Timescales for Detection of Super-Chandrasekhar White Dwarfs by Gravitational-wave Astronomy

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Abstract

Over the past two decades, the inference of the violation of the Chandrasekhar mass limit of white dwarfs (WDs) from indirect observation has been a revolutionary discovery in astronomy. Various researchers have already proposed different theories to explain this interesting phenomenon. However, such massive WDs usually possess very little luminosity, hence they so far cannot be detected directly by any observations. We have already proposed that the continuous gravitational wave may be one of the probes to detect them directly, and in the future various space-based detectors, such as LISA, DECIGO, and BBO, should be able to detect many of those WDs (provided they behave like pulsars). In this paper, we address various timescales related to the emission of gravitational as well as dipole radiations. This exploration sets a timescale for the detectors to observe the massive WDs.

Unified Astronomy Thesaurus concepts: White dwarf stars (1799); Rotation powered pulsars (1408); Gravitational waves (678); Astronomical radiation sources (89); Stellar magnetic fields (1610); Chandrasekhar limit (221); Gravitational wave astronomy (675); Gravitational wave sources (677); Pulsars (1306); Stellar luminosities (1609)

1. Introduction

In white dwarfs (WDs), the inward pressure due to gravity balances the outward pressure due to degenerate electron gas, thereby the WDs form a stable equilibrium. Chandrasekhar (1931, 1935) first proposed the idea of the existence of a mass limit of WDs. He showed that for a carbon–oxygen nonrotating nonmagnetized WD, the maximum possible mass is $\sim 1.4M_\odot$, popularly known as the Chandrasekhar mass limit. The theory of general relativity and basic quantum mechanics are sufficient to explain this mass limit, though Newton’s law is enough to understand its existence. Beyond this mass limit, the pressure balance is no longer sustained, and a WD blows up to produce a type Ia supernova (SN Ia). The luminosities of SNe Ia are very important as they are used as one of the standard candles to measure cosmological distances. However, during the past couple of decades, the inference of super-Chandrasekhar WDs has been considered one of the revolutionary discoveries in astrophysics. Howell et al. (2006) first reported an overluminous SN Ia, named SN 2003fg, with the content of nickel mass itself being $\sim 1.3M_\odot$, and thereby they predicted that the progenitor mass of the WD for that SN Ia is $\sim 2.1M_\odot$. Eventually, several similar overluminous SNe Ia have been discovered, which implies that the progenitor mass of WDs could be as high as $\sim 2.8M_\odot$ (Hicken et al. 2007; Yamanaka et al. 2009; Scalzo et al. 2010, 2012; Tanaka et al. 2010; Yuan et al. 2010; Silverman et al. 2011; Taubenberger et al. 2011). These WDs are eventually termed super-Chandrasekhar WDs, as they violate the Chandrasekhar mass limit significantly. This violation of the Chandrasekhar mass limit challenges use of the standard candle from the luminosities of SNe Ia.

While an argument for the existence of such a massive WD progenitor for SNe Ia was attempted by the double degenerate scenario, numerical simulations of the massive WD merger never lead to an observationally inferred progenitor mass as high as $2.8M_\odot$. Such double degenerate evolutions always produced the off-center ignition and formation of a neutron star rather than an (overluminous) SN Ia (e.g., Saio & Nomoto 2004; Martin et al. 2006). Although there are limitations in numerical simulations, including the chosen mass of component WDs, recently Wu et al. (2019) showed that the final outcome of WD mergers is practically not influenced by initial WD masses; it primarily depends on the mass-accretion rates during mergers. In a single degenerate scenario of accreting differentially rotating WDs in close binaries of a normal companion, Chen & Li (2009) showed that the formation of a very massive ($>1.7M_\odot$) progenitor is not possible. Hence, all the conventional pictures have failed to explain the existence of super-Chandrasekhar progenitor WDs. Kundu & Mukhopadhyay (2012) first showed that in the presence of a high magnetic field, which forms Landau levels (microscopic effect) in the plane perpendicular to the magnetic field axis, super-Chandrasekhar WDs are possible; and this leads to a new mass limit of $\sim 2.6M_\odot$ (Das & Mukhopadhyay 2013 and the references therein). Furthermore, Mukhopadhyay and his collaborators also showed that the macroscopic effect of the magnetic field (e.g., magnetic field pressure and magnetic field geometry) can also increase the mass of WDs significantly (Subramanian & Mukhopadhyay 2015; Kalita & Mukhopadhyay 2019). This idea was verified by, e.g., Franzon & Schramm (2015), Manreza Paretet al. (2015), and Bera & Bhattacharya (2016), to name a few. Similarly, many other researchers proposed different theories, such as modified gravity (Carvalho et al. 2017; Kalita & Mukhopadhyay 2018), the generalized Heisenberg uncertainty principle (Ong 2018), charged WDs (Liu et al. 2014), noncommutative geometry (Kalita et al. 2019), to mention a few, to explain the super-Chandrasekhar WDs. Each of these theories gives rise to different mass–radius relations for the WDs. However, since no super-Chandrasekhar WDs have so far been detected directly, the astroseismology of such WDs cannot be carried out. Hence, it has not yet been possible to single out which one of those theories is the theory behind the super-Chandrasekhar WDs. It
has already been argued that if one considers the idea of magnetized super-Chandrasekhar WDs, such WDs possess very little thermal luminosity (Bhattacharya et al. 2018), and hence they have not been detected so far by any of the surveys, such as GAIA, Kepler, or SDSS. The maximum observed magnetic field in an isolated WD is \( \sim 10^9 \) G (Heyl 2000; Ferrario et al. 2015). We have argued that if the magnetized WDs have a misalignment between the rotation and magnetic axes (same as the configuration of a pulsar), apart from dipole radiation, they can emit a significant amount of gravitational radiation, which might be detected by future space-based gravitational wave (GW) detectors, such as LISA, DECIGO, or BBO. Thereby it would be a direct detection of super-Chandrasekhar WDs (Kalita & Mukhopadhyay 2019). In this paper, we address the timescales related to the dipole and gravitational radiations for these pulsating WDs.

Unlike WDs, calculating various timescales for neutron star (NS) pulsars is not a new problem. Pulsars are generally rotating magnetized NSs with the magnetic and rotation axes not aligned with each other. Radio astronomers estimate the lifetime of a pulsar just by calculating its observed period \( (P) \) and the rate of change of period \( (\dot{P}) \). The characteristic age of a pulsar is therefore given by \( P/2\dot{P} \) (Lorimer 2008). However, this formula is valid if one considers that the angle between magnetic and rotation axes of a pulsar does not vary throughout its lifetime. However, in practice, due to emission of radiation, this angle is expected to change. Michel & Goldwire (1970) and Davis & Goldstein (1970) calculated the pulsar timescales simultaneously considering the variations of the angle as well as the spin period of the pulsar emitting dipole radiation, based on the torques calculated earlier by Deutsch (1955). Eventually, various researchers used this formalism to solve different properties of pulsars, such as, the braking index (Goldreich 1970; Fujimura & Kennel 1980; Heintzmann 1981; Good & Ng 1985), and the evolution of the pulsar magnetic field (Flowers & Ruderman 1977; Kundt 1981). Chau & Henriksen (1970) included the quadrupolar radiation along with the dipole radiation and recalculated the various aspects of NSs. All these calculations assumed spherical stars, which, however, are not true in the presence of magnetic field and rotation. Melatos (2000) generalized the equations and applied them for nonspherical NSs. Similarly, these formulae have again been modified considering a plasma filled magnetosphere rather than a vacuum magnetosphere (Spitkovsky 2006; Philippov et al. 2014). More recently, this formalism has been used to describe the highly magnetized NSs known as magnetars (Lü et al. 2018; Şaşmaz Muş et al. 2019; Lander & Jones 2020).

As mentioned earlier, in this paper, we investigate the time for which a WD pulsar can emit dipole and gravitational (quadrupole) radiation, i.e., the timescale after which either the magnetic and rotation axes align with each other, or the WD stops rotating. This exploration is essential because we argued in the earlier paper (Kalita & Mukhopadhyay 2019) that the future space-based GW detectors can detect the pulsating super-Chandrasekhar WDs. This raises an immediate question regarding the timescale over which we can observe such massive WDs, and in this paper we investigate such timescales for the first time in the case of WD pulsars. The plan of the paper is as follows. In Section 2, we discuss the fundamental physics behind dipole and quadrupole luminosities and thereby formulate the problem. In Section 3, we discuss the timescales of various possible types of pulsating WDs (regular as well as super-Chandrasekhar) based on our model and certain basic properties of GW emitted by the isolated magnetized WDs, before we conclude in Section 4.

### 2. Model of Pulsating WD

Since this paper is based on WD pulsars, we, hereafter, mostly concentrate on properties of WDs rather than NSs. It is, of course, well known that the number of detected WD pulsars is very little compared to that of NS pulsars. Some well known WD pulsars are, e.g., AE Aquarii (Bookbinder & Lamb 1987) and AR Scorpii (Marsh et al. 2016). Figure 1 shows a cartoon diagram of a pulsar with \( z' \) being the rotational axis and \( z \) the magnetic field axis.

![Cartoon diagram of a pulsar](image)

**Figure 1.** Cartoon diagram of a pulsar with \( z' \) being the rotational axis and \( z \) the magnetic field axis.
dimensionless amplitudes of the two polarizations of the GW at a
time $t$ are given by (Bonazzola & Gourgoulhon 1996; Zimmermann & Szedenski 1979)
\[ h_+ = h_0 \sin \left[ \frac{1}{2} \cos i \sin i \cos \chi \cos \Omega t \right. \]
\[ \left. - \frac{1 + \cos^2 i}{2} \sin \chi \cos 2 \Omega t \right], \]
\[ h_\times = h_0 \sin \chi \left[ \frac{1}{2} \sin i \cos \chi \sin \Omega t \right. \]
\[ \left. - \cos i \sin \chi \sin 2 \Omega t \right], \]
with
\[ h_0 = \frac{4G \Omega^2 \epsilon I_{xx}}{c^3 d}, \]
where $G$ is Newton’s gravitational constant, $c$ is the speed of light, $\Omega$ is the angular frequency, $d$ is the distance between the detector and the source, $i$ is the angle between the rotation axis of the object and our line of sight, and $\epsilon = I_{zz} - I_{x\times}/I_{x\times}$, with $I_{xx}$ and $I_{zz}$ being the moments of inertia of the WD about $x$- and $z$-axes, respectively. It is evident from Equation (1) that the amplitude of GW detected by the detector is always one to two orders of magnitude less than $h_0$ depending on the value of $\chi$ and $i$.

Since a pulsating WD can emit both dipole and quadrupolar luminosities, the dipole luminosity for an axisymmetric WD is given by (Melatos 2000)
\[ L_D = \frac{B_p R_p^6 \Omega^4}{2c^3} \sin^2 \chi F(x_0), \]
where $x_0 = R_0 \Omega/c$, $B_p$ is the strength of the magnetic field at the pole, $R_p$ is the radius of the pole, and $R_0$ is the average radius of the WD. The function $F(x_0)$ is defined as
\[ F(x_0) = \frac{x_0^4}{5(x_0^6 - 3x_0^4 + 36)} + \frac{1}{3(x_0^2 + 1)}. \]
Similarly, the quadrupolar GW luminosity is given by (Zimmermann & Szedenski 1979)
\[ L_{GW} = \frac{2G}{5c^5} (I_{zz} - I_{x\times})^2 \Omega^6 \sin^2 \chi (1 + 15 \sin^2 \chi). \]

It is important to note that this formula is valid if $\chi$ is very small. The total luminosity of a WD is due to both dipole and gravitational radiations. Hence, the changes in $\Omega$ and $\chi$ with respect to time are dependent both on $L_D$ and $L_{GW}$. The variations of $\Omega$ and $\chi$ with respect to time are given by (Chau & Henriksen 1970; Melatos 2000)
\[ \frac{d(\Omega I_{zz})}{dt} = -\frac{2G}{5c^5} (I_{zz} - I_{x\times})^2 \Omega^3 \sin^2 \chi (1 + 15 \sin^2 \chi) \]
\[ - \frac{B_p^2 R_p^6 \Omega^4}{2c^3} \sin^2 \chi F(x_0), \]
with
\[ \frac{d\chi}{dt} = -\frac{12G}{5c^5} (I_{zz} - I_{x\times})^2 \Omega^3 \sin^3 \chi \cos \chi \]
\[ - \frac{B_p^2 R_p^6 \Omega^4}{2c^3} \sin \chi \cos \chi F(x_0), \]
where $I_{zz'}$ is the moment of inertia of the body about the $z'$-axis. Considering small angle approximation, it can be expanded as
\[ I_{zz'} = I_{zz} \cos^2 \chi + I_{x\times} \sin^2 \chi. \]
The set of Equations (6) and (7) needs to be solved simultaneously to obtain the timescale over which a WD can radiate.

To solve Equations (6) and (7), one needs to supply the various quantities, such as $I_{xx}$, $I_{zz}$, $B_p$, and $R_p$ at the initial time. We use a numerical code named XNS, developed to study the structure of NSs primarily (Pili et al. 2014), which, however, was appropriately modified for WDs (Subramanian & Mukhopadhyay 2015). This code provides the axisymmetric equilibrium (not necessarily stable equilibrium) structure of a stellar body. The advantage of this code is that it can give an equilibrium solution of uniformly as well as differentially rotating WDs in the presence of toroidal or poloidal, or twisted-torus, magnetic fields. However, one needs to supply the equation of state (EoS) in the polytropic form, i.e., $P = K \rho^3$ with $P$ being the pressure and $\rho$ being the density. In the case of WDs with central density $\rho_c$ high and magnetic field $\lesssim 10^{15}$ G, the EoS becomes relativistic, which implies that $\Gamma \approx 4/3$ and $K \approx (1/8)(3/\pi)^{1/2}hc/(\mu_B m_H)^{1/3}$, where $h$ is Planck’s constant, $\mu_B$ is the mean molecular weight per electron, and $m_H$ is the mass of the hydrogen atom. Moreover, XNS assumes that the rotation and magnetic field axes are in the same direction, i.e., $\chi = 0$, due to its axisymmetric nature of the algorithm mentioned above. However, in the case of pulsars, since $\chi = 0$ is a necessary condition, we assume $\chi$ to be small so that the values of all the calculated quantities using the XNS code, such as mass, radius, and moment of inertia, are almost valid even if $\chi = 0$ (Kalita & Mukhopadhyay 2019).

3. Timescales of Magnetized WDs
Since massive WDs can only be formed when $\rho_c$ is high, we choose $\rho_c$ to be $10^9$, $10^{10}$, and $2 \times 10^{10}$ g cm$^{-3}$ for our calculations, where the relativistic EoS mentioned above, is perfectly valid. For each of these $\rho_c$, we choose various combinations of $\Omega$ and $B_p$ along with the initial angle $\chi$ to be $30^\circ$ so that we have an idea about the timescales for all possible types of WDs behaving as pulsars. Moreover, we first choose, in the following exploration, WDs possessing only purely poloidal magnetic field so that we can treat them as oscillating dipoles, and the formula for dipole luminosity is valid. Subsequently, we also choose a case with toroidal magnetic fields appropriately. While calculating the timescales, we define $t_{100}$, which is the time required for a WD to reach 10 orders of lesser luminosity than what it originally possessed at its birth.

All the different combinations of $\Omega$ and $B_p$ for different $\rho_c$ are given in Tables 1–3 with the respective mass $M$, $R_p$, and $h_0$. We assume throughout that the distance of the source from the detector is $d = 100$ pc. Here we primarily restrict the magnetic to gravitational energies ratio (ME/GE) as well as the kinetic to gravitational energies ratio (KE/GE) to less than $\sim 10^{-7}$ so that the magnetized WDs are surely stable (Komatsu et al. 1989;
Braithwaite (2009). It is important to note that with these values of ME/GE and KE/GE, a WD cannot possess mass significantly more than the Chandrasekhar mass limit. However, this limit may be relaxed in a suitable mixed field configuration leading to super-Chandrasekhar WDs, which is beyond the scope of the present work as XNS cannot handle a rotating star with a suitable and/or an equal fraction of mixed field configuration. It has of course been known for a long time that the stars containing purely toroidal or purely poloidal magnetic fields are unstable (Markey & Tayler 1973; Tayler 1973). However, in the present work, our aim is not to study the stability analysis, and the code we relied upon cannot handle a rotating star with suitable mixed field configurations. Hence, purely poloidal or purely toroidal magnetic fields, maintaining the ME/GE limit mentioned above, are valid approximations of poloidally dominated or toroidally dominated mixed field configurations. Below we discuss the time evolutions of the rotational period, the angle between magnetic and rotational axes, and the various luminosities of WDs.

### Table 1
Poloidal Magnetic Field with \( \rho_0 = 2 \times 10^{10} \) g cm\(^{-3} \)

| \( M (M_\odot) \) | \( R_p \) (km) | \( B_p \) (G) | \( P \) (s) | ME/GE | KE/GE | \( L_{GW} \) (erg s\(^{-1} \)) | \( L_D \) (erg s\(^{-1} \)) | \( h_0 \) | \( t_{10} \) (yr) |
|----------------|------------|-------------|----------|--------|-------|-----------------|----------------|------|---------|
| 1.42 | 1200.5 | 8.9 \( \times 10^{11} \) | 2.0 | 6.1 \( \times 10^{-4} \) | 2.5 \( \times 10^{-3} \) | 3.1 \( \times 10^{10} \) | 3.6 \( \times 10^{11} \) | 5.2 \( \times 10^{-22} \) | 1.1 \( \times 10^{4} \) |
| 1.42 | 1209.4 | 1.4 \( \times 10^{9} \) | 2.0 | 1.4 \( \times 10^{-9} \) | 2.5 \( \times 10^{-3} \) | 2.0 \( \times 10^{10} \) | 8.1 \( \times 10^{16} \) | 4.2 \( \times 10^{-22} \) | 5.4 \( \times 10^{4} \) |

Note. \( t_{10} \) is the time to decay \( L_{initial} \) to \( 10^{-10} L_{initial} \). \( t_D \) means the timescale is very large.

### Table 2
Poloidal Magnetic Field with \( \rho_0 = 10^{10} \) g cm\(^{-3} \)

| \( M (M_\odot) \) | \( R_p \) (km) | \( B_p \) (G) | \( P \) (s) | ME/GE | KE/GE | \( L_{GW} \) (erg s\(^{-1} \)) | \( L_D \) (erg s\(^{-1} \)) | \( h_0 \) | \( t_{10} \) (yr) |
|----------------|------------|-------------|----------|--------|-------|-----------------|----------------|------|---------|
| 1.44 | 1510.6 | 7.4 \( \times 10^{11} \) | 2.0 | 1.0 \( \times 10^{-3} \) | 5.1 \( \times 10^{-3} \) | 3.4 \( \times 10^{10} \) | 9.8 \( \times 10^{11} \) | 1.7 \( \times 10^{-21} \) | 6.7 \( \times 10^{6} \) |
| 1.44 | 1519.5 | 1.1 \( \times 10^{9} \) | 2.0 | 2.2 \( \times 10^{-9} \) | 5.1 \( \times 10^{-3} \) | 2.4 \( \times 10^{10} \) | 2.2 \( \times 10^{10} \) | 1.5 \( \times 10^{-21} \) | 3.8 \( \times 10^{6} \) |

Note. \( t_{10} \) is the time to decay \( L_{initial} \) to \( 10^{-10} L_{initial} \). \( t_D \) means the timescale is very large.

### Table 3
Poloidal Magnetic Field with \( \rho_0 = 10^{9} \) g cm\(^{-3} \)

| \( M (M_\odot) \) | \( R_p \) (km) | \( B_p \) (G) | \( P \) (s) | ME/GE | KE/GE | \( L_{GW} \) (erg s\(^{-1} \)) | \( L_D \) (erg s\(^{-1} \)) | \( h_0 \) | \( t_{10} \) (yr) |
|----------------|------------|-------------|----------|--------|-------|-----------------|----------------|------|---------|
| 1.48 | 3021.3 | 6.6 \( \times 10^{11} \) | 5.3 | 1.1 \( \times 10^{-2} \) | 7.4 \( \times 10^{-4} \) | 1.8 \( \times 10^{11} \) | 1.0 \( \times 10^{12} \) | 3.3 \( \times 10^{-21} \) | 4.6 \( \times 10^{6} \) |
| 1.46 | 3216.2 | 1.2 \( \times 10^{9} \) | 5.3 | 5.1 \( \times 10^{-8} \) | 7.4 \( \times 10^{-4} \) | 3.6 \( \times 10^{16} \) | 5.0 \( \times 10^{16} \) | 1.5 \( \times 10^{-21} \) | 9.4 \( \times 10^{5} \) |

Note. \( t_{10} \) is the time to decay \( L_{initial} \) to \( 10^{-10} L_{initial} \). \( t_D \) means the timescale is very large.

Since \( L_D \) increases with an increase in the magnetic field, it is understood from the tables that the WDs possessing high values of the magnetic field have \( L_D \gg L_{GW} \). Since the luminosity is dominated by \( L_D \) and \( L \propto dE/dt \), the timescale is governed by \( L_D \). Moreover, the total luminosity of a WD decreases with time either due to a decrease in \( \chi \) or a decrease in \( \Omega \). Whenever \( L_D \gg L_{GW} \), \( \chi \) decreases much faster as compared to \( \Omega \). For \( L_D \gg L_{GW} \), the Equations (6) and (7) can
be approximated as follows

\[ L_{c,c} \frac{d\Omega}{dt} = -\frac{B_p^2 R_p^6 \Omega^3}{2c^3} \sin^2 \chi F(x_0), \quad (9) \]

\[ L_{c,c} \frac{d\chi}{dt} = -\frac{B_p^2 R_p^6 \Omega^2}{2c^3} \sin \chi \cos F(x_0), \quad (10) \]

assuming \( L_{c,c} \) not changing with time. Let us denote the timescale for the change in \( \Omega \) to be \( T_\Omega \) and that for \( \chi \) to be \( T_\chi \). Integrating these two equations, we obtain

\[ T_\Omega \sim \left( \frac{2 L_{c,c} c^3}{B_p^2 R_p^6 \Omega^2 F(x_0)} \right) \frac{1}{2 \sin^2 \chi}, \quad (11) \]

\[ T_\chi \sim \left( \frac{2 L_{c,c} c^3}{B_p^2 R_p^6 \Omega^2 F(x_0)} \right) \ln \cot \chi. \quad (12) \]

In the range \( 0^\circ \leq \chi \leq 30^\circ \), we always have \( \ln \cot \chi \ll 1/2 \sin^2 \chi \), which implies \( T_\chi \ll T_\Omega \). This proves that \( \chi \) quickly becomes 0, and the WD starts rotating with a different angular velocity than it originally possesses. For example, if \( M = 1.42M_\odot, B_p = 8.9 \times 10^{11} \text{ G}, R_p = 1200 \text{ km} \), and at \( t = 0, \Omega = \pi \text{ rad s}^{-1} \), and \( \chi = 30^\circ \) such that \( L_D \gg L_{GW} \), then \( T_\Omega \approx 3 \text{ yr} \) and \( T_\chi \approx 0.2 \text{ yr} \). It can also be verified from Figure 2(a). Moreover, combining the Equations (9) and (10), we obtain the differential equation

\[ \frac{d\Omega}{d\chi} = \frac{\Omega \sin \chi}{\cos \chi}. \quad (13) \]

Solving this differential equation using the initial condition \( \chi = 30^\circ \), we obtain

\[ \Omega = \frac{\sqrt{3}}{2 \cos \chi} \Omega_0, \quad (14) \]

where \( \Omega_0 \) is the initial angular velocity of the WD. Using this formula, one can verify that the final time period would be \( \sim 2.3 \text{ s} \) if the initial time period is \( 2 \text{ s} \), and this is clearly evident from Figure 2(a).

3.2. Case II: \( L_{GW} \gg L_D \)

If the magnetic field is lower, but not the angular velocity, the WDs have \( L_{GW} \gg L_D \). In such a case, luminosity decreases slowly and a WD can radiate for a long period of time. For \( L_{GW} \gg L_D \), Equations (6) and (7) can be written as

\[ L_{c,c} \frac{d\Omega}{dt} = -\frac{2G}{5c^5}(I_{c,c} - I_{c,c})^2 \widehat{\Omega}^2 \sin^2 \chi (1 + 15 \sin^2 \chi), \quad (15) \]

\[ L_{c,c} \frac{d\chi}{dt} = -\frac{12G}{5c^5}(I_{c,c} - I_{c,c}) \Omega^4 \sin \chi \cos \chi. \quad (16) \]

Integrating these two equations, we obtain the timescales of changes in \( \Omega \) and \( \chi \), given by

\[ T_\Omega' \sim \left( \frac{5 L_{c,c} c^5}{2G(I_{c,c} - I_{c,c}) \Omega^4} \right) \frac{1}{4 \sin^2 \chi (1 + 15 \sin^2 \chi)}, \quad (17) \]

\[ T_\chi' \sim \left( \frac{5 L_{c,c} c^5}{2G(I_{c,c} - I_{c,c}) \Omega^4} \right) \frac{1}{12} \left( \frac{1}{\sin^2 \chi} + 2 \ln \cot \chi \right). \quad (18) \]

In the range \( 0^\circ \leq \chi \leq 30^\circ \), we have \( \Omega \) and \( \chi \) keep varying simultaneously for a long time before approaching to zero, which also can be verified from Figure 2(b). For instance, if \( M = 1.42M_\odot, B_p = 1.3 \times 10^{5} \text{ G}, R_p = 1209 \text{ km} \), and at \( t = 0, \Omega = \pi \text{ rad s}^{-1} \), and \( \chi = 30^\circ \) such that \( L_{GW} \gg L_D \), then \( T_\Omega' \approx 4 \times 10^{23} \text{ yr} \) and \( T_\chi' \approx 7 \times 10^{23} \text{ yr} \). Moreover, combining Equations (15) and (16), we obtain

\[ \frac{d\Omega}{d\chi} = \frac{1 + 15 \sin^2 \chi}{\sin \chi \cos \chi}. \quad (19) \]

Solving this equation using the initial condition \( \chi = 30^\circ \), we obtain

\[ \Omega = \frac{3^8 \sin \chi}{2^{13} \cos^{10} \chi} \Omega_0. \quad (20) \]

This proves that as \( \chi \to 0, \Omega \to 0 \); unlike the earlier case mentioned in Section 3.1. Hence the overall timescale is
determined by the change in both \( \Omega \) and \( \chi \), and in this case, it turns out to be much longer.

### 3.3. Super-Chandrasekhar WDs with Poloidal Magnetic Field

As we have mentioned above, the chosen values of magnetic field and rotation in Tables 1–3 cannot give super-Chandrasekhar WDs. However, at the time of its birth, a WD may possess a very high magnetic field (may be suitable mixed fields), which is even larger than the Schwinger limit of \( 4.414 \times 10^{13} \) G. This value of the magnetic field can make the WD significantly super-Chandrasekhar. If such a WD behaves like a pulsar, it can also emit a significant amount of gravitational radiation. However, due to high \( L_D \), these WDs cannot emit radiation for a longer duration, as \( \chi \) becomes zero much more quickly. This is evident from the first three rows of Table 4, where \( B_p \) is larger than \( 10^{13} \) G. Such WDs may also be detected by the future GW detectors just for a short duration of time (maybe momentarily). Recently, we have proposed the effect of noncommutativity on the EoS of the degenerate electrons (Kalita et al. 2019; Pal & Nandi 2019). We have shown that if noncommutativity is significant in a WD, it can have a mass of up to \( \sim 2.6 M_\odot \) even for a static nonmagnetized WD. If such WDs possess magnetic field and rotation, they can emit a significant amount of GW, which can also be detected by upcoming space-based detectors, such as LISA, DECIGO, and BBO. The last four rows of Table 4 show the timescales for the WDs if their EoS is governed by noncommutativity. It is evident that even if such WDs possess a lower magnetic field, they emit continuous gravitational radiation for a fairly longer duration. This will also be a valid test of the presence of noncommutativity in WD matter.

### 3.4. Effect of WD’s Birth Rate on Its Detection

It is important to note that the birth rate of WDs can also significantly affect the detection of massive WDs. The birth rate of a WD is \( \sim 10^{-12} \) pc\(^{-3}\) yr\(^{-1} \) (Guseinov et al. 1983), which means within a 100 pc radius, on average, only one WD is formed in 10\(^6\) yr. Hence if that particular WD is super-Chandrasekhar only due to the high magnetic field, particularly poloidally dominated (see below Section 3.5), it may not be detected by the detector, or at best, if one is lucky enough, it may be detected only for a short duration of time as it loses its spin-down luminosity very quickly. However, if noncommutativity prevails as compared to the magnetic field, as described in Section 3.3 above, such super-Chandrasekhar WDs (with weaker fields not really affecting the mass) emit a significant amount of GW for a more extended period, and the GW detectors can detect them for a longer duration. Of course, it is well known that the presence of a purely poloidal magnetic field in the WD makes it unstable (Markey & Tayler 1973). Hence, in reality, such WDs should have some toroidal magnetic field as well.

### 3.5. WDs with Toroidal Magnetic Field

Whatever calculations we have shown so far above are based on the simplistic assumption of purely poloidal field so that we can consistently use the formula of \( L_D \). In reality, a WD is stable only if it consists of both the toroidal and poloidal components suitably. However, such a suitable configuration is not possible to obtain with the help of the XNS code. XNS can capture a twisted-torus configuration, which contains significantly poloidally dominated fields. Hence we consider a few cases of WDs for \( \rho_c = 2 \times 10^{10} \) g cm\(^{-3}\) containing a toroidal magnetic field. Of course, in this case, we drop the contributions of the term \( L_D \). In other words, we assume that even if the WD possesses any dipole contribution, its effect is much smaller, which is similar to the case we have mentioned in Section 3.2 with specific estimates. Such a configuration possesses super-Chandrasekhar mass, because the toroidal field is dominant at the center and, at the surface, it may have a negligible contribution. WDs containing mostly toroidal field and a negligible poloidal component is indeed a stable configuration. Wickramasinghe et al. (2014) showed that such a configuration remains stable even after a long time, which satisfies the stability criteria given by Braithwaite (2009). They also proposed that the poloidal field is generated as a by-product of the decay of the toroidal field. As in this configuration \( L_{GW} \gg L_D \), such a magnetized super-Chandrasekhar WD can radiate for a long time.

### Table 4

Super-Chandrasekhar WDs Possessing Poloidal Magnetic Field for \( \rho_c = 2 \times 10^{10} \) g cm\(^{-3}\)

| \( M_\odot \) | \( R_p \) (km) | \( B_p \) (G) | \( P \) (s) | ME/GE | KE/GE | \( L_{GW} \) (erg s\(^{-1}\)) | \( L_D \) (erg s\(^{-1}\)) | \( h_0 \) | \( t_{\text{rot}} \) (yr) |
|---|---|---|---|---|---|---|---|---|---|
| 1.69 | 748.7 | \( 3.6 \times 10^{13} \) | 2.0 | \( 1.0 \times 10^{-1} \) | \( 2.3 \times 10^{-3} \) | \( 1.6 \times 10^{39} \) | \( 3.5 \times 10^{41} \) | \( 1.2 \times 10^{-20} \) | \( 1.5 \times 10^{-20} \) |
| 1.67 | 757.6 | \( 3.5 \times 10^{13} \) | 10.0 | \( 1.0 \times 10^{-1} \) | \( 9.1 \times 10^{-5} \) | \( 9.4 \times 10^{44} \) | \( 5.5 \times 10^{40} \) | \( 4.6 \times 10^{-22} \) | \( 3.7 \times 10^{-2} \) |
| 1.67 | 757.5 | \( 3.5 \times 10^{13} \) | 100.0 | \( 1.0 \times 10^{-1} \) | \( 9.1 \times 10^{-7} \) | \( 9.4 \times 10^{24} \) | \( 5.5 \times 10^{36} \) | \( 4.6 \times 10^{-24} \) | \( 3.7 \times 10^{-2} \) |
| 3.18 | 899.3 | \( 4.9 \times 10^{13} \) | 2.0 | \( 1.1 \times 10^{-1} \) | \( 2.3 \times 10^{-3} \) | \( 1.3 \times 10^{40} \) | \( 1.9 \times 10^{-4} \) | \( 3.4 \times 10^{-20} \) | \( 7.8 \times 10^{-2} \) |
| 3.15 | 917.0 | \( 4.6 \times 10^{13} \) | 10.0 | \( 1.1 \times 10^{-1} \) | \( 9.1 \times 10^{-5} \) | \( 7.8 \times 10^{35} \) | \( 3.0 \times 10^{41} \) | \( 1.3 \times 10^{-21} \) | \( 1.9 \times 10^{0} \) |
| 2.65 | 1492.9 | \( 1.1 \times 10^{8} \) | 2.0 | \( 6.7 \times 10^{-12} \) | \( 2.5 \times 10^{-3} \) | \( 1.6 \times 10^{37} \) | \( 2.1 \times 10^{34} \) | \( 1.2 \times 10^{-21} \) | \( t_{\text{rot}} \) |
| 2.64 | 1501.8 | \( 1.1 \times 10^{8} \) | 10.0 | \( 6.6 \times 10^{-12} \) | \( 9.6 \times 10^{-5} \) | \( 1.1 \times 10^{40} \) | \( 3.2 \times 10^{31} \) | \( 1.6 \times 10^{-24} \) | \( 1.3 \times 10^{10} \) |

Note. WDs in the first three rows follow Chandrasekhar EoS and the rest follow noncommutative EoS. \( t_{\text{rot}} \) means the timescale is very large.
however, are not possible to show in this paper due to the limitations of the code. Here purely poloidal and purely toroidal magnetic field configurations are replicas of poloidally dominated and toroidally dominated mixed field configurations, respectively.

### 3.6. Detectability of Isolated Magnetized WDs in Gravitational-wave Astronomy

Let us now briefly discuss the properties of GW strengths emitted by the magnetized WDs. A detailed discussion of GW strengths for various WDs with different sets of parameters is given by Kalita & Mukhopadhyay (2019). Figure 3 shows the dimensionless GW amplitudes for the WDs with respect to their frequencies, as given in Tables 1–5, along with the sensitivity curves of various detectors. Optimum \( t \) is chosen for \( \chi \) at \( t = 0 \).

In this work, we analyze the timescales related to pulsating WDs. We have considered both the dipole and GW luminosities emitted by pulsating WDs, which was, to our knowledge, not explored consistently before this work. We have used the XNS code to model the WDs, primarily containing the poloidal magnetic field such that we can treat them as oscillating dipoles. Our target has been to calculate the timescale for detecting the super-Chandrasekhar WDs through the GW detectors. We have shown that many of these massive WDs have higher GW amplitude, and they are well above the signal-to-noise ratio of the GW detectors. If the WDs are massive due to the high poloidal magnetic field, i.e., they possess high dipolar luminosities, they cannot be detected for a longer duration. However, if a massive WD possesses a high toroidal field at the center and very less magnetic field at the pole, it can be detected by the detectors for a long time. Moreover, if the WDs gain extra mass due to some other effects, such as noncommutative geometry, but possess some weaker fields, they can also emit gravitational radiation continuously for a long time, and the GW detectors should easily detect them.

S.K would like to thank Timothy Brandt of the University of California, Santa Barbara, for a useful discussion about the timescale for WDs. B.M. would like to thank Tom Marsh of the California, Santa Barbara, for a useful discussion about the noncommutative geometry, but possesses some weaker fields, they can also emit gravitational radiation continuously for a long time, and the GW detectors should easily detect them.

### 4. Conclusions

In this work, we analyze the timescales related to pulsating WDs. We have considered both the dipole and GW luminosities emitted by pulsating WDs, which was, to our knowledge, not explored consistently before this work. We have used the XNS code to model the WDs, primarily containing the poloidal magnetic field such that we can treat them as oscillating dipoles. Our target has been to calculate the timescale for detecting the super-Chandrasekhar WDs through the GW detectors. We have shown that many of these massive WDs have higher GW amplitude, and they are well above the signal-to-noise ratio of the GW detectors. If the WDs are massive due to the high poloidal magnetic field, i.e., they possess high dipolar luminosities, they cannot be detected for a longer duration. However, if a massive WD possesses a high toroidal field at the center and very less magnetic field at the pole, it can be detected by the detectors for a long time. Moreover, if the WDs gain extra mass due to some other effects, such as noncommutative geometry, but possess some weaker fields, they can also emit gravitational radiation continuously for a long time, and the GW detectors should easily detect them.

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**Software:** XNS code solves the time independent general relativistic magnetohydrodynamic (GRMHD) equations, see Pili et al. (2014) for development of its latest version, url: http://www.arcetri.astro.it/science/ahead/XNS/code.html.

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**Table 5**

| \( M (M_\odot) \) | \( R_e (\text{km}) \) | \( B_{\text{max}} (\text{G}) \) | \( P (\text{s}) \) | \( \text{ME/GE} \) | \( \text{KE/GE} \) | \( L_{\text{GW}} (\text{erg s}^{-1}) \) | \( h_0 \) | \( t_{d} (\text{yr}) \) |
|---------------|-----------------|----------------|---------|-----------|-----------|----------------|---------|-----------|
| 1.71          | 2095.4          | 2.6 \times 10^{14} | 2.0     | 1.0 \times 10^{-1} | 5.4 \times 10^{-3} | 3.1 \times 10^{-5} | 1.7 \times 10^{-20} | \( t_d \) |
| 1.44          | 1315.7          | 1.1 \times 10^{14} | 2.0     | 1.0 \times 10^{-2} | 2.7 \times 10^{-3} | 2.6 \times 10^{-5} | 4.6 \times 10^{-22} | \( t_d \) |
| 1.67          | 1767.6          | 2.6 \times 10^{14} | 10.0    | 1.0 \times 10^{-1} | 2.0 \times 10^{-4} | 2.3 \times 10^{-6} | 7.5 \times 10^{-22} | \( t_d \) |
| 1.43          | 1253.7          | 1.1 \times 10^{14} | 10.0    | 1.0 \times 10^{-2} | 1.0 \times 10^{-4} | 6.3 \times 10^{-2} | 3.7 \times 10^{-23} | \( t_d \) |

**Note.** Here \( B_{\text{max}} \) is the strength of the maximum magnetic field in the WD and \( R_e \) is the equatorial radius. \( t_d \) means the timescale is very large.

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**Figure 3.** Dimensionless GW amplitude for white dwarfs as a function of frequency, as given in Tables 1–5, along with the sensitivity curves of various detectors. Optimum \( \chi \) is chosen for \( t = 0 \).

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\[ \text{http://gwplotter.com/} \text{ and http://www.srl.caltech.edu/~shane/sensitivity/} \]
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