Calculating two- and three-body decays with *FeynArts* and *FormCalc*

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Abstract

The Feynman diagram generator *FeynArts* and the computer algebra program *FormCalc* allow for an automatic computation of $2 \rightarrow 2$ and $2 \rightarrow 3$ scattering processes in High Energy Physics. We have extended this package by four new kinematical routines and adapted one existing routine in order to accommodate also two- and three-body decays of massive particles. This makes it possible to compute automatically two- and three-body particle decay widths and decay energy distributions as well as resonant particle production within the Standard Model and the Minimal Supersymmetric Standard Model at the tree- and loop-level. The use of the program is illustrated with three standard examples: $h \rightarrow b \bar{b}$, $\mu \rightarrow e \bar{\nu}_e \nu_\mu$, and $Z \rightarrow \nu_e \bar{\nu}_e$.

Key words: Feynman diagrams, perturbative calculations, decays

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1 Introduction

The analytical calculation of Feynman diagrams and their subsequent numerical evaluation represent the standard technique for calculating cross sections of scattering processes and widths of particle decays in High Energy Physics. Unfortunately, the number of diagrams to be computed grows rapidly with the number of external particle legs, internal particle loops, and exchanged particles. This makes an automatic generation and computation of Feynman diagrams a highly desirable feature, not only to reduce the time needed for the calculations, but also to eliminate possible sources of errors.

For more than ten years, a well-tested and easy-to-use Feynman diagram generator has existed in the form of the *Mathematica* package *FeynArts* [1]. More recently, a fast and equally reliable computer algebra program for the evaluation of the Dirac and color traces inherent in the generated diagrams has
been implemented within *Form (FormCalc)* [2]. Together with *LoopTools* [2] (an adapted version of the *Fortran* program *FF* [3] for the numerical evaluation of tensor loop integrals), the *FeynArts* and *FormCalc* program package allows for an automatic computation of tree- and loop-level $2 \rightarrow 2$ and $2 \rightarrow 3$ scattering processes within the Standard Model and the Minimal Supersymmetric Standard Model. Since the phase space for each particular scattering process depends on the number of in- and out-going external particles, it is best evaluated directly within *Fortran* with specific kinematical routines. So far, only $2 \rightarrow 2$ and $2 \rightarrow 3$ scattering processes have been implemented within *FeynArts* and *FormCalc*. The aim of this Paper to extend their applicability also to two- and three-body decays of massive particles and to resonant particle production. We provide four new kinematical routines and one adapted routine, which allow for the computation of the total two- and three-body decay widths and of the energy distribution of one decay product for three-body decays.

The remainder of this Paper is organized as follows: In Sec. 2 we collect the well-known kinematical formulæ for two- and three-body decays. Their implementation in the new and modified *Fortran* routines is described in Sec. 3. As a test of the new routines, we calculate in Sec. 4 three standard examples for scalar, fermion, and vector boson decays. A short summary is given in Sec. 5.

### 2 Kinematics for two- and three-body decays

The decay width of a heavy particle with mass $m_1$,

$$d\Gamma_i = \frac{1}{F} |\mathcal{M}|^2 d\Phi_i,$$

depends on the flux factor $F$, i.e. the number of decaying particles per unit volume, the squared invariant scattering amplitude $|\mathcal{M}|^2$, and the phase space volume $d\Phi_i$ for $i$ final-state particles. In the rest frame of the decaying particle, $k_1 = (m_1, 0, 0, 0)$, the flux factor is simply

$$F = 2m_1.$$  

The four-momenta of two decay products with masses $m_2, m_3$,

$$k_2 = (E_2, |k_2| \sin \theta, 0, |k_2| \cos \theta),$$

$$k_3 = (E_3, -|k_3| \sin \theta, 0, -|k_3| \cos \theta),$$

$$m_2 = (m_2, 0, 0, 0),$$

$$m_3 = (m_3, 0, 0, 0),$$

and

$$k_1 = (m_1, 0, 0, 0).$$
are uniquely defined by their energies \( E_i^2 = |k_i|^2 + m_i^2 \), squared three-momenta

\[
|k_2|^2 = |k_3|^2 = \frac{(m_1^2 - m_2^2 + m_3^2)^2}{4m_1^2} - m_3^2,
\]

(4)

and by the center-of-mass scattering angle \( \theta \). The phase space volume for two-body decays is then [4]

\[
\begin{align*}
d\Phi_2 &= \frac{d^3k_2}{(2\pi)^32E_2} \frac{d^3k_3}{(2\pi)^32E_3} (2\pi)^4 \delta^{(4)}(k_1 - k_2 - k_3) \\
&= \frac{1}{(2\pi)^2} \frac{|k_2|}{4m_1} d\cos \theta d\phi,
\end{align*}
\]

(5)

where the dependence on the azimuthal angle \( \phi \) integrates to \( 2\pi \). The total two-body decay width is obtained by integrating over the polar angle \( \theta \). [Note that in the rest frame of the decaying particle, the distribution in the polar angle \( \theta \) depends on the arbitrary definition of a reference axis.]

For three-particle decays, the phase space volume is [4]

\[
\begin{align*}
d\Phi_3 &= \frac{d^3k_2}{(2\pi)^32E_2} \frac{d^3k_3}{(2\pi)^32E_3} \frac{d^3k_4}{(2\pi)^32E_4} (2\pi)^4 \delta^{(4)}(k_1 - k_2 - k_3 - k_4) \\
&= \frac{1}{(2\pi)^4} \frac{1}{8} dE_2 d\phi_2 dE_4 d\cos \theta_4.
\end{align*}
\]

(6)

Here, the \( \delta \)-function has been used to eliminate the integration over \( k_3 \) and \( \theta_2 \), and the trivial integration over the azimuthal angle \( \phi_4 \) has already been performed. For three-body decays, we will not only consider total decay widths, but also distributions in the decay energy \( E_4 \).

While the standard version of \textit{FeynArts} and \textit{FormCalc} diverges for \( 2 \to 2 \) and \( 2 \to 3 \) cross sections, which proceed through an intermediate \( s \)-channel resonance, our modified version makes it possible to calculate also these resonance cross sections using the formula

\[
\sigma_{ab \to R \to X}(\sqrt{s}) = \frac{4 \pi \Gamma(R \to ab) \Gamma(R \to X)}{(s - m_R^2)^2 + m_R^2 (\Gamma_{tot}^R)^2},
\]

(7)

where \( s \) is the squared center-of-mass energy of the initial state particles \( a \) and \( b \), \( m_R \) is the mass of the resonance \( R \), and the decay widths are calculated as described above.
3 Description of the new and modified Fortran routines

The phase space integrals described in the previous Section have been implemented in *FeynArts* and *FormCalc* with two new *Fortran* routines for two- and three-body decays,

- 1to2.F,
- 1to3.F,

and two corresponding common blocks,

- 1to2.h,
- 1to3.h.

The new *Fortran* routines have been adapted from the existing routines for $2 \to 2$ and $2 \to 3$ particle scattering by modifying variable and center-of-mass energy definitions, the flux factor, and the initial-state four-momenta. The *Fortran* routine

- num.F,

which handles the averaging and the summation over initial and final state spins, has been modified to accommodate single-particle initial states. For this purpose, the routine num.F checks if the new preprocessor variable DECAY has been defined in 1to2.F or 1to3.F or not.\[1\]

4 Three examples: $h \to b\bar{b}$, $\mu \to e \bar{\nu}_e \nu_\mu$, and $Z \to \nu_e \bar{\nu}_e$

In order to demonstrate that the new kinematical routines work properly within the extended package of *FeynArts* and *FormCalc* and are straightforward to use, we compute in this Section three standard decay widths. Note that the *FormCalc* routine process.h has to be adapted to every considered scattering process. In particular, the correct phase space generator has to be included at the end of process.h. However, the calling structure of the main program run.F has not been changed. For more details we refer the reader to the *FeynArts* [1] and *FormCalc* [2] manuals.

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1 The new *Fortran* routines can be obtained from klasen@mail.desy.de.
Table 1

| $m_H$/GeV | $\Gamma(H \to b\bar{b})$/MeV |
|-----------|-----------------------------|
| 100       | 4.15                        |
| 110       | 4.58                        |
| 120       | 5.00                        |
| 130       | 5.42                        |
| 140       | 5.85                        |
| 150       | 6.27                        |

Decay widths of a Standard Model Higgs boson into a pair of bottom quarks for six different Higgs masses $m_H$.

4.1 $h \to b\bar{b}$

The isotropic, scalar two-body decay $h \to b\bar{b}$ represents the dominant Higgs decay mode for a Standard Model Higgs boson with mass below $m_H = 140$ GeV. Using the default parameter settings of *FeynArts* and *FormCalc* and the Standard Model initialization file *sm.ini.F*, we obtain the decay widths listed in Table 1. Identical results are obtained with the tree-level formula [5]

$$\Gamma(H \to b\bar{b}) = \frac{\alpha N_C m_H m_b^2}{8 \sin^2 \theta_W m_W^2} \left(1 - \frac{4m_b^2}{m_H^2}\right)^{3/2}$$

where $1/\alpha = 137$ is the inverse of the electromagnetic fine structure constant, $\sin^2 \theta_W = 1 - m_W^2/m_Z^2$ is the electroweak mixing angle, $N_C = 3$ accounts for the color degree of freedom, $m_H$, $m_Z = 91.1882$ GeV, and $m_W = 80.419$ GeV are the masses of the Higgs, $Z$-, and $W$-bosons, and $m_b = 4.7$ GeV is the mass of the bottom quark. We have checked that the cross section differential in the scattering angle $\theta$ is indeed constant.

4.2 $\mu \to e\bar{\nu}_e\nu_\mu$

In the Standard Model, the three-body decay of muons with mass $m_\mu = 105.658$ MeV into electrons and neutrinos proceeds through a virtual $W$-boson and accounts for almost 100% of the muonic decay width. With the default parameter settings of *FeynArts* and *FormCalc*, we obtain

$$\Gamma(\mu \to e\bar{\nu}_e\nu_\mu) = 2.8 \times 10^{-10} \text{ eV}$$
which agrees with the result obtained from the lowest order formula [6]

\[ \Gamma(\mu \to e\bar{\nu}_e\nu_{\mu}) = \frac{G^2 m_\mu^5}{192 \pi^3}, \]  

(10)

where we have averaged over the initial-state spin, Fermi’s constant \( G \) has been computed from \( G = \frac{\pi \alpha}{\sqrt{2} \sin \theta_W^2 m_W^2} \), and the electron mass has been neglected. Furthermore, the calculated muon life time \( \tau = \frac{\hbar}{\Gamma} = 2.3 \times 10^{-6} \) s is in good agreement with the measured value of \( 2.2 \times 10^{-6} \) s. As mentioned above, for three-body decays we consider also the energy distribution of one of the decay products, in this case of the observed electron [6]

\[ \frac{d\Gamma}{dE_e}(\mu \to e\bar{\nu}_e\nu_{\mu}) = \frac{G^2 m_\mu^2}{12 \pi^3} E_e^2 \left( 3 - \frac{4 E_e}{m_\mu} \right). \]  

(11)

As can be seen in Fig. 1, the numerical evaluation of Eq. (11) agrees very well with the result of \textit{FeynArts} and \textit{FormCalc}.

4.3 \( Z \to \nu_e\bar{\nu}_e \)

In our final example, we consider the invisible branching ratio of the \( Z \)-boson in the Standard Model, which represents an important boundary condition for the number of neutrinos realized in nature. The tree-level formula [6]

\[ \Gamma(Z \to \nu_e\bar{\nu}_e) = \frac{\alpha m_Z}{24 \sin^2 \theta_W \cos^2 \theta_W} \]  

(12)

and our modified program package \textit{FeynArts} and \textit{FormCalc} unanimously lead to the numerical result

\[ \Gamma(Z \to \nu_e\bar{\nu}_e) = 160 \text{ MeV}. \]  

(13)

For three generations of neutrinos and employing the measured total width of the \( Z \)-boson, \( \Gamma_{\text{tot}} = 2.49 \) GeV, this translates into an invisible branching ratio of

\[ \text{BR}_{\text{inv}} = 3 \frac{\Gamma(Z \to \nu_e\bar{\nu}_e)}{\Gamma_{\text{tot}}} = 19.3\%, \]  

(14)

which is close to the experimental value of 20.0%.
Electron Energy Spectrum in $\mu \rightarrow e^{\pm} \nu_e \nu_\mu$

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{Electron energy spectrum of the decay width for muons decaying into electrons and neutrinos.}
\end{figure}

5 Summary

In this Paper, we have described the implementation of two- and three-body decays and resonance cross sections in the computer algebra package *FeynArts* and *FormCalc*. We have provided four new *Fortran* routines, which generate the phase space for two- and three-particle decays, and we have modified one existing routine in order to account for correct spin averages of one-particle initial states. While the new routines have been obtained in a straight-forward way and are easy to use, they will hopefully serve a large variety of purposes and a wide community in High Energy Physics to obtain interesting new results for tree- and loop-level particle decays or resonance cross sections.
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