Unitary Gauge,
Stückelberg Formalism
and Gauge Invariant Models
for Effective Lagrangians

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BI-TP 92/56
December 1992

Abstract

Within the framework of the path-integral formalism we reinvestigate the different methods of removing the unphysical degrees of freedom from spontaneously broken gauge theories. These are: construction of the unitary gauge by gauge fixing; $R_\xi$-limiting procedure; decoupling of the unphysical fields by point transformations. In the unitary gauge there exists an extra quartic divergent Higgs self-interaction term, which cannot be neglected if perturbative calculations are performed in this gauge. Using the Stückelberg formalism this procedure can be reversed, i.e., a gauge theory can be reconstructed from its unitary gauge. We also discuss the equivalence of effective-Lagrangian theories, containing arbitrary interactions, to (nonlinearly realized) spontaneously broken gauge theories and we show how they can be extended to Higgs models.

*Supported in part by Deutsche Forschungsgemeinschaft, Project No.: Ko 1062/1-2
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1 Introduction

The purpose of the present paper is primarily to reinvestigate the various approaches to the unitary gauge within quantized spontaneously broken gauge theories (SBGTs), thereby putting the emphasis on the connections between the different methods and their common basis. Although most of the described techniques are known (at least to several groups of experts) we find it worthwhile to clarify these different approaches and, especially, to analyze the powerful method of Stückelberg transformations. On the basis of this analysis we then utilize the Stückelberg formalism to connect effective vector-boson Lagrangians (containing standard or non-standard vector-boson self-interactions) with (linearly or nonlinearly realized) SBGTs.

SBGTs contain unphysical degrees of freedom, the pseudo-Goldstone scalars. On the classical level the unphysical fields can be removed by means of gauge transformations, i.e., for given values of the pseudo-Goldstone fields at each space-time point there exists a gauge transformation (with gauge parameters that have to be chosen as functions of these values) which maps the unphysical fields identically to zero. This gauge, which is characterized by the fact that the Lagrangian contains only “physical fields” is called the unitary gauge (U-gauge).

However, this naive definition of the U-gauge cannot be applied in quantum physics which is best seen if one uses for quantization the framework of Feynman’s path integral (PI) [1] and the Faddeev–Popov (FP) formalism [2]. On the quantum level it is the generating functional and not only the Lagrangian which contains the complete physical information. Since the gauge transformation which removes the unphysical fields is dependent on the values of these fields, it cannot be applied to the generating functional where a functional integration over all values of the fields is performed. In other words there is not a “universal” transformation which sets arbitrary pseudo-Goldstone fields equal to zero.

There are three (equivalent) ways of constructing a gauge without unphysical fields (i.e. without pseudo-Goldstone and without ghost fields) within the PI formalism. We discuss them mainly for the case of linearly and minimally realized SBGTs (i.e. those which contain physical Higgs scalars [3, 4] and which are renormalizable). The case of nonlinear and/or nonminimal realizations will also be explained at the end.

The first procedure for constructing the U-gauge is simply to impose the gauge-fixing condition that the pseudo-Goldstone fields are equal to zero [5] (which can be done because of the existence of the abovementioned gauge transformation). The corresponding FP δ-function is used to integrate out the unphysical scalars while the FP determinant can be exponentiated without introducing ghost fields and yields a quartic divergent (i.e. proportional to $\delta^4(0)$) nonpolynomial Higgs-self-coupling term.

The second method is to construct the $R_\xi$-gauge [6, 7, 8] in which unphysical fields are still present but with masses proportional to the free parameter $\sqrt{\xi}$,
and then to perform the limit $\xi \to \infty$. In this limit, the unphysical fields get infinite masses and decouple. However, the ghost-ghost-scalar couplings get infinite, too, with the consequence, that the ghost term does not completely vanish: there remains the abovementioned Higgs-self-coupling term.

The third way is most similar to the classical treatment: the unphysical fields are decoupled from the physical ones by point transformations. This procedure consists of two subsequent transformations; first the unphysical scalars are paramatrized nonlinearly and then they are decoupled and can be integrated out. Since in this formalism transformations of the functional integrand are performed, the Jacobian determinant due to the change of the integral measure has to be considered; it yields again the new Higgs-self-interaction term.

Thus, all three methods lead to a quantum level Lagrangian (in the U-gauge) which contains, in addition to the classical U-gauge Lagrangian, the extra non-polynomial quartic divergent Higgs-self-interaction term. The same term was derived by quantizing the classical U-gauge Lagrangian canonically where it emerges as a remnant of covariantrization. It has been shown on the one-loop level for three- and four-Higgs-interaction amplitudes that the quartic divergences cancel against quartic divergent N-Higgs-vertices which are quantum induced by gauge-boson loops in the U-gauge; this ensures renormalizability. In fact, a linear SBGT is renormalizable even in its U-gauge because of the equivalence of all gauges (although this is not expected by naive power-counting due to the bad high-energy behaviour of the gauge boson propagator in this gauge). So loop calculations can be performed consistently in the unitary gauge if the extra term (which does not contribute to the presently phenomenological most interesting processes at one-loop level) is taken into account. Loop calculations may be simpler in the U-gauge than in the $R_\xi$-gauge because there are less Feynman diagrams to be considered, on the other hand the resulting expressions can be more complex because of the form of the vector-boson propagator which is proportional to the zeroth instead of the inverse second power of the energy.

The third of the abovementioned procedures can be reversed, i.e. a SBGT can be “reconstructed” from its U-gauge Lagrangian (which is considered as an effective Lagrangian). To do this, scalar fields, which are initially completely decoupled, are introduced to the theory by multiplying an (infinite) constant to the generating functional, which contains the functional integration over these fields. The unphysical scalars are then coupled to the physical fields by an appropriate point transformation. At the next step unphysical and physical scalar fields are rewritten in a linearized form. This procedure has been described in and is formulated here within the PI formalism. The method of constructing SBGTs by such “field enlarging transformations” represents the non-Abelian version of the St"uckelberg formalism, which in its original form was studied only for theories without physical Higgs bosons, where it leads to the problem of nonpolynomial interactions and nonrenormalizability (in non-Abelian theories). The existence of physical scalars, however, enables a linearization of the scalar
sector, so that renormalizable Stückelberg models can be constructed.

One should remember in this connection that the classical U-gauge Lagrangian can be derived simply by the demand of tree unitarity (good high energy behaviour of tree level cross sections) \[12, 13, 18\]: tree unitarity implies a SBGT as stated in \[13\]. However, for being able to handle quantum effects, the Lagrangian must contain in addition the extra quartic divergent Higgs self-coupling term to compensate the Jacobian determinants that arise while performing the point transformations. This term cannot be derived from tree unitarity alone, but, as mentioned above, from the demand of vanishing quartic loop-implied Higgs self-couplings \[12\].

Recently the Stückelberg formalism has attracted attention in its original domain of constructing Higgs-less gauge-theories, because it has been shown that each effective-Lagrangian theory (containing arbitrary interactions) is equivalent to a gauge theory with (in general) nonlinearly realized symmetry \[19\]; i.e., there exists a (field enlarging) point transformation of the fields which makes the Lagrangian gauge invariant. For the case of massive spin-one particles with arbitrary (non-Yang–Mills) self-interactions this equivalence has been investigated in \[20\]. We will prove that the corresponding transformation is in fact a Stückelberg transformation. Within these models the gauge group acts nonlinearly on the unphysical fields, there are nonpolynomial interactions and they are nonrenormalizable. However, the gauge freedom enables the choice of the $R_\xi$-gauge (where the vector-boson propagators have a good high-energy behaviour) to perform loop calculations in an effective-Lagrangian theory, which shows that the loops in such a theory do not diverge as severely as one would expect. Besides, each effective-Lagrangian theory with massive vector bosons can even be extended to a SBGT with linearly realized symmetry by introducing a physical Higgs boson. This makes the loop corrections even smaller.

Within this paper, SBGTs are discussed by taking the example of the SU(2) × U(1) standard model (SM) of electroweak interaction \[21, 22\] since it is of greatest phenomenological interest and since it is a sufficiently general version of a SBGT (the gauge group is non-Abelian, a subgroup remains unbroken, there is gauge-boson mixing and the model contains fermions). Special simplifications that arise in simpler SBGTs do not take place here and results can be easily transferred to any other SBGT. Similarly, in our discussion of effective-Lagrangian theories we restrict ourselves to theories containing the electroweak vector bosons, which leads to SU(2) × U(1) gauge invariance.

The paper is organized as follows: In Sect. 2 we introduce our notation of the SM and of the FP quantization procedure. In the next three sections we describe the different methods of deriving the unitary gauge (within the SM): in Sect. 3 the removal of the unphysical scalars by gauge fixing, in Sect. 4 the limit $\xi \to \infty$ of the $R_\xi$-gauge, and in Sect. 5 the decoupling of these fields by point transformations. In Sect. 6 we explain the construction of the SM from its U-gauge Lagrangian and reformulate the Stückelberg formalism in the framework
of the PI formalism. In Sect. 7 we discuss the treatment of (nonrenormalizable) effective-Lagrangian theories within this formalism and explain the derivation of such models from SBGTs both with nonlinearly and with linearly realized symmetry. Sect. 8 is devoted to a summary of our results.

2 Preliminaries and Notation

The SM gauge fields corresponding to the gauge groups SU(2) and U(1), respectively, are $W_i \mu$ and $B_\mu$. For practical purposes the $W$ field is parametrized in terms of a $2 \times 2$ matrix:

$$W_\mu = \frac{1}{2} W_i \mu \tau_i. \quad (1)$$

The matrix valued field strength tensors are denoted by $W_{\mu \nu}$ and $B_{\mu \nu}$. The scalar fields $\tilde{h}$ and $\varphi_i (i = 1, 2, 3)$ are written as a $2 \times 2$ matrix as well:

$$\Phi = \frac{1}{\sqrt{2}} (\tilde{h} + i \tau_i \varphi_i). \quad (2)$$

Furthermore we consider one fermionic doublet (the generalization to more doublets works as usual) consisting of an up-type field $u$ and a down-type field $d$ (quark or lepton)

$$\Psi = \begin{pmatrix} u \\ d \end{pmatrix}, \quad \Psi_{L,R} = \frac{1}{2} (1 \mp \gamma_5) \Psi. \quad (3)$$

The fermion mass matrix is

$$M_f = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}. \quad (4)$$

With the help of the appropriate covariant derivatives $D_\mu \Phi$, $D_\mu \Psi_L$ and $D_\mu \Psi_R$, the gauge invariant Lagrangian of the SM takes the well known form \[21\], \[22\].
\[ \mathcal{L}_{\text{inv}} = -\frac{1}{2} \text{tr} (W^{\mu \nu} W_{\mu \nu}) - \frac{1}{4} (B^{\mu \nu} B_{\mu \nu}) \\
+ \frac{1}{2} \text{tr} \left[ (D^\mu \Phi)^\dagger \left( D_\mu \Phi \right) \right] - \frac{1}{2} \mu^2 \text{tr} (\Phi \Phi^\dagger) - \frac{1}{4} \lambda \text{tr} (\Phi \Phi^\dagger)^2 \\
+ i (\bar{\Psi}_L \gamma_\mu D_\mu \Psi_L + \bar{\Psi}_R \gamma_\mu D_\mu \Psi_R) - \frac{\sqrt{2}}{v} (\bar{\Psi}_L \Phi M_f \Psi_R + \bar{\Psi}_R M_f \Phi^\dagger \Psi_L) \] (5)

(with \( \mu^2 < 0 \) and \( \lambda > 0 \)). \( \mathcal{L}_{\text{inv}} \) is invariant under the local SU(2) \times U(1) gauge transformations

\[
W_\mu \rightarrow S(x) W_\mu S^\dagger(x) - \frac{i}{g} S(x) \partial_\mu S^\dagger(x), \\
B_\mu \rightarrow B_\mu - \partial_\mu \beta(x), \\
\Phi \rightarrow S(x) \Phi \exp \left( -\frac{i}{2} g' \beta(x) \tau_3 \right), \\
\Psi_L \rightarrow S(x) \exp \left( \frac{i}{2} g'(B - L) \beta(x) \right) \Psi_L, \\
\Psi_R \rightarrow \exp \left( \frac{i}{2} g' (\tau_3 + B - L) \beta(x) \right) \Psi_R, \] (6)

with

\[ S(x) = \exp \left( \frac{i}{2} g \alpha_i(x) \tau_i \right), \] (7)

(where \( g \) and \( g' \) are the SU(2) and U(1) coupling constants and \( B \) and \( L \) are the Baryon and the Lepton number of \( \Psi \), respectively). \( \alpha_i(x) \) and \( \beta(x) \) denote the four gauge parameters. The nonvanishing vacuum expectation value (VEV) of the scalar field \( \Phi \) is chosen as

\[ \langle \Phi \rangle_0 = \frac{v}{\sqrt{2}} \mathbf{1} \quad \text{with} \quad v = \sqrt{-\mu^2/\lambda}. \] (8)

By defining

\[ h = \tilde{h} - v, \] (9)

\( h \) and \( \varphi_i \) have vanishing VEV. \( h \) is the Higgs field and \( \varphi_i \) are the pseudo-Goldstone fields. The physical gauge-boson fields \( W^{\pm}_\mu \), \( Z_\mu \) and \( A_\mu \) (photon) are the well known combinations of \( W_i \mu \) and \( B_\mu \).

To quantize the theory one introduces the path integral \[1\]

\[ Z = \int \mathcal{D}W_{i \mu} \mathcal{D}B_\mu \mathcal{D}h \mathcal{D}\varphi_i \mathcal{D}\Psi \mathcal{D}\bar{\Psi} \exp \left( i \int d^4x \mathcal{L}_{\text{inv}} \right). \] (10)

It contains an infinite constant due to the gauge freedom, which is removed in the FP formalism \[2\]. Imposing the general gauge fixing conditions

\[ F_a(W_{i \mu}, B_\mu, \Phi) = C_a(x), \quad a = 1, \ldots, 4 \] (11)
(where $C_a(x)$ are arbitrary functions) $Z$ is rewritten as

$$Z = \int D\psi D\bar{\psi} D\varphi_i D\psi D\bar{\psi} \exp \left( i \int d^4x \mathcal{L}_{\text{inv}} \right) \det \left( \frac{\delta F_a(x)}{\delta \alpha_b(y)} \right) \exp \left( i \int d^4x \mathcal{L}_{\text{eff}} \right)$$

(12)

($\alpha_a = (\alpha_i, \beta)$). Since (12) is independent of the $C_a$ [5], one can perform the weighted average over them (with the weight functions $\exp \left( -\frac{i}{2\xi_a} \int d^4x C_a^2 \right)$, $\xi_a$ being a set of free parameters[7]) and one expresses the FP determinant through the ghost fields $\eta_a, \eta_a^*$ using

$$\det \left( \frac{\delta F_a(x)}{\delta \alpha_b(y)} \right) \propto \int D\eta_a D\eta_a^* \exp \left( -i \int d^4x \eta_a^* \frac{\delta F_a}{\delta \alpha_b} \eta_b \right)$$

(13)

As a result, (12) can be written in terms of a PI with an effective Lagrangian

$$Z = \int D\psi D\bar{\psi} D\varphi_i D\psi D\bar{\psi} D\eta_a D\eta_a^* \exp \left( i \int d^4x \mathcal{L}_{\text{eff}} \right),$$

(14)

where $\mathcal{L}_{\text{eff}}$ is given by

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{inv}} - \frac{1}{2\xi_a} F_a^2 - \eta_a^* \frac{\delta F_a}{\delta \alpha_b} \eta_b$$

$$\equiv \mathcal{L}_{\text{inv}} + \mathcal{L}_{\text{g.f.}} + \mathcal{L}_{\text{FP}}.$$  

(15)

Finally, source terms for all (physical and unphysical) fields have to be added to the Lagrangian to perform perturbative calculations [4].

### 3 Derivation of the U-Gauge by Gauge Fixing

In this section we explain the “direct way” of constructing the U-gauge within the FP formalism by setting the unphysical pseudo-Goldstone fields equal to zero from the beginning. This is done by imposing the gauge fixing conditions [4]

$$\varphi_i = 0,$$

$$\partial_\mu A^\mu = C(x).$$

(16)

This choice is possible since the $\varphi_i$ can be transformed to zero by gauge transformations. The second condition is necessary due to the unbroken symmetry

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1. Usually all $\xi_a$ are taken to be equal, but for our purposes we allow also different $\xi_a$.
2. In most of the literature the source terms are added before performing the FP procedure. We consider it as more consistent to add them afterwards because the source terms for the ghosts have to be added later, anyway. However, our analysis is independent of the different ways to treat this.
U(1)$_{\text{em}}$, which has to be fixed as well. (12) now takes the form

$$Z = \int D\varphi_i D\psi \delta^3(\varphi_i) \delta(\varphi_i - C(x)) \text{Det} \left( \frac{\delta F_a(x)}{\delta \alpha_b(y)} \right) \exp \left( i \int d^4x \ L_{\text{inv}} \right)$$

(17)

Only the second $\delta$-function is treated in the way explained above, leading to an appropriate $L_{g.f.}$. The other one enables to perform the $D\varphi_i$ integration with the result that all pseudo-Goldstone fields in the invariant Lagrangian and in the ghost term are set equal to zero. Thus, the unphysical pseudo-Goldstone bosons are removed from the theory. The effective Lagrangian becomes

$$L_{\text{eff}} = L_{\text{inv}} \big|_{\varphi_i=0} - \frac{1}{2 \xi_\gamma} (\partial_\mu A^\mu)^2 + L_{\text{FP}} \big|_{\varphi_i=0}. \quad (18)$$

Except for $L_{\text{FP}}$ this is identical to the classical U-gauge Lagrangian with fixed U(1)$_{\text{em}}$.

Let us now derive the ghost term and show that the ghost fields can be removed from the theory, too. Corresponding to the gauge boson mixing we define the parameters

$$\alpha_Z = \cos \theta_W \alpha_3 - \sin \theta_W \beta, \quad \alpha_\gamma = \sin \theta_W \alpha_3 + \cos \theta_W \beta \quad (19)$$

($\theta_W$ being the Weinberg angle defined by $\tan \theta_W = \frac{g'}{g}$). From (6) one finds the infinitesimal transformations of the $F_a$ for the present case:

$$\delta \varphi_1 = \frac{g}{2} (v + h) \delta \alpha_1 + O(\varphi_i),$$
$$\delta \varphi_2 = \frac{g}{2} (v + h) \delta \alpha_2 + O(\varphi_i),$$
$$\delta \varphi_3 = \frac{g}{2 \cos \theta_W} (v + h) \delta \alpha_Z + O(\varphi_i),$$
$$\delta (\partial_\mu A^\mu) = -\Box \delta \alpha_\gamma + O(\varphi_i) + O(W_\mu). \quad (20)$$

First, we see that all terms $O(\varphi_i)$, which are proportional to the pseudo-Goldstone fields $\varphi_i$, yield vanishing contributions to the ghost terms after integrating out the $\delta$-function, as explained above. Secondly, we note that $\eta_\gamma$ (the ghost belonging to the electromagnetic gauge freedom) is a physically inert field: it is not possible to construct a Feynman diagram with internal $\eta_\gamma$-lines, because (20) only yields (besides a kinetic term for $\eta_\gamma$) vertices with outgoing $\eta_\gamma$-lines (and incoming $\eta^\pm$ coupled to $W^\pm$) but no vertices with an incoming $\eta_\gamma$. Thus, the field $\eta_\gamma$ can be integrated out.

After removing all redundant terms, the resulting FP determinant can be expressed as

$$\text{Det} \left( \frac{\delta F_a(x)}{\delta \alpha_b(y)} \right) = \text{Det} \left( \frac{g}{2} (v + h) \text{diag} \left[ 1, 1, \frac{1}{\cos \theta_W} \right] \delta^4(x - y) \right). \quad (21)$$
Since the argument of the determinant is a local function we can express the functional determinant ("Det") in terms of the ordinary one ("det") using the relation

\[ \text{Det} \left( M_{ab}(x) \delta(x - y) \right) = \exp \left( \delta(0) \int dx \ln(\det M_{ab}(x)) \right). \] (22)

We therefore write

\[ \text{Det} \left( \frac{\delta F_a(x)}{\delta \alpha_b(y)} \right) = \exp \left( i \int dx \left( -3i \delta^4(0) \ln \left( 1 + \frac{h}{v} \right) - i \delta^4(0) \ln \frac{v^3}{\cos \theta_W} \right) \right). \] (23)

This means that here (in contrast to usual gauge fixing) the locality of the FP determinant enables exponentiation of this term without introducing unphysical ghost fields. As a result, we find (neglecting a constant and using $M_W = \frac{v g}{2}$) the ghostless FP term to the Lagrangian:

\[ \mathcal{L}_{FP} = -3i \delta^4(0) \ln \left( 1 + \frac{g}{2M_W} h \right). \] (24)

This, together with (18), shows that $\mathcal{L}_{\text{eff}}$ (in the gauge defined by (16)) contains no unphysical fields neither pseudo-Goldstone nor ghost fields. Instead, there is the extra term (24) describing a quartic divergent nonpolynomial Higgs self-interaction.

The extra term (24) can alternatively be derived from the Feynman diagrams obtained by expressing the determinant (21) via (13) in terms of usual ghost fields. We are going now to present this derivation also, since it makes the role of the new interaction term more transparent. In this formalism the ghost term is (introducing $\eta^\pm = \frac{1}{\sqrt{2}} (\eta_1 \mp i \eta_2)$)

\[ \mathcal{L}_{FP} = -M_W \eta^++\eta^+ - M_W \eta^- + \eta^- - M_Z \eta^*_Z \eta_Z^* \]
\[ - \frac{g}{2} \eta^+ h - \frac{g}{2} \eta^- h - \frac{g}{2 \cos \theta_W} \eta^*_Z \eta_Z h. \] (25)

There are no kinetic terms of the ghost fields, but only mass terms and couplings to the Higgs boson. This means that the ghost propagators simply are static ones, i.e. inverse masses. Figure 1 shows the Feynman rules derived from (24). Since the ghost fields only couple to the Higgs boson, they only contribute to Feynman diagrams with internal ghost loops connected to an arbitrary number of Higgs lines (Figure 2), which can be internal or external ones. Using the Feynman rules (Fig. 1) the contribution of such a loop with $N$ ghost propagators that is coupled

\[ \text{If the unbroken subgroup is non-Abelian the ghost fields belonging to this subgroup are still present. These can be removed by choosing the axial gauge } t_\mu A_\mu^a = C_\mu(x) \text{ for the massless gauge bosons instead of the last line of (16).} \]
to $N$ Higgs bosons to the amplitude is (considering a factor $(2\pi)^{-4}$ for the closed loop and one $(-1)$ due to the Fermi statistics of the ghosts)

$$-\int \frac{d^4p}{(2\pi)^4} \left(-\frac{g}{2M_W}\right)^N = -\delta^4(0) \left(-\frac{g}{2M_W}\right)^N$$

(26)

for an internal $\eta^\pm$ as well as for an internal $\eta_Z$. We see that such a ghost loop effectively provides for a quartic divergent $N$-Higgs self-coupling. Let us for a moment go to the one-loop level (where the Higgs lines connected to the ghost loop are “tree lines”) and consider all subdiagrams of type of Fig. 2 with a fixed number $N$ of Higgs lines. The sum of their contributions is

$$-3\delta^4(0)(N-1)! \left(-\frac{g}{2M_W}\right)^N$$

(27)

since there are three types of internal ghosts and (as one can easily verify by induction) $(N-1)!$ different possibilities to connect the $N$ Higgs lines to such a loop. So all the ghost loops with $N$ Higgs lines together can be replaced by an extra $N$-Higgs vertex (Figure [3]) with the quartic divergent vertex factor (27). Considering a combinatorial factor of $1/N!$ due to the $N!$ different possibilities to connect $N$ Higgs lines to the such a vertex, all the extra Higgs vertices (with all possible values of $N$) can be derived from a Lagrangian

$$\mathcal{L}_{extra} = 3i\delta^4(0) \sum_{N=1}^{\infty} \left(\frac{1}{N} \left(-\frac{g}{2M_W}\right)^N h^N\right) = -3i\delta^4(0) \ln \left(1 + \frac{g}{2M_W}h\right),$$

(28)

which is identical to (24).

This one loop derivation can easily be generalized to arbitrary loop order without changing the result: assuming that the Higgs lines in Figs. 2 and 3 are not “tree lines” but they are connected to loops among themselves or to other ghost loops or extra vertices of this type, the only thing in the above discussion that changes is the combinatorics. However, the combinatorial factors of $(N-1)!$ for the $N$-Higgs ghost loop and of $N!$ for the $N$-Higgs vertex change by the same extra factor so that this cancels. One easily can see that, no matter how the Higgs legs are connected, for each way to attach $N$ Higgs lines to a loop like Fig. 2 there are $N$ ways to attach them to a vertex like Fig. 3, corresponding to the $N$ cyclic permutations.

This alternative derivation of the extra term (24) to the Lagrangian (although it is more elaborate) shows explicitly the meaning of (24): $\delta^4(0)$ has to be interpreted as a quartic divergent integral stemming from (26) and can be expressed in terms of a cut-off $\Lambda$ as $\frac{1}{(2\pi)^4}$, the logarithm has to be evaluated in a power series as in (28) and represents a (nonpolynomial) self-interaction of an arbitrary number of Higgs bosons. We see that the unphysical ghost fields can be effectively removed from the theory by taking the U-gauge, but they do not completely
decouple (as the pseudo-Goldstone fields do): there remains the additional interaction term (24) as a remnant. However there are no more explicit ghost fields in this term.

4 \(R_\xi\)-Limiting Procedure

The second approach to the unitary gauge is to start from the SM in the \(R_\xi\)-gauge and then to perform the limit \(\xi \to \infty\). To construct this limit, one has to modify the general \(R_\xi\)-gauge a bit because the photon propagator

\[
\frac{-g^{\mu\nu} + (1 - \xi) p^\mu p^\nu}{p^2}
\]

would become infinite in this limit. We impose the usual gauge fixing conditions \([5, 6, 7, 8]\) but express them in terms of the mass eigenstates \(A^\mu\) and \(Z^\mu\) of the neutral sector instead of \(W_3^\mu\) and \(B^\mu\):
\[ \begin{align*}
\partial_\mu W^\mu_{1,2} - \xi M_W \varphi_{1,2} &= C_{1,2}(x), \\
\partial_\mu Z^\mu - \xi M_Z \varphi_3 &= C_3(x), \\
\partial_\mu A^\mu &= C_4(x). 
\end{align*} \] (30)

In order to obtain the corresponding \( \mathcal{L}_{g.f.} \), we make use of the possibility to introduce different parameters \( \xi_a \) in \( \mathcal{L}_{g.f.} \) for each \( F_a \) (\( a = 1, \ldots, 4 \)) (see footnote 1). The gauge fixing Lagrangian most convenient for our purposes is

\[ \mathcal{L}_{g.f.} = -\frac{1}{2\xi} \left[ \sum_{i=1}^{2} (\partial_\mu W^\mu_i - \xi M_W \varphi_i)^2 + (\partial_\mu Z^\mu - \xi M_Z \varphi_3)^2 \right] - \frac{1}{2\xi_\gamma}(\partial_\mu A^\mu)^2 \] (31)

with two free parameters \( \xi \) and \( \xi_\gamma \). As the ghost term depends only on \( \xi \) since (30) depends only on \( \xi \), the sole difference of our gauge to the usual \( R_\xi \)-gauge is that \( \xi \) in the photon propagator is replaced by \( \xi_\gamma \). Now we can take the limit \( \xi \to \infty \) (which does not affect physical observables since these do not depend on \( \xi \)) while the unbroken subgroup \( U(1)_{\text{em}} \) is fixed in an arbitrary gauge specified by a finite \( \xi_\gamma \) so that the photon propagator (29) remains finite.

A complete list of Feynman rules for the \( R_\xi \)-gauged SM is, e.g., given in [23]. The \( \xi \) dependent parts are:

- The propagators of the massive gauge bosons

\[ -g^{\mu\nu} + (1 - \xi) \frac{p^\mu p^\nu}{p^2 - \xi M_B^2} \] (32)

(\( M_B = M_W \) or \( M_Z \)), which become Proca propagators of massive spin-one particles for infinite \( \xi \).

- The propagators of the pseudo-Goldstone bosons and the ghost fields (except for \( \eta_\gamma \), which is massless)

\[ i \frac{1}{p^2 - \xi M_B^2}. \] (33)

For \( \xi \to \infty \), these particles acquire infinite mass and their propagators vanish. If there would be no \( \xi \) dependent couplings these particles would completely decouple.

- All couplings of a (physical or unphysical) scalar to a ghost pair. These are proportional to \( \xi \) and become infinite in the \( R_\xi \)-limiting procedure.

We now classify all Feynman diagrams containing lines corresponding to unphysical fields that do not vanish for \( \xi \to \infty \). Since the pseudo-Goldstone and the \( \eta^\pm, \eta_Z \) propagators behave as \( \xi^{-1} \) for large \( \xi \) their number has to equal the number of scalar–ghost–ghost vertices (\( \propto \xi \)) in such diagrams. This means:
• All propagators $\propto \xi^{-1}$ have to be coupled to $\xi$-dependent vertices at both ends, i.e., couplings of unphysical fields to gauge fields do not contribute in this limit.

• Only those $\xi$-dependent vertices yield nonvanishing contributions which couple to two $\xi$-dependent and one $\xi$-independent propagator. These are the $h\eta\eta^*$ and the $\varphi^{\pm}\eta^{\pm*}\eta_{\gamma}$ vertices. However, the latter do not contribute, since they exist only with an incoming $\eta_{\gamma}$ but not with an outgoing one, so it is not possible to construct closed ghost loops with them.

As a consequence, all graphs with pseudo-Goldstone lines vanish for $\xi \to \infty$. Therefore, the $\varphi_{\gamma}$-fields can be neglected altogether in this limit. Furthermore the only nonvanishing (sub)diagrams involving unphysical particles are those with ghost loops that are exclusively coupled to Higgs bosons (Fig. 2). The corresponding Feynman rules are given in Fig. 4.

The contribution of such a loop with $N$ external Higgs lines for $\xi \to \infty$ is (for internal $\eta^\pm$ as for $\eta_Z$)

$$
- \lim_{\xi \to \infty} \int \frac{d^4 p}{(2\pi)^4} \left( -\frac{i}{2} \xi g M_W \right)^N \prod_{i=1}^N \frac{i}{p_i^2 - \xi M_W^2} = - \int \frac{d^4 p}{(2\pi)^4} \left( -\frac{g}{2M_W} \right)^N = - \delta^4(0) \left( -\frac{g}{2M_W} \right)^N.
$$

(34)

(The momenta $p_i$ of the internal ghosts do not have to be specified to perform the limit.) This is identical to (26) and we can transfer the discussion of the previous section and find for the limit of the ghost term

$$
\lim_{\xi \to \infty} \mathcal{L}_{FP} = -3i\delta^4(0) \ln \left( 1 + \frac{g}{2M_W} \right),
$$

which is exactly the extra term (24).

Thus in the $\text{R}\xi$-gauge, after taking the limit $\xi \to \infty$, the pseudo-Goldstone fields and even the massless ghost field $\eta_{\gamma}$, which has a $\xi$-independent propagator, decouple completely\footnote{Remember footnote \ref{fn:note3}.}, while the contributions of the massive ghosts can be summarized in terms of the extra interaction Lagrangian (24). Therefore we obtain the same result for the U-gauge Lagrangian as in the previous section\footnote{In fact it is a priori not clear that the limit $\xi \to \infty$ can be performed before the loop integration in (34). The fact that the obtained result is identical to that of the alternative derivations justifies this treatment.}.

5 Decoupling the Unphysical Scalars

In this section we derive the U-gauge Lagrangian by applying appropriate point transformations to the invariant Lagrangian (5). We start with reparametrizing

\footnote{Remember footnote \ref{fn:note3}.}
(point transforming) the scalar sector of the theory \((\Box)\) nonlinearly \([3, 4, 5]\): \[
\Phi = \frac{1}{\sqrt{2}} \left( (v + h) \mathbf{1} + i\tau_i \varphi_i \right) = \frac{1}{\sqrt{2}} (v + \rho) \exp \left( \frac{i\zeta_i \tau_i}{v} \right).
\]

(36)

Here, \(\rho\) is the new Higgs field and the \(\zeta_i\) are the new pseudo-Goldstone fields. We see that in this parametrization the Lagrangian \((\Box)\) contains nonpolynomial interactions of the \(\zeta_i\) to the gauge bosons and to the fermions, which stem from expanding the exponential in the kinetic term of \(\Phi\) and in the Yukawa term. At the quantum level this is not the whole story: it is not the Lagrangian \((\Box)\) but the PI \((\Pi)\) which is the basis of quantization. Therefore we have to transform the integration measure, too, which yields a functional Jacobian determinant \([5, 10]\) according to \[
\mathcal{D}h \mathcal{D}\varphi_i = \mathcal{D}\rho \mathcal{D}\zeta_i \ \text{Det} \left[ \frac{\delta(h, \varphi_i)}{\delta(\rho, \zeta_i)} \right].
\]

(37)

The explicit form of the point transformation \((\Box)\) is (introducing \(\zeta = \sqrt{\zeta_1^2 + \zeta_2^2 + \zeta_3^2}, \ \tilde{\zeta}_i = \zeta_i / \zeta\) and \(\tilde{\zeta} = \zeta / v\))
\[
h = (v + \rho) \cos \tilde{\zeta} - v,
\]
\[
\varphi_i = (v + \rho) \tilde{\zeta}_i \sin \tilde{\zeta}.
\]

(38)

Since the Jacobian matrix is a local function, we can again use \((22)\) to express the functional determinant in terms of the ordinary one, which is given by \[
\det \frac{\partial(h, \varphi_i)}{\partial(\rho, \zeta_i)} = (v + \rho) \frac{3 \sin^2 \tilde{\zeta}}{v \zeta^2}.
\]

(39)

Exponentiating this, using \((22)\), we obtain the following extra terms to the Lagrangian due to the change of the functional measure (after dropping a constant)
\[
\mathcal{L}' = -3i\delta^4(0) \ln \left( 1 + \frac{g}{2M_W} \rho \right) - i\delta^4(0) \ln \left( \frac{\sin^2 \tilde{\zeta}}{\tilde{\zeta}^2} \right).
\]

(40)

The first term is just our well known quartic divergent nonpolynomial Higgs self-coupling term \((24)\), but there is also a quartic divergent nonpolynomial self-interaction term of the nonlinearly realized pseudo-Goldstone fields. The latter is not really important for practical purposes since the \(\zeta_i\) become decoupled in the next step to be performed below. (It has to be considered, however, if one actually wants to perform perturbative calculations with this parametrization of the scalars.)

Introducing the field combination
\[
U \equiv \exp \left( \frac{i\zeta_i \tau_i}{v} \right),
\]

(41)
one can deduce from (6) the behaviour of \( \rho \) and \( \zeta_i \) under SU(2) \( \times \) U(1) gauge transformations:

\[
\rho \rightarrow \rho, \\
U \rightarrow S(x)U \exp(-i g' \beta(x) \tau_3), \quad (42)
\]
i.e., the physical scalar \( \rho \) is a singlet. Consequently, the first term in (40) is gauge invariant while the second is not. This is, however, no serious problem since gauge invariance is not really destroyed, it is just not completely obvious due to the nonlinear parametrization\(^6\).

In order now to remove the unphysical scalars \( \zeta_i \) from the theory and thus obtaining the U-gauge Lagrangian, one can proceed in several ways. One possibility is to apply either methods of the previous sections. We do not perform this in detail but only mention the main features. If one imposes the gauge-fixing condition (16) (U-gauge) or (30) (R\(\xi\)-gauge) with \( \varphi_i \) replaced by \( \zeta_i \) one finds the analogous expressions for the ghost terms, except that there are additional interactions of more than one pseudo-Goldstone boson with the ghost fields (which do not affect the discussion) and that there are no couplings of the Higgs boson to the ghosts because the transformations of the \( \zeta_i \) (42) do not depend on \( \rho \). So no ghost loops as in Figure 2 can be constructed. Therefore, remembering our previous discussion, we see that the ghost terms vanish completely after integrating \( \int D\zeta_i \delta^3(\zeta_i) \) in the first case and after taking the limit \( \xi \rightarrow \infty \) in the second case, respectively.

Here, we choose one further possibility and apply one further point transformation, which affects the gauge and the fermion fields. It is just a reversed (non-Abelian) Stückelberg transformation [3, 13, 15, 17]:

\[
w_\mu = U^\dagger W_\mu U - \frac{i}{g} U^\dagger \partial_\mu U, \\
b_\mu = B_\mu, \\
\psi_L = U^\dagger \Psi_L, \\
\psi_R = \Psi_R. \quad (43)
\]
The Jacobian of this transformation is independent of the physical fields, since (43) is linear in them. Using (22) it yields the (non-gauge-invariant) extra term

\[
\mathcal{L}'' = -4i \delta^4(0) \ln(\sin^6 \tilde{\zeta} + \cos^6 \tilde{\zeta}). \quad (44)
\]

We now show that the unphysical fields \( \zeta_i \) decouple from the physical fields \( w_\mu, b_\mu, \rho \) and \( \psi \). The Lagrangian (3) originally contains couplings of pseudo-Goldstone bosons to the gauge bosons (in the kinetic term of \( \Phi \)) and to the

\(^6\)This can easily be visualized with the help of an example from quantum mechanics: if one studies a translational invariant Lagrangian and transforms the functional integration measure to polar coordinates one finds a non-translational-invariant extra term to the Lagrangian, although physics is still translational invariant.
express these terms by means of the new fields (36), (43) we find

$$\frac{1}{2} \text{tr} \left[ (D^\mu \Phi)^\dagger (D_\mu \Phi) \right] - \frac{\sqrt{2}}{v} (\bar{\Psi}_L \Phi M_f \Psi_R + \bar{\Psi}_R M_f \Phi^\dagger \Psi_L)$$

$$= \frac{1}{2} (\partial^\mu \rho) (\partial_\mu \rho) + \frac{1}{4} (v + \rho)^2 \text{tr} \left[ \left( gw_\mu - \frac{1}{2} g' b_\mu \tau_3 \right) \left( gw^\mu - \frac{1}{2} g' b^\mu \tau_3 \right) \right]$$

$$- \frac{v + \rho}{v} (\bar{\psi}_L M_f \psi_R + \bar{\psi}_R M_f \psi_L), \quad (45)$$

We see that the pseudo-Goldstone fields $\zeta_i$ have effectively disappeared here. They only emerge in the extra terms (40) and (44), i.e. without being coupled to any other fields. We therefore can integrate them out in the PI which yields a constant factor

$$\int D \zeta \exp \left( i \int d^4 x \tilde{L} \right) \quad (46)$$

with

$$\tilde{L} = -i \delta^4(0) \left( \ln \left( \frac{\sin^2 \tilde{\zeta}}{\tilde{\zeta}^2} \right) + 4 \ln (\sin^6 \tilde{\zeta} + \cos^6 \tilde{\zeta}) \right). \quad (47)$$

This can be removed by multiplying the PI with the compensating factor.

The effect of the point transformations (43) on the other terms in (5) is just to replace the fields $(W_i, B_\mu, \Psi)$ by $(w_i, b_\mu, \psi)$ due to the gauge invariance of these terms since (43) formally acts on the physical fields as a SU(2) gauge transformation.

In summary we see that the unphysical fields $\zeta_i$ have decoupled and we obtain the same result for the U-gauge Lagrangian as in the previous two sections; in particular the extra term (24) is again recovered.

Next we study the behaviour of the new fields under gauge transformations. From (3), (42), and (43) we find that all physical fields are invariant under the action of SU(2) and transform under U(1) as

$$w_\mu \rightarrow \exp \left( i g' \beta(x) \tau_3 \right) w_\mu \exp \left( -i g' \beta(x) \tau_3 \right) - \frac{1}{2} g' \partial_\mu \beta(x) \tau_3,$$

$$b_\mu \rightarrow b_\mu - \partial_\mu \beta(x),$$

$$\rho \rightarrow \rho,$$

$$\psi_{L,R} \rightarrow \exp \left( i \frac{g'}{2} \left( \tau_3 + B - L \right) \beta(x) \right) \psi_{L,R}. \quad (48)$$

Introducing the mass and charge eigenstates (which are now also assigned by small letters) $w_\mu \pm, z_\mu$ and $a_\mu$ and rescaling the gauge parameter as

$$g' \beta(x) = e \kappa(x) \quad (49)$$
one finds
\[ w_\mu^\pm \to \exp(\pm ie\kappa(x))w_\mu^\pm, \]
\[ z_\mu \to z_\mu, \]
\[ a_\mu \to a_\mu - \partial_\mu\kappa(x), \]
\[ \rho \to \rho, \]
\[ \psi \to \exp(ieQ_f\kappa(x))\psi, \]
(50)

where \( Q_f \) is the fermion charge matrix
\[ Q_f = \begin{pmatrix} q_u & 0 \\ 0 & q_d \end{pmatrix}. \]
(51)

This is just an electromagnetic gauge transformation. Thus, after having parametrized the fields such that all pseudo-Goldstone bosons decouple, the action of the whole gauge group on the physical fields reduces to a gauge transformation belonging to the unbroken subgroup. The remaining gauge freedom is only connected to the unphysical scalars and has been “removed” by dropping (46). Finally, the U(1)\textsubscript{em} gauge freedom has to be fixed by adding the term
\[ \mathcal{L}_{g.f.} = -\frac{1}{2\xi^\gamma}\partial_\mu a^\mu, \]
(52)

while the ghost belonging to it decouples, and we finally find the same effective Lagrangian as in the previous two sections.

This derivation of the U-gauge is similar to the classical treatment of removing the pseudo-Goldstone scalars by a means of a gauge transformation. In fact the Stückelberg transformation (43) formally acts as a SU(2) gauge transformation on the gauge and fermion fields where the gauge parameters \( \alpha_i \) are replaced by the pseudo-Goldstone fields \( -\zeta_i/M_W \). But there remains a principal difference between that gauge transformation and a Stückelberg transformation. Namely, the gauge transformation which transforms \( \zeta_i \) identically to zero depends on the numerical values of these fields at the various space-time points and it is not the same transformation for different functions \( \zeta_i \). The Stückelberg transformation (43), however, decouples the pseudo-Goldstone fields (independently of their functional forms) from the physical fields and so it can be applied to the PI, where an integration over all functions \( \zeta_i \) is performed. The only point where quantum physics enters the above discussion is the need of point transforming the function parameters.

\(^7\) On the first look this seems surprising since we performed a U(1)\textsubscript{Y} transformation to derive (54). However, due to the point transformation (43), this acts differently on the transformed fields than on the original ones; so it becomes a U(1)\textsubscript{em} transformation.

\(^8\) Remember footnote 3.

\(^9\) If one considers the source terms from the beginning (see footnote 3), the external sources become, after performing the point transformations, coupled to functions of the fields instead of the fields themselves. However this does not affect physical matrix elements.
The corresponding Jacobian determinants give rise to the extra term (24).

6 Construction of a Spontaneously Broken Gauge Theory Using the Stückelberg Formalism

The formalism of the last section also can be reversed: one can start from the U-gauge Lagrangian and construct the invariant Lagrangian by subsequent point transformations. Although it is clear from the previous section what do to, we explain this procedure a bit more in detail, since it is an alternative derivation of a SBGT, which illustrates some features of such a model quite well. Furthermore, the U-gauge Lagrangian of a SBGT can be motivated on a rather intuitive basis: it only involves “physical” fields (i.e. all the fields corresponding to observable particles) and it is the most general (effective) Lagrangian with massive vector bosons and scalars which guarantees tree-unitarity (i.e. $N$-particle $S$-matrix elements calculated on tree level decrease at least as $E^{4-N}$ for high energy $E$) \cite{12,13,18}. Therefore, it seems interesting to look whether and how the general (gauge invariant) structure of a SBGT can be reconstructed from its U-gauge Lagrangian. Tree unitarity implies, in particular, the need of physical scalars (Higgs bosons) with appropriate couplings to the other particles and to itself in order to ensure good high energy behaviour. It is clear that the extra term (24), which genuinely reflects quantum effects, is not obtained from tree level arguments, but it is inferred from the requirement of vanishing quartic divergent $N$-Higgs self-couplings (which has been shown for $N = 3, 4$ on 1-loop level in \cite{12}). In the following we take the term (24) as a given part of the U-gauge Lagrangian.

Starting from this full U-gauge Lagrangian (containing the fields $w_{\mu}, b_{\mu}, \rho$ and $\psi$ as in the previous section) one recognizes first the local $U(1)_{em}$ symmetry (50). Reversing the FP procedure for the unbroken subgroup by reintroducing the integration parameter as an (infinite) constant to the PI, one can remove the g. f. term (52) (and the ghost term if the unbroken subgroup is non-Abelian). Next, one can see that the Lagrangian is gauge invariant not only under $U(1)_{em}$ but also under the larger group $SU(2) \times U(1)$ except for the mass terms of the vector bosons and the fermions and the couplings of the physical scalar to these particles; the kinetic terms, the vector-boson–fermion interaction and vector-boson self-interaction terms represent exactly an unbroken gauge theory. The purpose of the Stückelberg formalism is now to introduce, by field enlarging transformations, unphysical degrees of freedom with an appropriate behaviour under $SU(2) \times U(1)$ gauge transformations that compensate the effect of the transformations of the physical fields in the non-gauge-invariant terms in the
U-gauge Lagrangian \[13, 14, 15, 16, 17\].

In the PI formalism, the field enlarging transformation is constructed as follows: First one introduces the completely decoupled fields \(\zeta_i\) by formally multiplying the (infinite) constant \((46)\) to the PI. This contains the functional integration over the unphysical fields \(\zeta_i\) (Stückelberg scalars) and an exponential, which is needed to remove the Jacobian determinant of the following point transformations. Then the \(\zeta_i\), parametrized in terms of the unitary matrix \(U\) \((41)\), are coupled to the physical fields by the transformations \((43)\). Defining the behaviour of \(U\) under SU(2) \(\times\) U(1) as in \((42)\), one finds that the transformations of \(W_{\mu}, B_{\mu}\) and \(\Psi\), which arise from the original U(1)_{em} gauge freedom of the \(w_{\mu}, b_{\mu}\) and \(\psi\) \((50)\) with \((49)\) and from the transformations of \(U\), are exactly the usual gauge transformations \((6)\). The Lagrangian is (except for the extra term contained in \((46)\)) gauge invariant under these transformations, because the fields appear only in the combinations \((43)\). The effect of an arbitrary gauge transformation \((41), (42)\) on these is just an electromagnetic one \((49)\). Since the Stückelberg transformation \((43)\) has the form of a SU(2) gauge transformation, its effect on the kinetic terms, vector-boson self-interaction term and vector-boson–fermion interaction terms is just to replace \(W_{\mu} \rightarrow w_{\mu}\), etc. (because these terms are already gauge invariant), while the mass and Higgs coupling terms give rise to a kinetic term for the \(\zeta_i\) (stemming from the gauge boson mass term) and to nonpolynomial interactions of the Stückelberg scalars to the physical particles. Thereby these terms have become replaced by gauge invariant expressions.

Two important points should be mentioned here.

- Using the Stückelberg formalism one can construct gauge theories with massive vector bosons. Such theories are usually described by using the formalism of spontaneous symmetry breaking (SSB). Note that the Stückelberg formalism does not avoid SSB, since the unitary matrix \(U\) has nonvanishing VEV,
  \[UU^\dagger = 1 \quad \Rightarrow \quad \langle U \rangle_0 = 1.\] \((53)\)
  But here, the nonvanishing VEV is not realized by introducing a scalar-self-interaction potential with a nontrivial minimum but by imposing \((53)\) as a constraint, as in the gauged nonlinear \(\sigma\)-model \([24, 25, 26]\).

- The starting U-gauge Lagrangian seems to be nonrenormalizable by simple power counting arguments (although we know it is renormalizable) due to the bad high energy behaviour of the massive-vector-boson propagators. The Stückelberg formalism introduces gauge freedom and enables

\[10^{1}

In \((16)\) one can find an alternative construction of the SM using the Stückelberg formalism. There the point transformations do not look like SU(2) transformations, as in our case, but like full SU(2) \(\times\) U(1) transformations with gauge parameters replaced by unphysical scalars, thus introducing four unphysical fields. However, there exists a reparametrization of these scalars (i.e., one more point transformation) that decouples one of these fields, so that one finally obtains the same result as we do.
so to perform calculations in the $R_\xi$-gauge where the vector boson propagator behaves well. However this formalism introduces nonpolynomial interactions of the St"uckelberg scalars $\zeta_i$. I. e., new interactions, arising from the expansion of $U$ (11) in powers of $\zeta_i$, have to be considered at each loop order, which again makes the model nonrenormalizable by naive power counting. So the problem of (possible) bad behaviour of loop corrections is only shifted from the physical to the unphysical sector of the theory.

In the original St"uckelberg formalism this is the end of the story, since point transformations like (43) are not applied to the U-gauge of a SBGT but to a Yang–Mills theory with mass terms added by hand. The St"uckelberg formalism transforms such a model to a gauged nonlinear $\sigma$-model, which is in fact nonrenormalizable [25, 26, 27], since it is not possible to reparametrize the unphysical scalars (11) in a linear form to avoid nonpolynomial interactions [28]. However, in our case the starting Lagrangian contains an extra Higgs boson (or several Higgs bosons in the case of more extended theories). It has been introduced to ensure good high energy behaviour on tree level and, in fact, it makes the model renormalizable, too, since physical and unphysical scalars together can be rewritten as a linear expression by means of the point transformation (36). So in the linear parametrization ((2) with (9)) the nonpolynomial interactions have been removed. It has been shown that for each tree unitary theory such a transformation exists [13]. The Jacobian determinant of (36) removes the extra term (24).

So finally we have “recovered” the SM from its unitary gauge by applying appropriate point transformations to the physical fields. On the level of the classical Lagrangian, this has originally been done in [13]. In a quantum field theoretical treatment performed here one has to consider the integration measure of the PI, too, so that two new features arise:

- Field enlarging is easily achieved by multiplying an infinite constant (as (46)) to the PI.
- The starting Lagrangian has necessarily to contain the extra term (24) to cancel against the Jacobian determinant which arises as a consequence of the transformation of the functional integration measure.

As the result of this section we see, that a Higgs model can be derived from the requirement of good high energy behaviour of tree-level amplitudes and of loops without explicitly using the Higgs mechanism (although SSB is implicitly included in the derivation). Thus, the physical sector of a SBGT “contains” the entire model. Non-Abelian St"uckelberg models are renormalizable if, in addition to the unphysical St"uckelberg scalars, also physical scalars with appropriate couplings to the other particles and themselves are present.
7 Effective-Lagrangian Theories

As mentioned before, the original purpose of the Stückelberg formalism was to construct (gauge) theories with massive gauge bosons but without physical scalars [14, 15, 16, 17, 28]. To construct a (non-Abelian) Stückelberg model one starts from a Yang-Mills theory, introduces a mass term by hand, thereby breaking gauge invariance explicitly, and then performs a field enlarging Stückelberg transformation as (43) (after introducing the functional integration over the unphysical Stückelberg scalars into the PI) to restore gauge invariance. Due to the Yang–Mills symmetry of the Lagrangian, except for the mass terms, the Stückelberg transformation, which formally looks like a gauge transformation with the gauge parameters being replaced by the unphysical scalar fields, affects only the mass terms in a nontrivial way thereby yielding, besides a kinetic term for the Stückelberg scalars, nonpolynomial interactions of the unphysical with the physical fields. In the Yang–Mills part of the Lagrangian the original physical fields simply are replaced by the transformed fields. Thus, the addition of unphysical scalars with suitable nonpolynomial couplings (in the non-Abelian case) to the physical fields embeds the original massive Yang–Mills theory into an equivalent gauge theory. The resulting model is a gauged nonlinear σ-model. As it is well known, this model can be derived in three alternative ways (described here for the case of the SU(2) × U(1) symmetry):

- By applying a Stückelberg transformation (43) to the massive Yang–Mills theory [14] as explained above.
- By imposing SSB in form of a constraint (53) on the scalar fields, which implies the parametrization (41), and gauging the broken symmetry (42) by coupling $\mathcal{U}$ minimally to the gauge fields [24].
- The gauged nonlinear σ-model also turns out to be the limit of the SM for infinite Higgs mass [25, 26]. To perform this limit, one has to substitute in the SM Lagrangian (5)

$$\Phi \rightarrow v \sqrt{2} \mathcal{U}$$

(with $\Phi$ and $\mathcal{U}$ given by (2), (9) and (41)), which automatically removes the scalar self-interaction term

$$-V(\Phi) = -\frac{1}{2} \mu^2 \text{tr}(\Phi \Phi^\dagger) - \frac{1}{4} \lambda \text{tr}(\Phi \Phi^\dagger)^2$$

and imposes the SSB (53).

Although the gauged nonlinear σ-model is nonrenormalizable due to the nonpolynomial interactions, gauge freedom enables perturbative calculations in the $R_\xi$-gauge and so one finds, that the loops do not diverge as severe as one would
expect from naive power counting. In fact, the one-loop divergences of the gauged nonlinear $\sigma$-model are only logarithmically cut-off dependent \cite{23, 25}.

During the last years electroweak effective-Lagrangian theories with massive vector bosons and non-Yang–Mills interactions have been studied as alternative models to the SM (for examples see \cite{29} and references in \cite{20}). These are also not gauge invariant, because the mass terms and the anomalous interactions violate gauge invariance. However, recently it has been found that each effective-Lagrangian theory is equivalent to a gauge theory \cite{19}, i.e., it can be written as a gauge theory by performing a field enlarging point transformation. For the case of theories containing electroweak vector bosons with arbitrary self-interactions, this was investigated in \cite{20}. The transformation which was used for reformulating such a theory as a SBGT can be identified with the Stückelberg transformation (13), as one can see by the same reasoning as above. Thereby, the spontaneous symmetry breaking is realized nonlinearly. Again, the effect of the non-gauge-invariant terms is cancelled by appropriate couplings of unphysical scalars but, in difference to a simple massive Yang–Mills theory, not only the mass terms but also the anomalous interaction terms give rise to new nonpolynomial interactions. So one can perform loop calculations within models containing arbitrary vector-boson self-interactions in the $R_\xi$-gauge and has better opportunities to subdue the divergences in such a nonrenormalizable model. The U-gauge of this SBGT becomes the original effective Lagrangian.

A method of deriving anomalous self-couplings of electroweak vector bosons from a gauge invariant Lagrangian with linearly realized symmetry (Higgs model) is to add to the the SM Lagrangian extra dimension six (or higher) SU(2) $\times$ U(1) invariant interaction terms, which contain the anomalous couplings \cite{31, 32, 33, 34, 36}. Although there are no nonpolynomial interactions, these models are nonrenormalizable, too, due to the higher dimension of the extra interaction terms. But it has recently been shown (for special cases with extra dimension six terms) that loop corrections depend (after renormalization) only logarithmically on the cut off due to the linear realized gauge invariance \cite{32, 33, 36}. Although these terms lead to observable deviations from the SM predictions (stemming from loop contributions to presently measurable quantities), the empirical limits on such deviations do not severely restrict the size of the anomalous couplings deriving from the extra terms because of the smallness of the loop contribution due to the weak cut-off dependence \cite{32, 33, 34}. In fact, most of the additional terms contain extra interactions of the Higgs boson to the gauge bosons and the Higgs boson contribution cancels the quadratic loop divergences \cite{33}. So we suppose that the Stückelberg models \cite{20} will yield larger loop corrections and thus be suppressed more strongly by present experiments since there is no Higgs scalar.

However, the two applied methods to get anomalous interactions out of gauge invariant models are in principle different. In \cite{30, 31, 32, 33, 34, 35, 36} terms are added to the SM Lagrangian, most of them contain not only vector-boson
self-interaction terms but also couplings to the physical Higgs bosons. So the original effective Lagrangian is extended to a gauge invariant model which is not equivalent to the original one, since there are additional Higgs couplings. In contrary, the generalized Stückelberg formalism \[20\], i.e., the introduction of nonlinearly realized symmetry enables to express an effective Lagrangian in terms of an equivalent gauge theory. From the above discussion it is clear how these two formalisms are connected. Given an effective Lagrangian for electroweak theory with arbitrary vector-boson self-interactions, one first performs the Stückelberg transformation \[(43)\] and finds the equivalent generalized gauged nonlinear $\sigma$-model. This can, like the usual gauged nonlinear $\sigma$-model, be understood as the limit $M_H \to \infty$ of a Higgs model with linearly realized symmetry \[23, 26\]. To recover the Higgs model one has to replace, reversing \[(54)\],
\[
U \to \sqrt{\frac{2}{v}} \Phi \tag{56}
\]
(with \[(2), (9)\]) and to add the Higgs potential \[(55)\], which implies a nonvanishing VEV and replaces the constraint \[(53)\]. As in the previous section the addition of a physical Higgs boson enables a linear parametrization of the scalar sector and removes the nonpolynomial interactions. By this simple formalism one can extend each effective-Lagrangian electroweak theory with arbitrary self-interactions to an SU(2) × U(1) invariant theory with linearly realized symmetry, which is expected to yield a more decent loop behaviour. This has explicitly been constructed for the special case of arbitrary cubic self-interactions in \[35\]. Here, we have given the general formalism to understand and perform this embedding for all types of vector boson self-interactions\[12\]. Such a model will in general contain extra interaction terms of even higher dimension than six \[35\], which are, however, supposed to yield higher loop divergences and so to be more suppressed by their indirect effects on present experiments than the dimension six terms.

8 Summary

In this paper we have reinvestigated the three different methods to reduce a SBGT to its physical sector (U-gauge Lagrangian) within the formalism of Lagrangian PI. In difference to the naive classical U-gauge, the quantum U-gauge Lagrangian contains an extra nonpolynomial quartic divergent self-interaction

\[\text{It should be clear that } (56) \text{ cannot be understood as a field enlarging point transformation like } (43), \text{ since a hermitian matrix } \Phi \text{ cannot in general not be expressed in terms of a unitary matrix } U. \text{ This shows that the step from the generalized gauged nonlinear } \sigma\text{-model to the generalized Higgs model is indeed an extension of the theory.}\]

\[\text{The investigation of this section concerning arbitrary vector-boson self-interactions also can be applied to get arbitrary fermionic interactions out of a (linearly or nonlinearly realized) SBGT. However, we do not stress this point here because this is of less phenomenological interest since the SM fermionic interactions are very well confirmed in experiments now.}\]
term of the Higgs boson(s) \((24)\), which is necessary to make the theory manifestly renormalizable even in the U-gauge. This term can be interpreted as a remnant of the ghost term if the U-gauge is constructed by gauge fixing or by \(R_\xi\)-limiting procedure or it derives from the functional Jacobian determinant if the U-gauge is constructed by Stückelberg transformations. Fortunately, for the phenomenologically most interesting processes the extra term \((24)\) is irrelevant in one-loop calculations.

On the basis of this analysis we have shown how to construct SBGTs from effective-Lagrangian models such that the original Lagrangian turns out to be the U-gauge. In fact, each effective-Lagrangian theory is equivalent to a SBGT with \textit{nonlinearly} realized symmetry and each effective-Lagrangian theory, which is tree unitary, thus containing (a) physical Higgs boson(s), and in which all quartic divergent loop implied Higgs self couplings are removed by extra counterterms, is equivalent to a SBGT with \textit{linearly} realized symmetry broken by the Higgs mechanism. Therefore, gauge freedom is nothing special for massive-vector-boson theories. This has important consequences for the present phenomenoloical discussion.

Within this context the old-fashioned Stückelberg formalism has acquired new importance since it can be implemented into the modern formalism of gauge theories: on the one hand it can be used to construct effective gauge theories with \textit{nonlinearly} realized symmetry, containing no physical Higgs bosons, which are extentions of the gauged nonlinear \(\sigma\)-model with anomalous gauge-boson self-intercations; on the other hand, the existence of physical Higgs bosons with appropriate coupling structure implied by the demand of tree unitarity (and of a suitable quartic divergent nonpolynomial Higgs self-interaction term) enables to find a linear representation of the scalar fields implying a renormalizable Stückelberg model. Introducing Higgs bosons by hand, the Stückelberg formalism can even be used to derive arbitrary vector-boson self-interactions from Higgs models with linearly realized symmetry.

For treating the point transformations of the fields in the present paper we always had to consider the Jacobian determinants of the functional integration measure. A more natural way would be to treat this subject in Hamiltonian instead of Lagrangian PI formalism since point tranformations are canonical transformations in phase space that do not change the functional integration measure if integration is performed over fields and conjugate fields. This will be the subject of a forthcoming paper.

**Acknowledgement**

One of us (C. G.-K.) thanks D. Schildknecht for arising his interest in the unitary gauge and the Stückelberg formalism. We thank J. Sladkowski and G. J. Gounaris for helpful discussions.
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Figure Captions

**Figure 1:** Feynman rules obtained from the ghost term (25) in the U-gauge. In all figures the solid lines represent the Higgs lines and the dotted ones the ghost lines.

**Figure 2:** Ghost loop connected to $N$ Higgs lines contributing to the Feynman diagrams in the U-gauge. The internal ghosts may be $\eta^\pm$ or $\eta_Z$.

**Figure 3:** Extra quartic divergent $N$-Higgs boson vertex.

**Figure 4:** Feynman rules for Fig. 2 in the $R_\xi$-gauge.