TWO SPECIES NONLOCAL DIFFUSION SYSTEMS WITH FREE BOUNDARIES

YIHONG DU
School of Science and Technology, University of New England
Armidale, NSW 2351, Australia

MINGXIN WANG*
School of Mathematics, Harbin Institute of Technology
Harbin 150001, China

MENG ZHAO
College of Mathematics and Statistics, Northwest Normal University
Lanzhou, Gansu 730070, China
School of Mathematics and Statistics, Lanzhou University
Lanzhou, Gansu 730000, China

(Communicated by Congming Li)

Abstract. We study a class of free boundary systems with nonlocal diffusion, which are natural extensions of the corresponding free boundary problems of reaction diffusion systems. As before the free boundary represents the spreading front of the species, but here the population dispersal is described by “nonlocal diffusion” instead of “local diffusion”. We prove that such a nonlocal diffusion problem with free boundary has a unique global solution, and for models with Lotka-Volterra type competition or predator-prey growth terms, we show that a spreading-vanishing dichotomy holds, and obtain criteria for spreading and vanishing; moreover, for the weak competition case and for the weak predation case, we can determine the long-time asymptotic limit of the solution when spreading happens. Compared with the single species free boundary model with nonlocal diffusion considered recently in [7], and the two species cases with local diffusion extensively studied in the literature, the situation considered in this paper involves several new difficulties, which are overcome by the use of some new techniques.

1. Introduction. Nonlocal diffusion has been widely used to describe diffusion processes where long range dispersal may play a significant role, a situation arising frequently in propagation questions in biology and ecology (see, e.g., [23]). Several well-known population models, where population dispersal was traditionally approximated by local diffusion, have been examined recently with the local diffusion operator in the model replaced by a nonlocal diffusion operator; see, for example,
A commonly used nonlocal diffusion operator has the form

\[ d(J * u - u)(t, x) := d \left( \int_{\mathbb{R}^N} J(x-y)u(t,y)dy - u(t,x) \right), \]

where the kernel function \( J : \mathbb{R} \to \mathbb{R} \) is continuous, nonnegative, even, and \( \int_{\mathbb{R}} J(x)dx = 1 \). The quantity \( J(x-y) \) is proportional to the probability that an individual member of the species (whose population density is \( u(t,x) \)) in location \( x \) moves to location \( y \) or vice versa.

In [7], such a nonlocal diffusion operator was applied to the free boundary model of [9], to investigate the spreading behaviour of a new or invasive species. The nonlocal diffusion model with free boundary in [7] has the form

\[
\begin{cases}
  u_t = d \int_{g(t)}^{h(t)} J(x-y)u(t,y)dy - du(t,x) + f(t,x,u), & t > 0, g(t) < x < h(t), \\
  u(t,g(t)) = u(t,h(t)) = 0, & t > 0, \\
  h'(t) = \mu \int_{g(t)}^{h(t)} \int_{g(t)}^{h(t)} J(x-y)u(t,x)dydx, & t > 0, \\
  g'(t) = -\mu \int_{g(t)}^{h(t)} \int_{-\infty}^{g(t)} J(x-y)u(t,x)dydx, & t > 0, \\
  u(0, x) = u_0(x), & |x| \leq h_0, \\
  h(0) = -g(0) = h_0,
\end{cases}
\]

where \( x = g(t) \) and \( x = h(t) \) are the moving boundaries to be determined together with \( u(t,x) \), which is always assumed to be identically 0 for \( x \in \mathbb{R} \setminus [g(t), h(t)] \); \( d, \mu \) and \( h_0 \) are given positive constants. The kernel function \( J : \mathbb{R} \to \mathbb{R} \) satisfies

\[ J \text{ is continuous, nonnegative and even, } J(0) > 0, \int_{\mathbb{R}} J(x)dx = 1, \sup_{\mathbb{R}} J < \infty. \]

The growth function \( f(t,x,u) \) is continuous, locally Lipschitz in \( u \), and \( f(t,x,0) \equiv 0 \).

In [7], the existence and uniqueness of a global solution were proved, and for the special case that \( f = f(u) \) is a logistic function, a spreading-vanishing dichotomy, criteria for spreading and vanishing, and long time behaviour of the solution were established. A series of new ideas and techniques appeared in [7].

In this paper we further develop the ideas and techniques in [7] to study systems of population models with nonlocal diffusion and free boundaries. It turns out that extra difficulties arise, and further new techniques are required. In order to keep the presentation transparent and ideas clear, we will restrict to systems with only two species.

We consider the case that the two species under consideration spread through a common spreading front, as in [14, 24, 27, 35]. Such a setting arises rather naturally in several situations; for example, when the two species are of predator-prey type, with the predator following (or driving) the spreading of the prey, or for two competing plant species whose spreading relies on the same group of animals (insects, birds etc.) carrying their seeds to new fields. Based on the free boundary conditions in (1.1) above, this free boundary problem with nonlocal diffusion can
be expressed in the form

\[
\begin{align*}
  u_{tt} &= d_i \int_{g(t)}^{h(t)} J_i(x-y)u_i(t,y)dy - d_i u_i + f_i(t,x,u_1,u_2), \quad t > 0, \quad g(t) < x < h(t), \\
  u_i(t,g(t)) &= u_i(t,h(t)) = 0, \quad t \geq 0, \\
  h'(t) &= \sum_{i=1}^{2} \mu_i \int_{g(t)}^{h(t)} J_i(x-y)u_i(t,x)dy, \quad t \geq 0, \\
  g'(t) &= -\sum_{i=1}^{2} \mu_i \int_{g(t)}^{h(t)} J_i(x-y)u_i(t,x)dy, \quad t \geq 0, \\
  u_i(0,x) &= u_{i0}(x), \quad h(0) = -g(0) = h_0, \quad |x| \leq h_0, \\
  i &= 1, 2,
\end{align*}
\]

where \( x = g(t) \) and \( x = h(t) \) are the moving boundaries to be determined together with \( u_1(t,x) \) and \( u_2(t,x) \), which are always assumed to be identically 0 for \( x \in \mathbb{R} \setminus [g(t), h(t)] \); \( d_i \) and \( \mu_i \) \((i = 1, 2)\) are positive constants. We assume that the initial function pair \((u_{i0}, u_{20})\) satisfies

\[
u_{i0} \in C([-h_0, h_0]), \quad u_{i0}(\pm h_0) = 0, \quad u_{i0} > 0 \quad \text{in} \quad (-h_0, h_0), \quad i = 1, 2, \quad (1.3)\]

with \([-h_0, h_0]\) representing the initial population range of the species. The kernel functions \( J_1 \) and \( J_2 \) satisfy the condition \((J)\).

The free boundary conditions in (1.2) mean that the expansion rate of the common population range of the two species is proportional to the outward flux of the population of the two species; some justifications of this assumption can be found in [7].

The growth terms \( f_i \) \((i = 1, 2)\) are assumed to be continuous and satisfy

\((f)\) \( f_1(t,x,0,u_2) = f_2(t,x,u_1,0) = 0, \) and \( f_i(t,x,u_1,u_2) \) is locally Lipschitz in \( u_1, u_2 \in \mathbb{R}^+ \), i.e., for any \( K_1, K_2 > 0 \), there exists a constant \( L(K_1, K_2) > 0 \) such that

\[
|f_i(t,x,u_1,u_2) - f_i(t,x,v_1,v_2)| \leq L(K_1, K_2)(|u_1 - v_1| + |u_2 - v_2|)
\]

for all \( u_1, v_1 \in [0, K_1] \), \( u_2, v_2 \in [0, K_2] \) and all \( (t,x) \in \mathbb{R}^+ \times \mathbb{R} \). When \( K_1 = K_2 \), we write \( L(K_1, K_2) = L(K_1) \);

\((f1)\) There exist \( k > 0 \) and \( r > 0 \) such that for all \( u_2 \geq 0 \) and \( (t,x) \in \mathbb{R}^+ \times \mathbb{R} \), there hold:

\[
f_1(t,x,u_1,u_2) < 0 \quad \text{when} \quad u_1 > k, \quad f_1(t,x,u_1,u_2) \leq ru_1 \quad \text{when} \quad 0 < u_1 \leq k;
\]

\((f2)\) For any given \( K > 0 \), there exists \( \Theta(K) > 0 \) such that \( f_2(t,x,u_1,u_2) < 0 \) for \( 0 \leq u_1 \leq K, \) \( u_2 \geq \Theta(K) \) and \( (t,x) \in \mathbb{R}^+ \times \mathbb{R} \).

We note that condition \((f)\) implies

\[
|f_1(t,x,u_1,u_2)| \leq L(K_1, K_2)|u_1|, \quad |f_2(t,x,u_1,u_2)| \leq L(K_1, K_2)|u_2|
\]

for all \( u_1 \in [0, K_1], \) \( u_2 \in [0, K_2] \) and all \( (t,x) \in \mathbb{R}^+ \times \mathbb{R} \).

It is easily seen that the conditions \((f), (f1)\) and \((f2)\) hold for the following classical Lotka-Volterra competition and predator-prey growth terms:

**Competition Model:** \( f_1 = u_1(a_1 - b_1 u_1 - c_1 u_2), f_2 = u_2(a_2 - b_2 u_2 - c_2 u_1), \) \((1.4)\)

**Predator-prey Model:** \( f_1 = u_1(a_1 - b_1 u_1 - c_1 u_2), f_2 = u_2(a_2 - b_2 u_2 + c_2 u_1), \) \((1.5)\)

where \( a_i, b_i, c_i \) \((i = 1, 2)\) are positive constants.

Unless otherwise stated, we always assume that \( f_1 \) and \( f_2 \) satisfy \((f), (f1)\) and \((f2)\), \( J_1, J_2 \) satisfy \((J)\), and \((1.3)\) is satisfied by the initial function pair. We will
write
\[ \|a, b\| \leq M \] to mean \( \|a\| \leq M, \|b\| \leq M. \)

The main results of this paper are the following theorems.

**Theorem 1.1.** Problem (1.2) has a unique solution \((u_1, u_2, g, h)\) defined for all \(t > 0\).

**Theorem 1.2** (Spreading-vanishing dichotomy). Assume further that \(J_i(x) > 0\), \(J_2(x) > 0\) in \(\mathbb{R}\), and that \((f_1, f_2)\) satisfies either (1.4) or (1.5). Let \((u_1, u_2, g, h)\) be the unique solution of (1.2). Then one of the following alternatives must happen:

(i) Spreading: \(\lim_{t \to \infty} [h(t) - g(t)] = \infty\),

(ii) Vanishing: \(\lim_{t \to \infty} (g(t), h(t)) = (\ell_\infty, \ell_\infty)\) is a finite interval and

\[ \lim_{t \to \infty} \max_{g(t) \leq x \leq h(t)} u_i(t, x) = 0, \quad i = 1, 2. \]

**Theorem 1.3** (Spreading-vanishing criteria). Under the conditions of Theorem 1.2, the following conclusions hold:

(i) If either \(a_1 \geq d_1\) or \(a_2 \geq d_2\), then spreading always happens.

(ii) If \(a_1 < d_1\) and \(a_2 < d_2\), then there exists a unique \(\ell_\ast > 0\) such that

(a) whenever vanishing happens, we have \(\ell_\infty - \ell_\ast \leq \ell_\ast\),

(b) spreading always happens when \(h_0 \geq \ell_\ast / 2\),

(c) if \(h_0 < \ell_\ast / 2\), then there exist two positive numbers \(\Lambda_\ast \geq \Lambda_\ast > 0\) such that vanishing happens when \(\mu_1 + \mu_2 \leq \Lambda_\ast\) and spreading happens when \(\mu_1 + \mu_2 > \Lambda_\ast\).

As we will see in Section 3 below, \(\ell_\ast\) depends only on \(a_i, d_i, J_i\), \(i = 1, 2\). On the other hand, \(\Lambda_\ast\) and \(\Lambda_\ast\) depend also on \(b_1, c_1\) and \(u_{i0}, i = 1, 2\).

Theorem 1.2 gives very little information on the long-time behaviour of the solution \((u, v, g, h)\) for the spreading case; it is even unclear whether \(\lim_{t \to \infty} [h(t) - g(t)] = \infty\) implies \(\lim_{t \to \infty} h(t) = \lim_{t \to \infty} [-g(t)] = \infty\) (see Remark 1.6 below for more details). We are able to determine the long-time behaviour of the solution for the case \(\lim_{t \to \infty} [h(t) - g(t)] = \infty\), in the following two special situations:

(a) The weak competition case: \((f_1, f_2)\) satisfies (1.4) with \(b_1/c_2 > a_1/a_2 > c_1/b_2\).

(b) The weak predation case: \((f_1, f_2)\) satisfies (1.5) with \(a_1 b_1 b_2 > a_2 b_1 c_1 + a_1 c_1 c_2\).

**Theorem 1.4** (Asymptotic limit). Let \((u_1, u_2, g, h)\) be the unique solution of (1.2) and suppose \(\lim_{t \to \infty} [h(t) - g(t)] = \infty\). Then

(i) in the weak competition case we have \(\lim_{t \to \infty} h(t) = - \lim_{t \to \infty} g(t) = \infty\) and

\[ \lim_{t \to \infty} (u_1(t, x), u_2(t, x)) = \left( \frac{a_1 b_2 - a_2 c_1}{b_1 b_2 - c_1 c_2}, \frac{a_2 b_1 - a_1 c_2}{b_1 b_2 - c_1 c_2} \right) \] locally uniformly for \(x \in \mathbb{R}\),

(ii) in the weak predation case we have \(\lim_{t \to \infty} h(t) = - \lim_{t \to \infty} g(t) = \infty\) and

\[ \lim_{t \to \infty} (u_1(t, x), u_2(t, x)) = \left( \frac{a_1 b_2 - a_2 c_1}{b_1 b_2 + c_1 c_2}, \frac{a_2 b_1 + a_1 c_2}{b_1 b_2 + c_1 c_2} \right) \] locally uniformly for \(x \in \mathbb{R}\).

**Remark 1.5.** We believe that the condition \(J_i(x) > 0\) in \(\mathbb{R}\) for \(i = 1, 2\) in Theorem 1.2 is unnecessary, though our proof of \(\lim_{t \to \infty} \max_{g(t) \leq x \leq h(t)} u_i(t, x) = 0\) in part (ii) of Theorem 1.2 makes essential use of this extra condition.
Remark 1.6. When spreading happens, it is a challenging task to determine the long-time limit of the solution for systems with Lotka-Volterra growth terms in general. Many technical difficulties arising here do not occur in the corresponding free boundary problems with local diffusion. It appears that new techniques are needed to handle most of the cases not covered in Theorem 1.4.

Remark 1.7. When the nonlocal diffusion term in (1.2) is replaced by the usual local diffusion term $d_i \partial_{xx} u_i$, for competition and predator-prey type Lotka-Volterra growth functions $(f_1, f_2)$, the problem was investigated in [14, 27, 35, 40]. The results in Theorem 1.4 indicate that when local diffusion is replaced by nonlocal diffusion for these special Lotka-Volterra systems with free boundary, the basic features of the model is not altered significantly. However, Theorem 1.3 (i) suggests that the dispersal rates $d_1$ and $d_2$ play a more dominant role in determining whether the species can spread successfully than in the local diffusion case [14, 27, 35, 40], reinforcing the phenomenon revealed in the single species case in [7].

The rest of the paper is arranged as follows. In Section 2 we prove Theorem 1.1, namely problem (1.2) has a unique global solution, by further developing the approach of [7]. As the situation here is more complicated, considerable changes are needed. In Section 3, we prove Theorems 1.2, 1.3 and 1.4. Here we encounter a new difficulty in understanding the vanishing case, i.e., whether $h_\infty - g_\infty < \infty$ implies $\lim_{t \to \infty} \max_{t \leq x \leq h(t)} u_i(t, x) = 0$ for $i = 1, 2$. In the single equation case with local or nonlocal diffusion, one can make use of the associated steady-state to show that the corresponding question has a positive answer. For the local diffusion systems with free boundary, one can use a general conclusion given in [29, Lemma 8.7] (see also [27, 35, 36]) to overcome this difficulty and give a positive answer to the above question. However, because nonlocal diffusion does not yield compactness for the solution, we do not know whether a similar conclusion to [29, Lemma 8.7] is still valid for the current nonlocal diffusion case. Indeed, for our present situation, we shall introduce a new technique to overcome this difficulty, and give a positive answer to the above question under the conditions $J_1(x) > 0, J_2(x) > 0$ in $\mathbb{R}$ (see details in the proof of Theorem 3.3). We believe these technical conditions are removable though don’t know how to drop them yet. The proof of the other conclusions is largely based on adequate adaptations of techniques developed for the local diffusion case in [27, 35, 36, 37, 40].

After this paper has been completed and made available in arXiv, several related papers have appeared ([11, 19, 20, 25, 26, 39]), where different issues were addressed, but the questions raised above remain open. In the case of local diffusion, competition and prey-predator models with free boundaries appearing in only one species were considered in [10, 12, 38], and with independent free boundaries for the two species were studied in [13, 15, 33, 34]. For two species models with local diffusion over a fixed bounded interval, and with a free boundary imposed in the interior of the interval, one may find interesting results in [21, 22, 28, 31]. These works rely on techniques rather different from this paper.

2. Global existence and uniqueness. In this section we prove that, for any given initial value $U_0 := (u_{10}, u_{20})$ satisfying (1.3), problem (1.2) has a unique solution defined for all $t > 0$. For convenience, we first introduce some notations. For given $T > 0$, define

$$\mathbb{H}_T = \{ h \in C^1 ([0, T]) : h(0) = h_0, \ h(t) \text{ is strictly increasing} \},$$
For \( g \in G_T, h \in H_T \) and \( U_0 = (u_{10}, u_{20}) \) satisfying (1.3), we denote
\[
D_{g,h}^T = \{(t,x) \in \mathbb{R}^2 : 0 < t \leq T, \ g(t) < x < h(t)\},
\]
\[
H_{g,h}^T = \varphi \in C(\overline{D}_{g,h}^T)^2 : \varphi \geq 0, \ \varphi|_{t=0} = U_0(x), \ \varphi|_{x=g(t),h(t)} = 0 \right\}.
\]

Here by \( \varphi = (\varphi_1, \varphi_2) \geq 0 \) we mean \( \varphi_1 \geq 0 \) and \( \varphi_2 \geq 0 \) in \( D_{g,h}^T \).

The following theorem, which contains the conclusion in Theorem 1.1, is the main result of this section.

**Theorem 2.1.** For any given initial value \( U_0 := (u_{10}, u_{20}) \) satisfying (1.3), problem (1.2) has a unique global solution \((u_1(t,x), u_2(t,x), g(t), h(t))\). Moreover, for any \( T > 0 \), we have \((g,h) \in G_T \times H_T, (u_1,u_2) \in H_{g,h}^T \), and
\[
\begin{cases}
0 < u_1 \leq \max \{\|u_{10}\|_\infty, k\} := A_1 & \text{in } D_{g,h}^T, \\
0 < u_2 \leq \max \{\|u_{20}\|_\infty, \Theta(A_1)\} := A_2 & \text{in } D_{g,h}^T, \\
g'(t) < 0, \quad h'(t) > 0, \quad \forall \ t > 0.
\end{cases}
\]

The rest of this section is devoted to the proof of Theorem 2.1. The following Maximum Principle will be used frequently in our analysis to follow.

**Lemma 2.2** (Maximum Principle [7]). Assume that \( J \) satisfies (J), and \((g,h) \in G_T \times H_T\). Suppose that \( \psi, \psi_i \in C(\overline{D}_{g,h}^T) \) and satisfies, for some \( c \in L^\infty(D_{g,h}^T) \),
\[
\begin{cases}
\psi_i(t,x) \geq d \int_{g(t)}^{h(t)} J(x-y)\psi(t,y)dy - d\psi + c(t,x)\psi, \quad t \in (0,T], \ g(t) < x < h(t), \\
\psi(t,g(t)) \geq 0, \quad \psi(t,h(t)) \geq 0, \quad t > 0, \\
\psi(0,x) \geq 0, \quad |x| \leq h_0.
\end{cases}
\]

Then \( \psi \geq 0 \) on \( D_{g,h}^T \). Moreover, if \( \psi(0,x) \neq 0 \) in \([-h_0,h_0]\), then \( \psi > 0 \) in \( D_{g,h}^T \).

The following result will play a crucial role in the proof of Theorem 2.1.

**Lemma 2.3.** For any \( T > 0 \) and \((g,h) \in G_T \times H_T\), the problem
\[
\begin{cases}
w_{it} = d_i \int_{g(t)}^{h(t)} J_i(x-y)w_i(t,y)dy - d_i w_i(t,x) + f_i(t,x,w_1,w_2), & 0 < t \leq T, \ g(t) < x < h(t), \\
w_i(t,g(t)) = w_i(t,h(t)) = 0, & 0 \leq t \leq T, \\
w_i(0,x) = u_{i0}(x), & |x| \leq h_0,
\end{cases}
\]
has a unique solution \( w_{g,h} = (w_{1,g,h}, w_{2,g,h}) \in H_{g,h}^T \), and \( w_{g,h} \) satisfies (2.1).

**Proof.** The idea of the proof comes from [7]. We break the proof into three steps. As \((g,h)\) is fixed in this proof, we write \( D_s = D_{g,h}^s \) and \( X_s = X_{g,h}^s \) simply for \( s > 0 \).

**Step 1.** A parametrized ODE problem. Define
\[
f^*_i(t,x,u_1,u_2) = \begin{cases}
f_i(t,x,u_1,u_2) & \text{if } u_1, u_2 \geq 0, \\
0 & \text{if } u_1, u_2 \leq 0,
\end{cases}
\]

For any \( A \) parametrized ODE problem.

\[
\begin{cases}
w_{i,t} = f^*_i(t,x,u_1,u_2), & 0 < t \leq T, \ g(t) < x < h(t), \\
w_i(t,g(t)) = w_i(t,h(t)) = 0, & 0 \leq t \leq T, \\
w_i(0,x) = u_{i0}(x), & |x| \leq h_0,
\end{cases}
\]

has a unique solution \( w_{i,g,h} = (w_{1,g,h}, w_{2,g,h}) \in H_{g,h}^T \), and \( w_{i,g,h} \) satisfies (2.1).
where we can deduce some interval $[g(T), h(T)]$ for any given $x \in [g(T), h(T)]$, set

$$
\bar{u}_{10}(x) = \begin{cases} 0, & |x| > h_0, \\
u_{10}(x), & |x| \leq h_0,
\end{cases} \quad \bar{u}_{20}(x) = \begin{cases} 0, & |x| > h_0, \\
u_{20}(x), & |x| \leq h_0,
\end{cases}
$$

$$
t_x = \begin{cases} t_{x,g} & \text{if } x \in [g(T), -h_0), x = g(t_{x,g}), \\
0 & \text{if } |x| \leq h_0, \\
t_{x,h} & \text{if } x \in (h_0, h(t)), x = h(t_{x,h}).
\end{cases}
$$

Clearly $t_x = T$ for $x = g(T)$ or $x = h(T)$, and $t_x \in [0, T)$ for $x \in (g(T), h(T))$.

For any given $0 < s \leq T$ and $\varphi = (\varphi_1, \varphi_2) \in X^s$, we first consider the initial value problem of the following ordinary differential system with parameter $x \in (g(s), h(s))$:

$$
\begin{align*}
\dot{p}_i &= d_i \int_{g(t)}^{h(t)} J_i(x - y) \varphi_i(t, y) \, dy - d_i p_i + f_i^*(t, x, p), \quad t_x < t \leq s, \\
p_i(t_x, x) &= \bar{u}_{i0}(x), \\
&\quad g(s) < x < h(s),
\end{align*}
$$

(2.4)

Denote

$$
F_i(t, x, p) = d_i \int_{g(t)}^{h(t)} J_i(x - y) \varphi_i(t, y) \, dy - d_i p_i + f_i^*(t, x, p), \quad i = 1, 2,
$$

$$
K_\varphi = 1 + A_1 + A_2 + \|\varphi_1, \varphi_2\|_{C(\overline{D}_x)}, \quad L_\varphi = \max \{d_1, d_2\} + L(K_\varphi),
$$

where $A_1$ and $A_2$ are given by (2.1). Then for any $p_i, q_i \in (-\infty, K_\varphi)$, $i = 1, 2$, we have

$$
|F_i(t, x, p_1, p_2) - F_i(t, x, q_1, q_2)| \leq |f_i^*(t, x, p_1, p_2) - f_i^*(t, x, q_1, q_2)| + d_i|p_i - q_i| \leq L_\varphi (|p_1 - q_1| + |p_2 - q_2|), \quad i = 1, 2.
$$

In other words, the function $F_i(t, x, p)$ is Lipschitz continuous in $p = (p_1, p_2)$ for $p_1, p_2 \in (-\infty, K_\varphi)$ with Lipschitz constant $L_\varphi$, uniformly for $t \in [0, T]$ and $x \in [g(s), h(s)]$, $i = 1, 2$. Additionally, $F_i(t, x, p)$ is continuous in all its variables in this range, $i = 1, 2$. Based on the Fundamental Theorem of ODEs, for every fixed $x \in (g(s), h(s))$, the problem (2.4) has a unique solution $p^\varphi = (p_1^\varphi, p_2^\varphi)$ defined in some interval $[t_x, T_x]$.

We will prove that $t \to p^\varphi(t, x)(t, x)$ can be uniquely extended to $[t_x, s]$. Clearly, it suffices to show that if $p^\varphi(t, x)$ is defined for $t \in [t_x, t_0]$ with $t_0 \in (t_x, s)$, then

$$
0 \leq p_1^\varphi(t, x) < K_\varphi \quad \text{for } t \in (t_x, t_0).
$$

(2.5)

We first show that $p_1^\varphi(t, x) < K_\varphi$ in $(t_x, t_0]$. If this inequality is not true, then, by $p_1^\varphi(t_x, x) = \bar{u}_{10}(x) \leq \|\varphi_1\|_{C(\overline{D}_x)} < K_\varphi$, there exists $t' \in (t_x, t_0)$ such that $p_1^\varphi(t, x) < K_\varphi$ in $(t_x, t')$ and $p_1^\varphi(t', x) = K_\varphi$. It follows that $(p_1^\varphi(t', x) \geq 0$ and $f_1^*(t', x, p_1^\varphi(t', x), p_2^\varphi(t', x)) < 0$ as $K_\varphi > k$. Hence from the equation satisfied by $p_1^\varphi$ we can deduce

$$
d_1 K_\varphi = d_1 p_1^\varphi(t', x) \leq d_1 \int_{g(t')}^{h(t')} J_1(x - y) \varphi_1(t', y) \, dy \leq d_1 \|\varphi_1\|_{C(\overline{D}_T)} \leq d_1 (K_\varphi - 1).
$$

This is a contradiction. Similarly, $p_2^\varphi(t, x) < K_\varphi$ in $(t_x, t_0]$.
We now prove the first inequality in (2.5). Since
\[ |f_1^\epsilon(t, x, p_1, p_2^\epsilon)| = |f_1^\epsilon(t, x, p_1, p_2^\epsilon) - f_1^\epsilon(t, x, 0, p_2^\epsilon)| \leq L(K\varphi)|p_1|, \forall \ p_1 \in (-\infty, K\varphi], \]
it follows that
\[ (p_1^\epsilon)_t \geq c(t, x)p_1^\epsilon + d_1 \int_{g(t)}^{h(t)} J_1(x - y)\varphi_1(t, y)dy \geq c(t, x)p_1^\epsilon, \ \forall \ t \in [t_x, t_0], \]
where \( c(t, x) = -L(K\varphi) - d_1 \) when \( p_1^\epsilon(t, x) \geq 0 \), and \( c(t, x) = L(K\varphi) \) when \( p_1^\epsilon(t, x) \leq 0 \). Notice \( p_1^\epsilon(t_x, 0) = \tilde{u}_{10}(x) \geq 0 \), the above inequality immediately gives \( p_1^\epsilon(t, x) \geq 0 \) in \([t_x, t_0] \). Similarly, \( p_2^\epsilon(t, x) \geq 0 \) in \([t_x, t_0] \). We have thus proved (2.5), and therefore the solution \( p^\epsilon(t, x) \) of (2.4) is uniquely defined for \( t \in [t_x, s] \).

**Step 2. A fixed point problem.** Recall \( p^\epsilon(0, x) = U_0(x) \) for \( |x| \leq h_0 \), and \( p^\epsilon(t, x) = 0 \) for \( x \in \{g(t), h(t)\} \) and \( t \in [0, s] \). Moreover, by the continuous dependence of the ODE solution on parameters, \( p^\epsilon \) is continuous in \( D_s \), and so \( p^\epsilon \in X^s \). Define a mapping \( \Gamma_s : X^s \to X^s \) by
\[ \Gamma_s \varphi = p^\epsilon. \]
Clearly, if \( \Gamma_s \varphi = \varphi \) then \( \varphi \) solves (2.4), and vice versa.

We will show that \( \Gamma_s \) has a unique fixed point in \( X^s \) when \( 0 < s \ll 1 \). This conclusion will be proved by the contraction mapping theorem, i.e., it will be shown that for such \( s \), \( \Gamma_s \) is a contraction on a closed subset of \( X^s \), and any fixed point of \( \Gamma_s \) in \( X^s \) lies in this closed subset.

Take \( C = \max \{2\|u_{10}\|_\infty, 2\|u_{20}\|_\infty, A_1, A_2\} \) and define
\[ X_0^s := \{\varphi = (\varphi_1, \varphi_2) \in X^s : \|\varphi_1, \varphi_2\|_{C(D_s)} \leq C\}. \]
Clearly \( X_0^s \) is a closed subset of \( X^s \). We will find a \( \delta > 0 \) small depending on \( C \) such that for every \( s \in (0, \delta] \), \( \Gamma_s \) maps \( X_0^s \) into itself, and is a contraction.

Let \( \varphi \in X_0^s \), and denote \( p^\varphi = \Gamma_s \varphi \). Then \( p^\varphi \) solves (2.4), and so (2.5) holds with \( t_0 \) replaced by \( s \). Thus, \( f_i^\epsilon(t, x, p^\varphi) = f_i(t, x, p^\varphi) \) for \( i = 1, 2 \). Now we prove that for \( 0 < s \ll 1 \),
\[ p_i^\epsilon(t, x, p_2^\epsilon(t, x) \leq C, \ \forall \ g(s) \leq x \leq h(s), \ t_x \leq t \leq s, \quad (2.6) \]
which is equivalent to \( \|p_i^\epsilon, p_2^\epsilon\|_{C(D_s)} \leq C \).

Note that \( p_i^\epsilon, p_2^\epsilon \geq 0 \) implies \( f_i(t, x, p_1^\epsilon, p_2^\epsilon) \leq rp_i^\epsilon \). It follows from the first equation of (2.4) that, for \( t \in [t_x, s] \) and \( x \in \{g(s), h(s)\} \),
\[ (p_1^\epsilon)_t \leq d_1 \int_{g(t)}^{h(t)} J_1(x - y)\varphi_1(t, y)dy + rp_1^\epsilon \leq d_1\|\varphi_1\|_{C(D_s)} + rp_1^\epsilon. \]

Multiplying this inequality by \( e^{-\tau t} \) and then integrating from \( t_x \) to \( t \) we obtain
\[ p_1^\epsilon(t, x) \leq e^{(t-t_x)}p_1^\epsilon(t_x, x) + d_1 \int_{t_x}^{t} e^{(t-\tau)}d\tau\|\varphi_1\|_{C(D_s)} \leq \|u_{10}\|_\infty e^{rs} + d_1 C e^{rs}. \]

Take \( \delta_1 > 0 \) such that \( d_1 \delta_1 e^{r\delta_1} \leq 1/4 \) and \( e^{r\delta_1} \leq 3/2 \). Then, for \( s \in (0, \delta_1] \), we have
\[ p_1^\epsilon(t, x) \leq (8\|u_{10}\|_\infty + C)/4 \leq C \quad \text{in} \quad D_s. \]
This combined with the properties of \( f_2 \) allows us to derive
\[ f_2(t, x, p_1^\epsilon, p_2^\epsilon) \leq L(C, \Theta(C))p_2^\epsilon := L^*p_2^\epsilon, \ \forall \ x \in \{g(s), h(s)\}, \ t \in [t_x, s]. \]

Similar to the above, take \( \delta_2 > 0 \) satisfying \( d_2\delta_2 e^{L^r\delta_2} \leq 1/4 \) and \( e^{L^r\delta_2} \leq 3/2 \), then
\[ p_2^\epsilon(t, x) \leq (8\|u_{20}\|_\infty + C)/4 \leq C \quad \text{in} \quad D_s. \]
for all \( s \in (0, \delta_2) \). Set \( \delta = \min \{ \delta_1, \delta_2 \} \). Then (2.6) holds for \( s \in (0, \delta] \).

Thus \( p^1 = \Gamma_s \varphi \in X_{\delta}^1 \), as desired. Next we show that by shrinking \( \delta \) if necessary, \( \Gamma_s \) is a contraction on \( X_{\delta}^1 \) for \( s \in (0, \delta] \). Let \( \varphi, \rho \in X_{\delta}^1 \), then \( p_1 = p_1^\rho - p_1^\varphi \) satisfy

\[
\begin{align*}
p_1(t, x) & = e^{-\int_0^t a(t, x) \, dt} \int_0^t e^{\int_0^s a(t, x) \, ds} \left( d_1 \int_{g(t)}^{h(t)} J_1(x - y) (\varphi_1 - \rho_1) (t, y) \, dy + b p_2 \right) \\
& \quad + b(l, x) p_2(l, x) \, dl.
\end{align*}
\]

Since \( (g(t), h(t)) \subseteq (g(s), h(s)) \) when \( t \leq s \), we deduce that, for \( x \in (g(t), h(t)) \),

\[
\begin{align*}
|p_1(t, x)| & \leq e^{L_1(l - t_x)} \int_{t_x}^t e^{L_1(l - t_s)} \, dl \left( d_1 \| \varphi_1 - \rho_1 \|_{C(\overline{\Omega})} + L_1 \| p_2 \|_{C(\overline{\Omega})} \right) \\
& \leq (t - t_x) e^{2L_1(l - t_x)} \left( d_1 \| \varphi_1 - \rho_1 \|_{C(\overline{\Omega})} + L_1 \| p_2 \|_{C(\overline{\Omega})} \right) \\
& \leq s e^{2L_1 s} \left( d_1 \| \varphi_1 - \rho_1 \|_{C(\overline{\Omega})} + L_1 \| p_2 \|_{C(\overline{\Omega})} \right).
\end{align*}
\]

This gives

\[
\| p_1 \|_{C(\overline{\Omega})} \leq s e^{2L_1 s} \left( d_1 \| \varphi_1 - \rho_1 \|_{C(\overline{\Omega})} + L_1 \| p_2 \|_{C(\overline{\Omega})} \right).
\]

Similarly,

\[
\| p_2 \|_{C(\overline{\Omega})} \leq s e^{2L_2 s} \left( d_2 \| \varphi_2 - \rho_2 \|_{C(\overline{\Omega})} + L_2 \| p_1 \|_{C(\overline{\Omega})} \right),
\]

where \( L_2 = d_2 + L(C) \). Set \( L = \max \{ L_1, L_2 \} \). Then

\[
\| \Gamma_s \varphi - \Gamma_s \rho \|_{C(\overline{\Omega})} \leq \frac{1}{2} \left( \| \varphi_1 - \rho_1 \|_{C(\overline{\Omega})} + \| \varphi_2 - \rho_2 \|_{C(\overline{\Omega})} \right), \quad \forall \ s \in (0, \sigma]
\]

provided that \( \sigma \in (0, \delta] \) satisfies

\[
\sigma L e^{2L_\sigma} \leq 1/2, \quad \sigma d_i e^{2L_\sigma} \leq 1/4, \quad i = 1, 2.
\]

For such \( s \) we may now apply the Contraction Mapping Theorem to conclude that \( \Gamma_s \) has a unique fixed point \( W \) in \( X_{\delta}^1 \). Thus, \( w = W \) solves (2.3) for \( 0 < t \leq s \).

If we can show that any solution \( w \) of (2.3) satisfies \( 0 \leq w_1, w_2 \leq C \) in \( D_s \), then \( w \) must coincide with the unique fixed point \( W \) of \( \Gamma_s \) in \( X_{\delta}^1 \). We next prove such an estimate for \( (w_1, w_2) \). Note that \( w_1, w_2 \geq 0 \) already follows from (2.5). It is enough to show \( w_1, w_2 \leq C \). We actually prove the following stronger inequality

\[
w_1(t, x) \leq A_1, \quad w_2(t, x) \leq A_2, \quad \forall \ g(s) \leq x \leq h(s), \quad t_x \leq t \leq s.
\]

We only prove \( w_1(t, x) \leq A_1 \) since \( w_2(t, x) \leq A_2 \) can be shown by the same way. It suffices to show that the above inequality holds with \( A_1 \) replaced by \( A_1 + \varepsilon \) for any given \( \varepsilon > 0 \). Suppose this is not true. Due to \( w_1(t_x, x) = w_{10}(x) \leq \| w_{10} \|_{\infty} < A^\varepsilon := A_1 + \varepsilon \), there exist \( x_0 \in (g(s), h(s)) \) and \( t_0 \in (t_{x_0}, s) \) such that

\[
w_1(t_0, x_0) = A^\varepsilon, \quad 0 \leq w_1(t, x) < A^\varepsilon \quad \text{for} \ g(t_0) \leq x \leq h(t_0), \quad t_x \leq t < t_0.
\]

\[
\| \Gamma_s \varphi - \Gamma_s \rho \|_{C(\overline{\Omega})} \leq \frac{1}{2} \left( \| \varphi_1 - \rho_1 \|_{C(\overline{\Omega})} + \| \varphi_2 - \rho_2 \|_{C(\overline{\Omega})} \right), \quad \forall \ s \in (0, \sigma]
\]

provided that \( \sigma \in (0, \delta] \) satisfies

\[
\sigma L e^{2L_\sigma} \leq 1/2, \quad \sigma d_i e^{2L_\sigma} \leq 1/4, \quad i = 1, 2.
\]
It follows that \( w_1(t_0, x_0) \geq 0 \) and \( f_1(t_0, x_0, w(t_0, x_0)) \leq 0 \) due to \( w_1(t_0, x_0) = A^e > A_1 \) and \( w_2(t_0, x_0) \geq 0 \). Hence from the first equation of (2.3) we obtain
\[
0 \leq w_1(t_0, x_0) \leq d_1 \int_{g(t_0)}^{h(t_0)} J_1(x_0 - y) w_1(t_0, y) dy - d_1 w_1(t_0, x_0).
\]
Since \( w_1(t_0, g(t_0)) = w_1(t_0, h(t_0)) = 0 \), we have \( w_1(t_0, y) < A^e \) for \( y \in (g(t_0), h(t_0)) \) but close to the boundary of this interval. It follows that
\[
d_1 A^e = d_1 w_1(t_0, x_0) \leq d_1 \int_{g(t_0)}^{h(t_0)} J_1(x_0 - y) w_1(t_0, y) dy < d_1 A^e \int_{g(t_0)}^{h(t_0)} J_1(x_0 - y) dy \leq d_1 A^e.
\]
This contradiction proves (2.7). Thus \((w_1, w_2)\) satisfies the wanted inequality and hence coincides with the unique fixed point of \( \Gamma_s \) in \( X_T^c \). We have now proved the fact that for every \( s \in (0, \sigma) \), \( \Gamma_s \) has a unique fixed point in \( X^s \).

**Step 3. Completion of the proof.** From Step 2 we know that (2.3) has a unique solution \( w \) defined for \( t \in [0, \sigma] \) and \( w \) satisfies (2.7) with \( s = \sigma \). Note that
\[
\max \left\{ \max_{[g(\sigma), h(\sigma)]} w_1(\sigma, x), k \right\} \leq \max \{ A_1, k \} = A_1,
\]
\[
\max \left\{ \max_{[g(\sigma), h(\sigma)]} w_2(\sigma, x), \Theta(A_1) \right\} \leq \max \{ A_2, \Theta(A_1) \} = A_2.
\]
Hence we may apply Step 2 to (2.3) but with the initial time \( t = 0 \) replaced by \( t = \sigma \) to conclude that the unique solution can be extended to a slightly larger domain \( D_\sigma \).

Moreover, by (2.7) and the definition of \( \sigma \) in Step 2, we see that \( \sigma \) depends only on \( d_1 \) and \( A_1 \), and it can take any value in \((0, 2\sigma]\). Furthermore, from the above proof of (2.7) we see that the extended solution \( w \) satisfies (2.7) in \( D_\sigma \). Thus the extension can be repeated. By repeating this process finitely many times, the solution \( w \) of (2.3) will be uniquely extended to \( D_T \). As explained above, now (2.7) holds with \( s = T \), and hence to prove that \( w \) satisfies (2.1), it only remains to show \( w_1 > 0 \), \( w_2 > 0 \) in \( D_T \). Recall \( w_1, w_2 \geq 0 \). Using the conditions (f), (f1), (f2) and the conclusion (2.7) we may write \( f_1(t, x, w) = b_1(t, x)w_1 \) and \( f_2(t, x, w) = b_2(t, x)w_2 \) with \( b_i \in L^\infty(D_T) \). Then Lemma 2.2 gives the desired result.

**Proof of Theorem 2.1.** By Lemma 2.3, for any \( T > 0 \) and \((g, h) \in G_T \times H_T \), we can find a unique \( w_{g,h} = (w_{1,g,h}, w_{2,g,h}) \in X_T^{g,h} \) that solves (2.3) and satisfies (2.1).

Using such a \( w_{g,h} \), we define the mapping \( F \) by \( F(g, h) = (\tilde{g}, \tilde{h}) \), where
\[
\tilde{h}(t) = h_0 + \sum_{i=1}^{2} \mu_i \int_0^t \int_{g(\tau)}^{h(\tau)} J_i(y, x) w_{i,g,h}(\tau, x) dy dx d\tau,
\]
\[
\tilde{g}(t) = -h_0 - \sum_{i=1}^{2} \mu_i \int_0^t \int_{g(\tau)}^{h(\tau)} J_i(y, x) w_{i,g,h}(\tau, x) dy dx d\tau
\]
for \( 0 < t \leq T \).

To prove this theorem, we first show that if \( T \) is small enough, then \( F \) maps a suitable closed subset \( \Sigma_T \) of \( G_T \times H_T \) into itself, and is a contraction mapping. This clearly implies that \( F \) has a unique fixed point in \( \Sigma_T \), which gives a solution \((w_{g,h}, g, h)\) of (1.2) defined for \( t \in (0, \sigma] \). Then we prove that any solution
(u_1, u_2, g, h) of (1.2) with (g, h) ∈ G_T × H_T must satisfy (g, h) ∈ Σ_T, and hence (g, h) must coincide with the unique fixed point of F in Σ_T, which then implies that the solution (u_1, u_2, g, h) of (1.2) is unique. Finally we extend this unique local solution to a global one. This plan will be carried out in several steps.

**Step 1. Properties of (g, h) and a closed subset of G_T × H_T.** Let (g, h) ∈ G_T × H_T.

Then ˜g, ˜h ∈ C^1([0, T]) and for 0 < t ≤ T,

\[
\begin{align*}
\tilde{h}'(t) &= \sum_{i=1}^{2} \mu_i \int_{g(t)}^{h(t)} \int_{h(t)}^{\infty} J_i(x-y)w_{i,g,h}(\tau, x)dydx, \\
\tilde{g}'(t) &= -\sum_{i=1}^{2} \mu_i \int_{g(t)}^{h(t)} \int_{-\infty}^{g(t)} J_i(x-y)w_{i,g,h}(\tau, x)dydx.
\end{align*}
\]  

(2.8)

These facts imply (˜g, ˜h) ∈ G_T × H_T. To show that F is a contraction, we need some further properties of ˜g and ˜h to be used in choosing a suitable closed subset of G_T × H_T, which is invariant under F, and on which F is a contraction mapping.

Denote w_i = w_{i,g,h} to simplify the notations. Since (w_1, w_2) solves (2.3) and satisfies (2.1), we obtain by using (F), (f_1), (f_2) that

\[
\begin{align*}
w_i(t, x) &\geq -d_iw_i(t, x) - L(A_1, A_2)w_i(t, x), \quad 0 < t \leq T, \quad g(t) < x < h(t), \\
w_i(0, x) &\geq 0, \quad 0 \leq t \leq T, \quad x = g(t), \quad h(t), \\
w_i(0, x) &\geq \nu_0(x), \quad |x| \leq \nu_0.
\end{align*}
\]

It follows that, for t ∈ (0, T] and |x| ≤ \nu_0,

\[
w_i(t, x) \geq e^{-[d_i + L(A_1, A_2)]t} \nu_0(x) \geq e^{-[d_i + L(A_1, A_2)]T} \nu_0(x).
\]  

(2.9)

By the condition (J), there exist constants \nu_0 ∈ (0, \nu_0/4) and \delta_0 > 0 such that

\[
J_i(x-y) \geq \delta_0 \quad \text{for} \quad |x-y| \leq \nu_0, \quad i = 1, 2.
\]  

(2.10)

Using (2.8) we easily see

\[
[\tilde{h}(t) - \tilde{g}(t)]' \leq (\mu_1 A_1 + \mu_2 A_2)[h(t) - g(t)], \quad \forall t \in [0, T].
\]

We can choose 0 < T ≪ 1, depending on \mu_i, A_i, \nu_0, \epsilon_0, such that h(T) - g(T) ≤ 2\nu_0 + \epsilon_0/4 and

\[
\tilde{h}(t) - \tilde{g}(t) \leq 2\nu_0 + T(\mu_1 A_1 + \mu_2 A_2)(2\nu_0 + \epsilon_0/4) \leq 2\nu_0 + \epsilon_0/4, \quad \forall t \in [0, T],
\]

h(t) ∈ [\nu_0, \nu_0 + \epsilon_0/4], \quad g(t) ∈ [-\nu_0 - \epsilon_0/4, -\nu_0], \quad \forall t \in [0, T].

Combining this with (2.9) and (2.10) we obtain that, for t ∈ (0, T],

\[
\sum_{i=1}^{2} \mu_i \int_{g(t)}^{h(t)} \int_{h(t)}^{\infty} J_i(x-y)w_i(t, x)dydx \\
\geq \sum_{i=1}^{2} \mu_i \int_{h(t)-\epsilon_0/4}^{h(t)} \int_{h(t)}^{h(t)+\epsilon_0/4} J_i(x-y)w_i(t, x)dydx \\
\geq \sum_{i=1}^{2} \mu_i e^{-LT} \int_{\nu_0-\epsilon_0/4}^{\nu_0} \int_{\nu_0+\epsilon_0/4}^{\nu_0} J_i(x-y)e^{-d_iT} \nu_0(x)dydx \\
\geq \frac{\epsilon_0}{4} \delta_0 e^{-(d_1 + d_2 + L)T} \sum_{i=1}^{2} \mu_i \int_{\nu_0-\epsilon_0/4}^{\nu_0} \nu_0(x)dx \\
=: \alpha_1 \mu_1 + \alpha_2 \mu_2,
\]

\[
\alpha_1, \alpha_2 \geq 0.
\]
where $L = L(A_1, A_2)$, $\alpha_1$, $\alpha_2$ are positive constants depending only on $J_i$, $f_i$ and $U_0$. Thus, for sufficiently small $T_0 = T(\mu_1, \mu_2, A_1, A_2, h_0, \varepsilon_0) > 0$,

$$h'(t) \geq \alpha_1 \mu_1 + \alpha_2 \mu_2 := h_* > 0, \quad t \in [0, T_0].$$

(2.11)

Similarly,

$$\tilde{g}'(t) \leq -(\hat{\alpha}_1 \mu_1 + \hat{\alpha}_2 \mu_2) := g_* < 0, \quad t \in [0, T_0]$$

(2.12)

for some positive constants $\hat{\alpha}_1$ and $\hat{\alpha}_2$ depending only on $J_i$, $f_i$ and $U_0$.

For $0 < T \leq T_0$, we define

$$\Sigma_T := \left\{ (g, h) \in \mathbb{G}_T \times \mathbb{H}_T : \frac{g(t_2) - g(t_1)}{t_2 - t_1} \leq g_*, \quad \frac{h(t_2) - h(t_1)}{t_2 - t_1} \geq h_* \right\}

\text{for } 0 \leq t_1 < t_2 \leq T, \text{and } h(t) - g(t) \leq 2h_0 + \varepsilon_0/4 \text{ for } 0 \leq t \leq T \right\}.$$

The above analysis shows that $\mathcal{F}(\Sigma_T) \subset \Sigma_T$.

**Step 2.** $\mathcal{F}$ is a contraction mapping on $\Sigma_T$ for $0 < T \ll 1$. For $(g_j, h_j) \in \Sigma_T$, $j = 1, 2$, we set

$$\Omega_T = D_{g_1, h_1}^T \cup D_{g_2, h_2}^T, \quad w_i^1 = w_{1,j}, \quad w_i^2 = w_{2,j}, \quad \mathcal{F}(g_j, h_j) = (\tilde{g}_j, \tilde{h}_j),$$

$$\hat{w}_j = w_j^1 - w_j^2, \quad \hat{g} = g_1 - g_2, \quad \hat{h} = h_1 - h_2, \quad \tilde{g} = \tilde{g}_1 - \tilde{g}_2, \quad \tilde{h} = \tilde{h}_1 - \tilde{h}_2.$$

Make the zero extension of $w_i^j$ in $([0, T] \times \mathbb{R}) \setminus D_{g_i, h_i}^T$. It then follows that

$$|\tilde{h}'(t)| \leq \sum_{i=1}^2 \frac{1}{\mu_i} \left| \int_{g_1(\tau)}^{h_1(\tau)} \int_{h_1(\tau)}^{h_2(\tau)} J_i(x - y) w_i^1(\tau, x) \, dy \, dx \right|

- \int_{g_2(\tau)}^{h_2(\tau)} \int_{h_2(\tau)}^{h_1(\tau)} J_i(x - y) w_i^2(\tau, x) \, dy \, dx

\leq \sum_{i=1}^2 \frac{1}{\mu_i} \left| \int_{g_1(\tau)}^{h_1(\tau)} \int_{h_1(\tau)}^{h_2(\tau)} J_i(x - y) |\hat{w}_i(\tau, x)| \, dy \, dx \right|

+ \sum_{i=1}^2 \frac{1}{\mu_i} \left| \left( \int_{g_1(\tau)}^{h_1(\tau)} \int_{h_1(\tau)}^{h_2(\tau)} J_i(x - y) w_i^1(\tau, x) \, dy \, dx \right) \right|

\leq \sum_{i=1}^2 \frac{1}{\mu_i} \left( 3h_0 \|\hat{w}_i\|_{C([\Omega_T])} + \|\hat{g}\|_{C([0, T])} A_i + (A_i + 3h_0 A_i \|J_i\|_{\infty}) \|\tilde{h}\|_{C([0, T])} \right),$$

and so

$$|\hat{h}(t)| \leq C_0 T \left( \|\hat{w}_1\|_{C([\Omega_T])} + \|\hat{g}\|_{C([0, T])} \right), \quad 0 < t \leq T,$$

where $C_0$ depends only on $h_0, \mu_i, A_i$ and $J_i$. Similarly,

$$|\hat{g}(t)| \leq C_0 T \left( \|\hat{w}_1\|_{C([\Omega_T])} + \|\tilde{h}\|_{C([0, T])} \right), \quad 0 < t \leq T.$$

Therefore,

$$\|\hat{g}, \hat{h}\|_{C([0, T])} \leq CT \left( \|\hat{w}_1\|_{C([\Omega_T])} + \|\hat{g}\|_{C([0, T])} \right).$$

(2.13)

Next, we estimate $\|\hat{w}_1, \hat{w}_2\|_{C([\Omega_T])}$. Fix $(s, x) \in \Omega_T$. We now estimate $|\hat{w}_1(s, x)|$ and $|\hat{w}_2(s, x)|$ in all the possible cases.

**Case 1.** $x \in (g_1(s), h_1(s)) \setminus (g_2(s), h_2(s))$. In such case, either $g_1(s) < x \leq g_2(s)$ or $h_2(s) \leq x < h_1(s)$, and $w_1^2(s, x) = w_2^2(s, x) = 0$.
When $h_2(s) \leq x < h_1(s)$, there exists $0 < s_1 < s$ such that $x = h_1(s_1)$, and so $h_1(s) > h_1(s_1) = x \geq h_2(s)$. Clearly, $g_1(t) < h_1(s_1) = x \leq h_1(t)$ for all $t \in [s_1, s]$. By integrating the equation satisfied by $w_1$ from $s_1$ to $s$ we obtain

$$|\hat{w}_1(s, x)| = w_1'(s, x) \leq \int_{s_1}^{s} \left( d_1 \int_{g_1(t)}^{h_1(t)} J_1(x - y)w_1(t, y)dy - d_1 w_1 + f_1(t, x, w_1, w_2) \right) dt$$

Hence

$$|\hat{w}_1(s, x)| = w_1'(s, x) \leq C_4 \|\hat{\hat{g}}\|_{C([0, T])}.$$

When $g_1(s) < x \leq g_2(s)$, we can analogously obtain

$$|\hat{w}_1(s, x)| = w_1'(s, x) \leq C_4 \|\hat{\hat{g}}\|_{C([0, T])}.$$
Using the conclusions of Case 1 and Case 2 we have $|\dot{w}_1(t_0, x)| \leq C_4 \|\dot{g}, \dot{h}\|_{C([0,T])}$. In view of (2.16), it is clear that (2.15) holds for all $t_0 < t \leq s$. Integrating (2.15) from $t_0$ to $s$ we obtain

$$\begin{align*}
|\dot{w}_1(s, x)| &\leq |\dot{w}_1(t_0, x)| + \left(2d_1 \|\dot{w}_1\|_{C(\Omega_T)} + d_1 A_1 \|J_1\|_\infty \|\dot{g}, \dot{h}\|_{C([0,T])}\right) \\
&\quad + L \|\dot{w}_1, \dot{w}_2\|_{C(\Omega_T)} (s - t_0) \\
&\leq C_4 \|\dot{g}, \dot{h}\|_{C([0,T])} + C_5 \left(\|\dot{g}, \dot{h}\|_{C([0,T])} + \|\dot{w}_1, \dot{w}_2\|_{C(\Omega_T)}\right) T \\
&=: C_6 \|\dot{g}, \dot{h}\|_{C([0,T])} + C_7 T \|\dot{w}_1, \dot{w}_2\|_{C(\Omega_T)}.
\end{align*}$$

In the same way one has

$$|\dot{w}_2(s, x)| \leq C_6 \|\dot{g}, \dot{h}\|_{C([0,T])} + C_7 T \|\dot{w}_2\|_{C(\Omega_T)}.$$ 

Summarizing the above discussions, we obtain

$$\|\dot{w}_1, \dot{w}_2\|_{C(\Omega_T)} \leq C' \|\dot{g}, \dot{h}\|_{C([0,T])} + C'T \|\dot{w}_1, \dot{w}_2\|_{C(\Omega_T)} \leq 2C' \|\dot{g}, \dot{h}\|_{C([0,T])}$$

if $C'T < 1/2$. This combined with (2.13) yields

$$\|\ddot{g}, \ddot{h}\|_{C([0,T])} \leq C(2C' + 1)T \|\ddot{g}, \ddot{h}\|_{C([0,T])} \leq \frac{1}{2} \|\ddot{g}, \ddot{h}\|_{C([0,T])}$$

if $C(2C' + 1)T \leq 1/2$. This shows that $\mathcal{F}$ is a contraction mapping on $\Sigma_T$.

**Step 3. Local existence and uniqueness.** By Step 2 and the Contraction Mapping Theorem we know that (1.2) has a solution $(u_1, u_2, g, h)$ defined for $t \in (0, T]$. If we can show that $(g, h) \in \Sigma_T$ holds for any solution $(u_1, u_2, g, h)$ of (1.2) defined for $t \in (0, T]$, then $(g, h)$ must coincide with the unique fixed point of $\mathcal{F}$ in $\Sigma_T$ and the uniqueness of the local solution $(u_1, u_2, g, h)$ to (1.2) would follow.

Let $(u_1, u_2, g, h)$ be an arbitrary solution of (1.2) defined for $t \in (0, T]$. Then

$$h'(t) = \sum_{i=1}^{2} \mu_i \int_{g(t)}^{h(t)} \int_{-\infty}^{\infty} J_i(x - y) u_i(t, x) dy dx,$$

$$g'(t) = -\sum_{i=1}^{2} \mu_i \int_{g(t)}^{h(t)} \int_{-\infty}^{\infty} J_i(x - y) u_i(t, x) dy dx.$$ 

In view of Lemma 2.3, $0 < u_1 \leq A_1$, $0 < u_2 \leq A_2$ in $D^{T}_{g,h}$. Thus

$$h'(t) - g'(t) = \sum_{i=1}^{2} \mu_i \int_{g(t)}^{h(t)} \left( \int_{-\infty}^{\infty} J_i(x - y) u_i(t, x) dy dx \right) \left( J_i(x - y) u_i(t, x) dy dx \right),$$

which implies

$$h(t) - g(t) \leq 2h_0 e^{(\mu_1 A_1 + \mu_2 A_2) t}, \quad t \in (0, T].$$

Shrink $T$ so that $2h_0 e^{(\mu_1 A_1 + \mu_2 A_2) T} \leq 2h_0 + \varepsilon_0/4$, then $h(t) - g(t) \leq 2h_0 + \varepsilon_0/4$ on $[0, T]$. Moreover, the proof of (2.11) and (2.12) gives $h'(t) \geq h$, and $g'(t) \leq g$, in $(0, T]$. Thus $(g, h) \in \Sigma_T$ as we required.

**Step 4. Global existence and uniqueness.** By Step 3, we see that the problem (1.2) has a unique solution $(u_1, u_2, g, h)$ for some time interval $(0, T]$. Moreover, for any fixed $s \in (0, T)$, there hold $u_1(s, x) > 0$, $u_2(s, x) > 0$ in $(g(s), h(s))$, and
$u_i(s, \cdot)$ ($i = 1, 2$) are continuous on $[g(s), h(s)]$. This implies that we can treat $(u_1(s, \cdot), u_2(s, \cdot))$ as an initial function and use Step 3 to extend the solution from $t = s$ to some $T' \geq T$. Suppose that $(0, T_0)$ is the maximal existence interval of $(u_1, u_2, g, h)$ obtained by such an extension process. We show that $T_0 = \infty$. Otherwise $T_0 \in (0, \infty)$ and we are going to derive a contradiction.

Firstly, (2.17) holds for $t \in (0, T_0)$. Since $h(t)$ and $g(t)$ are monotone in $[0, T_0)$, we may define

\[ h(t_0) := \lim_{t \to T_0} h(t), \quad g(T_0) := \lim_{t \to T_0} g(t) \quad \text{with} \quad h(T_0) = g(T_0) \leq 2h_0e^{(\mu_1A_1+\mu_2A_2)T_0}. \]

The free boundary conditions in (1.2), together with $0 \leq u_1 \leq A_1$ and $0 \leq u_2 \leq A_2$ indicate that $h', g' \in L^\infty([0, T_0])$ and hence $g, h \in C([0, T_0])$ with $g(T_0)$, $h(T_0)$ defined as above. It follows that the right-hand side of the first equation in (1.2) belongs to $L^\infty(D_{g,h}^{T_0})$, this implies $u_i \in L^\infty(D_{g,h}^{T_0})$. Thus for each $x \in (g(T_0), h(T_0))$ and $i = 1, 2$, the limit $u_i(T_0, x) := \lim_{t \to T_0} u_i(t, x)$ exists, and $u_i(\cdot, x)$ is continuous at $t = T_0$. We may now view $(u_1(t, x), u_2(t, x))$ as the unique solution of the ODE problem in Step 1 of the proof of Lemma 2.3 (with $\varphi = (u_1, u_2)$), which is defined over $[t_x, T_0]$. Since $t_x$, $J_1(x, y)$, and $f_1(t, x, u_1, u_2)$ are all continuous in $x$, by the continuous dependence of the ODE solution to the initial function and the parameters in the equation, we see that $u_i(t, x)$ ($i = 1, 2$) are continuous in $D_{g,h}^{T_0}$. By assumption, $u_1, u_2 \in C(D_s)$ for any $s \in (0, T_0)$. To show this also holds with $s = T_0$, it remains to show that $u_i(t, x) \to 0$ as $(t, x) \to (T_0, g(T_0))$ and as $(t, x) \to (T_0, h(T_0))$ from $D_{g,h}^{T_0}$. We only prove $u_1(t, x) \to 0$ as $(t, x) \to (T_0, g(T_0))$ from $D_{g,h}^{T_0}$ because of the other cases can be shown similarly. We note that $x \searrow g(T_0)$ implies $t_x \nearrow T_0$, and so

\[
|u_1(t, x)| = \left| \int_{t_x}^t \left( d_1 \int_{g(t)}^{h(t)} J_1(x, y)u_1(\tau, y)dy - d_1u_1(t, \tau) \right) d\tau \right|
\]

\[
+ f_1(\tau, x, u_1(\tau, x), u_2(\tau, x)) \right) d\tau \leq (t - t_x) \left[ 2d_1 + L(A_1, A_2) \right] A_1
\]

\[
\to 0 \quad \text{as} \quad D_{g,h}^{T_0} \ni (t, x) \to (T_0, g(T_0)).
\]

Thus we have shown that $u_i \in C(D_{g,h}^{T_0})$ and $(u_1, u_2, g, h)$ satisfies (1.2) for $t \in (0, T_0)$. Writing $f_i(t, x, u_1(t, x), u_2(t, x)) = b_i(t, x)u_i(t, x)$ with $b_i \in L^\infty(D_{g,h}^{T_0})$, and using Lemma 2.2 we have $u_i(T_0, x) > 0$ for $x \in (g(T_0), h(T_0))$. Thus we can regard $(u_1(T_0, \cdot), u_2(T_0, \cdot))$ as an initial function and apply Step 3 to conclude that the solution of (1.2) can be extended to some $(0, \tilde{T})$ with $\tilde{T} > T_0$. This contradicts the definition of $T_0$. Therefore we must have $T_0 = \infty$.

From the above proof we see that (2.1) and (2.2) hold, and the theorem is proved.

\[
\square
\]

3. **Spreading and vanishing.** In view of (2.2) we can define

\[
\lim_{t \to \infty} g(t) = g_\infty \in [-\infty, -h_0), \quad \lim_{t \to \infty} h(t) = h_\infty \in [h_0, \infty].
\]

Clearly we have either

\[ (i) \quad h_\infty - g_\infty < \infty, \quad (ii) \quad h_\infty - g_\infty = \infty. \]
We will call (i) the vanishing case, and call (ii) the spreading case.

The main purpose of this section is to determine when (i) or (ii) can occur, and to determine the long-time profile of \((u_1, u_2)\) if (i) or (ii) happens. It turns out that these are highly nontrivial tasks as many techniques worked in the corresponding local diffusion cases are not applicable anymore, and those worked in the one species nonlocal diffusion problem with free boundary in [7] are also lacking for treating the current two species situation. In subsection 3.1 below, we introduce some new techniques which are enough to treat the Lotka-Volterra cases (1.4) and (1.5). In subsection 3.2, we will further restrict the growth function classes in order to determine the long-time profile of \((u_1, u_2)\).

3.1. Criteria for vanishing and spreading. The following two simple lemmas provide some key ingredients for analysing the vanishing phenomenon.

**Lemma 3.1.** Let the condition \((J)\) hold for the kernel functions \(J_1, J_2, \) and \(\beta_1, \beta_2 > 0\) be constants. Suppose that \(g, h \in C^1([0, \infty))\), \(g(0) < h(0)\), \(g'(t) \leq 0, \ h'(t) \geq 0,\) and \(w_1, w_i \in C(D_{g,h}^\infty)\) for \(i = 1, 2,\) where \(D_{g,h}^\infty = \{ t > 0, g(t) < x < h(t) \} \). If \((w_1, w_2, g, h)\) satisfies

\[
\begin{aligned}
    h'(t) &= \sum_{i=1}^{2} \beta_i \int_{g(t)}^{h(t)} \int_{h(t)}^{\infty} J_i(x-y)w_i(t,x) dy dx, \quad t \geq 0, \\
    g'(t) &= -\sum_{i=1}^{2} \beta_i \int_{g(t)}^{h(t)} \int_{-\infty}^{g(t)} J_i(x-y)w_i(t,x) dy dx, \quad t \geq 0,
\end{aligned}
\]

and

\[
\lim_{t \to \infty} h(t) - \lim_{t \to \infty} g(t) < \infty, \tag{3.2}
\]

then

\[
\lim_{t \to \infty} g'(t) = \lim_{t \to \infty} h'(t) = 0.
\]

**Proof.** Set \(\lim_{t \to \infty} h(t) = h_\infty,\) \(\lim_{t \to \infty} g(t) = g_\infty.\) The condition \(3.2\) implies \(-\infty < g_\infty < h_\infty < \infty.\) Take \(M > 0\) such that \(\beta_i |w_i| \leq M, \ \beta_i |w_2| \leq M\) in \(D_{g,h}^\infty\) for \(i = 1, 2.\) It then follows from \(3.1\) that \(g'(t)\) and \(h'(t)\) are bounded. For any given \(t, s > 0,\) we have

\[
\begin{aligned}
    h'(t) - h'(s) &= \int_{g(t)}^{h(t)} \int_{h(t)}^{\infty} \sum_{i=1}^{2} \beta_i J_i(x-y)w_i(t,x) dy dx \\
    &- \int_{g(s)}^{h(s)} \int_{h(s)}^{\infty} \sum_{i=1}^{2} \beta_i J_i(x-y)w_i(s,x) dy dx \\
    &= \left\{ \int_{g(t)}^{h(t)} + \int_{h(t)}^{h(s)} \right\} \int_{h(t)}^{\infty} \sum_{i=1}^{2} \beta_i J_i(x-y)w_i(t,x) dy dx \\
    &- \int_{g(s)}^{h(s)} \int_{h(s)}^{h(t)} \sum_{i=1}^{2} \beta_i J_i(x-y)w_i(s,x) dy dx \\
    &+ \int_{g(s)}^{h(s)} \int_{h(s)}^{\infty} \sum_{i=1}^{2} \beta_i J_i(x-y)[w_i(t,x) - w_i(s,x)] dy dx
\end{aligned}
\]
Lemma 3.2. Let \( \beta_i \) be a nonnegative number for some positive constants \( \beta \). For any \( t \in (0, \infty) \),
\[
\begin{align*}
&= \left\{ \int_{g(t)}^{h(t)} g(s) \right\} \int_{h(t)}^{\infty} \sum_{i=1}^{2} \beta_i J_i(x-y)w_i(t,x)dydx \\
&- \int_{h(t)}^{\infty} \sum_{i=1}^{2} \beta_i J_i(x-y)w_i(s,x)dx \\
&+ \int_{h(t)}^{\infty} \sum_{i=1}^{2} \beta_i J_i(x-y)w_i(t,x)(t-s)dydx,
\end{align*}
\]
where \( \tau_i(x) \) is a number lying between \( s \) and \( t \). Therefore,
\[
|g'(t) - g'(s)| \leq 2M \left( |g(t) - g(s)| + |h(t) - h(s)| \right) \\
+ 2M(h(t) - h(s)) + 2M(h_\infty - g_\infty)(|t-s|)
\]
where \( M' = \|g'\|_\infty + \|h'\|_\infty \). This shows that \( h'(t) \) is Lipschitz continuous in \([0, \infty)\).
And hence, \( \lim_{t \to \infty} h'(t) = 0 \) due to \( h'(t) \geq 0 \) and \( \lim_{t \to \infty} h(t) = h_\infty \in (0, \infty) \). Similarly, \( \lim_{t \to \infty} g'(t) = 0 \).

\( \square \)

Lemma 3.2. Let \( J \) satisfy the condition (J) and \( J(x) > 0 \) in \( \mathbb{R} \). Suppose that \( g, h \in C^1([0, \infty)) \), \( g(0) < h(0) \), \( g'(t) \leq 0 \), \( h'(t) \geq 0 \), and (3.2) holds. If \((w, g, h)\) satisfies, for some positive constants \( \beta \) and \( M \),
\[
0 \leq w \leq M \text{ in } D_{g,h}, \quad w(t, g(t)) = w(t, h(t)) = 0, \quad \forall t \geq 0,
\]
\[
h'(t) \geq \beta \int_{g(t)}^{h(t)} \int_{h(t)}^{\infty} J(x-y)w(t,x)dydx, \quad \forall t > 0,
\]
and \( \lim_{t \to \infty} h'(t) = 0 \), then
\[
\lim_{t \to \infty} \int_{g(t)}^{h(t)} w(t,x)dx = 0, \quad \int_{0}^{\infty} \int_{g(t)}^{h(t)} w(t,x)dxdt < \infty.
\]

Proof. Since \( J(x) > 0 \) in \( \mathbb{R} \), we have, for any \( x \in (g(t), h(t)) \) and \( t > 0 \),
\[
\int_{h(t)}^{\infty} J(x-y)dy = \int_{-\infty}^{x-h(t)} J(z)dz \geq \int_{-\infty}^{h(h(t))} J(z)dz =: \sigma_0 > 0.
\]
It follows that
\[
\frac{1}{\beta} h'(t) \geq \int_{g(t)}^{h(t)} \int_{h(t)}^{\infty} J(x-y)w(t,x)dydx \geq \sigma_0 \int_{g(t)}^{h(t)} w(t,x)dx.
\]
Hence
\[
\lim_{t \to \infty} \int_{g(t)}^{h(t)} w(t,x)dx = 0, \quad \int_{0}^{\infty} \int_{g(t)}^{h(t)} w(t,x)dxdt \leq \frac{h_\infty - h(0)}{\beta \sigma_0}
\]
as \( \lim_{t \to \infty} h'(t) = 0 \). The proof is complete. \( \square \)

For \( a < b, i = 1, 2 \) and \( \theta \in C([a, b]) \), we define the operator \( \mathcal{L}_{d_i, (a,b)}^d + \theta \) by
\[
(\mathcal{L}_{d_i, (a,b)}^d + \theta) \varphi(x) := d_i \left( \int_{a}^{b} J_i(x-y)\varphi(y)dy - \varphi(x) \right) + \theta(x)\varphi(x), x \in [a, b], i = 1, 2.
\]
The generalized principal eigenvalue of \( L_{(a,b)}^d + \theta \) is given by

\[
\lambda_p(L_{(a,b)}^d + \theta) := \inf \{ \lambda \in \mathbb{R} : (L_{(a,b)}^d + \theta)\phi \leq \lambda \phi \text{ in } [a,b] \text{ for some } \phi \in C([a,b]), \phi > 0 \}.
\]

**Theorem 3.3.** Assume that \( J_1 \) and \( J_2 \) satisfy (J), \( J_1(x) > 0, J_2(x) > 0 \) in \( \mathbb{R} \), and that \((f_1, f_2)\) satisfies either (1.4) or (1.5). Let \((u_1, u_2, g, h)\) be the unique solution of (1.2). If \( h_\infty - g_\infty < \infty \), then

\[
\lim_{t \to \infty} \max_{g(t) \leq x \leq h(t)} u_i(t, x) = 0, \quad i = 1, 2; \tag{3.3}
\]

moreover,

\[
\lambda_p \left( L_{(g_\infty, h_\infty)}^d + a_i \right) \leq 0, \quad i = 1, 2. \tag{3.4}
\]

**Proof.** As \( h_\infty - g_\infty < \infty \) and \( u_1, u_2 \) are bounded, it follows from the first equation of (1.2) that \( u_{it} \in C(D_{g\_\infty}^\infty) \cap L^\infty(D_{g\_\infty}^\infty), \) \( i = 1, 2 \). Lemma 3.1 then infers that \( \lim_{t \to \infty} g'(t) = \lim_{t \to \infty} h'(t) = 0 \). Since \( u_i > 0 \) and \( \mu_i > 0 \), we have

\[
h'(t) \geq \mu_i \int_{g(t)}^{h(t)} \int_{g(t)}^{h(t)} J_i(x - y)u_i(t, x)dydx, \quad \forall t > 0, \quad i = 1, 2.
\]

Applying Lemma 3.2 we thus obtain

\[
\lim_{t \to \infty} \int_{g(t)}^{h(t)} u_i(t, x)dx = 0, \quad \int_0^\infty \int_{g(t)}^{h(t)} u_i(t, x)dxdt < \infty, \quad i = 1, 2.
\]

We extend \( u_i(t, x) \) by 0 for \( x \not\in [g(t), h(t)] \) and denote the extended function still by \( u_i(t, x) \). Then we may rewrite the above inequality as

\[
\int_0^\infty \int_{g(t)}^{h(t)} u_i(t, x)dxdt < \infty.
\]

By Fubini’s theorem we have

\[
\int_{g(t)}^{h(t)} \int_0^\infty u_i(t, x)dxdt = \int_0^\infty \int_{g(t)}^{h(t)} u_i(t, x)dxdt < \infty.
\]

Therefore, for \( i = 1, 2 \), the function

\[
U_i(x) := \int_0^\infty u_i(t, x)dt
\]

is finite for a.e. \( x \in (g_\infty, h_\infty) \). Since \( u_i(t, x) \geq 0 \) and \( u_{it} \in L^\infty(D_{g\_\infty}^\infty) \), it follows, in particular, that

\[
\lim_{t \to \infty} u_i(t, x) = 0 \text{ for almost every } x \in [g(0), h(0)]. \tag{3.5}
\]

Define, for \( i = 1, 2 \),

\[
M_i(t) := \max_{x \in [g(t), h(t)]} u_i(t, x).
\]

To complete the proof of (3.3), it suffices to show that

\[
\lim_{t \to \infty} M_i(t) = 0 \quad \text{for } i = 1, 2. \tag{3.6}
\]

To this end, we need to prove some useful properties of \( M_i(t) \) first. Clearly \( M_i(t) \) is continuous. Define

\[
X_i(t) := \{ x \in (g(t), h(t)) : u_i(t, x) = M_i(t) \}.
\]
Then $X_i(t)$ is a compact set for each $t > 0$. Therefore, there exist $\xi_i(t), \bar{\xi}_i(t) \in X_i(t)$ such that

$$u_{it}(t, \xi_i(t)) = \min_{x \in X_i(t)} u_{it}(t, x), \quad u_{it}(t, \bar{\xi}_i(t)) = \max_{x \in X_i(t)} u_{it}(t, x).$$

We claim that $M_i(t)$ satisfies, for each $t > 0$,

$$
\begin{cases}
M_i'(t + 0) := \lim_{s \to t, s > t} \frac{M_i(s) - M_i(t)}{s - t} = u_{it}(t, \bar{\xi}_i(t)), \\
M_i'(t - 0) := \lim_{s \to t, s < t} \frac{M_i(s) - M_i(t)}{s - t} = u_{it}(t, \xi_i(t)).
\end{cases} \tag{3.7}
$$

Indeed, for any fixed $t > 0$ and $s > t$, we have

$$u_i(s, \bar{\xi}_i(t)) - u_i(t, \bar{\xi}_i(t)) \leq M_i(s) - M_i(t) \leq u_i(s, \xi_i(s)) - u_i(t, \xi_i(s)).$$

It follows that

$$\liminf_{s \to t, s > t} \frac{M_i(s) - M_i(t)}{s - t} \geq u_{it}(t, \bar{\xi}_i(t)), \tag{3.8}$$

and

$$\limsup_{s \to t, s > t} \frac{M_i(s) - M_i(t)}{s - t} \leq \limsup_{s \to t, s > t} \frac{u_i(s, \xi_i(s)) - u_i(t, \xi_i(s))}{s - t}. \tag{3.9}$$

Let $s_n \searrow t$ satisfy

$$\lim_{n \to \infty} \frac{u_i(s_n, \bar{\xi}_i(s_n)) - u_i(t, \bar{\xi}_i(s_n))}{s_n - t} = \limsup_{s \to t, s > t} \frac{u_i(s, \xi_i(s)) - u_i(t, \xi_i(s))}{s - t}. \tag{3.10}$$

By passing to a subsequence if necessary, we may assume that $\bar{\xi}_i(s_n) \to \xi$ as $n \to \infty$. Then $u_i(t, \xi) = \lim_{n \to \infty} M_i(s_n) = M_i(t)$ and hence $\xi \in X_i(t)$. Due to the continuity of $u_{it}(t, x)$, it follows immediately that

$$\lim_{n \to \infty} \frac{u_i(s_n, \bar{\xi}_i(s_n)) - u_i(t, \bar{\xi}_i(s_n))}{s_n - t} = u_{it}(t, \xi) \leq u_{it}(t, \bar{\xi}_i(t)).$$

We thus obtain

$$\limsup_{s \to t, s > t} \frac{M_i(s) - M_i(t)}{s - t} \leq u_{it}(t, \bar{\xi}_i(t)).$$

Combining this with (3.8) we obtain

$$M_i'(t + 0) = u_{it}(t, \bar{\xi}_i(t)).$$

Analogously we can show

$$M_i'(t - 0) = u_{it}(t, \xi_i(t)).$$

Let us note from (3.7) that $M_i'(t - 0) \leq M_i'(t + 0)$ for all $t > 0$. Therefore if $M_i(t)$ has a local maximum at $t = t_0$, then $M_i'(t_0)$ exists and $M_i'(t_0) = 0$. Moreover, if $M_i(t)$ is monotone nondecreasing for all large $t$ and $\lim_{t \to \infty} M_i(t) = \sigma > 0$, then necessarily $M_i'(t_0) \to 0$ along some sequence $t_n \to \infty$; and if $M_i(t)$ is monotone nonincreasing for all large $t$ and $\lim_{t \to \infty} M_i(t) = \sigma > 0$, then necessarily $M_i'(t_n) \to 0$ along some sequence $s_n \to \infty$. These properties of $M_i(t)$ will be used below.

We are now ready to prove (3.6). We first consider the situation that $(f_1, f_2)$ satisfies (1.4). Arguing indirectly we assume that there exists $i \in \{1, 2\}$ such that
the desired identity above does not hold for \( M_i(t) \). For definiteness, we assume that \( i = 1 \). Then necessarily
\[
\sigma^* := \limsup_{t \to \infty} M_1(t) \in (0, \infty).
\]

By the above stated properties of \( M_i(t) \), there exists a sequence \( t_n > 0 \) increasing to \( \infty \) as \( n \to \infty \), and \( \xi_n \in \{ \xi_1(t_n), \xi_1(t_n) \} \) such that
\[
\lim_{n \to \infty} M_1(t_n) = \sigma^*, \quad \lim_{n \to \infty} u_{12}(t_n, \xi_n) = 0.
\]

By passing to a subsequence of \((t_n, \xi_n)\) if necessary, we may assume, without loss of generality,
\[
\lim_{n \to \infty} u_2(t_n, \xi_n) = \rho \in [0, \infty).
\]

Since
\[
\lim_{t \to \infty} \int_{g(t)}^{h(t)} u_1(t, x)dx = 0,
\]
and \( \sup_{x \in \mathbb{R}} J_1(x) < +\infty \), we have
\[
\lim_{t \to \infty} \int_{g(t)}^{h(t)} J_1(x - y)u_1(t, y)dy = 0 \quad \text{uniformly for } x \in \mathbb{R}.
\]

We now make use of the identity
\[
u_{12} = d_1 \int_{g(t)}^{h(t)} J_1(x - y)u_1(t, y)dy - d_1 u_1 + u_1(a_1 - b_1 u_1 - c_1 u_2)
\]
with \((t, x) = (t_n, \xi_n)\), and take \( n \to \infty \) to obtain
\[
0 = -d_1 \sigma^* + \sigma^*(a_1 - b_1 \sigma^* - c_1 \rho) < (a_1 - d_1)\sigma^*.
\]

It follows that \( a_1 > d_1 \). We show next that this leads to a contradiction.

Indeed, by (3.5), there exists \( x_0 \in (g(0), h(0)) \) such that
\[
\lim_{t \to \infty} u_i(t, x_0) = 0 \quad \text{for } i = 1, 2.
\]

Therefore we can find \( T > 0 \) large so that
\[
-d_1 + a_1 - b_1 u_1(t, x_0) - c_1 u_2(t, x_0) > (a_1 - d_1)/2 > 0 \quad \text{for } t \geq T.
\]

It then follows from the equation satisfied by \( u_1 \) that
\[
u_{12}(t, x_0) \geq \frac{a_1 - d_1}{2} u_1(t, x_0) \quad \text{for } t \geq T,
\]
which implies \( u_1(t, x_0) \to \infty \) as \( t \to \infty \), a contradiction to the boundedness of \( u_1 \).

This completes the proof of (3.3).

Next we consider the case that (1.5) is satisfied. The proof mainly follows the above argument for the competition case, though some small changes are needed. In this case we can similarly show that \( \lim_{t \to \infty} M_1(t) = 0 \). Hence
\[
\lim_{t \to \infty} \max_{x \in [g(t), h(t)]} u_1(t, x) = 0.
\]

With this at hand, we may now repeat the argument for the competition case to deduce that \( \lim_{t \to \infty} M_2(t) = 0 \). We have thus proved (3.3).
In the following we prove (3.4). Suppose on the contrary that \( \lambda_p(\mathcal{L}^d_{(g_\infty, h_\infty)} + a_1) > 0 \). Then \( \lambda_p(\mathcal{L}^d_{(g_\infty + \varepsilon, h_\infty - \varepsilon)} + a_1 - \varepsilon) > 0 \) for small \( \varepsilon > 0 \), say \( \varepsilon \in (0, \varepsilon_1) \). Due to (3.3), for such \( \varepsilon \), there exists \( T_\varepsilon > 0 \) such that

\[
c_1 u_\varepsilon(t, x) < \varepsilon, \quad \forall \ t \geq T_\varepsilon, \quad g(t) \leq x \leq h(t),
\]

and

\[
h(t) > h_\infty - \varepsilon, \quad g(t) < g_\infty + \varepsilon, \quad \forall \ t \geq T_\varepsilon.
\]

Consider the auxiliary problem

\[
\begin{aligned}
w_t &= d_1 \int_{g_\infty + \varepsilon}^{h_\infty - \varepsilon} J_1(x - y)w(t, y)dy - d_1 w + f_\varepsilon(w), \\
w(T_\varepsilon, x) &= u_1(T_\varepsilon, x), \quad x \in [g_\infty + \varepsilon, h_\infty - \varepsilon],
\end{aligned}
\]

(3.9)

where \( f_\varepsilon(w) = w(a_1 - \varepsilon - b_1 w) \). Since \( \lambda_p(\mathcal{L}^d_{(g_\infty + \varepsilon, h_\infty - \varepsilon)} + f_\varepsilon'(0)) > 0 \), it is well known (see [4, 8]) that the solution \( w_\varepsilon(t, x) \) of (3.9) converges to the unique positive steady state \( W_\varepsilon(x) \) of (3.9) uniformly in \( [g_\infty + \varepsilon, h_\infty - \varepsilon] \) as \( t \to \infty \). Moreover, a simple comparison argument yields

\[
u_1(t, x) \geq w_\varepsilon(t, x), \quad \forall \ t > T_\varepsilon, \quad x \in [g_\infty + \varepsilon, h_\infty - \varepsilon].
\]

Thus, there exists \( T_{1\varepsilon} > T_\varepsilon \) such that

\[
u_1(t, x) \geq \frac{1}{2} W_\varepsilon(x) > 0, \quad \forall \ t > T_{1\varepsilon}, \quad x \in [g_\infty + \varepsilon, h_\infty - \varepsilon].
\]

Clearly this is a contradiction to (3.3). The proof is complete.

**Corollary 3.4.** Suppose that \( J_1, J_2 \) and \( (f_1, f_2) \) satisfy the conditions in Theorem 3.3, and \( (u_1, u_2, g, h) \) is the unique solution of (1.2). If \( a_1 \geq d_1 \) or \( a_2 \geq d_2 \), then necessarily \( h_\infty - g_\infty = \infty \).

**Proof.** Arguing indirectly we assume that \( h_\infty - g_\infty < \infty \) and \( a_i \geq d_i \) for some \( i \in \{1, 2\} \). Thanks to [7, Proposition 3.4],

\[
\lambda_p \left( \mathcal{L}^d_{(g_\infty, h_\infty)} + a_i \right) > 0.
\]

This is a contradiction to (3.4).

We next consider the case that

\[
a_i < d_i \quad \text{for} \quad i = 1, 2.
\]

(3.10)

In this case, in view of [7, Proposition 3.4], \( \lambda_p(\mathcal{L}^d_{(0, \ell)} + a_i) < 0 \) if \( 0 < \ell \ll 1 \), and \( \lambda_p(\mathcal{L}^d_{(0, \ell)} + a_i) > 0 \) if \( \ell \gg 1 \), and there exist two positive constants \( \ell_1 \) and \( \ell_2 \) such that

\[
\lambda_p \left( \mathcal{L}^d_{(0, \ell)} + a_i \right) = 0, \quad i = 1, 2.
\]

Define

\[
\ell_* = \min\{\ell_1, \ell_2\}.
\]

We have the following result.
Assume that \( J_i(x) > 0 \) in \( \mathbb{R} \) for \( i = 1, 2 \), \((f_1, f_2)\) satisfies either (1.4) or (1.5), and (3.10) holds. Let \((u_1, u_2, g, h)\) be the unique solution of (1.2). Then the following conclusions hold:

(i) If \( h_\infty - g_\infty < \infty \), then \( h_\infty - g_\infty \leq \ell_* \).

(ii) If \( h_0 \geq \ell_*/2 \), then \( h_\infty - g_\infty = \infty \).

(iii) If \( h_0 < \ell_*/2 \), then there exist two positive numbers \( \Lambda^* \geq \Lambda_* > 0 \) such that \( h_\infty - g_\infty < \infty \) when \( 0 < \mu_1 + \mu_2 \leq \Lambda_* \) and \( h_\infty - g_\infty = \infty \) when \( \mu_1 + \mu_2 > \Lambda^* \).

It is easily seen that conclusions (i) and (ii) in Theorem 3.5 follow directly from the definition of \( \ell_* \) and (3.4). We prove (iii) by several lemmas.

**Lemma 3.6.** Under the assumptions of Theorem 3.5, there exists a positive number \( \Lambda_0 \), depending only on \( h_0, d_i, J_i, a_i \) and \( u_{i0}, i = 1, 2 \), such that \( h_\infty - g_\infty < \infty \) for any \( \mu_1, \mu_2 \) satisfying \( 0 < \mu_1 + \mu_2 < \Lambda_0 \).

We need some comparison results to prove this lemma. The proof of the following Lemmas 3.7 and 3.8 can be carried out by a combination of the proofs of [7, Theorem 3.1], [14, Lemma 5.1] and [27, Lemma 4.1]. Since the adaptation is rather straightforward, we omit the details here.

**Lemma 3.7.** For \( T \in (0, \infty) \), suppose that \( \bar{h}, \bar{g} \in C([0, T], \bar{u}_1, \bar{u}_2 \in C([0 \leq t \leq T, \bar{g}(t) \leq x \leq \bar{h}(t)]) \) and satisfy

\[
\begin{align*}
&\bar{u}_{it} \geq d_t \int_{\bar{g}(t)}^{\bar{h}(t)} J_1(x-y)\bar{u}_1(t,y)dy - d_t \bar{u}_1(a_1 - b_1 \bar{u}_1), \quad 0 < t \leq T, \quad \bar{g}(t) < x < \bar{h}(t), \\
&\bar{u}_{2t} \geq d_2 \int_{\bar{g}(t)}^{\bar{h}(t)} J_2(x-y)\bar{u}_2(t,y)dy - d_2 \bar{u}_2(a_2 - b_2 \bar{u}_2), \quad 0 < t \leq T, \quad \bar{g}(t) < x < \bar{h}(t), \\
&\bar{u}_i(t, \bar{g}(t)) \geq 0, \quad \bar{u}_i(t, \bar{h}(t)) \geq 0, \quad i = 1, 2, \quad 0 < t \leq T, \\
&\bar{h}'(t) \geq \sum_{i=1}^{\bar{h}(t)} J_1(x-y)\bar{u}_i(t, x)dydx, \quad 0 \leq t \leq T, \\
&\bar{g}'(t) \leq -\sum_{i=1}^{\bar{g}(t)} J_2(x-y)\bar{u}_i(t, x)dydx, \quad 0 \leq t \leq T, \\
&\bar{u}_i(0, x) \geq u_{i0}(x), \quad i = 1, 2, \quad \bar{h}(0) \geq h_0, \quad \bar{g}(0) \leq -h_0, \quad |x| \leq h_0.
\end{align*}
\]

Let \((u_1, u_2, g, h)\) be the unique solution of (1.2) with \((f_1, f_2)\) satisfying (1.4). Then \( u_1 \leq \bar{u}_1, \quad u_2 \leq \bar{u}_2, \quad g \geq \bar{g}, \quad h \leq \bar{h} \) in \( D_{p,b}^T \).

**Lemma 3.8.** In Lemma 3.7, if we replace the second inequality in (3.11) by

\[
\bar{u}_{2t} \geq d_2 \int_{\bar{g}(t)}^{\bar{h}(t)} J_2(x-y)\bar{u}_2(t,y)dy - d_2 \bar{u}_2(a_2 - b_2 \bar{u}_2 + c_2 \bar{u}_1)
\]

for \( 0 < t \leq T, \quad \bar{g}(t) < x < \bar{h}(t) \) and let \((u_1, u_2, g, h)\) be the unique solution of (1.2) with \((f_1, f_2)\) satisfying (1.5), then the conclusion still holds true.

**Proof of Lemma 3.6.** The idea of this proof comes from [7, Theorem 3.12], [27, Lemma 5.2] and [32, Lemma 4.4].

Since \( \lambda_p(\mathcal{L}^{d_i}_{(-h_0, h_0)} + a_i) < 0, \quad i = 1, 2 \), we can choose \( h_1 > h_0 \) such that

\[
\lambda_i := \lambda_p(\mathcal{L}^{d_i}_{(-h_1, h_1)} + a_i) < 0, \quad i = 1, 2.
\]
Case 1: The competition case. Suppose that \((f_1, f_2)\) satisfies \((1.4)\). Let \(w_i(t, x)\) be the unique solution of

\[
\begin{align*}
    w_{it} &= d_i \int_{-h_1}^{h_1} J_i(x-y)w_i(t, y)dy - d_iw_i + a_iw_i, \quad t > 0, \ |x| \leq h_1, \\
    w_i(0, x) &= w_{i0}(x), \quad |x| \leq h_0, \\
    w_i(0, x) &= 0, \quad |x| > h_0.
\end{align*}
\]

(3.12)

Let \(\varphi_i > 0\) be the corresponding normalized eigenfunction of \(\lambda_i\), namely \(\|\varphi_i\|_\infty = 1\) and

\[
\left(\mathcal{L}_{(-h_1, h_1)}^d + a_i\right)[\varphi_i](x) = \lambda_i \varphi_i(x), \quad \forall \ |x| \leq h_1.
\]

For \(C > 0\) and \(z_i(t, x) = Ce^{\lambda_i t/2} \varphi_i(x)\), it is easy to check that

\[
\begin{align*}
    d_i \int_{-h_1}^{h_1} J_i(x-y)z_i(t, y)dy - d_i z_i + a_i z_i - z_{it} \\
    &= Ce^{\lambda_i t/2} \left( d_i \int_{-h_1}^{h_1} J_i(x-y) \varphi_i(y)dy - d_i \varphi_i + a_i \varphi_i - \frac{\lambda_i}{2} \varphi_i \right) \\
    &= \frac{\lambda_i}{2} Ce^{\lambda_i t/2} \varphi_i(x) < 0, \quad \forall \ t > 0, \ |x| \leq h_1, \ i = 1, 2.
\end{align*}
\]

Choose \(C > 0\) large such that \(C \varphi_i > w_{i0}\) on \([-h_1, h_1]\). Then we can apply [7, Lemma 3.3] to \(w_i - z_i\) to conclude that

\[
w_i(t, x) \leq z_i(t, x) = Ce^{\lambda_i t/2} \varphi_i(x) \leq Ce^{\lambda_i t/2}, \quad \forall \ t > 0, \ |x| \leq h_1. \quad (3.13)
\]

Set

\[
\lambda = \max\{\lambda_1, \lambda_2\}, \quad r(t) = h_0 + 2(\mu_1 + \mu_2)Ch_1 \int_0^t e^{\lambda s/2} ds, \quad \eta(t) = -r(t), \quad t \geq 0.
\]

Then \(\lambda < 0\). We claim that \((w_1, w_2, \eta, r)\) is an upper solution of \((1.2)\) with \((f_1, f_2)\) satisfying \((1.4)\). Firstly, we compute for \(t > 0\),

\[
r(t) = h_0 - 2(\mu_1 + \mu_2)Ch_1 \frac{2}{\lambda} \left(1 - e^{\lambda t/2}\right) < h_0 - 2(\mu_1 + \mu_2)Ch_1 \frac{2}{\lambda} \leq h_1
\]

provided that

\[
0 < \mu_1 + \mu_2 \leq \Lambda_0 := \frac{-\lambda(h_1 - h_0)}{4Ch_1}.
\]

Similarly, for such \(\mu_1\) and \(\mu_2\), we have \(\eta(t) > -h_1\) for any \(t > 0\). Thus \((3.12)\) gives

\[
w_{it} \geq d_i \int_{\eta(t)}^{r(t)} J_i(x-y)w_i(t, y)dy - d_iw_i + w_i(a_i - b_i w_i), \quad t > 0, \ x \in [\eta(t), r(t)].
\]

Secondly, due to \((3.13)\), it is easy to check that

\[
\int_{\eta(t)}^{r(t)} \int_r^\infty J_i(x-y)w_i(t, x)dydx \leq 2Ch_1 e^{\lambda_i t/2} \leq 2Ch_1 e^{\lambda t/2}.
\]

Thus

\[
r'(t) = 2(\mu_1 + \mu_2)Ch_1 e^{\lambda t/2} \geq \sum_{i=1}^{2} \mu_i \int_{\eta(t)}^{r(t)} \int_r^\infty J_i(x-y)w_i(t, x)dydx.
\]
Similarly, one has
\[ \eta'(t) \leq -\sum_{i=1}^{2} \mu_i \int_{\eta(t)}^{r(t)} \int_{-\infty}^{\eta(t)} J_i(x-y)w_i(t,x)dydx. \] (3.14)

The above arguments show that \((w_1, w_2, \eta, r)\) is an upper solution of (1.2) with \((f_1, f_2)\) satisfying (1.4). By Lemma 3.7 we get
\[ u_1(t, x) \leq w_1(t, x), \quad u_2(t, x) \leq w_2(t, x), \quad g(t) \geq \eta(t), \quad h(t) \leq r(t) \]
for all \(t \geq 0, g(t) \leq x \leq h(t)\). Therefore
\[ h_{\infty} - g_{\infty} \leq \lim_{t \to \infty} [r(t) - \eta(t)] \leq 2h_1 < \infty. \]

Case 2: The prey-predator case. Suppose that \((f_1, f_2)\) satisfies (1.5). Inspired by [27, Lemma 5.2], let \(w_1(t, x)\) and \(w_2(t, x)\) be the unique solution of
\[
\begin{cases}
    w_{1t} = d_1 \int_{-h_1}^{h_1} J_1(x-y)w_1(t,y)dy - d_1 w_1 + a_1 w_1, & t > 0, \ |x| \leq h_1, \\
    w_1(0, x) = u_{10}(x), & |x| \leq h_0, \\
    w_1(0, x) = 0, & |x| > h_0,
\end{cases}
\]
and
\[
\begin{cases}
    w_{2t} = d_2 \int_{-h_1}^{h_1} J_2(x-y)w_2(t,y)dy - d_2 w_2 \\
    + w_2(a_2 - b_2 w_2 + c_2 w_1), & t > 0, \ |x| \leq h_1, \\
    w_2(0, x) = u_{20}(x), & |x| \leq h_0, \\
    w_2(0, x) = 0, & |x| > h_0,
\end{cases}
\]
respectively. Take \(\lambda_i\) and \(\varphi_i\) as above. Then there exists \(0 < \sigma \leq 1\) such that
\[ \sigma c_2 \varphi_1(x) \leq b_2 \varphi_2(x), \quad \forall \ |x| \leq h_1. \] (3.15)

Choose \(C > 0\) large such that
\[ \sigma C \varphi_1(x) > u_{10}(x), \quad C \varphi_2(x) > u_{20}(x), \quad \forall \ |x| \leq h_1. \]
Set \(z_1(t, x) = \sigma Ce^{\lambda_1 t/2} \varphi_1(x), \ \lambda = \max\{\lambda_1, \lambda_2\}\) and \(z_2(t, x) = Ce^{\lambda_2 t/2} \varphi_2(x)\). Similar to the above,
\[
\begin{align*}
    d_1 \int_{-h_1}^{h_1} J_1(x-y)z_1(t,y)dy - d_1 z_1 + a_1 z_1 - z_{1t} \\
    = \sigma Ce^{\lambda_1 t/2} \left( d_1 \int_{-h_1}^{h_1} J_1(x-y)\varphi_1(y)dy - d_1 \varphi_1 + a_1 \varphi_1 - \frac{\lambda_1}{2} \varphi_1 \right) \\
    = \frac{\lambda_1}{2} \sigma Ce^{\lambda_1 t/2} \varphi_1(x) < 0.
\end{align*}
\]

Now we consider \(z_2(t, x)\). Using \(\lambda = \max\{\lambda_1, \lambda_2\} < 0\) and (3.15), we obtain
\[
\begin{align*}
    d_2 \int_{-h_1}^{h_1} J_2(x-y)z_2(t,y)dy - d_2 z_2 + z_2(a_2 - b_2 z_2 + c_2 z_1) - z_{2t} \\
    = Ce^{\lambda_2 t/2} \left( d_2 \int_{-h_1}^{h_1} J_2(x-y)\varphi_2(y)dy - d_2 \varphi_2 + a_2 \varphi_2 - \frac{\lambda}{2} \varphi_2 \right) \\
    + C^2 e^{\lambda_2 t/2} \varphi_2 \left( \sigma c_2 e^{\lambda_1 t/2} \varphi_1 - b_2 e^{\lambda_2 t/2} \varphi_2 \right)
\end{align*}
\]
\[ = Ce^{\lambda t/2} \left( \frac{\lambda}{2} \phi_2 - \frac{\lambda}{2} \phi_2 \right) + C^2 e^{\lambda t/2} \left( \sigma c_2 e^{(\lambda_1 - \lambda)t/2} \phi_1 - b_2 \phi_2 \right) \]
\[ \leq \frac{\lambda}{2} Ce^{\lambda t/2} \phi_2(x) < 0. \]

Similar to the above, applying [7, Lemma 3.3] to \( w_i - z_i \) we have
\[
\begin{cases}
\forall t > 0, |x| 
\end{cases}
\]

\[ w_i(t, x) \leq z_i(t, x) = \sigma C e^{\lambda t/2} \phi_2(x) \leq \sigma C e^{\lambda t/2}, \quad \forall t > 0, |x| \leq h_1, \]
\[ w_2(t, x) \leq z_2(t, x) = C e^{\lambda t/2} \phi_2(x) \leq C e^{\lambda t/2}, \quad \forall t > 0, |x| \leq h_1. \]

Set
\[ r(t) = h_0 + 2(\sigma \mu_1 + \mu_2) Ch_1 \int_0^t e^{\lambda s/2} ds, \quad \eta(t) = -r(t), \quad t \geq 0. \]

For \( t > 0 \), we have
\[ r(t) = h_0 - 2(\sigma \mu_1 + \mu_2) Ch_1 \frac{2}{\lambda} \left( 1 - e^{\lambda t/2} \right) < h_0 - 2(\sigma \mu_1 + \mu_2) Ch_1 \frac{2}{\lambda} \leq h_1 \]
provided that
\[ 0 < \sigma \mu_1 + \mu_2 \leq \Lambda_0 := \left( \frac{-\lambda(h_1 - h_0)}{4Ch_1} \right). \]

Similarly, for such \( \mu_1 \) and \( \mu_2, \eta(t) > -h_1 \) for any \( t > 0 \). Thus we have
\[ w_1 \geq d_1 \int_{\eta(t)}^{r(t)} J_1(x - y) w_1(t, y) dy - d_1 w_1 + a_1 w_1, \]
\[ w_2 \geq d_2 \int_{-h_1}^{h_1} J_2(x - y) w_2(t, y) dy - d_2 w_2 + w_2(a_2 - b_2 w_2 + c_2 w_1) \]
for \( t > 0 \) and \( x \in [\eta(t), r(t)] \). On the other hand, due to (3.16), it is easy to check that
\[ \int_{\eta(t)}^{r(t)} \int_{r(t)}^{\infty} J_1(x - y) w_1(t, x) dy dx \leq 2\sigma Ch_1 e^{\lambda t/2}, \]
\[ \int_{\eta(t)}^{r(t)} \int_{r(t)}^{\infty} J_2(x - y) w_2(t, x) dy dx \leq 2Ch_1 e^{\lambda t/2}. \]

Thus
\[ r'(t) = 2(\sigma \mu_1 + \mu_2) Ch_1 e^{\lambda t/2} \geq \sum_{i=1}^2 \mu_i \int_{\eta(t)}^{r(t)} \int_{r(t)}^{\infty} J_i(x - y) w_i(t, x) dy dx. \]

Similarly, \( \eta(t) \) satisfies (3.14). We may now apply Lemma 3.8 to conclude that
\[ h_\infty - g_\infty \leq \lim_{t \to \infty} [r(t) - \eta(t)] \leq 2h_1 < \infty \]
when \( \sigma \mu_1 + \mu_2 \leq \Lambda_0 \). As \( \mu_1 + \mu_2 \leq \Lambda_0 \) implies \( \sigma \mu_1 + \mu_2 \leq \Lambda_0 \), the desired result is proved.

To complete the proof of Theorem 3.5, it remains to show that if \( \mu_1 + \mu_2 \) is large, then \( h_\infty - g_\infty = \infty \). We need the following lemma.

**Lemma 3.9.** Let (J) hold for the kernel function \( J \), and \( C > 0 \) be a constant. For any given constants \( s_0, H > 0 \), and any function \( w_0 \in C([0, s_0]) \) satisfying \( w_0(\pm s_0) = 0 \) and \( w_0 > 0 \) in \( (-s_0, s_0) \), there exists \( \mu^0 > 0 \), depending on \( J(x), d, C, w_0(x) \) and \( s_0 \), such that if \( \mu \geq \mu^0 \) and \( (w, s, c) \) satisfies
\[
\begin{aligned}
&\begin{cases}
w_t \geq d \int_{c(t)}^{s(t)} J(x-y)w(t,y)dy - dw - Cw, & t > 0, \ c(t) < x < s(t), \\
w(t,c(t)) = w(t,s(t)) = 0, & t > 0, \\
s'(t) \geq \mu \int_{c(t)}^{s(t)} \int_{s(t)}^{\infty} J(x-y)w(t,x)dydx, & t > 0, \\
c'(t) \leq -\mu \int_{c(t)}^{s(t)} \int_{-\infty}^{c(t)} J(x-y)w(t,x)dydx, & t > 0, \\
w(0,x) = w_0(x), & s(0) = -c(0) = s_0, |x| \leq s_0,
\end{cases}
\end{aligned}
\]
then \( \liminf_{t \to \infty} [s(t) - c(t)] > H. \)

**Proof.** We adapt the approach of [35, Lemma 3.2]. Firstly, the comparison principle gives
\[w(t, x) > 0, \quad \forall \ t > 0, \ c(t) < x < s(t).\]
It then follows that \( s'(t) > 0, \ c'(t) < 0 \) for \( t > 0. \)

Take a function \( b(t) \in C^1([0,1]) \) satisfying \( b(t) > 0 \) in \([0,1], b(0) = s_0 \) and \( b(1) = H, \) and set \( a(t) = -b(t) \). Consider the following problem
\[
\begin{aligned}
&\begin{cases}
z_t = d \int_{b(t)}^{b(t)} J(x-y)z(t,y)dy - dz - Cz, & 0 < t < 1, \ a(t) < x < b(t), \\
z(t,b(t)) = z(t,a(t)) = 0, & 0 < t < 1, \\
z(0,x) = w_0(x), & |x| \leq s_0.
\end{cases}
\end{aligned}
\]
In view of [7, Lemma 2.3], this problem has a unique solution \( z \) which is continuous on \([0 \leq t \leq 1, a(t) \leq x \leq b(t)]\) and satisfies \( z(t,x) > 0 \) for all \( t > 0 \) and \( a(t) < x < b(t) \). Thus the functions
\[
\begin{aligned}
r(t) := \int_{a(t)}^{b(t)} \int_{b(t)}^{\infty} J(x-y)z(t,x)dydx, \quad l(t) := \int_{a(t)}^{b(t)} \int_{-\infty}^{a(t)} J(x-y)z(t,x)dydx
\end{aligned}
\]
are positive and continuous on \([0,1], \) and so \( r(t), l(t) \geq \sigma > 0 \) on \([0,1] \) for some constant \( \sigma. \) Since \( a'(t) \) and \( b'(t) \) are bounded on \([0,1], \) we can find \( \mu^0 > 0 \) such that when \( \mu \geq \mu^0, \) there hold:
\[
\begin{aligned}
b'(t) \leq \mu r(t) - \mu \int_{a(t)}^{b(t)} \int_{b(t)}^{\infty} J(x-y)z(t,x)dydx, \\
a'(t) \geq -\mu l(t) - \mu \int_{a(t)}^{b(t)} \int_{-\infty}^{a(t)} J(x-y)z(t,x)dydx
\end{aligned}
\]
for all \( 0 \leq t \leq 1. \) Applying the comparison principle we get \( c(t) \leq a(t) \) and \( s(t) \geq b(t) \) for all \( 0 \leq t \leq 1, \) and so \( s(1) - c(1) \geq b(1) - a(1) = 2H \) when \( \mu \geq \mu^0. \) The desired conclusion now follows directly and the proof is complete. \( \square \)

**Completion of the proof of Theorem 3.5.** Since \( u_1 \) and \( u_2 \) are bounded, there exists \( C > 0 \) such that \( a_1 - b_1 u_1(t,x) - c_1 u_2(t,x) > -C. \) We thus have
see that, for any small $v$, we first consider the weak competition case. By a simple comparison with Proposition 3.10. the weak predation case as described in Theorem 1.4. simplicity, we only consider two situations, namely the weak competition case and $h$-Long-time behaviour in the case of spreading.

3.2. Long-time behaviour in the case of spreading. In this subsection, we examine the long-time behaviour of the solution to (1.2) when $h_\infty - g_\infty = \infty$. For simplicity, we only consider two situations, namely the weak competition case and the weak predation case as described in Theorem 1.4.

**Proposition 3.10.** Let $(u_1, u_2, g, h)$ be the unique solution of (1.2) with $h_\infty - g_\infty = \infty$. Then $h_\infty = -g_\infty = \infty$ in the case of weak competition and weak predation.

**Proof.** We first consider the weak competition case. By a simple comparison argument involving the ODE problem $v' = v(a_1 - b_1 v)$, $v(0) = \|u_{10}\|_\infty$, we easily see that, for any small $\epsilon > 0$, there exists $T = T_\epsilon > 0$ such that $u_1(t, x) \leq a_1/b_1 + \epsilon$ for $t \geq T$ and $x \in [g(t), h(t)]$. Since $b_1/c_2 > a_1/a_2$, we may assume $\bar{a}_2 := a_2 - c_2(a_1/b_1 + \epsilon) > 0$. Thus $(u_2, g, h)$ satisfies
\[
\begin{align*}
    u_{2t} &\geq d_2 \int_{g(t)}^{h(t)} J_2(x-y)u_2(t,y)dy - d_2u_2 + u_2(\bar{a}_2 - b_2u_2), \quad t \geq T, \quad g(t) < x < h(t), \\
    u_2(t, g(t)) = u_2(t, h(t)) = 0, \quad t \geq T, \\
    h'(t) &\geq \mu_2 \int_{g(t)}^{h(t)} \int_{h(t)}^{\infty} J_2(x-y)u_2(t,x)dydx, \quad t \geq T, \\
    g'(t) &\leq -\mu_2 \int_{g(t)}^{h(t)} \int_{-\infty}^{g(t)} J_2(x-y)u_2(t,x)dydx, \quad t \geq T.
\end{align*}
\]

Consider the following auxiliary problem
\[
\begin{align*}
    w_t &= d_2 \int_{\eta(t)}^{r(t)} J_2(x-y)w(t,y)dy - d_2w + w(\bar{a}_2 - b_2w), \quad t > T, \quad \eta(t) < x < r(t), \\
    w(t, \eta(t)) = w(t, r(t)) = 0, \quad t \geq T, \\
    r'(t) &= \mu_2 \int_{\eta(t)}^{r(t)} \int_{r(t)}^{\infty} J_2(x-y)w(t,x)dydx, \quad t \geq T, \\
    \eta'(t) &= -\mu_2 \int_{\eta(t)}^{r(t)} \int_{-\infty}^{\eta(t)} J_2(x-y)w(t,x)dydx, \quad t \geq T, \\
    \eta(T) &= g(T), \quad r(T) = h(T), \quad w(T, x) = u_2(T, x), \quad \eta(T) \leq x \leq r(T).
\end{align*}
\]
By the comparison principle, the unique solution \((w, \eta, r)\) of (3.18) satisfies
\[
w(t, x) \leq u_2(t, x), \quad \eta(t) \geq g(t), \quad r(t) \leq h(t).
\]

In view of [7, Theorem 1.3], if \(d_2 \leq \tilde{d}_2\), then spreading happens to (3.18) and hence \(\eta(t) \to -\infty, r(t) \to \infty\) as \(t \to \infty\), which imply that \(g_{\infty} = -\infty, h_{\infty} = \infty\).

If \(d_2 > \tilde{d}_2\), then [7, Theorem 1.3] infers the existence of a unique \(\ell_2\) such that spreading happens to (3.18) provided \(h(T) - g(T) \geq \ell_2\). The latter is guaranteed to happen if \(T\) is large enough, since \(h_{\infty} - g_{\infty} = \infty\). Therefore in either case, we must have \(g_{\infty} = -\infty, h_{\infty} = \infty\).

We now consider the predator-prey case. This time (3.17) holds for any \(T > 0\) with \(\tilde{a}_2\) replaced by \(a_2\). Hence the same argument shows that \(g_{\infty} = -\infty, h_{\infty} = \infty\). The proof is complete.

\[\square\]

**Remark 3.11.** The assumptions in Proposition 3.10 can be relaxed. From the above proof, it is easily seen that the conclusion holds whenever \((f_1, f_2)\) satisfies (1.5), and in the competition case (1.4), the conclusion holds if either \(b_1/c_2 > a_1/a_2\) or \(a_1/a_2 > c_1/b_2\).

The remaining part of this paper is devoted to the proof of Theorem 1.4. We start with several preparatory results. Consider the auxiliary problem

\[
\begin{aligned}
&u_t = d \int_{\mathbb{R}} J(x - y) u(t, y)dy - du + u(a(x) - bu), & t > 0, \ x \in \mathbb{R}, \\
u(0, x) = u_0(x), & x \in \mathbb{R},
\end{aligned}
\tag{3.19}
\]

where \(a \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})\), \(d\) and \(b\) are positive constants, and \(J\) satisfies (J).

**Proposition 3.12.** Let \(a, b, d\) and \(J\) be as given above. Then the following conclusions hold:

(i) For any bounded interval \(\Omega\), the principal eigenvalue \(\lambda_p(L^d_{\Omega} + a)\) is strictly increasing in \(a\).

(ii) If \(\lambda_p(L^d_{\Omega} + a) := \lim_{l \to \infty} \lambda_p(L^d_{[-l,l]} + a) > 0\), then problem (3.19) admits a unique positive steady state \(U(x)\). Moreover, for any non-negative initial function \(u_0 \in C(\mathbb{R}) \cap L^\infty(\mathbb{R}), u_0 \neq 0\), the unique solution of (3.19) satisfies

\[
\lim_{t \to \infty} u(t, x) = U(x) \text{ locally uniformly in } \mathbb{R}.
\]

**Proof.** Conclusion (i) follows from [8, Proposition 1.1 (ii)]. Conclusion (ii) can be obtained by similar arguments as in [6, Section 4]. \(\square\)

Next we consider the auxiliary problem

\[
\begin{aligned}
&u_t = d \int_{-l}^l J(x - y) u(t, y)dy - du + u(a_l - bu), & t > 0, \ x \in [-l, l], \\
u(0, x) = u_0(x), & x \in [-l, l],
\end{aligned}
\tag{3.20}
\]

where \(a_l, d, b, l\) are positive constants, \(J\) satisfies (J), and \(a_l\) satisfies

\[
\lim_{l \to \infty} a_l = a > 0.
\]

**Lemma 3.13.** Under the above assumptions, there exists \(L > 0\) large such that for any \(l > L\) and any \(u_0 \in C([-l, l])\) satisfying \(u_0 \geq, \neq 0\), the following conclusions hold:

(i) The unique solution of (3.20) satisfies

\[
\lim_{t \to \infty} u_l(t, x) = u_l(x) \text{ uniformly for } x \in [-l, l],
\]
where \( u_l(x) \) is the unique positive solution of
\[
d \int_{-l}^{l} J(x-y)u(y)dy - du + u(a_l - bu) = 0, \quad x \in [-l, l];
\]
(ii)
\[
\lim_{l \to \infty} u_l(x) = a/b \text{ locally uniformly in } x \in \mathbb{R}.
\]

**Proof.** Given any \( \varepsilon > 0 \) small, we can find \( L_0 > 0 \) such that
\[
0 < a - \varepsilon < a_l < a + \varepsilon \quad \text{for } l > L_0.
\]
It follows that, for such \( l \),
\[
\lambda_p \left( \mathcal{L}_d^{l,-l} + a_l \right) > \lambda_p \left( \mathcal{L}_d^{l,-l} + a + \varepsilon \right) \to a - \varepsilon > 0 \text{ as } l \to \infty.
\]
Therefore there exists \( L \geq L_0 \) such that
\[
\lambda_p \left( \mathcal{L}_d^{l,-l} + a_l \right) > 0 \quad \text{for } l > L,
\]
and by [4, 8] and Proposition 3.4 of [7] we can conclude that for \( l > L \), (i) holds.

To show (ii), we note that for \( l > L \), \( u_l(t,x) \) is a super-solution to (3.20) with \( a_l \) replaced by \( a - \varepsilon \), whose unique solution we denote by \( u_{l,\varepsilon}(t,x) \). Hence \( u_l(t,x) \geq u_{l,\varepsilon}(t,x) \). By [4, 8] again, we see that as \( t \to \infty \), \( u_{l,\varepsilon}(t,x) \to u_{l,\varepsilon}(x) \) uniformly in \([-l,l] \), with \( u_{l,\varepsilon}(x) \) the unique steady state of the problem. We thus obtain \( u_l(x) \geq u_{l,\varepsilon}(x) \). According to [7, Proposition 3.6], we have
\[
\lim_{l \to \infty} u_{l,\varepsilon}(x) = (a - \varepsilon)/b \text{ locally uniformly in } \mathbb{R}.
\]
It follows that \( \liminf_{l \to \infty} u_l(x) \geq (a - \varepsilon)/b \) locally uniformly in \( \mathbb{R} \). The arbitrariness of \( \varepsilon \) then infers that
\[
\liminf_{l \to \infty} u_l(x) \geq a/b \text{ locally uniformly in } \mathbb{R}.
\]

Analogously we can show
\[
\limsup_{l \to \infty} u_l(x) \leq a/b \text{ locally uniformly in } \mathbb{R}.
\]
Therefore (ii) holds.

When spreading happens, in the local diffusion case, to study the long-time behavior of diffusive population systems with free boundaries, a key tool is an iteration method, which has been widely used in, for example, [14, 30, 35, 36, 37]. To adapt this method to the nonlocal diffusion case here, we rely on the following technical lemma.

**Lemma 3.14.** Let \( g(t) < h(t) \) be two continuous functions satisfying
\[
\lim_{t \to \infty} g(t) = -\infty, \quad \lim_{t \to \infty} h(t) = \infty.
\]
Let \( K_0 \) be a positive constant, \( w \) be a continuous function satisfying \( |w(t,x)| \leq K_0 \) for \( t > 0, \ x \in [g(t), h(t)] \). Suppose that \( u \) satisfies
\[
\begin{cases}
u_t = d \int_{g(t)}^{h(t)} J(x-y)u(t,y)dy - du + u(a - bu - w(t,x)), & t > 0, \ x \in (g(t), h(t)), \\
u(t, g(t)) = u(t, h(t)) = 0, & t > 0,
\end{cases}
\]
\[
u(0, x) = u_0(x), \ -g(0) = h(0) = h_0, & x \in (-h_0, h_0),
\]
where \(a, b, d, h_0\) are positive constants, \(J\) satisfies (J), \(u_0 \in C([-h_0, h_0])\) is nonnegative and not identically 0. Then the following statements hold:

(i) If for some constant \(m \in [-K_0, K_0]\),

\[
\liminf_{t \to \infty} w(t, x) \geq m \quad \text{locally uniformly in } \mathbb{R},
\]

then

\[
\limsup_{t \to \infty} u(t, x) \leq [a - m]_+ / b \quad \text{locally uniformly in } \mathbb{R},
\]

where \([ \cdot ]_+\) is defined by \([\theta]_+ = \max\{\theta, 0\}\).

(ii) If \(a > M\) and

\[
\limsup_{t \to \infty} w(t, x) \leq M \quad \text{locally uniformly in } \mathbb{R},
\]

for some constant \(M\), then

\[
\liminf_{t \to \infty} u(t, x) \geq (a - M)/b \quad \text{locally uniformly in } \mathbb{R}.
\]

Proof: (i) For any integer \(n \geq 1\), it follows from (3.21) that there exist a sequence \(T_n\) increasing to \(\infty\) as \(n \to \infty\) such that

\[
w(t, x) \geq m - 1/n \quad \text{for } t \geq T_n \text{ and } x \in [-n - 1, n + 1].
\]

For any given small \(\varepsilon > 0\), define

\[
\sigma_n = \begin{cases} 
  a - m + 1/n, & a - m > 0, \\
  \varepsilon + 1/n, & a - m \leq 0,
\end{cases}
\]

and

\[
a_n(x) = \begin{cases} 
  \sigma_n, & |x| < n, \\
  \sigma_n + 2(a + K_0 + 1 - \sigma_n)(|x| - n), & n \leq |x| \leq n + 1/2, \\
  a + K_0 + 1, & |x| > n + 1/2.
\end{cases}
\]

Clearly \(a_n \in C(\mathbb{R})\), \(a - w(t, x) \leq a_n(x)\) for \(t > T_n\) and \(x \in \mathbb{R}\), \(a_n(x)\) is nonincreasing in \(n\) and

\[
\lim_{n \to \infty} a_n(x) = \sigma_\infty := \begin{cases} 
  a - m, & a - m > 0, \\
  \varepsilon, & a - m \leq 0.
\end{cases}
\]

Let \(K := \max\{(1 + a + K_0)/b, \|u_0\|_\infty\}\). It follows from the comparison principle ([7, Lemma 2.2]) that

\[
u(t, x) \leq K \quad \text{for } t \geq 0, \ x \in [g(t), h(t)].
\]

Let \(z_1\) and \(z_n\), with \(n \geq 2\), be the unique solutions of

\[
\begin{cases} 
  z_1t = \int_{\mathbb{R}} J(x - y)z_1(t, y)dy - dz_1 + z_1(a_1(x) - bz_1), \quad t > T_1, \ x \in \mathbb{R}, \\
  z_1(T_1, x) = K, \quad x \in \mathbb{R},
\end{cases}
\]

and

\[
\begin{cases} 
  z nt = \int_{\mathbb{R}} J(x - y)z_n(t, y)dy - dz_n + z_1(a_n(x) - bz_n), \quad t > T_n, \ x \in \mathbb{R}, \\
  z(T_n, x) = z_{n-1}(T_n, x), \quad x \in \mathbb{R},
\end{cases}
\]

(3.23)
respectively. Then clearly
\[
\begin{cases}
z_{1t} \geq d \int_{g(t)}^{b(t)} J(x-y) z_1(t,y)dy - dz_1 + z_1(a-w - bz_1), & t > T_1, \ x \in (g(t), h(t)), \\
z_1 \geq 0, \ z_1(t, h(t)) \geq 0, & t > T_1, \\
z_1(T_1, x) \geq u(T_1, x), & x \in [g(T_1), h(T_1)].
\end{cases}
\]

The comparison principle (see [7, Lemma 2.2]) then infers that
\[
z_1(t, x) \geq u(t, x) \text{ for } t \geq T_1 \text{ and } x \in [g(t), h(t)].
\]
Hence,
\[
\begin{cases}
z_{2t} \geq d \int_{g(t)}^{b(t)} J(x-y) z_2(t,y)dy - dz_2 + z_2(a-w - bz_2), & t > T_2, \ x \in (g(t), h(t)), \\
z_2(t, g(t)) \geq 0, \ z_2(t, h(t)) \geq 0, & t > T_2, \\
z_2(T_2, x) = z_1(T_2, x) \geq u(T_2, x), & x \in [g(T_2), h(T_2)].
\end{cases}
\]
and by the comparison principle (see [7, Lemma 2.2]) again, it yields
\[
z_2(t, x) \geq u(t, x) \text{ for } t \geq T_2 \text{ and } x \in [g(t), h(t)].
\]

By the inductive method we can get
\[
z_n(t, x) \geq u(t, x) \text{ for } t \geq T_n \text{ and } x \in [g(t), h(t)]. \tag{3.24}
\]
Since \(u(t, x) = 0\) for \(t \geq T_n\) and \(x \in \mathbb{R}\setminus(g(t), h(t))\), we have
\[
z_n(t, x) \geq u(t, x) \text{ for } t \geq T_n \text{ and } x \in \mathbb{R}.
\]
By Propositions 3.12 (i) and [7, Proposition 3.4(ii)], we have
\[
\lim_{t \to \infty} \lambda_p(L_{(-1,1)}^d + a_n(x)) \geq \lim_{t \to \infty} \lambda_p(L_{(-1,1)}^d + \sigma_\infty) = \sigma_\infty > 0.
\]
It follows from Proposition 3.12 (ii) that (3.23) admits a unique positive steady state \(\bar{z}_n \in C(\mathbb{R})\):
\[
d \int_{\mathbb{R}} J(x-y) \bar{z}_n(y)dy - d\bar{z}_n + \bar{z}_n(a_n(x) - b\bar{z}_n) = 0, \ x \in \mathbb{R}, \tag{3.25}
\]
and
\[
\lim_{t \to \infty} z_n(t, x) = \bar{z}_n(x) \text{ locally uniformly in } \mathbb{R}. \tag{3.26}
\]
Since \(\sigma_\infty/b\) is a lower solution of (3.25) and \(\sigma_\infty/b \leq K\), applying the comparison principle gives that \(z_n(t, x) \geq \sigma_\infty/b\) for \(t \geq T_n\) and \(x \in \mathbb{R}\). Similarly, we have \(z_n(t, x) \leq K\) for \(t \geq T_n\) and \(x \in \mathbb{R}\). Thus, \(\sigma_\infty/b \leq \bar{z}_n(x) \leq K\) for every \(x \in \mathbb{R}\). It follows from the monotonicity of \(a_n(x)\) in \(n\) that \(z_{n+1}(t, x) \leq z_n(t, x)\) for \(t \geq T_{n+1}\) and \(x \in \mathbb{R}\). Then \(\bar{z}_{n+1}(x) \leq \bar{z}_n(x)\) for every \(x \in \mathbb{R}\). Therefore, there exists \(\bar{z}_\infty(x)\) such that
\[
\lim_{n \to \infty} \bar{z}_n(x) = \bar{z}_\infty(x) \text{ for every } x \in \mathbb{R},
\]
where \(\bar{z}_\infty(x)\) satisfies \(\sigma_\infty/b \leq \bar{z}_\infty(x) \leq K\) in \(\mathbb{R}\). By the Lebesgue dominant convergence theorem, we can pass to the limit in (3.25) and obtain
\[
d \int_{\mathbb{R}} J(x-y) \bar{z}_\infty(y)dy - d\bar{z}_\infty + \bar{z}_\infty(\sigma_\infty - b\bar{z}_\infty) = 0, \ x \in \mathbb{R}.
\]
Since this problem has a unique positive solution, we necessarily have \(\bar{z}_\infty(x) \equiv \sigma_\infty/b\), which implies that
\[
\lim_{n \to \infty} \bar{z}_n(x) = \sigma_\infty/b \text{ for every } x \in \mathbb{R}.
\]
Since \( \tilde{z}_n \) is monotone in \( n \), thanks to Dini’s theorem, we have
\[
\lim_{n \to \infty} \tilde{z}_n(x) = \sigma_\infty / b \text{ locally uniformly in } \mathbb{R}.
\]
It follows from this fact, (3.24), (3.26) and the arbitrariness of \( \varepsilon \) that
\[
\limsup_{t \to \infty} u(t, x) \leq [a - m]_+ / b \text{ locally uniformly in } \mathbb{R}.
\]
(ii) For any given \( l > L_1 := 2 / (a - M) \). Clearly
\[
\lambda_p \left( \mathcal{L}_{(-l,l)}^d + a - (M + 1/l) \right) > \lambda_p \left( \mathcal{L}_{(-l,l)}^d + (a - M)/2 \right) .
\]
By [7, Proposition 3.4], there exists \( L_2 \geq 0 \) such that
\[
\lambda_p \left( \mathcal{L}_{(-l,l)}^d + (a - M)/2 \right) > 0 \text{ when } l > L_2.
\]
Take \( L := \max\{L_1, L_2\} \). Then
\[
\lambda_p \left( \mathcal{L}_{(-l,l)}^d + a - (M + 1/l) \right) > 0 \text{ when } l > L.
\]
For any such \( l \), it follows from (3.22) that there exists \( T_l \) such that
\[
(w(t, x)) \leq M + 1/l \text{ for } t \geq T_l, \ x \in [-l, l].
\]
Since \( h(t) \to \infty \) and \( g(t) \to -\infty \) as \( t \to \infty \), there exists \( e_T \geq T_l \) such that
\[
((-l, l) \subset (g(t), h(t)) \text{ for } t \geq e_T).
\]
Thus \( u \) satisfies
\[
u_t \geq d \int_{-l}^{l} J(x - y) u(t, y) dy - du + u [a - (M + 1/l) - bu], \ t > e_T, \ x \in [-l, l].
\]
Let \( u_l \) be the unique solution of
\[
\begin{align*}
u_t & = d \int_{-l}^{l} J(x - y) u(t, y) dy - du + u [a - (M + 1/l) - bu], \ t > e_T, \ x \in [-l, l], \\
u(e_T, x) & = u(e_T, x), \quad x \in [-l, l].
\end{align*}
\]
(3.27)
By the comparison principle [7, Lemma 3.3],
\[
u(t, x) \geq u_l(t, x) \text{ for } t \geq e_T, \ x \in [-l, l].
\]
By Lemma 3.13, we have that (3.27) has a unique positive steady state \( U_l(x) \) and
\[
\lim_{t \to \infty} u_l(t, x) = U_l(x) \quad \text{uniformly in } [-l, l],
\]
\[
\lim_{l \to \infty} U_l(x) = (a - M)/b \quad \text{locally uniformly in } \mathbb{R}.
\]
Consequently,
\[
\liminf_{t \to \infty} u(t, x) \geq (a - M)/b \quad \text{locally uniformly in } \mathbb{R}.
\]
This completes the proof.

We are now ready to prove Theorem 1.4.

Proof of Theorem 1.4 (i): The weak competition case.

Step 1. Let \( q(t) \) be the solution of
\[
\begin{align*}q'(t) & = q(a_1 - b_1 q), \quad t > 0, \\
q(0) & = \sup_{x \in \mathbb{R}} u_0(x).
\end{align*}
\]
Then $\lim_{t \to \infty} q(t) = a_1/b_1$. By the comparison principle ([7, Lemma 2.2]), we have $u_1(t, x) \leq q(t)$ for $t > 0$ and $x \in [g(t), h(t)]$. In view of $u_1(t, x) = 0$ for $t > 0$ and $x \in \mathbb{R}\setminus(g(t), h(t))$, we have $u_1(t, x) \leq q(t)$ for $t > 0$ and $x \in \mathbb{R}$. Hence,

$$\limsup_{t \to \infty} u_1(t, x) \leq a_1/b_1 =: \bar{A}_1 \text{ locally uniformly in } \mathbb{R}. \quad (3.28)$$

**Step 2.** By the condition $c_1/b_2 < a_1/a_2 < b_1/c_2$, we have

$$a_2 - c_2\bar{A}_1 = (a_2b_1 - a_1c_2)/b_1 > 0.$$

It follows from this fact, (3.28) and Lemma 3.14 that

$$\liminf_{t \to \infty} u_2(t, x) \geq (a_2 - c_2\bar{A}_1)/b_2 =: \bar{B}_1 \text{ locally uniformly in } \mathbb{R}. \quad (3.29)$$

The condition $c_1/b_2 < a_1/a_2 < b_1/c_2$ implies

$$a_1 - c_1\bar{B}_1 = a_1 - \frac{c_1}{b_1b_2}(a_2b_1 - a_1c_2) = a_1 - \frac{a_2c_1}{b_2} + \frac{a_1c_2c_1}{b_1b_2} > 0.$$

This fact combined with (3.29) and Lemma 3.14 allows us to derive

$$\limsup_{t \to \infty} u_1(t, x) \leq (a_1 - c_1\bar{B}_1)/b_1 =: \bar{A}_2 \text{ locally uniformly in } \mathbb{R}.$$

Furthermore, the condition $c_1/b_2 < a_1/a_2 < b_1/c_2$ implies

$$a_2 - c_2\bar{A}_2 = a_2 - \frac{a_1c_2}{b_1} + \frac{c_1c_2}{b_1b_2}(a_2b_1 - a_1c_2) > 0.$$

Similar to the above,

$$\liminf_{t \to \infty} u_2(t, x) \geq (a_2 - c_2\bar{A}_2)/b_2 =: \bar{B}_2 \text{ locally uniformly in } \mathbb{R}.$$

**Step 3.** Repeating the above procedure, we can find two sequences $\bar{A}_i$ and $\bar{B}_i$ such that

$$\limsup_{t \to \infty} u_1(t, x) \leq \bar{A}_i, \quad \liminf_{t \to \infty} u_2(t, x) \geq \bar{B}_i \text{ locally uniformly in } \mathbb{R},$$

and

$$\bar{A}_{i+1} = (a_1 - c_1\bar{B}_i)/b_1, \quad \bar{B}_i = (a_2 - c_2\bar{A}_i)/b_2, \quad i = 1, 2, \ldots.$$

Let

$$p := \frac{a_1}{b_1} - \frac{a_2c_1}{b_1b_2}, \quad q := \frac{c_1c_2}{b_1b_2}.$$

Then $p > 0, 0 < q < 1$ by the weak competition assumption. By direct calculation,

$$\bar{A}_{i+1} = p + q\bar{A}_i, \quad i = 1, 2, \ldots.$$

From $\bar{A}_2 < \bar{A}_1$ and the above iteration formula, we immediately obtain

$$0 < \bar{A}_{i+1} < \bar{A}_i, \quad i = 1, 2, \ldots,$$

from which it easily follows that

$$\lim_{i \to \infty} \bar{A}_i = \frac{a_1b_2 - a_2c_1}{b_1b_2 - c_1c_2}, \quad \lim_{i \to \infty} \bar{B}_i = \frac{a_2b_1 - a_1c_2}{b_1b_2 - c_1c_2}.$$

Thus we have

$$\limsup_{t \to \infty} u_1(t, x) \leq \frac{a_1b_2 - a_2c_1}{b_1b_2 - c_1c_2}, \liminf_{t \to \infty} u_2(t, x) \geq \frac{a_2b_1 - a_1c_2}{b_1b_2 - c_1c_2} \text{ locally uniformly in } \mathbb{R}.$$

Similarly, we can show

$$\liminf_{t \to \infty} u_1(t, x) \geq \frac{a_1b_2 - a_2c_1}{b_1b_2 - c_1c_2}, \limsup_{t \to \infty} u_2(t, x) \leq \frac{a_2b_1 - a_1c_2}{b_1b_2 - c_1c_2} \text{ locally uniformly in } \mathbb{R}.$$
Theorem 1.4 (i) is proved.

**Proof of Theorem 1.4 (ii): The weak predation case.**

**Step 1.** Similarly to the proof of Theorem 1.4 (i),

\[
\limsup_{t \to \infty} u_1(t, x) \leq a_1/b_1 =: \bar{A}_1 \text{ locally uniformly in } \mathbb{R}.
\]

Taking advantage of Lemma 3.14 one has

\[
\limsup_{t \to \infty} u_2(t, x) \leq (a_2 + c_2 \bar{A}_1)/b_2 =: \bar{B}_1 \text{ locally uniformly in } \mathbb{R}.
\]

The condition \(a_1 b_1 b_2 > a_2 b_1 c_1 + a_1 c_1 c_2\) implies

\[
a_1 - c_1 \bar{B}_1 = \frac{a_1 b_1 b_2 - a_2 b_1 c_1 - a_1 c_1 c_2}{b_1 b_2} > 0.
\]

Making use of Lemma 3.14, repeatedly, we have

\[
\liminf_{t \to \infty} u_1(t, x) \geq (a_1 - c_1 \bar{B}_1)/b_1 =: \underbar{A}_1 \text{ locally uniformly in } \mathbb{R},
\]

and

\[
\liminf_{t \to \infty} u_2(t, x) \geq (a_2 + c_2 \underbar{A}_1)/b_2 =: \underbar{B}_1 \text{ locally uniformly in } \mathbb{R}.
\]

Notice

\[
a_1 - c_1 \underline{B}_1 = \frac{b_1 b_2(a_1 b_1 b_2 - a_2 b_1 c_1 - a_1 c_1 c_2) + a_2 b_1 c_1^2 c_2 + a_1 c_1^2 c_2^2}{b_1 b_2} > 0,
\]

it follows from Lemma 3.14 that

\[
\limsup_{t \to \infty} u_1(t, x) \leq (a_1 - c_1 \underline{B}_1)/b_1 =: \bar{A}_2 \text{ locally uniformly in } \mathbb{R}.
\]

Similar to the above,

\[
\limsup_{t \to \infty} u_2(t, x) \leq (a_2 + c_2 \bar{A}_2)/b_2 =: \bar{B}_2 \text{ locally uniformly in } \mathbb{R}.
\]

**Step 2.** Repeating the above procedure, we can find four sequences \(A_i, \bar{A}_i, \underline{B}_i, \bar{B}_i\) and \(\bar{B}_i\) such that, for all \(i \geq 1\),

\[
A_i \leq \liminf_{t \to \infty} u_1(t, x) \leq \limsup_{t \to \infty} u_1(t, x) \leq \bar{A}_i,
\]

\[
\underline{B}_i \leq \liminf_{t \to \infty} u_2(t, x) \leq \limsup_{t \to \infty} u_2(t, x) \leq \bar{B}_i
\]

locally uniformly in \(\mathbb{R}\), and

\[
\bar{A}_1 = \frac{a_1}{b_1}, \quad \underline{B}_1 = \frac{a_2 + c_2 \bar{A}_1}{b_2}, \quad A_i = \frac{a_1 - c_1 \underline{B}_i}{b_1}, \quad B_i = \frac{a_2 + c_2 A_i}{b_2}, \quad \bar{A}_{i+1} = \frac{a_1 - c_1 B_i}{b_1},
\]

\(i = 1, 2, \cdots\). Define

\[
a := \frac{a_2 b_1 + a_1 c_2}{b_1 b_2}, \quad q := \frac{c_1 c_2}{b_1 b_2}.
\]

By direct calculation, we have

\[
\bar{B}_1 = a, \quad A_1 = \frac{a_1}{b_1} - \frac{c_1}{b_1} a, \quad \underline{B}_1 = a(1 - q),
\]

\[
\bar{A}_2 = \frac{a_1}{b_1} - \frac{c_1}{b_1} a(1 - q), \quad \underline{B}_2 = a(1 - q + q^2), \quad A_2 = \frac{a_1}{b_1} - \frac{c_1}{b_1} a(1 - q + q^2),
\]

and

\[
B_{i+1} = a(1 - q) + q^2 B_i, \quad \bar{B}_{i+1} = a(1 - q) + q^2 \underline{B}_i, \quad i \geq 1.
\]
Since \( a_1 b_2 > a_2 b_1 c_1 + a_1 c_1 c_2 \), we have \( 0 < q < 1 \) and
\[
B_2 > B_1 > 0, \quad \bar{B}_1 > \bar{B}_2 > 0.
\]
The above iteration formula then infers
\[
\bar{B}_{i+1} > \bar{B}_i > 0, \quad \bar{B}_i > \bar{B}_{i+1} > 0, \quad i \geq 1.
\]
From these we easily obtain
\[
\lim_{i \to \infty} \bar{B}_i = \lim_{i \to \infty} B_i = \frac{a_1 c_2 + a_2 b_1}{b_1 b_2 + c_1 c_2}, \quad (3.31)
\]
and subsequently
\[
\lim_{i \to \infty} \bar{A}_i = \lim_{i \to \infty} A_i = \frac{a_1 b_2 - a_2 c_1}{b_1 b_2 + c_1 c_2}. \quad (3.32)
\]

Theorem 1.4 (ii) clearly is a consequence of (3.30), (3.31) and (3.32). The proof is finished.

**Acknowledgments.** The authors would like to thank the anonymous referees for their helpful comments and suggestions.

**REFERENCES**

[1] X. Bai and F. Li, Classification of global dynamics of competition models with nonlocal dispersals I: Symmetric kernels, *Calc. Var. Partial Differential Equations*, 57 (2018), 35 pp.

[2] X. Bai and F. Li, Global dynamics of competition models with nonsymmetric nonlocal dispersals when one diffusion rate is small, *Discrete Contin. Dyn. Syst.*, 40 (2020), 3075–3092.

[3] X. Bao, W. T. Li and W. Shen, Traveling wave solutions of Lotka-Volterra competition systems with nonlocal dispersal in periodic habitats, *J. Differential Equations*, 260 (2016), 8590–8637.

[4] P. Bates and G. Zhao, Existence, uniqueness and stability of the stationary solution to a nonlocal evolution equation arising in population dispersal, *J. Math. Anal. Appl.*, 332 (2007), 428–440.

[5] H. Berestycki, J. Coville and H. H. Vo, On the definition and the properties of the principal eigenvalue of some nonlocal operators, *J. Funct. Anal.*, 271 (2016), 2701–2751.

[6] H. Berestycki, J. Coville and H.-H. Vo, Persistence criteria for populations with non-local dispersion, *J. Math. Biol.*, 72 (2016), 1693–1745.

[7] J. F. Cao, Y. Du, F. Li and W. T. Li, The dynamics of a Fisher-KPP nonlocal diffusion model with free boundaries, *J. Funct. Anal.*, 277 (2019), 2772–2814.

[8] J. Coville, On a simple criterion for the existence of a principal eigenfunction of some nonlocal operators, *J. Differential Equations*, 249 (2010), 2921–2953.

[9] Y. Du and Z. Lin, Spreading-Vanishing dichotomy in the diffusive logistic model with a free boundary, *SIAM J. Math. Anal.*, 42 (2010), 377–405.

[10] Y. Du and Z. Lin, The diffusive competition model with a free boundary: Invasion of a superior or inferior competitor, *Discrete Contin. Dyn. Syst. Ser. B*, 19 (2014), 3105–3132.

[11] Y. Du and W. Ni, Analysis of a West Nile virus model with nonlocal diffusion and free boundaries, *Nonlinearity*, 33 (2020), 4407–4448.

[12] Y. Du, M. Wang and M. Zhou, Semi-wave and spreading speed for the diffusive competition model with a free boundary, *J. Math. Pures Appl.*, 107 (2017), 253–287.

[13] Y. Du and C.-H. Wu, Spreading with two speeds and mass segregation in a diffusive competition system with free boundaries, *Calc. Var. Partial Differential Equations*, 57 (2018), 36 pp.

[14] J. S. Guo and C. H. Wu, On a free boundary problem for a two-species weak competition system, *J. Dynam. Differential Equations*, 24 (2012), 873–895.

[15] J. S. Guo and C. H. Wu, Dynamics for a two-species competition-diffusion model with two free boundaries, *Nonlinearity*, 28 (2015), 1–27.

[16] G. Hetzer, T. Nguyen and W. Shen, Coexistence and extinction in the Volterra-Lotka competition model with nonlocal dispersal, *Commun. Pure Appl. Anal.*, 11 (2012), 1699–1722.

[17] V. Hutson, S. Martinez, K. Mischaikow and G. T. Vickers, The evolution of dispersal, *J. Math. Biol.*, 47 (2003), 483–517.
[18] C. Y. Kao, Y. Lou and W. Shen, Random dispersal vs. non-local dispersal, *Discrete Contin. Dyn. Syst.*, **26** (2010), 551–596.

[19] L. Li, W. Sheng and M. Wang, Systems with nonlocal vs. local diffusions and free boundaries, *J. Math. Anal. Appl.*, **483** (2020), 27 pp.

[20] L. Li, J. Wang and M. Wang, The dynamics of nonlocal diffusion systems with different free boundaries, *Commun. Pure Appl. Anal.*, **19** (2020), 3651–3672.

[21] Z. Lin, A free boundary problem for a predator-prey model, *Nonlinearity*, **20** (2007), 1883–1892.

[22] M. Mimura, Y. Yamada and S. Yotsutani, Stability analysis for free boundary problems in ecology, *Hiroshima Math. J.*, **16** (1986), 477–498.

[23] R. Nathan, E. Klein, J. J. Robledo-Arnuncio and E. Revilla, Dispersal kernels: Review, *Dispersal Ecology and Evolution*, J. Clobert, M. Baguette, T. G. Benton and J. M. Bullock, eds., Oxford University Press, Oxford, UK, (2012), 187–210.

[24] J. Wang and M. Wang, The diffusive Beddington-DeAngelis predator-prey model with non-linear prey-taxis and free boundary, *Math. Methods Appl. Sci.*, **41** (2018), 6741–6762.

[25] J. Wang and M. Wang, Free boundary problems with nonlocal and local diffusions I: Global solution, *J. Math. Anal. Appl.*, **490** (2020), 24 pp.

[26] J. Wang and M. Wang, Free boundary problems with nonlocal and local diffusion II: Spreading-vanishing and long-time behavior, *Discrete Contin. Dyn. Syst. Ser. B*, **25** (2020), 4721–4736.

[27] M. Wang, On some free boundary problems of the prey-predator model, *J. Differential Equations*, **256** (2014), 3365–3394.

[28] M. Wang, Spreading and vanishing in the diffusive prey-predator model with a free boundary, *Commun. Nonlinear Sci. Numer. Simul.*, **23** (2015), 311–327.

[29] M. Wang, *Nonlinear Second Order Parabolic Equations*, Boca Raton: CRC Press, 2021.

[30] M. Wang and Q. Zhang, Dynamics for the diffusive Leslie-Gower model with double free boundaries, *Discrete Contin. Dyn. Syst.*, **38** (2018), 2591–2607.

[31] M. Wang and Y. Zhang, Two kinds of free boundary problems for the diffusive prey-predator model, *Nonlinear Anal. Real World Appl.*, **24** (2015) 73–82.

[32] M. Wang and Y. Zhang, The time-periodic diffusive competition models with a free boundary and sign-changing growth rates, *Z. Angew. Math. Phys.*, **67** (2016), 24 pp.

[33] M. Wang and Y. Zhang, Note on a two-species competition-diffusion model with two free boundaries, *Nonlinear Anal.*, **159** (2017), 458–467.

[34] M. Wang and Y. Zhang, Dynamics for a diffusive prey-predator model with different free boundaries, *J. Differential Equations*, **264** (2018), 3527–3558.

[35] M. Wang and J. Zhao, Free boundary problems for a Lotka-Volterra competition system, *J. Dynam. Differential Equations*, **26** (2014), 655–672.

[36] M. Wang and J. Zhao, A free boundary problem for the predator-prey model with double free boundaries, *J. Dynam. Differential Equations*, **29** (2017), 957–979.

[37] Y. Zhang and M. Wang, A free boundary problem of the ratio-dependent prey-predator model, *Appl. Anal.*, **94** (2015), 2147–2167.

[38] J. Zhao and M. Wang, A free boundary problem of a predator-prey model with higher dimension and heterogeneous environment, *Nonlinear Anal. Real World Appl.*, **16** (2014), 250–263.

[39] M. Zhao, Y. Zhang, W. T. Li and Y. Du, The dynamics of a degenerate epidemic model with nonlocal diffusion and free boundaries, *J. Differential Equations*, **269** (2020), 3347–3386.

[40] Y. Zhao and M. Wang, Free boundary problems for the diffusive competition system in higher dimension with sign-changing coefficients, *IMA J. Appl. Math.*, **81** (2016), 255–280.

Received May 2021; revised September 2021; early access September 2021.

*E-mail address:* ydu@une.edu.au
*E-mail address:* mxwang@hit.edu.cn
*E-mail address:* zhaom@nwnu.edu.cn