Higher dimensional Shimura varieties in the Prym loci of ramified double covers

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Abstract
In this paper, we construct Shimura subvarieties of dimension bigger than one of the moduli space of polarized abelian varieties of dimension $p$, which are generically contained in the Prym loci of (ramified) double covers. The idea is to adapt the techniques already used to construct Shimura curves in the Prym loci to the higher dimensional case, namely, to use families of Galois covers of $\mathbb{P}^1$.

The case of abelian covs is treated in detail, since in this case, it is possible to make explicit computations that allow to verify a sufficient condition for such a family to yield a Shimura subvariety of $\mathbb{A}_p^n$.

KEYWORDS
Galois covers, Prym loci, Shimura varieties

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1 | INTRODUCTION

The purpose of this paper is to construct Shimura subvarieties of dimension bigger than one of the moduli space of polarized abelian varieties of a given dimension, which are generically contained in the Prym loci of (ramified) double covers. The technique is similar to the one used in [13] and [18], where Shimura subvarieties of dimension 1 generically contained in the Prym loci were constructed using Galois covers of $\mathbb{P}^1$.

Let $R_{g,b}$ be the moduli space of isomorphism classes of triples $[(C, \eta, B)]$, where $C$ is a smooth complex projective curve of genus $g$, $B$ is a reduced effective divisor of degree $b$ on $C$, and $\eta$ is a line bundle on $C$ such that $\eta^2 = \mathcal{O}_C(B)$. This determines a double cover of $C$, $f: \tilde{C} \to C$ branched on $B$, with $\tilde{C} = \text{Spec}(\mathcal{O}_C \oplus \eta^{-1})$.

The Prym variety associated to $[(C, \eta, B)]$ is the connected component containing the origin of the kernel of the norm map $N_{f}: J\tilde{C} \to JC$. If $b > 0$, then $\ker N_{f}$ is connected. The Prym variety is a polarized abelian variety of dimension $g - 1 + \frac{b}{2}$, denoted by $P(C, \eta, B)$ or simply by $P(C, \eta)$.

Denote by $\Xi$ the restriction to $P(C, \eta)$ of the principal polarization on $J\tilde{C}$. For $b > 0$, the polarization $\Xi$ is of type $\delta = (1, \ldots, 1, 2, \ldots, 2)$. If $b = 0$ or 2, then $\Xi$ is of type $(2, \ldots, 2)$, thus it is twice of a principal polarization, hence we take a square root of $\Xi$ and we endow $P(C, \eta)$ with this principal polarization [37]. Let $\mathbb{A}_p^{\delta} = \mathbb{A}_p^{g-1+\frac{b}{2}}$ be the moduli space of abelian varieties of dimension $g - 1 + \frac{b}{2}$ with a polarization of type $\delta$. 

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The Prym map $P_{g,b} : R_{g,b} \rightarrow A^5_{g-1+b/2}$ is the map that associates to a point $[(C, \eta, B)]$ the polarized abelian variety $[(P(C, \eta, B), \Xi)]$. The Prym locus is the closure in $A^5_{g-1+b/2}$ of the image of the map $P_{g,b}$. The map $P_{g,b}$ is generically finite, if and only if $\dim R_{g,b} \leq \dim A^5_{g-1+b/2}$, and this holds if either $b \geq 6$ and $g \geq 1$, $b = 4$ and $g \geq 3$, $b = 2$ and $g \geq 5$, or $b = 0$ and $g \geq 6$.

If $b = 0$, the Prym map is generically injective for $g \geq 7$ [20, 25]. If $b > 0$, the generic injectivity in all the cases except for $b = 4, g = 3$ has been proven in [28, 29, 39]. The case $b = 4, g = 3$ has been previously studied in [2, 38], and in this case, the degree of the Prym map is 3. Recently, a global Prym–Torelli theorem was proved for all $g$ and $b \geq 6$ ([24] for $g = 1$ and [40] for all $g$).

In [13], a question about the existence of Shimura subvarieties of $A^5_{g-1+b/2}$ generically contained in the Prym loci in the cases $b = 0, 2$ was posed. The expectation is that, similarly to what is expected in the case of the Torelli map, for $g$ sufficiently high, they should not exist.

In fact, Shimura subvarieties of $A^5_{g-1+b/2}$ are totally geodesic with respect to the locally symmetric (orbifold) metric induced from the symmetric metric of the Siegel space [33, 36]. The expectation in the case of the Torelli map is coherent with the fact that the Torelli locus is very curved. Hence, a possible approach to investigate this problem is via the study of the second fundamental form of the Torelli map [12, 14, 19, 21, 22]. The second fundamental form of the Prym map has a very similar structure and similar properties as the one of the Torelli map and the results proven in [8–11] go in the direction of the expectation. Nevertheless, in [13] in the case $b = 0, 2$, and in [18] for all $b \geq 0$, examples of Shimura curves generically contained in the Prym loci were exhibited, using Galois covers of the projective line. However, the computations done in [13, 18] show that, for high genus, the number of examples that have been found drastically decreases. Here, we adapt this technique to investigate the existence of higher dimensional Shimura subvarieties in the (possibly ramified) Prym loci and we find several examples. Namely, we consider a family of Galois covers $\tilde{C}_t \rightarrow C_t / G \cong \mathbb{P}^1$, where $\tilde{C}_t$ is a smooth projective curve of genus $\tilde{g}$, $G$ is a group admitting a central involution $\sigma \in \tilde{G}$, and $\mathbb{P}^1 \cong \tilde{C}_t / \tilde{G} \cong C_t / G$, so that we have a factorization

$$\tilde{C}_t \xrightarrow{f_t} \mathbb{P}^1 \cong \tilde{C}_t / \tilde{G} \cong C_t / G$$

where $G = G / \langle \sigma \rangle$.

To such a family of Galois covers, we associate the Prym variety $P(\tilde{C}_t, C_t)$ of the double cover $f_t : \tilde{C}_t \rightarrow C_t$ and we ask whether this family of Pryms $P(\tilde{C}_t, C_t)$ yields a Shimura subvariety of $A^5_{g-1+b/2}$. Here, $g$ denotes the genus of $C_t$ and $b$ is the number of ramification points of the map $f_t$.

In [18, Theorem 3.2] (see Theorem 3.2), a sufficient condition is given for a family of abelian covers as above to yield a Shimura subvariety of $A^5_{g-1+b/2}$. This is called condition (B) in Section 3 and it is the natural generalization of the analogous sufficient condition used in [16, 34, 35] to determine Shimura subvarieties generically contained in the Torelli locus. Let us briefly explain (B). Denote by $V_{t-}$ the subspace of $V_t := H^0(\tilde{C}_t, \omega_{\tilde{C}_t})$, which is anti-invariant under the action of the involution $\sigma$. Set $W_t := H^0(\tilde{C}_t, \omega_{\tilde{C}_t}^2)$ and denote by $W_{t+} \subset W_t$ the invariant subspace under the action of $\sigma$. Consider the multiplication map $S^2 H^0(\tilde{C}_t, \omega_{\tilde{C}_t}) \rightarrow H^0(\tilde{C}_t, \omega_{\tilde{C}_t}^2)$. Denote by $m : (S^2 V_{t-})^G \rightarrow W_{t+}^G$ the restriction of the multiplication map to the space $(S^2 V_{t-})^G$ of symmetric tensors in $S^2 V_{t-}$, which are $G$-invariant. Condition (B) says that the map $m$ is an isomorphism.

The same condition was considered in [13] and [18], where many examples of one-dimensional families of covers satisfying condition (B) were given, thus producing Shimura curves in the Prym loci. In order to test this condition on a large amount of data, we use a script in Magma. Condition (B) clearly implies that $\dim (S^2 V_{t-})^G = \dim W_{t+}^G$ (condition (A) in Section 3) and the Magma script first verifies this equality. In the case of one-dimensional families of covers, once condition (A) is verified, condition (B) is equivalent to saying that the map $m$ should be nonzero, or equivalently that the differential of the Prym map is nonzero, that means that the family of Pryms is not constant. For higher dimensional families satisfying $\dim (S^2 V_{t-})^G = \dim W_{t+}^G > 1$, it is much harder to prove that $m$ is an isomorphism. However, the global Prym–Torelli theorem proved in [24, 40] is very helpful, since it ensures that conditions (A) and (B) are equivalent if $b \geq 6$. This is no longer true in the cases $b \leq 4$. 


Nevertheless, even in these cases, we are able to construct many examples satisfying condition (B). One technique is to verify another condition that we denoted by (B1), which implies condition (B), and that can be verified analyzing the representation of $\tilde{G}$ on $V_{1-}$ (see Section 3 for a precise statement).

This is done by the MAGMA script.

A class of examples for which we are able to verify condition (B) is the case of families of covers with abelian Galois group $\tilde{G}$. In fact, if $\tilde{G}$ is abelian, we are able to exhibit a basis for the space $(S^2V_{1-})^{\tilde{G}}$ and to compute the rank of the multipication map $m$. Thus, we have an explicit way to verify condition (B). This is explained in Section 4, and it is an application of results on abelian covers obtained in [31, 32, 43]. We apply this technique in some examples, some of which are described in Section 5.

In the case of families of nonabelian covers that do not satisfy condition (B1), we need to adopt ad hoc techniques, one of which is to use Theorem 3.8 of [17] (see, e.g., Example 5.4 in Section 5). This result is also used to give an example (Example 5.5 in Section 5) satisfying (A) but not (B) and to show that in fact this family of covers yields a subvariety of $A^5_4$ that is not totally geodesic, hence it is not Shimura. This example is interesting, since it shows that the weaker condition (A) alone is not sufficient to conclude that the subvariety arising from the family is special. Notice that, in the case of the Torelli map, the analogous of condition (A) is sufficient to ensure that the family yields a Shimura subvariety, as it is shown in [16, Theorem 3.9]. The problem in the case of the Prym maps is that positive dimensional fibers can exist, even if the Prym map is generically finite.

All the examples that we found satisfying condition (B) are listed in the table in the Appendix. Here, we summarize the results:

**Theorem 1.1.** There are 93 families of Galois covers of $\mathbb{P}^1$ with $r$ branch points, yielding Shimura subvarieties of $A^5_\rho$ of dimension $n := r - 3$, $2 \leq n \leq 6$, generically contained in the (possibly ramified) Prym loci. The highest value of $p$ is 16, obtained by a two-dimensional family with $\tilde{g} = 29$, $g = 13$, $b = 8$, and $\tilde{G} = Q_8 \circ D_8$. There are six families of dimension 6, which yield Shimura subvarieties either of $A^5_4$ (with $\tilde{g} = 6$, $g = 2$, $b = 6$, $\tilde{G} = (\mathbb{Z}/2\mathbb{Z})^2$), or of $A_5$ (with $\tilde{g} = 11$, $g = 6$, $b = 0$, $\tilde{G} = (\mathbb{Z}/2\mathbb{Z})^3$). These are all listed in the table in the Appendix and 92 of them satisfy condition (B). The family described in Example 5.4 is not in the table, since it is shown to yield a two-dimensional Shimura subvariety of $A_{12}$ by a different method and it has $\tilde{g} = 25$, $g = 13$, and $b = 0$.

The group $\tilde{G} = Q_8 \circ D_8$ is the central product of the group $Q_8$ of quaternions and of the dihedral group $D_8$ of order 16. It is a group of order 64, which has the following presentation:

$$Q_8 \circ D_8 = \langle a, b, c, d \mid a^4 = d^2 = 1, b^2 = c^4 = a^2, bab^{-1} = a^{-1}, ac = ca, ad = da, bc = cb, bd = db, dcd = a^2c^3 \rangle.$$

In the table, we only put the families that satisfy condition (B). Example 5.4 has a nonabelian Galois group and it is proved to yield a Shimura subvariety of $A_{12}$ of dimension 2 using results obtained in [17] and a geometric description of the family, as explained in Section 5.

Moreover for $r = 8, 9$, the families satisfying condition (B) for $\tilde{g} \leq 20$ and $\tilde{G}$ abelian are exactly the ones listed in the table. For $r = 7$, we have examples of families satisfying condition (B) with $b = 0, 2, 4, 6, 8$; for $r = 6$ with $b = 0, 2, 4, 6, 8, 16$; for $r = 5$ with $b = 0, 2, 4, 6, 8, 12$.

Our computations indicate that for large values of $\tilde{g}$, there might not exist such examples of Shimura subvarieties. In fact, as $\tilde{g}$ increases, the number of examples of families satisfying (B) decreases significantly. This is coherent with the computations done in [13, 18] and also with the estimates for the dimension of a germ of a totally geodesic submanifold of $A^5_\rho$ contained in the Prym loci found in [9–11].

The structure of the paper is as follows: In Section 2, we recall the basic definitions and properties of Prym varieties, Prym maps, and special (or Shimura) subvarieties of PEL type. In Section 3, we describe the construction of the families of Galois covers yielding Shimura subvarieties of $A^5_\rho$ contained in the Prym loci, and we explain conditions (A), (B), and (B1). In Section 4, we concentrate on the case of families abelian covers of the projective line, we explain their construction following [31, 32] and [43] and describe a basis of the space of holomorphic one-forms. This will be crucial to make explicit computations in the examples. In Section 5, we describe some examples of families of covers satisfying conditions (A). The first one has abelian Galois group $\tilde{G}$ and satisfies (B1) (hence (B)). The second and the third ones have abelian Galois group, they do not satisfy (B1), and we explicitly show that (B) holds using the techniques explained in Section 4. Example 5.4 has non-abelian Galois group and condition (B1) is not satisfied. In this case, we prove that it yields a Shimura subvariety of $A_{12}$ of dimension 2 using [17, Theorem 3.8] and a geometric description of the family. Finally, in Example 5.5, we show
that condition (A) is satisfied, but condition (B) is not and we show that it yields a variety, which is not totally geodesic, hence it is not Shimura. Notice that, in the case of the Torelli map conditions, (A) and (B) are equivalent (see condition (+) in [16]), but this does not hold for Prym maps, as it is shown by this example. The Appendix contains the table where all the examples found are listed, together with the link to the MAGMA script.

2 PRELIMINARIES ON PRYM VARIETIES AND ON SPECIAL SUBVARIETIES OF $A_g$

2.1 Prym varieties

Denote by $R_{g,b}$ the moduli space of isomorphism classes of triples $[[C,\eta,B]]$, where $C$ is a smooth complex projective curve of genus $g$, $B$ is a reduced effective divisor of degree $b$ on $C$, and $\eta$ is a line bundle on $C$ such that $\eta^2 = O_C(B)$. This determines a double cover of $C$, $f : \tilde{C} \to C$ branched on $B$, with $\tilde{C} = \text{Spec}(O_C \oplus \eta^{-1})$.

The Prym variety associated to $[[C,\eta,B]]$ is the connected component containing the origin of the kernel of the norm map $\text{Nm}_f : J\tilde{C} \to JC$. If $b > 0$, then $\ker \text{Nm}_f$ is connected. The Prym variety is a polarized abelian variety of dimension $g - 1 + \frac{b}{2}$, denoted by $P(C,\eta,B)$ or simply by $P(\tilde{C},C)$.

Denote by $\Xi$ the restriction to $P(\tilde{C},C)$ of the principal polarization on $J\tilde{C}$. For $b > 0$, the polarization $\Xi$ is of type $\delta = (1,...,1,2,...,2)$, $g$ times. If $b = 0$ or 2, then $\Xi$ is twice a principal polarization and $P(\tilde{C},C)$ is endowed with this principal polarization. Denote by $A^\delta_{g-1+b/2}$ the moduli space of abelian varieties of dimension $g - 1 + \frac{b}{2}$ with a polarization of type $\delta$.

The Prym map $P_{g,b}$ is defined as follows:

$$P_{g,b} : R_{g,b} \longrightarrow A^\delta_{g-1+b/2}, \quad [[C,\eta,B]] \longrightarrow [(P(C,\eta,B),\Xi)].$$

The Prym locus is the closure in $A^\delta_{g-1+b/2}$ of the image of the map $P_{g,b}$.

The dual of the differential of the Prym map $P_{g,b}$ at a generic point $[[C,\eta,B]]$ is given by the multiplication map

$$(dP_{g,b})^* : S^2H^0(C,\omega_C \otimes \eta) \to H^0(C,\omega_C^2 \otimes O_C(B)),$$

which is known to be surjective as soon as $\dim R_{g,b} \leq \dim A^\delta_{g-1+b/2}$ [26]. Hence $P_{g,b}$ is generically finite, if and only if $\dim R_{g,b} \leq \dim A^\delta_{g-1+b/2}$, and this holds if either $b \geq 6$ and $g \geq 1$, $b = 4$ and $g \geq 3$, $b = 2$ and $g \geq 5$, or $b = 0$ and $g \geq 6$.

If $b = 0$, the Prym map is generically injective for $g \geq 7$ [20,25]. If $b > 0$, in [28,29,39], the generic injectivity is proved in all the cases except for $b = 4, g = 3$. This case was previously studied in [2,38] and the degree of the Prym map is 3. Recently, a global Prym–Torelli theorem was proved for all $g$ and $b \geq 6$ ([24] for $g = 1$ and [40] for all $g$).

2.2 Special subvarieties of PEL type

Consider a rank $2g$ lattice $\Lambda \cong \mathbb{Z}^{2g}$ and an alternating form $Q : \Lambda \times \Lambda \to \mathbb{Z}$ of type $\delta = (1,...,1,2,...,2)$. There exists a basis of $\Lambda$ such that the corresponding matrix is $\begin{pmatrix} 0 & \Delta_g \\ -\Delta_g & 0 \end{pmatrix}$, where $\Delta_g$ is the diagonal matrix whose entries are $\delta = (1,...,1,2,...,2)$. Set $U : = \Lambda \otimes \mathbb{R}$. The Siegel space $\mathfrak{S}(U,Q)$ is defined as follows:

$$\mathfrak{S}(U,Q) := \{ J \in \text{GL}(U) : J^2 = -I, J^*Q = Q, Q(x,Jx) > 0, \forall x \neq 0 \}.$$

The symplectic group $\text{Sp}(\Lambda,Q)$ of the form $Q$ acts on the Siegel space $\mathfrak{S}(U,Q)$ by conjugation and this action is properly discontinuous. The moduli space of abelian varieties of dimension $g$ with a polarization of type $\delta$ is the quotient $A^\delta_g = \text{Sp}(\Lambda,Q) \backslash \mathfrak{S}(U,Q)$. It is a complex analytic orbifold. We will consider $A^\delta_g$ with the orbifold structure. To an element
Let \( T \in \mathfrak{S}(U, Q) \), we associate the real torus \( U / \Lambda \cong \mathbb{R}^{2g} / \mathbb{Z}^{2g} \) endowed with the complex structure \( J \) and the polarization \( Q \). This is a polarized abelian variety \( A_J \). On \( \mathfrak{S}(U, Q) \) there is a natural variation of rational Hodge structure, whose underlying local system is \( \mathfrak{S}(U, Q) \times \mathbb{Q}^{2g} \) and corresponds to the Hodge decomposition of \( \mathbb{C}^{2g} \) in \( \pm i \) eigenspaces for \( J \). This gives a variation of Hodge structure on \( A^\delta_{g} \) in the orbifold sense.

A special or Shimura subvariety \( Z \subseteq A^\delta_{g} \) is defined as a Hodge locus of this variation of Hodge structure on \( A^\delta_{g} \). Let us briefly recall the definition of Hodge loci (see [35, section 3.2.3] for more details). Denote \( \Lambda_{Q} := \Lambda \otimes \mathbb{Q} \). Given \( J \in \mathfrak{S}(U, Q) \), denote \( U_J := (U, J) \). We have \( U_J = U_{J,0} \oplus U_{J,1} \) and \( U_J = H^1(A_J, C), U_{J,0} = H^{1,0}(A_J, C), U_{J,1} = H^{0,1}(A_J, C) \). Fix \( d, e \in \mathbb{N}^m \) and consider the rational Hodge structure on \( \mathfrak{S}(U, Q) \) whose local system is given by

\[
T^{d,e}(\Lambda_{Q}) := \bigoplus_{j=1}^{m} \Lambda_{Q}^{\otimes d_j} \otimes (\Lambda_{Q}^*)^{\otimes e_j}.
\]

For any \( t \in T^{d,e}(\Lambda_{Q}) \), set \( Y(t) := \{ J \in \mathfrak{S}(U, Q) \mid t \in T^{d,e}(U_J) \} \). If \( t_1, \ldots, t_r \) are rational vectors in various tensorial constructions as in (2.2), define \( Y(t_1, \ldots, t_r) := Y(t_1) \cap \cdots \cap Y(t_r) \). We say that \( Y(t_1, \ldots, t_r) \) is proper if \( Y(t_1, \ldots, t_r) \neq \emptyset \).

A Hodge locus of \( A^\delta_{g} \) is an irreducible component of \( \pi(Y(t_1, \ldots, t_r)) \), where \( \pi : \mathfrak{S}(U, Q) \rightarrow A^\delta_{g} \) is the projection and \( Y(t_1, \ldots, t_r) \) is proper. Hodge loci of \( A^\delta_{g} \) are called special or Shimura subvarieties of \( A^\delta_{g} \).

Special subvarieties are totally geodesic and they contain a dense set of CM points [35, section 3.4(b)]. Conversely, an algebraic totally geodesic subvariety that contains a CM point is a special subvariety [33, Theorem 4.3], [36].

Let us now recall the definition of special subvarieties of PEL type [35, section 3.9]. Given \( J \in \mathfrak{S}(U, Q) \), set \( \text{End}_Q(A_J) := \{ f \in \text{End}(\Lambda_{Q}) \mid Jf = fJ \} \). Fix a point \( J_0 \in \mathfrak{S}(U, Q) \) and consider \( D := \text{End}_Q(A_{J_0}) \). The PEL-type special subvariety \( Z(D) \) is defined as the image of the connected component of the set \( \{ J \in \mathfrak{S}(U, Q) \mid D \subseteq \text{End}_Q(A_J) \} \) that contains \( J_0 \). In [16, section 3], it is shown that \( Z(D) \) is irreducible, since it is proven to be the image in \( A^\delta_{g} \) of a smooth complex connected submanifold of the Siegel space \( \mathfrak{S}(U, Q) \). In fact, if \( G \subseteq \text{Sp}(\Lambda, Q) \) is a finite subgroup, denote by \( \mathfrak{S}(U, Q)^G \) the subset of \( \mathfrak{S}(U, Q) \) of fixed points by the action of \( G \). Define

\[
D_G := \{ f \in \text{End}_Q(\Lambda \otimes \mathbb{Q}) \mid Jf = fJ, \forall J \in \mathfrak{S}(U, Q)^G \}.
\]

We have the following result proven in [16, section 3].

**Theorem 2.1.** The subset \( \mathfrak{S}(U, Q)^G \) is a connected complex submanifold of \( \mathfrak{S}(U, Q) \). The image of \( \mathfrak{S}(U, Q)^G \) in \( A^\delta_{g} \) coincides with the PEL subvariety \( Z(D_G) \). If \( J \in \mathfrak{S}(U, Q)^G \), then \( \dim Z(D_G) = \dim(S^2 \mathbb{R}^{2g})^G \), where \( S^2 \mathbb{R}^{2g} \) is endowed with the complex structure \( J \).

### 3 SPECIAL SUBVARIETIES IN THE PRYM LOCI

Let \( \Sigma_g \) be a compact connected oriented surface of genus \( g \). We denote by \( T_g := T(\Sigma_g) \) the Teichmüller space of \( \Sigma_g \). Denote by \( T_{0,\partial} \) the Teichmüller space in genus 0 with \( \partial \geq 4 \) marked points (see, e.g., [1, Chap.15]). Fix \( p_0, \ldots, p_r \in S^2 \) distinct points, denote by \( P := (p_1, \ldots, p_r) \). A point of \( T_{0,\partial} \) is an equivalence class of triples \( (P^1, x, [h]) \) where \( x = (x_1, \ldots, x_r) \) is an \( r \)-tuple of distinct points in \( P^1 \) and \( [h] \) is an isotopy class of orientation preserving homeomorphisms \( h : (P^1, x) \rightarrow (S^2, P) \). Two such triples \( (P^1, x, [h]), (P^1, x', [h']) \) are equivalent if there is a biholomorphism \( \varphi : P^1 \rightarrow P^1 \) such that \( \varphi(x_i) = x'_i \) for any \( i \) and \( [h] = [h' \circ \varphi] \).

Choosing the point \( p_0 \in S^2 - P \) as base point, we have an isomorphism

\[
\Gamma_r := \langle \gamma_1, \ldots, \gamma_r | \gamma_1 \cdots \gamma_r = 1 \rangle \cong \pi_1(S^2 - P, p_0).
\]

Take a point \( t = \{(P^1, x, [h])\} \in T_{0,\partial} \) and fix an epimorphism \( \theta : \Gamma_r \rightarrow G \). By Riemann's existence theorem, this gives a Galois cover \( C_t \rightarrow P^1 = C_t / G \) ramified over \( x \) whose monodromy is given by \( \theta \), which is endowed with an isotopy class of homeomorphisms \( [h_1] \) on a surface \( \Sigma_g \) covering \( S^2 \).

We get a map \( T_{0,\partial} \rightarrow T_g := \{ t \mid (\{C_t, [h_t]\}) \} \) and the group \( G \) acts on \( C_t \) and embeds in the mapping class group \( \text{Mod}_g \) of \( \Sigma_g \). Denote by \( G_{\Sigma_g} \) the image of \( G \) in \( \text{Mod}_g \). The image of the map \( T_{0,\partial} \rightarrow T_g \) is the subset \( T_{g,\partial}^G \) of \( T_g \) given of the fixed points by the action of \( G_{\Sigma_g} \) [23]. Nielsen realization theorem says that \( T_{g,\partial}^G \) is a complex submanifold of \( T_g \) of dimension \( r - 3 \).
The image of $\Gamma^{\mathcal{S}}$ in the moduli space $\mathcal{M}_g$ is an $(r - 3)$-dimensional algebraic subvariety (see, e.g., [6, 7, 23] and [5, Theorem 2.1]).

We denote this image by $\mathcal{M}_g(G, \delta)$. Notice that different data $(G, \delta)$ and $(G, \delta')$ may give rise to the same subvariety of $\mathcal{M}_g$. This is due to the fact that we have chosen the isomorphism $(3.1)$ $\Gamma_r \cong \pi_1(S^2 - P, p_0)$. The change from one choice of such an isomorphism to another one is described via an action of the Braid group $\mathbb{B}_r = \langle \sigma_1, \ldots, \sigma_{r - 1} | \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i - j| \geq 2, \sigma_i \sigma_{i+1} \sigma_i = \sigma_i \sigma_{i+1} \sigma_i \rangle$. The change from one choice of such an isomorphism to another one is described via an action of the Braid group $\mathbb{B}_r$ given by $\beta \cdot (G, \delta) := (G, \delta \circ \beta)$. We also have an action of the automorphism group $\text{Aut}(G)$ given by $\gamma \cdot (G, \delta) := (G, \gamma \circ \delta)$. The orbitsof the $\mathbb{B}_r \times \text{Aut}(G)$-action are called Hurwitz equivalence classes, and data in the same orbit give rise to the same subvariety of $\mathcal{M}_g$ (for more details, see [4, 6, 42]).

Definition 3.1. A Prym datum of type $(r, b)$ is a triple $\Xi = (\tilde{G}, \tilde{\delta}, \sigma)$, where $\tilde{G}$ is a finite group, $\tilde{\delta} : \Gamma_r \rightarrow \tilde{G}$ is an epimorphism, and $\sigma \in \tilde{G}$ is a central involution, and

$$b = \sum_{i : \sigma \in \langle \tilde{\delta}(\gamma_i) \rangle} \frac{|\tilde{G}|}{\text{ord}(\tilde{\delta}(\gamma_i))}.$$

Let us fix a Prym datum $\Xi = (G, \delta, \sigma)$, and set $G = G/\langle \sigma \rangle$. The composition of $\delta$ with the projection $\tilde{G} \rightarrow G$ is an epimorphism $\delta : \Gamma_r \rightarrow G$. To a point $t \in \mathcal{D}_{G, \delta}$ we can associate two Galois covers $\tilde{C}_t \rightarrow \mathbb{P}^1 \cong \tilde{C}_t/\tilde{G}$ and $C_t \rightarrow \mathbb{P}^1 \cong C_t/G$. Denote by $\tilde{g}$ the genus of $\tilde{C}_t$ and by $g$ the genus of $C_t$. We have a diagram

$$\tilde{C}_t \xrightarrow{f_t} C_t = \tilde{C}_t/\langle \sigma \rangle.$$

By the definition of $b$, we see immediately that the double covering $f_t$ has $b$ branch points. The covering $f_t$ is determined by its branch locus $B_t$ and by an element $\eta_t \in \text{Pic}^\sigma(C_t)$ such that $\eta_t^2 = O_{C_t}(B_t)$. Hence, we have a map $\mathcal{T}_{0,r} \rightarrow R_{g,b}$, which associates to $t$ the class $[(C_t, \eta_t, B_t)]$. This map depends on the datum $\Xi = (G, \delta, \sigma)$. Denote by $R(\Xi)$ its image in $R_{g,b}$.

In [23, p. 79], it is shown that there is an intermediate variety $\tilde{M}_g(\tilde{G}, \tilde{\delta})$ such that the projection $\mathcal{T}_g^{\tilde{G}, \tilde{\delta}} \rightarrow \tilde{M}_g(\tilde{G}, \tilde{\delta})$ factors through $\mathcal{T}_g^{\tilde{G}, \tilde{\delta}} \rightarrow \tilde{M}_g(\tilde{G}, \tilde{\delta}) \rightarrow M_g(G, \delta)$. The variety $\tilde{M}_g(\tilde{G}, \tilde{\delta})$ is the normalization of $M_g(G, \delta)$ and parameterizes equivalence classes of curves with an action of $G$ with topological type determined by the datum $\Xi$. Moreover, in [23, Theorem 4], it is proven that there exists a finite cover $X_g(\tilde{G}, \tilde{\delta}) \rightarrow \tilde{M}_g(\tilde{G}, \tilde{\delta})$, which admits a universal family $C(\tilde{G}, \tilde{\delta}) \rightarrow X_g(\tilde{G}, \tilde{\delta})$. In [23, section 4], they also prove that there is factorization $\mathcal{T}_g^{\tilde{G}, \tilde{\delta}} \rightarrow X_g(\tilde{G}, \tilde{\delta}) \rightarrow \tilde{M}_g(\tilde{G}, \tilde{\delta}) \rightarrow M_g(G, \delta)$.

So, we have a map $X_g(\tilde{G}, \tilde{\delta}) \rightarrow R(\Xi)$, which associates to the class $[C]$ of a curve with a fixed $G$-action the class of the cover $C \rightarrow C = \tilde{C}/\langle \sigma \rangle$, where $\sigma \in G$ is the central involution fixed by the datum. Clearly, the map $\mathcal{T}_g^{\tilde{G}, \tilde{\delta}} \rightarrow \tilde{M}_g(\tilde{G}, \tilde{\delta})$ is the composition $\mathcal{T}_g^{\tilde{G}, \tilde{\delta}} \rightarrow X_g(\tilde{G}, \tilde{\delta}) \rightarrow R(\Xi)$. Therefore, it has discrete fibers, since the map $\mathcal{T}_g^{\tilde{G}, \tilde{\delta}} \rightarrow X_g(\tilde{G}, \tilde{\delta})$ has discrete fibers, as shown in [23, section 4] and the map $X_g(\tilde{G}, \tilde{\delta}) \rightarrow R(\Xi)$ is finite. In fact, once we fix $[(C, \eta, B)] \in R(\Xi)$, the cover $\tilde{C} \rightarrow C$ is uniquely determined and there are only a finite number of $G$-actions on $\tilde{C}$. This shows that $\dim R(\Xi) = r - 3$. Moreover, the Prym map $\mathcal{P}_{g,b}$ lifts to a map $X_g(\tilde{G}, \tilde{\delta}) \rightarrow A_{\tilde{g} - g}^{\tilde{\delta}}$, which sends the class of a curve $[\tilde{C}]$ with an action of $\tilde{G}$ to $P(\tilde{C}, C)$. We still denote this map by $\mathcal{P}_{g,b} : X_g(\tilde{G}, \tilde{\delta}) \rightarrow A_{\tilde{g} - g}^{\tilde{\delta}}$. We have the following diagram:

$$\begin{array}{ccc}
\mathcal{T}_g^{\tilde{G}, \tilde{\delta}} & \xrightarrow{\cong} & \mathcal{T}_0,r \\ \downarrow & & \downarrow \\ M_g(G, \delta) & \xrightarrow{R(\Xi)} & M_g(G, \delta)
\end{array}$$

$$\begin{array}{ccc}
\tilde{M}_g(\tilde{G}, \tilde{\delta}) & \xrightarrow{\cong} & \tilde{M}_g(\tilde{G}, \tilde{\delta}) \\ \downarrow & & \downarrow \\ M_g(G, \delta) & \xrightarrow{R(\Xi)} & M_g(G, \delta)
\end{array}$$

$$\begin{array}{ccc}
\mathcal{T}_g^{\tilde{G}, \tilde{\delta}} & \xrightarrow{\cong} & \mathcal{T}_0,r \\ \downarrow & & \downarrow \\ X_g(\tilde{G}, \tilde{\delta}) & \xrightarrow{\mathcal{P}_{g,b}} & A_{\tilde{g} - g}^{\tilde{\delta}}
\end{array}$$
Given a Prym datum $\Xi = (\tilde{G}, \tilde{\theta}, \sigma)$, let us fix an element $\tilde{C}$ of the family $\mathbb{T}_{\tilde{G}}$ with the double covering $f : \tilde{C} \to C$. Set

$$V := H^0(\tilde{C}, \omega_{\tilde{C}}).$$

We have an action of $\langle \sigma \rangle$ on $V$ and an eigenspace decomposition for this action:

$$V := H^0(\tilde{C}, \omega_{\tilde{C}}) = V_+ \oplus V_-,$$

where $V_+ \cong H^0(C, \omega_C)$ and $V_- \cong H^0(C, \omega_C \otimes \eta)$ as $G$-modules. Similarly, set $W := H^0(\tilde{C}, \omega_{\tilde{C}}^2) = W_+ \oplus W_-$, $W_+ \cong H^0(C, \omega_C^2 \otimes \mathcal{O}_C(B))$, and $W_- \cong H^0(C, \omega_C^2 \otimes \eta)$.

The multiplication map $m : S^2V \to W$ is the dual of the differential of the Torelli map $\tilde{j} : \mathbb{M}_{\tilde{G}} \to \mathbb{A}_g$ at the point $[\tilde{C}] \in M_g$. This map is $\tilde{G}$-equivariant. We have the following isomorphisms:

$$(S^2V)^{\tilde{G}} = (S^2V_+)^{\tilde{G}} \oplus (S^2V_-)^{\tilde{G}}, \quad W^{\tilde{G}} = W_+^{\tilde{G}}.$$

Hence, the multiplication map $m$ maps $(S^2V)^{\tilde{G}}$ to $W_+^{\tilde{G}}$.

Notice that the codifferential of the Prym map at the point $[(C, \eta, B)]$ is the multiplication map in (2.1), which coincides with the restriction of the multiplication map $m$ to $S^2V_-$. Here, we are interested in the restriction of $m$ to $(S^2V_-)^{\tilde{G}}$, which we still denote by $m$ for simplicity:

$$m : (S^2V_-)^{\tilde{G}} \longrightarrow W_+^{\tilde{G}}. \quad \text{(3.3)}$$

By what we have said, this is the multiplication map

$$(S^2H^0(C, \omega_C \otimes \eta))^{\tilde{G}} \longrightarrow H^0(C, \omega_C^2 \otimes \eta^2)^{\tilde{G}} \cong H^0(C, \omega_C^2)^{\tilde{G}} \cong H^0(\tilde{C}, \omega_{\tilde{C}}^2)^{\tilde{G}}.$$

Now we recall Theorem 3.2 in [18], which is a generalization of Theorems 3.2 and 4.2 in [13].

**Theorem 3.2.** Let $\Xi = (G, \tilde{\theta}, \sigma)$ be a Prym datum. If for some $t \in T_{0,r}$ the map $m$ in (3.3) is an isomorphism, then the closure of $P_{g,b}(X_{g}(G, \tilde{\theta}))$ in $A^g_{g-1+b+\frac{h}{2}}$ is a special subvariety.

We would like to use Theorem 3.2 to construct Shimura varieties contained in the closure of $P_{g,b}(R_{g,b})$ in $A^g_{g-1+b+\frac{h}{2}}$ and intersecting $P_{g,b}(R_{g,b})$.

The hypothesis in Theorem 3.2 is the following condition:

$$m : (S^2V_-)^{\tilde{G}} \longrightarrow W_+^{\tilde{G}} \text{ is an isomorphism.} \quad \text{(B)}$$

This implies

$$\dim(S^2V_-)^{\tilde{G}} = r - 3. \quad \text{(A)}$$

A sufficient condition ensuring (B) is the following:

$$(S^2V_-)^{\tilde{G}} \cong Y_1 \otimes Y_2, \quad \text{where } \dim Y_1 = 1, \ \dim Y_2 = r - 3. \quad \text{(B1)}$$

In fact, by the requirement on the dimensions of $Y_i$, it is immediate that (B1) implies (A). Then, we have the following:

**Lemma 3.3.** Assume that (B1) holds, then condition (B) holds.
Proof. We want to show that \( m : (S^2V_\ldots)_G \rightarrow W^G_+ \) is an isomorphism. It is sufficient to show that \( m : (S^2V_\ldots)_G \rightarrow W^G_+ \) is injective, since by assumption condition (B1) holds, and (B1) implies (A). Assume (B1) holds and take bases for \( Y_1 \) and \( Y_2 : Y_1 = \langle u \rangle, Y_2 = \langle w_1, \ldots, w_{r-3} \rangle \). Then, a basis for \( (S^2V_\ldots)_G \cong Y_1 \otimes Y_2 \) is given by \( \{ v \otimes w_i \}, i = 1, \ldots, r-3 \). Hence, \( \forall x \in (S^2V_\ldots)_G \), we have \( x = \sum_{i=1}^{r-3} a_i (v \otimes w_i) = v \otimes (\sum_{i=1}^{r-3} a_i w_i) \) is a decomposable tensor. Therefore, \( m(x) = v \cdot (\sum_{i=1}^{r-3} a_i w_i) = 0 \) if and only if \( 0 = \sum_{i=1}^{r-3} a_i w_i \), so \( a_i = 0, \forall i = 1, \ldots, r-3 \), and \( x = 0 \). So \( m \) is injective.

\[ \square \]

Remark 3.4. Notice that, in the case of one-dimensional families (i.e., \( r = 4 \)), if the group \( \tilde{G} \) is abelian and condition (A) holds, then condition (B1) (hence also (B)) is automatically satisfied ([18, Remark 3.3], [13, Remark 3.4]). This is no longer true for higher dimensional families, as it is shown in Section 5, Example 5.5.

Finally, in [40], it is proven that if \( g > 0 \), and \( b \geq 6 \), \( P_{g,b} \) is an embedding. Hence, if \( b \geq 6 \) and \( g > 0 \) condition (A) implies condition (B).

4 | ABELIAN COVERS OF \( \mathbb{P}^1 \) AND THEIR PRYM MAP

In this section, we follow closely [13] and also [32] whose notations come mostly from [43]. More details about Prym varieties can be found in [3].

An abelian Galois cover of \( \mathbb{P}^1 \) is determined by a collection of equations in the following way:

Consider an \( m \times r \) matrix \( A = (r_{ij}) \) whose entries \( r_{ij} \) are in \( \mathbb{Z}/N\mathbb{Z} \) for some \( N \geq 2 \). Consider in the affine space \( \mathbb{A}^m+1 \) the algebraic curve \( C \) defined by the collection of equations

\[
\begin{align*}
    w_i^N = \prod_{j=1}^{r} (x-t_j)^{r_{ij}} \quad \text{for } i = 1, \ldots, m. 
\end{align*}
\]

(4.1)

Here, \( r_{ij} \) are considered integers in \( [0, N) \). In the following, when we consider elements of \( \mathbb{Z}/N\mathbb{Z} \) as integers, we will always assume that the representatives are in \( [0, N) \). We impose the condition that the sum of the columns of \( A \) is zero (when considered as a vector in \( (\mathbb{Z}/N\mathbb{Z})^m \)). This implies that the cover given by (4.1) is not ramified over the infinity. We call the matrix \( A \), the matrix of the covering. Notice that (4.1) gives a possibly singular affine curve \( C \). Using the Jacobian criterion, one sees easily that there are singularities over \( t_j \) if there is \( r_{ij} > 1 \) or, if \( t_j \) appears in more than one equation of the abelian cover, that is, if there are \( i \) and \( k \) such that \( r_{ij} \neq 0 \) and \( r_{kj} \neq 0 \). By Riemann’s existence theorem (cf. [30, chapter 3]) there exists a unique smooth projective compactification \( X \) of the normalization of \( C \), and the map to \( \mathbb{P}^1 \) extends to \( X \) as a Galois cover of \( \mathbb{P}^1 \).

The local monodromy around the branch point \( t_j \) is given by the column vector \( (r_{ij}, \ldots, r_{mj})^t \) and so the order of ramification over \( t_j \) is \( \frac{N}{\gcd(N, r_{ij}, \ldots, r_{mj})} \). Using this and the Riemann–Hurwitz formula, the genus \( g \) of the cover can be computed by:

\[
\begin{align*}
    g &= 1 + d \left( \frac{r - 2}{2} - \frac{1}{2N} \sum_{j=1}^{r} \gcd(N, r_{ij}, \ldots, r_{mj}) \right),
\end{align*}
\]

where \( d \) is the degree of the covering. Notice that \( d \) is equal to the size of the column span (equivalently row span) of the matrix \( A \), that is, the \( \mathbb{Z}/N\mathbb{Z} \)-module generated by the columns (resp. rows) of \( A \) as vectors. In fact, the Galois group \( G \) of the covering is isomorphic to the column span of the matrix \( A \) as an abelian group, see [43], section 2.2. Therefore, the Galois group will be considered as a subgroup of \( (\mathbb{Z}/N\mathbb{Z})^m \).

We remark that two families of abelian covers with matrices \( A \) and \( A' \) over the same \( \mathbb{Z}/N\mathbb{Z} \) such that \( A \) and \( A' \) have equal row spans are isomorphic.

By the construction of abelian covers of \( \mathbb{P}^1 \), one can compute the invariants of such a cover quite easily. Let \( G \) be a finite abelian group. We denote by \( G^* \) the group of the characters of \( G \), that is, \( G^* = \text{Hom}(G, \mathbb{C}^*) \). Consider a Galois covering \( \pi : X \rightarrow \mathbb{P}^1 \) with Galois group \( G \). The group \( G \) acts on the sheaves \( \pi_* (O_X) \) and \( \pi_* (C) \) via its characters and we get corresponding eigenspace decompositions \( \pi_* (O_X) = \bigoplus_{\chi \in G^*} \pi_* (O_X)_\chi \) and \( \pi_* (C) = \bigoplus_{\chi \in G^*} \pi_* (C)_\chi \). Let \( L_X^{-1} = \pi_* (O_X)_X \) and \( C_X = \pi_* (C)_X \) denote the eigensheaves corresponding to the character \( \chi \). \( L_X^{-1} \) is a line bundle and outside of the branch locus of \( \pi \), \( C_X \) is a local system of rank 1.
Remark 4.1. It is a standard fact that the groups $G$ and $G^*$ are (noncanonically) isomorphic. Throughout the paper, we fix an isomorphism $\varphi_G : G \rightarrow G^*$, which is given by identifying both groups with the product of groups of roots of unity $\mu_N$ in $\mathbb{C}^*$ as follows: For $G = \mathbb{Z}/N\mathbb{Z}$, the isomorphism $G \rightarrow \mu_N$ is given by $1 \mapsto \exp(2\pi i/N)$, while $G^* \rightarrow \mu_N$ is given by $\chi \mapsto \chi(1)$. We then extend these isomorphisms to the product of cyclic groups. In the sequel, we consider a character of $G$ as an element of the group without referring to the isomorphism $\varphi_G$.

Given an abelian cover with an $m \times r$ matrix $A$, let $\chi$ be a character of the Galois group $G \subseteq (\mathbb{Z}/N\mathbb{Z})^m$ and let $n = (n_1, \ldots, n_m) \in G$ be the element corresponding to $\chi$ by Remark 4.1. We set $\tilde{\alpha}_j = \sum_{i=1}^m n_i r_{ij} \in \mathbb{Z}$ (note that $\tilde{\alpha}_j$ is not necessarily in $\mathbb{Z} \cap [0,N)$).

Let us denote by $\omega_X$ the canonical sheaf of $X$. Similar to the case of $\pi_s(\mathcal{O}_X)$, the sheaf $\pi_s(\omega_X)_\chi$ decomposes according to the action of $G$. Using the notation of the last paragraph, we have the following formula for the line bundles $L_X$ corresponding to the character $\chi$ associated to the element $n \in G$ and $\pi_s(\omega_X)_\chi$:

**Lemma 4.2.** It holds that $L_X = \mathcal{O}_{\mathbb{P}^1}(\sum_{j=1}^r (-\frac{\tilde{\alpha}_j}{N}))$, where $(x)$ denotes the fractional part of the real number $x$ and

$$
\pi_s(\omega_X)_\chi = \omega_{\mathbb{P}^1} \otimes L_{X^{-1}} = \mathcal{O}_{\mathbb{P}^1}(-2 + \sum_{j=1}^r \langle -\frac{\tilde{\alpha}_j}{N} \rangle).
$$

Note that, since the sum of the columns of the matrix $A$ is zero modulo $N$, the above sum is an integer.

**Proof.** One can easily see that each section of the line bundle $\mathcal{O}_{\mathbb{P}^1}(\sum_{j=1}^r (-\frac{\tilde{\alpha}_j}{N}))$ is a function on which the Galois group acts as $\chi$ and conversely any such section must be a function of the above form. The rest of the lemma is [41], Proposition 1.2.

Consider now an abelian group $G \subseteq (\mathbb{Z}/N\mathbb{Z})^m$ containing a central involution $\sigma$ as in Section 3 and a $G$-abelian cover given by the Equation (4.1). Let $n \in G$ be the element $(n_1, \ldots, n_m) \in (\mathbb{Z}/N\mathbb{Z})^m$ under the inclusion $G \subseteq (\mathbb{Z}/N\mathbb{Z})^m$.

By Lemma 4.2, $d_n := \dim H^0(\tilde{C}, \omega_C)_n = -1 + \sum_{j=1}^r \langle -\frac{\tilde{\alpha}_j}{N} \rangle$. A basis for the $\mathbb{C}$-vector space $H^0(\tilde{C}, \omega_C)$ is given by the forms

$$
\omega_{n,v} = x^y w_1^{n_1} \cdots w_m^{n_m} \prod_{j=1}^r (x - t_j)^{-\langle \frac{\tilde{\alpha}_j}{N} \rangle} dx,
$$

where $0 \leq y \leq d_n - 1 = -2 + \sum_{j=1}^r \langle -\frac{\tilde{\alpha}_j}{N} \rangle$. The fact that the above elements constitute a basis can be seen in [32], proof of Lemma 5.1, where the dual version for $H^1(\tilde{C}, \mathcal{O}_C)$ is proved. Note that in [32], Proposition 2.8, the formula for $d_n$ has been proven using a different method.

Let us describe the action of the involution $\sigma$ using the formula (4.1) of an abelian cover of $\mathbb{P}^1$. The action of $\sigma$ on $\tilde{C}$ is given by $w_i \mapsto -w_i$ for every $i$ in some subset $I$ of $\{1, \ldots, m\}$ and by the trivial action on $w_j$ for every $j$ in $\{1, \ldots, m\} \setminus I$, that is, $w_j \mapsto w_j$. We may therefore assume, without loss of generality, that $\sigma(w_i) = -w_i$ for $i \in \{1, \ldots, k\}$ for some $k \leq m$ and $\sigma(w_i) = w_i$ for the $i > k$.

Let $n = (n_1, \ldots, n_m) \in G \subset (\mathbb{Z}/N\mathbb{Z})^m$. The following lemma has been proven in [31], Lemma 2.5 by using the above description of the action of $\sigma$.

**Lemma 4.3.** The group $\tilde{G}$ acts on $H^0(\tilde{C}, \omega_C)_-$ and for $n \in \tilde{G}$, it holds that $H^0(\tilde{C}, \omega_C)_{-n} = H^0(\tilde{C}, \omega_C)_n$ if $n_1 + \cdots + n_k$ is odd and $H^0(\tilde{C}, \omega_C)_{-n} = 0$ otherwise. Similar equalities hold for $H^1(\tilde{C}, \mathcal{O}_C)_{-n}$.

Families of abelian covers of $\mathbb{P}^1$ can be constructed as follows: Denote by $Y_r$ the complement of the big diagonals in $(A^1_C)^r$, that is, $Y_r = \{(t_1, \ldots, t_r) \in (A^1_C)^r \mid t_i \neq t_j, \forall i \neq j\}$. Over this affine open set, we define a family of abelian covers of $\mathbb{P}^1$ by Equation (4.1) with branch points $(t_1, \ldots, t_r) \in Y_r$, and $r_{ij}$ the lift of $r_{ij}$ to $\mathbb{Z} \cap [0,N)$ as before. Varying the branch points, we get a family $f : \tilde{C} \rightarrow Y_r$ of smooth projective curves whose fibers $\tilde{C}_r$ are abelian covers of $\mathbb{P}^1$ introduced above. The subvariety of the moduli space corresponding to this family of curves is $M_\delta(G, \delta)$, with $\delta : \Gamma_r \rightarrow \tilde{G} \subseteq (\mathbb{Z}/N\mathbb{Z})^m$ the epimorphism $\delta(\gamma_k) = A_k$, where $A_k$ is the $k$-th column of the matrix $A = (r_{ij})$. 
5 | EXAMPLES IN THE PRYM LOCUS

In this section, we describe some examples of families of covers satisfying condition (A). The first one has abelian Galois group $\tilde{G}$ and satisfies (B1) (hence (B)). The second and the third ones have abelian Galois group and do not satisfy (B1). We explicitly show that (B) holds using the techniques explained in Section 4. Example 5.4 has nonabelian Galois group and condition (B1) is not satisfied. In this case, we prove that it yields a Shimura subvariety of $A_{12}$ of dimension 2 using [17, Theorem 3.8] and a geometric description of the family. Example 5.5 satisfies condition (A) but not condition (B) and we show that it yields a variety, which is not totally geodesic, hence it is not Shimura.

To describe the examples, we first give the values of $\tilde{g}, g,$ and $b$. We also indicate a number $\#$ that refers to the numbering of the examples that we found with the MAGMA script with the given values of $r, \tilde{g}, g,$ and $b$. For the group $\tilde{G}$, we use the presentation and the name of the group given by MAGMA. In Example 5.5, we give the decomposition of $V_+$ and $V_-$ as a direct sum of irreducible representations of $\tilde{G}$, and also for the irreducible representations, we use the notation of MAGMA.

The first example that we describe satisfies condition (B1).

Example 5.1. $r = 5, \tilde{g} = 12, g = 6, b = 2$.

$\tilde{G} = G(10, 2) = \mathbb{Z}/10\mathbb{Z} = \langle g_1 \rangle$.

$(\tilde{\theta}(\gamma_1), \ldots, \tilde{\theta}(\gamma_5)) = (g_1, g_1, g_2, g_2, g_4), \quad \sigma = g_5^3$.

This is the family with equation given by $y^{10} = (x - t_1)(x - t_2)(x - t_3)^2(x - t_4)^2(x - t_5)^4$ and $\sigma(y) = -y$. Using the notation of the previous section, the matrix giving the monodromy is $(1, 1, 2, 2, 4)$. Hence, we have $d_1 = 3, d_3 = 2, d_7 = 1, d_9 = 0$, where $d_i = \dim(W_i), W_i = \{\omega \in H^0(K_{\tilde{C}}) \mid g_i(\omega) = \xi_{10}^i \omega\},$ with $\xi_{10}$ a primitive 10th root of unity. So we have $V_- = W_1 \oplus W_3 \oplus W_7$, hence $(S^2V_-)^{\tilde{G}} = W_3 \otimes W_7$ and $\dim(W_7) = 1$, therefore condition (B1) holds. So the family of Pryms yields a Shimura subvariety $\mathcal{P}_{6,2}(\mathcal{X}_{12}(\tilde{G}, \tilde{\theta}))$ of $\mathcal{A}_6$.

The next two examples do not satisfy condition (B1) but we show that they satisfy condition (B).

Example 5.2. $r = 6, \tilde{g} = 5, g = 3, b = 0$.

$G = G(8, 5) = \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2 = \langle g_1 \rangle \times \langle g_2 \rangle \times \langle g_3 \rangle$.

$(\tilde{\theta}(\gamma_1), \ldots, \tilde{\theta}(\gamma_6)) = (g_1, g_1, g_2, g_2, g_3, g_3), \quad \sigma = g_1g_2g_3$.

Hence, the matrix giving the monodromy is

$$A = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1
\end{pmatrix},$$

and the equations for the curves $\tilde{C}$ in the family are:

$$w_1^2 = (x - t_1)(x - t_2),$$
$$w_2^2 = (x - t_3)(x - t_4),$$
$$w_3^2 = (x - t_5)(x - t_6).$$

The action of $\sigma$ is in this case given by $w_i \mapsto -w_i$ for $i = 1, 2, 3$. One sees that $H^0(\tilde{C}, \omega_{\tilde{C}})_- = H^0(\tilde{C}, \omega_{\tilde{C}})_{-,(1,1,1)} = H^0(\tilde{C}, \omega_{\tilde{C}})_{(1,1,1)}$. By Lemma 4.3, we know that $d_{(1,1,1)} = \dim H^0(\tilde{C}, \omega_{\tilde{C}})_{-,(1,1,1)} = 2$. Hence, $\dim(S^2V_-)^{\tilde{G}} = 3$. We will show that condition (B) holds, hence, since $r = 6$, the family gives rise to a three-dimensional special subvariety in $A_2$, which therefore coincides with $A_2$. We have that

$$V_- = H^0(\tilde{C}, \omega_{\tilde{C}})_{(1,1,1)} = \langle \alpha_1 = w_1w_2w_3 \prod_{i=1}^{6}(x - t_i)^{-1}dx = \frac{dx}{w_1w_2w_3}, \alpha_2 = x\alpha_1 \rangle,$$
so that \((S^2V_-)^G = \langle \alpha_1 \odot \alpha_1, \alpha_1 \odot \alpha_2, \alpha_2 \odot \alpha_2 \rangle\). We have

\[
m(\alpha_1 \odot \alpha_1) = \frac{(dx)^2}{\prod_{i=1}^6(x-t_i)}, \quad m(\alpha_1 \odot \alpha_2) = \frac{x(dx)^2}{\prod_{i=1}^6(x-t_i)}, \quad m(\alpha_2 \odot \alpha_2) = \frac{x^2(dx)^2}{\prod_{i=1}^6(x-t_i)}.
\]

So \(v = a_1(\alpha_1 \odot \alpha_1) + a_2(\alpha_1 \odot \alpha_2) + a_3(\alpha_2 \odot \alpha_2) \in \ker(m)\) if and only if

\[
a_1 \frac{(dx)^2}{\prod_{i=1}^6(x-t_i)} + a_2 \frac{x(dx)^2}{\prod_{i=1}^6(x-t_i)} + a_3 \frac{x^2(dx)^2}{\prod_{i=1}^6(x-t_i)} = 0.
\]

It is straightforward to see that this holds if and only if \(a_1 = a_2 = a_3 = 0\). This shows that \(m\) is injective and by condition (A), it is an isomorphism, so condition (B) is satisfied. So we have shown that \(P_{3,0}(\mathbb{G}_5, \mathbb{G}_5) \subseteq \mathfrak{H}_2\) is a Shimura subvariety of dimension 3, hence it coincides with \(\mathfrak{H}_2\).

Notice that \(P(\tilde{C}, C) \sim J_2\), where \(C_2 = \tilde{C}/\langle g_1 g_2, g_1 g_3 \rangle\) is a genus 2 curve whose equation as a double cover of \(\mathbb{P}^1\) via the map \(C' \rightarrow C'/H, H = \mathbb{G}/\langle g_2 g_3, g_1 g_2, g_1 g_3, g_1 \rangle, \sigma = g_1 g_3\).

From the MAGMA script, we know that \(\dim(S^2V_-)^{\mathbb{G}} = 6\), hence condition (A) is satisfied, but (B1) is not. We will show that condition (B) holds. This shows that this family gives a six-dimensional Shimura subvariety of \(\mathfrak{H}_5\).

Denote by \((t_1, ..., t_9)\) the critical values of the covering \(\psi: \tilde{C} \rightarrow \tilde{C}/\mathbb{G} \cong \mathbb{P}^1\). One can easily show that \(P(\tilde{C}, C) \sim J_2\times F \times Y \times J_2\), where \(E, F, Y\) have genus 1, while \(X\) has genus 2. More precisely, the elliptic curve \(E\) is the quotient \(E = \tilde{C}/\langle g_1, g_2 \rangle\), and the double cover \(E \rightarrow E/L \cong \mathbb{P}^1\) where \(L = \mathbb{G}/\langle g_1, g_2, g_3 \rangle\) has an equation given by \(w_1 w_2 = (x-t_2)(x-t_6)(x-t_9)\). Finally, the genus 2 curve \(X\) is the quotient \(X = \tilde{C}/\langle g_1 g_2, g_3 \rangle\) and the double cover \(X \rightarrow X/L \cong \mathbb{P}^1\) where \(L = \mathbb{G}/\langle g_1 g_2, g_3 \rangle\) has an equation given by \(\theta = (x-t_1)(x-t_4)(x-t_5)(x-t_7)(x-t_8)(x-t_9)\).

Hence, we have \((S^2V_-)^{G} \cong S^2H^{1,0}(E) \oplus S^2H^{1,0}(F) \oplus S^2H^{1,0}(Y) \oplus S^2H^{1,0}(X)\).

The matrix giving the monodromy is the following:

\[
A = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0
\end{pmatrix}
\]

so the equations of \(C\) are

\[
w_1^2 = (x-t_1)(x-t_2)(x-t_6)(x-t_9),
\]
\[
w_2^2 = (x-t_2)(x-t_4)(x-t_5)(x-t_6)(x-t_7)(x-t_8),
\]
\[
w_3^2 = (x-t_2)(x-t_3)(x-t_5)(x-t_7).
\]

Notice that the first equation gives the elliptic curve \(F\), and the third one gives \(E\). The action of \(\sigma = g_1 g_3\) is the following \(\sigma(w_1) = -w_1, \sigma(w_2) = w_2, \sigma(w_3) = -w_3\). One immediately computes \(d_{g_1} = d_{(1,0,0)} = 1\) and \(\alpha_1 := \omega_{g_1,0} = \frac{dx}{w_1}\) is a generator of \(H^{1,0}(F)\). We have: \(d_{g_2} = 1\) and \(\alpha_2 := \omega_{g_2,0} = \frac{dx}{w_2}\) is a generator of \(H^{1,0}(E)\). We have \(d_{g_3} = 2\) and \(\alpha_3 := \omega_{g_3,0} = \frac{(x-t_2)(x-t_6)dx}{w_1 w_2}, \alpha_4 := \omega_{g_3,1} = \frac{x(x-t_2)(x-t_6)dx}{w_1 w_2} = x\alpha_3\). Moreover, \(H^{1,0}(X) = \langle \alpha_3, \alpha_4 \rangle\), since if we set \(\theta = \frac{w_1 w_2}{(x-t_2)(x-t_6)}\), we have \(\theta = (x-t_1)(x-t_4)(x-t_5)(x-t_7)(x-t_8)(x-t_9)\), hence \(\langle \alpha_3, \alpha_4 \rangle = \frac{dx}{\theta}, \alpha_4 = \frac{xdx}{\theta}\).
Finally, one computes $\frac{\omega_{2\Gamma_3}}{(x-t_3)(x-t_5)(x-t_7)}$ satisfies $\omega^2 = (x-t_3)(x-t_4)(x-t_6)(x-t_8)$ and $H^{1,0}(\Omega) = \langle \frac{dx}{\omega} = \alpha_5 \rangle$.

Therefore, we have

$$(S^2V_-)^\tilde{\Gamma} = \langle \alpha_1 \odot \alpha_2 \odot \alpha_2, \alpha_3 \odot \alpha_3, \alpha_3 \odot \alpha_4, \alpha_4 \odot \alpha_4, \alpha_5 \odot \alpha_5 \rangle.$$  

We have

$$m(\alpha_1 \odot \alpha_1) = \alpha_1^2 = \frac{(dx)^2}{w^2_1} = \frac{(dx)^2}{(x-t_1)(x-t_2)(x-t_6)(x-t_9)},$$

$$m(\alpha_2 \odot \alpha_2) = \alpha_2^2 = \frac{(dx)^2}{w^2_3} = \frac{(dx)^2}{(x-t_2)(x-t_3)(x-t_5)(x-t_7)},$$

$$m(\alpha_3 \odot \alpha_3) = \alpha_3^2 = \frac{(x-t_2)^2(x-t_6)^2(dx)^2}{w^2_1w^2_2} = \frac{(dx)^2}{(x-t_1)(x-t_4)(x-t_5)(x-t_7)(x-t_8)(x-t_9)},$$

$$m(\alpha_3 \odot \alpha_4) = \alpha_3\alpha_4 = x\alpha_3 \odot \alpha_3,$$

$$m(\alpha_4 \odot \alpha_4) = \alpha_4^2 = x^2m(\alpha_3 \odot \alpha_3),$$

$$m(\alpha_5 \odot \alpha_5) = \alpha_5^2 = \frac{(x-t_2)^2(x-t_3)^2(x-t_5)^2(dx)^2}{w^2_2w^2_3} = \frac{(dx)^2}{(x-t_3)(x-t_4)(x-t_6)(x-t_9)}.$$  

So, $v = a_1(\alpha_1 \odot \alpha_1) + a_2(\alpha_2 \odot \alpha_2) + a_3(\alpha_3 \odot \alpha_3) + a_4(\alpha_3 \odot \alpha_4) + a_5(\alpha_5 \odot \alpha_5) + a_6(\alpha_5 \odot \alpha_5)$ if and only if $a_1x^2 + a_2x^2 + a_3x^2 + a_4a_3a_4 + a_5x^2 + a_6x^2 = 0$ and this holds if and only if

$$a_1(x-t_1)(x-t_9)(x-t_3)(x-t_7)(x-t_6) + a_2(x-t_1)(x-t_4)(x-t_6)(x-t_9)(x-t_8) + a_3(x-t_2)(x-t_6)(x-t_9) + a_4x(x-t_2)(x-t_4)(x-t_6) + a_5x^2(x-t_3)(x-t_9) + a_6(x-t_1)(x-t_2)(x-t_3)(x-t_9)(x-t_7) = 0,$$

and one can easily show that the only solution is $a_i = 0, \forall i = 1, ..., 6$. This proves that $m$ is injective, hence by condition (A), it is an isomorphism, so condition (B) is satisfied.

Now we describe an example with nonabelian group $\tilde{\Gamma}$ that does not satisfy (B1), and we show that it gives a two-dimensional Shimura subvariety of $\mathcal{X}_{12}$ using [17, Theorem 3.8].

**Example 5.4.** $r = 5, g = 25, g = 13, b = 0$.  

$G = G(48, 32) = \mathbb{Z}/2 \times SL(2, 3) = \langle g_1 | g_1^2 = 1 \rangle \times \langle g_2, g_3, g_4, g_5 | g_1^2 = g_2^3 = 1, g_3^2 = g_4^2 = g_5, g_5^2 = 1, g_2^{-1}g_3g_2 = g_4, g_2^{-1}g_4g_2 = g_5, g_3^{-1}g_2g_3^{-1} = g_4, g_3^{-1}g_3g_3 = g_4g_3 \rangle$.

$(\tilde{\delta}(g_1) = g_1, \tilde{\delta}(g_2) = g_2, \tilde{\delta}(g_3) = g_3g_4, \tilde{\delta}(g_4) = g_2g_3g_4, \tilde{\delta}(g_5) = g_2, \sigma = g_5).$

Assume that the critical values of the map $\psi : C \to \tilde{C}/\tilde{\Gamma} \cong P^1$ are $(t_1, t_2, 0, 1, \infty)$. $V_4 = V_4 \oplus V_5 \oplus 2V_6 \oplus 3V_{13}$ and $V_- = 2V_7 \oplus 2V_8 \oplus V_9 \oplus V_{11}$, where $dimV_1 = 1$, $\forall i < 7$, $dimV_7 = dimV_8 = dimV_9 = dimV_{11} = 2$, $dimV_{13} = 3$. $(S^2V_-)_G \cong \Lambda^2V_7 \oplus \Lambda^2V_8$ has dimension 2. Thus condition (A) is satisfied, but (B1) is not. We want to show that the family yields a Shimura subvariety of $A_{12}$ of dimension 2. Notice that the center $Z$ of $G$ is the subgroup $Z = \langle g_1, g_5 \rangle$, and $V_4 = \text{Fix}(g_1, g_5)$, hence $E := \tilde{C}/Z$ is a genus 1 curve and it gives a degree 12 Galois cover $\varphi : E \to E/L = \tilde{C}/\tilde{\Gamma} \cong P^1$, where $L = \tilde{G}/Z$ with $H^{1,0}(E) = V_4$. It is immediate to check that the Galois cover $\varphi$ only ramifies over $(0, 1, \infty)$, hence the elliptic curve $E$ does not move.

We also have $\text{Fix}(g_1) = V_4 \oplus 2V_7 \oplus V_{11}$, $\text{Fix}(g_1g_5) = V_4 \oplus 2V_8 \oplus V_9$, hence if we set $C' = \tilde{C}/\langle g_1 \rangle$, $D' = \tilde{C}/\langle g_1g_5 \rangle$, we have $H^{1,0}(C') = V_4 \oplus 2V_7 \oplus V_{11}$, $H^{1,0}(D') = V_4 \oplus 2V_8 \oplus V_9$. Both $\langle g_1 \rangle$ and $\langle g_1g_5 \rangle$ are normal subgroups and both quotients $H := \tilde{G}/\langle g_1 \rangle$ and $K := \tilde{G}/\langle g_1g_5 \rangle$ are isomorphic to $SL(2, 3)$. So we have two $SL(2, 3)$-Galois covers
\(\alpha : C' \to C'/H = \tilde{C}/\tilde{G} \cong \mathbb{P}^1, \beta : D' \to C'/K = \tilde{C}/\tilde{G} \cong \mathbb{P}^1\) and a commutative diagram.

\[
\begin{array}{c}
\tilde{C}/\langle g_1 \rangle = C' \\
\alpha \downarrow \\
\tilde{C}/\langle g_1 g_5 \rangle = D' \\
\beta \downarrow \\
\tilde{C}/\tilde{G} = \mathbb{P}^1
\end{array}
\]

The two families of Galois covers \(\alpha : C'_1 \to C'/H \cong \mathbb{P}^1\) and \(\beta : D'_1 \to D'/K \cong \mathbb{P}^1\) yield Shimura curves. In fact, they both satisfy condition (\(*\)) of [16] since \((S^2H^0(K_{C'}))^H \cong \Lambda^2V_7\) and \((S^2H^0(K_{D'}))^K \cong \Lambda^2V_8\), which are both one dimensional, and they coincide with family (40) of [16]. The critical values of \(\alpha\) and \(\beta\) are, respectively, \((t_1, 0, 1, \infty)\), \((t_2, 0, 1, \infty)\) with the same monodromy. We have isogenies \(J_{C'_1} \sim E \times P(C'_1, E)\), \(J_{D'_1} \sim E \times P(D'_1, E)\), where \(E\) is fixed.

Therefore, the one-dimensional families of Pryms \(P(C'_1, E)\) and \(P(D'_1, E)\) yield Shimura curves in the moduli space of polarized abelian varieties \(\mathcal{A}_{g_6}^2\). Denote by \(W_1 := \mathcal{P}_{13,0}(\mathbb{Z}_2, \tilde{g}_1)\) and consider the two-dimensional Shimura subvariety \(W_2\) of \(\mathcal{A}_{g_6}^2 \times \mathcal{A}_{g_6}^2 \subset \mathcal{A}_{g_12}^2\) given by the family \(P(C'_1, E) \times P(D'_1, E)\). For any \((t_1, t_2) \in (\mathbb{P}^1 \setminus \{0, 1, \infty\}) \times (\mathbb{P}^1 \setminus \{0, 1, \infty\})\) with \(t_1 \neq t_2\), we have an isogeny \(P(\tilde{C}_{(t_1,t_2)}, C_{(t_1,t_2)}) \sim P(C'_1, E) \times P(D'_1, E)\). Hence, there exists an open subset \(U\) of \(W_2\) such that for any abelian variety \(A \in W_2\) there exists an abelian variety \(B \in W_1\) such that \(A\) is isogenous to \(B\). So, applying [17, Theorem 3.8], we see that \(2 = \dim(W_2) \leq \dim(W_1) \leq 2\), therefore \(\dim(W_1) = 2\) and by [17, Theorem 3.8], it is also totally geodesic. Moreover \(W_2\) is Shimura, hence it exist infinitely many \(t_1\) and \(t_2\) such that both \(P(C'_1, E)\) and \(P(D'_1, E)\) have complex multiplication. So also the family of Pryms \(P(\tilde{C}_{(t_1,t_2)}, C_{(t_1,t_2)})\) has a CM point, hence \(W_1\) is a Shimura subvariety of \(\mathcal{A}_{12}\) of dimension 2.

Finally, we give an example satisfying condition (A) but not condition (B) and we show that it yields a variety, which is not totally geodesic, hence it is not Shimura. This shows that condition (A) alone is not sufficient to ensure that the variety is a Shimura variety.

**Example 5.5.** \(r = 10, g = 7, g = 3, b = 4\).

\(\tilde{G} = G(4,2) = \mathbb{Z}/2 \times \mathbb{Z}/2 = \langle g_1 \rangle \times \langle g_2 \rangle\),

\((\tilde{\theta}(\gamma_1), ..., \tilde{\theta}(\gamma_{10})) = (g_1, g_1, g_2, g_2, g_1, g_1, g_1, g_2, g_2, g_1), \quad \sigma = g_1 g_2\).

From the MAGMA script, we know that \(\dim(S^2V_7)^{\tilde{G}} = 7\), hence condition (A) is satisfied, but (B1) is not. We will show that condition (B) does not hold and that the family is not totally geodesic, hence it is not Shimura.

Denote by \((t_1, ..., t_{10})\) the critical values of the covering \(\psi : \tilde{C} \to \tilde{C}/\tilde{G} \cong \mathbb{P}^1\). One can easily show that \(P(\tilde{C}, C) \sim E \times JC'\), where \(C'\) is genus 3, while \(E\) has genus 1. More precisely, the genus 3 curve \(C'\) is the quotient \(\tilde{C}/\langle g_2 \rangle\) and double cover \(C' \to C'/N \cong \mathbb{P}^1\), where \(N = \tilde{G}/\langle g_2 \rangle\) has an equation given by \(w_1^2 = (x - t_1)(x - t_2)(x - t_3)(x - t_5)(x - t_6)(x - t_7)(x - t_8)(x - t_9)(x - t_{10})\). The elliptic curve \(E\) is the quotient \(E = \tilde{C}/\langle g_1 \rangle\), and if we set \(H = \tilde{G}/\langle g_1 \rangle \cong \mathbb{Z}/2\), the double cover \(E \to E/H \cong \mathbb{P}^1\) ramifies over \([t_3, t_4, t_5, t_6]\), hence \(E\) has an equation given by \(w_2^2 = (x - t_3)(x - t_4)(x - t_5)(x - t_9)\).

Therefore, we have \((S^2V_7)^{\tilde{G}} \cong S^2H^{10}(E) \oplus S^2H^{10}(C')\) and it has dimension 7.

The matrix giving the monodromy is the following:

\[
A = \begin{pmatrix}
1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0
\end{pmatrix},
\]

so the equations of \(\tilde{C}\) are

\[
w_1^2 = (x - t_1)(x - t_2)(x - t_3)(x - t_5)(x - t_6)(x - t_7)(x - t_8)(x - t_{10}),
\]

\[
w_2^2 = (x - t_3)(x - t_4)(x - t_5)(x - t_9).
\]

Notice that the first equation gives the curve \(C'\), while the second one gives \(E\).
The action of \( \sigma = g_1 g_2 \) is the following: \( \sigma(w_1) = -w_1, \sigma(w_2) = -w_2 \). One immediately computes \( d_{g_2} = d_{(0,1)} = 1 \) and \( \alpha_1 := \omega_{g_1,0} = \frac{dx}{w_2} \) is a generator of \( H^{1,0}(E) \). We have: \( d_{g_1} = d_{(1,0)} = 3 \) and \( \{ \alpha_2 := \omega_{g_1,0} = \frac{dx}{w_1}, \alpha_3 := \omega_{g_1,1} = \frac{xdx}{w_1}, \alpha_4 := \omega_{g_1,2} = \frac{x^2 dx}{w_1} \} \) is a basis of \( H^{1,0}(C') \). Therefore, we have

\[
(S^2 V_-)^\hat{\gamma} = (\alpha_1 \otimes \alpha_1, \alpha_2 \otimes \alpha_2, \alpha_2 \otimes \alpha_3, \alpha_2 \otimes \alpha_4, \alpha_3 \otimes \alpha_3, \alpha_3 \otimes \alpha_4, \alpha_4 \otimes \alpha_4)
\]  

(5.2)

Moreover,

\[
m(\alpha_1 \otimes \alpha_1) = \alpha_1^2 = \frac{(dx)^2}{w_2^2} = \frac{(dx)^2}{(x-t_3)(x-t_6)(x-t_9)},
\]

\[
m(\alpha_2 \otimes \alpha_2) = \alpha_2^2 = \frac{(dx)^2}{(x-t_1)(x-t_2)(x-t_3)(x-t_5)(x-t_6)(x-t_7)(x-t_8)(x-t_{10})},
\]

\[
m(\alpha_2 \otimes \alpha_3) = \alpha_2 \alpha_3 = x \alpha_2, \quad m(\alpha_2 \otimes \alpha_4) = \alpha_2 \alpha_4 = x^2 \alpha_2, \quad m(\alpha_3 \otimes \alpha_3) = \alpha_3^2 = x^2 \alpha_2^2,
\]

\[
m(\alpha_3 \otimes \alpha_4) = \alpha_3 \alpha_4 = x^3 \alpha_2, \quad m(\alpha_4 \otimes \alpha_4) = \alpha_4^2 = x^4 \alpha_2^2.
\]

So, \( Q = a_1(\alpha_1 \otimes \alpha_1) + a_2(\alpha_2 \otimes \alpha_2) + a_3(\alpha_2 \otimes \alpha_3) + a_4(\alpha_2 \otimes \alpha_4) + a_5(\alpha_3 \otimes \alpha_3) + a_6(\alpha_4 \otimes \alpha_4) \in \ker(m) \) if and only if \( a_1 \alpha_1^2 + a_2 \alpha_2^2 + a_3 \alpha_2 \alpha_3 + a_4 \alpha_2 \alpha_4 + a_5 \alpha_3 \alpha_3 + a_6 \alpha_4 \alpha_4 = 0 \) and this holds if and only if

\[
a_1(x-t_1)(x-t_2)(x-t_3)(x-t_6)(x-t_7)(x-t_{10}) + (a_2 + a_3 x + (a_4 + a_5) x^2 + a_6 x^3 + a_7 x^4)(x-t_5)(x-t_9) = 0.
\]

One can verify that this holds if and only if \( a_1 = a_2 = a_3 = a_5 = a_7 = 0, a_4 = -a_6, \) hence

\[
\ker(m) = (Q = \alpha_2 \otimes \alpha_4 - \alpha_3 \otimes \alpha_3),
\]

so condition (B) is not satisfied.

Notice that since \( m \) has rank 6, the variety \( P_{3,4}(X_{\gamma}(G, \delta)) =: W_1 \) has dimension 6.

Now we show that \( W_1 \subset A^6_4 \) is not totally geodesic.

Consider the subvariety \( W_2 \) of \( A_4 \) given by \( A_1 \times \overline{j(HE_3)} \), where \( HE_3 \) denotes the hyperelliptic locus in \( M_3 \) and \( j : M_3 \to A_3 \) is the Torelli map. We have shown that there exists an open subset \( U \) of \( W_1 \) such that for every abelian variety \( A = P(\hat{C}, C) \in U \), there exists an abelian variety \( B \in W_2 \) such that \( A \) is isogenous to \( B = E \times JC' \). Since \( W_1 \) and \( W_2 \) have the same dimension, [17, Theorem 3.8] implies that \( W_1 \) is totally geodesic in \( A^6_4 \) if and only if \( W_2 \) is totally geodesic in \( A_4 \). So, assume by contradiction that \( W_1 \) is totally geodesic in \( A^6_4 \), then \( W_2 \subset A_1 \times A_3 \subset A_4 \) would be totally geodesic in \( A_4 \), hence it also would be totally geodesic in \( A_1 \times A_3 \), since \( A_1 \times A_3 \) is totally geodesic in \( A_4 \). This would imply that \( \overline{j(HE_3)} \) is totally geodesic in \( A_3 \), which is not true, as one can see applying [16, Corollary 5.14], since the hyperelliptic locus has codimension 1 in \( M_3 \).

Another way to prove that \( \overline{j(HE_3)} \) is not totally geodesic in \( A_3 \) is via a direct computation of the second fundamental form of the Torelli map that can be done as in [12, Proposition 5.4]. In fact, with the notation of [12, Proposition 5.4] one can show that \( \rho(Q)(v \otimes v) \neq 0 \), where \( v \) is a tangent vector to \( W_2 \) at \( JC' \) given by the sum of two Schiffer variations \( v = \xi_p + \xi_q \), at two general distinct points \( p, q \in C' \) that are exchanged by the hyperelliptic involution.

Hence, we have proven that \( W_1 = P_{3,4}(X_{\gamma}(G, \delta)) \) is not totally geodesic in \( A^6_4 \), therefore it is not Shimura.

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APPENDIX A

The MAGMA script used to find the examples is available at: http://mate.unipv.it/grosselli/publ/ and it is described in the appendix in [18]. In the script, we used some tricks to avoid computations following [15]. The script tests condition (A), next if either (B1) is met or $b \geq 6$ then condition (B) holds. The method described in Section 4 is used to test (B) for many specific abelian cases. In particular, in the cases with $r = 8, 9$, the families satisfying condition (B) for $g \leq 20$ and $\bar{G}$ abelian are exactly the ones listed in the table.

The following table lists all the calculated cases in which condition (B) holds. The data are listed sorted by $r, \bar{g},$ and $g$ (and thus $b$ and $p = \bar{g} – g$), then a progressive index (under column #) allows to distinguish cases with the same values. For each case, the group $\bar{G}$ and the id of $\bar{G}$ and $G$ as small group in the MAGMA Database is reported. A check mark tells if the conditions are met. Different examples with same data are grouped in the same row.

| $r$ | $\bar{g}$ | $g$ | $b$ | $p$ | # | $\bar{G}$ | $G$ 1d | $\bar{G}$ 1d | $G$ 1d | (B1) | $b \geq 6$ | (B) |
|-----|---------|-----|-----|-----|---|--------|------|---------|------|-----|-----------|-----|
| 5   | 2       | 0   | 6   | 2   | 1 | $C_2^2$ | $G(4,2)$ | $G(2,1)$ | ✔   | ✔   | ✔         |     |
| 5   | 3       | 0   | 8   | 3   | 1 | $C_4$   | $G(4,1)$ | $G(2,1)$ | ✔   | ✔   | ✔         |     |
| 5   | 4       | 1   | 6   | 3   | 1 | $C_4$   | $G(4,1)$ | $G(2,1)$ | ✔   | ✔   | ✔         |     |
| 5   | 4       | 1   | 6   | 3   | 2 | $C_6$   | $G(6,2)$ | $G(3,1)$ | ✔   | ✔   | ✔         |     |
| 5   | 4       | 1   | 6   | 3   | 3 | $C_1 \rtimes C_4$ | $G(12,4)$ | $G(6,1)$ | ✔   | ✔   | ✔         |     |
| 5   | 5       | 2   | 4   | 3   | 1 | $C_6$   | $G(6,2)$ | $G(3,1)$ | ✔   | ✔   | ✔         |     |
| 5   | 5       | 2   | 4   | 3   | 2 | $C_2 \rtimes C_4$ | $G(8,2)$ | $G(4,2)$ | ✔   | ✔   | ✔         |     |
| 5   | 5       | 1   | 8   | 4   | 1 | $C_2 \rtimes C_4$ | $G(8,2)$ | $G(4,2)$ | ✔   | ✔   | ✔         |     |
| 5   | 5       | 1   | 8   | 4   | 2 | $C_2 \rtimes C_4$ | $G(8,2)$ | $G(4,2)$ | ✔   | ✔   | ✔         |     |
| 5   | 5       | 1   | 8   | 4   | 3 | $D_8$   | $G(16,7)$ | $G(8,3)$ | ✔   | ✔   | ✔         |     |
| 5   | 5       | 1   | 8   | 4   | 4 | $C_1 \rtimes D_4$ | $G(16,11)$ | $G(8,5)$ | ✔   | ✔   | ✔         |     |
| 5   | 5       | 2   | 4   | 3   | 1,2 | $C_6$ | $G(6,2)$ | $G(3,1)$ | ✔   | ✔   | ✔         |     |
| 5   | 5       | 2   | 4   | 3   | 1,2,3 | $C_2 \rtimes C_6$ | $G(12,5)$ | $G(6,2)$ | ✔   | ✔   | ✔         |     |
| 5   | 7       | 4   | 0   | 3   | 1,2 | $C_2 \rtimes C_4$ | $G(8,2)$ | $G(4,1)$ | ✔   | ✔   | ✔         |     |
| 5   | 7       | 1   | 12  | 6   | 1 | $C_4 \rtimes D_4$ | $G(16,13)$ | $G(8,5)$ | ✔   | ✔   | ✔         |     |
| 5   | 7       | 2   | 8   | 5   | 1 | $C_2 \rtimes C_4$ | $G(8,2)$ | $G(4,2)$ | ✔   | ✔   | ✔         |     |
| 5   | 8       | 4   | 2   | 4   | 2 | $C_2 \rtimes C_6$ | $G(12,5)$ | $G(6,2)$ | ✔   | ✔   | ✔         |     |
| 5   | 9       | 5   | 0   | 4   | 1 | $C_2 \rtimes C_6$ | $G(12,5)$ | $G(6,2)$ | ✔   | ✔   | ✔         |     |
| 5   | 9       | 5   | 0   | 4   | 1,6 | $C_2^2 \rtimes C_4$ | $G(16,10)$ | $G(8,2)$ | ✔   | ✔   | ✔         |     |
| 5   | 9       | 3   | 8   | 6   | 1 | $C_8$   | $G(8,1)$ | $G(4,1)$ | ✔   | ✔   | ✔         |     |
| 5   | 9       | 3   | 8   | 6   | 2 | $C_8$   | $G(8,1)$ | $G(4,1)$ | ✔   | ✔   | ✔         |     |
| 5   | 9       | 3   | 8   | 6   | 3 | $C_2^2 \rtimes C_4$ | $G(16,10)$ | $G(8,5)$ | ✔   | ✔   | ✔         |     |
| 5   | 9       | 3   | 8   | 6   | 4 | $C_2^2 \rtimes C_4$ | $G(16,10)$ | $G(8,5)$ | ✔   | ✔   | ✔         |     |
| 5   | 9       | 3   | 8   | 6   | 5 | $C_4 \rtimes D_4$ | $G(16,13)$ | $G(8,5)$ | ✔   | ✔   | ✔         |     |
| r  | $g$ | $g$ | $b$ | $p$ | $b$ | $G = C_2 \times C_6$ | $G = G(12, 5)$ | $G = G(6, 2)$ |
|----|-----|-----|-----|-----|-----|----------------------|-------------------|-----------------|
| 5  | 10  | 5   | 2   | 5   | 1   | $C_2 \times C_6$    | $G(12, 5)$       | $G(6, 2)$       |
| 5  | 10  | 5   | 2   | 5   | 2,3 | $C_2 \times C_6$    | $G(12, 5)$       | $G(6, 2)$       |
| 5  | 10  | 4   | 6   | 6   | 1   | $C_6$               | $G(8, 1)$        | $G(4, 1)$       |
| 5  | 10  | 4   | 6   | 6   | 2   | $C_2 \times C_6$    | $G(12, 5)$       | $G(6, 2)$       |
| 5  | 10  | 4   | 6   | 6   | 3   | $Q_8 \rtimes C_2$   | $G(16, 8)$       | $G(8, 3)$       |
| 5  | 10  | 4   | 6   | 6   | 4   | $C_3 \rtimes D_4$   | $G(24, 8)$       | $G(12, 4)$      |
| 5  | 11  | 3   | 12  | 8   | 1   | $C_2 \times Q_8$    | $G(16, 12)$      | $G(8, 5)$       |
| 5  | 11  | 4   | 8   | 7   | 1   | $C_4 \rtimes C_4$   | $G(16, 4)$       | $G(8, 3)$       |
| 5  | 12  | 6   | 2   | 6   | 1,2 | $C_{10}$            | $G(10, 2)$       | $G(5, 1)$       |
| 5  | 13  | 7   | 0   | 6   | 3   | $C_4^2$             | $G(16, 22)$      | $G(8, 2)$       |
| 5  | 13  | 7   | 0   | 6   | 4   | $C_4^2$             | $G(16, 22)$      | $G(8, 2)$       |
| 5  | 13  | 5   | 8   | 8   | 1   | $C_2 \times C_8$    | $G(16, 5)$       | $G(8, 2)$       |
| 5  | 13  | 5   | 8   | 8   | 2   | $C_4 \rtimes D_4$   | $G(32, 42)$      | $G(16, 11)$     |
| 5  | 13  | 5   | 8   | 8   | 3,4 | $C_2 \times C_1\rtimes D_4$ | $G(32, 42)$ | $G(16, 14)$     |
| 5  | 15  | 12  | 10  | 1   | $C_6 \rtimes C_4$   | $G(24, 7)$       | $G(12, 4)$      |
| 5  | 15  | 7   | 4   | 8   | 2   | $C_{12}$            | $G(12, 2)$       | $G(6, 2)$       |
| 5  | 16  | 8   | 2   | 8   | 1   | $C_2 \times C_{10}$ | $G(20, 5)$       | $G(10, 2)$      |
| 5  | 21  | 9   | 8   | 12  | 1   | $C_8 \rtimes D_4$   | $G(32, 38)$      | $G(16, 10)$     |
| 5  | 29  | 13  | 8   | 16  | 1   | $Q_8 \rtimes D_8$   | $G(64, 259)$     | $G(32, 46)$     |
| 6  | 2   | 0   | 6   | 2   | 1   | $C_2$               | $G(2, 1)$        | $G(1, 1)$       |
| 6  | 3   | 1   | 4   | 2   | 1   | $C_4^2$             | $G(4, 2)$        | $G(2, 1)$       |
| 6  | 5   | 3   | 0   | 2   | 1   | $C_4^2$             | $G(4, 2)$        | $G(2, 1)$       |
| 6  | 5   | 1   | 8   | 4   | 1   | $C_4$               | $G(4, 1)$        | $G(2, 1)$       |
| 6  | 5   | 1   | 8   | 4   | 2   | $D_4$               | $G(8, 3)$        | $G(4, 2)$       |
| 6  | 6   | 2   | 6   | 4   | 1   | $C_4$               | $G(4, 1)$        | $G(2, 1)$       |
| 6  | 6   | 2   | 6   | 4   | 2   | $D_4$               | $G(8, 3)$        | $G(4, 2)$       |
| 6  | 7   | 2   | 8   | 5   | 1   | $C_2 \times C_4$    | $G(8, 2)$        | $G(4, 2)$       |
| 6  | 8   | 4   | 2   | 4   | 1   | $C_6$               | $G(6, 2)$        | $G(3, 1)$       |
| 6  | 9   | 5   | 0   | 4   | 3   | $C_2 \times C_4$    | $G(8, 2)$        | $G(4, 1)$       |
| 6  | 9   | 3   | 8   | 6   | 1   | $C_2 \times C_4$    | $G(8, 2)$        | $G(4, 2)$       |
| 6  | 10  | 4   | 6   | 6   | 1   | $C_6$               | $G(6, 2)$        | $G(3, 1)$       |
| 6  | 13  | 7   | 0   | 6   | 1   | $C_2 \times C_6$    | $G(12, 5)$       | $G(6, 2)$       |
| 6  | 13  | 5   | 8   | 8   | 1   | $C_4 \rtimes D_4$   | $G(16, 13)$      | $G(8, 5)$       |
| 6  | 14  | 6   | 6   | 8   | 1   | $C_8$               | $G(8, 1)$        | $G(4, 1)$       |
| 6  | 17  | 5   | 16  | 12  | 1   | $D_8 \rtimes Q_8$   | $G(32, 50)$      | $G(16, 14)$     |
| 7  | 4   | 1   | 6   | 3   | 1,2 | $C_2^2$             | $G(4, 2)$        | $G(2, 1)$       |
| 7  | 7   | 2   | 8   | 5   | 1   | $C_4$               | $G(4, 1)$        | $G(2, 1)$       |
| 7  | 9   | 4   | 4   | 5   | 2   | $C_2 \times C_4$    | $G(8, 2)$        | $G(4, 1)$       |
| 7  | 9   | 3   | 8   | 6   | 1   | $C_2 \times C_4$    | $G(8, 2)$        | $G(4, 2)$       |
| 7  | 10  | 5   | 2   | 5   | 1   | $C_6$               | $G(6, 2)$        | $G(3, 1)$       |
| 7  | 11  | 6   | 0   | 5   | 1,2,3 | $C_2 \times C_4$    | $G(8, 2)$       | $G(4, 1)$       |
| 7  | 13  | 7   | 0   | 6   | 2   | $C_2 \times C_6$    | $G(12, 5)$       | $G(6, 2)$       |
| 7  | 13  | 5   | 8   | 8   | 1   | $C_4 \rtimes D_4$   | $G(16, 13)$      | $G(8, 5)$       |
| 7  | 9   | 5   | 0   | 4   | 3–9 | $C_2^2$             | $G(8, 5)$        | $G(4, 2)$       |
| 8  | 9   | 3   | 8   | 6   | 1   | $C_4$               | $G(4, 1)$        | $G(2, 1)$       |
| 8  | 13  | 7   | 0   | 6   | 1   | $C_2 \times C_4$    | $G(8, 2)$        | $G(4, 1)$       |
| 9  | 6   | 2   | 6   | 4   | 1,2,3 | $C_2^2$             | $G(4, 2)$       | $G(2, 1)$       |
| 9  | 11  | 6   | 0   | 5   | 2,3,4 | $C_2^3$             | $G(8, 5)$       | $G(4, 2)$       |