Abstract—In recent years, a class of dictionaries have been proposed for multidimensional (tensor) data representation that exploit the structure of tensor data by imposing a Kronecker structure on the dictionary underlying the data. In this work, a novel algorithm called “STARK” is provided to learn Kronecker structured dictionaries that can represent tensors of any order. By establishing that the Kronecker product of any number of matrices can be rearranged to form a rank-1 tensor, we show that Kronecker structure can be enforced on the dictionary by solving a rank-1 tensor recovery problem. Because rank-1 tensor recovery is a challenging nonconvex problem, we resort to solving a convex relaxation of this problem. Empirical experiments on synthetic and real data show promising results for our proposed algorithm.

I. INTRODUCTION

Sparse representations of data have been widely used in a variety of information processing tasks such as data compression, feature extraction, data classification, signal denoising and inpainting, and audio processing [1]–[3]. One of the powerful techniques to obtain sparse representations is dictionary learning (DL) which can be formulated as

$$\min_{D,X} \frac{1}{2} \sum_{i=1}^{L} \|y_i - Dx_i\|_2^2, \text{ s.t. } \forall i \|x_i\|_0 \leq s. \quad (1)$$

We wish to find an overcomplete basis $D \in \mathbb{R}^{m \times p}$ with unit-norm columns and dictionary coefficient matrix $X = [x_1, \ldots, x_L] \in \mathbb{R}^{p \times L}$ such that each observation $y_i$ is represented by a linear combination of no more than $s$ columns of $D$. Since this problem is not convex, it is typically solved by alternating minimization; $X$ is updated using a fixed $D$ and then, $D$ is updated using a fixed $X$ [3].

In traditional DL literature, when dealing with multidimensional data, high order data $\{Y_i\}_{i=1}^L$ (tensors of order 2 or higher) are vectorized and stacked in columns of an observation matrix $Y = [\text{vec}(Y_1), \ldots, \text{vec}(Y_L)]$: the structure of the data is not considered in the dictionary underlying the data. In this case, any standard DL method can be used to find sparse representations of data. This simplistic method disregards the multidimensional structure in the data and does not capture the correlation and structure along different dimensions in each original “data point.”

On the other hand, in structured DL methods for tensor data, the multidimensional structure in data is taken into account. There exists a class of DL algorithms for tensor data that are based on the Tucker decomposition [4] of tensors. The resulting “Kronecker structured” DL methods (KS-DL) assume the dictionary consists of the Kronecker product [5] of smaller subdictionaries. Such algorithms represent tensor data using many fewer parameters compared to vectorized DL techniques [6]–[9]. This is due to the fact that the number of degrees of freedom in the KS-DL problem is significantly less than the traditional DL problem; this suggests that dictionary recovery is possible with smaller sample complexity using KS-DL methods [10], [11]. These works provide KS-DL algorithms to represent second order [6]–[8] and 3rd-order tensors [9]. In [6], for example, the KS-DL objective function is minimized using a Riemannian conjugate gradient method along with a nonmonotonic line search. In other methods, the subdictionaries composing the KS dictionary are updated alternately; in [7], an approach similar to K-SVD [3] is employed that uses higher-order SVD (HOSVD) [12] to alternately update coordinate dictionaries for second order tensor data and in [9], gradient descent is used to alternately update coordinate dictionaries for third order tensor data. The dictionary update stage in these algorithms involves solving a nonconvex minimization problem. The central challenge in the theoretical analysis of such KS-DL solvers is that the dictionary update stage is nonconvex. Furthermore, these explicit models are specialized to KS problems: they do not extend to more general structures in the underlying dictionary. In contrast, Dantas et al. [8] recently proposed an algorithm to learn the sum of KS dictionaries that represent second order tensor data by adding a regularizer in the objective function.

In this paper, we propose a novel algorithm called “STARK” to learn KS dictionaries for $N$th-order tensor data ($N \geq 2$). Our method involves adding a regularization term to the objective function of the DL problem defined in (1). The motivation for this term comes from the following realization: elements of any KS matrix can be rearranged to form a rank-1 tensor. Thus, enforcing a rank-1 constraint on such rearrangement of the dictionary results in a KS dictionary. To this end, we take advantage of low-rank tensor estimation literature [13]–[17] to add a convex regularizer that imposes low-rankness on the rearrangement tensor. This formulation has the advantage that it can be used to learn KS dictionaries as well as the case where the underlying dictionary is better approximated by sum of KS dictionaries. Our method can learn dictionaries of arbitrary tensor order; in the case of...
second order tensor data our general formulation coincides with that of Danita’s et al. [8].

We conduct numerical experiments to validate the performance of our algorithm. We use STARK for representation of third order synthetic and real tensor data and demonstrate that STARK outperforms vectorized DL technique K-SVD [3] and KS-DL technique K-HOSVD [7] for small sample sizes.

Notation Convention: Underlined bold upper-case, bold upper-case and lower-case letters are used to denote tensors, matrices and vectors, respectively. Lower-case letters denote scalars. We denote the Kronecker product and outer product by ⊗ and ⨽, respectively, while ×_n denotes the mode n product between a tensor and a matrix [18]. Norms are given by subscripts, so \( \|v\|_0 \) and \( \|v\|_2 \) are the \( \ell_0 \) and \( \ell_2 \) norms of \( v \), while \( \|X\|_2, \|X\|_F, \) and \( \|X\|_4 \) are the spectral, Frobenius, and nuclear norms of \( X \), respectively. A slice of a tensor is a 2-dimensional section defined by fixing all but two of its indices. Particularly, a frontal slice of a 3-dimensional tensor is defined by fixing the third index.

II. TUCKER-BASED KS-DL

According to the Tucker decomposition of tensors, an \( N \)-th order data tensor \( Y_i \in \mathbb{R}^{P_1 \times P_2 \times \cdots \times P_N} \) can be decomposed in the following form:

\[
Y_i = X_i \times_1 D_{1i} \times_2 D_{2i} \times_3 \cdots \times_N D_{Ni},
\]

where \( X_i \in \mathbb{R}^{P_1 \times P_2 \times \cdots \times P_N} \) denotes the core tensor and \( D_{1i}'s \) denote transformation matrices along each mode of \( Y_i \). The vectorized version of \( Y_i \) can be written as

\[
y_i = (D_N \otimes D_{N-1} \otimes \cdots \otimes D_1) x_i,
\]

where \( y_i = \text{vec}(Y_i) \) and \( x_i = \text{vec}(X_i) \) [18]. Here, the structure in tensor data is being exploited using the Kronecker product of transformation matrices. Stacking \( L \) vectorized data points \( \{y_i\}_{i=1}^L \) in columns of a matrix \( Y \), we get

\[
Y = (D_N \otimes D_{N-1} \otimes \cdots \otimes D_1) X,
\]

which is similar to the conventional DL problem, except that the dictionary \( D \) is Kronecker structured.

In the next section, we present our proposed KS-DL algorithm called “SStructured dictionAry learning through RanK-1 Tensor recovery” (STARK), which implicitly enforces Kronecker structure on the dictionary being learned by means of a regularizer in the DL objective function.

We note that STARK can be used to enforce a more general structure called low-separation-rank (LSR) structure. An LSR matrix can be written as sum of a few KS matrices:

\[
D = \sum_{k=1}^K D_{Ni}^k \otimes D_{N-1}^k \otimes \cdots \otimes D_1^k,
\]

where the factor matrices \( \{D_i^k \in \mathbb{R}^{m_i \times p_i}\}_{k=1}^K \) have the same size for a fixed \( i \), and \( K \) is the separation rank of \( D \) [19].

III. ENFORCING STRUCTURE VIA REGULARIZATION

To motivate the idea behind STARK, let us consider \( D = D_1 \otimes D_2 \). It turns out that the elements of \( D \) can be rearranged to form \( D^\pi = d_2 \circ d_1 \), where \( d_i = \text{vec}(D_i) \) for \( i = 1, 2 \) [5]. Figure 1 illustrates this rearrangement for \( D \). Similarly, for \( D = D_1 \otimes D_2 \otimes D_3 \), we can write \( D^\pi = d_3 \circ d_2 \circ d_1 \), where each frontal slice of the tensor \( D^\pi \) is a scaled copy of \( d_3 \circ d_2 \). Following a similar procedure, we can show that if \( D = \sum_{k=1}^K D_{N}^k \otimes D_{N-1}^k \otimes \cdots \otimes D_1^k \), then a certain “rearrangement” of \( D \) is the rank-\( K \) tensor \( D^\pi = \sum_{k=1}^K d_N^k \circ d_{N-1}^k \otimes \cdots \otimes d_1^k \), where \( d_i = \text{vec}(D_i) \) for \( i = 1 \) to [\( N \)]. This suggests that in the structured DL problem, we can impose the LSR structure (KS when \( K = 1 \)) on the dictionary \( D \) being learned by minimizing the rank of \( D^\pi \).

Since tensor rank is a nonconvex function, in order to make this DL problem convex with respect to \( D \), we use a commonly used convex proxy for the tensor rank function, the sum-trace-norm [15], which is defined as the average of the trace (nuclear) norms of the unfoldings of the tensor:

\[
\|D\|_{tr} = \frac{1}{N} \sum_{n=1}^N \|D_{(n)}\|_s.
\]

Using this convex relaxation for the rank function, the KS-DL problem has the following form:

\[
\min_{D,X} \frac{1}{2} \|Y - DX\|_F^2 + \lambda \|D^\pi\|_{tr}, \quad \text{s.t.} \quad \forall i \|x_i\|_0 \leq s,
\]

where the columns of \( D \) have unit norm. We use alternating minimization to solve this nonconvex problem. To minimize the objective function in (6) with respect to \( X \), we can use any of the standard sparse coding methods. In simulations, we use orthogonal matching pursuit (OMP) [20], [21]. To update the KS dictionary \( D \), we use the alternating direction method of multipliers (ADMM) [22]. We describe this dictionary update step in the next section.

IV. STRUCTURED DICTIONARY UPDATE USING ADMM

In this section, we discuss the dictionary update step of solving problem (6), which can be stated as

\[
\min_{D \in \mathbb{R}^{m \times p}} \frac{1}{2} \|Y - DX\|_F^2 + \lambda \sum_{n=1}^N \|D_{(n)}\|_s.
\]

The main issue in solving the convex dictionary update problem (7) is dealing with the interdependent nuclear norms. This makes optimization methods that use gradient information challenging. Inspired by many works in the literature on
our strategy for finding the permutation matrix is to define
\[ L_{\text{ADMM}} = \frac{1}{2} \| Y - DX \|_F^2 + \lambda \sum_{n=1}^{N} \| (W_n)_{(n)} \|_* \]
\[ \text{s.t.} \quad \forall n \quad W_n = D^{\pi} . \]  
(8)

In this formulation, although the nuclear norms are associated
with one another through the introduced constraint, we can
decouple the minimization problem into separate subproblems.
In particular, we can solve the objective function (8) using
ADMM, which involves decoupling the problem into inde-
pendent subproblems by forming the following augmented
Lagrangian function:
\[ \mathcal{L}_{\gamma}(D^{\pi}, \tilde{W}, \tilde{A}) = \frac{1}{2} \| Y - DX \|_F^2 + \sum_{n=1}^{N} \left( \lambda \| (W_n)_{(n)} \|_* - (A_n, D^{\pi} - W_n) + \frac{\gamma}{2} \| D^{\pi} - W_n \|_F^2 \right) . \]  
(9)

where \( \tilde{W} = [\tilde{W}_1^T, \cdots, \tilde{W}_N^T]^T \) and \( \tilde{A} = [\tilde{A}_1^T, \cdots, \tilde{A}_N^T]^T \).
Here, the inner product of two tensors is defined as the inner
product of their vectorizations.

Finally, to find the gradient of (9) with respect to \( D^{\pi} \), we
rewrite the Lagrangian function in the following form
\[ L_{\gamma}(D^{\pi}, \tilde{W}, \tilde{A}) = \frac{1}{2} \| Y - T(D^{\pi}) \|_F^2 + \sum_{n=1}^{N} \left( \lambda \| (W_n)_{(n)} \|_* - (A_n, D^{\pi} - W_n) + \frac{\gamma}{2} \| D^{\pi} - W_n \|_F^2 \right) . \]  
(10)

Here, we defined \( y = \text{vec}(Y) \) and \( T(D^{\pi}) = \text{vec}(DX) = \tilde{X}^T \Pi \text{vec}(D^{\pi}) \), where \( \tilde{X} = X \otimes I_m \) and \( \Pi \) is a permutation
matrix such that \( \Pi \text{vec}(D^{\pi}) = \text{vec}(D) \).

In the rest of this section, we briefly discuss derivation of
the permutation matrix as well as the update steps of ADMM.
Due to lack of space, we leave the details to an extended
version of this paper.

A. The Permutation Matrix

The permutation matrix \( \Pi \) represents a linear transformation
that maps the elements of \( \text{vec}(D) \) to \( \text{vec}(D^{\pi}) \). Given index \( l \)
of \( \text{vec}(D) \) and the corresponding mapped index \( l' \) of \( \text{vec}(D^{\pi}) \),
our strategy for finding the permutation matrix is to define \( l' \)
as a function of \( l \). To this end, we first find the the corresponding
row and column indices \( (i, j) \) of matrix \( D \) from the \( l \)th element
of \( \text{vec}(D) \). Then, we find the index of the element of interest
on the \( N \)th order rearranged tensor \( D^{\pi} \), and finally, we find
its location \( l' \) on \( \text{vec}(D^{\pi}) \). Note that the permutation matrix is
only a function of the dimensions of the factor matrices. We
leave the formal explanation of this procedure to an extended
version of this paper.

B. ADMM Update Rules

Recall that each iteration of ADMM consists of the following
steps [22]:
\[ D^{\pi}(t + 1) = \text{argmin}_{D^{\pi}} L_{\gamma}(D^{\pi}, \tilde{W}(t), \tilde{A}(t)), \]  
(11)
\[ \tilde{W}(t + 1) = \text{argmin}_{\tilde{W}} L_{\gamma}(\tilde{D}^{\pi}(t + 1), \tilde{W}, \tilde{A}(t)), \]  
(12)
\[ \tilde{A}(t + 1) = \tilde{A}(t) - \gamma \left( H \tilde{D}^{\pi}(t + 1) - \tilde{W}(t + 1) \right), \]  
(13)

where \( H \) is the vertical concatenation of \( N \) instances of \( I \), the
identity operator on \( \mathbb{R}^{m_1 p_1 \times \cdots \times m_N p_N} \).

The solution to problem (11) is found by taking the gradient
of \( \mathcal{L}_{\gamma}(\cdot) \) with respect to \( D^{\pi} \) and setting it to zero. Suppressing
the iteration index \( t \) for ease of notation, we have
\[ \frac{\partial L_{\gamma}}{\partial D^{\pi}} = T^*(T(D^{\pi}) - y) - \sum_{n=1}^{N} A_n + \sum_{n=1}^{N} \gamma (D^{\pi} - W_n) , \]
where \( T^* \) denotes the adjoint of the linear operator \( T \) [13].

Setting the gradient to zero results in
\[ T^*(T(D^{\pi})) + \gamma N D^{\pi} = T^*(y) + \sum_{n=1}^{N} (A_n + \gamma W_n) . \]  
(14)

Equivalently, we have,
\[ \text{vec}^{-1} \left( [\Pi^T \tilde{X} X^T \Pi + \gamma N I] \text{vec}(D^{\pi}) \right) = \text{vec}^{-1}(\Pi^T \tilde{X} y) + \sum_{n=1}^{N} (A_n + \gamma W_n) . \]  
(15)

Therefore, the update rule for \( D^{\pi} \) is
\[ D^{\pi}(t + 1) = \text{vec}^{-1} \left( [\Pi^T \tilde{X} X^T \Pi + \gamma N I]_{mp}^{-1} \right)^{-1} \cdot \left[ \Pi^T \tilde{X} y + \text{vec} \left( \sum_{n=1}^{N} (A_n(t) + \gamma W_n(t)) \right) \right] . \]  
(16)

To update \( \tilde{W} \), we can break the second step (12) into solving
\( N \) independent subproblems (suppressing the index \( t \)):
\[ \min_{W_n} \mathcal{L}_W = \lambda \| (W_n)_{(n)} \|_* + \frac{\gamma}{2} \| D^{\pi} - W_n \|_F^2 . \]

The objective function of this problem can be reformulated as
\[ \mathcal{L}_W = \lambda \| (W_n)_{(n)} \|_* + \frac{\gamma}{2} \| (W_n)_{(n)} - (D^{\pi}_{(n)} - \frac{(A_n)_{(n)}}{\gamma}) \|_F^2 + \text{const.} . \]  
(17)

The minimizer of an objective function of the form (17) is
\[ \text{shrink} \left( (D^{\pi})_{(n)} - \frac{1}{\gamma} (A_n)_{(n)}, \frac{\lambda}{\gamma} \right) , \]  
(18)

where \( \text{shrink}(M, \tau) \) is the shrinkage operator that applies
soft-thresholding at level \( \tau \) on the singular values of \( M \) (see
Theorem 2.1 in [23] for details). Therefore,
\[
\tilde{W}_n(t+1) = \text{refold} \left( \text{shrink} \left( \left( D^x(t+1) \right)_{(n)} \right) - \frac{1}{\gamma} \left( A_n(t) \right)_{(n)}, \frac{\lambda}{\gamma} \right) \right),
\]
where \text{refold}(\cdot) is the inverse of the unfolding operator. The summary of our DL method is provided in Algorithm 1.

V. NUMERICAL EXPERIMENTS

We compare the performance of STARK with two methods: KSVD [3], as a non-structured DL method, and K-HOSVD [7], which is a structured DL method that explicitly enforces Kronecker structure on the dictionary. We compare the performance of these methods are compared for synthetic 2-dimensional and 3-dimensional data as well as 3-dimensional real-world data.

a) Synthetic Data: For synthetic data, we randomly generate the dictionary \(D\) and the sparse coefficient matrix \(X\) to construct the observation matrix \(Y = DX\). We generate a KS dictionary as \(D = D_1 \otimes D_2 \otimes D_3\) (and \(D = D_1 \otimes D_2\) for 2-dimensional data) with unit-norm columns according to the following procedure. The elements of the subdictionaries \(D_1\) through \(D_3\) are chosen i.i.d from a Gaussian distribution \(\mathcal{N}(0, 1)\), and the columns of the subdictionaries are normalized. For generating \(X\), we select the locations of the \(s\) nonzero elements of each column uniformly at random. The values of those elements are sampled i.i.d from \(\mathcal{N}(0, 1)\). In the learning process, the dictionary \(D\) is initialized using random columns of the observation matrix \(Y\). The experiments were run for 20 Monte Carlo iterations and for various training sample sizes and the resulting dictionaries were tested on a set of 10000 test samples. For 2nd-order tensor data we selected \(m_1 = 4, p_1 = 12, m_2 = 6, p_2 = 8, s = 5\), and for 3rd-order tensor data we selected \(m_1 = 2, p_1 = 4, m_2 = 5, p_2 = 10, m_3 = 5, p_3 = 5\), and \(s = 10\).

The results of our experiments on synthetic data are shown in Figure 2. We compared our method to K-SVD and K-HOSVD. For K-HOSVD, we used algorithm provided by the authors, which actually enforces a Khatri-Rao structure on the dictionary rather than KS. STARK outperforms both K-SVD and K-HOSVD for all training sample sizes, especially when the number of training samples is small. The improvement over K-SVD can be attributed to the lower sample complexity of structured DL models. We conjecture that one reason for the improvement over K-HOSVD is that the dictionary update in STARK is a convex problem and thus the algorithm is less prone to getting stuck in a poor local minimum.

b) Real Data: For these experiments, we compare the denoising performance of the three methods on two RGB images, Peppers and Lena, which are \(512 \times 512 \times 2\) and \(512 \times 512 \times 3\) tensors, respectively. We corrupt the images using additive white Gaussian noise with \(\sigma = 50\). To construct the training data set, we extract overlapping \(6 \times 6\) patches from each image and treat each patch as a data point. Then we compare the denoising performances of the methods based on the resulting peak signal to noise ratio (PSNR) of the reconstructed images [24].

Figure 3 shows the results averaged over 10 Monte Carlo iterations. We can see the denoising performance of STARK is superior to both K-SVD and K-HOSVD for all training sample sizes. This is in part due to the fact that for real-world data, the underlying dictionaries may not be KS. STARK allows \(D^x\) to have rank higher than 1, meaning the algorithm can use a larger number of parameters (\(K\) times as many when \(\text{rank}(D^x) = K\)) to approximate the true dictionary.

VI. CONCLUSION

In this paper we showed that the Kronecker product of \(N\) matrices can be rearranged to form an \(N\)th order rank-1 tensor. Based on this, we proposed a novel structured dictionary learning method for multidimensional data that enforces LSR structure in the dictionary through imposing a low-rank structure on the rearranged tensor. In particular, Kronecker structure can be enforced by imposing a rank-
I constraint on the rearranged tensor. Since the low-rank tensor recovery problem is a nonconvex problem, we resort to solving its convex relaxation, namely, minimizing the sum-trace-norm of the rearranged tensor, which is a convex proxy for tensor rank. We used ADMM for solving the dictionary update stage of this structured DL problem. Our experiments on both synthetic and real data showed that when the sample size is small, our method considerably outperforms both K-SVD, which returns unstructured dictionaries, and K-HOSVD, a KS-DL method that directly finds the subdictionaries.

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