A new Kernel Regression approach for Robustified L$_2$ Boosting

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**ABSTRACT**

We investigate $L_2$ boosting in the context of kernel regression. Kernel smoothers, in general, lack appealing traits like symmetry and positive definiteness, which are critical not only for understanding theoretical aspects but also for achieving good practical performance. We consider a projection-based smoother (Huang and Chen, 2008) that is symmetric, positive definite, and shrinking. Theoretical results based on the orthonormal decomposition of the smoother reveal additional insights into the boosting algorithm. In our asymptotic framework, we may replace the full-rank smoother with a low-rank approximation. We demonstrate that the smoother’s low-rank $(d(n))$ is bounded above by $O(h^{-1})$, where $h$ is the bandwidth. Our numerical findings show that, in terms of prediction accuracy, low-rank smoothers may outperform full-rank smoothers. Furthermore, we show that the boosting estimator with low-rank smoother achieves the optimal convergence rate. Finally, to improve the performance of the boosting algorithm in the presence of outliers, we propose a novel robustified boosting algorithm which can be used with any smoother discussed in the study. We investigate the numerical performance of the proposed approaches using simulations and a real-world case.

1. Introduction

Boosting, also known as Gradient boosting, is a popular machine learning method. It also garnered a lot of interest from the statistics community. Schapire (1990) suggested boosting to improve the performance of a fitting method, called a weak learner. Following that, various attempts from both statistics and machine learning communities have been made to demystify the boosting method’s superior performance and resistance to overfitting (Freund, 1995; Freund and Schapire, 1996; Bartlett et al., 1998; Breiman, 1998, 1999; Schapire and Singer, 1999; Friedman et al., 2000; Friedman, 2001; Bühlmann and Yu, 2003; Park et al., 2009).

Theoretical underpinnings and practical implementations of boosting are predicated on the assumption that it may be thought of as a functional gradient descent algorithm (Breiman, 1999). For example, under appropriate risk functions, AdaBoost (Freund and Schapire, 1996) and LogitBoost (Friedman et al., 2000) algorithms may be seen as optimization problems. Friedman (2001) proposed least-squares boosting ($L_2$ boosting), a computationally simple variation of boosting, and explored some robust algorithms that used regression trees as weak learners.

We use a univariate nonparametric regression model in this work. Assume $(X_i, Y_i), i = 1, \ldots, n$, are $n$ independent copies of a random pair $(X, Y)$. The following regression model is of interest to us:

$$Y_i = m(X_i) + \epsilon_i, \quad i = 1, \ldots, n,$$

where $m(\cdot) = E(Y|X = \cdot)$ is the regression function and $\epsilon_i$’s are random variables with $E(\epsilon_i) = 0$ and $\text{var}(\epsilon_i) = \sigma^2$. Bühlmann and Yu (2003) investigated the properties of $L_2$ boosting and presented expressions for average squared bias and average variance of a boosting estimate for the model (1). These expressions involve eigenvalues and eigenvectors of the corresponding smoother. When the smoother’s eigenvalues are between 0 and 1, the squared bias decays exponentially rapidly and the variance increases exponentially small as the number of boosting iterations increases (Bühlmann and Yu, 2003). The number of boosting iterations is treated as a tuning or regularization parameter in this exponential bias-variance trade-off in the literature. In addition, Bühlmann and Yu (2003) shows that $L_2$ boosting with smoothing splines achieves optimal rate $n^{-2r/(2r+1)}$ if the iteration number $b$ is of order $O(n^{2r/(2r+1)})$ as the sample size $n$ goes to infinity, where $r$ is the order of the smoothing spline, and $\pi(> r)$ is the smoothness index of the regression function.

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One intriguing point of view for boosting, as mentioned in Di Marzio and Taylor (2008), is that it may be used as a bias reduction method, particularly in the kernel smoothing framework. This may be traced back to Tukey (1977) who refers to one-step boosting as “twicing”. Twicing a kernel smoother is asymptotically equivalent to employing a higher-order kernel (Stuetzle and Mittal, 1979), in the case of fixed equispaced n design points, i/n, i = 1, …, n. In the kernel smoothing framework, L₂ boosting has received little attention in the literature. Di Marzio and Taylor (2008) proposed Nadaraya-Watson L₂ boost algorithm and demonstrated its empirical performance. Their smoother is not symmetric, hence it does not provide positive characteristic roots for several common kernels such as Epanechnikov, Biweight, and Triweight. If the smoother has eigenvalues outside of (0, 1], boosting will not operate effectively (Bühlmann and Yu, 2003). As a result, their method works effectively for only kernels with strictly positive eigenvalues, such as Gaussian and Triangular. Di Marzio and Taylor (2008) additionally shows that their estimator achieves bias reduction after the first boosting iteration while maintaining the order of the variance asymptotically the same. Park et al. (2009) also considers Nadaraya-Watson L₂ boosting and shows that if the iteration number b is big enough and the bandwidth is appropriately set, h = O(n^−1/(2x+1)), it achieves the optimal rate of convergence.

Low-rank matrix approximation is widely studied in the literature, especially in machine learning and statistics. It is a popular technique in massive data analysis. For details, we refer to Kishore Kumar and Schneider (2017) and the references therein. Let A be n × n real, symmetric matrix, then we write its rank d(n)(< n) (low-rank) approximation as

\[ A_{n \times n} \approx B_{n \times (d(n))} B^T_{(d(n)) \times n} \]

where B is a rank d(n) matrix. This approximation is very economical for storage as it requires only nk elements to be stored instead of the original n² elements. In addition, the low-rank approximation can also be used to remove noise in the data. In applications, the rank to be removed often corresponds to the noise level where the signal-to-noise ratio is low (Chu et al., 2003). Boosting is known to produce superior performance if the base learner is weak. The low-rank approximation allows us to run the boosting method on a large sample size without having to store all of the smoother's elements. In Section 4 we provide the robustified boosting algorithm which is not large-memory consuming and does not require the full-rank smoother. Our asymptotic framework allows us to substitute the high-rank smoother with a low-rank approximation, allowing us to run the boosting method on a large sample size without having to store all of the smoother’s elements. We also provide a bound for the approximation error and show that it converges to zero when the low-rank approximation achieves the optimal convergence rate, n^−2x/(2x+1), for an appropriately chosen bandwidth, h = O(n^−1/(2x+1)), given the smoothness index x, under some regularity conditions.

The L₂ boosting, which employs a L₂ loss function, is sensitive to outliers in the data. Robustification of the boosting method has not received much attention in the literature. By using regression trees as base learners, Friedman (2001) described a few robust boosting algorithms. Lutz et al. (2008) suggested five robustification algorithms for linear regression using L₂ boosting. In this paper, we offer a novel robust boosting algorithm to estimate the model (1). The proposed method uses a pseudo-outcome approach (Cox, 1983; Oh et al., 2007, 2008), which converts the original problem of robust loss function optimization to the problem of least-squares loss function optimization. At each stage of the boosting algorithm, this procedure is repeated. We further demonstrate in Theorem 3 that the estimate based on the pseudo-outcome is asymptotically equivalent to the estimator obtained directly by optimizing the robust loss function. This robustified boosting algorithm is very general and can be used with any smoother. In our numerical study, we employed this algorithm to Nadaraya-Watson smoother and spline smoother in addition to the projection-based smoother.

The paper is organized as follows. Section 2 provides a brief introduction to the projection-based smoother matrix (Huang and Chen, 2008). Section 3 outlines the algorithm for L₂ boosting and discusses the asymptotic properties of the boosting estimate of a low-rank smoother. In Section 4 we provide the robustified boosting algorithm which is not
sensitive to outliers. Section 5 discusses the simulation results and Section 6 illustrates the usefulness of the proposed methods using the data from a real-world application. We summarize our findings in Section 7.

2. Background for Smoother Matrix

We consider the local linear modeling approach (Fan and Gijbels, 2018) which estimates the regression function $m(x)$ in model (1) using a first-order Taylor expansion $m(x) + m^{(1)}(x)(X - x)$ for $X$ in a neighborhood of $x$. Let

$$X^T_x = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ X_1 - x & X_2 - x & \cdots & X_n - x \end{bmatrix}$$

be a design matrix and $W_x = \text{diag}\{K_h(X_1 - x), \ldots, K_h(X_n - x)\}$ be a weight matrix with $K_h(\cdot) = K(\cdot/h)/h$ where $K(\cdot)$ is a positive and symmetric probability density function defined on a compact support, say $[-1, 1]$, and $h$ is a bandwidth. Denote the response vector $y = (Y_1, \ldots, Y_n)^T$ and coefficient vector $\beta = (\beta_0, \beta_1)^T$. Then

$$(\hat{m}(x), \hat{m}^{(1)}(x)) = \min_{\beta_0, \beta_1} \frac{1}{n} \sum_{i=1}^n (Y_i - \hat{\beta}_0 - \beta_1(X_i - x))^2 K_h(X_i - x)$$

$$= \min_{\beta_0, \beta_1} \left( y - X_x \beta \right)^T W_x \left( y - X_x \beta \right). \tag{2}$$

Suppose the random variable $X$ is compactly supported, say $[0, 1]$. Let $K_h(u, v)$ be the boundary corrected kernel defined in Mammen et al. (1999) as

$$K_h(u, v) = \frac{K_h(u - v)}{\int K_h(w - v)dw} I(u, v \in [0, 1]),$$

where $I(\cdot)$ is an indication function. The smoother matrix $H^*_1$ (for local linear) in Huang and Chen (2008) is based on integrating local least-squares errors (2)

$$\frac{1}{n} \int \sum_{i=1}^n (Y_i - \hat{m}(x) - \hat{m}^{(1)}(x)(X_i - x))^2$$

$$\times K_h(x, X_i)dx = \frac{1}{n} y^T \left( I - H^*_1 \right) y, \tag{3}$$

where $I$ is an $n$-dimensional identity matrix and

$$H^*_1 = \int W_x X_x (X^T_x W_x X_x)^{-1} X^T_x W_x dx$$

is a local linear smoother with its $(i, j)$th element, $w_j(X_i)$, is defined as

$$w_j(X_i) = \int \left[ K_h(x, X_i) (X_i - x) K_h(x, X_i) \right] \left( X^T_x W_x X_x \right)^{-1}$$

$$\times \left[ K_h(x, X_j) (X_j - x) K_h(x, X_j) \right] dx. \tag{4}$$

It is worth mentioning that $\sum_{j=1}^n w_j(X_i) = 1$, $\sum_{i=1}^n w_j(X_i) = 1$, and $\sum_{j=1}^n w_j^2(X_i) = O_p(n^{-1}h^{-1})$ for $i = 1, \ldots, n$. The $i$th fitted value can be written as

$$\hat{m}^{*}_{LL}(X_i) = \sum_{j=1}^n w_j(X_i) Y_j = \int \left( \hat{m}(x) + \hat{m}^{(1)}(x)(X_i - x) \right) K_h(x, X_i)dx. \tag{5}$$

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Boosting with kernel regression

The estimator \( \hat{m}_{LL}^*(X_i) \) (5) achieves bias reduction at interior points \([2h, 1−2h]\) (He and Huang, 2009). For this reason, the bias of \( \hat{m}_{LL}^*(X_i) \) is of order \( h^4 \) instead of \( h^2 \) which is the standard order of bias for local linear estimator \( \hat{m}(x) \). Huang and Chen (2008) and Huang and Chan (2014) show that \( H^* \) is symmetric, positive definite, and shrinking.

Similar arguments can be used to define the smoother \( H_0^* \) (local constant) and the \( i \)th fitted value \( \hat{m}_{LC}^*(X_i) \) for local constant modeling. The estimator \( \hat{m}_{LC}^*(X_i) \) takes the following form

\[
\hat{m}_{LC}^*(X_i) = \int \hat{m}_{NW}(x)K_h(x, X_i)dx,
\]

where \( \hat{m}_{NW}(x) \) is the Nadaraya-Watson estimator. The estimator \( \hat{m}_{LC}^*(X_i) \) has the bias of order \( h^2 \) at interior points.

3. Boosting with Kernel Regression

The \( L_2 \) boosting algorithm may be considered as an application of the functional gradient descent technique, as shown in Bühlmann and Yu (2003) and Park et al. (2009). We present a pseudo algorithm for estimating \( m(\cdot) \) in (1). Hereafter, notation \( H^* \) is used to denote both \( H_1^* \) and \( H_0^* \). Denote \( \hat{m}_a = (\hat{m}_a(X_1), \ldots, \hat{m}_a(X_n))^T \) for a given subscript \( a \).

(I). \( L_2 \) Boosting Algorithm:

Step 1 (Initialization): Given the data \( \{X_i, Y_i\}, i = 1, \ldots, n \), fit an initial estimate \( \hat{m}_0^* := H^*y \).

Step 2 (Iteration): Repeat for \( b = 1, \ldots, B \).

1. Compute the residual vector \( \delta := y - \hat{m}_{b−1}^* \).
2. Fit a nonparametric regression model to residual vector \( \delta \) to obtain \( \hat{m}_b^* := H^*\delta \).
3. Update \( \hat{m}_b^* = \hat{m}_{b−1}^* + \hat{m}_b^* \).

Di Marzio and Taylor (2008) and Park et al. (2009) investigated the properties of \( L_2 \) boosting for Nadaraya-Watson smoother, \( S_{NW} = ((K_h(X_i - X_j))/\sum_{j=1}^{n} K_h(X_i - X_j))_{i \neq j \in n} \). As shown in Bühlmann and Yu (2003), one of the requirements for bias reduction with boosting is to have eigenvalues are in \((0, 1)\). The smoother \( S_{NW} \), on the other hand, is not symmetric and may not contain all of the eigenvalues in \((0, 1)\). The requirements on kernel functions for which \( S_{NW} \) yields eigenvalues in \((0, 1)\) are provided by Cornillon et al. (2014). They prove that the spectrum of \( S_{NW} \) ranges between zero and one if the inverse Fourier-Stieltjes transform of a kernel is a real positive finite measure. While this is true for positive definite kernels like Gaussian and Triangular, it is not true for non-positive definite kernels like Epanechnikov and Uniform. Given that kernels are chosen considering their computational efficiency, this finding is unexpected. Smoothers using Epanechnikov kernels converged quicker than those with Gaussian kernels in our numerical analysis in Section 5.

The eigenvalues for both \( S_{NW} \) and \( H^* \) smoothers are shown in Figure 1 across different bandwidth values in \((0, 1)\). The eigenvalues of a cubic spline smoother for different values of smoothing parameters are also presented for comparison. Epanechnikov and Gaussian kernels are used in the first three graphs in the top and bottom rows, respectively. While \( S_{NW} \) produces some negative eigenvalues for the Epanechnikov kernel, it produces only positive eigenvalues for the Gaussian kernel. Figure 2 depicts the mean squared error \( (MSE) \)

\[
MSE = n^{-1}\sum_{i=1}^{n}(Y_i - \hat{m}^*(X_i))^2,
\]

values of boosting for a different number of boosting iterations. For different bandwidth and lambda (spline smoothing) values \{0.2, 0.4, 0.6, 0.8, 1\}, the \( MSE \) values are averaged over 50 samples. Because of the negative eigenvalues, it appears that boosting does not achieve bias reduction with smoother \( S_{NW} \) using Epanechnikov kernel. Figure 2 shows this with increasing \( MSE \) values in the top left plot.
According to Bühlmann and Yu (2003), the $L_2$ boost estimate in iteration $b$ of the Algorithm (I) is defined as

$$\hat{m}_b^* = \sum_{j=0}^{b} H^*(I - H^*)_j y = S_b y,$$

(8)

where $S_b = I - (I - H^*)^{b+1}$. The boosting update $\hat{m}^* := \hat{m}^*_{b-1} + H^* (y - \hat{m}^*_{b-1})$ becomes

$$S_b y = S_{b-1} y + H^* (I - S_{b-1}) y.$$

Let $0 < \lambda_k \leq 1$, $k = 1, \ldots, n$, be the eigenvalues of the smoother $H^*$. Theorem 4 of Huang and Chan (2014) describes the asymptotic properties of these eigenvalues and their related eigenvectors. We present the following assumptions for theoretical analysis before summarizing their findings.

(A.1). The covariate density $f(\cdot)$ of $X$ has a continuous $(p + 1)(b + 1)$-derivative on a compact support, $I = [0, 1]$, and $\inf_{x \in I} f(x) > 0$.

(A.2). The kernel $K(\cdot)$ is a bounded symmetric density function with bounded support $[-1, 1]$ and satisfies Lipschitz condition. The bandwidth $h \to 0$ and $nh/(\ln n)^2 \to \infty$ as $n \to \infty$.

(A.3). The function $m(\cdot)$ is $2(p + 1)(b + 1)$ times continuously differentiable.

In the nonparametric smoothing literature, the assumptions (A.1), (A.2), and (A.3) for $b = 0$ are conventional. They are mild conditions on design density and kernel functions, for example, please refer to (Di Marzio and Taylor, 2008; Park et al., 2009). Assumption (A.3) is required for the projection-based estimators to achieve bias reduction, please see Huang and Chen (2008); Huang and Chan (2014).

The conclusions of Theorem 4 in Huang and Chan (2014) are summarized in the following corollary.

**Corollary 1.** Suppose the Assumptions (A.1)–(A.3) hold for $b = 0$. Then conditioned on $x = (X_1, \ldots, X_n)^T$, we have the following results hold for $1 > \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n > 0$.

(a) Suppose $k \to \infty$ and $kh \to 0$. The eigenvalues $\lambda_k = 1 - (kh)^{2(p+1)} M_k$, for some positive constant $M_k$ and for $k = 1, 2, \ldots$, where $p = 0$ and $p = 1$ yield the respective results for $H^*_0$ and $H^*_1$. 

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**Figure 1:** Eigenvalues for (left) Nadaraya-Watson and (middle) $H^*$ smoothers, and (right) cubic spline smoother. Epanechnikov kernel: top row, Gaussian kernel: bottom row. Simulation design is based on the Example 1 in Section 5.1 with $n = 20$. 

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Figure 2: Average of mean squared error (MSE) \( \overline{\text{MSE}} \) values across 50 samples of size \( n = 500 \) for (left) Nadaraya-Watson and (middle) \( H^* \) smoothers, and (right) cubic spline smoothers. Epanechnikov kernel (top) and Gaussian kernel (bottom). Simulation design is based on Model (M1) in Section 5.1 with \( n = 500 \).

(a) Suppose \( k\ell \rightarrow C \) for some constant \( C \) or \( k\ell \rightarrow \infty \) as \( k \rightarrow \infty \). Then \( (k\ell)^{2(p+1)} M_k \rightarrow 1 \) and eigenvalues \( \lambda_k \rightarrow 0 \).

(b) Suppose \( k\ell \rightarrow C \) for some constant \( C \) or \( k\ell \rightarrow \infty \) as \( k \rightarrow \infty \). Then \( (k\ell)^{2(p+1)} M_k \rightarrow 1 \) and eigenvalues \( \lambda_k \rightarrow 0 \).

(c) The corresponding eigenvectors for \( \lambda_k \) are asymptotically the trigonometric polynomials \( \cos(2k\pi x) \) and \( \sin(2k\pi x) \).

Given that the eigenvalues are in \((0, 1]\), the proof follows immediately from Theorem 4 in Huang and Chan (2014). We may deduce from parts (a) and (b) of the Corollary 1 that the eigenvalues \( \lambda_k \) converge to 1 for values of \( k \) that fulfill \( k\ell \rightarrow 0 \), and the eigenvalues \( \lambda_k \) converge to zero for \( k\ell \rightarrow C \) for constant \( C \) or \( k\ell \rightarrow \infty \).

We now derive the asymptotic properties of the proposed boosting estimate in (9). We obtain the orthonormal decomposition as described in Proposition 2 of Bühlmann and Yu (2003),

\[
S_b = U'D_bU^T, \quad D_b = \text{diag}\{1 - (1 - \lambda_k)^{b+1}\},
\]

where \( U \) includes the orthonormal eigenvectors of \( H^* \). Denote the true regression function as \( m = (m(X_1), \ldots, m(X_n))^T \), and define \( \gamma = (\gamma_1, \ldots, \gamma_n)^T = U^T m \).

According to Bühlmann and Yu (2003) and Corollary 1, we obtain the squared bias and variance for \( L_2 \) boost estimate as

\[
\text{bias}^2(b; \hat{m}_b^*) = n^{-1} \sum_{k=1}^{n} \left( E[\hat{m}_b^*(X_k)] - m(X_k) \right)^2 = n^{-1} \sum_{k=1}^{n} \gamma_k^2 \left( 1 - \lambda_k \right)^{2(b+1)}
\]

\[
= n^{-1} \sum_{k=1}^{n} \gamma_k^2 \left( (kh)^{2(p+1)} M_k \right)^{2(b+1)}, \quad (10)
\]

and

\[
\text{var}(b; \hat{m}_b^*) = \sigma^2 n^{-1} \sum_{k=1}^{n} \text{var}(\hat{m}_b^*(X_k)) = \sigma^2 n^{-1} \sum_{k=1}^{n} \left( 1 - [1 - \lambda_k]^{b+1} \right)^2
\]

\[
= \sigma^2 n^{-1} \sum_{k=1}^{n} \left( 1 - [(kh)^{2(p+1)} M_k]^{b+1} \right)^2. \quad (11)
\]
We note that $L_2$ boosting improves asymptotic bias by $2(p+1)$ orders of magnitude provided the function is sufficiently smooth. For $p = 0$, this conclusion is consistent with the findings from Park et al. (2009); Di Marzio and Taylor (2008). The asymptotic variance exhibits a small increase with the increase in the number of boosting iterations ($b$) while maintaining the same asymptotic order remains.

We now consider a low-rank smoother. Let $H^*(d)$ be a rank $d(n)(< n)$ smoother of $H^*$. Without loss of generality, we assume that $1 > \lambda_1 > \lambda_2 > \ldots > \lambda_n > 0$. Therefore,

$$S_p(d) = U(d)_{n \times d(n)}D_p(d)U^T(d)_{d(n) \times n}, \quad D_p(d) = \text{diag} \{1 - (1 - \lambda_k)^{b+1}\}_{k=1}^{d(n)},$$

is the corresponding low-rank smoother for $S_p$. Suppose $\hat{\mathbf{m}}^*_b(d)$ is the boosting estimate at the $b$th iteration. The following theorem provides an upper bound for the low-rank, $d(n)$, and expressions for the approximation error of the smoother, $S_p(d)$, and the squared bias and variance of the boosting estimator $\hat{\mathbf{m}}^*_b(d)$.

**Theorem 1.** Suppose the Assumptions (A.1)–(A.3) hold. Then conditioned on $\{X_1, \ldots, X_n\}$, the following results hold.

(a) The low-rank $d(n)$ of a smoother $S_p(d)$ is bounded above by $O(h^{-1})$.

(b) The approximation error of the low-rank smoother, $S_p(d)$, which is defined as

$$\|S_p - S_p(d)\|_F^2 = \sum_{k=d(n)}^{n} \{1 - [(kh)^2(p+1)M_k]^{b+1}\}$$

where $\| \cdot \|_F$ is the Frobenius norm, goes to zero when $d(n) = O(h^{-1})$.

(c) The squared bias and variance for the low-rank $L_2$ boost estimate $\hat{\mathbf{m}}^*_b(d)$ are

$$\text{bias}^2 \{b; \hat{\mathbf{m}}^*_b(d)\} = n^{-1} \sum_{k=1}^{d(n)} \gamma_k^2 \{(kh)^2(p+1)M_k\}^{2(b+1)},$$

$$\text{var} \{b; \hat{\mathbf{m}}^*_b(d)\} = \sigma^2 n^{-1} \sum_{k=1}^{d(n)} \{1 - [(kh)^2(p+1)M_k]^{b+1}\}^2.$$

**Proof.** Without loss of generality, we assume that $1 > \lambda_1 > \ldots > \lambda_n > 0$.

(a) From Corollary 1, we know that $\lambda_k \rightarrow 0$ for $kh \rightarrow C$ for some constant $C$ or $kh \rightarrow \infty$. Therefore, for all $k > d(n) = O(h^{-1})$, $\lambda_k \rightarrow 0$. Hence, $d(n)$ is bounded above by $O(h^{-1})$.

(b) The result follows from the singular value decomposition of the smoother $S_p$. Again from Corollary 1, for all $k > d(n) = O(h^{-1})$, $(kh)^2(p+1)M_k \rightarrow 1$. Hence the approximation error goes to zero.

(c) The result follows from (10) and (11).
Figure 3: Average of training mean squared error (MSE) values (top row) and test MSE values (bottom row) across 25 samples of size $n = 50$ for model (M1) in Section 5.1 using $H^*_0$ (local constant) smoother. The first three plots use low-rank ($d(n) < n$) smoother. For the plots in bottom row the same test data is used for all the 25 training samples.

The following Sobolev space of $\pi$th-order smoothness, $\pi \in \mathbb{N}$,

$$
\mathcal{F}_2^\pi = \left\{ m : m \text{ is } (\pi - 1) \text{ times continuously differentiable and } \int [m^{(\pi)}(x)]^2 dx < \infty \right\}.
$$

\textbf{Theorem 2.} Suppose the Assumptions (A.1) – (A.3) hold. Assume that the true function $m(\cdot)$ belongs to $\mathcal{F}_2^\pi$ in (14). Let $b + 1 > \pi/2(p + 1)$ for $p = 0, 1$, and $d(n) = O(h^{-1})$ and $d(n) \to \infty$. Then, the $L_2$ boosting estimate, $\hat{m}^*_b(d)$, of a low-rank smoother, $S_b(d)$, achieves the optimal convergence rate, $n^{-2\pi/(2\pi + 1)}$, for the bandwidth $h = O(n^{-1/(2\pi + 1)})$.

\textbf{Proof.} Without loss of generality assume that the eigenvalues $1 > \lambda_1 > \cdots > \lambda_n > 0$. By definition, for the true function $m \in \mathcal{F}_2^\pi$

$$
\frac{1}{n} \sum_{k=1}^{d(n)} g^2_k \leq M < \infty,
$$

for some constant $M$. First, we bound the bias term in (12). Consider

$$
\text{bias}^2(b; \hat{m}^*_b(d)) = \frac{1}{n} \sum_{k=1}^{d(n)} \gamma_k^2 \{(kh)^{2(p+1)}M_k\}^{2(b+1)}
$$

$$
= \frac{1}{n} \sum_{k=1}^{d(n)} \gamma_k^2 \{(kh)^{2(p+1)}M_k\}^{2(b+1)}k^{-2\pi}
$$

$$
\leq \max_{k=1, \ldots, d(n)} \{(kh)^{2(p+1)}M_k\}^{2(b+1)}k^{-2\pi} \frac{1}{n} \sum_{k=1}^{d(n)} \gamma_k^2
$$

$$
= \max_{k=1, \ldots, d(n)} g(k) \frac{1}{n} \sum_{k=1}^{d(n)} \gamma_k^2 k^{2\pi},
$$
where \( g(k) = ((kh)^{2(p+1)}M_k)^{2(b+1)k^{-2\pi}} \) is an increasing function of \( k \) for \( b + 1 > \pi/2(p + 1) \). Since the number of basis functions \( d(n) \) is of order \( O(h^{-1}) \), we obtain

\[
\max_{k=1, \ldots, d(n)} g(k) \leq Ah^{2\pi},
\]

for some constant \( A \). Consequently,

\[
\text{bias}^2(b; \hat{m}_b^*(d)) \leq AMh^{2\pi},
\]

which is of order \( O(h^{2\pi}) \). We now consider the variance term. Since \( 1 - (1-x)^a \leq 1 - [1 - ax] = ax \) for any \( x \in [0, 1] \) and \( a \geq 1 \), we have

\[
\text{var}(b; \hat{m}_b^*(d)) = \frac{\sigma^2}{n} \sum_{k=1}^{d(n)} (1 - ((kh)^{2(p+1)}M_k)^{2(b+1)})^2
\]

\[
\leq \frac{\sigma^2}{n} \sum_{k=1}^{d(n)} (1 - (kh)^{2(p+1)}M_k)^{2(b+1)k^{-2\pi}}
\]

\[
\leq \frac{\sigma^2}{n} \sum_{k=1}^{d(n)} (1 - (kh)^{2(p+1)}M_k),
\]

where the last step follows because \( \sum_{k=1}^{d(n)} a_k^2 \leq \sum_{k=1}^{d(n)} a_k \) for \( 0 < a_k < 1 \). Since \( d(n) \) is \( O(h^{-1}) \),

\[
\text{var}(b; \hat{m}_b^*(d)) = O(n^{-1}h^{-1}).
\]

Consequently,

\[
n^{-1}\text{MSE}(b; \hat{m}_b^*(d)) \leq O(h^{2\pi}) + O(n^{-1}h^{-1}),
\]

for \( b + 1 > \pi/2(p + 1) \). Hence the result is proved.

We have the following remarks:

(i) The result in Theorem 2 is new to the kernel smoothing literature. The existing result on the optimal convergence rate by Park et al. (2009) uses only a full rank smoother. Moreover, their result is for the Nadaraya-Watson smoother which is different from the projection-based smoother used in our study.

(ii) Boosting may adopt to higher order smoothness since it refits several times, as detailed in Bühlmann and Yu (2003). The optimal rate for \( \pi = 4 \) is \( n^{-8/9} \) for the bandwidth \( n^{-1/9} \). This takes at least 1 and 2 iterations for local linear smoother \( (p = 1) \), \( H_1^* \), and local constant smoother \( (p = 0) \), \( H_0^* \), respectively.

3.1. Optimal Bandwidth and the number of iterations

Although, boosting is resistant to overfitting (Bühlmann and Yu, 2003), finding the correct number of boosting iterations \( (b) \) to avoid overfitting is critical. It is also worth mentioning that the learner’s bandwidth \( (h) \) has an impact on whether it is strong or weak. Boosting with a weak learner takes a lot of iterations, but boosting with a strong learner tends to overfit very quickly. As a result, choosing the best bandwidth parameter is essential.

The optimal pair \((b, h)\) is estimated from the data using the cross-validation procedure with mean square error, which is defined in (7). However, evaluation of the estimator \( \hat{m}^* \) at new or test data points is not straightforward. Denote \( X_{test} \) as the new test observation. From (5), we write

\[
\hat{m}^*(X_{test}) = H_{test}^*y = \int (\hat{m}(x) + \hat{m}^{(1)}(x)(X_{test} - x)) K_h(X_{test} - x)dx,
\]
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for \( \hat{m}(x) \) and \( \hat{m}^{(1)}(x) \) estimated using \{ \( X_i, Y_i \) \}, \( i = 1, \ldots, n \), observations, and

\[
H^*_\text{test} = \int K_h(X_{\text{test}} - x)[1 \ X_{\text{test}} - x](X_x W_x X_x)^{-1} X_x W_x dx.
\]

We now perform cross-validation for each pair of values \((b, h)\) using the above formula, and select the combination that has the minimum cross-validation mean square error.

4. Robustified Boosting

Because of the \( L_2 \) loss function, boosting is sensitive to outliers in the data. We robustify the boosting procedure developed in Section 3. The pseudo data technique detailed in Cox (1983); Oh et al. (2007) and Oh et al. (2008) is the key component of our methodology.

Let \( \rho(x) \) be a convex function that is symmetric at zero grows slower than \( x^2 \) as \( |x| \) becomes larger. The Huber loss function is a well-known example of such a function, which is stated as

\[
\rho(x) = \begin{cases} 
  x^2, & |x| \leq c \\
  2c|x| - c^2, & |x| > c,
\end{cases}
\]

where \( c > 0 \) is the cutoff that is determined based on the data. Let \( \psi = \rho' \) be the derivative of \( \rho \). From (3), we observe that the estimator \( \hat{m}^* = (\hat{m}^*(X_1), \ldots, \hat{m}^*(X_n))^T \) satisfies the following score equation under \( L_2 \) loss

\[
n^{-1} \sum_{i=1}^{n} (Y_i - \hat{m}^*(X_i)) = 0.
\]

(16)

Let \( \tilde{m}^* = (\tilde{m}^*_1, \ldots, \tilde{m}^*_n)^T \) be a robustified estimator for model (1) under the loss function \( \rho(\cdot) \). Therefore, from (16), it is reasonable to assume that the estimator \( \tilde{m}^* \) satisfies the following score equation

\[
n^{-1} \sum_{i=1}^{n} \psi(Y_i - \tilde{m}^*_i(X_i)) = 0.
\]

(17)

Similar to Oh et al. (2008), define the pseudo data

\[
Z_i = m(X_i) + \psi(\varepsilon_i)/2, \quad i = 1, \ldots, n,
\]

given \( \psi(\varepsilon_i) \) exists and has a finite variance. By taking the empirical pseudo data \( \tilde{Z}_\psi = \tilde{m}^*(X_i) + \psi(Y_i - \tilde{m}^*(X_i))/2 \), we can express the score function (17) as

\[
n^{-1} \sum_{i=1}^{n} 2\{ \tilde{Z}_\psi - \tilde{m}^*_i(X_i) \} = 0,
\]

(18)

which is equivalent to the score equation under \( L_2 \) loss (16) with \( \tilde{Z}_\psi \) as the response variable. This transformation serves as a motivation for the pseudo outcome approach described in Cox (1983); Oh et al. (2007). It facilitates a theoretical analysis and provides an easily computing algorithm for model estimation (Cox, 1983). Since \( \tilde{Z}_\psi \) involves \( \tilde{m}^*(X_i) \) which is unknown, in practice, an iterative algorithm is needed.

Let \( \tilde{m}^* = (\tilde{m}^*(X_1), \ldots, \tilde{m}^*(X_n))^T \) denote an iterative estimator that satisfies

\[
n^{-1} \sum_{i=1}^{n} 2\{ \tilde{Z}_\psi - \tilde{m}^*_i(X_i) \} = 0,
\]

(19)
for $\bar{Z}_i = \bar{m}^*(X_i) + \psi(Y_i - \bar{m}^*(X_i))/2$ for $i = 1, \ldots, n$. Because it employs the $L_2$ loss function, the properties of $\bar{m}^*(X_i)$ are comparatively easy to acquire. The goal now is to prove that $\bar{m}^*(X_i)$ is asymptotically equivalent to $\hat{m}_p^*(X_i)$ with the latter's properties remaining comparable to the former.

Let $\bar{Z} = (\bar{Z}_1, \ldots, \bar{Z}_n)^T$. We first present an algorithm for estimating $\bar{m}^*$ and then provide a robustified boosting algorithm.

**II. Pseudo data Algorithm for $\bar{m}^*$:**

1. Obtain initial estimate $\bar{m}^*_0 = H^* y$.
2. Set $\bar{Z}_0 = y$.
3. Repeat the following steps ($k = 1, \ldots$) until convergence
   (a) Compute $\bar{Z}_k = \bar{m}^*_{k-1} + \psi(y - \bar{m}^*_{k-1})$.
   (b) Compute the estimator $\hat{m}^*_k = H^* \bar{Z}_k$.
4. At convergence, take the final estimator as $\hat{m}^* := \hat{m}^*_k$.

The robustified $L_2$ boosting algorithm employs the above pseudo data algorithm (II) at every iteration. Mainly, it includes the following steps.

**III. Robustified $L_2$ Boosting Algorithm:**

Step 1 (Initialization): Given the data $\{X_i, Y_i\}, i = 1, \ldots, n$, obtain an initial estimate $\hat{m}^*_0 := \bar{m}^*$, from Algorithm II.

Step 2 (Iteration): Repeat for $b = 1, \ldots, B$.

1. Compute the residual vector $\delta := y - \hat{m}^*_{b-1}$.
2. Use Algorithm II with residuals as a response variable, $y = \delta$, to obtain $\hat{m}^*_b := \hat{m}^*$.
3. Update $\hat{m}^*_b := \hat{m}^*_{b-1} + \hat{m}^*_b$.

The concept of pseudo data has been successfully applied in the context of smoothing splines Cox (1983); Cantoni
and Ronchetti (2001) and wavelet regression Oh et al. (2007) to derive the asymptotic results as well as to facilitate a
computation algorithm. The key result is that the robust smoothing estimator is asymptotically equivalent to a least-
square smoothing estimator applied to the pseudo data. The proposed robustified boosting Algorithm (III) is general
in the sense that it can be used with any other smoother considered in the study. In our numerical study, in addition to
$H^*$, we also employ this algorithm with Nadaraya-Watson and spline smoothers.

We now state the following assumption which is required for Theorem 3. It is found in Oh et al. (2007).

(A.4). The function $\psi$ has a continuous second derivative and satisfies  
$$\sup_{-\infty < t < \infty} |\psi''(t)| < \infty.$$  
Assume $\psi$ is normalized such that $E[\psi(t)] = 0$, $E[\psi'(t)]/2 = 1$, $\text{var}\{\psi(t)\} < \infty$, and $\text{var}\{\psi'(t)\} < \infty$.

In Theorem 3, we show that the estimators $\hat{m}^*_p$ and $\hat{m}^*$ are asymptotically equivalent.

**Theorem 3.** Suppose the Assumptions of Theorem 2 and the Assumption (A.4) hold for $b = 0$. Let $C_n = n^{-1} E[\bar{m}^* - m^*_p]^2$, where $\| \cdot \|$ denotes the Euclidean norm, and assume that $C_n \to 0$ as $n \to \infty$. Then,

$$n^{-1}\|\bar{m}^* - \hat{m}^*_p\|^2 / \sqrt{C_n} \to 0$$

(20)  
in probability as $n \to \infty$. 

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The proof is a special case of the proof from Theorem 1 of Oh et al. (2007). Therefore, we omit the proof. It has mainly three basic parts: finding uniform bounds on the score functions (17) and (18), applying a fixed point argument, and evaluating pseudo data score function (18) at the robust estimator. Theorem 3 aids in establishing the asymptotic properties of the robustified boosting estimator \( \hat{m}^* \) in Algorithm (III). It is sufficient to deduce the asymptotic properties of \( \hat{m}^* \), which may be derived similarly to those of the boosting estimate given previously under conditions (A.1)–(A.3).

We note that obtaining the theoretical properties of the robustified boosting algorithm is not easy given the involvement of the nonlinear function \( \psi \). We defer this for future research.

5. Simulation Study

In this section, we use simulations to assess the finite sample performances of the proposed boosting methods. We investigate two scenarios, one with no outliers and the other with few outliers. All the computations were done using the software Julia (Bezanson et al., 2017) on a CentOS 7 machine.

5.1. Example 1: Without outliers

For this example, we mimic the simulation design in Bühlmann and Yu (2003). The following models are used to generate data:

\[
\begin{align*}
(M 1) & : Y_i = 0.8X_i + \sin(6X_i) + \epsilon_i, \\
(M 2) & : Y_i = 0.4\{3 \sin(4\pi X_i) + 2 \sin(3\pi X_i)\} + \epsilon_i,
\end{align*}
\]

where \( \epsilon_i \sim N(0, 2) \) and \( X_i \sim U(-0.5, 0.5) \) for \( i = 1, \ldots, n \). The function \( (M 2) \) is taken from Park et al. (2009). Using both \( H_1^* \) (local constant) and \( H_1^* \) (local linear) smoothers, we employ \( L_2 \) boosting to estimate (21). A fixed grid of length 200 is utilized to approximate the integrals for both of them. Nadaraya-Watson (Di Marzio and Taylor, 2008) and cubic spline (Bühlmann and Yu, 2003) smoothers are considered for comparison. Both Epanechnikov and Gaussian kernels are used for kernel and projection smoothers.

To assess the predictive performance of the above methods, we compute their out-of-sample prediction errors as follows:

1. For a sample of size \( n \), for each of the three kernels and one spline methods, we perform a 5-fold cross-validation for each pair of values \( (h, b) \) and \( (\lambda, b) \), respectively. Identify the optimal pairs \((\hat{h}, \hat{b})\) and \((\hat{\lambda}, \hat{b})\) which minimize the MSE.

2. We now simulate 100 datasets of size \( n \) and apply the above 4 boosting algorithms with their respective \((\hat{h}, \hat{b})\) and \((\hat{\lambda}, \hat{b})\).

3. Compute the average of

\[
MSE(T) = n^{-1} \sum_{i=1}^{n} \{m(X_i) - \hat{m}(X_i)\}^2
\]

across 100 datasets.

4. Repeat the above steps (1–3) 10 times and compute the average and standard deviation of the average \( MSE(T) \) values obtained in step (3).

For the cross-validation in step 1, we perform search on a grid of length 40 in the intervals \([0.1, 4]\) and \([0, 1000]\) for bandwidth \( h \) and spline smoothing parameter \( \lambda \), respectively. This search is performed for 5000 boosting iterations.

Table 1 shows the average \( MSE(T) \) results for model (M1) for increasing sample sizes \( n = 100, 200, 500, 1000 \). The results are consistent across the four smoothers: \( H^* \) with \( p = 0 \) and \( p = 1 \), Nadaraya-Watson, and cubic smoothing spline. The Nadaraya-Watson smoother provided relatively larger \( MSE(T) \) values for the Epanechnikov kernel than the Gaussian kernel. The results from \( H^* \) (local constant and local linear) are comparable across both kernels and similar to the results from spline smoother. The results for model (M2) are provided in Table 3. For model (M2), Nadaraya-Watson smoother produced slightly better results. Overall, the \( H^* \) smoothers performed well for both kernel functions and their results are comparable to the results from other smoothers.
Table 1
Model (M1): Average and standard deviations of 10 average \( \text{MSE}(T) \) values of 100 simulated datasets. LC: local constant, LL: local linear, NW: Nadaraya-Watson, SS: Smoothing Splines. Ep: Epanechnikov kernel, Ga: Gaussian Kernel

| Kernel | Sample size (n) | \( H_0^* \) (LC) | \( H_1^* \) (LL) | NW | SS |
|--------|----------------|------------------|------------------|----|----|
| Ep     | 100            | 0.1117 (0.045)   | 0.1174 (0.069)   | 0.1596 (0.075) | 0.1246 (0.053) |
|        | 200            | 0.0691 (0.023)   | 0.0646 (0.030)   | 0.0728 (0.037) | 0.0639 (0.024) |
|        | 500            | 0.0198 (0.013)   | 0.0273 (0.016)   | 0.0307 (0.025) | 0.0158 (0.009) |
|        | 1000           | 0.0165 (0.014)   | 0.0136 (0.005)   | 0.0137 (0.007) | 0.0114 (0.007) |

| Ga     | 100            | 0.1341 (0.057)   | 0.1249 (0.072)   | 0.1273 (0.055) | 0.1246 (0.053) |
|        | 200            | 0.0628 (0.025)   | 0.0599 (0.026)   | 0.0590 (0.022) | 0.0639 (0.024) |
|        | 500            | 0.0190 (0.018)   | 0.0228 (0.018)   | 0.0231 (0.028) | 0.0158 (0.009) |
|        | 1000           | 0.0119 (0.009)   | 0.0111 (0.005)   | 0.0107 (0.006) | 0.0114 (0.007) |

Table 2
Model (M1): Reduced rank smoother \( H_0^* \): Average and standard deviations of 10 average \( \text{MSE}(T) \) values (22) of 100 simulated datasets. \( d(n) \): number of basis functions (low-rank).

| Sample size (n) | \( d(n) = 2 \) | \( d(n) = 5 \) | \( d(n) = 10 \) | \( d(n) = 15 \) | \( d(n) = n \) |
|-----------------|----------------|----------------|----------------|----------------|----------------|
| 100             | 0.1141 (0.039) | 0.1201 (0.057) | 0.115 (0.047)  | 0.1149 (0.047) | 0.1149 (0.047) |
| 200             | 0.0769 (0.047) | 0.0664 (0.028) | 0.0683 (0.027) | 0.0685 (0.0271)| 0.0686 (0.0271)|
| 500             | 0.0171 (0.0065)| 0.0172 (0.0080)| 0.0191 (0.0128)| 0.0223 (0.0224)| 0.0320 (0.0387) |

We also evaluate the predictive performance of the low-rank smoothers. Table 2 shows the mean and standard deviation of 10 average \( \text{MSE}(T) \) values over 100 simulated datasets for smoother \( H_0^* \) (local constant). We find that for samples of sizes 100 and 200, only \( d(n) = 10 \) basis functions are required to approximate the full-rank smoother. Approximation gets better with increasing \( d(n) \). Furthermore, it is evident from the results that the low-rank smoother outperforms the full-rank smoother in terms of prediction accuracy. The results for model (M2) are provided in Table 4. The findings remain similar.

5.2. Example 2: With outliers
In this section, we evaluate the performance of the robustified \( L_2 \) boosting algorithm. The same simulation design of Section 5.1 is considered with one change; errors \( e_i \) are simulated from a \( t \)-distribution with 3 degrees of freedom. We consider the Huber loss function (15) to robustify the boosting algorithm. For comparison, the results of non-robust boosting algorithms are also provided. We followed the same procedure described in Example 1 to compute the average \( \text{MSE}(T) \) values. However, due to the involvement of the Huber loss function, the \( \text{MSE} \) values and cross-validation errors for the boosting iterations are computed as

\[
\text{MSE}(\rho) = \frac{\text{MSE}}{n} \sum_{i=1}^{n} \rho \left( \frac{Y_i - \hat{m}(X_i)}{\text{MSE}^{1/2}} \right)
\]
Table 3
Model \((M2)\): Average and standard deviations of 10 average MSEs for 100 simulated data for different sizes and using different kernel functions. LC: local constant, LL: local linear, NW: Nadaraya-Watson, SS: Smoothing Splines. Ep: Epanechnikov kernel, Ga: Gaussian Kernel

| Kernel | Sample size (n) | \(H^*\) (LC) | \(H^*\) (LL) | NW | SS |
|--------|----------------|--------------|--------------|----|----|
|       | 100            | 0.2233       | 0.2055       | 0.2316 | 0.3590 |
|       |                | (0.090)      | (0.148)      | (0.086) | (0.090) |
| Ep     | 200            | 0.1231       | 0.1395       | 0.1351 | 0.1628 |
|       |                | (0.047)      | (0.067)      | (0.041) | (0.041) |
|       | 500            | 0.0537       | 0.0511       | 0.0489 | 0.035 |
|       |                | (0.036)      | (0.028)      | (0.029) | (0.009) |
|       | 1000           | 0.0299       | 0.0305       | 0.0282 | 0.0178 |
|       |                | (0.016)      | (0.025)      | (0.018) | (0.010) |
| Ga    | 100            | 0.1796       | 0.1858       | 0.1926 | 0.3590 |
|       |                | (0.105)      | (0.126)      | (0.106) | (0.090) |
|       | 200            | 0.1045       | 0.1024       | 0.1045 | 0.1628 |
|       |                | (0.027)      | (0.040)      | (0.031) | (0.041) |
|       | 500            | 0.0391       | 0.0383       | 0.0341 | 0.0350 |
|       |                | (0.020)      | (0.025)      | (0.028) | (0.018) |
|       | 1000           | 0.0224       | 0.0232       | 0.0173 | 0.0178 |
|       |                | (0.018)      | (0.021)      | (0.012) | (0.010) |

Table 4
Model \((M2)\): Reduced rank smoother \(H^*_0\): Average and standard deviations of 10 average \(MSE(T)\) values \((22)\) of 100 simulated datasets. \(d(n)\): number of basis functions (low-rank).

| Sample size (n) | \(d(n) = 2\) | \(d(n) = 5\) | \(d(n) = 10\) | \(d(n) = 15\) | \(d(n) = n\) |
|----------------|---------------|---------------|----------------|----------------|----------------|
| 100            | 0.2248        | 0.2157        | 0.2163         | 0.2163         | 0.2163         |
|                | (0.1109)      | (0.1049)      | (0.1102)       | (0.1101)       | (0.1101)       |
| 200            | 0.1470        | 0.1338        | 0.1053         | 0.1039         | 0.1039         |
|                | (0.0460)      | (0.0608)      | (0.0284)       | (0.0291)       | (0.0291)       |
| 500            | 0.1066        | 0.0406        | 0.0394         | 0.0451         | 0.0507         |
|                | (0.0257)      | (0.0161)      | (0.0209)       | (0.0266)       | (0.0366)       |

where \(MSE\) is computed using \(L_2\) loss function. The Huber constant \(c\) is a tuning parameter that needs to be estimated from the data. In the following section where we analyze a real data, we consider the choice of \(\hat{c} = 1.345\hat{\sigma}\), where \(\hat{\sigma}\) is any robust estimator for the population standard deviation \(\sigma\) Huber (2004).

The results from the robustified boosting for the Huber constant \(c = 1\) and \(c = 2\) are reported in Table 5. The robustified boosting methods provided smaller \(MSE(T)\) values than their non-robust counterparts. This suggests that the robustified methods minimize the effect of outliers in the boosting algorithm. Surprisingly, \(H^*\) smoother provided smaller errors than the Nadaraya-Watson smoother. The results from \(H^*_0\) (local constant) and spline smoother are approximately similar. Moreover, we observe that the choice of the Huber constant \(c\) is not so critical for large samples.

Overall, it is shown that the projection-based smoothers \(H^*\) are useful tools for the boosting algorithm. Their nice theoretical properties allow us to look into the effect of low-rank smoothers on boosting. The robustified boosting algorithm provides much better performance in the presence of outliers.

6. Real Application

We consider the cps71 data from Ullah (1985); Pagan et al. (1999); Hayfield and Racine (2008) (1971 Canadian Public Use Tapes). It contains age and income information of 205 Canadian individuals. Individuals in the data were educated to the thirteenth grade 13, and schooling was believed to be equal for both genders.
We are interested in the following model:

\[ \ln(wage_i) = m(\text{age}_i) + \epsilon_i, \quad i = 1, \ldots, 250. \]  

(23)

The relation between the covariate age and the outcome variable logarithm of wage is depicted in Figure 4. Their association appears to be quadratic (Pagan et al., 1999) with several outliers, especially for the elderly. We used this data to test the proposed robust and non-robust boosting methods. To calculate smoother matrices \( H^* \) (both local constant and local linear) we utilize 200 grid points. We use the Huber’s constant, \( c = 1.345 \hat{\sigma} \), where \( \hat{\sigma} \) is a robust location-free scale estimate (Rousseeuw and Croux, 1993) based on the suggestion of Oh et al. (2007); Huber (2004). The \textit{gam} function in mgcv package (Wood and Wood, 2015) is used to fit model (23), and the residuals are used to compute \( \hat{\sigma} \). The calculations are run on a CentOS system using the program Julia (Bezanson et al., 2017).

![Figure 4: Scatter plot between the covariate Age and the response variable Log Wage for cps71 data.](image)

The smoothers \( H^* \) (local constant, local linear), Nadaraya-Watson, and cubic spline are used to estimate the model 23, using Algorithms (I) and (III) for non-robust and robust methods, respectively. We apply 5-fold cross-validation to find the best pair \((\hat{h}, \hat{b})\) for each smoother and algorithm combination. Grid search for bandwidth range \( h = [5, 20] \), smoothing parameter range \( \lambda = [0, 5000] \), and boosting iterations \( b = 1000 \) is performed.

The results are presented in Table 6 and Figure 5. We find that all the robust methods performed better than their non-robust counterparts. We also find that the results are not very sensitive to the value of Huber’s constant \( c \). This is much desirable in practice.

7. Summary and Conclusions

We present a novel kernel regression based \( L^2 \) boosting approach to estimate a univariate nonparametric model. In the context of \( L^2 \) boosting, the suggested method overcomes the shortcomings of existing kernel-based methods. The theory established for spline smoothing is easily applicable since the smoother \( H^* \) utilized in the study is symmetric and has eigenvalues in the range \((0, 1]\). Simultaneously, specific results for kernel smoothing may also be obtained. We consider low-rank smoothers instead of the original smoother in our asymptotic framework. Low-rank smoothers make the boosting algorithm scalable to big datasets. In addition to the computational gains, based on our numerical results, we find that the low-rank smoothers may outperform the full-rank smoothers in terms of test data prediction error. Furthermore, we also robustify the proposed boosting procedure to alleviate the effect of outliers.

The present study considers \( L^2 \) boosting for a univariate nonparametric model. The development of the \( L^2 \) boosting procedure for additive models is an intriguing area for future research. This topic may be very helpful in practice because the boosting approach also does variable selection.
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Figure 5: Robust (solid) and non-robust (dash) fits for the model (23). Huber constant $\hat{c} = 0.8638$. The optimal pair $(\hat{h}, \hat{b})$ for each model is chosen based on the 5-fold cross-validation.

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Table 5: Average and standard deviations of 10 average $MSE(T)$ values (22) for 100 simulated data of different sizes and using different Huber constant $c$ values.
LC: local constant, LL: local linear, NW: Nadaraya-Watson, SS: Smoothing Splines.

| Huber constant($c$) | Sample size(n) | $H^*_0$ (LC) | $H^*_1$ (LL) | NW | SS |
|----------------------|----------------|--------------|--------------|----|----|
|                      |                | robust       | non-robust   | robust | non-robust | robust | non-robust | robust | non-robust |
|                      | 100            | 0.1028       | 0.2018       | 0.1101  | 0.3954      | 0.1517  | 0.2150      | 0.1179  | 0.2183      |
|                      |                | (0.081)      | (0.168)      | (0.091) | (0.297)     | (0.088) | (0.095)     | (0.098) | (0.157)     |
|                      | 200            | 0.0428       | 0.0976       | 0.0550  | 0.2350      | 0.0508  | 0.0941      | 0.0364  | 0.0957      |
|                      |                | (0.024)      | (0.061)      | (0.051) | (0.239)     | (0.027) | (0.063)     | (0.027) | (0.068)     |
|                      | 500            | 0.0199       | 0.0325       | 0.0254  | 0.1530      | 0.0207  | 0.0469      | 0.0172  | 0.0327      |
|                      |                | (0.009)      | (0.016)      | (0.018) | (0.071)     | (0.011) | (0.015)     | (0.010) | (0.024)     |
|                      | 100            | 0.1309       | 0.2018       | 0.1158  | 0.3954      | 0.1981  | 0.2150      | 0.1452  | 0.2183      |
|                      |                | (0.111)      | (0.168)      | (0.083) | (0.297)     | (0.098) | (0.095)     | (0.109) | (0.157)     |
|                      | 200            | 0.0460       | 0.0976       | 0.0617  | 0.2350      | 0.0531  | 0.0941      | 0.0433  | 0.0957      |
|                      |                | (0.028)      | (0.061)      | (0.058) | (0.239)     | (0.031) | (0.063)     | (0.035) | (0.068)     |
|                      | 500            | 0.0203       | 0.0325       | 0.0237  | 0.1530      | 0.0215  | 0.0469      | 0.0178  | 0.0327      |
|                      |                | (0.009)      | (0.016)      | (0.014) | (0.071)     | (0.010) | (0.015)     | (0.009) | (0.024)     |
Table 6
Optimal smoothing parameter and the boosting number of iterations for the proposed robust and non-robust boosting algorithms for model 23. Here $\hat{\alpha} = 0.64$.

| Huber constant ($\hat{\alpha}$) | Smoother | $h$ or $\lambda$ | $b$ | MSE | $h$ or $\lambda$ | $b$ | MSE |
|---------------------------------|----------|------------------|-----|-----|------------------|-----|-----|
| 1.34$\hat{\alpha}$             | $H^*_0$  | 13.84            | 330 | 0.2738 | 12.69         | 255 | 0.2706 |
|                                 | $H^*_1$  | 20.00            | 227 | 0.2722 | 15.00         | 79  | 0.3028 |
|                                 | NW       | 5.76             | 19  | 0.2662 | 6.15          | 20  | 0.2744 |
|                                 | SS       | 128.20           | 1   | 0.2733 | 256.41        | 2   | 0.2702 |
| $\hat{\alpha}$                 | $H^*_0$  | 13.84            | 444 | 0.2744 | 12.69         | 254 | 0.2706 |
|                                 | $H^*_1$  | 20.00            | 218 | 0.2739 | 15.00         | 79  | 0.3028 |
|                                 | NW       | 5.76             | 20  | 0.2680 | 6.15          | 20  | 0.2744 |
|                                 | SS       | 128.20           | 1   | 0.2757 | 256.41        | 2   | 0.2702 |
| 1.6$\hat{\alpha}$              | $H^*_0$  | 13.84            | 305 | 0.2734 | 12.69         | 255 | 0.2706 |
|                                 | $H^*_1$  | 20.00            | 231 | 0.2716 | 15.00         | 79  | 0.3028 |
|                                 | NW       | 5.76             | 19  | 0.2665 | 6.15          | 20  | 0.2744 |
|                                 | SS       | 256.41           | 3   | 0.2693 | 256.41        | 2   | 0.2702 |