Symplectic extensions of the Kirillov-Kostant and Goldman Poisson structures and Fuchsian systems

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Abstract

We revisit symplectic properties of the monodromy map for Fuchsian systems on the Riemann sphere. We extend previous results of [19, 3, 22] where it was shown that the monodromy map is a Poisson morphism between the Kirillov-Kostant Poisson structure on the space of coefficients, on one side, and the Goldman bracket on the monodromy character variety on the other. The extension is provided by defining larger spaces on both sides which are equipped with symplectic structures naturally projecting to the canonical ones. On the coefficient side our symplectic structure corresponds to a non-degenerate quadratic Poisson structure expressed via the rational dynamical $r$-matrix; it reduces to the Kirillov-Kostant bracket when projected to the standard space. On the monodromy side we get a symplectic structure which induces the symplectic structure of [2] on the leaves of the Goldman Poisson bracket. We prove that the monodromy map provides a symplectomorphism using the formalism of Malgrange [24] and one of the authors [6, 7]. As a corollary we prove the recent conjecture by A.Its, O.Lisovyy and A.Prokhorov in its ”strong” version while the original ”weak” version is derived from previously known results. We show also that the isomonodromic Jimbo-Miwa tau-function is intimately related to a generating function of such transformation.

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1 Introduction

This paper is motivated by the theory of monodromy map for linear differential equation of second order on a Riemann surface. Symplectic aspects of such monodromy map were studied in several recent papers. In
it was proved that it is symplectic (under natural definition of symplectic structure on the space of potentials and on the monodromy character variety) when the potential is either holomorphic [21, 8] or has simple poles [23]. However, an important aspect of such symplectomorphism, namely, a complete understanding of the generating function of the monodromy map remains obscure. An importance of this problem lies in the theory of supersymmetric Yang-Mills equations (see [25]); various aspects and difficulties in description of such generating function were discussed in [25, 8].

In this paper we consider a different situation where generating functions of this type appear: the monodromy map for a system of linear differential equations on a Riemann sphere with poles of first order. On one hand this situation is more elementary than the monodromy map on a general Riemann surface; on the other hand it involves additional technical complications related to the degeneracy of some of Poisson structures and explicit dependence on moduli. We mention previous works [14, 19, 3, 22] and more recent papers [9, 12] where symplectic aspects of such monodromy map were studied.

In our present context the moduli of the punctured sphere (i.e. the positions of poles of coefficients of the equation) do not enter the Poisson structure and play the role of parameters; however, dependence on these parameters is of primary importance.

In this paper we address symplectic properties of a version of the monodromy map which is standard in the theory of isomonodromy deformations [26] but which is essentially different from the monodromy map as defined in [19, 3, 22] and other previous works on the subject.

Consider the Fuchsian equation

$$\frac{\partial \Psi}{\partial z} = \sum_{i=1}^{N} \frac{A_i}{z - t_i} \Psi$$

where $A_i \in \text{gl}(n)$ and

$$\sum_{i=1}^{N} A_i = 0$$

and impose an initial condition

$$\Psi(z = \infty) = 1$$

We assume also that eigenvalues of each $A_j$ are simple and furthermore do not differ by an integer. Choose a system of cuts $\gamma_1, \ldots, \gamma_N$ connecting $\infty$ with $t_1, \ldots, t_N$ respectively, and assume that the ends of these cuts emanating from $\infty$ are ordered as $(1, \ldots, N)$ counter-clockwise.

Then the solution $\Psi$ of (1.1) is uniquely defined in the simply connected domain $\mathbb{CP}^1 \setminus \{\gamma_j\}_{j=1}^{N}$. Denote the diagonal form of the matrix $A_j$ by $L_j$, $j = 1, \ldots, N$. Then the asymptotics of $\Psi$ near $t_j$ has the standard form [26]:

$$\Psi(z) = (G_j + O(z - t_j))(z - t_j)^{L_j}C_{j}^{-1}.$$  

The matrix $G_j$ is a diagonalizing matrix for $A_j$:

$$A_j = G_j L_j G_j^{-1}.$$

The matrices $C_{j}$ are called the connection matrices. Notice that the matrices $G_j$ and $C_j$ are not uniquely defined by equation (1.1) since a simultaneous transformation $G_j \rightarrow G_j D_j$ and $C_j \rightarrow C_j D_j$ with diagonal $D_j$’s changes neither the asymptotics (1.4) nor the equation (1.1).

Analytic continuation of $\Psi(z)$ around one of the points $t_j$ yields $\Psi(z)M_j$, where the monodromy matrix $M_j$ is related to the connection matrix $C_j$ and the exponent of monodromy $L_j$ by the relation:

$$M_j = C_j A_j C_j^{-1}, \quad A_j := e^{2i\pi L_j}.$$  

Our assumption about the ordering of the branch cuts $\gamma_j$ implies the relation

$$M_1 \cdots M_N = 1.$$  

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The monodromy map as proposed in [26] sends the set of pairs \((G_j, L_j)\) to the set of pairs \((C_j, \Lambda_j)\) for a given set of poles \(t_j\); it is this version of monodromy map whose symplectic properties we study in the paper.

The map between the set of coefficients \(A_j\) and the set of monodromy matrices \(M_j\) is a different (“weak”) version of monodromy map associated to equation (1.1); it was this version whose symplectic aspects were studied in [19, 3, 22].

Surprisingly enough, the symplectic formalism associated to monodromy map between \((G, L)\) and \((C, \Lambda)\) - spaces turns out to be rather different from the traditional one.

Let us introduce the following two spaces; the first space is the following quotient space

\[
\mathcal{A} = \left\{ (G_j, L_j)_{j=1}^N, \quad G_j \in GL(n), \quad L_j \in \mathfrak{h}_{ss}^nr, \forall j = 1, \ldots, N : \sum_{j=1}^N G_j L_j G_j^{-1} = 0 \right\} / \sim \tag{1.8}
\]

where \(\mathfrak{h}_{ss}^nr\) denotes the set of matrices with simple eigenvalues not differing by integers \((\text{non-resonant})\). The equivalence relation is given by the \(GL(n)\) action \(G_j \mapsto SG_j\) with \(S\) independent of \(j\). The second space is another quotient

\[
\mathcal{M} = \left\{ (C_j, \Lambda_j)_{j=1}^N, \quad C_j \in GL(n), \quad \Lambda_j \in T_{ss} : \prod_{j=1}^N C_j \Lambda_j C_j^{-1} = 1 \right\} / \sim \tag{1.9}
\]

where \(T_{ss}\) denotes the set of invertible diagonal matrices with distinct eigenvalues (an open-dense subset of the Cartan torus of \(GL(n)\)). Similarly to the above, the equivalence is given by the \(GL(n)\) action \(C_j \mapsto SC_j\) (with the same \(S\) for all \(j\)’s).

For a fixed set of poles \(\{t_j\}_{j=1}^N\) we denote the monodromy map induced by the Fuchsian ODE (1.1) by \(\mathcal{F}^t:\)

\[
\mathcal{F}^t : \mathcal{A} \to \mathcal{M} \tag{1.10}
\]

We observe that it is well defined independently of the choice of basepoint of normalization for the solution \(\Psi(z)\).

**Poisson and symplectic structures on \(\mathcal{A}\) and dynamical \(r\)-matrix.** To describe the natural Poisson bracket on \(\mathcal{A}\) we introduce the following Poisson structure on each pair \((G, L)\) where \(G \in GL(n)\) and \(L = \text{diag}(\lambda_1, \ldots, \lambda_n)\) (with \(\lambda_j \neq \lambda_k\)) as follows:

\[
\{ G, G \} = - \frac{1}{2} G^2 r(L), \quad \{ G, L \} = - G \Omega_h, \tag{1.11}
\]

where

\[
r(L) = \sum_{i<j} \frac{E_{i,j} \otimes E_{j,i} - E_{i,i} \otimes E_{j,j}}{\lambda_i - \lambda_j} \tag{1.12}
\]

for a diagonal matrix \(L = \text{diag}(\lambda_1, \ldots, \lambda_n)\) and \(\Omega_h = \sum_{i=1}^n E_{ii} \otimes E_{ii}\); we use the standard notation \(E_{ij}\) for the matrix with only one non-vanishing element equal to 1 in the \((i, j)\) entry. The matrix \(r(L)\) is a basic example of dynamical \(r\)-matrix [13]. We show in Appendix A that the above formula (1.11) is a symplectic Poisson bracket and we show in Theorem A.1 that it induces the standard Kirillov-Kostant Poisson bracket for \(A = GLG^{-1}\). However, in contrast to Kirillov-Kostant bracket the bracket (1.11) is non-degenerate; the corresponding symplectic form is given by

\[
\omega = - \text{tr}(LG^{-1}dG \wedge G^{-1}dG) + \text{tr}(dL \wedge G^{-1}dG) \tag{1.13}
\]

The form (1.13) is non-degenerate as long as the eigenvalues of \(L\) are distinct (Prop. A.1). Notice that \(\omega\) is in fact an exact form \(\omega = d\theta\) where

\[
\theta = \text{tr}(LG^{-1}dG) \tag{1.14}
\]
The bracket (1.11) can be used to define the Poisson structure on the space $\mathcal{A}$ as follows. First we define the Poisson structure of the unconstrained space $\mathcal{A}_0$ of pairs $(G_j, L_j)$ via

$$\{G_j, G_k\} = -G_j G_k \Omega(L_k) \delta_{jk}, \quad \{G_j, L_k\} = -G_k \Omega_h \delta_{jk},$$

(1.15)

The right $GL(n)$-action on the matrices $G_k$ is a Poisson action whose moment map is precisely

$$m = \sum_{j=1}^{N} G_j L_j G_j^{-1}$$

(1.16)

(Lemma 2.1, Theorem 2.1); then the standard symplectic reduction induces the bracket on the zero level set (1.8) of the moment map, quotient over the action of simultaneous left multiplication $G_j \to SG_j$ for $S \in GL(n)$. This Poisson structure on $\mathcal{A}$ turns out to be non-degenerate (see Theorem 2.1); the corresponding symplectic form is given by

$$\omega_{\mathcal{A}} = -\sum_{k=1}^{N} \text{tr}(L_k G_k^{-1} dG_k \wedge G_k^{-1} dG_k) + \sum_{k=1}^{N} \text{tr}(dL_k \wedge G_k^{-1} dG_k)$$

(1.17)

The natural choice of the symplectic potential for the form $\omega_{\mathcal{A}}$ is implied by (1.14):

$$\theta_{\mathcal{A}} = \sum_{k=1}^{N} \text{tr}(L_k G_k^{-1} dG_k)$$

(1.18)

The 1-form (1.18) is well-defined on the space $\mathcal{A}$; its invariance under simultaneous transformation $G_j \to SG_j$ is guaranteed by the vanishing of the moment map $m$ (1.16). Therefore, the form $\omega_{\mathcal{A}}$ is not only closed, but also exact on $\mathcal{A}$.

### Symplectic structure on $\mathcal{M}$.

We define the following 2-form on the space $\mathcal{M}$:

$$\omega_{\mathcal{M}} = \frac{1}{4\pi i} (\omega_1 + \omega_2)$$

(1.19)

where

$$\omega_1 = \text{tr} \sum_{\ell=1}^{N} \left( M_\ell^{-1} dM_\ell \wedge K_\ell^{-1} dK_\ell \right) + \text{tr} \sum_{\ell=1}^{N} \left( \Lambda_\ell^{-1} C_\ell^{-1} C_\ell^{-1} dC_\ell \right)$$

(1.20)

and

$$\omega_2 = 2 \sum_{\ell=1}^{N} \text{tr} \left( \Lambda_\ell^{-1} d\Lambda_\ell \wedge C_\ell^{-1} dC_\ell \right)$$

(1.21)

where $K_\ell = M_1 \cdots M_\ell$.

The form $\omega_{\mathcal{M}}$ is invariant under simultaneous transformation $C_j \to SC_j$ where $S$ is an $GL(n)$-valued function on the constraint surface $M_1, \ldots, M_N = 1$ (Theorem C.1) and therefore $\omega_{\mathcal{M}}$ is indeed defined on $\mathcal{M}$. The form $\omega_1/2$ in (1.19) coincides with the symplectic form on the symplectic leaves $\Lambda_j = \text{const}$ of the $GL(n)$ Goldman bracket (see (3.14) of [2] in the case $g = 0$; we notice that the term $\omega_2$ is analogous to the additional term in Alekseev-Malkin formula which appears in the higher genus version of (3.14)).

The first main result of this paper is the following

**Theorem 1.1** Given a set of fixed poles $\{t_j\}_{j=1}^{N}$ and a point $p_0 \in \mathcal{M}$ in a neighbourhood of which the monodromy map is invertible, the pullback of the form $\omega_{\mathcal{M}}$ under the map $F^t : \mathcal{A} \to \mathcal{M}$ coincides with $\omega_{\mathcal{A}}$, i.e.

$$(F^t)^* \omega_{\mathcal{M}} = \omega_{\mathcal{A}}$$

(1.22)

where the forms $\omega_{\mathcal{A}}$ and $\omega_{\mathcal{M}}$ are given by (1.17) and (1.19), respectively.
This theorem provides a generalization of results of [19, 3, 22] to our current setting. It implies the following

**Corollary 1.1** The form $\omega_M$ is closed and non-degenerate, and, therefore, defines a symplectic structure on $\mathcal{M}$.

Since, for given set of monodromy data, there is always a choice of poles for which the monodromy map is invertible [10], this also shows that the form $\omega_M$ is everywhere nondegenerate.

Theorem 1.1 is a generalization of the theorems in [19, 3, 22], where it was proved that the monodromy map between the smaller spaces - the space of coefficients $A_j$ with fixed eigenvalues and the $\text{GL}(n)$ character variety of $N$-punctured sphere is a symplectomorphism between corresponding symplectic leaves.

**Time dependence.** To assess the dependence of this picture on the $t_j$’s (the “times”) we extend the spaces $\mathcal{A}, \mathcal{M}$ to include also the coordinates $\{t_j\}$:

$$\tilde{\mathcal{A}} = \{(p, \{t_j\}_{j=1}^N) : p \in \mathcal{A}, \, t_j \in \mathbb{C}, \, t_j \neq t_k \}$$

$$\tilde{\mathcal{M}} = \{(p, \{t_j\}_{j=1}^N) : p \in \mathcal{M}, \, t_j \in \mathbb{C}, \, t_j \neq t_k \}$$

(1.23) (1.24)

The monodromy map $\mathcal{F}$ then naturally extends to the map

$$\mathcal{F} : \tilde{\mathcal{A}} \to \tilde{\mathcal{M}}$$

(1.25)

The map $\mathcal{F}$ can also be thought of as mapping to $\mathcal{M}$ by forgetting the time-variables; then $\mathcal{F}$ is surjective on $\mathcal{M}$ but not on $\tilde{\mathcal{M}}$. The locus in $\tilde{\mathcal{M}}$ where the map is not invertible is usually referred to as the Malgrange divisor. Denote the natural pullback of the form $\omega_A$ (1.17) from $\mathcal{A}$ to $\tilde{\mathcal{A}}$ by $\tilde{\omega}_A$ and the natural pullback of the form $\omega_M$ (1.19) from $\mathcal{M}$ to $\tilde{\mathcal{M}}$ by $\tilde{\omega}_M$ (notice that the forms $\tilde{\omega}_A$ and $\tilde{\omega}_M$ are closed but degenerate). Now we are in a position to formulate the next theorem

**Theorem 1.2** The following identity holds between two-forms on $\tilde{\mathcal{A}}$

$$\mathcal{F}^* \tilde{\omega}_M = \tilde{\omega}_A - \sum_{k=1}^N dH_k \wedge dt_k$$

(1.26)

where

$$H_k = \sum_{j \neq k} \text{tr} A_j A_k \frac{t_k - t_j}{t_k - t_j}$$

(1.27)

are the canonical Hamiltonians of the Schlesinger system.

We remind the reader that the Schlesinger equations [10] consist of the following system of PDEs for the coefficients of $A(z)$

$$\frac{\partial A_k}{\partial t_j} = \frac{[A_k, A_j]}{t_k - t_j}, \quad j \neq k; \quad \frac{\partial A_j}{\partial t_j} = -\sum_{k \neq j} \frac{[A_k, A_j]}{t_k - t_j}.$$  

(1.28)

and they express the deformations of the connection $A(z)$ which preserve the monodromy representation. They are Hamiltonian equations with respect to the standard Kirillov-Kostant Poisson bracket with time-dependent Hamiltonians $H_k$ as in (1.27).

The proof of Theorem 1.2 is rather technical and it is based on the formalism developed by Malgrange in [24] and one of the authors in [6, 7], see Section 3.
Tau functions and generating functions. Consider now some local symplectic potential $\theta_M$ for the form $\omega_M$ on the space $M$ and denote its pullback to $\tilde{M}$ by $\tilde{\theta}_M$. The potential for the form $\tilde{\omega}_A$ on $\tilde{A}$ is defined formally by the same formula (1.18):

$$
\tilde{\theta}_A = \sum_{k=1}^{N} \text{tr}(L_kG_k^{-1}dG_k) .
$$

Then (1.26) implies existence of a locally defined generating function $G$ on $\tilde{A}$ such that

**Definition 1.1** The generating function of monodromy map between spaces $\tilde{A}$ and $\tilde{M}$ is defined by

$$
dG = \tilde{\theta}_A - \sum_{j=1}^{N} H_j dt_j - \mathcal{F}^* \tilde{\theta}_M .
$$

This definition depends on the choice of symplectic potential $\theta_M$ on monodromy manifold $M$ (and, therefore, on its pullback $\tilde{\theta}_M$ to $\tilde{M}$). Change of the choice of $\theta_M$ adds a monodromy dependent term i.e. this change corresponds to a transformation $G \rightarrow G + f(\{C, L\})$.

The dependence of $G$ on $\{t_j\}$ is, however, completely fixed by (1.30). Namely, locally one can write (1.30) in the coordinate system where $\{t_j\}_{j=1}^{N}$ and $\{C_j, L_j\}_{j=1}^{N}$ are considered as independent variables. Then derivatives of $G_j$ on $\{t_k\}$ for constant $\{C_j, L_j\}_{j=1}^{N}$ i.e. for constant monodromy data, are given by Schlesinger equations of isomonodromic deformations:

$$
\frac{\partial G_k}{\partial t_j} = \frac{A_j G_k}{t_j - t_k}, \quad \frac{\partial G_k}{\partial t_k} = - \sum_{k \neq j}^{N} \frac{A_j G_k}{t_j - t_k} .
$$

A direct verification shows that the equations (1.31) are Hamiltonian

$$
\frac{\partial G_k}{\partial t_j} = \{H_j, G_k\}.
$$

with the extension of Kirillov-Kostant Poisson bracket (1.11) and the Hamiltonians (1.27). Then it is easy to compute that in $(t_j, C_j, L_j)$ coordinates the part of $\tilde{\theta}_A$ containing $dt_j$’s is given by $-2 \sum_{j=1}^{N} H_j dt_j$; together with (1.30) this implies

$$
\frac{\partial G}{\partial t_j} = H_j .
$$

Therefore, we get the following theorem:

**Theorem 1.3** For any choice of symplectic potential $\theta_M$ on $M$ the dependence of generating function $G$ (1.30) on $\{t_j\}_{j=1}^{N}$ coincides with $t_j$-dependence of the isomonodromic Jimbo-Miwa tau-function. In other words, $e^{-\tilde{\theta}_M}$ depends only on monodromy data $\{C_j, L_j\}_{j=1}^{N}$.

Theorem 1.3 shows that the generating function $G$ can be used to define the Jimbo-Miwa tau-function not only as a function of positions of singularities of the fuchsian differential equation but also as a function of monodromy matrices. The ambiguity built into this definition corresponds to the freedom to choose different symplectic potentials on the monodromy manifold.

**Conjecture by A.Its, O.Lisovyy and A.Prokhorov.** Theorem 1.2 emphasizes a close relationship of this paper with the recent work [20] where the issue of dependence of the Jimbo-Miwa tau-function on monodromy matrices was also addressed. In particular, the relevance of the Goldman bracket and corresponding symplectic form on its symplectic leaves was observed in [20] in the case of $2 \times 2$ system with four simple poles (the associate isomonodromic deformations give Painlevé 6 equation).

Moreover, the authors of [20] introduced a form $\mu$ (denoted by $\omega$ in (2.7) of [20] but we prefer to change the notation since $\omega$ is reserved for various two-forms in this paper; this form appeared in [20]
as a result of computation involving the 1-form introduced by Malgrange in [24], similarly to this work) which in our notations is given by

$$
\mu = \sum_{j<k} tr A_j A_k d \log (t_j - t_k) + \sum_{j=1}^N tr (L_j G_j^{-1} d^{(m)} G_j) \quad (1.34)
$$

where $d^{(m)}$ defines the differential with respect to monodromy data. Proposition 2.3 of [20] shows that the form (1.34) $d\mu$ is a closed 2-form independent of $\{t_j\}_{j=1}^N$. Furthermore, in Section 1.6 the authors of [20] formulate the following

**Conjecture 1** [Its-Lisovyy-Prokhorov] The form $d\mu$ coincides with the symplectic form on monodromy manifold.

There are two natural versions of this conjecture:

- **The ”weak” ILP conjecture.** In this version (which is really how this conjecture was formulated in [20]) the differential $d^{(m)}$ in (1.34) means the differential on a symplectic leaf $\{L_j = const\}_{j=1}^N$ of the $GL(n)$ character variety of $\pi_1(CP^1 \setminus \{t_j\}_{j=1}^N)$ (we denote this symplectic leave by $M_A$). The canonical symplectic form on $M_A$ is given by inversion of the $GL(n)$ Goldman’s bracket [17] and can be written explicitly in terms of monodromy data as shown in (1.1) for $g = 0$ and $k = 2\pi$.

  The coincidence of $d\mu$ (1.34) understood in this sense with the Goldman’s symplectic form on $M_A$ we call the ”weak” ILP conjecture.

The problem with this formulation is that the choice of matrices $G_j$ should be such that they satisfy the Schlesinger equations (1.31); this requirement is not natural from the symplectic point of view.

- **The ”strong” ILP conjecture.** In this version the differential $d^{(m)}$ in (1.34) means the differential on the full space $M$ (1.8) which contains both the eigenvalues of the monodromy matrices and the connection matrices. Then (ignoring the pullbacks) the strong IPL conjecture states that

$$
d\mu = \omega_M \quad (1.35)
$$

where $\omega_M$ is the symplectic form on $M$ given by (1.19).

The weak version of the ILP conjecture can be derived directly from known results of [19, 3] or [22], as shown in Section 4.

The strong version of the ILP conjecture is equivalent to our Theorem 1.2. To see this equivalence it is sufficient to write (1.30) in coordinates which are split into ”times” $\{t_j\}$ and some coordinates $\{m_k\}$ on the monodromy manifold $M$. Then the ”$t$-part” of the form $\tilde{\theta}_A$ is given by $2 \sum_{k=1}^N H_k dt_k$ (this follows from the isomonodromic equations (1.31) for $\{G_j\}$) and the ”$M$-part” coincides with the second term of the form (1.34) where the differential $d^{(m)}$ is understood as the differential on $M$. Now, taking the external derivative of (1.30) we come to (1.26) where the right-hand side coincides with the form $d\omega$ of [20].

The proof of Theorem 1.2 requires the full use of the formalism developed in this paper. The calculation relies on the following construction. Denote the solution of a Riemann-Hilbert problem on an embedded oriented graph $\Sigma$ with piecewise differentiable jump matrix $J(z)$ by $\Phi$; denote the boundary values of $\Phi$ on different sides of $\Sigma$ by $\Phi_{\pm}$ (see section 3 for more details). The following form was introduced by Malgrange [24]

$$
\Theta = \frac{1}{2\pi i} \int_{\Sigma} \tr \left( \Phi^{-1} \frac{d\Phi}{dz} dJ(z) J^{-1}(z) \right) dz \quad (1.36)
$$

where $dM$ means the differential with respect to deformation parameters.

Calculation of the form $\Theta$ and its exterior derivative $d\Theta$ in the case of Riemann-Hilbert problem associated to the system (1.1) leads to the proof of Theorem 1.1 and Theorem 1.2.
2 Poisson and symplectic structures on the space $\mathcal{A}$

Let $\mathcal{H} := GL(n, \mathbb{C}) \times \mathfrak{h}_{ss} = \{(G, L)\}$ and consider the following one-form

$$\theta := \text{tr}(LG^{-1}dG). \quad (2.1)$$

We prove in Prop. A.1 that $\omega = d\theta$ is a symplectic form and that the corresponding Poisson bracket is expressible in terms of the dynamical $r$-matrix structure given in the formulas (1.11), (1.12) (see Prop. A.2).

Denote by $A_0 = \bigotimes_{j=1}^N \mathcal{H}$ the space of pairs $\{(G_j, L_j)\}_{j=1}^N$ with the product symplectic structure, namely with the Poisson bracket (1.15). Consider the following action of the group $GL(n)$ on $A_0$:

$$\{G_j, L_j\}_{j=1}^N \rightarrow \{SG_j, L_j\}_{j=1}^N \quad (2.2)$$

for $S \in GL(n)$.

**Lemma 2.1** The moment map corresponding to the group action (2.2) is given by

$$\{G_j, L_j\}_{j=1}^N \rightarrow \sum_{j=1}^N G_j L_j G_j^{-1} \quad (2.3)$$

**Proof.** Consider the moment map of the $GL(n)$ action on a single copy $\mathcal{H}$. Define $A := GLG^{-1}$; a direct straightforward computation shows that

$$\{A_{ab}, G_{jk}\} = G_{ak} \delta_{bj}, \quad \{A_{ab}, \lambda_k\} = 0. \quad (2.4)$$

This directly implies that $\{\text{tr}(XA), G\} = XG$ for any fixed matrix $X$. Therefore $\text{Exp}(\{\text{tr}(XA), \bullet\})G = e^XG$ and thus the moment map of the group action $G \mapsto SG$ is the matrix $A = GLG^{-1}$. ■

The space $\mathcal{A}$ is then defined by (1.8) as the space of the orbits of the group action (2.2) in the zero level set of the moment map (2.3).

Now we are in a position to formulate the following theorem

**Theorem 2.1** The Poisson structure induced on $\mathcal{A}$ from Poisson structure (1.15) on $A_0$ via the reduction on the level set $\sum_{j=1}^N G_j L_j G_j^{-1} = 0$ of the moment map, corresponding to the group action $G_j \mapsto SG_j$, is non-degenerate and the corresponding symplectic form is given by

$$\omega_{\mathcal{A}} = - \sum_{k=1}^N \text{tr}(L_k G_k^{-1}dG_k \wedge G_k^{-1}dG_k) + \text{tr}(dL_k \wedge G_k^{-1}dG_k). \quad (2.5)$$

A symplectic potential $\theta_{\mathcal{A}}$ for $\omega_{\mathcal{A}}$ is given by (1.18).

The proof is the standard Hamiltonian reduction [4]. What remains to prove here is that the form $\omega_{\mathcal{A}}$ is invariant under a transformation (2.2) on the zero level set of the moment map. This follows from the following lemma which shows that $\omega_{\mathcal{A}}$ is exact also on the reduced space.

**Lemma 2.2** The one-form

$$\theta_{\mathcal{A}} = \sum_{k=1}^N \text{tr}(L_k G_k^{-1}dG_k) \quad (2.6)$$

on $A_0$ descends to a one-form on $\mathcal{A}$.

**Proof.** The potential $\theta_{\mathcal{A}}$ transforms as follows under the group action (2.2):

$$\theta_{\mathcal{A}} \rightarrow \theta_{\mathcal{A}} + \sum_{k=1}^N \text{tr}(L_k G_k^{-1}S^{-1}dSG_k) = \theta_{\mathcal{A}} + \sum_{k=1}^N (G_k L_k G_k^{-1}S^{-1}dS)$$

The last sum vanishes due to condition $\sum_{k=1}^N G_k L_k G_k^{-1} = 0$. ■
3 Symplectomorphism between $\mathcal{A}$ and $\mathcal{M}$ via Malgrange’s form

Let $\Sigma$ be an oriented embedded graph on $\mathbb{CP}^1$ whose edges are smooth oriented arcs meeting transversally at the vertices. We denote by $V$ the set of vertices of the graph. Suppose we are assigned a "jump matrix" datum, i.e. a function $J(z) : \Sigma \setminus V \to GL(n)$ that satisfies the following properties

**Assumption 3.1** 1. In a small neighbourhood of each point $z_0 \in \Sigma \setminus V$ the matrix $J(z)$ is given by a germ of analytic function;

2. for each $v \in V$, denote by $\gamma_1, \ldots, \gamma_n$, the edges incident at $v$ in a small disk centered thereof. Suppose first that all these edges are oriented away from $v$ and enumerated in counterclockwise order. Denote by $J^{(v)}_{\gamma_j}(z)$ the analytic restrictions of $J$ to $\gamma_j$. Then we assume that each $J^{(v)}_{\gamma_j}(z)$ admits an analytic extension to a full neighbourhood of $v$ and that these extensions satisfy the local no-monodromy condition

$$J^{(v)}_{\gamma_1}(z) \cdots J^{(v)}_{\gamma_n}(z) \equiv 1.$$  \hspace{1cm} (3.1)

If the orientation of an edge $\gamma_j$ is the opposite then $J^{(v)}_{\gamma_j}(z)$ is taken to be the inverse of $J(z)$.

Suppose now that $J(z)$ comes in an analytic family of data depending on some deformation parameters and satisfying Assumption 3.1, and consider the following family of Riemann–Hilbert problems on $\Sigma$ (we omit explicit reference to the deformation parameters).

**Riemann Hilbert Problem on the graph $\Sigma$.** Fix $z_0 \in \mathbb{C} \setminus \Sigma$; let $\Phi(z) : \mathbb{CP}^1 \setminus \Sigma \to GL_n(\mathbb{C})$ be a matrix–valued function, bounded everywhere and analytic on each face of $\Sigma$. We also assume that the boundary values on the two sides of each edge of $\Sigma$ are related by

$$\Phi_+(z) = \Phi_-(z)J(z), \quad \forall z \in \Sigma \setminus V, \quad \Phi(z_0) = 1.$$ \hspace{1cm} (3.2)

where the $+/-$ boundary value is from the left/right, respectively, of the oriented edge.

Then, following [24] one can define a natural one-form on the deformation space.

**Definition 3.1** The Malgrange form on the deformation space of Riemann-Hilbert problems with given contour $\Sigma$ is defined by

$$\Theta = \frac{1}{2i\pi} \int_{\Sigma} \text{tr} \left( \Phi^{-1} \frac{d\Phi}{dz} dJ(z) J^{-1}(z) \right) dz$$ \hspace{1cm} (3.3)

where $dJ$ denotes the total differential of $J$ in the space of deformation parameters for fixed $z$.

We observe that the form $\Theta$ is independent of the normalization point $z_0$ (the change of $z_0$ is equivalent to a left multiplication of $\Phi$ by an invertible matrix independent of $z$). Therefore we can also allow $z_0$ to be one of the vertices: in this case one needs then to specify which boundary value of $\Phi$ is then normalized to 1.

**Malgrange form and Schlesinger systems.** Let us now discuss how the form (3.3) can be used in the context of the Fuchsian equation (1.1) and the associated Riemann-Hilbert problem.

Assume that the solution to (1.1) is normalized to $\Psi(z_0) = 1$ (below we assume $z_0 = \infty$). Consider a set of non-intersecting cuts joining $z_0$ with the poles $t_j$; then $\Psi(z)$ is single valued on the complement. The graph $\Sigma$ is constructed as shown in Fig. 1. Introduce the piecewise analytic matrix on its faces

$$\Phi(z) = \begin{cases} 
\Psi(z) & z \in \mathbb{CP}^1 \setminus \Sigma \setminus \bigcup_{j=1}^N D_j \\
\Phi_j(z) := \Psi(z) C_j (z-t_j)^{-L_j} & z \in D_j
\end{cases}$$ \hspace{1cm} (3.4)

Then $\Phi$ solves a Riemann–Hilbert Problem on $\Sigma$ with the jump matrices on its edges as indicated in Fig. 1. Note that with these definitions the expression (3.3) only involves $\Psi(z)$ and its boundary values on the cuts and boundaries of the disks $D_j$.

In this context the deformation parameters involved in the expression (3.3) for $\Theta$ are $C_j, L_j$ subject to the monodromy relation $\prod_{j=1}^N C_j e^{2i\pi L_j} C_j^{-1} = 1$, and the locations of the poles $t_1, \ldots, t_N$. 

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Theorem 3.1 The form $\Theta$ (3.3) and the potential $\bar{\theta}_A$ (1.29) are related by

$$\Theta = \bar{\theta}_A - \sum_{j=1}^{N} H_j dt_j$$

(3.5)

where $H_j$ are the Hamiltonians 1.27.

Proof. The simplest way to see this is by using the localization formula [20]; the corresponding Riemann-Hilbert problem is defined on the set of contours in Fig. 1.

In the formula (3.3) the function $\Phi_-$ coincides with the boundary value of $\Psi$ on the main face $\mathbb{CP}^1 \setminus \bigcup_{j=1}^{N} \mathbb{D}_j$, which is the solution of the ODE (1.1). Therefore we have (we denote $d\Phi/dz$ by $\Phi'$):

$$\text{tr} \left( \Phi_-'^{-1} \Phi_- dJJ^{-1} \right) = \text{tr} \left( \Phi_-'^{-1} \Phi_-' dJJ^{-1} \Phi_-^{-1} \right) = \text{tr} \left( A(z) \Phi_-' dJJ^{-1} \Phi_-^{-1} \right)$$

(3.6)

In this last expression we have used the fact that $\Phi_-' = \Psi_-'$ and therefore $\Phi_-'^2 = A(z)$. Moreover we have

$$\Phi_- dJJ^{-1} \Phi_-^{-1} = d(\Phi_- J) J^{-1} \Phi_-^{-1} - d\Phi_- \Phi_-^{-1} = d\Phi_+ \Phi_+^{-1} - d\Phi_- \Phi_-^{-1}$$

(3.7)

since $\Phi_+ = \Phi_- J$. Thus an equivalent form of (3.3) is

$$\Theta = \frac{1}{2\pi i} \int_{\Sigma} \text{tr} \left( A(z)(d\Phi_+ \Phi_+^{-1} - d\Phi_- \Phi_-^{-1}) \right) dz.$$  

(3.8)

We can thus use Cauchy’s theorem; in the interior of the $\mathbb{D}_j$’s the solution $\Phi_j(z)$ of the Riemann-Hilbert problem is defined by (3.4) with $F_j(z) = G_j^{-1} \Phi_j(z)$ i.e.

$$\Psi(z) = G_j \left( 1 + O(z-t_j) \right)(z-t_j)^L C_j^{-1} =: G_j F_j(z)(z-t_j)^L C_j^{-1}.$$  

(3.9)

Thus the expression (3.8) reduces to

$$\Theta = \sum_{j} \text{res} \left( \text{tr} \left( A(z)d\Phi_j(z) \Phi_j^{-1}(z) \right) \right) dz$$

(3.10)

and since $A(z)$ has simple poles, the expression reduces to an evaluation. This is the expression that also appears in [20], formula (1.11).

There is now a subtlety in the further computation of (3.10) because the evaluation and the variation do not commute if the variation involves a variation of one of the positions of the poles. Namely

$$d\Phi_j(z)\Phi_j(z) \Bigg|_{z=t_j} = dG_j G_j^{-1} + G_j dF_j(z) F_j(z)^{-1} G_j^{-1} \Bigg|_{z=t_j}.$$  

(3.11)

Let us consider specifically the derivative w.r.t. one of the pole’s position $t_k$. Since $\partial_{t_k} \Psi = \frac{A_k}{z-t_k} \Psi$ we have

$$G_k^{-1} \partial_{t_k} G_k F_k(z) + \partial_{t_k} F_k(z) + F_k(z) \frac{L_k}{z-t_k} = \frac{G_k^{-1} A_k G_k}{z-t_k} F_k(z) \partial_{t_k} F_k(z) = \frac{[L_k, F_k(z)]}{z-t_k} - G_k^{-1} \partial_{t_k} G_k F_k(z).$$

Evaluating at $z = t_k$ gives

$$\partial_{t_k} F_k(z) \Bigg|_{z=t_k} = [L_k, F_k'(t_k)] - G_k^{-1} \partial_{t_k} G_k$$

(3.12)

and therefore

$$\partial_{t_k} \Phi_j(z) \Phi_j^{-1}(z) \Bigg|_{z=t_j} = \partial_{t_k} G_j G_j^{-1} - \partial_{t_k} G_k G_k^{-1} + \delta_{kj} [A_k, \Phi_j'(t_j) \Phi_j(t_j)^{-1}]$$

(3.13)
Thus
\[
\Theta = \sum_j \text{res}_{z=t_j} \left( A(z) d\Phi_j(z) \Phi_j^{-1}(z) \right)
\]
\[
= \sum_j \left( A_j dG_j G_j^{-1} \right) - \sum_j dt_j \text{tr} \left( A_j \partial_{t_j} \left( G_j G_j^{-1} - [A_k, \Phi_k'(t_k) \Phi_k^{-1}(t_k)] \right) \right)
\]
\[
= \sum_j \left( A_j dG_j G_j^{-1} \right) - \sum_j dt_j \text{tr} \left( A_j \partial_{t_j} G_j G_j^{-1} \right).
\]

Due to the Schlesinger equations for \( G_j \) (1.31) we get
\[
\Theta = \sum_j \text{tr} \left( A_j dG_j G_j^{-1} \right) - \sum_j dt_j \sum_{k \neq j} \frac{\text{tr} A_j A_k}{t_j - t_k}.
\]

Recalling that the Jimbo-Miwa Hamiltonians are given by \( H_j = \sum_{k \neq j} \frac{\text{tr} A_j A_k}{t_j - t_k} \) and that the first term equals the potential \( \tilde{\theta}_A \) (1.29) on \( \tilde{A} \), we get
\[
\Theta = \tilde{\theta}_A - \sum_j H_j dt_j.
\]

Let us now discuss the monodromy side.

**Definition 3.2** Define the following 2-form on \( \mathcal{M} \) (1.9):
\[
\omega_{\mathcal{M}} = \frac{1}{4\pi i} (\omega_1 + \omega_2)
\]
where
\[
\omega_1 = \sum_{\ell=1}^N \text{tr} \left( M_\ell^{-1} dM_\ell \wedge K_\ell^{-1} dK_\ell \right) + \sum_{\ell=1}^N \text{tr} \left( \Lambda_\ell^{-1} C_\ell^{-1} dC_\ell \wedge \Lambda_\ell C_\ell^{-1} dC_\ell \right),
\]
\[
\omega_2 = 2 \sum_{\ell=1}^N \text{tr} \left( \Lambda_\ell^{-1} d\Lambda_\ell \wedge C_\ell^{-1} dC_\ell \right)
\]

and \( K_\ell = M_1 \ldots M_\ell \).
On the monodromy manifold $M_1, \ldots, M_N = I$ the form $\omega_M$ is invariant under simultaneous transformation $C_j \to SC_j$ with $S$ is an arbitrary $GL(n)$-valued function on $M$ (this fact can be checked directly or deduced from the invariance of the form $\Theta$ proved in Theorem C.1).

**Remark 3.1** The restriction of the form $2i\pi \omega_M$ on the leaves $A_j = \text{constant}$ (under such restriction $\omega_2 = 0$ and hence $2i\pi \omega_M = \omega_1/2$) coincides with the symplectic form on the symplectic leaves of the $GL(n)$ Goldman bracket found in formula (3.14) of [2] (the relevant case of their formula corresponds to $k = 2n$ and $g = 0$ in the notation of [2]).

As we prove below in Corollary 3.1, the form $\omega_M$ is non-degenerate on the space $M$, which is a torus fibration (with fiber the product of $N$ copies of the $GL(n)$ torus of diagonal matrices) over the union of all the symplectic leaves of the Goldman bracket. The fact that $M$ is a torus fibration is simply due to the fact that the fibers of the map $(C_j, A_j) \to M_j = C_jA_jG_j^{-1}$ are obtained by multiplication of the $C_j$'s on the right by diagonal matrices.

Let us trivially extend the form $\omega_M$ to the space $\tilde{M}$ (1.24) which includes also the variables $t_j$. This extension is denoted by $\tilde{\omega}_M$. The following theorem was stated in [6] in slightly different notations without direct proof. The proof is a computation using Prop. B.1.

**Theorem 3.2** The exterior derivative of the form $\Theta$ is given by the pullback of the form $\tilde{\omega}_M$ (3.19) under the monodromy map:

$$d\Theta = [\tilde{F}]^* \tilde{\omega}_M$$

(3.22)

The proof is found in Appendix B.

This theorem immediately implies the following corollary.

**Corollary 3.1** The form $\omega_M$ (3.19) is closed and non-degenerate on the monodromy manifold $M$.

**Proof.** We expect this statement to have an intrinsic proof without the reference to the theory of Schlesinger equation, Malgrange form etc.

The proof we get in the framework of this paper relies on Theorem 3.1. For any fixed choice of positions of the poles $t_j$'s the monodromy map $F^t$ is known to be a local diffeomorphism between $A$ and $M$. Moreover, by making different choices of pole positions one can cover the whole $M$ by the images of $F^t(A)$; this is guaranteed by the original Plemelj theorem (see [10] for history and details) which says that the inverse monodromy map for Fuchsian systems exists for any choice of monodromy representation and generic choice of position of poles. Considering the slices $t_j = \text{const}$ and taking the derivative of both sides, in the right-hand side, we get the form $\omega_M$. In the left-hand side we get the extended Kirillov-Kostant form (2.5). Since the form (2.5) is symplectic due to Theorem 2.1 we conclude that the form $\omega_M$ is symplectic, too.

**Strong version of Its-Lisovyy-Prokhorov conjecture.** The theorem 3.2 proves the "strong" version of the ILP conjecture (1.34). To state this conjecture in the present setting we consider the form (1.11) or (2.7) of [20] which we denote by $\mu$ to avoid confusion with the notations of this paper (see also the identity (4.23) below):

$$\mu = \sum_{j<k}^N \text{tr}A_jA_k d \log(t_j - t_k) + \sum_{j}^N \text{tr}(L_jG_j^{-1}d^{(m)}G_j).$$

(3.23)

Although formally the form $\mu$ coincides with our form $\tilde{\Theta}_A$ (1.29), there is an essential difference. The way the formula (3.23) is understood in [20] is that the differential $d^{(m)}$ is with respect to coordinates on a given symplectic leaf $L_j = \text{const}$ of the monodromy manifold. Our form $\tilde{\Theta}_A$ contains derivatives with respect to all monodromy data on $M$. Moreover, the space denoted by $A$ in [20] is the space of coefficients of (1.1); therefore the form $\mu$ (3.23) is not well-defined on $A$ because it depends on the specific choice of diagonalizing matrices $G_j$ (i.e. it is not invariant under the group action $G_j \to G_jD_j$ with diagonal $D_j$).

The statement of theorem 3.2 is stronger than the statement originally conjectured in [20]. The original "weak" version of this conjecture is proved on the basis of known results [19, 3, 22] in the next section.
Generating function of monodromy map. The closure of $\omega_\mathcal{M}$ guarantees the local existence of a potential. Denoting any such local potential by $\theta_\mathcal{M}$ (such that $d\theta_\mathcal{M} = \omega_\mathcal{M}$) we define the generating function $\mathcal{G}$ as follows

$$d\mathcal{G} = \sum_{k=1}^{N} \text{tr}(L_k G_k^{-1} dG_k) - \sum_{j=1}^{N} H_k dt_k - \mathcal{F}^*[\tilde{\theta}_\mathcal{M}]$$

(3.24)

where $\tilde{\theta}_\mathcal{M}$ is the trivial pullback of the form $\theta_\mathcal{M}$ to $\tilde{\mathcal{M}}$.

To explain the reason for calling $\mathcal{G}$ a generating function, suppose to have chosen a maximal set of commuting functions $\{m_1, \ldots, m_d\}$ on $\mathcal{M}$ and a maximal set of commuting functions $\{q_1, \ldots, q_d\}$ on $\mathcal{A}$ (with $d = \frac{1}{2} \dim \mathcal{A} = \frac{1}{2} \dim \mathcal{M}$). If this choice is generic enough (locally) then the pullback of the functions $m_j$ to $\mathcal{A}$ provides a full set of local coordinates on $\mathcal{A}$ (we do not indicate this pullback here for simplicity). Then the function $\mathcal{G}$ can be written as

$$d\mathcal{G} = \sum_{j=1}^{d} p_j dq_j - \sum_{j=1}^{d} r_j dm_j - \sum_{j=1}^{N} H_k dt_k$$

(3.25)

where $(p_j, r_j)$ are appropriate functions of $(q_j, m_j)$. Then, for fixed times $t_j$ the functions $(p_j, r_j)$ are the other half of the corresponding Darboux coordinates; namely $\mathcal{G}$ is the generating function of the change of Darboux coordinates from $(p_j, q_j)$ and $(r_j, m_j)$.

The equation (3.24) can be used to extend the definition of Jimbo-Miwa tau-function to include its dependence on monodromies. However, unless we impose any additional global restrictions on the choice of $\theta_\mathcal{M}$, the generating function $\mathcal{G}$ is defined up to an arbitrary monodromy-dependent additive term. Irrespectively of the choice of $\theta_\mathcal{M}$, one gets the following theorem

**Theorem 3.3** For any choice of symplectom potential $\theta_\mathcal{M}$ on $\mathcal{M}$ the dependence of the generating function $\mathcal{G}$ (1.30) on $\{t_j\}_{j=1}^{N}$ coincides with $t_j$-dependence of the isomonodromic Jimbo-Miwa tau-function. In other words, $e^{-\mathcal{G}} \tau_{JM}$ depends only on monodromy data $\{C_j, L_j\}_{j=1}^{N}$.

Classical action of the Schlesinger system. This theorem confirms another conjecture formulated in Section 1.6 of [20] which states that the Jimbo-Miwa tau-function $\log \tau_{JM}$ is related to the "action" of the corresponding Schlesinger system which is the multi-time Hamiltonian system, computed at solutions of the equations of motion (i.e. solutions of the Schlesinger system).

We recall that the standard definition of the action of one-dimensional hamiltonian system is $S = \int \theta - H dt$ where $\theta = pdq$ is a symplectic potential for the symplectic form $dp \wedge dq$. The action minimizes on solutions of the equations of motion. For a multi-time Hamiltonian system the analog of the classical action would be $S = \int \theta - \sum_{i=1}^{N} H_i dt_i$; however, for this equation to have a solution we need to assume that the result of this integration does not depend on the choice of path in the $\{t_j\}$-space i.e. the Hamiltonians $H_i$ satisfy the equations $(H_i)_{t_i} - (H_j)_{t_j} + \{H_i, H_j\} = 0$.

This equation is satisfied by the Schlesinger Hamiltonians, which both Poisson-commute, $\{H_i, H_j\}$ and satisfy the equations $(H_i)_{t_i} = (H_j)_{t_j}$. Therefore, the classical action is well-defined in the context of the isomonodromic deformations, when computed on the space of solutions of the Schlesinger system.

If the right hand side of (3.24) is restricted to the space of solutions of the Schlesinger system, then $\mathcal{F}^*[\tilde{\theta}_\mathcal{M}] = 0$ and the function $\mathcal{G}$ can be interpreted as the classical action. To write it in the standard form one would need to find a set of Darboux coordinates $(p_i, q_i)$ for the form $\omega_\mathcal{A}$ such that $\theta_\mathcal{A} = \sum p_i dq_i$. Existence of such coordinates is guaranteed by the Darboux theorem; however, we don’t know how to find them explicitly.

4 Standard monodromy map and weak version of Its-Lisovyy-Prokhorov conjecture

Here we show that a weak version of Its-Lisovyy-Prokhorov conjecture can be derived in a simple way from previous results of [19, 3] or [22] where a symplectomorphism between the space of coefficients $\{A_j\}$
with given set of eigenvalues of the Fuchsian equation (1.1) and a symplectic leave of Goldman bracket was proved.

First, consider the submanifold $A_L$ of $A$ where we fix the diagonal form of each of the matrices $A_j$:

$$A_L = \{ \{ A_i \}_{i=1}^N, A_i \in O(L_i), \sum_{i=1}^N A_i = 0 \}/\sim \quad (4.1)$$

where $\sim$ is the equivalence over simultaneous adjoint transformation $A_i \to S A_i S^{-1}$ of all $A_i$ for $S \in GL(n)$; $L = (L_1, \ldots, L_N)$ where $L_j$ is the diagonal form of $A_j$ and $O(L)$ is the (co)-adjoint orbit of the diagonal matrix $L$. We assume that diagonal entries of each $L_j$ do not differ by an integer.

Consider similarly also the space $M_A$ which is the subspace of the $GL(n)$ character variety of $\pi_1(\mathbb{C}P^1 \setminus \{ t_j \}_{j=1}^N)$ such that the diagonal form of the matrix $M_j$ equals to $\Lambda_j = e^{2\pi i L_j}$.

The Kirillov-Kostant brackets (A.1) for each $A_j$:

$$\{ A_j, A_k \} = [A_j, \Pi] \delta_{jk} \quad (4.2)$$

can be equivalently rewritten in the $r$-matrix form

$$\{ A(z), \bar{A}(w) \} = \frac{1}{z - w} \left[ \Pi, A(z) + \bar{A}(w) \right] . \quad (4.3)$$

The Schlesinger equations for $A_j = G_j L_j G_j^{-1}$ which follow from the system (1.31) for $G_j$ take the form:

$$\frac{\partial A_k}{\partial t_j} = \frac{[A_k, A_j]}{t_k - t_j}, \quad \frac{\partial A_j}{\partial t_j} = -\sum_{k \neq j} \frac{[A_k, A_j]}{t_k - t_j} . \quad (4.4)$$

These equations are Hamiltonian,

$$\frac{\partial A_k}{\partial t_j} = \{ H_j, A_k \} ,$$

with the Poisson structure (4.3) and (time dependent) Hamiltonians (1.27). Notice that these Hamiltonians commute $\{ H_k, H_j \} = 0$ and moreover satisfy $\partial_{t_i} H_j = \partial_{t_j} H_i$.

After the symplectic reduction to the space of orbits of the global $Ad_{GL(N)}$ action and restriction to the level set $\sum_{j=1}^N A_j = 0$ of the corresponding moment map one gets a degenerate Poisson structure; its symplectic leaves coincide with $A_L$ [19]. The symplectic form on $A_L$ can be written as

$$\omega'_A = -\sum_{k=1}^N \text{tr}(L_k G_k^{-1} dG_k \wedge G_k^{-1} dG_k) \quad (4.5)$$

The form (4.5) is independent of the choice of matrices $G_j$ which diagonalize $A_j$; moreover, it is invariant under simultaneous transformation $A_j \to S A_j S^{-1}$ and thus it is indeed defined on the space $A_L$.

The $GL(n)$ character variety is equipped with the Poisson structure given by the Goldman bracket [17] defined as follows; for any two loops $\gamma, \tilde{\gamma} \in \pi_1(\mathbb{C}P^1 \setminus \{ t_i \}_{i=1}^N)$ the Poisson bracket between the traces of the corresponding monodromies is given by

$$\left\{ \text{tr} M_\gamma, \text{tr} M_{\tilde{\gamma}} \right\}_G = \sum_{p \in \gamma \cap \tilde{\gamma}} \nu(p) \text{tr} (M_{[\gamma, \tilde{\gamma}]}) ; \quad (4.6)$$

here $\nu(p) = \pm 1$ is the contribution of point $p$ to the intersection index of $\gamma$ and $\tilde{\gamma}$.

The space $M_A$ is a symplectic leaf of the Goldman bracket; the corresponding symplectic form is given by [2]:

$$w_G^L = \frac{1}{2} \omega'_1 \quad (4.7)$$

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where $\omega_1$ is given by (3.20). Denote also

$$\omega^L_M = \frac{1}{2\pi i} \omega^L_G.$$  

(4.8)

The study of the symplectic properties of the map (1.10) was initiated in [19, 3, 22]. In [19, 3] two different proofs were given of the fact that the monodromy map $F^t$ is a symplectomorphism i.e.

$$(F^t)^* \omega^L_M = \omega^L_A.$$  

(4.9)

In [22] the brackets between the monodromy matrices themselves were obtained starting from (4.3); the result is given by

$$\{M_i, 2_i\}^* = \pi i \Pi \left( M_i M_j - M_j M_i \right), \quad i < j$$  

(4.10)

$$\{M_i, 2_i\}^* = \pi i \Pi \left( M_i 2_i - 2_i M_i \right), \quad i < j$$  

(4.11)

where $\Pi$ is the matrix of permutation. The brackets (4.10), (4.11) were computed for the basepoint $z_0 = \infty$ on the level set $\sum_{j=1}^N A_j = 0$ of the moment map; thus the algebra (4.10), (4.11) does not satisfy the Jacobi identity. However, the Jacobi identity is restored for the algebra of $ad$-invariant objects i.e. for traces of monodromies; moreover, for any two loops $\gamma$ and $\tilde{\gamma}$ we have [27]

$$\{\text{tr} M_\gamma, \text{tr} M_\tilde{\gamma}\}^* = 2\pi i \{\text{tr} M_\gamma, \text{tr} M_\tilde{\gamma}\}_G$$  

(4.12)

which gives an alternative proof of (4.9).

Let us now show how (4.9) implies the weak version of the Its-Lisovyy-Prokhorov conjecture. Similarly to (1.23) and (1.24) we introduce the two spaces

$$\tilde{A}_L = \{(p, \{t_j\}^N_{j=1}, p \in A_L, t_j \in \mathbb{C}, t_j \neq t_k)\}$$  

(4.13)

$$\tilde{M}_\Lambda = \{(p, \{t_j\}^N_{j=1}, p \in M_\Lambda, t_j \in \mathbb{C}, t_j \neq t_k)\}$$  

(4.14)

Denote the pullback of the form $\omega^L_A$ with respect to the natural projection of $\tilde{A}_L$ to $A_L$ by $\tilde{\omega}_A$ and the pullback of the form $\omega^L_M$ with respect to the natural projection of $\tilde{M}_\Lambda$ to $M_\Lambda$ by $\tilde{\omega}_M$.

**Proposition 4.1** The following identity holds between two-forms on $\tilde{A}_L$

$$\tilde{F}^* [\tilde{\omega}^\Lambda_M] = \tilde{\omega}^L_A - \sum_{k=1}^N dH_k \wedge dt_k$$  

(4.15)

where $H_k$ are the Hamiltonians (1.27).

**Proof.** Denote by $2d$ the dimension of the spaces $A_L$ and $M_\Lambda$. Introduce some local Darboux coordinates $(p_i, q_i)$ on $A^L$ for the form $\omega^L_A$ (4.5) and also some Darboux coordinates $(P_j, Q_j)$ on $M^\Lambda$ for the form $\omega^\Lambda_M$ given by (4.8).

We are going to verify (4.15) using coordinates $\{t_j\}^N_{j=1}$ and $\{P_j, Q_j\}^d_{j=1}$. Let us split the operator $d$ into two parts:

$$d = d^{(0)} + d^{(m)}$$  

(4.16)

where $d^{(m)}$ is the differential with respect to $\{P_j, Q_j\}^d_{j=1}$.

Then relation (4.9) can be written as

$$\sum_{j=1}^d dP_j \wedge dQ_j = \sum_{j=1}^d d^{(m)} p_i \wedge d^{(m)} q_i$$  

(4.17)
The right-hand side can be further rewritten using the Hamilton equations \( \frac{\partial p_i}{\partial t} = -\frac{\partial H_k}{\partial q_i}; \frac{\partial q_i}{\partial t} = \frac{\partial H_k}{\partial p_i} \) (where the hamiltonians \( H_k \) are given by (1.27)). Using

\[
d^{(m)} p_i = dp_i + \sum_{k=1}^{N} \frac{\partial H_k}{\partial q_i} dt_k \quad \text{and} \quad d^{(m)} q_i = dq_i - \sum_{k=1}^{N} \frac{\partial H_k}{\partial p_i} dt_k
\]

one gets

\[
\sum_{i=1}^{d} d^{(m)} p_i \wedge d^{(m)} q_i = \sum_{i=1}^{d} dp_i \wedge dq_i + \sum_{k=1}^{N} dt_k \wedge \left( \sum_{i=1}^{d} \left( \frac{\partial H_k}{\partial q_i} dq_i + \frac{\partial H_k}{\partial p_i} dp_i \right) \right) - \sum_{\ell<k} \left( \frac{\partial H_\ell}{\partial q_k} \frac{\partial H_k}{\partial p_\ell} - \frac{\partial H_\ell}{\partial q_\ell} \frac{\partial H_k}{\partial p_k} \right) dt_\ell \wedge dt_k. \tag{4.18}
\]

To simplify the second sum in (4.18) we recall that

\[
dH_k = \sum_{i=1}^{d} \left( \frac{\partial H_k}{\partial q_i} dq_i + \frac{\partial H_k}{\partial p_i} dp_i \right) + \sum_{\ell=1}^{N} \left. \frac{\partial H_k}{\partial t_\ell} \right|_{p_i,q_i=\text{const}} dt_\ell
\]

thus the second sum can be written as

\[
\sum_{k=1}^{H} dt_k \wedge dH_k + \sum_{\ell,k,l<k} \left( \frac{\partial H_k}{\partial t_\ell} \bigg|_{p_i,q_i} - \frac{\partial H_\ell}{\partial t_k} \bigg|_{p_i,q_i} \right) dt_\ell \wedge dt_k.
\]

Adding them together we obtain

\[
\sum_{j=1}^{d} dP_j \wedge dQ_j = \sum_{i=1}^{d} dp_i \wedge dq_i + \sum_{k=1}^{N} dt_k \wedge dH_k - \sum_{\ell<k} \left( \frac{\partial H_\ell}{\partial q_k} \frac{\partial H_k}{\partial p_\ell} - \frac{\partial H_\ell}{\partial q_\ell} \frac{\partial H_k}{\partial p_k} \right) dt_\ell \wedge dt_k. \tag{4.19}
\]

The coefficient of \( dt_\ell \wedge dt_k \) vanishes because the Hamiltonians satisfy the zero-curvature equations implied by commutativity of the flows with respect to \( t_j \) and \( t_\ell \); in fact in this particular case they satisfy a stronger compatibility: \( \{ H_k, H_\ell \} = 0 \) and \( \partial_{t_\ell} H_k = \partial_{t_k} H_\ell \). Therefore we arrive at (4.15).

Let us show that (4.15) implies

**Proposition 4.2 (Weak IPL conjecture)** The following identity holds on the space \( \tilde{A}_L \):

\[
d\mu^L = [\tilde{F}]^* \omega^A_M
\]

where

\[
\mu^L = \sum_{j<k} \text{tr} A_j A_k dt_j - dt_k + \sum_{j=1}^{N} \text{tr} (L_j G_j^{-1} d^{(m)} G_j)
\]

and matrices \( G_j \) diagonalizing \( A_j \) are chosen to satisfy the Schlesinger equations (1.31); \( d^{(m)} \) denotes the differential with respect to monodromy coordinates on the space \( \tilde{A}_L \). The form \( \mu^L \) is the "weak" version of the form (1.34). The form \( \omega^A_M \) is the pullback of Alekseev-Malkin form (4.8) from \( M_A \) to \( \tilde{M}_A \).

**Proof.** The symplectic potential for the form \( \omega^A_M \) (4.5) can be written as

\[
\theta^A_M = \sum_{j=1}^{n} \text{tr}(L_j G_j^{-1} (d^{(t)} + d^{(m)}) G_j).
\]

We notice that the potential \( \theta^A_M \), in contrast to the form \( \omega^A_M \) itself, is not well-defined on the space \( \tilde{A}_L \) due to ambiguity \( G_j \to G_j D_j \) for diagonal \( D_j \) in the definition of \( G_j \). Under such transformation \( \theta^A_M \) changes
by an exact form. Therefore for the purpose of proving (4.20) one can pick any concrete representative for each $G_j$. The most natural choice is to assume that \{$G_j$\} satisfy the system (1.31). Then the "t"-part of potential (4.22) can be computed using (1.31) and the definition of the Hamiltonians (1.27) to give

$$
\sum_{j=1}^{n} \text{tr}(L_j G_j^{-1} d^{(t)} G_j) = 2 \sum_{j=1}^{N} H_j dt_j .
$$

(4.23)

Therefore, the relation (4.15) can be rewritten as

$$
\tilde{F}^* [\omega^L_{\mathcal{M}}] = d \left( \sum_{k=1}^{N} dH_k \wedge dt_k + \sum_{j=1}^{N} \text{tr}(L_j G_j^{-1} d^{(m)} G_j) \right) \tag{4.24}
$$

which coincides with (4.20).

As well as in the case of the previous section, for each choice of potential $\theta^L_{\mathcal{M}}$ such that $d\theta^L_{\mathcal{M}} = \omega^L_{\mathcal{M}}$ one can define the generating function $G_j$ similarly to (3.24). For the $SL(2)$ case (the formalism corresponding to $SL(n)$ case is identical to the $GL(n)$ case mainly treated in this paper) we know of two ways to construct a potential $\theta^L_{\mathcal{M}}$ (such that $d\theta^L_{\mathcal{M}} = \omega^L_{\mathcal{M}}$) on the space $\mathcal{M}^L$ explicitly; therefore, for constant matrices $L_j$ one can offer two ways to fix the dependence of function $G$ on monodromies. The first way is based on pants decomposition of the sphere with $n$ punctures and the use of complex Fenchel-Nielsen coordinates which are Darboux coordinates on symplectic leaves of Goldman bracket. The second way is to use $SL(2)$ Fock-Goncharov coordinates (which in this case were originally discovered by Thurston) (see [25, 18]).

For an arbitrary $n$ the construction of $SL(n)$ version of $\theta^L_{\mathcal{M}}$ can be carried out as follows. Let $\chi_k$ be a set of Fock-Goncharov coordinates [16] on the $SL(n)$ character variety of $\pi_1(\mathbb{C}P^1 \setminus \{t_j\}_{j=1}^{N})$ such that the symplectic form on symplectic leaves $L_j = \text{const}$ of the monodromy manifold is given by $\omega^L = \sum_{j,k} s_{ij} d \log \chi_j \wedge d \log \chi_k$ where $s_{ij}$ are integer constants. The the $\theta^L_{\mathcal{M}}$ can be written as

$$
\theta^L_{\mathcal{M}} = \sum_{j,k} s_{ij} \log \chi_j \wedge d \log \chi_k \tag{4.25}
$$

and used to define the generating function $G$, and, therefore, the Jimbo-Miwa tau-function. Of course, this construction would essentially depend on the chosen triangulation of the punctured sphere (as well as the construction based on complex Fenchel-Nielsen coordinates in $SL(2)$ case, which essentially depends on the chosen pants decomposition).

**Comparison of weak and strong ILP conjectures.** In spite of formal similarity, there is a significant difference between the statements of the weak and strong ILP conjectures. In the strong version the form $\sum \text{tr}(L_j dG_j G_j^{-1})$ is a well-defined form on the main moduli space $\mathcal{A}$ as well as on its extension $\tilde{\mathcal{A}}$.

In the weak version the same form is not defined on the space $\mathcal{A}^L$ since to get the equality (4.20) one needs to take the residues $A_j$ (which are given by a point of $\mathcal{A}^L$ up to a conjugation) and then diagonalize each $A_j$ into $G_j L_j G_j^{-1}$ in a way which is non-local in times $t_j$: the matrices $G_j$’s themselves must satisfy the Schlesinger system (1.31). This requirement can not be satisfied staying entirely within the space $\mathcal{A}^L$ and thus $G_j$’s can not be chosen as functionals of $A_j$’s only; their choice encodes a highly non-trivial $t_j$-dependence which fixes the freedom in the right multiplication of each $G_j$ by a diagonal matrix which also can be time-dependent.

The strong version of the ILP conjecture (Corollary 4.20) is a stronger statement since the form $\theta_{\mathcal{A}}$ is a 1-form defined on the phase space.
A Extension of Kirillov-Kostant bracket and dynamical \( r \)-matrix

A.1 Kirillov-Kostant bracket and symplectic form

Introduce the Kirillov-Kostant bracket on \( GL(n) \) which, in tensor notation, takes the form

\[
\{ A, A \} = [A, \Pi] = \Pi (A - A) \tag{A.1}
\]

where we use the customary notation for the Kronecker products

\[
\begin{align*}
\mathbf{1}_M &= M \otimes \mathbf{1}, \\
\mathbf{2}_M &= \mathbf{1} \otimes M
\end{align*}
\]

for any matrix \( M \). Here \( \Pi \) is the permutation matrix of size \( n^2 \times n^2 \) given by

\[
\Pi = \sum_{i,j=1}^{n} E_{ij} \otimes E_{ij} \tag{A.2}
\]

where \( E_{ij} \) is an \( n \times n \) matrix whose \( ij \) entry equals 1 while all other entries vanish.

The symplectic form of the Kirillov-Kostant bracket (A.1) on a symplectic leaf parametrized by diagonal matrix \( L \) is known to have the form (see [5], pp. 44, 45):

\[
\omega_{KK} = - \text{tr} \left( LG^{-1}dG \wedge G^{-1}dG \right) \tag{A.3}
\]

where \( G \) is any matrix diagonalizing \( A \) i.e. \( A = GLG^{-1} \).

It is easy to verify that the form \( \omega_{KK} \) is invariant under the transformation \( G \rightarrow GD \) where \( D \) is a diagonal matrix which may depend on \( G \); such transformation leaves \( A \) invariant.

A.2 Extension of Kirillov-Kostant bracket and symplectic form to the \( (G, L) \)-space

Let us introduce the space

\[
\mathcal{H} := GL(n, \mathbb{C}) \times \mathfrak{h}_{ss} \tag{A.4}
\]

where \( \mathfrak{h}_{ss} \) consists of diagonal matrices with distinct eigenvalues. We denote an element of \( \mathcal{H} \) by \( (G, L) \) where \( G \in GL(n) \) and \( L \in \mathfrak{h}_{ss} \).

Consider the following one-form on \( \mathcal{H} \):

\[
\theta := \text{tr}(LG^{-1}dG) \tag{A.5}
\]

Proposition A.1 The differential \( \omega = d\theta \) given by

\[
\omega = \text{tr} \left( dL \wedge G^{-1}dG - LG^{-1}dG \wedge G^{-1}dG \right) \tag{A.6}
\]

is a symplectic form on \( \mathcal{H} \).

Proof. The form \( \omega \) is clearly closed; to test the nondegeneracy we take two tangent vectors in \( T_{(G,L)} \mathcal{H} \) and write them as \( (X, D) \in \mathfrak{gl}(n) \oplus \mathfrak{h} \) where \( \mathfrak{h} \) denotes the Cartan subalgebra of \( \mathfrak{gl}(n) \) (i.e. diagonal matrices). Given two tangent vectors \( X_j \oplus D_j \), \( j = 1, 2 \) the evaluation of \( \omega \) on them yields:

\[
\omega((X_1, D_1), (X_2, D_2)) = \text{tr} \left( D_1 X_2 - D_2 X_1 - L[X_1, X_2] \right). \tag{A.7}
\]

We now show that this form is nondegenerate; using the cyclicity of the trace rewrite (A.7) as

\[
\omega((X_1, D_1), (X_2, D_2)) = \text{tr} \left( D_1 X_2 - (D_2 + [X_2, L])X_1 \right). \tag{A.8}
\]
Suppose that \((X_2, D_2)\) are chosen so that the result vanishes identically for all \((X_1, D_1)\); then, choosing \(D_1 = 0\) and \(X_1\) arbitrary, we have in particular \(\text{tr}(D_2 + [X_2, L])X_1 = 0\). But since \(X_1\) is arbitrary, it follows that \(D_2 + [X_2, L] = 0\); since \(L\) is diagonal, the commutator is diagonal free and hence \(D_2 = 0\); since \(L\) is semisimple (the eigenvalues are distinct), it follows that \(X_2\) must be diagonal.

Then, choosing \(X_1 = 0\) and \(D_1\) arbitrary we see that the diagonal part of \(X_2\) must vanish as well. Thus the pairing is nondegenerate and the form is symplectic. \(\blacksquare\)

We now address the corresponding Poisson bracket.

**Proposition A.2** The nonzero Poisson brackets corresponding to the symplectic form \(d\Theta\) are

\[
\{G_{bj}, G_{cl}\} = \frac{G_{bj}G_{cj}}{\lambda_j - \lambda_\ell}, \quad j \neq \ell, \quad \{G_{kk}, \lambda_\ell\} = -G_{kk}\delta_{\ell k}. \tag{A.9}
\]

**Proof.** The form \((A.7)\) defines a map \(\Phi_{(G,L)} : T_{(G,L)}\mathcal{H} \to T^*_{(G,L)}\mathcal{H}\) given by

\[
\langle \Phi_{(G,L)}(X, D), (Y, \bar{D}) \rangle := \omega((X, D), (Y, \bar{D})), \quad \forall (Y, \bar{D}) \in T_{(G,L)}\mathcal{H} \tag{A.10}
\]

\[
\Phi_{(G,L)}(X, D) = \left( -D - [X, L], X^D_0 \right) = (Q, \delta) \in T^*_{(G,L)}\mathcal{H} \tag{A.11}
\]

where \(X^D\) and \(X^{OD}\) denote the diagonal and off-diagonal parts of the matrix \(X\), respectively and the identification between matrices and dual is done with the trace pairing.

As usual the pairing between tangent and cotangent spaces is the Trace form. Given now \((Q, \delta) \in T^*_{(G,L)}\mathcal{H}\) we observe from the formula \((A.10)\) that \(D = -Q^D\) and \(X = \delta + \text{ad}_{L^{-1}}(Q^{OD})\). The inverse of \(\text{ad}_L(\bullet) = [L, \bullet]\) is given explicitly by

\[
\text{ad}_{L^{-1}}(M)_{ab} = \frac{M_{ab}}{L_{aa} - L_{bb}}, \quad a = b \tag{A.12}
\]

as a linear invertible map on the space of off–diagonal matrices.

Thus \(\Phi^{-1}_{(G,L)} : T^*_{(G,L)}\mathcal{H} \to T_{(G,L)}\mathcal{H}\) is given by

\[
\Phi^{-1}_{(G,L)}(Q, \delta) = \left( \delta + \text{ad}_{L^{-1}}(Q^{OD}), -Q^D \right) \tag{A.13}
\]

where \(Q^{OD}\) and \(Q^D\) denote the off-diagonal and diagonal parts, respectively. The Poisson tensor \(\mathbb{P} \in \bigwedge^2 T_{(G,L)}\mathcal{H} \simeq (\bigwedge^2 T_{(G,L)}\mathcal{H})^\vee\) is defined by

\[
\mathbb{P}_{(G,L)}\left((Q_1, \delta_1), (Q_2, \delta_2)\right) := \omega\left(\Phi^{-1}_{(G,L)}(Q_1, \delta_1), \Phi^{-1}_{(G,L)}(Q_2, \delta_2)\right). \tag{A.14}
\]

Plugging the definition \((A.7), (A.10)\) gives

\[
\mathbb{P}_{(G,L)}\left((Q_1, \delta_1), (Q_2, \delta_2)\right) \tag{A.15}
\]

\[
= \text{tr}\left( -Q^D_1 \left( \delta_2 + \text{ad}_{L^{-1}}(Q^{OD}_2) \right) + Q^D_2 \left( \delta_1 + \text{ad}_{L^{-1}}(Q^{OD}_1) \right) \right) \tag{A.16}
\]

\[
= \text{tr}\left( Q^D_2 \delta_1 - Q^D_1 \delta_2 + Q^{OD}_2 \text{ad}_{L^{-1}}(Q^{OD}_1) \right) \tag{A.17}
\]
To obtain the Poisson bracket between the matrix entries of \( G \) and \( L \) we express
\[ Q = G^{-1}dG \quad \text{and} \quad \delta = dL = \text{diag}(d\lambda_1, \ldots, d\lambda_n). \]
Choosing \( Q_1 = E_{jk}, \delta_1 = 0 \) and \( Q_2 = 0, \delta_2 = E_{\ell\ell} \) gives
\[
(G^{-1})_{jk} \{ G_{bk}, \lambda_{\ell} \} = \mathcal{P}((G^{-1}dG)_{jk}, d\lambda_{\ell}) = -\delta_{jk}\delta_{k\ell} \Rightarrow \{ G_{bk}, \lambda_{\ell} \} = -G_{bk}\delta_{k\ell}. \tag{A.18}
\]
Choosing \( Q_1 = E_{ij}, Q_2 = E_{k\ell}, \delta_1 = \delta_2 = 0 \) we get instead
\[
\mathcal{P}((G^{-1}dG)_{ij}, (G^{-1}dG)_{k\ell}) = (G^{-1})_{ib}(G^{-1})_{kc} \{ G_{bj}, G_{c\ell} \} = \frac{\delta_{jk}\delta_{i\ell}}{\lambda_j - \lambda_\ell}. \tag{A.19}
\]

**Proposition A.3** The bracket (A.9) can be alternatively written using the dynamical \( r \)-matrix ([13], p.4), which in the \(GL(n)\) case is written as follows
\[
r(L) = \sum_{i<j} E_{ij} \otimes E_{ji} - E_{ji} \otimes E_{ij} \tag{A.20}
\]
Introduce also the operator
\[
\Omega_h = \sum_{i=1}^n E_{ii} \otimes E_{ii} \tag{A.21}
\]
where \( E_{ii} \) is the diagonal matrix with 1 on \( i \)th place of the diagonal. Then we have
\[
\{ G, \frac{1}{2} G \} = - \frac{1}{2} G \frac{1}{2} G r(L) \tag{A.22}
\]
and
\[
\{ G, \frac{1}{2} L \} = - \frac{1}{2} G \Omega_h, \tag{A.23}
\]
in particular,
\[
\{ G, \lambda_j \} = -GE_{ii} \tag{A.24}
\]
Then Jacobi identity involving the brackets \( \{ \{ G^1, G^2 \}, G^3 \} \) implies (taking into account that \( i^j = -j^i \)) the classical dynamical Yang-Baxter equation (see (3) of [13]).
\[
[\frac{1}{12} r, \frac{1}{13} r] + [\frac{1}{12} r, \frac{23}{23} r] + [\frac{23}{23} r, \frac{31}{31} r] + \sum_{i=1}^n \frac{\partial}{\partial \lambda_i} \frac{\partial r(L)}{\partial \lambda_i} E_{ii} + \frac{\partial}{\partial \lambda_i} \frac{\partial r(L)}{\partial \lambda_i} E_{ii} + \frac{\partial}{\partial \lambda_i} \frac{\partial r(L)}{\partial \lambda_i} E_{ii} = 0 \tag{A.25}
\]

**A.3 From extended to regular Kirillov-Kostant bracket**

**Theorem A.1** The map \( (G, L) \mapsto A = GLG^{-1} \) is a Poisson morphism between the Poisson bracket (A.9) and Kirillov-Kostant Poisson bracket on \( A; \)
\[
\{ \text{tr}(AF), \text{tr}(AH) \}_{KK} = \text{tr} \left( A[H, F] \right), \quad \forall F, H \in \mathfrak{gl}_n \isom \mathfrak{gl}_n^\vee \tag{A.26}
\]
or, equivalently,
\[
\{ \frac{1}{2} A, \frac{1}{2} A \} = \{ A, \Pi \} \tag{A.27}
\]
Proof. Write $dA = [dGG^{-1}, A] + GdLG^{-1} = G \left( [X, L] + \Lambda \right) G^{-1}$.

Then
\[
\{(GLG^{-1})_{ab}, (GLG^{-1})_{cd}\} = \mathbb{P}_{(G, L)} \left( d(GLG^{-1})_{ab}, d(GLG^{-1})_{cd} \right) \quad (A.28)
\]
\[
= \mathbb{P}_{(G, L)} \left( \left( [G^{-1}dG, L] + dL \right) G^{-1}_{ab}, \left( [G^{-1}dG, L] + dL \right) G^{-1}_{cd} \right) \quad (A.29)
\]
\[
= G_{ai}(G^{-1})_{jb}G_{ck}(G^{-1})_{td}\mathbb{P}_{(G, L)} \left( \left( [G^{-1}dG, L] + dL \right)_{ij}, \left( [G^{-1}dG, L] + dL \right)_{k\ell} \right). \quad (A.30)
\]

We have seen above that
\[
\mathbb{P}((G^{-1}dG)_{jk}, d\lambda_k) = -\delta_{j\ell}\delta_{k}\mathbb{P}((G^{-1}dG)_{ij}, (G^{-1}dG)_{\ell k}) = \frac{\delta_{j\ell}\delta_{k}}{\lambda_j - \lambda_{\ell}}, \quad j \neq \ell, \quad (A.31)
\]
\[
\mathbb{P}(d\lambda_j, d\lambda_\ell) = 0. \quad (A.32)
\]

Plugging (A.32) in (A.30) the only terms that contribute are the following
\[
G_{ai}(G^{-1})_{jb}G_{ck}(G^{-1})_{td}\mathbb{P}_{(G, L)} \left( \left( [G^{-1}dG, L] + dL \right)_{ij}, \left( [G^{-1}dG, L] + dL \right)_{k\ell} \right) \quad (A.33)
\]
\[
= G_{ai}(G^{-1})_{jb}G_{ck}(G^{-1})_{td}(\lambda_i - \lambda_j)(\lambda_k - \lambda_\ell)\mathbb{P} \left( (G^{-1}dG)_{ij}, (G^{-1}dG)_{\ell k} \right) \quad (A.34)
\]
\[
= G_{ai}(G^{-1})_{jb}G_{ck}(G^{-1})_{td}(\lambda_i - \lambda_j)(\lambda_k - \lambda_\ell)\frac{\delta_{j\ell}}{\lambda_j - \lambda_{\ell}} \quad (A.35)
\]
\[
= G_{a\ell}(G^{-1})_{jb}G_{cj}(G^{-1})_{td}(\lambda_\ell - \lambda_j)(\lambda_j - \lambda_\ell) \frac{1}{\lambda_j - \lambda_{\ell}} \quad (A.36)
\]
\[
= G_{a\ell}(G^{-1})_{jb}G_{cj}(G^{-1})_{td}(\lambda_\ell - \lambda_j) = A_{ad}\delta_{bc} - A_{cb}\delta_{ad}. \quad (A.37)
\]

This precisely translates to the Poisson bracket (A.26). \hfill \square

Remark A.1 We were not able to find the complete construction of this appendix in the existing literature. However, in the special case of the $SL(2)$ group the Poisson algebra (A.22), (A.23) appeared in the work [1] in the context of classical Poisson geometry of $T^*SL(2)$, see formulas (2),(3) of that paper.

## B Proof of Theorem 3.2

In [6, 7] it was proved the following formula for the exterior derivative of $\Theta$ (3.5) in the general setting of a family of Riemann Hilbert problems satisfying Assumption 3.1:

**Theorem B.1 (Theorem 2.1 of [7])** In the general setting discussed in the beginning of this section 3 the exterior derivative of the 1-form $\Theta$ (3.5) is expressed by the following formula:

\[
d\Theta = -\frac{1}{2} \int_{\mathbb{Z} \times 2\pi} \frac{dz}{dz} \frac{dz}{2\pi} \text{tr} \left( \frac{d}{dz} (dJJ^{-1}) \wedge (dJJ^{-1}) \right) + \eta_{\nu} \quad (B.1)
\]

with
\[
\eta_{\nu} := -\frac{1}{4i\pi} \sum_{a \in \mathcal{V}} \sum_{\ell=2}^{\ell-1} \sum_{m=1}^{\ell-1} \text{tr} \left( J_{1, m-1}^{(v)} dJ_{m+1, \ell-1}^{(v)} J_{\ell+1, n}^{(v)} dJ_{\ell}^{(v)} J_{\ell+1, n}^{(v)} \right) \quad (B.2)
\]

where we have used the notation $J_{[a:b]} = J_a \cdot J_{a+1} \cdots J_b$. 

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Figure 2: On contribution of one of the loops to the form $d\Theta$

The term $\eta_v$ can be further simplified by a straightforward computation using the condition (3.1) as follows

$$\eta_v = -\frac{1}{4\pi i} \sum_{v \in V} \sum_{\ell = 1}^{n_v} \text{tr} \left( (J^{(v)}_{\ell})^{-1} \text{d}J^{(v)}_{\ell} \wedge \text{d}J^{(v)}_{[\ell + 1, n_v]} (J^{(v)}_{[\ell + 1, n_v]})^{-1} \right). \quad (B.3)$$

Here we give the proof of Theorem 3.2 by deriving it from the general formulas (B.1), (B.2).

**Corollary B.1** Let $\Phi$ be the solution of the RHP 3.2 given in (3.4) on the contour $\Sigma$ and jump matrices indicated in Fig. 1. Then the external derivative of the form $\Theta$ (3.3) (or alternatively given by (3.5)) is given by the formula

$$d\Theta = \omega M \quad (B.4)$$

where

$$\omega M = \frac{1}{4\pi i} (\omega_1 + \omega_2); \quad (B.5)$$

$$\omega_1 = \sum_{\ell = 1}^{N} \text{tr} \left( M^{-1}_{\ell} \text{d}M_{\ell} \wedge K^{-1}_{\ell} \text{d}K_{\ell} \right) + \sum_{\ell = 1}^{N} \text{tr} \left( \Lambda^{-1}_{\ell} C^{-1}_{\ell} \text{d}C_{\ell} \wedge \Lambda_{\ell} C^{-1}_{\ell} \text{d}C_{\ell} \right), \quad (B.6)$$

$$\omega_2 = 2 \sum_{\ell = 1}^{N} \text{tr} \left( \Lambda^{-1}_{\ell} \text{d}\Lambda_{\ell} \wedge C^{-1}_{\ell} \text{d}C_{\ell} \right) \quad (B.7)$$

and $K_{\ell} = M_1 \ldots M_{\ell}$.

**Proof.** The contour $\Sigma$ is depicted in Fig. 1, where also the jump matrices are indicated. The integral over $\Sigma$ in the formula (B.1) in this case reduces to a sum of integrals over $\partial D_{\ell}$’s because the jump matrix $J(z)$ on the cuts is constant with respect to $z$. We denote by $\beta_{\ell}$ the three-valent vertices where the circles around $t_{\ell}$ meet with the edges going towards $z_0$. Let us consider the contribution of one of the integrals over $\partial D_{\ell}$ to (B.3).

We will drop the index $\ell$ for brevity in the formulas below. Notice also that $dL \wedge dL = 0$ because the matrix $L$ is diagonal. Then we get

$$-\frac{1}{2} \int_{2\pi} \frac{dz}{2\pi i} \text{tr} \left( \frac{d}{dz} (d(C(z-t)^{-L}(z-t)^{L}C^{-1}) \wedge d(C(z-t)^{-L}(z-t)^{L}C^{-1}) \right) =$$

$$-\frac{1}{2} \int_{2\pi} \frac{dz}{2\pi i} \text{tr} \left( \left[ -\frac{dC\ln_{C^{-1}}}{z-t} - \frac{CdC^{-1}}{z-t} \right] \wedge \left[ dC\ln_{C^{-1}} + \frac{CdC^{-1}}{z-t} - \frac{CdC^{-1}}{z-t} \right] \right) =$$

$$= -\frac{1}{2} \int_{2\pi} \frac{dz}{2\pi i} \text{tr} \left( \frac{dL \wedge LdL \ln(z-t)}{(z-t)^2} - \frac{dL \wedge C^{-1}dC}{(z-t)} \right) = \frac{1}{2} \left( \frac{dL \wedge LdL}{(\beta-t)} + dL \wedge C^{-1}dC \right) \quad (B.8)$$

In the last integration we have used that

$$\int_{\beta}^{\beta} \frac{dz}{2\pi i} \frac{\ln(z-t)}{(z-t)^2} = -\frac{1}{\beta-t} \quad (B.9)$$
where the integration is along the circle \(|z - t| = |\beta - t|\) starting at \(z = \beta\). We now turn to the evaluation of the term \(\eta_\ell\) in (B.1). The set of vertices \(\mathbf{V}\) consists of \(\mathbf{V} = \{z_0, \beta_1, \ldots, \beta_N\}\). The contribution coming from the vertex \(z_0\) is precisely the first one in \(\omega_1\) (3.20) (in the formula we have simplified it using the local no-monodromy condition (3.1)).

To evaluate the contribution of the vertex \(\beta = \beta_\ell \in \mathbf{V}\) we observe that it is tri-valent and the jump matrices on the three incident arcs are

\[
J_1 = CA^{-1}C^{-1}, \quad J_2 = C(\beta - t)^{-L}, \quad J_3 = (\beta - t)^L e^{2i\pi L} C^{-1},
\]

where \(\Lambda := e^{2i\pi L}\). In the definition above we have used that \((z - t)^L\) is defined with a branch cut extending from \(t\) to \(\beta\). Since \(J_1 J_2 J_3 = 1\) the contribution of this vertex to (B.3) reduces to only the term \(
\frac{1}{4\pi i} \text{tr} (J_1 dJ_2 \wedge dJ_3) = \frac{1}{4\pi i} \text{tr} (J_2^{-1} dJ_2 \wedge dJ_3 J_3^{-1})\): here the two differentials are (recall that \(L, \Lambda\) are diagonal)

\[
J_2^{-1} dJ_2 = (\beta - t)^L C^{-1} dC(\beta - t)^{-L} + (\beta - t)^L \frac{Ldt}{\beta - t}(\beta - t)^{-L} - dL \ln(\beta - t),
\]

\[
dJ_3 J_3^{-1} = \frac{-dL}{(\beta - t)} + (\ln(\beta - t) + 2i\pi) dL - (\beta - t)^L \Lambda^{-1} dC \Lambda^{-1}(\beta - t)^{-L}.
\]

Plugging (B.11) into the expression and simplifying with straightforward algebra finally yields

\[
-\frac{1}{4\pi i} \text{tr} (J_2^{-1} dJ_2 \wedge dJ_3 J_3^{-1}) =
\]

\[
= -\frac{1}{4\pi i} \text{tr} \left( \left((\beta - t)^L C^{-1} dC(\beta - t)^{-L} + (\beta - t)^L \frac{Ldt}{\beta - t}(\beta - t)^{-L} - dL \ln(\beta - t)\right) \wedge 
\left( \frac{-dL}{(\beta - t)} + (\ln(\beta - t) + 2i\pi) dL - (\beta - t)^L \Lambda^{-1} dC \Lambda^{-1}(\beta - t)^{-L}\right) \right) = 
\]

\[
= -\frac{1}{4\pi i} \text{tr} \left(C^{-1} dC \wedge \frac{-dL}{(\beta - t)} + C^{-1} dC \wedge (\ln(\beta - t) + 2i\pi) dL - C^{-1} dC \wedge \Lambda^{-1} dC \Lambda^{-1} + 
\frac{Ldt}{\beta - t} \wedge (\ln(\beta - t) + 2i\pi) dL - \frac{Ldt}{\beta - t} \wedge C^{-1} dC - dL \ln(\beta - t) \wedge \frac{-dL}{(\beta - t)} + dL \ln(\beta - t) \wedge \Lambda^{-1} dC \right) = 
\]

\[
= -\frac{1}{4\pi i} \text{tr} \left(C^{-1} dC \wedge \Lambda^{-1} d\Lambda - C^{-1} dC \wedge \Lambda C^{-1} dC \Lambda^{-1} + 2i\pi \frac{Ldt}{\beta - t} \wedge dL\right) \quad (B.12)
\]

Adding (B.8) (contribution of the integral) with (B.12) (contribution coming from the vertex \(\beta = \beta_\ell\)) gives

\[
\frac{1}{2} \left( \frac{dL}{(\beta - t)} + dL \wedge C^{-1} dC \right) + \frac{1}{4\pi i} \text{tr} \left(C^{-1} dC \wedge \Lambda^{-1} d\Lambda - C^{-1} dC \wedge \Lambda C^{-1} dC \Lambda^{-1} + 2i\pi \frac{Ldt}{\beta - t} \wedge dL\right) = 
\]

\[
= \frac{1}{4\pi i} \text{tr} \left(-2C^{-1} dC \wedge \Lambda^{-1} d\Lambda + C^{-1} dC \wedge \Lambda C^{-1} dC \Lambda^{-1}\right) \quad (B.13)
\]

Summing over all contributions from vertices \(\beta_\ell\) yields the result. 

\[\square\]

\section{Invariance of \(\Theta\)}

The following theorem shows that the Malrange form is really defined on \(\tilde{A}\) (or \(\tilde{M}\) outside of the Malrange divisor).

\begin{theorem}
The Malrange form \(\Theta\) is invariant under the transformations \(C_j \mapsto SC_j\) where \(S\) is any matrix that depends on variables \(\{C_j, L_j, t_j\}\). Therefore, \(d\Theta = \tilde{\omega}_M\) is also invariant.
\end{theorem}
Proof. Under the transformation \( \widetilde{C}_j = SC_j \) the monodromy matrices transform as \( \widetilde{M}_j = SM_j S^{-1} \). The jumps of the RHP are presented in Fig. 1; notice that the jumps on the edges emanating from \( z_0 \) are conjugated by \( S \); on the boundaries of the disks \( \mathbb{D}_j \) the jumps are multiplied by \( S \) from the left. In view of this, we consider a related RHP for \( \tilde{\Phi} \) where

\[
\widetilde{\Phi}(z) = \begin{cases} 
\Phi(z) & z \not\in \bigcup \mathbb{D}_j \\
\Phi(z) C_j^{-1} & z \in \mathbb{D}_j.
\end{cases}
\]

(C.1)

The matrix \( \tilde{\Phi}(z) \) has jump \( C_j(z - t_j)^{-L_j}C_j^{-1} \) on the boundary of each disk \( \mathbb{D}_j \).

Let us study the effect of this transformation on the Malgrange form \( \Theta \) associated to \( \Phi \). All jump data \( J(z) \) of the new matrix \( \tilde{\Phi} \) are conjugated by \( S \) under our transformation i.e. \( \Phi \mapsto S\Phi S^{-1} \). As a consequence, the effect of this transformation on \( \Theta \) is as follows:

\[
\tilde{\Theta} \rightarrow \int_S \frac{dz}{2\pi i} \text{tr} \left( \tilde{\Phi}^{-1} \tilde{\Phi}' \left( dSJS^{-1} + SdJS^{-1} - SJ^{-1}S^{-1} \right) \right) = \\
\int_S \frac{dz}{2\pi i} \text{tr} \left( \Phi^{-1} \Phi'_- + \Phi^{-1} \Phi'_+ - \Phi^{-1} \Phi_+ dSS^{-1} \right) = \\
\tilde{\Theta} + \int_S \frac{dz}{2\pi i} \text{tr} \left( \tilde{J}dSS^{-1} \right) = \Theta - \sum_{j=1}^n \text{tr}(C_j L_j C_j^{-1} dSS^{-1}) . \tag{C.2}
\]

The difference between the forms \( \Theta \) and \( \tilde{\Theta} \) can be computed as follows

\[
\tilde{\Theta} - \Theta = \sum_j \int_{\partial \mathbb{D}_j} \frac{dz}{2\pi i} \text{tr} \left( \Phi^{-1} \Phi'_- C_j(z - t_j)^{-L_j} dC_j^{-1}(z - t_j) L_j C_j^{-1} \right) = \\
\sum_j \int_{\partial \mathbb{D}_j} \frac{dz}{2\pi i} \text{tr} \left( \Phi^{-1} \Phi'_- C_j(z - t_j)^{-L_j} d(C_j^{-1})C_j \right) . \tag{C.3}
\]

Since the term \( \Phi^{-1} \Phi'_- \) is analytic inside the disk, it gives a vanishing contribution to these integrals while the second term gives

\[
\tilde{\Theta}_M = \Theta_M - \sum_j \text{tr} \left( L_j C_j^{-1} dC_j \right) , \tag{C.4}
\]

Note now that the second term in (C.4) transforms as follows:

\[
- \sum_j \text{tr} \left( L_j C_j^{-1} dC_j \right) \rightarrow - \sum_j \text{tr} \left( L_j C_j^{-1} dC_j \right) - \sum_j \text{tr} \left( C_j L_j C_j^{-1} S^{-1} dS \right)
\]

Comparing with (C.2) we conclude that \( \Theta \) is invariant.

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