A mathematical theory of D-string world-sheet instantons, 
I: Compactness of the stack of $Z$-semistable Fourier-Mukai transforms from a compact family of nodal curves to a projective Calabi-Yau 3-fold

Chien-Hao Liu and Shing-Tung Yau

Abstract

In a suitable regime of superstring theory, D-branes in a Calabi-Yau space and their most fundamental behaviors can be nicely described mathematically through morphisms from Azumaya spaces with a fundamental module to that Calabi-Yau space. In the earlier work [L-L-S-Y] (D(2): arXiv:0809.2121 [math.AG], with Si Li and Ruifang Song) from the project, we explored this notion for the case of D1-branes (i.e. D-strings) and laid down some basic ingredients toward understanding the notion of D-string world-sheet instantons in this context. In this continuation, D(10), of D(2), we move on to construct a moduli stack of semistable morphisms from Azumaya nodal curves with a fundamental module to a projective Calabi-Yau 3-fold $Y$. In this Part I of the note, D(10.1), we define the notion of twisted central charge $Z$ for Fourier-Mukai transforms of dimension 1 and width $[0]$ from nodal curves and the associated stability condition on such transforms and prove that for a given compact stack of nodal curves $C_{\mathcal{M}}/\mathcal{M}$, the stack $\mathcal{FM}_{C_{\mathcal{M}}/\mathcal{M}}^{1,[0];Z-ss}(Y,c)$ of $Z$-semistable Fourier-Mukai transforms of dimension 1 and width $[0]$ from nodal curves in the family $C_{\mathcal{M}}/\mathcal{M}$ to $Y$ of fixed twisted central charge $c$ is compact. For the application in the sequel D(10.2), $C_{\mathcal{M}}/\mathcal{M}$ will contain $C_{\mathcal{M}}^g/\mathcal{M}_g$ as a substack and $\mathcal{FM}_{C_{\mathcal{M}}/\mathcal{M}}^{1,[0];Z-ss}(Y,c)$ in this case will play a key role in defining stability conditions for morphisms from arbitrary Azumaya nodal curves (with the underlying nodal curves not necessary in the family $C_{\mathcal{M}}/\mathcal{M}$) to $Y$.

Key words: D-string, world-sheet instanton; Azumaya nodal curve, fundamental module, morphism; central charge, stability condition, wall, chamber structure; compactness of moduli.

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Chien-Hao Liu dedicates this subseries D(10), to be completed, to his uncle Prof. Pin-Hsingu Liu*, aunt Ms. Rui-Be Lin, and cousin Master Guo-Guang Shr (Te-Ru Liu) for their hospitality in his first year of college, years of communications with Te-Ru, and lots of cherished memories.

†Deceased, winter 2004. (1925 – 2004)

*(From C.H.L.) From the preschool years when I was beginning to be aware of this world, the big family of my father’s generation seemed to already fall apart, like the story in the Chinese classic “Hong Lou Meng (Dream of the Red Chamber)”, composed by Hsue-Chin Tsao in the middle of the 18th century. My third uncle somehow stood out as the most educated and also the most independent among my father’s eleven siblings; and left the big shadow of my grandfather – being a medical doctor and a local elite – to pursue his own study and dream on anthropology. Later he got a Harvard Yenching Institute scholarship to visit here as a scholar during 1964-1965. There, through the interaction with a linguist John Harvey, he got deep insight toward a resolution of a kinship problem that had puzzled anthropologists for long, using *combinatorial group theory* (!!!); [L1]. Starting from there, *mathematic anthropology* became his marking territory, a field that would likely deter most anthropologists; [L2]. I remember in my first year of high school, he came to visit us and taught my brother and me to play ‘go’. But my real contact with him won’t begin until I got to college and moved in to his house and lived together with my aunt and cousin for a year. I observed a scholar that is driven not by fame but by his dream. While writing this, lots of cherished memories with my aunt and, particularly, my cousin also re-emerge. I thank them all for shredding away the clumsiness of a country kid in a big city. In preparing this note, I read through parts of [L-G-H] again. In the last sentence of the last article of the collection, it was pondered upon what legacy my uncle had left behind. This is a question that likely can have only personal answers. But to me, in view of being in the middle of an extremely demanding project that still has a long way to go, the answer couldn’t be more obvious: He has given me a role model of perseverance and persistence to pursue one’s dream. I thus dedicate this D(10), yet to be completed, to the memory of my uncle, and to my aunt and cousin Te-Ru/Master Guo-Guang.

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[L2] ———, *Foundations of kinship mathematics*, Mono. Ser. A, no. 28, Institute of Ethnology, Academic Sinica, Taipei, Taiwan, 1986.

[L-G-H] M.-r. Lin, P.-y. Guo and C.-H. Huang eds., *Essays in honor of Professor Pin-Hsingu Liu*, Institute of Ethnology, Academic Sinica, Taipei, Taiwan, 2008.
0. Introduction and outline.

In a suitable regime of superstring theory, D-branes in a Calabi-Yau space and their most fundamental behaviors can be nicely described mathematically through morphisms from Azumaya spaces with a fundamental module to that Calabi-Yau space. In the earlier work [L-L-S-Y] (D(2): arXiv:0809.2121 [math.AG], with Si Li and Ruifang Song) from the project, we explored this notion for the case of D1-branes (i.e. D-strings) and laid down some basic ingredients toward understanding the notion of D-string world-sheet instantons in this context. In this continuation, D(10), of D(2), we move on to construct a moduli stack of semistable morphisms from Azumaya nodal curves with a fundamental module to a projective Calabi-Yau 3-fold $Y$.

In this Part I of the note, D(10.1), we define the notion of twisted central charge $Z$ for Fourier-Mukai transforms of dimension 1 and width $[0]$ from nodal curves and the associated stability condition on such transforms and prove that for a given compact stack of nodal curves $C_{\mathcal{M}}/\mathcal{M}$, the stack $\mathcal{FM}^{1,[0];Z,ss}_{C_{\mathcal{M}}/\mathcal{M}}(Y,c)$ of $Z$-semistable Fourier-Mukai transforms of dimension 1 and width $[0]$ from nodal curves in the family $C_{\mathcal{M}}/\mathcal{M}$ to $Y$ of fixed twisted central charge $c$ is compact. For the application in the sequel D(10.2), $C_{\mathcal{M}}/\mathcal{M}$ will contain $C_{\mathcal{M}}^{g}/\mathcal{M}^{g}$ as a substack and $\mathcal{FM}^{1,[0];Z,ss}_{C_{\mathcal{M}}/\mathcal{M}}(Y,c)$ in this case will play a key role in defining stability conditions for morphisms from arbitrary Azumaya nodal curves (with the underlying nodal curves not necessary in the family $C_{\mathcal{M}}/\mathcal{M}$) to $Y$.

Remark 0.1. [general Fourier-Mukai transform]. We should remark that it is very natural to anticipate a generalized setting and compactness result of the current note D(10.1), in which general Fourier-Mukai transforms (cf. [Hu]) associated to objects in the bounded derived category $D^b(Coh(C \times Y))$ of coherent sheaves are considered and the general stability conditions from the works [Bay], [Br], [G-K-R], [Ru], a generalization of the Kleiman’s Boundedness Criterion ([Kl]) and of the related inequalities that bounds $h^0$ by slopes ([LP], [Ma2], [Si]), and the result of valuative criterion from Dan Abramovich and Alexander Polishchuk [A-P] and Jason Lo [Lo1], [Lo2] are taken as the foundation.

Convention. Standard notations, terminology, operations, facts in (1) string theory/D-branes, supersymmetry; (2) stacks; (3) Fourier-Mukai transforms; (4) moduli spaces of sheaves can be found respectively in (1) [Po1], [Po2], [Bac], [Jo]; [Ar], [W-B]; (2) [L-MB]; (3) [Hu] (4) [H-L].

- All varieties, schemes and their products are over $\mathbb{C}$; a ‘curve’ means a 1-dimensional proper scheme over $\mathbb{C}$.
- The ‘support’ $\text{Supp}(\mathcal{F})$ of a coherent sheaf $\mathcal{F}$ on a scheme $Y$ means the scheme-theoretical support of $\mathcal{F}$ unless otherwise noted; $\mathcal{I}_Z$ denotes the ideal sheaf of a subscheme of $Z$ of a scheme $Y$.
- The ‘polarization class group’ of a curve $C$ means the same as the ‘degree class group’ $\text{DCG}(C)$ of $C$. The former emphasizes that its elements are represented by ample line bundles while the latter emphasizes its discrete nature.
- For the projection maps $pr_1: C \times Y \to C$ and $pr_2: C \times Y \to Y$, $pr_1^*L$ (resp. $pr_2^*(B+\sqrt{-1}J)$) may be denoted still by $L$ (resp. $B+\sqrt{-1}J$) for the simplicity of expressions when there is no chance of confusion. Here, $L$ is a line bundle or a line bundle class on $C$ and $B+\sqrt{-1}J$ is a complexified Kähler class on $Y$. 

1
· A curve class in $A_1(C \times Y)$ and its image in $N_1(C \times Y)_Z$ are denoted the same when we need the latter.

· **Central charge functional $Z$ vs. variety/scheme or cycle/cycle class $Z$.**

· **Line bundle or line-bundle class $L$ vs. calibrated cycle $L$,** (especially, for special Lagrangian).

· The word ‘twist’ in this note means an alteration by a line-bundle, except in Remark 1.3, where it means an effect due to $B$-field. (The two concepts are completely different.)

· The current note continues the study in [L-L-S-Y] (arXiv:0809.2121 [math.AG], D(2)). A partial review of D-branes and Azumaya noncommutative geometry is given in [L-Y3] (arXiv:1003.1178 [math.SG], D(6)) and [LiuCH] (arXiv:1112.4317 [math.AG]). Notations and conventions follow these early works when applicable.

**Outline.**

0. Introduction.

1. D-string world-sheet instantons, morphisms from Azumaya nodal curves with a fundamental module, and Fourier-Mukai transforms from nodal curves.
   · D-branes and morphisms from Azumaya spaces with a fundamental module.
   · Morphisms from Azumaya schemes with a fundamental module versus Fourier-Mukai transforms.
   · Stable morphisms from Azumaya nodal curves with a fundamental module as D-string world-sheet instantons.

2. Twisted central charges and stability conditions on Fourier-Mukai transforms from nodal curves.

   2.1 Twisted central charges of D-string world-sheet instantons.
      · Central charges of D-branes in a Calabi-Yau 3-fold and why we need a twist.
      · Twisted central charges of D-string world-sheet instantons.
      · Basic properties.

   2.2 Stability conditions on Fourier-Mukai transforms of dimension 1 and width $[0]$ from nodal curves.
      · Stability conditions defined by a central charge functional $Z$.
      · The Harder-Narasimhan filtration with respect to $Z$.
      · Jordan-Hölder filtrations and $S$-equivalence with respect to $Z$.

   2.3 A chamber structure on the space of stability conditions.
      · The space of stability conditions $Stab^{1, [0]}_1(C \times Y)$.
      · Walls and chambers on $Stab^{1, [0]}_1(C \times Y)$.
      · Local finiteness of walls.
      · Behavior of the moduli stack of stable objects when crossing an actual wall: a conjecture.

3. Compactness of the moduli stack $\mathcal{FM}^{1, [0]; Z_{ss}}_{C_M/\mathcal{M}}(Y, c)$ of $Z$-semistable Fourier-Mukai transforms.
   · Boundedness of $\mathcal{FM}^{1, [0]; Z_{ss}}_{C_M/\mathcal{M}}(Y, c)$.
   · Completeness of $\mathcal{FM}^{1, [0]; Z_{ss}}_{C_M/\mathcal{M}}(Y, c)$.
1 D-branes, morphisms from Azumaya spaces with a fundamental module, Fourier-Mukai transforms, and D-string world-sheet instantons

In this section, we review tersely how we come to this to set up basic terminologies and notations and bring out the notion of ‘D-string world-sheet instantons’. Readers are referred to [L-Y1] (D(1)), [L-L-S-Y] (D(2)) for more thorough related discussions, [L-Y3] (D(6)) and [LiuCH] for a review of the project up to D(9.1), and [Hu] for Fourier-Mukai transforms.

D-branes and morphisms from Azumaya spaces with a fundamental module.

The open-string-induced matrix-valued-type enhancement of the scalar fields on a D-brane world-volume $X$ for coincident D-branes in a space-time $Y$ that describe deformations of the brane world-volume in $Y$ motivated string-theorists, Pei-Ming Ho and Yong-Shi Wu [H-W] in particular, to propose a fundamental matrix-type noncommutative geometry on the D-brane world-volume (i.e. D-branes as ‘quantum space’ in the language of [H-W]); see also related discussion of Michael Douglas [Dou1]. This appearance of matrix-type noncommutative structure on the D-brane world-volume $X$ (rather than directly on the target space-time $Y$) turns out to be also a most natural interpretation of what’s going on from Grothendieck’s viewpoint of algebraic geometry and his theory of schemes and morphisms between them. A D-brane in this content is then described by a morphism from a scheme with a matrix-type noncommutative structure sheaf (i.e. an Azumaya scheme $(X,\mathcal{O}_X^{\text{Az}})$) together with a fundamental $\mathcal{O}_X^{\text{Az}}$-module $\mathcal{E}$ to $(Y,\mathcal{O}_Y)$, where $\mathcal{O}_Y$ is the structure sheaf of $Y$ in either commutative or noncommutative setting; in notation/symbol,

$$\varphi : (X,\mathcal{O}_X^{\text{Az}};\mathcal{E}) \rightarrow (Y,\mathcal{O}_Y),$$

with a built-in isomorphism $\mathcal{O}_X^{\text{Az}} \simeq \text{End}_X(\mathcal{E})$. In true contents, this means a contravariant gluing system of ring-homomorphisms

$$\mathcal{O}_X^{\text{Az}} \leftarrow \mathcal{O}_Y : \varphi^\sharp,$$

which in general does not induce any morphisms directly from $X$ to $Y$. It is through $\varphi^\sharp$ that the $\mathcal{O}_X^{\text{Az}}$-module $\mathcal{E}$ can be pushed forward to an $\mathcal{O}_Y$-module, in notation $\varphi_*\mathcal{E}$, on $Y$. Despite the language difference and different level of developments, the same idea applies to nonsupersymmetric D-branes, (supersymmetric) D-branes of B-type (cf. the above setting in algebraic geometry), and (supersymmetric) D-branes of A-type ([L-Y4] (D(7))).

Morphisms from Azumaya schemes with a fundamental module versus Fourier-Mukai transforms.

When the target space $Y$ is a commutative scheme and $\mathcal{E}_X$ is locally free $\mathcal{O}_X$-module, then associated to a morphism $\varphi : (X,\mathcal{O}_X^{\text{Az}};\mathcal{E}) \rightarrow (Y,\mathcal{O}_Y)$ is the following diagram

$$\begin{array}{ccc}
\mathcal{O}_X^{\text{Az}} = \text{End}\mathcal{E} & \xrightarrow{\varphi^\sharp} & \mathcal{O}_Y \\
\uparrow & & \downarrow \\
\mathcal{A}_\varphi := \text{Im}\varphi^\sharp & \xleftarrow{} & \mathcal{O}_Y
\end{array},$$
which defines a subscheme $X_\varphi := \text{Spec} A_\varphi \subset X \times Y$ together with a coherent sheaf $\tilde{E}_\varphi$ supported on $X_\varphi$, which is simply the $\mathcal{O}^{\tilde{E}}_X$-module $\mathcal{E}$ regarded as an $A_\varphi$-module. $\tilde{E}_\varphi$ is called the graph of the morphism $\varphi$. It is a coherent sheaf on $X \times Y$ that is flat over $X$, of relative dimension 0. Conversely, given such a coherent sheaf on $X \times Y$, a morphism $\varphi : (X, \mathcal{O}^{\tilde{E}}_X; \mathcal{E}) \to Y$ can be constructed from $\tilde{E}$ by taking

- $\mathcal{E} = pr_1^* \tilde{E}$,
- $\mathcal{O}^{\tilde{E}}_X = \text{End}_{\mathcal{O}_X}(\mathcal{E})$, and
- $\varphi^2 : \mathcal{O}_Y \to \mathcal{O}^{\tilde{E}}_X$ is defined by the composition

$$\mathcal{O}_Y \xrightarrow{pr_2^*} \mathcal{O}_{X \times Y} \xrightarrow{i^*} \mathcal{O}_{\text{Supp}(\tilde{E})} \hookrightarrow \mathcal{O}^{\tilde{E}}_X.$$  

Here, $X \xleftarrow{pr_1} X \times Y \xrightarrow{pr_2}$ are the projection maps, $i : \text{Supp}(\tilde{E}) \to X \times Y$ is the embedding of the subscheme, and note that $\text{Supp}(\tilde{E})$ is affine over $X$. The subseries D(10) studies issues in the special case $X$ is a nodal curve parameterized by a compact Artin stack $\mathcal{M}$.

Resume the general discussion. Treating $\tilde{E}$ as an object in the bounded derived category $D^b(\text{Coh}(X \times Y))$ of coherent sheaves on $X \times Y$, $\tilde{E}$ defines a Fourier-Mukai transform $\Phi_{\tilde{E}} : D^b(\text{Coh}(X)) \to D^b(\text{Coh}(Y))$, in short name, a Fourier-Mukai transform from $X$ to $Y$. In this way, the data that specifies a morphism $\phi : (X, \mathcal{O}^{\tilde{E}}_X; \mathcal{E}) \to Y$ is matched to a data that specifies a special kind of Fourier-Mukai transform.

Definition 1.1. [support, dimension, width of Fourier-Mukai transform]. For a general $\mathcal{F}^\bullet \in D^b(\text{Coh}(X \times Y))$, we define the (scheme-theoretical) support $\text{Supp}(\mathcal{F})$ of $\mathcal{F}^\bullet$ to be the (scheme-theoretical) support of $\bigoplus_i H^i(\mathcal{F}^\bullet)$, the dimension $\text{dim} \mathcal{F}^\bullet$ of $\mathcal{F}^\bullet$ to be the dimension $\text{dim}(\text{Supp}(\mathcal{F}^\bullet))$, and the width of $\mathcal{F}^\bullet$ to be the interval $[i, j]$ such that $H^i(\mathcal{F}^\bullet) \neq 0$, $H^j(\mathcal{F}^\bullet) \neq 0$, and $H^k(\mathcal{F}^\bullet) = 0$, for $k \notin [i, j]$. We’ll denote the width $[i, i]$ by $|i|.$

Thus, for $X$ fixed of pure dimension $d$, the stack of morphisms $(X, \mathcal{O}^{\tilde{E}}_X; \mathcal{E}) \to Y$ is embedded in the stack of Fourier-Mukai transforms from $X$ to $Y$ of dimension $d$ and width $[0]$; the latter is identical to the stack of $d$-dimensional coherent sheaves on $X \times Y$. Similar statement holds for $X$ not fixed.

After the above review, let us turn to the focus of this subseries D(10): The case of D1-branes (i.e. D-strings).

Stable morphisms from Azumaya nodal curves with a fundamental module as D-string world-sheet instantons.

An instanton in a field theory is by definition a field configuration that is localized both in space and in time. Thus, when an (Euclidean) D$p$-brane world-volume wraps around a $(p+1)$-cycle in a Calabi-Yau 3-fold, it looks like an instanton in the 4-dimensional effective field theory from the compactification of a Type II superstring theory on that Calabi-Yau 3-fold. Due to its origin, it is called a D-brane world-volume instanton in the 4-dimensional field theory. (See e.g., [B-B-S] for various dimensional brane world-volume instantons and [B-C-K-W] for a review of D-instantons in various contents.) Recall now how the Gromov-Witten theory of stable maps from nodal curves (resp. bordered Riemann surface) to a Calabi-Yau space $Y$ (resp. a Calabi-Yau space $Y$ with
a calibrated or holomorphic cycle \( L \subset Y \) is related to string-theorists’ (fundamental) closed (resp. open) string world-sheet instantons, e.g. [C-K], [C-dI0-G-P], [D-S-W-W], [K-K-L-MG], [K-L], [LiuCC] and [O-V]. Now that we have realized the moving of a D-string in a space-time as a morphism from an Azumaya nodal curve with a fundamental module (i.e. Azumaya Riemann surface with nodes in the complex geometry language) to that space-time, it is very natural to regard a stable morphism from an Azumaya nodal curve with a fundamental module to a Calabi-Yau space \( Y \) as giving a D-string world-sheet instanton of the 4-dimensional effective field theory, if the notion of ‘stable morphism’ can be defined appropriately for our objects. (See Remark 1.2 below for more detailed explanations.) Once that is achieved in such a way that the corresponding moduli space of stable morphisms in our contents behaves good enough to have a reasonable tangent-obstruction theory or its extension as long as any kind of intersection theory or theory of constructible functions can apply, then one can define and compute in good cases (a version of) ‘D-string world-sheet instanton numbers’ exactly as the moduli stack of stable maps in Gromov-Witten theory did for the fundamental closed or open string. This is the topic of D(10). It turns out that ‘stable D-branes’ in string-theory contents as referred to whether they can decay or not has to do with the notion of (BPS) central charges of D-branes. Furthermore, besides this crucial notion from superstring theory, when one allows the topology of a D-string world-sheet to vary under deformations, additional mathematical issues come in if one wants to obtain a compact moduli stack of stable morphisms. The latter issue forces us to separate the D-string world-sheet to a major subcurve and a minor subcurve in such a way that the former with the fundamental module in general is pushed forward under the morphism to a 1-dimensional coherent sheaf on the target Calabi-Yau space \( Y \) while the latter consists of a collection of \( \mathbb{P}^1 \)-trees and, with the fundamental module, is pushed forward only to a 0-dimensional coherent sheaf on \( Y \). The former carries the main information of central charge while the latter is to be constrained by hand by requiring good properties on the induced morphism of related moduli stacks. (See [L-Y5] for details). This leads us to construct first an auxiliary moduli stack, i.e. the moduli stack of semistable (with respect to a central charge) Fourier-Mukai transforms of dimension 1 and width [0] from a compact family of nodal curves to the Calabi-Yau manifold \( Y \). These two sub-topics, central charges and the auxiliary moduli stack, are the focus of this note D(10.1). In particular, we prove that such stacks are compact and, hence, provide us with a collection of reasonable reference stacks to begin with, (Sec. 3, Theorem 3.1).

Remark 1.2. [where is the special connection?] D-brane (or M-brane) instantons are generally more complicated than (fundamental) open or closed string world-sheet instantons, since a D-brane or M-brane carries more structure thereupon. The most basic such structure for D-branes (in the simplest case) is an (open-string-induced ) bundle with a gauge connection. Besides the D-brane world-volume instantons addressed above, a D-brane of dimension \( \geq 3 \) (i.e. world-volume of dimension \( \geq 4 \) ) can wrap around a cycle of an internal Calabi-Yau 3-space to create another copy of \( d = 4 \) effective space-time. Dimension reduction of the full (open-string-induced) field theory on the D-brane world-volume thus creates another sector to the complete 4-dimensional effect field theory from the compactification of a 10-dimensional Type II superstring theory. Quantity from this section in general may be subject to open-string world-sheet instanton corrections. This sector contains, in particular, a 4-dimensional gauge theory, which may itself has instanton solutions (in the sense of a gauge theory). All these different types of 4-dimensional instantons arising from D-branes say the importance of special gauge connections in the definition of D-brane instantons in whatever contents. Thus, alert string-theorists may legitimately question our algebro-geometric inclined formalism:
Q. One has the bundle $\mathcal{E}$ on the D-string world-sheet, but where is the connection on $\mathcal{E}$ in this setting that, as for any supersymmetric D-brane, must satisfy supersymmetry-induced equations of motion?

To answer this, one has to reason as follows. In our data, we specify only holomorphic structures on $\mathcal{E}$. In general there are nonunique connections on $\mathcal{E}$ that are compatible with the given holomorphic structure on $\mathcal{E}$. Stability condition on the morphism comes in to distinguish string-theory-allowed $\mathcal{E}$ while, at the same time, select a unique compatible connection on it through a Donaldson-Uhlenbeck-Yau-type Theorem (cf. [Don], [Ja], [Le], [U-Y]), though the latter cannot be spelled out literally yet in most cases. Thus, we have the following correspondence

- Azumaya curve with a fundamental module $(C, O^A_{\text{Az}}; \mathcal{E})$
  (Euclidean) D-string world-sheet with open-string-induced structures

- morphism to Calabi-Yau space $\varphi : (C, O^A_{\text{Az}}; \mathcal{E}) \rightarrow Y$ \iff wrapping of (Euclidean) D-string world-sheet on a holomorphic 1-cycle in $Y$

- stability condition on $\varphi$ distinguished holomorphic bundle with a unique special connection thereupon on D-string world-sheet

which would then justify a stable morphism in our content as a D-string world-sheet instanton. This is the same working philosophy behind many literatures on D-branes of B-type.

Remark 1.3. [effect of B-field]. Alert readers who bridge well between mathematics and string theory will notice immediately an incompleteness of our setting in this note: While we take into account the effect of a $B$-field to the central charge of D-branes, we completely ignore its effect to the twisting of the Chan-Paton sheaf on the D-brane. With the language of twisted sheaves (cf. [Că] of Andrei Căldăraru), D-branes in a space-time with a $B$-field background can be formulated as a morphism from a general Azumaya space $(X, O^A_X)$ – whose associated class in the Brauer group $Br(X)$ of $X$ is non-zero – with a compatible twisted fundamental $O^A_X$-module $\mathcal{E}$ ([L-Y2] (D(5))); and the above two effects to D-branes (in particular, to D-string world-sheet instantons) can then be taken care of simultaneously. For the current note, we follow the TASI 2003 Lecture Notes of Paul Aspinwall [As] to take the former effect into account but suppress the latter for simplicity. This definitely leaves room for future works toward a complete treatment.

2 Twisted central charges and stability conditions on Fourier-Mukai transforms from nodal curves

In this section we define the stability conditions associated to central charges we will use in our problem, and study their basic properties and the chamber structure in the space of stability conditions. A conjecture is made on the wall-crossing behavior of the related moduli space.
2.1 Twisted central charges of D-string world-sheet instantons

We recall in this subsection the origin of (BPS) central charge of a D-brane in a Calabi-Yau 3-fold in superstring theory and then explain why we need a twist of it in our problem. After that, we define precisely the central charge functional to be used for our problem and study its basic properties.

Central charges of D-branes in a Calabi-Yau 3-fold and why we need a twist.

The $d = 4$, $N = 2$ supersymmetry algebra $\mathcal{A}$ (over $\mathbb{C}$) has a 1-dimensional center. Let $Z$ be a generator of this center. Then, for an irreducible representation $\mathcal{H}$ of $\mathcal{A}$, $Z$ acts on $\mathcal{H}$ as $c \cdot Id_{\mathcal{H}}$, where $c \in \mathbb{C}$ is a constant and $Id_{\mathcal{H}}$ is the identity map on $\mathcal{H}$. $c$ is called the central charge of the representation. We also say that the elements in the representation $\mathcal{H}$ carry a central charge $c$. When the Type IIA or the Type IIB 10-dimensional superstring theory is compactified on a general Calabi-Yau 3-fold $Y$, i.e. the whole space-time becomes now $\mathbb{R}^{3+1} \times Y$, the resulting effective field theory on the 4-dimensional Minkowski space-time $\mathbb{R}^{3+1}$ has a $d = 4$, $N = 2$ supersymmetry $\mathcal{A}_{d=4,N=2}$. When one adds a D-brane $X$ (with structures thereupon kept implicit) to the internal Calabi-Yau 3-fold $Y$ over a $p \in \mathbb{R}^{3+1}$, $X$ would be effectively a point-like particle sitting at $p$ from the $\mathbb{R}^{3+1}$ aspect. When $X$ in $\mathbb{R}^{3+1} \times Y$ evolves along with time, we obtain then an embedding of the world-volume $X \times \mathbb{R}$ of the D-brane in $\mathbb{R}^{3+1} \times Y$. From the effective 4-dimension aspect, this is an embedding $\mathbb{R}^{0+1} \hookrightarrow \mathbb{R}^{3+1}$ of the world-line of a particle in the Minkowski space-time. In general, such a configuration in a compactification of a superstring theory renders no supersymmetry left in the 4-dimensional effective theory. For a special D-brane (i.e. those wrapping a calibrated or holomorphic cycle, as is in our case), this will reduce the previous $d = 4$, $N = 2$ supersymmetry only to $d = 4$, $N = 1$ supersymmetry in the 4-dimensional effective theory and the corresponding particle is thus a BPS particle, by definition. In terms of quantum mechanics of a particle in $\mathbb{R}^{3+1}$, it corresponds to a state in an irreducible representation $\mathcal{H}$ of $\mathcal{A}_{d=4,N=2}$ whose annihilator $\mathcal{A}_0$ is a $d = 4$, $N = 1$ subalgebra of $\mathcal{A}_{d=4,N=2}$. The central charge $c$ of $\mathcal{H}$ (or equivalently this BPS state) is defined to be the central charge of the D-brane $X$ in the Calabi-Yau 3-fold $Y$. $c$ determines $\mathcal{A}_0$. This is the 4-dimensional effective space-time aspect of the central charge of a D-brane $X$ in $Y$.

From the superstring world-sheet aspect, the nonlinear $\sigma$-model on a closed string world-sheet $\Sigma_{\text{closed}}$ with target a general Calabi-Yau 3-fold $Y$ defines a $d = 2$, $N = (2, 2)$ superconformal field theory on $\Sigma_{\text{closed}}$. When one adds a D-brane $X$ to $Y$ and requires the open-string to have their end-points lying in $X$, the resulting $d = 2$ open-string world-sheet theory in general has no longer any supersymmetry. Again, for a special D-brane $X$, the nonlinear $\sigma$-model with target $(Y, X)$ gives a $d = 2$, $N = (1, 1)$ superconformal field theory on the open string world-sheet $(\Sigma_{\text{open}}, \theta \Sigma_{\text{open}})$. As an abstract open-and-closed conformal field theory, a $d = 2$, $N = (1, 1)$ superconformal field theory with boundary is obtained from a $d = 2$, $N = (2, 2)$ superconformal field theory by specifying how its holomorphic sector and its antiholomorphic sector should be matched along the boundary $\partial \Sigma_{\text{open}}$. The rule is specified exactly by the D-brane $X$ (i.e. a boundary condition to the $d = 2$ theory) and the phase of the central charge of $X$ enters the rule. With more technicality, the central charge of the D-brane $X \subset Y$ can be reproduced from the expansion of the boundary state $|B_X\rangle$ associated to $X$ in terms of Ishibashi states in the $d = 2$ superconformal field theory. This gives a meaning of the (BPS) central charge of a D-brane from the open-string world-sheet aspect.

Mathematicians are referred to [Ar], [A-B-C-D-G-K-M-S-S-W], [B-B-S], [H-K-K-P-T-V-V-Z], and [O-O-Y] for more detailed explanations.

Back to the mathematical world, Given a Calabi-Yau 3-fold $Y$ with a complexified Kähler class $B + \sqrt{-1}J$ and a D-brane of B-type, realized as a coherent sheaf $\mathcal{F}$ (or more generally
an object in the derived category $D^b(Coh(Y))$ of coherent sheaves) on $Y$, after several string-theorists’ works, e.g. [As], [C-Y], [F-W], [G-H-M], [Harv], [M-M], [O-O-Y], the formula of the central charge of D-branes in this case is given by

$$Z^{B+\sqrt{-1}J}(F) = \int_Y e^{-(B+\sqrt{-1}J)} ch(F) \sqrt{td(T_Y)} + O(\alpha'),$$

where $O(\alpha')$ is a stringy correction to $Z(F)$ that tends to zero as $\alpha' \to 0$ (i.e. fundamental string tension $\alpha_s \to \infty$). Naively, in our situation we have a coherent $\tilde{F}$ on $C \times Y$ and one would like to define the central charge of $\tilde{F}$ to be

$$Z^\text{naive}_{\sqrt{-1}J}(F) = \int_{C \times Y} pr_2^* \left( \frac{e^{-(B+\sqrt{-1}J)}}{\sqrt{td(T_Y)}} \right) \tau_{C \times Y}(\tilde{F}) + O(\alpha').$$

When $\text{Supp}(\tilde{F})$ defines an embedding $\iota : C \hookrightarrow Y$, then $Z^\text{naive}_{\sqrt{-1}J}(\tilde{F}) = Z^{B+\sqrt{-1}J}(\iota_* \mathcal{E})$, where $\mathcal{E} := pr_{1*} \tilde{F}$. However, e.g. by considering the case $Y = S \times E$, where $S$ is an algebraic $K3$-surface that contains an embedded $\mathbb{P}^1$ and $E$ is a smooth curve of genus 1, it can happen that under a deformation of $\tilde{F}$, $\tilde{F}$ is deformed to $\tilde{F}' = \tilde{F}_1 + \tilde{F}_2$ such that $\text{Supp}(\tilde{F}_1) \cap \text{Supp}(\tilde{F}_2) = \emptyset$ and that $pr_{2*} \tilde{F}_2$ is 0-dimensional. In this case, any coherent sheaf on $\text{Supp}(\tilde{F}_2)$ is $Z^\text{naive}_{\sqrt{-1}J}$-semistable and, hence, the moduli stack of 1-dimensional $Z^\text{naive}_{\sqrt{-1}J}$-semistable coherent sheaves on $C \times Y$ is fixed central charge won’t be bounded in general. To cure this, one needs to recover the positivity of $-\text{Im} Z$ by introducing a twist from a positive degree class on $C$.

**Twisted central charges of D-string world-sheet instantons.**

Let $C$ be a nodal curve with a polarization class $L$ and $Y$ be a projective Calabi-Yau manifold with a complexified Kähler class $B + \sqrt{-1}J$.

**Definition 2.1.1. [twisted central charge of Fourier-Mukai transform].** Let $\tilde{F}$ be a coherent sheaf of dimension 1 on $C \times Y$ and $\Phi_{\tilde{F}}$ be the Fourier-Mukai transform $\tilde{F}$ defines. Then, the **twisted central charge** of $\Phi_{\tilde{F}}$ associated to the data $(B + \sqrt{-1}J, L)$ is defined to be

$$Z^{B+\sqrt{-1}J,L}(\Phi_{\tilde{F}}) := Z^{B+\sqrt{-1}J,L}(\tilde{F}) := \int_{C \times Y} pr_2^* \left( \frac{e^{-(B+\sqrt{-1}J)}}{\sqrt{td(T_Y)}} \right) \cdot pr_1^* e^{-\sqrt{-1}L} \cdot \tau_{C \times Y}(\tilde{F}),$$

where $\tau_{C \times Y}(\tilde{F}) := ch(\tilde{F}) \cdot td(T_{C \times Y})$ is the $\tau$-class of $\tilde{F}$.

**Lemma 2.1.2. [twisted central charge: explicit form].** Continuing the above notation. Let

$$\beta(\tilde{F}) := \sum_i d_i [\zeta_i] \in A_1(C \times Y),$$

where $\zeta_i$ runs through the generic points of $\text{Supp}(\tilde{F})$ and $d_i$ is the dimension of $\tilde{F}|_{\zeta_i}$ as a $k_{\zeta_i}$-vector space. Then,

$$Z^{B+\sqrt{-1}J,L}(\tilde{F}) = \left( \chi(\tilde{F}) - B \cdot \beta(\tilde{F}) \right) - \sqrt{-1} \left( (J + L) \cdot \beta(\tilde{F}) \right).$$

In particular, for non-zero coherent sheaves on $C \times Y$ of dimension $\leq 1$, $Z^{B+\sqrt{-1}J,L}$ takes its values in the partially completed lower-half complex plane

$$\mathbb{H}^- := \{ z \in \mathbb{C} \mid \text{either Im} z < 0 \text{ or Im} z = 0 \text{ with } Re z > 0 \}.$$
Proof. The design of the functorial $\tau$-class $\tau(\cdot)$ for coherent sheaves on singular varieties to fit well with the Grothendieck-Riemann-Roch Theorem and the behavior of $\tau$-class in a flat family imply that
\[
\tau_{C \times Y}(\tilde{F}) = \tilde{\beta}(\tilde{F}) + Z_0
\]
with $Z_0 \in A_0(C \times Y)$ of degree $\chi(\tilde{F})$. The lemma follows. \qed

Remark 2.1.3. [general case and stringy correction]. Though our main interest motivated from superstring theory is the case $Y$ is a Calabi-Yau 3-fold. For a general $Y$ with $c_1(Y) \neq 0$ and of general dimensions, the expression for $Z^{B+\sqrt{-1}J,L}$ in Lemma 2.1.2 is modified to
\[
Z^{B+\sqrt{-1}J,L}(\tilde{F}) = \left(\chi(\tilde{F}) - (B + \frac{1}{4}c_1(Y)) \cdot \tilde{\beta}(\tilde{F})\right) - \sqrt{-1} \left((J + L) \cdot \tilde{\beta}(\tilde{F})\right).
\]
Purely mathematically, this may be taken as the starting point of the current note. All the statements remain to hold after being mildly modified accordingly if necessary. Furthermore, when the stringy correction $O(\alpha')$ to the central charge $Z^{B+\sqrt{-1}J,L}$ is taken into account, the evaluation of differential 2-forms on (real 2-)cycles in the above expression should be replaced by a quantum evaluation from Gromov-Witten theory.

Definition 2.1.4. [$Z$-slope $\mu^Z$]. Continuing Definition 2.1.1. We define the $Z$-slope for a non-zero coherent sheaf $\tilde{F}$ on $C \times Y$ of dimension $\leq 1$ to be
\[
\mu^Z(\tilde{F}) := -\text{Re} \left(\frac{Z^{B+\sqrt{-1}J,L}(\tilde{F})}{\text{Im} \left(\frac{Z^{B+\sqrt{-1}J,L}(\tilde{F})}{(J + L) \cdot \tilde{\beta}(\tilde{F})}\right)}\right)
\]
We collect here a few basic properties of a central charge functional $Z^{B+\sqrt{-1}J}$ we need for later discussions.

Lemma 2.1.5. [invariant of flat family]. Let $(C_S, L_S)$ be a flat family of nodal curves with a polarization class over $S$ and $\tilde{F}_s$ be a coherent sheaf on $C_S \times Y$ that is flat and of relative dimension 1 over $S$. Then $Z^{B+\sqrt{-1}J,L_s}(\tilde{F}_s) \in \mathbb{C}$, $s \in S$, is locally constant on $S$.

Proof. By considering the restriction of the coherent sheaf $\tilde{F}_s$ to generic points of $\text{Supp}(\tilde{F}_s)$, one observes that, under flat deformations of $(C, L, \tilde{F})$, $\tilde{\beta}(\cdot)$ form a flat family of 1-cycles on the family $(C_S \times Y)/S$. Since, in addition, $\chi(\cdot)$ are constant, the lemma follows. \qed

Lemma 2.1.6. [additivity]. Let $0 \rightarrow \tilde{F}_1 \rightarrow \tilde{F}_2 \rightarrow \tilde{F}_3 \rightarrow 0$ be a short exact sequence of coherent sheaves on $C \times Y$ of dimension $\leq 1$. Then,
\[
Z^{B+\sqrt{-1}J,L}(\tilde{F}_2) = Z^{B+\sqrt{-1}J,L}(\tilde{F}_1) + Z^{B+\sqrt{-1}J,L}(\tilde{F}_3).
\]
Proof. The $\tau$-class is additive with respect to a short exact sequence. In the explicit form, both $\tilde{\beta}(\bullet)$ and $\chi(\bullet)$ are additive with respect to a short exact sequence.

Lemma 2.1.7. [finite set of corresponding Hilbert polynomials]. Given a bounded flat family $(C_S, L_S)$ of nodal curves with a polarization class over $S$, let $c \in \hat{H}_-$, $H$ be a relative hyperplane class on $(C_S \times Y)/S$ and $\text{Poly}^H(Z^{B+\sqrt{-1}J,L} = c)$ be the set of Hilbert polynomials of coherent sheaves $\tilde{F}$ of dimension $\leq 1$ on fibers of $(C_S \times Y)/S$ with $Z^{B+\sqrt{-1}J,L}(\tilde{F}) = c$. Then, $\text{Poly}^H(Z^{B+\sqrt{-1}J,L} = c)$ is a finite set.

Proof. Recall that the Hilbert polynomial defined by $H$ (in variable $m$) of a coherent sheaf on fibers of $(C_S \times Y)/S$ of dimension $\leq 1$ is given by

$$P^H(\tilde{F}) = (H \cdot \tilde{\beta}(\tilde{F}))m + \chi(\tilde{F}).$$

Let $\iota_S : C_S \times Y \to \mathbb{P}^N_S$ be an $S$-embedding of $C_S \times Y$ into the projective-space bundle $\mathbb{P}^N_S$ over $S$ determined by $H$. Then $N_1(\mathbb{P}^N_S/S) \cong \mathbb{P}^N_S \times \mathbb{Z}$ is a trivial bundle of free abelian group of rank $1$, equipped with a canonical trivialization from the generating effective curve class in fibers of $\mathbb{P}^N_S/S$. Let $L_{S,*} : N_1((C_S \times Y)/S) \to \mathbb{Z}$ be the composition $N_1((C_S \times Y)/S) \to N_1(\mathbb{P}^N_S/S) \to \mathbb{Z}$ of the induced bundle homomorphism and the projection map. Consider the subset of effective curve classes in fibers of $(C_S \times Y)/S$:

$$\Xi_{(C_S \times Y)/S,1}^{J+L,c} := \{ E \in N_{E_1}((C_S \times Y)/S) \mid (J + L) \cdot E = -\text{Im} c \}$$

Then, it follows from Kleiman’s Ampleness Criterion that $L_{S,*}(\Xi_{(C_S \times Y)/S,1}^{J+L,c})$ is a finite subset of $\mathbb{Z}$. Consequently, both the set

$$\left\{ H \cdot \tilde{\beta}(\tilde{F}) \mid \tilde{F} \text{ is a coherent sheaf on fibers of } (C_S \times Y)/S \right\} \leq 1 \text{ with } Z^{B+\sqrt{-1}J,L}(\tilde{F}) = c \subset \mathbb{Z}$$

and the set

$$\left\{ pr_{2,*}(\tilde{\beta}(\tilde{F})) \mid \tilde{F} \text{ is a coherent sheaf on fibers of } (C_S \times Y)/S \right\} \leq 1 \text{ with } Z^{B+\sqrt{-1}J,L}(\tilde{F}) = c \subset N_1(Y)_{\mathbb{Z}}$$

are finite. It follows that the set

$$\left\{ B \cdot \tilde{\beta}(\tilde{F}) \mid \tilde{F} \text{ is a coherent sheaf on fibers of } (C_S \times Y)/S \right\} \leq 1 \text{ with } Z^{B+\sqrt{-1}J,L}(\tilde{F}) = c \subset \mathbb{R}$$

and, hence, the set

$$\left\{ \chi(\tilde{F}) \mid \tilde{F} \text{ is a coherent sheaf on fibers of } (C_S \times Y)/S \right\} \leq 1 \text{ with } Z^{B+\sqrt{-1}J,L}(\tilde{F}) = c \subset \mathbb{Z}$$

are also finite. This proves the lemma.

The following lemma relates $\mu^Z$ with the more familiar $\mu^P$:
Lemma 2.1.8. [P-slope vs. Z-slope]. Let $L$ be a relative polarization class on the universal curve $C_M/M$ and $H$ be a relative hyperplane class on $(C_M \times Y)/M$. Then there exist constants $c_1, c_2, c_3, c_4 \in \mathbb{R}$, with $c_1, c_3 > 0$, that depend only on $B + \sqrt{-1}J$, $L$, and $H$ such that

$$c_1 \mu^P(\tilde{F}) + c_2 \leq \mu^Z(\tilde{F}) \leq c_3 \mu^P(\tilde{F}) + c_4$$

for all nodal curves $C$ in the family $C_M/M$ and all 1-dimensional coherent sheaves $\tilde{F}$ on $C \times Y$.

Here

$$P(\tilde{F}) = (H \cdot \tilde{\beta}(\tilde{F}))m + \chi(\tilde{F})$$

is the Hilbert polynomial (in $m$) of $\tilde{F}$ defined by $H$, and

$$\mu^P(\tilde{F}) := \chi(\tilde{F})/(H \cdot \tilde{\beta}(\tilde{F}))$$

is the slope of $\tilde{F}$ defined by $P$.

Proof. Let $S$ be an atlas of $M$. Under the same setting as in the proof of Lemma 2.1.7,

$$\frac{B \cdot (\bullet)}{(J + L) \cdot (\bullet)} : \mathbb{P}(\overline{NE_1((C_S \times Y)/S)}) \to \mathbb{R}$$

has image in a bounded subset of $\mathbb{R}$ while

$$\frac{H \cdot (\bullet)}{(J + L) \cdot (\bullet)} : \mathbb{P}(\overline{NE_1((C_S \times Y)/S)}) \to \mathbb{R}_{>0}$$

has image in a compact subset of $\mathbb{R}_{>0}$. The lemma follows.

\[\square\]

Proposition 2.1.9. [properness/projectivity of Quot-scheme]. Given a family of nodal curves $C_S/S$ over $S$ with a relative polarization class $L$ and a 1-dimensional coherent sheaf $\tilde{F}_S$ on $(C_S \times Y)/S$. Then the Quot-scheme $\text{Quot}^{Z_{B+\sqrt{-1}J,L}}(C_S \times Y)/S(\tilde{F}_S, c)$ of quotient sheaves $\tilde{F}_S \to \tilde{Q}_S \to 0$ that is flat over $S$ with $Z_{B+\sqrt{-1}J,L}(\tilde{Q}_S) = c$ is projective over $S$.

Proof. Let $H$ be a relative ample line bundle on $(C_S \times Y)/S$. Since both the Hilbert polynomial $P^H$ and the central charge $Z_{B+\sqrt{-1}J,L}$ are invariant under flat deformations, connected components of $\text{Quot}^{Z_{B+\sqrt{-1}J,L}}(C_S \times Y)/S(\tilde{F}_S, c)$ coincide with some connected components of $\text{Quot}^H(C_S \times Y)/S(\tilde{F}_S)$. It follows from Lemma 2.1.7 that

$$\text{Quot}^{Z_{B+\sqrt{-1}J,L}}(C_S \times Y)/S(\tilde{F}_S,c) = \coprod_{P_i \in \text{Poly}^H(Z_{B+\sqrt{-1}J,L}=c)} \text{Quot}^H(C_S \times Y)/S(\tilde{F}_S,P_i)$$

is a finite disjoint union. The latter is known to be projective over $S$.

\[\square\]
2.2 Stability conditions on Fourier-Mukai transforms of dimension 1 and width \([0]\) from nodal curves

Recall the compact family \(C_M/M\) of nodal curves with a relative ample class \(L\) and the projective Calabi-Yau 3-fold \(Y\) with a complexified Kähler class \(B + \sqrt{-1}J\). We introduce the notion of \(Z\)-(semi)stability of Fourier-Mukai transforms of dimension 1 and width \([0]\) from a fiber of \(C_M/M\) to \(Y\) and their Harder-Narasimhan filtration with respect to \(Z\), and collect their basic properties in this subsection. Though the technique here remains pre-Bridgeland, readers are recommended to [Br] of Tom Bridgeland for the generalization to objects in the derived category of coherent sheaves and to [Dou2] and [Dou3] of Michael Douglas for the string-theoretical motivation and reason of the design.

Stability conditions defined by a central charge functional \(Z\).

**Definition 2.2.1.** [\(Z\)-semistable, \(Z\)-stable, \(Z\)-unstable, strictly \(Z\)-semistable]. Let \(|C| \in M\). A 1-dimensional coherent sheaf \(\tilde{F}\) on \(C \times Y\) is said to be \(Z\)-semistable (resp. \(Z\)-stable) if \(\tilde{F}\) is pure and \(\mu^Z(\tilde{F}') \leq \mu^Z(\tilde{F})\) for any nonzero proper subsheaf \(\tilde{F}' \subset \tilde{F}\). Such \(\tilde{F}\) is called \(Z\)-unstable if it is not \(Z\)-semistable, and is called strictly \(Z\)-semistable if it is \(Z\)-semistable but not \(Z\)-stable. When the central charge functional \(Z\) is known and fixed either explicitly or implicitly, we may use the terminology: semistable, stable, unstable, strictly semistable, for simplicity.

**Notation 2.2.2.** [\(Z\)-(semi)stable]. (Cf. [H-L: Notation 1.2.5].) Following Huybrechts and Lehn [H-L], we’ll rephrase, for example, the above definition in a combined statement and notation “\(\tilde{F}\) is (semi)stable if it is pure and \(\mu^Z(\tilde{F}') \leq \mu^Z(\tilde{F})\) for any nonzero proper subsheaf \(\tilde{F}' \subset \tilde{F}\).” to cover both the \(Z\)-semistable and the \(Z\)-stable situation.

The following statements from [H-L: Sec. 1.2] remain to be true by the same argument.

**Proposition 2.2.3.** [equivalent form]. (Cf. [H-L: Proposition 1.2.6].) Let \(\tilde{F}\) be a purely 1-dimensional coherent sheaf on \(C \times Y\). Then the following statements are equivalent:

1. \(\tilde{F}\) is (semi)stable.
2. \(\mu^Z(\tilde{F}') \leq \mu^Z(\tilde{F})\) for all nonzero proper saturated subsheaves \(\tilde{F}' \subset \tilde{F}\).
3. \(\mu^Z(\tilde{F}) \leq \mu^Z(\tilde{F}'')\) for all nonzero proper 1-dimensional quotient sheaves \(\tilde{F} \to \tilde{F}''\).
4. \(\mu^Z(\tilde{F}'') \leq \mu^Z(\tilde{F})\) for all nonzero proper purely 1-dimensional quotient sheaves \(\tilde{F} \to \tilde{F}''\).

**Proposition 2.2.4.** [\(Z\)-slope and homomorphism]. (Cf. [H-L: Proposition 1.2.7].) Let \(\tilde{F}\) and \(\tilde{G}\) be purely 1-dimensional coherent sheaf on \(C \times Y\). If \(\mu^Z(\tilde{F}) > \mu^Z(\tilde{G})\), then \(\text{Hom}(\tilde{F}, \tilde{G}) = 0\). If \(\mu^Z(\tilde{F}) = \mu^Z(\tilde{G})\) and \(h: \tilde{F} \to \tilde{G}\) is nonzero, then \(h\) is injective if \(\tilde{F}\) is \(Z\)-stable and surjective if \(\tilde{G}\) is \(Z\)-stable. If \(Z(\tilde{F}) = Z(\tilde{G})\), then any nonzero homomorphism \(h: \tilde{F} \to \tilde{G}\) is an isomorphism provided \(\tilde{F}\) or \(\tilde{G}\) is \(Z\)-stable.

**Corollary 2.2.5.** [\(Z\)-stable \(\Rightarrow\) simple]. (Cf. [H-L: Corollary 1.2.8].) If \(\tilde{F}\) is a \(Z\)-stable sheaf, then \(\text{End}(\tilde{F}) \simeq \mathbb{C}\).
The Harder-Narasimhan filtration with respect to $Z$.

**Definition 2.2.6. [Harder-Narasimhan filtration]**. Let $\tilde{F}$ be a purely 1-dimensional coherent sheaf on $C \times Y$. A **Harder-Narasimhan filtration** of $\tilde{F}$ (with respect to a central charge $Z$) is an increasing filtration

$$0 = HN^Z_0(\tilde{F}) \subset HN^Z_1(\tilde{F}) \subset \cdots \subset HN^Z_l(\tilde{F}) = \tilde{F},$$

such that the factors $gr^Z_i := HN^Z_i(\tilde{F})/HN^Z_{i-1}(\tilde{F})$, for $i = 1, \ldots, l$, are $Z$-semistable of dimension 1 with $Z$-slopes $\mu_i$ satisfying

$$\mu^Z_{\text{max}}(\tilde{F}) := \mu_1 > \mu_2 > \cdots > \mu_l =: \mu^Z_{\text{min}}(\tilde{F}).$$

**Lemma 2.2.7. [$\mu^Z_{\text{min}}, \mu^Z_{\text{max}}$, and homomorphism]**. (Cf. [H-L: Lemma 1.3.3].) If $\tilde{F}$ and $\tilde{G}$ are purely 1-dimensional coherent sheaves on $C \times Y$ with $\mu^Z_{\text{min}}(\tilde{F}) > \mu^Z_{\text{max}}(\tilde{G})$, then $\text{Hom}(\tilde{F}, \tilde{G}) = 0$.

**Lemma 2.2.8. [subsheaf with maximal $Z$-slope]**. (Cf. [H-L: Lemma 1.3.5].) Let $\tilde{F}$ be a purely 1-dimensional coherent sheaf on $C \times Y$. Then there exists a subsheaf $\tilde{G} \subset \tilde{F}$ such that for all subsheaves $\tilde{F}' \subset \tilde{F}$, one has $\mu^Z(\tilde{F}') \leq \mu^Z(\tilde{G})$, and in case of equality $\tilde{F}' \subset \tilde{G}$. Furthermore, $\tilde{G}$ is uniquely determined by $Z$ and is $Z$-semistable.

**Definition 2.2.9. [maximal destabilizing subsheaf]**. Continuing Lemma 2.2.8. $\tilde{G}$ is called the **maximal $Z$-destabilizing subsheaf** of $\tilde{F}$.

**Theorem 2.2.10. [existence and uniqueness of Harder-Narasimhan filtration]**. (Cf. [H-L: Theorem 1.3.4].) Every purely 1-dimensional coherent sheaf $\tilde{F}$ on $C \times Y$ has a unique Harder-Narasimhan filtration with respect to $Z$.

**Theorem 2.2.11. [Harder-Narasimhan filtration stable under base field extension]**. (Cf. [H-L: Theorem 1.3.7].) Let $\tilde{F}$ be a purely 1-dimensional coherent sheaf on $C \times Y$ and $K$ be a field extension of $k \simeq \mathbb{C}$. Then

$$HN_\bullet(\tilde{F} \otimes_k K) = HN_\bullet(\tilde{F}) \otimes_k K.$$

**Corollary 2.2.12. [semistability under field extension]**. (Cf. [H-L: Corollary 1.3.8].) If $\tilde{F}$ is a $Z$-semistable 1-dimensional coherent sheaf on $C \times Y$ and $K$ is a field extension of $k \simeq \mathbb{C}$, then $\tilde{F} \otimes_k K$ is $Z$-semistable as well.
Jordan-H"older filtrations and $S$-equivalence with respect to $Z$.

**Definition 2.2.13. [Jordan-H"older filtration of semistable object].** Let $\tilde{F}$ be $Z$-semistable coherent sheaf of dimension 1 on $C \times Y$. A Jordan-H"older filtration of $\tilde{F}$ with respect to $Z$ is a filtration

$$0 = \tilde{F}_0 \subset \tilde{F}_1 \subset \cdots \subset \tilde{F}_l = \tilde{F}$$

such that the factors $gr_i(\tilde{F}) := \tilde{F}_i / \tilde{F}_{i-1}$ are $Z$-stable with $Z$-slope $\mu^Z(\tilde{F})$.

**Proposition 2.2.14. [existence of JH filtration/uniqueness of graded object].** (Cf. [H-L: Proposition 1.5.2].) Continuing Definition 2.2.13. Jordan-H"older filtrations always exist. The graded object $gr(\tilde{F}) := \oplus_i gr_i(\tilde{F})$ does not depend on the choice of the Jordan-H"older filtration.

**Definition 2.2.15. [S-equivalence of semistable objects].** Two $Z$-semistable coherent sheaves $\tilde{F}_1$ and $\tilde{F}_2$ of dimension 1 on $C \times Y$ are called $S$-equivalent if $gr(\tilde{F}_1) \simeq gr(\tilde{F}_2)$. In notation, $\tilde{F}_1 \simeq \tilde{F}_2$.

### 2.3 A chamber structure on the space of stability conditions

We discuss in this subsection a chamber structure on the space of stability conditions used in this note. As we remain in the case before Tom Bridgeland [Br], readers are referred to the classical work [Qin] of Zhenbo Qin for related early discussions and references.

**The space of stability conditions $\text{Stab}^{[1,0]}(C \times Y)$.**

Let $DCG(C)$ be the degree class group of the nodal curve $C$, $DCG^+(C) \subset DCG(C)$ be the semigroup of effective classes, $DCG(C)_{\mathbb{R}} := DCG(C) \otimes_{\mathbb{Z}} \mathbb{R}$, and $DCG^+(C)_{\mathbb{R}} \subset DCG(C)_{\mathbb{R}}$ be the cone of effective classes spanned by $\mathbb{R}_{>0}$-rays through elements in $DCG^+(C) \subset DCG(C)_{\mathbb{R}}$, and $KCone(Y)$ be the K"ahler cone of $Y$. Then, the data $(B + \sqrt{-1}J, L)$ that defines a central charge functional $Z_B + \sqrt{-1}J, L$ is parameterized by the set

$$\text{Stab}^{[1,0]}(C \times Y) := H_2(Y; \mathbb{R}) \times \sqrt{-1} \cdot KCone(Y) \times DCG^+(C).$$

The natural topology on $\text{Stab}^{[1,0]}(C \times Y)$ as a subset in a vector space by definition coincides with the topology on $\text{Stab}^{[1,0]}(C \times Y)$ defined by treating its elements $(B + \sqrt{-1}J, L)$ as $\mathbb{H}$-valued functionals $Z_B + \sqrt{-1}J, L$ on the set of 1-dimensional coherent sheaves on $C \times Y$. Denote also

$$\text{Stab}^{[1,0]}(C \times Y)_{\mathbb{R}} := H_2(Y; \mathbb{R}) \times \sqrt{-1} \cdot KCone(Y) \times DCG^+(C)_{\mathbb{R}},$$

with multi-variable coordinates $(x_B + \sqrt{-1}x_J, x_L)$.

**Definition 2.3.1. [space of stability conditions].** The set $\text{Stab}^{[1,0]}(C \times Y)$ equipped with the above topology is called the space of stability conditions on the category of Fourier-Mukai transforms of dimension 1 and width [0] from $C$ to $Y$. 

14
In this subsection, we shall fix a class \( \chi \) and consider only \( 1 \)-dimensional coherent sheaves on \( C \times Y \) with Euler characteristic \( \chi \) and curve class \( \beta \).

Given \( (B_1 + \sqrt{-1} J_1, L_1), (B_2 + \sqrt{-1} J_2, L_2) \in \text{Stab}^{1,0}(C \times Y) \), let \( Z_1 := Z^{B_1 + \sqrt{-1} J_1, L_1} \) and \( Z_2 := Z^{B_2 + \sqrt{-1} J_2, L_2} \). Suppose that there exists a \( 1 \)-dimensional coherent sheaf \( \tilde{F} \) with \( \chi(\tilde{F}) = \chi_0 \) and \( \beta(\tilde{F}) = \tilde{\beta}_0 \) such that \( \tilde{F} \) is \( Z_1 \)-stable but \( Z_2 \)-unstable. Then, there exists a saturated proper subsheaf \( \tilde{F}' \subset \tilde{F} \) such that

\[
\mu^{Z_1}(\tilde{F}') < \mu^{Z_1}(\tilde{F}) \quad \text{while} \quad \mu^{Z_2}(\tilde{F}') > \mu^{Z_2}(\tilde{F}).
\]

In the explicit form, this says that

\[
\frac{\chi(\tilde{F}') - B_1 \cdot \tilde{\beta}(\tilde{F}')}{(J_1 + L_1) \cdot \tilde{\beta}(\tilde{F}')} < \frac{\chi_0 - B_1 \cdot \tilde{\beta}_0}{(J_1 + L_1) \cdot \tilde{\beta}_0} \quad \text{while} \quad \frac{\chi(\tilde{F}') - B_2 \cdot \tilde{\beta}(\tilde{F}')}{(J_2 + L_2) \cdot \tilde{\beta}(\tilde{F}')} > \frac{\chi_0 - B_2 \cdot \tilde{\beta}_0}{(J_2 + L_2) \cdot \tilde{\beta}_0}
\]

Note that \( 0 < \tilde{\beta}(\tilde{F}) < \tilde{\beta}(\tilde{F}) = \tilde{\beta}_0 \). This motivates the following definition:

**Definition 2.3.3. [numerical-class-defined walls and chambers in \( \text{Stab}^{1,0}(C \times Y) \)]** For a fixed \( (\chi_0, \tilde{\beta}_0) \in \mathbb{Z} \times \text{NE}(C \times Y) \), let \((e, \xi) \in \mathbb{Z} \times \text{NE}(C \times Y)\) with \( 0 < \xi < \tilde{\beta}_0 \). Let

\[
Q_{(e, \xi)}^{(\chi_0, \tilde{\beta}_0)}(x_B, x_J, x_L) := \left( e - x_B \cdot \xi \right) \left( (x_J, x_L) \cdot \tilde{\beta}_0 \right) - \left( \chi_0 - x_B \cdot \tilde{\beta}_0 \right) \left( (x_J, x_L) \cdot \xi \right),
\]

a quadratic polynomial that is affine in the multivariable \( x_B \) and linear in the combined multivariable \( (x_J, x_L) \). Define the numerical-class-defined wall \( W_{(e, \xi)}^{(\chi_0, \tilde{\beta}_0)} \) in \( \text{Stab}^{1,0}(C \times Y)_\mathbb{R} \) to be

\[
W_{(e, \xi)}^{(\chi_0, \tilde{\beta}_0)} := \left\{ (x_B + \sqrt{-1} x_J, x_L) \in \text{Stab}^{1,0}(C \times Y)_\mathbb{R} \mid Q_{(e, \xi)}^{(\chi_0, \tilde{\beta}_0)}(x_B, x_J, x_L) = 0 \right\}.
\]

A connected component of

\[
\text{Stab}^{1,0}(C \times Y)_\mathbb{R} - \bigcup_{(e, \xi) \in \mathbb{Z} \times \text{NE}(C \times Y)} W_{(e, \xi)}^{(\chi_0, \tilde{\beta}_0)}
\]

with \( 0 < \xi < \tilde{\beta}_0 \)

is called a numerical-class-defined chamber of \( \text{Stab}^{1,0}(C \times Y)_\mathbb{R} \). For a given numerical-class-defined wall \( W_{(e_0, \xi_0)}^{(\chi_0, \tilde{\beta}_0)} \subset \text{Stab}^{1,0}(C \times Y)_\mathbb{R} \), the intersection

\[
W_{(e_0, \xi_0)}^{(\chi_0, \tilde{\beta}_0)} \cap \bigcup_{(e, \xi) \in \mathbb{Z} \times \text{NE}(C \times Y)} W_{(e, \xi)}^{(\chi_0, \tilde{\beta}_0)}
\]

with \( 0 < \xi < \tilde{\beta}_0 \)

induces a stratification of \( W_{(e_0, \xi_0)}^{(\chi_0, \tilde{\beta}_0)} \) by manifolds, which is called the numerical-class-defined stratification of the wall \( W_{(e_0, \xi_0)}^{(\chi_0, \tilde{\beta}_0)} \).
For a fixed $(\chi_0, \tilde{\beta}_0)$, the above wall-and-chamber structure on $\text{Stab}^{1,0}(C \times Y)_\mathbb{R}$ depends only on the structure of $\mathbb{Z} \times \text{NE}(C \times Y)$ and the pairing $H^2(Y; \mathbb{C}) \times \text{NE}(Y) \to \mathbb{C}$; and, hence, the name. However, it should be noted that as long as distinguishing stable conditions is concerned, a numerical-class-defined wall $W^{(\chi_0, \tilde{\beta}_0)}_{(e, \xi)}$ can truly separate two stability conditions whose associated subcategory of semistable objects are different if, in addition, there exists a 1-dimensional coherent sheaf $\tilde{F}$ on $C \times Y$ that contains a proper subsheaf $\tilde{F}'$ such that $\chi(\tilde{F}') = e$ and $\tilde{\beta}(\tilde{F}') = \tilde{\xi}$. I.e. the numerical equivalence class $(e, \tilde{\xi})$ is actually realized by some coherent subsheaf in our category.

**Definition 2.3.4. [actual walls and chambers in $\text{Stab}^{1,0}(C \times Y)$].** A numerical-class-defined wall $W^{(\chi_0, \tilde{\beta}_0)}_{(e, \xi)}$ is called an *actual wall* if $(e, \xi)$ is realized by some coherent subsheaf of a coherent sheaf in our category. A connected component of

$$\text{Stab}^{1,0}(C \times Y)_\mathbb{R} = \bigcup_{(e, \xi) \in \mathbb{Z} \times \text{NE}(C \times Y)} W^{(\chi_0, \tilde{\beta}_0)}_{(e, \xi)}$$

with

$$0 < \xi < \tilde{\beta}_0$$

is called an *actual chamber* of $\text{Stab}^{1,0}(C \times Y)_\mathbb{R}$. For a given actual wall $W^{(\chi_0, \tilde{\beta}_0)}_{(e_0, \tilde{\xi}_0)} \subset \text{Stab}^{1,0}(C \times Y)_\mathbb{R}$, the intersection

$$W^{(\chi_0, \tilde{\beta}_0)}_{(e_0, \tilde{\xi}_0)} \cap \bigcup_{(e, \xi) \in \mathbb{Z} \times \text{NE}(C \times Y)} W^{(\chi_0, \tilde{\beta}_0)}_{(e, \xi)}$$

with

$$0 < \xi < \tilde{\beta}_0$$

and $(e, \xi)$ is realized by a subobject $(e, \tilde{\xi}) \neq (e_0, \tilde{\xi}_0)$

induces a stratification of $W^{(\chi_0, \tilde{\beta}_0)}_{(e_0, \tilde{\xi}_0)}$ by manifolds, which is called the *actual stratification* of the wall $W^{(\chi_0, \tilde{\beta}_0)}_{(e_0, \tilde{\xi}_0)}$.

By construction, the numerical-class-defined wall-and-chamber structure on $\text{Stab}^{1,0}(C \times Y)_\mathbb{R}$ is a refinement of the actual wall-and-chamber structure on $\text{Stab}^{1,0}(C \times Y)_\mathbb{R}$. The former depends only on the intersection theory on $C \times Y$ while the latter is much harder to decide.

Two immediate consequences follow:

**Lemma 2.3.5. [coincidence of semistability and stability].** For $(B + \sqrt{-1}J, L)$ in an actual chamber of $\text{Stab}^{1,0}(C \times Y)_\mathbb{R}$, $Z$-semistable implies $Z$-stable and, hence, the notion of $Z$-semistability and of $Z$-stability coincide for objects in our category.

**Lemma 2.3.6. [equivalent central charges].** If $Z_1$ and $Z_2$ lie in the same actual chamber of $\text{Stab}^{1,0}(C \times Y)_\mathbb{R}$, then an object $\tilde{F}$ in our category is $Z_1$-(semi)stable if and only if it is $Z_2$-(semi)stable.
Example 2.3.7. [chamber structure on $\text{Stab}^{1,[0]}(\mathbb{P}^1 \times Y)_{\mathbb{R}}$, $Y$ quintic Calabi-Yau 3-fold]. Let $Y$ be a quintic Calabi-Yau 3-fold in $\mathbb{P}^4$. Then, $\dim H_2(Y, \mathbb{R}) = h^{1,1} = 1$; $\text{DCG}^+(\mathbb{P}^1) = \mathbb{Z}_{>0}$ and, hence; $\text{Stab}^{1,[0]}(\mathbb{P}^1 \times Y)_{\mathbb{R}} \simeq \mathbb{R} \times \mathbb{R}_{>0} \times \mathbb{R}_{>0}$, whose coordinates will be denoted by $(x_B, x_J, x_L)$; and $\text{NE}(\mathbb{P}^1 \times Y) = \mathbb{Z}_{>0} \oplus \mathbb{Z}_{>0}$. Let $(e, \xi) = (e; m, n) \in \mathbb{Z} \times (\mathbb{Z}_{>0} \oplus \mathbb{Z}_{>0})$. For a fixed $(\chi_0, \tilde{\beta}_0) = (\chi_0; m_0, n_0)$, the wall $W_{(e, m, n)}^{(\chi_0, \tilde{\beta}_0)}$, with $0 < m < m_0$ and $0 < n < n_0$, is given by zero-locus of the function in $(x_B, x_J, x_L)$

$$(e - n x_B)(m_0 x_L + n_0 x_J) - (\chi_0 - n_0 x_B)(m x_L + n x_J)$$

which is a truncated hyperbolic paraboloid (i.e. saddle-shaped quadric surface) in $\mathbb{R} \times \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ if $\left| \frac{e}{\chi_0} \frac{n}{n_0} \right| \neq 0$, and the union of two truncated hyperplanes, described by the equation $x_L \left( \left| \frac{e}{\chi_0} \frac{m}{m_0} \right| + \left| \frac{m}{m_0} \frac{n}{n_0} \right| x_B \right) = 0$ if $\left| \frac{e}{\chi_0} \frac{n}{n_0} \right| = 0$. The latter can occur only for finitely many $(e, n)$'s. In general, when $e \to \pm \infty$, the hyperbolic paraboloid tends to the hyperplane $n_0 x_J + m_0 x_L = 0$, which has no intersection with $\text{Stab}^{1,[0]}(C \times Y)_{\mathbb{R}}$. Thus we have a locally finite system of walls that defines the numerical-class-defined chamber structure of $\text{Stab}^{1,[0]}(C \times Y)_{\mathbb{R}}$. This turns out to be a general feature.

Local finiteness of walls.

Proposition 2.3.8. [local finiteness of walls]. For any $p \in \text{Stab}^{1,[0]}(C \times Y)_{\mathbb{R}}$, there exists an open neighborhood $U$ of $p$ in $\text{Stab}^{1,[0]}(C \times Y)_{\mathbb{R}}$ such that there are only finitely many $(e, \xi)$ with $W_{(e, \xi)}^{(\chi_0, \tilde{\beta}_0)} \cap U \neq \emptyset$.

Proof. Recall the defining equation for a wall:

$$Q_{(e, \xi)}^{(\chi_0, \tilde{\beta}_0)}(x_B, x_J, x_L) := \left( e - x_B \cdot \xi \right) \left( (x_J, x_L) \cdot \tilde{\beta}_0 \right) - \left( \chi_0 - x_B \cdot \tilde{\beta}_0 \right) \left( (x_J, x_L) \cdot \xi \right) = 0.$$

Since $\chi_0$ and $\tilde{\beta}_0$ are fixed and there are only finitely many choices of $\tilde{\xi}$, we only need to focus on the behavior of equation when $e \to \pm \infty$. In which case, the equations are approximating the fixed equation $(x_J, x_L) \cdot \tilde{\beta}_0 = 0$. But $(x_J, x_L) \cdot \tilde{\beta}_0 = 0$ has no solution in $\text{Stab}^{1,[0]}(C \times Y)_{\mathbb{R}}$. This proves the proposition. 

Behavior of the moduli stack of stable objects when crossing an actual wall:

a conjecture.

Let $\Delta^-$ and $\Delta^+$ be two actual chambers in $\text{Stab}^{1,[0]}(C \times Y)_{\mathbb{R}}$ such that both $\Delta^- \cap \text{Stab}^{1,[0]}(C \times Y)$ and $\Delta^+ \cap \text{Stab}^{1,[0]}(C \times Y)$ are non-empty and that the intersection $\Delta^- \cap \Delta^+$ of their closure contains a maximal stratum of a wall $W_{(\chi_0, \tilde{\beta}_0)}^{(\chi_0, \tilde{\beta}_0)}$. Then, $W_{(e_0, \xi_0)}^{(\chi_0, \tilde{\beta}_0)}$ is the only wall that intersects the interior $\text{Int}(\Delta^- \cup \Delta^+)$ of $\Delta^- \cup \Delta^+$. Recall the defining equation $Q_{(e_0, \xi_0)}^{(\chi_0, \tilde{\beta}_0)}$ of $W_{(e_0, \xi_0)}^{(\chi_0, \tilde{\beta}_0)}$. Let
Conjecture 2.3.9. [wall-crossing of moduli stacks]. There exists projective schemes $S^-$, $S^0$, and $S^+$, with built-in morphisms $S^- \rightarrow S^-$, $S^0 \rightarrow S^0$, $S^+ \rightarrow S^+$ that parameterize $S$-equivalence classes of semistable objects in the related moduli stack, and birational surjections $S^- \rightarrow S^0 \leftarrow S^+$ such that the following diagram commute.

Remark 2.3.10. [stringy correction/deformation to walls/chambers]. It should be noted that when the stringy correction $O(\alpha')$ to our central charge functional $Z$ is taken into account, the walls and, hence, the chamber structure on $\text{Stab}^{1,0}(C \times Y)_{\mathbb{R}}$ described in this subsection are subject to a corresponding stringy correction/deformation as well.
3 Compactness of the moduli stack \( \mathcal{F} \mathcal{M}^{1,0;Z-ss}_{C_M/M}(Y; c) \) of Z-semistable Fourier-Mukai transforms

We now prove the main theorem of this note:

**Theorem 3.1.** \( \mathcal{F} \mathcal{M}^{1,0;Z-ss}_{C_M/M}(Y; c) \) compact. Let \((Y, B + \sqrt{-1}J)\) be a projective Calabi-Yau 3-fold with a fixed complexified Kähler class, \(M\) be a compact stack of nodal curves, \(C_M/M\) be the associated universal curve over \(M\) with a fixed relative polarization class \(L\). Then the moduli stack \( \mathcal{F} \mathcal{M}^{1,0;Z-ss}_{C_M/M}(Y; c) \) of \( \mathbb{Z} + \sqrt{-1}L \)-semi-stable Fourier-Mukai transforms of dimension 1, width \([0]\), and central charge \(c \in \mathbb{H} \) from fibers of \(C_M/M\) to \(Y\) is compact.

**Boundedness of \( \mathcal{F} \mathcal{M}^{1,0;Z-ss}_{C_M/M}(Y; c) \).**

Recall first the Kleiman’s Boundedness Criterion:

**Theorem 3.2.** [Kleiman’s boundedness criterion]. (Cf. [H-L: Theorem 1.7.8], [Kl].) Let \(F_s\) be a family of coherent sheaves on \(X\) with the same Hilbert polynomial \(P\). Then this family is bounded if and only if there are constants \(c_i, i = 0, \ldots, d = \deg(P)\) such that for every \(F_s\) there exists an \(F_s\)-regular sequence of hyperplane sections \(H_1, \ldots, H_d\) such that \(h^0(F_s|_{\bigcap_{j \leq d} H_j}) \leq c_i\), for all \(i = 0, \ldots, d\).

Let \(H\) be a relative ample class of \((C_M \times Y)/M\). Then, it follows from Lemma 2.1.7 that

\[
\mathcal{F} \mathcal{M}^{1,0;Z-ss}_{C_M/M}(Y; c) = \coprod_{P^H \in \text{Poly}^H(\mathbb{Z} + \sqrt{-1}L = c)} \mathcal{F} \mathcal{M}^{1,0;Z-ss}_{C_M/M}(Y; c)^{P^H}
\]

is a finite disjoint union of substacks \( \mathcal{F} \mathcal{M}^{1,0;Z-ss}_{C_M/M}(Y; c)^{P^H} \), with the latter the union of connected components of \( \mathcal{F} \mathcal{M}^{1,0;Z-ss}_{C_M/M}(Y; c) \) whose elements have the same Hilbert polynomial \(P^H\). Recall the following estimate:

**Proposition 3.3.** [bound on \(h^0(\mathcal{F})\) via \(P\)-slope]. (H-L: Theorem 3.3.1), [LP], [Ma2], [Si].) Let \(\mathcal{F}\) be a purely 1-dimensional coherent sheaf on \(C \times Y\) for \([C] \in M\). Then,

\[
h^0(\mathcal{F}) \leq \left( H \cdot \tilde{\beta}(\mathcal{F}) \right) \left[ \mu_{\text{max}}(\mathcal{F})^P + \frac{1}{2} \left( H \cdot \tilde{\beta}(\mathcal{F}) \right) \left( H \cdot \tilde{\beta}(\mathcal{F}) + 1 \right) - 1 \right]_+,\]

where \([\bullet] := \max\{0, \bullet\}\).

Combined with Lemma 2.1.8, one has then the estimate:

**Proposition 3.4.** [bound on \(h^0(\mathcal{F})\) via \(Z\)-slope]. There exist constants \(a_1 > 0\) and \(a_0 \in \mathbb{R}\) that depend only on \((Y, B + \sqrt{-1}J), (C_M, M), L\), and \(H\) such that for all \(\mathcal{F} \in \mathcal{F} \mathcal{M}^{1,0;Z-ss}_{C_M/M}(Y; c)\),

\[
h^0(\mathcal{F}) \leq \left( H \cdot \tilde{\beta}(\mathcal{F}) \right) \left[ \left( a_1 \frac{\text{Re}c}{-\text{Im}c} + a_0 \right) + \frac{1}{2} \left( H \cdot \tilde{\beta}(\mathcal{F}) \right) \left( H \cdot \tilde{\beta}(\mathcal{F}) + 1 \right) - 1 \right]_+,\]

where \([\bullet] := \max\{0, \bullet\}\).
On the other hand, \( h^0(\tilde{\mathcal{F}}|_H) = H \cdot \beta(\tilde{\mathcal{F}}) \) is the leading coefficient of the Hilbert polynomial \( P_H(\tilde{\mathcal{F}}) \) of \( \tilde{\mathcal{F}} \) and, hence, is constant for all \( [\tilde{\mathcal{F}}] \in \mathcal{FM}_{C_M/M}^{1,0}; Z_{ss} \) (\( Y; c \)) \( P_H \).

It follows now from the Kleiman’s Boundedness Criterion that each \( \mathcal{FM}_{C_M/M}^{1,0}; Z_{ss} \) (\( Y; c \)) \( P_H \) is bounded and, hence, that \( \mathcal{FM}_{C_M/M}^{1,0}; Z_{ss} \) (\( Y; c \)) is bounded.

**Completeness of \( \mathcal{FM}_{C_M/M}^{1,0}; Z_{ss} \) (\( Y; c \)).**

From the basic properties of twisted central charges and of the associated stability conditions and Harder-Narasimhan filtrations given in Sec. 2, the completeness of the stack \( \mathcal{FM}_{C_M/M}^{1,0}; Z_{ss} \) (\( Y; c \)) follows from the Langton’s argument [La] through elementary modifications, with (e.g., for the modified and generalized presentation in [H-L: Sec. 2.B]) Hilbert polynomials and reduced Hilbert polynomials replaced by twisted central charges and associated slope respectively and using the propenseness of the related relative Quot-schemes. We convert the proof of Stacy Langton [La] and Daniel Huybrechts and Manfred Lehn [H-L: Theorem 2.B.1] to our case below for the thoroughness of the discussion. Readers are referred to [La] and [H-L] for the original ideas and proofs. (Note that the inductive proof of Huybrechts and Lehn via quotient category of coherent sheaves becomes superficial when one concerns only with 1-dimensional coherent sheaves.)

Let \( R \) be a discrete valuation ring with maximal ideal \( m = (t) \), residue field \( R/(t) \cong k \cong \mathbb{C} \) and quotient field \( K \). When needed, a scheme over \( \text{Spec} \, R \) (resp. the generic point \( \text{Spec} \, K \) and the closed point \( \text{Spec} \, k \in \text{Spec} \, R \)) will be denoted by \( X_R \) (resp. \( X_K \) and \( X_k \)); and similarly for a coherent sheaf.

**Proposition 3.5.** [valuative criterion]. Let \((C_K, L_K) \) be a nodal curve with a polarization class over \( \text{Spec} \, K \) from a morphism \( \text{Spec} \, K \to M \) and \( \tilde{\mathcal{F}}_K \) be \( Z \)-semistable coherent sheaf on \((C_K \times Y)/K \). Then, up to a field extension \( K \subset K' \) of finite degree, \( \tilde{\mathcal{F}}_K \) extend to a coherent sheaf \( \tilde{\mathcal{F}}_{R'} \) on \((C_{R'} \times Y)/R' \) that is flat over \( R' \) and has \( \tilde{\mathcal{F}}_k \) \( Z \)-semistable. Here, \( R' \leftarrow R \) is the discrete valuation ring with residue field \( k \) and quotient field \( K' \).

We proceed to prove the proposition. Since \( M \) is compact, up to a field extension (still denoted by \( K \) and the associated discrete valuation ring by \( R \) for simplicity of notations), \((C_K, L_K) \) extends to \((C_R, L_R) \) from a morphism \( \text{Spec} \, R \to M \) that extends the given (or induced under a field extension) \( \text{Spec} \, K \to M \). (Note that there is an embedding \( C_R \hookrightarrow \mathbb{P}^N_R \) over \( R \), for some \( N > 0 \). Thus, one can treat our problem exchangeably as a problem of coherent sheaves on the fixed smooth \( \mathbb{P}^N_R \times Y \) with their support contained in the closed subscheme \( C_R \times Y \) if one prefers.) One can extend \( \tilde{\mathcal{F}}_K \) to a coherent sheaf \( \tilde{\mathcal{F}}_R =: \tilde{\mathcal{F}}_R^1 \) of coherent sheaves on \((C_R \times Y)/R \) that is flat over \( R \). (Cf. [Hart: II, Exercise 5.15] for an extension inside \( j_* \tilde{\mathcal{F}}_K \), which must be flat over \( R \). Here, \( j : C_K \times Y \to C_R \times Y \) is the built-in open immersion.)

If \( \tilde{\mathcal{F}}_k^1 \) is \( Z \)-semistable, then we are done. Otherwise, let \( \mathcal{B}^1 \) be the maximal destabilizing subsheaf of \( \tilde{\mathcal{F}}_k^1 \), \( \tilde{\mathcal{G}}^1 := \tilde{\mathcal{F}}_k^1/\mathcal{B}^1 \), and

\[
\tilde{\mathcal{F}}_R^2 := \text{Ker}(\tilde{\mathcal{F}}_R^1 \to \tilde{\mathcal{G}}^1)
\]

be the elementary modification of \( \tilde{\mathcal{F}}_R^1 \) at \( \tilde{\mathcal{G}}^1 \). Here, \( \tilde{\mathcal{F}}_R^1 \to \tilde{\mathcal{G}}^1 \) is the composition \( \tilde{\mathcal{F}}_R^1 \to \tilde{\mathcal{F}}_k^1 \to \tilde{\mathcal{G}}^1 \) of quotient-sheaf homomorphisms. If \( \tilde{\mathcal{F}}_k^2 \) is semistable, then we are done. Otherwise, one can iterate the above elementary modification now on \( \tilde{\mathcal{F}}_R^2 \) at \( \tilde{\mathcal{G}}^2 := \tilde{\mathcal{F}}_k^2/\mathcal{B}^2 \), where \( \mathcal{B}^2 \) is the maximal destabilizing subsheaf of \( \tilde{\mathcal{F}}_k^2 \). If \( \mathcal{B}^i = \tilde{\mathcal{F}}_k^i \) for some \( i \geq 1 \), then we are done.
Assume that this is not true, we obtain then a sequence of proper subsheaves that are flat over $R$:

$$
\cdots \subseteq \tilde{F}^i_{R} \subseteq \tilde{F}^{i+1}_{R} \subseteq \cdots \subseteq \tilde{F}^1_{R}.
$$

Note that $t\tilde{F}^i_{R} \subseteq \tilde{F}^i_{R} \subseteq \tilde{F}^{i-1}_{R}$ and, hence, $t^{i-1}\tilde{F}^i_{R} \subseteq \tilde{F}^i_{R} \subseteq \tilde{F}^{1}_{R}$. By construction, one has the built-in exact sequence

$$
0 \rightarrow \tilde{B}^i \rightarrow \tilde{F}^i_k \rightarrow \tilde{G}^i \rightarrow 0,
$$

for all $i$.

**Lemma 3.6. [short exact sequence].** There is also an exact sequence

$$
0 \rightarrow \tilde{G}^{i-1} \rightarrow \tilde{F}^i_k \rightarrow \tilde{B}^{i-1} \rightarrow 0,
$$

for all $i$.

**Proof.** The short exact sequence

$$
0 \rightarrow \tilde{F}^i_R \rightarrow \tilde{F}^{i-1}_R \rightarrow \tilde{G}^{i-1} \rightarrow 0
$$

induces a long exact sequence of $\mathcal{O}_{C_R \times Y}$-modules

$$
\cdots \rightarrow \text{Tor}_1^{CR \times Y}(\mathcal{O}_{C_k \times Y}, \tilde{F}^{i-1}_R) \rightarrow \text{Tor}_1^{CR \times Y}(\mathcal{O}_{C_k \times Y}, \tilde{G}^{i-1})
\rightarrow \tilde{F}^i_k \rightarrow \tilde{F}^{i-1}_k \rightarrow \tilde{G}^{i-1} \rightarrow 0.
$$

The lemma follows from the observations that

$$
\text{Tor}_1^{CR \times Y}(\mathcal{O}_{C_k \times Y}, \tilde{F}^{i-1}_R) = 0,
$$

$$
\text{Tor}_1^{CR \times Y}(\mathcal{O}_{C_k \times Y}, \tilde{G}^{i-1}) \simeq (\tilde{t}) \otimes_{C_R \times Y} \tilde{G}^{i-1} \simeq \tilde{G}^{i-1},
$$

$$
\text{Im}(\tilde{F}^i_k \rightarrow \tilde{F}^{i-1}_k) = \tilde{B}^{i-1}.
$$

These two sets of exact sequences give rise to two sequences of homomorphisms of coherent sheaves on $C_k \times Y$:

$$
\cdots \rightarrow \tilde{B}^{i+1} \rightarrow \tilde{B}^i \rightarrow \cdots \rightarrow \tilde{B}^1
$$

and

$$
\tilde{G}^1 \rightarrow \cdots \rightarrow \tilde{G}^i \rightarrow \tilde{G}^{i+1} \rightarrow \cdots.
$$

**Lemma 3.7. [stationary behavior of $\{\tilde{B}^i\}$, $\{\tilde{G}^i\}$, and $\{\tilde{F}^i_k\}$.** There exists an $i_0 \geq 1$ such that for all $i \geq i_0$, the homomorphisms $\tilde{B}^{i+1} \rightarrow \tilde{B}^i$ and $\tilde{G}^{i+1} \rightarrow \tilde{G}^i$ are isomorphisms and $\tilde{F}^i_k \simeq \tilde{B} \oplus \tilde{G}$, where $\tilde{B} \simeq \tilde{B}^i$ and $\tilde{G} \simeq \tilde{G}^i$ for any $i \geq i_0$.

**Proof.** Observe first that

$$
\{\tilde{\beta}(\tilde{F}') \mid \tilde{F}' \text{ is a subsheaf of } \tilde{F}^i_k \text{ for some } i, i \geq 1\}
$$

is a finite subset of $N_1(C_k \times Y)_Z$. Thus, for any subsheaf of $\tilde{F}^i_k$, $i \geq 1$, $\mu^{2}(\tilde{F}')$ takes values in a discrete affine lattice of rank 1 (i.e. a shift – due to the effect of $B$ in $B + \sqrt{-1}J$ – of an additive
subgroup isomorphic to $\mathbb{Z}$) in $\mathbb{R}$ that is independent of $i$. Back to our problem, since $\tilde{B}^i$ are $Z$-semistable,
\[ \cdots \leq \mu^Z(\tilde{B}^{i+1}) \leq \mu^Z(\tilde{B}^i) \leq \cdots \leq \mu^Z(\tilde{B}^1). \]
Since $\mu^Z(\tilde{B}^i) > \mu^Z(\tilde{F}_k^i) = \mu^Z(\tilde{F}_k)$ for all $i$, the above observation implies that there exists an $i_0 \geq 1$ such that
\[ \cdots \leq \mu^Z(\tilde{B}^{i+1}) = \mu^Z(\tilde{B}^i) = \cdots = \mu^Z(\tilde{B}^{i_0}) \]
for all $i \geq i_0$. On the other hand, note that $\tilde{B}^{i+1}/(\tilde{B}^{i+1} \cap \tilde{G}^i)$ is isomorphic to a non-zero subsheaf of $\tilde{B}^i$ and, since $\tilde{B}^i$'s are $Z$-semistable,
\[ \mu^Z(\tilde{B}^{i+1}) \leq \mu^Z(\tilde{B}^{i+1}/(\tilde{B}^{i+1} \cap \tilde{G}^i)) \leq \mu^Z(\tilde{B}^i) \]
with the equalities hold if and only if $\tilde{B}^{i+1} \cap \tilde{G}^i = 0$. It follows that $\tilde{B}^{i+1} \cap \tilde{G}^i = 0$ for $i \geq i_0$, which implies that the homomorphisms, $\tilde{B}^{i+1} \to \tilde{B}^i$ and $\tilde{G}^i \to \tilde{G}^{i+1}$, are injective for $i \geq i_0$. Being non-zero subsheaves, $\bar{\beta}(\tilde{B}^{i+1})$ is an effective sub-1-cycle of $\bar{\beta}(\tilde{B}^i)$ and $\bar{\beta}(\tilde{G}^i)$ is an effective sub-1-cycle of $\bar{\beta}(\tilde{G}^{i+1})$. Since all are effective sub-1-cycles of $\bar{\beta}(\tilde{F}_k)$, there exists an $i_1 \geq 0$ such that
\[ \bar{\beta}(\tilde{B}^{i+1}) = \bar{\beta}(\tilde{B}^i) \quad \text{and} \quad \bar{\beta}(\tilde{G}^i) = \bar{\beta}(\tilde{G}^{i+1}), \]
for all $i \geq i_1$. It follows that
\[ Z^{B+\sqrt{-1}J,L}(\tilde{B}^{i+1}) = Z^{B+\sqrt{-1}J,L}(\tilde{B}^i) \quad \text{and, hence,} \quad Z^{B+\sqrt{-1}J,L}(\tilde{G}^i) = Z^{B+\sqrt{-1}J,L}(\tilde{G}^{i+1}), \]
for all $i \geq i_1$, and that the inclusions of the purely 1-dimensional coherent sheaves, $\tilde{B}^{i+1} \to \tilde{B}^i$ and $\tilde{G}^i \to \tilde{G}^{i+1}$, are isomorphisms in dimension-1, for $i \geq i_1$. Now
\[ \tilde{G}^{i_1} \subset \cdots \subset \tilde{G}^i \subset \tilde{G}^{i+1} \subset \cdots \]
is an inclusion sequence of purely 1-dimensional coherent sheaves which are isomorphic in dimension 1. In particular, their reflexive hulls $(\tilde{G}^i)^{DD} := \mathcal{E}xt_3^{C_k \times Y}(\mathcal{E}xt_3^{C_k \times Y}(\tilde{G}^i, \omega_{C_k \times Y}), \omega_{C_k \times Y})$ are canonically isomorphic, induced by the inclusions. Here, $\omega_{C_k \times Y}$ is the dualizing sheaf of $C_k \times Y$ and the superscript 3 is the codimension of $\tilde{G}^i$ in $C_k \times Y$. It follows that there exists an $i_2 \geq i_1$ such that $\tilde{G}^i \simeq \tilde{G}^{i+1}$ for all $i \geq i_2$. Recall the exact sequence $0 \to \tilde{G}^i \to \tilde{F}_k^i \to \tilde{B}^i \to 0$; one has thus in addition that the exact sequence $0 \to \tilde{B}^i \to \tilde{F}_k^i \to \tilde{G}^i \to 0$ splits and $\tilde{B}^{i+1} \simeq \tilde{B}^i$, for all $i \geq i_2 + 1$.

With $i_2 + 1$ re-denoted by $i_0$, this proves the lemma.

\[ \square \]

After removing finitely many terms and relabelling, we may assume without loss of generality that $i_0 = 1$ in the following discussion.

Define now $\tilde{Q}^i = \tilde{F}_R/\tilde{F}_k^i$. Then $\tilde{Q}^i \simeq \tilde{G}$ and there are exact sequences
\[ 0 \to \tilde{G} \to \tilde{Q}^{i+1} \to \tilde{Q}^i \to 0 \]
for all $i$. It follows from the Local Flatness Criterion [H-L:Lemma 2.1.3] that $\tilde{Q}^i$ is a $R/(t^i)$-flat quotient of $\tilde{F}_R/((t^i)\tilde{F}_R)$ for all $i$. Hence the image of the proper morphism
\[ \pi_R : \text{Quot} Z^{B+\sqrt{-1}J,L}(\tilde{F}_R, Z^{B+\sqrt{-1}J,L}(\tilde{G})) \to \text{Spec} R \]
contains the closed subscheme $\text{Spec} (R/(t^i))$ for all $i$. This implies that $\pi_R$ is actually surjective and, hence, there exists a field extension $K \subset K'$ of $K$ such that $\tilde{F}_{K'}$ admits a destabilizing quotient with central charge $Z^{B+\sqrt{-1}J,L}(\tilde{G})$. This contradicts the semistability assumption on $\tilde{F}_K$ since $\tilde{F}_K$ is $Z$-semistable if and only if $\tilde{F}_{K'}$ is $Z$-semistable. This proves Proposition 3.5 and hence the completeness of the stack $\mathcal{F}\mathcal{M}_{C_{\mathcal{M}}/\mathcal{M}}^{1,0};Z^{ss}(Y; e)$.

Altogether, this proves the compactness of $\mathcal{F}\mathcal{M}_{C_{\mathcal{M}}/\mathcal{M}}^{1,0};Z^{ss}(Y; e)$, as claimed in Theorem 3.1.
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