Nonlinear Stochastic Estimators on the Special Euclidean Group SE(3) using Uncertain IMU and Vision Measurements

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Abstract—Two novel robust nonlinear stochastic full pose (i.e., attitude and position) estimators on the Special Euclidean Group SE(3) are proposed using the available uncertain measurements. The resulting estimators utilize the basic structure of the deterministic pose estimators adopting it to the stochastic sense. The proposed estimators for six degrees of freedom (DOF) pose estimations consider the group velocity vectors to be contaminated with constant bias and Gaussian random noise, unlike nonlinear deterministic pose estimators which disregard the noise component in the estimator derivations. The proposed estimators ensure that the closed loop error signals are semi-globally uniformly ultimately bounded in mean square. The equivalent quaternion representation and complete implementation steps of the proposed filters are presented. The efficiency and robustness of the proposed estimators are demonstrated by the numerical results which test the estimators against high levels of noise and bias associated with the group velocity and body-frame measurements and large initialization error.

Index Terms—Nonlinear stochastic filter, pose, position, attitude, Ito, stochastic differential equations, Brownian motion process, adaptive estimate, feature, inertial measurement unit, inertial vision system, 6 DOF, IMU, SE(3), SO(3).

I. INTRODUCTION

LANDMARK-BASED navigation is an integral part of robotics and control applications due to its ability to identify the pose (i.e., attitude and position) of a rigid-body in three-dimensional (3D) space. Applications requiring accurate 3D pose information include, but are not limited to, sensor calibration [1], manipulation and registration [2], and tracking control of autonomous vehicles [3–5]. The orientation of a rigid-body, also known as attitude, cannot be measured directly, instead, it has to be reconstructed using one of the following methods [6]: static reconstruction [7,8], Gaussian filter estimation [9–11], or nonlinear-based estimators [12–15]. The static methods of attitude reconstructions such as QUEST [7] or singular value decomposition (SVD) [8] utilize two or more known non-collinear observations in the inertial-frame and their sensor measurements in the body-frame. Nonetheless, it is worth noting that sensor measurements are vulnerable to bias and noise components causing the algorithms in [7,8] to produce poor results, especially if the vehicle is equipped with low-cost inertial measurement units (IMU).

Conventionally, the attitude estimation problem is predominantly addressed using Gaussian filters, for instance, Kalman filter (KF) [11], extended KF (EKF) [9], multiplicative EKF (MEKF) [10], and for good survey of Gaussian attitude estimator visit [5,14]. Gaussian filters generate reliable attitude estimates when the rigid-body is equipped with high quality measurement units. Despite all the benefits offered by Gaussian filters, high quality measurement units have multiple disadvantages, namely large size, heavy weight, and high cost. The recent rise of micro-elector-mechanical systems (MEMS) allowed for development of IMU, which are relatively inexpensive, small in size, and light-weight. However, the output of the low-cost IMU is contaminated with noise resulting in unsatisfactory performance of Gaussian attitude filters [5,14,16]. Consequently, numerous nonlinear complementary estimators evolved directly on the Special Orthogonal Group SO(3) have been proposed, for example [12–17]. Nonlinear complementary estimators have been proven to outperform Gaussian filters in multiple respects, namely, 1) nonlinear complementary estimator design accounts for the nonlinear nature of the attitude problem, 2) their derivation and representation is considerably simpler, 3) they require less computational cost, and 4) they perform better tracking performance [5,12,14]. Pose estimation is also best approached in nonlinear sense (on the Lie group of the Special Euclidean Group SE(3)), since nonlinear attitude estimation is an integral component of pose estimation.

The structure of nonlinear pose estimators developed on SE(3) relies on angular and translational velocity measurements, vector measurements, landmark(s) measurements, and estimates of the uncertain components associated with the velocity measurements (for example [1,4,5,18–21]). With the aim of improving the convergence behavior, several nonlinear deterministic pose estimators have been proposed [1,4,19–23]. An early implementation of nonlinear deterministic pose estimator with an inertial vision system was introduced in [1]. It was followed by a semi-direct deterministic pose estimator on SE(3) which required pose reconstruction [19]. The work in [19] has been modified to obtain a direct deterministic pose estimator on SE(3) [21] which utilizes the measurements directly, thus obviating the necessity for pose reconstruction. The noteworthy feature of the nonlinear deterministic pose

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estimators in [1,4,18–23] is the guarantee of the almost global asymptotic stability of the pose error achieved by disregarding the random noise attached to the group velocity vector. However, it is common for the group velocity vector measurements to be contaminated with constant bias and random noise. Bias and noise have the potential to compromise the estimation process and lead to poor results, in particular, if the vehicle is fitted with low-cost inertial vision system which includes an IMU module and a vision system. Several nonlinear stochastic estimators have been developed that address the sensitivity to measurement noise, for instance, [24], and [25] and bias estimation problem [26].

Concluding the introductory overview of the pose problem, it is important to emphasize two critical considerations. Firstly, the pose problem is naturally nonlinear on the Lie group of SE (3). Secondly, the group velocity vectors are not only corrupted with constant bias but also with random noise. The two nonlinear stochastic pose estimators on the Lie group of SE (3) proposed in this paper take into account the above-mentioned considerations and use data extracted by an IMU module and a vision system. In case when the group velocity vector is contaminated with constant bias and Gaussian random noise, the advantages of the proposed estimators are as follows: 1) The closed loop error signals are guaranteed to be almost semi-globally uniformly ultimately bounded in mean square. 2) The noise contamination of the estimator dynamics is minimized. 3) Unlike previously proposed nonlinear deterministic estimators, the proposed stochastic estimators produce reliable pose estimate and successfully handle irregular behavior of the measurement noise as well as large initialization error.

The rest of the paper is organized as follows: Section II introduces SO (3) and SE (3) preliminaries and mathematical notation. In Section III the pose problem is presented in stochastic sense. Section IV proposes two nonlinear stochastic pose estimators on SE (3) including related stability analysis. Section V illustrates the effectiveness and robustness of the proposed estimation schemes. Finally, Section VI concludes the work.

II. PRELIMINARIES AND MATH NOTATION

Throughout the paper, the set of non-negative real numbers, real n-dimensional space, and real m × n dimensional space are referred to as $\mathbb{R}_+$, $\mathbb{R}^n$, and $\mathbb{R}^{m\times n}$, respectively. For any $x \in \mathbb{R}^n$, $[x]_D$ denotes a diagonal matrix of $x$ and $\top$ denotes a transpose of a component. $\|x\| = \sqrt{x^\top x}$ stands for the Euclidean norm of $x \in \mathbb{R}^n$. The n-by-n identity matrix is referred to as $I_n$. $C^n$ stands for the nth continuous partial derivative of a continuous function. $\mathcal{K}$ describes a set of continuous and strictly increasing functions which follows $\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and is zero only at the origin. $\mathcal{K}_\infty$, despite being a class $\mathcal{K}$ function, is unbounded. $\text{Tr} \{\cdot\}$, $\mathbb{P} \{\cdot\}$, and $\mathbb{E} \{\cdot\}$ denote trace, probability, and an expected value of a component, respectively. $\{\mathcal{B}\}$ denotes the body-frame and $\{\mathcal{I}\}$ denotes the inertial-frame.

The orthogonal group $\mathbb{O} (3)$ is a Lie group and a subgroup of the 3-dimensional general linear group, characterized by smooth multiplication and inversion and defined by

$$\mathbb{O} (3) = \{ M \in \mathbb{R}^{3\times3} | M^\top M = MM^\top = I_3 \}$$

where $I_3 \in \mathbb{R}^{3\times3}$ is the identity matrix. The Special Orthogonal Group $\mathbb{SO} (3)$ is a subgroup of $\mathbb{O} (3)$ and is given by

$$\mathbb{SO} (3) = \{ R \in \mathbb{R}^{3\times3} | RR^\top = R^\top R = I_3, \det (R) = +1 \}$$

where $\det (\cdot)$ is a determinant of a matrix, and $R \in \mathbb{SO} (3)$ describes the orientation, commonly known as attitude, of a rigid-body in the body-frame relative to the inertial-frame in 3D space. The Special Euclidean Group $\mathbb{SE} (3)$ is a subset of the affine group defined by

$$\mathbb{SE} (3) = \{ T = \begin{bmatrix} R & P \end{bmatrix} \in \mathbb{R}^{4\times4} | R \in \mathbb{SO} (3), P \in \mathbb{R}^3 \}$$

where $T \in \mathbb{SE} (3)$ is a homogeneous transformation matrix that describes the pose of a rigid-body in 3D space, while $P \in \mathbb{R}^3$ stands for position, $R \in \mathbb{SO} (3)$. The Lie-algebra of the group $\mathbb{SO} (3)$ is termed $\mathfrak{s}o (3)$ and expressed as

$$\mathfrak{s}o (3) = \{ [x]_x \in \mathbb{R}^{3\times3} | [x]_x = -[x]_x \}$$

with $[x]_x$ being a skew symmetric matrix such that the map $[\cdot]_x : \mathbb{R}^3 \rightarrow \mathfrak{s}o (3)$ is given by

$$[x]_x = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \in \mathfrak{s}o (3), \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Define $[x]_y = x \times y$ where $\times$ denotes the cross product for all $x, y \in \mathbb{R}^3$, $\mathfrak{s}e (3)$ is a Lie-algebra of $\mathbb{SE} (3)$ such that

$$\mathfrak{s}e (3) = \{ \mathcal{Y}_\lambda, \lambda \in \mathbb{R}^{1\times4} | \forall y_1, y_2 \in \mathbb{R}^3 : [\mathcal{Y}]_\lambda = \begin{bmatrix} [y_1]_x \\ 0_{3\times1} \end{bmatrix} \}$$

where the wedge map $[\cdot]_\lambda : \mathbb{R}^6 \rightarrow \mathfrak{s}e (3)$ is defined by

$$[\mathcal{Y}]_\lambda = \begin{bmatrix} [y_1]_x \\ [y_2]_x \end{bmatrix} \in \mathfrak{s}e (3), \quad \mathcal{Y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathbb{R}^6$$

On the other side, the inverse of $[\cdot]_x$ is $\text{vex} : \mathfrak{s}o (3) \rightarrow \mathbb{R}^3$, such that for $\alpha \in \mathbb{R}^3$ one has

$$\text{vex}(\alpha)_x = \alpha$$

(1)

The anti-symmetric projection on the Lie-algebra of $\mathfrak{s}o (3)$ is defined by $\mathcal{P}_a$ and its mapping follows $\mathcal{P}_a : \mathbb{R}^{3\times3} \rightarrow \mathfrak{s}o (3)$ such that

$$\mathcal{P}_a (M) = \frac{1}{2} (M - M^\top) \in \mathfrak{s}o (3), \quad M \in \mathbb{R}^{3\times3}$$

(2)

Let $\mathcal{Y}_a (\cdot)$ represent the composition mapping $\mathcal{Y}_a = \text{vex} \circ \mathcal{P}_a$. Accordingly, for $M \in \mathbb{R}^{3\times3}$ one has

$$\mathcal{Y}_a (M) = \text{vex}(\mathcal{P}_a (M)) \in \mathbb{R}^3$$

(3)

The normalized Euclidean distance of the attitude matrix $R \in \mathbb{SO} (3)$ is defined as follows

$$\| R \|_1 = \frac{1}{4} \text{Tr} \{ I_3 - R \} \in [0, 1]$$

(4)

The orientation of any rigid-body can be established knowing its angle of rotation $\alpha \in \mathbb{R}$ about the unit-axis $u \in \mathbb{R}^3$ in the sphere $S^2$. This method of attitude representation is generally
termed as angle-axis parameterization [27]. The mapping of angle-axis parameterization to $SO(3)$ is defined by $R_\alpha: \mathbb{R} \times \mathbb{R}^3 \rightarrow SO(3)$ with

$$R_\alpha(\alpha, u) = I_3 + \sin(\alpha)[u]_x + (1 - \cos(\alpha))[u]^2$$

(5)

For $x, y \in \mathbb{R}^3, R \in SO(3), A \in \mathbb{R}^{3 \times 3}$, and $B = B^T \in \mathbb{R}^{3 \times 3}$ the following mathematical identities will be used in the subsequent derivations

$$[x \times y]_x = yx^T - xy^T$$

(6)

$$[Rx]_x = R[x]_x R^T$$

(7)

$$[x]^2_x = -x^T x I_3 + xx^T$$

(8)

$$B [x]_x + [x]_x B = \text{Tr} \{ B \} [x]_x - [Bx]_x$$

(9)

$$\text{Tr} \{ A [x]_x \} = \text{Tr} \{ P_a(A) [x]_x \} = -2 \text{ve}(P_a(A))^T x$$

(11)

III. PROBLEM FORMULATION

The pose estimation problem involves a set of vector measurements made with respect to the inertial- and body-frames of reference. In this section the pose problem is defined and the associated measurements are presented.

Attitude and position are the two elements necessary to describe the pose of a rigid-body in 3D space. Therefore, producing reliable estimates of these two elements is at the core of this work. The orientation of a rigid-body is termed attitude $R \in SO(3)$ and defines the body orientation in the body-frame relative to the inertial-frame $R \in \{B\}$. The translation of the rigid-body is represented by $P \in \mathbb{R}^3$ where $P$ is defined relative to the inertial-frame $P \in \{I\}$. Fig. 1 illustrates the pose estimation problem of a rigid-body in 3D space. Thus, the pose of a rigid-body is represented by the following homogeneous transformation matrix $T \in SE(3)$:

$$T = \begin{bmatrix} R & P \\ 0_{3 \times 1} & 1 \end{bmatrix}$$

(12)

![Fig. 1. Pose estimation problem of a rigid-body in 3D space.](image)

For clarity, the superscripts $B$ and $I$ are used to differentiate components of body-frame and inertial-frame, respectively.

From one side, the attitude can be extracted given the availability of $N_R$ known non-collinear observations in the inertial-frame and their measurements in the body-frame. The body-frame measurements can be obtained, for instance, by low-cost IMU, and the $i$th measurement can be represented by

$$v_i^{B(R)} = T^{-1} \begin{bmatrix} v_i^{I(R)} \\ 0 \end{bmatrix} + b_i^{B(R)} + \omega_i^{B(R)}$$

(13)

More simply put,

$$v_i^{B(R)} = R^T v_i^{I(R)} + b_i^{B(R)} + \omega_i^{B(R)}$$

(13)

where $v_i^{I(R)}$, $v_i^{I(R)}$, and $\omega_i^{B(R)}$ are the $i$th known inertial-frame vector, unknown constant bias, and unknown random noise, respectively, $v_i^{I(R)}$, $v_i^{I(R)}$, $b_i^{B(R)}$, $\omega_i^{B(R)} \in \mathbb{R}^3$ and $i = 1, 2, \ldots, N_R$. Both $v_i^{I(R)}$ and $v_i^{B(R)}$ in (13) can be normalized as

$$v_i^{I(R)} = \frac{v_i^{I(R)}}{\| v_i^{I(R)} \|}, \quad v_i^{B(R)} = \frac{v_i^{B(R)}}{\| v_i^{B(R)} \|}$$

(14)

In that case, $v_i^{I(R)}$ and $v_i^{B(R)}$ in (14) can be utilized to extract the body’s attitude instead of $v_i^{I(R)}$ and $v_i^{B(R)}$. Define the following two sets

$$\{ \upsilon^{I(R)} \} = \left\{ v_1^{I(R)}, v_2^{I(R)}, \ldots, v_{N_I}^{I(R)} \right\} \subseteq \{I\}$$

(15)

$$\{ \upsilon^{B(R)} \} = \left\{ v_1^{B(R)}, v_2^{B(R)}, \ldots, v_{N_B}^{B(R)} \right\} \subseteq \{B\}$$

(15)

where $v_1^{I(R)}, v_{N_B}^{B(R)} \in \mathbb{R}^{3 \times N_R}$ contain the normalized vectors introduced in (14). From the other side, the rigid-body’s position can be determined if the body’s attitude is available and there are $N_L$ known landmarks identified, for instance, by a low-cost inertial vision system such that the $i$th body-frame measurement is given by

$$v_i^{B(L)} = T v_i^{I(L)} + b_i^{B(L)} + \omega_i^{B(L)}$$

(16)

with $v_i^{I(L)}$ being the $i$th known landmark placed in the inertial-frame, $b_i^{B(L)}$ being the additive unknown constant bias, and $\omega_i^{B(L)}$ being the additive unknown random noise vector, for all $v_i^{I(L)}, v_{N_B}^{B(L)} \in \mathbb{R}^3$ and $i = 1, 2, \ldots, N_L$. The inertial-frame and body-frame vectors in (16) are divided into the following two sets

$$\{ \upsilon^{I(L)} \} = \left\{ v_1^{I(L)}, \ldots, v_{N_L}^{I(L)} \right\} \subseteq \{I\}$$

(17)

$$\{ \upsilon^{B(L)} \} = \left\{ v_1^{B(L)}, \ldots, v_{N_B}^{B(L)} \right\} \subseteq \{B\}$$

(17)

where $v_1^{I(L)}, v_{N_B}^{B(L)} \in \mathbb{R}^{3 \times N_L}$. For the case when more than one landmark is available for measurement, weighted geometric center approach can be employed

$$P_c^I = \frac{1}{\sum_{i=1}^{N_B} s_i} \sum_{i=1}^{N_B} s_i v_i^{I(L)}$$

(18)

$$P_c^B = \frac{1}{\sum_{i=1}^{N_B} s_i} \sum_{i=1}^{N_B} s_i v_i^{B(L)}$$

(19)
where $s_i^I$ refers to the confidence level of the $i$th measurement.

**Assumption 1.** The pose of a rigid-body can be obtained provided that the set in (15) has rank $3$ and the rank of the set in (17) is nonzero such that there are at least two non-collinear vectors in (14) ($N_\Omega \geq 2$) and one landmark in (16) ($N_\xi \geq 1$) available. For $N_\Omega = 2$, the third vector can be obtained through $v_3^{(R)} = v_1^{(R)} \times v_2^{(R)}$ and $v_3^{(L)} = v_1^{(L)} \times v_2^{(L)}$.

Accordingly, the homogeneous transformation matrix $T$ is obtainable if Assumption 1 is valid. Moreover, vectors of velocity measurements, bias, and noise as a Gaussian process results in a Gaussian process ([28,29]), one vector has zero mean and is bounded. Since the derivative of $t$ respectively, $\beta$, $b$, $\omega$ be expressed, respectively, as

\begin{equation}
\Omega = \Omega + 3 \upsilon \upsilon^\top (22)
\end{equation}

\begin{equation}
V = V + b + b + \omega V \in \{ B \}
\end{equation}

where $\Omega$ and $V$ denote the true angular velocity, $V \in \mathbb{R}^3$ denotes the translational velocity of the moving body, and $Y = [\Omega^T, V^T]^T \in \mathbb{R}^6$ denotes the group velocity vector. The measurements of angular and translational velocities can be expressed, respectively, as

\begin{align}
\Omega_m &= \Omega + b + \omega \Omega \in \{ B \} \\
V_m &= V + b + \omega V \in \{ B \}
\end{align}

where $b_\Omega$ and $b_V$ stand for constant bias vectors, while $\omega_\Omega$ and $\omega_V$ refer to unknown random noise attached to the measurement, $\forall b_\Omega, b_V, \omega_\Omega, \omega_V \in \mathbb{R}^3$. Define the group vectors of velocity measurements, bias, and noise as $Y_m = [\Omega_m^T, V_m^T]^T$, $b = [b_\Omega, b_V]^T$, and $\omega = [\omega_\Omega, \omega_V]^T$, respectively, $\forall Y_m, b, \omega \in \mathbb{R}^6$. $\omega$ being a random Gaussian noise vector has zero mean and is bounded. Since the derivative of a Gaussian process results in a Gaussian process ([28,29]), one could define $\omega$ as a function of a Brownian motion process vector such that

\begin{equation}
\omega = \Omega d\beta \quad \text{with} \quad \omega_\Omega = \Omega d\beta_\Omega \quad \text{and} \quad \omega_V = \Omega d\beta_V
\end{equation}

where $\beta = [\beta_\Omega^T, \beta_V^T]^T \in \mathbb{R}^6$, and $Q = \begin{bmatrix} \Omega_\Omega & 0_{3 \times 3} \\ 0_{3 \times 3} & \Omega_V \end{bmatrix} \in \mathbb{R}^{6 \times 6}$ is a diagonal matrix whose diagonal includes unknown time-variant non-negative components for all $\beta_\Omega, \beta_V \in \mathbb{R}^3$ and $Q_\Omega, Q_V \in \mathbb{R}^{3 \times 3}$. Brownian motion process signal is characterized by the following properties ([29–31])

\[ P \{ \beta(0) = 0 \} = 1, \quad \mathbb{E}[d\beta/dt] = 0, \quad \mathbb{E}[^3] = 0 \]

In the light of the identity in (11), the expression of $||R||_1$ in (4), and the expressions in (22) and (24), the true attitude dynamics in (20) can be written in terms of (4) in incremental form as

\begin{align}
d||R||_1 &= -\frac{1}{4} \text{Tr} \{ dR \} \\
&= -\frac{1}{4} \text{Tr} \{ \mathcal{P}_a(R) [ \Omega ]_x \} dt \\
&= \frac{1}{2} \text{vec} \{ \mathcal{P}_a(R) \}^\top (\Omega_m - b_\Omega) dt - Q_\Omega d\beta_\Omega (25)
\end{align}

Define $X = [||R||_1, P^T]^T \in \mathbb{R}^4$. Thus, from (25), the pose dynamics in (21) are written in vector form as a stochastic differential equation

\begin{equation}
dX = F dt - G d\beta
\end{equation}

\begin{equation}
G = \begin{bmatrix} \frac{1}{2} \mathcal{P}_a(R) & 0_{3 \times 3} \\ 0_{3 \times 3} & R \end{bmatrix}
\end{equation}

\begin{equation}
F = G (Y_m - b)
\end{equation}

where both $G$ and $F$ are locally Lipschitz.

**Remark 1.** Define $S_0 \subseteq \mathbb{S}^3(3) \times \mathbb{R}^3$ as a non-attractive, forward invariant unstable set:

\[ S_0 = \{ (R, 0, P) || \text{Tr} \{ R \} = -1, P = 0_{3 \times 3} \} \]

where the only three possible scenarios for $\text{Tr} \{ R \} = -1$ are: $R = \text{diag}(-1, -1, -1)$, $R = \text{diag}(-1, 1, -1)$, and $R = \text{diag}(-1, -1, 1)$.

The stochastic differential equation of the system in (26) has a solution on $t \in [t(0), T] \forall t(0) \leq T < \infty$ and $R(0) \notin S_0$ in the mean square sense. Additionally, for any $X(t)$ where $t \neq t(0)$, $X - \bar{X}(0)$ is independent of $\beta(t)$ $\forall t \geq t$, and $\forall t \in [t(0), T)$ (Theorem 4.5 [29]). The goal of this work is to design a reliable pose estimator that achieves adaptive stabilization and accounts for unknown constant bias and unknown time-variant covariance matrix attached to velocity measurements. Let the upper-bound of the diagonal entries in $Q_\Omega^2$ and $Q_V^2$, be $\sigma$ and $\xi$, respectively, with $\sigma, \xi \in \mathbb{R}^3$ such that

\begin{equation}
\sigma = \max\{ Q_\Omega^2(1,1), \max\{ Q_\Omega^2(2,2), \max\{ Q_\Omega^2(3,3) \} \}^T \\
\xi = \max\{ Q_V^2(1,1), \max\{ Q_V^2(2,2), \max\{ Q_V^2(3,3) \} \}^T
\end{equation}

with $\max \{ \cdot \}$ being the maximum value of the element.

**Assumption 2.** Consider $b, \sigma$, and $\xi$ to be upper-bounded by $\Gamma$ and to belong to a compact set $\Delta$ with $\Gamma \in \mathbb{R}^+$ and $\| \Delta \| \leq \Gamma < \infty$.

**Definition 1.** ([22] For $X = [||R||_1, P^T]^T$ in the stochastic differential system (26), define a compact set $\Theta \in \mathbb{R}^4$ and $X(0) = \bar{X}(t(0))$: If there exists a positive constant $c$ and a time constant $t_c = t_c(c, \bar{X}(0))$ with $E[\|X\|] < c, \forall t > t(0) + t_c$, the trajectory of $X$ is semi-globally uniformly ultimately bounded (SGUUB).

**Definition 2.** Consider the stochastic dynamics in (26) and let $V(X)$ be a given function which is twice differentiable such that $V(X) \in C^2$. The differential operator of $V(X)$ is defined by

\begin{equation}
\mathcal{L}V(X) = V'_X F + \frac{1}{2} \text{Tr} \{ \mathcal{P}_a^2 G^\top V_{XX} \}
\end{equation}

where $V_{X} = \partial V/\partial X$ and $V_{XX} = \partial^2 V/\partial X^2$. 

Lemma 1. [31–33] Consider the stochastic dynamics in (26) and suppose that there exists a potential function $V(\mathbf{x})$ that satisfies $V \in C^2$ with $V : \mathbb{R}^4 \rightarrow \mathbb{R}_+$. Suppose there are a class $K_{\infty}$ function $\bar{v}_1(\cdot)$ and $\bar{v}_2(\cdot)$, constants $c > 0$ and $k \geq 0$, and a non-negative function $\mathcal{N}(\|\mathbf{x}\|)$ such that

$$\bar{v}_1(\|\mathbf{x}\|) \leq V(\mathbf{x}) \leq \bar{v}_2(\|\mathbf{x}\|)$$

(30)

$$LV(\mathbf{x}) = \frac{1}{2} \text{Tr} \{ \mathcal{G} \mathcal{Q}^2 \mathcal{G}^T V_{\mathbf{x}\mathbf{x}} \} \leq -c \mathcal{N}(\|\mathbf{x}\|) + k$$

(31)

Then for $\mathbf{x}(0) \in \mathbb{R}^4$ and $R(0) \notin S_0$ defined in Remark 1, there exists almost a unique strong solution on $[0, \infty)$ for the dynamic system in (26). Also, the solution $\mathbf{X}$ of the stochastic system in (26) is bounded in probability satisfying

$$E(V(\mathbf{x})) \leq V(\mathbf{x}(0)) \exp(-ct) + \frac{k}{c}$$

(32)

with $\mathbf{X} \in \mathbb{R}^4$ being SGUUB.

The proof of Lemma 1 can be found in [31]. For $R \in SO(3)$, the set $S_0$ is unstable and forward invariant for the stochastic system described in (21) and (26) [27]. From almost any initial condition given that $R(0) \notin S_0$, we have $-1 < \text{Tr} \{ R(0) \} \leq 3$ and the trajectory of $\mathbf{X}$ is SGUUB.

Lemma 2. (Young’s inequality) Suppose there are two real vectors $x$ and $y$ with $x, y \in \mathbb{R}^n$. For any $a > 0$ and $b > 0$ that satisfy $\frac{1}{a} + \frac{1}{b} = 1$, there is

$$x^T y \leq (1/a) \|x\|^a + (1/b) \|y\|^b$$

(33)

where $q$ is a small positive constant.

Lemma 3. Consider $R \in SO(3)$, $\mathbf{M}_R = \mathbf{M}_R^T \in \mathbb{R}^{3 \times 3}$ with a rank of 3, $\text{Tr} \{ \mathbf{M}_R \} = 3$, and $\tilde{\mathbf{M}}_R = \text{Tr} \{ \mathbf{M}_R \} \mathbf{I}_3 - \mathbf{M}_R$ with the minimum singular value of $\mathbf{M}_R$ being $\lambda_1 = \lambda(\mathbf{M}_R)$. Then, the following holds:

$$\frac{2 \| \text{vex}(P_a (R)) \|^2}{\lambda_1 + \text{Tr} \{ R \mathbf{M}_R \mathbf{M}_R^{-1} \}} \geq \| R \mathbf{M}_R \|_1$$

(34)

(35)

Proof. See Appendix A.

IV. NONLINEAR STOCHASTIC POSE ESTIMATORS ON SE(3)

This section presents two nonlinear stochastic pose estimators evolved directly on $\text{SE}(3)$ designed with reliability as the primary consideration. The first estimator is termed a semidirect pose estimator since it requires the attitude and position to be reestablished using vector measurements in (15) and (17) and the group velocity measurements described in (22) and (23). Whereas, the second pose estimator is referred to as direct and is designed to use the above-mentioned measurements directly. Define the estimate of the homogeneous transformation matrix by

$$\hat{T} = \begin{bmatrix} \hat{R}^T & \hat{P} \\ 0_{3 \times 1} & 1 \end{bmatrix} \in \text{SE}(3)$$

The proposed pose estimators are evolved on $\text{SE}(3)$ and their structure follows

$$\dot{\hat{T}} = \hat{T} \hat{\mathbf{y}}$$

(36)

where $\hat{\mathbf{y}} = [\hat{\Omega}^T, \hat{V}^T]^T \in \mathbb{R}^6$ such that $\dot{\hat{R}} = \hat{R} [\hat{\Omega} \times]$ and $\dot{\hat{P}} = \hat{R} \hat{V}$. Consider the error of the homogeneous transformation matrix estimation to be given by

$$\hat{T} - \mathbf{T}^{-1} = \begin{bmatrix} \hat{R} & \hat{\mathbf{P}} \\ 0_{3 \times 1} & 1 \end{bmatrix}$$

(37)

where $\hat{R} = \hat{R} R^T$ and $\hat{\mathbf{P}} = \hat{P} - \hat{R} \hat{P}$ are the orientation and the position error, respectively, between the rigid-body-frame and the estimator-frame. As such, driving $\hat{T} \rightarrow \mathbf{T}$ ensures that $\hat{P} \rightarrow 0_{3 \times 1}$ and $\hat{R} \rightarrow \mathbf{I}_3$, or equivalently, $\| \hat{R} \|_1 = \frac{1}{4} \text{Tr} \{ \mathbf{I}_3 - \hat{R} \} \rightarrow 0$, which implies driving $\hat{T} \rightarrow \mathbf{I}_3$.

Consider the estimates of the unknown parameters $b$ and $\sigma$ to be denoted, respectively, by $\hat{b} = \begin{bmatrix} \hat{b}_\Omega, \hat{b}_V \end{bmatrix}^T$ and $\hat{\sigma} = \hat{\sigma}$ for all $\hat{b}_\Omega, \hat{b}_V, \hat{\sigma} \in \mathbb{R}^3$. Consider the error in $b$ and $\sigma$ to be

$$\hat{\sigma} = \hat{b} - \tilde{b}$$

(38)

where $\tilde{b} = [\tilde{b}_\Omega, \tilde{b}_V]^T$ for all $\tilde{b}_\Omega, \tilde{b}_V, \tilde{\sigma} \in \mathbb{R}^3$.

A. Semi-direct Nonlinear Stochastic Pose Estimator on SE(3)

Let the reconstructed matrix of the true homogeneous transformation matrix be denoted by $\mathbf{T}_y = \begin{bmatrix} R_y & P_y \\ 0_{3 \times 1} & 1 \end{bmatrix}$. In this context, $R_y$ refers to uncertain attitude which can be reconstructed, for instance [7,8] and for attitude construction methods visit [6]. From (18) and (19), $P_y$ can be reconstructed using $P_y = \frac{1}{\sum_{i=1}^{N_i} k_i} \sum_{i=1}^{N_i} k_i t_i \hat{v}_i^{(L)} - R_y V_i^{(L)}$. From (36) and in view of the pose dynamics in (26), one can rewrite the error in vector form as

$$\mathcal{E} = [\mathcal{E}_R, \mathcal{E}_P]^T = \begin{bmatrix} ||\hat{R}||_1, \hat{P} \end{bmatrix}^T \in \mathbb{R}^4$$

(39)

where $\hat{R} = \hat{R} R_y^T$, $\mathcal{E}_R = ||\hat{R}||_1 = \frac{1}{4} \text{Tr} \{ \mathbf{I}_3 - \hat{R} \}$ as defined in (4), and $\mathcal{E}_P = \hat{P} - \hat{R} P_y$. Consider the following nonlinear pose estimator on $\text{SE}(3)$

$$\begin{bmatrix} \dot{\hat{R}} \\ \dot{\hat{P}} \\ 0_{3 \times 1} \end{bmatrix} = \begin{bmatrix} \hat{R} & \hat{P} \\ 0_{3 \times 1} & 1 \end{bmatrix} \begin{bmatrix} \Omega_m - \hat{b}_\Omega - W_\Omega \\ V_m - \hat{b}_V - W_V \end{bmatrix}$$

(40)
where $\mathcal{E}$ is given in (39), $\mathcal{Y}_m(\hat{R}) = \text{vec}(\mathcal{P}_a(\hat{R}))$ is defined in (3), and $[\cdot]_D$ is a diagonal matrix of a vector. $k_w$, $\gamma_b$, and $\gamma_\sigma$ are positive constants, $\hat{b}$ is the estimate of $b$ and $\sigma$ is the estimate of $\tilde{\sigma}$. The equivalent quaternion representation and complete implementation steps of the semi-direct filter are given in Appendix B.

**Theorem 1.** Consider the pose dynamics in (21) combined with the group velocity measurements $\mathcal{Y}_m = [\mathcal{O}_m^T, \mathcal{V}_m^T]^T$ in (22) and (23). Let Assumption 1 hold. Suppose that $\mathcal{T}_y$ is reconstructed based on the vector measurements in (16) and (14), and geared with the estimator in (40), (41), (42), (43), (44), and (45). Suppose that the design parameters are selected as follows: $\gamma_b > 0$, $\gamma_\sigma > 0$, $k_b > 0$, $k_\sigma > 0$, $\varrho > 0$, and $k_w > 9/8$ with $\varrho$ being selected sufficiently small, and recall the set in Remark 1. In case where $\mathcal{Y}_m$ is biased and contaminated by random Gaussian noise ($\omega \neq 0$), and $\hat{R}(0) \notin S_0$, all the closed-loop signals are semi-globally uniformly ultimately bounded in mean square. Additionally, the filter errors could be minimized by the appropriated selection of the design parameters.

**Proof.** Recall the true and the estimated attitude dynamics in (20) and (40), respectively. Considering that $\hat{R} = RR^T$, the error in attitude dynamics is

\[
d\hat{R} = d\hat{R}R^T + \hat{R}dR^T
\]

In the light of (20) and (25), and with the aid of the identity in (11), the error dynamics in (46) can be expressed in terms of normalized Euclidean distance

\[
d\| \hat{R} \|_1 = -\frac{1}{4} \text{Tr} \left\{ [\hat{R}(b_\Omega - W_\Omega) dt + \hat{R}Q_\Omega dB_\Omega] \times \hat{R} \right\} = \frac{1}{2} \mathcal{Y}_a(\hat{R}) \hat{R} \left( (b_\Omega - W_\Omega) dt + Q_\Omega dB_\Omega \right)
\]

Given that $\hat{P} = P - \hat{R} \hat{P}$, the position dynamics error can be found in the following way

\[
d\hat{P} = d\hat{P} - d\hat{R}P - \hat{R}dP = \hat{R}(b_\Omega - W_\Omega) dt + \hat{R}Q_\Omega dB_\Omega \times \hat{P} + \hat{R}Q_V dB_V - \hat{R}(b_\Omega - W_\Omega) dt + \hat{R}Q_\Omega dB_\Omega \times \hat{P} + [\hat{P} - \hat{P}] \times \hat{R}(b_\Omega - W_\Omega) dt + [\hat{P} - \hat{P}] \times \hat{R}Q_\Omega dB_\Omega + \hat{R}Q_V dB_V
\]

Defining $\mathcal{E} = [\mathcal{E}_R, \mathcal{E}_P] = \{\|\hat{R}\|_1, \hat{R}^T\}$ as in (39) and combining it with (26), the following set of equations is obtained

\[
d\mathcal{E} = \tilde{\mathcal{F}} dt + \tilde{\mathcal{G}} Q dB
\]

For $V := V(\mathcal{E}, \hat{b}, \hat{\sigma})$, consider the following Lyapunov candidate function

\[
V = V(\mathcal{E}_R, \mathcal{E}_P) + \frac{1}{2} \|\mathcal{E}_P\|^2 + \frac{1}{2} \|\hat{b}\|^2 + \frac{1}{2} \hat{\sigma}^T \hat{\sigma} = \frac{1}{2} \text{Tr} \left\{ \tilde{\mathcal{G}} Q^2 \tilde{\mathcal{G}}^T V(\mathcal{E}) - \frac{1}{\gamma_1} \hat{b}^T \hat{b} - \frac{1}{\gamma_2} \hat{\sigma}^T \hat{\sigma} \right\}
\]

Thus, using (52) and (53), the differential operator $\mathcal{L}V$ in (51) can be rewritten as

\[
\mathcal{L}V = \frac{1}{2} \text{Tr} \left\{ \|\mathcal{E}_P\|^2 I_3 + 2 \mathcal{E}_P \mathcal{E}_P^T \hat{R} \tilde{Q}_V \hat{R}^T \tilde{Q}_V^T \mathcal{E}_p \right\} + \frac{1}{2} \text{Tr} \left\{ \|\mathcal{E}_P\|^2 I_3 + 2 \mathcal{E}_P \mathcal{E}_P^T \hat{R} \tilde{Q}_V \hat{R}^T \right\}
\]

Since $\hat{R} \tilde{Q}_V \hat{R}^T$ is positive semi-definite, the last trace component in (54) is negative semi-definite. Also, in the light of the fact that $\hat{P}^T [\hat{P}] = 0_{1 \times 3}$, the differential operator in (54)
can take a form of an inequality
\[ \mathcal{L}V \leq \frac{1}{2}(1 + E_R) \exp(E_R) \mathbf{Y}_a^T \left( \hat{R} \right) \hat{R}(\hat{b}_\Omega - W\Omega) + \frac{3}{8}(2 + E_R) \exp(E_R) \mathbf{Y}_a^T \left( \hat{R} \right) \hat{R}(\hat{\sigma}) \hat{R}^T \mathbf{Y}_a \left( \hat{R} \right) + ||E_P||^2 \mathbf{P}^T \left( \hat{P}_\times \hat{R}(\hat{b}_\Omega - W\Omega) + \hat{R}(\hat{b}_V - W_V) \right) + \frac{1}{2} \mathbf{Tr} \left\{ ||E_P||^2 I_3 + 2E_P E_P^T \right\} \hat{R} \hat{\xi}_D \hat{R}^T + \frac{1}{\gamma_b} \hat{b}^T \hat{b} - \frac{1}{\gamma_\sigma} \hat{\sigma}^T \hat{\sigma} \right\} \]  
(55)

Due to the fact that \( \mathbf{Tr} \{ \hat{R} \hat{\xi}_D \hat{R}^T \} = \sum_i^3 \xi_i \), define \( \hat{\xi} = \sum_i^3 \xi_i \). As such, one may obtain
\[ \frac{1}{2} \mathbf{Tr} \left\{ \left( ||E_P||^2 I_3 + 2E_P E_P^T \right) [\hat{\xi} I_3 \right\} \leq \frac{3}{2} ||E_P||^2 \hat{\xi} \]
Combining the above expression with Young’s inequality produces the following results
\[ \frac{3}{2} ||E_P||^2 \hat{\xi} \leq \frac{9}{8\sigma} ||E_P||^4 + \frac{\sigma^2}{2} \]  
(56)

with \( \sigma \) being a small positive constant. Combining (56) with (55) yields
\[ \mathcal{L}V \leq \frac{1}{2}(1 + E_R) \exp(E_R) \mathbf{Y}_a^T \left( \hat{R} \right) \hat{R}(\hat{b}_\Omega - W\Omega) + \frac{3}{8}(2 + E_R) \exp(E_R) \mathbf{Y}_a^T \left( \hat{R} \right) \hat{R}(\hat{\sigma}) \hat{R}^T \mathbf{Y}_a \left( \hat{R} \right) + ||E_P||^2 \mathbf{P}^T \left( \hat{P}_\times \hat{R}(\hat{b}_\Omega - W\Omega) + \hat{R}(\hat{b}_V - W_V) \right) + \frac{9}{8\sigma} ||E_P||^4 + \frac{\sigma^2}{2} \]  
(57)

With direct substitution for the correction factor \( W\Omega \) and \( W_V \) in (41) and (42), respectively, and the differential operators \( \hat{b} \) and \( \hat{\sigma} \) in (43), (44), and (45), respectively, into (57) yields
\[ \mathcal{L}V \leq - \left( k_w - \frac{3}{4} \right) \frac{1 + E_R}{1 - E_R} \exp(E_R) \mathbf{Y}_a^T \left( \hat{R} \right) \hat{R}(\hat{\sigma}) \hat{R}^T \mathbf{Y}_a \left( \hat{R} \right) - \frac{1}{\rho} \left( k_w - \frac{9}{8} \right) ||E_P||^4 - k_b ||\hat{b}||^2 - k_\sigma ||\hat{\sigma}||^2 + k_b \hat{b}^T \hat{b} + k_\sigma \hat{\sigma}^T \hat{\sigma} + \frac{\sigma^2}{2} \]  
(58)

which implies that
\[ \mathcal{L}V \leq - \left( \frac{4k_w - 3}{4} \right) \frac{1 + E_R}{1 - E_R} \exp(E_R) \mathbf{Y}_a^T \left( \hat{R} \right) \hat{R}(\hat{\sigma}) \hat{R}^T \mathbf{Y}_a \left( \hat{R} \right) - \frac{9}{8\rho} ||E_P||^4 - k_b ||\hat{b}||^2 - k_\sigma ||\hat{\sigma}||^2 + k_b \hat{b}^T \hat{b} + k_\sigma \hat{\sigma}^T \hat{\sigma} + \frac{\sigma^2}{2} \]  
(59)

Consequently, the result in (59) becomes
\[ \mathcal{L}V \leq - \left( 4k_w - 3 \right) \frac{1 + E_R}{1 - E_R} \exp(E_R) \mathbf{Y}_a^T \left( \hat{R} \right) \hat{R}(\hat{\sigma}) \hat{R}^T \mathbf{Y}_a \left( \hat{R} \right) - \frac{9}{8\rho} ||E_P||^4 - k_b ||\hat{b}||^2 - k_\sigma ||\hat{\sigma}||^2 + k_b \hat{b}^T \hat{b} + k_\sigma \hat{\sigma}^T \hat{\sigma} + \frac{\sigma^2}{2} \]  
(60)

Recall that \( b \) and \( \sigma \) are bounded as defined in Assumption 2. Setting \( \gamma_b, \gamma_\sigma, k_b, k_\sigma > 0 \), \( k_w > 9/8 \), and the positive constant \( \rho \) sufficiently small, the operator \( \mathcal{L}V \) in (60) becomes similar to (34) in Lemma 1. Define \( \lambda = \lambda(\sigma) \),
\[ \mathbf{h} = \frac{k_w}{\rho} ||\hat{b}||^2 + \frac{k_\sigma}{\rho} ||\hat{\sigma}||^2 + \frac{\sigma^2}{2} \]  
and
\[ \mathbf{H} = \left[ \begin{array}{cccccc} (4k_w - 3)\lambda_2 & 0_{1 \times 3} & 0_{1 \times 6} & 0_{1 \times 3} \\ 0_{3 \times 1} & \frac{1}{\rho} \left( 8k_w - 9 \right) I_3 & 0_{3 \times 6} & 0_{3 \times 3} \\ 0_{6 \times 1} & 0_{6 \times 3} & \gamma_b k_b I_3 & 0_{6 \times 3} \\ 0_{13 \times 1} & 0_{13 \times 3} & 0_{13 \times 6} & \gamma_\sigma k_\sigma I_3 \end{array} \right] \]
where \( \mathbf{H} \in \mathbb{R}^{13 \times 13} \). Thereby, the differential operator in (60) is equivalent to
\[ \mathcal{L}V \leq - \lambda(H) V + \mathbf{k} \]  
(61)

with \( \lambda(H) \) being the minimum eigenvalue of \( \mathbf{H} \). As such, it can be found that
\[ \frac{d(\mathbb{E}[V])}{dt} = \mathbb{E}[\mathcal{L}V] \leq - \lambda(H) \mathbb{E}[V] + \mathbf{k} \]
such that \( \frac{d(\mathbb{E}[V])}{dt} \leq 0 \forall \mathbb{E}[V] \geq \frac{k}{\lambda(H)} \). Thus, in consistence with Lemma 1, the following result is obtained
\[ 0 \leq \mathbb{E}[V] \leq V(0) \exp(-\lambda(H) t) + \frac{k}{\lambda(H)} \forall t \geq 0 \]  
(62)

Considering \( \mathbf{Y} = [\mathbf{E}^T, \hat{b}^T, \hat{\sigma}^T]^T \in \mathbb{R}^{13} \) and bearing in mind the result in (62), it can be easily shown that \( \mathbb{E}[V] \) is eventually ultimately bounded by \( k/\lambda(H) \). Accordingly, \( \mathbf{Y} \) is SGUBB in the mean square. For a rotation matrix \( \hat{R} \in \mathbb{S}(3) \), \( \mathbb{E}_R \) and \( \mathbb{E}_P \), define \( \mathcal{U}_0 \subseteq \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^6 \times \mathbb{R}^3 \) such that \( \mathcal{U}_0 = \left\{ \left( \hat{R}_0, \hat{P}_0, \hat{b}_0, \hat{\sigma}_0 \right) | \mathbb{E}_R(0) = +1, \hat{P}_0 = 0, \hat{b}_0 = 0, \hat{\sigma}_0 = 0 \right\} \). The set \( \mathcal{U}_0 \) is forward invariant and unstable for the pose dynamics in (21). Thus, from almost any initial condition that satisfies \( \mathbb{E}_R(0) \notin \mathcal{U}_0 \), or equivalently, \( \mathbf{Tr} \{ \hat{R}_0 \} \neq -1 \), the trajectory of \( \mathbf{Y} \) is SGUBB in the mean square.

### B. Direct Nonlinear Stochastic Pose Estimator on \( \mathbb{S}(3) \)

The reconstructed matrix \( \mathbf{T}_y \) given in Subsection IV-A contains two elements: \( R_y \) and \( P_y \). In spite of the fact that \( R_y \) can be easily reconstructed, for instance, through QUEST [7], or SVD [8], the previously proposed methods of static reconstruction increase the processing cost [5,14]. The nonlinear stochastic estimator introduced in this Subsection
circumvents the need for \( R_y \) reconstruction by directly utilizing the measurements obtained from the inertial and body-frame units. Consider

\[
\mathcal{M}_T = \left[ \begin{array}{ccc}
M_T & m_v & m_e
\end{array} \right] = \sum_{i=1}^{N_{I}} s_{i}^{R} \left[ \begin{array}{cc}
v_i^{T(R)} & 0
\end{array} \right] + \sum_{j=1}^{N_{I}} s_{j}^{L} \left[ \begin{array}{cc}
v_j^{T(L)} & 0
\end{array} \right]^{T}
\]

with \( s_{i}^{R} \geq 0 \) and \( s_{j}^{L} \geq 0 \) being the constant gains associated with the confidence level of the \( i \)th and \( j \)th sensor measurements, respectively, also, \( M_T = M_R + M_L \) and

\[
\begin{align*}
M_R &= \sum_{i=1}^{N_{I}} s_{i}^{R} v_i^{T(R)} \left( v_i^{T(R)} \right)^{T} \\
M_L &= \sum_{j=1}^{N_{I}} s_{j}^{L} v_j^{T(L)} \left( v_j^{T(L)} \right)^{T} \\
m_v &= \sum_{i=1}^{N_{I}} s_{i}^{R} v_i^{B(R)} \\
m_e &= \sum_{j=1}^{N_{I}} s_{j}^{L} v_j^{B(L)}
\end{align*}
\]

Also, define

\[
\mathcal{K}_T = \left[ \begin{array}{ccc}
K_T & k_v & m_e
\end{array} \right] = \sum_{i=1}^{N_{I}} s_{i}^{R} v_i^{B(R)} \left( v_i^{T(R)} \right)^{T} + \sum_{j=1}^{N_{I}} s_{j}^{L} v_j^{B(L)} \left( \nu_j^{T(L)} \right)^{T}
\]

where \( m_v = \sum_{j=1}^{N_{I}} s_{j}^{L} v_j^{T(L)} \), \( m_e = \sum_{i=1}^{N_{I}} s_{i}^{R} v_i^{B(R)} \), and

\[
\begin{align*}
K_T &= \sum_{i=1}^{N_{I}} s_{i}^{R} v_i^{B(R)} \left( v_i^{T(R)} \right)^{T} + \sum_{j=1}^{N_{I}} s_{j}^{L} v_j^{B(L)} \left( \nu_j^{T(L)} \right)^{T} \\
k_v &= \sum_{i=1}^{N_{I}} s_{i}^{R} v_i^{B(R)}
\end{align*}
\]

It is worth mentioning that \( s_{i}^{R} \) is selected such that it satisfies \( \sum_{i=1}^{N_{I}} s_{i}^{R} = 3 \) and \( \sum_{j=1}^{N_{I}} s_{j}^{L} \neq 0 \). It is clear that \( M_R \) is symmetric. Letting Assumption 1 hold implies that \( M_R \) is nonsingular with \( \text{rank}(M_R) = 3 \). The three eigenvalues of \( M_R \) are \( \lambda(M_R) = \{ \lambda_1, \lambda_2, \lambda_3 \} \). Thereby, \( \lambda_1, \lambda_3 \), and \( \lambda_3 \) are greater than zero. Let \( \tilde{M}_R = \text{Tr}(M_R)I_3 - M_R \in \mathbb{R}^{3 \times 3} \), provided that \( \text{rank}(M_R) = 3 \). The subsequent statements are true ( [34] page: 553):

1) The matrix \( \tilde{M}_R \) is positive-definite.
2) The eigenvalues of \( \tilde{M}_R \) are \( \lambda(\tilde{M}_R) = \{ \lambda_3 + \lambda_2, \lambda_3 + \lambda_2 + \lambda_1 \} \) with \( \Delta(\tilde{M}_R) > 0 \) being the minimum singular value of the set.

To guarantee that these two statements remain true, it is considered that \( \text{rank}(M_R) = 3 \) in the rest of this subsection. Let

\[
\hat{v}_i^{B(R)} = \hat{R} v_i^{T(R)}
\]

Define the homogeneous transformation matrix error \( \tilde{T} = \hat{T} \tilde{T}^{-1} \) as in (36). It follows that \( \hat{R} = \hat{R} \hat{R}^{T} \) and \( \hat{P} = \hat{P} - \hat{R} \hat{P} \). The error in \( b \) and \( \sigma \) as in (37) and (38), respectively. To introduce the direct stochastic pose estimator on \( \mathbb{SE}(3) \), it is necessary to define a set of expressions in terms of vector measurements. Therefore, let us define the following terms: \( \text{vex}(\mathcal{P}_a(\hat{R}M_R)) \), \( \hat{R}M_R \), \( ||\hat{R}M_R||_1 \), and \( \hat{P} \). From the identities in (6) and (7), one obtains

\[
\begin{align*}
&\hat{R} = \frac{1}{2} \hat{R} \hat{R}^{T} M_R - \frac{1}{2} M_R \hat{R} \hat{R}^{T} = \mathcal{P}_a(\hat{R}M_R)
&(\hat{R}M_R)\mathbf{c} = \mathcal{P}_a(\hat{R}M_R) \mathbf{c}
&\text{where} \quad \mathcal{Y}_a(\hat{R}M_R) = \frac{1}{4} \text{Tr} \left\{ (I_3 - \hat{R})M_R \right\}
&= \frac{1}{4} \text{Tr} \left\{ I_3 - \hat{R} \sum_{i=1}^{N_{I}} \left( \hat{R}_i^{R} v_i^{B(R)} \left( v_i^{T(R)} \right)^{T} \right) \right\}
&= \frac{1}{4} \text{Tr} \left\{ \sum_{i=1}^{N_{I}} \left( 1 - \left( \hat{v}_i^{B(R)} \right)^{T} v_i^{B(R)} \right) \right\}
\end{align*}
\]

From Appendix A, it becomes apparent that

\[
1 - ||\hat{R}||_1 = \frac{1}{4} (1 - \text{Tr} \{ \hat{R} \hat{R} M_R^{-1} \})
\]

From (71), one has

\[
\text{Tr} \left\{ \hat{R} M_R M_R^{-1} \right\} = \text{Tr} \left\{ \sum_{i=1}^{N_{I}} \left( \hat{R}_i^{R} v_i^{B(R)} \left( v_i^{T(R)} \right)^{T} \right) \left( \sum_{i=1}^{N_{I}} \hat{R}_i^{R} v_i^{B(R)} \left( v_i^{T(R)} \right)^{T} \right)^{-1} \right\}
\]

From (63) and (64), one has

\[
\tilde{T} \mathcal{M} = \left[ \begin{array}{ccc}
\hat{R} M_T + \hat{P} m_v^{T} & \hat{R} m_v + m_e \hat{P}
\end{array} \right] m_v
\]

The expression in (73) can be transformed as follows

\[
\tilde{T} \mathcal{M} = \left[ \begin{array}{ccc}
\hat{R} & \hat{P} I \end{array} \right] \left[ \begin{array}{ccc}
K_T & k_v \\
m_v & m_e
\end{array} \right]
\]

From (73) and (74), the position error can be evaluated in view of the vector measurements as

\[
\hat{P} = \hat{P} + \frac{1}{m_e} \left( \hat{R} k_v - \hat{R} M_R m_e \right)
\]
where $\tilde{R}M_R$ is calculated as in (69). As such, in all the subsequent derivations and calculations vex$(P_a(\tilde{R}M_R))$, $\tilde{R}M_R$, $||\tilde{R}M_R||$, $\{\tilde{R}M_R\}$, and $\tilde{P}$ are extracted via a set of vector measurements as defined in (68), (69), (70), (72), and (75), respectively. Modify the vector error in (39) and redefine it as follows

$$E = [E_R, E_P]^T = \left[||\tilde{R}M_R||, \tilde{P}^T\right]^T$$

(76)

where $E_R = ||\tilde{R}M_R||$, and $E_P = \tilde{P}$ are defined in (70) and (75), respectively. Consider the following estimator design

$$\dot{\tilde{P}} = \left[\begin{array}{c}
\tilde{R}^T \tilde{P} \\
0^3 \\
\end{array}\right] = \left[\begin{array}{c}
\tilde{R}^T (\tilde{P} - \gamma \dot{\Pi}_1 - \Omega) \\
1 \\
\end{array}\right] = \left[\begin{array}{c}
\Omega_m - \dot{b}_\Omega - W_\Omega \\
V_m - \dot{b}_V - W_V \\
\end{array}\right]$$

(77)

$$W_\Omega = \frac{4}{\lambda_1} \gamma w \left[\begin{array}{c}
\tilde{R}^T \Theta_\omega (\tilde{R}M_R) \\
\end{array}\right]_D$$

(78)

$$W_V = -\tilde{R}^T \left[\begin{array}{c}
\tilde{P} \\
\end{array}\right] \tilde{W}_\Omega + \frac{k_\eta}{\eta} \tilde{R}^T E_P$$

(79)

$$\dot{b}_\Omega = \frac{\gamma_k}{2} (1 + E_R) \exp (E_R) \tilde{R}^T \Theta_\omega (\tilde{R}M_R)$$

(80)

$$\dot{b}_V = \frac{\gamma_k}{2} ||E_P||^2 \tilde{R}^T E_P - \gamma_k b_V$$

(81)

$$K_\omega = \frac{\gamma a}{1 + E_R R M_R \tilde{R}^T} \exp (E_R)$$

(82)

$$\dot{\sigma} = \frac{2k_\omega}{\lambda_1} K_\omega \left[\begin{array}{c}
\tilde{R}^T \Theta_\omega (\tilde{R}M_R) \\
\end{array}\right]_D \tilde{R}^T \Theta_\omega (\tilde{R}M_R)$$

with $[E_R, E_P]^T = \left[||\tilde{R}M_R||, \tilde{P}^T\right]^T$ and $\Theta_\omega (\tilde{R}M_R)$ being specified in (70), (75), and (68), respectively. $[.]_D$ is a diagonal matrix of the associated vector, $\lambda_1 = \lambda(M_R)$ is the minimum singular value of $M_R$, $k_\omega$, $\gamma_k$, and $\gamma_a$ are positive constants, while $b = [b^T, b^T]^T$ and $\sigma$ are the estimates of $b$ and $\sigma$, respectively. The equivalent quaternion representation and complete implementation steps of the direct filter are given in Appendix B.

**Theorem 2.** Consider the pose estimator in (77), (78), (79), (80), (81), and (82) geared with the vector measurements in (15) and (17), and the velocity measurements in (22) and (23). Let Assumption 1 hold and assume that the selected parameters fulfill the following conditions: $\gamma_k > 0$, $\gamma_k > 0$, $k_\omega > 0$, $\sigma_k > 0$, and $k_\omega > 9/8$. Let $\dot{\omega} > 0$ be selected sufficiently small. Consider the set in Remark 1. In the event of $\gamma_m$ are corrupted with unknown constant bias and random noise ($\omega \neq 0$), and $\tilde{R}(0) \notin S_0$, the vector $[E^T, \hat{\omega}^T, \hat{\sigma}^T]^T$ is semi-globally uniformly ultimately bounded in mean square. Additionally, the filter errors could be minimized by the appropriated selection of the design parameters.

**Proof.** Let the error of $T$, $b$, and $\sigma$ be defined as in (36), (37), and (38), respectively. As such, the error in attitude dynamics is analogous to (46). $M_R = 0_{3 \times 3}$ due to the ith inertial vector $\gamma^T(R)$ being constant. Hence, from (46), the derivative of $||\tilde{R}M_R||$ becomes

$$\frac{d}{dt} ||\tilde{R}M_R|| = \frac{-1}{4} \text{Tr} \left\{ \left[\tilde{R}b_\Omega - W_\Omega\right]dt + \tilde{R}Q_\Omega \frac{d\Omega}{\gamma_\eta} \right\} \times \tilde{R}M_R$$

(83)

with $\text{Tr}\{W_\Omega\} \tilde{R}M_R = -2\text{vex} (P_a(\tilde{R}M_R))^T W_\Omega$ as defined in identity (11). It can be demonstrated that the derivative of $\tilde{P}$ in incremental form is identical to (48). As such, one has

$$d\dot{E} = \tilde{F} dt + \tilde{G} Q d\tilde{\sigma}$$

(84)

with $E$ being defined in terms of vector measurements in (76) and $b = [\tilde{b}_\Omega, \tilde{b}_V]^T$. Let $V := V(E, \tilde{b}, \tilde{\sigma})$ and consider the following Lyapunov candidate function

$$V = \left|\begin{array}{c}
\frac{1}{4} ||E_P||^4 + \frac{1}{2\gamma_b} ||\tilde{b}||^2 + \frac{1}{2\gamma_\sigma} ||\tilde{\sigma}||^2 \\
\end{array}\right.$$
Due to the fact that $\text{Tr}\{\tilde{R} [\xi_{\text{d}} \tilde{R}^T]\} = \sum_i^3 \xi_i$, and define $\xi = \sum_i^3 \xi_i$ to obtain the following

$$\frac{1}{2} \text{Tr} \left\{ \left( \| \mathcal{E}_P \|^2 I_3 + 2 \mathcal{E}_P \mathcal{E}_P^T \right) [\xi_{\text{d}}] \right\} \leq \frac{3}{2} \| \mathcal{E}_P \|^2 \| \xi \|
$$

With the aid of the Young’s inequality, one obtains

$$\frac{3}{2} \| \mathcal{E}_P \|^2 \| \xi \| \leq \frac{9}{8\varrho} \| \mathcal{E}_P \|^4 + \frac{\varrho}{2} \xi^2 \tag{88}$$

Combining the result in (88) with (87) and substituting $W_{\text{f}}, W_{\text{v}}, \hat{b}$ and $\hat{\sigma}$ with their definitions in (41) and (42), (43), (44), and (45), respectively, yields

$$\mathcal{L} V \leq - \lambda_2 \left( \frac{2k_w}{\lambda_1} - \frac{3}{4} \right) \frac{(1 + \mathcal{E}_R) \exp(\mathcal{E}_R)}{1 + \text{Tr}\{\tilde{R}M_1 \tilde{R}^{-1}\}}$$

$$\times \left(\| \mathcal{Y}_a (R \tilde{R}^{\dagger}) \|^2 - \frac{1}{\varrho} \frac{8k_w - 9}{8} \| \mathcal{E}_P \|^4 - \frac{k_b}{2} \| \tilde{b} \|^2 - k_{\sigma} \| \tilde{\sigma} \|^2 + k_b \hat{b}^T b + k_{\sigma} \tilde{\sigma}^T \sigma + \frac{\varrho}{2} \xi^2 \right) \tag{89}$$

which results in

$$\mathcal{L} V \leq - \lambda_2 \left( \frac{2k_w}{\lambda_1} - \frac{3}{4} \right) \frac{(1 + \mathcal{E}_R) \exp(\mathcal{E}_R)}{1 + \text{Tr}\{R \tilde{R} M_1^{-1}\}}$$

$$\times \left(\| \mathcal{Y}_a (R \tilde{R}^T) \|^2 - \frac{1}{\varrho} \frac{8k_w - 9}{8} \| \mathcal{E}_P \|^4 - \frac{k_b}{2} \| \tilde{b} \|^2 - k_{\sigma} \| \tilde{\sigma} \|^2 \right) \tag{90}$$

where $\| \mathcal{Y}_a (R \tilde{R}^T) \|^2 = \| \tilde{R}^T \mathcal{Y}_a (R \tilde{R}^T) \|^2$, while $\lambda_2 = \lambda ([\sigma]_{\text{d}})$ and $\lambda_2 = \lambda([\sigma]_{\text{d}})$ refer to the minimum value of $\| [\sigma]_{\text{d}} \|$ and $\text{Tr}\{R \tilde{R} M_1^{-1}\}$, respectively. According to Young’s inequality, it can be shown that

$$k_b \hat{b}^T b \leq \frac{k_b}{2} \| b \|^2 + \frac{k_b}{2} \| \tilde{b} \|^2$$

$$k_{\sigma} \tilde{\sigma}^T \sigma \leq \frac{k_{\sigma}}{2} \| \sigma \|^2 + \frac{k_{\sigma}}{2} \| \tilde{\sigma} \|^2$$

Also, from (35) in Lemma 3, one has

$$\frac{d}{dt} \| \mathcal{E}_R \| \geq \frac{d}{dt} [\mathcal{E}_R \mathcal{E}_R^T] \geq \mathcal{E}_R$$

Thus, the result in (90) can be expressed as

$$\mathcal{L} V \leq - \lambda_2 \left( \frac{8k_w - 3\lambda_1}{8} \right) (1 + \mathcal{E}_R) \exp(\mathcal{E}_R)$$

$$\times \left(\| \mathcal{E}_P \|^4 - \frac{k_b}{2} \| \tilde{b} \|^2 - k_{\sigma} \| \tilde{\sigma} \|^2 \right) \tag{91}$$

It worth mentioning that $b$ and $\sigma$ are bounded as defined in Assumption 2. Letting $\gamma_0, \gamma_0, k_b, k_b < 0, k_w > \frac{3}{2} \lambda_1$ and setting $\varrho$ as a sufficiently small positive constant, leads to the differential operator $\mathcal{L} V$ in (91) eventually becoming similar to (34) in Lemma 1. Let

$$k = \frac{k_b}{2} \| b \|^2 + \frac{k_{\sigma}}{2} \| \sigma \|^2 + \frac{\varrho}{2} \xi^2$$

and

$$\mathcal{H} = \begin{bmatrix} \lambda_2 \frac{8k_w - 3\lambda_1}{8} & 0_{1 \times 3} & 0_{1 \times 6} & 0_{1 \times 3} \\ 0_{3 \times 1} & \frac{1}{2} \frac{8k_w - 3\lambda_1}{8} I_3 & 0_{3 \times 6} & 0_{3 \times 3} \\ 0_{6 \times 1} & 0_{6 \times 3} & \gamma_0 k_b I_6 & 0_{6 \times 3} \\ 0_{3 \times 1} & 0_{3 \times 3} & \gamma_0 k_{\sigma} k_b I_3 & \end{bmatrix}$$

where $\mathcal{H} \in \mathbb{R}^{13 \times 13}$. Accordingly, $\mathcal{L} V$ in (91) can be written as

$$\mathcal{L} V \leq - \lambda (\mathcal{H}) V + k \tag{92}$$

where $\lambda (\mathcal{H})$ is the minimum eigenvalue of the matrix $\mathcal{H}$. Based on (92), it can be found that

$$\frac{d}{dt} [\mathbb{E} [V]] = \mathbb{E} [\mathcal{L} V] \leq - \lambda (\mathcal{H}) \mathbb{E} [V] + k \tag{93}$$

and according to Lemma 1 the following inequality holds

$$0 \leq \mathbb{E} [V] \leq V(0) \exp(-\lambda (\mathcal{H}) t) + \frac{k}{\lambda (\mathcal{H})}, \forall t \geq 0 \tag{94}$$

which signifies that $\mathbb{E} [V]$ eventually becomes ultimately bounded by $k/\lambda (\mathcal{H})$. Let $Y = [\mathcal{E}_R, \hat{b}^T, \hat{\sigma}^T] \in \mathbb{R}^{13 \times 1}$. According to the result in (94), $Y$ is SGUUB in the mean square. For $R \in \mathbb{S}^O(3)$, define the following forward invariant and unstable set $U_0 \subseteq \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$ for the pose dynamics in (21) such that

$$U_0 = \{ (\tilde{R}_0, \tilde{b}_0, \tilde{\sigma}_0) | \mathcal{E}_R(0) = +1, \tilde{b}_0 = 0, \tilde{\sigma}_0 = 0 \}$$

From almost any initial condition such that $\mathcal{E}_R(0) \notin U_0$, the trajectory of $Y$ is SGUUB in the mean square.

V. SIMULATION RESULTS

This section presents and compares the performance of the two nonlinear stochastic estimators on $\mathbb{S}^O(3)$. Both estimators are tested against high levels of unknown bias and noise attached to the measurements of the group velocity vector and the body-frame vectors and against large initialization error. Let us begin by defining the homogeneous transformation matrix $T$ as in (21). Consider the angular velocity $(\text{rad/sec})$ to be given by

$$\Omega = \begin{bmatrix} \sin \left( \frac{t}{2} \right) & 0.7 \sin \left( \frac{t}{4} + \pi \right) & \frac{1}{2} \sin \left( 0.4 t + \frac{\pi}{3} \right) \end{bmatrix}^T$$

and the translational velocity to be

$$V = \begin{bmatrix} \sin \left( \frac{t}{2} \right) & 0.6 \sin \left( \frac{5 \pi}{4} \right) & \sin \left( 0.4 t + \frac{\pi}{3} \right) \end{bmatrix} (\text{m/sec})$$

with $R(0) = I_3$ and $P(0) = 0_{3 \times 3}$, respectively, such that $T = I_3$. Let $\Omega_{\text{m}} = \Omega + \omega_{\text{v}} + \omega_{\text{n}}$ with the unknown constant bias $b_{\Omega} = 0.1 [1, -1, 1]^T$ and the unknown random noise vector $\omega_{\text{n}}$ having zero mean and standard deviation (STD) 0.15 (rad/sec). Consider $v_{\text{n}} = V + b_{\text{v}} + \omega_{\text{n}}$ where the unknown constant bias $b_{\text{v}} = 0.1 [2, 5, 1]^T$ and the random noise vector $\omega_{\text{n}}$ has zero mean and STD = 0.15 (m/sec). Assume there is a landmark available for measurement $v_{\text{f}}(L) = \left[ \frac{2}{\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}} \right]^T$ and its body-frame measurement is as (16). The associated bias is $b_{\text{f}}^{(L)} = 0.1 [0.3, 0.2, -0.2]^T$ and the noise vector $\omega_{\text{f}}^{(L)}$ has zero mean and STD = 0.1 (m/sec). To incorporate uncertain measurements obtained from an IMU module, let us consider two non-collinear inertial-frame vectors $v_{\text{f}}^{(L)} = \frac{1}{\sqrt{3}} [1, -1, 1]^T$ and $v_{\text{f}}^{(L)} = [0, 0, 1]^T$ and
Define body-frame vectors $v_1^{B(R)}$ and $v_2^{B(R)}$ according to (13) for $i = 1, 2$. The bias associated with the two body-frame measurements are $b_1^{B(R)} = 0.1 \begin{bmatrix} -1, 1, 0.5 \end{bmatrix}^T$ and $b_2^{B(R)} = 0.1 \begin{bmatrix} 0, 0, 1 \end{bmatrix}^T$, while the noise vectors $\omega_i^{B(R)}$ and $\sigma_i^{B(R)}$ have zero mean and STD = 0.1 (m/sec). The third inertial and body-frame vectors are defined by $v_3^{I(R)} = v_1^{I(R)} \times v_2^{I(R)}$ and $v_4^{B(R)} = v_1^{B(R)} \times v_2^{B(R)}$. It is worth noting that $\hat{v}_i^{B(R)}$ and $\hat{v}_i^{I(R)}$ are normalized to $v_i^{B(R)}$ and $v_i^{I(R)}$, respectively, for all $i = 1, 2, 3$ using (14). Hence, Assumption 1 holds.

For the pose estimator design presented in Subsection IV-A, $R_y$ is determined using SVD [8], for complete survey visit [6]. The simulation time is set to 25 seconds. Let us set the attitude estimate using the angle-axis parameterization method outlined in (5) as $\hat{R}(0) = R_\alpha(\alpha, u/\|u\|)$. Define $\alpha = 170$ (deg) and $u = [3, 10, 8]^T$, letting $\|\hat{R}(0)\|_1$ be very close to the unstable equilibrium (+1) and setting the initial position of the estimator as $\hat{P}(0) = [4, -3, 5]^T$. The initial conditions are given below:

$$
T(0) = I_4, \quad \hat{T}(0) = \begin{bmatrix} -0.8816 & 0.2386 & 0.4074 & 4 \\ 0.4498 & 0.1625 & 0.8782 & -3 \\ 0.1433 & 0.9574 & -0.2505 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Design parameters and initial estimates are chosen as follows: $\gamma_1 = 1$, $\gamma_2 = 1$, $k_w = 8$, $k_b = 0.1$, $k_p = 0.1$, $\varrho = 0.2$, $b(0) = 0_{6 \times 1}$ and $\hat{\sigma}(0) = 0_{3 \times 1}$. Also, the following color notation is adopted: black color describes the true value, magenta refers to a measured value, red illustrates the performance of the proposed nonlinear stochastic semi-direct pose estimator (S-DIR), while blue demonstrates the performance of the proposed nonlinear stochastic direct pose estimator (DIR).

Fig. 2, 3 and 4 illustrate angular velocity, translational velocity and body-frame vector measurements corrupted with high values of bias and noise plotted against the true values. Fig. 5 demonstrates impressive tracking performance of the Euler angles which are subject to large initialization error. Similarly, Fig. 6 depicts remarkable tracking performance of the rigid-body’s position in 3D space when large initialization error is present. Additionally, the upper portion of Fig. 7 shows that $\|\hat{R}\|_1 = \frac{1}{3}\text{Tr}\{I_3 - \hat{R}R^T\}$ initiated very close to the unstable equilibria approximated as (0.99) and was regulated to the close proximity of the origin. In the same vein, the lower portion of Fig. 7 demonstrates how $\|\hat{P} - \hat{P}\|_2$ initiated at a high value and steered to the close neighborhood of the origin. The impressive tracking performance presented in Fig. 5, 6, and 7 illustrates the robustness of the proposed estimators against the high values of bias, noise and initialization errors inherent to the angular velocity, translational velocity, and body-frame vector measurements.

To compliment the estimator performance demonstrated in Fig. 5, 6, and 7 with statistical analysis over the steady-state performance, Table I lists the mean and STD of $\|\hat{R}\|_1$ and $\|\hat{P} - \hat{P}\|_2$ over the period of (8-25 sec) of the proposed stochastic estimators. It can be noticed that both errors of the proposed stochastic estimators exhibit small values of mean as well as STD which confirms the results presented in Fig. 5, 6, and 7. However, the semi-direct stochastic pose estimator displays smaller mean and STD in comparison with the direct stochastic pose estimator.

Accordingly, the simulation results confirm the outstanding
have been assumed to be contaminated not only with unknown uncertain data extracted from low-cost IMU and landmark measurements. Although the semi-direct estimator has smaller values of convergence capability of the proposed estimators considering large initialized value of pose error and high levels of unknown random noise and constant bias associated with velocity measurements.

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**APPENDIX A**

**Proof of Lemma 3**

Define the rotational matrix of a rigid-body in space by \( R \in SO(3) \). Let \( \rho \in \mathbb{R}^3 \) be a Rodriguez parameters vector commonly used for attitude representation [27,35]. The mapping from vector form to a 3-by-3 matrix \( R_\rho : \mathbb{R}^3 \rightarrow SO(3) \) is equivalent to

\[
R_\rho(\rho) = \frac{1}{1 + ||\rho||^2} \left( (1 - ||\rho||^2) I_3 + 2\rho\rho^\top + 2 [\rho]_x \right) \quad (95)
\]

Combining (95) and (4) one obtains

\[
||R||_1 = ||\rho||^2/(1 + ||\rho||^2) \quad (96)
\]

For \( R_\rho = R_\rho(\rho) \), the anti-symmetric projection on the Lie-algebra of \( so(3) \) is given by

\[
P_a (R) = \frac{1}{2} (R_\rho - R_\rho^\top) = 2 \frac{1}{1 + ||\rho||^2} [\rho]_x
\]

Accordingly, the vex of \( P_a (R) \) is equivalent to

\[
vex (P_a (R)) = 2\rho/(1 + ||\rho||^2) \quad (97)
\]

Thus, using (96) yields

\[
(1 - ||R||_1) ||R||_1 = \frac{||\rho||^2}{(1 + ||\rho||^2)^2} \quad (98)
\]

and (97) shows that

\[
||\text{vex} (P_a (R))||_2^2 = 4 \frac{||\rho||^2}{(1 + ||\rho||^2)^2} \quad (99)
\]

As such, (98) and (99) justify (34) in Lemma 3. According to Subsection 4-B \( \sum_{i=1}^n s_i = 3 \) in order to satisfy \( \text{Tr} (M_R) = 3 \). Consider \( ||R_\rho M_R||_1 = \frac{1}{4} \text{Tr} \left\{ (I_3 - R) M_R \right\} \) and the angle-axis parameterization in (5). For \( M_R = (M_R)^\top \in \mathbb{R}^3 \), one has \( \text{Tr} \left\{ [u]_x M_R \right\} = 0 \) as given in (10). Thus, it could be found that

\[
||R_\rho M_R||_1 = \frac{1}{4} \text{Tr} \left\{ -\left( \sin(\theta) [u]_x + (1 - \cos(\theta)) [u]^2 \right) M_R \right\} = -\frac{1}{4} \text{Tr} \left\{ (1 - \cos(\theta)) [u]^2 M_R \right\} \quad (100)
\]

Accordingly, the following holds [36]

\[
||R||_1 = \frac{1}{2} (1 - \cos (\theta)) = \sin^2 (\theta/2) \quad (101)
\]
Hence, the unit axis vector is equivalent to [27]
\[ u = \cot \left( \frac{\theta}{2} \right) \rho \]
Using \[ |u|^2 = -\|u\|^2 I_3 + uu^T \] in identity (8), one could rewrite the expression in (100) as
\[ \|RM_R\|_1 = \frac{1}{2} \|R\|_1 u^T M_R u \]
\[ = \frac{1}{2} \|R\|_1 \cos^2 \left( \frac{\theta}{2} \right) \rho^T M_R \rho \]
Based on (101), \( \cos^2 \left( \frac{\theta}{2} \right) = 1 - \|R\|_1 \) such that
\[ \tan^2 \left( \frac{\theta}{2} \right) = \frac{\|R\|_1}{1 - \|R\|_1} \]
which means that \( \|RM_R\|_1 \) formulated in terms of \( \rho \) is
\[ \|RM_R\|_1 = \frac{1}{2} \left( 1 - \|R\|_1 \right) \rho^T M_R \rho \]
Using (6) and (9), the anti-symmetric projection operator of \( RM_R \) is equivalent to
\[ \mathcal{P}_a (RM_R) = \frac{\rho^T M_R - M_R \rho^T + M_R [\rho]_x + [\rho]_x M_R}{1 + \|\rho\|^2} \]
\[ = \frac{\left( \text{Tr} \{ M_R \} I_3 - M_R - [\rho]_x M_R \right) \rho}{1 + \|\rho\|^2} \]
such that
\[ \text{vex} (\mathcal{P}_a (RM_R)) = \frac{(I_3 [\rho]_x) \rho}{1 + \|\rho\|^2} \]
One can verify that the 2-norm of the above result is
\[ \|\text{vex} (\mathcal{P}_a (RM_R))\|^2 = \frac{\rho^T M_R (I_3 - [\rho]_x) \rho}{1 + \|\rho\|^2} \]
From identity (8) \( |\rho|^2 = -\|\rho\|^2 I_3 + \rho \rho^T \). Thus, the following inequality holds
\[ \|\text{vex} (\mathcal{P}_a (RM_R))\|^2 \leq \frac{\rho^T M_R (I_3 - [\rho]_x) \rho}{(1 + \|\rho\|^2)^2} \]
\[ \geq \frac{\lambda}{2 \rho^T M_R \rho} \leq \frac{1}{\lambda \|\rho\|^2} \]
\[ \lambda = \lambda(M_R) \text{ is the minimum singular value of } M_R. \]
\[ \lambda(M_R) \text{ has rank 3, one finds} \]
\[ 1 - \|R\|_1 = \text{Tr} \left\{ \frac{1}{2} I_3 + \frac{1}{4} R \right\} \]
\[ = \text{Tr} \left( \frac{1}{2} I_3 + \frac{1}{4} RM_R M_R^{-1} \right) \]
Based on (104) and (105), the following inequality holds
\[ \|\text{vex} (\mathcal{P}_a (RM_R))\|^2 \]
\[ \geq \frac{\lambda}{2} \left( 1 + \text{Tr} \{ RM_R M_R^{-1} \} \right) \|RM_R\|_1 \]
which proves (35) in Lemma 3.

**APPENDIX B**

**Quaternion Representation**

Define \( Q = [q_0, q^T]^T \in S^3 \) as a unit-quaternion with \( q_0 \in \mathbb{R} \) and \( q \in \mathbb{R}^3 \) such that \( S^3 = \{ Q \in \mathbb{R}^4 \| Q \| = \sqrt{q_0^2 + q^T q} = 1 \} \). \( Q^{-1} = [q_0, -q^T]^T \in S^3 \) denotes the inverse of \( Q \). Define \( \circ \) as a quaternion product where the quaternion multiplication of \( Q_1 = [q_0_1, q_1^T]^T \in S^3 \) and \( Q_2 = [q_0_2, q_2^T]^T \in S^3 \) is \( Q_1 \circ Q_2 = [q_0_1 q_0_2 - q_1^T q_2, q_0_1 q_2 + q_0_2 q_1 + [q_1] \times [q_2]^T] \). The mapping from unit-quaternion \( S^3 \) to \( SO(3) \) is described by \( R_Q : S^3 \rightarrow SO(3) \)
\[ R_Q = (q_0^2 - ||q||^2) I_3 + 2qq^T + 2q_0 [q]_x \in SO(3) \]
(106)
The quaternion identity is described by \( Q_1 = [\pm 1, 0, 0, 0]^T \) with \( R_{Q_1} = I_3 \). For more information visit [35]. Define the estimate of \( Q = [\hat{q}_0, \hat{q}^T]^T \in S^3 \) as \( \hat{Q} = [\hat{q}_0, \hat{q}^T]^T \in S^3 \) with \( \hat{R}_Q = (\hat{q}_0^2 - ||\hat{q}||^2)I_3 + 2\hat{q}\hat{q}^T + 2\hat{q}_0 [\hat{q}]_x \), see the map in (106). For any \( x \in \mathbb{R}^3 \) and \( Q \in S^3 \), define the map
\[ \Pi = [0, x]^T \in \mathbb{R}^4 \]
\[ \tilde{Y}(Q^{-1}, x) = \left[ \begin{array}{c} 0 \\ Y(Q^{-1}, x) \end{array} \right] = Q^{-1} \circ \left[ \begin{array}{c} 0 \\ x \end{array} \right] \circ Q \\ \tilde{Y}(Q, x) = \left[ \begin{array}{c} 0 \\ Y(Q, x) \end{array} \right] = Q \circ \left[ \begin{array}{c} 0 \\ x \end{array} \right] \circ Q^{-1} \]

The equivalent quaternion representation and complete implementation steps of the filter in (40), (41), (42), (43), (44), and (45) is:
\[ \left( \begin{array}{c} v_i^S \\ Q_y \\ \bar{Q}_E \\ \bar{P}_y \\ \bar{E}_P \\ \bar{Q} \\ \bar{P} \\ W_{\Omega} \\ W_V \\ \hat{b}_0 \\ \hat{b}_V \\ \hat{K}_E \\ \hat{\sigma} \end{array} \right) = \mathcal{Y}(Q^{-1}, v_i^T) \]
\[ \mathcal{Y}(Q^{-1}, v_i) \text{: Reconstructed by QUEST algorithm} \]
\[ \bar{Q} = [\hat{q}_0, \hat{q}^T]^T = \hat{Q} \circ Q^{-1} \]
\[ \bar{E}_P = \sum_{k=l}^{3} e_k^T \left( v_i^{(L)} - \tilde{Y}(q_i, v_i^{(L)}) \right) \]
\[ \bar{P}_y = \sum_{k=l}^{3} e_k^T \left( v_i^{(L)} - \tilde{Y}(q_i, v_i^{(L)}) \right) \]
\[ \bar{Q} = \sum_{k=l}^{3} e_k^T \left( v_i^{(L)} - \tilde{Y}(q_i, v_i^{(L)}) \right) \]
\[ \bar{P} = \sum_{k=l}^{3} e_k^T \left( v_i^{(L)} - \tilde{Y}(q_i, v_i^{(L)}) \right) \]
\[ W_{\Omega} = \frac{2\gamma_b}{\epsilon E_R} \left( Y(Q^{-1}, \hat{\hat{P}}) + \frac{\epsilon_b}{\epsilon} Y(Q^{-1}, \bar{E}_P) \right) \]
\[ W_V = \gamma_b ||\bar{E}_P||^2 \left( Y(Q^{-1}, \bar{P}) + \frac{\epsilon_b}{\epsilon} Y(Q^{-1}, \bar{E}_P) \right) \]
\[ \hat{b}_0 = \gamma_b (1 + \epsilon E_R) \tilde{Y}(Q^{-1}, \hat{\hat{P}}) \]
\[ \hat{b}_V = \gamma_b ||\bar{E}_P||^2 \left( Y(Q^{-1}, \bar{P}) + \frac{\epsilon_b}{\epsilon} Y(Q^{-1}, \bar{E}_P) \right) \]
\[ \hat{K}_E = \gamma_b \left( 1 + \frac{\epsilon E_R}{\epsilon} \right) \tilde{Y}(Q^{-1}, \hat{\hat{P}}) \]
\[ \hat{\sigma} = 4k_w \epsilon_b \left( Y(Q^{-1}, \hat{\hat{P}}) + \frac{\epsilon_b}{\epsilon} Y(Q^{-1}, \bar{E}_P) \right) \]

The equivalent quaternion representation and complete implementation steps of the filter in (77), (78), (79), (80), (81), and
(82) is:

$$
\begin{bmatrix}
0 \\
\upsilon_i^B \\
\hat{\upsilon}_i^B
\end{bmatrix} =
\begin{bmatrix}
0 \\
\mathbf{Y}(Q^{-1}, \upsilon_i^T) \\
0
\end{bmatrix} = Q^{-1} \circ \begin{bmatrix}
0 \\
\upsilon_i^T \\
0
\end{bmatrix} \circ \hat{Q}
$$

$$
\mathbf{Y} = \mathcal{R} \sum_{i=1}^{N_R} \left( \frac{1}{2} \upsilon_i^B(R) \times \upsilon_i^B(R) \right)
$$

$$
\mathcal{E}_R = \frac{1}{2} \sum_{i=1}^{N_R} \left( 1 - \left( \upsilon_i^B(R) \right)^T \upsilon_i^B(R) \right)
$$

$$
M_1 = \sum_{i=1}^{N_R} R_i^B(R) \upsilon_i^B(R)^T
$$

$$
M_2 = \left( \sum_{i=1}^{N_R} R_i^B(R) \upsilon_i^B(R)^T \right)^{-1}
$$

$$
\mathcal{E}_P = \hat{P} = \hat{P} + \frac{1}{m_v} \left( \mathbf{Y} \left( \hat{Q}, k_v \right) - M_1 M_2 m_v \right)
$$

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