Backward Monge Potential and Monge-Ampère Equation

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Abstract

In this paper, Monge-Kantorovich problem is considered in the infinite dimension on an abstract Wiener space \((W, H, \mu)\), where \(H\) is Cameron-Martin space and \(\mu\) is the Gaussian measure. We study the regularity of optimal transport maps with a quadratic cost function assuming that both initial and target measures have a strictly positive Radon-Nikodym density with respect to \(\mu\). Under conditions on the density functions, the forward and backward transport maps can be written in terms of Sobolev derivative of so-called Monge-Brenier maps, or Monge potentials. We show Sobolev regularity of the backward potential under the assumption that the density of the initial measure is log-concave and prove that it solves Monge-Ampère equation.

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1 Introduction

Monge problem is motivated by the application of moving given piles of sand to fill up holes of the same volume with a minimum total cost of transportation [1, 22]. Piles and holes are modeled by probability measures $\rho$ and $\nu$ defined on some measurable sets $X$ and $Y$, respectively, with a measurable and non-negative cost function $c : X \times Y \to \mathbb{R}_+ \cup \{+\infty\}$. The aim is to find a transport map $T : X \to Y$, with $T \rho = \nu$, that minimizes the expected cost of moving the sand. A solution to Monge problem may not always exist, as it does not allow split of mass. Monge-Kantorovich problem overcomes this difficulty by the relaxation that mass from location $x \in X$ can be moved possibly to several locations in $Y$ with the objective of finding a transference plan $\gamma^* \in \Gamma(\rho, \nu)$ that minimizes the functional

$$J[\gamma] = \int_{X \times Y} C(x, y) d\gamma(x, y)$$

where $\Gamma(\mu, \nu)$ is the set of joint probability measures on $X \times Y$ with first and second marginals $\rho$ and $\nu$, respectively. If $X$ and $Y$ are Polish spaces and $c$ is lower semi-continuous, then Monge-Kantorovich problem admits a minimizer [2, Thm.1.5]. In the infinite dimension, Monge-Kantorovich problem has been studied on Wiener space first in [15] with a quadratic cost function. Later in [10, 13, 20], different cost functions are considered with other initial and target measures.

In this paper, we consider an abstract Wiener space $(W, H, \mu)$, where $H$ is Cameron-Martin space and $\mu$ is the Gaussian measure, and define $d\rho = e^{-f} d\mu$ and $d\nu = e^{-g} d\mu$ for measurable $f, g : W \to \mathbb{R}$. Under conditions on $f$ and $g$, there exists so-called forward Monge potential or Monge-Brenier map $\varphi$ such that $T = I_W + \nabla \varphi$ is the optimal transport map, where $\nabla$ denotes the Gross-Sobolev derivative operator. Moreover, the inverse $S$ of $T$ is given by $S = I_W + \nabla \psi$ for some $\psi : W \to \mathbb{R}$, called backward Monge potential, satisfying $(S \times I_W) \rho = (I_W \times T) \nu = \gamma$. Sobolev regularity of the Monge-Brenier maps is an important issue in order to write Jacobian functions associated with transformations $T$ and $S$. In Theorem 4.1, assuming $f$ is $(1 - c)$-convex, equivalently $e^{-f}$ is log-concave, we show that $\nabla^2 \psi$ is in $L^2(\nu, H \otimes H)$ and provide an upper estimate for its norm as

$$\mathbb{E}_\nu \left[ |\nabla^2 \psi|^2 \right] \leq \frac{3}{c} \left( \mathbb{E}_\rho \left[ |\nabla \varphi|^2 \right] + \mathbb{E}_\nu \left[ |\nabla g|^2 \right] + \mathbb{E}_\rho \left[ |\nabla f|^2 \right] \right),$$

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where $H \otimes H$ denotes the space of Hilbert-Schmidt operators on $H$ and $\| \cdot \|_2$ denotes the Hilbert-Schmidt norm. We use the approximation approach of [24], where the initial measure $\rho$ is taken as $\mu$ and the target measure $\nu$ is assumed to be absolutely continuous with respect to $\mu$.

Monge potentials are closely related to Monge-Ampère equation defined in Wiener space. On $\mathbb{R}^n$, with $F, G : \mathbb{R}^n \to \mathbb{R}_+$, Monge-Ampère equation in $T$ can be written as

$$F = G \circ T \det J_T$$

(1)

where $J_T$ is the Jacobian of $T$. When $F$ and $G$ are the densities of $\rho$ and $\nu$ with respect to Lebesgue measure, respectively, the corresponding transport map $T$ is a solution. In finite dimension, regularity of solution of Monge-Kantorovich problem and Monge-Ampère equation have been studied in [9, 11, 12, 19]. For the aim of defining an analogous equation in Wiener space, suppose $F = e^{-f}$ and $G = e^{-g}$. In other words, Radon-Nikodym derivatives of $\rho$ vs $\nu$ with respect to Lebesgue measure are proportional to $e^{-f - \frac{|x|^2}{2}}$ and $e^{-g - \frac{|x|^2}{2}}$, respectively, for $x \in \mathbb{R}^n$ and (1) becomes

$$e^{-f(x) - \frac{|x|^2}{2}} = e^{-g \circ T(x) - \frac{|T(x)|^2}{2}} \det J_T(x).$$

On $\mathbb{R}^n$, we have $T = I_W + J_\varphi$, where $J_\varphi$ denotes the Jacobian of forward Monge potential $\varphi$. Using the identities $J_T = I_{\mathbb{R}^n} + \nabla \varphi$ and $\det(I_{\mathbb{R}^n} + \cdot) = \det_2(I_{\mathbb{R}^n} + \cdot) e^{\text{tr} (\cdot)}$, where $\det_2$ denotes Carleman-Fredholm determinant, $\text{tr}$ denotes the trace of a matrix and $\Delta$ denotes the Laplacian, we get

$$e^{-f} = e^{-g \circ T} \det_2(I_{\mathbb{R}^n} + \nabla^2 \varphi) e^{\text{tr}(\nabla^2 \varphi)} e^{-x J_\varphi} e^{-\frac{|J_\varphi|^2}{2}}.$$ 

In view of the identity $-\mathcal{L} \varphi = \Delta \varphi - x J_\varphi = \text{tr}(\nabla^2 \varphi) - x J_\varphi$, where $\mathcal{L}$ is the generator of the Ornstein–Uhlenbeck process on $\mathbb{R}^n$, Monge-Ampère equation can now be written as

$$e^{-f} = e^{-g \circ T} \det_2 (I_{\mathbb{R}^n} + \nabla^2 \varphi) \exp \left[ -\mathcal{L} \varphi - \frac{1}{2} |J_\varphi|^2 \right].$$

Since only $\det_2$ is well-defined in the infinite dimension, this form can be used to define Monge-Ampère equation in Wiener space by replacement of $| \cdot |$ with the norm in $H$, $J_\varphi$ with Gross-Sobolev derivative $\nabla \varphi$, and $\Delta \varphi$ with $\nabla^2 \varphi$. In [15], it is proved that Monge
potential $\varphi$ solves Monge-Ampère equation under the condition $f$ and $g$ are two positive random variables with values in a bounded interval $[a, b]$. Monge-Ampère equation for $\varphi$ has been obtained in [16, 17] with $f = 0$ and $g$ an $H$-convex Wiener function. In [5, 6], the authors have shown that $\varphi$ satisfies Monge-Ampère equation for $g \in L^1(\mu)$ and $ge^{-g} \in L^1(\mu)$ as well as the case when $e^{-f/2} \in D_{2,1}$ and $g = 0$. The case when both initial and target measures are absolutely continuous with respect to $\mu$ is studied in [13] for bounded $f$ and lower bounded $g$ with the additional assumption that second Sobolev derivative of $g$ exists.

In the settings of the present paper, the backward potential is shown to be regular in Theorem 4.1, that is, $\nabla^2 \psi \in L^2(\nu, H \otimes H)$ under the assumption that the initial measure $\rho$ has a log-concave density with respect to Gaussian measure $\mu$. Therefore, we prove in Theorem 4.2 that backward potential $\psi$ solves Monge-Ampère equation given by

$$e^{-g} = e^{-f \circ S} \det_2 (I_H + \nabla^2 \psi) \exp \left[ -\mathcal{L}_\psi - \frac{1}{2} |\nabla \psi|^2_H \right]$$

$\nu$-almost surely.

The organization of the paper is as follows. In Section 2, we review the preliminary definitions and introduce the notation used in the paper. Section 3 focuses on the finite dimensional and smooth case as a basis for the infinite dimension. In Section 4, the regularity of backward potential is shown and Monge-Ampère equation is considered.

## 2 Preliminaries

Let $(W, H, \mu)$ be an abstract Wiener space. The corresponding Cameron-Martin space is denoted by $H$. The norm in $H$ will be denoted by $| \cdot |_H$. If $h \in H$, then there exists $(l_n) \subset W^*$ such that image of this sequence under injection $W^* \hookrightarrow H$, say $(\tilde{l}_n)$, converges to $h$ in $H$. Therefore, the sequence of random variables $(\langle l_n, \cdot \rangle_H)$ is Cauchy in $L^p(\mu)$ for any $p \geq 0$. We denote its limit by $\delta h$, which is $N(0, |h|^2_H)$ random variable.

The function $F : W \to \mathbb{R}$ is called a cylindrical Wiener functional if it is of the form

$$F(\omega) = f(\delta h_1(\omega), \ldots, \delta h_n(\omega)),$$

for some $n \in \mathbb{N}$ and $S(\mathbb{R}^n)$ denotes the Schwartz space of rapidly decreasing functions on $\mathbb{R}^n$ [15, 23]. We denote the collection of Wiener functionals by $\mathcal{S}(W)$. For such $F \in \mathcal{S}(W)$
and \( h \in H \), we define
\[
\nabla_h F(\omega) = \frac{d}{d \epsilon} F(\omega + \epsilon h) \bigg|_{\epsilon=0}
\]
For fixed \( \omega \in W \), \( h \mapsto \nabla_h F(\omega) \) is continuous and linear on \( H \). Therefore, there exists an element in \( H \), say \( \nabla F \) such that \( \nabla_h F = \langle \nabla F, h \rangle_H \). The operator \( \nabla : F \mapsto \nabla F \) is linear from \( \mathcal{S}(W) \) into the space of \( H \)-valued Wiener functionals \( L^p(\mu; H) \) for any \( p > 1 \). The operator \( \nabla \) is closable from \( L^p(\mu) \) into \( L^p(\mu; H) \) for any \( p > 1 \). The completion of \( \mathcal{S}(W) \) under the norm
\[
\| \cdot \|_{p,1} = \| \cdot \|_{L^p(\mu)} + \| \cdot \|_{L^p(\mu; H)}
\]
is denoted by \( \mathbb{D}_{p,1} \), which is a Banach space with norm \( \| \cdot \|_{p,1} \) [23]. The definition can be also extended to the collection of Wiener functionals which take values in a separable Hilbert space \( \mathcal{X} \). The completion of the collection of \( \mathcal{X} \)-valued Wiener functionals \( \mathcal{S}(W; \mathcal{X}) \) under norm
\[
\| \cdot \|_{L^p(\mu; \mathcal{X})} + \| \cdot \|_{L^p(\mu; \mathcal{X} \otimes H)}
\]
is denoted by \( \mathbb{D}_{p,1}(\mathcal{X}) \). Higher order derivatives can also be defined, e.g. we say \( F \in \mathbb{D}_{p,2} \) if \( \nabla F \in \mathbb{D}_{p,1}(H) \) and write \( \nabla^2 F = \nabla(\nabla F) \).

Let \( \nu \) be measure on \( W \) absolutely continuous with respect to \( \mu \) with Radon-Nykodm derivative \( L \). If \( \int_W |\nabla L|^2 e^{-L} \, d\mu < \infty \), then operator \( \nabla \) is closable over \( \mathcal{S}(W) \) under the norm
\[
\| \cdot \|_{L^p(\nu)} + \| \cdot \|_{L^p(\nu; H)}
\]
We will denote the completion by \( \mathbb{D}_{p,1}(\nu) \). The adjoint of continuous and linear operator \( \nabla \) will be denoted by \( \delta \). That is, for suitable \( \xi : W \to H \) and any \( F \in \mathbb{D}_{p,1} \), we have
\[
\mathbb{E} \left[ \langle \nabla F, \xi \rangle_H \right] = \mathbb{E} [\varphi \cdot \delta \xi].
\]
The operator \( \delta : \xi \mapsto \delta \xi \) is called the divergence operator. We can also define \( \delta_\nu \) by this procedure, i.e, \( \delta_\nu \) is the adjoint of the Sobolev derivative \( \nabla \) under the measure \( \nu \).

For measurable \( f : W \to \mathbb{R} \) and \( t \geq 0 \) the Ornstein-Uhlenbeck semi-group \( (P_t)_{t \geq 0} \) is given by
\[
(P_t f)(x) = \int_W f \left( e^{-t}x + \sqrt{1 - e^{-2t}}y \right) d\mu(y).
\]
Its infinitesimal generator is denoted by $-\mathcal{L}$ and we call $\mathcal{L}$ the Ornstein–Uhlenbeck operator. The norm given by $\|(I + \mathcal{L})^{r/2}(\cdot)\|_{L^p(\mu)}$ is equivalent to the norm $\| \cdot \|_{p,r}$ for any $p > 1$ and $r \in \mathbb{N}$. Completion of $\mathcal{S}((W))$ with respect to the norm $\|(I + \mathcal{L})^{r/2}(\cdot)\|_{L^p(\mu)}$ again denoted by $\mathcal{D}_{p,r}$ which has $\mathcal{D}_{q,-r}, q^{-1} = 1 - p^{-1}$, as its continuous dual for any $p > 1$, $r \in \mathbb{R}$. Similarly, we can define $\mathcal{D}_{p,r}(\mathcal{X})$ as a completion of $\mathcal{S}((W; \mathcal{X}))$ with respect to the norm $\|(I + \mathcal{L})^{k/2}F\|_{L^p(\mu; \mathcal{X})}$ and its continuous dual is $\mathcal{D}_{q,-r}(\mathcal{X}^*)$, where $\mathcal{X}^*$ is the dual of $\mathcal{X}$. The space $\mathcal{D}(\mathcal{X}) = \bigcap_{p,r} \mathcal{D}_{p,r}(\mathcal{X})$ is dense in $\mathcal{D}_{p,r}(\mathcal{X})$, for any $p > 1$ and $r \in \mathbb{R}$. If the sequence $(F_n)$ converges to zero in $\mathcal{D}_{p,r}(\mathcal{X})$, for any $p > 1$ and $r \in \mathbb{R}$, we say that it converges to zero in $\mathcal{D}(\mathcal{X})$. Under this topology $\mathcal{D}(\mathcal{X})$ is a complete, locally convex topological vector space. The continuous dual $\mathcal{D}^*(\mathcal{X}^*)$ of $\mathcal{D}(\mathcal{X})$ is called Meyer-Watanabe distributions on $W$ with values in $\mathcal{X}^*$.

A measurable function $f : W \to \mathbb{R} \cup \{\infty\}$ is called $\alpha$-convex, $\alpha \in \mathbb{R}$, if the map

$$h \mapsto f(\cdot + h) + \frac{\alpha}{2} |h|^2_H$$

is convex on the Cameron-Martin space $H$ with values in $L^0(\mu)$ a.s.

For Monge-Kantorovich problem in the Wiener space, the cost function $C(x, y) : W \times W \to \mathbb{R}^+ \cup \{+\infty\}$ is taken as

$$C(x, y) = \begin{cases} |x - y|^2_H & \text{if } x - y \in H \\ +\infty & \text{otherwise.} \end{cases}$$

in [15]. The following result is the starting point of the present work [13, Thm.1.1].

**Theorem 2.1.** Let $\rho$, $\nu$ be the probability measures on $W$ such that $d\rho = F d\mu$ and $d\nu = G d\mu$, where $\mu$ is the Wiener measure, $F : W \to \mathbb{R}$ and $G : W \to \mathbb{R}$ are measurable functions such that

$$\int_W \frac{|
abla F|^2_H}{F} d\mu < \infty \text{ and } \int_W \frac{|
abla G|^2_H}{G} d\mu < \infty$$

and the function $F$ satisfies Poincaré inequality, that is, for every cylindrical functional $\xi : W \to \mathbb{R}$

$$\int_W (1 - c) (\xi - E_\rho[\xi])^2 F d\mu \leq \int_W |
abla \xi|^2_H F d\mu$$

(2)
for some \( c \in [0,1) \). Then, there exists a \( \varphi \in D_{2,1}(\rho) \) such that \( T = I_W + \nabla \varphi \) is the unique solution of Monge problem for \((\rho,\nu)\) and the probability measure \( \gamma \) given by \( \gamma = (I_W \times T)\rho \) is the unique solution of Monge-Kantorovich problem for \((\rho,\nu)\). Moreover, \( T \) is \( \rho \)-a.s invertible and the inverse map \( S = T^{-1} \) has the form \( S = I_W + \nabla \psi \), where \( \psi \in D_{2,1}(\nu) \).

Finally, let \( \mathcal{P}(X) \) denote the probability measures on a measurable space \( X \). For \( m_1, m_2 \in \mathcal{P}(X) \) on a Polish space \((X,d)\), the Wasserstein distance of order \( p \in [1,\infty) \) between \( m_1 \) and \( m_2 \) is defined by

\[
d_p(m_1, m_2) = \left( \inf_{\gamma \in \Gamma(m_1,m_2)} \int_X d(x,y)^p \, d\gamma(x,y) \right)^{1/p}.
\] (3)

Note that \( d_p \) is not a metric, since it can take the value of \(+\infty\). However, its restriction on a subset of \( \mathcal{P}(X) \times \mathcal{P}(X) \) where it takes finite values leads to a a complete metric space. The relative entropy of \( m_1 \) with respect to \( m_2 \) given by

\[
H(m_1 \mid m_2) = \begin{cases} 
\int_X \frac{dm_1}{dm_2} \log \left( \frac{dm_1}{dm_2} \right) \, dm_2 & \text{if } m_1 \ll m_2 \\
+\infty & \text{otherwise}.
\end{cases}
\] (4)

We take \( X = W \) and \( d(x,y) = |x - y|_H \) and \( p = 2 \). The following form of Talagrand’s inequality, derived in [15, Thm.3.1], provides a nice relation between Wasserstein distance and the relative entropy. For any probability measure \( \mu_0 \) on \( W \), it holds that

\[
d_2^2(\mu_0, \mu) \leq 2H(\mu_0 \mid \mu).
\] (5)

3 An Estimate for Smooth Backward Monge Potential

In this section, we assume that forward Monge potential \( \varphi \) is smooth, that is, \( \varphi \in D_{2,k} \) for all \( k \in \mathbb{N} \). We first prove an identity for forward Monge potential \( \varphi \) and then find an upper bound for \( D_{2,2}(\nu) \)-norm of \( \psi \), which will be auxiliary for regularity in the infinite dimension as shown in the next section. Similar results have been obtained in [24] when the initial measure is Gaussian and the target measure is absolutely continuous with respect to Gaussian measure.
Remark 3.1. When \( f \) and \( g \) are smooth and bounded from below, and \( \mu \) is a standard Gaussian measure on \( \mathbb{R}^n \), then \( \varphi \) is smooth. This would follow from [18] and [22, Thm.4.14].

Proposition 3.1. When \( W \) is finite dimensional and \( \varphi \) is smooth, Monge potential \( \varphi \) satisfies the relation

\[
\nabla \varphi + \nabla f \circ T - \nabla g = \delta_\rho \left[ (I_H + \nabla^2 \varphi)^{-1} - I_H \right].
\]

Proof. From [7, Prop.2.5], it is known that for bounded \( f, g \)

\[
- \log \int_W e^{-g+f} d\rho = \inf \left( \int_W (g-f) \, dm + H(\gamma|\rho) : m \in \mathcal{P}(W) \right)
\]

The infimum is attained at \( \nu \) given that \( H(\nu|\rho) < \infty \). Moreover, since \( T\rho = \nu \), we have

\[
- \log \int_W e^{-g+f} d\rho = \inf \left( \int_W (g-f) \circ (I_W + \nabla a) \, d\rho + H((I_W + \nabla a)\rho | \rho) : a \in D_{2,1}(\rho) \right)
\]

\[
\geq \inf \left( \int_W (g-f) \circ (I_W + \xi) \, d\rho + H((I_W + \xi)\rho | \rho) : \xi \in D_{2,0}(\rho) \right)
\]

\[
\geq \inf \left( \int_W (g-f) \, dm + H(m | \rho) : m \in \mathcal{P}(W) \right).
\]

Therefore, \( \inf \{ K_{g-f}(\xi) : \xi \in D_{2,0}(\rho) \} \) is attained at \( \xi = \nabla \varphi \), where

\[
K_{g-f}(\xi) = \int_W (g-f) \circ (I_W + \xi) \, d\rho + H((I_W + \xi)\rho | \rho).
\]

We have \( \nu \ll \rho \) with \( d\nu/d\rho = e^{-g}/e^{-f} \). It follows that

\[
H(T\rho | \rho) = H(\nu|\rho)
\]

\[
= \int_W e^{-g}(-g+f) \, d\mu
\]

\[
= \int_W e^{-g_{0T}}(-g+f) \circ T \Lambda_{\varphi} \, d\mu
\]

(7)

where \( \Lambda_{\varphi} \) is the Gaussian Jacobian of \( T = I_W + \nabla \varphi \) given by

\[
\Lambda_T = \det_2(I_H + \nabla^2 \varphi) \exp \left( -\mathcal{L}_{\varphi} - \frac{1}{2} |\nabla \varphi|_H^2 \right).
\]

Since we have smooth \( \varphi \in D_{2,1}(\rho) \), it solves Monge-Ampère equation, that is,

\[
e^{-f} = e^{-g_{0T}} \det_2(I_H + \nabla^2 \varphi) \exp \left( -\mathcal{L}_{\varphi} - \frac{1}{2} |\nabla \varphi|_H^2 \right)
\]
If we substitute this to (7), we get

\[ H(T\rho | \rho) = \int_W e^{-f}(-f - \log \Lambda \varphi + f \circ T) \, d\mu \]

Consider the map \( T_t = I_W + t\nabla \varphi \) for \( t \in [0,1) \). Here, \( T_t \) is strongly \((1-t)\)-monotone shift \([25, \text{Lem.6.2.1}]\). Moreover, for \( \xi \in \mathbb{D}_{2,1}(H) \) with \( \|\nabla \xi\|_2 \) is in the space of bounded random variable \( L^\infty(\rho) \) and sufficiently small \( \epsilon > 0 \), the shift \( T_{t,\epsilon} = I_W + t\nabla \varphi + \epsilon \xi \) is still strongly monotone. Therefore, \( \Lambda_t \nabla \varphi + \epsilon \xi > 0 \) holds a.s. \([25]\). Moreover, \( T_{t,\epsilon}\rho \) is absolutely continuous with respect to \( \rho \) \([15, \text{Thm.7.3}]\) and we have

\[ e^{-f} = e^{-g_{t,\epsilon} \circ T_{t,\epsilon} \Lambda_{T_{t,\epsilon}}} \]

where \( e^{g_{t,\epsilon}} = dT_{t,\epsilon}\rho / d\rho \) \([19]\). A similar calculation shows that

\[ H(T_{t,\epsilon}\rho | \rho) = \int_W e^{-f}(-f + f \circ T_{t,\epsilon} - \log \Lambda_{T_{t,\epsilon}}) \, d\mu \]  

(8)

and

\[ K_{g_{t-f}}(t\nabla \varphi + \epsilon \xi) = \int_W g_t \circ T_{t,\epsilon} - f - \log \Lambda_{T_{t,\epsilon}} \, d\rho, \]  

(9)

where \( e^{g_t} = dT_t\rho / d\rho \). Note that \( t\varphi \) is unique (up to a constant) Monge potential of MKP for \( \Sigma(\rho, T_t\rho) \) and it minimizes \( K_{f_{t-g}} \) among all absolutely continuous shifts. Therefore, we have

\[ \left. \frac{d}{d\epsilon} K_{g_{t-f}}(t\nabla \varphi + \epsilon \xi) \right|_{\epsilon=0} = 0. \]

Observe that

\begin{align*}
\left. \frac{d}{d\epsilon} \left( \int_W g \, d\rho \right) \right|_{\epsilon=0} &= 0, \\
\left. \frac{d}{d\epsilon} \left( \int_W g_t \circ T_{t,\epsilon} \, d\rho \right) \right|_{\epsilon=0} &= \int_W (\nabla g_t \circ T_{t,\epsilon} \xi) \, d\rho, \\
\left. \frac{d}{d\epsilon} \left( \int_W -\log \Lambda_{T_{t,\epsilon}} \, d\rho \right) \right|_{\epsilon=0} &= \int_W -\text{trace} \left( ((I_H + t\nabla^2 \varphi)^{-1} - I_H) \cdot \nabla \xi \right) + \delta \xi + (\nabla \varphi, \xi) \, d\rho, 
\end{align*}

for each \( \xi \in \mathbb{D}_{2,1}(H) \) with \( \|\xi\|_2 \in L^\infty(\rho) \). Here, the first derivative is 0 since it does not depend on \( \epsilon \), the second derivative result comes from definition and for the last derivative we have used \([25, \text{Thm A.2.2}]\). The set

\[ \Theta = \{ \xi \in \mathbb{D}_{2,1}(H) : \|\xi\|_2 \in L^\infty(\rho) \} \]
is dense in $L^p(\rho)$, we have

$$t\nabla \varphi + \nabla g_t \circ T_t - \nabla f = \delta_\rho \left[ (I_H + t\nabla^2 \varphi)^{-1} - I_H \right]$$

Since

$$e^{-f} = e^{-g_t \circ T_t} \Lambda_t \nabla \varphi$$

$g_t \circ T_t$ converges in probability to $\nabla g \circ T$ as $t \to 1$, and we get

$$\nabla \varphi + \nabla g \circ T - \nabla f = \delta_\rho \left[ (I_H + \nabla^2 \varphi)^{-1} - I_H \right].$$

□

We need the following technical lemmas in order to prove the regularity of backward Monge potential. We refer to [24] for their proofs.

**Lemma 3.1.** Suppose $W$ is finite dimensional and $\varphi$ is smooth. Let $K = (I_H + \nabla^2 \varphi)^{-1}$ and $h \in H$, then we have

$$\text{trace}(K\nabla^3 \varphi K \cdot K\nabla^3 \varphi K h) \geq 0 \quad \mu - a.s.$$  

**Lemma 3.2.** If $\xi : W \to H$ is smooth, then $\delta_\rho \xi = \delta \xi + \langle \nabla f, \xi \rangle_H$.

**Lemma 3.3.** If $\xi : W \to H$ is smooth, then

$$\mathbb{E}_\rho[(\delta_\rho \xi)^2] = \mathbb{E}_\rho[(I_H + \nabla^2 g, \xi \otimes \xi)_{H \otimes H} + \text{trace}(\nabla \xi \cdot \nabla \xi)].$$

**Proposition 3.2.** Suppose $W$ is finite dimensional, $\varphi$ is smooth, and the function $f$ is $(1 - c)$-convex for some $c \in [0, 1)$. Then, we have

$$c\mathbb{E}_\nu \left[ |\nabla^2 \psi|_H^2 \right] \leq 3 \left( \mathbb{E}_\rho \left[ |\nabla \varphi|_H^2 \right] + \mathbb{E}_\nu \left[ |\nabla g|_H^2 \right] + \mathbb{E}_\rho \left[ |\nabla f|_H^2 \right] \right)$$

**Proof.** Taking expectation of the second moment of the norm of the expression in Proposition 3.1 yields

$$\mathbb{E}_\rho \left[ |\delta_\rho \left( (I_H + \nabla^2 \varphi)^{-1} - I_H \right)|_H^2 \right] \leq 3 \left( \mathbb{E}_\rho \left[ |\nabla \varphi|_H^2 \right] + \mathbb{E}_\rho \left[ |\nabla g \circ T|_H^2 \right] + \mathbb{E}_\rho \left[ |\nabla f|_H^2 \right] \right)$$

$$= 3 \left( \mathbb{E}_\rho \left[ |\nabla \varphi|_H^2 \right] + \mathbb{E}_\nu \left[ |\nabla g|_H^2 \right] + \mathbb{E}_\rho \left[ |\nabla f|_H^2 \right] \right) \quad (10)$$
Denote \((I_H + \nabla^2 \varphi)^{-1}\) by \(M\). If we apply Lemma 3.3, then
\[
E_\rho [\delta_\rho (M - I_H)^2] = \sum_{k=1}^{\infty} E_\rho [(\delta_\rho (M - I_H)(e_k))^2] = \sum_{k=1}^{\infty} E_\rho \left[(I_H + \nabla^2 f, (M - I_H)e_k \otimes (M - I_H)e_k)_{H \otimes H}\right] + \sum_{k=1}^{\infty} E_\rho [\text{trace}(\nabla (M e_k) \cdot \nabla (M e_k))]
\]
By Lemma 3.1, the second term in the last equality is positive. So, we have
\[
E_\rho [\delta_\rho (M - I_H)^2] \geq \sum_{k=1}^{\infty} E_\rho \left[(I_H + \nabla^2 f, (M - I_H)e_k \otimes (M - I_H)e_k)_{H \otimes H}\right] \geq c \sum_{k=1}^{\infty} E_\rho \left[| (M - I_H) e_k |^2_H \right] = c E_\rho \left[| (M - I_H) e_k |^2_H \right] \geq \sum_{k=1}^{\infty} E_\rho \left[| (M - I_H) e_k |^2_H \right].
\]
(11)
Since \(T = I_W + \nabla \varphi\) and \(S = I_W + \nabla \psi\) are inverses of each other, we have
\[
(I_H + \nabla^2 \varphi)^{-1} = (I_H + \nabla^2 \psi) \circ T.
\]
(12)
Combining (10), (11) and (12), we get
\[
c E_\nu \left[| \nabla^2 \psi |^2_H \right] \leq 3 \left( E_\rho \left[| \nabla \varphi |^2_H \right] + E_\nu \left[| \nabla g |^2_H \right] + E_\rho \left[| \nabla f |^2_H \right] \right).
\]
\[\square\]

4 Regularity and Monge-Ampère Equation

Recall that \(f : W \to \mathbb{R}\) and \(g : W \to \mathbb{R}\) are measurable functions such that \(f, g \in D_{2,1}\) and
\[
\int_W |\nabla f|^2 e^{-f} \, d\mu < \infty \quad \text{and} \quad \int_W |\nabla g|^2 e^{-g} \, d\mu < \infty
\]
(13)
with \(e^{-f}\) satisfying Poincaré inequality (2), and the initial and target measures of Monge-Kantorovich problem are \(d\rho = e^{-f} d\mu\) and \(d\nu = e^{-g} d\mu\). In this section, we will show that backward Monge potential \(\psi\) solves Monge-Ampère equation, that is, we have
\[
e^{-g} = e^{-f \circ S} \det_2 \left( I_H + \nabla^2 \psi \right) \exp \left[-\mathcal{L} \psi - \frac{1}{2} |\nabla \psi|^2_H \right]
\]
where $S = T^{-1} = I + \nabla \psi$. However, we only know $\psi \in D_{2,1}(\nu)$ so far. Therefore, we first establish an upper bound for $D_{2,2}(\nu)$-norm of $\psi$, which implies that the second Sobolev derivative of $\psi$ is well-defined, in subsections 4.1 and 4.2. Monge-Ampère equation is considered in Subsection 4.3.

### 4.1 Approximation Lemmas

In order to show the regularity of backward Monge potential $\psi$ in the general setting of this section, we will approximate $f$ and $g$ of (13). This will be accomplished through several lemmas. In Lemma 4.1, we define a sequence of probability measures $\rho_n$ and $\nu_n$ which are absolutely continuous with respect to the standard Gaussian measure on $\mathbb{R}^n$, that is, in finite dimension.

**Lemma 4.1.** Let $f, g \in D_{2,1}$ satisfy (13) with $e^{-f}$ satisfying Poincaré inequality (2). Let $(\varphi, \psi)$ be the Monge potentials associated to the Monge-Kantorovitch problem $(\rho, \nu)$, where $d\rho = e^{-f} d\mu$ and $d\nu = e^{-g} d\mu$. Define $g_n$ and $f_n$ as

$$e^{-f_n} = E[e^{-f}|V_n] \quad \text{and} \quad e^{-g_n} = E[e^{-g}|V_n]$$

where $V_n$ is generated by $\{\delta e_1, \ldots, \delta e_n\}$ and $\{e_i, \ i \geq 1\}$ is an orthogonal basis of $H$. Let $(\varphi_n, \psi_n)$ be the Monge potentials associated with Monge-Kantorovitch problem $(\rho_n, \nu_n)$, where

$$d\rho_n = e^{-f_n} d\beta \quad \text{and} \quad d\nu_n = e^{-g_n} d\beta$$

and $\beta$ is the standard Gaussian measure on $\mathbb{R}^n$. Then, $(\varphi_n)$ converges to $\varphi$ in $D_{2,1}(\rho)$, $(\psi_n)$ converges to $\psi$ in $L^1(\nu)$ and $(\nabla \psi_n)$ converges to $\nabla \psi$ in $L^2(\nu)$.

**Proof.** This lemma is proved in [13, Thm.1.1]

As the next step, we will define smooth functions $f_m$ and $g_m$ in the following lemma using the Ornstein-Uhlenbeck semigroup to approximate functions $f$ and $g$ given in the finite dimension. The sequences $(f_m)$ and $(g_m)$ will be used later to approximate the sequences of Lemma 4.1, which are in finite dimension.
Lemma 4.2. Let $\beta$ be the standard Gaussian measure on $\mathbb{R}^n$, $f \in D_{2,1}(\beta)$ and $g \in D_{2,1}(\beta)$ such that
\[ \int_{\mathbb{R}^d} |\nabla f|^2 e^{-f} \, d\beta < \infty \quad \text{and} \quad \int_{\mathbb{R}^d} |\nabla g|^2 e^{-g} \, d\beta < \infty. \]

Let $(\varphi, \psi)$ be the Monge potentials associated with the Monge-Kantorovich problem $\Gamma(\rho, \nu)$, where
\[ d\rho = e^{-f} \, d\beta \quad \text{and} \quad d\nu = e^{-g} \, d\beta. \]

Define $f_m$ and $g_m$ as
\[ e^{-f_m} = Q_{\frac{1}{m}} e^{-f} \quad \text{and} \quad e^{-g_m} = Q_{\frac{1}{m}} e^{-g} \]
where $(Q_t, t \geq 0)$ is the Ornstein-Uhlenbeck semigroup on $\mathbb{R}^d$. Let $(\varphi_m, \psi_m)$ be the Monge potentials associated with Monge-Kantorovich problem $\Gamma(\rho_m, \nu_m)$, where
\[ d\rho_m = e^{-g_m} \, d\beta \quad \text{and} \quad d\nu_m = e^{-g_m} \, d\beta. \]

Then, $(\varphi_m)$ converges to $\varphi$ in $D_{2,1}(\rho_m)$, $(Q_{\frac{1}{m}} \psi_m)$ converges to $\psi$ in $L^1(\nu)$ and $(Q_{\frac{1}{m}} \nabla \psi_m)$ converges to $\nabla \psi$ in $L^2(\nu)$.

Proof. Let $\gamma_m$ and $\gamma$ be solutions of Monge-Kantorovich problems for $(\rho_m, \nu_m)$ and $(\rho, \nu)$, respectively. Then, we have
\[ \int_{\mathbb{R}^d} |\nabla \varphi_m|^2 e^{-f_m} \, d\beta = d_2^2(\rho_m, \nu_m) \leq 4 \left( H(\rho_m \mid \beta) + H(\nu_m \mid \beta) \right) \]
\[ \leq 4 \left( H(\rho \mid \beta) + H(\nu \mid \beta) \right) \]
\[ \leq 2 \left( \int_{\mathbb{R}^d} |\nabla f|^2 e^{-f} \, d\beta \right) \left( \int_{\mathbb{R}^d} |\nabla g|^2 e^{-g} \, d\beta \right) < \infty \]
where we have used Talagrand’s inequality in the first line, Jensen’s inequality in the second and Logarithmic Sobolev inequality in the third. Therefore, we have
\[ \sup_m \int_{\mathbb{R}^d} |\nabla \varphi_m|^2 e^{-f_m} \, d\beta < \infty \]
Using Poincaré inequality, we get
\[ \int_{\mathbb{R}^d} |\varphi_m - E_{\rho_m} [\varphi_m]|^2 e^{-f_m} \, d\beta \leq \int_{\mathbb{R}^d} |\nabla \varphi_m|^2 e^{-f_m} \, d\beta. \]
Then, we replace \( \varphi_m \) with \( \varphi_m - E_{\rho_m}[\varphi_m] \) and use the fact that \( \rho_m \) converges to \( \rho \) weakly to get

\[
\sup_m \|\varphi_m\|_{D_{2,1}(\rho)} < \infty
\]

which implies \( (\varphi_m) \) converges weakly in \( D_{2,1}(\rho) \). As in the proof of [15, Thm.4.1], we have

\[
\varphi_m(x) + \psi_m(y) + \frac{1}{2}|x - y|^2 \geq 0.
\]

Applying Ornstein–Uhlenbeck semigroup with respect to \( x \) and then \( y \), we get

\[
Q_{\frac{1}{m}} \varphi_m(x) + Q_{\frac{1}{m}} \psi_m(y) + \frac{1}{2}Q_{\frac{1}{m}} Q_{\frac{1}{m}}(|x - y|^2) \geq 0
\]

in view of the positivity improving property of Ornstein–Uhlenbeck semigroup. Observe that

\[
\lim_m \int \left[ Q_{\frac{1}{m}} \varphi_m(x) + Q_{\frac{1}{m}} \psi_m(y) + \frac{1}{2}Q_{\frac{1}{m}} Q_{\frac{1}{m}}(|x - y|^2) \right] \, d\gamma
\]

\[
= \lim_m \int Q_{\frac{1}{m}} \varphi_m(x) \, d\rho + \int Q_{\frac{1}{m}} \psi_m(y) \, d\nu + \int \frac{1}{2}Q_{\frac{1}{m}} Q_{\frac{1}{m}}(|x - y|^2) \, d\gamma
\]

We want to show that \( Q_{\frac{1}{m}} Q_{\frac{1}{m}}(|x - y|^2) \) converges to \( |x - y|^2 \) in \( L^1(\gamma) \). Observe that

\[
Q_{\frac{1}{m}} Q_{\frac{1}{m}}(|x - y|^2) = \int \int \left| e^{\frac{1}{m}} x + \sqrt{1 - e^{\frac{2}{m}} t} - e^{\frac{1}{m}} y - \sqrt{1 - e^{\frac{2}{m}} z} \right|^2 \, d\beta(z) \, d\beta(t).
\]

If we integrate with respect to \( \gamma \), we get

\[
\int Q_{\frac{1}{m}} Q_{\frac{1}{m}}(|x - y|^2) \, d\gamma = \int \int \int \left| e^{\frac{1}{m}} x - e^{\frac{1}{m}} y \right|^2 (1 - e^{-\frac{2}{m}})|t|^2 + (1 - e^{-\frac{2}{m}})|z|^2 \, d\beta(z) \, d\beta(t) \, d\gamma(x, y).
\]

It is easy to see that

\[
\lim_{m \to \infty} \int (1 - e^{-\frac{2}{m}})|z|^2 \, d\beta(z) = \lim_{m \to \infty} \int (1 - e^{-\frac{2}{m}})|t|^2 \, d\beta(t) = 0
\]

and

\[
\left| e^{\frac{1}{m}} x + \sqrt{1 - e^{-\frac{2}{m}} t} - e^{\frac{1}{m}} y - \sqrt{1 - e^{-\frac{2}{m}} z} \right|^2 \leq C \left( |x|^2 + |y|^2 + |z|^2 + |t|^2 \right).
\]

Moreover, from a version of Young’s inequality given in [3, pg. 15], we get

\[
\int |y|^2 \, d\gamma(x, y) \leq \int e^{\alpha |y|^2} \, d\beta(y) + \frac{1}{\alpha} H(\nu|\beta)
\]

\[
\int |x|^2 \, d\gamma(x, y) \leq \int e^{\alpha |x|^2} \, d\beta(x) + \frac{1}{\alpha} H(\rho|\beta).
\]
From dominated convergence theorem, \( Q_m Q_m (|x - y|^2) \) converge to \(|x - y|^2\) in \( L^1(\gamma) \). Therefore, Equation (15) is equal to

\[
\lim_m \int Q_m \varphi_m(x) \, d\rho + \int Q_m \psi_m(y) \, d\nu + \int \frac{1}{2} |x - y|^2 \, d\gamma
= \lim_m \int \varphi_m(x) \, d\rho_m + \int \psi_m(y) \, d\nu_m + \frac{1}{2} d^2(\rho, \nu)
\]

\[
= \lim_m -\frac{1}{2} d^2(\rho_m, \nu_m) + \frac{1}{2} d^2(\rho, \nu)
\]

\[
\leq \lim_m -\frac{1}{2} d^2(\rho_m, \rho) + \frac{1}{2} d^2(\nu_m, \nu).
\]

(16)

Observe that

\[
\lim_m \int |x|^2 \, d\rho_m = \lim_m \int \int |x|^2 Q_m e^{-f} \, d\beta
= \lim_m \int (Q_m \lambda |x|^2) \frac{e^{-f}}{\lambda} \, d\beta = \int |x|^2 \, d\rho
\]

where we have used weak convergence of \( \rho_m \) to \( \rho \) in the first line and dominated convergence theorem in second. Indeed, Young’s inequality implies \( Q_m (\lambda |x|^2) \frac{e^{-f}}{\lambda} \leq \exp(Q_m \lambda |x|^2) + H(\rho|\beta) \) and for \( \lambda < \frac{1}{2} \) Jensen’s inequality yields

\[
\int \exp(Q_m \lambda |x|^2) d\beta \leq \int Q_m \exp(\lambda |x|^2) = \int \exp(\lambda |x|^2) d\beta < \infty.
\]

Similarly, \( \lim \int |y|^2 \, d\nu_m = \int |y|^2 \, d\nu \). So \( d_2(\rho_m, \rho) \to 0 \) and \( d_2(\nu_m, \nu) \to 0 \) by using [4, Lem.8.3]. Combining this result with Equation (16), we get \( (Q_m \varphi_m(x) + Q_m \psi_m(y) + \frac{1}{2} Q_m Q_m (|x - y|^2)) \) converges to 0 in \( L^1(\gamma) \) and it is uniformly integrable with respect to \( \gamma \). Moreover, \( (Q_m \varphi_m) \) is uniformly integrable, so \( (Q_m \psi_m) \) is uniformly integrable. Let \( a' \) and \( b' \) be weak limit points of \( (Q_m \varphi_m) \) and \( (Q_m \psi_m) \). The Cesaro means

\[
Q_m \varphi'_m = \frac{1}{n} \sum_{i=1}^{n} Q_m \varphi_m \quad \text{and} \quad Q_m \psi'_m = \frac{1}{n} \sum_{i=1}^{n} Q_m \psi_m.
\]

converge to \( a' \), up to a subsequence, and \( b' \) in \( L^1(\gamma) \), respectively. A as result, we have

\[
a' + b' + \frac{1}{2} |x - y|^2 = 0 \quad \gamma\text{-a.s.}
\]

Let \( a(x) = \lim \sup \varphi'_m(x) \) and \( b(y) = \lim \sup \psi'_m(y) \). Then \( a' = a \) and \( b' = b \) \( \gamma\)-a.s. and

\[
a(x) + b(y) + \frac{1}{2} |x - y|^2 \geq 0, \quad \text{for any } (x, y) \in \mathbb{R}^d \times \mathbb{R}^d
\]

\[
a(x) + b(y) + \frac{1}{2} |x - y|^2 = 0, \quad \gamma\text{-a.s.}
\]
By uniqueness of solutions, we get \( a = \varphi \) and \( b = \psi \). Therefore, we deduce that \( (Q_m \varphi_m) \) converges weakly to \( \varphi \) in \( L^1(\gamma) \).

On the other hand, from (14), there exists \( \varphi' \in L^2(\beta) \) such that \( (\varphi_m) \) converges weakly to \( \varphi' \) in \( L^2(\rho) \) and in \( L^2(\gamma) \). For \( h \in L^2(\gamma) \), we have

\[
\int (\varphi_m - \varphi) h \, d\gamma = \int (\varphi_m - Q_m \varphi_m) h \, d\gamma + \int (Q_m \varphi_m - \varphi) h \, d\gamma.
\]

Since both integrals on the right hand side converge to zero as \( m \to \infty \), \( (\varphi_m) \) converges weakly to \( \varphi \) in \( L^2(\gamma) \) and in \( L^2(\rho) \). Moreover, \( \lim \mathbb{E}_\rho[[\nabla \varphi_m]^2] = \mathbb{E}_\rho[[\nabla \varphi]^2] \), which implies \( (\varphi_m) \) converges to \( \varphi \) in \( \mathbb{D}_{2,1}(\rho) \). Similarly, \( (Q_m \psi_m) \) converges to \( \psi \) in \( L^1(\nu) \), and since \( \nabla \) is closable in \( L^p(\nu) \) for \( p \geq 1 \), \( (\nabla Q_m \psi_m) \) converges weakly to \( \nabla \psi \) in \( L^2(\nu) \). In addition, we have

\[
\lim_m \mathbb{E}_{\nu_m} [||\nabla \psi_m||^2] = \lim_m d^2(\rho_m, \nu_m) = d^2(\rho, \nu) = \mathbb{E}_\nu [||\nabla \psi||^2]
\]

which implies that \( Q_m \nabla \psi_m \) converges to \( \nabla \psi \) in \( L^2(\nu) \).

Now, let \( \rho \) and \( \nu \) be probability measures on \( \mathbb{R}^n \) defined by

\[
d\rho = F \, d\beta, \quad d\nu = G \, d\beta.
\]

where \( \beta \) is a Gaussian measure on \( \mathbb{R}^n \) and \( F, G \in L^1(\beta) \). Suppose that \( \nu \) is also absolutely continuous with respect to \( \rho \) with \( \frac{d\nu}{d\rho} = L \). Define

\[
F_k = \frac{\theta_k F}{\mathbb{E}[\theta_k F]}, \quad G_k = \frac{\theta_k G}{\mathbb{E}[\theta_k G]}
\]

where \( \theta_k \in C_c^\infty(\mathbb{R}^n) \) is a smooth function with compact support satisfying \( \theta_k(x) = 0 \) if \( |x| \geq k \), \( \theta_k(x) = 1 \) if \( |x| \leq k - 1 \) with \( 0 \leq \theta_k \leq 1 \) for each \( k \in \mathbb{N} \), and \( \sup_k ||(\nabla \theta_k)^2|| \leq 1 \). Consider probability measures \( \rho_k \) and \( \nu_k \) given by

\[
d\rho_k = F_k \, d\beta, \quad d\nu_k = G_k \, d\beta.
\]

It is easy to see that \( \nu_k \) is absolutely continuous with respect to \( \rho_k \) with Radon-Nikodym derivative \( d\nu_k/d\rho_k = L_k \), where

\[
L_k = \begin{cases} 
\frac{\theta_k}{\theta_k} & \text{on } \{ \theta_k \neq 0 \} \\
0 & \text{on } \{ \theta_k = 0 \}
\end{cases}
\]
Observe that \( L_k = b_k/a_k L \) on the set \( \{ \theta_k \neq 0 \} \), where \( a_k = 1/E[\theta_k F] \) and \( b_k = 1/E[\theta_k G] \). Therefore, the relative entropy of \( \nu_k \) with respect to \( \rho_k \), \( H(\nu_k|\rho_k) \), is finite if and only if \( H(\nu|\rho) \) is finite. Note that, \( F_k \) and \( G_k \) are bounded, being continuous functions with compact support. Using the following lemma, we will be able to approximate the smooth density functions of Lemma 4.2 by smooth and bounded sequences.

**Lemma 4.3.** Let \( (\rho, \nu) \) and \( (\rho_k, \nu_k) \) be probability measures on \( \mathbb{R}^n \) defined as in (17) and (18), respectively, with \( H(\nu|\rho) < \infty \). Let \( (\varphi, \psi) \) and \( (\varphi_k, \psi_k) \) be the Monge potentials associated with Monge-Kantorovich problems for \( (\rho, \nu) \) and \( (\rho_k, \nu_k) \), respectively, with quadratic cost. Then \( (\varphi_k) \) converges to \( \varphi \) in \( D_{2,1}(\rho) \), \( (\theta_k \psi_k) \) converges to \( \psi \) in \( L^1(\nu) \) and \( (\sqrt{\theta_k} \nabla \psi_k) \) converges to \( \nabla \psi \) in \( L^2(\nu) \).

**Proof.** Let \( \gamma \) and \( \gamma_k \) be the optimal transport plans to the Monge-Kantorovich problems for \( (\rho, \nu) \) and \( (\rho_k, \nu_k) \). From [8, Prop.3.1], replacing \( \varphi_k \) with \( \varphi_k - E[\varphi_k] \) and \( \psi_k \) with \( \psi_k + E[\varphi_k] \), we have

\[
F_k(x, y) = \varphi_n(x) + \psi_n(y) + \frac{1}{2} |x - y|^2 \geq 0
\]

for all \( x, y \in \mathbb{R}^n \). Since \( 0 \leq \theta_k \), we have

\[
0 \leq \int_{\mathbb{R}^n \times \mathbb{R}^n} \theta_k(y) F_k(x, y) \, d\gamma(x, y)
= \int \theta_k(y) \varphi_k(x) \, d\gamma + \int \theta_k(y) \psi_k(y) \, d\gamma + \frac{1}{2} \int \theta_k(y) |x - y|^2 \, d\gamma.
\]

In the second integral above, we have

\[
\int \theta_k(y) \psi_k(y) \, d\gamma = \int \theta_k(y) \psi_k(y) \, d\nu
= b_k^{-1} \int \psi_k(y) \, d\nu_k(y)
= b_k^{-1} \int \psi_k(y) \, d\gamma_k(y)
= b_k^{-1} \int -\varphi_k(x) - \frac{1}{2} |x - y|^2 \, d\gamma_k(x, y)
= b_k^{-1} \left[ \int -\varphi_k(x) \rho_k(x) - \frac{1}{2} \int |x - y|^2 \, d\gamma_k(x, y) \right]
= \frac{a_k}{b_k} \int -\theta_k(x) \varphi(x) \, d\rho - \frac{1}{2b_k} \int |x - y|^2 \, d\gamma_k(x, y)
\]
If we substitute (21) into (20), we get
\[
\int \theta_k(y)F_k(x, y) \, d\gamma = \int \left( \theta_k(y) - \frac{a_k}{b_k} \theta_k(x) \right) \varphi_k(x) \, d\gamma \\
+ \frac{1}{2} \int \theta_k(y)|x - y|^2 \, d\gamma(x, y) - \frac{1}{2b_k} \int |x - y|^2 \, d\gamma_k(x, y)
\]
=: \(I_k + II_k - III_k\)

Sequences \((a_k)\) and \((b_k)\) given by \(a_k = \mathbb{E}[\theta_k F]^{-1}\) and \(b_k = \mathbb{E}[\theta_k G]^{-1}\) converge to 1 decreasingly. Therefore, for given \(\epsilon > 0\) there exists \(0 < k_\epsilon\) such that \(0 < \frac{b_k}{a_k} \leq \frac{1}{1-\epsilon}\) for each \(k \geq k_\epsilon\). Hence, we have
\[
\lim_k \int |\varphi_k|^2 \, d\gamma = \lim_k \int |\varphi_k|^2 \, d\beta \leq \lim_k \int |\nabla \varphi_k|^2 \, d\rho \\
\leq \lim_k 2H(\nu_k|\rho_k) \leq 2 \int \left( \frac{b_k}{a_k} L \log L + \frac{b_k}{a_k} L \log \frac{b_k}{a_k} \right) \, d\rho \\
\leq \frac{2}{1 - \epsilon} \log \frac{2}{1 - \epsilon} < \infty
\]
for sufficiently small \(\epsilon\). We have used Poincaré inequality in the first line and Talagrand’s inequality in the second. Therefore, from Cauchy–Schwarz inequality and dominated convergence theorem, \(I_k \to 0\) as \(k \to \infty\). From monotone convergence theorem, \(II_k\) converges to \(\frac{1}{2} \int |x - y|^2 \, d\gamma\). Moreover, \(III_k\) also converges to \(\frac{1}{2} \int |x - y|^2 \, d\gamma\). Indeed, Young’s inequality implies that
\[
|x|^2 \theta_k(x)F \leq \frac{1}{1 - \epsilon} \left( e^{\epsilon|x|^2} + \frac{1}{\epsilon} F \log F \right) \\
|y|^2 \theta_k(y)G \leq \frac{1}{1 - \epsilon} \left( e^{\epsilon|y|^2} + \frac{1}{\epsilon} G \log G \right) .
\]
In other words, the sequences \(|x|^2 \theta_k(x)F, k \geq k_\epsilon\) \(|y|^2 \theta_k(y)G, k \geq k_\epsilon\) are bounded by a \(\beta\)-integrable function. Using dominated convergence theorem, we get
\[
\lim_{k \to \infty} \int |x|^2 F_k \, d\beta = \lim_{k \to \infty} \int |x|^2 \, d\rho_k = \int |x|^2 \, d\rho \\
\lim_{k \to \infty} \int |y|^2 G_k \, d\beta = \lim_{k \to \infty} \int |y|^2 \, d\nu_k = \int |y|^2 \, d\nu .
\]
Combining the above with \([4, \text{Lem.8.3}]\) yields
\[
\lim_{k \to \infty} \frac{1}{2b_k} \int |x - y|^2 \, d\gamma_k(x, y) = \lim_{k \to \infty} \frac{1}{2b_k} d^2_2(\rho_k, \nu_k) \\
= \frac{1}{2} d^2_2(\rho, \nu) = \frac{1}{2} \int |x - y|^2 \, d\gamma(x, y).
\]
Therefore, we get
\[
\lim_{k \to \infty} \int \theta_k(y) F_k(x, y) \, d\gamma = 0. \tag{23}
\]
The sequence \((\varphi_k)\) is bounded in \(\mathbb{D}_{2,1}(\rho)\). Therefore, it converges weakly to some \(\varphi' \in \mathbb{D}_{2,1}(\rho)\) up to a subsequence. Moreover, the sequence \((\theta_k(y)\varphi_k(x))\) is uniformly integrable and for any \(h \in L^\infty(\gamma)\)
\[
\int (\theta_k(y)\varphi_k(x) - \varphi'(x)) \, h \, d\gamma = \int (\theta_k(y) - 1) \varphi_k(x) \, h \, d\gamma + \int (\varphi_k(x) - \varphi'(y)) \, h \, d\gamma.
\]
The second integral on the right hand side converges to zero since \((\varphi_k)\) converges to \(\varphi'\) weakly and the first integral converges to zero since \((\theta_k - 1)\) converges to zero in \(L^2(\gamma)\). Therefore, \((\theta_k \varphi_k)\) converges weakly to \(\varphi'\) in \(L^2(\gamma)\). On the other hand, \((\theta_k F_k)\) converges to zero in \(L^1(\gamma)\) as given in (23), so it is uniformly integrable. Since \((\theta_k \varphi_k)\) is uniformly integrable, \((\theta_k(y)\psi_k(x))\) is also uniformly integrable and converge weakly to some \(\psi'\) in \(L^1(\gamma)\) up to a subsequence. There exists a further subsequence such that the Cesaro means
\[
\varphi'_k = \frac{1}{k} \sum_{i=1}^{k} \theta_i(y) \varphi_i(x) \quad \text{and} \quad \psi'_n = \frac{1}{k} \sum_{i=1}^{k} \theta_i \psi_i.
\]
converge to \(\varphi'\) in \(L^2(\gamma)\) and \(\psi'\) in \(L^1(\gamma)\), respectively. Let \(a(x) = \lim \sup \varphi'_k(x)\) and \(b(y) = \lim \sup \psi'_k(y)\). Then, \(\varphi' = a\) and \(\psi' = b\ \gamma\text{-a.s. Therefore,} \varphi' = a\) and \(\psi' = b\ \gamma\text{-a.s. and}
\[
a(x) + b(y) + \frac{1}{2}|x - y|^2 \geq 0, \quad \text{for any } (x, y) \in \mathbb{R}^d \times \mathbb{R}^d
\]
\[
a(x) + b(y) + \frac{1}{2}|x - y|^2 = 0, \quad \gamma\text{-a.s.}
\]
By uniqueness of solutions, we have \(\varphi' = \varphi\) and \(\psi' = \psi\). Combining this result with \(\lim \mathbb{E}_\rho[|\nabla \varphi_k|^2] = \mathbb{E}_\rho[|\nabla \varphi|^2]\), we see that \((\varphi_k)\) converges strongly to \(\varphi\) in \(\mathbb{D}_{2,1}(\rho)\). Finally, since \((\theta_k F_k)\) converges to zero and \((\frac{1}{2}\theta_k|x - y|^2)\) converges to \(\frac{1}{2}|x - y|^2\) in \(L^1(\gamma)\), \(\theta_k \psi_k\) converges to \(\psi\) in \(L^1(\gamma)\). The only remaining part is to show the convergence of the sequence \((\sqrt{\theta_k} \nabla \psi_k)\). Observe that
\[
\int \theta_k |\nabla \psi_k|^2 \, d\gamma = \int \theta_k |\nabla \psi_k|^2 \, d\nu = \int b_k^{-1} |\nabla \psi_k|^2 \, d\nu(y)
\]
\[
= \int b_k^{-1} |\nabla \varphi_k|^2 \, d\rho_k(x) = \int \frac{a_k}{b_k} \theta_k |\nabla \varphi_k|^2 \, d\rho(x)
\]
If we take the limit of both sides, we get
\[
\lim_{k \to \infty} \int \theta_k |\nabla \psi_k|^2 \, d\gamma = \int |\nabla \varphi|^2 \, d\rho(x) = \int |\nabla \psi|^2 \, d\nu(y). \tag{24}
\]

Next, we will show that the sequence \((\sqrt{\theta_k} \nabla \psi_k)\) converges weakly to \(\nabla \psi\). Let \(\xi\) be a bounded smooth vector field, then
\[
\int \langle \sqrt{\theta_k} \nabla \psi_k, \xi \rangle \, d\gamma = \int_{\{\theta_k < c\}} \langle \sqrt{\theta_k} \nabla \psi_k, \xi \rangle \, d\gamma + \int_{\{\theta_k \geq c\}} \langle \sqrt{\theta_k} \nabla \psi_k, \xi \rangle \, d\gamma. \tag{25}
\]

Equation (24) implies that the sequence \((\sqrt{\theta_k} \nabla \psi_k)\) is uniformly integrable with respect to \(\gamma\). Combining this with the boundedness of \(\xi\) and the fact that \(\theta_k\) converges to 1 imply that the first integral converges to 0 for \(c < 1\). On the other hand, the second integral in (25) can be written as
\[
\int_{\{\theta_k \geq c\}} \langle \sqrt{\theta_k} \nabla \psi_k, \xi \rangle \, d\nu_k = b_k^{-1} \int_{\{\theta_k \geq c\}} \frac{\langle \nabla \psi_k, \xi \rangle}{\sqrt{\theta_k}} \, d\nu_k
\]
\[
= b_k^{-1} \int_{\{\theta_k \circ T_k \geq c\}} -\frac{\langle \nabla \varphi_k, \xi \circ T_k \rangle}{\sqrt{\theta_k \circ T_k}} \, d\rho_k
\]
\[
= \frac{a_k}{b_k} \int_{\{\theta_k \circ T_k \geq c\}} -\frac{\langle \nabla \varphi_k, \xi \circ T_k \rangle}{\sqrt{\theta_k \circ T_k}} \, d\rho_k
\]

where \(T_k = I_{\mathbb{R}^n} + \nabla \varphi_k\) is the optimal transport map of the Monge problem. Since the sequence \((L_k)\) is uniformly integrable, the sequence \((T_k)\) is equi-concentrated on a compact set. Hence, \(\lim_{k \to \infty} \theta_k \circ T_k = 1\) and \(\lim_{k \to \infty} \xi \circ T_k = \xi \circ T\) in \(\rho\)-probability. By using dominated convergence theorem and \(L^2(\rho)\)-boundedness of the sequence of \((\nabla \varphi_k)\), we get
\[
\lim_{k \to \infty} \frac{a_k}{b_k} \int_{\{\theta_k \circ T \geq c\}} -\frac{\langle \nabla \varphi_k, \xi \circ T_k \rangle}{\sqrt{\theta_k \circ T}} \, d\rho_k = \int -\langle \nabla \varphi, \xi \circ T \rangle \, d\rho
\]
\[
= \int (\nabla \psi_k \circ T, \xi \circ T) \, d\rho
\]
\[
= \int (\nabla \psi, \xi) \, d\gamma.
\]

As a result, the limit of (25) is
\[
\lim_{k \to \infty} \int \langle \sqrt{\theta_k} \nabla \psi_k, \xi \rangle \, d\gamma = \int \langle \nabla \psi_k, \xi \rangle \, d\gamma
\]

which implies that the sequence \(\sqrt{\theta_k} \nabla \psi_k\) converges weakly to \(\nabla \psi\) with respect to \(\gamma\). In view of (24), \((\sqrt{\theta_k} \nabla \psi_k)\) converges to \(\nabla \psi\) in \(L^2(\nu)\). ☐
In the next lemma, we will use strictly positive sequences to approximate density functions such as those used in Lemma 4.3.

**Lemma 4.4.** Let \((\rho, \nu)\) be probability measures on \(\mathbb{R}^n\) defined as in (17) with \(H(\rho \mid \beta) < \infty\). Define for each \(\epsilon > 0\)

\[
d\rho_\epsilon = \frac{F + \epsilon}{1 + \epsilon} \, d\beta \quad \text{and} \quad d\nu_\epsilon = \frac{G + \epsilon}{1 + \epsilon} \, d\beta.
\]

Let \((\varphi, \psi)\) and \((\varphi_\epsilon, \psi_\epsilon)\) be the Monge potentials associated with Monge-Kantorovitch problems \((\rho, \nu)\) and \((\rho_\epsilon, \nu_\epsilon)\), respectively with quadratic cost. Then, as \(\epsilon\) goes to zero \((\varphi_\epsilon, \epsilon > 0)\) converges to \(\varphi\) in \(D_{2,1}(\rho)\), \((\psi_\epsilon, \epsilon > 0)\) converges to \(\psi\) in \(L^1(\nu)\) and \((\nabla \psi_\epsilon, \epsilon > 0)\) converges to \(\nabla \psi\) in \(L^2(\nu)\).

**Proof.** Let \(\gamma\) and \(\gamma_\epsilon\) be the optimal transport plans of Monge-Kantorovitch problems for \((\rho, \nu)\) and \((\rho_\epsilon, \nu_\epsilon)\), respectively. If we replace \(\varphi_\epsilon\) with \(\varphi_\epsilon - \mathbb{E}_{\rho_\epsilon}[\varphi_\epsilon]\) and \(\psi_\epsilon\) with \(\psi_\epsilon + \mathbb{E}_{\rho_\epsilon}[\varphi_\epsilon]\), we have

\[
F_\epsilon(x, y) = \varphi_\epsilon(x) + \psi_\epsilon(y) + \frac{1}{2}|x - y|^2 \geq 0 \quad \text{for any} \ x, y \in \mathbb{R}^n,
\]

\[
F_\epsilon(x, y) = \varphi_\epsilon(x) + \psi_\epsilon(y) + \frac{1}{2}|x - y|^2 = 0 \quad \text{}\gamma_\epsilon\text{-a.s.}
\] (26)

for all \(x, y \in \mathbb{R}^n\). Observe that

\[
0 \leq \int F_\epsilon(x, y) \, d\gamma + \epsilon \int \varphi_\epsilon(x) \, d\beta(x) + \epsilon \int \psi_\epsilon(y) \, d\beta(y)
\]

\[
= (1 + \epsilon) \int \varphi_\epsilon(x) \, d\rho_\epsilon(x) + (1 + \epsilon) \int \psi_\epsilon(y) \, d\nu_\epsilon(y) + \frac{1}{2} \int |x - y|^2 \, d\gamma
\]

\[
= (1 + \epsilon) \int \psi_\epsilon(y) \, \nu_\epsilon(y) + \frac{1}{2} \int |x - y|^2 \, d\gamma. \quad (27)
\]

Note that last inequality follows from the fact that \(\mathbb{E}_{\rho_\epsilon}[\varphi_\epsilon] = 0\). Using Young’s inequality and [4, Lem.8.3], we get

\[
\lim_{\epsilon \to 0}(1 + \epsilon) \int \psi_\epsilon \, d\nu_\epsilon = \lim_{\epsilon \to 0} \frac{1 + \epsilon}{2} \int |x - y|^2 \, d\gamma_\epsilon
\]

\[
= \lim_{\epsilon \to 0} -d_2^2(\rho_\epsilon, \nu_\epsilon) = -d_2^2(\rho, \nu) = -\frac{1}{2} \int |x - y|^2 \, d\gamma
\]

Therefore, we have

\[
\lim_{\epsilon \to 0} \int F_\epsilon(x, y) \, d\gamma + \epsilon \int \varphi_\epsilon(x) \, d\beta(x) + \epsilon \int \psi_\epsilon(y) \, d\beta(y) = 0.
\]
On the other hand, \( F_\epsilon(x, x) = \varphi_\epsilon(x) + \psi_\epsilon(x) \geq 0 \). Hence, \( \epsilon \int \varphi_\epsilon \beta + \epsilon \int \psi_\epsilon \beta \geq 0 \), which implies
\[
\lim_{\epsilon \to 0} \int F_\epsilon(x, y) \, d\gamma = 0
\]
and we conclude that \((F_\epsilon, \epsilon > 0)\) is uniformly integrable with respect to \( \gamma \). Observe that
\[
\int |\nabla \varphi_\epsilon|^2 \, d\rho_\epsilon = d_2^2(\rho_\epsilon, \nu_\epsilon)
\leq 4 \left( H(\rho_\epsilon | \beta) + H(\nu_\epsilon | \beta) \right).
\]
Moreover, \( V_\epsilon = \frac{V}{1+\epsilon} + \frac{\epsilon}{1+\epsilon} \) and the convexity of the function \( x \to x \log x \) implies
\[
H(\rho_\epsilon | \beta) = \int F_\epsilon \log F_\epsilon \, d\beta \leq \frac{1}{1+\epsilon} H(\rho | \beta)
\]
Similarly, \( H(\nu_\epsilon | \beta) \leq \frac{1}{1+\epsilon} H(\nu | \beta) \) and
\[
\sup_{\epsilon > 0} \int |\nabla \varphi_\epsilon|^2 \, d\rho_\epsilon < \infty.
\]
Combining the above result with \( \int |\nabla \varphi_\epsilon|^2 \, d\rho_\epsilon \geq \frac{1}{1+\epsilon} \int |\nabla \varphi_\epsilon|^2 \, d\rho, \) we get \( \sup_{\epsilon > 0} \int |\nabla \varphi_\epsilon|^2 \, d\rho < \infty \). Applying Poincaré inequality yields \( \sup_{\epsilon > 0} \| \varphi_\epsilon \|_{D_{2,1}(\rho)} < \infty \). Since the sequence \((\varphi_\epsilon, \epsilon > 0)\) is bounded in \( D_{2,1}(\rho) \), it is also uniformly integrable with respect to \( \rho \) and \( \gamma \). We also know that the sequence \((F_\epsilon, \epsilon > 0)\) is uniformly integrable. Hence, \((\psi_\epsilon, \epsilon > 0)\) is uniformly integrable with respect to \( \gamma \). The rest of the proof goes along the proof of the previous lemmas.

4.2 Regularity of Backward Monge Potential

Our approach for proving regularity of backward Monge potential in a more general setting will be to approximate the functions \( f \) and \( g \) of (13) by appropriate sequences that will enable the use of Proposition 3.2. Explicitly, we first consider \( f_n \) on \( \mathbb{R}^n \) by \( e^{-f_n} = \mathbb{E}[e^{-f}\mid V_n] \), where \( V_n \) is generated by \( \{\delta e_1, \ldots, \delta e_n\} \) for an orthogonal basis \( \{e_i, i \geq 1\} \) of \( H \). In the second step, \( f_{nm} \) are chosen as smooth functions using the Ornstein-Uhlenbeck semigroup \((P_{t/m})\) by \( e^{-f_{nm}} = P_{t/m}(e^{-f_n}) \), for each \( n \). Then, \( F_{nmk} \) are continuous and compact supported functions given by \( F_{nmk} = \frac{\theta_{k} e^{-f_{nm}}}{\mathbb{E}[\theta_{k} e^{-f_{nm}}]} \) for fixed \( m \) and \( k \). Finally, a strictly positive sequence given by
\[ e^{-f_{nmkl}} = \frac{F_{nmkl} + \frac{1}{1 + \frac{1}{T}}}{1 + \frac{1}{T}} \] is formed to be compatible with the density \( e^{-f} > 0 \). Similarly, a sequence \((g_{nmkl})\) is also defined. As a result, forward Monge potential \( \varphi_{nmkl} \) of the Monge-Kantorovich problem with respect to \((\rho_{nmkl}, \nu_{nmkl})\) is smooth and Proposition (3.2) is applicable. We will work with a subsequence of \((\varphi_{nmkl})\), which has the form \((\varphi_n, l, m, k_n, m_k)\) and can be extracted by applying the diagonal method three times. We will denote the corresponding sequences with \((\varphi_n), (f_n)\) and \((\rho_n)\). Similarly, \((\psi_n)\) will be formed by the diagonal method and its indices can be matched with those of \((\varphi_n)\). We will work with this subsequence by relabeling as \((\varphi_n, \psi_n)\). After this simplification, \((\varphi_n)\) converges to \(\varphi\) in \(D_2, 1(\rho)\), \((\theta_n P_{1/n} \psi_n)\) converges to \(\psi\) in \(L_1(\nu)\) and \((\sqrt{\theta_n P_{1/n}} \nabla \psi_n)\) converges to \(\nabla \psi\) in \(L_2(\nu)\).

Lemma 4.5. Let \(f, g \in D_2, 1\) satisfy (13) with \(e^{-f}\) satisfying Poincaré inequality (2). Then, we have

\[
\lim_n \nabla f_n = \nabla f, \quad \lim_n \nabla g_n \circ T_n = \nabla g \circ T
\]

in \(L^2(\rho, H)\).

Proof. It will be convenient to use the multi-index sequence defined above as

\[
e^{-f_{nmkl}} = \frac{\theta_k e^{-f_{nm}} + \frac{1}{1 + \frac{1}{T}}}{1 + \frac{1}{T}}.
\]

We have

\[
\mathbb{E} \left[|\nabla f_{nmkl}|^2 e^{-f_{nmkl}}\right] = 4 \mathbb{E} \left[|\nabla e^{-f_{nmkl}/2}|^2\right] = \mathbb{E} \left[\frac{\nabla e^{-f_{nmkl}}}{e^{-f_{nmkl}/2}}|^2\right] \\
= \mathbb{E} \left[1_{\{\theta_k=0\}} \frac{\nabla e^{-f_{nmkl}}}{e^{-f_{nmkl}/2}}|^2\right] + \mathbb{E} \left[1_{\{0<\theta_k<1\}} \frac{\nabla e^{-f_{nmkl}}}{e^{-f_{nmkl}/2}}|^2\right] \\
+ \mathbb{E} \left[1_{\{\theta_k=1\}} \frac{\nabla e^{-f_{nmkl}}}{e^{-f_{nmkl}/2}}|^2\right] \\
= I_{nmkl} + II_{nmkl} + III_{nmkl}
\]

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On the set \( \{ \theta_k = 1 \} \), we have

\[
III_{nmkl} = \frac{l}{l+1} \mathbb{E} \left[ 1_{\{ \theta_k = 1 \}} \left| \nabla e^{-f_{nm}} \right|^2 \left( e^{-f_{nm}} + \frac{1}{m} \right)^2 \right]
\]

\[
\leq \mathbb{E} \left[ 1_{\{ \theta_k = 1 \}} \left| \nabla P_{\pm} (e^{-f_n}) \right|^2 \left( P_{\pm} (e^{-f_n}) \right)^{-1} \right]
\]

\[
\leq e^{-2m} \mathbb{E} \left[ 1_{\{ \theta_k = 1 \}} \left| \nabla f_n e^{-f_n} \right|^2 \left( P_{\pm} (e^{-f_n}) \right)^{-1} \right]
\]

\[
= \mathbb{E} \left[ 1_{\{ \theta_k = 1 \}} P_{\pm} \left( \left| \nabla f_n e^{-f_n} \right|^2 \right) \left( P_{\pm} (e^{-f_n}) \right)^{-1} \right]
\]

\[
= \mathbb{E} \left[ 1_{\{ \theta_k = 1 \}} P_{\pm} \left( \left| \nabla f_n \right|^2 e^{-f_n} \right) \frac{P_{\pm} (e^{-f_n})}{P_{\pm} (e^{-f_n})} \right]
\]

\[
\leq \mathbb{E} \left[ 1_{\{ \theta_k = 1 \}} \left| \nabla e^{-f_n} \right|^2 \right]
\]

\[
\leq \mathbb{E} \left[ 1_{\{ \theta_k = 1 \}} \frac{\nabla e^{-f | V_n} \left| | V_n \right|}{\mathbb{E} e^{-f | V_n}} \right] \leq \mathbb{E} \left[ 1_{\{ \theta_k = 1 \}} \frac{\nabla e^{-f} | V_n} \mathbb{E} e^{-f | V_n} \right] \leq \mathbb{E} \left[ \frac{\nabla e^{-f}}{\mathbb{E} e^{-f}} \right]
\]

On the set \( \{ 0 < \theta_k < 1 \} \), we get

\[
II_{nmkl} = \frac{l}{1+l} \mathbb{E} \left[ 1_{\{ 0 < \theta_k < 1 \}} \left( \nabla \theta_k e^{-f_{nm}} + \theta_k \nabla e^{-f_{nm}} \right)^2 \right]
\]

\[
\leq 2 \mathbb{E} \left[ 1_{\{ 0 < \theta_k < 1 \}} \left| \nabla \theta_k e^{-f_{nm}} \right|^2 \right] + 2 \mathbb{E} \left[ \left| \nabla e^{-f_{nm}} \right|^2 \right]
\]

\[
\leq 2 \mathbb{E} \left[ 1_{\{ 0 < \theta_k < 1 \}} \left| \nabla \theta_k e^{-f} \right|^2 \right] + 2 \mathbb{E} \left[ \left| \nabla e^{-f} \right|^2 \right]
\]

Therefore, \( II_{nmkl} \) is bounded uniformly. Moreover, \( \lim_{k} \mu(\{ 0 < \theta_k < 1 \}) = 0 \), which implies \( \lim_{k} II_{nmkl} = 0 \). When \( \{ \theta_k = 0 \} \), as the function \( e^{-f_{nmkl}} \) is constant, its derivative is zero and \( \lim_{k} I_{nmkl} = 0 \) as well. As a result, \( (\nabla e^{-f_{nmkl}}/2) \) is uniformly bounded in \( L^2(\mu, H) \) and

\[
\limsup \mathbb{E} \left[ \left| \nabla e^{-f_{nmkl}/2} \right|^2 \right] \leq \mathbb{E} \left[ \left| \nabla e^{-f/2} \right|^2 \right].
\]

On the other hand, \( (f_{nmkl}) \) converges to \( f \) in \( L^0(\mu) \), that is, in probability, \( \mathbb{E} \left[ \left| e^{-f_{nmkl}/2} \right|^2 \right] \) converges to \( \mathbb{E} \left[ \left| e^{-f/2} \right|^2 \right] \), which imply that \( e^{-f_{nmkl}/2} \) converges to \( e^{-f/2} \) in \( L^2(\mu) \) and \( \nabla e^{-f_{nmkl}/2} \) converges weakly to \( \nabla e^{-f/2} \) in \( L^2(\mu, H) \). By the weak lower semi-continuity of the norm, we
get
\[ \mathbb{E} \left[ \left| \nabla e^{-f/2} \right|^2 \right] \leq \lim \inf \mathbb{E} \left[ \left| \nabla e^{-f_{nmkl}/2} \right|^2 \right] \]
which implies that \( \lim_{n,l,k,m} \mathbb{E} \left[ \left| \nabla e^{-f_{nmkl}/2} \right|^2 \right] = \mathbb{E} \left[ \left| \nabla e^{-f/2} \right|^2 \right] \), and also \( \left( \nabla e^{-f_{nmkl}/2} \right) \) converges to \( \nabla e^{-f/2} \) in \( L^2(\mu, H) \) since it converges weakly. Hence,
\[
\lim_{n,l,k,m} \mathbb{E} \left[ \left| \nabla f_{nmkl} \right|^2 e^{-f} \right] = 4 \mathbb{E} \left[ \left| \nabla e^{-f/2} \right|^2 \right] = \mathbb{E} \left[ \left| \nabla f \right|^2 e^{-f} \right]
\]
and we get \( \lim_{n,l,k,m} \mathbb{E} \left[ \left| e^{-f} \nabla f_{nmkl} \right|^2 \right] = \mathbb{E} \left[ \left| e^{-f} \nabla f \right|^2 \right] \). Moreover, \( \left( \nabla f_{nmkl} \right) \) converges to \( \nabla f \) in \( L^0(\rho) \). Hence, \( \left( \nabla f_{nmkl} \right) \) converges to \( \nabla f \) in \( L^2(\rho) \). Similar calculations show that \( \left( \nabla f_{nmkl} \circ T_{nmkl} \right) \) converges to \( \nabla g \circ T \) in \( L^2(\rho) \). \( \square \)

We are ready to prove the regularity of backward Monge potential. In the next theorem, note that we do not assume Poincaré inequality since it is implied by \( (1 - c) \) convexity [14, Thm.6.2].

**Theorem 4.1.** Let \( (W, H, \mu) \) be an abstract Wiener space, \( g \in D_{2,1} \) and \( f \in D_{2,1} \) such that
\[
\int_W |\nabla f|^2 e^{-f} \, d\mu < \infty \quad \text{and} \quad \int_W |\nabla g|^2 e^{-g} \, d\mu < \infty
\] (28)
and the function \( f \) is \( (1 - c) \)-convex function for some \( c \in [0, 1) \). Let \( (\varphi, \psi) \) be forward and backward potentials to the Monge-Kantorovich problem with initial measure and target measure
\[
d\rho = e^{-f} \, d\mu, \quad d\nu = e^{-g} \, d\mu
\]
and quadratic cost given by
\[
C(x, y) = \begin{cases} |x - y|^2_H & \text{if } x - y \in H \\ \infty & \text{if } x - y \notin H. \end{cases}
\]
Then, \( \nabla^2 \psi \in L^2(\nu, H \otimes H) \) and can be estimated by
\[
\mathbb{E}_\nu \left[ \left| \nabla^2 \psi \right|^2 \right] \leq \frac{3}{c} \left( \mathbb{E}_\rho \left[ \left| \nabla \varphi \right|^2_H \right] + \mathbb{E}_\nu \left[ \left| \nabla g \right|^2_H \right] + \mathbb{E}_\rho \left[ \left| \nabla f \right|^2_H \right] \right),
\]
where \( H \otimes H \) denotes the space of Hilbert-Schmidt operators on \( H \).
Proof. Thanks to condition (28), the Sobolev derivative is closable in $L^2(\rho)$ and $L^2(\nu)$. Define

$$d\rho_n = e^{-f_n} d\mu, \quad d\nu_n = e^{-g_n} d\mu$$

with $f_n$ and $g_n$ as described before Lemma 4.5. The new probability measures $\rho_n$ and $\nu_n$ satisfy the sufficient conditions for forward Monge potential to be smooth. Hence, if we apply Proposition 3.2, we have

$$c\mathbb{E}_{\nu_n} \left[|\nabla^2 \psi_n|^2 \right] \leq 3 \left( \mathbb{E}_{\rho_n} \left[|\nabla \varphi_n|^2_H \right] + \mathbb{E}_{\nu_n} \left[|\nabla g_n|^2_H \right] + \mathbb{E}_{\rho_n} \left[|\nabla f_n|^2_H \right] \right).$$

Lemmas 4.1, 4.2, 4.3, 4.4 and 4.5 imply that the limits on right hand side exist and

$$\mathbb{E}_{\nu_n} \left[|\nabla^2 \psi_n|^2 \right] \leq \frac{3}{c} \left( \mathbb{E}_{\rho} \left[|\nabla \varphi|^2_H \right] + \mathbb{E}_{\nu} \left[|\nabla g|^2_H \right] + \mathbb{E}_{\rho} \left[|\nabla f|^2_H \right] \right).$$

Those lemmas also imply that $(\sqrt{\theta_n}P_{1/n}\nabla \psi_n)$ converges weakly to $\nabla^2 \psi$. In view of the weak lower continuity of the norms, if we take a weak limit with respect to $n$, we obtain

$$\mathbb{E}_{\nu} \left[|\nabla^2 \psi|^2 \right] \leq \liminf_{n} \mathbb{E} \left[|\sqrt{\theta_n} \nabla^2 P_{1/n} \psi_n|^2 e^{-g} \right]$$

$$\leq \sup_n e^{-\frac{g}{2}} \mathbb{E} \left[\theta_n P_{1/n} |\nabla^2 \psi_n|^2 e^{-g_n} \right]$$

$$\leq \sup_n \mathbb{E} \left[|\nabla^2 \psi_n|^2 e^{-9n} \right]$$

$$\leq \frac{3}{c} \left( \mathbb{E}_{\rho} \left[|\nabla \varphi|^2_H \right] + \mathbb{E}_{\nu} \left[|\nabla g|^2_H \right] + \mathbb{E}_{\rho} \left[|\nabla f|^2_H \right] \right)$$

and the result follows.

Remark 4.1. The result of Lemma 4.5 is more than what is needed in the proof of Theorem 4.1. Indeed, it holds that since $\mathbb{E}_{\nu_n} \left[|\nabla g_n|^2_H \right] \leq 8\mathbb{E}_{\nu} \left[|\nabla g|^2_H \right]$ and $\mathbb{E}_{\rho_n} \left[|\nabla f_n|^2_H \right] \leq 8\mathbb{E}_{\rho} \left[|\nabla f|^2_H \right]$.

4.3 Monge-Ampère Equation

We will prove that backward potential $\psi$ solves Monge-Ampère equation. The idea of the proof is to start with finite dimension and take limits, as accomplished through the following lemmas.

Lemma 4.6. Under assumptions of Theorem 4.1, $(L\psi_n)$ converges to $L\psi$ in the sense of distributions and $L\psi \in L^1(\nu)$. 

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Proof. The proof of convergence in the sense of distributions follows from duality. Once we show that \((L\psi_n)\) is uniformly integrable, we are done. We will use an idea from [6]. First we will show
\[
\sup_n \int \frac{(L\psi_n)^2}{1 + |\nabla \psi_n|^2} \, d\rho < M < \infty
\]
then we will show the uniform integrability of \((L\psi_n)\) by using the convergence of \((\nabla \psi_n)\) in \(L^2(\nu)\). Let \(u\) be a decreasing function on \([0, +\infty]\). Observe that
\[
\int (L\psi_n)^2 \, u(|\nabla \psi_n|^2) e^{-g} \, d\mu = \int \langle \nabla \psi_n, \nabla (L\psi_n u(|\nabla \psi_n|^2) e^{-g}) \rangle_{H} \, d\mu
\]
\[
= \int \langle \nabla \psi_n, \nabla (L\psi_n u(|\nabla \psi_n|^2)) \rangle_{H} e^{-g} \, d\mu
\]
\[
- \int \langle \nabla \psi_n, \nabla g \rangle_{H} L\psi_n u(|\nabla \psi_n|^2) e^{-g} \, d\mu
\]
\[
= \int \langle \nabla \psi_n, \nabla (L\psi_n) \rangle_{H} u(|\nabla \psi_n|^2) e^{-g} \, d\mu
\]
\[
+ 2 \int \langle \nabla \psi_n, \nabla^2 \psi (\nabla \psi) \rangle_{H} u'(|\nabla \psi_n|^2) L\psi_n e^{-g} \, d\mu
\]
\[
- \int \langle \nabla \psi_n, \nabla g \rangle_{H} L\psi_n u(|\nabla \psi_n|^2) e^{-g} \, d\mu
\]
\[
= \int \langle \nabla \psi_n, L\nabla \psi_n \rangle_{H} u(|\nabla \psi_n|^2) e^{-g} \, d\mu
\]
\[
+ \int \langle \nabla \psi_n, \nabla \psi_n \rangle_{H} u(|\nabla \psi_n|^2) e^{-g} \, d\mu
\]
\[
+ 2 \int \langle \nabla \psi_n, \nabla^2 \psi (\nabla \psi) \rangle_{H} u'(|\nabla \psi_n|^2) L\psi_n e^{-g} \, d\mu
\]
\[
- \int \langle \nabla \psi_n, \nabla g \rangle_{H} L\psi_n u(|\nabla \psi_n|^2) e^{-g} \, d\mu
\]

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Finally, if we write the first line of the last term, we see that

\[
\int (L\psi_n)^2 u(|\nabla\psi_n|^2)e^{-g} \, d\mu = \int |\nabla^2\psi_n|^2 u(|\nabla\psi_n|^2)e^{-g} \, d\mu \tag{I}
\]

\[
- \int \langle \nabla^2\psi_n(\nabla\psi_n), \nabla g \rangle_H u(|\nabla\psi_n|^2)e^{-g} \, d\mu \tag{II}
\]

\[
+ 2 \int |\nabla^2\psi_n(\nabla\psi_n)|^2_H u'(|\nabla\psi_n|^2)e^{-g} \, d\mu \tag{III}
\]

\[
+ \int \langle \nabla\psi_n, \nabla\psi_n \rangle_H u(|\nabla\psi_n|^2)e^{-g} \, d\mu \tag{IV}
\]

\[
+ 2 \int \langle \nabla\psi_n, \nabla^2\psi(\nabla\psi) \rangle_H u'(|\nabla\psi_n|^2)\mathcal{L}\psi_n e^{-g} \, d\mu \tag{V}
\]

\[
- \int \langle \nabla\psi_n, \nabla g \rangle_H \mathcal{L}\psi_n u(|\nabla\psi_n|^2)e^{-g} \, d\mu \tag{VI}
\]

We know that \(I, IV\) are bounded and \(III\) is negative since \(u\) is decreasing. We will show that the other integrals are also bounded when we choose \(u\) properly. Observe that for all \(\epsilon > 0\) the Cauchy–Schwarz inequality implies

\[
|II| \leq \sqrt{\frac{1}{4\epsilon} \int |\nabla g|^2 e^{-g} \, d\mu} \sqrt{4\epsilon \int |\nabla^2\psi(\nabla\psi)|^2 u^2(|\nabla\psi_n|^2)e^{-g} \, d\mu}
\]

\[
\leq \frac{1}{4\epsilon} \int |\nabla g|^2 e^{-g} \, d\mu + \epsilon \int |\nabla^2\psi(\nabla\psi)|^2 u^2(|\nabla\psi_n|^2)e^{-g} \, d\mu
\]

\[
|VI| \leq \sqrt{\frac{1}{4\epsilon} \int |\nabla g|^2 e^{-g} \, d\mu} \sqrt{4\epsilon \int |\nabla\psi_n|^2(L\psi_n)^2 u^2(|\nabla\psi_n|^2)e^{-g} \, d\mu}
\]

\[
\leq \frac{1}{4\epsilon} \int |\nabla g|^2 e^{-g} \, d\mu + \epsilon \int |\nabla\psi_n|^2(L\psi_n)^2 u^2(|\nabla\psi_n|^2)e^{-g} \, d\mu
\]

and

\[
|V| \leq \sqrt{\frac{1}{\epsilon} \int (L\psi_n)^2 u(|\nabla\psi_n|^2)e^{-g} \, d\mu} \sqrt{4\epsilon \int \frac{(u')^2}{u} (|\nabla\psi_n|^2)|\nabla^2\psi(\nabla\psi)|^2 |\nabla\psi|^2 e^{-g} \, d\mu}
\]

\[
\leq \epsilon \int (L\psi_n)^2 u(|\nabla\psi_n|^2)e^{-g} \, d\mu + \frac{1}{\epsilon} \int \frac{(u')^2}{u} (|\nabla\psi_n|^2)|\nabla^2\psi(\nabla\psi)|^2 |\nabla\psi|^2 e^{-g} \, d\mu.
\]

If we take \(u(t) = \frac{1}{1+t}\) and \(\epsilon = \frac{1}{4}\), we get

\[
\frac{(u')^2}{u} (|\nabla\psi_n|^2)|\nabla\psi_n|^4 \leq 1, \quad |\nabla\psi_n|^2 u(|\nabla\psi_n|^2) \leq 1
\]

and

\[
\epsilon u^2(|\nabla\psi_n|^2) + 2u'(|\nabla\psi_n|^2) \leq 0
\]
Hence, we have

$$\sup_n \int \frac{(L\psi_n)^2}{1 + |\nabla \psi_n|^2} \, d\rho < M < \infty$$

where

$$M = \int |\nabla g|^2 e^{-g} \, d\mu + 2 \sup_n \int |\nabla \psi_n|^2 e^{-g} \, d\mu + 10 \sup_n \int |\nabla^2 \psi_n|^2 e^{-g} \, d\mu < \infty$$

Note that $(\nabla \psi_n)$ is uniformly integrable with respect to $\nu$. So for every $\epsilon > 0$ there exists $\delta > 0$ such that $\rho(E) < \delta$ implies $E_{\nu}[|\nabla \psi|^2_{1E}] < \frac{\epsilon^2}{M+1}$.

Define

$$1_{E_1} = \left\{ |L\psi_n| \leq \frac{\epsilon}{2M} \frac{|L|\psi_n^2}{1 + |\nabla \psi_n|^2} \right\}$$

$$1_{E_2} = \left\{ |L\psi_n| > \frac{\epsilon}{2M} \frac{|L|\psi_n^2}{1 + |\nabla \psi_n|^2} \right\}$$

We have $\mathbb{E}_\nu[|L\psi_n|_{1E}] = \mathbb{E}_\nu[|L\psi_n|_{1E_1}] + \mathbb{E}_\nu[|L\psi_n|_{1E_2}] < \frac{\epsilon}{2} + \frac{\epsilon}{2}$. Hence $(L\psi_n)$ is uniformly integrable with respect to $\nu$. Therefore, $(L\psi_n)$ converges weakly to $L\psi$ in $L^1(\nu)$.

**Lemma 4.7.** Under assumptions of Theorem 4.1, $(f_n \circ S_n)$ converges to $f \circ S$ in $L^1(\nu)$, where $S = T^{-1} = I_W + \nabla \psi$.

**Proof.** We have

$$\int |f_n \circ S_n - f \circ S| \, d\nu \leq \int |f_n \circ S_n - f \circ S|_H \, d\nu + \int |f \circ S_n - f \circ S|_H \, d\nu$$

By Fatou’s lemma

$$\lim \int |f_n \circ S_n - f \circ S| \, d\nu \leq \lim \int |f_n - f| e^{-f_n} \, d\mu = 0$$

Let $\hat{f} \in C_b(W)$ such that $|f - \hat{f}|_{L^1(\rho)} < \epsilon$. Then, again using Fatou’s lemma

$$\lim \int |f \circ S_n - f \circ S| \, d\nu \leq \lim \int |f \circ S_n - \hat{f} \circ S_n| e^{-\hat{f}_n} \, d\mu + \lim \int |\hat{f} \circ S_n - \hat{f} \circ S e^{-\hat{f}_n} \, d\mu$$

$$+ \int |\hat{f} \circ S - \hat{f} \circ S e^{-\hat{f}} \, d\mu$$

$$\leq \lim \int |f - \hat{f}| e^{-f_n} \, d\mu + \lim \int |\hat{f} \circ S_n - \hat{f} \circ S e^{-\hat{f}_n} \, d\mu$$

$$+ \int |\hat{f} - f| e^{-\hat{f}} \, d\mu$$

$$\leq 2|f - \hat{f}|_{L^1(\rho)} < 2\epsilon.$$
Lemma 4.8. Under assumptions of Theorem 4.1,

$$\lim_n E_\nu [\|\nabla^2 \psi - \nabla^2 \psi_n\|_2] = 0$$

where $\| \cdot \|_2$ denotes the Hilbert-Schmidt operator norm.

Proof. Let $m, n$ be two integers such that $m > n$ and $\pi^n_m$ be the orthogonal projection from $H_m$ onto $H_n$. Then [13, Thm.2.5] implies

$$|\nabla^2 (\psi_n \circ \pi^n_m) - \nabla^2 \psi_n|^2_{L^1(\rho_m, H_m \otimes H_m)} \leq C_1 \int_{H_m} (f_m - f_n \circ \pi^n_m) \, d\rho_m$$

$$+ \frac{C_2}{\epsilon} \int_{H_m} |\nabla g_m - \nabla (g_n \circ \pi^n_m)|^2 \, d\nu_m$$

for some constants $C_1, C_2 > 0$. If we take the limit with respect to $m$ first, and then with respect to $n$, we get the result. \qed

Theorem 4.2. Let $(W, H, \mu)$ be an abstract Wiener space, $f \in D_{2,1}$ and $g \in D_{2,1}$ such that

$$\int_{\mathbb{R}^d} |\nabla f|^2 e^{-f} \, d\mu < \infty \quad \text{and} \quad \int_{\mathbb{R}^d} |\nabla g|^2 e^{-g} \, d\mu < \infty$$

the function $f$ is $(1 - c)$-convex function for some $c \in [0, 1)$. Let $\psi$ be backward potential of Monge-Kantorovich problem with initial and target measures

$$d\rho = e^{-f} \, d\mu, \quad d\nu = e^{-g} \, d\mu$$

and quadratic cost. Then, the backward potential $\psi$ solves Monge-Ampère equation

$$e^{-g} = e^{-f \circ S} \det_2 (I_H + \nabla^2 \psi) \exp \left[ -L \psi - \frac{1}{2} |\nabla \psi|^2_H \right]$$

$\nu$-a.s., where $S = T^{-1} = I_W + \nabla \psi$.

Proof. By the finite dimensional result in [22], we have

$$e^{-g_n} = e^{-f_n \circ S_n} \det_2 (I_H + \nabla^2 \psi_n) \exp \left[ -L \psi_n - \frac{1}{2} |\nabla \psi_n|^2_H \right].$$

We already know that $e^{-g_n} \to e^{-g}$. From approximation lemmas in Subsection 4.1, as $n \to \infty$, $\sqrt{\theta_n} P_{1/n} \nabla \psi_n \to \nabla \psi$ $\nu$-a.s. Due to the continuity of $P_t$ in $t$ [21, Prop.2.1] and since $\theta_n \to 1$, it follows that $\nabla \psi_n \to \nabla \psi$ $\nu$-a.s. By lemmas 4.7 and 4.8, we also know that
\[ e^{-f_{\pi}^T} \rightarrow e^{-f^T}, \text{ and } \nabla^2 \psi_n \rightarrow \nabla^2 \psi \text{ and the sequence } (-\mathcal{L} \psi_n) \text{ has a limit, say } A, \nu\text{-a.s.} \]

Therefore, we have

\[ e^{-g} = e^{-f \circ S} \text{det}_2 \left( I_H + \nabla^2 \psi \right) \exp \left[ A - \frac{1}{2} |\nabla \psi|^2_H \right]. \]

Once we show that \(-\mathcal{L} \psi = A\) holds \(\nu\)-a.s., we are done. Indeed, since \((-\mathcal{L} \psi_n)\) is uniformly integrable, for every cylindrical function \(\xi\), we have

\[
\int A \xi e^{-g} \, d\mu = \lim_n \int -\mathcal{L} \psi_n \xi e^{-g} \, d\mu \\
= \lim_n \int -\langle \nabla \psi_n, \nabla \xi - \xi \nabla g \rangle_H \psi_n e^{-g} \, d\mu \\
= \lim_n \int -\langle \nabla \psi, \nabla \xi - \xi \nabla g \rangle_H \psi_n e^{-g} \, d\mu \\
= \int -\mathcal{L} \psi \xi e^{-g} \, d\mu
\]

so \(-\mathcal{L} \psi = A\) holds \(\nu\)-a.s. \(\square\)

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