Bistable systems, as their name implies, are systems which may reside in one of two states. Typically, these states are extremely stable, with rare transitions only occurring through the effects of noise (intrinsic or extrinsic) or external perturbations.

The standard theoretical approach used to investigate bistability is to begin by modeling the system deterministically though a set of differential or difference equations. In the deterministic system there can be no transitions between steady states without the addition of noise to move the system from one state to the other. The theoretical literature examining this effect is enormous, with very many variants of this basic scenario having been investigated in considerable detail. The majority of these theoretical studies fail to use the noise structure appropriate to the system under consideration, and reverse the logical sequence of model building: the deterministic equations together with the correct form of the noise should follow from a model constructed at the microscale (see for instance [2] or [3]).

A bottom-up approach such as this is required to understand unexpected and non-intuitive results such as those seen when a chemical system with a single stable fixed point is driven to bistability at low molecule numbers. This recently discovered mechanism for bistability, so far only investigated in the context of biochemical reactions, is a result of the non-linear nature of the intrinsic noise. In this type of bistability, the noise is responsible for the existence of the bistable states, as well as causing the transitions between them, in contrast to the conventional picture of bistability in which the role of the noise is simply to induce transitions. A distinguishing feature of these noise-induced bistable states is the presence of a critical system size, which bistability does not occur. Evidence for the effect was first found numerically in a study of autocatalytic reactions in a cell. Subsequent analytical studies proposed that the phenomenon is due to the multiplicative nature of the noise, and this was later confirmed by the estimation of the critical system size.

An experimentally testable biological system that exhibits bistability may be found in the foraging behavior of an ant colony. Here we consider a classic experiment, in which a colony of ants is exposed to two identical sources of food. The foraging ants, rather than distributing equally between the two sources instead favor only one source. After a period of time they appear to turn their attention to the other option, so that the majority of ants then start to collect their food from the other source. The models initially used to explain this result were typically rather detailed. However, Kirman observed that analogous behavior also occurs in other systems involving populations, for instance queuing and stock market trading. This suggests a common mechanism depending only on shared properties of the different systems. It is generally agreed that the autocatalytic dynamics present in all of these systems is a key ingredient required for their bistability.

In this Letter we propose that the underlying mechanism for the bistability observed in the experiment described above is the same as that found in the biochemical reactions previously mentioned. To study this, we use a simple model of autocatalytic recruitment and review the estimation of the critical system size, using stationary analysis, for our system. However, the expression obtained for is not easy to experimentally test in our system. We therefore extend our analysis to study the time-dependent behavior of the system, by calculating the mean switching time between the two bistable states for different population sizes. This provides a means to measure experimentally and can be used to test our hypothesized mechanism for bistability.

Our model consists of a colony of ants collecting food from two identical sources, labeled 1 and 2. Ants which collect food from source 1 are denoted by and those which collect food from source 2 by . The fraction of ants which choose source is denoted by . An ant collecting food from one source can be recruited by an ant collecting food from the other. The recruitment of ants is thus autocatalytic, in that the more ants collecting from any particular source, the higher the rate of recruitment to that source. An ant may also spontaneously...
choose to use the other source. We may summarize the model through the following reaction scheme:

$$
\begin{align*}
X_1 + X_2 &\xrightarrow{\epsilon} 2X_1, \quad X_2 + X_1 \xrightarrow{r} 2X_2, \\
X_2 &\xrightarrow{1} X_1, \quad X_1 \xrightarrow{1} X_2.
\end{align*}
$$

(1)

This model is already known in the context of chemical reactions \[15\], obtained as a simplification of the Togashi-Kaneko scheme \[1]. Ant recruitment is dominant so that $0 < \epsilon \ll r$, and we assume $r = 1$ without loss of generality by noting that $\epsilon$ may always be rescaled, as discussed in the supplementary material (SM). We note that the number of ants is conserved so that $x_1 + x_2 = 1$ for all time, and hence the system is fully described by a single independent variable.

To fully specify the model we now give the probability of transition, $T(a|b)$ from state $b$ to state $a$. Invoking mass action \[15\]

$$
T_1 \equiv T(x_1 + \frac{1}{N}, \quad x_2 - \frac{1}{N}|x_1, x_2) = rx_1x_2 + \epsilon x_2,
$$

$$
T_2 \equiv T(x_1 - \frac{1}{N}, \quad x_2 + \frac{1}{N}|x_1, x_2) = rx_1x_2 + \epsilon x_1.
$$

(2)

We use the transition rates to write down the master equation for the probability density function (PDF), $P(x_1, x_2, t)$ \[15\]:

$$
\partial_t P(x_1, x_2, t) = \sum_{(x_1', x_2') \neq (x_1, x_2)} [T(x_1, x_2|x_1', x_2')P(x_1', x_2', t) - T(x_1', x_2'|x_1, x_2)P(x_1, x_2, t)].
$$

(3)

The scheme of reactions \[1\] was simulated using the Gillespie algorithm \[16\] and a typical time series for $z = x_1 - x_2$ is shown in Fig. \[1\]. Regardless of the initial condition, the system settles into one of the steady states $z \approx \pm 1$, indicating that the majority of ants favor one food source. After some time, the system then switches to the other state, $z \approx \mp 1$, where the majority of ants favor the other source.

Unlike other forms of bistability (for example, a Brownian particle in a double-well potential \[1\]), this type of bistability cannot be understood from the fixed points of the corresponding deterministic equations. Indeed, if we take the limit $N \to \infty \[15\]$ to eliminate stochastic effects, we obtain the equation $\dot{z} = -2\epsilon z$ (see SM). This equation has a unique stable fixed point at $z^* = 0$, which is not seen in simulations of the full system. Thus the bistability observed in the stochastic system is not reflected in the deterministic equations.

To understand the origin of the bistability, we expand the master equation \[3\] in powers of the inverse population size, $N^{-1}$ (see SM). After rescaling time, $2\epsilon t/N = \tau$, we find that the system is approximated by the following stochastic differential equation (SDE) \[17\]:

$$
z' = -z + \frac{N_c}{N} \sqrt{1 + 2\epsilon - z^2} \eta(\tau),
$$

(4)

where $N_c \equiv 1/\epsilon$ and $\eta(\tau)$ is Gaussian white noise with zero mean and correlator $\langle \eta(\tau)\eta(\tau') \rangle = \delta(\tau - \tau')$. As shown in \[6\], Eq. \[4\] underlies a broad class of systems featuring an autocatalytic network and a slow linear reaction. The variable $z = x_1 - x_2$ ranges over the interval $[-1, 1]$, whose extrema correspond to all ants collecting food from a single source. Equation \[4\] for $\epsilon = 0$ is equivalent to the Wright-Fisher model with mutation, under the change of variable $x = (1 + z)/2 \[18\].$

We see from Eq. \[4\] that the strength of the intrinsic system noise is proportional to $\sqrt{1 + 2\epsilon - z^2}$. The noise therefore has maximum strength at the deterministic steady state $z = z^* = 0$, pushing the system away from this point and towards $z = \pm \sqrt{1 + 2\epsilon}$. Since $z$ is defined in the interval $[-1, 1]$ the system cannot cross these boundaries. Bistability originates from the dependence of the noise strength on the variable $z$. At $z = \pm 1$ the noise term is at a minimum, whilst the deterministic term $-z$ attracts the system back towards $z^*$. As the trajectory leaves $z = \pm 1$ the noise term regains strength and once again kicks the system towards one of the bistable steady states $z = \pm 1$. These combined effects are seen in the dynamics of Fig. \[1\].

A distinguishing characteristic of noise-induced bistable states is the existence of a critical system size, above which bistability ceases to occur. This should be contrasted with the bistability in which the system moves between two fixed points due to the presence of noise, where varying the noise strength merely affects the characteristic time spent in each bistable state. We may therefore predict that if the bistable states are noise-induced then there should exist a critical population size above which the behavior ceases to occur.

As shown in previous studies \[5, 8, 18\], the transition between the regime which shows bistable behavior and the one that does not, can be understood from the
Fokker-Planck equation corresponding to Eq. (4). Taking $\partial_t P = 0$ and imposing zero-flux boundary conditions at $z = \pm 1$, we obtain the stationary probability distribution

$$P_s(z) = \frac{C_0}{(1 + 2\epsilon - z^2)^{1 - Nc}},$$

(5)

where $C_0$ is a normalisation constant, found by requiring that the integral of $P_s(z)$ over the interval $[-1, 1]$ is unity.

FIG. 2. (Color online) Equation (5) (solid colors) is compared against simulations of scheme (1) (symbols). Simulations are obtained by taking the normalised histogram of a time series of length $\tau = 2.5 \times 10^3$. We have used $\epsilon = 10^{-3}$ and $N = 1500$ (blue line, triangles), $N = N_c \equiv 1000$ (red line, squares) and $N = 500$ (purple line, circles).

The stationary distribution predicts the normalised long-time frequency histogram of $z$ and is plotted against simulation data in Fig. 2 for different population sizes. For $N < N_c$, $P_s(z)$ has a U-shape, diverging at $z = \pm \sqrt{1 + 2\epsilon}$. Below the critical population size, the system therefore spends most of the time close to the bistable states. In contrast, for $N > N_c$, the steady state distribution, $P_s(z)$ has an inverted U-shape, centred on the deterministic fixed point $z = z^* = 0$. This latter regime is the only one that is captured by the linear noise approximation technique (the van Kampen expansion) [15, 17, 19].

To estimate the critical population size requires knowledge of the parameters $r$ and $\epsilon$ (recall that we set $r = 1$ by rescaling $\epsilon$). However, these reaction constants are difficult to measure experimentally. An alternative way to estimate $N_c$ is provided by calculating the time taken for the system to move from one bistable state ($z = -1$, say) to the other ($z = 1$). This time is a stochastic variable whose mean (over many realizations) is denoted by $\mathcal{T}_c$. Using Eq. (4), we may find this mean switching time [11] (see the SM for details). In the rescaled time variable, $\tau$, this is given by

$$\mathcal{T}_c = \frac{4N}{(1 + 2\epsilon)N_c} \frac{2F_1}{1} \left( \frac{1}{2}, 1 - Nc, \frac{3}{2}; 1; 1 + 2\epsilon \right) \times \frac{2F_1}{1} \left( \frac{1}{2}, Nc, \frac{3}{2}; \frac{1}{2}; 1 + 2\epsilon \right),$$

(6)

where the function $2F_1$ is the hypergeometric function [20]. Equation (6) agrees with simulations of the reaction scheme (1) only for $N$ in the neighborhood of $N_c$ (Fig. 3) and for $N > N_c$ (this latter result is not shown). Results are shown for different values of $\epsilon$ using different symbols. Note that for small $N$ the simulation results merge so that the mean time is independent of $\epsilon$. Since time was rescaled by $\epsilon$, however, an $\epsilon$ dependence is retained in the definition of $\mathcal{T}_c$.

FIG. 3. (Color online) Equation (6) (solid lines) is compared against stochastic simulations (symbols). Parameter used: $\epsilon = 1/50$ (blue line, triangles), $\epsilon = 1/100$ (green line, squares) and $\epsilon = 1/2000$ (red line, circles). Each symbol has been obtained by averaging over 500 simulations.

At small population sizes, as the simulation results become independent of $\epsilon$, Eq. (6) breaks down and does not capture the system behavior. The failure of Eq. (6) in this regime is due to assumptions made in the derivation of Eq. (4), which is no longer representative of the system at small population sizes. Instead the terms neglected in the expansion of the master equation must be retained.

Indeed, in our derivation, the noise strength in Eq. (4) diverges as $N \to 0$, so that the time taken to move from one bistable state to the other shrinks to zero. In contrast, the simulated switching times do not go to zero as $N \to 0$. However, we see from Fig. 3 that the range of $N$ where our prediction holds differs for different values of $\epsilon$. The agreement improves for smaller $\epsilon$, suggesting that the limiting value of Eq. (6) as $\epsilon \to 0$ may capture the system dynamics at small population sizes.

Taking $\epsilon \to 0$ (see SM), Eq. (6) reduces to:

$$\mathcal{T}_0 = 2\pi \frac{N}{N_c - 2N} \cot \left( \frac{\pi N}{N_c} \right).$$

(7)
Equation (7) agrees well with simulation data for small population sizes (Fig. 5). Since the mean switching time depends strongly on $\epsilon$ for larger population sizes (Fig. 3), we do not expect $T_0$ to accurately predict the simulation data for larger $N$. Indeed, as $N \to N_c$, Eq. (7) diverges and thus does not capture the behavior of the system (see SM).

The analysis may be further extended by considering the full distribution of times to move between the bistable states, rather than using only the mean time. In this way it would be possible to assess any skewness of the distribution and determine how representative the mean time is of the full distribution.

Our results do not only apply to the model described here, as Eq. (4) is the reduced one-dimensional equation for many stochastic systems, such as the Togashi-Kaneko model (6). We believe that the mechanism for noise-induced bistability, in which the changing noise strength at different system states leads to substantially different behavior from the deterministic approximation, will be applicable to a wide variety of systems.

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The derivation of the equation for the $z$ variable

The model is defined by the two transition rates:

$$
T_1 \equiv T(x_1 + \frac{1}{N}, x_2 - \frac{1}{N}|x_1, x_2) = rx_1x_2 + \epsilon x_2,
$$

$$
T_2 \equiv T(x_1 - \frac{1}{N}, x_2 + \frac{1}{N}|x_1, x_2) = rx_1x_2 + \epsilon x_1.
$$

We rewrite the master equation,

$$
\partial_t P(x_1, x_2, t) = \sum_{(x'_1 \neq x_1, x'_2 \neq x_2)} \left[ T(x_1, x_2| x'_1, x'_2)P(x'_1, x'_2, t) - T(x'_1, x'_2| x_1, x_2)P(x_1, x_2, t) \right],
$$

using the step operators, $\varepsilon_i^\pm$, which represent the creation or destruction of a molecule of species $X_i$ ($i = 1, 2$). Taylor expanding in $1/N$, the inverse of the population size:

$$
\varepsilon_i^\pm f(x_i) = f(x_i \pm \frac{1}{N}) \approx \left( 1 \pm \frac{1}{N} \partial x_i + \frac{1}{2N^2} \partial^2 x_i \right) f(x_i),
$$

where $f(x_i)$ is a general function of the fraction of the $i$-th species, $x_i$. The master equation (9) can be approximated using Eq. (10) to give

$$
\partial_t P(x_1, x_2, t) = \left[ (\varepsilon_1^+ \varepsilon_2^- - 1) T_1 + (\varepsilon_1^+ \varepsilon_2^- - 1) T_2 \right] P(x_1, x_2, t)
$$

$$
\approx \left[ \frac{1}{N} (\partial x_2 - \partial x_1) T_1 + \frac{1}{N} (\partial x_1 - \partial x_2) T_2 + \frac{1}{2N^2} (\partial x_1 - \partial x_2)^2 (T_1 + T_2) \right] P(x_1, x_2, t),
$$

neglecting terms of $O(1/N^3)$.

Rescaling time by $t/N \rightarrow t$ and inserting the expressions of the transition rates (8) gives the Fokker-Planck equation

$$
\partial_t P(x_1, x_2, t) = \left[ -\partial x_1 A_1 - \partial x_2 A_2 + \frac{1}{2N} \sum_{i,j=1}^2 \partial x_i \partial x_j B_{ij} \right] P(x_1, x_2, t),
$$

where $A_1 = -A_2 = \epsilon(x_2 - x_1)$ and $B_{ij} = (2r(x_1x_2 + \epsilon(x_1 + x_2))(-1)^{i+j}$. This is equivalent to the following system of SDEs (11) in which the noises have zero mean:

$$
\dot{x}_1 = A_1 + \frac{1}{\sqrt{N}} \xi_1, \quad \dot{x}_2 = A_2 + \frac{1}{\sqrt{N}} \xi_2, \quad \langle \xi_i(t)\xi_j(t') \rangle = B_{ij} \delta(t-t').
$$

We make the transformation $\xi_i = \sum_{j=1}^2 G_{ij} \eta_j$, where $G_{ij} = (-1)^{i+j+1} \sqrt{2r(x_1x_2 + \epsilon(x_1 + x_2))}$, hence the new noises are delta-correlated, that is, $\langle \eta_i(t)\eta_j(t') \rangle = \delta_{ij} \delta(t-t')$. This can be proved using the expression of the correlator for $\xi_i$ and the fact that $B = GG^T$. System (13) then becomes:

$$
\dot{x}_1 = \epsilon(x_2 - x_1) + \sqrt{2r} \frac{x_1x_2 + \epsilon(x_1 + x_2)}{2N} (\eta_2 - \eta_1) = -\dot{x}_2.
$$

We now introduce new variables, $w = x_1 + x_2$ and $z = x_1 - x_2$, which satisfy equations obtained by summing and subtracting the equations for $x_1$ and $x_2$:

$$
\dot{w} = 0, \quad \dot{z} = -2\epsilon z + \sqrt{\frac{1}{N}} \sqrt{r(w^2 - z^2)} (\eta_2 - \eta_1).
$$

The $z$ equation can be simplified as follows. First, we use the sum rule for Gaussian variables, so that $\eta_1 - \eta_2 = \sqrt{2} \eta$, where $\eta$ is normalised Gaussian white noise (14). Then, we rescale time by $2\epsilon t \rightarrow t$. Note that the coefficient which multiplies the noise scales with a square root law, as expected (15). The overall time scaling is given by $\tau = 2\epsilon t/N$ and we obtain

$$
w' = 0, \quad z' = -z + \sqrt{\frac{r}{N\epsilon}} \sqrt{(w^2 - z^2) + 2\epsilon w \eta(\tau)},
$$
where the prime sign indicates the time derivative with respect to \( \tau \). Without loss of generality we set \( r = 1 \), since we may rescale \( \epsilon \) to absorb \( r \). Since the transition rates (8) do not alter the total number of ants, \( N \) and \( w \) are conserved quantities with \( w = 1 \). Hence

\[
z' = -z + \sqrt{\frac{N_c}{N}} \sqrt{1 - z^2 + 2\epsilon \eta(\tau)}.
\]

(17)

where \( N_c = 1/\epsilon \).

The mean switching time

We wish to find the mean time for the system to leave \( z = -1 \) and reach \( z = 1 \). To derive this quantity we consider the mean time, \( \langle T_\epsilon \rangle \), for a system starting at \( z \) to leave the interval \([-1, 1]\). This is derived from \( G(z, t) \), the density of probability that a system beginning at \( z \) has not left the interval \([-1, 1]\) by time \( t \). Then \( G(z, t) \) satisfies the backward Fokker-Planck equation corresponding to Eq. (17) \[1\]

\[
\frac{\partial G}{\partial t} = -z \frac{\partial G}{\partial z} + \frac{1}{2\lambda} (1 - z^2 + 2\epsilon) \frac{\partial^2 G}{\partial z^2},
\]

(18)

where \( \lambda = N/N_c \), with a reflecting boundary condition at \( z = -1 \) and an absorbing boundary condition at \( z = 1 \).

Now the probability density function for the system beginning at \( z \) and reaching the boundary at \( z = 1 \) at time \( t \) (where it is thus removed from the interval) is given by

\[
\langle T_\epsilon \rangle = -\int_0^\infty t \frac{\partial G}{\partial t}(z, t) \, dt = \int_0^\infty G(z, t) \, dt,
\]

(19)

assuming \( G(z, t) \) is well behaved as \( t \to \infty \). Integrating Eq. (18) over \( t \), we obtain

\[
\frac{1}{2\lambda} (1 - z^2 + 2\epsilon) \langle T_\epsilon \rangle'' - z \langle T_\epsilon \rangle' + 1 = 0,
\]

(20)

since the system must start in the interval \([-1, 1]\) so that

\[
\int_0^\infty \partial_t G(z, t) \, dt = -G(z, 0) = -1.
\]

(21)

We may solve Eq. (20) by first writing it as

\[
\frac{d}{dz} (1 + 2\epsilon - z^2) \lambda \langle T_\epsilon \rangle' = -2\lambda (1 + 2\epsilon - z^2) \lambda^{-1}.
\]

(22)

To integrate the right hand side we need the following integral

\[
I_1(\mu) = \int (1 + 2\epsilon - z^2)^\mu dz,
\]

\[
= \frac{z(1 - z^2 + 2\epsilon)^{1+\mu}}{1 + 2\epsilon} \mathbf{2F1} \left( \frac{1}{2} + \mu, \frac{3}{2} \frac{z^2}{1 + 2\epsilon} \right),
\]

(23)

where \( \mathbf{2F1} \) is the hypergeometric function \[20\]. Equality (23) can be seen by expanding \((1 + 2\epsilon - z^2)^\mu\) as the binomial series, integrating term-by-term and using the series definition for the hypergeometric function. Integrating both sides of Eq. (22):

\[
\langle T_\epsilon \rangle' = -\frac{2\lambda z}{1 + 2\epsilon} \mathbf{2F1} \left( 1, \frac{1}{2} + \lambda, \frac{3}{2} \frac{z^2}{1 + 2\epsilon} \right) + C_1 (1 + 2\epsilon - z^2)^{-\lambda},
\]

(24)

where \( C_1 \) is an integration constant. Integrating again, we obtain

\[
\langle T_\epsilon \rangle = I_2 + C_1 I_1(-\lambda) + C_2,
\]

(25)
where

\[
I_2 = -\frac{2\lambda}{1+2\epsilon} \int z_2 F_1 \left( \frac{1}{2}, \lambda; \frac{3}{2}, \frac{z^2}{1+2\epsilon} \right) dz,
\]

\[
= -\frac{\lambda z^2}{1+2\epsilon} F_2 \left( 1, 1, \frac{1}{2} + \lambda; \frac{3}{2}, \frac{z^2}{1+2\epsilon} \right),
\]

since \(z = 1\) and

\[
\int z^{\alpha-1} z_2 F_1 \left( a, b, c; z \right) dz = \frac{z^{\alpha} z_2}{\alpha} F_2 \left( a, b, c, 1 + \alpha, z \right),
\]

where \(_2 F_1(x_1, \ldots, x_q; y_1, \ldots, y_p; z)\) indicates the generalised hypergeometric function. \(\text{[20, 21]}\)

The final expression is therefore

\[
\langle T_\epsilon \rangle = C_2 + z C_1 z_2 F_1 \left( \frac{1}{2}, \lambda; \frac{3}{2}, \frac{z^2}{1+2\epsilon} \right) - \frac{z^2 \lambda}{1+2\epsilon} F_2 \left( 1, 1, \frac{1}{2} + \lambda; \frac{3}{2}, \frac{z^2}{1+2\epsilon} \right),
\]

where \(C_1 = C_1/(1+2\epsilon)\) and we have used the Euler transformation, \((1-w)^{a+b-c} z_2 F_1(\alpha, \beta; \gamma; w) = \gamma z_2 F_1(\alpha-a, \beta-c; \gamma; w)\), to simplify the second term.

We take a reflecting boundary condition at \(z = -1\) and an absorbing boundary condition at \(z = 1\). Hence with the initial condition \(z = -1\), we model a trajectory that begins at \(z = -1\) and is stopped at \(z = 1\). These two boundary conditions determine the constants \(C_1\) and \(C_2\) and are given by \(\text{[1]}\):

\[
\langle T_\epsilon \rangle \big|_{z=1} = 0, \quad \langle T_\epsilon \rangle' \big|_{z=-1} = 0.
\]

To determine the constant \(C_2\) we use the absorbing boundary condition in Eq. \(\text{[28]}\), so that

\[
C_2 = -C_1 z_2 F_1 \left( \frac{1}{2}, \lambda; \frac{3}{2}, \frac{1}{1+2\epsilon} \right) + \frac{\lambda}{1+2\epsilon} F_2 \left( 1, 1, \frac{1}{2} + \lambda; \frac{3}{2}, \frac{1}{1+2\epsilon} \right).
\]

Thus, \(C_2\) is fully determined once \(C_1\) has been found.

To satisfy the reflecting boundary condition, we differentiate Eq. \(\text{[28]}\):

\[
\langle T_\epsilon \rangle' = \left( 1 - \frac{z^2}{1+2\epsilon} \right)^{-\lambda} \left[ C_1 - \frac{2\lambda z}{1+2\epsilon} z_2 F_1 \left( \frac{1}{2}, 1 - \lambda; \frac{3}{2}, \frac{z^2}{1+2\epsilon} \right) \right].
\]

The reflecting boundary condition is satisfied if the term in the square brackets converges to zero as \(z \to -1\). This yields:

\[
C_1 = -\frac{2\lambda}{1+2\epsilon} z_2 F_1 \left( \frac{1}{2}, 1 - \lambda; \frac{3}{2}, \frac{1}{1+2\epsilon} \right).
\]

Inserting \(C_1\) into Eq. \(\text{[30]}\) leads to an expression for \(C_2\):

\[
C_2 = \frac{\lambda}{1+2\epsilon} \left[ 2 z_2 F_1 \left( \frac{1}{2}, 1 - \lambda; \frac{3}{2}, \frac{1}{1+2\epsilon} \right) z_2 F_1 \left( \frac{1}{2}, \lambda; \frac{3}{2}, \frac{1}{1+2\epsilon} \right) + z_2 F_1 \left( 1, 1, \frac{1}{2} + \lambda; \frac{3}{2}, \frac{1}{1+2\epsilon} \right) \right].
\]

We now use the expressions in Eq. \(\text{[32]}\) and Eq. \(\text{[33]}\) in Eq. \(\text{[28]}\). We also set the initial condition, \(z = -1\). Hence the final formula for the mean time for the system to leave \(z = -1\) and reach \(z = 1\) is

\[
T_\epsilon \equiv \langle T_\epsilon \rangle \big|_{z=1} = \frac{4\lambda}{1+2\epsilon} z_2 F_1 \left( \frac{1}{2}, 1 - \lambda; \frac{3}{2}, \frac{1}{1+2\epsilon} \right) z_2 F_1 \left( \frac{1}{2}, \lambda; \frac{3}{2}, \frac{1}{1+2\epsilon} \right).
\]

Note that as \(\epsilon \to 0\) the expression of the mean time may be written as

\[
T_0 \equiv \langle T_0 \rangle \big|_{z=1} = 4\lambda \frac{\Gamma \left( \frac{3}{2} \right)^2 \Gamma \left( \frac{3}{2} \right) \Gamma \left( 1 - \frac{3}{2} \right)}{\Gamma \left( \frac{3+1}{2} \right) \Gamma \left( \frac{3-1}{2} \right)} = \pi \lambda \frac{\Gamma \left( \frac{3}{2} \right) \Gamma \left( 1 - \frac{3}{2} \right)}{\Gamma \left( \frac{3+1}{2} \right) \Gamma \left( \frac{3-1}{2} \right)},
\]

where \(\Gamma(x)\) is the Gamma function.
then the trajectory cannot reach $z \in \mathbb{R}$ starting from Eq. (20) for $\epsilon > 0$ the point $z = 1$ is not contained in $[-1,1]$, the interval of definition for $z$.

This concludes the analytical treatment of the mean time. As a final remark, note that if one wants to derive Eq. (37) starting from Eq. (20) for $\epsilon = 0$, more care is required in formulating the boundary conditions. In fact, Eq. (20) for $\epsilon = 0$ becomes singular as $z \to -1$, in the sense that the coefficient which multiplies the second derivative vanishes in that limit. The boundary conditions must therefore be modified to

$$\langle T_0 \rangle (z) = 0, \quad \lim_{z \to -1} (1 - z^2) \langle T_0 \rangle (z) = 0,$$

representing the absorbing boundary at $z = 1$ and reflecting boundary at $z = -1$. The calculation may then be carried out analogously to the previous derivation. However, we find that the reflecting boundary condition is inherently satisfied regardless of the choice of $C_1$. To determine this constant, we must require an additional condition, namely that the derivative of the mean time remain finite as $z \to -1$. This condition is automatically satisfied when $\epsilon$ is non-zero (as can be seen from Eq. (31)). Using all these conditions one can derive Eq. (37).

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