EXTENSION OF LEVI-FLAT HYPERSURFACES PAST CR BOUNDARIES

JIŘÍ LEBL

ABSTRACT. Local conditions on boundaries of $C^\infty$ Levi-flat hypersurfaces, in case the boundary is a generic submanifold, are studied. For nontrivial real analytic boundaries we get an extension and uniqueness result, which forces the hypersurface to be real analytic. This allows us to classify all real analytic generic boundaries of Levi-flat hypersurfaces in terms of their normal coordinates. For the remaining case of generic real analytic boundary we get a weaker extension theorem. We find examples to show that these two extension results are optimal. Further, a class of nowhere minimal real analytic submanifolds is found, which is never the boundary of even a $C^2$ Levi-flat hypersurface.

1. INTRODUCTION

The question we wish to ask is when is a generic codimension 2 submanifold $M \subset \mathbb{C}^N$ locally the boundary of a Levi-flat hypersurface $H$. In particular, we will ask the following questions. When does $H$ extend as a Levi-flat hypersurface past $M$? When is $H$ unique? How does the regularity of $H$ depend on the regularity of $M$? We will answer these questions fully when $M$ is real analytic and $H$ is smooth.

The results here are motivated by Dolbeault, Tomassini and Zaitsev [8], who consider the global situation under additional assumptions on $M$. These results are also related to results of Straube and Sucheston [14]. This paper can also be seen as a natural extension of the results in [10], as we will mostly concern ourselves with the situation when $M$ is real analytic. In the non-CR case, which is not considered here, similar questions are considered in $\mathbb{C}^2$ for example by Bishop [4], Moser and Webster [12], or Bedford and Gaveau [2]. For further discussion of the non-CR case and more references see [8].

In the following, by submanifold we always mean embedded submanifold, by hypersurface a submanifold of codimension 1, and by real analytic subvariety of an open set $U$, a set closed in $U$ and locally defined by the vanishing of a family of real analytic functions.

Let $M \subset \mathbb{C}^N$ be a real codimension 2 connected submanifold. Let $J$ be the complex structure on $\mathbb{C}^N$, and let $T^p_\mathbb{C}M = J(T^p_pM) \cap T^p_pM$. A real submanifold is called CR if the dimension of $T^p_\mathbb{C}M$ is constant as $p$ varies in $M$. The smallest germ (in terms of dimension) of a CR submanifold $N$ of $M$ through $p$ such that the $T^p_\mathbb{C}N = T^p_\mathbb{C}M$ is called the local CR orbit at $p$, and is guaranteed to exist by the Nagano theorem [13] in case $M$ is real analytic, or the Sussmann theorem [15] if $M$ is only smooth. $M$ is said to be minimal at $p$ (in the sense of Tumanov [10])
if the local CR orbit through \( p \) is of codimension 0 in \( M \). If \( M \) is not minimal at any point then \( M \) is said to be nowhere minimal. See [1, 5, 7] for more details.

We will say that \( M \) is generic if and only if \( T_p M + J(T_p M) = T_p C^N \) for all \( p \in M \), where \( J \) is the complex structure on \( C^N \). If \( M \) is a real analytic CR submanifold of codimension 2, this just means that \( M \) is not a complex analytic submanifold near any point. We will always assume that \( 0 \in M \).

A set \( H \subset C^N \) is a \( C^k \) hypersurface with boundary, if there is a subset \( \partial H \subset H \), such that \( \partial H \subset H \); \( H \setminus \partial H \) is a \( C^k \) hypersurface (submanifold of codimension 1), and for each point \( p \in \partial H \), there exists a neighbourhood \( p \in U \subset C^N \), a \( C^k \) diffeomorphism \( \varphi: U \rightarrow R^{2N} \), such that \( \varphi(H \cap U) = \{ x \in R^{2N} \mid x_{2N-1} \geq 0, x_{2N} = 0 \} \), and such that \( \varphi(\partial H \cap U) = \{ x \in R^{2N} \mid x_{2N-1} = 0, x_{2N} = 0 \} \). Hence, \( \partial H \) is a \( C^k \) submanifold of codimension 2 in \( C^N \). We will call \( H^o := H \setminus \partial H \) the interior of \( H \). As we are concerned with only local questions, we can assume that there exists just one such \( U \), and such that \( \partial H, H \subset U \). We can further assume that \( \partial H \) and \( H \) are closed subsets of \( U \). We can extend \( H \) to \( \hat{H} \), a full \( C^k \) submanifold near 0, by just pulling back a neighbourhood of 0 in \( R^{2N} \) by \( \varphi \).

A \( C^k \) \((k \geq 2)\) hypersurface \( H \) is said to be Levi-flat if the bundle \( T^c H \) is involutive. An equivalent definition is to say that near every point of \( H \), there exists a one parameter local foliation of \( H \) by complex hypersurfaces, which is called the Levi foliation. To see why these are equivalent, note that if \( T^c H \) is involutive the Frobenius theorem gives us a \( C^{k-1} \) foliation with the leaves being complex hypersurfaces (they are locally the graphs of holomorphic functions). If \( H \) is a hypersurface with boundary as defined above, then we will say it is Levi-flat when \( H^o \) is Levi-flat. If \( H \) is a real analytic subvariety of codimension 1, then we say it is Levi-flat, if it is Levi-flat as a submanifold at all the nonsingular points.

We can now state our main result.

**Theorem 1.1.** Let \( M \subset C^N \) be a connected real analytic generic submanifold of codimension 2 through the origin, such that not all local CR orbits of \( M \) are of codimension 2 in \( M \). Suppose that there exists a connected Levi-flat \( C^\infty \) hypersurface \( H \) with boundary, where \( M \subset \partial H \). Then there exists a neighbourhood \( U \) of the origin and a nonsingular real analytic Levi-flat hypersurface \( \mathcal{H} \) such that \( H \cap U \subset \mathcal{H} \).

Further, the germ \((\mathcal{H}, 0)\) is unique in the sense that if \((\mathcal{H}', 0)\) is a germ of an irreducible real analytic Levi-flat subvariety of codimension 1 such that \((M, 0) \subset (\mathcal{H}', 0)\), then \((\mathcal{H}', 0) = (\mathcal{H}, 0)\).

First, note that the condition that \( M \) is real analytic is necessary for the extension to hold. See Example [4] for a counterexample in case \( M \) is \( C^\infty \).

The condition on the local CR orbits is necessary for the conclusion that the extension \( \mathcal{H} \) is unique and real analytic. If \( M \) is the boundary of a Levi-flat hypersurface, then all local CR orbits must be of positive codimension in \( M \), see Lemma 2.1. If all the local CR orbits are of codimension 1 in \( M \), then the theorem follows easily by known results, see Lemma 2.3. Finally, if all local CR orbits would be of codimension 2 in \( M \), then \( M \subset C^N \) would be locally biholomorphic to \( C^{N-2} \times R^2 \), and we will give (Example 4.2) an example of a bona fide \( C^\infty \) (i.e. not contained in a real analytic subvariety) Levi-flat hypersurface which contains such an \( M \). Hence the theorem is, in this respect, optimal. In [3] we will prove the following weaker

\footnote{This is equivalent to saying that the Levi form vanishes identically at every point, which is the usual definition.}
extension theorem for such submanifolds, which is also optimal in view of the above examples. In the sequel, when we consider $\mathbb{C}^{N-2} \times \mathbb{R}^2$ as a subset of $\mathbb{C}^N$, we mean the natural embedding.

**Theorem 1.2.** Suppose $H \subset \mathbb{C}^N$ is a $C^\infty$ Levi-flat hypersurface with boundary, and $0 \in \partial H \subset \mathbb{C}^{N-2} \times \mathbb{R}^2$. Then for some neighbourhood $U$ of the origin, there exists a $C^\infty$ Levi-flat hypersurface $\mathcal{H}$ (without boundary) such that $H \cap U \subset \mathcal{H}$.

Further, the germ $(\mathcal{H}, 0)$ is unique in the sense that if $(\mathcal{H}', 0)$ is another a germ of a $C^\infty$ Levi-flat hypersurface such that $(H, 0) \subset (\mathcal{H}', 0)$, then $(\mathcal{H}', 0) = (\mathcal{H}, 0)$.

Note that the uniqueness in Theorem 1.2 is much weaker as $\mathcal{H}$ depends on $H$, whereas in Theorem 1.1 $\mathcal{H}$ depends only on $M$.

Theorem 1.1 says that in particular, there exists a holomorphic function defined near the origin with nonzero gradient that is real valued on $M$. In other words, $M$ is locally the boundary of a Levi-flat $C^\infty$ hypersurface if and only if $M$ has local defining functions in $(z, w) \in \mathbb{C}^{N-2} \times \mathbb{C}^2$ of the form:

\[
\begin{align*}
\text{Im } w_1 &= \varphi(z, \bar{z}, \text{Re } w), \\
\text{Im } w_2 &= 0,
\end{align*}
\]

for some $\varphi$ such that $\varphi(0, \bar{z}, s) \equiv \varphi(z, 0, s) \equiv 0$ (i.e. these are normal coordinates, see [1] for example). The classification of Levi-flat boundaries that are generic and real analytic is therefore simple.

**Corollary 1.3.** Let $M \subset \mathbb{C}^N$ be a connected real analytic generic submanifold of codimension 2 through the origin. The following are equivalent:

(i) There exists a Levi-flat $C^\infty$ hypersurface $H$ with boundary, such that $0 \in \partial H \subset M$.

(ii) There exists a real analytic Levi-flat hypersurface (submanifold) $H$ defined in a neighbourhood $U$ of the origin such that $M \cap U \subset H$.

(iii) There exist local holomorphic coordinates (near the origin) such that $M$ is defined by an equation of the form (1).

(iv) There exists a real analytic foliation of codimension 1 in $M$, defined in a neighbourhood of the origin, such that the leaves are unions of (representatives of) local CR orbits of $M$.

When the Levi-flat hypersurface is only $C^2$ rather than smooth, then we will be able to prove that the individual leaves of the Levi foliation extend across $M$. See Lemma 5.1. As an application of this lemma we prove the following theorem. First, we must define the property of being almost minimal (see [10]). Let $M$ be a real analytic, generic submanifold through the origin. Suppose that for every $U$ a neighbourhood of the origin there exists a $p \in U \cap M$ such that the local CR orbit (take a representative of this germ) of $M$ through $p$ is not contained in any complex analytic subvariety of $U$, then $M$ is almost minimal at the origin. An example of this kind of manifold can be found in [6].

**Theorem 1.4.** Let $M \subset \mathbb{C}^N$ be a connected real analytic generic submanifold of codimension 2 through the origin, which is almost minimal at the origin. Let $H$ be a connected $C^2$ hypersurface with boundary and $M \subset \partial H$. Then $H$ is not Levi-flat.

If $H$ would be $C^\infty$ then the above result follows at once from Theorem 1.1. Further, not being almost minimal is a necessary, but not sufficient, condition to being a boundary of a $C^2$ Levi-flat hypersurface.
The organization of this paper is as follows. In [2] we discuss boundaries of Levi-flat hypersurfaces in general and prove Theorem 1.1. In [3] we prove Theorem 1.2. In [4] we give examples that show that Theorems 1.1 and 1.2 are optimal. In [5] we prove Theorem 1.3. In [6] we give an example almost minimal submanifold which does not “bound” (in a very weak sense) even a singular Levi-flat real analytic subvariety. Finally, in [7] we discuss the existence of subanalytic Levi-flat hypersurfaces.

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2. Locally flat boundaries

We prove some basic results about locally flat boundaries. For the rest of this section, we assume that $H$ is a hypersurface with boundary, that $M = \partial H$, and that $M$ is a generic submanifold through the origin.

**Lemma 2.1.** Let $M$ be $C^\infty$ and $H$ be $C^2$, and suppose that $H$ is Levi-flat, then $M$ is nowhere minimal.

**Proof.** We can just extend $H$ to $\tilde{H}$ as in the introduction and assume $\rho$ is a defining function for $\tilde{H}$. Then $\theta = i(\partial \rho - \bar{\partial} \rho)$ is a real $C^1$ one-form that vanishes on $T^*H$. On $H$, as $H$ is Levi-flat, $d\theta \wedge \theta = 0$ and by continuity this happens on $M$ as well if we restrict $\theta$ to $M$. $\theta$ can’t vanish on $M$ as that would make $M$ have a complex tangency (it would be tangent to $T^*\tilde{H}$). Hence there exists (locally at near every point) a foliation of $M$ by CR submanifolds of smaller dimension with the same CR dimension as $M$, and so $M$ cannot be minimal at any point. \hfill \Box

**Lemma 2.2.** Let $M$ and $H$ be $C^k$ ($2 \leq k \leq \infty$), and suppose that $H$ is Levi-flat, then the Levi foliation of $H^o$ extends to a foliation of $H$. That is, in a perhaps smaller neighbourhood of the origin, there exists a $C^{k-1}$, real valued, function $f$ on $H$ (including $M$) with nonvanishing differential ($f|_M$ also has nonvanishing differential), such that $f$ is constant along leaves of the Levi foliation of $H^o$. If $M$ and $H$ are $C^\infty$, then $f$ is $C^\infty$.

**Proof.** If $k = \infty$, then by $C^{k-1}$ we will mean $C^\infty$ below. For convenience we change notation slightly. We straighten out the boundary, and assume $H$ is the upper half plane $\{x \in \mathbb{R}^n \mid x_1 \geq 0\}$ and $M$ is defined by $x_1 = 0$ (where $n = 2N - 1$). The $C^{k-1}$ 1-form given in the proof of Lemma 2.1 that vanishes on the vectors in $T^*_pH$ induces a $C^{k-1}$ 1-form $\theta$ on the upper half plane in $\mathbb{R}^n$. $\theta$ does not vanish on the tangent vectors to $x_1 = 0$ (else $M$ would have a complex tangency). We can easily extend $\theta$ to all of $\mathbb{R}^n$ (or at least a neighbourhood of the origin) as a $C^{k-1}$ 1-form. We now follow the proof of the Frobenius theorem in [9], to show that there exists a real valued function with nonvanishing differential at 0 that is constant on the Levi foliation of $H^o$. That is, we just need to show that we can modify $\theta$ on the set $x_1 < 0$, such that the modification is completely integrable. We have that $d\theta \wedge \theta = 0$ for $x_1 \geq 0$. It is not hard to see that there exists a $C^{k-2}$ 1-form $\alpha$ defined near the origin such that $d\theta = \theta \wedge \alpha$ for $x_1 \geq 0$. 

As $\theta$ does not vanish near the origin (and does not vanish identically on $T_0 M$), we may assume that $\theta = dx_n + \sum_{j=1}^{n-1} A_j dx_j$. Fix a point $a$ in $x'$ space, where $x' = (x_1, \ldots, x_{n-1})$. We consider the equation $\theta = 0$ on the hyperplane where $x_j = a_j t$ for $t \in \mathbb{R}$. We solve this ODE for $x_n$, with the initial condition $x_n(0) = c$, for some constant $c$. That is, we find the unique solution of

$$\frac{\partial F}{\partial t}(t, a, c) = -\sum_j A_j(at, F(t, a, c))a_j,$$

$$F(0, a, c) = c. \tag{2}$$

We note that we can change scale $F(t, a, c) = F(kt, a/k, c)$, and hence setting $k = 1/t$, we get $F(t, a, c) = F(1, ta, c)$. We change variables to $(u, v) \in \mathbb{R}^{n-1} \times \mathbb{R}$ by

$$x' = u,$n

$$x_n = F(1, u, v). \tag{3}$$

It is not hard to check that this is a change of coordinates. In these new coordinates we write

$$\theta = \sum_j P_j du_j + B dv. \tag{4}$$

Now we define

$$\tilde{\theta} = B dv. \tag{5}$$

If we show that the $P_j$ vanish for $x_1 \geq 0$ ($u_1 \geq 0$), then we are done. We know that $\sum P_j(ta, v)a_j = 0$. This implies that if we consider the mapping $\varphi(t, a, v) := (ta, v)$, we get

$$\varphi^* \theta = \sum \tilde{P}_j(t, a, v)da_j + \tilde{B}(t, a, v)dv. \tag{6}$$

In particular, $\varphi^* \theta$ does not depend on $dt$. Further $\tilde{P}_j(t, a, v) = tP_j(ta, v)$ so $\tilde{P}_j(0, a, v) = 0$. Now suppose that $a_1 \geq 0$ and $t \geq 0$, then we have that $d(\varphi^* \theta) = (\varphi^* \alpha) \wedge (\varphi^* \theta)$. We set $D$ so that $\varphi^* \alpha = D(t, a, v)dt + \ldots$. From this equation we obtain

$$\frac{\partial \tilde{P}_j}{\partial t} = D \tilde{P}_j. \tag{7}$$

By the uniqueness theorem for ODEs and the fact that $\tilde{P}_j(0, a, v) = 0$ this implies that $\tilde{P}_j$ is identically zero, and hence $P_j$ is identically zero. This was true for $a_1 \geq 0$ ($t \geq 0$) and hence on the upper half plane and hence on $H$. We therefore have $\theta = \tilde{\theta}$ on the upper half plane and $\tilde{\theta}$ is closed and thus exact. We get our $f$ of class $C^{k-1}$ (or $C^\infty$ if $k = \infty$) by taking $v$ as a function of $x$.  

**Lemma 2.3.** Let $M$ be real analytic and $H$ be $C^2$, and suppose that the local CR orbits of $M$ are all of codimension 1 in $M$. Then there exists a neighbourhood $U$ of the origin such that $(U \cap H) \subset \mathcal{H}$, where $\mathcal{H}$ is the unique Levi-flat real analytic hypersurface in $U$ that contains $M$.

Note that $\mathcal{H}$ is the union of the intrinsic complexifications of the local CR orbits of $M$. Where the intrinsic complexification is the smallest complex submanifold containing the local CR orbit.
Proof. Since $M$ is real analytic and the local CR orbits are all of codimension 1 in $M$, we can therefore apply the analytic Frobenius theorem to get a real analytic real valued function on some small neighbourhood $U$ of the origin in $M$ with nonvanishing differential that is constant along the local CR orbits of $M$. Such a function is CR and hence extends to be holomorphic and the vanishing of its imaginary part defines a Levi-flat hypersurface $\mathcal{H}$.

Assume that $H \subset U$. We must show that $H \subset \mathcal{H}$. By Lemma 2.2 we have that the Levi foliation of $H^o$ extends to $M$ (by perhaps making $U$ smaller still). That is, we have complex submanifolds of $\mathbb{C}^n$ with boundary on $M$. It is not hard to see by the arguments used above that a leaf $L \subset H$ extended to the boundary intersects $M$ precisely on a local CR orbit (by dimension). The function that defines the corresponding leaf of the Levi foliation of $\mathcal{H}$ is of course holomorphic on $L$ and zero on the boundary of $L$, hence $L \subset \mathcal{H}$, and so $H \subset \mathcal{H}$. 

Proof of Theorem 1.1. Let $L_k$, $k = 1, \ldots, 2N - 4$, be a basis of real analytic vector fields spanning $T^c_p M$ defined near the origin. As not all local CR orbits are of codimension 2 in $M$, then there must exist an iterated commutator $K$ of the $L_k$, which is not identically zero. As $M$ is nowhere minimal (by Lemma 2.1), then by dimension, $K$ together with $L_k$ span the tangent space of the CR orbit whenever $K$ is nonzero.

By Lemma 2.2 we have a $C^\infty$ codimension 1 foliation on $M$. Hence, by forgetting for a moment the CR structure of $M$, we can reduce to a situation where we have a $C^\infty$ codimension 1 foliation on a small neighbourhood $U \subset \mathbb{R}^{2N-2}$, given by a $C^\infty$ submersion $\varphi: U \to \mathbb{R}$, and real analytic vector fields $L_k$ and $K$, which are tangent to the leaves of the foliation, $L_k$ never vanish and $K$ does not vanish identically. To see that the foliation must be real analytic, we only need to look at $TU$, the tangent bundle of $U$, and look at the normal bundle of the foliation:

$$\{(x, v) \in U \times \mathbb{R}^{2N-2} = TU \mid \nabla \varphi(x) = tv, t \in \mathbb{R}\},$$

which is a $C^\infty$ submanifold of dimension $2N - 1$. We define a larger real analytic subvariety of the same dimension:

$$\{(x, v) \in U \times \mathbb{R}^{2N-2} = TU \mid K(x) \cdot v = 0, L_k(x) \cdot v = 0, k = 1, \ldots, 2N - 4\},$$

where we view $L_k$ and $K$ as an $\mathbb{R}^{2N-2}$ valued function, and the dot is the usual dot product. Hence by a theorem of Malgrange (see [11] Chapter VI, Proposition 3.11), we see that the normal bundle to the foliation must be a real analytic submanifold.

Therefore there must exist (locally near the origin, by Frobenius) a real valued, real analytic submersion $f: M \to \mathbb{R}$ defining the foliation. This submersion is constant along the local CR orbits of $M$ and hence must be a CR function. All real analytic CR functions extend uniquely to holomorphic functions in $\mathbb{C}^N$. Thus $f$ is really a holomorphic function with a nonvanishing gradient on $M$, which is real valued on $M$. Hence the equation $\text{Im } f = 0$ defines a real analytic Levi-flat hypersurface $\mathcal{H}$, which contains $M$. $\mathcal{H}$ must contain $H$ since it must contain the leaves of the Levi foliation of $H$ by Lemma 2.3 and the leaves of $H$ are given by the foliation given by Lemma 2.2. Actually, Lemma 2.3 only tells us about leaves that pass through points of $M$ where the codimension in $M$ of the local CR orbit is 1. However, the remaining points lie on a real analytic subvariety of $M$, and hence leaves that only pass through these points are isolated and thus must also lie in $\mathcal{H}$, since it is locally closed.
The uniqueness of $\mathcal{H}$ is one of the conclusions of Theorem 1.1 in [10].

\[\square\]

3. Extension across flat boundaries

When the local CR orbits of $M$ are all of codimension 2 in $M$, the situation is different. In this section we will prove Theorem 1.2. First we will prove this result in $\mathbb{C}^2$, and then reduce the general case to this. In [4] we will see that a $C^\infty$ extension is the best we can do. Suppose that $\tau$ is the complex conjugation function.

**Theorem 3.1.** Suppose that $H \subset \mathbb{C}^2$ is a Levi-flat $C^\infty$ hypersurface with boundary, with $0 \in \partial H \subset \mathbb{R}^2$. Then there exists a neighbourhood $U \subset \mathbb{C}^2$ of 0, with $U = \tau(U)$ such that $(H \cap U) \cup \tau(H \cap U)$ is a $C^\infty$ Levi-flat hypersurface (without boundary).

The idea is to extend the leaves of the Levi foliation of $H$ across $\mathbb{R}^2$. Because $H$ has a boundary on $\mathbb{R}^2$, the leaves must be subvarieties of $U \setminus \mathbb{R}^2$, and further can be extended to be complex submanifolds of $U$.

**Proof.** Let $(z, w) \in \mathbb{C}^2$ be the coordinates. As in the introduction, let $\tilde{H}$ be any $C^\infty$ hypersurface without boundary through 0, such that $H \subset \tilde{H}$. Since $\partial H \subset \mathbb{R}^2$, then either the $z$ or $w$ imaginary axis is not tangent to $\tilde{H}$, so let us assume it is the $w$ imaginary axis. Then there exists a polydisc $U$ with center at 0 such that $\tilde{H} \cap U$ is graph of a real valued continuous function over $\{w = 0\} \cap U$ (i.e. a function in $z$ and $\text{Re } w$). Further we can choose $U$ such that $\tilde{H} \cap U$ and $H \cap U$ are closed in $U$. To simplify notation, assume $\tilde{H} = \tilde{H} \cap U$ and $H = H \cap U$. Now $\mathbb{R}^2 \cap U$ is a subset of $\tilde{H}$, and thus $H$ is without loss of generality a graph over $\{\text{Im } w = 0 \text{ and } \text{Im } z \geq 0\} \cap U$. This means that $\tau(H) \cap H \subset \mathbb{R}^2$. Hence $\tau(H) \cup H$ is a graph of a continuous real function over $\{\text{Im } w = 0\} \cap U$. Thus it remains to be shown that near every point $p \in \partial H$, the union $\tau(H) \cup H$ is a smooth submanifold.

By Lemma 2.2 the foliation of $H$ extends up to $\mathbb{R}^2$. In particular the leaves (extended to the boundary) are closed subsets of $H$. Let $L \subset U$ be a leaf of the foliation of $H$ extended to the boundary of $H$ as a submanifold with boundary. Pick $p = (z_0, w_0) \in \partial H \subset \mathbb{R}^2$. Since $L$ is a submanifold with boundary, we look at any $\tilde{L}$ being an arbitrary $C^\infty$ extension of $L$ and we can assume $\tilde{L} \subset \tilde{H}$. We know from before that $\tilde{L}$ is not tangent to the $w$ imaginary axis at $p$ as $\tilde{H}$ is not. Since $T_p\tilde{L} = T_pL$, we know that $\tilde{L}$ is not tangent to the real $w$ axis near $p$ either. Hence there exists a small polydisc $V$ with center $(z_0, w_0)$, such that $\tilde{L} \cap V$ is a graph of a continuous complex valued function of $z$ over $\{w = w_0\} \cap V$. Since $L \cap V$ lies above $\{w = w_0 \text{ and } \text{Im } z \geq 0\} \cap V$ because $L \subset H$, then as before $\tau(L \cap V)$ is a graph over $\{w = w_0 \text{ and } \text{Im } z \leq 0\} \cap V$ and $(L \cup \tau(L)) \cap V$ is a graph of a continuous complex valued function of $z$ over $\{w = w_0\}$. Further this function is real valued when Im $z = 0$, and holomorphic when Im $z > 0$. Hence by Schwarz reflection principle it is holomorphic everywhere and $(L \cup \tau(L)) \cap V$ is a complex submanifold. Since this is true near all $p \in \mathbb{R}^2$, then $L \cup \tau(L)$ is a complex submanifold.

We can therefore foliate the set $H \cup \tau(H)$ by the complex submanifolds $L \cup \tau(L)$. Since the leaves of the foliation are complex submanifolds (and hence $C^\infty$) and are not tangent to $\mathbb{R}^2$, then $H \cup \tau(H)$ must be a $C^\infty$ submanifold. To see this, we recall that $H \cup \tau(H)$ is a graph of a real function $f$ over $\{\text{Im } w = 0\} \cap U$ and let $\pi$ be the projection onto this plane. Further, note that $H$ and $\tau(H)$ are $C^\infty$ up to the boundary and hence all partial derivatives of $f|_{\pi(H)}$ extend to $\mathbb{R}^2$ and similarly for
We only need to check that they match up. For this note that derivatives along $\mathbb{R}^2$ are all zero. Hence we only need to check one remaining direction. There we know that this is along one of the leaves of $H \cup \tau(H)$, which we know are $C^\infty$ submanifolds. \hfill \Box

To finish the proof of Theorem 1.2 we can just apply the following lemma. We will use coordinates $(z, w) \in \mathbb{C}^{N-2} \times \mathbb{C}^2$.

**Lemma 3.2.** Suppose that $U = U_z \times U_w \subset \mathbb{C}^{N-2} \times \mathbb{C}^2$ is a connected neighbourhood, and $\tau(U_w) = U_w$. $H \subset U$ is a connected Levi-flat $C^\infty$ hypersurface with boundary, with $\partial H \subset \mathbb{C}^{N-2} \times \mathbb{R}^2$. Then $H \subset \mathbb{C}^{N-2} \times H_w \subset \mathbb{C}^{N-2} \times \mathbb{C}^2$, where $H_w \subset \mathbb{C}^2$ is a $C^\infty$ Levi-flat hypersurface with boundary such that $\partial H_w \subset \mathbb{R}^2$.

**Proof.** We have already seen that the leaves of the foliation induced on $\partial H$ are unions of CR orbits. Here the CR orbits are just given by $\{(z, w) \mid w = w^0\}$ for a fixed $w^0$. So take one leaf $L$ of the Levi foliation on $H$ extended to the boundary. It is then easy to see that $L \cap \partial H$ is equal to (after perhaps extending in the $z$ direction) to $\mathbb{C}^{N-2} \times A$ for some submanifold $A \subset \mathbb{R}^2$.

Fix some $p = (z^0, w^0) \in \mathbb{C}^{N-2} \times \mathbb{R}^2$, such that $p \in L$. It is not hard to see that if we let $L_w := \{w \mid (z^0, w) \in L\}$, then $L_w \setminus \mathbb{R}^2$ is a codimension 1 complex analytic subvariety of $V \setminus \mathbb{R}^2$, for some small neighbourhood $V$ of $w^0$. Further, $(\mathbb{C}^{N-2} \times L_w) \cap \partial H = A$, and one component of $L_w$ is path connected to $A$. This is because of how $L$ is defined. If $\tilde{H}$ is any $C^\infty$ submanifold extending $H$ (as noted in the introduction), then $L$ can be extended to a real $C^\infty$ submanifold of $\tilde{H}$. Further, this extension meets $\mathbb{C}^{N-2} \times \mathbb{R}^2$ transversely in $H$, and all the derivatives in the $z$ and $\bar{z}$ directions of the defining functions of $L$ must vanish at $p$, since $L \cap \partial H = \mathbb{C}^{N-2} \times A$. Hence, $L_w$ is a submanifold with boundary in some small neighbourhood of $w^0$. By dimension, $L$ is then equal to $\mathbb{C}^{N-2} \times L_w$ in some small neighbourhood of $p$. So near some point, $L$ can be defined by an equation not depending on $z$. Since $L$ is a connected complex analytic submanifold, this is true everywhere on $L$. $H$ is a union of such $L$ and the lemma follows. \hfill \Box

The uniqueness in Theorem 1.2 is obvious in view of the fact that the extension (near the origin) is given by extension of the leaves of the Levi foliation and complex submanifolds have unique continuation.

4. **Counterexamples**

In this section we will give examples to show that the assumptions in Theorems 1.1 and 1.2 are indeed optimal.

**Example 4.1.** It is obvious that Levi-flat hypersurfaces which contain $\mathbb{R}^2 \subset \mathbb{C}^2$ cannot be unique since for example if we have coordinates $(z, w) \in \mathbb{C}^2$, then both the hypersurfaces $\text{Im } z = 0$ and $\text{Im } w = 0$ contain $\mathbb{R}^2$.

**Example 4.2.** We can find a $C^\infty$ Levi-flat hypersurface in $\mathbb{C}^2$ which contains $\mathbb{R}^2$, but which is not real analytic (not contained in a real analytic subvariety of the same dimension). First let

$$
\varphi(x) := \begin{cases} 
e^{-1/x} & x > 0, \\ 0 & x \leq 0. \end{cases}
$$

(10)
Then define $H$ by looking at
\[ \rho_t(z, w) := \varphi(t)z^2 + t - w. \] (11)

On $\mathbb{R}^2$ this defines a $C^\infty$ (but not real analytic) family of real analytic curves, and it therefore cannot be induced by a real analytic Levi-flat hypersurface. We need to show that as $(z, w)$ range over some neighbourhood of the origin in $\mathbb{C}^2$, and $t$ ranges over a small interval, $\rho_t = 0$ defines a Levi-flat hypersurface. It suffices to show that it is a submanifold near zero. It is automatically Levi-flat since it is given by a 1 parameter family of complex analytic subvarieties. First, we check that if $z$, $w$, and $t$ are kept small, then the complex analytic subvarieties do not intersect for different $t$. By direct calculation this can be seen to be the case as long as $|z| < 1$.

We look at $\text{Re} \rho_t$ and $\text{Im} \rho_t$, and notice $\text{Re} \rho_t$ as a function of $(\text{Re} z, \text{Im} z, \text{Re} w, t)$ satisfies the real analytic implicit function theorem at 0 and hence we can find a real analytic solution $t = \alpha(\text{Re} z, \text{Im} z, \text{Re} w)$, then we have a smooth hypersurface defined by
\[ 0 = \text{Im} \rho_\alpha = \varphi(\alpha(\text{Re} z, \text{Im} z, \text{Re} w)) \text{Im}(z^2) - \text{Re} w. \] (12)

Thus the requirement in Theorem 1.1 that not all local CR orbits are of codimension 2 in $M$ is necessary. This is because the above example extends to $\mathbb{C}^N$ by just letting $M = \mathbb{C}^{N-2} \times \mathbb{R}^2$.

**Example 4.3.** The methods of this paper revolve around extending the Levi foliation of the hypersurface and thereby extending $H$. Such methods are bound to fail in general when $M$ has a complex tangent and therefore is not a CR submanifold. In the following example, we show that even if we can extend a Levi-flat hypersurface past a CR singular boundary, the extension need not be unique, even in the sense of Theorem 1.2.

Let $(z, w) \in \mathbb{C}^2 \times \mathbb{C}$ be our coordinates. For a fixed $t$, let $H_t$ be a Levi-flat hypersurface defined by
\[ \text{Im} w = t\varphi(-\text{Re} w), \] (13)
where $\varphi$ is as before. Then define $M$ by
\[ \text{Re} w = |z_1|^2 + |z_2|^2 \quad \text{and} \quad \text{Im} w = 0. \] (14)

Outside of the origin, $M$ is a CR submanifold, where the codimension in $M$ of the CR orbits must be 1, as $M$ contains no complex analytic subvarieties. But then we have a whole family of Levi-flat hypersurfaces which contain $M$.

**Example 4.4.** If $M$ would be only $C^\infty$, then no general extension theorem like Theorem 1.1 nor Theorem 1.2 holds. First, let $\sqrt[2\pi]{\cdot}$ denote the principal branch of the square root, and note that the function $\xi \mapsto e^{-1/\sqrt[2\pi]{\xi}}$, holomorphic for $\text{Re} \xi > 0$, can be extended to be $C^\infty$ on $\text{Re} \xi \geq 0$. Suppose that in coordinates $(z, w_1, w_2) \in \mathbb{C}^3$ we define a $C^\infty$ Levi-flat hypersurface with boundary by
\[ \text{Re} w_1 \geq |z|^2 \quad \text{and} \quad \text{Re} w_2 = \text{Re} e^{-1/\sqrt[2\pi]{\cdot}}. \] (15)

$M$ is defined similarly by $\text{Re} w_1 = |z|^2$ and $\text{Re} w_2 = \text{Re} e^{-1/\sqrt[2\pi]{\cdot}}$. It is easy to check that $M$ is a generic $C^\infty$ submanifold. Further, since $M$ contains no complex analytic subvarieties, the CR orbits of $M$ can be seen to be of codimension 1 in $M$. At an interior point, $H$ is given by a vanishing of the real part of a holomorphic function and so $H$ is Levi-flat.

However, $H$ cannot possibly extend across $M$ since that would mean that the leafs of the Levi foliation of $H$ would have to extend. The leaf of $H$ that goes
through the origin is given by \( w_2 = e^{-1/\sqrt{w_1}} \). Since this subvariety is given as a graph, if we could possibly extend this complex analytic subvariety across the origin, we could extend the function \( e^{-1/\sqrt{w_1}} \) across \( w_1 = 0 \), and we know this is not possible.

5. Almost minimal submanifolds

We will now prove Theorem 1.4. Recall that a real analytic generic submanifold \( M \) is almost minimal at 0 if for every neighbourhood \( U \) of 0, there exists a point \( p \in M \cap U \) such that (some representative of) the local CR orbit at \( p \) is not contained in a proper complex analytic subvariety of \( U \). Let us restate Theorem 1.4 for reader convenience.

**Theorem.** Let \( M \) be a connected real analytic generic submanifold of codimension 2 through the origin, which is almost minimal at the origin. Let \( H \) be a connected \( C^2 \) hypersurface with boundary and \( M \subset \partial H \). Then \( H \) is not Levi-flat.

Theorem 1.4 is a consequence of the following more general result.

**Lemma 5.1.** Let \( M \) be a connected real analytic generic codimension 2 submanifold through the origin and let \( H \) be a connected \( C^2 \) hypersurface with boundary, and \( M \subset \partial H \). Suppose that there exists a point on \( M \) where the local CR orbits are of codimension 1 in \( M \). Then there exists some neighbourhood \( U \) of the origin such that the leaves of the Levi foliation of \( H^0 \) extend to be closed complex analytic subvarieties of \( U \).

We will need the following lemma from [10]. Here \( X_p \) is the intrinsic complexification of the local CR orbit at \( p \), that is, the smallest germ of a complex analytic submanifold that contains the local CR orbit at \( p \). When we say that \( M \) is given in normal coordinates, we mean local holomorphic coordinates \((z, w) \in \mathbb{C}^{N-2} \times \mathbb{C}^2\), such that \( M \) is given near the origin by

\[
\begin{align*}
\text{Im} w_1 &= \varphi_1(z, \bar{z}, \text{Re } w), \\
\text{Im} w_2 &= \varphi_2(z, \bar{z}, \text{Re } w),
\end{align*}
\]

(16)

where \( \varphi_j(0, \bar{z}, s) \equiv \varphi_j(z, 0, s) \equiv 0 \) for \( j = 1, 2 \). Thus \( M \) is locally a graph over \( \mathbb{C}^{N-2} \times \mathbb{R}^2 \).

**Lemma 5.2.** Given \( M \subset U \) in normal coordinates, then there is a small neighbourhood of the origin \( V \) such that for \( p \in M \cap V \), \( X_p \) contains \( \{(z, w) \in U \mid w = w^0 \} \) as germs at any \((z^0, w^0) \in X_p \).

We now prove Lemma 5.1 and therefore Theorem 1.4. The method of this proof together with Theorem 3.1 could be used to give a different (but longer) proof of Theorem 1.4.

**Proof of Lemma 5.1.** We first write \( M \) in terms of normal coordinates \((z, w) \in \mathbb{C}^{N-2} \times \mathbb{C}^2\), and take \( U \) to be the neighbourhood small enough to apply Lemma 5.2 (that is \( U = V \) in the Lemma).

If the local CR orbits are of codimension 1 in \( M \) somewhere on \( M \), they are of codimension 1 outside a proper real analytic subvariety of \( M \). Let \( p \) be one of the points where local CR orbits of \( M \) are of codimension 1 in \( M \).

We note that if \( L \) is a leaf of the Levi foliation of \( H \) (we extend this foliation to \( M \) as above) such that \( p \in L \), then by Lemma 2.3 applied in a suitably small
neighbourhood of \( p \), we see that as germs \((L, p) \subset X_p\). Hence we can extend \( L \) to a small neighbourhood of \( p \), and it will agree with some representative of \( X_p \). By Lemma 5.2, we see that near \( p \), \( L \) is defined by equations independent of \( z \). Since \( L \) is a connected complex submanifold of \( U \), then at each point it is defined by equations independent of \( z \). Hence there exists a submanifold \( \tilde{L} \) of the same dimension, such that \( L \subset \tilde{L} \) and \( \tilde{L} = C^{N-2} \times \tilde{L}_w \) where \( \tilde{L}_w \) is a complex hypersurface of \( C^2 \). Now, if we fix \( z = z^0 \) and look at \( M \cap \{ z = z_0 \} \), we see that this is a maximally totally real submanifold of \( C^2 \), and hence locally biholomorphic to \( R^2 \). We can apply the same reasoning as in the proof of 3.1 to apply Schwarz reflection principle to extend this complex hypersurface across \( R^2 \). We can therefore assume that \( \tilde{L}_w \) is a subvariety of \( U \cap \{ z = z_0 \} \) (for a perhaps smaller \( U \)) and hence \( \tilde{L} \) is a complex analytic subvariety of \( U \).

6. Almost Minimal Example

Let \( M_\lambda \), \( \lambda \in R \), be the generic, nowhere minimal submanifold of \( C^3 \), with holomorphic coordinates \((z, w_1, w_2)\) defined by

\[
\begin{align*}
\bar{w}_1 &= e^{i\bar{z}z}w_1, \\
\bar{w}_2 &= e^{i\bar{z}z}w_2.
\end{align*}
\]

When \( \lambda \) is irrational, this submanifold is almost minimal at 0, and thus not contained in any Levi-flat real analytic subvariety of codimension 1 in \( C^3 \), see [10]. As we will see below, the intrinsic complexification for a generic point \( p = (z^0, w^0_1, w^0_2) \) is given by

\[
\{ (z, w) \in C \times C^2 \mid w_1 = \overline{w^0_1}e^{i(\omega + \theta)}, w_2 = \overline{w^0_2}e^{i(\omega + \theta)} \},
\]

where \( \theta = \arg w^0_1 \), and \( \omega \) varies over \( C \). It is not hard to see that these sets cannot be contained in complex analytic subvarieties for any neighbourhood at the origin. To see this note that if we let \( \omega \) vary over \( C \), for any point \( p^1 = (z^1, w^1_1, w^1_2) \) in the set we can (by adding 2\( \pi \) to \( \omega \)) get a dense set of rotations of \( w^1_2 \) to also be in the set. This means that the closure of the set will in general be 5 real dimensional.

When \( \lambda = a/b \) is rational, \( M_\lambda \) is contained in a Levi-flat subvariety of codimension 1, as the meromorphic function \( w^0_1/w^0_2 \) is real valued on \( M_\lambda \).

By Theorem 3.4, \( M_\lambda \) is not a boundary of a \( C^2 \) Levi-flat hypersurface for \( \lambda \) irrational. We prove the following theorem to show that it can’t be a “boundary” of a real analytic Levi-flat subvariety, even if we allow singularities.

**Theorem 6.1.** Let \( \lambda \) be irrational and let \( M_\lambda \subset C^3 \) be as above. Suppose \( H \) is a codimension 1 real analytic subvariety of \( D - M_\lambda \), where \( D \) is a polydisc in \( C^3 \) centered at the origin. Suppose that there exists a point \( p \in M_\lambda \), and a connected \( C^2 \) hypersurface \( N \) with boundary, such that \( p \in \partial N \subset M_\lambda \), and \( N^0 \subset H \). Then \( H \) is not Levi-flat.

In fact, if \( H \) is irreducible, then \( H \) is not Levi-flat at any nonsingular point of top dimension.

**Proof.** Assume for contradiction that \( H \) is Levi-flat. In particular this means that \( N^0 \cap H^* \) is Levi-flat, where \( H^* \) are the nonsingular points of hypersurface dimension. Thus \( N^0 \) is Levi-flat on an open dense set. Since being Levi-flat means a certain \( C^1 \) 1-form is integrable, then it is integrable on all of \( N^0 \) by continuity.
Pick a point \( p = (z, w_1, w_2) \) on \( M = M_\lambda \), such that \( p \in N \), and the local CR orbits of \( M \) are of codimension 1 in \( M \) in a neighbourhood \( U \subset M \) of \( p \).

If we take \( q \in U \) and \( X_q \) is (some representative of the germ of) intrinsic complexification of the local CR orbit, then \( M \cap X_q \) is a hypersurface in \( X_q \) and hence divides \( X_q \) into two connected sets (we can pick a representative of \( X_q \) small enough).

Hence we can write \( X_q \) as a disjoint union of three connected sets as follows:

\[
X_q = X_q^+ \cup (M \cap X_q) \cup X_q^-.
\]

By Lemma 2.3 we see that either \( X_q^+ \subset N^o \) or \( X_q^- \subset N^o \). So suppose \( X_q^+ \subset N^o \subset H \).

Now we will find a parametrization of \( X_q \) and hence of \( X_q^+ \). We will construct this parametrization of \( X_q \) by the use of Segre sets. We can compute the third Segre set at \( q = (z^0, w_1^0, w_2^0) \), where \( w_1^0 \neq 0 \) and \( w_2^0 \neq 0 \), by the following mapping (see [1])

\[
\varphi(t_1, t_2, t_3) := (t_3, \overline{w_1^0}e^{i(t_2t_3 - t_2t_1 + t_1^2)}, \overline{w_2^0}e^{\lambda(t_3t_2 - t_2t_1 + t_1^2)}).
\]

That is, the image of this mapping agrees with \( X_q \) as germs at \( p \). We must be careful to stay within the polydisc \( D \subset \mathbb{C}^3 \) in the image. So let us suppose that \( M \) is only defined in \( D \).

Let \( \theta := \arg w_1^0 \). On \( M \), \( \arg w_1^0 = \frac{1}{2} \arg w_2^0 = \theta \). Changing variables by precomposing with \((ξ, ω) \mapsto (0, \frac{w_1^0}{ξ}, ξ) \) we get the map:

\[
\tilde{ϕ}(ξ, ω) := (ξ, w_1^0e^{i(ω+θ)}, w_2^0e^{i(ω+θ)}).
\]

The image of this map is on \( M \) when

\[
\overline{w_1^0}e^{i(ω+θ)} = e^{i|ξ|^2}w_1^0e^{-i(ω+θ)}, \quad \overline{w_2^0}e^{\lambda(ω+θ)} = e^{i\lambda|ξ|^2}w_2^0e^{-i(ω+θ)}.
\]

That is, the pullback of the CR orbit at \( q \) by \( \tilde{ϕ} \) is

\[
\text{Re} \omega = \frac{1}{2}|ξ|^2.
\]

Let \( S \) be hypersurface in the parameter space \((ξ, ω)\) defined by (23). So if, in the parameter space \((ξ, ω)\), we stay on one or the other side of \( S \), we are parametrizing either \( X_q^+ \) or \( X_q^- \).

Let us also vary \( w_1^0 \) and \( w_2^0 \), while keeping \( q = (z^0, w_1^0, w_2^0) \) within \( M \). That is let \( w_1^0 = re^{ιθ} \) and \( w_2^0 = se^{iλθ} \), and now let \( r \) and \( s \) vary. We define a map \( ψ \) by adding the parameters \( r \) and \( s \) to \( \tilde{ϕ} \)

\[
ψ(ξ, ω, r, s) := (ξ, re^{-ιθ}e^{i(ω+θ)}, se^{-iλθ}e^{iλ(ω+θ)}) = (ξ, re^{ιω}, se^{iλω}).
\]

As \( r \) and \( s \) vary over a small interval and \( ξ \) and \( ω \) vary over some small connected open set, such that the image of \( ψ \) never leaves \( D \), and further, such that \( ξ \) and \( ω \) stay on one side of \( S \), we get a parametrization of an open part of \( H \). This is because as we vary \( r \) and \( s \), we vary \( q \), and then as we vary \( ξ \) and \( ω \), we parametrize \( X_q^+ \) (as long as \( ξ \) and \( ω \) stay on one side of \( S \)).

We will make the parametrization an immersion by restricting \( ω \) to be real. Then for a small open set \( V \subset \mathbb{C} \times \mathbb{R}^3 \), \( ψ|_V \) is an immersion. We pick this \( V \) such that \( ψ(V) \subset H \). Now pick any connected open \( V' \subset \mathbb{C} \times \mathbb{R}^3 \), such that \( V \subset V' \) and for all \((ξ, ω, r, s) \in V' \) we have \( ω > \frac{1}{2}|ξ|^2 \) (or \( ω < \frac{1}{2}|ξ|^2 \)) and \( r, s \in (0, ε) \) (where \( ε \) is the radius of \( D \)). It is clear that \( ψ(V') \subset D \setminus M \). Further, \( ψ(V') \subset H \), since \( H \) is
Without loss of generality suppose we can let $\omega$ within $H$ but arbitrarily close to 0. Let $\xi$ get a subvariety of $V$ 5 dimensional set. Now we can start adding $2\pi \omega$ in some small open interval. For a bounded interval of $\omega$ we will parametrize a 5 dimensional set. Now we can start adding $2\pi$ to $\omega$ and we add a dense set of rotations to the third component in [24], without changing the first two. Thus the image of $\psi$ must be dense near some point and this contradicts $H$ being a subvariety of codimension 1.

Now suppose that $H$ is irreducible. Let $H^*$ be the nonsingular points of top dimension.

In [6] (Lemmas 2.1 and 2.2), Burns and Gong prove the following. Let $K$ be a subvariety of codimension 1 ($0 \in K$) defined by $r(z, \bar{z}) = 0$, for $r$ an irreducible real analytic real valued function. Then for some small neighbourhood $U$ of 0, $r$ complexifies (the Taylor series $r(z, w)$ converges for $z \in U$, $w \in U$) and is irreducible as a holomorphic function. Further, if $K^* \cap U$ is Levi-flat at a single point, then $K^* \cap U$ is Levi-flat at all points.

We can use this to show that if $H^*$ is Levi-flat at one point and $H$ is irreducible in $D \setminus M$, then $H^*$ is Levi-flat at all points and hence $H$ is Levi-flat by our definition. By the above result we can find a collection of open neighbourhoods $U_j$ and for each $U_j$ we find irreducible branches $A_{j1}, \ldots, A_{jn}$ of $H$ in $U_j$, and assume that each $A_{jk} \subset U_j$ satisfies the above property. Now take $H'$ be a union of those $A_{jk}$ such that $A_{11} \subset H'$ and if $A_{jk} \subset H'$ and $A_{lm} \cap A_{jk}$ is of codimension 1, then $A_{lm} \subset H'$. It is clear that $H'$ is a subvariety of $D \setminus M$ and since $H$ is irreducible then $H' = H$. It is clear that all the $A_{jk}$ are Levi-flat if and only if $A_{11}$ is Levi-flat, and we are done.

7. Subanalytic hypersurfaces

If we allow subanalytic hypersurfaces (see [3]), then we have the following result.

**Theorem 7.1.** Let $M$ be a real analytic, codimension 2, generic submanifold that is nowhere minimal. Then there exists a subanalytic hypersurface $H$, which is Levi-flat at nonsingular points, such that $M \subset H$. Further, if $H^*$ are the nonsingular points of top dimension of $H$, then $M \cap H^*$ is dense in $M$.

**Proof.** If all CR orbits of $M$ are of codimension 2 in $M$, this is trivial. Otherwise, intersect with a small ball around any point in which normal coordinates $(z, \bar{w})$ are defined. Then take the projection $\pi_w$ onto the $w$ factor. $\pi_w(M)$ is a subanalytic hypersurface in general. Apply Lemma 6.2 to see that all the $X_p$ are product sets, and $\pi_w(X_p)$ is contained in $\pi_w(M)$. If $\pi_w(M)$ is of codimension 2, $M$ had CR orbits of only codimension 2 in $M$. If $\pi_w(M)$ is of codimension 0, then $M$ must have been minimal. Hence $\pi_w(M)$ must have been a subanalytic hypersurface foliated by complex analytic subvarieties (the projections of the CR orbits), since $M$ is nowhere minimal. Thus $\pi_w(M)$ is the subanalytic hypersurface we are looking for. See [10] for more details of this method. □
Note that we must intersect with a small ball first, else the image of the projection need not be subanalytic. The submanifold $M_\lambda$ for $\lambda$ irrational from §6, when projected onto the $w$ factor without restricting the $z$ to be bounded, will be a dense set in $C^2$ which is not subanalytic. On the other hand, if we intersect $M_\lambda$ with the set $|z| \leq 1$, and we look at $\pi_w(M_\lambda)$, we get the following subanalytic hypersurface:

$$\left\{ w \in C^2 \mid \arg w_1 = t, \quad \arg w_2 = \lambda t, \quad -\frac{1}{2} \leq t \leq 0 \right\}. \quad (25)$$

Note that $H^*$ (the nonsingular points of top dimension of $H$) is again a subanalytic set and hence a locally finite union of real analytic submanifolds. Thus we have that a nowhere minimal $M$ is contained in the closure of a locally finite union of real analytic Levi-flat hypersurfaces. If the points where the CR foliation of $M$ is of codimension 1 are connected, then we need only take one hypersurface. However, $H$ need not have smooth boundary nor does the boundary need to be equal to $M$ if it does. Thus we cannot apply Theorem 1.1.

If we allow hypersurfaces with singularities all the way up to $M$ in the sense of [8], then the above result suggests that, at least locally, any nowhere minimal submanifold could conceivably bound such a singular hypersurface.

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Department of Mathematics, University of California at San Diego, La Jolla, CA 92093-0112, USA

E-mail address: jlebl@math.ucsd.edu