(\Delta \tau)_{\beta} = \rho^\xi_{\beta} \tau_\xi, \]
which is
\[2\rho(X,T) = -X(\Delta \tau), \quad X \in \mathfrak{X}M. \]

QED

Remark 4.3. The equalities
\[\nabla_\alpha \tau_\beta = 0, \quad \nabla_\alpha \tau_\beta - \nabla_\beta \tau_\alpha = 0\]
imply \(JT\) is an analytic and Killing vector field. Hence \(JT\) generates a local one-parameter group of local holomorphic motions.

Proposition 4.4. On any Bochner-Kähler manifold the function
\[\|\rho\|^2 - \frac{\tau^2}{2(n+1)} + \frac{\Delta \tau}{n+1} \]
is a constant.

Proof. From the equality (4.3) we obtain successively
\[\|\rho\|^2_\alpha = 4\rho^\beta\gamma \nabla_\alpha \rho_{\beta\gamma} = \frac{1}{n+1} (\tau^2 - \Delta \tau)_{\alpha}. \]

Hence
\[\|\rho\|^2 - \frac{\tau^2}{2(n+1)} + \frac{\Delta \tau}{n+1} = \text{const.} \]

We set
\[\|\rho\|^2 - \frac{\tau^2}{2(n+1)} + \frac{\Delta \tau}{n+1} = \mathfrak{B} \]

QED
and call $B$ the Bochner constant of the manifold.

It is clear that the 1-form $d\tau$ is of basic importance in the above formulas. The equality (4.3) shows that the conditions $\tau = \text{const}$ and $\nabla \rho = 0$ are equivalent on a Bochner-Kähler manifold. Because of the structural theorem in [13] the case $B(R) = 0$, $d\tau = 0$ can be considered as well-studied.

According to [9] it follows that any compact Bochner-Kähler manifold satisfies the condition $d\tau = 0$ ($\nabla R = 0$).

Further we consider the general case of Bochner-Kähler manifolds

$$d\tau \neq 0 \quad \text{for all points} \ p \in M.$$  

This condition allows us to introduce the frame field

$$\left\{ \xi = \frac{\text{grad} \tau}{\|d\tau\|} = \frac{T}{\|d\tau\|}, \quad J\xi = \frac{J\text{grad} \tau}{\|d\tau\|} = \frac{JT}{\|d\tau\|} \right\}$$

and consider the $J$-invariant distributions $D_\tau$ and $D_\tau = \text{span}\{\xi, J\xi\}$.

Thus our approach to the local theory of Bochner-Kähler manifolds is to investigate them as Kähler manifolds $(M, g, J, D_\tau)$ endowed with a $J$-invariant distribution $D_\tau$ generated by the Kähler structure $(g, J)$. In what follows we call this distribution the scalar distribution of the manifold.

The scalar distribution $D_\tau$ of any Bochner-Kähler manifold carries the functions

$$\kappa := R(\xi, J\xi, J\xi, \xi), \quad \sigma := \rho(\xi, \xi) = \rho(J\xi, J\xi).$$

Then $\kappa$, $\sigma$ and $\tau$ determine the functions

$$a = \frac{\tau - 4\sigma + 2\kappa}{n(n-1)}, \quad b = \frac{-2\tau + 4(n+2)\sigma - 4(n+1)\kappa}{n(n-1)},$$

$$c = \frac{\tau - 4(n+1)\sigma + (n+1)(n+2)\kappa}{n(n-1)}.$$

Calculating $\kappa$ from (4.1) we obtain

$$0 = \tau - 4(n+1)\sigma + (n+1)(n+2)\kappa = n(n-1)\ c.$$  

Below we establish some properties of the distributions $D_\tau$, $D_\tau$ and $\Delta_\tau$ ($\perp \xi$).

Let $\{Z_\lambda\}$, $\lambda = 1, \ldots, n-1$ be a basis for $D^{1,0}(m)$. The basis $\{Z_0, Z_\lambda\}$, $\lambda = 1, \ldots, n-1$, where $Z_0 = \frac{\xi - i J\xi}{2}$, is said to be a special complex basis for $T_m^{1,0}M$. Then $\{Z_0, Z_\lambda\}$, $\lambda = 1, \ldots, n-1$, is a special complex basis for $T_m^{0,1}M$. The Greek indices $\lambda, \mu, \nu, \kappa, \sigma$ will run through $1, \ldots, n-1$.

With respect to special complex bases the 1-forms $\eta$ and $\tilde{\eta}$ have the following components:

$$\eta_\lambda = g_{\alpha\theta}, \quad \eta_\lambda = g_{\alpha\theta}, \quad \eta_\lambda = -i\eta_\lambda, \quad \eta_\lambda = i\eta_\lambda,$$

$$\eta_\lambda = \eta_\lambda = 0, \quad \eta_\lambda = \eta_\lambda = g_{\alpha\theta} = \frac{1}{2}.$$

We introduce the following functions and 1-forms associated with the vector fields $\nabla_\xi \xi$ and $\nabla_{J\xi} J\xi$:

$$p = g(\nabla_\xi \xi, J\xi), \quad p^* = g(\nabla_{J\xi} J\xi, \xi),$$

$$\theta(X) = g(\nabla_\xi \xi, X) - p \tilde{\eta}(X), \quad \theta^*(X) = g(\nabla_{J\xi} J\xi, X) - p^* \eta(X), \quad X \in T_m M.$$  

It is clear that $\theta(X) = \theta(x_0)$, $\theta^*(X) = \theta^*(x_0)$, where $x_0 = X - \tilde{\eta}(X) J\xi - \eta(X) \xi$. 


Taking into account that \( Z_0 = \frac{\xi - iJ\xi}{2}, \ Z_0 = \frac{\xi + iJ\xi}{2} \) we find

\[
\nabla_0 \eta_\lambda = \frac{1}{2} (\theta_\lambda + \theta_\lambda^*), \quad \nabla_0 \bar{\eta}_\lambda = \frac{1}{2} (\theta_\lambda - \theta_\lambda^*),
\]

(4.14)

\[
\nabla_0 \eta_\mu = \frac{p^* - ip}{4}, \quad \nabla_0 \bar{\eta}_0 = \frac{-p^* + ip}{4}.
\]

(4.15)

Since the distribution \( \Delta_\tau \) is involutive then

\[
\nabla_\lambda \eta_0 = -\nabla_\lambda \bar{\eta}_0 = \frac{1}{2} \theta_\lambda^*.
\]

(4.16)

From \( d\tau = \|d\tau\| \eta \) we have

\[
(\nabla_X d\tau) Y = \|d\tau\| (\nabla_X \eta) Y + X(\|d\tau\|) \eta(Y), \quad X, Y \in \mathfrak{X}M.
\]

Then (4.7) and (4.17) imply that

\[
\|d\tau\| \alpha \eta_\beta + \|d\tau\| \nabla_\alpha \eta_\beta = 0.
\]

The last equality, (4.10), (4.14), (4.16) and (4.15) give

\[
\nabla_\lambda \eta_\mu = 0;
\]

(4.18)

\[
\theta_\lambda + \theta_\lambda^* = 0, \quad \theta_\lambda = (\ln \|d\tau\|)_\lambda; \quad p = -J\xi (\ln \|d\tau\|), \quad p^* = -\xi (\ln \|d\tau\|).
\]

In a similar way the equality \( \nabla_\alpha \tau_\beta = \nabla_\beta \tau_\alpha \) implies

\[
\nabla_\lambda \eta_\mu = \nabla_\mu \eta_\lambda; \quad p = J\xi (\ln \|d\tau\|).
\]

(4.19)

From (4.17) we calculate \( div \xi = \frac{\Delta_\tau}{\|d\tau\|} + p^* \). On the other hand \( div_0 \xi = div \xi - p^* \). Hence

\[
div_0 \xi = \frac{\Delta_\tau}{\|d\tau\|}.
\]

Summarizing the conditions (4.18) and (4.19) we get

**Proposition 4.5.** The scalar distribution of any Bochner-Kähler manifold has the following properties:

1. \( \nabla_\lambda \eta_\mu = 0, \ \nabla_\lambda \eta_\mu = \nabla_\mu \eta_\lambda, \)
2. \( \theta_\lambda + \theta_\lambda^* = 0, \ \theta_\lambda = (\ln \|d\tau\|)_\lambda, \)
3. \( p = 0 = J\xi (\ln \|d\tau\|), \quad p^* = -\xi (\ln \|d\tau\|), \)
4. \( div_0 \xi = \frac{\Delta_\tau}{\|d\tau\|} \)

(4.20)

Thus we have

The local theory of any Bochner-Kähler manifold with scalar distribution \( D_\tau \) is determined by: the symmetric tensor \( \nabla_\lambda \eta_\mu \), the 1-form \( \theta_\lambda = (\ln \|d\tau\|)_\lambda \) and the function \( p^* = -\xi (\ln \|d\tau\|) \).

Any additional conditions for the above mentioned objects give rise to special classes of Bochner-Kähler manifolds.
5. Bochner-Kähler manifolds whose scalar distribution is a $B_0$-distribution

The first step in our study of Bochner-Kähler manifolds with respect to their scalar distribution is to study the basic class of Bochner-Kähler manifolds whose scalar distribution is a $B_0$-distribution. We start with the following curvature characterization of these manifolds.

**Proposition 5.1.** Let $(M, g, J)$ (dim $M = 2n \geq 6$) be a Bochner-Kähler manifold with respect to their scalar distribution. We start with the following curvature characterization of these manifolds.

$$R = a\pi + b\Phi, \quad b \neq 0.$$ 

**Proof.** Let $D_\tau$ be a $B_0$-distribution, i.e. we have to add the conditions

$$\nabla_\lambda \eta_\mu = \frac{k}{2} g_{\lambda\mu}, \quad k = \frac{\text{div}_0 \xi}{n - 1} \neq 0,$$

$$\theta_\lambda = (\ln \|d\tau\|)_\lambda = 0$$

to the equations (4.20).

Then (2.5) implies that:

$$R(X, Y)\xi = -\frac{1}{4} (k^2 + 2kp^*) \{\eta(X)Y - \eta(Y)X$$

$$- \tilde{\eta}(X)JY + \tilde{\eta}(Y)JX + 2g(JX, Y)J\xi\}$$

$$- \frac{1}{2k} \xi(k^2 + 2kp^*) \{\eta(X)\tilde{\eta}(Y) - \eta(Y)\tilde{\eta}(X)\}J\xi,$$

$X, Y \in \mathfrak{X}M$.

Taking a trace in (5.2) we find

$$\rho(X, \xi) = \frac{1}{2k} \xi(k^2 + 2kp^*) + \frac{n+1}{2} (k^2 + 2kp^*),$$

From (5.2) and (5.3) it follows that

$$\kappa = R(\xi, J\xi, J\xi, \xi) = \frac{1}{2k} \xi(k^2 + 2kp^*) + (k^2 + 2kp^*),$$

$$\sigma = \rho(\xi, \xi) = \frac{1}{2k} \xi(k^2 + 2kp^*) + \frac{n+1}{2} (k^2 + 2kp^*).$$

The equalities (4.1) and (5.2) give two expressions for the component $R_{\eta\mu\nu\sigma}$ of the curvature tensor $R$. Comparing these expressions and taking into account (4.11) we obtain

$$\rho_{\nu\mu} = \frac{\tau - 2\sigma}{2(n - 1)} g_{\nu\mu}.$$ 

The conditions (5.3) and (5.6) imply

$$\rho = \frac{\tau - 2\sigma}{2(n - 1)} g + \frac{2n\sigma - \tau}{2(n - 1)} (\eta \otimes \eta + \tilde{\eta} \otimes \tilde{\eta}),$$

which combined with (4.1) gives

$$R = a\pi + b\Phi.$$
Thus we obtained that \((M, g, J, D_\tau)\) is a Kähler QCH-manifold with \(B_0\)-distribution. Applying the second Bianchi identity to (5.8) it follows that (cf Theorem 3.5 in [6])

\[
da = \frac{1}{2} b k \eta, \quad db = b k \eta.
\]

From (5.8) we calculate

\[
\tau = (n + 1)(na + b).
\]

On the other hand, it follows from (5.9) that

\[
d(2a - b) = 0.
\]

Combining (5.10) and (5.11) we get

\[
d\tau = \frac{1}{2} (n + 1)(n + 2) db \neq 0,
\]

which implies that \(b \neq 0\).

QED

Taking into account (5.11), we denote

\[
b_0 = \frac{2a - b}{2} = \text{const}.
\]

Thus any Bochner-Kähler manifold whose scalar distribution is a \(B_0\)-distribution has two geometric constants: the Bochner constant \(B\) given by (4.9) and \(b_0\) given by (5.13).

Next we find expressions for the functions \(\kappa\), \(\sigma\), \(a\) and \(b\) by \(\tau\).

**Lemma 5.2.** Let \((M, g, J)\) (dim \(M = 2n \geq 6\)) be a Bochner-Kähler manifold whose scalar distribution is a \(B_0\)-distribution. Then

\[
\kappa = \frac{3\tau}{(n + 1)(n + 2)} - \frac{2(n - 1)b_0}{n + 2}, \quad \sigma = \frac{\tau}{n + 1} - \frac{(n - 1)b_0}{2},
\]

\[
a = \frac{\tau}{(n + 1)(n + 2)} + \frac{2b_0}{n + 2}, \quad b = \frac{2\tau}{(n + 1)(n + 2)} - \frac{2nb_0}{n + 2},
\]

\[
k^2 = \frac{(n + 1)^2(n + 2)^2\mathcal{R} - \{2\tau - (n + 1)(n - 2)b_0\}^2}{4(n + 1)(n + 2)\{\tau - n(n + 1)b_0\}^2},
\]

where

\[
\mathcal{R} = \frac{4B + n^2(2n + 1)b_0^2}{n(n + 2)}.
\]

**Proof.** Taking into account (2.3), (4.11) and (5.13) we find (5.14) and (5.15).

From (5.2), (5.4) and (5.5) we get

\[
R(x_0, \xi, \xi, x_0) = \frac{\sigma - \kappa}{2(n - 1)} = \frac{1}{4} (k^2 + 2kp^*), \quad x_0 \in D, \quad g(x_0, x_0) = 1.
\]

On the other hand, from (5.8) we have

\[
R(x_0, \xi, \xi, x_0) = \frac{2a + b}{8} = \frac{\sigma - \kappa}{2(n - 1)}, \quad x_0 \in D, \quad g(x_0, x_0) = 1.
\]
Comparing the above equalities in view of (5.13) and (2.6) we find

\[(5.18)\]
\[\xi(k) = \frac{1}{2} (k^2 + b + b_0).\]

Then (5.9) implies that

\[(5.19)\]
\[\frac{dk^2}{db} = \frac{2k \xi(k)}{\xi(b)} = - \frac{k^2 + b + b_0}{b}.\]

The general solution of the last equation is

\[(5.20)\]
\[k^2 = \frac{1}{2b} \{K - (b + b_0)^2\}\]

where \(K = \text{const.}\)

We shall express the constant \(K\) by means of the constants \(B\) and \(b_0\) of the manifold under consideration.

By using (5.12) and (5.9) we have

\[(5.21)\]
\[\|d\tau\| = \xi(\tau) = \frac{\sqrt{2}(n+1)(n+2)}{2} k b.\]

Differentiating the equality \(\tau_\alpha = \|d\tau\| \eta_\alpha\) we find

\[(5.22)\]
\[\Delta \tau = \frac{(n+1)(n+2)}{2b} \{2\xi(k) + (n+1)k^2\}.\]

Further we replace \(\xi(k)\) from (5.18) and \(k^2\) from (5.19) into (5.22) and obtain

\[(5.23)\]
\[\frac{\Delta \tau}{n+1} = \frac{n+2}{4} \{nK - (n+2)b^2 - 2(n+1)b_0^2 - n b_0^2\}.\]

From (5.10) and (5.13) it follows that

\[(5.24)\]
\[\frac{\tau^2}{2(n+1)} = \frac{n+1}{8} \{(n+2)^2b^2 + 4n(n+2)b_0 + 4n^2b_0^2\}.\]

In order to calculate the constant \(\mathfrak{B}\) it remains to find \(\|\rho\|^2\). By using the equality

\[\rho_{\alpha\bar{\beta}} = \frac{(n+2)b + 2(n+1)b_0}{4} g_{\alpha\bar{\beta}} + \frac{n+2}{2} b_0 \eta_{\alpha\bar{\beta}}\]

we get

\[(5.25)\]
\[\|\rho\|^2 = \frac{(n+2)^2(n+3)}{8} b^2 + \frac{(n+1)^2(n+2)}{2} b_0 + \frac{n(n+1)}{2} b_0^2.\]

Counting (5.23), (5.24) and (5.25) we find

\[\mathfrak{B} = \|\rho\|^2 - \frac{\tau^2}{2(n+1)} + \frac{\Delta \tau}{n+1} = \frac{n(n+2)}{4} K - \frac{n^2(2n+1)}{4} b_0^2.\]

Hence

\[K = \frac{4\mathfrak{B} + n^2(2n+1)b_0^2}{n(n+2)}.\]

Now (5.19), (5.15) and (5.17) imply the assertion. QED
It follows from (5.17) that the pair \((B, b_0)\) determines the pair \((\bar{B}, b_0)\) and vice versa. Further in our considerations we shall use the pair of geometric constants \((\bar{B}, b_0)\).

From (5.13) and (5.19) we get

\[
a + k^2 = \frac{(n + 1)(n + 2)(\bar{B} - b_0^2)}{4\{\tau - n(n + 1)b_0\}}.
\]

In [7] we have shown that a Kähler manifold with curvature tensor satisfying (2.2) is Bochner flat if and only if \(c = 0\).

Another characterization is given by the following

**Lemma 5.3.** Let \((M, g, J, D)\) \((\dim M = 2n \geq 6)\) be a Kähler QCH-manifold with \(B_0\)-distribution \(D\). Then the following conditions are equivalent:

\(i\) \(B(R) = 0\),

\(ii\) \(2a - b = \text{const}\).

**Proof.** Under the conditions of the lemma the curvature tensor \(R\) has the form (2.2). Then the second Bianchi identity applied to \(R\) (Theorem 3.5 [6]) implies that

\[d(2a - b) = -4kc \eta.\]

Now the statement of the lemma follows immediately. QED

### 6. Warped Product Bochner-Kähler Metrics

From Theorem 3.7 and Proposition 5.1 it follows that the local theory of Bochner-Kähler manifolds whose scalar distribution is a \(B_0\)-distribution is equivalent to the local theory of warped product Bochner-Kähler manifolds.

In this section we give a complete description of the warped product Kähler metrics of quasi-constant holomorphic sectional curvatures, which are Bochner flat.

Let \((N, g, J, \xi)\) \((\dim M = 2n \geq 6)\) be a warped product Kähler manifold given as in (3.15). Next we shall consider the metric (3.15) with respect to the function \(p(t), \, t \in I \subset \mathbb{R}\), satisfying the initial conditions \(p(t_0) = 1, \, \frac{dp}{dt}(t_0) = \alpha_0\), in the form

\[
g = p^2(t)\left\{g_0 + \left(\frac{1}{\alpha_0} \frac{dp}{dt} - 1\right) \bar{\eta}_0 \otimes \bar{\eta}_0\right\} + \eta \otimes \eta.
\]

Thus any warped product Kähler metric of quasi-constant holomorphic sectional curvatures is determined uniquely by the function \(p(t)\) and the underlying \(\alpha_0\)-Sasakian space form \(Q_0\).

Now we shall find the functions \(p(t)\) for which the metric (6.1) is Bochner-Kähler.

The functions \(a(s)\) and \(b(s)\) are given by (3.28). According to Lemma 5.3 the manifold \((N, g, J, \xi)\) is Bochner flat if and only if \(2a - b = \text{const} = 2b_0\). We denote

\[d_0 := H_0 + 3\alpha_0^2.\]

Taking into account (5.18), (5.19), (3.17) and (5.13) we find

\[
\left(\frac{p'}{p}\right)' - \left(\frac{p'}{p}\right)^2 = \frac{b_0}{4} - \frac{d_0}{2p^2}.
\]
and

\( (6.3) \)

\[
\left\{ \left( \frac{p'}{p} \right)' - \left( \frac{p'}{p} \right)^2 \right\}^2 = \frac{\mathcal{K}}{16} + \frac{d_0}{p^2} \left( \frac{p'}{p} \right)'.
\]

Warped product Bochner-Kähler metrics with \( a + k^2 \neq 0 \) \((d_0 = H_0 + 3\alpha_0^2 \neq 0)\).

The equalities (6.2) and (6.3) imply

\( (6.4) \)

\[
16d_0 p'^2 = (b_0^2 - \mathcal{K}) p^4 - 4b_0d_0 p^2 + 4d_0^2.
\]

Next we treat the function \( p(s), s \in I_0 \) with respect to the parameter \( t \in I \). Keeping in mind Remark 3.4, we have

\[
p' = \sqrt{\alpha_0} \frac{dp}{dt}.
\]

Then (6.4) becomes

\( (6.5) \)

\[
16\alpha_0 d_0 \frac{dp}{dt} = (b_0^2 - \mathcal{K}) p^4 - 4b_0d_0 p^2 + 4d_0^2.
\]

The initial conditions \( p(t_0) = 1 \) and \( \frac{dp}{dt}(t_0) = \alpha_0 \) give the following relation between the geometric constants of the manifold:

\( (6.6) \)

\[
16\alpha_0^2 d_0 + \mathcal{K} = (b_0 - 2d_0)^2.
\]

Then from (6.5) we have

\( (6.7) \)

\[
t = 16\alpha_0 d_0 \int \frac{dp}{(b_0^2 - \mathcal{K}) p^4 - 4b_0d_0 p^2 + 4d_0^2}.
\]

Analyzing (6.7) and taking into account (6.6) we obtain the following solutions of the equation (6.5):

1. The case \( a + k^2 > 0 \).

This case is determined by \( d_0 > 0 \), which is equivalent to \( H_0 > -3\alpha_0^2 \).

Type 1. \( \mathcal{K} > 0, \ -\sqrt{\mathcal{K}} < b_0 < \sqrt{\mathcal{K}} \) \((b_0^2 - \mathcal{K} < 0)\).

\[
t = 2\sqrt{2} \alpha_0 \frac{\sqrt{\mathcal{K} - b_0}}{\sqrt{d_0\mathcal{K}}} \text{arctan} \frac{\sqrt{\mathcal{K} - b_0}}{\sqrt{2d_0}} p - 2\sqrt{2} \alpha_0 \frac{\sqrt{\mathcal{K} + b_0}}{\sqrt{d_0\mathcal{K}}} \ln \left| p - \sqrt{\frac{2d_0}{\sqrt{\mathcal{K} + b_0}}} \right| + \sqrt{\frac{2d_0}{\sqrt{\mathcal{K} + b_0}}} + \sqrt{\frac{2d_0}{\sqrt{\mathcal{K} + b_0}}} p,
\]

\[
0 < p^2 < \frac{2d_0}{\sqrt{\mathcal{K} + b_0}}.
\]

Type 2. \( \mathcal{K} > 0, \ -\frac{16\alpha_0^4 + \mathcal{K}}{8\alpha_0^2} \leq b_0 < -\sqrt{\mathcal{K}} \) \((b_0^2 - \mathcal{K} > 0)\).

\[
t = 2\sqrt{2} \alpha_0 \frac{\sqrt{\mathcal{K} - b_0}}{\sqrt{d_0\mathcal{K}}} \text{arctan} \frac{\sqrt{\mathcal{K} - b_0}}{\sqrt{2d_0}} p - 2\sqrt{2} \alpha_0 \frac{\sqrt{-(\mathcal{K} + b_0)}}{\sqrt{d_0\mathcal{K}}} \text{arctan} \frac{-\sqrt{\mathcal{K} + b_0}}{\sqrt{2d_0}} p,
\]

\[
p^2 > 0.
\]
Type 3. \( \Re > 0, \ b_0 > \sqrt{\Re} \ (b_0^2 - \Re > 0) \).

\[
t = \frac{\sqrt{2} \alpha_0 \sqrt{b_0 - \sqrt{\Re}}}{\sqrt{d_0 \Re}} \ln \frac{|p - \sqrt{2d_0 \frac{b_0 - \sqrt{\Re}}{b_0 - \sqrt{\Re}}}|}{p + \sqrt{2d_0 \frac{b_0 - \sqrt{\Re}}{b_0 - \sqrt{\Re}}}} - \frac{\sqrt{2} \alpha_0 \sqrt{b_0 + \sqrt{\Re}}}{\sqrt{d_0 \Re}} \ln \frac{|p - \sqrt{2d_0 \frac{b_0 + \sqrt{\Re}}{b_0 + \sqrt{\Re}}}|}{p + \sqrt{2d_0 \frac{b_0 + \sqrt{\Re}}{b_0 + \sqrt{\Re}}}},
\]

\( p^2 \in \left( 0, \frac{2d_0}{b_0 + \sqrt{\Re}} \right) \cup \left( \frac{2d_0}{b_0 - \sqrt{\Re}}, \infty \right) \).

Type 4. \( \Re = 0, \ -2\alpha_0^2 \leq b_0 < 0 \).

\[
t = \frac{4\alpha_0 p}{2d_0 - b_0 p^2} + \frac{2\sqrt{2} \alpha_0 \arctan \frac{-b_0}{\sqrt{2d_0}}}{\sqrt{-b_0 d_0}}, \quad p^2 > 0.
\]

Type 5. \( \Re = 0, \ b_0 > 0 \).

\[
t = \frac{\sqrt{2} \alpha_0}{\sqrt{b_0 d_0}} \ln \frac{p + \sqrt{2d_0 \frac{b_0 - \sqrt{\Re}}{b_0 - \sqrt{\Re}}}}{|p - \sqrt{2d_0 \frac{b_0 - \sqrt{\Re}}{b_0 - \sqrt{\Re}}}|} + \frac{4\alpha_0 p}{2d_0 - b_0 p^2}, \quad p^2 \in \left( 0, \frac{2d_0}{b_0} \right) \cup \left( \frac{2d_0}{b_0}, \infty \right).
\]

Type 6. \(-16\alpha_0^2 d_0 \leq \Re < 0, \ -2\alpha_0^2 - \frac{\Re}{8\alpha_0^2} \leq b_0 \).

\[
t = \frac{2\alpha_0}{\mu} \ln \frac{\lambda p^2 - \mu p + \nu}{\lambda p^2 + \mu p + \nu} + \frac{4\alpha_0}{\bar{\mu}} \left( \arctan \frac{2\lambda p - \mu}{\sqrt{4\lambda^2 p^2 - \mu^2}} + \arctan \frac{2\lambda p + \mu}{\sqrt{4\lambda^2 p^2 + \mu^2}} \right),
\]

\( p^2 > 0 \).

where

\[
\lambda = \sqrt{b_0^2 - \Re}, \quad \mu = 2\sqrt{d_0} \sqrt{b_0^2 - \Re + b_0}, \quad \bar{\mu} = 2\sqrt{d_0} \sqrt{b_0^2 - \Re - b_0}, \quad \nu = 2d_0.
\]

II. The case \( a + k^2 < 0 \).

This case is determined by \( d_0 < 0 \), which is equivalent to \( H_0 < -3\alpha_0^2 \).

Type 7. \( \Re \geq -16\alpha_0^2 d_0, \ -2\alpha_0^2 - \frac{\Re}{8\alpha_0^2} \leq b_0 < -\sqrt{\Re} \ (b_0^2 - \Re > 0) \).

\[
t = \frac{\sqrt{2} \alpha_0 \sqrt{\sqrt{\Re} - b_0}}{\sqrt{-d_0 \Re}} \ln \frac{|p - \sqrt{2d_0 \frac{\sqrt{\Re} - b_0}{\sqrt{-d_0 \Re}}}|}{p + \sqrt{2d_0 \frac{\sqrt{\Re} - b_0}{\sqrt{-d_0 \Re}}}} - \frac{\sqrt{2} \alpha_0 \sqrt{-(b_0 + \sqrt{\Re})}}{\sqrt{-d_0 \Re}} \ln \frac{|p - \sqrt{2d_0 \frac{b_0 + \sqrt{\Re}}{b_0 + \sqrt{\Re}}}|}{p + \sqrt{2d_0 \frac{b_0 + \sqrt{\Re}}{b_0 + \sqrt{\Re}}}},
\]

\[
\frac{2d_0}{b_0 - \sqrt{\Re}} < p^2 < \frac{2d_0}{b_0 + \sqrt{\Re}}.
\]

Type 8. \( \Re \geq -16\alpha_0^2 d_0, \ -\sqrt{\Re} < b_0 < \sqrt{\Re} \ (b_0^2 - \Re < 0) \).

\[
t = \frac{\sqrt{2} \alpha_0 \sqrt{\sqrt{\Re} - b_0}}{\sqrt{-d_0 \Re}} \ln \frac{|p - \sqrt{2d_0 \frac{\sqrt{\Re} - b_0}{\sqrt{-d_0 \Re}}}|}{p + \sqrt{2d_0 \frac{\sqrt{\Re} - b_0}{\sqrt{-d_0 \Re}}}} - 2\sqrt{2} \alpha_0 \sqrt{\sqrt{\Re} + b_0} \arctan \frac{\sqrt{\Re + b_0}}{\sqrt{-2d_0}} p,
\]
\[ p^2 \in \left( \frac{2d_0}{b_0 - \sqrt{k}}, \infty \right). \]

Warped product Bochner-Kähler metrics with \( a + k^2 = 0 \) \((d_0 = H_0 + 3\alpha_0^2 = 0)\).

In this case (5.26) gives that \( b_0^2 = k \), which implies that the equations (6.2) and (6.3) coincide.

Considering the positive function \( p(s) \) with respect to the parameter \( t \) the equality (6.2) becomes

\[ \frac{d^2 p}{d t^2} - \frac{4}{p} \left( \frac{d p}{d t} \right)^2 - \frac{b_0}{2\alpha_0} p \frac{d p}{d t} = 0. \]  

Taking into account the initial conditions \( p(t_0) = 1, \frac{d p}{d t}(t_0) = \alpha_0 \) we find the general solution of (6.8)

\[ t = 4\alpha_0 \int \frac{d p}{p^2 \left\{ (4\alpha_0^2 - b_0)p^2 + b_0 \right\}}. \]

From (6.9) we obtain the following functions generating warped product Bochner-Kähler metrics:

**Type 9.** \( b_0 = k = 0 \).

\[ p(t) = \frac{1}{\sqrt[3]{1 - 3\alpha_0(t - t_0)}}, \quad t \in \left( -\infty, \frac{1}{3\alpha_0} \frac{1 + 3\alpha_0 t_0}{\alpha_0} \right). \]

**Type 10.** \( b_0 = -\sqrt{k} < 0 \).

\[ t = \frac{4\alpha_0}{-b_0 p} + \frac{2\alpha_0}{-b_0 \sqrt{-b_0}} \sqrt{4\alpha_0^2 - b_0} \ln \left\{ \frac{\sqrt{4\alpha_0^2 - b_0} p - \sqrt{-b_0}}{\sqrt{4\alpha_0^2 - b_0} p + \sqrt{-b_0}} \right\}, \]

\[ p \in \left( 0, \frac{\sqrt{-b_0}}{\sqrt{4\alpha_0^2 - b_0}} \right) \cup \left( \frac{\sqrt{-b_0}}{\sqrt{4\alpha_0^2 - b_0}}, \infty \right). \]

**Type 11.** \( b_0 = \sqrt{k} > 0, \ 4\alpha_0^2 - b_0 > 0 \).

\[ t = -\frac{4\alpha_0}{b_0 p} - \sqrt{4\alpha_0^2 - b_0} \arctan \frac{\sqrt{4\alpha_0^2 - b_0}}{\sqrt{b_0}} p, \quad p > 0. \]

**Type 12.** \( b_0 = \sqrt{k} > 0, \ 4\alpha_0^2 - b_0 = 0 \).

\[ p(t) = \frac{1}{1 - \alpha_0(t - t_0)}, \quad t \in \left( -\infty, \frac{1}{\alpha_0} \frac{1 + \alpha_0 t_0}{\alpha_0} \right). \]

**Type 13.** \( b_0 = \sqrt{k} > 0, \ 4\alpha_0^2 - b_0 < 0 \).

\[ t = -\frac{4\alpha_0}{b_0 p} - \sqrt{\frac{b_0 - 4\alpha_0^2}{2\sqrt{b_0}}} \ln \left\{ \frac{\sqrt{b_0 - 4\alpha_0^2} p - \sqrt{b_0}}{\sqrt{b_0 - 4\alpha_0^2} p + \sqrt{b_0}} \right\}, \]

\[ p \in \left( 0, \frac{\sqrt{b_0}}{\sqrt{b_0 - 4\alpha_0^2}} \right) \cup \left( \frac{\sqrt{b_0}}{\sqrt{b_0 - 4\alpha_0^2}}, \infty \right). \]
Summarizing the above results and combining with Theorem 3.7 we obtain the main theorem in this section.

**Theorem 6.1.** Any Bochner-Kähler manifold whose scalar distribution is a $B_0$-distribution locally has the structure of a warped product Kähler manifold with metric given by (6.1) and function $p(t)$ (or $t(p)$) of type 1. - 13.

**Remark 6.2.** In [5] we proved that a Kähler metric is of quasi-constant holomorphic sectional curvatures satisfying the condition $a + k^2 > 0$ if and only if it locally has the form $\partial\bar{\partial}f(r^2)$, $r^2$ being the distance function from the origin in $\mathbb{C}^n$. According to Theorem 3.7 the Bochner-Kähler metrics of type $\partial\bar{\partial}f(r^2)$ (see [13, 3]) can be considered as warped product Bochner-Kähler metrics satisfying the condition $a + k^2 > 0$. These metrics are described explicitly in the types 1. - 6.

**Remark 6.3.** In [7] we proved that a Kähler metric is of quasi-constant holomorphic sectional curvatures satisfying the condition $a + k^2 < 0$ if and only if it locally has the form $\partial\bar{\partial}f(-r^2)$, $-r^2$ being the time-like distance function from the origin in $\mathbb{T}^{n-1}$. According to Theorem 3.7 the Bochner-Kähler metrics of type $\partial\bar{\partial}f(-r^2)$ can be considered as warped product Bochner-Kähler metrics satisfying the condition $a + k^2 < 0$. These metrics are described explicitly in the types 7. - 8.

Finally we shall describe the complete warped product Bochner-Kähler metrics.

According to [11] a warped product Kähler metric (6.1) is complete if and only if the base $Q_0$ is a complete $\alpha_0$-Sasakian space form and the function $f(t)$ is defined in the interval $I = \mathbb{R}$. Investigating the functions of type 1. - 13. we obtain four families of complete metrics:

**The family** $(\mathfrak{A} = 0, \ b_0 > 0, \ d_0 > 0)$.

Given a complete $\alpha_0$-Sasakian space form $Q_0$ satisfying the condition $d_0 = H_0 + 3\alpha_0^2 > 0$ (cf [14]) and a constant $b_0 > 0$. Then the function (Type 5.)

$$u = \frac{1}{t} = \left\{ \frac{\sqrt{2}\alpha_0}{\sqrt{b_0}d_0} \ln \frac{p + \sqrt{2\alpha_0}d_0}{\sqrt{b_0}d_0} + \frac{4\alpha_0 p}{2b_0 - b_0 p^2} \right\}^{-1}, \quad p > 0$$

determines a one-parameter family (with respect to $b_0$) of complete metrics (cf [3]).

**The family** $(\mathfrak{A} \geq -16\alpha_0^2d_0, \ -2\alpha_0^2 - \frac{\mathfrak{A}}{8\alpha_0^2} \leq b_0 < -\sqrt{\mathfrak{A}}, \ d_0 < 0)$.

Given a complete $\alpha_0$-Sasakian space form $Q_0$ satisfying the condition $d_0 = H_0 + 3\alpha_0^2 < 0$ (cf [14]) and two constants $\mathfrak{A} \geq -16\alpha_0^2d_0$ and $-2\alpha_0^2 - \frac{\mathfrak{A}}{8\alpha_0^2} \leq b_0 < -\sqrt{\mathfrak{A}}$. Then the function (Type 7.)

$$t = \frac{\sqrt{2}\alpha_0\sqrt{\mathfrak{A}} - b_0}{\sqrt{-d_0\mathfrak{A}}} \ln \frac{|p - \sqrt{2\alpha_0}d_0|}{p + \sqrt{2\alpha_0}d_0} - \frac{\sqrt{2}\alpha_0\sqrt{-b_0 + \sqrt{\mathfrak{A}}}}{\sqrt{-d_0\mathfrak{A}}} \ln \frac{|p - \sqrt{2\alpha_0}d_0|}{p + \sqrt{2\alpha_0}d_0}$$

$$\frac{2d_0}{b_0 - \sqrt{\mathfrak{A}}} < p^2 < \frac{2d_0}{b_0 + \sqrt{\mathfrak{A}}}$$
determines a two-parameter family (with respect to \( b_0 \) and \( K \)) of complete metrics.

The family \((b_0 = -\sqrt{K} < 0, \ d_0 = 0)\).

Given a complete \( \alpha_0 \)-Sasakian space form \( Q_0 \) satisfying the condition \( d_0 = H_0 + 3\alpha_0^2 = 0 \) (cf [14]) and a constant \( b_0 < 0 \). Then the function (Type 10.)

\[
t = \frac{4\alpha_0}{-b_0 p} + \frac{2\alpha_0 \sqrt{4\alpha_0^2 - b_0^2}}{-b_0 \sqrt{-b_0}} \ln \left\{ \frac{\sqrt{4\alpha_0^2 - b_0^2} p - \sqrt{-b_0}}{\sqrt{4\alpha_0^2 - b_0^2} p + \sqrt{-b_0}} \right\}, \quad p \in \left(0, \frac{\sqrt{-b_0}}{\sqrt{4\alpha_0^2 - b_0^2}}\right)
\]
determines a one-parameter family (with respect to \( b_0 \)) of complete metrics.

The family \((4\alpha_0^2 < b_0 = \sqrt{K}, \ d_0 = 0)\).

Given a complete \( \alpha_0 \)-Sasakian space form \( Q_0 \) satisfying the condition \( d_0 = H_0 + 3\alpha_0^2 = 0 \) and a constant \( b_0 > 4\alpha_0^2 \). Then the function (Type 13.)

\[
t = -\frac{4\alpha_0}{b_0 p} - \frac{\sqrt{b_0 - 4\alpha_0^2}}{2\sqrt{b_0}} \ln \left\{ \frac{\sqrt{b_0 - 4\alpha_0^2} p - \sqrt{b_0}}{\sqrt{b_0 - 4\alpha_0^2} p + \sqrt{b_0}} \right\}, \quad p \in \left(0, \frac{\sqrt{b_0}}{\sqrt{b_0 - 4\alpha_0^2}}\right)
\]
determines a one-parameter family (with respect to \( b_0 \)) of complete metrics.

Notes on the geometry of Kähler manifolds with \( J \)-invariant distributions of codimension two

We note that every Kähler manifold \((M, g, J)\) (dim \( M = 2n \geq 4 \)) with \( J \)-invariant distribution \( D \) carries the tensor

\[
P = \frac{2}{(n + 1)(n + 2)} \pi - \frac{4}{n + 2} \Phi + \Psi,
\]
which is the unique (up to a factor) invariant tensor with zero Ricci trace.

We can draw a parallel between the Kähler manifolds \((M, g, J)\) whose structural group is \( U(n) \) and the Kähler manifolds \((M, g, J, D)\) whose structural group is \( U(n - 1) \times U(1) \). We compare a class of Kähler manifolds \((M, g, J, D)\) with any of the basic classes of Kähler manifolds \((M, g, J)\). The correspondence between the curvature identities, which characterize these classes is given as follows

\[
(M, g, J) \leftrightarrow (M, g, J, D)
\]

\[
R = \frac{\tau}{n(n + 1)} \pi \leftrightarrow R = a\pi + b\Phi + c\Psi,
\]

\[
B(R) = 0 \leftrightarrow B(R) = cP,
\]

\[
\rho = \frac{\tau}{2n} \leftrightarrow \rho = \frac{\tau - 2\sigma}{2(n - 1)} \left( \eta \triangleleft \eta + \tilde{\eta} \triangleleft \tilde{\eta} \right).
\]

The next natural proposition is valid.

A Kähler manifold \((M, g, J, D)\) (dim \( M = 2n \geq 4 \)) is of quasi-constant holomorphic sectional curvatures if and only if the following curvature identities hold good

\[
(i) \quad B(R) = cP;
\]

\[
(ii) \quad \rho = \frac{\tau - 2\sigma}{2(n - 1)} g + \frac{2n\sigma - \tau}{2(n - 1)} (\eta \triangleleft \eta + \tilde{\eta} \triangleleft \tilde{\eta}).
\]
Let $(M, g, J)$ be a Kähler manifold with $d\tau \neq 0$ and $\eta = \frac{d\tau}{\|d\tau\}$.

Suppose that

\[(*)\quad 1) (\text{grad } \tau)^{1,0} \text{ is a holomorphic vector field; } 2) \nabla_\lambda \eta_\bar{\mu} = \frac{k}{2} g_{\lambda\bar{\mu}}.\]

Under the conditions \((*)\) any biconformal transformation of the structure $(g, \eta)$ again gives a Kähler structure (cf [6]).

In Kähler geometry it seems that the following statement is true.

\begin{quote}
Let $(M, g, J)$ be a Kähler manifold whose scalar distribution satisfies the conditions \((*)\).

Then the tensor $B(R) - cP$ is a biconformal invariant.
\end{quote}

References

[1] Bishop, R.; O’Neil, B. Manifolds of negative curvature, Trans. Amer. Math. Soc., 145 (1969), 1-49.

[2] Boju, V.; Popesku, M. Espaces à courbure quasi-constante, J. Diff. Geom., 13 (1978), 373-383.

[3] Bryant, R. Bochner-Kähler metrics, J. Amer. Math. Soc., 14 (2001), 623-715.

[4] Gray, A.; Hervella, L. The sixteen classes of almost Hermitian manifolds and their linear invariants, Ann. di Mat. Pura ed Appl., 123 (1980), 35-58.

[5] Ganchev, G.; Mihova, V. Riemannian manifolds of quasi-constant sectional curvature, J. reine und angew. Math., 522 (2000), 119-141.

[6] Ganchev, G.; Mihova, V. Kähler manifolds of quasi-constant holomorphic sectional curvatures, ArXiv: math.DG/0505671 to appear.

[7] Ganchev, G.; Mihova, V. Kähler metrics generated by functions of the time-like distance in the flat Kähler-Lorentz space, ArXiv: math.DG/0510468 to appear.

[8] Janssens, D.; Vanhecke, L. Almost contact structures and curvature tensors, Kodai Math. J., 4 (1981), 1-27.

[9] Kamishima, Y. Uniformization of Kähler manifolds with vanishing Bochner tensor, Acta Math., 172 (1994), 299-308.

[10] Kobayasi, S.; Nomizu, K. Foundations of Differential Geometry, II, Interscience Publishers, New Yourk, 1969.

[11] Ogiue, K. On almost contact manifolds admitting axiom of planes or axiom of free mobility, Kodai Math. Sem. Rep., 16 (1964), 223-232.

[12] Okumura, M. On infinitesimal conformal and projective transformations of normal contact spaces, Tôhoku Math. J., 14 (1962), 398-412.

[13] Tachibana, S.; Liu, R.C. Notes on Kählerian metrics with vanishing Bochner curvature tensor, Kodai Math. Sem. Rep., 22 (1970), 313-321.

[14] Tanno, S. Sasakian manifolds with constant $\varphi$-holomorphic sectional curvature, Tôhoku Math. J., 21 (1969), 501-507.

[15] Tashiro, Y. On contact structures on hypersurfaces in almost complex manifolds I, Tôhoku Math. J., 15 (1963), 62-79.

[16] Tashiro, Y. On contact structures on hypersurfaces in almost complex manifolds II, Tôhoku Math. J., 15 (1963), 167-175.

[17] Tricerri, F.; Vanhecke, L. Curvature tensors on almost Hermitian manifolds, Trans. Amer. Math. Soc., 267 (1981), 365-398.