Abstract

We study the dependence of quark condensate $\Sigma$ on an external magnetic field. For weak fields, it rises linearly:

$$\Sigma(H) = \Sigma(0) \left[ 1 + \frac{eH \ln 2}{16\pi^2 F_\pi^2} + O\left(\frac{e^2 H^2}{F_\pi^4}\right) \right]$$

(1)

$M_\pi$ and $F_\pi$ are also shifted so that the Gell-Mann – Oakes – Renner relation is satisfied.

In the strong field region, $\Sigma(H) \propto (eH)^{3/2}$.

1 Introduction.

Phase structure of QCD is now a subject of an intense discussion. The question of how the properties of the system are modified by non-zero temperature was studied especially well (see e.g. [1] for a recent review). We know now that, in the theory with massless quarks, phase transition with restoration of spontaneously broken chiral symmetry occurs at some temperature $T = T_c$. That means that the order parameter of the symmetry
breaking, the quark condensate $\Sigma = -\langle \bar{q}q \rangle$ falls down as temperature increases and turns to zero at $T \geq T_c$. When temperature is small compared to characteristic hadron scale $\mu_{hadr}$, the dependence $\Sigma(T)$ is known exactly $[2]$. In the case of two massless flavors, 

$$\Sigma(T) = \Sigma(0) \left[ 1 - \frac{T^2}{8F_\pi^2} - \frac{T^4}{384F_\pi^4} - \cdots \right]$$

(2)

The derivation of this formula relies on the fact that a lukewarm heat bath involves mainly pions — other degrees of freedom are not excited yet. The pion interaction at small energies is known from the effective chiral Lagrangian:

$$L = \frac{F_\pi^2}{4} \text{Tr}\{\partial_\mu U \partial^\mu U^\dagger\} + \Sigma \text{Re} \text{Tr}\{\mathcal{M}U^\dagger\} + \text{higher order terms}$$

(3)

Here $U$ is a unitary SU(2) matrix (it may be parameterized as $U = \exp\{i\tau^a \phi^a / F_\pi\}$ where $\phi^a$ is the pion field) and we included also the mass term involving quark mass matrix $\mathcal{M}$. $F_\pi = 93$ MeV is the pion decay constant and the parameter $\Sigma$ has the meaning of quark condensate $\Sigma = |\langle \bar{u}u \rangle| = |\langle \bar{d}d \rangle|$. The particular formula (3) of the Lagrangian is dictated by chiral symmetry (see $[3]$ for a nice pedagogical review).

But temperature is not the only external parameter which can affect the properties of the system. One can consider equally well a cold but dense system with non-zero mean baryon charge density related to the chemical potential. This system is less studied, but there are good reasons to believe that also in this case chiral symmetry is restored when the chemical potential exceeds some critical value.

In this paper, we address the issue of how the properties of QCD vacuum state depend on an external magnetic field. A naive expectation based on the analogy with superconductivity and with the situation in QCD at non-zero temperature and/or chemical potential could be that the condensate decreases as the magnetic field increases and melts down completely at some critical value $H = H_c$ above which the chiral symmetry is restored. We will see (and that is our main conclusion) that it is not so. Quark condensate rises with the increase of magnetic field and no phase transition occurs.

The behavior of a hadron system in magnetic field was studied earlier by Klevansky and Lemmer in the framework of Nambu-Jona-Lasinio (NJL) model $[4]$. Solving gap equation in the presence of an external magnetic field they found that the order parameters of the spontaneously broken chiral symmetry, the dynamical fermion mass and the chiral condensate, rise with the field. This conclusion was confirmed in later studies of NJL model and related theories $[5]$. The shift of fermion mass and of the condensate is quadratic in field

$$\Sigma(H) = \Sigma(0) \left[ 1 + c \frac{e^2 H^2}{\Sigma^4} + o \left( \frac{e^2 H^2}{\Sigma^4} \right) \right]$$

(4)

So, in this model, chiral symmetry is not restored by magnetic field. Qualitatively, NJL method behaves in the same way as QCD. However, we shall see later that the quantitative predictions are different.
Figure 1: Vacuum energy in a weak magnetic field. Solid lines stand for charged pions.

2 Weak field.

We will consider QCD with two massless flavors. In the leading order of chiral perturbation theory, the masses of $u$- and $d$- quarks appear in the vacuum energy only via its dependence on $M^2_\pi$, which is proportional to the sum $m_u + m_d$. It means that in this order $\langle \bar{u}u \rangle$ and $\langle \bar{d}d \rangle$ condensates will not differ and we can define the quark condensate as

$$\Sigma(H) = -\frac{\partial \epsilon_{\text{vac}}(m_u, m_d, H)}{\partial m_u} \bigg|_{m_u=m_d=0} = -\frac{\partial \epsilon_{\text{vac}}(m_u, m_d, H)}{\partial m_d} \bigg|_{m_u=m_d=0}$$  \hspace{1cm} (5)$$

where the small quark masses $m_u, m_d$ are introduced as external probes.

To find $\epsilon_{\text{vac}}$, we need to calculate vacuum loops in the presence of magnetic field. When the field is weak $eH \ll \mu^2_{\text{hadr}} \sim (2\pi F_\pi)^2$ (however, we will always assume $eH \gg M^2_\pi$), characteristic momenta in the loops are small and the theory is adequately described by the effective low energy chiral Lagrangian (3). In the leading order, pion interactions can be neglected whatsoever, and the field dependent part in $\epsilon_{\text{vac}}$ is given by one loop graphs depicted in Fig. 1.

It is instructive to consider first the graph with two photon legs. The corresponding contribution in the vacuum energy is

$$\epsilon^{(2)}_{\text{vac}}(H, M^2_\pi) = \frac{e^2 H^2}{96\pi^2} \ln \frac{\Lambda^2}{M^2_\pi},$$ \hspace{1cm} (6)$$

where $\Lambda$ is the ultraviolet cutoff. By virtue of Gell-Mann - Oakes - Renner relation,

$$F^2_\pi M^2_\pi = \Sigma(m_u + m_d),$$ \hspace{1cm} (7)$$

one can relate the derivatives over $m_u(m_d)$ and over $M^2_\pi$. Taking into account Eqs. (3), (5), and (7), we obtain

$$\Delta \Sigma^{(2)}(H) = \frac{\Sigma}{F^2_\pi} \frac{\epsilon^2 H^2}{96\pi^2 M^2_\pi^2}. \hspace{1cm} (8)$$

This expression diverges in the chiral limit $M^2_\pi \to 0$ and makes as such little sense. It is easy to trace back the origin of this divergence. The corresponding graph for $\epsilon_{\text{vac}}$ involves the logarithmic divergence both in the ultraviolet and in the infrared. After
differentiating, it gives the contribution in the condensate which diverges as a power in infrared.

But one is not allowed, of course, to restrict oneself by the graphs with only two external field insertions. All other graphs are also important. The more is the number of legs, the more severe are the infrared singularities. For example, the graph with four legs involves the singularity $\sim (eH)^4/M_\pi^4$ in the vacuum energy which gives the singularity $\propto 1/M_\pi^6$ in the condensate, etc. All such graphs should be summed up. The result is the analog of the Euler-Heisenberg Lagrangian for scalar particles which was found long time ago by Schwinger [6]. We have

$$
\epsilon_{\text{vac}} = -\frac{1}{16\pi^2} \int_0^\infty \frac{dz}{z^2} \left\{ \frac{z}{\sinh(z)} - 1 \right\} = -\ln 2 \quad (9)
$$

We see that the integral (9) is regular in the infrared (large $s$ region). The infrared cutoff is provided now not by $M_\pi^2$, but by the field $eH$ itself. All infrared divergent pieces in the graphs in Fig. 1 are summed up into a finite expression.

To find the shift of the condensate, we have to differentiate Eq. (9) over quark mass and substitute it into the definition (5). We arrive at the expression proportional to the simple integral

$$
I = \int_0^\infty \frac{dz}{z^2} \left\{ \frac{z}{\sinh(z)} - 1 \right\} = -\ln 2 \quad (10)
$$

which can be done as a sum of residues of the poles on, say, positive imaginary $z$ axis. Our final result is

$$
\Sigma(H) = \Sigma(0) \left[ 1 + \frac{eH \ln 2}{16\pi^2 F_\pi^2} + \ldots \right]. \quad (11)
$$

We see that the shift is positive and linear in $H$. The latter is easy to understand if substituting the actual infrared cutoff $\sim eH$ for $M_\pi^2$ in Eq. (8).

It is also clear now why NJL model gave the shift which was quadratic rather than linear in field: the corresponding calculation involved the loop of massive quarks rather than the loop of massless pions and was infrared finite in any order in $H$. A side remark is that a simple-minded NJL calculation is also not able to reproduce the result (4) for the temperature dependence of the condensate. At small temperatures, the density of massive quarks in the heat bath is exponentially suppressed, while massless pions are in abundance.

The correction $\propto eH$ in Eq. (11) is of order 1 when the field is comparatively large $\sqrt{eH} \sim 1.4$ GeV. Well before that, our calculation based on the effective pion Lagrangian (3) loses validity. In principle, one can trace the deviation from the leading order result (11) at intermediate values of $H$ in the framework of chiral perturbation theory [7] taking into account nonlinear pion interactions. An example of the graph contributing in the

\(^1\text{This is the unrenormalized vacuum energy. More precisely, the quartic divergence which corresponds to the loop without legs and is field independent is subtracted, but the logarithmic ultraviolet divergence associated with the graph with two legs is not. The expression usually found in the textbooks corresponds to subtracting also the two-leg graph, so that the expansion of } \epsilon_{\text{vac}}^\text{ren}(H, M_\pi) \text{ in } H \text{ starts from the term } \propto H^4. \text{ The two-leg graph is then absorbed in the charge (} \equiv \text{field}) \text{ renormalization, and the derivative of } H^2_{\text{ren}}(M_\pi)/2 + \epsilon_{\text{vac}}^\text{ren}(H, M_\pi) \text{ over mass would, of course, be the same as that of } \epsilon_{\text{vac}}^\text{unren}.\)
Figure 2: A two-loop graph contributing to the vacuum energy.

vacuum energy in the next order is depicted in Fig. 2. Also higher order terms in the chiral Lagrangian which we did not specify and did not discuss here would become important. The estimate for two-loop correction to the condensate is \( \propto \Sigma(0)(eH)^2/(2\pi F_\pi)^4 \) (the same estimate \( \propto \Sigma(0)(eH)^2/\mu_{\text{had}}^4 \) is obtained if taking into account massive charged particles such as \( K \) and \( \rho \) meson). In this order, one should not expect the corrections to \( |<\bar{u}u>| \) and \( |<\bar{d}d>| \) to be the same (the magnetic field breaks down isotopic invariance). The calculation of the coefficients (it is not yet clear whether also some logarithmic factor \( \propto \ln(eH/\mu_{\text{had}}) \) appears) is now in progress.

**3 \( M_\pi(H), F_\pi(H) \) and Gell-Mann - Oakes - Renner relation.**

As the electric charges of \( u^- \) and \( d^- \) quarks are different, flavor symmetry is broken in an external magnetic field. In particular, the axial \( SU_A(2) \) symmetry is broken down to \( U_A^3(1) \) corresponding to chiral rotation of \( u^- \) and \( d^- \) quarks with opposite phases (the singlet axial symmetry is broken already in the absence of the field due to the anomaly). The formation of the condensate breaks down this remnant \( U_A^3(1) \) symmetry spontaneously leading to appearance of a Goldstone boson, the \( \pi^0 \)-meson. Indeed, charged pions acquire a gap in the spectrum \( \propto \sqrt{eH} \) and are not goldstones anymore. If quarks are endowed a small non-zero mass \( m \), pions are not exactly massless, their mass being related to the quark condensate by the Gell-Mann - Oakes - Renner relation (7).

It is interesting to study the question how the mass and residue of \( \pi^0 \) depend on the external field.

Let us find first the mass shift. We have to calculate the polarization operator of \( \pi^0 \) given by the sum of graphs of the kind drawn in Fig. 3.

The four-pion vertex can be found from the expansion of the effective chiral Lagrangian. In the exponential parameterization \( U = \exp\{i\phi^a\tau^a/F_\pi\} \), the corresponding terms in the Lagrangian have the form

\[
L^{(4)} = \frac{1}{6F_\pi^2}[(\phi^a\partial_\mu\phi^a)^2 - (\phi^a\phi^b)(\partial_\mu\phi^b)^2] + \frac{M_\pi^2}{24F_\pi^2}(\phi^a\phi^a)^2. \tag{12}
\]

We need also the expression of charged pion propagator in a magnetic field. It can be inferred from the results of Ref. [6] where an explicit expression for the fermion propagator in a magnetic field at non-zero chemical potential \( \mu \) has been found. Basically, one has to take the integral multiplying the factor \( -\mu\gamma^0 \) in Eq. (4.9) of Ref. [6] without the factor \( \cos(eHs) + \gamma^1\gamma^2\sin(eHs) \equiv \exp\{ie(\sigma F)s/2\} \) in the integrand. We arrive at the following
expression for the Euclidean scalar propagator

\[ D_H(x, y) = \exp \left\{ ie \int_y^x A_\mu(\xi) d\xi_\mu \right\} \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} D_H(k), \]

where the integral in the phase factor is done along the straight line connecting \( x \) and \( y \), and

\[ D_H(k) = \int_0^\infty \frac{ds}{\cosh(es)} \exp \left\{ -s \left[ k^2_\parallel + k^2_\perp \frac{\tanh(es)}{es} + M^2_\pi \right] \right\} \]

with \( k^2_\parallel = k^2_1 + k^2_3 \), \( k^2_\perp = k^2_1 + k^2_2 \) (the magnetic field is aligned along the third axis).

Substituting (14) and the vertex inferred from (12) in the graph in Fig. 3 (as the propagator enters with coinciding initial and final points, the phase factor disappears), and subtracting the similar expression at \( H = 0 \), we obtain

\[ \Pi_H(p^2) - \Pi_0(p^2) = -\frac{1}{3F^2_\pi} \int \frac{d^4k}{(2\pi)^4} (M^2_\pi + 2p^2 + 2k^2) \int_0^\infty ds \left\{ \exp\left\{ -s \left[ k^2_\parallel + k^2_\perp \frac{\tanh(es)}{es} + M^2_\pi \right] \right\} - \exp\left\{ -s(k^2_\parallel + k^2_\perp + M^2_\pi) \right\} \right\}. \]

The mass shift is given by the shift (14) of the (Euclidean) polarization operator at the point \( p^2 = -M^2_\pi \). Note first of all that the shift is zero for massless pions. An exact goldstone remains the exact goldstone also when \( H \neq 0 \). The shift \( \Delta_H M^2_\pi \) is proportional to \( M^2_\pi(H = 0) \). Taking two first terms of the expansion of \( M^2_\pi \) and calculating first the integral over \( d^4k = \pi^2 k^2_\parallel dk^2_\perp \) and then over \( ds \) (the latter has the same structure as the integral (10) which we encountered when calculating the shift of the condensate and involves also an extra piece \( \sim \int_0^\infty ds [\cosh(s)/\sinh^2(s) - 1/s^2] \) which is, however, zero), we arrive at the simple result

\[ \Pi_H(p^2) - \Pi_0(p^2) = -p^2 \frac{eH \ln 2}{24\pi^2 F^2_\pi} + M^2_\pi \frac{eH \ln 2}{48\pi^2 F^2_\pi} + \ldots \]

which implies

\[ M^2_\pi(H) = M^2_\pi(0) \left[ 1 - \frac{eH \ln 2}{16\pi^2 F^2_\pi} + \ldots \right] \]

Figure 3: An example of the graph contributing to the polarization operator of \( \pi^0 \) in a magnetic field.
Figure 4: Axial currents correlator in a magnetic field. The graphs contain exact propagator of charged pions in the constant magnetic field.
Let us find now the renormalization of the residue brought about by the field. We need to calculate the one-loop graphs depicted in Fig. 4 which contribute to the pole structure in the axial current correlator \( \int dxe^{ipx} \langle A^3_\mu(x)A^3_\nu(0) \rangle_H \sim p_\mu p_\nu F^2_\pi(H)/(p^2 + M^2_\pi) \). As \( F_\pi \) and its shift are non-zero in the chiral limit, we can set \( M^2_\pi = 0 \). The graph in Fig. 4a has already been calculated (the first term in Eq. (16)). To calculate the graphs in Fig. 4b,c, we need the vertex \( <0|A^3_\mu|\pi^0\pi^+\pi^-> \) which can be found by “covariantizing” the derivative in Eq. (3) according to the rule \[ \partial^\mu U \rightarrow \partial^\mu U - i(A^\mu U + U A^\mu) \] where \( A^\mu = A^\mu_t a \). The final result is
\[
F^2_\pi(H) = F^2_\pi(0) \left[ 1 + \frac{eH \ln 2}{8\pi^2 F^2_\pi} + ... \right]
\]
Note that the contributions of the individual graphs Fig. 4a and Fig. 4b,c depend on the chosen exponential parameterization, but the physical pion residue \( F^2_\pi(H) \) depending on their sum does not. Also the physical pole position (17) does not depend on parametrization while the individual terms in (16) do.

The results (11), (17), and (18) are very much analogous to Eq. (2) and the similar known expressions for \( M^2_\pi(T) \) and \( F^2_\pi(T) \). Only here the physical situation is just opposite compared to the thermal case. A nonzero temperature suppresses the condensate and residue, and leads to the increase of the mass. A nonzero magnetic field brings about the increase in the condensate and residue, and suppresses the mass.

In the thermal case, the renormalized condensate, pion mass, and residue still satisfy the relation (12) in the leading order in \( T^2 \). Likewise, the expressions (11), (17), and (18) satisfy the Gell-Mann – Oakes – Renner relation in the leading order in \( eH \).

4 Strong field.

When \( eH \gg \mu^2_{\text{hadr}} \), characteristic momenta in the vacuum loops are high, and the system is adequately described in terms of quarks and gluons rather than in terms of pions and other low lying hadron states. Due to asymptotic freedom, the effective coupling constant \( \alpha_s(eH) \) is small, and we can try to treat strong interaction effects perturbatively.

For sure, in QCD with massless quarks, the condensate cannot appear in any finite order of perturbation theory, irrespectively of whether a magnetic field is present or not — both electromagnetic and strong interaction vertices respect chirality.

It was recently discovered, however, that the condensate is generated in the strong field limit [9]. To see that, one has to sum up an infinite set of relevant graphs. The authors of [9] studied the Bethe-Salpeter equation which implements such a resummation for massless QED and showed that the equation admits a nontrivial solution with dynamically generated mass. Later this result was reproduced in the language of Schwinger-Dyson equation [10]. The derivation applies also in QCD without essential modifications. For clarity sake, let us say here few words about it.

When strong interaction is disregarded whatsoever, we have a system of free charged quarks in a magnetic field. The spectrum presents the set of Landau levels
\[
\epsilon_\pm(n, \sigma, k_3) = \pm \sqrt{|e_q H|(2n + \sigma + 1) + k_3^2 + m_q^2},
\]
where \( n = 0, 1, \ldots \), \( \sigma = \pm 1 \) marks the spin orientation, and \( k_3 \) is the momentum along the magnetic field direction. Negative energies describe the Dirac sea. The corresponding eigenstates are localized in the transverse direction. The characteristic size of the orbits is \( \sim 1/\sqrt{eH} \) (so that for \( \sqrt{eH} \sim 1.4 \) GeV they are already pretty small). In infinite space, each state \([13]\) is infinitely degenerate, the degeneracy being associated with the position of the center of the orbit. For a finite box of size \( L \), the level of degeneracy is

\[
N_\perp = \frac{|e\gamma H|}{2\pi} L^2,
\]

and also \( k_3 \) is quantized to \( 2\pi n/L \). For \( m_q = 0 \), the ground states have zero energy.

We need in the following the fermion Green’s function in a magnetic field. Its explicit expression was found in \([11]\) and, in a somewhat more convenient form, in \([8]\) with Schwinger technique. First of all, one can write

\[
G_H(x, y) = \exp \left\{ ie_q \int_y^x A_\mu(\xi) d\xi^\mu \right\} \hat{G}_H(x - y)
\]

where the integral in the phase factor is done along the straight line connecting \( x \) and \( y \). The Fourier image of \( \hat{G}_H(x - y) \) presents a complicated integral over proper time. Fortunately, we do not need the full expression, but only its asymptotic form in the region of small momenta \( k \ll \sqrt{eH} \). In this region, it suffices to retain only the lowest Landau levels (LLL) with \( n = 0, \sigma = -1 \) in the spectral decomposition of the Green’s function, and the latter acquires the simple form \([11, 9]\)

\[
\hat{G}_H(k) = ie^{-k_\perp/|e\gamma H|} \frac{\hat{k}_\parallel + m_q}{k^2_\parallel - m^2_q} (1 - i\gamma^1\gamma^2),
\]

where \( k_\parallel^2 = k_0^2 - k_3^2 \) and \( k_\perp^2 = k_1^2 + k_2^2 \).

Basically, this Green’s function describes a free motion of the states with \( \sigma = -1 \) in longitudinal direction. Strictly speaking, retaining the exponential factor \( \exp(-k_\perp/|e\gamma H|) \) is not quite consistent - the dominance of LLL on which the derivation of Eq. \([22]\) was based is justified only in the region where both \( k_\perp^2 \) and \( k_\parallel^2 \) are small compared to \( |e\gamma H| \) and the exponential factor is not efficient. We will see later, however, that the generation of condensate is related to the infrared region \( k \ll \sqrt{|e\gamma H|} \). Really, in the opposite limit \( k \gg \sqrt{|e\gamma H|} \), the Green’s function \( G_H(x, y) \) tends to the free fermion Green’s function, and we cannot expect a nontrivial dynamic phenomenon like the condensate generation to be associated with that region. The exponential factor in \([22]\) is convenient to retain as an effective momentum cutoff.

We are interested in the theory with massless quarks, so that \( m_q = 0 \) in the first place. We expect, however, the dynamical mass generation, and our Ansatz for the exact Green’s function at low momenta is Eq. \([22]\) with nonzero and, generally speaking, momentum-dependent \( m_q \).

Next, we substitute this Ansatz in the Schwinger-Dyson equations schematically presented in the Fig. 5. The equation was solved in the approximation where the loop corrections to the vertices and also to the gluon propagator (the latter is actually a rather strong and not so innocent assumption) were disregarded.
Doing the spinor trace and performing Wick rotation, one can see after some transformations that the equation admits a factorized solution

$$m_q(p_{\parallel}, p_{\perp}) = \hat{m}_q(p_{\parallel}^2) \exp\{-p_{\perp}^2/|e_q H|\},$$

where $\hat{m}_q(p_{\parallel}^2)$ satisfies the following 2-dimensional integral equation

$$\hat{m}_q(p_{\parallel}^2) = \frac{\alpha_s c_F}{2\pi^2} \int \frac{d^2k_{\parallel}}{k_{\parallel}^2 + \hat{m}_q^2(k_{\parallel}^2)} \int \frac{d^2k_{\perp}}{(k_{\parallel} - p_{\parallel})^2 + k_{\perp}^2} \exp\{-k_{\perp}^2/(2|e_q H|)\}.\quad(24)$$

The integral (24) is double logarithmic receiving a main support from the region $\hat{m}_q^2(0) \ll k_{\parallel}^2 \ll k_{\perp}^2 \ll |e_q H|$.

Let us first tentatively neglect the dependence of $\hat{m}_q$ on longitudinal momenta. We would obtain

$$\hat{m}_q(0) \sim \hat{m}_q(0) \frac{\alpha_s c_F}{4\pi} \ln^2 \frac{|e_q H|}{m_q^2(0)}.\quad(25)$$

The corresponding calculation was actually first done in [12] where a correction to the electron mass in the presence of the external magnetic field $\Delta m(H) = \frac{\alpha m(0)}{4\pi} \ln^2(eH/m^2(0))$ was found. In that (and other) earlier papers, the equation (25) was not treated, however, as a self-consistent equation allowing to unravel the dynamical generation of mass even if the lagrangian electron mass is zero. The solution of this equation is

$$\hat{m}_q(0) \sim \sqrt{|e_q H|} \exp\left\{-\frac{\pi}{\alpha_s c_F} \right\}.$$\quad(26)

The exponential factor displays a truly nonpertubative nature of the result. An accurate analysis of Ref. [1] which takes into account the momentum dependence of the mass leads to the result

$$\hat{m} \sim \sqrt{|e_q H|} \exp\left\{-\frac{\pi}{2\sqrt{2\alpha_s c_F}} \right\}.$$\quad(27)

The quark condensate is defined as

$$<\bar{q}q>_H = -\int \frac{d^4p}{(2\pi)^4} Tr \hat{G}_H(p) = -\frac{4}{(2\pi)^4} \int d^2p_{\perp} d^2p_{\parallel} \frac{m_q(p_{\parallel}^2, p_{\perp}^2)}{p_{\parallel}^2 + p_{\perp}^2 + m_q^2(p_{\parallel}^2, p_{\perp}^2)}.$$\quad(28)

The double logarithmic structure of the integral is easy to understand if disregarding the phase factor in Eq. (21) which makes the calculations elementary. That gives qualitatively the same form of the integral as in Eq. (24). Quantitative results, however, would be wrong. In particular, the coefficient in Eq. (24) would be 4 times larger than the actual one.
Assuming the exponential fall-off of dynamical mass at $p_\perp^2 \gg |e_q H|$, $p_\parallel^2 \gg |e_q H|$ (cf. Eq. (23)), we finally obtain

$$\Sigma_H = - \langle \bar{q} q \rangle_H = F(\alpha_s)|e_q H|^{3/2} \exp \left\{ -\frac{\pi}{2} \sqrt{\frac{\alpha_s}{2\alpha_s(|e_q H|)c_F}} \right\}$$

where $F(\alpha_s)$ is some function not involving an exponential dependence.

We see that the condensate increases with the field, and no phase transition occurs.

5 Discussion.

Our main result (11) is an exact theorem of QCD. It has the same status as Eq. (2) and some exact results for density of eigenvalues of Euclidean Dirac operator at finite volume [14] and in thermodynamic limit [15]. It would be very interesting to check this and other similar exact theorems in lattice calculations with dynamical fermions and/or vacuum model calculations [3].

The derivation of the result (29) in the strong field limit reviewed in the previous section is similar in spirit to the derivation of spontaneous symmetry breaking in the NJL model. In both cases, a self-consistent gap equation for the fermion propagator is solved. This method of derivation seems to be rather reliable [4], but it is indirect. It would be very nice to understand the appearance of the quark condensate in strong fields in plain and direct terms.

We have in mind the following. According to the famous Banks and Casher theorem [17], the quark condensate is related to the spectral density of Euclidean massless Dirac operator $\rho(\lambda)$ at $\lambda = 0$

$$\Sigma = \pi \rho(0).$$

Nonzero spectral density $\rho(0)$ means that the number of Dirac operator eigenvalues with $\lambda \leq \lambda_0 \ll \mu_{hadr}$ is estimated as $N(\lambda \leq \lambda_0) = \rho(0)\lambda_0 V$, where $V$ is the Euclidean volume. For free massless fermions, the spectrum is $\lambda_n = 2\pi|\mathbf{n}|/L$ where $\mathbf{n}$ is a four-dimensional integer vector. $\rho_{free}(\lambda) \sim \lambda^3$, $\rho_{free}(0) = 0$ and the condensate is zero. For free massless fermions in a magnetic field, the infrared branch of the spectrum is

$$\lambda_m = \frac{2\pi}{L}|\mathbf{m}|,$$

where $\mathbf{m}$ is a two-dimensional integer vector. The eigenvalues (31) correspond to free motion in the longitudinal direction. Each level (31) involves the high level of degeneracy (20). The spectral density is then estimated as $\rho_{free}^H(\lambda) \sim |e_q H|\lambda$. This is ”better”

3Unfortunately, lattice calculations are very difficult here. The Euclidean lattice should be roomy enough to accommodate pions: the size of the box should be considerably larger than the pion Compton wavelength. Calculations in the framework of the instanton liquid model [16] allow for larger boxes and are more promising.

4There is, however, a not quite clear for us at the moment question on whether the loop corrections to the gauge boson line in the Schwinger – Dyson equation in Fig.5 could modify the result. According to [13], the effective mass which gauge bosons acquire in a magnetic field brings about an additional infrared cutoff which kills one of the logarithms in the strong field asymptotic of the fermion mass operator.
than in the absence of the field, but still $\Sigma = \pi \rho(0) = 0$. A nonzero condensate means that, after taking into account the interaction between quarks, the Euclidean Dirac spectrum is substantially modified, and small eigenvalues with characteristic spacing $\sim 1/(\sqrt{|e_q H| L^2})$ appear. We are not able now to display the presence of such small eigenvalues explicitly.

An external magnetic field increases the condensate which means that it should make the chiral restoration phase transition in temperature and/or in baryon chemical potential more difficult. That means, in particular, that critical temperature $T_c$ (at $H = 0$, it is estimated to be of order 200 MeV) should increase with $H$. This question was studied in recent for massless QED. Translating their result in QCD language, we obtain

$$T_c \sim \alpha_s \sqrt{|e_q H|}$$

in the strong field region. The result is easy to understand. $\sqrt{|e_q H|}$ appears by dimensional reasons and the factor $\alpha_s$ appears because the generation of the condensate in a strong magnetic field is driven by strong perturbative inter-quark interaction.

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