Global well-posedness and large-time behavior of 1D compressible Navier-Stokes equations with density-depending viscosity and vacuum in unbounded domains

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Abstract We consider the Cauchy problem for one-dimensional (1D) barotropic compressible Navier-Stokes equations with density-depending viscosity and large external forces. Under a general assumption on the density-depending viscosity, we prove that the Cauchy problem admits a unique global strong (classical) solution for the large initial data with vacuum. Moreover, the density is proved to be bounded from above time-independently. As a consequence, we obtain the large time behavior of the solution without external forces.

Keywords 1D compressible Navier-Stokes equations, global well-posedness, large initial data, vacuum, density-depending viscosity

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1 Introduction and main results

The motion of one-dimensional (1D) viscous compressible barotropic fluid is governed by the following compressible Navier-Stokes equations:

$$\begin{cases}
\rho_t + (\rho u)_x = 0, \\
(\rho u)_t + (\rho u^2)_x + [P(\rho)]_x = [\mu(\rho)u_x]_x + \rho f.
\end{cases} \quad (1.1)$$

Here, $t \geq 0$ is time, $x \in \mathbb{R} = (-\infty, \infty)$ is the spatial coordinate, and $\rho(x, t) \geq 0$, $u(x, t)$ and $P(\rho) = A\rho^\gamma$ ($A > 0$, $\gamma > 1$) are the fluid density, velocity and pressure, respectively. Without loss of generality, it is assumed that $A = 1$. The viscosity $\mu(\cdot)$ is a function of the density $\rho$. The external force $f = f(x)$

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is a known function. We look for the solutions \((\rho, u)\) to the Cauchy problem for (1.1) with the initial condition
\[
(\rho, u)|_{t=0} = (\rho_0(x), u_0(x)), \quad x \in \mathbb{R},
\]
and the following far field behavior:
\[
u(x, t) \to 0, \quad \rho(x, t) \to \hat{\rho} > 0, \quad \text{as } |x| \to \infty,
\]
where \(\hat{\rho}\) is a given positive constant.

There is huge literature on the studies of the global existence and large time behavior of solutions to the 1D compressible Navier-Stokes equations. For constant viscosity \(\mu\) and the initial density away from vacuum, the problems were addressed by Kanel [10] for sufficiently smooth data, and by Serre [19, 20] and Hoff [6] for discontinuous initial data. On the other hand, if \(\mu\) depends on \(\rho\) and admits a positive constant lower bound, the global well-posedness and large time behavior of solutions without initial vacuum were discussed in [1, 2, 21] and the references therein. However, when the vacuum is allowed initially, as emphasized in many papers related to compressible fluid dynamics [3, 5, 7-9, 16, 22], the presence of vacuum is one of the major difficulties in discussing the well-posedness of solutions to the compressible Navier-Stokes equations. In the presence of vacuum, Ding et al. [4] considered the global existence of classical solutions to 1D compressible Navier-Stokes equations in bounded domains, provided that \(\mu \in C^2[0, \infty)\) satisfies
\[
0 < \bar{\mu} \leq \mu(\rho) \leq C(1 + P(\rho)).
\]
Recently, Liu et al. [17] established not only the global existence but also the large-time behavior for classical solutions to the initial boundary value problem for 1D compressible Navier-Stokes equations with more general \(\mu(\rho)\). For the Cauchy problem (1.1)–(1.3) without the external force \((f = 0)\), Ye [23] studied the global classical large solutions under the following restriction on \(\mu(\rho)\):
\[
\mu(\rho) = 1 + \rho^\beta, \quad 0 \leq \beta < \gamma.
\]
However, both the uniform upper bound of density and the large time behavior of solutions are not obtained in [23].

In this paper, for more general density-depending viscosity (see (1.6)) and large external forces, we will derive the uniform upper bound of density and thus prove the global well-posedness of strong (classical) large solutions to the Cauchy problem (1.1)–(1.3) with initial vacuum. Before stating the main results, we first explain the notations and conventions used throughout this paper. We set
\[
D_t \triangleq \frac{\partial}{\partial t} + u \frac{\partial}{\partial x}, \quad \dot{\nu} \triangleq \nu_t + uv_x.
\]
For \(1 \leq r \leq \infty\) and \(k \geq 1\), we adopt the following simplified notations:
\[
L^r = L^r(\mathbb{R}), \quad W^{k,r} = W^{k,r}(\mathbb{R}), \quad H^k = W^{k,2}(\mathbb{R}).
\]

The first result is the global existence of strong solutions to the Cauchy problem (1.1)–(1.3).

**Theorem 1.1.** Suppose that \(\int_{-\infty}^{\infty} f(y)dy \in H^2\) and that the viscosity \(\mu(\rho) \in C^1[0, \infty)\) satisfies
\[
0 < \bar{\mu} \leq \mu(\rho) \leq \lambda_0 \int_1^\rho \mu(s)ds + \lambda_1
\]
for some constants \(\bar{\mu} > 0, \lambda_0 \geq 0\) and \(\lambda_1 > 0\). Let the initial data \((\rho_0, u_0)\) satisfy
\[
\rho_0 \geq 0, \quad \rho_0 - \hat{\rho} \in H^1, \quad u_0 \in H^1.
\]
Then there exists a unique global strong solution \((\rho, u)\) to the Cauchy problem (1.1)–(1.3) satisfying that for any \(0 < T < \infty\),
\[
\begin{cases}
\rho - \hat{\rho} \in C([0, T]; H^1), \quad \rho_t \in L^\infty(0, T; L^2), \\
u \in L^\infty(0, T; H^1) \cap L^2(0, T; H^2),
\end{cases}
\]
\[
\sqrt{\nu}u \in L^\infty(0, T; H^2), \quad \sqrt{\nu}u_t \in L^2(0, T; H^1).
\]
Moreover, the density remains uniformly bounded for all time, i.e.,
\[
\sup_{0 \leq t < \infty} \|\rho(\cdot, t)\|_{L^\infty} < \infty. \tag{1.9}
\]

Similar to [17, Theorem 1.2], we will show that the strong solutions obtained by Theorem 1.1 are indeed classical ones, provided that the initial data \((\rho_0, u_0)\) satisfy some additional compatibility conditions.

**Theorem 1.2.** In addition to the conditions of Theorem 1.1, suppose that \(\mu(\rho) \in C^2(0, \infty)\) and that the initial data \((\rho_0, u_0)\) satisfy
\[
(\rho_0 - \tilde{\rho}, \, P(\rho_0) - P(\tilde{\rho})) \in H^2, \quad u_0 \in H^2 \tag{1.10}
\]
and the compatibility condition
\[
[\mu(\rho_0)u_{0x}]_x - [P(\rho_0)]_x = \sqrt{\rho_0}g(x), \quad x \in \mathbb{R} \tag{1.11}
\]
for a given function \(g \in L^2\). Then the strong solutions obtained in Theorem 1.1 become classical and satisfy that for any \(0 < \tau < T < \infty\),
\[
\begin{align*}
(\rho - \tilde{\rho}, \, P(\rho) - P(\tilde{\rho})) & \in C([0, T]; H^2), \\
u & \in C([0, T]; H^2) \cap L^2(\tau, T; H^3), \\
u_t & \in L^2(0, T; H^3), \quad \sqrt{\rho}u \in L^\infty(0, T; H^3), \\
u_t u_t & \in L^\infty(0, T; H^1) \cap L^2(0, T; H^2), \quad \sqrt{\rho}u_t \in L^2(0, T; L^2), \\
u_t^2 u_{tt} & \in L^\infty(0, T; L^2), \quad t\sqrt{\rho}u_{tt} \in L^2(0, T; H^1).
\end{align*} \tag{1.12}
\]

When there is no external force, i.e., \(f \equiv 0\) in (1.1), we can obtain the large time behavior of the strong solutions to the Cauchy problem (1.1)–(1.3).

**Theorem 1.3.** Let \((\rho, u)\) be the strong solutions to the Cauchy problem (1.1)–(1.3) with \(f \equiv 0\) obtained in Theorem 1.1. Then the following large time behavior holds:
\[
\lim_{t \to \infty} (\|\rho - \tilde{\rho}\|_{L^p} + \|u_x\|_{L^2 \cap L^p}) = 0, \quad \forall \, p > 2. \tag{1.13}
\]
Moreover, if there exists some point \(x_0 \in (-\infty, \infty)\) such that \(\rho_0(x_0) = 0\), the spatial gradient of the density has to blow up as \(t \to \infty\) in the following sense,
\[
\lim_{t \to \infty} \|\rho_x(\cdot, t)\|_{L^2} = \infty. \tag{1.14}
\]

A few remarks are in order:

**Remark 1.4.** It should be noted here that we obtain a completely new time-independent upper bound of the density (see (1.9)), which is in sharp contrast to [4, 23] where the upper bound of the density is time-dependent. Indeed, this is crucial to the study of the large time behavior of the solutions.

**Remark 1.5.** We want to point out that the restriction on \(\mu(\rho)\) in (1.6) is more general in comparison with those in [4, 23] (see (1.5) and (1.4)). More precisely, we list some special cases satisfying (1.6) as follows:
- \(\mu(\rho) = \mu\) is a positive constant. Let \(\lambda_0 = 0\) and \(\lambda_1 = \tilde{\mu}\). Then \(\lambda_0 \int_1^\rho \mu(s)ds + \lambda_1 = \tilde{\mu} = \mu(\rho)\).
- \(\mu(\rho) = 1 + \rho^\alpha\) for any \(\alpha \geq 0\). Choose \(\lambda_0 = 1 + a\) and \(\lambda_1 = 4 + a\). It holds that
  \[
  \lambda_0 \int_1^\rho \mu(s)ds + \lambda_1 = (1 + a)\rho + \rho^{1+a} + 2 \geq 1 + \rho^\alpha = \mu(\rho).
  \]
- \(\mu(\rho) = e^\rho\). Let \(\lambda_0 = 1\) and \(\lambda_1 = e\). One has
  \[
  \lambda_0 \int_1^\rho \mu(s)ds + \lambda_1 = e^\rho = \mu(\rho).
  \]
It is clear that the density-dependent viscosity \(\mu(\rho)\) studied in [4, 23] is all included in our results.
Remark 1.6. To obtain the global existence of strong solutions in Theorem 1.1, we do not need the additional compatibility condition (1.11), which is required for discussing the global classical solutions in Theorem 1.2. This shows how the compatibility condition (1.11) plays its role in studying the well-posedness of solutions with initial vacuum.

Remark 1.7. It should be noted here that the solutions \((\rho, u)\) obtained in Theorem 1.2 are actually classical. Indeed, the Sobolev embedding theorem together with the regularities of \((\rho, u)\) in (1.12) shows that
\[
(\rho, P, u) \in C([0, T]; C^{1+\frac{1}{2}}), \quad \rho_t \in C([0, T]; C^{\frac{1}{2}}).
\]
(1.15)

Furthermore, it is easy to deduce from (1.12) that for any \(0 < \tau < T\),
\[
u \in L^\infty(\tau, T; H^3), \quad u_t \in L^\infty(\tau, T; H^1) \cap L^2(\tau, T; H^2), \quad u_{tt} \in L^2(\tau, T; H^1),
\]
which yields that
\[
u \in C([\tau, T]; C^2[\infty, \infty]), \quad u_t \in C([\tau, T]; C[\infty, \infty]).
\]
(1.16)

Hence, the combination of (1.15) with (1.16) shows that \((\rho, u)\) are the classical solutions to (1.1)–(1.3).

We now make some comments on the analysis of this paper. Note that for initial density away from vacuum, the local existence and uniqueness of classical solutions to the Cauchy problem (1.1)–(1.3) have been obtained in [3] (see also Lemma 2.1 below). To extend the local solutions globally in time, one needs some global a priori estimates which are independent of the lower bound of density. Indeed, the key issue is to derive both the time-independent upper bound of the density and some necessary a priori estimates of the solutions. However, there are some difficulties due to the initial vacuum, density-depending viscosity, large external force, and the unboundedness of the domain. Thanks to [11], we first localize the problem on bounded domains. Then, motivated by Lü et al. [17], we can get the upper bound of the density independent of \(x\) and thus the uniform bound of density on the whole \(\mathbb{R}\) due to the arbitrariness of \(x\) (see Lemma 2.3). Furthermore, using the far field behavior of the density, we can bound the \(L^2\)-norm of \(u\) in terms of \(\|\rho^{1/2} u\|_{L^2}\) and \(\|u_x\|_{L^2}\) (see (2.23)). Following the methods used in [14, 15], we use the material derivative \(\dot{u}\) instead of the usual \(u_t\) and succeed in obtaining the derivative estimates on \((\rho, u)\). Note that all these global a priori estimates obtained are independent of the lower bound of density (see Section 2). Hence, we can prove Theorems 1.1 and 1.2 for any time \(T > 0\) (see Section 3).

Finally, for the case without the external force \((f \equiv 0)\), we can establish the time-independent lower order estimates (see (4.1) and (4.2)). Using the methods due to [9, 15] and the key time-independent a priori estimates, we prove Theorem 1.3 in Section 4.

2 A priori estimates

In this section, we establish some a priori bounds for classical solutions \((\rho, u)\) to the Cauchy problem (1.1)–(1.3) whose existence is guaranteed by the following Lemma 2.1, which can be proved by similar arguments to those in [3, 18].

Lemma 2.1. Let \(\mu(\rho) \in C^2[0, \infty)\). Assume that \(\int_{-\infty}^{X} f(y)dy \in H^2\) and that the initial data \((\rho_0, u_0)\) satisfy
\[
(\rho_0 - \tilde{\rho}, \ P(\rho_0) - P(\tilde{\rho}), u_0) \in H^2, \quad \inf_{x \in \mathbb{R}} \rho_0(x) > 0,
\]
where \(\tilde{\rho} > 0\) is a given constant. Then there exists a small time \(T_0 > 0\) such that the Cauchy problem (1.1)–(1.3) has unique classical solutions \((\rho, u)\) on \(\mathbb{R} \times (0, T_0]\) satisfying
\[
\begin{align*}
\rho - \tilde{\rho}, & \ P(\rho) - P(\tilde{\rho}) \in C([0, T_0]; H^2), \\
u & \in C([0, T_0]; H^3) \cap L^2(0, T_0; H^3), \\
\rho_t & \in C([0, T_0]; H^1), \quad u_t \in C([0, T_0]; L^2) \cap L^2(0, T_0; H^1).
\end{align*}
\]
2.1 *A priori* estimates (I): Time-independent *a priori* estimates

In this subsection, we use the convention that $C$ denotes a generic positive constant depending only on the initial data $(\rho_0, u_0)$ and some known constants but independent of both $\inf_{x \in \mathbb{R}} \rho_0(x)$ and $T$, and use $C(\alpha)$ to emphasize that $C$ depends on $\alpha$.

We start with the following energy estimate for the solutions $(\rho, u)$.

**Lemma 2.2.** There is a positive constant $C$ depending on $\gamma$, $\bar{\mu}$, $\bar{\rho}$, $\|\rho_0 - \tilde{\rho}\|_{H^1}$, $\|u_0\|_{H^1}$ and $\|\tilde{f}\|_{H^2}$ such that
\[
\sup_{0 \leq t \leq T} \int_{-\infty}^{+\infty} \left( \frac{1}{2} \rho u^2 + G(\rho) \right) dx + \int_0^T \int_{-\infty}^{+\infty} \mu(\rho) u_x^2 dx dt \leq C, \tag{2.1}
\]
where, and in what follows, $\tilde{f} \triangleq \int_{-\infty}^{T} f(y) dy$ and $G$ denotes the potential energy density given by
\[
G(\rho) = \rho \int_{\tilde{\rho}}^{\rho} \frac{P(s) - P(\tilde{\rho})}{s^2} ds = \frac{1}{\gamma - 1} [P(\rho) + \rho^{\gamma - 1} - P(\tilde{\rho}) - \tilde{\rho}^{\gamma - 1}(\rho - \tilde{\rho})]. \tag{2.2}
\]

**Proof.** Multiplying (1.1)_1 and (1.1)_2 by $G'(\rho)$ and $u$ respectively, we obtain after using integration by parts and the far field condition (1.3) that
\[
\frac{d}{dt} \int_{-\infty}^{+\infty} \left( \frac{1}{2} \rho u^2 + G(\rho) \right) dx + \int_{-\infty}^{+\infty} \mu(\rho) u_x^2 dx = \frac{d}{dt} \int_{-\infty}^{+\infty} (\rho - \bar{\rho}) \tilde{f} dx,
\]
which yields
\[
\sup_{0 \leq t \leq T} \int_{-\infty}^{+\infty} \left( \frac{1}{2} \rho u^2 + G(\rho) \right) dx + \int_0^T \int_{-\infty}^{+\infty} \mu(\rho) u_x^2 dx dt \leq C + \int_{-\infty}^{+\infty} \rho \tilde{f} dx - \int_{-\infty}^{+\infty} (\rho_0 - \bar{\rho}) \tilde{f} dx
\]
\[
\leq C + \int_{-\infty}^{+\infty} (\rho - \bar{\rho}) \tilde{f} dx + C \|\rho_0 - \bar{\rho}\|_{L^2} + C \|\tilde{f}\|_{L^2}^2
\]
\[
\leq C + \int_{R_1} (\rho - \bar{\rho}) \tilde{f} dx + \int_{\mathbb{R} \setminus R_1} (\rho - \bar{\rho}) \tilde{f} dx \tag{2.3}
\]
with
\[
R_1 \triangleq \{ x \in \mathbb{R} : \rho^{\gamma - 1} < \gamma^2 (\rho^{\gamma - 1} + 1) \}. \tag{2.4}
\]

Then, on the one hand, when $0 \leq \rho \leq M$ for some positive constant $M > 0$, there are positive constants $K_1$ and $K_2$ depending only on $\bar{\rho}$ and $M$ such that
\[
K_1 (\rho - \bar{\rho})^2 \leq G(\rho) \leq K_2 (\rho - \bar{\rho})^2, \tag{2.5}
\]
which together with Young’s inequality implies that
\[
\int_{R_1} |\rho - \bar{\rho}| \tilde{f} dx \leq \varepsilon \|\rho - \tilde{\rho}\|_{L^2(R_1)}^2 + C(\varepsilon) \|\tilde{f}\|_{L^2(R_1)}^2 \leq C \varepsilon \int_{-\infty}^{+\infty} G(\rho) dx + C(\varepsilon). \tag{2.6}
\]

On the other hand, since $\rho > \tilde{\rho}$ in $\mathbb{R} \setminus R_1$, we have
\[
\int_{\mathbb{R} \setminus R_1} |\rho - \tilde{\rho}| \tilde{f} dx \leq 2 \int_{\mathbb{R} \setminus R_1} \rho \tilde{f} dx
\]
\[
\leq \varepsilon \int_{\mathbb{R} \setminus R_1} \rho^{\frac{2\gamma - 1}{\gamma}} dx + C(\varepsilon) \int_{\mathbb{R} \setminus R_1} \tilde{f}^{\frac{2\gamma - 1}{\gamma}} dx
\]
\[
\leq \varepsilon \int_{\mathbb{R} \setminus R_1} \rho^{\frac{2\gamma - 1}{\gamma}} dx + C(\varepsilon) \|\tilde{f}\|_{L^{\infty}(\mathbb{R} \setminus R_1)} \|\tilde{f}\|_{L^2(\mathbb{R} \setminus R_1)}^2
\]
\[
\leq \varepsilon \int_{-\infty}^{+\infty} G(\rho) dx + C(\varepsilon), \tag{2.7}
\]
where in the last inequality one has used the following fact:
\[
\rho^{\frac{2\gamma-1}{\gamma}} \leq \rho \left( \frac{1}{\gamma} \rho^{\gamma-1} + \frac{\gamma-1}{\gamma} \right) \leq \frac{1}{\gamma-1} (\rho^{\gamma} - \gamma \hat{\rho}^{\gamma-1} \rho) \leq G(\rho), \quad x \in \mathbb{R} \setminus R_1
\]
due to Young’s inequality, (2.4) and (2.2).

Substituting (2.6) and (2.7) into (2.3), we obtain (2.1) after choosing \( \varepsilon \) suitably small. This completes the proof of Lemma 2.2.

Next, motivated by Kazhikhov [11], we will derive the key uniform (in time) upper bound of the density, which is crucial to obtain both the derivative estimates and the large time behavior of the solutions.

**Lemma 2.3.** There is a positive constant \( \tilde{\rho} \) depending only on \( \gamma, \bar{\mu}, \hat{\rho}, \|\rho_0 - \hat{\rho}\|_{H^1}, \|u_0\|_{H^1} \) and \( \|\tilde{f}\|_{H^2} \) such that
\[
0 \leq \rho(x, t) \leq \tilde{\rho}, \quad \forall (x, t) \in \mathbb{R} \times [0, T].
\]  

**Proof.** Let \( x \in [N, N+1] \) for any integer \( N \). Integrating (1.1) over \( (N, x) \) with respect to \( x \) leads to
\[
-\mu(\rho)u_x + P + \int_N^x \rho dy = (\mu(\rho)u_x + P + \rho u^2)(N, t) + \int_N^x \rho f dy,
\]  
which, in particular, implies
\[
(\mu(\rho)u_x + P + \rho u^2)(N, t)
\]
\[
= \int_N^{N+1} \left( -\mu(\rho)u_x + P + \rho u^2 - \int_N^x \rho f dy \right) dx + \left( \int_N^{N+1} \int_N^x \rho dy dx \right)
\]
\[
\leq -\int_N^{N+1} \mu(\rho)u_x dx + C + \left( \int_N^{N+1} \int_N^x \rho dy dx \right)
\]  
(2.10)
due to (2.1) and the following simple fact:
\[
\int_N^{N+1} \rho^\gamma dx \leq C + \int_N^{N+1} 1_{(\rho^\gamma \geq 2\gamma \hat{\rho}^\gamma - 1)} \rho^\gamma dx \leq C + 2(\gamma - 1) \int_N^{N+1} G(\rho) dx \leq C.
\]  
(2.11)
Denoting \( F(\rho) \triangleq \int_0^\rho \mu(s)s^{-1}ds \), one deduces from (1.1)_1 that \( D_t F(\rho) = -\mu(\rho)u_x \). This together with (2.9) and (2.10) gives
\[
D_t (F(\rho) + b_1(t)) + P \leq -\int_N^{N+1} \mu(\rho)u_x dx + C
\]
\[
\leq \int_N^{N+1} \mu(\rho)dx \int_N^{N+1} \mu(\rho)u_x^2 dx + C
\]
\[
\leq \sup_{\rho \geq 0} \int_N^{N+1} \int_N^{N+1} (\rho + 1) dx \int_N^{N+1} \mu(\rho)u_x^2 dx + C
\]
\[
\leq C \sup_{\rho \geq 0} \int_N^{N+1} \mu(\rho)u_x^2 dx + C,
\]  
(2.12)
where
\[
b_1(t) \triangleq \int_N^x \rho dy - \int_N^{N+1} \int_N^x \rho dy dx
\]
satisfies
\[
|b_1(t)| \leq C \int_N^{N+1} \rho u|dx| \leq C \int_N^{N+1} \rho dx + C \int_N^{N+1} \rho u^2 dx \leq C_1
\]  
(2.13)
owing to (2.11) and (2.1).

Since \( \mu \) satisfies (1.6), it holds
\[
\sup_{\rho \geq 0} \frac{\mu(\rho)}{\rho + 1} \leq \lambda_0 \sup_{\rho \geq 0} \left( \frac{1}{\rho + 1} \int_1^\rho \mu(s)ds \right) + \lambda_1 \leq \lambda_0 \sup_{\rho \geq 0} \int_1^{\max(\rho, 1)} \frac{\mu(s)}{s} ds + \lambda_1,
\]
which together with (2.12) leads to
\[ D_t(F(\rho) + b_1(t)) + P \leq C_2 + C \left( \sup_{x \in \mathbb{R}} \int_{1}^{\max\{\rho, 1\}} \frac{\mu(s)}{s} ds + 1 \right) \int_{-\infty}^{\infty} \mu(\rho) u_x^2 dx \tag{2.14} \]
with some constant \( C_2 > 1 \).

Choosing a constant \( \nu \geq C_2^{1/\gamma} \) such that for all \( \rho \geq \nu \),
\[ P(\rho) - C_2 \geq 0, \tag{2.15} \]
and then multiplying (2.14) by
\[ H \triangleq (F(\rho) + b_1(t) - F(\nu) - C_1)_+, \tag{2.16} \]
we obtain after using (2.13) and (2.15) that
\[ D_t H^2 \leq CH \left( \sup_{x \in \mathbb{R}} \int_{1}^{\max\{\rho, 1\}} \frac{\mu(s)}{s} ds + 1 \right) \int_{-\infty}^{\infty} \mu(\rho) u_x^2 dx \
\leq C \sup_{x \in \mathbb{R}} H^2 \int_{-\infty}^{+\infty} \mu(\rho) u_x^2 dx + C \int_{-\infty}^{+\infty} \mu(\rho) u_x^2 dx. \tag{2.17} \]

Integrating (2.17) over \((0, t)\) gives that for \( x \in [N - 1, N] \),
\[ H^2(x, t) \leq \bar{C} + \bar{C} \int_{0}^{t} \sup_{x \in \mathbb{R}} H^2 \int_{-\infty}^{+\infty} \mu(\rho) u_x^2 dx dt, \tag{2.18} \]
where \( \bar{C} \) is a positive constant independent of \( N \). Note that \( N \) is arbitrary, so the inequality (2.18) holds for all \( x \in \mathbb{R} \), i.e.,
\[ \sup_{x \in \mathbb{R}} H^2 \leq C + C \int_{0}^{t} \sup_{x \in \mathbb{R}} H^2 \int_{-\infty}^{+\infty} \mu(\rho) u_x^2 dx dt, \]
which combined with Gronwall’s inequality and (2.1) yields that
\[ \sup_{(x, t) \in \mathbb{R} \times (0, \infty)} H^2 \leq C. \tag{2.19} \]

Consequently, the desired (2.8) is a direct consequence of (2.16), (2.19) and (2.13). This finishes the proof of Lemma 2.3.

With the uniform upper bound of the density (2.8) at hand, we will derive the following time-independent bound on \( \|\rho - \tilde{\rho}\|_{L^2}^2 \).

**Corollary 2.4.** There is a positive constant \( C \) depending on \( \gamma, \tilde{\mu}, \tilde{\rho}, \|\rho_0 - \tilde{\rho}\|_{H^1}, \|u_0\|_{H^1} \) and \( \|\tilde{f}\|_{H^2} \) such that
\[ \sup_{0 \leq t \leq T} (\|\rho - \tilde{\rho}\|_{L^2}^2 + \|P(\rho) - P(\tilde{\rho})\|_{L^2}^2) \leq C. \tag{2.20} \]

**Proof.** It follows from (2.1) and (2.5) that
\[ \|\rho - \tilde{\rho}\|_{L^2}^2 + \|P(\rho) - P(\tilde{\rho})\|_{L^2}^2 \leq C \int_{-\infty}^{+\infty} G(\rho) dx + \|\tilde{\rho}^{-1}\|_{L^2}^2 (\gamma - 1)G(\rho) + \gamma \tilde{\rho}\gamma^{-1} \tilde{\rho} - \gamma \tilde{\rho}\gamma^{-1} \|_{L^2}^2 \]
\[ \leq C \int_{-\infty}^{+\infty} G(\rho) dx \leq C, \]
which completes the proof of Corollary 2.4. 
\[ \square \]
2.2 A priori estimates (II): Time-dependent a priori estimates

In this subsection, we now proceed to derive the derivative estimates of the solutions \((\rho, u)\) to the Cauchy problem (1.1)–(1.3).

**Lemma 2.5.** There is a positive constant \(C\) depending on \(T, \gamma, \tilde{\mu}, \|\rho_0 - \tilde{\rho}\|_{H^1}, \|u_0\|_{H^1}\) and \(\|f\|_{H^2}\) such that

\[
\sup_{0 \leq t \leq T} \|u\|_{H^1}^2 + \int_0^T \|\rho^{1/2} \dot{u}\|_{L^2}^2 dt \leq C(T). \tag{2.21}
\]

**Proof.** First, multiplying (1.1)_2 by \(\dot{u}\) and integrating the resulting equation by parts, we yield

\[
\frac{1}{2} \frac{d}{dt} \int_{-\infty}^{+\infty} \mu(\rho) u_x^2 dx + \int_{-\infty}^{+\infty} \rho \dot{u}^2 dx
= \frac{d}{dt} \left( \int_{-\infty}^{+\infty} [P(\rho) - P(\tilde{\rho})] u_x dx + \int_{-\infty}^{+\infty} \rho f u_x dx \right)
- \frac{1}{2} \left( \int_{-\infty}^{+\infty} \mu(\rho + \mu'(\rho)\rho) u_{xx}^2 dx + \gamma \int_{-\infty}^{+\infty} P(\rho) u_x^2 dx - \mu \int_{-\infty}^{+\infty} \rho u^2 f_x dx \right)
\leq \frac{d}{dt} \left( \int_{-\infty}^{+\infty} [P(\rho) - P(\tilde{\rho})] u_x dx + \int_{-\infty}^{+\infty} \rho f u_x dx \right)
+ C\|u_x\|_{L^2}^3 + C\|u_x\|_{L^2}^2 + C\|u\|_{L^\infty} \rho^{1/2} \|u\|_{L^2} \|f_x\|_{L^2}
\leq \frac{d}{dt} \left( \int_{-\infty}^{+\infty} [P(\rho) - P(\tilde{\rho})] u_x dx + \int_{-\infty}^{+\infty} \rho f u_x dx \right)
+ C\|u_x\|_{L^\infty} \|u_x\|_{L^2}^2 + C\|u_x\|_{L^2}^2 + C, \tag{2.22}
\]

where in the last inequality one has used the following fact:

\[
\|u\|_{L^2}^2 + \|u\|_{L^\infty} = \tilde{\rho}^{-1} \left( \int_{-\infty}^{+\infty} \mu \rho^2 dx + \int_{-\infty}^{+\infty} \rho (\rho - \tilde{\rho})^2 dx \right) + \|u\|_{L^\infty}
\leq C \|\rho^{1/2} u\|_{L^2}^2 + C\|\rho - \tilde{\rho}\|_{L^2} \|u\|_{L^2} \|u\|_{L^\infty} + C\|u\|_{L^\infty}
\leq C \|\rho^{1/2} u\|_{L^2}^2 + C\|u\|_{L^2}^3 \|u_x\|_{L^2}^{1/2} + C\|u\|_{L^2}^{1/2} \|u_x\|_{L^2}^{1/2}
\leq C + C \|u_x\|_{L^2}^2 + C \|u_x\|_{L^2}^2 + C, \tag{2.23}
\]

owing to (2.20), (2.1), and the Sobolev inequality. Furthermore, using (1.6), (2.1), (1.1)_2 and (2.20), we get

\[
\|u_x\|_{L^\infty} \leq C \|\mu(\rho) u_x - P(\rho) + P(\tilde{\rho})\|_{L^\infty} + C\|P(\rho) - P(\tilde{\rho})\|_{L^\infty}
\leq C \|\mu(\rho) u_x - [P(\rho) - P(\tilde{\rho})] u_{xx} + C\|u_x\|_{L^2} + C\|\rho f\|_{L^2} + C
\leq C \|\mu(\rho) u_x\|_{L^2} + C\|P(\rho) - P(\tilde{\rho})\|_{L^2} + C\|\rho \dot{u}\|_{L^2} + C\|\rho f\|_{L^2} + C
\leq C \sqrt{\|\mu(\rho) u_x\|_{L^2}} + C \|\rho^{1/2} \dot{u}\|_{L^2} + C. \tag{2.24}
\]

Then, putting (2.24) into (2.22), we obtain after using Young’s inequality that

\[
\frac{d}{dt} B(t) + \frac{1}{2} \int_{-\infty}^{+\infty} \rho \dot{u}^2 dx \leq C + C \|\sqrt{\mu(\rho)} u_x\|_{L^2}^2 + C \|\sqrt{\mu(\rho)} u_x\|_{L^2}^2, \tag{2.25}
\]

where

\[
B(t) \triangleq \frac{1}{2} \int_{-\infty}^{+\infty} \mu(\rho) u_x^2 dx - \int_{-\infty}^{+\infty} [P(\rho) - P(\tilde{\rho})] u_x dx - \int_{-\infty}^{+\infty} \rho f u_x dx \tag{2.26}
\]

satisfies

\[
\frac{1}{4} \|\sqrt{\mu(\rho)} u_x\|_{L^2}^2 - C \leq B(t) \leq C \|\sqrt{\mu(\rho)} u_x\|_{L^2}^2 + C. \tag{2.27}
\]
Indeed, it follows from (1.6), (2.1) and (2.20) that
\[
\int_{-\infty}^{+\infty} [P(\rho) - P(\hat{\rho})] u_x dx + \int_{-\infty}^{+\infty} \rho f u dx \\
\leq \frac{1}{4} \int_{-\infty}^{+\infty} \mu(\rho) u_x^2 dx + C \|P(\rho) - P(\hat{\rho})\|_{L^2}^2 + C \|\rho^{1/2} u\|_{L^2} \|f\|_{L^2}
\] 
which together with (2.26) yields (2.27).

Finally, Gronwall’s inequality combined with (2.25), (2.27), (1.6), (2.1) and (2.23) yields (2.21). The proof of Lemma 2.5 is completed.

**Lemma 2.6.** There is a positive constant \( C \) depending on \( T, \gamma, \bar{\mu}, \bar{\rho}, \|\rho_0 - \hat{\rho}\|_{H^1}, \|u_0\|_{H^1} \) and \( \|f\|_{H^2} \) such that \[
\sup_{0 \leq t \leq T} \left( \|\rho_x\|_{L^2}^2 + \|\rho_t\|_{L^2}^2 \right) \leq C(T).
\] (2.28)

**Proof.** Differentiating (1.1), with respect to \( x \) gives
\[
\rho_{tx} + \rho_{xx} u + 2\rho_x u_x + \rho u_{xx} = 0.
\] (2.29)

Multiplying (2.29) by \( \rho_x \), one obtains after using integration by parts and (2.8) that
\[
\frac{d}{dt} \|\rho_x\|_{L^2} \leq C \|u_x\|_{L^\infty} \|\rho_x\|_{L^2} + C \|u_{xx}\|_{L^2}.
\] (2.30)

Then, it follows from (1.1) that \( \rho \rho_x u_{xx} = \rho \hat{u} + P_x - \rho f - \mu'(\rho) \rho_x u_x \), which together with (1.6) and (2.8) yields that
\[
\|u_{xx}\|_{L^2} \leq C \|\rho^{1/2} \hat{u}\|_{L^2} + C \|\rho_x\|_{L^2} + C \|f\|_{L^2} + C \|\rho_x\|_{L^2} \|u_x\|_{L^\infty}.
\] (2.31)

Substituting (2.31) into (2.30), one obtains after using (2.24), (2.21), and Gronwall’s inequality that
\[
\sup_{0 \leq t \leq T} \|\rho_x\|_{L^2} \leq C,
\] (2.32)

which together with (1.1), (2.8), (2.23) and (2.21) leads to
\[
\|\rho_t\|_{L^2} \leq C \|u\|_{L^\infty} \|\rho_x\|_{L^2} + C \|u_x\|_{L^2} \leq C.
\]

Combining this with (2.32) shows (2.28) and thus finishes the proof of Lemma 2.6.

**Lemma 2.7.** There is a positive constant \( C \) depending on \( T, \gamma, \bar{\mu}, \bar{\rho}, \|\rho_0 - \hat{\rho}\|_{H^1}, \|u_0\|_{H^1} \) and \( \|f\|_{H^2} \) such that \[
\sup_{0 \leq t \leq T} \left( \sigma(t) \|u_{xx}\|_{L^2}^2 + \int_0^T (\|u_{xx}\|_{L^2}^2 + \sigma \|u_t\|_{H^1}^2) dt \right) \leq C(T)
\] (2.33)

with \( \sigma(t) \triangleq \min\{1, t\} \).

**Proof.** First, operating \( \partial / \partial_t + (u \cdot )_x \) to (1.1) yields
\[
\rho \hat{u}_t + \rho \hat{u} u_x - [\mu(\rho) \hat{u}_x]_x = -\gamma [P(\rho) u_x|_x - [(\mu(\rho) + \mu'(\rho) \rho) u^2_x]|_x + \rho u f_x.
\] (2.34)

Multiplying (2.34) by \( \hat{u} \), one gets after using (2.23), (2.24) and (2.21) that
\[
\frac{1}{2} \frac{d}{dt} \int_{-\infty}^{+\infty} \rho |\hat{u}|^2 dx + \int_{-\infty}^{+\infty} \mu(\rho) |\hat{u}_x|^2 dx \\
= \gamma \int_{-\infty}^{+\infty} P(\rho) u_x \hat{u}_x dx + \int_{-\infty}^{+\infty} (\mu(\rho) + \mu'(\rho) \rho) u^2_x \hat{u}_x dx + \int_{-\infty}^{+\infty} \rho u f_x \hat{u} dx
\]
which gives
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\[ C\| \dot{u}_x \|^2_{L^2} + C\| u_x \|^2_{L^2} + C\| u \|^2_{L^2} + f \| \rho \|^2_{L^2} + \rho^{1/2} \| \dot{u} \|^2_{L^2} \]
\[ \leq \frac{1}{2} \| \sqrt{\mu(\rho)} \dot{u}_x \|^2_{L^2} + C\| \rho^{1/2} \dot{u} \|^2_{L^2} + C. \]  
(2.35)

Then, multiplying (2.35) by \( \sigma \) and integrating the resulting inequality over \( (0, T) \), we obtain after using (2.21) that
\[ \sup_{0 \leq t \leq T} \sigma \| \rho^{1/2} \dot{u} \|^2_{L^2} + \int_0^T \sigma \int_{-\infty}^{+\infty} \mu(\rho) |u_x|^2 \, dx \, dt \leq C(T), \]  
(2.36)

which together with (2.23) and (2.24) implies that
\[ \sup_{0 \leq t \leq T} \sigma \| u_x \|^2_{L^2} \leq C(T). \]  
(2.37)

Finally, it follows from some direct calculations, (2.21) and (2.23) that
\[ \| \rho^{1/2} u_t \|^2_{L^2} \leq C\| \rho^{1/2} \dot{u} \|^2_{L^2} + C\| \rho \|^2_{L^2} \| u \|^2_{L^2} \| u_x \|^2_{L^2} \leq C\| \rho^{1/2} \dot{u} \|^2_{L^2} + C, \]
which gives
\[ \| u_t \|^2_{H^1} \leq C \int |\hat{\rho} - \rho| u_t^2 \, dx + C \int \rho u_t^2 \, dx + C\| u_x \|^2_{L^2} \]
\[ \leq C\| \rho - \hat{\rho} \|^2_{L^2} \| u_t \|^2_{L^2} \| u_x \|^2_{L^2} + C\| \rho^{1/2} \dot{u} \|^2_{L^2} + C\| u_x \|^2_{L^2} \]
\[ \leq \frac{1}{2} \| u_t \|^2_{L^2} + C\| \rho^{1/2} \dot{u} \|^2_{L^2} + C + C\| u_x \|^2_{L^2} \]
\[ = \frac{1}{2} \| u_t \|^2_{L^2} + C\| \rho^{1/2} \dot{u} \|^2_{L^2} + C + \| (\dot{u} - uu_x) \|^2_{L^2} \]
\[ \leq \frac{1}{2} \| u_t \|^2_{L^2} + C\| \rho^{1/2} \dot{u} \|^2_{L^2} + C + C\| u_x \|^2_{L^2} + \| u_x \|^2_{L^2} + \| u \|^2_{L^2} \]
\[ \leq \frac{1}{2} \| u_t \|^2_{L^2} + C\| \dot{u}_x \|^2_{L^2} + C\| \rho^{1/2} \dot{u} \|^2_{L^2} + C, \]  
(2.38)

where in the last inequality we have used (2.23), (2.21), (2.31), (2.28) and (2.24). Hence, it is easy to deduce from (2.38), (2.31), (2.23), (2.36) and (2.37) that (2.33) holds. The proof of Lemma 2.7 is completed. \( \square \)

3 Proof of Theorem 1.1

For \((\rho_0, u_0, \tilde{f})\) satisfying the conditions in Theorem 1.1, we first construct the smooth approximate data as follows:
\[ \rho^{\delta, \eta} = \frac{\rho_0 * j_\delta + \eta \hat{\rho}}{1 + \eta}, \quad u^{\delta, \eta} = u_0 * j_\delta, \quad \tilde{f}^{\delta, \eta} = \tilde{f} * j_\delta, \]  
(3.1)
where \( \delta \in (0, 1), \eta \in (0, 1), \) and \( j_\delta(x) \) is the standard mollifier with width \( \delta \). It is easy to check that
\[ 0 < \frac{\eta \hat{\rho}}{1 + \eta} \leq \rho^{\delta, \eta} \leq \frac{\sup_{x \in \mathbb{R}} \rho_0 + \eta \hat{\rho}}{1 + \eta} < \infty, \]  
\[ \lim_{\delta, \eta \to 0} \left( \| \rho^{\delta, \eta} - \rho_0 \|_{H^1} + \| u^{\delta, \eta} - u_0 \|_{L^2} + \| \tilde{f}^{\delta, \eta} - \tilde{f} \|_{H^2} \right) = 0. \]  
(3.2)

Choosing \( \mu_\eta \in C^2[0, \infty) \) satisfying \( \lim_{\eta \to 0} \| \mu_\eta - \mu \|_{C^1[0, M]} = 0 \) for any \( M > 0 \), we consider the Cauchy problem (1.1)–(1.3) with \( \mu \) replaced by \( \mu_\eta \) and the data \((\rho^{\delta, \eta}, u^{\delta, \eta}, \tilde{f}^{\delta, \eta})\) satisfying (3.1)–(3.2). It follows from Lemma 2.1 that there exists a unique solution \((\rho, u)\) to problem (1.1)–(1.3) on \( \mathbb{R} \times [0, T_{\delta, \eta}] \). Moreover, the estimates obtained in Lemmas 2.2–2.3, Corollary 2.4, and Lemmas 2.5–2.6 show that the solutions \((\rho, u)\) satisfy that for any \( 0 < T \leq T_{\delta, \eta} \),
\[ \sup_{0 \leq t \leq T} \left( \| (\rho - \hat{\rho}, \mu(\rho)) - (\mu(\hat{\rho}), P(\rho) - P(\hat{\rho})) \|_{H^1} + \| \rho_t \|_{L^2} + \| \rho u_t \|_{L^2} + \| u \|_{H^1} + \sqrt{t} \| \sqrt{\rho} u_t \|_{L^2} \right) \]
\[ + \sqrt{t} \|u_{xx}\|_{L^2} + \int_0^T (\|u\|_{H^2}^2 + t\|u_{xt}\|_{L^2}^2) dt \leq \bar{C}, \]

(3.3)

where \(\bar{C}\) is independent of \(\delta\) and \(\eta\). Furthermore, similar to [17], we can prove that there exists some \(\tilde{C}\) depending on \(\delta\) and \(\eta\) such that

\[
\sup_{0 \leq t \leq T} (\|\rho\|_{L^2}^2 + \|P_{xx}\|_{L^2}^2 + \|\rho_{xt}\|_{L^2}^2 + \|P_{xt}\|_{L^2}^2 + t\|u_{xt}\|_{L^2}^2 + t\|u_{xxxx}\|_{L^2}^2) \leq \tilde{C},
\]

which, in particular, implies that

\[
\rho - \tilde{\rho}, \ P(\rho) - P(\tilde{\rho}) \in C([0, T]; H^2), \quad u \in C([0, T]; H^2) \cap L^2(0, T; H^3),
\]
\[
u_t \in L^2(0, T; H^1), \quad t^{1/2}u \in L^\infty(0, T; H^3),
\]
\[
t^{1/2}u_t \in L^\infty(0, T; H^1) \cap L^2(0, T; H^2), \quad t^{1/2}\rho^{1/2}u_{tt} \in L^2(0, T; L^2).
\]

(4.4)

Then, we will extend the existence time \(T_{\delta, \eta}\) to be infinity. Indeed, let \(T^*\) be the maximal time of the existence for the strong solutions. Then, \(T^* \geq T_{\delta, \eta}\). If \(T^* < \infty\), defining

\[
(\rho^*, u^*) = \lim_{t \to T^*} (\rho, u)(x, t),
\]

we can derive from (4.4) that \((\rho^*, P^*, u^*)\) satisfy the initial condition (1.7) at \(t = T^*\). Therefore, one can take \((\rho^*, u^*)\) as the initial data at \(t = T^*\) and then use the local existence theory (Lemma 2.1) to extend the strong solutions beyond the maximal existence time \(T^*\). This contradicts the assumption on \(T^*\). Thus, \(T^* = \infty\).

Finally, we denote the global strong solutions on \(\mathbb{R} \times [0, \infty)\) obtained above by \((\rho^{\delta, \eta}, u^{\delta, \eta})\). With the estimates (3.3) at hand, letting first \(\delta \to 0\) and then \(\eta \to 0\), we have the sequence \((\rho^{\delta, \eta}, u^{\delta, \eta})\) converges, up to the extraction of subsequences, to some limit \((\rho, u)\) in the obvious weak sense. Then we deduce from (3.3) that \((\rho, u)\) is a strong solution to (1.1)–(1.3) on \(\mathbb{R} \times (0, T)\) (for any \(0 < T < \infty\)) satisfying (1.8). In addition, the uniqueness of the strong solutions \((\rho, u)\) is guaranteed by the regularities (1.8); see [12] for the detailed proof. The proof of Theorem 1.1 is completed.

4 Proof of Theorem 1.3

In this section, we consider (1.1) with \(f \equiv 0\). The proof of Theorem 1.3 is divided into three steps.

Step 1. Modifying slightly the arguments as Lemmas 2.5–2.7, we can also derive some time-independent estimates as follows:

\[
\sup_{0 \leq t \leq T} (\|1\|_{L^2} + \|\sqrt{\mu(\rho)}u_x\|_{L^2}^2) + \int_0^T \|\rho^{1/2}u_t\|_{L^2}^2 dt \leq C,
\]

(4.1)

\[
\sup_{0 \leq t \leq T} \|u_x\|_{L^\infty}^2 + \|\rho^{1/2}u_t\|_{L^2}^2 + \int_0^T \|\sqrt{\mu}u_x\|_{L^2}^2 dt \leq C,
\]

(4.2)

where and in what follows, \(C\) is independent of \(T\).

Indeed, first, multiplying (1.1) by \(u\) and integrating by parts, we get from the same arguments as the proof of (2.22) that

\[
\frac{1}{2} \frac{d}{dt} \int_{-\infty}^{\infty} \mu(\rho)u_x^2 dx + \int_{-\infty}^{\infty} \rho u_t^2 dx \\
\leq \frac{d}{dt} \int_{-\infty}^{\infty} [P(\rho) - P(\tilde{\rho})]u_x dx + C\|u_x\|_{L^\infty} \|u_x\|_{L^2}^2 + C\|u_x\|_{L^4}^2
\]
\[
\leq \frac{d}{dt} \int_{-\infty}^{\infty} [P(\rho) - P(\tilde{\rho})]u_x dx + \frac{1}{2} \|\rho^{1/2}u_t\|_{L^2}^2 + C\|u_x\|_{L^2}^4 + C\|u_x\|_{L^2}^2,
\]

(4.3)
where in the last inequality one has used (2.24) and Young’s inequality. Since
\[
\int_{-\infty}^{+\infty} [P(\rho) - P(\rho)]u_x dx \leq \frac{\mu}{4} \|u_x\|_{L^2}^2 + \|P(\rho) - P(\rho')\|_{L^2}^2 \leq \frac{\mu}{4} \|u_x\|_{L^2}^2 + C \tag{4.4}
\]
due to (2.20), Gronwall’s inequality together with (4.4), (4.3) and (2.1) yields (4.1).

Next, operating \((\partial_t + (u \cdot x))\dot{u}\) to (1.1) and integrating the resulting equation by parts, we deduce from the same calculations as (2.34)–(2.35) that
\[
\frac{1}{2} \frac{d}{dt} \int_{-\infty}^{+\infty} \rho |\dot{u}|^2 dx + \int_{-\infty}^{+\infty} \mu(\rho)|\dot{u}|^2 dx = \gamma \int_{-\infty}^{+\infty} P(\rho)u_x \dot{u}_x dx + \int_{-\infty}^{+\infty} (\mu(\rho) + \mu'(\rho)\rho)u_x^2 \dot{u}_x dx
\]
\[
\leq \frac{\mu}{2} \|\dot{u}_x\|_{L^2}^2 + C \|u_x\|_{L^2}^2 \|\rho^{1/2} \dot{u}\|_{L^2}^2 + C \|u_x\|_{L^2}^2 + C \|u_x\|_{L^2}^2. \tag{4.5}
\]

Multiplying (4.5) by \(\sigma\), one gets after using Gronwall’s inequality, (1.11), (2.1) and (4.1) that
\[
\sup_{0 \leq t \leq T} (\sigma \|\rho^{1/2} \dot{u}\|_{L^2}^2) + \int_0^T \sigma \|\dot{u}_x\|_{L^2}^2 dt \leq C, \tag{4.6}
\]
which together with (2.23), (2.24) and (4.1) implies that
\[
\sup_{0 \leq t \leq T} (\|u\|_{L^2 \cap L^\infty} + \sigma \|u_x\|_{L^\infty}) \leq C. \tag{4.7}
\]
The combination of (4.7) with (4.6) gives (4.2).

Step 2. We will show that for any \(p \geq 2\),
\[
\lim_{t \to 0} \|u_x\|_{L^p} = 0. \tag{4.8}
\]
First, we claim that
\[
\lim_{t \to 0} \int_{t-1}^t \left( \|u_x\|_{L^2}^2 + \left| \frac{d}{dr} \|u_x\|_{L^2}^2 \right| \right) dr = 0. \tag{4.9}
\]

For any \(t > 1\) and \(s \in (t-1, t)\), it holds
\[
\|u_x\|_{L^2}^2(t) - \|u_x\|_{L^2}^2(s) = \int_s^t \frac{d}{dr} \|u_x\|_{L^2}^2 dr \leq \int_s^t \left| \frac{d}{dr} \|u_x\|_{L^2}^2 \right| dr,
\]
i.e.,
\[
\|u_x\|_{L^2}^2(t) \leq \int_{t-1}^t \left( \|u_x\|_{L^2}^2 + \left| \frac{d}{dr} \|u_x\|_{L^2}^2 \right| \right) dr,
\]
which together with (4.9) yields
\[
\lim_{t \to \infty} \|u_x\|_{L^2}^2(t) = 0. \tag{4.10}
\]
It is easy to deduce from (4.7) that for any \(p > 2\),
\[
\|u_x\|_{L^p}^p \leq \|u_x\|_{L^2}^2 \|u_x\|_{L^\infty}^{p-2} \leq C \|u_x\|_{L^2}^2, \quad \forall t > 1. \tag{4.11}
\]
The combination of (4.10) with (4.11) yields (4.8).

Now, it remains to prove (4.9). Indeed, it follows from integration by parts and (4.7) that
\[
\frac{d}{dt} \|u_x\|_{L^2}^2 = 2 \int_{-\infty}^{+\infty} u_x u_{xt} dx = 2 \int_{-\infty}^{+\infty} u_x (\dot{u}_x - (u_x)_{xt}) dx
\]
\[
\leq 2 \int_{-\infty}^{+\infty} u_x \dot{u}_x dx - \int_{-\infty}^{+\infty} u_x^2 dx
\]
\[
\leq C \|\dot{u}_x\|_{L^2}^2 + C(1 + \|u_x\|_{L^\infty}) \|u_x\|_{L^2}^2
\]
\[
\leq C \|\dot{u}_x\|_{L^2}^2 + C \|u_x\|_{L^2}^2.
\]
This together with (2.1) and (4.2) yields
\[
\int_{1}^{\infty} \left( \|u_x\|_{L^2}^2 + \left| \frac{d}{dt} \|u_x\|_{L^2}^2 \right| \right) dt \leq C,
\]
which yields directly (4.9). The proof of Step 2 is completed.

**Step 3.** Using the methods due to [9, 15], we are now in a position to prove that for any \( p > 2 \),
\[
\lim_{t \to \infty} \|\rho - \tilde{\rho}\|_{L^p} = 0. 
\]

First, we claim that
\[
\int_{0}^{\infty} \|\rho - \tilde{\rho}\|_{L^6}^6 dt \leq C. 
\]

It follows from integration by parts, (1.1) that
\[
\frac{d}{dt} \|\rho - \tilde{\rho}\|_{L^6}^6 = 6 \int_{-\infty}^{+\infty} (\rho - \tilde{\rho})^5 (\rho - \tilde{\rho}) dx 
= -6 \int_{-\infty}^{+\infty} (\rho - \tilde{\rho})^5 (\rho - \tilde{\rho})_x u dx - 6 \int_{-\infty}^{+\infty} (\rho - \tilde{\rho})^5 \rho u_x dx 
\leq C\|\rho - \tilde{\rho}\|_{L^6}^3 \|u_x\|_{L^2} + C\|\rho - \tilde{\rho}\|_{L^6}^3 \|u_x\|_{L^2} 
\leq C\|\rho - \tilde{\rho}\|_{L^6}^6 + C\|u_x\|_{L^2}^2,
\]
which together with (4.13) and (2.1) gives \( \int_{0}^{\infty} \|\rho - \tilde{\rho}\|_{L^6}^6 dt \leq C \). Combining this with (4.13) implies that \( \lim_{t \to \infty} \|\rho - \tilde{\rho}\|_{L^6} = 0 \), which together with (2.20) and (2.8) leads to the desired (4.12).

Next, we will prove (4.13). Denote
\[
Q \triangleq \mu(\rho) u_x - (P(\rho) - P(\tilde{\rho})).
\]

It follows from (1.1) that \( Q_x = \rho \tilde{u} \), which gives
\[
\|Q_x\|_{L^2} = \|\rho \tilde{u}\|_{L^2} \leq C\|\rho^{1/2} \tilde{u}\|_{L^2}.
\]

It deduces from (1.1) that
\[
(\rho - \tilde{\rho})_t + (\rho - \tilde{\rho})_x u + (\rho - \tilde{\rho}) u_x + \tilde{\rho} u_x = 0.
\]

Multiplying (4.16) by \( 6(\rho - \tilde{\rho})^5 \) and integrating by parts imply that
\[
\frac{d}{dt} \|\rho - \tilde{\rho}\|_{L^6}^6 + 6 \int_{-\infty}^{+\infty} \tilde{\rho}(\mu(\rho))^{-1}(\rho - \tilde{\rho})^5 (P(\rho) - P(\tilde{\rho})) dx 
= -6 \int_{-\infty}^{+\infty} \tilde{\rho}(\mu(\rho))^{-1}(\rho - \tilde{\rho})^5 Q dx - 5 \int_{-\infty}^{+\infty} (\rho - \tilde{\rho})^6 u_x dx 
\leq C\|\rho - \tilde{\rho}\|_{L^6}^2 \|Q\|_{L^6} + C\|\rho - \tilde{\rho}\|_{L^6} \|\rho - \tilde{\rho}\|_{L^6} \|u_x\|_{L^2} 
\leq C\|\rho - \tilde{\rho}\|_{L^6}^6 + C(\varepsilon) \|\rho^{1/2} \tilde{u}\|_{L^2}^2 + C(\varepsilon) \|u_x\|_{L^2}^2 
\leq C\|\rho - \tilde{\rho}\|_{L^6}^6 + C(\varepsilon) \|\rho^{1/2} \tilde{u}\|_{L^2}^2 + C(\varepsilon) \|u_x\|_{L^2}^2,
\]
where one has used (4.14), (1.6), (2.8), (4.15), (4.2) and (2.20). Furthermore, the direct calculations combined with (1.6) and (2.8) show that for some 0 < \( \alpha < 1 \),
\[
6\tilde{\rho}(\mu(\rho))^{-1}(\rho - \tilde{\rho})^5 (P(\rho) - P(\tilde{\rho})) = 6\tilde{\rho}(\mu(\rho))^{-1}(\rho - \tilde{\rho})^5 P'(\alpha \rho + (1 - \alpha)\tilde{\rho}) \geq C_0(\rho - \tilde{\rho})^6,
\]
where the positive constant \( C_0 \) depends only on \( \gamma, \tilde{\mu}, \tilde{\rho} \) and \( \tilde{\rho} \). Substituting (4.18) into (4.17) and choosing \( \varepsilon \) suitably small yield
\[
\frac{d}{dt} \|\rho - \tilde{\rho}\|_{L^6}^6 + C_0\|\rho - \tilde{\rho}\|_{L^6}^6 \leq C\|\rho^{1/2} \tilde{u}\|_{L^2}^2 + C\|u_x\|_{L^2}^2.
\]
Thus, one can derive the desired (4.13) from (4.19), (2.1), (2.20) and (4.1).

Finally, notice that
\[
\|\rho - \hat{\rho}\|_{C(\overline{\mathbb{R}})} \leq C \|\rho - \hat{\rho}\|_{L^{3/2}}^{3/4} \|\rho\|_{L^2}^{1/4}
\]

The proof of (1.14) is similar to that of [13, Theorem 1.2] (see also [9]). The proof of Theorem 1.3 is finished.

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