A quantum bound on the thermodynamic description of gravity

Shahar Hod

The Ruppin Academic Center, Emeq Hefer 40250, Israel

and

The Hadassah Institute, Jerusalem 91010, Israel

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Abstract

The seminal works of Bekenstein and Hawking have revealed that black holes have a well-defined thermodynamic description. In particular, it is often stated in the physical literature that black holes, like mundane physical systems, obey the first law of thermodynamics: \( \Delta S = \Delta E/T_{\text{BH}} \), where \( T_{\text{BH}} \) is the Bekenstein-Hawking temperature of the black hole. In the present work we test the regime of validity of the thermodynamic description of gravity. In particular, we provide compelling evidence that, due to quantum effects, the first law of thermodynamics breaks down in the low-temperature regime \( T_{\text{BH}} \times r_H \lesssim (\hbar/r_H)^2 \) of near-extremal black holes (here \( r_H \) is the radius of the black-hole horizon).
I. INTRODUCTION

It is well known [1] that the thermodynamic description of mundane physical systems breaks down in the low-temperature regime $T \sim \hbar / R$ [2, 3], when the characteristic thermal wavelengths $\lambda_{\text{thermal}} \sim \hbar / T$ are no longer small on the scale $R$ set by the spatial size of the system. The physical properties of these low-temperature systems are dominated by quantum (rather than thermodynamic) effects. Thus, the thermodynamic description of mundane physical systems is known to be restricted to the high-temperature regime $[1]$

$$T \times R \gg \hbar .$$

Interestingly, black holes are unique in this respect. It is well known that the Bekenstein-Hawking temperature [4, 5] of the Schwarzschild black hole is given by $T_{\text{BH}} = \hbar / 4 \pi r_{\text{H}}$, where $r_{\text{H}} = 2M$ is the radius of the black-hole horizon. Thus, Schwarzschild black holes are characterized by the relation $T_{\text{BH}} \times r_{\text{H}} \sim \hbar$.

Moreover, the Bekenstein-Hawking temperature of Kerr black holes is given by

$$T_{\text{BH}} = \frac{\hbar (r_+ - r_-)}{4 \pi r_+^2},$$

where $r_\pm = M + (M^2 - a^2)^{1/2}$ are the black-hole (outer and inner) horizon radii [6]. Thus, rapidly-rotating Kerr black holes in the near-extremal $r_+ - r_- \ll r_+$ regime are characterized by the strong inequality

$$T_{\text{BH}} \times r_+ \ll \hbar .$$

It is quite remarkable that black holes have a well defined thermodynamic behavior in the low-temperature regime [3], where mundane physical systems are governed by quantum effects and no longer have a self-consistent thermodynamic description.

One naturally wonders whether the thermodynamic description of black holes is valid all the way down to the zero temperature $T \times r_+ \to 0$ limit? In order to address this interesting question, we shall analyze in this paper the regime of validity of the first law of thermodynamics for black holes.
II. BLACK HOLES AND THE FIRST LAW OF THERMODYNAMICS

For a closed physical system with a well defined temperature $T$, a change $\Delta E$ in the energy of the system results with a change

$$\Delta S = \frac{\Delta E}{T}$$

in the entropy of the system. This famous differential relation, known as the first law of thermodynamics, is one of the most important features of the thermodynamic description of mundane physical systems.

It is often stated in the physical literature (see e.g., [9]) that black holes, like ordinary thermodynamic systems, also obey this law. However, the regime of validity of the standard thermodynamic description of black holes has never been discussed in the literature. In the present paper we would like to raise the following intriguing question: Do black holes always obey the first law of thermodynamics?

In order to address this interesting question, it proves useful to examine carefully the assumptions made in the physical literature in deriving the first law of black-hole thermodynamics. The characteristic Bekenstein-Hawking entropy of a black hole is given by a quarter of its horizon area [4, 5]:

$$S_{BH} = \frac{A}{4\hbar}.$$  

(5)

Remembering that the surface area of Kerr black holes [10] is given by $A = 4\pi(r_+^2 + a^2) = 8\pi M r_+$, one finds [11]

$$S_{BH} = \frac{2\pi M \{ M + \sqrt{M^2 - (J/M)^2} \}}{\hbar}.$$  

(6)

Consider now a physical process which changes the energy (mass) of the black hole by a small amount $\Delta E \ll M$ [12]. From Eq. (6) one finds that the resulting change in the Bekenstein-Hawking entropy of the black hole is given by the rather cumbersome expression

$$\frac{\hbar}{2\pi} \Delta S_{BH} = (M + \Delta E) \left\{ M + \Delta E + \left[ (M + \Delta E)^2 - \left( \frac{J}{M + \Delta E} \right)^2 \right]^{1/2} \right\} - M \left\{ M + \sqrt{M^2 - \left( \frac{J}{M} \right)^2} \right\} \frac{1}{\hbar}.$$  

(7)

A careful inspection of Eq. (7) reveals that if $M \Delta E \ll (r_+ - r_-)^2$ then, to leading-order in the small ratio $M \Delta E / (r_+ - r_-)^2$, the changes in the black-hole entropy and energy are related to each other by the standard first law of thermodynamics,

$$\Delta S_{BH} = \frac{\Delta E}{T_{BH}},$$  

(8)
where \( T_{\text{BH}} \) as given by (2) is the familiar Bekenstein-Hawking temperature of the black hole.

On the other hand, in the opposite regime \( M \Delta E \gg (r_+ - r_-)^2 \) one finds from (7) the non-standard relation

\[
\Delta S_{\text{BH}} = \frac{\sqrt{8\pi}}{\hbar} M^{3/2} \sqrt{\Delta E}
\]

(9)

between the changes in the physical parameters (entropy and energy) of the black hole.

One therefore concludes that, for black holes in Einstein gravity, the validity of the standard first law of thermodynamics, Eq. (8), is restricted to the regime

\[
\frac{(r_+ - r_-)^2}{r_+} \gg \Delta E \geq \Delta E_{\text{min}} \, ,
\]

(10)

where \( \Delta E_{\text{min}} \) is the smallest possible change in the energy (mass) of the black hole. Thus, in order to determine the regime of validity of the law in the context of black-hole physics, one should first determine the value of the fundamental physical parameter \( \Delta E_{\text{min}} \).

How small can \( \Delta E_{\text{min}} \), the minimal change in the energy (mass) of a black hole, be made? The answer to this question at the classical level was given by Christodoulou and Ruffini [13]: the capture of a point particle by a black hole is characterized by the relation \( \Delta E_{\text{min}} = 0 \) if the particle is captured at the black-hole horizon from a radial turning point of its motion. In this scenario the energy (as measured by asymptotic observers) of the absorbed particle is completely red-shifted. Substituting \( \Delta E_{\text{min}} = 0 \) into (10) one deduces that, at the classical level, the first law of thermodynamics may be valid all the way down to the extremal limit \( T_{\text{BH}} \to 0 \).

However, as emphasized by Bekenstein in his seminal work [4], the classical limit of a perfectly localized particle (a point particle) is physically unacceptable in a self-consistent quantum theory of relativity. In particular, due to the quantum uncertainty principle [14], the particle cannot be localized at the black-hole horizon without having a non-zero radial momentum (kinetic energy). Specifically, the Heisenberg uncertainty principle sets a lower bound on the smallest possible energy delivered to the black hole by the captured particle [4]:

\[
\Delta E_{\text{min}} = 2\pi T_{\text{BH}} \, .
\]

(11)

Substituting (11) into (10) one finds that, within the framework of a self-consistent quantum theory of gravity, the validity of the first law of thermodynamics, Eq. (8), is restricted to the regime [15]

\[
T_{\text{BH}} \times r_+ \gg \left( \frac{\hbar}{r_+} \right)^2 \, .
\]

(12)
III. SUMMARY

The seminal works of Bekenstein and Hawking have revealed that gravity has a well-defined thermodynamic description. In particular, it is often stated in the physical literature that black holes, like mundane physical systems, obey the first law of thermodynamics: \( \Delta E = T_{BH} \Delta S \), where \( T_{BH} \) is the Bekenstein-Hawking temperature of the black hole. In the present paper we have explored the regime of validity of this law. In particular, we have shown that, due to quantum effects, the first law of thermodynamics breaks down in the low-temperature regime

\[
T_{BH} \times r_H \lesssim \left( \frac{\hbar}{r_H} \right)^2
\]

of near-extremal black holes.

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[1] J. D. Bekenstein, Phys. Rev. D 23, 287 (1981).
[2] Here $R$ is the effective radius of the system.
[3] We shall use gravitational units in which $G = c = k_B = 1$.
[4] J. D. Bekenstein, Phys. Rev. D 7, 2333 (1973).
[5] S. W. Hawking, Commun. Math. Phys. 43, 199 (1975).
[6] Here $M$ and $a \equiv J/M$ are the mass and angular-momentum per unit mass of the black hole, respectively.
[7] Here we refer to closed physical systems with fixed volumes.
[8] L. D. Landau and E. M. Lifshitz, Statistical Physics (Addison-Wesley, Reading, Mass., 1969).
[9] B. Carter, in Black Holes, edited by C. M. DeWitt and B. S. DeWitt (Gordon and Breach, New York, 1973).
[10] For concreteness, we shall henceforth consider the physics of astrophysically realistic Kerr black holes. However, it is worth emphasizing that our physical conclusions (to be presented below) can easily be generalized to other types of black holes.
[11] Here we have used the relation $J = Ma$ for the black-hole angular momentum.
[12] For example, one can throw into the black hole a small body whose total energy (as measured by asymptotic observers) is $\Delta E$.
[13] D. Christodoulou and R. Ruffini, Phys. Rev. D 4, 3552 (1971).
[14] M. Born, Atomic Physics (Blackie, London, 1969).
[15] Here we have used the relation $(r_+ - r_-)^2/r_+ \sim (T_{\text{BH}}/\hbar)^2 r_+^3$ in Eq. (10).