Exact solution for scalar field collapse

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Abstract

We give an exact spherically symmetric solution for the Einstein-scalar field system. The solution may be interpreted as an inhomogeneous dynamical scalar field cosmology. The spacetime has a timelike conformal Killing vector field and is asymptotically conformally flat. It also has black or white hole-like regions containing trapped surfaces. We describe the properties of the apparent horizon and comment on the relevance of the solution to the recently discovered critical behaviour in scalar field collapse.

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The gravitational collapse of distributions of matter is one of the most important research areas in general relativity. The essential question posed is whether and under what initial conditions a black hole or naked singularity forms in the collapse. One of the aims of studying such problems is to test the cosmic censorship conjecture [1], one form of which states that gravitational collapse produces black holes.

A model system for studying this question is provided by the Einstein equations minimally coupled to a massless scalar field. While this full system appears to be intractable, the simplified set of equations obtained by imposing spherical symmetry is easier to handle. Without any matter fields, the spherically symmetric metric does not contain any field degrees of freedom. Therefore, with a scalar field, the system is effectively a two dimensional field theory and it can be described by a single two dimensional nonlinear differential equation

\(2,3\).

There are a number of exact solutions known for this system, almost all of which are either static or depend only upon the time coordinate [4,5]. The first non-static solutions have been given by Roberts [6] (which are different from the one we give below). The equations have been studied in detail by Christodoulou [2] who established, among other things, that there exist regular solutions for arbitrarily long times for particular types of initial data.

The model has also been studied numerically and there are a number of interesting numerical results. The first results obtained by Goldwirth and Piran [7] indicated that there is a class of initial data that leads to black hole formation. More recently it has been shown by Choptuik [8] that, for large classes of initial data, there is critical behaviour at the onset of black hole formation: the black hole mass \(M_{BH}\) is given by the equation

\[M_{BH} = K|c - c_*|^\gamma,\]

where \(K\) is a constant, \(c\) is any one of the parameters in the initial data for the scalar field, \(c_*\) is a critical value of the parameter, and \(\gamma \sim .37\) is a universal exponent. The remarkable feature of this result is that \(\gamma\) appears to be independent of a set of particular shapes of the initial data, and is universal in this sense. It has been shown by Abraham and Evans [9] that the same mass equation is obtained for the axisymmetric
collapse of gravitational radiation. Thus the critical behaviour appears to be independent
of not only the type of matter fields, but also the symmetries of the system.

It would be very useful to understand the universality of this result analytically. A
modest approach is to attempt to find an exact solution describing scalar field collapse and
to see if one can read off the critical behaviour by calculating the mass of the black hole.

Here we describe an exact solution for scalar field collapse and discuss some of its prop-
erties. While the solution we present does not describe a realistic collapse corresponding
to the classes of initial data used in the numerical work mentioned above, it appears to be
among the few exact non-static solutions known for this system.

The Einstein-scalar field equations we consider (in units $G = c = 1$) are

$$G_{\mu\nu} = 8\pi T_{\mu\nu}; \quad T_{\mu\nu} = \phi_\mu \phi_\nu - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \phi_\alpha \phi_\beta. \quad (1)$$

which may be written in the form

$$R_{\mu\nu} = 8\pi \phi_\mu \phi_\nu, \quad (2)$$

(where the subscript on $\phi$ denotes partial differentiation).

The spherically symmetric solution we obtain is

$$ds^2 = (at + b) (-f^2(r)dt^2 + f^{-2}(r)dr^2) + R^2(r, t)(d\theta^2 + \sin^2 \theta d\phi^2), \quad (3)$$

where

$$f^2(r) = (1 - 2c/r)^\alpha$$

$$R^2(r, t) = (at + b)r^2(1 - 2c/r)^{1-\alpha}. \quad (4)$$

The scalar field is

$$\phi(r, t) = \pm \frac{1}{4 \sqrt{\pi}} \ln[d (1 - 2c/r)^{\frac{\alpha}{3}} (at + b)^{\frac{2}{\sqrt{3}}}], \quad (5)$$

where $a, b, c, d$ are constants,

$$\alpha = \pm \sqrt{3}/2,$$ and the overall sign of $\phi$ is independent
of \( \alpha \). The coordinate ranges and the values of the constants are coupled (for the metric to be Lorentzian): \(-b/a \leq t \leq \infty\) and \(2c \leq r \leq \infty\) \((c > 0)\).

We note that for \( a \neq 0 \) there is a coordinate transformation, that eliminates the constant \( b \). However, since we will also be interested in the metrics for which \( a = 0 \), it is useful to keep the general form (3). We note also that when \( b = 0 \), there are two ranges for \( t \), namely \(0 \leq t \leq \infty\) for \( a > 0 \), and \(-\infty \leq t \leq 0\) for \( a < 0 \). As shown below these correspond to white and black hole-like solutions respectively, and simply reflect the choice of the arrow of time.

When \( a \neq 0 \), the coordinate transformation \( T = at + b \) does not eliminate \( a \) in the metric, which is an overall scale. The constant \( d \) in the scalar field is a trivial additive constant.

The only Killing vectors of the metric (3) are the three associated with spherical symmetry. The metric also has the conformal Killing vector field \( V = \partial/\partial t \) such that

\[
\mathcal{L}_V g_{\mu\nu} = \frac{a}{at + b} g_{\mu\nu}.
\]

Asymptotically \((r \rightarrow \infty)\), the metric is conformal to the Minkowski metric. The locus of points at \( r = 2 \) is a timelike curvature

singularity, and so the ‘horizon’ is shrunk to a point as in the static metric given by Janis, Newman and Winicour (JNW), and others \([4,5]\). There is also a spacelike singularity at \( t = -b/a \). The solution may be interpreted as a scalar field cosmology, since it is not asymptotically flat.

For the JNW metric, the functions \( f(r) \) have arbitrary exponents specified by integration constants rather than the fixed \( \alpha = \pm \sqrt{3}/2 \) above. Thus our metric is conformal to the JNW metric with this exponent fixed, and with \((at + b)\) as the conformal factor. In fact for \( a = 0 \) we recover one of the JNW metrics. Therefore \( a \) distinguishes static from non-static metrics. This is similar to the parameter \( k \) in

Friedman-Robertson-Walker cosmologies, which discretely distinguishes the spatial curvatures of the metrics.

The case \( c = 0 \) gives a homogeneous time dependent solution. This is equivalent to the
$r \to \infty$ limit of (3). The parameter $c$ therefore distinguishes homogeneous from inhomogeneous time dependent solutions.

For comparison with recent numerical work [8,9], where the scalar field and its time derivative are specified as part of the initial data on a spacelike hypersurface, we note that this data for our solution is

$$
\phi(r, t = t_0) = \frac{1}{4} \sqrt{\frac{\alpha}{\pi}} \ln \left( d \left( 1 - \frac{2c}{r} \right) \sqrt{\frac{\alpha}{3}} \left( at_0 + d \right)^{\pm \sqrt{3}} \right),
$$
$$
\dot{\phi}(r, t = t_0) = \frac{1}{4(\alpha t_0 + b)} \sqrt{\frac{3}{\pi}}.
$$

(7)

The asymptotic behaviour of $\phi$ is

$$
\phi(r = \infty, t = t_0) = \frac{1}{4} \sqrt{\frac{\alpha}{\pi}} \ln \left( d \left( at_0 + b \right)^{\pm \sqrt{3}} \right).
$$

(8)

These are not the initial data associated with the standard collapse situation [8,9], where the data is typically an ingoing pulse with a specified amplitude and width.

The Ricci scalar is

$$
\mathcal{R} = \frac{12ca^2(r - c) - 3a^2r^2}{2r^2(at + b)^3} \left( 1 - \frac{2c}{r} \right)^{-2-\alpha} + \frac{2c^2(1 - \alpha^2)}{(at + b)r^4} \left( 1 - \frac{2c}{r} \right)^{-2+\alpha}
$$

(9)

which shows that curvature singularities are present at $r = 2c$, and at $t = -b/a$.

Since the metric is not static, it is of interest to investigate the existence and properties of the apparent horizon and to calculate the mass function. The apparent horizon is the 3-surface on which outgoing or ingoing null rays are momentarily stationary. The presence of the horizon for asymptotically flat spacetimes indicates that there is a black or white hole, and a measure of the mass within it is given by the mass function evaluated at the horizon. This measure of the black hole mass has been used in recent numerical work on the collapse of ingoing matter pulses [8,9]. Although the metric (3) is not asymptotically flat, we can nevertheless see whether there are horizons.

The apparent horizon surface is given by
\[
g^{\alpha\beta} R_{\alpha\beta} = 0, \tag{10}
\]

which for our metric gives the equation

\[
\frac{a}{at_{AH} + b} = \frac{2}{r^2} [r - c(1 + \alpha)](1 - 2c/r)^{\alpha-1} \tag{11}
\]

This equation has no non-trivial solution for \(a = 0\) which corresponds to the static JNW metric, whereas for \(a \neq 0\) there is always an evolving apparent horizon.

The apparent horizon can in general be spacelike, null or timelike in different spacetime regions. This is easily determined by calculating the ratio of the slopes of the apparent horizon and the outgoing null ray. This ratio for our metric (with \(a \neq 0\)) is

\[
\frac{t_{AH,r}}{t_{N,r}} = 1 - \frac{(1 - 2c/r)}{2[1 - c(1 + \alpha)/r]^2}. \tag{12}
\]

The second term on the right hand side is always positive, therefore the apparent horizon is spacelike for all \(r > 2c\). It is null only at \(r = 2c\).

The scalar field is not singular at the apparent horizon.

Figures 1 and 2 are plots of the horizon for \(\alpha = \pm\sqrt{3}/2\) and \(c = 1, b = 0\). The main features remain unaltered for all values of \(b, c\). The horizon forms at \(r = 2c, t = 0\) and grows in size forever. For \(\alpha = \sqrt{3}/2\), the light cones are collapsed to a vertical line at \(r = 2c\) and open up to a slope of \(\pm 1/b^2\) as \(r \to \infty\), whereas for \(\alpha = -\sqrt{3}/2\) the cones collapse to a horizontal line at \(r = 2c\).

It is of interest to note a number of other features of the apparent horizon. By computing the expansions of the spacelike symmetry 2-spheres

\[
ds^2 = R^2(r, t)(d\theta^2 + \sin^2\theta d\phi^2) \tag{13}
\]

along future pointing null directions orthogonal to the spheres, one can determine whether the apparent horizon is past or future, and inner or outer, and what region is trapped (see [10] for a general discussion of apparent horizons).

The expansions \(\theta_{\pm}\) of the area 2-form \(\omega = R^2(r, t) \sin \theta \, d\theta \wedge d\phi\) of the 2-spheres are defined by
\[ \mathcal{L}_{l_{\pm}}\omega = \theta_{\pm}\omega, \]  

(14)

where \( \mathcal{L} \) denotes the Lie derivative and

\[ l_+ := \frac{\partial}{\partial t} + f^2 \frac{\partial}{\partial r} \quad l_- := \frac{\partial}{\partial t} - f^2 \frac{\partial}{\partial r} \]  

(15)

are the outgoing and ingoing future pointing null directions. With \( b = 0 \) and \( a > 0 \), (so that \( 0 \leq t \leq \infty \)), we find

\[ \theta_{\pm} = \frac{1}{t} \pm \frac{1}{t_{AH}}, \]  

(16)

with \( t_{AH} \) as given in Eq. (11). Thus it is the ingoing expansion \( \theta_- \) that vanishes at the apparent horizon, while the outgoing expansion is \( \theta_+ = 2/t_{AH} > 0 \). This implies that the horizon is a past horizon. For a given value of \( r \), the symmetry 2-spheres are trapped surfaces for \( t < t_{AH} \), since this is the region where both the ingoing and outgoing light rays have positive expansions. Similarly, the region \( t > t_{AH} \) is a normal region where the outgoing light expansion (\( \theta_+ \)) is positive and the ingoing one (\( \theta_- \)) is negative.

We note also that if \( \mathcal{L}_{l_{+}}\theta_{-}|_{AH} < 0 \) the horizon is an outer one (otherwise it is inner). For the metric (3), we find

\[ \mathcal{L}_{l_{+}}\theta_{-}|_{AH} = \frac{1}{t_{AH}^3} \left[ \frac{t_{AH,r}}{t_{N,r}} - 1 \right]. \]  

(17)

From (12) it follows that this is always less than zero except at the singularity where it is zero.

Summarizing the above results, the apparent horizon is a past outer one and is spatial everywhere except at \( r = 2c \), where it is null. The scalar field flows from the past trapped region \( t < t_{AH} \) (white hole) into the untrapped region \( t > t_{AH} \). An observer in the untrapped region sees both the \( t = 0 \) initial singularity and the one at \( r = 2c \).

When \( b = 0 \), there is a change for the time reversed case which corresponds to \( a < 0 \) and \(-\infty \leq t \leq 0 \). The horizon is now a future outer one and corresponds to a black hole situation. The future singularity at \( t = 0 \) is covered by the spacelike horizon and the region is a black hole.
We now turn to a discussion of the mass function defined by

\[ m(r, t) = \frac{R}{2} (1 - g^{\alpha\beta} R_{\alpha \beta}). \]  

(18)

This function for static or stationary asymptotically flat spacetimes gives the ADM mass in the asymptotic limit. Its value on the apparent horizon is obtained by substituting \( t_{AH}(r) \) from (11) into (18):

\[ M := \frac{R_{AH}}{2} = m(r, t)|_{AH} = \sqrt{\left| a \right| \frac{r^2 (1 - 2 c/r)^{1-\alpha}}{8 \sqrt{r - c (1 + \alpha)}}}. \]  

(19)

The mass \( M \) is zero at \( r = 2c \) and grows as \( r^{3/2} \) for large \( r \). We note that it is independent of \( b \). For \( a = 0 \), \( M = 0 \), which corresponds to the static JNW solution. This corresponds to the fact that there is no solution to the apparent horizon equation (11) for \( a = 0 \).

Superficially, the mass function may be viewed as describing a form of critical behaviour since for fixed \( r \) we can write \( M = constant (a - a_*)^{1/2} \), with the ‘critical’ value \( a_* = 0 \), and with one half as the ‘exponent’. Similarly, one can do this for the parameter \( c \) in (19), which gives a different exponent. This, however, does not shed light on the collapse of initial pulses of scalar field, since the initial data (7) does not correspond to this situation.

The metric is nevertheless an exact solution and may be viewed as a model in which the formation and evolution of an apparent horizon can be studied exactly. The value of the parameter \( a \) in the solution determines whether an apparent horizon forms. It would be of much interest to find an exact solution corresponding to a realistic collapse, as this would provide an analytical model for viewing the scaling and critical behaviour obtained in recent numerical work [8,9].

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FIGURE CAPTIONS

Figure 1: The apparent horizon for $\alpha = +\frac{\sqrt{3}}{2}$ and $c = 1, b = 0$. The light cones illustrate the spacelike character of the horizon. The singularities are at $r = 2c$ and $t = 0$. The trapped region is $t < t_{AH}$.

Figure 2: The apparent horizon for $\alpha = -\frac{\sqrt{3}}{2}$ and $c = 1, b = 0$. 
This figure "fig1-1.png" is available in "png" format from:

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