Sequence of potentials interpolating between the U(5) and E(5) symmetries

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Abstract

It is proved that the potentials of the form $\beta^{2n}$ (with $n$ being integer) provide a “bridge” between the U(5) symmetry of the Bohr Hamiltonian with a harmonic oscillator potential (occurring for $n = 1$) and the E(5) model of Iachello (Bohr Hamiltonian with an infinite well potential, materialized for $n \to \infty$). Parameter-free (up to overall scale factors) predictions for spectra and B(E2) transition rates are given for the potentials $\beta^4$, $\beta^6$, $\beta^8$, corresponding to $R_4 = E(4)/E(2)$ ratios of 2.093, 2.135, 2.157 respectively, compared to the $R_4$ ratios 2.000 of U(5) and 2.199 of E(5). Hints about nuclei showing this behaviour, as well as about potentials “bridging” the E(5) symmetry with O(6) are briefly discussed. A note about the appearance of Bessel functions in the framework of E(n) symmetries is given as a by-product.

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1. Introduction

The recently introduced E(5) [1] and X(5) [2] models are supposed to describe shape phase transitions in atomic nuclei, the former being related to the transition from U(5) (vibrational) to O(6) (γ-unstable) nuclei, and the latter corresponding to the transition from U(5) to SU(3) (prolate deformed) nuclei. In both cases the original Bohr collective Hamiltonian [3] is used, with an infinite well potential in the collective β-variable, after separating variables in two different ways. The selection of an infinite well potential in the β-variable in both cases is justified by the fact that the potential is expected to be flat around the point at which a shape phase transition occurs. In both models the predictions for nuclear spectra (normalized to the excitation energy of the first excited state) and B(E2) transition rates (normalized to the B(E2) transition rate connecting the first excited state to the ground state) do not contain any free parameters, thus providing useful benchmarks for nuclei in these two critical regions.

In the present paper we study a sequence of potentials building a “bridge” between the U(5) symmetry of the Bohr Hamiltonian (corresponding to a five-dimensional (5-D) harmonic oscillator [4]) and the E(5) model. The potentials, which are of the form $u_{2n}(\beta) = \frac{\beta^{2n}}{2}$, with $n$ being integer, are shown in Fig. 1. The Bohr Hamiltonian is obtained for $n = 1$, while E(5) (which corresponds to an infinite well potential) occurs for $n \to \infty$ (in practice $n = 4$ is already quite close to E(5)). Parameter-independent predictions for the spectra and B(E2) values (up to the overall scales mentioned above) are obtained for the potentials $\beta^4$, $\beta^6$, $\beta^8$. In addition to providing a number of models giving predictions directly comparable to experiment, the present sequence of potentials shows the way for approaching the E(5) symmetry starting from U(5) and gives a hint on how to approach the E(5) symmetry starting from O(6).

It should be pointed out that the $\beta^4$ potential has already received attention [5], since it turns out to be the critical potential for the U(5) to O(6) shape phase transition in the realm of an appropriate Interacting Boson Model [6] Hamiltonian.

In Section 2 of the present paper a sequence of potentials providing a “bridge” between the U(5) model of Bohr [3, 4] and the E(5) model of Iachello [1] is introduced. Numerical results for spectra and B(E2) transition rates are shown in Sections 3 and 4 respectively, while Section 5 contains a note on the appearance of Bessel functions in the framework of E(n). Perspectives for further experimental and theoretical work are discussed in Section 6, while in Section 7 the conclusions are summarized.

2. E(5), U(5), and a sequence of potentials between them

The original Bohr Hamiltonian [3] is

\[ H = -\frac{\hbar^2}{2B} \left[ \frac{1}{\beta^4} \frac{\partial}{\partial \beta} \frac{\partial^4}{\partial \beta^4} + \frac{1}{\beta^2 \sin 3\gamma} \frac{\partial}{\partial \gamma} \sin 3\gamma \frac{\partial}{\partial \gamma} - \frac{1}{4\beta^2} \sum_{k=1,2,3} \frac{Q_k^2}{\sin^2 \left( \gamma - \frac{2}{3} \pi k \right)} \right] + V(\beta, \gamma), \]

(1)
where $\beta$ and $\gamma$ are the usual collective coordinates describing the shape of the nuclear surface, $Q_k$ ($k = 1, 2, 3$) are the components of angular momentum, and $B$ is the mass parameter.

Assuming that the potential depends only on the variable $\beta$, i.e. $V(\beta, \gamma) = U(\beta)$, one can proceed to separation of variables in the standard way [3, 7], using the wavefunction

$$\Psi(\beta, \gamma, \theta_i) = f(\beta)\Phi(\gamma, \theta_i),$$

where $\theta_i$ ($i = 1, 2, 3$) are the Euler angles describing the orientation of the deformed nucleus in space.

In the equation involving the angles, the eigenvalues of the second order Casimir operator of SO(5) occur, having the form $\Lambda = \tau(\tau + 3)$, where $\tau = 0, 1, 2, \ldots$ is the quantum number characterizing the irreducible representations (irreps) of SO(5), called the “seniority” [8]. This equation has been solved by Bes [9].

The “radial” equation can be simplified by introducing [1] reduced energies $\epsilon = \frac{2B}{\hbar^2}E$ and reduced potentials $u = \frac{2B}{\hbar^2}U$, as well as by making the transformation [1] $\phi(\beta) = \beta^{3/2}f(\beta)$, leading to

$$\phi'' + \frac{\phi'}{\beta} + \left[\epsilon - u(\beta) - \frac{\left(\tau + \frac{3}{2}\right)^2}{\beta^2}\right]\phi = 0.$$  

(3)

For $u(\beta) = \beta^2/2$ one obtains the original solution of Bohr [3, 10], which corresponds to a 5-dimensional (5-D) harmonic oscillator characterized by the symmetry $\text{U}(5) \supset \text{SO}(5) \supset \text{SO}(3) \supset \text{SO}(2)$ [4], the eigenfunctions being proportional to Laguerre polynomials [11]

$$F^\tau_\nu(\beta) = \left[\frac{2^\nu\nu!}{\Gamma(\nu + \tau + \frac{5}{2})}\right]^{1/2}\beta^\tau L^\tau_{\nu + \frac{3}{2}}(\beta^2)e^{-\beta^2/2},$$

(4)

where $\Gamma(n)$ stands for the $\Gamma$-function, and the spectrum having the simple form

$$E_N = N + \frac{5}{2}, \quad N = 2\nu + \tau, \quad \nu = 0, 1, 2, 3, \ldots$$

(5)

For $u(\beta)$ being a 5-D infinite well

$$u(\beta) = \begin{cases} 0 & \text{if } \beta \leq \beta_W \\ \infty & \text{for } \beta > \beta_W \end{cases}$$

(6)

one obtains the E(5) model of Iachello [1], in which the eigenfunctions are Bessel functions $J_{\tau+3/2}(z)$ (with $z = \beta k$, $k = \sqrt{\epsilon}$), while the spectrum is determined by the zeros of the Bessel functions

$$E_{\xi,\tau} = \frac{\hbar^2}{2B}k^2_{\xi,\tau}, \quad k_{\xi,\tau} = \frac{x_{\xi,\tau}}{\beta_W}$$

(7)
where \( x_{\xi,\tau} \) is the \( \xi \)-th zero of the Bessel function \( J_{\tau+3/2}(z) \). The relevant symmetry in this case is \( \text{E}(5) \supset \text{SO}(5) \supset \text{SO}(3) \supset \text{SO}(2) \), where the Euclidean algebra in 5 dimensions, \( \text{E}(5) \), is generated by the 5-D momenta \( \pi_\mu \) and the 5-D angular momenta \( L_{\mu\nu} \), while \( \text{SO}(5) \) is generated by the \( L_{\mu\nu} \) alone [2]. \( \tau \), \( L \), and \( M \) are the quantum numbers characterizing the irreps of \( \text{SO}(5) \), \( \text{SO}(3) \), and \( \text{SO}(2) \) respectively. The values of angular momentum \( L \) contained in each irrep of \( \text{SO}(5) \) (i.e. for each value of \( \tau \)) are given by the algorithm [6]

\[
\tau = 3\nu_\Delta + \lambda, \quad \nu_\Delta = 0, 1, \ldots, \quad (8)
\]

\[
L = \lambda, \lambda + 1, \ldots, 2\lambda - 2, 2\lambda \quad (9)
\]

(with 2\( \lambda - 1 \) missing), where \( \nu_\Delta \) is the missing quantum number in the reduction \( \text{SO}(5) \supset \text{SO}(3) \), and are listed in Table 1.

The spectra of the \( u(\beta) = \beta^2/2 \) potential and of the \( \text{E}(5) \) model become directly comparable by establishing the formal correspondence

\[
\nu = \xi - 1. \quad (10)
\]

It should be emphasized that the quantum numbers appearing in Eq. (10) have different origins, \( \nu \) being an oscillator quantum number labeling the number of zeros of a Laguerre polynomial, while \( \xi \) is labeling the order of a zero of a Bessel function. Eq. (10) establishes a formal one-to-one correspondence between the states in the two spectra and allows one to continue using for the states the notation \( L_{\xi,\tau} \) (where \( L \) is the angular momentum), as in Ref. [1], although a notation \( L_{\nu,\tau} \) would have been equally appropriate. The ground state band corresponds to \( \xi = 1 \) (or, equivalently, \( \nu = 0 \)).

The two cases mentioned above are the only ones in which Eq. (3) is exactly soluble, giving spectra characterized by \( R_4 = E(4)/E(2) \) ratios 2.00 and 2.20 respectively. However, the numerical solution of Eq. (3) for potentials other than the ones mentioned above is a straightforward task [12], in which one uses the chain \( \text{U}(5) \supset \text{SO}(5) \supset \text{SO}(3) \supset \text{SO}(2) \) for the classification of the states.

Not all potentials can be used in Eq. (3), though, since they have to obey the restrictions imposed by the 24 transformations mentioned in [3] and listed explicitly in [13]. These restrictions allow the presence of even powers of \( \beta \) in the potentials, while odd powers of \( \beta \) should be accompanied by \( \cos 3\gamma \) [14].

A particularly interesting sequence of potentials is given by

\[
u_{2n}(\beta) = \frac{\beta^{2n}}{2}, \quad (11)
\]

with \( n \) being an integer. For \( n = 1 \) the Bohr case (\( \text{U}(5) \)) is obtained, while for \( n \to \infty \) the infinite well of \( \text{E}(5) \) is obtained [15]. Therefore this sequence of potentials provides a “bridge” between the \( \text{U}(5) \) symmetry and the \( \text{E}(5) \) model, using their common \( \text{SO}(5) \supset \text{SO}(3) \) chain of subalgebras for the classification of the spectra.
3. Spectra

Numerical results for the spectra of the $\beta^4$, $\beta^6$, and $\beta^8$ potentials have been obtained through two different methods. In one approach, the representation of the position and momentum operators in matrix form [16] has been used, while in the other the direct integration method [17] has been applied. In the latter, the differential equation is solved for each value of $\tau = 0, 1, 2, \ldots$ separately, the successive eigenvalues for each value of $\tau$ labeled by $\xi = 1, 2, 3, \ldots$ (or, equivalently, by $\nu = 0, 1, 2, \ldots$). The two methods give results mutually consistent, the second one appearing of more general applicability. The results are shown in Table 2, where excitation energies relative to the ground state, normalized to the excitation energy of the first excited state, are exhibited.

In Table 2 the labels $E(5)\beta^4$, $E(5)\beta^6$, $E(5)\beta^8$ have been used for the above-mentioned potentials, their meaning being that $E(5)\beta_2^m$ corresponds to the potential $\beta_2^m/2$ plugged in the differential equation obtained in the framework of the $E(5)$ model. In this notation $E(5)\beta^2$ coincides with the original U(5) model of Bohr [3], while $E(5)\beta_2^n$ with $n \rightarrow \infty$ is simply the original $E(5)$ model [1].

From Table 2 it is clear that in all bands and for all values of the angular momentum, $L$, the potentials $\beta^4$, $\beta^6$, $\beta^8$ gradually lead from the U(5) case to the E(5) results in a smooth way. The same conclusion is drawn from Fig. 2(a), where several levels of the ground state band of each model are shown vs. the angular momentum $L$, as well as from Fig. 2(b), where the bandheads of several excited bands are shown for each model as a function of the index $\xi$.

It is instructive to compare the results obtained with the potentials of Eq. (11) to the ones provided by the potentials [1, 18]

$$u(\beta) = \frac{1}{2}(1 - \eta)\beta^2 + \frac{\eta}{4}(1 - \beta^2)^2,$$

(12)

where $\eta$ is a control parameter. Results for the spectra of these potentials (for $\eta = 1/4, 1/2, 3/4, 1$) are shown in Table 3, while for $\eta = 0$ it is clear that the Bohr U(5) case is reproduced. The following observations can be made:

1) For $\eta = 1/2$ the results coincide with these of E(5)$\beta^4$, as expected, since for $\eta = 1/2$ Eq. (12) gives $u(\beta) = (\beta^4 + 1)/8$, while in Tables 2 and 3 excitation energies relative to the ground state and normalized to the excitation energy of the first excited state are shown.

2) Giving to the control parameter $\eta$ the values $0, 1/4, 1/2, 3/4, 1$, one obtains spectra characterized by $R_4$ ratios 2.00, 2.06, 2.09, 2.11, 2.13 respectively.

3) It has been noticed in [1] that the potential should exhibit a flat behaviour when the system undergoes a phase transition. The only flat potential contained in the family of potentials of Eq. (12) is the above mentioned potential $u(\beta) = (b^4 + 1)/8$, which occurs for $\eta = 1/2$, giving $R_4 = 2.09$. In contrast, the sequence of potentials given in Eq. (11) is indeed a series of gradually flatter (with increasing $n$) potentials, giving the infinite well potential of E(5) (with $R_4 = 2.20$) as a limiting case. These potentials therefore do provide a complete “bridge” between U(5) and E(5).
4. B(E2) transition rates

In nuclear structure it is well known that electromagnetic transition rates are quantities sensitive to the details of the underlying microscopic structure, as well as to details of the theoretical models, much more than the corresponding spectra. It is therefore a must to calculate B(E2) ratios (normalized to $B(E2; 2^+_1 \rightarrow 0^+_1) = 100$) for the potentials of Eq. (11).

The quadrupole operator has the form [7]

$$ T^{(E2)}_{\mu} = t \alpha_\mu = t \beta \left( D^{(2)}_{\mu,0}(\theta_i) \cos \gamma + \frac{1}{\sqrt{2}} (D^{(2)}_{\mu,2}(\theta_i) + D^{(2)}_{\mu,-2}(\theta_i)) \sin \gamma \right) , \quad (13) $$

where $t$ is a scale factor and $D(\theta_i)$ denote Wigner functions of the Euler angles, while the B(E2) transition rates are given by

$$ B(E2; \varrho_i L_i \rightarrow \varrho_f L_f) = \frac{1}{2 L_i + 1} |\langle \varrho_f L_f | T^{(E2)} | \varrho_i L_i \rangle|^2 $n\frac{2 L_f + 1}{2 L_i + 1} B(E2; \varrho_f L_f \rightarrow \varrho_i L_i), \quad (14) $$

where by $\varrho$ quantum numbers other than the angular momentum $L$ are denoted.

The states with $\nu_\Delta = 0$ and $L = 2 \tau$ can be written in the form dictated by Eq. (2)

$$ |\xi, \tau, \nu_\Delta = 0, L = 2 \tau, M = L \rangle = f_{\xi \tau}(\beta) \Phi_{\nu_\Delta = 0}^{\tau, M = L}(\gamma, \theta_i) = f_{\xi \tau}(\beta) \phi_\tau(\gamma, \theta_i), \quad (15) $$

where the functions $\phi_\tau(\gamma, \theta_i)$ have the form [9]

$$ \phi_\tau(\gamma, \theta_i) = \frac{1}{\sqrt{A_\tau}} \left( \frac{\alpha_2}{\beta} \right)^\tau, \quad (16) $$

with $\alpha_2$ defined in Eq. (13) and with the normalization factor

$$ A_\tau = \frac{\tau!}{(2 \tau + 3)!!} (4 \pi)^2 \quad (17) $$

determined from the normalization condition

$$ \int_{\gamma=0}^{\pi} \int_{\theta_1=0}^{\pi} \int_{\theta_2=0}^{\pi} \int_{\theta_3=0}^{\pi} \phi_\tau^*(\gamma, \theta_i) \phi_\tau(\gamma, \theta_i) \sin 3 \gamma \sin \theta_2 \sin \theta_3 d\gamma d\theta_1 d\theta_2 d\theta_3 = 1. \quad (18) $$

From Eqs. (14) and (16) one obtains

$$ B(E2; L_{\xi, \tau} \rightarrow (L + 2)_{\xi', \tau + 1}) = \frac{(\tau + 1)(4 \tau + 5)}{(2 \tau + 5)(4 \tau + 1)} \frac{\tau^2 I^2_{\xi', \tau + 1; \xi, \tau}}{I^2_{\xi, \tau}}; \quad L = 2 \tau, \quad (19) $$

$$ B(E2; (L + 2)_{\xi', \tau + 1} \rightarrow L_{\xi, \tau}) = \frac{\tau + 1}{2 \tau + 5} \frac{\tau^2 I^2_{\xi', \tau + 1; \xi, \tau}}{I^2_{\xi, \tau}}; \quad L = 2 \tau. \quad (20) $$
where
\[ I_{\xi',\tau+1;\xi,\tau} = \int_0^\infty \beta f_{\xi',\tau+1}(\beta) f_{\xi\tau}(\beta) \beta^4 d\beta. \]  
(21)

In the special case of the potential being a 5-D infinite well the eigenfunctions are
\[ f_{\xi\tau}(\beta) = \frac{1}{\sqrt{C_{\xi\tau}}} \beta^{-3/2} J_{\tau-3/2}(x_{\xi\tau} \frac{\beta}{\beta_W}), \]  
with
\[ C_{\xi\tau} = \frac{2}{\beta^2 W^2 J_{\tau+3/2}^2(x_{\xi\tau})}, \]  
(23)
where \( x_{\xi\tau} \) is the \( \xi \)-th zero of the Bessel function \( J_{\tau+3/2}(z) \), while the constants \( C_{\xi\tau} \) are obtained from the normalization condition
\[ \int_0^{\beta_W} f_{\xi\tau}^2(\beta) \beta^4 d\beta = 1. \]  
(24)

In this case the integrals of Eq. (21) take the form
\[ I_{\xi',\tau+1;\xi,\tau} = \int_0^{\beta_W} \beta f_{\xi',\tau+1}(\beta) f_{\xi\tau}(\beta) \beta^4 d\beta \]
\[ = (C_{\xi',\tau+1} C_{\xi\tau})^{-1/2} \beta_W^3 \int_0^{1} z^2 J_{\tau+5/2}(x_{\xi',\tau+1} z) J_{\tau+3/2}(x_{\xi\tau} z) dz. \]  
(25)

In the case of the oscillator potential \( u(\beta) = \beta^2/2 \) the eigenfunctions are given by Eq. (4). In this case the integrals \( I_{n',\tau+1; n,\tau} \) appearing in Eq. (21) (where \( n = \xi - 1 \) and \( n' = \xi' - 1 \), as mentioned in Eq. 10) in the cases \( n' = n, n \pm 1 \) are found to be
\[ I_{n,\tau+1; n,\tau} = \sqrt{n + \tau + 5/2}, \]  
(26)
\[ I_{n+1,\tau+1; n,\tau} = 0, \quad n \geq 0, \]  
(27)
\[ I_{n-1,\tau+1; n,\tau} = \sqrt{n}, \quad n \geq 1, \]  
(28)
leading to the ratios
\[ \frac{B(E2; (L+2)\xi_{\tau+1} \rightarrow L_{\xi,\tau})}{B(E2; 2_{1,1} \rightarrow 0_{1,0})} = \frac{(\tau + 1)}{(2\tau + 5)} \frac{(2\xi + 2\tau + 3)}{(2\xi - 2)}, \quad L = 2\tau, \; \xi \geq 1, \]  
(29)
\[ \frac{B(E2; L_{\xi,\tau} \rightarrow (L+2)\xi_{\tau+1})}{B(E2; 2_{1,1} \rightarrow 0_{1,0})} = \frac{(\tau + 1)(4\tau + 5)}{(2\tau + 5)(4\tau + 1)} \frac{(2\xi - 2)}{(2\xi + 2\tau + 3)}, \quad L = 2\tau, \; \xi \geq 2. \]  
(30)

The results of the calculations for intraband transitions are shown in Table 4, while interband transitions are shown in Table 5. In addition, the normalized B(E2) transition rates within the ground state band of each model are shown in Fig. 2(c). In all cases a smooth evolution from U(5) to E(5) is seen. Furthermore, the results are in agreement to general qualitative expectations: the more rotational the nucleus, the less rapid the increase.
(with increasing initial angular momentum) of the B(E2)s within the ground state band should be (in the absence of bandcrossings). Indeed the most rapid increase is seen in the case of U(5), while the slowest increase is observed in the case of E(5). The E(5) results reported in Tables 4 and 5 are in good agreement with the results given in Ref. [19].

Finally, in Fig. 3 the lowest part of the spectrum, which is of interest for comparisons with experimental data, together with all relevant B(E2) transition rates is shown for the models E(5)-β^4, E(5)-β^6 and E(5)-β^8. The models E(5) and U(5) are also included for comparison.

5. A note on E(n) and Bessel functions

Concerning the appearance of Bessel functions in the case of E(5), the following mathematical remarks can be made in the general case of the Euclidean algebra in n dimensions, E(n), which is the semidirect sum \[20\] of the algebra \( T_n \) of translations in n dimensions, generated by the momenta

\[
P_j = -i \frac{\partial}{\partial x_j},
\]

and the SO(n) algebra of rotations in n dimensions, generated by the angular momenta

\[
L_{jk} = -i \left( x_j \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_j} \right),
\]

symbolically written as \( E(n) = T_n \oplus_s SO(n) \) [21]. The generators of E(n) satisfy the commutation relations

\[
[P_i, P_j] = 0, \quad [P_i, L_{jk}] = i(\delta_{ik}P_j - \delta_{ij}P_k), \quad (33)
\]

\[
[L_{ij}, L_{kl}] = i(\delta_{ik}L_{jl} + \delta_{jl}L_{ik} - \delta_{il}L_{jk} - \delta_{jk}L_{il}). \quad (34)
\]

From these commutation relations one can see that the square of the total momentum, \( P^2 \), is a second order Casimir operator of the algebra, while the eigenfunctions of this operator satisfy the equation

\[
\left( -\frac{1}{r^{n-1}} \frac{\partial}{\partial r} r^{n-1} \frac{\partial}{\partial r} + \frac{\omega(\omega + n - 2)}{r^2} \right) F(r) = k^2 F(r), \quad (35)
\]

in the left hand side of which the eigenvalues of the Casimir operator of SO(n), \( \omega(\omega + n - 2) \) appear [11]. Putting

\[
F(r) = r^{(2-n)/2} f(r), \quad (36)
\]

and

\[
\nu = \omega + \frac{n - 2}{2}, \quad (37)
\]

Eq. (35) is brought into the form

\[
\left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + k^2 - \frac{\nu^2}{r^2} \right) f(r) = 0, \quad (38)
\]
the eigenfunctions of which are the Bessel functions \( f(r) = J_\nu(kr) \) [22]. We see therefore that the Bessel functions appear in general in this type of problems when the potential is vanishing, so that only the kinetic energy term appears in the Hamiltonian.

A similar result for the case of the \( n \)-dimensional harmonic oscillator has been obtained in Ref. [8] and developed in more detail in Ref. [11], showing the appearance of Laguerre polynomials in the eigenfunctions of the harmonic oscillator in all dimensions.

6. Perspectives

It is interesting to examine if there is any experimental evidence supporting the E(5)-\( \beta^{2n} \) predictions. It is clear that the first regions to be considered are the ones around the nuclei which have been identified as good candidates for E(5), i.e. \(^{134}\text{Ba} \), \(^{104}\text{Ru} \), \(^{102}\text{Pd} \) [25]. A very preliminary search indicates that \(^{98}\text{Ru} \) can be a candidate for E(5)-\( \beta^6 \), while \(^{100}\text{Pd} \) can be a candidate for E(5)-\( \beta^4 \). Existing data for the ground state bands of these nuclei are compared to the theoretical predictions in Table 6. However, much more detailed information on the spectra and B(E2) transitions of these nuclei are required before final conclusions can be reached.

Concerning future theoretical work, at least two directions open up:

1) One should study a similar sequence of potentials serving as a “bridge” between U(5) and X(5) [2]. This task has been carried out in Ref. [28].

2) One should try to find a sequence of potentials interpolating between O(6) and E(5), as well as between SU(3) and X(5). In other words, one should try to approach E(5) and X(5) “from the other side”. From the classical limit of the O(6) and SU(3) symmetries of the Interacting Boson Model [6] it is clear that for this purpose potentials with a minimum at \( \beta \neq 0 \) should be considered, the Davidson-like potentials [29]

\[
u^D_{2n}(\beta) = \beta^{2n} + \frac{\beta^{4n}}{\beta^{2n}}
\]

being strong candidates. The Davidson potential, corresponding to \( n = 1 \), is known to be exactly soluble [29, 30]. Work in these directions is in progress.

3) Another candidate for the task described in 2) is the sextic oscillator with a centrifugal barrier [31], recently considered in Ref. [32], which contains \( \beta^2, \beta^4, \beta^6 \) and \( \beta^{-2} \) terms with coefficients interrelated in an appropriate way in order to guarantee that the potential is a Quasi-Exactly Soluble one [33, 34, 35]. The sextic oscillator with a centrifugal barrier contains two free parameters, and it is capable of producing potentials with minima at \( \beta \neq 0 \) [32].

7. Conclusion

It has been proved that the potentials \( \beta^{2n} \) (with \( n \) being integer) provide a complete “bridge” between the U(5) symmetry of the Bohr Hamiltonian with a harmonic oscillator potential (occurring for \( n = 1 \)) and the E(5) model of Iachello, which is obtained from the Bohr Hamiltonian when an infinite well potential is plugged in it (materialized for \( n \to \infty \)). Parameter-free (up to overall scale factors) predictions for spectra and B(E2)
transition rates have been given for the potentials $\beta^4$, $\beta^6$, $\beta^8$, called the E(5)-$\beta^4$, E(5)-$\beta^6$, and E(5)-$\beta^8$ models, respectively. Hints about nuclei showing this behaviour, as well as about potentials approaching E(5) “from the other side” (i.e. providing a “bridge” between O(6) and E(5)) have been briefly discussed. A mathematical note on the appearance of Bessel functions in the framework of E(n) models has been given as a by-product.

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Figure captions

**Fig. 1.** The potentials $\beta^{2n}$, with $n = 1$ (harmonic oscillator, solid line), $n = 2$ (dash line), $n = 3$ (dash dot), $n = 4$ (dot), $n = 8$ (dash dot dot), $n = 16$ (short dash dot), $n = 32$ (short dot), gradually approaching (with increasing $n$) the infinite well potential.

**Fig. 2** (Color online) (a) Levels of the ground state bands of the models $E(5)-\beta^{2n}$ with $n = 2-4$ and of the $U(5)$ and $E(5)$ models, vs. the angular momentum $L$. In each model all levels are normalized to the energy of the first excited state. See Section 3 for further discussion. (b) Bandhead energies of excited bands of the same models and with the same normalization, vs. the band index $\xi$. See Section 3 for further discussion. (c) $B(E2; L_f + 2 \rightarrow L_f)$ transition rates within the ground state bands of the same models, vs. the angular momentum of the final state, $L_f$. In each model all rates are normalized to the one between the lowest states, $B(E2; 2 \rightarrow 0)$. See Section 4 for further discussion.

**Fig. 3** Lowest part of the spectrum, together with the relevant $B(E2)$ transition rates, for the models $U(5)$ (labeled as $\beta^2$), $E(5)-\beta^4$ (labeled as $\beta^4$), $E(5)-\beta^6$ ($\beta^6$), $E(5)-\beta^8$ ($\beta^8$), and $E(5)$ (labeled as infinite well). See Section 4 for further discussion. The results for $E(5)-\beta^4$ and $E(5)$ compare well with prior work reported in Refs. [5] and [19] respectively.
Table 1: Quantum numbers appearing in the $\text{SO}(5) \supset \text{SO}(3)$ reduction [6], occurring from Eqs. (8) and (9).

| $\tau$ | $\nu_{\Delta}$ | $\lambda$ | $L$   |
|-------|----------------|---------|------|
| 0     | 0              | 0       | 0    |
| 1     | 0              | 1       | 2    |
| 2     | 0              | 2       | 4,2  |
| 3     | 0              | 3       | 6,4,3|
| 3     | 1              | 0       | 0    |
| 4     | 0              | 4       | 8,6,5,4|
| 4     | 1              | 1       | 2    |
| 5     | 0              | 5       | 10,8,7,6,5|
| 5     | 1              | 2       | 4,2  |
| 6     | 0              | 6       | 12,10,9,8,7,6|
| 6     | 1              | 3       | 6,4,3|
| 6     | 2              | 0       | 0    |
Table 2: Spectra of the E(5)-$\beta^4$, E(5)-$\beta^6$, and E(5)-$\beta^8$ models, compared to the predictions of the U(5) (Eq. (5)) and E(5) (Eq. (7)) models. For each value of $\tau$, only the maximum value of $L$ occurring for it, $L_{\text{max}}$, is reported. The rest of the allowed values of $L$ for each value of $\tau$, indicating states having the same energy as the state with $L_{\text{max}}$, can be read from Table 1. The lowest four levels in each band of E(5)-$\beta^4$ are in good agreement with results already published in Ref. [5].

| band $\xi$ | $\tau$ | $L_{\text{max}}$ | U(5) | E(5)-$\beta^4$ | E(5)-$\beta^6$ | E(5)-$\beta^8$ | E(5) |
|-----------|--------|------------------|------|----------------|----------------|----------------|------|
| $\xi = 1$ | 0      | 0.000            | 0.000| 0.000          | 0.000          | 0.000          | 0.000|
|           | 1      | 1.000            | 1.000| 1.000          | 1.000          | 1.000          | 1.000|
|           | 2      | 2.000            | 2.093| 2.135          | 2.157          | 2.199          | 2.199|
|           | 3      | 3.000            | 3.265| 3.391          | 3.459          | 3.590          | 3.590|
|           | 4      | 4.000            | 4.508| 4.757          | 4.894          | 5.169          | 5.169|
|           | 5      | 5.000            | 5.813| 6.225          | 6.456          | 6.934          | 6.934|
|           | 6      | 6.000            | 7.176| 7.788          | 8.138          | 8.881          | 8.881|
|           | 7      | 7.000            | 8.592| 9.442          | 9.935          | 11.009         | 11.009|
|           | 8      | 8.000            | 10.057| 11.180        | 11.841         | 13.316         | 13.316|
|           | 9      | 9.000            | 11.569| 13.000        | 13.854         | 15.799         | 15.799|
|           | 10     | 10.000           | 13.124| 14.898        | 15.968         | 18.459         | 18.459|
|           | 11     | 11.000           | 14.720| 16.871        | 18.182         | 21.294         | 21.294|
|           | 12     | 12.000           | 16.355| 18.916        | 20.492         | 24.302         | 24.302|
|           | 13     | 13.000           | 18.028| 21.031        | 22.896         | 27.484         | 27.484|
|           | 14     | 14.000           | 19.737| 23.213        | 25.391         | 30.837         | 30.837|
|           | 15     | 15.000           | 21.480| 25.460        | 27.975         | 34.363         | 34.363|
| $\xi = 2$ | 0      | 2.000            | 2.390| 2.619          | 2.756          | 3.031          | 3.031|
|           | 1      | 3.000            | 3.625| 4.012          | 4.255          | 4.800          | 4.800|
|           | 2      | 4.000            | 4.918| 5.499          | 5.874          | 6.780          | 6.780|
|           | 3      | 5.000            | 6.266| 7.075          | 7.607          | 8.967          | 8.967|
|           | 4      | 6.000            | 7.666| 8.738          | 9.450          | 11.357         | 11.357|
|           | 5      | 7.000            | 9.115| 10.483         | 11.400         | 13.945         | 13.945|
| $\xi = 3$ | 0      | 4.000            | 5.153| 5.887          | 6.364          | 7.577          | 7.577|
|           | 1      | 5.000            | 6.563| 7.588          | 8.269          | 10.107         | 10.107|
|           | 2      | 6.000            | 8.015| 9.363          | 10.274         | 12.854         | 12.854|
|           | 3      | 7.000            | 9.509| 11.213         | 12.379         | 15.814         | 15.814|
|           | 4      | 8.000            | 11.043| 13.134        | 14.580         | 18.983         | 18.983|
|           | 5      | 9.000            | 12.617| 15.125        | 16.875         | 22.359         | 22.359|
| $\xi = 4$ | 0      | 6.000            | 8.213| 9.698          | 10.707         | 13.639         | 13.639|
|           | 1      | 7.000            | 9.764| 11.661         | 12.966         | 16.928         | 16.928|
|           | 2      | 8.000            | 11.349| 13.687        | 15.316         | 20.436         | 20.436|
|           | 3      | 9.000            | 12.967| 15.776        | 17.753         | 24.161         | 24.161|
|           | 4      | 10.000           | 14.619| 17.928        | 20.278         | 28.100         | 28.100|
|           | 5      | 11.000           | 16.304| 20.111        | 22.888         | 32.250         | 32.250|
Table 3: Same as Table 2, but for spectra of the potentials of Eq. (12) for different values of the control parameter $\eta$, compared to the predictions of the U(5) ($\eta = 0$, Eq. (5)) and E(5) (Eq. (7)) models.

| band | $\tau$ | $L_{\text{max}}$ | $\xi = 1$ | $\xi = 2$ | $\xi = 3$ | $\xi = 4$ |
|------|--------|------------------|----------|----------|----------|----------|
|      |        |                  | 0        | 1/4      | 1/2      | 3/4      | 1        | E(5)     |
| 0    | 1      | 0.000            | 0.000    | 0.000    | 0.000    | 0.000    | 0.000    |
| 2    | 4      | 5.069            | 5.813    | 5.982    | 6.115    | 6.934    |
| 3    | 6      | 3.000            | 3.183    | 3.265    | 3.323    | 3.368    | 3.590    |
| 0    | 1      | 6.000            | 6.828    | 7.176    | 7.415    | 7.605    | 8.881    |
| 2    | 4      | 8.000            | 8.127    | 8.592    | 9.164    | 11.009   |
| 3    | 6      | 10.000           | 10.057   | 10.464   | 10.786   | 13.316   |
| 4    | 8      | 12.000           | 12.124   | 13.124   | 13.727   | 14.204   | 18.459   |
| 5    | 10     | 14.000           | 14.720   | 15.432   | 15.994   | 21.294   |
| 0    | 1      | 5.000            | 5.569    | 5.813    | 5.982    | 6.115    | 6.934    |
| 2    | 4      | 7.000            | 7.412    | 8.015    | 8.438    | 8.779    | 9.542    |
| 3    | 6      | 9.000            | 9.581    | 10.266   | 10.657   | 11.061   | 13.945   |
| 4    | 8      | 11.000           | 11.043   | 11.649   | 11.133   | 13.943   |
| 5    | 10     | 13.000           | 13.324   | 15.814   | 15.814   | 20.436   |
| 0    | 1      | 7.000            | 7.548    | 8.213    | 8.681    | 9.061    | 13.639   |
| 2    | 4      | 9.000            | 9.851    | 10.764   | 10.788   | 10.788   |
| 3    | 6      | 11.000           | 11.349   | 12.019   | 12.556   | 14.366   |
| 4    | 8      | 13.000           | 13.322   | 14.366   | 14.366   | 18.218   |
| 5    | 10     | 15.000           | 14.831   | 16.304   | 17.311   | 18.111   | 22.500   |
Table 4: Intraband B(E2) transition rates for the E(5)-β⁴, E(5)-β⁶, and E(5)-β⁸ models, compared to the predictions of the U(5) and E(5) models. Some of the E(5)-β⁴ transitions have been reported in Ref. [5], with very similar values. See Section 4 for details.

|       | (L_{ξ,τ})_i | (L_{ξ,τ})_f | U(5)  | E(5)-β⁴ | E(5)-β⁶ | E(5)-β⁸ | E(5)  |
|-------|--------------|--------------|-------|---------|---------|---------|-------|
| (ξ = 1) → (ξ = 1) |              |              |       |         |         |         |       |
| 2_{1,1} | 0_{1,0}      | 100.00       | 100.00| 100.00  | 100.00  | 100.00  | 100.00|
| 4_{1,2} | 2_{1,1}      | 200.00       | 183.20| 176.60  | 173.32  | 167.40  |       |
| 6_{1,3} | 4_{1,2}      | 300.00       | 256.37| 239.80  | 231.64  | 216.88  |       |
| 8_{1,4} | 6_{1,3}      | 400.00       | 322.73| 294.27  | 280.39  | 255.20  |       |
| 10_{1,5}| 8_{1,4}      | 500.00       | 384.12| 342.57  | 322.51  | 286.01  |       |
| 12_{1,6}| 10_{1,5}     | 600.00       | 441.65| 386.26  | 359.74  | 311.47  |       |
| 14_{1,7}| 12_{1,6}     | 700.00       | 496.11| 426.36  | 393.25  | 332.95  |       |
| 16_{1,8}| 14_{1,7}     | 800.00       | 548.02| 463.57  | 423.80  | 351.39  |       |
| 18_{1,9}| 16_{1,8}     | 900.00       | 597.78| 498.40  | 451.94  | 367.44  |       |
| 20_{1,10}| 18_{1,9}    | 1000.00     | 645.69| 531.23  | 478.10  | 381.56  |       |
| 22_{1,11}| 20_{1,10}   | 1100.00     | 692.00| 562.35  | 502.58  | 394.10  |       |
| 24_{1,12}| 22_{1,11}   | 1200.00     | 736.89| 592.00  | 525.62  | 405.34  |       |
| 26_{1,13}| 24_{1,12}   | 1300.00     | 780.52| 620.35  | 547.41  | 415.48  |       |
| 28_{1,14}| 26_{1,13}   | 1400.00     | 823.01| 647.55  | 568.12  | 424.68  |       |
| 30_{1,15}| 28_{1,14}   | 1500.00     | 864.47| 673.73  | 587.86  | 433.09  |       |
| (ξ = 2) → (ξ = 2) |              |              |       |         |         |         |       |
| 2_{2,1} | 0_{2,0}      | 140.00       | 112.64| 98.97   | 91.24   | 75.22   |       |
| 4_{2,2} | 2_{2,1}      | 257.14       | 197.92| 170.97  | 156.06  | 124.32  |       |
| 6_{2,3} | 4_{2,2}      | 366.67       | 271.04| 230.57  | 208.71  | 161.52  |       |
| 8_{2,4} | 6_{2,3}      | 472.73       | 336.84| 282.53  | 253.85  | 191.58  |       |
| 10_{2,5}| 8_{2,4}      | 576.92       | 397.56| 329.12  | 293.70  | 216.77  |       |
| 2_{2,2} | 2_{2,1}      | 257.14       | 197.92| 170.97  | 156.06  | 124.32  |       |
| (ξ = 3) → (ξ = 3) |              |              |       |         |         |         |       |
| 2_{3,1} | 0_{3,0}      | 180.00       | 126.58| 103.69  | 91.64   | 65.73   |       |
| 4_{3,2} | 2_{3,1}      | 314.29       | 214.91| 173.97  | 152.67  | 106.63  |       |
| 6_{3,3} | 4_{3,2}      | 433.33       | 288.38| 230.96  | 201.40  | 137.44  |       |
| 8_{3,4} | 6_{3,3}      | 545.45       | 353.71| 280.48  | 243.22  | 162.57  |       |
| 10_{3,5}| 8_{3,4}      | 653.85       | 413.72| 325.01  | 280.39  | 183.95  |       |
| 2_{3,2} | 2_{3,1}      | 314.29       | 214.91| 173.97  | 152.67  | 106.63  |       |
| (ξ = 4) → (ξ = 4) |              |              |       |         |         |         |       |
| 2_{4,1} | 0_{4,0}      | 220.00       | 140.44| 109.56  | 94.03   | 60.68   |       |
| 4_{4,2} | 2_{4,1}      | 371.43       | 232.42| 179.66  | 153.33  | 96.89   |       |
| 6_{4,3} | 4_{4,2}      | 500.00       | 306.70| 235.08  | 199.63  | 123.79  |       |
| 8_{4,4} | 6_{4,3}      | 618.18       | 371.85| 282.79  | 239.04  | 145.70  |       |
| 10_{4,5}| 8_{4,4}      | 730.77       | 431.31| 325.60  | 274.06  | 164.42  |       |
Table 5: Same as Table 4, but for interband B(E2) transitions.

| bands | $(L_{\xi,\tau})_i$ | $(L_{\xi,\tau})_f$ | U(5) | E(5)-$\beta^4$ | E(5)-$\beta^6$ | E(5)-$\beta^8$ | E(5) |
|-------|------------------|------------------|------|----------------|----------------|----------------|------|
| $(\xi = 2) \rightarrow (\xi = 1)$ | | | | | | | |
| 0,0   | 2,1,1            | 200.00           | 141.77 | 118.98         | 107.57        | 86.79          |      |
| 2,1   | 4,1,2            | 102.86           | 66.10  | 52.62          | 46.00         | 33.82          |      |
| 2,1   | 0,1,0            | 0.00             | 0.16   | 0.30           | 0.38          | 0.47           |      |
| 4,2   | 6,1,3            | 96.30            | 57.33  | 43.78          | 37.263        | 25.17          |      |
| 4,2   | 2,1,1            | 0.00             | 0.24   | 0.45           | 0.56          | 0.69           |      |
| 6,2   | 8,1,4            | 95.11            | 53.20  | 39.26          | 32.68         | 20.44          |      |
| 6,2   | 4,1,2            | 0.00             | 0.28   | 0.52           | 0.65          | 0.79           |      |
| $(\xi = 3) \rightarrow (\xi = 2)$ | | | | | | | |
| 0,0   | 2,2,1            | 400.00           | 257.90 | 205.27         | 178.52        | 123.22         |      |
| 2,3   | 4,2,2            | 205.71           | 123.14 | 94.54          | 80.50         | 51.57          |      |
| 2,3   | 0,2,0            | 0.00             | 0.22   | 0.38           | 0.46          | 0.54           |      |
| 4,3   | 6,2,3            | 192.59           | 108.39 | 80.68          | 67.46         | 40.44          |      |
| 4,3   | 2,2,1            | 0.00             | 0.34   | 0.58           | 0.69          | 0.79           |      |
| 6,3   | 8,2,4            | 190.21           | 101.58 | 73.59          | 60.54         | 34.16          |      |
| 6,3   | 4,2,2            | 0.00             | 0.42   | 0.71           | 0.84          | 0.92           |      |
| $(\xi = 4) \rightarrow (\xi = 3)$ | | | | | | | |
| 0,0   | 2,3,1            | 600.00           | 358.53 | 273.82         | 232.05        | 144.02         |      |
| 2,4   | 4,3,2            | 308.57           | 173.79 | 129.12         | 107.67        | 62.88          |      |
| 2,4   | 0,3,0            | 0.00             | 0.26   | 0.43           | 0.51          | 0.56           |      |
| 4,4   | 6,3,3            | 288.89           | 154.60 | 112.08         | 92.13         | 50.93          |      |
| 4,4   | 2,3,1            | 0.00             | 0.41   | 0.66           | 0.77          | 0.81           |      |
| 6,4   | 8,3,4            | 285.31           | 145.99 | 103.53         | 84.01         | 44.16          |      |
| 6,4   | 4,3,2            | 0.00             | 0.51   | 0.82           | 0.94          | 0.96           |      |

Table 6: Experimental spectra of the ground state bands of $^{100}$Pd [27] and $^{98}$Ru [26], compared to the predictions of the E(5)-$\beta^4$ and E(5)-$\beta^6$ models respectively.

| L   | $^{100}$Pd | E(5)-$\beta^4$ | $^{98}$Ru | E(5)-$\beta^6$ |
|-----|-----------|----------------|-----------|----------------|
| 2   | 1.000     | 1.000          | 1.000     | 1.000          |
| 4   | 2.128     | 2.093          | 2.142     | 2.135          |
| 6   | 3.290     | 3.265          | 3.406     | 3.391          |
| 8   | 4.489     | 4.508          | 4.792     | 4.757          |
| 10  | 5.814     | 5.813          | 6.091     | 6.225          |
| 12  | 7.154     | 7.176          | 7.788     |                |
| 14  | 8.574     | 8.592          | 9.442     |                |
| 16  | 10.425    | 10.057         | 11.180    |                |
energy $E$ vs. angular momentum $L$

- U(5)
- $E(5) - \beta^4$
- $E(5) - \beta^6$
- $E(5) - \beta^8$
- $E(5)$
\( B(\text{E2}) \) transition rate

\[ \text{angular momentum } L_f \]

(c)

- \( U(5) \)
- \( E(5) - \beta^4 \)
- \( E(5) - \beta^6 \)
- \( E(5) - \beta^8 \)
- \( E(5) \)