Research Article

A New Iterative Construction for Approximating Solutions of a Split Common Fixed Point Problem

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1.Introduction

Let H1 and H2 be the Hilbert spaces and C and Q be nonempty closed and convex subsets of H1 and H2, respectively.

The split feasibility problem (SFP) is known to find

\[ x \in C, \quad \text{such that } Ax \in Q, \]

where \( A: H_1 \rightarrow H_2 \) is a linear bounded operator.

In [1], the split feasibility problem (SFP) in the finite-dimensional Hilbert spaces was introduced by Censor and Elfving. This problem is equivalent to a number of nonlinear optimization problems and finds numerous real applications, such as signal processing and medical imaging (see, e.g., [2–7]).

For this split problem, simultaneous multiprojections algorithm was employed by Censor and Elfving in the finite-dimensional space \( \mathbb{R}^n \) to obtain the algorithm as follows:

\[ x_{n+1} = A^{-1}P_{Q}P_{A(C)}Ax_n, \quad n \geq 0, \quad \text{(2)} \]

where both C and Q are convex and closed subsets of \( \mathbb{R}^n \), the linear bounded operator A of \( \mathbb{R}^n \) is an \( n \times n \) matrix, and \( P_Q \) is the orthogonal projection operator onto the sets Q.

The above algorithm (2) involves the matrix \( A^{-1} \) (one always assumes the existence of \( A^{-1} \)) at every iterative step. Calculating \( A^{-1} \) is very much time-consuming, if the dimensions are large scale, in particular, and thus it does not become popular.

In order to overcome the fault, Byrne [2, 8] proposed the following novel algorithm CQ, which is under the spotlight of recent research

\[ x_{n+1} = P_C(x_n - yA^*(I - P_Q)Ax_n), \quad n \geq 0, \quad \text{(3)} \]

where \( P_C \) and \( P_Q \) are the orthogonal projection operators onto the sets C and Q, respectively, and \( 0 < y < (2/\rho) \) with \( \rho \) being the spectral radius of the composite mapping \( A^*A \). But, the CQ algorithm’s step-size is fixed, and it is related to spectral radius of \( A^*A \). On the other hand, the orthogonal projection onto the subsets C and Q in Hilbert space \( H_1 \) is not easily calculated generally except the special cases, such as balls and polyhedrals. With the real applications (intensity-modulated radiation therapy and medical imaging) of the SFP in signal processing, the SFP has obtained much attention. Now, the approximate solutions of the SFP have been studied extensively by scholars and engineers (see, e.g., [9–13]).
In (1), if C and Q are the intersections of fixed point sets of finite many nonlinear operators, the SFP becomes the split common fixed point problem (SCFPP). The SCFPP was studied first by Censor and Segal [14] in 2009, which consists of finding an element \( x \in H_1 \) with

\[
x \in \bigcap_{i=1}^{m} \text{Fix}(T_i), \quad \text{s.t.} \ Ax \in \bigcap_{j=1}^{n} \text{Fix}(S_j),
\]

where \( \text{Fix}(T_i) \) denotes the fixed point set of \( T_i \); \( H_1 \rightarrow H_1 \); and \( \text{Fix}(S_j) \) denotes the fixed point sets of \( S_j \); \( H_2 \rightarrow H_2 \), respectively.

In particular, if \( m = n = 1 \), then

\[
x \in \text{Fix}(T), \quad \text{s.t.} \ Ax \in \text{Fix}(S),
\]

and \( T: H_1 \rightarrow H_1, \ S: H_2 \rightarrow H_2 \), and \( \text{Fix}(T) \) denotes the fixed point set of \( T \), and \( \text{Fix}(S) \) denotes the fixed point set of \( S \).

The SCFPP becomes a specific case of SFP and closely related to SFP. To solve this problem, the original algorithm for the directed operator was introduced by Censor and Segal [14] in 2009 as follows:

\[
x_{n+1} = T(x_n - \rho A^*(I - S)Ax_n), \quad n \geq 0,
\]

where \( \rho \) satisfies the constraint condition \( 0 < \rho < (2/\|A\|^2) \), and the authors got the weak convergence of the sequence \( \{x_n\} \) for solving the SCFPP (5) if the SCFPP consists, that is, its solution set is nonempty.

Recently, Cui and Wang [15] studied the following algorithm, and they got the weak convergence of the sequence \( \{x_n\} \) for solving the SCFPP (5):

\[
x_{n+1} = U_{\lambda}(x_n - \rho_n A^*(I - T)Ax_n),
\]

where \( U_{\lambda} = (1-\lambda)I + \lambda U \) and \( \rho_n \) is given in the following pattern:

\[
\rho_n = \begin{cases} 
\frac{(1-\tau)\|A^*(I - T)Ax_n\|^2}{2\|A^*(I - T)Ax_n\|^2}, & A_x \neq T(Ax_n), \\
0, & \text{otherwise}
\end{cases}
\]

The step-size of this algorithm \( \rho_n \) does not depend on the norm of the operator \( A \) and searches automatically.

In 2015, Boikanyo [16] extended the main results of Cui and Wang [15] and constructed the Halpern-type algorithm for demicontractive operators that converge to a solution of the SCFPP (5) strongly:

\[
x_{n+1} = \alpha_n u + (1-\alpha_n)U_{\lambda}(x_n - \rho_n A^*(I - T)Ax_n),
\]

where \( \rho_n \) is given as (8). In this result, the resolvent \( I - \rho_n A^*(I - T)A \) plays an important role. Indeed, the techniques of resolvents is quite popular, and it acts as a bridge between fixed point problems and a number of optimization problems (see, e.g., [17–21] and the references therein).

Motivated by the above results, we propose a novel algorithm on demicontractive operators for approximating a solution of the SCFPP (5):

\[
\begin{align*}
\mu_n &= x_n - \rho_n A^*(I - T)Ax_n, \\
x_{n+1} &= (1 - \alpha_n)[(1 - \xi_n)(1 - \eta_n)I + \eta_n U][(1 - \eta_n)I + \eta_n U]u_n + \alpha_n u_n,
\end{align*}
\]

where \( \rho_n \) is also obtained by (8). Our algorithm is also based on the Halpern iteration. Indeed, it is a core for many algorithms in split problems (see, e.g., [22–26]). We get the strong convergence of the iterative sequence \( \{x_n\} \) generated by (10) for solving the SCFPP (5). Our main results are in two folds. First, we construct a novel iterative algorithm to solve the split common fixed point problem for the demicontractive operators. Second, we permit step-size to be selected self-adaptively by the self-adaptive method, which avoids to depend on the norm of the nonlinear operator \( A \).

Our results extend and improve some results of Boikanyo [16], Cui and Wang [15], Yao et al. [27], and many others.

### 2. Preliminaries

In this section, we will present some lemmas, which are useful to prove our main results as follows.

Let \( H \) be a Hilbert space, which is endowed with the inner product \( \langle \cdot, \cdot \rangle \), norm \( \| \cdot \| \). Then, the following inequalities hold:

\[
\|u + v\|^2 \leq 2\|u\|^2 + 2\langle u, v \rangle, \quad \forall u, v \in H,
\]

\[
\|tu + (1-t)v\|^2 = t\|u\|^2 + (1-t)\|v\|^2 - (1-t)t\|u - v\|^2, \quad \forall t \in R \text{ and } \forall u, v \in H.
\]

**Definition 1.** Let \( T: H \rightarrow H \) be an operator, then \( I - T \) called demiclosed at zero, if the following implication holds for any \( \{x_n\} \) in \( H \):

\[
x_n \rightarrow x, \quad (I - T)x_n \rightharpoonup 0 \quad \Rightarrow \quad x = Tx.
\]

Note that the nonexpansive operator is demiclosed at zero [28].

**Lemma 1** (see [29]). Let \( \{\alpha_n\} \) be a sequence of real nonnegative numbers with

\[
\alpha_n \leq (1 - \gamma_n)\alpha_n + \delta_n,
\]

where \( \{\gamma_n\} \) is a sequence in \((0,1)\) and \( \{\delta_n\} \) is a real sequence such that

\[
(i) \sum_{n=1}^{\infty} \gamma_n = \infty
\]

\[
(ii) \limsup_{n \rightarrow \infty} \frac{\delta_n}{\gamma_n} \leq 0 \text{ or } \sum_{n=1}^{\infty} \delta_n < \infty
\]

Then, \( \lim_{n \rightarrow \infty} \alpha_n = 0 \).

**Lemma 2** (see [15]). Let \( A: H_1 \rightarrow H_2 \) be a linear bounded operator and \( T: H_2 \rightarrow H_2 \) a \( \tau \)-demicontactive mapping with \( \tau < 1 \). If \( A^{-1}\text{Fix}(T) \neq \emptyset \), then it is as follows:
Definition 4. (see [31])

Lemma 4

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for all $x, y \in H_1$.

(b) In addition, for $z \in A^{-1}\text{Fix}(T)$,

$$
\|x - z - \rho A^*(I - T)Ax\|^2 + \frac{(1 - \rho)^2\|A^*(I - T)Ax\|^2}{2\|A^*(I - T)Ax\|^2} \leq \|x - z\|^2,
$$

where $x \in H_1$, $Ax \neq T(Ax)$ and

$$
\rho = \frac{(1 - \rho)^2\|A^*(I - T)Ax\|^2}{2\|A^*(I - T)Ax\|^2}.
$$

Lemma 3 (see [30]). Let $H$ be a Hilbert space and let $T$ be an $L$-Lipschitzian mapping defined on $H$ with the module $L \geq 1$.

Set

$$
K = \xi T(\eta T + (1 - \eta)I) + (1 - \xi)I.
$$

If $0 < \xi < \eta < (1/1 + \sqrt{1 + L^2})$, then the following conclusions hold:

(1) $K$ is demiclosed at zero point 0, if $T$ is demiclosed at 0

(2) $\text{Fix}(T) = \text{Fix}(T(\eta T + (1 - \eta)I)) = \text{Fix}(K)$

(3) If $T : H \rightarrow H$ is a quasi-pseudo-contractive operator, then the operator $K$ is quasi-non-expansive

Definition 5. An operator $T : H \rightarrow H$ is said to be firmly nonexpansive if and only if

$$
\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|T(x - (I - T)y)\|^2,
$$

for all $x, y \in H$.

Definition 6. An operator $T : H \rightarrow H$ is said to be firmly quasi-non-expansive if and only if $\text{Fix}(T) \neq \emptyset$ and

$$
\|Tx - z\|^2 \leq \|x - z\|^2 - \|T(x - (I - T)y)\|^2,
$$

for all $x, y \in H, z \in \text{Fix}(T)$.

Definition 7. An operator $T : H \rightarrow H$ is said to be pseudocontractive if and only if

$$
\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2,
$$

for all $x, y \in H$.

Note that $T$ is pseudocontractive if and only if the operator $I - T$ is monotone. There is also an alternative definition for pseudocontractive operators, that is, $T$ is said to be pseudocontractive if and only if

$$
\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|T(x - (I - T)y)\|^2,
$$

for all $x, y \in H$.

Definition 8. An operator $T : H \rightarrow H$ is said to be quasi-pseudo-contractive if and only if $\text{Fix}(T) \neq \emptyset$ and

$$
\|Tx - x^*\|^2 \leq \|x - x^*\|^2 + \|Tx - x\|^2,
$$

for all $x \in H, x^* \in \text{Fix}(T)$.

Definition 9. An operator $T : H \rightarrow H$ is said to be strictly pseudocontractive if and only if there exists $k \in [0, 1)$ such that

$$
\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|T(x - (I - T)y)\|^2,
$$

for all $x, y \in H$.

Definition 10. A operator $T : H \rightarrow H$ is said to be directed if and only if

$$
\langle z - Tx, x - Tx \rangle \leq 0,
$$

for all $x \in H, z \in \text{Fix}(T)$.

Definition 11. An operator $T : H \rightarrow H$ is said to be $\tau$-demicontractive with $\tau < 1$ if and only if

$$
\|Tx - z\|^2 \leq \|x - z\|^2 + \tau\|x - Tx\|^2,
$$

for all $x \in H, z \in \text{Fix}(T)$.

It is easy to obtain that (29) is equivalent to
\[ \|z - Tx\|^2 + \|x - Tx\|^2 - \|x - z\|^2 \leq 0, \]
\[ \forall x \in H, \forall z \in \text{Fix}(T). \]

**Remark 1.** The classes of \( k \)-demicontractive mappings, directed mappings, quasi-non-expansive mappings, and nonexpansive mappings are closely related. By the above definitions, we obtain the following conclusion relations easily (see Figures 1–7).

1. The nonexpansive mapping with \( \text{Fix}(T) \neq \emptyset \) is quasi-non-expansive mapping
2. The quasi-non-expansive mapping is 0-demicontractive mapping
3. The firmly nonexpansive mapping is nonexpansive mapping
4. The firmly quasi-non-expansive mapping is quasi-non-expansive mapping
5. The firmly nonexpansive mapping is firmly quasi-non-expansive mapping
6. The directed mapping is demicontractive mapping
7. The demicontractive mapping is quasi-pseudocontractive mapping
8. The strictly pseudocontractive mapping is quasi-contractive mapping
9. The pseudocontractive mapping is quasi-pseudocontractive mapping

### 4. Main Results

In this section, some assumptions are as follows:

1. \( H_1 \) and \( H_2 \) are two Hilbert spaces, \( A: H_1 \rightarrow H_2 \) is a linear bounded operator, and \( A^* \) is the adjoint of \( A \)
2. \( U: H_1 \rightarrow H_1 \) and \( T: H_2 \rightarrow H_2 \) are two \( L \)-Lipschitzian operators with \( L \geq 1 \), \( \text{Fix}(U) \neq \emptyset \), and \( \text{Fix}(T) \neq \emptyset \)
3. \( U: H_1 \rightarrow H_1 \) is a \( \kappa \)-demicontractive operator \((\kappa < 1)\), and \( T: H_2 \rightarrow H_2 \) is a \( \tau \)-demicontractive operator \((\tau < 1)\)
4. \( I - U \) and \( I - T \) are two demiclosed operators at \( O \)
5. The set of solutions of SCFPP (5), denoted by \( S \), is nonempty

The strong convergence of a sequence \( \{x_n\} \) to a point \( x \in H \) is denoted by \( x_n \rightharpoonup x \).

Now, we give the new algorithm to find \( x^* \in S \), where \( A \) is a bounded and linear mapping, \( A^* \) is the adjoint of operator \( A \), and \( \rho_n \) is obtained as follows:

\[
\rho_n = \begin{cases} 
\frac{(1 - \tau)(I - T)Ax_n^2}{2\|A^*(I - T)Ax_n\|^2}, & A x_n \neq T(A x_n), \\
0, & \text{otherwise. }
\end{cases}
\]

**Algorithm 1.** \( H_1 \) is a real Hilbert space, and \( \text{Fix}(U) \neq \emptyset \). Take an initial point \( x_0 \in H_1 \) arbitrarily, and fix \( u \in H_1 \) and \( \{\theta_n\} \subset (0, 1) \). If the \( n \)-th iteration \( x_n \) is available, then the \((n + 1)\)-th iteration is constructed via the following formula:

\[
\begin{aligned}
\rho_n &= x_n - \rho_n A^*(I - T)Ax_n, \\
x_{n+1} &= \theta_n u + (1 - \theta_n)(1 - \mu_n)I + \mu_n U[(1 - \nu_n)I + \nu_n U]u_n,
\end{aligned}
\]

**Lemma 5.** Assume that \( H_1 \) is a Hilbert space, \( U: H_1 \rightarrow H_1 \) is a \( \kappa \)-demicontractive operator with \( \kappa \leq 1 \), \( L \)-Lipschitzian...
mappings \((L \geq 1)\), and \(\text{Fix}(U) \neq \emptyset\). Denote \(U_{\mu,\nu} \equiv (1 - \mu)I + \mu U[(1 - \nu)I + \nu U]\) with \(0 < \mu < \nu < (2 - \kappa)/1 + \sqrt{1 + L^2(2 - \kappa)}\). Then, for all \(x \in H_1\),
\[
\|z - U_{\mu,\nu}\|^2 \leq \|x - z\|^2 - \nu(2 - 2\nu - \kappa - \nu^2L^2\|Ux - x\|^2),
\]
where \(z \in \text{Fix}(U)\). Moreover,
\[
\|z - U_{\mu,\nu}\| \leq \|z - x\|.
\] (35)
That is, \(U_{\mu,\nu}\) is quasi-non-expansive.

**Proof.** Since \(z \in \text{Fix}(U)\), we get from (30) that
Based on the fact that $U$ is $L$-Lipschitzian, we get
\[ \|U[(1 - \nu)I + \nu U]x - z\|^2 \leq \|U[(1 - \nu)I + \nu U]x - z\|^2 + \kappa \|U[(1 - \nu)I + \nu U]x\|^2. \] (36)

Also, from (30) and (12), we can get
\[ \|Ux - U[(1 - \nu)I + \nu U]x\| \leq \nu L \|x - Ux\|. \] (37)

\[ \|(1 - \nu)(x - z) + \nu(Ux - z)\|^2 \]
\[ = (1 - \nu)\|x - z\|^2 + \nu\|Ux - z\|^2 - \nu(1 - \nu)\|x - Ux\|^2 \]
\[ \leq (1 - \nu)\|x - z\|^2 + \nu\left(\|x - z\|^2 + \kappa\|Ux - x\|^2\right) \\
- \nu(v - 1)\|x - Ux\|^2 \]
\[ = \|x - z\|^2 + \nu(v + \kappa - 1)\|Ux - x\|^2. \] (38)

By (12) and (37), we get
\[ \|[1 - \nu]I + \nu U\|x - U[(1 - \nu)I + \nu U]x\|^2 \]
\[ = \|(1 - \nu)(x - U[(1 - \nu)I + \nu U]x) + \nu(Ux - U[(1 - \nu)I + \nu U]x)\|^2 \]
\[ = (1 - \nu)\|x - U[(1 - \nu)I + \nu U]x\|^2 + \nu\|Ux - U[(1 - \nu)I + \nu U]x\|^2 \\
- \nu(1 - \nu)\|x - Ux\|^2 \]
\[ \leq (1 - \nu)\|x - U[(1 - \nu)I + \nu U]x\|^2 + \nu\nu^2 L^2 \|Ux - x\|^2 \\
- \nu(1 - \nu)\|x - Ux\|^2 \]
\[ = (1 - \nu)\|x - U[(1 - \nu)I + \nu U]x\|^2 - \nu(1 - \nu - \nu^2 L^2)\|x - Ux\|^2. \] (39)
Substituting (38) and (39) into (36), we have
\[
\|U[(1-\nu)I+\nu U]x-z\|^2 \\
\leq \|x-z\|^2 + \nu(\nu + \kappa - 1)\|Ux-x\|^2 \\
+ (1-\nu)\|x-U[(1-\nu)I+\nu U]x\|^2 \\
- \nu(1-\nu - \nu^2 L^2)\|x-Ux\|^2 \\
= \|x-z\|^2 + (1-\nu)\|x-U[(1-\nu)I+\nu U]x\|^2 \\
- \nu(2 - 2\nu - \kappa - \nu^2 L^2)\|x-Ux\|^2.
\]
(40)

Since \( \mu < \nu \), combining (12) and (40), we get
\[
\|(1-\mu)x + \mu U[(1-\nu)I+\nu U]x-z\|^2 \\
= \|(1-\mu)x - z\|^2 + \mu\|U[(1-\nu)I+\nu U]x-z\|^2 \\
= (1-\mu)\|x-z\|^2 + \mu\|U[(1-\nu)I+\nu U]x-z\|^2 \\
- \mu(1-\mu)\|U[(1-\nu)I+\nu U]x-x\|^2 \\
= (1-\mu)\|x-z\|^2 - \mu(1-\mu)\|U[(1-\nu)I+\nu U]x-x\|^2 \\
+ \mu\|U[(1-\nu)I+\nu U]x-x\|^2 \\
- \nu(2 - 2\nu - \kappa - \nu^2 L^2)\|x-Ux\|^2 \\
= \|x-z\|^2 + \mu(\mu - \nu)\|U[(1-\nu)I+\nu U]x\|^2 \\
- \nu(2 - 2\nu - \kappa - \nu^2 L^2)\|x-Ux\|^2 \\
\leq \|x-z\|^2 - \nu(2 - 2\nu - \kappa - \nu^2 L^2)\|x-Ux\|^2.
\]
(41)

Since \( \nu < (2 - \kappa/1 + \sqrt{1+L^2(2-\kappa)} \), we deduce
\[ 2 - 2\nu - \kappa - \nu^2 L^2 > 0. \]
(42)

Hence,
\[
\|(1-\mu)x + \mu U[(1-\nu)I+\nu U]x-z\|^2 \leq \|x-z\|^2.
\]
(43)

That is, \( U_{\mu,\nu} \) is quasi-non-expansive. \( \square \)

**Theorem 1.** Assume that problem (5) is consistent \((S \neq \emptyset)\).
Let \( H_1, H_2, A, U, T, \{x_n\} \) be the same as above. If \( \theta_n \in (0,1) \) satisfies \( \lim_{n \to \infty} \theta_n = 0 \) and \( \sum_{n=0}^{\infty} \theta_n = \infty \), where \( a \) and \( b \) are constants and \( \{\mu_n\} \) and \( \{\nu_n\} \) satisfies \( 0 < a < \mu_n < \nu_n < b < (2 - \kappa/1 + \sqrt{1+L^2(2-\kappa)}) \), \( \forall n \geq 1 \), then the sequence \( \{x_n\} \) converges to a point \( x \in S \) in norm and \( x \) is the nearest point \( S \) to \( u \) \((x = tP_S u)\).

**Proof.** This proof is split into three parts as follows. \( \square \)

**Step 1.** Prove that \( \{x_n\} \) is a bounded sequence.

Take \( p \in S \). From Theorem 1, we know that \( U_{\mu,\nu} \) is quasi-non-expansive. From (32), we have
\[
\|x_{n+1} - p\|^2 = \|\theta_n u + (1 - \theta_n)U_{\mu,\nu}u_n - p\|^2 \\
= \|\theta_n (u-p) + (1 - \theta_n)(U_{\mu,\nu}u_n - p)\|^2 \\
\leq \theta_n\|u-p\|^2 + (1 - \theta_n)\|U_{\mu,\nu}u_n - p\|^2 \\
\leq \theta_n\|u-p\|^2 + (1 - \theta_n)\|u_n - p\|^2 \\
\leq \theta_n\|u-p\|^2 + (1 - \theta_n)\|x_n - p\|^2.
\]
(44)

By induction, we get
\[
\|x_n - p\|^2 \leq \max\{\|u-p\|, \|x_0 - p\|\}.
\]
(45)
Thus, \( \{x_n\} \) is bounded.

**Step 2**
\[
\|x_n - x\|^2 \leq (1 - \theta_n)\|x_n - x\|^2 + 2\theta_n\langle u - x, x_n - x\rangle.
\]
(46)

Consider the case \( \rho_n \neq 0 \). From (32), (35), and (11), we get
\[
\|x_n - x\|^2 = \|\theta_n u + (1 - \theta_n)U_{\mu,\nu}u_n - x\|^2 \\
= \|\theta_n (u-x) + (1 - \theta_n)(U_{\mu,\nu}u_n - x)\|^2 \\
\leq (1 - \theta_n)^2\|U_{\mu,\nu}u_n - x\|^2 + 2\theta_n\langle u-x, x_n - x\rangle \\
\leq (1 - \theta_n)\|U_{\mu,\nu}u_n - x\|^2 + 2\theta_n\langle u-x, x_n - x\rangle \\
\leq (1 - \theta_n)\|u_n - x\|^2 + 2\theta_n\langle u-x, x_n - x\rangle \\
\leq (1 - \theta_n)\|x_n - x\|^2 - (1 - \theta_n)^2\|\langle I-T\rangle A x_n\|^2/4 \\
+ 2\theta_n\langle u-x, x_n - x\rangle \\
\leq (1 - \theta_n)\|x_n - x\|^2 + 2\theta_n\langle u-x, x_n - x\rangle.
\]
(47)

Hence,
\[
\|x_{n+1} - x\|^2 \leq (1 - \theta_n)\|x_n - x\|^2 + 2\theta_n\langle u-x, x_{n+1} - x\rangle.
\]
(48)

Consider the case \( \rho_n = 0 \). From (32) and (11), we get
\[
\|x_n - x\|^2 = \|\theta_n u + (1 - \theta_n)U_{\mu,\nu}u_n - x\|^2 \\
= \|\theta_n (u-x) + (1 - \theta_n)(U_{\mu,\nu}u_n - x)\|^2 \\
\leq (1 - \theta_n)^2\|U_{\mu,\nu}u_n - x\|^2 + 2\theta_n\langle u-x, x_n - x\rangle \\
\leq (1 - \theta_n)\|U_{\mu,\nu}u_n - x\|^2 + 2\theta_n\langle u-x, x_n - x\rangle \\
\leq (1 - \theta_n)\|u_n - x\|^2 + 2\theta_n\langle u-x, x_{n+1} - x\rangle \\
\leq (1 - \theta_n)\|x_n - x\|^2 + 2\theta_n\langle u-x, x_{n+1} - x\rangle.
\]
(49)
Hence,
\[
\|x_{n+1} - \bar{x}\|^2 \leq (1 - \theta_n)\|x_n - \bar{x}\|^2 + 2\theta_n \langle u - \bar{x}, x_{n+1} - \bar{x} \rangle.
\]  
(50)

Step 3. Prove that \(x_n \rightarrow \bar{x}\) as \(n \rightarrow \infty\).

This step is divided into two cases. Denote \(s_n := \|x_n - \bar{x}\|^2\).

Case 1. Assume there exists a positive integer \(n_0\) and the sequence \(\{s_n\}\) is decreasing for any \(n \geq n_0\). Then, \(\{s_n\}\) converges to some point strongly by the monotonic bounded principle.

First, we show that
\[
\limsup_{n \to \infty} \langle u - \bar{x}, x_n - \bar{x} \rangle \leq 0.
\]  
(51)

Using the choice (33) of the step-size \(\rho_n\), (32), (34), (35), and (11), we get
\[
\|x_{n+1} - \bar{x}\|^2 \leq \theta_n \|u + (1 - \theta_n)U_{\mu_n,s_n}u - \bar{x}\|^2 + 2\theta_n \langle u - \bar{x}, x_{n+1} - \bar{x} \rangle
\]
\[
\leq (1 - \theta_n)^2 \|U_{\mu_n,s_n}u - \bar{x}\|^2 + 2\theta_n \langle u - \bar{x}, x_{n+1} - \bar{x} \rangle
\]
\[
\leq \|U_{\mu_n,s_n}u - \bar{x}\|^2 + 2\theta_n \langle u - \bar{x}, x_{n+1} - \bar{x} \rangle
\]
\[
\leq \|u_n - \bar{x}\|^2 - \mu_n \gamma_n(2 - 2\gamma_n - \kappa - \gamma_n^2 L^2)\|Uu_n - u_n\|^2 + 2\theta_n \langle u - \bar{x}, x_{n+1} - \bar{x} \rangle
\]
\[
\leq \|u_n - \bar{x}\|^2 - \|Uu_n - u_n\|^2 - \|Uu_n - u_n\|^2 \frac{(1 - \tau)^2}{4 A^\top(I - T)Ax_n}
\]
\[
- \mu_n \gamma_n(2 - 2\gamma_n - \kappa - \gamma_n^2 L^2)\|Uu_n - u_n\|^2 + 2\theta_n \langle u - \bar{x}, x_{n+1} - \bar{x} \rangle.
\]  
(52)

So,
\[
\mu_n \gamma_n(2 - 2\gamma_n - \kappa - \gamma_n^2 L^2)\|Uu_n - u_n\|^2 \leq s_n - s_{n+1} + \theta_n L,
\]
\[
0 \leq \frac{(1 - \tau)^2}{4 A^\top(I - T)Ax_n} \|Uu_n - u_n\|^2 \leq s_n - s_{n+1} + \theta_n L.
\]  
(53)

where \(L\) is a nonnegative real constant such that \(\sup_{n,N} \{\langle f(x_n) - \bar{x}, x_{n+1} - \bar{x} \rangle\} \leq L\). Based on the fact that \(\{s_n\}\) is convergent, we have

\[
\|u_n - Uu_n\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty,
\]  
(54)

\[
\|A^\top(I - T)Ax_n\|^2 \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.
\]  
(55)

Moreover,
\[
\frac{\|I - T)Ax_n\|^2}{\|A^\top(I - T)Ax_n\|^2} \geq \|I - T)Ax_n\|^2 \geq \|I - T)Ax_n\|^2 \frac{(1 - \tau)^2}{4 A^\top(I - T)Ax_n}
\]
\[
\|Ax_n - TAx_n\| \longrightarrow 0.
\]  
(57)

Since
\[
\|x_n - u_n\| = \rho_n \|A^\top(I - T)Ax_n\|
\]
\[
= (1 - \tau)^2 \|I - T)Ax_n\|^2 \frac{(1 - \tau)^2}{2 A^\top(I - T)Ax_n} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.
\]  
(58)

Since \(x_n \rightarrow q\), we have \(u_n \rightarrow q\) due to (58). From (54) and as \(I - U\) is demiclosed at zero, we have
\[
q \in \text{Fix}(U).
\]  
(59)

From (55) and \(I - T\) is demiclosed at zero, we have
\[
Aq \in \text{Fix}(T).
\]  
(60)

Thus, \(q \in S\) by (59) and (60). Hence, it follows from \(\bar{x} = P_Su\) that
\[
\limsup n \longrightarrow \infty \langle u - \bar{x}, x_n - \bar{x} \rangle
\]
\[
= \langle u - \bar{x}, q - \bar{x} \rangle \leq 0.
\]  
(61)

Secondly, we show that
\[
\|x_{n+1} - x_n\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.
\]  
(62)

From (32), we have
\[
\|U_{\gamma_n \gamma_n}u_n - u_n\| = \|\mu_n u_n - U[(1 - \gamma_n)I + \gamma_n U]u_n\|
\]
\[
= \|\mu_n u_n - Uu_n + Uu_n - U[(1 - \gamma_n)I + \gamma_n U]u_n\|
\]
\[
\leq \|\mu_n u_n - Uu_n\| + \|Uu_n - U[(1 - \gamma_n)I + \gamma_n U]u_n\|
\]
\[
\leq \|\mu_n u_n - Uu_n\| + \|Uu_n - U[(1 - \gamma_n)I + \gamma_n U]u_n\|
\]
\[
= \|\mu_n u_n - Uu_n\| + \|\mu_n \gamma_n L\| u_n - Uu_n\|
\]
\[
= \|\mu_n(1 + \gamma_n L)\| u_n - Uu_n\|
\]  
(63)

From the above equation and (32), (54), and (58), we have
Applying Lemma 2 to (46), which together with the assumption of \(\{\theta_n\}\) and (66), we get \(x_n \to x\) as \(n \to \infty\) easily.

Case 2. Assume that there is no positive integer \(n_0\) and a decreasing sequence \(\{s_k\}\) for any \(n \geq n_0\). That is, there is a subsequence \(\{s_{k_i}\}\) of \(\{s_k\}\) such that \(s_{k_i} \leq s_{k_i+1}\) for any \(i \in N\).

From Lemma 4, we can define a nondecreasing sequence \(\{m_k\} \subset N\) such that \(m_k \to \infty\) as \(k \to \infty\) and
\[
\lim_{n \to \infty} \langle u - x, x_{m_k} - x \rangle \leq 0.
\]

It follows from (52) and (67) and the boundedness of \(\{x_{m_k}\}\) that

\[
\mu_{m_k}^{-1} \left( 2 - 2\rho_{m_k} - \alpha - \rho_{m_k}^2 \right) \left\| U u_{m_k} - u_{m_k} \right\|^2 \leq s_{m_k} - s_{m_k+1} + \alpha_{m_k} L
\]
\[
\leq \alpha_{m_k} L,
\]

Thus,
\[
\left\| U u_{m_k} - u_{m_k} \right\| \to 0, \quad \text{as} \quad n \to \infty,
\]
\[
\left\| (I - T)A x_{m_k} \right\|^2 \to 0, \quad \text{as} \quad n \to \infty.
\]

Moreover,
\[
\frac{1}{\|A\|} \left\| (I - T)A x_{m_k} \right\|^2 \leq \left\| (I - T)A x_{m_k} \right\|^2 \leq \left\| A^+ (I - T)A x_{m_k} \right\|^2
\]

Hence,
\[
\left\| A x_{m_k} - T A x_{m_k} \right\| \to 0,
\]
due to
\[
\left\| x_{m_k} - u_{m_k} \right\| = \rho_{m_k} \left\| A^+ (I - T)A x_{m_k} \right\|
\]
\[
= \left( 1 - \rho \right) \left\| (I - T)A x_{m_k} \right\|^2 \to 0, \quad \text{as} \quad n \to \infty.
\]

Since \(x_{m_k} \to q\), then \(u_{m_k} \to q\). So, we have \(q \in S\) by the similar proofs in Case 1. Hence, it follows from \(x = P_S u\) that
\[
\limsup_{n \to \infty} \langle u - x, x_{m_k} - x \rangle = \langle u - x, q - x \rangle \leq 0.
\]

Secondly, we show
\[
\left\| x_{m_k+1} - x_{m_k} \right\| \to 0, \quad \text{as} \quad k \to \infty.
\]

From (32), we have
\[
\left\| U_{\rho_{m_k} \rho_{m_k}} u_{m_k} \right\|
\]
\[
= \mu_{m_k} \left\| u_{m_k} - U \left[ (1 - \rho_{m_k}) I + \rho_{m_k} U \right] u_{m_k} \right\|
\]
\[
= \mu_{m_k} \left\| u_{m_k} - U u_{m_k} + U u_{m_k} - U \left[ (1 - \rho_{m_k}) I + \rho_{m_k} U \right] u_{m_k} \right\|
\]
\[
\leq \mu_{m_k} \left\| u_{m_k} - U u_{m_k} \right\| + \mu_{m_k} \left\| U u_{m_k} - U \left[ (1 - \rho_{m_k}) I + \rho_{m_k} U \right] u_{m_k} \right\|
\]
\[
\leq \mu_{m_k} \left\| u_{m_k} - U u_{m_k} \right\| + \mu_{m_k} L \left\| u_{m_k} - \left[ (1 - \rho_{m_k}) I + \rho_{m_k} U \right] u_{m_k} \right\|
\]
\[
= \mu_{m_k} \left\| u_{m_k} - U u_{m_k} \right\| + \mu_{m_k} \left\| u_{m_k} - \left[ (1 - \rho_{m_k}) I + \rho_{m_k} U \right] u_{m_k} \right\|
\]
\[
= \mu_{m_k} \left( 1 + \rho_{m_k} L \right) \left\| u_{m_k} - U u_{m_k} \right\|.
\]
\[ x_{m+1} - x_m \|
\leq \alpha_m \|u - x_m\| + (1 - \alpha_m) \|x_m - U_{\mu_m} x_m u_m\| \\
\leq \alpha_m \|u - x_m\| + \|x_m - u_m\| + \|u_m - U_{\mu_m} x_m u_m\| \\
\leq \alpha_m \|u - x_m\| + \|x_m - u_m\| + \mu_m (1 + \rho_m L) \|u_m - U u_m\| \\
\leq \alpha_m \|u - x_m\| + \|x_m - u_m\| + b (1 + \rho L) \|u_m - U u_m\|. \\
\] (77)

Combining (54) and the (58), we get
\[ \|x_{m+1} - x_m\| \to 0, \quad \text{as } n \to \infty. \] (78)

Thirdly, we show that \( x_m \to \bar{x} \) as \( n \to \infty \).

Using (68) and (75), we get
\[ \limsup\limits_{n \to \infty} \langle u - \bar{x}, x_{m+1} - \bar{x} \rangle \leq 0. \] (79)

Based on \( s_m \leq s_{m+1}, \forall k \in N \) and (46), we get
\[ \alpha_m s_{m+1} + (1 - \alpha_m) (s_{m+1} - s_m) \leq 2 \alpha_m \langle u - \bar{x}, x_{m+1} - \bar{x} \rangle. \] (80)

So,
\[ \alpha_m s_{m+1} \leq 2 \alpha_m \langle u - \bar{x}, x_{m+1} - \bar{x} \rangle, \] (81)

that is,
\[ s_{m+1} \leq 2 \langle u - \bar{x}, x_{m+1} - \bar{x} \rangle. \] (82)

Taking the limit \( k \to \infty \), using (79), we obtain
\[ s_{m+1} \to 0, \quad \text{as } k \to \infty. \] (83)

Thus,
\[ s_k \to 0, \quad \text{as } k \to \infty, \] (84)
due to \( s_k \leq s_{m+1} \). The proof is completed.

5. Numerical Example

In the section, we present a numerical experiment to demonstrate the convergence of this algorithm.

Assume \( H_1 = H_2 = (R^3, \| \cdot \|_2) \) and \( T, U : R^3 \to R^3 \) is defined by
\[
\begin{align*}
T \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} x \\ \frac{1}{3} y \\ z \end{pmatrix}, \\
U \begin{pmatrix} a \\ b \\ c \end{pmatrix} &= \begin{pmatrix} 0 \\ a \\ b \end{pmatrix}.
\end{align*}
\] (85)

Let the bounded linear operator \( A \) be defined by
\[
\begin{align*}
A &= \begin{pmatrix} 5 & -5 & -7 \\ -4 & 2 & -4 \\ -7 & -4 & 5 \end{pmatrix}.
\end{align*}
\] (86)

Clearly, both \( U \) and \( T \) are 0–demicontractive mappings. Choose the parameters as follows:
\[
\begin{align*}
\theta_n &= \frac{1}{n}, \\
\mu_n &= \frac{1}{n}, \\
\gamma_n &= \frac{1}{\sqrt{n}}, \quad \forall n \geq 1.
\end{align*}
\] (87)

\( \rho_n \) is chosen in the following way:
\[
\rho_n = \begin{cases} 
\frac{1}{2} \| (I - T) A x_n \| \| (I - T) A x_n \|, & A x_n \neq T (A x_n), \\
0, & \text{otherwise},
\end{cases}
\] (88)

where \( A \) is a bounded and linear mapping and \( A^* \) is its adjoint. Then, the iterative algorithm (10) becomes as follows:
\[
\begin{align*}
u_n &= x_n - \rho_n A^* (I - T) A x_n, \\
x_{n+1} &= \frac{1}{n} u + \frac{1}{n} \left( (1 - \frac{1}{n}) I + \frac{1}{n} U \right) \left( (1 - \frac{1}{\sqrt{n}}) I + \frac{1}{\sqrt{n}} U \right) u_n,
\end{align*}
\] (89)

where \( u = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \) is a fixed point in \( R^3 \), and the initial point 
\[
\begin{align*}
x_1 &= \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} \quad \text{and} \quad x_n &= \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix}
\end{align*}
\] is generated by the algorithm (10). We plot the numbers of iterations and
The value of $a(n), b(n), c(n)$

\[ \|x_{n+1} - x_n\|_2 \] in the following graphs (Figures 8 and 9), the numbers of iterations and \( \{x_n\} = \{a_n, b_n, c_n\} \).

6. Conclusion

In this paper, we proposed a new iteration algorithm (10) and we obtained the strong convergence of the sequence \( \{x_n\} \) for split common fixed point problems (5). The main result is an extension of the related results announced in [15, 16, 27]. The research highlights of this paper are novel algorithms and their analysis techniques. The improvement on the extension of the operator, such as the demicontractive mappings, the directed operators, the quasi-non-expansive operators, and quasi-pseudo-contractive operators will be of interest for further research in the future.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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