UNCONDITIONALITY OF PERIODIC ORTHONORMAL SPLINE SYSTEMS IN $L^p$

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Abstract. Given any natural number $k$ and any dense point sequence $(t_n)$ on the torus $T$, we prove that the corresponding periodic orthonormal spline system of order $k$ is an unconditional basis in $L^p$ for $1 < p < \infty$.

1. Introduction

In this work, we are concerned with periodic orthonormal spline systems of arbitrary order $k$ with arbitrary partitions. We let $(s_n)_{n=1}^\infty$ be a dense sequence of points in the torus $T$ such that each point occurs at most $k$ times. Such point sequences are called admissible.

For $n \geq k$, we define $\hat{S}_n$ to be the space of polynomial splines of order $k$ with grid points $(s_j)_{j=1}^n$. For each $n \geq k+1$, the space $\hat{S}_{n-1}$ has codimension 1 in $\hat{S}_n$ and, therefore, there exists a function $\hat{f}_n \in \hat{S}_n$ that is orthonormal to the space $\hat{S}_{n-1}$. Observe that this function $\hat{f}_n$ is unique up to sign. In addition, let $(\hat{f}_n)_{n=1}^k$ be an orthonormal basis for $\hat{S}_k$. The system of functions $(\hat{f}_n)_{n=1}^\infty$ is called periodic orthonormal spline system of order $k$ corresponding to the sequence $(s_n)_{n=1}^\infty$. We remark that if a point $x$ occurs $m$ times in the sequence $(s_n)_{n=1}^\infty$ before index $N$, the space $\hat{S}_N$ consists of splines that are in particular $(k-1-m)$ times continuously differentiable at $x$, where here for $k-1-m \leq -1$ we mean that no restrictions at the point $x$ are imposed. This means that if $m = k$ and also $s_N = x$, we have $\hat{S}_{N-1} = \hat{S}_N$ and therefore it makes no sense to consider non-admissible point sequences.

The main result of this article is the following

Theorem 1.1. Let $k \in \mathbb{N}$ and $(s_n)_{n \geq 1}$ be an admissible sequence of knots in $T$. Then the corresponding periodic orthonormal spline system of order $k$ is an unconditional basis in $L^p(T)$ for every $1 < p < \infty$.

This is the periodic version of the main result in [13]. We now give a few comments on the history of this result. We can similarly define the spaces $S_n$ corresponding to an admissible point sequence $(t_n)$ on the interval $[0, 1]$. A celebrated result of A. Shadrin [16] states that the orthogonal projection operator onto those spaces $S_n$ is bounded on $L^\infty[0, 1]$ by a constant that depends only on the spline order $k$. As a consequence, $(f_n)_{n=1}^k$ (also similarly defined to $\hat{f}_n$) is a Schauder basis in $L^p[0, 1]$, $1 \leq p < \infty$ and in the space of continuous functions $C^0[0, 1]$. There are various results on the unconditionality of spline systems restricting either the spline order $k$ or the partition $(t_n)_{n \geq 0}$. The
first result in this direction is [1], who proves that the classical Franklin system—that is orthonormal spline systems of order 2 corresponding to dyadic knot sequence \((1/2, 1/4, 3/4, 1/8, 3/8, \ldots)\)—is an unconditional basis in \(L^p[0, 1] \), \(1 < p < \infty\). This argument was extended in [3] to prove unconditionality of orthonormal spline systems of arbitrary order, but still restricted to dyadic knots. Considerable effort has been made in the past to weaken the restriction to dyadic knot sequences. In the series of papers [9, 11, 10] this restriction was removed step-by-step for general Franklin systems, with the final result that it was shown for each admissible point sequence \((t_n)_{n \geq 0}\) with parameter \(k = 2\), the associated general Franklin system forms an unconditional basis in \(L^p[0, 1]\), \(1 < p < \infty\). Combining the methods used in [11, 10] with some new inequalities from [15] it was proved in [13] that non-periodic orthonormal spline systems are unconditional bases in \(L^p[0, 1]\), \(1 < p < \infty\), for any spline order \(k\) and any admissible point sequence \((t_n)\).

The periodic analogue of Shadrin’s theorem can be obtained from Shadrin’s result [16] using [5]. Alternatively, [14] gives a direct proof. In case of dyadic knots, J. Domsta [8] obtained exponential decay for the inverse of the Gram matrix of periodic B-splines, which were exploited to prove the unconditionality of the periodic orthonormal spline systems with dyadic knots in \(L^p\) for \(1 < p < \infty\). In [12] it was proved that for any admissible point sequence the corresponding periodic Franklin system (i.e. the case \(k = 2\)) forms an unconditional basis in \(L^p[0, 1]\), \(1 < p < \infty\). Here we obtain an estimate for general periodic orthonormal spline functions, which combined with the methods developed in [10] yield the unconditionality of periodic orthonormal spline systems in \(L^p(\mathbb{T})\).

The main idea of the proofs of \((f_n)\) or \((\hat{f}_n)\) being an unconditional basis in \(L^p\), \(p \in (1, \infty)\) in the articles [10, 12, 13] is that corresponding to one single function \(f_n\), a grid point interval is associated on which most of the mass of \(f_n\) is concentrated. In case of Haar functions \(h_n\), its support splits into two intervals \(I\) and \(J\), where on \(I\), the function \(h_n\) is positive and on \(J\), \(h_n\) is negative. As the associated interval, we could just use the largest one of \(I\) and \(J\).

The organization of the present article is as follows. In Section 2 we give some preliminary results concerning polynomials, splines and non-periodic orthonormal spline functions. Section 3 develops crucial estimates for the periodic orthonormal spline functions \(\hat{f}_n\) and gives several relations between \(\hat{f}_n\) and its non-periodic counterpart. In Section 4 we prove a few technical lemmata used in the proof of Theorem 1.1 and Section 5 finally proves Theorem 1.1.

We remark that the results and most of the proofs in Sections 4 and 5 follow closely [10]. Let us also remark that the proof of the crucial Lemma 4.4 is new and much shorter than it was in [10].

2. Preliminaries

Let \(k\) be a positive integer. The parameter \(k\) will always be used for the order of the underlying polynomials or splines. We use the notation \(A(t) \sim B(t)\) to indicate the existence of two constants \(c_1, c_2 > 0\) that depend only on \(k\), such that \(c_1B(t) \leq A(t) \leq c_2B(t)\) for all \(t\), where \(t\) denotes all implicit and explicit dependences that the expressions \(A\) and \(B\) might have. If the constants \(c_1, c_2\) depend on an additional parameter \(p\), we write this as \(A(t) \sim_p B(t)\). Correspondingly, we use the symbols
For a subset $E$ of the real line, we denote by $|E|$ the Lebesgue measure of $E$ and by $1_E$ the characteristic function of $E$.

We will need the classical Remez inequality:

**Theorem 2.1** (Remez). Let $V \subset \mathbb{R}$ be a compact interval in $\mathbb{R}$ and $E \subset V$ a measurable subset. Then, for all polynomials $p$ of order $k$ on $V$,

$$
\|p\|_{L^\infty(V)} \leq \left( \frac{4|V|}{|E|} \right)^{k-1} \|p\|_{L^\infty(E)}.
$$

This immediately yields the following corollary:

**Corollary 2.2.** Let $p$ be a polynomial of order $k$ on a compact interval $V \subset \mathbb{R}$. Then

$$
\left\{ x \in V : |p(x)| \geq 8^{-k+1}\|p\|_{L^\infty(V)} \right\} \geq |V|/2.
$$

**Proof.** This is a direct application of Theorem 2.1 with the set $E$ := $\{ x \in V : |p(x)| \leq 8^{-k+1}\|p\|_{L^\infty(V)} \}$.

Let $T = (0 = \tau_{-k} = \cdots = \tau_0 \leq \cdots \leq \tau_{n-1} = \tau_n = \cdots = \tau_{n+k-1} = 1)$ be a partition of $[0, 1]$ consisting of knots of multiplicity at most $k$, that means $\tau_i < \tau_{i+k}$ for all $0 \leq i \leq n - 1$. Let $S_T$ be the space of polynomial splines of order $k$ with knots $T$. The basis of $L^\infty$-normalized B-spline functions in $S_T$ is denoted by $(N_{i,k})_{i=-k}^{n-k}$ or for short $(N_i)_{i=-k}^{n-k}$. Corresponding to this basis, there exists a biorthogonal basis of $S_T$, which is denoted by $(N_i^*)_{i=-k}^{n-k}$ or $(N_i^*)_{i=-k}^{n-k}$. Moreover, we write $\nu_i = \tau_i - \tau_{i-1} = |\text{supp} N_i|$. We continue with recalling a few important results for B-splines $N_i$ and their dual functions $N_i^*$.

**Theorem 2.3** (Shadrin [16]). Let $P$ be the orthogonal projection operator onto $S_T$ with respect to the canonical inner product in $L^2[0,1]$. Then, there exists a constant $C_k$ depending only on the spline order $k$ such that

$$
\|P : L^\infty[0,1] \rightarrow L^\infty[0,1]\| \leq C_k.
$$

**Proposition 2.4** (B-spline stability). Let $1 \leq p \leq \infty$ and $g = \sum_{j=-k}^{n-1} a_j N_j$ be a linear combination of B-splines. Then,

$$
|a_j| \lesssim |L_j|^{-1/p}\|g\|_{L^p(L_j)}, \quad -k \leq j \leq n - 1,
$$

where $L_j$ is a subinterval $[\tau_i, \tau_{i+1}]$ of $[\tau_j, \tau_{j+k}]$ of maximal length. Additionally,

$$
\|g\|_p \sim \left( \sum_{j=-k}^{n-1} |a_j|^p \nu_j \right)^{1/p} = \| (a_j \nu_j^{1/p} )_{j=-k}^{n-1} \|_{\ell^p}.
$$

Moreover, if $h = \sum_{j=-k}^{n-1} b_j N_j^*$,

$$
\|h\|_p \sim \left( \sum_{j=-k}^{n-1} |b_j|^p \nu_j^{1-p} \right)^{1/p} = \| (b_j \nu_j^{1/p-1} )_{j=-k}^{n-1} \|_{\ell^p}.
$$

The two inequalities (2.2) and (2.3) are Lemma 4.1 and Lemma 4.2 in [7] Chapter 5, respectively. Inequality (2.4) is a consequence of Theorem 2.3. For a deduction of the lower estimate in (2.4) from this result, see [3] Property P.7. The proof of the upper estimate uses a simple duality argument which we shall present here:
Proof of the upper estimate in (2.4). Just consider the case \( p < \infty \) and w.l.o.g. we assume \( b_j \geq 0 \). Let \( N_{j,p} = \nu_j^{-1/p}N_j \) be the \( p \)-normalized B-spline function and correspondingly \( N_{j,p}^* = \nu_j^{1/p}N_j^* \) be the \( p \)-normalized dual B-spline function. By definition, the system \( N_{j,p}^* \) forms a dual basis to the system of functions \( N_{j,p} \). By choosing \( p' = p/(p-1) \) and \( \alpha = 2/p' \) (so \( 2 - \alpha = 2/p \)), we obtain by (2.3)

\[
\sum_j b_j^2 = \left( \sum_j b_j^\alpha N_{j,p}^* \right) \sum_j b_j^{2-\alpha} N_{j,p} \leq \| \sum_j b_j^{2-\alpha} N_{j,p} \|_p \| \sum_j b_j^\alpha N_{j,p}^* \|_{p'}
\]

\[
= \| \sum_j b_j^{2-\alpha} \nu_j^{-1/p} N_j \|_p \| \sum_j b_j^\alpha \nu_j^{1/p} N_j^* \|_{p'}
\]

\[
\lesssim \left( \sum_j b_j^2 \right)^{1/p'} \| \sum_j b_j^\alpha \nu_j^{1/p} N_j^* \|_{p'}
\]

So we get

(2.5) \[
\left( \sum_j b_j^2 \right)^{1/p'} \lesssim \| \sum_j b_j^\alpha \nu_j^{1/p} N_j^* \|_{p'}
\]

Setting \( a_j = b_j^\alpha \nu_j^{1/p} \), we see that \( b_j^2 = (a_j \nu_j^{1/p})^{2/\alpha} = a_j^{p'/p} \nu_j^{p'-p} = a_j^{p'/p} \nu_j^{p'-1} \) and therefore, we may write (2.5) as

\[
\left( \sum_j a_j^{p'/p} \nu_j^{1-p} \right)^{1/p'} \lesssim \| \sum_j a_j N_j^* \|_{p'}
\]

which is the upper estimate in (2.4). \( \square \)

It can be shown that Shadrin’s theorem actually implies the following estimate on the B-spline Gram matrix inverse:

**Theorem 2.5 ([15]).** Let \( k \in \mathbb{N} \), the partition \( \mathcal{T} \) be defined as in (2.1) and \( (a_{ij})_{i,j=-k}^{n-1} \) be the inverse of the Gram matrix \((N_i, N_j)_{i,j=-k}^{n-1}\) of B-spline functions \( N_i = N_{i,k} \) of order \( k \) corresponding to the partition \( \mathcal{T} \). Then,

\[
|a_{ij}| \leq C \frac{q^{1-j}}{|\text{conv}(\text{supp} N_i \cup \text{supp} N_j)|}, \quad -k \leq i, j \leq n - 1,
\]

where the constants \( C > 0 \) and \( 0 < q < 1 \) depend only on the spline order \( k \) and where by \( \text{conv}(U) \) for \( U \subset [0, 1] \) we denote the smallest subinterval of \([0, 1]\) that contains \( U \).

Let \( f \in L^p[0,1] \) for some \( 1 \leq p < \infty \). Since the orthonormal spline system \((f_n)_{n \geq -k+2}\) is a basis in \( L^p[0,1] \), we can write \( f = \sum_{n=-k+2} a_n f_n \). Based on this expansion, we define the maximal function \( Mf := \sup_m \left| \sum_{n \leq m} a_n f_n \right| \). Given a measurable function \( g \), we denote by \( \mathcal{M}g \) the Hardy-Littlewood maximal function of \( g \) defined as

\[
\mathcal{M}g(x) := \sup_{I \ni x} |I|^{-1} \int_I |g(t)| \, dt,
\]

where the supremum is taken over all intervals \( I \) containing the point \( x \).

A corollary of Theorem 2.5 is the following relation between \( M \) and \( \mathcal{M} \):

**Theorem 2.6 ([15]).** If \( f \in L^1[0,1] \), we have

\[
Mf(t) \lesssim \mathcal{M}f(t), \quad t \in [0, 1].
\]
2.1. **Orthonormal spline functions, non-periodic case.** This section recalls some facts about orthonormal spline functions \( f_n = f_n^{(k)} \) for fixed \( k \in \mathbb{N} \) and \( n \geq 2 \) induced by the admissible sequence \( (t_n) \).

We consider again the mesh \( \mathcal{T} \) from before:

\[
\mathcal{T} = (0 = \tau_{-k} = \cdots = \tau_{-1} \leq \cdots \leq \tau_{i_0} \leq \cdots \leq \tau_{n-1} < \tau_n = \cdots = \tau_{n+k-1} = 1),
\]

where we singled out the point \( \tau_{i_0} \) and the partition \( \tilde{T} \) is defined to be the same as \( \mathcal{T} \), but with \( \tau_{i_0} \) removed. In the same way we denote by \( (N_i : -k \leq i \leq n - 1) \) the B-spline functions corresponding to \( \mathcal{T} \) and by \( (\tilde{N}_i : -k \leq i \leq n - 2) \) the B-spline functions corresponding to \( \tilde{T} \). Böhm’s formula \([2]\) gives us the following relationship between \( N_i \) and \( \tilde{N}_i \):

\[
\left\{\begin{array}{ll}
\tilde{N}_i(t) = N_i(t) & \text{if } -k \leq i \leq i_0 - k - 1, \\
\tilde{N}_i(t) = \frac{\tau_{i_0} - \tau_i}{\tau_{i+k} - \tau_i} N_i(t) + \frac{\tau_{i+k+1} - \tau_{i_0}}{\tau_{i+k+1} - \tau_{i+1}} N_{i+1}(t) & \text{if } i_0 - k \leq i \leq i_0 - 1, \\
\tilde{N}_i(t) = N_{i+1}(t) & \text{if } i_0 \leq i \leq n - 2.
\end{array}\right.
\]

In order to calculate the orthonormal spline function corresponding to the partitions \( \tilde{T} \) and \( \mathcal{T} \), we first determine a function \( g \in \text{span}\{N_i : -k \leq i \leq n - 1\} \) such that \( g \perp \tilde{N}_j \) for all \(-k \leq j \leq n - 2\). The function \( g \) is of the form (up to a multiplicative constant)

\[
g = \sum_{j=i_0-k}^{i_0} \alpha_j N_j^*,
\]

where \( (N_j^* : -k \leq j \leq n - 1) \) is the biorthogonal system to the functions \( (N_i : -k \leq i \leq n - 1) \) and

\[
\alpha_j = (-1)^{j-i_0+k} \left( \prod_{\ell=i_0-k+1}^{i_0} \frac{\tau_{i_0} - \tau_{\ell}}{\tau_{i+k} - \tau_{\ell}} \right) \left( \prod_{\ell=j+1}^{i_0-1} \frac{\tau_{\ell+k} - \tau_{i_0}}{\tau_{\ell+k} - \tau_{\ell}} \right), \quad i_0 - k \leq j \leq i_0.
\]

Alternatively, the coefficients \( \alpha_j \) can be described by the recursion

\[
\alpha_{i_0+1} \frac{\tau_{i+k+1} - \tau_{i_0}}{\tau_{i+k+1} - \tau_{i+1}} + \alpha_i \frac{\tau_{i_0} - \tau_i}{\tau_{i+k} - \tau_{i}} = 0.
\]

In order to give estimates for \( g \) and a fortiori, for the normalized function \( f = g/\|g\|_2 \), we assign to each function \( g \) a characteristic interval that is a grid point interval \([\tau_i; \tau_{i+1}]\) and lies in the proximity of the newly inserted point \( \tau_{i_0} \).

**Definition 2.7** ([13], Characteristic interval for non-periodic sequences). Let \( \mathcal{T}, \tilde{T} \) be as above and \( \tau_{i_0} \) be the new point in \( \mathcal{T} \) that is not present in \( \tilde{T} \). We define the **characteristic interval** \( J \) corresponding to \( \tau_{i_0} \) as follows.

1. Let

\[
\Lambda^{(0)} := \{i_0 - k \leq j \leq i_0 : ||[\tau_j; \tau_{j+k}]|| \leq 2 \min_{i_0 - k \leq \ell \leq i_0} ||[\tau_{\ell}; \tau_{\ell+k}]||\}
\]

be the set of all indices \( j \) for which the corresponding support of the B-spline function \( N_j \) is approximately minimal. Observe that \( \Lambda^{(0)} \) is nonempty.
Moreover, let the definition of the new grid point associated to the mesh point \( \tau \) the function from (13) Lemma 2.8, where \( J \) and therefore are defined to consist of the grid points \((t, s)\) spline functions \((t, s)\) torial lemma concerning the collection of characteristic intervals co rresponding to all \( x < \infty \) Combinatorics of characteristic intervals.

2.2. Similarly, for \( x > \sup J \), we have

\[
\| f \|_{L^p(0,x)} \lesssim \frac{q^{d_T(x)}|J|^{1/2}}{(|J| + \text{dist}(x,J))^{1-1/p}}, \quad 1 \leq p \leq \infty.
\]

Similarily, for \( x > \sup J \),

\[
\| f \|_{L^p(x,1)} \lesssim \frac{q^{d_T(x)}|J|^{1/2}}{(|J| + \text{dist}(x,J))^{1-1/p}}, \quad 1 \leq p \leq \infty.
\]

2.2. Combinatorics of characteristic intervals. We additionally have a combinatorial lemma concerning the collection of characteristic intervals corresponding to all grid points of an admissible sequence \((t_n)\) of points and the corresponding orthonormal spline functions \((f_n)_{n=-k+2}^\infty\) of order \( k \). For \( n \geq 2 \), the partitions \( \mathcal{T}_n \) associated to \( f_n \) are defined to consist of the grid points \((t_j)_{j=-n}^{n}\), the knots \( t_{-1} = 0 \) and \( t_0 = 1 \) having both multiplicity \( k \) in \( \mathcal{T}_n \) and we enumerate them as

\[
\mathcal{T}_n = (0 = \tau_{n,-k} = \cdots = \tau_{n,-1} < \tau_{n,0} \leq \cdots \leq \tau_{n,n-1} < \tau_{n,n} = \cdots = \tau_{n,n+k-1} = 1).
\]

If \( n \geq 2 \), we denote by \( J_n^{(0)} \) and \( J_n \) the characteristic intervals \( J_n^{(0)} \) and \( J \) from Definition 2.7 associated to the new grid point \( t_n \), which is defined to be the characteristic
interval associated to \((T_{n-1}, T_n)\). If \(n\) is in the range \(-k + 2 \leq n \leq 1\), we additionally set \(J_n := [0, 1]\).

**Lemma 2.9** (\cite{[13]}) Let \(V\) be an arbitrary subinterval of \([0, 1]\) and let \(\beta > 0\). Then there exists a constant \(F_{k,\beta}\) only depending on \(k\) and \(\beta\) such that

\[
\text{card}\{n : J_n \subseteq V, |J_n| \geq \beta |V|\} \leq F_{k,\beta},
\]

where \(\text{card} E\) denotes the cardinality of the set \(E\).

### 3. Periodic splines

In this section, we give estimates for periodic orthonormal spline functions \((\hat{f}_n)\) similar to the ones contained in Lemma 2.8 for non-periodic orthonormal splines. The main difficulty in proving such estimates is that we do not have a periodic version of Theorem 2.5 at our disposal. Instead, we estimate the differences between \(\hat{f}_n\) and two suitable non-periodic orthonormal spline functions \(f_n\).

Let \(n \geq k\) and \((\hat{N}_i)_{i=0}^{n-1}\) be periodic B-spline functions of order \(k\) with arbitrary admissible grid \((\sigma_j)_{j=0}^{n-1}\) on \(\mathbb{T}\) canonically identified with \([0, 1]\): \(\hat{T} = (0 \leq \sigma_0 \leq \sigma_1 \leq \cdots \leq \sigma_{n-2} \leq \sigma_{n-1} < 1)\).

Moreover, let \((\hat{N}^*_i)_{i=0}^{n-1}\) be the dual basis to \((\hat{N}_i)_{i=0}^{n-1}\) and \(\hat{S}_{\hat{T}}\) be the linear span of \((\hat{N}_i)_{i=0}^{n-1}\).

First, we recall that we have a periodic version of Shadrin’s theorem:

**Theorem 3.1.** Let \(\hat{P}\) be the orthogonal projection operator onto \(\hat{S}_{\hat{T}}\) with respect to the canonical inner product in \(L^2(\mathbb{T})\). Then, there exists a constant \(C_k\) depending only on the spline order \(k\) such that \(\|\hat{P} : L^\infty(\mathbb{T}) \to L^\infty(\mathbb{T})\| \leq C_k\).

We refer to the articles \cite{[16],[5]} for a proof of this result for infinite knot sequences on the real line, which then can be carried over to \(\mathbb{T}\). Alternatively, we refer to \cite{[14]} for a direct proof.

Next, note that B-spline stability carries over to the periodic setting:

**Proposition 3.2.** Let \(n \geq 2k\) and \(1 \leq p \leq \infty\). Then, for \(g = \sum_{j=0}^{n-1} a_j \hat{N}_j\), we have

\[
\|g\|_p \sim \left(\sum_{j=0}^{n-1} |a_j|^p |\text{supp} \hat{N}_j|\right)^{1/p} = \|(a_j \cdot |\text{supp} \hat{N}_j|^{1/p})_{j=0}^{n-1}\|_p.
\]

If we define the matrix \((\hat{a}_{ij})_{i,j=0}^{n-1} = (\langle \hat{N}^*_i, \hat{N}^*_j \rangle)^{-1}_{i,j=0}^{n-1}\), we have the following geometric decay inequality, which is a consequence of Theorem 3.1 on the uniform boundedness of the periodic orthogonal spline projection operator:

**Proposition 3.3.** Let \(n \geq 2k\). Then, there exists a constant \(q \in (0, 1)\) depending only on the spline order \(k\) such that

\[
|\hat{a}_{ij}| \lesssim \frac{q^\hat{d}(i,j)}{\max(|\text{supp} \hat{N}_i|, |\text{supp} \hat{N}_j|)}, \quad 0 \leq i, j \leq n - 1,
\]

where \(\hat{d}\) is the periodic distance function on \(\{0, \ldots, n - 1\}\).
The proof of this proposition follows along the same lines as in the non-periodic case, where B-spline stability and Demko’s theorem [6] on the geometric decay of inverses of band matrices is used. Its proof in the non-periodic case can be found in [4].

Observe that the estimate contained in this proposition for periodic splines is not as good as the one from Theorem 2.5 for non-periodic splines due to the different term in the denominator. Next, we also get stability of the periodic dual B-spline functions ($N_j^*$):

**Proposition 3.4.** Let $n \geq 2k$, $1 \leq p \leq \infty$ and $h = \sum_{j=0}^{n-1} b_j \hat{N}_j^*$, then

$$\|h\|_p \sim \left( \sum_{j=0}^{n-1} |b_j|^p \text{ supp } \hat{N}_j^{1-p} \right)^{1/p} = \|(b_j \cdot \text{ supp } \hat{N}_j^{1/p-1})_{j=0}^{n-1}\|_{lp}.$$  

**Proof.** We only prove the assertion for $p \in (1, \infty)$. The boundary cases follow from obvious modifications of the proof. By Propositions 3.3, 3.2 and Hölder’s inequality

$$\left\| \sum_j a_j \nu_j^p \hat{N}_j^* \right\|_p^p = \left\| \sum_j \left( \sum_i a_j \nu_j \hat{a}_{ij} \hat{N}_i \right) \right\|_p^p \leq \sum_i \left( \sum_j |a_j|^{1/p} \hat{a}_{ij} \right)^p \leq \sum_i \left( \sum_j |a_j|^{1/p} \hat{a}_{ij} \right)^p \leq \sum_i \left( \sum_j |a_j|^{1/p} \hat{a}_{ij} \right)^p \leq \sum_i \left( \sum_j |a_j|^{1/p} \hat{a}_{ij} \right)^p \leq \|a\|_p.$$ 

Setting $b_j = a_j \nu_j^{1/p'}$ yields the first inequality of dual B-spline stability. The other inequality is proved similarly to the result for the non-periodic case in Proposition 2.4. □

### 3.1. Periodic orthonormal spline functions.

We now consider the same situation as for the non-periodic case: Let

$$\hat{T} = (0 \leq \sigma_0 \leq \sigma_1 \leq \cdots \leq \sigma_{i_0} \leq \cdots \leq \sigma_{n-2} \leq \sigma_{n-1} < 1)$$

be a partition of $\mathbb{T}$ canonically identified with $[0, 1)$ and $\hat{\mathbb{T}}$ be the same partition, but with $\sigma_{i_0}$ removed. Similarly, we denote by $(\hat{N}_j)_{j=0}^{n-1}$ the periodic B-spline functions of order $k$ with respect to $\hat{T}$ and by $(\tilde{N}_j)_{j=0}^{n-2}$ the periodic B-spline functions of order $k$ with respect to $\hat{\mathbb{T}}$. Here, we use the notation of periodic extension of the sequence $(\sigma_j)_{j=0}^{n-1}$, i.e. $\sigma_{rn+j} = r + \sigma_j$ for $j \in \{0, \ldots, n-1\}$ and $r \in \mathbb{Z}$ and in the subindices of the B-spline functions, we take the indices modulo $n$. 
In order to calculate the periodic orthonormal spline functions corresponding to the above grids, we determine a function \( \hat{g} \in \text{span}\{\tilde{N}_i : 0 \leq i \leq n - 1\} \) such that \( \hat{g} \perp \tilde{N}_j \) for all \( 0 \leq j \leq n - 2 \). That is, we assume that \( \hat{g} \) is of the form

\[
\hat{g} = \sum_{k=0}^{n-1} \hat{\alpha}_j \tilde{N}_j^*,
\]

where \( (\tilde{N}_j^* : 0 \leq j \leq n - 1) \) is the system biorthogonal to the functions \( (\tilde{N}_i : 0 \leq i \leq n - 1) \) and \( \hat{\alpha}_j = \langle \hat{g}, \tilde{N}_j \rangle \). In order for \( \hat{g} \) to be orthogonal to \( \tilde{N}_j \) for \( 0 \leq j \leq n - 2 \), it has to satisfy the identities

\[
0 = \langle \hat{g}, \tilde{N}_i \rangle = \sum_{j=0}^{n-1} \hat{\alpha}_j \tilde{N}_j^* \tilde{N}_i, \quad 0 \leq i \leq n - 2.
\]

We can look at the indices \( j \) here periodically meaning that \( \hat{\alpha}_j \neq 0 \) only for \( j \in \{i_0 - k, \ldots, i_0\} \). Observing that the formulas for B-splines are local and thus, we are able to use formula (2.6) for \( \tilde{g} \) and \( \tilde{\sigma} \) to satisfy the identities

\[
\alpha (3.1) \quad \hat{\alpha}_j = \langle \hat{g}, \tilde{N}_j \rangle, \quad 0 \leq i \leq n - 2.
\]

Now, similarly to Definition 2.7, we are able to define characteristic intervals for periodic grids as follows.

**Definition 3.5** (Characteristic interval for periodic sequences). Let \( \tilde{T} \) be as above and \( \tilde{\sigma}_{i_0} \) be the new point in \( \tilde{T} \) that is not present in \( \tilde{T} \). Under the restriction \( n \geq 2k \), we define the characteristic interval \( \tilde{J} \) corresponding to \( \tilde{\sigma}_{i_0} \) as follows.

1. Let \( \Lambda^{(0)} := \{i_0 - k \leq j \leq i_0 : [[\sigma_j, \sigma_{j+k}]] \leq 2 \min_{i_0-k \leq \ell \leq i_0} [[\sigma_\ell, \sigma_{\ell+k}]]\} \)

be the set of all indices \( j \) in the vicinity of the index \( i_0 \) for which the corresponding support of the periodic B-spline function \( \tilde{N}_j \) is approximately minimal. Observe that \( \Lambda^{(0)} \) is nonempty.

2. Define \( \Lambda^{(1)} := \{j \in \Lambda^{(0)} : |\hat{\alpha}_j| = \max_{\ell \in \Lambda^{(0)}} |\hat{\alpha}_\ell| \} \).

For an arbitrary, but fixed index \( j^{(0)} \in \Lambda^{(1)} \), set \( \tilde{J}^{(0)} := [\sigma_{j^{(0)}}, \sigma_{j^{(0)}+k}] \).

\[
\hat{\alpha}_{i_0-k} = \prod_{\ell=i_0-k+1}^{i_0-1} \frac{\sigma_{\ell+k} - \sigma_i}{\sigma_{\ell+k} - \sigma_\ell}, \quad i_0 - k \leq j \leq i_0.
\]
The interval $\hat{J}^{(0)}$ can now be written as the union of $k$ grid intervals

$$\hat{J}^{(0)} = \bigcup_{\ell=0}^{k-1} [\sigma_{j^{(0)}+\ell}, \sigma_{j^{(0)}+\ell+1}]$$

with $j^{(0)}$ as above.

Define the characteristic interval $\hat{J} = \hat{J}(\sigma_{i_0})$ to be one of the above $k$ intervals that has maximal length.

### 3.2. $L^p$ norms of $\hat{g}$.

**Proposition 3.6.** Let $n \geq 2k + 2$. Then,

$$\|\hat{g}\|_p \sim |\hat{J}|^{1/p-1}, \quad 1 \leq p \leq \infty.$$  

**Proof.** We are able to arrange the periodic point sequence $(\sigma_j)_{j=0}^{n-1}$ such that $\sigma_0 > 0$ and $i_0 = \lfloor n/2 \rfloor$. Corresponding to this point sequence, we define a non-periodic point sequence $(\tau_j)_{j=0}^{n+k-1}$ by $\tau_j = \sigma_j$ for $j \in \{0, \ldots, n-1\}$, $\tau_{-k} = \cdots = \tau_{-1} = 0$ and $\tau_n = \cdots = \tau_{n+k-1} = 1$. With this choice and the assumption $n \geq 2k + 2$, the conditions $i_0 \leq k$ and $i_0 \leq n - k - 1$ are satisfied. Therefore, by comparing (2.8) to (3.2), we get $\alpha_j = \hat{\alpha}_j$ for $i_0 - k \leq j \leq i_0$, which yields

$$\hat{g} = \sum_{j=i_0-k}^{i_0} \hat{\alpha}_j \hat{N}_j^*, \quad g = \sum_{j=i_0-k}^{i_0} \hat{\alpha}_j N_j^*.$$  

Also, comparing the two definitions of $J$ and $\hat{J}$, in the present case we see that $|J| = |\hat{J}|$ and thus, we use B-spline stability to get

$$\|\hat{g}\|_p \sim \sum_{j=i_0-k}^{i_0} |\hat{\alpha}_j|^p \supp N_j |1-p| \sim \|g\|_p \sim |\hat{J}|^{p-1}.$$  

where the last equivalence follows from Lemma 2.8. \qed

**Lemma 3.7.** Let $n \geq 2k + 2$. If we write $\hat{g} = \sum_{i=0}^{n-1} \hat{w}_i \hat{N}_i$, we can estimate the coefficients $\hat{w}_i$ by

$$|\hat{w}_i| \lesssim q^{\hat{d}(i,i_0)} \frac{\max_{i_0-k \leq j \leq i_0} 1}{\max(|\supp \hat{N}_i|, |\supp \hat{N}_j|)}$$

where we take the index $j$ to be modulo $n$ and $\hat{d}$ is the periodic distance function on $\{0, \ldots, n-1\}$.

**Proof.** By looking at formula (3.2), we see that $|\hat{\alpha}_j| \leq 1$ for all $j$ and therefore, by Proposition 3.3

$$|w_i| = \left| \sum_{j=i_0-k}^{i_0} \hat{\alpha}_j \hat{a}_{ij} \right| \lesssim \sum_{j=i_0-k}^{i_0} |\hat{a}_{ij}| \lesssim \sum_{j=i_0-k}^{i_0} \max(|\supp \hat{N}_i|, |\supp \hat{N}_j|) \cdot q^{\hat{d}(i,j)}.$$  

This readily implies the assertion. \qed

**Proposition 3.8.** There exists an index $N(k)$ that depends only on $k$ such that for all partitions $\mathcal{T}$ with $n \geq N(k)$, we have

$$\|\hat{g}\|_{L^p(\mathcal{T})} \gtrsim |\hat{J}|^{1/p-1}, \quad p \in [1, \infty].$$  

Proof. Assuming again that $i_0 = \lfloor n/2 \rfloor$ and $n \geq 2k + 2$, we begin by considering the difference between the periodic function $\hat{g}$ to the non-periodic function $g$ corresponding to the partition $T = (\tau_j)_{j=-k}^{n+k-1}$ with $\tau_j = \sigma_j$ for $j \in \{0, \ldots, n-1\}$, $\tau_{-k} = \cdots = \tau_{-1} = 0$ and $\tau_n = \cdots = \tau_{n+k-1} = 1$:

$$u := g - \hat{g} = \sum_{j=-k}^{n-1} \beta_j N^*_j,$$

where the coefficients $\beta_j$ are chosen so that this equation is true. This is possible since both $g$ and $\hat{g}$ are contained in the linear span of the functions $(N^*_j)$. By defining the set of boundary indices $B$ in $T$ by

$$B = \{-k, \ldots, -1\} \cup \{n-k, \ldots, n-1\} \subset \{-k, \ldots, n-1\},$$

we see that for $j \in B$,

$$\beta_j = \langle u, N_j \rangle = \langle g - \hat{g}, N_j \rangle = \langle g, N_j \rangle - \langle \hat{g}, \hat{N}_j \rangle = \alpha_j - \hat{\alpha}_j = 0,$$

where the last equation follows from the fact that $\alpha_j = \hat{\alpha}_j$ for all indices $j$ in our current definition of $T$. Therefore, the function $u = g - \hat{g}$ can be expressed as

$$u = \sum_{j \in B} \beta_j N^*_j.$$

Now, we estimate the coefficients $\beta_j$ for $j \in B$ by Lemma 3.7:

$$|\beta_j| = |\langle g - \hat{g}, N_j \rangle| = |\langle \hat{g}, N_j \rangle|$$

$$= \sum_{i=0}^{n-1} w_i \langle \hat{N}_i, N_j \rangle \lesssim \sum_{i=0}^{n-1} |w_i| \cdot |\text{supp} \hat{N}_i \cap \text{supp} N_j|$$

$$\lesssim \sum_{i=0}^{n-1} q^{d(i,i_0)} \max_{m:i_0-k} \frac{1}{\text{max}(|\text{supp} \hat{N}_i|, |\text{supp} N_m|)} \cdot |\text{supp} \hat{N}_i \cap \text{supp} N_j|$$

$$\leq \sum_{i: |\text{supp} \hat{N}_i \cap \text{supp} N_j| > 0} q^{d(i,i_0)}$$

and, since $j \in B = \{-k, \ldots, -1\} \cup \{n-k, \ldots, n-1\}$,

$$(3.4) \quad |\beta_j| \lesssim q^{d(0,i_0)} \lesssim q^{n/2}, \quad j \in B.$$

So, we estimate for $x \in \hat{J}$:

$$|u(x)| = \left| \sum_{j \in B} \beta_j N^*_j(x) \right| = \left| \sum_{j \in B} \beta_j \sum_{i=-k}^{n-1} a_{ij} N_i(x) \right|$$

$$= \left| \sum_{j \in B} \sum_{i: j \subset \text{supp } N_i} a_{ij} N_i(x) \right|$$

$$\lesssim \sum_{j \in B} |\beta_j| \max_{i: j \subset \text{supp } N_i} |a_{ij}|.$$

So, by (3.4) and the estimate in Theorem 2.5 for the non-periodic matrix $(a_{ij})$

$$|u(x)| \lesssim q^{n/2} \max_{i: j \subset \text{supp } N_i} \max_{j \in B} \frac{q^{i-j}}{h_{ij}}.$$
where $h_{ij} = |\text{conv}(\text{supp} N_i \cup \text{supp} N_j)|$. Since $\hat{J} \subset \text{supp} N_i$ for the above indices $i$, we have $h_{ij} \geq |\hat{J}| = |J|$ and therefore,

$$|u(x)| = |(g - \hat{g})(x)| \lesssim q^n|J|^{-1}.$$  

This means that on $J$, we can estimate $\hat{g}$ from below: let $x \in J$ be a point such that $|g(x)| \geq \|g\|_{L^\infty(J)}/2$, then $|g(x)| \gtrsim |J|^{-1}$ by Lemma 2.8 and we get

$$|\hat{g}(x)| = |g(x) - (g(x) - \hat{g}(x))|$$

$$\geq |g(x)| - |g(x) - \hat{g}(x)|$$

$$\geq C_1|J|^{-1} - C_2|J|^{-1}q^n,$$

where $C_1$ and $C_2$ are constants that only depend on $k$ and $q < 1$. So there exists an index $N(k)$ such that for all $n \geq N(k)$

$$\|\hat{g}\|_{L^\infty(\hat{J})} \gtrsim |\hat{J}|^{-1}.$$  

Since $\hat{g}$ is a polynomial on $\hat{J}$, by Corollary 2.2 we now get for any $p \in [1, \infty]$

$$\|\hat{g}\|_{L^p(\hat{J})} \gtrsim |\hat{J}|^{1/p-1},$$

which is the assertion. \qed

3.3. More estimates for $\hat{g}$. We now change our point of view slightly and compare the function $\hat{g}$ with a non-periodic function $g$ where we shift the sequence $\hat{T} = (\sigma_j)_{j=0}^{n-1}$ in such a way that we split in the middle of a largest grid point interval:

$$\sigma_0 = 1 - \sigma_{n-1} = \frac{1}{2} \max_{0 \leq j \leq n-1} (\sigma_j - \sigma_{j-1}),$$

and, like before, choose $T = (\tau_j)_{j=-k}^{n+k-1}$ such that $\tau_j = \sigma_j$ for $j \in \{0, \ldots, n-1\}$ so that we have

$$\tau_0 - \tau_{-1} = \tau_n - \tau_{n-1} = \frac{1}{2} \max_{0 \leq j \leq n-1} (\sigma_j - \sigma_{j-1}).$$

We refer to this choice of $T$ as the maximal splitting of $\hat{T}$. Similar to above, we define $\hat{T}$ and $\hat{T}$ to be the partitions $\hat{T}$ and $T$ respectively, with the grid points $\sigma_{i_0}$ and $\tau_{i_0}$ removed.

If we work under this assumption, it is not necessarily the case that $|J| = |\hat{J}|$ as there is the possibility that $J$ lies near $\tau_0$ or $\tau_n$, but we have

**Proposition 3.9.** Let $J$ be the characteristic interval corresponding to the point sequences $(T, \hat{T})$ and $\hat{J}$ be the periodic one corresponding to $(\hat{T}, \hat{T})$ with the above maximal splitting. Then

$$|J| \sim |\hat{J}|.$$  

**Proof.** The definition of the characteristic intervals $J$ (Definition 2.7) and $\hat{J}$ (Definition 3.3) yield that

$$|J| \sim \min_{\tau_0-k \leq j \leq \tau_0} |\text{supp} N_j|, \quad |\hat{J}| \sim \min_{\tau_0-k \leq j \leq \tau_0} |\text{supp} \hat{N}_j|,$$

where

$$h_{ij} = |\text{conv}(\text{supp} N_i \cup \text{supp} N_j)|.$$
where the periodic indices are interpreted in the sense of the usual periodic continuation of the subindices. Then, the very definition of the point sequence $\mathcal{T}$ in terms of the point sequence $\mathcal{T}$ implies
\[
|\text{supp } N_j| \leq |\text{supp } \hat{N}_j|, \quad -k \leq j \leq n - 1,
\]
so, in combination with (3.5), we get the first inequality $|J| \lesssim |\hat{J}|$. In order to show the converse inequality, we show
\[
\min_{i_0 - k \leq j \leq i_0} |\text{supp } \hat{N}_j| \lesssim \min_{i_0 - k \leq j \leq i_0} |\text{supp } N_j|.
\]
We assume that $j_0$ is an index such that $|\text{supp } N_{j_0}| = \min_{i_0 - k \leq j \leq i_0} |\text{supp } N_j|$. If $j_0 \notin B = \{-k, \ldots, -1\} \cup \{n-k, \ldots, n-1\}$, we even have $|\text{supp } N_{j_0}| = |\text{supp } \hat{N}_{j_0}|$. If $j_0 \in B$, we have due to the choice of the maximal splitting
\[
|\text{supp } N_{j_0}| \geq \frac{1}{2} \max_{0 \leq j \leq n-1} (\sigma_{j+1} - \sigma_j) \geq \frac{1}{2k} |\text{supp } \hat{N}_{j_0}|
\]
for all indices $j$. So, in particular, (3.6) holds. Thus we have shown the converse inequality $|\hat{J}| \lesssim |J|$ as well and the proof is complete. □

We also have the following relation between the dual B-spline coefficients of $g$ and $\hat{g}$:

**Proposition 3.10.** For the maximal splitting, there exists a constant $c \sim 1$ such that for all $j \notin B$,
\[
\alpha_j = c \cdot \hat{\alpha}_j.
\]

**Proof.** Comparing the recursion formulas (2.9) for $\alpha_j$ and (3.1) for $\hat{\alpha}_j$, we see that for $j \in \{i_0 - k, \ldots, i_0 - 1\}$,
\[
\hat{\alpha}_{j+1} \alpha_j = \frac{\alpha_{j+1}}{\alpha_j}, \quad \{j, j+1\} \subset B^c
\]

since by definition $\tau_i = \sigma_i$ for $0 \leq i \leq n - 1$. So, now take an arbitrary $j \in B^c$. Looking at the formulas for $\alpha_j$ and $\hat{\alpha}_j$ we write
\[
\frac{\hat{\alpha}_j}{\alpha_j} = \left( \prod_{\ell = i_0 - k + 1}^{j-1} \frac{\sigma_{i_0} - \sigma_{\ell}}{\tau_{i_0} - \tau_{\ell}} \right) \left( \prod_{\ell = i_0 - k + 1}^{j-1} \frac{\tau_{\ell+k} - \tau_{\ell}}{\sigma_{\ell+k} - \sigma_{\ell}} \right) \left( \frac{\alpha_{j+1}}{\alpha_j} \right) \left( \frac{\alpha_{j+1}}{\alpha_j} \right) \left( \frac{\alpha_{j+1}}{\alpha_j} \right).
\]

Note that for every $s, t \in \{i_0 - k + 1, \ldots, i_0 + k - 1\}$ such that $0 < s - t \leq k$ either $\sigma_s - \sigma_t = \tau_s - \tau_t$ or $\sigma_s - \sigma_t > \tau_s - \tau_t$, and the latter can only happen when $[\tau_{i-1}, \tau_0]$ or $[\tau_{n-1}, \tau_n]$ is a subset of $[\tau_i, \tau_{i+1}]$, so
\[
\sigma_s - \sigma_t \geq \tau_s - \tau_t \geq \frac{1}{2} \max_{0 \leq j \leq n-1} (\sigma_{j+1} - \sigma_j) \geq \frac{1}{2k} (\sigma_s - \sigma_t).
\]

Hence we obtain $\sigma_s - \sigma_t \sim \tau_s - \tau_t$. Therefore $\alpha_j \sim \hat{\alpha}_j$. This, in combination with (3.7) proves the proposition. □

**Proposition 3.11.** Let $x \in [\sigma_\ell, \sigma_{\ell+1}]$. Then, there exists an interval $C = C(x) \subset \mathbb{T}$ which is minimal under the inclusion relation with
\[
\hat{J} \cup [\sigma_\ell, \sigma_{\ell+1}] \subset C
\]
such that if $K(C)$ is the number of grid points of $\hat{T}$ contained in $C$,

$$|\hat{g}(x)| \lesssim \frac{\hat{q}^{K(C)}}{|C|},$$

where $\hat{q} \in (0, 1)$ depends only on $k$.

Proof. In order to estimate $\hat{g}$, we consider the difference $u := g - c \cdot \hat{g}$ and $g$ separately, where $g$ is the orthogonal spline function corresponding to $(\tilde{T}, T)$ that arises from the maximal splitting and $c \sim 1$ denotes the constant from Proposition 3.10. We can write

$$u = \sum_{j=-k}^{n-1} \beta_j N_j^*$$

for some coefficients $\beta_j$. This is possible since $g$ as well as $\hat{g}$ is contained in the linear span of $(N_j^*)_{j=-k}^{n-1}$. We can write

$$\beta_j = \langle g - c \cdot \hat{g}, N_j \rangle = \langle g, N_j \rangle - c \cdot \langle \hat{g}, \hat{N}_j \rangle = \alpha_j - c \cdot \hat{\alpha}_j = 0,$$

where the last equality follows from Proposition 3.10. Therefore, the function $u = g - c \cdot \hat{g}$ can be expressed as

$$u = \sum_{j \in B} \beta_j N_j^*$$

and its coefficients $\beta_j$ can be estimated by

$$|\beta_j| = |\langle g - c \cdot \hat{g}, N_j \rangle| \lesssim |\langle g, N_j \rangle| + |\langle \hat{g}, N_j \rangle|$$

$$= \left| \sum_{i=-k}^{n-1} w_i \langle N_i, N_j \rangle \right| + \left| \sum_{i=0}^{n-1} \hat{w}_i \langle \hat{N}_i, N_j \rangle \right| =: \Sigma_1 + \Sigma_2.$$

Now, we estimate the term $\Sigma_1$ by using inequality (2.10) and the fact that $j \in B$:

$$\Sigma_1 \leq \sum_{i=-k}^{n-1} |w_i| |\text{supp } N_i \cap \text{supp } N_j|$$

$$\lesssim \sum_{i=-k}^{n-1} \frac{q^{d_T(\tau_i)}}{|J| + \text{dist}(\text{supp } N_i, J) + |\text{supp } N_i|} |\text{supp } N_i \cap \text{supp } N_j|$$

$$\leq \sum_{i: |\text{supp } N_i \cap \text{supp } N_j| > 0} q^{d(i, i_0)} \leq q^{d(0, i_0)}.$$
Combining the estimates for $\Sigma_1$ and $\Sigma_2$, we get $|\beta_j| \lesssim q^{d(0,i_0)}$. So, we continue to estimate $u(x)$ for $x \in [\tau_\ell, \tau_{\ell+1})$:

$$|u(x)| = \left| \sum_{j \in B} \beta_j N_j^*(x) \right| = \left| \sum_{j \in B} \beta_j \sum_{i=-k}^{n-1} a_{ij} N_i(x) \right|$$

$$= \left| \sum_{j \in B} \beta_j \sum_{i=\ell-k+1}^\ell a_{ij} N_i(x) \right|$$

$$\leq \sum_{j \in B} |\beta_j| \max_{i=\ell-k+1} |a_{ij}|.$$

So, by the above calculation and the estimate for the non-periodic Gram matrix inverse in Theorem 2.5,

$$|u(x)| \lesssim q^{d(0,i_0)} \max_{i=\ell-k+1} \max_{j \in B} \frac{q^{i-j}}{h_{ij}},$$

where $h_{ij} = |\text{conv}(\text{supp } N_i \cup \text{supp } N_j)|$. Since for $j \in B$, either $h_{ij} \geq \tau_0 - \tau_1$ or $h_{ij} \geq \tau_n - \tau_{n-1}$, we have by the defining property of the maximal splitting that $h_{ij} \geq \frac{1}{2} \max_{0 \leq m \leq n} (\sigma_m - \sigma_{m-1})$, and therefore,

$$|u(x)| \lesssim q^{d(i_0,\ell)} \frac{\max_m (\sigma_m - \sigma_{m-1})}{\max_m (\sigma_m - \sigma_{m-1})},$$

(3.9)

since we also have $\hat{d}(i_0, \ell) \leq \hat{d}(i_0, 0) + \hat{d}(0, \ell) \leq \hat{d}(i_0, 0) + 2k + \min_{j \in B, \ell-k+1 \leq i \leq \ell} |i - j|$. Thus, we conclude by Lemma 2.8 and (3.9)

$$|\dot{g}(x)| \leq c^{-1} |g(x)| + |\dot{g}(x) - c^{-1} g(x)| \lesssim \frac{q^{d(i_0,\ell)}}{\max_m (\sigma_m - \sigma_{m-1})} + \frac{q^{d(i_0,\ell)}}{\max_m (\sigma_m - \sigma_{m-1})},$$

which, with the use of Proposition 3.9 and the definitions of the characteristic intervals $J$ and $\dot{J}$, finishes the proof. \qed

So, by defining the normalized orthonormal spline function $\hat{f} = \hat{g}/\|\hat{g}\|_2$, we immediately obtain

**Corollary 3.12.** Let $U$ be an arbitrary subset of $\mathbb{T}$. Then,

$$\int_U |\hat{f}(x)|^p \, dx \lesssim |\hat{J}|^{p/2} \sum_{\ell, [\sigma_\ell, \sigma_{\ell+1}] \cap U \neq \emptyset} \frac{\hat{q}^{pK(C(\sigma_\ell))}}{|C(\sigma_\ell)|^p} |U \cap [\sigma_\ell, \sigma_{\ell+1}]|$$

where $\hat{q} \in (0, 1)$ depends only on $k$.

We will also need the pointwise estimate of the maximal spline projection operator by the Hardy-Littlewood maximal function in the periodic case, which is true in the non-periodic case by Theorem 2.6.

**Theorem 3.13.** Let $\hat{P}$ be the orthogonal projection onto $\hat{S}_P$. Then

$$|\hat{P}h(t)| \lesssim \hat{M}h(t), \quad h \in L^1(\mathbb{T}),$$

where $\hat{M}h(t) = \sup_{I \ni t} |I|^{-1} \int_I |h(y)| \, dy$ is the periodic Hardy-Littlewood maximal function operator and the sup is taken over all intervals $I \subset \mathbb{T}$ containing the point $t$. 


Proof. Let \( h \) be such that \( \text{supp} \ h \subset [\sigma_r, \sigma_{r+1}] \) for some \( \ell \in \{0, \ldots, n-1\} \). The first thing we show is that for any index \( r \),
\[
\| \hat{P} h \|_{L^1[\sigma_r, \sigma_{r+1}]} \lesssim q^{d(r, \ell)} \| h \|_{L^1}.
\]
For this, we write in the case \( t \in [\sigma_r, \sigma_{r+1}] \)
\[
\hat{P} h(t) = \sum_{j: \text{supp} \ N_j \ni t} \sum_{i: \text{supp} \ N_i \ni [\sigma_r, \sigma_{r+1}]} \hat{a}_{ij} \langle h, \hat{N}_i \rangle \hat{N}_j(t).
\]
After using Proposition 3.3 and a simple Hölder, this is less than
\[
\| h \|_{L^1} \sum_{j: \text{supp} \ N_j \ni t} \sum_{i: \text{supp} \ N_i \ni [\sigma_r, \sigma_{r+1}]} q^{d(i, j)} \frac{1}{\max(\| \text{supp} \ N_i \|, \| \text{supp} \ N_j \|)} \| \hat{N}_j(t) \|.
\]
Integrating this estimate over \([\sigma_r, \sigma_{r+1}]\), we get
\[
(3.10) \quad \| \hat{P} h \|_{L^1[\sigma_r, \sigma_{r+1}]} \lesssim \| h \|_{L^1} \cdot q^{d(\ell, r)}.
\]
The same can be proved for the non-periodic projection operator \( P \), since here we can use the same estimates.

Now, we take an arbitrary function \( h \) and localize it by setting
\[
h_\ell = h \cdot \mathbb{1}_{[\sigma_r, \sigma_{r+1}]}.
\]
We fix a point \( t \in [\sigma_m, \sigma_{m+1}] \) and associate to \( \hat{P} \) the non-periodic projection operator \( P \) corresponding to the maximal splitting. Then
\[
(3.11) \quad \hat{P} h(t) = P h(t) + (\hat{P} h(t) - P h(t)).
\]
In order to show \( \hat{P} h(t) \lesssim \hat{M} h(t) \), we first recall that Theorem 2.6 yields \( |P h(t)| \lesssim \hat{M} h(t) \leq M h(t) \). For the second term \( (\hat{P} - P) h \), we write
\[
(\hat{P} - P) h = \sum_{\ell=0}^{n-1} (\hat{P} - P) h_\ell
\]
and prove an estimate for \( g_\ell(t) := (\hat{P} - P) h_\ell \). Observe that
\[
g_\ell(t) = \sum_{i \in B} \langle g_\ell, N_i \rangle N_i^* (t) = \sum_{j=m-k+1}^{m} \sum_{i \in B} a_{ij} \langle g_\ell, N_i \rangle N_j(t),
\]
since the range of both \( \hat{P} \) and \( P \) is contained in the linear span of the functions \( (N_i^*) \) and \( h_\ell - \hat{P} h_\ell \) and \( h_\ell - P h_\ell \) are both orthogonal to the span of \( N_i, i \notin B \). Therefore, by using Theorem 2.5 for \( a_{ij} \),
\[
|g_\ell(t)| \lesssim \sum_{j=m-k+1}^{m} \sum_{i \in B} q_{ij}^{d(i, \ell)} \| g_\ell \|_{L^1(\text{supp} N_i)}.
\]
Consequently, by (3.10) and its non-periodic counterpart,
\[
(3.12) \quad |g_\ell(t)| \lesssim \sum_{j=m-k+1}^{m} \sum_{i \in B} q_{ij}^{d(i, \ell)} \| h_\ell \|_{L^1}.
\]
Since we performed the maximal splitting for our periodic partition, we get that
\[ h_{ij} \geq \frac{1}{2} \max_{\nu} (\sigma_\nu - \sigma_{\nu-1}), \quad i \in B. \]
Denoting by \( C_{\ell m} \) the convex set that contains \([\sigma_\ell, \sigma_{\ell+1}] \cup [\sigma_m, \sigma_{m+1}]\) containing the least grid points, we get
\[ h_{ij} \gtrsim \frac{|C_{\ell m}|}{\hat{d}(\ell, m)}, \quad i \in B. \]
Thus, we estimate (3.12) by
\[
\sum_{j=m-k+1}^{m} \sum_{i \in B} q^{i-j+\hat{d}(i,\ell)} \hat{d}(\ell, m) \frac{\|h\|_{L^1[\sigma_\ell,\sigma_{\ell+1}]]}}{|C_{\ell m}|} \lesssim \hat{d}(\ell, m) \max_{i \in B} (q^{i-m}+\hat{d}(i,\ell)) \cdot \hat{M}h(t)
\]
for all \( t \in [\sigma_m, \sigma_{m+1}] \). By the triangle inequality, \( \hat{d}(\ell, m) \leq \hat{d}(i, m) + \hat{d}(i, \ell) \leq |i-m| + \hat{d}(i, \ell) \) and thus we can estimate further
\[ |g_\ell(t)| \lesssim \max_{i \in B} \alpha^{i-m} \cdot \hat{M}h(t), \]
where \( \alpha \) can be chosen as \( q^{1/2} \). Summing this over \( \ell \), we finally obtain
\[ |(\hat{P} - P)h(t)| \lesssim \alpha^{\hat{d}(0,m)} \hat{M}h(t) \leq \hat{M}h(t), \]
which, in combination with (3.11) and the result for \( Ph(t) \) yields the assertion of the theorem. \( \square \)

3.4. Combinatorics of characteristic intervals. Similarly to the non-periodic case we are able to analyze the combinatorics of subsequent characteristic intervals. Let \( (s_n)_{n=1}^\infty \) be an admissible sequence of points in \( T \) and \((\hat{f}_n)_{n=1}^\infty \) be the corresponding periodic orthonormal spline functions of order \( k \). For \( n \geq 1 \), the partitions \( \hat{T}_n \) associated to \( \hat{f}_n \) are defined to consist of the grid points \((s_j)_{j=1}^n\) and we enumerate them as
\[ \hat{T}_n = (0 \leq \sigma_{n,0} \leq \cdots \leq \sigma_{n,n-1} < 1). \]
If \( n \geq 2k \), we denote by \( \hat{J}_n^{(0)} \) and \( \hat{J}_n \) the characteristic intervals \( \hat{J}^{(0)} \) and \( \hat{J} \) from Definition 3.5 associated to the new grid point \( s_n \), which is defined to be the characteristic interval associated to \((\hat{T}_{n-1}, \hat{T}_n)\). For any \( x \in T \), let \( C_n(x) \) be the interval from Proposition 3.11 associated to \( \hat{J}_n \). We define \( \hat{d}_n(x) \) to be the number of grid points in \( \hat{T}_n \) between \( x \) and \( \hat{J}_n \) contained in \( C_n(x) \) counting \( x \) and endpoints of \( \hat{J}_n \). Moreover, for a subinterval \( V \) of \( T \), we denote \( \hat{d}_n(V) = \min_{x \in V} \hat{d}_n(x) \).
An immediate consequence of the definition of \( \hat{J}_n \) is that the sequence of characteristic intervals \( (\hat{J}_n) \) forms a nested collection of sets, i.e., two sets in it are either disjoint or one is contained in the other.
Since the definition of the characteristic interval \( \hat{J}_n \) only involves local properties of the point sequence \((s_j)\) and the definition of \( \hat{J}_n \) is the same as the definition of \( J_n \) for any identification of \( T \) with \([0, 1]\) that has the property that between the newly inserted point \( s_0 \) and 0 or 1 are more than \( k \) grid points of \( T_n \), we also get the periodic version of Lemma 2.9.
Lemma 3.14. Let $V$ be an arbitrary subinterval of $\mathbb{T}$ and let $\beta > 0$. Then there exists a constant $F_{k,\beta}$ only depending on $k$ and $\beta$ such that
\[
\text{card}\{n \geq 2k : \hat{J}_n \subseteq V, |\hat{J}_n| \geq \beta |V|\} \leq F_{k,\beta}.
\]

Additionally, Lemma 3.14 has the following corollary:

Corollary 3.15. Let $(\hat{J}_n)_{n=1}^\infty$ be a decreasing sequence of characteristic intervals, i.e. $\hat{J}_{n+1} \subseteq \hat{J}_n$. Then, there exists a number $\kappa \in (0,1)$ and a constant $C_k$, both depending only on $k$ such that
\[
|\hat{J}_n| \leq C_k \kappa^i |\hat{J}_n|, \quad i \in \mathbb{N}.
\]

4. Technical estimates

The lemmas proved in this section are similar to the corresponding results in [10] or [13] and also the proofs are more or less the same. The exception is Lemma 3.14 for which we give a renewed and shorter proof.

Lemma 4.1. Let $N(k)$ be the number given by Proposition 3.8, $f = \sum_{n \geq N(k)}^\infty a_n \hat{f}_n$ and $V$ be a subinterval of $\mathbb{T}$. Then,
\[
(4.1) \quad \int_{V^c} \sum_{j \in \Gamma} |a_j \hat{f}_j(t)| \, dt \lesssim \int_V \left( \sum_{j \in \Gamma} |a_j \hat{f}_j(t)|^2 \right)^{1/2} \, dt,
\]

where
\[
\Gamma := \{ j : \hat{J}_j \subset V \text{ and } N(k) \leq j < \infty \}.
\]

Proof. First, assume that $|V| = 1$. Then (4.1) holds trivially. In the following, we assume that $|V| < 1$. Fixing $n \in \Gamma$, Corollary 3.12 and Proposition 3.8 then imply
\[
(4.2) \quad \int_{V^c} |\hat{f}_n(t)| \, dt \lesssim \hat{q}^{\alpha_n(V^c)} |\hat{J}_n|^{1/2} \lesssim \hat{q}^{\alpha_n(V^c)} \int_{\hat{J}_n} |\hat{f}_n(t)| \, dt.
\]

Now choose $\beta = 1/4$ and let $\hat{J}_n^{\beta}$ be the unique closed interval that satisfies
\[
|\hat{J}_n^{\beta}| = \beta |\hat{J}_n| \quad \text{and} \quad \text{center}(\hat{J}_n^{\beta}) = \text{center}(\hat{J}_n).
\]

Since $f_n$ is a polynomial of order $k$ on the interval $J_n$, we apply Corollary 2.2 to (4.2) and estimate further
\[
(4.3) \quad \int_{V^c} |a_n \hat{f}_n(t)| \, dt \lesssim \hat{q}^{\alpha_n(V^c)} \int_{\hat{J}_n^{\beta}} |a_n \hat{f}_n(t)| \, dt \leq \hat{q}^{\alpha_n(V^c)} \int_{\hat{J}_n^{\beta}} \left( \sum_{j \in \Gamma} |a_j \hat{f}_j(t)|^2 \right)^{1/2} \, dt.
\]

Define $\Gamma_s := \{ j \in \Gamma : \hat{d}_j(V^c) = s \}$ for $s \geq 0$. If $(\hat{J}_n_j)_{j=1}^N$ is a decreasing sequence of characteristic intervals with $n_j \in \Gamma_s$, we can split $(\hat{J}_n_j)$ into at most two groups so that for each group one endpoint of $\hat{J}_n_j$ coincides for each $j \in \Gamma$ since $\hat{J}_j \subset V$ for all $j \in \Gamma$.

So, Lemma 3.14 implies that there exists a constant $F_k$ only depending on $k$, such that each point $t \in V$ belongs to at most $F_k$ intervals $\hat{J}_j^{\beta}, j \in \Gamma_s$. Thus, summing over
Proof. First, we observe that in the case $g > \lambda$ by Lemma 4.3. Let $t \in E_{\lambda} \setminus B_{\lambda;r}$.

Let $g$ be a real-valued function defined on the torus $T$. In the following, we denote by $[g > \lambda]$ the set $\{x \in T : g(x) > \lambda\}$ for any number $\lambda > 0$.

**Lemma 4.2.** Let $f = \sum_{n=1}^{\infty} a_n \hat{f}_n$ with only finitely many nonzero coefficients $a_n$, $\lambda > 0$, $r < 1$ and

$$E_{\lambda} = [Sf > \lambda], \quad B_{\lambda;r} = [\hat{M} \mathbb{1}_{E_{\lambda}} > r],$$

where $Sf(t)^2 = \sum_{n=1}^{\infty} a_n^2 \hat{f}_n(t)^2$ is the spline square function. Then we have $E_{\lambda} \subset B_{\lambda;r}$.

**Proof.** Let $t \in E_{\lambda}$ be fixed. The square function $Sf(t) = \left( \sum_{n=1}^{\infty} |a_n \hat{f}_n|^2 \right)^{1/2}$ is continuous except possibly at finitely many grid points, where $Sf$ is at least continuous from one side. As a consequence, for $t \in E_{\lambda}$, there exists an interval $I \subset E_{\lambda}$ such that $t \in I$. This implies the following estimate:

$$(\hat{M} \mathbb{1}_{E_{\lambda}})(t) = \sup_{t \in U} |U|^{-1} \int_U \mathbb{1}_{E_{\lambda}}(x) \, dx$$

$$= \sup_{t \in U} \frac{|E_{\lambda} \cap U|}{|U|} \geq \frac{|E_{\lambda} \cap I|}{|I|} = \frac{|I|}{|I|} = 1 > r.$$ 

The above inequality shows $t \in B_{\lambda;r}$, proving the lemma. \(\square\)

**Lemma 4.3.** Let $f = \sum_{n \geq N(k)} a_n \hat{f}_n$ with only finitely many nonzero coefficients $a_n$, $\lambda > 0$ and $r < 1$, where $N(k)$ is the number given by Proposition 3.8. Then we define

$$E_{\lambda} := [Sf > \lambda], \quad B_{\lambda;r} := [\hat{M} \mathbb{1}_{E_{\lambda}} > r],$$

where $Sf(t)^2 = \sum_{n \geq N(k)} a_n^2 \hat{f}_n(t)^2$ is the spline square function. If

$$\Lambda = \{n : \hat{J}_n \not\subset B_{\lambda;r} \text{ and } N(k) \leq n < \infty\} \quad \text{and} \quad g = \sum_{n \in \Lambda} a_n \hat{f}_n,$$ 

we have

$$\int_{E_{\lambda}} Sg(t)^2 \, dt \lesssim \int_{E_{\lambda}} Sg(t)^2 \, dt \quad \quad (4.4)$$

**Proof.** First, we observe that in the case $B_{\lambda;r} = T$, the index set $\Lambda$ is empty and thus, (4.4) holds trivially. So let us assume $B_{\lambda;r} \neq T$. Then, we start the proof of (4.4) with an application of Proposition 3.6 and Proposition 3.8 to obtain

$$\int_{E_{\lambda}} Sg(t)^2 \, dt = \sum_{n \in \Lambda} \int_{E_{\lambda}} |a_n \hat{f}_n(t)|^2 \, dt \lesssim \sum_{n \in \Lambda} \int_{E_{\lambda}} |a_n \hat{f}_n(t)|^2 \, dt.$$
We split the latter expression into the parts

\[ I_1 := \sum_{n \in \Lambda} \int_{J_n \cap E^c_n} |a_n \hat{f}_n(t)|^2 \, dt, \quad I_2 := \sum_{n \in \Lambda} \int_{\hat{J}_n \cap E_n} |a_n \hat{f}_n(t)|^2 \, dt. \]

For \( I_1 \), we clearly have

\[ I_1 \leq \sum_{n \in \Lambda} \int_{E^c_n} |a_n \hat{f}_n(t)|^2 \, dt = \int_{E^c_n} Sg(t)^2 \, dt. \]  

(4.5)

It remains to estimate \( I_2 \). First we observe that by Lemma 4.2, \( E_n \subset B_{\lambda, r} \). Since the set \( B_{\lambda, r} = [\hat{\mathcal{M}} \mathbb{1}_{E_n} > r] \) is open in \( \mathbb{T} \), we decompose it into a countable collection of disjoint open subintervals \( (V_j)_{j=1}^\infty \) of \( \mathbb{T} \). Utilizing this decomposition, we estimate

\[ I_2 \leq \sum_{n \in \Lambda} \sum_{j : |\hat{J}_n \cap V_j| > 0} \int_{\hat{J}_n \cap V_j} |a_n \hat{f}_n(t)|^2 \, dt. \]  

(4.6)

If the indices \( n \) and \( j \) are such that \( n \in \Lambda \) and \( |\hat{J}_n \cap V_j| > 0 \), then, by definition of \( \Lambda \), \( \hat{J}_n \) is an interval containing at least one endpoint \( x \) of \( V_j \) for which

\[ \mathcal{M} \mathbb{1}_{E_n}(x) \leq r. \]

This implies

\[ |E^c_n \cap \hat{J}_n \cap V_j| \leq r \cdot |\hat{J}_n \cap V_j| \quad \text{or equivalently} \quad |E^c_n \cap \hat{J}_n \cap V_j| \geq (1 - r) \cdot |\hat{J}_n \cap V_j|. \]

Using this inequality and that \( |\hat{f}_n|^2 \) is a polynomial of order \( 2k - 1 \) on \( \hat{J}_n \) allows us to use Corollary 2.2 to conclude from (4.6)

\[ I_2 \lesssim_r \sum_{n \in \Lambda} \sum_{j : |\hat{J}_n \cap V_j| > 0} \int_{E^c_n \cap \hat{J}_n \cap V_j} |a_n \hat{f}_n(t)|^2 \, dt \]

\[ \leq \sum_{n \in \Lambda} \int_{E^c_n \cap \hat{J}_n \cap B_{\lambda, r}} |a_n \hat{f}_n(t)|^2 \, dt \]

\[ \leq \sum_{n \in \Lambda} \int_{E^c_n} |a_n \hat{f}_n(t)|^2 \, dt = \int_{E^c_n} Sg(t)^2 \, dt. \]

The latter inequality combined with (4.5) completes the proof the lemma.

\[ \square \]

**Lemma 4.4.** Let \( V \) be an open subinterval of \( \mathbb{T} \) and \( f = \sum_n \hat{a}_n \hat{f}_n \in L^p(\mathbb{T}) \) for \( p \in (1, \infty) \) with \( \text{supp} \, f \subset V \). Then, there exists a number \( R > 1 \) depending only on \( k \) such that

\[ \sum_n R^{\beta d_n(V)} |\hat{a}_n|^p \| \hat{f}_n \|_{L^p(\hat{V}^c)}^p \lesssim_{p, R} \| f \|_p^p, \]

(4.7)

with \( \hat{V} \) being the interval with the same center as \( V \) but with three times the diameter.

**Proof.** We observe first that we can assume that \( |V| \leq 1/3 \), since otherwise \( |\hat{V}^c| = 0 \) and the left hand side of (4.7) is zero.

We start by estimating \( |\hat{a}_n| \). Depending on \( n \), we partition \( V \) into intervals \( (A_{n,j})_{j=1}^{N_n} \), where except at most two intervals at the boundary of \( V \), we choose \( A_{n,j} \) to be a grid point interval in the grid \( \hat{F}_n \). Let \( I_{n,\ell} := [\sigma_{n,\ell}, \sigma_{n,\ell+1}] \) be the \( \ell \)th grid point interval in
\( \hat{T}_n \). Moreover, for a grid point interval \( I \) in grid \( \hat{T}_n \) and all subsets \( E \subset I \), we set \( C_n(E) \) to be the interval given by Proposition 3.11 that satisfies
\[
C_n(E) \supset I \cup \hat{J}_n
\]
and \( K_n(C_n(I)) \) denotes the number of grid points from \( \hat{T}_n \) that are contained in the set \( C_n(I) \). Next, we define \( r_n = \min_{\ell_n \in \mathbb{C}} K_n(C_n(I_\ell)) \), \( a_{n,j} = K_n(C_n(A_{n,j})) \) and we choose a number \( S > 1 \) which we will specify later and estimate by Hölder’s inequality with the dual exponent \( p' = p/(p-1) \) to \( p \),
\[
|\hat{a}_n| = |\langle f, \hat{f}_n \rangle| = |\sum_{j=1}^{N_n} \int_{A_{n,j}} f(t) \hat{f}_n(t) \, dt|
\leq \sum_{j=1}^{N_n} \left( \int_{A_{n,j}} |f(t)|^p \, dt \right)^{1/p} \left( \int_{A_{n,j}} |\hat{f}_n(t)|^{p'} \, dt \right)^{1/p'}
= \sum_{j=1}^{N_n} S^{-a_{n,j}} S^{a_{n,j}} \left( \int_{A_{n,j}} |f(t)|^p \, dt \right)^{1/p} \left( \int_{A_{n,j}} |\hat{f}_n(t)|^{p'} \, dt \right)^{1/p'}
\leq \left( \sum_{j=1}^{N_n} S^{-p'a_{n,j}} \int_{A_{n,j}} |f(t)|^p \, dt \cdot \left( \int_{A_{n,j}} |\hat{f}_n(t)|^{p'} \, dt \right)^{p-1} \right)^{1/p}.
\]
Since the first sum above is a geometric series and by using Corollary 3.12 on the interval of \( \hat{f}_n \), we obtain
\[
|\hat{a}_n| \leq \left( \sum_{j=1}^{N_n} S^{pa_{n,j}} \int_{A_{n,j}} |f(t)|^p \, dt \cdot |\hat{J}_n|^{p/2} \frac{q_{pa_{n,j}}(A_{n,j})^{p-1}}{|C_n(A_{n,j})|^p} \right)^{1/p}.
\]
We also estimate \( \|\hat{f}_n\|_{L^p(\hat{V}^c)} \) by Corollary 3.12 and get
\[
\|\hat{f}_n\|_{L^p(\hat{V}^c)} \lesssim |\hat{J}_n|^{p/2} q^{pr_n} \sum_{\ell_n \in \hat{V}^c \cap I_{n,\ell} \neq \emptyset} \frac{1}{|C_n(I_{n,\ell})|^p} \lesssim |\hat{J}_n|^{p/2} q^{pr_n} \int_{\hat{V}^c} \sum_{\ell_n \in \hat{V}^c \cap I_{n,\ell} \neq \emptyset} \frac{1}{|C_n(I_{n,\ell})|^p} \, dt.
\]
By integration of the function \( t \mapsto t^{-p} \), this is dominated by
\[
|\hat{J}_n|^{p/2} q^{pr_n} \min_{\ell_n \in \hat{V}^c \cap I_{n,\ell} \neq \emptyset} |C_n(I_{n,\ell})|^{p-1}.
\]
For an arbitrary set \( E \subset \mathbb{T} \), let \( \ell_0(E) \) be an index such that \( I_{n,\ell_0(E)} \cap E \neq \emptyset \) and
\[
|C_n(I_{n,\ell_0(E)})| = \min_{\ell_n \in I_{n,\ell} \cap E \neq \emptyset} |C_n(I_{n,\ell})|.
\]
Then, we introduce one more notation and set \( B_n(E) \subset C_n(I_{n,\ell_0(E)}) \) to be the largest interval \( B \) such that \( B \cap E = \hat{J}_n \cap E \). Obviously \( B_n(E) \supset \hat{J}_n \) for every \( E \). Using this notation, we estimate \( \|\hat{f}_n\|_{L^p(\hat{V}^c)} \) and conclude
\[
\|\hat{f}_n\|_{L^p(\hat{V}^c)} \lesssim \frac{|\hat{J}_n|^{p/2} q^{pr_n}}{|B_n(\hat{V}^c)|^{p-1}}.
\]
Combining (4.8) and (4.10) yields
\[
\sum_n R^{pd_n(V)}|\hat{\alpha}_n|^p\|\hat{f}_n\|_{L^p(\hat{V})}^p \\
\lesssim \sum_n |\hat{J}_n|^p \hat{q}^{p\alpha_n} R^{pd_n(V)}|B_n(\hat{V})|^{1-p} \cdot \left(\sum_{j=1}^{\frac{N_n}{n}}|\hat{q}S|^p a_{n,j} \int_{A_{n,j}} |f(t)|^p \, dt \cdot \frac{|A_{n,j}|^{p-1}}{|C_n(A_{n,j})|^p}\right).
\]
Since \((A_{n,j})_{n=1}^N\) is a partition of \(V\) for any \(n\), we further write
\[
\sum_n R^{pd_n(V)}|\hat{\alpha}_n|^p\|\hat{f}_n\|_{L^p(\hat{V})}^p \\
\lesssim \int_V \sum_n \left(\frac{\hat{J}_n}{|\hat{B}_n(\hat{V})|}\right)^{p-1} \hat{q}^{p\alpha_n} R^{pd_n(V)} \sum_{j=1}^{\frac{N_n}{n}}|\hat{q}S|^p a_{n,j} \frac{|\hat{J}_n| |A_{n,j}|^{p-1}}{|C_n(A_{n,j})|^p} 1_{A_{n,j}}(t) |f(t)|^p \, dt.
\]
In order to estimate this by \(\int_V |f(t)|^p \, dt\), we estimate pointwise for fixed \(t \in V\). To do this, we first observe that we have to estimate the expression
\[
\sum_n \left(\frac{\hat{J}_n}{|\hat{B}_n(\hat{V})|}\right)^{p-1} \hat{q}^{p\alpha_n} R^{pd_n(V)} |\hat{q}S|^p a_{n,j} \frac{|\hat{J}_n| |A_{n,j}|^{p-1}}{|C_n(A_{n,j})|^p},
\]
where \(A_{n,j(n)}\) is just the interval \(A_{n,j}\) such that \(t \in A_{n,j}\). Next, we split the summation index set into \(\cup T_s\), where
\[
T_s = \{n : r_n + a_{n,j(n)} = s\}.
\]
Since \(\hat{d}_n(V) \leq a_{n,j(n)}\), we get that if \(R, S > 1\) are such that \(RS\hat{q} < 1\), there exists \(\alpha < 1\) depending only on \(k\), such that the above expression is \(\lesssim\)
\[
\sum_{s=0}^{\infty} \alpha^s \sum_{n \in T_s} \left(\frac{\hat{J}_n}{|\hat{B}_n(\hat{V})|}\right)^{p-1} \frac{|\hat{J}_n| |A_{n,j(n)}|^{p-1}}{|C_n(A_{n,j(n)})|^p}.
\]
Now, we split the analysis of this expression into two cases:

**Case 1:** \(T_{s,1} = \{n \in T_s : |\hat{B}_n(\hat{V})| \leq |B_n(V)|\) or \(|V| \leq |\hat{J}_n|\).

We want to estimate the inner sum in (4.11) over \(n \in T_{s,1}\), which in the present case is immediately estimated by
\[
\sum_{n \in T_{s,1}} \frac{|\hat{J}_n| |A_{n,j(n)}|^{p-1}}{|C_n(A_{n,j(n)})|^p}.
\]
In order to estimate this sum, we further split the set \(T_{s,1}\) into
\[
S_1 = \{n \in T_{s,1} : \hat{J}_n \text{ contains at least one of the two endpoints of } V\};
\]
\[
S_2 = T_{s,1} \setminus S_1.
\]
By the conditions of Case 1 and the definition of \(\hat{V}\), if \(n \in S_1\), we have \(|\hat{J}_n| \geq |V|\) and a geometric decay in the length of \(\hat{J}_n\) by Corollary 3.15, therefore,
\[
\sum_{n \in S_1} \frac{|\hat{J}_n| |A_{n,j(n)}|^{p-1}}{|C_n(A_{n,j(n)})|^p} \leq \sum_{n \in S_1} \frac{|\hat{J}_n||V|^{p-1}}{|C_n(A_{n,j(n)})|^p} \leq \sum_{n \in S_1} \left(\frac{|V|}{|\hat{J}_n|}\right)^{p-1} \lesssim 1.
\]
Next, observe that under the conditions in Case 1 and the definition of \(S_2\), we have \(|\hat{J}_n \cap V| = 0\) for \(n \in S_2\). Since additionally \((A_{n,j(n)})\) is a decreasing family of subsets
of $V$ and since $r_n + a_{n,j(n)} = s$ for $n \in T_s$, we can split $S_2$ into two subsets $S_{2,1}$ and $S_{2,2}$ such that for two different indices $n_1, n_2 \in S_{2,i}$ for $i \in \{1, 2\}$, we have that the corresponding intervals $\hat{J}_{n_1}$ and $\hat{J}_{n_2}$ are either disjoint or share an endpoint.

If $n \in S_{2,2}$, then an endpoint $a$ of $B_n(V)$ coincides with an endpoint of $V$ (since $\hat{J}_n \subset V$). In this case, we let $B_n(t) \subset B_n(V)$ be the interval with the endpoints $t$ and $a$ for $t \in B_n(V)$. Let $\hat{J}_n^\beta$ for $\beta = 1/4$ be the interval characterized by the properties

$$\hat{J}_n^\beta \subset \hat{J}_n, \text{ center}(\hat{J}_n^\beta) = \text{center}(\hat{J}_n), \quad |\hat{J}_n^\beta| = |\hat{J}_n|/4.$$  

By Lemma 3.14 for each point $u \in T$, there exist at most $F_k$ indices in $S_{2,i}$ such that $u \in \hat{J}_n^\beta$. We now enumerate the intervals $\hat{J}_n$ with $n \in S_{2,i}$ in the following way: Since those are nested, we write $\hat{J}_{n}^\ell$ for the maximal ones under the inclusion relation and we enumerate as $\hat{J}_{n}^\ell$ such that $\hat{J}_{n}^\ell \subset \hat{J}_{n}$ for all $j$. Since an endpoint of $\hat{J}_{n}^\ell \subset \hat{J}_{n}$ for $n_1, n_2 \in S_{2,i}$ coincides, for each maximal interval $\hat{J}_{n}^\ell$, we have at most two sequences of this form.

Using this enumeration, we write

$$\sum_{n \in S_{2,i}} \frac{|A_{n,j(n)}|^{p-1}|\hat{J}_n^\beta|}{C_n(A_{n,j(n)})^{p}} \leq 2\beta^{-1}|V|^{p-1} \sum_{\ell,j} \int_{\hat{J}_n^\ell} \frac{dt}{|B_{n\ell,j}(t)|^{p}}.$$  

Observe that the function

$$x \mapsto |\{\ell, j, t : t \in \hat{J}_n^\ell, x = B_{n\ell,j}(t)\}|$$  

is uniformly bounded by $4F_k$ for all $x \geq 0$. Since we also have the estimate

$$|V|/2 \leq |B_n(t)|, \quad n \in S_{2,i}, t \in \hat{J}_n^\beta,$$

we conclude

$$\beta^{-1}|V|^{p-1} \sum_{\ell,j} \int_{\hat{J}_n^\ell} \frac{dt}{|B_{n\ell,j}(t)|^{p}} \leq 4F_k \beta^{-1}|V|^{p-1} \int_{|V|/2}^{\infty} \frac{dx}{x^p} \leq C_k$$

where $C_k$ is a constant only depending on $k$. This finishes the proof in the case $n \in T_{s,1}$.

**Case 2:** $T_{s,2} = \{n \in T_s : |B_n(V)| \leq |B_n(\tilde{V})| \text{ and } |\hat{J}_n| \leq |V|\}$:

Observe that for $n \in T_{s,2}$, we have $\hat{J}_n \subset \tilde{V}$. Next, we subdivide $T_{s,2}$ into generations $G_{s,\ell}$ such that for two indices $n_1, n_2$ in the same generation, the corresponding characteristic intervals $\hat{J}_{n_1}$ and $\hat{J}_{n_2}$ are disjoint. We observe that from the geometric decay of characteristic intervals, we get that $|\hat{J}_n|/|V| \lesssim \kappa^\ell$ for some $\kappa < 1$ and $n \in G_{s,\ell}$. Therefore, by introducing $\beta < 1$ such that $\beta(p-1) < 1$ we continue estimating by using the inequality $|V| \lesssim |B_n(\tilde{V})|$ for $n \in T_{s,2}$,

$$\sum_{n \in T_{s,2}} \left( \frac{|\hat{J}_n|}{|B_n(\tilde{V})|} \right)^{p-1} \frac{|\hat{J}_n|A_{n,j(n)}|^{p-1}}{C_n(A_{n,j(n)})^{p}} \lesssim \sum_{\ell=0}^{\infty} \kappa^\ell(1-\beta)(p-1) \sum_{n \in G_{s,\ell}} \frac{|\hat{J}_n|^{1+\beta(p-1)}|A_{n,j(n)}|^{p-1}}{|V|^{\beta(p-1)}C_n(A_{n,j(n)})^{p}}.$$  

We further split $G_{s,\ell}$ into two collections $G_{s,\ell}^{(i)}$, where

$$G_{s,\ell}^{(1)} = \{n \in G_{s,\ell} : |C_n(A_{n,j(n)})| \geq 1 - 2|V|\}, \quad G_{s,\ell}^{(2)} = G_{s,\ell} \setminus G_{s,\ell}^{(1)}.$$
Since we have \(|V| \leq 1/3\) and the \(\hat{J}_n\)'s in the collection \(\mathcal{G}_{s,\ell}^{(1)}\) are disjoint, for the sum over the first collection, we immediately see that \(\sum_{n \in \mathcal{G}_{s,\ell}^{(1)}} |\hat{J}_n| |A_{n,j(n)}|^{-1} \leq 1\), so we next consider

\[
(4.12) \sum_{n \in \mathcal{G}_{s,\ell}^{(2)}} \frac{|\hat{J}_n|^{1+\beta(p-1)} |A_{n,j(n)}|^{p-1}}{|V|^{\beta(p-1)} |C_n(A_{n,j(n)})|^p}.
\]

To analyze this expression, we define \(\tilde{C}_n'(A_{n,j(n)})\) as \(C_n(A_{n,j(n)})\) if \(\partial C_n(A_{n,j(n)}) \cap \overline{A_{n,j(n)}} \neq \emptyset\) and as the smallest interval which is a subset of \(C_n(A_{n,j(n)})\) that contains \(\hat{J}_n\) and \(\partial V \cap \overline{A_{n,j(n)}}\) if \(\partial C_n(A_{n,j(n)}) \cap \overline{A_{n,j(n)}} = \emptyset\). The canonical case is the first one, the second case can only occur if \(A_{n,j(n)}\) is not a grid point interval in grid \(n\) which happens only if \(A_{n,j(n)}\) lies at the boundary of \(V\). With this definition, we consider the set of different endpoints of \(\tilde{C}_n'(A_{n,j(n)})\) intersecting \(\overline{A_{n,j(n)}}\) as

\[
E_{s,\ell} = \{ x \in \partial \tilde{C}_n'(A_{n,j(n)}) \cap \overline{A_{n,j(n)}} : n \in \mathcal{G}_{s,\ell}^{(2)} \},
\]

e numerate the set \(E_{s,\ell}\) by the sequence \((x_r)_{r=1}^{\infty}\) which by definition is entirely contained in \(\overline{V}\) and split the collection \(\mathcal{G}_{s,\ell}^{(2)}\) according to those different endpoints into

\[
\mathcal{G}_{s,\ell,r}^{(2)} = \{ n \in \mathcal{G}_{s,\ell}^{(2)} : r \text{ is minimal with } x_r \in \partial \tilde{C}_n'(A_{n,j(n)}) \cap \overline{A_{n,j(n)}} \}.
\]

If we set \(\Lambda_{s,\ell} = \{ r : \mathcal{G}_{s,\ell,r}^{(2)} \neq \emptyset \}\), we can write and estimate \((4.12)\) as

\[
\sum_{r \in \Lambda_{s,\ell}} \sum_{n \in \mathcal{G}_{s,\ell,r}^{(2)}} |\hat{J}_n|^{1+\beta(p-1)} |A_{n,j(n)}|^{p-1} \lesssim \frac{1}{|V|^{\beta(p-1)}} \sum_{r \in \Lambda_{s,\ell}} \sum_{n \in \mathcal{G}_{s,\ell,r}^{(2)}} |\hat{J}_n|^{1-\beta(p-1)}.
\]

Since the \(\hat{J}_n\)'s in the above sum are disjoint, \(\hat{J}_n \subset \overline{V}\) and \(x_r\) is an endpoint of \(\tilde{C}_n'(A_{n,j(n)})\) for all \(n \in \mathcal{G}_{s,\ell,r}^{(2)}\), we can estimate by integration:

\[
\frac{1}{|V|^{\beta(p-1)}} \sum_{r \in \Lambda_{s,\ell}} \sum_{n \in \mathcal{G}_{s,\ell,r}^{(2)}} \frac{|\hat{J}_n|}{|C_n'(A_{n,j(n)})|^{1-\beta(p-1)}} \lesssim \frac{1}{|V|^{\beta(p-1)}} \sum_{r \in \Lambda_{s,\ell}} \int_{0}^{2|V|} \frac{1}{t^{1-\beta(p-1)}} dt \lesssim |\Lambda_{s,\ell}|.
\]

In order to finish our estimate, we show that \(|\Lambda_{s-1,\ell}| < 8s^2 + 1 =: N\). If we assume the contrary, let \((n_i)_{i=1}^{N}\) be an increasing sequence such that

\[
n_i \in \mathcal{G}_{s,\ell,r_{n_i}}^{(2)}
\]

for some different values \(r_{n_i}\). Consider \(F := A_{n:N,j(n:N)}\) and since the \(\hat{J}_n\)'s corresponding to \(n_i\) are disjoint, one of the two connected components of \(\overline{V} \setminus F\) contains \((N-1)/2 = 4s^2\) intervals \(\hat{J}_{m_1}, \ldots, \hat{J}_{m_{(N-1)/2}}\). Enumerate them as \(\hat{J}_{m_1}, \ldots, \hat{J}_{m_{(N-1)/2}}\).

Since any real sequence of length \(s^2 + 1\) has a monotone subsequence of length \(s\), we only have the following two possibilities:

1. There is a subsequence \((\ell_i)_{i=1}^{s}\) of the sequence \((m_i)\) such that, for each \(i\),
   \[
   \text{conv}(\hat{J}_{\ell_i} \cup F) \subset \text{conv}(\hat{J}_{\ell_{i+1}} \cup F)
   \]
2. There is a subsequence \((\ell_i)_{i=1}^{s}\) of the sequence \((m_i)\) such that, for each \(i\),
   \[
   \text{conv}(\hat{J}_{\ell_{i+1}} \cup F) \subset \text{conv}(\hat{J}_{\ell_i} \cup F),
   \]
where by \(\text{conv}(U)\) for \(U \subset \tilde{V}\) we mean the smallest interval contained in \(\tilde{V}\) that contains \(U\).

We observe that \(\text{conv}(\hat{J}_{n_i} \cup F) \subset C(A_{n_i,j}(n_i))\) for all \(i\) since the sequence \((A_{n_i,j}(n_i))_i\) is decreasing and therefore, in case (1), we have \(a_{\ell_i,j(j)} \geq i\) and therefore \(a_{\ell_i,j(j)} \geq s\) which is in conflict with the definition of \(T_{s-1,2}\).

We now recall that \(r_n = \min_{r \in \mathcal{I}_n(\tilde{V}^c)} K_n(C_n(I_n,t))\). We let \(i(n)\) be an index such that

\[
\begin{align*}
  r_n &= K_n(C_n(I_n,i(n))).
\end{align*}
\]

In case (2), we distinguish the two cases

(a) \(C_{\ell_i}(I_{\ell_i,i(\ell_i)}) \supset \bigcup_{j=1}^s \hat{J}_{t_j}\),
(b) \(C_{\ell_i}(I_{\ell_i,i(\ell_i)})\) contains the set of points \(\{x_{\ell ta_1}, \ldots, x_{\ell ta_s}\}\).

If we are in case (a), we have of course \(r_n \geq s\) in contradiction to the definition of \(T_{m,2}\). If we are in case (b), since the points \(x_{\ell t_i}\) are all different by definition of \(G_{s,\ell,r}^{(2)}\) and they are all (except possibly the two endpoints of \(V\)) part of the grid points in the grid corresponding to the index \(\ell\), we have here as well that \(r_n \geq s\), which shows that \(|A_{s-1,\ell}| \leq 8s^2 + 1\) and therefore, by collecting all estimates and summing geometric series over \(\ell\) and \(s\),

\[
\begin{align*}
  \sum_n R^{i_\omega n}(V) |\hat{a}_n|^p \|\hat{f}_n\|_{L^p(\tilde{V}^c)}^p &\lesssim \|f\|_p^p,
\end{align*}
\]

which finishes the proof of the lemma. 

\(\square\)

5. Proof of the Main Theorem

In this section, we prove our main result Theorem 1.1 that is unconditionality of periodic orthonormal spline systems corresponding to an arbitrary admissible point sequence \((s_n)_{n \geq 1}\) in \(L^p(\mathbb{T})\) for \(p \in (1, \infty)\).

Proof of Theorem 1.1. We recall the notation

\[
\begin{align*}
  Sf(t) &= \left( \sum_{n \geq N(k)} |a_n \hat{f}_n(t)|^2 \right)^{1/2}, \quad Mf(t) = \sup_{m \geq N(k)} \left| \sum_{n = N(k)}^m a_n \hat{f}_n(t) \right|
\end{align*}
\]

when

\[
  f = \sum_{n \geq N(k)} a_n \hat{f}_n.
\]

Since \((\hat{f}_n)_{n=1}^\infty\) is a basis in \(L^p(\mathbb{T})\), \(1 \leq p < \infty\), by Theorem 3.1 for showing its unconditionality, it suffices to show that \((\hat{f}_n)_{n \geq N(k)}\) is an unconditional basis sequence in \(L^p(\mathbb{T})\). Khintchine’s inequality implies that a necessary and sufficient condition for this is

\[
\begin{align*}
  \|Sf\|_p &\sim_p \|f\|_p, \quad f \in L^p(\mathbb{T}).
\end{align*}
\]

We will prove (5.1) for \(1 < p < 2\) since the cases \(p > 2\) then follow by a duality argument.

We first prove the inequality

\[
\begin{align*}
  \|f\|_p &\lesssim_p \|Sf\|_p.
\end{align*}
\]

...
To begin with, let \( f \in L^p(\mathbb{T}) \) with \( f = \sum_{n=N(k)}^{\infty} a_n f_n \). Without loss of generality, we may assume that the sequence \((a_n)_{n \geq N(k)}\) has only finitely many nonzero entries. We will prove (5.2) by showing the inequality \( \|Mf\|_p \lesssim \|Sf\|_p \) and we first observe that

\[
\|Mf\|_p^p = p \int_0^\infty \lambda^{p-1} \psi(\lambda) \, d\lambda,
\]

with \( \psi(\lambda) := [Mf > \lambda] := \{ t \in \mathbb{T} : Mf(t) > \lambda \} \). Next we decompose \( f \) into the two parts \( \varphi_1, \varphi_2 \) and estimate the corresponding distribution functions \( \psi_i(\lambda) := [M \varphi_i > \lambda/2], i \in \{1, 2\} \), separately. We continue with the definition of the functions \( \varphi_i \). For \( \lambda > 0 \), we define

\[
E_\lambda := [Sf > \lambda], \quad B_\lambda := [\hat{M}1_{E_\lambda} > 1/2],
\]

\[
\Gamma := \{ n : \hat{J}_n \subset B_\lambda, N(k) \leq n < \infty \}, \quad \Lambda := \Gamma^c,
\]

where we recall that \( \hat{J}_n \) is the characteristic interval corresponding to the grid point \( s_n \) and the function \( \hat{f}_n \). Then, let

\[
\varphi_1 := \sum_{n \in \Gamma} a_n \hat{f}_n \quad \text{and} \quad \varphi_2 := \sum_{n \in \Lambda} a_n \hat{f}_n.
\]

Now we estimate \( \psi_1 = [M \varphi_1 > \lambda/2] \):

\[
\psi_1(\lambda) = |\{ t \in B_\lambda : M \varphi_1(t) > \lambda/2 \}| + |\{ t \notin B_\lambda : M \varphi_1(t) > \lambda/2 \}|
\]

\[
\leq |B_\lambda| + \frac{2}{\lambda} \int_{B_\lambda} M \varphi_1(t) \, dt
\]

\[
\leq |B_\lambda| + \frac{2}{\lambda} \int_{B_\lambda} \sum_{n \in \Gamma} |a_n \hat{f}_n(t)| \, dt.
\]

We decompose the open set \( B_\lambda \) into a disjoint collection of open subintervals of \( \mathbb{T} \) and apply Lemma 4.1 to each of those intervals to conclude from the latter expression

\[
\psi_1(\lambda) \lesssim |B_\lambda| + \frac{1}{\lambda} \int_{B_\lambda} Sf(t) \, dt
\]

\[
= |B_\lambda| + \frac{1}{\lambda} \int_{B_\lambda \setminus E_\lambda} Sf(t) \, dt + \frac{1}{\lambda} \int_{E_\lambda \cap B_\lambda} Sf(t) \, dt
\]

\[
\leq |B_\lambda| + |B_\lambda \setminus E_\lambda| + \frac{1}{\lambda} \int_{E_\lambda} Sf(t) \, dt,
\]

where in the last inequality, we simply used the definition of \( E_\lambda \). Since the Hardy-Littlewood maximal function operator \( \hat{M} \) is of weak type \((1,1)\), \( |B_\lambda| \lesssim |E_\lambda| \) and thus we obtain finally

\[
\psi_1(\lambda) \lesssim |E_\lambda| + \frac{1}{\lambda} \int_{E_\lambda} Sf(t) \, dt.
\]
We now estimate $\psi_2(\lambda)$ and obtain from Theorem 3.13 and the fact that $\mathcal{M}$ is a bounded operator on $L^2[0,1]$

$$\psi_2(\lambda) \lesssim \frac{1}{\lambda^2} \left| \mathcal{M}_\lambda \varphi_2 \right| \lesssim \frac{1}{\lambda^2} \| \varphi_2 \|_2 = \frac{1}{\lambda^2} \left( \int_{E_\lambda} S \varphi_2(t)^2 \, dt + \int_{E_\lambda^c} S \varphi_2(t)^2 \, dt \right).$$

We apply Lemma 4.3 to the former expression to get

(5.5)

$$\psi_2(\lambda) \lesssim \frac{1}{\lambda^2} \int_{E_\lambda^c} S \varphi_2(t)^2 \, dt$$

Thus, combining (5.4) and (5.5),

$$\psi(\lambda) \leq \psi_1(\lambda) + \psi_2(\lambda) \lesssim |E_\lambda| + \frac{1}{\lambda} \int_{E_\lambda} S f(t) \, dt + \frac{1}{\lambda^2} \int_{E_\lambda^c} S f(t)^2 \, dt.$$

Inserting this inequality into (5.3),

$$\| Mf \|_p^p \lesssim p \int_0^{\infty} \lambda^{p-1} |E_\lambda| \, d\lambda + p \int_0^{\infty} \lambda^{p-2} \int_{E_\lambda} S f(t) \, dt \, d\lambda$$

$$+ p \int_0^{\infty} \lambda^{p-3} \int_{E_\lambda^c} S f(t)^2 \, dt \, d\lambda$$

$$= \| Sf \|_p^p + p \int_0^1 S f(t) \int_0^{S f(t)} \lambda^{p-2} \, d\lambda \, dt$$

$$+ p \int_0^1 S f(t)^2 \int_0^{\infty} \lambda^{p-3} \, d\lambda \, dt,$$

and thus, since $1 < p < 2$,

$$\| Mf \|_p \lesssim_p \| Sf \|_p.$$ 

So, the inequality $\| f \|_p \lesssim_p \| Sf \|_p$ is proved.

We now turn to the proof of the inequality

(5.6) $\| Sf \|_p \lesssim_p \| f \|_p$, \quad $1 < p < 2$.

It is enough to show that the operator $S$ is of weak type $(p,p)$ for each exponent $p$ in the range $1 < p < 2$. This is because $S$ is (clearly) also of strong type 2 and we can use the Marcinkiewicz interpolation theorem to obtain (5.6). Thus we have to show

(5.7) $\| Sf > \lambda \| \lesssim_p \frac{\| f \|_p^p}{\lambda^p}$, \quad $f \in L^p(\mathbb{T})$, \quad $\lambda > 0$.

We fix the function $f$ and the parameter $\lambda > 0$. To begin with the proof of (5.7), we define $G_\lambda := [\mathcal{M} f > \lambda]$ for $\lambda > 0$ and observe that

(5.8) $|G_\lambda| \lesssim_p \| f \|_p \frac{\| f \|_p^p}{\lambda^p}$,

since $\mathcal{M}$ is of weak type $(p,p)$, and, by the Lebesgue differentiation theorem,

(5.9) $|f| \leq \lambda$ \quad a.e. on $G_\lambda^c$. 

We decompose the open set $G_\lambda \subset [0, 1]$ into a collection $(V_j)_{j=1}^\infty$ of disjoint open subintervals of $[0, 1]$ and split the function $f$ into the two parts $h$ and $g$ defined by

$$ h := f \cdot 1_{G_\lambda^c} + \sum_{j=1}^\infty T_{V_j} f, \quad g := f - h, $$

where for fixed index $j$, $T_{V_j} f$ is the projection of $f \cdot 1_{V_j}$ onto the space of polynomials of order $k$ on the interval $V_j$.

We treat the functions $h, g$ separately and begin with $h$. The definition of $h$ implies

$$ ||h||_2^2 = \int_{G_\lambda^c} |f(t)|^2 \, dt + \sum_{j=1}^\infty \int_{V_j} (T_{V_j} f)(t)^2 \, dt, $$

(5.10)

since the intervals $V_j$ are disjoint. We apply the following argument to the second summand: by Corollary 2.2.

$$ \int_{V_j} (T_{V_j} f)(t)^2 \, dt \sim |V_j|^{-1} \left( \int_{V_j} |f(t)|^2 \, dt \right)^2. $$

Since $T_{V_j}$ is a bounded operator on $L^1$ (this can be seen as a very special instance of Shadrin’s theorem, Theorem 2.3),

$$ \int_{V_j} (T_{V_j} f)(t)^2 \, dt \lesssim |V_j|^{-1} \left( \int_{V_j} |f(t)| \, dt \right)^2 \lesssim (\hat{M} f(x))^2 |V_j| \lesssim \lambda^2 |V_j|, $$

where $x$ is a boundary point of $V_j$ and the last inequality follows from the defining property of $V_j$. So, by using this estimate, we obtain from (5.10)

$$ ||h||_2^2 \lesssim \lambda^{2-p} \int_{G_\lambda^c} |f(t)|^p \, dt + \lambda^2 |G_\lambda|, $$

and thus, in view of (5.8),

$$ ||h||_2^2 \lesssim_p \lambda^{2-p} ||f||_p^p. $$

This inequality allows us to estimate

$$ ||[Sh > \lambda/2]| \leq \frac{4}{\lambda^2} ||Sh||_2^2 = \frac{4}{\lambda^2} ||h||_2^2 \lesssim_p \frac{||f||_p^p}{\lambda^p}, $$

which concludes the proof of (5.7) for the part $h$.

We turn to the proof of (5.7) for the function $g$. Since $p < 2$, we have

$$ (5.11) \quad Sg(t)^p = \left( \sum_{n \geq N(k)} |\langle g, f_n \rangle|^2 f_n(t)^2 \right)^{p/2} \leq \sum_{n \geq N(k)} |\langle g, f_n \rangle|^p |f_n(t)|^p $$

For each index $j$, we define $\tilde{V}_j$ to be the open interval with the same center as $V_j$ but with 5 times its length. Then, set $\tilde{G}_\lambda := \bigcup_{j=1}^\infty \tilde{V}_j$ and observe that $|\tilde{G}_\lambda| \leq 5|G_\lambda|$. We get

$$ ||[Sg > \lambda/2]| \leq |\tilde{G}_\lambda| + \frac{2^p}{\lambda^p} \int_{\tilde{G}_\lambda} Sg(t)^p \, dt. $$

By (5.8) and (5.11), this becomes

$$ ||[Sg > \lambda/2]| \lesssim_p \lambda^{-p} \left( ||f||_p^p + \sum_{n \geq N(k)} \int_{\tilde{G}_\lambda} |\langle g, \hat{f}_n \rangle|^p |\hat{f}_n(t)|^p \, dt \right). $$
But by definition of $g$ and the fact that $T_{V_j}$ is a bounded operator on $L^p$,

$$
\|g\|^p_p = \sum_j \int_{V_j} |f(t) - T_{V_j} f(t)|^p \, dt \lesssim_p \sum_j \int_{V_j} |f(t)|^p \lesssim \|f\|^p_p,
$$

so in order to prove the inequality $|[Sg > \lambda/2]| \leq \lambda^{-p} \|f\|^p_p$, it is enough to show the inequality

$$
(5.12) \quad \sum_{n \geq N(k)} \int_{\mathbb{R}^+} |\langle g, \hat{f}_n \rangle|^p |\hat{f}_n(t)|^p \, dt \lesssim \|g\|^p_p.
$$

We now let $g_j := g \cdot 1_{V_j}$. The supports of $g_j$ are therefore disjoint and we have $\|g\|^p_p = \sum_{j=1}^\infty \|g_j\|^p_p$. Furthermore $g = \sum_{j=1}^\infty g_j$ with convergence in $L^p$. Thus for each $n$, we obtain

$$
\langle g, \hat{f}_n \rangle = \sum_{j=1}^\infty \langle g_j, \hat{f}_n \rangle,
$$

and it follows from the definition of $g_j$ that

$$
\int_{V_j} g_j(t)p(t) \, dt = 0
$$

for each polynomial $p$ on $V_j$ of order $k$. This implies that $\langle g_j, \hat{f}_n \rangle = 0$ for $n < n(V_j)$, where

$$
n(V) := \min\{n : \hat{f}_n \cap V \neq \emptyset\}.
$$

Thus we obtain for all $R > 1$ and for every $n$

$$
(5.13) \quad |\langle g, \hat{f}_n \rangle|^p \leq \left( \sum_{j \geq n(V_j)} R^{d_n(V_j)} |\langle g_j, \hat{f}_n \rangle| R^{-d_n(V_j)} \right)^p \leq \left( \sum_{j \geq n(V_j)} R^{p d_n(V_j)} |\langle g_j, \hat{f}_n \rangle|^p \right)^{p/p'} \left( \sum_{j \geq n(V_j)} R^{-p' d_n(V_j)} \right)^{p'/p'},
$$

where $p' = p/(p - 1)$. If we fix $n \geq n(V_j)$, there is at least one point of the partition $\hat{f}_n$ contained in $V_j$. This implies that for each fixed $s \geq 0$, there are at most two indices $j$ such that $n \geq n(V_j)$ and $\hat{d}_n(V_j) = s$. Therefore,

$$
\left( \sum_{j \geq n(V_j)} R^{-p' d_n(V_j)} \right)^{p'/p'} \lesssim_p 1,
$$

thus we obtain from (5.13),

$$
|\langle g, \hat{f}_n \rangle|^p \lesssim_p \sum_{j \geq n(V_j)} R^{p d_n(V_j)} |\langle g_j, \hat{f}_n \rangle|^p.
$$
Now we insert this inequality in (5.12) to get
\[
\sum_{n=N(k)}^{\infty} \int_{\tilde{G}_\lambda} |\langle g, \hat{f}_n \rangle|^p |\hat{f}_n(t)|^p \, dt \\
\lesssim_p \sum_{n=N(k)}^{\infty} \sum_{j:n \geq n(V_j)} \mathcal{R}^{p_{dn}(V_j)} |\langle g_j, \hat{f}_n \rangle|^p \int_{\tilde{G}_\lambda} |\hat{f}_n(t)|^p \, dt \\
\leq \sum_{n=N(k)}^{\infty} \sum_{j:n \geq n(V_j)} \mathcal{R}^{p_{dn}(V_j)} |\langle g_j, \hat{f}_n \rangle|^p \int_{\tilde{V}_j} |\hat{f}_n(t)|^p \, dt \\
\leq \sum_{j=1}^{\infty} \sum_{n \geq n(V_j)} \mathcal{R}^{p_{dn}(V_j)} |\langle g_j, \hat{f}_n \rangle|^p \int_{\tilde{V}_j} |\hat{f}_n(t)|^p \, dt
\]
We choose \( R > 1 \) such that we can apply Lemma 4.4 to obtain
\[
\sum_{n=N(k)}^{\infty} \int_{\tilde{G}_\lambda} |\langle g, \hat{f}_n \rangle|^p |\hat{f}_n(t)|^p \, dt \lesssim_p \sum_{j=1}^{\infty} ||g_j||^p_p = ||g||^p_p,
\]
proving (5.12) and with it the inequality \( \|Sf\|_p \lesssim_p \|f\|_p^p \). Thus the proof of Theorem 1.1 is completed. \( \square \)

Acknowledgments. K. Keryan was supported by SCS RA grant 15T-1A006 and M. Passenbrunner was supported by the FWF, project number P27723. Part of this work was done while K. Keryan was visiting the Department of Analysis, J. Kepler University Linz in January 2017.

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