Gravity in Causal Perturbative Quantum Field Theory

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Abstract

We extend the general framework of perturbative quantum field theory developed for the pure Yang-Mills model to gravity. First we present a variant of the elimination procedure of the anomalies in the second order of perturbation theory. After that we prove that gauge invariance restricts severely the possible finite renormalizations, so at least in the second order of the perturbation theory, the model is renormalizable.

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1 Introduction

The most natural way to arrive at the Bogoliubov axioms of perturbative quantum field theory (pQFT) is by analogy with non-relativistic quantum mechanics [6], [13], [3], [4]: in this way one arrives naturally at Bogoliubov axioms [1], [5], [17], [18]. We prefer the formulation from [4] and as presented in [8]; for every set of monomials $A_1(x_1), \ldots, A_n(x_n)$ in some jet variables (associated to some classical field theory) one associates the operator-valued distributions $T^{A_1,\ldots,A_n}(x_1,\ldots,x_n)$ called chronological products; it will be convenient to use another notation: $T(A_1(x_1),\ldots,A_n(x_n))$.

The Bogoliubov axioms, presented in Section 2.2 express essentially some properties of the scattering matrix understood as a formal perturbation series with the “coefficients” the chronological products: (1) (skew)symmetry properties in the entries $A_1(x_1),\ldots,A_n(x_n)$; (2) Poincaré invariance; (3) causality; (4) unitarity; (5) the “initial condition” which says that $T(A(x))$ is a Wick polynomial.

So we need some basic notions on free fields and Wick monomials. One can supplement these axioms by requiring (6) power counting; (7) Wick expansion property.

It is a highly non-trivial problem to find solutions for the Bogoliubov axioms, even in the simplest case of a real scalar field.

There are, at least to our knowledge, three rigorous ways to do that; for completeness we remind them following [9]: (a) Hepp axioms [13]; (b) Polchinski flow equations [14], [16]; (c) the causal approach due to Epstein and Glaser [5], [6] which we prefer.

The procedure of Epstein and Glaser is a recursive construction for the basic objects $T(A_1(x_1),\ldots,A_n(x_n))$ and reduces the induction procedure to a distribution splitting of some distributions with causal support. In an equivalent way, one can reduce the induction procedure to the process of extension of distributions [15].

An equivalent point of view uses retarded products [20] instead of chronological products. For gauge models one has to deal with non-physical fields (the so-called ghost fields) and impose a supplementary axiom namely gauge invariance, which guarantees that the physical states are left invariant by the chronological products.

We will extend the analysis from [11] to the case of (pure and massless) gravity. We will consider only the second order of the perturbation theory i.e. only chronological products of the type $T(T(x_1),T(x_2))$.

All these chronological products can be split in the loop and tree contributions. The loop contributions have been analysed in detail in [10] where it was proved that they are trivial i.e. they are coboundaries.

In the next Section we give the basic facts on perturbative quantum field theory of quantum gravity: the construction of Wick monomials, Bogoliubov axioms and the framework of perturbative quantum gravity. In Section 3 we investigate the tree contributions. As is known, they produce anomalies; we compute in detail these anomalies for the basic chronological products of the type $T(T(x_1),T(x_2))$. Then we show an elementary way of eliminating the anomalies by redefinitions of the chronological products. In Section 4 we consider the question of the renormalizability of quantum gravity. It is asserted in the literature that quantum gravity is a non-renormalizable theory, this meaning that the arbitrariness of the chronological products
(given by quasi-local operators) increases with the order of the perturbation theory. We consider this problem in the second order of perturbation theory. After fixing gauge invariance in the form
\[ sT(T(x_1), T(x_2)) \equiv d_Q T(T(x_1), T(x_2)) - i \partial_1^\mu T(T^\mu(x_1), T(x_2)) - i \partial_2^\mu T(T(x_1), T^\mu(x_2)) = 0 \] (1.1)
we are left with an arbitrariness of the form
\[ R(T^I(x_1), T^J(x_2)) = \delta(x_1 - x_2) N(T^I, T^J)(x_2) + \partial_\mu \delta(x_1 - x_2) N(T^I, T^J)^\mu(x_2) + \partial_\mu \partial_\nu (x_1 - x_2) N(T^I, T^J)^\mu\nu(x_2). \] (1.2)
However this arbitrariness is constrained by the requirement that it preserve the gauge invariance just established so it must verify:
\[ sR(T(x_1), T(x_2)) = 0. \] (1.3)
Also, not all solutions of this (cohomological) equation are physically semnificative: we have trivial solutions (coboundaries) of the form
\[ R(T(x_1), T(x_2)) = \bar{s}B(T(x_1), T(x_2)) = d_Q B(T(x_1), T(x_2)) + i \partial_1^\mu B(T^\mu(x_1), T(x_2)) + i \partial_2^\mu B(T(x_1), T^\mu(x_2)) \] (1.4)
because such finite renormalizations give null contributions when restricted to the physical subspace. The new result is that any solution of the type (1.2) quadri-linear in the fields (and their derivatives) of the cocyle equation (1.3) is a coboundary i.e. of the form (1.4). So we are left only with the solution tri-linear in the variables - see (2.41), (2.42). Such terms can be eliminated by a redefinition of the coupling constant.

It natural to expect that such a result will be true in arbitrary orders of the perturbation theory, making perturbative quantum gravity a bona fidae theory like the standard model.
2 Perturbative Quantum Field Theory

There are two main ingredients in the construction of a perturbative quantum field theory (pQFT): the construction of the Wick monomials and the Bogoliubov axioms. For a pQFT of Yang-Mills theories one needs one more ingredient, namely the introduction of ghost fields and gauge charge.

2.1 Wick Products

We follow the formalism from [8]. We consider a classical field theory on the Minkowski space \( \mathcal{M} \simeq \mathbb{R}^4 \) (with variables \( x^\mu, \mu = 0, \ldots, 3 \) and the metric \( \eta \) with \( \text{diag}(\eta) = (1, -1, -1, -1) \)) described by the Grassmann manifold \( \Xi_0 \) with variables \( \xi_a, a \in \mathcal{A} \) (here \( \mathcal{A} \) is some index set) and the associated jet extension \( J^r(\mathcal{M}, \Xi_0) \), \( r \geq 1 \) with variables \( x^\mu, \xi_{a; \mu_1, \ldots, \mu_n}, n = 0, \ldots, r \); we denote generically by \( \xi^p, p \in P \) the variables corresponding to classical fields and their formal derivatives and by \( \Xi_r \) the linear space generated by them. The variables from \( \Xi_r \) generate the algebra \( \text{Alg}(\Xi_r) \) of polynomials.

To illustrate this, let us consider a real scalar field in Minkowski space \( \mathcal{M} \). The first jet-bundle extension is \( J^1(\mathcal{M}, \mathbb{R}) \simeq \mathcal{M} \times \mathbb{R} \times \mathbb{R}^4 \) with coordinates \( (x^\mu, \phi, \phi_\mu) \), \( \mu = 0, \ldots, 3 \).

If \( \varphi : \mathcal{M} \rightarrow \mathbb{R} \) is a smooth function we can associate a new smooth function \( j^1 \varphi : \mathcal{M} \rightarrow J^1(\mathcal{M}, \mathbb{R}) \) according to \( j^1 \varphi(x) = (x^\mu, \varphi(x), \partial_\mu \varphi(x)) \).

For higher order jet-bundle extensions we have to add new real variables \( \phi_{\{\mu_1, \ldots, \mu_r\}} \) considered completely symmetric in the indexes. For more complicated fields, one needs to add supplementary indexes to the field i.e. \( \phi \rightarrow \phi_a \) and similarly for the derivatives. The index \( a \) carries some finite dimensional representation of \( \text{SL}(2, \mathbb{C}) \) (Poincaré invariance) and, maybe a representation of other symmetry groups. In classical field theory the jet-bundle extensions \( j^r \varphi(x) \) do verify Euler-Lagrange equations. To write them we need the formal derivatives defined by

\[
   d_\nu \phi_{\{\mu_1, \ldots, \mu_r\}} \equiv \phi_{\{\nu, \mu_1, \ldots, \mu_r\}}.
\]

We suppose that in the algebra \( \text{Alg}(\Xi_r) \) generated by the variables \( \xi_p \) there is a natural conjugation \( A \rightarrow A^\dagger \). If \( A \) is some monomial in these variables, there is a canonical way to associate to \( A \) a Wick monomial: we associate to every classical field \( \xi_a, a \in \mathcal{A} \) a quantum free field denoted by \( \xi^\text{quant}_a(x), a \in \mathcal{A} \) and determined by the 2-point function

\[
   \langle \Omega, \xi^\text{quant}_a(x), \xi^\text{quant}_b(y)\Omega \rangle = -i \ D_{ab}^{(+)}(x-y) \times 1.
\]

Here

\[
   D_{ab}(x) = D_{ab}^{(+)}(x) + D_{ab}^{(-)}(x)
\]

is the causal Pauli-Jordan distribution associated to the two fields; it is (up to some numerical factors) a polynomial in the derivatives applied to the Pauli-Jordan distribution. We understand by \( D_{ab}^{(\pm)}(x) \) the positive and negative parts of \( D_{ab}(x) \). From \( (2.2) \) we have

\[
   [\xi_a(x), \xi_b(y)] = -i \ D_{ab}(x-y) \times 1
\]
where by $[\cdot, \cdot]$ we mean the graded commutator.

The $n$-point functions for $n \geq 3$ are obtained assuming that the truncated Wightman functions are null: see \cite{2}, relations (8.74) and (8.75) and proposition 8.8 from there. The definition of these truncated Wightman functions involves the Fermi parities $|\xi_p|$ of the fields $\xi_p, p \in P$.

Afterwards we define

$$\xi_{a_{\mu_1, \ldots, \mu_n}}^\text{quant}(x) \equiv \partial_{\mu_1} \cdots \partial_{\mu_n} \xi_a(x), a \in A$$

which amounts to

$$[\xi_{a_{\mu_1, \ldots, \mu_n}}(x), \xi_{b_{\nu_1, \ldots, \nu_n}}(y)] = (-1)^n i \partial_{\mu_1} \cdots \partial_{\mu_n} \partial_{\nu_1} \cdots \partial_{\nu_n} D_{ab}(x - y) \times 1. \quad (2.5)$$

More sophisticated ways to define the free fields involve the GNS construction.

The free quantum fields are generating a Fock space $\mathcal{F}$ in the sense of the Borchers algebra: formally it is generated by states of the form $\xi_{1}^\text{quant}(x_1) \cdots \xi_{n}^\text{quant}(x_n) \Omega$ where $\Omega$ the vacuum state. The scalar product in this Fock space is constructed using the $n$-point distributions and we denote by $\mathcal{F}_0 \subset \mathcal{F}$ the algebraic Fock space.

One can prove that the quantum fields are free, i.e. they verify some free field equation; in particular every field must verify Klein Gordon equation for some mass $m$

$$(\Box + m^2) \xi_a^\text{quant}(x) = 0 \quad (2.6)$$

and it follows that in momentum space they must have the support on the hyperboloid of mass $m$. This means that they can be split in two parts $\xi_{a_{\mu_1, \ldots, \mu_n}}^\text{quant}(\pm)$ with support on the upper (resp. lower) hyperboloid of mass $m$. We convene that $\xi_{a_{\mu_1, \ldots, \mu_n}}^\text{quant}(+) \text{ resp. } \xi_{a_{\mu_1, \ldots, \mu_n}}^\text{quant}(-)$ correspond to the creation (resp. annihilation) part of the quantum field. The expressions $\xi_{p_{\nu_1, \ldots, \nu_n}}^\text{quant}(\pm)$ resp. $\xi_{p_{\nu_1, \ldots, \nu_n}}^\text{quant}(\mp)$ for a generic $\xi_p, p \in P$ are obtained in a natural way, applying partial derivatives. For a general discussion of this method of constructing free fields, see ref. \cite{2} - especially prop. 8.8. The Wick monomials are leaving invariant the algebraic Fock space. The definition for the Wick monomials is contained in the following Proposition.

**Proposition 2.1** The operator-valued distributions $N(\xi_{q_1}(x_1), \ldots, \xi_{q_n}(x_n))$ are uniquely defined by:

$$N(\xi_{q_1}(x_1), \ldots, \xi_{q_n}(x_n)) \Omega = \xi_{q_1}^{(+)}(x_1) \cdots \xi_{q_n}^{(+)}(x_n) \Omega \quad (2.7)$$

$$[\xi_p(y), N(\xi_{q_1}(x_1), \ldots, \xi_{q_n}(x_n))]$$

$$= \sum_{m=1}^{n} \prod_{l<m} (-1)^{|\xi_p||\xi_{q_l}|} \left[ [\xi_p(y), \xi_{q_{m}}(x_{m})] N(\xi_{q_1}(x_1), \ldots, \xi_{q_m}(x_m)) \right]$$

$$= -i \sum_{m=1}^{n} \prod_{l<m} (-1)^{|\xi_p||\xi_{q_l}|} D_{pq_m}(y - x_m) \ N(\xi_{q_1}(x_1), \ldots, \xi_{q_m}(x_m)) \quad (2.8)$$
\[ N(\emptyset) = I. \] (2.9)

The expression \( N(\xi_1(x_1), \ldots, \xi_n(x_n)) \) is (graded) symmetrical in the arguments.

The expressions \( N(\xi_1(x_1), \ldots, \xi_n(x_n)) \) are called Wick monomials. There is an alternative definition based on the splitting of the fields into the creation and annihilation part for which we refer to [8].

It is a non-trivial result of Wightman and Gårding [21] that in \( N(\xi_1(x_1), \ldots, \xi_n(x_n)) \) one can collapse all variables into a single one and still gets a well-defined expression (see Prop. 2.2 of [11]). In this way we can associate to every monomial \( A \) in the jet variables a quantum operator \( N(A) \) which is called the associated Wick monomial.

One can prove that

\[ [N(A(x)), N(B(y))] = 0, \quad (x - y)^2 < 0 \] (2.10)

where by \([\cdot, \cdot]\) we mean the graded commutator. This is the most simple case of causal support property. Now we are ready for the most general setting. We define for any monomial \( A \in \text{Alg}(\Xi_r) \) the derivation

\[ \xi \cdot A \equiv (-1)^{|\xi||A|} \frac{\partial}{\partial \xi} A \] (2.11)

for all \( \xi \in \Xi_r \). Here \(|A|\) is the Fermi parity of \( A \) and we consider the left derivative in the Grassmann sense. So, the product \( \cdot \) is defined as an map \( \Xi_r \times \text{Alg}(\Xi_r) \to \text{Alg}(\Xi_r) \). An expression \( E(A_1(x_1), \ldots, A_n(x_n)) \) is called of Wick type iff verifies:

\[ [\xi_p(y), E(A_1(x_1), \ldots, A_n(x_n))] \]

\[ = \sum_{m=1}^{n} \prod_{l=1}^{m} (-1)^{|\xi_p||A_l|} \sum_q [\xi_p(y), \xi_q(x_m)] E(A_1(x_1), \ldots, \xi_q \cdot A_m(x_m), \ldots, A_n(x_n)) \]

\[ = -i \sum_{m=1}^{n} \prod_{l=1}^{m} (-1)^{|\xi_p||A_l|} \sum_q D_{pq}(y - x_m) E(A_1(x_1), \ldots, \xi_q \cdot A_m(x_m), \ldots, A_n(x_n)) \] (2.12)

\[ E(A_1(x_1), \ldots, A_n(x_n), 1) = E(A_1(x_1), \ldots, A_n(x_n)) \] (2.13)

\[ E(1) = 1. \] (2.14)
2.2 Bogoliubov Axioms

Suppose the monomials $A_1, \ldots, A_n \in \text{Alg}(\Xi_r)$ are self-adjoint: $A_j^\dagger = A_j$, $\forall j = 1, \ldots, n$ and of Fermi number $f_i$.

The chronological products

$$T(A_1(x_1), \ldots, A_n(x_n)) \equiv T^{A_1 \ldots A_n}(x_1, \ldots, x_n) \quad n = 1, 2, \ldots$$

are some distribution-valued operators leaving invariant the algebraic Fock space and verifying the following set of axioms:

- **Skew-symmetry** in all arguments:
  $$T(\ldots, A_i(x_i), A_{i+1}(x_{i+1}), \ldots) = (-1)^{f_j f_{j+1}} T(\ldots, A_{i+1}(x_{i+1}), A_i(x_i), \ldots) \quad (2.15)$$

- **Poincaré invariance**: we have a natural action of the Poincaré group in the space of Wick monomials and we impose that for all $g \in \text{inSL}(2, \mathbb{C})$ we have:
  $$U_g T(A_1(x_1), \ldots, A_n(x_n)) U_g^{-1} = T(g \cdot A_1(x_1), \ldots, g \cdot A_n(x_n)) \quad (2.16)$$
  where in the right hand side we have the natural action of the Poincaré group on $\Xi$.

Sometimes it is possible to supplement this axiom by other invariance properties: space and/or time inversion, charge conjugation invariance, global symmetry invariance with respect to some internal symmetry group, supersymmetry, etc.

- **Causality**: if $y \cap (x + V^+) = \emptyset$ then we denote this relation by $x \succeq y$. Suppose that we have $x_i \succeq x_j$, $\forall i \leq k$, $j \geq k + 1$; then we have the factorization property:
  $$T(A_1(x_1), \ldots, A_n(x_n)) = T(A_1(x_1), \ldots, A_k(x_k)) \cdot T(A_{k+1}(x_{k+1}), \ldots, A_n(x_n)) \quad (2.17)$$

- **Unitarity**: We define the *anti-chronological products* using a convenient notation introduced by Epstein-Glaser, adapted to the Grassmann context. If $X = \{j_1, \ldots, j_s\} \subset N \equiv \{1, \ldots, n\}$ is an ordered subset, we define
  $$T(X) \equiv T(A_{j_1}(x_{j_1}), \ldots, A_{j_s}(x_{j_s})). \quad (2.18)$$

Let us consider some Grassmann variables $\theta_j$, of parity $f_j$, $j = 1, \ldots, n$ and let us define

$$\theta_X \equiv \theta_{j_1} \cdots \theta_{j_s}. \quad (2.19)$$

Now let $(X_1, \ldots, X_r)$ be a partition of $N = \{1, \ldots, n\}$ where $X_1, \ldots, X_r$ are ordered sets. Then we define the (Koszul) sign $\epsilon(X_1, \ldots, X_r)$ through the relation

$$\theta_1 \cdots \theta_n = \epsilon(X_1, \ldots, X_r) \theta_{X_1} \cdots \theta_{X_r} \quad (2.20)$$

and the antichronological products are defined according to

$$(-1)^n T(N) \equiv \sum_{r=1}^n (-1)^r \sum_{I_1, \ldots, I_r \in \text{Part}(N)} \epsilon(X_1, \ldots, X_r) T(X_1) \cdots T(X_r) \quad (2.21)$$

Then the unitarity axiom is:

$$T(N) = T(N)^\dagger. \quad (2.22)$$
• The “initial condition”:

\[ T(A(x)) = N(A(x)). \] (2.23)

• Power counting: We can also include in the induction hypothesis a limitation on the order of singularity of the vacuum averages of the chronological products associated to arbitrary Wick monomials \( A_1, \ldots, A_n \); explicitly:

\[ \omega(< \Omega, T^{A_1, \ldots, A_n}(X) \Omega >) \leq \sum_{l=1}^{n} \omega(A_l) - 4(n - 1) \] (2.24)

where by \( \omega(d) \) we mean the order of singularity of the (numerical) distribution \( d \) and by \( \omega(A) \) we mean the canonical dimension of the Wick monomial \( W \).

• Wick expansion property: In analogy to (2.12) we require

\[
\begin{align*}
&[\xi_p(y), T(A_1(x_1), \ldots, A_n(x_n))] \\
&= \sum_{m=1}^{n} \prod_{l \leq m} (-1)^{|\xi_p||A_l|} \sum_q [\xi_p(y), \xi_q(x_m)] T(A_1(x_1), \ldots, \xi_q \cdot A_m(x_m), \ldots, A_n(x_n)) \\
&= -i \sum_{m=1}^{n} \prod_{l \leq m} (-1)^{|\xi_p||A_l|} \sum_q D_{pq}(y - x_m) T(A_1(x_1), \ldots, \xi_q \cdot A_m(x_m), \ldots, A_n(x_n))
\end{align*}
\] (2.25)

In fact we can impose a sharper form:

\[
\begin{align*}
&[\xi_p^{(\epsilon)}(y), T(A_1(x_1), \ldots, A_n(x_n))] \\
&= -i \sum_{m=1}^{n} \prod_{l \leq m} (-1)^{|\xi_p||A_l|} \sum_q D_{pq}^{(\epsilon)}(y - x_m) T(A_1(x_1), \ldots, \xi_q \cdot A_m(x_m), \ldots, A_n(x_n))
\end{align*}
\] (2.26)

Up to now, we have defined the chronological products only for self-adjoint Wick monomials \( W_1, \ldots, W_n \) but we can extend the definition for Wick polynomials by linearity.

The chronological products \( T(A_1(x_1), \ldots, A_n(x_n)) \) are not uniquely defined by the axioms presented above. They can be modified with quasi-local expressions i.e. expressions localized on the big diagonal \( x_1 = \cdots = x_n \); such expressions are of the type

\[ N(A_1(x_1), \ldots, A_n(x_n)) = P_j(\partial)\delta(X) W_j(X) \] (2.27)

where \( \delta(X) \equiv \delta(x_1 - x_n), \ldots, \delta(x_{n-1} - x_n) \), the expressions \( P_j(\partial) \) are polynomials in the partial derivatives and \( W_j(X) \) are Wick polynomials. There are some restrictions on these quasi-local expressions such that the Bogoliubov axioms remain true. One such consistency relations comes from Wick expansion property.
2.3 Quantum Gravity in the Linear Approximation

If we consider pure massless gravity then the jet variables are \((H_{\mu\nu}, \Phi, u_\rho, \bar{u}_\sigma)\) where \(H_{\mu\nu}\) and \(\Phi\) are Grassmann even and \(u_\rho, \bar{u}_\sigma\) are Grassmann odd variables. Also \(H_{\mu\nu}\) is symmetric and traceless. The interaction Lagrangian is determined by gauge invariance. Namely we define the gauge charge operator by

\[
d_Q H_{\mu\nu} = -\frac{i}{2} \left( d_\mu u_\nu + d_\nu u_\mu - \frac{1}{2} \eta_{\mu\nu} d_\rho u^\rho \right), \quad d_Q \Phi = \frac{i}{2} d_\rho u^\rho
\]

\[
d_Q u_\rho = 0, \quad d_Q \bar{u}_\sigma = i \left( d^\lambda H_{\sigma\lambda} + \frac{1}{2} d_\sigma \Phi \right)
\]

(2.28)

where \(d^\mu\) is the formal derivative. The gauge charge operator squares to zero:

\[
d_Q^2 \simeq 0 \quad \text{(2.29)}
\]

where by \(\simeq\) we mean, modulo the equation of motion.

Then we define the associated Fock space by the non-zero 2-point distributions are

\[
\langle \Omega, H_{\mu\nu}(x_1) H_{\rho\sigma}(x_2) \Omega \rangle = -\frac{i}{2} \left( \eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\rho} - \frac{1}{2} \eta_{\mu\nu} \eta_{\rho\sigma} \right) D_0^+(x_1 - x_2),
\]

\[
\langle \Omega, \Phi(x_1) \Phi(x_2) \Omega \rangle = i D_0^+(x_1 - x_2),
\]

\[
\langle \Omega, u_\rho(x_1) \bar{u}_\sigma(x_2) \Omega \rangle = i \eta_{\rho\sigma} D_0^+(x_1 - x_2),
\]

\[
\langle \Omega, \bar{u}_\sigma(x_1) u_\rho(x_2) \Omega \rangle = -i \eta_{\rho\sigma} D_0^+(x_1 - x_2).
\]

(2.30)

Here we are using the Pauli-Jordan distribution

\[
D_m(x) = D_m^{(+)}(x) + D_m^{(-)}(x)
\]

(2.31)

where

\[
D_m^{(\pm)}(x) = \pm \frac{i}{(2\pi)^3} \int d^4 p e^{-ip \cdot x} \theta(\pm p_0) \delta(p^2 - m^2)
\]

(2.32)

and

\[
D^{(-)}(x) = -D^{(+)}(-x).
\]

(2.33)

We stress that in (2.28) we are dealing with the classical jet variables, but in the previous relation we have the associated quantum variables; we have omitted the super-script “quant” for simplicity.

The definitions above are describing a system of massless particles of helicity 2 as it is proved in [7].

If we define the derivative on the jet variables by

\[
H_{\mu\nu} \cdot H_{\rho\sigma} \equiv \frac{1}{2} \left( \eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\rho} - \frac{1}{2} \eta_{\mu\nu} \eta_{\rho\sigma} \right)
\]

(2.34)

then we have for any monomial \(A\)

\[
[H_{\mu\nu}(x), A(y)] = [H_{\mu\nu}(x), H_{\rho\sigma}(y)] H^{\rho\sigma} \cdot A(y)
\]

(2.35)
so we do not have to worry about overcounting. It is more convenient to define

$$ h_{\mu\nu} \equiv H_{\mu\nu} + \frac{1}{2} \eta_{\mu\nu} \Phi $$

(2.36)

and then we have

$$ < \Omega, h_{\mu\nu}(x_1)h_{\rho\sigma}(x_2)\Omega > = -\frac{i}{2} \left( \eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\rho} - \eta_{\mu\nu} \eta_{\rho\sigma} \right) D_0^+(x_1 - x_2) $$

(2.37)

$$ d_Q h_{\mu\nu} = -\frac{i}{2} \left( d_{\mu} u_{\nu} + d_{\nu} u_{\mu} - \eta_{\mu\nu} d_{\rho} u^{\rho} \right), $$

$$ d_Q u_{\mu} = 0, \quad d_Q \tilde{u}_{\sigma} = i d^\lambda h_{\sigma\lambda} $$

(2.38)

and

$$ h_{\mu\nu} \cdot h_{\rho\sigma} \equiv \frac{1}{2} \left( \eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\rho} \right) $$

(2.39)

In [18], [12] it is proved that the relation

$$ d_Q T \sim \text{total divergence} $$

(2.40)

fixes uniquely (up to a coboundary) the expression $T$ if we require: (a) Lorentz covariance; (b) $T$ should be tri-linear in the jet variables; (c) $\omega(T) \leq 5$; (there are no solutions with $\omega(T) \leq 4$).

We can choose, up to a coboundary

$$ T = \sum_{j=1}^{9} T_j $$

(2.41)

where

$$ T_1 \equiv -h_{\alpha\beta} d^\alpha h d^\beta h $$

$$ T_2 \equiv 2 h^{\alpha\beta} d_{\alpha} h_{\mu\nu} d_{\beta} h^{\mu\nu} $$

$$ T_3 \equiv 4 h_{\alpha\beta} d_{\nu} h^{\beta\mu} d_{\mu} h^{\alpha\nu} $$

$$ T_4 \equiv 2 h_{\alpha\beta} d_{\mu} h^{\alpha\beta} d^\mu h $$

$$ T_5 \equiv -4 h_{\alpha\beta} d_{\nu} h^{\alpha\mu} d^\nu h^{\beta\mu} $$

$$ T_6 \equiv -4 u^\mu d_{\beta} \tilde{u}_{\nu} d_{\mu} h^{\nu\beta} $$

$$ T_7 \equiv 4 d_{\nu} u^{\beta} d_{\mu} \tilde{u}_{\beta} h^{\mu\nu} $$

$$ T_8 \equiv -4 d^\nu u_{\nu} d_{\alpha} \tilde{u}_{\beta} h^{\alpha\beta} $$

$$ T_9 \equiv 4 d_{\nu} u^{\mu} d_{\mu} \tilde{u}_{\beta} h^{\nu\beta} $$

(2.42)

Then the expression gives a Wick polynomial $T_{\text{quant}}$ formally the same, but: (a) the jet variables must be replaced by the associated quantum fields; (b) the formal derivative $d^\mu$ goes in the true derivative in the coordinate space; (c) Wick ordering should be done to obtain well-defined operators. We also have an associated gauge charge operator in the Fock space given by the graded commutator

$$ d_Q A \equiv [Q, A]. $$

(2.43)
One can be proved that $Q^2 = 0$ and

$$[Q, T^\text{quant}(x)] = \text{total divergence} \quad (2.44)$$

where the equations of motion are automatically used because the quantum fields are on-shell.

From now on we abandon the super-script \textit{quant} because it will be obvious from the context if we refer to the classical expression or to its quantum counterpart.

We conclude our presentation with a generalization of (2.44). In fact, it can be proved that (2.44) implies the existence of Wick polynomials $T^\alpha, T^{\alpha\beta}, T^{\alpha\beta\gamma}$ such that we have:

$$[Q, T^I] = i\partial_\mu T^{I\mu} \quad (2.45)$$

for any multi-index $I$ with the convention $T^\emptyset \equiv T$. Explicitly:

$$T^\alpha = \sum_{j=1}^{19} T_j^\alpha \quad (2.46)$$

\[
\begin{align*}
T_1^\alpha &\equiv 4 \, u^\mu \, d_\mu h_\beta \, d^\beta h^{\alpha\nu} \\
T_2^\alpha &\equiv -2 \, u^\mu \, d_\mu h^{\beta\nu} \, d^\alpha h_\beta \\
T_3^\alpha &\equiv -2 \, u^\alpha \, d^\beta h^{\mu\nu} \, d_\mu h_\beta \\
T_4^\alpha &\equiv -4 \, d_\nu u_\beta \, d_\mu h^{\alpha\beta} \, h^{\mu\nu} \\
T_5^\alpha &\equiv 4 \, d_\nu u_\nu \, d^\mu h^{\alpha\beta} \, h_\beta \\
T_6^\alpha &\equiv u^\alpha \, d_\beta h^{\mu\nu} \, d^\beta h^{\mu\nu} \\
T_7^\alpha &\equiv -2 \, d_\nu u_\nu \, h_{\mu\beta} \, d^\alpha h^{\mu\beta} \\
T_8^\alpha &\equiv -\frac{1}{2} \, u^\alpha \, d_\mu h \, d^\mu h \\
T_9^\alpha &\equiv d^\nu u_\nu \, h \, d^\alpha h \\
T_{10}^\alpha &\equiv u^\nu \, d^\alpha h \, d_\nu h \\
T_{11}^\alpha &\equiv -2 \, d_\nu u_\mu \, h^{\mu\nu} \, d^\alpha h \\
T_{12}^\alpha &\equiv 4 \, d^\nu u_\mu \, d^\alpha h^{\mu\beta} \, h_\beta \\
T_{13}^\alpha &\equiv -4 \, d^\nu u^\mu \, d_\mu h^{\alpha\beta} \, h_\beta \\
T_{14}^\alpha &\equiv -2 \, u^\mu \, d_\mu u_\nu \, d^\gamma \tilde{u}^\nu \\
T_{15}^\alpha &\equiv 2 \, u^\mu \, d^\nu u^\alpha \, d_\mu \tilde{u}^\nu \\
T_{16}^\alpha &\equiv -2 \, u^\alpha \, d_\nu u_\mu \, d^\mu \tilde{u}^\nu \\
T_{17}^\alpha &\equiv 2 \, d^\nu u_\nu \, d^\mu u^\alpha \, \tilde{u}_\mu \\
T_{18}^\alpha &\equiv 2 \, u^\mu \, d_\mu d_\nu u^\nu \, \tilde{u}^\alpha \\
T_{19}^\alpha &\equiv -2 \, u^\alpha \, d_\mu d_\nu u^\nu \, \tilde{u}^\alpha \\
\end{align*}
\]

\[T^{\alpha\beta} = \sum_{j=1}^{5} T_j^{\alpha\beta} \quad (2.47)\]
\[ T_{\alpha\beta}^1 \equiv 2 \, u^\mu \, d_\mu u_\nu \, d^\beta h^{\alpha\nu} - (\alpha \leftrightarrow \beta) \]
\[ T_{\alpha\beta}^2 \equiv 2 \, u^\mu \, d_\nu u^\alpha \, d_\mu h^{\beta\nu} - (\alpha \leftrightarrow \beta) \]
\[ T_{\alpha\beta}^3 \equiv -2 \, u^\alpha \, d_\nu u_\mu \, d^\mu h^{\beta\nu} - (\alpha \leftrightarrow \beta) \]
\[ T_{\alpha\beta}^4 \equiv 4 \, d^\mu u^\alpha \, d^\nu u^\beta \, h_{\mu\nu} \]
\[ T_{\alpha\beta}^5 \equiv 2 \, d_\nu u^\nu \, d_\mu u_\alpha h^{\mu\beta} - (\alpha \leftrightarrow \beta) \]
\[ T_{\alpha\beta} = \sum_{j=1}^{6} T_{\alpha\beta}^j \]

Finally we give the relation expressing gauge invariance in order \( n \) of the perturbation theory. We define the operator \( \delta \) on chronological products by:
\[ \delta T(T^{I_1}(x_1), \ldots, T^{I_n}(x_n)) \equiv \sum_{m=1}^{n} (-1)^{s_m} \partial^m T(T^{I_1}(x_1), \ldots, T^{I_m}(x_m), \ldots, T^{I_n}(x_n)) \]
with
\[ s_m = \sum_{p=1}^{m-1} |I_p|, \]
then we define the operator
\[ s \equiv d_Q - i \delta \]
Gauge invariance in an arbitrary order is then expressed by
\[ sT(T^{I_1}(x_1), \ldots, T^{I_n}(x_n)) = 0. \]
This concludes the necessary prerequisites.
We notice that in \((2.38)\) and in \((2.45)\) we have a pattern of the type:

\[
d_Q A = \text{total divergence.} \quad (2.56)
\]

This pattern remains essentially true for Wick submonomials if we use the definition \((2.11)\). We consider the expressions \((2.46), (2.48), (2.50)\) and define:

\[
B_{\mu \nu} \equiv \tilde{u}_{\mu, \nu} \cdot T = 4 (u^\rho d_\rho h_{\mu \nu} - d_\rho u_{\rho \mu} h_{\nu \rho} - d^\rho u_{\nu} h_{\rho \mu} - d^\rho u_{\rho} h_{\mu \nu})
\]

\[
C_{\mu \nu} \equiv h_{\mu \nu} \cdot T = -d_\mu h d_\nu h + 2d_\mu h_{\rho \sigma} d_\nu h^{\rho \sigma} + 4d^\rho h_{\mu \rho} d_\nu h_{\nu \sigma} + 2d_\nu h_{\mu \rho} d_\rho h_{\nu \sigma} + 2(d_\mu u^\rho d_\nu \tilde{u}_\rho + d_\nu u^\rho d_\mu \tilde{u}_\rho) - 2d^\nu u_{\rho} (d_\nu \tilde{u}_\mu + d_\mu \tilde{u}_\nu) + 2(d_\mu u^\rho d_\nu \tilde{u}_\rho + d_\nu u^\rho d_\mu \tilde{u}_\rho)
\]

\[
D_\mu \equiv u_\mu \cdot T = -4d^\sigma \tilde{u}^\beta d_\mu h_{\alpha \beta}
\]

\[
E_{\mu, \rho} \equiv h_{\mu \nu} \cdot T = 4(h_{\rho \sigma} d^\sigma h_{\mu \nu} - h^\rho \sigma d_\rho h_{\nu \sigma} - h^\nu \sigma d_\nu h_{\mu \rho}) + 2d_\mu \rho d \nu h + 4(h_\mu \nu d_\nu h_{\rho \sigma} + h^\nu \sigma d_\mu h_{\rho \sigma}) - 2u_\mu (d_\nu \tilde{u}_\nu + d_\nu \tilde{u}_\mu) + 2\eta_{\mu \nu} (-h_{\rho \sigma} d^\sigma h + h_{\alpha \beta} d_\rho h_{\alpha \beta})
\]

\[
F_{\mu, \nu} \equiv u_{\mu, \nu} \cdot T = 4 [(d^\rho \tilde{u}_\mu + d_\mu \tilde{u}^\rho) h_{\nu \rho} - \eta_{\mu \nu} d_\sigma \tilde{u}_\rho h_{\sigma \rho}]
\]

\[
G_\mu \equiv \tilde{u}_\mu \cdot T = 0
\]

\[
H_{\mu, \alpha \beta} \equiv u_{\mu, \alpha \beta} \cdot T = 0 \quad (2.57)
\]

We also have:

\[
B_{\mu, \nu, \rho} \equiv \tilde{u}_{\mu, \nu} \cdot T = 2(-u_{\nu} d_\mu u_{\rho} + u_{\rho} d_\mu u_{\nu}) + 2\eta_{\mu \nu} u^\sigma d_\sigma u_{\mu}
\]

\[
C_{\mu, \nu, \rho} \equiv h_{\mu \nu} \cdot T = 2(-d_\mu u^\rho d_\rho h_{\nu \sigma} - d_\nu u^\sigma d_\mu h_{\rho \sigma} + d_\mu u^\sigma d_\rho h_{\nu \sigma} + d_\nu u^\sigma d_\rho h_{\mu \sigma} - d_\mu u^\sigma d_\nu h_{\sigma} - d_\nu u^\sigma d_\mu h_{\sigma} - (d_\mu u_{\nu} + d_\nu u_{\rho}) d_{\rho} h)
\]

\[
D_{\mu, \rho} \equiv u_\mu \cdot T = -4d_\mu h^\alpha \beta d_\sigma h_{\alpha \beta} + 2d_\mu h^\alpha \beta d_\rho h_{\alpha \beta} - d_\mu h d_\rho h + 2(d_\mu u_\sigma d_\nu \tilde{u}_{\sigma} - d_\nu u_\sigma d_\mu \tilde{u}_{\sigma} - d_\mu u_\nu d_\rho \tilde{u}_{\nu}) + \eta_{\mu \rho} \left(2d^\alpha d^\beta h_{\sigma \alpha} - d_\nu h_{\rho \sigma} d^\alpha d^\beta h_{\sigma \alpha} + \frac{1}{2} d_\nu h d^\alpha d^\beta h + 2d_\mu d_\alpha \sigma d^\beta \tilde{u}_{\sigma} + 2d_\nu d_\sigma \alpha d^\beta \tilde{u}_{\sigma} \right)
\]

\[
E_{\mu, \nu, \lambda, \rho} \equiv h_{\mu, \nu, \lambda} \cdot T = 2u_{\lambda} (d_{\lambda} h_{\rho \mu} + d_{\mu} h_{\rho \nu} - d_{\nu} h_{\rho \mu}) - 2u_{\rho} (d_{\nu} h_{\lambda \mu} + d_{\mu} h_{\lambda \nu} - d_{\lambda} h_{\mu \nu}) + 2\eta_{\mu \rho} (u_\lambda d_\sigma h_{\rho \nu} - d_\nu h_{\rho \sigma} - d_\sigma h_{\rho \nu} - d_\nu h_{\rho \sigma}) + (\mu \leftrightarrow \nu)
\]

\[
F_{\mu, \nu, \rho} \equiv u_{\mu, \nu} \cdot T = 4 (d^\rho h_{\mu \nu} h_{\rho \sigma} - d_\rho h_{\mu \nu} h_{\nu \sigma} + d_\nu h_{\rho \sigma} h_{\mu \sigma}) + 2h_{\mu \nu} d_\rho h - 2(u_\nu d_\mu \tilde{u}_{\rho} + u_{\rho} d_\mu \tilde{u}_{\nu}) + \eta_{\mu \nu} (-4d_\beta h_{\rho \alpha} h_{\alpha \beta} + 2d_\rho h_{\alpha \beta} h_{\alpha \beta} - h_{\rho \beta} h_{\alpha \beta} - 2d_\alpha h_{\beta \alpha})
\]

\[
G_{\mu, \rho} \equiv \tilde{u}_\mu \cdot T = 2(-d_\nu u_\nu d_\rho u_\rho + u_\rho d_\mu d_\nu u_\nu - \eta_{\mu \rho} u_\sigma d_\alpha d_\beta u^\beta) + 2\eta_{\mu \rho} (u_\nu d_\alpha \tilde{u}_{\sigma} + d_\alpha u_\nu \tilde{u}_{\sigma})
\]

\[
H_{\mu, \alpha \beta, \rho} \equiv u_{\mu, \alpha \beta} \cdot T = \eta_{\mu \beta} (u_{\alpha} \tilde{u}_{\rho} - u_{\rho} \tilde{u}_{\alpha}) + (\alpha \leftrightarrow \beta) \quad (2.58)
\]
Then we try to extend the structure (2.56) to the Wick submonomials defined above. We have the formal derivative

\[ \delta A \equiv \partial_{\mu} A^{\mu} \]  

(2.61) and used in the definition of gauge invariance (2.52) + (2.54); we also define the derivative \( \delta' \) by

\[ \delta' B_{\mu, \nu} = G_{\mu, \nu} \]

\[ \delta' E_{\mu \nu, \lambda} = C_{\mu \nu, \lambda} \]

\[ \delta' F_{\mu, \nu} = D_{\mu, \nu} - C_{\mu, \nu} + \frac{1}{2} \eta_{\mu \nu} C, \quad C \equiv \eta^{\rho \sigma} C_{\rho \sigma} \]

\[ \delta' H_{\mu, \alpha \beta} = \frac{1}{2} \left( F_{\mu, \alpha \beta} - E_{\mu \alpha, \beta} + \frac{1}{2} \eta_{\mu \alpha} E_{\beta} \right) + (\alpha \leftrightarrow \beta), \quad E_{\beta} \equiv \eta^{\rho \sigma} E_{\rho \sigma, \beta} \]

\[ \delta' E_{\mu \nu, \lambda, \rho} = -C_{\mu \nu, \lambda, \rho} + \frac{1}{2} \left( \eta_{\mu \lambda} G_{\nu, \rho} + \eta_{\nu \lambda} G_{\mu, \rho} \right) \]

\[ \delta' F_{\mu, \nu, \rho} = -D_{\mu, \nu, \rho} + C_{\mu, \nu, \rho} - \frac{1}{2} \eta_{\mu \nu} C_{\rho}, \quad C_{\rho} \equiv \eta^{\alpha \beta} C_{\alpha \beta, \rho} \]

\[ \delta' H_{\mu, \alpha \beta, \rho} = \frac{1}{2} \left( -F_{\mu, \alpha \beta, \rho} + E_{\mu \alpha, \beta, \rho} - \frac{1}{2} \eta_{\mu \alpha} E_{\beta, \rho} \right) + (\alpha \leftrightarrow \beta), \quad E_{\beta, \rho} \equiv \eta^{\rho \sigma} E_{\rho \sigma, \beta, \rho} \]

\[ \delta' F_{\mu, \nu, \alpha \beta} = D_{\mu, \alpha \beta, \nu} - C_{\mu \nu, \alpha \beta} + \frac{1}{2} \eta_{\mu \nu} \tilde{C}_{\alpha \beta}, \quad \tilde{C}_{\alpha \beta} \equiv \eta^{\rho \sigma} C_{\rho \sigma, \alpha \beta} \]  

(2.62)
and 0 for the other Wick submonomials (2.57). Finally

\[ s \equiv d_Q - i\delta, \quad s' \equiv s - i\delta' = d_Q - i(\delta + \delta'). \]  

(2.63)

Then we have the structure

\[ s'A = 0 \]  

(2.64)

for all expressions \( A = T^I, B_{a\mu}, C_{a\mu}, \text{ etc.} \) and also for the basic jet variables \( v_{a\mu}, u_a, \bar{u}_a. \)
3 Tree Contributions

The Hopf structure of pure gravity is similar to the the Hopf structure of the pure Yang-Mills model given in [11].

We implement the Wick theorem for the pure massless gravity model. First we use the more precise form of Wick theorem for the expression and compute the expressions

\[ T(T^I(x_1)^{(k)}, A_2, \ldots, A_n). \]

Because loop contributions are trivial, only the case \( k = 2 \) i.e. when we “pull out” two factors from the first entry \( T^I \) is relevant. We have:

\[
T(T(x_1)^{(2)}, A_2(x_2), \ldots, A_n(x_n)) = \\
: C^\mu\nu(x_1) \ T(h^\mu\nu(x_1)^{(0)}, A_2(x_2), \ldots, A_n(x_n)) : \\
+ : E^\mu\nu,\lambda(x_1) \ T(h^\mu\nu,\lambda(x_1)^{(0)}, A_2(x_2), \ldots, A_n(x_n)) : \\
- : D^\mu(x_1) \ T(u^\mu(x_1)^{(0)}, A_2(x_2), \ldots, A_n(x_n)) : \\
- : F^\mu,\nu(x_1) \ T(u^\mu,\nu(x_1)^{(0)}, A_2(x_2), \ldots, A_n(x_n)) : \\
- : B^\mu,\nu(x_1) \ T(\tilde{u}^\mu,\nu(x_1)^{(0)}, A_2(x_2), \ldots, A_n(x_n)) : \\
\] (3.1)

\[
T(T^\rho(x_1)^{(2)}, A_2(x_2), \ldots, A_n(x_n)) = \\
: C^\mu,\nu,\rho(x_1) \ T(h^\mu,\nu,\rho(x_1)^{(0)}, A_2(x_2), \ldots, A_n(x_n)) : \\
+ : E^\mu,\nu,\lambda,\rho(x_1) \ T(h^\mu,\nu,\lambda,\rho(x_1)^{(0)}, A_2(x_2), \ldots, A_n(x_n)) : \\
- : D^\mu,\rho(x_1) \ T(u^\mu,\rho(x_1)^{(0)}, A_2(x_2), \ldots, A_n(x_n)) : \\
- : F^\mu,\nu,\rho(x_1) \ T(u^\mu,\nu,\rho(x_1)^{(0)}, A_2(x_2), \ldots, A_n(x_n)) : \\
- : H^\mu,\nu,\alpha,\beta(x_1) \ T(u^\mu,\nu,\alpha,\beta(x_1)^{(0)}, A_2(x_2), \ldots, A_n(x_n)) : \\
- : G^\mu,\nu,\rho(x_1) \ T(\tilde{u}^\mu,\nu,\rho(x_1)^{(0)}, A_2(x_2), \ldots, A_n(x_n)) : \\
- : B^\mu,\nu,\rho(x_1) \ T(\tilde{\tilde{u}}^\mu,\nu,\rho(x_1)^{(0)}, A_2(x_2), \ldots, A_n(x_n)) : \\
\] (3.2)

\[
T(T^{\rho\sigma}(x_1)^{(2)}, A_2(x_2), \ldots, A_n(x_n)) = \\
: C^\mu,\nu,\rho,\sigma(x_1) \ T(h^\mu,\nu,\rho,\sigma(x_1)^{(0)}, A_2(x_2), \ldots, A_n(x_n)) : \\
+ E^\mu,\nu,\lambda,\rho,\sigma(x_1) \ T(h^\mu,\nu,\lambda,\rho,\sigma(x_1)^{(0)}, A_2(x_2), \ldots, A_n(x_n)) : \\
- : D^\mu,\rho,\sigma(x_1) \ T(u^\mu,\rho,\sigma(x_1)^{(0)}, A_2(x_2), \ldots, A_n(x_n)) : \\
- : F^\mu,\nu,\rho,\sigma(x_1) \ T(u^\mu,\nu,\rho,\sigma(x_1)^{(0)}, A_2(x_2), \ldots, A_n(x_n)) : \\
\] (3.3)

and

\[
T(T^{\rho\sigma\tau}(x_1)^{(2)}, A_2(x_2), \ldots, A_n(x_n)) = \\
: D^\mu,\rho,\sigma,\tau(x_1) \ T(u^\mu(x_1)^{(0)}, A_2(x_2), \ldots, A_n(x_n)) : \\
+ F^\mu,\nu,\rho,\sigma,\tau(x_1) \ T(u^\mu,\nu,\rho,\sigma,\tau(x_1)^{(0)}, A_2(x_2), \ldots, A_n(x_n)) : \\
\] (3.4)
Here the expressions $T(A_1(x_1), A_2(x_2), \ldots, A_n(x_n))$ are of Wick type only in $A_2, \ldots, A_n$. There are various signs — because we have a true Grassmann structure: the variables $u_\mu, \tilde{u}_\nu$ are of ghost number 1 (resp. $-1$); the Wick submonomials $B_{\mu\nu}, C_{\mu\rho}, E_{\mu\nu, \lambda\rho}, D_{\mu, \alpha\beta}, F_{\mu, \nu, \alpha\beta}$ are of ghost number 1 and $D_{\mu}, F_{\mu, \nu}$ are of ghost number $-1$.

Now we study tree contributions. If we iterate the preceding formulas we get:

$$T(T(x_1) , T(x_2)) =: C^{\mu_1\nu_1}(x_1) C^{\mu_2\nu_2}(x_2) : T(h_{\mu_1\nu_1}(x_1), h_{\mu_2\nu_2}(x_2)) + \cdots$$

$$T(T^\rho(x_1) , T(x_2)) =: C^{\mu_1\nu_1, \rho}(x_1) C^{\mu_2\nu_2}(x_2) : T(h_{\mu_1\nu_1}(x_1), h_{\mu_2\nu_2}(x_2)) + \cdots$$

$$T(T^\rho(x_1) , T^\sigma(x_2)) =: C^{\mu_1\nu_1, \rho}(x_1) C^{\mu_2\nu_2, \sigma}(x_2) : T(h_{\mu_1\nu_1}(x_1), h_{\mu_2\nu_2}(x_2)) + \cdots$$
\[ T(T^{\rho\sigma}(x_1)^{(2)}, T(x_2)^{(2)}) = C^{\mu_1,\nu_1,\rho\sigma}(x_1) C^{\mu_2,\nu_2}(x_2) : T(h_{\mu_1\nu_1}(x_1)^{(0)}, h_{\mu_2\nu_2}(x_2)^{(0)}) + : C^{\mu_1,\nu_1,\rho\sigma}(x_1) E^{\mu_2,\nu_2,\lambda}(x_2) : T(h_{\mu_1\nu_1}(x_1)^{(0)}, h_{\mu_2\nu_2,\lambda}(x_2)^{(0)}) \\
+ : E^{\mu_1,\nu_1,\lambda,\rho\sigma}(x_1) C^{\mu_2,\nu_2}(x_2) : T(h_{\mu_1\nu_1,\lambda}(x_1)^{(0)}, h_{\mu_2\nu_2}(x_2)^{(0)}) + : E^{\mu_1,\nu_1,\lambda,\rho\sigma}(x_1) E^{\mu_2,\nu_2,\lambda_2}(x_2) : T(h_{\mu_1\nu_1,\lambda}(x_1)^{(0)}, h_{\mu_2\nu_2,\lambda_2}(x_2)^{(0)}) \\
- : D^{\mu_1,\rho\sigma}(x_1) B^{\mu_2,\nu}(x_2) : T(u_{\mu_1}(x_1)^{(0)}, \bar{u}_{\mu_2,\nu}(x_2)^{(0)}) - : F^{\mu_1,\nu_1,\rho\sigma}(x_1) B^{\mu_2,\nu_2}(x_2) : T(u_{\mu_1,\nu_1}(x_1)^{(0)}, \bar{u}_{\mu_2,\nu_2}(x_2)^{(0)}) \] (3.8)

\[ T(T^{\rho\sigma}(x_1)^{(2)}, T^{\tau}(x_2)^{(2)}) = C^{\mu_1,\nu_1,\rho\sigma}(x_1) C^{\mu_2,\nu_2,\tau}(x_2) : T(h_{\mu_1\nu_1}(x_1)^{(0)}, h_{\mu_2\nu_2}(x_2)^{(0)}) + : C^{\mu_1,\nu_1,\rho\sigma}(x_1) E^{\mu_2,\nu_2,\lambda,\tau}(x_2) : T(h_{\mu_1\nu_1}(x_1)^{(0)}, h_{\mu_2\nu_2,\lambda,\tau}(x_2)^{(0)}) \\
+ : E^{\mu_1,\nu_1,\lambda,\rho\sigma}(x_1) C^{\mu_2,\nu_2,\tau}(x_2) : T(h_{\mu_1\nu_1,\lambda}(x_1)^{(0)}, h_{\mu_2\nu_2}(x_2)^{(0)}) + : E^{\mu_1,\nu_1,\lambda_1,\rho\sigma}(x_1) E^{\mu_2,\nu_2,\lambda_2,\tau}(x_2) : T(h_{\mu_1\nu_1,\lambda_1}(x_1)^{(0)}, h_{\mu_2\nu_2,\lambda_2,\tau}(x_2)^{(0)}) \\
+ : D^{\mu_1,\rho\sigma}(x_1) C^{\mu_2,\nu,\tau}(x_2) : T(u_{\mu_1}(x_1)^{(0)}, \bar{u}_{\mu_2,\nu,\tau}(x_2)^{(0)}) + : F^{\mu_1,\nu_1,\rho\sigma}(x_1) C^{\mu_2,\nu_2,\tau}(x_2) : T(u_{\mu_1,\nu_1}(x_1)^{(0)}, \bar{u}_{\mu_2,\nu_2}(x_2)^{(0)}) \\
+ : F^{\mu_1,\nu_1,\rho\sigma}(x_1) B^{\mu_2,\nu_2,\tau}(x_2) : T(u_{\mu_1,\nu_1}(x_1)^{(0)}, \bar{u}_{\mu_2,\nu_2}(x_2)^{(0)}) \] (3.9)

\[ T(T^{\rho\sigma\tau}(x_1)^{(2)}, T(x_2)^{(2)}) = - : D^{\mu_1,\rho\sigma\tau}(x_1) B^{\mu_2,\nu}(x_2) : T(u_{\mu_1}(x_1)^{(0)}, \bar{u}_{\mu_2,\nu}(x_2)^{(0)}) \\
- : F^{\mu_1,\nu_1,\rho\sigma\tau}(x_1) B^{\mu_2,\nu_2}(x_2) : T(u_{\mu_1,\nu_1}(x_1)^{(0)}, \bar{u}_{\mu_2,\nu_2}(x_2)^{(0)}) \] (3.10)

\[ T(T^{\rho_1\sigma_1}(x_1)^{(2)}, T^{\rho_2\sigma_2}(x_2)^{(2)}) = C^{\mu_1,\nu_1,\rho_1,\sigma_1}(x_1) C^{\mu_2,\nu_2,\rho_2,\sigma_2}(x_2) : T(h_{\mu_1\nu_1}(x_1)^{(0)}, h_{\mu_2\nu_2}(x_2)^{(0)}) \\
+ : E^{\mu_1,\nu_1,\lambda_1,\rho_1,\sigma_1}(x_1) E^{\mu_2,\nu_2,\lambda_2,\rho_2,\sigma_2}(x_2) : T(h_{\mu_1\nu_1,\lambda_1}(x_1)^{(0)}, h_{\mu_2\nu_2,\lambda_2}(x_2)^{(0)}) \\
+ : C^{\mu_1,\nu_1,\rho_1,\sigma_1}(x_1) E^{\mu_2,\nu_2,\rho_2,\sigma_2}(x_2) : T(h_{\mu_1\nu_1}(x_1)^{(0)}, h_{\mu_2\nu_2,\lambda}(x_2)^{(0)}) \\
+ : F^{\mu_1,\nu_1,\rho_1,\sigma_1}(x_1) C^{\mu_2,\nu_2}(x_2) : T(u_{\mu_1}(x_1)^{(0)}, \bar{u}_{\mu_2,\nu}(x_2)^{(0)}) \\
+ : F^{\mu_1,\nu_1,\rho_1,\sigma_1}(x_1) B^{\mu_2,\nu_2}(x_2) : T(u_{\mu_1,\nu_1}(x_1)^{(0)}, \bar{u}_{\mu_2,\nu_2}(x_2)^{(0)}) \] (3.11)

\[ T(T^{\rho\sigma\tau}(x_1)^{(2)}, T^{\nu}(x_2)^{(2)}) = : D^{\mu_1,\rho\sigma\tau}(x_1) G^{\mu_2,\nu}(x_2) : T(u_{\mu_1}(x_1)^{(0)}, \bar{u}_{\mu_2,\nu}(x_2)^{(0)}) \\
+ : D^{\mu_1,\rho\sigma\tau}(x_1) B^{\mu_2,\lambda,\nu}(x_2) : T(u_{\mu_1}(x_1)^{(0)}, \bar{u}_{\mu_2,\lambda,\nu}(x_2)^{(0)}) \\
+ : F^{\mu_1,\lambda,\rho\sigma\tau}(x_1) C^{\mu_2,\nu}(x_2) : T(u_{\mu_1,\lambda}(x_1)^{(0)}, \bar{u}_{\mu_2,\nu}(x_2)^{(0)}) \\
+ : F^{\mu_1,\lambda,\rho\sigma\tau}(x_1) B^{\mu_2,\lambda_2,\nu}(x_2) : T(u_{\mu_1,\lambda_1}(x_1)^{(0)}, \bar{u}_{\mu_2,\lambda_2}(x_2)^{(0)}) \] (3.12)
We must give the values of the chronological products \(T(\xi^0_\rho(x_1), \xi^0_q(x_2))\) which are not unique. For the pure massless gravity model we consider the causal commutators of the basic fields and perform the causal splitting; we obtain:

\[
T(h_{\mu\nu}(x_1^0), h_{\rho\sigma}(x_2^0)) = -\frac{i}{2} \left( \eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\rho} - \frac{1}{2} \eta_{\mu\nu} \eta_{\rho\sigma} \right) D^F_0(x_1 - x_2)
\]

\[
T(u_\mu(x_1^0), \bar{u}_\nu(x_2^0)) = i \eta_{\mu\nu} D^F_0(x_1 - x_2),
\]

\[
T(\bar{u}_\mu(x_1^0), u_\nu(x_2^0)) = -i \eta_{\mu\nu} D_0(x_1 - x_2). \quad (3.13)
\]

and, according to (2.5):

\[
T(\xi_{a,\mu}(x_1^0), \xi_{b,\nu}(x_2^0)) = i \partial_\mu^1 \partial_\nu^2 T(\xi_a(x_1^0), \xi_b(x_2^0)) \quad (3.14)
\]

e tc. This gives:

\[
T(T(x_1)^{(2)}, T(x_2)^{(2)}) = -i D^F_0(x_1 - x_2) \left[ : C^{\mu\nu}(x_1) C_{\mu\nu}(x_2) : - \frac{1}{2} : C(x_1) C(x_2) : \right]
\]

\[
+ i \partial_\lambda D^F_0(x_1 - x_2) \left\{ : C_{\mu\nu}(x_1) E^{\mu\nu,\lambda}(x_2) : - \frac{1}{2} : C(x_1) E^\lambda(x_2) : \right\}
\]

\[
+ i \partial_\lambda \partial_\mu D^F_0(x_1 - x_2) \left[ : E^{\mu,\lambda_1}(x_1) E_{\mu,\lambda_2}(x_2) : - \frac{1}{2} : E^\lambda_1(x_1) E^\lambda_2(x_2) : \right]
\]

\[
+ : F^{\mu,\lambda_1}(x_1) B_{\mu,\lambda_2}(x_2) - B_{\mu,\lambda_1}(x_1) F^{\mu,\lambda_2}(x_2) \right] \quad (3.15)
\]

\[
T(T^\rho(x_1)^{(2)}, T(x_2)^{(2)}) =
\]

\[
- i D^F_0(x_1 - x_2) \left[ : C^{\mu\nu,\rho}(x_1) C_{\mu\nu}(x_2) : - \frac{1}{2} : C^\rho(x_1) C(x_2) : - : G^{\mu,\rho}(x_1) D_\mu(x_2) : \right]
\]

\[
+ i \partial_\lambda D^F_0(x_1 - x_2) \left[ : C^{\mu\nu}(x_1) E_{\mu\nu,\lambda}(x_2) : - \frac{1}{2} : C^\rho(x_1) E^\lambda(x_2) : \right]
\]

\[
+ i \partial_\lambda \partial_\mu D^F_0(x_1 - x_2) \left[ : E^{\mu,\lambda_1}(x_1) E_{\mu,\lambda_2}(x_2) : - \frac{1}{2} : E^\lambda_1(x_1) E^\lambda_2(x_2) : \right]
\]

\[
+ : F^{\mu,\lambda_1}(x_1) B_{\mu,\lambda_2}(x_2) - B_{\mu,\lambda_1}(x_1) F^{\mu,\lambda_2}(x_2) \right] \quad (3.16)
\]
\[ T(T^{\rho}(x_1)^{(2)}, T^{\sigma}(x_2)^{(2)}) = \]
\[-i \ D^{F}_0(x_1 - x_2) \left[ : C^{\mu,\rho}(x_1) C^{\mu,\sigma}(x_2) : - \frac{1}{2} : C^\rho(x_1) C^\sigma(x_2) : \right.\]
\[ - : D^{\mu,\rho}(x_1) G^\rho_\mu(x_2) : + G^\rho_\mu(x_1) D^{\mu,\rho}(x_1) : \]
\[ + i \ \partial_\lambda D^{F}_0(x_1 - x_2) \left[ : C^{\mu,\rho}(x_1) E^{\mu,\lambda,\rho}(x_2) : - \frac{1}{2} : C^\rho(x_1) E^{\lambda,\rho}(x_2) : \right.\]
\[ - : D^{\mu,\rho}(x_1) B^{\mu,\lambda,\rho}(x_2) : + F^{\mu,\lambda,\rho}(x_1) G^\rho_\mu(x_2) : \]
\[- : E^{\mu,\lambda,\rho}(x_1) C^{\mu,\rho}(x_2) : + \frac{1}{2} : E^{\lambda,\rho}(x_1) C^\rho(x_2) : \]
\[- : B^{\mu,\lambda,\rho}(x_1) D^{\mu,\rho}(x_2) : + : G^\rho_\mu(x_2) F^{\mu,\lambda,\rho}(x_1) : \]
\[ + i \ \partial^{\lambda}_1 \partial^{\lambda}_2 D^{F}_0(x_1 - x_2) \left[ : E^{\mu,\lambda_1,\rho}(x_1) E^{\mu,\lambda_2,\sigma}(x_2) : - \frac{1}{2} : E^{\lambda_1,\rho}(x_1) E^{\lambda_2,\sigma}(x_2) : \right.\]
\[ - : F^{\mu,\lambda_1,\rho}(x_1) B^{\mu,\lambda_2,\sigma}(x_2) + B^{\mu,\lambda_1,\rho}(x_1) F^{\mu,\lambda_2,\sigma}(x_2) \]
\[ + : H^{\mu,\lambda_1,\lambda_2,\rho}(x_1) G^\rho_\mu(x_2) : - : G^\rho_\mu(x_1) H^{\mu,\lambda_1,\lambda_2,\rho}(x_1) : \]
\[- i \ \partial^{\lambda}_1 \partial^{\lambda}_2 \partial^{\lambda}_3 D^{F}_0(x_1 - x_2) \left[ : H^{\mu,\lambda_1,\lambda_2,\rho}(x_1) B^{\mu,\lambda_3,\sigma}(x_2) : - : B^{\mu,\lambda_3,\rho}(x_1) H^{\mu,\lambda_1,\lambda_2,\rho}(x_1) : \right] \] (3.17)

\[ T(T^{\rho\sigma}(x_1)^{(2)}, T^{\sigma}(x_2)^{(2)}) = \]
\[-i \ D^{F}_0(x_1 - x_2) \left[ : C^{\mu,\rho\sigma}(x_1) C^{\mu,\rho\sigma}(x_2) : - \frac{1}{2} : C^{\rho\sigma}(x_1) C(x_2) : \right.\]
\[ + i \ \partial_\lambda D^{F}_0(x_1 - x_2) \left[ : C^{\mu,\rho\sigma}(x_1) E^{\mu,\lambda}(x_2) : - \frac{1}{2} : C^{\rho\sigma}(x_1) E^{\lambda}(x_2) : \right.\]
\[ - : E^{\mu,\lambda,\rho\sigma}(x_1) C^{\mu,\rho\sigma}(x_2) : + \frac{1}{2} : E^{\lambda,\rho\sigma}(x_1) C(x_2) : \]
\[ + : D^{\mu,\rho\sigma}(x_1) B^{\mu,\lambda}(x_2) : \]
\[ + i \ \partial^{\lambda}_1 \partial^{\lambda}_2 D^{F}_0(x_1 - x_2) \left[ : E^{\mu,\lambda_1,\rho\sigma}(x_1) E^{\mu,\lambda_2,\rho\sigma}(x_2) : - \frac{1}{2} : E^{\lambda_1,\rho\sigma}(x_1) E^{\lambda_2}(x_2) : \right.\]
\[ + F^{\mu,\lambda_1,\rho\sigma}(x_1) B^{\mu,\lambda_2}(x_2) \] (3.18)
\( T(T^{\mu\sigma}(x_1)^{(2)}, T^\nu(x_2)^{(2)}) = \)

\(-i \, D_0^F(x_1 - x_2) \left[ : C^{\mu\nu,\rho\sigma}(x_1) C_{\mu\nu,\rho\sigma}(x_2) : \right. - \frac{1}{2} : C^{\mu\sigma}(x_1) C^{\nu\tau}(x_2) : \\
\left. - : D^{\mu,\rho\sigma}(x_1) G_{\mu,\rho\sigma}(x_2) : \right] \)

\(+i \, \partial_\lambda D_0^F(x_1 - x_2) \left[ : C^{\mu\nu,\rho\sigma}(x_1) E_{\mu\nu,\lambda,\tau}(x_2) : \right. - \frac{1}{2} : C^{\rho\sigma}(x_1) E^{\lambda,\tau}(x_2) : \\
\left. - : E^{\mu\nu,\lambda,\rho\sigma}(x_1) C_{\mu\nu,\tau}(x_2) : \right. + \frac{1}{2} : E^{\lambda,\rho\sigma}(x_1) C^{\nu,\tau}(x_2) : \\
\left. - : D^{\mu,\rho\sigma}(x_1) B_{\mu,\lambda,\tau}(x_2) : \right. + : F^{\mu,\lambda,\rho\sigma}(x_1) G^{\mu,\tau}(x_2) : \)

\(+i \, \partial_\lambda \partial_\delta D_0^F(x_1 - x_2) \left[ : E^{\mu\nu,\lambda,\rho\sigma}(x_1) E_{\mu\nu,\lambda,\tau}(x_2) : \right. - \frac{1}{2} : E^{\lambda,\rho\sigma}(x_1) E^{\nu,\tau}(x_2) : \\
\left. - : F^{\mu,\lambda,\rho\sigma}(x_1) B_{\mu,\lambda,\tau}(x_2) : \right] \) (3.19)

\( T(T^{\mu\sigma}(x_1)^{(2)}, T^\nu(x_2)^{(2)}) = i \, \partial_\lambda D_0^F(x_1 - x_2) : D^{\mu,\rho\sigma}(x_1) B_{\mu,\lambda}(x_2) : \\
+ i \, \partial_\lambda \partial_\delta D_0^F(x_1 - x_2) : F^{\mu,\lambda,\rho\sigma}(x_1) B_{\mu,\lambda,\nu}(x_2) : \) (3.20)

\( T(T^{\rho_1\sigma_1}(x_1)^{(2)}, T^{\rho_2\sigma_2}(x_2)^{(2)}) = \)

\(-i \, D_0^F(x_1 - x_2) \left[ : C^{\mu\nu,\rho_1\sigma_1}(x_1) C_{\mu\nu,\rho_2\sigma_2}(x_2) : \right. - \frac{1}{2} : C^{\rho_1\sigma_1}(x_1) C^{\rho_2\sigma_2}(x_2) : \\
\left. + i \, \partial_\lambda D_0^F(x_1 - x_2) \left[ : C^{\mu\nu,\rho_1\sigma_1}(x_1) E_{\mu\nu,\lambda,\rho_2\sigma_2}(x_2) : \right. - \frac{1}{2} : C^{\rho_1\sigma_1}(x_1) E^{\lambda,\rho_2\sigma_2}(x_2) : \\
\left. - : E^{\mu\nu,\lambda,\rho_1\sigma_1}(x_1) C_{\mu\nu,\rho_2\sigma_2}(x_2) : \right. + \frac{1}{2} : E^{\lambda,\rho_1\sigma_1}(x_1) C^{\rho_2\sigma_2}(x_2) : \\
\left. + i \, \partial_\lambda \partial_\delta D_0^F(x_1 - x_2) \left[ : E^{\mu\nu,\lambda,\rho_1\sigma_1}(x_1) E_{\mu\nu,\lambda,\rho_2\sigma_2}(x_2) : \right. - \frac{1}{2} : E^{\lambda,\rho_1\sigma_1}(x_1) E^{\nu,\rho_2\sigma_2}(x_2) : \\
\left. - : F^{\mu,\lambda,\rho_1\sigma_1}(x_1) B_{\mu,\lambda,\nu}(x_2) : \right] \) (3.21)

\( T(T^{\mu\sigma}(x_1)^{(2)}, T^\nu(x_2)^{(2)}) = i \, D_0^F(x_1 - x_2) : D^{\mu,\rho\sigma}(x_1) G_{\mu,\nu}(x_2) : \\
+ i \, \partial_\lambda D_0^F(x_1 - x_2) \left[ : D^{\mu,\rho\sigma}(x_1) B_{\mu,\lambda}(x_2) : \right. + : F^{\mu,\lambda,\rho\sigma}(x_1) G^{\mu,\tau}(x_2) : \\
\left. - i \, \partial_\lambda \partial_\delta D_0^F(x_1 - x_2) : F^{\mu,\lambda,\rho\sigma}(x_1) B_{\mu,\lambda,\nu}(x_2) : \right] \) (3.22)

Here

\[
C \equiv \eta^{\mu\nu} C_{\mu\nu}, \quad C_{\rho} \equiv \eta^{\mu\nu} C_{\mu\nu,\rho}, \quad C_{\rho\sigma} \equiv \eta^{\mu\nu} C_{\mu\nu,\rho\sigma}, \\
E_{\lambda} \equiv \eta^{\mu\nu} E_{\mu\nu,\lambda}, \quad E_{\lambda,\rho} \equiv \eta^{\mu\nu} E_{\mu\nu,\lambda,\rho}, \quad E_{\lambda,\rho\sigma} \equiv \eta^{\mu\nu} C_{\mu\nu,\lambda,\rho\sigma}.
\] (3.23)
From these formulas we can determine now if gauge invariance is true; in fact, we have anomalies, as it is well known. The cancelations are more subtle than in the case of pure Yang-Mills case. One has to decompose the expressions from (2.57) - (2.60) according to their tensorial character; we need:

\[
E^{\mu\nu,\rho} = E_1^{\mu\nu,\rho} + \eta^{\mu\nu} E_2^\rho \\
B_1^{\mu,\nu,\rho} = B_1^{\mu,\nu,\rho} + \eta^{\mu\rho} B_1^\nu
\]

\[
E^{\mu\nu,\lambda,\rho} = E_1^{\mu\nu,\lambda,\rho} + \eta^{\mu\rho} E_2^{\lambda\nu} + \eta^{\nu\rho} E_2^{\mu\lambda} + \eta^{\lambda\rho} E_3^{\mu\nu} + \eta^{\mu\nu} E_4^{\lambda\rho}
\]

\[
E^{\lambda,\rho} \equiv \eta_{\mu\nu} E^{\mu\nu,\lambda,\rho} = E_1^{\lambda,\rho} + 2 E_2^{\lambda\nu} + \eta^{\lambda\rho} E_3 + 4 E_4^{\lambda\rho}
\]

\[
H^{\mu,\alpha\beta,\rho} = \eta_{\mu\beta} H^{\alpha\rho} + (\alpha \leftrightarrow \beta)
\]

\[
E^{\mu\nu,\lambda,\alpha\beta} = (\eta^{\mu\beta} \tilde{E}_1^{\nu,\lambda,\alpha} + \eta^{\nu\beta} \tilde{E}_1^{\mu,\lambda,\alpha} + \eta^{\mu\alpha} \eta^{\rho\lambda} \tilde{E}_2^{\nu} + \eta^{\nu\alpha} \eta^{\rho\lambda} \tilde{E}_2^{\mu}) - (\alpha \leftrightarrow \beta)
\]

\[
F^{\mu,\nu,\alpha\beta\gamma} = \eta^{\mu\alpha} F_1^{\nu,\beta\gamma} + \eta^{\mu\beta} F_1^{\nu,\gamma\alpha} + \eta^{\nu\alpha} F_1^{\mu,\beta\gamma} + \eta^{\nu\beta} F_1^{\mu,\gamma\alpha}
\]

\[
+ \eta^{\mu\alpha} F_2^{\nu,\beta\gamma} + \eta^{\mu\beta} F_2^{\nu,\gamma\alpha} + \eta^{\nu\alpha} F_2^{\mu,\beta\gamma} + \eta^{\nu\beta} F_2^{\mu,\gamma\alpha}
\]

\[
+(\eta^{\mu\alpha} \eta^{\nu\beta} - \eta^{\mu\beta} \eta^{\nu\alpha}) F_3^\gamma + (\eta^{\mu\beta} \eta^{\nu\gamma} - \eta^{\mu\gamma} \eta^{\nu\beta}) F_3^\alpha + (\eta^{\nu\gamma} \eta^{\mu\alpha} - \eta^{\mu\alpha} \eta^{\nu\gamma}) F_3^\beta
\]

(3.24)

where

\[
E_1^{\mu\nu,\rho} = 4(\sigma^\rho d_\sigma h^{\mu\nu} - h^{\mu\sigma} d_\sigma h^{\nu\sigma} - h^{\nu\sigma} d_\sigma h^{\mu\sigma}) + 2 h^{\mu\nu} d^\rho h + 4(h^{\mu\sigma} d^\rho h^{\rho\sigma} + h^{\nu\sigma} d^\rho h^{\sigma\rho}) - 2 u^{\rho}(d^{\mu}\tilde{u}^{\nu} + d^{\nu}\tilde{u}^{\mu})
\]

\[
E_2^\rho = 2(-h^{\sigma\rho} d_\sigma h + h_{\alpha\beta} d_\sigma h^{\alpha\beta})
\]

\[
B_1^{\mu,\nu,\rho} = 2(-u^{\nu} d^\mu u^{\rho} + u^{\rho} d^\mu u^{\nu})
\]

\[
B_1^\mu = 2 u_\sigma d^\sigma u^{\mu}
\]

\[
E_1^{\mu\nu,\lambda,\rho} = 2(u^{\lambda} (d^\rho h^{\mu\nu} + d^{\mu} h^{\rho\nu} - d^{\nu} h^{\mu\rho}) - u^{\rho} (d^\nu h^{\lambda\mu} + d^{\mu} h^{\lambda\nu} - d^{\lambda} h^{\mu\nu}))
\]

\[
E_2^{\lambda\nu} = 2(u_\sigma d^\sigma h^{\rho\lambda} - d_\sigma u^{\nu} h^{\sigma\lambda} + d^\sigma u_\sigma h^{\rho\lambda} - d_\sigma u^{\lambda} h^{\rho\nu})
\]

\[
E_3^{\mu\nu} = 2(-u_\sigma d^\sigma h^{\mu\nu} - d_\sigma u^{\nu} h^{\mu\sigma} + d_\sigma u^{\mu} h^{\nu\sigma} + d_\sigma u^{\nu} h^{\mu\sigma}) + \eta^{\mu\nu} (d_\sigma u^{\sigma} h + u^{\sigma} d_\sigma h - 2 d_\mu u_{\alpha} h^{\alpha\beta})
\]

\[
E_4^{\lambda\rho} = -u^{\rho} d^\lambda h + u^{\lambda} d^{\rho} h
\]

\[
E_1^{\lambda,\rho} \equiv \eta_{\mu\nu} E_1^{\mu\nu,\lambda,\rho}, \quad E_3 = \eta_{\mu\nu} E_3^{\mu\nu}
\]

\[
H^{\alpha\rho} = u^\alpha \tilde{u}^\rho - u^\rho \tilde{u}^\alpha
\]

\[
\tilde{E}_1^{\nu,\lambda,\alpha} = u^{\lambda} d^\nu u^\alpha - u^\alpha d^\nu u^{\lambda}
\]

\[
\tilde{E}_2^{\nu} = u_\rho d^\nu u^{\rho}
\]

\[
F_1^{\nu,\beta\gamma} = d^\alpha u^{\nu} u^{\beta} - d^\alpha u^{\beta} u^{\nu} - (\alpha \leftrightarrow \beta)
\]
\[ F_2^{\mu,\beta\gamma} = d^\mu u^\beta u^\alpha - (\alpha \leftrightarrow \beta) \]
\[ F_3^\alpha = u_\rho d^\rho u^\alpha \] (3.25)

Then we have, with the notations introduced above:

**Theorem 3.1** The following formulas are true

\[ sT(T(x_1)^{(2)}, T(x_2)^{(2)}) = \delta(x_1 - x_2) \left( - : B_1_\sigma C_{\rho\sigma} : + \frac{1}{2} : B_1^\rho C : + C^{\mu,\rho} E_{3\mu\nu} : - \frac{1}{2} : C^\rho E_3 : - : D^{\mu,\rho} B_1^\mu : \right)(x_2) : + \partial_\lambda \delta(x_1 - x_2) \left[ - : E_{3\mu\nu}(x_1) E^{\mu,\lambda}(x_2) : + \frac{1}{2} : E_3(x_1) E^\lambda(x_2) : - B_1^\mu(x_1) F^{\mu,\lambda}(x_2) : \right] - [x_1 \leftrightarrow x_2] \] (3.26)

\[ sT(T^\rho(x_1)^{(2)}, T(x_2)^{(2)}) = \delta(x_1 - x_2) \left( - : B_1_\sigma C_{\rho\sigma} : + \frac{1}{2} : B_1^\rho C : + C^{\mu,\rho} E_{3\mu\nu} : - \frac{1}{2} : C^\rho E_3 : - : D^{\mu,\rho} B_1^\mu : \right)(x_2) : + \partial_\lambda \delta(x_1 - x_2) \left[ - : E_{3\mu\nu}(x_1) E^{\mu,\lambda}(x_2) : + \frac{1}{2} : E_3(x_1) E^\lambda(x_2) : - B_1^\mu(x_1) F^{\mu,\lambda}(x_2) : \right] - \partial_\lambda_1 \partial_\lambda_2 \delta(x_1 - x_2) : H^{\mu,\lambda,\lambda_2,\rho}(x_1) B_1^\mu(x_2) : \] (3.27)

\[ sT(T^\rho(x_1)^{(2)}, T^\sigma(x_2)^{(2)}) = \delta(x_1 - x_2) \left[ ( : C^{\mu,\rho} B_1^\mu : - \frac{1}{2} : C^\rho B_1^\rho : ) (x_2) - (\rho \leftrightarrow \sigma) \right] + \partial_\lambda \delta(x_1 - x_2) \left[ - : E^{\sigma,\lambda,\rho}(x_1) B_1^\mu(x_2) : + \frac{1}{2} : E^\lambda B_1^\rho(x_1) B_1^\sigma(x_2) : \right] + (x_1 \leftrightarrow x_2, \rho \leftrightarrow \sigma) \] (3.28)

\[ sT(T^\rho(x_1)^{(2)}, T(x_2)^{(2)}) = \delta(x_1 - x_2) \left( - : C^{\mu,\rho} E_{3\mu\nu} : + \frac{1}{2} : C^{\rho} E_3 : \right)(x_2) : + \partial_\lambda \delta(x_1 - x_2) \left[ - : E^{\mu,\lambda,\rho\sigma}(x_1) E_{3\mu\nu}(x_2) : + \frac{1}{2} : E^{\lambda,\rho\sigma}(x_1) E_3(x_2) : \right] + : \tilde{E}_1^{\mu,\rho\sigma}(x_1) B_1^\mu(x_2) : + : \tilde{E}_2^\rho(x_1) B_1^\sigma(x_2) : - : \tilde{E}_2^\rho(x_1) B_1^\rho(x_2) : \] (3.29)

\[ sT(T^\rho(x_1)^{(2)}, T^\tau(x_2)^{(2)}) = \frac{1}{2} \delta(x_1 - x_2) \left( : B_1^{\mu,\rho\sigma} G_\mu^\tau : + : B_1^\rho G_{\sigma,\tau}^\tau : - : B_1^\rho G^{\mu,\tau} : - 2 : C^{\mu,\rho,\tau} B_1^\mu : + : C_{\rho,\sigma,\tau} B_1^\tau : \right)(x_2) \] (3.30)

\[ sT(T^\rho(x_1)^{(2)}, T(x_2)^{(2)}) = - \delta(x_1 - x_2) : D^{\mu,\rho,\sigma}(x_1) B_1^\mu(x_2) : - \partial_\lambda \delta(x_1 - x_2) : F^{\mu,\lambda,\rho,\sigma}(x_1) B_1^\mu(x_2) : \] (3.31)
\[ sT(T^{\rho_1\sigma_1}(x_1)^{(2)}, T^{\rho_2\sigma_2}(x_2)^{(2)}) = 0 \]  \hspace{1cm} (3.32)

\[ sT(T^{\rho_1\sigma_2}(x_1)^{(2)}, T^{\nu}(x_2)^{(2)}) = 0 \]  \hspace{1cm} (3.33)

**Proof:** We consider for illustration only the first relation. We have to compute the expressions
\[ A = d\mathcal{Q}T(T(x_1)^{(2)}, T(x_2)^{(2)}), \]
\[ B = -i\partial_1 T(T^{\mu_1}(x_1)^{(2)}, T(x_2)^{(2)}), C = -i\partial_2 T(T(x_1)^{(2)}, T^{\mu_2}(x_2)^{(2)}) \]
using the relations. The expression in the left hand side of is \[ A + B + C. \] There are a lot of cancelations and only the terms with \( \Box \) acting on \( D_0^\mu(x_1 - x_2), \partial_\mu D_0^\mu(x_1 - x_2) \) survive.

Some identities are valid and must be used to obtain the compensations:
\[ E_1^\mu\nu,\lambda,\rho = -(\lambda \leftrightarrow \rho), \quad E_4^\lambda,\rho = -(\lambda \leftrightarrow \rho), \quad E_2^\lambda,\rho = \frac{1}{2} B^\lambda,\rho \]
\[ S_\lambda\rho(E_1^\mu\nu,\rho) = S_\lambda\rho(F_1^\mu,\lambda,\rho) \]
\[ E_0^\mu,\lambda,\sigma = -(\lambda \leftrightarrow \sigma) \]
\[ B_1^\mu,\lambda,\rho = -2 E_1^\mu,\lambda,\rho \]
\[ B_1^\mu = 2 E_2^\mu \]
\[ F_1^\mu,\rho\sigma = E_1^\mu,\rho\sigma - \tilde{E}_1^\mu,\rho\sigma \]
\[ F_2^\mu,\rho\sigma = \tilde{E}_1^\mu,\rho\sigma \]
\[ F_3^\mu = \tilde{E}_2^\mu. \]  \hspace{1cm} (3.34)

\[ \Box \]

From (3.26) we can obtain a simple form of the anomaly:
\[ sT(T(x_1)^{(2)}, T(x_2)^{(2)}) = \delta(x_1 - x_2) A(T, T)(x_2) \]  \hspace{1cm} (3.35)

where
\[ A(T, T) \equiv 2 : E_3^\mu C_\mu C : - : E_3 C : +2 : B_1^\mu D_\mu : - : \partial_\lambda B_1^\mu F_1^{\mu,\lambda} : - : B_1^\mu \partial_\lambda F_1^{\mu,\lambda} : + : \partial_\lambda E_3^\mu E_1^{\mu,\lambda} : - \frac{1}{2} : \partial_\lambda E_3 E_2^\lambda : - : E_3^\mu \partial_\lambda E_1^{\mu,\lambda} : + \frac{1}{2} : E_3 \partial_\lambda E_2^\lambda : \]  \hspace{1cm} (3.36)

The previous anomaly can be written in a simpler form:
\[ A(T, T) = : B_1^\mu C_\mu C : +2 : B_1^\mu D_\mu : - : B_1^\mu \partial_\lambda E_1^{\mu,\lambda} : -2 : B_1^\mu \partial^\lambda F_1^{\mu,\lambda} : - \partial_\lambda \left( \frac{1}{2} : B_1^\mu E_1^{\mu,\lambda} : + : B_1^\mu F_1^{\mu,\lambda} : \right) \]  \hspace{1cm} (3.37)

We consider only the first contribution
\[ A'(T, T) = : B_1^\mu (C_\mu - \partial_\lambda E_1^{\mu,\lambda}) : + 2 : B_1^\mu (D_\mu - \partial_\lambda F_1^{\mu,\lambda}) : \]  \hspace{1cm} (3.38)
It is useful to consider separately the contributions trilinear and linear in \( h \) namely \( A_{u\ hhh} \) and \( A_{u\ u\ h} \) respectively. After some calculations one get:

\[
A_{u\ hhh} = u_\rho \ A^\rho + u_\rho,\rho \ A + u_{\mu,\nu} \ A^{\mu\nu} \tag{3.39}
\]

where

\[
A_\rho \equiv 4 \ h_\rho \ D + 4 \ h_{\mu,\rho} \ D^{\mu\nu} \\
A \equiv 4 \ h \ D + 4 \ h_{\mu,\nu} \ D^{\mu\nu} \\
A_{\mu\nu} \equiv -8 \ h_{\mu\nu} \ D - 8 \ h_{\nu,\lambda} \ D_{\mu\lambda} \tag{3.40}
\]

where

\[
D \equiv 2(\ h^{\rho_\sigma,\lambda} h_{\rho_\sigma} + h^{\rho_\sigma} h_{\rho_\sigma,\lambda} - h^{\rho,\sigma,\lambda} h_{\rho_\sigma,\lambda}) \tag{3.41}
\]

and

\[
D_{\mu\nu} \equiv -h_\mu \ h_\nu + 2 \ h^{\rho_\sigma\mu} h_{\rho_\sigma,\nu} + 4 \ (h_{\mu,\sigma} h_{\nu}^{\sigma,\rho} + h_{\mu,\rho} h_{\nu}^{\rho,\sigma}) - 4 \ (h_{\mu,\sigma} h^{\rho_\sigma,\nu} + h_{\nu,\sigma} h^{\rho_\sigma,\mu} + h_{\mu,\nu} h^{\rho_\sigma,\rho\sigma} + h_{\nu,\rho} h^{\rho_\sigma,\mu\sigma}). \tag{3.42}
\]

To obtain a simpler form for \( A_{u\ u\ h} \) it is useful to eliminate total derivatives and exhibit it in the form: \( A_{u\ u\ h} = \) total divergence + \( h_{\mu,\nu} \ B^{\mu\nu} \) i.e. to eliminate the derivatives on \( h_{\mu,\nu} \). The end result is very nice i.e. \( B^{\mu\nu} = 0 \) so \( A_{u\ u\ h} \) is a total derivative:

\[
A_{u\ u\ h} = d_\mu \ [4 \ u_\mu h^{\rho_\sigma} C^{\rho_\sigma} - 16 \ u_\sigma u^{\rho_\sigma} (\tilde{u}_{\rho,\lambda} + \tilde{u}_{\lambda,\rho}) h^{\mu\lambda}] \tag{3.43}
\]

where

\[
C_{\rho_\sigma} = 2 \ [u^{\lambda,\rho} (\tilde{u}_{\lambda,\sigma} + \tilde{u}_{\sigma,\lambda}) + u^{\lambda} \tilde{u}_{\rho,\sigma\lambda}] + (\rho \leftrightarrow \sigma). \tag{3.44}
\]

Now we can try to eliminate the anomaly \( A_{u\ hhh} \) using finite renormalizations:

\[
T^{\text{ren}}(T^I(x_1), T^J(x_2)) = T(T^I(x_1), T^J(x_2)) + \delta(x_1 - x_2) \ N(T^I, T^J)(x_2) \tag{3.45}
\]

with the polynomials \( N(T^I, T^J) \) verify the symmetry property:

\[
N(T^I, T^J) = (-1)^{|I||J|} N(T^J, T^I) \tag{3.46}
\]

and

\[
gh(N(T^I, T^J)) = |I| + |J|, \quad \omega(N(T^I, T^J)) \leq 6 \tag{3.47}
\]

Because the anomaly \( A_{u\ hhh} \) is quadri-linear, we are looking for an expression \( N(T, T) \) which is also quadri-linear. The result is \([12]\):

**Theorem 3.2** The anomaly \( A_{u\ hhh} \) can be eliminated if one takes

\[
N(T, T) = i \ -16 \ h^{\mu\nu} h^{\rho_\sigma} d_\rho h_{\mu\nu} d_\sigma h + 8 \ h^{\mu\nu} h^{\rho_\sigma} d_\lambda h_{\mu\nu} d^\lambda h_{\rho_\sigma} \\
+ 32 \ h^{\mu\nu} h_{\nu,\rho} d^\alpha h^{\rho_3\beta} d_\beta h_{\mu,\alpha} + 32 \ h^{\mu\nu} h^{\rho_\sigma} d_\mu h_{\nu,\rho} d_{\nu} h_{\sigma,\alpha} \\
- 32 \ h^{\mu\nu} h_{\nu,\rho} d^\alpha h_{\mu,\beta} d_{\alpha} h^{\rho_3} \\
- 16h^{\mu\nu} h^{\rho_\sigma} d_\lambda h_{\mu,\rho} d^\lambda h_{\nu,\sigma} + 16h^{\mu\nu} h^{\rho_\sigma} d_{\lambda} h_{\mu,\rho} d^\lambda h_{\nu,\sigma}) \tag{3.48}
\]
Proof: We consider finite renormalizations of the type

\[ R(T^I(x_1), T^J(x_2)) = \delta(x_1 - x_2) \, N(T^I, T^J)(x_2) \]  

(3.49)

where the polynomials \( N(T^I, T^J) \) verify the symmetry property:

\[ N(T^I, T^J) = (-1)^{|I||J|} \, N(T^J, T^I). \]  

(3.50)

Then we have by direct computation that the expression

\[
\begin{align*}
 sR(T^I(x_1), T^J(x_2)) &\equiv d_Q R(T^I(x_1), T^J(x_2)) \\
&- i \, \partial^I_{\mu} R(T^{I\mu}(x_1), T^J(x_2)) - i \, (-1)^{|I|} \, \partial^J_{\mu} R(T^I(x_1), T^{J\mu}(x_2))
\end{align*}
\]  

(3.51)

is:

\[
\begin{align*}
 sR(T^I(x_1), T^J(x_2)) &= \delta(x_1 - x_2) \, [d_Q N(T^I, T^J) - i \, (-1)^{|I|} \, \partial_{\mu} N(T^I, T^{J\mu})](x_2) \\
&\quad - i \, \partial_{\mu} \delta(x_1 - x_2) \, [N(T^{I\mu}, T^J) - (-1)^{|I|} \, N(T^I, T^{J\mu})](x_2)
\end{align*}
\]  

(3.52)

In particular, due to the symmetry property

\[ sR(T(x_1), T(x_2)) = \delta(x_1 - x_2) \, [d_Q N(T, T) - i \, \partial_{\mu} N(T, T^{\mu})](x_2) \]  

(3.53)

Taking into account the form of the anomaly (3.37) it follows that we must have

\[ \mathcal{A}(T, T) + d_Q N(T, T) - i \, \partial_{\mu} N(T, T^{\mu}) = 0 \]  

(3.54)

and from here with (3.38)

\[ \mathcal{A}'(T, T) + d_Q N(T, T) - i \, \partial_{\mu} N'(T, T^{\mu}) = 0 \]  

(3.55)

and finally with (3.43)

\[ \mathcal{A}_{uhhh} + d_Q N(T, T) - i \, \partial_{\mu} N''(T, T^{\mu}) = 0. \]  

(3.56)

It is useful to transform the expression (3.39) of \( \mathcal{A}_{uhhh} \) such that we eliminate, up to a total derivative, all terms with derivatives on the ghost field i.e. we write

\[ \mathcal{A}_{uhhh} = u_\rho \, \tilde{\mathcal{A}}^\rho + d_\rho \mathcal{B}^\rho. \]  

(3.57)

Indeed, we obtain from (3.39)

\[
\tilde{\mathcal{A}}^\rho = \mathcal{A}^\rho - d_\rho \mathcal{A} - d^\rho \mathcal{A}_{\rho\sigma}, \quad \mathcal{B}^\rho = u^\rho \, \mathcal{A} + u_\sigma \, \mathcal{A}^{\sigma\rho}.
\]  

(3.58)

The strategy is to find an expression \( N \) such that

\[ d_Q N = i \, (u_\rho \, \tilde{\mathcal{A}}^\rho + d_\rho N^\rho). \]  

(3.59)
For this purpose, we make an ansatz for $N$, as an expression quadri-linear in $h$; it is sufficient to consider terms of the type $hhdhdh$. There are 43 terms of this type so we have an ansatz

$$N = \sum_{j=1}^{43} a_j N_j$$

(3.60)

We do not provide the complete list, but only give

\begin{align*}
N_1 &= h^{\mu\nu} h^{\rho\sigma} d_\rho h_{\mu\nu} \ d_\sigma h \\
N_2 &= h^{\mu\nu} h^{\rho\sigma} d_\lambda h_{\mu\nu} \ d_\lambda h_{\rho\sigma} \\
N_3 &= h^{\mu\nu} h_{\nu\rho} \ d^\alpha h^{\rho\beta} \ d_\beta h_{\mu\alpha} \\
N_4 &= h^{\mu\nu} h^{\rho\sigma} d_\mu h_{\rho\alpha} \ d_\nu h_{\sigma\alpha} \\
N_5 &= h^{\mu\nu} h_{\nu\rho} \ d^\alpha h_{\mu\beta} \ d_\alpha h^{\rho\beta} \\
N_6 &= h^{\mu\nu} h^{\rho\sigma} d_\lambda h_{\mu\rho} \ d^\lambda h_{\nu\sigma} \\
N_7 &= h^{\mu\nu} h_{\nu\rho} \ d^\lambda h_{\mu\rho} \ d_\lambda h
\end{align*}

(3.61)

By direct computation we can exhibit the expressions $d_Q N_j$ in the form

$$d_Q N_j = i(u_\rho \ A^\rho_j + d_\rho N^\rho_j)$$

(3.62)

and we impose the equation

$$\sum a_j A^\rho_j = \tilde{A}^\rho.$$

(3.63)

This equation is equivalent to a system of linear equations for the coefficients $a_j$; the solution is not unique. If we consider our lucky guess i.e. only the coefficients $a_j \ j = 1, \ldots, 7$ then we get an unique solution, namely the expression (3.48) from the statement. ■
4 Finite Renormalizations

We have mentioned in the preceding Section that the solution for the finite renormalization $N$ is not unique. Here we investigate how big is this non-uniqueness. In other words, we suppose that we have a solution $T(T^I(x_1), T^J(x_2))$ such that Bogoliubov axioms and gauge invariance i.e. the cocyle relations

$$sT(T^I(x_1), T^J(x_2)) = 0$$ (4.1)

are fulfilled. Then the arbitrariness is given by finite renormalizations of the type:

$$R(T^I(x_1), T^J(x_2)) = \delta(x_1 - x_2) N(T^I, T^J)(x_2) + \partial_\mu \delta(x_1 - x_2) N(T^I, T^J)\mu(x_2)$$

$$+ \partial_\nu \delta(x_1 - x_2) N(T^I, T^J)\nu(x_2)$$ (4.2)

where the polynomials $N$ are Lorentz covariant, from power counting we have

$$\omega(N(T^I, T^J)) \leq 6, \quad \omega(N(T^I, T^J)^\mu) \leq 5, \quad \omega(N(T^I, T^J)^\mu\nu) \leq 4$$ (4.3)

and we also have

$$gh(N(T^I, T^J)) = |I| + |J|$$ (4.4)

and we can suppose that $N(T^I, T^J)^\mu\nu$ is symmetric in $\mu \leftrightarrow \nu$.

If we want to preserve the symmetry properties of the chronological products (see Section 2.2) we must impose

$$R(T^I(x_1), T^J(x_2)) = (-1)^{|I||J|} R(T^J(x_2), T^I(x_1))$$ (4.5)

which is equivalent to the following system of equations:

$$N(T^I, T^J) = (-1)^{|I||J|} \left[ N(T^J, T^I) + d_\mu N(T^J, T^I)^\mu + d_\nu d_\mu N(T^J, T^I)^\mu\nu \right]$$

$$N(T^I, T^J)^\mu = (-1)^{|I||J|} \left[ N(T^J, T^I)^\mu + 2d_\nu N(T^J, T^I)^\mu\nu \right]$$

$$N(T^I, T^J)^\mu\nu = (-1)^{|I||J|} N(T^J, T^I)^\mu\nu.$$ (4.6)

More important, if we want that gauge invariance $sT(T^I(x_1), T^J(x_2)) = 0$ is preserved we must also have the cocyle relations

$$sR(T^I(x_1), T^J(x_2)) = 0$$ (4.7)

which is equivalent to the following system of equations:

$$d_\rho N(T^I, T^J) = i (-1)^{|I|} d_\mu N(T^I, T^J)$$

$$d_\rho N(T^I, T^J)^\mu = i \left\{ N(T^I, T^J)^\mu - (-1)^{|I|} \left[ N(T^I, T^J)^\mu - d_\nu N(T^I, T^J)^\mu\nu \right] \right\}$$

$$d_\rho N(T^I, T^J)^\rho\sigma = i \left\{ S_{\rho\sigma} \left\{ N(T^I, T^J)^\rho - (-1)^{|I|} \left[ N(T^I, T^J)^\rho - d_\mu N(T^I, T^J)^\mu\rho \sigma \right] \right\}$$

$$- S_{\mu\rho\sigma} \left[ N(T^I, T^J)^\mu\rho - (-1)^{|I|} N(T^I, T^J)^\mu\rho\sigma \right] = 0.$$ (4.8)

The restrictions from the preceding relations are rather severe and we start to analyse this cohomology problem.
Theorem 4.1 Suppose we have a cocyle $R(T^I(x_1), T^J(x_2))$ of the type (4.2) and verifying the cocyle equation (4.7). Then $R$ is cohomologous with a cocycle verifying
\[ N(T^I, T^J)^{\mu\nu} = 0. \] (4.9)

Proof: (i) We first define the cochain
\[ S(T^\mu, T) = \partial_\nu \delta(x_1 - x_2) \ S^{\mu\nu}(x_2) \] (4.10)
and compute the coboundary
\[ \delta S(T, T) = \delta(x_1 - x_2) \ d_\mu d_\nu S^{\mu\nu}(x_2) - 2 \partial_\mu \delta(x_1 - x_2) \ d_\nu S^{\mu\nu}(x_2) + 2 \partial_\mu \partial_\nu \delta(x_1 - x_2) \ S^{\mu\nu}(x_2). \] (4.11)

So, if we choose
\[ S^{\mu\nu} = \frac{1}{2} N(T, T)^{\mu\nu} \] (4.12)
the expression
\[ R'(T, T) \equiv R(T, T) - \delta S(T, T) \] (4.13)
which is cohomologous to $R(T, T)$ has a simpler form i.e.
\[ R'(T, T) = \delta(x_1 - x_2) \ R(x_2) + \partial_\nu \delta(x_1 - x_2) \ R^\mu(x_2) \] (4.14)
and this means that we can take
\[ N(T, T)^{\mu\nu} = 0. \] (4.15)

(ii) We now define the cochain
\[ S(T^\mu, T^\nu) = \partial_\rho \delta(x_1 - x_2) \ S^{(\mu\nu),\rho}(x_2) - 2 \delta(x_1 - x_2) \partial_\rho S^{(\mu\nu),\rho}(x_2) \]
\[ S(T^\mu, T) = \partial_\rho \delta(x_1 - x_2) \ S^{[\mu\nu],\rho}(x_2) \] (4.16)
such that we have the symmetry property
\[ S(T^\mu, T^\nu) = -S(T^\nu, T^\mu). \] (4.17)

Now we compute the coboundary
\[ \delta S(T^\mu, T) = \partial^\nu S(T^{\mu\nu}, T) - \partial^\nu S(T^\mu, T^\nu) \] (4.18)
and we find
\[ \delta S(T^\mu, T) = \partial_\rho \partial_\sigma S^{\mu\nu,\rho\sigma}(x_2) + \cdots \] (4.19)
where
\[ S^{\mu\nu,\rho} \equiv S^{(\mu\nu),\rho} + S^{[\mu\nu],\rho} \] (4.20)
and \cdots are terms with one and no derivatives on $\delta$. If we choose
\[ S^{\mu\nu,\rho} = N(T^\mu, T)^{\nu\rho} \] (4.21)
then it follows that the cocyle $R(T^\mu, T)$ is cohomologous with a cocyle having

$$N(T^\mu, T)^{\rho\sigma} = 0.$$  \hfill (4.22)

(iii) Let us define the cochain

$$S(T^{\mu\nu}, T^\rho) = \partial_\sigma \delta(x_1 - x_2) \, S^{[\mu\nu][\rho] \sigma}(x_2)$$  \hfill (4.23)

and compute the coboundary

$$\delta S(T^\mu, T^\nu) = \partial_\rho^1 S(T^{\mu\rho}, T^\nu) - \partial_\rho^2 S(T^\mu, T^{\nu\rho}).$$  \hfill (4.24)

After some computations we find

$$\delta S(T^\mu, T^\nu) = \partial_\rho \partial_\sigma \delta(x_1 - x_2) \left( S^{[\mu\nu][\rho] \sigma} - S^{[\nu\rho][\mu] \sigma} \right)(x_2) + \cdots$$  \hfill (4.25)

where \cdots are terms with one and no derivatives on $\delta$. We choose

$$S^{[\mu\nu][\rho] \sigma} = \frac{1}{2} [N(T^\mu, T^\nu)^{\rho\sigma} + N(T^\mu, T^\rho)^{\nu\sigma} + N(T^\nu, T^\rho)^{\mu\sigma}].$$  \hfill (4.26)

From the last relation \hfill (4.26) we find out that

$$N(T^\mu, T^\nu)^{\rho\sigma} = -N(T^\nu, T^\mu)^{\rho\sigma}$$  \hfill (4.27)

so the preceding definition is consistent: we have the anti-symmetry property in $\mu \leftrightarrow \nu$. Moreover, we easily obtain:

$$S^{[\mu\rho][\nu] \sigma} = S^{[\nu\rho][\mu] \sigma} = N(T^\mu, T^\nu)^{\rho\sigma}$$  \hfill (4.28)

so

$$\delta S(T^\mu, T^\nu) = \partial_\rho \partial_\sigma \delta(x_1 - x_2) \, N(T^\mu, T^\nu)^{\rho\sigma}(x_2) + \cdots$$  \hfill (4.29)

It follows that the cocyle $R(T^\mu, T^\nu)$ is cohomologous with a cocyle having

$$N(T^\mu, T^\nu)^{\rho\sigma} = 0.$$  \hfill (4.30)

(iv) We choose the cochain

$$S(T^{\mu\nu\rho}, T) = \partial_\sigma \delta(x_1 - x_2) \, S^{[\mu\nu\rho] \sigma}(x_2)$$

$$S(T^{\mu\nu}, T^\rho) = 0$$  \hfill (4.31)

so in this way we preserve the preceding relation \hfill (4.30). We compute the coboundary

$$\delta S(T^{\mu\nu}, T) = \partial_\rho^1 S(T^{\mu\nu\rho}, T) + \partial_\rho^2 S(T^{\mu\nu}, T^\rho)$$  \hfill (4.32)

and we find immediately

$$\delta S(T^{\mu\nu}, T) = \partial_\rho \partial_\sigma \delta(x_1 - x_2) \, S^{[\mu\nu\rho] \sigma}(x_2)$$  \hfill (4.33)
Let us take
\[
S^{[\mu\nu\rho][\sigma} \equiv N(T^{\mu\nu}, T)^{\rho\sigma} + N(T^{\mu\rho}, T)^{\nu\sigma} + N(T^{\nu\mu}, T)^{\rho\sigma}
\] (4.34)
such that we have complete anti-symmetry in the indexes \( \mu, \nu, \rho \). From the last relation (4.38) we find out that
\[
S_{\nu\rho\sigma}[N(T^{\mu\nu}, T)^{\rho\sigma} + N(T^{\mu\rho}, T)^{\nu\sigma}] = 0
\] (4.35)
If we use (4.30) we are left with
\[
S_{\nu\rho\sigma} N(T^{\mu\nu}, T)^{\rho\sigma} = N(T^{\mu\nu}, T)^{\rho\sigma} + N(T^{\mu\rho}, T)^{\nu\sigma} + N(T^{\nu\mu}, T)^{\rho\sigma} = 0.
\] (4.36)
If we use this relation we end up with
\[
S^{[\mu\nu\rho]} \equiv N(T^{\mu\nu}, T)^{\rho\sigma} + N(T^{\mu\rho}, T)^{\nu\sigma}
\] (4.37)
which implies
\[
\delta S(T^{\mu\nu}, T) = \partial_\rho \partial_\sigma \delta(x_1 - x_2) N(T^{\mu\nu}, T^{\rho\sigma})(x_2).
\] (4.38)
It follows that \( R(T^{\mu\nu}, T) \) is cohomologous with a cocyle having
\[
N(T^{\mu\nu}, T)^{\rho\sigma} = 0.
\] (4.39)

(v) We define the cochain
\[
S(T^{\mu\nu\rho}, T^{\sigma}) = \partial_\lambda \delta(x_1 - x_2) S^{[\mu\nu\rho][\sigma] \lambda}(x_2)
\]
\[
S(T^{\mu\nu\rho}, T^{\sigma}) = 0
\] (4.40)
and compute the coboundary
\[
\delta S(T^{\mu\nu\rho}, T) = \partial_1 S(T^{\mu\nu\rho}, T^{\sigma}) - \partial_2 S(T^{\mu\nu\rho}, T^{\sigma}).
\] (4.41)
We immediately get:
\[
\delta S(T^{\mu\nu\rho}, T) = \partial_\lambda \partial_\sigma \delta(x_1 - x_2) S^{[\mu\nu\rho][\sigma] \lambda}(x_2) + \cdots
\] (4.42)
We choose
\[
S^{[\mu\nu\rho][\sigma] \lambda} \equiv N(T^{\mu\nu\rho}, T)^{\lambda\sigma}
\] (4.43)
and get
\[
\delta S(T^{\mu\nu\rho}, T) = \partial_\lambda \partial_\sigma \delta(x_1 - x_2) N(T^{\mu\nu\rho}, T)^{\lambda\sigma}(x_2) + \cdots
\] (4.44)
so \( R(T^{\mu\nu\rho}, T) \) is cohomologous with a cocyle having
\[
N(T^{\mu\nu\rho}, T)^{\alpha\beta} = 0.
\] (4.45)

(vi) We use the cochain
\[
S(T^{\mu\nu}, T^{\rho\sigma}) = \partial_\lambda \delta(x_1 - x_2) S^{[\mu\nu][\rho\sigma] \lambda}(x_2)
\]
\[
S(T^{\mu\nu}, T^{\rho\sigma}) = 0
\] (4.46)
and this preserves (4.43). We compute the coboundary
\[ \delta S(T^{\mu\nu}, T^\rho) = \partial_\sigma S(T^{\mu\nu\sigma}, T^\rho) + \partial_\sigma S(T^{\mu\nu}, T^{\rho\sigma}) \] (4.47)
and we immediately obtain
\[ \delta S(T^{\mu\nu}, T^\rho) = -\partial_\lambda \partial_\sigma \delta(x_1 - x_2) S^{[\mu\nu][\rho\sigma]}(x_2) + \cdots \] (4.48)
Let us choose
\[ S^{[\mu\nu][\rho\sigma]} = -\frac{1}{2} [N(T^{\mu\nu}, T^\rho)^{\sigma\lambda} - N(T^{\mu\nu}, T^\sigma)^{\rho\lambda}] \] (4.49)
From the last relation (4.8) we find out that
\[ S_{\rho\sigma\lambda}(N(T^{\mu\nu\rho}, T)^{\sigma\lambda} - N(T^{\mu\nu}, T^\rho)^{\sigma\lambda}) = 0 \] (4.50)
and if we use (4.45) we are left with
\[ S_{\rho\sigma\lambda} N(T^{\mu\nu\rho}, T)^{\sigma\lambda} = 0 \quad \Leftrightarrow \quad S_{\sigma\lambda} N(T^{\mu\nu}, T^\sigma)^{\rho\lambda} = -N(T^{\mu\nu}, T^\rho)^{\sigma\lambda}. \] (4.51)
Then we have
\[ \delta S(T^{\mu\nu}, T^\rho) = \partial_\lambda \partial_\sigma \delta(x_1 - x_2) N(T^{\mu\nu}, T^\rho)^{\sigma\lambda}(x_2) + \cdots \] (4.52)
so \( R(T^{\mu\nu}, T^\rho) \) is cohomologous with a cocycle having
\[ N(T^{\mu\nu}, T^\rho)^{\alpha\beta} = 0. \] (4.53)

(vii) We consider the cochain
\[ S(T^{\mu\nu\rho\sigma}, T^\lambda) = \partial_\alpha \delta(x_1 - x_2) S^{[\mu\nu\rho\sigma][\lambda\alpha]}(x_2) \] (4.54)
and compute the coboundary
\[ \delta S(T^{\mu\nu\rho\sigma}, T) = \partial_\lambda S(T^{\mu\nu\rho\sigma}, T^\lambda). \] (4.55)
The result is
\[ \delta S(T^{\mu\nu\rho\sigma}, T) = -\partial_\lambda \partial_\sigma \delta(x_1 - x_2) S^{[\mu\nu\rho\sigma][\lambda\alpha]}(x_2) + \cdots \] (4.56)
If we choose
\[ S^{[\mu\nu\rho\sigma][\lambda\alpha]} = -N(T^{\mu\nu\rho\sigma}, T)^{\lambda\alpha} \] (4.57)
then we have
\[ \delta S(T^{\mu\nu\rho\sigma}, T) = \partial_\lambda \partial_\sigma \delta(x_1 - x_2) N(T^{\mu\nu\rho\sigma}, T)^{\lambda\alpha}(x_2) + \cdots \] (4.58)
so \( R(T^{\mu\nu\rho\sigma}, T) \) is cohomologous with a cocycle having
\[ N(T^{\mu\nu\rho\sigma}, T)^{\alpha\beta} = 0. \] (4.59)

(viii) We start from the cochain
\[ S(T^{\mu\nu}, T^{\sigma\lambda}) = \partial_\alpha \delta(x_1 - x_2) S^{[\mu\nu][\sigma\lambda\alpha]}(x_2) \]
\[ S(T^{\mu\nu\rho\sigma}, T^\lambda) = 0 \] (4.60)
so \((4.59)\) is preserved. We compute the coboundary

\[
\delta S(T^{\mu\nu\rho}, T^\sigma) = \partial_\lambda S(T^{\mu\nu\rho\lambda}, T^\sigma) - \partial^2_\lambda S(T^{\mu\nu\rho}, T^{\sigma\lambda})
\]  

(4.61)

and the result is:

\[
\delta S(T^{\mu\nu\rho}, T^\sigma) = \partial_\lambda \partial_\alpha \delta(x_1 - x_2) S^{[\mu\nu\rho][\sigma\lambda]\alpha}(x_2) + \cdots
\]

(4.62)

We choose

\[
S^{[\mu\nu\rho][\sigma\lambda]} \equiv \frac{a}{2} \left[ N(T^{\mu\nu\rho}, T^\sigma)^{\alpha\lambda} - N(T^{\mu\nu\rho}, T^{\sigma\lambda}) \right]
\]

(4.63)

From the last relation \((4.8)\) we find out that

\[
S_{\sigma\alpha\beta}[N(T^{\mu\nu\rho}, T^\sigma)^{\alpha\beta} + N(T^{\mu\nu\rho}, T^\alpha)^{\sigma\beta} + N(T^{\mu\nu\rho}, T^\beta)^{\sigma\alpha}] = 0
\]

(4.64)

and if we use \((4.59)\) we are left with

\[
S_{\sigma\alpha\beta}N(T^{\mu\nu\rho}, T^\sigma)^{\alpha\beta} = 0 \quad \Leftrightarrow \quad N(T^{\mu\nu\rho}, T^\sigma)^{\alpha\beta} + N(T^{\mu\nu\rho}, T^\alpha)^{\sigma\beta} + N(T^{\mu\nu\rho}, T^\beta)^{\sigma\alpha} = 0.
\]

(4.65)

It follows that:

\[
\delta S(T^{\mu\nu\rho}, T^\sigma) = \frac{3a}{4} \partial_\lambda \partial_\alpha \delta(x_1 - x_2) N(T^{\mu\nu\rho}, T^\sigma)^{\lambda\alpha}(x_2) + \cdots
\]

(4.66)

If we choose \(a = \frac{4}{3}\) it follows that \(R(T^{\mu\nu\rho}, T^\sigma)\) is cohomologous with a cocyle having

\[
N(T^{\mu\nu\rho}, T^\sigma)^{\alpha\beta} = 0.
\]

(4.67)

(ix) We are left with the last step, namely the consideration of \(N(T^{\mu\nu}, T^{\rho\sigma})^{\alpha\beta}\). To prove that one can fix this expression to zero, is not as easy as in the previous steps. We need an ansatz for this expression. The most general quadri-linear expression of ghost number 4 and with the required symmetry properties is:

\[
N(T^{\mu\nu}, T^{\rho\sigma})^{\alpha\beta} = a_1 \eta^{\alpha\beta} u^\mu u^\nu u^\rho u^\sigma + a_2 S_{\alpha\beta} (\eta^{\mu\alpha} u^\nu u^\rho u^\sigma u^\beta - \eta^{\nu\alpha} u^\mu u^\rho u^\sigma u^\beta + \eta^{\rho\alpha} u^\mu u^\nu u^\sigma u^\beta - \eta^{\sigma\alpha} u^\mu u^\nu u^\rho u^\beta)
\]

(4.68)

From the last relation \((4.8)\) we find out that

\[
S_{\rho\alpha\beta}[N(T^{\mu\nu\rho}, T^\sigma)^{\alpha\beta} - N(T^{\mu\nu\sigma}, T^{\rho\sigma\alpha})] = 0
\]

(4.69)

and if use \((4.67)\) we are left with

\[
S_{\rho\alpha\beta}N(T^{\mu\nu}, T^{\rho\sigma})^{\alpha\beta} = 0.
\]

(4.70)

If we introduce here \((4.68)\) we find out \(a_1 = a_2\) so the generic form becomes:

\[
N(T^{\mu\nu}, T^{\rho\sigma})^{\alpha\beta} = a S_{\alpha\beta} (\eta^{\alpha\beta} u^\mu u^\nu u^\rho u^\sigma + \eta^{\mu\alpha} u^\nu u^\rho u^\sigma u^\beta - \eta^{\nu\alpha} u^\mu u^\rho u^\sigma u^\beta + \eta^{\rho\alpha} u^\mu u^\nu u^\sigma u^\beta - \eta^{\sigma\alpha} u^\mu u^\nu u^\rho u^\beta)
\]

(4.71)
We try to compensate it using a cochain of the form
\[ S(T^{\mu\nu\lambda}, T^{\rho\sigma}) = \partial_\alpha \delta(x_1 - x_2) \, S[^{\mu\nu\lambda][\rho\sigma]]^\alpha(x_2) \] (4.72)
and the generic form of \( S \) is
\[ S[^{\mu\nu\lambda}[\rho\sigma]]^\alpha = b_1 (\eta^{\mu\alpha} u^\nu u^\rho u^\sigma u^\lambda + \eta^{\nu\alpha} u^\mu u^\rho u^\sigma u^\lambda + \eta^{\lambda\alpha} u^\mu u^\rho u^\sigma u^\nu) \]
\[ + b_2 (\eta^{\rho\alpha} u^\mu u^\nu u^\sigma u^\lambda - \eta^{\sigma\alpha} u^\mu u^\rho u^\sigma u^\lambda) \]
\[ + b_3 [(\eta^{\mu\rho} u^\nu u^\lambda u^\sigma u^\alpha + \eta^{\nu\rho} u^\mu u^\sigma u^\alpha + \eta^{\lambda\rho} u^\mu u^\nu u^\sigma u^\alpha) - (\rho \leftrightarrow \sigma)] \] (4.73)
We want to preserve (4.67) so we must have
\[ S_{\alpha\beta} S[^{\mu\nu\lambda}[\rho\sigma]]^\alpha S[^{\mu\nu\lambda}[\rho\sigma]]^\beta = 0. \] (4.74)
If we insert the previous expression we get \( b_1 = b_3 = b, \ b_2 = 0 \). Now we compute the coboundary
\[ \delta S(T^{\mu\nu}, T^{\rho\sigma}) = \partial_\lambda S(T^{\mu\nu\lambda}, T^{\rho\sigma}) + \partial_\lambda S(T^{\mu\nu}, T^{\rho\sigma\lambda}) \] (4.75)
After some standard work we get
\[ \delta S(T^{\mu\nu}, T^{\rho\sigma}) = 2 \partial_\alpha \partial_\beta \delta(x_1 - x_2) \, S[^{\mu\alpha\nu\sigma]^{\beta]} (x_2) + \cdots \] (4.76)
Using the previous expression for \( S[^{\mu\nu\lambda}[\rho\sigma]]^\alpha \) with \( b = \frac{a}{2} \) we get
\[ \delta S(T^{\mu\nu}, T^{\rho\sigma}) = \partial_\alpha \partial_\beta \delta(x_1 - x_2) \, N(T^{\mu\nu}, T^{\rho\sigma})^{\alpha\beta}(x_2) + \cdots \] (4.77)
It follows that \( R(T^{\mu\nu}, T^{\rho\sigma}) \) is cohomologous with a cocycle having
\[ N(T^{\mu\nu}, T^{\rho\sigma})^{\alpha\beta} = 0. \] (4.78)
This finishes the proof. ■

**Remark 4.2** From the previous proof it follows that we do not need all relations (4.8) to obtain the result, but only the last relation of (4.8).

As a consequence, we have:

**Theorem 4.3** Suppose the cochain \( R \) - see (4.2) verifies
\[ N(T^I, T^J)^{\mu\nu} = 0 \] (4.79)
i.e.
\[ R(T^I(x_1), T^J(x_2)) = \delta(x_1 - x_2) \, N(T^I, T^J)(x_2) + \partial_\mu \delta(x_1 - x_2) \, N(T^I, T^J)^{\mu}(x_2) \] (4.80)
Then the symmetry properties (4.6) are equivalent to
\[ N(T^I, T^J) = (-1)^{|I||J|} \, [N(T^J, T^I) + d_\mu N(T^J, T^I)^{\mu}] \]
\[ N(T^I, T^J)^{\mu} = -(-1)^{|I||J|} \, N(T^J, T^I)^{\mu} \] (4.81)
The gauge invariance condition

\[ sR(T^I(x_1), T^J(x_2)) = 0 \]  \hspace{1cm} (4.82)

is equivalent to the following system of equations:

\[
d_Q N(T^I, T^J) = i (-1)^{|I|} d_\mu N(T^I, T^{J\mu})
\]

\[
d_Q N(T^I, T^J)_\mu = i \left[ N(T^{I\mu}, T^J) - (-1)^{|I|} \left[ N(T^I, T^{J\mu}) - d_\nu N(T^I, T^{J\nu}) \right] \right]
\]

\[
S_{\rho\sigma} \left[ N(T^\rho, T)^{\sigma} - (-1)^{|I|} N(T^I, T^{J\rho})^{\sigma} \right] = 0.
\]  \hspace{1cm} (4.83)

The next step is:

**Theorem 4.4** Suppose we have a cocyle of the type \((4.80)\). Then \(R\) is cohomologous to a cocyle verifying

\[ N(T^I, T^J)_\mu = 0. \]  \hspace{1cm} (4.84)

**Proof:** (i) We have from the last equation \((4.83)\) for \(I = J = \emptyset\):

\[ N(T, T)_\mu = -N(T, T)^\mu \]  \hspace{1cm} (4.85)

i.e.

\[ N(T, T)_\mu = 0. \]  \hspace{1cm} (4.86)

(ii) Let us define the cochain

\[ S(T^{\mu\nu}, T) = \delta(x_1 - x_2) S_{[\mu\nu]}(x_2) \]

\[ S(T^{\mu}, T^{\nu}) = 0 \]  \hspace{1cm} (4.87)

and compute the coboundary

\[ \delta S(T^\mu, T) = \partial^1_\nu S(T^{\mu\nu}, T) - \partial^2_\nu S(T^{\mu}, T^{\nu}). \]  \hspace{1cm} (4.88)

We obtain:

\[ \delta S(T^\mu, T) = \partial_\nu \delta(x_1 - x_2) S_{[\mu\nu]}(x_2). \]  \hspace{1cm} (4.89)

We have from the last equation \((4.83)\):

\[ S_{\rho\sigma} \left[ N(T^\rho, T)^{\sigma} - N(T, T^{\rho})^{\sigma} \right] = 0 \]  \hspace{1cm} (4.90)

and from the last relation \((4.81)\)

\[ N(T^\mu, T)^\nu = -N(T, T^{\mu})^\nu \]  \hspace{1cm} (4.91)

so we have in fact

\[ S_{\rho\sigma} N(T^\rho, T)^{\sigma} = 0 \quad \Leftrightarrow \quad N(T^\rho, T)^{\sigma} = -N(T^{\sigma}, T)^{\rho} \]  \hspace{1cm} (4.92)
It follows that the choice

\[ S^{[\mu \nu]} = N(T^\mu, T)\nu \]  

(4.93)
is consistent. It leads to

\[ \delta S(T^\mu, T) = \partial_\nu \delta(x_1 - x_2) \ N(T^\mu, T)^\nu(x_2) \]  

(4.94)
and it follows that \( R(T^\mu, T) \) is cohomologous with a cocyle having

\[ N(T^\mu, T)^\nu = 0. \]  

(4.95)

(iii) We define the cochain

\[ S(T^{\mu \nu}, T^\rho) = \delta(x_1 - x_2) \ S^{[\mu \nu][\rho]}(x_2) \]  

(4.96)
and compute the coboundary

\[ \delta S(T^\mu, T^\nu) = \partial_\rho S(T^{\mu \rho}, T^\nu) - \partial_\rho S(T^\mu, T^{\nu \rho}). \]  

(4.97)
We obtain:

\[ \delta S(T^\mu, T^\nu) = \partial_\rho \delta(x_1 - x_2) \ (S^{[\mu \rho][\nu]} + S^{[\nu \rho][\mu]})(x_2) + \cdots \]  

(4.98)
where by \( \cdots \) we mean an expression \( \sim \delta(x_1 - x_2) \). We take

\[ S^{[\mu \nu][\rho]} = \frac{1}{2} \left[N(T^\mu, T^\nu)^\rho - N(T^\nu, T^\mu)^\rho + N(T^{\mu \nu}, T^\rho)^\rho\right] \]  

(4.99)
such that the anti-symmetry property in \( \mu \leftrightarrow \nu \) is true. From the last relation we have

\[ N(T^\mu, T^\nu)^\rho = N(T^\nu, T^\mu)^\rho \]  

(4.100)
and from the last relation we have:

\[ S_{\rho \sigma} [N(T^{\mu \rho}, T^\sigma)^\nu - N(T^\mu, T^\rho)^\nu] = 0. \]  

(4.101)
If we use these last two relations we obtain

\[ S^{[\mu \rho][\nu]} + S^{[\nu \rho][\mu]} = N(T^\mu, T^\nu)^\rho \]  

(4.102)
so

\[ \delta S(T^\mu, T^\nu) = \partial_\rho \delta(x_1 - x_2) \ N(T^\mu, T^\nu)^\rho(x_2) + \cdots \]  

(4.103)
It follows that \( R(T^\mu, T^\nu) \) is cohomologous with a cocyle having

\[ N(T^\mu, T^\nu)^\rho = 0. \]  

(4.104)

(iv) We define the cochain

\[ S(T^{\mu \nu \rho}, T) = \delta(x_1 - x_2) \ S^{[\mu \nu \rho]}(x_2) \]  

(4.105)
such that we preserve (4.104). Next, we compute the coboundary
\[ \delta S(T^{\mu \nu}, T) = \partial_\rho S(T^{\mu \nu \rho}, T). \] (4.106)
and get:
\[ \delta S(T^{\mu \nu}, T) = \partial_\rho \delta(x_1 - x_2) S^{[\mu \nu \rho]}(x_2). \] (4.107)
We take
\[ S^{[\mu \nu \rho]} = \frac{1}{3} [N(T^{\mu \nu}, T)^\rho + N(T^{\nu \rho}, T)^\mu + N(T^{\rho \mu}, T)^\nu] \] (4.108)
such that we have complete anti-symmetry in the indexes \( \mu, \nu, \rho \). We use now the last formula (4.83) with the choice (4.104) and have:
\[ S_{\rho \sigma} N(T^{\mu \rho}, T)^\sigma = 0. \] (4.109)
If we use the preceding formula we obtain that all three terms in the expression of \( S^{[\mu \nu \rho]} \) are equal, so we have
\[ S^{[\mu \nu \rho]} = N(T^{\mu \nu}, T)^\rho \] (4.110)
and from here
\[ \delta S(T^{\mu}, T^{\nu}) = \partial_\rho \delta(x_1 - x_2) N(T^{\mu \nu}, T)^\rho(x_2). \] (4.111)
It follows that \( R(T^{\mu}, T) \) is cohomologous with a cocycle having
\[ N(T^{\mu}, T)^\rho = 0. \] (4.112)
(v) We define the cochain
\[ S(T^{\mu \nu \rho}, T^\sigma) = \delta(x_1 - x_2) S^{[\mu \nu \rho][\sigma]}(x_2) \] (4.113)
and compute the coboundary
\[ \delta S(T^{\mu \nu}, T) = -\partial_\sigma S(T^{\mu \nu \rho}, T^\sigma). \] (4.114)
We get:
\[ \delta S(T^{\mu \nu}, T) = \partial_\sigma \delta(x_1 - x_2) S^{[\mu \nu \rho][\sigma]}(x_2) + \cdots \] (4.115)
so if we choose
\[ S^{[\mu \nu \rho][\sigma]} = N(T^{\mu \nu \rho}, T)^\sigma \] (4.116)
we have
\[ \delta S(T^{\mu \nu}, T) = \partial_\sigma \delta(x_1 - x_2) N(T^{\mu \nu \rho}, T)^\sigma(x_2) + \cdots \] (4.117)
It follows that \( R(T^{\mu \nu}, T) \) is cohomologous with a cocycle having
\[ N(T^{\mu \nu}, T)^\sigma = 0. \] (4.118)
(vi) We define the cochain
\[
S(T^{\mu \nu}, T^{\rho \sigma}) = \delta(x_1 - x_2) S^{[\mu \nu [\rho \sigma]}(x_2) \\
S(T^{\mu \nu \rho}, T^{\sigma}) = 0
\] (4.119)
such that we preserve (4.118). We compute the coboundary
\[
\delta S(T^{\mu\nu}, T^\rho) = \partial_1^1 S(T^{\mu\nu\sigma}, T^\rho) + \partial_2^2 S(T^{\mu\nu}, T^{\rho\sigma})
\]
and we get:
\[
\delta S(T^{\mu\nu}, T^\rho) = -\partial_\sigma \delta(x_1 - x_2) \ S^{[\mu\nu][\rho\sigma]}(x_2) + \cdots
\]
We take
\[
S^{[\mu\nu][\rho\sigma]} = N(T^{\mu\nu}, T^\rho)^\sigma.
\]
From the last relation (4.81) we have:
\[
N(T^{\mu\nu}, T^\rho)^\sigma = -N(T^\rho, T^{\mu\nu})^\sigma
\]
and from the last relation (4.83) we get
\[
S_{\rho\sigma}[N(T^{\mu\nu\rho}, T)^\sigma - N(T^{\mu\nu}, T^\rho)^\sigma] = 0.
\]
Taking into account (4.118) we are left with:
\[
S_{\rho\sigma} N(T^{\mu\nu}, T^\rho)^\sigma = 0
\]
and this ensures that the expression $S^{[\mu\nu][\rho\sigma]}$ defined above is anti-symmetric to $\rho \leftrightarrow \sigma$. Also from the last relation (4.83) we get
\[
S_{\rho\sigma}[N(T^{\mu\nu\rho}, T^\nu)^\rho + N(T^\mu, T^{\nu\rho})^\sigma] = 0.
\]
Using this relation we obtain after some computations that
\[
N(T^{\mu\nu}, T^\rho)^\sigma = N(T^{\rho\sigma}, T^\mu)^\nu
\]
so the expression $S^{[\mu\nu][\rho\sigma]}$ defined above is symmetric to $\mu \leftrightarrow \rho, \nu \leftrightarrow \sigma$. So the definition of $S^{[\mu\nu][\rho\sigma]}$ is consistent and we also have
\[
\delta S(T^{\mu\nu}, T^\rho) = -\partial_\sigma \delta(x_1 - x_2) \ N(T^{\mu\nu}, T^\rho)^\sigma(x_2) + \cdots
\]
It follows that $R(T^{\mu\nu}, T^\rho)$ is cohomologous with a cocycle having
\[
N(T^{\mu\nu}, T^\rho)^\sigma = 0.
\]
(vii) We define the cochain
\[
S(T^{\mu\nu\rho\sigma}, T^\lambda) = \delta(x_1 - x_2) \ S^{[\mu\nu\rho\sigma][\lambda]}(x_2)
\]
and compute the coboundary
\[
\delta S(T^{\mu\nu\rho\sigma}, T) = \partial^2_\lambda S(T^{\mu\nu\rho\sigma}, T^\lambda).
\]
We get:
\[
\delta S(T^{\mu\nu\rho\sigma}, T) = -\partial_\lambda \delta(x_1 - x_2) \, S^{[\mu\nu\rho\sigma]}(x_2) + \cdots
\]
(4.132)

If we take
\[
S^{[\mu\nu\rho\sigma]}(x) = N(T^{\mu\nu\rho\sigma}, T)^\lambda
\]
we prove that \(R(T^{\mu\nu\rho\sigma}, T)\) is cohomologous with a cocycle having
\[
N(T^{\mu\nu\rho\sigma}, T)^\lambda = 0.
\]
(4.134)

(viii) We define the cochain
\[
S(T^{\mu\nu\rho}, T^{\sigma\lambda}) = \delta(x_1 - x_2) \, S^{[\mu\nu\rho][\sigma\lambda]}(x_2)
\]
\[
S(T^{\mu\nu\rho\sigma}, T^{\lambda}) = 0
\]
(4.135)
such that we preserve (4.134). We compute the coboundary
\[
\delta S(T^{\mu\nu\rho}, T^{\sigma\lambda}) = \partial_\lambda \delta(x_1 - x_2) \, S^{[\mu\nu\rho][\sigma\lambda]}(x_2) + \cdots
\]
(4.136)
and we get:
\[
\delta S(T^{\mu\nu\rho}, T^{\sigma}) = \partial_\lambda \delta(x_1 - x_2) \, S^{[\mu\nu\rho][\sigma\lambda]}(x_2) + \cdots
\]
(4.137)
We take
\[
S^{[\mu\nu\rho][\sigma\lambda]} = N(T^{\mu\nu\rho}, T^{\sigma})^\lambda
\]
(4.138)
and must prove the consistency of this definition. From the last relation (4.83) we have:
\[
S_{\sigma\lambda}[N(T^{\mu\nu\rho\sigma}, T)^\lambda + N(T^{\mu\nu\rho}, T^{\sigma})^\lambda] = 0
\]
(4.139)
and if we use (4.134) we are left with
\[
S_{\sigma\lambda}N(T^{\mu\nu\rho}, T^{\sigma})^\lambda = 0
\]
(4.140)
and this gives the desired consistency of the definition of \(S^{[\mu\nu\rho][\sigma\lambda]}\). Because
\[
\delta S(T^{\mu\nu\rho}, T^{\sigma}) = \partial_\lambda \delta(x_1 - x_2)N(T^{\mu\nu\rho}, T^{\sigma})^\lambda(x_2) + \cdots
\]
(4.141)
it follows that \(R(T^{\mu\nu\rho}, T^{\sigma})\) is cohomologous with a cocycle having
\[
N(T^{\mu\nu\rho}, T^{\sigma})^\lambda = 0.
\]
(4.142)

(ix) We still have to analyse the expression \(N(T^{\mu\nu}, T^{\rho\sigma})^\lambda\) and as in theorem (4.1) we need a general ansatz. Beside the properties of anti-symmetry in \(\mu \leftrightarrow \nu\) and \(\rho \leftrightarrow \sigma\) we must use the last relation (4.81) to get
\[
N(T^{\mu\nu}, T^{\rho\sigma})^\lambda = -N(T^{\rho\sigma}, T^{\mu\nu})^\lambda
\]
(4.143)
so the general ansatz is:

\[ N(T^{\mu\nu}, T^{\rho\sigma})^\lambda = a_1 (d^\lambda u^\nu u^\rho u^\sigma - d^\lambda u^\nu u^\mu u^\sigma - d^\lambda u^\rho u^\mu u^\nu) + a_2 (d^\mu u^\nu u^\rho u^\sigma - d^\mu u^\nu u^\mu u^\sigma - d^\mu u^\rho u^\mu u^\nu + d^\rho u^\mu u^\nu) + a_3 (d^\nu u^\rho u^\sigma u^\lambda - d^\nu u^\rho u^\sigma u^\lambda - d^\nu u^\rho u^\mu u^\lambda + d^\nu u^\rho u^\mu u^\lambda) + a_4 (d^\rho u^\nu u^\sigma u^\lambda - d^\rho u^\nu u^\mu u^\lambda - d^\rho u^\nu u^\sigma u^\mu + d^\rho u^\nu u^\sigma u^\mu) - d^\rho u^\nu u^\sigma u^\lambda + d^\rho u^\nu u^\sigma u^\lambda - d^\rho u^\nu u^\mu u^\lambda - d^\rho u^\nu u^\mu u^\lambda) + a_5 (\eta^{\rho\lambda} u^\nu u^\sigma u^\lambda - \eta^{\rho\lambda} u^\nu u^\sigma u^\lambda - \eta^{\rho\lambda} u^\nu u^\sigma u^\mu + \eta^{\rho\lambda} u^\nu u^\sigma u^\mu) - d_\alpha u^\alpha + a_6 (\eta^{\rho\lambda} u^\nu u^\sigma u^\lambda - \eta^{\rho\lambda} u^\nu u^\sigma u^\lambda - \eta^{\rho\lambda} u^\nu u^\sigma u^\mu + \eta^{\rho\lambda} u^\nu u^\sigma u^\mu) ) \] (4.144)

Form the last relation (4.83) we also have

\[ S_{\sigma\lambda}[N(T^{\mu\nu\sigma}, T^{\rho})^\lambda - N(T^{\mu\nu}, T^{\rho\sigma})^\lambda] = 0 \] (4.145)

and if we use (4.142) we are left with

\[ S_{\sigma\lambda}N(T^{\mu\nu}, T^{\rho\sigma})^\lambda = 0. \] (4.146)

If we substitute the preceding ansatz we get after some computations that \( a_j = 0, \ j = 1, \ldots, 6 \)
so in fact we have

\[ N(T^{\mu\nu}, T^{\rho\sigma})^\lambda = 0 \] (4.147)

and this finishes the proof. \( \square \)

As a consequence, we have:

**Theorem 4.5** Suppose the cochain \( R \) - see (4.2) verifies

\[ N(T^I, T^J)^{\mu\nu} = 0, \quad N(T^I, T^J)^{\mu} = 0. \] (4.148)

i.e.

\[ R(T^I(x_1), T^J(x_2)) = \delta(x_1 - x_2) N(T^I, T^J)(x_2) \] (4.149)

Then the symmetry properties (4.7) are equivalent to

\[ N(T^I, T^J) = (-1)^{|I||J|} N(T^J, T^I) \] (4.150)

and gauge invariance condition

\[ sR(T^I(x_1), T^J(x_2)) = 0 \] (4.151)

is equivalent to the following system of equations:

\[ d_Q N(T^I, T^J) = i (-1)^{|I|} d_\mu N(T^I, T^J^\mu) \]

\[ N(T^I^\mu, T^J) = (-1)^{|I|} N(T^I, T^J^\mu). \] (4.152)

We have proved in [11] that in this case we have

**Theorem 4.6** These finite renormalizations do not produce anomalies iff there exists expressions \( N^I \) such that

\[ N(T^I, T^J) = N^{JI} \] (4.153)

and

\[ d_Q N^I = i d_\mu N^{I\mu}. \] (4.154)
5 The Renormalizablity of Quantum Gravity

It is a comon lore in the literature that the gravity is not renormalizable i.e. the arbitrariness of the chronological products increases with the order of the perturbation theory. We prove here that this assertion is wrong, at least in the second order of the perturbation theory. As we have proved in the preceding Section, the arbitrariness is severely constricted by the gauge invariance condition. In fact, gauge invariance is a cocyle condition, and we have proved that such a cocyle is cohomologous with one of the form

\[ R(T^I(x_1), T^I(x_2)) = \delta(x_1 - x_2) \, N^{II}(x_2) \] (5.1)

and we must have:

\[ \omega(N) = 6, \quad gh(N^I) = |I| \] (5.2)

and the gauge invariance condition

\[ d_Q N^I = i \, d_\mu N^{I\mu}. \] (5.3)

Here we prove that such a cocyle is trivial in the quadri-linear sector i.e. it is a coboundary \( N = d_Q B + d_\mu B^\mu \). First we must give a list of all possible terms. We start with \( N \) which is of ghost number 0. We need the list of all terms up to a total derivative. We have three sectors:

(a) Terms quadri-linear in \( h_{\mu\nu} \): it is sufficient to consider terms of the type \( hhhdh \); there are 43 terms of this type. Terms of the type \( hhddh \) can be written, up to a total derivative, as a linear combination of these 43 terms.

\[ N_1 = h^2 \, d_\rho h d^\rho h \]

\[ N_{2,1} = h \, d_\rho h d_\sigma h h^{\rho\sigma} \quad N_{2,2} = h^2 \, d_\rho h d_\sigma h^{\rho\sigma} \]

\[ N_{3,1} = d_\rho h h^\rho h h^{\alpha\beta} h_{\alpha\beta} \quad N_{3,2} = d_\rho h d_\sigma h h^{\rho\sigma} h_{\alpha\beta} \]
\[ N_{3,3} = h d_\rho h h_{\alpha\beta} d^\rho h^{\alpha\beta} \quad N_{3,4} = h d_\rho h h_{\alpha\beta} d^\rho h^{\alpha\rho} \]
\[ N_{3,5} = h d_\rho h h^{\rho\sigma} d^\lambda h_{\sigma\lambda} \quad N_{3,6} = h^2 d_\rho h_{\alpha\beta} d^\rho h^{\alpha\beta} \]
\[ N_{3,7} = h^2 d_\beta h_{\rho\sigma} d^\rho h^{\rho\beta} \quad N_{3,8} = h^2 d^\alpha h_{\alpha\beta} d_\beta h^{\mu\beta} \]

\[ N_{4,1} = d_\rho h h^{\rho\sigma} h^{\alpha\beta} d_\sigma h_{\alpha\beta} \quad N_{4,2} = d_\rho h h^{\rho\sigma} h^{\alpha\beta} d_\beta h_{\alpha\sigma} \]
\[ N_{4,3} = d_\rho h h^{\rho\alpha} h_{\sigma\lambda} d_\alpha h^{\lambda\beta} \quad N_{4,4} = d_\rho h h_{\lambda\alpha} h^{\lambda\beta} d_\beta h^{\alpha\beta} \]
\[ N_{4,5} = d_\rho h h_{\lambda\alpha} h^{\lambda\beta} d^\rho h^{\alpha\beta} \quad N_{4,6} = h h^{\alpha\beta} d_\alpha h_{\rho\sigma} d_\beta h^{\rho\sigma} \]
\[ N_{4,7} = h h^{\alpha\beta} d_\sigma h_{\rho\sigma} d_\beta h^{\rho\sigma} \quad N_{4,8} = h h^{\alpha\beta} d_\sigma h_{\alpha\rho} d^\sigma h^{\beta\rho} \]
\[ N_{4,9} = h h^{\alpha\beta} d_\rho h_{\rho\sigma} d^\rho h^{\sigma\beta} \quad N_{4,10} = h h^{\alpha\beta} d_\rho h_{\alpha\rho} d^\sigma h^{\beta\rho} \]
\[ N_{4,11} = h h^{\alpha\beta} d_\mu h_{\alpha\beta} d_\nu h^{\mu\nu} \quad N_{4,12} = h h^{\alpha\beta} d_\alpha h_{\beta\mu} d_\nu h^{\mu\nu} \]
\[ N_{4,13} = d_\mu h d_\nu h^{\mu\nu} h^{\alpha\beta} h_{\alpha\beta} \]
\[ N_{5,1} = h_{\lambda\rho} h^{\alpha\beta} d^\beta h^{\alpha\beta} \]
\[ N_{5,2} = h_{\lambda\rho} h^{\alpha\beta} d^\beta h^{\alpha\beta} \]
\[ N_{5,3} = h_{\lambda\rho} h^{\alpha\beta} d^\beta h^{\alpha\beta} \]
\[ N_{5,4} = h_{\lambda\rho} h^{\alpha\beta} d^\beta h^{\alpha\beta} \]
\[ N_{5,5} = h_{\lambda\rho} h^{\alpha\beta} d^\beta h^{\alpha\beta} \]
\[ N_{5,6} = h_{\lambda\rho} h^{\alpha\beta} d^\beta h^{\alpha\beta} \]
\[ N_{5,7} = h_{\lambda\rho} h^{\alpha\beta} d^\beta h^{\alpha\beta} \]
\[ N_{5,8} = h_{\lambda\rho} h^{\alpha\beta} d^\beta h^{\alpha\beta} \]
\[ N_{5,9} = h_{\lambda\rho} h^{\alpha\beta} d^\beta h^{\alpha\beta} \]
\[ N_{5,10} = h_{\lambda\rho} h^{\alpha\beta} d^\beta h^{\alpha\beta} \]
\[ N_{5,11} = h_{\lambda\rho} h^{\alpha\beta} d^\beta h^{\alpha\beta} \]
\[ N_{5,12} = h_{\lambda\rho} h^{\alpha\beta} d^\beta h^{\alpha\beta} \]
\[ N_{5,13} = h_{\lambda\rho} h^{\alpha\beta} d^\beta h^{\alpha\beta} \]
\[ N_{5,14} = h_{\lambda\rho} h^{\alpha\beta} d^\beta h^{\alpha\beta} \]
\[ N_{5,15} = h_{\lambda\rho} h^{\alpha\beta} d^\beta h^{\alpha\beta} \]
\[ N_{5,16} = h_{\lambda\rho} h^{\alpha\beta} d^\beta h^{\alpha\beta} \]
\[ N_{5,17} = h_{\lambda\rho} h^{\alpha\beta} d^\beta h^{\alpha\beta} \]
\[ N_{5,18} = h_{\lambda\rho} h^{\alpha\beta} d^\beta h^{\alpha\beta} \]
\[ N_{5,19} = h_{\lambda\rho} h^{\alpha\beta} d^\beta h^{\alpha\beta} \]

(b) Terms of the type \( u \bar{u} h h \): we have terms of the type \( u \bar{u} h h d h d h \), \( u \bar{u} h h d h d h \), \( u \bar{u} h h d h d h \), \( u \bar{u} h h d h d h \); there are 83 terms of these types. The terms with derivatives on \( u \) can be written, up to a total derivative, as linear combination of these 83 terms.

\[ N_{6,1} = u^\mu \bar{u}_\mu d_\rho d^\rho h \]
\[ N_{6,2} = u^\mu \bar{u}_\mu d_\rho d^\rho h \]
\[ N_{6,3} = u^\mu \bar{u}_\mu d_\rho h^{\alpha\beta} d^\beta h^{\alpha\beta} \]
\[ N_{6,4} = u^\mu \bar{u}_\mu h^{\alpha\beta} d^\beta h^{\alpha\beta} \]
\[ N_{6,5} = u^\mu \bar{u}_\mu h^{\alpha\beta} d^\beta h^{\alpha\beta} \]
\[ N_{6,6} = u^\mu \bar{u}_\mu h^{\alpha\beta} d^\beta h^{\alpha\beta} \]
\[ N_{6,7} = u^\mu \bar{u}_\mu h^{\alpha\beta} d^\beta h^{\alpha\beta} \]
\[ N_{6,8} = u^\mu \bar{u}_\mu h^{\alpha\beta} d^\beta h^{\alpha\beta} \]
\[ N_{6,9} = u^\mu \bar{u}_\mu h^{\alpha\beta} d^\beta h^{\alpha\beta} \]
\[ N_{6,10} = u^\mu \bar{u}_\mu h^{\alpha\beta} d^\beta h^{\alpha\beta} \]
\[ N_{6,11} = u^\mu \bar{u}_\mu h^{\alpha\beta} d^\beta h^{\alpha\beta} \]
\[ N_{6,12} = u^\mu \bar{u}_\mu h^{\alpha\beta} d^\beta h^{\alpha\beta} \]
\[ N_{6,13} = u^\mu \bar{u}_\mu h^{\alpha\beta} d^\beta h^{\alpha\beta} \]
\[ N_{6,14} = u^\mu \bar{u}_\mu h^{\alpha\beta} d^\beta h^{\alpha\beta} \]
\[ N_{6,15} = u^\mu \bar{u}_\mu h^{\alpha\beta} d^\beta h^{\alpha\beta} \]
\[ N_{6,16} = u^\mu \bar{u}_\mu h^{\alpha\beta} d^\beta h^{\alpha\beta} \]
\[ N_{6,17} = u^\mu \bar{u}_\mu h^{\alpha\beta} d^\beta h^{\alpha\beta} \]
\[ N_{6,18} = u^\mu \bar{u}_\mu h^{\alpha\beta} d^\beta h^{\alpha\beta} \]
\[ N_{6,19} = u^\mu \bar{u}_\mu h^{\alpha\beta} d^\beta h^{\alpha\beta} \]
\[ N_{6,20} = u^\mu \bar{u}_\mu h^{\alpha\beta} d^\beta h^{\alpha\beta} \]
\( N_{8,1} = u_\mu d^\mu \bar{u}^\nu h_{\nu\rho} d^\rho h \)
\( N_{8,2} = u_\mu d^\mu \bar{u}^\nu h_{\nu\rho} d_\rho h^{\mu\alpha} \)
\( N_{8,3} = u_\mu d^\mu \bar{u}^\nu h d_\nu h \)
\( N_{8,4} = u_\mu d^\mu \bar{u}^\nu h^{\alpha\beta} d_\alpha h_{\beta} \)
\( N_{8,5} = u_\mu d^\mu \bar{u}^\nu h d^\rho h_{\nu\rho} \)
\( N_{8,6} = u_\mu d^\mu \bar{u}^\nu h^{\alpha\beta} d_\alpha h_{\beta} \)
\( N_{8,7} = u_\mu d^\nu \bar{u}^\nu h_{\nu\rho} d^\rho h \)
\( N_{8,8} = u_\mu d^\nu \bar{u}^\nu h^{\alpha\beta} d_\rho h_{\nu\rho} \)
\( N_{8,9} = u_\mu d^\nu \bar{u}^\nu h d_\rho h \)
\( N_{8,10} = u_\mu d^\nu \bar{u}^{\alpha\beta} d_\alpha h_{\beta} \)
\( N_{8,11} = u_\mu d^\nu \bar{u}^\nu h d^\rho h_{\nu\rho} \)
\( N_{8,12} = u_\mu d^\nu \bar{u}^{\alpha\beta} d_\alpha h_{\beta} \)
\( N_{8,13} = u_\mu d \cdot \bar{u} h^{\mu\nu} d_\nu h \)
\( N_{8,14} = u_\mu d \cdot \bar{u} h^{\mu\nu} d_\nu h_{\rho} \)
\( N_{8,15} = u_\mu d_\alpha \bar{u}_\beta h^{\mu\alpha} d_\beta h \)
\( N_{8,16} = u_\mu d_\alpha \bar{u}_\beta h^{\mu\alpha} d_\beta h_{\rho} \)
\( N_{8,17} = u_\mu d_\alpha \bar{u}_\beta h^{\rho\lambda} d_\lambda h \)
\( N_{8,18} = u_\mu d_\alpha \bar{u}_\beta h^{\rho\lambda} d_\lambda h_{\rho} \)
\( N_{8,19} = u_\mu d \cdot \bar{u} h d^{\beta} h \)
\( N_{8,20} = u_\mu d \cdot \bar{u} h_{\alpha\beta} d^{\beta} h \)
\( N_{8,21} = u_\mu d_\alpha \bar{u}_\beta h^{\alpha\beta} d^{\mu} h \)
\( N_{8,22} = u_\mu d_\alpha \bar{u}_\beta h^{\alpha\beta} d^{\mu} h_{\rho} \)
\( N_{8,23} = u_\mu d_\alpha \bar{u}_\beta h^{\lambda} d^{\mu\lambda\beta} d^{\mu} h \)
\( N_{8,24} = u_\mu d_\alpha \bar{u}_\beta h^{\lambda} d^{\mu\lambda\beta} d^{\mu} h_{\rho} \)
\( N_{8,25} = u_\mu d \cdot \bar{u} h_{\alpha\beta} d^{\mu\lambda} d^{\mu\beta} h \)
\( N_{8,26} = u_\mu d \cdot \bar{u} h_{\alpha\beta} d^{\mu\lambda} d^{\mu\beta} h_{\rho} \)
\( N_{8,27} = u_\mu d_\alpha \bar{u}_\beta h^{\alpha\beta} d^{\mu\beta} h \)
\( N_{8,28} = u_\mu d_\alpha \bar{u}_\beta h^{\alpha\beta} d^{\mu\beta} h_{\rho} \)
\( N_{8,29} = u_\mu d_\alpha \bar{u}_\beta h^{\alpha\beta} d_\nu h d^{\mu\lambda} d^{\mu\beta} h \)
\( N_{8,30} = u_\mu d_\alpha \bar{u}_\beta h^{\alpha\beta} d_\nu h d^{\mu\lambda} d^{\mu\beta} h_{\rho} \)
\( N_{8,31} = u_\mu d_\alpha \bar{u}_\beta h^{\lambda\beta} d_\nu h d^{\mu\alpha} h \)
\( N_{8,32} = u_\mu d_\alpha \bar{u}_\beta h^{\lambda\beta} d_\nu h d^{\mu\alpha} h_{\rho} \)
\( N_{8,33} = u_\mu d_\alpha \bar{u}_\beta h^{\lambda\beta} d_\nu h d^{\mu\lambda} h \)
\( N_{8,34} = u_\mu d_\alpha \bar{u}_\beta h^{\lambda\beta} d_\nu h d^{\mu\lambda} h_{\rho} \)
\( N_{8,35} = u_\mu d_\alpha \bar{u}_\beta h^{\lambda \beta} d_\nu h d^{\mu\alpha} h \)
\( N_{8,36} = u_\mu d_\alpha \bar{u}_\beta h^{\lambda \beta} d_\nu h d^{\mu\alpha} h_{\rho} \)

(c) Terms of the type \( uu\bar{u}\bar{u} : \) we have terms of the type \( uud\bar{u}, uu\bar{d}\bar{u}, ud\bar{u}\bar{u}, u\bar{d}u\bar{u} \); there are 23 terms of these types. The terms of the type \( d\bar{u}u\bar{u}\bar{u} \) can be written, up to a total derivative, as linear combination of these 23 terms.

\( N_{10,1} = u_\mu u_\rho d^\mu \bar{u}^\nu d \cdot \bar{u} \)
\( N_{10,2} = u_\mu u_\rho d^\mu \bar{u}^\nu d^\rho \bar{u} \)
\( N_{10,3} = u_\mu u_\rho d^\mu \bar{u}^\nu d_\rho \bar{u} \)
\( N_{10,4} = u_\mu u_\rho d^\rho \bar{u}^\nu d^\mu \bar{u} \)
\( N_{11,1} = u_\mu u_\nu \bar{u}^\mu d^\nu \bar{u} \)
\( N_{11,2} = u_\mu u_\nu d^\nu \bar{u}^\mu \bar{u} \)
\( N_{11,3} = u_\mu u_\nu \bar{u}^\nu d^\mu \bar{u} \)
\( N_{11,4} = u_\mu u_\nu d^\mu \bar{u}^\nu \bar{u} \)
\( N_{12,1} = u_\mu d^\mu \bar{u}^\nu \bar{u}_\rho d \cdot \bar{u} \)
\( N_{12,2} = u_\mu d^\mu \bar{u}^\nu \bar{u}_\rho d^\rho \bar{u} \)
\( N_{12,3} = u_\mu d^\mu \bar{u}^\nu \bar{u}_\rho d^\nu \bar{u}_\nu \)
\( N_{12,4} = u_\mu d^\nu \bar{u}^\nu \bar{u}_\rho d \cdot \bar{u} \)
\( N_{12,5} = u_\mu d^\nu \bar{u}^\nu \bar{u}_\rho d_\nu \bar{u} \)
\( N_{12,6} = u_\mu d^\rho \bar{u}^\nu \bar{u}_\rho d^\nu \bar{u}_\nu \)
\[ \begin{align*}
N_{12,7} &= u_\mu \cdot u \tilde{u}^\mu \cdot \tilde{u}^\rho \\
N_{12,9} &= u_\mu d_\alpha u_\beta \tilde{u}^\mu \tilde{d}^\beta \tilde{u}^\alpha \\
N_{12,11} &= u_\mu d_\nu u_\rho \tilde{d}^\nu \tilde{u}^\rho \\
N_{12,13} &= u_\mu \cdot u \tilde{u}_\mu d^\rho \tilde{u}^\rho \\
N_{12,8} &= u_\mu d_\alpha u_\beta \tilde{u}^\mu d^\alpha \tilde{u}^\beta \\
N_{12,10} &= u_\mu \cdot u \tilde{u}^\nu d^\mu \tilde{u}_\nu \\
N_{12,12} &= u_\mu d_\rho u_\nu \tilde{d}^\rho \tilde{u}^\nu \\
N_{12,14} &= u_\mu d_\nu u_\rho \tilde{d}^\nu \tilde{u}^\rho \\
N_{12,15} &= u_\mu d_\rho u_\nu \tilde{d}^\rho \tilde{u}^\nu \\
N_{13,1} &= u_\mu d^\mu \tilde{d}^\rho \tilde{u}^\alpha \tilde{u}_\beta \\
N_{13,2} &= u_\mu d^\nu d \cdot u \tilde{u}_\mu \tilde{u}^\nu. 
\end{align*} \]

The next step is to consider the gauge variations \( dQ N_j \) for all \( 43 + 83 + 23 = 149 \) terms from the three sectors described above. Some of these terms can be eliminated with coboundaries of the type \( dQ B \); there are 26 such coboundaries: we can make

\[ c_{2,2} = 0 \]
\[ c_{3,k} = 0, \quad k = 5, 8 \]
\[ c_{4,k} = 0, \quad k = 3, 10, 11, 12 \]
\[ c_{5,k} = 0, \quad k = 14, \ldots, 19 \]
\[ c_{6,k} = 0, \quad k = 4, 5, 13 \]
\[ c_{7,k} = 0, \quad k = 1, 3, 4, 6, 8, 11, 12, 16, 17 \]

so we are left with 123 independent terms. We impose the condition that the sum \( \sum c_{j,k} dQ N_{j,k} \) is a total derivative. In principle we should make an ansatz for the total derivative also, but we can simplify the analysis is we observe that one can write

\[ dQ N = i u_\mu X^\mu + i h_{\alpha\beta} X^{\alpha\beta} + \text{total derivative}; \]

indeed for the terms in the sector (a) we have to “move” all derivatives appearing on the \( u \) factor on the \( h \) factors, up to a total derivative. In the (c) sector we “move” all derivatives appearing on the \( h_{\alpha\beta} \) factor on the \( u, \tilde{u} \) factors, up to a total derivative. Sector (b) is a mixed one i.e. we have to perform both tricks on various terms.

Let us make a general ansatz

\[ N^\mu = u_\nu R^{\mu\nu} + d_\nu u_\rho R^{\mu\nu\rho} + h_{\alpha\beta} S^{\mu,\alpha\beta} + d_\nu h_{\alpha\beta} S^{\mu\nu,\alpha\beta} \]

with \( R^{\mu\nu}, R^{\mu\nu\rho} \sim hhh \) and \( S^{\mu,\alpha\beta}, S^{\mu\nu,\alpha\beta} \sim uu\tilde{u} \). It is convenient to split

\[ R^{\mu\nu\rho} = R^{\mu\nu\rho}_+ + R^{\mu\nu\rho}_- \]

where \( R^{\mu\nu\rho}_\pm \) are symmetric (resp. anti-symmetric) in \( \mu \leftrightarrow \nu \). Then we write

\[ d_\nu u_\rho R^{\mu\nu\rho}_- = d_\nu (u_\rho R^{\mu\nu\rho}_-) - u_\rho d_\nu R^{\mu\nu\rho}_- \]

and notice that the first term can be neglected because it gives a null contribution in \( d_\mu N^\mu \) and the second term can be absorbed in the first term of (5.9). So, we can assume that \( R^{\mu\nu\rho}_- \)
is symmetric in $\mu \leftrightarrow \nu$. In a similar way we can argue that $S^{\mu \nu, \alpha \beta}$ can be chosen symmetric in $\mu \leftrightarrow \nu$. It means that we can write

$$R^{\mu \rho} = \eta^{\mu \nu} R^{\nu \rho} + \tilde{R}^{\mu \rho}$$
$$S^{\mu \nu, \alpha \beta} = \eta^{\mu \nu} S^{\alpha \beta} + \tilde{S}^{\mu \nu, \alpha \beta}$$

(5.11)

where $\tilde{R}^{\mu \rho}$ and $\tilde{S}^{\mu \nu, \alpha \beta}$ do not contain the factor $\eta^{\mu \nu}$.

Then the equation (5.3) becomes equivalent to the following system:

$$X^{\mu} = d_{\nu} R^{\nu \mu}$$
$$R^{\mu \nu} + d_{\rho} R^{\rho \mu \nu} = 0$$
$$\tilde{R}^{\mu \rho} = 0$$
$$X^{\alpha \beta} = d_{\mu} S^{\mu \alpha \beta}$$
$$S^{\mu \alpha \beta} + d_{\nu} S^{\nu \mu \alpha \beta} = 0$$
$$\tilde{S}^{\mu \nu, \alpha \beta} = 0.$$  

(5.12)

It easy to derive from this system that

$$X^{\mu} = -\Box R^{\mu}, \quad X^{\alpha \beta} = -\Box S^{\alpha \beta}.$$  

(5.13)

So, we must consider first, the most general forms for $R^{\mu}$ and $S^{\alpha \beta}$. We have

$$R^{\mu} = \sum \lambda_{j} R_{j}^{\mu}$$

(5.14)

where

$$R_{1}^{\mu} = d^{\mu} h h^{2}, \quad R_{2}^{\mu} = d_{\mu} h h h^{\mu \nu},$$
$$R_{3}^{\mu} = d_{\mu} h^{\mu \nu} h^{2}, \quad R_{4}^{\mu} = d^{\mu} h h_{\alpha \beta} h^{\alpha \beta},$$
$$R_{5}^{\mu} = d_{\mu} h^{\alpha \beta} h_{\alpha \beta} h, \quad R_{6}^{\mu} = d^{\alpha \beta} h_{\alpha \beta} h,$$
$$R_{7}^{\mu} = d^{\nu} h_{\mu \nu} h h^{\mu \nu}, \quad R_{8}^{\mu} = d^{\nu} h^{\mu \rho} h u_{\rho},$$
$$R_{9}^{\mu} = d^{\mu} h^{\alpha \beta} h_{\alpha \beta} h^{\lambda}, \quad R_{10}^{\mu} = d_{\mu} h^{\alpha \beta} h_{\alpha \beta} h^{\lambda},$$
$$R_{11}^{\mu} = d^{\alpha \beta} h_{\alpha \beta} h_{\alpha \beta} h^{\lambda}, \quad R_{12}^{\mu} = d_{\nu} h^{\alpha \beta} h^{\mu \nu} h_{\alpha \beta},$$
$$R_{13}^{\mu} = d_{\alpha} h_{\nu \beta} h^{\mu \nu} h^{\alpha \beta}, \quad R_{14}^{\mu} = d_{\nu} h^{\alpha \beta} h^{\mu \nu} h_{\nu \beta}.$$  

(5.15)

Also we have

$$S^{\alpha \beta} = \sum \omega_{j} S_{j}^{\alpha \beta}$$

(5.16)

where

$$S_{1}^{\alpha \beta} = S_{\alpha \beta} (u^{\alpha} d^{\beta} u^{\mu} \tilde{u}_{\mu}), \quad S_{2}^{\alpha \beta} = S_{\alpha \beta} (u^{\alpha} d^{\mu} u^{\beta} \tilde{u}_{\mu}),$$
$$S_{3}^{\alpha \beta} = S_{\alpha \beta} (u^{\alpha} d \cdot u \tilde{u}^{\beta}), \quad S_{4}^{\alpha \beta} = S_{\alpha \beta} (u^{\alpha} d^{\beta} u^{\mu} \tilde{u}_{\mu}),$$
$$S_{5}^{\alpha \beta} = S_{\alpha \beta} (u^{\alpha} d^{\beta} u \tilde{u}^{\beta}), \quad S_{6}^{\alpha \beta} = S_{\alpha \beta} (u_{\mu} d^{\alpha} u^{\beta} \tilde{u}^{\beta}),$$
$$S_{7}^{\alpha \beta} = \eta^{\alpha \beta} u^{\mu} d_{\mu} u_{\nu} \tilde{u}^{\nu}, \quad S_{8}^{\alpha \beta} = \eta^{\alpha \beta} u^{\mu} d_{\nu} u_{\mu} \tilde{u}^{\nu},$$
$$S_{9}^{\alpha \beta} = \eta^{\alpha \beta} u^{\mu} d \cdot u \tilde{u}^{\beta}, \quad S_{10}^{\alpha \beta} = S_{\alpha \beta} (u^{\alpha} u_{\mu} d^{\beta} \tilde{u}^{\beta}),$$
$$S_{11}^{\alpha \beta} = S_{\alpha \beta} (u^{\alpha} u^{\mu} d_{\mu} \tilde{u}^{\beta}), \quad S_{12}^{\alpha \beta} = \eta^{\beta} u^{\mu} u^{\nu} d_{\nu} \tilde{u}^{\mu}.$$  

(5.17)
Next, we are left with a routine operation: to consider all 123 variables $c_{j,k}$ from the expression of $N$ and the 14 + 12 variables $\lambda_j, \omega_k$, exhibit $d_Q N$ in the form (5.18) and solve the system (5.13). We are left with a highly over-determined system of 256 equations for the 149 variables $c_{j,k}, \lambda_j, \omega_k$. After tedious computations one is left with 14 solutions

$$N = a_1 N_1 + a_2 (N_{3,1} + 2N_{3,3}) + a_3 (2N_{3,3} + N_{3,6}) + a_4 N_{4,5} + a_5 N_{4,8} + a_6 (2N_{5,2} + N_{5,8}) + a_7 (2N_{5,9} + N_{5,12}) + a_8 (N_{6,1} + 2N_{8,9}) + a_9 (N_{6,2} + 2N_{8,10}) + a_{10} (N_{6,16} + N_{8,17} + N_{8,27}) + a_{11} (N_{6,18} + N_{8,33} + N_{8,35}) + a_{12} (N_{10,4} + N_{12,8} - N_{12,14}) + a_{13} (-N_{10,1} + N_{11,2} + N_{12,12} - N_{12,15}) + a_{14} (N_{12,7} + N_{12,11} - N_{12,12} - N_{12,13} - N_{13,1} + N_{13,2})$$

and one can prove that they are total derivatives:

$$N = d_\mu N^\mu$$

where

$$N_1^\mu = \frac{1}{3} h^3 d^\mu h, \quad N_2^\mu = h d^\mu h h_{\alpha \beta} h^{\alpha \beta},$$

$$N_3^\mu = h^2 h_{\alpha \beta} d^\mu h^{\alpha \beta}, \quad N_4^\mu = \frac{1}{3} d^\mu h h_{\alpha \beta} h^{\alpha \beta},$$

$$N_5^\mu = \frac{1}{2} \left( h h_{\alpha \beta} h^\lambda h^{\alpha \beta} - \frac{1}{3} d^\mu h h_{\alpha \beta} h^{\alpha \beta} \right), \quad N_6^\mu = d^\mu h h_{\alpha \beta} h_{\rho \sigma} h^{\rho \sigma},$$

$$N_7^\mu = h^{\alpha \beta} h_{\beta \rho} h_{\alpha \beta} d^\mu h^{\rho \beta}, \quad N_8^\mu = \frac{1}{2} \left( u_{\nu \mu} d^\nu h^2 + 2 u_{\nu \mu} \tilde{u}^\nu d^\mu h - d^\mu u^\nu \tilde{u}^\nu h^2 \right),$$

$$N_9^\mu = \frac{1}{2} \left( u_{\nu \mu} d^\nu h^2 + 2 u_{\nu \mu} \tilde{u}^\nu d^\mu h - d^\mu u^\nu \tilde{u}^\nu h^2 \right),$$

$$N_{10}^\mu = \frac{1}{2} \left( u_{\nu \mu} d^\nu h_{\rho \sigma} + u_{\nu \mu} \tilde{u}^\nu d^\rho h_{\sigma \rho} + u_{\nu \mu} \tilde{u}^\nu d^\rho h_{\sigma \rho} - d^\mu u_{\nu \mu} \tilde{u}^\nu h_{\rho \sigma} \right),$$

$$N_{11}^\mu = \frac{1}{2} \left( u_{\rho \mu} d^\rho d^\mu + u_{\rho \mu} \tilde{u}^\rho d^\mu + u_{\rho \mu} \tilde{u}^\rho d^\mu - d^\mu u_{\rho \mu} \tilde{u}^\rho \tilde{u}^\rho \right),$$

$$N_{12}^\mu = \frac{1}{2} \left( u_{\rho \mu} d^\rho d^\mu + u_{\rho \mu} \tilde{u}^\rho d^\mu + u_{\rho \mu} \tilde{u}^\rho d^\mu - d^\mu u_{\rho \mu} \tilde{u}^\rho \tilde{u}^\rho \right),$$

$$N_{13}^\mu = u_{\alpha \beta} d^\mu u_{\alpha \beta} \tilde{u}^\mu \tilde{u}^\beta, \quad N_{14}^\mu = u_{\alpha \beta} \tilde{u}^\mu \tilde{u}^\beta - u_{\mu} d_{\alpha} u_{\alpha \beta} \tilde{u}^\beta. \quad (5.20)$$

The expresions $N_{1} - N_{12}$ from (5.18) are grouping in a clever way all terms of the form

$$a_1 a_2 d_\mu a_3 d^\mu a_4$$

from the lists (5.4), (5.5) and (5.6) with the exception of $N_{12,5}$. In fact the general structure is

$$a_1 (d^\mu a_2 d_\mu a_3 d^\mu a_4 + a_2 d_\mu a_3 d^\mu a_4 + d^\mu a_2 d_\mu a_4)$$

$$\simeq \frac{1}{2} a_1 \Box (a_2 a_3 a_4) = \text{total divergence} + \frac{1}{2} \Box a_1 a_2 a_3 a_4 \simeq \text{total divergence} \quad (5.21)$$

where, as before, $\simeq$ means modulo equations of motion (Klein-Gordon).
So we have the important result:

**Theorem 5.1** Let $N$ be a quadri-linear polynomial in the variables verifying

$$\omega(N) = 6, \quad gh(N) = 0$$

and the gauge invariance condition

$$d_QN = i d_\mu N^\mu$$ (5.23)

for some $N^\mu$. Then $N$ is a coboundary i.e.

$$N \simeq d_QB + d_\mu B^\mu.$$ (5.24)

This theorem proves that, after we eliminate the anomaly (3.37) with the finite renormalization (3.48), the arbitrariness of the chronological product $T(T(x_1), T(x_2))$ is trivial in the quadri-linear sector i.e. a coboundary. Presumably this is true also for all $N^I$. So, the only arbitrariness left is in the tri-linear sector where we obtain the interaction Lagrangian (2.41) + (2.42). This arbitrariness amounts to a redefinition of the coupling constant. The details are in

**Theorem 5.2** Let $N$ be a tri-linear polynomial in the variables verifying

$$\omega(N) = 6, \quad gh(N) = 0$$

and the gauge invariance condition (5.23). Then $N$ is a coboundary i.e.

$$N = d_QB + d_\mu B^\mu.$$ (5.26)

**Proof:** (i) First, we note that for tri-linear finite renormalizations we can have in (4.2) a supplementary term

$$R(T^I(x_1), T^J(x_2)) = \cdots + \partial_\mu \partial_\nu \partial_\rho (x_1 - x_2) \, N(T^I, T^J)^{\mu\nu\rho}(x_2)$$ (5.27)

where

$$\omega(N(T^I, T^J)^{\mu\nu\rho}) = 3, \quad gh(N(T^I, T^J)^{\mu\nu\rho}) = |I| + |J|. \quad (5.28)$$

First we consider the expression $N(T, T)^{\mu\nu\rho}$; because is of ghost number 0 it must have the form:

$$N(T, T)^{\mu\nu\rho} \sim hhh + hu\tilde{u}. \quad (5.29)$$

The existence of such an expression is prevented by Lorentz covariance, so

$$N(T, T)^{\mu\nu\rho} = 0. \quad (5.30)$$

Similarly we must have

$$N(T^\alpha, T)^{\mu\nu\rho} \sim uhh + uu\tilde{u}$$ (5.31)

and, again, the existence of such an expression is prevented by Lorentz covariance, so

$$N(T^\alpha, T)^{\mu\nu\rho} = 0.$$ (5.32)
Next we have
\[ N(T^\alpha, T^\beta)_{\mu\nu} \sim uu \] \hfill (5.33)
and, as before, the existence of such an expression is prevented by Lorentz covariance, so
\[ N(T^\alpha, T^\beta)_{\mu\nu} = 0, \quad N(T^\alpha, T^\beta)_{\mu\nu} = 0. \] \hfill (5.34)
Finally,
\[ N(T^\alpha, T^\beta)_{\mu\nu} \sim uu \] \hfill (5.35)
and the existence of such an expression is prevented by Lorentz covariance, so
\[ N(T^\alpha, T^\beta)_{\mu\nu} = 0, \quad N(T^\alpha, T^\beta)_{\mu\nu} = 0. \] \hfill (5.36)

So we have:
\[ N(T^I, T^J)_{\mu\nu} = 0. \] \hfill (5.37)

(ii) Next, we observe that the assertion of theorem 4.1 stays true for the case when the expression \( N(T^I, T^J)_{\mu\nu} \) is tri-linear in the fields variables. Indeed because gh\((N(T^I, T^J)_{\mu\nu}) = |I| + |J| \) we must have
\[ N(T^I, T^J)_{\mu\nu} = 0, \quad |I| + |J| \geq 4 \] \hfill (5.38)
because an expression tri-linear in the fields can have the ghost number at most 3. It follows that we have only the cases (i) - (vi) of theorem 4.1, but in these cases we did not used the explicit form of \( N(T^I, T^J)_{\mu\nu} \); we have proved that we can fix
\[ N(T^I, T^J)_{\mu\nu} = 0 \] \hfill (5.39)
only through general cohomological arguments. The same observation applies to the expressions \( N(T^I, T^J)_{\mu} \); we need only the steps (i) - (vi) of theorem 4.4 so we also have
\[ N(T^I, T^J)_{\mu} = 0 \] \hfill (5.40)
in this case. It follows that we are reduced to the study of equation (5.3) for a tri-linear expression \( N \). Such an equation leads to a solution cohomological to (2.41) + (2.42) - see [18], [12].

So, at least in the second order of perturbation theory, we have proved that quantum gravity is renormalizable as in the pure Yang-Mills case!

These is a slight discrepancy with the pure Yang-Mills case. In [11] we have proved that the finite renormalizations used to eliminate the anomalies in the second order of the perturbation theory can be obtained by a clever redefinition of the chronological products \( T(\xi_{a,\mu}(x_1), \xi_{b,\nu}(x_2)) \).

In the quantum gravity formalism used above we have the follow possible redefinitions:
\[
N_1(h_{\mu_1\nu_1,\lambda_1}(x_1), h_{\mu_2\nu_2,\lambda_2}(x_2)) = \frac{1}{2} \left( \eta_{\mu_1\mu_2} \eta_{\nu_1\nu_2} + \eta_{\mu_1\nu_2} \eta_{\nu_1\mu_2} \right) \eta_{\lambda_1\lambda_2} \delta(x_1 - x_2)
\]
\[
N_2(h_{\mu_1\nu_1,\lambda_1}(x_1), h_{\mu_2\nu_2,\lambda_2}(x_2)) = \eta_{\mu_1\nu_1} \eta_{\mu_2\nu_2} \eta_{\lambda_1\lambda_2} \delta(x_1 - x_2)
\]
\[
N_3(h_{\mu_1\nu_1,\lambda_1}(x_1), h_{\mu_2\nu_2,\lambda_2}(x_2)) = \frac{1}{4} \left( \eta_{\mu_1\lambda_1} \eta_{\mu_2\lambda_2} \eta_{\nu_1\nu_2} + \eta_{\nu_1\lambda_1} \eta_{\mu_2\lambda_2} \eta_{\mu_1\nu_2} \right)
\]
\[ N_4(h_{\mu_1\nu_1,\lambda_1}(x_1), h_{\mu_2\nu_2,\lambda_2}(x_2)) = \frac{1}{4} (\eta_{\mu_1\lambda_2} \eta_{\mu_2\lambda_1} \eta_{\nu_1\nu_2} + \eta_{\nu_1\lambda_2} \eta_{\mu_2\lambda_1} \eta_{\mu_1\nu_2} + \eta_{\mu_1\lambda_2} \eta_{\nu_2\lambda_2} + \eta_{\nu_1\lambda_2} \eta_{\nu_2\lambda_1} \eta_{\mu_1\mu_2}) \delta(x_1 - x_2) \]

\[ N_5(h_{\mu_1\nu_1,\lambda_1}(x_1), h_{\mu_2\nu_2,\lambda_2}(x_2)) = \frac{1}{4} (\eta_{\mu_1\nu_1} \eta_{\mu_2\lambda_2} \eta_{\nu_2\lambda_2} + \eta_{\mu_1\nu_1} \eta_{\nu_2\lambda_2} \eta_{\nu_1\lambda_2} + \eta_{\mu_2\nu_2} \eta_{\mu_1\lambda_2} \eta_{\nu_1\lambda_2}) \delta(x_1 - x_2) \] (5.41)

\[ N_6(u_{\mu_1,\nu_1}(x_1), \bar{u}_{\mu_2,\nu_2}(x_2)) = \eta_{\mu_1\mu_2} \eta_{\nu_1\nu_2} \delta(x_1 - x_2) \]

\[ N_7(u_{\mu_1,\nu_1}(x_1), \bar{u}_{\mu_2,\nu_2}(x_2)) = \eta_{\mu_1\nu_2} \eta_{\nu_1\mu_2} \delta(x_1 - x_2) \]

\[ N_8(u_{\mu_1,\nu_1}(x_1), \bar{u}_{\mu_2,\nu_2}(x_2)) = \eta_{\mu_1\nu_1} \eta_{\mu_2\nu_2} \delta(x_1 - x_2) \] (5.42)

\[ N_9(u_{\mu,\alpha\beta}(x_1), \bar{u}_\nu(x_2)) = \eta_{\mu\nu} \eta_{\alpha\beta} \delta(x_1 - x_2) \]

\[ N_{10}(u_{\mu,\alpha\beta}(x_1), \bar{u}_\nu(x_2)) = \frac{1}{2} (\eta_{\mu\alpha} \eta_{\nu\beta} + \eta_{\mu\beta} \eta_{\nu\alpha}) \delta(x_1 - x_2) \] (5.43)

\[ N_{11}(u_{\mu,\alpha\beta}(x_1), \bar{u}_{\nu,\rho}(x_2)) = \frac{1}{2} \eta_{\mu\nu} (\eta_{\alpha\rho} \partial_\beta + \eta_{\beta\rho} \partial_\alpha) \delta(x_1 - x_2) \]

\[ N_{12}(u_{\mu,\alpha\beta}(x_1), \bar{u}_{\nu,\rho}(x_2)) = \frac{1}{2} \eta_{\mu\rho} (\eta_{\alpha\nu} \partial_\beta + \eta_{\beta\nu} \partial_\alpha) \delta(x_1 - x_2) \]

\[ N_{13}(u_{\mu,\alpha\beta}(x_1), \bar{u}_{\nu,\rho}(x_2)) = \frac{1}{2} \eta_{\nu\rho} (\eta_{\alpha\mu} \partial_\beta + \eta_{\beta\mu} \partial_\alpha) \delta(x_1 - x_2) \]

\[ N_{14}(u_{\mu,\alpha\beta}(x_1), \bar{u}_{\nu,\rho}(x_2)) = \eta_{\mu\nu} \eta_{\alpha\beta} \partial_\rho \delta(x_1 - x_2) \]

\[ N_{15}(u_{\mu,\alpha\beta}(x_1), \bar{u}_{\nu,\rho}(x_2)) = \eta_{\mu\rho} \eta_{\alpha\beta} \partial_\nu \delta(x_1 - x_2) \]

\[ N_{16}(u_{\mu,\alpha\beta}(x_1), \bar{u}_{\nu,\rho}(x_2)) = \eta_{\nu\rho} \eta_{\alpha\beta} \partial_\mu \delta(x_1 - x_2) \] (5.44)

However, we did not succeed to exhibit the finite renormalization (3.48) using these redefinitions, as in the pure YM case.
6 Conclusions

The main result of this paper is that perturbative quantum gravity is in fact a renormalizable theory, at least in the second order of the perturbation theory. Mathematically it means that a certain (relative) cohomology problem is trivial. The proof is brute force, based on the elaboration of a long list of possible terms appearing in the generic form of a cocycle and reducing the cocycle equation to a long system of linear equations which must be solved. It is clear that this method becomes unmanageable in higher orders of the perturbation theory, so the next goal would be to find more sophisticated ways to solve the cohomology problem appearing in higher orders of perturbation theory.

References

[1] N. N. Bogoliubov, D. Shirkov, “Introduction to the Theory of Quantized Fields”, John Wiley and Sons, 1976 (3rd edition)

[2] N. N. Bogolubov, A. A. Logunov, A.I. Oksak, I. Todorov, “General Principles of Quantum Field Theory”, Kluwer 1989

[3] M. Dütsch, “From Classical Field Theory to Perturbative Quantum Field Theory”, Progress in Mathematical Physics 74, Springer 2019

[4] M. Dütsch, K. Fredenhagen, “Algebraic Quantum Field Theory, Perturbation Theory, and the Loop Expansion”, arXiv: hep-th/0001129, Commun. Math. Phys. 219 (2001) 5 - 30

[5] H. Epstein, V. Glaser, “The Rôle of Locality in Perturbation Theory”, Ann. Inst. H. Poincaré 19 A (1973) 211-295

[6] V. Glaser, “Electrodynamique Quantique”, L’enseignement du 3e cycle de la physique en Suisse Romande (CICP), Semestre d’hiver 1972/73

[7] D. R. Grigore, “On the Quantization of the Linearized Gravitational Field”, [hep-th/9905190], Class. Quant. Grav. 17 (2000) 319-344

[8] D. R. Grigore, “A Generalization of Gauge Invariance”, arxiv: hep-th/1612.04998, Journal of Mathematical Physics 58 (2017) 082303

[9] D. R. Grigore, “Anomaly-Free Gauge Models: A Causal Approach”, hep-th/1804.08276, Romanian Journ. Phys. 64 (2019) 102

[10] D. R. Grigore, “On the Super-Renormalizability of Quantum Gravity in the Linear Approximation”, [arXiv:1905.05410v1 [hep-th]], Romanian Journ. Phys. 65 (2020) 101

[11] D. R. Grigore, “Wick Theorem and Hopf Algebra Structure in Causal Perturbative Quantum Field Theory”, [arXiv:2202.08056v2 [hep-th]]
[12] D. R. Grigore, G. Scharf, “Massive Gravity as a Quantum Gauge Theory”, arXiv: hep-th/0404157, General Relativity and Gravitation 37 (2005) 1075-1096

[13] K. Hepp, “Renormalization Theory”, in “Statistical Mechanics and Quantum Field Theory” pp. 429 - 500, (Les Houches 1970), C. DeWitt-Morette, Raymond Stora (eds.), Gordon and Breach 1971

[14] J. Polchinski, “Renormalization and Effective Lagrangians”, Nucl. Phys. B 231 (1984) 269 - 295

[15] G. Popineau, R. Stora, “A Pedagogical Remark on the Main Theorem of Perturbative Renormalization Theory”, Nuclear Physics B 912 (2016) 70 - 78

[16] M. Salmhofer, “Renormalization: An Introduction”, (Theoretical and Mathematical Physics) Springer 1999

[17] G. Scharf, “Finite Quantum Electrodynamics: The Causal Approach”, (second edition) Springer, 1995; (third edition) Dover, 2014

[18] G. Scharf, “Quantum Gauge Theories. A True Ghost Story”, John Wiley, 2001, “Quantum Gauge Theories - Spin One and Two”, Google books, 2010 and “Gauge Field Theories: Spin One and Spin Two, 100 Years After General Relativity”, Dover 2016

[19] R. Stora, “Lagrangian Field Theory”, Les Houches lectures, Gordon and Breach, N.Y., 1971, C. De Witt, C. Itzykson eds.

[20] O. Steinmann, “Perturbation Expansions in Axiomatic Field Theory”, Lect. Notes in Phys. 11, Springer, 1971

[21] A. S. Wightman, L. Gårding, “Fields as Operator-Valued Distributions in Relativistic Quantum Field Theory”, Arkiv Fysik 28 (1965) 129-184