HANDLE HOMOLOGY OF MANIFOLDS

SEBASTIAN DURST, HANSJÖRG GEIGES, AND MARC KEGEL

Abstract. We give an entirely geometric proof, without recourse to cellular homology, of the fact that $\partial^2 = 0$ in the chain complex defined by a handle decomposition of a given manifold. Topological invariance of the resulting ‘handle homology’ is a consequence of Cerf theory.

1. Introduction

It is well known that any handle decomposition of a given smooth manifold gives rise to a cell complex of the same homotopy type. This allows one to compute the singular homology of the manifold as the cellular homology of that associated complex. In this situation, the boundary operator in the cellular chain complex can be interpreted as a boundary operator on handles, defined in terms of intersection numbers between the attaching spheres of $k$-handles and the belt spheres of $(k-1)$-handles.

In the present note we take this geometric interpretation of the boundary operator $\partial$ as our starting point, and we prove $\partial^2 = 0$ by geometric means, rather than by relating $\partial$ to the boundary operator on the cellular chain complex. As a benefit, the resulting ‘handle homology’ no longer relies, neither explicitly nor implicitly, on singular homology theory. The topological invariance of the ‘handle homology’ thus constructed is proved by appealing to Cerf theory [1], according to which any two handle decompositions of a given manifold are related by some simple moves, cf. [3, Theorem 4.2.12]: handle slides and the creation or annihilation of cancelling handle pairs.

The definition of the homology of a manifold in terms of a handle decomposition gives a very simple proof of Poincaré duality; our reasoning puts this proof on a purely geometric footing. Apart from the geometric proof of $\partial^2 = 0$, which we have not found in the literature, this note is largely expository, expanding on some aspects of [3, Section 4.2]. It is instructive to compare our arguments for dealing with sign issues in the proof of $\partial^2 = 0$ with the discussion of orientations in Morse homology [5, 6].

2. Handle decompositions

We assume that the reader is familiar with the basics of handle decompositions of manifolds at the level of [3, Sections 4.1]; see also [2] for an elementary introduction. Here we only recall the parts of this theory necessary to set up notation.

2010 Mathematics Subject Classification. 55N35, 57R65.
Key words and phrases. manifold, handle decomposition, homology theory.
2.1. Handles. An $n$-dimensional $k$-handle is a copy of $h_k := D^k \times D^{n-k}$, attached along its lower boundary $\partial_- h_k := \partial D^k \times D^{n-k}$ to the boundary of a smooth $n$-dimensional manifold $X$ by an embedding

$$\varphi: \partial_- h_k \to \partial X,$$

see Figure 1. The number $k \in \{0, \ldots, n\}$ is called the index of the handle.

After smoothing the corner $\varphi(\partial D^k \times \partial D^{n-k})$, the resulting space $X \cup \varphi h_k$ is a smooth manifold. Its boundary is given by removing $\varphi(\partial_- h_k)$ from $\partial X$ and replacing it with the upper boundary $\partial_+ h_k := D^k \times \partial D^{n-k}$.

We write $A_k := \partial D^k \times \{0\} \equiv \varphi(\partial D^k \times \{0\}) \cong S^{k-1}$ for the attaching sphere of the $k$-handle $h_k$, and $B_k := \{0\} \times \partial D^{n-k} \cong S^{n-k-1}$ for its belt sphere.

We shall usually identify $A_k$ and $\partial_- h_k$ with their respective images in $\partial X$ under the embedding $\varphi$.

2.2. Nice handle decompositions. Let $M$ be a smooth compact $n$-manifold with boundary $\partial M = \partial_- M \cup \partial_+ M$, where either collection $\partial_\pm M$ of boundary components may be empty. If $M$ is oriented, the boundaries are oriented such that $\partial M = \partial_- M \cup \partial_+ M$. A handle decomposition of $M$ relative to $\partial_- M$ is an identification of $M$ with a manifold obtained by successively attaching handles to $[0,1] \times \partial_- M$ along $\{1\} \times \partial_- M$, where $\{0\} \times \partial_- M$ is identified with $\partial_- M$. Attaching a 0-handle amounts to the disjoint union with an $n$-disc $D^n = \{0\} \times D^n$. Morse theory implies that such a handle decomposition always exists.

If we attach a $k$-handle $h_k$ to an $n$-manifold $X$ with boundary, followed by an $\ell$-handle $h_\ell$ with $\ell \leq k$, the attaching sphere $A_\ell$ in $\partial (X \cup h_k)$ can be made disjoint from the belt sphere $B_k$ by an isotopy, since

$$\dim A_\ell + \dim B_k = (\ell - 1) + (n - k - 1) < n - 1 = \dim \partial (X \cup h_k).$$

This allows one to push $\partial_- h_\ell$ away from $\partial_+ h_k$ by an isotopy that flows radially outward (in the $D^k$-factor) on $\partial_+ h_k$. It follows that handles may always be attached in the order of increasing index, and this will be assumed from now on. We write $M_k$
for the manifold obtained from $[0, 1] \times \partial_- M$ by attaching handles up to and including index $k$. We also assume without loss of generality that the lower boundaries of the $k$-handles are (disjointly) embedded in $\partial M_{k-1} \setminus \partial_- M$, rather than some lower boundaries intersecting the upper boundaries of other $k$-handles.

Now consider the attaching of a $k$-handle $h_k$, followed by a $(k+1)$-handle $h_{k+1}$. Since $\dim A_{k+1} + \dim B_k = k + (n - k - 1) = n - 1$, we may assume after an isotopy that $A_{k+1}$ and $B_k$ intersect each other transversely in finitely many points. The embedding of the lower boundary $\partial_- h_{k+1} = \partial D^{k+1} \times D^{n-k-1}$ defines a tubular neighbourhood of $A_{k+1}$, and we can take this to intersect $B_k$ in finitely many copies of $\{\ast\} \times D^{n-k-1}$, with $\ast \in \partial D^{k+1}$.

By flowing out radially from $B_k$ on the upper boundary $\partial_+ h_k$ we may further assume that $A_{k+1}$ intersects $\partial_+ h_k$ in finitely many copies of $D^k \times \{\ast\}$, with $\ast \in \partial D^{n-k}$.

Finally, we consider the intersection of the attaching sphere $A_{k+1}$ with the belt sphere $B_{k-1}$ of any $(k-1)$-handle in $\partial M_k$. The dimensions of $A_{k+1}$ and $B_{k-1}$ add up to $n$, so after making the intersection transverse, it will be a 1-dimensional manifold with boundary; the boundary points lie in the corners of the $k$-handles.

These assumptions on the handle decomposition being sufficiently ‘nice’, illustrated in Figure 2, will be taken for granted from now on. The superscripts $\mu$ and $\nu$ are used to label the $k$- and $(k-1)$-handles, respectively. Beware that, due to lack of dimensions, this figure is a little misleading. The intersection $A_{k+1} \cap B_{k-1}^\nu$ is indeed 1-dimensional, but the intersection $A_{k+1} \cap \partial_+ h_k^\mu$ is $k$-dimensional. Also, because of $k = 1$ in the figure, the belt sphere $B_{k-1}^\nu$ coincides with a component of $\partial M_{k-1}$.

Let $c$ be a boundary point of the 1-dimensional manifold $A_{k+1} \cap B_{k-1}^\nu$. This point lies in the corner $\partial D^k \times \partial D^{n-k}$ of a $k$-handle $h_k^\mu$, and we write it as $c = (c_k, c_{n-k})$ with respect to this product structure. Since our handle decomposition is nice, we have

$$\{c_k\} \times D^{n-k} \subset B_{k-1}^\nu \cap \partial_- h_k^\mu$$
and
\[ D^k \times \{ c_{n-k} \} \subset A_{k+1} \cap \partial_+ h^\mu_k. \]
This determines a pair of points \((a,b)\) with
\[ a = (c_k, 0) \in B^\nu_{k-1} \cap A^\mu_k \]
and
\[ b = (0, c_{n-k}) \in A_{k+1} \cap B^\mu_k. \]
Conversely, any pair of points \((a,b)\) determines a unique point \(c\) in the boundary of \(A_{k+1} \cap B^\nu_{k-1}\). So there is a one-to-one correspondence between pairs \((a,b)\) and points \(c \in \partial(A_{k+1} \cap B^\nu_{k-1})\).

In Morse homology, each pair \((a,b)\) (or the respective point \(c\)) corresponds to a broken trajectory of the negative gradient flow, connecting two critical points of index difference 2, broken at a critical point of intermediate index.

2.3. Orientations. (1) If the manifold \(M\) is oriented, we orient the \(k\)-handles in the relative handle decomposition of \((M, \partial_+ M)\) as follows. Choose any identification of the core disc of \(h_k\) with \(D^k \times \{ 0 \}\), equipped with its standard orientation as unit disc in \(\mathbb{R}^k\) (see Figure 1). Then identify the belt disc of \(h_k\) with \(\{ 0 \} \times D^{n-k}\) (and hence \(h_k\) with \(D^k \times D^{n-k}\)) in such a way that the orientation of \(h_k\) induced by the orientation of \(M\) coincides with the standard orientation of \(D^k \times D^{n-k}\) as subset of \(\mathbb{R}^n\).

The attaching sphere \(A_k\) is oriented as the boundary of the core disc \(D^k \times \{ 0 \}\) according to the dictum ‘outward normal first’; the belt sphere \(B_k\), as the boundary of the belt disc \(\{ 0 \} \times D^{n-k}\).

The successive handlebodies \(M_k\) are oriented as equidimensional submanifolds of \(M\); the hypersurface \(\partial M_k\) is oriented as the boundary of \(M_k\).

Notice that for the \(n\)-handles the core disc coincides with the full handle. In particular, here the orientation of the core disc is determined by the ambient orientation, and \(A_n\) carries the opposite orientation of \(\partial M_{n-1}\).

(2) If \(M\) is non-orientable, we can still choose an orientation for each core disc and give the attaching sphere the induced orientation. This orientation of the \(D^k\)-factor in \(h_k = D^k \times D^{n-k}\) defines a coorientation on the belt sphere \(B_k = \{ 0 \} \times \partial D^{n-k}\) as submanifold of the upper boundary \(\partial_+ h_k\), i.e. an orientation of the normal bundle of \(B_k \subset \partial_+ h_k\). This, as we shall see, is sufficient for defining the relevant intersection numbers over the integers.

3. Handle homology

We now use the handle decomposition of \(M\) relative to \(\partial_+ M\) to define relative homology groups \(H_\ast(M, \partial_+ M)\). We first formulate everything under the assumption that \(M\) is oriented, where we can work with integral coefficients. The arguments in Sections 2.1 and 2.2 likewise apply to non-orientable manifolds if one works over \(\mathbb{Z}/2\). The necessary modifications to define integral handle homology for non-orientable manifolds will be explained in Section 3.3.
3.1. Definition of the handle chain complex. Let \( C_k(M, \partial_- M) \), be the free abelian group generated by the oriented \( k \)-handles in a handle decomposition of \( M \) relative to \( \partial_- M \). Of course, \( C_k(M, \partial_- M) \) depends on the choice of handle decomposition, but we suppress this from the notation.

The boundary operator is then defined by

\[
\partial_k : \quad C_k(M, \partial_- M) \quad \mapsto \quad C_{k-1}(M, \partial_- M)
\]

\[
\quad \mapsto \quad (-1)^{k-1} \sum_\nu (A_k \bullet B^\nu_{k-1}) h^\nu_{k-1}.
\]

Here \( A_k \bullet B^\nu_{k-1} \) is the intersection number of \( A_k \) and \( B^\nu_{k-1} \) in the oriented manifold \( \partial M_{k-1} \).

The sign \((-1)^{k-1}\) in this definition is chosen for consistency with the boundary operator in cellular homology. For a point \( * \in \partial D^{n-k+1} \), the intersection number of \( D^{k-1} \times \{*\} \equiv D^{k-1} \) with the belt sphere \( \{0\} \times \partial D^{n-k+1} \equiv \partial D^{n-k+1} \) in \( \partial h_{k-1} \) equals

\[
(D^{k-1} \times \{*\}) \cdot \partial D^{n-k+1} = (-1)^{k-1},
\]

since the orientation of \( D^{k-1} \) followed by the orientation of \( \partial D^{n-k+1} \) equals \((-1)^{k-1}\) times the boundary orientation of \( \partial h_{k-1} \).

To see this formally, we introduce the notation \([\ldots]\) for orientations determined by submanifolds or frame fields whose dimensions add up to the dimension of the ambient manifold. For instance, in \( h_{k-1} \) we have \([D^{k-1}, D^{n-k+1}] = 1\). Writing \( n \) for the outer normal to \( \partial_+ h_{k-1} = D^{k-1} \times \partial D^{n-k+1} \), we have \([n, \partial_+ h_{k-1}] = 1\).

The intersection number in question is then computed as

\[
D^{k-1} \cdot \partial D^{n-k+1} = [n, D^{k-1}, \partial D^{n-k+1}]
\]

\[
= (-1)^{k-1} [D^{k-1}, n, \partial D^{n-k+1}]
\]

\[
= (-1)^{k-1} [D^{k-1}, D^{n-k+1}] = (-1)^{k-1}.
\]

Remark 3.1. In [3] p. 111], the boundary operator \( \partial_k \) is defined without the factor \((-1)^k\). This does not change the homology of the chain complex.

3.2. Proof of \( \partial^2 = 0 \). Applying the boundary operator twice, we find

\[
\partial_k (\partial_{k+1} h_{k+1}) = - \sum_{\mu, \nu} (A_{k+1} \bullet B^\mu_k)(A^\mu_k \bullet B^\nu_{k-1}) h^\nu_{k-1}.
\]

We therefore need to show that

\[
\sum_{\mu} (A_{k+1} \bullet B^\mu_k)(A^\mu_k \bullet B^\nu_{k-1}) = 0.
\]

In the notation of Section 2.2, this amounts to

\[
\sum \text{sign(a) sign(b)} = 0,
\]

where the sum is over all intersection points \( a \in A^\mu_k \cap B^\nu_{k-1} \) and \( b \in A_{k+1} \cap B^\mu_k \), and \( \text{sign} \in \{\pm 1\} \) denotes the sign induced by the orientations of the relevant submanifolds intersecting transversely in the given point.

Now, each pair of points \((a, b)\) corresponds to a boundary point \( c \) of the compact 1-dimensional manifold \( A_{k+1} \cap B^\nu_{k-1} \). This already proves \( \partial^2 = 0 \) over \( \mathbb{Z}_2 \). Over the integers, our aim is to show that \( A_{k+1} \cap B^\nu_{k-1} \) can be oriented in such a way that any boundary point \( c \), with \( \text{sign}(c) \) denoting the boundary orientation, satisfies

\[
\text{sign}(c) = \text{sign}(a) \text{sign}(b).
\]
Then the sum $\sum \text{sign}(a) \text{sign}(b)$ amounts to a sum over the signed boundary points $c$, and hence equals zero.

Consider a component of the 1-manifold $C := A_{k+1} \cap B_{k-1}$ (we suppress the superscript $\nu$ from now on) with a boundary point $c$. This point lies on the corner $S^{k-1} \times S^{n-k-1}$ of a $k$-handle $h_k$. This corner is a separating hypersurface in $\partial M_{k-1}$, with the lower boundary $\partial_{-} h_k$ to one side of it; see Figure 3.

We now orient the 1-manifold $C = A_{k+1} \cap B_{k-1}$ by a tangent vector $u$ such that transverse frames $v = (v_2, \ldots, v_k)$ in $A_{k+1}$ and $w = (w_2, \ldots, w_{n-k})$ in $B_{k-1}$ can be chosen subject to the following orientation conventions:

(i) $v, u$ is a positive frame for $A_{k+1}$ along $C$;
(ii) $u, w$ is a negative frame for $B_{k-1}$ along $C$;
(iii) $v, u, w$ is a negative frame for $\partial M_{k-1}$ along $C$.

If $1 < k < n - 1$, in which case both $v$ and $w$ contain at least one vector, these conventions indeed determine the orientation $[u]$ of $C$. Simply choose a tangent vector $u$ along $C$, and extend to frames $v, u$ and $u, w$ subject to conditions (i) and (ii). If (iii) is satisfied by $v, u, w$, this $[u]$ is the orientation we take for $C$; if not, replace $u$ by $-u$, and one vector each in $v$ and $w$ by its negative.

If one or both of $v, w$ are empty frames, i.e. if $k = 1$ or $k = n - 1$, conditions (i) to (iii) are consistent: $A_n$ has the opposite orientation of $\partial M_{n-2}$; the belt sphere $B_0$ is a component of $\partial M_0$.

The point $c$ is then oriented as a boundary point of $C$. Figure 3 shows the case $\text{sign}(c) = -1$.

**Lemma 3.2.** $\text{sign}(c) = \text{sign}(a) \text{sign}(b)$. 

**Figure 3.** Orienting $A_{k+1} \cap B_{k-1}$. 

---

[Image: Figure 3. Orienting $A_{k+1} \cap B_{k-1}$.]
Proof. For simplicity of notation, assume that the intersections $A_k \cap B_{k-1}$ and $A_{k+1} \cap B_k$ consist of single points

$$A_k \cap B_{k-1} = \{a\}, \quad A_{k+1} \cap B_k = \{b\}.$$ 

Let $c \in \partial C$ be the corresponding boundary point of $C$.

First we are going to relate the sign of $b$ to the orientation of the $D^k$-factor in the $k$-handle $h_k$. Write $n$ for the outer normal to $\partial M_{k-1}$. We may assume that $h_k$ is attached vertically to $M_{k-1}$, so that $n$ may be thought of as a tangent vector to $\partial_t h_k$ at $c$; see Figure 4 which shows the situation for $\text{sign}(c) = 1$.

According to our conventions, $[v, u]$ is the positive orientation of $A_{k+1}$ along $C$. The transverse frame $v$ can be chosen such that at $c$ it is tangent to the $S^{k-1}$-factor of the corner of $h_k$, see Figure 3. Then $[v, n]$ is likewise a frame of $A_{k+1}$ at $c$, when we consider the part of $A_{k+1}$ lying in $\partial_t h_k$. The orientations defined by these two frames are related by a factor $\text{sign}(c)$.

The part of $A_{k+1}$ lying in $\partial_t h_k$ is a $k$-disc $D^k \times \{\ast\}$, with $\ast \in \partial D^{n-k}$, passing through the point $b = A_{k+1} \cap B_k$ on the belt disc of $h_k$. By definition, we have $\text{sign}(b) = A_{k+1} \cdot B_k$. From the observation in Section 3.1 that $(D^k \times \{\ast\}) \cdot B_k = (-1)^k$, we then deduce that the positive orientation of $D^k \times \{\ast\}$ is given by

$$(-1)^k \text{sign}(b) \text{sign}(c)[v, n] = - \text{sign}(b) \text{sign}(c)[n, v].$$

Next we relate the sign of $a$ to the orientation of the $D^{n-k}$-factor in $h_k$. The part of $B_{k-1}$ lying in $\partial_t h_k$ is an $(n - k)$-disc $\{\ast\} \times D^{n-k}$, with $\ast \in \partial D^k$, passing through the point $a = A_k \cap B_{k-1}$. By definition, we have $\text{sign}(a) = A_k \cdot B_{k-1}$, where the orientation of the intersection point is determined with respect to the ambient orientation of $\partial M_{k-1}$. Thus, regarded as an intersection in $\partial_t h_k$, with respect to the orientation as boundary of $h_k$, the sign of the intersection point is $-\text{sign}(a)$. 

**Figure 4.** $u$, $n$ and $\text{sign}(c)$.
Now, we have $\partial D_k \bullet \{\ast\} \times D^{n-k} = \partial_- h_k$. Recall that our convention (ii) says that $B_{k-1}$ is oriented by $-[u, w]$, and $A_k$ is oriented as $\partial D^k$. It follows that the $D^{n-k}$-factor of $h_k$ is oriented by

$$\text{sign}(a)[u, w].$$

Assembling this information, we find that the orientation of $h_k$ is given by

$$-\text{sign}(a)\text{sign}(b)\text{sign}(c)[n, v, u, w].$$

On the other hand, the orientation of $M_{k-1}$, by (iii), is given by $-\{n, v, u, w\}$. We conclude $\text{sign}(a)\text{sign}(b)\text{sign}(c) = 1$. \hfill $\square$

### 3.3. Integral homology of non-orientable manifolds.

We now show that the orientation convention from Section 2.3 (2) suffices to define integral homology over the integers, even if the manifold is not orientable.

Let $A, B$ be submanifolds of a manifold $N$ with a transverse intersection $A \cap B$. The manifold $N$ is not assumed to be orientable. If $A$ is oriented and $B$ is cooriented (i.e. the normal bundle of $B$ in $N$ is oriented), the submanifold $A \cap B$ inherits an orientation, since the normal bundle of $A \cap B$ in $A$ can be identified with the normal bundle of $B$ in $N$, restricted to $A \cap B \subset B$. The intersection $A \cap B$ is oriented by this rule: the coorientation of $B$, followed by the positive orientation of $A \cap B$, defines the orientation of $A$.

This means that we can still make sense of the intersection numbers $A_k \bullet B_{k-1}$ of the submanifolds $A_k, B_{k-1} \subset \partial M_{k-1}$. So we may define the boundary operator $\partial_k$ as in Section 3.1 (With the natural coorientation of the $B'_n$, the factor $(-1)^{k-1}$ in the definition of $\partial_k$ can be removed; see Remark 3.3)

Likewise, the 1-manifold $C = A_{k+1} \cap B_{k-1}$ inherits an orientation $[u]$. If $k = 1$ (when $\dim A_{k+1} = 1$), we simply take $u$ as a vector field defining the orientation of $A_2$. If $k > 1$ we choose $u$ by the following rule. As before, $v = (v_2, \ldots, v_k)$ denotes a transverse frame along $C$ in $A_{k+1}$, which we now interpret as a coframe for $B_{k-1}$ in $\partial M_{k-1} \cap C$.

(i) $v$ is a positive coframe for $B_{k-1}$ along $C$;
(ii) $v, u$ is a positive frame for $A_{k+1}$ along $C$.

We can now prove Lemma 3.2 in this setting (up to an irrelevant sign, see Remark 3.3). As before, we assume for notational simplicity that we are dealing with a pair $(a, b)$ of single intersection points corresponding to a boundary point $c$ of $C$. The signs of these three points are given by

$$\text{sign}(a) = A_k \bullet B_{k-1},$$

$$\text{sign}(b) = A_{k+1} \bullet B_k,$$

$$[v, u] = \text{sign}(c)[v, u].$$

Here the first two equations hold by definition; for the third the argument is as in the proof of Lemma 3.2

The coorientation of $B_k$ in $\partial M_k$ is given by the orientation of $D^k \times \{\ast\}$ in the boundary of the $k$-handle $h_k$. Since the outer normal along the boundary of that disc is $-n$ (see Figure 4), this coorientation is $[-n, A_k]$. On the other hand, by (ii) and the definition of $\text{sign}(b)$, this coorientation is also given by $\text{sign}(b)[v, u]$, hence

$$[v, u] = \text{sign}(b)[-n, A_k].$$
Similarly, by (i) and the definition of sign(a), we have
\[ [A_k] = \text{sign}(a)[v]. \]
We then compute
\[
[v, u] = \text{sign}(b)[-n, A_k] \\
= -\text{sign}(a)\text{sign}(b)[u, v] \\
= (-1)^k\text{sign}(a)\text{sign}(b)[v, n] \\
= (-1)^k\text{sign}(a)\text{sign}(b)\text{sign}(c)[v, u],
\]
which proves that \[ \text{sign}(c) = (-1)^k\text{sign}(a)\text{sign}(b). \] Hence \[ \sum \text{sign}(a)\text{sign}(b) = 0 \] also in the non-orientable setting.

**Remark 3.3.** If the belt sphere \( \{0\} \times \partial D^{n-k} \) in \( h_k = D^k \times D^{n-k} \) is cooriented by the orientation of \( D^k \), as seems natural, then \( D^k \cdot \partial D^{n-k} = 1 \), whereas in the oriented setting we computed this intersection number as \( (-1)^k \), see Section 3.1. This accounts for the extra sign in the above computation.

**4. Topological invariance**

We define the ‘handle homology’ \( H_*(M, \partial_- M) \) as the homology of the chain complex \( C_*(M, \partial_- M) \). By construction, it is isomorphic to the cellular homology of the cell complex associated with the handle decomposition, and hence a homotopy invariant. It has already been observed in [3, Section 4.2] that, alternatively, the following theorem of Cerf [1] may be used to show that this handle homology does not depend on the choice of relative handle decomposition.

**Theorem 4.1** (Cerf). Any two relative handle decompositions of a compact manifold pair \( (M, \partial_- M) \), both ordered by increasing index, are related by a finite sequence of handle slides and the creation or annihilation of cancelling handle pairs.

Cerf actually deals with homotopies of Morse functions. Cancelling pairs of critical points in such a homotopy translate into a cancelling handle pair; connecting trajectories between critical points of equal index correspond to the intermediate stage of a handle slide when the attaching sphere of a \( k \)-handle passes through the belt sphere of another \( k \)-handle.

As sketched in [3, p. 112], these moves do not affect the handle homology. This means that \( H_*(M, \partial_- M) \) is a diffeomorphism invariant. For completeness, we provide a few details.

**4.1. Handle slides.** Given two \( k \)-handles \( h_k \) and \( h'_k \) attached to the boundary of an \( n \)-manifold \( X \), a handle slide of \( h_k \) over \( h'_k \) is defined by isotoping the attaching sphere \( A_k \) of \( h_k \) over a \( k \)-disc \( D^k \times \{*\} \) in the upper boundary \( \partial_+ h'_k \) of \( h'_k \), as sketched in Figure 5. Depending on the relative orientations of \( A_k \) and \( \partial(D^k \times \{*\}) \), this amounts to replacing \( A_k \) by the connected sum \( A_k \#(\pm\partial(D^k \times \{*\})) \), which in \( \partial X \) is isotopic to \( A_k^\text{new} := A_k \#(\pm A'_k) \). For the \( k \)-handle \( h_k^\text{new} \) after the handle slide this implies
\[
\partial_k h_k^\text{new} = \partial_k (h_k \pm h'_k).
\]
In order to understand the effect of such a handle slide on the handle homology, we need to take into account that the \( (k+1) \)-handles that intersect the upper boundary \( \partial_+ h_k \) of \( h_k \) will be transformed by the handle slide. A very simple
example is shown in Figures 6 and 7. With respect to the orientations of the 1-handles shown in the figures, we can interpret $h_1^{\text{new}}$ as the sum of $h_1$ and $h'_1$. Indeed, we have $\partial_1 h_1^{\text{new}} = \partial_1(h_1 + h'_1)$.

The 2-handle $h_2$ with attaching circle $A_2$ shown in Figure 6 before the handle slide, with the standard orientation of $\mathbb{R}^2$, satisfies $\partial_2 h_2 = -h_1$. After the handle slide we have $\partial_2 h_2^{\text{new}} = h'_1 - h_1^{\text{new}}$. Notice that we can also write $\partial_2 h_2 = h'_1 - (h_1 + h'_1)$. In other words, the boundary operator before the handle slide has exactly the same form as the boundary operator after the handle slide, when in the former we replace the basis element $h_1$ by $h_1 + h'_1$. Similar relations hold for the 2-handle with attaching circle $A'_2$.

We now show that this is the general picture. Consider the attaching sphere $A_{k+1}$ of a $(k+1)$-handle $h_{k+1}$. The part $A_{k+1} \cap \partial_+ h_k$ in the upper boundary of the sliding handle $h_k$, which is a collection of $m := |A_{k+1} \cap B_k|$ copies of $D^k$, is simply moved along with $h_k$. The part $A_- := A_{k+1} \setminus \text{Int}(\partial_+ h_k)$ of the attaching sphere is a surface of genus zero with $m$ boundary circles. The orientation of each of these circles, as boundary of $A_-$, is the opposite of its orientation as the boundary of the respective component $D^k$ of $A_{k+1} \cap \partial_+ h_k$. After the handle slide, we obtain the boundary connected sum of $A_-$ with $m$ copies of $D^k \times \{\ast\}$ in $\partial_+ h'_k$, where the orientation of $D^k \times \{\ast\}$ depends on the orientation of the corresponding boundary component of $A_-$. Our comment on the orientation of $\partial A_-$ means that for the
Figures 6 and 7.

We now want to interpret this boundary connected sum as the successive attaching computation of the handle homology, after the handle slide $h \mapsto h + h'$, we may write

$$A_{\text{new}} = A_{-} \{ \mp (-1)^k(A_{k+1} \cdot B_{k})(D^k \times \{ * \}) \}.$$ 

Notice the slight abuse of notation: we have $m = |A_{k+1} \cap B_k|$ disjoint copies of $D^k \times \{ * \} \subset \partial h'$, but for the computation of the boundary operator, only the signed count $(-1)^k(A_{k+1} \cdot B_{k})$ of these discs matters.

Since $(-1)^k(D^k \times \{ * \}) \cdot B'_k = 1$, the boundary $\partial_{k+1}h_{k+1}^{\text{new}}$ is given by

$$\partial_{k+1}h_{k+1}^{\text{new}} = (-1)^k(A_{k+1} \cdot B_k)(h_k \mp h'_k) + (-1)^k(A_{k+1} \cdot B'_k)h'_k + \ldots$$

On the other hand, we have

$$\partial_{k+1}h_{k+1} = (-1)^k(A_{k+1} \cdot B_k)h_k + (-1)^k(A_{k+1} \cdot B'_k)h'_k + \ldots$$

$$= (-1)^k(A_{k+1} \cdot B_k)(h_k \pm h'_k) + (-1)^k(A_{k+1} \cdot B'_k)h'_k + \ldots$$

Thus, as in our simple example, the new chain complex is given by replacing the basis element $h_k$ with $h_k \pm h'_k$ in the old one.

One further comment is in order. In the case $k = n - 1$, the upper boundary $\partial h'_k = \partial h'_{n-1}$ is disconnected. If $D^{n-1} \times \{ * \} \subset \partial h'_{n-1}$ is a part of $A_{-}$, the slide of $h'_{n-1}$ over $h'_{n-1}$, which formally would lead to a boundary connected sum of $A_{-}$ with $D^{n-1} \times \{ * \}$ with the reversed orientation, amounts to a subtraction of $D^{n-1} \times \{ * \}$ from $A_{-}$. This is illustrated by the handle with attaching circle $A'_2$ in Figures 8 and 7.

4.2. Cancelling handle pairs. A cancelling pair of a $(k - 1)$- and a $k$-handle, attached to the boundary of an $n$-manifold $X$, can be described as follows, see Figure 8.

Write the boundary of the $k$-disc $D^k$ as the union of two discs:

$$\partial D^k = D^k_{+1} \cup D^k_{-1}.$$ 

The boundary connected sum of $X$ with an $n$-ball, which we think of as $D^k \times D^{n-k}$, along $D^{n-1} \times D^{n-k}$ does not change $X$ up to diffeomorphism, that is,

$$X \simeq X \cup D^{n-1} \times D^{n-k} (D^k \times D^{n-k}).$$

We now want to interpret this boundary connected sum as the successive attaching of a $(k - 1)$- and a $k$-handle. To this end, think of $D^k$ as the union of a smaller disc...
\(D^k_0\), whose boundary intersects \(D^{k-1}_0\) in a slightly smaller disc, and a rim \(D^{k-1}_+ \times D^1\), where \(D^{k-1}_+ \times \{1\}\) is identified with \(D^{k-1}_-\), and \(D^{k-1}_- \times \{-1\}\) with \(\partial D^0_0 \setminus D^{k-1}_-\):

\[
D^k = D^k_0 \cup D^{k-1}_- \times \{-1\} (D^{k-1}_+ \times D^1).
\]

Then

\[
X \cup D^{k-1}_- \times D^{n-k} (D^k \times D^{n-k}) \cong (X \cup \partial D^{k-1}_+ \times D^1 \times D^{n-k} (D^{k-1}_+ \times D^1 \times D^{n-k}))
\]

\[
\cup \partial D^k_0 \times D^{n-k} (D^k_0 \times D^{n-k}).
\]

Observe that the attaching sphere \(\partial D^k_0 \times \{0\}\) of the \(k\)-handle intersects the belt sphere \(\{0\} \times \partial (D^1 \times D^{n-k})\) of the \((k-1)\)-handle in a single point \((0, -1, 0) \in D^{k-1}_+ \times D^1 \times D^{n-k}\).

Thus, if the boundary connected sum with the ball is done away from the other handles, this interpretation as the introduction of a cancelling handle pair does not change the handle homology.

Conversely, if a \((k-1)\)-handle and a \(k\)-handle have been attached such that \(A_k\) intersects \(B_{k-1}\) transversely in a single point, provided the attaching is done ‘nicely’ in the sense of Section 2.2, one can identify the handle attachment with this standard model and hence remove this pair of handles.

We now want to describe the effect on the handle homology of removing a cancelling handle pair \((h^{\mu_0}_k, h^{\nu_0}_{k-1})\). Here we need to take into account that the attaching sphere \(A_k\) may also pass over \((k-1)\)-handles other than \(h^{\nu_0}_{k-1}\). Homotopically, this removal amounts to shrinking the core disc \(D^k_0 \times \{0\}\) of \(h^{\nu_0}_k\) to a point, so in the new chain complex we simply set \(h^{\nu_0}_k = 0\). Since \(A^{\mu_0}_k \bullet B^{\nu_0}_{k-1} = \pm 1\), in the old chain complex we have

\[
\partial_k h^{\nu_0}_k = \pm h^{\nu_0}_{k-1} + (-1)^{k-1} \sum_{\nu \neq \nu_0} (A^{\mu_0}_k \bullet B^{\nu}_{k-1}) h^{\nu}_{k-1}.
\]

The handle \(h^{\nu_0}_{k-1}\) is removed from the chain complex by setting the right-hand side of this equation equal to zero. Indeed, the diffeomorphism \(X \cong X \cup h^{\nu_0}_k \cup h^{\mu_0}_k\)

**Figure 8.** A cancelling handle pair.
amounts homotopically to isotoping the core disc
\[ D_{+}^{k-1} \equiv D_{+}^{k-1} \times \{(0, 0)\} \subset D_{+}^{k-1} \times D \times D^{n-k} \]
of \(h_{k-1}^{\nu} \) rel boundary to \(D_{+}^{k-1}\), where, as part of the attaching sphere of \(h_{k-1}^{\nu} \), it represents \(\mp (-1)^{k-1} \sum_{\nu \neq \nu_0} (A_{k}^{\nu} \bullet B_{k-1}^{\nu}) h_{k-1}^{\nu} \) in the chain complex.

How does this removal of the cancelling handle pair affect the other \(k\)-handles? Any \(h_{k}^{\mu} \), \(\mu \neq \mu_0\), may be assumed to intersect \(\partial_{+}h_{k-1}^{\nu_0} \) in discs of the form
\[ D_{+}^{k-1} \times \{(1, *)\} \subset D_{+}^{k-1} \times D \times D^{n-k}. \]

Again, the diffeomorphism \(X \cong X \cup h_{k-1}^{\nu_0} \cup h_{k}^{\mu_0} \) amounts homotopically to replacing each of these discs by \(D_{+}^{k-1}\). In other words, in the boundary \(\partial_{k} h_{k}^{\mu} \) for \(\mu \neq \mu_0\), we likewise have to replace \(h_{k-1}^{\nu_0} \) by \(\mp (-1)^{k-1} \sum_{\nu \neq \nu_0} (A_{k}^{\nu} \bullet B_{k-1}^{\nu}) h_{k-1}^{\nu} \) in the new chain complex.

We are now ready to show that this leaves the handle homology unchanged. Write \(C_{k}\) for the chain complex before, and \(C_{k}^{\prime}\) for the handle complex after removing the cancelling handle pair. Let
\[ m_{\mu} := (-1)^{k-1} (A_{k}^{\mu} \bullet B_{k-1}^{\nu_0}) \]
be the coefficient of \(h_{k-1}^{\nu_0} \) in the expansion of \(\partial_{k} h_{k}^{\mu} \). As basis for \(C_{k}\) we may use
\[ \{ h_{k}^{\nu} + m_{\mu} h_{k}^{\mu_0} \text{ for } \mu \neq \mu_0; h_{k}^{\mu_0} \}; \]
as basis for \(C_{k-1}\) we choose
\[ \{ h_{k-1}^{\nu} \text{ for } \nu \neq \nu_0; \pm h_{k-1}^{\nu_0} + (-1)^{k-1} \sum_{\nu \neq \nu_0} (A_{k}^{\nu} \bullet B_{k-1}^{\nu}) h_{k-1}^{\nu} \}; \]
The quotient groups \(C_{k}^{\prime}\) and \(C_{k-1}^{\prime}\) of \(C_{k}\) and \(C_{k-1}\) under the new relations can be identified with the subgroups given by removing the last vector in either basis. Then the boundary operator \(\partial_{k} : C_{k} \to C_{k-1}\) splits as
\[ \partial_{k} = \partial_{k}^{\prime} \oplus \text{id} : C_{k} \oplus \mathbb{Z} \to C_{k-1}^{\prime} \oplus \mathbb{Z}, \]
which shows that \(H_{k}^{\prime} = H_{k}\).

4.3. Orientations. In the case where \(M\) is orientable, we made a choice of orientations of \(M\) and the core discs of the handles to define the boundary operator over the integers, but the resulting homology is independent of this choice.

Changing the orientation of \(M\), while keeping the orientations of the core discs, will change the orientation of all belt spheres. This will leave the intersection numbers \(A_{k} \bullet B_{k-1}\) unchanged, because they are now computed with respect to the opposite ambient orientation.

Changing the orientation of the core disc of a handle \(h_{k}\), while keeping the ambient orientation, will also change the orientation of its belt sphere. So this change amounts to replacing \(h_{k}\) by \(-h_{k}\) as a generator of \(C_{k}\).

When \(M\) is not orientable, the argument why the choice of (co-)orientations for the \(A_{k}\) and \(B_{k}\), respectively, does not affect the resulting integral homology is analogous.
4.4. **Euler characteristic.** The Euler characteristic $\chi$ of a handle decomposition can be defined as the alternating sum over the number of $k$-handles in the decomposition. The topological invariance of $\chi$ is an immediate consequence of Theorem 4.1. This is a more direct argument for the invariance of $\chi$ than the usual algebraic reasoning [4, p. 146], based on showing that $\chi$ equals the alternating sum over the ranks of the homology groups.

5. **Poincaré duality**

A relative handle decomposition of $(M, \partial_- M)$ can be read ‘upside down’ as a relative handle decomposition of $(M, \partial_+ M)$. Any $k$-handle in the former becomes an $(n-k)$-handle in the latter, with the roles of lower and upper boundary, and that of attaching and belt sphere, reversed.

As has been observed before, see [3, p. 112], this yields a quick proof of Poincaré duality. We formulate it in the oriented case for integral (co-)homology. For non-orientable manifolds it remains true over $\mathbb{Z}_2$.

**Theorem 5.1.** Let $M$ be a compact, oriented $n$-manifold with boundary $\partial M = \partial_- M \sqcup \partial_+ M$. Then

$$H^k(M, \partial_- M) \cong H_{n-k}(M, \partial_+ M).$$

**Proof.** Reading the handle decomposition of $(M, \partial_- M)$ upside down gives a dual chain complex $C'_*(M, \partial_+ M)$ and the following commutative diagram:

$$
\begin{array}{ccc}
C_k(M, \partial_- M) & \xrightarrow{\partial_k} & C_{k-1}(M, \partial_- M) \\
\| & & \| \\
C'_n(M, \partial_+ M) & \xrightarrow{\partial'_{n-k+1}} & C'_{n-k+1}(M, \partial_+ M).
\end{array}
$$

The lower line computes $H_{n-4}(M, \partial_+ M)$.

Since, in the dual handle decomposition, the roles of attaching and belt spheres is reversed, the boundary operator $\partial'_{n-k+1}$ is, up to sign, simply the transpose of $\partial_k$. Therefore, the lower line may be read as

$$\text{Hom}(C_k(M, \partial_- M), \mathbb{Z}) \xleftarrow{\partial'_k} \text{Hom}(C_{k-1}(M, \partial_- M), \mathbb{Z}).$$

So the lower line also computes $H^*(M, \partial_- M)$.

**Remark 5.2.** In the non-orientable situation, $\partial'_{n-k+1}$ is not, in general, the transpose of $\partial_k$ (even up to sign). For instance, in the standard handle decomposition of the real projective plane $\mathbb{RP}^2$ with a single 0, 1- and 2-handle each, $\partial_2, \partial'_2: \mathbb{Z} \to \mathbb{Z}$ are multiplication by 2, while $\partial_1$ and $\partial'_1$ are the zero homomorphism, so no new information is gained. The reason is that the attaching sphere of the 1-handle $h_1$ counts as two points of opposite sign, while the coorientations of the two points making up the belt sphere of $h_1$ both correspond to the same orientation of the attaching circle $A_2$ of the 2-handle; see also the next section. When turning this handle decomposition upside down, this whole picture becomes reversed.
6. Orientability and top homology

Let $M$ be a connected, closed $n$-dimensional manifold. We claim that the top-dimensional integral handle homology of $M$ is given by

$$H_n(M) \cong \begin{cases} \mathbb{Z} & \text{if } M \text{ is orientable,} \\ 0 & \text{otherwise.} \end{cases}$$

By handle cancellation we may assume that $M$ contains a single 0-handle $h_0$. Each 1-handle $h_1$ is attached by an embedding $\partial_- h_1 \to \partial h_0$. If each of these embeddings is orientation preserving for the boundary orientation of $\partial_- h_1 \subset h_1$ (and a suitable orientation of $h_1$), the manifold $M$ is orientable. If there is at least one embedding $\partial_- h_1 \to \partial h_0$ where the embeddings of the two components of $\partial_- h_1$ have opposite orientation behaviour, $M$ is not orientable.

We now compute the homology by turning this handle decomposition upside down as in Section 5. The crucial observation, as in Remark 5.2, is that $\partial_- h_1 = \{\pm 1\} \times D^{n-1}$, where the two $(n-1)$-discs inherit opposite orientations, while in the interpretation of $h_1$ as an $(n-1)$-handle in the reverse handle decomposition, $h_1 = D^1 \times D^{n-1} = h'_{n-1}$, the belt sphere $B'_{n-1} = (\pm 1, 0)$ consists of two points with the same coorientation. Thus, if $\partial_- h_1 \to \partial h_0$ is orientation preserving, we have $A'_{n-1} \bullet B'_{n-1} = 0$; if $\partial_- h_1 \to \partial h_0$ maps the two components with opposite orientation behaviour (with respect to the boundary orientation), we have $A'_{n-1} \bullet B'_{n-1} = \pm 2$.

We conclude that $h'_{n-1}$ is a cycle generating the top-dimensional homology in the orientable case, while $\partial'_{n-1} \neq 0$ in the non-orientable case.

Acknowledgements. We thank the referee for very constructive comments. In particular, the discussion of handle homology over the integers in the case of non-orientable manifolds is based on a suggestion by the referee. H. G. is partially supported by the SFB/TRR 191 “Symplectic Structures in Geometry, Algebra and Dynamics”, funded by the Deutsche Forschungsgemeinschaft. M. K. is supported by the Berlin Mathematical School.

References

[1] J. Cerf, La stratification naturelle des espaces de fonctions différentiables réelles et le théorème de la pseudo-isotopie, Inst. Hautes Études Sci. Publ. Math. 39 (1970), 5–173.
[2] H. Geiges, How to depict 5-dimensional manifolds, Jahresber. Dtsch. Math.-Ver. 119 (2017), 221–247.
[3] R. E. Gompf and A. I. Stipsicz, 4-Manifolds and Kirby Calculus, Grad. Stud. Math. 20, American Mathematical Society, Providence, RI (1999).
[4] A. Hatcher, Algebraic Topology, Cambridge University Press, Cambridge (2002).
[5] M. Schwarz, Morse Homology, Progr. Math. 111, Birkhäuser Verlag, Basel (1993).
[6] J. Weber, The Morse–Witten complex via dynamical systems, Expo. Math. 24 (2006), 127–159.