Violating Bell’s inequality using a number state and a beam splitter

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Passing a photon number state through a balanced beam splitter will produce an entangled state in which the phases of the two output modes are highly correlated. We show that Bell’s inequality can be violated using this entangled state and two distant interferometers. The output modes of the beam splitter can be viewed as a generalized form of an entangled Schrodinger cat state, which may have practical applications in quantum communications.

I. Introduction

Quantum mechanics violates Bell inequality, which rules out the possibility of local hidden variable theories [1-5] as an alternative to quantum mechanics. The earliest experimental tests of Bell’s inequality were based on entanglement between the polarizations or spins of two particles [6-12]. It was subsequently shown that Bell’s inequality could be violated using continuous degrees of freedom, such as energy-time entanglement combined with two distant interferometers [13]. Here we note that a photon number state incident on a balanced beam splitter will produce an entangled state in which the phases of the two output modes are highly correlated [14,15]. We show that Bell’s inequality can be violated using this entangled state combined with two distant interferometers, which may have practical applications in quantum communications.

It is well known that photon number states are highly nonclassical states of light [16] and that they are a useful resource for generating other kinds of nonclassical states. For example, a number state incident on a beam splitter has been used to herald an approximate squeezed cat state in one output mode by post-selecting on the results of a homodyne measurement in the other output mode [17]. It has previously been shown that Bell’s inequality can be violated using a variety of continuous variable states, homodyne measurements, or NOON states [18-21]. The approach described here is somewhat similar to earlier nonlocal interferometers [13,22], but the source of the entangled state is very different.

This paper is organized as follows. Section II describes the basic approach that can be used to violate Bell’s inequality using a photon number state and a beam splitter. Section III derives the form of the quantum state at the output of the beam splitter. Section IV calculates the form of the nonlocal interference produced in two distant interferometers. Section V discusses the results of the analysis and shows that Bell’s inequality can be violated. Section VI provides a summary and conclusions.

II. Basic approach

A number state has a totally uncertain phase because the number of photons and the phase of the field are conjugate variables. One might ask whether or not a number state has a specific phase that is transferred to both output ports of a beam splitter, even though the value of the phase is uncertain [14]. This is illustrated in Fig. 1, where a number state $|N\rangle$ containing a large number $N$ of photons is incident on a beam splitter with the vacuum state $|0\rangle$ incident in the other input port. Suppose that we split off a small amount of the field in path 1 and measure its phase $\phi_1$ using a series of homodyne measurements. It can be shown that this process will collapse the state $|\psi_1\rangle$ of the remaining field in beam 1 to an approximate coherent state with the corresponding phase $\phi_1$. Moreover, the measurement in path 1 will also collapse the state of the field in path 2 to an approximate coherent state with a phase $\phi_2 = \phi_1$ [14].

This example shows that a number state $|N\rangle$ can be thought of as having a specific phase that is transferred to both output ports of the beam splitter, even though its value is uncertain. As a result, the fields in path 1 and 2 correspond to an entangled superposition state in which there is a strong correlation between the phases in the two beams as illustrated in Fig. 2. This will be shown to be the case in more detail in the next section.

The entangled phases of the two beams after the beam splitter can be used to implement a nonlocal interferometer [13,22] as illustrated in Fig. 3. Suppose that we can produce an equal probability amplitude for shifting
the phases of the two beams through an angle of $\pm \theta$. This can be done using a pair of single-photon interferometers containing a Kerr medium as illustrated in Fig. 4. We also assume that the final states in the two beams are post-selected on having a specific final phase, such as $\pi/2$ as illustrated by the dashed blue circle in the phase space diagram of Fig. 3.

A final phase of $\pi/2$ can occur in two different ways. One possibility is that the two fields originally had a phase of $\pi/2 + \theta$, corresponding to the red arrow in Fig. 3, after which a Kerr phase shift of $-\theta$ gave a final phase of $\pi/2$. The other possibility is that the two fields originally had a phase of $\pi/2 - \theta$, followed by a Kerr phase shift of $\theta$. Quantum interference between these two probability amplitudes will produce nonlocal interference effects that can violate Bell’s inequality.

A phase shift of $\pm \theta$ can be produced by passing a single photon through an interferometer with a Kerr medium in one path, as illustrated in Fig. 4 [22,23]. Fixed phase shifts of $\sigma_1$ and $\sigma_2$ are inserted in the single-photon interferometers labelled A and B in paths 1 and 2, respectively, which allows the phase of the nonlocal quantum interference to be varied. Post-selection is applied based on the single photons being detected in detectors $D_2$ and $D_4$ as well as phase measurements of $\pi/2$ in both paths. It should be noted that the homodyne measurements actually measure the x-quadrature of the fields, and a second homodyne measurement of the p-quadrature would be required to completely determine the phase as in Fig. 3. Similar nonlocal interference effects are obtained in either case as will be shown in more detail below.

![FIG. 1. A photon number state $|N\rangle$ with $N >> 1$ is incident on a beam splitter. Operator $\hat{\Phi}_M$ represents a phase measurement performed by splitting off a small fraction of the field and using it to measure phases $\phi_1$ and $\phi_2$. It can be shown [14] that such a measurement of $\phi_1$ in path 1 will collapse the output states $|\psi_1\rangle$ and $|\psi_2\rangle$ to approximate coherent states with $\phi_1 = \phi_2$. This suggests that the incident number state can be viewed as having a specific but unknown phase that is transferred to both of the output beams as illustrated in Fig. 2.](image1)

![FIG. 2. Interpretation of the results of the phase measurements of Fig. 1 in phase space, where $x$ and $p$ represent the position and momentum in the Wigner distribution. A phase measurement $\hat{\Phi}_M$ in path 1 collapses the states in the two beams to approximate coherent states with equal phases [14]. Two possible results are illustrated by the two sets of arrows. As a result, the state of the system leaving the beam splitter can be viewed as an entangled state in which there is an equal probability amplitude of all values of $\phi_1 = \phi_2$.](image2)

![FIG. 3. Nonlocal quantum interference produced by applying a phase shift of $\pm \theta$ to the two output modes of the beam splitter shown in Fig. 1. This can be done using two single-photon interferometers containing a Kerr medium in one path as illustrated in Fig. 4. The results are post-selected on obtaining a final phase of $\pi$ in both beams. One probability amplitude for this process to occur corresponds to an initial phase of $\pi + \theta$ in both beams, followed by a phase shift of $-\theta$ from the single-photon interferometers. A second probability amplitude corresponds to an initial phase of $\pi - \theta$ in both beams, followed by a phase shift of $\theta$ from the single-photon interferometers. Quantum interference between these two probability amplitudes can produce a violation of Bell’s inequality.](image3)
These arguments suggest that Bell’s inequality could be violated using a number state $|N\rangle$ incident on a beam splitter as the source of an entangled state. A more detailed analysis is presented in the following sections.

![Diagram of a nonlocal interferometer](image)

**FIG. 4.** A nonlocal interferometer that uses a number state $|N\rangle$ incident on a beam splitter to create an entangled state with correlated phases as illustrated in Fig. 3. A phase shift of $\pm \theta$ can be applied to each of the beams using a pair of single-photon interferometers A and B with a Kerr medium located in one path combined with a constant bias phase shift (not shown). The results are post-selected on the detection of a single photon in detectors D$_2$ and D$_4$, which creates a coherent superposition of states with phase shifts of $\pm \theta$. The results are also post-selected on measuring a specific value of the x quadrature of the two beams using homodyne detectors. Quantum interference between the two probability amplitudes is illustrated in Fig. 3. The phase of the interference pattern can be controlled using the variable phase shifts $\sigma_1$ and $\sigma_2$, which allows a violation of Bell’s inequality.

### III. Entangled state after the beam splitter

The effect of a balanced beam splitter can be described as usual by the unitary transformation

$$\hat{a}_1^\dagger \rightarrow \frac{\hat{a}_1^\dagger + i\hat{a}_2^\dagger}{\sqrt{2}}$$

(1)

and

$$\hat{a}_2^\dagger \rightarrow \frac{\hat{a}_2^\dagger + i\hat{a}_1^\dagger}{\sqrt{2}}.$$  

Here $\hat{a}_1^\dagger$ and $\hat{a}_2^\dagger$ are the photon creation operators in the two input/output modes and we have used the convention that the reflected component undergoes a phase shift of $\pi/2$.

The initial state $|\psi\rangle$ incident on the beam splitter is given by

$$|\psi\rangle = |N, 0\rangle.$$  

(3)

where $|i, j\rangle$ will denote a state with $i$ photons in one input mode and $j$ photons in the other mode. This initial state can be written in terms of the creation operators as

$$|\psi\rangle = \frac{(\hat{a}_1^\dagger)^N}{\sqrt{N!}} |0, 0\rangle.$$  

(4)

After passing through the beam splitter, Eq. (1) can be used to write the transformed state as

$$|\psi\rangle = \frac{(\hat{a}_1^\dagger + i\hat{a}_2^\dagger)^N}{\sqrt{N! 2^N}} |0, 0\rangle.$$  

(5)

Equation (5) can be rewritten using the binomial expansion as

$$|\psi\rangle = \sum_{n=0}^{N} \sqrt{\frac{N!}{n!}} |N-n, n\rangle$$

(6)

where $\binom{N}{n}$ are the binomial coefficients. Eq. (6) can be used to obtain an analytic form for the output of the interferometer, as described in the Appendix.

It is more straightforward and informative, however, to use a different approach based on the fact that a number state can be written in the form [15]

$$|N\rangle = \int_0^{2\pi} d\phi f_\phi |R e^{i\phi}\rangle.$$  

(7)

where $f_\phi$ is defined by

$$f_\phi = e^{i\phi/2} e^{-|\phi|^2} \sqrt{N!}.$$  

(8)

Here $|R e^{i\phi}\rangle$ denotes a coherent state with amplitude $R$ and phase $\phi$. $R$ is an arbitrary constant, but it will be convenient here to choose the value $R = \sqrt{N}$. 
Eq. (7) can be used to write the initial state of the system before the beam splitter as

$$\left| \psi \right\rangle = \int_{0}^{2\pi} d\phi f_{\phi} \left| Re^{i\phi}, 0 \right\rangle.$$  

(9)

Here $\left| Re^{i\phi}, 0 \right\rangle$ denotes a coherent state with amplitude $Re^{i\phi}$ in one input to the beam splitter and a coherent state with zero amplitude in the other input port.

It is well known that a coherent state incident on a beam splitter will produce a coherent state in the two output modes with amplitudes equal to the corresponding classical fields. As a result, the beam splitter transforms the state of the system in Eq. (9) into

$$\left| \psi \right\rangle = \int_{0}^{2\pi} d\phi f_{\phi} \frac{R}{\sqrt{2}} e^{i\phi} \left| Re^{i\phi}, \sqrt{2} \right\rangle.$$  

(10)

Here we have applied a phase shift of $-\pi/2$ in path 2 after the beam splitter to compensate for the factor of $i$ in Eq. (1).

The origin of the phase entanglement is apparent in Eq. (10), which is qualitatively consistent with the results of Ref. [14] as well as Figs. 1 and 2. The rapidly oscillating factor of $e^{-i\pi\phi}$ in the definition of $f_{\phi}$ can have a significant impact on the nature of the nonlocal interference, however, which is not reflected in Figs. 1 through 3.

IV. Calculation of the Nonlocal Interference

The phase-entangled state of Eq. (10) can be used to produce nonlocal interference effects by including a pair of two-photon interferometers as described previously in Section II and Fig. 4. The interferometers inserted into paths 1 and 2 will be labelled by A and B, respectively. The state $|i,j\rangle_A$ will denote the case in which there are $i$ photons in the left path of interferometer A with $j$ photons in the right path, while $|i,j\rangle_B$ will denote the corresponding state in interferometer B. Including the single photons, the complete state of the system before the photons have entered the interferometers is given by

$$\left| \psi \right\rangle = \int_{0}^{2\pi} d\phi f_{\phi} \frac{R}{\sqrt{2}} e^{i\phi} \left| Re^{i\phi}, \sqrt{2} \right\rangle \left| 10 \right\rangle_A \left| 10 \right\rangle_B.$$  

(11)

After the single photons have entered their respective interferometers and passed through the first beam splitter, the state of the system becomes

$$\left| \psi \right\rangle = \int_{0}^{2\pi} d\phi f_{\phi} \frac{R}{\sqrt{2}} e^{i\phi} \left| Re^{i\phi}, \sqrt{2} \right\rangle \times \left( \left| 10 \right\rangle_A + i \left| 01 \right\rangle_A \right) \left( \left| 10 \right\rangle_B + i \left| 01 \right\rangle_B \right).$$  

(12)

The presence of a single photon in the path with the Kerr media will produce a nonlinear phase shift and we assume that a constant phase shift is also applied so that the net phase shift is $\pm \theta$. As a result, the state of the system after the Kerr media can be written in the form

$$\left| \psi \right\rangle = \int_{0}^{2\pi} d\phi f_{\phi} \frac{R}{\sqrt{2}} e^{i\phi}$$

$$\times \left( \left| + \right\rangle_{\phi} \left| 1010 \right\rangle + i \left| + \right\rangle_{\phi} \left| 0101 \right\rangle \right)$$

$$+ i \left| - \right\rangle_{\phi} \left| 0110 \right\rangle + i^2 \left| - \right\rangle_{\phi} \left| 0101 \right\rangle.$$

(13)

Here we have introduced the notation

$$\left| + \right\rangle_{\phi} = \frac{R}{\sqrt{2}} e^{i\phi} \left| Re^{i\phi}, \sqrt{2} \right\rangle.$$  

(14)

with analogous definitions for $\left| - \right\rangle_{\phi}$, $\left| + \right\rangle_{\phi}$, and $\left| - \right\rangle_{\phi}$.

We have also used the more compact notation $\left| 1010 \right\rangle_A = \left| 10 \right\rangle_A \left| 01 \right\rangle_A$, and so forth.

The single photons encounter the variable phase shifts $\sigma_1$ and $\sigma_2$ depending on which path they traverse as shown in Fig. 1. This transforms the state of Eq. (14) into

$$\left| \psi \right\rangle = \int_{0}^{2\pi} d\phi f_{\phi} \frac{R}{\sqrt{2}}$$

$$\times \left( e^{i(\sigma_1 + \sigma_2)} \left| + \right\rangle_{\phi} \left| 1010 \right\rangle + i e^{i\sigma_2} \left| + \right\rangle_{\phi} \left| 0101 \right\rangle \right)$$

$$+ i e^{i\sigma_1} \left| - \right\rangle_{\phi} \left| 0110 \right\rangle + i^2 \left| - \right\rangle_{\phi} \left| 0101 \right\rangle.$$

(15)
Finally, the single photons exit the interferometers through another set of beam splitters which gives the state

\[
|\psi\rangle = \int_{0}^{2\pi} d\phi \frac{f_{\phi}}{4} \\
\times \left( e^{i(\sigma_{1}+\sigma_{2})} |++\rangle_{\phi} \right.
\left. + e^{i\sigma_{1}} |+-\rangle_{\phi} + e^{i\sigma_{2}} |+--\rangle_{\phi} + |--\rangle_{\phi} \right).
\]

The four terms in Eq. (17) correspond to the possible phase shifts in the two beams before further post-selection is performed using the results of the homodyne detectors, as illustrated in Fig. 3.

A single mode of the electromagnetic field is mathematically equivalent to a harmonic oscillator, and a homodyne measurement of the x-quadrature can be represented by the operator \( \hat{x} = (\hat{a} + \hat{a}^\dagger) / \sqrt{2} \) with a suitable choice of units. As a result, it is convenient to use the position representation, where the usual wave function \( \psi(x) \) is given by

\[
\psi(x) = \langle x | \psi \rangle.
\]

It can be shown [24] that the wave function \( \psi_{\sigma}(x) \) for a coherent state \( |\sigma, e^{i\theta}\rangle \) of the field corresponds to a Gaussian wave packet of the form

\[
\psi_{\sigma}(x) = \frac{1}{\sqrt{\pi}^{1/4}} e^{i0_{0}x} e^{-x^{2}/2} e^{-i\sigma_{0}/2}.
\]

Here \( x_{0} = \sqrt{2}\alpha_{0} \cos(\varphi) \) and \( p_{0} = \sqrt{2}\alpha_{0} \sin(\varphi) \). The overall phase factor of \( e^{-i0_{0}p_{0}/2} \) is sometimes ignored, but it plays an important role [25] in superposition states such as in Eq. (12).

In the coordinate representation, Eq. (17) gives

\[
\psi(x_{1}, x_{2}) = \left( x_{1}, x_{2} | \psi \right) = \psi_{++}(x_{1}, x_{2}) + \psi_{--}(x_{1}, x_{2}) + \psi_{-+}(x_{1}, x_{2}) + \psi_{+-}(x_{1}, x_{2}).
\]

where \( \psi_{\pm}(x_{1}, x_{2}) \) correspond to the four terms in Eq. (17).

It will be convenient to choose the phase shift \( \theta \) so that \( N\theta = m(2\pi) \), where \( m \) is an integer. In that case, Eqs. (13) and (19) can be used to show that

\[
\psi_{++}(x_{1}, x_{2}) = -\frac{1}{4\sqrt{\pi}} e^{i(\sigma_{1}+\sigma_{2})} \int_{0}^{2\pi} d\phi f_{\phi} e^{iR\sin(\phi+\theta)x_{1}} e^{iR\sin(\phi+\theta)x_{2}}
\times e^{-[x_{1}-R\cos(\phi+\theta)]^{2}/2} \times e^{-[x_{2}-R\cos(\phi+\theta)]^{2}/2}
\times e^{-[R\sin(\phi+\theta)R\cos(\phi+\theta)]^{2}/2}.
\]

With \( N\theta = m(2\pi) \), \( \psi_{--} = \psi_{++} \) aside from the phase shift of \( e^{i(\sigma_{1}+\sigma_{2})} \). The cross-terms are then given by

\[
\psi_{-+}(x_{1}, x_{2}) = -\frac{1}{4\sqrt{\pi}} e^{i\sigma_{1}} \int_{0}^{2\pi} d\phi f_{\phi} e^{iR\sin(\phi+\theta)x_{1}} e^{iR\sin(\phi+\theta)x_{2}}
\times e^{-[x_{1}-R\cos(\phi+\theta)]^{2}/2} \times e^{-[x_{2}-R\cos(\phi+\theta)]^{2}/2}
\times e^{-[R\sin(\phi+\theta)R\cos(\phi+\theta)]^{2}/2}.
\]

with a similar expression for \( \psi_{+-} \).

The probability \( P(x_{1M}, x_{2M}) \) of obtaining the quadrature values \( x_{1} = x_{1M} \) and \( x_{2} = x_{2M} \) within a small interval \( \Delta x_{1}\Delta x_{2} \) in the homodyne measurements is given by

\[
P(x_{1M}, x_{2M}) = |\psi(x_{1M}, x_{2M})|^{2} \Delta x_{1}\Delta x_{2}.
\]

As we will show in the next section, it is possible to choose values of \( x_{1M} \) and \( x_{2M} \) such that \( \psi_{--}(x_{1M}, x_{2M}) \) and \( \psi_{++}(x_{1M}, x_{2M}) \) are negligible. In that case, Eq. (21) and the corresponding equation for \( \psi_{-+}(x_{1M}, x_{2M}) \) can be used to show that
\[ P(x_{1M}, x_{2M}) = \left[ e^{i(\sigma_1 + \sigma_2)} + 1 \right] |\psi_{++}(x_{1M}, x_{2M})|^2 \Delta x_1 \Delta x_2 = \gamma \cos^2[(\sigma_1 + \sigma_2)/2]. \]  

(23)

Here \( \gamma \) is a constant that depends on the choice of \( x_{1M} \) and \( x_{2M} \). The success rate for the post-selection process (coincidence counting rate) depends on the value of \( \gamma \) as will be discussed in the next section.

Eq. (23) shows that the nonlocal interference is proportional to \( \cos^2[(\sigma_1 + \sigma_2)/2] \), which is somewhat similar to the nonlocal interferometer of Ref. [13]. Eqs (21) and (22) can be integrated numerically to show that \( \psi_{++}(x_{1M}, x_{2M}) \) and \( \psi_{--}(x_{1M}, x_{2M}) \) can be neglected for appropriate values of \( x_{1M} \) and \( x_{2M} \). The form of the wave functions in the coordinate representation are plotted and discussed in the next section. Alternatively, the properties of the Hermite polynomials can be combined with Eq. (6) to derive an analytic form for the nonlocal interference, as is described in the appendix. Both approaches give the same result.

V. Violations of Bell's inequality

The nonlocal dependence on the sum of the phases \( \sigma_1 \) and \( \sigma_2 \) in Eq. (23) suggests that it should be possible to violate Bell’s inequality. Roughly speaking, a violation of Bell’s inequality would show that the output of the measurement apparatus in path 1 cannot be determined solely from the value of the phase shift \( \sigma_1 \) applied in that path. Instead, the output of each measurement device depends on both \( \sigma_1 \) and \( \sigma_2 \).

The simple form of Eq. (23) depends on the assumption that the cross-terms \( \psi_{++}(x_{1M}, x_{2M}) \) and \( \psi_{--}(x_{1M}, x_{2M}) \) can be neglected. In order to investigate this possibility, the value of \( |\psi_{++}(x_1, x_2)|^2 = |\psi_{--}(x_1, x_2)|^2 \) is plotted as a function of \( x_1 \) and \( x_2 \) in Fig. 5(a). These results correspond to \( N = 24 \) and \( \theta = \pi/4 \), which satisfies the condition that \( N\theta = m(2\pi) \). It can be seen that the phases of the two fields are highly correlated as expected. The magnitude squared of the wave function also shows an oscillatory behavior extending towards the origin, which is due to the rapidly varying phase factor of \( e^{-iN\phi} \) in the definition of \( f_\phi \).

For comparison, Fig. 5(b) shows the magnitude squared of the cross-terms \( |\psi_{++}(x_1, x_2)|^2 = |\psi_{--}(x_1, x_2)|^2 \) as a function of \( x_1 \) and \( x_2 \). A phase shift of \( \theta = \pi/4 \) causes the phases of the two beams to become uncorrelated. In addition, the wave function is only appreciable inside a ring with a relatively narrow width. It can be seen that there are many choices of \( x_{1M} \) and \( x_{2M} \) where the cross-terms would be negligible compared to \( |\psi_{++}(x_1, x_2)| \), which would give high-visibility nonlocal interference as described by Eq. (23). There are also regions where \( |\psi_{++}(x_1, x_2)| \) is negligible compared to \( |\psi_{--}(x_1, x_2)| \), which would also allow high-visibility quantum interference between the \( \psi_{--}(x_{1M}, x_{2M}) \) and \( \psi_{++}(x_{1M}, x_{2M}) \) terms.

FIG. 5. Plots of the magnitude squared of the wave function in the coordinate representation as a function of \( x_1 \) and \( x_2 \). (a) Plot of \( |\psi_{++}(x_1, x_2)|^2 = |\psi_{--}(x_1, x_2)|^2 \). (b) Plot of the cross-terms \( |\psi_{++}(x_1, x_2)|^2 = |\psi_{--}(x_1, x_2)|^2 \). These results correspond to \( N = 24 \) and \( \theta = \pi/4 \), which satisfies the condition that \( N\theta = m(2\pi) \) where \( m \) is an integer.
The normalized probability \( P(x_{1M}, x_{2M}) \) is shown in Fig. 6 as a function of the phase shift \( \sigma_1 \) in interferometer A for several values of the phase shift \( \sigma_2 \) in interferometer B. Here the post-selected quadratures \( x_{1M} \) and \( x_{2M} \) were chosen to be \( x_{1M} = x_{2M} = \sqrt{N} \), for which \( \psi_{+}(x_{1M}, x_{2M}) \) and \( \psi_{-}(x_{1M}, x_{2M}) \) are negligibly small. Bell’s inequality [3] is violated if the visibility of an interference pattern described by Eq. (23) is greater than \( 1/\sqrt{2} \) [26], which is the case for the results shown in Fig. 6. Due to the use of post-selection, this violation of Bell’s inequality relies on the fair sampling assumption as is the case in all experiments with limited detection efficiency [27].

![FIG. 6. The normalized probability \( P(x_{1M}, x_{2M}) \) of a successful detection event for \( x_{1M} = x_{2M} = \sqrt{N} \), plotted as a function of the phase shift \( \sigma_1 \) in interferometer A. (a) Phase shift \( \sigma_2 = 0 \) in interferometer B. (b) Phase shift \( \sigma_2 = \pi \) in interferometer B. These results correspond to \( N = 24 \) and \( \theta = \pi / 4 \) as in Fig. 5. It can be seen that Bell’s inequality can be violated using a uniform state and a beam splitter, subject to the fair sampling assumption.]

There can be a significant contribution from the \( \psi_{-}(x_{1M}, x_{2M}) \) and \( \psi_{+}(x_{1M}, x_{2M}) \) terms for smaller values of \( N \) or \( \theta \). In that case, the interference pattern is no longer described by Eq. (23) and we must make use of the CHSH form of Bell’s inequality introduced by Clauser, Horne, Shimony, and Holt [3]. The CHSH inequality requires two sets of measurement settings, which will be denoted by \( \sigma_1 \) or \( \sigma_1' \) in interferometer A and \( \sigma_2 \) or \( \sigma_2' \) in interferometer B. The result \( a \) of the measurement obtained using \( \sigma_1 = \sigma_1' \) in interferometer A will be assigned the value \( a = 1 \) if a photon is detected in detector 1, while it will be assigned the value \( a = -1 \) if a photon is detected in detector 2 [3]. The results obtained in interferometer A using \( \sigma_1' \) will be denoted \( a' = \pm 1 \) in a similar way, while the results obtained in interferometer B will be denoted \( b = \pm 1 \) or \( b' = \pm 1 \), depending on the choice of \( \sigma_2 \).

The parameter \( S \) in the CHSH form of the inequality is then defined as

\[
S = \langle ab \rangle + \langle a'b \rangle + \langle ab' \rangle - \langle a'b' \rangle.
\]

The inequality \( |S| \leq 2 \) holds for all local hidden-variable theories.

In the example of interest here, the results are post-selected on having obtained a specific outcome from the homodyne measurements. The properly normalized expectation values are therefore given by [30]

\[
\langle ab \rangle = \frac{[\langle a_1 = 1, b_1 = 1 \rangle^2 - \langle a_1 = 1, b_1 = -1 \rangle^2 - \langle a_1 = -1, b_1 = 1 \rangle^2 + [\langle a_1 = -1, b_1 = -1 \rangle^2]^2}{\langle a_1 = 1, b_1 = 1 \rangle^2 + \langle a_1 = 1, b_1 = -1 \rangle^2 + \langle a_1 = -1, b_1 = 1 \rangle^2 + [\langle a_1 = -1, b_1 = -1 \rangle^2]^2},
\]

with analogous results for the other expectation values. Here we have used the notation \( \psi_{a_1 = 1, b_1 = 1} = \langle x_1, x_2 | \otimes_{\theta} (10 | 10) \rangle | \psi \rangle \).

Fig. 7 shows a plot of \(|S|\) as a function of \( \sigma_A' \) and \( \sigma_B' \), where the other measurement settings were held fixed at \( \sigma_A = 0 \) and \( \sigma_B = \pi \). These results correspond to a relatively large photon number of \( N = 24 \) and \( \theta = \pi / 4 \), as was used in Figs. 5 and 6. It can be seen that there are regions of the plot where \(|S| > 2 \) and Bell’s inequality is violated, as would be expected from Fig. 6.

![FIG. 7. A plot of the absolute value of the CHSH parameter \( S \) as a function of the measurement settings \( \sigma_A' \) and \( \sigma_B' \). The other measurement settings \( \sigma_A \) and \( \sigma_B \) were held fixed at values of 0 and \( \pi \), respectively, while \( N = 24 \), \( \theta = \pi / 4 \), and \( x_{1M} = x_{2M} = \sqrt{N} \). It can be seen that there are large regions of the parameter space where \(|S| > 2 \) and the CHSH form of Bell’s inequality is violated.]

\[0.2, 0.4, 0.6, 0.8, 1.0\]

\[0.2, 0.4, 0.6, 0.8, 1.0\]
Fig. 8 shows a similar plot of $|S|$ for a more realistic value of $N = 4$. Although the interference pattern would no longer have the simple form shown in Eq. (23), it can be seen that there are still values of $\sigma'_A$ and $\sigma'_B$ where Bell’s inequality can be violated. It can be shown that Bell’s inequality can be violated in a similar way for $N = 2$ as well.

FIG. 8. Another plot of the absolute value of the CHSH parameter $S$ as a function of the measurement settings $\sigma'_A$ and $\sigma'_B$, where here the number of photons corresponds to $N = 4$. All of the other parameters are the same as in Fig. 7. It can be seen that there are still regions of the parameter space where $|S| > 2$ and the CHSH form of Bell’s inequality is violated, although the choice of parameters is more restricted than in Fig. 7.

In order to obtain a reasonable counting rate in an actual experiment, it would be necessary to post-select events in which the results of the homodyne measurements lie in a range $\Delta x_1$ and $\Delta x_2$ about the desired results $x_{1m}$ and $x_{2m}$. It can be shown that a choice of $\Delta x_1 = \Delta x_2 = 0.10\sqrt{N}$ is sufficient to violate the CHSH form of Bell’s inequality with a probability of success for the post-selection process of 0.27% per pulse. These results correspond to $N = 4$, $\sigma'_A = 0$, $\sigma'_B = 0.9$, $\sigma_B = \pi$, and $\sigma'_B = -2.2$, for which $|S| = 2.3$. Although the probability of success is relatively small, it should be acceptable for an experimental test.

Perhaps the most difficult aspect of an experimental test of Bell’s inequality using these techniques is the need to implement a Kerr phase shift of $\pi/4$ at the single-photon level. Single-photon nonlinear phase shifts as large as $\pi/2$ have been demonstrated experimentally [30-37] but experiments of that kind remain challenging. Further improvements in those techniques would probably be required for practical applications of these results.

VI. Summary and conclusions

A photon number state is one of the most basic examples of a nonclassical state of light. Our results were motivated by the observation that a photon number state $|N\rangle$ has a totally uncertain phase, but passing a number state through a 50/50 beam splitter will produce two entangled beams whose phases are highly correlated. A measurement of the phase in one beam will collapse the state of the other beam to an approximate coherent state with the same phase, as illustrated in Fig. 1 [14].

We have shown that this property can be used to violate Bell’s inequality by creating a superposition of states where the phases in the two output beams were shifted by $\pm \theta$. That can be done using two single-photon interferometers containing a Kerr cell in one path as shown in Fig. 4. Post-selection based on homodyne measurements will give a probability amplitude for a successful outcome that is a superposition of terms corresponding to the two possible values of $\theta$, as illustrated in Figs. 3 and 4. Quantum interference between these two probability amplitudes will produce a nonlocal dependence on the single-photon phases $\sigma_1$ and $\sigma_2$ introduced in the two interferometers, in violation of Bell’s inequality.

Under ideal conditions, this approach can violate Bell’s inequality with 100% visibility of the quantum interference patterns, as illustrated in Fig. 6. Bell’s inequality can be violated in this way for relatively small photon numbers, including $N = 2$. It is also possible to violate the CHSH form of Bell’s inequality under more realistic conditions where it is necessary to accept the output of the homodyne measurements over a range of values. Our approach requires a Kerr phase shift at the single-photon level, which is challenging but has been demonstrated experimentally [30-37].

Somewhat similar violations of Bell’s inequality have previously been proposed using entangled Schrodinger cat states [22,29]. The main difference is that most entangled Schrodinger cat states only contain a superposition of two possible phases, whereas the states produced by a number state and a beam splitter contain a continuous range of correlated phases as illustrated in Fig. 2. As a result, the states produced from a number state and a
beam splitter can be viewed as a generalized form of entangled cat state with a continuous range of possible phases.

A number state is one of the most fundamental forms of a nonclassical state and the fact that it can be used in this way to violate Bell’s inequality is of basic scientific interest. In addition, number states with moderate values of \( N \) are relatively straightforward to produce and the effects described here may be of practical use in quantum communications.

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Appendix

The results in the main text were derived using Eqs. (7) and (8), which express a photon number state as a superposition of coherent states with all possible phases. In this appendix, we give an alternative derivation based on the expansion of a coherent state as a superposition of number states as in Eqs. (4) through (6). The results are the same as in the main text, but they provide further insight into the nature of the nonlocal quantum interference as well as a check on the earlier results. In addition, an analytic form can be obtained for the most important results.

We can simplify each of the terms in Eq. (17) by integrating over the phase \( \phi \). The results are that
\[ |\psi_k\rangle = -p_k \frac{1}{4} \sum_{n=0}^{N} q^n \sqrt{\frac{N}{2^n}} |N-n,n\rangle, \quad (26) \]

where \( p_1 = e^{i(\sigma_1 + \sigma_2 + \pi N \theta)} \), \( p_2 = e^{i(\sigma_1 - \pi N \theta)} \), \( p_3 = e^{i(\sigma_2 + \pi N \theta)} \), \( p_4 = e^{-\pi N \theta} \), \( q_1 = q_4 = 1 \) and \( q_2 = q_3 = e^{2i\theta} \). The index \( k \) corresponds to each of the four terms in Eq. (17).

The dimensionless position-basis representation of the final state is given by the inner product \( \langle x_1, x_2 | \psi \rangle = \varphi(x_1, x_2) \). We use the dimensionless position-basis representation of the number states,

\[ \langle x | n \rangle = \sqrt{\frac{e^{-x^2}}{n!2^n \sqrt{\pi}}} H_n(x), \quad (27) \]

to obtain

\[ \psi_k (x_1, x_2) = -p_k \frac{e^{-(x_1^2 + x_2^2)/2}}{4 \sqrt{\pi N!}} \sum_{n=0}^{N} q^n \sqrt{\frac{N}{2^n}} H_{N-n}(x_1) H_n(x_2). \quad (28) \]

Here \( H_n(x) \) is the \( n^{th} \) Hermite polynomial.

We can further simplify Eq. (28) for \( k = 1 \) and \( k = 4 \) to the analytical form

\[ \psi_k (x_1, x_2) = -p_k \frac{e^{-(x_1^2 + x_2^2)/2}}{4 \sqrt{2^n \pi N!}} \sqrt{\frac{N}{2^n}} H_N \left( \frac{x_1 + x_2}{\sqrt{2}} \right). \quad (29) \]

When the cross-terms terms corresponding to \( k = 2 \) and \( k = 3 \) are negligible, Eq. (29) gives an interference pattern that is equivalent to Eq. (23) in the text, but without any complicated integrals over \( \phi \).