A NEW GENERALIZATION OF $\theta$-CONGRUENT NUMBERS

SAJAD SALAMI AND ARMAN SHAMSI ZARGAR

ABSTRACT. In this paper, we introduce a new generalization of $\theta$-congruent numbers by defining the notion of $\theta$-parallelogram envelops for an angle $\theta \in (0, \pi)$ with rational cosine and a positive integer. We study more closely some problems related to $\theta$-parallelogram envelops, using the arithmetic of algebraic curves. Our results generalize the recent work of T. Ochiai, where only the case $\theta = \pi/2$ was considered, and answer to the open questions contained in his paper and their generalization for any Pythagorean angles.

Subjclass 2010: Primary 11G05; Secondary 14H52
keywords: $\theta$-parallelogram envelop, $\theta$-congruent number, elliptic curve..

1. Introduction

A positive integer $n$ is called a congruent number if it is equal to the area of a right triangle with rational sides. Equivalently, if there exist positive rational numbers $a, b,$ and $c$ such that $a < b < c,$ and

$$a^2 + b^2 = c^2, \quad ab = 2n. \quad (1.1)$$

Determining all the congruent numbers is an old problem in Number Theory. There are various sorts of generalizations of the problem, see for example [3, 4, 10, 14, 15, 16].

In [3], M. Fujiwara introduced and studied an interesting generalization of congruent numbers, called $\theta$-congruent numbers. Afterwards, several authors ([1, 4, 6, 9, 11] studied this new concept with different approaches using the arithmetic of elliptic curves [10, 13]. In order to describe this generalization, let $\theta \in (0, \pi)$ be an angle with rational cosine, i.e., it satisfies $\cos(\theta) = s/r$ with $r, s \in \mathbb{Z}$ such that $0 \leq |s| < r$ and $\gcd(r, s) = 1$. A positive integer $n$ is called a $\theta$-congruent number if there exists a triangle with rational sides and area equal to $n\sqrt{r^2 - s^2}$. Equivalently, if there are three positive rational numbers $a, b,$ and $c$ satisfying $a \leq b < c,$ and

$$a^2 + b^2 - \frac{2s}{r}ab = c^2, \quad ab = 2rn. \quad (1.2)$$

We denote such a triangle by $(a, b, c)$ and call it a rational $\theta$-triangle for $n$. It is clear that if a positive integer $n$ is $\theta$-congruent with a $\theta$-triangle $(a, b, c)$, then $nm^2$ is also a $\theta$-congruent with the $\theta$-triangle $(ma, mb, mc)$. Hence, one may concentrate on the square-free positive integers. The problem of determining $\theta$-congruent numbers is related to finding non-2-torsion points on the elliptic curve given by the following Weierstrass equation,

$$E_\theta^n : y^2 = x(x + (r + s)n)(x - (r - s)n), \quad (1.3)$$

where $r$ and $s$ are as above. Consult [3, 4, 9] for more details.

In a recent work, T. Ochiai [11] generalized the congruent numbers in a new way. Indeed, he considered the set $\mathcal{E}$ of all positive integers $n$ such that there is an envelope for $n$, i.e., a quintuple $(a, b, c, d, e)$ of positive rational numbers satisfying

$$a^2 + b^2 = c^2, \quad a^2 + d^2 = e^2, \quad a(b + d) = n. \quad (1.4)$$
We note that any congruent number belongs to the set $\mathcal{E}$, because if we let $b = d$, then the equations (1.4) are equivalent to (1.1). It is proved that $n$ belongs to $\mathcal{E}$ if either $n$ or $2n$ is a congruent number. Moreover, it is shown that $n$ belongs to $\mathcal{E}$ if and only if a certain set of algebraic equations has some rational solutions, see Theorem 2 in [11]. Given any $m \geq 1$, T. Ochiai also considered the set $\mathcal{E}(m)$ consisting of all positive integers $n \in \mathcal{E}$ such that there exists an envelope $(a, b, c, d, e)$ for $n$ with $d = mb$. By studying the set of rational points on certain family of elliptic curves and a genus five algebraic curve, he provided certain conditions which lead to decide whether a given positive integer $n$ belongs to $\mathcal{E}(m)$. In the end of his paper [11], T. Ochiai asked the following questions concerning the sets $\mathcal{E}$ and $\mathcal{E}(m)$.

**Question 1.1.** Notation being as above, one may ask that:

(i) Is $\mathcal{E} = \mathbb{N}$?

(ii) Given any $n \in \mathcal{E}$, are there infinitely many distinct envelopes for $n$?

(iii) Given any $n \in \mathcal{E}$, are there infinitely many rational numbers $m > 1$ such that $n$ belongs to $\mathcal{E}(m)$?

In this paper, by generalizing the notion of envelopes, we introduce naturally real-life geometric objects for a given positive $n$ which is called a $\theta$-parallelogram envelop for $n$. This new notion can be viewed as a generalization of $\theta$-congruent numbers which is involved by the rational $\theta$-triangles. We will study the set of $\theta$-parallelogram envelops for positive numbers by investigating the set of rational points on a certain defined algebraic curve. Moreover, we shall respond affirmatively to Question 1.1 in a more general setting related to the set of $\theta$-parallelograms which arises through this study.

### 2. The notion of $\theta$-parallelogram envelops

Consider the angle $\theta \in (0, \pi)$ with rational cosine, i.e., $\cos(\theta) = s/r$ with $r, s \in \mathbb{Z}$ such that $0 \leq |s| < r$ and $\gcd(r, s) = 1$. We call $\theta$ a *Pythagorean angle* if its sine is also rational, say $\sin(\theta) = t/r$ for which $0 \leq |t| = \sqrt{r^2 - s^2} < r$ and $\gcd(r, t) = 1$. We start by the following definition.

**Definition 2.1.** Let $\mathbb{N}$ be the set of natural numbers and let $\mathcal{N}_\theta$ consist of all $n \in \mathbb{N}$ such that there exist positive rational numbers $a, b, c, d,$ and $e$ satisfying

\[
a^2 + b^2 - \frac{2s}{r}ab = c^2, \quad a^2 + d^2 + \frac{2s}{r}ad = e^2, \quad a(b + d) = rn. \tag{2.1}
\]

For any $n \in \mathcal{N}_\theta$, we denote such a quintuple of positive rational numbers satisfying the equations (2.1) by $(a, b, c, d, e)_\theta$ and call it a $\theta$-parallelogram envelop for $n$. This notion with $\theta = \pi/2$ can be seen as a generalization of the envelopes defined by T. Ochiai in [11]. Figure 1 shows a $\theta$-parallelogram envelop for $n$.

![Figure 1. A $\theta$-parallelogram envelop for $n$](image)
Remark 2.2. The following facts can be easily deduced from the definition.

(1) A natural number \( n \) belongs to \( N_\theta \) if and only if \( nm^2 \in N_\theta \) for any integer \( m \). Hence, the area of a \( \theta \)-parallelogram envelop for \( n \) is equal to \( n\sqrt{r^2-s^2} \mod (Q^*)^2 \).

(2) A quintuple \((a,b,c,d,e)\) is a \( \theta \)-parallelogram envelop for \( n \) if and only if the quintuple \((a,d,e,b,c)\) is a \((\pi-\theta)\)-parallelogram envelop for \( n \).

(3) If \( n \) is both \( \theta \)- and \((\pi-\theta)\)-congruent number, then \( n \) belongs to \( N_\theta \). But, the converse is not true, see the following example.

By the above remarks, we restrict our study to the case \( \theta \in (0,\pi/2] \) and the square-free natural numbers \( n \) throughout this work.

Example 2.3. For \( \theta = \pi/3 \), all the square-free natural numbers \( 1 \leq n \leq 40 \) which are neither \( \pi/3 \)- nor \( 2\pi/3 \)-congruent numbers but belong to \( N_{\pi/3} \) are \( n = 2, 3, 7, 26, 31, \) and \( 38 \). See Table 2 in the appendix for corresponding \( \pi/3 \)-parallelogram for all of the square-free natural numbers \( 1 \leq n \leq 50 \). For instant, \( n = 7 \) is neither a \( \pi/3 \)-congruent number nor a \( 2\pi/3 \)-congruent number, but there is a \( \pi/3 \)-parallelogram envelop for \( n = 7 \) given by

\[
(a, b, c, d, e)_{\pi/3} = \left( \frac{56}{27}, \frac{256}{99}, \frac{856}{297}, \frac{1649}{396}, \frac{6053}{1188} \right).
\]

In the next definition, we consider a subset of \( N_\theta \) which involves with the \( \theta \)-parallelograms \((a,b,c,d,e)\) having an angle \( \tau \) with rational cosine between the sides \( c \) and \( e \).

Definition 2.4. Let \( \tau \in (0,\pi) \) be an angle with rational cosine. Denote by \( N_\theta(\tau) \) the set of all \( n \in N_\theta \) such that there exist positive rational numbers \( a, b, c, d, \) and \( e \) satisfying (2.1) and

\[
\cos \tau = \frac{a^2 - bd - sn}{ce}.
\]

For any \( n \in N_\theta(\tau) \), we denote such a quintuple by \((a,b,c,d,e)_{\tau} \) and call it a \( \theta \)-parallelogram envelope with an angle \( \tau \) for \( n \).

Figure 2 shows a \( \theta \)-parallelogram envelop with an angle \( \tau \) for \( n \in N_\theta(\tau) \).

![Figure 2](image)

Figure 2. A \( \theta \)-parallelogram envelop with angle \( \tau \) for \( n \).

The following theorem gives a relation between the set \( N_\theta(\tau) \) and the \( \tau \)-congruent numbers, which is proved in Section 7.

Theorem 2.5. Let \( \theta \in (0,\pi/2] \) and \( \tau \in (0,\pi) \) be angles with rational cosines satisfying \( \sin(\theta)/\sin(\tau) \equiv n' \mod (Q^*)^2 \) for some positive integer \( n' \). Then, \( n \in \mathbb{N} \) belongs to \( N_\theta(\tau) \) if and only if \( 2nn' \) is a \( \tau \)-congruent number. In particular, \( n \) belongs to \( N_\theta(\theta) \) if and only if \( 2n \) is a \( \theta \)-congruent number. Equivalently, \( n \) is a \( \theta \)-congruent number if and only if \( 2n \) belongs to \( N_\theta(\theta) \).
3. The $\theta$-congruent number elliptic curve and $\mathcal{N}_\theta$

In this section, we provide an algebraically necessary and sufficient condition for $n \in \mathbb{N}$ to be an element of $\mathcal{N}_\theta$ using the $\theta$-congruent number elliptic curves.

**Theorem 3.1.** A natural number $n$ belongs to $\mathcal{N}_\theta$ if and only if the equations,

$$
\begin{align*}
E_w^\theta : y^2 &= x(x + (r + s)w)(x - (r - s)w), \\
F_N^\theta : v^2 &= u(u - (r + s)N)(u + (r - s)N), \\
C_R : xv &= uy,
\end{align*}
$$

have a rational solution $(u, v, w, x, y)$ where $N = 2n - w$, $yv \neq 0$ and $0 < w \leq n$.

Taking $w = n$ in Theorem [3.1] leads to the fact that if a natural number $n$ is simultaneously $\theta$- and $(\pi - \theta)$-congruent number, then $n$ belongs to $\mathcal{N}_\theta$.

The existence of a rational solution $(u, v, w, x, y)$ in the above theorem means that there exists a natural number $0 < w \leq n$ such that both the elliptic curves $E_w^\theta$ and $F_N^\theta$ have some non-2-torsion points $(x, y)$ and $(u, v)$ satisfying $x/y = u/v$, where the letter “R” in $C_R$ refers to the word “Ratio”.

We note that the ratio $b/d$ of a $\theta$-parallelogram envelope $(a, b, c, d, e)$ for $n$ depends on the parameter $w$. When $w$ varies from 0 up to $n$, the point $P_w$ moves from the point $A_1$ to the middle point of the side $A_1A_2$, which is denoted by $P_n$. Moreover, the point $P_w$ moves proportionally, namely for each rational number $t \geq 1$ the point $P_{n/w}$ is the point which divides $A_1A_2$ internally in the ratio $1 : 2w - 1$, see Figure 3 below.

![Figure 3](image)

**Figure 3.** The role of $w$ in $\theta$-parallelogram envelop for $n$

In order to reduce the number of variables in [3.1], we let $v = uy/x$ in $F_N^\theta$ to obtain the following equivalence for Theorem [3.1]
Theorem 3.2. A natural number \( n \) belongs to \( N_\theta \) if and only if the following equations,
\[
\begin{align*}
E^w_\theta : y^2 &= x(x + (r + s)w)(x - (r - s)w), \\
G^N_\theta : z^2 &= (x^2 + 2s(N + w)x - (r^2 - s^2)w^2)^2 + 4(r^2 - s^2)N^2x^2,
\end{align*}
\]
have a rational solution \((w, x, y, z)\) where \( N = 2n - w, y \neq 0, \) and \( 0 < w \leq n. \)

The proof of Theorems 3.1 and 3.2 are included in Section 8.

4. The \( \theta \)-parallelograms with ratio \( m \)

In this section, we are going to study \( N_{\theta,m} \) a certain subset of \( N_\theta \) associated to a given rational number \( m \geq 1. \)

Definition 4.1. Given a rational number \( m \geq 1, \) let \( N_{\theta,m} \) denote the subset of \( N_\theta \) consisting of all natural numbers \( n \) such that there exist positive rational numbers \( a, b, c, d, e \) satisfying (2.1) and \( d = mb. \) We call such a quintuple a \( \theta \)-parallelogram envelop with ratio \( m \) for \( n \) and denote by \((a, b, c, d, e)^m_\theta, \) see Figure 4.

\[ \text{Figure 4. A } \theta \text{-parallelogram envelop with ratio } m \text{ for } n \]

We note that \( N_{\theta,1} \) is nothing but \( N_\theta. \) For a positive integer \( m, \) a natural number \( n \) belongs to \( N_{\theta,m} \) if and only if \( mn \) belongs to \( N_{\theta,m}. \) Indeed, if \((a, b, c, d, e)^m_\theta \) is a \( \theta \)-parallelogram envelop with ratio \( m \) for \( n, \) then we obtain a \( \theta \)-parallelogram envelop with ratio \( m \) as \((d, a, e, ma, mc)^m_\theta \) for \( mn, \) by multiplying the first equation of (2.1) by \( m^2 \) and the third by \( m, \) see Figure 5.

The following theorem is a consequence of Theorems 3.1 and 3.2.

Theorem 4.2. A natural number \( n \) belongs to \( N_{\theta,m} \) for some rational number \( m \geq 1 \) if and only if the simultaneous equations
\[
\begin{align*}
E^{(m,n)}_\theta : y^2 &= x\left(x + \frac{2n(r + s)}{m + 1}\right)\left(x - \frac{2n(r - s)}{m + 1}\right), \\
G^{(m,n)}_\theta : z^2 &= x^4 + b_1x^3 + b_2x^2 + b_3x + b_4,
\end{align*}
\]
have a rational solution \((x, y, z)\) with \( y \neq 0, \) where
\[
\begin{align*}
b_1 &= 8ns, & b_2 &= \frac{8n^2(2m^2r^2 + 4s^2m + 3s^2 - r^2)}{(m + 1)^2}, \\
b_3 &= -\frac{32n^3s(r^2 - s^2)}{(m + 1)^2}, & b_4 &= \frac{16n^4(r^2 - s^2)^2}{(m + 1)^4}.
\end{align*}
\]
5. The elliptic curve $G^m_\theta$ related to $N_{\theta,m}$

In order to investigate the set of rational points satisfying the equations (4.1), we have the following two theorems on the quartic curve $G^{(m,n)}_\theta$. We will provide their detailed proofs in Section 9.

**Theorem 5.1.** The quartic curve $G^{(m,n)}_\theta$ in Theorem 4.2 can be birationally transformed to an elliptic curve given by the following cubic equation,

$$G^m_\theta: \quad Y^2 = X^3 + (r^2m^2 + 2s^2m + r^2)X^2 + m^2(r^2 - s^2)^2X.$$  \hspace{1cm} \text{(5.1)}

Moreover, for $\theta \in (0, \pi/2]$ with rational cosine, the rank $r_\theta(m)$ of the Mordell-Weil group $G^m_\theta(\mathbb{Q})$ is at least one for all but finitely many $m \in \mathbb{Q}$ with an independent point given by

$$P = (-(r^2 - s^2)^2m^2, s(r^2 - s^2)m^2(m + 1)).$$

The discriminant and $j$-invariant of the curve $G^m_\theta$ are given respectively as follows:

$$\Delta^m_\theta = 16r^2(r^2 - s^2)^4(m + 1)^2m^4(r^2m^2 + 2(2s^2 - r^2)m + r^2),$$

$$j^m_\theta = 256\frac{(r^4m^4 + 4r^2s^2m^3 + (s^4 + 6r^2s^2 - r^4)m^2 + 4r^2s^2m + r^4)^3}{(m + 1)^2(r^2 - s^2)^4(r^2m^2 + 2(2s^2 - r^2)m + r^2)m^4r^2}.$$  \hspace{1cm} \text{(5.2)}

It is easy to check the following isomorphisms over rational numbers,

$$G^m_\theta \cong G_\theta^{1/m}, \quad \text{and} \quad G^{-m}_\theta \cong G^{-1/m}_\theta.$$  \hspace{1cm} \text{(5.3)}

Hence, we will only deal with $\theta$-parallelogram envelops for a rational number $m \geq 1$ in the rest of the paper.

In the next result we determine all of the possibilities for the torsion subgroup $G^m_\theta(\mathbb{Q})_{tors}$ of $G^m_\theta$. 

---

**Figure 5.** Two $\theta$-parallelogram envelops with ratio $m$ for $n$ and $mn$
Theorem 5.2. Given $\theta \in (0, \pi/2]$ with $\cos(\theta) = s/r \in \mathbb{Q}^*$ and a rational number $m \ge 1$, let $M_0 = m(r^2 - s^2)$, $M_1 = r(m + r)(r(m + 1) + 2\sqrt{M_0})$, $M_2 = r(m + r)(r(m + 1) - 2\sqrt{M_0})$. Then, we have:

$$G_\theta^m(\mathbb{Q})_{\text{tors}} \cong \begin{cases} 
\mathbb{Z} & \text{if } \sqrt{M_0} \text{ and either } \sqrt{M_1} \text{ or } \sqrt{M_2} \in \mathbb{Q}^* \\
\mathbb{Z} \times \mathbb{Z} & \text{if } \sqrt{M_0}, \sqrt{M_1}, \text{ and } \sqrt{M_2} \in \mathbb{Q}^* \\
\mathbb{Z} / 4\mathbb{Z} & \text{otherwise.}
\end{cases}$$

The points $(M_0, \pm r(m + 1)M_0)$ are of order 4 and points with the following $X$-coordinates

$$X_1 = M_0 + (r(m + 1) + \sqrt{M_1}) \sqrt{M_0}, \quad X_2 = M_0 + (r(m + 1) - \sqrt{M_1}) \sqrt{M_0},$$

$$X_3 = M_0 - (r(m + 1) + \sqrt{M_2}) \sqrt{M_0}, \quad X_4 = M_0 - (r(m + 1) - \sqrt{M_2}) \sqrt{M_0},$$

are of order 8 in the torsion subgroup of $G_\theta^m$.

Moreover, for $\theta \neq \pi/2$, the cases $\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/8\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ happen for infinitely many $m$, but the case $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$ occurs only for finitely many $m$.

The following table shows examples for the torsion subgroups of $G_\theta^m(\mathbb{Q})$ respect to the conditions in the above theorem. For more data on the elliptic curve $G_\theta^m$, one can see Table 3 in the appendix. We end this section with noting to the fact that $G_\theta^m$ is isomorphic to the elliptic curve

$$E(m) : U^2 = X(X - m^2)(X - (m - 1)(m + 1)),$$

by changing the variables $X = X - m^2$ and $Y = U$, which is given and studied in [11].

| $\{r, s, m\}$ | $\{\sqrt{M_0}, \sqrt{M_1}, \sqrt{M_2}\}$ | $G_\theta^m(\mathbb{Q})_{\text{tors}}$ |
|---------------|------------------------------------------|-----------------------------|
| $\{2, 1, 2\}$ | $\{\sqrt{6}, 2\sqrt{9} + 3\sqrt{6}, 2\sqrt{9} - 3\sqrt{6}\}$ | $\mathbb{Z}/2\mathbb{Z}$ |
| $\{2, 1, 3\}$ | $\{3, 3\sqrt{7}, 4\}$ | $\mathbb{Z}/8\mathbb{Z}$ |
| $\{2, 1, 1\}$ | $\{\sqrt{3}, 2\sqrt{3} + 2, 2\sqrt{3} - 2\}$ | $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ |
| $\{25, 7, 1\}$ | $\{24, 70, 10\}$ | $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$ |

Table 1. Examples for the torsion subgroups of $G_\theta^m(\mathbb{Q})$ in Theorem 5.2.

6. The main results related to $N_{\theta, m}$ and $N_\theta$

Given $\theta \in (0, \pi/2]$ with $\cos(\theta) = s/r \in \mathbb{Q}^*$ and a rational number $m \ge 1$, we have the following facts on $N_{\theta, m}$.

Theorem 6.1. (i) If $\sqrt{m} \sin(\theta)$ and $\sqrt{m + 1} \in \mathbb{Q}^*$, then $\sqrt{M_0} \mod (\mathbb{Q}^*)^2$ belongs to $N_{\theta, m}$, and some points of order 8 in $G_\theta^m(\mathbb{Q})$ give some $\theta$-parallelogram envelop with ratio $m$.

(ii) Otherwise, there exists a $\theta$-parallelogram envelop $(a, b, c, d, e)^m_\theta \in N_{\theta, m}$ if and only if $r_\theta(m) \ge 1$.

Using the above theorem, we obtain the next result.

Theorem 6.2. (i) The set $N_{\theta, m}$ is empty if and only if $\sqrt{m} \sin(\theta)$ and $\sqrt{m + 1}$ are not rational numbers and $r_\theta(m) = 0$. 
Let \( \cos(\tau) = p/q \in \mathbb{Q} \) with \( p, q \in \mathbb{Z} \) satisfying \( 0 < p < q \) and \( \gcd(p, q) = 1 \). By the assumption, we have \( \sqrt{r^2 - s^2} = n'v^2 \sqrt{q^2 - p^2} \), where \( n' \) is a positive integer and \( v \) is a rational number.

If \( n \in \mathcal{N}_\theta(\tau) \), then \( 4n \) belongs to \( \mathcal{N}_\theta(\tau) \), and there exists a \( \theta \)-parallelogram \( (a, b, c, d, e)_\theta^* \) with angle \( \tau \) for \( 4n \), i.e., there exists a quintuple \( (a, b, c, d, e)_\theta^* \) of positive rational numbers satisfying
\[
c^2 = a^2 + b^2 - \frac{2sab}{r}, \quad a(b + d) = 4rn,
\]
\[
e^2 = a^2 + d^2 + \frac{2sab}{r}, \quad \cos(\tau) = \frac{a^2 - bd - 4sn}{ce}.
\]

These equations imply that
\[
c^2 + e^2 - \frac{2pce}{q} = (b + d)^2.
\]

Hence, we have a rational \( \tau \)-triangle \( (c, e, b + d) \) with area
\[
2n \sqrt{r^2 - s^2} = 2nn'v^2 \sqrt{q^2 - p^2}.
\]
Dividing all sides by \( v \) leads to a \( \tau \)-triangle \( (c/v, e/v, (b + d)/v) \) with area \( 2nn' \sqrt{q^2 - p^2} \), see Figure 6. This means that \( 2nn' \) is a \( \tau \)-congruent number as desired.
A NEW GENERALIZATION OF $\theta$-CONGRUENT NUMBERS

Conversely, since $2nn'$ is a $\tau$-congruent number, there exists a rational $\tau$-triangle $(a, b, c)$ for $2nn'$. Then, as described in the following, one may find a $\theta$-parallelogram envelope with angle $\tau$ for $n$ given by

$$
\left( \frac{rab}{2qcn'}, \frac{a^2}{2c} - \frac{(pn' - s)ab}{2qcn'}, \frac{a^2 - a^2}{2c} + \frac{(pn' - s)ab}{2qcn'}, \frac{b}{2} \right)^\tau.
$$

We first note that the $\tau$-triangle $(a, b, c)$ for $2nn'$ satisfies

$$
c^2 = a^2 + b^2 - \frac{2pab}{q}, \quad ab = 4qnn'.
$$

Hence, to find a $\theta$-parallelogram envelope with angle $\tau$ for $4n$, we have to solve

$$
a^2 = x^2 + y^2 - \frac{2sxy}{r},
$$

$$
b^2 = x^2 + (c - y)^2 + \frac{2sx(c - y)}{r},
$$

$$
xc = 4rn.
$$

Substituting (7.3) in (7.4) and using (7.2), we get $a^2 - cy - sab/r + scx/r = 0$ which gives

$$
y = \frac{a^2}{c} - \frac{pab}{qc} + \frac{sx}{r}.
$$

From (7.2) and (7.3), we have $x = rab/n'qc$ and hence $y = a^2/c - (pn' - s)ab/qcn'$ by (7.6). Therefore, we have found the $\theta$-parallelogram envelop $(x, y, a, c - y, b)$ for $4n$. Dividing all components by 2 leads to a $\theta$-parallelogram envelop for $n$ defined by (7.1), see Figure 7.

Figure 6. A $\tau$-triangle for $2nn'$ obtained from a $\theta$-parallelogram envelop for $n$

Figure 7. A $\theta$-parallelogram envelop with angle $\tau$ for $n$ obtained from a rational $\tau$-triangle for $2nn'$
Checking that it is really a \(\theta\)-parallelogram envelop with angle \(\tau\) for \(n\) is left as an exercise to the reader. For \(\tau = \pi/2\), we have a \(\theta\)-parallelogram envelop with angle \(\pi/2\) given by

\[
\left(\frac{rab}{2cn^2/2c} + \frac{sab}{2cn^2/2} \cdot \frac{a^2 - c^2}{2c} - \frac{sb}{2cn^2/2}\right)_{\pi/2}.
\]

(7.7)

In the case \(\tau = \theta\) we obtain the following \(\theta\)-parallelogram envelop with angle \(\theta\) given by

\[
\left(\frac{ab}{2cn^2/2c} \cdot \frac{a^2 - c^2}{2c} - \frac{b}{2cn^2/2}\right)_{\theta}
\]

(7.8)

using a rational \(\theta\)-triangle \((a, b, c)\). The last statement is trivial considering Remark 2.2 (1) and the definition of the \(\tau\)-congruent numbers. Therefore, the proof of Theorem 2.5 is completed.

8. Proofs of Theorems 3.1, 3.2 and 4.2

The proof of Theorem 3.1 is a consequence of the following one-to-one correspondence between \(R_\theta\) the set of all rational solutions of (3.1), and \(P_\theta\) the set of all \(\theta\)-parallelogram envelopes for \(n\). Indeed, for any \((u, v, w, x, y) \in R_\theta\), we have a \(\theta\)-parallelogram envelope for \(n\) as follows

\[
\begin{align*}
& a = \frac{y}{2x} = \frac{v}{2u}, \quad b = \frac{rwx}{y}, \quad c = \frac{x^2 + (r^2 - s^2)w^2}{2y}, \\
& d = \frac{r(2n - w)u}{v}, \quad e = \frac{u^2 + (r^2 - s^2)(2n - w)^2}{2v}.
\end{align*}
\]

(8.1)

Conversely, any \(\theta\)-parallelogram envelope \((a, b, c, d, e)_\theta\) for \(n\) corresponds to a rational solution \((u, v, w, x, y)\) defined by

\[
\begin{align*}
& x = 2a \left( a + c - \frac{s}{r} \right), \quad y = 4a^2 \left( a + c - \frac{s}{r} \right), \quad w = \frac{2ab}{r}, \\
& u = 2a \left( a + e + \frac{2s}{r} \right), \quad v = 4a^2 \left( a + e + \frac{s}{r} \right).
\end{align*}
\]

(8.2)

In order to show Theorem 3.2 we first assume that \(n\) belongs to \(N_\theta\). Then, by Theorem 3.1 there exists a rational solution \((u, v, w, x, y) \in R_\theta\) for the equations (3.1) satisfying \(yv \neq 0\) and \(0 < w \leq n\). Substituting \(uy/x\) for \(v\) in the equation of \(F_\theta^N\) gives that

\[
\frac{u^2y^2}{x^2} = u(u + (r - s)N)(u - (r + s)N),
\]

where \(N = 2n - w\). Multiplying both sides of the latter equation by \(x^2\) and then dividing by \(u\), we obtain

\[
x^2u^2 - 2x^2Ns - (r^2 - s^2)N^2x^2 - y^2u = 0.
\]

Substituting \(y^2\) with \(x^3 + 2sxw - (r^2 - s^2)w^2x\) in the above equation and then dividing both sides of the resulting equation by \(x\) give a quadratic equation in terms of \(u\),

\[
xu^2 - (x^2 + 2xs(N + w) - (r^2 - s^2)w^2)u - (r^2 - s^2)N^2x = 0.
\]

Solving this equation, we obtain

\[
u = \frac{x^2 + 2xs(N + w) - (r^2 - s^2)w^2}{2x} \pm \sqrt{\frac{(x^2 + 2xs(N + w) - (r^2 - s^2)w^2)^2 + 4(r^2 - s^2)N^2x^2}{2x}}.
\]

(8.3)
Since $x$, $u$, and $w$ are all rational, so there exists some $z \in \mathbb{Q}$ such that
\[
z^2 = (x^2 + 2xs(N + w) - (r^2 - s^2)w^2)^2 + 4(r^2 - s^2)N^2x^2.
\] (8.4)
Thus, we obtain the desired rational solution $(w, x, y, z)$ for the equation (3.2). Conversely, if we assume that $(w, x, y, z)$ with $y \neq 0$ and $0 < w \leq n$ is a solution for the equation (3.2), then we obtain a rational solution $(u, v, w, x, y)$ to (3.1), where $u$ is given by the equation (8.3) and $v = yu/x \neq 0$, respectively. Therefore, the proof of Theorem 3.2 is completed.

By the condition $d = mb$, we obtain that
\[
a(b + d) = rn \Longleftrightarrow ab(m + 1) = rn \Longleftrightarrow \frac{2ab}{r} = \frac{2n}{m + 1},
\]
and the aforementioned correspondence with $w = 2n/(m+1)$ implies that a natural number $n$ belongs to $N_{q,m}$ if and only if the simultaneous equations (3.1) with $N = 2nm/(m+1)$ have a rational solution $(u, v, x, y)$ with $yu \neq 0$. Now, using Theorem 3.2 and some simple algebraic simplifications on the equation (8.4) we obtain the desired equation (4.1). This completes the proof of Theorem 4.2.

9. Proofs of Theorems 5.1 and 5.2

We fix the natural number $n$ and the rational number $m \geq 1$. One can transmit the quartic $G_{\theta}^{(m,n)}$ to the cubic $G_{\theta}^m$ by the following birational transformations:

\[
x = \frac{-2n(r^2 - s^2)(\mathcal{X} + s(m + 1)\mathcal{X})}{(m + 1)\mathcal{X}(\mathcal{X} + r^2 - s^2)},
\]
\[
z = \frac{4n^2(r^2 - s^2)}{(m + 1)^2\mathcal{X}(\mathcal{X} + r^2 - s^2)}(\mathcal{X}^3 + (2d_2 - (r^2 - s^2))\mathcal{X}^2 + (3d_1 + 2s(m + 1)\mathcal{Y})\mathcal{X} + (r^2 - s^2)d_1),
\] (9.1)
where $d_1 = m^2(r^2 - s^2)^2$ and $d_2 = (r^2m^2 + 2s^2m + r^2)$ are the coefficients of elliptic curve $G_{\theta}^m$, and

\[
\mathcal{X} = \frac{-(r^2 - s^2)\left((m + 1)^2x^2 + 4sn(m + 1)^2x + (m + 1)^2z - 4n^2(r^2 - s^2)\right)}{2(m + 1)^2x^2},
\]
\[
\mathcal{Y} = \frac{(r^2 - s^2)\left(s(m + 1)^4x^3 + c_2x^2 + c_1x + c_0\right)}{2(m + 1)^3x^3},
\] (9.2)
where
\[
c_0 = -2n(r^2 - s^2)\left((m + 1)^2z - 4n^2(r^2 - s^2)\right),
\]
\[
c_1 = s(m + 1)^2\left((m + 1)^2z - 12n^2(r^2 - s^2)\right),
\]
\[
c_2 = 2n(m + 1)^2(2d_2 - 3(r^2 - s^2)).
\]
The points $(x, z) = (0, \pm 2n(r^2 - s^2)(m + 1))$ on the quartic $G_{\theta}^{(m,n)}$ map to the point at infinity on $G_{\theta}^m$ by the above change of variables. The Mordell-Weil group of rational points on $G_{\theta}^m$ has generic rank one, because one can easily examine by SAGE software that $[n]P$ are not $\infty$ for $2 \leq n \leq 16$, where
\[
P = (-r^2 - s^2)^2m^2, s(r^2 - s^2)m^2(m + 1)).
\]
Hence, it is of infinite order and generates $G_{\theta}^m(\mathbb{Q})$. This completes the proof of Theorem 5.1.
In what follows we classify the possibilities for $G^n_{\theta}(Q)_{\text{tors}}$, the torsion subgroup of $G^n_{\theta}$. It has trivially the point $(0,0)$, which is of order 2. Hence, by the celebrated Mazur’s theorem, we have the following possibilities:

$$\frac{Z}{2Z}, \frac{Z}{4Z}, \frac{Z}{6Z}, \frac{Z}{8Z}, \frac{Z}{10Z}, \frac{Z}{12Z}, \frac{Z}{2Z\times 4Z}, \frac{Z}{2Z\times 6Z}, \frac{Z}{2Z\times 8Z}.$$ 

For any point $T = (X,Y)$ on the curve $G^n_{\theta}$, the duplication formula of $G^n_{\theta}(Q)$ leads to

$$X([2]T) = \left(\frac{X^2 - (r^2 - s^2)m^2}{4Y^2}\right). \quad (9.3)$$

Hence the $X$-coordinates of the points $[2n]T$, for $n = 2, 3, 4, 5, 6$, are also squares.

If $T$ is a point of order 4, then $X([2]T) = 0$. By the duplication formula and the fact that $(0,0)$ is the unique point of order 2 on $G^n_{\theta}$, we find out two points of order 4 as $T$ and $-T$, where

$$T := \left((r^2 - s^2)m, r(r^2 - s^2)m(m+1)\right).$$

Thus, the cases $Z/2Z, Z/6Z$ and $Z/10Z$ cannot happen. Hence, the possibilities for torsion subgroup reduce to

$$\frac{Z}{4Z}, \frac{Z}{8Z}, \frac{Z}{12Z}, \frac{Z}{22Z\times 4Z}, \frac{Z}{22Z\times 6Z}, \frac{Z}{22Z\times 8Z}.$$ 

Let us rule out the two possibilities $Z/12Z$ and $Z/22Z \times Z/6Z$. To this end, it suffices to show the non-existence of any point of order 3. By contrariwise, we assume that $T = (X,Y)$ is such a point. Then, the equality $[2]T = -T$ implies that

$$-3X^4 + (-4r^2 - 8ms^2 - 4mr^2s^2)X^3 - 6m^2(r-s)^2(r+s)^4(r+s)^4 = 0,$$

which is unsolvable over $Q$. Hence, the cases $Z/12Z$ and $Z/22Z \times Z/6Z$ can never happen as torsion subgroup of $G^n_{\theta}$ over $Q$. Considering these observations and applying the Mazur’s theorem show that

$$G^n_{\theta}(Q)_{\text{tors}} \cong \frac{Z}{4Z}, \text{ or } \frac{Z}{8Z}, \text{ or } \frac{Z}{22Z\times 4Z}, \text{ or } \frac{Z}{22Z\times 6Z}, \text{ or } \frac{Z}{22Z\times 8Z}.$$ 

Now, if we assume that $T$ is a point of order 8, then $[2]T$ (resp. $[4]T$) will be a point of order 4 (resp. 2). Solving $X([2]T) = (r^2 - s^2)m$ is equivalent to finding the solutions of the following quartic equation,

$$X^4 - 4m(r^2 - s^2)X^3 + 2(m(r^2 - s^2))(2r^2 + mr^2 + 2m^2r^2 + 3ms^2)X^2 - 4m^3(r^2 - s^2)^3X + m^4(r^2 - s^2)^4 = 0,$$

which can be rewritten as $(X - m(r^2 - s^2))^4 = 4m^2(r^2 - s^2)(m+1)^2X^2$. Solving this equation leads to the following list of $X$-coordinates of $T$,

$$X_1 = M_0 + (r(m+1) + \sqrt{M_1})\sqrt{M_0}, \quad X_3 = M_0 - (r(m+1) - \sqrt{M_1})\sqrt{M_0},$$

$$X_2 = M_0 + (r(m+1) + \sqrt{M_2})\sqrt{M_0}, \quad X_4 = M_0 - (r(m+1) - \sqrt{M_2})\sqrt{M_0}, \quad (9.4)$$

where $M_0 = m(r^2 - s^2), M_1 = (rm+r)(rm + r + 2\sqrt{M_0}), M_2 = (rm+r)(rm + r - 2\sqrt{M_0})$.

Using the duplication formula (9.3), one may write down the $Y$-coordinates as follow:

$$Y_1 = \frac{1}{2}M_0 \left(r(m+1) + \sqrt{M_1}\right)\left(r(m+1) + \sqrt{M_1} + 2\right),$$

$$Y_2 = \frac{1}{2}M_0 \left(r(m+1) - \sqrt{M_1}\right)\left(r(m+1) - \sqrt{M_1} + 2\right),$$

$$Y_3 = \frac{1}{2}M_0 \left(r(m+1) + \sqrt{M_2}\right)\left(r(m+1) + \sqrt{M_2} - 2\right),$$

$$Y_4 = \frac{1}{2}M_0 \left(r(m+1) - \sqrt{M_2}\right)\left(r(m+1) + \sqrt{M_2} - 2\right). \quad (9.5)$$
We note that $M_0$, $M_1$, and $M_2$ in the above formulas are all positive numbers. Evidently $M_0$, and $M_1 > 0$, since $0 \leq |s| < r$ and $m \geq 1$. To show $M_2 > 0$, it is then enough to observe

$$rm + r - 2\sqrt{M_0} > 0 \iff r^2(m + 1)^2 > 4m(r^2 - s^2) \iff r^2(m - 1)^2 + 4ms^2 > 0$$

which is the case. Thus, in order to have $E_0^m(\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$, we need to assume that all the numbers $\sqrt{M_0}$, $\sqrt{M_1}$, and $\sqrt{M_2}$ are rational. If one of the values $\sqrt{M_1}$ and $\sqrt{M_2}$, does not belong to $\mathbb{Q}$, then $E_0^m(\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/8\mathbb{Z}$ provided that $\sqrt{M_0} \in \mathbb{Q}^*$. When the last condition fails to be true, we will have $E_0^m(\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$.

In order to prove the last statement of Theorem 5.2, we assume that $m = m_0^2/(r^2 - s^2)$, which yields

$$M_1 = \frac{r(m_0^2 + r^2 - s^2)(r(m_0^2 + (r^2 - s^2)) + 2m_0(r^2 - s^2))}{(r^2 - s^2)^2},$$

$$M_2 = \frac{r(m_0^2 + r^2 - s^2)(r(m_0^2 + (r^2 - s^2)) - 2m_0(r^2 - s^2))}{(r^2 - s^2)^2}.$$  \hspace{1cm} (9.6)

(9.7)

We let $m_i = m_0^2(r^2 - s^2)\sqrt{M_i}$ for $i = 1, 2$. Then,

$$m_1 = r(m_0^2 + r^2 - s^2)(r(m_0^2 + (r^2 - s^2)) + 2m_0(r^2 - s^2)),$$

$$m_2 = r(m_0^2 + r^2 - s^2)(r(m_0^2 + (r^2 - s^2)) - 2m_0(r^2 - s^2)).$$  \hspace{1cm} (9.8)

(9.9)

Each of the equations \((9.8)\) and \((9.9)\) defines an elliptic curve over $\mathbb{Q}$ for a given angle $\theta \in (0, \pi/2]$, which is birational to

$$E_0 : Y^2 = X^3 - 108r^2(r^2 - s^2)^2(r^2 + 3s^2)X + 432r^4(r^2 - s^2)^3(r^2 - 9s^2),$$

by the following change of variables:

$$\begin{cases}
X = \frac{6r (r^2 - s^2)m_0^2 + 3s(r^2 - s^2)m_0 + 3(m_i + r)}{m_0^2}, \\
Y = \frac{54r(r^2 - s^2)m_0 (r(r^2 - s^2)m_0^2 + 2r^2m_0 + (m_i + 3r)) + 2er(m_i + r)}{m_0^3},
\end{cases}$$  \hspace{1cm} (9.10)

and

$$\begin{cases}
m_0 = \frac{6r \left( Y + 3s(r^2 - s^2)(X + 12r^2(r^2 - s^2)) \right)}{\left( X + 12r^2(r^2 - s^2) \right) \left( X - 24r^2(r^2 - s^2) \right)}, \\
m_i = \frac{r \left( X^3 + e_1X + e_0 \right)}{\left( X + 12r^2(r^2 - s^2) \right) \left( X - 24r^2(r^2 - s^2) \right)^2},
\end{cases}$$  \hspace{1cm} (9.11)

where $e = (-1)^{i-1}$ for $i = 1, 2$, and

$$e_0 = -72r^2(r^2 - s^2)^2 (48r^4(r^2 - s^2) + \varepsilon Y),$$

$$e_1 = 108r^2(r^2 - s^2)(5r^2 - 9s^2) \left( (r^2 - s^2) + \varepsilon Y \right).$$

The torsion subgroup of $E_0$ is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ containing the point at infinity and

$$P_0 = (-12r^2(r^2 - s^2), 0), \quad P_1 = (-6r(r - s)(r^2 - 9s^2), 0), \quad P_2 = (-6r(r + 3s)(r^2 - s^2), 0).$$

Moreover, the Mordell-Weil rank of $E_0(\mathbb{Q})$ is at least one and an independent point of infinite order is given by

$$Q = (-3(r^2 - s^2)(r^2 + 3s^2), 27(r^2 - s^2)^3).$$
10. Proofs of the main results on $N_{\theta,m}$

10.1. Proof of Theorem 6.1. Let $\sqrt{m+1} = v$ and $\sqrt{m}\sin(\theta) = N'u^2$, where $N'$ is a positive integer and $\alpha, u \in \mathbb{Q}$. Then, $\sqrt{M_0} = N'ru^2$ and

$$\sqrt{M_1} = \frac{rv}{u}\sqrt{(v/u)^2 + 2N'}, \quad \sqrt{M_2} = \frac{rv}{u}\sqrt{(v/u)^2 - 2N'}.$$  

Hence, both the numbers $\sqrt{M_1}$ and $\sqrt{M_2}$ are rational if and only if there are rational numbers $\alpha$ and $\beta$ such that $(v/u)^2 + 2N' = \alpha^2$ and $(v/u)^2 - 2N' = \beta^2$, which is equivalent to saying $2N'$ is a congruent number with a right triangle $(\alpha - \beta, \alpha + \beta, 2\gamma)$, where $\gamma = v/u$. Hence, for such a right triangle we have $\sqrt{M_1} = r\alpha\gamma$ and $\sqrt{M_2} = r\beta\gamma$. In order to prove the part (i) of Theorem 6.1 one needs to do cumbersome computations using a series of transformations which we only give a sketch and leave the detailed computations to the interested readers. For $i = 1, \ldots, 4$, writing the order 8 points $(x_i, \pm y_i)$ in terms of $N', u, v, \alpha, \beta$, transforming them to the points $(x_i, z_i)$ on $G_{\theta}^{(m,n)}$ by (9.1), and finding $y_i$-coordinates on $E_{\theta}^{(m,n)}$ using $x_i$'s, we obtain rational solutions $(x_i, y_i, z_i)$ for the equation (4.1). Calculating $u_i$ from $x_i$ and $z_i$ using the formula (8.3) and then putting $v_i = u_iy_i/x_i$ lead to the solutions $(u_i, v_i, w, x_i, y_i)$ with $w = 2N'/(N'u^2 + 1)$ for the equations (5.2).

Now, using the correspondence (5.1), one may obtain $\theta$-parallelograms with ratio $m$ for $\sqrt{M_0} \mod (\mathbb{Q}^*)^2$, which means that it belongs to $N_{\theta,m}$ as desired.

If one or both $\sqrt{m+1}$ and $\sqrt{m}\sin(\theta)$ do not belong to $\mathbb{Q}$, then at least one of the $\sqrt{M_i}$ will be a non-rational number, hence the torsion subgroup of $G_{\theta}^{m}$ cannot have a point of order 8. Thus, it will be isomorphic to $\mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$, by Theorem 5.2. The order 4 point in $G_{\theta}^{m}(\mathbb{Q})_{\text{tors}}$ is given by $T = (m(r^2 - s^2), r(r^2 - s^2)m(m+1))$, which leads to a rational point $(x, z)$ on $G_{\theta}^{(m,n)}$ with $x$-coordinate $-2n(r+s)/(m+1)$. But, this is a zero of the cubic defining $E_{\theta}^{(m,n)}$. Thus no points of order 4 in $G_{\theta}^{m}(\mathbb{Q})_{\text{tors}}$ give us a $\theta$-parallelogram. Therefore, the existence of a $\theta$-parallelogram with ration $m$ for $n$, which one or both $\sqrt{m+1}$ and $\sqrt{m}\sin(\theta)$ are not rational, means that $G_{\theta}^{m}(\mathbb{Q})$ has a point of infinite order.

Conversely, we let $(\mathcal{X}, \mathcal{Y}) \in G_{\theta}^{(m,n)}(\mathbb{Q})$ be a point of infinite order. Then, the $\mathcal{Y}$-coordinates cannot be either 0 or $\pm r(m+1)\mathcal{X}$ since it is neither of order 2 nor 4. By substituting

$$x = \frac{-2n(r^2 - s^2)(\mathcal{Y} + s(m+1)\mathcal{X})}{(m+1)\mathcal{X}(\mathcal{X} + r^2 - s^2)},$$

one can write the defining equation of $E_{\theta}^{(m,n)}$ as

$$y^2 = \left(\frac{2n(r^2 - s^2)}{(m+1)\mathcal{X}(\mathcal{X} + r^2 - s^2)}\right)^2 \cdot A_{\theta}^{(m,n)}.$$
where
\[
A^{(m,n)}_g = \frac{2n(Y + s(m + 1)X)(X^2 - (r - s)(ms - r)X - (r - s)Y)}{(m + 1)X(X + s^2)}
\times (X^2 + (r + s)(ms - r)X + (r + s)Y).
\]

If required, one can choose the sign of \(Y\) in a way such that \(A^{(m,n)}_g\) is a positive rational number. To see this, we may assume \(0 < s < r\). Since \((X, Y)\) is not of finite order and \(G^m_n\) goes from the origin of coordinate system, so we have \(X > 0\) which implies that \(Y + s(m + 1)X > 0\). Hence, the expression \(A^{(m,n)}_g\) is positive if and only if the following two expressions are of the same sign,
\[
A_1 := X^2 - (r - s)(ms - r)X - (r - s)Y, \quad A_2 := X^2 + (r + s)(ms - r)X + (r + s)Y.
\]

For \(Y > 0\) we have two cases:

(i) \(A_1, A_2 > 0\): The relation \(A_1 > 0\) holds if and only if
\[
X > (r - s)(ms - r) \quad \text{and} \quad Y < \frac{X(X - (r - s)(ms - r))}{r - s},
\]
and \(A_2 > 0\) if and only if
\[
X > (r + s)(r - ms) \quad \text{and} \quad Y > 0,
\]
or
\[
X > (r + s)(r - ms) \quad \text{and} \quad Y > -\frac{X(X + (r + s)(ms - r))}{r + s}.
\]

(ii) \(A_1, A_2 < 0\): The relation \(A_1 < 0\) holds if and only if
\[
X < (r - s)(ms - r) \quad \text{and} \quad Y > 0,
\]
or
\[
X < (r - s)(ms - r) \quad \text{and} \quad Y > \frac{X(X - (r - s)(ms - r))}{r - s},
\]
and \(A_2 < 0\) if and only if
\[
X < (r + s)(r - ms) \quad \text{and} \quad Y < -\frac{X(X + (r + s)(ms - r))}{r + s}.
\]

For \(Y < 0\) we have two cases:

(i) \(A_1, A_2 > 0\): The relation \(A_1 > 0\) holds if and only if
\[
X > (r - s)(ms - r) \quad \text{and} \quad Y < 0,
\]
or
\[
X < (r - s)(ms - r) \quad \text{and} \quad Y < \frac{X(X - (r - s)(ms - r))}{r - s},
\]
and \(A_2 > 0\) if and only if
\[
X > (r + s)(r - ms) \quad \text{and} \quad Y > -\frac{X(X + (r + s)(ms - r))}{r + s}.
\]

(ii) \(A_1, A_2 < 0\): The relation \(A_1 < 0\) holds if and only if
\[
X < (r - s)(ms - r) \quad \text{and} \quad Y > \frac{X(X - (r - s)(ms - r))}{r - s}
\]
and \(A_2 > 0\) if and only if
\[
X < (r + s)(r - ms) \quad \text{and} \quad Y < 0.
or
\[ X > (r + s)(r - ms) \quad \text{and} \quad Y < \frac{X(X + (r + s)(ms - r))}{r + s}. \]

Therefore, any \( n \in \mathbb{N} \) for which \( A_{\theta}^{(m,n)} \in (\mathbb{Q}^*)^2 \) belongs to \( \mathcal{N}_{\theta,m} \) by Theorem 6.2.

10.2. **Proof of Theorem 6.2.** The part (i) and the sufficiency of (ii) are direct consequences of Theorem 6.2. The necessity of (iii) is clear by sufficiencies of (i) and (ii).

For the sufficiency of (iii), we assume that \( r_\theta(m) \geq 1 \) and \((X, Y)\) is a point of infinite order. Let \( \varphi: \mathcal{G}_{\theta}^{(m,n)}(\mathbb{Q}) \to \mathbb{Q} \) be defined by \( \varphi(\infty) = 0 \) and

\[
\varphi(X, Y) = A_{\theta}^{(m,n)} \cdot \left( \frac{m + 1}{2n} \right) X(X + r^2 - s^2) = (Y + s(m + 1)X)(X^2 - (r - s)(ms - r)X - (r - s)Y) \times (X^2 + (r + s)(ms - r)X + (r + s)Y).
\]

We also define the map \( \pi: \mathbb{Q} \to \mathbb{Q}^*/(\mathbb{Q}^*)^2 \cup \{0\} \) by

\[
\pi(q) = \begin{cases} q \mod \mathbb{Q}^* & \text{if } q \neq 0, \\ 0 & \text{if } q = 0. \end{cases}
\]

Then, the set \( \mathcal{N}_{\theta,m} \) has infinite element if and only if the image of \( \mathcal{G}_{\theta}^{(m,n)}(\mathbb{Q}) \) under the composite map \( \pi \circ \varphi \) is infinite by the proof of part (ii) of Theorem 6.1. We define \( \Sigma \) to be the following set

\[
\left\{ (\mathbb{Q}(\varphi(P)) : P = (X, Y) \in \mathcal{G}_{\theta}^{(m,n)}(\mathbb{Q}) \right\}.
\]

Then, the image of \( \mathcal{G}_{\theta}^{(m,n)}(\mathbb{Q}) \) by \( \pi \circ \varphi \) is infinite if and only if \( \Sigma \) is infinite. Because, if we assume that \( \Sigma \) is finite, then by the pigeonhole principal there is a field \( \mathbb{K} \in \Sigma \) and an infinite subset \( \mathcal{H} \subset \mathcal{G}_{\theta}^{(m,n)}(\mathbb{Q}) \) such that \( \mathbb{K} = \mathbb{Q}(\varphi(P)) \) for all \( P \in \mathcal{H} \). This implies that the algebraic surface \( \mathcal{S} \) defined by the equation \( Z^2 = \varphi(X, Y) \) has infinitely many \( \mathbb{K} \)-rational points \((X, Y, Z) = (X(P), Y(P), \sqrt{\varphi(P)})\) for \( P \in \mathcal{H} \). Furthermore, \( P \in \mathcal{G}_{\theta}^{(m,n)}(\mathbb{Q}) \) so that the following algebraic curve

\[
\mathcal{C}_{\theta}^m \coloneqq \left\{ \begin{array}{l} Z^2 = \varphi(X, Y), \\ Y^2 = X^3 + (r^2m^2 + 2s^2m + r^2)X^2 + m^2(r^2 - s^2)^2X \end{array}\right.
\]

has also infinitely many \( \mathbb{K} \)-rational points as above. But, this is a contradiction with the Faltings' theorem [8] on the set of rational points on algebraic curves with genus \( \geq 2 \), since we have the following result.

**Lemma 10.1.** The genus of \( \mathcal{C}_{\theta}^m \) is equal to 9.

**Proof.** Let \( \omega \) be the canonical sheaf of \( \mathcal{C}_{\theta}^m \). By the exercise (II.8.4.e) in [7], it is isomorphic to the invertible sheaf \( \mathcal{O}(4) \), and hence \( \deg(\omega) = 4 \cdot \deg(\mathcal{C}_{\theta}^m) \). Using the classical version of the Bezout's theorem, see Proposition 8.4 in [8] or Example 1 on page 198 of [12], the degree of \( \mathcal{C}_{\theta}^m \) over \( \mathbb{C} \) is equal to 4 and so \( \deg(\omega) = 16 \). As a consequence of the Riemann-Roch theorem, it is well known that the degree of canonical sheaf of any algebraic curve of genus \( g \) is equal to \( 2g - 2 \), see the example 1.3.3 in chapter IV of [7]. Therefore, the genus of \( \mathcal{C}_{\theta}^m \) is equal to 9. \( \square \)
10.3. **Proof of Theorem 6.3.** Putting \( Z = (m + 1)X(X + r^2 - s^2)y/(2n(r^2 - s^2)) \) and using the variable changes given by \((9.1)\), the simultaneous equations defining \( E_{\theta}^{(m,n)} \) and \( G_{\theta}^{(m,n)} \) give us the space curve \( C_{\theta}^{(m,n)} \) given in the statement of Theorem 6.3. On the other hand, it is easy to check that there exists a solution \((x, y, z)\) of the equation \((4.1)\) if and only if \( C_{\theta}^{(m,n)} \) has a solution \((X, Y, Z)\) with \( Y \neq 0\). This shows the part (i) of Theorem 6.3.

By a similar argument as given in the proof of Lemma 10.1, one can show that \( C_{\theta}^{(m,n)} \) is an algebraic curve of genus 9. Hence, it contains only finitely many rational solutions. Therefore, applying Theorem 4.2 gives us the part (ii) of 6.3.

10.4. **Proof of Theorem 6.4.** First, we prove the part (i) in the following two steps.

**Step 1:** Let \( T \) be a positive rational number. If \((c, e, f)\) is a rational triangle with area \( 2T \), then there exists a \( \theta \)-parallelogram envelop \((a, b, c, d, e)_{\theta} \) for \( 4T \), where

\[
a = \frac{4rT}{f \cdot \sqrt{r^2 - s^2}}, \quad b = \frac{f^2 + c^2 - e^2}{2f} + \frac{as}{r}, \quad d = \frac{f^2 + e^2 - c^2}{2f} - \frac{as}{r}.
\]

To show this, we show the existence of a rational \( \theta \)-parallelogram envelop with base \( f \) and the same height of the rational triangle \((c, e, f)\) like in Figure 8.

![Figure 8](image)

**Figure 8.** An envelop for \( 4T \) made from a triangle with area \( 2T \)

Let us to denote by \( \tau \) the angle between the sides \( c \) and \( e \) of the triangle \((c, e, f)\) with area \( T \). Then, by the area formula for the \( \theta \)-parallelogram envelop \((a, b, c, d, e)_{\theta} \), we have

\[
a = \frac{4rT}{f \cdot \sqrt{r^2 - s^2}}.
\]

Using the internal angles \( \alpha \) and \( \beta \), opposites to the sides \( c \) and \( e \) respectively, we have

\[
a \sin(\theta) = e \sin(\alpha) = c \sin(\beta), \quad b - a \cos(\theta) = c \cos(\beta), \quad d + a \cos(\theta) = e \cos(\alpha).
\]

By the law of cosines for the angles \( \alpha \) and \( \beta \), we have

\[
c^2 = e^2 + f^2 - 2ef \cos(\alpha), \quad e^2 = c^2 + f^2 - 2cf \cos(\beta).
\]

Considering the above equalities leads to

\[
b = \frac{f^2 + c^2 - e^2}{2f} + a \cos(\theta), \quad d = \frac{f^2 + e^2 - c^2}{2f} - a \cos(\theta).
\]

Since we have supposed that \( \theta \) is a Pythagorean angle with \( \cos(\theta) = s/r \), so its sine is rational number, namely \( \sin(\theta) = \sqrt{r^2 - s^2}/r \). Thus, the \( \theta \)-parallelogram envelop \((a, b, c, d, e)_{\theta} \) determined as above has rational components.

**Step 2:** For any positive rational number \( T \) there exists a rational triangle with area \( T \).
Indeed, this is proved by N. J. Fine in Theorem 2 of [2]. Considering his proof, the desired rational triangle with area $T$ is given by

$$
(c_e,f) = \left(\left|\frac{y}{x}\right|, \left|\frac{T^2 x^2 + 1}{y}\right|, \left|\frac{T^2 x^4 + 1}{xy}\right|\right),
$$

(10.3)

where the point $P = (x, y)$ with

$$
x = \frac{T^2 + 64}{8(T^2 - 8)}, \quad y = \frac{-3T(T^4 - 160T^2 - 512)}{64(T^2 - 8)^2},
$$

(10.4)

satisfies the following genus one quartic curve

$$
C_T : y^2 = T^2 x^4 + T^2 x^3 - x - 1.
$$

(10.5)

Using the equations (10.2) and (10.3), one may obtain that all components of the corresponding $\theta$-parallelogram envelop $(a, b, c, d, e)$ in terms of $T$. Now, to show that $N = N_\theta$ it is enough to consider $T = n\sqrt{r^2 - s^2}$ for any Pythagorean angle $\theta \in (0, \pi/2]$ with $\cos(\theta) = s/r$. One needs to divide all the rational numbers $a, b, c, d, e,$ and $f$ by 2 to get a $\theta$-parallelogram envelop for $n \geq 1$.

In order to prove the part (ii), we investigate more on the curve $C_T$. This curve is birational to the elliptic curve in short Weierstrass equation

$$
E_T : y^2 = X^3 + 3T^2 X - T^2(T^2 - 1),
$$

(10.6)

by the following birational maps

$$
\begin{align*}
x &= \frac{X + 1}{T^2 - X}, & y &= \frac{(T^2 + 1)Y}{(T^2 - X)^2} \\
X &= \frac{T^2 x - 1}{x + 1}, & Y &= \frac{(T^2 + 1)y}{(x + 1)^2}.
\end{align*}
$$

(10.7)

Under these maps, the point $(x, y)$ given by (10.4) can be transformed into $Q$ with

$$
X(Q) = \frac{T^4 + 56T^2 + 64}{9T^2}, \quad Y(Q) = \frac{-3T(T^4 - 160T^2 - 512)}{27T^3}.
$$

(10.8)

By the transformation maps (10.7), the point $Q_0 = (T^2, T(T^2 + 1))$ corresponds to the point at the infinity on $C_T$. Also, the point $P_0 = (-1, 0)$ on $C_T$ corresponds to the point at the infinity on $E_T$. Moreover, one can easily check that the point

$$
Q_1 = \left(\frac{T^2}{4}, \frac{T(T^2 - 8)}{8}\right)
$$

is another point on the elliptic curve $E_T$ with corresponds to non-torsion point

$$
P_1 = \left(\frac{T^2 + 4 + 2(T^2 + 1)(T^2 - 8)}{3T^2}, \frac{2(T^2 + 1)(T^2 - 8)}{9T^3}\right).
$$

The point $P_1$ gives us the following triangle

$$
(c_e,f)_1 = \left(\left|\frac{T^2 + 16}{2(T^2 - 8)}\right|, \left|\frac{T^6 + 96T^4 + 256}{6T^2(T^2 - 8)(T^2 + 4)}\right|, \left|\frac{2(T^2 + 1)(T^2 - 8)}{3T(T^2 + 4)}\right|\right).
$$

(10.9)

For any integer $\ell \geq 1$, we let $P_\ell = [\ell]P$ be the $\ell$-th multiple of $P$ and denote its corresponding rational triangle by $(c_e,f)_\ell$. Then, applying the equation (10.2) and letting $T = n\sqrt{r^2 - s^2}$, one can obtain infinitely many distinct $\theta$-parallelogram envelopes for each $n \geq 1$ and for any Pythagorean angle $\theta \in (0, \pi/2]$ with $\cos(\theta) = s/r$. Therefore, the part (ii) is proved. The
A NEW GENERALIZATION OF $\theta$-CONGRUENT NUMBERS

last part is a direct consequence of the part (ii) if we define $m = b/d$ for the infinitely many $\theta$-parallelogram envelopes corresponding to the points $[\ell]P_1$ for $\ell \geq 1$.

APPENDIX

| $n$ | $[r_1, t_1]$ | $[r_2, t_2]$ | $(a, b, c, d, e) = \frac{b}{d}$ |
|-----|--------------|--------------|--------------------------------|
| 1   | [0, 8]       | [0, 4]       | $\left(\frac{3}{4}, \frac{5}{3}, \frac{19}{12}, \frac{1}{3}, \frac{117}{12}\right)$ |
| 2   | [0, 4]       | [0, 4]       | $\left(\frac{24}{13}, \frac{128}{65}, \frac{152}{65}, \frac{77}{39}, \frac{73}{90}\right)$ |
| 3   | [0, 4]       | [0, 4]       | $\left(\frac{34}{13}, \frac{575}{182}, \frac{589}{33}, \frac{560}{458}, \frac{481}{55}\right)$ |
| 5   | [0, 4]       | [1, 4]       | $\left(\frac{5}{3}, \frac{4}{3}, \frac{19}{12}, \frac{20}{3}, \frac{12}{33}\right)$ |
| 6   | [1, 4]       | [0, 4]       | $\left(\frac{330}{117}, \frac{135}{28}, \frac{19}{3}, \frac{1585}{49}, \frac{131}{28}\right)$ |
| 7   | [0, 4]       | [0, 4]       | $\left(\frac{56}{27}, \frac{256}{99}, \frac{856}{297}, \frac{1649}{991}, \frac{4053}{1188}\right)$ |
| 10  | [1, 4]       | [1, 4]       | $\left(\frac{7}{3}, \frac{35}{11}, \frac{133}{71}, \frac{715}{112}, \frac{2461}{306}\right)$ |
| 11  | [1, 4]       | [0, 4]       | $\left(\frac{264}{71}, \frac{121}{28}, \frac{1661}{364}, \frac{137}{42}, \frac{7063}{546}\right)$ |
| 13  | [1, 4]       | [0, 4]       | $\left(\frac{32}{21}, \frac{13}{1}, \frac{209}{84}, \frac{20}{4}, \frac{691}{84}\right)$ |
| 14  | [0, 4]       | [2, 4]       | $\left(\frac{1115}{456}, \frac{2432}{105}, \frac{152}{3077}, \frac{25725}{23961}, \frac{3924}{390}\right)$ |
| 15  | [0, 4]       | [1, 4]       | $\left(\frac{80}{27}, \frac{115}{7}, \frac{23}{8}, \frac{893}{168}\right)$ |
| 17  | [1, 4]       | [1, 4]       | $\left(\frac{119}{12}, \frac{115}{11}, \frac{119}{67}, \frac{482}{35}, \frac{1025}{54}\right)$ |
| 19  | [0, 4]       | [1, 4]       | $\left(\frac{456}{77}, \frac{282}{77}, \frac{456}{991}, \frac{3193}{924}, \frac{3924}{921}\right)$ |
| 21  | [1, 4]       | [1, 4]       | $\left(\frac{7}{3}, \frac{7}{2}, \frac{7}{3}, \frac{8}{3}\right)$ |
| 22  | [1, 4]       | [1, 4]       | $\left(\frac{8}{3}, \frac{8}{3}, \frac{3}{2}, \frac{17}{2}, \frac{17}{2}\right)$ |
| 23  | [1, 4]       | [1, 4]       | $\left(\frac{98553}{63019}, \frac{404320}{189067}, \frac{437171}{189067}, \frac{56467478762}{2070237169}, \frac{57362384614}{2070237169}\right)$ |
| 26  | [0, 4]       | [0, 4]       | $\left(\frac{208}{39}, \frac{392}{247}, \frac{89}{89}, \frac{1037}{131}\right)$ |
| 29  | [0, 4]       | [1, 4]       | $\left(\frac{8}{3}, \frac{5}{2}, \frac{19}{6}, \frac{77}{12}, \frac{241}{12}\right)$ |
| 30  | [1, 4]       | [0, 4]       | $\left(\frac{8}{3}, \frac{4}{3}, \frac{7}{2}, \frac{17}{2}\right)$ |
| 31  | [0, 4]       | [0, 4]       | $\left(\frac{4464}{655}, \frac{744}{655}, \frac{5952}{6840}, \frac{6607}{47880}, \frac{344331}{1744}\right)$ |
| 33  | [0, 4]       | [1, 4]       | $\left(\frac{264}{115}, \frac{1408}{33}, \frac{88}{38}, \frac{86943}{2129}, \frac{38733}{2129}\right)$ |
| 34  | [1, 4]       | [1, 4]       | $\left(\frac{15}{2}, \frac{40}{11}, \frac{115}{32}, \frac{352}{1879}, \frac{1759}{740}\right)$ |
| 35  | [1, 4]       | [0, 4]       | $\left(\frac{15}{35}, \frac{35}{4}, \frac{10}{7}, \frac{23}{3}, \frac{123}{5}\right)$ |
| 37  | [1, 4]       | [0, 4]       | $\left(\frac{7077}{11716}, \frac{164983}{23432}, \frac{23569}{2929}, \frac{23377611}{44776502}, \frac{49885834}{5597619}\right)$ |
| 38  | [0, 4]       | [0, 4]       | $\left(\frac{2128}{999}, \frac{1995}{899}, \frac{2527}{899}, \frac{572441}{25172}, \frac{769403}{769403}\right)$ |
| 39  | [2, 4]       | [1, 4]       | $\left(\frac{78}{17}, \frac{416}{95}, \frac{494}{95}, \frac{189}{95}, \frac{474}{95}\right)$ |

Table 2. Some $\pi/3$-parallelograms for square-free $1 \leq n \leq 50$

REFERENCES

[1] A. Dujella, A. S. Janfada, C. J. Peral, S. Salami, On the high rank $\pi/3$ and $2\pi/3$-congruent number elliptic curves, Rocky Mountain J. Math. 44 (2014) 1867–1880.

[2] N. J. Fine, On rational triangles, Amer. Math. Month. 83 (1976) 517–521.
| $m$ | $r_{2}(m)$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | $\mathbb{Z}/4\math{
A NEW GENERALIZATION OF θ-Congruent Numbers

[14] J. Top, N. Yui, Congruent number problems and their variants, in: Algorithmic number theory, Math. Sci. Res. Inst. Publ. 44, Cambridge University Press,.

[15] S.-I. Yoshida, Some variant of the congruent number problem, I, Kyushu J. Math. 55 (2001) 387–404.

[16] S.-I. Yoshida, Some variant of the congruent number problem, II, Kyushu J. Math. 56 (2002) 147–165.

Instituto da Matemática e Estatística, Universidade Estadual do Rio de Janeiro (UERJ), Rio de Janeiro, Brazil
Email address: sajad.salami@ime.uerj.br

Department of Mathematics and Applications, Faculty of Basic Sciences, University of Mohaghegh Ardabili, Ardabil 56199-11367, Iran
Email address: zargar@uma.ac.ir