Rotating vacuum wormhole

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Abstract

We investigate whether self-maintained vacuum traversible wormhole can exist described by stationary but nonstatic metric. We consider metric being the sum of static spherically symmetric one and a small nondiagonal component which describes rotation sufficiently slow to be taken into account in the linear approximation. We study semiclassical Einstein equations for this metric with vacuum expectation value of stress-energy of physical fields as the source. In suggestion that the static traversible wormhole solution exists we reveal possible azimuthal angle dependence of angular velocity of the rotation (angular velocity of the local inertial frame) that solves semiclassical Einstein equations. We find that in the macroscopic (in the Plank scale) wormhole case a rotational solution exists but only such that, first, angular velocity depends on radial coordinate only and, second, the wormhole connects the two asymptotically flat spacetimes rotating with angular velocities different in asymptotic regions.
1. Introduction. The possibility of existence of static spherically-symmetrical traversible wormhole as topology-nontrivial solution to the Einstein equations has been first studied by Morris and Thorne in 1988 [1]. Since that time much activity has been developed in studying the wormhole subject (see, e.g., review by Visser [2]). Rather interesting is the possibility of existence of self-consistent wormhole solutions to semiclassical Einstein equations. Checking this possibility requires finding vacuum expectation value of the stress-energy tensor as functional of geometry and solving the Einstein equations with quantum backreaction, i.e. with such the induced stress-energy as a source.

Recently some arguments in favour of this possibility has been given. In Refs. [3, 4, 5] the gravity induced vacuum stress-energy tensor in the wormhole background has been found to violate energy conditions just as it is required for this tensor itself be the source for such the wormhole metric [1, 6]. (We consider physical vacuum of spin 1 and 1/2 massless fields in these papers). In Ref. [7] self-consistent spherically-symmetrical wormhole solution has been found numerically for the quantised scalar field vacuum playing the role of a source for gravitation.

The problem of existence of self-consistent static wormhole is a particular case of the more fundamental problem of self-consistent solutions to semiclassical Einstein equations with vacuum expectations of stress-energy tensor of physical fields as a source. In Ref. [8] it has been found that such the problem linearised over metric perturbations off Minkowski spacetime gives solutions (apart from some unphysical ones) coinciding with those for classical gravity wave problem. If, however, topology is not Minkowski one as in the wormhole case at hand, new solutions can appear such as static wormhole itself. The natural next step may be the search for stationary but nonstatic topology nontrivial solution, namely rotating vacuum wormhole. In the case of slow rotation one simply adds small nondiagonal polar-angle-time component of metric to the self-maintained static wormhole metric:

\[ ds^2 = \exp(2\Phi)dt^2 - d\rho^2 - r^2\left[d\theta^2 + \sin^2\theta \left(d\phi^2 + 2h d\phi dt\right)\right] \] (1)

where \( h(\rho, \theta) \) has the sense of angular velocity of the local inertial frame and will be called the angular velocity of rotation in what follows. The static metric \( \Phi(\rho), r(\rho) \) is assumed to exist as a self-consistent solution of static semiclassical Einstein equations, and solution for \( h \) is to be found. One can take \( h \) arbitrarily small in order to limit oneself to the theory linearised in \( h \). In practice, in order that the quadratic in \( h \) terms in the Riemann tensor could be disregarded, the space derivatives of \( h \) should be negligible as compared to the derivatives of the static part of metric in the scale of typical wormhole size. Note that in the linear approximation the only component of Einstein equations being new compared to the static case is the \( t\phi \) one, as it follows from symmetry considerations. The role of the source is played there by the induced vacuum energy flow (angle-time stress-energy component).

In the given note we show that the rotational solution exists at least in the case when the coefficient at the Weyl term in the induced stress-energy is large and if one can neglect other terms. As noted in Refs. [4, 5], this corresponds to the wormhole of macroscopic size (that is, large in Plank units). The rotational solution which we argue should exist
is such that $h$ depends on radial distance $\rho$ only and has different asymptotic limits in the two asymptotically flat spacetimes connected by wormhole. This, in particular, means that these two spacetimes cannot be "glued" together in asymptotic region, so that this wormhole cannot be considered as that connecting the two regions of the same asymptotically flat spacetime.

As for the general case when we do not assume the Weyl term to dominate, we find the two kinds of azimuthal angle $\theta$ dependence of angular velocity $h$ for which the radial and angular variables $\rho, \theta$ are separated and the Einstein equation with quantum backreaction for $h$ reduces to that for the function of purely $\rho$. One possibility is the above mentioned angle-independent angle velocity; another one is proportional to $\cos \theta$ velocity.

2. Classical rotation. By classical we mean rotation considered without taking into account corresponding quantum backreaction, i.e. induced vacuum energy flow. However, it is implied that the static wormhole problem is already solved (the static metric is found) with taking into account corresponding backreaction. We show (in the linearised in $h$ theory used throughout the paper) that only solution for $h$ not depending on $\theta$ is physically acceptable such that it has different finite limits at $\rho \to +\infty$ and at $\rho \to -\infty$.

The $t\phi$ Einstein equation of interest can be conveniently written using the tetrad components introduced like the following basic 1-forms $\omega^a = e^a_{\mu}dx^\mu$ [9]:

$$
\omega^0 = \exp(\Phi)dt, \ \omega^1 = (d\phi + h dt)r \sin \theta, \ \omega^2 = d\rho, \ \omega^3 = r d\theta.
$$

(2)

Taking the expressions for the Riemann tensor presented in Ref. [9] we find for the equation of interest:

$$
-R_{10} = -\frac{\exp(\Phi)}{r \sin \theta} R^t_{\phi} = \frac{1}{2r^3} \frac{\partial}{\partial \rho} \exp(-\Phi) r^2 \frac{\partial h}{\partial \rho} \sin \theta
+ \frac{1}{2r} \exp(-\Phi) \frac{1}{\sin^2 \theta} \frac{\partial}{\partial \theta} \sin^3 \theta \frac{\partial h}{\partial \theta} = 0.
$$

(3)

Remind that we neglect the induced vacuum energy flow $T^t_{\phi}$ in this section. Separating the variables, $h = f(\rho)Y(\theta)$, we find hypergeometric function for the angle dependence:

$$
Y \sim F(a, 3 - a; 2; \frac{1 - \cos \theta}{2}),
$$

(4)

$a = \text{const.}$ This function diverges at $\theta = \pi$ unless $a = -k, k = 0, 1, 2, \ldots$; in the latter case it reduces to the Gegenbauer polynomial $C^{(3/2)}_k(\cos \theta)$. Then the radial part satisfies the equation

$$
(r^4 \exp(-\Phi)f')' = k(k + 3)r^2 \exp(-\Phi)f.
$$

(5)

By asymptotical flatness $\Phi \to 0$, $r \to \rho$ at $\rho \to \pm \infty$ and thus $f \sim \rho^{-k-3}$ at large distances. Another solution, $f \sim \rho^k$, should be discarded at $k \geq 1$ as unphysical one. Therefore if we choose some large $L > 0$, we shall have solutions at $|\rho| > L$ parametrised.
by two constants $C_{\pm}$ ($f \to C_{\pm}\rho^{-k-3}$ at $\rho \to \pm\infty$), whereas in the intermediate region $|\rho| < L$ the Eq. (3) is regular in the wormhole geometry and has a solution parametrised by two constants $C_1, C_2$. The four constants $C_1, C_2, C_+, C_-$ are subject to four uniform equations which are matching conditions for $f$ and its derivative $f'$ at $\rho = +L$ and at $\rho = -L$. Requiring for this system to have nonzero solution, i.e. zero determinant, we get some constraint on the already known metric functions $r(\rho), \Phi(\rho)$. Therefore the set of possible solutions for $f(\rho)$ at $k \geq 1$ has zero measure as compared to the set of possible static solutions $r, \Phi$. At $k = 0$ the Eq. (5) can be easily integrated and leads to physically admissible $h$ not depending on $\theta$ and being monotonic function of $\rho$ which has finite but different limits at $\rho \to \pm\infty$: $h(-\infty) \neq h(+\infty)$. Because of the latter circumstance the two asymptotically flat spacetimes connected by the wormhole channel cannot be "glued" together in asymptotic region (glueing at a shorter distance would spoil spherical symmetry of the background static solution described by two functions $r, \Phi$) so we cannot derive from such the rotating wormhole the wormhole connecting the two distinct regions of the same asymptotically flat spacetime.

3. Quantum backreaction and angle dependence of rotation. Now consider possible dependence of the angle velocity $h$ on the azimuthal angle $\theta$ that could solve the semiclassical Einstein equation (with backreaction). Here we show that the only two versions of the angle dependence for this equation to be solved by separation of variables are the following ones: $h = f(\rho)$ or $h = f(\rho)\cos \theta$.

Evidently, the problem reduces to studying the angle dependence of $T_{10}$ for a given angle dependence of $h$. Given any physical field, we should solve equations of motion for this field in curved spacetime and sum vacuum contributions into stress-energy from all the eigenmodes. The resulting expression can be regularised by, e.g., covariant geodesic point separation and renormalised by subtracting the divergent parts known for physical fields [10]. Choosing such separation in the radial direction we avoid discussing the renormalisation issue as far as the angle dependence is concerned.

Let us consider general structure of the equations of motion and stress-energy for arbitrary field in the metric (1) and illustrate this by the case of massless fields of spin 1 (electromagnetic) and 1/2 (neutrino). Most natural to display effect of rotation in the stationary axisymmetrical nonstatic metric is to use the complex Newman-Penrose formalism in analogy with that applied to Kerr metric [9]. In particular, up to the linear order in $h$, we can choose isotropic tetrad of real $l^\mu, n^\mu$ and complex $m^\mu, m^{\*\mu}$ (asterics means complex conjugation) of which $l^\mu, n^\mu$ are tangential to some geodesics and $m^\mu$ (and thus $m^{\*\mu}$) are orthogonal to $l^\mu, n^\mu$. Besides that, normalisation can be chosen such that $l_\mu n^\mu = 1, m^\mu m^{\*\mu} = -1$. These are the distinctive properties of the Newman-Penrose tetrad which can be written with the help of the derivatives over directions as

$$
\begin{align*}
l^\mu \partial_\mu &= \exp (-2\Phi)[\partial_t + \exp (\Phi)\partial_\rho - h \partial_\phi], \\
n^\mu \partial_\mu &= \frac{1}{2}[\partial_t - \exp (\Phi)\partial_\rho - h \partial_\phi], \\
m^\mu \partial_\mu &= (r\sqrt{2})^{-1}[\partial_\theta + i (\sin \theta)^{-1}\partial_\phi],
\end{align*}
$$

(6)
\[ m^{\mu \nu} \partial_\mu = (r \sqrt{2})^{-1} [\partial_\theta - i (\sin \theta)^{-1} \partial_\phi]. \]

Calculation gives the following values for the standard 12 complex spin-connection coefficients \( \kappa, \nu, \sigma, \lambda, \varepsilon, \varrho, \mu, \tau, \pi, \alpha, \beta, \gamma \)\[9\]:

\[
\begin{align*}
\kappa &= \nu = 0, \quad \sigma = 2 \exp (-2\Phi) \lambda = -2\varepsilon = \frac{i}{2} \exp (-2\Phi) h_\theta \sin \theta, \\
\varrho &= 2\mu = -\frac{r'}{r} \exp (-\Phi), \quad -\tau = \pi = \frac{i}{2\sqrt{2}} \exp (-\Phi) h_\rho r \sin \theta, \\
-\alpha &= \beta = \frac{1}{2\sqrt{2}} \cot \theta - \frac{i}{4\sqrt{2}} \exp (-\Phi) h_\rho r \sin \theta, \quad \gamma = \frac{1}{2} \Phi' \exp (\Phi) - \frac{i}{8} h_\theta \sin \theta
\end{align*}
\]

where subscript on \( h \) means corresponding derivative; prime on \( r, \Phi \) means derivative over \( \rho \). The main feature of the covariant equations of motion for an arbitrary physical field is therefore occurrence of \( h \) in the form \( h_\rho \sin \theta, h_\theta \sin \theta \) and \( h \partial_\phi \) there. In the diagrammatic language, \( h \)-field-field vertex is combination of these expressions. Besides that, in the Newman-Penrose formalism operators acting on the angle variables appear in the form of spin raising and lowering operators

\[
L_s = \partial_\theta - \frac{i \partial_\phi}{\sin \theta} + s \cot \theta, \quad L_s^+ = \partial_\theta + \frac{i \partial_\phi}{\sin \theta} + s \cot \theta,
\]

\( 0 \leq s \leq s_0 \), \( s_0 \) being spin of the field. These operators simplify in the basis of spin spherical harmonics proportional to the elements of rotation matrix \( D_{sm}^l(\phi, \theta, 0) \)\[4\]: \( L_s \) and \( L_s^+ \) transform \( D_{sm}^l \) to \( D_{s+m}^{l+1} \) and \( D_{s-m}^{l-1} \), respectively. In this basis the expression for \( T_{10} \) turns out to be a value of spin weight \( s = \pm 1 \) (and \( m = 0 \)), that is, combination of \( D_{10}^k \) (or \( D_{10}^{-k} \)) for different \( l \). Quantum contribution to \( T_{10} \) can be viewed as some loop diagram with external \( T_{10} \) and \( h \)-legs. Since \( h \) can be expanded in the Legendre polynomials \( P_k \sim D_{00}^k \), we can take \( D_{00}^k \) as probe function for the angle dependence of \( h \). Then the expressions \( h_\rho \sin \theta \) and \( h_\theta \sin \theta \) appearing in the vertex on \( h \)-line are combinations of the values \( D_{10}^{k+1} \) and \( D_{10}^{k-1} \). This corresponds to the angular momenta \( k + 1 \) and \( k - 1 \) flowing through the \( h \)-line. It is less evident but shown at the end of this section that the vertex \( h \partial_\phi \) corresponds to the combination of angular momenta \( k + 1, k - 1, k - 3, \ldots \). By conservation of angular momentum the \( T_{10} \) also should be combination of \( D_{k+2n}^{k+1-2n} \), \( n = 0, 1, 2, \ldots \). In particular, at \( k = 0, 1 \) the only term \( (D_{10}^1 \text{ or } D_{10}^2) \) remains and \( T_{10} \) factorises into the functions of \( \rho \) and of \( \theta \). Moreover, since \( D_{10}^1 \sim C_k^{1/2} \) just for \( k = 0, 1 \), the same \( \theta \)-dependence also factors out in the LHS of Einstein equation. Therefore we conclude: \( h = f(\rho) \) or \( h = f(\rho) \cos \theta \) solves for the \( \theta \)-dependence of semiclassical Einstein equation.

Finally, let us illustrate the above said by the examples of electromagnetic and neutrino fields; for more detail on the Newman-Penrose description of these fields see Ref.\[3\]. Electromagnetic field is described by three complex functions \( f_0, f_1, f_2 \); for our choice of complex tetrad \[6\] these are related to the electromagnetic field strength tensor \( F_{\mu \nu} \) as follows:

\[
2f_0 = F_{t\theta} + h F_{\theta \phi} + \exp (\Phi) F_{\rho \theta} + [F_{t \phi} + \exp (\Phi) F_{\rho \phi}] \frac{i}{\sin \theta},
\]
Analogously, massless fermion field is described by two complex values $g_1$, $g_2$ obeying the field equations

\begin{align}
2f_1 &= -r^2 \exp (-\Phi)(F_{\rho\phi} + hF_{\rho\phi}) + F_{\phi\phi} \frac{i}{\sin \theta}, \\
2f_2 &= -(F_{\theta\phi} + hF_{\theta\phi}) + [F_{t\phi} - \exp (\Phi)F_{\rho\phi}] \frac{i}{\sin \theta}.
\end{align}

The eight real Maxwell equations can be recast into the following four complex ones:

\begin{align}
\partial_+ f_1 - \mathcal{L}_1 f_0 &= h\partial_\phi f_1, \\
\partial_+ f_1 + \mathcal{L}_1^+ f_0 &= -h\partial_\phi f_1, \\
\partial_+ f_0 + \frac{\exp (2\Phi)}{r^2} \mathcal{L}_0^+ f_1 &= i\frac{f_0 - f_2}{2} h\sin \theta - i\hbar \exp (\Phi) \sin \theta f_1 - h\partial_\phi f_0, \\
\partial_+ f_2 - \frac{\exp (2\Phi)}{r^2} \mathcal{L}_0 f_1 &= -i\frac{f_0 - f_2}{2} h\sin \theta + i\hbar \exp (\Phi) \sin \theta f_1 + h\partial_\phi f_2,
\end{align}

where $\partial_{\pm} \equiv \exp (\Phi)\partial_{\rho} \mp \partial_{\theta}, \, i\partial_t = \omega$ being energy. Each mode should be normalised so that its full energy

\begin{align}
\int T^t_t r^2 \exp (\Phi) d\rho \sin \theta d\theta d\phi &= \int \sin \theta d\theta d\phi \exp (-\Phi) d\rho \left\{ f_0^* f_0 + f_2^* f_2 \\
&\quad + 2 \frac{\exp (2\Phi)}{r^2} f_1^* f_1 - 2 h \sin \theta \Im [(f_0 - f_2)^* f_1] \right\}
\end{align}

be equal to the vacuum value $\omega/2$ and then substituted into the expression for the stress-energy component studied,

\begin{align}
T_{10} \left( = \frac{\exp (\Phi)}{r \sin \theta} T^t_t \right) = -2 \frac{\exp (-\Phi)}{r^3} \Im [(f_0 - f_2)^* f_1].
\end{align}

Analogously, massless fermion field is described by two complex values $g_1$, $g_2$ obeying the field equations

\begin{align}
\partial_- g_1 + \frac{\exp (\Phi)}{r} \mathcal{L}_{1/2} g_2 &= \frac{i}{4} \sin \theta (g_1 h_{\theta} - r h_{\rho} g_2) + h\partial_\phi g_1, \\
\frac{\exp (\Phi)}{r} \mathcal{L}_{1/2}^+ g_1 - \partial_+ g_2 &= \frac{i}{4} \sin \theta (-r h_{\rho} g_1 - g_2 h_{\theta}) + h\partial_\phi g_2.
\end{align}

Expression for the energy of each mode takes the form (on the field equations)

\begin{align}
\int T^t_t r^2 \exp (\Phi) d\rho \sin \theta d\theta d\phi &= \int \sin \theta d\theta d\phi \exp (-\Phi) d\rho \cdot 4\omega (g_1^* g_1 + g_2^* g_2)
\end{align}

while the stress-energy component of interest is

\begin{align}
T_{10} &= \frac{\exp (-\Phi)}{r^3} \left\{ g_1^* \mathcal{L}_{1/2}^+ g_1 + (\mathcal{L}_{1/2}^+ g_1)^* g_1 - g_2^* \mathcal{L}_{1/2} g_2 - (\mathcal{L}_{1/2} g_2)^* g_2 - \partial_\theta (g_1^* g_1 - g_2^* g_2) \\
&\quad + (r\Phi' - \Phi')(g_2^* g_1 + g_1^* g_2) + 2 r \exp (-\Phi) [g_2^* (\partial_\theta - h\partial_\phi) g_1 - g_1^* (\partial_\theta - h\partial_\phi) g_2] \right\}.
\end{align}
Solutions to the above equations of motion can be constructed iteratively. In zero order one takes
\[
\begin{pmatrix}
f_0 \\
f_1 \\
f_2
\end{pmatrix}^{(0)} = \begin{pmatrix}
\partial_- R D^{l-1,m}_1 \\
-l(l+1) R D^{l}_{0m} \\
\partial_+ R D^{l+1}_1
\end{pmatrix}
\] (16)
which are the well-known TE-modes for the electromagnetic field (TM-modes follow by multiplying this by \(i = \sqrt{-1}\)) and
\[
\begin{pmatrix}
g_1 \\
g_2
\end{pmatrix}^{(0)} = \begin{pmatrix}
Z_1 D^{l+1/2,m}_1 \\
Z_2 D^{l-1/2,m}_1
\end{pmatrix}
\] (17)
for the neutrino field. The \(R\) and \(Z_1, Z_2\) are some radial functions. Substituting Eqs. (16) and (17) into the RHS of the equations of motion we find for the first \(O(h)\) correction an expression of the type
\[
\begin{pmatrix}
f_0 \\
f_1 \\
f_2
\end{pmatrix}^{(1)} = \sum_j i (-1)^m \begin{pmatrix}
\ldots D^{l-1,m}_j \\
\ldots D^{l}_{0m} \\
\ldots D^{l+1}_j
\end{pmatrix} \begin{pmatrix}
l k + 1 & j \ \\
0 & m \ - m
\end{pmatrix} + \begin{pmatrix}
\ldots D^{l-1,m}_j \\
\ldots D^{l}_{0m} \\
\ldots D^{l+1}_j
\end{pmatrix} \begin{pmatrix}
l k - 1 & j \ \\
0 & m \ - m
\end{pmatrix}
\] (18)
and quite analogous one for the fermion field. The dots mean (real) factors which do not depend on \(\theta, \phi\) and \(m\). The linear order in \(h\) of interest comes from interplay in the bilinear \(T_{10}\) between zero order (16) and first correction (18). By properties of 3j-symbols and rotation matrix elements \(D^{l}_{mn}\) summation over \(m\) just yields combination of the harmonics \(D^{l+p}_{10}, p = 1, -1, -3, \ldots\). Important is that the last term in (18) which stems from \(h\partial_\phi\) operator in the equations of motion is representable as combination of 3j-symbols as
\[
m \begin{pmatrix}
l k & j \ \\
m \ 0 \ - m
\end{pmatrix} = \sum_{n \geq 0} A_n \begin{pmatrix}
l k + 1 & 2n \ j \ \\
m \ 0 \ - m
\end{pmatrix}
\] (19)
where \(A_n\) does not depend on \(m\). In \(T_{10}\) this and other terms enter multiplied by \(D^{l}_{0m}(D^{l}_{\pm 1,m})^*\) and summed over \(m\) thus giving just combination of \(D^{l+p-2n}_{10}, n \geq 0\).

4. Macroscopic wormhole and radial dependence of rotation. Usually, if one does not assume existence of fundamental scales in the theory other than the Plank scale one expects the typical wormhole size be of the Plank scale too. However, a new scale can exist connected with coefficient of the Weyl term in the effective action. This coefficient is subject to renormalisation in both infrared (if massless fields are present in the theory) and ultraviolet regions. Possible large value of this coefficient can enable existence of the wormhole of macroscopic size.

Here we argue that if the Weyl term coefficient is large and one can disregard other terms in the effective action then the conclusion concerning the existence of rotating wormholes resembles that for the case of classical rotation in Sect. 2.
The effective gravity Lagrangian density with taking into account the Weyl term can be written as proportional to \( R + (2\mu^2)^{-1} C_{\mu\nu\lambda\rho} C^{\mu\nu\lambda\rho} \) [12]. Up to the full derivative, the Weyl term \( C_{\mu\nu\lambda\rho} C^{\mu\nu\lambda\rho} \) is equivalent to \( 2(R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2) \). We calculate the latter up to the second order in \( h \) (required to get the first order in the equations of motion) using Riemann tensor given in Ref. [3] in the tetrad components. Varying in \( h \) gives the desired \( t\phi \)-component of the Einstein equations. Consider both versions of \( \theta \)-dependence of \( h \) found in Sect. 3, \( h \sim 1 \) and \( h \sim \cos \theta \), and introduce new variable \( z \) via \( dz = \exp (-\Phi) d\rho \) and the function \( \tilde{r} = r \exp (-\Phi) \). For \( h \) not depending on \( \theta \), \( h = f(\rho) = f(\rho(z)) \), the result reads

\[
(\tilde{r}^4 f_z \exp(2\Phi))_z = \frac{1}{\mu^2} \left\{ \tilde{r}^4 \left[ \frac{1}{\tilde{r}^4} (\tilde{r}^4 f_z)_z \right] + \left[ \frac{10}{3} (\tilde{r}^2 - 1) + \frac{2}{3} \tilde{r}_{zz} \right] \tilde{r}^2 f_z \right\}_z \tag{20}
\]

(subscript \( z \) means differentiation over \( z \)). For \( h = f(\rho) \cos \theta \) we find

\[
(\tilde{r}^4 f_z \exp(2\Phi))_z - 4 \exp(2\Phi) \tilde{r}^2 f = \frac{1}{\mu^2} \left\{ \tilde{r}^4 \left[ \frac{1}{\tilde{r}^4} (\tilde{r}^4 f_z)_z \right] + \left( \frac{10}{3} \tilde{r}^2 + \frac{2}{3} \tilde{r}_{zz} \right) \tilde{r}^2 f_z \right\}_z + \frac{8}{3} (\tilde{r}_{zz} \tilde{r} + \tilde{r}_{z}^2 + 8) f \tag{21}
\]

The infrared contribution to the coefficient \( \mu^{-2} \) goes from the massless fields. In the considered case of \( \mu^{-2} \) large in the Plank scale the typical wormhole size \( r_0^2 \) is defined just by \( \mu^{-2} \) as \( r_0^2 = (3\mu^2)^{-1} \) [4, 5]. For example, in the vacuum of \( N_1 \) spin 1 and \( N_{1/2} \) spin 1/2 massless fields we have

\[
(3\mu^2)^{-1} = r_0^2 = \frac{G}{120\pi} (4N_1 + N_{1/2}) \ln \left( \frac{120\pi}{G} \frac{\Lambda^2}{4N_1 + N_{1/2}} \right), \tag{22}
\]

\( \Lambda \) being infrared cut off.

Consider first Eq. (21) which upon integrating both parts and denoting \( f_z \equiv g \) reduces to the second order one:

\[
\mu^2 \exp(2\Phi) \tilde{r}^4 g + C = \tilde{r}^4 \left[ \frac{1}{\tilde{r}^4} (\tilde{r}^4 g)_z \right] + \left[ \frac{10}{3} (\tilde{r}^2 - 1) + \frac{2}{3} \tilde{r}_{zz} \right] \tilde{r}^2 g \tag{23}
\]

where \( C = \text{const} \). Asymptotical \( \rho \to \pm\infty \) form of this equation reads

\[
\frac{d}{d\rho} \left( \frac{1}{\rho^4} \frac{d}{d\rho} \rho^4 g - \mu^2 g \right) = \frac{C}{\rho^4}. \tag{24}
\]

The general solution is the sum of a particular one which behaves as \( g \sim \rho^{-4} + O(\rho^{-6}) \) at \( \rho \to \pm \infty \) and arbitrary combination of the two independent solutions to the uniform equation. Of the latter two one exponentially grows at \( \rho \to +\infty \) or at \( \rho \to -\infty \) and should be omitted as unphysical solution while another one proportional to \( \exp(-\mu \rho) \) (at \( \rho \to +\infty \)) or \( \exp(\mu \rho) \) (at \( \rho \to -\infty \)) should be kept. Therefore, if we choose, as in Sect. 2, some large \( L > 0 \) we shall have physically acceptable solutions at \( |\rho| > L \) parametrised by three constants, one of which is \( C \). Meanwhile, in the intermediate
region $|\rho| < L$ the equation (23) is regular in the wormhole geometry and has solution parametrised by maximal set of three constants, one of which is $C$. The overall set of five constants is subject to four uniform equations which are matching conditions for $g$ and for its derivative at $\rho = +L$ and at $\rho = -L$. This defines all five constants up to an overall factor. Note that imposing additional condition $\int_{-\infty}^{+\infty} gdz = 0$ (that is, $h(-\infty) = h(+\infty)$) is, generally speaking, contradictory since it would be condition not on a freely chosen constant, but on the already defined static metric $\Phi(\rho)$, $r(\rho)$.

Next consider Eq. (21) which has asymptotic form (at $\Phi = 0$, $r = \rho$)

$$\rho^4 \frac{d}{d\rho} \frac{1}{\rho^4} \frac{d}{d\rho} \rho f = \frac{1}{\mu^2} \rho^4 \left( \frac{d}{d\rho} \frac{1}{\rho^4} \frac{d}{d\rho} \rho^4 f \right)^2.$$  

Assuming this form at $|\rho| > L$ we find

$$\frac{d}{d\rho} \frac{1}{\rho^4} \frac{d}{d\rho} \rho^4 f - \mu^2 f = C_{-1}^\pm + \frac{C_{+4}^\pm}{\rho^4}$$

where $C_{-1}^\pm$, $C_{+4}^\pm$ ($C_{-1}^-, C_{+4}^-$) are some constants which parametrise the solution at $\rho > L$ (at $\rho < -L$). To get physical solution we put $C_{-1}^\pm = 0$. Of the two solutions of uniform equation we discard exponentially growing one in each region $\rho > L$ or $\rho < -L$ and retain exponentially falling off another solution. Thus, in each region $\rho > L$ or $\rho < -L$ the solution is specified by two constants. At the same time, the regular fourth order differential equation has solution parametrised by four constants at $|\rho| < L$. The overall number of constants is eight. These should ensure validity of eight matching conditions for $f$, $f'$, $f''$ and $f'''$ at $\rho = \pm L$. The determinant of this uniform system should be zero. This imposes a constraint on the already known static metric $\Phi$, $r$. Therefore the subset of rotating wormhole solutions with $h = f(\rho) \cos \theta$ should have zero measure w.r.t. the set of spherically symmetrical static wormhole solutions $\Phi(\rho)$, $r(\rho)$.

5. Conclusion. We have shown that if the coefficient at the Weyl term is large (infrared cut off is large) and one can discard other terms in the effective action then the rotation existing for any static wormhole background $\Phi(\rho)$, $r(\rho)$ is that which proceeds with the angular velocity $h$ not depending on the azimuthal angle $\theta$ and having the different finite limits $h(+\infty)$ and $h(-\infty)$ in the asymptotic region $\rho \to \pm \infty$. We see that despite that the structure of the equations is drastically changed because of enhancing the maximal order of derivatives upon taking into account vacuum polarisation, the result practically does not differ from that for the classical case of Sect. 2.

Note that the sign of infrared divergent coefficient $\mu^{-2}$ at the Weyl tensor squared in the effective action is crucial for existence (or, rather, nonexistence) of more rotational solutions in our case. Were $\mu^{-2}$ substituted by negative value, the equations above would have oscillating instead of monotonic exponential solutions, and we would not have to omit some of them as unphysical ones. Then the general solution of interest would be parametrised by more constants, and the set of such solutions would be larger.

Also we can say that if we denote $\mu^2 \equiv x$ and extend the equations to arbitrary real $x$, the problem will be singular at $x = 0$. 

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In the case if the infrared logarithm is not large we do not have macroscopic vacuum wormhole, and the following interesting question arises: whether microscopic wormhole can rotate so that it would have the macroscopic "tail" of rotation (when \( h \) falls off in power law in asymptotic region). Answering this question implies rather complicated problem of calculation and analysis of the terms in stress-energy other than the Weyl term.

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References

[1] M.S.Morris and K.S.Thorne, \textit{Amer.J.Phys.} 56 (1988) 395.

[2] M.Visser, \textit{Lorentzian Wormholes: from Einstein to Hawking} (American Institute of Physics, Woodbury, 1995).

[3] V.M.Khatsymovskyy, \textit{Phys.Lett.} B320 (1994) 234.

[4] V.M.Khatsymovskyy, \textit{Phys.Lett.} B399 (1997) 215.

[5] V.M.Khatsymovskyy, \textit{Phys.Lett.} B403 (1997) 203.

[6] M.S.Morris, K.S.Thorne and U.Yurtsever, \textit{Phys.Rev.Lett.} 61 (1988) 1446.

[7] D.Hochberg, A.Popov and S.V.Sushkov, \textit{Self-consistent Wormhole Solutions of Semiclassical Gravity} (Preprint LAEFF 96/25, KSPU-96-03, [gr-qc/9701064]), \textit{Phys.Rev.Lett.} 78 (1997) 2050.

[8] G.T.Horowitz, \textit{Phys.Rev.} D21 (1980) 1445.

[9] S.Chandrasekhar, \textit{The mathematical theory of black holes} (Clarendon Press, Oxford, 1983).

[10] S.M.Christensen, \textit{Phys.Rev.} D17 (1978) 946.

[11] J.N.Goldberg, A.J.Macfarlane, E.T.Newman, F.Rohrlich, E.C.G.Sudarshan, \textit{J.Math.Phys.} 8 (1967) 2155.

[12] B.S.DeWitt, \textit{Phys.Rep.} C19 (1975) 295.