Dilogarithm identities, fusion rules and structure constants of CFTs

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Abstract

Recently dilogarithm identities have made their appearance in the physics literature. These identities seem to allow to calculate structure constants like, in particular, the effective central charge of certain conformal field theories from their fusion rules. In Nahm, Recknagel, Terhoeven (1992) a proof of identities of this type was given by considering the asymptotics of character functions in the so-called Rogers-Ramanujan sum form and comparing with the asymptotics predicted by modular covariance. Refining the argument, we obtain the general connection of quantum dimensions of certain conformal field theories to the arguments of the dilogarithm function in the identities in question and an infinite set of consistency conditions on the parameters of Rogers-Ramanujan type partitions for them to be modular covariant.
1 Introduction and Overview

Recently, the dilogarithm function has reappeared (Babujia (1983)) in the physics of two-dimensional quantum field theories and lattice models (Bazhanov, Reshetikhin (1989), Klassen, Melzer (1990), Klümper, Pearce (1991), Zamolodchikov (1990)). In Nahm, Recknagel, Terhoeven (1992) a connection of fusion rules to thermodynamic Bethe Ansatz (TBA) type equations was conjectured. In this paper, we make this connection precise in the framework of CFT.

Crucial in the latter approach of the subject is to (re)write the characters of the conformal field theory (CFT) in the Rogers-Ramanujan sum form (cf. (1.1) below). In Terhoeven (1992), Kuniba, Nakanishi, Suzuki (1992) this form was conjectured to exist for general parafermionic theories. Allowing a slight generalization, one obtains formulas of similar type for characters of all unitary Virasoro minimal models (cf. Dashmahapatra, Kedem, Klassen, McCoy, Melzer (1993) and references therein, also for the so-called quasi-particle picture). Thus it seems possible that the class of theories allowing their characters to be written in a suitable form is not too far away from the set of all CFTs. In this paper, we find an infinite set of consistency conditions to be satisfied by all partitions of the form (1.1), which is a first step in the classification of this type of partitions, possibly relevant for a classification of all rational CFTs.

In the following, we consider partition functions of the Rogers-Ramanujan (or Euler) sum form

\[ Z(q) = \sum_{m_1, \ldots, m_n \in \mathbb{Z} \geq 0} \frac{q^{Bm^t + b \cdot m + \beta}}{(q)_m} e^{c \cdot m = \gamma \mod 1} \tag{1.1} \]

where \((q)_m = (q)_{m_1} \cdots (q)_{m_n}\), \((q)_m = (1 - q)(1 - q^2) \cdots (1 - q^m)\), \(B_{ij}, b_i, c_i, \beta\) are rational numbers, \(\gamma \in \mathbb{Z}_K/K\) (\(K \in \mathbb{N}\)) and \(B\) is symmetric and positive definite. Let \(q = e^{2\pi i \tau} = e^{-2\pi t}\) and \(\bar{q} = e^{-2\pi i/\tau} = e^{-2\pi/t}\), thus \(1/\log(q) = \log(\bar{q})/4\pi^2\) and in the limit \(t = -i\tau \searrow 0\) we have \(q \nearrow 1\).

More accurately, we consider the asymptotic of (1.1) in the above limit. Refining the argument of Nahm, Recknagel, Terhoeven (1992) and requiring (1.1) to be modular...
covariant, we obtain not only a dilogarithm identity for the lowest energy state in the spectrum. More than that: our main results are the connection of fusion rules to TBA-type equations (meaning a formula relating quantum dimensions and a certain solution of the TBA-type equations, the arguments of the dilogarithm function in the formula for the effective central charge) and an infinite set of consistency equations restricting the possible choices of $B$, $b$ and $\beta$ (cf. the discussion in section four).

The outline of the paper is as follows:

In the rather technical second section we study the asymptotics of (1.1) for $q \searrow 1$ using an integral representation and integral transform of the partition function following Meinardus (1954) and a higher order saddle point approximation, preferable diagrammatically (à la Feynman).

In the third section we compare the partition asymptotics calculated explicitly with the one predicted by modular covariance, and thus find the results mentioned above.

Finally, we outline future work and as a short application present a classification of modular covariant partitions of the Rogers-Ramanujan form in the case rank $B = n = 1$, which is found from the first few consistency equations in this particular case.

2 Partition asymptotics

In this section, we will extract the first term of the modular transform of $Z(q)$ – a power series in $\tilde{q}$ – from the asymptotic behavior of (1.1) when $q \searrow 1$.

Following Meinardus (1954), we write (1.1) in an integral representation using Cauchy’s theorem

$$Z(q) = \int \prod_j \frac{dw_j}{2\pi i} w_j^{-\beta} K \left( \sum_{\mu \in \mathbb{Z}^n \text{ such that } l \in \mathbb{Z}_K} q^{B \mu} \exp \left\{ -2\pi i l \cdot (b \cdot c - \gamma) \right\} \prod_{\mu_j} w_j^{-\mu_j} \right) \prod_j \left( \sum_{m_j \geq 0} \frac{q^{h_j m_j}}{(q)_{m_j}} w_j^{m_j} \right).$$

(2.1)
On the first bracket inside the multiple integral in (2.1) we apply the well-known Jacobi inversion formula (Gunning (1962)) (\(|q| < 1\))

\[
\sum_{\mu \in \mathbb{Z}^n, \ell \in \mathbb{Z}_K} q^{\mu B \mu} e^{-2\pi i l (\mu \cdot c - \gamma)} w^\mu \frac{1}{\sqrt{\det(2B) t^n}} \sum_{\ell \in \mathbb{Z}_K} e^{2\pi i l \gamma} \sum_{\mu \in \mathbb{Z}^n} q^{(\mu + \log(w)/2\pi i l + c) B^{-1}(\mu + \log(w)/2\pi i l + c)/4},
\]

(2.2)

where \(w^\mu = \prod_j w_j^{\mu_j}\). The terms in the second bracket of (2.1) require some more explicit calculations. First, we use a relation going back to Euler (Andrews (1976)) (\(|q| < 1, \ |w| < |q|^{-b}\))

\[
\sum_{m \geq 0} \frac{q^{bm}}{(q)_m} w^m = \prod_{n \geq 0} (1 - w q^{n+b})^{-1}.
\]

(2.3)

Second, we apply \(\text{id} = \exp \circ \log\) and the Taylor series expansion of \(\log(1 - x)\) to obtain

\[
\exp \{ \sum_{r=1}^{\infty} \frac{w^r}{r} \sum_{n=0}^{\infty} e^{-2\pi t(n+b)r} \}.
\]

(2.4)

Using inverse Mellin transformation (cf. e.g. Davies (1978)), the exponent in (2.4) can be written as

\[
\frac{1}{2\pi i} \int_{3/2-i\infty}^{3/2+i\infty} ds \Gamma(s) D(s+1, w) \zeta(s, b) (2\pi t)^{-s},
\]

(2.5)

where \(\Gamma(s)\) is the Gamma function with \(\text{Res}_{s=-n} \Gamma(s) = \frac{(-1)^n}{n!} \quad (n > 0)\), \(\zeta(s, b)\) is the generalized Riemann \(\zeta\)-function \(\zeta(s, b) = \sum_{n=0}^{\infty} (n + b)^{-s}\) with \(\text{Res}_{s=-n} \zeta(s, b) = -\frac{\varphi_{n+1}(b)}{(n+1)} \quad (n > 0)\), where \(\varphi_n(x)\) are the Bernoulli polynomials \((\varphi_0(x) = 1, \varphi_1(x) = x - 1/2, \varphi_2(x) = x^2 - x + 1/6, \ldots)\) generated by \(\frac{te^{xt}}{(e^t-1)} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \varphi_n(x)\) and \(D(s, w) = \sum_{r=1}^{\infty} \frac{w^r}{r^s}\) (\(|w| < 1\)).

Third, shifting the contour of integration in (2.5) along the real axis, we pick up one by one the residues of the integrand at \(s = 1, 0, -1, \ldots\), the first coming from the pole of \(\zeta\), the others from poles of \(\Gamma\). This gives the following expression for the exponent of (2.4)

\[
-\frac{1}{\log(q)} \sum_{p=0}^{\infty} \log^p(q) \frac{\varphi_p(b)}{p!} (w \partial_w)^p \sum_{r=1}^{\infty} \frac{w^r}{r^2},
\]

(2.6)
which, using the generating function of the Bernoulli polynomials and the series definition of the dilogarithm \( \text{Li}_2(w) = \sum_{r=1}^{\infty} \frac{w^r}{r^2} \), can be written more compactly as

\[
-w \partial_w \frac{q^{bw} \partial_w}{q_w} - 1 \text{Li}_2(w).
\]  

(2.7)

Taking (2.6) and (2.2), we can write (2.1) in the form

\[
Z(q) = \oint \prod_j \frac{dw_j}{2\pi i w_j} \frac{1}{\sqrt{\det(2B)}} \sum_{\mu \in \mathbb{Z}^n, \ell \in \mathbb{Z}^K} \exp(\mathcal{E}_{\mu, \ell}(q, w)),
\]

(2.8a)

where

\[
\mathcal{E}_{\mu, \ell}(q, w) = \log(q) \beta + 2\pi i \ell \gamma + \frac{4\pi^2}{\log(q)} \left( \frac{1}{4} (\mu + \frac{\log(w)}{2\pi i} + \ell c) B^{-1}(\mu + \frac{\log(w)}{2\pi i} + \ell c)^t \right.
\]

\[
- \frac{1}{\log(q)} \sum_{p=0}^{\infty} \log^p(q) \sum_{a=1}^{n} \frac{\varphi_p(b_a)}{p!} (w_a \partial_{w_a})^p \sum_{r_a=1}^{\infty} \frac{w_a^{r_a}}{r_a^2},
\]

(2.8b)

which in lowest order in \( \log(q) \) reduces to

\[
\frac{1}{\log(q)} \left( 4\pi^2 \frac{1}{4} (\mu + \frac{\log(w)}{2\pi i} + \ell c) B^{-1}(\mu + \frac{\log(w)}{2\pi i} + \ell c)^t - \sum_a \text{Li}_2(w_a) \right).
\]

(2.9)

The sum over \( \mu \) together with the contour integral can be understood as an infinite contour integral over all branches of the analytically continued logarithm \( \log(w) \). Thus, we forget (the sum over) \( \mu \) in the following and understand the contour integral as the appropriate infinite integral.

We continue with the saddle point approximation of (2.8) (in short: the value of the integral is approximated by the integrand at the saddle point):

A necessary condition for a minimum in a point \( w \) (in lowest \( \log(q) \) order, only then \( w \) is independent of \( q \), i.e. a number) is given by

\[
0 = w_a \partial_{w_a} \mathcal{E}_t(q, w) + \mathcal{O}(\log^0(q))
\]

\[
= \frac{1}{\log(q)} \left( - 2\pi i B^{-1}_{aa'} \left( \frac{\log(w)}{2\pi i} + \ell c \right)_{a'}/2 + \log(1 - w_a) \right)
\]

(2.10)

or

\[
\frac{1}{2} B^{-1}_{aa'} \left( \frac{\log(w)}{2\pi i} + \ell c \right)_{a'} = \frac{\log(1 - w_a)}{2\pi i}
\]

(2.11)
or in exponentiated form (ignoring for the moment possible phases)

\[ w_{a'} = \prod_{a=1}^{n} (1 - w_a)^{2B_{aa'}}, \quad (2.12) \]

which appeared in the physical literature as certain limits of TBA-type (thermodynamic Bethe Ansatz) equations. We will assume in the following that a unique solution of \((2.10,11)\) with \(0 < w_a < 1\) exists and that furthermore \(l = 0\) for this solution, which is indeed true for all examples known to us.

Substituting \((2.11)\) into \((2.8b)\) and introducing Rogers’ dilogarithm function

\[ L(z) = \text{Li}_2(z) + \frac{1}{2} \log(z) \log(1 - z), \quad (2.13) \]

we obtain the exponent function at the saddle point \(w = w_s(l = 0)\) (repeated indices are summed over)

\[ \mathcal{E}_l(q, w = w_s) = \log(\tilde{q}) \left\{ -\frac{1}{24} \frac{L(1)}{L(1)} \sum_{a=1}^{n} L(w_a) \right\} \]

\[ + \log^0(q) \left\{ \varphi_1(b_a) \log(1 - w_a) \right\} \]

\[ + \log^1(q) \left\{ \beta - \frac{\varphi_2(b_a)}{2} \frac{w_a}{1 - w_a} \right\} + \mathcal{O}(\log^2(q)). \quad (2.14) \]

The \(\log(\tilde{q})\) part of \((2.14)\) yields the first and crudest approximation of the integral. To extract higher order corrections, we change the variable of integration in \((2.8)\) by the substitution \(w_a = e^{\sqrt{-2\pi t} v_a} = e^{\sqrt{\log(q)} v_a} (\partial v_a = \sqrt{\log(q)} w_a \partial w_a)\) and expand \(\mathcal{E}_l(q, v)\) in a Taylor series around the saddle point \(v_s\) to obtain

\[ Z(q) = \int \frac{1}{K \sqrt{\det(2B)}} \sum_{l \in \mathbb{Z}_K} \exp(\mathcal{E}_l(q, v_s + \Delta)) \prod_j \frac{d\Delta_j}{\sqrt{2\pi}}, \quad (2.15a) \]

where

\[ \mathcal{E}_l(q, v_s + \Delta) = \mathcal{E}_l(q, v)|_{v=v_s} + \sum_a \Delta_a \partial_{v_a} \mathcal{E}_l(q, v)|_{v=v_s} \]

\[ + \frac{1}{2} \sum_{a,a'} \Delta_a \Delta_{a'} \partial_{v_a} \partial_{v_{a'}} \mathcal{E}_l(q, v)|_{v=v_s} \]

\[ + \frac{1}{6} \sum_a \Delta_a^3 \partial^3_{v_a} \mathcal{E}_l(q, v)|_{v=v_s} + \ldots . \quad (2.15b) \]
The first derivative of $\mathcal{E}_i(q, v)$ is given by

$$\partial_{v_a} \mathcal{E}_i(q, v) \big|_{v = v_s} = -\sum_{p=1}^{\infty} \log^{p-1/2}(q) \frac{\varphi_p(b_a)}{p!} \sum_{r_a=1}^{\infty} w_a^{r_a} r_a^{p-1},$$

(2.16)

the second by

$$\partial_{v_a} \partial_{v_{a'}} \mathcal{E}_i(q, v) \big|_{v = v_s} = -(2B)^{-1}_{aa'} \frac{w_a}{1 - w_a}$$

$$- \delta_{a,a'} \sum_{p=1}^{\infty} \log^{p}(q) \frac{\varphi_p(b_a)}{p!} \sum_{r_a=1}^{\infty} w_a^{r_a} r_a^{p},$$

(2.17)

and the higher ones for $n \geq 3$ by

$$\partial^{n}_{v_a} \mathcal{E}_i(q, v) \big|_{v = v_s} = -\sum_{p=0}^{\infty} \log^{p-1+n/2}(q) \frac{\varphi_p(b_a)}{p!} \sum_{r_a=1}^{\infty} w_a^{r_a} r_a^{p+n-2}.$$  

(2.18)

The next log($q$) order of our expression of $Z(q)$ can then be found by evaluating the integral as a gaussian integral. Namely, for any symmetric, real, positive definite matrix $A$, one has

$$\int e^{-\frac{1}{2} x A x - d \cdot x - c} \frac{d^n x}{(2\pi)^{n/2}} = \frac{1}{\sqrt{\det A}} e^{\frac{1}{2} d^t A^{-1} d - c},$$

(2.19)

leading to

$$Z(q) = \exp \left( \log(\tilde{q}) \left\{ -\frac{1}{24L(1)} \sum_{a=1}^{n} L(w_a) \right\} \right) \exp \left( \sum_{a=1}^{n} \varphi_1(b_a) \log(1 - w_a) \right) / K \sqrt{\det(2B \cdot A)}$$

$$\left( 1 + \mathcal{O}(\log(q)) \right),$$

(2.20)

where $d_a = \varphi_1(b_a) \frac{w_a}{1-w_a} \sqrt{\log(q)}$ and

$$A_{aa'} = (2B)^{-1}_{aa'} + \delta_{a,a'} \frac{w_a}{1 - w_a}.$$  

(2.21)

The next order in log($q$) leads to replacing $\left( 1 + \mathcal{O}(\log(q)) \right)$ by (sums over all indices
\( a, a' = 1, \ldots n \) implied

\[
\exp \left[ \log^1(q) \left( \left\{ \beta - \frac{\varphi_2(b_a)}{2} \frac{w_a}{1-w_a} \right\} \right. \right.

\left. + \frac{1}{2} \varphi_1(b_a) \frac{w_a}{1-w_a} A_{aa'}^{-1} \varphi_1(b_{a'}) \frac{w_{a'}}{1-w_{a'}} \right.

\left. - \frac{1}{2} \varphi_1(b_a) \frac{w_a}{(1-w_a)^2} A_{aa}^{-1} \right.

\left. + \frac{1}{2} \frac{w_a}{(1-w_a)^2} A_{aa}^{-1} A_{aa'}^{-1} \varphi_1(b_{a'}) \frac{w_{a'}}{1-w_{a'}} \right.

\left. - \frac{1}{8} \frac{w_a(1+w_a)}{(1-w_a)^3} \left( A_{aa}^{-1} \right)^2 \right.

\left. + \frac{1}{12} \frac{w_a}{(1-w_a)^2} \left( A_{aa}^{-1} \right)^3 \frac{w_{a'}}{(1-w_{a'})^2} \right.

\left. + \frac{1}{8} \frac{w_a}{(1-w_a)^2} A_{aa}^{-1} A_{aa'}^{-1} A_{a'a'}^{-1} \frac{w_{a'}}{(1-w_{a'})^2} \right]

\left. \left( 1 + \mathcal{O}(\log^2(q)) \right) \right)
\]

in (2.20). It can be found by taking derivatives of (2.19) with respect to \( d \) and then collecting all terms of order \( \log(q) \). Once written down, one also recognizes it as the exponential of the generating function of the connected Feynman diagrams of order \( \log(q) \) (with corresponding factors of internal symmetry) allowed by the Feynman rules encoded in (2.15).

This insight allows one to write down the next order by hand (with 35 ’new’ diagrams, meaning diagrams not constructed out of ‘dressings’ \( p \mapsto p + 1 \) in (2.16-18)) of (2.22)). The next to next order already is a bit unwieldly with 367 diagrams before choosing possible dressings. Higher orders can in principle be written down. However, the proliferation of diagrams is enormous and we have looked beyond \( \log(q) \) only in the comparably simple case where rank \( B = 1 \).

3 Modular covariance

In this section we compare the asymptotics found above with the prediction of modular covariance.
In the following, we suppose that the partition $Z(q)$ given in (1.1) equals a character of some CFT with effective central charge $c_{\text{eff}} = c - 24 h_{\min}$ (the index min refers to the primary field of minimal conformal dimension) – more accurately: $Z(q)$ equals the character corresponding to the primary field $\phi_\lambda$ of conformal dimension $h_\lambda$, thus

$$Z(\lambda)(q) = q^{h_\lambda - c/24} \sum_{m \in \mathbb{Z}_{\geq 0}} d_m^{(\lambda)} q^m.$$  \hfill (3.1)

Then by modular covariance we have

$$Z(\lambda)(q) = \sum_{\lambda'} S_{\lambda,\lambda'} Z(\lambda')(\tilde{q}) = \sum_{\lambda'} S_{\lambda,\lambda'} \tilde{q}^{h_{\lambda'} - c/24} \sum_{m \in \mathbb{Z}_{\geq 0}} d_m^{(\lambda')} \tilde{q}^m, \hfill (3.2)$$

where the $d_m^{(\lambda)}$ are the multiplicities of states at grade $m$.

Comparing (1.1) with (3.1), we see that

$$h_\lambda - c/24 = \min_{m \geq 0} (mBm^t + b \cdot m + \beta). \hfill (3.3)$$

Comparing the first term (in powers of $\tilde{q}$) of the right hand side of (3.2) with (2.20,22), we obtain

$$c_{\text{eff}} = \sum_{a=1}^{n} \frac{L(w_a)}{L(1)} \hfill (A)$$

$$S_{\lambda,\min} = \frac{\exp \left( \sum_{a=1}^{n} \varphi_1(b_a^{(\lambda)}) \log(1 - w_a) \right)}{K \sqrt{\det(2B \cdot A)}} \hfill (B)$$

$$0 = \text{exponent of } (2.22)\big|_{\beta = \beta^{(\lambda)}, b = b^{(\lambda)}} \hfill (C)$$

$$0 = \text{higher orders in } \log(q) \text{ of the expansion of } Z(\lambda)(q), \hfill (D)$$

where as before

$$A_{aa'} = (2B)^{-1}_{aa'} + \delta_{a,a'} \frac{w_a}{1 - w_a}.$$  

In words:

$(A)$ gives the effective central charge of the theory in terms of a dilogarithmic expression with arguments $w_a$ given by a certain solution of the TBA-type equations (2.12) which only depend on $B$. 

8
(B) gives an expression for certain elements of the modular S-matrix in terms of the $w_a$. From (B) we obtain the following relation between quantum dimensions and solutions of the TBA-type equations

$$\frac{S_{\lambda, \text{min}}}{S_{0, \text{min}}} = \prod_{a=1}^{n} (1 - w_a)^{b_a^{(\lambda)} - b_a^{(0)}},$$

where 0 denotes the vacuum of the theory.

(C) fixes $\beta$ in terms of $w_a$ (i.e. B) and $b_a$, thus with (3.3) gives an expression for the conformal dimensions of the theory. Finally, (D) is an innocent looking short-hand for an infinite set of complicated consistency conditions for modular covariance putting constraints on $B$ and $b_a$. However, trying to solve them is highly non trivial and of the type to solve (C) for given $\beta$ in terms of $B$ and $b_a$. For the case rank $B = 1$ we refer the reader to the outlook.

4 Outlook

Assuming the existence of a character with $b = 0$, we have classified the partitions of type (1.1) with rank $B = n = 1$:

We found numerically the zeroes of the first non-trivial consistency condition (order $\log^2(q)$) in $w \in ]0, 1[$, namely $\tau$, $1/2$, $\tau^2 (\tau^2 + \tau - 1 = 0)$, which correspond to the (3, 5)—, (3, 4)— and (2, 5)—model in the series of Virasoro minimal models, with $B = 1/4, 1/2, 1$ respectively. Given $w$ and thus $B$, we could solve the first consistency conditions for other possible $b \neq 0$ and found the known results.

Certainly one should try to attack the higher rank case by the same method and to extend the argument to the more general partitions mentioned in the introduction.
Also, we would like to understand, in a way similar to the discussion given, other
dilogarithm identities leading to the other effective conformal dimensions or possibly
(cf. Kuniba, Nakanishi, Suzuki (1992)) to the whole spectrum of CFTs, i.e. we would
like to be able to explicitly construct the whole modular transform of partitions in the
Rogers-Ramanujan form and not only the first term. A ‘naive’ analytic continuation of
the (di)logarithm in (2.20) leads to a form surprisingly similar to (1.1) again. However, a
closer look reveals certain inconsistencies and substantially reduces the predictive power
(only (3.C) seems to hold without any ‘tuning’). This problem is solved by W. Nahm
(1993).

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