THE CAUCHY PROBLEM FOR A GENERALIZED $b$-EQUATION WITH HIGHER-ORDER NONLINEARITIES IN CRITICAL BESOV SPACES AND WEIGHTED $L^p$ SPACES

SHOUMING ZHOU
College of Mathematics Science, Chongqing Normal University
Chongqing 401331, China

(Communicated by Adrian Constantin)

Abstract. This paper deals with the Cauchy problem for a generalized $b$-equation with higher-order nonlinearities $y_t + u^{m+1}y_x + bu^mu_x y = 0$, where $b$ is a constant and $m \in \mathbb{N}$, the notation $y := (1 - \partial_x^2)u$, which includes the famous $b$-equation and Novikov equation as special cases. The local well-posedness in critical Besov space $B^3_{2,1}$ is established. Moreover, a lower bound for the maximal existence time is derived. Finally, the persistence properties in weighted $L^p$ spaces for the solution of this equation are considered, which extend the work of Brandolese [L. Brandolese, Breakdown for the Camassa-Holm equation using decay criteria and persistence in weighted spaces, Int. Math. Res. Not. 22 (2012), 5161-5181] on persistence properties to more general equation with higher-order nonlinearities.

1. Introduction. The present paper focuses on the Cauchy problem for the following shallow water equation with high-order nonlinearities

$$
\begin{align*}
&\begin{cases}
  y_t + u^{m+1}y_x + bu^mu_x y = 0, & x \in \mathbb{R}, t > 0, \\
  u(x, 0) = u_0(x), & x \in \mathbb{R},
\end{cases}
\end{align*}
$$

where $b$ is a constant and $m \in \mathbb{N}$, the notation $y := (1 - \partial_x^2)u$. It is easy to see that model (1.1) contains the two kinds of famous shallow water equation, that is, $b$-equation and Novikov equation. Our main purpose of this paper is to establish the well-posedness in critical Besov space $B^3_{2,1}$ and persistence in weighted Sobolev space.

Obviously, if $m = 0, b \in \mathbb{R}$, the Eq. (1.1) becomes a $b$-equation,

$$
\begin{align*}
u_t - u_{xx} + (b + 1)uu_x = bu_x u_{xx} + uu_{xxx},
\end{align*}
$$

which can be derived as the family of asymptotically equivalent shallow water wave equations that emerges at quadratic order accuracy for any $b \neq -1$ by an appropriate Kodama transformation. For the case $b = -1$, the corresponding Kodama transformation is singular and the asymptotic ordering is violated (see [29, 30, 31]).

2010 Mathematics Subject Classification. Primary: 35G25, 35L05; Secondary: 35Q50.

Key words and phrases. Persistence properties, local well-posedness, blow-up, weighted $L^p$ spaces.

This work is supported in part by NSFC grant No.11301573 and in part by the found of Chongqing Normal University grant No.13XLZ08 and in part by the Program of Chongqing Innovation Team Project in University under Grant No.KJTD201308.
Equation (1.2) belongs to the following family of nonlinear dispersive partial differential equations
\[ u_t - \gamma u_{xxx} - \alpha^2 u_{xxt} = (c_1 u^2 + c_2 u_x^2 + c_3 uu_{xx})_x, \]
where \( \gamma, \alpha, c_1, c_2 \) and \( c_3 \) are real constants. By using Painlevé analysis in \([26, 28, 45]\), there are only three asymptotically integrable within this family: the KdV equation, the Camassa-Holm (Equation (1.2) with \( b = 2 \)) equation and the Degasperis-Procesi equation (Equation (1.2) with \( b = 3 \)). The solutions of the \( b \)-equation were studied numerically for various values of \( b \) in \([41, 42]\), where \( b \) was taken as a bifurcation parameter. The necessary conditions for integrability of the \( b \)-equation were investigated in \([55]\). The \( b \)-equation also admits peakon solutions for any \( b \in \mathbb{R} \) (see \([28, 41, 42]\)). The well-posedness, blow-up phenomena and global solutions for the \( b \)-equation were shown in \([34, 37, 58]\).

In fact, the Camassa-Holm and Degasperis-Procesi equations are the only integrable members of the \( b \)-equation family with a bi-Hamiltonian structure \([45]\). The Camassa-Holm equation arises in a variety of different contexts. In 1981, it was originally derived as a bi-Hamiltonian equation with infinitely many conservation laws by Fokas and Fuchssteiner \([35]\). It has been widely studied since 1993 when Camassa and Holm \([6]\) proposed it as a model for the unidirectional propagation of shallow water waves over a flat bed. Such as, Camassa-Holm equation has also a bi-Hamiltonian structure \([35, 50]\) and is completely integrable \([6, 11, 46]\), and it possesses an infinity of conservation laws and is solvable by its corresponding inverse scattering transform (cf.\([5, 9, 16]\)). The stability of the smooth solitons was considered in \([22]\), and the orbital stability of the peaked solitons were proved in \([21]\). It is worth pointing out that solutions of this type are not mere abstractions: the peakons replicate a feature that is characteristic for the waves of great height-waves of largest amplitude that are exact solutions of the governing equations for irrotational water waves (see \([15]\) and references therein). The explicit interaction of the peaked solitons were given in \([2]\). It has been shown that this problem is locally well-posed for initial data \( u_0 \in H^s \) with \( s > \frac{3}{2} \) (cf.\([12, 51, 23]\)). Moreover, Camassa-Holm equation not only has global strong solutions, but also admits finite time blow-up solutions \([10, 12, 13, 51, 17]\), and the blow-up occurs in the form of breaking waves, namely, the solution remains bounded but its slope becomes unbounded in finite time. On the other hand, it also has global weak solutions in \( H^1 \) (see \([3, 14, 20, 64]\)). The advantage of the Camassa-Holm equation in comparison with the KdV equation lies in the fact that the Camassa-Holm equation has peaked solitons and models the peculiar wave breaking phenomena (cf.\([7, 13]\)).

In 1999, Degasperis and Procesi \([26]\) derived a nonlinear dispersive equation, which can be regarded as a model for nonlinear shallow water dynamics and its asymptotic accuracy is the same as that for the Camassa-Holm shallow water equation, and the Degasperis-Procesi equation can be obtained from the shallow water elevation equation by an appropriate Kodama transformation (cf.\([19, 29, 30]\)). Degasperis, Holm and Hone \([27]\) proved the formal integrability of this equation by constructing a Lax pair. They also showed that this equation has a bi-Hamiltonian structure and an infinite sequence of conserved quantities, and admits exact peakon solutions. Lundmark and Szmigielski \([54]\) presented an inverse scattering approach for computing \( n \)-peakon solutions, and the direct and inverse scattering approach pursued recently in \([18]\). The traveling wave solutions for the Degasperis-Procesi equation was studied by Vakhnenko and Parkes in \([61]\). Similar to the Camassa-Holm equation, the local well-posedness, the precise blow-up scenario and the global...
existence of strong solutions to Degasperis-Procesi were derived in [32, 53, 66, 67]. On the other hand, it also has global weak solutions in $H^1$ (see [32, 66]) and global entropy weak solutions belonging to the class $L^2 \cap BV$ and to the class $L^2 \cap L^4$ (see [8]). A special property of the Degasperis-Procesi equation is the existence of discontinuous shock wave and periodic shock wave solutions (see [33] and references therein).

On the other hand, taking $m = 1$, $b = 3$ in (1.1) we arrived the Novikov equation,

\[ u_t - u_{xxt} + 4u^2 u_x = 3uu_xu_{xx} + u^2 u_{xxx}, \tag{1.3} \]

which was recently discovered by Novikov in a symmetry classification of nonlocal PDEs with quadratic or cubic nonlinearity [59]. The perturbative symmetry approach [55] yields necessary conditions for a PDE to admit infinitely many symmetries. Using this approach, Novikov was able to isolate the Equation (1.3) and find its first few symmetries, and he subsequently found a scalar Lax pair for it, proving that the equation is integrable. By using the prolongation algebra method, Hone and Wang [43] gave a matrix Lax pair and many conserved densities and a bi-Hamiltonian structure of the Novikov equation, and showed how it was related by a reciprocal transformation to a negative flow in the Sawada-Kotera hierarchy. The explicit formulas for multipeakon solutions of Novikov equation were derived in [44]. Recently, by the transport equations theory and the classical Friedrichs regularization method, the authors proved that the Cauchy problem for the Novikov equation is locally well-posed in the Besov spaces $B^s_{p,r}$ (with $1 \leq p, r \leq +\infty$ and $s > \max\{1 + 1/p, 3/2\}$ in [62, 65], and with the critical index $s = 3/2, p = 2, r = 1$ in [56]). It is also shown in [56] that the Novikov equation associated with the initial value is locally well-posed in Sobolev space $H^s$ with $s > 3/2$ by using the abstract Kato Theorem. Two results about the persistence properties of the strong solution for Equ.(1.3) were established in [56]. A Galerkin-type approximation method was used to establish the well-posedness of Novikov equation in the Sobolev space $H^s$ with $s > 3/2$ on both the line and the circle [39], and in [48, 68] the authors proved that the data-to-solution map is not globally uniformly continuous on $H^s$ for $s < 3/2$, this result supplements Himonas and Holliman’s works [39]. Tiglay [60] shown the local well-posedness of the problem in Sobolev spaces and existence and uniqueness of solutions for all time using orbit invariants. For analytic initial data, the existence and uniqueness of analytic solutions for Equ.(1.3) were also obtained in [60]. Analogous to the Camassa-Holm equation, the Novikov equation possesses blow-up phenomenon [47, 68] and global weak solutions [49, 63].

Motivated by the results mentioned above, in [69, 70], we considered the Cauchy problem for a weakly dissipative shallow water equation with higher-order nonlinearities

\[
\begin{cases}
y_t + u^{m+1} y_x + bu^m u_x y + \lambda y = 0, & x \in \mathbb{R}, t > 0, \\
u(x, 0) = u_0(x), & x \in \mathbb{R},
\end{cases}
\tag{1.4}
\]

where $\lambda, b$ are constants and $m \in \mathbb{N}$, the notation $y := (1 - \partial_x^2)u$. The local well-posedness of solutions for the Cauchy problem in Besov space $B^s_{p,r}$ with $1 \leq p, r \leq +\infty$ and $s > \max\{1 + 1/p, 3/2\}$ is obtained in [70], and the global existence and blow-up phenomenon propagation behaviors of compactly supported solutions are also established in [70]. The persistence properties of the strong solutions, the existence of weak solutions and the explicit formulas for peakon and multipeakon for Equ.(1.4) with $\lambda = 0$ are studied in [69]. The local and global existence and
Let $T_0$ denote the lifespan of the solution, then there exists a lifespan $T_0 > 0$, which shows that $H^s$ and $B^s_{2,2}$ are quite close, so, here, Motivated by the argument of the approximate solutions in Dauchin [24], and Wu et al. [62]. We first use the classical Friedrichs regularization method and the transport equations theory to establish the local well-posedness of the Equation (1.1) in critical Besov space $B^{3/2}_{2,1}$.

**Theorem 1.1.** Assume that the initial data $u_0(x) \in B^3_{2,1}$. Then there exists a unique solution $u(x,t)$ and a maximal time $T = T(u_0) > 0$ to the Cauchy problem (1.1) such that

$$u = u(\cdot, u_0) \in C([0,T]; B^{3}_{2,1}) \cap C^1([0,T]; B^{3}_{2,1}).$$

Moreover, the solution depends continuously on the initial data, i.e. the mapping $u_0 \mapsto u(\cdot, u_0) : B^{3}_{2,1} \mapsto C([0,T]; B^{3}_{2,1}) \cap C^1([0,T]; B^{3}_{2,1})$ is continuous.

**Remark 1.1.** Since

$$||fg||_{-\frac{1}{p}} \leq C||f||_{-\frac{1}{p}}||g||_{\frac{1}{p}} \quad \text{and} \quad ||f||_{\frac{1}{p}} \leq C||f||_{\frac{1}{p}} \log \left( e + \frac{||f||_{\frac{1}{p}}}{||g||_{\frac{1}{p}}(\log(1 + 1/p))} \right)$$

holds for $1 \leq p \leq \infty$ (see [24]), Theorem 1.1 holds true in the case of $B^{1+1/p}_{p,1}$ with $1 \leq p \leq \infty$. Besides, using the similar arguments in [23], Theorem 1.1 can also hold true in the case of $B^s_{p,r}$ with $s > \max(1+1/p, 3/2)$. Furthermore, the existence of solutions to Equation (1.1) holds as the initial data belong to $B^s_{p,r} \cap \text{Lip}$ with $s > 1$, which improves the corresponding result in [70].

For Theorem 1.1, we obtain local well-posedness of solution in $B^{3/2}_{2,1}$ to Equation (1.1). In next theorem, we shall show that local well-posedness of Equation (1.1) in $B^{3/2}_{2,1}$ fails by an example, thus $s = \frac{3}{2}$ is critical for $H^s$. More precisely, we will use the solitary wave solution $u_c(t,x) = e^{\frac{4}{3-s}} e^{-|x-ct|}$ with $c > 0$ to prove the following result.

**Theorem 1.2.** Assume that $u_0 = e^{\frac{4}{3-s}} e^{-|x|}$ with $c > 0$. There exists a unique global solution $u(x,t) \in B^{3/2}_{2,\infty}$ to Equation (1.1) with the initial data $u_0$. Moreover, any $\epsilon > 0$ and $T > 0$, there exists a solution $v \in L^\infty([0,T]; B^{3/2}_{2,\infty})$ such that

$$||u_0 - v_0||_{B^{3/2}_{2,\infty}} < \epsilon \quad \text{and} \quad ||u - v||_{B^{3/2}_{2,\infty}} > 2.$$  (1.5)

**Remark 1.2.** From Theorem 1.1 and 1.2, it is easy to see that the question of local well-posedness of Equation (1.1) in $H^{3/2}$ remains open. Actually, this is still an open problem for the $b$-equation and Novikov equation.

Our main blow-up criterion reads:

**Theorem 1.3.** Let $u_0 \in \text{Lip} \cap B^s_{p,r}$ with $1 \leq p, r \leq \infty$ and $s > \max(3/2, 1+1/p)$, then there exists a lifespan $T_{u_0} > 0$ such that

$$T_{u_0} < \infty \implies \int_0^{T_{u_0}} ||u(\tau)||_{L^{3+s-1}}^2 d\tau = \infty.$$
Moreover, if $b = m + 2$ and $u_0 \in H^1$, then
\[ T^*_u < \infty \implies \int_0^{T^*_u} \|u_x(\tau)\|_{L^\infty}^{m+1} d\tau = \infty. \]

Next, we get a lower bound for the maximal existence time which depending only on $\|u_0\|_{L^p}$. 

**Theorem 1.4.** Assume that $u_0 \in \text{Lip} \cap B^s_{p,r}$, $s > \max\{3/2, 1 + 1/p\}$. Let $T^*$ be the maximal existence time of the solution $u$ to Equation (1.1) with the initial data $u_0$. Then $T^*$ satisfies
\[ T^* \geq \frac{1}{a(2\|u_0\|_{L^\infty} + \|u_{0,x}\|_{L^\infty})^{m+1}}, \]
where $a$ depending only on $m$ and $|b|$.

In [4, 38, 40, 56, 57, 69], the spacial decay rate for the strong solution to the Camassa-Holm [4, 40, 57], $b$-equation [38], Novikov equation [56] were established. We can now state our main result on admissible weights.

**Theorem 1.5.** Let $T > 0$, $s > 3/2$, and $2 \leq p < \infty$. Let also $u \in C([0, T], H^s(\mathbb{R}))$ be a strong solution of the Cauchy problem for Equ.(1.1), such that $u|_{t=0} = u_0$ satisfies
\[ u_0 \phi \in L^p(\mathbb{R}) \text{ and } (\partial_x u_0) \phi \in L^p(\mathbb{R}), \]
where $\phi$ is an admissible weight function for the Equ.(1.1). Then, for all $t \in [0, T]$, we have the estimate
\[ \|u(t)\|_p + \|\partial_x u(t)\|_p \leq (\|u_0\|_p + \|\partial_x u_0\|_p) e^{CMt} \]
for some constant $C > 0$ depending only on $v, \phi$ (through the constants $A, C_0, \text{inf}_\mathbb{R} v$, and $\int_\mathbb{R} \frac{v(x)}{e^{mt}} dx < \infty$), and
\[ M \equiv \sup_{t \in [0, T]} (\|u(t)\|_\infty + \|\partial_x u(t)\|_\infty) < \infty. \]
The basic example of the application of Theorem 1.5 is obtained by taking the standard weights \( \phi = \phi_{a,b,c,d}(x) = e^{a|x|^c}(1 + |x|)^d \) with the following conditions:

\[
a \geq 0, c, d \in \mathbb{R}, 0 \leq b \leq 1, ab < 1.
\]

The restriction \( ab < 1 \) guarantees the validity of condition (1.6) for a multiplicative function \( v(x) \geq 1 \). Indeed, for \( a < 0 \) one has \( \phi(x) \to 0 \) as \( |x| \to \infty \): the conclusion of the Theorem 1.5 remains true but it is not interesting in this case. We interesting the following two special cases:

**Remark 1.3.** (1) Take \( \phi = \phi_{0,0,c,0} \) with \( c > 0 \), and choose \( p = \infty \). In this case the Theorem 1.6 states that the condition

\[
|u_0(x)| + |\partial_x u_0(x)| \leq C(1 + |x|)^{-c}
\]

implies the uniform algebraic decay in \([0, T]:\)

\[
|u(x, t)| + |\partial_x u(x, t)| \leq C(1 + |x|)^{-c}.
\]

Thus, Theorem 1.5 generalizes the main result of Ni and Zhou [57] on algebraic decay rates of strong solutions to the Equ.(1.1).

(2) Choose \( \phi = \phi_{a,1,0,0} \) if \( x \geq 0 \) and \( \phi(x) = 1 \) if \( x \leq 0 \) with \( 0 \leq a < 1 \). It is easy to see that such weight satisfies the admissibility conditions of Definition 1.1. Let further \( p = \infty \) in Theorem 1.5. Then one deduces that Equ.(1.1) preserve the pointwise decay \( O(e^{-ax}) \) as \( x \to +\infty \) for any \( t > 0 \). Similarly, we have persistence of the decay \( O(e^{-ax}) \) as \( x \to -\infty \). A corresponding result on persistence of strong solutions of the Camassa-Holm, b-equation, Novikov equation and Equ.(1.1) can be found in [38, 40, 56, 69].

Clearly, the limit case \( \phi = \phi_{1,1,c,d} \) is not covered by Theorem 1.5. In the following theorem however we may choose the weight \( \phi = \phi_{1,1,c,d} \) with \( c < 0, d \in \mathbb{R} \), and \( \frac{1}{|c|} < p \leq \infty \), or more generally when \( (1 + |\cdot|)^c \log(e + |\cdot|)^d \in L^p(\mathbb{R}) \). See Theorem 1.6 below, which covers the case of such fast growing weights. In other words, we want to establish a variant of Theorem 1.5 that can be applied to some \( v \)-moderate weights \( \phi \) for which condition (1.6) does not hold. Instead of assuming (1.6), we now put the weaker condition

\[
ve^{-|\cdot|} \in L^p(\mathbb{R}), 
\]

where \( 2 \leq p \leq \infty \).

**Theorem 1.6.** Let \( 2 \leq p \leq \infty \) and \( \phi \) be a \( v \)-moderate weight function as in Definition 1.1 satisfying condition (1.7) instead of (1.6). Let also \( u|_{t=0} = u_0 \) satisfy

\[
u_0 \phi \in L^p(\mathbb{R}), u_0 \phi \frac{1}{p+2} \in L^{m+2}(\mathbb{R}),
\]

and

\[
(\partial_x u_0) \phi \in L^p(\mathbb{R}), (\partial_x u_0) \phi \frac{1}{p+2} \in L^{m+2}(\mathbb{R}).
\]

Let also \( u \in C([0,T], H^s(\mathbb{R})), s > 3/2 \) be the strong solution of the Cauchy problem for Equation (1.1), emanating from \( u_0 \). Then,

\[
\sup_{t \in [0,T]} (||u(t)\phi||_{L^p} + ||\partial_x u(t)\phi||_{L^p})
\]

and

\[
\sup_{t \in [0,T]} \left(||u(t)\phi \frac{1}{p+2}||_{L^{m+2}} + ||\partial_x u(t)\phi \frac{1}{p+2}||_{L^{m+2}} \right)
\]

are finite.
Choosing \( \phi(x) = \phi_{1,1,0,0}(x) = e^{x^2} \) and \( p = \infty \) in Theorem 1.6. It follows that if \( |u_0(x)| \) and \( |\partial_x u_0(x)| \) are both bounded by \( ce^{-|x|} \), then the strong solution satisfies

\[
|u(x, t)| + |\partial_x u(x, t)| \leq Ce^{-|x|}
\]

uniformly in \([0, T]\). In the following result we compute the spatial asymptotic profiles of solutions with exponential decay. As a further consequence we may infer that the peakon-like decay \( O(e^{-|x|}) \) mentioned above is the fastest possible decay for a nontrivial solution \( u \) of Eqn.(1.1) to propagate.

\[ \text{Theorem 1.7.} \quad \text{Let the initial data } u_0 \in H^s(s > 3/2), u_0 \neq 0 \text{ and satisfy that} \\
\sup_{x \in \mathbb{R}} e^{x^2/(m+2)}(1 + |x|)^{1/(m+2)} \log(e + |x|) d(\{u_0(x)\} + \{|\partial_x u_0(x)|\}) < \infty, \tag{1.9} \]

with some \( d > 1/(m+2) \). Then the condition (1.9) uniformly in the time interval \([0, T]\) for the strong solution \( u \in C([0, T], H^s) \) of Eqn.(1.1) starting from \( u_0 \).

Moreover, suppose that the functions \( \Phi(t) \) and \( \Psi(t) \) satisfy

\[
c_1 \leq \Phi(t) \leq c_2, c_1 \leq \Psi(t) \leq c_2, \tag{1.10} \]

with some constants \( c_1, c_2 > 0 \) independent on \( t \), then the following asymptotic profiles hold:

\[
\begin{aligned}
&\{ u(x, t) = u_0(x) + e^{-x}t[\Phi(t) + \epsilon(x, t)], \\
&u(x, t) = u_0(x) - e^x t[\Psi(t) + \epsilon(x, t)], \\
&\text{with } \lim_{x \to +\infty} \epsilon(x, t) = 0, \\
&\text{with } \lim_{x \to -\infty} \epsilon(x, t) = 0, \tag{1.11} \\
&\text{for all } t \in [0, T] \text{ provided that either }
\end{aligned}
\]

\( (i) \) \( m = 0 \) and \( 0 < b < 3 \), or

\( (ii) \) \( b = m + 1 \) and \( m \) is a even.

The plan of this paper is organized as follows. In the next section, the local well-posedness in critical Besov space \( B^{3/2}_{2,1} \) is considered, and prove Theorem 1.1-1.2. the blow-up criteria is obtained in Section 3, and prove Theorem 1.3-1.4. In the last section, the persistence properties in weighted spaces for the solution of Eqn.(1.1) are considered, and prove Theorem 1.5-1.7.

2. Local well-posedness in critical Besov space. In this section, we shall establish the local well-posedness of the Eqn.(1.1) in critical Besov spaces. More precisely, we give the proof of Theorem 1.1. First, we rewrite model (1.1) in the following Transports equation form:

\[
\begin{aligned}
&u_t + u^{m+1}u_x + \frac{m(b-m-1)}{2} \partial_x^2(1 - \partial_x^2)^{-1} u^{m-1} (\partial_x u)^3 \\
&+ (1 - \partial_x^2)^{-1} \partial_x \left( \frac{b}{m+2} u^{m+2} + \frac{3m+3-b}{2} \frac{u^m (\partial_x u)^2}{2} \right) = 0, \tag{2.1} \\
&u(x, 0) = u_0(x). 
\end{aligned}
\]

By the arguments similar to the case \( u_0 \in B^s_{p,r}, s > \max\{1 + \frac{1}{p}, \frac{3}{2}\} \) (see [24, 70]), we can easy get the following two lemmas:

\[ \text{Lemma 2.1.} \quad \text{Let } u_0(x) \in B^3_{2,1}. \quad \text{Then there exists a time } T > 0 \text{ such that the Cauchy problem (1.1) has a solution } u \in C([0, T]; B^3_{2,1}) \cap C^1([0, T]; B^1_{2,1}). \]

\[ \text{Lemma 2.2.} \quad \text{Assume that } u_0 \text{ (respectively } v_0) \in B^3_{2,1} \text{ such that } u \text{ (respectively } v) \in L^\infty([0, T]; B^3_{2,1}) \cap C([0, T]; B^3_{2,1}) \text{ is a solution to the Cauchy problem (1.1) with} \]


initial data \( u_0 \) (respectively \( v_0 \)). Then for every \( t \in [0, T] \):

\[
\|u(t) - v(t)\|_{B^s_{p,r}} \leq \|u_0 - v_0\|_{B^s_{p,r}} \times \exp \left\{ C \int_0^T \left( \sum_{j=0}^{m+1} \|u_j\|_{B^s_{p,r}}^j \right) \right\}. 
\]

(2.2)

**Lemma 2.3.** For any \( u_0 \in B^s_{2,1} \), there exists a neighborhood \( V \) of \( u_0 \) in \( B^s_{2,1} \) and a time \( T > 0 \) such that for any solution of the Cauchy problem (1.1) \( v \in V \), the map

\[
\Phi : v_0 \mapsto v(\cdot, v_0) : V \subset B^s_{2,1} \mapsto C([0, T]; B^s_{2,1}) \cap C^1([0, T]; B^s_{2,1})
\]

is continuous.

**Proof.** Firstly, we prove the continuity of the map \( \Phi \) in \( C([0, T]; B^s_{2,1}) \). Fix \( u_0 \in B^s_{2,1} \) and \( \delta > 0 \). Now we claim that there exists a \( T > 0 \) and \( M > 0 \) such that for any \( v_0 \in B^s_{2,1} \) with \( \|v_0 - u_0\|_{B^s_{2,1}} \leq \delta \), the solution \( v = \Phi(v) \) of the Cauchy problem (1.1) belongs to \( C([0, T]; B^s_{2,1}) \) and satisfies \( \|v\|_{L^\infty([0,T]; B^s_{2,1})} \leq M \). In fact, according to the proof of the local well-posedness, we have that if we fix a \( T > 0 \) such that

\[
0 < T < \min \left\{ \frac{1}{2(m+1)C\|u_0\|^{m+1}}, \frac{1}{2C} \right\}
\]

then

\[
\|u(t)\|_{B^s_{2,1}} \leq \frac{\|v_0\|_{B^s_{2,1}}}{\left( 1 - 2(m+1)C\|u_0\|_{B^s_{2,1}}^{m+1} \right)^{m+1}}. 
\]

(2.3)

As \( \|v_0 - u_0\|_{B^s_{2,1}} \leq \delta \), then \( \|v_0\|_{B^s_{2,1}} \leq \|u_0\|_{B^s_{2,1}} + \delta \). Here, one can choose some suitable constant \( C \) such that

\[
T = \frac{1}{4C(m+1)(\|u_0\|_{B^s_{2,1}} + \delta)^{m+1}} < \min \left\{ \frac{1}{2(m+1)C\|v_0\|_{B^s_{2,1}}^{m+1}}, \frac{1}{2C} \right\}
\]

and

\[
M = 2^{\frac{1}{m+1}} \left( \|u_0\|_{B^s_{2,1}} + \delta \right). 
\]

Now combining the above uniform bounds with Lemma 2.2, we get that

\[
\|\Phi(v_0) - \Phi(u_0)\|_{L^\infty([0,T]; B^s_{2,1})} \leq \delta e^{C(m+1)MmT}.
\]

Hence \( \Phi \) is Hölder continuous from \( B^s_{2,1} \) into \( C([0, T]; B^s_{2,1}) \).

Next we prove the continuity of the map \( \Phi \) in \( C([0, T]; B^s_{2,1}) \). Let \( u^n_0 \in B^s_{2,1} \) and \((u_0^{(n)})_{n \in \mathbb{N}} \to u_0^\infty \) in \( B^s_{2,1} \). Let \( u^{(n)} \) be the solution of the Cauchy problem (1.1) with the initial data \( u_0^{(n)} \). From above argument, we deduce that for any \( n \in \mathbb{N}, t \in T \)

\[
\sup_{n \in \mathbb{N}} \|u^{(n)}\|_{L^\infty([0,T]; B^s_{2,1})} \leq M. 
\]

(2.4)

Note that to prove \( u^{(n)} \to u^\infty \) in \( C([0, T]; B^s_{2,1}) \) means to prove \( u^{(n)} \to u^\infty \) in \( C([0, T]; B^s_{2,1}) \).
Recall that \((v^{(n)}) = \partial_x u^{(n)}\) solves the linear transport equation:

\[
\begin{split}
&\begin{cases} \\
\partial_t (v^{(n)}) + (u^{(n)})^{m+1} \partial_x (v^{(n)}) = f^{(n)}, \\
v^{(n)} |_{t=0} = \partial_x(u_0^{(n)}), \end{cases}
\end{split}
\]

where

\[
f^{(n)} = (m + 1)(u^{(n)})^m \left( \partial_x u^{(n)} \right)^2 - \frac{m(b - m - 1)}{2} \partial_x (1 - \partial_x^2)^{-1} \left( u^{(n)} \right)^{m-1} \left( \partial_x u^{(n)} \right)^3 - (1 - \partial_x^2)^{-1} \partial_x^2 \left( \frac{b}{m + 2} \left( u^{(n)} \right)^{m+2} + \frac{3m + 3 - b}{2} \left( u^{(n)} \right)^m \left( \partial_x u^{(n)} \right)^2 \right).
\]

Thanks to the Kato theory [23], we decompose \(v^{(n)}\) into \(v^{(n)} = z^{(n)} + w^{(n)}\) with

\[
\begin{split}
&\begin{cases} \\
\partial_t (z^{(n)}) + (u^{(n)})^{m+1} \partial_x (z^{(n)}) = f^{(n)} - f^\infty, \\
v^{(n)} |_{t=0} = \partial_x(u_0^{(n)}) - \partial_x(u_0^{\infty}), \end{cases}
\end{split}
\]

and

\[
\begin{split}
&\begin{cases} \\
\partial_t (w^{(n)}) + (u^{(n)})^{m+1} \partial_x (w^{(n)}) = f^\infty, \\
w^{(n)} |_{t=0} = \partial_x(u_0^{\infty}). \end{cases}
\end{split}
\]

According to the first step, we have that the sequence \((u^{(n)})_{n \in \mathbb{N}} (\mathbb{N} = \mathbb{N} \cup \{\infty\})\) is uniformly bounded in \(C([0,T]; B^{\frac{3}{2}}_{2,1})\) and tends to \(u^{\infty}\) in \(C([0,T]; B^{\frac{3}{2}}_{2,1})\), thus we can use Proposition 3 in [24], which implies that \(w^{(n)}\) tends to \(w^{\infty}\) in \(C([0,T]; B^{\frac{3}{2}}_{2,1})\), i.e. for any \(\varepsilon > 0\), \(\|w^{(n)} - w^{\infty}\|_{B^{\frac{3}{2}}_{2,1}} \leq \varepsilon\).

On the other hand, applying Lemma 2.3 in [56] and the product law in the Besov spaces to equation (2.5), one may get that

\[
\|z^{(n)}\|_{B^{\frac{3}{2}}_{2,1}} \leq \exp \left\{ C \int_0^t \| (u^{(n)})^{m+1} (\tau) \|_{B^{\frac{3}{2}}_{2,1}} d\tau \right\} \cdot \left( \|\partial_x(u_0^{(n)}) - \partial_x(u_0^{\infty})\|_{B^{\frac{1}{2}}_{2,1}} + \int_0^t \|f^{(n)} - f^\infty\|_{B^{\frac{1}{2}}_{2,1}} d\tau \right).
\]

Using the properties of Besov spaces exhibited in [24], one easily checks that \((f^{(n)})_{n \in \mathbb{N}}\) is uniformly bounded in \(C([0,T]; B^{\frac{3}{2}}_{1,2})\). Moreover,

\[
\|f^{(n)} - f^\infty\|_{B^{\frac{3}{2}}_{1,2}} \leq C \left( m, \|u_0^{(n)}\|_{B^{\frac{3}{2}}_{2,1}}, \|u^{\infty}\|_{B^{\frac{3}{2}}_{2,1}} \right) \left( \|\partial_x(u^{(n)}) - \partial_x(u^{\infty})\|_{B^{\frac{1}{2}}_{2,1}} + \|u^{(n)} - u^{\infty}\|_{B^{\frac{1}{2}}_{2,1}} \right). \tag{2.7}
\]

Hence, combining the convergence of \(z^{(n)}\) in \(C([0,T]; B^{\frac{3}{2}}_{2,1})\) with estimates (2.4)-(2.7), we deduce that for large enough \(n \in \mathbb{N}\)

\[
\|\partial_x(u^{(n)}) - \partial_x(u^{\infty})\|_{B^{\frac{1}{2}}_{2,1}} \leq \varepsilon + C \left( m, \|u_0^{(n)}\|_{B^{\frac{3}{2}}_{2,1}}, \|u^{\infty}\|_{B^{\frac{1}{2}}_{2,1}} \right) e^{C(m+1)M^{m+1}T} \left[ \|\partial_x(u_0^{(n)}) - \partial_x(u_0^{\infty})\|_{B^{\frac{1}{2}}_{2,1}} + \int_0^t \|u^{(n)} - u^{\infty}\|_{B^{\frac{1}{2}}_{2,1}} d\tau \right. \\
\left. + \int_0^t \|\partial_x(u^{(n)}) - \partial_x(u^{\infty})\|_{B^{\frac{1}{2}}_{2,1}} d\tau \right].
\]

THE CAUCHY PROBLEM FOR A GENERALIZED b-EQUATION ... 4975
Thanks to the Gronwall’s inequality, we have

$$
\|\partial_x (u^{(n)}) - \partial_x (u^\infty)\|_{L^\infty(0,T;B^\frac{1}{2})} \leq C(M, m, T) \left( \|\partial_x (u_0^{(n)}) - \partial_x (u_0^\infty)\|_{B^\frac{1}{2}} + \varepsilon \right)
$$

for some constant $C$ depending only on $m, M$ and $|b|$. We have complete the continuity of the map $\Phi$ in $C([0, T]; B^\frac{1}{2})$.

Now applying $\partial_t$ to the Equ.(1.1) and by the same argument to the resulting equation in terms of $\partial_t u$, we may checking the continuity of the map $\Phi$ in $C^1([0, T]; B^\frac{1}{2})$.

**Proof of Theorem 1.1**. Combining the result in Lemma 2.1 with that in Lemma 2.2, one gets the existence and uniqueness of the solution of the Cauchy problem (1.1). And Lemma 2.3 shows that the solution of the Cauchy problem (1.1) depends continuously on the initial data. This completes the proof of Theorem 1.1. □

**Proof of Theorem 1.2**. By Theorem 1.6 in [69], $u_c(t, x) = c^{\frac{1}{m+1}} e^{-|x-ct|}$ is the unique solution to Equ.(1.1) with the initial data $u_0$, and $u_c(x, t)$ is the peakon solitary wave to Equ.(1.1). Its Fourier transform in $x$ is

$$
\hat{u}_c(t, \xi) = c^{\frac{1}{m+1}} \int_{\mathbb{R}} e^{-i\xi x} e^{-|x-ct|} dx = c^{\frac{1}{m+1}} \left[ e^{-ct} \int_{-\infty}^{ct} e^{-i\xi x} dx + e^{ct} \int_{ct}^{\infty} e^{-i\xi x} dx \right]
$$

$$
= c^{\frac{1}{m+1}} e^{-ict\xi} \left( \frac{1}{1-i\xi} + \frac{1}{1+i\xi} \right) = \frac{2c^{\frac{1}{m+1}} e^{-ict\xi}}{1+|\xi|^2}
$$

then by the definition of $B^\frac{s}{2}$, we have

$$
||u_c||_{B^\frac{s}{2}} = \max \left\{ \int_1^\infty (1+\xi^2)^\frac{s}{2} |\hat{u}(t, \xi)|^2 d\xi, \sup_{q \in \mathbb{R}} \int_{2^q}^{2^{q+1}} 2(1+\xi^2)^\frac{s}{2} |\hat{u}(t, \xi)|^2 d\xi \right\}
$$

$$
= 4c^{\frac{1}{m+1}} \max \left\{ \int_0^1 \frac{d\xi}{\sqrt{1+\xi^2}}, \sup_{q \in \mathbb{R}} \int_{2^q}^{2^{q+1}} \frac{d\xi}{\sqrt{1+\xi^2}} \right\} = 8c^{\frac{1}{m+1}} \log(1+\sqrt{2})
$$

Moreover, one can easily check that $u_c \notin B^\frac{s}{2}$, $r \in [1, \infty)$.

Similarly, we can get

$$
||u_{c_2}(0) - u_{c_1}(0)||_{B^\frac{2}{2}} = (c_{2}^{\frac{1}{m+1}} - c_{1}^{\frac{1}{m+1}})^2 ||e^{-|x|}||_{B^\frac{2}{2}}
$$

$$
= 8(c_{2}^{\frac{1}{m+1}} - c_{1}^{\frac{1}{m+1}})^2 \log(1+\sqrt{2}).
$$

(2.8)

Let $\theta c = c_2 - c_1$. Note that

$$
\left| c_{2}^{\frac{1}{m+1}} e^{-ic_2 t\xi} - c_{1}^{\frac{1}{m+1}} e^{-ic_1 t\xi} \right|^2 = c_{2}^{\frac{1}{m+1}} + c_{1}^{\frac{1}{m+1}} - 2(c_{2} c_{1})^{\frac{1}{m+1}} \cos((c_{2} - c_{1})t\xi))
$$

$$
\geq 2(c_{2} c_{1})^{\frac{1}{m+1}} (1 - \cos(\theta t\xi)).
$$

Thus, we deduce

$$
||u_{c_2}(t) - u_{c_1}(t)||_{B^\frac{2}{2}}
$$

$$
= 8 \max \left\{ \int_0^1 \frac{|c_{2}^{\frac{1}{m+1}} e^{-ic_2 t\xi} - c_{1}^{\frac{1}{m+1}} e^{-ic_1 t\xi}|^2}{\sqrt{1+\xi^2}} d\xi \right\},
$$
Combining (2.9) with (2.10), we deduce the result.

**Lemma 3.1.** This section is devoted to proof of Theorem 1.3-1.4. Theorem 1.3-1.4 are based on we have
\[
\sup_{q \in \mathbb{N}} \int_{2^q}^{2^{q+1}} \left| m^+ \sqrt{c_1 c_2} \frac{e^{-ic_2 t \xi} - e^{-ic_1 t \xi}}{1 + \xi^2} \right|^2 d\xi \geq 16 m^+ \sqrt{c_1 c_2} \sup_{q \in \mathbb{N}} \int_{2^q}^{2^{q+1}} \frac{1 - \cos(\theta c t \xi)}{1 + \xi^2} d\xi.
\]

Choose \( c_1 = 1 \) and \( c_2 = 1 + \frac{\pi}{2T} \). Then \( \theta c = \frac{\pi}{2T} \). Therefore,
\[
\|u_{c_2}(T) - u_{c_1}(T)\|_{B^{3/2}} \geq 16 m^+ \sqrt{c_1 c_2} \sup_{q \in \mathbb{N}} \int_{2^q}^{2^{q+1}} \frac{1 - \cos(2^{-q} \pi \xi)}{1 + \xi^2} d\xi
\]
\[
\geq 16 m^+ \sqrt{c_1 c_2} \sup_{q \in \mathbb{N}} \int_{2^q}^{2^{q+1}} \frac{1 - \cos(2^{-q} \pi \xi)}{1 + 2^{2q+2}} d\xi
\]
\[
\geq 16 m^+ \sqrt{c_1 c_2} \sup_{q \in \mathbb{N}} \frac{2^q}{1 + 2^{2q+2}} > 4.
\]

In view of (2.8) and (1 + \( \frac{\pi}{2T} \))\(^{2q+1} \) \( \leq 1 + \frac{1}{m+1} \frac{\pi}{2T} \) for fixed \( T > 0 \) large enough \( q \), we can get
\[
\|u_{c_2}(0) - u_{c_1}(0)\|_{B^{3/2}} = 8(c_2 \frac{\pi}{2T} - c_1 \frac{\pi}{2T})^2 \log(1 + \sqrt{2})
\]
\[
\leq 8 \left( \frac{\pi}{(m+1)(2T)} \right)^2 \log(1 + \sqrt{2}) < \epsilon^2.
\]

Combining (2.9) with (2.10), we deduce the result. \( \square \)

**3. Blow-up scenario and lower bound for the maximal existence time.**

This section is devoted to proof of Theorem 1.3-1.4. Theorem 1.3-1.4 are based on the following lemma:

**Lemma 3.1.** Let \( u_0 \in B^s_{p,r} \) with \( 1 \leq p, r \leq \infty \) and \( s > 1 \), let \( u \in L^\infty([0,T]; B^s_{p,r}) \) solving Equ.(1.1) on \([0,T] \times \mathbb{R}\) with the initial datum \( u_0 \). There exist a constant \( C_1 \) depending only on \( s \) and \( p \), and a universal constant \( C_2 \) such that for all \( t \in [0,T) \), we have
\[
\|u(t)\|_{B^s_{p,r}} \leq \|u_0\|_{B^s_{p,r}} \exp \left( C_1 \int_0^t \|u(\tau)\|_{L^p}^{m+1} d\tau \right),
\]
(3.1)
\[
\|u(t)\|_{L^p} \leq \|u_0\|_{L^p} \exp \left( C_2 \int_0^t \|u(\tau)\|_{L^p}^{m+1} d\tau \right).
\]
(3.2)

**Proof.** Applying the last of Lemma 2.4 in [70] to Equ.(1.1) and using the fact that \( (1 - \partial_x^2)^{-1} \) is a multiplier of order \(-2\) yields
\[
\exp \left( -C \int_0^t \|u^m \partial_x u(\tau)\|_{L^\infty} d\tau \right) \|u(t)\|_{B^s_{p,r}} \leq \|u_0\|_{B^s_{p,r}} + C \int_0^t \exp \left( -C \int_0^\tau \|u^m \partial_x u(\tau')\|_{L^\infty} d\tau' \right)
\]
\[
\cdot \left( \|u^{m+2}\|_{B^{s-1}_{p,r}} + \|u^m u_x^2\|_{B^{s-1}_{p,r}} + \|u^{m-1} u_x^3\|_{B^{s-2}_{p,r}} \right) d\tau,
\]
As \( s - 1 > 0 \), according to Lemma 2.5 in [70], one gets
\[
\|u^{m+2}\|_{B^{s-1}_{p,r}} + \|u^m u_x^2\|_{B^{s-1}_{p,r}} + \|u^{m-1} u_x^3\|_{B^{s-2}_{p,r}} \leq C \|u\|_{L^p} \|u\|_{B^s_{p,r}}.
\]
Therefore
\[
\exp \left( -C \int_0^t \| u^m \partial_x u(\tau) \|_{L^\infty} d\tau \right) \| u(t) \|_{B^s_{p,r}} \\
\leq \| u_0 \|_{B^s_{p,r}} + C \int_0^t \exp \left( -C \int_0^\tau \| u^m \partial_x u(\tau') \|_{L^\infty} d\tau' \right) \| u \|_{L^p_{\text{lip}}}^{m+1} \| u \|_{B^s_{p,r}} d\tau,
\]
Applying Gronwall lemma completes the proof of (3.1).

By differentiating once equation (1.1) with respect to \( x \), and applying the \( L^\infty \) estimate for transport equations, we easily prove that
\[
\exp \left( -C \int_0^t \| u^m \partial_x u(\tau) \|_{L^\infty} d\tau \right) \| u(t) \|_{L^p_{\text{lip}}}
\leq \| u_0 \|_{L^p_{\text{lip}}} + C \int_0^t \exp \left( -C \int_0^\tau \| u^m \partial_x u(\tau') \|_{L^\infty} d\tau' \right) \\
\cdot \| (1 - \partial_x^2)^{-1} u^{m-1}(\partial_x u)^3 + (1 - \partial_x^2)^{-1} \partial_x (u^{m+2} + u^m (\partial_x u)^2) \|_{L^p_{\text{lip}}} d\tau
\]
Since \( (1 - \partial_x^2)^{-1} f = \frac{1}{2} e^{-|x|} * f \) and the Young inequality, we get
\[
\| (1 - \partial_x^2)^{-1} u^{m-1}(\partial_x u)^3 + (1 - \partial_x^2)^{-1} \partial_x (u^{m+2} + u^m (\partial_x u)^2) \|_{L^p_{\text{lip}}} \leq C' \| u \|_{L^p_{\text{lip}}}^{m+2}
\]
for some universal constant \( C' \). Hence Gronwall lemma gives inequality (3.2).

**Proof of Theorem 1.3.** Applying Theorem 1.1, there exists a unique solution \( u \) to Equ.(1.1) with the initial data \( u_0 \). Assume that \( u \) satisfies
\[
\int_0^{T^*} \| u(\tau) \|_{L^p_{\text{lip}}}^{m+1} d\tau < \infty.
\]
Hence, for all \( t \in [0, T^*) \), (3.1) insures that
\[
\| u(t) \|_{B^s_{p,r}} \leq M_{T^*} := \| u_0 \|_{B^s_{p,r}} \exp \left( C_1 \int_0^t \| u(\tau) \|_{L^p_{\text{lip}}}^{m+1} d\tau \right) < \infty, \quad (3.3)
\]
Let \( \epsilon > 0 \) be such that \( 2(m+1)CeM_{T^*}^2 < 1 \) where \( C \) stands for the constants used in the proof of Proposition 3.1 in [70]. We then have a solution \( \tilde{u}(t) \in E^s_{p,r}(\epsilon) \) to Equ.(1.1) with initial datum \( u(T^* - \epsilon/2) \). For the sake of uniqueness, \( \tilde{u}(t) = u(t + T^* - \epsilon/2) \) on \([0, \epsilon/2)\) so that \( \tilde{u} \) extends the solution \( u \) beyond \( T^* \). We conclude that \( T^* < T_{u_0}^* \).

Multiply Equ.(1.1) by \( u \), we have
\[
u u_t - uu_{txx} = -(b+1)u^{m+2}u_x + bu^{m+1}u_x u_{xx} + u^{m+2}u_{xxx}.
\]
Integrating by parts on \( \mathbb{R} \),
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} (u^2 + u_x^2)dx = -(m+2-b) \int_{\mathbb{R}} (u^{m+1}u_x u_{xx})dx = - \frac{(m+2-b)(m+1)}{2} \int_{\mathbb{R}} u^{m+2}u_x^2 dx.
\]
Clearly, if \( b = m+2 \), we get \( \| u \|^2_{H^1} = \| u_0 \|^2_{H^1} \), it follows that
\[
\| u(t) \|_{L^\infty} \leq \frac{\| u_0 \|_{H^1}}{\sqrt{2}}.
\]
Similar to the above argument, we have
\[
T_{u_0}^* < \infty \implies \int_0^{T_{u_0}^*} \| u_x(\tau) \|_{L^\infty}^{m+1} d\tau = \infty.
\]
This completes the proof of Theorem 1.3. \[\square\]

**Proof of Theorem 1.4.** Multiplying Equation (2.1) by \(u^{2n-1}\) with \(n \in \mathbb{Z}^+\) and integrating by parts, we obtain

\[
\int_{\mathbb{R}} u^{2n-1}(u_t + u^{m+1}u_x + F) \, dx = 0
\]

with

\[
F = \frac{m(b - m - 1)}{2} (1 - \partial_x^2)^{-1} u^{m-1}(\partial_x u)^3
\]

\[
+ (1 - \partial_x^2)^{-1} \partial_x \left( \frac{b}{m + 2} u^{m+2} + \frac{3m + 3 - b}{2} u^m (\partial_x u)^2 \right).
\]

Note that the estimates

\[
\int_{\mathbb{R}} u^{2n-1}u_t \, dx = \frac{1}{2n} \frac{d}{dt} \|u(x, t)\|_{L^{2n}}^2 = \|u(x, t)\|_{L^{2n}}^2 \frac{d}{dt} \|u(x, t)\|_{L^{2n}},
\]

and

\[
\left| \int_{\mathbb{R}} u^{2n-1}(u^{m+1}u_x) \, dx \right| \leq \|u^{m}u_x(x, t)\|_{L^\infty} \|u(x, t)\|_{L^{2n}}^2
\]

are true. Moreover, using Hölder’s inequality

\[
\left| \int_{\mathbb{R}} u^{2n-1}F \, dx \right| \leq \|u(x, t)\|_{L^{2n-1}}^2 \|F\|_{L^n}.
\]

Thus, we can obtain

\[
\frac{d}{dt} \|u(x, t)\|_{L^n} \leq \|u^{m}u_x(x, t)\|_{L^\infty} \|u(x, t)\|_{L^{2n}} + \|F\|_{L^n}.
\]

Since \(\|f\|_{L^n} \to \|f\|_{L^\infty}\) as \(n \to \infty\) for any \(f \in L^\infty \cap L^2\) and the operator \((1 - \partial_x^2)^{-1} = \frac{1}{2} e^{-|x|}\).

Form above inequality we deduce that

\[
\frac{d}{dt} \|u(x, t)\|_{L^\infty} \leq \|u_x\|_{L^\infty} \|u\|_{L^\infty}^{m+1} + \frac{mb - m - 1}{2} \|u(\tau)\|_{L^{2n-1}}^2 \|u_x(\tau)\|_{L^\infty}^2
\]

\[
+ \left( \frac{b}{m + 2} \|u(\tau)\|_{L^{2n}}^2 + \frac{3m + 3 - b}{2} \|u(\tau)\|_{L^\infty} \|u_x(\tau)\|_{L^\infty}^2 \right) \tag{3.4}
\]

Next, we will give estimates on \(\|u_x(x, t)\|_{L^\infty}\). Differentiating (2.1) with respect to \(x\)-variable produces the equation

\[
u_{xt} + u^{m+1}u_{xx} + (m + 1)u^m u_x^2 + \partial_x F = 0,
\]

where

\[
\partial_x F = \frac{m(b - m - 1)}{2} \partial_x (1 - \partial_x^2)^{-1} u^{m-1}(\partial_x u)^3 - \frac{b}{m + 2} u^{m+2} + \frac{3m + 3 - b}{2} u^m (\partial_x u)^2
\]

\[
+ (1 - \partial_x^2)^{-1} \left( \frac{b}{m + 2} u^{m+2} + \frac{3m + 3 - b}{2} u^m (\partial_x u)^2 \right).
\]

Similar to the estimate of (3.4), we deduce that

\[
\frac{d}{dt} \|u_x(t)\|_{L^\infty} \leq 2\|u_x\|_{L^\infty} \|u\|_{L^\infty}^2 + \frac{mb - m - 1}{2} \|u(\tau)\|_{L^{2n-1}}^2 \|u_x(\tau)\|_{L^\infty}^2
\]

\[
+ \left( \frac{2b}{m + 2} \|u(\tau)\|_{L^{2n}}^2 + \frac{3m + 3 - b}{2} \|u(\tau)\|_{L^\infty} \|u_x(\tau)\|_{L^\infty}^2 \right) \tag{3.5}
\]
Choose
\[ H(t) := 2||u(t)||_{L^\infty} + ||u_x(t)||_{L^\infty}. \]
Combining (3.4) with (3.5), we get
\[ \frac{d}{dt} H(t) \leq \frac{a}{m+1} H^{m+2}(t), \]
where \( a \) depending only on \( m \) and \( |b| \). Define \( T := \frac{1}{a(2||u_0||_{L^\infty} + ||u_0,x||_{L^\infty})^{m+1}}. \) By (3.6), then for all \( t < \min\{T, T^*\} \), one can easily get
\[ H(t) \leq \frac{H(0)}{\frac{m+1}{\sqrt{1-aH^{m+1}(0)t}}}. \]

Theorem 1.3 follows that \( T^* \geq T \). This completes the proof of Theorem 1.4. \( \square \)

4. Analysis of the Novikov equation in weighted spaces. In this section, for the convenience of the readers, we first present some standard definitions. In general a weight function is simply a non-negative function. A weight function \( \nu : \mathbb{R}^n \to \mathbb{R} \) is called sub-multiplicative if
\[ \nu(x+y) \leq \nu(x)\nu(y), \text{ for all } x, y \in \mathbb{R}^n. \]

Given a sub-multiplicative function \( \nu \), a positive function \( \phi \) is \( \nu \)-moderate if and only if
\[ \exists C_0 > 0 : \phi(x+y) \leq C_0\nu(x)\phi(y), \text{ for all } x, y \in \mathbb{R}^n. \]

If \( \phi \) is \( \nu \)-moderate for some sub-multiplicative function \( \nu \), then we say that \( \phi \) is moderate. This is the usual terminology in time-frequency analysis papers [1]. Let us recall the most standard examples of such weights. Let
\[ \phi(x) = \phi_{a,b,c,d}(x) = e^{a|x|^b}(1+|x|)^c\log(e+|x|)^d. \]
We have (see [4]) the following conditions:
(i) For \( a, c, d \geq 0 \) and \( 0 \leq b \leq 1 \) such weight is sub-multiplicative.
(ii) If \( a, c, d \in \mathbb{R} \) and \( 0 \leq b \leq 1 \), then \( \phi \) is moderate. More precisely, \( \phi_{a,b,c,d} \) is \( \phi_{a,\beta,\gamma,\delta} \)-moderate for \( |a| \leq \alpha, |b| \leq \beta, |c| \leq \gamma \) and \( |d| \leq \delta \).

The elementary properties of sub-multiplicative and moderate weights can be find in [4]. Next, we prove Theorem 1.5.

**Proof of Theorem 1.5.** We define
\[ E(u) = \frac{m(b-m-1)}{2}u^{m-1}(\partial_x u)^3 + \partial_x \left( \frac{b}{m+2}u^{m+2} + \frac{3m+3-b}{2}u^m(\partial_x u)^2 \right) \]
We also introduce the kernel \( G(x) = \frac{1}{4}e^{-|x|}. \) Then the Equ.(1.1) can be rewritten as
\[ u_t + u^{m+1}\partial_x u + G \ast E(u) = 0, \]
Note that, from the assumption \( u \in C([0,T], H^s), s > 3/2 \), we get
\[ M \equiv \sup_{t \in [0,T]}(||u(t)||_{L^\infty} + ||\partial_x u(t)||_{L^\infty}) < \infty. \]

For any \( N \in \mathbb{Z}^+ \) let us consider the \( N \)-truncations
\[ f(x) = f_N(x) = \begin{cases} \phi(x), & \text{if } \phi(x) \leq N, \\ N, & \text{if } \phi(x) > N. \end{cases} \]
Observe that \( f : \mathbb{R} \to \mathbb{R} \) is a locally absolutely continuous function such that
\[ ||f||_\infty \leq N, \quad |f'(x)| \leq A|f(x)| \quad \text{a.e.} \]
In addition, if \( C_1 = \max\{C_0, \alpha^{-1}\} \), where \( \alpha = \inf_{x \in \mathbb{R}} v(x) > 0 \), then
\[
f(x + y) \leq C_1 v(x) f(y), \quad \forall x, y \in \mathbb{R}.
\]

Indeed, let us introduce the set \( U_N = \{ x : \phi(x) \leq N \} \), if \( y \in U_N \) then \( f(x + y) \leq \phi(x + y) \leq C_0 v(x) f(y) \); if \( y \notin U_N \), then \( f(x + y) \leq N = f(y) \leq \alpha^{-1} v(x) f(y) \).

The constant \( C_1 \) being independent on \( N \), this shows that the \( N \)-truncations of a \( v \)-moderate weight are uniformly \( v \)-moderate with respect to \( N \).

We start considering the case \( 2 \leq p < \infty \). Multiplying the Equation (4.1) by \( f \), and then by \( |uf|^{p-2}(uf) \) we get, after integration,
\[
\int_{\mathbb{R}} |uf|^{p-2}(uf) (\partial_t uf) dx + \int_{\mathbb{R}} |uf|^p (u^m \partial_x u) dx + \int_{\mathbb{R}} |uf|^{p-2}(uf) (f \cdot G * E(u)) dx = 0.
\]

Note that the estimates
\[
\int_{\mathbb{R}} |uf|^{p-2}(uf) (\partial_t uf) dx = \frac{1}{p} \frac{d}{dt} ||uf||_{L^p}^p = ||uf||_{L^p}^{p-1} \frac{d}{dt} ||uf||_{L^p},
\]

and
\[
\int_{\mathbb{R}} |uf|^p (u^m \partial_x u) dx \leq ||u^m \partial_x u||_{L^\infty} ||uf||_{L^p}^p \leq M^{m+1} ||uf||_{L^p}^p
\]
are true. Moreover, we get
\[
\left| \int_{\mathbb{R}} |uf|^{p-2}(uf) [f \cdot (G * E(u))] dx \right|
\leq ||uf||_{L^p}^{p-1} \left| f \cdot \left[ G \left( \frac{m(b - m - 1)}{2} u^{m-1} (\partial_x u)^3 
+ \partial_x \left( \frac{b}{m + 2} u^{m+2} + \frac{3m + 3 - b}{2} u^m (\partial_x u)^2 \right) \right] \right|_{L^p}
\leq c ||uf||_{L^p}^{p-1} \left( \| (\partial_x G) v \|_{L^1} \| f (u^{m+2} + u^m (\partial_x u)^2) \|_{L^p} + \| G v \|_{L^1} \| f u^{m-1} (\partial_x u)^3 \|_{L^p} \right)
\leq CM^{m+1} ||uf||_{L^p}^{p-1} (||uf||_{L^p} + ||(\partial_x u)f||_{L^p}).
\]

In the first inequality we used Hölder’s inequality, and in the second inequality we applied Proposition 3.1 and 3.2 in [4], and the last we used condition (1.6). Here, \( C \) depends only on \( v \) and \( \phi \). Form (4.2) we can obtain
\[
\frac{d}{dt} ||uf||_{L^p} \leq C_1 M^{m+1} ||uf||_{L^p} + C_2 M^{m+1} ||(\partial_x u)f||_{L^p}. \tag{4.3}
\]

Next, we will give estimates on \( u_x f \). Differentiating (4.1) with respect to \( x \)-variable, next multiplying by \( f \) produces the equation
\[
\partial_t [(\partial_x u)f] + u^{m+1} f \partial_x^2 u + [(\partial_x u)f] (u^m \partial_x u) + f [\partial_x G * E(u)] = 0.
\]

Multiplying this equation by \( |f \partial_x u|^{p-2}(f \partial_x u) \) with \( p \in \mathbb{Z}^+ \), integrating the result in the \( x \)-variable, and note that
\[
\int_{\mathbb{R}} |f \partial_x u|^{p-2}(f \partial_x u) \partial_t [(\partial_x u)f] dx = ||f \partial_x u||_{L^p}^{p-1} \frac{d}{dt} ||f \partial_x u||_{L^p},
\]

and then by \( |uf|^{p-2}(uf) \) we get, after integration,
and
\[
\left| \int_\mathbb{R} |f \partial_x u|^{p-2} (f \partial_x u) [f \partial_x (G * E(u))] dx \right|
\leq \||f \partial_x u||_{L^p}^{p-1} ||f \partial_x (G * E(u))||_{L^p}
\leq CM^{m+1} ||f \partial_x u||_{L^p}^{p-1} (||uf||_{L^p} + ||(\partial_x u)f||_{L^p})
\]

In the third inequality we applied the pointwise bound $|\partial_x G(x)| \leq \frac{1}{2} e^{-|x|}$ and the condition.

\[
\left| \int_\mathbb{R} |f \partial_x u|^{p-2} (f \partial_x u) u^{m+1} |\partial_x|^2 u dx \right|
= \left| \int_\mathbb{R} |f \partial_x u|^{p-2} (f \partial_x u) u^{m+1} (\partial_x f) - (\partial_x u)(\partial_x f) dx \right|
= \int_\mathbb{R} u^{m+1} \partial_x \left( \frac{|\partial_x u|^p}{p} \right) - \int_\mathbb{R} |f \partial_x u|^{p-2} (f \partial_x u) u^{m+1} (\partial_x f) dx
\leq 2/pM^{m+1} ||f \partial_x u||_{L^p}^{p} + AM^{m+1} ||f \partial_x u||_{L^p}^{p}
\]

In the last inequality we used $|\partial_x f(x)| \leq Af(x)$ for a.e. $x$. Thus, we get

\[
\frac{d}{dt} ||f \partial_x u||_{L^p} \leq C_3 M^{m+1} ||uf||_{L^p} + C_4 M^{m+1} ||(\partial_x u)f||_{L^p}.
\tag{4.4}
\]

Now, combining the inequalities (4.3) with (4.4) and then integrating yields,

\[
||u(t)f||_{L^p} + ||(\partial_x u)(t)f||_{L^p} \leq (||u_0f||_{L^p} + ||\partial_x u_0f||_{L^p}) \exp(CM^{m+1}t)
\]

for all $t \in [0,T]$.

Since $f(x) = f_N(x) \uparrow \phi(x)$ as $N \to \infty$ for a.e. $x \in \mathbb{R}$. Recall that $u_0 \phi \in L^p(\mathbb{R})$ and $\partial_x u_0 \phi \in L^p(\mathbb{R})$, we get

\[
||u(t)\phi||_{L^p} + ||(\partial_x u)(t)\phi||_{L^p} \leq (||u_0\phi||_{L^p} + ||\partial_x u_0\phi||_{L^p}) \exp(CM^{m+1}t)
\]

for all $t \in [0,T]$.

At last, we treat the case $p = \infty$. We have $u_0, \partial_x u_0 \in L^2 \cap L^\infty$ and $f(x) = f_N(x) \in L^\infty$. Hence, we have

\[
||u(t)f||_{L^\infty} + ||(\partial_x u)(t)f||_{L^\infty} \leq (||u_0f||_{L^\infty} + ||\partial_x u_0f||_{L^\infty}) \exp(CM^{m+1}t), q \in [2,\infty).
\tag{4.5}
\]

The last factor in the right-hand side is independent on $q$. Since $||f||_{L^p} \to ||f||_{L^\infty}$ as $p \to \infty$ for any $f \in L^\infty \cap L^2$, implies that

\[
||u(t)f||_{L^\infty} + ||(\partial_x u)(t)f||_{L^\infty} \leq (||u_0f||_{L^\infty} + ||\partial_x u_0f||_{L^\infty}) \exp(CM^{m+1}t).
\]

The last factor in the right-hand side is independent on $N$. Now taking $N \to \infty$ implies that estimate (4.5) remains valid for $p = \infty$.

\[\square\]

Proof of Theorem 1.6. In [4], the author consider the case $m = 0, b = 2$. So, We only need to deal with the case $m \in Z^+$. Arguing as in the proof of Theorem 1.5, we can easy get,

\[
\frac{d}{dt} ||uf||_{L^p} \leq M^{m+1} ||uf||_{L^p} + ||f(G * E(u))||_{L^p}, \text{ for } p < \infty,
\tag{4.6}
\]

and

\[
\frac{d}{dt} ||f \partial_x u||_{L^p} \leq CM^{m+1} ||(\partial_x u)f||_{L^p} + ||f(\partial_x G * E(u))||_{L^p}, \text{ for } p < \infty,
\tag{4.7}
\]
implies that $v^\frac{1}{m+2}$-moderate weight such that $(\phi^\frac{1}{m+2})'(x) \leq \frac{A}{m+2} \phi^\frac{1}{m+2}(x)$. Moreover, $\inf_{\mathbb{R}} v^{\frac{1}{m+2}} > 0$. By condition (1.7), $v^{\frac{1}{m+2}} e^{-|x|/(m+2)} \in L^{m+2}(\mathbb{R})$, hence Hölder’s inequality implies that $v^{\frac{1}{m+2}} e^{-|x|} \in L^1(\mathbb{R})$. Then Theorem 1.5 applies with $p = m + 2$ to the weight $\phi^{\frac{1}{m+2}}$ yielding

$$||u(t)\phi^{\frac{1}{m+2}}||_{L^{m+2}} + ||(\partial_x u)(t)\phi^{\frac{1}{m+2}}||_{L^{m+2}} \leq \left(||u_0\phi^{\frac{1}{m+2}}||_{L^{m+2}} + ||\partial_x u_0\phi^{\frac{1}{m+2}}||_{L^{m+2}}\right) \exp(C M^{m+1} t).$$

Therefore

$$||f(G \ast E(u))||_{L^p} \leq C(||\phi^{\frac{1}{m+2}} u||_{L^{m+2}}^m + ||\phi u^m(\partial_x u)^2||_{L^1} + ||\phi u^{m-1}(\partial_x u)^3||_{L^1})$$

$$\leq C(||\phi^{\frac{1}{m+2}} u||_{L^{m+2}}^m + ||\phi^\frac{1}{m+2} u^m||_{L^{m+2}}^m + ||\phi^\frac{1}{m+2} (\partial_x u)^2||_{L^{m+2}}^m$$

$$+ ||\phi^\frac{1}{m+2} u^{m-1}||_{L^{m+2}}^m + ||\phi^{\frac{1}{m+2}} (\partial_x u)^3||_{L^{m+2}}^m)$$

$$\leq C(||\phi^{\frac{1}{m+2}} u||_{L^{m+2}}^m + ||\phi^{\frac{1}{m+2}} u||_{L^{m+2}}^m + ||\phi^{\frac{1}{m+2}} (\partial_x u)^2||_{L^{m+2}}^m + ||\phi^{\frac{1}{m+2}} (\partial_x u)^3||_{L^{m+2}}^m)$$

$$\leq C_0 \exp((m+2) CM^{m+1} t),$$

where the constants $C_0$ depending only on $\phi, b, m$ and the initial data.

Similarly, recalling that $\partial_x G \leq \frac{1}{2} e^{-|x|}$ and $\partial_x^2 G = G - \delta$, $\partial_x G \leq E(u)$

$$||f(\partial_x G \ast E(u))||_{L^p} \leq c||f(G \ast (u^{m+2} + u^m(\partial_x u)^2))||_{L^p} + c||f(u^{m+2} + u^m(\partial_x u)^2)||_{L^p} + c||f(\partial_x G \ast (u^{m-1}(\partial_x u)^3))||_{L^p}$$

$$\leq C_1 \exp((m+2) CM^{m+1} t) + CM^{m+1} (||uf||_{L^p} + ||(\partial_x u)f||_{L^p}).$$

Plugging the two last estimates in (4.6)-(4.7), and summing we obtain

$$\frac{d}{dt} (||u(t)f||_{L^p} + ||(\partial_x u)(t)f||_{L^p}) \leq K_1 M^{m+1} (||u_0 f||_{L^p} + ||\partial_x u_0 f||_{L^p})$$

$$+ (m+1) K_0 \exp((m+2) CM^{m+1} t).$$

Integrating and finally letting $N \to \infty$ yields the conclusion in the case $2 \leq p < \infty$. The rest argument is fully similar to that of Theorem 1.5.

**Proof of Theorem 1.7.** Conservation of (1.9) follows from Theorem 1.5 with $p = \infty$ and $\phi(x) = e^{x/(m+2)}(1 + |x|)^{1/(m+2)} \log(e + |x|)^{d}$. Integration of (4.1) yields

$$u(x, t) = u_0(x) - \int_0^t u^{m+1} \partial_x u(x, s)ds - \int_0^t G \ast E(u)(x, s)ds. \quad (4.8)$$

Using the fact that condition in Theorem 1.5 is conserved uniformly in $[0, T]$, we get

$$\int_0^t u^{m+1} \partial_x u(x, s)ds \leq C te^{-|x|}(1 + |x|)^{-1} \log(e + |x|)^{(m+2)d}.$$
Next, we show that the last term in (4.8) can be included inside the lower order terms of the asymptotic profiles (1.11). The assumption $m = 0$ or $b = m + 1$ ensures the validity of the following equality:

$$
\int_0^t (G * E(u))(x, \tau) d\tau = \int_0^t \partial_x G * \left( \frac{b}{m+2} u^{m+2} + \frac{3m+3-b}{2} u^m (\partial_x u)^2 \right) d\tau
$$

with $f(x, t) = \frac{1}{t} \int_0^t \left( \frac{b}{m+2} u^{m+2} + \frac{3m+3-b}{2} u^m (\partial_x u)^2 \right) ds$ for $t \in (0, T]$. It is easy to see that the function $(1 + |\cdot|)^{-1/(m+2)} \log(e + |\cdot|)^{-d}$ belongs to $L^{m+2}(\mathbb{R})$. From the given condition and Theorem 1.6 with $p = m + 2$ and $\phi(x) = e|x|/(m+2)$, we know $\int_{-\infty}^\infty e^{y|f(y,t)|} dy < \infty$.

Now, we denote

$$
\Phi = \frac{1}{2} \int_{-\infty}^{\infty} e^y f(y, t) dy, \quad \Psi = \frac{1}{2} \int_{-\infty}^{\infty} e^{-y} f(y, t) dy.
$$

By the arguments in [4], this achieves the asymptotic representation of $u$.

**Acknowledgments.** The authors are very grateful to the anonymous reviewers for their careful read and useful suggestions, which greatly improved the presentation of the paper.

**REFERENCES**

[1] A. Aldroubi and K. Gröchenig, Nonuniform sampling and reconstruction in shift-invariant spaces, *SIAM Rev.*, **43** (2001), 585–620.

[2] R. Beals, D. Sattinger and J. Szmigielski, Acoustic scattering and the extended Korteweg-de Vries hierarchy, *Adv. Math.*, **140** (1998), 190–206.

[3] A. Bressan and A. Constantin, Global conservative solutions of the Camassa-Holm equation, *Arch. Rat. Mech. Anal.*, **183** (2007), 215–239.

[4] L. Brandolese, Breakdown for the Camassa-Holm equation using decay criteria and persistence in weighted spaces, *Int. Math. Res. Not.*, **22** (2012), 5161–5181.

[5] A. Boutet de Monvel and D. Shepelsky, Riemann-Hilbert approach for the Camassa-Holm equation on the line, *C. R. Math. Acad. Sci. Paris*, **343** (2006), 627–632.

[6] R. Camassa and D. Holm, An integrable shallow water equation with peaked solitons, *Phys. Rev. Letters*, **71** (1993), 1661–1664.

[7] R. Camassa, D. Holm and J. Hyman, A new integrable shallow water equation, *Adv. Appl. Mech.*, **31** (1994), 1–33.

[8] G. M. Coclite and K. H. Karlsen, On the well-posedness of the Degasperis-Procesi equation, *J. Funct. Anal.*, **233** (2006), 60–91.

[9] A. Constantin, On the inverse spectral problem for the Camassa-Holm equation, *J. Funct. Anal.*, **155** (1998), 352–363.

[10] A. Constantin, Existence of permanent and breaking waves for a shallow water equation: A geometric approach, *Ann. Inst. Fourier (Grenoble)*, **50** (2000), 321–362.

[11] A. Constantin, On the scattering problem for the Camassa-Holm equation, *Proc. Roy. Soc. London*, **457** (2001), 953–970.

[12] A. Constantin and J. Escher, Global existence and blow-up for a shallow water equation, *Ann. Scuola Norm. Super. Pisa-Cl. Sci.*, **26** (1998), 303–328.

[13] A. Constantin and J. Escher, Wave breaking for nonlinear nonlocal shallow water equations, *Acta Mathematica*, **181** (1998), 229–243.

[14] A. Constantin and J. Escher, Global weak solutions for a shallow water equation, *Indiana. Univ. Math. J.*, **47** (1998), 1527–1545.

[15] A. Constantin and J. Escher, Analyticity of periodic traveling free surface water waves with vorticity, *Ann. of Math.*, **173** (2011), 559–568.

[16] A. Constantin, V. Gerdjikov and R. Ivanov, Inverse scattering transform for the Camassa-Holm equation, *Inverse Problems*, **22** (2006), 2197–2207.
THE CAUCHY PROBLEM FOR A GENERALIZED $b$-EQUATION ... 4985

[17] A. Constantin and R. Ivanov, On an integrable two-component Camassa-Holm shallow water system, Phys. Lett. A, 372 (2008), 7129–7132.
[18] A. Constantin, R. I. Ivanov and J. Lenells, Inverse scattering transform for the Degasperis-Procesi equation, Nonlinearity, 23 (2010), 2559–2575.
[19] A. Constantin and D. Lannes, The hydrodynamical relevance of the Camassa-Holm and Degasperis-Procesi equations, Arch. Ration. Mech. Anal., 192 (2009), 165–186.
[20] A. Constantin and L. Molinet, Global weak solutions for a shallow water equation, Comm. Math. Phys., 211 (2000), 45–61.
[21] A. Constantin and W. A. Strauss, Stability of peakons, Comm. Pure Appl. Math., 53 (2000), 603–610.
[22] A. Constantin and W. A. Strauss, Stability of the Camassa-Holm solitons, J. Nonlinear. Sci., 12 (2002), 415–422.
[23] R. Danchin, A few remarks on the Camassa-Holm equation, Differential Integral Equations, 14 (2001), 953–988.
[24] R. Danchin, A note on well-posedness for Camassa-Holm equation, J. Differential Equations, 192 (2003), 429–444.
[25] R. Danchin, Fourier Analysis Methods for PDEs, Lecture Notes, 14 November, 2003.
[26] A. Degasperis and M. Procesi, Asymptotic integrability, in: Symmetry and Perturbation Theory, World Scientific, Singapore, (1999), 23–37.
[27] A. Degasperis, D. Holm and A. Hone, A new integral equation with peakon solutions, Theoret. Math. Phys., 133 (2002), 1463–1474.
[28] A. Degasperis, D. D. Holm and A. N. W. Hone, Integral and non-integrable equations with peakons, Nonlinear physics: Theory and experiment, II (Gallipoli 2002), World Sci. Publ., River Edge, NJ, (2003), 37–43.
[29] H. R. Dullin, G. A. Gottwald and D. D. Holm, Camassa-Holm, Korteweg-de Vries-5 and other asymptotically equivalent equations for shallow water waves, Fluid Dyn. Res., 33 (2003), 73–95.
[30] H. R. Dullin, G. A. Gottwald and D. D. Holm, On asymptotically equivalent shallow water wave equations, Phys. D., 190 (2004), 1–14.
[31] H. R. Dullin, G. A. Gottwald and D. D. Holm, An integrable shallow water equation with linear and nonlinear dispersion, Phys. Rev. Letters, 87 (2001), 4501–4504.
[32] J. Escher, Y. Liu and Z. Y. Yin, Global weak solutions and blow-up structure for the Degasperis-Procesi equation, J. Funct. Anal., 241 (2006), 457–485.
[33] J. Escher, Y. Liu, Z. Yin, Shock waves and blow-up phenomena for the periodic Degasperis-Procesi equation, Indiana Univ. Math. J., 56 (2007), 87–117.
[34] J. Escher and Z. Yin, Well-posedness, blow-up phenomena, and global solutions for the $b$-equation, J. reine Angew. Math., 624 (2008), 51–80.
[35] A. Fokas and B. Fuchssteiner, Symplectic structures, their Bäcklund transformation and hereditary symmetries, Phys. D., 4 (1981/82), 47–66.
[36] K. Gröchenig Weight functions in time-frequency analysis, Pseudo-differential operators: Partial differential equations and time-frequency analysis, Fields Inst. Commun., Amer. Math. Soc., Providence, RI, 52 (2007), 343–366.
[37] G. L. Gui, Y. Liu and T. X. Tian, Global existence and blow-up phenomena for the peakon $b$-family of equations, Indiana Univ. Math. J., 57 (2008), 1209–1234.
[38] D. Henry, Persistence properties for a family of nonlinear partial differential equations, Nonlinear Anal., 70 (2009), 1565–1573.
[39] A. Himonas and C. Holliman, The Cauchy problem for the Novikov equation, Nonlinearity, 25 (2012), 449–479.
[40] A. A. Himonas, G. Misiolek, G. Ponce and Y. Zhou, Persistence properties and unique continuation of solutions of the Camassa-Holm equation, Comm. Math. Phy., 271 (2007), 511–522.
[41] D. D. Holm and M. F. Staley, Wave structure and nonlinear balances in a family of evolutionary PDEs, SIAM J. Appl. Dyn. Syst., 2 (2003), 323–380.
[42] D. D. Holm and M. F. Staley, Nonlinear balance and exchange of stability in dynamics of solitons, peakons, ramps/cliffs and leftons in a 1-1 nonlinear evolutionary PDE, Phys. Lett. A, 308 (2003), 437–444.
[43] A. N. W. Hone and J. P. Wang, Integrable peakon equations with cubic nonlinearity, J. Phys. Appl. Math. Theor., 41 (2008), 372002, 10pp.
[44] A. N. W. Hone, H. Lundmark and J. Szmigielski, Explicit multipeakon solutions of Novikov’s cubically nonlinear integrable Camassa-Holm equation, Dyn. Partial Differ. Equ., 6 (2009), 253–289.
[45] R. I. Ivanov, Water waves and integrability, Philos. Trans. Roy. Soc. London A, 365 (2007), 2267–2280.
[46] R. Ivanov, Extended Camassa-Holm hierarchy and conserved quantities, Z. Naturforsch. A, 61 (2006), 133–138.
[47] Z. H. Jiang and L. D. Ni, Blow-up phenomena for the integrable Novikov equation, J. Math. Anal. Appl., 385 (2012), 551–558.
[48] K. Grayshan, Peakon solutions of the Novikov equation and properties of the data-to-solution map, J. Math. Anal. Appl., 397 (2013), 515–521.
[49] S. Y. Lai, N. Li and Y. H. Wu, The existence of global strong and weak solutions for the Novikov equation, J. Math. Anal. Appl., 399 (2013), 682–691.
[50] J. Lenells, Conservation laws of the Camassa-Holm equation J. Phys. (A), 38 (2005), 869–880.
[51] Y. A. Li and P. J. Olver, Well-posedness and blow-up solutions for an integrable nonlinearly dispersive model wave equation, J. Diff. Equ., 162 (2000), 27–63.
[52] N. Li, S. Y. Lai, S. Li and M. Wu, The local and global existence of solutions for a generalized Camassa-Holm equation, Abstr. Appl. Anal., (2012), 26 pp.
[53] Y. Liu and Z. Yin, Global existence and blow-up phenomena for the Degasperis-Procesi equation, Comm. Math. Phys., 267 (2006), 801–820.
[54] H. Lundmark and J. Szmigielski, Multi-peakon solutions of the Degasperis-Procesi equation, Inverse Problems, 19 (2003), 1241–1245.
[55] A. V. Mikhailov and V. S. Novikov, Perturbative symmetry approach, J. Phys. (A), 35 (2002), 4775–4790.
[56] L. D. Ni and Y. Zhou, Well-posedness and persistence properties for the Novikov equation, J. Differential Equations, 250 (2011), 3002–3021.
[57] L. D. Ni and Y. Zhou, A new asymptotic behavior of solutions to the Camassa-holm equation, Proc. Amer. Math. Soc., 140 (2012), 607–614.
[58] W. Niu and S. Zhang, Blow-up phenomena and global existence for the nonuniform weakly dispersive b-equation, J. Math. Anal. Appl., 374 (2011), 166–177.
[59] V. S. Novikov, Generalizations of the Camassa-Holm equation, J. Phys. A, 42 (2009), 342002 (14pp).
[60] F. Tiglay, The periodic cauchy problem for novikov’s equation, Int. Math. Res. Not., 20 (2011), 4633–4648.
[61] V. O. Vakhnenko and E. J. Parkes, Periodic and solitary-wave solutions of the Degasperis-Procesi equation, Chaos Solitons Fractals, 20 (2004), 1059–1073.
[62] X. L. Wu and Z. Y. Yin, A note on the Cauchy problem of the Novikov equation, Appl. Anal., 92 (2013), 1116–1137.
[63] S. Y. Wu and Z. Y. Yin, Global weak solutions for the Novikov equation, J. Phys. A: Math. Theor., 44 (2011), 055202 (17pp).
[64] Z. P. Xin and P. Zhang, On the weak solutions to a shallow water equation, Comm. Pure Appl. Math., 53 (2000), 1411–1433.
[65] W. Yan, Y. Li and Y. Zhang, The Cauchy problem for the integrable Novikov equation, J. Differential Equations, 253 (2012), 298–318.
[66] Z. Y. Yin, Global solutions to a new integrable equation with peakons, Indiana. Univ. Math. J., 53 (2004), 1189–1209.
[67] Z. Yin, On the Cauchy problem for an integrable equation with peakon solutions, Illinois J. Math., 47 (2003), 649–666.
[68] W. Yan, Y. S. Li and Y. M. Zhang, The cauchy problem for the Novikov equation, Nonlinear Differ. Equ. Appl., 20 (2013), 1157–1169.
[69] S. M. Zhou and C. L. Mu, The properties of solutions for a generalized b-family equation with higher-order nonlinearities and peakons, J. Nonlinear Sci., 23 (2013), 863–889.
[70] S. M. Zhou, C. L. Mu and L. C. Wang, Well-posedness, blow-up phenomena and global existence for the generalized b-equation with higher-order nonlinearities and weak dissipation, Discrete Contin. Dyn. Syst. Ser. A, 34 (2014), 843–867.

Received August 2013; revised March 2014.

E-mail address: zhoushouming76@163.com