S1. HIGH-FREQUENCY EXPANSION FOR THE FLOQUET-LINDBLAD EQUATION

Here we provide the detailed derivation of the high-frequency expansion of the Floquet-Lindblad equation. The Lindblad equation that we discuss in this work is symbolically represented as

$$\partial_t \rho(t) = \mathcal{L}(t) \rho(t),$$  \hspace{1cm} (S1)

where the time-dependent Liouvillian $\mathcal{L}(t)$ is defined by

$$\mathcal{L}(t) \rho = -i[H(t), \rho] + \mathcal{D}(\rho),$$  \hspace{1cm} (S2)

where $H(t) = H(t + T)$ is the periodic Hamiltonian and $\mathcal{D}(\rho)$ denotes the dissipation term represented by the Lindblad operators. We introduce the Fourier series for the Liouvillian as

$$\mathcal{L}(t) = \sum_m \mathcal{L}_m e^{-i\omega_m t}.$$  \hspace{1cm} (S3)

Since the Lindblad operators $L_{ij}$ are time-independent in this work, each Fourier component is given as follows:

$$\mathcal{L}_0 \rho = -i[H_0, \rho] + \mathcal{D}(\rho); \quad \mathcal{L}_m \rho = -i[H_m, \rho] \quad (m \neq 0).$$  \hspace{1cm} (S4)

The formal solution of Eq. (S1) is obtained as $\rho(t) = \mathcal{V}(t, t') \rho(t')$ with the propagator

$$\mathcal{V}(t, t') = \mathcal{T} \exp \left[ \int_{t'}^t \mathcal{L}(s) ds \right],$$  \hspace{1cm} (S5)

where $\mathcal{T} \exp$ denotes the time-ordered exponential. The determining equations for $\mathcal{V}$ are

$$\partial_s \mathcal{V}(t, t') = \mathcal{L}(t) \mathcal{V}(t, t'),$$  \hspace{1cm} (S6)

$$\mathcal{V}(t', t') = 1.$$  \hspace{1cm} (S7)

The high-frequency expansion in terms of the van Vleck approach makes the following ansatz:

$$\mathcal{V}(t, t') = e^{\mathcal{G}(t)} e^{(t-t')\mathcal{L}_{\text{eff}}} e^{-\mathcal{G}(t')}.$$  \hspace{1cm} (S8)

where $\mathcal{G}(t)$ is periodic in time and $\mathcal{L}_{\text{eff}}$ is time-independent. This ansatz satisfies Eq. (S7) for any choices of $\mathcal{G}(t)$ and $\mathcal{L}_{\text{eff}}$, and what determines these two is Eq. (S6). As we will see below, Eq. (S6) only determines the derivative of $\mathcal{G}(t)$ and thus we further impose $\int_0^T \mathcal{G}(t) dt = 0$ to fix the constant of integration.

To obtain the determining equations for $\mathcal{G}(t)$ and $\mathcal{L}_{\text{eff}}$, we substitute Eq. (S8) into Eq. (S6), having

$$\partial_t (e^{\mathcal{G}(t)}) + e^{\mathcal{G}(t)} \mathcal{L}_{\text{eff}} = \mathcal{L}(t) e^{\mathcal{G}(t)}.$$  \hspace{1cm} (S9)

To rewrite the first term on the left-hand side we invoke the Wilcoxon formula $\partial_t e^{-\beta H} = -\alpha e^{-\beta H} (\partial_\alpha H) e^{-(\beta - \alpha) H} du$ for $H = H(\lambda)$. We replace $\lambda$, $H$, and $\beta$ by $t$, $-\mathcal{G}$, and 1, respectively, obtaining

$$\partial_t (e^{\mathcal{G}(t)}) = \left\{ \int_0^1 e^{u \mathcal{G}(t)} [\partial_t \mathcal{G}(t)] e^{-u \mathcal{G}(t)} du \right\} e^{\mathcal{G}(t)} = \left\{ \int_0^1 e^{u \gamma} du [\partial_t \mathcal{G}(t)] \right\} e^{\mathcal{G}(t)} = \{\phi(\gamma \mathcal{G})[\partial_t \mathcal{G}(t)]\} e^{\mathcal{G}(t)}.$$  \hspace{1cm} (S10)

Here $\gamma \mathcal{G}$ is defined by $\gamma \mathcal{G} = [\mathcal{G}(t), \gamma]$ and $\phi(x) \equiv (e^x - 1)/x$. We substitute Eq. (S10) into Eq. (S9) and have

$$\partial_t \mathcal{G}(t) = \phi^{-1}(\gamma \mathcal{G}) \mathcal{L}(t) - \phi^{-1}(\gamma \mathcal{G}) e^{\gamma \mathcal{G}} \mathcal{L}_{\text{eff}}.$$  \hspace{1cm} (S11)
Now we notice $\phi^{-1}(x)e^x = \phi^{-1}(-x)$ and make use of the Taylor expansion of $\phi^{-1}(x)$: $\phi^{-1}(x) = \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k$, where $B_k$ denotes the Bernoulli number ($B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6 \cdots$). Then we obtain

$$\partial_t \mathcal{G}(t) = \sum_{k=0}^{\infty} \frac{B_k}{k!} (\text{ad}_\rho)^k \left[ \mathcal{L}(t) + (-1)^{k+1} \mathcal{L}_{\text{eff}} \right].$$ \hfill (S12)

Now we determine $\mathcal{G}(t)$ and $\mathcal{L}_{\text{eff}}$ from Eq. (S12) by the series expansions

$$\mathcal{G}(t) = \sum_{k=1}^{\infty} \mathcal{G}^{(k)}(t); \quad \mathcal{L}_{\text{eff}} = \sum_{k=1}^{\infty} \mathcal{L}_{\text{eff}}^{(k)}.$$ \hfill (S13)

We substitute these expansions into Eq. (S12) and find the order-by-order solutions, where we assign an order 1 for $\mathcal{L}(t)$ and $k$ for $\mathcal{G}^{(k)}(t)$ and $\mathcal{L}_{\text{eff}}^{(k)}$ (see Ref. (6) for the case of unitary dynamics).

The first-order equation leads to

$$\partial_t \mathcal{G}^{(1)}(t) = \mathcal{L}(t) - \mathcal{L}_{\text{eff}}^{(1)}.$$ \hfill (S14)

To obtain $\mathcal{L}_{\text{eff}}^{(1)}$, we integrate Eq. (S14) over $0 \leq t \leq T$. With the periodicity $\mathcal{G}(T) = \mathcal{G}(0)$, we obtain

$$\mathcal{L}_{\text{eff}}^{(1)} = \int_0^T \frac{dt}{T} \mathcal{L}(t) = \mathcal{L}_0.$$ \hfill (S15)

To obtain $\mathcal{G}(t)$, we integrate Eq. (S14), having

$$\mathcal{G}^{(1)}(t) - \mathcal{G}^{(1)}(0) = \int_0^t \mathcal{L}(s)ds - t\mathcal{L}_{\text{eff}}^{(1)} = t\mathcal{L}_0 + \sum_{m \neq 0} \frac{e^{-im\omega t} - 1}{-im\omega} \mathcal{L}_m - t\mathcal{L}_{\text{eff}}^{(1)},$$ \hfill (S16)

which means

$$\mathcal{G}^{(1)}(t) = \frac{i}{\omega} \sum_{m \neq 0} \frac{e^{-im\omega t}}{m} \mathcal{L}_m.$$ \hfill (S17)

Note that $\mathcal{L}_{\text{eff}}^{(1)}$ is $O(\omega^0)$ and $\mathcal{G}^{(1)}(t)$ is $O(\omega^{-1})$.

The second-order equation leads to

$$\partial_t \mathcal{G}^{(2)}(t) = -\frac{1}{2} [\mathcal{G}^{(1)}(t), \mathcal{L}(t) + \mathcal{L}_{\text{eff}}^{(1)}] - \mathcal{L}_{\text{eff}}^{(2)}.$$ \hfill (S18)

To obtain $\mathcal{L}_{\text{eff}}^{(2)}$, we integrate Eq. (S18) over $0 \leq t \leq T$. Upon this, we note that $\mathcal{G}^{(2)}(t)$ is periodic and $\int_0^T \mathcal{G}^{(1)}(t)dt = 0$. Then we have

$$\mathcal{L}_{\text{eff}}^{(2)} = -\frac{1}{2} \int_0^T \frac{dt}{T} [\mathcal{G}^{(1)}(t), \mathcal{L}(t)] = -\frac{i}{2\omega} \sum_{m \neq 0} \frac{[\mathcal{L}_m, \mathcal{L}_{-m}]}{m} = -\frac{i}{\omega} \sum_{m > 0} \frac{[\mathcal{L}_m, \mathcal{L}_{-m}]}{m}.$$ \hfill (S19)

By straightforward calculations, one can obtain $\mathcal{G}^{(2)}(t)$ by integrating Eq. (S18) from 0 to $t$. Likewise, one could systematically build the higher order solutions although we do not go further here.

Let us rewrite $\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{eff}}^{(1)} + \mathcal{L}_{\text{eff}}^{(2)}$ in terms of the effective Hamiltonian $H_{\text{eff}}$ \hfill (5, 6)

$$H_{\text{eff}} = H_0 + \frac{1}{\omega} \sum_{m > 0} \frac{[H_{-m}, H_m]}{m} + O(\omega^{-2}).$$ \hfill (S20)

To do this, we consider the action of $\mathcal{L}_{\text{eff}}^{(2)}$ onto a density operator $\rho$. From Eqs. (S21) and (S14), we have

$$\mathcal{L}_{\text{eff}}^{(2)} \rho = -\frac{i}{\omega} \sum_{m > 0} \frac{1}{m} ([\mathcal{L}_m \mathcal{L}_{-m}, \rho] - [\mathcal{L}_{-m} \mathcal{L}_m, \rho]) = -\frac{i}{\omega} \sum_{m > 0} \frac{1}{m} ([H_m, [H_m, \rho]] - [H_{-m}, [H_{-m}, \rho]])$$

$$= -\frac{i}{\omega} \sum_{m > 0} \frac{1}{m} [[H_{-m}, H_m], \rho],$$ \hfill (S21)

where we have used the Jacobi identity $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$. Combining Eqs. (S4), (S15), (S21), and (S20), we obtain

$$\mathcal{L}_{\text{eff}} \rho = -i[H_{\text{eff}}, \rho] + \mathcal{D}(\rho) + O(\omega^{-2}).$$ \hfill (S22)

We remark that $\mathcal{L}_{\text{eff}} \rho$ is not equal to $-i[H_{\text{eff}}, \rho] + \mathcal{D}(\rho)$ at higher orders because $\mathcal{L}_{\text{eff}}$ involves contributions of $\mathcal{D}$ from $O(\omega^{-2})$. 
S2. GENERALIZATION TO THE DEGENERATE ENERGY SPECTRA

Here we generalize our main results [Eqs. (9)–(11)] to the cases in which the energy spectrum \( \{ E_i \}_{i=1}^{N} \) is degenerate. To deal with such a spectrum, we introduce new notations for the eigenenergies and eigenstates of \( H_0 \) given by \( E_i^\alpha \) and \( |E_i^\alpha\rangle \) with \( H_0 |E_i^\alpha\rangle = E_i^\alpha |E_i^\alpha\rangle \). Here, \( i = 1, \ldots, M \) labels the distinct eigenenergies and \( \alpha = 1, 2, \ldots, N_i \) does the degenerate eigenstates, and we assume the orthonormality \( \langle E_i^\alpha | E_j^\beta \rangle = \delta_{ij} \delta_{\alpha\beta} \). We remark that the choice of the degenerate eigenstates has arbitrariness up to unitary transformation for each degenerate subspace:

\[
|E_i^\alpha\rangle \rightarrow |\tilde{E}_i^\alpha\rangle = \sum_{\beta=1}^{N_i} |E_i^\beta\rangle U_{\beta\alpha}^{(i)},
\]

where \( U_{\beta\alpha}^{(i)} \) is an \( N_i \times N_i \) unitary matrix. We should be aware that the following formulation needs to be invariant under the unitary transformation (S23).

The Lindblad operators with the detailed balance condition are generalized as follows: \( L_{ij} \rightarrow L_{i\alpha,j\alpha'} \equiv |E_i^\alpha\rangle \langle E_j^{\alpha'}| \). The corresponding transition rates are written as \( \Gamma_{i\alpha,j\alpha'} \), which are assumed independent of the degeneracy labels \( \alpha \) or \( \alpha' \) and to satisfy the detailed balance condition:

\[
\Gamma_{i\alpha,j\alpha'} e^{-\beta E_j} = \Gamma_{j\alpha',i\alpha} e^{-\beta E_i} \quad \text{(for} i \neq j),
\]

and \( \Gamma_{i\alpha,j\alpha'} = 0 \) for \( i = j \). We also assume that the transition rates \( \Gamma_{i\alpha,j\alpha'} \) are irreducible, which is satisfied, for example, if \( \Gamma_{i\alpha,j\alpha'} > 0 \) for all pairs of \( i \) and \( j \). Then the dissipation term in the Lindblad equation is given by

\[
\mathcal{D}(\rho) = \sum_{i\alpha,j\beta \ (i \neq j)} \Gamma_{i\alpha,j\beta} \left( L_{i\alpha,j\beta} \rho L_{i\alpha,j\beta}^\dagger - \frac{1}{2} \{ L_{i\alpha,j\beta}^\dagger L_{i\alpha,j\beta}, \rho \} \right).
\]  

As one can check easily, the dissipation term (S25) is invariant under Eq. (S23).

Now that we have the Lindblad equation, we can repeat the high-frequency-expansion arguments in Sec. S1 to obtain Eq. (S22) for the generalized \( \mathcal{D} \) term (S25). Thus, we move on to deriving the counterparts of the main results [Eqs. (9)–(11)] by generalizing the arguments in Materials and Methods.

Let us solve \( \mathcal{L}_{\text{eff}} \rho'_\infty = 0 \) for \( \rho'_\infty \) at the leading order of \( \omega^{-1} \). The solution \( \rho'_\infty \) is necessarily written in the following form:

\[
\rho'_\infty = \rho'^{(d)}_\infty + \rho'^{(od)}_\infty,
\]

\[
\rho'^{(d)}_\infty = \sum_{k,\alpha,\beta} \rho'^{(k,\alpha,\beta)} \langle E_k^\alpha | E_k^\beta \rangle,
\]

\[
\rho'^{(od)}_\infty = \sum_{k,\alpha,\beta} \rho'^{(k,\alpha,\beta)} |E_k^\alpha\rangle \langle E_k^\beta|.
\]

Since we have arbitrariness of choosing the degenerate eigenstates as noted above, we can assume without loss of generality that \( \rho'^{(d)}_\infty \) is diagonal

\[
\rho'^{(d)}_{\infty, k\alpha, k\beta} = q_{k\alpha} \delta_{\alpha\beta},
\]

where \( q_{k\alpha} \geq 0 \).

First, we focus on the off-diagonal matrix elements of \( \mathcal{L}_{\text{eff}} \rho'_\infty = 0 \): \( \langle E_k^\alpha | \mathcal{L}_{\text{eff}} \rho'_\infty | E_l^\beta \rangle = 0 \). Repeating similar arguments in deriving Eq. (30), we have

\[
\rho'^{(od)}_{\infty, k\alpha, l\beta} = \frac{\langle E_k^\alpha | \Delta H_{\text{eff}} | E_l^\beta \rangle}{(E_k - E_l) - i\gamma_{kl}} (q_{k\alpha} - q_{l\beta}) + O(\omega^{-2}),
\]

where we have introduced the working hypothesis \( q_{k\alpha} = O(\omega^0) \) and \( \gamma_{kl} = \sum_{\gamma} (\Gamma_{\gamma,k\alpha} + \Gamma_{\gamma,l\beta})/2 \) (Remember that \( \Gamma_{k\alpha,l\beta} \) is independent of \( \alpha \) or \( \beta \)).

Next, we consider the diagonal elements of \( \mathcal{L}_{\text{eff}} \rho'_\infty = 0 \): \( \langle E_k^\alpha | \mathcal{L}_{\text{eff}} \rho'_\infty | E_k^\beta \rangle = 0 \). While, for \( \alpha \neq \beta \), we have irrelevant equations of \( O(\omega^{-2}) \), for \( \alpha = \beta \), we have

\[
\sum_{l,\beta} (\Gamma_{k\alpha,l\beta} q_{l\beta} - \Gamma_{l\beta,k\alpha} q_{k\alpha}) = 0.
\]
According to the irreducibility of $\Gamma_{\alpha\beta}$, this equation has the unique positive solution, which is given by

$$q_{\alpha} = p^{(k)}_{\text{can}} = \frac{e^{-\beta E_k}}{Z},$$

with $Z = \sum_{k,\alpha} e^{-\beta E_k}$. One can confirm this by using the detailed balance condition. From the above argument, we obtain

$$\rho_\infty = \rho_{\text{can}} + \sigma_{\text{FE}} + O(\omega^{-2}),$$

where

$$\langle E^\alpha_k | \sigma_{\text{FE}} | E^\beta_l \rangle = \frac{\langle E^\alpha_k | \Delta H_{\text{eff}} | E^\beta_l \rangle}{(E_k - E_l) - t\gamma_{kl}} (p^{(k)}_{\text{can}} - p^{(l)}_{\text{can}}) \quad (k \neq l)$$

and $\langle E^\alpha_k | \sigma_{\text{FE}} | E^\beta_l \rangle = 0$ for $k = l$.

Finally, we calculate the time-dependent density matrix by $\rho(t) = e^{\hat{H}t} \rho_\infty$, obtaining

$$\rho(t) = \rho_{\text{can}} + \sigma_{\text{MM}}(t) + \sigma_{\text{FE}} + O(\omega^{-2}),$$

where $\sigma_{\text{MM}}(t)$ is the same as Eq. (10) for the nondegenerate case.

To summarize, our main results [Eqs. (9)–(11)] are generalized in a straightforward manner. Among the three terms on the right-hand side of Eq. (S35), the first two $\rho_{\text{can}}$ and $\sigma_{\text{MM}}(t)$ are expressed exactly in the same way for the degenerate case, and the third one $\sigma_{\text{FE}}$ is naturally generalized as in Eq. (S34).

**S3. All the Observables in the Effective Model for the NV Center**

Although we have discussed the two observables $S_z$ and $\{S_x, S_y\}$, there are in total 8 observables including these two (since we are considering a spin-1 system represented by $3 \times 3$ matrices): the spins along one direction, $S_x, S_y$ and $S_z$, and the nematics $S_x^2, S_y^2, S_x S_y + S_y S_x, S_y S_z + S_z S_y$, and $S_z S_x + S_x S_z$. In this section, we consider all these observables and validate our main results [Eqs. (9)–(11)].

**A. Vanishing one-cycle averages due to dynamical symmetry**

We compare the one-cycle average $\langle A(\omega) \rangle$ of an observable $A$ for the actual dynamics with that from our formulas [Eqs. (9)–(11)] and the FGS. Upon this comparison, we note that the average vanishes for $A = S_x, S_y, S_y S_z + S_z S_y$, and $S_z S_x + S_x S_z$. The common property shared by these observables is that they are all odd under the $\pi$-rotation around the $S_z$ axis:

$$U_{\pi}^z A U_{\pi}^{-z} = -A \quad \text{for} \quad A = S_x, S_y, S_y S_z + S_z S_y, S_z S_x + S_x S_z \quad (S36)$$

Another important property is the dynamical symmetry associated with this unitary operation:

$$U_{\pi}^z H_{\text{NV}}(t + T/2) U_{\pi}^{-z} = H_{\text{NV}}(t). \quad (S37)$$

As we see below, Eqs. (S36) and (S37) imply that the one-cycle averages for these observables vanish in the actual calculation, our formulas [Eqs. (9)–(11)], and the FGS, respectively.

First, we discuss the actual dynamics governed by the Lindblad equation:

$$\partial_t \rho(t) = -i[H(t), \rho(t)] + D[\rho(t)]. \quad (S38)$$

We try to have some implication of the dynamical symmetry (S37) to this equation. For this purpose, we shift $t \rightarrow t + T/2$ in the equation and apply $U_{\pi}^z$ from left and $U_{\pi}^{-z}$ from right to the both sides of the equation, having

$$\partial_t \rho^{U^z}(t) = -i[H_{\text{NV}}(t), \rho^{U^z}(t)] + D'[\rho^{U^z}(t)],$$

where $\rho^{U^z}(t) = U_{\pi}^z \rho(t + T/2) U_{\pi}^{-z}$, $D'$ is defined by $L_{ij} \rightarrow L_{ij} = U_{\pi}^z L_{ij} U_{\pi}^{-z}$ in $D$, and we have used Eq. (S37). In fact, $D' = D$ holds true because the time-independent part $H_{\text{NV}}^0$ of $H_{\text{NV}}(t)$ is invariant under $U_{\pi}^z$: $[U_{\pi}^z, H_{\text{NV}}^0] = 0$ and hence the energy eigenstates $|E_k\rangle$ are the simultaneous eigenstates for $H_{\text{NV}}^0$ and $U_{\pi}^z$ (recall that $L_{ij}$ appears together with $L_{ij}^1$ in $D$). Therefore, we have

$$\partial_t \rho^{U^z}(t) = -i[H_{\text{NV}}(t), \rho^{U^z}(t)] + D[\rho^{U^z}(t)], \quad (S39)$$
which is the same as Eq. (S38). As is the case in the high-frequency expansion, we assume that Eq. (S38) leads to the unique time-periodic NESS $\rho_{\text{NESS}}(t) = \rho_{\text{NESS}}(t + T)$ in $t \gg \gamma^{-1}$. Then we have

$$
\rho_{\text{NESS}}(t) = U_z^* \rho_{\text{NESS}}(t + T/2) U_z^T.
$$

(S40)

From this equation, we have the one-cycle average of an observable in Eq. (S36) as

$$
\bar{A} = \int_0^T \frac{dt}{T} \text{tr} [\rho_{\text{NESS}}(t) A] = \int_0^T \frac{dt}{T} \text{tr} [\rho_{\text{NESS}}(t + T/2) U_z^T A U_z] = \int_0^T \frac{dt}{T} \text{tr} [\rho_{\text{NESS}}(t) (-A)] = -\bar{A},
$$

(S41)

which means $\bar{A} = 0$ for the NESS. To obtain this, we have used, the cyclic property of trace, the periodicity of $\rho_{\text{NESS}}(t)$, and Eq. (S36).

Second, we show that those one-cycle averages vanish in our formula [Eq. (9)] as well. Recall that the micromotion part $\sigma_{\text{MM}}(t)$ does not contribute and neither $\rho_{\text{can}}$ nor $\sigma_{\text{FE}}$ depends on time. Thus we are to prove $\text{tr} [\rho_{\text{can}} A] = \text{tr} [\sigma_{\text{FE}} A] = 0$. The first equation $\text{tr} [\rho_{\text{can}} A] = 0$ follows from the invariance of the static Hamiltonian $[U_z^T, H_{\text{NV}}^0] = 0$ and Eq. (S36). To show the second one $\text{tr} [\sigma_{\text{FE}} A] = 0$, we translate the dynamical symmetry [Eq. (S37)] into the Fourier components:

$$
(-1)^m U_z^T H_{\text{m}} U_z^T = H_m,
$$

(S42)

which is obtained by Fourier-expanding both sides of Eq. (S37). This relation implies that the effective Hamiltonian is invariant under the unitary transformation: $U_z^T H_{\text{eff}} U_z^T = H_{\text{eff}}$ and hence $U_z^T \Delta H_{\text{eff}} U_z^T = \Delta H_{\text{eff}}$. In fact, this relation leads to the invariance of the Floquet-engineering part $\sigma_{\text{FE}}$:

$$
U_z^T \sigma_{\text{FE}} U_z^T = \sigma_{\text{FE}}.
$$

(S43)

To show Eq. (S43), we compare the matrix elements in the energy eigenbasis. This basis is convenient because $U_z^T |E_k\rangle = e^{i\theta_k} |E_k\rangle$ holds true. The left-hand side of Eq. (S43) gives

$$
\langle E_k | U_z^T \sigma_{\text{FE}} U_z^T | E_l \rangle = e^{i\theta_k} \langle E_k | \sigma_{\text{FE}} | E_l \rangle e^{-i\theta_l} = \frac{e^{i\theta_k}}{(E_k - E_l) - i\gamma_{kl}} (p_{\text{can}}^{(k)} - p_{\text{can}}^{(l)}) \langle E_k | \Delta H_{\text{eff}} | E_l \rangle + \frac{e^{-i\theta_l}}{(E_k - E_l) + i\gamma_{kl}} (p_{\text{can}}^{(l)} - p_{\text{can}}^{(k)}) \langle E_k | \Delta H_{\text{eff}} | E_l \rangle = \langle E_k | \sigma_{\text{FE}} | E_l \rangle,
$$

(S44)

which thus equals the right-hand side of Eq. (S43). Thus Eq. (S43) has been proved and leads to $\text{tr} [\sigma_{\text{FE}} A] = 0$ together with Eq. (S36). Therefore, the one-cycle averages for the observables in Eq. (S36) vanish in our formula (9). Finally, we show that the one-cycle averages for those observables vanish in the FGS. In fact, a stronger statement holds true: The one-cycle average vanishes for each Floquet state,

$$
\int_0^T \frac{dt}{T} \langle u_i(t) | A | u_i(t) \rangle = \int_0^T \frac{dt}{T} \text{tr} [u_i(t) \langle u_i(t) | A | u_i(t) \rangle] = 0.
$$

(S45)

Thanks to Eq. (S36), it is sufficient to show that the one-cycle-averaged Floquet state

$$
\bar{\rho}_i^{\text{FS}} = \int_0^T \frac{dt}{T} |u_i(t)\rangle \langle u_i(t)|.
$$

(S46)

is invariant under $U_z^T$ for each $i$. This invariance follows from the dynamical symmetry (S37) as follows. Let us remember the defining equation of the Floquet state

$$
\left[ H_{\text{NV}}(t) - i \frac{d}{dt} \right] |u_i(t)\rangle = \epsilon_i |u_i(t)\rangle.
$$

(S47)

By applying $U_z^T$ from left, shifting time as $t \to t + T/2$, and making use of the dynamical symmetry (S37), we have

$$
\left[ H_{\text{NV}}(t) - i \frac{d}{dt} \right] U_z^T |u_i(t + T/2)\rangle = \epsilon_i U_z^T |u_i(t + T/2)\rangle.
$$

(S48)

Thus $U_z^T |u_i(t + T/2)\rangle$ is also the Floquet state with quasienergy $\epsilon_i$. Assuming that the quasienergies are not degenerate, we obtain

$$
U_z^T |u_i(t + T/2)\rangle = e^{i\epsilon_i \gamma_i} |u_i(t)\rangle
$$

(S49)
for some phase $\varphi_i$. Noticing the periodicity $|u_i(t + T)| = |u_i(t)|$, we obtain

$$\rho_i^{FS} = \frac{1}{T} \int_0^T ds u_i(s) \langle u_i(s) | U_T^{-1} | u_i(s + T/2) \rangle U_T^{-1} = U_T \rho_i^{FS} U_T^\dagger,$$

which means $\rho_i^{FS}$ is invariant under the unitary transform $U_T$ and hence $\text{tr}[\rho_i^{FS}A] = 0$. By taking the weighted average with $p^{(i)}_{FG} = e^{-\beta\epsilon_i}/Z_{FG}$, we obtain

$$\int_0^T dt \frac{1}{T} \text{tr}[\rho_{FG}(t)A] = \sum_i p^{(i)}_{FG} \text{tr}[\rho_i^{FS}A] = 0$$

for $A$ in Eq. (S36). We note that, by replacing $p^{(i)}_{FG}$ by $p^{(i)}_{can}$, we obtain the same-type equation for the canonical Floquet steady state.

### B. Nonvanishing one-cycle averages

We have shown that the one-cycle averages for the four observables in Eq. (S36) vanish for the actual dynamics, our formulas [Eqs. (9)–(11)], and the FGS, respectively. In other words, our formulas and the FGS both respect the dynamical symmetry (S37) and give precise descriptions for these observables.

Thus, for the complete comparison, we are to discuss the remaining four observables: $S_z, S_g^2 - S_y^2, S_y$, and $\{S_x, S_y\}$. In Fig. S1, we plot the deviation of the one-cycle average calculated by our formula and the FGS (as well as the canonical Floquet steady state for future reference) from that of the actual dynamics. While the deviation of the FGS is $O(\omega^{-1})$ for all these observables, that of our formula is $O(\omega^{-2})$. Thus our formula correctly describes all the observables at $O(\omega^{-1})$.

![FIG. S1. Difference of the one-cycle average calculated from the actual dynamics. The difference is calculated for our formula [Eq. (9)] (circle), the FGS (square), and the CFSS (triangle) and plotted against the driving frequency $\omega$. Each panel shows the result for the observables (a) $S_z$, (b) $S_g^2 - S_y^2$, (c) $S_y$, and (d) $\{S_x, S_y\}$. The solid and dashed lines are the guides to the eye showing the lines with slopes $-2$ and $-1$, respectively.](image)

### C. One-cycle standard deviations

In the paper, we have discussed the difference of the one-cycle standard deviation $\Delta \Sigma_A(\omega)$ for the representative two observables $A = S_z$ and $\{S_x, S_y\}$. Here we supplement the data, plotting $\Delta \Sigma_A(\omega)$ for all the eight observables calculated with our formula [Eqs. (9) and (10)], the FGS (as well as the CFSS for future reference) in Fig. S2.

The difference $\Delta \Sigma_A(\omega)$ between the actual dynamics and our formula is $O(\omega^{-2})$ for all observables as shown in Fig. S2. This result supports that our micromotion part $\sigma_{MM}(t)$ properly describes the NESS at $O(\omega^{-1})$. Quantitatively, $\Delta \Sigma_A(\omega)$ is smaller for the FGS, where all-order contributions in $\omega^{-1}$ are included. We could improve the accuracy of our formula by extending our formula to higher orders.
FIG. S2. Difference of the one-cycle standard deviation calculated from the actual dynamics. The difference is calculated for our formula [Eq. (9)] (circle), the FGS (square), and the CFSS (triangle) and plotted against the driving frequency \( \omega \). Each panel shows the result for the observables (a) \( S_x \), (b) \( S_y \), (c) \( S_z \), (d) \( S_x^2 - S_y^2 \), (e) \( S_z^2 \), (f) \( \{S_x, S_y\} \), (g) \( \{S_y, S_z\} \) and (h) \( \{S_z, S_x\} \). The solid and dashed lines are the guides to the eye showing the lines with slopes \(-2\) and \(-1\), respectively.

S4. BREAKDOWN OF ANTIUNITARY DYNAMICAL SYMMETRY

We supplement the argument in the paper that the one-cycle average of \( A = \{S_x, S_y\} \) vanishes for the FGS but does not for the actual dynamics and our formulas [Eqs. (9)-(11)]. In the paper, we have shown that the antiunitary operator \( V \) and the associated dynamical symmetry

\[
V H_{NV} (T - t) V^\dagger = H_{NV}(t)
\]  

(S52)

lead to the vanishing one-cycle average for the FGS. Let us see how such an antiunitary dynamical symmetry does not constrain the actual dynamics or our formula due to dissipation.

First, we discuss the actual dynamics described by the Lindblad equation (S38). To utilize the antiunitary dynamical symmetry, we substitute \( t \) by \( T - t \) and multiply \( V \) from left and \( V^\dagger \) from right. Then, we have

\[
-\partial_t \rho^{V}(t) = i[H_{NV}(t), \rho^{V}(t)] + D''[\rho^{V}(t)],
\]  

(S53)

where we have used Eq. (S52), \( \rho^{V}(t) \equiv V \rho(T - t) V^\dagger \), and \( D'' \) is defined by \( L_{ij} \rightarrow L''_{ij} = V L_{ij} V^\dagger \) in \( D \). We notice that \( D'' = D \) because the time-independent Hamiltonian \( H_{NV} \) is invariant under the antiunitary transform \( V \) similarly to the argument in Sec. S3A. Therefore, Eq. (S53) leads to

\[
\partial_t \rho^{V}(t) = -i[H_{NV}(t), \rho^{V}(t)] - D[\rho^{V}(t)].
\]  

(S54)

We note that the sign of the \( D \) term has changed from the original Lindblad equation (S38) and \( \rho^{V}(t) \) cannot be related directly to \( \rho(t) \). Thus the antiunitary dynamical symmetry (S52) does not constrain the actual dynamics in the presence of dissipation.

Second, we show that our formula is not constrained by the antiunitary dynamical symmetry (S52). More concretely, we have \( V \sigma_{FE} V^\dagger \neq \sigma_{FE} \) unlike the case of unitary transformations. To show this, we first notice that the dynamical symmetry (S52) leads to \( V H_m V^\dagger = H_m \) for the Fourier components and to \( V H_{eff} V^\dagger = H_{eff} \) and hence \( V \Delta H_{eff} V^\dagger = \Delta H_{eff} \). We second notice \( V = K U^z_r \), where \( K \) is the complex conjugate operator. Then, we consider the matrix
elements of $V\sigma_{FE} V^\dagger$ in the energy eigenbasis:

$$
\langle E_k | V\sigma_{FE} V^\dagger | E_l \rangle = e^{i\theta_k} \langle E_k | K\sigma_{FE} K | E_l \rangle e^{-i\theta_l} = e^{i\theta_k} \langle E_k | \sigma_{FE} | E_l \rangle^* e^{-i\theta_l}
$$

$$
= e^{i\theta_k} \langle E_k | \Delta H_{eff} | E_l \rangle^* e^{-i\theta_l} \left( \langle p_{\text{can}}^{(k)} - p_{\text{can}}^{(l)} \rangle \right) = \frac{\langle E_k | U^{\dagger} \Delta H_{eff} U | E_l \rangle^*}{(E_k - E_l) + i\gamma_{kl}} \left( \langle p_{\text{can}}^{(k)} - p_{\text{can}}^{(l)} \rangle \right).
$$

(S55)

Although $\langle E_k | \Delta H_{eff} | E_l \rangle^* = \langle E_k | \Delta H_{eff} | E_l \rangle$ in fact, the sign of $\gamma_{kl}$ has changed from $\langle E_k | \Delta H_{eff} | E_l \rangle$. Thus, in the presence of dissipation, $V\sigma_{FE} V^\dagger \neq \sigma_{FE}$ and $\text{tr}(\sigma_{FE} A) \neq 0$ in general even if $VAV^\dagger = -A$.

**S5. CANONICAL FLOQUET STEADY STATE (CFSS)**

Here we introduce the canonical Floquet steady state (CFSS)

$$
\rho_{\text{CFSS}}(t) = \frac{1}{Z} \sum_i e^{-\beta E_i} |u_i(t)\rangle \langle u_i(t)| = \frac{1}{Z} \sum_i e^{-\beta E_i} |\psi_i(t)\rangle \langle \psi_i(t)|,
$$

(S56)

where $Z = \sum_i e^{-\beta E_i}$, $|u_i(t)\rangle$ is the Floquet state, and $|\psi_i(t)\rangle = e^{-i\epsilon_i t} |u_i(t)\rangle$ is the corresponding solution of the time-dependent Schrödinger equation with $\epsilon_i$ being the quasienergy. Here, we have assumed that the driving frequency $\omega$ is so large and $|u_i(t)\rangle$ is so close to $|E_i\rangle$ that there is the one-to-one correspondence between $|E_i\rangle$ and $|u_i(t)\rangle$ for each index $i$.

The difference between the FGS and CFSS is the weight factor. This is defined by the quasienergy $\epsilon_i$ for the FGS whereas by the real energy $E_i$ for the CFSS. This difference is quantitatively important because $E_i - \epsilon_i = O(\omega^{-1})$ and the FGS and CFSS can give different scalings in $\omega$ at high frequency.

The difference of the one-cycle-averaged observables calculated for the actual dynamics and the CFSS is shown in Fig. S1. For the two observables $S_z$ and $S_z^2$, the CFSS gives the appropriate $\omega^{-2}$ scaling which is not captured by the FGS. For the other two $S_x^2$ and $S_y^2$, the CFSS deviates from the actual value at $O(\omega^{-1})$ and fails to describe the actual dynamics at $O(\omega^{-1})$. The CFSS thus provide partly improved descriptions for some observables than the FGS. It is noteworthy that the CFSS does not involve any information about the system-bath coupling like the FGS.

The difference of the one-cycle standard deviations $\Delta \Sigma_A(\omega)$ calculated for the actual dynamics and the CFSS is shown in Fig. S2. At high-frequency, the CFSS leads to more rapid decreases of $\Delta \Sigma_A(\omega)$ than the FGS for most observables. Thus the CFSS gives improved descriptions of the NESS than the FGS.

**S6. EQUIVALENCE OF OUR FORMULA AND CFSS IN $\Gamma_{ij} \to 0$**

Here we show that, in the weak dissipation limit $\Gamma_{ij} \to 0$, our formula [Eqs. (9)–(11)] coincides with the CFSS rather than the FGS. Since the extension to the degenerate case is straightforward, we consider the case where $H_0$ is nondegenerate for simplicity.

The weak dissipation limit of our formula is obtained just by replacing $\gamma_{ij}$ with 0 in $\sigma_{FE}$:

$$
\rho(t) = \rho_{\text{can}} + \sigma_{MM}(t) + \sigma_{FE} + O(\omega^{-2}),
$$

(S57)

with

$$
\langle E_k | \sigma_{FE} | E_l \rangle = \frac{\langle E_k | \Delta H_{eff} | E_l \rangle}{E_k - E_l} \langle p_{\text{can}}^{(k)} - p_{\text{can}}^{(l)} \rangle \quad (k \neq l)
$$

(S58)

and $\langle E_k | \sigma_{FE} | E_k \rangle = 0$. We will show that $\rho_{\text{CFSS}}(t)$ coincides with the above $\rho(t)$ by considering its high-frequency expansion.

This is achieved by finding the solution $|\psi_i(t)\rangle$ within the high-frequency expansion. According to Ref. (5), $|\psi_k(t)\rangle$ can be represented as

$$
|\psi_k(t)\rangle = e^{G(t)} |\psi_k(0)\rangle,
$$

(S59)
where
\[ G(t) = \frac{1}{\omega} \sum_{m \neq 0} e^{-im\omega t} m H_m + O(\omega^{-2}), \]  
(S60)
and \(|\psi_k(0)\rangle\) is the eigenstate of the effective Hamiltonian \(H_{\text{eff}} = H_0 + \Delta H_{\text{eff}}\) with eigenvalue \(\epsilon_k = E_k + O(\omega^{-1})\). Since \(\Delta H_{\text{eff}} = O(\omega^{-1})\) as shown in Sec. S1, \(|\psi_k(0)\rangle\) can be obtained by the standard perturbation technique as
\[ |\psi_k(0)\rangle = |E_k\rangle + \sum_{l \neq k} |E_l\rangle \frac{\langle E_l | \Delta H_{\text{eff}} | E_k \rangle}{E_k - E_l} + O(\omega^{-2}). \]  
(S61)
Substituting this equation into Eq. (S56), we obtain
\[
\rho_{\text{CFSS}}(t) = \sum_k p_{\text{can}}^{(k)} e^{G(t)} \left[ |E_k\rangle \langle E_k| + \sum_{l \neq k} \frac{\langle E_k | \Delta H_{\text{eff}} | E_l \rangle}{E_k - E_l} |E_k\rangle \langle E_l| + \sum_{l \neq k} \frac{\langle E_l | \Delta H_{\text{eff}} | E_k \rangle}{E_k - E_l} |E_l\rangle \langle E_k| \right] e^{-G(t)} + O(\omega^{-2}) \\
= e^{G(t)} \rho_{\text{can}} e^{-G(t)} + \sum_{k,l} \frac{p_{\text{can}}^{(k)}}{E_k - E_l} |E_k\rangle \langle E_l| + p_{\text{can}}^{(l)} \frac{\langle E_l | \Delta H_{\text{eff}} | E_k \rangle}{E_k - E_l} |E_l\rangle \langle E_k| + O(\omega^{-2}) \\
= e^{G(t)} \rho_{\text{can}} + \sum_{k,l} \frac{p_{\text{can}}^{(k)} - p_{\text{can}}^{(l)}}{E_k - E_l} |E_k\rangle \langle E_l| + O(\omega^{-2}) \\
= \rho_{\text{can}} + \sigma_{\text{MM}}(t) + \sigma_{\text{FE}} + O(\omega^{-2}),
\]  
(S62)
which is equal to our formula (S57). Here, \(G(t)\) was defined in Sec. S1 and we have used the Baker-Campbell-Hausdorff formula. We note that the FGS deviates from the CFSS in general by \(O(\omega^{-1})\) since \(E_i - \epsilon_i = O(\omega^{-1})\). Thus, in the small dissipation limit, the NESS coincides with the CFSS rather than the FGS.