ON QUASISYMMETRIC EMBEDDINGS OF THE BROWNIAN MAP AND CONTINUUM TREES

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ABSTRACT. The Brownian map is a model of random geometry on the sphere and as such an important object in probability theory and physics. It has been linked to Liouville Quantum Gravity and much research has been devoted to it. One open question asks for a canonical embedding of the Brownian map into the sphere or other, less abstract, metric spaces. Similarly, Liouville Quantum Gravity has been shown to be “equivalent” to the Brownian map but the exact nature of the correspondence (i.e. embedding) is still unknown. In this article we show that any embedding of the Brownian map or continuum random tree into $\mathbb{R}^d$, $\mathbb{S}^d$, $\mathbb{T}^d$, or more generally any doubling metric space, cannot be quasisymmetric. We achieve this with the aid of dimension theory by identifying a metric structure that is invariant under quasisymmetric maps and which implies infinite Assouad dimension.

We show, using elementary methods, that this structure is almost surely present in the Brownian continuum random tree and the Brownian map. We further show that snowflaking the metric is not sufficient to find an embedding and discuss continuum trees as a tool to studying “fractal functions”.

1. Introduction

Over the past few years two important models of random geometry of the sphere $S^2$ emerged. One was originally motivated by string theory and conformal field theory in physics and is known as Liouville Quantum Gravity. The other is known as the Brownian map, which originated from the study of large random planar graphs. The Brownian map turned out to be a universal limit of random planar graphs of $S^2$ and both models have attracted a great deal of interest over the past few years.

Let $Q_n$ be the set of all quadrangulations of the sphere $S^2$ (up to homeomorphism) with $n$ vertices and a unique, distinguished root. This set is finite and we randomly choose a specific quadrangulation $q_n \in Q_n$ with the uniform measure. Let $d_{gr}$ be the graph metric. It was independently proven by Le Gall [15] and Miermont [19] that for $c = (9/8)^{1/4}$ the sequence $(q_n, c\cdot n^{-1/4}d_{gr})$ converges in distribution to a random limit object $(\mathfrak{m}_\infty, D)$ in the space of all compact sets $(K, d_{GH})$ equipped with the Gromov-Hausdorff distance. Remarkably, Le Gall also established that this object is universal in the sense that not just quadrangulations, but also all triangulation and $2n$-angulations ($n \geq 2$) converge in distribution to the same object, where the normalisation constant $c$ only depends on the type of $n$-angulation, see [15] and the survey [16]. Notably, the Brownian map is homeomorphic to $S^2$ but has Hausdorff dimension 4, indicating that the homeomorphism is highly singular. Finding such a canonical map is still an active research area and in this article we show, using elementary facts from probability and dimension theory, that the Brownian map has an almost sure metric property that we coin starry. This property implies that the Brownian map and its images under quasisymmetric maps has infinite Assouad dimension and hence cannot be embedded by quasisymmetric maps into finite dimensional manifolds.

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The other important model of random surfaces is known as Liouville Quantum Gravity (LQG) which is defined in terms of a real parameter $\gamma$. The LQG is a random geometry based on the Gaussian Free Field (GFF), a random construction that can be considered a higher dimensional variant of Brownian motion. Crucially, the law of the GFF is conformally invariant. When $\gamma = \sqrt{8/3}$, the resulting random surface has long been conjectured to be equivalent to the Brownian map. Very recently, work began to unify the two models and although the exact nature of this equivalency (that is, a canonical embedding) is still unknown, major progress was made by Miller and Sheffield in their series of works [21, 22, 23]; see also the recent survey [20]. In particular, they showed that the Brownian map and LQG share the same axiomatic properties and that there exists a homeomorphism of the Brownian map onto $\mathbb{S}^2$ which realises LQG. As above, the quasisymmetrically invariant starry property implies that no such embedding can be quasisymmetric.

Before describing our results and background in detail, we remark that the Brownian map can also be obtained from another famous random space: the Brownian continuum random tree. The Brownian continuum random tree (CRT) is a continuum tree that was introduced and studied by Aldous in [2, 3, 4]. It appears in many seemingly disjoint contexts such as the scaling limit of critical Galton-Watson trees and Brownian excursions using a “least intermediate point” metric. This ubiquity led to the CRT becoming an important object in probability theory in its own right. The Brownian map can also be constructed from the CRT by changing the metric in a random fashion, further strengthening the role of the Brownian map as a universal limit object. This description allows access to a very different set of probabilistic tools which we will directly exploit to prove our results. In fact, we first prove that the CRT is starry and thus also cannot be embedded into finite dimensional manifolds almost surely. Note that this is also in stark contrast with the work on conformal weldings of the CRT in [17] and implies much higher singularity of embeddings than previously known.

This article is structured as follows: In Section 2, we recall the definition of the Assouad dimension and its use in embedding theory. We will also introduce the starry property for metric spaces in Section 2.2 and show that it is invariant under quasisymmetric maps and implies infinite Assouad dimension. In Section 3, we define the Brownian continuum random tree via the Brownian excursion and show that the CRT is starry almost surely. We conclude that it cannot be embedded into $\mathbb{R}^d$ for any $d \in \mathbb{N}$ using quasisymmetric maps. In Section 4, we define the Brownian map through the CRT and prove that the Brownian map is starry, also. Theorems 2.4, 3.5, and 4.1 are our main results. Section 5 finishes this article by containing a discussion of our results from a fractal geometric point of view. Throughout, we postpone proofs until the end of their respective section.

2. Assouad dimension and embeddings

The Assouad dimension is an important tool in the study of embedding problems. It was first introduced by Patrice Assouad in [5]. More recently, the exact determination of the Assouad dimension for random and deterministic subsets of Euclidean space has revealed intricate relations with separation properties in the study of fractal sets. Notably, the Assouad dimension of random sets tends to be “as big as possible”, see e.g. [11], and that for self-conformal subsets of $\mathbb{R}$ Ahlfors regular is equivalent to the Assouad dimension coinciding with the Hausdorff dimension, see [1].

2.1. Assouad dimension. Formally, let $(X, d)$ be a metric space and write $N(X, r)$ for the minimal number of sets of diameter at most $r$ needed to cover $X$. We set $N(X, r) = \infty$ if no such collection exists.
Let $B(x,r)$ be the closed ball in $X$ of radius $r > 0$. The Assouad dimension is then given by
\[
\dim_A(X) = \inf \left\{ \alpha : (\exists C > 0) \left( \forall 0 < r < R < 1 \right) \sup_{x \in X} N(B(x,R),r) \leq C \left( \frac{R}{r} \right)^\alpha \right\}.
\]
(2.1)

There are several important generalisations and variations of the Assouad dimension such as the quasi-Assouad dimension and the Assouad spectrum. The latter fixes the relationship between $r$ and $R$ by a parameter $\theta$, i.e. $r = R^{1/\theta}$, whereas the former is a slightly more regularised version of the Assouad dimension. One could certainly ask the questions that arise here of these variants and we forward the interested reader to the survey [10] for an overview.

The Assouad dimension is always an upper bound to the Hausdorff dimension but coincides in many “natural” examples such as $k$-dimensional Riemannian manifolds. However, they can also differ widely in general metric spaces and it is possible to construct a space $X$ such that $X$ is countable with Hausdorff dimension 0 but has infinite Assouad dimension; see e.g. Proposition 2.7. These sets are however somewhat pathological and it would be interesting to find ‘natural’ metric spaces that have low Hausdorff and box-counting dimension but are ‘big’ in the sense of Assouad dimension. A natural candidate are random sets since they tend to have a regular average behaviour, giving low Hausdorff and box-counting dimension but are ‘thick’ regions; see [11, 12, 24, 25].

The Assouad dimension is strongly related to the metric notion of doubling: a metric space has finite Assouad dimension if and only if it is doubling. Further, the Assouad dimension is invariant under bi-Lipschitz maps and is a useful indicator when a space is or is not embeddable. Because of this invariance, a metric space $(X,d)$ with Assouad dimension $s_a = \dim_A X$ cannot be embedded into $\mathbb{R}^{[s_a]-1}$ with bi-Lipschitz maps. The converse is not quite true, but “snowflaking” the metric by some $\alpha > 0$ allows a bi-Lipschitz embedding [5].

**Theorem 2.1** (Assouad Embedding Theorem). Let $(X,d)$ be a metric space with finite Assouad dimension. Then there exists $C > 1$, $N \in \mathbb{N}$, $1/2 < \alpha < 1$, and an injection $\phi : X \to \mathbb{R}^N$ such that
\[
C^{-1}d(x,y)^\alpha \leq |\phi(x) - \phi(y)| \leq Cd(x,y)^\alpha.
\]
Explicit bounds on $N$ and $\alpha$ can be obtained from the Assouad dimension, see e.g. [8].

### 2.2. Starry metric spaces.

While the Assouad dimension is invariant under bi-Lipschitz maps, this is not true for the more general notion of quasiisymmetric maps. We recall the definition of a quasiisymmetric map.

**Definition 2.2.** Let $(X,d_X)$ and $(Y,d_Y)$ be metric spaces. A homeomorphism $\phi : X \to Y$ is called $\Psi$-quasiisymmetric if there exists an increasing function $\Psi : (0, \infty) \to (0, \infty)$ such that for any three distinct points $x, y, z \in X$,
\[
d_Y(\phi(x),\phi(y)) \leq \Psi \left( \frac{d_X(x,y)}{d_X(x,z)} \right) d_Y(\phi(x),\phi(z)).
\]

In Euclidean space these correspond to quasi-conformal mappings. That is, if $\phi : \Omega \to \Omega'$, where $\Omega, \Omega' \subset \mathbb{R}^d$ are open, is $\Psi$-quasiisymmetric then $\phi$ is also $K$-quasiconformal for some $K$ depending only on the function $\Psi$. A similar statement holds in the other direction.
In [18], Mackay and Tyson study the Assouad dimension under symmetric maps. In particular, it is possible to lower the Assouad dimension by quasisymmetric maps (see also [20]) and they introduce the notion of the conformal Assouad dimension. The conformal Assouad dimension of a metric space \((X,d)\) is defined as the infimum of the Assouad dimension of all quasisymmetric images of \(X\). This notion has been subsequently explored for many examples of deterministic sets, such as self-affine carpets in [14].

Here we define the structure of an approximate \(n\)-star, which, heuristically, is the property that a space contains \(n\) distinct points that are roughly equidistant to a central point with every geodesic between them going through the centre.

**Definition 2.3.** A metric space \((X,d)\) is said to contain an \((A,\eta)\)-**approximate \(n\)-star** if there exists \(A > 1, 0 < \eta, \rho > 0\), and a set of points \(\{x_0, \ldots, x_n\} \subseteq (X,d)\) such that \(A - \eta > 1 + \eta\) and

\[
(A - \eta)\rho \leq d(x_i, x_j) \leq A\rho \text{ for all } i, j \in \{1, \ldots, n\} \text{ with } i \neq j
\]

and

\[
\rho \leq d(x_0, x_i) \leq (1 + \eta)\rho \text{ for all } i \in \{1, \ldots, n\}.
\]

We say that a metric space \((X,d)\) is **starry** if there exists \(A > 1, \eta > 0\) and a strictly increasing sequence \((n_k)_{k \in \mathbb{N}}\) of integers such that \((X,d)\) contains an \((A,\eta)\)-approximate \(n_k\)-star for all \(k\).

Our main theorem in this section states that all quasisymmetric images of starry metric spaces (including the identity) have infinite Assouad dimension. Essentially, being starry means that the conformal Assouad dimension is maximal.

**Theorem 2.4.** Let \((X,d)\) be a starry metric space and let \(\phi\) be a quasisymmetric map \(\phi : (X,d) \to (\phi(X),d_\phi)\). Then \(\dim_A \phi(X) = \infty\).

Given that the Assouad dimension of a metric space is finite if and only if it is doubling we obtain

**Corollary 2.5.** Let \((X,d_X)\) be a starry metric space and \((Y,d_Y)\) be a doubling metric space. Then \(X\) is not doubling and any embedding \(\phi : X \to Y\) cannot be quasisymmetric.

It is a simple exercise to see that any \(s\)-Ahlfors regular space\(^1\) has Assouad (and Hausdorff) dimension equal to \(s\) and so \(\mathbb{R}^d\) is \(d\)-Ahlfors regular. Similarly, any \(d\)-dimensional Riemannian space is \(d\)-Ahlfors regular, as it supports a \(d\)-regular volume measure. Our result immediately implies that any starry metric space cannot be embedded into such finite dimensional spaces by quasisymmetric (and hence bi-Lipschitz) maps.

Careful observation of the estimates in the proof of Theorem 2.4 shows that snowflaking does not allow a quasisymmetric or bi-Lipschitz embedding. That is, there is no analogy of the Assouad embedding theorem (Theorem 2.1) for starry metric spaces and we get the stronger statement

**Corollary 2.6.** Let \((X,d_X)\) be a starry metric space and let \((Y,d_Y)\) be doubling metric space (such as an \(s\)-Ahlfors regular space with \(s \in [0,\infty)\)). Let \(\alpha \in (0,1]\), \(\Psi : (0,\infty) \to (0,\infty)\) be an increasing function, and \(\phi : X \to Y\) be an embedding of \(X\) into \(Y\). Then \(\phi\) cannot satisfy

\[
\frac{d_Y(\phi(x),\phi(y))}{d_Y(\phi(x),\phi(z))} \leq \Psi \left( \frac{d_X(x,y)^\alpha}{d_X(x,z)^\alpha} \right)
\]

for distinct \(x, y, z \in X\).

\(^1\)A metric space \(X\) is \(s\)-Ahlfors regular if it supports a Radon measure \(\mu\) such that \(B(x,r) \sim r^s\) for all \(x \in X\) and \(0 < r < \text{diam} X\).
In analogy to the observation that every set $X \subset \mathbb{R}^d$ with Assouad dimension $d \in \mathbb{N}$ must contain $[0,1]^d$ as a weak tangent, see [13], one might think that a metric space with infinite Assouad dimension must also be starry. However, that is not true.

**Proposition 2.7.** There exists a countable and bounded metric space with infinite Assouad dimension that is not starry.

We will give an example in the next section.

### 2.3. Proofs for Section 2.

**Proof of Theorem 2.4.** We argue by contradiction. Assume $\dim_{A} \phi(X) < \infty$. Then there exists $s > 0$ and $C > 1$ such that $N(B_{d_{\phi}}(x,R), r) \leq C \cdot (R/r)^s$ for all $0 < r < R < \text{diam}_{d_{\phi}} \phi(X)$ and $x \in \phi(X)$. We assume that $X$ is starry and thus there exist $A$ and $\eta$ such that $X$ has $(A, \eta)$-approximate $n_k$-stars. Pick $k_0$ such that $n_{k_0} > C \cdot (4\Psi(1)\Psi(1 + \eta))^s$, where $\Psi : (0, \infty) \to [0, \infty]$ is the scale distortion function of the quasisymmetric map $\phi$.

Let $x_i$, $\varphi$, be the points and size of the approximate $n_k$-star. First note that distances relative to the centre $x_0$ are preserved. That is, for all $i \in \{1, \ldots, n_k\}$,

$$
\frac{d_{\phi}(\phi(x_0), \phi(x_i))}{d_{\phi}(\phi(x_0), \phi(x_j))} \leq \Psi \left( \frac{d(x_0, x_i)}{d(x_0, x_j)} \right) \leq \Psi \left( \frac{1 + \eta}{\eta} \right) \leq \Psi(1 + \eta).
$$

One can similarly obtain a lower bound by considering reciprocals and we obtain

$$
\frac{1}{\Psi(1 + \eta)} \leq \frac{d_{\phi}(\phi(x_0), \phi(x_j))}{d_{\phi}(\phi(x_0), \phi(x_i))} \leq \Psi(1 + \eta)
$$

for all $i, j \in \{1, \ldots, n_k\}$. Note that $B_{d_{\phi}}(\phi(x_0), R) = \{y \in \phi(X) : d_{\phi}(\phi(x_0), y) \leq R\}$ contains $\{\phi(x_0), \ldots, \phi(x_n)\}$ for $R = \Psi(1 + \eta) \min_k d_{\phi}(\phi(x_0), \phi(x_k))$.

Let $i \neq j \in \{1, \ldots, n_k\}$. We estimate

$$
\frac{d_{\phi}(\phi(x_0), \phi(x_i))}{d_{\phi}(\phi(x_i), \phi(x_j))} \leq \Psi \left( \frac{d(x_0, x_i)}{d(x_i, x_j)} \right) \leq \Psi \left( \frac{1 + \eta}{A - \eta} \right) \leq \Psi(1).
$$

and so obtain $d_{\phi}(\phi(x_i), \phi(x_j)) \geq d_{\phi}(\phi(x_0), \phi(x_i))/\Psi(1) \geq (1/\Psi(1)) \min_k d_{\phi}(\phi(x_0), \phi(x_k))$.

Set $r = (1/(4\Psi(1))) \min_k d_{\phi}(\phi(x_0), \phi(x_k))$ and let $y_i, y_j \in \phi(X)$ be such that $\phi(x_i) \in B_{d_{\phi}}(y_i, r)$ and $\phi(x_j) \in B_{d_{\phi}}(y_j, r)$ for $i \neq j \in \{1, \ldots, n_k\}$. Then,

$$
B_{d_{\phi}}(y_i, r) \cap B_{d_{\phi}}(y_j, r) = \{z \in \phi(X) : d_{\phi}(z, y_i) < r \text{ and } d_{\phi}(z, y_j) < r\} \\
\subseteq \{z \in \phi(X) : d_{\phi}(z, \phi(x_i)) < 2r \text{ and } d_{\phi}(z, \phi(x_j)) < 2r\} \\
\subseteq \{z \in \phi(X) : d_{\phi}(\phi(x_i), z) + d_{\phi}(\phi(z, \phi(x_j)) < 4r = (1/\Psi(1)) \min_k d_{\phi}(\phi(x_0), \phi(x_i))\}
$$

by the triangle inequality and our estimate for $d_{\phi}(\phi(x_i), \phi(x_j))$\footnote{We conclude that any $r$-cover of $\{\phi(x_1), \ldots, \phi(x_{n_k})\}$ must consist of at least $n_k$ balls as no single $r$-ball can cover two distinct points. Hence,

$$
N(B_{d_{\phi}}(\phi(x_0), R), r) \geq n_k \quad \text{and} \quad \frac{R}{r} = 4\Psi(1)\Psi(1 + \eta).
$$

But then

$$
C(4\Psi(1)\Psi(1 + \eta))^s < n_k \leq N(B_{d_{\phi}}(\phi(x_0), R), r) \leq C\left(\frac{R}{r}\right)^s = C(4\Psi(1 + \eta)\Psi(1))^s,
$$

a contradiction. \qed

This also shows that the starry property is invariant under quasisymmetric maps.
Proof of Proposition 2.7. We construct an explicit example of a metric space that is bounded, has infinite Assouad dimension, but is not starry. Consider the set of points \((n, m) \in \mathbb{N} \times \mathbb{N}\) with the pseudometric

\[
d((n, m), (n', m')) = \begin{cases} 
0 & \text{if } n = n' \text{ and } m = m' \\
0 & \text{if } n = n', \min\{m, m'\} = 1 \text{ and } \max\{m, m'\} > n \\
0 & \text{if } n = n' \text{ and } \min\{m, m'\} > n \\
2^{-n} & \text{if } n = n' \text{ and none of the above} \\
2 \sum_{k = \min\{n, n'\}}^{\max\{n, n'\}} k^{-2} & \text{if } n \neq n'.
\end{cases}
\]

Since \(d((n, m), (n, m)) = 0\) and the distance function is symmetric, we only need to check the triangle inequality. Let \((n, m), (n', m'), (n'', m'')\) be points in our space. If \(n \neq n''\), then

\[
d((n, m), (n'', m'')) = 2^{-\max\{n, n''\}} \sum_{k = \min\{n, n''\}}^{\max\{n, n''\}} k^{-2} \leq 2 \sum_{k = \min\{n', n''\}}^{\max\{n', n''\}} k^{-2} = d((n, m), (n', m')) + d((n', m'), (n'', m'')).
\]

If, however \(n = n''\), we may assume that \(m \neq m''\) and \(\min\{m, m''\} \leq n\) and \(\min\{m, m''\} \neq 1\) or \(\max\{m, m''\} \leq n\) as otherwise the triangle inequality is trivially satisfied. If \(n' \neq n\), then

\[
d((n, m), (n'', m'')) = 2^{-n} \leq 2n^{-2} \leq 2 \sum_{k = \min\{n, n''\}}^{\max\{n, n''\}} k^{-2} = d((n, m), (n', m')).
\]

When \(n' = n\) then \(m' \neq m\) or \(m' \neq m''\). Therefore, at least one of \(d((n, m), (n', m'))\) and \(d((n', m'), (n'', m''))\) is equal to \(2^{-n}\). Again we obtain

\[
d((n, m), (n'', m'')) = 2^{-n} \leq d((n, m), (n', m')) + d((n', m'), (n'', m''))
\]

and \(d\) is a pseudometric. In fact, identifying points with distance zero we get the metric space \(((\mathbb{N} \times \mathbb{N})/\sim, d)\) where all \((n, m)\) get identified with \((1, 1)\) for \(m > n\).

To see that this space has infinite Assouad dimension one can consider balls \(B_n\) centred at \((n, 1)\) with diameter \(R = 2^{-n}\). This ball contains \(n\) distinct points \((n, 1), (n, 2), \ldots, (n, n)\) each at distance \(R\) from each other. Letting \(r = R/2\) we need \(n\) balls of diameter \(r\) to cover \(B_n\). However, there is no uniform constants \(C, s\) such that \(n \leq C(2)^s\) and hence the Assouad dimension is infinite.

The fact that this metric space is not starry follows from the property that within each cluster of fixed \(n\), all elements are equidistant and there is no centre point. These clusters are spaced very far apart relative to the distances within them and hence no star can be formed that way. Lastly, it is bounded as the entire set is contained in \(B((1, 1), 2\pi^2/6)\). \(\square\)

3. Result for Brownian Continuum Random Trees

In this section we introduce the concept of an \(\mathbb{R}\)-tree and a pseudometric on \([0, 1]\) which is defined in terms of an excursion function \(f\). This pseudometric gives rise to an \(\mathbb{R}\)-tree that we call the continuum tree (of function \(f\)). Letting \(f\) be a generic realisation of the Brownian excursion we obtain the Brownian continuum random tree (CRT). We will show that the CRT is a starry metric space and thus has infinite Assouad dimension. In Section 5 we ask whether this is true for a larger class of function.
3.1. \( \mathbb{R} \)-trees and excursion functions. An \( \mathbb{R} \)-tree is a continuum variant of a tree. It is a metric space that satisfies the following properties.

**Definition 3.1.** A metric space \((X,d)\) is an \( \mathbb{R} \)-tree if, for every \( x, y \in X \),

1. there exists a unique isometric map \( f_{(x,y)} : [0,d(x,y)] \to X \) such that \( f_{(x,y)}(0) = x \) and \( f_{(x,y)}(d(x,y)) = y \),
2. if \( f : [0,1] \to X \) is injective with \( f(0) = x \) and \( f(1) = y \), then \( f([0,1]) = f_{(x,y)}([0,d(x,y)]) \).

We further say that \((X,d)\) is **rooted** if there is a distinguished point \( x \in X \) that we call the root.

Heuristically, \( X \) is a connected, but potentially uncountable, union of line segments. Every two points \( x, y \in X \) are connected by a unique arc (or geodesic) that is isomorphic to a line segment. We call any \( x \in X \) a leaf if \( X \setminus \{x\} \) is still connected. In Section 4, we will further introduce a labelling on the trees that is used to identify (or glue) certain leaves. The resulting space will be the Brownian map.

There are several canonical ways of generating \( \mathbb{R} \)-trees, such as the "stick-breaking model" but here we will focus on continuum trees generated by excursion functions.

**Definition 3.2.** Let \( f : [0,1] \to [0,\infty) \) be a continuous function. We say that \( f \) is an **excursion function** if \( f(0) = f(1) = 0 \) and \( f(t) > 0 \) for all \( t \in (0,1) \).

These excursion functions are now used to define a new metric on \([0,1]\) that gives an \( \mathbb{R} \)-tree called the continuum tree.

**Definition 3.3.** Let \( f \) be an excursion function. We call \( T_f = ([0,1]/\sim, d_f) \) the **continuum tree** with excursion \( f \), where \( d_f \) is the (pseudo-)metric given by

\[
d_f(x, y) = f(x) + f(y) - 2 \min\{f(t) : \min\{x, y\} \leq t \leq \max\{x, y\}\},
\]

and the equivalence relation \( \approx \) on \([0,1]\) is: \( x \approx y \) if and only if \( d_f(x, y) = 0 \).

The resulting metric space is of cardinality the continuum for non-trivial \( f \) but can be considered a tree with root vertex \( 0 \approx 1 \), where the join (closest common ancestor) of \( x < y \) is given by the unique \( t_0 \) such that \( f(t_0) = \min\{f(t) : x \leq t \leq y\} \). We note that the minimum might not be unique in \([0,1]\) with the Euclidean metric, but is unique in the metric space \( T_f = ([0,1]/\sim, d_f) \), where \( d_f \) is a bona fide metric.

3.2. The Brownian Continuum Random Tree. The Brownian continuum random tree was first studied in the comprehensive work of Aldous [2,3,4]. It is a random metric space that can be obtained with the continuum tree metric described above by choosing the Brownian excursion \( e(t) \) as the excursion function. The Brownian excursion can be defined in terms of a Brownian bridge \( B(t) \) with parameter \( T \), which is a Wiener process conditioned on \( B(0) = B(1) = 0 \). Further, it is well-known that the Brownian bridge (with \( T = 1 \)) can be expressed as \( B(t) = W(t) - tW(1) \) where \( B(t) \) is independent of \( W(1) \). The graph \((t, W(t))\) and hence \((t, B(t))\) are compact almost surely and so \( B(t_{\min}) = \min\{B(t) : 0 \leq t \leq 1\} \) exists and is unique almost surely. Cutting the Brownian bridge at the minimum and translating, one obtains a Brownian excursion

\[
e(t) = \begin{cases} 
B(t + t_{\min}) - B(t_{\min}) & 0 \leq t \leq 1 - t_{\min}, \\
B(t - 1 + t_{\min}) - B(t_{\min}) & 1 - t_{\min} < t \leq 1.
\end{cases}
\]

**Definition 3.4.** Let \( e \) be a Brownian excursion. The random metric space \((T_e, d_e) = ([0,1]/\sim, d_e)\) is called the **Brownian continuum random tree (CRT)**.
The CRT also appears as the limit object of the stick-breaking model and rescaled critical Galton-Watson trees as the number of nodes is taken to infinity. As such, the CRT is an important object in probability theory. It also appears in the construction of the Brownian map, which we will recall in Section 3.

Our main result in this section is that the CRT is starry. However, we establish a slightly stronger result below, which we will need in the proof that the Brownian map is starry.

**Theorem 3.5.** Let \( E \) be a Brownian excursion and \( T_e \) be the associated Brownian continuum random tree. Then, for every \( n \), \( T_e \) contains infinitely many approximate \( n \)-stars, almost surely. In particular, \( T_e \) is starry almost surely.

Therefore, by Theorem 2.4 and Corollary 2.5, the Brownian excursion has infinite Assouad dimension and cannot be embedded into finite dimensional manifolds using quasisymmetric mappings.

We end this section by noting that, from a fractal geometry standpoint, the continuum tree metric could hold the key to a better understanding of “fractal functions” such as the Weierstraß functions and general self-affine functions, see Section 5.

### 3.3. Proofs of Section 3

To prove that \( T_e \) and later that the Brownian map, is starry, we partition \([1/2, 3/4]\) into countably many disjoint intervals \( A_{nk}^n \) and show that the processes on these intervals are “almost” independent. Let \( A_{nk}^n = [a(n,k), a(n,k) + 2^{-((n+k+2)/2)}] \), where \( a(n,k) = {3\over 4} - 2^{-(n+1)}(3 \cdot 2^n - 2^{1-k} - 1) \). Then the interiors of \( A_{nk}^n \) and \( A_{nk'}^n \) are disjoint when \( n \neq n' \) or \( k' \neq k \). Further, \( \bigcup_{n,k \in \mathbb{N}} A_{nk}^n = [1/2, 3/4] \) and the half-open intervals \([\inf A_{nk}^n, \sup A_{nk}^n]\) form a countable partition of \([1/2, 3/4]\). We require the following well-known result that allows us to estimate a Wiener process with a given function. In particular, this follows directly from the construction of the Classical Wiener measure on the classical Wiener space and the fact that this measure is strictly positive.

**Lemma 3.6.** Let \( W(t) \) be a Wiener process on \([0,1]\). Then, for every continuous function \( f \in C[0,1] \) and \( \varepsilon > 0 \), there exists \( p(f,\varepsilon) > 0 \) such that

\[
\mathbb{P}(\|W(t) + f(0) - f(t)\|_{\infty} < \varepsilon) \geq p(f,\varepsilon) > 0.
\]

We will use this lemma with a “zig-zag” function that will give the correct structure on \([0,1]\) when applied with the continuum tree metric. Let \( F_n(x) : [0,1] \to \mathbb{R}_{\geq 0} \), where

\[
F_n(x) = \begin{cases} 
2nx, & 0 \leq x < 1/(2n); \\
-nx + 2k + 3/2, & 2k + 1/2n \leq x < (k+1)/2n \text{ and } 0 \leq k \leq n - 2; \\
(nx - k - 1/2, & k + 1/n \leq x < 2k + 3/2n \text{ and } 0 \leq k \leq n - 2; \\
-2nx + 2n, & 2n - 1/2n \leq x \leq 1.
\end{cases}
\]

The function \( F_n \) is an excursion function \((F_n(0) = 0 = F_n(1))\) that is continuous and linear between the local maxima \( F_n((k + 1)/n) = 1 \) and local minima \( F_n((2k + 1)/(2n)) = 1/2 \) in \((0,1)\). We next show that the Brownian excursion contains arbitrarily good approximations to this function for all \( n \).

**Lemma 3.7.** Let \( E(s) \) be a Brownian excursion. Then, almost surely, for every \( n \) large enough, there exists infinitely many intervals \([s_{nk}^k, t_{nk}^k]\) in \([0,1]\), \( k \in \mathbb{N} \), such that

\[
\sup_{s \in [s_{nk}^k, t_{nk}^k]} |E(s) - E(s_{nk}^k) - \sqrt{t_{nk}^k - s_{nk}^k} \cdot F_n(s)| < \sqrt{t_{nk}^k - s_{nk}^k} \cdot 2^{1-n}
\]

where all intervals \([s_{nk}^k, t_{nk}^k]\) are disjoint.
Further, since $\mathbb{E}^{n} \rightarrow |A_{n}^{k}|x + \inf A_{n}^{k}$. We next show that $(W \circ H_{n}^{k}(s) - W \circ H_{n}^{k}(0))/\sqrt{|A_{n}^{k}|}$ is independent of $(W \circ H_{n}^{k}(t) - W \circ H_{n}^{k}(0))/\sqrt{|A_{n}^{k'}|}$ for all $s, t \in [0, 1]$ and at least one of $n \neq n'$ or $k \neq k'$. Without loss of generality assume $\sup A_{n}^{k} \leq \inf A_{n'}^{k'}$ and so $H_{n}^{k}(s) \leq H_{n'}^{k'}(t)$. Then, using that the Wiener process is a Gaussian process with covariance kernel $K(x, y) = \min\{x, y\}$,

$$\text{Cov} \left( (W \circ H_{n}^{k}(s) - W \circ H_{n}^{k}(0))/\sqrt{|A_{n}^{k}|}, (W \circ H_{n'}^{k'}(t) - W \circ H_{n'}^{k'}(0))/\sqrt{|A_{n'}^{k'}|} \right)$$

$$\leq \left( |A_{n}^{k}| \cdot |A_{n'}^{k'}| \right)^{-1/2} \left[ \text{Cov}(W \circ H_{n}^{k}(s), W \circ H_{n'}^{k'}(t)) - \text{Cov}(W \circ H_{n}^{k}(s), W \circ H_{n'}^{k'}(0)) \right.$$

$$\left. - \text{Cov}(W \circ H_{n}^{k}(0), W \circ H_{n'}^{k'}(t)) + \text{Cov}(W \circ H_{n}^{k}(0), W \circ H_{n'}^{k'}(0)) \right]$$

$$= \left( |A_{n}^{k}| \cdot |A_{n'}^{k'}| \right)^{-1/2} \left[ H_{n}^{k}(s) - H_{n}^{k}(s) - H_{n}^{k}(0) + H_{n}^{k}(0) \right] = 0.$$

Further, since $H_{n}^{k}(s)$ is linear, $(W \circ H_{n}^{k}(s) - W \circ H_{n}^{k}(0))/\sqrt{|A_{n}^{k}|}$ is a centered Gaussian process with variance

$$\text{Var} \left( (W \circ H_{n}^{k}(s) - W \circ H_{n}^{k}(0))/\sqrt{|A_{n}^{k}|} \right)$$

$$= |A_{n}^{k}|^{-1} \left( \text{Cov}(W \circ H_{n}^{k}(s), W \circ H_{n}^{k}(s)) - 2 \text{Cov}(W \circ H_{n}^{k}(s), W \circ H_{n}^{k}(0)) \right.$$

$$\left. + \text{Cov}(W \circ H_{n}^{k}(0), W \circ H_{n}^{k}(0)) \right)$$

$$= |A_{n}^{k}|^{-1} \left( H_{n}^{k}(s) - 2H_{n}^{k}(0) + H_{n}^{k}(0) \right) = |A_{n}^{k}|^{-1}(H_{n}^{k}(s) - H_{n}^{k}(0)) = s.$$

We conclude that $W_{n}^{k}(s) = (W \circ H_{n}^{k}(s) - W \circ H_{n}^{k}(0))/\sqrt{|A_{n}^{k}|}$ is equal in distribution to $W(s)$. Therefore, $\mathbb{P}\{|W_{n}^{k} - F_{n}|_{\infty} < 2^{-n}\} > p_{n}$ and the events are independent if at least one of $n, n'$ or $k, k'$ differ. A simple application of the Borel-Cantelli lemma then shows that for every $n$ there are infinitely many $k$ such that $\|W_{n}^{k} - F_{n}\|_{\infty} < 2^{-n}$ almost surely.
Let $B(s) = W(s) - W(1)$ be a Brownian bridge and let

$$B^k_n(s) = (B \circ H^k_n(s) - B \circ H^k_n(0))/\sqrt{|A^k_n|}$$

$$= (W \circ H^k_n(s) - W(1) \cdot H^k_n(s) - W \circ H^k_n(0) + W(1) \cdot H^k_n(0))/\sqrt{|A^k_n|}.$$ 

While it is not true that $B^k_n(s)$ is independent of $B^k_n(t)$, the problematic $W(1)$ plays a diminishing role in approximating $B^k_n$ by $F^k_n$ for $n,k$ large.

Fix a realisation such that $|W(1)| < \infty$ and for all $n$ there are infinitely many $k$ such that $\|W_n - F_n\| < 2^{-n}$. Clearly this realisation is generic as the event has full measure. Then, for this realisation,

$$\|B^k_n - F_n\|_{\infty}$$

$$= \left\| \left( W \circ H^k_n(s) - W(1) \cdot H^k_n(s) - W \circ H^k_n(0) + W(1) \cdot H^k_n(0) \right) / \sqrt{|A^k_n|} - F_n \right\|_{\infty}$$

$$\leq \left\| W^k_n - F_n \right\|_{\infty} + \left\| W(1) \cdot (H^k_n(0) - H^k_n) \right\|_{\infty} / \sqrt{|A^k_n|}$$

$$\leq 2^{-n} + \sqrt{|A^k_n| \cdot |W(1)|} = 2^{-n} + 2^{-(n+k)/2-1}|W(1)|.$$ 

Since $W(1)$ is fixed for this realisation we obtain that for every $n \in \mathbb{N}$ there are infinitely many $k \in \mathbb{N}$ such that $\|B^k_n - F_n\| < 2^{-(n-1)}$. Finally, $e$ is a simple cut and translate transformation of $B$, where the cut is the (almost surely) unique $t_{\text{min}}$ with $B(t_{\text{min}}) = \min_{t \in [0,1]} B(t)$. There are two cases to consider. Either $t_{\text{min}} \geq 3/4$ or $t_{\text{min}} < 3/4$. In the former we define the interval $I^k_n = A^k_n + 1 - t_{\text{min}}$ and the latter we define $I^k_n = A^k_n - t_{\text{min}}$ for $n$ large enough such that $\text{inf} A^k_n > t_{\text{min}}$. Analogous to $H^k_n$, define $G^k_n : [0,1] \rightarrow I^k_n$ to be the unique orientation preserving similarity mapping $[0,1]$ into $I^k_n$. Therefore, almost surely, for large enough $n$ there exist infinitely many $k$ such that $\sup_{s \in [0,1]} |e \circ G^k_n(s) - e \circ G^k_n(0)| / \sqrt{|I^k_n| - F_n(s)} < 2^{1-n}$. This is the required conclusion. \hfill \Box

Equipped with this lemma, we prove that the CRT is starry.

Proof of Theorem 3.5. Let $G^k_n$ be as in the proof of Lemma 3.7 and assume that $e$ is a generic realisation. Then,

$$e \circ G^k_n(0) + \sqrt{|I^k_n| \cdot (F_n(s) - 2^{1-n})} \leq e \circ G^k_n(s) \leq e \circ G^k_n(0) + \sqrt{|I^k_n| \cdot (F_n(s) + 2^{1-n})} \quad (3.1)$$

for all $0 \leq s \leq 1$, where the existence of $k$ for large enough $n$ is guaranteed by Lemma 3.7.

Let $z = 1/(4n)$. Further, let $y_p = (2p + 1)/2n$ for $0 \leq p \leq n - 2$. We now estimate the distances between $G^k_n(z)$ and $G^k_n(y_p)$ with respect to the $d_e$ metric. First, by (3.1)

$$d_e(G^k_n(z), G^k_n(y_p)) = e( G^k_n(z) ) + e( G^k_n(y_p) ) - 2 \min_{t \in [z, y_p]} e( G^k_n(t) )$$

$$\leq \sqrt{|I^k_n| \left( F_n(z) + F_n(y_p) + 2 \cdot 2^{1-n} - 2 \min_{t \in [z, 1-z]} (F_n(t) - 2^{1-n}) \right)$$

$$= \sqrt{|I^k_n| \left( 2^{3-n} + 1/2 + 1 - 2 \cdot 1/2 \right) = \sqrt{|I^k_n| \left( 1/2 + 2^{3-n} \right)$$

and

$$d_e(G^k_n(z), G^k_n(y_p)) \geq \sqrt{|I^k_n| \left( F_n(z) + F_n(y_p) - 2 \cdot 2^{1-n} - 2 \min_{t \in [z, 1-z]} (F_n(t) + 2^{1-n}) \right)$$

$$= \sqrt{|I^k_n| \left( -2^{3-n} + 1/2 + 1 - 2 \cdot 1/2 \right) = \sqrt{|I^k_n| \left( 1/2 - 2^{3-n} \right).$$


Similarly, for \( q \neq p, \)
\[
d_{e}(G^k_n(y_q), G^k_n(y_p)) = e(G^k_n(y_q)) + e(G^k_n(y_p)) - 2 \min_{t \in [y_q,y_p]} e(G^k_n(t))
\]
\[
\leq \sqrt{|I^k_n|} \left( F_n(y_q) + F_n(y_p) + 2 \cdot 2^{1-n} - 2 \min_{t \in [z,1-z]} (F_n(t) - 2^{1-n}) \right)
\]
\[
= \sqrt{|I^k_n|} (2^{3-n} + 1 + 2 \cdot 1/2) = \sqrt{|I^k_n|} (1 + 2^{3-n})
\]
and
\[
d_{e}(G^k_n(y_q), G^k_n(y_p)) \geq \sqrt{|I^k_n|} (1 - 2^{3-n}) .
\]
Let \( \varrho = (1 - 2^{4-n}) \frac{1}{2} \sqrt{|T^k_n|}, A = (2 + 2^{1-n})/(1 - 2^{4-n}), \) and \( \eta = (1 + 2^{4-n})/(1 - 2^{4-n}) - 1. \) For \( n \) sufficiently large, we obtain \( A - \eta = (2 - 2^{1-n})/(1 - 2^{4-n}) > (1 + 2^{4-n})/(1 - 2^{4-n}) = 1 + \eta. \) Therefore the points \( x_0 = G^k_n(z), x_p = G^k_n(y_p) (1 \leq p \leq n) \) form an \((A, \eta)\)-approximate \((n - 1)\)-star for sufficiently large \( n \) and so, almost surely, \( T_\varrho \) is starry. \( \square \)

4. The Brownian Map

As mentioned in the introduction, the Brownian map is a model of random geometry of the sphere that arises as the limit of triangulations or \( 2n \)-angulations \((n \geq 2)\) of \( S^2, \) as the number of vertices is increased. We refer the reader to the surveys [16] and [20] for a detailed description of the Brownian map and its relation to other random geometries, such as Liouville Quantum Gravity.

In this section we define the Brownian map in terms of the Brownian continuum random tree and show that the Brownian map is starry almost surely.

4.1. Definition of the Brownian map. To obtain the Brownian map from the CRT, one defines a random pseudometric on the CRT, which in turn results in a (quotient) metric space that is the Brownian map. This metric is defined in terms of a Gaussian process defined on the CRT.

Let \( Z(t) \) be a centred Gaussian process conditioned on \( e \) such that the covariance satisfies
\[
\mathbb{E}(Z(s)Z(t) | e) = \min\{e(u) : \min\{s, t\} \leq u \leq \max\{s, t\}\}.
\]
Note that this completely determines the Gaussian process and that the second moment of distances is equal to the distance on the CRT, that is \( \mathbb{E}((Z(s) - Z(t))^2 | e) = d_e(s, t). \) One may imagine this process as a Wiener process along geodesics of the CRT, where branches evolve independently up to the common join value.

We use this process to define a pseudo metric on \([0, 1]\) by first setting
\[
D^o(s, t) = Z(s) + Z(t) - 2 \max \left\{ \min_{u \in [s, t]} Z(u), \min_{u \in [t, s]} Z(u) \right\} .
\]
We can also define \( D^o \) directly on the tree \( T_\varrho \) by defining
\[
D^o(x, y) = \min\{D^o(s, t) : s \approx x \text{ and } t \approx y\}.
\]
Note that \( D^o \) does not satisfy the triangle inequality and we further define
\[
D(s, t) = \inf \left\{ \sum_{i=0}^{n-1} D^o(u_i, u_{i+1}) \right\} ,
\]
where the infimum is taken over all finite choices of \( u_i \) satisfying \( s = u_0 \text{ and } t = u_n. \) As with \( D^o \) we make no distinction between differences of points on \([0, 1]\) or their corresponding points in \( T_\varrho. \) It can be verified that \( D \) is indeed a pseudometric on \([0, 1]. \) Considering \( m_\infty = [0, 1]/\approx, \) where \( x \approx y \) if and only if \( D(x, y) = 0 \) we obtain the Brownian map \((m_\infty, D). \)
The Lemma 3.7 gives an insight in the extremal structure of $T_e$ and we will see that it also implies extremal behaviour for the Brownian map. The approximate star for the CRT (see Figure 2 (left)) gives rise to a line segment of length approximately $\varrho$ to which $n-1$ line segments of length approximately $\varrho$ are glued near the end.

By a similar argument to the one in Lemma 3.7 we will see that there exists positive probability that the Gaussian defined on this substructure follows a similar “zig-zag” pattern that gives rise to a further approximate $(n-1)$-star, see Figure 2 (right).

**Theorem 4.1.** Let $f : [0, 1] \to \mathbb{R}$ be an excursion function such that for infinitely many $n \in \mathbb{N}$, there exists infinitely many intervals $[s^k_n, t^k_n] \in [0, 1]$, $k \in \mathbb{N}$, such that

$$\sup_{s \in [s^k_n, t^k_n]} |f(s) - f(s^k_n) - \sqrt{t^k_n - s^k_n} \cdot F_n((s - s^k_n)/(t^k_n - s^k_n))| < \sqrt{t^k_n - s^k_n} \cdot 2^{1-n}$$

(4.1)

where all intervals $[s^k_n, t^k_n]$ are disjoint. Then $(\mathfrak{m}_\infty, D)$ is starry almost surely conditioned on the excursion $f$.

In particular, this is true for almost every realisation of the Brownian excursion, see Lemma 3.7.

**Corollary 4.2.** The Brownian map $(\mathfrak{m}_\infty, D)$ is starry almost surely and thus for all $0 < \alpha \leq 1$, doubling metric spaces $(X, d)$ and increasing functions $\Psi : (0, \infty) \to (0, \infty)$ any homeomorphism $\phi : (\mathfrak{m}_\infty, D) \to (X, d)$ cannot satisfy

$$\frac{d(\phi(x), \phi(y))}{d(\phi(x), \phi(z))} \leq \Psi \left( \frac{D(x, y)^\alpha}{D(x, z)^\alpha} \right)$$

for all $x, y, z \in \mathfrak{m}_\infty$.

Note further that the intervals at which we chose to “zoom in” on the CRT and Brownian map are arbitrary. We can easily make the stronger statement that there cannot be a single neighbourhood at which the homeomorphism can be quasisymmetric, i.e. the map cannot even be locally quasisymmetric.
4.2. Proofs of Section 4

Proof of Theorem 4.7. Temporarily fix n, k such that (4.1) holds. Define

\[ x_0 = x_n = \max \left\{ \min \{ f(s) : 0 \leq s - s_n^k \leq (t_n^k - s_n^k)/(2n) \}, \quad \min \{ f(s) : (1 - 1/(2n))(t_n^k - s_n^k) \leq s - s_n^k \leq (t_n^k - s_n^k)/2 \} \right\}, \]

\[ x_p = \min \left\{ f(s) : p - 1/2 \leq (t_n^k - s_n^k) \leq s - s_n^k \leq p + 1/2 \right\} \quad \text{for} \quad 1 \leq p \leq n - 1, \]

and

\[ y_p = \max \left\{ f(s) : p(t_n^k - s_n^k) \leq s - s_n^k \leq p + 1 \right\} \quad \text{for} \quad 0 \leq p \leq n - 1.\]

Consider the subtree \(([s_n^k, t_n^k], d_f) \subset (T_f, d_f)\) and note that it contains \(n + 1\) lines \(L_i\) of length comparable to \(1/(2 \sqrt{n})\). They are

\[ L_i = \{ \max s \in [x_i, y_i] : f(s) = t \} : t \in \{ \max f(x_i), f(x_{i+1}), f(y_i) \}. \]

and we have \(f(L_i) = \max f(x_i), f(x_{i+1}), f(y_i)\), sup \(L_i \leq \inf L_j\) for \(i < j\), \(\min_{s \in L_i} f(s) = f(\inf L_i)\), and \(f (L_i)\) is a bijection. This structure can easily be seen in Figure 2 (left). The point \(x_0 = x_0\) is the root of the tree, the \(x_i\) are the branch points of line segment \(L_i\) that ends at the leaf \(y_i\).

The Gaussian \(Z\) on \([0, 1]\) has the property that \(Z|_{L_i}\) is independent of \(Z|_{L_j}\) if \(i \neq j\) and is a Wiener process on \(f(L_i)\). Let \(s \in L_i, t \in L_j\) and assume without loss of generality that \(s < t\). Then,

\[ \text{Cov}(Z(s) - Z(\inf L_i), Z(t) - Z(\inf L_j)) = \text{Cov}(Z(s), Z(t)) - \text{Cov}(Z(s), Z(\inf L_j)) - \text{Cov}(Z(t), Z(\inf L_i)) + \text{Cov}(Z(\inf L_i), Z(\inf L_j)) \]

\[ = \min_{u \in [s, t]} f(u) - \min_{u \in [s, \inf L_i]} f(u) - \min_{u \in [\inf L_i, \inf L_j]} f(u) + \min_{u \in [\inf L_i, \inf L_j]} f(u) = 0. \]

Now let \(s, t \in L_i\). Then,

\[ \text{Cov}(Z(s), Z(t)) = \min \left\{ f(u) : \min \{ s, t \} \leq u \leq \max \{ s, t \} \right\} = \min \{ f(s), f(t) \}. \]

Let \(G_i\) be the unique orientation preserving similarity mapping the interval \([0, 1]\) onto \(f(L_i) = \max \{ f(x_i), f(x_{i+1}) \}, f(y_i)\). Then,

\[ Z_i(s) = (Z \circ f^{-1} \circ G_i(s) - Z \circ f^{-1} \circ G_i(0)) / \sqrt{|f(L_i)|} \]

is equal in distribution to a Wiener process \(W\) and as \(|f(L_i)| \sim |f(L_j)|\) there exists \(p_n\) independent of \(i\) such that

\[ \mathbb{P}\{ |Z_i(s) - s| \leq 2^{-n} \} \geq p_n > 0. \]

We can argue similarly for all the joined pieces \(J_i\). They are independent (conditioned on starting value) and have length at most \((1 + 2^{1-n})|f(L_0)|\). Hence the probability that the process deviates at most \(2^{-n}\) is also comparable to \(p_n\) and we assume without loss of generality that \(p_n\) is a lower bound. Thus, the probability that this holds for all \(i\) and all connecting tree pieces is at least \(p_n^{n+1+\log_2(n+1)} > 0\).

We now establish that these events are independent for distinct intervals. Let \(x_0(n, k)\) be as in (4.2) and let \([s_0(n, k), t_0(n, k)] \subseteq [s_n^k, t_n^k]\) be the smallest interval such that \(f(s_0(n, k)) = f(t_0(n, k)) = x_0\). Therefore \(f(u) \geq x_0\) for all \(u \in (s_0(n, k), t_0(n, k))\). Assume that at least one
of \( n \neq n' \) and \( k \neq k' \) is given, then the following holds for all \( u_1, u_2 \in [s_0(n, k), t_0(n, k)] \) and \( v_1, v_2 \in [s_0(n', k'), t_0(n', k')] \), assuming without loss of generality that \( t_0(n, k) \leq s_0(n', k') \),

\[
\text{Cov}(Z(u_1) - Z(u_2), Z(v_1) - Z(v_2)) \leq \min_{u \in [u_1, v_1]} f(u) - \min_{u \in [u_2, v_2]} f(u) - \min_{u \in [v_1, u_2]} f(u) + \min_{u \in [v_2, u_1]} f(u)
\]

Thus the (relative) Gaussian processes chosen earlier are independent if at least one of \( n, n' \) or \( k, k' \) differ. Thus a standard Borel-Cantelli argument shows that there are infinitely many intervals where this occurs for every \( n \), almost surely.

The last thing to show is that these structures give rise to \((n - 1)\)-stars. This involves estimating the Brownian map metric as well as the following estimates. Let \( x, y \in \mathbf{m}_\infty \), then

\[
Z(x) + Z(y) - 2Z(x \wedge y) \leq D(x, y) \leq D^o(x, y),
\]

where \( x \wedge y \) is the join (least common ancestor) of \( x \) and \( y \) in \( T_e \) and \( \min_{z \in [x, y]} Z(z) \geq \min_{z \in [0, 1]} Z(z) \). We thus estimate, for \( i \neq j \),

\[
0 \leq D(x_i, x_j) \leq D^o(x_i, x_j) = Z(x_i) + Z(x_j) - 2 \min_{s \in [x_i, x_j]} Z(s)
\]

\[
\leq 2 \max_k Z(x_k) - 2 \min_k Z(x_k) \leq 4 \cdot 2^{-n} \max_k \sqrt{|f(L_k)|}
\]

as well as for the branches

\[
D(x_i, y_i) \leq D(x_i, x_j) + D(x_j, y_i)
\]

\[
\leq 4 \cdot 2^{-n} \max_k \sqrt{|f(L_k)|} + (1 + 2 \cdot 2^{-n}) \sqrt{|f(L_i)|}
\]

\[
\leq (1 + 6 \cdot 2^{-n}) \max_k \sqrt{|f(L_k)|}
\]

and

\[
D(y_i, y_j) \leq D(x_i, x_j) + D(x_j, y_i) + D(x_j, y_j)
\]

\[
\leq (2 + 16 \cdot 2^{-n}) \max_k \sqrt{|f(L_k)|}
\]

The lower bounds are given by

\[
D(x_i, y_i) \geq Z(y_i) + Z(x_i) - 2Z(x_i) = Z(y_i) - Z(x_i)
\]

\[
\geq Z(y_i) - Z(x_i) - |Z(x_i) - Z(x_i)|
\]

\[
\geq \sqrt{|f(L_i)|} - 2 \cdot 2^{-n} \max_k \sqrt{|f(L_k)|}
\]

\[
\geq (1 - 6 \cdot 2^{-n}) \max_k \sqrt{|f(L_k)|}
\]

and

\[
D(y_i, y_j) \geq Z(y_i) + Z(y_j) - 2Z(y_i \wedge y_j) \geq (2 - 16 \cdot 2^{-n}) \max_k \sqrt{|f(L_k)|}
\]

Analogous to the proof of Theorem 3.5 we see that the collection of points \( y_i \) together with centre \( x_1 \), for an approximate \((n - 1)\)-star. Since this is true, almost surely, for every \( n \) we have shown that \((\mathbf{m}_\infty, D)\) is starry. \(\square\)
5. The continuum tree as a dual space

While it is not generally true that the operator mapping continuous excursion functions on \([0,1]\) to continuum trees is invertible, one can see the tree associated to a function as an indicator of its irregularity. For instance, local maxima of the excursion function are exactly the leaves of the associated continuum tree (with the exception of the root). One might conjecture a strong link between dimensions of the graph of an excursion function and its associated continuum tree.

**Question 5.1.** Let \(f\) be an excursion function on \([0,1]\) with graph \((t, f(t))_{t \in [0,1]} \subseteq \mathbb{R}^2\). Suppose additionally that \(\dim(x, f(x)) = s\). What can we say about the dimension\(^3\) \(\dim T_f\)?

Given the extremal nature of the Assouad dimension one might further conjecture that functions whose graph has Assouad dimension strictly larger than 1 have an associated continuum tree which is starry.

**Conjecture 5.2.** Let \(f\) be an excursion function such that \(\dim_A(t, f(t)) > 1\). Then \(T_f\) is starry.

This is certainly not a necessary condition as the Example 5.1 below shows.

Establishing such relationships between dimensions and irregular curves is at the heart of fractal geometry, see e.g. [9] and [7]. The dimension theory of the family of Weierstraß functions has only recently been analysed and self-affine curves are still not fully understood, see e.g. [9]. One of the major difficulty for affine functions are complicated relationships between the scaling in the horizontal as well as vertical axis and the continuum tree might get around this problem by separating the two scales in terms of placement of nodes and length of subtrees. These may give rise to a self-similar set in an abstract space other than \(\mathbb{R}^d\) that may be easier to understand. Knowing bounds such as those asked in Question 5.1 may lead to a better understanding of the self-affine theory and this “dual space” could solve many open questions for singular functions.

5.1. Example: Excursion with low Assouad dimension whose continuum tree is infinite dimensional. Let \(s_n(x)\) be the positive triangle function with slope 2 and \(n\) maxima,

\[ s_n(x) = \begin{cases} 2(x - \frac{k}{n}) & x \in [\frac{k}{n}, \frac{k+1}{n}) \text{ and } 0 \leq k < n \\ -2(x - \frac{k+1}{n}) & x \in [\frac{k}{n} + \frac{1}{2n}, \frac{k+1}{n}) \text{ and } 0 \leq k < n \\ 0 & x = 1 \end{cases} \]

Let \(a_n = 1/2 - 2^{-(n+1)}\) and \(b_n = a_n + 2^{-(n+2)}\) be decreasing and disjoint dyadic intervals \([a_n, b_n]\) in \([0,1/2]\). Note that \(a_{n+1} - b_n = b_n - a_n\), and so, the gap between \([a_n, b_n]\) and \([a_{n+1}, b_{n+1}]\) is equal in size to the length of the former interval. We define

\[ s(x) = \sum \chi_{[a_n, b_n]}(x) \cdot (b_n - a_n) \cdot s(n+1)\times((x - a_n)/(b_n - a_n)) \]

and note that this function is triangular on \([a_n, b_n]\) with slope 2 but decreasing amplitude and increasing frequency, such that it has \((n + 1)^n\) maxima on \([a_n, b_n]\). The map \(S : (x, y) \mapsto (x, y + s(x))\) is easily shown to be bi-Lipschitz on \([0,1] \times \mathbb{R}\) with Lipschitz constant \(\sqrt{5}\). Further, let

\[ g_1(x) = \int_0^x \left( 1 - \sum_{n=1}^{\infty} \chi_{[a_n, b_n]}(y) \right) \, dy. \]

\(^3\)Here, dimension refers to any of the commonly considered dimensions: Hausdorff, packing, box-counting, Assouad, and lower dimension.
It is clear that \( g_1(x) \) is a non-decreasing function with slope 0 in \([a_n,b_n]\) and slope 1 otherwise. Let \( g_2(x) = -2g_1(1/2)x + 2g_1(1/2) \) and

\[
g(x) = \begin{cases} 
g_1(x) & 0 \leq x < 1/2 \\
g_2(x) & 1/2 \leq x \leq 1
\end{cases}
\]

Similarly, \( G : (x, y) \mapsto (x, y + g(x)) \) is bi-Lipschitz on \([0, 1] \times \mathbb{R}\) with Lipschitz constant \( \sqrt{2} \). It can be checked that \( g(x) + s(x) \) is an excursion function on \([0, 1]\). The graph \((x, g(x) + s(x))\) is the image of \((x, 0)\) under \( S \circ G \) and hence is a bi-Lipschitz image of the unit line. Therefore the Assouad dimension of the excursion graph \( g(x) + s(x) \) is 1, i.e. as low as possible for a continuous function. To see that the resulting continuum tree \( T_{g(x) + s(x)} \) is starry, one only needs to observe that due to the triangle function there are approximate \((n+1)^n\)-stars centred at \( a_n \).

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