Modules with minimax Cousin cohomologies*

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Abstract. Let $R$ be a commutative Noetherian ring with non-zero identity and let $X$ be an arbitrary $R$-module. In this paper, we show that if all the cohomology modules of the Cousin complex for $X$ are minimax, then the following hold for any prime ideal $p$ of $R$ and for every integer $n$ less than $X$—the height of $p$:

(i) the $n$th Bass number of $X$ with respect to $p$ is finite;
(ii) the $n$th local cohomology module of $X_p$ with respect to $pR_p$ is Artinian.

Introduction

Throughout $R$ will denote a commutative Noetherian ring with non-zero identity, $X$ an arbitrary $R$-module which is not necessarily finite (i.e., finitely generated), and $M$ a non-zero finite $R$-module. For basic results, notations and terminology not given in this paper, the reader is referred to [2], [3], and [12].

The notion of the Cousin complex for an $R$-module $X$ was introduced by Sharp [13] as an analogue of Hartshorne [8]. The Cousin cohomologies (i.e., the cohomology modules of the Cousin complex) have been studied by several authors. Sharp used the vanishing of Cousin cohomologies for investigating the Cohen-Macaulay property, Serre’s $S_n$-condition, and the vanishing of Bass numbers of $X$ in [13], [14], and [15]. Dibaei, Tousi, Jafari, and Kawasaki, in [4], [5], [6], [7], and [10], worked on the finiteness of

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Cousin cohomologies and, in [11, Proposition 9.3.5], Lipman, Nayak, and Sastry generalized their results to complexes on formal schemes.

Sharp, in [14, Theorem 2.4], showed that $M$ is Cohen-Macaulay if and only if the Cousin complex for $M$ is exact. Thus we get the following theorem.

**Theorem 1.** Let $M$ be a non-zero finite $R$-module such that all the cohomology modules of the Cousin complex for $M$ are zero. Then the followings hold for any prime ideal $p$ of $R$ and for every integer $n$ less than $X$—the height of $p$.

(i) The $n$th Bass number of $M$ with respect to $p$ is zero;
(ii) The $n$th local cohomology module of $M_p$ with respect to $pR_p$ is zero.

Now, it is natural to ask whether a similar statement is valid if ‘zero’ is replaced by ‘finite’.

**Question 1.** Let $X$ be an arbitrary $R$-module such that all the cohomology modules of the Cousin complex for $X$ are finite. Do the followings hold for any prime ideal $p$ of $R$ and for every integer $n$ less than $X$–height of $p$?

(i) The $n$th Bass number of $X$ with respect to $p$ is finite;
(ii) The $n$th local cohomology module of $X_p$ with respect to $pR_p$ is finite.

In this paper, we answer the above question. We show that the first part of Question 1 is true. In fact, in Theorem 2, we prove that the $n$th Bass number of $X$ with respect to $p$ is finite for any prime ideal $p$ of $R$ and for every integer $n$ less than $X$–height of $p$, when all the cohomology modules of the Cousin complex for $X$ are minimax. Even though the second part of Question 1 is false in general, we show in Theorem 3 that if all the cohomology modules of the Cousin complex for $X$ are minimax, then the $n$th local cohomology module of $X_p$ with respect to $pR_p$ is Artinian for any prime ideal $p$ of $R$ and for every integer $n$ less than $X$–height of $p$.

1. **Main results**

Suppose that $X$ is an arbitrary $R$-module. Recall that, for a prime ideal $p$ of $\text{Supp}_R(X)$, the $X$–height of $p$ is defined to be $\text{ht}_X(p) = \dim_R(X_p)$. Let $i$ be a non-negative integer and set $U^i(X) = \{ p \in \text{Supp}_R(X) : \text{ht}_X(p) \geq i \}$. Then $\text{Supp}_R(X) = U^0(X)$, $U^i(X) \supseteq U^{i+1}(X)$, and $U^i(X) - U^{i+1}(X) (= \{ p \in \text{Supp}_R(X) : \text{ht}_X(p) = i \})$ is low with respect to $U^i(X)$ (i.e., each member of $U^i(X) - U^{i+1}(X)$ is a minimal member of $U^i(X)$ with respect to inclusion). The Cousin complex $C_R(X)$ for $X$ is of the form

$$C_R(X) : 0 \xrightarrow{d^{-2}} X \xrightarrow{d^{-1}} X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{i-2}} X^{i-1} \xrightarrow{d^{i-1}} X^i \xrightarrow{d^i} \cdots$$
where, for all $i \geq 0$,

- $X^i = \bigoplus_{p \in U^i(X)} (\text{Coker } d^{i-2})^i_p$ and
- $d^{i-1}(x) = \left\{ \frac{x + \text{Im} d^{i-2}}{1} \right\}_{p \in U^i(X) - U^{i+1}(X)}$ for every element $x$ of $X^{i-1}$; and satisfies

- $\text{Supp}_R(X^i) \subseteq U^i(X)$,
- $\text{Supp}_R(\text{Coker } d^{i-2}) \subseteq U^i(X)$, and
- $\text{Supp}_R(\text{H}^{i-1}(C_R(X))) \subseteq U^{i+1}(X)$

(see [13] for details). Here, we use the notations $C^{i-2} := \text{Coker } d^{i-2}$ and $\text{H}^{i-1} := \text{H}^{i-1}(C_R(X))$ for all $i \geq 0$.

Recall that an $R$-module $X$ is said to be minimax, if there is a finite submodule $X'$ of $X$ such that $\frac{X}{X'}$ is Artinian [3]. Thus the class of minimax modules includes all finite and all Artinian modules. Note that, for any short exact sequence

$$0 \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow 0$$

of $R$-modules, $X$ is minimax if and only if $X'$ and $X''$ are both minimax [1, Lemma 2.1].

In the following, we state our first main result. Note that, for an $R$-module $X$ and a prime ideal $p$ of $R$, the number

$$\mu^n(p, X) = \dim_{\frac{R_p}{pR_p}} (\text{Ext}^n_{\frac{R_p}{pR_p}} (\frac{R_p}{pR_p}, X_p))$$

is the $n$th Bass number of $X$ with respect to $p$.

**Theorem 2.** Let $X$ be an arbitrary $R$-module such that $\text{H}^i$ is minimax for all $i$. Then $\mu^n(p, X)$ is finite for all prime ideals $p$ of $R$ and all $n < \text{ht}_X(p)$.

**Proof.** Let $p$ be a prime ideal of $R$ and let $n < \text{ht}_X(p)$. Let $i$ be an integer such that $0 \leq i \leq n$. By considering the short exact sequences

$$0 \rightarrow \frac{C^{i-2}}{H^{i-1}} \rightarrow X^i \rightarrow C^{i-1} \rightarrow 0 \quad (1)$$

and

$$0 \rightarrow H^{i-1} \rightarrow C^{i-2} \rightarrow \frac{C^{i-2}}{H^{i-1}} \rightarrow 0, \quad (2)$$

we have the long exact sequences

$$0 \rightarrow \text{Hom}_R(\frac{R}{p}, \frac{C^{i-2}}{H^{i-1}}) \rightarrow \text{Hom}_R(\frac{R}{p}, X^i) \rightarrow \text{Hom}_R(\frac{R}{p}, C^{i-1})$$
\[ \cdots \to \Ext^1_R\left( \frac{R}{p}, C^{i-2} \right) \to \Ext^1_R\left( \frac{R}{p}, X^i \right) \to \Ext^1_R\left( \frac{R}{p}, C^{i-1} \right) \]

\[ \to \cdots \]

\[ \cdots \to \Ext^n_{R}\left( \frac{R}{p}, H^{i-1} \right) \to \Ext^n_{R}\left( \frac{R}{p}, C^{i-2} \right) \to \Ext^n_{R}\left( \frac{R}{p}, C^{i-1} \right) \]

Since $H^i$ is minimax for all $i$, $\Ext^n_{R}(\frac{R}{p}, H^{i-1})$ is minimax for all $0 \leq i \leq n$. On the other hand, by [13, Lemma 4.5], $\Ext^n_{R}(\frac{R}{p}, X^i) = 0$ for all $0 \leq i \leq n$. Thus, from the above long exact sequences, $\Ext^n_{R}(\frac{R}{p}, C^{i-2})$ is minimax whenever $\Ext^n_{R}(\frac{R}{p}, C^{i-1})$ is minimax. Hence $\Ext^n_{R}(\frac{R}{p}, C^{-2})$ is minimax. Therefore $\Ext^n_{R}(\frac{R}{p}, X)$ is minimax. Thus there is a finite submodule $E'$ of $\Ext^n_{R}(\frac{R}{p}, X)$ such that $\frac{\Ext^n_{R}(\frac{R}{p}, X)}{E'}$ is Artinian. Since $\mu^n(p, X)$ is finite as desired.

For an $R$-module $X$ and an ideal $a$ of $R$, we write $H^n_a(X)$ as the $n$th local cohomology module of $X$ with respect to $a$. An important problem in commutative algebra is to determine when $H^n_a(X)$ is Artinian. In the second main result of this paper, we show that for an arbitrary $R$-module $X$ (not necessarily finite) with minimax Cousin cohomologies, $H^n_{pR_p}(X_p)$ is Artinian for all prime ideals $p$ of $R$ and all $n < \text{ht}_X(p)$, which is related to the third of Huneke’s four problems in local cohomology modules [9].
Theorem 3. Let $X$ be an arbitrary $R$-module such that $H^i$ is minimax for all $i$. Then $H^n_{pR_p}(X_p)$ is Artinian for all prime ideals $p$ of $R$ and all $n < \text{ht}_X(p)$.

Proof. The proof is similar to that of Theorem 2. We bring it here for the sake of completeness. Let $p$ be a prime ideal of $R$ and let $n < \text{ht}_X(p)$. Let $i$ be an integer such that $0 \leq i \leq n$. By considering the short exact sequences (1) and (2), we have the long exact sequences

\[0 \to \Gamma_{pR_p}(C^i_{p-2}) \to \Gamma_{pR_p}(X^i_p) \to \Gamma_{pR_p}(C^i_{p-1}) \to \cdots \]

\[0 \to \Gamma_{pR_p}(H^i_{p-1}) \to \Gamma_{pR_p}(X^i_p) \to \Gamma_{pR_p}(C^i_{p-1}) \to \cdots \]

and

\[0 \to \Gamma_{pR_p}(H^i_{p-1}) \to \Gamma_{pR_p}(C^i_{p-2}) \to \Gamma_{pR_p}(C^i_{p-1}) \to \cdots \]

\[0 \to \Gamma_{pR_p}(H^i_{p-1}) \to \Gamma_{pR_p}(C^i_{p-2}) \to \Gamma_{pR_p}(H^i_{p-1}) \to \cdots \]

Since $H^i$ is minimax for all $i$, there is a finite submodule $H''^i$ of $H^i$ such that $H''^i$ is Artinian. Therefore, from the exact sequence

\[H^n_{pR_p}(H^i_{p-1}) \to H^n_{pR_p}(H^i_{p-1}) \to H^n_{pR_p}(H^i_{p-1}) \to \cdots \]
$H_{pR_p}^{n-i}(H_{p}^{i-1})$ is Artinian for all $0 \leq i \leq n$. On the other hand, by [13, Lemma 4.5], for all $0 \leq i \leq n$ and all $j \geq 0$, $\text{Ext}_{R}^{n-i}(R_j, X^i) = 0$ and so $H_{pR_p}^{n-i}(X^i) \cong (H_{p}^{n-i}(X^i))_p = 0$ because

$$H_{p}^{n-i}(X^i) \cong \lim_{j \to 0} \text{Ext}_{R}^{n-i}(R_j, X^i).$$

Thus, from the above long exact sequences, $H_{pR_p}^{n-i}(C_{p}^{i-2})$ is Artinian whenever $H_{pR_p}^{n-i-1}(C_{p}^{i-1})$ is Artinian. Hence $H_{pR_p}^{n}(C_{p}^{2})$ is Artinian. Therefore $H_{pR_p}^{n}(X_p)$ is Artinian.

The following corollaries are immediate applications of the above theorems.

**Corollary 1.** Let $X$ be an arbitrary $R$-module such that $H^i$ is finite for all $i$. Then

(i) $\mu^n(p, X)$ is finite and

(ii) $H_{pR_p}^{n}(X_p)$ is Artinian

for all prime ideals $p$ of $R$ and all $n < \text{ht}_X(p)$.

**Corollary 2.** Let $X$ be an arbitrary $R$-module such that $H^i$ is Artinian for all $i$. Then

(i) $\mu^n(p, X)$ is finite and

(ii) $H_{pR_p}^{n}(X_p)$ is Artinian

for all prime ideals $p$ of $R$ and all $n < \text{ht}_X(p)$.

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