Leggett-Garg Inequality for a Two-Level System under Decoherence: A Broader Range of Violation

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We consider a macroscopic quantum system in a tilted double-well potential. By solving Hamiltonian equation, we obtain tunneling probabilities which contain oscillation effects. To show how one can decide between quantum mechanics and the implications of macrorealism assumption, a given form of Leggett-Garg inequality is used. The violation of this inequality occurs for a broader range of decoherence effects, compared to previous results obtained for two-level systems.

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INTRODUCTION

Extrapolating the laws of quantum mechanics QM, up to the scale of everyday objects, means that objects composed of many atoms exist in quantum superpositions of macroscopically distinct states. In 1935, Schrodinger attempted to demonstrate the limitations of QM using a thought experiment in which a cat is put in a quantum superposition of alive and dead states [1]. The idea remained theoretical until 1980s, when much progress has been made in demonstrating the macroscopic quantum behavior of various systems such as superconductors [2–5], nanoscale magnets [6, 7], laser-cooled trapped ions [8], photons in a microwave cavity [9] and C60 molecules [10].

A typical double-well potential system provides a unique opportunity to study the fundamental behavior of a macroscopic quantum system (MQS), such as macroscopic quantum tunneling and quantum coherence. In the context of a double-well potential, Schrodinger’s cat describes a state in which one system simultaneously occupies both wells. There are also studies focused on decoherence effects in double-well potentials. Huang et al. [11] showed that decoherence due to the interactions of atoms with the electromagnetic vacuum can cause the collapse of Schrodinger cat-like states. Thermal effects [12] and dissipation [13] constitute other sources of decoherence and can suppress tunneling between wells [14, 15]. In addition, double-well potential is used to describe some special phenomena like ammonia flipping. The resulting quantum tunneling have been extensively applied in many branches of physics. For example, it appears in the dynamics of Bose-Einstein condensates, the recent developments of ion trap technology, the ultracold trapped atoms theory and its applications [16–20].

Such a situation brings in mind the question of how the everyday macroscopic world works. The Leggett-Garg inequality (LGI) provides a method to investigate the existence of macroscopic coherence, to test the applicability of QM as we scale from the micro- to the macro-world [21, 22]. In this fashion, we can test the correlations of a single system measured at different times. Violation of LGI implies either the absence of a realistic description of the system or the impossibility of measuring the system without disturbing it. QM violates different forms of LGIs on both accounts. A number of experimental tests and violations of these inequalities have been demonstrated in recent years [23–24]. Leggett and Garg initially proposed an rf-SQUID flux qubit as a promising system to test their inequalities [22], which was later improved by Tesche [25]. The first measured violation of a type of LGI was reported by Palacios-Laloy and coworkers [26]. Palacios-Laloy et al. found that LGI was violated by their qubit, albeit with a single data point, with the conclusion that their system could not admit a realistic, non-invasively-measurable description. Recently, several experimental tests of LGIs were implemented, all of which confirm the predicted violations in accordance with the fundamental laws of QM [20, 23]. Most of these experiments were weak measurements, where the effects of the measured back-action in a sequential setup are minimized [34].

In this article we calculate the so-called time correlations in a tilted double-well potential. To do this, we consider the effect of the environment as a perturbation on the system. Then, we use the time correlations to test a given type of LGI. For a symmetric double-well potential considered as a two-level quantum system undergoing coherent oscillations between the two states, it has been shown that QM violates different forms of LGI [35]. Morover, no violation occurs, when strong decoherence is at work. According to our calculations for a tilted double-well potential, however, it is possible to see the violation, even for significant effects of decoherence.

The structure of our paper is as follows. In section 2, we focus on a tilted double-well model, to introduce its Hamiltonian, considering the effects of the environment on it. Then we calculate the tunneling probabilities to obtain time correlations. In section 3, we show the violation of a given LGI under decoherence. Finally, in section 4, we
conclude the results.

TILTED DOUBLE-WELL POTENTIAL

We consider a typical tilted double-well, where its asymmetric form is measured by the amount of the parameter \( \varepsilon \). Here \(|-\rangle\) (\(|+\rangle\)) denotes the state in which the macrosystem is localized in the left (right) well. They also describe the ground states in the left and right wells, respectively.

The macroscopic feature of the system is identified by dimensionless equations. We define dimensionless variables for the momentum \( p \), the position \( q \), the Hamiltonian \( \hat{H}_S \), the potential \( V(q) \) and the time \( t \) as the following relations (35, p.18):

\[
q := \frac{R}{R_0} ; \quad t := \frac{t_{\text{conv}}}{\tau_0} ; \quad \hat{H}_S := \frac{\hat{H}_{\text{conv}}}{U_0} ; \quad V(q) := \frac{U(R)}{U_0}
\]

where,

\[
\tau_0 := \frac{R_0}{(U_0/M)^{1/2}} ; \quad U_0 = M(R_0/\tau_0)^2
\]

Here \( R_0 \) and \( U_0 \) are the characteristic length and the characteristic energy of the system, respectively. Also, \( \tau_0 \) is the characteristic time, estimated by the time needed for a particle of mass \( M \) to pass the distance \( R_0 \) with kinetic energy of the order of \( U_0 \). Regarding the relations (1) and (2), one can define a new dimensionless parameter \( \tilde{\hbar} \) instead of plank’s constant in units of action \( U_0\tau_0 \):

\[
\tilde{\hbar} = \frac{\hbar}{P_0R_0}
\]

The new constant \( \tilde{\hbar} \) quantitatively shows the macroscopic behavior of the system. So that, for smaller values of \( \tilde{\hbar} \), the situation is more quasi-classical. Yet, to detect the quantum tunneling effect, \( \tilde{\hbar} \) shouldn’t be too small. Considering a tilted double-well potential, we assume that the value of \( \tilde{\hbar} \) is about 0.1 to support the macroscopic quantum trait of the system, in a quasi-classical situation.

When the macrosystem is isolated from its environment, it can be described effectively by the following Hamiltonian:

\[
H = \frac{\tilde{\hbar}}{2} \left( \frac{\delta}{\Delta} \frac{\Delta}{\Delta - \delta} \right)
\]

where \( \delta = (E_- - E_+)/\tilde{\hbar} \) is a measure of the tilt \( \varepsilon \), and \( \Delta \) is a measure of the strength of the tunneling between the two wells. The eigenvalues of the Hamiltonian (4) are \( \pm \frac{\tilde{\hbar}}{2}(\delta^2 + \Delta^2)^{1/2} \) and the eigenstates of this Hamiltonian are obtained as:

\[
|0\rangle = \left( \frac{1}{\Delta^2 + B^2} \right)^{1/2}(\Delta|+\rangle + B|-\rangle) \quad B = (\Delta^2 + \delta^2)^{1/2} + \delta
\]

\[
|1\rangle = \left( \frac{1}{\Delta^2 + A^2} \right)^{1/2}(\Delta|+\rangle - A|-\rangle) \quad A = (\Delta^2 + \delta^2)^{1/2} - \delta
\]

For our next purposes, we refine the eigenstates (5a) and (5b) as the following:

\[
|0\rangle = -\cos\theta|+\rangle + \sin\theta|-\rangle \quad (6a)
\]

\[
|1\rangle = \sin\theta|+\rangle + \cos\theta|-\rangle \quad (6b)
\]

where \( \sin\theta = \sqrt{\frac{B}{A + B}} \) and \( \cos\theta = \sqrt{\frac{A}{A + B}} \). Also we define

\[
|+\rangle = a|0\rangle + b|1\rangle \quad (7a)
\]

\[
|-\rangle = a'|0\rangle + b'|1\rangle \quad (7b)
\]

where \( a = -\cos\theta, b = \sin\theta, a' = \sin\theta \) and \( b' = \cos\theta \). One can show that the probability of the tunneling from the left to the right well is

\[
P_+ = \frac{\Delta^2}{\Delta^2 + \delta^2} \sin^2\left( (\Delta^2 + \delta^2)^{1/2} \frac{t}{2} \right)
\]
which is independent of $\tilde{h}$ and contains oscillation effects. Nevertheless, to deal with real systems, the inevitable effects of the environment should be considered. So, in order to retain oscillation effects and therefore the macroscopic quantum coherence, we consider the effects of the environment as a kind of perturbation on the system. For the Hamiltonian of the system, $\hat{H}_s$ we have

$$\hat{H}_s|n\rangle = E_n|n\rangle; \quad E_0 < E_1 < E_2 < ...$$

(9)

where $\{|n\rangle\}$ denotes the complete set of eigenstates of $\hat{H}_s$ with $n = 0, 1, 2, ...$ and $E_n$ is the energy of the system. We also define $|\ast\rangle$ and $\varepsilon_\ast$ as the energy eigenstates and the energy eigenvalues of the environment, respectively. Here,

$$\hat{H}_e|\ast\rangle = \varepsilon_\ast|\ast\rangle, \quad \hat{H}_e|\text{vac}\rangle = 0$$

(10)

and $|\text{vac}\rangle$ is the vacuum state. The state of the entire system could be written as $|n,\ast\rangle \equiv |n\rangle|\ast\rangle$. Apparently, the environment is assumed to be a bosonic field. The ground state of $\hat{H}_e$ is $|\text{vac}\rangle$ and $|\alpha\rangle = b_\alpha^\dagger|\text{vac}\rangle$ is the state with a single boson $\alpha$. The state $|n,\text{vac}\rangle$ is an eigenstate of $\hat{H}_0 = \hat{H}_s + \hat{H}_e$ with energy $E_n$.

We define $\delta E_n$ as the related shift due to the perturbation of the interaction Hamiltonian $\hat{H}_{ic}$ between the system and its environment. Then, the stationary perturbation theory gives:

$$\delta E_n \approx \delta E^{(1)}_n + \sum_{m, \ast} \frac{|\langle m, \ast|\hat{H}|n,\text{vac}\rangle|^2}{E_n - (E_m + \varepsilon_\ast)}$$

(11)

where $\delta E^{(1)}_n = \langle \langle n, \text{vac}|\hat{H}_{ic}|n,\text{vac}\rangle \rangle$ is the contribution of the first order perturbation. Considering the frequency distribution of the environmental oscillators $J(\omega)$, one can define (11) as ([35], Ch.6):

$$\delta E_n = \frac{1}{\pi} \sum_m |f_{mn}|^2 \Omega_{mn} \mathcal{P} \int_0^\infty d\omega \frac{J(\omega)}{\omega + \Omega_{mn}}$$

(12)

Similarly, we have:

$$\Gamma_n = \frac{2}{h} \sum_m |f_{mn}|^2 J(\Omega_{nm}) \theta(\Omega_{nm})$$

(13)

where $\Gamma_n^{-1}$ is the life time of the shifted energy $E_n + \delta E_n$. The symbol $\mathcal{P}$ in relation (12) denotes the principal value and $\theta(\Omega_{nm})$ is a step function which indicates that the macrosystem initially in an excited state is allowed to make transition to the lower states only. We also define $f_{mn} = \langle m|f(q)|n\rangle$ where $f(q)$ is an arbitrary function of $q$, depending on how the macrosystem exerts force on the environmental oscillators and $\Omega_{nm} = (E_n - E_m)/h$. Here, $\Omega_{10} = \Delta$. Supposing that the macrosystem is initially in the state $|-,\text{vac}\rangle$, we investigate the time evolution of the entire system with perturbation theory, which indicates the preservation of the macroscopic quantum coherence.

To do so, we are going to calculate the probability of finding the macrosystem in each well. This could be defined as

$$P_+ = |\langle +|\Psi(t)\rangle|^2$$

(14)

where $|\Psi(t)\rangle$, is the quantum state of the entire system at time $t$:

$$|\Psi(t)\rangle = \sum_n |n\rangle\langle n|e^{-i\hat{H}_s t/h}\hat{U}_I|\Psi(0)\rangle$$

(15)

Here, $\hat{U}_I$ is the time-evolution operator in the interaction picture, given by

$$\hat{U}_I(t) = \exp(-i\hat{H}_0 t/h)\exp(-i\hat{H}_t t/h)$$

(16)

The relation (15) could be written in the following form:

$$|\Psi(t)\rangle = \sum_n e^{-iE_n t/h}|n\rangle|\chi_n(t)\rangle$$

(17)

where $n = \pm$ and $|\chi_n(t)\rangle = \langle n|e^{-i\hat{H}_s t/h}\hat{U}_I|\Psi(0)\rangle$. Hence, we have

$$P_+ = \langle \chi_+(t)|\chi_+(t)\rangle$$

(18)
The time evolution operator $\hat{U}_f$ could be expanded up to the second order with respect to the interaction Hamiltonian $H_{sc}$ as:

$$
\hat{U}_f(t) \simeq 1 - \frac{i}{\hbar} \int_0^t dt_1 \hat{H}_{sc}(t_1) - \frac{1}{\hbar^2} \int_0^t dt_2 \int_0^{t_2} dt_1 \hat{H}_{sc}(t_2) \hat{H}_{sc}(t_1)
$$

(19)

where $\hat{H}_{sc}(t) = e^{iH_{0t}/\hbar} \hat{H}_{sc} e^{-iH_{0t}/\hbar}$. In (19), $\hat{U}_f(t)$ contains the following terms:

$$
\hat{U}_{vac}(t) = -\frac{i}{\hbar} \int_0^t \delta \hat{V}(t_1) dt_1 - \frac{1}{2\hbar} \sum_\alpha \int_0^t dt_2 \int_0^{t_2} dt_1 \hat{f}_\alpha(t_2) e^{-i(t_2-t_1)\omega_\alpha} \hat{f}_\alpha(t_1)
$$

(20)

$$
\hat{U}_\alpha(t) = \frac{i}{\sqrt{2}h} \int_0^t dt_1 e^{-i\omega_\alpha t} \hat{f}_\alpha(t_1)
$$

(21)

$$
\hat{U}_{\alpha\beta}(t) = -\frac{1}{2h} \int_0^t dt_2 \int_0^{t_2} dt_1 \hat{f}_\beta(t_2) e^{-i\omega_\beta t_2+i\omega_\alpha t_1} \hat{f}_\alpha(t_1)
$$

(22)

where $\delta \hat{V}(t) = \frac{1}{2} \sum_\alpha \omega_\alpha^2 (\hat{f}_\alpha(q(t)))^2$, $\omega_\alpha$ is the frequency of the particle $\alpha$ in the environment. Assuming that the environmental oscillator $\alpha$ is displaced by $f_\alpha(q)$, we use the separable model in which $f_\alpha(q)$ is independent of $\alpha$, $f_\alpha(q) = \gamma_\alpha f(q)$, where $\gamma_\alpha$ is a positive constant. All time-operators $\delta \hat{V}(t), \hat{f}_\alpha(t), ...$ in (20) to (22) are also defined in the interaction picture.

Using the relations (20)-(22) one can show that:

$$
|\chi_+(t)\rangle = (|\psi_+(0)\rangle + \sum_\alpha \alpha \langle \psi_+(0) | e^{-i\omega_\alpha t} | \alpha \rangle \langle \psi_+(0) | e^{-i\omega_\alpha t} | \alpha \rangle + \sum_\alpha \beta \langle \psi_+(0) | e^{-i(\omega_\alpha+\omega_\beta) t} | \alpha \beta \rangle \langle \psi_+(0) | e^{-i(\omega_\alpha+\omega_\beta) t} | \alpha \beta \rangle)
$$

(23)

If $\psi_+(0) = |\rangle$, we get:

$$
P_+ = \langle \chi_+ | \chi_+ \rangle = a^2 b^2 |\langle 0 | \hat{U}_{vac}(0) | 0 \rangle|^2 + a^2 b^2 |\langle 1 | \hat{U}_{vac}(1) | 1 \rangle|^2 + 2 a^2 b^2 |\langle 0 | \hat{U}_{vac}(1) | 1 \rangle|^2 + a^2 b^2 |\langle 1 | \hat{U}_{vac}(0) | 1 \rangle|^2
$$

(24)

where $\Re$ denotes the real part. We have also used the relations (7a) and (7b) for the states $|\pm\rangle$.

In the symmetric double-well the following two assumptions could be considered:

A1: The potential term $V(q)$ and so $\delta V(q)$ are even. Then, all elements $\langle n | \hat{U}_{vac}| n \rangle = 0$, when $m - n$ is odd.

A2: The function $f(q)$ is odd, so all elements $\langle n | \hat{U}_{vac}| n \rangle = 0$, when $m - n$ is even.

In the tilted double-well, also, our calculations show that the elements $\langle 0 | \hat{U}_{vac}| 1 \rangle$, $\langle 0 | \hat{U}_{vac}| 0 \rangle$ and $\langle 1 | \hat{U}_{vac}| 1 \rangle$ should be zero again, similar to what is resulted from A1 and A2 for a symmetric model. The detailed results are given in Appendix A. There are also some other assumptions, appropriate in our case:

A3. The higher orders of $f_\alpha^2$ can be neglected, so $\hat{U}_{\alpha\beta} \simeq 0$.

A4. The frequency distribution of the environment can be supposed as ohmic. This means that $J(\Delta) = \eta \Delta$ where $\eta$ is a measure of the strength of the interaction between the macrosystem and the environment ($\eta \ll \hbar$) and $\Omega_{10} = \Delta$.

As a consequence, in (13) $\Gamma_1 = (2/\hbar) |f_{01}|^2 J(\Delta) \simeq (2\eta/\hbar) |f_{01}|^2 \Delta < \Delta$. Also $\Gamma_0 = 0$. So all the terms with $\Gamma_1/\Delta$ are negligible, in our calculations.

A5. The distribution $J(\Omega_{mn})$ is always positive. Thus $J(0) = 0$ and $J(-\Delta) = 0$ where $\Omega_{10} = -\Delta$.

With all these assumptions in mind, the tunneling probability can be obtained as (see Appendix B):

$$
P_{-\rightarrow +} = \sin^2 \theta + (\sin^2 \theta \cos 2\theta) e^{-\Gamma_1 t} - 2 \sin^2 \theta \cos^2 \theta \cos(\Omega_{10} t) e^{-\Gamma_1 t/2}
$$

(25)

where $\Omega_{10} = (\delta E_1 - \delta E_0) \hbar$. This result shows that there is a decay factor $e^{-\Gamma_1 t/2}$ in the term containing $\cos(\Omega_{10} t)$ which reduces the strength of the oscillation due to the decoherence (dissipation) effects. In order to diminish the effect of $e^{-\Gamma_1 t/2}$, we consider the principal time domain, which requires that $\Gamma_1 t \ll 1$.

With the same approach, we can calculate other probabilities. For example, when the macrosystem is in the state
For the symmetric double well potential and any other two-level system studied, these calculations show a maximum
is defined as the following for the time sequences (25) to (28), one can show that:
\[
P_{+\rightarrow -} = \cos^2 \theta - \cos^2 \theta \cos 2 \theta e^{-\Gamma_1 t} - 2 \sin^2 \theta \cos^2 \theta \cos(\Omega_{10} t)e^{-\Gamma_1 t/2}
\]
(26)
\[
P_{+\rightarrow +} = \cos^2 \theta - \sin^2 \theta \cos 2 \theta e^{-\Gamma_1 t} + 2 \sin^2 \theta \cos^2 \theta \cos(\Omega_{10} t)e^{-\Gamma_1 t/2}
\]
(27)
\[
P_{-\rightarrow -} = \sin^2 \theta + (\cos^2 \theta \cos 2 \theta)e^{-\Gamma_1 t} + 2 \sin^2 \theta \cos^2 \theta \cos(\Omega_{10} t)e^{-\Gamma_1 t/2}
\]
(28)

VIOLATION OF LEGGETT-GARG INEQUALITY UNDER DECOHERENCE

In order to show the detection of a cat state, the experimental results in question should be compatible with QM
but incompatible with macrorealism (MR). The assumption of MR demands that, first, one can assign definite
states to a macrosystem, so that it could be actually in one of these states independent of any observation. Second,
it requires the non-invasive measurability of such macrostates which should not be affected, when they are measured.
LGI serves to examine quantitatively whether the theories satisfying MR are compatible with QM or not. For this,
we use the following LGI:
\[
K_1 \equiv |C_{32} - C_{31}| + C_{21} \leq 1
\]
(29)
where the time-correlation function for the two-value variables \(r\) and \(q\) \((r, q = \pm 1)\) at three moments of time \(t_3 > t_2 > t_1\)
is defined as the following for the time sequences \((i, j) = (3, 2), (3, 1), (2, 1)\):
\[
C_{ij} = \sum_{r,q=\pm 1} rqP_{r t_i q t_j}
\]
(30)
For the symmetric double well potential and any other two-level system studied, these calculations show a maximum
violation of \(K = 3/2\), when the effect of decoherence is negligible [22]. Now, let us assume that:
\[
t_3 - t_2 = t_2 - t_1 = \frac{\tau}{\Omega_{10}}, \quad \frac{\Gamma_1}{\Omega_{10}} = \gamma, \quad z = e^{-\gamma \tau}
\]
(31)
Then, the estimation of a maximum value of \(\gamma\) that violates LGI gives \(\gamma = 0.31\) [35, Ch.9]. We also choose \(\tau = \frac{\pi}{3}\), so
that \(\cos \Omega_{10}(t_2 - t_1) = \frac{1}{2}\). Then, we have:
\[
K_1 = |P_{+t_3|+t_2}P_{+t_2} + P_{+t_3|-t_2}P_{-t_2} - P_{+t_3|-t_2}P_{-t_2} - P_{+t_3|+t_2}P_{+t_2} - P_{+t_3|-t_2}P_{-t_2} + P_{+t_3|+t_2}P_{+t_2}|
\]
(32)
where, e.g., \(P_{+t_3|+t_2} = P_{+t_2\rightarrow +t_3}\) is the conditional probability that when the macrosystem is in the state \(+\) at \(t_2\),
it can be found in \(+\) at \(t_3(t_3 > t_2)\). Generally, we have \(P_{r t_i, q t_j} = P_{q t_j|r t_i}P_{r t_i}\), due to Bayesian rule and \(P_{r t_i}\) is the
single variable probability for the system being in the state \(|r\) at \(t_i\) \((i = 1, 2, 3)\). Conditional probabilities are given in
relations (25) to (28), albeit without time labeling. Let us suppose that the macrosystem is initially in the state \(-\),
so that \(P_{+t_1} = 0\). Accordingly \(P_{+t_2}\) is obtained from the following relation:
\[
P_{+t_2} = P_{+t_2|+t_1}P_{+t_1} + P_{+t_2|-t_1}P_{-t_1} = P_{+t_2|-t_1}
\]
(33)
Having into account the above considerations and using the relations (25) to (28), one can find that:
\[
K_1 = |(\sin^2 \theta - \cos^2 \theta + 2 \cos^2 \theta \cos 2 \theta z + 2 \sin^2 \theta \cos^2 \theta z^2)\]
\[
(\sin^2 \theta + \cos^2 \theta \cos 2 \theta z + \sin^2 \theta \cos^2 \theta z^2)\]
\[
+ (\cos^2 \theta - \sin^2 \theta - 2 \sin^2 \theta \cos 2 \theta z + 2 \sin^2 \theta \cos^2 \theta z^2)\]
\[
(\cos^2 \theta - \cos^2 \theta \cos 2 \theta z - \sin^2 \theta \cos^2 \theta z^2)\]
\[
- (\sin^2 \theta - \cos^2 \theta + 2 \cos^2 \theta \cos 2 \theta z^2 - 2 \sin^2 \theta \cos^2 \theta z)\]
\[
+ \sin^2 \theta - \cos^2 \theta + 2 \cos^2 \theta \cos 2 \theta z + 2 \sin^2 \theta \cos^2 \theta z^2\]
(34)
If we consider \( \sin^2 \theta = 0.2 \) and \( \cos^2 \theta = 0.8 \), at \( z = 1 \) the inequality is violated, maximally. This situation is analogous to negligible decoherence. Generally, the important result is that for \( 0.5 < z < 1 \), the inequality will be violated. This yields \( 0 < \gamma < 0.66 \) which shows a broader range of violation compared to \( \gamma = 0.31 \) for the symmetric double well potential and/or other proposed two-level systems \([35][37]\). In Fig.1, \( K_1 \) in (34) is plotted against \( z \) for \( \theta = 26.6^\circ \). It is obvious that \( K_1 \) increases as \( z \) increases from 0 to 1. In Fig.2 \( K_1 \) is plotted against \( \sin^2 \theta \) for \( z = 0.6 \) (upper curve), \( z = 0.5 \) (middle curve) and \( z = 0.4 \) (lower curve).

CONCLUSION

Considering a macro-system prepared in a quasi-classical situation described in a tilted double-well potential, we studied the effect of the environment as a perturbation source. In this regime, the decoherence (dissipative) effects are reduced according to the so-called principal time domain in which \( t \ll \Gamma_1^{-1} \) in (13) for \( n = 1 \). Calculations of the tunneling probabilities show that the coherence effects are present, in spite of the interaction with the environment. To decide between the predictions of \( QM \) and the requirements of \( MR \), a type of LGI (\( K_1 \leq 1 \)) is examined in (29), when decoherence is assumed to be present, but not so dominant. The violation of this inequality shows that the quantum behavior of a macro-system could be present in more realistic situations. So, the key parameter \( \gamma \) (characterizing the effect of dissipation) in (31) is improved from \( \gamma = 0.31 \) in previous works to \( \gamma = 0.66 \). This improvement is crucial for showing the violation of LGIs in proposed future experiments. When the classical trait of the system is increased, which is illustrated by the larger values of \( \gamma \) in (31), the assumption of non-invasive measurement becomes more determinate. This means that time-correlations could be assumed to be achieved by higher time-ordered probabilities at the macro-level [35]. Due to the quantum calculations, however, this should be denied, since no three-varibale joint probability could be defined for our model in quantum formalism from which one can obtain two-variable time-correlations. So, for broader ranges of violation due to increased values of \( \gamma \), \( \gamma \sim \Gamma_1 \sim (\hbar)^{-1} \) which shows more classicality of the system, the violation of Leggett-Garg inequality features the violation of non-invasive measurability of the system in a more concrete fashion. Yet, it is an open question, if in practice the same violation could be approved. If so, a consistency of \( MR \) with an invasive-measurement account should be envisaged more seriously.
FIG. 2: The amount of $K_1$ vs. $\sin^2 \theta$ for three different values of $z = 0.6$ (upper curve), $z = 0.5$ (middle curve) and $z = 0.4$ (lower curve).

Appendix A

We calculate $\langle 0 | \hat{U}_{\text{vac}} | 1 \rangle$ and $\langle 0 | \hat{U}_\alpha | 0 \rangle$ here to show that they are approximately equal to zero, even for non-symmetric double-well potentials. First, for $\langle 0 | \hat{U}_{\text{vac}} | 1 \rangle$, we have:

$$\langle 0 | \hat{U}_{\text{vac}} | 1 \rangle = -\frac{i}{\hbar} \langle 0 | \delta V(t) | 1 \rangle - \frac{1}{2\hbar} \langle 0 | g | 1 \rangle \quad (A-1)$$

where $g$ is defined as:

$$g = -\frac{1}{2\hbar} \sum_\alpha \int_0^t dt_2 \int_0^{t_2} dt_1 \hat{f}_\alpha(t_2) e^{-i(t_2-t_1)\omega_\alpha} \hat{f}_\alpha(t_1) \quad (A-2)$$

For the first term, one can show that it is equal to:

$$\frac{1}{2} \sum_\alpha \omega_\alpha^2 \langle 0 | f_\alpha^2 | 1 \rangle = \frac{1}{2} \sum_{\alpha,m=0,1} \omega_\alpha^2 \langle 0 | f_\alpha | m \rangle \langle m | f_\alpha | 1 \rangle$$

$$= \frac{1}{2} \sum_{\alpha} \omega_\alpha^2 (\langle 0 | f_\alpha | 0 \rangle \langle 0 | f_\alpha | 1 \rangle + \langle 0 | f_\alpha | 1 \rangle \langle 1 | f_\alpha | 1 \rangle)$$

$$= \frac{1}{2} \sum_{\alpha} \frac{\gamma_\alpha^2}{\omega_\alpha} (f_{00} f_{01} + f_{01} f_{11}) = \frac{t}{\pi} (f_{00} f_{01} + f_{01} f_{11}) \int_0^\omega d\omega J(\omega)$$

$$\times \frac{\Omega_{10}}{\Omega_{10}} \times \frac{f_{01}}{f_{01}} \Rightarrow \frac{t}{\pi} \frac{f_{00} + f_{11}}{f_{01} \Omega_{10}} \frac{f_{01}^{2} \Omega_{10}}{\int_0^\omega d\omega J(\omega)} = \frac{t}{\pi} \frac{f_{00} + f_{11}}{f_{01} \Omega_{10}} \Gamma_1 \quad (A-3)$$

which is negligible, because $\Gamma_1 t/\Omega_{10} \ll 1$. The second term is zero, because the following integrals have meaningful values, only when the terms in denominator are equal to zero (i.e., $\Omega_{10} + \omega_\alpha = 0$), which is impossible since $\omega_\alpha > 0$ and $\Omega_{10} > 0$, so the entire term vanishes. To show this, we have
\[-\frac{1}{2\hbar} \sum_{\alpha,m} \int_0^t dt_2 \int_0^{t_2} dt_1 \langle 0 | f_\alpha(t_2) | m \rangle \langle m | f_\alpha(t_1) | 1 \rangle e^{-i\omega_\alpha(t_2-t_1)} \]

\[= -\frac{1}{2\hbar} \int_0^t dt_2 \int_0^{t_2} dt_1 e^{-i(\Omega_{10}-\omega_\alpha)t_1} \langle 0 | f_\alpha | 0 \rangle \langle 0 | f_\alpha | 1 \rangle e^{-i\omega_\alpha t_2} + e^{i\omega_\alpha t_1} \langle 0 | f_\alpha | 1 \rangle \langle 1 | f_\alpha | 1 \rangle e^{-i(\Omega_{10}+\omega_\alpha)t_2} \]

\[= -\frac{1}{2\hbar} \sum_{\alpha} \int_0^t dt_2 \left( \frac{1}{-i(\Omega_{10}+\omega_\alpha)} (e^{-i(\Omega_{10}+\omega_\alpha)t_2} - 1) \langle 0 | f_\alpha | 0 \rangle \langle 0 | f_\alpha | 1 \rangle e^{-i\omega_\alpha t_2} + \frac{1}{i\omega_\alpha} (e^{i\omega_\alpha t_2} - 1) \langle 0 | f_\alpha | 1 \rangle \langle 1 | f_\alpha | 1 \rangle e^{-i(\Omega_{10}+\omega_\alpha)t_2} \right) \]

\[= -\frac{1}{2\hbar} \sum_{\alpha} \{ \langle 0 | f_\alpha | 0 \rangle \langle 0 | f_\alpha | 1 \rangle \frac{1}{-i(\Omega_{10}+\omega_\alpha)} \left( \frac{1}{-i\Omega_{10}} (e^{-i\Omega_{10}t_1} - 1) + \frac{1}{-i\omega_\alpha} (e^{-i\omega_\alpha t_1} - 1) \right) \}

\[\langle 0 | f_\alpha | 1 \rangle \langle 1 | f_\alpha | 1 \rangle \frac{1}{i\omega_\alpha} \{ \frac{1}{-i\Omega_{10}} (e^{-i\Omega_{10}t_1} - 1) + \frac{1}{-i\omega_\alpha} (e^{-i(\Omega_{10}+\omega_\alpha)t_1} - 1) \} \} \approx 0 \quad (A-4)\]

For \( \langle 0 | \hat{U}_\alpha | 0 \rangle \), one can show that

\[\langle 0 | \hat{U}_\alpha | 0 \rangle = \frac{i}{\sqrt{2\hbar}} \int_0^t dt_1 \langle 0 | f_\alpha(t_1) | 0 \rangle e^{i\omega_\alpha t_1} = \frac{i}{\sqrt{2\hbar}} \hat{\xi}_\alpha f_{mn} \hat{D}_1(\omega_\alpha) e^{i\omega_\alpha t/2} \quad (A-5)\]

where

\[\hat{D}_1(\omega; t) = \frac{1}{2\pi} \frac{\sin(\omega t/2)}{\omega/2} \quad (A-6)\]

Then

\[|\langle 0 | \hat{U}_\alpha | 0 \rangle|^2 = \frac{2t}{\hbar} f_{\bar{a}0} \int_0^\infty d\omega J(\omega) \hat{D}_2(\omega) \quad (A-7)\]

where

\[\hat{D}_2(\omega; t) = \frac{1}{2\pi t} \left( \frac{\sin(\omega t/2)}{\omega/2} \right)^2 \quad (A-8)\]

We work in the principal time domain for which \( \hat{D}_2(\omega, t) \sim \delta(\omega) \). So the relation (A-7) is equal to zero, since \( J(0) = 0 \). So, the term \( \langle 0 | \hat{U}_\alpha | 0 \rangle \) could be neglected. The same situation holds true for the element \( \langle 1 | \hat{U}_\alpha | 1 \rangle \) with relations similar to \( \langle 0 | \hat{U}_\alpha | 0 \rangle \).

**Appendix B**

Here, we calculate the term \( P_{-1} \) as an instance. Other probabilities can be obtained in the same way. We need to calculate some terms at first and then put them in the main formula. To show this, we have:

\[P_{-1} = a^2 b^2 |\langle 0 | \hat{U}_{vac} | 0 \rangle|^2 + a^2 b^2 |\langle 1 | \hat{U}_{vac} | 1 \rangle|^2 + 2 a a' b b' \Re \langle 0 | \hat{U}_{vac} | 0 \rangle \langle 1 | \hat{U}_{vac} | 1 \rangle + a^2 b^2 |\langle 1 | \hat{U}_0 | 0 \rangle|^2 \quad (B-1)\]

The terms \( \langle 0 | \hat{U}_{vac} | 0 \rangle \), \( \langle 1 | \hat{U}_{vac} | 1 \rangle \), \( \langle 1 | \hat{U}_0 | 0 \rangle \) and \( \langle 0 | \hat{U}_0 | 1 \rangle \) are calculated in [35]. So, we have

\[|\langle 0 | \hat{U}_{vac} | 0 \rangle|^2 = 1, \quad |\langle 1 | \hat{U}_{vac} | 1 \rangle|^2 = e^{-\Gamma_1 t} \quad (B-2)\]

\[\langle m | \hat{U}_\alpha | n \rangle = \frac{i}{\sqrt{2\hbar}} \hat{\xi}_\alpha f_{mn} \hat{D}_1(\omega_\alpha + \Omega_{mn}; t) e^{i(\Omega_{mn}+\omega_\alpha)t/2} \quad (B-3)\]
where $D_1(\omega; t) = \frac{1}{2\pi} \frac{\sin(\omega t/2)}{\omega/2}$. For $|\langle \hat{U}_\alpha | 0 \rangle|^2$, we have:

$$|\langle \hat{U}_\alpha | 0 \rangle|^2 = \frac{t\pi}{\hbar} f_{10}^2 \int_0^\infty d\omega J(\omega) D_2(\omega - \Omega_{10})$$

$$= \frac{t}{\hbar} f_{10}^2 J(\Omega_{10}) = \Gamma_1 t \approx 1 - e^{-\Gamma_1 t}$$

(B-4)

where $D_2(\omega - \Omega_{10}) \sim \delta(\omega - \Omega_{10})$.

All the terms that produced by multiplying the terms containing $\hat{U}_\alpha$ are zero, because there is $\frac{\Gamma_1}{\Omega_{10}}$ ratio in all of them.

There is also one non-zero multiplying term as the following:

$$\Re \langle 0 | \hat{U}_{vac} | 0 \rangle^* \langle 1 | \hat{U}_{vac} | 1 \rangle = e^{-\Gamma_1 t/2} \cos(\Omega_{10}t)$$

(B-5)

Finally, we obtain:

$$P_{+\rightarrow -} = \cos^2 \theta - \cos^2 \theta \cos 2\theta e^{-\Gamma_1 t} - 2 \sin^2 \theta \cos^2 \theta \cos(\Omega_{10}t)e^{-\Gamma_1 t/2}$$

(B-6)

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