A note on the integrable discretization of the nonlinear Schrödinger equation

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Abstract. We revisit integrable discretizations for the nonlinear Schrödinger equation due to Ablowitz and Ladik. We demonstrate how their main drawback, the non-locality, can be overcome. Namely, we factorize the non-local difference scheme into the product of local ones. This must improve the performance of the scheme in the numerical computations dramatically. Using the equivalence of the Ablowitz–Ladik and the relativistic Toda hierarchies, we find the interpolating Hamiltonians for the local schemes and show how to solve them in terms of matrix factorizations.
1 Introduction

In [1]–[4] the author pushed forward a new method of finding integrable
discretizations for integrable differential equations, based on the notion of $r$–matrix
hierarchies and the related mathematical apparatus. The main idea of
this approach is to seek for integrable discretizations in the same hierarchies
where their continuous counterparts live.

In fact, this method is not quite new. The first integrable discretizations
which can be treated as an application of this method go back as far as to
the year 1976, to the work of Ablowitz and Ladik [6]. In the present note
we revisit the Ablowitz–Ladik discretizations, improving them both from the
aesthetical (theoretical) and the practical (computational) point of view.

In [5] Ablowitz and Ladik proposed the following remarkable system of
ordinary differential equations:

$$
\begin{align*}
\dot{q}_k &= q_{k+1} - 2q_k + q_{k-1} - q_k(r_{k+1} + r_{k-1}) \\
\dot{r}_k &= -r_{k+1} + 2r_k - r_{k-1} + q_k r_k (r_{k+1} + r_{k-1}).
\end{align*}
$$

(1.1)

It may be considered either on an infinite lattice ($k \in \mathbb{Z}$) under the boundary
conditions of a rapid decay ($|q_k|, |r_k| \to 0$ as $k \to \pm\infty$), or on a finite lattice
($1 \leq k \leq N$) under the periodic boundary conditions ($q_0 \equiv q_N$, $r_0 \equiv r_N$, $q_{N+1} \equiv q_1$, $r_{N+1} \equiv r_1$). In any case we shall denote by $q$ ($r$) the (infinite- or
finite-dimensional) vector with the components $q_k$ (resp. $r_k$).

In [3] the system (1.1) appeared as a space discretization of the following
system of partial differential equations:

$$
\begin{align*}
q_t &= q_{xx} - 2q^2 r, \\
r_t &= -r_{xx} + 2qr^2.
\end{align*}
$$

(1.2)

(to perform the corresponding continuous limit, one has first to rescale in
(1.1) $t \mapsto \epsilon^{-2} t$, $q_k \mapsto \epsilon q_k$, $r_k \mapsto \epsilon r_k$, and then to send
$\epsilon \to 0$).

It is important to notice that upon the change of the independent variable
t $\mapsto it$, $i = \sqrt{-1}$, the system (1.2) allows a reduction

$$
r = \pm q^*,
$$

(1.3)

leading to the nonlinear Schrödinger equation

$$
- iq_t = q_{xx} + 2|q|^2 q.
$$

(1.4)
(In (1.3) and below we use the asterisque $\ast$ to denote the complex conjugation). The same reduction is admissible also by the Ablowitz–Ladik system (1.1), leading to

$$-i\dot{q}_k = q_{k+1} - 2q_k + q_{k-1} \mp |q_k|^2(q_{k+1} + q_{k-1}). \quad (1.5)$$

Ablowitz and Ladik found also a commutation representation for the system (1.1) – a semi-discrete version of the zero–curvature representation:

$$\dot{L}_k = M_{k+1}L_k - L_kM_k \quad (1.6)$$

with $2 \times 2$ matrices $L_k, M_k$ depending on the variables $q, r$ and on the additional (spectral) parameter $\lambda$:

$$L_k = L_k(q, r) = \begin{pmatrix} \lambda & q_k \\ r_k & \lambda^{-1} \end{pmatrix}, \quad (1.7)$$

$$M_k = M_k(q, r) = \begin{pmatrix} \lambda^2 - 1 - q_kr_{k-1} & \lambda q_k - \lambda^{-1}q_{k-1} \\ \lambda r_{k-1} - \lambda^{-1}r_k & -\lambda^{-2} + 1 + q_{k-1}r_k \end{pmatrix}. \quad (1.8)$$

Note that the linear problem associated with the matrix $L_k$,

$$\Psi_{k+1} = L_k\Psi_k, \quad (1.9)$$

is a discretization of the linear Zakharov–Shabat problem, associated with the system (1.2),

$$\Psi_x = \begin{pmatrix} i\zeta & q \\ r & -i\zeta \end{pmatrix} \Psi. \quad (1.10)$$

In (6) Ablowitz and Ladik made also the next step in discretizing the system (1.2): they constructed a family of time discretizations of the system (1.1). Although it was not stressed very explicitly, this time their approach to discretization was fundamentally different: they did not modify the linear problem (1.9) any more, restricting themselves with a choice of a suitable (discrete)–time evolution of the wave function $\Psi_k$. Hence the basic feature of the time–discretizations in (6) is following: they admit a discrete analog of the zero–curvature representation,

$$\tilde{L}_kV_k = V_{k+1}L_k \quad (1.11)$$
with the same matrix $L_k$ as the underlying continuous time system. (In (1.11) and below we use the tilde to denote the $h$–shift in the discrete time $hZ$). In a more modern language, the maps generated by the discretizations in [6] belong to the same integrable hierarchy as the continuous time system (1.1).

The results of [6] may be formulated as follows.

**Proposition 0.** Let the matrix $L_k$ be given by (1.7), and let the entries of the matrix

$$
V_k = \begin{pmatrix}
A_k & B_k \\
C_k & D_k \\
\end{pmatrix}
\tag{1.12}
$$

have the following $\lambda$–dependence:

$$
A_k = 1 + h\lambda^2 A_k^{(2)} + hA_k^{(0)} + h\lambda^{-2} A_k^{(-2)},
$$

$$
D_k = 1 + h\lambda^2 D_k^{(2)} + hD_k^{(0)} + h\lambda^{-2} D_k^{(-2)},
$$

$$
B_k = h\lambda B_k^{(1)} + h\lambda^{-1} B_k^{(-1)},
$$

$$
C_k = h\lambda C_k^{(1)} + h\lambda^{-1} C_k^{(-1)}.
$$

Then the discrete zero–curvature equation (1.11) implies the following expressions:

$$
A_k = 1 - h\alpha_0 + h\alpha_+(\lambda^2 - A_k) + h(\alpha_-\lambda^{-2} - \delta_-\bar{q}_k r_{k-1})\Lambda_k, \tag{1.13}
$$

$$
D_k = 1 + h\delta_0 - h\delta_+(\lambda^2 - D_k) - h(\delta_-\lambda^2 - \alpha_- q_{k-1}\bar{r}_k)\Lambda_k, \tag{1.14}
$$

$$
B_k = h(\alpha_+\lambda q_k - \delta_+\lambda^{-1}\bar{q}_{k-1}) + h(\delta_-\lambda\bar{q}_k - \alpha_-\lambda^{-1} q_{k-1})\Lambda_k, \tag{1.15}
$$

$$
C_k = h(\alpha_+\lambda\bar{r}_{k-1} - \delta_+\lambda^{-1} r_k) + h(\delta_-\lambda r_{k-1} - \alpha_-\lambda^{-1}\bar{r}_k)\Lambda_k, \tag{1.16}
$$

Together with the equations of motion:

$$
(\bar{q}_k - q_k)/h = \alpha_+ q_{k+1} + \alpha_0 q_k - \delta_0\bar{q}_k + \delta_+\bar{q}_{k-1} - (\alpha_+ q_k A_{k+1} + \delta_+\bar{q}_k D_k) + (\delta_-\bar{q}_{k+1} + \alpha_- q_{k-1})(1 - \bar{q}_k r_k)\Lambda_k \tag{1.17}
$$

$$
(\bar{r}_k - r_k)/h = -\delta_+ r_{k+1} + \delta_0 r_k + \alpha_0\bar{r}_k - \alpha_+\bar{r}_{k-1} + (\delta_+ r_k D_{k+1} + \alpha_+\bar{r}_k A_k) - (\alpha_-\bar{r}_{k+1} + \delta_- r_{k-1})(1 - \bar{q}_k r_k)\Lambda_k
$$

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Here $\alpha_0$, $\alpha_+$, $\alpha_-$, $\delta_0$, $\delta_+$, $\delta_-$ are constants, and the functions $A_k$, $D_k$, $\Lambda_k$ satisfy the following difference relations

\begin{align*}
A_{k+1} - A_k &= q_{k+1}r_k - \tilde{q}_k\tilde{r}_{k-1}, \quad (1.18) \\
D_{k+1} - D_k &= q_kr_{k+1} - \tilde{q}_{k-1}\tilde{r}_k, \quad (1.19) \\
\Lambda_{k+1}(1 - q_kr_k) &= \Lambda_k(1 - \tilde{q}_k\tilde{r}_k). \quad (1.20)
\end{align*}

Remark. In the case of the rapidly decreasing boundary conditions there exists a canonical way to single out certain solutions of the difference equations (1.18)–(1.20) above, namely, by the conditions

\begin{align*}
A_k, \ D_k &\to 0, \ \Lambda_k \to 1 \quad \text{as} \quad k \to \pm \infty,
\end{align*}

which results in

\begin{align*}
A_k &= q_k r_{k-1} + \sum_{j=-\infty}^{k-1} (q_j r_{j-1} - \tilde{q}_j \tilde{r}_{j-1}), \quad (1.21) \\
D_k &= q_{k-1} r_k + \sum_{j=-\infty}^{k-1} (q_{j-1} r_j - \tilde{q}_{j-1} \tilde{r}_j), \quad (1.22) \\
\Lambda_k &= \prod_{j=-\infty}^{k-1} \frac{1 - q_j \tilde{r}_j}{1 - q_j r_j}. \quad (1.23)
\end{align*}

In the case of the periodic boundary conditions to choose a particular solution one can use the same formulas with the sums and product starting from $j = 0$ instead of $j = -\infty$.

The numbers $\alpha_0$, $\delta_0$, $\alpha_+$, $\alpha_-$, $\delta_+$, $\delta_-$, playing the role of constants of summation, are defined as soon as certain solutions $A_k$, $D_k$, $\Lambda_k$ have been fixed.

The formulas (1.17), (1.21)–(1.23) define the map which we shall denote

\[ \mathcal{T}_{\text{AL}}(h; \alpha_0, \alpha_+, \alpha_-; \delta_0, \delta_+, \delta_-) : \ (q, r) \mapsto (\tilde{q}, \tilde{r}) \]

It is important to notice that for the pure imaginary values of $h$ this map obviously allows the reduction

\[ r = \pm q^*. \]
provided \( A_k, D_k, \Lambda_k \) are chosen as in (1.21)–(1.23), and
\[
\delta_0 = \alpha_0^*, \quad \delta_+ = \alpha_+^*, \quad \delta_- = \alpha_-^*,
\]
so that there remains a three–parameter family of difference schemes satisfying this condition.

The expressions (1.21)–(1.23) serve as a source of a non-locality of the difference scheme, which is its major drawback. This feature makes any numerical realization of the numerical scheme extremely time–consuming.

The numerical experiments reported in [7] showed that even despite this drawback the Ablowitz–Ladik difference schemes are the best among the class of finite difference methods, being surpassed only by certain spectral numerical methods.

In the present note we shall demonstrate how to factorize the non-local scheme (1.17) into the product of very simple (in particular, local) schemes, which surely can speed up the performance of this scheme considerably.

2 Ablowitz–Ladik hierarchy and its simplest flows

From the modern point of view, the Ablowitz–Ladik system (1.1) is a representative of a whole hierarchy of commuting Hamiltonian flows. Considering, for notational simplicity, the finite dimensional case, we define the Poisson bracket on the space \( \mathbb{R}^{2N}(q, r) \) by the formula
\[
\{ q_k, r_j \} = (1 - q_k r_k) \delta_{jk}, \quad \{ q_k, q_j \} = \{ r_k, r_j \} = 0.
\]
(2.1)
The Hamiltonians of the commuting flows are the coefficients in the Laurent expansion of the trace \( \text{tr} \, T_N(q, r, \lambda) \) where \( T_N \) is the monodromy matrix
\[
T_N = L_N \cdot L_{N-1} \cdots L_2 \cdot L_1,
\]
(2.2)
supplied by the function
\[
H_0(q, r) = \log \det T_N = \sum_{k=1}^N \log(1 - q_k r_k).
\]
(2.3)
The involutivity of all integrals of motion follows from the fundamental $r$-matrix relation:

$$\{L(\lambda) \otimes L(\mu)\} = [L(\lambda) \otimes L(\mu), \rho(\lambda, \mu)], \quad (2.4)$$

where

$$\rho(\lambda, \mu) = \begin{pmatrix}
\frac{1}{2} \frac{\lambda^2 + \mu^2}{\lambda^2 - \mu^2} & 0 & 0 & 0 \\
0 & \frac{1}{2} \frac{\lambda^2 + \mu^2}{\lambda^2 - \mu^2} & -\lambda \mu & 0 \\
0 & \lambda \mu & \frac{1}{2} \frac{\lambda^2 - \mu^2}{\lambda^2 - \mu^2} & 0 \\
0 & 0 & 0 & \frac{1}{2} \frac{\lambda^2 + \mu^2}{\lambda^2 - \mu^2}
\end{pmatrix}. \quad (2.5)$$

It is easy to see that the following two functions belong to the involutive family generated by $\text{tr} T_N$:

$$H_+(q, r) = \sum_{k=1}^{N} q_{k+1} r_k, \quad H_-(q, r) = \sum_{k=1}^{N} q_k r_{k+1}. \quad (2.6)$$

The corresponding Hamiltonian flows are described by the differential equations

$$\begin{align*}
\mathcal{F}_+ : \quad & \dot{q}_k = q_{k+1}(1 - q_k r_k), \quad \dot{r}_k = -r_{k-1}(1 - q_k r_k), \quad (2.7) \\
\mathcal{F}_- : \quad & \dot{q}_k = q_{k-1}(1 - q_k r_k), \quad \dot{r}_k = -r_{k+1}(1 - q_k r_k), \quad (2.8)
\end{align*}$$

The flow generated by the Hamiltonian function $(2.3)$ is described, up to the factor 2, by the differential equations

$$\mathcal{F}_0 : \quad \dot{q}_k = -2q_k, \quad \dot{r}_k = 2r_k. \quad (2.9)$$

The Ablowitz–Ladik flow proper is an obvious linear combination of these more fundamental and simple flows. According to the general theory $[3]$, each of the flows $\mathcal{F}_\pm$, $\mathcal{F}_0$ is described by the zero–curvature representation $[10]$ with the same matrix $L_k$, but with different matrices $M_k$. The corresponding matrices $M_k$ are given by:

$$\begin{align*}
\mathcal{F}_+ : \quad M_k^{(+)} &= \begin{pmatrix}
\lambda^2 - q_k r_{k-1} & \lambda q_k \\
\lambda r_{k-1} & 0
\end{pmatrix}, \quad (2.10)
\end{align*}$$
\[ F_\pm : \quad M_k^{(-)} = \begin{pmatrix} 0 & -\lambda^{-1}q_{k-1} \\ -\lambda^{-1}r_k & -\lambda^{-2} + q_{k-1}r_k \end{pmatrix}. \quad (2.11) \]

\[ F_0 : \quad M_k^{(0)} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (2.12) \]

3 Local discretizations for \( F_\pm \)

We demonstrate now an unexpected fact. Namely, let only one of the four parameters \( \alpha_+, \delta_+, \alpha_-, \delta_- \) of the Ablowitz–Ladik scheme not vanish (so that the resulting scheme approximates one of the flows \( F_\pm \) rather than the original system (1.1)). Then it is possible to render the scheme local. This results in four integrable maps \((q,r) \mapsto (\tilde{q},\tilde{r})\) which are described by local equations of motion and belong to the Ablowitz–Ladik hierarchy, i.e. admit commutation representations with the matrix \( L_k \) from (1.7). The commutation representations for the maps given in the following four Propositions could be proved by an easy and direct check, but we prefer to trace back the relations between our maps and the original formulation by Ablowitz and Ladik.

**Proposition 1.** The Ablowitz–Ladik scheme (1.17) with the parameters \( \alpha_0 = \delta_0 = \alpha_- = \delta_- = 0, \quad \alpha_+ = 1 \)

is equivalent to the following map:

\[ T_\pm (h) : \begin{cases} (\tilde{q}_k - q_k)/h = q_{k+1}(1 - q_k \tilde{r}_k), \\ (\tilde{r}_k - r_k)/h = -\tilde{r}_{k-1}(1 - q_k \tilde{r}_k) \end{cases} \quad (3.1) \]

approximating the flow \( F_\pm \). This map has the commutation representation

\[ T_+(h) : \quad \bar{L}_k V_k^{(+)} = V_{k+1}^{(+)} L_k \]

with the matrix

\[ V_k^{(+)} = V_k^{(+)}(q,\tilde{r},h) = \begin{pmatrix} 1 + h\lambda^2 - hq_k \tilde{r}_{k-1} & h\lambda q_k \\ h\lambda \tilde{r}_{k-1} & 1 \end{pmatrix}. \quad (3.2) \]
Proof. According to the Proposition 0, the matrix $V_k^{(+)}$ for the scheme $\mathcal{T}_{AL}(h; 0, 1, 0; 0, 0, 0)$ has the form

$$V_k^{(+)} = \begin{pmatrix} 1 + h\lambda^2 - hA_k & h\lambda q_k \\ h\lambda \tilde{r}_{k-1} & 1 \end{pmatrix},$$

while the equations of motion read:

$$\tilde{q}_k - q_k)/h = q_{k+1} - q_k A_{k+1}, \quad (\tilde{r}_k - r_k)/h = -\tilde{r}_{k-1} + \tilde{r}_k A_k. \quad (3.3)$$

Here $A_k$ is the solution of the difference relation

$$A_{k+1} - A_k = q_{k+1} r_k - \tilde{q}_k \tilde{r}_{k-1} \quad (3.4)$$

tending to 0 as $k \to \pm \infty$ in the case of the rapidly decaying boundary conditions. The Proposition will be demonstrated if we prove the following formula for $A_k$:

$$A_k = q_k \tilde{r}_{k-1} \quad (3.5)$$

which comes on the place of the non-local expression (1.21). To this end multiply the first equation in (3.3) by $\tilde{r}_{k-1}$, the second one by $q_{k+1}$, and add the two resulting equations:

$$\tilde{q}_k \tilde{r}_{k-1} - q_{k+1} r_k + q_{k+1} \tilde{r}_k - q_k \tilde{r}_{k-1} = h q_{k+1} \tilde{r}_k A_k - h q_k \tilde{r}_{k-1} A_{k+1}.$$ 

Using (3.4) we obtain:

$$-A_{k+1} + A_k + q_{k+1} \tilde{r}_k - q_k \tilde{r}_{k-1} = h q_{k+1} \tilde{r}_k A_k - h q_k \tilde{r}_{k-1} A_{k+1},$$

which is equivalent to

$$\frac{1 - hA_k}{1 - hq_k \tilde{r}_{k-1}} = \frac{1 - hA_{k+1}}{1 - q_{k+1} \tilde{r}_k} = \text{const.}$$

As the both quantities $A_k, q_k \tilde{r}_{k-1}$ tend to 0 by $k \to \pm \infty$, this constant has to be equal to 1, which ends the proof.

Proposition 2. The Ablowitz–Ladik scheme (1.17) with the parameters

$$\alpha_0 = \delta_0 = \alpha_- = \delta_- = \alpha_+ = 0, \quad \delta_+ = 1$$

ends the proof.
is equivalent to the following map:
\[
\mathcal{T}_-(h) : \begin{cases} 
(\bar{q}_k - q_k)/h = \bar{q}_{k-1}(1 - \bar{q}_k r_k), \\
(\bar{r}_k - r_k)/h = -r_{k+1}(1 - \bar{q}_k r_k)
\end{cases}
\]  
approximating the flow $\mathcal{F}_-$. This map has the commutation representation
\[
\mathcal{T}_-(h) : \bar{L}_k V_k^{(-)} = V_{k+1}^{(-)} L_k
\]
with the matrix
\[
V_k^{(-)} = V_k^{(-)}(\bar{q}, r, h) = \begin{pmatrix} 1 & -h\lambda^{-1}\bar{q}_{k-1} \\
-h\lambda^{-1}r_k & 1 - h\lambda^{-2} + h\bar{q}_{k-1}r_k \end{pmatrix}.
\]  
(3.7)

**Proof.** of this Proposition is completely parallel to that of the previous one and is therefore omitted.

**Proposition 3.** The Ablowitz–Ladik scheme (1.17) with the parameters
\[
\alpha_0 = \delta_0 = \alpha_+ = \delta_- = 0, \quad \alpha_- = 1
\]
is equivalent to the following map:
\[
\mathcal{T}_-^{-1}(-h) : \begin{cases} 
(\bar{q}_k - q_k)/h = q_{k-1}(1 - q_k \bar{r}_k), \\
(\bar{r}_k - r_k)/h = -\bar{r}_{k+1}(1 - q_k \bar{r}_k)
\end{cases}
\]  
approximating the flow $\mathcal{F}_-$. This map has the commutation representation
\[
\mathcal{T}_-^{-1}(-h) : \bar{L}_k W_k^{(-)} = W_{k+1}^{(-)} L_k
\]
with the matrix
\[
W_k^{(-)} = W_k^{(-)}(q, \bar{r}, h) = \frac{1}{1 - hq_{k-1}\bar{r}_k} \begin{pmatrix} 1 + h\lambda^{-2} - h\bar{q}_{k-1}r_k & -h\lambda^{-1}q_{k-1} \\
-h\lambda^{-1}\bar{r}_k & 1 \end{pmatrix}.
\]  
(3.9)

**Proof.** According to the Proposition 0, the matrix $W_k^{(-)}$ for the difference scheme $\mathcal{T}_{AL}(h; 0, 0, 1; 0, 0, 0)$ has the form
\[
W_k^{(-)} = \begin{pmatrix} 1 + h\lambda^{-2}\Lambda_k & -h\lambda^{-1}q_{k-1}\Lambda_k \\
-h\lambda^{-1}\bar{r}_k\Lambda_k & 1 + hq_{k-1}\bar{r}_k\Lambda_k \end{pmatrix}.
\]
The equations of motion for this scheme may be presented as

\[
\left( \tilde{q}_k - q_k \right)/h = q_{k-1}(1 - \tilde{q}_k \bar{r}_k)\Lambda_k, \quad \left( \bar{r}_k - r_k \right)/h = -\bar{r}_{k+1}(1 - q_k r_k)\Lambda_{k+1}. \tag{3.10}
\]

Here \( \Lambda_k \) is the solution of the difference equation

\[
\Lambda_{k+1}(1 - q_k r_k) = \Lambda_k(1 - \tilde{q}_k \bar{r}_k) \tag{3.11}
\]
tending to 1 by \( k \to \pm \infty \) in the case of the rapidly decaying boundary conditions. It is easy to check that the proposition will be proved if we demonstrate the following formula for \( \Lambda_k \):

\[
\Lambda_k = \frac{1}{1 - hq_{k-1}\bar{r}_k} \tag{3.12}
\]

which replaces in this case the general non-local expression (1.23). To do this, we first re-write (3.10) as

\[
\tilde{q}_k(1 + hq_{k-1}\bar{r}_k\Lambda_k) = q_k + hq_{k-1}\Lambda_k, \quad r_k(1 + hq_k\bar{r}_{k+1}\Lambda_{k+1}) = \bar{r}_k + h\bar{r}_{k+1}\Lambda_{k+1}.
\]

Now multiply the first of these equations by \( \bar{r}_k \), the second one by \( q_k \) and subtract the two resulting equations:

\[
(1 - \tilde{q}_k \bar{r}_k)(1 + hq_{k-1}\bar{r}_k\Lambda_k) = (1 - q_k r_k)(1 + hq_k\bar{r}_{k+1}\Lambda_{k+1}).
\]

According to (3.11), this is equivalent to

\[
\frac{\Lambda_{k+1}}{\Lambda_k} = \frac{1 + hq_k\bar{r}_{k+1}\Lambda_{k+1}}{1 + hq_{k-1}\bar{r}_k\Lambda_k},
\]

or

\[
\frac{1}{\Lambda_k} + hq_{k-1}\bar{r}_k = \frac{1}{\Lambda_{k+1}} + hq_k\bar{r}_{k+1} = \text{const.}
\]

Taking the \( k \to \pm \infty \) limit, we see that this constant has to be equal to 1, which finishes the proof.

**Proposition 4.** The Ablowitz–Ladik scheme (1.17) with the parameters

\[
\alpha_0 = \delta_0 = \alpha_+ = \delta_+ = \alpha_- = 0, \quad \delta_- = 1
\]

is equivalent to the following map:

\[
\mathcal{T}_{-1}^{-1}(-h) : \begin{cases} 
(\bar{q}_k - q_k)/h = \bar{q}_{k+1}(1 - \bar{q}_k r_k), \\
(\bar{r}_k - r_k)/h = -r_{k-1}(1 - \bar{q}_k r_k)
\end{cases} \tag{3.13}
\]
approximating the flow $F_+$. This map has the commutation representation

$$T_+^{-1}(-h) : \quad \tilde{L}_k W_k^{(+)} = W_{k+1}^{(+)} L_k$$

with the matrix

$$W_k^{(+)}(\tilde{q}, \tilde{r}, h) = \frac{1}{1 + h\tilde{q}_k r_k - 1} \begin{pmatrix} 1 & h\lambda \tilde{q}_k \\ h\lambda r_k - 1 & 1 - h\lambda^2 + h\tilde{q}_k r_k - 1 \end{pmatrix}.$$ 

**(3.14)**

*Proof* of this Proposition is omitted, because it is completely analogous to that of the previous one.

**Remark.** It is important to notice the following relations:

$$W_k^{(+)}(\tilde{q}, r, h) = (1 - h\lambda^2) \left(V_k^{(+)}(\tilde{q}, r, -h)\right)^{-1},$$

$$W_k^{(-)}(q, \tilde{r}, h) = (1 + h\lambda^{-2}) \left(V_k^{(-)}(q, \tilde{r}, -h)\right)^{-1}.$$ 

These relations are very difficult to guess and to prove, if one remains by the original formulation of the Proposition 0.

We have demonstrated so far that the maps $T_+(h), T_-(h)$ have commutation representations with the same matrix $L_k$ as their continuous time counterparts, and hence they share all the integrals of motion. To claim that the maps belong to the Ablowitz–Ladik hierarchy, we still need to show the Poisson property.

**Proposition 5.** The both maps $T_+(h), T_-(h)$ are Poisson with respect to the Poisson bracket (2.1).

*Proof.* The Poisson property of a map $(q, r) \mapsto (\tilde{q}, \tilde{r})$ with respect to the bracket (2.1) is equivalent to the preservation of the corresponding 2-form:

$$\sum_{k=1}^{N} \frac{d\tilde{q}_k \wedge d\tilde{r}_k}{1 - \tilde{q}_k \tilde{r}_k} = \sum_{k=1}^{N} \frac{d q_k \wedge d r_k}{1 - q_k r_k}$$

**(3.15)**

(remaining for simplicity by the finite dimensional case with the periodic boundary conditions). We shall prove this identity only for the map $T_+(h)$, since for $T_-(h)$ everything is completely analogous. Differentiating the equations of motion (3.1), we obtain the following expressions:

$$d\tilde{q}_k = (1 - h q_{k+1} \tilde{r}_k) d q_k + h(1 - q_k \tilde{r}_k) d q_{k+1} - h q_{k+1} q_k d \tilde{r}_k,$$ 

**(3.16)**
\begin{equation}
(1 - h q_k \tilde{r}_{k-1}) \, d \tilde{r}_k = dr_k - h(1 - q_k \tilde{r}_k) \, d \tilde{r}_{k-1} + h \tilde{r}_{k-1} \, d q_k. \tag{3.17}
\end{equation}

Using successively these two formulas, we obtain:

\begin{equation}
d \tilde{q}_k \wedge d \tilde{r}_k = \left( 1 - h q_{k+1} \tilde{r}_k \right) dq_k \wedge d \tilde{r}_k + h(1 - q_k \tilde{r}_k) \, dq_{k+1} \wedge d \tilde{r}_k - h \tilde{r}_{k-1} \, dq_k \wedge d \tilde{r}_k - h \tilde{r}_{k-1} \, dq_{k+1} \wedge d \tilde{r}_k. \tag{3.18}
\end{equation}

To make the last step, we observe that the equations of motion (3.1) may be equivalently re-written as

\begin{equation}
1 - \tilde{q}_k \tilde{r}_k = (1 - q_k \tilde{r}_k)(1 - h q_{k+1} \tilde{r}_k), \tag{3.19}
\end{equation}

\begin{equation}
1 - q_k r_k = (1 - q_k \tilde{r}_k)(1 - h q_k \tilde{r}_{k-1}). \tag{3.20}
\end{equation}

Upon use of (3.19), (3.20) the identity (3.18) may be presented as

\begin{equation}
\frac{d \tilde{q}_k \wedge d \tilde{r}_k}{1 - \tilde{q}_k \tilde{r}_k} = \frac{dq_k \wedge dr_k}{1 - q_k r_k} + \frac{h \, dq_{k+1} \wedge d \tilde{r}_k}{1 - h q_{k+1} \tilde{r}_k} - \frac{h \, dq_k \wedge d \tilde{r}_{k-1}}{1 - h q_k \tilde{r}_{k-1}}. \tag{3.21}
\end{equation}

Summation over \( k \) results in (3.15). The proof is complete.

It should be mentioned that each of the pairs \( (\mathcal{T}_+(h), \mathcal{T}^{-1}_-(-h)) \) and \( (\mathcal{T}^{-1}_+(-h), \mathcal{T}_-(h)) \) may be seen as generated by one of the two simplest partitioned Runge–Kutta methods \cite{4} when applied to the pair of differential systems \( (\mathcal{F}_+, \mathcal{F}_-) \). Recall that the both of these partitioned Runge–Kutta methods are symplectic when applied to canonical Hamiltonian systems \cite{10}.

The Proposition 5 shows that this is still true for some systems which are Hamiltonian with respect to nonlinear Poisson brackets. However, unlike the canonical case, this statement cannot be extended to arbitrary systems Hamiltonian with respect to the bracket (2.1): the concrete form of the right-hand side is essential for the validity of the Proposition 5.

We finish this Section by constructing a discretization for the linear flow \( \mathcal{F}_0 \). This is a much more simple task. Among many reasonable discretizations of the flow \( \mathcal{F}_0 \) we choose (for the reasons which will become clear in the next Section)

\begin{equation}
\mathcal{T}_0(h) : \quad \tilde{q}_k = \frac{1 - h}{1 + h} q_k, \quad \tilde{r}_k = \frac{1 + h}{1 - h} r_k. \tag{3.21}
\end{equation}

(Note that \( \mathcal{T}^{-1}_0(-h) = \mathcal{T}_0(h) \).
Proposition 6. The linear map $T_0$ is Poisson with respect to the bracket \((2.1)\) and has the commutation representation

$$\tilde{L}_k V^{(0)} = V^{(0)} L_k$$

with the constant matrix

$$V^{(0)} = V^{(0)}(h) = \begin{pmatrix} 1 - h & 0 \\ 0 & 1 + h \end{pmatrix}.$$ \hspace{1cm} (3.22)

4 Local discretizations for $\mathcal{F}_0 \circ \mathcal{F}_- \circ \mathcal{F}_+$

Recall that, when considering the system \((1.1)\) as a space discretization of the nonlinear Schrödinger equation \((1.2)\), the following reduction is of the primary interest:

$$r = \pm q^*,$$ \hspace{1cm} (4.1)

(it is admissible in the case of pure imaginary values of time $t$, that is, after the change of the independent variable $t \mapsto it$, $i = \sqrt{-1}$).

The flows $\mathcal{F}_+, \mathcal{F}_-$ alone do not allow this reduction any more, as well as their time discretizations $T_+(h), T_-(h)$. Nevertheless, we shall demonstrate now that the composition of these maps does again have this attractive property.

Proposition 7. The Ablowitz–Ladik scheme \((1.17)\) with the parameters

$$\alpha_0 = \delta_0 = \alpha_- = \delta_- = 0, \quad \alpha_+ = \delta_+ = 1$$

may be presented as the composition

$$T_-(h) \circ T_+(h).$$

Proposition 8. The Ablowitz–Ladik scheme \((1.17)\) with the parameters

$$\alpha_0 = \delta_0 = \alpha_+ = \delta_+ = 0, \quad \alpha_- = \delta_- = 1$$

may be presented as the composition

$$T_-(\pm h)^{-1} \circ T_+(\pm h)^{-1}.$$
Proof of the Proposition 7. Let
\[ T_+(h) : (q, r) \mapsto (\tilde{q}, \tilde{r}), \quad T_-(h) : (\tilde{q}, \tilde{r}) \mapsto (\tilde{q}, \tilde{r}), \]
so that, according to (3.1), (3.6),
\[ (\hat{q}_k - q_k)/h = q_{k+1}(1 - q_k\tilde{r}_k), \quad (\hat{r}_k - r_k)/h = -\hat{r}_{k-1}(1 - q_k\tilde{r}_k), \quad (4.2) \]
\[ (\hat{q}_k - q_k)/h = \hat{q}_{k-1}(1 - \hat{q}_k\tilde{r}_k), \quad (\hat{r}_k - \tilde{r}_k)/h = -\hat{r}_{k+1}(1 - \hat{q}_k\tilde{r}_k). \quad (4.3) \]
It follows from the Propositions 1,2 that the composition \( T_-(h) \circ T_+(h) \) allows the commutation representation
\[ \tilde{L}_k V_k = V_{k+1} L_k \quad (4.4) \]
with the matrix
\[ V_k = V_k^{-}(\tilde{q}, \tilde{r}) V_k^{(+)}(q, \hat{r}). \quad (4.5) \]
We calculate now the entries of the matrix \( V_k \) (denoting them according to (1.12)) in order to show that they have the form (1.13)–(1.16). From (4.5), (3.2), (3.7) we obtain:
\[ B_k = h\lambda q_k - h\lambda^{-1}\hat{q}_{k-1}, \]
\[ C_k = -h\lambda^{-1}\hat{r}_k(1 + h\lambda^2 - hq_k\tilde{r}_k) + h\lambda\hat{r}_k(1 - h\lambda^{-2} + h\hat{q}_{k-1}\tilde{r}_k) \]
\[ = h\lambda(\hat{r}_{k-1} - h\tilde{r}_k(1 - \hat{q}_{k-1}\tilde{r}_{k-1})) - h\lambda^{-1}(\hat{r}_k + h\hat{r}_{k-1}(1 - q_k\tilde{r}_k)) \]
\[ = h\lambda\hat{r}_{k-1} - h\lambda^{-1}\tilde{r}_k \]
(the last equality follows from the equations of motion (1.12), (4.3)),
\[ A_k = 1 + h\lambda^2 - hA_k, \]
\[ D_k = 1 - h\lambda^{-2} + hD_k, \]
where
\[ A_k = (q_k + h\hat{q}_{k-1})\hat{r}_{k-1}, \quad D_k = (\hat{q}_{k-1} - hq_k)\hat{r}_k. \quad (4.6) \]
So, the entries of the matrix \( V_k \) has exactly the form (1.13)–(1.16) with the parameters \( \alpha_0 = \delta_0 = \alpha_- = \delta_- = 0, \alpha_+ = \delta_+ = 1 \). We may conclude that the quantities \( A_k, D_k \) satisfy the difference relations (1.18), (1.19). (One could as well derive these difference relations (1.18), (1.19) directly from the definitions (4.6) and the equations of motion (1.2), (1.3).) In the case of
the rapidly decaying boundary conditions we have obviously \( A_k, D_k \to 0 \) as \( k \to \pm \infty \), so that the quantities \( A_k, D_k \) may be alternatively represented as in (1.21), (1.22). This finishes the proof.

**Proof of the Proposition 8.** This time let
\[
\mathcal{T}_+^{-1}(-h) : (q, r) \mapsto (\tilde{q}, \tilde{r}), \quad \mathcal{T}_-^{-1}(-h) : (\bar{q}, \bar{r}) \mapsto (\tilde{q}, \tilde{r}),
\]
so that, according to (3.13), (3.8),
\[
(\tilde{q}_k - q_k)/h = \tilde{q}_{k+1}(1 - \tilde{q}_k r_k), \quad (\tilde{r}_k - r_k)/h = -r_{k-1}(1 - \tilde{q}_k r_k), \quad (4.7)
\]
\[
(\tilde{q}_k - q_k)/h = \tilde{q}_{k-1}(1 - \tilde{q}_k \bar{r}_k), \quad (\tilde{r}_k - r_k)/h = -\bar{r}_{k+1}(1 - \tilde{q}_k \bar{r}_k). \quad (4.8)
\]
It follows from the Propositions 3, 4 that the composition \( \mathcal{T}_-^{-1}(-h) \circ \mathcal{T}_+^{-1}(-h) \) allows the commutation representation
\[
\tilde{L}_k W_k = W_{k+1} L_k \quad (4.9)
\]
with the matrix
\[
W_k = W_k(-)(\tilde{q}, \tilde{r}) W_k(+) (\tilde{q}, r). \quad (4.10)
\]
We calculate now the entries of the matrix \( W_k \) (denoting them again according to (1.12)) in order to show that they, as in the previous case, have the form (1.13)—(1.16). Denoting
\[
\Lambda_k = \frac{1}{(1 - h\tilde{q}_{k-1} \tilde{r}_k)(1 + h\tilde{q}_k \bar{r}_{k-1})}, \quad (4.11)
\]
we obtain from (4.10), (3.14), (3.9):
\[
\mathcal{C}_k \Lambda_k^{-1} = h\lambda \tilde{r}_{k-1} - h\lambda^{-1} \tilde{r}_k,
\]
\[
\mathcal{B}_k \Lambda_k^{-1} = h\lambda \tilde{q}_k(1 + h\lambda^{-2} - h\tilde{q}_{k-1} \tilde{r}_k) - h\lambda^{-1} \tilde{q}_{k-1}(1 - h\lambda^2 + h\tilde{q}_k \bar{r}_{k-1})
\]
\[
= h\lambda \big( \tilde{q}_k + h\tilde{q}_{k-1}(1 - \tilde{q}_k \bar{r}_k) \big) - h\lambda^{-1} \big( \tilde{q}_{k-1} - h\tilde{q}_k(1 - \tilde{q}_k \bar{r}_{k-1}) \big)
\]
\[
= h\lambda \tilde{q}_k - h\lambda^{-1} \tilde{q}_{k-1}
\]
(the last equality being based on the equations of motion (1.7), (1.8)),
\[
\mathcal{A}_k \Lambda_k^{-1} = 1 - h\tilde{q}_{k-1} \tilde{r}_k - h^2 \tilde{q}_{k-1} \bar{r}_{k-1} + h\lambda^{-2}
\]
\[
= \Lambda_k^{-1} - h\tilde{r}_{k-1} \big( \tilde{q}_k + h\tilde{q}_{k-1}(1 - \tilde{q}_k \bar{r}_k) \big) + h\lambda^{-2}
\]
\[
= \Lambda_k^{-1} - h\tilde{q}_k \bar{r}_{k-1} + h\lambda^{-2},
\]
\[
\mathcal{D}_k \Lambda_k^{-1} = 1 + h\tilde{q}_k \bar{r}_{k-1} - h^2 \tilde{q}_k \bar{r}_k - h\lambda^2
\]
\[
= \Lambda_k^{-1} + h\tilde{r}_k \big( \tilde{q}_{k-1} - h\tilde{q}_k(1 - \tilde{q}_k \bar{r}_{k-1}) \big) - h\lambda^2
\]
\[
= \Lambda_k^{-1} + h\tilde{q}_{k-1} \tilde{r}_k - h\lambda^2
\]
(using again repeatedly the equations of motion (4.7), (4.8)) and the definition (4.11)).

We see that this time the entries of the matrix $W_k$ have exactly the form (1.13)–(1.16) with the parameters $\alpha_0 = \delta_0 = \alpha_+ = \delta_+ = 0$, $\alpha_- = \delta_- = 1$. It follows that the quantity $\Lambda_k$ satisfies the difference relation (1.20) (which could be as well derived from the definition (4.11) and the equations of motion (4.7), (4.8)). In the case of the rapidly decaying boundary conditions we have obviously $\Lambda_k \to 1$ as $k \to \pm\infty$, so that $\Lambda_k$ may be alternatively represented as in (1.23). The proof is finished.

The maps constructed in these two Propositions have the desired property (1.24), assuring that the reduction (1.1) is admissible for them. However, they still do not approximate the Ablowitz–Ladik system (1.1). In order to achieve this, we have to commute them with $T_0(h)$. The following lemma, following directly from the formulas (1.17), allows to control the parameters $\alpha_0, \delta_0$ of the Ablowitz–Ladik discretizations.

**Lemma.** Let $T'(h) = T_{AL}(h; 0, \alpha_+, \alpha_-; 0, \delta_+, \delta_-)$, and let $T''(h)$ be the linear map

$$
(q, r) \mapsto \left( \frac{1 - h\alpha_0}{1 + h\delta_0} q, \frac{1 + h\delta_0}{1 - h\alpha_0} r \right).
$$

Then

$$
T'(h) \circ T''(h) = T''(h) \circ T'(h) = T_{AL}(h; \alpha_0, \alpha_+(1 - h\alpha_0), \alpha_-(1 - h\alpha_0); \delta_0, \delta_+(1 + h\delta_0), \delta_-(1 + h\delta_0)).
$$

According to this lemma, we derive from the Propositions 7,8 the following fundamental statement.

**Proposition 9.** The Ablowitz–Ladik scheme (1.17) with the parameters

$$
\alpha_0 = \delta_0 = 1, \quad \alpha_+ = 1 - h, \quad \delta_+ = 1 + h, \quad \alpha_- = \delta_- = 0
$$

may be presented as the composition

$$
T_0(h) \circ T_-(h) \circ T_+(h).
$$

The Ablowitz–Ladik scheme (1.17) with the parameters

$$
\alpha_0 = \delta_0 = 1, \quad \alpha_+ = \delta_+ = 0, \quad \alpha_- = 1 - h, \quad \delta_- = 1 + h
$$
may be presented as the composition

\[ T_0(h) \circ T_+^{-1}(-h) \circ T_-^{-1}(-h). \]

These two maps, approximating the system (1.17), have the property (1.24) under the condition \( h^* = -h \), i.e. \( h \) pure imaginary. Hence they both may serve as honest time discretizations of the reduced version (1.3) of the Ablowitz–Ladik system (1.1). Note that all maps in each of these compositions commute.

5 Connection with relativistic Toda

As noted in [11], the Ablowitz–Ladik hierarchy is in principle nothing but the relativistic Toda hierarchy and vice versa. We first give a Hamiltonian interpretation of this statement, and then use the results on the discrete time relativistic Toda lattice [4] to clarify the place of the maps \( T_+ \), \( T_- \) in the Ablowitz–Ladik hierarchy.

Define new variables \( c_k, d_k \) on the phase space of the Ablowitz–Ladik hierarchy:

\[ d_k = \frac{q_{k-1}}{q_k}, \quad c_k = \frac{q_{k-1}}{q_k}(q_k r_k - 1). \]  

A direct computation shows that the only nonvanishing Poisson brackets between these functions are:

\[ \{ c_k, d_{k+1} \} = c_k d_{k+1}, \quad \{ c_k, d_k \} = -c_k d_k, \quad \{ c_k, c_{k+1} \} = c_k c_{k+1}. \]  

One immediately recognizes in these relations the quadratic Poisson brackets underlying the relativistic Toda hierarchy. Moreover, one sees immediately that the simplest Hamiltonians of the Ablowitz–Ladik hierarchy \( H_\pm, H_0 \) may be expressed in the variables \( c_k, d_k \) as

\[ H_+ = \sum_{k=1}^{N} q_{k+1} r_k = \sum_{k=1}^{N} \frac{c_k + d_k}{d_k d_{k+1}}, \]  

\[ H_- = \sum_{k=1}^{N} q_k r_{k+1} = \sum_{k=1}^{N} (c_k + d_k), \]  

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\[ H_0 = \sum_{k=1}^{N} \log(1 - q_k r_k) = \sum_{k=1}^{N} \log(c_k/d_k), \quad (5.5) \]

and in \( H_\pm \) we recognize the two basic Hamiltonians of the relativistic Toda hierarchy. We demonstrate now that our maps \( \mathcal{T}_+, \mathcal{T}_- \), being expressed in the variables \( c_k, d_k \), also coincide with the discrete time flows of the relativistic Toda lattice introduced in [2].

**Proposition 10.** Consider the map \( \mathcal{T}_+(h) \). Let the variables \( c_k, d_k \) be defined by (5.1), and define the auxiliary function \( \vartheta_k \) by

\[ \vartheta_k = q_k \tilde{r}_k - 1. \quad (5.6) \]

Then the following relations hold:

\[ \frac{c_k}{\vartheta_k} = d_k - h - h\vartheta_{k-1}, \quad (5.7) \]

\[ \tilde{d}_k = d_{k+1} \frac{d_k - h\vartheta_{k-1}}{d_{k+1} - h\vartheta_k}, \quad \tilde{c}_k = c_{k+1} \frac{c_k + h\vartheta_k}{c_{k+1} + h\vartheta_{k+1}}. \quad (5.8) \]

**Proposition 11.** Consider the map \( \mathcal{T}_-(h) \). Let the variables \( c_k, d_k \) be defined by (5.1), and define the auxiliary function \( a_k \) by

\[ a_k = \frac{q_{k-1}}{\tilde{q}_{k-1}} + \frac{hq_{k-1}}{q_k}. \quad (5.9) \]

Then the following relations hold:

\[ a_k = 1 + hd_k + \frac{hc_{k-1}}{a_{k-1}}, \quad (5.10) \]

\[ \tilde{d}_k = d_k \frac{a_{k+1} - hd_{k+1}}{a_k - hd_k}, \quad \tilde{c}_k = c_k \frac{a_{k+1} + hc_{k+1}}{a_k + hc_k}. \quad (5.11) \]

**Proof of the Proposition 10.** We shall use in the proof the equations of motion in the form (3.19), (3.20) and also two additional auxiliary identities. The first equation of motion in (3.1), re-written with the help of definitions for \( d_{k+1} \) and \( \vartheta_k \), reads:

\[ \frac{\tilde{q}_k}{q_{k+1}} = d_{k+1} - h\vartheta_k, \quad (5.12) \]
Further, from (3.19), (3.20) and the definition (5.6) it follows:

\[
1 - \tilde{q}_k \tilde{r}_k
\]
\[
1 - q_{k+1} \tilde{r}_{k+1} = \frac{\vartheta_k}{\vartheta_{k+1}},
\]

(5.13)

Using the definitions (5.1) for \( c_k \) and (5.6) for \( d_k \), then the identity (3.20), and again the definitions (5.1) for \( d_k \) and (5.6) for \( d_{k-1} \), we get:

\[
c_k \frac{1 - q_k \tilde{r}_k}{q_k} = \frac{1 - h \tilde{q}_k \tilde{r}_{k-1}}{1 - q_k \tilde{r}_k} = \frac{1 - h \tilde{q}_k \tilde{r}_{k-1}}{1 - q_k \tilde{r}_k} = d_k - h - h \vartheta_{k-1},
\]

which is the recurrent relation (5.7).

To prove the first equality in (5.8), we use the definition of \( d_k \) and the identity (5.12):

\[
\frac{\tilde{q}_{k-1}}{q_k} \frac{1 - q_k \tilde{r}_k}{1 - q_k \tilde{r}_k} = \frac{\tilde{q}_{k-1}}{q_k} = \frac{d_k - h \vartheta_{k-1}}{d_{k+1} - h \vartheta_k}.
\]

Finally, to prove the second equality in (5.8), we use the definition of \( c_k \), the identities (5.12), (5.13), and the recurrent relation (5.7):

\[
\frac{\tilde{c}_k}{c_{k+1}} = \frac{\tilde{q}_{k-1}}{q_k} \frac{1 - q_k \tilde{r}_k}{1 - q_k \tilde{r}_k} = \frac{d_k - h \vartheta_{k-1}}{d_{k+1} - h \vartheta_k} \frac{\vartheta_k}{\vartheta_{k+1}} = \frac{c_k + h \vartheta_k}{c_{k+1} + h \vartheta_{k+1}}.
\]

The Proposition 10 is proved.

Proof of the Proposition 11. We start again with re-writing the equations of motion (3.6) in the equivalent form:

\[
1 - q_k r_k = (1 - \tilde{q}_k r_k)(1 + h \tilde{q}_k r_{k-1}),
\]

(5.14)

\[
1 - \tilde{q}_k \tilde{r}_k = (1 - \tilde{q}_k r_k)(1 + h \tilde{q}_k r_{k+1}).
\]

(5.15)

Note that the first equation of motion in (3.6) may be represented also in another equivalent form:

\[
\tilde{q}_k (1 + h \tilde{q}_{k-1} r_k) = q_k + h \tilde{q}_{k-1}.
\]

(5.16)

We shall need also two additional auxiliary identities. The definitions (5.9), (5.1) immediately imply:

\[
a_k - h d_k = \frac{q_{k-1}}{q_{k-1}},
\]

(5.17)
\[ a_k + hc_k = \frac{q_{k-1}}{q_k}(1 + h\tilde{q}_{k-1}r_k). \] (5.18)

Now to prove the recurrent relation (5.10) we use (5.18) in conjunction with (5.9), and then (5.16) and (5.17):

\[ 1 + \frac{hc_k}{a_k} = \frac{q_k(1 + h\tilde{q}_{k-1}r_k)}{q_k + h\tilde{q}_{k-1}} = \frac{q_k}{q_k} = a_{k+1} - h\tilde{d}_{k+1}. \]

The first equation of motion in (5.11) follows from the definition of \(d_k\) and the formula (5.17):

\[ \tilde{d}_k = \frac{\tilde{q}_{k-1} q_k}{q_{k-1} - q_k} = \frac{a_{k+1} - h\tilde{d}_{k+1}}{a_k - h\tilde{d}_k}. \]

Finally, to prove the second equation of motion in (5.11), we use the definition of \(c_k\), the formulas (5.14), (5.15), and then (5.16):

\[ \tilde{c}_k = \frac{\tilde{q}_{k-1} q_k}{q_{k-1} - q_k} = \frac{\tilde{q}_{k-1} q_k}{q_{k-1} - q_k} = \frac{a_{k+1} + hc_{k+1} + h\tilde{c}_{k+1}}{a_k + hc_k}. \]

The Proposition 11 is proved.

Now one immediately recognizes in (5.7), (5.8) and in (5.10), (5.11) two discrete time flows of the relativistic Toda hierarchy introduced and studied in [2]. Applying the results in [2], we get the following statement (formulated, as before, for the periodic case for the sake of notational simplicity).

**Proposition 12.** The maps \(T_\pm\) have Lax representations with either of the \(N \times N\) Lax matrices

\[ T_+(\mathbf{q}, \mathbf{r}, \lambda) = L(\mathbf{q}, \mathbf{r}, \lambda)U^{-1}(\mathbf{q}, \mathbf{r}, \lambda) \quad \text{or} \quad T_-(\mathbf{q}, \mathbf{r}, \lambda) = U^{-1}(\mathbf{q}, \mathbf{r}, \lambda)L(\mathbf{q}, \mathbf{r}, \lambda), \]

where

\[ L(\mathbf{q}, \mathbf{r}, \lambda) = \sum_{k=1}^{N} \frac{q_{k-1}}{q_k} E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k}, \]

\[ U(\mathbf{q}, \mathbf{r}, \lambda) = \sum_{k=1}^{N} E_{kk} + \lambda^{-1} \sum_{k=1}^{N} \frac{q_{k-1}}{q_k} (1 - q_k r_k) E_{k,k+1}. \]

These maps are interpolated by the flows with the Hamiltonian functions

\[ -\text{tr}_0(\Phi(-T^{-1}_\pm)) = H_+ + O(h) \quad \text{and} \quad \text{tr}_0(\Phi(T_\pm)) = H_- + O(h), \]

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respectively, where
\[
\Phi(\xi) = h^{-1} \int_0^\xi \log(1 + h\eta) \frac{d\eta}{\eta} = \xi + O(h).
\]

The initial value problems for the maps $T_\pm$ may be solved in terms of the matrix factorization problem for the matrices
\[
\left( I - hT_\pm^{-1}(t = 0) \right)^n \quad \text{and} \quad \left( I + hT_\pm(t = 0) \right)^n,
\]
respectively.

(In the formulation above $\text{tr}_0$ stands for the free term of the Laurent expansion for the trace; the detailed definition of the matrix factorization problem in a loop group mentioned in this Proposition, may be found in [2]).

6 Conclusion

Re-considering the Ablowitz–Ladik discretizations from the modern point of view, undertaken in this paper, turned out to be rather fruitful. We factorized a highly non-local scheme into the product of very simple (local) ones, each of them approximating a more simple and fundamental flow of the Ablowitz–Ladik hierarchy. These local schemes may be studied exhaustively. In particular, we found in this paper the interpolating Hamiltonian flows for them, as well as the solution in terms of factorization problem in a loop group. We guess that also in the practical computations our variant of the difference scheme will exceed considerably the old one. It would be interesting and important to carry out the corresponding numerical experiments.

It seems also promising to re-consider from this point of view other non-local integrable discretizations derived and tested in [7].

We note also that our maps are ideal building blocks for applying the Ruth–Yoshida–Suzuki techniques [12], which will result in higher order integrable discretizations for the Ablowitz–Ladik system. This point will be reported in detail elsewhere.

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