Lemaître-Tolman-Bondi Static Universe in Rastall-like Gravity

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ABSTRACT

In this work, we try to construct a stable Lemaître-Tolman-Bondi (LTB) static universe, which is spherically symmetric and radially inhomogeneous. However, this is not an easy task, and fails in general relativity (GR) and various modified gravity theories, because the corresponding LTB static universes must reduce to the Friedmann-Robertson-Walker (FRW) static universes. We find a way out in a new kind of modified gravity theory, namely Rastall-like gravity, in which the conservation of energy and momentum is broken. Actually, in order to construct a successful LTB static universe, we have proposed a new variant of the Rastall gravity theory. The stability of LTB static universe against both the homogeneous and the inhomogeneous scalar perturbations is also discussed in details. Finally, we successfully get a stable LTB static universe.

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I. INTRODUCTION

As is well known, modern cosmology began with the application of general relativity (GR) to the universe, soon after the birth of GR. The first cosmological model developed by Einstein himself in 1917 [1] is the well-known Einstein static universe, which is homogeneous, isotropic, and spatially closed. It can be static by the help of a positive cosmological constant counteracting the attractive effects of gravity on ordinary matter. However, in 1929, Hubble found that the universe is expanding (rather than static), by examining the relation between distance and redshift of galaxies [2]. On the other hand, in 1930, Eddington [3] argued that the Einstein static universe is unstable with respect to homogeneous and isotropic scalar perturbations in GR, and hence the universe cannot be static in the presence of perturbations. Therefore, Einstein abandoned the idea of static universe (and the cosmological constant as the “biggest blunder” of his life [4]).

Recently, the Einstein static universe has been revived to avoid the big bang singularity in the emergent universe scenario [5, 6]. In such kind of scenario, the Einstein static universe is the initial state for a past-eternal inflationary cosmological model and then evolves to an inflationary era. So, there is no big bang singularity, and no exotic physics is involved. The quantum gravity regime can even be avoided, if the size of Einstein static universe is big enough. In fact, it is argued that the Einstein static state is favored by entropy considerations as the initial state for the universe [7–9].

Motivated by the emergent universe scenario, in the past decade, the Einstein static universe was extensively studied in many gravity theories. It has been reconsidered in GR, and the Einstein static universe can be stable against perturbations, if the universe contains a perfect fluid with \( c_s^2 > 1/5 \) [7, 8, 10]. It is also very interesting to consider the Einstein static universe in various modified gravity theories, for example, loop quantum cosmology [11], \( f(R) \) theory [12, 13], \( f(T) \) theory [15, 16], modified Gauss-Bonnet gravity (\( f(G) \) theory) [17, 18], Brans-Dicke theory [19, 20], Horava-Lifshitz theory [21, 22], massive gravity [23, 24], braneworld scenario [25, 26], Einstein-Cartan theory [31], \( f(R,T) \) gravity [32], hybrid metric-Palatini gravity [33], Eddington-inspired Born-Infeld theory [34], degenerate massive gravity [35], and so on [36–43].

We note that (almost) all relevant works in the literature by now only considered the Einstein static universe, which is homogeneous and isotropic, and hence is described by a Friedmann-Robertson-Walker (FRW) metric. In other words, they assumed the cosmological principle.

However, as a tenet, the cosmological principle is not born to be true. Actually, this assumption has not yet been well proven on cosmic scales \( \gtrsim 1 \) Gpc [44]. Therefore, it is still of interest to test both the homogeneity and the isotropy of the universe carefully. In fact, they could be violated in some theoretical models, such as Lemaître-Tolman-Bondi (LTB) model [45] (see also e.g. [46, 47] and references therein) violating the cosmic homogeneity, and the exotic Gödel universe [50] (see also e.g. [51] and references therein), most of the Bianchi type I \( \sim \) IX universes [52], Finsler universe [53], violating the cosmic isotropy.

On the other hand, many observational hints of the cosmic inhomogeneity and/or anisotropy have been claimed in the literature (see e.g. [47–49] for brief reviews), including type Ia supernovae (SNIa), cosmic microwave background (CMB), baryon acoustic oscillations (BAO), gamma-ray bursts (GRBs), integrated Sachs-Wolfe effect, Sunyaev Zeldovich effect, quasars, radio galaxies, and so on. Therefore, on both the theoretical and the observational sides, it is reasonable to consider the cosmological models violating the cosmological principle.

It is natural to ask “why must the initial state for the universe be homogeneous and isotropic?” If the initial state is random, it has great probability to be inhomogeneous and/or anisotropic. Therefore, it is interesting to consider a static universe violating the cosmological principle.

In the present work, we are interested in the well-known LTB model [45] (see also e.g. [46, 47] and references therein). In this model, the universe is spherically symmetric and radially inhomogeneous, and we are living in a locally underdense void centered nearby our location. As is well known, without invoking dark energy or modified gravity, it is possible to explain the apparent cosmic acceleration discovered in 1998 by using the LTB model [46, 54–60]. This fact further justifies the motivation to consider a LTB static universe.

However, it is not an easy task to construct a stable LTB static universe in GR and various modified gravity theories. We will briefly discuss this issue in Sec. [III] Knowing the cause of failure, we find a way out in a new kind of modified gravity theory, namely Rastall-like gravity, which will be briefly introduced in Sec. [III] Then, in Secs. [IV] and [V] we construct the LTB static universe in Rastall-like gravity without
and with a cosmological constant, respectively. The stability of LTB static universe against both the homogeneous and the inhomogeneous scalar perturbations is also discussed in details. In Sec. VII some brief concluding remarks are given.

II. THE FAILURE OF LTB STATIC UNIVERSE IN GR AND VARIOUS MODIFIED GRAVITY THEORIES

The LTB metric, in comoving coordinates \((r, \theta, \phi)\) and synchronous time \(t\), is given by

\[
ds^2 = -dt^2 + \frac{A^2(r, t)}{1 - K(r)} dr^2 + A^2(r, t) d\Omega^2,
\]

where \(d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2\), and a prime denotes a derivative with respect to \(r\). \(K(r)\) is an arbitrary function of \(r\), playing the role of spatial curvature. In general, \(A(r, t)\) is an arbitrary function of \(r\) and \(t\), playing the role related to the scale factor. Obviously, the LTB metric reduces to the well-known FRW metric if \(A(r, t) = a(t) r\) and \(K(r) = Kr^2\).

Let us consider the LTB static universe, in which \(A = A_0(r)\) is independent of the time \(t\), and the subscript “0” indicates the quantities related to the static solution \(s\). In this case, \(\dot{A} = 0\) and \(\ddot{A} = 0\), where a dot denotes a derivative with respect to \(t\). Now, by definition, the Einstein tensor is given by

\[
G^{\nu\mu} = \text{diag} \left( -\frac{K_0}{A_0^2}, \frac{K_0}{2A_0A_0'}, \frac{K_0}{2A_0A_0'}, -\frac{K_0}{A_0A_0'} \right),
\]

where the Greek indices \(\mu, \nu\) run over 0, 1, 2, 3. We note that in general the last three diagonal components \(G^{11} \neq G^{22} = G^{33}\). On the other hand, if the universe contains a perfect fluid, the corresponding energy-momentum tensor reads

\[
T^{\mu\nu} = (\rho + p) U^\mu U^\nu + p g^{\mu\nu},
\]

or equivalently

\[
T^\nu_\mu = \text{diag} (-\rho, p, p, p),
\]

where 4-velocity vector \(U^\mu = (1, 0, 0, 0)\) satisfying \(g_{\mu\nu}U^\mu U^\nu = -1\), while \(\rho\) and \(p\) are energy density and pressure, respectively. In general, \(\rho\) and \(p\) are functions of \(r\) and \(t\). In the case of LTB static universe, \(\rho = \rho_0(r)\) and \(p = p_0(r)\) are independent of the time \(t\).

In GR, the field equations are \(G^{\mu\nu} = 8\pi G_N T^{\mu\nu}\), where \(G_N\) is the Newtonian gravitational constant. Since the last three diagonal components of \(T^{\mu\nu}\) are equal, we have \(G^{11} = G^{22} = G^{33} = 8\pi G_N p\). In the case of LTB static universe, from Eq. (2), it requires

\[
\frac{K_0}{A_0^2} = \frac{K_0'}{2A_0A_0'} \frac{dK_0}{dA_0^2},
\]

which means \(K_0/A_0^2 = \mathcal{K} = \text{const.}\). Introducing \(a_0 = |\mathcal{K}|^{-1/2}, \tilde{r} = A_0(r)/a_0, \tilde{K} = \mathcal{K} a_0^2 = \pm 1\) if \(\mathcal{K} \neq 0\), or introducing \(\tilde{r} = A_0(r)/a_0, \tilde{K} = 0, a_0 = \text{const.}\) \(\neq 0\) if \(\mathcal{K} = 0\), the metric of LTB static universe becomes

\[
ds^2 = -dt^2 + a_0^2 \left( \frac{d\tilde{r}^2}{1 - \tilde{K}\tilde{r}^2} + \tilde{r}^2 d\tilde{\Omega}^2 \right),
\]

which is nothing but the one of FRW static universe. So, the LTB static universe in GR fails. In addition, we have \(8\pi G_N p_0 = G^{11} = -K_0/A_0^2 = -\mathcal{K} = \text{const.}\), which is also independent of \(r\). Actually, one can find \(\partial_r p_0 = 0\) from the conservation equations \(T^{\mu\nu}_{\mu,\nu} = 0\). In fact, this gives an instructive hint for the failure of LTB static universe in GR.

Furthermore, we have also considered the LTB static universes in various modified gravity theories, such as \(f(R)\) theory, \(f(T)\) theory, Brans-Dicke theory, modified Gauss-Bonnet gravity \((f(G)\) theory), and
they all failed because the corresponding LTB static universes all reduced to the FRW static universes. In fact, one can always recast the field equations of modified gravity theory as the form of

\[ G^{\nu \mu} + M^{\nu \mu} = 8\pi G N T^{\nu \mu}, \tag{7} \]

where \( M^{\nu \mu} \) is the modification term with respect to GR. In the case of LTB static universe, if we require that the conservation equations \( T^{\nu \mu} = 0 \) hold (which are actually equivalent to \( M^{\nu \mu} = 0 \) because \( G^{\nu \mu} = 0 \) always), the last three diagonal components of \( M^{\nu \mu} \) should be equal in various modified gravity theories such as \( f(R) \) theory, \( f(T) \) theory, Brans-Dicke theory, and modified Gauss-Bonnet gravity \( (f(G) \) theory). Since the last three diagonal components of \( T^{\nu \mu} \) are also equal, it is necessary to require \( G^{11} = G^{22} = G^{33} \). Following the similar derivations in GR, the LTB static universes in these modified gravity theories also fail, because they must reduce to the FRW static universes.

III. THE RASTALL-LIKE GRAVITY THEORY

A. The original Rastall gravity theory

From the discussions in Sec. II, it is easy to see that the conservation equations \( T^{\nu \mu} = 0 \) might be responsible for the failure of LTB static universe in GR and various modified gravity theories. In order to construct a successful LTB static universe, a possible way out might be breaking the conservation equations \( T^{\nu \mu} = 0 \). In the literature, there exists some models breaking the conservation of energy and momentum. For instance, the famous steady state model of the expanding universe proposed by Hoyle, Bondi and Gold in 1948 \[61, 62\] requires continuous creation of matter (particle creation) from nothing. In fact, it is not completely unacceptable to consider such kind of models.

In this work, we are interested in the so-called Rastall gravity theory proposed in 1972 \[63\]. Rastall argued that the fundamental assumption \( T^{\nu \mu} = 0 \) of GR is questionable in fact. All one can assert with fair confidence is \[63\]

\[ T^{\nu \mu} = X^{\mu}, \tag{8} \]

In \[63\], Rastall proposed to consider the assumption

\[ T^{\nu \mu} = X^{\mu} = \lambda R^{\mu}, \tag{9} \]

where \( \lambda \) is a constant, and \( R = R^{\mu \mu} \). Since \( G^{\nu \mu} = 0 = \kappa (T^{\nu \mu} - \lambda \delta^{\nu \mu} R) \), always, the assumption in Eq. (9) is consistent with the modified field equations \[63\]

\[ G^{\nu \mu} = \kappa (T^{\nu \mu} - \lambda \delta^{\nu \mu} R); \quad \text{or equivalently} \quad R_{\mu \nu} + \left( \kappa \lambda - \frac{1}{2} \right) g_{\mu \nu} R = \kappa T_{\nu \mu}, \tag{10} \]

where \( \kappa \) is a non-zero constant. Rastall \[63\] argued that

\[ \frac{\kappa}{4\kappa \lambda - 1} \left( 3\kappa \lambda - \frac{1}{2} \right) = 4\pi G_N, \tag{11} \]

while the model parameters \( \kappa \lambda = 1/4 \) should be excluded. When \( \lambda = 0 \), we have \( \kappa = 8\pi G_N \), and GR is recovered. Obviously, Rastall gravity theory is a new kind of modified gravity theory violating the conservation of energy and momentum.

In the past decade, Rastall gravity has been extensively studied in the literature (Ref. \[63\] has been cited more than 110 times by now), and we refer to \[64–93\] for example. In particular, very recently, it is claimed in \[93\] that Rastall gravity is strongly favored by 118 galaxy-galaxy strong gravitational lensing systems, with \( \kappa \lambda = 0.163 \pm 0.001 \). This new observational evidence further justified the serious studies of Rastall gravity. In fact, the corresponding cosmology, black hole, compact star, wormhole, thermodynamics in Rastall gravity have been extensively considered in the literature. It attracted much attention in the recent three years, and grows rapidly now.
However, the original Rastall gravity theory is not suitable for our purpose. This can be clearly seen from Eq. (10). Since the last three diagonal components of the term $\lambda \delta^\nu_\mu R$ are equal, and the last three diagonal components of $T^\nu_\mu$ are also equal for a perfect fluid, we must have $G^{11} = G^{22} = G^{33}$ again. Then, following the similar discussions in Sec. II, the LTB static universe in the original Rastall gravity theory also fails. So, we should introduce a new variant of the Rastall gravity theory.

In Rastall gravity, Eq. (8) is firm, but the choice of $X^\mu_\nu$ might be changed. For simplicity, we propose to consider a fairly general form

$$T^\nu_\mu = X^\mu_\nu = Y^\nu_\mu,$$

where $Y^\nu_\mu \neq T^\nu_\mu$ (note that if $Y^\nu_\mu = \lambda \delta^\nu_\mu R$, the original Rastall gravity theory [63] can be recovered). In this new form, since $G^\nu_\mu = T^\nu_\mu - Y^\nu_\mu$ always, the assumption in Eq. (12) is consistent with the modified field equations

$$G^\nu_\mu + \kappa Y^\nu_\mu = \kappa T^\nu_\mu,$$

where $\kappa$ is a non-zero constant. Contracting Eq. (13), we find that the trace of $Y^\nu_\mu$ is given by

$$\kappa Y = R + \kappa T,$$

where $Y = Y^\mu_\mu$, $R = R^\mu_\mu$ and $T = T^\mu_\mu$. So far, the choice of $Y^\nu_\mu$ is still pending. Before we make a particular choice of $Y^\nu_\mu$, here are some general remarks:

(R1) Regardless of the matter distribution in the universe, if $Y^\nu_\mu = 0$, this Rastall-like gravity theory reduces to GR.

(R2) If the matter distribution in the universe is isotropic and homogeneous, and if $Y^\nu_\mu$ is also isotropic and homogeneous (namely its last three diagonal components are equal and independent of spatial coordinates), the universe should be described by a FRW metric, and the Rastall-like gravity theory equivalently reduces to GR with an effective $T^\nu_\mu, \text{eff} = T^\nu_\mu - Y^\nu_\mu$.

(R3) If the matter distribution in the universe is isotropic and homogeneous, but $Y^\nu_\mu$ is anisotropic and/or inhomogeneous (namely its last three diagonal components are not equal and/or depend on spatial coordinates), because the anisotropic and/or inhomogeneous energy-momentum-exchange between matter and geometry will change the matter distribution in the universe, the universe becomes anisotropic and/or inhomogeneous (and hence is not described by a FRW metric).

(R4) If the matter distribution in the universe is anisotropic and/or inhomogeneous, regardless of $Y^\nu_\mu$, the universe is not described by a FRW metric.

Next, let us step forward, and try to specify the choice of $Y^\nu_\mu$. We assume that the universe contains a perfect fluid whose energy-momentum tensor is given by Eqs. (3) or (4). To construct a successful LTB static universe, $Y^\nu_\mu$ should satisfy three conditions:

(C1) $Y^\nu_\mu$ is diagonal, because $G^\nu_\mu$ and $T^\nu_\mu$ are both diagonal (n.b. Eqs. (2) and (11)).

(C2) Its trace should be related to geometry and matter according to Eq. (14).

(C3) Due to the discussions in Sec. II $Y^{11} \neq Y^{22} = Y^{33}$ is required to construct a successful LTB static universe, while $G^{11} \neq G^{22} = G^{33}$ and $T^{11} = T^{22} = T^{33}$ (n.b. Eqs. (2) and (11)).

Obviously, there is a large room for the choice of $Y^\nu_\mu$. The simplest one is given by

$$\kappa Y^\nu_\mu = \text{diag} (0, R + \kappa T, 0, 0),$$

and we will briefly mention other reasonable choices in Sec. VI. When $R + \kappa T = 0$, this Rastall-like gravity theory reduces to GR (if $\kappa \neq 8\pi G_N$, one can simply rescale $T^\nu_\mu, \text{new} = \kappa T^\nu_\mu / (8\pi G_N)$, but $\kappa > 0$ is required to ensure the energy density $\rho_{\text{new}} \propto \kappa \rho \geq 0$).
IV. LTB STATIC UNIVERSE IN RASTALL-LIKE GRAVITY

In this section, we consider the LTB static universe in Rastall-like gravity. The field equations are given by Eq. (13), and \( Y_{\nu \mu} \) is given by Eq. (15). We assume that the universe contains a perfect fluid whose energy-momentum tensor is given by Eqs. (3) or (4), and \( p = w \rho \), where the equation-of-state parameter \( w \) is a constant.

A. LTB static solutions

In the case of LTB static universe, \( A, \rho \) and \( p \) are all independent of the time \( t \), and hence they are functions only depending on the spatial coordinate \( r \), namely \( A_0^0(r), \rho_0(r) \) and \( p_0(r) \), while we also denote \( K = K_0(r) \). Due to the non-minimal coupling between geometry and matter, \( T_{\nu \mu};_{\nu} = Y_{\nu \mu};_{\nu} \neq 0 \), one can find that \( \rho_0' \neq 0 \) and \( p_0' \neq 0 \), namely they are not homogeneous. The static solutions are determined by the field equations in Eq. (13), namely

\[
-\frac{K_0}{A_0^2} - \frac{K_0'}{A_0 A_0'} = -\kappa \rho_0, \tag{16}
\]

\[
-\frac{K_0}{A_0^2} + R_0 + \kappa T_0 = \kappa p_0, \tag{17}
\]

\[
-\frac{K_0'}{2A_0 A_0'} = \kappa p_0. \tag{18}
\]

Noting \( R_0 = 2K_0/A_0^2 + 2K_0'/A_0 A_0' \) and \( T_0 = 3p_0 - \rho_0 \), Eq. (17) is not independent of Eqs. (16) and (18). Multiplying Eq. (16) by \( w \), and then adding Eq. (18), we have

\[
\frac{wK_0}{A_0^2} + \frac{K_0'}{2A_0 A_0'} (1 + 2w) = 0. \tag{19}
\]

Noting \( K_0'/A_0' = dK_0/dA_0 \), Eq. (19) can be regarded as an ordinary differential equation of \( K_0 \) with respect to \( A_0 \), and its solution is given by

\[
K_0 = CA_0^{-2w/(1+2w)}, \tag{20}
\]

where \( C \) is an integral constant, and we require \( w \neq -1/2 \). Obviously, \( K_0/A_0^2 \) is not a constant if \( w \neq -1/3 \) and \( C \neq 0 \), and hence the LTB static universe does not reduce to the FRW static universe. Multiplying Eq. (18) by 2, and then subtracting Eq. (16), we have

\[
\kappa (1 + 2w) \rho_0 = \frac{K_0}{A_0^2}. \tag{21}
\]

Substituting Eq. (20) into Eq. (21), it is easy to get

\[
\rho_0 = \frac{C}{\kappa (1 + 2w)} A_0^{-2(1+3w)/(1+2w)}. \tag{22}
\]

Eqs. (20) and (22) are the explicit expressions of the LTB static solutions.

B. Stability analysis

To become a successful LTB static universe, it should be stable against perturbations. Fortunately, the comprehensive perturbation theory in LTB cosmology has been developed in [94]. Because of the spherical symmetry of the LTB spacetime, perturbations can be decoupled into two independent modes, namely the polar and the axial modes [94, 95]. Since we are interested in the evolution of the density perturbations, we focus on the polar mode [96]. Following e.g. [96] and Sec. III of [94], a first approximation is to
neglect the mode-mixing, and focus only on the scalar perturbations. In the Regge-Wheeler (RW) gauge, the perturbed metric is given by

\[ g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \]

where \( \bar{g}_{\mu\nu} \) is the background (static) metric, and

\[ h_{\mu\nu} = \text{diag}(-2\Phi, 2\Psi, 2\Psi, 2\Psi). \]

The perturbation of energy-momentum tensor is given by

\[ \delta T_{\mu\nu} = (\rho + p) (h_{\mu\sigma}U^\sigma U^\nu + \bar{g}_{\mu\sigma}\delta U^\sigma U^\nu + \bar{g}_{\mu\sigma}\delta\bar{U}^\sigma U^\nu) + (\delta\rho + \delta p) \bar{g}_{\mu\sigma}U^\sigma U^\nu + \delta p \delta_{\mu\nu}. \]

The perturbation of Ricci tensor induced by the perturbation of the metric is given by

\[ \delta R_{\mu\sigma} = \frac{1}{2} \left( \nabla_\lambda \nabla_\mu h_\sigma - \nabla_\lambda \nabla_\sigma h_\mu \right) - \frac{1}{2} g^{\alpha\beta} \nabla_\alpha \nabla_\beta h_{\mu\sigma}. \]

The perturbation of Ricci scalar reads

\[ \delta R = g^{\mu\sigma} \delta R_{\mu\sigma} - h^{\mu\sigma} R_{\mu\sigma} = \nabla_\nu h^{\mu\nu} - \Box h - h^{\mu\nu} R_{\mu\nu} \]

where \( \Box \) is the d'Alembertian. Substituting them into the perturbation of the field equation, namely

\[ \delta R_{\mu\nu} - \frac{1}{2} \delta R \delta_{\mu\nu} + \kappa \delta Y_{\mu\nu} = \kappa \delta T_{\mu\nu}, \]

its space-space “i ≠ j” components tell us

\[ \Phi = \Psi. \]

Substituting Eq. (29) into the diagonal components and the time-space “0 i” components of Eq. (28), they become

\[ 4 \ddot{\Psi} + \Box \Psi - \frac{1}{2} \delta R = -\kappa \delta \rho, \]

\[ -\Box \Psi - \frac{2K_0'}{A_0 A_1'} \Psi + \frac{1}{2} \delta R + \kappa (3 \delta p - \delta \rho) = \kappa \delta p, \]

\[ -\Box \Psi - \frac{K_0}{A_0} \Psi - \frac{K_0'}{2A_0 A_1'} \psi - \frac{1}{2} \delta R = \kappa \delta p, \]

\[ 2 \partial_i \partial_i \Psi = (\rho_0 + p_0) \delta U_i. \]

One can check that Eq. (31) is not independent of Eqs. (30) and (32). Substituting Eq. (29) into Eq. (27), it is easy to get

\[ \delta R = 4 \ddot{\Psi} - \frac{4K_0}{A_0} \Psi - \frac{4K_0'}{A_0 A_1'} \psi - 2 \Box \Psi. \]

Noting \( \Box \Psi = -\ddot{\Psi} + \nabla^2 \Psi \) in the case of LTB static universe (\( \nabla^2 \) is the Laplacian), and then substituting Eq. (34) into Eq. (30), we obtain

\[ -\kappa \delta \rho = 2 \left( \frac{K_0}{A_0} + \frac{K_0'}{A_0 A_1'} \right) \Psi + 2 \nabla^2 \Psi. \]
So, once $\Psi$ is available, $\delta\rho$ and $\delta U_i$ are ready by using Eqs. (35) and (33), respectively. If $\Psi$ is stable, they are also stable. Multiplying Eq. (30) by $w$, and then adding Eq. (32), we have

$$\ddot{\Psi} - \left[ \left( \frac{K_0}{A_0} + \frac{K_0'}{A_0 A_0'} \right) w + \frac{K_0'}{2A_0 A_0'} \right] \Psi - w \nabla^2 \Psi = 0.$$ 

(36)

Using Eqs. (16) and (18), it becomes

$$\ddot{\Psi} - w \nabla^2 \Psi = 0.$$ 

(37)

Considering a harmonic decomposition [94] (see also e.g. [18, 34, 97]),

$$\Psi = \sum_{n=0}^{\infty} \psi_n(t) \Upsilon_n(r, \theta, \phi) = \sum_{n=0}^{\infty} \psi_n(t) \sum_{m=-n}^{n} Y_{m}^{n}(\theta, \phi),$$

(38)

Eq. (37) can be separated into two differential equations, namely

$$\nabla^2 \Upsilon_n = -k^2 \Upsilon_n,$$ 

(39)

$$\ddot{\psi}_n + wk^2 \psi_n = 0,$$ 

(40)

where the wave number $k$ is related to the degree $n$ [94, 97, 98]. One can obtain the spatial part of $\Psi$ by solving Eq. (39) following e.g. [94, 97], but it does not determine the stability of the perturbation $\Psi$. In fact, the stability of the perturbation $\Psi$ depends on the temporal part, $\psi_n(t)$, which can be stable on the condition $wk^2 > 0$ (n.b. Eq. (40)). The LTB static universe is stable against the inhomogeneous ($k^2 > 0$) scalar perturbation if $w > 0$. From Eq. (21), it is easy to see that $w > 0$ requires a closed ($K_0 > 0$) universe. Unfortunately, the LTB static universe is unstable against the homogeneous ($k^2 = 0$) scalar perturbation, since $\psi_n(t) \propto t$ diverges when $t \to \infty$. So, the LTB static universe fails.

V. LTB STATIC UNIVERSE IN RASTALL-LIKE GRAVITY WITH A COSMOLOGICAL CONSTANT

Let us come back to the discussions in Sec. III B. In fact, a cosmological constant $\Lambda$ can be allowed in the Rastall-like gravity theory. Since $\Lambda = 0$, it is easy to see that $(G^\nu_\mu + \Lambda \delta^\nu_\mu)_,\mu = 0 = \kappa (T^\nu_\mu - Y^\nu_\mu)_,\mu$ always holds. So, Eq. (12) is also consistent with the modified field equations

$$G^\nu_\mu + \Lambda \delta^\nu_\mu + \kappa Y^\nu_\mu = \kappa T^\nu_\mu.$$ 

(41)

Contracting Eq. (41), we find that the trace of $Y^\nu_\mu$ is given by

$$\kappa Y^\nu_\mu = \text{diag} (0, R + \kappa T - 4\Lambda, 0, 0).$$ 

(42)

The general remarks (R1) ~ (R4) in Sec. III B are still valid. To construct a successful LTB static universe, the conditions (C1) and (C3) in Sec. III B are still valid, while the condition (C2) should be changed to Eq. (42). Again, there is a large room for the choice of $Y^\nu_\mu$. The simplest one is given by

$$\kappa Y^\nu_\mu = \text{diag} (0, R + \kappa T - 4\Lambda, 0, 0),$$ 

(43)

and we will briefly mention other reasonable choices in Sec. VI. When $R + \kappa T - 4\Lambda = 0$, this Rastall-like gravity theory reduces to GR (if $\kappa \neq 8\pi G_N$, one can simply rescale $T^\nu_\mu_{\text{new}} = \kappa T^\nu_\mu/(8\pi G_N)$, but $\kappa > 0$ is required to ensure the energy density $\rho_{\text{new}} \propto \kappa \rho \geq 0$).

A. LTB static solutions

We assume that the universe contains a perfect fluid whose energy-momentum tensor is given by Eqs. (3) or (4), and $p = w\rho$, where the equation-of-state parameter $w$ is a constant. In the case of LTB static
universe, \( A, \rho \) and \( p \) are all independent of the time \( t \), and hence they are functions only depending on the spatial coordinate \( r \), namely \( A_0(r) \), \( \rho_0(r) \) and \( p_0(r) \), while we also denote \( K = K_0(r) \). Due to the non-minimal coupling between geometry and matter, \( T^\nu_{\mu\nu} = Y^\nu_{\mu\nu} \neq 0 \), one can find that \( \rho_0 \neq 0 \) and \( p_0 \neq 0 \), namely they are not homogeneous. The static solutions are determined by the field equations in Eq. (41), namely

\[
\frac{K_0}{A_0^2} - \frac{K_0'}{A_0 A_0'} + \Lambda = -\kappa \rho_0 ,
\]

\[
-\frac{K_0}{A_0^2} + R_0 + \kappa T_0 - 3\Lambda = \kappa p_0 ,
\]

\[
-\frac{K_0'}{2A_0 A_0'} + \Lambda = \kappa p_0 .
\]

Noting \( R_0 = 2K_0/A_0^2 + 2K_0'/(A_0 A_0') \) and \( T_0 = 3p_0 - \rho_0 \), Eq. (45) is not independent of Eqs. (44) and (46). Multiplying Eq. (44) by \( w \), and then adding Eq. (46), we have

\[
\frac{wK_0}{A_0^2} + \frac{K_0'}{2A_0 A_0'} (1 + 2w) = (1 + w) \Lambda .
\]

Noting \( K_0'/A_0' = dK_0/dA_0 \), Eq. (47) can be regarded as an ordinary differential equation of \( K_0 \) with respect to \( A_0 \), and its solution is given by

\[
K_0 = \mathcal{C} A_0^{-2w/(1+2w)} + \left(1 + \frac{w}{1 + 3w}ight) \Lambda A_0^2 ,
\]

where \( \mathcal{C} \) is an integral constant, and we require \( w \neq -1/2 \) and \( w \neq -1/3 \). Obviously, \( K_0/A_0^2 \) is not a constant if \( w \neq -1/3 \) and \( \mathcal{C} \neq 0 \), and hence the LTB static universe does not reduce to the FRW static universe. Multiplying Eq. (46) by \( 2 \), and then subtracting Eq. (44), we have

\[
\kappa (1 + 2w) \rho_0 = \Lambda + \frac{K_0}{A_0^2} .
\]

Substituting Eq. (48) into Eq. (49), it is easy to get

\[
\rho_0 = \frac{\mathcal{C}}{\kappa (1 + 2w)} A_0^{-2(1+3w)/(1+2w)} + \frac{2\Lambda}{\kappa (1 + 3w)} .
\]

Eqs. (48) and (50) are the explicit expressions of the LTB static solutions. Note that if \( \Lambda = 0 \), all the results obtained here reduce to the ones in Sec. IV A But a non-zero \( \Lambda \) makes difference.

B. Stability analysis

To become a successful LTB static universe, it should be stable against perturbations. Similar to Sec. IV B, we consider the perturbed metric given by Eqs. (23) and (24). Accordingly, the perturbations \( \delta T_{\mu\nu} , \delta R_{\mu\nu} , \delta R \) are the same given by Eqs. (25) ~ (27). Since \( \delta \Lambda = 0 \), the perturbation of the field equation in Eq. (11) is still the same given in Eq. (28). Once again, its space-space “\( i \neq j \)” components tell us \( \Phi = \Psi \) as in Eq. (29). Then, its diagonal components and the time-space “\( 0\)” components become the ones given in Eqs. (30) ~ (32). Of course, Eqs. (31) ~ (30) still hold. However, the static solutions in Eqs. (44) and (46) make difference. Substituting Eqs. (44) and (46) into Eq. (36), it becomes

\[
\ddot{\Psi} - (1 + w) \Lambda \Psi - w \nabla^2 \Psi = 0 ,
\]

which is different from Eq. (37) if \( \Lambda \neq 0 \) and \( w \neq -1 \). Considering the harmonic decomposition given in Eq. (38), we can separate Eq. (51) into two differential equations, namely

\[
\nabla^2 \gamma_n = -k^2 \gamma_n ,
\]

\[
\ddot{\psi}_n + \left[ w k^2 - (1 + w) \Lambda \right] \psi_n = 0 ,
\]
where the wave number $k$ is related to the degree $n \ [94, 97, 98]$. From Eq. (53), it is easy to see that the stability condition for the LTB static universe reads

$$w k^2 - (1 + w) \Lambda > 0.$$  

The LTB static universe is stable against the homogeneous ($k^2 = 0$) scalar perturbation if

$$(1 + w) \Lambda < 0.$$  

On the other hand, the LTB static universe is also stable against the inhomogeneous ($k^2 > 0$) scalar perturbation if Eq. (54) is satisfied for all possible modes with $k^2 > 0$. To ensure that Eq. (54) is valid even when $k^2 \to \infty$, it is necessary to require

$$w \geq 0.$$  

Combining Eqs. (55) and (56), the LTB static universe can be stable against both the homogeneous ($k^2 = 0$) and the inhomogeneous ($k^2 > 0$) scalar perturbations if

$$w \geq 0 \quad \text{and} \quad \Lambda < 0.$$  

From Eq. (49), we find that $w \geq 0$ and $\Lambda < 0$ require a closed ($K_0 > 0$) universe. So far, we successfully get a stable LTB closed static universe on the conditions given by Eq. (57).

**VI. CONCLUDING REMARKS**

In this work, we try to construct a stable LTB static universe, which is spherically symmetric and radially inhomogeneous. However, this is not an easy task, and fails in GR and various modified gravity theories, because the corresponding LTB static universes must reduce to the FRW static universes. We find a way out in a new kind of modified gravity theory, namely Rastall-like gravity, in which the conservation of energy and momentum is broken. Actually, in order to construct a successful LTB static universe, we have proposed a new variant of the Rastall gravity theory. The stability of LTB static universe against both the homogeneous and the inhomogeneous scalar perturbations is also discussed in details. Finally, we successfully get a stable LTB static universe.

As is well known, the stability conditions for many FRW (Einstein) static universes in various modified gravity theories are fairly complicated, and usually require exotic matter with $w < 0$, in particular dark energy ($w < -1/3$) or even phantom ($w < -1$). On the contrary, the stability conditions for the LTB static universe given in Eq. (57) is very simple. Obviously, the condition $w \geq 0$ can be easily satisfied by using ordinary matter, such as radiation ($w = 1/3$) or dust matter ($w = 0$). On the other hand, a negative cosmological constant ($\Lambda < 0$) is also welcome in e.g. string theory. Although the current accelerated expansion of the universe requires dark energy ($w < -1/3$) or a positive cosmological constant ($\Lambda > 0$), this is not the case of LTB static universe (as the initial state for the past-eternal early universe in the emergent universe scenario).

It is worth noting that the (effective) gravitational forces provided by the ordinary matter ($w \geq 0$) and a negative cosmological constant ($\Lambda < 0$) are attractive. This means that the effective force contributed by the non-minimal coupling between matter and geometry $Y_{\mu,\nu}$ is repulsive, which can also be seen from the first equation of $T^\nu_{\mu,\nu} = X_\mu = Y^\nu_{\mu,\nu}$, namely

$$\dot{\rho} + \left(\frac{2 A'}{A} + \frac{A''}{A}\right) (\rho + p + p_{\text{eff}}) = 0,$$  

with a negative effective pressure $p_{\text{eff}} < 0$ coming from $X_\mu = Y^\nu_{\mu,\nu}$ (note that Eq. (58) in the LTB cosmology corresponds to the familiar $\dot{\rho} + 3H (\rho + p + p_{\text{eff}}) = 0$ in the FRW cosmology). So, when the (effective) attractive forces are stably balanced by the effective repulsive force, the LTB static universe can be accomplished.

In this work, we assume that the universe contains a perfect fluid. Actually, one can also extend our discussions to a non-perfect fluid, for example, van der Waals fluid, viscous fluid, and Newtonian fluid.
Of course, the role of matter can also be played by a scalar field or a vector field. In Rastall-like gravity, we expect that a stable LTB static universe can also be accomplished in these cases. For simplicity, in this work we only consider the simplest choices of $Y_{\nu\mu}$, as given in Eqs. (15) and (43).

Actually, there is a large room for other reasonable choices. In the case without a cosmological constant, one might instead consider, for example,

$$\kappa Y_{\nu\mu} = \text{diag} \left( R, \kappa T, 0, 0 \right), \quad \text{or} \quad \kappa Y_{\nu\mu} = \text{diag} \left( \kappa T, R, 0, 0 \right),$$

or even the more general choice

$$\kappa Y_{\nu\mu} = \text{diag} \left( \alpha R + \beta \kappa T, (1 - \alpha) R + (1 - \beta) \kappa T, 0, 0 \right),$$

where $\alpha$ and $\beta$ are any constants. If we are willing to involve the other two spatial components, it is also possible to choose

$$\kappa Y_{\nu\mu} = \text{diag} \left( (1 - \alpha_1 - \alpha_2) R + (1 - \beta_1 - \beta_2) \kappa T, \alpha_1 R + \beta_1 \kappa T, \alpha_2 R + \beta_2 \kappa T \right),$$

where $\alpha$ and $\beta$ are any constants. Of course, in the case with a cosmological constant $\Lambda$, the choices are similar, while one should appropriately insert $\Lambda$ into Eqs. (59) $\sim$ (61). Note that the choices mentioned above all satisfy the conditions (C1) $\sim$ (C3) to construct a successful LTB static universe, as in Sec. IIIB or Sec. VI. However, when we consider other topics rather than static universe, the conditions (C1) and (C3) can be abandoned, and then we can instead adopt other suitable $Y_{\nu\mu}$.

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In fact, when one separates the spatial and the temporal parts, the constant $k^2$ in principle can be any constant, say $\epsilon$, which might be positive, zero and negative. However, if $\epsilon < 0$ in the Poisson equation for $\xi_n(r)$, the solution of $\xi_n(r)$ will diverge when $r \to \infty$ (or equivalently $\xi_n(A_0)$ will diverge when $A_0 \to \infty$). Therefore, $\epsilon = k^2 \geq 0$ is necessary for a stable perturbation $\Psi$. Alternatively, one can also arrive at the same result by arguing a plane waves expansion $Y_n \propto \exp(ik \cdot x)$ in the locally flat spacetime [77].