Neutrino Mixing and Leptogenesis in $\mu-\tau$ Symmetry

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Abstract

We study the consequences of the $Z_2$-symmetry behind the $\mu-\tau$ universality in neutrino mass matrix. We then implement this symmetry in the type-I seesaw mechanism and show how it can accommodate all sorts of lepton mass hierarchies and generate enough lepton asymmetry to interpret the observed baryon asymmetry in the universe. We also show how a specific form of a high-scale perturbation is kept when translated via the seesaw into the low scale domain, where it can accommodate the neutrino mixing data. We finally present a realization of the high scale perturbed texture through addition of matter and extra exact symmetries.

Keywords: Neutrino Physics; Flavor Symmetry; Matter-anti-matter

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1 Introduction

Flavor symmetry is commonly used in model building seeking to determine the nine free parameters characterizing the effective neutrino mass matrix $M_\nu$, namely the three masses ($m_1, m_2$ and $m_3$), the three mixing angles ($\theta_{23}, \theta_{12}$ and $\theta_{13}$), the two Majorana-type phases ($\rho$ and $\sigma$) and the Dirac-type phase ($\delta$). Incorporating family symmetry at the Lagrangian level leads generally to textures of specific forms, and one may then study whether or not these specific textures can accommodate the experimental data involving the above mentioned parameters ([1] and references therein). The recent observation of a non-zero value for $\theta_{13}$ from the T2K[2], MINOS[3], and Double Chooz[4] experiments puts constraints on models based on flavor symmetry (see Table 1 where the most recent updated mixing angles are taken from [5]). In this regard, recent, particularly simple, choices for discrete and continuous flavor symmetry addressing the non-vanishing $\theta_{13}$ question were respectively worked out ([6] and references therein). The

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μ-τ symmetry [7, 8] is enjoyed by many popular mixing patterns such as tri-bimaximal mixing (TBM) [9], bimaximal mixing (BM) [10], hexagonal mixing (HM) [11] and scenarios of $A_5$ mixing [12], and it was largely studied in the literature [13]. Any form of the neutrino mass matrix respects a $(Z_2)^2$ symmetry [14], and we can define the $\mu-\tau$ symmetry by fixing one of the two $Z_2$’s to express exchange between the second and third families, whereas the second $Z_2$ factor is to be determined later by data or, equivalently, by $M_\nu$ parameters. The whole $(Z_2)^2$ symmetry might turn out to be a subgroup of a larger discrete group imposed on the whole leptonic sector. In realizing $\mu-\tau$ symmetry we have two choices namely $(S_-, S_+, S_+)$, as explained later), and thus we have two textures corresponding to $\mu-\tau$ symmetry. It is known that both of these textures lead to a vanishing $\theta_{13}$ (with $S_-$ achieving this in a less natural way), and thus perturbations are needed to get remedy of this situation[15]. In [16] we studied the perturbed $\mu-\tau$ neutrino symmetry and found the four patterns, obtained by disentangling the effects of the perturbations, to be phenomenologically viable.

Table 1: Results for the neutrino mixing angles taken from the global fit to neutrino oscillation data [5].
(NH, IH) denote respectively Normal and Inverted Hierarchies.

| Parameter                  | Best fit | 1σ range         |
|----------------------------|----------|------------------|
| $\sin^2 \theta_{12}/10^{-1}$ (NH or IH) | 3.08     | 2.91 – 3.25      |
| $\sin^2 \theta_{13}/10^{-2}$ (NH)    | 2.34     | 2.15 – 2.54      |
| $\sin^2 \theta_{13}/10^{-2}$ (IH)    | 2.40     | 2.18 – 2.59      |
| $\sin^2 \theta_{23}/10^{-1}$ (NH)    | 4.37     | 4.14 – 4.70      |
| $\sin^2 \theta_{23}/10^{-1}$ (IH)    | 4.24     | 5.94 – 6.11      |

In this work, we re-examine the question of exact $\mu-\tau$-symmetry and implement it in a complete setup of the leptonic sector. Then, within type-I seesaw scenarios, we show the ability of exact symmetry to accommodate lepton mass hierarchies. Upon studying its effect on leptogenesis we find, in contrast to other symmetries studied in [6] and [17] that it can account for it. The reason behind this fact is that fixing just one $Z_2$ in $\mu-\tau$ symmetry leaves one mixing angle free which can be adjusted differently in the Majorana and Dirac neutrino mass matrices ($M_R$ and $M_D$), thus allowing for different diagonalizing matrices. For the mixing angles and in order to accommodate data, we introduce perturbations at the seesaw high scale and study their propagations into the low scale effective neutrino mass matrix. As in [16], we consider that the perturbed texture arising at the high scale keeps its form upon RG running which, in accordance with [18], does not affect the results in many setups. As to the origin of the perturbations, we shall not introduce explicitly symmetry breaking terms into the Lagrangian [19], but rather follow [16], and enlarge the symmetry with extra matter and then spontaneously break the symmetry by giving vacuum expectation values (vev) to the involved Higgs fields.

The plan of the paper is as follows: In Section 2, we review the standard notation for the neutrino mass matrix and the definition of the $\mu-\tau$ symmetry. In Section 3, we introduce the type-I seesaw scenario. In Subsection 3.1, we address the charged lepton sector, whereas in Subsection 3.2 we study the different neutrino mass hierarchies. In Subsection 3.3, we study the generation of lepton asymmetry, and in Subsection 3.4 we examine the mixing angles in a particular perturbed texture describing approximate $\mu-\tau$-symmetry. In Subsection 3.5 we present a theoretical realization of the perturbed texture. We end by discussion and summary in Section 4.

2 Notations

In the Standard Model (SM) of particle interactions, there are 3 lepton families. The charged-lepton mass matrix linking left-handed (LH) to their right-handed (RH) counterparts is arbitrary, but can always be diagonalized by a bi-unitary transformation:

$$V_L M_{1} (V_R^\dagger)^\dagger = \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix}.$$  (1)
Likewise, we can diagonalize the symmetric Majorana neutrino mass matrix by just one unitary transformation:

$$V^{\nu} M_{\nu} V^{\nu \dagger} = \left( \begin{array}{ccc} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{array} \right),$$

with $m_i$ (for $i = 1, 2, 3$) real and positive.

The observed neutrino mixing matrix comes from the mismatch between $V^l$ and $V^\nu$ in that

$$V_{PMNS} = (V^l_L)^\dagger V^\nu.$$ 

If the charged lepton mass eigen states are the same as the current (gauge) eigen states, then $V^l_L = 1$ (the unity matrix) and the measured mixing comes only from the neutrinos $V_{PMNS} = V^\nu$. We shall assume this saying that we are working in the “flavor” basis. As we shall see, corrections due to $V^l_L \neq 1$ are expected to be of order of ratios of the hierarchical charged lepton masses, which are small enough to justify our assumption of working in the flavor basis. However, one can treat these corrections as small perturbations and embark on a phenomenological analysis involving them [19].

We shall adopt the parametrization of [20], related to other ones by simple relations [1], where the $V_{PMNS}$ is given in terms of three mixing angles ($\theta_{12}, \theta_{23}, \theta_{13}$) and three phases ($\delta, \rho, \sigma$), as follows.

$$V_{PMNS} = U P,$$

$$P = \text{diag} \left( e^{i\rho}, e^{i\sigma}, 1 \right),$$

$$U = R_{23} (\theta_{23}) R_{13} (\theta_{13}) \text{ diag} \left( 1, e^{-i\delta}, 1 \right) R_{12} (\theta_{12}),$$

$$= \begin{pmatrix} c_{12} c_{13} & s_{12} c_{13} & s_{13} \\ -c_{12} s_{23} s_{13} - s_{12} c_{23} e^{-i\delta} & -s_{12} s_{23} s_{13} + c_{12} c_{23} e^{-i\delta} & s_{23} c_{13} \\ -c_{12} c_{23} s_{13} + s_{12} s_{23} e^{-i\delta} & -s_{12} c_{23} s_{13} - c_{12} s_{23} e^{-i\delta} & c_{23} c_{13} \end{pmatrix},$$

where $R_{ij} (\theta_{ij})$ is the rotation matrix in the $(i, j)$-plane by angle $\theta_{ij}$, and $s_{12} = \sin \theta_{12} \ldots$.

In this parametrization, and in the flavor basis, the neutrino mass matrix elements are given by:

$$M_{\nu \, 1} = m_1 c_{12} c_{13} e^{2i\rho} + m_2 s_{12} c_{13} e^{2i\sigma} + m_3 s_{13}^2,$$

$$M_{\nu \, 12} = m_1 \left(-c_{13} s_{13} c_{12} c_{23} e^{2i\rho} - c_{13} c_{12} s_{12} c_{23} e^{(2\rho - \delta)} \right) + m_2 \left(-c_{13} s_{13} s_{12} s_{23} c_{23} e^{2i\sigma} + c_{13} c_{12} s_{12} c_{23} e^{(2\sigma - \delta)} \right) + m_3 c_{13} s_{13} s_{23},$$

$$M_{\nu \, 13} = m_1 \left(-c_{13} s_{13} c_{12} c_{23} e^{2i\rho} + c_{13} c_{12} s_{12} s_{23} e^{(2\rho - \delta)} \right) + m_2 \left(-c_{13} s_{13} s_{12} s_{23} c_{23} e^{2i\sigma} - c_{13} c_{12} s_{12} s_{23} e^{(2\sigma - \delta)} \right) + m_3 c_{13} s_{13} c_{23},$$

$$M_{\nu \, 22} = m_1 \left(c_{12} s_{13} s_{23} e^{i\rho} + c_{23} s_{12} e^{i(\rho - \delta)} \right)^2 + m_2 \left(s_{12} s_{13} s_{23} e^{i\sigma} - c_{23} c_{12} e^{i(\sigma - \delta)} \right)^2 + m_3 c_{13}^2 s_{23}^2,$$

$$M_{\nu \, 33} = m_1 \left(c_{12} s_{13} c_{23} e^{i\rho} - c_{23} s_{12} e^{i(\rho - \delta)} \right)^2 + m_2 \left(s_{12} s_{13} c_{23} e^{i\sigma} + s_{23} c_{12} e^{i(\sigma - \delta)} \right)^2 + m_3 c_{13}^2 c_{23}^2,$$

$$M_{\nu \, 23} = m_1 \left(e_{12} c_{23} s_{23} s_{13} e^{2i\rho} + s_{13} c_{12} s_{12} \left(c_{23}^2 - s_{23}^2 \right) e^{(2\rho - \delta)} - c_{23} s_{23} s_{12} e^{2i(\rho - \delta)} \right) + m_2 \left(s_{12} c_{23} s_{23} s_{13} e^{2i\sigma} + s_{13} c_{12} s_{12} \left(s_{23}^2 - c_{23}^2 \right) e^{(2\sigma - \delta)} - c_{23} s_{23} c_{12} e^{2i(\sigma - \delta)} \right) + m_3 s_{23} c_{23} c_{13}^2.$$ 

As said before, any form of $M_{\nu}$ satisfies a $Z_2^2$-symmetry. This means that there are two commuting unitary $Z_2$-matrices (squared to unity) ($S_1, S_2$) which leave $M_{\nu}$ invariant:

$$S^T M_{\nu} S = M_{\nu}.$$ 

(6)
For a non-degenerate mass spectrum, the form of the $Z_2$-matrix $S$ is given by [17]:

$$S = V^\nu \text{diag}(\pm 1, \pm 1, \pm 1) V^\nu \dagger$$

(7)

where the two $S$’s correspond to having, in diag$(\pm 1, \pm 1, \pm 1)$, two pluses and one minus, the position of which differs in the two $S$’s (the third $Z_2$-matrix, corresponding to the third position of the minus sign, is generated by multiplying the two $S$’s and noting that the form invariance formula Eq.(6) is invariant under $S \rightarrow -S$).

In practice, however, we follow a reversed path, in that if we assume a ‘real’ orthogonal $Z_2$-matrix (and hence symmetric with eigenvalues $\pm 1$) satisfying Eq.(6), then it commutes with $M_\nu$, and so both matrices can be simultaneously diagonalized. Quite often, the form of $S$ is simpler than $M_\nu$, so one proceeds to solve the eigensystem problem for $S$, and find a ‘real’ orthogonal diagonalizing matrix $\tilde{U}$:

$$\tilde{U} \dagger S \tilde{U} = \text{diag}(\pm 1, \pm 1, \pm 1)$$

(8)

This matrix $\tilde{U}$ can ‘commonly’ be identified with, or related simply to, the matrix $V$ satisfying the ‘Takagi’ decomposition of Eq.(2) *. In this case, and in the flavor basis, the $V_{\text{PMNS}}$ would be real and equal to $U = R_{23}(\theta_{23})R_{13}(\theta_{13})R_{12}(\theta_{12})$. Determining the eigenvectors of the $S$ matrices helps thus to determine the neutrino mixing angles.

The $\mu$–$\tau$ symmetry is defined when one of the two $Z_2$-matrices corresponds to switching between the $2^{\text{nd}}$ and the $3^{\text{rd}}$ families. We have, up to a global irrelevant minus sign (see again Eq.6), two choices, which would lead to two textures at the level of $M_\nu$.

2.1 The $\mu$–$\tau$ symmetry manifested through $S_-$: ($M_{\nu 12} = M_{\nu 13}$ and $M_{\nu 22} = M_{\nu 33}$)

The $Z_2$-symmetry matrix is given by:

$$S_- = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

(9)

The invariance of $M_\nu$ under $S_-$ (Eq.6) forces the symmetric matrix $M_\nu$ to have a texture of the form:

$$M_\nu = \begin{pmatrix} A_\nu & B_\nu & B_\nu \\ B_\nu & C_\nu & D_\nu \\ B_\nu & D_\nu & C_\nu \end{pmatrix}$$

(10)

Since $S_-$ and $M_\nu$ commute, they have common eigenvectors. The normalized eigen vectors of $S_-$ are: $\{v_1 = (0, 1, \sqrt{2}, 1/\sqrt{2})^T, v_2 = (1, 0, 0)^T, v_3 = (0, 1/\sqrt{2}, -1/\sqrt{2})^T\}$ corresponding respectively to the eigenvalues $(1, 1, -1)$. Since the eigenvalue 1 is two-fold degenerate, then there is still a freedom for rotation by angle $\varphi$ in its eigenspace to get the eigen vectors:

$$\begin{pmatrix} v_1(\varphi) \\ v_2(\varphi) \end{pmatrix} = \begin{pmatrix} c_\varphi & s_\varphi \\ -s_\varphi & c_\varphi \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$  

(11)

We have three choices as to how we order the eigenvectors forming the diagonalizing matrix $U$.

- Eigenvalues $(1, -1, 1)$

The matrix $U_-$ which diagonalizes $S_-$ can be cast into the form:

$$U_- = [v_2(\varphi) \ v_3 \ v_1(\varphi)] = R_{23}(\theta_{23})R_{13}(\theta_{13})R_{12}(\theta_{12}) = \begin{pmatrix} c_\varphi & 0 & s_\varphi \\ -s_\varphi/\sqrt{2} & 1/\sqrt{2} & c_\varphi/\sqrt{2} \\ -s_\varphi/\sqrt{2} & -1/\sqrt{2} & c_\varphi/\sqrt{2} \end{pmatrix}. \quad (12)$$

*In fact, up to an irrelevant sign and dropping the trivial $S = 1$ (identity) case, one can restrict the study to an $S$ with eigen values $(-1, +1, +1)$. The eigenvector of $S$ corresponding to the eigenvalue $(-1)$ is an eigenvector of $M_\nu$, and the 2–dim eigenspace of $S$ corresponding to the multiple eigenvalue $+1$ is ‘stable’ under $M_\nu$. The restriction of the symmetric $M_\nu$ to this eigenspace is symmetric (since $\tilde{U}$ is orthogonal and so $\tilde{U}^{-1}M\tilde{U}$ is symmetric) and assumed to be diagonalizable by a ‘real’ rotation with no complex phases. Thus we end up with a real orthogonal matrix diagonalizing both $S$ and $M_\nu$ having a free rotation angle defined by the $M_\nu$ parameters.
In order to enforce $U_-$ to be a matrix which diagonalizes $M_\nu$ as given in Eq.(2), we need the free parameter $\varphi$ to be expressed in terms of the mass parameters as follows,

$$\tan (2\varphi) = \frac{2\sqrt{2}B_\nu}{C_\nu + D_\nu - A_\nu}. \quad (13)$$

Comparing Eq.(12) with Eq.(4) we find that the $\mu-\tau$ symmetry forces the following mixing angles:

$$\theta_{23} = \pi/4, \theta_{12} = 0, \theta_{13} = \varphi \quad (14)$$

We can get, as phenomenology suggests, a small value for $\theta_{13}$ assuming

$$B_\nu \ll (A_\nu, C_\nu, D_\nu). \quad (15)$$

and then the mass spectrum turns out to be:

$$m_1 = (C - D + A) \csc^2 \phi - \sqrt{2} s_{2\phi} B + C + D \simeq A \quad (16)$$

$$m_2 = C - D$$

$$m_3 = (C + D - A) \csc^2 \phi + \sqrt{2} s_{2\phi} B + A \simeq C + D.$$ 

Inverting these relations to express the mass parameters in terms of the mass eigenvalues we get

$$A \approx m_1, \quad C \approx \frac{m_2 + m_3}{2}, \quad D \approx \frac{m_3 - m_2}{2}. \quad (17)$$

We can easily see using Eq.(16) that all mass spectra can be accommodated by properly adjusting the parameters $A, B, C$ and $D$. To get a glimpse of a better analytical understanding, we show that all kinds of possible mass hierarchies can be generated as follows:

(i) Normal hierarchy ($m_2 > m_1, m_3 \gg m_2, m_1$). It is sufficient to have

$$C > D, \quad A \ll C, D. \quad (18)$$

(ii) Inverted hierarchy ($m_2 > m_1, m_3 \ll m_2, m_1$). It is sufficient to have

$$C \approx -D, \quad A \gg C, D. \quad (19)$$

(iii) Degenerate case [$m_1 \approx m_2 \approx m_3, (m_2 > m_1)$] It is sufficient to have

$$A \approx C, \quad D \ll C. \quad (20)$$

We see that this choice is not viable phenomenologically. Whereas the value of $\theta_{23}$ is acceptable corresponding to maximal atmospheric mixing, and that one can assume a hierarchy in the mass parameters to accommodate the small mixing angle $\theta_{13}$, and that the neutrino mass hierarchies can be accounted for, however the vanishing value of the angle $\theta_{12}$ is far from its experimental value $\simeq 33^\circ$.

- Eigenvalues $(-1, 1, 1)$: This choice leads again to a maximal mixing for $\theta_{23} = \pi/4$, and an adjustable value for $\theta_{13} = \varphi$, but the value of $\theta_{12}$ is predicted to be $\pi/2$ far from its experimental value $\simeq 33^\circ$:

$$U_- = [-v_3, v_2(\varphi), v_1(\varphi)] = R_{23} (\theta_{23} = \pi/4) R_{13} (\theta_{13} = \pi/2) R_{12} (\theta_{12} = \varphi). \quad (21)$$

- Eigenvalues $(1, 1, -1)$: One can check that this choice will lead to a free adjustable mixing angle $\theta_{12} = \pi + \varphi$ and either to ($\theta_{23} = -\pi/4, \theta_{13} = 0$) or to ($\theta_{23} = 3\pi/4, \theta_{13} = \pi$), in that we have respectively:

$$U_- = [-v_2(\varphi), -v_1(\varphi), -v_3] = R_{23} (\theta_{23} = -\pi/4) R_{13} (\theta_{13} = 0) R_{12} (\theta_{12} = \pi + \varphi)$$

OR

$$U_- = [v_2(\varphi), v_1(\varphi), -v_3] = R_{23} (\theta_{23} = 3\pi/4) R_{13} (\theta_{13} = \pi) R_{12} (\theta_{12} = \pi + \varphi). \quad (22)$$

Whereas one might argue that this choice is viable phenomenologically, however, we shall not use the phase ambiguity to put all mixing angles in the first quadrant. Rather, we prefer to find a symmetry leading directly, dropping all the phases, to mixing angles in the first quadrant. This can be carried out in the second texture expressing the $\mu-\tau$ symmetry materialized through $S_+$. 5
2.2 The $\mu$–$\tau$ symmetry manifested through $S_+$: $(M_{\nu 12} = -M_{\nu 13}$ and $M_{\nu 22} = M_{\nu 33}$)

The $\mathbb{Z}_2$-symmetry matrix is given by:

$$S_+ = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

(23)

The invariance of $M_{\nu}$ under $S_+$ (Eq. 6) forces the symmetric matrix $M_{\nu}$ to have a texture of the form:

$$M_{\nu} = \begin{pmatrix} A_{\nu} & B_{\nu} & -B_{\nu} \\ B_{\nu} & C_{\nu} & D_{\nu} \\ -B_{\nu} & D_{\nu} & C_{\nu} \end{pmatrix}$$

(24)

As $S_+$ and $M_{\nu}$ commute, they have common eigenvectors. The normalized eigenvectors of $S_+$ are: $\{v_1 = (0, -1/\sqrt{2}, 1/\sqrt{2})^T, v_2 = (1, 0, 0)^T, v_3 = (0, 1/\sqrt{2}, 1/\sqrt{2})^T\}$ corresponding respectively to the eigenvalues $\{-1, -1, 1\}$. Since the eigenvalue $-1$ is two-fold degenerate, then there is still a freedom for rotation by angle $\varphi$, and we define the eigenvectors $v_1(\varphi)$ and $v_2(\varphi)$ as in Eq.(11). We can, as in the last subsection, discuss different forms for the $S_+$-diagonalizing matrix $U_+$ shuffling through its columns and taking various values for $\varphi$, but to fix the ideas we fix the order of the eigenvalues as mentioned above, and cast $U_+$ in the form:

$$U_+ = R_{23}(\theta_{23}) R_{13}(\theta_{13}) R_{12}(\theta_{12}) = [v_1(\pi/2 - \varphi), v_2(\pi/2 - \varphi), v_3]$$

$$= \begin{pmatrix} c_\varphi \sqrt{2} & s_\varphi / \sqrt{2} & 0 \\ -s_\varphi / \sqrt{2} & c_\varphi \sqrt{2} & 1/\sqrt{2} \\ s_\varphi / \sqrt{2} & -c_\varphi / \sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

(25)

Again, one can express the free parameter $\varphi$ in terms of the neutrino mass matrix parameters by forcing $U_+$ to diagonalize $M_{\nu}$ satisfying Eq.(2), so to get:

$$\tan(2\varphi) = \frac{2 \sqrt{2} B_{\nu}}{C_{\nu} - D_{\nu} - A_{\nu}}.$$ 

(26)

and the mass spectrum is given by:

$$m_1 = A_{\nu} c_\varphi^2 - \sqrt{2} s_\varphi B_{\nu} + (C_{\nu} - D_{\nu}) s_\varphi^2$$

$$m_2 = A_{\nu} s_\varphi^2 + \sqrt{2} s_\varphi B_{\nu} + (C_{\nu} - D_{\nu}) c_\varphi^2$$

$$m_3 = C_{\nu} + D_{\nu}.$$ 

(27)

Comparing Eq.(25) with Eq.(4) we find that the $\mu$–$\tau$ symmetry forces the following mixing angles:

$$\theta_{23} = \pi/4, \theta_{13} = 0, \theta_{12} = \varphi$$

(28)

These predictions are phenomenologically viable, and furthermore do not need a special adjustment for the parameters $A_{\nu}, B_{\nu}, C_{\nu}, D_{\nu}$ which can be of the same order, in contrast to Eq.(15), and still accommodate the experimental value of $\theta_{12} \approx 33^\circ$.

The various neutrino mass hierarchies can also be produced as can be easily seen from Eq.(27) where the three masses are given in terms of four parameters. Again, for the sake of a better analytical understanding, we show that all kinds of possible hierarchies can be be obtained as follows (with $\varphi$ fixed around its phenomenologically acceptable value leading to $s_\varphi^2 \approx 0.3$):

(i) Normal hierarchy $(m_2 > m_1, m_3 \gg m_2, m_1)$. It is sufficient to have

$$C \approx D, \quad A \approx B, \quad C \gg A.$$ 

(29)

(ii) Inverted hierarchy $(m_2 > m_1, m_3 \ll m_2, m_1)$. It is sufficient to have

$$C \approx -D, \quad A \approx B, \quad C \gg A.$$ 

(30)

(iii) Degenerate case $[m_1 \approx m_2 \approx m_3, (m_2 > m_1)]$. It is sufficient to have

$$C \approx D, \quad A \approx 4C, \quad B \approx 0.6 C.$$ 

(31)
3 The seesaw mechanism and the $\mu - \tau$ symmetry

We impose now the $\mu - \tau$-symmetry, defined by the matrix $S = S_+$, at the Lagrangian level within a model for the Leptons sector. Then, we shall invoke the type-I see-saw mechanism to address the origin of the effective neutrino mass matrix, with consequences on leptogenesis. The procedure has already been done in [17] for other $Z_2$-symmetries.

3.1 The charged lepton sector

We start with the part of the SM Lagrangian responsible for giving masses to the charged leptons:

$$\mathcal{L}_1 = Y_{ij} \overline{L}_i \phi \ell^c_j,$$

where the SM Higgs field $\phi$ and the right handed (RH) leptons $\ell^c_j$ are assumed to be singlet under $S$, whereas the left handed (LH) leptons transform in the fundamental representation of $S$:

$$L_i \rightarrow S_{ij} L_j.$$  

(33)

Invariance under $S$ implies:

$$S^T Y = Y,$$

(34)

and this forces the Yukawa couplings to have the form:

$$Y = \begin{pmatrix} 0 & 0 & 0 \\ a & b & c \\ a & b & c \end{pmatrix},$$

(35)

which leads, when the Higgs field acquires a vev $v$, to a charged lepton squared mass matrix of the form:

$$M_l M_l^\dagger = v^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} |a|^2 + |b|^2 + |c|^2 \end{pmatrix}.$$  

(36)

As the eigenvectors of $M_l M_l^\dagger$ are $(0, 1/\sqrt{2}, 1/\sqrt{2})^T$ with eigenvalue $2v^2 \begin{pmatrix} |a|^2 + |b|^2 + |c|^2 \end{pmatrix}$ and $(0, 1/\sqrt{2}, -1/\sqrt{2})^T$ with a degenerate eigenvalue 0, then the charged lepton mass hierarchy can not be accommodated. Moreover, the nontrivial diagonalizing matrix, illustrated by non-canonical eigenvectors, means we are no longer in the flavor basis. To remedy this, we introduce SM-singlet scalar fields $\Delta_k$ coupled to the lepton LH doublets through the dimension-5 operator:

$$\mathcal{L}_2 = f_{ikr} \overline{L}_i \phi \Delta_k \ell^c_r.$$  

(37)

This way of adding extra SM-singlets is preferred, for suppressing flavor–changing neutral currents, than to have additional Higgs fields. Also, we assume the $\Delta_k$’s transform under $S$ as:

$$\Delta_i \rightarrow S_{ij} \Delta_j.$$  

(38)

Invariance under $S$ implies,

$$S^T f_r S = f_r,' \quad \text{where} \quad (f_r)_{ij} = f_{ijr},$$

(39)

and thus we have the following form

$$f_r = \begin{pmatrix} A^r & B^r & -B^r \\ E^r & C^r & D^r \\ -E^r & D^r & C^r \end{pmatrix},$$  

(40)

when the fields $\Delta_k$ and the neutral component of the Higgs field $\phi^\circ$ take vevs ($\langle \Delta_k \rangle = \delta_k, \quad v = \langle \phi^\circ \rangle$) we get a charged lepton mass matrix:

$$(M_l)_{ir} = \frac{vf_{ikr}}{\Lambda} \delta_k,$$

(41)
if $\delta_1, \delta_2 \ll \delta_3$ then

$$
(M_i)_{ij} \simeq \frac{v f_{ij}}{\Lambda} \delta_3 \simeq \begin{pmatrix}
-B^1 & -B^2 & -B^3 \\
D^1 & D^2 & D^3 \\
C^1 & C^2 & C^3
\end{pmatrix},
$$

(42)

with $f_{13j} = -B^1$, $f_{23j} = D^1$, $f_{33j} = C^1$ for $j = 1, 2, 3$. In Ref. [17], a charged lepton matrix of exactly the same form was shown to represent the lepton mass matrix in the flavor basis with the right charged lepton mass hierarchies, assuming just the ratios of the magnitudes of the vectors comparable to the lepton mass ratios.

### 3.2 Neutrino mass hierarchies

The effective light LH neutrino mass matrix is generated through the seesaw mechanism formula

$$
M_\nu = M_D M_R^{-1} M_D^T,
$$

(43)

where the Dirac neutrino mass matrix $M_D$ comes from the Yukawa term

$$
g_{ij} \bar{\nu}_i \tau_2 \Phi^* v_R j,
$$

(44)

upon the Higgs field acquiring a vev, whereas the symmetric Majorana neutrino mass matrix $M_R$ comes from a term ($C$ is the charge conjugation matrix)

$$
\frac{1}{2} v_{ij}^T C^{-1} (M_R)_{ij}^T v_R j.
$$

(45)

We assume the RH neutrino to transform under $S$ as:

$$
v_R j \rightarrow S_{jr} v_R r,
$$

(46)

and thus the $S$-invariance leads to

$$
S^T g S = g, \quad S^T M_R S = M_R.
$$

(47)

This forces the following textures:

$$
M_D = v \begin{pmatrix}
A_D & B_D & -B_D \\
E_D & C_D & D_D \\
-E_D & D_D & C_D
\end{pmatrix}, \quad M_R = \Lambda_R \begin{pmatrix}
A_R & B_R & -B_R \\
B_R & C_R & D_R \\
-B_R & D_R & C_R
\end{pmatrix},
$$

(48)

where $\Lambda_R$ is a high energy scale characterizing the heavy RH Majorana neutrinos. The resulting effective matrix $M_\nu$ will have the form of Eq.(24) with

$$
A_\nu = [(C_R^2 - D_R^2) A_D^2 - 4 B_R (C_R + D_R) A_D B_D + 2 A_R (C_R + D_R) B_D^2]/\det M_R,
$$

$$
B_\nu = -(C_R + D_R) \{ (D_D - C_D) B_D A_D + (D_R - C_R) E_D A_D + [A_D (C_D - D_D) + 2 B_D E_D] B_R \}/\det M_R,
$$

$$
C_\nu = \{ (A_R C_R - B_R^2) D_D^2 + [-2 (A_R D_R + B_R^2) C_D + 2 B_R (C_R + D_R) E_D] D_D \\
+(A_R C_R - B_R^2) C_D^2 - 2 B_R (C_R + D_R) E_D C_D + E_D^2 (C_R^2 - D_R^2) \}/\det M_R,
$$

$$
D_\nu = \{ (A_R D_R + B_R^2) D_D^2 + [-2 (-A_R C_R + B_R^2) C_D - 2 B_R (C_R + D_R) E_D] D_D \\
-(A_R D_R + B_R^2) C_D^2 + 2 B_R (C_R + D_R) E_D C_D - E_D^2 (C_R^2 - D_R^2) \}/\det M_R,
$$

(49)

$$
\det M_R = (C_R + D_R) [A_R (C_R - D_R) - 2 B_R^2].
$$

Constraints imposed on the mass parameters of $(M_D, M_R)$ can find their way through the seesaw formula to similar constraints on the mass parameters of $M_\nu$, which can be those stated in the preceding section. This helps to generate all neutrino mass hierarchies assuming their origin resides in the seesaw high energy domain as follows.
\[ C_{R,D} \simeq D_{R,D}, \quad C_{R,D} \gg B_{R,D}, \quad A_{R,D} \simeq B_{R,D}, \quad \text{and} \quad E_D \simeq B_D \simeq A_D, \quad (50) \]

we find
\[ A_\nu \simeq B_\nu \simeq \frac{A^2_D}{B_R}, \quad C_\nu \simeq D_\nu \simeq \frac{C^2_D}{C_R} \quad \Rightarrow \quad \frac{C_\nu}{A_\nu} \gg 1. \quad (51) \]

in accordance with Eq.(29).

- Inverted hierarchy: Assuming
\[ C_{R,D} \simeq -D_{R,D}, \quad A_{R,D} \simeq B_{R,D}, \quad C_{R,D} \ll B_{R,D}, \quad A_{R,D}, \quad C_{R,D} + D_R \simeq C_D + D_D, \quad \text{and} \quad E_D \simeq A_D \simeq B_D \quad (52) \]

we find
\[ A_\nu \simeq B_\nu \simeq \frac{A^2_D}{B_R}, \quad C_\nu \simeq -D_\nu \simeq -\frac{E^2_D C_R}{B^2_R} \quad \Rightarrow \quad |C_\nu| \ll |A_\nu|, |B_\nu| \quad (53) \]

in accordance with Eq.(30), and one can arrange to have also \( A_\nu < 6B_\nu \) (while keeping \( A_\nu \simeq B_\nu \)) in order to impose \( m_2 > m_1 \).

- Degenerate case: Assuming:
\[ C_{R,D} \simeq D_{R,D}, \quad A_{R,D} \simeq 4C_{R,D} \quad B_{R,D} \simeq 0.6 \, C_{R,D} \quad \text{and} \quad E_D \simeq 0.6 \, C_D \quad (54) \]

we get:
\[ A_\nu \simeq \frac{A_D C_D}{C_R}, \quad B_\nu \simeq \frac{C_D E_D}{C_R}, \quad C_\nu \simeq D_\nu \simeq \frac{C^2_D}{C_R} \quad \Rightarrow \quad B_\nu \simeq 0.6 \, C_\nu, \quad A_\nu \simeq 4 \, C_\nu \quad (55) \]

in accordance with Eq.(31).

3.3 Leptogenesis

In this kind of models, the unitary matrix diagonalizing \( M_R \) is not necessarily diagonalizing \( M_D \). In fact, the Majorana and Dirac neutrino mass matrices have different forms dictated by the \( S \)-symmetry and the angle \( \varphi \) in Eq.(26) depends on the corresponding mass parameters. This point is critical in generating lepton asymmetry, in contrast to other symmetries [17] where no freedom was left for the mixing angles leading to the same form on \( M_R \) and \( M_D \) with identical diagonalizing matrices. This is important when computing the lepton asymmetry induced by the lightest RH neutrinos, since it involves explicitly the unitary matrix diagonalizing \( M_R \):

\[ \varepsilon \simeq \frac{3}{16 \pi v^2} \frac{1}{(\tilde{M}^D \tilde{M}^D)_{11}} \sum_{j=2,3} \text{Im} \left\{ \left[ (\tilde{M}^D \tilde{M}^D)_{1j} \right]^2 \right\} \frac{M_{R1}}{M_{Rj}} \quad (56) \]

with \( \tilde{M}^D \) is the Dirac neutrino mass matrix in the basis where the RH neutrinos are mass eigenstates.

\[ \tilde{M}^D = M^D V^*_R F_0 \quad (57) \]

where \( V_R \) the unitary matrix, defined up to a phase diagonal matrix, that diagonalizes the symmetric matrix \( M_R \), and \( F_0 \) is a phase diagonal matrix chosen such that the eigenvalues of \( M_R \) are real and positive. In our case where the \( S \)-symmetry imposes a particular form on \( M_R \) (Eq. 48), we can take \( V_R \) as being the rotation matrix \( U_+ \) of Eq.(25) corresponding to

\[ \theta_{23} = \pi/4, \quad \theta_{12} = \varphi = \tan^{-1} \left( \frac{2 \sqrt{2} B_R}{C_R - A_R - D_R} \right), \quad \text{and} \quad \theta_{13} = 0. \quad (58) \]
As to the diagonal phase matrix, \( F_0 = \text{diag} \left( e^{-i\alpha_1}, e^{-i\alpha_2}, e^{-i\alpha_3} \right) \), it can be chosen according to Eq.(27) to be
\[
(\alpha_1, \alpha_2, \alpha_3) = \frac{1}{2} \arg \left( A_R c_\varphi^2 - \sqrt{2} s_2 \varphi B_R + (C_R - D_R) s_\varphi^2, A_R s_\varphi^2 + \sqrt{2} s_2 \varphi B_R + (C_R - D_R) c_\varphi^2, C_R + D_R \right).
\]

Note here that had the matrix \( V_R \) diagonalized \( M^D \), which would have meant that \( N = V_R^\dagger M^D V_R \) is diagonal, then we would have reached a diagonal \( \tilde{M}^D \) equaling a product of diagonal matrices, and no leptogenesis:
\[
\tilde{M}^D = F_0^\dagger \left( V_R^\dagger M^D V_R \right) F_0 = F_0^\dagger N F_0 \quad \text{(60)}
\]
In contrast, we get in our case:
\[
\begin{align*}
\left( \tilde{M}^D \right)_{12} & = e^{i(\alpha_1 - \alpha_2)} \left[ \sqrt{2} (A_D B_D + B_D A_D - E_D D_D - E_D D_D + E_D C_D^* + E_D C_D^* - C_D^* C_D^*) c_\varphi^2 \\
&\quad + (-2B_D B_D - C_D C_D^* - D_D D_D^* + 2E_D E_D + C_D D_D^* + C_D D_D^* + A_D A_D^*) s_\varphi c_\varphi \\
&\quad - \sqrt{2} (B_D^* A_D + C_D^* E_D - D_D^* E_D) \right] \\
\left( \tilde{M}^D \right)_{13} & = 0 \\
\left( \tilde{M}^D \right)_{11} & = (-2B_D B_D - C_D C_D^* - D_D D_D^* + 2E_D E_D + C_D D_D^* + D_D^* C_D + A_D A_D^*) c_\varphi^2 \\
&\quad + \sqrt{2} (A_D B_D + A_D B_D^* + E_D E_D + E_D C_D^* - D_D^* E_D - D_D E_D) s_\varphi c_\varphi \\
&\quad + D_D^* D_D + 2B_D^* B_D - C_D^* D_D - C_D D_D^* + C_D^* C_D \\
\end{align*}
\]
We see that \( \left( \tilde{M}^D \right)_{12} \) is complex in general, and the question is asked whether or not one can tune it to produce the correct lepton asymmetry. Clearly, when the mass parameters \( (A_D, B_D, C_D, D_D, E_D) \) are all real then we get
\[
\epsilon \propto \sin 2(\alpha_1 - \alpha_2). \quad \text{(62)}
\]
We deduce that by adjusting the phase difference \( (\alpha_1 - \alpha_2) \), one can generate enough lepton asymmetry, to be transformed later, via sphelarons, into the observed baryon/antibaryons asymmetry observed in the universe [17].

### 3.4 Neutrino mixing

We saw that exact \( \mu - \tau \)-symmetry implied a vanishing value for the mixing angle \( \theta_{13} \). Recent oscillation data pointing to a small but non-vanishing value for this angle suggest then a deviation on the exact symmetry texture in order to account for the observed mixing. We showed in [16] how “minimal” perturbed textures disentangling the effects of the perturbations can account for phenomenology. We shall consider now, within the scheme of type-I seesaw, a specific perturbed texture imposed on Dirac neutrino mass matrix \( M_D \), and parameterized by only one small parameter \( \alpha \), and show how it can resurface on the effective neutrino mass matrix \( M_e \), which is known to be phenomenologically viable. We compute then the “perturbed” eigenmasses and mixing angles to first order in \( \alpha \), whereas we address the question of realizing the perturbed texture of \( M_D \) in the next subsection. Thus, we assume a perturbed \( M_D \) of the form
\[
M_D = \begin{pmatrix}
A_D & B_D (1 + \alpha) & -B_D \\
E_D & C_D & D_D \\
-E_D & D_D & C_D
\end{pmatrix}
\quad \text{(63)}
\]
The small parameter \( \alpha \) affects only one condition defining the exact \( S \)-symmetry texture, and can be expressed as:
\[
\alpha = \frac{(M_D)_{12} + (M_D)_{13}}{(M_D)_{13}} \quad \text{(64)}
\]
Applying the seesaw formula Eq.(43) with $M_R$ given by Eq.(48) then we get:

\[
\begin{align*}
M_{\nu}^{(1, 1)} &= M_{\nu}^{0} (1, 1) + \alpha \chi \frac{B_D^2 (C_R A_R - B_R^2)}{\det M_R} + \alpha \frac{2 B_D (C_R + D_R) (A_R B_D - B_R A_D)}{\det M_R} \\
M_{\nu}^{(1, 2)} &= M_{\nu}^{0} (1, 2) + \alpha \chi \frac{B_D [A_R (C_R C_D - D_R D_D) - B_R^2 (D_D + C_D) - E_D B_R (D_R + C_R)]}{\det M_R} \\
M_{\nu}^{(1, 3)} &= M_{\nu}^{0} (1, 3) + \alpha \chi \frac{B_D [A_R (C_R D_D - D_R C_D) - B_R^2 (D_D + C_D) + E_D B_R (D_R + C_R)]}{\det M_R} \\
M_{\nu}^{(2, 2)} &= M_{\nu}^{0} (2, 2) = M_{\nu}^{0} (3, 3) = M_{\nu} (3, 3) \\
M_{\nu}^{(2, 3)} &= M_{\nu}^{0} (2, 3)
\end{align*}
\]  

(65)

where $M_{\nu}^0$ is the ‘unperturbed’ effective neutrino mass matrix (corresponding to $\alpha = 0$) and thus can be diagonalized by $U_+$ of Eq.(25) corresponding to the following angles,

\[\theta_{23} = \pi/4, \theta_{12} = \varphi = \tan^{-1} \left( \frac{2 \sqrt{2} B_\nu}{C_\nu - A_\nu - D_\nu} \right) \bigg|_{\alpha = 0} \text{ and } \theta_{13} = 0.\]  

(66)

We see that $M_{\nu}$ has exactly the following form:

\[
M_{\nu} = \begin{pmatrix}
A_\nu & B_\nu (1 + \chi) & -B_\nu \\
B_\nu (1 + \chi) & C_\nu & D_\nu \\
-B_\nu & D_\nu & C_\nu
\end{pmatrix}
\]  

(67)

where the perturbation parameter $\chi$ is given by:

\[
\chi \equiv -\frac{(M_{\nu})_{13} + (M_{\nu})_{13}}{(M_{\nu})_{13}} = \frac{\alpha N_1}{G_1 + \alpha G_2} : \text{ where,}
\]

\[
\begin{align*}
N_1 &= B_D (C_R + D_R) \left[ A_R (D_R - C_R) + 2 B_R^2 \right] \\
G_1 &= (C_R + D_R) \left\{ B_D (D_D - C_D) A_R + A_D E_D (D_R - C_R) + B_R [2 B_D E_D - A_D (D_D - C_D)] \right\}, \\
G_2 &= B_D \left[ A_R (C_R D_D - D_R C_D) - B_R^2 (C_D + D_D) + B_R E_D (C_R + D_R) \right].
\end{align*}
\]  

(68)

The two parameters $\chi$ and $\alpha$ are of the same order provided we do not have unnatural cancellations between the mass parameters of $M_D$ and $M_R$.

In order to compute the new eigenmasses and mixing angles of $M_{\nu}$, we write it in the following form working only to first order in $\alpha$:

\[
M_{\nu} = M_{\nu}^0 + M_{\alpha},
\]  

(69)

where the matrix $M_{\alpha}$ is given as,

\[
M_{\alpha} = \begin{pmatrix}
\alpha_{11} & \alpha_{12} & \alpha_{13} \\
\alpha_{12} & 0 & 0 \\
\alpha_{13} & 0 & 0
\end{pmatrix},
\]  

(70)

and the non-vanishing entries of $M_{\alpha}$ are found to be,

\[
\begin{align*}
\alpha_{11} &= \frac{2 \alpha B_D (C_R + D_R) (A_R B_D - B_R A_D)}{\det M_R}, \\
\alpha_{12} &= \frac{\alpha B_D [A_R (C_R C_D - D_R D_D) - B_R^2 (D_D + C_D) - E_D B_R (D_R + C_R)]}{\det M_R}, \\
\alpha_{13} &= \frac{\alpha B_D [A_R (C_R D_D - D_R C_D) - B_R^2 (D_D + C_D) + E_D B_R (D_R + C_R)]}{\det M_R}.
\end{align*}
\]  

(71)

Note here that $M_{\nu} (1, 1)$ get distorted by terms of order $\alpha$ and $\alpha^2$, however this will not “perturb” the relations defining $\mu–\tau$ symmetry, which are expressed only through $M_{\nu} (1, 2), M_{\nu} (1, 3), M_{\nu} (2, 2)$ and $M_{\nu} (3, 3)$.
We seek now a unitary matrix $Q$ diagonalizing $M_\nu$, and we write it in the form:

$$Q = U_+ (1 + \varepsilon) : \varepsilon = \begin{pmatrix} 0 & \varepsilon_1 & \varepsilon_2 \\ -\varepsilon_1^* & 0 & \varepsilon_3 \\ -\varepsilon_2^* & -\varepsilon_3^* & 0 \end{pmatrix},$$

(72)

where $\varepsilon$ is an antihermitian matrix due to the unitarity of $Q$. Imposing the diagonalization condition on $M_\nu$, knowing that $U_+$ diagonalizes $M_\nu^0$ we have:

$$Q^T M_\nu Q = M_\nu^{\text{diag}} = \text{diag} (m_1, m_2, m_3) \quad , \quad U_+^T M_\nu^0 U_+ = M_\nu^{\text{diag}} = \text{diag} (m_1^0, m_2^0, m_3^0).$$

(73)

If we restrict to the real case for the matrix $\varepsilon$, then we get the condition:

$$i, j \in \{1, 2, 3\}, i \neq j \implies \left[\varepsilon, M_\nu^{\text{diag}}\right]_{ij} = (U_+^T M_\alpha U_+)^{ij}.$$ (74)

One can solve analytically for $\varepsilon_1$, $\varepsilon_2$, $\varepsilon_3$ to get:

$$\varepsilon_1 = \frac{s_{2\varphi} \alpha_{11} + \sqrt{2} c_{2\varphi} \alpha_{12} - \sqrt{2} c_{2\varphi} \alpha_{12}}{2 (m_2^0 - m_4^0)}, \quad \varepsilon_2 = \frac{c_{\varphi} (\alpha_{12} + \alpha_{13})}{\sqrt{2} (m_3^0 - m_4^0)}, \quad \varepsilon_3 = \frac{s_{\varphi} (\alpha_{12} + \alpha_{13})}{\sqrt{2} (m_3^0 - m_4^0)}.$$

(75)

The new eigenmasses are given as

$$m_1 = m_1^0 + c_{\varphi}^2 \alpha_{11} - \frac{s_{2\varphi} (\alpha_{12} - \alpha_{13})}{\sqrt{2}},$$

$$m_2 = m_2^0 + s_{\varphi}^2 \alpha_{11} + \frac{s_{2\varphi} (\alpha_{12} - \alpha_{13})}{\sqrt{2}},$$

$$m_3 = m_3^0.$$

(76)

Computing now $Q = U_+ (1 + \varepsilon)$ and comparing to Eq.(4) we find the new mixing angles

$$t_{12} \simeq t_{\varphi} \left(1 + \frac{2 \varepsilon_1}{s_{2\varphi}}\right),$$

$$t_{13} \simeq \varepsilon_2 c_{\varphi} + \varepsilon_3 s_{\varphi},$$

$$t_{23} \simeq 1 - 2 \varepsilon_2 s_{\varphi} + 2 \varepsilon_3 c_{\varphi}.$$

(77)

These formulae show that the deviations from the mixing values predicted by the exact $\mu$-$\tau$ symmetry depend on the perturbation parameter $\alpha$. We assumed $\alpha$ real for these formulae, but one can extend the study to the complex case in order to investigate any effect on the phases. From this simple analysis, the parameter $\alpha$ (or equivalently $\chi$) – when it is real – is responsible for producing the correct mixing, while the phases of the RH Majorana neutrino fields are responsible for producing the lepton asymmetry. Introducing complex-valued $\alpha(\chi)$ can have an effect on the lepton asymmetry.

### 3.5 Realization of perturbed texture

As we saw, perturbed textures are needed in order to account for phenomenology. We have two ways to seek models for achieving these perturbations. The first method consists of introducing a term in the Lagrangian which breaks explicitly the symmetry [19], and then of expressing the new perturbed texture in terms of this breaking term. The second method is to keep assuming the exact symmetry, but then we break it spontaneously by introducing new matter and enlarging the symmetry. We follow here the second approach in order to find a realization of the forms given in Eq.(63) for $M_D$ and in Eq.(48) for $M_R$, while assuring that we work in the flavor basis. However, for the sake of minimum added matter, we shall not force the most general forms of $M_R$ and $M_D$, but rather be content with special forms of them leading to an effective mass matrix $M_\nu$ of the desired perturbed texture (Eq. 67). In [16] a realization was given for a perturbed texture corresponding to the $S_-$-symmetry, whereas here we treat the more phenomenologically motivated $S_+$-symmetry (we shall drop henceforth the +suffix). We present two ways, not meant by whatsoever to be restrictive but rather should be looked at as proof of existence.
tools, to get the three required conditions of a “perturbed” \( M_D \), non-perturbed \( M_R \) and diagonal \( M_l M_l^\dagger \). Both ways add new matter, but whereas the first approach adds just a \((Z_2)^2\) factor to the \( S \)-symmetry while requiring some Yukawa couplings to vanish, the second approach enlarges the symmetry larger to \( S \times Z_8 \) but without need to equate Yukawa couplings to zero by hand. Some “form invariance” relations are in order:

\[
\left\{ \left( M = M^T \right) \wedge \left[ S^T \cdot M \cdot S = M \right] \right\} \Leftrightarrow \exists A, B, C, D : M = \begin{pmatrix} A & B & -B \\ B & C & D \\ -B & D & C \end{pmatrix}, \tag{78}
\]

\[
\left\{ \left( M = M^T \right) \wedge \left[ S^T \cdot M \cdot S = -M \right] \right\} \Leftrightarrow \exists B, C : M = \begin{pmatrix} 0 & B & B \\ B & C & 0 \\ B & 0 & -C \end{pmatrix}, \tag{79}
\]

\[
\left[ S^T \cdot M \cdot S = M \right] \Leftrightarrow \exists A, B, C, D : M = \begin{pmatrix} A & B & -B \\ E & C & D \\ -E & D & C \end{pmatrix}, \tag{80}
\]

\[
\left[ S^T \cdot M \cdot S = -M \right] \Leftrightarrow \exists B, C, D : M = \begin{pmatrix} 0 & B & B \\ E & C & D \\ E & -D & -C \end{pmatrix}, \tag{81}
\]

We denote \( L^T = (L_1, L_2, L_3) \) with \( L_i \)'s \((i = 1, 2, 3)\) are the components of the \( i \)-th-family LH lepton doublets (we shall adopt this notation of ‘vectors’ in flavor space even for other fields, like \( l^c \) the RH charged lepton singlets, \( \nu_R \) the RH neutrinos, . . .).

### 3.5.1 \( S \times Z_2 \times Z_2^l \)-flavor symmetry

- **Matter content and symmetry transformations**

We have three SM-like Higgs doublets \((\phi_i, i = 1, 2, 3)\) which would give mass to the charged leptons and another three Higgs doublets \((\phi'_i, i = 1, 2, 3)\) for the Dirac neutrino mass matrix. All the fields are invariant under \( Z_2^l \) except the fields \( \phi' \) and \( \nu_R \) which are multiplied by \(-1\), so that we assure that neither \( \phi \) can contribute to \( M_D \), nor \( \phi' \) to \( M_l \). The fields transformation are as follows.

\[
\nu_R \xrightarrow{Z_2^l} \text{diag}(1, -1, -1) \nu_R, \quad \phi' \xrightarrow{Z_2^l} \text{diag}(1, -1, -1) \phi', \tag{82}
\]

\[
L \xrightarrow{Z_2^l} \text{diag}(1, -1, -1) L, \quad l^c \xrightarrow{Z_2^l} \text{diag}(1, 1, -1) l^c, \quad \phi \xrightarrow{Z_2^l} \text{diag}(1, -1, -1) \phi, \tag{83}
\]

\[
\nu_R \xrightarrow{S} S \nu_R, \quad \phi' \xrightarrow{S} \text{diag}(1, 1, -1) \phi', \tag{84}
\]

\[
L \xrightarrow{S} S L, \quad l^c \xrightarrow{S} l^c, \quad \phi \xrightarrow{S} S \phi, \tag{85}
\]

- **Charged lepton mass matrix-flavor basis**

The Lagrangian responsible for \( M_l \) is given by:

\[
\mathcal{L}_2 = f_{ik}^l \bar{\nu}_i l^c_j \tag{86}
\]

The transformations under \( S \) and \( Z_2^l \), with the “form invariance” relations Eqs. (78–81), lead to:

\[
f^{(1)} = \begin{pmatrix} A^1 & 0 & 0 \\ 0 & C^1 & D^1 \\ 0 & D^1 & C^1 \end{pmatrix}, \quad f^{(2)} = \begin{pmatrix} A^2 & 0 & 0 \\ 0 & C^2 & D^2 \\ 0 & D^2 & C^2 \end{pmatrix}, \quad f^{(3)} = \begin{pmatrix} 0 & B^3 & -B^3 \\ E^3 & 0 & 0 \\ -E^3 & 0 & 0 \end{pmatrix} \tag{87}
\]

where \( f_{ik}^l \) is the \((i, k)^{th}\)-entry of the matrix \( f^{(j)} \). Assuming \((v_3 \gg v_1, v_2)\) we get:

\[
M_l = v_3 \begin{pmatrix} 0 & 0 & -B^3 \\ D^1 & D^2 & 0 \\ C^1 & C^2 & 0 \end{pmatrix} \Rightarrow M_l M_l^\dagger = v_3^2 \begin{pmatrix} |B|^2 & 0 & 0 \\ |D|^2 & D \cdot C \\ 0 & 0 & |C|^2 \end{pmatrix}, \tag{88}
\]
where \( B = (0, 0, -B^3)^T \), \( D = (D^1, D^2, 0)^T \) and \( C = (C^1, C^2, 0)^T \), and where the dot product is defined as \( D \cdot C = \sum_{i=1}^{3} D^i C^i \). Under the reasonable assumption that the magnitudes of the Yukawa couplings come in ratios proportional to the lepton mass ratios as \(|B| : |C| : |D| \sim m_e : m_\mu : m_\tau\), one can show, as was done in [16], that the diagonalization of the charged lepton mass matrix can be achieved by infinitesimally rotating the LH charged lepton fields, which justifies working in the flavor basis to a good approximation.

- **Majorana neutrino mass matrix**
  The mass term is directly present in the Lagrangian

\[
\mathcal{L}_R = M_{Rij} \nu_{Ri} \nu_{Rj} .
\]

The invariance under \( Z_2 \) is trivially satisfied while the one under \( S \times Z_2 \) is more involved. The symmetry \( S \) constrains \( M_R \) to satisfy

\[
S^T M_R S = M_R,
\]

whereas the restrictions due to \( Z_2 \) are imprinted in the bilinear of \( \nu_{Ri} \nu_{Rj} \) determining their transformations under \( Z_2 \) as:

\[
\nu_{Ri} \nu_{Rj} \overset{Z_2}{\sim} B = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}
\]

which means:

\[
\nu_{Ri} \nu_{Rj} \overset{Z_2}{\sim} Z_2(\nu_{Ri} \nu_{Rj}) = B_{ij} \nu_{Ri} \nu_{Rj} (\text{no sum})
\]

Thus the symmetry through Eqs.(78,90,91) entails that \( M_R \) would assume the following form,

\[
M_R = \begin{pmatrix} A_R & 0 & 0 \\ 0 & C_R & D_R \\ 0 & D_R & C_R \end{pmatrix},
\]

which is of the general form (Eq. 48) with \( B_R = 0 \).

- **Dirac neutrino mass matrix**
  The Lagrangian responsible for the neutrino mass matrix is

\[
\mathcal{L}_D = \chi^k_{Ri} \bar{\nu}_i \bar{\nu}_{Rj} , \text{ where } \bar{\phi}' = i \sigma_2 \phi'^* 
\]

Because of the fields transformations under \( S \) and \( Z_2 \) we get:

\[
S^T g^{(k=1,2)} S = g^{(k=1,2)}, \quad S^T g^{(k=3)} S = -g^{(k=3)}, \quad \bar{\nu}_{Rj} \overset{Z_2}{\sim} \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}
\]

where \( g^{(k)} \) is the matrix whose \((i, j)\)th-entry is the Yukawa coupling \( g^k_{ij} \). Then, the “form invariance” relations (Eqs.78–81) lead to:

\[
g^{(1)} = \begin{pmatrix} A^1 & 0 & 0 \\ 0 & C^1 & D^1 \\ 0 & D^1 & C^1 \end{pmatrix}, \quad g^{(2)} = \begin{pmatrix} 0 & B^2 & -B^2 \\ E^2 & 0 & 0 \\ -E^2 & 0 & 0 \end{pmatrix}, \quad g^{(3)} = \begin{pmatrix} 0 & B^3 & B^3 \\ E^3 & 0 & 0 \\ E^3 & 0 & 0 \end{pmatrix}
\]

Upon acquiring vevs \((v'_i, i = 1, 2, 3)\) for the Higgs fields \((\phi'_i)\), we get for Dirac neutrino mass matrix the form:

\[
M_D = \begin{pmatrix} v'_1 A^1 & v'_2 B^2 + v'_3 B^3 & -v'_2 B^2 + v'_3 B^3 \\ v'_2 E^2 + v'_3 E^3 & v'_1 C^1 & v'_1 D^1 \\ -v'_2 E^2 + v'_3 E^3 & v'_1 D^1 & v'_1 C^1 \end{pmatrix}
\]
which can be put into the form,

$$ M_D = \begin{pmatrix} A_D & B_D (1 + \alpha) & -B_D \\ E_D (1 + \beta) & C_D & D_D \\ -E_D & D_D & C_D \end{pmatrix}. $$  

(98)

with

$$ \alpha = \frac{2v_3^2 B}{v_2 B - v_3^2 B^3}, \quad \beta = \frac{2v_3^2 E}{v_2 E^2 - v_3^2 E^3}. $$

(99)

If the vevs satisfy $v_3^2 \ll v_2^2$ and the Yukawa couplings are of the same order, then we get perturbative parameters $\alpha, \beta \ll 1$.

The deformations appearing in the Dirac mass matrix as described in Eqs.(97–99) would resurface in the effective light neutrino mass matrix $M_\nu$ through the seesaw formula (Eq.43) with $M_R$ given in Eq.(93). The resulting deformations in $M_\nu$ can be described by two parameters:

$$ \chi \equiv -\frac{M_\nu (1, 2) + M_\nu (1, 3)}{M_\nu (1, 3)}, \quad \xi \equiv \frac{M_\nu (2, 2) - M_\nu (3, 3)}{M_\nu (3, 3)}. $$

(100)

One can repeat now the analysis of the last subsection in order to compute $\chi, \xi$ in terms of $\alpha, \beta$ and other mass parameters to get:

$$ \chi = -\frac{\alpha A_R B_D (C_R - D_R) (C_D + D_D) + \beta A_D E_D (C_R^2 - D_R^2)}{\alpha A_R B_D (C_R D_R - D_R C_D) + B_D A_R (D_R + C_R) (D_D - C_D) - E_D A_D (C_R^2 - D_R^2)} $$

(101)

$$ \xi = \frac{\beta (\beta - 2) E_D^2 (C_R^2 - D_R^2)}{A_R [C_R (D_R^2 + C_D^2) - 2 C_D D_R D_R] + E_D^2 (C_R^2 - D_R^2)}. $$

We note here that we do not get in general the desired pattern (Eq. 67) corresponding to disentanglement of the perturbations ($\xi = 0$). However, for specific choices of Yukawa couplings, for e.g. $E^3 = 0$ leading to $\beta = 0$ and hence $\xi = 0$, we get this form, in which case $M_D$ is of the form (Eq.63) and $\chi$ of Eq.(101) would also be given by Eq.(68) with $B_R = 0$.

3.5.2 $S \times Z_8$-flavor symmetry

- Matter content and symmetry transformations

In addition to the left doublets ($L_i, i = 1, 2, 3$), the RH charged singlets ($l^c_j, j = 1, 2, 3$), the RH neutrinos ($\nu_{Rj}, j = 1, 2, 3$) and the SM-Higgs three doublets ($\phi_i, i = 1, 2, 3$) responsible for the charged lepton masses, we have now four Higgs doublets ($\phi_j', j = 1, 2, 3, 4$) giving rise when acquiring a vev to Dirac neutrino mass matrix, and also two Higgs singlet scalars ($\Delta_k, k = 1, 2$) related to Majorana neutrino mass matrix. We denote the octic root of the unity by $\omega = e^{\frac{i\pi}{4}}$. The fields transform as follows.

$$ L \xrightarrow{S} SL, \quad l^c \xrightarrow{S} l^c, \quad \phi \xrightarrow{S} S\phi, $$

(102)

$$ \nu_R \xrightarrow{S} S\nu_R, \quad \phi' \xrightarrow{S} \text{diag} (1, 1, 1, -1) \phi', \quad \Delta \xrightarrow{S} \Delta $$

(103)

$$ L \xrightarrow{Z_8} \text{diag} (1, -1, -1, 1), \quad l^c \xrightarrow{Z_8} \text{diag} (1, 1, -1, -1), \quad \phi \xrightarrow{Z_8} \text{diag} (1, -1, -1), $$

(104)

$$ \nu_R \xrightarrow{Z_8} \text{diag} (\omega, \omega^3, \omega^3), \quad \phi' \xrightarrow{Z_8} \text{diag} (\omega, \omega^3, \omega^3, \omega^3), \quad \Delta \xrightarrow{Z_8} \text{diag} (\omega^6, \omega^2) \Delta $$

(105)

Note here that we have the following transformation rule for $\phi' \equiv i \sigma_2 \phi^*:

$$ \phi' \xrightarrow{S} \text{diag} (1, 1, 1, -1), \quad \phi' \xrightarrow{Z_8} \text{diag} (\omega^7, \omega^5, \omega, \omega^5) \phi' $$

(106)

- Charged lepton mass matrix-flavor basis

The symmetry restriction in constructing the charged lepton mass Lagrangian as given by Eq.(86) is similar to what is obtained in the case of $(S \times Z_2 \times Z_2^\prime)$. The similarity originates from the fact
that the charges assigned to the fields \((L, l^c, \phi)\) corresponding to the factor \(Z_2\) (of \(S \times Z_2 \times Z'_2\)) and that of \(Z_8\) (of \(S \times Z_8\)) are the same. Thus we end up, assuming again a hierarchy in the Higgs \(\phi's\) fields vevs \((v_3 \gg v_2, v_1)\), with a charged lepton mass matrix adjustable to be approximately in the flavor basis. Note also here that the symmetry forbids the term \(\bar{L_i} \phi'_k l^c_j\) since we have:

\[
\begin{pmatrix}
1 & 1 & -1 \\
-1 & -1 & 1 \\
-1 & 1 & 1
\end{pmatrix} \Rightarrow \varnothing i, j, k : \bar{L_i} \phi'_k l^c_j = Z_8(\bar{L_i} \phi'_k l^c_j)
\]

(107)

- **Dirac neutrino mass matrix**

The Lagrangian responsible for the Dirac neutrino mass matrix is given by Eq. (94). By means of fields transformations we have:

\[
S^T g^{(k=1,2,3)} S = g^{(k=1,2,3)}, \quad S^T g^{(k=4)} S = -g^{(k=4)}, \bar{L_i} \nu_R j \sim
\begin{pmatrix}
\omega & \omega^3 & \omega^3 \\
\omega^5 & \omega^7 & \omega^7 \\
\omega^5 & \omega^7 & \omega^7
\end{pmatrix}
\]

(108)

where \(g^{(k)}\) is the matrix whose \((i, j)\)-th entry is the Yukawa coupling \(g_{ij}^k\). Then, the “form invariance” relations impose the following forms:

\[
\begin{align*}
g^{(1)} &= \begin{pmatrix} A^1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
g^{(2)} &= \begin{pmatrix} 0 & B^2 & -B^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
g^{(3)} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & C^3 & D^3 \\ 0 & D^3 & C^3 \end{pmatrix}, \\
g^{(4)} &= \begin{pmatrix} 0 & B^4 & B^4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\end{align*}
\]

(109)

When the Higgs fields \(\phi'_l\) get vevs \((v'_i, i = 1, 2, 3, 4)\), we obtain:

\[
M_D = \sum_{k=1}^{4} v'_k g^{(k)} = \begin{pmatrix} v'_1 A^1 & v'_2 B^2 + v'_4 B^4 & -v'_2 B^2 + v'_4 B^4 \\ 0 & v'_3 C^3 & v'_3 D^3 \\ 0 & v'_3 D^3 & v'_3 C^3 \end{pmatrix}
\]

(110)

which is of the form of Eq.(63) with \(E_D = 0\):

\[
M_D = \begin{pmatrix} A_D & B_D (1 + \alpha) & -B_D \\ 0 & C_D & D_D \\ 0 & D_D & C_D \end{pmatrix},
\]

(111)

where

\[
\alpha = \frac{2v'_4 B^4}{v'_2 B^2 - v'_4 B^4}
\]

(112)

If the vevs satisfy \(v'_4 \ll v'_2\) and the Yukawa couplings are of the same order then we get a perturbative parameter \(\alpha \ll 1\).

- **Majorana neutrino mass matrix**

The mass term is generated from the Lagrangian

\[
\mathcal{L}_R = h_{ij}^k \Delta_k \nu_R i \nu_R j
\]

(113)

Under \(Z_8\) we have the bilinear:

\[
\begin{pmatrix}
\omega^2 & \omega^4 & \omega^4 \\
\omega^4 & \omega^6 & \omega^6 \\
\omega^4 & \omega^6 & \omega^6
\end{pmatrix} \Rightarrow \varnothing i, j, k : \nu_R i \nu_R j
\]

\[
\bar{\nu}_R i \nu_R j \sim
\begin{pmatrix}
\omega^2 & \omega^4 & \omega^4 \\
\omega^4 & \omega^6 & \omega^6 \\
\omega^4 & \omega^6 & \omega^6
\end{pmatrix}
\]

\[
\mathcal{L}_R = h_{11}^1 \Delta_1 \nu_{R1} \nu_{R1} + h_{22}^2 \Delta_2 \nu_{R2} \nu_{R2} + h_{22}^3 \Delta_2 \nu_{R2} \nu_{R3} + h_{23}^2 \Delta_2 \nu_{R3} \nu_{R2} + h_{33}^2 \Delta_2 \nu_{R3} \nu_{R3}
\]

(114)
If we call \( h^{(k)} \) the matrix whose \((i,j)\)-th entry is the coupling \( h_{ij}^k \), then we have (the cross sign denote a non-vanishing entry):

\[
h^{(1)} = \begin{pmatrix}
  \times & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 0
\end{pmatrix},
\]

\[
h^{(2)} = \begin{pmatrix}
  0 & 0 & 0 \\
  0 & \times & \times \\
  0 & \times & \times
\end{pmatrix},
\]

(115)

Then the “form invariance” relations lead to:

\[
S^T h^{(k)} S = h^{(k)}, \quad \text{Eqs. 78, 115}
\]

\[
h^{(1)} = \begin{pmatrix}
  a_R & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 0
\end{pmatrix},
\]

\[
h^{(2)} = \begin{pmatrix}
  0 & 0 & 0 \\
  0 & c_R & d_R \\
  0 & d_R & c_R
\end{pmatrix},
\]

(116)

Thus when the Higgs singlets \( \Delta \) acquire vevs \( (\Delta_1^0, \Delta_2^0) \) we get the following form for \( M_R \),

\[
M_R = \begin{pmatrix}
  \Delta_1^0 a_R & 0 & 0 \\
  0 & \Delta_2^0 c_R & \Delta_2^0 d_R \\
  0 & \Delta_2^0 d_R & \Delta_2^0 c_R
\end{pmatrix},
\]

(117)

which of the form of Eq.(48) with \( B_R = 0 \). The analysis of the last subsection shows then that the deformation \( \alpha \) in \( M_D \) resurfaces as a ‘sole’ perturbation \( \chi \) in \( M_L \) which would get the desired form of Eq.(67) with \( \chi \) given by Eq.(68) after putting \( B_R = E_D = 0 \):

\[
\chi = \frac{\alpha \left( d_R - c_R \right) (C_D + D_D)}{(D_D - C_D) (c_R + d_R) + \alpha (c_R D_D - d_R C_D)}.
\]

(118)

Before ending this section, we would like to mention that having multiple Higgs doublets in our constructions might display flavor-changing neutral currents. However, the effects are calculable and in principle one can adjust the Yukawa couplings so that to suppress processes like \( \mu \rightarrow e\gamma \) [21]. Moreover, the constructions are carried out at the seesaw high scale, but the RG running effects are expected to be small when multiple Higgs doublets are present, and so we expect the predictions of the symmetry will still be valid at low scale.

4 Discussion and summary

We studied the properties of the \( Z_2 \) symmetry behind the \( \mu - \tau \) neutrino universality. We singled out the texture \( (S_+) \) which imposes naturally a maximal atmospheric mixing \( \theta_{23} = \pi/4 \) and vanishing \( \theta_{13}. \) The remaining mixing angle \( \theta_{12} \) remains free, and the other \( Z_2 \) necessary to characterize the neutrino mass matrix can be used to fix it at its experimentally measured value (~ 33°). We showed how the \( S_+ \)-texture accommodates all the neutrino mass hierarchies. Later, we implemented the \( S_+ \)-symmetry in the whole lepton sector, and showed how it can accommodate the charged lepton mass hierarchies with small mixing angles of order of the ‘acute’ charged lepton mass hierarchies. We computed, within type-I seesaw, the lepton asymmetry generated by the symmetry and found that the phases of the RH Majorana fields may be adjusted to produce enough lepton asymmetry. The fact that the \( \mu-\tau \) symmetry does not determine fully the mixing angles, but leaves \( \theta_{12} \) as a free parameter able to take different values in \( M_R \) and \( M_D \) is crucial for obtaining leptogenesis within type-I seesaw scenarios. We found also that “real-valued” perturbations on Dirac neutrino mass matrix can account for the correct neutrino mixing angles. However, introducing “complex-valued” perturbations on \( M_D \) can have an effect on lepton asymmetry as well. Finally, we presented a theoretical realization of the perturbed Dirac mass matrix, where the symmetry is broken spontaneously and the perturbation parameter originates from ratios of different Higgs fields vevs.
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