Asymptotic Scaling in the Two-Dimensional $SU(3)$ $\sigma$-Model at Correlation Length $4 \times 10^5$

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Abstract

We carry out a high-precision simulation of the two-dimensional $SU(3)$ principal chiral model at correlation lengths $\xi$ up to $\approx 4 \times 10^5$, using a multi-grid Monte Carlo (MGMC) algorithm. We extrapolate the finite-volume Monte Carlo data to infinite volume using finite-size-scaling theory, and we discuss carefully the systematic and statistical errors in this extrapolation. We then compare the extrapolated data to the renormalization-group predictions. For $\xi \gtrsim 10^3$ we observe good asymptotic scaling in the bare coupling; at $\xi \approx 4 \times 10^5$ the nonperturbative constant is within 2–3% of its predicted limiting value.

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A key tenet of modern elementary-particle physics is the asymptotic freedom of four-dimensional nonabelian gauge theories [1]. However, the nonperturbative validity of asymptotic freedom has been questioned [2]; and numerical studies of lattice gauge theory have thus far failed to detect asymptotic scaling in the bare coupling [3]. Even in the simpler case of two-dimensional nonlinear $\sigma$-models [4], numerical simulations at correlation lengths $\xi \sim 10–100$ have often shown discrepancies of order 10–50% from asymptotic scaling. In a recent paper [5] we employed a finite-size-scaling extrapolation method [6, 7, 8, 9] to carry simulations in the $O(3)$ $\sigma$-model to correlation lengths $\xi \approx 10^5$; the discrepancy from asymptotic scaling decreased from $\approx 25\%$ to $\approx 4\%$. In the present Letter we apply a similar technique to the $SU(3)$ principal chiral model, reaching correlation lengths $\xi \approx 4 \times 10^5$ with errors $\lesssim 2\%$. For $\xi \gtrsim 10^3$ we observe good asymptotic scaling in the bare parameter $\beta$; moreover, at $\xi \approx 4 \times 10^5$ the nonperturbative ratio $\xi_{\text{observed}}/\xi_{\text{theor};3\text{-loop}}$ is within 2–3% of the predicted limiting value.

We study the lattice $\sigma$-model taking values in the group $SU(N)$, with nearest-neighbor action $\mathcal{H}(U) = -\beta \sum \text{Re \, tr}(U_x^\dagger U_y)$. Perturbative renormalization-group computations predict that the infinite-volume correlation lengths $\xi^{(\exp)}$ and $\xi^{(2)}$ behave as

$$\xi^\#(\beta) = C_{\xi^\#} e^{4\pi\beta/N} \left(\frac{4\pi\beta}{N}\right)^{-1/2} \left[1 + \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \cdots\right]$$

as $\beta \to \infty$. Three-loop perturbation theory yields [12]

$$a_1 = -0.121019N + 0.725848N^{-1} - 1.178097N^{-3}.$$  

The nonperturbative constant $C_{\xi^{(\exp)}}$ has been computed using the thermodynamic Bethe Ansatz [13]:

$$C_{\xi^{(\exp)}} = \frac{\sqrt{e}}{16\sqrt{\pi}} \frac{\pi/N}{\sin(\pi/N)} \exp\left(-\pi \frac{N^2 - 2}{2N^2}\right).$$

The nonperturbative constant $C_{\xi^{(2)}}$ is unknown, but Monte Carlo studies indicate
that $C_{\xi(2)}/C_{\xi(\exp)}$ lies between $\approx 0.985$ and 1 for all $N \geq 2$ \cite{14}; for $N = 3$ it is $0.987 \pm 0.002$ \cite{12}. Monte Carlo studies \cite{16, 17, 18, 12} of the $SU(3)$ model up to $\xi \approx 35$ have failed to observe asymptotic scaling \cite{14}; the discrepancy from \cite{14} is of order 10–20%.

Our extrapolation method \cite{8} is based on the finite-size-scaling (FSS) Ansatz

$$\frac{O(\beta, sL)}{O(\beta, L)} = F_O\left(\xi(\beta, L)/L; s\right) + O\left(\xi^{-\omega}, L^{-\omega}\right),$$

(4)

where $O$ is any long-distance observable, $s$ is a fixed scale factor (here $s = 2$), $L$ is the linear lattice size, $F_O$ is a universal function, and $\omega$ is a correction-to-scaling exponent. We make Monte Carlo runs at numerous pairs $(\beta, L)$ and $(\beta, sL)$; we then plot $O(\beta, sL)/O(\beta, L)$ versus $\xi(\beta, L)/L$, using those points satisfying both $\xi(\beta, L) \geq$ some value $\xi_{\text{min}}$ and $L \geq$ some value $L_{\text{min}}$. If all these points fall with good accuracy on a single curve, we choose a smooth fitting function $F_O$. Then, using the functions $F_\xi$ and $F_O$, we extrapolate the pair $(\xi, O)$ successively from $L \rightarrow sL \rightarrow s^2L \rightarrow \ldots \rightarrow \infty$. See \cite{8} for how to calculate statistical error bars on the extrapolated values.

We have chosen to use functions $F_O$ of the form

$$F_O(x) = 1 + a_1e^{-1/x} + a_2e^{-2/x} + \ldots + a_ne^{-n/x}.$$  

(5)

We increase $n$ until the $\chi^2$ of the fit becomes essentially constant; the resulting $\chi^2$ value provides a check on the systematic errors arising from corrections to scaling and/or from inadequacies of the form \cite{5}. The discrepancies between the extrapolated values from different $L$ at the same $\beta$ can also be subjected to a $\chi^2$ test. Further details on the method can be found in \cite{8, 5}.

We simulated the two-dimensional $SU(3)$ $\sigma$-model using an $XY$-embedding multigrid Monte Carlo (MGMC) algorithm \cite{11}. We ran on lattices $L = 8, 16, 32, 64, 128, 256$ at 184 different pairs $L$ in the range $1.65 \leq \beta \leq 4.35$ (corresponding to $5 \approx \xi_{\infty} \approx 4 \times 10^5$). Each run was between $4 \times 10^5$ and $5 \times 10^6$ iterations, and the total CPU time was one year on a Cray C-90 \cite{20}. The raw data will appear in \cite{21}. 

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Our FSS data cover the range $0.08 \lesssim x \equiv \xi(L)/L \lesssim 1.12$, and we found tentatively that for $\mathcal{O} = \xi$ a thirteenth-order fit (3) is indicated: see Table I. There are significant corrections to scaling in the regions $x \lesssim 0.84$ (resp. 0.64, 0.52, 0.14) when $L = 8$ (resp. 16, 32, 64): see the deviations plotted in Figure [I]. We therefore investigated systematically the $\chi^2$ of the fits, allowing different cuts in $x$ for different values of $L$: see again Table I. A reasonable $\chi^2$ is obtained when $n \geq 13$ and $x_{\text{min}} \geq (0.80, 0.70, 0.60, 0.14, 0)$ for $L = (8, 16, 32, 64, 128)$. Our preferred fit is $n = 13$ and $x_{\text{min}} = (\infty, 0.90, 0.65, 0.14, 0)$: see Figure 2, where we compare also with the perturbative prediction

$$F_\xi(x; s) = s \left[ 1 - \frac{aw_0 \log s}{2} x^{-2} - a^2 \left( \frac{w_1 \log s}{2} + \frac{w_0^2 \log^2 s}{8} \right) x^{-4} + O(x^{-6}) \right]$$

valid for $x \gg 1$, where $a = 2N/(N^2 - 1)$, $w_0 = N/(8\pi)$ and $w_1 = N^2/(128\pi^2)$.

The extrapolated values $\xi^{(2)}_\infty$ from different lattice sizes at the same $\beta$ are consistent within statistical errors: only one of the 58 $\beta$ values has a $\chi^2$ too large at the 5% level; and summing all $\beta$ values we have $\chi^2 = 64.28$ (103 DF, level = 99.9%).

In Table II we show the extrapolated values $\xi^{(2)}_\infty$ from our preferred fit and some alternative fits. The deviations between the different fits (if larger than the statistical errors) can serve as a rough estimate of the remaining systematic errors due to corrections to scaling. The statistical errors in our preferred fit are of order 0.5% (resp. 0.9%, 1.1%, 1.3%, 1.5%) at $\xi_\infty \approx 10^2$ (resp. $10^3$, $10^4$, $10^5$, $4 \times 10^5$), and the systematic errors are of the same order or smaller. The statistical errors at different $\beta$ are strongly positively correlated.

In Figure 3 (points + and ×) we plot $\xi^{(2)}_\infty, \text{estimate}_{(\infty, 0.90, 0.65)}$ divided by the two-loop and three-loop predictions (I)–(3) for $\xi^{(\text{exp})}$. The discrepancy from three-loop asymptotic scaling, which is $\approx 13\%$ at $\beta = 2.0$ ($\xi_\infty \approx 25$), decreases to 2–3% at $\beta = 4.35$ ($\xi_\infty \approx 3.7 \times 10^5$). For $\beta \gtrsim 2.2$ ($\xi_\infty \gtrsim 60$) our data are consistent with convergence to a limiting value $C_{\xi^{(2)}}/C_{\xi^{(\text{exp})}} \approx 0.99–1$ with the expected $1/\beta^2$ corrections.
We can also try an “improved expansion parameter” \[22, 12\] based on the energy $E = N^{-1} \langle \text{Re} \, \text{tr}(U_0^* U_1) \rangle$. First we invert the perturbative expansion \[12\]

$$E(\beta) = 1 - \frac{N^2 - 1}{4N\beta} \left[ 1 + \frac{N^2 - 2}{16N\beta} + \frac{0.0756 - 0.0634 N^2 + 0.01743 N^4}{N^2\beta^2} + O(1/\beta^3) \right]$$

(7)

and substitute into (4); this gives a prediction for $\xi$ as a function of $1 - E$. For $E$ we use the value measured on the largest lattice (which is usually $L = 128$); the statistical errors and finite-size corrections on $E$ are less than $5 \times 10^{-4}$, and they induce an error less than $0.85\%$ on the predicted $\xi_\infty$ (less than $0.55\%$ for $\beta \geq 2.2$). The corresponding observed/predicted ratios are also shown in Figure 3 (points □ and ◆). The “improved” 3-loop prediction is extremely flat, and again indicates a limiting value $\approx 0.99$.

Further discussion of the conceptual basis of our analysis can be found in [5]. Details of this work, including an analysis of the susceptibility $\chi$, will appear elsewhere [21].

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[10] Here $\xi^{(\exp)}$ is the exponential correlation length (= inverse mass gap), and $\xi^{(2)}$ is the second-moment correlation length defined by (4.11)–(4.13) of Ref. [11]. Note
that $\xi^{(2)}$ is well-defined in finite volume as well as in infinite volume; where necessary we write $\xi^{(2)}(L)$ and $\xi^{(2)}_\infty$, respectively. In this paper, $\xi$ without a superscript denotes $\xi^{(2)}$.

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Figure 1: Deviation of points from fit to $F_{\xi}$ with $s = 2$, $n = 13$, $x_{\text{min}} = (\infty, \infty, \infty, 0.14, 0)$. Symbols indicate $L = 8$ (●), 16 (×××××), 32 (+). Error bars are one standard deviation. Curves near zero indicate statistical error bars (± one standard deviation) on the function $F_{\xi}(x)$. 
Figure 2: $\xi(\beta, 2L)/\xi(\beta, L)$ versus $\xi(\beta, L)/L$. Symbols indicate $L = 8$ (±), 16 (×××), 32 (+), 64 (×), 128 (□). Error bars are one standard deviation. Solid curve is a thirteenth-order fit in (5), with $x_{\text{min}} = (\infty, 0.90, 0.65, 0.14, 0)$ for $L = (8, 16, 32, 64, 128)$. Dashed curve is the perturbative prediction (8).
Figure 3: $\xi_{\infty,\text{estimate}}(\infty,0.90,0.65)/\xi_{\infty,\text{theor}}^{(\exp)}$ versus $\beta$. Error bars are one standard deviation (statistical error only). There are four versions of $\xi_{\infty,\text{theor}}^{(\exp)}$: standard perturbation theory in $1/\beta$ gives points + (2-loop) and $\times$ (3-loop); “improved” perturbation theory in $1 - E$ gives points $\square$ (2-loop) and $\diamond$ (3-loop). Dotted line is the Monte Carlo prediction $C_{\xi^{(2)}}/C_{\xi^{(\exp)}} = 0.987 \pm 0.002$ [12].
Table 1: Degrees of freedom (DF), $\chi^2$, $\chi^2$/DF and confidence level for the $n^{th}$-order fit (5) of $\xi(\beta, 2L)/\xi(\beta, L)$ versus $\xi(\beta, L)/L$. The indicated $x_{\min}$ values apply to $L = 8, 16, 32$, respectively; we always take $x_{\min} = 0, 0$ for $L = 64, 128$. Our preferred fit is shown in italics; other good fits are shown in sans-serif; bad fits are shown in roman.

| $x_{\min}$       | $n = 11$ | $n = 12$ | $n = 13$ | $n = 14$ | $n = 15$ |
|-------------------|----------|----------|----------|----------|----------|
| $(0.50,0.40,0)$   | 180 718.80 | 179 626.60 | 178 560.20 | 177 558.60 | 176 558.30 |
|                   | 3.99 0.0%  | 3.50 0.0%  | 3.15 0.0%  | 3.16 0.0%  | 3.17 0.0%  |
| $(\infty,0.40,0)$| 154 673.80 | 153 566.30 | 152 533.00 | 151 532.10 | 150 531.80 |
|                   | 4.38 0.0%  | 3.70 0.0%  | 3.51 0.0%  | 3.52 0.0%  | 3.55 0.0%  |
| $(\infty,\infty,0)$ | 108 236.00 | 107 172.40 | 106 154.80 | 105 154.70 | 104 153.40 |
|                   | 3.99 0.0%  | 2.19 0.0%  | 1.61 0.0%  | 1.46 0.1%  | 1.47 0.1%  |
| $(0.70,0.55,0.45)$| 162 288.30 | 161 219.20 | 160 183.00 | 159 182.50 | 158 182.30 |
|                   | 1.78 0.0%  | 1.36 0.2%  | 1.14 10.3% | 1.15 9.8%  | 1.15 9.0%  |
| $(0.75,0.60,0.50)$| 150 222.40 | 149 172.20 | 148 129.90 | 147 129.80 | 146 129.80 |
|                   | 1.48 0.0%  | 1.16 9.4%  | 0.88 85.6% | 0.88 84.3% | 0.89 82.9% |
| $(0.80,0.70,0.60)$| 129 173.90 | 128 135.00 | 127 96.30  | 126 96.28  | 125 94.31  |
|                   | 1.35 0.5%  | 1.05 32.0% | 0.76 98.1% | 0.76 97.7% | 0.75 98.1% |
| $(0.95,0.85,0.60)$| 111 150.30 | 110 107.20 | 109 77.62  | 108 77.62  | 107 75.67  |
|                   | 1.35 0.8%  | 0.97 55.8% | 0.71 99.0% | 0.72 98.8% | 0.71 99.1% |
| $(1.00,0.90,0.60)$| 105 139.20 | 104 100.90 | 103 70.74  | 102 70.73  | 101 67.50  |
|                   | 1.33 1.4%  | 0.97 56.7% | 0.69 99.4% | 0.69 99.2% | 0.67 99.6% |
| $(\infty,0.90,0.65)$ | 92 130.00 | 91 77.01   | 90 60.85   | 89 58.66   | 88 58.31   |
|                   | 1.41 0.6%  | 0.85 85.2% | 0.68 99.2% | 0.66 99.5% | 0.66 99.4% |
| $(\infty,\infty,0.65)$ | 78 96.09  | 77 56.51  | 76 49.55  | 75 46.63  | 74 45.94  |
|                   | 1.23 8.1%  | 0.73 96.2% | 0.65 99.2% | 0.62 99.6% | 0.62 99.6% |
| $(\infty,\infty,\infty)$ | 52 55.85  | 51 25.23  | 50 25.17  | 49 24.11  | 48 24.10  |
|                   | 1.07 33.2% | 0.49 99.9% | 0.50 99.9% | 0.49 99.9% | 0.50 99.8% |
Table 2: Estimated correlation lengths $\xi^{(2)}_\beta$ as a function of $\beta$, from various extrapolations. Error bar is one standard deviation (statistical errors only). All extrapolations use $s = 2$ and $n = 13$. The indicated $x_{min}$ values apply to $L = 8, 16, 32$, respectively; we always take $x_{min} = 0.14, 0$ for $L = 64, 128$. Our preferred fit is shown in italic; other good fits are shown in sans-serif; bad fits are shown in roman.