Deterministic Pod Repositioning Problem in Robotic Mobile Fulfillment Systems

Ruslan Krenzler\textsuperscript{a}, Lin Xie\textsuperscript{a} and Hanyi Li\textsuperscript{b}

\textsuperscript{a}Leuphana University of Lüneburg, Universitätsallee 1, 21335 Lüneburg
\textsuperscript{b}Hanning ZN Tech Co., Ltd.

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Abstract

In a robotic mobile fulfillment system, robots bring shelves, called pods, with storage items from the storage area to pick stations. At every pick station there is a person – the picker – who takes parts from the pod and packs them into boxes according to orders. Usually there are multiple shelves at the pick station. In this case, they build a queue with the picker at its head. When the picker does not need the pod any more, a robot transports the pod back to the storage area. At that time, we need to answer a question: “Where is the optimal place in the inventory to put this pod back?”. It is a tough question, because there are many uncertainties to consider before answering it. Moreover, each decision made to answer the question influences the subsequent ones. The goal of this paper is to answer the question properly. We call this problem the Pod Repositioning Problem and formulate a deterministic model. This model is tested with different algorithms, including binary integer programming, cheapest place, fixed place, random place, genetic algorithms, and a novel algorithm called \textit{tetris}.

Keywords: logistics, applied combinatorial optimization, robotic mobile fulfillment system, warehouse

ORCID IDs and email addresses:
Ruslan Krenzler \copyright https://orcid.org/0000-0002-6637-1168, ruslan.krenzler@leuphana.de,
Lin Xie \copyright https://orcid.org/0000-0002-3169-4922, xie@leuphana.de
## Contents

1 Introduction .......................................................... 3

2 The robotic mobile fulfillment system .............................. 4

3 Warehouse game ......................................................... 6
   3.1 Simplifications .................................................. 6
   3.2 Parameters and states ......................................... 8
   3.3 Dynamics ......................................................... 9
   3.4 Costs ............................................................ 10
   3.5 Binary integer programming .................................. 11

4 Test systems .................................................................. 12
   4.1 Small test system ................................................ 12
   4.2 Storage-area chart .............................................. 14
   4.3 Medium-size test system ..................................... 14

5 Algorithms .................................................................. 15
   5.1 Cheapest place .................................................. 16
      5.1.1 Properties of cheapest-place algorithms .......... 17
      5.1.2 Advantages and disadvantages ...................... 18
   5.2 Fixed place ....................................................... 18
      5.2.1 Properties of fixed-place algorithms .......... 19
      5.2.2 Advantages and disadvantages .................. 20
   5.3 Random place ................................................... 20
   5.4 Iterative binary integer programming ..................... 21
   5.5 Genetic algorithms .............................................. 22
      5.5.1 First approach ........................................... 22
      5.5.2 Second approach ....................................... 23
      5.5.3 Properties of genetic algorithms ................. 25
      5.5.4 Advantages and disadvantages .................. 26
   5.6 Tetris ............................................................. 27
      5.6.1 Extension and variations of the algorithm ....... 29
      5.6.2 Properties of the tetris algorithm ............... 29
      5.6.3 Advantages and disadvantages .................. 29

6 Computational results ................................................ 30

7 The problem with multiple stations ................................ 31

8 Conclusion .................................................................. 35
1 Introduction

Robotic mobile fulfillment systems (RMFS) are a new type of warehousing system which is becoming popular due to increasing growth in the e-commerce sector: see Banker (2016). In such systems, robots carry mobile shelves – called pods – with items from the storage area to human operators – the pickers – at pick stations. At each station, the picker picks the items according to pick orders. A pick order is basically the content of your shopping cart, when you buy something online. Usually, there are multiple pods at every pick station. In this case, they build a queue with the picker at its head. After the picker has picked all the items, the robot carries the pod back to the storage area and selects another pod.

Such a fulfillment system contains many different operational decision problems, see Merschformann et al. (2018b, Section 2.1). In this work, we concentrate on one question: “Where to put the pod back to, after the pick station?” To support an intuitive understanding we sketch this problem with the help of the following small example.

Example Figure 1 shows possible destinations – A, B, C and D – for a pod (marked green) which leaves the upper pick station. We want to minimize the traveling distance of the robots. Depending on the situation, each of the places A, B, C and D can be optimal:

- A is optimal if we want to move the pod to the closest place.
- B is optimal if we want to move the pod later to the upper pick station.
- C is optimal if we want to move the pod later to the lower pick station.
- D is optimal if we will not use the green pod any-more. Then we can use places A, B, and C for other more frequently used pods.
We call this decision problem the Pod Repositioning Problem (in short: PRP). There is only one publication about this problem, since this problem appears firstly with an RMFS: In Merschformann (2018), Merschformann analyzed PRP in regard to passive and active repositioning. Passive repositioning means that the pods find a better location after visiting a pick station; while active repositioning means that the pods move from their current locations to better ones without visiting any pick station in between. That paper concentrates on active repositioning. The author uses simulations to investigate the effects of three different repositioning mechanisms. He shows that the best strategy is to use the nearest available storage location in terms of the path time. He also shows that the positive effects can be achieved during a nightly down period, but not in all cases, especially if the passive repositioning already keeps the storage area sorted well. Moreover, the active repositioning during the nightly down period faces additional costs such as energy costs. The PRP is a problem of operations planning. An overview of decision problems in an RMFS is listed in Azadeh et al. (2017), where publications about other problems are also mentioned, such as system analysis and design optimizations.

The contribution of our work are listed as follows:

• Our work is the first paper to concentrate on the passive case of PRP, because we expect better performance of an RMFS and save additional repositioning costs.

• We create a deterministic mathematical model to better understand PRP.

• We analyze classical solving methods: binary integer programming, cheapest place, fixed place, random place, iterative binary integer programming, and genetic algorithms.

• We create a heuristic to solve large instances quickly.

• From the model, we derive some general rules, which one can apply in the real world.

• We critically discuss if and how our algorithms can be implemented in the real world.

The remainder of the paper is structured as follows. First, we define the process of an RMFS and the decision problems in it in Section 2. After that, we formulate the PRP as a deterministic model under some simplifications in Section 3. We introduce two test systems for algorithm analysis in Section 4 and provide suitable exact and heuristic solution approaches in Section 5. We briefly analyze the computational results in Section 6 and discuss real-world implementation of the solvers in Section 7. Finally, Section 8 concludes the paper.

2 The robotic mobile fulfillment system

Before we explain the essential processes in an RMFS, we define several terms which we did not mention in the introduction:
• **workstations** – places where the persons interact with the pods; a special type of a workstation is a pick station

• **replenishment station** – a workstation where the persons replenish the pods

• stock keeping unit (**SKU**)  

• an **order line** including one SKU with number

• a **pick order** including a set of order lines from an order of a customer

• a **replenishment order** consisting of a number of physical units of one SKU

The processes of an RMFS are illustrated in Figure 2. The robots carry pods between the storage area and workstations. Two processes are included:

• **retrieval process**: After a replenishment order has arrived, robots carry selected pods to a replenishment station, where the units are stored in these pods. We assume that a shared storage policy is applied (such as in Bartholdi and Hackmann (2017)), which means SKUs of the same type are not stored together in a unique pod, but are spread over several pods.

• **storage process**: After a pick order has arrived, we calculate which pods we need to process the order lines. Robots carry required pods to a pick station, where the picker picks the SKUs according to the order lines. We assume it is unlikely that a pick order can be completed with only one pod, unless there is only one order line or the association policy was applied in the retrieval process (in other words, all SKUs are stored together in one pod, if they are often ordered together by the same customer).

Then, after a pod has been processed at one or more stations, it is brought back to the storage area.

In the following sections we will use following notations:

• **\( \mathbb{N} \)**; natural numbers 1, 2, 3, …; \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \); \( \mathbb{R}^+ \) positive real numbers

• **\([a,b)\)**; half opened interval \( \{ x \in \mathbb{R} : a \leq x < b \} \)

• **\( a_t \)** some value \( a \) at time \( t \), \( a_t^{\mathbb{N}} \) sequence of values \( (a_t, a_{t+1}, a_{t+2}, \ldots a_N) \)

• \( \subset \) subset; it also includes the equal set

• **\( 1_{\{expression\}} \)**; is an indicator function; it is 1 if the **expression** is true and 0 otherwise

• **\([a_0,a_1,\ldots]\)**; ordered set (list). The element in the beginning has index **zero**.
3 Warehouse game

A real robotic warehouse is very complex. Even its simulation is still too complex to be completely modeled mathematically. Therefore we create a simplified mathematical model which reflects only the parts of an RMFS essential for the Pod Repositioning Problem.

3.1 Simplifications

No explicit pick orders In the real world, a customer orders one or multiple items from an online shop. In the shop’s warehouse, these items are in a single pod or in multiple pods in the storage area. The warehouse commands the robots to move the corresponding pods to a pick station where a picker will pack the items for the customer. For the Pod Repositioning Problem it is not important what items the customer has ordered and what items are in the pods. For us, only the direct consequence of the order is important: at some point in time a particular pod is at the pick station and then it returns back to the storage space.

No explicit robots In the real world, we need to know exactly where each particular robot is and which pod it carries. For our mathematical model, the only important properties are the position of the robots’ pods in the pick-station queue and the number of robots. This number is equal to the maximal number of pods at all pick stations.
No replenishment In this model we neglect replenishment of the pods, because it happens much less frequently than emptying of the pods at the pick stations.

Keep queues full We require our algorithms to keep the queues at the pick stations as full as possible. The reason for this requirement is: When we have many pods at the pick stations, we have many free places in the storage area. More free places in the storage area mean more possibilities to select cheaper places and to reduce the costs.

Time discretization In the real world, many pods are moving simultaneously between the storage area and the pick stations and within the pick stations. To simplify the model, we split the time into discrete steps and we combine different movements of the pod into a single step. Each time step combines the movement of a pod from the storage area to the pick-station queue, movement within the pick-station queue, and the movement of a pod out of the station. See Figure 3.

Remark 1. In the “Keep queues full” requirement, we have showed that free places in the storage area are important for optimization. In a real warehouse, there are at least two approaches how to create them: The first one is to have more places than the total number of pods in the system. The disadvantage of this approach is long traveling distances of the pods and costs for the unused space in the storage. The second approach is to keep many pods out of the storage area by moving them into long queues at the picking station or into a drift space. For drift space, see Amazon’s patent.
3.2 Parameters and states

- Time space $T_{\text{time}} = \{0, 1, 2, \ldots, N\} \subset \mathbb{N}_0$, where $N$ is the maximal number of steps.
- Finite set of pods $S_{\text{pod}} = \{1, 2, \ldots\} \subset \mathbb{N}$.
- Finite set of places $P = \{1, 2, \ldots\} \subset \mathbb{N}$.
- Finite set of stations $S_{\text{stn}} = \{1, 2, \ldots\} \subset \mathbb{N}$. Each station $s \in S_{\text{stn}}$ contains maximal $M_{\text{stn}}(s)$ pods.
- $q_{\text{stn}}(s) = [h_0, h_1, h_2, \ldots]$ of the station $s$. It is a first-in, first-out (FIFO) queue with pod $h_0 \in S_{\text{pod}}$ at the head, followed by pod $h_1 \in S_{\text{pod}}$, and so on. 
  - $q_{\text{stn}}(s, i)$ is the $i$-th element of the queue counted from the head. The head has index 0.
  - $\text{enq}$ is the enqueue operator:
    \[
    \text{enq}([h_0, h_1, \ldots, h_L, h_{L+1}]) = \begin{cases} 
    [h_0, \ldots, h_L, h_{L+1}], & \text{if queue was not full} \\
    [h_1, \ldots, h_L, h_{L+1}], & \text{if queue was full}
    \end{cases}
    \]
  - $\text{deq}$ is the dequeue operator:
    \[
    \text{deq}([h_0, h_1, \ldots, h_L]) = [h_1, \ldots, h_L].
    \]
- $s_{\text{strg}} : P \rightarrow S_{\text{pod}} \cup \{0\}$, $s_{\text{strg}}(p) = h \in S_{\text{pod}}$ means place $p$ is occupied by pod $h$. $s_{\text{strg}}(p) = 0$ means place $p$ in the storage area is free. The storage can be represented as an indexed family with index set $S_{\text{pod}}$. For example $(4, 5, 0, 0, 1, 0)$ means there is pod 4 at place 1, pod 5 at place 2, and pod 1 at place 5; all other places are free.
- $b_t$ describes pod departure from the storage space. A value $b_t = (h, s)$ means that at time $t + 1$ a pod $h$ will go to station $s$.
  - $b_t|_\text{pod}$ is the pod component of $b_t$. This means it is $h$ for $b_t = (h, s)$.
  - $b_t|_\text{station}$ is the station component of $b_t$. This means it is $s$ for $b_t = (h, s)$.
  - $b$ is a sequence of all future departures. At time $t$ it is $b^N_{t:t-1} = (b_t, b_{t+1}, \ldots, b_{N-1})$.
- The state of the system $s_{\text{sys}}$ consists of the state of the storage $s_{\text{strg}}$, current state of the station queues $(q_{\text{stn}}(s))_{s \in S_{\text{stn}}}$ and the future departures $b$.
- $a \in P \cup \{0\}$ is an action in the sense of discrete-time dynamic programming. It is a decision to move a pod from the head of the queue at a station $b|_\text{station}$ to some free place in the storage. $a = 0$ means no pod leaves the stations.
3.3 Dynamics

We describe the changes of the system over the time with discrete-time dynamic programming. See Hinderer et al. (2016, Chapters 2 and 3).

Following the requirement of full pick-station queues on page 7, during the first time steps we fill the queues. Formally, assume at time $t$ the system is in state $s_{\text{sys}} = \left( s_{\text{strg}}, (q_{\text{stn}}(s))_{s \in S_{\text{stn}}, b_t^{N-1}} \right)$. We select a pod $b_t|_{\text{pod}}$ at place $p$ and put it into not-full station $s := b_t|_{\text{station}}$. No pod is allowed to leave the queue and the action $a_t$ must be zero. At time $t + 1$, the station changes to the state $q_{s_{\text{stn}}}'(s) := \text{enq}(q_{s_{\text{stn}}}(\cdot), b_t|_{\text{pod}})$ and the storage area gets a gap at position $p$ and changes to $s'_{\text{strg}} := (\ldots, 0, \ldots)$. Most of the time the system runs with full queues. A newly arrived pod “pushes” another pod out of this queue. More precisely, assume at time $t$ a system is in state $s_{\text{sys}} = \left( s_{\text{strg}}, (q_{s_{\text{stn}}}(s))_{s \in S_{\text{stn}}, b_t^{N-1}} \right)$. We select a pod $b_t|_{\text{pod}}$ and put it into the corresponding station $s := b_t|_{\text{station}}$. Because station $s$ is full, the pod at the head of queue $q_{s_{\text{stn}}}(s)$ departs immediately and station $s$ changes its state to $q_{s_{\text{stn}}}'(s) := \text{enq}(q_{s_{\text{stn}}}(s), b_t|_{\text{pod}})$. The remaining stations stay unchanged.

In the same instant of time $t$, an action $a$ decides where to put pod $h := q_{s_{\text{stn}}}(s, 0)$ from the head of station $s$. This action can choose only free places in storage $s_{\text{strg}}$ and the place that has been newly freed by the departed pod $b_t|_{\text{pod}}$:

$$a \in D(s_{\text{sys}}) := \{ p \in P : \begin{array}{ll} s_{\text{strg}}(p) = 0 \lor s_{\text{strg}}(p) = b_t|_{\text{pod}} \end{array}.\$$

The storage state changes from $s_{\text{strg}} = (\ldots, 0, \ldots)$ or $s_{\text{strg}} = (\ldots, b_t|_{\text{pod}}, \ldots)$ to $s'_{\text{strg}} := (\ldots, h, \ldots)$. For the following system state, we do not need information about pods departed previously to time $t$, and the sequence of the future departures changes to $b_{t+1}^{N-1}$. The new system state is $s'_{\text{sys}} := \left( s'_{\text{strg}}, (q_{s_{\text{stn}}}(s))_{s \in S_{\text{stn}}, b_{t+1}^{N-1}} \right)$.

We summarize the dynamics described in this section as a transition function

$$T : S_{\text{sys}} \times A \rightarrow S_{\text{sys}}, \quad (s_{\text{sys}}, a) \mapsto s'_{\text{sys}}. \quad (1)$$

**Example 1.** Given six pods $S_{\text{pod}} = \{1, 2, 3, 4, 5, 6\}$, two stations $S = \{1, 2\}$. Each station has maximal length $M_{\text{stn}}(1) = M_{\text{stn}}(2) = 2$. The system starts at time $t = 0$ and ends at time $t = N := 2$. The initial system state is

$$s_{\text{sys}, 0} := (s_{\text{strg}, 0}, q_{s_{\text{stn}}, 0}(1), q_{s_{\text{stn}}, 0}(2), b_0)$$
with \( s_{\text{strg},0} := (1, 0, 3, 0, 0, 0) \), \( q_{\text{stn},0}(1) = [5, 2] \), \( q_{\text{stn},0}(2) = [4, 6] \) and \( b_0^1 = ((3, 2), (1, 2)) \). The action at time 0 is \( a_0 = 3 \).

Because of departure \( b_0 = (3, 2) \), the system puts pod 3 into station 2. The action \( a_0 = 3 \) moves pod 4 from the head of queue \( q_{\text{stn},0}(2) = [4, 6] \) to place 3 in the storage state \( s_{\text{strg},0} \). The following state \( s_{\text{sys},1} \) then becomes

\[
s_{\text{sys},1} := (s_{\text{strg},1}, q_{\text{stn},1}(1), q_{\text{stn},1}(2), b_1^1)
\]

with \( s_{\text{strg},1} := (1, 4, 0, 0, 0, 0) \), \( q_{\text{stn},1}(1) = [5, 2] \), \( q_{\text{stn},1}(2) = [6, 3] \) and \( b_1^1 = ((1, 2)) \).

### 3.4 Costs

Figure 4: Two types of costs, caused by the pod movements in a single time step.

Each movement of a pod causes non-negative costs. They depend on the location of the pod and consist of two parts – see Figure 4:

1. Moving a pod from the storage to a station costs

\[
c_{\text{to stn}} : P \times S_{\text{stn}} \rightarrow \mathbb{R}^+_0.
\]  

2. Moving a pod from a station to the storage costs

\[
c_{\text{from stn}} : S_{\text{stn}} \times P \rightarrow \mathbb{R}^+_0.
\]

Using the terminology of dynamical programming, the costs for an action \( a \) applied to a system state \( s_{\text{sys}} = \left( s_{\text{strg}}, (q_{\text{stn}}(s))_{s \in S_{\text{stn}}}, b_t^{N-1} \right) \) are described by a function

\[
c : A \times S_{\text{sys}} \rightarrow \mathbb{R}.
\]

If departure \( b_t \) sends a pod \( h \) from place \( p \) to station \( s \), it holds

\[
c(a, s_{\text{sys}}) = c_{\text{to stn}}(p, s) + c_{\text{from stn}}(s, a).
\]

10
The total costs for a decision sequence \( y := a_0^{N-1} \) for the initial state \( s_{\text{sys},0} \) correspond to the \( N \)-stage objective function

\[
C_{Ny}(s_{\text{sys},0}) := \sum_{t=0}^{N-1} c(a_t, s_{\text{sys},t}) + C_0(s_{\text{sys},N}),
\]

(5)

where \( s_{\text{sys},t} \) are generated from \( s_{\text{sys},0} \) by \( y \) using the transition function \( T \) from (1) and \( C_0 \) is the **terminal cost function**.

In our model we work with two types of terminal costs:

1. Zero terminal costs. We use this for most algorithms when \( N \) is large and therefore \( C_0 \) is negligible.
2. (Estimated) future costs, when \( N \) is small. It forces us to make good strategic decisions before we stop the system and resume it from the last state. We use these costs in the iterative Binary Integer Programming (BIP): see Remark 7.

Other terminal costs are also possible. For example:

3. The minimal costs of carrying all the pods in the output station back to the storage place.

**Remark 2.** The more general form of an \( N \)-stage objective function is

\[
C_{Ny}(s_{\text{sys},0}) := \sum_{t=0}^{N-1} \beta^t c(a_t, s_{\text{sys},t}) + \beta^N C_0(s_{\text{sys},N})
\]

with discount factor \( \beta \in \mathbb{R}^+ \). For our analysis, the simpler version (5) is sufficient.

## 3.5 Binary integer programming

In the binary formulation of the problem, we mainly focus on the case when the terminal costs are zero. That is \( C_0 \equiv 0 \).

**Decision variables** \( x_{tp} \in \{0,1\} \). \( x_{tp} = 1 \) means at time \( t \in T_{\text{time}} \setminus \{N\} \) we decide to put the current pod in place \( p \in P \). In other words \( x_{tp} \) corresponds to action \( a_t \):

\[
x_{tp} = 1 \iff a_t = p.
\]

**Parameters** At time \( t \in T_{\text{time}} \setminus \{N\} \) we decide to put a pod into the storage area. This pod leaves station \( s_{\text{from}}(t) \). It arrives into the storage area at time \( B_{\text{start},t} := t + 1 \) and leaves it at time \( B_{\text{end},t} \in T_{\text{time}} \). Hence, the pod stays there for a time interval \([B_{\text{start},t}, B_{\text{end},t})\). We set \( B_{\text{end},t} := \max(T_{\text{time}}) + 1 \) if the pod does not leave within the timespan \( T_{\text{time}} \). Later, the pod leaves the storage area and goes to station \( s_{\text{to}}(t) \). We set \( s_{\text{to}}(t) \) equal to zero when the pod does not leave the storage area within the timespan \( T_{\text{time}} \).

\( B_{\text{init},p} \) is the first time a place \( p \) becomes free within the timespan \( T_{\text{time}} \).
Cost function  In the cost function, we ignore the costs of moving pods to stations at the beginning of the game, because we cannot influence them. These are pods whose position was not decided within the timespan $T_{\text{time}} \setminus \{N\}$.

During the timespan $T_{\text{time}}$ almost every pod assigned to a place will also leave this place during the same timespan $T_{\text{time}}$. The exceptions are the pods which are marked by the parameter $s_{to}(t) = 0$. Thus, every decision $x_{tp}$ causes costs $c_{\text{from stn}}(s_{\text{from}}(t), p) + c_{\text{to stn}}(p, s_{to}(t)) \cdot 1_{\{s_{to}(t) \neq 0\}}$. Consequently, the total costs for $C_0 \equiv 0$ are

$$
\sum_{t \in T_{\text{time}} \setminus \{N\}} \sum_{p \in P} (c_{\text{from stn}}(s_{\text{from}}(t), p) + c_{\text{to stn}}(p, s_{to}(t)) \cdot 1_{\{s_{to}(t) \neq 0\}}) \cdot x_{tp}.
$$

Constraints  A pod must be assigned to only one place at time:

$$
\sum_{p \in P} x_{tp} = 1.
$$

The pod may not arrive in a busy place:

$$
B_{\text{init},p} \leq B_{\text{start},t} \cdot x_{tp} + M_{\text{big}} \cdot (1 - x_{tp}) \quad \forall t \in T_{\text{time}} \setminus \{N\}, p \in P,
$$

$$
B_{\text{end},\tau} x_{tp} \leq B_{\text{start},t} \cdot x_{tp} + M_{\text{big}} \cdot (1 - x_{tp}) \quad \forall t \in T_{\text{time}} \setminus \{N\}, \tau < t, p \in P,
$$

with a large value $M_{\text{big}}$. $M_{\text{big}}$ deactivates both constrains (7) and (8) when at time $t$ we do not decide to put the current pod to place $p$ – this is when $x_{tp} = 0$. Constraints (7) consider the busy places resulting from the initial state of the system. Constraints (8) consider the busy places resulting from the previous decisions.

We can significantly reduce the number of constraints (8) if we consider only occupation time intervals which may overlap with current decision $x_{tp}$. Formally, we need to consider only $\tau$ with $B_{\text{end},\tau} > B_{\text{start},t}$:

$$
B_{\text{end},\tau} x_{tp} \leq B_{\text{start},t} \cdot x_{tp} + M_{\text{big}} (1 - x_{tp}) \quad \forall t \in T_{\text{time}} \setminus \{N\}, \tau < t,
$$

$$
p \in P, B_{\text{end},\tau} > B_{\text{start},t}.
$$

4 Test systems

To test all our algorithms, we will mainly use two systems: a small system for qualitative analysis and a medium-size test system for more realistic tests.

4.1 Small test system

For the qualitative analysis of the Pod Storage Problem and its solving algorithms we need a very small and simple system. See Figure 5. It has only 10 places and 10 pods.
The costs are Manhattan distances between places in the storage area and places right in front of pickers. For example $c_{\text{to stn}}(1, s) = 5$ and $c_{\text{from stn}}(s, 5) = 9$ for every station $s \in \{1, 2\}$. The storage space is one-dimensional – it makes it easy to identify the cheapest (nearest) and the most expensive (most distant) places. We use two stations, because according to our experience, a system with only one pick station behaves very differently from a system with multiple pick stations. And, to our knowledge, there is no RMFS in the real world with only one pick station. As a further simplification, we use equal pick stations which we place symmetrically to the storage area. Because of this layout, the distance from any place in the storage area to every pick station is the same.

To create a sequence of departures $b = b_0^{N-1}$, we randomly choose a pod and a pick station. In the real world, some pods are used more frequently than others. Also, due to different pickers, some pick stations have higher throughput than others. To emulate this behavior we assign to each pod a positive weight $w_{\text{pod}, h} \in \mathbb{R}^+$, $h \in S_{\text{pod}}$, and to each station a positive weight $w_{\text{stn}, s} \in \mathbb{R}^+$, $s \in S_{\text{stn}}$, with $\sum_{s \in S_{\text{stn}}} w_{\text{stn}, s} = 1$.

To decide on the next departure $b$, we look at all pods in the storage $s_{\text{strg}}$, and send pod $h$ to station $s$ with probability $w_{\text{stn}, s} w_{\text{pod}, h} / \sum_{h \in s_{\text{strg}}} w_{\text{pod}, h}$. Here, $s_{\text{strg}}$ means the set of all pods in the storage area.

For our small system, we use pod weights equal to the density of the truncated geometric distribution

$$w_{\text{pod}, h} = C^{-1} w_{\text{pod}, 1}^h$$

with normalization constant $C := \sum_{h \in S_{\text{pod}}} w_{\text{pod}, h}$.

We choose $w_{\text{pod}, 1}$ in such a way that the weight of the most frequently used pod is 20
times\textsuperscript{1} higher than the weight of the least frequently used pod

\[
\frac{w_{\text{pod},1}}{w_{\text{pod},10}} = 20 \implies w_{\text{pod},1} = 0.29365446.
\]

See Figure 6.

For our simple system we use equal station weights: \( w_{\text{stn},1} = w_{\text{stn},10} = 1/2 \). The number of time steps is \( N = 1000 \) and we use the same sequence \( \mathbf{b} \) for every algorithm.

We start the small system with pre-sorted pods. A more frequently used pod is closer to the pick stations than a less frequently used pod.

### 4.2 Storage-area chart

For visual analysis of algorithms in Section 5 we will use special graphs – storage-area charts. They show how the small system changes in time. The vertical axis shows places. See detailed explanation in Figure 7. The horizontal axis shows time. The colors show which pod occupies the current place. Every pod has its own unique color all the time. We use colors from dark blue to dark red: from least used to most used pods. If the place is empty, we mark it white.

\textit{Remark 3.} The storage-area chart is a Gantt chart. The places are machines. The not interrupted sequences of squares, which belongs to the same pod, are the jobs. The chart shows that Pod Repositioning Problem is a special case of an NP-complete Problem \textit{Interval Scheduling with Required Jobs}, see Kolen et al. (2007, Subsection 2.2.1).

### 4.3 Medium-size test system

For more realistic tests we create a larger system with 504 places and 441 pods. The total number of places at the pick stations is equal to the minimal number of robots we would expect in a warehouse of this size. To make the problem harder we use asymmetric

\textsuperscript{1}There is no particular reason why this number is 20 and not 19 or \( 3\pi + \sqrt{2} \). The number should be high enough to have very different frequencies of the pods, but not too high, because we want to use even the rarest pod during the test run.
Figure 7: Section of storage-area chart. The red-framed section is storage state at the
time 165 or during the time span [160, 173).

Figure 8: Medium-size test system with asymmetric pick stations.

pick stations. See Figure 8. The costs between places and pick stations correspond to
distances that a real robot would travel. The initial positions of the pods at time $t = 0$
are random.

Like in the small system, we use for the pod weights the truncated geometric distribu-
tion with weight $w_{pod,1}$ selected in such a way that $w_{pod,1}/w_{pod,441} = 20$. Thus, the weight
of the most frequently used pod is $w_{pod,1} = 0.0071399315$. For the stations we assume
that station 1 works faster than station 2 and we set their weights to $w_{stn,1} := 0.6$ and
$w_{stn,2} := 0.4$. (A higher station weight means more throughput through that station.)
The number of time steps is 20000.

5 Algorithms

In this section we will analyze several algorithms to optimize our warehouse model:
three simple, like cheapest place, fixed place and random place; two classical solving
approaches, like iterative BIP and Genetic Algorithms; and finally we will create our
own heuristic. We will not only use the algorithms for optimization, but we will also use them to better understand the underlying Pod Repositioning Problem. At the end of each analysis we will briefly list the advantages and disadvantages of each algorithm.

5.1 Cheapest place
As the name suggests, a cheapest-place algorithm selects for a current pod the cheapest available place in the storage. It completely ignores how this decision will influence the costs of other pods in the system. In the real world, “cheapest” often refers to distance or time. In the RAWSim-O simulation (see Merschformann (2018)) this algorithm is called Nearest.

Even for this very simple algorithm we have the first technical problem: “What do costs mean?” Here we talk about abstract costs – not about their physical meaning like money, time, or distance. We introduce three possibilities:

1. **Cheapest to storage.** In this case we consider only the costs of moving the current pod from a pick station to the storage, that is \( c_{\text{to stn}}(p, s) \) only. We ignore how much it will cost to move this pod to a pick station again. The NearestPod algorithm in RAWSim-O works in this way. Sometimes it is reasonable to use these costs. This is when the fact that it would be cheap to move a pod into a place implies that it would also be cheap to move it out of this place.

2. **Cheapest on average.** In this case we assign to each place average costs

\[
\forall p \in P, \sum_{s \in S_{\text{stn}}} (c_{\text{to stn}}(p, s) + c_{\text{from stn}}(s, p)) \cdot v_{\text{stn},s} \tag{10}
\]

where \( v_{\text{stn},s} \) is the proportion of pods who leaves the storage area to station \( s \)

\[
v_{\text{stn},s} := \frac{1}{N} \sum_{i=0}^{N-1} 1\{b_{\text{station}=s}\}.
\]

The average costs can be helpful if we want to “globally rank” the places – independently from the pick stations and from the pods. We will use these costs for a fixed-place approximation and for the genetic algorithm later.

3. **Cheapest decision.** In these costs we use the known future station \( s_{\text{to}} \), which the pod will move later to. The special value \( s_{\text{to}} = 0 \) means that the pod will not leave the storage

\[
c_{\text{decis}} : P \times S_{\text{stn}} \times (S_{\text{stn}} \cup \{0\}) \rightarrow \mathbb{R}
(p, s_{\text{from}}, s_{\text{to}}) \mapsto c_{\text{to stn}}(p, s_{\text{from}}) + c_{\text{from stn}}(s_{\text{to}}, s_{\text{from}}) \cdot 1\{s_{\text{to}} \neq 0\}. \tag{11}
\]

To keep the cheapest-decision costs simple, we ignore the future costs caused by the terminal costs \( C_0 \).
5.1.1 Properties of cheapest-place algorithms

We analyze the storage-area chart created by a cheapest-place algorithm. In our small system, it does not matter which of the three costs we use to determine the cheapest place: the result is the same.

A cheapest-place algorithm generates a distinct pattern in storage-area charts – it uses as few places as possible. See Figure 9. The algorithm appears to be optimal – but it is not. The problem is that it often puts rarely used pods close to the output stations. There, these pods waste resources for a long time.

![Cheapest-place algorithm](image)

Figure 9: Cheapest-place algorithm. Changing of the storage area during the time span [160, 196). (From dark blue to dark red: least used to most used pods.)

The final question we want to answer in this subsection is: “Is the cheapest-place algorithm optimal?” Our experiments suggest that, in general, it is not. But under special conditions – uniform usage of pods – it provides pretty good results. To be more precise, our experiments show that “pretty good results” actually mean that the cheapest-place algorithm provides similar results for different distributions of pods, but under the special condition – uniform usage of pods – the other algorithms become worse.

**Remark 4.** A cheapest-place solution is closer to an optimal solution the more uniformly the pods are used: see Figure 10. There we have tested cheapest-place algorithm on the small test system with different degrees of uniformity:

- **random geometric** The departures of the pods are random and not uniformly distributed. We use geometric weights from the small test system as described in Subsection 4.1. The stations are randomly selected.

- **random uniform** The departures are random and uniformly distributed. We use equal weights for pod selection. The stations are randomly selected.

- **periodic random** The departures are blocks of random permutations of pods. For example (1, 4, 5, 6, 7, 3, 9, 2, 8, 10), (4, 8, 7, 1, 3, 2, 4, 5, 6, 9, 10),... Each permutation is uniformly distributed. If a pod, that needs to depart according to the generated sequence, is not in the storage area, we use some simple correction. The stations are randomly selected.

- **periodic** The departures are deterministic and uniformly distributed. The departures are repeating sequences (1, 2, 3, 4, 5, 6, 7, 8, 9, 10) and the destination
Figure 10: Optimality of the cheapest place algorithm for pod departures with different degree of uniform usage.

stations are repeating sequences (1, 2).

With periodic departures, the cheapest-place solution is optimal.

5.1.2 Advantages and disadvantages

A cheapest-place algorithm has several advantages and disadvantages:

⊕ It is simple. The cheapest-to-storage version of the algorithm can be easily applied for more complex simulations without modification.

⊕ It appears to work well, when the pods are used similarly often.

⊕ It appears to provide good results in the RAWSim-O simulations if combined with optimization policies for other RMFS decision problems, see Merschformann et al. (2018a, Section 8).

⊖ It does not work well when the pods are used with very different frequencies.

⊖ If the pods are used with different frequencies and they are already optimally ordered in the storage area, the cheapest-place algorithm will destroy this order and proceed to work not optimally.

To evaluate the quality of our algorithms we need some reference algorithms. The best reference is the exact solution, but unfortunately it is only available for small and simplified models. More practical reference algorithms are: fixed place and random place.

5.2 Fixed place

A fixed-place algorithm always assigns the same place to the same pod. That means we look for an optimal function

\[ a_{\text{fix}} : S_{\text{pod}} \rightarrow P \]

\[ h \mapsto a_{\text{fix}}(h). \]
which assigns a place to every pod $h$ which will leave the pick stations. We calculate an optimal function $a_{\text{fix}}$ with BIP. To prevent conflicts, we assume that in the initial state all the pods in the storage area have been assigned by $a_{\text{fix}}$ too.

For BIP, we represent the function $a_{\text{fix}}$ with decision variables $x_{hp} \in \{0, 1\}$. $x_{hp} = 1$ means “assign pod $h$ to place $p$,” or more formally: $x_{hp} = 1 \iff a_{\text{fix}}(h) := p$. From departures $b$, we count how frequently each pod $h$ visits station $s$ and how frequently it comes back. We store this information in $f_{\text{to stn}}(h, s)$ and $f_{\text{from stn}}(h, s)$. Now we can define costs $c_{hp}$ if pod $h$ will always return to place $p$

$$c_{hp} := \sum_{s \in S_{\text{stn}}} (f_{\text{to stn}}(h, s) \cdot c_{\text{to stn}}(p, s_{\text{from}}) + f_{\text{from stn}}(h, s) \cdot c_{\text{from stn}}(s, s_{\text{to}})).$$

We solve the minimization problem

$$\min \sum_{h \in S_{\text{pod}}} \sum_{p \in P} c_{hp} x_{hp}$$

subject to constraints: assign every pod

$$\sum_{p \in P} x_{hp} = 1, \quad \forall h \in S_{\text{pod}}, \quad (12)$$

and no more than one pod per place

$$\sum_{h \in S_{\text{pod}}} x_{hp} \leq 1, \quad \forall p \in P. \quad (13)$$

Remark 5. Determining of the optimal places for the fixed-place algorithm is an assignment problem. The pods are tasks and the places are the agents. When we assume that the pods go with the same relative frequency to the pick stations we can find very fast solution with following algorithm: Sort pods by their usage frequency and write them to a list. Sort places by their average costs (10) and write them to another list. Assign the $i$-th pod to the $i$-th place in the corresponding lists.

Remark 6. The fixed-place algorithm is an extreme case of zoning strategy. Zoning means, we divide the storage area into different zones. Pods with different frequencies of usage go to different zones: see Lamballais et al. (2017). The fixed-place algorithm has as many zones as there are pods in the system.

5.2.1 Properties of fixed-place algorithms

We analyze a storage-area chart of a system with a fixed-place policy. See Figure 11. The typical pattern of the fixed-place algorithm is that there is only one non-white color in every line. We see that the algorithm does not use a lot of cheap places. This means that the fixed-place algorithm is in general not optimal. The effect of unused places is very clear in the small test system, because most of the pods are at the output stations. In contrast, in a more realistic system, where there are many more pods in the storage area than in the pick stations, we expect the fixed-place algorithm to be pretty good.
When we know the future departures for a long period of time, we can use the fixed-place algorithm as a good upper bound for the long-term cost. Here we assume, that we need only a small fraction of time and costs to bring a system from some arbitrary initial state, where pods are not optimally distributed, to an optimal state, where places are assigned by \( a_{\text{fix}} \). Or if we assume that we can actively reposition the pods before applying the algorithm, such as in Merschformann (2018), for a small fraction of the total costs. Unfortunately, this does not work for many real-world settings, because we cannot predict future departure of the pods for a long period of time. On one day, we start to move the system to some optimal state, but on the next day the customers order different things and we need to move to another optimal state. In this case, the fixed-place algorithm becomes a good lower bound for the optimal costs.

### 5.2.2 Advantages and disadvantages

Here are advantages and disadvantages of the fixed-place algorithm:

- It is simple.
- We expect that it works well for larger systems.
- It requires an (expensive) rearrangement of pods before we can use the algorithm.
- It is not practical if we have seasonal changes in pod usage.

### 5.3 Random place

We use the random place assignment as a performance reference for other algorithms. In this subsection random means: every time, we select one place from all available free places with equal probability independently, from our previous decisions. From a modeling point of view the random strategy stands for “we don’t know” and for “we don’t care.” It shows what happens if we do not optimize. The performance of
the random algorithm is usually bad, but in some cases stochastic magic happens and random becomes optimal.

For the sake of completeness, we provide a storage-area chart of the random algorithm. See Figure 12. It shows, how the resources are wasted: The algorithm does not use places close to the output stations efficiently. It puts frequently used pods too far away.

![Figure 12: Random algorithm. Changing of the storage area in time.](image)

Finally, here is a brief list of advantages and disadvantages of the random algorithm:

⊕ It is simple.
⊕ It is a good reference for performance comparison between different algorithms.
⊖ It provides bad results.

In Subsection 3.5 we described how to solve the Pod Repositioning Problem exactly with BIP. This approach works well for the small test system but fails for more realistic systems, like for example the medium-size test system. The reason for the failure is a lot of constraints (4 158 636 886), about $|P| \cdot |S_{pod}| \cdot |T_{time}| = 504 \cdot 441 \cdot 20 000$, and possible NP-hardness of the Pod Repositioning Problem. In the next subsection we will use three heuristic methods, which should still provide good results.

### 5.4 Iterative binary integer programming

A calculation of an optimal solution is computationally intensive. Instead of calculating the optimal solution for the entire time $T_{time}$, we split the time into $K$ intervals $I_k = [t_k, t_{k+1})$ with $t_0 = 0$ and $t_{K+1} = N$. We call the set of these intervals $\mathcal{I}$. Then we minimize (6) restricted to an interval

$$\sum_{t \in I_k} \sum_{p \in P} (c_{from \ stan(t),p} + c_{to \ stan(p,s_{to}(t))} \cdot 1_{\{s_{to}(t)=0\}}) \cdot x_{tp} \quad \forall I_k \in \mathcal{I}$$

one by one.

The conditions for the initially busy places (7) become

$$B_{init,p}(t_k) \leq \underbrace{B_{start,t}}_{=t+1} x_{tp} + M_{big} (1 - x_{tp}) \quad \forall t \in T_{time} \setminus \{0\}, p \in P,$$
where the end of the busy periods $B_{\text{init},p}(t_k)$ is defined as

$$B_{\text{init},p}(0) := B_{\text{init},p}$$

$$B_{\text{init},p}(t) := \max\{B_{\text{init},p}(t-1), B_{\text{end},t-1}(t-1)x_{t-1,p}\}.$$ 

The conditions for previously busy places (8) become

$$B_{\tau x tp} \leq B_{\text{start},t \cdot x tp} + M_{\text{big},k}(1 - x tp) \quad \forall t \in I_k, \tau \in I_k, \tau < t, p \in P$$

with $M_{\text{big},k} = \max(I_k) + 1$.

**Remark 7.** One may see the iterative BIP as a sequence of solving of small Pod Repositioning Problems for each time interval $I_k$. In each small problem, the terminal costs consist of the future costs for placing pods which will leave the storage area outside of $I_k$. The final state becomes the initial state of the subsequent problem on $I_{k+1}$. The fact, that we ignore costs from the previous decisions on $I_{k-1}$, does not change the solution on $I_k$, because these costs are only a constant subtracted from the total costs on $I_k$.

The main advantages and disadvantages of the iterative BIP method are:

- It provides good results.
- It is still computationally intensive.
- It is not flexible.

### 5.5 Genetic algorithms

A solution for a deterministic warehouse problem is a sequence of actions $a_0^{N-1}$ which optimally assigns resources (places) under complex conditions. This kind of combinatorial problem suggests the use of genetic algorithms.

The goal of this section is not to find the best genetic algorithm or the best set of parameters. It is rather an attempt to find out how one may use genetic algorithms for the Pod Repositioning Problem, what problems may occur, and what we can learn from the first results.

We implemented genetic solvers with a python package, DEAP; see Fortin et al. (2012).

#### 5.5.1 First approach

For a genetic algorithm we must represent our solution as a chromosome. The first natural attempt is to use as the chromosome a sequence of actions. That means a sequence of places $p \in P$ and zeros. As you may recall, a zero action means: “do not send any pod to storage.” In this paper, we will call this algorithm *genetic 1*.

Every mutation of a chromosome is a random change of some action. The probability that a chromosome element will mutate is $3/N$, where $N$ is the total number of actions. This leads to three changes in the entire chromosome on average. We do not want to have too many changes, because they easily create an unfeasible solution. We also do
not want to have too few changes, because then the improvement rate sinks. If an action randomly changes, it changes to a uniformly distributed value $p \in P$. We use a random two-point cross-over.

As an initial solution we use a random feasible solution. We tried to use the cheapest-place solution, but our genetic algorithm could not improve it. The fitness value of a chromosome is the average costs per time step.

We ran the algorithm until the last 100 generations could not improve the best results of the previous generations. The population size per generation was 100 individuals. In our small test system, the algorithm converged very slowly: only after 3761 generations did it become better than the cheapest-place algorithm. See Figure 13.

![Figure 13: Minimal costs by a genetic algorithm, whose chromosome is places. See Subsection 5.5.1.](image)

### 5.5.2 Second approach

In our first implementation we had a chromosome made of places, this caused a lot of not feasible solutions (ca. 23 % for the small test system). The reason for this was that the algorithm randomly chose a place but did not care whether this place was free. We improve this by constructing a chromosome which considers only free places. In this paper, we will call this algorithm genetic 2.

For this new algorithm, we need a particular order of places $\gamma$. That means, a bijective function

$$\gamma : \{0, \ldots, |P| - 1\} \rightarrow P$$

$$p \rightarrow \gamma(p).$$

Every chromosome element for time $t$ is an index from 0 to $|D(s_{sys}(t))| - 1$. It shows which free place from $D(s_{sys}(t))$ – sorted by $\gamma$ – to choose. The element at the beginning has index 0. The following example demonstrates how the new algorithm works.
Example 2. Consider a chromosome $\chi_{\text{close}} := (1, 1)$. We order the places by their distances to the stations. The closest one is in the front

$$\gamma_{\text{close}} := [1, 2, 3, 4, 5, 6, 7, 8, 9, 10].$$ \hspace{1cm} (14)

At time $t = 0$ the system has a state shown in Figure 14a. As you may recall, the decision to occupy a free place at time $t$ is made at time $t - 1$. At time $t = 1$ a pod at place 9 will arrive at the left queue. The sequence of allowed actions ordered by $\gamma_{\text{close}}$ is then $D(s_{\text{sys}}(0)) = [3, \hspace{1cm} 4, 5, 6, 7, 8, 9]$. The red pod goes to place 4. The new state is shown in Figure 14b.

At time $t = 2$ a pod from place 10 will arrive at the right queue. The ordered sequence of allowed actions at time $t = 1$ is $D(s_{\text{sys}}(1)) = [3, \hspace{1cm} 5, 6, 7, 8, 9, 10]$. Following
5.5.3 Properties of genetic algorithms

We focus on analysis of the improved genetic algorithm from Subsection 5.5.2. As we can see in Figure 15, within the plotted time span, the algorithm puts a rarely used pod in the far end of the storage onto place 9. It keeps other pods close to the output station and uses the closest place – place 1 – very intensively for different pods.

Place order $\gamma$ matters In our algorithm, a mutation is a uniformly distributed variable. It seems that the place order $\gamma$ does not matter. And this is true for a single mutated element of the chromosome (a permutation of uniformly distributed variables has the same distribution). But a chain effect on the subsequent actions depends a lot on the place order $\gamma$. The following example demonstrates this.

Example 3. Consider three systems. All of them have the same initial state and the same dynamics at times 0, 1 and 2 as shown in Figures 14a, 14b and 14c. The orders of
the systems are:
\[
\gamma_{\text{close}} := [1, 2, 3, 4, 5, 6, 7, 8, 9, 10],
\gamma_{\text{far}} := [10, 9, 8, 7, 6, 5, 4, 3, 2, 1],
\gamma_{\text{zigzag}} := [1, 6, 2, 7, 3, 8, 4, 9, 5, 10].
\]

The system changes following chromosomes
\[
\chi_{\text{close}} := (1, 1), \quad \chi_{\text{far}} := (5, 5), \quad \text{and} \quad \chi_{\text{zigzag}} := (4, 5).
\]

While the dynamics of the “close” and “far” systems are more or less clear, the behavior of the “zigzag” system is less obvious. We explain it step by step: At time \(t = 0\) the sequence of allowed actions, ordered by \(\gamma_{\text{zigzag}}\), is \([6, 7, 3, 8, 4, 9, 5]\). The system chooses the fourth element – place 4. At time \(t = 1\) the ordered sequence of actions is \([6, 7, 3, 8, 4, 9, 5, 10]\). The system chooses the fifth element – place 5.

Now the first element of the chromosome mutates and the red pod moves at time \(t = 1\) onto place 5. See Figure 14d. This corresponds to the new mutated chromosomes \(\chi_{\text{mut}}_{\text{close}} := (2, 1), \quad \chi_{\text{mut}}_{\text{far}} := (5, 5), \quad \text{and} \quad \chi_{\text{mut}}_{\text{zigzag}} := (6, 5).\) The meaning of the second element in \(\chi_{\text{close}}, \chi_{\text{far}}\) and \(\chi_{\text{zigzag}}\) changes. In \(\chi_{\text{close}}\) and \(\chi_{\text{far}}\) the values 1 and 5 now mean: “move green pod to place 4.”; see Figure 14e. This mutated state at time \(t = 2\) is not very different from the non-mutated state in Figure 14c. But for the system ordered by \(\gamma_{\text{zigzag}}\) the change is more significant: The value 5 in \(\chi_{\text{mut}}_{\text{zigzag}}\) now means “move green pod to place 9” (fifth free place from \([6, 7, 3, 8, 4, 9, 10]\)). The resulting state at time \(t = 2\) on Figure 14f differs much from the non-mutated state in Figure 14c.

Example 3 shows how important the order \(\gamma\) is. We suggest selecting an order in such a way that the neighbor places have similar costs. We made a numerical experiment with different orders. We ran 20 optimizations for every order type, each with 100 individuals per generation for 100 generations. The results in Figure 16 confirm the advantages of a “smooth”\(^3\) and disadvantages of a “jumping” order. To speed up the calculation we sort the places by their average costs (10).

5.5.4 Advantages and disadvantages

Here are advantages and disadvantages of the genetic algorithm. We only consider the efficient version described in Subsection 5.5.2:

- It is flexible.
- Decisions based only on free places are robust. When a place is busy, the algorithm automatically resolves the problem by choosing the next place.
- It is slow.

\(^3\)In this subsection the word smooth means that a small change in one decision does not cause a large change in the subsequent decision and therefore total costs.
5.6 Tetris

Every pod-repositioning algorithm assigns a time interval and a place to a pod. We call these time intervals occupation intervals, because they show when the pod will occupy a place in the storage area. These algorithms cannot change the length and the horizontal position of the occupation intervals and they must ensure that the occupation intervals do not overlap.

Figure 17: Comparison between two algorithms: cheapest-place and BIP. Both algorithms are applied to the same instance of our small test problem. Pods are colored according to their usage. Note, the cheapest-place algorithm uses fewer places, but it is less optimal than the BIP. This is because it is not important how many places an algorithm uses, but rather how frequently it uses the cheapest places.

Figure 17 explains this visually: It shows two storage-area charts of two different algorithms. Each block represents an occupation interval and the block’s color refers to a particular pod. Both charts show that each algorithm moves blocks with the same color up and down. The algorithms cannot move the blocks to the left or to the right. They cannot split the blocks of the same color. It cannot let the blocks overlap.

The visual comparison leads to a new algorithm. It is inspired by a computer game, tetris.

Figure 18 explains the algorithm visually. In the very first part of the algorithm, we create a feasible solution with a lot of free cheap places. See Figure 18a. The easiest way is to use a reverse version of the cheapest-place algorithm – the most-expensive-place algorithm. There, every time a pod leaves an output station, it will go to the most
expensive available place according to $c_{\text{decis}}$ costs.

In the second part, we improve the initial solution: We select the most frequently used pod and move all its occupation intervals to the cheapest free places. See Figure 18b. If there is no cheaper place, we keep the old place. We start with earlier occupation intervals and continue with later ones. We assign the remaining pods in the same way. See Figures 18c and 18d. A more formal pseudocode is in Algorithm 1.

In this paper, we call this algorithm 	extit{tetris}. This single word explains the idea quite well and is slightly shorter than its German single-word alternative: Regalhäufigkeitspriorisierungs-zukunftsverhaltungsberücksichtigungs-algorithmus.

Figure 18: Some steps from the 	extit{tetris} algorithm from Subsection 5.6. Each color shows how frequently a corresponding pod is used during the whole optimization time.
Algorithm 1 Tetris algorithm.

1: function Tetris
2: \( \mathcal{I} \leftarrow \text{MostExpensivePlace()} \)
   \( \mathcal{I} \subset P \times S_{\text{pod}} \times T_{\text{time}} \times T_{\text{time}} \)
   \( \triangleright \) Improve the initial solution.
3: \( \mathcal{I} \leftarrow \text{sort}(\mathcal{I}) \) \( \triangleright \) Sort the occupation intervals by the pod frequency and then by the arrival times.
4: for each \((p, h, t_{\text{begin}}, t_{\text{end}}) \in \mathcal{I}\) do
5:   for each \(p^* \in P\) do
6:     if \(\text{costs}(p^*) < \text{costs}(p)\) and \(p^*\) is free for \([t_{\text{begin}}, t_{\text{end}}]\) then
7:       \((p, h, t_{\text{begin}}, t_{\text{end}}) \leftarrow (p^*, h, t_{\text{begin}}, t_{\text{end}})\)
8:       \(a(t_{\text{begin}} - 1) \leftarrow p^*\)
9:   end if
10: end for
11: end for
12: return \(a_1^{N-1}\) \( \triangleright \) Return the sequence of actions.
13: end function

5.6.1 Extension and variations of the algorithm

Sort by occupation duration The first version of the algorithm relies on pod frequencies during the whole time. This makes it less suitable for seasonal changes of pod frequencies. To make it more robust against the seasonal changes, we modify the second part of the algorithm: Instead of assigning the most frequently used pods first, we first assign pods with shorter occupation durations. According to our experiments, when we do not have seasonal data, the algorithm based on the duration time is a little less efficient than the frequency-based one. In contrast, when we use seasonal data, the algorithm based on occupation duration is better.

5.6.2 Properties of the tetris algorithm

Figure 20 shows the storage-area chart produced by the tetris algorithm. The algorithm puts frequently used pods to the front and pods which spend a lot of time in the storage area to the end of the storage. It tries to use the cheapest places even for less frequently used pods when a more frequent pod cannot use them. See for example the rarely used pod during the time span \([198, 199]\) at place 1 in Figure 20.

5.6.3 Advantages and disadvantages

⊕ It appears to be good.
⊕ It is fast.
Figure 19: Frequency-based tetris vs occupation-duration-based tetris applied to seasonal data. Costs for a system with 504 places and 411 pods during the time $T = \{1, \ldots, 10000\}$. Every 2000 time steps, the probability weights for the pods change randomly. We made 20 tests with different random initial pod positions and different pod departures.

Figure 20: Tetris algorithm. Changing of the storage area during the time span $[160, 196]$.

⊕ The version which is based on the length of occupation intervals produces good results for seasonal pod usage.

⊖ This algorithm is a heuristic which is made under some assumptions about the usage frequency of the pods. We do not know how well this algorithm will perform for a particular real data set.

6 Computational results

We tested different algorithms on two types of system, the small one from Subsection 4.1 and the medium-size test system from Subsection 4.3. For each system we used the same instance for all tested algorithms. For BIP we used open-source library COIN-OR CBC over the python PuLP interface and Gurobi. For fixed-place algorithm we used COIN-OR CBC.

We use different algorithms for a small system with 10 places and 10 pods. We ran the test on a notebook with an Intel Core i7-7600 CPU, 2.80 GHz, two cores (four logical cores), 16 GB RAM and Ubuntu 18.04. The results are in Figure 21. They show that tetris provides pretty good results and it is also very fast.
For a medium-size test system with 504 places and 441 pods we used only the random-place, cheapest-place, iterative BIP, genetic 2, tetris and fixed-place algorithms. We did not use genetic 1 algorithm because it demonstrates bad results for a small system. For computationally-intensive genetic 2 algorithm we used a faster cluster with four Intel Xeon E5-4670, 32 cores, 2.7GHz, with 64 GB RAM. For iterative BIP with interval size 100 we used a cluster with two Intel Xeon X5650, 16 cores, 2.67GHz, 32 GB RAM and Gurobi 7.5.2. Most of the time, the iterative BIP solver spent for adding constraints. The results are in Figure 22. They show that tetris provides a good solution with 20 000 decisions within less than one minute.

Note, the fixed-place algorithms rearrange pods of the system with 504 before starting the system. But we do not consider any additional costs for this improvement of the initial state. This corresponds to the active pod repositioning for free. In contrast, all the other algorithms are not allowed to change the initial, randomly generated, positions of the pods before the system starts. They must use only the passive pod repositioning. That is why we cannot compare other algorithms with both fix-place algorithms directly. Nevertheless we use the fixed-place results as a reference, it shows what would happen if the system would starts with perfect pod positions and keep them the whole time.

7 The problem with multiple stations

The previous section demonstrates that it is important to know the occupation time intervals of the pods in the storage area. The algorithms with the most optimal costs – BIP and tetris – use this information. Unfortunately, in real life we do not have exact information about the occupation time intervals. This is because the picking times (and to a smaller extent the transport times) introduce randomness to the system. Our occupation time intervals are only estimations with estimation errors.

Although it is hard to determine the occupation time intervals exactly, we do not really need it. What we really need is to know the order in which pods leave the storage area and the order in which pods return to the storage area. This information is partially available in the real-world warehouses because they process customers’ orders in batches. For example, a warehouse collects customers’ orders for one hour without fulfilling them. Then it calculates an efficient order in which the pods will go to particular stations to process the customers’ orders. See for example the algorithms in Boysen et al. (2017).

Single pick station For a single station, the information, about the order in which the pods will go to the pick station is fully sufficient for most algorithms. For example, when pods $A$, $B$, $C$ and $D$ leave the storage area in the order $ABCD$, they also arrive in the same order, $ABCD$. No matter how fast or slow the picker is, the order stays unchanged.

If we know the order of departures, we can expand them to rescaled occupation time intervals. The basic idea is as follows: Imagine you record two warehouses simultaneously. One uses algorithm $A$ for optimization and the other uses algorithm $B$. The pods depart in both systems simultaneously and the picking times are the same. Algorithm
Figure 21: Computation results for the small system with 10 places and 10 pods from Subsection 4.1.
Figure 22: Computation results for the medium-size test system with 504 places and 441 pods from Subsection 4.3.
A is better than algorithm B. Algorithm A will still be better than algorithm B even if you play back your recording at different speeds. Now you can speed up and slow down the playback to have exactly one unit of time between the departures and between the arrivals; these will make the occupation time intervals suitable for our algorithms.

To keep the things simple, we focus on the case when the queue at the pick station is full: that means it contains $M_{\text{stn}}$ elements. Then, for the $i$-th pod we use the time interval $[i + M_{\text{stn}}, j)$ as the occupation time interval in the storage area, where $j$ is the index of the same pod when it will depart from the storage area next time. We give this information to one of our algorithms and the algorithm returns the sequence of decisions. The precise time of the pod assignment does not matter, only its order.

When we know the order of the decisions, small timing problems may still occur, but we can easily resolve them: It may happen that pod $A$ leaves the pick station too early and the assigned free place is still occupied by another pod $B$. To resolve this conflict, the robot with pod $A$ just waits until another robot removes pod $B$ according to the previously calculated sequence\(^4\). When pod $A$ arrives at its place too late, the place can be taken away. To prevent this, we reserve this place for $A$. We can apply a similar conflict resolution to the pick station. When the pod comes too early, it must wait for other pods. When the pod arrives too late, another pod waits for it.

The conflict resolution requires additional time and therefore it may have negative consequences for the throughput at the pick station. To dampen these consequences we suggest having a queue with several pods at the output station. This queue acts like a buffer. It dampens time fluctuations and reorganizes the pods into the correct order until they actually reach the picker.

**Multiple pick stations** The situation becomes much more complicated, as soon as we have more than one pick station. Also in this situation, the random fluctuations in picking times and in transport times change the order in which the pods leave and arrive at the storage area. But the consequences are now different. For example we may plan that pod $A$ and then pod $B$ go to station 1, and pod $C$ and then pod $D$ go to station 2. For our calculation we assume that the pods will leave the storage area in the order $ABCD$ and then return to the storage area in the same order. But in reality the actual departure order could be any of the following: $ABCD, ACBD, ACDB, CDAB, CADB$ or $CABD$. We also can surely determine the order in which the pods will return. The rescaling of the time, like we did in the single-station system, will not fix the departure and return orders.

If the fluctuations are not very high, then we still use a waiting strategy to stick to the plan. To correct higher fluctuations we can use long queues at the pick stations or drift spaces. But if we plan 1000 pod assignments in advance, the difference between the planned pod order and the actual pod order can become too large. The output station will not be able to compensate time differences and the whole system will slow down. Instead of waiting we can use other strategies for larger timing errors:

\(^4\)In a rare case, when pod $B$ requires the robot of pod $A$, we need more steps to exchange the pods, but it is still fast and physically possible.
Strategies for multiple pick stations  Often we will not be able to apply deterministic algorithms directly in real-world settings: instead we can use them as concepts. For example, the genetic algorithm, with its reaction on mutation and its focus on free places, provides two possible strategies for pod assignment:

1. Calculate the solution with estimated picking times. When a previously assigned place is occupied in reality, then go to the next best (or next worst) free place according to some order.

2. Calculate the solution with estimated picking times and translate the solution into free-place indices, like the genome of the genetic algorithm in Subsection 5.5.2. For example, translate “pod A goes to place 117” to “pod A goes to the 14th free available place” according to some preference list.

We can derive another way to solve the problem from the tetris and cheapest-place algorithms. We can look at the tetris algorithm as a priority algorithm. In tetris, the most frequently used pods selects the best places first. That means these pods have the highest priority. Also, a pod cannot select a place and occupy it for a period of time if another pod with higher priority wants to have this place within this period of time. Also, the cheapest-place algorithm is a sort of priority algorithm: a different one. It gives to the current pod the highest priority to select any possible place. Based on these priority concepts, we can create a new priority algorithm which better suits the random occupation time intervals.

3. Estimate how frequently every pod will be used. Assign a priority to every pod based on its frequency. When a pod leaves a pick station, assign a free place to it based on its priority. That means: do not take the best places if other pods with higher priority want to use them too. Apply the cheapest-place algorithm among pods with similar priorities.

8 Conclusion

We have shown that a very small mathematical model helps us to understand essential parts of the Pod Repositioning Problem. It also provides a fast heuristic – the tetris algorithm. This algorithm appears to be robust and flexible enough to be applied to larger instances. We critically analyzed how implication from our mathematical models can be or are already used in the real world.

In Section 7, we showed that for a system with multiple pick stations, a deterministic model is not sufficient for real-world application. Unfortunately, to our knowledge, there is no real warehouse with only one pick station. That means that the deterministic model is only a starting point for a more realistic stochastic model of a robotized warehouse. We also showed that in a stochastic system we should focus on the estimation of the departure and arrival order of the pods from and into the storage area.
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