Large and Moderate Deviation Principles for McKean-Vlasov SDEs with Jumps

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Abstract
In this paper, we consider McKean-Vlasov stochastic differential equations (MVSDEs) driven by Lévy noise. By identifying the right equations satisfied by the solutions of the MVSDEs with shifted driving Lévy noise, we build up a framework to fully apply the weak convergence method to establish large and moderate deviation principles for MVSDEs. In the case of ordinary SDEs, the rate function is calculated by using the solutions of the corresponding skeleton equations simply replacing the noise by the elements of the Cameron-Martin space. It turns out that the correct rate function for MVSDEs is defined through the solutions of skeleton equations replacing the noise by smooth functions and replacing the distributions involved in the equation by the distribution of the solution of the corresponding deterministic equation (without the noise). This is somehow surprising. With this approach, we obtain large and moderate deviation principles for much wider classes of MVSDEs in comparison with the existing literature see Dos Reis et al. (Ann. Appl. Probab. 29, 1487–1540, 2019).

Keywords Large deviation · Moderate deviation · Weak convergence method · McKean-Vlasov equation · Lévy noise

Mathematics Subject Classification (2010) 60F10 · 60H10 · 60H15 · 60J75 · 37L55

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1 Introduction

The purpose of this paper is to establish large and moderate deviation principles for the following McKean-Vlasov stochastic differential equations (MVSDEs) driven by Lévy noise as the parameter $\epsilon$ tends to 0,

$$dX^\epsilon(t) = b_\epsilon(t, X^\epsilon, \mu^\epsilon)dt + \sqrt{\epsilon} \sigma_\epsilon(t, X^\epsilon, \mu^\epsilon)dW(t) + \epsilon \int_Z G_\epsilon(t, X^\epsilon, \mu^\epsilon, z)\tilde{N}^\epsilon^{-1}(dz, dt), \quad t \in [0, T], \epsilon \in (0, 1],$$

(1.1)

where $\mu^\epsilon$ is the law of $X^\epsilon$, $W$ is a Brownian motion (BM in short), $N^\epsilon^{-1}$ is a Poisson random measure (PRM in short) on $[0, T] \times Z$ with a $\sigma$-finite intensity measure $-\epsilon \text{Leb}_T \otimes \nu$, and $\tilde{N}^\epsilon^{-1}([0, t] \times B) = N^\epsilon^{-1}([0, t] \times B) - \epsilon^{-1}t \nu(B), \forall B \in B(Z)$ with $\nu(B) < \infty$, is the compensated PRM. The precise assumptions on (1.1) will be given in Section 2. We notice that $b_\epsilon$, $\sigma_\epsilon$, $G_\epsilon$ are functionals of the path $X^\epsilon$ and the law of $X^\epsilon$. (1.1) in particular includes the MVSDE:

$$dX(t) = b(t, X(t), \mu_t)dt + \sqrt{\epsilon} \sigma(t, X(t), \mu_t)dW(t) + \epsilon \int_Z G(t, X(t), \mu_t, z)\tilde{N}(dz, dt), \quad t \in [0, T], \epsilon \in (0, 1],$$

(1.2)

where $b_\epsilon(t, \cdot, \cdot)$, $\sigma_\epsilon(t, \cdot, \cdot)$ and $G_\epsilon(t, \cdot, \cdot, \cdot)$ depend only on the value and the law of the process $X^\epsilon$ at time $t$. We stress that the general framework (1.1) covers MVSDEs driven by Lévy noise, McKean-Vlasov stochastic partial differential equations (MVSPDEs) driven by Lévy noise, and MVSDEs/MVSPDEs with delay/memory driven by Lévy noise, etc.

MVSDEs were first suggested by Kac [37, 38] as a stochastic toy model for the Vlasov kinetic equation of plasma, and then introduced by McKean [47] to model plasma dynamics. These equations describe limiting behaviors of individual particles in an interacting particle system of mean-field type when the number of particles goes to infinity (so-called propagation of chaos). For this reason MVSDEs are also referred as mean-field SDEs. The theory and applications of MVSDEs and associated interacting particle systems have been extensively studied by a large number of researchers under various settings due to their wide range of applications in several fields, including physics, chemistry, biology, economics, financial mathematics etc. One can refer to [13, 26, 32, 50, 52] and the references therein for the existence and uniqueness of solutions to MVSDEs, [2, 21, 34, 48, 49, 57] for propagation of chaos, [22–25, 30, 40, 43–45, 55] for exponential ergodicity and functional inequalities, [1, 20, 28, 42, 56] and the references therein for large deviation principles.

On the other hand, real world models in finance, physics, biology, etc., sometimes cannot be well represented by Gaussian noise. And from the point of view of particle systems, in many scenarios, the individual particles and the related whole population will demonstrate some sudden jumps. Lévy-type perturbations come to the stage to reproduce the performance of these natural phenomena, and a PRM is a good and natural model to express these jumps. Thus, it is natural to consider MVSDEs driven by BM and PRM as follows:

$$dX(t) = b(t, X(t), \mu_t)dt + \sigma(t, X(t), \mu_t)dW(t) + \int_Z G(t, X(t-), \mu_t, z)\tilde{N}(dz, dt).$$

(1.3)

Compared with the MVSDEs driven by BM, the MVSDEs with Lévy noise have been much less studied. In [36], Jourdain et al. studied the existence, uniqueness and particle approximations for MVSDEs driven by Lévy noise. Recently in [48, 51], the authors considered the well-posedness and propagation of chaos results for MVSDEs with delay driven...
by Lévy noise. In analogy to the case of Gaussian noise (see \cite{12, 31}), nonlinear and nonlocal integral Fokker-Planck PDEs can be related to MVSDEs with Lévy noise (see \cite{27, 36, 39}).

Large and moderate deviation principles can provide an exponential estimate for tail probability (or the probability of a rare event) in terms of some explicit rate function. In the case of stochastic processes, the heuristics underlying large and moderate deviations theory is to identify a deterministic path around which the diffusion is concentrated with overwhelming probability, so that the stochastic motion can be seen as a small random perturbation of this deterministic path.

Large and moderate deviation principles for classical stochastic evolution equations and SPDEs driven by BM and/or PRM have been extensively investigated in recent years. Among the approaches to deal with these problems, the weak convergence method based on a variational representation for positive measurable functionals of a BM and/or PRM (see \cite{4, 6–9}) is proved to be a powerful tool to establish large and moderate deviation principles for various dynamical systems driven by Gaussian noise and/or PRM. The reader is referred to \cite{4, 6–11, 14, 17–19, 46, 54, 58, 60, 61} and the references therein. The key components of the variational representation are the controlled BM and the controlled PRM. The controlled BM basically shifts the mean, while the controlled PRM plays the role of a thinning function. We refer to \cite{5} for an excellent review of the advances on the weak convergence method during the past decade.

Assume that there is a unique strong solution $X^\varepsilon$ to MVSDE (1.1). Then, there exists a measurable map $G^\varepsilon$ such that the solution $X^\varepsilon$ can be represented as

$$X^\varepsilon = G^\varepsilon(\sqrt{\varepsilon} W, \varepsilon N^{-1}). \tag{1.4}$$

One key step to establish the LDP is to prove the weak convergence of the perturbations $X^{\varepsilon, u} := G^\varepsilon(\sqrt{\varepsilon} W + \int_0^t \phi_\varepsilon(s) ds, \varepsilon N^{-1}\psi_\varepsilon)$ as $\varepsilon \to 0$, here $u_\varepsilon = (\phi_\varepsilon, \psi_\varepsilon)$. It is therefore important to identify the correct equation satisfied by $X^{\varepsilon, u}$ in this setting. It would be natural to think that $X^{\varepsilon, u}$ is the solution to the following controlled SDE:

$$dX^{\varepsilon, u}(t) = b(t, X^{\varepsilon, u}, \mu^{\varepsilon, u}) dt + \sqrt{\sigma(t, X^{\varepsilon, u}, \mu^{\varepsilon, u})} dW(t) + \sigma(t, X^{\varepsilon, u}, \mu^{\varepsilon, u}) \phi_\varepsilon(t) dt + \int_Z G_\varepsilon(z, X^{\varepsilon, u}, \mu^{\varepsilon, u}, z) \left( \varepsilon N^{-1}\psi_\varepsilon(dz, dt) - \nu(dz) dt \right), \quad t \in [0, T], \tag{1.5}$$

where $\mu^{\varepsilon, u}$ is the distribution of the solution $X^\varepsilon$ to (1.1). The reason is that when perturbing the BM and PRM in the arguments of the map $G^\varepsilon(\cdot, \cdot)$, $\mu^{\varepsilon}$ is already deterministic and hence it is not affected by the perturbation, as the following example shows.

Indeed, this was the claim in \cite{15}, in which Cai et al. attempted to apply fully weak convergence method to obtain the LDP for MVSDEs with Lévy noise. Unfortunately this claim is wrong, which leads to a wrong rate function of the LDP.

One of the contributions of this paper is to find the correct equation satisfied by $X^{\varepsilon, u}$. In fact we find out that $X^{\varepsilon, u}$ is actually the unique solution to the following SDE:

$$dX^{\varepsilon, u}(t) = b(t, X^{\varepsilon, u}, \mu^{\varepsilon}) dt + \sqrt{\sigma(t, X^{\varepsilon, u}, \mu^{\varepsilon})} dW(t) + \sigma(t, X^{\varepsilon, u}, \mu^{\varepsilon}) \phi_\varepsilon(t) dt + \int_Z G_\varepsilon(z, X^{\varepsilon, u}, \mu^{\varepsilon}, z) \left( \epsilon N^{-1}\psi_\varepsilon(dz, dt) - \nu(dz) dt \right), \quad t \in [0, T], \tag{1.6}$$

where $\mu^{\varepsilon}$ is the distribution of the solution $X$ to (1.1). The reason is that when perturbing the BM and PRM in the arguments of the map $G^\varepsilon(\cdot, \cdot)$, $\mu^{\varepsilon}$ is already deterministic and hence it is not affected by the perturbation, as the following example shows.
Example 1.1 Consider the following simple MVSDE:

$$X^\varepsilon(t) = x_0 + \int_0^t \mathbb{E}(X^\varepsilon(s))ds + \sqrt{\varepsilon} W(t), \ t \in [0, T], \varepsilon \in (0, 1]. \quad (1.7)$$

Here $x_0 \in \mathbb{R}$, $W$ is a one-dimensional BM. Due to the existence and uniqueness of the strong solution, there exists a map $G^\varepsilon$ such that $X^\varepsilon = G^\varepsilon(\sqrt{\varepsilon} W)$. To see the equation satisfied by $Y^\varepsilon := G^\varepsilon(\sqrt{\varepsilon} W + \int_0^t \phi_\varepsilon(s)ds)$, we take expectation on both sides of (1.7) to get

$$\mathbb{E}(X^\varepsilon(t)) = x_0 + \int_0^t \mathbb{E}(X^\varepsilon(s))ds, \ t \in [0, T], \varepsilon \in (0, 1].$$

Hence $\mathbb{E}(X^\varepsilon(t)) = x_0 e^{\varepsilon t}$. Thus we have

$$X^\varepsilon(t) = x_0 + \int_0^t x_0 e^{\varepsilon s} ds + \sqrt{\varepsilon} W(t) = G^\varepsilon(\sqrt{\varepsilon} W)(t), \ t \in [0, T], \varepsilon \in (0, 1].$$

Therefore, $Y^\varepsilon := G^\varepsilon(\sqrt{\varepsilon} W + \int_0^t \phi_\varepsilon(s)ds)$ is the solution of the equation:

$$Y^\varepsilon(t) = x_0 + \int_0^t x_0 e^{\varepsilon s} ds + \sqrt{\varepsilon} W(t) + \int_0^t \phi_\varepsilon(s)ds$$

$$= x_0 + \int_0^t \mathbb{E}(X^\varepsilon(s))ds + \sqrt{\varepsilon} W(t) + \int_0^t \phi_\varepsilon(s)ds, \ t \in [0, T], \varepsilon \in (0, 1]. \quad (1.8)$$

$Y^\varepsilon$ does NOT satisfy the following controlled SDE:

$$Y^\varepsilon(t) = x_0 + \int_0^t \mathbb{E}(Y^\varepsilon(s))ds + \sqrt{\varepsilon} W(t) + \int_0^t \phi_\varepsilon(s)ds, \ t \in [0, T], \varepsilon \in (0, 1]. \quad (1.9)$$

Large and moderate deviations for MVSDEs and MVSPDEs, especially driven by Lévy noise, have been much less studied. For the MVSDE Eq. 1.1 without jumps, Herrmann et al. [28] obtained the large deviation principle (LDP) in path space equipped with the uniform norm, assuming the superlinear growth of the drift but imposing coercivity condition, and a constant diffusion coefficient. Dos Reis et al. [20] obtained LDPs in path space topologies under the assumption that coefficients $b$ and $\sigma$ have some extra regularity with respect to time. The approach in [28] and [20] is to first replace the distribution $\mu^\xi$ of $X^\xi$ in the coefficients with a Dirac measure $\delta_{X^0(t)}$, where $X^0$ is the solution to the following ordinary differential equation

$$dX^0(t) = b(t, X^0(t), X^0(t))dt, \quad (1.10)$$

and then to use discretization, approximation and exponential equivalence arguments. Carrying out similar arguments, Adams et al. [1] studied a class of reflected McKean-Vlasov diffusions. These techniques require more stringent conditions on the coefficients.

Recently Suo and Yuan [56] obtained a moderate deviation principle (MDP) for MVSDEs driven by BM, assuming the Lipschitz conditions on coefficients $b, \sigma$ and on the Lyons derivative of the coefficients $b$. They used the weak convergence approach to first establish the MDP for the following SDEs

$$dY^\varepsilon(t) = b_\varepsilon(t, Y^\varepsilon(t), \delta_{X^0(t)})dt + \sqrt{\varepsilon} \sigma_\varepsilon(t, Y^\varepsilon(t), \delta_{X^0(t)})dW(t), \ t \in [0, T], \quad (1.11)$$

where $X^0$ is given in Eq. 1.10, and then proved the exponential equivalence of $X^\varepsilon$ and $Y^\varepsilon$. However stronger conditions on the coefficients are required in this approach.

One of the main contributions in this article is the identification of the correct controlled equations for MVSDEs when perturbing the driving BM and PRM. This is the key for us to fully use the weak convergence method to establish LDPs and MPDs, which leads to the
very natural Lipschitz conditions on the coefficients without the extra assumptions appearing in the literatures mentioned above. The discretization and approximation techniques can not be applied to the case of Lévy driving noise and also require stronger conditions on the coefficients even in the Gaussian case. We formulate the abstract results in infinite dimensions so that they can be applied both to finite and infinite dimensions. Due to the length of the current paper, we present only an application in finite dimensions. However, we like to point out the article [29] where the authors already applied our results to infinite dimensional situations.

The paper is organized as follows. In Section 2, we introduce notations, cylindrical Brownian motion, Poisson random measures as well as various associated spaces. In Section 3, we introduce the setup for McKean-Vlasov stochastic differential equations on Banach spaces. The framework is sufficiently general to include also MVSDEs/MVSPDEs with delay/memory driven by Lévy noise. In Section 4, we formulate two abstract results on large deviation and moderate deviation principles for the MVSDEs. In Section 5, we consider MVSDEs in $\mathbb{R}^d$. We obtain a large deviation principle and a moderate deviation principle for the MVSDEs under much weaker conditions than those existing in the literature.

2 Framework

Set $\mathbb{N} := \{1, 2, 3, \cdots\}$, $\mathbb{R} := (-\infty, \infty)$ and $\mathbb{R}_+ := [0, \infty)$. For a metric space $S$, the Borel $\sigma$-field on $S$ will be written as $\mathcal{B}(S)$. We denote by $C_c(S)$ the space of real-valued continuous functions with compact supports. Let $C([0, T], S)$ be the space of continuous functions $f : [0, T] \to S$ endowed with the uniform convergence topology. Let $D([0, T], S)$ be the space of all càdlàg functions $f : [0, T] \to S$ endowed with the Skorokhod topology. For an $S$-valued measurable map $X$ defined on some probability space $(\Omega, \mathcal{F}, P)$ we will denote by $\text{Law}(X)$ the measure induced by $X$ on $(S, \mathcal{B}(S))$. For a measurable space $(U, \mathcal{U})$, let $Pr(U)$ be the class of probability measures on this space. We use the symbol $\Rightarrow$ to denote the convergence in distribution.

For a locally compact Polish space $S$, the space of all Borel measures on $S$ is denoted by $M(S)$, and $M_{FC}(S)$ denotes the set of all $\mu \in M(S)$ with $\mu(O) < \infty$ for each compact subset $O \subseteq S$. We endow $M_{FC}(S)$ with the weakest topology such that for each $f \in C_c(S)$ the mapping $\mu \in M_{FC}(S) \to \int_S f(s) \mu(ds)$ is continuous. This topology is metrizable such that $M_{FC}(S)$ is a Polish space, see [6] for more details.

We fix $T > 0$ throughout this paper. Let $K$ be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_K$ and norm $\| \cdot \|_K$. Assume that $Z$ is a locally compact Polish space with a $\sigma$-finite measure $\nu \in M_{FC}(Z)$. The probability space $(\Omega, \mathcal{F}, \mathbb{P} := \{\mathcal{F}_t\}_{t \in [0, T]}, P)$ is specified as

$$\Omega := C([0, T], K) \times M_{FC}([0, T] \times Z \times \mathbb{R}_+), \quad \mathcal{F} := \mathcal{B}(\Omega).$$

We introduce the coordinate mappings

$$W : \Omega \to C([0, T], K), \quad W(\alpha, \beta)(t) = \alpha(t), \quad t \in [0, T];$$

$$N : \Omega \to M_{FC}([0, T] \times Z \times \mathbb{R}_+), \quad N(\alpha, \beta) = \beta.$$

Define for each $t \in [0, T]$ the $\sigma$-algebra

$$\mathcal{G}_t := \sigma \left( \left\{ (W(s), N((0, s] \times A)) : 0 \leq s \leq t, A \in \mathcal{B}(Z \times \mathbb{R}_+) \right\} \right).$$

For the given $\nu$, it follows from [33, Sec.I.8] that there exists a unique probability measure $P$ on $(\Omega, \mathcal{F})$ such that:
(a) \( W \) is a \( K \)-cylindrical BM;
(b) \( N \) is a PRM on \([0, T] \times Z \times \mathbb{R}_+\) with intensity measure \( \text{Leb}_T \otimes \nu \otimes \text{Leb}_\infty \), where \( \text{Leb}_T \) and \( \text{Leb}_\infty \) stand for the Lebesgue measures on \([0, T]\) and \( \mathbb{R}_+ \) respectively;
(c) \( W \) and \( N \) are independent.

We denote by \( \mathbb{F} := \{ \mathcal{F}_t \}_{t \in [0, T]} \) the \( \mathcal{P} \)-completion of \( \{ \mathcal{G}_t \}_{t \in [0, T]} \) and \( \mathcal{P} \) the \( \mathbb{F} \)-predictable \( \sigma \)-field on \([0, T] \times \Omega \). The cylindrical BM \( W \) and the PRM \( N \) will be defined on the (filtered) probability space \(( \Omega, \mathcal{F}, \mathcal{P} := \{ \mathcal{F}_t \}_{t \in [0, T]} )\). The corresponding compensated PRM will be denoted by \( \tilde{N} \).

Denote \( \mathbb{R}_+ := \{ \phi : [0, T] \times Z \to \mathbb{R}_+ : \phi \) is \(( \mathcal{P} \otimes \mathcal{B}(Z)) / \mathcal{B}(\mathbb{R}_+) \)-measurable \} \).

For any \( \varphi \in \mathbb{R}_+ \), \( N^\varphi : \Omega \to M_{FC}([0, T] \times Z) \) is a counting process on \([0, T] \times Z \) defined by

\[
N^\varphi((0, t] \times A) = \int_{(0, t] \times A \times \mathbb{R}_+} 1_{[0, \varphi(s, z)]}(r) N(ds, dz, dr), \quad 0 \leq t \leq T, \quad A \in \mathcal{B}(Z). \tag{2.1}
\]

\( N^\varphi \) can be viewed as a controlled random measure, with \( \varphi \) selecting the intensity.

Analogously, \( \tilde{N}^\varphi \) is defined by replacing \( N \) with \( \tilde{N} \) in (2.1). When \( \varphi \equiv c > 0 \), we write \( N^\varphi = N^c \) and \( \tilde{N}^\varphi = \tilde{N}^c \).

For each \( f \in L^2([0, T], K) \), we introduce the quantity

\[
Q_1(f) := \frac{1}{2} \int_0^T \| f(s) \|^2_K ds,
\]

and for each \( m > 0 \), denote

\[
S^m_1 := \{ f \in L^2([0, T], K) : Q_1(f) \leq m \}.
\]

Equipped with the weak topology, \( S^m_1 \) is a compact subset of \( L^2([0, T], K) \).

For each measurable function \( g : [0, T] \times Z \to [0, \infty) \), define

\[
Q_2(g) := \int_{[0, T] \times Z} \ell(g(s, z)) \nu(dz) ds,
\]

where \( \ell(x) = x \log x - x + 1 \), \( \ell(0) := 1 \). For each \( m > 0 \), denote

\[
S^m_2 := \{ g : [0, T] \times Z \to [0, \infty) : Q_2(g) \leq m \}.
\]

Any measurable function \( g \in S^m_2 \) can be identified with a measure \( \hat{g} \in M_{FC}([0, T] \times Z) \), defined by

\[
\hat{g}(A) = \int_A g(s, z) \nu(dz) ds, \quad \forall A \in \mathcal{B}([0, T] \times Z). \tag{2.2}
\]

This identification induces a topology on \( S^m_2 \) under which \( S^m_2 \) is a compact space (see the Appendix of [9]).

Denote

\[
S := \bigcup_{m \in \mathbb{N}} \left( S^m_1 \times S^m_2 \right), \tag{2.3}
\]

and equip it with the usual product topology.

For any \( m \in (0, \infty) \), let \( S^m_1 \) be a space of stochastic processes on \( \Omega \) defined by

\[
S^m_1 := \{ \varphi : [0, T] \times \Omega \to K : \mathbb{F}\text{-predictable and } \varphi(\cdot, \omega) \in S^m_1 \text{ for } P\text{-a.e. } \omega \in \Omega \}.\]
Let \( \{Z_n\}_{n \in \mathbb{N}} \) be a sequence of compact sets satisfying that \( Z_n \subseteq Z \) and \( Z_n \nearrow Z \). For each \( n \in \mathbb{N} \), let

\[
\mathcal{R}_{b,n} = \left\{ \psi \in \mathcal{R}_+ : \psi(t, z, \omega) \in \left[ \frac{1}{n}, n \right], \quad \text{if } z \in Z_n \right\}
\]

and \( \mathcal{R}_b = \bigcup_{n=1}^{\infty} \mathcal{R}_{b,n} \). For any \( m \in (0, \infty) \), let \( S_2^m \) be a space of stochastic processes on \( \Omega \) defined by

\[
S_2^m := \{ \psi \in \mathcal{R}_b : \psi(\cdot, \cdot, \omega) \in S_2^m \text{ for } P\text{-a.e. } \omega \in \Omega \}.
\]

### 3 McKean-Vlasov SDES

Now we are in the position to introduce the framework of distribution-dependent SDEs on Banach spaces. Our framework is sufficiently general to also include SPDEs.

Let \( H \) be a separable Hilbert space with inner product \( \langle \cdot, \cdot \rangle_H \) and norm \( \| \cdot \|_H \). Let \( V \) and \( E \) be separable Banach spaces with norms \( \| \cdot \|_V \) and \( \| \cdot \|_E \) such that \( V \subset H \subset E \) continuously and densely. In the setting of SPDEs, one often takes \( V = H_0^{1,2}(D) \), \( H = L^2(D) \) and \( E = H_0^{-1,2}(D) \), where \( D \) is an open domain in \( \mathbb{R}^d \), \( H_0^{1,2}(D) \) is the Sobolev space of order one, and \( H_0^{-1,2}(D) \) is the dual space of \( H_0^{1,2}(D) \).

\( \| \cdot \|_V \) is extended to a functional on \( H \) by setting \( \| x \|_V := \infty \) if \( x \in H \setminus V \). Note that this extension is \( \mathcal{B}(H) \)-measurable and lower semi-continuous. Hence, the following path space is well defined:

\[
\mathbb{D} := \{ x \in D([0, T], H) : \int_0^T \| x(t) \|_V \, dt < \infty \}
\]

endowed with the metric

\[
d(x_1, x_2) := \int_0^T \| x_1(t) - x_2(t) \|_V \, dt + d_T(x_1, x_2),
\]

where \( d_T(x_1, x_2) \) is the Skorokhod distance on \( D([0, T], H) \). It is easy to see that \( (\mathbb{D}, d) \) is a Polish space.

Let \( L_2(K, H) \) be the space of all Hilbert-Schmidt operators from \( K \) to \( H \) equipped with the usual Hilbert-Schmidt norm \( \| \cdot \|_{L_2} \). Denote by \( \mathcal{B}_s(\mathbb{D}) \) the \( \sigma \)-algebra generated by all maps \( \pi_s : \mathbb{D} \to H, s \in [0, t] \), where \( \pi_s(w) := w(s), w \in \mathbb{D} \).

**Assumption 3.1** Throughout this paper we always assume that, for any fixed \( J \in Pr(D([0, T], H)) \):

- \( b(\cdot, \cdot, J) : [0, T] \times \mathbb{D} \to E \) is \( \mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbb{D}) \otimes \mathcal{B}(E) \)-measurable;
- \( \sigma(\cdot, \cdot, J) : [0, T] \times \mathbb{D} \to L_2(K, H) \) is \( \mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbb{D}) \otimes \mathcal{B}(L_2(K, H)) \)-measurable;
- \( G(\cdot, \cdot, J) : [0, T] \times \mathbb{D} \otimes Z \to H \) is \( \mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbb{D}) \otimes \mathcal{B}(Z) \otimes \mathcal{B}(H) \)-measurable;
- \( b(t, \cdot, J) \) is \( \mathcal{B}_t(\mathbb{D}) / \mathcal{B}(E) \)-measurable, for any \( t \in [0, T] \);
- \( \sigma(t, \cdot, J) \) is \( \mathcal{B}_s(\mathbb{D}) / \mathcal{B}(L_2(K, H)) \)-measurable, for any \( t \in [0, T] \);
- \( G(t, \cdot, J, z) \) is \( \mathcal{B}_t(\mathbb{D}) / \mathcal{B}(H) \)-measurable, for any \( t \in [0, T] \) and \( z \in Z \).
• $G(\cdot, \cdot, J, \cdot) \text{ is predictable with respect to } \{\mathcal{B}_t(\mathbb{D}), \ t \in [0, T]\}. \footnote{See Definition 3.3 in [33], page 61. An $H$-valued function $f(t, x, z)$ defined on $[0, T] \times \mathbb{D} \times Z$ is called $(\mathcal{B}_t(\mathbb{D}), \ t \in [0, T])$-predictable if the mapping $(t, x, z) \rightarrow f(t, x, z)$ is $\mathcal{I}/\mathcal{B}(H)$-measurable where $\mathcal{I}$ is the smallest $\sigma$-field on $[0, T] \times \mathbb{D} \times Z$ such that all real valued function $g$ having the following properties are measurable:

\begin{enumerate}
  \item for each $t > 0$, $(x, z) \rightarrow g(t, x, z)$ is $\mathcal{B}_t(\mathbb{D}) \otimes \mathcal{B}(Z)/\mathcal{B}(\mathbb{R})$-measurable;
  \item for each $(x, z)$, $t \rightarrow g(t, x, z)$ is left continuous.
\end{enumerate}}$

In this section, we consider the following general distribution-dependent SDEs with jumps

$$dY(t) = b(t, Y, J)dt + \sigma(t, Y, J)dW(t) + \int_Z G(t, Y, J, z)\tilde{N}^1(dz, dt), \ 0 \leq t \leq T \ (3.1)$$

with initial value $Y(0) = h \in H$.

Remark 3.1 We stress that the abstract formulation of SDEs (3.1) is general enough to cover many types of SPDEs, such as SPDEs with delay, distribution-dependent SPDEs, McKean-Vlasov SDEs, etc.

We now introduce some definitions related to the solutions of (3.1).

Definition 3.2 For a fixed $J \in \mathcal{P}_r(D([0, T], H))$, $Y$ is called a solution of (3.1) if

\begin{enumerate}
  \item $Y = \{Y(t), t \in [0, T]\}$ is an $\mathbb{F}$-adapted process with paths in $\mathbb{D}$,
  \item $\int_0^T \|b(t, Y, J)\|_E dt + \int_0^T \|\sigma(t, Y, J)\|_{L^2}^2 dt + \int_0^T \int_Z \|G(t, Y, J, z)\|_{L^2}^2 H \nu(dz) dt < \infty$, $P$-a.s.,
  \item as a stochastic process on $E$, $Y$ satisfies
$$Y(t) = h + \int_0^t b(s, Y, J)ds + \int_0^t \sigma(s, Y, J)dW(s) + \int_0^t \int_Z G(s, Y, J, z)\tilde{N}^1(dz, ds), \ t \in [0, T], \ P$-a.s. \ (3.2)$$
\end{enumerate}

Remark 3.3 If (a) holds, the assumptions on $b$, $\sigma$ and $G$ imply that

\begin{enumerate}
  \item $\{b(s, Y(\omega), J), s \in [0, T], \omega \in \Omega\}$ is an $E$-valued $\mathbb{F}$-adapted process,
  \item $\{\sigma(s, Y(\omega), J), s \in [0, T], \omega \in \Omega\}$ is a $L_2(K, H)$-valued $\mathbb{F}$-adapted process,
  \item $\{G(s, Y(\omega), J, z), s \in [0, T], \omega \in \Omega, z \in Z\}$ is a $H$-valued $\mathbb{F}$-predictable process.\footnote{An $H$-valued function $f(t, x, z)$ defined on $[0, T] \times \Omega \times Z$ is called $\mathbb{F}$-predictable if the mapping $(t, x, z) \rightarrow f(t, x, z)$ is $\mathcal{I}_\mathcal{F}/\mathcal{B}(H)$-measurable where $\mathcal{I}_\mathcal{F}$ is the smallest $\sigma$-field on $[0, T] \times \Omega \times Z$ with respect to which all real valued function $g$ having the following properties are measurable:

\begin{enumerate}
  \item for each $t > 0$, $(x, z) \rightarrow g(t, x, z)$ is $\mathcal{F}_t \otimes \mathcal{B}(Z)/\mathcal{B}(\mathbb{R})$-measurable,
  \item for each $(x, z)$, $t \rightarrow g(t, x, z)$ is left continuous.
\end{enumerate}}

Furthermore, if (b) holds, then $\int_0^T b(s, Y, J)ds$, $\int_0^T \sigma(s, Y, J)dW(s)$ and $\int_0^T \int_Z G(s, Y, J, z)\tilde{N}^1(ds, dz)$ are well-defined as the Lebesgue-Stieltjes integral and the Itô integrals respectively. The reader is referred to [53] and [63] for more details.
**Definition 3.4** The pathwise uniqueness is said to hold for (3.1) with the fixed \( J \in \text{Pr}(D([0, T], H)) \), if for any two solutions \( Y_1 \) and \( Y_2 \) of (3.1),
\[
Y_1(t) = Y_2(t), \ t \in [0, T], \ P\text{-a.s.}
\]
Now consider the McKean-Vlasov equation:
\[
dX(t) = b(t, X, \text{Law}(X))\,dt + \sigma(t, X, \text{Law}(X))\,dW(t) + \int_Z G(t, X, \text{Law}(X), z)\,\widetilde{N}^1(dz, dt) \tag{3.3}
\]
with initial value \( X(0) = h \in H \).
Notice that a stochastic process \( X = \{X(t)\}_{0 \leq t \leq T} \) is a solution to equation (3.3) if \( X \) is a solution of (3.1) with \( J = \text{Law}(X) \).

**Definition 3.5** The pathwise uniqueness is said to hold for (3.3), if for any two solutions \( X_1 \) and \( X_2 \) of (3.3),
\[
X_1(t) = X_2(t), \ t \in [0, T] \ P\text{-a.s.},
\]
and hence \( \text{Law}(X_1) = \text{Law}(X_2) \).

Now we state a perturbation result.

**Theorem 3.6** Fix \( J \in \text{Pr}(D([0, T], H)) \). Suppose that \( Y \) is a solution of (3.1), and pathwise uniqueness holds with the fixed \( J \). Then there exists a unique map \( \Gamma_J : C([0, T], K) \times M_{FC}([0, T] \times Z) \to \mathbb{D} \) such that
\[
Y = \Gamma_J(W, N^1).
\]
Moreover for any \( m \in (0, \infty) \) and \( u = (\phi, \psi) \in S^m_1 \times S^m_2 \), let
\[
Y^u := \Gamma_J(W + \int_0^\cdot \phi(s)\,ds, N^\psi), \tag{3.4}
\]
then we have
(a) \( Y^u = \{Y^u(t), t \in [0, T]\} \) is an \( \mathcal{F} \)-adapted process with paths in \( \mathbb{D} \),
(b) \[
\int_0^T \|b(t, Y^u, J)\|_E\,dt + \int_0^T \|\sigma(t, Y^u, J)\|_{L^2}^2\,dt + \int_0^T \|\sigma(t, Y^u, J)\phi(t)\|_H\,dt
\]
\[
+ \int_0^T \int_Z \|G(t, Y^u, J, z)\|_H^2\psi(t, z)\nu(dz)\,dt + \int_0^T \int_Z \|G(t, Y^u, J, z)(\psi(t, z) - 1)\|_H\nu(dz)\,dt
\]
< \infty, \ P\text{-a.s.},
(c) as a stochastic equation on \( E \), \( Y^u \) satisfies
\[
Y^u(t) = h + \int_0^t b(s, Y^u, J)\,ds + \int_0^t \sigma(s, Y^u, J)\,dW(s) + \int_0^t \sigma(s, Y^u, J)\phi(s)\,ds
\]
\[
+ \int_0^t \int_Z G(s, Y^u, J, z)\left(N^\psi(dz, ds) - \nu(dz)\,ds\right) , \ t \in [0, T], \ P\text{-a.s.} \tag{3.5}
\]
Moreover, \( Y^u \) is the unique stochastic process satisfying (a)-(c).
Remark 3.7 Note that Eq. 3.5 is equivalent to
\[ Y^u(t) = h + \int_0^t b(s, Y^u, J)ds + \int_0^t \sigma(s, Y^u, J)dW(s) + \int_0^t \int_G G(s, Y^u, J, z)\tilde{N}(dz, ds) + \int_0^t \sigma(s, Y^u, J)\phi(s)ds, \ t \in [0, T], \ P\text{-a.s.} \]

where
\[ \int_0^t \int_G G(s, Y^u, J, z)\tilde{N}(dz, ds) = \int_0^t \int_0^{\infty} G(s, Y^u, J, z)1_{[0,\psi(s,z)]}(r)\tilde{N}(dr, dz, ds). \] (3.6)

Although the claim in Theorem 3.6 is not surprising, its rigorous proof requires the careful use of the Girsanov Theorem for the mixture of Brownian motion and Poisson random measures. The sketch of the proof of Theorem 3.6 is provided in the appendix.

Applying Theorem 3.6 and Remark 3.7, we immediately have

Theorem 3.8 Assume that \( X \) is a solution of (3.3) with initial value \( X(0) = h \in H \), and that the pathwise uniqueness holds for (3.1) with \( J = \text{Law}(X) \). Then \( X = \Gamma_{\text{Law}(X)}(W, N^1) \), where \( \Gamma_{\text{Law}(X)} \) is the map \( \Gamma_J \) in Theorem 3.6 with \( J = \text{Law}(X) \).

Moreover for any \( m \in (0, \infty) \) and \( u = (\phi, \psi) \in S^m_1 \times S^m_2 \), let \( X^u := \Gamma_{\text{Law}(X)}(W + \int_0^t \phi(s)ds, N^\psi) \), then we have

(a) \( X^u = [X^u(t), t \in [0, T]] \) is an \( \mathbb{F} \)-adapted process with paths in \( \mathbb{D} \),

(b) as a stochastic equation on \( E \), \( X^u \) satisfies
\[ X^u(t) = h + \int_0^t b(s, X^u, \text{Law}(X))ds + \int_0^t \sigma(s, X^u, \text{Law}(X))dW(s) + \int_0^t \int_G G(s, X^u, \text{Law}(X), z)\bar{N}^\psi(dz, ds) + \int_0^t \sigma(s, X^u, \text{Law}(X))\phi(s)ds, \ t \in [0, T], \ P\text{-a.s.} \] (3.7)

4 Large and Moderate Deviation Principles

In this section, we will consider the large and moderate deviation principles of the solutions:
\[ dX^\epsilon(t) = b_\epsilon(t, X^\epsilon, \text{Law}(X^\epsilon))dt + \sqrt{\epsilon}\sigma_\epsilon(t, X^\epsilon, \text{Law}(X^\epsilon))dW(t) + \epsilon \int_G G_\epsilon(t, X^\epsilon, \text{Law}(X^\epsilon), z)\bar{N}^{\epsilon^{-1}}(dz, dt) \] (4.1)

with initial value \( X^\epsilon(0) = h \in H \), as \( \epsilon \downarrow 0 \).

4.1 Large Deviation Principle

Let us first recall the definition of a rate function and LDP.

Let \( E \) be a Polish space with the Borel \( \sigma \)-field \( \mathcal{B}(E) \). Recall
Definition 4.1 (Rate function) A function $I : E \to [0, \infty]$ is called a rate function on $E$, if for each $M < \infty$, the level set $\{ x \in E : I(x) \leq M \}$ is a compact subset of $E$.

Definition 4.2 (Large deviation principle) Let $I$ be a rate function on $E$. Given a collection \{ $h(\varepsilon)$ \}$_{\varepsilon > 0}$ of positive reals, a family \{ $X^\varepsilon$ \}$_{\varepsilon > 0}$ of $E$-valued random elements is said to satisfy a LDP on $E$ with speed $h(\varepsilon)$ and rate function $I$ if the following two claims hold.

(a) (Upper bound) For each closed subset $C$ of $E$,
$$\limsup_{\varepsilon \to 0} h(\varepsilon) \log P(X^\varepsilon \in C) \leq - \inf_{x \in C} I(x).$$

(b) (Lower bound) For each open subset $O$ of $E$
$$\liminf_{\varepsilon \to 0} h(\varepsilon) \log P(X^\varepsilon \in O) \geq - \inf_{x \in O} I(x).$$

Introduce the hypothesis:

(S0) For any fixed $\varepsilon > 0$ and $J \in Pr(D([0, T], H))$, the maps $b_\varepsilon(\cdot, \cdot, J) : [0, T] \times D \to E$, $\sigma_\varepsilon(\cdot, \cdot, J) : [0, T] \times D \to L^2(K, H)$ and $G_{\varepsilon}(\cdot, \cdot, J, \cdot) : [0, T] \times D \times Z \to H$ satisfy the Assumption 3.1;

(S1) (4.1) has a unique solution $X^\varepsilon$ as stated in Definition 3.5;

(S2) Pathwise uniqueness holds for the following SDE with the fixed $J$ replaced by $Law(X^\varepsilon)$ as stated in Definition 3.4,
$$dY^\varepsilon(t) = b_\varepsilon(t, Y^\varepsilon, J)dt + \sqrt{\varepsilon}\sigma_\varepsilon(t, Y^\varepsilon, J)dW(t) + \varepsilon \int_{Z} G_{\varepsilon}(t, Y^\varepsilon, J, z)N^{-1}(dz, dt) \quad (4.2)$$

with initial value $Y^\varepsilon(0) = h \in H$.

Theorem 3.8 states that there exists a map $\Gamma^\varepsilon_{Law(X^\varepsilon)}$ such that $X^\varepsilon = \Gamma^\varepsilon_{Law(X^\varepsilon)}(\sqrt{\varepsilon}W(\cdot), \varepsilon N^{-1})$, that is, $X^\varepsilon$ can be represented by a functional of a PRM and an infinite-dimensional BM. Moreover, for any $m \in (0, \infty), u_\varepsilon = (\phi_\varepsilon, \psi_\varepsilon) \in S^m_1 \times S^m_2$, let
$$Z^{u_\varepsilon} := \Gamma^\varepsilon_{Law(X^\varepsilon)}(\sqrt{\varepsilon}W(\cdot) + \int_0^t \phi_\varepsilon(s)ds, \varepsilon N^{-1}\psi_\varepsilon), \quad (4.3)$$
then $Z^{u_\varepsilon}$ is the unique solution of the equation:
$$Z^{u_\varepsilon}(t) = h + \int_0^t b_\varepsilon(s, Z^{u_\varepsilon}, Law(X^\varepsilon))ds + \sqrt{\varepsilon} \int_0^t \sigma_\varepsilon(s, Z^{u_\varepsilon}, Law(X^\varepsilon))dW(s)$$
$$+ \int_0^t \sigma_\varepsilon(s, Z^{u_\varepsilon}, Law(X^\varepsilon))\phi_\varepsilon(s)ds + \varepsilon \int_0^t \int_{Z} G_{\varepsilon}(s, Z^{u_\varepsilon}, Law(X^\varepsilon), z)N^{-1}\psi_\varepsilon(dz, ds)$$
$$+ \int_0^t \int_{Z} G_{\varepsilon}(s, Z^{u_\varepsilon}, Law(X^\varepsilon), z)(\psi_\varepsilon(s, z) - 1)v(dz)ds, \quad t \in [0, T], \ P\text{-a.s.} \quad (4.4)$$

Theorem 2.4 in [9] and Theorem 4.2 in [6] provide sufficient conditions/criteria to establish LDP for functionals of a PRM and an infinite-dimensional BM. The statements and the proofs of the theorems do not require specific forms of the functionals of PRM and BM. Therefore, they can be applied to establish LDP for solutions of Mckean-Vlasov SDEs which are indeed functionals of PRM and BM. The following is a re-formulation of these two results in the current setting.
Theorem 4.3 Assume that (S0)-(S2) hold. Suppose that there exists a measurable map \( \Gamma^0 : S \to D^3 \) such that

(a) For any \( m \in (0, \infty) \) and any family \( \{u_\epsilon = (\phi_\epsilon, \psi_\epsilon) ; \ \epsilon > 0\} \subset S_1^m \times S_2^m \) satisfying that \( u_\epsilon \) converges in law as \( S_1^m \times S_2^m \)-valued random elements to some element \( u = (\phi, \psi) \) as \( \epsilon \to 0 \), \( Z^{u_\epsilon} \) converges in law to \( \Gamma^0(\phi, \psi) \).

(b) For every \( m \in (0, \infty) \), the set

\[ \left\{ \Gamma^0(\phi, \psi); (\phi, \psi) \in S_1^m \times S_2^m \right\} \]

is a compact subset of \( D \).

Then the family \( \{X^\epsilon\}_{\epsilon > 0} \) satisfies a LDP in \( D \) with speed \( \epsilon \) and the rate function \( I \) given by

\[ I(g) := \inf_{(\phi, \psi) \in S, g = \Gamma^0(\phi, \psi)} \{Q_1(\phi) + Q_2(\psi)\}, \ g \in D, \quad (4.5) \]

with the convention \( \inf\{\emptyset\} = \infty \).

The next theorem provides a convenient, sufficient condition for verifying the assumptions in Theorem 4.3.

Theorem 4.4 Assume that (S0)-(S2) hold. Suppose that there exists a measurable map \( \Gamma^0 : S \to D^3 \) such that

(a) For any \( m \in (0, \infty) \), any family \( \{u_\epsilon = (\phi_\epsilon, \psi_\epsilon) ; \ \epsilon > 0\} \subset S_1^m \times S_2^m \), and any \( \delta > 0 \),

\[ \lim_{\epsilon \to 0} P(d(Z^{u_\epsilon}, \Gamma^0(\phi_\epsilon, \psi_\epsilon)) > \delta) = 0. \]

(b) For any \( m \in (0, \infty) \) and any family \( \{(\phi_n, \psi_n) \in S_1^m \times S_2^m, n \in \mathbb{N}\} \) satisfying that \( (\phi_n, \psi_n) \) converges to some element \( (\phi, \psi) \) in \( S_1^m \times S_2^m \) as \( n \to \infty \), \( \Gamma^0(\phi_n, \psi_n) \) converges to \( \Gamma^0(\phi, \psi) \) in the space \( D \).

Then the family \( \{X^\epsilon\}_{\epsilon > 0} \) satisfies a LDP in \( D \) with speed \( \epsilon \) and the rate function \( I \) given by (4.5).

The proof of Theorem 4.4 is very similar to that of Theorem 3.2 in [46], so we omit it here.

4.2 Moderate deviation principle

Theorem 3.8 can also be applied to establish a MDP of the solution \( X^\epsilon \) to (4.1) as \( \epsilon \) decreases to 0.

Introduce the following conditions.

---

3The existence of \( \Gamma^0 \) implies that there exists a measurable map \( \tilde{\Gamma}^0 : C([0, T], K) \times M_{FC}([0, T] \times Z) \to D \) such that, for any \( u = (\phi, \psi) \in S \),

\[ \tilde{\Gamma}^0(\int_0^T \phi(s)ds, \hat{\psi}) := \Gamma^0(u). \]

The definition of \( \hat{\psi} \) can be found in (2.2).
(S3) As the parameter $\epsilon$ tends to zero, the solution $X^\epsilon$ of (4.1) will converge in probability in $D([0, T], H)$ to $X^0$ given as the solution of the following deterministic equation

$$dX^0(t) = b_0(t, X^0, \text{Law}(X^0))dt$$

with initial value $X^0(0) = h \in H$.

(S4) (4.6) has a unique solution $X^0 = \{X^0(t), t \in [0, T]\}$.

Assume $a(\epsilon) > 0$, $\epsilon > 0$ satisfies

$$a(\epsilon) \to 0, \quad \epsilon/a^2(\epsilon) \to 0, \quad \text{as } \epsilon \to 0. \quad (4.7)$$

Let

$$M^\epsilon(t) = \frac{1}{a(\epsilon)}(X^\epsilon(t) - X^0(t)), \quad t \in [0, T].$$

Then $M^\epsilon = \{M^\epsilon(t), t \in [0, T]\}$ satisfies the following SDEs

$$dM^\epsilon(t) = \frac{1}{a(\epsilon)} \left(b_\epsilon(t, a(\epsilon)M^\epsilon + X^0, \text{Law}(X^\epsilon)) - b_0(t, X^0, \text{Law}(X^0))\right)dt$$

$$+ \frac{\sqrt{\epsilon}}{a(\epsilon)} \sigma_\epsilon(t, a(\epsilon)M^\epsilon + X^0, \text{Law}(X^\epsilon))dW(t)$$

$$+ \frac{\epsilon}{a(\epsilon)} \int Z G_\epsilon(t, a(\epsilon)M^\epsilon + X^0, \text{Law}(X^\epsilon), z)\tilde{N}^\epsilon^{-1}(dz, dt), \quad (4.8)$$

with $M^\epsilon(0) = 0$.

Denote

$$R := \{\phi : [0, T] \times Z \times \Omega \to \mathbb{R}; \phi \text{ is } (\mathcal{P} \otimes \mathcal{B}(Z))/\mathcal{B}(\mathbb{R})\text{-measurable}. \}$$

For any given $\epsilon > 0$ and $m \in (0, \infty)$, denote

$$S^m_{+, \epsilon} := \{g : [0, T] \times Z \to [0, \infty) \mid \text{Q}_2(g) \leq ma^2(\epsilon)\},$$

$$S^m_\epsilon := \{\phi : [0, T] \times Z \to \mathbb{R} \mid \phi = (g - 1)/a(\epsilon), \; g \in S^m_{+, \epsilon}\},$$

$$S^m_{+, \epsilon} := \{g \in R_b \mid g(\cdot, \cdot, \omega) \in S^m_{+, \epsilon}, \; \text{for } P\text{-a.e. } \omega \in \Omega\},$$

$$S^m_{\epsilon} := \{\phi \in R \mid \phi(\cdot, \cdot, \omega) \in S^m_{\epsilon}, \; \text{for } P\text{-a.e. } \omega \in \Omega\}. $$

Denote $L^2(\nu_T)$ the space of all $\mathcal{B}([0, T]) \otimes \mathcal{B}(Z)/\mathcal{B}(\mathbb{R})$ measurable functions $f$ satisfying that

$$\|f\|_2^2 := \int_0^T \int_Z |f(s, z)|^2 \nu(dz)ds < +\infty.$$  

Then $(L^2(\nu_T), \| \cdot \|_2)$ is a Hilbert space. Denote by $B_2(r)$ the ball of radius $r$ centered at 0 in $L^2(\nu_T)$. Throughout this paper, $B_2(r)$ is equipped with the weak topology of $L^2(\nu_T)$ and therefore compact.

Suppose $g \in S^m_{+, \epsilon}$. By Lemma 3.2 in [4], there exists a constant $\kappa_2(1) > 0$ (independent of $\epsilon$) such that $\varphi_{1/|\varphi| \leq 1/a(\epsilon)} \in B_2(\sqrt{m\kappa_2(1)})$, where $\varphi = (g - 1)/a(\epsilon)$.

Assume that (S0)–(S2) hold. Recall the map $\Gamma^\epsilon_{\text{Law}(X^\epsilon)}$ defined in (4.3). Set

$$\Upsilon^\epsilon_{\text{Law}(X^\epsilon)}(\cdot, \cdot) := \frac{1}{a(\epsilon)} \left(\Gamma^\epsilon_{\text{Law}(X^\epsilon)}(\cdot, \cdot) - X^0\right).$$

Then, by the property of $\Gamma^\epsilon_{\text{Law}(X^\epsilon)}$, we have
(a) \( \Upsilon_{\text{Law}(X^\epsilon)}^\epsilon \) is a measurable map from \( C([0, T], K) \times M_{FC}([0, T] \times Z) \rightarrow \mathbb{D} \) such that
\[
M^\epsilon = \Upsilon_{\text{Law}(X^\epsilon)}^\epsilon \left( \sqrt{\epsilon} W(\cdot), \epsilon N^\epsilon \right),
\]
that is, \( M^\epsilon \) can be represented by a functional of a PRM and an infinite-dimensional BM.

(b) for any \( m \in (0, \infty) \), \( u_\epsilon = (\phi_\epsilon, \psi_\epsilon) \in S^1_1 \times S^{m, \epsilon}_+ \), let
\[
M^{u_\epsilon} := \Upsilon_{\text{Law}(X^\epsilon)}^\epsilon \left( \sqrt{\epsilon} W(\cdot) + a(\epsilon) \int_0^\cdot \phi_\epsilon(s) ds, \epsilon N^\epsilon \phi_\epsilon \right),
\]
then \( M^{u_\epsilon} \) is the unique solution of the following SDE: for each \( t \in [0, T] \),
\[
M^{u_\epsilon}(t) = \frac{1}{a(\epsilon)} \int_0^t \left( b_\epsilon(s, a(\epsilon) M^{u_\epsilon} + X^0, \text{Law}(X^\epsilon)) - b_0(s, X^0, \text{Law}(X^0)) \right) ds
+ \frac{\sqrt{\epsilon}}{a(\epsilon)} \int_0^t \sigma_\epsilon(s, a(\epsilon) M^{u_\epsilon} + X^0, \text{Law}(X^\epsilon)) dW(s)
+ \int_0^t \sigma_\epsilon(s, a(\epsilon) M^{u_\epsilon} + X^0, \text{Law}(X^\epsilon)) \phi_\epsilon(s) ds
+ \frac{\epsilon}{a(\epsilon)} \int_0^t \int_Z G_\epsilon(s, a(\epsilon) M^{u_\epsilon} + X^0, \text{Law}(X^\epsilon), z) \tilde{N}^\epsilon \psi_\epsilon(dz, ds)
+ \frac{1}{a(\epsilon)} \int_0^t \int_Z G_\epsilon(s, a(\epsilon) M^{u_\epsilon} + X^0, \text{Law}(X^\epsilon), z) \left( \psi_\epsilon(s, z) - 1 \right) v(dz) ds.
\]

Theorem 2.3 in [4] provides sufficient conditions/criteria to establish MDP for functionals of a PRM and an infinite-dimensional BM. The statement and the proof of the theorem do not require specific forms of the functionals of PRM and BM. Therefore, they can be applied to establish MDP for \( M^\epsilon \) which are functionals of PRM and BM. Next we state the result on MDP which is a re-formulation of [4, Theorem 2.3] in the present setting.

**Theorem 4.5** Assume that (S0)-(S4) hold. Suppose that there exists a measurable map \( \Upsilon^0 : L^2([0, T], K) \times L^2(\nu_T) \rightarrow \mathbb{D} \) such that

\begin{enumerate}
  \item [(MDP 1)] for any given \( m \in (0, \infty) \), the set
  \[ \Upsilon^0(\phi, \varphi); (\phi, \varphi) \in S^m_1 \times B_2(m) \]
  is a compact subset of \( \mathbb{D} \);
  \item [(MDP 2)] for any given \( m \in (0, \infty) \) and any family \( \{ (\phi_\epsilon, \psi_\epsilon); \epsilon > 0 \} \subset S^m_1 \times S^{m, \epsilon}_+ \)
satisfying that \( \phi_\epsilon \rightarrow \phi \) in \( S^1_1 \) and for some \( \beta \in (0, 1] \), \( \varphi_\epsilon \in L^1(\mathbb{R}; a(\epsilon)^{-1}) \rightarrow \varphi \) in
  \[ B_2(\sqrt{m \kappa_2(1)}) \] where \( \varphi_\epsilon = (\psi_\epsilon - 1)/a(\epsilon) \), then,
  \[ M^{u_\epsilon} \Rightarrow \Upsilon^0(\phi, \varphi) \text{ in } \mathbb{D}, \]
\end{enumerate}

where \( M^{u_\epsilon} \) is the solution to (4.9).

---

\({}^4\)The existence of \( \Upsilon^0 \) implies that there exists a measurable map \( \tilde{\Upsilon}^0 : C([0, T], K) \times L^2(\nu_T) \rightarrow \mathbb{D} \) such that, for any \( u = (\phi, \varphi) \in L^2([0, T], K) \times L^2(\nu_T) \),
\[
\tilde{\Upsilon}^0(\int_0^\cdot \phi(s) ds, \varphi) := \Upsilon^0(u).
\]
Then \( \{M^\varepsilon(t), \ t \in [0, T]\}_{\varepsilon>0} \) satisfies a LDP with speed \( \varepsilon/a^2(\varepsilon) \) and the rate function \( I \) given by

\[
I(g) := \inf_{(\phi,\varphi) \in L^2([0,T],K) \times L^2(\nu_T)} \left\{ \frac{1}{2} \int_0^T \|\phi(s)\|^2_K \, ds + \frac{1}{2} \int_0^T |\varphi(s,z)|^2 \nu(dz) \, ds \right\}, \ g \in \mathbb{D},
\]

with the convention \( \inf \emptyset = \infty \).

Here is a sufficient condition for verifying the assumptions in Theorem 4.5.

**Theorem 4.6** Assume that (S0)-(S4) hold. Suppose that there exists a measurable map \( \Upsilon^0 : L^2([0,T],K) \times L^2(\nu_T) \to \mathbb{D} \) such that

(MDP 1') Given \( m \in (0, \infty) \), for any \( (\phi_n, \varphi_n) \in S^m_1 \times B_2(m), n \in \mathbb{N} \) and \( \phi_n \to \phi \) in \( S^m_1 \), \( \varphi_n \to \varphi \) in \( B_2(m) \), as \( n \to \infty \), then

\[ \Upsilon^0(\phi_n, \varphi_n) \to \Upsilon^0(\phi, \varphi) \text{ in } \mathbb{D}; \]

(MDP 2') Given \( m \in (0, \infty) \), let \( \{(\phi_\varepsilon, \psi_\varepsilon)\}_{\varepsilon>0} \) be such that for every \( \varepsilon > 0 \), \( (\phi_\varepsilon, \psi_\varepsilon) \in S^m_\varepsilon \times S^m_\varepsilon, \) and for some \( \beta \in (0, 1], \varphi_\varepsilon \in \{ |\varphi_\varepsilon| \leq \beta/a(\varepsilon) \} \in B_2(\sqrt{mk_2(1)}) \) where \( \varphi_\varepsilon = (\psi_\varepsilon - 1)/a(\varepsilon) \), for any \( \sigma > 0 \),

\[ \lim_{\varepsilon \to 0} \mathbb{P}(d(M^{\varepsilon\varepsilon}, \Upsilon^0(\phi_\varepsilon, \varphi_\varepsilon 1_{\{ |\varphi_\varepsilon| \leq \beta/a(\varepsilon) \}})) > \sigma) = 0. \]

Then \( \{M^\varepsilon(t), \ t \in [0, T]\}_{\varepsilon>0} \) satisfies a LDP with speed \( \varepsilon/a^2(\varepsilon) \) and the rate function \( I \) defined by (4.10).

The proof of Theorem 4.6 is also similar to that of Theorem 3.2 in [46] and we omit it here.

## 5 Applications

In this section, we will apply the abstract formulation in Section 4 to establish a LDP and a MDP for MVSDDEs in \( \mathbb{R}^d \). For this end, we set \( K = V = H = E = \mathbb{R}^d, \ d \in \mathbb{N} \). Then the notations \( C([0, T], K), \Omega, W, S^m_1, S^m_1, \mathcal{L}_2(K, H) \) and many others in Section 4 should be replaced correspondingly. For example \( C([0, T], K) \) will be replaced by \( C([0, T], \mathbb{R}^d), \)

\( W \) is a \( d \)-dimensional standard Brownian motion, \( \mathcal{L}_2(K, H) \) is replaced by \( \mathcal{L}_2(\mathbb{R}^d, \mathbb{R}^d) = \mathbb{R}^d \otimes \mathbb{R}^d \).

In the Euclidean space \( \mathbb{R}^d \), the inner product and norm are denoted by \( | \cdot | \) and \( \langle \cdot, \cdot \rangle \), respectively. The Dirac measure concentrated at a point \( x \in \mathbb{R}^d \) is denoted by \( \delta_x \).

Denote by \( \mathcal{P}(\mathbb{R}^d) \) the collection of probability measures on \( (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \). Define

\[
\mathcal{P}_2 := \left\{ \mu \in \mathcal{P}(\mathbb{R}^d) : \int_{\mathbb{R}^d} |y|^2 \mu(dy) < \infty \right\}.
\]

Then \( \mathcal{P}_2 \) is a Polish space equipped with the Wasserstein distance

\[
\mathbb{W}_2(\mu_1, \mu_2) := \inf_{\pi \in \mathcal{E}(\mu_1, \mu_2)} \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 \pi(dx, dy) \right)^{\frac{1}{2}},
\]

where \( \mathcal{E}(\mu_1, \mu_2) \) is the set of all couplings for \( \mu_1 \) and \( \mu_2 \).
Remark 5.1 For any $\mathbb{R}^d$-valued random variables $X$ and $Y$,

$$\mathbb{W}_2(Law(X), Law(Y)) \leq [\mathbb{E}(X - Y)^2]^{\frac{1}{2}}.$$ 

Let $\epsilon > 0$. For measurable maps

$$b_{\epsilon} : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2 \to \mathbb{R}^d, \quad \sigma_{\epsilon} : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2 \to \mathbb{R}^d \otimes \mathbb{R}^d,$$
and

$$G_{\epsilon} : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2 \times Z \to \mathbb{R}^d,$$

consider the following MVSDEs on $\mathbb{R}^d$:

$$X^\epsilon(t) = h + \int_0^t b_{\epsilon}(s, X^\epsilon(s), Law(X^\epsilon(s)))ds + \sqrt{\epsilon} \int_0^t \sigma_{\epsilon}(s, X^\epsilon(s), Law(X^\epsilon(s)))dW(s)
+ \epsilon \int_0^t \int_Z G_{\epsilon}(s, X^\epsilon(s-), Law(X^\epsilon(s)), z) \tilde{N}_t^{\epsilon, -1}(dz, ds), \quad t \in [0, T],$$ (5.1)

where $h$ is an element of $\mathbb{R}^d$.

We assume that

(A0) For any $\epsilon > 0$, there exists a unique solution $X^\epsilon$ to (5.1).

The aim of this section is to establish the large and moderate deviation principles for the solutions $\{X^\epsilon, \epsilon > 0\}$ to (5.1) as $\epsilon$ decreases to 0.

Let

$$b : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2 \to \mathbb{R}^d, \quad \sigma : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2 \to \mathbb{R}^d \otimes \mathbb{R}^d,$$
and

$$G : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2 \times Z \to \mathbb{R}^d,$$

be measurable maps.

Introduce the following assumptions.

There are $L > 0$ and $q \geq 1$ such that for each $t \in [0, T], x, x' \in \mathbb{R}^d$ and $\mu, \mu' \in \mathcal{P}_2$,

(A1)

$$\langle x - x', b(t, x, \mu) - b(t, x', \mu) \rangle \leq L|x - x'|^2,$$

$$|b(t, x, \mu) - b(t, x, \mu')| \leq L \mathbb{W}_2(\mu, \mu'),$$

$$|b(t, x, \mu) - b(t, x', \mu)| \leq L(1 + |x|^{q-1} + |x'|^{q-1})|x - x'|,$$

$$\|\sigma(t, x, \mu) - \sigma(t, x', \mu')\|_{\mathcal{L}_2} \leq L(|x - x'| + \mathbb{W}_2(\mu, \mu')),$$

$$\int_0^T \left( |b(t, 0, \delta_0)| + \|\sigma(t, 0, \delta_0)\|^2_{\mathcal{L}_2} \right) dt < \infty.$$

(A2)

$$\int_Z |G(t, x, \mu, z) - G(t, x', \mu', z)|^2v(dz) \leq L(|x - x'|^2 + \mathbb{W}_2^2(\mu, \mu')),$$

$$\int_0^T \int_Z |G(t, 0, \delta_0, z)|^2v(dz)dt < \infty.$$
As $\epsilon \downarrow 0$, the maps $b_\epsilon$ and $\sigma_\epsilon$ converge uniformly to $b$ and $\sigma$ respectively, that is, there exist nonnegative constants $\varphi_{b,\epsilon}$ and $\varphi_{\sigma,\epsilon}$ converging to 0 as $\epsilon \downarrow 0$ such that

$$\sup_{(t,x,\mu)\in[0,T]\times\mathbb{R}^d\times\mathcal{P}_2} \left( |b_\epsilon(t,x,\mu) - b(t,x,\mu)| \right) \leq \varphi_{b,\epsilon}, \tag{5.2}$$

$$\sup_{(t,x,\mu)\in[0,T]\times\mathbb{R}^d\times\mathcal{P}_2} \left( \|\sigma_\epsilon(t,x,\mu) - \sigma(t,x,\mu)\|_2 \right) \leq \varphi_{\sigma,\epsilon}. \tag{5.3}$$

For each fixed $J \in Pr(D([0,T],\mathbb{R}^d))$, pathwise uniqueness holds for the following SDE:

$$Y^\epsilon(t) = h + \int_0^t b_\epsilon(s, Y^\epsilon(s), J(s)) ds + \sqrt{\epsilon} \int_0^t \sigma_\epsilon(s, Y^\epsilon(s), J(s)) dW(s) + \epsilon \int_0^t \int_Z G_\epsilon(s, Y^\epsilon(s-), J(s), z) \tilde{N}^\epsilon(ds, dz), \quad t \in [0,T],$$

where $J(s)$ denotes the marginal distribution of $J$ at time $s$.

**Remark 5.2** Condition (A0) and (A4) are not the same statements. The uniqueness in condition (A0) concerns the solution pair $(X^\epsilon, \text{Law}(X^\epsilon))$ where $\text{Law}(X^\epsilon)$ is a part of the solution. For the uniqueness mentioned in (A4), the law is given IN Advance as $J$. So the uniqueness in (A4) is the uniqueness of solutions of an ordinary stochastic differential equations.

Notice that (A0) and (A4) are conditions on the coefficients $b_\epsilon$, $\sigma_\epsilon$ and (A1)-(A3) are conditions on the coefficients $b$, $\sigma$ (without the parameter $\epsilon$). In general, (A1)-(A3) do not imply (A0) and (A4).

A trivial example for (A0), (A3) and (A4) to hold is that $b_\epsilon = b$, $\sigma_\epsilon = \sigma$, $G_\epsilon = G$, i.e. the coefficients do not depend on $\epsilon$. In this particular case, we only need to assume (A1) and (A2).

**Remark 5.3** Let us point out an application for the varying coefficients $b_\epsilon$ and $\sigma_\epsilon$:

Consider a small time LDP for

$$X(t) = h + \int_0^t b(s, X(s), \text{Law}(X(s))) ds + \int_0^t \sigma(s, X(s), \text{Law}(X(s))) dW(s), \quad t \in [0,T],$$

that is to study the limiting behavior of the solution in time interval $[0,t]$ as $t$ goes to zero. By the scaling property of the Brownian motion, it is easy to see that $X(\epsilon t)$ coincides in law with the solution of the following equation:

$$u^\epsilon(t) = h + \epsilon \int_0^t b(es, u^\epsilon(s), \text{Law}(u^\epsilon(s))) ds + \sqrt{\epsilon} \int_0^t \sigma(es, u^\epsilon(s), \text{Law}(u^\epsilon(s))) dW(s), \quad t \in [0,1].$$

The small time LDP for $X$ is equivalent to the LDP for $u^\epsilon$ whose equations have coefficients depending on $\epsilon$.

For more details, we refer the reader to [59] and [62].

The following result was proved in [20, Theorem 3.3].

**Proposition 5.4** Assume that (A1) holds. There exists a unique function $X^0 = \{X^0(t), t \in [0,T]\}$ such that

- $X^0 \in C([0,T],\mathbb{R}^d)$,
- $\int_0^T |b(s, X^0(s), \text{Law}(X^0(s)))| ds < \infty,$
• $X^0$ satisfies
\[ X^0(t) = h + \int_0^t b(s, X^0(s), \text{Law}(X^0(s))) ds, \quad \forall t \in [0, T]. \tag{5.4} \]

Note that $X^0$ is a deterministic path and $\text{Law}(X^0(s)) = \delta_{X^0(s)}$. In the sequel, we will always use $X^0$ to denote the unique solution to (5.4).

### 5.1 Large deviations principle

In order to obtain the LDP, we need the following notations and assumptions.

Set $L^2(\nu) := \{ f : Z \to \mathbb{R} | f \text{ is } \mathcal{B}(Z)/\mathcal{B}(\mathbb{R})\text{-measurable and } \int_Z |f(z)|^2 \nu(dz) < \infty \}$, and

$$
\mathcal{H} = \left\{ g : Z \to \mathbb{R}_+ | g \text{ is Borel measurable and there exists } c > 0 \text{ such that } \int_O e^{cg^2(z)} \nu(dz) < \infty \text{ for all } O \in \mathcal{B}(Z) \text{ with } \nu(O) < \infty. \right\}
$$

(B1) There exist $L_1, L_2, L_3 \in \mathcal{H} \cap L^2(\nu)$ such that for all $t \in [0, T], x, x' \in \mathbb{R}^d, \mu, \mu' \in \mathcal{P}_2$ and $z \in Z$,

$$
|G(t, x, \mu, z) - G(t, x', \mu', z)| \leq L_1(z) \left( |x - x'| + \mathbb{W}_2(\mu, \mu') \right),
$$

and there exists nonnegative constant $G_0, \delta_0$ converging to 0 such that

$$
\sup_{(t,x,\mu) \in [0,T] \times \mathbb{R}^d \times \mathcal{P}_2} |G(t, x, \mu, z)| \leq \varrho_G e L_3(z).
$$

It is easy to see that (B1) implies (A2).

Before stating the main result in this subsection, we need the following result. Recalling $S^n_1, S^n_2$ and $S$ defined in Section 2, we have

**Proposition 5.5** Assume that (A1) and (B1) hold. Then for any $u = (\phi, \psi) \in S$, there exists a unique solution $Y^u = \{Y^u(t), t \in [0, T]\} \in C([0, T], \mathbb{R}^d)$ to the following equation:

$$
Y^u(t) = h + \int_0^t b(s, Y^u(s), \text{Law}(X^0(s))) ds + \int_0^t \sigma(s, Y^u(s), \text{Law}(X^0(s)))\phi(s) ds
$$

$$
+ \int_0^t \int Z G(s, Y^u(s), \text{Law}(X^0(s)), z)(\psi(s, z) - 1) \nu(dz) ds, \quad t \in [0, T]. \tag{5.5}
$$

Moreover, for any $m > 0$,

$$
\sup_{u=(\phi, \psi) \in S^m_1 \times S^m_2} \sup_{t\in[0,T]} |Y^u(t)| < \infty. \tag{5.6}
$$

**Proof** Without loss of generality, we assume that $u = (\phi, \psi) \in S^m_1 \times S^m_2$.

We first prove that there exists a unique solution to (5.5). Set

$$
b_u(t, x) := b(t, x, \text{Law}(X^0(t))), \quad \sigma_u(t, x) := \sigma(t, x, \text{Law}(X^0(t)))\phi(t),
$$

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and
\[ G_\psi(t, x, z) := G(t, x, \text{Law}(X^0(t)), z)(\psi(t, z) - 1), \quad G_\psi^Z(t, x) := \int_Z G_\psi(t, x, z)\nu(dz). \]

By (A1), Remark 5.1, and the fact that \( X^0 \in C([0, T], \mathbb{R}^d) \), we have that for each \( t \in [0, T] \), \( x, x' \in \mathbb{R}^d \),
\[
\langle x - x', b_u(t, x) - b_u(t, x') \rangle \leq L|x - x'|^2, \tag{5.7}
\]
\[
|b_u(t, x) - b_u(t, x')| \leq L(1 + |x|^{q-1} + |x'|^{q-1})|x - x'|, \tag{5.8}
\]
\[
|\sigma_\phi(t, x) - \sigma_\phi(t, x')| \leq L|x - x'||\phi(t)|, \tag{5.9}
\]
\[
\int_0^T |b_u(t, 0)|dt \leq \int_0^T |b(t, 0, \text{Law}(X^0(t))) - b(t, 0, \delta_0)|dt + \int_0^T |b(t, 0, \delta_0)|dt \\
\leq L \int_0^T \mathbb{W}_2(\text{Law}(X^0(t)), \delta_0)dt + \int_0^T |b(t, 0, \delta_0)|dt \\
\leq L \int_0^T |X^0(t)|dt + \int_0^T |b(t, 0, \delta_0)|dt < \infty, \tag{5.10}
\]

and
\[
\int_0^T |\sigma_\phi(t, 0)|dt \\
\leq \int_0^T |\sigma(t, 0, \text{Law}(X^0(t)))\phi(t) - \sigma(t, 0, \delta_0)\phi(t)|dt + \int_0^T |\sigma(t, 0, \delta_0)\phi(t)|dt \\
\leq L \int_0^T \mathbb{W}_2(\text{Law}(X^0(t)), \delta_0)|\phi(t)|dt + \int_0^T |\sigma(t, 0, \delta_0)\phi(t)|dt \\
\leq L \sup_{t \in [0, T]} |X^0(t)| \left( \int_0^T |\phi(t)|^2 dt + T \right) + \int_0^T \|\sigma(t, 0, \delta_0)\|^2_{S_1} dt \\
+ \int_0^T |\phi(t)|^2 dt < \infty, \tag{5.11}
\]

where we use the condition \( \phi \in S_1^m \).

By (B1), Remark 5.1, and the fact that \( X^0 \in C([0, T], \mathbb{R}^d) \), we get for each \( t \in [0, T] \), \( x, x' \in \mathbb{R}^d \),
\[
|G_\psi^Z(t, x) - G_\psi^Z(t, x')| \\
\leq \int_Z |G(t, x, \text{Law}(X^0(t)), z)(\psi(t, z) - 1) - G(t, x', \text{Law}(X^0(t)), z)(\psi(t, z) - 1)|\nu(dz) \\
\leq \int_Z L_1(z)|\psi(t, z) - 1|\nu(dz)|x - x'|, \tag{5.12}
\]

and
\[
\int_0^T |G_\psi^Z(t, 0)|dt \\
\leq \int_0^T \int_Z |G(t, 0, \text{Law}(X^0(t)), z)(\psi(t, z) - 1) - G(t, 0, \delta_0, z)(\psi(t, z) - 1)|\nu(dz)dt \\
+ \int_0^T \int_Z |G(t, 0, \delta_0, z)(\psi(t, z) - 1)|\nu(dz)dt
\]

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\[ \begin{align*}
&\leq \int_{0}^{T} \int_{Z} L_{1}(z) \mathbb{W}_{2}(\text{Law}(X^{0}(t)), \delta_{0})|\psi(t, z) - 1| \nu(dz)dt \\
&\quad + \int_{0}^{T} \int_{Z} L_{2}(z)|\psi(t, z) - 1| \nu(dz)dt \\
&\leq \sup_{t \in [0, T]} |X^{0}(t)| \int_{0}^{T} \int_{Z} L_{1}(z)|\psi(t, z) - 1| \nu(dz)dt \\
&\quad + \int_{0}^{T} \int_{Z} L_{2}(z)|\psi(t, z) - 1| \nu(dz)dt. 
\end{align*} \]

(5.13)

By Lemma 3.4 in [9], we have

\[ \sup_{\varphi \in S_{m}} \int_{0}^{T} \int_{Z} (L_{1}(z) + L_{2}(z) + L_{3}(z))|\varphi(t, z) - 1| \nu(dz)dt < \infty. \] (5.14)

In view of (5.7)–(5.14), using the classical fixed point arguments, we can deduce that there exists a unique function \( Y^{u} = \{Y^{u}(t), t \in [0, T]\} \in C([0, T], \mathbb{R}^{d}) \) which is the solution to (5.5).

Next we prove (5.6). By the chain rule,

\[ |Y^{u}(t)|^{2} = |h|^{2} + 2 \int_{0}^{t} \langle b(s, Y^{u}(s), \text{Law}(X^{0}(s))), Y^{u}(s) \rangle ds \\
\quad + 2 \int_{0}^{t} \langle \sigma(s, Y^{u}(s), \text{Law}(X^{0}(s)))\phi(s), Y^{u}(s) \rangle ds \\
\quad + 2 \int_{0}^{t} \int_{Z} \langle G(s, Y^{u}(s), \text{Law}(X^{0}(s)), z)(\psi(s, z) - 1), Y^{u}(s) \rangle \nu(dz)ds \\
:= |h|^{2} + I_{1}(t) + I_{2}(t) + I_{3}(t). \] (5.15)

By (A1) and Remark 5.1,

\[ I_{1}(t) = 2 \int_{0}^{t} \langle b(s, Y^{u}(s), \text{Law}(X^{0}(s))) - b(s, 0, \text{Law}(X^{0}(s))), Y^{u}(s) \rangle ds \\
\quad + 2 \int_{0}^{t} \langle b(s, 0, \text{Law}(X^{0}(s))) - b(s, 0, \delta_{0}), Y^{u}(s) \rangle ds \\
\quad + 2 \int_{0}^{t} \langle b(s, 0, \delta_{0}), Y^{u}(s) \rangle ds \\
\leq 2L \int_{0}^{t} |Y^{u}(s)|^{2} ds + 2L \int_{0}^{t} |Y^{u}(s)||X^{0}(s)| ds + 2 \int_{0}^{t} \langle b(s, 0, \delta_{0})||Y^{u}(s)| ds \\
\leq \int_{0}^{t} (3L + |b(s, 0, \delta_{0})|)|Y^{u}(s)|^{2} ds + L \int_{0}^{t} |X^{0}(s)|^{2} ds + \int_{0}^{t} |b(s, 0, \delta_{0})| ds, \] (5.16)

and

\[ I_{2}(t) = 2 \int_{0}^{t} \langle \sigma(s, Y^{u}(s), \text{Law}(X^{0}(s)))\phi(s) - \sigma(s, 0, \delta_{0})\phi(s), Y^{u}(s) \rangle ds \\
\quad + 2 \int_{0}^{t} \langle \sigma(s, 0, \delta_{0})\phi(s), Y^{u}(s) \rangle ds \]
\[
\begin{align*}
&\leq 2L \int_0^t |Y^u(s)| + |X^0(s)||\phi(s)||Y^u(s)| ds + 2 \int_0^t \|\sigma(s, 0, \delta_0)\|_{L_2} |\phi(s)||Y^u(s)| ds \\
&\leq L \int_0^t |Y^u(s)|^2 \left(1 + |\phi(s)|^2\right) ds + L \sup_{l \in [0, T]} |X^0(l)| \int_0^t |Y^u(s)|^2 + |\phi(s)|^2 ds \\
&\quad + \int_0^t \left(\|\sigma(s, 0, \delta_0)\|_{L_2}^2 + |\phi(s)|^2\right) \left(|Y^u(s)|^2 + 1\right) ds \\
&\leq (L + 1) \int_0^t |Y^u(s)|^2 \left(1 + \sup_{l \in [0, T]} |X^0(l)| + |\phi(s)|^2 + \|\sigma(s, 0, \delta_0)\|_{L_2}^2\right) ds \\
&\quad + (1 + L \sup_{l \in [0, T]} |X^0(l)|) \int_0^t \|\sigma(s, 0, \delta_0)\|_{L_2}^2 + |\phi(s)|^2 ds. \quad (5.17)
\end{align*}
\]

By (B1) and Remark 5.1,
\[
I_3(t) = 2 \int_0^t \int_Z \left(G(s, Y^u(s), Law(X^0(s)), z) - G(s, 0, \delta_0, z)\right) (\psi(s, z) - 1) Y^u(s) \nu(dz) ds \\
+ 2 \int_0^t \int_Z \left(G(s, 0, \delta_0, z)(\psi(s, z) - 1), Y^u(s)\right) \nu(dz) ds \\
\leq 2 \int_0^t \int_Z L_1(z) \left(|Y^u(s)| + |X^0(s)|\right) |\psi(s, z) - 1||Y^u(s)||\nu(dz) ds \\
+ 2 \int_0^t \int_Z L_2(z)(\psi(s, z) - 1)|Y^u(s)||\nu(dz) ds \\
\leq \int_0^t \int_Z |Y^u(s)|^2 \left(3L_1(z) + L_2(z)\right) |\psi(s, z) - 1||\nu(dz) ds \\
\quad + \sup_{l \in [0, T]} |X^0(l)|^2 \int_0^t \int_Z L_1(z)|\psi(s, z) - 1||\nu(dz) ds + \int_0^t \int_Z L_2(z)|\psi(s, z) - 1||\nu(dz) ds. \quad (5.18)
\]

Set
\[
\Theta_1(u) := |h|^2 + \int_0^T L|X^0(s)|^2 + |b(s, 0, \delta_0)| ds \\
\quad + \left(1 + L \sup_{l \in [0, T]} |X^0(l)|\right) \int_0^T \|\sigma(s, 0, \delta_0)\|_{L_2}^2 + |\phi(s)|^2 ds \\
\quad + \sup_{l \in [0, T]} |X^0(l)|^2 \int_0^T \int_Z L_1(z)|\psi(s, z) - 1||\nu(dz) ds \\
\quad + \int_0^T \int_Z L_2(z)|\psi(s, z) - 1||\nu(dz) ds, \quad (5.19)
\]

and
\[
\Theta_2(u) := \int_0^T \left(3L + |b(s, 0, \delta_0)|\right) ds + \int_0^T \int_Z \left(3L_1(z) + L_2(z)\right) |\psi(s, z) - 1||\nu(dz) ds \\
\quad + (L + 1) \int_0^T \left(1 + \sup_{l \in [0, T]} |X^0(l)| + |\phi(s)|^2 + \|\sigma(s, 0, \delta_0)\|_{L_2}^2\right) ds. \quad (5.20)
\]
It follows from (A1), (B1) and (5.14) that there exists a constant $\tilde{C}_m$ such that

$$\sup_{u=(\phi, \psi) \in S_1^m \times S_2^m} \left( \Theta_1(u) + \Theta_2(u) \right) \leq \tilde{C}_m < \infty.$$ (5.21)

Then by combining (5.15)-(5.21) together and using Gronwall’s inequality, we get

$$\sup_{u=(\phi, \psi) \in S_1^m \times S_2^m} |Y_u(t)|^2 \leq \sup_{u=(\phi, \psi) \in S_1^m \times S_2^m} \Theta_1(u)e^{\Theta_2(u)} \leq \tilde{C}_m e^{\tilde{C}_m} < \infty,$$ (5.22)

which completes the proof.

We now state the main result in this subsection.

**Theorem 5.6** Assume that (A0), (A1), (B1), (A3) and (A4) hold. Then the solutions \{X_\epsilon,t > 0\} to (5.1) satisfy a LDP on $D([0,T], \mathbb{R}^d)$ with speed $\epsilon$ and the rate function $I$ given by

$$I(g) := \inf \left\{ Q_1(\phi) + Q_2(\psi) : u = (\phi, \psi) \in S, Y_u = g \right\}, g \in D([0,T], \mathbb{R}^d),$$ (5.23)

where for $u = (\phi, \psi) \in S$, $Y_u$ is the unique solution of (5.5). Here we use the convention that the infimum of an empty set is $\infty$.

**Remark 5.7** The LDP for (5.1) with $G \equiv 0$ was established in [20]. Besides the assumptions of Theorem 5.6, they require that the coefficients $b$ and $\sigma$ satisfy the following additional restrictions (for more details see Assumption 4.1 of [20]):

1. there exists $M > 0$ such that $\sigma$ is bounded by $M$,
2. there exists $\beta \in (0, 1]$ such that for any $s, s' \in [0, 1]$, for any $y \in \mathbb{R}^d$ and for all $\mu \in \mathcal{P}_2$,

$$\|\sigma(s, y, \mu) - \sigma(s', y, \mu)\|_{L_2} \leq L|s - s'|^{\beta}, \text{ and } |b(s, y, \mu) - b(s', y, \mu)| \leq L|s - s'|^{\beta}.$$ 

Thus, our assumptions are clearly weaker than those of [20].

**Proof** We will apply Theorem 4.4. Proposition 5.5 allows us to define a map

$$\Gamma^0 : S \ni u = (\phi, \psi) \mapsto Y_u \in D([0,T], \mathbb{R}^d),$$ (5.24)

here $Y_u$ is the unique solution of (5.5).

By (4.3) and (4.4), for any $\epsilon > 0$, $m \in (0, \infty)$ and $u_\epsilon = (\phi_\epsilon, \psi_\epsilon) \in S_1^m \times S_2^m$, there exists a unique solution $\{Z^{u_\epsilon}(t)\}_{t \in [0,T]}$ to the following SDE

$$dZ^{u_\epsilon}(t) = b_\epsilon(t, Z^{u_\epsilon}(t), Law(X^\epsilon(t)))dt + \sqrt{\epsilon}\sigma_\epsilon(t, Z^{u_\epsilon}(t), Law(X^\epsilon(t)))dW(t)$$

$$+ \epsilon \int_Z G_\epsilon(t, Z^{u_\epsilon}(t), Law(X^\epsilon(t)), z)\tilde{N} \psi^{-1}(\epsilon)dz, \quad (5.25)$$

with the initial data $Z^{u_\epsilon}(0) = h$ and $X^\epsilon$ is the solution to (5.1).
According to Theorem 4.4, to complete the proof of the theorem, it is sufficient to verify the following two claims:

(LDP1) For any given $m \in (0, \infty)$, let $u_n = (\phi_n, \psi_n)$, $n \in \mathbb{N}$, $u = (\phi, \psi) \in S^m_1 \times S^m_2$ be such that $u_n \to u$ in $S^m_1 \times S^m_2$ as $n \to \infty$. Then

$$
\lim_{n \to \infty} \sup_{t \in [0, T]} |\Gamma^0(u_n)(t) - \Gamma^0(u)(t)| = 0.
$$

(LDP2) For any given $m \in (0, \infty)$, let $\{u_\epsilon = (\phi_\epsilon, \psi_\epsilon), \epsilon > 0\} \subset S^m_1 \times S^m_2$. Then

$$
\lim_{\epsilon \to 0} \mathbb{E} \left( \sup_{t \in [0, T]} |Z^{u_\epsilon}(t) - \Gamma^0(u_\epsilon)(t)|^2 \right) = 0.
$$

The verification of (LDP1) will be given in Proposition 5.8. (LDP2) will be established in Proposition 5.10.

Recall the map $\Gamma^0$ in (5.24).

**Proposition 5.8** For any given $m \in (0, \infty)$, let $u_n = (\phi_n, \psi_n)$, $n \in \mathbb{N}$, $u = (\phi, \psi) \in S^m_1 \times S^m_2$ be such that $u_n \to u$ in $S^m_1 \times S^m_2$ as $n \to \infty$. Then

$$
\lim_{n \to \infty} \sup_{t \in [0, T]} |\Gamma^0(u_n)(t) - \Gamma^0(u)(t)| = 0.
$$

**Proof** Let $Y^u$ be the the solution of (5.5) and $Y^u_n$ be the solution of (5.5) with $u$ replaced by $u_n$. By the definition of $\Gamma^0$, $Y^u_n = \Gamma^0(u_n)$ and $Y^u = \Gamma^0(u)$. For simplicity, denote $Y_n = Y^u_n$ and $Y = Y^u$. Note that $Y, Y_n \in C([0, T], \mathbb{R}^d), \forall n \in \mathbb{N}$.

The proof is divided into two steps.

**Step 1:** We first prove that $\{Y_n\}_{n \geq 1}$ is pre-compact in $C([0, T], \mathbb{R}^d)$. It suffices to show that $\{Y_n\}_{n \geq 1}$ is uniformly bounded and equi-continuous in $C([0, T], \mathbb{R}^d)$.

It follows from (5.6) that $\{Y_n\}_{n \geq 1}$ is uniformly bounded, i.e.

$$
\sup_{n \geq 1} \sup_{t \in [0, T]} |Y_n(t)| =: C_m < \infty. \tag{5.26}
$$

Next, we will prove that $\{Y_n\}_{n \geq 1}$ is equi-continuous in $C([0, T], \mathbb{R}^d)$. For $t > s$,

$$
Y_n(t) - Y_n(s) = \int_s^t b(r, Y_n(r), \text{Law}(X^0(r)))dr + \int_s^t \sigma(r, Y_n(r), \text{Law}(X^0(r)))\phi_n(r)dr \\
+ \int_s^t \int Z G(r, Y_n(r), \text{Law}(X^0(r)), z)(\psi_n(r, z) - 1)\nu(dz)dr. \tag{5.27}
$$

By (A1), (5.26) and Remark 5.1 we have

$$
\int_s^t |b(r, Y_n(r), \text{Law}(X^0(r)))|dr \leq \int_s^t |b(r, Y_n(r), \text{Law}(X^0(r))) - b(r, 0, \text{Law}(X^0(r)))|dr \\
+ \int_s^t |b(r, 0, \text{Law}(X^0(r))) - b(r, 0, \delta_0)|dr + \int_s^t |b(r, 0, \delta_0)|dr
$$
\[
\leq \int_s^t L(1 + |Y_n(r)|^{q-1})|Y_n(r)|dr + L \int_s^t |X^0(r)|dr + \int_s^t |b(r, 0, \delta_0)|dr \\
\leq L \left(C_m (1 + C_m^{q-1}) + \sup_{r \in [0,T]} |X^0(r)| \right) |t - s| + \int_s^t |b(r, 0, \delta_0)|dr, \tag{5.28}
\]
and
\[
\int_s^t |\sigma(r, Y_n(r), Law(X^0(r)))\phi_n(r)|dr \\
\leq \int_s^t |\sigma(r, Y_n(r), Law(X^0(r)))\phi_n(r) - \sigma(r, 0, \delta_0)\phi_n(r)|dr + \int_s^t |\sigma(r, 0, \delta_0)\phi_n(r)|dr \\
\leq \int_s^t L(1 + |Y_n(r)| + |X^0(r)|)|\phi_n(r)|dr + \int_s^t \|\sigma(r, 0, \delta_0)\|_{L^2} |\phi_n(r)|dr \\
\leq L(1 + C_m + \sup_{r \in [0,T]} |X^0(r)|) \left( \int_0^T |\phi_n(r)|^2 dr \right)^{1/2} |t - s|^{1/2} \\
+ \left( \int_s^t |\sigma(r, 0, \delta_0)|^2 dr \right)^{1/2} \left( \int_0^T |\phi_n(r)|^2 dr \right)^{1/2}. \tag{5.29}
\]

By (B1), (5.26) and Remark 5.1, we have
\[
\int_s^t \int_Z |G(r, Y_n(r), Law(X^0(r)), z) (\psi_n(r, z) - 1)|\nu(dz)dr \\
\leq \int_s^t \int_Z |G(r, Y_n(r), Law(X^0(r)), z) (\psi_n(r, z) - 1) - G(r, 0, \delta_0, z)(\psi_n(r, z) - 1)|\nu(dz)dr \\
+ \int_s^t \int_Z |G(r, 0, \delta_0, z)(\psi_n(r, z) - 1)|\nu(dz)dr \\
\leq \int_s^t \int_Z L_1(z)|Y_n(r)| + |X^0(r)|)|\psi_n(r, z) - 1|\nu(dz)dr + \int_s^t \int_Z L_2(z)|\psi_n(r, z) - 1|\nu(dz)dr \\
\leq (C_m + \sup_{r \in [0,T]} |X^0(r)| + 1) \int_s^t \int_Z (L_1(z) + L_2(z))|\psi_n(r, z) - 1|\nu(dz)dr. \tag{5.30}
\]

To show that the right side of (5.30) is uniformly small, we need the following result, see [61, (3.3) of Lemma 3.1] or [60, Remark 2], or [9, (3.5) of Lemma 3.4].

For every \( \theta > 0 \), there exists some \( \beta > 0 \) such that for any \( A \in \mathcal{B}([0, T]) \) with \( Leb_T(A) \leq \beta \),
\[
\sup_{i=1,2,3} \sup_{\varphi \in \mathcal{S}_2} \int_A \int_Z L_i(z)|\varphi(s, z) - 1|\nu(dz)ds \leq \theta. \tag{5.31}
\]

On the other hand, (A1) implies that, for every \( \theta > 0 \), there exists some \( \beta > 0 \) such that if \( A \in \mathcal{B}([0, T]) \) satisfies \( Leb_T(A) \leq \beta \), then
\[
\int_A |b(r, 0, \delta_0)|dr + \int_A \|\sigma(r, 0, \delta_0)\|_{L^2}^2 dr \leq \theta. \tag{5.32}
\]
Combining (5.27)-(5.32) together, we can deduce that \( \{Y_n\}_{n \geq 1} \) is equi-continuous on \([0, T]\). So \( \{Y_n\}_{n \geq 1} \) is pre-compact in \( C([0, T], \mathbb{R}^d) \).

**Step 2:** Let \( \gamma \) be a limit of some subsequence of \( \{Y_n\}_{n \geq 1} \). We will show that \( \gamma = Y \), completing the proof of the proposition. Without loss of generality, we simply assume
\[
\lim_{n \to \infty} \sup_{t \in [0,T]} |\gamma(t) - Y_n(t)| = 0. \tag{5.33}
\]
First, we note that
\[ \sup_{t \in [0,T]} |\gamma(t)| \leq \sup_{n \geq 1} \sup_{t \in [0,T]} |Y_n(t)| = C_m < \infty. \quad (5.34) \]

By (A1), (5.33) and (5.34),
\[
\int_0^T |b(s, Y_n(s), \text{Law}(X^0(s))) - b(s, \gamma(s), \text{Law}(X^0(s)))| ds \\
\leq L \int_0^T (1 + |Y_n(s)|^{q-1} + |\gamma(s)|^{q-1}) |Y_n(s) - \gamma(s)| ds \\
\leq LT(1 + 2C_m^{q-1}) \sup_{s \in [0,T]} |Y_n(s) - \gamma(s)| \to 0, \quad n \to \infty.
\]
Hence, for each \( t \in [0, T] \),
\[
\int_0^t b(s, Y_n(s), \text{Law}(X^0(s))) ds \to \int_0^t b(s, \gamma(s), \text{Law}(X^0(s))) ds, \quad n \to \infty. \quad (5.35)
\]

Due to (A1), \( X^0 \in C([0,T], \mathbb{R}^d) \), Remark 5.1 and (5.34), it is not difficult to prove that
\[
\int_0^T \|\sigma(s, \gamma(s), \text{Law}(X^0(s)))\|^2_{L^2} ds < \infty.
\]

Since \( \phi_n \) converges to \( \phi \) weakly in \( L^2([0,T], \mathbb{R}^d) \), for any \( e \in \mathbb{R}^d \),
\[
\int_0^t \langle \sigma(s, \gamma(s), \text{Law}(X^0(s)))\phi_n(s), e \rangle ds \to \int_0^t \langle \sigma(s, \gamma(s), \text{Law}(X^0(s)))\phi(s), e \rangle ds, \quad n \to \infty.
\]

Hence,
\[
\int_0^t \sigma(s, \gamma(s), \text{Law}(X^0(s)))\phi_n(s) ds \to \int_0^t \sigma(s, \gamma(s), \text{Law}(X^0(s)))\phi(s) ds, \quad n \to \infty. \quad (5.36)
\]

By (A1) and (5.33) we have
\[
\int_0^T |\sigma(s, Y_n(s), \text{Law}(X^0(s)))\phi_n(s) - \sigma(s, \gamma(s), \text{Law}(X^0(s)))\phi_n(s)| ds \\
\leq L \int_0^T |Y_n(s) - \gamma(s)| |\phi_n(s)| ds \\
\leq LT^{1/2} \sup_{s \in [0,T]} |Y_n(s) - \gamma(s)| \sup_{i \geq 1} \left( \int_0^T |\phi_i(s)|^2 ds \right)^{1/2} \to 0, \quad n \to \infty \quad (5.37)
\]
where we use \( \phi_i \in S^m_1 \), i.e. \( \frac{1}{2} \int_0^T |\phi_i(s)|^2 ds \leq m \), in the last inequality.

Therefore, (5.36) and (5.37) imply that
\[
\int_0^t \sigma(s, Y_n(s), \text{Law}(X^0(s)))\phi_n(s) ds \to \int_0^t \sigma(s, \gamma(s), \text{Law}(X^0(s)))\phi(s) ds, \quad n \to \infty. \quad (5.38)
\]

Applying (B1), (5.14) and (5.33), we have
\[
\int_0^t \int_Z |G(s, Y_n(s), \text{Law}(X^0(s)), z) - G(s, \gamma(s), \text{Law}(X^0(s)), z)||\psi_n(s, z) - 1|\nu(\text{d}z) ds \\
\leq \int_0^t \int_Z |Y_n(s) - \gamma(s)||L_1(z)||\psi_n(s, z) - 1|\nu(\text{d}z) ds
\]
By (B1), Remark 5.1, (5.14) and (5.34), we see that
\[
\int_0^T \int_Z |G(s, \gamma(s), \text{Law}(X_0^0(s)), z)^2 \, \nu(dz) \, ds < \infty.
\]
Combining the above inequality with \(\psi_n \to \psi\) in \(S^m\), and using Lemma 3.11 in [9], we deduce that
\[
\lim_{n \to \infty} \int_0^T \int_Z G(s, \gamma(s), \text{Law}(X_0^0(s)), z)(\psi_n(s, z) - 1) \, \nu(dz) \, ds = \int_0^T \int_Z G(s, \gamma(s), \text{Law}(X_0^0(s)), z)(\psi(s, z) - 1) \, \nu(dz) \, ds.
\]
This, together with (5.39), yields that
\[
\lim_{n \to \infty} \int_0^T \int_Z G(s, Y_n(s), \text{Law}(X_0^0(s)), z)(\psi_n(s, z) - 1) \, \nu(dz) \, ds = \int_0^T \int_Z G(s, \gamma(s), \text{Law}(X_0^0(s)), z)(\psi(s, z) - 1) \, \nu(dz) \, ds.
\]
Recall that \(Y_n\) is the solution of (5.5) with \(u\) replaced by \(u_n\):
\[
Y_n(t) = h + \int_0^t b(s, Y_n(s), \text{Law}(X_0^0(s))) \, ds + \int_0^t \sigma(s, Y_n(s), \text{Law}(X_0^0(s))) \phi_n(s) \, dW(s)
\]
\[+ \int_0^t \int_Z G(s, Y_n(s), \text{Law}(X_0^0(s)), z)(\psi_n(s, z) - 1) \, \nu(dz) \, ds, \quad t \in [0, T].
\]
Letting \(n \to \infty\) and taking into account (5.33), (5.35), (5.38) and (5.40), we see that \(\gamma\) is a solution to (5.5), and the uniqueness of the solutions of (5.5) implies that \(\gamma = Y\), which completes the proof. \(\square\)

To verify (LDP2), we need the following result.

**Lemma 5.9** There exists some \(\epsilon_0 > 0\) and a constant \(C_T\) independent of \(\epsilon\) such that
\[
\mathbb{E} \left( \sup_{t \in [0, T]} |X^\epsilon(t) - X_0^0(t)|^2 \right) \leq C_T \left( \epsilon + \epsilon^2 \tilde{\varrho}_{b, \epsilon} + \epsilon^2 \tilde{\varrho}_{\sigma, \epsilon} + \epsilon^2 \tilde{\varrho}_{G, \epsilon} \right), \quad \forall \epsilon \in (0, \epsilon_0]
\]
where \(\tilde{\varrho}_{b, \epsilon}\) and \(\tilde{\varrho}_{\sigma, \epsilon}\) are the constants given in (A3), and \(\tilde{\varrho}_{G, \epsilon}\) in (B1).

**Proof** By Itô’s formula,
\[
|X^\epsilon(t) - X_0^0(t)|^2
\]
\[= 2 \int_0^t \langle b(\epsilon, X^\epsilon(s), \text{Law}(X^\epsilon(s))) - b(s, X_0^0(s), \text{Law}(X_0^0(s))), X^\epsilon(s) - X_0^0(s) \rangle \, ds
\]
\[+ 2\sqrt{\epsilon} \int_0^t \langle \sigma(\epsilon, X^\epsilon(s), \text{Law}(X^\epsilon(s))), X^\epsilon(s) - X_0^0(s) \rangle \, dW(s)
\]
\[+ 2\epsilon \int_0^t \int_Z \langle G(\epsilon, X^\epsilon(s-), \text{Law}(X^\epsilon(s))), z \rangle, X^\epsilon(s-) - X_0^0(s-) \rangle \, N^{-1}(dz, ds)
\]
\[+ \epsilon \int_0^t \|\sigma(\epsilon, X^\epsilon(s), \text{Law}(X^\epsilon(s)))\|_{L_2}^2 \, ds
\]
\[ +\varepsilon^2 \int_0^T \int_Z |G_\varepsilon(s, X^\varepsilon(s^-), Law(X^\varepsilon(s)), z)|^2 N^{-1}(dz, ds) \]
\[ =: I_1(t) + I_2(t) + I_3(t) + I_4(t) + I_5(t). \]

By (A1), (A3) and Remark 5.1, we have

\[ I_1(t) \]
\[ \leq 2 \int_0^T |b_\varepsilon(s, X^\varepsilon(s), Law(X^\varepsilon(s))) - b(s, X^\varepsilon(s), Law(X^\varepsilon(s))), X^\varepsilon(s) - X^0(s)| ds \]
\[ + 2 \int_0^T |b(s, X^\varepsilon(s), Law(X^\varepsilon(s))) - b(s, X^0(s), Law(X^\varepsilon(s))), X^\varepsilon(s) - X^0(s)| ds \]
\[ + 2 \int_0^T |b(s, X^0(s), Law(X^\varepsilon(s))) - b(s, X^0(s), Law(X^0(s))), X^\varepsilon(s) - X^0(s)| ds \]
\[ \leq 2\varepsilon_\varepsilon \int_0^T |X^\varepsilon(s) - X^0(s)| ds + 2L \int_0^T |X^\varepsilon(s) - X^0(s)|^2 ds \]
\[ + 2L \int_0^T \mathbb{W}_2^2(Law(X^\varepsilon(s)), Law(X^0(s))) |X^\varepsilon(s) - X^0(s)| ds \]
\[ \leq (3L + 1) \int_0^T |X^\varepsilon(s) - X^0(s)|^2 ds + L \int_0^T \mathbb{W}_2^2(Law(X^\varepsilon(s)), Law(X^0(s))) ds + \varepsilon_{\varepsilon, b}^2 T \]
\[ \leq (3L + 1) \int_0^T |X^\varepsilon(s) - X^0(s)|^2 ds + L \int_0^T \mathbb{E}(|X^\varepsilon(s) - X^0(s)|^2) ds + \varepsilon_{\varepsilon, b}^2 T. \quad (5.42) \]

Hence,

\[ \mathbb{E}(\sup_{t \in [0, T]} I_1(t)) \leq (4L + 1) \mathbb{E} \int_0^T |X^\varepsilon(s) - X^0(s)|^2 ds + \varepsilon_{\varepsilon, b}^2 T. \quad (5.43) \]

Also by (A1), (A3) and Remark 5.1,

\[ \mathbb{E} \left( \sup_{t \in [0, T]} |I_4(t)| \right) \]
\[ = \varepsilon \mathbb{E} \int_0^T \|\sigma_\varepsilon(s, X^\varepsilon(s), Law(X^\varepsilon(s)))\|_{\mathbb{L}_2}^2 ds \]
\[ \leq C\varepsilon \mathbb{E} \int_0^T \|\sigma_\varepsilon(s, X^\varepsilon(s), Law(X^\varepsilon(s))) - \sigma(s, X^\varepsilon(s), Law(X^\varepsilon(s)))\|_{\mathbb{L}_2}^2 ds \]
\[ + C\varepsilon \mathbb{E} \int_0^T \|\sigma(s, X^\varepsilon(s), Law(X^\varepsilon(s))) - \sigma(s, X^0(s), Law(X^\varepsilon(s)))\|_{\mathbb{L}_2}^2 ds \]
\[ + C\varepsilon \int_0^T \|\sigma(s, X^0(s), Law(X^0(s)))\|_{\mathbb{L}_2}^2 ds \]
\[\begin{align*}
&\leq C\epsilon^2 \mathbb{E}_{\sigma, \epsilon} T + CL^2 \epsilon \mathbb{E} \int_0^T \left( |X^\epsilon(s) - X^0(s)|^2 + \mathbb{V}^2(X^\epsilon(s), X^0(s)) \right) ds \\
&\quad + C \epsilon \int_0^T \|\sigma(s, X^0(s), X^0(s))\|_{L^2}^2 ds \\
&\leq C\epsilon^2 \mathbb{E}_{\sigma, \epsilon} T + CL^2 T \epsilon \mathbb{E} \left( \sup_{t \in [0, T]} |X^\epsilon(s) - X^0(s)|^2 \right) \\
&\quad + C \epsilon \int_0^T \|\sigma(s, X^0(s), X^0(s))\|_{L^2}^2 ds. \\
\end{align*}\] (5.44)

By Burkholder-Davis-Gundy’s inequality, Young’s inequality and (5.44), we have

\[\begin{align*}
&\mathbb{E} \left( \sup_{t \in [0, T]} |I_2(t)| \right) \\
&\leq C \epsilon \sqrt{\epsilon} \mathbb{E} \left( \int_0^T \|\sigma(s, X^\epsilon(s), X^0(s))\|_{L^2}^2 |X^\epsilon(s) - X^0(s)|^2 ds \right)^{1/2} \\
&\leq \frac{1}{5} \mathbb{E} \left( \sup_{t \in [0, T]} |X^\epsilon(s) - X^0(s)|^2 \right) + C \epsilon \mathbb{E} \int_0^T \|\sigma(s, X^\epsilon(s), X^0(s))\|_{L^2}^2 ds. \\
&\leq \frac{1}{5} + CL^2 T \epsilon \mathbb{E} \left( \sup_{t \in [0, T]} |X^\epsilon(s) - X^0(s)|^2 \right) + C \epsilon^2 \mathbb{E}_{\sigma, \epsilon} T \\
&\quad + C \epsilon \int_0^T \|\sigma(s, X^0(s), X^0(s))\|_{L^2}^2 ds. \\
\end{align*}\] (5.45)

It follows from Remark 5.1 and (B1) that

\[\begin{align*}
&\mathbb{E} \left( \sup_{t \in [0, T]} |I_2(t)| \right) \\
&= \epsilon \mathbb{E} \left( \int_0^T \int_Z |G_{\epsilon}(s, X^\epsilon(s), X^0(s), z)|^2 \nu(dz) ds \right) \\
&\leq C \epsilon \mathbb{E} \left( \int_0^T \int_Z |G_{\epsilon}(s, X^\epsilon(s), X^0(s), z) - G(s, X^\epsilon(s), X^0(s), z)|^2 \nu(dz) ds \right) \\
&\quad + C \epsilon \mathbb{E} \left( \int_0^T \int_Z |G(s, X^\epsilon(s), X^0(s), z) - G(s, X^0(s), X^0(s), z)|^2 \nu(dz) ds \right) \\
&\quad + C \epsilon \int_0^T \int_Z |G(s, X^0(s), X^0(s), z)|^2 \nu(dz) ds \\
&\leq C \epsilon^2 \mathbb{E}_{\sigma, \epsilon} \epsilon \int Z L^2(z) \nu(dz) + C \epsilon \int_0^T \int_Z |G(s, X^0(s), X^0(s), z)|^2 \nu(dz) ds \\
&\quad + C \epsilon \mathbb{E} \left( \int_0^T \int_Z \left( |X^\epsilon(s) - X^0(s)|^2 + \mathbb{V}^2(X^\epsilon(s), X^0(s)) \right) \nu(dz) ds \right) \\
&\leq C \epsilon^2 \mathbb{E}_{\sigma, \epsilon} \epsilon \int Z L^2(z) \nu(dz) + C \epsilon \int_0^T \int_Z |G(s, X^0(s), X^0(s), z)|^2 \nu(dz) ds \\
&\quad + C \epsilon T \int Z L^2(z) \nu(dz) \mathbb{E} \left( \sup_{t \in [0, T]} |X^\epsilon(s) - X^0(s)|^2 \right). \\
\end{align*}\] (5.46)
By Burkholder-Davis-Gundy’s inequality, Young’s inequality and (5.46), one can obtain

\[
E \left( \sup_{t \in [0,T]} |I_3(t)| \right) 
\leq C \epsilon E \left( \int_0^T \int_Z |G_\epsilon(s, X^\epsilon(s), \text{Law}(X^\epsilon(s)), z)|^2 |X^\epsilon(s) - X^0(s)|^2 N^{-1} (dz, ds) \right)^{\frac{1}{2}} 
\leq \frac{1}{5} E \left( \sup_{s \in [0,T]} |X^\epsilon(s) - X^0(s)|^2 \right) + C \epsilon E \left( \int_0^T \int_Z |G_\epsilon(s, X^\epsilon(s), \text{Law}(X^\epsilon(s)), z)|^2 \nu(dz, ds) \right) 
\leq \left( \frac{1}{5} + C \epsilon T \int_Z L_1^2(z) \nu(dz) \right) E \left( \sup_{s \in [0,T]} |X^\epsilon(s) - X^0(s)|^2 \right) 
+ C \epsilon \int_0^T \int_Z |G(s, X^0(s), \text{Law}(X^0(s)), z)|^2 \nu(dz, ds) + C \epsilon \int_0^T \int_Z L_3^2(z) \nu(dz) T. 
\]

(5.47)

Based on the above estimates, it holds

\[
\left( \frac{3}{5} - CL^2 T \epsilon - C \epsilon T \int_Z L_1^2(z) \nu(dz) \right) E \left( \sup_{t \in [0,T]} |X^\epsilon(t) - X^0(t)|^2 \right) 
\leq (4L + 1) E \int_0^T |X^\epsilon(s) - X^0(s)|^2 ds + C \epsilon \int_0^T \|\sigma(s, X^0(s), \text{Law}(X^0(s)))\|^2 L_2 ds 
+ C \epsilon \int_0^T \int_Z |G(s, X^0(s), \text{Law}(X^0(s)), z)|^2 \nu(dz, ds) 
+ \epsilon^2_{\sigma, \epsilon} T + C \epsilon \int_0^T \int_Z L_3^2(z) \nu(dz) T. 
\]

(5.48)

By (A1), (B1) and using the fact that \( f_2(L_1^2(z) + L_2^2(z)) \nu(dz) < \infty \), we can prove that

\[
\int_0^T (\|\sigma(s, X^0(s), \text{Law}(X^0(s)))\|^2 _{L_2} + \int_Z |G(s, X^0(s), \text{Law}(X^0(s)), z)|^2 \nu(dz, ds) ) ds < \infty, 
\]

(5.49)

and there exists \( \epsilon_0 > 0 \) small enough such that, for any \( \epsilon \in (0, \epsilon_0) \),

\[
\frac{3}{5} - C L^2 T \epsilon - C \epsilon T \int_Z L_1^2(z) \nu(dz) \geq \frac{1}{5}. 
\]

(5.50)

Hence, by (5.48), (5.49), (5.50) and Gronwall’s inequality, there exists some constant \( C_T > 0 \) such that, for any \( \epsilon \in (0, \epsilon_0) \),

\[
E \left( \sup_{t \in [0,T]} |X^\epsilon(t) - X^0(t)|^2 \right) \leq C_T (\epsilon + \epsilon^2_{\sigma, \epsilon} + \epsilon^2 G, \epsilon),
\]

which is the desired result.

Next we will verify (LDP2).

**Proposition 5.10** For any given \( m \in (0, \infty) \), let \( \{u_\epsilon = (\phi_\epsilon, \psi_\epsilon), \epsilon > 0\} \subset S^m_1 \times S^m_2 \). Then, for the solution \( Z^{u_\epsilon} \) to (5.25),

\[
\lim_{\epsilon \to 0} E \left( \sup_{t \in [0,T]} |Z^{u_\epsilon}(t) - \Gamma^0(u_\epsilon)(t)|^2 \right) = 0.
\]

\( \square \) Springer
Proof Let $Y^u$ be the solution of (5.5) with $u$ replaced by $u$. Then $\Gamma^0(u) = Y^u$.
By (5.25) and (5.5), we have

\[
Z^u(t) - Y^u(t)
= \int_0^t \left( b_\varepsilon(s, Z^u(s), \text{Law}(X^\varepsilon(s))) - b(s, Y^u(s), \text{Law}(X^0(s))) \right) ds
+ \sqrt{\varepsilon} \int_0^t \sigma_\varepsilon(s, Z^u(s), \text{Law}(X^\varepsilon(s))) dW(s)
+ \int_0^t \left( \sigma_\varepsilon(s, Z^u(s), \text{Law}(X^\varepsilon(s))) - \sigma(s, Y^u(s), \text{Law}(X^0(s))) \right) \phi_\varepsilon(s) ds
+ \varepsilon \int_0^t \left( G_\varepsilon(s, Z^u(s), \text{Law}(X^\varepsilon(s)), z) - G(s, Y^u(s), \text{Law}(X^0(s)), z) \right) \left( \psi_\varepsilon(s, z) - 1 \right) \nu(dz) ds
+ \varepsilon \int_0^t \left( G_\varepsilon(s, Z^u(s-), \text{Law}(X^\varepsilon(s)), z) \right) N^{-1} \psi_\varepsilon(dz, ds).
\]

By Itô’s formula,

\[
|Z^u(t) - Y^u(t)|^2
= 2 \int_0^t \left( Z^u(s) - Y^u(s), b_\varepsilon(s, Z^u(s), \text{Law}(X^\varepsilon(s))) - b(s, Y^u(s), \text{Law}(X^0(s))) \right) ds
+ 2 \sqrt{\varepsilon} \int_0^t \left( Z^u(s) - Y^u(s), \sigma_\varepsilon(s, Z^u(s), \text{Law}(X^\varepsilon(s))) dW(s) \right)
+ 2 \int_0^t \left( Z^u(s) - Y^u(s), \left( \sigma_\varepsilon(s, Z^u(s), \text{Law}(X^\varepsilon(s))) - \sigma(s, Y^u(s), \text{Law}(X^0(s))) \right) \phi_\varepsilon(s) ds \right)
+ 2 \int_0^t \left( Z^u(s) - Y^u(s), \left( G_\varepsilon(s, Z^u(s), \text{Law}(X^\varepsilon(s)), z) - G(s, Y^u(s), \text{Law}(X^0(s)), z) \right) \left( \psi_\varepsilon(s, z) - 1 \right) \nu(dz) ds \right)
+ 2 \varepsilon \int_0^t \left( |Z^u(s) - Y^u(s)|^2 \right) N^{-1} \psi_\varepsilon(dz, ds)
\]

=: J_1(t) + J_2(t) + J_3(t) + J_4(t) + J_5(t) + J_6(t) + J_7(t).

(5.51)

By (B1), Remark 5.1 and (5.14),

\[
J_4(t)
\leq 2 \int_0^t \int_Z |Z^u(s) - Y^u(s)| |G_\varepsilon(s, Z^u(s), \text{Law}(X^\varepsilon(s)), z) - G(s, Z^u(s), \text{Law}(X^\varepsilon(s))), z)|
\left| \psi_\varepsilon(s, z) - 1 \right| \nu(dz) ds
+ C \int_0^t \left( |Z^u(s) - Y^u(s)|^2 + |Z^u(s) - Y^u(s)| \sqrt{\nu} (\text{Law}(X^\varepsilon(s)), \text{Law}(X^0(s))) \right)
\cdot L_1(z) |\psi_\varepsilon(s, z) - 1| \nu(dz) ds
\]
\[ \begin{align*}
\mathbb{E} \sup_{t \in [0,T]} |Z^{u\epsilon}(t) - Y^{u\epsilon}(t)|^2 & \leq C \mathbb{E} \left( \sup_{s \in [0,T]} |X^{\epsilon}(s) - X^0(s)|^2 \right) \mathbb{E} \sup_{\varphi \in \mathcal{S}_2} \int_0^T \int_Z |L_1(z)| \varphi(s, z) - 1 |\nu(\text{d}z) \text{d}s \\
& \quad + C \varepsilon \mathbb{E} \left( \sup_{s \in [0,T]} |X^{\epsilon}(s) - X^0(s)|^2 + |Z^{u\epsilon}(s) - Y^{u\epsilon}(s)|^2 \right) \mathbb{E} \sup_{\varphi \in \mathcal{S}_2} \int_0^T \int_Z |L_1(z)| \varphi(s, z) - 1 |\nu(\text{d}z) \text{d}s \\
& \quad + \frac{\varepsilon^2}{2} \mathbb{E} \sup_{\varphi \in \mathcal{S}_2} \int_0^T \int_Z |L_3(z)| \varphi(s, z) - 1 |\nu(\text{d}z) \text{d}s \\
& \leq C \int_0^T \int_Z |Z^{u\epsilon}(s) - Y^{u\epsilon}(s)|^2 (|L_1(z)| + |L_3(z)|) |\varphi(s, z) - 1 |\nu(\text{d}z) \text{d}s \\
& \quad + \frac{\varepsilon^2}{2} \left( C \mathbb{E} \sup_{s \in [0,T]} |X^{\epsilon}(s) - X^0(s)|^2 + |Z^{u\epsilon}(s) - Y^{u\epsilon}(s)|^2 \right).
\end{align*} \]

Set
\[ J := \sup_{t \in [0,T]} \left( J_1(t) + J_2(t) + J_3(t) + J_4(t) + J_5(t) + J_6(t) + J_7(t) \right). \]

Plugging the above inequality into (5.51), by Gronwall’s inequality and (5.14), we arrive at
\[ \mathbb{E} \sup_{t \in [0,T]} |Z^{u\epsilon}(t) - Y^{u\epsilon}(t)|^2 \leq C \exp\left( C \sup_{\varphi \in \mathcal{S}_2} \int_0^T \int_Z (|L_1(z)| + |L_3(z)|) |\varphi(s, z) - 1 |\nu(\text{d}z) \text{d}s \right) \times \left( C \left( \mathbb{E} \left( \sup_{s \in [0,T]} |X^{\epsilon}(s) - X^0(s)|^2 \right) + \frac{\varepsilon^2}{2} \right) + J \right) \]
\[ \leq C \left( \mathbb{E} \left( \sup_{s \in [0,T]} |X^{\epsilon}(s) - X^0(s)|^2 \right) + \frac{\varepsilon^2}{2} + J \right). \] (5.52)

By using (A1), Lemma 5.9, (A3), (B1) and Burkholder-Davis-Gundy’s inequality, it follows from (5.52) that
\[ \mathbb{E} \left( \sup_{t \in [0,T]} |Z^{u\epsilon}(t) - Y^{u\epsilon}(t)|^2 \right) \leq C (\varepsilon + \varepsilon^2 \mathbb{E} \sup_{\varphi \in \mathcal{S}_2} \int_0^T |Z^{u\epsilon}(s) - Y^{u\epsilon}(s)|^2 \text{d}s ) \]
\[ + C \mathbb{E} \left( \int_0^T \left( |Z^{u\epsilon}(s) - Y^{u\epsilon}(s)|^2 + |Z^{u\epsilon}(s) - Y^{u\epsilon}(s)| \mathcal{W}_2(Law(X^{\epsilon}(s)), Law(X^0(s))) \right) \text{d}s \right) \]
\[ + C \sqrt{\varepsilon} \mathbb{E} \left( \int_0^T |Z^{u\epsilon}(s) - Y^{u\epsilon}(s)|^2 \left\| \sigma(s, Z^{u\epsilon}(s), Law(X^{\epsilon}(s))) \right\|_{\mathcal{L}_2}^2 \text{d}s \right)^{\frac{1}{2}} \]
\[ + C \mathbb{E} \sup_{\varphi \in \mathcal{S}_2} \int_0^T |Z^{u\epsilon}(s) - Y^{u\epsilon}(s)| |\varphi(s)| \text{d}s \]
\[ + C \mathbb{E} \left( \int_0^T |Z^{u\epsilon}(s) - Y^{u\epsilon}(s)| \left\| \sigma(s, Z^{u\epsilon}(s), Law(X^{\epsilon}(s))) - \sigma(s, Y^{u\epsilon}(s), Law(X^0(s))) \right\|_{\mathcal{L}_2} |\varphi(s)| \text{d}s \right) \]
\[ + C \mathbb{E} \left( \int_0^T |Z^{u\epsilon}(s) - Y^{u\epsilon}(s)| \left\| \sigma(s, Z^{u\epsilon}(s), Law(X^{\epsilon}(s))) - \sigma(s, Y^{u\epsilon}(s), Law(X^0(s))) \right\|_{\mathcal{L}_2} \right)^{\frac{1}{2}} \]
\[ + C \mathbb{E} \left( \int_0^T \left\| \sigma(s, Z^{u\epsilon}(s), Law(X^{\epsilon}(s))) \right\|_{\mathcal{L}_2}^2 \text{d}s \right). \]
\[ + C \epsilon E \int_0^T \int_Z |G_\epsilon(s, Z^\mu(s), \text{Law}(X^\mu(s)), z)|^2 |\psi_\epsilon(s, z)| \nu(\mathrm{d}z) \mathrm{d}s \]

\[ := C(\epsilon + \epsilon \sigma_\epsilon^2 + \epsilon \sigma_G^2 + \epsilon \sigma_{G, \epsilon}^2) \]

\[ + I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8, \quad \forall \epsilon \in (0, \epsilon_0). \] (5.53)

where \(\epsilon_0\) is the constant appearing in Lemma 5.9.

In the sequel, let \(\epsilon \in (0, \epsilon_0)\).

Keeping in mind that \(\phi_\epsilon \in S_{1}^m\), we have

\[ I_1 + I_4 \leq C \int_0^T \left( \sup_{\epsilon \in [0,sl]} |Z^{\mu_\epsilon}(r) - Y^{\mu_\epsilon}(r)|^2 \right) \mathrm{d}s + C \sigma_{b,\epsilon}^2 T + C \sigma_{\sigma,\epsilon}^2 \int_0^T |\phi_\epsilon(s)|^2 \mathrm{d}s \]

\[ \leq C \int_0^T \left( \sup_{\epsilon \in [0,sl]} |Z^{\mu_\epsilon}(r) - Y^{\mu_\epsilon}(r)|^2 \right) \mathrm{d}s + C \sigma_{b,\epsilon}^2 T + C \sigma_{\sigma,\epsilon}^2. \] (5.54)

By Young’s inequality and Lemma 5.9,

\[ I_2 \leq C \int_0^T \left( \sup_{\epsilon \in [0,sl]} |Z^{\mu_\epsilon}(r) - Y^{\mu_\epsilon}(r)|^2 \right) \mathrm{d}s + C \int_0^T \sup_{\epsilon \in [0,sl]} |Z^{\mu_\epsilon}(r) - Y^{\mu_\epsilon}(r)|^2 \mathrm{d}s + C \text{E} \left( \sup_{\epsilon \in [0,sl]} |X_\epsilon(s) - X^0(s)|^2 \right) \]

\[ \leq C \int_0^T \left( \sup_{\epsilon \in [0,sl]} |Z^{\mu_\epsilon}(r) - Y^{\mu_\epsilon}(r)|^2 \right) \mathrm{d}s + C \text{E} \left( \sup_{\epsilon \in [0,sl]} |X_\epsilon(s) - X^0(s)|^2 \right) \]

\[ \leq C \int_0^T \left( \sup_{\epsilon \in [0,sl]} |Z^{\mu_\epsilon}(r) - Y^{\mu_\epsilon}(r)|^2 \right) \mathrm{d}s + C(\epsilon + \sigma_{b,\epsilon}^2 + \sigma_{\sigma,\epsilon}^2 + \sigma_{G,\epsilon}^2). \] (5.55)

Using (A3) and (5.6), we have

\[ I_3 + I_7 \]

\[ \leq \frac{1}{10} \text{E} \left( \sup_{\epsilon \in [0,T]} |Z^{\mu_\epsilon}(t) - Y^{\mu_\epsilon}(t)|^2 \right) + C \epsilon \text{E} \int_0^T \|\sigma_\epsilon(s, Z^{\mu_\epsilon}(s), \text{Law}(X^\mu(s)))\|_{L_2}^2 \mathrm{d}s \]

\[ \leq \frac{1}{10} \text{E} \left( \sup_{\epsilon \in [0,T]} |Z^{\mu_\epsilon}(t) - Y^{\mu_\epsilon}(t)|^2 \right) \]

\[ + C \epsilon \text{E} \int_0^T \|\sigma_\epsilon(s, Z^{\mu_\epsilon}(s), \text{Law}(X^\mu(s))) - \sigma(s, Z^{\mu_\epsilon}(s), \text{Law}(X^\mu(s)))\|_{L_2}^2 \mathrm{d}s \]

\[ + C \epsilon \text{E} \int_0^T \|\sigma(s, Z^{\mu_\epsilon}(s), \text{Law}(X^\mu(s))) - \sigma(s, Y^{\mu_\epsilon}(s), \text{Law}(X^\mu(0)))\|_{L_2}^2 \mathrm{d}s \]

\[ \leq \frac{1}{10} \text{E} \left( \sup_{\epsilon \in [0,T]} |Z^{\mu_\epsilon}(t) - Y^{\mu_\epsilon}(t)|^2 \right) + C \epsilon \sigma_{\sigma,\epsilon}^2 T + C \epsilon \text{E} \int_0^T \sup_{\epsilon \in [0,sl]} |Z^{\mu_\epsilon}(s) - Y^{\mu_\epsilon}(s)|^2 \mathrm{d}s \]

\[ + C \epsilon \text{E} \left( \sup_{\epsilon \in [0,T]} |X_\epsilon(s) - X^0(s)|^2 \right) \]

\[ \leq \frac{1}{10} + C \epsilon \text{E} \left( \sup_{\epsilon \in [0,T]} |Z^{\mu_\epsilon}(t) - Y^{\mu_\epsilon}(t)|^2 \right) + C(\epsilon + \epsilon^2 + \sigma_{b,\epsilon}^2 + \sigma_{\sigma,\epsilon}^2 + \sigma_{G,\epsilon}^2). \] (5.56)
Again by (A1), Young’s inequality and Lemma 5.9, and using the fact that \( \phi \in S^m_1 \), we have

\[
I_5 \leq C \mathbb{E}\left\{ \sup_{t \in [0,T]} |Z^{u*}(t) - Y^{u*}(t)| \left( \int_0^T |\phi(s)|^2 ds \right)^{\frac{1}{2}} \right\}
\]

\[
\left( \int_0^T \|\sigma(s, Z^{u*}(s), Law(X^*(s))) - \sigma(s, Y^{u*}(s), Law(X^0(s)))\|_{L^2_z}^2 ds \right)^{\frac{1}{2}} \right\}
\]

\[
\leq \frac{1}{10} \mathbb{E}\left( \sup_{t \in [0,T]} |Z^{u*}(t) - Y^{u*}(t)|^2 \right) + C \int_0^T \mathbb{E}|Z^{u*}(s) - Y^{u*}(s)|^2 ds
\]

\[
+ C \int_0^T \mathbb{E}|\nabla_2^z (Law(X^*(s)), Law(X^0(s)))| ds
\]

\[
\leq \frac{1}{10} \mathbb{E}\left( \sup_{t \in [0,T]} |Z^{u*}(t) - Y^{u*}(t)|^2 \right) + C \int_0^T \mathbb{E}\left( \sup_{r \in [0,s]} |Z^{u*}(r) - Y^{u*}(r)|^2 \right) ds
\]

\[
+ C \left( \epsilon + \Theta_\delta^2 + \epsilon \Theta_\delta^2 + \epsilon \Theta_\delta^2 \right). \quad (5.57)
\]

By Hölder’s inequality, (B1), (5.6) and Lemma 5.9,

\[
I_6 + I_8
\]

\[
\leq \frac{1}{10} \mathbb{E}\left( \sup_{t \in [0,T]} |Z^{u*}(t) - Y^{u*}(t)|^2 \right) + C \mathbb{E}\left( \int_0^T |G_*(s, Z^{u*}(s), Law(X^*(s)), z)|^2 |\psi_*(s, z)| |\nu(\cdot)| dz \right) ds
\]

\[
+ C \mathbb{E}\left( \int_0^T \int_Z |G_*(s, Z^{u*}(s), Law(X^*(s)), z) - G(s, Z^{u*}(s), Law(X^0(s)), z)|^2 |\psi_*(s, z)| |\nu(\cdot)| dz \right) ds
\]

\[
+ C \mathbb{E}\left( \int_0^T \int_Z |G(s, Y^{u*}(s), Law(X^0(s)), z) - G(s, 0, \delta_0(z))^2 |\psi_*(s, z)| |\nu(\cdot)| dz \right) ds
\]

\[
+ C \mathbb{E}\left( \int_0^T \int_Z |G(s, 0, \delta_0(z))^2 |\psi_*(s, z)| |\nu(\cdot)| \right) ds
\]

\[
\leq \frac{1}{10} \mathbb{E}\left( \sup_{t \in [0,T]} |Z^{u*}(t) - Y^{u*}(t)|^2 \right) + C \Theta_m \mathbb{E}\left( \sup_{t \in [0,T]} |Z^{u*}(t) - Y^{u*}(t)|^2 \right)
\]

\[
+ C \Theta_m \left\{ \mathbb{E}\left( \sup_{t \in [0,T]} |Y^{u*}(t)|^2 \right) + \mathbb{E}\left( \sup_{t \in [0,T]} |X^*(t) - X^0(t)|^2 \right) \right\}
\]

\[
\leq \left( \frac{1}{10} + C \Theta_m \right) \mathbb{E}\left( \sup_{t \in [0,T]} |Z^{u*}(t) - Y^{u*}(t)|^2 \right) + C \Theta_m \left( 1 + \epsilon^2 + \epsilon \Theta_\delta^2 + \epsilon \Theta_\delta^2 + \epsilon \Theta_\delta^2 + \epsilon \Theta_\delta^2 \right). \quad (5.58)
\]

Here

\[
\Theta_m := \sup_{\psi \in S^m_z} \int_0^T \int_Z \left( L_1^2(z) + L_2^2(z) + L_3^2(z) \right) (\psi(s, z) + 1) \nu(\cdot) \right) ds < \infty.
\]
See (3.3) in [9, Lemma 3.4]. Combining (5.53)–(5.58) together, we arrive at

\[
\left( \frac{7}{10} - C \epsilon - C \epsilon \Theta_m \right) \mathbb{E} \left( \sup_{t \in [0,T]} |Z^{\mu \epsilon}(t) - Y^{\mu \epsilon}(t)|^2 \right) \leq C \epsilon + C \int_0^T \mathbb{E} \left( \sup_{r \in [0,s]} |Z^{\mu \epsilon}(r) - Y^{\mu \epsilon}(r)|^2 \right) ds + C (q_{b, \epsilon}^2 + q_{\sigma, \epsilon}^2 + q_{G, \epsilon}^2).
\]

Hence, by Gronwall’s inequality we have

\[
\lim_{\epsilon \to 0} \mathbb{E} \left( \sup_{t \in [0,T]} |Z^{\mu \epsilon}(t) - Y^{\mu \epsilon}(t)|^2 \right) = 0,
\]

which completes the proof. \( \square \)

5.2 Moderate Deviation Principle

For any \( t \in [0, T] \) and \( \mu \in \mathcal{P}_2 \), let \( b'_2(t, x, \mu) \) denote the derivative of \( b(t, x, \mu) \) with respect to the variable \( x \). In order to obtain the MDP for the solution \( \{X^\epsilon, \epsilon > 0\} \) to (5.1), we give the following assumption.

**(B2)** There are \( L', q' \geq 0 \) such that for each \( x, x' \in \mathbb{R}^d \),

\[
|b'_2(s, x, \text{Law}(X^0(s))) - b'_2(s, x', \text{Law}(X^0(s)))| \leq L'(1 + |x|^{q'} + |x'|^{q'})|x - x'|,
\]

and

\[
\int_0^T |b'_2(t, X^0(t), \text{Law}(X^0(t)))| dt < \infty.
\]

Before stating the main result, we first present the following result.

**Proposition 5.11** Assume that (A1), (B1) and (B2) hold. Then for any fixed \( m \in (0, \infty) \) and \( u = (\phi, \varphi) \in S^m_1 \times B_2(m) \), there is a unique solution \( K^u = \{K^u(t), t \in [0, T]\} \in C([0, T], \mathbb{R}^d) \) to the following equation,

\[
\begin{align*}
\frac{dK^u(t)}{dt} &= b'_2(t, X^0(t), \text{Law}(X^0(t)))K^u(t)dt + \sigma(t, X^0(t), \text{Law}(X^0(t)))\phi(t)dt \\
&\quad + \int_Z G(t, X^0(t), \text{Law}(X^0(t)), z)\varphi(t, z)\nu(dz)dt \\
K^u(0) &= 0.
\end{align*}
\]

Moreover,

\[
\sup_{u \in S^m_1 \times B_2(m)} \sup_{t \in [0,T]} |K^u(t)| =: \mathcal{E}_m < \infty.
\]

**Proof** By using (5.49) and the fact that \( u \in S^m_1 \times B_2(m) \), we have

\[
\int_0^T |\sigma(t, X^0(t), \text{Law}(X^0(t)))\phi(t)| dt \leq \left( \int_0^T |\sigma(t, X^0(t), \text{Law}(X^0(t)))|^2 dt \right)^{\frac{1}{2}} \left( \int_0^T |\phi(t)|^2 dt \right)^{\frac{1}{2}} \leq \left( \int_0^T |\sigma(t, X^0(t), \text{Law}(X^0(t)))|^2 dt \right)^{\frac{1}{2}} (2m)^{\frac{1}{2}} < \infty,
\]

and

\[
\int_0^T \int_Z |G(t, X^0(t), \text{Law}(X^0(t)), z)\varphi(t, z)|\nu(dz)dt < \infty.
\]
LDP and MDP for DDSPDEs with Jumps

\[
\begin{align*}
&\leq \left( \int_0^T \int_Z |G(t, X^0(t), \text{Law}(X^0(t)), z)|^2 \nu(dz) dt \right)^{\frac{1}{2}} \left( \int_0^T \int_Z |\varphi(t, z)|^2 \nu(dz) dt \right)^{\frac{1}{2}} \\
&< \infty.
\end{align*}
\] (5.63)

With these two estimates above it is standard to show that the linear equation (5.60) has a unique solution \( \{K^u(t)\}_{t \in [0, T]} \). The estimate (5.61) follows by using Gronwall’s inequality.

Recall \( a(\epsilon) \) in (4.7). For any \( \epsilon \in (0, 1) \), define

\[
M^\epsilon(t) = \frac{1}{a(\epsilon)} (X^\epsilon(t) - X^0(t)), \quad t \in [0, T].
\]

Due to (5.1) and (5.4), \( M^\epsilon \) satisfies

\[
\begin{align*}
M^\epsilon(t) &= \frac{1}{a(\epsilon)} \int_0^t \left( b(\epsilon, a)M^\epsilon(s) + X^0(s), \text{Law}(X^0(s))) - b(s, X^0(s), \text{Law}(X^0(s))) \right) ds \\
&\quad + \frac{\sqrt{\epsilon}}{a(\epsilon)} \int_0^t \sigma(\epsilon, a)M^\epsilon(s) + X^0(s), \text{Law}(X^0(s))) dW(s) \\
&\quad + \frac{\epsilon}{a(\epsilon)} \int_0^t \int_Z G(\epsilon, a)M^\epsilon(s-) + X^0(s-), \text{Law}(X^0(s), z)\mathcal{N}^{-1}(dz, ds).
\end{align*}
\] (5.64)

We introduce the following assumption.

(B3) \( \lim_{\epsilon \to 0} \frac{\partial b_\epsilon}{\partial \epsilon} = 0 \), where \( \partial b_\epsilon \) is given in (A3).

Now we state the main result in this subsection.

**Theorem 5.12** Assume that (A0), (A1), (A3), (A4), (B1), (B2) and (B3) hold. Then \( \{M^\epsilon, \epsilon > 0\} \) satisfies a LDP on \( D([0, T], \mathbb{R}^d) \) with speed \( \varepsilon \) and the rate function \( I \) given by for any \( g \in D([0, T], \mathbb{R}^d) \)

\[
I(g) := \inf_{u=(\phi, \varphi) \in L^2([0, T], \mathbb{R}^d) \times L^2(\nu_T), K^u=g} \left\{ \frac{1}{2} \int_0^T \phi(s)^2 ds + \frac{1}{2} \int_0^T \int_Z \varphi(s, z)^2 \nu(dz) ds \right\},
\]

where for \( u = (\phi, \varphi) \in L^2([0, T], \mathbb{R}^d) \times L^2(\nu_T), K^u \) is the unique solution of (5.60). Here we use the convention that the infimum of an empty set is \( \infty \).

**Proof** By Proposition 5.11, we can define a map

\[
\gamma^0 : L^2([0, T], \mathbb{R}^d) \times L^2(\nu_T) \ni u = (\phi, \varphi) \mapsto K^u \in D([0, T], \mathbb{R}^d),
\] (5.65)

where \( K^u \) is the unique solution of (5.60).
For any \( \epsilon > 0, m \in (0, \infty) \) and \( u_\epsilon = (\phi_\epsilon, \psi_\epsilon) \in S^m_1 \times S^m_{+,\epsilon} \), recall that \( \{M^{u_\epsilon}(t)\}_{t \in [0, T]} \) (see (4.9)) is the solution to the following SDE

\[
\begin{align*}
dM^{u_\epsilon}(s) &= \frac{1}{a(\epsilon)} \left( b_\epsilon(s, a(\epsilon)M^{u_\epsilon}(s) + X^0(s), \text{Law}(X^\epsilon(s))) - b(s, X^0(s), \text{Law}(X^0(s))) \right) \, ds \\
&\quad + \frac{\epsilon}{a(\epsilon)} \sigma_\epsilon(s, a(\epsilon)M^{u_\epsilon}(s) + X^0(s), \text{Law}(X^\epsilon(s))) \, dW(s) \\
&\quad + \frac{\epsilon}{a(\epsilon)} \int_Z G_\epsilon(s, a(\epsilon)M^{u_\epsilon}(s) + X^0(s), \text{Law}(X^\epsilon(s)), z) \mathcal{N}^{-1}(\psi_\epsilon)(dz, ds) \\
&\quad + \frac{1}{a(\epsilon)} \int_Z G_\epsilon(s, a(\epsilon)M^{u_\epsilon}(s) + X^0(s), \text{Law}(X^\epsilon(s)), z)(\psi_\epsilon(s, z) - 1) \nu(dz) \, ds, \\
M^{u_\epsilon}(0) &= 0.
\end{align*}
\]

(5.66)

According to Theorem 4.6, it is sufficient to verify the following two claims:

- **(MDP1)** For any given \( m \in (0, \infty) \), let \( u_n = (\phi_n, \varphi_n), n \in \mathbb{N}, u = (\phi, \psi) \in S^m_1 \times B_2(m) \) be such that \( u_n \to u \) in \( S^m_1 \times B_2(m) \) as \( n \to \infty \). Then
  \[
  \lim_{n \to \infty} \sup_{t \in [0, T]} |Y^0(u_n)(t) - Y^0(u)(t)| = 0.
  \]

- **(MDP2)** For any given \( m \in (0, \infty) \), let \( \{u_\epsilon = (\phi_\epsilon, \psi_\epsilon), \epsilon > 0 \} \subset S^m_1 \times S^m_{+,\epsilon} \), and for some \( \beta \in (0, 1], \varphi_\epsilon 1_{|\varphi_\epsilon| \leq \beta / a(\epsilon)} \in B_2(\sqrt{m}\kappa_2(1)) \) where \( \varphi_\epsilon = (\psi_\epsilon - 1) / a(\epsilon) \).
  Set
  \[
  \tilde{u}_\epsilon := (\phi_\epsilon, \varphi_\epsilon 1_{|\varphi_\epsilon| \leq \beta / a(\epsilon)}).
  \]
  Then for any \( \varpi > 0 \),
  \[
  \lim_{\epsilon \to 0} P \left( \sup_{t \in [0, T]} |M^{u_\epsilon}(t) - Y^0(\tilde{u}_\epsilon)(t)| > \varpi \right) = 0.
  \]

The verification of (MDP1) and (MDP2) will be given in Propositions 5.13 and 5.16 respectively.

Next proposition is the verification of (MDP1).

**Proposition 5.13** For any given \( m \in (0, \infty) \), let \( u_n = (\phi_n, \varphi_n), n \in \mathbb{N}, u = (\phi, \psi) \in S^m_1 \times B_2(m) \) be such that \( u_n \to u \) in \( S^m_1 \times B_2(m) \) as \( n \to \infty \). Then

\[
\lim_{n \to \infty} \sup_{t \in [0, T]} |Y^0(u_n)(t) - Y^0(u)(t)| = 0.
\]

**Proof** Recall that \( K^u = Y^0(u), K^{u_n} = Y^0(u_n) \) are the corresponding solutions to (5.60). We need to prove the following result:

\[
\lim_{n \to \infty} \sup_{t \in [0, T]} |K^{u_n}(t) - K^u(t)| = 0.
\]

The proof is similar to that of Proposition 5.8 and we just give a sketch here. We first show that \( \{K^{u_n}\}_{n \geq 1} \) is pre-compact in \( C([0, T], \mathbb{R}^d) \). (5.61) implies that \( \{K^{u_n}\}_{n \geq 1} \) is uniformly bounded, i.e.

\[
C_m := \sup_{n \geq 1} \sup_{t \in [0, T]} |K^{u_n}(t)| < \infty.
\]

(5.68)

For any \( s, t \in [0, T] \) with \( s < t \), by (5.62), (5.63) and (5.68),

\[
|K^{u_n}(t) - K^{u_n}(s)|
\]
This, together with (5.49) and (B2), implies that \(\{K_{n}\}_{n \geq 1}\) is equi-continuous in \(C([0, T], \mathbb{R}^d)\).

Hence, \(\{K_{n}\}_{n \geq 1}\) is pre-compact in \(C([0, T], \mathbb{R}^d)\).

Let \(K\) be any limit of some subsequence of \(\{K_{n}\}_{n \geq 1}\) in \(C([0, T], \mathbb{R}^d)\). Using the similar arguments as in the proof of Proposition 5.8, we can show \(K = K^a\), which completes the proof.

In order to verify (MDP2), we need the following two lemmas. The first one is taken from Lemma 4.2, Lemma 4.3 and Lemma 4.7 in [3].

**Lemma 5.14** Fix \(m \in (0, \infty)\).

(a) There exists \(\varsigma_m \in (0, \infty)\) such that for all \(I \in \mathcal{B}([0, T])\) and \(\varepsilon \in (0, \infty)\),

\[
\sup_{\phi \in S_{m}} \int_{Z \times I} \left( L_{1}^2(y) + L_{2}^2(y) + L_{3}^2(z) \right) \phi(y, s) \nu(dy)ds \leq \varsigma_m (a^2(\varepsilon) + \text{Leb}_T(I)). \tag{5.69}
\]

(b) There exist \(m, \rho_m : (0, \infty) \rightarrow (0, \infty)\) such that \(m(s) \downarrow 0\) as \(s \uparrow \infty\), and for all \(I \in \mathcal{B}([0, T])\) and \(\beta, \epsilon \in (0, \infty)\),

\[
\sup_{\phi \in S_{\epsilon}} \int_{Z \times I} \left( L_{1}(z) + L_{2}(z) + L_{3}(z) \right) |\phi(y, s)| 1_{\{|p| \geq \beta \epsilon \}}(y, s) \nu(dy)ds \\
\leq m_\beta (1 + \sqrt{\text{Leb}_T(I)}),
\tag{5.70}
\]

and

\[
\sup_{\phi \in S_{\epsilon}} \int_{Z \times I} \left( L_{1}(z) + L_{2}(z) + L_{3}(z) \right) |\phi(y, s)| \nu(dy)ds \\
\leq \rho_m (\beta) \sqrt{\text{Leb}_T(I)} + \Gamma_m (\beta) a(\epsilon). \tag{5.71}
\]

(c) For any \(\beta > 0\),

\[
\lim_{\varepsilon \to 0} \sup_{\phi \in S_{\epsilon}} \int_{Z \times [0, T]} \left( L_{1}(z) + L_{2}(z) + L_{3}(z) \right) |\phi(y, s)| 1_{\{|p| > \beta \varepsilon \}}(y, s) \nu(dy)ds = 0. \tag{5.72}
\]

**Lemma 5.15** Let \(M^{a\epsilon}\) be the solution to (5.66). Then there exists \(\kappa_0 > 0\) such that

\[
\sup_{t \in \{0, \ldots, T\}} \mathbb{E} \sup_{\epsilon \in \{0, \ldots, \kappa_0\}} |M^{a\epsilon}(t)|^2 < \infty. \tag{5.73}
\]

---

Note: The reference [4] is the published version of [3], and the paper [4] considered a little more general assumptions than those in [3], see (2.13) in [4] and (2.13) in [3]. Hence some of the a priori estimates are different between [3] and [4], for example, Lemma 4.2, Lemma 4.3, Lemma 4.7 and Lemma 4.8 of [4] are different than Lemma 4.2, Lemma 4.3, Lemma 4.7 and Lemma 4.8 of [3]. In this paper, we use the same assumption with those in [3], and hence, we use the a priori estimates in [3].
Proof By Itô’s formula, for any \( t \in [0, T] \),

\[
|M^\varepsilon(t)|^2 = \frac{2}{a(\varepsilon)} \int_0^t |b_\varepsilon(s, a(\varepsilon))M^\varepsilon(s) + X^0(s), Law(X^\varepsilon(s)) - b(s, X^0(s), Law(X^0(s)))| M^\varepsilon(s) ds \\
+ \frac{2\sqrt{\varepsilon}}{a(\varepsilon)} \int_0^t \langle b_\varepsilon(s, a(\varepsilon))M^\varepsilon(s) + X^0(s), Law(X^\varepsilon(s)) \rangle dW(s) \\
+ \frac{\varepsilon}{a^2(\varepsilon)} \int_0^t \| \sigma_\varepsilon(s, a(\varepsilon))M^\varepsilon(s) + X^0(s), Law(X^\varepsilon(s)) \|^2_{L_2} ds \\
+ 2 \int_0^t \langle \sigma_\varepsilon(s, a(\varepsilon))M^\varepsilon(s) + X^0(s), Law(X^\varepsilon(s)) \rangle \phi_\varepsilon(s, M^\varepsilon(s)) ds \\
+ \frac{2\varepsilon}{a(\varepsilon)} \int_0^t \int_Z \langle G_\varepsilon(s, a(\varepsilon))M^\varepsilon(s) + X^0(s), Law(X^\varepsilon(s)) \rangle d\psi_\varepsilon(s) ds \\
+ \frac{\varepsilon^2}{a^2(\varepsilon)} \int_0^t \int_Z \| G_\varepsilon(s, a(\varepsilon))M^\varepsilon(s) + X^0(s), Law(X^\varepsilon(s)) \|^2_{L_2} d\psi_\varepsilon(s) ds \\
+ \frac{2}{a(\varepsilon)} \int_0^t \int_Z \langle G_\varepsilon(s, a(\varepsilon))M^\varepsilon(s) + X^0(s), Law(X^\varepsilon(s)) \rangle (\psi_\varepsilon(s, z) - 1) d\nu(s) ds \\
=: I_1(t) + I_2(t) + I_3(t) + I_4(t) + I_5(t) + I_6(t) + I_7(t).
\]

Equation (4.7), (A3), (B1) and (B3) imply that there exists \( \varepsilon_1 > 0 \) such that

\[
\frac{\varepsilon}{a^2(\varepsilon)} \vee a(\varepsilon) \vee \varrho_{b, \varepsilon} \vee \varrho_{\sigma, \varepsilon} \vee \varrho_{G, \varepsilon} \vee \frac{\varrho_{b, \varepsilon}}{a(\varepsilon)} \in (0, \frac{1}{2}], \forall \varepsilon \in (0, \varepsilon_1].
\]

Recall the constant \( \varepsilon_0 \) in Lemma 5.9. Set \( \varepsilon_2 = \varepsilon_0 \land \varepsilon_1 \land \frac{1}{2} \).

Note that we always use \( C \) to denote a generic constant which may change from line to line and is independent of \( \varepsilon \).

By (A1), (A3), Lemma 5.9 and (5.49), for any \( \varepsilon \in (0, \varepsilon_2] \),

\[
I_1(t) \leq \frac{2\varrho_{b, \varepsilon}}{a(\varepsilon)} \int_0^t |M^\varepsilon(s)| ds \\
+ 2L \int_0^t |M^\varepsilon(s)|^2 ds + \frac{2}{a(\varepsilon)} \int_0^t |M^\varepsilon(s)| W^2(Law(X^\varepsilon(s)), Law(X^0(s))) ds \\
\leq C \int_0^t |M^\varepsilon(s)|^2 ds + C,
\]

(5.76)

\[
I_3(t) \leq \frac{C\varrho_{\sigma, \varepsilon}}{a^2(\varepsilon)} + C \int_0^t |M^\varepsilon(s)|^2 ds \\
+ \frac{C\varepsilon}{a^2(\varepsilon)} \left( \int_0^T W^2(Law(X^\varepsilon(s)), Law(X^0(s))) + \| \sigma(s, X^0(s), Law(X^0(s))) \|^2_{L_2} ds \right) \\
\leq C \int_0^t |M^\varepsilon(s)|^2 ds + C,
\]

(5.77)

and

\[
I_4(t) \leq 2\varrho_{\sigma, \varepsilon} \int_0^t |\phi_\varepsilon(s)| |M^\varepsilon(s)| ds \\
+ C \left( a(\varepsilon)|M^\varepsilon(s)| + W^2(Law(X^\varepsilon(s)), Law(X^0(s))) \\
+ \| \sigma(s, X^0(s), Law(X^0(s))) \|^2_{L_2} \right)
\]
\[ |\phi_\epsilon(s)||M^{\mu_\epsilon}(s)|ds \leq Ca(\epsilon) \int_0^t |M^{\mu_\epsilon}(s)|^2 |\phi_\epsilon(s)|ds + C \left( E(\sup_{s\in[0,T]}|X^\epsilon(s) - X^0(s)|^2)^{1/2} + 1 \right) \int_0^t |\phi_\epsilon(s)||M^{\mu_\epsilon}(s)|ds \\
+ C \int_0^t \|\sigma(s, X^0(s), Law(X^0(s)))\|_2 |\phi_\epsilon(s)||M^{\mu_\epsilon}(s)|ds \\
\leq C \int_0^t |M^{\mu_\epsilon}(s)|^2 (|\phi_\epsilon(s)|^2 + 1) ds \\
+ C \int_0^T \|\sigma(s, X^0(s), Law(X^0(s)))\|^2_2 ds + C \int_0^T |\phi_\epsilon(s)|^2 ds \\
\leq C \int_0^t |M^{\mu_\epsilon}(s)|^2 (|\phi_\epsilon(s)|^2 + 1) ds + C, \quad (5.78) \]

where the last inequality follows from the fact that \( \phi_\epsilon \in S^m_1 \).

Recall \( \varphi_\epsilon(s, z) = \frac{\psi_\epsilon(s, z) - 1}{a(\epsilon)} \). By (B1) and Lemma 5.9, for any \( \epsilon \in (0, \epsilon_2] \),

\[ I_7(t) = 2 \int_0^t \int_Z \langle G_\epsilon(s, a(\epsilon)M^{\mu_\epsilon}(s) + X^0(s), Law(X^\epsilon(s)), z) \frac{\psi_\epsilon(s, z) - 1}{a(\epsilon)}, M^{\mu_\epsilon}(s) \rangle v(dz) ds \\
\leq 2 \int_0^t \int_Z \varrho_{G, \epsilon} L_3(z)|\varphi_\epsilon(s, z)||M^{\mu_\epsilon}(s)||v(dz)| ds \\
+ 2 \int_0^t \int_Z \left( G(s, a(\epsilon)M^{\mu_\epsilon}(s) + X^0(s), Law(X^\epsilon(s)), z) - G(s, 0, 0_0, z) \right) \varphi_\epsilon(s, z), M^{\mu_\epsilon}(s) \rangle v(dz) ds \\
+ 2 \int_0^t \int_Z \langle G(s, 0, 0_0, z)\varphi_\epsilon(s, z), M^{\mu_\epsilon}(s) \rangle v(dz) ds \\
\leq C \int_0^t \int_Z L_1(z) \left( a(\epsilon)|M^{\mu_\epsilon}(s)| + |X^0(s)| + W_2(Law(X^\epsilon(s)), 0_0) \right) |\varphi_\epsilon(s, z)||M^{\mu_\epsilon}(s)||v(dz)| ds \\
+ C \int_0^t \int_Z (L_2(z) + L_3(z))|\varphi_\epsilon(s, z)||M^{\mu_\epsilon}(s)||v(dz)| ds \\
\leq C \int_0^t \int_Z (L_1(z) + L_2(z) + L_3(z))|\varphi_\epsilon(s, z)||v(dz)||M^{\mu_\epsilon}(s)||^2 ds \\
+ C \int_0^t \int_Z (L_1(z) + L_2(z) + L_3(z))|\varphi_\epsilon(s, z)||v(dz)| ds. \quad (5.79) \]

To deduce the last inequality, the following facts have been used

- \( X^0 \in C([0, T], \mathbb{R}^d) \),
- \( W_2(Law(X^\epsilon(s)), 0_0) \leq W_2(Law(X^\epsilon(s)), Law(X^0(s))) + |X^0(s)| \).

Set

\[ D_\epsilon := \int_0^T (|\phi_\epsilon(s)|^2 + 1) ds + \int_0^T \int_Z (L_1(z) + L_2(z) + L_3(z))|\varphi_\epsilon(s, z)||v(dz)| ds. \]

By substituting (5.76)-(5.79) back into (5.74) and applying Gronwall’s inequality, we obtain

\[ |M^{\mu_\epsilon}(t)|^2 \leq e^{C_D} \left\{ C D_\epsilon + \sup_{s\in[0,T]} |I_2(s) + I_5(s) + I_6(s)| \right\}, \forall \epsilon \in (0, \epsilon_2], \ t \in [0, T]. \quad (5.80) \]
Since \((\phi_\ɛ, \varphi_\ɛ) \in S^m_1 \times S^m_2\) P-a.s., we have
\[
\frac{1}{2} \int_0^T |\phi_\ɛ(s)|^2 \, ds \leq m, \quad \text{P-a.s. } \forall \ɛ \in (0, \varepsilon_2].
\]  
(5.81)

Hence, by (5.71), (5.80) and (5.81), there exists a constant \(\Delta \in (0, \infty)\) such that for each \(\varepsilon \in (0, \varepsilon_2],\)
\[
\mathbb{E}\left( \sup_{t \in [0,T]} |M^{\mu_\varepsilon}(t)|^2 \right) \Delta \left\{ 1 + \mathbb{E}\left( \sup_{t \in [0,T]} |I_2(t)| \right) + \mathbb{E}\left( \sup_{t \in [0,T]} |I_3(t)| \right) + \mathbb{E}\left( \sup_{t \in [0,T]} |I_6(t)| \right) \right\}. \tag{5.82}
\]

By Burkholder-Davis-Gundy’s inequality, (A1), (A3), Young’s inequality, Lemma 5.9, (5.49) and (5.77), for any \(\varepsilon \in (0, \varepsilon_2],\)
\[
\mathbb{E}\left( \sup_{t \in [0,T]} |I_2(t)| \right) \leq \frac{C\sqrt{T}}{a(\varepsilon)} \mathbb{E}\left( \int_0^T |M^{\mu_\varepsilon}(s)|^2 \|\sigma_\varepsilon(s, a(\varepsilon)M^{\mu_\varepsilon}(s) + X^0(s), \text{Law}(X^t(s)))|_{\mathcal{Z}_2}^2 \, ds \right)^{\frac{1}{2}}
\leq \frac{C\sqrt{T}}{a(\varepsilon)} \mathbb{E}\left( \sup_{t \in [0,T]} |M^{\mu_\varepsilon}(s)|^2 \right)^{\frac{1}{2}} + \frac{C\sqrt{T}}{a(\varepsilon)} \mathbb{E}\left( \int_0^T \|\sigma_\varepsilon(s, a(\varepsilon)M^{\mu_\varepsilon}(s) + X^0(s), \text{Law}(X^t(s)))|_{\mathcal{Z}_2}^2 \, ds \right)^{\frac{1}{2}}
\leq \frac{C\sqrt{T}}{a(\varepsilon)} \mathbb{E}\left( \sup_{t \in [0,T]} |M^{\mu_\varepsilon}(s)|^2 \right)^{\frac{1}{2}} + C. \tag{5.83}
\]

Similarly, using (B1), we have for any \(\varepsilon \in (0, \varepsilon_2],\)
\[
\Delta \left( \mathbb{E}\left( \sup_{t \in [0,T]} |I_3(t)| \right) + \mathbb{E}\left( \sup_{t \in [0,T]} |I_6(t)| \right) \right) \leq \frac{C\sqrt{T}}{a(\varepsilon)} \mathbb{E}\left( \int_0^T |G_\varepsilon(s, a(\varepsilon)M^{\mu_\varepsilon}(s) + X^0(s), \text{Law}(X^t(s)), z)|^2 |M^{\mu_\varepsilon}(s)|^2 N^{-1}\psi_\varepsilon(z, z) \, ds \right)^{\frac{1}{2}}
\leq \frac{C\sqrt{T}}{a(\varepsilon)} \mathbb{E}\left( \sup_{t \in [0,T]} |M^{\mu_\varepsilon}(s)|^2 \right)^{\frac{1}{2}} + C. \tag{5.84}
\]
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\[ + C \sup_{\psi \in \mathcal{S}^K_{+}, T_0} \int_{Z} (L^1(z) + L^2(z) + L^3(z)) \psi(s, z) \nu(dz) ds \]

\[ + \left( 1 + \sup_{s \in [0, T]} |X^0(s)|^2 + \mathbb{E} \left( \sup_{s \in [0, T]} |X^* - X^0(s)|^2 \right) \right) \]

\[ \leq \frac{1}{10} \mathbb{E} \left( \sup_{s \in [0, T]} |M^{u*}(s)|^2 \right) \]

\[ + C \epsilon \sup_{\psi \in \mathcal{S}^K_{+}, T_0} \int_{Z} \int_{0}^{T} L(t, \psi(s, z)) \nu(dz) ds \mathbb{E} \left( \sup_{s \in [0, T]} |M^{u*}(s)|^2 \right) \]

\[ + C \sup_{\psi \in \mathcal{S}^K_{+}, T_0} \int_{Z} (L^1(z) + L^2(z) + L^3(z)) \psi(s, z) \nu(dz) ds \]

(5.84)

By (5.82)–(5.84), we have for \( \forall \epsilon \in (0, \epsilon_2] \),

\[ \left( \frac{9}{10} - C \sqrt{\epsilon a(\epsilon)} + C \sqrt{\epsilon a(\epsilon)} - C \epsilon \sup_{\psi \in \mathcal{S}^K_{+}, T_0} \int_{Z} \int_{0}^{T} L^2 \psi(s, z) \nu(dz) ds \right) \mathbb{E} \left( \sup_{s \in [0, T]} |M^{u*}(s)|^2 \right) \]

\[ \leq C \left( 1 + \sup_{\psi \in \mathcal{S}^K_{+}, T_0} \int_{Z} \int_{0}^{T} (L^1(z) + L^2(z) + L^3(z)) \psi(s, z) \nu(dz) ds \right) \]

In view of (5.69) and (4.7), this yields that there exists \( \kappa_0 > 0 \) such that

\[ \sup_{\epsilon \in (0, \kappa_0]} \mathbb{E} \left( \sup_{s \in [0, T]} |M^{u*}(s)|^2 \right) < \infty. \]

The proof of this lemma is completed. \( \square \)

The verification of (MDP2) is given in the next proposition.

Recall \( \bar{u}_\epsilon \) in (5.67).

**Proposition 5.16** For any \( \alpha > 0 \), \( \lim_{\epsilon \to 0} P \left( \sup_{t \in [0, T]} |M^{u*}(t) - K^{u*}(t)| > \alpha \right) = 0. \)

**Proof** For each fixed \( \epsilon > 0 \) and \( j \in \mathbb{N} \), define a stopping time

\[ \tau^j_\epsilon = \inf \{ t \geq 0 : |M^{u*}(t)| \geq j \} \land T. \]

By Lemma 5.15, we have

\[ P(\tau^j_\epsilon < T) \leq \frac{\mathbb{E} \left( \sup_{t \in [0, T]} |M^{u*}(t)|^2 \right)}{j^2} \leq \frac{C}{j^2}, \quad \forall \epsilon \in (0, \kappa_0] \] (5.85)

where \( \kappa_0 \) is the same as in Lemma 5.15.
Let $Q^\epsilon(s) = M^\epsilon(s) - K^\epsilon(s)$ for each $s \in [0, T]$. Notice that the corresponding equations $M^\epsilon$ and $K^\epsilon$ satisfy are distribution-independent SDEs. By Itô’s formula, we have

$$|Q^\epsilon(t \wedge \tau^\epsilon_\varepsilon)|^2 = 2 \int_0^{t \wedge \tau^\epsilon_\varepsilon} \left( \frac{1}{a(\epsilon)} \left( b(s, a(\epsilon)M^\epsilon(s) + X^0(s), \text{Law}(X^\epsilon(s))) - b(s, X^0(s), \text{Law}(X^0(s))) \right) \right. \\
- b'_2(s, X^0(s), \text{Law}(X^0(s)))K^\epsilon(s), Q^\epsilon(s))ds \\
+2 \sqrt{\varepsilon} \int_0^{t \wedge \tau^\epsilon_\varepsilon} \langle Q^\epsilon(s), \sigma_\epsilon(s, a(\epsilon)M^\epsilon(s) + X^0(s), \text{Law}(X^\epsilon(s)))dW(s) \rangle \\
+ \frac{\varepsilon}{a(\epsilon)} \int_0^{t \wedge \tau^\epsilon_\varepsilon} \|\sigma_\epsilon(s, a(\epsilon)M^\epsilon(s) + X^0(s), \text{Law}(X^\epsilon(s)))\|_Z^2 ds \\
+2 \int_0^{t \wedge \tau^\epsilon_\varepsilon} \langle (\sigma_\epsilon(s, a(\epsilon)M^\epsilon(s) + X^0(s), \text{Law}(X^\epsilon(s))) \\
- \sigma(s, X^0(s), \text{Law}(X^0(s)))\phi(\epsilon, \sigma\epsilon(s), Q^\epsilon(s))ds \\
+ \frac{2\varepsilon}{a(\epsilon)} \int_0^{t \wedge \tau^\epsilon_\varepsilon} \int_Z \langle G_\epsilon(s, a(\epsilon)M^\epsilon(s) - X^0(s), \text{Law}(X^\epsilon(s)), z), Q^\epsilon(s) \rangle \phi^{\epsilon^{-1}}\varepsilon(\psi^z) (dz, ds) \\
+ \frac{\varepsilon^2}{a^2(\epsilon)} \int_0^{t \wedge \tau^\epsilon_\varepsilon} \int_Z \langle G_\epsilon(s, a(\epsilon)M^\epsilon(s) - X^0(s), \text{Law}(X^\epsilon(s)), z)\|\phi^z(\psi) (dz, ds) \\
+2 \int_0^{t \wedge \tau^\epsilon_\varepsilon} \int_Z \langle G_\epsilon(s, a(\epsilon)M^\epsilon(s) + X^0(s), \text{Law}(X^\epsilon(s)), z)\|\phi^{\epsilon}(\psi) (dz, ds) \\
- G(s, X^0(s), \text{Law}(X^0(s)), z)\|\phi^{\epsilon}(\psi) (dz, ds) \rangle 1_{B^0(\epsilon) \subseteq \beta(\epsilon)B^0 \subseteq B^0(\epsilon)} (s, z), Q^\epsilon(s))d\psi^z)ds \\
= I_{1}(t) + I_{2}(t) + I_{3}(t) + I_{4}(t) + I_{5}(t) + I_{6}(t) + I_{7}(t). \quad (5.86)$$

Due to (5.61) and the fact that $\tilde{u}_\epsilon \in S^m_1 \times B_2(\sqrt{mK_2}(1))$, there exists some $\Omega^0 \in \mathcal{F}$ with $P(\Omega^0) = 1$ such that

$$\kappa := \sup_{\epsilon \in (0, \kappa_0)} \sup_{\omega \in \Omega^0} |K^\epsilon(t)(\omega)| < \infty. \quad (5.87)$$

Recall the constant $\epsilon_2$ appearing in (5.75). Set $\epsilon_3 = \epsilon_2 \wedge \kappa_0$. By the mean value theorem and (B2), for any $\epsilon \in (0, \epsilon_3]$, there exists $\theta_\epsilon(s) \in [0, 1]$ such that

$$I_{1, 1}(t) := \int_0^{t \wedge \tau^\epsilon_\varepsilon} \left( \frac{b(s, a(\epsilon)M^\epsilon(s) + X^0(s), \text{Law}(X^\epsilon(s))) - b(s, X^0(s), \text{Law}(X^0(s)))}{a(\epsilon)} \right. \\
- b'_2(s, X^0(s), \text{Law}(X^0(s)))M^\epsilon(s), Q^\epsilon(s))ds \\
\leq \int_0^{t \wedge \tau^\epsilon_\varepsilon} \left| b'_2(s, a(\epsilon)M^\epsilon(s)\theta_\epsilon(s) + X^0(s), \text{Law}(X^0(s))) \\
- b'_2(s, X^0(s), \text{Law}(X^0(s)))\|M^\epsilon(s)\|Q^\epsilon(s)\|ds \\
\leq La(\epsilon) \int_0^{t \wedge \tau^\epsilon_\varepsilon} (1 + \sup_{s \in [0, T]} |X^0(s)|q' + \sup_{s \in [0, T]} |X^0(s)|q') |M^\epsilon(s)|^2 |Q^\epsilon(s)| ds \\
\leq Ca(\epsilon),$$

where

$$C_j = L \left( 1 + \sup_{s \in [0, T]} |X^0(s)|q' + \sup_{s \in [0, T]} |X^0(s)|q' \right) j^2 (j + \kappa) T,$$

which is independent of $\epsilon$.

In the sequel, $C_j$ will denote generic constants which are independent of $\epsilon$. 

\[ \text{ Springer} \]
Hence, by Lemma 5.9, (A1) and the above inequality, for any $\epsilon \in (0, \epsilon_3]$, 

$$I_1(t) = 2I_{1,1}(t) + 2\int_0^{t \wedge \tau_j} b_2(s, a(\epsilon)M_\epsilon(s) + X^0(s), Law(X^s(s))) - b(s, a(\epsilon)M_\epsilon(s) + X^0(s), Law(X^s(s)))\frac{Q^\epsilon(s)}{a(\epsilon)} ds$$

$$+ 2\int_0^{t \wedge \tau_j} b_2(s, X^0(s), Law(X^0(s)))M^\epsilon(s) - b_2(s, X^0(s), Law(X^0(s)))K^\epsilon(s), Q^\epsilon(s)) ds$$

$$+ 2\int_0^{t \wedge \tau_j} b(s, a(\epsilon)M_\epsilon(s) + X^0(s), Law(X^s(s))) - b(s, a(\epsilon)M_\epsilon(s) + X^0(s), Law(X^s(s)))\frac{Q^\epsilon(s)}{a(\epsilon)} ds$$

$$\leq 2\frac{kb_2}{a(\epsilon)} \int_0^{t \wedge \tau_j} |Q^\epsilon(s)| ds + 2L \frac{a(\epsilon)}{a(\epsilon)} \int_0^{t \wedge \tau_j} (E|X^s(s) - X^0(s)|^2)^{1/2} |Q^\epsilon(s)| ds$$

$$+ C_j a(\epsilon) + 2\int_0^t b_2^2(s \wedge \tau_j^i, X^0(s) \wedge \tau_j^i), Law(X^0(s) \wedge \tau_j^i))||Q^\epsilon(s \wedge \tau_j^i)||^2 ds$$

$$\leq C_j \left( \frac{kb_2}{a(\epsilon)} + \frac{(\epsilon + \epsilon q_{i,\epsilon} + \epsilon q_{i,\epsilon}^2 + \epsilon q_{i,\epsilon}^2)^{1/2}}{a(\epsilon)} + a(\epsilon) \right)$$

$$+ 2\int_0^t b_2^2(s \wedge \tau_j^i, X^0(s) \wedge \tau_j^i), Law(X^0(s) \wedge \tau_j^i))||Q^\epsilon(s \wedge \tau_j^i)||^2 ds. \quad (5.88)$$

Inserting the inequality (5.88) into (5.86), and using Gronwall’s inequality, we deduce that, for any $\epsilon \in (0, \epsilon_3]$ and $t \in [0, T]$,

$$\sup_{t \in [0, T]} |Q^\epsilon(t \wedge \tau_j^i)|^2 \leq \exp \left\{ 2\int_0^T b_2^2(s, X^0(s), Law(X^0(s))) ds \right\} \times \left\{ C_j \left( \frac{kb_2}{a(\epsilon)} + \frac{(\epsilon + \epsilon q_{i,\epsilon} + \epsilon q_{i,\epsilon}^2 + \epsilon q_{i,\epsilon}^2)^{1/2}}{a(\epsilon)} + a(\epsilon) \right) + \sum_{i=1}^{\infty} \sup_{t \in [0, T]} |I_i(t)| \right\}. \quad (5.89)$$

Set

$$\Lambda := \exp \left\{ 2\int_0^T b_2^2(s, X^0(s), Law(X^0(s))) ds \right\}.$$ 

By Burkholder-Davis-Gundy’s inequality and (5.49), using similar arguments as in the proofs of (5.77) and (5.83), one can obtain for each $\epsilon \in (0, \epsilon_3]$,

$$\Lambda \left( \mathbb{E} \left( \sup_{t \in [0, T]} |I_2(t)| \right) + \mathbb{E} \left( \sup_{t \in [0, T]} |I_3(t)| \right) \right)$$

$$\leq \frac{1}{10} \mathbb{E} \left( \sup_{t \in [0, T]} |Q^\epsilon(t \wedge \tau_j^i)|^2 \right) + \frac{C_2}{a^2(\epsilon)} \mathbb{E} \left( \int_0^{T \wedge \tau_j^i} \|\sigma_\epsilon(s, a(\epsilon)M_\epsilon(s) + X^0(s), Law(X^s(s)))\|^2_{C^2} ds \right)$$

$$\leq \frac{1}{10} \mathbb{E} \left( \sup_{t \in [0, T]} |Q^\epsilon(t \wedge \tau_j^i)|^2 \right) + \frac{C_2 q_{i,\epsilon}^2}{a^2(\epsilon)}$$
\[
+ \frac{C\epsilon}{a^2(\epsilon)} \mathbb{E}\left( \int_0^{T \wedge t^j} |M^{\mu^*}(s)|^2 + \mathbb{E}(|X^\epsilon(s) - X^0(s)|^2)ds \right) \\
+ \frac{C\epsilon}{a^2(\epsilon)} \mathbb{E}\left( \int_0^T \|\sigma(s, X^0(s), \text{Law}(X^0(s)))\|^2_{L_2} ds \right) \\
\leq \frac{1}{10} \mathbb{E}\left( \sup_{t \in [0,T]} |Q^\epsilon(t \wedge t^j)|^2 \right)
\]
\[
+ C_j \frac{\epsilon}{a^2(\epsilon)} (1 + \epsilon + \epsilon\rho^2_{\sigma,\epsilon} + \epsilon\rho^2_{b,\epsilon} + \epsilon\rho^2_{\sigma,\epsilon} + \epsilon\rho^2_{\tilde{G},\epsilon}).
\]

(5.90)

By (A1), (5.87) and Lemma 5.9, remembering that \(\phi_\epsilon \in S_1^m\), we have for any \(\epsilon \in (0, \epsilon_3)\),

\[
\Delta \mathbb{E}\left( \sup_{t \in [0,T]} |I_5(t)| \right) \\
\leq C\rho_{\sigma,\epsilon} \mathbb{E}\int_0^{T \wedge t^j} |\phi_\epsilon(s)||Q^\epsilon(s)|ds \\
\]
\[
+ C\mathbb{E}\int_0^{T \wedge t^j} (|a(\epsilon)M^{\mu^*}(s)| + \mathbb{V}_2(\text{Law}(X^\epsilon(s)), \text{Law}(X^0(s))))|\phi_\epsilon(s)||Q^\epsilon(s)|ds \\
\leq C_j (\rho_{\sigma,\epsilon} + a(\epsilon) + (\epsilon + \rho^2_{b,\epsilon} + \epsilon\rho^2_{\sigma,\epsilon} + \epsilon\rho^2_{\tilde{G},\epsilon})^{1/2}) \mathbb{E}\left( \int_0^T |\phi_\epsilon(s)|^2 ds \right)^{1/2} \\
\leq C_j (\rho_{\sigma,\epsilon} + a(\epsilon) + (\epsilon + \rho^2_{b,\epsilon} + \epsilon\rho^2_{\sigma,\epsilon} + \epsilon\rho^2_{\tilde{G},\epsilon})^{1/2}).
\]

(5.91)

By Burkholder-Davis-Gundy’s inequality, (5.69) and (5.75), using similar arguments as in the proof of (5.84), one has for \(\forall \epsilon \in (0, \epsilon_3)\),

\[
\Delta \mathbb{E}\left( \sup_{t \in [0,T]} |I_5(t)| \right) + \mathbb{E}\left( \sup_{t \in [0,T]} |I_6(t)| \right) \\
\leq \frac{1}{10} \mathbb{E}\left( \sup_{t \in [0,T]} |Q^\epsilon(t \wedge t^j)|^2 \right) \\
+ \frac{C\epsilon}{a^2(\epsilon)} \mathbb{E}\left( \int_0^{T \wedge t^j} \int_Z |G_\epsilon(s, a(\epsilon)M^{\mu^*}(s) + X^0(s), \text{Law}(X^\epsilon(s)), z)|^2\psi_\epsilon(s, z)\nu(dz)ds \right) \\
\leq \frac{1}{10} \mathbb{E}\left( \sup_{t \in [0,T]} |Q^\epsilon(t \wedge t^j)|^2 \right) + \frac{C\epsilon\rho^2_{G,\epsilon}}{a^2(\epsilon)} \mathbb{E}\left( \int_0^{T \wedge t^j} \int_Z |L^2_1(z)\psi_\epsilon(s, z)\nu(dz)ds \right) \\
+ \frac{C\epsilon}{a^2(\epsilon)} \mathbb{E}\left( \int_0^{T \wedge t^j} \int_Z (|a(\epsilon)M^{\mu^*}(s) + X^0(s)|^2 + \mathbb{E}(|X^\epsilon(s)|^2)\psi_\epsilon(s, z)\nu(dz)ds \right) \\
\leq \frac{1}{10} \mathbb{E}\left( \sup_{t \in [0,T]} |Q^\epsilon(t \wedge t^j)|^2 \right) \\
+ \frac{C\epsilon}{a^2(\epsilon)} \sup_{\psi \in \mathcal{S}_{1,\epsilon}} \left( \int_0^T \int_Z (L^2_1(z) + L^2_2(z) + L^2_3(z))\psi_\epsilon(s, z)\nu(dz)ds \right) \\
\leq \frac{1}{10} \mathbb{E}\left( \sup_{t \in [0,T]} |Q^\epsilon(t \wedge t^j)|^2 \right) + \frac{C\epsilon}{a^2(\epsilon)}. \tag{5.92}
\]

Note that

\[
I_7(t) = 2 \int_0^{t \wedge t^j} \int_Z ((G_\epsilon(s, a(\epsilon)M^{\mu^*}(s) + X^0(s), \text{Law}(X^\epsilon(s)), z) \\
- G(s, a(\epsilon)M^{\mu^*}(s) + X^0(s), \text{Law}(X^\epsilon(s)), z))\psi_\epsilon(s, z, Q^\epsilon(s))\nu(dz)ds
\]
\[ +2 \int_0^{\tau_1} \int_{Z} \{(G(s, a(\epsilon)M^{a(\epsilon)}(s)) + X^0(s), \text{Law}(X^0(s)), z) \mid \mathcal{F}(s), \mathcal{Q}(s)\} \nu(dz)ds \\
- G(s, X^0(s), \text{Law}(X^0(s)), z) \mid \mathcal{F}(s), \mathcal{Q}(s)\} \nu(dz)ds \\
+2 \int_0^{\tau_2} \int_{Z} \{(G(s, X^0(s), \text{Law}(X^0(s)), z) - G(s, 0, \delta_0, z) \mid \mathcal{F}(s), \mathcal{Q}(s)\} \nu(dz)ds \\
+2 \int_0^{\tau_2} \int_{Z} \{G(s, 0, \delta_0, z) \mid \mathcal{F}(s), \mathcal{Q}(s)\} \nu(dz)ds. \]

Hence, by (B1), (5.87), Remark 5.1 and Lemma 5.9,

\[
\Lambda \mathbb{E} \left( \sup_{t \in [0, T]} |L_I(t)| \right) \leq C_0 \mathbb{E} \left( \int_0^{\tau_1} \int_{Z} L_3(z) \mid \mathcal{F}(s), \mathcal{Q}(s)\} \nu(dz)ds \right) \\
+ C \mathbb{E} \left( \int_0^{\tau_1} \int_{Z} L_1(z) \mid (a(\epsilon)M^{a(\epsilon)}(s)) \mid \mathcal{F}(s), \mathcal{Q}(s)\} \nu(dz)ds \right) \\
+ \left( \mathbb{E}(X^\epsilon(s) - X^0(s))^2 \right)^{\frac{1}{2}} \mathbb{E} \left( \mid \mathcal{F}(s), \mathcal{Q}(s)\} \nu(dz)ds \right) \\
+ C \mathbb{E} \left( \int_0^{\tau_1} \int_{Z} L_1(z) \mid X^0(s) \mid \mathcal{F}(s), \mathcal{Q}(s)\} \nu(dz)ds \right) \\
+ C \mathbb{E} \left( \int_0^{\tau_1} \int_{Z} \mid \mathcal{F}(s), \mathcal{Q}(s)\} \nu(dz)ds \right) \\
\leq C \left( q_{G, \epsilon} + a(\epsilon) + (\epsilon + \theta_{h, \epsilon}^2 + \epsilon \theta_{G, \epsilon}^2 + \epsilon \theta_{G, \epsilon}^2) \right) \sup_{\varphi \in \mathbb{S}^n} \int_0^T \int_{Z} \mid \mathcal{F}(s), \mathcal{Q}(s)\} \nu(dz)ds \\
+ L_3(z) \mid \mathcal{F}(s), \mathcal{Q}(s)\} \nu(dz)ds \\
+ C \sup_{\varphi \in \mathbb{S}^n} \int_0^T \int_{Z} \mid \mathcal{F}(s), \mathcal{Q}(s)\} \nu(dz)ds \\
\leq C \left( q_{G, \epsilon} + a(\epsilon) + (\epsilon + \theta_{h, \epsilon}^2 + \epsilon \theta_{G, \epsilon}^2 + \epsilon \theta_{G, \epsilon}^2) \right) \left( \int_0^T \int_{Z} \mid \mathcal{F}(s), \mathcal{Q}(s)\} \nu(dz)ds \right). \tag{5.93} \]

By combining (5.75) and (5.86)-(5.93) together, we obtain for any \( \epsilon \in (0, \epsilon_3] \),

\[
\frac{8}{10} \mathbb{E} \left( \sup_{t \in [0, T]} \mid \mathcal{F}(s \land \tau_\epsilon) \right)^2 \\
\leq C \left( q_{G, \epsilon} + a(\epsilon) + \frac{\theta_{h, \epsilon}}{a(\epsilon)} + \frac{(\epsilon + \theta_{h, \epsilon}^2 + \epsilon \theta_{G, \epsilon}^2 + \epsilon \theta_{G, \epsilon}^2)}{a(\epsilon)} + \frac{\epsilon}{a(\epsilon)} \right) + a(\epsilon) + (\epsilon + \theta_{h, \epsilon}^2 + \epsilon \theta_{G, \epsilon}^2 + \epsilon \theta_{G, \epsilon}^2) \left( \int_0^T \int_{Z} \mid \mathcal{F}(s), \mathcal{Q}(s)\} \nu(dz)ds \right). \tag{5.94} \]

In view of (B3), (4.7) and (5.72) it follows that

\[
\lim_{\epsilon \to 0} \mathbb{E} \left( \sup_{t \in [0, T]} \mid M^{a(\epsilon)}(t \land \tau_\epsilon) - K^{a(\epsilon)}(t \land \tau_\epsilon) \right)^2 = \lim_{\epsilon \to 0} \mathbb{E} \left( \sup_{t \in [0, T]} \mid \mathcal{F}(s \land \tau_\epsilon) \right)^2 = 0. \tag{5.95} \]
Now for any $\sigma > 0$, $\epsilon \in (0, \epsilon_3]$ and $j \in \mathbb{N}$, we have

$$P\left( \sup_{t \in [0,T]} |M^{\mu_{\epsilon}}(t) - K^{\check{\mu}_{\epsilon}}(t)| \geq \sigma \right) \leq P\left( \left( \sup_{t \in [0,T]} |M^{\mu_{\epsilon}}(t \wedge \tau_j^t) - K^{\check{\mu}_{\epsilon}}(t \wedge \tau_j^t)| \geq \sigma \right) \cap \left( \tau_j^t \geq T \right) \right)$$

$$+ P(\tau_j^T < T)$$

$$\leq \frac{1}{\sigma^2} \mathbb{E}\left( \sup_{t \in [0,T]} |M^{\mu_{\epsilon}}(t \wedge \tau_j^t) - K^{\check{\mu}_{\epsilon}}(t \wedge \tau_j^t)|^2 \right) + \frac{C}{j^2}.$$ 

By letting $\epsilon \to 0$ first and then $j \to \infty$, we get

$$\lim_{\epsilon \to 0} P\left( \sup_{t \in [0,T]} |M^{\mu_{\epsilon}}(t) - K^{\check{\mu}_{\epsilon}}(t)| \geq \sigma \right) = 0,$$

which is the desired result.

**Appendix**

**The Proof of Theorem 3.6**

For the fixed $J \in Pr(D([0,T], H))$, because of the existence of the strong solution, by the Yamada-Watanabe theorem (see [53] for the Wiener case and [63] for the PRM case), there exists a unique map $\Gamma_J : C([0,T], K) \times M_{FC}([0,T] \times Z) \to \mathbb{D}$ such that for any $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}, \{\bar{\mathcal{F}}_t, t \in [0,T]\}, \bar{W}, \eta)$ satisfying that

- $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$ is a complete probability space;
- $\{\bar{\mathcal{F}}_t, t \in [0,T]\}$ is a right continuous filtration on $\{\bar{\Omega}, \bar{\mathcal{F}}\}$ augmented by the $\bar{P}$-zero sets;
- on the stochastic basis $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}, \{\bar{\mathcal{F}}_t, t \in [0,T]\}, \bar{W} = \{\bar{W}(t), t \in [0,T]\})$ is a cylindrical Brownian motion taking values in $K$, $\eta$ is a PRM with intensity measure $Leb_T \otimes \nu$;
- $\bar{W}$ and $\eta$ are independent;

the following properties hold:

(A0) $(\Gamma_J(\bar{W}, \eta)(t), t \in [0,T])$ is an $\{\bar{\mathcal{F}}_t, t \in [0,T]\}$-adapted process with paths in $\mathbb{D}$;

(A1) $\int_0^T \|b(t, \Gamma_J(\bar{W}, \eta), J)\|_E dt + \int_0^T \|\sigma(t, \Gamma_J(\bar{W}, \eta), J)\|_{L_2^Z} dt + \int_0^T \int_Z \|G(t, \Gamma_J(\bar{W}, \eta), J, z)\|_H^2 \, dz \, dt$ $< \infty$, $\bar{P}$-a.s.;

(A2) as a stochastic equation on $E$ one has

$$\Gamma_J(\bar{W}, \eta)(t) = h + \int_0^t b(s, \Gamma_J(\bar{W}, \eta), J) ds + \int_0^t \sigma(s, \Gamma_J(\bar{W}, \eta), J) d\bar{W}(s)$$

$$+ \int_0^t \int_Z G(s, \Gamma_J(\bar{W}, \eta), J, z) \eta(dz, ds), t \in [0,T], \bar{P}-\text{a.s.}$$

where $\bar{\eta}$ is the corresponding compensated PRM with respect to $\eta$.

Therefore we have $Y = \Gamma_J(W, N^1)$, since $(Y, J)$ is a solution of (3.1) and pathwise uniqueness holds with the fixed $J$. 
For any given $m \in (0, \infty)$ and $u = (\phi, \psi) \in S_1^m \times S_2^m$, \( \forall t \in [0, T] \), let

$$
\mathcal{M}_t(\phi) := \exp \left( - \int_0^t (\phi(s), dW(s))_K - \frac{1}{2} \int_0^t \|\phi(s)\|^2_K ds \right)
$$

and

$$
\mathcal{E}_t(\psi) := \exp \left( \int_0^t \int_{[0,\psi]} \log \varphi(s, z) N(dr, dz, ds) + \int_0^t \int_{[0,\psi]} (-\varphi(s, z) + 1) dr \nu(dz) ds \right),
$$

where $\varphi := \frac{1}{\psi}$. Then we have, on the probability space $(\Omega, \mathcal{F}, P)$

- $\{\mathcal{M}_t(\phi), t \in [0, T]\}$ is a $\mathbb{F}$-martingale;
- $\{\mathcal{E}_t(\psi), t \in [0, T]\}$ is a $\mathbb{F}$-martingale by Theorem 6.1 in [10];
- moreover, $\{\mathcal{M}_t(\phi)\mathcal{E}_t(\psi), t \in [0, T]\}$ is a $\mathbb{F}$-martingale on $(\Omega, \mathcal{F}, P)$, thanks to the independence of $W$ and $N$ on $(\Omega, \mathcal{F}, P)$.

Let

$$
Q(O) := \int_O \mathcal{M}_T(\phi)\mathcal{E}_T(\psi) dP, \quad \forall O \in \mathcal{F},
$$

then

- $(\Omega 1)$ $Q$ is a probability measure on $(\Omega, \mathcal{F})$;
- $(\Omega 2)$ the measures $Q$ and $P$ are equivalent;
- $(\Omega 3)$ By the Girsanov Theorem (see, e.g., [35, Theorem III.3.24], [16, Appendix A.1]), under the probability space $(\Omega, \mathcal{F}, \mathbb{F}, Q)$, $(W(\cdot) + \int_0^\cdot \phi(s) ds, N^\psi)$ has the same law as that of $(W(\cdot), N^1)$ on $(\Omega, \mathcal{F}, \mathbb{F}, P)$.

Let

$$
Y^u := \Gamma_J(W(\cdot) + \int_0^\cdot \phi(s) ds, N^\psi).
$$

By the property of $\Gamma_J$, it follows that (under the probability $Q$),

- $(B 0)$ $Y^u = \{Y^u(t), t \in [0, T]\}$ is an $\mathbb{F}$-adapted process with paths in $\mathbb{D}$;
- $(B 1)$ $\int_0^T \|b(t, Y^u, J)\|_E dt + \int_0^T \|\sigma(t, Y^u, J)\|^2_{L^2} dt + \int_0^T \int Z \|G(t, Y^u, J, z)\|^2_H v(dz) dt < \infty, \quad Q$-a.s.;
- $(B 2)$ as a stochastic equation on $E$ one has

$$
Y^u(t) = h + \int_0^t b(s, Y^u, J) ds + \int_0^t \sigma(s, Y^u, J) d\left( W(s) + \int_0^s \phi(l) dl \right)
+ \int_0^t \int Z G(s, Y^u, J, z) \left( N^\psi(dz, ds) - \nu(dz) ds \right), \quad t \in [0, T], \quad Q$-a.s. (5.98)

Since stochastic integrals against semimartingales remain the same with respect to a class of equivalent probability measures and since $Q$ and $P$ are equivalent, we conclude that under the probability $P$, $Y^u$ fulfills the equation (5.98) as well. This completes the proof of Theorem 3.6.

Data Availability No datasets were generated or analysed during the current study.
References

1. Adams, D., Reis, G.D., Ravaille, R., Salkeld, W., Tugaut, J.: Large Deviations and Exit-times for reflected McKean-Vlasov equations with self-stabilizing terms and superlinear drifts. arXiv:2005.10057v1 (2020)
2. Andreis, L., Dai Pra, P., Fischer, M.: McKean-vlasov limit for interacting systems with simultaneous jumps. Stoch. Anal. Appl. 36, 960–995 (2018)
3. Budhiraja, A., Dupuis, P., Ganguly, A.: Moderate Deviation Principles for Stochastic Differential Equations with Jumps, arXiv:1401.7316v1 (2014)
4. Budhiraja, A., Dupuis, P., Ganguly, A.: Moderate deviation principles for stochastic differential equations with jumps. Ann. Probab. 44, 1723–1775 (2016)
5. Budhiraja, A., Dupuis, P.: Analysis and approximation of rare events: representations and weak convergence methods. Probability Theory and Stochastic Modeling, Volume 94 Springer (2019)
6. Budhiraja, A., Dupuis, P., Maroulas, V.: Variational representations for continuous time processes. Ann. Inst. Henri poincaré, Probab. Stat. 47, 725–747 (2011)
7. Budhiraja, A., Dupuis, P.: A variational representation for positive functionals of an infinite dimensional Brownian motion. Probab. Math. Stat. 20, 39–61 (2000)
8. Budhiraja, A., Dupuis, P., Maroulas, V.: Large deviations for infinite dimensional stochastic dynamical systems continuous time processes. Ann. Probab. 36, 1390–1420 (2008)
9. Budhiraja, A., Chen, J., Dupuis, P.: Large deviations for stochastic partial differential equations driven by a Poisson random measure. Stoch. Proc. Appl. 123, 523–560 (2013)
10. Brzeźniak, Z., Peng, X., Zhai, J.: Well-posedness and large deviations for 2-D Stochastic Navier-Stokes equations with jumps. arXiv:1908.06228 (2019)
11. Brzeźniak, Z., Manna, U., Zhai, J.: Large Deviations for a Stochastic Landau-Lifshitz-Gilbert Equation Driven by Pure Jump Noise. in preparation
12. Barbu, V., Röckner, M.: From nonlinear Fokker-Planck equations to solutions of distribution dependent SDE. Ann. Probab. 48, 1902–1920 (2020)
13. Buckdahn, R., Li, J., Peng, S., Rainer, C.: Mean-field stochastic differential equations and associated PDEs. Ann. Probab. 45, 824–878 (2017)
14. Brzaniak, Z., Goldys, B., Jegaraj, T.: Large deviations and transitions between equilibria for stochastic Landau-Lifshitz-Gilbert equation. Arch. Ration. Mech. Anal. 226(2), 497–558 (2017)
15. Cai, Y., Huang, J., Maroulas, V.: Large deviations of mean-field stochastic differential equations with jumps. Statist. Probab. Lett. 96, 1–9 (2015)
16. Da Prato, G., Flandoli, F., Priola, E., Röckner, M.: Strong uniqueness for stochastic evolution equations in Hilbert spaces perturbed by a bounded measurable drift. Ann. Probab. 41(5), 3306–3344 (2013)
17. Dong, Z., Xiong, J., Zhai, J., Zhang, T.: A moderate deviation principle for 2-D stochastic Navier-Stokes equations driven by multiplicative lévy noises. J. Funct. Anal. 272, 227–254 (2017)
18. Dong, Z., Zhai, J., Zhang, R.: Large deviation principles for 3D stochastic primitive equations. J. Differential Equations 263(5), 3110–3146 (2017)
19. Dong, Z., Wu, J., Zhang, R., Zhang, T.: Large deviation principles for first-order scalar conservation laws with stochastic forcing. Ann. Appl. Probab. 30(1), 324–367 (2020)
20. Dos Reis, G., Salkeld, W., Tugaut, J.: Freidlin-wentzell LDPs in path space for McKean-Vlasov equations and the functional iterated logarithm law. Ann. Appl. Probab. 29, 1487–1540 (2019)
21. Durmus, A., Eberle, A., Guillin, A., Zimmer, R.: An elementary approach to uniform in Time propagation of chaos. Proc. Amer. Math. Soc. 148, 5387–5398 (2020)
22. Eberle, A.: Reflection couplings contraction rates for diffusions. Probab. Theory Relat. Fields 166, 851–886 (2016)
23. Eberle, A., Guillin, A., Zimmer, R.: Quantitative Harris type theorems for diffusions and McKean-Vlasov processes. Trans. Amer. Math. Soc. 371, 7135–7173 (2019)
24. Guillin, A., Liu, W., Wu, L., Zhang, C.: Poincaré and logarithmic Sobolev inequalities for particles in mean field interactions. arXiv:1909.07051, to appear in Annals of applied probability
25. Guillin, A., Liu, W., Wu, L., Zhang, C.: The kinetic Fokker-Planck equation with mean field interaction. J. Math. Pures Appl. 150, 1–23 (2021)
26. Hammersley, W., Siska, D., Szpruch, L.: McKean-vlasov SDEs under measure dependent Lyapunov conditions. Ann. Inst. H. Poincaré, Probab. Statist. 57, 1032–1057 (2021)
27. Hao, T., Li, J.: Mean-field SDEs with jumps and nonlocal integral-PDEs. Nonlinear Diff. Equ. Appl. 23, Art. 17, 51 (2016)
28. Herrmann, S., Imkeller, P., Peithmann, D.: Large deviations and a Kramers’ type law for self-stabilizing diffusions. Ann. Appl. Probab. 18, 1379–1423 (2008)
29. Hong, W., Li, S., Liu, W.: Large deviation principle for McKean-Vlasov quasilinear stochastic evolution equations. Appl. Math. Optim. suppl. 84(1), S119–S147 (2021)
30. Huang, X., Song, Y.: Well-posedness and regularity for distribution dependent SPDEs with singular drifts. Nonlinear Analysis 203, 11217 (2021)
31. Huang, X., Röckner, M., Wang, F.Y.: Nonlinear Fokker-Planck equations for probability measures on path space and path-distribution dependent SDEs. Discrete Contin. Dyn. Syst. 39, 3017–3035 (2019)
32. Huang, X., Wang, F.Y.: Distribution dependent SDEs with singular coefficients. Stoch. Proc. Appl. 129, 4747–4770 (2019)
33. Ikeda, N., Watanabe, S.: Stochastic differential equations and diffusion processes amsterdam: North-Holland Publishing Company (1981)
34. Jabin, P.E., Wang, Z.: Quantitative estimates of propagation of chaos for stochastic systems with $\omega^{-1,\infty}$ kernels. Invent. Math. 214, 523–591 (2018)
35. Jacod, J., Shiryaev, A.N.: Limit Theorems for Stochastic Processes. Springer, Berlin (1987)
36. Jourdain, B., Méléard, S., Woyczynski, W.A.: Nonlinear SDEs driven by Lévy processes and related PDEs. ALEA Lat. Am. J. Probab. Math. Stat. 4, 1–29 (2008)
37. Kac, M.: Foundations of kinetic theory. Proc. 3rd Berkeley Symp. Math. Statist. Probability 3, 171–197 (1956)
38. Kac, M.: Probability and Related Topics in the Physical Sciences. Interscience Publishers, New York (1958)
39. Li, J.: Mean-field forward and backward SDEs with jumps and associated nonlocal quasi-linear integral-PDEs. Stoch. Proc. Appl. 128, 3118–3180 (2018)
40. Liang, M., Majka, B., Wang, J.: Exponential ergodicity for SDEs and McKean-Vlasov processes with lévy noise. Ann. Inst. H. Poincaré, Probab. Statist. 57, 1665–1701 (2021)
41. Liu, W., Rockner, M.: Stochastic partial differential equations: an introduction. Universitext. Springer, Cham, 2015. vi+266 pp. ISBN: 978-3-319-22353-7; 978-3-319-22354-4
42. Liu, W., Wu, L.: Large deviations for empirical measures of mean-field gibbs measures. Stoch. Proc. Appl. 130, 503–520 (2020)
43. Liu, W., Wu, L., Zhang, C.: Long-time behaviors of mean-field interacting particle systems related to McKean-Vlasov equations. Commun. Math. Phys. 387, 179–214 (2021)
44. Malrieu, F.: Logarithmic sobolev inequalities for some nonlinear PDE's. Stoch. Proc. Appl. 95, 109–132 (2001)
45. Malrieu, F.: Convergence to equilibrium for granular media equations and their Euler schemes. Ann. Appl. Probab. 13, 540–560 (2003)
46. Matoussi, A., Sabbagh, W., Zhang, T.: Large deviation principle of obstacle problems for Quasilinear Stochastic PDEs. Appl. Math. Optim. 83, 849–879 (2021)
47. McKean, H.P.: A class of Markov processes associated with nonlinear parabolic equations. Proc. Nat. Acad. Sci. U.S.A. 56, 1907–1911 (1966)
48. Mehri, S., Scheutzow, M., Stannat, W., Zangeneh, B.Z.: Propagation of chaos for stochastic spatially structured neuronal networks with fully path dependent delays and monotone coefficients driven by jump diffusion noise. Ann. Appl. Probab. 30(1), 175–207 (2020)
49. Méléard, S.: Asymptotic behaviour of some interacting particle systems; McKean-Vlasov and Boltzmann models. Lect. Notes Math, p. 1996. Springer, Berlin (1627)
50. Mishura, Y.S., Veretennikov, A.Y.: Existence and uniqueness theorems for solutions of McKean-Vlasov stochastic equations. Theor. Probability Math. Statist. 103, 59–101 (2020)
51. Neelima, D., Biswas, S., Kumar, C., dos Reis, G., Reisinger, C.: Well-posedness and tamed Euler schemes for McKean-Vlasov equations driven by Lévy noise. arXiv:2010.08585
52. Röckner, M., Zhang, X.: Well-posedness of distribution dependent SDEs with singular drifts. Bernoulli 27, 1131–1158 (2021)
53. Röckner, M., Schmuald, B., Zhang, X.: Yamada-watanabe theorem for stochastic evolution equations in infinite dimensions. Condensed Matter Physics 11, 247–259 (2008)
54. Ren, J., Zhang, X.: Freidlin-wentzell’s large deviations for stochastic evolution equations. J. Funct. Anal. 254, 3148–3172 (2008)
55. Song, Y.: Gradient estimates exponential ergodicity for mean-field SDEs with jumps. J. Theoret. Probab. 33, 201–238 (2020)
56. Rao, Y., Yuan, C.: Central Limit Theorem and Moderate Deviation Principle for McKean-Vlasov SDEs. Acta Applicandae Mathematicae 175(16), 19 (2021)
57. Sznitman, A.S.: Topics in propagation of chaos. In École d’Été de probabilités de Saint-Flour XIX-1989. Lecture Notes in Math. 1464, 165–251 (1991)
58. Wang, R., Zhai, J., Zhang, T.: A moderate deviation principle for 2-D stochastic Navier-Stokes equations. J. Differential Equations 258, 3363–3390 (2015)
59. Xu, T., Zhang, T.: On the small time asymptotics of the two-dimensional stochastic Navier-Stokes equations. Ann. Inst. Henri poincaré Probab. Stat. 45(4), 1002–1019 (2009)
60. Yang, X., Zhai, J., Zhang, T.: Large deviations for SPDEs of jump type. Stochastics and Dynamics, 15 (2015) Article ID 1550026, 30 pages, https://doi.org/10.1142/S0219493715500264
61. Zhai, J., Zhang, T.: Large deviations for 2-D stochastic Navier-Stokes equations driven by multiplicative lévy noises. Bernoulli 21, 2351–2392 (2015)
62. Zhang, T.: On the small time asymptotics of diffusion processes on Hilbert spaces. Ann. Probab. 28(2), 537–557 (2000)
63. Zhao, H.: Yamada-watanabe theorem for stochastic evolution equation driven by Poisson Random Measure. ISRN Probability and Statistics 2014, 7 (2014). Article ID 982190

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