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Families of polynomials of every degree with no rational preperiodic points

Familles de polynômes de degré arbitraire sans points prépériodiques rationnels

Mohammad Sadek

Abstract. Let $K$ be a number field. Given a polynomial $f(x) \in K[x]$ of degree $d \geq 2$, it is conjectured that the number of preperiodic points of $f$ is bounded by a uniform bound that depends only on $d$ and $[K : Q]$. However, the only examples of parametric families of polynomials with no preperiodic points are known when $d$ is divisible by either 2 or 3 and $K = Q$. In this article, given any integer $d \geq 2$, we display infinitely many parametric families of polynomials of the form $f_t(x) = x^d + c(t), c(t) \in K(t)$, with no rational preperiodic points for any $t \in K$.

Résumé. Soit $K$ un corps de nombres. Étant donné un polynôme $f(x) \in K[x]$ de degré $d \geq 2$, il est conjecturé que le nombre de points prépériodiques de $f$ est borné par une constante ne dépendant que de $d$ et $[K : Q]$. Cependant, les seuls exemples de familles paramétriques de polynômes sans points prépériodiques supposent 2|d ou 3|d et $K = Q$. Dans cet article, étant donné un entier $d \geq 2$, nous démontrons qu’il existe une infinité de familles paramétriques de polynômes de la forme $f_t(x) = x^d + c(t), c(t) \in K(t)$, sans points prépériodiques rationnels pour tout $t \in K$.

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1. Introduction

An arithmetic dynamical system over a number field $K$ consists of a rational function $f : \mathbb{P}^n(K) \to \mathbb{P}^n(K)$ of degree at least 2 with coefficients in $K$ where the $m^{th}$ iterate of $f$ is defined recursively by $f^1(x) = f(x)$ and $f^m(x) = f(f^{m-1}(x))$ when $m \geq 2$. A point $P \in \mathbb{P}^n(K)$ is said to be a periodic point for $f$ if there exists a positive integer $m$ such that $f^m(P) = P$. If $N$ is the smallest positive
integer such that \( f^N(P) = P \), then the periodic point \( P \) is said to be of **exact period** \( N \). A point \( P \in \mathbb{P}^n(K) \) is said to be a **preperiodic** point for \( f \) if the orbit \( \{ f^i(P) : i = 0, 1, 2, \ldots \} \) of \( P \) is finite, i.e., if some iterate \( f^i(P) \) is periodic.

The following conjecture was proposed by Morton and Silverman in p. 4 of [6].

**Conjecture 1.** There exists a bound \( B(D, n, d) \) such that if \( K/\mathbb{Q} \) is a number field of degree \( D \), and \( f : \mathbb{P}^n(K) \to \mathbb{P}^n(K) \) is a morphism of degree \( d \geq 2 \) defined over \( K \), then the number of \( K \)-rational preperiodic points of \( f \) is bounded by \( B(D, n, d) \).

When \( f \) is taken to be a quadratic polynomial over \( \mathbb{Q} \), the following conjecture was suggested in [2, Conjecture 2] and [9, Conjecture 2].

**Conjecture 2.** If \( N \geq 4 \), then there is no quadratic polynomial \( f(x) \in \mathbb{Q}[x] \) with a rational point of exact period \( N \).

The conjecture has been proved when \( N = 4 \), see [5, Theorem 4], and \( N = 5 \), see [2, Theorem 1]. A conditional proof for the case \( N = 6 \) was given in [10, Theorem 7].

Although polynomials described by an equation of the form \( x^d + c, d \geq 2, c \in K \), with rational preperiodic points are scarce, examples of parametric families of polynomials with no preperiodic points are very few in the literature. In Theorem 4 of [3], families of such polynomials were given when \( d \) is even or when \( d \) is divisible by 3, and \( K = \mathbb{Q} \). The main finding of this article can be described as follows. Let \( K \) be a number field. Given an arbitrary integer \( d \geq 2 \), we prove the existence of infinitely many parametric families of polynomials of degree \( d \) with no \( K \)-rational preperiodic points. This is achieved using some recent results on the non existence of rational points on certain twisted superelliptic curves.

## 2. Parametric families of polynomials with no periodic points

In what follows \( K \) will denote a number field with ring of integers \( \mathcal{O}_K \). The following proposition is [8, Lemma 1].

**Proposition 3.** Let \( f(x) = x^d + c \), where \( d \geq 2 \) is an integer and \( c \in K \setminus \{0\} \). If \( f \) has a \( K \)-rational periodic point, then there exist \( a, b \in \mathcal{O}_K \) such that \( c = a/b^d \), and \((a\mathcal{O}_K, b^d\mathcal{O}_K) = I^d \) for some ideal \( I \) in \( \mathcal{O}_K \).

Proposition 3 shows that polynomials described by an equation of the form \( x^d + c, d \geq 2, c \in K \), with \( K \)-rational preperiodic points are rare. However, up to the knowledge of the author, the only such family is given as Theorem 4 in [3] where \( K = \mathbb{Q} \). The statement of the latter theorem is as follows.

**Theorem 4.** Let \( 2 \mid d \) and \( m \geq 4 \); or \( 3 \mid d \) and \( m \geq 3 \). Then for \( t \in \mathbb{Q} \), the polynomial

\[
x^d + \frac{1}{1 + t^m}
\]

has no \( \mathbb{Q} \)-rational preperiodic points.

Now we state the main result of this work.

**Theorem 5.** Let \( K \) be a number field with ring of integers \( \mathcal{O}_K \). Let \( d \geq 2 \) be an integer. Let \( P(T) \in \mathcal{O}_K[T] \) be of degree \( N \) a multiple of \( d \) such that the multiplicity of each of its roots is at most \( d - 1 \). Assume moreover that the Galois group of \( P(T) \) over \( K \) has an element fixing no root of \( P(T) \). Then there exists \( w \in \mathcal{O}_K \setminus \{0\} \) such that the polynomial

\[
f(x) = x^d + \frac{1}{w \cdot P(t)}
\]

has no \( K \)-rational preperiodic points for any \( t \in K \).
Proof. According to [4, Theorem 3.1], given that the Galois group of \( P(T) \) over \( K \) has an element fixing no root of \( P(T) \), it follows that there exists \( w \in \mathcal{O}_K \setminus \{0\} \) such that the twisted superelliptic curve defined by \( y^d = w \cdot P(T) \) has no \( K \)-rational points. In other words, there exists no \((y, t, s) \in K^3 \setminus \{(0,0,0)\}\) such that \( y^d = w \cdot Q(t,s) \), where \( Q(T,S) = S^N \cdot P(T/S) \). In view of Proposition 3, \( f \) has no \( K \)-rational periodic points of any period, hence no \( K \)-rational preperiodic points for any \( t \in K \). \( \square \)

Remark 6. One knows that the proportion of degree \( N \) polynomials \( P(T) \in \mathcal{O}_K[T] \) with height bounded by \( H \) and such that the Galois group of \( P(T) \) over \( K \) is isomorphic to the symmetric group \( S_N \) tends to 1 as \( H \) tends to \( \infty \), see for example [1, Theorem 2.1]. Consequently, the proportion of fixed degree polynomials \( P(T) \) introduced in Theorem 5 with height bounded by \( H \) tends to 1 as \( H \) tends to \( \infty \).

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