I. INTRODUCTION

Stabilizer states and Clifford group operations play a central role in quantum error correction, quantum computing, and entanglement distillation. A stabilizer state is a state of an n-qubit system that is a simultaneous eigenvector of a commutative subgroup of the Pauli group. The latter consists of all tensor products of n single-qubit Pauli operations. The Clifford group is the group of unitary operations that map the Pauli group to itself under conjugation. In quantum error correction these concepts play a central role in the theory of stabilizer codes [1]. Although a quantum computer working with only stabilizer states and Clifford group operations is not powerful enough to disallow efficient simulation on a classical computer [2, 3], it is not unlikely that possible new quantum algorithms will exploit the rich structure of this group. In [4], we also showed the relevance of a quotient group of the Clifford group in mixed state entanglement distillation.

In this paper, we link stabilizer states and Clifford operations with binary linear algebra and binary quadratic forms (over GF(2)). The connection between multiplication of Pauli group elements and binary addition is well known as is the connection between commutability of Pauli group operations and a binary symplectic inner product [1]. In [4] we extended this connection to a link between a quotient group of the Clifford group and binary symplectic matrices (there termed P orthogonal). In this paper we give a binary characterization of the full Clifford group, by adding quadratic forms to the symplectic operations. In addition we show how the coefficients, with respect to a standard basis, of both stabilizer states and Clifford operations can also be described using binary quadratic forms. Our results also lead to efficient ways for decomposing Clifford group operations in a product of 2-qubit operations.

II. CLIFFORD GROUP OPERATIONS AND BINARY LINEAR AND QUADRATIC OPERATIONS

In this section, we show how the Clifford group is isomorphic to a group that can be entirely described in terms of binary linear algebra, by means of symplectic linear operations and quadratic forms.

We use the following notation for Pauli matrices.

\[
\begin{align*}
\sigma_{00} &= \tau_{00} = \sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\
\sigma_{01} &= \tau_{01} = \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \\
\sigma_{10} &= \tau_{10} = \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \\
\sigma_{11} &= \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \\
\tau_{11} &= i\sigma_y = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.
\end{align*}
\]

We also use vector indices to indicate tensor products of Pauli matrices. If \( v, w \in \mathbb{Z}_2^n \) and \( a = \begin{bmatrix} v \\ w \end{bmatrix} \in \mathbb{Z}_2^{2n} \), then we denote

\[
\begin{align*}
\sigma_a &= \sigma_{v_1 w_1} \otimes \ldots \otimes \sigma_{v_n w_n}, \\
\tau_a &= \tau_{v_1 w_1} \otimes \ldots \otimes \tau_{v_n w_n}.
\end{align*}
\tag{1}
\]

If we define the Pauli group to contain all tensor products of Pauli matrices with an additional complex phase in \( \{1, i, -1, -i\} \), an arbitrary Pauli group element can be represented as \( i^\delta (-1)^\epsilon \tau_u \), where \( \delta, \epsilon \in \mathbb{Z}_2 \) and \( u \in \mathbb{Z}_2^n \). The separation of \( \delta \) and \( \epsilon \), rather than having \( i^\gamma \) with \( \gamma \in \mathbb{Z}_4 \), is deliberate and will simplify formulas below. Throughout this paper exponents of \( i \) will always be binary. As a result \( i^{\delta_1 + \delta_2} = i^{\delta_1 + h_2 (-1)^{\delta_1} \delta_2} \). Multiplication of two Pauli group elements can now be translated into binary terms in the following way:

**Lemma 1** If \( a_1, a_2 \in \mathbb{Z}_2^n \), \( \delta_1, \delta_2, \epsilon_1, \epsilon_2 \in \mathbb{Z}_2 \) and \( \tau \) is
defined as in Eq. 4, then
\[
i^{\delta_1}(-1)^{\epsilon_1} \tau_{a_1} i^{\delta_2}(-1)^{\epsilon_2} \tau_{a_2} = i^{\delta_{12}}(-1)^{\epsilon_{12}} \tau_{a_{12}}
\]
with \( \delta_{12} = \delta_1 + \delta_2 \)
\[
\epsilon_{12} = \epsilon_1 + \epsilon_2 + \delta_1 \delta_2 + a_2^T U a_1
\]
\[
a_{12} = a_1 + a_2,
\]
\[
U = \begin{bmatrix} 0_n & I_n \\ 0_n & 0_n \end{bmatrix},
\]
where multiplication and addition of binary variables is modulo 2.

These formulas can easily be verified for \( n = 1 \) and then generalized for \( n > 1 \). The term \( a_k^T U a_1 \) “counts” (modulo 2) the number of positions \( k \) where \( w_{1k} = 1 \) and \( v_{2k} = 1 \) (with \( a_1 = \begin{bmatrix} v_1 \\ w_1 \end{bmatrix} \) and \( a_2 = \begin{bmatrix} v_2 \\ w_2 \end{bmatrix} \)), as only these positions get a minus sign in the following derivation:
\[
\tau_{v_1,w_1} \tau_{v_2,w_2} = \sigma_x^{w_1 \times w_1} \sigma_y^{v_1 \times v_2} \sigma_z^{v_2 \times w_2} = (-1)^{w_1 \times v_1 + v_2 \times w_2} \sigma_x^{w_1 + w_2}.
\]

A Clifford group operation \( Q \), by definition, maps the Pauli group to itself under conjugation:
\[
Q \tau_a Q^\dagger = i^{\delta}(-1)^{\epsilon} \tau_b
\]
for some \( \delta, \epsilon, \tau \), function of \( a \).

Because \( Q \tau_a Q^\dagger = (Q \tau_a Q^\dagger)(Q \tau_a Q^\dagger) \), it is sufficient to know the image of a generating set of the Pauli group to know the image of all Pauli group elements and define \( Q \) (up to an overall phase). In binary terms it is sufficient to know the image of \( \tau_{b_k}, k = 1, \ldots, n \) where \( b_k, k = 1, \ldots, n \) form a basis of \( \mathbb{Z}_2^{2n} \).

For this purpose it is possible to work with Hermitian Pauli group elements only as the image of a Hermitian matrix under \( X \rightarrow QXQ^\dagger \) will again be Hermitian (and the images of Hermitian Pauli group elements are sufficient to derive the images of non Hermitian ones). In our binary language Hermitian Pauli group elements are described as
\[
i^{a^T U a} (-1)^{\tau_a}
\]
as \( a^T U a \) counts (modulo 2) the number of \( \tau_{11} \) in the tensor product \( \tau_a \). For \( \tau_{11} \) is the only non-Hermitian (actually skew Hermitian) of the four matrices and multiplication with \( i \) makes it Hermitian.

Now we take the standard basis of \( \mathbb{Z}_2^{2n} \), \( e_k, k = 1, \ldots, n \) where \( e_k \) is the \( k \)-th column of \( I_{2n} \), and consider the generating set of Hermitian operators \( \tau_{e_k} \). These correspond to single-qubit operations \( \sigma_x \) and \( \sigma_z \). We denote their images under \( X \rightarrow QXQ^\dagger \) by \( i^{d_k}(-1)^{h_k} \tau_{c_k} \) and assemble the vectors \( c_k \) in a matrix \( C \) (with columns \( e_k \)) and the scalars \( d_k \) and \( h_k \) in the vectors \( d \) and \( h \). As the images are Hermitian, \( d_k = c^T_k U c_k \) or \( d = \text{diag}(C^T U C) \) (with \( \text{diag}(X) \) the vector with the diagonal elements of \( X \)).

Now, given \( C \), \( d \) and \( h \), defining the Clifford operation \( Q \), the image \( i^{d_k}(-1)^{h_k} \tau_{c_k} \) of \( i^{d_1}(-1)^{h_1} \tau_{b_1} \) under \( X \rightarrow QXQ^\dagger \) can be found by multiplying those operators \( i^{d_k}(-1)^{h_k} \tau_{c_k} \) for which \( b_{1k} = 1 \). By repeated application of Lemma 1 this yields
\[
b_2 = C b_1
\]
\[
h_2 = h_1 + d^T b_1
\]
\[
\epsilon_2 = \epsilon_1 + h^T b_1 + b_1^T \text{low}(C^T U C + dd^T)b_1 + \delta_1 d^T b_1
\]
where \( \text{low}(X) \) is the strictly lower triangular part of \( X \). These formulas can be simplified by introducing the following notation
\[
C = \begin{bmatrix} C & 0 \\ d^T & 1 \end{bmatrix}
\]
\[
U = \begin{bmatrix} U & 0 \\ 0 & 1 \end{bmatrix}
\]
\[
h = \begin{bmatrix} h \\ 0 \end{bmatrix}
\]
\[
b = \begin{bmatrix} b \\ \delta \end{bmatrix}
\]
\[
\tau_{b_1} = \tau_1^{\tau_{b_1}}, \quad \tau_{b_2} = \tau_2^{\tau_{b_2}}
\]

We then get the following theorem

**Theorem 1** Given \( C \) and \( h \), defining the Clifford operation \( Q \) as above, the image under \( X \rightarrow QXQ^\dagger \) of \( (-1)^{x_1} \tau_{b_1} \) is \( (-1)^{x_2} \tau_{b_2} \) with
\[
b_2 = C b_1,
\]
\[
\epsilon_2 = \epsilon_1 + h^T b_1 + b_1^T \text{low}(C^T U C)b_1
\]

With this theorem we can also compose two Clifford operations using the binary language. To this end we have to find the images under the second operation of the images under the first operation of the standard basis vectors. This can be done using Theorem 2.

**Theorem 2** Given \( C_1 \), \( h_1 \), \( C_2 \) and \( h_2 \), defining two Clifford operations \( Q_1 \) and \( Q_2 \) as above, the product \( Q_{21} = Q_2 Q_1 \) is represented by \( C_{21} \) and \( h_{21} \) given by
\[
C_{21} = C_2 C_1
\]
\[
h_{21} = h_1 + C_1^T h_2 + \text{diag}(C_1^T \text{low}(C_2^T U C_2) C_1)
\]

The next question is which \( C \) and \( h \) or \( C \) and \( d \) and \( h \) can represent a Clifford operation. The answer is that \( C \) has to be a symplectic matrix (and \( d \) has to be equal to \( \text{diag}(C^T U C) \) as above). If we define \( P \) to be \( U + U^T \), we call a matrix symplectic if \( C^T PC = P \). One way to see that \( C \) has to be symplectic is through the connection of the symplectic inner product \( b^T P a \) with commutability of Pauli group elements:
\[
\tau_a \tau_b = (-1)^{b^T P a} \tau_b \tau_a
\]

Since the map \( X \rightarrow QXQ^\dagger \) preserves commutability, \( a \) and \( b \) have to represent commutate Pauli group elements (\( b^T P a = 0 \)) if and only if \( Ca \) and \( Cb \) represent commutate elements (\( b^T C^T PCa = 0 \)). This implies that \( C \) has to be symplectic.
That symplecticity is also sufficient was first implied by Theorem 1 of [2] (almost, as this result was set in the context of entanglement distillation where the signs ε play no significant role). The idea is to give a constructive way of realizing the Clifford operation Q by Ĉ and h. This can be done using only one and two-qubit operations, which makes the result also of practical use. In Sec. [IV] we give two such decompositions that are more transparent than the results of [2].

First, to conclude this section, we complete the binary group picture by a formula for the inverse of a Clifford group element, given in binary terms.

**Theorem 3** Given Ĉ₁ and h₁, defining a Clifford operation Q₁ as above, the inverse Q₂ = Q₁⁻¹ is represented by

\[
\begin{aligned}
\bar{C}_2 &= \bar{C}_1^{-1} = \begin{bmatrix} C_1^{-1} & \Pi \end{bmatrix} = \begin{bmatrix} PC_1^T P & 0 \\ d_1^T PC_1^T P & 1 \end{bmatrix} \\
h_2 &= C^{-T} h + \text{diag}(C^{-T} \text{diag}(C^T UC)C^{-1})
\end{aligned}
\]

These formulas can be verified using Theorem 2. Finally note that since the Clifford operations form a group and the matrices Ĉ are simply multiplied when composing Clifford group operations, the matrices Ĉ with C symplectic and d = diag(C^T UC) must form a group of (2n + 1) × (2n + 1) matrices that is isomorphic to the symplectic group of 2n × 2n matrices. This can be easily verified by showing that

\[
\text{diag}(C_1^T C_2^T UC_2 C_1) = C_1^{-T} \text{diag}(C_1^T UC_2) + \text{diag}(C_1^T UC_1)
\]

This follows from the fact that C^T UC + U is symmetric when C^T PC = P and x^T S x = x^T \text{diag}(S) when S is symmetric. In a similar way it can be proven that \text{diag}(C^{-T}UC^{-1}) = C^{-T} \text{diag}(C^T UC).

### III. SPECIAL CLIFFORD OPERATIONS IN THE BINARY PICTURE

In this section we consider a selected set of Clifford group operations and their representation in the binary picture of Sec. [I].

First, we consider the Pauli group operations Q = τₐ as Clifford operations. Note that a global phase cannot be represented as it does not affect the action X → QXQ⁻¹. To construct C and h we have to consider the images of the operators τₐ representing one-qubit operations σₓ and σᶻ. One can easily verify that τₐ is represented by

\[
\begin{aligned}
C &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
h &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\end{aligned}
\]

Second, note that Clifford operations acting on a subset \(α \subset \{1, \ldots, n\}\) consist of a symplectic matrix on the rows and columns with indices in \(α \cup (α + n)\), embedded in an identity matrix (that is, with ones on positions \(C_{k,k} = 1, k \notin α \cup (α + n)\) and \(C_{k,l} = 0\) if \(k \neq l\) and \(k, l \notin α \cup (α + n)\).) Also \(h_k = 0\) if \(k \notin α \cup (α + n)\).

Third, qubit permutations, are represented by

\[
\begin{aligned}
C &= \begin{bmatrix} \Pi & 0 \\ 0 & \Pi \end{bmatrix} \\
h &= \begin{bmatrix} 1 \end{bmatrix}
\end{aligned}
\]

where \(\Pi\) is a permutation matrix.

Fourth, the conditional not or CNOT operation on two qubits is represented by

\[
\begin{aligned}
C &= \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
h &= \begin{bmatrix} 0 \end{bmatrix}
\end{aligned}
\]

Fifth, by composing qubit permutations and CNOT operations on selected qubits any linear transformation of the index space \(|x⟩ → |Rx⟩\) can be realized, where \(x ∈ Z^2\) labels the standard basis states \(|x⟩ = |x_1⟩ \otimes \ldots \otimes |x_n⟩\) and \(R ∈ Z^{2n×n}\) is an invertible matrix (modulo 2). This operation is represented in the symplectic picture by

\[
\begin{aligned}
C &= \begin{bmatrix} R^T & 0 \\ 0 & R \end{bmatrix} \\
h &= \begin{bmatrix} 0 \end{bmatrix}
\end{aligned}
\]

The qubit permutations and CNOT operation discussed above are special cases of such operations as qubit permutations can be represented as \(|x⟩ → |Πx⟩\) and the CNOT operation as \(|x⟩ → |1 \ 0 \ 1 \ 1⟩ x⟩\).

Decomposing a general linear transformation \(R\) into CNOTS and qubit permutations can be done by Gauss elimination (a well known technique for the solution of systems of linear equations). In this process \(R\) is operated on the left by CNOTS and qubit permutations to be gradually transformed in an identity matrix. The process operates on \(R\), column by column, first moving a nonzero element into the diagonal position by a qubit permutation, then zeroing the rest of the column by CNOTS. The inverses of the applied operations yield a decomposition of \(R\).

Sixth, we consider Hadamard operations. The Hadamard operation on a single qubit \(Q = H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\) is represented by \(C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\) and \(h = 0\). A Hadamard operation on a selected set of qubits is represented by the embedding of such matrices in an identity matrix as explained above. As a special case we mention the Hadamard operation on all qubits, which is represented by \(C = P\) and \(h = 0\).

Seventh, we consider operations \(e^{i(\pi/4)τₐ} = \frac{1}{\sqrt{2}} (I + iτₐ)\) where \(a ∈ Z^2\) and \(a = \begin{bmatrix} a & a^TUa \end{bmatrix}\), and \(τₐ = iσ^T U a τₐ\). These operations are represented by

\[
\begin{aligned}
C &= I + aa^T P \\
h &= C^T U a
\end{aligned}
\]
This is proved in the Appendix.

Finally, we mention that real Clifford operations have \( d = 0 \).

IV. DECOMPOSITIONS OF CLIFFORD OPERATIONS IN ONE AND TWO-QUBIT OPERATIONS

In this section we write general Clifford group operations as products of one and two-qubit operations using the binary picture. This does not only complete the results of Sec. II showing that every symplectic \( C \) and arbitrary \( h \) represent a Clifford operation. It is also of practical use for quantum computing applications as well as entanglement distillation applications since two-qubit operations can be realized relatively easily and the number of two-qubit operations needed is “only” quadratical in the number of qubits. We give two different schemes.

First, for both schemes, we observe that the main problem is realizing \( C \), not \( h \). For once a Clifford operation represented by \( C \) and \( h' \) is realized, we can realize \( h \) by doing an extra operation \( Q = \tau_C P (h + h') \) on the left or \( Q = \tau_D Q (h + h') \) on the right. This can be proved by using Eq. (2) and Theorem 2.

The first scheme realizes \( C \) by two-qubit operations, acting on qubit \( k \) and \( l \) of the type \( e^{i(\pi/4)\tau_0} \) with symplectic matrices \( (I + aa^T) \) where \( a \) can be nonzero (i.e. one) only at positions \( k, l, n + k \) and \( n + l \). The scheme works by reducing a given symplectic matrix \( C \) to the identity matrix by operating on the left with two-qubit operations. The product of the inverses of these two-qubit matrices is then equal to \( C \). The reduction to the identity matrix is done by working on two columns \( m \) and \( n + m \) at a time, for \( m = 1, \ldots, n \). First columns \( 1 \) and \( n + 1 \) are reduced to columns \( 1 \) and \( n + 1 \) of the identity matrix. Because through all the operations \( C \) remains symplectic, one can show that as a result also rows \( 1 \) and \( n + 1 \) are reduced to rows \( 1 \) and \( n + 1 \) of the identity matrix. Then one can repeat the same process on the submatrix of \( C \) obtained by dropping rows and columns \( 1 \) and \( n + 1 \), until the whole matrix is reduced to the identity matrix.

Let \( \alpha = \{1, 1 + n\} \) and \( \beta = \{l, l + n\} \). The first step in reducing columns \( 1 \) and \( n + 1 \) of \( C \) to the corresponding columns of the identity matrix is a qubit permutation, exchanging qubit \( 1 \) with some qubit \( k \) to make \( C_{\alpha, \alpha} \) invertible. This can be done for if all \( C_{\beta, \alpha} \) would be rank deficient, we would have \( c_{\gamma}^T P c_{\gamma+1} = 0 \) which is in conflict with the symplecticity of \( C \). (Note that a 2 \( \times \) 2-matrix is invertible if and only if it is symplectic). Next, we perform two-qubit operations \( e^{i(\pi/4)\tau_0} \) on qubits \( 1 \) and \( l \) with \( a_\alpha = c_{\alpha, n + 1} \) and \( a_\beta = c_{\beta, n + 1} \), for \( l = 2, \ldots, n \). Such an operation changes \( C \) through multiplication with \( I + aa^T P \). For the first column this means that \( c_1 \) is replaced by \( c_1 + a_1 \) as \( a^T P c_1 = c_{\alpha, n + 1} T_2 c_{\alpha, 1} + c_{\beta, n + 1} c_{\beta, 1} = 1 + 0 = 1 \), where \( P_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). This way \( c_{\beta, 1} \) is reduced to 0. \( C_{\alpha, \alpha} \) is changed at every step but remains invertible (and symplectic). Note that through these operations also the other columns of \( C \) are changed. After the first column has been zeroed on all positions except \( \alpha \), we tackle column \( n + 1 \) with operations \( e^{i(\pi/4)\tau_0} \) on qubits 1 and \( l \) with \( a_\alpha = c_{\alpha, 1} \) and \( a_\beta = c_{\beta, n + 1} \), \( l = 2, \ldots, n \). These operations have no effect on \( c_1 \) because \( a^T P c_1 = e_{\alpha, 1} T_2 c_{\alpha, 1} + 0 = 0 \), and reduce \( c_{\beta, n + 1} \) to 0 in the same way as was done for the first column. After these operations we are left with \( c_1 \) and \( c_{n + 1} \) all 0 except for \( C_{\alpha, \alpha} \) which equals an invertible matrix. This matrix can be transformed into an identity matrix by a one-qubit symplectic operation on qubit 1. One-qubit Clifford operations can be easily made by one-qubit operations of type \( e^{i(\pi/4)\tau_0} \).

An advantage of this scheme is that it is efficient if only some columns of \( C \) (or rows, as one can also work on the right) are specified while the other columns do not matter. This is the case in the entanglement distillation protocols of \( \text{I} \).

The second scheme also takes a number of steps that is quadratic in \( n \). It is based on the following theorem, which will also be of importance in Sec. II, and for which we give a constructive proof.

**Theorem 4** If \( C \in \mathbb{Z}_{2}^{n \times n} \) is a symplectic matrix \( (C^T P C = P) \), it can be decomposed as

\[
C = \left[ \begin{array}{cc} T_1^{T} & 0 \\ 0 & T_1 \end{array} \right] \times \left[ \begin{array}{cccc} 0 & 0 & 0 & 0 \\ I_{n-r} & V_1 & Z_3 + V_1 V_1^T & V_2 + V_1 Z_2 \\ 0 & Z_1 & V_1^T + Z_1 V_1^T & V_2 + Z_1 Z_2 \\ 0 & 0 & I_{n-r} & 0 \\ 0 & I_r & V_2^T & Z_2 \end{array} \right] \left[ \begin{array}{c} T_2^{-T} 0 \\ 0 & T_2 \end{array} \right]
\]

\[
T_1 = \left[ \begin{array}{cc} T_1^{T} & 0 \\ 0 & T_1 \end{array} \right] \times \left[ \begin{array}{cccc} 0 & 0 & 0 & 0 \\ I_r & V_1 & Z_1 & V_2 \\ 0 & I_r & V_2 & Z_2 \\ 0 & 0 & I_{n-r} & 0 \\ 0 & 0 & I_{n-r} & 0 \\ 0 & 0 & I_{n-r} & 0 \\ 0 & 0 & I_{n-r} & 0 \\ 0 & 0 & I_r & 0 \end{array} \right] \times \left[ \begin{array}{c} T_2^{-T} 0 \\ 0 & T_2 \end{array} \right]
\]

where \( T_1 \) and \( T_2 \) are invertible \( n \times n \) matrices, \( Z_1 \) and \( Z_2 \) are symmetric \( r \times r \) matrices, \( Z_3 \) is a symmetric \((n-r) \times (n-r)\) matrix, \( V_1 \) and \( V_2 \) are \((n-r) \times r \) matrices and the zero blocks have appropriate dimensions.

**Proof:** To prove this theorem we consider \( C \) as a block matrix \( C = \begin{bmatrix} E' & F' \\ G' & H' \end{bmatrix} \).

Then, we find invertible \( R_1 \) and \( R_2 \) in \( \mathbb{Z}_{2}^{n \times n} \) such that

\[
R_1^{-1} G' R_2 = \begin{bmatrix} 0 & 0 \\ 0 & I_r \end{bmatrix},
\]

where \( r \) is the rank of \( G' \). This is a standard linear algebra technique and can be realized (for example) by (1) setting the first \( n - r \) columns of...
$R_2$ equal to a basis of the kernel of $G'$, (2) choosing the other columns of $R_2$ as to make it invertible, (3) setting the last $r$ columns of $R_1$ equal to the last $r$ columns of $R_0$ multiplied on the left by $G'$ (This yields a basis of the range of $G'$), and (4) choosing the other columns of $R_1$ as to make it invertible. By construction, this implies

$$G'R_2 = R_1 \begin{bmatrix} 0 & 0 & \vdots & 0 \\ 0 & I_r & \end{bmatrix}.$$  

Now we set

$$
\begin{bmatrix}
R_1^T & 0 & 0 & 0 \\
0 & R_1^{-1} & 0 & 0 \\
E_{11} & E_{12} & F_{11} & F_{12} \\
E_{21} & E_{22} & F_{21} & F_{22} \\
0 & 0 & H_{11} & H_{12} \\
0 & 0 & H_{21} & H_{12}
\end{bmatrix}
C
\begin{bmatrix}
R_2 & 0 & 0 & 0 \\
0 & R_2^{-T} & 0 & 0 \\
E_{11} & E_{12} & F_{11} & F_{12} \\
E_{21} & E_{22} & F_{21} & F_{22} \\
0 & 0 & H_{11} & H_{12} \\
0 & 0 & H_{21} & H_{12}
\end{bmatrix}
$$  

(7)

Because the three matrices in the left-hand side of Eq. (7) are symplectic, so is the right-hand side. This leads to the following relations between its submatrices:

$$
E_{11}^T H_{11} + E_{12}^T H_{21} = I
$$  

(8)

$$
E_{11}^T H_{12} + E_{12}^T H_{22} = 0
$$  

(9)

$$
E_{12}^T + E_{22} = 0
$$  

(10)

$$
E_{12}^T H_{11} + E_{22}^T H_{21} + F_{21} = 0
$$  

(11)

$$
E_{12}^T H_{12} + E_{22}^T H_{22} + F_{22} = I
$$  

(12)

$$
F_{11}^T H_{12} + F_{21}^T H_{21} + H_{11}^T F_{12} + H_{21}^T F_{22} = 0
$$  

(13)

$$
F_{11}^T H_{12} + F_{21}^T H_{22} + H_{12}^T F_{12} + H_{22}^T F_{22} = 0
$$  

(14)

With Eq. (8) and Eq. (9) we find $H_{11} = E_{11}^T$. Now, if we replace $R_2$ by $R_2 \begin{bmatrix} E_{11}^{-1} & 0 & 0 & 0 \\
0 & 0 & I_r & \end{bmatrix}$, both $H_{11}$ and $E_{11}$ are replaced by $I_{n-r}$. We will assume that this choice of $R_2$ was taken from the start. Then, from Eq. (8) and Eq. (10) we find $H_{12} = 0$. From Eq. (11) we learn that $E_{22}$ is symmetric. From Eq. (12) and Eq. (13) we find $F_{21} = E_{12}^T + E_{22}^T H_{21}$ and $F_{22} = I + E_{22} H_{22}$. Substituting these equations in Eqs. (14), (15), and (16), we find that $F_{11} + B_{21}^T B_{21}$ is symmetric, $F_{12} = E_{21}^T + E_{22}^T H_{22}$, and $H_{22}$ is symmetric. Setting $T_1 = R_1$, $T_2 = R_2^T$ (with $R_2$ chosen as to make $E_{11} = H_{11} = I$), $V_1 = E_{12}$, $V_2 = H_{21}$, $Z_1 = E_{22}$, $Z_2 = H_{22}$ and $Z_3 = F_{11} + V_1 V_2^T$, we obtain Eq. (6). Note that $Z_3$ is symmetric because $F_{11} + V_1 V_2^P$ and $V_2 V_1 + V_1 V_2^T$ are symmetric. Finally Eq. (6) can be easily verified. This completes the proof. $\square$

To find a decomposition of $C$ in one and two-qubit operations we concentrate on the five matrices in the right-hand side of Eq. (6), all of which are symplectic. Clearly the first and last matrix are linear index space transformations as discussed in Sec. III. These can be decomposed into CNOTs and qubit permutations. The middle matrix corresponds to Hadamard operations on the last $r$ qubits. We will now show that the second and fourth matrix can be realized by one and two-qubit operations of the type $e^{i(\pi/4)q_a}$. First note that both matrices are of the form $\begin{bmatrix} I & Z \\
0 & I \end{bmatrix}$ with $Z$ symmetric. These matrices form a commutative subgroup of the symplectic matrices with

$$
\begin{bmatrix} I & Z_a \\
0 & I \end{bmatrix} \begin{bmatrix} I & Z_b \\
0 & I \end{bmatrix} = \begin{bmatrix} I & Z_a + Z_b \\
0 & I \end{bmatrix}.
$$

Now, we realize $\begin{bmatrix} I & Z \\
0 & I \end{bmatrix}$ with one and two-qubit operations by first realizing the ones on off-diagonal positions in $Z$ and then realizing the diagonal. Entries $Z_{k,l} = Z_{l,k} = 1$ are realized by operations $e^{i(\pi/4)q_a}$ with $a_k = a_l = 1$ and $a_m = 0$ if $m \neq k$ and $m \neq l$. These are two-qubit operations which realize the off-diagonal part of $Z$ and as a by-product produce some diagonal. Now this diagonal can be replaced by the diagonal of $Z$ by one-qubit operations $e^{i(\pi/4)q_a}$ with $a_k = 1$ and $a_m = 0$ if $m \neq k$, which affect only the diagonal entries $Z_{k,k}$. This completes the construction of $C$ by means of one and two-qubit operations.

V. DESCRIPTION OF STABILIZER STATES AND CLIFFORD OPERATIONS USING BINARY QUADRATIC FORMS

In this section we use our binary language to get further results on stabilizer states and Clifford operations. First, we take the binary picture of stabilizer states and their stabilizers and show how Clifford operations act on stabilizer states in the binary picture. We also discuss the binary equivalent of replacing one set of generators of a stabilizer by another. Then we move to two seemingly unrelated results. One is the expansion of a stabilizer state in the standard basis, describing the coefficients with binary quadratic forms. The other is a similar description of the entries of the unitary matrix of a Clifford operation with respect to the same standard basis.

A stabilizer state $|\psi\rangle$ is the simultaneous eigenvector, with eigenvalues 1, of $n$ commutable Hermitian Pauli group elements $i^k (-1)^{b_k} \tau_{s_k}$, $k = 1, \ldots, n$, where $s_k \in \mathbb{Z}_2^n$, $k = 1, \ldots, n$ are linearly independent, $f_k, b_k \in \mathbb{Z}_2$ and $f_k = s_k^T U s_k$. The $n$ Hermitian Pauli group elements generate a commutative subgroup of the Pauli group, called the stabilizer $S$ of the state. We will assemble the vectors $s_k$ as the columns of a matrix $S \in \mathbb{Z}_2^{2n \times n}$ and the scalars $f_k$ and $b_k$ in vectors $f$ and $b \in \mathbb{Z}_2^n$. This binary representation of stabilizer states is common in the literature of stabilizer codes $\mathcal{G}$. The fact that the Pauli group elements are commutable is reflected by $S^T P S = 0$. One can think of $S$, $f^T$ and $b^T$ as the “left half” of $C$, $d^T$ and $h^T$ of Sec. III in the style of that section we also define \begin{align*}
\tilde{S} &= \begin{bmatrix} S \\
f^T \end{bmatrix}.
\end{align*}

If $|\psi\rangle$ is operated on by a Clifford operation $Q$, $Q|\psi\rangle$ is a new stabilizer state whose stabilizer is given by $Q S Q^T$. As a result, the new set of generators, represented by $\tilde{S}'$
and $b'$ can be found by acting with $\tilde{C}$ and $h$, representing $Q$, as in Theorem 1 and Theorem 2. One finds

$$\tilde{S}' = \tilde{C} \tilde{S}, \quad b' = b + S^T h + \text{diag}(\tilde{S}^T \text{low}(\tilde{C}^T U \tilde{C}) \tilde{S})$$

The representation of $\mathcal{S}$ by $\tilde{S}$ and $b$ is not unique as they only represent one set of generators of $\mathcal{S}$. In the binary language a change from one set of generators to another is represented by an invertible linear transformation $R$ acting on the right on $\tilde{S}$ and acting appropriately on $b$. By repeated application of Lemma 1 one finds that $\tilde{S}$ and $b$ can be transformed as

$$\tilde{S}' = \tilde{S} R, \quad b' = R b + \text{diag}(R \text{low}(\tilde{S}^T U \tilde{S}) R)$$

Below we will refer to such a transformation as a stabilizer basis change.

Before we state the main results of this section, we show how binary linear algebra can also be used to describe the action of a Pauli matrix on a state, expanded in the standard basis.

$$\tau_a \sum_{x \in \mathbb{Z}_2^n} \psi_x |x\rangle = \sum_{x \in \mathbb{Z}_2^n} (-1)^{\tau_a x} \psi_{x+w} |x\rangle \tag{17}$$

where $a = \begin{bmatrix} v \\ w \end{bmatrix}$. This is proved as follows. From $\sigma_x |b\rangle = |b+1\rangle$ with $b \in \mathbb{Z}_2$, we have $\tau \begin{bmatrix} v \\ w \end{bmatrix} \sum_x \psi_x |x\rangle = \sum_x \psi_x |x+w\rangle$. From $\sigma_x |b\rangle = (-1)^{b}|b\rangle$, we then find Eq. (17).

Now we explore our binary language to get results about the expansion in the standard basis of a stabilizer state as summarized in the following theorem, for which we give a constructive proof.

**Theorem 5** (i) If $\tilde{S}$ and $b$ represent a stabilizer state $|\psi\rangle$ as described above, $\tilde{S}$ and $b$ can be transformed by an invertible index space transformation $|x\rangle \rightarrow |T^{-1}x\rangle$ with $T \in \mathbb{Z}_2^{nxn}$ and an invertible stabilizer basis change $R \in \mathbb{Z}_2^{nxn}$ into the form

$$\tilde{S}' = \begin{bmatrix} T^T & 0 & 0 \\ 0 & T^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tilde{S} R = \begin{bmatrix} Z & 0 & 0 \\ 0 & 0 & I_{r_a} \\ 0 & I_{r_b} & 0 \\ f_a^T & 0 & 0 \end{bmatrix} \tag{18}$$

$$b' = \begin{bmatrix} b_{ab} \\ b_c \end{bmatrix}$$

where $Z$ is full rank and symmetric and $f_a = \text{diag}(Z)$. (ii) The state $|\psi\rangle$ can be expanded in the standard basis as

$$|\psi\rangle = (1/\sqrt{2^{(r_a+r_b)}}) \sum_{y \in \mathbb{Z}_2^{(r_a+r_b)}} (-1)^{f_a^T y_a} (-1)^{(y_b^T \text{low}(Z+f_a f_a^T) + b_{ab} y)} |T \begin{bmatrix} y \\ b \end{bmatrix}\rangle$$

where $y = \begin{bmatrix} y_a \\ y_b \end{bmatrix}$ with $y_a \in \mathbb{Z}_2^{r_a}$ and $y_b \in \mathbb{Z}_2^{r_b}$.

In words this theorem reads as follows. If the coefficients of a stabilizer state $|\psi\rangle$, with respect to the standard basis $\{|x\rangle | x \in \mathbb{Z}_2^n\}$, are considered as a function of the binary basis label $x$, this function is nonzero in an $r_a + r_b$ dimensional plane (a coset of a subspace of $\mathbb{Z}_2^n$) and the nonzero elements are (up to a global scaling factor) equal to $1, i, -1$ or $-i$, where the signs are given by a binary quadratic function over the plane and $i$'s appear either in a subspace of codimension one or nowhere (if $f_a = 0$).

**Proof:** We first write $S$ as a block matrix

$$S = \begin{bmatrix} V \\ W \end{bmatrix}$$

with $V, W \in \mathbb{Z}_2^{n \times n}$. Then we perform a first stabilizer basis change $R_1$, transforming $W$ to $W^{(1)} = W R_1 = \begin{bmatrix} W^{(1)} & 0 \\ 0 & 0 \end{bmatrix}$, where $W^{(1)} \in \mathbb{Z}_2^{n \times (r_a+r_b)}$ and $r_a + r_b = \text{rank}(W)$. This is achieved by setting the last columns of $R_1$ equal to a basis of the kernel of $W$ and choosing the other columns as to make it invertible. As a result the columns of $W^{(1)}_{ab}$ are a basis of the range of $W$. We also write the transformation of $V$ in block form as $V^{(1)} = VR_1 = [V^{(1)}_{ab} V^{(1)}_c]$. Because $S^{(1)}$ is full rank, $V^{(1)}_c$ must also be full rank.

Now we perform a second stabilizer basis change $R_2 = \begin{bmatrix} R^{ab,ab} & 0 \\ R^{c,ab} & I_{rc} \end{bmatrix}$, transforming $V^{(1)} = [V^{(1)}_{ab} V^{(1)}_c]$ to $V^{(2)} = V^{(1)} R_2 = [V^{(2)}_{ab} V^{(2)}_c]$, where $V^{(2)}_{ab} \in \mathbb{Z}_2^{n \times r_a}$ and $r_a + r_c = \text{rank}(V)$. This is achieved by setting the columns $r_a + 1$ till $r_a + r_b$ of $R_2$ equal to a basis of the kernel of $V^{(1)}$ and choosing the first $r_a$ columns as to make it invertible. (Note that the last $r_c$ columns of $R_2$ are equal to the corresponding columns of the identity matrix and no linear combination of them can be in the kernel of $V^{(1)}$ as $V^{(1)}_c$ is full rank). As a result the columns of $[V^{(2)}_{ab} V^{(2)}_c]$ are a basis of the range of $V$. We also write the transformation of $W^{(1)}$ in block form as $W^{(2)} = W^{(1)} R_2 = [W^{(2)}_{ab} W^{(2)}_c] = 0$.

Next we perform an index space transformation $|x\rangle \rightarrow |T^{-1}x\rangle$ with $T = [W^{(2)}_{ab} W^{(2)}_c]$, where the columns $W^{(2)}_c$ are chosen as to make $T$ invertible. As a result $V^{(2)}$ is transformed to $V^{(3)} = T^T V^{(2)} = [V^{(3)}_{ab} V^{(3)}_c]$, $W^{(2)}$ is transformed to $W^{(3)} = T^{-1} W^{(2)} = [I_{r_a+r_b} 0 \\ 0 0]$. Because $S^{(3)} = \begin{bmatrix} V^{(3)} \\ W^{(3)} \end{bmatrix}$ satisfies $S^{(3)}T P S^{(3)} = 0$, one also finds $V^{(3)} = \begin{bmatrix} Z & 0 \\ 0 & 0 \\ V^{(3)}_{ab} & 0 \\ 0 & V^{(3)}_{cc} \end{bmatrix}$ where $Z$ is symmetric and $V^{(3)}_{cc}$ is full rank. A final stabilizer basis change $R_3 = \begin{bmatrix} I_{r_a} & 0 \\ 0 & I_{r_b} \\ V^{(3)}_{ab} & 0 \\ V^{(3)}_{cc} \end{bmatrix}$ transforms $V^{(3)}$ to
\[ V' = V^{(3)}R_3 = \begin{bmatrix} Z & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_c \end{bmatrix} \] and leaves \( W^{(3)} = W' \) unchanged. Through all the transformations we also have to keep track of \( f \) and \( b' \). We find \( f' = \text{diag}(S'U'S') = \begin{bmatrix} \text{diag}(Z) \\ 0 \end{bmatrix} \). Setting \( R = R_1R_2R_3 \) we find \( b' = R^Tb + \text{diag}(R^T\text{low}(V'TW + dd'R))R \).

We still have to prove that \( Z \) is full rank. First note that \( Z = W^{(2)}_b = V^{(2)} \). From \( S'^T = PS^{(2)} = 0 \), and the fact that \( [V_a^{(2)} V_c^{(2)}] \) and \( [W_b^{(2)} W_b^{(2)}] \) are full rank, it follows that the columns of \( W^{(2)}_b \) span the orthogonal complement of \( [V_a^{(2)} V_c^{(2)}] \) and the columns of \( V^{(2)} \) span the orthogonal complement of \( [W_b^{(2)} W_b^{(2)}] \). Assume now that there exists some \( x \in \mathbb{Z}_2^c \) with \( x \neq 0 \) and \( Zx = 0 \), then \( V^{(2)}_b \) is orthogonal to the columns of \( W^{(2)}_b \). And \( V^{(2)}_c \) is also orthogonal to the columns of \( W^{(2)}_b \). Therefore \( V^{(2)}_a \) is a linear combination of the columns of \( V^{(2)}_b \). This is in contradiction with the fact that \( [V_a^{(2)} V_c^{(2)}] \) is full rank. Therefore, \( Z \) is full rank. This completes the proof of part (i).

To prove part (ii), first observe that applying \( |x\rangle \rightarrow |T^{-1}x\rangle \) to \( |\psi\rangle \) simply replaces \( |T\begin{bmatrix} y \\ b \end{bmatrix}\rangle \) by \( |\begin{bmatrix} y \\ b \end{bmatrix}\rangle \), and stabilizer basis transformations only change the description of a stabilizer state but not the state itself. Therefore, we have to prove that

\[ |\phi\rangle = \sum_{y \in \mathbb{Z}_2^n} \psi(y) |\begin{bmatrix} y \\ b \end{bmatrix}\rangle \]

is an eigenvector with eigenvalue one of the operators \( i f_k^{(r)}(-1)^{k} \tau_k^{t} \) described by \( S' \) and \( b' \). For \( k = 1, \ldots, r_a \), we have

\[ s'_k = \begin{bmatrix} Zc_k \\ 0 \\ e_k \end{bmatrix}, \quad f'_k = f_{ak} = \begin{bmatrix} x_k,k \\ 0 \\ 0 \end{bmatrix}, \quad b'_k = b_{abk} \]

where \( e_k \) is the \( k \)-th column of \( I_{r_a} \). With Eq. (17) we find

\[ i f_k^{(r)}(-1)^{k} \tau_k^{t} |\phi\rangle = \sum_{y \in \mathbb{Z}_2^n} \psi(y) |\begin{bmatrix} y \\ b \end{bmatrix}\rangle \]

\[ = \sum_{y \in \mathbb{Z}_2^n} \psi(y) f_{ak}(-1)(Zc_k)^{t}yu_{-i}(-1)^{k}f_{k}^{(r)}(yu_{+e_k}) \times
\]

\[ (-1)^{k}(yu_{+e_k})^{t}\text{low}(Z + f_{a}f_{k}^{(r)}(yu_{+e_k}) + b_{abk}(yu_{+e_k}) + b_{abk}(yu_{+e_k})) \times
\]

\[ |\begin{bmatrix} y \\ b \end{bmatrix}\rangle \]

\[ = \sum_{y \in \mathbb{Z}_2^n} \psi(y) f_{ak}(-1)(Zc_k)^{t}yu_{-i}(-1)^{k}f_{k}^{(r)}(yu_{+e_k}) \times
\]

\[ (-1)^{k}(yu_{+e_k})^{t}\text{low}(Z + f_{a}f_{k}^{(r)}(yu_{+e_k}) + b_{abk}(yu_{+e_k}) + b_{abk}(yu_{+e_k})) \times
\]

\[ (-1)(c_{k}^{t}(Z + f_{a}f_{k}^{(r)}(yu_{+e_k}) + b_{abk}(yu_{+e_k}) + b_{abk}(yu_{+e_k})) |\begin{bmatrix} y \\ b \end{bmatrix}\rangle \]

\[ = |\phi\rangle \]

For \( k = r_a + 1, \ldots, r_b \) we have

\[ s'_k = \begin{bmatrix} 0 \\ 0 \\ e_k \end{bmatrix}, \quad f'_k = 0, \quad b'_k = b_{abk} \]

where now \( e_k \) is the \( k \)-th column of \( I_{(r_a+r_b)} \). With Eq. (17) we find

\[ i f_k^{(r)}(-1)^{k} \tau_k^{t} |\phi\rangle \]

\[ = \sum_{y \in \mathbb{Z}_2^n} \psi(y) f_{ak}(-1)(Zc_k)^{t}yu_{-i}(-1)^{k}f_{k}^{(r)}(yu_{+e_k}) \times
\]

\[ (-1)^{k}(yu_{+e_k})^{t}\text{low}(Z + f_{a}f_{k}^{(r)}(yu_{+e_k}) + b_{abk}(yu_{+e_k}) + b_{abk}(yu_{+e_k})) |\begin{bmatrix} y \\ b \end{bmatrix}\rangle \]

\[ = |\phi\rangle \]

For \( k = r_b + 1, \ldots, n \), we find with Eq. (17) that

\[ i f_k^{(r)}(-1)^{k} \tau_k^{t} |x\rangle = (-1)^{x_k+k} |x\rangle. \]

The state |\phi\rangle is clearly an eigenstate of this operator as \( x_k+b'_k = 0 \) for all states \( |x\rangle = |\begin{bmatrix} y \\ b \end{bmatrix}\rangle \) and \( k = r_b + 1, \ldots, n \). This completes the proof.

Finally, we show how also the entries of a Clifford matrix can be described with binary quadratic forms, by using Theorem 4. This leads to the following theorem for which we give a constructive proof.

**Theorem 6**

Given a Clifford operation \( Q \), represented by \( C \) and \( h \) (or \( C,d \) and \( h \)) as in Sec. 17, \( Q \) can be written as

\[ Q = \frac{1}{\sqrt{2^n}} \sum_{x_{br} \in \mathbb{Z}_2^{n-r}} \sum_{x_{bc} \in \mathbb{Z}_2^{n-r}} \psi(x_{bc}) |T_1^{x_{bc}}T_2^{x_{bc}+t}⟩ \]

where \( x_{br} = \begin{bmatrix} x_b \\ x_r \end{bmatrix} \) and \( x_{bc} = \begin{bmatrix} x_b \\ x_{c} \end{bmatrix} \), \( T_1, T_2 \in \mathbb{Z}_2^{n-r} \) are invertible matrices, \( Z_{br}, Z_{bc} \in \mathbb{Z}_2^{n-r} \) are symmetric, \( d_{br} = \text{diag}(Z_{br}) \), \( d_{bc} = \text{diag}(Z_{bc}) \) and \( h_{bc}, t \in \mathbb{Z}_2^2 \).

**Proof:** The proof is based on the decomposition of \( C \) as a product of five matrices as in Theorem 4. Due to the isomorphism between the group of symplectic matrices \( C \) and the extended matrices \( C \) as defined in Sec. 11 this decomposition can be converted into a decomposition of \( C \) as follows.

\[ \bar{C} = \bar{C}^{(1)} \bar{C}^{(2)} \bar{C}^{(3)} \bar{C}^{(4)} \bar{C}^{(5)} \]

\[ = \begin{bmatrix} I_n & Z_{bc} & 0 \\ 0 & I_{n-r} & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

\[ = \begin{bmatrix} T_2^1 & 0 & 0 \\ 0 & T_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

\[ = \begin{bmatrix} I_{n-r} & 0 & 0 \\ 0 & I_r & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

\[ = \begin{bmatrix} I_{n} & 0 & 0 \\ 0 & I_{n-r} & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

\[ = \begin{bmatrix} I_{n} & 0 & 0 \\ 0 & I_{n-r} & 0 \\ 0 & 0 & 1 \end{bmatrix} \]
\[ Z_{br} = \begin{bmatrix} Z_3 & V_1 \\ V_1^T & Z_1 \end{bmatrix}, \quad Z_{bc} = \begin{bmatrix} 0 & V_2 \\ V_2^T & Z_2 \end{bmatrix}, \quad d_{br} = \text{diag}(Z_{br}) \quad \text{and} \quad d_{bc} = \text{diag}(Z_{bc}). \]

If we define Clifford operations \( Q^{(k)} \) and \( h^{(k)} = 0 \), \( k = 1, \ldots, 5 \), the operation \( Q^{(k)} Q^{(2)} Q^{(3)} Q^{(4)} \) is represented by \( C \) and some vector \( h' \), that can be found by repeated application of Theorem 2. The vector \( h \) of the given Clifford operation \( Q \) can then be realized by an extra operation \( Q^{(k)} \) to the right with \( C^{(0)} = I \) and \( h^{(0)} = h + h' \). Now, \( Q^{(3)} \) is a Hadamard operation on the last \( r \) qubits. Because a Hadamard operation on one qubit can be written as \( H_1 = (1 / \sqrt{2}) \sum_{\{b, b'\} \in \mathbb{Z}_2} (-1)^{b b'} | b \rangle \langle b' | \), the Hadamard operation on \( r \) qubits can be written as \( H_r (1 / \sqrt{2^r}) \sum_{x_r, x_r \in \mathbb{Z}_2^r} (-1)^{x_r^T x_r} | x_r \rangle \langle x_r | \) and, including the \( n - r \) qubits that are not operated on, as

\[ Q^{(3)} = (1 / \sqrt{2^r}) \sum_{x_r, x_r \in \mathbb{Z}_2^r} \sum_{x_r, x_r \in \mathbb{Z}_2^r} (-1)^{x_r^T x_r} | x_r \rangle \langle x_r |. \]  

Considered as a matrix this is a block diagonal matrix with \( 2^n \times 2^n \) identical \( 2^r \times 2^r \) blocks with entries that are 1 or -1. The index \( x_r \) addresses the blocks and the indices \( x_r \) and \( x_r \) address the columns and rows inside the blocks. Now we will show that the matrix \( Q \) can be derived from this matrix by multiplying on the left and the right with a diagonal matrix and a permutation matrix representing an affine index space transformation. First we concentrate on \( Q^{(2)} \) and \( Q^{(4)} \). \( C^{(2)} \) and \( C^{(4)} \) have the form

\[
\tilde{C} = \begin{bmatrix} I & \tilde{Z} & 0 \\ 0 & I & 0 \\ 0 & 0 & d \end{bmatrix}. \]

We show that such a matrix (together with \( \tilde{h} = 0 \)) represents a diagonal Clifford operation

\[ \tilde{Q} = \sum_{x \in \mathbb{Z}_2^n} (-i)^{x^T \tilde{Z} x} | x \rangle \langle x |. \]  

This result can be derived using the decomposition in (diagonal) one and two-qubit operations given in Sec. IV, but can more easily be proved by showing that the Pauli group elements \( \tau_{ek} \), with \( e_k \) the \( k \)-th column of \( I_{2n} \), are mapped to operators represented by the columns of \( \tilde{C} \) under \( X \rightarrow Q X Q^\dagger \). Clearly, for \( k = 1, \ldots, n \), \( Q \tau_{ek} \tilde{Q}^\dagger = \tau_{ek} \tilde{Q} \tilde{Q}^\dagger = \tau_{ek} \) (as \( Q \) and \( \tilde{Q} \) are diagonal). For \( k = n + 1, \ldots, 2n \), let \( e_k \) again be the \( k \)-th column of \( I_{2n} \) and \( \tilde{e}_k \) the \( k \)-th column of \( I_n \). Then we have

\[
\tilde{Q} \tau_{ek} \tilde{Q}^\dagger \tau_{ek} = \sum_{x} \left[ (-i)^{x^T \tilde{Z} x} (-1)^{x^T \tilde{Z} x} | x \rangle \langle x | \right] \times \sum_{x} \left[ (-i)^{x^T \tilde{Z} x} | x \rangle \langle x | \right] \times \sum_{x} \left[ (i)^{x^T \tilde{Z} x} | x \rangle \langle x | \right] \times \sum_{x} \left[ (i)^{x^T \tilde{Z} x} | x \rangle \langle x | \right] \times \sum_{x} \left[ (-1)^{x^T \tilde{Z} x} | x \rangle \langle x | \right] = (-1)^{x^T (2 \tilde{Z} + \tilde{e}_k) | e_k \rangle = i^{d_{ke_k} \tau} \begin{bmatrix} Z_{ke_k} \\ V_{ke_k} \\ 0 \end{bmatrix}. \]

Bringing the second \( \tau_{ek} \) from the left-hand side to the right-hand side we finally prove Eq. (21).

Combining Eqs. (20) and (21), we find

\[
Q^{(2)} Q^{(3)} Q^{(4)} = (1 / \sqrt{2^r}) \sum_{x_r, x_r \in \mathbb{Z}_2^r} \sum_{x_r, x_r \in \mathbb{Z}_2^r} (-1)^{x_r^T x_r} \langle x_r | x_r \rangle. \]

To take into account the index space transformation \( C^{(1)} \) we simply have to replace \( | x_{br} \rangle \) by \( | T_1 x_{br} \rangle \). For \( C^{(5)} \) and \( C^{(6)} \) we first define \( t \) and \( h_{bc} \in \mathbb{Z}_2^n \) by writing \( h^{(6)} \) as

\[
Q^{(6)} = \begin{bmatrix} t \\ T_2^T h_{bc} \end{bmatrix}. \]

Then, with Eqs. (2) and (7) we find

\[
\langle x_{bc} | C^{(5)} = \langle x_{bc} | C^{(6)} =\langle x_{bc} | T_2^{-1} x_{bc} + t \rangle. \]

This completes the proof. \( \square \)

**VI. CONCLUSION**

We have shown the relevance of binary linear algebra (over \( GF(2) \)) for the theory of stabilizer states and Clifford group operations. We have described how the Clifford group is isomorphic to a group that can be entirely described in terms of binary linear algebra. This has led to two schemes for the decomposition of Clifford group operations in a product of one and two-qubit operations, and to the description of standard basis expansions of both stabilizer states and Clifford group operations with binary quadratic forms.

**APPENDIX: PROOF OF EQUATION (41)**

Let \( e_k \) be the \( k \)-th column of \( I_{2n} \), \( k = 1, \ldots, 2n \). Then we have to find the images of \( \tau_{ek} \) (Hermitian matrices) under \( X \rightarrow Q X Q^\dagger \) with \( Q = e^{i(\pi/4)\tau_a} = \frac{1}{\sqrt{2}}(I + i \tau_a) \) to yield the \( k \)-th column \( c_k = C e_k \) of \( C \) and the \( k \)-th entry \( h_k = e_k^T h \) of \( h \). We find

\[
i e_k^T U c_k (-1)^{h_k \tau_{ek}} = \frac{1}{\sqrt{2}}(I + i \tau_a) e_k^T U \tau_{ek} = \frac{1}{\sqrt{2}}(I + i \tau_a) e_k^T U \tau_{ek} = \frac{1}{\sqrt{2}}(I + (-1)^{e_k^T P \tau_{ek}}) e_k^T (1 + (-1)^{e_k^T P \tau_{ek}}) \tau_{ek} \tau_{ek}, \]

where in the last step we used \( \tau_a^2 = I \) and \( \tau_a \tau_a = (-1)^{e_k^T P \tau_{ek}} \tau_{ek} \tau_{ek} \) as follows from Lemma 3. When \( e_k^T P \tau_{ek} = 0 \) we find \( c_k = e_k \) and \( h_k = 0 \). When \( e_k^T P \tau_{ek} = 1 \) we find

\[
i e_k^T U c_k (-1)^{h_k \tau_{ek}} = i \tau_{ek} \tau_a e_k^T U \tau_a e_k = i a^T U a (-1)^{e_k^T U a e_k} e_k + e_k \]

From this formula it can be read that \( c_k = a + e_k \). With \( i a^T U a = i a^T U a + (-1) a^T U a \) (with the addition in the exponents modulo 2) and \( (a + e_k)^T U (a + e_k) \) and \( a^T U a + e_k^T P a + e_k^T U e_k = a^T U a + 1 \), we also find that \( h_k = a^T U a + e_k^T P a \).

Combining the two cases \( e_k^T P \tau_{ek} = 0 \) and \( e_k^T P \tau_{ek} = 1 \) we find \( c_k = e_k + a(e_k^T P a) = (I + a a^T P) e_k \), yielding
$C = (I + aa^T P)$. For $h$ we find $h_k = (e_k^T P a)(a^T U a + e_k^T U a)$. With $(e_k^T P a)(e_k^T U a) = e_k^T U a$ this reduces to $h_k = e_k^T (P a a^T U a + U a)$ and $h = (I + aa^T P)^T U a$. This completes the proof. □

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