Testing Many Restrictions Under Heteroskedasticity*

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Abstract

We propose a hypothesis test that allows for many tested restrictions in a heteroskedastic linear regression model. The test compares the conventional F statistic to a critical value that corrects for many restrictions and conditional heteroskedasticity. This correction uses leave-one-out estimation to correctly center the critical value and leave-three-out estimation to appropriately scale it. The large sample properties of the test are established in an asymptotic framework where the number of tested restrictions may be fixed or may grow with the sample size, and can even be proportional to the number of observations. We show that the test is asymptotically valid and has non-trivial asymptotic power against the same local alternatives as the exact F test when the latter is valid. Simulations corroborate these theoretical findings and suggest excellent size control in moderately small samples, even under strong heteroskedasticity.

Keywords: linear regression, ordinary least squares, many regressors, leave-out estimation, hypothesis testing, high-dimensional models.

JEL codes: C12, C13, C21

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1 Introduction

One of the central tenets in modern economic research is to consider models that allow for flexible specifications of heterogeneity and to establish whether meaningful heterogeneity is present or absent in a particular empirical setting. For example, Abowd et al. (1999) study whether there is firm-specific heterogeneity in a linear model for individual log-wages, Card et al. (2016, 2018) ask if this heterogeneity varies by the individual’s gender or education, and Lachowska et al. (2022) investigate whether the firm-specific heterogeneity is constant over time. Other work relies on similarly flexible models to investigate the presence of heterogeneity in health economics (Finkelstein et al., 2016) and to study neighborhood effects (Chetty and Hendren, 2018). In all these examples, the absence of a particular dimension of heterogeneity corresponds to a hypothesis that imposes hundreds or thousands of restrictions on the model of interest. The present paper provides a tool to conduct a test of such hypotheses. In contemporary work, Kline et al. (2022) apply our proposed test in a study of discrimination among U.S. employers.

We develop a test for hypotheses that impose multiple restrictions and establish its asymptotic validity in a heteroskedastic linear regression model where the number of tested restrictions may be fixed or increasing with the sample size. In particular, we allow for the number of restrictions and the sample size to be proportional. The exact F test, which compares the F statistic to a quantile of the F distribution, fails to control size in this environment. Instead, our proposed test rejects the null hypothesis if the F statistic exceeds a critical value that corrects for many restrictions and conditional heteroskedasticity. This critical value is a recentered and rescaled quantile of what is naturally called the $F$-bar distribution as it describes the distribution of a chi-bar-squared random variable divided by an independent chi-squared random variable over its degrees of freedom.\footnote{Chi-bar-squared is a standard name used to describe a mixture of chi-squared distributions. See, e.g., Dykstra (1991) who studies asymptotic properties of chi-bar-squared distributions.} This family of distributions can approximate both the finite sample properties of the F statistic under homoskedastic normal errors and—after recentering and rescaling—the asymptotic distribution of the F statistic in the presence of conditional heteroskedasticity and few or many restrictions.
The large sample validity of our proposed test holds uniformly in the number of regressors and tested restrictions. In combination with the F-bar distribution, the key to this uniformity is our proposed location and variance estimators that are used to recenter and rescale the critical value. The location estimator utilizes unbiased leave-one-out estimators for individual error variances, while the variance estimator utilizes unbiased leave-three-out estimators for products of these variances. While the product of leave-one-out estimators is biased for the product of variances because of mutual dependence, dropping three observations in successive fashion breaks the dependence between estimators of individual error variances, and hence provides unbiasedness of their products. The use of leave-three-out estimation to implement this idea is novel in the literature. Because the essential elements of the test are built on leave-out machinery, we will at times and for brevity refer to the proposed test using the acronym LO.

The LO test has exact asymptotic size when the regression design has full rank after leaving any combination of three observations out of the sample. This condition is satisfied in models with many continuous regressors and only a few discrete ones. However, the condition can fail when many discretely valued regressors are included, as occurs for models with fixed individual or group effects. With group effects, in particular, leave-three-out may not exist when group sizes are two or three. To handle such cases, the proposed test uses estimators for the products of individual error variances that are intentionally biased upward when the unbiased leave-three-out estimators do not exist. This construction ensures large sample validity but can potentially lead to a slightly conservative test when a large fraction of the leave-three-out estimators do not exist.

Using both theoretical arguments and simulations, Huber (1973) and Berndt and Savin (1977) have highlighted the importance of allowing the number of regressors and potentially the number of tested restrictions to increase with sample size when studying asymptotic properties of inference procedures. The latter paper specifically documents cases where asymptotically equivalent classical tests yield opposite outcomes when the number of tested restrictions is somewhat large. Despite these early cautionary tales, most inference procedures that allow for proportionality between the number of regressors, sample size, and potentially the number of restrictions, are of a more recent vintage. Here, we survey the ones most relevant to
the current paper and refer to Anatolyev (2019) for a more extensive review of the literature.\footnote{In analysis of variance contexts, which are special cases of linear regression, Akritas and Papadatos (2004) and Zhou et al. (2017) propose heteroskedasticity robust tests for equality of means that are, however, specific to their models. An expanding literature considers (outlier) robust estimation of linear high-dimensional regressions (e.g., El Karoui et al., 2013) but does not provide valid tests of many restrictions.}

In homoskedastic regression models, Anatolyev (2012) and Calhoun (2011) propose various corrections to classical tests that restore asymptotic validity in the presence of many restrictions. In heteroskedastic regressions with one tested restriction and many regressors, Cattaneo et al. (2018b) show that the use of conventional Eicker-White standard errors and their “almost-unbiased” variations (see MacKinnon, 2013) does not yield asymptotic validity. This failure may be viewed as a manifestation of the incidental parameters problem. To overcome this problem, Cattaneo et al. (2018b) and subsequently Anatolyev (2018) propose new versions of the Eicker-White standard errors, which restore size control in large samples. However, these proposals rely on the inversion of n-by-n matrices (n denotes sample size) that may fail to be invertible in examples of practical interest (Horn et al., 1975; Verdier, 2020). Rao (1970)’s unbiased estimator for individual error variances is closely related to Cattaneo et al. (2018b)’s proposal and suffers from the same existence issue.\footnote{A recent use of Rao (1970)’s MINQUE estimator is Juhl and Lugovskyy (2014), where it is incorporated into a test of heterogeneity in short panels with fixed effects. Here, the MINQUE estimator is applied to each cross-sectional unit, and its use therefore imposes a growing number of invertibility requirements.}

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Kline, Saggio, and Sølvsten (2020) propose instead a version of the Eicker-White stan-
standard errors that relies only on leave-one-out estimators of individual error variances and show that its use leads to asymptotic size control when testing a single restriction. While this conclusion extends to hypotheses that involve a fixed and small number of restrictions through the use of a heteroskedasticity-robust Wald test, it fails to hold in cases of many restrictions. When testing many coefficients equal to zero, Kline et al. (2020) note that those leave-one-out individual variance estimators can be used to center the conventional F statistic and propose a rescaling of the statistic that relies on successive sample splitting (Kline et al., 2020, section 5) as a tool of breaking dependence among different estimates when error variances enter as pairwise products. However, first, sample splitting places restrictions on the data that will fail when the number of regressors is larger than half of the sample size. Second, sample splitting means that the error variances are estimated only from a part of the sample, which is clearly inefficient and undesirable for conditional objects that require the use of as many observations as possible. Third, sample splitting may be undesirable because different ways of splitting the sample can lead to opposite conclusions.

We instead take another route and utilize information for estimation of error variances in the whole sample. In order to remove the dependence among individual estimates, we appeal to leave-three-out estimation instead of sample splitting. Importantly, leave-three-out estimation places much fewer restrictions on the number of regressors than sample splitting, exploits available sample information more efficiently, and does not require a researcher to choose a way to split the sample. Additionally, the robustified version of the LO test using the F-bar distribution enables asymptotic size control uniformly in the number of restrictions.

We provide a theoretical study of the power properties under local and global alternatives. Under local alternatives, the asymptotic power curve of the proposed LO test is parallel to that of the exact F test when the latter is valid, e.g., under homoskedastic normal errors. While the curves are parallel, the LO test tends to have power somewhat below the exact

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4Jochmans (2022) additionally uses simulations to investigate the behavior of this variance estimator.  
5The use of leave-one-out estimation has a long tradition in the literature on instrumental variables (see, e.g., Phillips and Hale, 1977), and our test shares an algebraic representation with the adjusted J test analyzed in Chao et al. (2014) (see Kline et al., 2020, for a discussion). An attractive feature of relying on leave-one-out is that challenging estimation of higher order error moments can be avoided, which is in contrast to the tests of Calhoun (2011) and Anatolyev (2013).  
6This phenomenon is akin to the situation in Cattaneo et al. (2018b), in which the ‘Hadamard square’ of the orthogonal projection matrix may be non-invertible. Cattaneo et al. (2018b) rule out this possibility by imposing a sufficient condition such that the number of covariates is no larger than half of the sample size.
F test. This loss in power stems from the estimation of individual error variances and can be viewed as a cost of using a test that is robust to general heteroskedasticity. This cost is largely monotone in the number of tested restrictions and disappears when the number of restrictions is small relative to sample size.

We also conduct a simulation study that documents excellent performance of the LO test in small and moderately sized samples. We document that the LO test delivers nearly exact size control in samples as small as 100 observations in both homoskedastic and heteroskedastic environments. On the other hand, conventional tools such as the Wald test and the exact F test can exhibit severe size distortions and reject a true null with near certainty for some configurations. These findings are documented using two simulation settings: one with continuous regressors only, and one with a mix of both continuous and discrete regressors. In the latter setting, roughly 7% of observations cause a full rank failure when leaving up to three observations out, but the proposed test shows almost no conservatism even in this adverse environment. When both the LO and exact F tests are valid, the simulations document a power loss that varies between being negligible and up to roughly 15 percentage points, depending on the type of deviation from the null and sample size. For many applications, this range of power losses is a small cost to incur for being robust to heteroskedasticity.

The paper is organized as follows. Section 2 introduces the setup and the proposed critical value in samples where all the leave-three-out estimators exist, while Section 3 analyzes the asymptotic size and power of the LO test for such samples. Section 4 describes the critical value for use in samples where the design loses full rank after leaving certain triples of observations out. Section 5 discusses the results of simulation experiments, and Section 6 concludes. Proofs of theoretical results and some clarifying but technical details are collected in the online supplemental Appendix. An R package (Anatolyev and Sølvsten, 2020) that implements the proposed test is available online.
2 Leave-out test

Consider a linear regression model

\[ y_i = x_i' \beta + \varepsilon_i, \quad E[\varepsilon_i|x_i] = 0, \]

where an intercept is included in the regression function \( x_i' \beta \) and the \( n \) observed random vectors \( \{(y_i, x_i')\}_{i=1}^{n} \) are independent across \( i \). The dimension of the regressors \( x_i \in \mathbb{R}^m \) may be large relative to sample size with \( m < n \), and there is conditional heteroskedasticity in the unobserved errors:

\[ E[\varepsilon_i^2|x_i] = \sigma^2(x_i) \equiv \sigma_i^2. \]

The conditional variances are assumed to exist with no restrictions placed on the functional form, as in Kline et al. (2020).

The hypothesis of interest involves \( r \leq m \) linear restrictions

\[ H_0 : R \beta = q, \]

where the matrix \( R \in \mathbb{R}^{r \times m} \) has full row rank \( r \), and \( q \in \mathbb{R}^r \). Both \( R \) and \( q \) are specified by the researcher. Specifically, they are assumed to be known and are allowed to depend on the observed regressors. The space of alternatives is \( H_A : R \beta \neq q \).

The attention of the paper is on settings where the design matrix \( S_{xx} = \sum_{i=1}^{n} x_i x_i' \) has full rank so that \( \hat{\beta} = S_{xx}^{-1} \sum_{i=1}^{n} x_i y_i \), the ordinary least squares (OLS) estimator of \( \beta \), is defined. For compact reference, we define the degrees-of-freedom adjusted residual variance

\[ \hat{\sigma}_\varepsilon^2 = \frac{1}{n - m} \sum_{i=1}^{n} (y_i - x_i' \hat{\beta})^2. \]

Remark 1. We maintain in this paper that observations are independent across \( i \), as it facilitates simplicity when discussing some of our high-level conditions. We conjecture that the results of the paper continue to hold under the weaker assumption that the error terms are mean zero and independent across \( i \) when conditioning on all the regressors \( \{x_i\}_{i=1}^{n} \).
2.1 Test statistic

Our proposed test rejects $H_0$ for large values of Fisher’s F statistic,

$$F = \frac{(\hat{R}\hat{\beta} - q)'(RS_{xx}^{-1}R')^{-1}(\hat{R}\hat{\beta} - q)}{r\hat{\sigma}_e^2},$$

which is a monotone transformation of the likelihood ratio statistic when the regression errors are homoskedastic normal. Since we do not impose normality, $F$ may be viewed as a quasi likelihood ratio statistic. The behavior of the test is governed by the numerator of $F$, which we denote by $\mathcal{F}$:

$$\mathcal{F} = (\hat{R}\hat{\beta} - q)'(RS_{xx}^{-1}R')^{-1}(\hat{R}\hat{\beta} - q).$$

(1)

By taking this statistic as a point of departure, we are able to construct a critical value that ensures size control in the presence of heteroskedasticity and an arbitrary number of restrictions.\footnote{An alternative approach might have taken a heteroskedasticity-robust Wald statistic $RS_{xx}^{-1}(\sum^n_{i=1} x_i x_i'\hat{\varepsilon}_i^2)S_{xx}^{-1}R'$, where $\{\hat{\varepsilon}_i\}_{i=1}^n$ are OLS residuals, in a similar attempt to ensure validity when the number of restrictions is proportional to the sample size. However, in such environments, any heteroskedasticity-robust Wald statistic relies on the inverse of a high-dimensional covariance matrix estimator, a feature that presents substantial challenges when attempting to control size. Specifically, the randomness in the residuals induced into this $r \times r$-matrix persists in large samples and is therefore a threat to valid inference. Our conjecture is that some regularization of the covariance matrix may be helpful in mitigating the noise arising from this estimated covariance matrix. In addition, it is not clear if the weighting behind the heteroskedasticity-robust Wald statistic and hence a potential test will preserve optimality under asymptotics where the number of restrictions is proportional to the sample size. We leave investigation of these difficult but interesting questions to future research.}

The proposed critical value yields asymptotic validity under two asymptotic frameworks, one where the number of restrictions is fixed, and one where the number of restrictions may grow as fast as proportionally to the sample size. To achieve such uniformity with respect to the number of restrictions, we rely on an auxiliary distribution, the F-bar distribution, that helps unite these two frameworks.

2.2 F-bar distribution

Our test rejects $H_0$ if Fisher’s F exceeds a linearly transformed quantile of a distribution, which we call the F-bar distribution. We define this family of distributions and discuss its
role before we turn to a description of the linear transformation mentioned above.

**Definition 1** (F-bar distribution). Let \( \mathbf{w} = (w_1, \ldots, w_r) \) be a collection of non-negative weights summing to one, and \( df \) be a positive real number. The F-bar distribution with weights \( \mathbf{w} \) and degrees of freedom \( df \), denoted by \( \bar{F}_{w,df} \), is a distribution of

\[
\frac{\sum_{\ell=1}^r w_\ell Z_\ell}{Z_0/df},
\]

where \( Z_0, Z_1, \ldots, Z_r \) are mutually independent random variables with \( Z_0 \sim \chi^2_{df} \) and \( Z_\ell \sim \chi^2_1 \) for \( 1 \leq \ell \leq r \). Here, \( \chi^2_\kappa \) denotes a chi-squared distribution with \( \kappa > 0 \) degrees of freedom.

The name attached to this family originates from its close relationship to both the chi-bar-squared distribution and to Snedecor’s F distribution, which we denote as \( \bar{\chi}^2_w \) and \( F_{r,df} \), respectively. In the Appendix, we show the following three essential properties of this family to be used later. First, Snedecor’s F is a special case when the entries of \( \mathbf{w} \) are all equal. Second, the limiting case of \( \bar{F}_{w,df} \) when \( df \to \infty \) is \( \bar{\chi}^2_w \). Third, the standard normal distribution, whose CDF is denoted as \( \Phi \), is also a limiting case since, as \( df \to \infty \) and \( \max_{1 \leq \ell \leq r} w_\ell \to 0 \),

\[
\frac{q_\tau(\bar{F}_{w,df}) - 1}{\sqrt{2 \sum_{\ell=1}^r w_\ell^2 + 2/df}} \to q_\tau(\Phi)
\]

for \( \tau \in (0, 1) \), where \( q_\tau(G) \) denotes the \( \tau \)-th quantile of the distribution \( G \). The centering and rescaling in (3) are done according to the limiting mean and variance of the underlying random variable from Definition 1 following the \( F_{w,df} \) distribution, while asymptotic normality results from mixing over infinitely many independent chi-squared variables.

Our reliance on the F-bar distribution is tied to its three properties described in the previous paragraph and three closely related observations about the F statistic. These observations are: (i) the F statistic is distributed as \( F_{r,n-m} \) if the errors are homoskedastic normal, (ii) the F statistic converges in distribution (after rescaling) to a chi-bar-squared if the number of restrictions \( r \) is fixed, and (iii) the F statistic converges in distribution (after centering and rescaling) to a standard normal as \( r \) grows. Therefore, the class of F-bar distributions serves as a roof designed both to match the finite sample distribution of the F statistic in an important special case and to approximate each of the possible limiting
distributions after a suitable linear transformation.

2.3 Critical value

The proposed critical value for the F statistic $F$ at a nominal size $\alpha \in (0, 1)$ is a linear transformation of $q_{1-\alpha}(\hat{F}_{\hat{w}, n-m})$ given by

$$\hat{c}_\alpha = \frac{1}{r\sigma^2} \left( \hat{E}_F + \hat{V}_F^{-1/2} \frac{q_{1-\alpha}(\hat{F}_{\hat{w}, n-m}) - 1}{\sqrt{2 \sum_{\ell=1}^{r} \hat{w}^2_{\ell} + 2/(n-m)}} \right).$$

The hat over $c_\alpha$ emphasizes that this value is data-dependent. The quantities $\hat{E}_F$ and $\hat{V}_F$ are related to $F$ in (1), the numerator of the F statistic. From this point forward, all means and variances are conditional on the regressors $\{x_i\}_{i=1}^n$, and those with a subscript 0 are calculated under $H_0$. The quantity $\hat{E}_F$ is an unbiased estimator of the conditional mean $E_{0}[F]$, while $\hat{V}_F$ is either an unbiased or positively biased estimator of the conditional variance $V_{0}[F - \hat{E}_F]$ as explained further below. The estimated weights $\hat{w} = (\hat{w}_1, \ldots, \hat{w}_r)$ are constructed to be consistent for weights $w_F$ in those cases where $F/E_{0}[F]$ converges in distribution to $\hat{\chi}^2_{w_F}$.

The critical value $\hat{c}_\alpha$ ensures asymptotic size control irrespective of whether $r$ is viewed as fixed or growing with the sample size $n$. To explain why $\hat{c}_\alpha$ provides such uniformity, we consider first the case where $r$ grows. In this case, it is illuminating to rewrite the rejection rule as an equivalent event

$$\hat{V}_F^{-1/2}(F - \hat{E}_F) > \frac{q_{1-\alpha}(\hat{F}_{\hat{w}, n-m}) - 1}{\sqrt{2 \sum_{\ell=1}^{r} \hat{w}^2_{\ell} + 2/(n-m)}}.$$

Since $\hat{V}_F^{-1/2}(F - \hat{E}_F)$ is asymptotically normal under the null, the validity in large samples follows from the relationship between the F-bar and standard normal distributions given in (3).

When instead $r$ is viewed as asymptotically fixed, it is more informative to express the
rejection region through the inequality

\[
\frac{F}{\hat{E}_F} > q_{1-\alpha}(\hat{F}_{\hat{w},n-m}) + (q_{1-\alpha}(\hat{F}_{\hat{w},n-m}) - 1) \left( \frac{\hat{V}_1^{1/2}}{\hat{E}_F} \sqrt{2 \sum_{t=1}^{r} \hat{w}_t^2 + 2/(n-m)} - 1 \right).
\]  (4)

Note that rejecting when \(F/\hat{E}_F\) exceeds the quantile \(q_{1-\alpha}(\hat{F}_{\hat{w},n-m})\) suffices for validity; for the case of a single restriction such an approach corresponds to the standard practice of comparing squares of a heteroskedasticity robust t statistic and the \((1-\alpha)\)-th quantile of Student’s t distribution with \(n-m\) degrees of freedom.\(^8\) The last term on the right hand side of (4) can then be viewed as a finite sample correction that adjusts the critical value up or down depending on the relative size of the variance estimator for the ratio \(F/\hat{E}_F\), which is \(\hat{V}_F/\hat{E}_F^2\), and the variance of the approximating distribution \(\hat{F}_{\hat{w},n-m}\), which is roughly \(2 \sum_{t=1}^{r} \hat{w}_t^2 + 2/(n-m)\). As the ratio of these variances converges to unity when the number of restrictions is fixed, this term does not affect first order asymptotic validity.

Finally, note that if one is willing to rest on the assumption that the restrictions are numerous and the few restriction framework is superfluous, one might use the following simplified critical value not robust to few restrictions:\(^9\)

\[
\hat{c}_\alpha = \frac{1}{r\hat{\sigma}_e^2} \left( \hat{E}_F + \hat{V}_F^{1/2} q_{1-\alpha}(F_{r,n-m}) \frac{1}{\sqrt{2/r + 2/(n-m)}} \right). \]  (5)

To complete the description of the proposed critical value, definitions of the quantities \(\hat{E}_F\), \(\hat{V}_F\) and \(\hat{w}\) are needed. Section 2.5 describes how we rely on leave-one-out OLS estimators to construct \(\hat{E}_F\) and \(\hat{w}\). For \(\hat{V}_F\), Section 2.6 provides the corresponding definition when it is possible to rely on leave-three-out OLS estimators, while Section 4 introduces the form of \(\hat{V}_F\) for settings where some of the leave-three-out estimators cease to exist. In the former case, it is possible to ensure that \(\hat{V}_F\) is unbiased, while the latter introduces a (small) positive bias.

We initially consider the former case, a framework where the design matrix has full rank.

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\(^8\)When testing a single restriction, \(\hat{w}\) must equal unity so that \(\hat{F}_{\hat{w},n-m} = F_{1,n-m} = \hat{t}_{n-m}^2\), and in this case \(F/\hat{E}_F\) is the square of the t statistic studied in Kline et al. (2020, Theorem 1).

\(^9\)Such settings occur, for example, if the null of interest involves thousands of restrictions, in which case the two critical values \(\hat{c}_\alpha\) and \(\check{c}_\alpha\) are essentially equivalent but \(\check{c}_\alpha\) is computationally simpler to construct as it circumvents computation of \(\hat{w}\).
when any three observations are left out of the sample, and relax this condition in Section 4.

**Assumption 1.** \( \sum_{\ell \neq i,j,k} x_\ell x_\ell' \) is invertible for every \( i, j, k \in \{1, \ldots, n\} \).

When \( x_i \) is identically and continuously distributed with unconditional second moment \( \mathbb{E}[x_i x_i'] \) of full rank, Assumption 1 holds with probability one whenever \( n - m \geq 3 \). The asymptotic framework considers a setting where \( n - m \) diverges so that Assumption 1 must hold in sufficiently large samples with continuous regressors. This conclusion also applies when \( x_i \) includes a few discrete regressors and, in particular, an intercept. In settings with many discrete regressors, Assumption 1 may fail to hold, even in large samples. For that reason, Section 4 introduces the version of \( \hat{V}_F \) for empirical settings where the full rank condition is satisfied when any one observation is left out, but not necessarily when leaving two or three observations out.

### 2.4 Leave-out algebra

Before describing \( \hat{E}_F, \hat{V}_F, \) and \( \hat{w} \) in detail, we will reformulate Assumption 1 using leave-out algebra. That is, we will derive an equivalent way of expressing this assumption while introducing notation that is essential for the construction of the critical value and for stating the asymptotic regularity conditions.

When \( S_{xx} \) has full rank, a direct implication of the Sherman-Morrison-Woodbury identity (Sherman and Morrison, 1950; Woodbury, 1949, SMW) is that the leave-one-out design matrix \( \sum_{j \neq i} x_j x_j' \) is invertible if and only if the statistical leverage of the \( i \)-th observation \( P_{ii} = x_i' S_{xx}^{-1} x_i \) is less than one. Letting \( M_{ij} = 1\{i = j\} - x_i' S_{xx}^{-1} x_j \) be elements of the residual projection matrix \( M \) associated with the regressor matrix, this condition on the leverage is equivalently stated as \( M_{ii} > 0 \) holds, we can additionally use SMW to represent the inverse of \( \sum_{j \neq i} x_j x_j' \) as

\[
(\sum_{j \neq i} x_j x_j')^{-1} = S_{xx}^{-1} + \frac{S_{xx}^{-1} x_i x_i' S_{xx}^{-1}}{M_{ii}}, \tag{6}
\]

which highlights the role of a non-zero \( M_{ii} \).

The representation in (6) can also be used to understand when the leave-two-out design matrix \( \sum_{k \neq i,j} x_k x_k' \) has full rank, since (6) can be used to compute leverages in a sample.
that excludes \( i \). After leaving observation \( i \) out, the leverage of a different observation \( j \) is 
\[
x_j'\left(\sum_{k \neq i} x_k x_k'\right)^{-1} x_j.
\]
To see when this leverage is less than one, note that (6) yields
\[
1 - x_j'\left(\sum_{k \neq i} x_k x_k'\right)^{-1} x_j = M_{jj} - \frac{M_{ij}^2}{M_{ii}},
\]
so that a necessary and sufficient condition for a full rank of \( \sum_{k \neq i,j} x_k x_k' \) is that
\[
D_{ij} > 0,
\]
where
\[
D_{ij} = \begin{vmatrix} M_{ii} & M_{ij} \\ M_{ij} & M_{jj} \end{vmatrix} = M_{ii}M_{jj} - M_{ij}^2,
\]
and \(|\cdot|\) denotes the determinant.

Extending the previous argument to the case of leaving three observations out, we find that the invertibility of \( \sum_{\ell \neq i,j,k} x_\ell x_\ell' \) for \( i, j, \) and \( k \), all of which are different, is equivalent to \( D_{ijk} > 0 \), where
\[
D_{ijk} = \begin{vmatrix} M_{ii} & M_{ij} & M_{ik} \\ M_{ij} & M_{jj} & M_{jk} \\ M_{ik} & M_{jk} & M_{kk} \end{vmatrix} = M_{ii}D_{jk} - (M_{jj}M_{ik}^2 + M_{kk}M_{ij}^2 - 2M_{jk}M_{ij}M_{ik}).
\]

This discussion reveals that Assumption 1 can equivalently be stated as requiring full rank of \( S_{xx} \) and
\[
D_{ijk} > 0 \text{ for every } i, j, k \in \{1, \ldots, n\} \text{ with } i \neq j \neq k \neq i.
\] (7)

In addition to facilitating an algebraic description of Assumption 1, the quantities \( M_{ii}, D_{ij}, \) and \( D_{ijk} \) also play a role in the computation of the proposed critical value. Specifically, they can be used to avoid explicitly computing the OLS estimates after leaving one, two, or three observations out. Additionally, since construction of \( \hat{E}_X, \hat{V}_X, \) and \( \hat{w} \) relies on dividing by \( M_{ii}, D_{ij}, \) and \( D_{ijk} \), the study of the asymptotic size of the proposed testing procedure imposes a slight strengthening of (7), which bounds the smallest \( D_{ijk} \) away from zero.
2.5 Location estimator

Recall that we recenter the numerator of the F statistic (\( F \) in (1)) by using \( \hat{E}_F \), which is an unbiased estimator of the conditional mean \( \mathbb{E}_0[F] \) under \( H_0 \). This mean equals

\[
\mathbb{E}_0[F] = \sum_{i=1}^{n} B_{ii} \sigma_i^2,
\]

where the values \( B_{ij} = x_i' S_{xx}^{-1} R' \left( R S_{xx}^{-1} R' \right)^{-1} R S_{xx}^{-1} x_j \) are observed and satisfy \( \sum_{i=1}^{n} B_{ii} = r \). Furthermore, the exact null distribution of \( \frac{F}{\mathbb{E}_0[F]} \), under the additional condition of normally distributed regression errors, is \( \chi^2_{w_F} \), with \( w_F \) containing the eigenvalues of the matrix\(^{10} \)

\[
\Omega(\sigma_1^2, \ldots, \sigma_n^2) = \frac{1}{\sum_{i=1}^{n} B_{ii} \sigma_i^2} \left( R S_{xx}^{-1} R' \right)^{-1} R S_{xx}^{-1} \left( \sum_{i=1}^{n} x_i' x_i \sigma_i^2 \right) S_{xx}^{-1} R'.
\]

Both \( \mathbb{E}_0[F] \) and \( w_F \) are thus functions of \( \{\sigma_i^2\}_{i=1}^{n} \), and the relevance of the vector \( w_F \) for asymptotic size control transcends the normality assumption on the errors that we used in order to introduce it.

As shown in Kline et al. (2020), the individual specific error variances can be estimated without bias for any value of \( \beta \) using leave-one-out estimators. Let the leave-\( i \)-out OLS estimator of \( \beta \) be \( \hat{\beta}_{-i} = \left( \sum_{j \neq i} x_j x_j' \right)^{-1} \sum_{j \neq i} x_j y_j \), and construct

\[
\hat{\sigma}_i^2 = y_i (y_i - x_i' \hat{\beta}_{-i}).
\]

With these leave-one-out estimators, we can estimate the null mean of \( F \) using

\[
\hat{E}_F = \sum_{i=1}^{n} B_{ii} \hat{\sigma}_i^2,
\]

\(^{10}\) Under error normality, the exact null distribution of \( R \hat{\beta} - q \) is \( \mathcal{N}(0, \mathbb{V}[R \hat{\beta}]) \), and it follows from Lemma 3.2 of Vuong (1989) that \( \frac{F}{\mathbb{E}_0[F]} \) is distributed as a weighted sum of chi-squares with weights that are eigenvalues of \( \left( R S_{xx}^{-1} R' \right)^{-1} \mathbb{V}[R \hat{\beta}] / \mathbb{E}_0[F] = \Omega(\sigma_1^2, \ldots, \sigma_n^2) \). Note that the eigenvalues of \( \Omega(\sigma_1^2, \ldots, \sigma_n^2) \) are all real and non-negative as they can be expressed as the eigenvalues of the symmetric and positive semidefinite matrix \( \left( R S_{xx}^{-1} R' \right)^{-1/2} \mathbb{V}[R \hat{\beta}] \left( R S_{xx}^{-1} R' \right)^{-1/2} / \mathbb{E}_0[F] \). Furthermore, the entries of \( w_F \) sum to one as \( \mathbb{E}_0[F] \) is the trace of \( \left( R S_{xx}^{-1} R' \right)^{-1} \mathbb{V}[R \hat{\beta}] \).
which ensures that the first moment of $F - \hat{E}_F$ is zero under the null. Since $\hat{\sigma}_i^2$ is unbiased for any value of $\beta$, this centered statistic still has its expectation minimized under $H_0$, so that large values of the statistic can be taken as evidence against the null. Following the same approach, we can estimate $w_F$ using the sample analog $\tilde{w} = (\tilde{w}_1, \ldots, \tilde{w}_r)'$, where $\tilde{w}_\ell$ is the $\ell$-th eigenvalue of $\Omega(\hat{\sigma}_1^2, \ldots, \hat{\sigma}_n^2)$. However, $\tilde{w}$ may not have non-negative entries summing to one, so we ensure that these conditions hold by letting $\hat{w} = (\hat{w}_1, \ldots, \hat{w}_r)'$, where $\hat{w}_\ell = \tilde{w}_\ell \lor 0$.

While our construction of $\hat{E}_F$ implies that the first moment of $F - \hat{E}_F$ is known when $H_0$ holds, its second moment still depends heavily on unknown parameters. Under $H_0$, $V_0[F - \hat{E}_F] = \sum_{i=1}^{n} \sum_{j \neq i} U_{ij} \sigma_i^2 \sigma_j^2 + \sum_{i=1}^{n} \left( \sum_{j \neq i} V_{ij} x_j' \beta \right)^2 \sigma_i^2$, (9)

where $U_{ij} = 2(B_{ij} - M_{ij}(B_{ii}/M_{ii} + B_{jj}/M_{jj})/2)^2$ and $V_{ij} = M_{ij}(B_{ii}/M_{ii} - B_{jj}/M_{jj})$ are known quantities. This representation of the null-variance stems from writing $F - \hat{E}_F$ as a second order $U$-statistic with squared kernel weights of $U_{ij}/2$ plus a linear term with weights $\sum_{j \neq i} V_{ij} x_j' \beta$ (see the Appendix for details).

### 2.6 Variance estimator

This subsection describes the construction of an unbiased estimator of the conditional variance $V_0[F - \hat{E}_F]$. As is evident from the representation in (9), this variance depends on products of second moments such as the product $\sigma_i^2 \sigma_j^2$. While $\hat{\sigma}_i^2$ and $\hat{\sigma}_j^2$ are unbiased for $\sigma_i^2$ and $\sigma_j^2$, their product is not unbiased, as the estimation error is correlated across the two estimators. Some of this dependence can be removed by leaving both $i$ and $j$ out, but a bias remains as the remaining sample is used in estimating both $\sigma_i^2$ and $\sigma_j^2$. We therefore propose a leave-three-out estimator of the variance product $\sigma_i^2 \sigma_j^2$. The product $x_j' \beta x_i' \beta \sigma_i^2$ appearing in the second component of $V_0[F - \hat{E}_F]$ can similarly be estimated without bias using leave-three-out estimators.

Towards this end, let $\hat{\beta}_{-ijk} = (\sum_{\ell \neq i,j,k} x_\ell x_\ell')^{-1} \sum_{\ell \neq i,j,k} x_\ell y_\ell$ denote the OLS estimator
of $\beta$ applied to the sample that leaves observations $i$, $j$, and $k$ out. Then, define a leave-three-out estimator of $\sigma_i^2$ as

$$\hat{\sigma}^2_{i,-jk} = y_i (y_i - x'_i \hat{\beta}_{-ijk}).$$

When $j$ and $k$ are identical, only two observations are left out, and we also write $\hat{\beta}_{-ij}$ and $\hat{\sigma}^2_{i,-j}$. To construct an estimator of $\sigma_i^2 \sigma_j^2$, we first write the leave-two-out variance estimator $\hat{\sigma}^2_{i,-j}$ as a weighted sum (see Section 2.7 for details)

$$\hat{\sigma}^2_{i,-j} = y_i \sum_{k \neq j} \tilde{M}_{ik,-ij} y_k \quad \text{where} \quad \tilde{M}_{ik,-ij} = \frac{M_{jj} M_{kk} - M_{ij} M_{jk}}{D_{ij}}. \quad (10)$$

Then we multiply each summand above by a leave-three-out variance estimator $\hat{\sigma}^2_{j,-ik}$, which leads to an unbiased estimator of $\sigma_i^2 \sigma_j^2$:

$$\hat{\sigma}^2_i \hat{\sigma}^2_j = y_i \sum_{k \neq j} \tilde{M}_{ik,-ij} y_k \cdot \hat{\sigma}^2_{j,-ik}. \quad (11)$$

While this construction appears to treat $i$ and $j$ in an asymmetric fashion, we show to the contrary that (11) is invariant to a permutation of the indices; $\hat{\sigma}^2_i \hat{\sigma}^2_j = \hat{\sigma}^2_j \hat{\sigma}^2_i$.

To understand why this proposal is unbiased for $\sigma_i^2 \sigma_j^2$, it is useful to highlight that $\hat{\sigma}^2_{j,-ik}$ is conditionally independent of $(y_i, y_k)$ and unbiased for $\sigma_j^2$, which, when coupled with (10), leads to unbiasedness immediately:

$$\mathbb{E} [ \hat{\sigma}^2_i \hat{\sigma}^2_j ] = \sum_{k \neq j} \mathbb{E} [y_i \tilde{M}_{ik,-ij} y_k] \cdot \mathbb{E} [\hat{\sigma}^2_{j,-ik}] = \mathbb{E} [\hat{\sigma}^2_{i,-jk}] \sigma_j^2 = \sigma_i^2 \sigma_j^2.$$

An unbiased estimator of the variance expression in (9) that utilizes the variance product estimator in (11) is

$$\hat{V}_F = \sum_{i=1}^{n} \sum_{j \neq i} (U_{ij} - V_{ij}^2) \cdot \hat{\sigma}^2_i \hat{\sigma}^2_j + \sum_{i=1}^{n} \sum_{j \neq i} \sum_{k \neq i} V_{ij} y_j \cdot V_{ik} y_k \cdot \hat{\sigma}^2_{i,-jk}. \quad (12)$$

Note that the product of the $(j = k)$-th terms in the second component generate, for each $i$, a term not present in (9) and whose non-zero expectation contains $V_{ij}^2 \sigma_i^2 \sigma_j^2$; hence the use
of $U_{ij} - V_{ij}^2$ instead of $U_{ij}$ in the first component.

**Remark 2.** In the process of establishing the asymptotic validity of the proposed test, we show that the variance estimator $\hat{V}_F$ is close to the null variance $V_0 [F - \hat{E}_F]$. In particular, this property implies that the variance estimator is positive with probability approaching one in large samples. However, negative values may still emerge in small samples. In such cases, we propose to replace the variance estimator with an upward biased alternative that uses squared outcomes as estimators of all the error variances. This replacement is guaranteed positive, as is detailed in the Appendix, and therefore ensures that the critical value is always defined. Relatedly, Section 4 considers settings where the design matrix may turn rank deficient after leaving certain triples of observations out of the sample. There, we similarly propose to use squared outcomes as estimators of some error variances, namely those whose observations cause rank deficiency when left out of the sample.

**Remark 3.** Note that in finite samples, the proposed critical value $\hat{c}_\alpha$ is not invariant to the value of $\beta$. In practice, this means that finite sample size and power may be influenced by the size of the regression coefficients. In Section 5 we analyze via simulations the impact of the average signal size on the finite sample size and power through the regression $R^2$.

**Remark 4.** The null restrictions $R\beta = q$ can be imposed during the estimation of auxiliary quantities (i.e., $\sigma_i^2$, $\sigma_i^2\sigma_j^2$, ...), and the restricted estimates may be used in place of our proposals that do not impose those restrictions (i.e., $\hat{\sigma}_i^2$, $\hat{\sigma}_i^2\hat{\sigma}_j^2$, ...).\footnote{We thank an anonymous referee for raising this point.} In the Appendix, we show how this idea can be implemented, and leave further investigation to future research. Incorporating restricted estimates may make the the procedure more complex and slow down computations, but it may also improve the efficiency of the auxiliary estimates.

### 2.7 Computational remarks

While the previous subsections introduced the location estimator $\hat{E}_F$, variance estimator $\hat{V}_F$, and empirical weights $\hat{w}$ using leave-out estimators of $\beta$, we note here that direct computation of $\hat{\beta}_{-i}$, $\hat{\beta}_{-ij}$, and $\hat{\beta}_{-ijk}$ can be avoided by using the Sherman-Morrison-Woodbury
(SMW) identity. Specifically, (6) implies that

\[ y_i - x_i' \hat{\beta}_{-i} = \frac{y_i - x_i' \hat{\beta}}{M_{ii}}, \]

so that computation of \( \hat{\beta}_{-i} \) can be avoided when constructing the leave-one-out variance estimator \( \hat{\sigma}_i^2 = y_i(y_i - x_i' \hat{\beta}_{-i}) \). Similarly, it is possible to show that for \( i \) and \( j \) not equal,

\[ y_i - x_i' \hat{\beta}_{-ij} = \frac{M_{ij}(y_i - x_i' \hat{\beta}) - M_{ij}(y_j - x_j' \hat{\beta})}{D_{ij}}, \]

which leads to (10), and, for \( i, j, \) and \( k \), all of which are different,

\[ y_i - x_i' \hat{\beta}_{-ijk} = \frac{(y_i - x_i' \hat{\beta}) - M_{ij}(y_j - x_j' \hat{\beta}_{-jk}) - M_{ik}(y_k - x_k' \hat{\beta}_{-jk})}{D_{ijk}/D_{jk}}. \]

These relationships allow for recursive computation of the leave-out residuals and therefore for simple construction of the variance estimators \( \hat{\sigma}_i^2, \hat{\sigma}_{i-j}^2, \) and \( \hat{\sigma}_{i-j-k}^2 \) needed to compute the components of the critical value \( c_\alpha \). In particular, the location estimator \( \hat{E}_F \) and empirical weights \( \hat{w} \), which require only the leave-one-out residuals, can be computed without explicit loops, by relying instead on elementary matrix operations applied to the matrices containing \( M_{ij} \) and \( B_{ij} \) as well as the data matrices. Similarly, all doubly indexed objects entering the variance estimator \( \hat{V}_F \) can be computed by elementary matrix operations. Those objects are \( D_{ij}, V_{ij}, U_{ij} \), and the leave-two-out residuals. The remaining objects entering \( \hat{V}_F \) can be computed by a single loop across \( i \) with matrices containing \( D_{ijk} \) and leave-three-out residuals renewed at each iteration.

Additionally, the above representations of leave-out residuals demonstrate how \( M_{ii}^{-1}, D_{ij}^{-1} \) and \( D_{ijk}^{-1} \) enter the critical value, and thus highlight the need for bounding \( D_{ijk} \) away from zero when analyzing the large sample properties of the proposed test.

Remark 5. The quantile \( q_{1-\alpha}(\hat{F}_{\hat{w}, n-m}) \) can easily be constructed by simulating the distribution of the random variable in (2) conditional on the realized value of \( \hat{w} \).
3 Asymptotic size and power

This section studies the asymptotic properties of the proposed test. Specifically, we provide a set of regularity conditions under which the test has a correct asymptotic size and non-trivial power against local alternatives. All limits are taken as the sample size $n$ approaches infinity. In studying asymptotic size, we allow for the number of restrictions $r$ and/or number of regressors $m$ to be fixed or diverging with $n$, and show that the asymptotic size is controlled uniformly over the two situations by the test, which is therefore robust to the type of asymptotics. When studying asymptotic power, we focus on the case of many restrictions, i.e., $r$ diverging with $n$. The ordering $r \leq m < n - 3$ is maintained throughout.

3.1 Assumptions

To establish asymptotic validity of the proposed test, we impose some regularity conditions. We begin by outlining the assumptions for the sampling scheme.

**Assumption 2.** \{($y_i, x_i')_{i=1}^n$ are i.i.d., \( E[\varepsilon_i | x_i] = 0 \), \( \max_{1 \leq i \leq n} (E[\varepsilon_i^4 | x_i] + \sigma_i^{-2}) = O_p(1) \).

Assumption 2 places restrictions on the error conditional moments: an upper bound on the conditional fourth moments and lower bound on the skedastic function. Such restrictions are typically required when heteroskedasticity is allowed (see, e.g., Chao et al., 2012; Cattaneo et al., 2018b; Kline et al., 2020).

Next, we impose regularity conditions on the regressors to ensure convergence of the centered statistic $\mathcal{F} - \hat{E}_\mathcal{F}$ to the normal distribution when the number of restrictions grow large. These conditions restrict the weights on the regression errors in the bilinear form of $\mathcal{F} - \hat{E}_\mathcal{F}$ \(^{12}\). These weights contain various functions of regressors (in particular, potentially unbounded regression function values $x_i' \beta$), and the purpose of Assumption 3 is to restrict their asymptotic behavior.

**Assumption 3.** There exists a sequence $\epsilon_n \to 0$ such that (i) $\epsilon_n^{1/3} \max_{1 \leq i \leq n} (x_i' \beta)^2 = O_p(1)$ and (ii) at least one of the following two conditions is satisfied:

(a) $\max_{1 \leq i \leq n} B_{ii} = o_p(1)$.

\(^{12}\)See, e.g., expansion (13) below and its extended version (17) in the Appendix.
\[(b) \max_{1 \leq i \leq n} (\sum_{j \neq i} V_{ij} x'_j \beta)^2 / r = o_p(1) \text{ and } \epsilon_n r \to \infty.\]

Part \(i\) of Assumption 3, which places bounds on regression function values, is used to control the variance of the leave-out estimators \(\hat{\sigma}_i^2\) and \(\hat{\sigma}_{i,j-k}^2\). This condition places a rate bound on extreme outliers among the individual signals, and its role is primarily to control certain higher order terms. The condition \(i\) is used to establish both size control and local power properties, so we stress that it pertains to the actual data generating process, not just the hypothesized value of \(\beta\). Also note that Assumption 3\(i\) differs from Assumption 1\(iii\) of Kline et al. (2020) in that we allow \(x'_i \beta\) to have an unbounded support so the maximum over \(i\) may be slowly diverging with \(n\). This relaxation is important, as it allows regressors with unbounded support and associated non-zero regression coefficients.

Part \(ii\) of Assumption 3 is an analogue of the Lindeberg condition in the central limit theorem for weighted sums of independent random variables, in that it also controls the collective asymptotic behavior of weights, but the weights in a bilinear form of independent regression errors. When the number of tested restrictions is fixed, this condition implies that the estimator of the tested contrasts \(R \hat{\beta} - q\) is asymptotically normal. When the number of restrictions is growing, this condition is weaker and involves only a high-level transformation of the regressors \(\sum_{j \neq i} V_{ij} x'_j \beta\), which enters \(F - \hat{E}_F\) as a weight on the \(i\)-th error term \(\varepsilon_i\). To ensure that the asymptotic distribution of \(F - \hat{E}_F\) does not depend on the unknown distribution of any one error term, we therefore require that no squared \(\sum_{j \neq i} V_{ij} x'_j \beta\) dominates the variance \(V_0[F - \hat{E}_F]\), which in turn is proportional to \(r\). Assumption 3\(ii\) can be verified in particular applications of interest. For example, the Appendix shows that part \(ii)(b)\) of Assumption 3 holds in models characterized by group specific regressors.

The next assumption imposes the previously discussed regularity condition that the determinant \(D_{ijk}\) is bounded away from zero for any \(i, j,\) and \(k\), all of which are different. This condition will be relaxed in Section 4, where such a version of \(\hat{V}_F\) is introduced that exists even when leaving two or three observations out leads to rank deficiency of the design.

**Assumption 4.** \(\max_{i \neq j \neq k \neq i} D_{ijk}^{-1} = O_p(1)\).
3.2 Asymptotic size

Under the regularity conditions in Assumptions 3 and 4, the following theorem establishes the asymptotic validity of the proposed testing procedure.

**Theorem 3.1.** If Assumptions 1, 2, 3, and 4 hold, then, under $H_0$,

$$\lim_{n \to \infty} \mathbb{P}(F > \hat{c}_\alpha) = \alpha.$$  

A discussion of the structure of the decision event $F > \hat{c}_\alpha$ may aid in understanding why size control occurs in the critical case of many restrictions, $r \to \infty$, and where the challenges come from. The critical value is then asymptotically close to $\hat{c}_\alpha$ defined in (5), and the decision event is equivalent to $\hat{V}_\alpha^{-1/2}(F - \hat{E}_F) > (q_{1-\alpha}(Fr,n-m) - 1)/\sqrt{2/r + 2/(n-m)}$.

The right side is asymptotically standard normal, so the size control rests on asymptotic standard normality of $\hat{V}_\alpha^{-1/2}(F - \hat{E}_F)$, which, in turn, is shown using a central limit theorem for $F - \hat{E}_F$ and its asymptotically correct standardization by $\hat{V}_\alpha^{1/2}$.

The demeaned statistic $F - \hat{E}_F$ has a representation of a bilinear form in independent, not necessarily identically distributed, random variables (see equation (3) in the Appendix):

$$F - \hat{E}_F = \sum_{i=1}^{n} \sum_{j \neq i} C_{ij}u_i v_j + \sum_{j=1}^{n} c_{\beta_j} w_j,$$

where, in our case, $u_j = v_j = w_j = \varepsilon_j$ are regression errors, and the coefficients $c_{\beta_j}$ in the second, linear component, explicitly depend on the unknown parameters $\beta$. Note that the quadratic term in (13) has a jackknife structure and lacks terms of the type $C_{ii}\varepsilon_i^2$, whose presence would introduce higher-order moments into the variance.\(^3\) This further highlights the importance of demeaning the original statistic $F$.

The econometric theory literature is populated by central limit theorems (CLTs) handling the asymptotics of (13) under various assumptions, starting from an early CLT in Kelejian and Prucha (2001) that was originated in a spatial regression environment. More recent examples are the CLT of Chao et al. (2012), formulated in the many-weak-instrument

\(^3\)See also Calhoun (2011) for an example of a structure where higher-order moments do arise and need to be tediously estimated.
context, the CLT of Kline et al. (2020) designed for many-regressor models, and the CLT of Anatolyev and Mikusheva (2021) targeting big factor models. Cattaneo et al. (2018a) presented a unifying framework leading to the description of asymptotical behavior of V-statistics that further generalize bilinear forms like (13). Finally, Kuersteiner and Prucha (2020) provide a CLT for expressions like (13), allowing for data-dependent $C_{ij}$ and $c_{\beta j}$, heteroskedasticity, and some forms of dependence across $j$. In all these setups, the asymptotic normality of (13) eventually results in an asymptotically normal test statistic when (13) is suitably standardized.

The next challenge is converting the asymptotic normality of (13) into an asymptotically valid test. To obtain an asymptotically pivotal statistic, (13) needs to be standardized by a consistent estimate of its (rescaled) variance (9). Constructing such an estimate is difficult because of explicit dependence of the coefficients $c_{\beta j}$ and implicit dependence of the regression errors $\varepsilon_i$ on $\beta$, a high-dimensional parameter when regressors are many. The estimator (12) based on leave-three-out error variance estimates accomplishes this goal.

As our treatment covers two asymptotically different frameworks under one roof, we use two CLTs in this paper. One, formulated as Lemma B.1 in Kline et al. (2020), which builds on Lemmas A2.1 and A2.2 in Sølvsten (2020a), pertains to the case of asymptotically growing $r$. The other is the regular Lyapounov central limit theorem, which pertains to the case of asymptotically fixed $r$.

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14 Often, the bilinear form (13) has a tighter structure, which simplifies the emergence of asymptotic normality and further pivotization, removing the challenges handled in the present paper. For example, simplification may come from a slow growth of incidental parameters’ dimensionality (Hong and White, 1995), absence of the quadratic component in (13) (Breitung et al., 2016), or assumption of conditional homoskedasticity (Anatolyev, 2012). Likewise, in the many weak instrument literature, a variety of tests are also based on asymptotic normality of bilinear forms, with simplifying deviations from the general form (13). In J type tests for validity of many instruments and thus many restrictions (Anatolyev and Gospodinov, 2011; Lee and Okui, 2012; Chao et al., 2014), the dependence on a parameter of asymptotically fixed dimensionality can be handled using its plug-in estimate. In Anderson-Rubin type tests for few parameter restrictions (Anatolyev and Gospodinov, 2011; Crudu et al., 2021; Mikusheva and Sun, 2022), the number of parameter restrictions is asymptotically fixed, and one can use restricted null values of the parameters. In our situation, in contrast, both the restriction numerosity and parameter dimensionality are asymptotically increasing, at least when $r$ is asymptotically growing.
3.3 Asymptotic power

To describe the power of the proposed test, we introduce a drifting sequence of local alternatives indexed by a deviation $\delta$ from the null times $(RS_{xx}^{-1}R')^{1/2}$, which specifies the precision the tested linear restrictions can be estimated with in the given sample. Thus, we consider alternatives of the form

$$H_\delta : R \beta = q + (RS_{xx}^{-1}R')^{1/2} \cdot \delta,$$

for $\delta \in \mathbb{R}^r$ satisfying the limiting condition

$$\lim_{n,r \to \infty} r^{1/4} \|\delta\| = \Delta_\delta \in [0, \infty].$$

Below we show that the power of the test is monotone in $\Delta_\delta$, with power equal to size when $\Delta_\delta = 0$ and power equal to one when $\Delta_\delta = \infty$.

The role of $(RS_{xx}^{-1}R')^{1/2}$ in indexing the local alternatives is analogous to that of $n^{-1/2}$ often used in parametric problems. However, in settings with many regressors some linear restrictions may be estimated at rates that are substantially lower than the standard parametric one. Therefore, we index the deviations from the null by the actual rate of $(RS_{xx}^{-1}R')^{1/2}$ instead of $n^{-1/2}$.

The alternative is additionally indexed by $\delta$, which in standard parametric problems is typically fixed. However, fixed $\delta$ is less natural here, as the dimension of $\delta$ increases with sample size. Instead, we fix the limit of its Euclidean norm when scaled by $r^{1/4}$. This approach allows us to discuss different types of alternatives and how the numerosity of the tested restrictions affects the test’s ability to detect deviations from the null. Specifically, note that when the deviation $\delta$ is sparse, i.e., only a bounded number of its entries are non-zero, then the test has a non-trivial power against alternatives, whose individual elements on average diverges at a rate that is $r^{1/4}$ lower than when only a fixed number of restrictions is tested. This observation highlights the cost for the power of including many irrelevant restrictions in the hypothesis. On the other hand, if $\delta$ is dense, e.g., with all entries bounded away from zero, then the test can detect local deviations, in which an individual element on average shrinks at a rate that is $r^{1/4}$ greater than the usual. This means that if the tested
restrictions can be estimated at the parametric rate and they are all relevant, then the test can detect deviations from the null of order $n^{-1/2}r^{-1/4}$.

The following theorem states the asymptotic power under sequences of local alternatives of the form given in (14) and discussed above.

**Theorem 3.2.** If Assumptions 1, 2, 3, and 4 hold, then, under $H_δ$,

$$\lim_{n,r \to \infty} P(F > \hat{c}_\alpha) - \Phi \left( \Phi^{-1}(\alpha) + \Delta_δ^2 \left( \mathbb{V}_0[F - \hat{E}_\mathcal{F}] / r \right)^{-1/2} \right) = 0,$$

where $\Phi$ denotes the cumulative distribution function of the standard normal and $\Phi(\infty) = 1$.

**Remark 6.** It is instructive to compare the power curve documented in Theorem 3.2 with the asymptotic power curve of the exact F test when both tests are valid. When the individual error terms are homoskedastic normal with variance $\sigma^2$, the asymptotic power of the exact F test is the limit of (Anatolyev, 2012)

$$\Phi \left( \Phi^{-1}(\alpha) + \Delta_δ^2 \left( 2\sigma^4 + 2\sigma^4 r/(n-m) \right)^{-1/2} \right).$$

Thus, the relative asymptotic power of the proposed LO test and the exact F test is determined by the limiting ratio of $r^{-1}\mathbb{V}_0[F - \hat{E}_\mathcal{F}]$ to $2\sigma^4(1 + r/(n-m))$. The Appendix shows that this ratio approaches one in large samples if the number of tested restriction is small relative to the sample size or if the limiting variability of $B_{ii}/M_{ii}$ is small (Kline et al., 2020, calls this a balanced design). When neither of these conditions holds, the proposed test will, in general, have a slightly lower power than the exact F test, which we also document in the simulations in Section 5.

**Remark 7.** The order of the numerosity of alternatives that can be detected with the proposed test is optimal in the minimax sense when the alternatives are moderately sparse to dense, i.e., when $O(\sqrt{r})$ or more of the tested restrictions are violated (Arias-Castro et al., 2011). However, if the alternative is strongly sparse so that at most $o(\sqrt{r})$ tested restrictions are violated, a higher power can be achieved by tests that redirect their power towards those alternatives. Such tests typically focus their attention on a few largest t statistics (i.e., smallest p values) and are often described as multiple comparison procedures (Donoho and Jin,
While such tests can control size when the error terms are homoskedastic normal, it is not clear whether they can do so in the current semiparametric framework with an unspecified error distribution. The issue is that the size control for multiple comparisons relies on knowing the (normal or t) distributions of individual t statistics, but in the current framework with many regressors those distributions are not necessarily known (even asymptotically).

4 If leave-three-out fails

This section extends the definition of the critical value $c_\alpha$ to settings where the design matrix may turn rank deficient after leaving certain pairs or triples of observations out of the sample. When Assumption 1 fails in this way, $\hat{E}_F$ is still an unbiased estimator of $E_0[F]$, but the unbiased variance estimator introduced in Section 2.6 does not exist. For this reason, we propose an adjustment to the variance estimator that introduces a positive bias for pairs of observations where we are unable to construct an unbiased estimator of the variance product $\sigma_i^2 \sigma_j^2$ and for triples of observations where we are unable to construct an unbiased estimator of $x'_j \beta x'_k \beta \sigma_i^2$. This introduction of a positive bias to the variance estimator ensures asymptotic size control, even when Assumption 1 fails.

Since this section considers a setup where Assumption 1 may fail, we introduce a weaker version of the assumption, which only imposes the full rank of the design matrix after dropping any one observation.

**Assumption 1’.** $\sum_{j \neq i} x_j x'_j$ is invertible for every $i \in \{1, \ldots, n\}$.

One can always satisfy this assumption by appropriately pruning the sample, the model, and the hypothesis of interest. For example, if $S_{xx}$ does not have full rank, then one can remove unidentified parameters from both the model and hypothesis of interest, and proceed by testing the subset of restrictions in $H_0$ that are identified by the sample. Similarly, if $\sum_{j \neq i} x_j x'_j$ does not have full rank for some observation $i$, then there is a parameter in the model which is identified only by this observation. Therefore, one can proceed as in the case of rank deficiency of $S_{xx}$, by dropping observation $i$ from the sample and by removing the parameter that determines the mean of this observation from the model and null hypothesis.
When doing this for any observation $i$ such that $\sum_{j \neq i} x_j x'_j$ is non-invertible, one obtains a sample that satisfies Assumption 1' and can be used to test the restrictions in $H_0$ that are identified by this leave-one-out sample.

4.1 Variance estimator

When Assumption 1 fails, some of the unbiased estimators $\hat{\sigma}^2_{i,-jk}$ and $\hat{\sigma}_i^2 \hat{\sigma}_j^2$ cease to exist. For such cases, the variance estimator $\hat{V}_F$ utilizes replacements that are either also unbiased or positively biased, depending on the cause of the failure. Assumption 1 fails if $D_{ijk} = 0$ for some triple of observations, and we say that this failure of full rank is caused by $i$ if $D_{jk} > 0$ or $D_{ij}D_{ik} > 0$, i.e., if the design retains full rank when only observations $j$ and $k$ are left out or if leaving out observations $(i, j)$ or $(i, k)$ leads to rank deficiency. Our replacement for $\hat{\sigma}^2_{i,-jk}$ is biased when $i$ causes $D_{ijk} = 0$, while the replacement for $\hat{\sigma}_i^2 \hat{\sigma}_j^2$ is biased when both $i$ and $j$ cause $D_{ijk} = 0$ for some $k$.

To introduce the replacement for $\hat{\sigma}^2_{i,-jk}$, we consider the case when it does not exist, or equivalently, when $D_{ijk} = 0$. If $i$ causes this leave-three-out failure, then our replacement is the upward biased estimator $y^2_i$. When this failure of leave-three-out is not caused by $i$, the leave-two-out estimators $\hat{\sigma}^2_{i,-j}$ and $\hat{\sigma}^2_{i,-k}$ are equal and independent of both $y_j$ and $y_k$ (as shown in the Appendix). These properties imply that $y_jy_k \hat{\sigma}^2_{i,-j}$ is an unbiased estimator of $x'_j \beta x'_k \sigma_i^2$, and we therefore use $\hat{\sigma}^2_{i,-j}$ as a replacement for $\hat{\sigma}^2_{i,-jk}$. To summarize, we let

$$\hat{\sigma}^2_{i,-jk} = \begin{cases} 
\hat{\sigma}^2_{i,-jk}, & \text{if } D_{ijk} > 0, \\
\hat{\sigma}^2_{i,-j}, & \text{if } D_{jk} = 0 \text{ and } D_{ij}D_{ik} > 0, \\
y^2_i, & \text{otherwise.}
\end{cases}$$

When $j$ is equal to $k$, we consider pairs of observations, and the definition only involves the last two lines since $D_{ijj} = 0$. In this case, we also write $\hat{\sigma}^2_{i,-j}$ for $\hat{\sigma}^2_{i,-jj}$.

For the replacement of $\hat{\sigma}_i^2 \hat{\sigma}_j^2 = y_i \sum_{k \neq j} \tilde{M}_{ik,-ij} y_k \cdot \hat{\sigma}^2_{j,-ik}$, we similarly consider the case where this estimator does not exist, i.e., where $D_{ijk} = 0$ for a $k$ not equal to $i$ or $j$. When any such rank deficiency is caused by both $i$ and $j$, we rely on the upward biased replacement $y^2_i \hat{\sigma}^2_{j,-i}$. When none of the leave-three-out failures are caused by both $i$ and $j$, the replacement
uses $\sigma_i^2_{-jk}$ in place of $\sigma_i^2_{-ij}$. To summarize, we define

$$\overline{\sigma_i^2 \sigma_j^2} = \begin{cases} 
y_i \sum_{k \neq j} \tilde{M}_{ik,-ij} y_k \cdot \overline{\sigma_j^2_{-ik}}, & \text{if } D_{ij} > 0 \text{ and } (D_{ijk} > 0 \text{ or } D_{ik}D_{jk} = 0 \text{ for all } k), \\
y_i^2 \sigma_j^2_{-i}, & \text{otherwise.} 
\end{cases}$$

This estimator is unbiased for $\sigma_i^2 \sigma_j^2$ when none of the leave-three-out failures are caused by both $i$ and $j$, i.e., when the first line of the definition applies. Unbiasedness holds because the presence of a bias in $\overline{\sigma_j^2_{-ik}}$ implies that $j$ is causing the leave-three-out failure. Therefore, $i$ cannot be the cause, which yields that $\overline{\sigma_i^2_{-j}}$ is independent of $y_k$, or equivalently, that $\tilde{M}_{ik,-ij} = 0$.

Now, we describe how these replacement estimators enter the variance estimator $\hat{V}_F$. When $\overline{\sigma_i^2 \sigma_j^2}$ or $\overline{\sigma_i^2_{-jk}}$ are biased and would enter the variance estimator with a negative weight, we remove these terms, as they would otherwise introduce a negative bias. For $\overline{\sigma_i^2 \sigma_j^2}$, the weight is $U_{ij} - V_{ij}^2$, so a biased variance product estimator is removed when $U_{ij} - V_{ij}^2 < 0$. For $\overline{\sigma_i^2_{-jk}}$, the weight is $V_{ijy_j} \cdot V_{iky_k}$, but $\overline{\sigma_i^2_{-jk}}$ does not depend on $j$ and $k$ when it is biased, so we sum these weights across all such $j$ and $k$, and we remove the term if this sum is negative.

The following variance estimator extends the definition of $\hat{V}_F$ to settings where leave-three-out may fail:

$$\hat{V}_F = \sum_{i=1}^{n} \sum_{j \neq i} (U_{ij} - V_{ij}^2) \cdot G_{ij} \cdot \overline{\sigma_i^2 \sigma_j^2} + \sum_{i=1}^{n} \sum_{j \neq i} \sum_{k \neq i} V_{ijy_j} \cdot V_{iky_k} \cdot G_{i,-jk} \cdot \overline{\sigma_i^2_{-jk}},$$

where the indicators $G_{ij}$ and $G_{i,-jk}$ remove biased estimators with negative weights:

$$G_{ij} = \begin{cases} 0, & \text{if } \overline{\sigma_i^2 \sigma_j^2} = y_i^2 \sigma_j^2_{-i} \text{ and } U_{ij} - V_{ij}^2 < 0, \\
1, & \text{otherwise}, \end{cases}$$

$$G_{i,-jk} = \begin{cases} 0, & \text{if } \overline{\sigma_i^2_{-jk}} = y_i^2 \text{ and } \sum_{j \neq i} \sum_{k \neq i} V_{ijy_j} \cdot V_{iky_k} \cdot 1\{\overline{\sigma_i^2_{-jk}} = y_i^2\} < 0, \\
1, & \text{otherwise}. \end{cases}$$
4.2 Asymptotic size

In order to establish that the proposed test controls asymptotic size when there are some failures of leave-three-out, we replace the regularity condition in Assumption 4 with an analogous version that allows for some of the determinants $D_{ij}$ and $D_{ijk}$ to be zero. Otherwise, the role of Assumption 3’ below is the same as Assumption 4 in that it rules out denominators that are arbitrarily close to zero.

**Assumption 3’.** (i) $\max_{1 \leq i \leq n} M^{-1}_{ii} = O_p(1)$, and (ii) $\max_{i,j: D_{ij} \neq 0} D_{ij}^{-1} + \max_{i,j,k: D_{ijk} \neq 0} D_{ijk}^{-1} = O_p(1)$.

When computing $\hat{V}_F$, one must account for machine zero imperfections while comparing $D_{ij}$ and $D_{ijk}$ with zero in the definitions of $\bar{\sigma}_{i,j}^2$ and $\bar{\sigma}_{i,j,k}^2$. Such imperfections are typically of order $10^{-15}$; however, we propose to compare $D_{ij}$ to $10^{-4}$ and $D_{ijk}$ to $10^{-6}$. Doing so will replace any potential case of a small denominator with an upward biased alternative and ensures that Assumption 3’(ii) is automatically satisfied.

The following theorem establishes the asymptotic validity of the proposed leave-out test in settings where Assumption 1 fails. The theorem pertains to a nominal size below 0.31, as the upward biased variance estimator may not ensure validity in cases where a nominal size above 0.31 is desired. This happens because the quantile $q_{1-\alpha}(\hat{w}_{w,n-m})$ may fall below 1 when $\alpha$ is greater than 0.31.

**Theorem 4.1.** If $\alpha \in (0, 0.31]$ and Assumptions 1’, 2, 3, and 3’ hold, then, under $H_0$,

$$\limsup_{n \to \infty} \mathbb{P}(F > \hat{c}_\alpha) \leq \alpha.$$ 

An important difference between this result and that of Theorem 3.1 is that the asymptotic size may be smaller than desired, which can happen when leave-three-out fails for a large fraction of possible triples. When such conservatism materializes, there will be a corresponding loss in power relative to the result in Theorem 3.2. Otherwise, the power properties are analogous to those reported in Theorem 3.2 and we therefore omit a formal result.

**Remark 8.** Before turning to a study of the finite sample performance of the proposed test, we describe an adjustment to the test which is based on finite sample considerations. This adjustment is to rely on demeaned outcome variables in the definitions of $\hat{E}_F$, $\hat{V}_F$, and $\hat{w}$.
The benefit of relying on demeaned outcomes is that it makes the critical value invariant to the location of the outcomes. On the other hand, this adjustment removes the exact unbiasedness used to motivate the estimators of $\mathbb{E}_0[\mathcal{F}]$ and $\mathbb{V}_0[\mathcal{F} - \hat{E}_\mathcal{F}]$. However, one can show that the biases introduced by demeaning vanish at a rate that ensures asymptotic validity. Therefore, we deem the gained location invariance sufficiently desirable that we are willing to introduce a small finite sample bias to achieve it. We refer to the Appendix for exact mathematical details but note that this adjustment is used in the simulations that follow; we also probe the version that uses non-demeaned outcomes (see the end of Section 5).

5 Simulation evidence

This section documents finite sample performance of the leave-out test and compares it with that of benchmark tests that could be used by a researcher in the present context:

1. The proposed leave-out test, which will be marked as LO in the resulting tables.

2. The exact F test, marked as EF, which uses critical values from the F distribution to reject when $F > q_{1-\alpha}(F_{r,n-m})$. This test has actual size equal to nominal size in finite samples under conditionally homoskedastic normal errors for any number of regressors and restrictions. It is also asymptotically valid with conditional homoskedasticity and non-normality under certain regressor homogeneity conditions (Anatolyev, 2012), but not under general regressor designs (Calhoun, 2011).

3. Three Wald tests that reject when a heteroskedasticity-robust Wald statistic exceeds the $(1-\alpha)$-th quantile of a $\chi^2_r$ distribution, i.e., when $W > q_{1-\alpha}(\chi^2_r)$ for

$$W = (\mathbf{R}\hat{\beta} - q)' \left( \mathbf{R}S_{xx}^{-1} \left( \sum_{i=1}^{n} x_i x_i' \tilde{\sigma}_i^2 \right) S_{xx}^{-1} \mathbf{R}' \right)^{-1} (\mathbf{R}\hat{\beta} - q).$$

The three Wald tests differ only by how one constructs variance estimates $\{\tilde{\sigma}_i^2\}_{i=1}^n$, and are only palliatives:

(a) $W_1$ most closely corresponds to the original Wald test, but with the degrees-of-freedom adjustment (MacKinnon, 2013): $\tilde{\sigma}_i^2 = (y_i - x_i'\hat{\beta})^2 n/(n-m)$,
(b) $W_K$ uses variance estimates of Cattaneo et al. (2018b),

(c) $W_L$ uses leave-one-out estimates $\hat{\sigma}_i^2 = \tilde{\sigma}_i^2$ as in (8).

Asymptotically, the Wald tests $W_L$ and $W_K$ are valid with many regressors under arbitrary heteroskedasticity but not necessarily with many restrictions, while $W_1$ is valid only with few regressors and few restrictions under arbitrary heteroskedasticity.\footnote{That the baseline version $W_1$ is invalid with many restrictions was noticed empirically in Berndt and Savin (1977) and shown in Anatolyev (2012) under homoskedasticity; one can hardly expect that such measures as simply altering estimation of individual variances is able to solve the matters in a more complex heteroskedastic situation.}

By their comparison with the LO test one can see Wald test’s potential to control size, and how much distortions are due to its wrong structure when restrictions are many.

4. The test based on the split-sample idea of Kline et al. (2020) is not going to be available for our simulations, because it requires regressor numerosity to be at most a half of the sample size, which is not satisfied in the simulation design.

5.1 Simulation design

The simulation setup borrows elements of MacKinnon (2013) and adapts it to the case of many regressors as in Richard (2019) but with richer heterogeneity in the design. The outcome equation is

$$y_i = \beta_1 + \sum_{k=2}^{m} \beta_k x_{ik} + \varepsilon_i, \quad i = 1, \ldots, n,$$

where data is drawn $i.i.d.$ across $i$. Following MacKinnon (2013), the sample sizes take the values 80, 160, 320, 640, and 1280. The number of unknown coefficients is $m = 0.8n$ throughout to demonstrate the validity of the proposed test even with very many regressors. The null restricts the values of the last $r$ coefficients using $R = \begin{bmatrix} 0_{r \times (m-r)}, I_r \end{bmatrix}$. We consider both a design that contains only continuous regressors and a mixed one that also includes some discrete regressors.

In the continuous design, the regressors $x_{i2}, \ldots, x_{im}$ are products of independent standard log-normal random variables and a common multiplicative mean-unity factor drawn independently from a shifted standard uniform distribution, i.e., $0.5 + u_i$ where $u_i$ is standard
uniform. This common factor induces dependence among the regressors and rich heterogeneity in the statistical leverages of individual observations. For this design, we consider \( r = 3 \) and \( r = 0.6n \).

When also including discrete regressors, we let \( x_{i2}, \ldots, x_{i,m-r} \) be as above and let the last \( r \) regressors be group dummies. This mixed design corresponds to random assignment into \( r + 1 \) groups with the last group effect removed due to the presence of an intercept in the model. The assigned group number is the integer ceiling of \((r+1)(u_i + u_i^2)/2\), where \( u_i \) is the multiplicative factor used to generate dependence among the continuous regressors. By reusing \( u_i \) we maintain dependence between all regressors, and by using a nonlinear transformation of \( u_i \) we induce systematic variability among the \( r + 1 \) expected group sizes. We let \( r = 0.15n \), which leads the expected group sizes to vary between 4 and 13 with an average group size of about 6.5. The null corresponds to a hypothesis of equality of means across all groups.

Each regression error is a product of a standard normal random variable and an individual specific standard deviation \( \sigma_i \). The standard deviation is generated by

\[
\sigma_i = z_\zeta (1 + s_i)^\zeta, \quad i = 1, \ldots, n,
\]

where \( s_i > 0 \) depends on the design and the multiplier \( z_\zeta \) is such that the mean of \( \sigma_i^2 \) is unity. The parameter \( \zeta \in [0, 2] \) indexes the strength of heteroskedasticity, with \( \zeta = 0 \) corresponding to homoskedasticity. We consider only the two extreme cases of \( \zeta \in \{0, 2\} \). In the continuous design, we let \( s_i = \sum_{k=2}^{m} x_{ik} \), and in the mixed design, \( s_i = \sum_{k=2}^{m-r} x_{ik} + z_u u_i \). The factor \( z_u = 2r \exp(1/2) \) ensures that \( s_i \) has the same mean in both designs.

Under the null, the coefficients on the continuous regressors are all equal to \( \varrho \), where \( \varrho \) is such that the coefficient of determination, \( R^2 \), equals 0.16. The coefficients on the included group dummies are zero, which correspond to the null of equality across all groups. The intercept is chosen such that the mean of the outcomes is unity. For the continuous design this yields an intercept of \( 1 - (m-1)\varrho \exp(1/2) \), while the intercept is \( 1 - (m - r - 1)\varrho \exp(1/2) \) in the mixed design. With these parameter values, the null is \( (\beta_{m-r}, \ldots, \beta_r)' = q \), where \( q = (\varrho, \ldots, \varrho)' \in \mathbb{R}^r \) in the continuous design, and \( q = (0, \ldots, 0)' \in \mathbb{R}^r \) in the mixed design.

To document power properties, we consider both a sparse and dense deviations from the
null, and focus on the settings where \( r \) is proportional to \( n \). In parallel to the theoretical power analysis in Section 3, we consider deviations for the last \( r \) coefficients that are parameterized using

\[
(\beta_{m-r}, \ldots, \beta_m)' = q + (R\mathbb{E}[S_{xx}]^{-1}R)^{1/2} \delta,
\]

where we use the lower triangular square-root matrix. This choice of square-root implies that the alternative is sparse when only the last few entries of \( \delta \) are non-zero. As shown in Section 3, asymptotic power is governed by the norm of \( \delta \) over \( r^{1/4} \), but whether an alternative is fixed or local, additionally depends on the rate at which the tested coefficients are estimated. This rate is governed by \( \mathbb{E}[S_{xx}] \), which is reported in the Appendix.

In the continuous design, the tested coefficients are estimated at the standard parametric rate of \( n^{-1/2} \). To specify a fixed sparse alternative we therefore use \( \delta = 0.5n^{1/2}(0, \ldots, 0, 1)' \in \mathbb{R}^r \), for which \( \beta_m \) differs from the null value by approximately 0.2 (here and hereafter, the scaling is chosen so that the power is bounded away from the size and away from unity for the sample sizes we consider). Since the norm of \( \delta \) grows faster than \( r^{1/4} \), the power will be an increasing function of the sample size. For the dense alternative, we consider instead \( \delta = 0.5n^{1/2}r^{-1/2} \iota_r \), where \( \iota_r = (1, \ldots, 1)' \in \mathbb{R}^r \), for which all deviations between the tested coefficients and \( q \) shrink at the standard parametric rate of \( n^{-1/2} \). Here, power is again increasing in the sample size due to numerous deviations from the null.

In the mixed design, the group effects are not estimated consistently as the group sizes are bounded. A possible fixed sparse alternative is then \( \delta = (0, \ldots, 0, 6)' \in \mathbb{R}^r \), for which \( \beta_m \) differs from the null value of zero by roughly 3. In contrast to the continuous design, the power will decrease with sample size as the precision, with which \( \beta_m \) can be estimated, does not increase with \( n \). For the dense alternative, we use \( \delta = 1.5\iota_r \), which corresponds to a fixed alternative for every tested coefficient. Here, the power will be increasing in \( n \) due to the numerosity of deviations.
5.2 Simulation results

We present rejection rates based on 10000 Monte-Carlo replications and consider tests with nominal sizes of 1%, 5% and 10%. Furthermore, we report the frequency with which the proposed variance estimate $\hat{V}_F$ is negative and therefore replaced by the upward biased and positive alternative introduced in Remark 2. For the design that includes discrete regressors, we also report the average fraction of observations that cause a failure of leave-three-out full rank, and for which we therefore rely on an upward biased estimator of the corresponding error variance. For all sample sizes, this fraction is around 7% in the mixed design, which corresponds to the percentage of observations that belong to groups of size 2 or 3. The fraction is zero in the design that only involves continuous regressors.

Table 1 contains the actual rejection rates under the null for both the continuous and mixed designs. In settings with many regressors and restrictions, the considered versions of the “heteroskedasticity-robust” Wald test fail to control size irrespective of the design, presence of heteroskedasticity, and nominal size. The failure of the conventional Wald test, $W_1$, is spectacular, with type I error rates close to one for the continuous design, but the two versions that are robust to many regressors, $W_K$ and $W_L$, also exhibit size well above the nominal level. With few restrictions, the Wald tests show a more moderate inability to match actual size with nominal size, and the table suggests that the leave-one-out version, $W_L$, can control size in samples that are somewhat larger than considered here. Under homoskedasticity, the table reports that the exact F test indeed has exact size. However, in the heteroskedastic environments with many restrictions the exact F test is oversized with a type I error rate that approaches unity as the sample size increases.

By contrast, the proposed leave-out test exhibits nearly flawless size control as it is oversized by at most one percent across nearly all designs, nominal sizes, and whether heteroskedasticity is present or not. In the smallest sample for the continuous design, the test is somewhat conservative, presumably due to the relatively high rate of negative variance estimates (20% with homoskedasticity and 13% with heteroskedasticity) that are replaced by a strongly upward biased alternative. This rate diminishes quickly with sample size, and the fraction of negative variance estimates is already essentially zero in samples with 640 observations and 512 regressors. In the mixed design, negative variance estimates are even
### Table 1: Empirical size (in percent)

| Nominal size | Test | 1% | 5% | 10% |
|--------------|------|----|----|-----|
|              | LO   | EF | W₁ | Wₖ | W₇ | LO | EF | W₁ | Wₖ | W₇ | NEG |
| Homoskedasticity |
| n = 80   | r = 3 | 2  | 1  | 4  | 14 | 11 | 7  | 5  | 10 | 19 | 18 | 12 | 10 | 15 | 23 | 23 | 8.7 |
| n = 160  | r = 3 | 1  | 1  | 2  | 14 | 7  | 6  | 5  | 6  | 19 | 14 | 12 | 10 | 10 | 23 | 20 | 2.0 |
| n = 320  | r = 3 | 1  | 1  | 1  | 15 | 4  | 6  | 5  | 4  | 22 | 10 | 11 | 10 | 8  | 27 | 15 | 0.6 |
| n = 640  | r = 3 | 1  | 1  | 1  | 15 | 2  | 6  | 5  | 3  | 23 | 8  | 11 | 10 | 7  | 29 | 13 | 0.1 |
| n = 1280 | r = 3 | 1  | 1  | 1  | 9  | 2  | 6  | 5  | 3  | 17 | 7  | 11 | 10 | 7  | 24 | 12 | 0.0 |

| Mixed design |
| n = 80   | r = 12 | 2  | 1  | 21 | 25 | 19 | 7  | 5  | 33 | 29 | 23 | 12 | 10 | 41 | 31 | 26 | 4.8 |
| n = 160  | r = 24 | 1  | 1  | 25 | 27 | 23 | 6  | 5  | 38 | 30 | 27 | 12 | 10 | 47 | 31 | 29 | 0.4 |
| n = 320  | r = 48 | 1  | 1  | 33 | 29 | 29 | 5  | 5  | 49 | 31 | 32 | 11 | 10 | 58 | 32 | 34 | 0.1 |
| n = 640  | r = 96 | 1  | 1  | 47 | 31 | 32 | 5  | 5  | 64 | 33 | 35 | 11 | 10 | 72 | 33 | 36 | 0.0 |
| n = 1280 | r = 192 | 1 | 1 | 69 | 31 | 35 | 5  | 5  | 82 | 33 | 37 | 10 | 10 | 87 | 33 | 38 | 0.0 |

| Heteroskedasticity |
| n = 80   | r = 48 | 1  | 1  | 100 | 54 | 16 | 3  | 5  | 100 | 54 | 18 | 7  | 10 | 100 | 54 | 19 | 19.8 |
| n = 160  | r = 96 | 2  | 1  | 100 | 54 | 20 | 5  | 5  | 100 | 54 | 21 | 10 | 10 | 100 | 54 | 22 | 6.4 |
| n = 320  | r = 192 | 2 | 1 | 100 | 53 | 20 | 6 | 5 | 100 | 54 | 21 | 11 | 10 | 100 | 54 | 21 | 1.5 |
| n = 640  | r = 384 | 1 | 1 | 100 | 54 | 22 | 6 | 5 | 100 | 54 | 23 | 11 | 10 | 100 | 54 | 23 | 0.3 |
| n = 1280 | r = 768 | 1 | 1 | 100 | 53 | 22 | 5 | 5 | 100 | 53 | 23 | 11 | 10 | 100 | 53 | 23 | 0.0 |

Note: LO: leave-out test, EF: exact F test, W₁: heteroskedastic Wald test with degrees-of-freedom correction, Wₖ: heteroskedastic Wald test with Cattaneo et al. (2018b) correction, W₇: heteroskedastic Wald test with Kline et al. (2020) correction; NEG: fraction of negative variance estimates for LO (in percent). Results from 10000 Monte-Carlo replications.
Table 2: Empirical power (in percent) corresponding to 5% and 10% size

| Deviation | Homoskedasticity | Heteroskedasticity |
|-----------|------------------|--------------------|
|           | 5% | 10% | 5% | 10% | 5% | 10% | 5% | 10% |
| Test      | LO | EF | LO | EF | LO | EF | LO | EF |
| Continuous design |     |     |     |     |     |     |     |     |
| n = 80    | r = 48 | 6  | 15 | 12 | 25 | 5  | 15 | 10 | 25 |
| n = 160   | r = 96 | 16 | 23 | 26 | 34 | 12 | 21 | 22 | 34 |
| n = 320   | r = 192| 29 | 35 | 43 | 48 | 26 | 36 | 39 | 51 |
| n = 640   | r = 384| 49 | 55 | 63 | 69 | 44 | 57 | 58 | 71 |
| n = 1280  | r = 768| 74 | 80 | 84 | 88 | 68 | 84 | 81 | 92 |
| Mixed design |     |     |     |     |     |     |     |     |
| n = 80    | r = 12 | 18 | 23 | 30 | 36 | 17 | 19 | 29 | 30 |
| n = 160   | r = 24 | 18 | 18 | 29 | 28 | 27 | 28 | 41 | 41 |
| n = 320   | r = 48 | 13 | 13 | 22 | 22 | 40 | 42 | 54 | 56 |
| n = 640   | r = 96 | 10 | 10 | 18 | 18 | 60 | 65 | 73 | 77 |
| n = 1280  | r = 192| 8  | 8  | 16 | 15 | 87 | 91 | 94 | 95 |

NOTE: LO: leave-out test, EF: exact F test. Results from 10000 Monte-Carlo replications.

less prevalent, potentially due to the fact that the test uses some upward biased variance estimators for 7% of observations. Perhaps somewhat surprisingly, having 7% of observations causing failure of leave-three-out is not sufficient to bring about any discernible conservativeness in the leave-out test for this design.

Table 2 contains simulated rejection rates for the continuous and mixed designs under alternatives where the parameters deviate from their null values in one of two ways – either one tested coefficient deviates (sparse) or all tested coefficients deviate (dense). The table reports these power figures for tests with a nominal size of 5% and 10% that also control the size well, i.e., the LO and exact F tests under homoskedasticity and the LO test under heteroskedasticity.

For the continuous design, the power of the tests increases from slightly above nominal size to somewhat below unity as the number of observations increases from 80 to 1280. This pattern largely holds irrespective of the type of deviation and presence of heteroskedasticity, although the LO test is a bit more responsive to sparse deviations than to dense ones. Along this stretch of the power curve, the LO test exhibits a power loss that varies between 4 and 16 percentage points when compared to the exact F test, and in relative terms, this gap
in power shrinks as the sample size grows. Given that the number of tested restrictions in this setting is above half of the sample size, we conjecture that these figures are towards the high end of the power loss that a typical practitioner would incur in order to be robust with respect to heteroskedasticity.

In the mixed design, the fixed dense alternative exhibits similar power figures as in the continuous design, while the fixed sparse deviation generates a power function that decreases with sample size. The reason for the latter is, as discussed in the previous subsection, that the deviating group effect is not estimated more precisely as additional groups are added to the data. Upon comparison of the LO and exact F tests, we see that the differences in the power figures are only 0–7 percentage points. In light of Remark 6, which explains that there is no power difference between the LO and exact F tests when \( r/n \) is small, it is natural to attribute this almost non-existent power loss to the fact that there are four times fewer tested restrictions in this mixed design than in the continuous one.

We have also run additional simulation experiments with our baseline continuous regressor design, where we track the impact of the relative numerosities of regressors and restrictions \( r/m \) and \( m/n \) and of the coefficient of determination \( R^2 \) on the positivity failure rate of \( \hat{V}_F \), the empirical size of the test, and its empirical power. The two numerosity ratios show the severity of deviations from the standard regression testing setup, while the coefficient of determination summarizes the magnitude of regression coefficients relative to the size of error variances. The results are relegated to the Appendix (Tables A1 and A2), and here we give a brief summary. The general observation is that the percentage of negative \( \hat{V}_F \) positively varies with all three parameters, with \( R^2 \) having most pronounced impact and the ratio \( r/m \) having smallest impact. The percentage, however, is still kept within 0.1-0.2% for all combinations when \( n = 640 \), and the negativity issue is practically non-existent when \( n = 1280 \). Next, while the three parameters do affect some of the wrongly sized tests from the existing literature, they do not influence the actual empirical size of our proposal, except for minor variation in very small samples. The empirical power of the proposed test, however, is non-trivially affected by all the three parameters, whose higher values imply somewhat smaller power. The coefficient of determination, in particular, has such an effect because a higher signal relative to noise increases the variability of the individual error vari-

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ance estimators relative to their targets, and so the power tends to be negatively affected by large(r) coefficients. The numerosity ratios also have a negative effect on power because the signal gets dispersed across a larger number of regressors or restrictions as the ratios increase, which naturally reduces power. These tendencies are shared by the EF test when it is appropriately sized.

Finally, we have examined the differences that result from the use of non-demeaned outcomes when estimating individual variances and their products (see Remark 8). The general impression from those simulations is, first, the use of non-demeaned outcomes makes size control less stable; in particular, for smaller sample sizes, the LO test is undersized. Second, it seriously decreases power at all sample sizes. In practice, we therefore recommend exploiting the version with demeaned outcomes.

6 Concluding remarks

This paper develops an inference method for use in a linear regression with conditional heteroskedasticity where the objective is to test a hypothesis that imposes many linear restrictions on the regression coefficients. The proposed test rejects the null hypothesis if the conventional F statistic exceeds a linearly transformed quantile from the F-bar distribution. The central challenges for construction of the test is estimation of individual error variances and their products, which requires new ideas when the number of regressors is large. We overcome these challenges by using the idea of leaving up to three observations out when estimating individual error variances and their products. In some samples the variance estimate used for rescaling of the critical value may either be negative or cease to exist due to the presence of many discrete regressors. For both of these issues, we propose an automatic adjustment that relies on intentionally upward biased estimators which in turn leaves the resulting test somewhat conservative. Simulation experiments show that the test controls size in small samples, even in strongly heteroskedastic environments, and only exhibits very limited adjustment-induced conservativeness. The simulations additionally illustrate good power properties that signal a manageable cost in power from relying on a test that is robust to heteroskedasticity and many restrictions.

Bootstrapping and closely related resampling methods are often advocated as automatic
approaches for construction of critical values. However, in the context of linear regression with proportionality between the number of regressors and sample size, multiple papers (Bickel and Freedman, 1983; El Karoui and Purdom, 2018; Cattaneo et al., 2018b) demonstrate invalidity of standard bootstrap schemes even when inferences are made on a single regression coefficient. Under additional assumptions of homoskedasticity and restrictions on the design, El Karoui and Purdom (2018) and Richard (2019) show that problem-specific corrections to bootstrap methods can restore validity. We leave it to future research to determine whether bootstrap or other resampling methods can be corrected to ensure validity in our context of a heteroskedastic regression with many regressors and tested restrictions.

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Appendix A Leave-out test

A.1 F-bar distribution

First, when the entries of $w$ are all equal and thus equal to $1/r$, we have $\sum_{\ell=1}^{r} w_{\ell} Z_{\ell} = r^{-1} \sum_{\ell=1}^{r} Z_{\ell}$. Because $Z_1, \ldots, Z_r$ are independent $\chi^2_1$, we have $\sum_{\ell=1}^{r} Z_{\ell} \overset{d}{=} \chi^2_r$ that is independent of $Z_0$, and thus

$$
\bar{F}_{w, df} \overset{d}{=} \frac{\sum_{\ell=1}^{r} w_{\ell} Z_{\ell}}{Z_0/df} = \frac{df \chi^2_r}{r \chi^2_{df}} \overset{d}{=} F_{r, df},
$$

so Snedecor’s F distribution is a special case.

Second, as $df \to \infty$, by the law of large numbers, $Z_0/df \overset{d}{=} df^{-1} \sum_{\ell=1}^{df} z_{\ell}^2 \overset{P}{\to} E[z_{\ell}^2] = 1$, where $z_1, \ldots, z_{df}$ are independent standard normals. Hence, $\bar{F}_{w, df} \overset{d}{\to} \chi^2_w$, and so the limiting case of $\bar{F}_{w, df}$ when $df \to \infty$ is $\chi^2_w$.

Third, note that $\sum_{\ell=1}^{r} w_{\ell} Z_{\ell}$ converges in probability to its expectation of $\sum_{\ell=1}^{r} w_{\ell} = 1$ by Kolmogorov’s law of large numbers for sums of independent heterogeneous random variables. Then,

$$
\bar{F}_{w, df} \overset{d}{=} \frac{1 + (\sum_{\ell=1}^{r} w_{\ell} Z_{\ell} - 1)}{1 + (Z_0/df - 1)}
= \left(1 + \left(\sum_{\ell=1}^{r} w_{\ell} Z_{\ell} - 1\right)\right) \left(1 - \left(\sum_{\ell=1}^{df} z_{\ell}^2 / df - 1\right)(1 + o_p(1))\right)
= 1 + \left(\sum_{\ell=1}^{r} w_{\ell} Z_{\ell} - \sum_{\ell=1}^{df} z_{\ell}^2 / df\right)(1 + o_p(1)).
$$

Now, the expression in the first pair of brackets is distributed as $\chi^2_{w, -(df^{-1}, \ldots, df^{-1})}$, with mean $\sum_{\ell=1}^{r} w_{\ell} - \sum_{\ell=1}^{df} df^{-1} = 1 - 1 = 0$ and variance $\sum_{\ell=1}^{r} 2w_{\ell}^2 + \sum_{\ell=1}^{df} 2df^{-2} = 2 \sum_{\ell=1}^{r} w_{\ell}^2 + 2/df$. Using Lyapounov’s central limit theorem for sums of independent heterogeneous random
variables, we obtain that
\[
\frac{\bar{F}_{w,df} - 1}{\sqrt{2\sum_{\ell=1}^r w_\ell^2 + 2/df}} \overset{d}{\to} N(0, 1),
\]
and (3) follows.

\subsection*{A.2 Leave-out algebra}

For an arbitrary triple \((i, j, k)\) with \(i \neq j \neq k \neq i\), the following shows that \(\sum_{\ell \neq i, j, k} x_\ell x'_\ell\) is invertible if and only if \(D_{ijk} > 0\). By SMW it suffices to show that \(D_{ijk} > 0\) is equivalent to
\[
1 - x'_i \left(\sum_{\ell \neq j, k} x_\ell x'_\ell\right)^{-1} x_i > 0 \text{ when } D_{jk} > 0.
\]
Now, we have
\[
\left(\sum_{\ell \neq j, k} x_\ell x'_\ell\right)^{-1} = S_{xx}^{-1} + S_{xx}^{-1} \left(x'_j M_{jj} M_{jk} x'_k\right) \left(M_{ij} M_{jk} M_{kk}\right)^{-1} \left(x'_j M_{ij} x'_k\right) S_{xx}^{-1},
\]
thus
\[
1 - x'_i \left(\sum_{\ell \neq j, k} x_\ell x'_\ell\right)^{-1} x_i = M_{ii} - \left(M_{ij} M_{ik}\right)^{-1} \left(M_{ij} M_{jk} M_{kk}\right)^{-1} \left(M_{ij}\right) = \frac{D_{ijk}}{D_{jk}}.
\]
Therefore, \(D_{ijk} > 0\) if and only if \(1 - x'_i \left(\sum_{\ell \neq j, k} x_\ell x'_\ell\right)^{-1} x_i > 0\).

\subsection*{A.3 Location estimator}

The following shows that \(E_0[F] = \sum_{i=1}^n B_{ii} \sigma_i^2\) and that \(E[\hat{E}_F] = E_0[F]\) which yields that \(F - \hat{E}_F\) is centered at zero under the null. When \(H_0\) holds so that \(R\beta = q\), we have
\[
\hat{R}\hat{\beta} = q + RS_{xx}^{-1} \sum_{i=1}^n x_i \varepsilon_i.
\]
Inserting this relationship into the definition of \(F\) yields
\[
F = \left(RS_{xx}^{-1} \sum_{i=1}^n x_i \varepsilon_i\right)' \left(RS_{xx}^{-1} R\right)^{-1} \left(RS_{xx}^{-1} \sum_{i=1}^n x_i \varepsilon_i\right) = \sum_{i=1}^n \sum_{j=1}^n B_{ij} \varepsilon_i \varepsilon_j,
\]
where $B_{ij} = x_i' S_{xx}^{-1} R' (RS_{xx}^{-1} R')^{-1} R S_{xx}^{-1} x_j$. Independent sampling and exogenous regressors yield $\mathbb{E}[\varepsilon_i \varepsilon_j] = 0$ whenever $i \neq j$, so

$$
\mathbb{E}_0[\mathcal{F}] = \mathbb{E}_0 \left[ \sum_{i=1}^{n} \sum_{j=1}^{n} B_{ij} \varepsilon_i \varepsilon_j \right] = \sum_{i=1}^{n} B_{ii} \sigma_i^2.
$$

The matrix $B = (B_{ij})$ is a projection matrix, so it is symmetric and satisfies $r = \text{tr}(I_r) = \text{tr}(B) = \sum_{i=1}^{n} B_{ii}$ as claimed in the main text. It follows from (Kline et al., 2020, Lemma 1) that $\mathbb{E}[\hat{\sigma}_i^2] = \sigma_i^2$, so $\mathbb{E}[\sum_{i=1}^{n} B_{ii} \hat{\sigma}_i^2] = \mathbb{E}_0[\mathcal{F}]$ since $B_{11}, \ldots, B_{nn}$ are known.

Next, we show that the conditional variance of $\mathcal{F} - \hat{E}_{\mathcal{F}}$ satisfies the relation given in (9). Since $\hat{\sigma}_i^2 = \hat{y}_i (y_i - x_i' \hat{\beta}) / M_{ii} = \sum_{j=1}^{n} \frac{M_{ij}}{M_{ii}} y_i \varepsilon_j$, we have that, under $H_0$,

$$
\mathcal{F} - \hat{E}_{\mathcal{F}} = \sum_{i=1}^{n} \sum_{j=1}^{n} B_{ij} \varepsilon_i \varepsilon_j - \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{B_{ii}}{M_{ii}} M_{ij} y_i \varepsilon_j
$$

$$
= \sum_{i=1}^{n} \sum_{j=1}^{n} \left( B_{ij} - \frac{B_{ii}}{M_{ii}} M_{ij} \right) \varepsilon_i \varepsilon_j - \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{B_{ij}}{M_{ii}} M_{ij} x_i' \beta \varepsilon_j
$$

$$
= \sum_{i=1}^{n} \sum_{j \neq i} C_{ij} \varepsilon_i \varepsilon_j - \sum_{j=1}^{n} \left( \sum_{i=1}^{n} \left( \frac{B_{ii}}{M_{ii}} - \frac{B_{ij}}{M_{jj}} \right) M_{ij} x_i' \beta \right) \varepsilon_j,
$$

where $C_{ij} = B_{ij} - \frac{M_{ij}}{2} \left( \frac{B_{ii}}{M_{jj}} + \frac{B_{jj}}{M_{ii}} \right)$ is a set of symmetric weights, i.e., $C_{ij} = C_{ji}$. Note that we have subtracted off $n$ zeroes in the form of $\varepsilon_j x_i / M_{ij} \sum_{i=1}^{n} M_{ij} x_i' \beta$, which exploits the identity $\sum_{i=1}^{n} M_{ij} x_i = 0$. Independent sampling yields $\mathbb{E}[\varepsilon_i \varepsilon_j \varepsilon_k] = 0$ whenever $i \neq j$ for any $k$, so the two components in this representation of $\mathcal{F} - \hat{E}_{\mathcal{F}}$ are uncorrelated. A straightforward variance calculation for each component leads to the variance expression in (9):

$$
\nabla_0[\mathcal{F} - \hat{E}_{\mathcal{F}}] = 2 \sum_{i=1}^{n} \sum_{j \neq i} C_{ij}^2 \sigma_i^2 \sigma_j^2 + \sum_{i=1}^{n} \left( \sum_{j \neq i} \left( \frac{B_{ii}}{M_{jj}} - \frac{B_{ij}}{M_{ji}} \right) M_{ij} x_j' \beta \right)^2 \sigma_i^2
$$

$$
= \sum_{i=1}^{n} \sum_{j \neq i} U_{ij} \sigma_i^2 \sigma_j^2 + \sum_{i=1}^{n} \left( \sum_{j \neq i} V_{ij} x_j' \beta \right)^2 \sigma_i^2,
$$

where $U_{ij} = 2 C_{ij}^2$ and $V_{ij} = M_{ij} \left( \frac{B_{ii}}{M_{ji}} - \frac{B_{ij}}{M_{ji}} \right)$. 

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A.4 Variance estimator

First, we show that \( \hat{\sigma}_i^2 \sigma_j^2 = \hat{\sigma}_j^2 \sigma_i^2 \). To establish this equality we introduce some notation used to describe \( \hat{\sigma}_i^2 \) and \( \hat{\sigma}_i^2 \). Define

\[
\tilde{M}_{ij, -i} = \frac{M_{ij}}{M_{ii}},
\]
\[
\tilde{M}_{ik, -ij} = \frac{M_{ik} - M_{ij}\tilde{M}_{jk, -j}}{D_{ij}/M_{jj}},
\]
and

\[
\tilde{M}_{il, -ijk} = \frac{M_{il} - M_{ij}\tilde{M}_{il, -jk} - M_{ik}\tilde{M}_{k\ell, -jk}}{D_{ij}/D_{jk}},
\]

where the indices following the commas are all different and their ordering is irrelevant (note that \( \tilde{M}_{ik, -ij} \) was also introduced in the main text). In addition, we will also at times write \( \tilde{M}_{i\ell, -ijj} \) for \( \tilde{M}_{i\ell, -ij} \). With these definitions we now have

\[
\hat{\sigma}_i^2 = y_i \left( y_i - x_i' \hat{\beta}_{-i} \right) = y_i \sum_{j=1}^{n} \tilde{M}_{ij, -i} y_k,
\]
\[
\hat{\sigma}_i^2 \sigma_j^2 = y_i \left( y_i - x_i' \hat{\beta}_{-ij} \right) = y_i \sum_{k=1}^{n} \tilde{M}_{ik, -ij} y_k,
\]
and

\[
\hat{\sigma}_i^2 \sigma_j^2 = y_i \left( y_i - x_i' \hat{\beta}_{-ijk} \right) = y_i \sum_{\ell \neq k} \tilde{M}_{i\ell, -ijk} y_\ell.
\]

To see why these relationships hold note that \( \tilde{M}_{ii, -ij} = 1 \), \( \tilde{M}_{ij, -ij} = 0 \), and

\[
-x_i' \left( \sum_{\ell \neq i, j} x_\ell x_\ell' \right)^{-1} x_k = M_{ik} - \left( M_{ii} - 1 \right) \begin{bmatrix} M_{ii} & M_{ij} \\ M_{ij} & M_{jj} \end{bmatrix}^{-1} \begin{bmatrix} M_{ik} \\ M_{jk} \end{bmatrix} = \tilde{M}_{ik, -ij},
\]

(18)
where the first equality follows from (15). Similarly, note that \( \tilde{M}_{ii,-ijk} = 1 \), \( \tilde{M}_{ij,-ijk} = \tilde{M}_{ik,-ijk} = 0 \), and use SMW and (16) to see that

\[
\left( \sum_{l \neq i,j,k} x_l x'_l \right)^{-1} = \left( \sum_{l \neq j,k} x_l x'_l \right)^{-1} + \frac{\left( \sum_{l \neq j,k} x_l x'_l \right)^{-1} x_i x'_i \left( \sum_{l \neq j,k} x_l x'_l \right)^{-1}}{D_{ijk} / D_{jk}}
\]

which together with (15) yields

\[
-x'_i \left( \sum_{l \neq i,j,k} x_l x'_l \right)^{-1} x_\ell = -\frac{x_i \left( \sum_{l \neq j,k} x_l x'_l \right)^{-1} x_\ell}{D_{ijk} / D_{jk}} = \tilde{M}_{i\ell,-ijk}.
\]

Relying on the newly defined \( \tilde{M}_{ik,-ij} \) and \( \tilde{M}_{i\ell,-ij} \) we can write

\[
\tilde{\sigma}_j^2 \tilde{\sigma}_i^2 = y_i y_j \sum_{k=1}^{n} \sum_{\ell \neq k} \tilde{M}_{jk,-ij} \tilde{M}_{ik,-ijk} y_k y_\ell \quad \text{and} \quad \tilde{\sigma}_i^2 \tilde{\sigma}_j^2 = y_i y_j \sum_{k=1}^{n} \sum_{\ell \neq k} \tilde{M}_{ik,-ij} \tilde{M}_{jk,-ijk} y_k y_\ell,
\]

from which \( \tilde{\sigma}_j^2 \tilde{\sigma}_i^2 = \tilde{\sigma}_i^2 \tilde{\sigma}_j^2 \) will follow if

\[
\tilde{M}_{jk,-ij} \tilde{M}_{i\ell,-ijk} + \tilde{M}_{j\ell,-ij} \tilde{M}_{ik,-ij\ell} = \tilde{M}_{ik,-ij} \tilde{M}_{jk,-ij\ell} + \tilde{M}_{i\ell,-ij} \tilde{M}_{jk,-ijk}.
\]

That this equality holds follows immediately from the observation that

\[
\tilde{M}_{i\ell,-ijk} = \tilde{M}_{i\ell,-ij} - \tilde{M}_{ik,-ij} \tilde{M}_{k\ell,-ijk},
\]

which shows equality between

\[
\tilde{M}_{jk,-ij} \tilde{M}_{i\ell,-ijk} + \tilde{M}_{j\ell,-ij} \tilde{M}_{ik,-ij\ell} = \tilde{M}_{jk,-ij} (\tilde{M}_{i\ell,-ij} - \tilde{M}_{ik,-ij} \tilde{M}_{k\ell,-ijk})
\]

+ \( \tilde{M}_{j\ell,-ij} (\tilde{M}_{ik,-ij} - \tilde{M}_{i\ell,-ij} \tilde{M}_{k\ell,-ijk}) \)

and

\[
\tilde{M}_{ik,-ij} \tilde{M}_{j\ell,-ijk} + \tilde{M}_{i\ell,-ij} \tilde{M}_{jk,-ij\ell} = \tilde{M}_{ik,-ij} (\tilde{M}_{j\ell,-ij} - \tilde{M}_{jk,-ij} \tilde{M}_{k\ell,ij\ell})
\]

+ \( \tilde{M}_{i\ell,-ij} (\tilde{M}_{jk,-ij} - \tilde{M}_{j\ell,-ij} \tilde{M}_{k\ell,ij\ell}) \).
Now we derive that \( \hat{V}_F \) is a conditionally unbiased estimator of the null variance given in (9). That \( \hat{\sigma}_i^2 \hat{\sigma}_j^2 \) is conditionally unbiased for \( \sigma_i^2 \sigma_j^2 \) was given in the main text, so here we elaborate on the bias introduced by the second component. Note that \((y_j, y_k)\) is conditionally independent of \(\hat{\sigma}_{i-jk}^2\) and \(\mathbb{E}[y_j y_k] = x'_j \beta x_k \beta + \sigma_j^2 1_{\{j=k\}}\) so that

\[
\mathbb{E}\left[ \sum_{i=1}^{n} \sum_{j \neq i} \sum_{k \neq i} V_{ij} y_j \cdot V_{ik} y_k \cdot \hat{\sigma}_{i-jk}^2 \right] = \sum_{i=1}^{n} \sum_{j \neq i} \sum_{k \neq i} V_{ij} V_{ik} \cdot \mathbb{E}[y_j y_k] \cdot \mathbb{E}[\hat{\sigma}_{i-jk}^2] \\
= \sum_{i=1}^{n} \left( \sum_{j \neq i} V_{ij} x'_j \beta \right)^2 \sigma_i^2 + \sum_{i=1}^{n} \sum_{j \neq i} V_{ij}^2 \sigma_j^2 \sigma_i^2.
\]

The first component of this expectation is equal to the corresponding second part of the target variance \( \mathbb{V}_0[F - \bar{E}_F] \), but the second component is a bias which we correct for by using \(\sum_{i=1}^{n} \sum_{j \neq i} (U_{ij} - V_{ij}) \hat{\sigma}^2_i \hat{\sigma}^2_j\) instead of \(\sum_{i=1}^{n} \sum_{j \neq i} U_{ij} \hat{\sigma}^2_i \hat{\sigma}^2_j\) as an estimator of the first part in \( \mathbb{V}_0[F - \bar{E}_F] \).

Now we derive the test statistic that relies on restricted variance estimates. Denote by \(S_{-i} = \sum_{j \neq i} x_j x'_j\) leave-one-out analogs of \(S_{xx}\). The restricted LO estimates are \(\tilde{\beta}_{-i} = \hat{\beta}_{-i} - S_{-i}^{-1} R' (R S_{-i}^{-1} R')^{-1} (R \hat{\beta}_{-i} - q)\), the restricted LO residuals are \(y_i - x'_i \tilde{\beta}_{-i}\), and the resulting restricted individual variance estimates are \(\hat{\sigma}_i^2 = y_i (y_i - x'_i \tilde{\beta}_{-i})\), which are conditionally unbiased under \(H_0\).

To derive the restricted estimator \(\bar{E}_F\) of \(\mathbb{E}_0[F]\) and null conditional variance \(\mathbb{V}_0[F - \bar{E}_F]\) of \(F - \bar{E}_F\), note first that the restricted LO residuals are

\[
y_i - x'_i \tilde{\beta}_{-i} = y_i - x'_i \tilde{\beta}_{-i} - \sum_{j \neq i} \gamma_{ij} \epsilon_j,
\]

where \(\gamma_{ij} = x'_i S_{-i}^{-1} R' (R S_{-i}^{-1} R')^{-1} R S_{-i}^{-1} x_j\). By the Woodbury matrix identity,

\[
\gamma_{ij} = \frac{x'_i S_{-i}^{-1} R' (R S_{-i}^{-1} R')^{-1} R S_{-i}^{-1} x_j}{M_{ii}} \\
= \frac{x'_i S_{-i}^{-1} R' (R S_{-i}^{-1} R')^{-1} R S_{-i}^{-1}}{M_{ii} + x'_i S_{-i}^{-1} R' (R S_{-i}^{-1} R')^{-1} R S_{-i}^{-1} x_i} \left( x_j - \frac{M_{ij}}{M_{ii}} x_i \right).
\]
Note that $\Upsilon_{ii} = 0$. As a result, the restricted variance estimates are

$$\tilde{\sigma}_i^2 = \hat{\sigma}_i^2 - (x_i'\beta_i + \varepsilon_i) \sum_{j \neq i} \Upsilon_{ij} \varepsilon_j,$$

and the restricted estimator of $E_0[F]$ is

$$\tilde{E}_F = \sum_{i=1}^n B_{ii} \tilde{\sigma}_i^2 = \hat{E}_F - \sum_{i=1}^n B_{ii} (x_i'\beta_i + \varepsilon_i) \sum_{j \neq i} \Upsilon_{ij} \varepsilon_j,$$

so that

$$F - \tilde{E}_F = \sum_{i=1}^n \sum_{j \neq i} (C_{ij} - B_{ii} \Upsilon_{ij}) \varepsilon_i \varepsilon_j - \sum_{i=1}^n \left( \sum_{j \neq i} (V_{ij} + B_{jj} \Upsilon_{ji}) x_j'\beta \right) \varepsilon_j,$$

and so

$$V_0 \left[ F - \tilde{E}_F \right] = \sum_{i=1}^n \sum_{j \neq i} \tilde{U}_{ij} \sigma_i^2 \sigma_j^2 + \sum_{i=1}^n \left( \sum_{j \neq i} \tilde{V}_{ij} x_j'\beta \right)^2 \sigma_i^2,$$

where $\tilde{U}_{ij} = 2 (C_{ij} - B_{ii} \Upsilon_{ij})^2$ and $\tilde{V}_{ij} = V_{ij} + B_{jj} \Upsilon_{ji}$. The estimate $\tilde{V}_F$ can be constructed similarly to $\hat{V}_F$ using coefficients $\tilde{U}_{ij}$ and $\tilde{V}_{ij}$ in place of $U_{ij}$ and $V_{ij}$ and restricted leave-three-out variance estimates $\tilde{\sigma}_{i, -jk}^2 = y_i (y_i - x_i'\tilde{\beta}_{-ijk})$, where $\tilde{\beta}_{-ijk} = \hat{\beta}_{-ijk} - S_{-ijk}^{-1} R (RS_{-ijk}^{-1} R')^{-1} (R \hat{\beta}_{-ijk} - q)$ are restricted leave-three-out parameter estimates, and $S_{-ijk} = \sum_{i \neq j, k} x_i x_j' x_k'$ are leave-three-out analogs of $S_{xx}$.

Note that restricted estimation of error variances when the null is imposed may in fact facilitate existence of leave-three-out estimators. For example, suppose that the null’s parameters are $R = (0_{m_2 \times m_1}, I_{m_2})$ and $q = 0_{m_2 \times 1}$, where $m_1 + m_2 = m$ is the total regressor dimensionality, so that one tests for joint insignificance of the last $m_2$ parameters. Suppose that the first $m_1$ regressors are continuously distributed, while the last $m_2$ regressors are discrete. Then, when the null is imposed, the discrete regressors do not enter the design matrix, and there is no problem with the existence of leave-out estimators.
A.5 Computational remarks

The representation of leave-one-out residuals and individual leave-one-out variance estimators given in the main text follows immediately from (6). Here we derive the representation of the leave-two-out and leave-three-out residuals given in the main text and used in implementation of the testing procedure. In (18), we showed for $j \neq i$ that $y_i - x_i' \hat{\beta}_{-ij} = \sum_{k=1}^{n} M_{ik, -ij} y_k$ where $M_{ik, -ij} = \frac{M_{ik} - M_{ij} M_{jk} / M_{jj}}{D_{ij} / M_{jj}}$. Thus it follows that

$$y_i - x_i' \hat{\beta}_{-ij} = \sum_{k=1}^{n} \frac{M_{jj} M_{ik} - M_{ij} M_{jk}}{D_{ij}} y_k = \frac{M_{jj} \sum_{k=1}^{n} M_{ik} y_k - M_{ij} \sum_{k=1}^{n} M_{jk} y_k}{D_{ij}}$$

$$= \frac{M_{jj} (y_i - x_i' \hat{\beta}) - M_{ij} (y_j - x_j' \hat{\beta})}{D_{ij}},$$

as claimed.

To break the monotonicity of the constant reliance on SMW, we establish the representation of the leave-three-out residuals using blockwise inversion. For $i \neq j \neq k \neq i$, $y_i - x_i' \hat{\beta}_{-ijk}$ is the first entry of the vector

$$\begin{bmatrix} M_{ii} & M_{ij} & M_{ik} \\ M_{ij} & M_{jj} & M_{jk} \\ M_{ik} & M_{jk} & M_{kk} \end{bmatrix}^{-1} \begin{bmatrix} y_i - x_i' \hat{\beta} \\ y_j - x_j' \hat{\beta} \\ y_k - x_k' \hat{\beta} \end{bmatrix}$$

which by blockwise inversion equals

$$\begin{bmatrix} M_{ii} - \left( \begin{bmatrix} M_{ij} \\ M_{ik} \end{bmatrix} \right)' \begin{bmatrix} M_{jj} & M_{jk} \\ M_{jk} & M_{kk} \end{bmatrix} \begin{bmatrix} M_{ij} \\ M_{ik} \end{bmatrix} \right)^{-1} \begin{bmatrix} y_i - x_i' \hat{\beta} - \begin{bmatrix} M_{ij} \\ M_{ik} \end{bmatrix} \left( \begin{bmatrix} M_{jj} & M_{jk} \\ M_{jk} & M_{kk} \end{bmatrix} \right)^{-1} \begin{bmatrix} y_j - x_j' \hat{\beta} \\ y_k - x_k' \hat{\beta} \end{bmatrix} \right) \\ = D_{ijk} / D_{jk} \end{bmatrix}$$

which in turn is the representation provided in the main text.
Appendix B  Asymptotic size and power

B.1 Asymptotic size

As a preliminary observation, note that \( \max_i B_{ii} = O_p(\epsilon n) \), Assumption 3(i), and Assumption 4 imply that \( \max_i (\sum_{j \neq i} V_{ij} x_j' \beta)^2 / r = o_p(1) \). This follows from the idempotency of \( M \) through

\[
\max_i \frac{1}{r} \left( \sum_{j \neq i} V_{ij} x_j' \beta \right)^2 \leq \frac{1}{r} \sum_{i=1}^n \left( \sum_{j=1}^n M_{ij} B_{jj} x_j' \beta \right)^2 \\
\leq \max_i \frac{(x_i' \beta)^2}{M_{ii}^2} \frac{1}{r} \sum_{j=1}^n B_{jj}^2 \leq \max_i \frac{(x_i' \beta)^2}{M_{ii}^2} \max_j B_{jj} = o_p(1).
\]

Thus, we have \( \max_i (\sum_{j \neq i} V_{ij} x_j' \beta)^2 / r = o_p(1) \) under either of the two possible conditions in Assumption 3(ii). Similarly, we have that \( \max_i B_{ii} / (\epsilon_n r) = o_p(1) \) under either of the two possible conditions in Assumption 3(ii). Finally, we will repeatedly rely on the simple bound that \( \max_i \sum_{j=1}^n M_{ij} B_{jj} x_j' \beta \leq 1 / r \max_i (x_i' \beta)^2 M_{ii}^2 \) holds.

Finally, as a further motivation of the high-level condition \( \max_i (\sum_{j \neq i} V_{ij} x_j' \beta)^2 / r = o_p(1) \) we provide a simple example where it holds with \( r \) proportional to \( n \). This example is characterized by

1. \( n/r = O(1) \) and \( \max_i \sum_{j=1}^n \mathbf{1}\{M_{ij} \neq 0\} = O_p(\epsilon_n n^{1/2}) \).

This example focus on settings where the number of restrictions is large relative to sample size, and covers any model with group specific regressors only and maximal group sizes that grow slower than \( n^{1/2} \). This is so since \( M_{ij} = 0 \) for any two observations in different groups. Here, we have

\[
\max_i \frac{1}{r} \left( \sum_{j \neq i} V_{ij} x_j' \beta \right)^2 = \frac{1}{r} \max_i \left( \sum_{j=1}^n M_{ij} B_{jj} x_j' \beta \right)^2 \\
\leq \frac{1}{r} \max_i \frac{(x_i' \beta)^2}{M_{ii}^2} \left( \sum_{j=1}^n \mathbf{1}\{M_{ij} \neq 0\} \right)^2 = o_p(1),
\]

where the order statement use 1., Assumption 3(i), and Assumption 4.

**Proof of Theorem 3.1.** The proof naturally separates into three parts. In the first two parts,
we consider an infeasible version of the test that relies on $V_0[F - \hat{E}_F]$ instead of $\hat{V}_F$. The first part then establishes asymptotic size control when $r$ grows to infinity with $n$, while the second part establishes size control when $r$ is fixed in the asymptotic regime. The third part shows consistency of the proposed variance estimator, i.e., $\hat{V}_F / V_0[F - \hat{E}_F] \xrightarrow{p} 1$. Together, these results and the continuous mapping theorem lead to the conclusion of the theorem irrespective of how $r$ is viewed in relation to the sample size.

**Asymptotic size control when $r$ is growing** Using (17) and defining the vector $\tilde{x}_i = \sum_{j=1}^n M_{ij}\bar{B}_{jj}x_j = -\sum_{j\neq i} V_{ij}x_j$, we can write

$$F - \hat{E}_F = \sum_{i=1}^n \sum_{j\neq i} C_{ij}\varepsilon_i\varepsilon_j - \sum_{i=1}^n \tilde{x}_i^t\beta\varepsilon_i.$$ 

Under Assumption 2, it follows (see Kline et al., 2020, Lemma B.1 and its proof) that $F - \hat{E}_F$ scaled down by $V_0[F - \hat{E}_F]^{1/2}$ is asymptotically standard normal provided that

$$\frac{\text{trace}(C^4)}{V_0[F - \hat{E}_F]^2} = o_p(1) \quad \text{and} \quad \frac{\max_i(\tilde{x}_i^t\beta)^2}{V_0[F - \hat{E}_F]} = o_p(1),$$

where $C$ is a matrix with $C_{ij}$ as its $(i,j)$-th entry. To show that (a) and (b) holds, we first note that $C = B - \frac{1}{2}(D_{B\otimes M}M + MD_{B\otimes M})$, where $B$ has $B_{ij}$ as its $(i,j)$-th entry and $D_{B\otimes M}$ is a diagonal matrix with $B_{ii}/M_{ii}$ as its $(i,i)$-th entry. Note also that for even $p$, $\text{trace}(C^p) = \text{trace}(B) + \text{trace}(2^{-p}(D_{B\otimes M}M + MD_{B\otimes M})^p)$, as $B$ and $M$ are idempotent and orthogonal. Since $V_0[F - \hat{E}_F] \geq \min_i \sigma_i^4 \sum_{i=1}^n \sum_{j\neq i} 2C_{ij}^2$, these observations yield

$$\sum_{i=1}^n \sum_{j\neq i} 2C_{ij}^2 = 2\text{trace}(C^2) \geq 2\text{trace}(B) = 2r.$$ 

For (a), we can now observe that

$$\text{trace}(C^4) = \text{trace}(B) + \frac{1}{16} \text{trace}((D_{B\otimes M}M + MD_{B\otimes M})^4) \leq r + \sum_{i=1}^n \frac{\sigma_i^4}{M_{ii}} \leq r \left(1 + (\max_i B_{ii})^3 (\max_i M_{ii}^{-1})^{1/4}\right).$$
Since Assumption 4 implies that \( \max_i M_{ii}^{-1} = O_p(1) \), (a) therefore holds with a rate of \( 1/r \).

Condition (b) follows immediately from Assumption 3(ii) and that the variance \( \mathbb{V}_0[\mathcal{F} - \hat{E}_\mathcal{F}] \) is at least of order \( r \).

As the above establishes asymptotic standard normality of \( \mathcal{F} - \hat{E}_\mathcal{F} \) scaled down by \( \sqrt{\mathbb{V}_0[\mathcal{F} - \hat{E}_\mathcal{F}]} \), it now suffices for asymptotic size control to show that

\[
\frac{q_{1-\alpha}(F_{w,n-m}) - 1}{\sqrt{2\sum_{\ell=1}^r \hat{w}_\ell^2 + 2/(n-m)}} \overset{p}{\rightarrow} q_{1-\alpha}(\Phi)
\]

which by (3) follows provided that \( \max_\ell \hat{w}_\ell \overset{p}{\rightarrow} 0 \) and \( n - m \rightarrow \infty \). The latter condition is implied by \( \max_i M_{ii}^{-1} = O_p(1) \) as it leads to \( \limsup_{n \rightarrow \infty} m/n < 1 \).

We prove that \( \max_\ell \hat{w}_\ell \overset{p}{\rightarrow} 0 \), by establishing that entries of \( w_\mathcal{F} = (w_1, \ldots, w_r)' \) converges to zero when \( r \rightarrow \infty \) and that

\[
\max_\ell (\hat{w}_\ell - w_\ell)^2 = O \left( \frac{\max_i B_{ii}^{1/2}}{\epsilon r^{1/2}} \right)
\]

where the entries of both \( w_\mathcal{F} \) and \( \hat{w} \) are sorted by magnitude. Since \( \max_\ell \hat{w}_\ell \leq \max_\ell \bar{w}_\ell \) and \( \frac{\max_i B_{ii}}{\epsilon n r} = o_p(1) \) these observations yield the desired conclusion.

First, have that

\[
\max_\ell w_\ell \leq \|w_\mathcal{F}\| = \sqrt{\sum_{i=1}^n \sum_{j=1}^n B_{ij}^2 \sigma_i^2 \sigma_j^2} \leq \frac{\max_i \sigma_i^2 \sqrt{r}}{\min_i \sigma_i^2 r} = o_p(1).
\]

Second, we have that

\[
\max_\ell (\hat{w}_\ell - w_\ell)^2 \leq \left( \frac{\mathbb{E}_0[\mathcal{F}]}{\hat{E}_\mathcal{F}} \right)^2 \sum_{i=1}^n (\hat{\sigma}_i^2 - \sigma_i^2) \sum_{j=1}^n B_{ij}^2 (\hat{\sigma}_j^2 - \sigma_j^2) \left( \sum_{i=1}^n B_{ii} \sigma_i^2 \right)^2.
\]

It follows from the first part of this proof that \( \hat{E}_\mathcal{F}/\mathbb{E}_0[\mathcal{F}] \overset{p}{\rightarrow} 1 \), and from an application of
v theorem implies that $F_{q}$ in probability (as we can otherwise argue along subsequences) and will use this limit. The entries of this limit are necessarily strictly positive. It follows from \(r\) Asymptotic size control when \(n\) also have that $2w$ it follows that the proposed variance estimator satisfies the decomposition Cauchy-Schwarz that

$$\mathbb{E} \left[ \sum_{i=1}^{n} (\hat{\sigma}_{i}^{2} - \sigma_{i}^{2}) \sum_{j=1}^{n} B_{ij}^{2} (\hat{\sigma}_{j}^{2} - \sigma_{j}^{2}) \right] \leq \sum_{i=1}^{n} \mathbb{V} \left[ \hat{\sigma}_{i}^{2} \right]^{1/2} \mathbb{V} \left[ \sum_{j=1}^{n} B_{ij}^{2} (\hat{\sigma}_{j}^{2} - \sigma_{j}^{2}) \right]$$

$$= O \left( \max_{i} \frac{(x_{i}/B)^{2}}{M_{ii}} \sum_{i=1}^{n} \sqrt{\sum_{j=1}^{n} B_{ij}^{4}} \right) = O \left( \max_{i} \frac{B_{ii}^{1/2}}{\varepsilon_{i}^{1/2}} \right).$$

**Asymptotic size control when \(r\) is fixed** For \(r\) fixed, it must be that max, $B_{ii} = o_{p}(1)$ by Assumption 2. When \(r\) is fixed, we can without loss of generality suppose that $w_{\mathcal{F}}$ converges in probability (as we can otherwise argue along subsequences) and will use $\tilde{w}_{\mathcal{F}}$ to denote this limit. The entries of this limit are necessarily strictly positive. It follows from \(19\), that $\tilde{w} \overset{p}{\to} \tilde{w}_{\mathcal{F}}$ and thus also that $\tilde{w} \overset{p}{\to} \tilde{w}_{\mathcal{F}}$. This conclusion naturally implies that we also have $\tilde{F}_{\mathcal{F}}/\mathbb{E}_{0}[\mathcal{F}] \overset{p}{\to} 1$.

Lyapounovs central limit theorem and max, $B_{ii} = o_{p}(1)$ implies that $\mathbb{V}[R\hat{\beta}]^{-1/2}(R\hat{\beta} - q) \overset{d}{\to} N(0, I_{r})$ which when coupled with the conclusions above and the continuous mapping theorem implies that $\mathcal{F}/\tilde{F}_{\mathcal{F}} \overset{d}{\to} \tilde{X}^{2}_{\tilde{w}_{\mathcal{F}}}$. Finally, we have that $\mathbb{V}_{0}[\mathcal{F} - \tilde{F}_{\mathcal{F}}]/\mathbb{E}_{0}[\mathcal{F}]^{2} = V_{0}[\mathcal{F}]/\mathbb{E}_{0}[\mathcal{F}]^{2} + o_{p}(1) = 2\|\tilde{w}_{\mathcal{F}}\|^{2} + o_{p}(1)$ and due to the continuous nature of this variance we also have that $2\|\tilde{w}\|^{2} + 2/(n - m) = 2\|\tilde{w}_{\mathcal{F}}\|^{2} + o_{p}(1)$. Thus it follows that

$$\lim_{n \to \infty} \mathbb{P} \left( \frac{\mathcal{F}}{\tilde{F}_{\mathcal{F}}} > q_{1-\alpha}(\tilde{F}_{\tilde{w}_{n-m}}) + (q_{1-\alpha}(\tilde{F}_{\tilde{w}_{n-m}}) - 1) \left( \frac{\mathbb{V}_{0}[\mathcal{F} - \tilde{F}_{\mathcal{F}}]^{1/2} - \mathbb{E}_{\mathcal{F}}}{\sqrt{2 \sum_{i=1}^{n} \tilde{w}_{i}^{2}/(n - m) - 1}} \right) \right) = 1 - \alpha.$$ 

To finish the proof we only need to establish that $\tilde{V}_{\mathcal{F}}/\mathbb{V}_{0}[\mathcal{F} - \tilde{F}_{\mathcal{F}}] \overset{p}{\to} 1$.

**Consistency of variance estimator** In the remainder of this proof $\sum_{i \neq j}$ is shorthand for the double sum $\sum_{i=1}^{n} \sum_{j \neq i}$, $\sum_{i \neq j \neq k}$ denotes $\sum_{i=1}^{n} \sum_{j \neq i} \sum_{k \neq i, j}$, and $\sum_{i \neq j \neq k \neq \ell}$ abbreviates $\sum_{i=1}^{n} \sum_{j \neq i} \sum_{k \neq i,j} \sum_{\ell \neq i,j,k}$. Similarly, we use $\sum_{i,j}^{n}$ to denote the double sum $\sum_{i=1}^{n} \sum_{j=1}^{n}$. Note that $\sum_{j \neq i}$ (without a raised $n$) will still denote a single sum that excludes $i$.

From the algebraic manipulations of the leave-out estimators provided in Appendix A.4, it follows that the proposed variance estimator satisfies the decomposition

$$\tilde{V}_{\mathcal{F}} = \sum_{i \neq j} U_{ij} y_{i}^{2} y_{j} \varepsilon_{j} + \sum_{i \neq j \neq k} a_{ijk} y_{i}^{2} y_{j} \varepsilon_{k} + b_{ijk} y_{i} \varepsilon_{i} y_{j} y_{k} + \sum_{i \neq j \neq k \neq \ell} b_{ijk \ell} y_{i} y_{j} y_{k} \varepsilon_{\ell}$$

(20)

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where the weights in (20) are

\[ a_{ijk} = U_{ij} \tilde{M}_{jk,-ij}, \quad b_{ijk} = (U_{ij} - V_{ij}^2) \tilde{M}_{jk,-ij} + V_{ij}V_{ik}, \quad \text{and} \quad b_{ijk\ell} = b_{ijk} \tilde{M}_{\ell i,-ijk}. \]

Appendix A.4 already showed that \( \hat{V}_F \) is conditionally unbiased, so consistency follows if the conditional variance of \( \hat{V}_F \) is small relative to the squared estimand \( V_0 \left[ F - \hat{E}_F \right]^2 \). The derivations further below establish that this is the case by working with the four components of (20) one at a time.

An essential algebraic trick that is used repeatedly below is that the property \( M^2 = M \) or \( \sum_{k=1}^n M_{jk}M_{\ell k} = M_j = M_{\ell j} \) translate into similar statements regarding the leave-out analogs \( \tilde{M}_{ij,-i} \), \( \tilde{M}_{jk,-ij} \) and \( \tilde{M}_{\ell i,-ijk} \):

\[
\sum_{j=1}^n \tilde{M}_{ij,-i} \tilde{M}_{ij,-i} = \frac{M_{ij,-i}}{M_{ii}} = \frac{M_{ij}}{M_{ii}}
\]

for leave-one-out,

\[
\sum_{k=1}^n \tilde{M}_{jk,-ij} \tilde{M}_{jk,-ij} = \frac{M_{ij,-i} - M_{ij}}{D_{ij}/M_{ii}} = \frac{M_{ij,-i} - M_{ij}}{D_{ij}/M_{ii}} \quad \text{(21)}
\]

for leave-two-out, and in the case of leave-three-out:

\[
\sum_{\ell=1}^n \tilde{M}_{\ell i,-ijk} \tilde{M}_{\ell i,-ijk} = \frac{M_{ij,-i} - M_{ij} \tilde{M}_{jk,-ij} - \tilde{M}_{ij,-i} \tilde{M}_{\ell i}}{D_{ij}/M_{ii}} \quad \text{(22)}
\]

Beyond these identities, the remaining arguments rely on well-known inequalities such as Cauchy-Schwarz, Minkowski, and Courant-Fischer.
First component of $\hat{V}_F$. For the first component of (20), we have

$$
\mathbb{V} \left[ \sum_{i \neq j}^n U_{ij} y_i^2 \varepsilon_j \right] = \sum_{i \neq j}^n U_{ij}^2 \left( \mathbb{V} [y_i^2 y_j \varepsilon_j] + \mathbb{C} [y_i^2 y_j \varepsilon_j, y_j^2 y_i \varepsilon_i] \right)
+ \sum_{i \neq j \neq k}^n U_{ij} U_{jk} \left( \mathbb{C} [y_i^2 y_j \varepsilon_j, y_k^2 y_j \varepsilon_j] + \mathbb{C} [y_i^2 y_j \varepsilon_j, y_j^2 y_k \varepsilon_k] \right)
+ \sum_{i \neq j \neq k}^n U_{ik} U_{jk} \left( \mathbb{C} [y_i^2 y_j \varepsilon_j, y_k^2 y_i \varepsilon_k] + \mathbb{C} [y_i^2 y_j \varepsilon_j, y_k^2 y_i \varepsilon_k] \right)
\leq \max_{i,j} \mathbb{E} [y_i^4] \mathbb{E} [y_j^2 \varepsilon_j^2] 4 \sum_{i=1}^n (\sum_{j \neq i} U_{ij})^2.
$$

The upper bound on this variance is 4 times a product between a conditional moment $\max_{i,j} \mathbb{E} [y_i^4] \mathbb{E} [y_j^2 \varepsilon_j^2]$ which is $O_p(\epsilon_n^{-1})$ by Assumptions 2 and 3(i) and a sum of squared influences $\sum_{i=1}^n (\sum_{j \neq i} U_{ij})^2$. The latter term we can write as four times trace($C^2 \odot C^2$), where $\odot$ denotes Hadamard (elementwise) product. This representation immediately yields

$$
\frac{\sum_{i=1}^n (\sum_{j \neq i} U_{ij})^2}{\mathcal{V}_Q [F - E_F]^2} \leq \frac{4 \text{trace}(C^2) \cdot \max_{i,j} \sum_{j \neq i} U_{ij}}{\mathcal{V}_Q [F - E_F]^2} = O\left( \frac{\max_{i,j} U_{ij}}{(\sum_{i \neq j}^n C^2)^2} \right) = O\left( \frac{\max_i B_{ii}}{\epsilon_n r^1} \right),
$$

where the last two equalities follow from the asymptotic normality step of this proof.

Second component of $\hat{V}_F$. The second component of (20) we further decompose into two parts

$$
\sum_{i \neq j \neq k} a_{ijk} y_i^2 \varepsilon_j \varepsilon_k + a_{ijk} x_j \beta y_j^2 \varepsilon_k
$$

Proceeding with variance calculations and bounds for the first part we have

$$
\mathbb{V} \left[ \sum_{i \neq j \neq k} a_{ijk} y_i^2 \varepsilon_j \varepsilon_k \right] = \sum_{i \neq j \neq k} a_{ijk} (a_{ijk} + a_{ijk}) \mathbb{E} [y_i^4 \varepsilon_j^2 \varepsilon_k^2]
+ \sum_{i \neq j \neq k} a_{ijk} (a_{ijk} + a_{jki}) + a_{jki} (a_{jki} + a_{jki}) \mathbb{E} [y_i^2 \varepsilon_j \varepsilon_j \varepsilon_k^2]
+ \sum_{i \neq j \neq k} a_{ijk} (a_{ijk} + a_{kij}) \mathbb{E} [y_i^2 \varepsilon_j \varepsilon_j \varepsilon_k^2]
\leq \max_{i,j} \mathbb{E} [y_i^4] \sigma_j^4 8 \sum_{i \neq j \neq k} a_{ijk}^2 + \max_{i,j} \sigma_i^4 \sum_{j \neq k} \left( \sum_{i \neq j} a_{ijk} \mathbb{E} [y_i^2] \right)^2
$$

where we utilize $a_{iji} = 0$. Now observe that a special case of (21)

$$
\sum_{k=1}^n M_{jk,-ij} \tilde{M}_{jk,-ij} = \frac{1 - M_{ij,-i} M_{ij,-ij}}{D_{ij}/M_{ij}} = \frac{1 - M_{ij,-i} M_{ij,-ij}}{D_{ij}/M_{ij}} = \frac{M_{ij} M_{ij} + M_{ij} M_{ij} - M_{ij}^2 M_{ij} - M_{ij}^2 M_{ij}}{D_{ij}/M_{ij}}
$$

55
is bounded in absolute value by $4D_{ij}^{-1}D_{ij}^{-1}$, and that a further special case of (21) yields the bound $\sum_{k=1}^{n} \tilde{M}_{jk,-ij}^{2} = \frac{M_{i}}{D_{ij}} \leq \frac{1}{D_{ij}}$. In turn these bounds lead to
\[
\sum_{i \neq j}^{n} \frac{a_{i,k}^{2} \beta_{ij}}{\nu_{d}} \leq \max_{i \neq j} \frac{D_{ij}^{-1}}{D_{ij}} \times \frac{\sum_{i \neq j}^{n} \left( \sum_{j \neq k}^{n} U_{ij} \right)^{2}}{\nu_{d} \left( \sum_{j \neq k}^{n} U_{ij} \right)^{2}} = O \left( \max_{i \neq j} B_{ij} \right),
\]
\[
\sum_{i \neq j}^{n} \frac{\sum_{i \neq j}^{n} \left( \sum_{j \neq k}^{n} U_{ij} \right)^{2}}{\nu_{d} \left( \sum_{j \neq k}^{n} U_{ij} \right)^{2}} \leq \max_{i \neq j} \frac{D_{ij}^{-1}}{D_{ij}} \times \frac{\sum_{i \neq j}^{n} \left( \sum_{j \neq k}^{n} U_{ij} \right)^{2}}{\nu_{d} \left( \sum_{j \neq k}^{n} U_{ij} \right)^{2}} = O \left( \max_{i \neq j} B_{ij} \right),
\]
where the order statement of the first line stems from Assumption 4 from which it follows that $\max_{i \neq j} D_{ij}^{-1} = O(1)$, while the second order statement additionally utilizes Assumptions 2 and 3(i) from which we obtain $\max_{i \neq j} \frac{\sum_{i \neq j}^{n} \left( \sum_{j \neq k}^{n} U_{ij} \right)^{2}}{D_{ij}^{2}} = O \left( \epsilon_{n}^{-1} \right)$.

Turning to a variance calculation for the second part of (23) we have
\[
\forall \left[ \sum_{i \neq j \neq k}^{n} a_{ijk} x_{i}^{j} \beta_{ij} y_{i}^{2} z_{k} \right] = \sum_{i \neq j \neq k}^{n} a_{i,k}^{2} \beta_{ij} y_{i}^{2} z_{k} + a_{i,k} a_{i,k} \beta_{ij} y_{i}^{2} z_{k} \leq 2 \max_{i,j} \beta_{ij} y_{i}^{2} z_{k} \sum_{i,j}^{n} a_{i,k}^{2} + \max_{i} \sigma_{i}^{2} \sum_{k=1}^{n} \left( \sum_{i=1}^{n} a_{i,k} \beta_{ij} y_{i}^{2} z_{k} \right)^{2}, \tag{24}
\]
where $a_{i,k} = \sum_{j \neq i}^{n} a_{i,j} x_{j}^{i} \beta_{ij}$. From (21) we obtain the special case
\[
\sum_{k=1}^{n} \tilde{M}_{jk,-ij} \tilde{M}_{jk,-ij} = \frac{M_{i,j,-ij}}{D_{ij}/M_{ij}} = \frac{M_{i,j,-ij}}{D_{ij}/M_{ij}} = \frac{M_{i,j,-ij}}{D_{ij}/M_{ij}} = \frac{M_{i,j,-ij}}{D_{ij}/M_{ij}}.
\]
where a coarse bound on the absolute value of this expression is $2D_{ij}^{-1}D_{ij}^{-1}$. Utilizing this coarse bound, we immediately see that the first part of the variance in (24) satisfies
\[
\sum_{i \neq j}^{n} \frac{a_{i,k}^{2}}{\nu_{d} \left( \sum_{j \neq k}^{n} U_{ij} \right)^{2}} \leq \max_{i \neq j} \frac{\sum_{i \neq j}^{n} \left( \sum_{j \neq k}^{n} U_{ij} \right)^{2}}{\nu_{d} \left( \sum_{j \neq k}^{n} U_{ij} \right)^{2}} \leq \max_{i \neq j} \frac{\sum_{i \neq j}^{n} \left( \sum_{j \neq k}^{n} U_{ij} \right)^{2}}{\nu_{d} \left( \sum_{j \neq k}^{n} U_{ij} \right)^{2}} = O \left( \max_{i \neq j} B_{ij} \right),
\]
For the second part of (24) we instead rely on the full generality of (21)
\[
\sum_{k=1}^{n} \tilde{M}_{jk,-ij} \tilde{M}_{jk,-ij} = \frac{M_{i,j,-ij} + M_{i,j,-ij} M_{i,j,-ij} - M_{i,j,-ij} M_{i,j,-ij}}{D_{ij}/M_{ij}}.
\]
When coupled with the observation that the eigenvalues of $M$ belong to $\{0, 1\}$, this leads to

$$
\frac{\sum_{k=1}^{n}(\sum_{i=1}^{n} a_{i,k} \mathbb{E}[y_i])^2}{\text{Var}[\hat{F} - E_F]^2} \leq \max_{i \neq j} \frac{\mathbb{E}[y_i^2] (\mathbf{x}_i' \beta)^2}{D_{ij}^2} \frac{4 \sum_{i \neq j} U_{ij}}{\text{Var}[\hat{F} - E_F]^2} = O\left(\frac{\max_i B_i}{n^{1/2}}\right),
$$

where the last relation follows from $\max_{i \neq j} D_{ij}^{-2} \mathbb{E}[y_i^2] (\mathbf{x}_j' \beta)^2 = O_p(\epsilon_n^{-1})$ which holds by Assumptions 2 and 3(i) and Assumption 4.

**Third component of $\hat{V}_F$.** For the third component of (20) we similarly employ a decomposition

$$
\sum_{i \neq j \neq k}^{n} b_{ijk} (\mathbf{x}_j' \beta y_i \varepsilon_i y_k + \mathbf{x}_k' \beta y_i \varepsilon_j y_k + y_i \varepsilon_i \varepsilon_j \varepsilon_k)
$$

(25)

where the variance of the first part satisfies

$$
\text{Var}\left[\sum_{i \neq j \neq k}^{n} b_{ijk} \mathbf{x}_j' \beta y_i \varepsilon_i y_k\right] = \sum_{i \neq k}^{n} b_{i,k}^2 \text{Var}[y_i \varepsilon_i y_k] + b_{i,k} b_{k,i} \text{C}[y_i \varepsilon_i y_k, y_k \varepsilon_k y_i]
$$

$$
+ \sum_{i \neq j \neq k}^{n} b_{i,k} b_{k,j} \text{C}[y_i \varepsilon_i y_k, y_i \varepsilon_j y_k] + b_{i,k} b_{i,j} \text{C}[y_i \varepsilon_i y_k, y_i \varepsilon_j y_k]
$$

$$
+ 2 \sum_{i \neq j \neq k}^{n} b_{i,k} b_{k,j} \text{C}[y_i \varepsilon_i y_k, y_k \varepsilon_k y_k]
$$

$$
\leq \max_{i,j} \mathbb{E}[y_i^2] \mathbb{E}[y_j^2] 6 \sum_{i,k}^{n} b_{i,k}^2
$$

$$
+ \max_{i,j} \left(\mathbb{E}[y_i^2] + |\text{C}[y_i, y_i \varepsilon_i]|\right) \sum_{i,k}^{n} \left(\sum_{i=1}^{n} b_{i,k} \sigma_i^2\right)^2
$$

$$
+ \max_{i,j} \left(\mathbb{E}[y_i^2 \varepsilon_i^2] + |\text{C}[y_i, y_i \varepsilon_i]|\right) \sum_{i,k}^{n} b_{i,k}^2.
$$

for $b_{i,k} = \sum_{j \neq i,k} b_{ijk} \mathbf{x}_j' \beta$, $b_{i..} = \sum_{k=1}^{n} b_{i,k} \mathbf{x}_k' \beta$ and we have used that $b_{iji} = 0$. From the representation

$$
b_{i,k} = -\mathbf{x}_i' \beta V_{ik} + \sum_{j \neq i,k} U_{ij} \tilde{M}_{jk,-ij} \mathbf{x}_j' \beta - \sum_{j \neq i} V_{ij}^2 \tilde{M}_{jk,-ij} \mathbf{x}_j' \beta
$$

and the previously derived bound $|\sum_{k=1}^{n} \tilde{M}_{jk,-ij} \tilde{M}_{ik,-ij}| \leq 2 D_{ij}^{-1} D_{ik}^{-1}$, we immediately obtain

$$
\frac{\sum_{i,k}^{n} b_{i,k}^2}{\text{Var}[\hat{F} - E_F]^2} \leq \max_{i \neq j} \frac{(\mathbf{x}_i' \beta)^2}{D_{ij}^2} 3 \sum_{i=1}^{n} (\sum_{j \neq i} U_{ij})^2 + 3 \sum_{i=1}^{n} (\sum_{j \neq i} V_{ij}^2)^2 + 3 \sum_{i=1}^{n} (\mathbf{x}_i' \beta)^2 \sum_{i=1}^{n} V_{ij}^2.
$$

Since $\mathbf{M}^2 = \mathbf{M}$ and the largest eigenvalue of $\mathbf{M} \odot \mathbf{M}$ is bounded by one (a consequence of
the Gershgorin circle theorem), it follows that

\[
\frac{\sum_{i=1}^{n} \left( \sum_{j \neq i}^{n} \frac{v_{ij}^2}{v_{ij}[F - E_{ij}]} \right)^2}{\sqrt[2]{[F - E_{ij}]}} \leq \frac{16 \sum_{i=1}^{n} B_{ii}^2/M_{ii}^2}{\sqrt[2]{[F - E_{ij}]}} = O\left( \frac{\max_i B_{ii}}{r} \right),
\]

and since \( \forall_0[F - \hat{E}_F] \geq \min_i \sigma_i^2 \sum_{i=1}^{n} (\hat{x}_i^2) \sigma_i^2 \) we similarly have that

\[
\frac{\sum_{i=1}^{n} \left( \sum_{j \neq i}^{n} \frac{v_{ij}^2}{v_{ij}[F - E_{ij}]} \right)^2}{\sqrt[2]{[F - E_{ij}]}} \leq \max_i M_{ii}^{-2} \sum_{i=1}^{n} \frac{(\hat{x}_i^2)^2}{\sqrt[2]{[F - E_{ij}]}} = o_p(1).
\]

Turning to the second part of this variance we reuse the expression in (21) to derive the bound

\[
\frac{\sum_{i=1}^{n} \sum_{i \neq j}^{n} \frac{b_{ij}^2}{v_{ij}[F - E_{ij}]}}{\sqrt[2]{[F - E_{ij}]}} \leq \frac{2 \sum_{i=1}^{n} (\hat{x}_i^2)^4}{\sqrt[2]{[F - E_{ij}]}} + \max_i (\hat{x}_i^2) \frac{2 \sum_{i=1}^{n} \left( \sum_{j \neq i}^{n} \frac{U_{ij}^2}{v_{ij}[F - E_{ij}]^2} \right)^2}{\sqrt[2]{[F - E_{ij}]}} = O_p(1)
\]

Finally, since \( \sum_{k=1}^{n} \alpha_{j} \sum_{i=1}^{n} x_{ij} \) we have that \( b_{ii} = \frac{(\hat{x}_i^2) \sigma_i^2}{\sqrt[2]{[F - E_{ij}]}} \) so

\[
\sum_{i=1}^{n} \frac{b_{ij}^2}{v_{ij}[F - E_{ij}]^2} \leq \sum_{i=1}^{n} (\hat{x}_i^2)^2 \frac{2 \sum_{i=1}^{n} \left( \sum_{j \neq i}^{n} \frac{U_{ij}^2}{v_{ij}[F - E_{ij}]^2} \right)^2}{\sqrt[2]{[F - E_{ij}]}} = o_p(1)
\]

where the order statement regarding the first part follows from Assumption 3(ii) and the derivation in the asymptotic normality part of this proof.

For the second part of (25) we have

\[
\forall \left[ \sum_{i \neq j}^{n} b_{ij} x_{ij} \beta y_i \varepsilon_i \varepsilon_j \right] = \sum_{i \neq j}^{n} b_{ij}^2 \left( y_i^2 \varepsilon_i^2 \varepsilon_j \right) + b_{ij} \sum_{j \neq i}^{n} \left( y_i \varepsilon_i \varepsilon_j \right) + \sum_{i \neq j}^{n} b_{ij} \sum_{j \neq i}^{n} b_{ij} \left( y_i \varepsilon_i \varepsilon_j \right) \leq \max_i \left[ y_i^2 \varepsilon_i^2 \right] \sigma_i^2 \sum_{i \neq j}^{n} b_{ij}^2 \sum_{j \neq i}^{n} \left( \sum_{i \neq j}^{n} b_{ij} \right)^2
\]

for \( b_{ij} = \sum_{k \neq j}^{n} b_{ijk} x_{ij} \). We have \( b_{ij} = -U_{ij} x_{ij} \beta + \hat{x}_i \beta V_{ij} \), which leads to

\[
\frac{\sum_{i \neq j}^{n} b_{ij}^2}{\sqrt[2]{[F - E_{ij}]}} \leq \max_i (\hat{x}_i^2) \frac{2 \sum_{i \neq j}^{n} U_{ij}^2}{\sqrt[2]{[F - E_{ij}]}} + \max_i \frac{4 \sum_{i \neq j}^{n} (\hat{x}_i^2)^2}{\sqrt[2]{[F - E_{ij}]}} = O_p(1),
\]

\[
\frac{\sum_{i \neq j}^{n} \left( \sum_{i \neq j}^{n} b_{ij} \sigma_i^2 \right)^2}{\sqrt[2]{[F - E_{ij}]}} \leq \max_i (\hat{x}_i^2) \frac{2 \sum_{i \neq j}^{n} U_{ij}^2}{\sqrt[2]{[F - E_{ij}]}} + \max_i \frac{4 \sum_{i \neq j}^{n} (\hat{x}_i^2)^2}{\sqrt[2]{[F - E_{ij}]}} = O_p(1).
\]
Turning to the third and final part of (25) we have

\[
\mathbb{V} \left[ \sum_{i \neq j \neq k}^{n} b_{ijk} y_{i} \varepsilon_{j} \varepsilon_{k} \right] = \sum_{i \neq j \neq k}^{n} b_{ijk} (b_{ijk} + b_{ijk}) \mathbb{E} \left[ y_{i}^{2} \varepsilon_{j}^{2} \varepsilon_{k}^{2} \right] \\
+ \sum_{i \neq j \neq k}^{n} (b_{ijk} (b_{ijk} + b_{jk}) + b_{ijk} (b_{ijk} + b_{jk})) \mathbb{E} \left[ y_{i} \varepsilon_{j} \varepsilon_{j}^{2} \varepsilon_{k}^{2} \right] \\
+ \sum_{i \neq j \neq k}^{n} b_{ijk} (b_{ijk} + b_{jk}) \mathbb{E} \left[ y_{i} \varepsilon_{j} \varepsilon_{j} \varepsilon_{k}^{2} \right] \\
\leq \max_{i,j} \mathbb{E} \left[ y_{i}^{2} \varepsilon_{j}^{2} \varepsilon_{k}^{2} \right] \sigma_{i}^{2} \mathbb{E} \left[ b_{ijk}^{2} + \max_{i} \sigma_{i}^{2} \sum_{j \neq k}^{n} \left( \sum_{i \neq j}^{n} b_{ijk} \sigma_{i}^{2} \right)^{2} \right].
\]

By reusing the inequalities \(|\sum_{k=1}^{n} M_{jk,-i} \bar{M}_{jk,-i}| \leq \frac{4}{D_{ij} D_{ij}}\) and \(|\sum_{k=1}^{n} \bar{M}_{jk}^{2} \leq \frac{1}{D_{ij}}\), one can show that

\[
\sum_{i \neq j \neq k}^{n} b_{ijk} \mathbb{E} \left[ y_{i}^{2} \varepsilon_{j}^{2} \varepsilon_{k}^{2} \right] \leq \max_{i \neq j} \frac{1}{D_{ij}} \left[ \sum_{i \neq j}^{n} \left( \sum_{k \neq l}^{n} \mathbb{E} \left[ y_{i}^{2} \varepsilon_{j}^{2} \varepsilon_{k}^{2} \right] \right)^{2} + \max_{i} \frac{1}{M_{i}^{4}} \sum_{k=1}^{n} \mathbb{E} \left[ y_{i}^{2} \varepsilon_{j}^{2} \varepsilon_{k}^{2} \right]^{2} \right] \\
= O_p \left( \max_{i \neq j} \frac{B_{i}}{M_{i}^{2}} \right),
\]

\[
\sum_{i \neq j \neq k}^{n} \mathbb{E} \left[ b_{ijk}^{2} \right] \leq \max_{i \neq j} \frac{1}{D_{ij}} \left[ \sum_{i \neq j}^{n} \mathbb{E} \left[ y_{i}^{2} \varepsilon_{j}^{2} \varepsilon_{k}^{2} \right]^{2} \right] + \max_{i} \frac{1}{M_{i}^{4}} \sum_{k=1}^{n} \mathbb{E} \left[ y_{i}^{2} \varepsilon_{j}^{2} \varepsilon_{k}^{2} \right]^{2} \\
= O_p \left( \max_{i \neq j} \frac{B_{i}}{M_{i}^{2}} \right).
\]

**Fourth component of \(\hat{V}_{\mathcal{F}}\).** The fourth and final component of (20) we can rewrite as

\[
\sum_{i \neq j \neq k \neq \ell}^{n} b_{ijk\ell} \left( x_{i}^{\prime} \beta x_{j}^{\prime} \beta y_{i} \varepsilon_{j} \varepsilon_{k} \varepsilon_{\ell} + x_{i}^{\prime} \beta y_{i} \varepsilon_{j} \varepsilon_{k} \varepsilon_{\ell} + x_{j}^{\prime} \beta y_{i} \varepsilon_{j} \varepsilon_{k} \varepsilon_{\ell} + x_{i}^{\prime} \beta \varepsilon_{j} \varepsilon_{k} \varepsilon_{\ell} \right) \leq \frac{1}{D_{ij} D_{ij}} \left[ \sum_{i \neq j}^{n} \mathbb{E} \left[ y_{i}^{2} \varepsilon_{j}^{2} \varepsilon_{k}^{2} \varepsilon_{\ell}^{2} \right] \right]^{2} + \max_{i} \frac{1}{M_{i}^{4}} \sum_{k=1}^{n} \mathbb{E} \left[ y_{i}^{2} \varepsilon_{j}^{2} \varepsilon_{k}^{2} \varepsilon_{\ell}^{2} \right]^{2}
\]

where the variance of the first term satisfies

\[
\mathbb{V} \left[ \sum_{i \neq j \neq k \neq \ell}^{n} b_{ijk\ell} x_{i}^{\prime} \beta x_{j}^{\prime} \beta y_{i} \varepsilon_{j} \varepsilon_{k} \varepsilon_{\ell} \right] = \sum_{i \neq \ell}^{n} b_{i \ell}^{2} \mathbb{E} \left[ y_{i}^{2} \varepsilon_{j}^{2} \varepsilon_{k}^{2} \varepsilon_{\ell}^{2} \right] + \sum_{i \neq \ell}^{n} b_{i \ell}^{2} \mathbb{E} \left[ y_{i} \varepsilon_{j} \varepsilon_{k} \varepsilon_{\ell} \right] \\
+ \sum_{i \neq \ell}^{n} b_{i \ell} \mathbb{E} \left[ y_{i} \varepsilon_{j} \varepsilon_{k} \varepsilon_{\ell} \right] \\
\leq \max_{i,j} \mathbb{E} \left[ y_{i}^{2} \varepsilon_{j}^{2} \varepsilon_{k}^{2} \varepsilon_{\ell}^{2} \right] \sigma_{i}^{2} \mathbb{E} \left[ b_{i \ell}^{2} + \max_{i} \sigma_{i}^{2} \sum_{\ell=1}^{n} \left( \sum_{i \neq \ell}^{n} b_{i \ell} x_{i}^{\prime} \beta \right)^{2} \right].
\]

for \(b_{i \ell} = \sum_{j \neq \ell} b_{ij \ell} x_{j}^{\prime} \beta\) and \(b_{ij \ell} = \sum_{k \neq i,j} b_{ijk\ell} x_{j}^{\prime} \beta\) and we have used that \(b_{ijk} = b_{ijk} = 0\)
To further upper bound the first part of this final expression, we rely on (22) which yields

$$\sum_{\ell=1}^{n} \tilde{M}_{i\ell,\, -ijk} \tilde{M}_{i\ell,\, -ijk} = \frac{1 - \tilde{M}_{ij,\, -ijk} \tilde{M}_{ij,\, -ijk} - \tilde{M}_{i\ell,\, -ijk} \tilde{M}_{i\ell,\, -ijk}}{D_{ijk}/D_{jk}}.$$

Using this identity in conjunction with Cauchy-Schwarz and $M$ having all its eigenvalues in \{0, 1\} we obtain that

$$\frac{\sum_{i\neq j} b_{ij}^2}{\mathbb{V}[F - \hat{F}^2]} \leq \frac{\sum_{i\neq j} b_{ij}^2}{\mathbb{V}[F - \hat{F}^2]} + \max_{i\neq j\neq k \neq i} \frac{(x'_i \beta)^2}{D_{ijk}^2} \max_{i\neq j\neq k \neq i} \frac{(x'_i \beta)^2}{D_{ijk}^2} \frac{12 \sum_{i\neq j\neq k} b_{ijk}^2}{\mathbb{V}[F - \hat{F}^2]} + 2 \sum_{i=1}^{n} (\sum_{j\neq i} b_{ij} x'_j x'_j \beta M_{kk}/D_{ijk})^2 \frac{1}{\mathbb{V}[F - \hat{F}^2]} + \max_{i} (x'_i \beta)^2 \frac{6 \sum_{j\neq i} (\sum_{k\neq i} b_{ijk} x'_i \beta M_{kk}/D_{ijk})^2}{\mathbb{V}[F - \hat{F}^2]} \frac{1}{\mathbb{V}[F - \hat{F}^2]} + \max_{i} (x'_i \beta)^2 \frac{6 \sum_{j\neq i} (\sum_{k\neq i} b_{ijk} x'_i \beta M_{kk}/D_{ijk})^2}{\mathbb{V}[F - \hat{F}^2]} \frac{1}{\mathbb{V}[F - \hat{F}^2]}.$$

The first two terms of this bound were shown to be $o_p(1)$ in the treatment of the third component of (20), so here we focus on the latter three. The identities

\[
\begin{align*}
\frac{M_{ij}}{D_{ij}} &= \frac{1}{D_{ij}} + \frac{M_{ij} M_{ij, -ik} + M_{ik} M_{ik, -ik}}{D_{ij}}, \\
\frac{M_{ik}}{D_{ik}} &= \frac{1}{D_{ik}} + \frac{M_{ij} M_{ij, -ik} + M_{ik} M_{ik, -ij}}{D_{ik}}, \\
\frac{D_{ij}}{D_{ik}} &= \frac{1}{M_{ii}} + \frac{M_{ij} M_{ii} + M_{ik} M_{ik} - 2 M_{ij} M_{ik} M_{ik}}{M_{ii} D_{ijk}},
\end{align*}
\]

immediately leads to

\[
\begin{align*}
\frac{\sum_{i\neq j} (\sum_{k\neq i} b_{ijk} x'_i x'_{ik} \beta M_{kk}/D_{ijk})^2}{\mathbb{V}[F - \hat{F}^2]} &\leq \max_{i\neq j} \frac{1}{D_{ij}^2} \frac{2 \sum_{i\neq j} b_{ij}^2}{\mathbb{V}[F - \hat{F}^2]} + \max_{i\neq j\neq k \neq i} \frac{(x'_i \beta)^2}{D_{ijk}^2} \frac{4 \sum_{i\neq j\neq k} b_{ijk}^2}{\mathbb{V}[F - \hat{F}^2]} = o_p(1) \\
\frac{\sum_{i\neq k} (\sum_{j\neq i} b_{ijk} x'_i x'_{ik} \beta M_{kk}/D_{ijk})^2}{\mathbb{V}[F - \hat{F}^2]} &\leq \max_{i\neq j} \frac{1}{D_{ij}^2} \frac{2 \sum_{i\neq j} b_{ij}^2}{\mathbb{V}[F - \hat{F}^2]} + \max_{i\neq j\neq k \neq i} \frac{(x'_i \beta)^2}{D_{ijk}^2} \frac{4 \sum_{i\neq j\neq k} b_{ijk}^2}{\mathbb{V}[F - \hat{F}^2]} = o_p(1)
\end{align*}
\]
and
\[
\sum_{t=1}^{n} \left( \sum_{i \neq j} \sum_{k \neq l} b_{ij \ell} x'_i \beta y_{ij \ell} \beta D_{ijk} / D_{ijk} \right)^2 \leq \max_{i,j,k} \frac{1}{M_{ij}} \sum_{t=1}^{n} b_{ij \ell}^2 \sum_{i,j} \left( \frac{\beta x'_i \beta x'_j \beta x'_k \beta x'_l \beta D_{ijk} / D_{ijk}}{D_{ijk}} \right)^2 + \max_{i,j,k} \frac{1}{M_{ij}} \sum_{t=1}^{n} b_{ij \ell}^2 \sum_{i,j} \left( \frac{\beta x'_i \beta x'_j \beta x'_k \beta x'_l \beta D_{ijk} / D_{ijk}}{D_{ijk}} \right)^2
\]
\[
+ \max_{i,j,k} \frac{1}{M_{ij}} \sum_{t=1}^{n} b_{ij \ell}^2 \sum_{i,j} \left( \frac{\beta x'_i \beta x'_j \beta x'_k \beta x'_l \beta D_{ijk} / D_{ijk}}{D_{ijk}} \right)^2 = o_p(1),
\]
where the order statements were established in the treatment of the third component of (20).

To deal with \( \sum_{t=1}^{n} \left( \sum_{i \neq j} b_{i \cdot \ell} x'_i \beta x'_j \beta x'_k \beta x'_l \beta D_{ijk} / D_{ijk} \right)^2 \), we use the full generality of (22)
\[
\sum_{t=1}^{n} \sum_{i,j} \sum_{k \neq l} b_{i \cdot \ell} \beta x'_i \beta x'_j \beta x'_k \beta x'_l \beta D_{ijk} / D_{ijk} = M_{i-,j-} \cdot D_{ijk} / D_{ijk}
\]
from which we derive that
\[
\left( \sum_{i \neq j} b_{i \cdot \ell} x'_i \beta x'_j \beta x'_k \beta x'_l \beta D_{ijk} / D_{ijk} \right)^2 \leq \max_{i,j,k} \frac{1}{M_{ij}} \sum_{t=1}^{n} b_{ij \ell}^2 \sum_{i,j} \left( \frac{\beta x'_i \beta x'_j \beta x'_k \beta x'_l \beta D_{ijk} / D_{ijk}}{D_{ijk}} \right)^2
\]
\[
+ \max_{i,j,k} \frac{1}{M_{ij}} \sum_{t=1}^{n} b_{ij \ell}^2 \sum_{i,j} \left( \frac{\beta x'_i \beta x'_j \beta x'_k \beta x'_l \beta D_{ijk} / D_{ijk}}{D_{ijk}} \right)^2
\]
\[
+ \max_{i,j,k} \frac{1}{M_{ij}} \sum_{t=1}^{n} b_{ij \ell}^2 \sum_{i,j} \left( \frac{\beta x'_i \beta x'_j \beta x'_k \beta x'_l \beta D_{ijk} / D_{ijk}}{D_{ijk}} \right)^2 = o_p(1).
\]

For the second part of (27), we have the following variance expression and bound
\[
\text{Var} \left[ \sum_{i \neq j \neq k \neq \ell} b_{i \cdot \ell} x'_i \beta y_{i \cdot \ell} \epsilon_{i \cdot \ell} \right] = \sum_{i \neq j \neq \ell} \sum_{i \neq j \neq \ell} b_{i \cdot \ell} \cdot (b_{ij \ell} + b_{ij \ell} + b_{ij \ell} + b_{ij \ell}) \text{Var} \left[ y_i^2 \epsilon_{i \cdot \ell} \right]
\]
\[
+ \sum_{i \neq j \neq \ell} \sum_{i \neq j \neq \ell} b_{i \cdot \ell} (b_{ij \ell} + b_{ij \ell} + b_{ij \ell} + b_{ij \ell}) \text{Var} \left[ \epsilon_{i \cdot \ell} \right]
\]
\[
+ \sum_{i \neq j \neq \ell} \sum_{i \neq j \neq \ell} b_{i \cdot \ell} (b_{ij \ell} + b_{ij \ell} + b_{ij \ell} + b_{ij \ell}) \text{Var} \left[ y_i^2 \epsilon_{i \cdot \ell} \right]
\]
\[
\leq \max_{i,j} \text{Var} \left[ y_i^2 \right] \sigma^2 \sum_{i \neq j \neq \ell} b_{ij \ell}^2 + \max_{i} \sigma^2 \sum_{i \neq j \neq \ell} \left( \sum_{i \neq j \neq \ell} b_{ij \ell} \right)^2
\]
and we have used that \(b_{ijk} = b_{ijk}\). For this variance bound we utilize (22) to obtain

\[
\sum_{\ell=1}^n \hat{\mathbf{M}}_{i\ell,-ijk} \hat{\mathbf{M}}_{i\ell,-ijk} = \frac{\hat{\mu}_{l,-ijk} - \hat{\mu}_{i,jk} \hat{\mu}_{k,l-jk}}{D_{ijk}/D_{jk}},
\]

\[
\sum_{\ell=1}^n \hat{\mathbf{M}}_{i\ell,-ijk} \hat{\mathbf{M}}_{i\ell,-ijk} = \frac{1-\hat{\mu}_{l,-ijk} \hat{\mu}_{i,jk} \hat{\mu}_{k,l-jk}}{D_{ijk}/D_{jk}}.
\]

When combined with Cauchy-Schwarz and \(\mathbf{M}\) having its eigenvalues in \(\{0, 1\}\), we therefore obtain the further bounds

\[
\frac{\sum_{i,j \neq k}^n b_{ij}^2}{v_0[F - \mathbf{E}_x]} \leq \frac{\sum_{i,j \neq k}^n b_{ij}^2}{v_0[F - \mathbf{E}_x]} + \max_{i \neq j \neq k \neq \ell} \frac{(x_i'\beta)^2}{D_{ijk}^2} \left( \frac{5\sum_{i,j \neq k}^n b_{ij}^2}{v_0[F - \mathbf{E}_x]^2} + \frac{3\sum_{i,j \neq k}^n (\sum_{k \neq i,j} b_{ij} x_i'\beta M_{kk}/D_{ijk})^2}{v_0[F - \mathbf{E}_x]^2} \right) = o_p(1)
\]

and

\[
\frac{\sum_{i,j \neq k}^n (\sum_{i,j \neq k} b_{ij} x_i'\beta)^2}{v_0[F - \mathbf{E}_x]^2} \leq \max_{i \neq j \neq k \neq \ell} \frac{(x_i'\beta)^2}{D_{ijk}^2} \left( \frac{11\sum_{i,j \neq k}^n b_{ij}^2}{v_0[F - \mathbf{E}_x]^2} \right) + \max_i (x_i'\beta)^2 \left( \frac{7\sum_{i,j \neq k}^n (\sum_{k \neq i,j} b_{ij} x_i'\beta M_{kk}/D_{ijk})^2}{v_0[F - \mathbf{E}_x]^2} \right) = o_p(1)
\]

where the order statement follows from arguments given for the third component of (20) and the first part of (27).

The variance of the third part of (27) satisfies

\[
\sqrt{\sum_{i,j \neq k \neq \ell}^n b_{ij\ell} x_i'\beta y_i \varepsilon_k \varepsilon_\ell} = \sum_{i,j \neq k \neq \ell}^n b_{i-k\ell} (b_{i-k\ell} + b_{i-k} E[y_i^2 \varepsilon_k^2 \varepsilon_\ell^2]) + \sum_{i,j \neq k \neq \ell}^n b_{i-k\ell} (b_{i-k\ell} + b_{i-k\ell} + b_{k\ell}) E[y_i^2 y_j^2 \varepsilon_k^2 \varepsilon_\ell^2] + \sum_{i,j \neq k \neq \ell}^n b_{i-k\ell} (b_{j-k\ell} + b_{j-k\ell} + b_{j-k\ell}) E[y_i^2 \varepsilon_k^2 \varepsilon_\ell^2] \leq \max_{i,j} E[y_i^2] \sigma_i^4 \sum_{i,j \neq k \neq \ell}^n b_{i-k\ell}^2 + \sigma_j^4 \sum_{i,j \neq \ell}^n \left( \sum_{i \neq k} b_{i-k\ell} x_i'\beta \right)^2,
\]

where \(b_{i-k\ell} = \sum_{j \neq i,k} b_{ij\ell} x_i'\beta\). In complete analogy with the preceding argument, we use (22)
to obtain
\[
\sum_{\ell=1}^{n} \tilde{M}_{\ell,-ijk} \tilde{M}_{\ell,-ijk} = \frac{M_{n,-ijk} - M_{ijk} - M_{ijk} - M_{ijk}}{D_{ijk}/D_{jk}},
\]
\[
\sum_{\ell=1}^{n} M_{\ell,-ijk} \tilde{M}_{\ell,-ijk} = \frac{1 - M_{n,-ijk} M_{ijk}}{D_{ijk}/D_{jk}},
\]
which leads to
\[
\frac{\sum_{i \neq j \neq k \neq \ell} b_{ijk}^2}{\sqrt{\sum_{\ell} b_{ijk}^2}} \leq \frac{\sum_{i \neq j \neq k \neq \ell} b_{ijk}^2}{\sqrt{\sum_{\ell} b_{ijk}^2}} + \max_{i \neq j \neq k \neq \ell} \frac{(x_i^\prime \beta)^2 \sum_{i \neq j \neq k \neq \ell} b_{ijk}^2}{\sqrt{\sum_{\ell} b_{ijk}^2}} + \max_{i \neq j \neq k \neq \ell} \frac{(x_i^\prime \beta)^2 \sum_{i \neq j \neq k \neq \ell} b_{ijk}^2}{\sqrt{\sum_{\ell} b_{ijk}^2} \sum_{i \neq j \neq k \neq \ell} b_{ijk}^2} = o_p(1)
\]
and
\[
\frac{\sum_{i \neq j \neq k \neq \ell} b_{ijk}^2}{\sqrt{\sum_{\ell} b_{ijk}^2}} \leq \frac{\sum_{i \neq j \neq k \neq \ell} b_{ijk}^2}{\sqrt{\sum_{\ell} b_{ijk}^2}} + \max_{i \neq j \neq k \neq \ell} \frac{(x_i^\prime \beta)^2 \sum_{i \neq j \neq k \neq \ell} b_{ijk}^2}{\sqrt{\sum_{\ell} b_{ijk}^2} \sum_{i \neq j \neq k \neq \ell} b_{ijk}^2} = o_p(1)
\]
where the order statement follows from arguments given for the third component of (20) and the first part of (27).

Now the fourth term of (27) satisfies that
\[
\forall \left[ \sum_{i \neq j \neq k \neq \ell} b_{ijk} x_i^\prime \beta \varepsilon_j \varepsilon_k \varepsilon_\ell \right] = \sum_{j \neq k \neq \ell} b_{j k \ell} (b_{j k \ell} + b_{j k \ell}) \mathbb{E} \left[ \varepsilon_j^2 \varepsilon_k^2 \varepsilon_\ell^2 \right]
\]
\[
+ \sum_{i \neq \ell} b_{i j k} (b_{i j k} + b_{i j k}) \mathbb{E} \left[ \varepsilon_j^2 \varepsilon_k^2 \varepsilon_\ell^2 \right]
\]
\[
\leq \max_{i \neq j \neq \ell} \sigma_i^6 \left( \sum_{j \neq k \neq \ell} b_{ijk}^2 + \sum_{j \neq k \neq \ell} b_{j k \ell}^2 \right)
\]
for \( b_{j k \ell} = \sum_{i \neq j, k} b_{ijk} x_i^\prime \beta \) and we have used that \( b_{i j k} = b_{i j k} \). The first term was dealt with in (26). For the second term we use a special case of (22)
\[
\sum_{\ell=1}^{n} \tilde{M}_{\ell,-ijk} \tilde{M}_{\ell,-ijk} = \frac{M_{n,-ijk} - M_{ijk} - M_{ijk} - M_{ijk}}{D_{ijk}/D_{jk}},
\]
\[
= \frac{D_{jk} (M_{ij} D_{jk} - (M_{ij} M_{ik} M_{jk} + M_{ik} M_{ij} M_{jk} - M_{ijk} (M_{ij} M_{ik} + M_{ik} M_{ij})))}{D_{ijk} D_{jk}}
\]

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so that it follows from the largest eigenvalue of $M$ being one that

$$\frac{\sum_{i\neq j\neq k\neq \ell} b_{ijkl}^2}{\nu_0[F - \hat{E}]^2} \leq \max_{i\neq j\neq k\neq \ell} \frac{(x_i^\prime \beta)^2 \sum_{i\neq j\neq k\neq \ell} b_{ijkl}^2}{D_{ijk}^2 \nu_0[F - \hat{E}]^2} = o_p(1).$$

Finally, the variance of the fifth term of (27) satisfies the bound

$$\forall \left[\sum_{i\neq j\neq k\neq \ell} b_{ijkl} \varepsilon_i \varepsilon_j \varepsilon_k \varepsilon_\ell \right] = \sum_{i\neq j\neq k\neq \ell} b_{ijkl} \mathbb{E} \left[ \sum_{n=1}^n \varepsilon_i^2 \varepsilon_j^2 \varepsilon_k^2 \varepsilon_\ell^2 \right] \times \left( b_{ijkl} + b_{ij\ell k} + b_{ik\ell j} + b_{i\ell jk} + b_{i\ell kj} + b_{jik\ell} + b_{jik\ell} + b_{j\ell ik} + b_{j\ell ki} + b_{kij\ell} + b_{kij\ell} + b_{k\ell ij} + b_{k\ell ji} \right) \leq \max_i \sigma_i^8 24 \sum_{i\neq j\neq k\neq \ell} b_{ijkl}^2.$$  

Since a special case of (22) is $\sum_{\ell=1}^n \hat{M}_{i\ell, -ijk}^2 = \frac{1}{D_{ijk}/D_{jk}} \leq \frac{1}{D_{ijk}}$, we have that

$$\frac{\sum_{i\neq j\neq k\neq \ell} b_{ijkl}^2}{\nu_0[F - \hat{E}]^2} \leq \max_{i\neq j\neq k\neq \ell} D_{ijk}^{-2} \frac{\sum_{i\neq j\neq k\neq \ell} b_{ijkl}^2}{\nu_0[F - \hat{E}]^2} = o_p(1)$$

which completes our proof that the variance estimator $\hat{V}_0[F - \hat{E}]$ is consistent. 

### B.2 Asymptotic power

**Proof of Theorem 3.2.** First, we define the null vector $\beta_0$ corresponding to the parameter vector $\beta$ under $H_\delta$. That is, we let

$$\beta_0 = \beta - S_{xx}^{-1} R (R S_{xx}^{-1} R')^{-1/2} \delta,$$

where we see that $\beta_0$ satisfies the null, $R \beta_0 = q$, since $\beta$ is generated by a local alternative with $R \beta = q + (R S_{xx}^{-1} R')^{1/2} \delta$. Furthermore, we denote by $[F - \hat{E}]_\delta$ the value of $F - \hat{E}$ under $H_\delta$, and by $[F - \hat{E}]_0$ its value when the parameter vector $\beta$ is equal to $\beta_0$. 

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The value of $\mathcal{F} - \hat{\mathcal{E}}_F$ under $H_\delta$ can be represented in accordance with (28) as

$$
\begin{align*}
\mathcal{F} - \hat{\mathcal{E}}_F &= \left( R S^{-1}_{x x} \sum_{i=1}^{n} x_i \varepsilon_i + R \beta - q \right)' \left( R S^{-1}_{x x} R \right)^{-1} \left( R S^{-1}_{x x} \sum_{i=1}^{n} x_i \varepsilon_i + R \beta - q \right) \\
&\quad - \sum_{i=1}^{n} \frac{B_i}{M_i} \left( x_i' \beta_0 + x_i' (\beta - \beta_0) + \varepsilon_i \right) \sum_{j=1}^{n} M_{ij} \varepsilon_j \\
&= [\mathcal{F} - \hat{\mathcal{E}}_F]_0 + \delta' \delta + 2(\beta - \beta_0)' \sum_{i=1}^{n} x_i \varepsilon_i - \sum_{j=1}^{n} \tilde{x}_j' (\beta - \beta_0) \varepsilon_j.
\end{align*}
$$

Note that the third and fourth terms in this representation of $[\mathcal{F} - \hat{\mathcal{E}}_F]_\delta$ are both of smaller order than the sum of the first two when $r \to \infty$. Indeed, both have conditional mean zero, while the third term has variance

$$
\mathbb{V}[(\beta - \beta_0)' \sum_{i=1}^{n} x_i \varepsilon_i] = (\beta - \beta_0)' \sum_{i=1}^{n} x_i x_i' (\beta - \beta_0) \leq \max_i \sigma^2_i \delta' \delta,
$$

which implies that the third term is $o_p(||\delta||)$ and therefore either of a smaller magnitude than the first term in $[\mathcal{F} - \hat{\mathcal{E}}_F]_\delta$ if $\delta' \delta$ is bounded, or of a smaller magnitude than the second if $\delta' \delta \to \infty$. For the fourth term we similarly have that

$$
\mathbb{V}\left[ \sum_{j=1}^{n} \tilde{x}_j' (\beta - \beta_0) \varepsilon_j \right] = \sum_{j=1}^{n} (\tilde{x}_j' (\beta - \beta_0))^2 \sigma^2_j \leq \max_i \sigma^2_i \sum_{j=1}^{n} (\tilde{x}_j' (\beta - \beta_0))^2 \\
= \max_i \sigma^2_i \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{B_i}{M_i} x_i' (\beta - \beta_0) B_{ji} \tilde{x}_j' (\beta - \beta_0) M_{ij} \\
\leq \max_i \sigma^2_i \sum_{i=1}^{n} \frac{B^2_i}{M_i} (\tilde{x}_i' (\beta - \beta_0))^2 \leq \max_{i,j} \frac{\sigma^2_i}{M_{ij}} \delta' \delta,
$$

and since Assumptions 2 and 4 imply that $\max_{i,j} \frac{\sigma^2_i}{M_{ij}} = O_p(1)$ the argument applied to the third term applies here as well.

Hence, we have that

$$
\frac{[\mathcal{F} - \hat{\mathcal{E}}_F]_\delta}{\sqrt{\mathbb{V}_0[\mathcal{F} - \hat{\mathcal{E}}_F]_r}} = \frac{\delta' \delta}{\sqrt{r}} \left( \frac{\mathbb{V}_0[\mathcal{F} - \hat{\mathcal{E}}_F]}{r} \right)^{-1/2} = \frac{[\mathcal{F} - \hat{\mathcal{E}}_F]_0}{\sqrt{\mathbb{V}_0[\mathcal{F} - \hat{\mathcal{E}}_F]_r}} + o_p(1) \xrightarrow{d} \mathcal{N}(0,1),
$$

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and since $\frac{\lfloor \mathcal{F} - \hat{E}_F \rfloor}{r}$ is bounded and bounded away from zero, it also follows that

$$\left( \frac{\Delta^2}{\mathcal{F}} - \Delta^2 \right)^{1/2} = o_p(1), \quad \text{if } \Delta < \infty,$$

$$\frac{\lfloor \mathcal{F} - \hat{E}_F \rfloor}{\sqrt{\mathcal{F}}} \xrightarrow{p} \infty, \quad \text{if } \Delta = \infty,$$

from which the statement of the theorem follows if $\mathcal{V}_F / \mathcal{V}_0 \left[ \mathcal{F} - \hat{E}_F \right] \xrightarrow{P} 1$.

Next we argue that the variance estimator remains consistent under the sequence of alternatives characterized by $H_\delta$. A recall of the argumentation in Appendix A.4 reveals that $\mathcal{V}_F$ is an unbiased estimator of $\mathcal{V}_0 \left[ \mathcal{F} - \hat{E}_F \right]$ for any value of $\beta$. Similarly, an inspection of the proof of Theorem 3.1 reveals that the variance bounds derived for components of $\mathcal{V}_F$ do not depend on the particular value for $\beta$. Thus, it suffices that the sequence of local alternatives satisfy Assumption 3(i) as assumed.

Finally, we substantiate the comparison between the power of our proposed LO test and the exact F test provided in Remark 6. Specifically, we show that the ratio

$$\frac{1}{r} \mathcal{V}_0 \left[ \mathcal{F} - \hat{E}_F \right] \xrightarrow{2\sigma^4 + \frac{2r}{n-m} \sigma^4}$$

converges in probability to unity under either of the following two conditions;

1. $\frac{1}{r} \sum_{i=1}^{n} B_{ii}^2 = o_p(1)$ which effectively covers settings with $\sum_{i=1}^{n} B_{ii}^2 \sim 0$.

2. $\frac{1}{r} \sum_{i=1}^{n} \left( \frac{B_{ii}}{M_{ii}} - \mu_n \right)^2 = o_p(1)$ for $\mu_n = \frac{1}{n} \sum_{i=1}^{n} \frac{B_{ii}}{M_{ii}}$ which corresponds to settings with approximately balanced $\frac{B_{ii}}{M_{ii}}$ across observations.

To see why that convergence holds, note first that

$$\frac{1}{r} \mathcal{V}_0 \left[ \mathcal{F} - \hat{E}_F \right] \xrightarrow{2\sigma^4 + \frac{2r}{n-m} \sigma^4} \frac{1}{r} \left( \text{trace}(C^2) + \frac{1}{2r} \sum_{i=1}^{n} \left( \sum_{j \neq i} V_{ij} a_j' \beta \right)^2 \right).$$
The second component is negligible under either 1. or 2., as

\[
\frac{1}{r} \sum_{i=1}^{n} \left( \sum_{j \neq i} V_{ij} x'_{j} \beta \right)^{2} \leq \min \left\{ \max_{i} \left( \frac{x'_{i} \beta}{M_{ii}} \right)^{2} \sum_{i=1}^{n} B_{ii}^{2} \cdot \frac{1}{r} \sum_{i=1}^{n} \left( \frac{B_{ii}}{M_{ii}} - \frac{B}{M} \right)^{2} \right\}. 
\]

For the first component we use \( C = B - \frac{1}{2} \left( D_{B \odot M} M + M D_{B \odot M} \right) \), which leads to

\[
\text{trace}(C^{2}) = r + \frac{1}{2} \sum_{i=1}^{n} B_{ii}^{2} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{B_{ii}}{M_{ii}} \frac{B_{jj}}{M_{jj}} M_{ij}^{2} \\
= r (1 + \mu_{n}) + \frac{1}{2} \sum_{i=1}^{n} B_{ii} \left( \frac{B_{ii}}{M_{ii}} - \mu_{n} \right) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left( B_{ii} - \mu_{n} \right) \frac{B_{jj}}{M_{jj}} M_{ij}^{2}. 
\]

To see that the last two terms in this expression for \( \text{trace}(C^{2}) \) are \( o_{p}(r) \) under either 1. or 2., we note that

\[
\left| \frac{1}{r} \sum_{i=1}^{n} B_{ii} \left( \frac{B_{ii}}{M_{ii}} - \mu_{n} \right) \right| \leq \left( \frac{1}{r} \sum_{i=1}^{n} B_{ii}^{2} \cdot \frac{1}{r} \sum_{i=1}^{n} \left( \frac{B_{ii}}{M_{ii}} - \mu_{n} \right) \right)^{1/2} = o_{p}(1) \\
\left| \frac{1}{r} \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \frac{B_{ii}}{M_{ii}} - \mu_{n} \right) \frac{B_{jj}}{M_{jj}} M_{ij}^{2} \right| \leq \max_{i} \left( \frac{1}{r} \sum_{i=1}^{n} B_{ii}^{2} \cdot \frac{1}{r} \sum_{i=1}^{n} \left( \frac{B_{ii}}{M_{ii}} - \mu_{n} \right) \right)^{1/2} = o_{p}(1). 
\]

Thus the claim of Remark 6 follows if \( \frac{r}{n - m} - \frac{\mu_{n}}{1 + \frac{r}{n - m}} = o_{p}(1) \) which follows from

\[
\left( \frac{r - m - \mu_{n}}{1 + \frac{r}{n - m}} \right)^{2} = \frac{1}{(1 + \frac{r}{n - m})^{2}} \left( \sum_{i=1}^{n} \frac{B_{ii}}{M_{ii}} - \mu_{n} \right)^{2} = \frac{1}{(1 + \frac{r}{n - m})^{2}} \left( \sum_{i=1}^{n} \frac{M_{ii}}{M_{ii}} \left( \frac{B_{ii}}{M_{ii}} - \mu_{n} \right) \right)^{2} \\
\leq \frac{1}{(1 + \frac{r}{n - m})^{2}} \sum_{i=1}^{n} \left( \frac{B_{ii}}{M_{ii}} - \mu_{n} \right)^{2}. 
\]

The last expression is \( o_{p}(1) \) under either 1. or 2., as Assumption 4 implies that \( \frac{n}{n - m} = O(1) \) and 1. implies that \( \frac{r}{n} = o(1) \) so that \( \frac{r}{n - m} = o(1) \) under 1.
Appendix C  If leave-three-out fails

C.1 Variance estimator

Here we show that if \( i \) does not cause \( D_{ijk} > 0 \), i.e., if \( D_{ij}D_{ik} > 0 \) and \( D_{jk} = 0 \), then \( \hat{\sigma}^2_{i,-j} = \sigma^2_{i,-k} \) and \( \hat{\sigma}^2_{i,-j} \) is independent of both \( y_j \) and \( y_k \).

When \( D_{jk} = 0 \), the leverage of observation \( k \) after leaving \( j \) out is one. This, in particular, implies that the weight given to observation \( k \) in \(-\mathbf{x}_i'\hat{\mathbf{\beta}} - \mathbf{x}_j\) is zero, i.e.,

\[
0 = -\mathbf{x}_i'\left(\sum_{\ell \neq j} \mathbf{x}_\ell \mathbf{x}_\ell'\right)^{-1} \mathbf{x}_k = M_{ik} - \frac{M_{ij}M_{jk}}{M_{jj}} = \frac{D_{ij}}{M_{jj}} \hat{M}_{ik,-ij} \tag{29}
\]

where the second equality follows from (6). Thus \( \hat{M}_{ik,-ij} \), the weight given to \( y_k \) in \( \hat{\sigma}^2_{i,-j} \), is zero since \( D_{ij} > 0 \). Hence \( \hat{\sigma}^2_{i,-j} \) is independent of both \( y_j \) and \( y_k \).

We now have that \( \hat{\sigma}^2_{i,-j} = \hat{\sigma}^2_{i,-k} \) if \( \hat{M}_{i\ell,-ij} = \hat{M}_{i\ell,-ik} \) for all \( \ell \) different from \( j \) and \( k \). Note that under \( D_{jk} = 0 \), equation (29) shows that \( M_{jj}M_{ik} - M_{ij}M_{jk} = 0 \), and reversing the roles of \( j \) and \( k \) also leads to \( M_{kk}M_{ij} - M_{ik}M_{kj} = 0 \). By rearranging terms we then obtain

\[
M_{jj}D_{ik} = M_{jj}D_{ik} - M_{ij}(M_{kk}M_{ij} - M_{ik}M_{jk}) = M_{kk}D_{ij} - M_{ik}(M_{jj}M_{ik} - M_{ij}M_{jk}) = M_{kk}D_{ij}.
\]

which implies that \( D_{ij} > 0 \) if and only if \( D_{ik} > 0 \). Since (29) also applies when \( i \) is replaced by \( \ell \), we have that \( M_{jj}M_{\ell k} - M_{\ell j}M_{jk} = 0 \) which in turn implies that

\[
M_{kk}M_{ij}M_{j\ell} = M_{ik}M_{kj}M_{j\ell} = M_{ik}M_{jj}M_{k\ell}.
\]

From the two previous highlighted equations it follows that \( \hat{M}_{i\ell,-ij} = \hat{M}_{i\ell,-ik} \) since

\[
M_{kk}D_{ij}\hat{M}_{i\ell,-ij} = M_{kk}M_{jj}M_{i\ell} - M_{kk}M_{ij}M_{j\ell} = M_{jj}M_{kk}M_{i\ell} - M_{jj}M_{ik}M_{k\ell} = M_{jj}D_{ik}\hat{M}_{i\ell,-ik}.
\]

Finally, we clarify that the first line in the definition of \( \sigma^2_i \sigma^2_j \) correspond to the case where none of the leave-three-out failures are caused by both \( i \) and \( j \). This statement is

\[
D_{ijk} > 0 \text{ or } (D_{ij}D_{ik} > 0 \text{ and } D_{jk} = 0) \text{ or } (D_{ij}D_{jk} > 0 \text{ and } D_{ik} = 0) \text{ for all } k
\]
and since \( D_{ij} > 0 \) if and only if \( D_{ik} > 0 \) when \( D_{jk} = 0 \) this statement is equivalent to

\[
D_{ijk} > 0 \text{ or } (D_{ij} > 0 \text{ and } D_{jk} = 0) \text{ or } (D_{ij} > 0 \text{ and } D_{ik} = 0) \text{ for all } k
\]

which is easily seen to be equivalent to

\[
D_{ij} > 0 \text{ and } (D_{ijk} > 0 \text{ or } D_{ik} D_{jk} = 0 \text{ for all } k).
\]

### C.2 Asymptotic size

**Proof of Theorem 4.1.** It suffices to show that \( \hat{V}_F \) is a non-negatively biased estimator of the relevant target \( V_0[F - \hat{E}_F] \), that this adjusted variance estimator concentrates around its expectation, and that \( \liminf_{n \to \infty} q_{1-\alpha}(\bar{F}_{w,n-m}) \geq 1 \) in probability when \( r \) is fixed. To establish the required properties regarding \( \hat{V}_F \), it is useful to let \( H_{i,jk} \) and \( H_{ij} \) be indicators for presence of bias in the error variance estimators, i.e., let

\[
H_{i,jk} = 1 \{ D_{ij} D_{ik} = 0 \text{ or } (D_{ijk} = 0, D_{jk} > 0) \}, \\
H_{ij} = 1 \{ D_{ij} = 0 \text{ or } \exists k : D_{ijk} = 0, D_{ik} D_{jk} > 0 \}
\]

We can then write \( \hat{V}_F = A_1 + A_2 + B_1 + B_2 \) for

\[
A_1 = \sum_{i=1}^{n} \sum_{j \neq i} (1 - H_{ij}) (U_{ij} - V_{ij}^2) \bar{\sigma}_i^2 \bar{\sigma}_j^2, \\
A_2 = \sum_{i=1}^{n} \sum_{j \neq i} H_{ij} (U_{ij} - V_{ij}^2) + y_i^2 \bar{\sigma}_j^2, \\
B_1 = \sum_{i=1}^{n} \sum_{j \neq i} \sum_{k \neq i} (1 - H_{i,jk}) V_{ij} V_{ik} y_j y_k \bar{\sigma}_{i,-jk}^2, \\
B_2 = \sum_{i=1}^{n} \left( \sum_{j \neq i} \sum_{k \neq i} H_{i,jk} V_{ij} V_{ik} y_j y_k \right) y_i^2
\]

where \((\cdot)_+\) stands for taking a positive part.

First we consider the expectation of \( \hat{V}_F \). The bias in \( \hat{V}_F \) stems from \( A_2 \) and \( B_2 \) so we
have that
\[
\begin{align*}
\mathbb{E}[\hat{V}_F] - \mathbb{V}_0[F - \hat{E}_F] &= \sum_{i=1}^{n} \sum_{j \neq i} H_{ij} \left((U_{ij} - V_{ij}^2) + \mathbb{E}[y_{ij}^2] \mathbb{E}[\sigma_{j,-i}^2] - (U_{ij} - V_{ij}^2) \sigma_i^2 \sigma_j^2 \right) + \mathbb{E}[B_2] - \sum_{i=1}^{n} \sum_{j \neq i} \sum_{k \neq i} H_{i,j,k} V_{ij} V_{ik} \mathbb{E}[y_{jk} \sigma_i^2],
\end{align*}
\]

The first component of this bias is non-negative since \((U_{ij} - V_{ij}^2) + \mathbb{E}[y_{ij}^2] \mathbb{E}[\sigma_{j,-i}^2] \) is never smaller than \((U_{ij} - V_{ij}^2) \sigma_i^2 \sigma_j^2 \). For the second line we use that the mapping \((\cdot)_+\) is convex and larger than its argument. These two properties yield
\[
\mathbb{E}[B_2] = \sum_{i=1}^{n} \mathbb{E} \left[ \left( \sum_{j \neq i} \sum_{k \neq i} H_{i,j,k} V_{ij} y_{jk} V_{ik} \right)_+ \right] \mathbb{E}[y_{i}^2] \\
\geq \sum_{i=1}^{n} \sum_{j \neq i} \sum_{k \neq i} H_{i,j,k} V_{ij} V_{ik} \mathbb{E}[y_{jk} \sigma_i^2] + \sum_{i=1}^{n} \mathbb{E} \left[ \left( \sum_{j \neq i} \sum_{k \neq i} H_{i,j,k} y_{ij} y_{jk} V_{ik} \right)_+ \right] (x'_i \beta)^2,
\]

and since the second part of this lower bound is non-negative, we conclude that the second component of the bias in \(\hat{V}_F\) is also greater than equal to zero.

We now show that \(\hat{V}_F\) concentrates around its expectation, i.e., \((\hat{V}_F - \mathbb{E}[\hat{V}_F])/\mathbb{E}[\hat{V}_F] \xrightarrow{p} 0\). Since \(\mathbb{E} \mathbb{V}_0[F - \hat{E}_F]^{-1} = O_p(\frac{1}{r})\), it suffices for this conclusion to show that \(\hat{V}_F - \mathbb{E}[\hat{V}_F] = o_p(r)\). Since \(A_1, A_2, \) and \(B_2\) are quartic functions of the outcome variables it can be shown that \(A_1 - \mathbb{E}[A_1] = o_p(r)\), \(A_2 - \mathbb{E}[A_2] = o_p(r)\), and \(B_1 - \mathbb{E}[B_1] = o_p(r)\) by the same argumentation as in the proof of Theorem 3.1.

\(B_2\) involves additional non-linearities due to the presence of outcome variables inside the positive part function. For this reason we handle this term using Sølvsten (2020b), Lemma A2.2, which is a version of the Efron-Stein inequality. Letting \(\Delta_\ell B_2 = B_2 - B_{2,-\ell}\) where
\[
B_{2,-\ell} = \sum_{i=1}^{n} \left( \sum_{j \neq i} \sum_{k \neq i} H_{i,j,k} V_{ij} y_{jk} y_{i,-\ell} y_{k,-\ell} \right) y_{i,-\ell}^2 + y_{i,-\ell} = \begin{cases} y_i, & \text{if } i \neq \ell, \\ x'_i \beta, & \text{if } i = \ell \end{cases}
\]
it follows from Lemma A2.2 of Sølvsten (2020b) that \(B_1 - \mathbb{E}[B_1] = o_p(r)\) provided that
\[ \sum_{\ell=1}^{n} \mathbb{E}[(\Delta_{\ell}B_{2})^2] = o_p(r^2). \] That \( \sum_{\ell=1}^{n} \mathbb{E}[(\Delta_{\ell}B_{2})^2] = o_p(r^2) \) holds can be established following the argumentation in the proof of Theorem 3.1 and we therefore omit the details.

To show that \( \lim \inf_{n \to \infty} q_{1-\alpha}(F_{w,n-m}) \geq 1 \) in probability when \( r \) is fixed, we can argue along subsequences where \( \hat{w} \overset{p}{\to} \overline{w}_F \) and simply treat the limit problem of showing that \( q_{1-\alpha}(\hat{\chi}_2^2) \geq 1 \) whenever \( \alpha \leq 0.31 \). Now use \( G \) to denote the distribution function of a \( \chi_1^2 \) random variable and note that \( G \) is a concave function. Therefore, we have that (without loss of generality let \( w_1 > 0 \))

\[
\mathbb{P} \left( \sum_{\ell=1}^{r} w_{\ell}Z_{\ell} \leq 1 \right) = \mathbb{E} \left[ G \left( \frac{1 - \sum_{\ell=2}^{r} w_{\ell}Z_{\ell}}{w_1} \right) \right] \leq G \left( \mathbb{E} \left[ \frac{1 - \sum_{\ell=2}^{r} w_{\ell}Z_{\ell}}{w_1} \right] \right) = G(1) < 0.69.
\]

Thus it follows that \( q_{1-\alpha}(\hat{\chi}_2^2) > 1 \) for any value of \( \overline{w}_F \). \( \square \)

### C.3 Location invariance

The test statistic considered in the simulation study of Section 5 relies on demeaned outcomes \( \hat{y}_i = y_i - \frac{1}{n} \sum_{i=1}^{n} y_i \) when centering and studentizing \( F \). This is done to ensure that the critical value is invariant to shifts in location of the outcomes. Specifically, we estimate \( \mathbb{E}_0[F] \) using \( \hat{E}_F = \sum_{i=1}^{n} B_{ii} \hat{\sigma}^2_i \), where \( \hat{\sigma}^2_i = \hat{y}_i(y_i - x_i' \hat{\beta}_{-i}) \). Furthermore, we let \( \hat{w} = (\hat{w}_1, \ldots, \hat{w}_n)' \), where \( \hat{w}_\ell = \frac{\hat{w}_\ell}{\sum_{\ell=1}^{n} (\hat{w}_\ell' v_0)} \) and \( \hat{w}_\ell \) is the \( \ell \)-th eigenvalue of \( \Omega(\hat{\sigma}_1^2, \ldots, \hat{\sigma}_n^2) \).

The variance estimator similarly relies on demeaned outcomes in its construction. In analogy with the above definition, we let

\[
\hat{\sigma}_{i,-jk}^2 = \begin{cases} 
\hat{y}_i(y_i - x_i' \hat{\beta}_{-ijk}), & \text{if } D_{ijk} > 0, \\
\hat{y}_i(y_i - x_i' \hat{\beta}_{-ij}), & \text{if } D_{ij}D_{jk} > 0 \text{ and } D_{jk} = 0, \\
y_i^2, & \text{otherwise},
\end{cases}
\]

where we also write \( \hat{\sigma}_{i,-k}^2 \) when \( j \) is equal to \( k \). We use \( \hat{\sigma}_{i,-jk}^2 \) in the construction of the
variance product estimator

\[ \tilde{\sigma}_{ij}^2 = \begin{cases} 
\hat{y}_i \sum_{k \neq j} \tilde{M}_{ik,j} \cdot \tilde{\sigma}_{j,-i}^2, & \text{if } D_{ij} > 0 \text{ and } (D_{ijk} > 0 \text{ or } D_{ik}D_{jk} = 0 \text{ for all } k), \\
\hat{y}_i \tilde{\sigma}_{j,-i}^2, & \text{otherwise.}
\end{cases} \]

This leads to the variance estimator

\[ \tilde{V}_F = \sum_{i=1}^{n} \sum_{j \neq i} (U_{ij} - V_{ij}^2) \cdot G_{ij} \cdot \tilde{\sigma}_i^2 \tilde{\sigma}_j^2 + \sum_{i=1}^{n} \sum_{j \neq i} \sum_{k \neq i} V_{ij} \hat{y}_j \cdot V_{ik} \hat{y}_k \cdot G_{i,-jk} \cdot \tilde{\sigma}_{i,-jk}^2, \]

where the indicators \( G_{ij} \) and \( G_{i,-jk} \) remove biased estimators with negative weights:

\[ G_{ij} = H_{ij} \{ U_{ij} - V_{ij}^2 < 0 \}, \quad G_{i,-jk} = H_{i,jk} \{ \sum_{j \neq i} V_{ij} \hat{y}_j \cdot V_{ik} \hat{y}_k \cdot H_{i,jk} < 0 \} \]

for \( H_{ij} \) and \( H_{i,jk} \) as introduced in the proof of Theorem 4.1.

While \( \tilde{V}_F \) is positive with probability approaching one in large samples, it may be negative in small samples. When this occurs we instead rely on a variance estimator that estimates all error variances unconditionally. This guarantees positivity of the variance estimator:

\[ \tilde{V}_F^+ = \begin{cases} 
\tilde{V}_F, & \text{if } \tilde{V}_F > 0, \\
\sum_{i=1}^{n} \sum_{j \neq i} (U_{ij} - V_{ij}^2) + \hat{y}_i^2 \hat{y}_j^2 + \sum_{i=1}^{n} \left( \sum_{j \neq i} V_{ij} \hat{y}_j \right)^2 \hat{y}_i^2, & \text{otherwise.}
\end{cases} \]

The test considered in the simulations rejects when

\[ F > \frac{1}{\tau \tilde{\sigma}_e^2} \left( \tilde{E}_F + (\tilde{V}_F^+)^{1/2} \frac{\hat{q}_{1-\alpha}(\tilde{F}_{\tilde{w},n-m} - 1)}{\sqrt{2 \sum_{\ell=1}^{r} \tilde{w}_\ell^2 + 2/(n-m)}} \right), \]

and \( \hat{q}_{1-\alpha}(\tilde{F}_{\tilde{w},n-m}) \) is the \((1 - \alpha)\)-th quantile among 49,999 independent draws of \( \sum_{j=1}^{r} \tilde{w}_j \tilde{Z}_j \).
Appendix D  Simulation evidence

D.1 Simulation design

For $k \in \{2, \ldots, m-1\}$, let the $k$-th continuous regressor for observation $i$ have the representation $x_{ik} = \left(\frac{1}{2} + u_i\right)x_{ik}^0$, where $u_i \sim \text{IID } U[0,1]$ with $E[\frac{1}{2} + u_i] = 1$, $E[\left(\frac{1}{2} + u_i\right)^2] = \frac{13}{12}$, and $x_{ik}^0 \sim \text{IID } LN$ with $E[x_{ik}^0] = e^{1/2}$ and $\forall x_{ik}^0 = e(e - 1)$. For the mixed design, let additionally $d_{i\ell}$ be the $\ell$-th discrete regressor for observation $i$, where $\ell = 1, \ldots, r$.

To report $E[S_{xx}]$, we note that for the continuous regressors $E[x_{ik}^2] = \frac{13}{12}e^2$, $E[x_{ik}] = e^{1/2}$, and $E[x_{ik}x_{ik}'] = \frac{13}{12}e$ for $1 \neq k \neq k' \neq 1$. For discrete regressors, we have

$$s_{dd,\ell} \equiv E[d_{i\ell}^2] = \mathbb{P}\left\{ \frac{\ell - 1}{r + 1} \leq \frac{u_i + u_i^2}{2} < \frac{\ell}{r + 1} \right\} = \sqrt{\frac{1}{4} + 2\frac{\ell - 1}{r + 1}} - \sqrt{\frac{1}{4} + 2\frac{\ell - 1}{r + 1}},$$

$E[d_{i\ell}] = s_{dd,\ell}$, and $E[d_{i\ell}d_{i\ell'}] = 0$ for $1 \leq \ell \neq \ell' \leq r$. For the cross-moments between continuous and discrete regressors, we have

$$s_{xd,\ell} \equiv E[x_{ik}d_{i\ell}] = e^{1/2}E\left[\left(\frac{1}{2} + u_i\right) 1\left\{ \frac{\ell - 1}{r + 1} \leq \frac{u_i + u_i^2}{2} < \frac{\ell}{r + 1} \right\} \right] = \frac{e^{1/2}}{r + 1} \equiv s_{xd},$$

which does not depend on $\ell$.

The matrix $E[S_{xx}]$ is therefore structured as follows. In the continuous design,

$$E[S_{xx}] = n \begin{bmatrix}
1 & e^{1/2}l'_{m-1} \\
 e^{1/2}l_{m-1} & 13/12 e \cdot l_{m-1}l'_{m-1} + 13/12 e (e - 1) I_{m-1} \\
\end{bmatrix},$$

while in the mixed design,

$$E[S_{xx}] = n \begin{bmatrix}
1 & e^{1/2}l'_{m-r-1} \\
 e^{1/2}l_{m-r-1} & 13/12 e \cdot l_{m-r-1}l'_{m-r-1} + 13/12 e (e - 1) I_{m-r-1} \\
(s_{dd,1}, \ldots, s_{dd,r})' & (s_{dd,1}, \ldots, s_{dd,r}) \\
\end{bmatrix} \begin{bmatrix}
1 & e^{1/2}l'_{m-r-1} \\
 e^{1/2}l_{m-r-1} & 13/12 e \cdot l_{m-r-1}l'_{m-r-1} + 13/12 e (e - 1) I_{m-r-1} \\
(s_{dd,1}, \ldots, s_{dd,r})' & (s_{dd,1}, \ldots, s_{dd,r}) \\
\end{bmatrix}.$$
The link between the value of the regression coefficients under the null, $\varrho$, and the $R^2$ is

$$
\varrho = \frac{1}{\sqrt{m-1}} \sqrt{\frac{R^2}{1 - R^2} \frac{12}{13e^2 + (m - 14)e}}
$$

for the continuous design. In the mixed design, $m$ is replaced by $m - r$ as all coefficients for the discrete regressors are equal to zero under the null.

### D.2 Simulation results

Tables A1-A2 contain results from additional simulations where we, starting from the baseline continuous regressor design, for which the results are reported in the main text, vary parameters related to the strength of the signal relative to the noise: the coefficient of determination $R^2$, and the relative numerosities of regressors and restrictions $r/m$ and $m/n$. Table A1 contains figures on empirical size and positivity failure rate of $\hat{V}_F$ for the heteroskedastic setup, and Table A2 contains figures on empirical power for the homoskedastic setup with dense deviations from the null.
Table 3: Empirical size (in percent)

| Test | LO | EF | $W_1$ | $W_K$ | $W_L$ | NEG | LO | EF | $W_1$ | $W_K$ | $W_L$ | NEG | LO | EF | $W_1$ | $W_K$ | $W_L$ | NEG |
|------|----|----|-------|-------|-------|-----|----|----|-------|-------|-------|-----|----|----|-------|-------|-------|-----|
| Baseline |   |    |       |       |       |     |    |    |       |       |       |     |    |    |       |       |       |     |
| $n = 80$ | 5 47 100 | 52 16 | 12.8 | 4 47 100 | 52 16 | 15.3 | 3 47 100 | 51 15 | 18 |     |
| $n = 160$ | 5 61 100 | 50 16 | 3.7 | 5 81 100 | 50 17 | 4.4 | 6 61 100 | 50 17 | 5.3 |     |
| $n = 320$ | 6 81 100 | 49 18 | 0.9 | 5 81 100 | 50 17 | 1.0 | 6 81 100 | 50 18 | 1.2 |     |
| $n = 640$ | 5 96 100 | 49 19 | 0.1 | 5 96 100 | 49 18 | 0.2 | 5 96 100 | 49 18 | 0.2 |     |
| $n = 1280$ | 5 100 100 | 48 19 | 0.0 | 5 100 100 | 49 18 | 0.0 |     |      |    |     |

R$^2 = \frac{0}{6}$, $r/m = \frac{9}{12}$, $m/n = \frac{4}{5}$

| R$^2 = \frac{0}{6}$ | R$^2 = \frac{2}{6}$ | R$^2 = \frac{3}{6}$ |
|-------------------|-------------------|-------------------|
| $n = 80$ | 5 18 66 | 34 18 | 9.8 | 5 26 91 | 40 16 | 10.6 | 5 36 99 | 46 16 | 12.2 |     |
| $n = 160$ | 6 24 78 | 34 21 | 2.5 | 5 34 98 | 41 18 | 2.7 | 6 49 100 | 46 17 | 3.4 |     |
| $n = 320$ | 6 31 90 | 33 23 | 0.5 | 5 49 100 | 42 20 | 0.6 | 7 68 100 | 46 18 | 0.7 |     |
| $n = 640$ | 5 46 98 | 35 27 | 0.0 | 5 72 100 | 41 22 | 0.2 | 5 88 100 | 45 20 | 0.1 |     |
| $n = 1280$ | 5 67 100 | 35 27 | 0.0 | 5 92 100 | 41 23 | 0.0 | 5 99 100 | 45 21 | 0.0 |     |

$r/m = \frac{3}{12}$

| $r/m = \frac{3}{12}$ | $r/m = \frac{5}{12}$ | $r/m = \frac{7}{12}$ |
|----------------------|----------------------|----------------------|
| $n = 80$ | 7 78 81 | 86 35 | 2.5 | 6 72 96 | 76 32 | 3.6 | 6 64 100 | 61 25 | 5.3 |     |
| $n = 160$ | 6 85 87 | 92 45 | 1.0 | 6 82 99 | 76 39 | 1.5 | 5 77 100 | 60 26 | 1.8 |     |
| $n = 320$ | 5 91 94 | 97 57 | 0.5 | 5 92 100 | 76 43 | 0.4 | 5 92 100 | 60 29 | 0.5 |     |
| $n = 640$ | 5 97 99 | 99 68 | 0.1 | 5 99 100 | 78 46 | 0.1 | 5 99 100 | 58 30 | 0.1 |     |
| $n = 1280$ | 5 100 100 | 100 78 | 0.0 | 5 100 100 | 79 48 | 0.0 | 5 100 100 | 59 32 | 0.0 |     |

$\frac{m}{n} = \frac{1}{5}$

| $\frac{m}{n} = \frac{1}{5}$ | $\frac{m}{n} = \frac{2}{6}$ | $\frac{m}{n} = \frac{3}{6}$ |
|---------------------------|---------------------------|---------------------------|
| $n = 80$ | 5 78 81 | 86 35 | 2.5 | 6 72 96 | 76 32 | 3.6 | 6 64 100 | 61 25 | 5.3 |     |
| $n = 160$ | 6 85 87 | 92 45 | 1.0 | 6 82 99 | 76 39 | 1.5 | 5 77 100 | 60 26 | 1.8 |     |
| $n = 320$ | 5 91 94 | 97 57 | 0.5 | 5 92 100 | 76 43 | 0.4 | 5 92 100 | 60 29 | 0.5 |     |
| $n = 640$ | 5 97 99 | 99 68 | 0.1 | 5 99 100 | 78 46 | 0.1 | 5 99 100 | 58 30 | 0.1 |     |
| $n = 1280$ | 5 100 100 | 100 78 | 0.0 | 5 100 100 | 79 48 | 0.0 | 5 100 100 | 59 32 | 0.0 |     |

NOTE: All size results are for 5% nominal size under the continuous design with heteroskedasticity as described in Section 5 of the paper. The first panel repeats baseline results from Table 1 with a coefficient of determination R$^2$ at 1/6, a fraction of tested restrictions relative to number of regressors $r/m$ at 9/12, and a fraction of regressors relative to sample size $m/n$ at 4/5. The remaining three panels make ceteris paribus deviations by varying either R$^2$, $r/m$, or $m/n$. LO: leave-out test, EF: exact F test, $W_1$: heteroskedastic Wald test with degrees-of-freedom correction, $W_K$: heteroskedastic Wald test with Cattaneo et al. (2018b) correction, $W_L$: heteroskedastic Wald test with Kline et al. (2020) correction; NEG: fraction of negative variance estimates for LO (in percent). Results from 10000 Monte-Carlo replications.
### Table 4: Empirical power (in percent) corresponding to 5% size

| Test     | LO | EF | LO | EF | LO | EF |
|----------|----|----|----|----|----|----|
| Baseline | n = 80 | 5 | 15 |  |  |  |
|          | n = 160 | 12 | 21 |  |  |  |
|          | n = 320 | 26 | 36 |  |  |  |
|          | n = 640 | 44 | 57 |  |  |  |
|          | n = 1280 | 68 | 84 |  |  |  |

| R² = \(\frac{0}{6}\) | R² = \(\frac{2}{6}\) | R² = \(\frac{3}{6}\) |
|----------------------|----------------------|----------------------|
| n = 80               | 7                    | 4                    | 3                    |
| n = 160              | 16                   | 11                   | 10                   |
| n = 320              | 29                   | 23                   | 21                   |
| n = 640              | 48                   | 41                   | 39                   |
| n = 1280             | 73                   | 65                   | 63                   |

| \(\frac{r}{m} = \frac{3}{12}\) | \(\frac{r}{m} = \frac{5}{12}\) | \(\frac{r}{m} = \frac{7}{12}\) |
|-----------------------|-----------------------|-----------------------|
| n = 80                | 9                     | 7                     | 6                     |
| n = 160               | 19                    | 16                    | 15                    |
| n = 320               | 33                    | 28                    | 27                    |
| n = 640               | 55                    | 48                    | 46                    |
| n = 1280              | 81                    | 74                    | 70                    |

| \(\frac{m}{n} = \frac{1}{5}\) | \(\frac{m}{n} = \frac{2}{5}\) | \(\frac{m}{n} = \frac{3}{5}\) |
|-----------------------------|-----------------------------|-----------------------------|
| n = 80                      | 37                         | 23                         | 14                         |
| n = 160                     | 71                         | 47                         | 28                         |
| n = 320                     | 96                         | 81                         | 52                         |
| n = 640                     | 100                        | 98                         | 81                         |
| n = 1280                    | 100                        | 100                        | 98                         |

**NOTE:** All power results are for 5% nominal size under the continuous design with homoskedasticity and dense deviations as described in Section 5 of the paper. The first panel repeats baseline results from Table 2 with a coefficient of determination R² at 1/6 under the null, a fraction of tested restrictions relative to number of regressors \(\frac{r}{m}\) at 9/12, and a fraction of regressors relative to sample size \(\frac{m}{n}\) at 4/5. The remaining three panels make ceteris paribus deviations by varying either R², \(\frac{r}{m}\), or \(\frac{m}{n}\). LO: leave-out test, EF: exact F test. Results from 10000 Monte-Carlo replications.