HOMOTOPY METHOD FOR SOLVING GENERALIZED NASH EQUILIBRIUM PROBLEM WITH EQUALITY AND INEQUALITY CONSTRAINTS

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Abstract. In this paper, we utilize a new homotopy method to solve generalized Nash equilibrium problem with equality and inequality constraints on unbounded sets. Based on the existing homotopy method, we establish a new homotopy equation by introducing a suitable perturbation on the equality constraint, the existence and the global convergence of homotopy path under certain assumptions have also been proved. In the proposed method, the initial point only needs to satisfy the inequality constraints. Compared with the existing homotopy method, this method expands the scope of the initial points and provides the convenience for solving generalized Nash equilibrium problem. The numerical results illustrate the effectiveness of this method.

1. Introduction. Nash equilibrium theory is one of the important scientific developments in the twentieth century. Initially, Nash equilibrium theory was widely applied to economic and social problems. In recent years, Nash equilibrium theory has also been introduced to the management of science, computer science, mathematics, mechanics, physics, economics, statistics, transportation biology and other science communities [8].

The generalized Nash equilibrium problem (GNEP for short) is an extension of the classical Nash equilibrium problem proposed by Nash [18] in that the objective function and the feasible set of each decision maker is dependent on other players’ strategies. It has recently attracted much attention. The reader is referred to the survey paper on GNEP given by Facchinei and Kanzow [8] and the references therein for more applications, some theoretical results and an overview of existing methods for the solution of GNEPs. In fact, the solution of a GNEP in this general form is still little investigated. Finding a solution of such a GNEP is a very hard problem; see [3, 10] for a detailed discussion. Due to its difficulty, only very few results are available for the solution of a GNEP at the level of generality described above. In order to solve the GNEP and to have an efficient computational method, some reformulations of the GNEP have been proposed in the literature. Among these methods, one of the most common means is to rewrite the GNEP as a nonsmooth structured system that derives from the KKT systems for the player’s optimization.

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problems. In fact, this nonsmooth system is a quasi-variational inequality (QVI). However, the application of the Newton method to such a system normally requires some nonsingularity assumption on the Jacobian. But this nonsingularity condition is almost never met in certain cases (see, e.g., [4]). In [5], based on KKT systems of the GNEP, a globally convergent interior-point algorithm was proposed. However, this method assumes that the Jacobian is positive semidefinite and usually does not permit to compute solutions with high precision. Facchinei and Kanzow [9] proposed a semismooth Newton method to solve the nonsmooth system of the GNEP. However, the deficiency of this method is that it requires fairly strong assumptions and has to calculate the B-subdifferentials of the nonsmooth function.

Recently, a so called LP-Newton method has been developed to solve the nonsmooth systems of equations with non-isolated solutions [6, 7]. In [4], a LP-Newton method for solving generalized Nash equilibrium problems has been described. Indeed, although the LP-Newton method leads to fast local solutions in this case, it seems that it is only suitable to problems whose dimension is not too large, otherwise, the computational cost of each iteration can become expensive.

Another approach for the solution of QVIs is based on a reformulation as a constrained or unconstrained global optimization problem with the help of so-called gap functions, see [2, 3, 11, 12, 13, 16, 20] for some contributions in this direction. In general, these gap functions are nonsmooth which makes it difficult to solve the corresponding optimization problem. In some cases, the gap function is smooth, cf. [3, 12, 13], but this is true only for certain classes of QVIs, and global convergence is still difficult to guarantee.

The so-called relaxation method is another popular one, which is known to be globally convergent under a set of assumptions. Some of these assumptions, however, are rather strong or somewhat difficult to understand [14, 15].

In order to overcome these mentioned disadvantages of the above algorithms, we consider the homotopy method to solve the GNEPs. In 2009, Xu et al. [21] ever used the homotopy method to solve the GNEP in a bounded region. And the method illustrates the excellent convergence result and numerical result. In this paper, we propose a new homotopy method for the GNEP based on [21]. This method extends the GNEP to the unbounded set, and expands the scope of the initial point only to satisfy the given constraints of the inequalities. The existence and the global convergence of homotopy path under certain assumptions has also been proved. The numerical results illustrate the effectiveness of the method.

The remainder of this paper is organized as follows. In Section 2, the main results of the paper are given. A new homotopy equation for solving generalized Nash equilibrium problem is formulated. Furthermore, the existence and the global convergence of homotopy path under certain assumptions have also been proved. The numerical experiment results show that our method is effective and efficient in Section 3. In the last section, the conclusion of this paper is mentioned.

2. Generalized Nash equilibrium problem and homotopy method. Consider an $N$-person non-cooperative game in which each player’s strategy set depends on the other players’ strategies. Given a set-valued map $X^i : R^{n_{i-1}} \to R^{n_i}$, where $n_i$ is a positive integer with $n \equiv \sum_{i=1}^N n_i, n_{-i} = n - n_i$. To emphasize the $i$-th player’s variables within $x$, we write the vector $x$ in the partitioned form

$$x = (x^1, \cdots, x^i, \cdots, x^N) = (x^{-i}, x^i), x^i \in R^{n_i}.$$
For each \( x^{-i} \in \mathbb{R}^{n-i} \), \( g^i : \mathbb{R}^n \to \mathbb{R}^{n_i}, h^i : \mathbb{R}^{n_i} \to \mathbb{R}^{l_i} \), the strategy set of player \( i \) which is a subset of \( \mathbb{R}^{n_i} \) is denoted by \( X^i(x^{-i}) = \{ x^i \in \mathbb{R}^{n_i} : g^i(x) \leq 0, h^i(x^i) = 0 \} \). The index set is defined as

\[
I_i(x) = \{ j \in \{1, \ldots, m_i \} : g^i_j(x) = 0 \}, \quad I_i^i(x) = \{ j \in \{1, \ldots, l_i \} : h^i_j(x^i) = 0 \},
\]

where \( x = (x^{-i}, x^i) \in \mathbb{R}^n, x^i = (x_1^i, \ldots, x_{n_i}^i) \in \mathbb{R}^{n_i}, g^i(x) = (g^i_1(x), \ldots, g^i_{m_i}(x))^T, h^i(x^i) = (h^i_1(x^i), \ldots, h^i_{l_i}(x^i))^T \). The interior of \( X^i(x^{-i}) \) denotes \( X^i(x^{-i})^0 = \{ x^i \in \mathbb{R}^{n_i} : g^i(x) < 0, h^i(x^i) = 0 \} \), and the boundary set is \( \partial(X^i(x^{-i})) = \{ x^i \in \mathbb{R}^{n_i} : g^i_j(x) = 0, h^i_j(x^i) = 0, j \in I_i(x) \} \).

For the GNEP, each player’s strategy must belong to a set \( X^i(x^{-i}) \) that depends on the rival players’ strategies. The aim of player \( i \), given the other players’ strategies \( x^{*-i} \), is to choose a strategy \( x^i \) that solves the minimization problem

\[
\min_{x^i} u_i(x^{*-i}, x^i)
\]

\[
st. \ x^i \in X^i(x^{*-i}),
\]

where \( u_i : \mathbb{R}^n \to \mathbb{R} \) is often called payoff function of player \( i \).

For simplicity, the following notation is quoted

\[
\begin{align*}
&u(x) = \begin{pmatrix} u_1(x) \\ \vdots \\ u_N(x) \end{pmatrix}_{N \times 1}, \quad \square u(x) = \begin{pmatrix} \nabla_{x^i} u_1(x) \\ \vdots \\ \nabla_{x^i} u_N(x) \end{pmatrix}_{n \times 1}, \\
g(x) = \begin{pmatrix} g^1(x) \\ \vdots \\ g^N(x) \end{pmatrix}_{m \times 1}, \quad \square g(x) = \begin{pmatrix} \nabla_{x^i} g^1(x) \\ \vdots \\ \nabla_{x^i} g^N(x) \end{pmatrix}_{n \times m}, \\
h(x) = \begin{pmatrix} h^1(x^i) \\ \vdots \\ h^N(x^N) \end{pmatrix}_{l \times 1}, \quad \square h(x) = \begin{pmatrix} \nabla_{x^i} h^1(x^i) \\ \vdots \\ \nabla_{x^i} h^N(x^N) \end{pmatrix}_{n \times l},
\end{align*}
\]

where \( m = m_1 + \cdots + m_N, l = l_1 + \cdots + l_N, n = n_1 + \cdots + n_N \).

In 2009, Xu et al. [21] used the homotopy method to solve the generalized Nash equilibrium problem in a bounded region whose form is as follows:

\[
C = \{ x = (x^1, \ldots, x^N) \in \mathbb{R}^n : g^i(x) \leq 0, h^i(x^i) = 0, i = 1, \ldots, N \}, \quad \text{and correspondingly, } \ C^0 = \{ x = (x^1, \ldots, x^N) \in \mathbb{R}^n : g^i(x) < 0, h^i(x^i) = 0, i = 1, \ldots, N \} \text{ is its strictly feasible region.}
\]

Based on the following KKT system

\[
\nabla_{x^i} u_i(x) + \sum_{j=1}^{m_i} \lambda^i_j \nabla_{x^i} g^i_j(x) + \sum_{j=1}^{l_i} \beta^i_j \nabla_{x^i} h^i_j(x^i) = 0,
\]

\[
h^i(x^i) = 0
\]

\[
\lambda^i_j \geq 0, g^i(x) \leq 0,
\]

\[
\lambda^i_j g^i_j(x) = 0, j = 1, \ldots, m_i, i = 1, \ldots, N.
\]
where $\lambda^i = (\lambda^i_1, \cdots, \lambda^i_{m_i})^T$ and $\beta^i = (\beta^i_1, \cdots, \beta^i_l)^T$ are Lagrange multipliers, the homotopy equation is constructed as follows

$$
H(w, w^{(0)}, \mu) = \begin{pmatrix}
(1 - \mu) \left( \Box u(x) + \Box g(x) V e_m + \Box h(x) U e_l + \mu (x - x^{(0)}) \right) \\
V g(x) - \mu V^{(0)} g(x^{(0)}) \\
h(x)
\end{pmatrix} = 0,
$$

(3)

where $w = (x, \lambda, \beta) \in C \times R^m \times R^l$, $w^{(0)} = (x^{(0)}, \lambda^{(0)}, \beta^{(0)}) \in C^0 \times R^m \times R^l$, $\lambda = (\lambda^1, \cdots, \lambda^N)^T \in R^m$, $\beta = (\beta^1, \cdots, \beta^N)^T \in R^l$, $V = \text{diag}(\lambda) \in R^{m \times m}$, $V^{(0)} = \text{diag}(\lambda^{(0)}) \in R^{m \times m}$, $U = \text{diag}(\beta) \in R^{l \times l}$, $e_l = (1, \cdots, 1)^T \in R^l$, $e_m = (1, \cdots, 1)^T \in R^m$.

Under the following assumptions, the existence and convergence of the homotopy path are obtained.

**Assumption 1.**

(C1) Each function $h^i_j(x^i)$ is a real-valued differentiable affine function. For each $x^i \in R^{n-i}$, the functions $u_i(x^{-i}, \cdot)$ and $g^i_j(x^{-i}, \cdot)$ are three times continuously differentiable and convex in the argument $x^i$ for each $j = 1, \cdots, m_i$.

(C2) For each $i = 1, \cdots, N$ and $x \in C$, $\{\nabla_x h^i_j(x^i) : j \in I_i(x^i), \nabla_x g^i_j(x^{-i}, x^i) : j \in I_i(x^i)\}$ are independent.

(C3) $C^0$ is nonempty and $C$ is bounded.

(C4) For each $i = 1, \cdots, N$, $\{\nabla_x h^i(x^i)\}$ has the full column rank.

(C5) For each $i = 1, \cdots, N$ and $(x^{-i}, x^i) \in \{(x^{-i}, x^i) : h^i(x^i) = 0, g^i_j(x^{-i}, x^i) = 0, j \in I_i(x^i)\}$,

$$
x^i + \sum_{j \in I_i(x^i)} \lambda^i_j \nabla_x g^i_j(x^{-i}, x^i) + \sum_{j \in I_i(x^i)} \beta^i_j \nabla_x h^i_j(x^i) : \lambda^i_j \geq 0 \cap X^i(x^{-i}) = 0.
$$

In this paper, a new homotopy method for GNEP is given. In this method, we enlarge the scope of feasible set in that it is constrained only by inequality

$$
\tilde{C} = \{x = (x^1, \cdots, x^N) \in R^n : g^i(x) \leq 0, i = 1, \cdots, N\},
$$

and correspondingly, its strictly feasible region is $\tilde{C}^0 = \{x = (x^1, \cdots, x^N) \in R^n : g^i(x) < 0, i = 1, \cdots, N\}$. The homotopy equation is constructed as follows:

$$
H(w, w^{(0)}, \mu) = \begin{pmatrix}
(1 - \mu) \left( \Box u(x) + \Box g(x) V e_m + \Box h(x) U e_l + \mu (x - x^{(0)}) \right) \\
V g(x) - \mu V^{(0)} g(x^{(0)}) \\
h(x)
\end{pmatrix} = 0,
$$

(4)

where $w = (x, \lambda, \beta) \in \tilde{C} \times R^m \times R^l$, $w^{(0)} = (x^{(0)}, \lambda^{(0)}, \beta^{(0)}) \in \tilde{C}^0 \times R^m \times R^l$, $\lambda = (\lambda^1, \cdots, \lambda^N)^T \in R^m$, $\beta = (\beta^1, \cdots, \beta^N)^T \in R^l$, $V = \text{diag}(\lambda) \in R^{m \times m}$, $V^{(0)} = \text{diag}(\lambda^{(0)}) \in R^{m \times m}$, $U = \text{diag}(\beta) \in R^{l \times l}$, $e_l = (1, \cdots, 1)^T \in R^l$, $e_m = (1, \cdots, 1)^T \in R^m$.

Especially, when $\mu = 1$, the homotopy equation is written by

$$
x - x^{(0)} = 0
$$

$$
V g(x) - V^{(0)} g(x^{(0)}) = 0
$$

$$
h(x) - U e_l = 0.
$$
Therefore, we obtain $x = x^{(0)}$, $\beta = \beta^{(0)} = h(x^{(0)})$. Due to $g(x^{(0)}) < 0$, we can get $\lambda = \lambda^{(0)}$. Hence, the equation $H(w, w^{(0)}, 1) = 0$ has only one solution $w = w^{(0)}$. When $\mu = 0$, the above homotopy equation reduces to the KKT system of GNEP.

To derive the main results, the following lemmas from differential topology are introduced.

Let $\tilde{U} \subset R^n$ be an open set, and $\phi : \tilde{U} \to R^n$ be a $C^{\alpha} (\alpha > \max\{0, n - p\})$ mapping. We say that $y \in R^n$ is a regular value for $\phi$, if

$$\text{Range} \left[ \frac{\partial \phi(x)}{\partial x} \right] = R^n, \forall x \in \phi^{-1}(y).$$

**Lemma 2.1.** (Parameterized Sard Theorem [1]). Let $\tilde{V} \subset R^n$, $\tilde{U} \subset R^n$ be open sets and let $\phi : \tilde{V} \times \tilde{U} \to R^k$ be a $C^{\alpha}$ mapping, where $\alpha > \max\{0, m - k\}$. If $0 \in R^k$ is a regular value of $\phi$, then for almost all $a \in \tilde{V}$, $0$ is a regular value of $\phi_a = \phi(a, \cdot)$.

**Lemma 2.2.** (Inverse Image Theorem [19]). Let $\phi : \tilde{U} \subseteq R^n \to R^n$ be a $C^{\alpha}$ ($\alpha > \max\{0, n - p\}$) mapping. If $0$ is a regular value of $\phi$, then $\phi^{-1}(0)$ consists of some $(n - p)$-dimensional $C^{\alpha}$ manifolds.

**Lemma 2.3.** (Classification Theorem of One-Dimensional Smooth Manifolds [19]). A one-dimensional smooth manifold is homeomorphic to a unit circle or a unit interval.

Under the following assumptions, the existence and convergence of the homotopy path are obtained.

**Assumption 1’ (C1’)** Each function $h^i_j(x^i)$ is a real-valued differentiable affine function. For each $x^{-i} \in R^{n-i}$, the functions $u_i(x^{-i}, \cdot)$ and $g^i_j(x^{-i}, \cdot)$ are three times continuously differentiable and convex in the argument $x^i$ for each $j = 1, \cdots, m_i$.

**Assumption 2’ (C2’)** For each $i = 1, \cdots, N$ and $x \in C$, $\{\nabla_{x^i} g^i_j(x^{-i}, x^i) : j \in I_i(x), \nabla_{x^i} h^i_j(x^i) : j \in I_i(x^i)\}$ are independent.

**Assumption 3’ (C3’)** $C^0$ is nonempty.

**Assumption 4’ (C4’)** There exists $z \in C$ such that, for any sequence $x^{(k)} \in \tilde{C}$ with $\|x^{(k)}\| \to \infty$ as $k \to \infty$

$$\left(x^{(k)} - z\right)^T \Box u(x^{(k)}) > 0$$

holds for sufficiently large $k$.

For a given initial point $w^{(0)} = (x^{(0)}, \lambda^{(0)}, \beta^{(0)}) \in C^0 \times R^n_+ \times R^l$, we rewrite $H(w, w^{(0)}, \mu)$ as $H_{w^{(0)}}(w, \mu)$. The zero point set of $H_{w^{(0)}}$ is

$$H_{w^{(0)}}^{-1}(0) = \{(w, \mu) \in \tilde{C} \times R_+ \times R^l \times (0, 1) \subseteq R^{n+m+l+1} : H_{w^{(0)}}(w, \mu) = 0\}.$$

**Lemma 2.4.** If Assumption(C1’) holds, then for almost all $w^{(0)} = (x^{(0)}, \lambda^{(0)}, \beta^{(0)}) \in C^0 \times R^n_+ \times R^l$, $0$ is a regular value of $H_{w^{(0)}} : \tilde{C} \times R_+ \times R^l \times (0, 1) \subseteq R^{n+m+l+1} \to R^{n+m+l+1}$, $H_{w^{(0)}}^{-1}(0)$ consists of some smooth curves, among them, there exists a smooth curve $\Gamma_{w^{(0)}}$ starting from $(w^{(0)}, 1)$.

**Proof.** We denote the homotopy mapping as $H(w, w^{(0)}, \mu)$ by regarding $w^{(0)}$ also as a variate. For any $(w, w^{(0)}, \mu) \in H_{w^{(0)}}^{-1}(0)$, we have

$$\frac{\partial H(w, w^{(0)}, \mu)}{\partial (x^{(0)}, \lambda^{(0)}, \beta)} = \begin{pmatrix} -\mu I_n & 0 & (1 - \mu) \Box h(x)^T \\ -\mu V^{(0)} \nabla x^{(0)} g(x^{(0)}) & -\mu G(x^{(0)}) & 0 \\ 0 & 0 & -\mu I_l \end{pmatrix}$$
where $I_n$ and $I_l$ denote an $n \times n$ and $l \times l$ identity matrix, respectively. As $\mu \in (0,1]$ and $G(x^{(0)}) = \text{diag}(g(x^{(0)}))$ with $g(x^{(0)}) < 0$, $\mu G(x^{(0)})$ is a nonsingular matrix. Hence, $\partial H(w, w^{(0)}, \mu)/\partial (x^{(0)}, \lambda^{(0)}, \beta)$ is of the full row rank, so $0$ is a regular value of $H(w, w^{(0)}, \mu)$. Following from Lemma 2.1, Lemma 2.2 and Assumption (C1'), we have $H^{-1}_{w^{(0)}}(0)$ consists of some smooth curves. And since $H^{-1}_{w^{(0)}}(w^{(0)}, 1) = 0$, there must exist a smooth curve denoted by $\Gamma_{w^{(0)}}$ in $H^{-1}_{w^{(0)}}(0)$, which starts from $(w^{(0)}, 1)$.

**Lemma 2.5.** If Assumption (C1')-(C4') hold, then for almost all $w^{(0)} = (x^{(0)}, \lambda^{(0)}, \beta^{(0)}) \in \mathcal{C}^0 \times \mathbb{R}^m_+ \times \mathbb{R}^l$, the component $x$ of $w$ is bounded.

**Proof.** Assume that there exists $\{(w^{(k)}, \mu_k)\} \subset \Gamma_{w^{(0)}}$ such that $\|x^{(k)}\| \rightarrow \infty$, when $k \rightarrow \infty$. By the homotopy equation (4), we have

\[
(1 - \mu_k) \left( \Box u(x^{(k)}) + \Box g(x^{(k)}) V^{(k)} e_m \right) + \Box h(x^{(k)}) U^{(k)} e_l + \mu_k (x^{(k)} - x^{(0)}) = 0
\]

Then, reformulating the first equation of (5) and multiplying $(x^{(k)} - z)$ on its both sides for $z \in C$, we have

\[
(1 - \mu_k) \left( \Box u(x^{(k)}) \right) - \frac{\mu_k}{1 - \mu_k} (x^{(k)} - z)^T (x^{(k)} - x^{(0)}) - \frac{\mu_k}{1 - \mu_k} (x^{(k)} - z)^T \Box g(x^{(k)}) V^{(k)} e_m - \frac{\mu_k}{1 - \mu_k} (x^{(k)} - z)^T \Box h(x^{(k)}) U^{(k)} e_l.
\]

That is

\[
(1 - \mu_k) \left( \Box u(x^{(k)}) \right) \geq \frac{\mu_k}{1 - \mu_k} (x^{(k)} - z)^T (x^{(k)} - x^{(0)}) + \mu_k g(x^{(k)}) V^{(0)} e_m - g(z)^T V^{(0)} e_m
\]

\[
+ \frac{\mu_k}{1 - \mu_k} (x^{(k)} - z)^T \Box h(x^{(k)}) U^{(k)} e_l
\]

\[
= \frac{\mu_k}{1 - \mu_k} (x^{(k)} - z)^T (x^{(k)} - x^{(0)}) + \mu_k g(x^{(k)}) V^{(0)} e_m - g(z)^T V^{(0)} e_m
\]

\[
+ \mu_k e_l^T U^{(k)} e_l
\]

\[
= \frac{\mu_k}{1 - \mu_k} \left[ (x^{(k)} - z)^T (x^{(k)} - x^{(0)}) + (1 - \mu_k) g(x^{(0)}) V^{(0)} e_m
\]

\[
+ (1 - \mu_k) e_l^T U^{(k)} e_l \right]
\]

\[
= \frac{\mu_k}{1 - \mu_k} \left[ \left\| x^{(k)} - z + x^{(0)} \right\|^2 / 2 - \left\| z - x^{(0)} \right\|^2 / 4
\right]
\]
$+(1 - \mu_k)g(x^{(0)})^T V^{(0)} e_{m} + (1 - \mu_k) e_1^T U^{(k)} U^{(k)} e_1$

where the first inequality is based on the convexity of $g$ and the linearity of $h$, that is

$$(x^{(k)} - z)^T \Box g(x^{(k)}) = \left(\begin{array}{c} (x^{(k,1)} - z^1)^T \nabla x^1 g^1(x^{(k,1)}) \\ \vdots \\ (x^{(k,N)} - z^N)^T \nabla x^N g^N(x^{(k,N)}) \end{array}\right) \geq \left(\begin{array}{c} g^1(x^{(k,1)}) - g^1(z^1) \\ \vdots \\ g^N(x^{(k,N)}) - g^N(z^N) \end{array}\right) = g(x^{(k)}) - g(z)$$

$$(x^{(k)} - z)^T \Box h(x^{(k)}) = \left(\begin{array}{c} (x^{(k,1)} - z^1)^T \nabla x^1 h^1(x^{(k,1)}) \\ \vdots \\ (x^{(k,N)} - z^N)^T \nabla x^N h^N(x^{(k,N)}) \end{array}\right) = \left(\begin{array}{c} h^1(x^{(k,1)}) - h^1(z^1) \\ \vdots \\ h^N(x^{(k,N)}) - h^N(z^N) \end{array}\right) = h(x^{(k)}) - h(z).$$

The second equality is derived from the second and third equation of (5). And the detailed derivation is as follows:

$$(g(x^{(k)}) - g(z))^T V^{(k)} e_m = \left(\begin{array}{c} g_1(x^{(k)}) - g_1(z) \\ \vdots \\ g_m(x^{(k)}) - g_m(z) \end{array}\right) \begin{bmatrix} V_1^{(k)} \\ \vdots \\ V_m^{(k)} \end{bmatrix} = \sum_{i=1}^m (g_i(x^{(k)}) - g_i(z)) V_i^{(k)},$$

$$\begin{bmatrix} V_1 g_1(x) \\ \vdots \\ V_m g_m(x) \end{bmatrix} = V g(x) = \mu V^{(0)} g(x^{(0)}) = \mu \begin{bmatrix} V_1^{(0)} g_1(x^{(0)}) \\ \vdots \\ V_m^{(0)} g_m(x^{(0)}) \end{bmatrix},$$

hence we have $V_i g_i(x) = \mu V_i^{(0)} g_i(x^{(0)})$. Since $g(x^{(k)}) \leq 0$ and $V^{(k)} e_m \geq 0$, the second inequality can also be obtained. When $k \to \infty$, we have $\mu_k \in [0, 1]$ and $\|x^{(k)} - x^{(0)}\| \to \infty$. Hence, for sufficiently large $k$, there exists $M > 0$ such that

$$\left\|x^{(k)} - \frac{z + x^{(0)}}{2}\right\|^2 - \left\|z - x^{(0)}\right\|^2 = (1 - \mu_k) g(x^{(0)})^T V^{(0)} e_m + (1 - \mu_k) e_1^T U^{(k)} U^{(k)} e_1 > M.$$ 

So we get $(z - x^{(k)})^T \Box u(x^{(k)}) > 0$, that is $(x^{(k)} - z)^T \Box u(x^{(k)}) < 0$, which contradicts with Assumption (C4’). Hence, the component $x$ of $w$ is bounded. 

For the following optimization problem:

$$\begin{align*}
\min \quad & p(x) \\
\text{s.t.} \quad & q_i(x) \leq 0, i = 1, \cdots, m,
\end{align*}$$
the constrained set is denoted by
\[ W = \{ x \in \mathbb{R}^n : q_i(x) \leq 0, i = 1, \cdots, m \}, \]
the active indices set is
\[ I(x) = \{ i = 1, \cdots, m | q_i(x) = 0 \}. \]
Following from [17], we have the following result:

**Lemma 2.6.** If \( q_i(x), i = 1, \cdots, m \) are convex, then \( W^0 \) is non-empty if and only if there exists \( x \in W \), \( \{ \nabla q_i(x), i \in I(x) \} \) is positive independent, i.e. \( \sum_{i \in I(x)} \lambda_i \nabla q_i(x) = 0, \lambda_i \geq 0, i \in I(x) \Rightarrow \lambda_i = 0, i \in I(x) \).

**Lemma 2.7.** If Assumption \((C1')-(C4')\) hold, then for almost all \( w^{(0)} = (x^{(0)}, \lambda^{(0)}, \beta^{(0)}) \in C^0 \times R^m_+ \times R^l \), \( 0 \) is a regular value of \( H_{w^{(0)}}(w, 1) = 0 \), then \( \Gamma_{w^{(0)}} \) is a bounded curve in \( C^0 \times R^m_+ \times R^l \).

**Proof.** Assume that \( \Gamma_{w^{(0)}} \in C^0 \times R^m_+ \times R^l \) is an unbounded curve. If Assumption \((C4')\) holds, according to Lemma 2.5, then there exists a sequence of points \( \{(w(k), \mu_k)\} \subseteq \Gamma_{w^{(0)}} \) with \( \| (\lambda^{(k)}, \beta^{(k)}) \| \to \infty \), as \( k \to \infty \). For index \( i \), we consider the first equality of Equation (5) and obtain
\[ \mu_k(x^{(k,i)} - x^{(0,i)}) + (1 - \mu_k) \left( \nabla_{x^i} u_k(x^{(k,i)}, x^{(0,i)}) + \nabla_{x^i} g^i(x^{(k,i)}, x^{(0,i)}) \lambda^{(k,i)} + \nabla_{x^i} h^i(x^{(k,i)}) \beta^{(k,i)} \right) = 0. \]

At first, we discuss the boundedness of \( \beta^{(k)} \). For \( \mu^* \in [0, 1] \), there are two subcases to discuss:

1. For \( \mu^* \in (0, 1] \), by homotopy equation (5), we have \( \beta^{(k)} = \frac{1}{\mu_k} h(x^{(k)}) \) is bounded.
2. For \( \mu^* = 0 \), assume that \( \{ \beta^{(k)} \} \) is an unbounded sequence. Let \( J_i(x^*) = \{ j \in \{1, \cdots, m_i\} : \lim_{k \to \infty} \lambda_j^{(k,i)} = \infty \} \), obviously, \( J_i(x^*) \subseteq I_i(x^*) \). That is \( g^i_j(x^{(k,i)}, x^{(0,i)}) = 0, j \in J_i(x^*). \)

Rewriting Equation (6), we have
\[ (1 - \mu_k) \nabla_{x^i} u_k(x^{(k,i)}, x^{(0,i)}) + \mu_k(x^{(k,i)} - x^{(0,i)}) = (1 - \mu_k) \left( \sum_{j \notin J_i(x^{(k)})} \nabla_{x^i} g^i_j(x^{(k,i)}, x^{(0,i)}) \lambda_j^{(k,i)} \right) + (1 - \mu_k) \left( \sum_{j \in J_i(x^{(k)})} \nabla_{x^i} g^i_j(x^{(k,i)}, x^{(0,i)}) \lambda_j^{(k,i)} + \nabla_{x^i} h^i(x^{(k,i)}) \beta^{(k,i)} \right) = 0. \]

Dividing the two sides of (7) by \( \| (\lambda^{(k)}, \beta^{(k)}) \| \) and taking the limit for \( k \to \infty \), we yield \( \{ \nabla_{x^i} g^i_j(x^{(k,i)}, x^{(0,i)}), j \in J_i(x^*), \nabla_{x^i} h^i_j(x^{(0,i)}) \} \) is linearly dependent, which contradicts with Assumption \((C2')\). Hence, \( \{ \beta^{(k)} \} \) is bounded.

In the following, we devote to proving the boundedness of the sequence \( \{ \lambda^{(k)} \} \). Assume that \( \{ \lambda^{(k)} \} \) is an unbounded sequence. That is \( J_i(x^*) = \{ j \in \{1, \cdots, m_i\} : \lim_{k \to \infty} \lambda_j^{(k,i)} = \infty \} \) is nonempty.

For \( \mu^* \in [0, 1] \), it is divided into two cases to discuss:
\( \text{(a) } \mu^* = 1. \text{ Equation (6) is rewritten as} \)

\[
\sum_{j \in J_i(x^{(k)})} (1 - \mu_k) \nabla_x g_j^i(x^{(k,i)}, x^{(k,i)}) \lambda_j^i + \mu_k (x^{(k,i)} - x^{(0,i)}) = \\
-(1 - \mu_k) \left[ \sum_{j \notin J_i(x^{(k)})} \nabla_x g_j^i(x^{(k,i)}, x^{(k,i)}) \lambda_j^i + \nabla_x u_i(x^{(k,i)}, x^{(k,i)}) \right] \\
+ \sum_{j=1}^{\gamma_j} \nabla_x h^j(x^{(k,i)}) \beta_{j}^{(k,i)} (8) \]

Due to Lemma 2.5 and the above analysis, let \( k \to \infty \) and \( x^{(k)} \to x^*, \beta^{(k)} \to \beta^* \), then the above equation can be written as

\[
x^{(s,i)} + \sum_{j \in J_i(x^*)} \lim_{k \to \infty} (1 - \mu_k) \lambda_j^i \nabla_x g_j^i(x^{(s,-i)}, x^{(s,i)}) = x^{(0,i)} (9) \]

As \( J_i(x^*) \) is nonempty, therefore, \( x^{(s,i)} \in \partial(X_i(x^{-i})) \). Since \( g_j^i(x^{-i}, \cdot) \) is convex, \( \lim_{k \to \infty} ((1 - \mu_k) \lambda_j^i) \geq 0 \) and \( x^{(0,i)} \in X_i(x^{-i})^0 \), we have Equation (9) is impossible.

\( \text{(b) } \mu^* \in [0,1] \). Let \( \alpha^{(k,i)} = \max\{|\lambda_j^i|, j \in J_i(x^{(k)})\} \), then \( \alpha^{(k,i)} \to \infty, k \to \infty \).

Dividing the two sides of Equation (8) by \( \alpha^{(k,i)} \) and taking the limit for \( k \to \infty \), we have

\[
\sum_{j \in J_i(x^*)} \lim_{k \to \infty} \gamma_j^i \nabla_x g_j^i(x^{(s,-i)}, x^{(s,i)}) = 0, \]

by Lemma 2.6, it is a contradiction with the convexity of \( g_j^i(x^{(s,-i)}, \cdot) \) in the argument \( x^{(s,i)} \) for each \( j = 1, \ldots, m_i \).

Consequently, \( \Gamma_{w^{(0)}} \) is a bounded curve. \( \square \)

**Theorem 2.8.** Suppose that Assumption \((C1)-(C4)\) hold. Then, for almost all \( w^{(0)} \in C^0 \times R_m^m \times R^l \), \( H^{-1}(w^{(0)}, 0) \) contains a smooth curve \( \Gamma_{w^{(0)}} \subset \tilde{C} \times R_m^m \times R^l \times (0,1) \), which starts from \( (w^{(0)}, 1) \). When \( \mu \to 0 \), the limit set \( T \times \{0\} \subset \tilde{C} \times R_m^m \times R^l \times \{0\} \) is nonempty, and the \( x \)-component \( x^* \) of every point in \( T \) solves (1).

**Proof.** By Lemma 2.4, for almost all \( w^{(0)} \in C^0 \times R_m^m \times R^l \), we have 0 is a regular value of \( H(w, w^{(0)}, \mu) \). By the parameterized Sard theorem, for almost all \( w^{(0)} \in C^0 \times R_m^m \times R^l \), 0 is a regular value of mapping \( H_{w^{(0)}} : R^{n+m+l+1} \to R^{n+m+l} \). By the inverse image theorem, homotopy equation (4) generates some smooth curves. Among them, a smooth curve \( \Gamma_{w^{(0)}} \) starts from \( (w^{(0)}, 1) \). For this curve, only the following three cases are possible:

1. \( \text{Return to } (w^{(0)}, 1); \)

2. \( \text{End at or approach to a point on the boundary of } \tilde{C} \times R_m^m \times R^l \times (0,1); \)

3. \( \text{Go to infinity.} \)

Since

\[
\frac{\partial H_{w^{(0)}}(w^{(0)}, 1)}{\partial w} = \left. \frac{\partial H_{w^{(0)}}(w^{(0)}, 1)}{\partial (x, \lambda, \beta)} \right|_{w=w^{(0)}} \left. \begin{array}{c}
I_n \\
V \nabla_x g(x^{(0)})^T \\
h(x^{(0)})^T \\
0 \\
0 \\
-1_l
\end{array} \right) ,
\]

and for \( x^{(0)} \in \tilde{C}^0 \), we get \( G(x^{(0)}) \) is full row rank. Hence,

\[
\text{det} \left( \frac{\partial H_{w^{(0)}}(w^{(0)}, 1)}{\partial w} \right) = -\text{det}(G(x^{(0)})) \neq 0
\]
Then the matrix \( \frac{\partial H_{w(0)}(w(0), 1)}{\partial w} \) is nonsingular. By the classification theorem of the one-dimensional smooth manifold, \( \Gamma_{w(0)} \subset \hat{C} \times R^m_{+} \times R^l \times (0, 1] \) is homeomorphic to a unit circle or the unit interval \([0, 1]\). When \( \mu \to 0 \), all the limit points of \( \Gamma_{w(0)} \) must lie in \( \partial(\hat{C} \times R^m_{+} \times R^l \times (0, 1]) \). Let \((w^*, \mu^*)\) be a limit point of \( \Gamma_{w(0)} \), then only the following four cases are possible:

(i) \((w^*, \mu^*) \in \hat{C} \times R^m_{+} \times R^l \times \{1\}\);
(ii) \((w^*, \mu^*) \in \partial(\hat{C} \times R^m_{+} \times R^l) \times \{1\}\);
(iii) \((w^*, \mu^*) \in \partial(\hat{C} \times R^m_{+} \times R^l) \times (0, 1)\);
(iv) \((w^*, \mu^*) \in \hat{C} \times R^m_{+} \times R^l \times \{0\}\).

Since \( H_{w(0)}(w, 1) = 0 \) has only one solution \((w(0), 1)\) in \( \hat{C}^0 \times R^m_{+} \times R^l \times \{1\} \) and \( \frac{\partial H_{w(0)}(w(0), 1)}{\partial w} \) is nonsingular, so case (i) is impossible. In case (ii) and case (iii), there must exist a sequence of \( \{(x^{(k)}), (k^{(k)}), (\beta^{(k)}), (\mu^{(k)})\} \subset \Gamma_{w(0)} \) such that \( g^l(x^{(k,i)}) \to 0 \) with \( 1 \leq i \leq N \). From the second equality of (5), we have \( \|\lambda^{(k)}\| \to \infty \), which contradicts Lemma 2.7. As a conclusion, case (iv) is the only possible. And \((w^*, \mu^*)\) is a solution of the homotopy equation (3).

By Theorem 1, for almost all \( w^{(0)} \in \hat{C}^0 \times R^m_{+} \times R^l \), if Assumption 1' holds, then the homotopy equation (3) generates a smooth curve \( \Gamma_{w^{(0)}} \), which is called as the homotopy pathway. Tracing numerically \( \Gamma_{w^{(0)}} \) from \((w^{(0)}, 1)\) until \( \mu \to 0 \), one can find a solution of (1). Letting \( s \) be the arc-length of \( \Gamma_{w^{(0)}} \), we can parameterize \( \Gamma_{w^{(0)}} \) with respect to \( s \). That is, there exist continuously differentiable functions \( w(s), \mu(s) \) such that

\[
\begin{align*}
H(w(s), \mu(s)) &= 0, \\
\|\dot{w}(s), \dot{\mu}(s)\| &= 1, \\
w(0) = w^{(0)}, \mu(0) &= 1.
\end{align*}
\]

Differentiating the first equality of (10), we obtain the following result.

**Theorem 2.9.** The homotopy path \( \Gamma_{w^{(0)}} \) is determined by the following initial value problem to the system of ordinary differential equations

\[
DH_{w^{(0)}}(w(s), \mu(s)) \left( \begin{array}{c} \dot{w} \\ \dot{\mu} \end{array} \right) = 0,
\]

\[
\|\dot{w}(s), \dot{\mu}(s)\| = 1,
\]

\[
w(0) = w^{(0)}, \mu(0) = 1.
\]

And the \( x \)–component of the solution point \((w(s^*), \mu(s^*))\) of (10), for \( \mu(s^*) = 0 \), is the solution of GNEP(1).

By Theorem 2.8 and Theorem 2.9, the standard predictor-corrector procedure [1] can be applied to trace the homotopy path \( \Gamma_{w^{(0)}} \) until \( \mu \to 0 \). We can find a solution of the homotopy equation (3).

3. **Numerical experiments.** In this section, by implementing the Euler Newton algorithm in Matlab, we trace the homotopy pathway generated by homotopy equation (3) and make a comparison with the existing homotopy method in [21] . For all test problems, we take the accuracy parameters \( \epsilon_1 = 10^{-4}, \epsilon_2 = 10^{-2}, \epsilon_3 = 10^{-6}, \mu_0 = 1.0 \). And the numerical results are listed in the following tables, where \( \textbf{A1} \) denotes our homotopy method in this paper, \( \textbf{A2} \) denotes the homotopy method
in [21]. IT denotes the number of iterations, CPU denotes the total cost time in seconds for solving the problem, $x^*$ denotes the approximate solution of the considered problem, and $\mu^*$ denotes the value when the algorithm terminates. For the initial point $x^{(0)}$, we choose two groups of the same interior points to compare method A1 with A2 and an external point for method A1 is implemented. Numerical tests indicate that our homotopy method for solving the GNEP is effective.

**Example 3.1.** ([21])

\[
\begin{align*}
\min_{(x_1,x_2)} & \ u_1(x) = -x_1 x_2 x_3 x_4 \\
\text{s.t.} & \ 3x_1^2 + 2x_1 x_2 + x_3 + 3x_4 \leq 10, \quad x_1^2 + x_2^2 + x_3^2 + x_4^2 \leq 8, \quad x_1 + x_2 = 1; \\
\min_{(x_3,x_4)} & \ u_2(x) = -x_2 (x_1^2 x_3 + x_4^2 x_4) \\
\text{s.t.} & \ x_1^2 + 3x_2^2 + 2x_3 + 3x_4 \leq 4, \quad x_3^2 - x_4 - x_2 \leq 7, \quad x_3 + x_4 = 1;
\end{align*}
\]

In this example, for method A1, we choose two different interior points and an exterior point which only satisfies the inequality constraint as initial points to compare with method A2. The numerical results are listed in Table 1.

**Table 1: The numerical results of Example 3.1**

| $x_0$          | method | CPU    | IT    | $x^*$               | $\mu^*$     |
|---------------|--------|--------|-------|---------------------|-------------|
| $(0.8,0.2,0.5,0.5)^*$ | A1     | 0.049842 | 19    | $(0.5000,0.5000,0.7743,0.2257)^*$ | 5.6755 $\times 10^{-7}$ |
|               | A2     | 0.068653 | 23    | $(0.5000,0.5000,0.7743,0.2257)^*$ | 4.8879 $\times 10^{-7}$ |
| $(0.6,0.4,0.5,0.5)^*$ | A1     | 0.032000 | 11    | $(0.5000,0.5000,0.7743,0.2257)^*$ | 1.2632 $\times 10^{-7}$ |
|               | A2     | 0.047000 | 12    | $(0.5000,0.5000,0.7743,0.2257)^*$ | 8.5698 $\times 10^{-7}$ |
| $(0.3,0.4,0.6,0.5)^*$ | A1     | 0.013000 | 12    | $(0.5000,0.5000,0.7743,0.2257)^*$ | 1.3661 $\times 10^{-7}$ |
|               | A2     | 0.016000 | 12    | $(0.5000,0.5000,0.7743,0.2257)^*$ | 5.6057 $\times 10^{-7}$ |

**Example 3.2.**

\[
\begin{align*}
\min_{(x_1,x_2)} & \ u_1(x) = -x_1 x_2 x_3 x_4 \\
\text{s.t.} & \ 2x_1^2 - x_1 x_2 x_3 - x_3^2 + x_4 \leq 12, \quad x_1^2 - x_2^2 + x_3^2 + x_4^2 \leq 9, \quad x_1 + x_2 = 1; \\
\min_{(x_3,x_4)} & \ u_2(x) = -x_2 (x_1 x_3 + x_3 x_4) \\
\text{s.t.} & \ x_1^2 + 4x_2^2 - 2x_3^2 + x_4 \leq 6, \quad x_1 - 2x_2^2 + x_3^2 - x_4^2 \leq 7, \quad x_3 + x_4 = 1;
\end{align*}
\]

In this example, for method A1, we choose two different interior points and an exterior point which only satisfies the inequality constraint as initial points to compare with method A2. The numerical results are listed in Table 2.

**Table 2: The numerical results of Example 3.2**

| $x_0$          | method | CPU    | IT    | $x^*$               | $\mu^*$     |
|---------------|--------|--------|-------|---------------------|-------------|
| $(0.7,0.3,0.3,0.7)^*$ | A1     | 0.017811 | 20    | $(0.5000,0.5000,0.7500,0.2500)^*$ | 5.5142 $\times 10^{-7}$ |
|               | A2     | 0.052286 | 23    | $(0.5000,0.5000,0.7500,0.2500)^*$ | 1.2044 $\times 10^{-7}$ |
| $(0.7,0.3,0.5,0.5)^*$ | A1     | 0.016000 | 11    | $(0.5000,0.5000,0.7500,0.2500)^*$ | 1.7536 $\times 10^{-7}$ |
|               | A2     | 0.032000 | 11    | $(0.5000,0.5000,0.7500,0.2500)^*$ | 5.0169 $\times 10^{-7}$ |
| $(0.6,0.3,0.6,0.5)^*$ | A1     | 0.016000 | 10    | $(0.5000,0.5000,0.7500,0.2500)^*$ | 2.0758 $\times 10^{-7}$ |
|               | A2     | 0.015000 | 9     | $(0.5000,0.5000,0.7500,0.2500)^*$ | 8.4904 $\times 10^{-7}$ |
Example 3.3.

\[
\begin{align*}
\min_{(x_1, x_2)} u_1(x) &= -x_1 x_2 x_3 - x_2 x_3 x_4 \\
\text{s.t.} &\quad 2x_1^2 - x_2 x_3 + x_3^2 + x_1 x_4 \leq 11, \\
&\quad x_1^2 - x_2^2 + x_3 + x_4^2 \leq 8, \\
&\quad x_1 + x_2 = 1; \\
\min_{(x_3, x_4)} u_2(x) &= -x_1^2 x_2 x_3 - x_1 x_3^2 x_4 \\
\text{s.t.} &\quad 2x_1^2 + x_2^2 + x_3 x_4 + x_2 x_3 \leq 6, \\
&\quad x_1 + x_2 - x_3^2 + x_4 \leq 7, \\
&\quad x_3 + x_4 = 1;
\end{align*}
\]

In this example, for method A1, we choose two different interior points and an exterior point which only satisfies the inequality constraint as initial points to compare with method A2. The numerical results are listed in Table 3.

| $x_0$ | method | CPU | IT | $x^*$ | $\mu^*$ |
|-------|--------|-----|-----|-------|---------|
| (0.8, 0.2, 0.2, 0.8)$^*$ | A1     | 0.066919 | 27  | (0.2165, 0.7835, 0.4331, 0.5669)$^*$ | 2.9945 x $10^{-7}$ |
|       | A2     | 0.102540 | 102 | (0.2165, 0.7835, 0.4331, 0.5669)$^*$ | 3.3410 x $10^{-7}$ |
| (0.9, 0.1, 0.1, 0.9)$^*$ | A1    | 0.063579 | 33  | (0.2165, 0.7835, 0.4331, 0.5669)$^*$ | 4.5863 x $10^{-7}$ |
|       | A2    | 0.097978 | 128 | (0.2165, 0.7835, 0.4331, 0.5669)$^*$ | 6.9601 x $10^{-7}$ |
| (0.3, 0.8, 0.4, 0.5)$^*$ | A1    | 0.050146 | 17  | (0.2165, 0.7835, 0.4331, 0.5669)$^*$ | 8.8414 x $10^{-7}$ |
|       | A2    | 0.032000 | 22  | (0.2165, 0.7835, 0.4331, 0.5669)$^*$ | 1.6179 x $10^{-7}$ |

Example 3.4.

\[
\begin{align*}
\min_{(x_1, x_2)} u_1(x) &= -x_1 x_2 x_4 - x_1 x_2^2 x_3 x_4 \\
\text{s.t.} &\quad 2x_1^2 - x_1 x_3 + x_2 x_3^2 + x_1 x_4 \leq 12, \\
&\quad x_1 - x_2^2 + x_3 + x_4^2 \leq 8, \\
&\quad x_1 + x_2 = 1; \\
\min_{(x_3, x_4)} u_2(x) &= -x_1^2 x_2 x_3 x_4 - x_2 x_3^2 x_4 \\
\text{s.t.} &\quad 2x_1^2 - x_2^2 x_3 + x_3 x_4 + x_2 x_3 x_4 \leq 6, \\
&\quad x_1^2 - x_2 - x_3^2 + x_4 \leq 7, \\
&\quad x_3 + x_4 = 1;
\end{align*}
\]

In this example, for method A1, we choose two different interior points and an exterior point which only satisfies the inequality constraint as initial points to compare with method A2. The numerical results are listed in Table 4.

| $x_0$ | method | CPU | IT | $x^*$ | $\mu^*$ |
|-------|--------|-----|-----|-------|---------|
| (0.8, 0.2, 0.2, 0.8)$^*$ | A1     | 0.090294 | 34  | (0.4420, 0.5580, 0.6384, 0.3616)$^*$ | 9.9066 x $10^{-7}$ |
|       | A2     | 0.170305 | 79  | (0.4420, 0.5580, 0.6384, 0.3616)$^*$ | 8.9245 x $10^{-7}$ |
| (0.7, 0.3, 0.3, 0.7)$^*$ | A1     | 0.074322 | 31  | (0.4420, 0.5580, 0.6384, 0.3616)$^*$ | 9.4454 x $10^{-7}$ |
|       | A2     | 0.096164 | 64  | (0.4420, 0.5580, 0.6384, 0.3616)$^*$ | 6.9529 x $10^{-7}$ |
| (0.3, 0.8, 0.4, 0.7)$^*$ | A1     | 0.071024 | 32  | (0.4420, 0.5580, 0.6384, 0.3616)$^*$ | 7.7431 x $10^{-7}$ |
|       | A2     | 0.051600 | 17  | (0.4420, 0.5580, 0.6384, 0.3616)$^*$ | 1.3352 x $10^{-7}$ |

4. Conclusion. In this paper, we propose a new homotopy method for GNEP with equality and inequality constraints. Comparing with the existing homotopy method for GNEP, on the one hand, we generalize the GNEP to the unbounded set; On the other hand, we expand the scope of the initial points which only need to satisfy the inequality constraint. Hence, this method provides the convenience to solve the GNEP. The numerical results illustrate that it is an effective method.
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