THE UNION OF TWO COMPACT SETS IN $\mathbb{R}^2$ WITH CONNECTED COMPLEMENT HAS A CONNECTED COMPLEMENT

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Abstract. In the papers from Chui and Parnes (1971) and Luh (1972), as well on the paper from V. Nestoridis (1996) on the Universal Taylor series, it is used, without proof, that the union of two compact sets in $\mathbb{R}^2$ with connected complement has a connected complement. In this work we present a rigorous proof of this fact, by studying some important topological properties of the plane.

Preliminaries

The following known results will be used in proving the main theorem.

Proposition 1. Let $A \subset C$ open. Then $A$ is path-connected if and only if $A$ is connected.

Proof. Every path-connected subset of $C$ is connected, so it suffices to show the other direction. Let $A \subset C$ be open and connected. That implies that if the exist $U, V \subset A$ open, such as $U \cup V = A$ and $U \cap V = \emptyset$, then either $U = \emptyset$ or $V = \emptyset$. Assume $A \neq \emptyset$ and $x_0 \in A$, and define

$$U := \{ x \in A \mid \exists \gamma: [0,1] \to A, \text{ continuous}, \text{ such as } C(0) = x_0 \text{ and } C(1) = x \}.$$ 

Then $U$ is open. Indeed, if $x \in U$ then there exists an $\varepsilon > 0$, such that $B(x, \varepsilon) \subset U$ because $A$ is open. Now, for all $x_1 \in B(x, \varepsilon)$, there exists $C_1: [0,1] \to A$, such as $C_1(0) = x, C_1(1) = x_1$ continuous (the line segment $xx_1$), and so

$$C_2 := C \oplus C_1 = \begin{cases} C(2s) & \text{ if } s \in [0, \frac{1}{2}], \\ C_1(2s - 1) & \text{ if } s \in [\frac{1}{2}, 1] \end{cases},$$

is continuous and $C_2(0) = x_0, C_2(1) = x_1$. So $x_1 \in U$ which implies $B(x, \varepsilon) \subset U$ and so $U$ is open. However, $V := A \setminus U$ is also open: If $x \in V$ then there exists an $\varepsilon_2 > 0 : B(x, \varepsilon_2) \subset A$. If there exists an $x_2 \in B(x, \varepsilon_2) : x_2 \in U$ then $x \in U$, which leads to a contradiction. So $B(x, \varepsilon_2) \subset V$ which implies $V$ open. As $U, V$ are open and $U \cup V = A, U \cap V = \emptyset$ it follows that $V = \emptyset$ and $A = U$, which means $A$ is path-wise connected.

Remark 1. The proof above can be slightly modified in order to be valid for topological space which is locally path-connected. In particular, Proposition 1 holds for any open $A \subset C_{\infty} = C \cup \{ \infty \}$, where $C_{\infty}$ the Riemann sphere.

Proposition 2. Let $A \subset C \ (or \ C_{\infty})$ bounded. Then $A^c$ has exactly one unbounded component.

Proof. Because $A$ is bounded, then there exists $M > 0 : A \subset B(0, M) \Rightarrow A \supset (B(0, M))^c$ and so $A^c$ is not bounded. Furthermore, $(B(0, M))^c$ is connected and $A^c \setminus (B(0, M))^c$ is bounded or empty. Therefore $A^c$ has exactly one unbounded component and every other component (if any exist) are contained in $B(0, M)$.

Proposition 3. Let $A \subset C$ bounded. Then $C \setminus A$ is connected if and only if $C_{\infty} \setminus A$ is connected.

Proof. Let $A$ bounded. Then $C \setminus A$ has exactly one unbounded component, $U_{A^c}$.

If $C \setminus A$ is connected, then $C \setminus A = U_{A^c}$ and therefore every neighborhood of infinity is a subset of $U_{A^c} \cup \{ \infty \}$. We conclude that $C_{\infty} \setminus A = U_{A^c} \cup \{ \infty \}$ is connected.

Lets now assume $C \setminus A$ disconnected. Then, because $A$ is bounded, there exists $B \neq \emptyset$, a bounded
component of \( \mathbb{C} \setminus A \) \((B \subset (\mathbb{C} \setminus A) \setminus U_{A^c})\).

Then \( B \cap (U_{A^c} \cup \{\infty\}) = \emptyset \) and therefore \( \mathbb{C}_\infty \setminus A \) disconnected. \(\square\)

**Simply Connected Subsets of the Plane**

The following theorem is a characterization of simply connected subsets of the plane.

**Theorem 1.** An open and connected \( A \subset \mathbb{C} \) is simply connected if and only if \( \mathbb{C}_\infty \setminus A \) is connected.

In order to prove this, we will need the following \(^1\).

**Lemma 1.** Let \( X \) be a connected topological manifold, and \( x, y \in X \) such that \( x \neq y \). Then there exists an injective curve \( \sigma : [0, 1] \to X \) with \( \sigma(0) = x \) and \( \sigma(1) = y \).

**Proof.** (of lemma \(^1\))

Let \( x_0 \in X \) and define \( W = \{ x \in X : x \neq x_0 \} \) and \( x_0 \) is connected to \( x \) by an injective curve \( \gamma \).

Now \( W \) is both open and closed. Indeed, let \( x_1 \in W \), and \( V \) an open neighbourhood of \( x_1 \) which is homeomorphic to the unit ball \( B(0, 1) \), with \( p : V \to B(0, 1) \) a homeomorphism with \( p(x_1) = 0 \).

Let \( x_2 \in V \). Without any loss of generality, we can assume that \( x_2 \) does not belong to the injective curve \( \gamma_1 \) which connects \( x_0 \) and \( x_1 \). If \( r = ||p(x_2)|| < 1 \), let

\[
\delta = \text{dist}(p(x_2), \overline{B}(0, r) \cap p(\gamma [1])) > 0.
\]

Let \( y_3 \in \overline{B}(0, r) \), such that \( ||y_3 - p(x_2)|| = \delta \) (such \( y_3 \) exists since \( \overline{B}(0, r) \cap p(\gamma [1]) \) is compact), \( J \) be the segment connecting \( y_3 \) and \( p(x_2) \) and \( x_3 = p^{-1}(y_3) \). Then the union of \( p^{-1}(J) \) and the part of \( \gamma_1 \) which connects \( x_0 \) with \( x_3 \) is an injective curve connecting \( x_0 \) and \( x_2 \), and therefore \( W \) is open.

Using a similar approach to the one we just used, it is readily obtained that a limit point of \( W \) also belong to \( W \).

Since \( X \) is assumed connected, and \( W \neq \emptyset \), we have that \( W = X \) and, therefore such an injective curve exists for any \( x, y \in X \) with \( x \neq y \). \(\square\)

**Proposition 4.** Let \( A \subset \mathbb{C} \) be open, connected and not simply connected. Then there exists in \( A \) a simple closed curve, which is not \( A \)-null-homotopic.

**Proof.** Let \( X \) be the universal cover of \( A \) and \( \pi : X \to A \) be the universal covering map.

Since \( A \) is not null-homotopic, there exists (by definition) a closed curve \( \gamma : [0, 1] \to A \) which is not \( A \)-null-homotopic.

Let \( \tilde{\gamma} : [0, 1] \to X \) be a lift of \( \gamma \). Since \( \gamma \) is not \( A \)-null-homotopic, we have \( \tilde{\gamma}(1) \neq \tilde{\gamma}(0) \) \(^4\). Since \( X \) is a connected topological manifold, by the previous lemma, there exists an injective curve \( \sigma : [0, 1] \to X \) such that \( \sigma(0) = \tilde{\gamma}(0) \) and \( \sigma(1) = \tilde{\gamma}(1) \). Next define

\[
s = \inf \{ t \in (0, 1) : (\exists u \in [0, t) \} : (\pi(\sigma(t))) = \pi(\sigma(u))) \}.
\]

We claim that there is an \( u \in [0, s] \) with \( \pi(\sigma(s)) = \pi(\sigma(u)) \): Choose a neighbourhood \( V \) of \( \sigma(s) \) on which \( \pi \) is injective. There is an \( \varepsilon > 0 \) such that \( \sigma(t) \in V \) for all \( t \in [s - \varepsilon, s + \varepsilon] \). We have sequences \( t_n \) and \( u_n \) with \( \pi(\sigma(t_n)) = \pi(\sigma(u_n)) \) where \( t_n \geq t_{n+1} \geq s \) and \( u_n < t_n \). Without loss of generality, we can assume \( t_n < s + \varepsilon \) for all \( n \). Since \( \pi \) is injective on \( V \), we have \( u_n < s - \varepsilon \) for all \( n \). After taking a subsequence, we can assume that \( u_n \) converges to some \( u \leq s - \varepsilon \). By continuity,

\[
\pi(\sigma(s)) = \lim_{n \to \infty} \pi(\sigma(t_n)) = \lim_{n \to \infty} \pi(\sigma(u_n)) = \pi(\sigma(u)).
\]

Now define \( \tilde{\sigma}(t) = \sigma((1 - t)u + ts) \) and \( \tilde{\gamma} = \pi \circ \tilde{\sigma} \). By construction, \( \tilde{\gamma} \) is a simple closed curve in \( U \), and since \( \sigma(u) \neq \sigma(s) \), it is not null-homotopic. \(\square\)

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\(^1\) I am indebted to Y. Smyrlis for suggesting this line of proof.
We are now ready to prove theorem [1]

Proof. of Theorem 1

Let \( A \) be not simply connected. Then there exists \( \gamma : [0, 1] \to A \) simple, continuous and closed that is not \( A \)-null-homotopic.

Let \( \gamma^* := \gamma([0, 1]) \). From the Jordan-Schoenflies curve theorem, we have that \( \mathbb{C}_\infty \setminus \gamma^* \) consists of exactly two components, the interior \( \text{int}(\gamma) \), which is the bounded one, and the exterior, \( \text{ext}(\gamma) \), both of which are homeomorphic to the unit disk.

Therefore, if \( \text{int}(\gamma) \subset A \), it would follow that \( \gamma \) is \( A \)-null-homotopic. Therefore \( \exists z_1 \in \mathbb{C} : z_1 \in \text{int}(\gamma) \) and \( z_1 \notin A \). Then, on the Riemann sphere, \( z_1 \) and \( \infty \) belong on different components of \( \mathbb{C}_\infty \setminus \gamma^* \) and hence \( \mathbb{C}_\infty \setminus A \) is disconnected.

Conversely, let \( \mathbb{C}_\infty \setminus A \) be disconnected. Let \( K \) be a component of \( \mathbb{C}_\infty \setminus A \) such that \( \infty \notin K \). Then \( K \) is compact. Moreover, let \( \Omega := \mathbb{C} \setminus U_{A^c} \). Since \( U_{A^c} \) is closed, \( \Omega \) is open and since \( K \subset \Omega \), there exists \( \varepsilon > 0 \) such that the \( \text{dist}(K, U_{A^c}) = 2\varepsilon \).

If \( z \in K \) then there exists a continuous and closed curve, \( \alpha : [0, 1] \to A \), such as and \( z \) belongs in a bounded component of \( \mathbb{C} \setminus \alpha^* \). A detail construction of such a curve can be found in [5]. We begin by constructing a grid of horizontal and vertical lines such as the distance between any two adjacent horizontal lines and any two adjacent vertical lines is \( \varepsilon \). From the squares (closed 2-cells) formed by the grid, we can choose those that intersect with \( K \). Since \( K \) is compact, only finitely many are required. By parameterizing the edges of these squares carefully, we arrive at a representative of an element of the fundamental group of \( A \) which is different from the identity.

\[ \Box \]

**Corollary 1.** Let \( A \subset \mathbb{C} \) open. Then every component of \( A \) is simply connected if and only \( \mathbb{C}_\infty \setminus A \) connected.

**Proof.** First let's assume that \( \mathbb{C}_\infty \setminus A \) is connected. If \( A_0 \subset A \) be one connected component of \( A \), it suffices to show that \( \mathbb{C}_\infty \setminus A_0 \) connected. We can assume that \( A_0 \neq A \) (this is the case of theorem [1]) and so let \( z \in A \setminus A_0 \). If \( z_0 \in A_0 \), then if \( \gamma : [0, 1] \to \mathbb{C} \) is continuous such as \( \gamma(0) = z_0 \) and \( \gamma(1) = z \) we have \( \gamma^* \notin A \) (where \( \gamma^* := \gamma([0, 1]) \) the image of \( \gamma \)). Therefore \( \exists s_1 \in (0, 1) \) and \( z_1 \notin A \) such as \( \gamma(s_1) = z_1 \) and consequently \( z \) belongs in the only component of \( \mathbb{C}_\infty \setminus A_0 \) which is therefore connected.

Now, from theorem [1] \( A_0 \) is simply connected.

Let now assume that \( \mathbb{C}_\infty \setminus A \) disconnected. So, and because the connected components of \( A \) are pairwise disjoint, \( \exists z \in \mathbb{C} \) which is contained in the bounded component of the compliment of at least one connected component of \( A \). Let \( A_1 \) be one such component of \( A \). Then \( \mathbb{C}_\infty \setminus A_1 \) disconnected and therefore, from theorem [1] \( A_1 \) is not simply connected. \[ \Box \]
Main Theorem

Having the results above established, we can now prove the main theorem.

**Theorem 2** (Characterization). Let \( K \subset \mathbb{C} \) compact. Then \( \mathbb{C} \setminus K \) connected if and only if for all open \( A \supset K \) there exists \( V \) open such that \( K \subset V \subset A \) and every component of \( V \) is simply connected.

**Proof.** Let \( K^c := \mathbb{C} \setminus K \) connected. Since \( K \) is compact, \( K^c \) is open and therefore path-wise connected. (Proposition 1)

Let \( A \supset K \). If the conclusion is true for every \( A \) open and bounded and \( A_1 \) is an open set containing \( K \), then \( K \subset A_1 \cap B(0, M) \subset A_1 \), for a sufficiently large \( M > 0 \). Since \( A_1 \cap B(0, M) \) is bounded there is an open \( V \) such as \( K \subset V \subset A_1 \cap B(0, M) \subset A_1 \) and every component of \( V \) is simply connected.

Therefore, we can assume, without a loss of generality, that \( A \) is bounded.

Since \( A \) is bounded, \( A^c \) has exactly one unbounded component, \( U_{A^c} \). (Proposition 2)

If \( U_{A^c} \) is the only component of \( A^c \), then \( A^c \) is connected and therefore connected on the Riemann sphere (Proposition 3) and consequently (Theorem 1) \( A \) is simply connected and the conclusion follows.

We can, therefore, assume that \( A^c \) has bounded components.

Since \( K \subset A \) and \( A \) open, \( \exists \varepsilon > 0 : B(z, \varepsilon) \subset AV \in K \) (i.e. \( \varepsilon < \text{dist}(\partial A, K) \)). \( K \) is compact, hence only finite such balls \( B_1, ..., B_n, n \in \mathbb{N} \) with radius \( \varepsilon \) suffice to cover \( K: K \subset \bigcup_{i=1}^{n} B_i = W \subset A \)

Now, \( W^c \) has a finite number of components, \( N \in \mathbb{N} \). If \( z_1, z_2, ..., z_N \in \mathbb{C} \) each belong to a different component of \( W^c \), then \( z_1, z_2, ..., z_N \in K^c \) and since \( K^c \) is path-wise connected, \( K^c \) is also path-wise connected on the Riemann sphere (Proposition 3). Therefore, for \( i = 1, 2, ..., N \) there exist \( \gamma_i : [0, 1] \rightarrow (\mathbb{C} \setminus K) \) continuous, such as \( \gamma_i(0) = z_i, \gamma_i(1) = \infty \). Without a loss of generality, we can assume that \( \gamma_i(s) \in K \forall s \in (0, 1) \) and \( \forall i \in \{1, 2, ..., N\} \)

We now define \( G_i := \gamma_i([0, 1]) \subset \mathbb{C} \) for all \( i \in \{1, 2, ..., N\} \) and notice that \( G_i \) is closed \( \forall i \in \{1, 2, ..., N\} \)

Hence \( V := W \cap \bigcup_{i=1}^{N} G_i \subset A \) is open and \( V^c \) is path-wise connected. Therefore \( V^c \) path-wise connected on Riemann sphere and hence every component of \( V \) is simply connected (Corollary of 1).

Conversely, let \( K^c \) be disconnected.

If \( U_{K^c} \) is the unbounded component of \( K^c \), then there exists a \( z_0 \in \mathbb{C} \) such that \( z_0 \notin K \) and \( z_0 \notin U_{K^c} \). Since \( K^c \) is open, there exist an \( \varepsilon > 0 : B(z, \varepsilon) \subset K^c \). For this \( \varepsilon \), it is straightforward to show that \( B(z, \varepsilon) \cap U_{K^c} = \emptyset \) and \( \overline{B}(z_0, \frac{\varepsilon}{2}) \subset K^c \).

Since \( K \) is compact, \( K^c \) is bounded and therefore there exists an \( M > 0 \) such that \( B(z_0, M) \subset K \)

Let \( A := \{ \{ \in \mathbb{C} : \frac{\varepsilon}{2} < |z - z_0| < M \} \} \) the annulus centered at \( z_0 \), with radii \( \frac{\varepsilon}{2} \) and \( M \). Then there is no open subset \( V \) of \( A \) such as \( K \subset V \) such as all components of \( V \) are simply connected. Indeed, for any \( \gamma : [0, 1] \rightarrow \mathbb{C} \) continuous such as \( \gamma(0) = z_0 \) and \( \gamma(1) = \infty \), we have \( \gamma^* \cap K \neq \emptyset \) and since \( K \subset V \), \( \gamma^* \cap V \neq \emptyset \). Therefore \( V^c \) is not path-wise connected on Riemann sphere and the conclusion follows from Theorem 1.

Our main result follows as a corollary from the characterization.

**Corollary 2.** Let \( K, L \subset \mathbb{C} \) be disjoint and compact, such as \( K^c \) and \( L^c \) are connected. Then \( (K \cup L)^c \) is connected.

**Proof.** Let \( A \) open such as \( K \cup L \subset A \). Since \( K, L \) are disjoint and closed, there are \( A_1, A_2 \subset A \) open and disjoint such as \( K \subset A_1 \subset L \subset A_2 \). From the characterization (Theorem 2), \( \exists V_1, V_2 \) open such as \( K \subset V_1 \subset A_1 \), \( L \subset V_2 \subset A_2 \) and every component of \( V_1 \) and \( V_2 \) is simply connected.

Then \( K \cup L \subset V_1 \cup V_2 \cup A \) and every component of \( V := V_1 \cup V_2 \) is simply connected. Now, it is obvious that \( V \) is open and since \( A \) was arbitrary, the conclusion follows from the characterization.

\[ \text{This holds for any open and convex subsets of } \mathbb{R}^n: \text{ Let } V_1, \ldots, V_n \text{ be open, bounded and convex subsets of } \mathbb{R}^n. \text{ Then the complement } F = \mathbb{R}^2 \setminus \bigcup_{i=1}^{n} V_i \text{ possesses only finitely many connected components.} \]
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