A Geometric Approach to Sample Compression

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**Abstract**

The Sample Compression Conjecture of Littlestone & Warmuth has remained unsolved for over two decades. While maximum classes (concept classes meeting Sauer’s Lemma with equality) can be compressed, the compression of general concept classes reduces to compressing maximal classes (classes that cannot be expanded without increasing VC-dimension). Two promising ways forward are: embedding maximal classes into maximum classes with at most a polynomial increase to VC dimension, and compression via operating on geometric representations. This paper presents positive results on the latter approach and a first negative result on the former, through a systematic investigation of finite maximum classes. Simple arrangements of hyperplanes in Hyperbolic space are shown to represent maximum classes, generalizing the corresponding Euclidean result. We show that sweeping a generic hyperplane across such arrangements forms an unlabeled compression scheme of size VC dimension and corresponds to a special case of peeling the one-inclusion graph, resolving a recent conjecture of Kuzmin & Warmuth. A bijection between finite maximum classes and certain arrangements of Piecewise-Linear (PL) hyperplanes in either a ball or Euclidean space is established. Finally we show that PL arrangements can be swept by a moving hyperplane to unlabeled $d$-compress any finite maximum class, forming a peeling scheme as conjectured by Kuzmin & Warmuth. A corollary is that some $d$-maximal classes cannot be embedded into any maximum class of VC dimension $d+k$, for any constant $k$. The construction of the PL sweeping involves Pachner moves on the one-inclusion graph, corresponding to moves of a hyperplane across the intersection of $d$ other hyperplanes. This extends the well known Pachner moves for triangulations to cubical complexes.

**Keywords:** Sample Compression, Hyperplane Arrangements, Hyperbolic and Piecewise-Linear Geometry, One-Inclusion Graphs

1. Introduction

*Maximum* concept classes have the largest cardinality possible for their given VC dimension. Such classes are of particular interest as their special recursive structure underlies all general
sample compression schemes known to-date (Floyd, 1989; Warmuth, 2003; Kuzmin and Warmuth, 2007). It is this structure that admits many elegant geometric and algebraic topological representations upon which this paper focuses.

Littlestone and Warmuth (1986) introduced the study of sample compression schemes, defined as a pair of mappings for given concept class \( C \): a compression function mapping a \( C \)-labeled \( n \)-sample to a subsequence of labeled examples and a reconstruction function mapping the subsequence to a concept consistent with the entire \( n \)-sample. A compression scheme of bounded size—the maximum cardinality of the subsequence image—was shown to imply learnability. The converse—that classes of VC dimension \( d \) admit compression schemes of size \( d \)—has become one of the oldest unsolved problems actively pursued within learning theory (Floyd, 1989; Helmbold et al., 1992; Ben-David and Litman, 1998; Warmuth, 2003; Hellerstein, 2006; Kuzmin and Warmuth, 2007; Rubinstein et al., 2007, 2009; Rubinstein and Rubinstein, 2008). Interest in the conjecture has been motivated by its interpretation as the converse to the existence of compression bounds for PAC learnable classes (Littlestone and Warmuth, 1986), the basis of practical machine learning methods on compression schemes (Marchand and Shawe-Taylor, 2003; von Luxburg et al., 2004), and the conjecture’s connection to a deeper understanding of the combinatorial properties of concept classes (Rubinstein et al., 2009; Rubinstein and Rubinstein, 2008). Recently Kuzmin and Warmuth (2007) achieved compression of maximum classes without the use of labels. They also conjectured that their elegant Min-Peeling Algorithm constitutes such an unlabeled \( d \)-compression scheme for \( d \)-maximum classes.

As discussed in our previous work (Rubinstein et al., 2009), maximum classes can be fruitfully viewed as cubical complexes. These are also topological spaces, with each cube equipped with a natural topology of open sets from its standard embedding into Euclidean space. We proved that \( d \)-maximum classes correspond to \( d \)-contractible complexes—topological spaces with an identity map homotopic to a constant map—extending the result that 1-maximum classes have trees for one-inclusion graphs. Peeling can be viewed as a special form of contractibility for maximum classes. However, there are many non-maximum contractible cubical complexes that cannot be peeled, which demonstrates that peelability reflects more detailed structure of maximum classes than given by contractibility alone.

In this paper we approach peeling from the direction of simple hyperplane arrangement representations of maximum classes. Kuzmin and Warmuth (2007) Conjecture 1) predicted that \( d \)-maximum classes corresponding to simple linear hyperplane arrangements could be unlabeled \( d \)-compressed by sweeping a generic hyperplane across the arrangement, and that concepts are min-peeled as their corresponding cell is swept away. We positively resolve the first part of the conjecture and show that sweeping such arrangements corresponds to a new form of corner-peeling, which we prove is distinct from min-peeling. While min-peeling removes minimum degree concepts from a one-inclusion graph, corner-peeling peels vertices that are contained in unique cubes of maximum dimension.

We explore simple hyperplane arrangements in Hyperbolic geometry, which we show correspond to a set of maximum classes, properly containing those represented by simple linear Euclidean arrangements. These classes can again be corner-peeled by sweeping. Citing the proof of existence of maximum unlabeled compression schemes due to Ben-David and Litman (1998), Kuzmin and Warmuth (2007) ask whether unlabeled compression schemes for infinite classes such as positive half spaces can be constructed explicitly. We present
constructions for illustrative but simpler classes, suggesting that there are many interesting
infinite maximum classes admitting explicit compression schemes, and under appropriate
conditions, sweeping infinite Euclidean, Hyperbolic or PL arrangements corresponds to
compression by corner-peeling.

Next we prove that all maximum classes in \(\{0, 1\}^n\) are represented as simple arrange-
ments of Piecewise-Linear (PL) hyperplanes in the \(n\)-ball. This extends previous work
by G"artner and Welzl (1994) on viewing simple PL hyperplane arrangements as maximum
classes. The close relationship between such arrangements and their Hyperbolic versions
suggests that they could be equivalent. Resolving the main problem left open in the prelim-
inary version of this paper, (Rubinstein and Rubinstein, 2008), we show that sweeping of
d-contractible PL arrangements does compress all finite maximum classes by corner-peeling,
completing (Kuzmin and Warmuth, 2007, Conjecture 1).

We show that a one-inclusion graph \(\Gamma\) can be represented by a \(d\)-contractible PL hyper-
plane arrangement if and only if \(\Gamma\) is a strongly contractible cubical complex. This motivates
the nomenclature of \(d\)-contractible for the class of arrangements of PL hyperplanes. Note
then that these one-inclusion graphs admit a corner-peeling scheme of the same size \(d\) as
the largest dimension of a cube in \(\Gamma\). Moreover if such a graph \(\Gamma\) admits a corner-peeling
scheme, then it is a contractible cubical complex. We give a simple example to show that
there are one-inclusion graphs which admit corner-peeling schemes but are not strongly
contractible and so are not represented by a \(d\)-contractible PL hyperplane arrangement.

Compressing maximal classes—classes which cannot be grown without an increase to
their VC dimension—is sufficient for compressing all classes, as embedded classes trivially
inherit compression schemes of their super-classes. This reasoning motivates the attempt
to embed \(d\)-maximal classes into \(O(d)\)-maximum classes (Kuzmin and Warmuth, 2007,
Open Problem 3). We present non-embeddability results following from our earlier counter-
examples to Kuzmin & Warmuth’s minimum degree conjecture (Rubinstein et al. 2009),
and our new results on corner-peeling. We explore with examples, maximal classes that
can be compressed but not peeled, and classes that are not strongly contractible but can
be compressed.

Finally, we investigate algebraic topological properties of maximum classes. Most no-
tably we characterize \(d\)-maximum classes, corresponding to simple linear Euclidean ar-
rangements, as cubical complexes homeomorphic to the \(d\)-ball. The result that such classes’
boundaries are homeomorphic to the \((d - 1)\)-sphere begins the study of the boundaries
of maximum classes, which are closely related to peeling. We conclude with several open
problems.

2. Background

2.1 Algebraic Topology

**Definition 1** A homeomorphism is a one-to-one and onto map \(f\) between topological spaces
such that both \(f\) and \(f^{-1}\) are continuous. Spaces \(X\) and \(Y\) are said to be homeomorphic if
there exists a homeomorphism \(f : X \rightarrow Y\).

**Definition 2** A homotopy is a continuous map \(F : X \times [0, 1] \rightarrow Y\). The initial map is \(F\)
restricted to \(X \times \{0\}\) and the final map is \(F\) restricted to \(X \times \{1\}\). We say that the initial
and final maps are homotopic. A homotopy equivalence between spaces \( X \) and \( Y \) is a pair of maps \( f : X \to Y \) and \( g : Y \to X \) such that \( f \circ g \) and \( g \circ f \) are homotopic to the identity maps on \( Y \) and \( X \) respectively. We say that \( X \) and \( Y \) have the same homotopy type if there is a homotopy equivalence between them. A deformation retraction is a special homotopy equivalence between a space \( X \) and a subspace \( A \subseteq X \). It is a continuous map \( r : X \to X \) with the properties that the restriction of \( r \) to \( A \) is the identity map on \( A \), \( r \) has range \( A \) and \( r \) is homotopic to the identity map on \( X \).

**Definition 3** A cubical complex is a union of solid cubes of the form \([a_1,b_1] \times \ldots \times [a_m,b_m]\), for bounded \( m \in \mathbb{N} \), such that the intersection of any two cubes in the complex is either a cubical face of both cubes or the empty-set.

**Definition 4** A contractible cubical complex \( X \) is one which has the same homotopy type as a one point space \( \{p\} \). \( X \) is contractible if and only if the constant map from \( X \) to \( p \) is a homotopy equivalence.

### 2.2 Concept Classes and their Learnability

A concept class \( C \) on domain \( X \), is a subset of the power set of set \( X \) or equivalently \( C \subseteq \{0,1\}^X \). We primarily consider finite domains and so will write \( C \subseteq \{0,1\}^n \) in the sequel, where it is understood that \( n = |X| \) and the \( n \) dimensions or colors are identified with an ordering \( \{x_i\}_{i=1}^n = X \).

The one-inclusion graph \( G(C) \) of \( C \subseteq \{0,1\}^n \) is the graph with vertex-set \( C \) and edge-set containing \( \{u,v\} \subseteq C \) iff \( u \) and \( v \) differ on exactly one component (Haussler et al., 1994); \( G(C) \) forms the basis of a prediction strategy with essentially-optimal worst-case expected risk. \( G(C) \) can be viewed as a simplicial complex in \( \mathbb{R}^n \) by filling in each face with a product of continuous intervals (Rubinstein et al., 2009). Each edge \( \{u,v\} \) in \( G(C) \) is labeled by the component on which the two vertices \( u, v \) differ.

A \( d \)-complete collection is a union of \( d \)-subcubes in \( \{0,1\}^n \), one with each choice of \( d \) colors from \( n \).

Probably Approximately Correct learnability of a concept class \( C \subseteq \{0,1\}^X \) is characterized by the finiteness of the Vapnik-Chervonenkis (VC) dimension of \( C \) (Blumer et al., 1989). One key to all such results is Sauer’s Lemma.

**Definition 5** The VC-dimension of concept class \( C \subseteq \{0,1\}^X \) is defined as \( VC(C) = \sup \left\{ n \mid \exists Y \subseteq \binom{X}{n}, \forall y \in Y, \Pi_Y(C) = \{0,1\}^n \right\} \) where \( \Pi_Y(C) = \{\{c(x_1), \ldots, c(x_n)\} \mid c \in C \subseteq \{0,1\}^n\} \) is the projection of \( C \) on sequence \( Y = (x_1, \ldots, x_n) \).

**Lemma 6** (Vapnik and Chervonenkis, 1971, Sauer, 1972, Shelah, 1972) The cardinality of any concept classes \( C \subseteq \{0,1\}^n \) is bounded by \( |C| \leq \sum_{i=1}^{VC(C)} \binom{n}{i} \).

Motivated by maximizing concept class cardinality under a fixed VC-dimension, which is related to constructing general sample compression schemes (see Section 2.3), Welzl (1987) defined the following special classes.

**Definition 7** Concept class \( C \subseteq \{0,1\}^X \) is called maximal if \( VC(C \cup \{c\}) > VC(C) \) for all \( c \in \{0,1\}^X \setminus C \). Furthermore if \( \Pi_Y(C) \) satisfies Sauer’s Lemma with equality for each
Y ∈ \( \binom{X}{n} \), for every \( n \in \mathbb{N} \), then \( C \) is termed maximum. If \( C \subseteq \{0,1\}^n \) then \( C \) is maximum (and hence maximal) if \( C \) meets Sauer’s Lemma with equality.

The reduction of \( C \subseteq \{0,1\}^n \) with respect to \( i \in [n] = \{1, \ldots, n\} \) is the class \( C^i = \Pi_{[n] \setminus \{i\}}(\{c \in C \mid i \notin I_{G(C)}(c)\}) \) where \( I_{G(C)}(c) \subseteq [n] \) denotes the labels of the edges incident to vertex \( c \); a multiple reduction is the result of performing several reductions in sequence. The tail of class \( C \) is \( \text{tail}_i(C) = \{c \in C \mid i \notin I_{G(C)}(c)\} \). Welzl showed that if \( C \) is \( d \)-maximum, then \( \Pi_{[n] \setminus \{i\}}(C) \) and \( C^i \) are maximum of VC-dimensions \( d \) and \( d - 1 \) respectively.

The results presented below relate to other geometric and topological representations of maximum classes existing in the literature. Under the guise of ‘forbidden labels’, Floyd (1989) showed that maximum \( C \subseteq \{0,1\}^n \) of VC-dim \( d \) is the union of a maximally overlapping \( d \)-complete collection of cubes [Rubinstein et al. 2009]—defined as a collection of \( d \)-cubes which project onto all \( \binom{n}{d} \) possible sets of \( d \) coordinate directions. (An alternative proof was developed by Neylon 2006.) It has long been known that VC-1 maximum classes have one-inclusion graphs that are trees [Dudley 1985]; we previously extended this result by showing that when viewed as complexes, \( d \)-maximum classes are contractible \( d \)-cubical complexes [Rubinstein et al. 2009]. Finally the cells of a simple linear arrangement of \( n \) hyperplanes in \( \mathbb{R}^d \) form a VC-\( d \) maximum class in the \( n \)-cube [Edelsbrunner 1987], but not all finite maximum classes correspond to such Euclidean arrangements [Floyd 1989].

## 2.3 Sample Compression Schemes

Littlestone and Warmuth (1986) showed that the existence of a compression scheme of finite size is sufficient for learnability of \( C \), and conjectured the converse, that \( \text{VC}(C) = d < \infty \) implies a compression scheme of size \( d \). Later Warmuth (2003) weakened the conjectured size to \( O(d) \). To-date it is only known that maximum classes can be \( d \)-compressed (Floyd 1989). Unlabeled compression was first explored by Ben-David and Litman (1998); Kuzmin and Warmuth (2007) defined unlabeled compression as follows, and explicitly constructed schemes of size \( d \) for maximum classes.

**Definition 8** Let \( C \) be a \( d \)-maximum class on a finite domain \( X \). A representation mapping \( r \) of \( C \) satisfies:

1. \( r \) is a bijection between \( C \) and subsets of \( X \) of size at most \( d \); and
2. [non-clashing]: \( c | (r(c) \cup r(c')) \neq c' | (r(c) \cup r(c')) \) for all \( c, c' \in C, c \neq c' \).

As with all previously published labeled schemes, all previously known unlabeled compression schemes for maximum classes exploit their special recursive projection-reduction structure and so it is doubtful that such schemes will generalize. Kuzmin and Warmuth (2007, Conjecture 2) conjectured that their Min-Peeling Algorithm constitutes an unlabeled \( d \)-compression scheme for maximum classes; it iteratively removes minimum degree vertices from \( G(C) \), representing the corresponding concepts by the remaining incident dimensions in the graph. The authors also conjectured that sweeping a hyperplane in general position across a simple linear arrangement forms a compression scheme that corresponds to min-peeling the associated maximum class (Kuzmin and Warmuth 2007, Conjecture 1).
A particularly promising approach to compressing general classes is via their maximum-embeddings: a class $C$ embedded in class $C'$ trivially inherits any compression scheme for $C'$, and so an important open problem is to embed maximal classes into maximum classes with at most a linear increase in VC-dimension (Kuzmin and Warmuth, 2007, Open Problem 3).

3. Preliminaries

3.1 Constructing All Maximum Classes

The aim in this section is to describe an algorithm for constructing all maximum classes of VC-dimension $d$ in the $n$-cube. This process can be viewed as the inverse of mapping a maximum class to its $d$-maximum projection on $[n]\setminus\{i\}$ and the corresponding $(d-1)$-maximum reduction.

**Definition 9** Let $C, C' \subseteq \{0, 1\}^n$ be maximum classes of VC-dimensions $d, d-1$ respectively, so that $C' \subset C$, and let $C_1, C_2 \subset C$ be $d$-cubes, i.e., $d$-faces of the $n$-cube $\{0, 1\}^n$.

1. $C_1, C_2$ are connected if there exists a path in the one-inclusion graph $G(C)$ with endpoints in $C_1$ and $C_2$; and

2. $C_1, C_2$ are said to be $C'$-connected if there exists such a connecting path that further does not intersect $C'$.

The $C'$-connected components of $C$ are the equivalence classes of the $d$-cubes of $C$ under the $C'$-connectedness relation.

The recursive algorithm for constructing all maximum classes of VC-dimension $d$ in the $n$-cube, detailed as Algorithm 1, considers each possible $d$-maximum class $C$ in the $(n-1)$-cube and each possible $(d-1)$-maximum subclass $C'$ of $C$ as the projection and reduction of a $d$-maximum class in the $n$-cube, respectively. The algorithm lifts $C$ and $C'$ to all possible maximum classes in the $n$-cube. Then $C' \times \{0, 1\}$ is contained in each lifted class; so all that remains is to find the tails from the complement of the reduction in the projection. It turns out that each $C'$-connected component $C_i$ of $C$ can be lifted to either $C_i \times \{0\}$ or $C_i \times \{1\}$ arbitrarily and independently of how the other $C'$-connected components are lifted. The set of lifts equates to the set of $d$-maximum classes in the $n$-cube that project-reduce to $(C, C')$.

**Lemma 10** MaximumClasses($n, d$) (cf. Algorithm 1) returns the set of maximum classes of VC-dimension $d$ in the $n$-cube for all $n \in \mathbb{N}, d \in [n]$.

**Proof** We proceed by induction on $n$ and $d$. The base cases correspond to $n \in \mathbb{N}, d \in \{0, n\}$ for which all maximum classes, enumerated as singletons in the $n$-cube and the $n$-cube respectively, are correctly produced by the algorithm. For the inductive step we assume that for $n \in \mathbb{N}, d \in [n-1]$ all maximum classes of VC-dimension $d$ and $d-1$ in the $(n-1)$-cube are already known by recursive calls to the algorithm. Given this, we will show that MaximumClasses($n, d$) returns only $d$-maximum classes in the $n$-cube, and that all such classes are produced by the algorithm.
Let classes $C \in \text{MaximumClasses}(n-1, d)$ and $C' \in \text{MaximumClasses}(n-1, d-1)$ be such that $C' \subseteq C$. Then $C$ is the union of a $d$-complete collection and $C'$ is the union of a $(d-1)$-complete collection of cubes that are faces of the cubes of $C$. Consider a concept class $C^*$ formed from $C$ and $C'$ by Algorithm 1. The algorithm partitions $C$ into $C'$-connected components $C_1, \ldots, C_k$ each of which is a union of $d$-cubes. While $C'$ is lifted to $C' \times \{0,1\}$, some subset of the components $\{C_i\}_{i \in S_0}$ are lifted to $\{C_i \times \{0\}\}_{i \in S_0}$ while the remaining components are lifted to $\{C_i \times \{1\}\}_{i \notin S_0}$. By definition $C^*$ is a $d$-complete collection of cubes with cardinality equal to $\binom{n}{d}$ since $|C^*| = |C'| + |C|$. So $C^*$ is $d$-maximum (Rubinstein et al., 2009, Theorem 34).

If we now consider any $d$-maximum class $C^* \subseteq \{0,1\}^n$, its projection on $[n]\{i\}$ is a $d$-maximum class $C \subseteq \{0,1\}^{n-1}$ and $C^*$ is the $(d-1)$-maximum projection $C' \subseteq C$ of all the $d$-cubes in $C^*$ which contain color $i$. It is thus clear that $C^*$ must be obtained by lifting parts of the $C'$-connected components of $C$ to the 1 level and the remainder to the 0 level, and $C'$ to $C' \times \{0,1\}$. We will now show that if the vertices of each component are not lifted to the same levels, then while the number of vertices in the lift match that of a $d$-maximum class in the $n$-cube, the number of edges are too few for such a maximum class. Define a lifting operator on $C$ as $\ell(v) = \{v\} \times \ell_v$, where $\ell_v \subseteq \{0,1\}$ and

$$|\ell_v| = \begin{cases} 2, & \text{if } v \in C' \\ 1, & \text{if } v \in C \setminus C'. \end{cases}$$

Consider now an edge $\{u,v\}$ in $G(C)$. By the definition of a $C'$-connected component there exists some $C_j$ such that either $u, v \in C_j \setminus C'$, $u, v \in C'$ or WLOG $u \in C_j \setminus C', v \in C'$. In the first case $\ell(u) \cup \ell(v)$ is an edge in the lifted graph iff $\ell_u = \ell_v$. In the second case $\ell(u) \cup \ell(v)$ contains four edges and in the last it contains a single edge. Furthermore, it is clear that this accounts for all edges in the lifted graph by considering the projection of an edge in the lifted product. Thus any lift other than those produced by Algorithm 1 induces strictly too few edges for a $d$-maximum class in the $n$-cube (cf. Kuzmin and Warmuth 2007, Corollary 7.5).
3.2 Corner-Peeling

Kuzmin and Warmuth (2007) conjectured that their simple Min-Peeling procedure is a valid unlabeled compression scheme for maximum classes. Beginning with a concept class \( C_0 = C \subseteq \{0, 1\}^n \), Min-Peeling operates by iteratively removing a vertex \( v_t \) of minimum-degree in \( G(C_t) \) to produce the peeled class \( C_{t+1} = C_t \setminus \{v_t\} \). The concept class corresponding to \( v_t \) is then represented by the dimensions of the edges incident to \( v_t \) in \( G(C_t) \), \( I_{G(C_t)}(v_t) \subseteq [n] \). Providing that no-clashing holds for the algorithm, the size of the min-peeling scheme is the largest degree encountered during peeling. Kuzmin and Warmuth predicted that this size is always at most \( d \) for \( d \)-maximum classes. We explore these questions for a related special case of peeling, where we prescribe which vertex to peel at step \( t \) as follows.

**Definition 11** We say that \( C \subseteq \{0, 1\}^n \) can be corner-peeled if there exists an ordering \( v_1, \ldots, v_{|C|} \) of the vertices of \( C \) such that, for each \( t \in [|C|] \) where \( C_0 = C \),

1. \( v_t \in C_{t-1} \) and \( C_t = C_{t-1} \setminus \{v_t\} \);
2. There exists a unique cube \( C'_t \) of maximum dimension over all cubes in \( C_{t-1} \) containing \( v_t \);
3. The neighbors \( \Gamma(v_t) \) of \( v_t \) in \( G(C_{t-1}) \) satisfy \( \Gamma(v_t) \subseteq C'_t \); and
4. \( C_{|C|} = \emptyset \).

The \( v_t \) are termed the corner vertices of \( C_{t-1} \) respectively. If \( d \) is the maximum degree of each \( v_t \) in \( G(C_{t-1}) \), then \( C \) is \( d \)-corner-peeled.

Note that we do not constrain the cubes \( C'_t \) to be of non-increasing dimension. It turns out that an important property of maximum classes is invariant to this kind of peeling.

**Definition 12** We call a class \( C \subseteq \{0, 1\}^n \) shortest-path closed if for any \( u, v \in C \), \( G(C) \) contains a path connecting \( u, v \) of length \( \|u - v\|_1 \).

**Lemma 13** If \( C \subseteq \{0, 1\}^n \) is shortest-path closed and \( v \in C \) is a corner vertex of \( C \), then \( C \setminus \{v\} \) is shortest-path closed.

**Proof** Consider a shortest-path closed \( C \subseteq \{0, 1\}^n \). Let \( c \) be a corner vertex of \( C \), and denote the cube of maximum dimension in \( C \), containing \( c \), by \( C' \). Consider \( \{u, v\} \subseteq C \setminus \{c\} \). By assumption there exists a \( u \)-\( v \)-path \( p \) of length \( \|u - v\|_1 \) contained in \( C \). If \( c \) is not in \( p \) then \( p \) is contained in the peeled product \( C \setminus \{c\} \). If \( c \) is in \( p \) then \( p \) must cross \( C' \) such that there is another path of the same length which avoids \( c \), and thus \( C \setminus \{c\} \) is shortest-path closed.

### 3.2.1 Corner-Peeling Implies Compression

**Theorem 14** If a maximum class \( C \) can be corner-peeled then \( C \) can be \( d \)-unlabeled compressed.

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The invariance of the shortest-path closed property under corner-peeling is key. The corner-peeling unlabeled compression scheme represents each \( v_t \in C \) by \( r(v_t) = I G(C_{t-1}) \), the colors of the cube \( C_{t-1} \) which is deleted from \( C_t \) when \( v_t \) is corner-peeled. We claim that any two vertices \( v_s, v_t \in C \) have non-clashing representatives. WLOG, suppose that \( s < t \). The class \( C_{s-1} \) must contain a shortest \( v_s \)-\( v_t \)-path \( p \). Let \( i \) be the color of the single edge contained in \( p \) that is incident to \( v_s \). Color \( i \) appears once in \( p \), and is contained in \( r(v_s) \). This implies that \( v_{s,i} \neq v_{t,i} \) and that \( i \in r(v_s) \cup r(v_t) \), and so \( v_s| (r(v_s) \cup r(v_t)) \neq v_t| (r(v_s) \cup r(v_t)) \). By construction, \( r(\cdot) \) is a bijection between \( C \) and all subsets of \([n]\) of cardinality \( \leq VC(C) \).

If the oriented one-inclusion graph, with each edge directed away from the incident vertex represented by the edge’s color, has no cycles, then that representation’s compression scheme is termed acyclic [Floyd, 1989; Ben-David and Litman, 1998; Kuzmin and Warmuth, 2007].

**Proposition 15** All corner-peeling unlabeled compression schemes are acyclic.

**Proof** We follow the proof that the Min-Peeling Algorithm is acyclic [Kuzmin and Warmuth, 2007]. Let \( v_1, \ldots, v_{|C|} \) be a corner vertex ordering of \( C \). As a corner vertex \( v_t \) is peeled, its unoriented incident edges are oriented away from \( v_t \). Thus all edges incident to \( v_1 \) are oriented away from \( v_1 \) and so the vertex cannot take part in any cycle. For \( t > 1 \) assume \( V_t = \{v_s \mid s < t \} \) is disjoint from all cycles. Then \( v_t \) cannot be contained in a cycle, as all incoming edges into \( v_t \) are incident to some vertex in \( V_t \). Thus the oriented \( G(C) \) is indeed acyclic.

### 3.3 Boundaries of Maximum Classes

We now turn to the geometric boundaries of maximum classes, which are closely related to corner-peeling.

**Definition 16** The boundary \( \partial C \) of a \( d \)-maximum class \( C \) is defined as all the \((d-1)\)-subcubes which are the faces of a single \( d \)-cube in \( C \).

Maximum classes, when viewed as cubical complexes, are analogous to soap films (an example of a minimal energy surface encountered in nature), which are obtained when a wire frame is dipped into a soap solution. Under this analogy the boundary corresponds to the wire frame and the number of \( d \)-cubes can be considered the area of the soap film. An important property of the boundary of a maximum class is that all lifted reductions meet the boundary multiple times.

**Theorem 17** Every \( d \)-maximum class has boundary containing at least two \((d-1)\)-cubes of every combination of \( d-1 \) colors, for all \( d > 1 \).

**Proof** We use the lifting construction of Section 3.1. Let \( C^* \subseteq \{0,1\}^n \) be a 2-maximum class and consider color \( i \in [n] \). Then the reduction \( C^{*i} \) is an unrooted tree with at least two leaves, each of which lifts to an \( i \)-colored edge in \( C^* \). Since the leaves are of degree
1 in $C^*$, the corresponding lifted edges belong to exactly one 2-cube in $C^*$ and so lie in $\partial C^*$. Consider now a $d$-maximum class $C^* \subseteq \{0,1\}^n$ for $d > 2$, and make the inductive assumption that the projection $C = \Pi_{[n-1]}(C^*)$ has two of each type of $(d-1)$-cube, and that the reduction $C' = C^*|_n$ has two of each type of $(d-2)$-cube, in their boundaries. Pick $d-1$ colors $I \subseteq [n]$. If $n \in I$ then consider two $(d-2)$-cubes colored by $I \setminus \{x_n\}$ in $\partial C'$. By the same argument as in the base case, these lift to two $I$-colored cubes in $\partial C^*$. If $n \notin I$ then $\partial C$ contains two $I$-colored $(d-1)$-cubes. For each cube, if the cube is contained in $C'$ then it has two lifts one of which is contained in $\partial C^*$, otherwise its unique lift is contained in $\partial C^*$. Therefore $\partial C^*$ contains at least two $I$-colored cubes.

Having a large boundary is an important property of maximum classes that does not follow from contractibility.

**Example 1** Take a 2-simplex with vertices $A, B, C$. Glue the edges $AB$ to $AC$ to form a cone. Next glue the end loop $BC$ to the edge $AB$. The result is a complex $D$ with a single vertex, edge and 2-simplex, which is classically known as the dunce hat (cf. Figure 1). It is not hard to verify that $D$ is contractible, but has no (geometric) boundary.

Although Theorem 17 will not be explicitly used in the sequel, we return to boundaries of maximum complexes later.

### 4. Euclidean Arrangements

**Definition 18** A linear arrangement is a collection of $n \geq d$ oriented hyperplanes in $\mathbb{R}^d$. Each region or cell in the complement of the arrangement is naturally associated with a concept in $\{0,1\}^n$; the side of the $i^{th}$ hyperplane on which a cell falls determines the concept’s $i^{th}$ component. A simple arrangement is a linear arrangement in which any subset of $d$ planes has a unique point in common and all subsets of $d+1$ planes have an empty mutual intersection. Moreover any subset of $k < d$ planes meet in a plane of dimension $d-k$. Such a collection of $n$ planes is also said to be in general position.

Many of the familiar operations on concept classes in the $n$-cube have elegant analogues on arrangements.
• Projection on \([n] \setminus \{i\}\) corresponds to removing the \(i^{th}\) plane;

• The reduction \(C_i\) is the new arrangement given by the intersection of \(C\)'s arrangement with the \(i^{th}\) plane; and

• The corresponding lifted reduction is the collection of cells in the arrangement that adjoin the \(i^{th}\) plane.

A \(k\)-cube in the one-inclusion graph corresponds to a collection of \(2^k\) cells, all having a common \((d-k)\)-face, which is contained in the intersection of \(k\) planes, and an edge corresponds to a pair of cells which have a common face on a single plane. The following result is due to Edelsbrunner (1987).

**Lemma 19** The concept class \(C \subseteq \{0,1\}^n\) induced by a simple linear arrangement of \(n\) planes in \(\mathbb{R}^d\) is \(d\)-maximum.

**Proof** Note that \(C\) has VC-dimension at most \(d\), since general position is invariant to projection i.e., no \(d+1\) planes are shattered. Since \(C\) is the union of a \(d\)-complete collection of cubes (every cell contains \(d\)-intersection points in its boundary) it follows that \(C\) is \(d\)-maximum (Rubinstein et al., 2009).

**Corollary 20** Let \(A\) be a simple linear arrangement of \(n\) hyperplanes in \(\mathbb{R}^d\) with corresponding \(d\)-maximum \(C \subseteq \{0,1\}^n\). The intersection of \(A\) with a generic hyperplane corresponds to a \((d-1)\)-maximum class \(C' \subseteq C\). In particular if all \(d\)-intersection points of \(A\) lie to one side of the generic hyperplane, then \(C'\) lies on the boundary of \(C\); and \(\partial C\) is the disjoint union of two \((d-1)\)-maximum sub-classes.

**Proof** The intersection of \(A\) with a generic hyperplane is again a simple arrangement of \(n\) hyperplanes but now in \(\mathbb{R}^{d-1}\). Hence by Lemma 19 \(C'\) is a \((d-1)\)-maximum class in the \(n\)-cube. \(C' \subseteq C\) since the adjacency relationships on the cells of the intersection are inherited from those of \(A\).

Suppose that all \(d\)-intersections in \(A\) lie in one half-space of the generic hyperplane. \(C'\) is the union of a \((d-1)\)-complete collection. We claim that each of these \((d-1)\)-cubes is a face of exactly one \(d\)-cube in \(C\) and is thus in \(\partial C\). A \((d-1)\)-cube in \(C'\) corresponds to a line in \(A\) where \(d-1\) planes mutually intersect. The \((d-1)\)-cube is a face of a \(d\)-cube in \(C\) iff this line is further intersected by a \(d^{th}\) plane. This occurs for exactly one plane, which is closest to the generic hyperplane along this intersection line. For once the \(d\)-intersection point is reached, when following along the line away from the generic plane, a new cell is entered. This verifies the second part of the result.

Consider two parallel generic hyperplanes \(h_1, h_2\) such that all \(d\)-intersection points of \(A\) lie in between them. We claim that each \((d-1)\)-cube in \(\partial C\) is in exactly one of the concept classes induced by the intersection of \(A\) with \(h_1\) and \(A\) with \(h_2\). Consider an arbitrary \((d-1)\)-cube in \(\partial C\). As before this cube corresponds to a region of a line formed by a mutual intersection of \(d-1\) planes. Moreover this region is a ray, with one end-point at a \(d\)-intersection. Because the ray begins at a point between the generic hyperplanes \(h_1, h_2\), it
follows that the ray must cross exactly one of these.

**Corollary 21** Let $A$ be a simple linear arrangement of $n$ hyperplanes in $\mathbb{R}^d$ and let $C \subseteq \{0, 1\}^n$ be the corresponding $d$-maximum class. Then $C$ considered as a cubical complex is homeomorphic to the $d$-ball $B^d$; and $\partial C$ considered as a $(d-1)$-cubical complex is homeomorphic to the $(d-1)$-sphere $S^{d-1}$.

**Proof** We construct a Voronoi cell decomposition corresponding to the set of $d$-intersection points inside a very large ball in Euclidean space. By induction on $d$, we claim that this is a cubical complex and the vertices and edges correspond to the class $C$. By induction, on each hyperplane, the induced arrangement has a Voronoi cell decomposition which is a $(d-1)$-cubical complex with edges and vertices matching the one-inclusion graph for the tail of $C$ corresponding to the label associated with the hyperplane. It is not hard to see that the Voronoi cell defined by a $d$-intersection point $p$ on this hyperplane is a $d$-cube. In fact, its $(d-1)$-faces correspond to the Voronoi cells for $p$, on each of the $d$ hyperplanes passing through $p$. We also see that this $d$-cube has a single vertex in the interior of each of the $2^d$ cells of the arrangement adjacent to $p$. In this way, it follows that the vertices of this Voronoi cell decomposition are in bijective correspondence to the cells of the hyperplane arrangement. Finally the edges of the Voronoi cells pass through the faces in the hyperplanes. So these correspond bijectively to the edges of $C$, as there is one edge for each face of the hyperplanes. Using a very large ball, containing all the $d$-intersection points, the boundary faces become spherical cells. In fact, these form a spherical Voronoi cell decomposition, so it is easy to replace these by linear ones by taking the convex hull of their vertices. So a piecewise linear cubical complex $C$ is constructed, which has one-skeleton (graph consisting of all vertices and edges) isomorphic to the one-inclusion graph for $C$.

Finally we want to prove that $C$ is homeomorphic to $B^d$. This is quite easy by construction. For we see that $C$ is obtained by dividing up $B^d$ into Voronoi cells and replacing the spherical boundary cells by linear ones, using convex hulls of the boundary vertices. This process is clearly given by a homeomorphism by projection. In fact, the homeomorphism preserves the PL-structure so is a PL homeomorphism.

The following example demonstrates that not all maximum classes of VC-dimension $d$ are homeomorphic to the $d$-ball. The key to such examples is branching.

**Example 2** A simple linear arrangement in $\mathbb{R}$ corresponds to points on the line—cells are simply intervals between these points and so corresponding 1-maximum classes are sticks. Any tree that is not a stick can therefore not be represented as a simple linear arrangement in $\mathbb{R}$ and is also not homeomorphic to the 1-ball which is simply the interval $[-1, 1]$.

As Kuzmin and Warmuth (2007) did previously, consider a generic hyperplane $h$ sweeping across a simple linear arrangement $A$. $h$ begins with all $d$-intersection points of $A$ lying in its positive half-space $\mathcal{H}_+$. The concept corresponding to cell $c$ is peeled from $C$ when $|\mathcal{H}_+ \cap c| = 1$, i.e., $h$ crosses the last $d$-intersection point adjoining $c$. At any step in the
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|   | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|---|-------|-------|-------|-------|
| $v_0$ | 0     | 0     | 0     | 0     |
| $v_1$ | 1     | 0     | 0     | 0     |
| $v_2$ | 0     | 1     | 0     | 0     |
| $v_3$ | 0     | 0     | 1     | 0     |
| $v_4$ | 1     | 0     | 1     | 0     |
| $v_5$ | 1     | 1     | 0     | 0     |
| $v_6$ | 0     | 1     | 1     | 0     |
| $v_7$ | 1     | 0     | 0     | 1     |
| $v_8$ | 1     | 1     | 0     | 1     |
| $v_9$ | 0     | 1     | 0     | 1     |
| $v_{10}$ | 0 | 1 | 1 | 1 |

Figure 2: A 2-maximum class in $\{0,1\}^4$ having a simple linear line arrangement in $\mathbb{R}^2$.

Figure 3: The 2-maximum class in the 4-cube, enumerated in Figure 2.

Figure 4: A simple linear line arrangement corresponding to the class in Figure 2 swept by the dashed line. Each cell has a unique vertex.

process, the result of peeling $j$ vertices from $C$ to reach $C_j$, is captured by the arrangement $\mathcal{H}_+ \cap A$ for the appropriate $h$.

Example 3 Figure 2 enumerates the 11 vertices of a 2-maximum class in the 4-cube. Figures 4 and 3 display a hyperplane arrangement in Euclidean space and its Voronoi cell decomposition, corresponding to this maximum class. In this case, sweeping the vertical dashed line across the arrangement corresponds to a partial corner-peeling of the concept.
Next we resolve the first half of Kuzmin and Warmuth 2007 Conjecture 1.

**Theorem 22** Any $d$-maximum class $C \subseteq \{0, 1\}^n$ corresponding to a simple linear arrangement $A$ can be corner-peeled by sweeping $A$, and this process is a valid unlabeled compression scheme for $C$ of size $d$.

**Proof** We must show that as the $j$th $d$-intersection point $p_j$ is crossed, there is a corner vertex of $C_{j-1}$ peeled away. It then follows that sweeping a generic hyperplane $h$ across $A$ corresponds to corner-peeling $C$ to a $(d - 1)$-maximum sub-class $C' \subseteq \partial C$ by Corollary 20. Moreover $C'$ corresponds to a simple linear arrangement of $n$ hyperplanes in $\mathbb{R}^{d-1}$.

We proceed by induction on $d$, noting that for $d = 1$ corner-peeling is trivial. Consider $h$ as it approaches the $j$th $d$-intersection point $p_j$. The $d$ planes defining this point intersect $h$ in a simple arrangement of hyperplanes on $h$. There is a compact cell $\Delta$ for the arrangement on $h$, which is a $d$-simplex and shrinks to a point as $h$ passes through $p_j$. We claim that the cell $c$ for the arrangement $A$, whose intersection with $h$ is $\Delta$, is a corner vertex $v_j$ of $C_{j-1}$. Consider the lines formed by intersections of $d - 1$ of the $d$ hyperplanes, passing through $p_j$. Each is a segment starting at $p_j$ and ending at $h$ without passing through any other $d$-intersection points. So all faces of hyperplanes adjacent to $c$ meet $h$ in faces of $\Delta$. Thus, there are no edges in $C_{j-1}$ starting at the vertex corresponding to $p_j$, except for those in the cube $C'_{j-1}$, which consists of all cells adjacent to $p_j$ in the arrangement $A$. So $c$ corresponds to a corner vertex $v_j$ of the $d$-cube $C'_{j-1}$ in $C_{j-1}$. Finally, just after the simplex is a point, $c$ is no longer in $\mathcal{H}_+$ and so $v_j$ is corner-peeled from $C_{j-1}$.

Theorem 14 completes the proof that this corner-peeling of $C$ constitutes unlabeled compression. 

**Corollary 23** The sequence of cubes $C'_0, \ldots, C'_{|C|}$, removed when corner-peeling by sweeping simple linear arrangements, is of non-increasing dimension. In fact, there are $\binom{n}{d}$ cubes of dimension $d$, then $\binom{n}{d-1}$ cubes of dimension $d - 1$, etc.

While corner-peeling and min-peeling share some properties in common, they are distinct procedures.

**Example 4** Consider sweeping a simple linear arrangement corresponding to a 2-maximum class. After all but one 2-intersection point has been swept, the corresponding corner-peeled class $C_t$ is the union of a single 2-cube with a 1-maximum stick. Min-peeling applied to $C_t$ would first peel a leaf, while corner-peeling must begin with the 2-cube.

The next result follows from our counter-examples to Kuzmin & Warmuth’s minimum degree conjecture Rubinstein et al. 2009.

**Corollary 24** There is no constant $c$ so that all maximal classes of VC dimension $d$ can be embedded into maximum classes corresponding to simple hyperplane arrangements of dimension $d + c$.

1. A topological simplex—the convex hull of $d + 1$ affinely independent points in $\mathbb{R}^d$. 

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5. Hyperbolic Arrangements

We briefly discuss the Klein model of hyperbolic geometry [Ratcliffe 1994, pg. 7]. Consider the open unit ball $H^k$ in $\mathbb{R}^k$. Geodesics (lines of shortest length in the geometry) are given by intersections of straight lines in $\mathbb{R}^k$ with the unit ball. Similarly planes of any dimension between 2 and $k-1$ are given by intersections of such planes in $\mathbb{R}^k$ with the unit ball. Note that such planes are completely determined by their spheres of intersection with the unit sphere $S^{k-1}$, which is called the ideal boundary of hyperbolic space $\mathbb{H}^k$. Note that in the appropriate metric, the ideal boundary consists of points which are infinitely far from all points in the interior of the unit ball.

We can now see immediately that a simple hyperplane arrangement in $H^k$ can be described by taking a simple hyperplane arrangement in $\mathbb{R}^k$ and intersecting it with the unit ball. However we require an important additional property to mimic the Euclidean case. Namely we add the constraint that every subcollection of $d$ of the hyperplanes in $H^k$ has mutual intersection points inside $H^k$, and that no $(d+1)$-intersection point lies in $H^k$. We need this requirement to obtain that the resulting class is maximum.

**Definition 25** A simple hyperbolic $d$-arrangement is a collection of $n$ hyperplanes in $H^k$ with the property that every sub-collection of $d$ hyperplanes mutually intersect in a $(k-d)$-dimensional hyperbolic plane, and that every sub-collection of $d+1$ hyperplanes mutually intersect as the empty set.

**Corollary 26** The concept class $C$ corresponding to a simple $d$-arrangement of hyperbolic hyperplanes in $\mathbb{H}^k$ is $d$-maximum in the $k$-cube.

**Proof** The result follows by the same argument as before. Projection cannot shatter any $(d+1)$-cube and the class is a complete union of $d$-cubes, so is $d$-maximum.

The key to why hyperbolic arrangements represent many new maximum classes is that they allow flexibility of choosing $d$ and $k$ independently. This is significant because the unit ball can be chosen to miss much of the intersections of the hyperplanes in Euclidean space. Note that the new maximum classes are embedded in maximum classes induced by arrangements of linear hyperplanes in Euclidean space.

A simple example is any 1-maximum class. It is easy to see that this can be realized in the hyperbolic plane by choosing an appropriate family of lines and the unit ball in the appropriate position. In fact, we can choose sets of pairs of points on the unit circle, which will be the intersections with our lines. So long as these pairs of points have the property that the smaller arcs of the circle between them are disjoint, the lines will not cross inside the disk and the desired 1-maximum class will be represented.

Corner-peeling maximum classes represented by hyperbolic hyperplane arrangements proceeds by sweeping, just as in the Euclidean case. Note first that intersections of the hyperplanes of the arrangement with the moving hyperplane appear precisely when there is a first intersection at the ideal boundary. Thus it is necessary to slightly perturb the collection of hyperplanes to ensure that only one new intersection with the moving hyperplane occurs at any time. Note also that new intersections of the sweeping hyperplane with the various lower dimensional planes of intersection between the hyperplanes appear similarly at the
Figure 5: 2-maximum classes in \( \{0,1\}^4 \) that can be represented as hyperbolic arrangements but not as Euclidean arrangements.

ideal boundary. The important claim to check is that the intersection at the ideal boundary between the moving hyperplane and a lower dimensional plane, consisting entirely of \( d \) intersection points, corresponds to a corner-peeling move. We include two examples to illustrate the validity of this claim.

**Example 5** In the case of a 1-maximum class coming from disjoint lines in \( \mathbb{H}^2 \), a cell can disappear when the sweeping hyperplane meets a line at an ideal point. This cell is indeed a vertex of the tree, i.e., a corner-vertex.

**Example 6** Assume that we have a family of 2-planes in the unit 3-ball which meet in pairs in single lines, but there are no triple points of intersection, corresponding to a 2-maximum class. A corner-peeling move occurs when a region bounded by two half disks and an interval disappears, in the positive half space bounded by the sweeping hyperplane. Such a region can be visualized by taking a slice out of an orange. Note that the final point of contact between the hyperplane and the region is at the end of a line of intersection between two planes on the ideal boundary.

We next observe that sweeping by generic hyperbolic hyperplanes induces corner-peeling of the corresponding maximum class, extending Theorem 22. As the generic hyperplane sweeps across hyperbolic space, not only do swept cells correspond to corners of \( d \)-cubes but also to corners of lower dimensional cubes as well. Moreover, the order of the dimensions of the cubes which are corner-peeled can be arbitrary—lower dimensional cubes may be corner-peeled before all the higher dimensional cubes are corner-peeled. This is in contrast
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to Euclidean sweepouts (cf. Corollary \[23\]). Similar to Euclidean sweepouts, hyperbolic sweepouts correspond to corner-peeling and not min-peeling.

**Theorem 27** Any \(d\)-maximum class \(C \subseteq \{0,1\}^n\) corresponding to a simple hyperbolic \(d\)-arrangement \(A\) can be corner-peeled by sweeping \(A\) with a generic hyperbolic hyperplane.

**Proof** We follow the same strategy of the proof of Theorem 22. For sweeping in hyperbolic space \(H^k\), the generic hyperplane \(h\) is initialized as tangent to \(H^k\). As \(h\) is swept across \(H^k\), new intersections appear with \(A\) just after \(h\) meets the non-empty intersection of a subset of hyperplanes of \(A\) with the ideal boundary. Each \(d\)-cube \(C'\) in \(C\) still corresponds to the cells adjacent to the intersection \(I_{C'}\) of \(d\) hyperplanes. But now \(I_{C'}\) is a \((k-d)\)-dimensional hyperbolic hyperplane. A cell \(c\) adjacent to \(I_{C'}\) is corner-peeled precisely when \(h\) last intersects \(c\) at a point of \(I_{C'}\) at the ideal boundary. As for simple linear arrangements, the general position of \(A \cup \{h\}\) ensures that corner-peeling events never occur simultaneously.

For the case \(k = d+1\), as for the simple linear arrangements just prior to the corner-peeling of \(c\), \(H_+ \cap c\) is homeomorphic to a \((d+1)\)-simplex with a missing face on the ideal boundary. And so as in the simple linear case, this \(d\)-intersection point corresponds to a corner \(d\)-cube. In the case \(k > d+1\), \(H_+ \cap c\) becomes a \((d+1)\)-simplex (as before) multiplied by \(\mathbb{R}^{k-d-1}\). If \(k = d\), then the main difference is just before corner-peeling of \(c\), \(H_+ \cap c\) is homeomorphic to a \(k\)-simplex which may be either closed (hence in the interior of \(H^k\)) or with a missing face on the ideal boundary. The rest of the argument remains the same, except for one important observation.

Although swept corners in hyperbolic arrangements can be of cubes of differing dimensions, these dimensions never exceed \(d\) and so the proof that sweeping simple linear arrangements induces \(d\)-compression schemes is still valid. □

**Example 7** Constructed with lifting, Figure 5 completes the enumeration, up to symmetry, of the \(2\)-maximum classes in \(\{0,1\}^4\) begun with Example 3. These cases cannot be represented as simple Euclidean linear arrangements, since their boundaries do not satisfy the condition of Corollary 21 but can be represented as hyperbolic arrangements as in Figure 6. Figures 7 and 8 display the sweeping of a general hyperplane across the former arrangement and the corresponding corner-peeling. Notice that the corner-peeled cubes’ dimensions decrease and then increase.

**Corollary 28** There is no constant \(c\) so that all maximal classes of VC dimension \(d\) can be embedded into maximum classes corresponding to simple hyperbolic hyperplane arrangements of VC dimension \(d + c\).

This result follows from our counter-examples to Kuzmin & Warmuth’s minimum degree conjecture [Rubinstein et al. 2009].

Corollary 26 gives a proper superset of simple linear hyperplane arrangement-induced maximum classes as hyperbolic arrangements. We will prove in Section 7 that all maximum classes can be represented as PL hyperplane arrangements in a ball. These are the topological analogue of hyperbolic hyperplane arrangements. If the boundary of the ball is removed, then we obtain an arrangement of PL hyperplanes in Euclidean space.
6. Infinite Euclidean and Hyperbolic Arrangements

We consider a simple example of an infinite maximum class which admits corner-peeling and a compression scheme analogous to those of previous sections.

Example 8 Let $\mathcal{L}$ be the set of lines in the plane of the form $L_{2m} = \{(x, y) \mid x = m\}$ and $L_{2n+1} = \{(x, y) \mid y = n\}$ for $m, n \in \mathbb{N}$. Let $v_{00}$, $v_{0n}$, $v_{m0}$, and $v_{mn}$ be the cells bounded by the lines $\{L_2, L_3\}$, $\{L_2, L_{2n+1}, L_{2n+3}\}$, $\{L_{2m}, L_{2m+2}, L_3\}$, and $\{L_{2m}, L_{2m+2}, L_{2n+1}, L_{2n+3}\}$, respectively. Then the cubical complex $C$, with vertices $v_{mn}$, can be corner-peeled and hence compressed, using a sweepout by the lines $\{(x, y) \mid x + (1 + \epsilon)y = t\}$ for $t \geq 0$ and any small fixed irrational $\epsilon > 0$. $C$ is a 2-maximum class and the unlabeled compression scheme is also of size 2.

To verify the properties of this example, notice that sweeping as specified corresponds to corner-peeling the vertex $v_{00}$, then the vertices $v_{10}, v_{01}$, then the remaining vertices $v_{mn}$. The lines $x + (1 + \epsilon)y = t$ are generic as they pass through only one intersection point of $\mathcal{L}$ at a time. Additionally, representing $v_{00}$ by $\emptyset$, $v_{0n}$ by $\{L_{2n+1}\}$, $v_{m0}$ by $\{L_{2m}\}$ and $v_{mn}$ by $\{L_{2m}, L_{2n+1}\}$ constitutes a valid unlabeled compression scheme. Note that the compression scheme is associated with sweeping across the arrangement in the direction of decreasing $t$. This is necessary to pick up the boundary vertices of $C$ last in the sweepout process, so that they have either singleton representatives or the empty set. In this way, similar to Kuzmin and Warmuth (2007), we obtain a compression scheme so that every labeled sample of size 2 is associated with a unique concept in $C$, which is consistent with the sample. On the
Figure 7: The simple hyperbolic arrangement of Figure 6(a) with a generic sweeping hyperplane shown in several positions before and after it sweeps past four cells.

Figure 8: The 2-maximum class in \{0, 1\}^4 of Figure 5(a), with the first four corner-vertices peeled by the hyperbolic arrangement sweeping of Figure 7. Notice that three 2-cubes are peeled, then a 1-cube (all shown) followed by 2-cubes.

other hand to obtain corner-peeling, we need the sweepout to proceed with \( t \) increasing so that we can begin at the boundary vertices of \( C \).

In concluding this brief discussion, we note that many infinite collections of simple hyperbolic hyperplanes and Euclidean hyperplanes can also be corner-peeled and compressed, even if intersection points and cells accumulate. However a key requirement in the Euclidean case is that the concept class \( C \) has a non-empty boundary, when considered as a cubical complex. An easy approach is to assume that all the \( d \)-intersections of the arrangement lie in a half-space. Moreover, since the boundary must also admit corner-peeling, we require more conditions, similar to having all the intersection points lying in an octant.

**Example 9** In \( \mathbb{R}^3 \), choose the family of planes \( \mathcal{P} \) of the form \( P_{3n+i} = \{ x \in \mathbb{R}^3 \mid x_{i+1} = 1 - 1/n \} \) for \( n \geq 1 \) and \( i \in \{0, 1, 2\} \). A corner-peeling scheme is induced by sweeping a generic plane \( \{ x \in \mathbb{R}^3 \mid x_1 + \alpha x_2 + \beta x_3 = t \} \) across the arrangement, where \( t \) is a parameter and \( 1, \alpha, \beta \) are algebraically independent (in particular, no integral linear combination is rational) and \( \alpha, \beta \) are both close to 1. This example has similar properties to Example 8: the compression scheme is again given by decreasing \( t \) whereas corner-peeling corresponds to increasing \( t \). Note that cells shrink to points, as \( x \to 1 \) and the volume of cells converge to zero as \( n \to \infty \), or equivalently any \( x_i \to 1 \).
Example 10  In the hyperbolic plane $\mathbb{H}^2$, represented as the unit circle centered at the origin in $\mathbb{R}^2$, choose the family of lines $\mathcal{L}$ given by $L_{2n} = \{(x,y) \mid x = 1 - 1/n\}$ and $L_{2n+1} = \{(x,y) \mid x + ny = 1\}$, for $n \geq 1$. This arrangement has corner-peeling and compression schemes given by sweeping across $\mathcal{L}$ using the generic line $\{y = t\}$.

7. Piecewise-Linear Arrangements

A PL hyperplane is the image of a proper piecewise-linear homeomorphism from the $(k-1)$-ball $B^{k-1}$ into $B^k$, i.e., the inverse image of the boundary $S^{k-1}$ of the $k$-ball is $S^{k-2}$, (Rourke and Sanderson, 1982). A simple PL $d$-arrangement is an arrangement of $n$ PL hyperplanes such that every subcollection of $j$ hyperplanes meet transversely in a $(k-j)$-dimensional PL plane for $2 \leq j \leq d$ and every subcollection of $d+1$ hyperplanes are disjoint.

7.1 Maximum Classes are Represented by Simple PL Hyperplane Arrangements

Our aim is to prove the following theorem by a series of steps.

Theorem 29  Every $d$-maximum class $C \subseteq \{0,1\}^n$ can be represented by a simple arrangement of PL hyperplanes in an $n$-ball. Moreover the corresponding simple arrangement of PL hyperspheres in the $(n-1)$-sphere also represents $C$, so long as $n > d + 1$.

7.1.1 Embedding a $d$-Maximum Cubical Complex in the $n$-cube into an $n$-ball.

We begin with a $d$-maximum cubical complex $C \subseteq \{0,1\}^n$ embedded into $[0,1]^n$. This gives a natural embedding of $C$ into $\mathbb{R}^n$. Take a small regular neighborhood $\mathcal{N}$ of $C$ so that the boundary $\partial \mathcal{N}$ of $\mathcal{N}$ will be a closed manifold of dimension $n-1$. Note that $\mathcal{N}$ is contractible because it has a deformation retraction onto $C$ and so $\partial \mathcal{N}$ is a homology $(n-1)$-sphere (by a standard, well-known argument from topology due to [Mazur, 1961]). Our aim is to prove that $\partial \mathcal{N}$ is an $(n-1)$-sphere and $\mathcal{N}$ is an $n$-ball. There are two ways of proving this: show that $\partial \mathcal{N}$ is simply connected and invoke the well-known solution to the generalised Poincaré conjecture ([Smale, 1961]), or use the cubical structure of the $n$-cube and $C$ to directly prove the result. We adopt the latter approach, although the former works fine. The advantage of the latter is that it produces the required hyperplane arrangement, not just the structures of $\partial \mathcal{N}$ and $\mathcal{N}$.

7.1.2 Bisecting Sets

For each color $i$, there is a hyperplane $P_i$ in $\mathbb{R}^n$ consisting of all vectors with $i^{th}$ coordinate equal to 1/2. We can easily arrange the choice of regular neighborhood $\mathcal{N}$ of $C$ so that $\mathcal{N}_i = P_i \cap \mathcal{N}$ is a regular neighborhood of $C \cap P_i$ in $P_i$. (We call $\mathcal{N}_i$ a bisecting set as it intersects $C$ along the ‘center’ of the reduction in the $i^{th}$ coordinate direction, see Figure 9.) But then since $C \cap P_i$ is a cubical complex corresponding to the reduction $C^i$, by induction on $n$, we can assert that $\mathcal{N}_i$ is an $(n-1)$-ball. Similarly the intersections $\mathcal{N}_i \cap \mathcal{N}_j$ can be arranged to be regular neighborhoods of $(d-2)$-maximum classes and are also balls of dimension $n-2$, etc. In this way, we see that if we can show that $\mathcal{N}$ is an $n$-ball, then the
induction step will be satisfied and we will have produced a PL hyperplane arrangement (the system of $N_i$ in $\mathcal{N}$) in a ball.

7.1.3 Shifting

To complete the induction step, we use the technique of shifting [Alon, 1983; Frankl, 1983; Haussler, 1995]. In our situation, this can be viewed as the converse of lifting. Namely if a color $i$ is chosen, then the cubical complex $C$ has a lifted reduction $C'$ consisting of all $d$-cubes containing the $i^{th}$ color. By shifting, we can move down any of the lifted components, obtained by splitting $C$ open along $C'$, from the level $x_i = 1$ to the level $x_i = 0$, to form a new cubical complex $C^*$. We claim that the regular neighborhood of $C$ is a ball if and only if the same is true for $C^*$. But this is quite straightforward, since the operation of shifting can be thought of as sliding components of $C$, split open along $C'$, continuously from level $x_i = 1$ to $x_i = 0$. So there is an isotopy of the attaching maps of the components onto the lifted reduction, using the product structure of the latter. It is easy then to check that this does not affect the homeomorphism type of the regular neighborhood and so the claim of shift invariance is proved.

But repeated shifting finishes with the downwards closed maximum class consisting of all vertices in the $n$-cube with at most $d$ coordinates being one and the remaining coordinates all being zero. It is easy to see that the corresponding cubical complex $\tilde{C}$ is star-like, i.e., contains all the straight line segments from the origin to any point in $\tilde{C}$. If we choose a regular neighborhood $\tilde{N}$ to also be star-like, then it is obvious that $\tilde{N}$ is an $n$-ball, using radial projection. Hence our induction is complete and we have shown that any $d$-maximum class in $\{0, 1\}^n$ can be represented by a family of PL hyperplanes in the $n$-ball.

7.1.4 Ideal Boundary

To complete the proof of Theorem 29, let $\partial N = S^{n-1}$ denote the boundary of the $n$-ball $N$ constructed above (cf. Figures 10 and 11). Each PL hyperplane intersects this sphere in a PL hypersphere of dimension $n-2$. It remains to show this arrangement of hyperspheres gives the same cubical complex as $C$, unless $n = d + 1$. 

Figure 9: A 1-maximum class (thick solid lines) with its fattening (thin solid lines with points), bisecting sets (dashed lines) and induced complementary cells.
Suppose that \( n > d + 1 \). Then it is easy to see that each cell \( c \) in the complement of the PL hyperplane arrangement in \( \mathcal{N} \) has part of its boundary on the ideal boundary \( \partial \mathcal{N} \). Let \( \partial c = \partial c_+ \cup \partial c_- \), where \( \partial c_+ \) is the intersection of \( c \) with the ideal boundary and \( \partial c_- \) is the closure of \( \partial c \setminus \partial c_+ \).

It is now straightforward to verify that the face structure of \( \partial c_+ \) is equivalent to the face structure of \( \partial c_- \). Note that any family of at most \( d \) PL hyperplanes meet in a PL ball properly embedded in \( \mathcal{N} \). Since \( n > d + 1 \), the smallest dimension of such a ball is two, and hence its boundary is connected. Then \( \partial c_- \) has faces which are PL balls obtained in this way of dimension varying between \( n - d \) and \( n - 1 \). Each of these faces has boundary a PL sphere which is a face of \( \partial c_+ \). So this establishes a bijection between the faces of \( \partial c_+ \) and those of \( \partial c_- \). It is easy to check that the cubical complexes corresponding to the PL hyperplanes and to the PL hyperspheres are the same.

Note that if \( n = d + 1 \), then any \( d \)-maximum class \( C \subseteq \{0, 1\}^{d+1} \) is obtained by taking all the \( d \)-faces of the \((d+1)\)-cube which contain a particular vertex. So \( C \) is a \( d \)-ball and the ideal boundary of \( \mathcal{N} \) is a \( d \)-sphere. The cubical complex associated with the ideal boundary is the double \( 2C \) of \( C \), i.e., two copies of \( C \) glued together along their boundaries. The proof of Theorem \ref{thm:embedding} is now complete.

**Example 11** Consider the bounded below \( 2 \)-maximum class \( \tilde{C} \subseteq \{0, 1\}^5 \). We claim that \( \tilde{C} \) cannot be realized as an arrangement of PL hyperplanes in the \( 3 \)-ball \( B^3 \). Note that our method gives \( \tilde{C} \) as an arrangement in \( B^5 \) and this example shows that \( B^4 \) is the best one might hope for in terms of dimension of the hyperplane arrangement.

For suppose that \( \tilde{C} \) could be realized by any PL hyperplane arrangement in \( B^3 \). Then clearly we can also embed \( \tilde{C} \) into \( B^3 \). The vertex \( v_0 = \{0\}^5 \) has link given by the complete graph \( K \) on \( 5 \) vertices in \( \tilde{C} \). (By link, we mean the intersection of the boundary of a small ball in \( B^3 \) centered at \( v_0 \) with \( \tilde{C} \).) But as is well known, \( K \) is not planar, i.e., cannot be embedded into the plane or 2-sphere. This contradiction shows that no such arrangement is possible.
7.2 Maximum Classes with Manifold Cubical Complexes

We prove a partial converse to Corollary 21 if a $d$-maximum class has a ball as cubical complex, then it can always be realized by a simple PL hyperplane arrangement in $\mathbb{R}^d$.

**Theorem 30** Suppose that $C \subseteq \{0, 1\}^n$ is a $d$-maximum class. Then the following properties of $C$, considered as a cubical complex, are equivalent:

(i) There is a simple arrangement $A$ of $n$ PL hyperplanes in $\mathbb{R}^d$ which represents $C$.

(ii) $C$ is homeomorphic to the $d$-ball.

(iii) $C$ is a $d$-manifold with boundary.

**Proof** To prove (i) implies (ii), we can use exactly the same argument as Corollary 21. Next (ii) trivially implies (iii). So it remains to show that (iii) implies (i). The proof proceeds by double induction on $n, d$. The initial cases where either $d = 1$ or $n = 1$ are very easy.

Assume that $C$ is a manifold. Let $p$ denote the $i$th coordinate projection. Then $p(C)$ is obtained by collapsing $C^i \times [0, 1]$ onto $C^i$, where $C^i$ is the reduction. As before, let $P_i$ be the linear hyperplane in $\mathbb{R}^n$, where the $i$th coordinate takes value $1/2$. Viewing $C$ as a manifold embedded in the $n$-cube, since $P_i$ intersects $C$ transversely, we see that $C^i \times \{1/2\}$ is a proper submanifold of $C$. But it is easy to check that collapsing $C^i \times [0, 1]$ to $C^i$ in $C$ produces a new manifold which is again homeomorphic to $C$. (The product region $C^i \times [0, 1]$ in $C$ can be expanded to a larger product region $C^i \times [-\epsilon, 1 + \epsilon]$ and so collapsing shrinks the larger region to one of the same homeomorphism type, namely $C^i \times [-\epsilon, \epsilon]$.) So we conclude that the projection $p(C)$ is also a manifold. By induction on $n$, it follows that there is a PL hyperplane arrangement $A$, consisting of $n - 1$ PL hyperplanes in $B^d$, which represents $p(C)$.

Next, observe that the reduction $C^i$ can be viewed as a properly embedded submanifold $M$ in $B^d$, where $M$ is a union of some of the $(d - 1)$-dimensional faces of the Voronoi cell decomposition corresponding to $A$, described in Corollary 21. By induction on $d$, we conclude that $C^i$ is also represented by $n$ PL hyperplanes in $B^{d-1}$. But then since condition (i) implies (ii), it follows that $M$ is PL homeomorphic to $B^{d-1}$, since the underlying cubical complex for $C^i$ is a $(d - 1)$-ball. So it follows that $A \cup \{M\}$ is a PL hyperplane arrangement in $B^d$ representing $C$. This completes the proof that condition (iii) implies (i). □

8. Corner-Peeling 2-Maximum Classes

We give a separate treatment for the case of 2-maximum classes, since it is simpler than the general case and shows by a direct geometric argument, that representation by a simple family of PL hyperplanes or PL hyperspheres implies a corner-peeling scheme.

**Theorem 31** Every 2-maximum class can be corner-peeled.

**Proof** By Theorem 29 we can represent any 2-maximum class $C \subseteq \{0, 1\}^n$ by a simple family of PL hyperspheres $\{S_i\}$ in $S^{n-1}$. Every pair of hyperspheres $S_i, S_j$ intersects in an
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\[(n - 3)\)-sphere \(S_{ij}\) and there are no intersection points between any three of these hyperspheres. Consider the family of spheres \(S_{ij}\), for \(i\) fixed. These are disjoint hyperspheres in \(S_i\) so we can choose an innermost one \(S_{ik}\) which bounds an \((n - 2)\)-ball \(B_1\) in \(S_i\) not containing any other of these spheres. Moreover there are two balls \(B_2, B_3\) bounded by \(S_{ik}\) on \(S_k\). We call the two \((n - 1)\)-balls \(Q_2, Q_3\) bounded by \(B_1 \cup B_2, B_1 \cup B_3\) respectively in \(S^{n-1}\), which intersect only along \(B_1\), quadrants.

Assume \(B_2\) is innermost on \(S_k\). Then the quadrant \(Q_2\) has both faces \(B_1, B_2\) innermost. It is easy to see that such a quadrant corresponds to a corner vertex in \(C\) which can be peeled. Moreover, after peeling, we still have a family of PL hyperspheres which give an arrangement corresponding to the new peeled class. The only difference is that cell \(Q_2\) disappears, by interchanging \(B_1, B_2\) on the corresponding spheres \(S_i, S_k\) and then slightly pulling the faces apart. (If \(n = 3\), we can visualize a pair of disks on two intersecting spheres with a common boundary circle. Then peeling can be viewed as moving these two disks until they coincide and then pulling the first past the second). So it is clear that if we can repeatedly show that a quadrant can be found with two innermost faces, until all the intersections between the hyperspheres have been removed, then we will have corner-peeled \(C\) to a \(1\)-maximum class, i.e., a tree. So peeling will be established.

Suppose neither of the two quadrants \(Q_2, Q_3\) has both faces innermost. Consider \(Q_2\) say and let \(\{S_\alpha\}\) be the family of spheres intersecting the interior of the face \(B_2\). Amongst these spheres, there is clearly at least one \(S_\beta\) so that the intersection \(S_{k\beta}\) is innermost on \(S_k\). But then \(S_{k\beta}\) bounds an innermost ball \(B_4\) in \(S_k\) whose interior is disjoint from all the spheres \(\{S_\alpha\}\). Similarly, we see that \(S_{k\beta}\) bounds a ball \(B_5\) which is the intersection of the sphere \(S_\beta\) with the quadrant \(Q_2\). We get a new quadrant bounded by \(B_4 \cup B_5\) which is strictly smaller than \(Q_2\) and has at least one innermost face. But clearly this process must terminate—we cannot keep finding smaller and smaller quadrants and so a smallest one must have both faces innermost.

9. Corner-Peeling Finite Maximum Classes

Above, simple PL hyperplane arrangements in the \(n\)-ball \(B^n\) are defined. For the purposes of this section, we study a slightly more general class of arrangements.

**Definition 32** Suppose that a finite arrangement \(P\) of PL hyperplanes \(\{P_\alpha\}\), each properly embedded in an \(n\)-ball \(B^n\), satisfies the following conditions:

1. Each \(k\)-subcollection of hyperplanes either intersects transversely in a PL plane of dimension \(n - k\), or has an empty intersection; and
2. The maximum number of hyperplanes which mutually intersect is \(d \leq n\).

Then we say that the arrangement \(P\) is \(d\)-contractible.

The arrangements in Definition 32 are called \(d\)-contractible because we prove later that their corresponding one-inclusion graphs are strongly contractible cubical complexes of dimension \(d\). Moreover we now prove that the corresponding one-inclusion graphs have VC dimension exactly \(d\).
Lemma 33 The one-inclusion graph $\Gamma$ corresponding to a $d$-contractible arrangement $\mathcal{P}$ has VC dimension $d$.

**Proof** We observe first of all, that since $\mathcal{P}$ has a subcollection of $d$ hyperplanes which mutually intersect, the corresponding one-inclusion graph $\Gamma$ has a $d$-subcube, when considered as a cubical complex. But then the VC dimension of $\Gamma$ is clearly at least $d$. On the other hand, suppose that the VC dimension of $\Gamma$ was greater than $d$. Then there is a projection of $\Gamma$ which shatters some $(d+1)$-cube. But this projection can be viewed as deleting all the hyperplanes of $\mathcal{P}$ except for a subcollection of $d+1$ hyperplanes. However, by assumption, such a collection cannot have any mutual intersection points. It is easy to see that any such an arrangement has at most $2^{d+1} - 1$ complementary regions and hence cannot represent the $(d+1)$-cube. This completes the proof. \hfill $\blacksquare$

**Definition 34** A one-inclusion graph $\Gamma$ is strongly contractible if it is contractible as a cubical complex and moreover, all reductions and multiple reductions of $\Gamma$ are also contractible.

**Definition 35** The complexity of a PL hyperplane arrangement $\mathcal{P}$ is the lexicographically ordered pair $(r,s)$, where $r$ is the number of regions in the complement of $\mathcal{P}$, and $s$ is the smallest number of regions in any half space on one side of an individual hyperplane in $\mathcal{P}$.

We allow several different hyperplanes to be used for a single sweeping process. So a hyperplane $P$ may start sweeping across an arrangement $\mathcal{P}$. One of the half spaces defined by $P$ can become a new ball $B_+$ with a new arrangement $\mathcal{P}_+$ defined by restriction of $\mathcal{P}$ to the half space $B_+$. Then a second generic hyperplane $P'$ can start sweeping across this new arrangement $\mathcal{P}_+$. This process may occur several times. It is easy to see that sweeping a single generic hyperplane as in Theorem 27 applies to such a multi-hyperplane process. Below we show that a suitable multiple sweeping of a PL hyperplane arrangement $\mathcal{P}$ gives a corner-peeling sequence of all finite maximum classes.

The following states our main theorem.

**Theorem 36** Assume that $\mathcal{P}$ is a $d$-contractible PL hyperplane arrangement in the $n$-ball $B^n$. Then there is a $d$-corner-peeling scheme for this collection $\mathcal{P}$.

**Corollary 37** There is no constant $k$ so that every finite maximal class of VC dimension $d$ can be embedded into a maximum class of VC dimension $d+k$.

**Proof** By Theorem 36 every maximum class has a peeling scheme which successively removes vertices from the one-inclusion graph, so that the vertices being discarded never have degree more than $d$. But Rubinstein et al. (2007) gave examples of maximal classes of VC dimension $d$ which have a core of the one-inclusion graph of size $d+k$ for any constant $k$. Recall that a core is a subgraph and its size is the minimum degree of all the vertices. Having a peeling scheme gives an upper bound on the size of any core and so the result follows. \hfill $\blacksquare$
9.1 Pachner Moves

Pachner (1987) showed that triangulations of manifolds which are combinatorially equivalent after subdivision are also equivalent by a series of moves which are now referred to as Pachner moves. For the main result of this paper, we need a version of Pachner moves for cubical structures rather than simplicial ones. The main idea of Pachner moves remain the same.

A Pachner move replaces a topological $d$-ball $U$ divided into $d$-cubes, with another ball $U'$ with the same $(d-1)$-cubical boundary but with a different interior cubical structure. In dimension $d = 2$, for example, such an initial ball $U$ can be constructed by taking three 2-cubes forming a hexagonal disk and in dimension $d = 3$, four 3-cubes form a rhombic dodecahedron, which is a polyhedron $U$ with 12 quadrilateral faces in its boundary. The set $U'$ of $d$-cubes is attached to the same boundary as for $U$, i.e., $\partial U = \partial U'$, as cubical complexes homeomorphic to the $(d-1)$-sphere. Moreover, $U'$ and $U$ are isomorphic cubical complexes, but the gluing between their boundaries produces the boundary of the 3- or 4-cube, as a 2- or 3-dimensional cubical structure on the 2- or 3-sphere respectively.

To better understand this move, consider the cubical face structure of the boundary $V$ of the $(d+1)$-cube. This is a $d$-sphere containing $2d + 2$ cubes, each of dimension $d$. There are many embeddings of the $(d-1)$-sphere as a cubical subcomplex into $V$, dividing it into a pair of $d$-balls. One ball is combinatorially identical to $U$ and the other to $U'$.

There are a whole series of Pachner moves in each dimension $d$, but we are only interested in the ones where the pair of balls $U, U'$ have the same numbers of $d$-cubes. In Figures 12 and 13 a change in a hyperplane arrangement is shown, which corresponds to a Pachner move on the corresponding one-inclusion graph (considered as a cubical complex).
9.2 Proof of Main Theorem

The proof is by induction on the complexity of $\mathcal{P}$. Since we are dealing with the class of $d$-contractible PL hyperplane arrangements, it is easy to see that if any such $\mathcal{P}$ is split open along some fixed hyperplane $P_1$ in the arrangement (see Figure 9.2), then the result is two new arrangements $\mathcal{P}_+, \mathcal{P}_-$ each of which contains fewer hyperplanes and also fewer complementary regions than the initial one. The new arrangements have smaller complexity than $\mathcal{P}$ and are $k-, k'$-contractible for some $k, k' \leq d$. This is the key idea of the construction.

To examine this splitting process in detail, first note that each hyperplane $P_\alpha$ of $\mathcal{P}$ is either disjoint from $P_1$ or splits along $P_1$ into two hyperplanes $P_\alpha^+, P_\alpha^-$. We can now construct the new PL hyperplane arrangements $\mathcal{P}_+, \mathcal{P}_-$ in the balls $B_+, B_-$ obtained by splitting $B$ along $P_1$. Note that $\partial B_+ = P_1 \cup D_+$ and $\partial B_- = P_1 \cup D_-$ where $D_+, D_-$ are balls of dimension $n-1$ which have a common boundary with $P_1$. It is easy to verify that $\mathcal{P}_+, \mathcal{P}_-$ satisfy similar hypotheses to the original arrangement. Observe that the maximum number of mutually intersecting hyperplanes in $\mathcal{P}_+, \mathcal{P}_-$ may decrease relative to this number for $\mathcal{P}$, after the splitting operation. The reason is that the hyperplane $P_1$ ‘disappears’ after splitting and so if all maximum subcollections of $\mathcal{P}$ which mutually intersect, all contain $P_1$, then this number is smaller for $\mathcal{P}_+, \mathcal{P}_-$ as compared to the initial arrangement $\mathcal{P}$. This number shows that $\mathcal{P}_+, \mathcal{P}_-$ can be $k$- or $k'$-contractible, for $k, k' < d$ as well as the cases where $k, k' = d$.

Start the induction with any arrangement with one hyperplane. This gives two regions and complexity $(2, 1)$. The corresponding graph has one edge and two vertices and obviously can be corner-peeled.

We now describe the inductive step. There are two cases. In the first, assume the arrangement has complexity $(r, 1)$. The corresponding graph has a vertex which belongs to only one edge, so can be corner-peeled. This gives an arrangement with fewer hyperplanes.
and clearly the complexity has decreased to \((r - 1, s)\) for some \(s\). This completes the inductive step for the first case.

For the second case, assume that all \(d\)-contractible hyperplane arrangements with complexity smaller than \((r, s)\) have corner-peeling sequences and \(s > 1\). Choose any \(d\)-contractible hyperplane arrangement \(\mathcal{P}\) with complexity \((r, s)\). Select a hyperplane \(P_1\) which splits the arrangement into two smaller arrangements \(\mathcal{P}_+, \mathcal{P}_-\) in the balls \(B_+, B_-\). By our definition of complexity, it is easy to see that however we choose \(P_1\), the complexity of each of \(\mathcal{P}_+, \mathcal{P}_-\) will be less than that of \(\mathcal{P}\). However, a key requirement for the proof will be that we select \(P_1\) so that it has precisely \(s\) complementary regions for \(\mathcal{P}_+\), i.e., \(P_1\) has fewest complementary regions in one of its halfspaces, amongst all hyperplanes in the arrangement.

Since \(\mathcal{P}_+\) has smaller complexity than \((r, s)\), by our inductive hypothesis, it can be corner-peeled (cf. Figure 15). If any of the corner-peeling moves of \(\mathcal{P}_+\) is a corner-peeling move for \(\mathcal{P}\), then the argument follows. For any corner-peeling move of \(\mathcal{P}\) gives a PL hyperplane arrangement with fewer complementary cells than \(\mathcal{P}\) and thus smaller complexity than \((r, s)\). Hence by the inductive hypothesis, it follows that \(\mathcal{P}\) can be corner-peeled.

Next, suppose that no corner-peeling move of \(\mathcal{P}_+\) is a corner-peeling move for \(\mathcal{P}\). In particular, the first corner-peeling move of \(\mathcal{P}_+\) must occur for a cell \(R_+\) in the complement of \(\mathcal{P}\), which is adjacent to \(P_1\). (Clearly any corner-peeling move for \(\mathcal{P}_+\), which occurs at a region \(R_1\) with a face on \(D_+\), will be a corner-peeling move for \(\mathcal{P}\).) \(R_+\) must be a product of a \(d'\)-simplex \(\Delta\) with a copy of \(\mathbb{R}^{n-d'}\), with one face on \(P_1\) and the other faces on planes of \(\mathcal{P}\). This is because a corner-peeling move can only occur at a cell with this type of face structure, as described in Theorem 27. The corresponding effect on the one-inclusion graph is peeling of a vertex which is a corner of a \(d'\)-cube in the binary class corresponding to the arrangement \(\mathcal{P}_+\), where \(d' \leq d\).

Now even though such a cell \(R_+\) does not give a corner-peeling move for \(\mathcal{P}\), we can push \(P_1\) across \(R_+\). The effect of this is to move the complementary cell \(R_+\) from \(B^+\) to \(B^-\).
Figure 16: A 2-maximum complex in the 3-cube. After a Pachner move vertices $v_1, v_2, v_3$, etc. can be corner-peeled.

Moreover, since we assumed that the hyperplane $P_1$ satisfies $P^+$ has a minimum number $s$ of complementary regions, it follows that the move pushing $P_1$ across $R_+$ produces a new arrangement $P^*$ with smaller complexity $(r, s - 1)$ than the original arrangement $P$. Hence by our inductive assumption, $P^*$ admits a corner-peeling sequence.

To complete the proof, we need to show that existence of a corner-peeling sequence for $P^*$ implies that the original arrangement $P$ has at least one corner-peeling move. Recall that $R_+$ has face structure given by $\Delta \times \mathbb{R}^{n-d'}$, with one face on $P_1$ and the other faces on planes of $P$. Consider the subcomplex $U$ of the one-inclusion graph consisting of all the regions sharing a vertex or face of dimension $k$ for $1 \leq k \leq n - 1$ with $R_+$. It is not difficult to see that $U$ is a $d'$-ball consisting of $d' + 1$ cubes, each of dimension $d'$. (As examples, if $d' = 2$, $U$ consists of 3 2-cubes forming a hexagon and if $d' = 3$, $U$ consists of 4 3-cubes with boundary a rhombic dodecahedron.)

Consider the first corner-peeling move on the arrangement $P^*$. Note that the one-inclusion graphs of $P^*$ and $P$ differ precisely by replacing $U$ with $U'$, i.e., by a Pachner move. Hence this first corner-peeling move must occur at a vertex $v_1$ whose degree is affected by this replacement, since otherwise, the corner-peeling move would also apply to $P$ and the proof would be complete. In fact, if $v_1$ has the same number of adjacent edges before and after the Pachner move, then it must belong to the same single maximum dimension cube before and after the Pachner move. (The only cubes altered by the Pachner move are the ones in $U$.) It is easy to see that, $v_1$ must belong to $\partial U = \partial U'$ and must have degree $d'$ in $P^*$. So $v_1$ is a corner of a single $d'$-cube for $U'$ and does not belong to any other edges or cubes of the one inclusion graph for $P^*$. In $U$ (and hence also in $P$), $v_1$ belongs to $d'$-cubes of dimension $d'$ and so has degree $d' + 1$. After peeling away $v_1$ and its corresponding $d'$-cube, we still have a $d'$-ball with only $d'$-cubes, (cf. Figure 16).

Consider the next corner-peeling move. We claim that it must again be at a vertex $v_2$ belonging to $\partial U'$. The reason is that only vertices belonging to $U'$ have degree reduced by our first corner-peeling move. So if this second move did not occur at a vertex of $U'$, then it could be used as a corner-peeling move of our initial arrangement $P$. There may be several choices for $v_2$. For example, if $d' = 2$, then $U'$ is a hexagonal disk and removing one 2-cube
from $U'$ gives a choice which could be either of the two vertices which are corners of a single 2-cube in $U'$, (cf. Figure 16). Note that a vertex which is a corner of a single cube in $U'$ remains so after corner-peeling at $v_1$. Note also that $v_2$ cannot belong to any edges of the one-inclusion graph which are not in $U'$, as for $v_1$, if $v_2$ can be used for corner-peeling.

We can continue examining corner-peeling moves of $P^*$ and find that all must occur at vertices in $\partial U'$, until the unique interior vertex is ready to be peeled, i.e., belongs to a single cube. (See Figure 16) The key to understanding this is that firstly, when we initially peel only vertices in $\partial U'$, these are not adjacent to any vertices of the one-inclusion graph outside $U'$ and so cannot produce any new opportunities for corner-peeling of vertices not in $U'$. Secondly, if the unique interior vertex $v$ of $U'$ can be corner-peeled, after sufficiently many vertices in $\partial U'$ have been peeled, then new vertices in $\partial U'$ become candidates for peeling. For although these latter vertices may be adjacent to vertices outside $U'$, after $v$ has been peeled, they may become a corner vertex of a unique maximal cube.

But now a final careful examination of this situation shows that there must be at least one vertex of $U$ which belongs to a single $d'$-cube in $U$ and to no other edges in $P$. So this will give our initial corner-peeling move of $P$.

To elaborate, we can describe $U$ as the set of $d'$-cubes which share the vertex $(0, 0, \ldots, 0)$ in the $(d'+1)$-cube $\{0, 1\}^{d'+1}$. Then $U'$ consists of all the $d'$-cubes in $\{0, 1\}^{d'+1}$ which contain the vertex $(1, 1, \ldots, 1)$. Now assume that an initial sequence of corner-peeling of vertices in $\partial U'$ allows the next step to be corner-peeling of the unique interior vertex $v$. Note that in the notation above, $v$ corresponds to the vertex $(1, 1, \ldots, 1)$.

As in Figure 16, we may assume that after the corner-peeling corresponding to the initial sequence of vertices in $\partial U'$, that there is a single $d'$-cube left in $U'$ containing $v$. Without loss of generality, suppose this is the cube with vertices with $x_1 = 1$ where the coordinates are $x_1, x_2, \ldots, x_{d'+1}$ in $\{0, 1\}^{d'+1}$. But then, it follows that there are no vertices outside $U'$ adjacent to any of the initial sequence of vertices, which are all the vertices in $\{0, 1\}^{d'+1}$ with $x_1 = 0$, except for $(0, 0, \ldots, 0)$. But now the vertex $(0, 1, \ldots, 1)$ has the property that we want - it is contained in a unique $d'$-cube in $U$ and is adjacent to no other vertices outside $U$. This completes the proof.

### 9.3 Peeling Classes with Generic Linear or Generic Hyperbolic Arrangements

**Definition 38** A linear or hyperbolic hyperplane arrangement $P$ in $\mathbb{R}^n$ or $\mathbb{H}^n$ respectively, is called generic, if any subcollection of $k$ hyperplanes of $P$, for $2 \leq k \leq n$ has the property that there are no intersection points or the subcollection intersects transversely in a plane of dimension $n - k$.

**Corollary 39** Suppose $P$ is a generic linear or hyperbolic hyperplane arrangement in $\mathbb{R}^n$ or $\mathbb{H}^n$ and amongst all subcollections of $P$, the largest with an intersection point in common, has $d$ hyperplanes. Then $P$ admits a $d$-corner-peeling scheme.

**Remark 40** The proof of Corollary 39 is immediate since it is obvious that any generic linear or hyperbolic hyperplane arrangement is a $d$-contractible PL hyperplane arrangement, where $d$ is the cardinality of the largest subcollection of hyperplanes which mutually intersect. Note that many generic linear, hyperbolic or $d$-contractible PL hyperplane arrangements do not embed in any simple linear, hyperbolic or PL hyperplane arrangement. For if there
are two hyperplanes in $\mathcal{P}$ which are disjoint, then this is an obstruction to enlarging the arrangement by adding additional hyperplanes to obtain a simple arrangement. Hence this shows that Theorem 36 produces compression schemes, by corner-peeling, for a considerably larger class of one-inclusion graphs than just maximum one-inclusion graphs. However it seems possible that $d$-contractible PL hyperplanes always embed in $d$-maximum classes, by ‘undoing’ the operation of sweeping and corner-peeling, which pulls apart the hyperplanes.

10. Peeling Infinite Maximum Classes with Finite Dimensional Arrangements

We seek infinite classes represented by arrangements satisfying the same conditions as above. Note that any finite subclass of such an infinite class then satisfies these conditions and so can be corner-peeled. Hence any such a finite subclass has a complementary region $R$ which has face structure of the product of a $d'$-simplex with a copy of $\mathbb{R}^{n-d}$ with one face on the boundary of $B^n$. To find such a region in the complement of our infinite collection $\mathcal{P}$, we must impose some conditions.

One convenient condition (cf the proof of Theorem 36) is that a hyperplane $P_\alpha$ in $\mathcal{P}$ can be found which splits $B^n$ into pieces $B^+_\alpha, B^-_\alpha$ so that one, say $B^+_\alpha$ gives a new arrangement for which the maximum number of mutually intersecting hyperplanes is strictly less than that for $\mathcal{P}$. Assume that the new arrangement satisfies a similar condition, and we can keep splitting until we get to disjoint hyperplanes.

It is not hard to prove that such arrangements always have peeling sequences. Moreover the peeling sequence does give a compression scheme. This sketch establishes the following.

**Theorem 41** Suppose that a countably infinite collection $\mathcal{P}$ of PL hyperplanes $\{P_\alpha\}$, each properly embedded in an $n$-ball $B^n$, satisfies the following conditions:

i. $\mathcal{P}$ satisfies the conditions of $d$-contractible arrangements as in Definition 32 and

ii. There is an ordering of the planes in $\mathcal{P}$ so that if we split $B^n$ successively along the planes, then at each stage, at least one of the two resulting balls has an arrangement with a smaller maximum number of planes which mutually intersect.

Then there is a $d$-corner-peeling scheme for $\mathcal{P}$, and this provides a $d$-unlabeled compression scheme.

**Example 12** [Rubinstein and Rubinstein 2008] give an example that satisfies the assumptions of Theorem 41. Namely in $\mathbb{R}^n$ choose the positive octant $O = \{(x_1, x_2, \ldots, x_n) : x_i \geq 0\}$. Inside $O$ choose the collection of hyperplanes given by $x_i = m$ for all $1 \leq i \leq n$ and $m \geq 1$ a positive integer. There are many more examples, we present only a very simple model here. Take a graph inside the unit disk $D$ with a single vertex of degree 3 and the three end vertices on $\partial D$. Now choose a collection of disjoint embedded arcs representing hyperplanes with ends on $\partial D$ and meeting one of the edges of the graph in a single point. We choose finitely many such arcs along two of the graph edges and an infinite collection along one arc. This gives a very simple family of hyperplanes satisfying the hypotheses of Theorem 41. Higher dimensional examples with intersecting hyperplanes based on arbitrary trees can be constructed in a similar manner.
11. Contractibility, Peeling and Arrangements

In this section, we characterize the concept classes which have one-inclusion graphs representable by $d$-contractible PL hyperplane arrangements.

**Theorem 42** Assume that $C$ is a concept class in the binary $n$-cube and $d$ is the largest dimension of embedded cubes in its one-inclusion graph $\Gamma$. The following are equivalent.

1. $\Gamma$ is a strongly contractible cubical complex.

2. There is a $d$-contractible PL hyperplane arrangement $P$ in an $n$-ball which represents $\Gamma$.

**Proof** To prove that i implies ii, we use some important ideas in the topology of manifolds. The cubical complex $C$ is naturally embedded into the binary $n$-cube, which can be considered as an $n$-ball $B^n$. A regular neighborhood $N$ of $C$ homotopy retracts onto $C$ and so is contractible. Now we can use a standard argument from algebraic and geometric topology to prove that $N$ is a ball. Firstly, $\partial N$ is simply connected, assuming that $n - d > 2$. For given a loop in $\partial N$, it bounds a disk in $N$ by contractibility. Since $C$ is a $d$-dimensional complex and $n - d > 2$ it follows that this disk can be pushed off $C$ by transversality and then pushed into $\partial N$. But now we can follow a standard argument using the solution of the Poincaré conjecture in all dimensions (Perelman, 2002; Freedman, 1982; Smale, 1961). By duality, it follows that $\partial N$ is a homotopy $(n - 1)$-sphere and so by the Poincaré conjecture, $\partial N$ is an $(n - 1)$-sphere. Another application of the Poincaré conjecture shows that $N$ is an $n$-ball.

Next, the bisecting planes of the binary $n$-cube meet the $n$-ball $N$ in neighborhoods of reductions. Hence the assumption that each reduction is contractible enables us to conclude that these intersections are also PL hyperplanes in $N$. Therefore the PL hyperplane arrangement has been constructed which represents $\Gamma$. It is easy to see that this arrangement is indeed $d$-contractible, since strong contractibility implies that all multiple reductions are contractible and so intersections of subfamilies of PL hyperplanes are either empty or are contractible and hence planes, by the same argument as the previous paragraph. (Note that such intersections correspond to multiple reductions of $\Gamma$.)

Finally to show that ii implies i, by Theorem 36, a $d$-contractible PL hyperplane arrangement $P$ has a peeling sequence and so the corresponding one-inclusion graph $\Gamma$ is contractible. This follows since a corner-peeling move can be viewed as a homotopy retraction. But then reductions and multiple reductions are also represented by $d'$-contractible hyperplane arrangements, since these correspond to the restriction of $P$ to the intersection of a finite subfamily of hyperplanes of $P$. It is straightforward to check that these new arrangements are $d'$-contractible, completing the proof.

**Remark 43** Note that any one-inclusion graph $\Gamma$ which satisfies the hypotheses of Theorem 42 admits a corner-peeling sequence. From the proof above, $\Gamma$ must be contractible if it has a peeling sequence. However $\Gamma$ does not have to be strongly contractible. A simple example can be found in the binary 3-cube, with coordinate directions $x, y, z$. Define $\Gamma$ to
be the union of four edges, labeled $x, y, z, x$. It is easy to see that $\Gamma$ has a peeling sequence and is contractible but not strongly contractible. For the bisecting hyperplane transverse to the $x$ direction meets $\Gamma$ in two points, so the reduction $\Gamma^x$ is a pair of vertices, which is not contractible.

Note that all maximum classes are strongly contractible, as are also all linear and hyperbolic arrangements, by Corollary 39 and Theorem 42.

12. Future Directions: Compression Schemes for Maximal Classes

In this section, we compare two maximal classes of VC dimension 2 in the binary 4-cube. For the first, we show that the one-inclusion graph is not contractible and therefore there is no peeling or corner peeling scheme. There is an unlabeled compression scheme, but this is not associated with either peeling or a hyperplane arrangement. For the second, the one-inclusion graph is contractible but not strongly contractible. However there are simple corner peeling schemes and a related compression scheme. Note that the relation between the compression scheme and the corner peeling scheme is not as straightforward as in our main result above. Finally for the second example, there is a non simple hyperplane arrangement consisting of lines in the hyperbolic plane which represents the class. It would be interesting to know if there are many maximal classes which admit such non simple representations and if there is a general procedure to find associated compression schemes.

Example 13 Let $C$ be the maximal class of VC dimension 2 in the 4-cube with concepts and labels shown in Figure 17. This forms an unlabeled compression scheme. Note that the one-inclusion graph is not connected, consisting of four 2-cubes with common vertex at the origin 0000 and an isolated vertex at 1111. So since a contractible complex is connected, the one-inclusion graph cannot be contractible. Moreover any hyperplane arrangement represents a connected complex so there cannot be such an arrangement for this example. This example is the same class (up to flipping coordinate labels) as in (Kuzmin and Warmuth, 2007, Table 2) but there appear to be some errors there in describing the compression scheme.
Example 14 Let $C$ be the maximal class of VC dimension 2 in the 4-cube with concepts and labels defined in Figure 18. The class is enlarged by adding an extra vertex $1111$ to complete the labeling.

This forms an unlabeled compression scheme and is the same as in \cite[Table 1]{Kuzmin and Warmuth:2007}. The one-inclusion graph is contractible, consisting of three 2-cubes with common vertex $0100$ and three edges attached to these 2-cubes. It is easy to form a hyperbolic line arrangement consisting of three lines meeting in three points forming a triangle and three further lines near the boundary of the hyperbolic plane which do not meet any other line.

It is easy to see that there is a corner peeling sequence, but there is not such an obvious way of using this to form a compression scheme. The idea is that the label $x_1x_4$ comes from picking the origin at 0000 and considering the shortest path to the origin as giving the label. There are numerous ways of corner peeling this one-inclusion complex. The only other comment is that the final vertices $0111$, $1011$, $1101$ and $1111$ are labeled in a different manner. Namely putting the origin at 0000 means that $0111$ has shortest path with label $x_2x_3x_4$. We replace this by the label $x_1x_4$ since clearly this satisfies the no-clashing condition. Then the final vertex $1111$ has the remaining label $x_4$ to uniquely specify it.

13. Conclusions and Open Problems

We saw in Corollary 21 that $d$-maximum classes represented by simple linear hyperplane arrangements in $\mathbb{R}^d$ have underlying cubical complexes that are homeomorphic to a $d$-ball. Hence the VC dimension and the dimension of the cubical complex are the same. Moreover in Theorem 30 we proved that $d$-maximum classes represented by PL hyperplane arrangements in $\mathbb{R}^d$ are those whose underlying cubical complexes are manifolds or equivalently $d$-balls.

Question 44 Does every simple PL hyperplane arrangement in $B^d$, where every subcollection of $d$ planes transversely meet in a point, represent the same concept class as some simple linear hyperplane arrangement?

Question 45 What is the connection between the VC dimension of a maximum class induced by a simple hyperbolic hyperplane arrangement and the smallest dimension of hyperbolic space containing such an arrangement? In particular, can the hyperbolic space dimension be chosen to only depend on the VC dimension and not the dimension of the binary cube containing the class?

We gave an example of a 2-maximum class in the 5-cube that cannot be realized as a hyperbolic hyperplane arrangement in $\mathcal{H}^3$. Note that the Whitney embedding theorem \cite{Rourke and Sanderson:1982} proves that any cubical complex of dimension $d$ embeds in $\mathbb{R}^{2d}$. Can such an embedding be used to construct a hyperbolic arrangement in $\mathcal{H}^{2d}$ or a PL arrangement in $\mathbb{R}^{2d}$?

The structure of the boundary of a maximum class is strongly related to corner-peeling. For Euclidean hyperplane arrangements, the boundary of the corresponding maximum class is homeomorphic to a sphere by Corollaries 20 and 21.
Questions

**Question 46** Is there a characterization of the cubical complexes that can occur as the boundary of a maximum class? Characterize maximum classes with isomorphic boundaries.

**Question 47** Does a corner-peeling scheme exist with corner vertex sequence having minimum degree?

Theorem 29 suggests the following.

**Question 48** Can any \(d\)-maximum class in \(\{0,1\}^n\) be represented by a simple arrangement of hyperplanes in \(\mathbb{H}^n\)?

**Question 49** Which compression schemes arise from sweeping across simple hyperbolic hyperplane arrangements?

Kuzmin and Warmuth (2007) note that there are unlabeled compression schemes that are cyclic. In Proposition 15 we show that corner-peeling compression schemes (like min-peeling) are acyclic. So compression schemes arising from sweeping across simple arrangements of hyperplanes in Euclidean or Hyperbolic space are also acyclic. Does acyclicity characterize such compression schemes?

We have established peeling of all finite maximum and a family of infinite maximum classes by representing them as PL hyperplane arrangements and sweeping by multiple generic hyperplanes. A larger class of arrangements has these properties—namely those which are \(d\)-contractible—and we have shown that the corresponding one-inclusion graphs are precisely the strongly contractible ones. Finally we have established that there are \(d\)-maximal classes that cannot be embedded in any \((d + k)\)-maximum classes for any constant \(k\). Some important open problems along these lines are the following.

**Question 50** Prove peeling of maximum classes using purely combinatorial arguments

**Question 51** Can all maximal classes be peeled by representing them by hyperplane arrangements and then using a sweeping technique (potentially solving the Sample Compressibility conjecture)? The obvious candidate for this approach is to use \(d\)-contractible PL hyperplane arrangements.

**Question 52** What about more general collections of infinite maximum classes, or infinite arrangements?

**Question 53** Is it true that any \(d\)-contractible PL hyperplane arrangement is equivalent to a Hyperbolic hyperplane arrangement?

**Question 54** Is it true that all strongly contractible classes, with largest dimension \(d\) of cubes can be embedded in maximum classes of VC dimension \(d\)?

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