Transformer Vs. MLP-Mixer: Exponential Expressive Gap For NLP Problems

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Abstract

Vision-Transformers are widely used in various vision tasks. Meanwhile, there is another line of works starting with the MLP-mixer trying to achieve similar performance using mlp-based architectures. Interestingly, until now those mlp-based architectures have not been adapted for NLP tasks. Additionally, until now, mlp-based architectures have failed to achieve state-of-the-art performance in vision tasks. In this paper, we analyze the expressive power of mlp-based architectures in modeling dependencies between multiple different inputs simultaneously, and show an exponential gap between the attention and the mlp-based mechanisms. Our results suggest a theoretical explanation for the mlp inability to compete with attention-based mechanisms in NLP problems, they also suggest that the performance gap in vision tasks may be due to the mlp relative weakness in modeling dependencies between multiple different locations, and that combining smart input permutations with mlp architectures may not be enough to close the performance gap alone.

1. Introduction

Since ViT proposed in a seminal paper by Dosovitskiy et al. [10] attention-based architectures [4, 26, 32] are widely used for various vision tasks, [10, 27] and achieve state of the art results in many benchmarks including the Imagenet-1k benchmark [9, 30, 35]. Bit later Radford et al. [27] followed by [11, 25, 37] suggested that simple mlp-based models combined with input permutations can achieve similar performance for the attention-based mechanisms. The heart of the mlp-mixer approach is to permute the input each time before applying the mlp-layer. Their idea is that permuting the inputs would allow the mlp-based architecture to mix information from different tokens in a similar way to the attention mechanism.

It is only natural to ask whether the mlp-based approaches combined with some permutations can compete with the attention-based mechanisms also in NLP tasks. Interestingly until now the MLP-Mixer have not been adapted for NLP tasks. Additionally, until now mlp-based approaches failed to achieve the state of the art performance on vision tasks but they are competitive with a small margin. In this paper, we seek to improve our theoretical understanding of the difference between the mlp and the attention-based architectures in their expressive power to model problems in different domains namely NLP and Vision.

We will do it by answering to some extent the following three questions (1) Can mlp-based models compete with the attention-based mechanisms also in NLP-based tasks. (2) Is the gap between the mlp to the attention-based mechanisms on vision tasks can be closed or a result of a gap in the expressive power, and hence can not be closed without architectural changes. (3) What differences between NLP and Vision cause the change in the ability of MLP-models to compete with the attention-based ones between these different fields. To answer those questions we estimate the expressive power of the different models to model connections between multiple variables simultaneously. This would allow us to compare the architecture’s ability to compete on NLP problems since in NLP problems the relevant information does not necessarily lie in the nearest neighbors and hence modeling multi-variable connections is necessary to get all the relevant information. This metric would also differentiate us from the vision case where the nearest neighbors contain most of the relevant information and their small number suggests that modeling multi-variable connections are less important.

Estimating network ability to model multi-variable connections will require us to define some metric to capture this notion, and for this, we will adapt the separation-rank metric [1, 2, 7], for comparing the expressive power of different classes of architectures. To do this we will further develop the notion of the separation-rank, for functions with multi-dimensional range. We will define what the separation-rank of a class of architectures means. Then we will define the notion of expressive-gap between different architectural classes. This expressiveness definition would capture
Finally, we will establish the relevant bounds on the mlp and attention-based architectures and will show the higher expressiveness of attention-based architectures relative to mlp-based architectures for NLP tasks. We will also show that when fixing the parameter budget, mlp-based models have lower expressivity than transformers and that there is an exponential gap in their expressive power as long as they are not able to replace each multi-head-attention layer at the transformer with at least 1.58 mlp-layers. This means that mlp-based models should be significantly deeper, to achieve the same level of expressiveness.

Using our theory we will suggest theoretical answers to the above three questions. (1) Since mlp-based models are significantly less expressive in their ability to model multi-variable connections, we will suggest that they are not fitted for NLP problems, including the mlp-mixer-based architectures. (2) Since in vision also it is reasonable that there is some importance in modeling multi-variable connections, we suggest it as a possible reason for the existing gap between the attention-based and the mlp-based architectures in vision tasks. (3) As for the difference between the NLP and the vision tasks, our results suggest that the mlp-based architectures, may be competitive for vision tasks due to the lower importance of modeling the multi-variable connections there and the higher importance of the nearest neighbors and their limited number. In NLP however, this is no longer true and mlp-based models would no longer be expressive enough to obtain competitive results.

Using our theory we predict bounds on the optimal depth-to-width ratio for mlp-mixer models. These bounds are different from the bounds for transformer architectures. We will test our predictions by comparing the accuracy of mlp-mixer models with a varied depth-to-width ratio on a variety of vision and NLP datasets. We further predict that mlp-mixer, due to its weaker expressive power, would require longer training, and larger data size to decrease the gap, as seen in many cases when training models with the same architecture but different budgets that larger models tend to converge faster. And assess these predictions by the experiments reported by Tolstikhin et al. [29].

To sum up, our contributions using an exact mathematical analysis we show an exponential gap in expressive power between mlp-mixer and attention-based architectures. Our results show the expressive weakness of mlp and mlp-mixer architectures, for NLP problems, and suggest that also for vision problems mlp-based architectures, including mlp-mixer, are weaker in modeling complicated connections between multiple variables simultaneously. We extended the separation-rank definition further into the multi-dimensional and the class of architectures cases, and define formally how to compare the expressive power between different architectural classes in terms of the separation-rank metric. Finally, we establish a few basic lemmas about the separation rank properties and came up with a new way to bound the separation rank of complicated deep learning architectures in a recursive way.

2. Related works

Modeling in computer vision has long been dominated by convolutional neural networks (CNNs). Beginning with AlexNet [18] and its revolutionary performance on the ImageNet image classification challenge. CNN architectures have evolved to become increasingly powerful through greater scale [14, 38], more extensive connections [17], and more sophisticated forms of convolution [8, 36, 39], with CNN serving as the backbone networks for a variety of vision tasks. These architectural advances have led to performance improvements that have broadly lifted the entire field. On the other hand, the evolution of network architectures in natural language processing (NLP) has taken a different path, where the prevalent architecture today instead is the transformer [31] designed for sequence modeling and transduction tasks. The transformer is notable for its use of attention to model long-range dependencies in the data. Its tremendous success in the language domain has led researchers to investigate its adaptation to computer vision, where it has recently demonstrated promising results on certain tasks, specifically image classification [10], and joint vision-language modeling [27].

There is another line of works, started by [27], trying to improve the mlp-based architecture for vision purposes. Existing MLP-like models share a similar macro framework, but have different block designs, MLP-like models usually divide one input image into patches like in vision transformers, and then perform two main steps, especially token-mixing steps are different from the existing methods. ViP [16] mixes information along the height and width dimensions, by summing permutations on those dimensions before applying the mlp layer, S2-MLP [37] uses another spatial shift permutation step to enable information interaction among tokens, Hire-MLP permutes tokens within a local region and cross local regions, and in common all of these MLP-like methods rely on permutation matrices followed by the linear operator.

The current state of the art, however, is achieved by attention-based models, and although when training on large-scale data-sets, such as JFT-300M [28], MLP-mixer attains similar accuracy when moving into medium-scale data-sets such as ImageNet-1k there is a clear performance gap. Specifically, Mixer-Base-16 [15] achieves only a 76.44, whereas ViT-Base-16 [10] achieves a 79.67.

The research on the expressive power of NN has a long history, in 2016 Cohen and Shashua [6] introduced the separation-rank metric, to quantify the expressive power of
CNNs and to mathematically quantify the difference between vision and NLP that creates the relative success of CNNs in vision vs. NLP. This work has started a line of works Cohen et al. [5], Levine et al. [22], Weiss et al. [33] that use and develop those tools to mathematically quantify the effectiveness of different NN architectures and training regimes [20]. In this work, we continue this line of work further by comparing the expressive power of transformer and mlp-like architectures in modeling multi-variable dependencies. Our results show the superiority of attention-based architectures in modeling such dependencies.

3. Problem formulation

In this section, we will present a formal definition for the MLP-mixer architecture followed by some relaxations on the analyzed models, during our analysis we will use the \( \sigma_2 (x) = (ABS(x))^2 \) activation as a relaxing assumption, we will justify this assumption later in this section (3.2).

3.1. MLP-mixer formulation

Definition 3.1 Let \( y_p^2 \) be a fully connected network with residual connections, depth \( p \) and \( \sigma_2 \) activation. Then it can be written as \( y_p^2 = L_2^p \circ \ldots \circ L_2^1 (X) \) where \( L_2^i \) denotes the \( i \) layer and can be written as

\[
L_2^i (X) = \sigma_2 (W_i X) + 1_R [i] X
\]

where \( R \subset [m] \) is the set of the indices of all the layers with residual connections.

The MLP-mixer is defined by applying a linear layer on the rows and the columns iteratively. This can be formulated as transposing the input before each even layer, as done in the following definition

Definition 3.2 Let \( y_{p,m,n}^{2,MM} : \mathbb{R}^{n \times m} \rightarrow \mathbb{R}^{n \times m} \) be an MLP-mixer architecture with residual connections no normalization layers and with \( \sigma_2 (x) = x^2 \) activations. Then, it can be written in the form \( y_{p,m,n}^{2,MM} (X) = L_{p}^{2,MM} \circ \ldots \circ L_{1}^{2,MM} (X) \), where \( L_{2}^{i,MM} : \mathbb{R}^{n \times m} \rightarrow \mathbb{R}^{n \times m} \) denotes the \( i \) layer and is defined by

\[
L_{1}^{2,MM} (X) = 1_O [k] \cdot \sigma_2 (W_o^k X) + 1_E [k] \cdot \sigma_2 (X W_e^k) + 1_R [k] X
\]

where \( X \) is the input, and \( W_o^k \) is the weights matrix when \( k \) is odd, while \( W_e^k \) is the weights matrix where \( k \) is even. More formally, \( X \in \mathbb{R}^{n \times m} \) while \( W_o^k \), \( W_e^k \in \mathbb{R}^{n \times n} \), where \( E \) and \( O \) are the sets of even and odd indices correspondingly i.e. \( E := 2N \cap [p] \) while \( O := (2N + 1) \cap [p] \).

More generally, if more general permutations are combined, which are not necessarily transposes, then a more general formulation would be

\[
L_{1}^{2,MM} (X) = 1_O [k] \cdot \sigma_2 (W_o^k \pi_o (X)) + 1_E [k] \cdot \sigma_2 (\pi_e (X) W_e^k) + 1_R [k] \pi_r (X)
\]

Where \( \pi_o, \pi_e, \pi_r \in S_{n \times m} \) are permutations over the input matrix elements, and \( R \subseteq [p] \) is the subset containing the indices of all the layers with residual connections.

Remark 3.1 In the last definition (3.2) the first equation captures only the MLP-mixer properties (2), while the second equation (3) intended to capture the properties of some of the variants like the model described in [37]. It of course captures also the original MLP-mixer properties, since it can be that \( \pi_e = \pi_o = \pi_r = e \), where \( e \) is the identity element of \( S_{n \times m} \). Hence we would refer to equation (3) when talking about MLP-mixer from here on since it is more general.

Remark 3.2 Although equation (3) is intended to capture some more variants, it still does not captures all of them, like the variant introduced in [16] which sums up a few different permutations each time before applying the mlp. However, it did capture the essence, and the proof can be extended also for those more sophisticated variants.

3.2. Relaxing assumptions

In this subsection, we will state some relaxations on the analyzed models that would make our analysis simpler, while preserving the validity of our findings at the same time.

Transformer relaxations. Following [20,21,33] we will assume that all the mlp layers are at the end, will remove all the normalization layers, and omit the ReLU and softmax non-linearities. We refer the reader to Levine et al. [21], Wies et al. [33] for a discussion on the impact of these relaxations. Essentially, they are shown to weaken the overall network power but still allow a meaningful comparison of the self-attention integration abilities.

However, in this work our main goal is to lower bound the transformer expressivity, and show that this lower bound is still higher than the appropriate upper bound we establish for the mlp-based architectures. Hence analyzing a weaker version of the transformer, and showing that even this weaker version is stronger than the mlp-based architectures, doesn’t weaken our results.

Mixer relaxations. For easiness of our analysis we will assume the \( \sigma_2 (x) = (ABS(x))^2 \) activation. Notice that the \( (ABS(x))^2 \) activation is universal from the universality of \( ABS(x) \) [3] and that assuming positivity does not affect the network information mixing abilities as measured by the separation-rank metric [20, p. 14]. Further justification for this relaxation is provided by the first experiment (6.1).
4. Separation-rank

4.1. Introducing the separation rank

The separation rank, introduced in [2] for high-dimensional numerical analysis, was employed for various applications, e.g., chemistry [13], particle engineering [12], and machine learning [1]. More recently, the separation rank has been established as a measure of dependencies modeled by deep convolutional and recurrent networks w.r.t. their inputs [5,7,19]. More recently, [22,34] employed this measure for studying the expressivity of a self-attention architecture with respect to its input.

For a function \( y(A,B) \) over variables \( A = \{ a_j \in \mathcal{X} \}_{j=1}^M \) and \( B = \{ b_j \in \mathcal{X} \}_{j=1}^M \), the separation rank w.r.t. \((A,B)\) is the minimal number of summands that together sum up to equal \( y(A,B) \), where each summand is multiplicatively separable w.r.t. \((A,B)\), i.e., is equal to a product of two functions – one that intakes only \( A \) variables and another that intakes only \( B \) variables. Formally, the separation rank of \( y : \mathcal{X}^{2M} \to \mathbb{R} \) w.r.t. \((A,B)\) is defined as:

\[
\text{sep}_{(A,B)}(y) := \min_{R \in \mathbb{N}} \left\{ \right. \begin{array}{l}
X = \{ g_{i,j} \in \mathcal{X}^{M} \}_{i,j=1}^R \rightarrow \mathbb{R} \\
y(A,B) = \sum_{i=1}^R g_{i}^{A}(A) \cdot g_{i}^{B}(B) \left. \right\}
\]

If the separation rank of a function w.r.t. \((A,B)\) is 1, the function is multiplicatively separable w.r.t. \((A,B)\), meaning it cannot take into account consistency between \( A \) and \( B \). In a statistical setting, if \( y \) is a probability density function, this would mean that \( A \) and \( B \) are statistically independent. The higher \( \text{sep}_{(A,B)}(y) \) is, the farther is \( y \) from this situation, i.e., the more it models dependency between \( A \) and \( B \).

4.2. Extending the separation rank

In our case, we have the architecture \( y_{\Theta} : \mathbb{R}^{k \times 1} \rightarrow \mathbb{R}^{m \times n} \) with \( \Theta \) as the parameters. We will denote by \( y_{\Theta} \) a transformer architecture and by \( y_{\Theta}^{\text{MLP}} \) an mlp-based architecture. The architecture output is given in matrix form, and we are interested in measuring the ability of our network to model dependencies between different locations of the input. As we move further into NLP tasks, the connections we will be interested in modeling will be connections between multiple different and not necessarily close positions. Hence we will adopt a balanced partition of the inputs, i.e., we would take \(|A| = |B|\), and then \( \text{sep}_{(A,B)}(y) \) will just measure the ability to model connections between different places at the input that are not necessarily close to each other.

Finally, it is shown at [23] that for transformer architecture \( \text{sep}_{(A,B)}(y) \) is invariant under the different balanced partitions. However, this property may not be true when handling mlp-based architectures. Hence we will define the supermom-separation-rank to be the maximal separation rank an architecture can achieve relative to some balanced partition, i.e. \( \sup_{(A,B) \in \mathcal{P}} \text{sep}_{(A,B)}(y) = \sup_{(A,B) \in \mathcal{P}} \text{sep}_{(A,B)}(y) \). Similarly, we will define the infimum-separation-rank to be the infimum separation rank the architecture can get relative to some balanced partition, i.e. \( \inf_{(A,B) \in \mathcal{P}} \text{sep}_{(A,B)}(y) \).

However, since we are dealing with multidimensional architectures we will expand our definition further into the multidimensional case. Denote by \( y : \mathcal{X}^{2M} \rightarrow \mathbb{R}^{n \times m} \) a multi-dimensional architecture, we will define the supremum-separation-rank as \( \sup_{(A,B) \in \mathcal{P}} \text{sep}_{(A,B)}(y) \). Similarly, the infimum-separation-rank would be extended for the multi-dimensional case to be the minimal inf-sep-rank achieved by some of the components, and more formally \( \inf_{i,j \in [n] \times [m]} \text{sep}_{(i,j)}(y) \).

4.3. Expressive gap definition

In this subsection, we will define how to compare the expressive power of different architectures using the \( \inf - \text{sep} \) and \( \sup - \text{sep} \) defined thus far. So denote by \( y_{1,\Theta}, y_{2,\Theta} : \mathcal{X}^{2M} \rightarrow \mathbb{R}^{n \times m} \) two architectures with \( \Theta \) as learned parameters, we will say that \( y_{2,\Theta} \) is more expressive than \( y_{1,\Theta} \), if \( \sup_{y_{1,\Theta}} - \text{sep}(y_{1,\Theta}) < \inf_{y_{2,\Theta}} - \text{sep}(y_{2,\Theta}) \). Similarly, we will say that \( y_{2,\Theta} \) is asymptotically more expressive than \( y_{1,\Theta} \) and will denote it by \( y_{1,\Theta} \prec y_{2,\Theta} \) if \( \lim_{|\Theta| \rightarrow \infty} \sup_{y_{p,\Theta}} - \text{sep}(y_{p,\Theta}) = \infty \) holds, when \(|\Theta|\) denotes the number of parameters. Assuming further that the depth is varied we will compare the expressiveness as follows.

**Definition 4.1** Let \( y_{p,\Theta}^{1}, y_{p,\Theta}^{2} : \mathcal{X}^{2M} \rightarrow \mathbb{R}^{n \times m} \) be two architectures with parameters \( \Theta \) and budget dependent architectural parameter \( p \), let’s say the depth of the network. Assume further that there is some monotone increasing function \( f : \mathbb{N} \rightarrow \mathbb{R} \) with \( \lim_{p \rightarrow \infty} f(p) = \infty \) s.t. \( \lim_{|\Theta| \rightarrow \infty} \frac{\log \inf_{y_{1,\Theta}} - \text{sep}(y_{1,\Theta})}{\log \sup_{y_{2,\Theta}} - \text{sep}(y_{2,\Theta})} \) is going to \( \infty \) faster than \( f(p) \), and more formally \( \lim_{p \rightarrow \infty} f(p) = \infty \) and \( \frac{\log \inf_{y_{1,\Theta}} - \text{sep}(y_{p,\Theta}^{1})}{\log \sup_{y_{2,\Theta}} - \text{sep}(y_{p,\Theta}^{2})} = \Omega\left(f(p)\right)\). Then we would say that \( y_{p,\Theta}^{1} \) is \( f \)-asymptotically more expressive than \( y_{p,\Theta}^{2} \), and will denote it by \( y_{p,\Theta}^{1} \prec f y_{p,\Theta}^{2} \).

Finally denoting by \( \mathcal{F}_{B} = \{ y_{\Theta}^{P} | P \in P \land |\Theta| \leq B \} \) a class of architectures with budget \( B \) and architectural parameters \( p \), where \( p \) is the parameters of the architecture shape, like the depth-to-width ratio, the embedding dimension, and the number of heads. We will define the separation rank of the class \( \mathcal{F}_{B} \) as the separation rank the wisest architectural parameters choice can give to us.
within the class, i.e \( \text{sep}(\mathcal{F}_B) = \sup_{p \in P} \text{sep}(y_B^p) \). Similarly, for the supremum and the infimum separation ranks, we would have \( \sup - \text{sep}(\mathcal{F}_B) = \sup_{p \in P} \sup - \text{sep}(y_B^p) \) and \( \inf - \text{sep}(\mathcal{F}_B) = \sup_{p \in P} \inf - \text{sep}(y_B^p) \). And exactly like in the case of architecture, we will define the dominance between classes of architectures as follows:

**Definition 4.2** Let \( \mathcal{F}_{B,P} \), \( \mathcal{G}_{B,P} \) be two different classes of architectures we say that \( \mathcal{F}_{B,P} \) is asymptotically more expressive than \( \mathcal{G}_{B,P} \), and will denote it by \( \mathcal{G}_{B,P} \prec \mathcal{F}_{B,P} \), if \( \lim_{B \to \infty} \frac{\log \inf - \text{sep}(\mathcal{F}_{B,P})}{\log \sup - \text{sep}(\mathcal{G}_{B,P})} = \infty \), where \( B \) denotes the number of parameters. If furthermore, there exist some budget dependent architectural parameter \( p \), let say the depth of the network as a function of the parameter \( p \), then we would say that the class \( \mathcal{F}_{B,P} \) is \( f \)-asymptotically more expressive than the class \( \mathcal{G}_{B,P} \), and will denote it by \( \mathcal{G}_{B,P} \prec_{f} \mathcal{F}_{B,P} \).

5. Separation-rank upper-bounds

In the following subsections, we will develop tools for proving the following theorem which is also the main result of this paper.

**Theorem 5.1** Let \( \mathcal{F}_{B,P}^{\mathcal{T}} \) be the class of all the transformers architectures with up to \( B \) parameters and depth \( p \), and let \( \mathcal{F}_{B,P}^{\mathcal{M,M}} \) be the class of all the mlp-architectures, possibly with permutations of the input before each mlp-layer, and with up to \( B \) parameters and depth \( p \). Then we have the following asymptotic relation \( \frac{\log \inf - \text{sep}(\mathcal{F}_{B,P}^{\mathcal{M,M}})}{\log \sup - \text{sep}(\mathcal{F}_{B,P}^{\mathcal{T}})} = \Omega \left( \left( \frac{2}{k} \right)^p \right) \), and more formally we have \( \mathcal{F}_{B,P}^{\mathcal{M,M}} \prec \left( \frac{2}{k} \right)^p \mathcal{F}_{B,P}^{\mathcal{T}} \).

**Proof Idea 5.1** In the proof, we upper bound \( \sup - \text{sep}(\mathcal{F}_{B,P}^{\mathcal{M,M}}) \) while lower bounding \( \inf - \text{sep}(\mathcal{F}_{B,P}^{\mathcal{T}}) \). Then, we compare these two bounds asymptotically to get the desired asymptotic relation. The lower bound is obtained mainly, by relying on a similar lower bound taken from theorem 7.1 at [23]. While upper bounding \( \sup - \text{sep}(\mathcal{F}_{B,P}^{\mathcal{M,M}}) \), is obtained by using a recursive argument of bounding the \( \text{sep} - \text{rank} \) of all of the small components of the network first. Then recursively bound the \( \text{sep} - \text{rank} \) of larger and larger components until we reach a bound for all of the architecture. □

**Elementary operations bound.** We will start with some simple lemmas about the behavior of the \( \text{sep} - \text{rank} \) under the basic operations involved in each layer.

Being more formal, let \( f, g : \mathbb{R}^{k \times l} \to \mathbb{R}^{n \times m} \) and \( h : \mathbb{R}^{n \times m} \to \mathbb{R}^{r \times s} \) be matrix functions, where \( h \) is some function of \( f, g \), and we want to bound the separation rank of \( h \), i.e \( \text{sep} - \text{rank} (h) \) in terms of \( \text{sep} - \text{rank} (f) \) and \( \text{sep} - \text{rank} (g) \). Specifically, the \( h \) of interest for us are the basic operations involved in the network definition, or just \( \sigma_2 (X) \cdot f (X) \circ g (X) \), \( f (X) + g (X) \), \( W f (X) \), \( f \circ g (X) \)

And for each such form of \( h \)-function we will establish a bound on \( \text{sep} - \text{rank} (h) \) of the form

\[ \text{sep} - \text{rank} (h) \leq \phi (\text{sep} - \text{rank} (f) , \text{sep} - \text{rank} (g)) \]

when \( \phi : \mathbb{N}^2 \to \mathbb{N} \) is scalar function. All these bounds are proved in the appendices, and result in the following sequence of upper bounds:

**Lemma 5.2** Let \( f, g : \mathbb{R}^{k \times l} \to \mathbb{R}^{n \times m} \) be a matrix function, and let \( k_f := \text{sep} - \text{rank} (f (X)) \) be the separation-rank of \( f \). Then we have the following properties

(i) Separation rank is a sub-additive operator

\[ \text{sep} - \text{rank} (f (X) + g (X)) \leq \text{sep} - \text{rank} (f (X)) + \text{sep} - \text{rank} (g (X)) \]  

(ii) Separation-rank is invariant under permutations.

More formally, let \( \pi \in S_{n \times m} \) be a permutation over the entries of \( n \times m \) matrices, and let \( f : \mathbb{R}^{k \times l} \to \mathbb{R}^{n \times m} \) be a matrix function. Then the following equality holds

\[ \text{sep} - \text{rank} (\pi \circ f (X)) = \text{sep} - \text{rank} (f (X)) \]  

(iii) For \( \text{Id} : M_{n \times m} (\mathbb{R}) \to M_{n \times m} (\mathbb{R}) \) we have

\[ \text{sep} - \text{rank} [\text{Id} (X)] \leq 2 \]  

(iv) The following inequality holds

\[ \text{sep} - \text{rank} \left ( f (X)^{\odot 2} \right ) \leq \left ( \frac{k_f + 1}{2} \right ) \]

where \( \odot \) is the Hadmard product and is defined by \( (A \odot k)_{ij} = (A_{ij})^k \).

**Proof 5.3** We will bring the proof of clause (i), (ii), (iii) here, the proof of clause (iv) is left for the appendices.

(i) Denoting \( k_{f_i} := \text{sep} - \text{rank} (f_i) \) and using the \( \text{sep} - \text{rank} \) definition there exists \( (f_1)^{k_f} \), \( (f_1)^{n k_f} \) s.t.

\[ f (X) = \sum_{i=1}^{k_f} f_i^l (X_A) \circ f_i^{n l} (X_B) \]

similarly denoting \( k_g := \text{sep} - \text{rank} (g) \) there exist \( (g_1)^{k_g} \), \( (g_1)^{n k_g} \) s.t.

\[ g (X) = \sum_{i=1}^{k_g} g_i^l (X_A) \circ g_i^{n l} (X_B) \]
In particular
\[ f(X) + g(X) = \sum_{i=1}^{k_f} f_i(X_A) \otimes g_i''(X_B) + \sum_{i=1}^{k_g} g_i'(X_A) \otimes g_i''(X_B) \]

Is valid decomposition for \( f + g \) with \( k_f + k_g \) elements, and hence we have
\[ \text{sep} - \text{rank}(f(X) + g(X)) = k_f + k_g \]
As needed \( \square \)

(ii) And indeed we have
\[ \text{sep} - \text{rank}(\pi \circ f(X)) \]
\[ = \max_{i,j} \text{sep} - \text{rank}\left( (\pi \circ f(X))_{i,j} \right) \]
\[ = \max_{i,j} \text{sep} - \text{rank}\left( (f(X))_{i,j} \right) \]
\[ = \text{sep} - \text{rank}(f(X)) \]
As needed \( \square \)

(iii) The decomposition is of order 2
\[ \text{Id}(X) = X = X_A + X_B \]
\[ = X_A \otimes 1_{n \times m} + 1_{n \times m} \otimes X_B \]
when \( 1_{n \times m} \) is the \( n \times m \) matrix with 1 on all of its entries. \( \square \)

(iv) See Subappendix A.1

MLP-mixer bounds. Relying on the last lemma (5), the next lemma presents an upper bound on the separation-rank of mixer layer.

Lemma 5.4 Let \( L_{\text{mlp}} \) be a general mlp layer as defined in equation (3)
\[ L_{i, m, n}^{2, \text{mlp}}(X) = 1_D[k] \cdot \sigma_2(W_k \pi_o(X)) + 1_E[k] \cdot \sigma_2(\pi_e(X) W_k E) + 1_R[k] \pi_r(X) \]
and let \( f: X^M \rightarrow \mathbb{R}^{n \times m} \) be a general function. Then, the following upper bound on the separation rank holds
\[ \text{sep} - \text{rank}\left( L_{i, k, m, n}^{2, \text{mlp}} \circ f(X) \right) \leq n^2 \cdot \text{sep} - \text{rank}\left(f(X)\right)^2 + \text{sep} - \text{rank}\left(f(X)\right) \]
Proof. See Appendix B.

Applying the last lemma (5.4) recursively, we may get the following upper bound on the separation-rank of a full mlp-mixer architecture:

\[ \text{Theorem 5.5} \text{ Let } y_{p, m, n}^{2, \text{mlp}} : \mathbb{R}^{n \times m} \rightarrow \mathbb{R}^{n \times m} \text{ be an mlp-based architecture with depth } p \text{ of the form } y_{p, m, n}^{2, \text{mlp}}(X) = L_{p, m, n}^{2, \text{mlp}} \circ \ldots \circ L_{1, m, n}^{2, \text{mlp}}(X), \text{ then we have the following bound on the separation-rank of the entire model.} \]
\[ \text{sep} - \text{rank}(y_{p, m, n}^{2, \text{mlp}}) \leq (2H \cdot m^2 \cdot n^2)^p \]
writing differently, we have
\[ \log_3 \left( \text{sep} - \text{rank}\left( y_{p, m, n}^{2, \text{mlp}} \right) \right) \]
\[ \leq \log_3 \left( 2H \cdot m^2 \cdot n^2 \right) \cdot 2^p \]
Proof. See Appendix C.

Transformer bounds. The main thing left for us to do in order to conduct expressiveness comparisons, between the transformer and the mlp-based architectures, is to develop similar lower bounds for attention-based mechanisms, and then show that the found lower bound for the transformer is asymptotically larger than the corresponding upper bound (9) we established thus far.

We will start by presenting an equivalent upper bound for transformer architectures for the one we just established for the mlp-based architectures (9). Getting such an upper bound is useful in order to show the tightness of our lower bound. Such tightness results would mean that our expressiveness gap result could not be widened by achieving a better lower bound for the transformer and that if someone manages to show that the upper bound for the mlp-mixer is tight, then at least under the relaxed model’s assumptions the gap we found is exact in the sense that no larger gap exists.

Theorem 5.6 Let \( y_{p, H}^R : \mathbb{R}^{n \times m} \rightarrow \mathbb{R}^{n \times m} \), be transformer architecture without activations and normalization layers with depth \( p \) and residual connections of the form
\[ y_{p, H}^R(X) = L_{p, H}^R \circ \ldots \circ L_{1, H}^R(X) \]
with all the layers and matrices having the same dimensions, \( W_k^p \in M_{m \times n}(\mathbb{R}) \). Then, the following bound on the separation rank holds
\[ \text{sep} - \text{rank}\left( y_{p, H}^R \right) \leq (2H \cdot m^2 \cdot n^2)^{3p} \]
Proof. See Appendix D.

Finally, to get a lower bound, we are relying on theorem (7.1) from the book ([23]) to get that for linear transformers without residual connections the following holds

Theorem 5.7 For \( p < \log_3 m \) there is a weights assignment such that our upper bound
\[ \log_3 \text{ sep} - \text{rank}\left( y_{p, H}^R \right) \]
\[ \leq 3^p \cdot \left[ \log_3 (2H) + 2 \log_3 m + 2 \log_3 n \right] \]
Figure 1. Mixer Performance for different depth-to-width ratios. Results obtained on the CIFAR10, SVHN and MNLI datasets when trained for 40 epochs, with 9 different budgets averaged over 5 seeds. We use smaller budgets for easier datasets. Only runs with a standard deviation smaller than 0.15 are reported. We can see that the best performance is obtained for $1 < \frac{p}{\log_2 d} < 2$.

is asymptotically tight in the sense

\[ \log_3 \text{sep} - \text{rank} \left( y^R_{p,H} \right) \geq 3^{p-2} \left( \log_3 (m - H) - p + 2 - \log_3 2 \right) \]  

(15)

Proof. See Appendix E.

Results. Comparing the obtained bounds, we may end with the following theorems regarding the expressive gap between the transformer and the mlp-mixer architectures.

Conclusion 5.8 We got that for $\mathcal{F}^T_B$, $\mathcal{F}^\text{mlp}_B$ the classes of transformer and mlp-based architectures with up to $B$ parameters respectively. It holds that $
abla^\text{sep} - \text{rank} (\mathcal{F}^T_B) = \Omega \left( \left( \frac{3}{2} \right)^p \right)$. More formally, there is a monotonicity relation of the form $\mathcal{F}^\text{mlp}_B \prec \left( \frac{3}{2} \right)^p \mathcal{F}^T_B$.

Proof. See Appendix F.

Conclusion 5.9 For $p < \log_3 m$ and assuming $p >> \log_3 \log_3 m$, $n < m^2$, $H < \frac{m}{2}$ and $p \geq 13$. Then, every mlp-based architecture has a strictly smaller expressive power in modeling multi-variable dependencies than any attention-based architecture, when fixing the depth and the parameters budget. Also, for $\log_3 m < p < \log_2 m$, then still, transformers enjoy strictly higher expressive power than mlp-based architectures for large enough $p$, and when moving into the depth efficiency regime $p < \log_3 m$ the gap becomes asymptotically exponential in $p$.

Proof. See Appendix G.

Remark 5.1 The difference between the last two conclusions is that the first conclusion (5.8) states that the wisest choice of transformer architecture is better than the wisest choice of mlp-architecture, whereas the second conclusion (5.9) states that every transformer with a good depth-to-width ratio is superior to every mlp-based architecture.
Proposition 5.10 Conclusion (5.8) states dominance relation between transformer and mlp classes with the same depth. When comparing classes of different depth $F_{B,p_n,mlp}^p$, $F_{B,p_n,T}^T$ then as long as $\alpha = \limsup_{n \to \infty} \frac{p_n^{mlp}}{p_n^{T}} < \log_3 2 \approx 1.584$ the following dominance relation holds $F_{B,p_n,mlp}^p \prec (\frac{\alpha}{\alpha^*})^p F_{B,p_n}^T$.

Proof. See Appendix H.

Remark 5.2 The last (5.10) proposition leaves open the possibility that if someone can scale mlp-architectures $\approx 1.58$ deeper than transformer architectures while using the same budget, then it may be possible that the mlp-architectures would have a higher ability to model multivariable dependencies. However, our upper bound over the separation rank of mlp architectures is not necessarily tight, so we did not claim it but we leave this possibility open for further research.

6. Experiments

To assess our theory we derived a few predictions from it and assess them in experiments as shown below. The first experiment is also intended to support the $\sigma_2$ relaxation performed above (3.2), by using the separation-rank of the relaxed MLP-mixer to predict the optimal depth-to-width ratio for the MLP-mixer model and assessing it by experiments.

6.1. Depth to width ratio

Our first prediction is about the optimal depth-to-width ratio for the mixer architecture, when coming to this issue, then for the transformer architectures as shown in the appendices and relying on [22, 23] it holds that the optimal depth to width ratio for transformers architectures is $p \approx \log_3 d$ where $p$ and $d$ denote the transformer depth and width respectively.

In general, as shown in appendixes (I) for every architecture $y_{p,d}$ with

$$\log_3 | sep - rank (y_{p,d}) | = \Theta (Q_1 (p,d) \cdot \alpha^p) \quad (16)$$

for $p < \log_3 d$ and

$$\log_3 | sep - rank (y_{p,d}) | = \Theta (Q_2 (p,d)) \quad (17)$$

for $p > \log_3 d$ where $Q_1, Q_2 : \mathbb{N}^2 \to \mathbb{N}$ is some multinomial with a finite degree, and $1 < \alpha \in \mathbb{R}$ is the exponent basis when fixing a budget $B$ the optimal depth to width ratio satisfies $1 < \frac{p}{\log_3 d}$ and hence in the mixer case since we manage to show that

$$\log_2 | sep - rank (y_{p,d}) | = O (Q_1 (p,d) \cdot 2^p) \quad (18)$$

for $p < \log_3 d$ and

$$\log_2 | sep - rank (y_{p,d}) | = O (Q_2 (p,d)) \quad (19)$$

but we did not show the appropriate lower bound then we may hypothesize that for the mixer it also holds $p^* = \log_2 \alpha_{mixer} d^*$ when $1 \leq \alpha_{mixer} < 2$ and in particular

$$2 = \alpha_{mixer} < \alpha_{transformer} = 3 \quad (20)$$

We tested this hypothesis by examining the accuracy of multiple different models with the same parameter budget, but with different depth-to-width ratios on the CIFAR10, SVHN and MNLI datasets when trained for 40 epochs.

As we can see (1) the pick performance is obtained for

$$1 < \frac{p}{\log_2 d} < 2 \quad (21)$$

and note that

$$\frac{p}{\log_3 d} = \frac{p}{\log_2 d} \cdot \log_2 3 \approx 1.58 \cdot \frac{p}{\log_2 d} \quad (22)$$

hence for the transformer it just holds that

$$\frac{p_{transformer}}{\log_2 d_{transformer}} \approx \frac{5}{8} < 1 < \frac{p_{mixer}}{\log_2 d_{mixer}} \quad (23)$$

6.2. Data-size and training-time

It has been shown by Li et al. [24] deeper RoBERTa models tend to converge faster, [24] also shows that larger and more expressive models usually converge faster, unless there are overfitting issues. Their results assass our theory about the larger effective depth of the transformer architectures relative to the mlp-based ones, which results in slower convergence of the mixer models as indicated by [29].

7. Conclusions and discussion

To conclude, we showed the existence of an exponential gap in the expressive power between MLP-based architectures and attention-based ones in their ability to model multi-variable dependencies. This may explain the performance gap in vision tasks as well as the nonexistence of mlp-based architectures for NLP tasks. This also suggests that mlp-based architectures are indeed inferior to the attention-based ones and that although permutations-based strategies may give some improvements they may not suffice to close the gap since those architectures have degraded expressive power in the sense of modeling long dependencies. We also showed that this gap sustains as long as the mlp-architecture, with the same budget, is not 1.58 times deeper. However, we leave the question open, of how much depth increase is required for the mlp to achieve the same expressive power as the transformer. Say it differently, the transformer can achieve a larger effective depth using fewer layers relative to the mlp. This suggests some more explanation for the wide success of the attention-based mechanisms for various different tasks.
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