On subdivision invariant actions for random surfaces

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Abstract. We consider a subdivision invariant action for dynamically triangulated random surfaces that was recently proposed [1] and show that it is unphysical: The grand canonical partition function is infinite for all values of the coupling constants. We conjecture that adding the area action to the action of [1] leads to a well-behaved theory.

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The most straightforward way to discretize the functional integral for bosonic strings with the Nambu-Goto action is to sum over all triangulations with the weight $\exp(-\mu S_A)$ where $S_A$ is the sum of the areas of individual triangles. This discretization was studied in [2] and the partition function was found to be divergent due to spikes that are not suppressed by the weight factor. Since then the most widely studied random surface theory is the one with Gaussian action on random triangulations [2, 3, 4] which is a natural discretization of the Polyakov functional integral.

The area action has the nice feature that it is subdivision invariant, i.e. if we have a triangulated surface imbedded in $\mathbb{R}^d$ and refine the triangulation by adding new vertices and links that lie in the original surface then the action does not change. This implies that two surfaces which are ”close to each other” in imbedding space have almost the same action. In other words, the area action is a continuous function on the space of imbedded triangulated surfaces. The Gaussian action is not continuous in this sense, but one should keep in mind that it is by no means clear that such continuity is a correct requirement of a well-behaved physical theory.

The Gaussian action leads to a string tension which does not scale to zero at the critical point [5], a pathology which is believed to be caused by the dominance of branched polymer-like surfaces [3, 4, 8]. In [9, 10] it was argued that extrinsic curvature terms should be added to the Gaussian action in order to prevent the dominance of branched polymers. Recent numerical work [11, 12, 13, 14] indicates that this modification results in a scaling string tension, but a conclusive proof is still lacking.

In a recent paper [1] a new subdivision invariant action for random surfaces was proposed. The purpose of this letter is to point out that this action suffers from a similar problem as the area action, i.e. the partition function is divergent. The action of [1] depends on the angle between neighbouring triangles, like the extrinsic curvature action does, but in a subdivision invariant fashion which we now describe. Let $T$ be a closed triangulation with one vertex marked. A piecewise linear surface $S$ triangulated by $T$ is given by a mapping $i \mapsto x_i$ from the vertices of $T$ into $\mathbb{R}^d$. We assume that the marked vertex $i_0$ is mapped to 0 in order to remove translational invariance. If $(i, j)$ is a pair of nearest neighbour vertices in $T$ we denote by $\alpha_{ij}$ the angle between the planes of the two triangles in imbedding space which share the link $(x_i, x_j)$ (i.e. $\alpha_{ij} = 0$ if the triangles lie in the same plane with the same orientation). The action for $S$ proposed in [1] is defined by

$$A_T = \frac{1}{2} \sum_{(i,j)} |x_i - x_j| |\alpha_{ij}|$$

2This notion of continuity can of course be made precise but for our present purposes this is not necessary.
where the sum is over all nearest neighbour pairs of vertices in \( T \). Since the action associated with a link vanishes if the adjoining triangles are parallel and is linear in the length of links, \( A_T \) is readily seen to be subdivision invariant in the sense described above.

The partition function associated with \( T \) is defined, for \( \lambda > 0 \), by

\[
Z_T(\lambda) = \int e^{-\lambda A_T} \prod_{i \in T \setminus \{i_0\}} dx_i.
\]  

(2)

The grand canonical partition function is defined for \( \mu > 0 \) and \( \lambda > 0 \) by

\[
Z(\mu, \lambda) = \sum_T e^{-\mu |T|} Z_T(\lambda),
\] 

(3)

where the sum is over all closed triangulations with a fixed topology and \( |T| \) is the number of vertices in \( T \).

We now prove that the sum defining \( Z(\mu, \lambda) \) is divergent for all values of \( \mu \) and \( \lambda \) and suggest a modification of the action (1) whose associated grand canonical partition function we expect to be well defined. This divergence is caused by surfaces which are almost planar with a large distance between vertices. For simplicity we restrict ourselves to triangulations with spherical topology. Consider a square triangulation with \( n \) vertices on each edge and regular inside with all interior vertices of order 6, see Fig. 1. Next take an identical square triangulation rotated through \( 90^\circ \) and glue the two together along their boundaries.

![Fig. 1: A regular square triangulation with 6 vertices on each edge.](image)

Then we obtain a closed triangulation of \( S^2 \) with \( 2n^2 - 4n + 4 \) vertices, all of which are of order 6, except 4 (the corners) which are of order 3. Denote this triangulation by \( T_n \) and let the marked vertex be one of the corners.
Consider the surface $S_n$ triangulated by $T_n$ defined such that the marked vertex $i_0$ is mapped to 0 and the other vertices fall on a planar hexagonal grid, all of whose links have length $a$, in such a way that the ”upper half” of the surface overlaps the ”lower half”. Let $B_{n,i}$ be an ellipsoid in $\mathbb{R}^d$ centered on the vertex $x_i$ in $S_n$ with axes of length $a/3$ in the plane of the surface and axes of length 1 in the $d-2$ coordinate directions perpendicular to this plane. We assume that $a \gg 1$ and let $B_n \subset \mathbb{R}^d(\left|T_n\right|-1)$ denote the Cartesian product over $i$ of all the $B_{n,i}$’s except $B_{n,i_0}$.

We now obtain a lower bound on $Z_{T_n}(\lambda)$ by restricting the integration domain in (2), with $T = T_n$, to $B_n$. This means that we only take into account perturbations of the surface $S_n$ for which $x_i \in B_{n,i}$ for $i \in T \setminus i_0$. For such surfaces we evidently have $\alpha_{ij} = O(a^{-1})$, i.e. there is a constant $c_1$ such that $\alpha_{ij} \leq c_1/a$, for all links $(i, j)$ except the ”boundary links” where the surface bends back upon itself. It follows that

$$A_{T_n} \leq 6\lambda(n-2)^2(a + 2a/3)c_1a^{-1} + \frac{1}{2}\lambda(4n - 4)(a + 2a/3)\pi$$

The first term on the right hand side of (4) is a bound on the contribution of interior links to the action while the second term is a bound on the boundary contribution. The volume $|B_n|$ of $B_n$ fulfills

$$|B_n| \geq (a^2c_2)^{|T_n|-1}$$

where $c_2$ is a constant which only depends on $d$. Hence,

$$Z_{T_n}(\lambda) \geq (a^2c_2)^{|T_n|-1} \exp\left(-6\lambda(n-2)^2(a + 2a/3)c_1/a - \lambda(2n - 2)(a + 2a/3)\pi\right)$$

$$\geq c_3 \exp\left((4 \ln a - c_4 - c_5\lambda)n^2 - (8 \ln a - 2c_4 + c_5\lambda a)n\right),$$

where $c_3$, $c_4$ and $c_5$ are positive constants independent of $n$ and $a$. From (3) and (4) it follows that

$$Z(\mu, \lambda) \geq \sum_{n=2}^{\infty} e^{-\mu|T_n|} Z_{T_n}(\lambda)$$

$$\geq c'_3 \sum_{n=2}^{\infty} e^{(-4\mu+4 \ln a - c_4 - c_5\lambda)n^2 - (8 \ln a - 2c_4 + c_5\lambda a)n},$$

for a positive constant $c'_3$. Choosing $a$ sufficiently large the right hand side of (7) is clearly divergent. We have therefore shown that the contribution of a particular class of ”smooth” slowly undulating surfaces gives a divergent contribution to the partition function. This could be expected from the outset since the action $A_T$ does not at all counteract the entropy of vertices in the plane of the surface.

It is easy to convince oneself that the above construction can be generalized to surfaces with an arbitrary boundary and genus. It also generalizes to actions of the
form
\[ A_T = \sum_{(i,j)} |x_i - x_j| \theta(\alpha_{ij}) \] (8)
provided
\[ 0 \leq \theta(\alpha) \leq \text{constant} |\alpha|^\varepsilon \] (9)
with \( \varepsilon > (d - 2)/d \).

It is natural to ask whether it is the unhappy fate of any subdivision invariant action to lead to a divergent partition function. In order to address this question we recall that it was shown in [2] that the partition functions associated with a certain class of triangulations are divergent if the model is defined by the area action. More generally, it was proven that for any triangulation the expectation value of a sufficiently high power of the mean extent of the surface is divergent. These divergences are caused by a high probability for the formation of spikes which extend far out but have small area. For example, in \( d = 2 \), the integration over a vertex of order 3 gives rise to a nonintegrable singularity in the distances between the three neighbouring vertices. On the other hand, for regular triangulations like \( T_n \), the partition function was not shown to be divergent. On the contrary, numerical simulations indicate that it is finite [15].

Thus the following picture emerges: The action (1) clearly suppresses spiky surfaces. It is easy to see that the integration over a single vertex in (2) does not give rise to a singularity in the distances between the remaining vertices. But, as demonstrated above, the action (1) is not coercive enough to compete successfully with the entropy of almost flat surfaces, no matter how large we choose the coupling \( \lambda \). In contrast the area action is well behaved for regular surfaces but fails to suppress spikes.

On the basis of these observations it does not seem unreasonable to expect proper thermodynamical behaviour of a model whose action is a linear combination of the area action and the action (1), i.e.
\[ A'_T = \kappa \sum_{\Delta \in T} |\Delta| + \lambda \sum_{(i,j)} |x_i - x_j| |\alpha_{ij}|, \] (10)
where the first summation is over all triangles in \( T \), \( |\Delta| \) denotes the area of the triangle \( \Delta \) in the surface and the coupling constants \( \kappa \) and \( \lambda \) are in a suitable range.

Let us denote the partition functions for this model by \( Z_T(\kappa, \lambda) \) and \( Z(\mu, \kappa, \lambda) \), defined in analogy with (2) and (3), respectively. It is easy to see that the proof given above of the divergence of \( Z(\mu, \lambda) \) breaks down for \( Z(\mu, \kappa, \lambda) \), if \( \kappa \) is chosen large enough for given \( \mu \) and \( \lambda \).
We do not have a proof that $Z(\mu, \kappa, \lambda)$ is finite for any values of the coupling constants. For this one would need an exponential bound

$$Z_T(\kappa, \lambda) \leq C(\kappa, \lambda)^{|T|}$$

for some $C(\kappa, \lambda) > 0$ and $T$ of fixed topology. A proof of (11) by successive integration over vertices is not straightforward.

Let us finally mention that provided $Z(\mu, \kappa, \lambda)$ is finite in some region $\Omega$ of coupling constants, one can prove by standard methods elementary properties of the model such as convexity of $\Omega$, non-negativity and concavity of the mass and the string tension as well as linear upper bounds on these quantities (see [2, 6, 7, 8, 10]). In this model it appears, however, to be just as difficult to establish the existence of a critical point at which both the mass and the string tension scale as it is in the Gaussian models with extrinsic curvature terms added to the action.

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