A gravito-electromagnetic analogy based on tidal tensors

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Abstract

We propose a new approach to a physical analogy between General Relativity and Electromagnetism, based on tidal tensors of both theories. Using this approach we write a covariant form for the gravitational analogues of the Maxwell equations. The following realisations of the analogy are given. The first one matches linearised gravitational tidal tensors to exact electromagnetic tidal tensors in Minkowski spacetime. The second one matches exact magnetic gravitational tidal tensors for ultra-stationary metrics to exact magnetic tidal tensors of electromagnetism in curved spaces. In the third we show that our approach leads to two-step exact derivation of the Papapetrou force on a gyroscope. We then establish a new proof for a class of tensor identities that define invariants of the type $\vec{E}^2 - \vec{B}^2$ and $\vec{E} \cdot \vec{B}$, and we exhibit the invariants built from tidal tensors in both gravity and electromagnetism. We contrast our approach with the two gravito-electromagnetic analogies commonly found in the literature, and argue that it sheds light on the debate about the limit of validity of one of the analogies, and clarifies issues concerning the physical interpretation of the other.

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1 Introduction

Two rather different analogies between classical Electromagnetism and General Relativity have been presented in the literature, both of which have been dubbed \textit{gravito-electromagnetism} (see for instance \cite{5} and \cite{22}).

\begin{itemize}
  \item The first one draws an analogy between some components of the spacetime metric and the electromagnetic potentials, using linearised theory;
\end{itemize}
The second one makes a parallelism between a decomposition of the Weyl and the Maxwell tensors in electric and magnetic parts;

Besides these two analogies there is a third interesting connection between relativistic gravity and electromagnetism: the Klein Gordon equation in ultra-stationary metrics can be mapped to a non-relativistic Schrödinger problem in a time independent magnetic field in a curved space [38, 39].

The first analogy relies on the obvious similarity between the linearised Einstein equations, in the harmonic gauge, and Maxwell’s equations in the Lorentz gauge. It is, therefore, presented for perturbations around Minkowski spacetime and it is clearly not covariant.

The second analogy, on the other hand, relies on two facts. Firstly that we can do an irreducible splitting into electric and magnetic parts for both the Maxwell tensor and the Weyl tensor. Secondly, that we can find a (formally) similar set of equations for both the electromagnetic parts of the Maxwell tensor and the electromagnetic parts of the Weyl tensor. Moreover, in analogy with their electromagnetic counterparts, the electric and magnetic parts of the Weyl tensor define “invariant quantities” (in a sense to be precised below) of the type $E^2 - B^2$ and $E \cdot B$. Thus, this analogy is covariant and it is exact (i.e it does not rely on linearised theory).

1.1 The need for a new, physically transparent approach

Within each of the approaches presented in literature there are issues needing a clarification. In the case of the analogy based on linearised theory, its limit of validity has been under debate, and there is no consensus about it. While some authors limit the analogy to stationary configurations [1]-[4], others argue it can be extended to time dependent setups [5]-[12]. On this last version of the analogy, a set of Maxwell like equations are derived, which predict the existence of gravitational induction effects similar to the electromagnetic ones; experiments to detect those induced fields have been proposed [11] and, recently, such an experiment has actually been performed [13]. However, due to the symmetries of gravitational tidal tensors, as we shall see, such phenomena cannot take place in gravity.

In the case of the second analogy, it is its physical content that is not clear in literature. While the electric part of the Weyl tensor (hereafter denoted by $E_{\mu\nu}$) is regarded as a generalisation of the Newtonian tidal tensor [23]-[34], its magnetic counterpart (hereafter denoted by $H_{\mu\nu}$) is not well understood [23, 32, 33], but claimed to be associated with rotation [20, 21, 22, 24, 27] and gravitational radiation [26]-[34]. However, immediately contradictions arise [20, 21, 22]: there are many known examples of rotating spacetimes where the magnetic part of the Weyl tensor vanishes; amongst them is the notorious example of the Gödel Universe. It is also clear that gravitational waves cannot be the sole source for $H_{\mu\nu}$, since the latter is generically non-vanishing in most stationary spacetimes. Another topic under debate is the Newtonian limit of $H_{\mu\nu}$; while some authors assert it must vanish in that limit (e.g. [31, 36]), it has been argued it does not [34, 35].

The physical connection between these two analogies is another matter needing for a clarification. They are not only distinct in their approach; they seem to refer to different phenomena and (since the physical content of the second approach is still an open question in literature) one might even be led to apparently contradictory conclusions. That is what happens, for instance, in the cases of the the Lense and Thirring or the Heisenberg spacetimes. According to the first approach,
their “gravito-electric field” is zero, while their “gravito-magnetic field” is finite and uniform. But in the second approach, these spacetimes are classified as “purely electric spacetimes” since the Weyl tensor is electric.

1.2 A new approach to gravito-electromagnetism based on tidal tensors

The purpose of this paper is to propose a new approach to gravito-electromagnetism. We claim that a physical analogy steams from the tidal tensors of the two theories. This analogy stands on universal, covariant equations: the geodesic deviation equation and its analogue electromagnetic worldline deviation equation; the Papapetrou force applied on a gyroscope and the electromagnetic force exerted on a magnetic dipole. The identification of the correct tidal tensors allows us to write an explicitly covariant form for the Maxwell equations, and derive their gravitational analogues. From these tidal tensors we also define, whenever they exist, analogous electromagnetic and gravitational invariants. Explicit evidence for this analogy is additionally given by a comparison of the electromagnetic and gravitational tidal tensors, as well as the invariants from them constructed, in an elementary example of analogous physical systems (a spinning mass in gravity, and a spinning charge in electromagnetism); in generic electromagnetic fields and linearised gravitational perturbations; and by a suggestive application of the Klein-Gordon equation in ultra-stationary spacetimes.

The approach proposed herein clarifies several issues concerning the gravito-electromagnetic approaches found in the literature:

★ While embodying all the correct predictions from the usual linear perturbation approach, our approach reveals, in an unambiguous way, its regime of validity (for which, as mentioned before, there is no consensus in the literature);

★ It sheds light into the debate about the physical content of the second approach. In analogy with the electromagnetic tidal tensors, we give a simple physical interpretation for the electric and magnetic parts of the Weyl tensor, which trivially solves the contradictions discussed in the literature;

★ Finally, the third connection mentioned above becomes but another realisation of this analogy, even somewhat more surprising because there is an exact matching between (magnetic) tidal tensors of a non-linear theory (gravity) and the ones of a linear theory (electromagnetism).

This paper is organised as follows. Our proposal, of a physical analogy between gravity and electromagnetism based on tidal tensors is presented in section 2, where a covariant form for the gravitational analogues of the Maxwell equations is derived and its physical implications discussed. Three realisations follow. In the first we study the analogy for small perturbations around Minkowski spacetime, and compare gravitational and electromagnetic tidal tensors in an example of physically analogous setups. The second gives an interpretation for the physical similarities of the

1The magnetic and electric parts of the Weyl tensor are observer dependent. Both in the Lense and Thirring spacetime and the Heisenberg group manifold there are observers for which the magnetic part of the Weyl tensor vanishes, while the electric part never vanishes. For this reason, these spacetimes are classified as “purely electric” (see for instance [17]). The electric character of the spacetime is equivalently revealed by the invariants, which using the notation of section 3 are $\mathcal{L} > 0$, $\mathcal{M} = 0$. 
Klein-Gordon in ultra-stationary spacetimes with the Schrödinger equation in some curved spaces. Finally, in the third we show that within the framework of our approach, the exact Papapetrou equation for the force applied on gyroscope is obtained in a “two step” derivation, avoiding the lengthy original computation, and gaining physical insight. In sections 3 and 4 the aforementioned two approaches to gravito-electromagnetism are reviewed and contrasted with our proposal. The second analogy is used to introduce gravitational invariants of the type of $\vec{E}^2 - \vec{B}^2$ and $\vec{E} \cdot \vec{B}$, which are ‘deconstructed’ in section 5 where the reason for such scalar invariants to be observer independent is dissected and new invariants, the ones that play a similar role in the gravito-electromagnetic analogy based on tidal tensors are constructed. We close with a discussion.

2 The physical gravito-electromagnetic analogy, based on tidal tensors

An analogy with physical content between electromagnetism and general relativity must, if it exists, be based on physical quantities common to both theories. Taking the perspective that, in general relativity, the only ‘physical forces’ (in the sense of being covariant) are the tidal forces, described by the curvature tensor, the starting point of the analogy should be the tidal tensors. In this section we build an analogy between the gravitational and electromagnetic tidal tensors. For this purpose we start by defining these tensors.

2.1 Gravitational and Electromagnetic Tidal Tensors

Tidal forces in gravity are described in a covariant way by the geodesic deviation equation:

$$\frac{D^2 \delta x^\alpha}{D\tau^2} = -R^\alpha_{\mu\beta\sigma} U^\mu U^\sigma \delta x^\beta$$

where $D/D\tau$ denotes covariant differentiation along a curve parameterised by $\tau$. This equation gives the relative acceleration of two neighbouring particles with the same 4-velocity $U^\alpha$ (see Appendix A). In order to find the electromagnetic analogue to (1), one must first notice a very intrinsic difference between the two interactions: while the ratio between gravitational and inertial mass is universal, the same does not apply to the ratio between electrical charge and inertial mass; i.e., there is no electromagnetic counterpart of the equivalence principle. Therefore, the analogue electromagnetic problem will be to consider two neighbouring particles with the same 4-velocity $U^\alpha$ in an electromagnetic field on Minkowski spacetime, with the extra condition that the two particles have the same $q/m$ ratio. Under these conditions one obtains the worldline deviation equation (see Appendix A):

$$\frac{D^2 \delta x^\alpha}{D\tau^2} = \frac{q}{m} F^\alpha_{\mu\beta} U^\mu \delta x^\beta$$

where $F_{\alpha\beta}$ is the Maxwell tensor. The comparison of (1) and (2) suggests a physical analogy between the two rank-2 tensors:

$$E_{\alpha\beta} \equiv R_{\alpha\mu\beta\sigma} U^\mu U^\sigma \longleftrightarrow E_{\alpha\beta} \equiv F_{\alpha\mu\beta} U^\mu$$

The tensor $E_{\alpha\beta}$ is the covariant derivative of the electric field $E^\alpha = F^{\alpha\beta} U_\beta$ seen by the observer of (fixed) 4-velocity field $U^\alpha$; for this reason we will refer to it as the electric tidal tensor and its
gravitational counterpart $E_{\alpha\beta}$, which is known in literature as the electric part of the Riemann tensor (cf. section 4), as the electric gravitational tidal tensor. The different signs in (1) and (2) reflect the different character (attractive or repulsive) of the interaction between masses or charges of the same sign. Given our definition of the electric tidal tensor, it is straightforward to define the magnetic tidal tensor

$$B_{\alpha\beta} \equiv \star F_{\alpha\mu\beta\gamma} U^\mu = \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} F_{\gamma\delta} U^\mu$$

where $\star$ denotes the Hodge dual and $\epsilon_{\alpha\beta\gamma\delta}$ is the Levi-Civita tensor. This tensor measures the tidal effects produced by the magnetic field $B^\alpha = \star F_{\alpha\beta} U^\beta$ seen by the observer of 4-velocity $U^\gamma$. An analogous procedure applied to the Riemann tensor yields the so called magnetic part of the Riemann tensor (cf. section 4)

$$H_{\alpha\beta} \equiv \star R_{\alpha\mu\beta\sigma} U^\mu U^\sigma = \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} R_{\gamma\delta} U^\mu U^\sigma$$

which we claim, and give evidence throughout this paper, to be the physical gravitational analogue of $B_{\alpha\beta}$:

$$B_{\alpha\beta} \leftrightarrow H_{\alpha\beta}$$

For this reason $H_{\alpha\beta}$ will be herein referred to as the magnetic gravitational tidal tensor. In (5) the Hodge dual was taken with respect to the first pair of indices of the Riemann tensor; a different choice amounts to changing the order of the indices in $H_{\alpha\beta}$. Note that this tensor is generically not symmetric, only in vacuum.

### 2.1.1 Maxwell equations as tidal tensor equations

Maxwell equations are tidal equations; indeed, an explicitly covariant form for them can be obtained in a simple fashion from the above defined electromagnetic tidal tensors.\(^2\)

\[\begin{align*}
\text{i) } E^\alpha_{\alpha} &= 4\pi \rho_c \\
\text{ii) } E_{[\alpha\beta]} &= \frac{1}{2} F_{\alpha\beta\gamma\delta} U^\gamma \\
\text{iii) } B^\alpha_{\alpha} &= 0 \\
\text{iv) } B_{[\alpha\beta]} &= \frac{1}{2} \star F_{\alpha\beta\gamma\delta} U^\gamma - 2\pi \epsilon_{\alpha\beta\gamma\delta} j^\delta U^\gamma
\end{align*}\]

where $j^\alpha$ and $\rho_c = -j^\alpha U_\alpha$ denote, respectively, the (charge) current 4-vector and the charge density as measured by the observer of 4-velocity $U^\alpha$. These equations can be completely expressed in terms of tidal tensors and sources, by noting that:

\[\begin{align*}
F_{\alpha\beta\gamma} U^\gamma &= 2 U_{[\alpha E_{\beta\gamma}]} U^\gamma + \epsilon_{\alpha\beta\mu\sigma} U^\sigma B^\mu U^\gamma \\
\star F_{\alpha\beta\gamma} U^\gamma &= 2 U_{[\alpha B_{\beta\gamma}]} U^\gamma - \epsilon_{\alpha\beta\mu\sigma} U^\sigma E^\mu U^\gamma
\end{align*}\]

which follows from decomposition (58) given in section 4 by a straightforward computation. Equations (i) and (ii) are the covariant forms of $\nabla \cdot \vec{E} = \rho_c$ and $\nabla \cdot \vec{B} = 0$; equations (iii) and (iv) are covariant forms for $\nabla \times \vec{E} = -\partial \vec{B} / \partial t$ and $\nabla \times \vec{B} = \partial \vec{E} / \partial t + 4\pi \vec{j}$, respectively.\(^2\)

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\(^2\)Here and throughout the paper (except in sec. 3) we use $c = 1 = G$. 

2.1.2 The Gravitational Analogue of Maxwell equations

In what follows it proves useful to introduce the decomposition of the Riemann tensor (eg. [18]):

\[ R_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} + g_{\alpha[\gamma} R_{\delta]\beta} + g_{\beta[\delta} R_{\gamma]\alpha} + \frac{1}{3} g_{\alpha[\delta} g_{\gamma]\beta} R \]  

(8)

where \( C_{\alpha\beta\gamma\delta} \) is the Weyl tensor, which (by definition) is traceless and exhibits the property:

\[ \ast C_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta}. \]

\( E_{\alpha\beta} \) and \( H_{\alpha\beta} \) then become:

\[ E_{\alpha\beta} = E_{\alpha\beta} + \frac{1}{2} \left[ g_{\alpha\beta} R_{\gamma\delta} U^\gamma U^\delta - R_{\alpha\beta} - 2 U_{(\alpha} R_{\beta)\delta} U^\delta \right] + \frac{R}{6} \left[ g_{\alpha\beta} + U_\alpha U_\beta \right] \]  

(9)

\[ H_{\alpha\beta} = H_{\alpha\beta} + \frac{1}{2} \epsilon_{\alpha\beta\sigma\gamma} R^\sigma_{\delta} U^\gamma U^\delta \]  

(10)

where

\[ E_{\alpha\beta} \equiv C_{\alpha\gamma\beta\sigma} U^\gamma U^\sigma, \quad H_{\alpha\beta} \equiv \ast C_{\alpha\gamma\beta\sigma} U^\gamma U^\sigma, \]  

(11)

are, respectively, the electric and magnetic parts of the Weyl tensor, both of which are symmetric and traceless.

By repeating the same procedure that led to equations \( (6) \), i.e, by taking the traces and anti-symmetric parts of the tidal tensors, we obtain the analogue set of equations:

i) \( E^\alpha_\alpha = 4\pi \left( 2\rho_m + T_\alpha^\alpha \right) \)  

ii) \( E_{[\alpha\beta]} = 0 \)  

iii) \( H^\alpha_\alpha = 0 \)  

iv) \( H_{[\alpha\beta]} = -4\pi \epsilon_{\alpha\beta\sigma\gamma} J^\sigma U^\gamma \)  

(12)

where \( T_{\alpha\beta} \) denotes the energy momentum tensor, and \( J^\alpha = -T^\alpha_\beta U^\beta \) and \( \rho_m = T_{\alpha\beta} U^\alpha U^\beta \) are, respectively, the mass/energy current density and the mass/energy density as measured by the observer of four velocity \( U^\alpha \).

2.1.3 Gravity versus Electromagnetism

Equations \( (6) \) are strikingly similar to equations \( (12) \) when the setups are stationary in the observer’s rest frame. Otherwise, comparing the two sets of equations tells us that gravitational and electromagnetic interactions must differ significantly, since the tidal tensors do not have the same symmetries. We shall now discuss each equation in detail.

Comparing \( (6) \) and \( (12) \), we see that the gravitational analogue of the electric charge density \( \rho_c \) is \( 2\rho_m + T_\alpha^\alpha \). Thus, changing the sign of the latter combination amounts to changing the character of the gravitational source from attractive (positive) to repulsive (negative); requiring it to be positive is the statement of the strong energy condition. The analogy gets more enlightening if we use the energy momentum tensor of a perfect fluid (energy density \( \rho_m \), pressure \( p \)); in that case we have the correspondence \( \rho_c \leftrightarrow \rho_m + 3p \), manifesting the contribution of pressure as a source of the gravitational field. The combination \( \rho_m + 3p \) is well known from the Raychaudhuri equation of FRW models, determining if the expansion of the universe is decelerated or accelerated.

A perfect analogy exists in the case of right eqs. of \( (6) \) and \( (12) \): the trace of \( B_{\alpha\beta} \) is zero by virtue of the electromagnetic Bianchi identity; likewise, the trace of \( H_{\alpha\beta} \) vanishes by virtue of the first Bianchi identities.
Equations (6vi) and (12vi) reveal a fundamental difference between $E_{\mu\nu}$ and $E_{\nu\gamma}$: while the former is always symmetric, the latter is only symmetric if the Maxwell tensor is covariantly constant along the observer’s worldline. The physical content of these equations depends crucially on these symmetries: since (6vi) is a covariant form of $\nabla \times \vec{E} = -\partial \vec{B} / \partial t$, the statement encoded in the equation $E_{[\alpha\beta]} = 0$ is that there is no gravitational analogue to Faraday’s law of induction.

There is a clear gravitational counterpart to the Ampère law: in stationary (in the observer rest frame) configurations, equations (6v) and (12v) match up to a factor of 2; therefore, currents of mass/energy source gravitomagnetism just like currents of charge source magnetism. The extra factor of 2 in (12iv) reflects the different spin of the gravitational and electromagnetic interactions; in some literature the gravito-magnetic charge $Q_B = 2m$ (e.g. [17, 5]) has been defined to account for this factor. In generic dynamics, again, the physical content of these equations will be drastically different, due to the presence of the induction term $\star F_{\alpha\beta\gamma\delta} U^\delta$ in (6v), which has no counterpart in (12v). This means that there is no gravitational analogue to the magnetic fields induced, for instance, by the time varying electric field between the plates of a charging/discharging capacitor.

The different symmetries of the gravitational and electromagnetic tidal tensors are related to a fundamental difference in their tensorial structure: while the former are *not* symmetries of the Riemann tensor. In electromagnetism, we have a different situation. On one hand, $E_{\alpha\beta} U^\alpha = 0$, which is trivial consequence of the symmetries of $F_{\alpha\beta}$; but, on the other hand, $E_{\alpha\beta} U_\beta^\alpha$ is non-zero when the field varies with the observer proper time. The same applies to $B_{\alpha\beta}$. These contractions determine the temporal projections of the electromagnetic tidal tensors, and they are indeed at the origin of the extra terms in (6v) and (6vi) as compared to (12vi) and (12v), as can be seen from expressions (17).

### 2.2 Small perturbations around Minkowski spacetime

Let us start by considering a generic electromagnetic field, described by the potential one-form

$$A = -\phi(t, x^i) dt + A_j(t, x^i) dx^j ,$$

in Minkowski space, with metric

$$ds^2 = -dt^2 + \hat{g}_{ij}(x^k) dx^i dx^j ,$$

where $\hat{g}_{ij}$ is an arbitrary spatial metric on $\mathbb{R}^3$. It follows that

$$F = dA = (\dot{A}_i + \dot{\phi}) dt \wedge dx^i + A_j dx^i \wedge dx^j ,$$

$$\star F = A^i \epsilon_{ijk} dx^k \wedge dt + \frac{1}{2} \dot{\epsilon}_{ijk}(\phi^i + \dot{\phi}^i) dx^j \wedge dx^k ,$$

where dots represent time derivatives, the semi-colon represents covariant derivatives with respect to $\hat{g}_{ij}$ (covariant derivatives in $\hat{g}_{ij}$ commute since the metric is flat) and $\dot{\epsilon}_{ijk}$ are the components of the Levi-Civita tensor on $\mathbb{R}^3$ in coordinates $\{x^i\}$. We take the orientation defined by $\epsilon_{0123} = -1$. The electric tidal tensor $E_{\alpha\beta}$ (13) is, for an observer with four velocity $U^\alpha = (u^0, u^i)$, given by

$$E_{00} = (\dot{\phi}^i + \dot{A}_i) u^i , \quad E_{ij} = - (\phi_{;ij} + \dot{A}_{ij}) u^0 + 2 \dot{A}_{[i;j]} u^k ,$$

$$E_{i0} = - (\dot{\phi}^i + \dot{A}_i) u^0 + 2 \dot{A}_{[i;j]} u^j , \quad E_{0i} = (\phi_{;ki} + \dot{A}_{ki}) u^k .$$
Similarly, the magnetic tidal tensor \( B_{ij} \) is given by

\[
B_{00} = -\epsilon_{ijk} \dot{A}^j i u^k, \quad B_{ij} = \epsilon_{lmi} A^{m, j} i u^0 + \epsilon_{ikk} \left( \phi^{j, i}_j + \dot{A}^j i_j \right) u^k, \\
B_{i0} = \epsilon_{kji} \dot{A}^j j u^0 + \epsilon_{jik} \left( \phi^{i, j}_j + \dot{A}^j j \right) u^k, \quad B_{0i} = -\epsilon_{ijk} A^{k, i}_j u^k.
\]

(16)

 Obviously, these quantities would look simpler and more familiar if written in terms of the electric and magnetic fields. But writing them in this fashion makes the comparison with the gravitational case explicit.

Now consider linearised perturbations of Minkowski spacetime \([44]\). We take the perturbed metric in the form\(4\)

\[
ds^2 = -c^2(1 - 2\phi(t, x^i))dt^2 - 4cA_j(t, x^i)dt dx^j + \left[ 1 + 2\frac{C(t, x^k)}{c^2} \right] \hat{g}_{ij}(x^k) + 2\frac{\xi_{ij}(t, x^k)}{c^2} dx^i dx^j,
\]

(17)

where, as before, \( \hat{g}_{ij} \) is an arbitrary spatial metric on \( \mathbb{R}^3 \) and \( \xi_{ij} \) is traceless. A simple computation reveals that (again, the semi-colon represents covariant derivatives with respect to \( \hat{g}_{ij} \)):

\[
R^0_{i0j} = \ddot{C} \hat{g}_{ij} + \phi_{ij} + 2\dot{A}_{(i)j} + \ddot{\xi}_{ij}, \quad R^0_{ij0} = -2\ddot{g}_{[i]j} \ddot{C}^{[j]} + 2A_{[ij]i} + 2\ddot{\xi}_{[ij]},
\]

\[
R^i_{jkl} = -2C_{[ij]l}^{[j]} + \ddot{g}_{[k]} C_{[il]}^{[l]} + 2\xi_{[ijl]} - 2\xi_{[jikl]}.
\]

The perturbations actually separate into scalar, vector and tensor parts \([44]\), but such separation will not be needed here. The dual Riemann tensor components are:

\[
\star R_{000} = -\epsilon_{lik} \dot{C}^k i + \epsilon_{lik} \left( A^{k, j}_j + \ddot{\xi}^{k, j}_j \right), \quad \star R_{i0j0} = \epsilon_{ij} \left( \dddot{C} \hat{g}_{ij} + \phi_{ij} + 2\dot{A}_{(k)j} + \ddot{\xi}_{ik} \right),
\]

\[
\star R_{0ij0} = \epsilon_{ij} \left( \dddot{g}_{ij} [C_{ij}] - C_{im}[j]^{[m]} + \xi_{im}^{[m]} - \xi_{[m]ij} \right), \quad \star R_{ijkl} = 2\epsilon_{ij} \left( \dddot{g}_{[kl]} \dddot{C}_{[ij]} - A_{[kl]m} + \ddot{\xi}_{[kl]} \right).
\]

The electric gravitational tidal tensor \( E_{\alpha \beta} \) is, for an observer with four velocity \( U^\alpha = (u^0, u^i) \), given by

\[
E_{00} = -\left( \dddot{C} \hat{g}_{ij} + \phi_{ij} + 2\dot{A}_{(i)j} + \ddot{\xi}_{ij} \right) u^i u^j,
\]

\[
E_{i0} = E_{0i} = \left( \dddot{C} \hat{g}_{ij} + \phi_{ij} + 2\dot{A}_{(i)j} + \ddot{\xi}_{ij} \right) u^0 u^j + 2 \left( \dddot{g}_{ij} \dddot{C}_{ij} - A_{[ij]} - \ddot{\xi}_{ij} \right) u^k u^j,
\]

\[
E_{ij} = -\left( \dddot{C} \hat{g}_{ij} + \phi_{ij} + 2\dot{A}_{(i)j} + \ddot{\xi}_{ij} \right) (u^0)^2 + 2 \left( \dddot{g}_{ij} \dddot{C}_{ij} - \dddot{g}_{ij} \dddot{C}_{ij} - A_{[ij]k} + \ddot{\xi}_{ij} \right) u^0 u^k + \left( \ddot{g}_{ij} \dddot{C}_{ij} + \dddot{g}_{ij} \dddot{C}_{ij} - 2\xi_{ij} \right) u^k u^j.
\]

(19)

The magnetic gravitational tidal tensor \( H_{\alpha \beta} \) is,

\[
H_{00} = \epsilon_{imm} \left( A_{n,m}^{n} j + \ddot{\xi}_{j m}^{n} \right) u^i u^j, \quad H_{ij} = \left( -\epsilon_{ikk} \dot{C}^0 \dddot{C}_{ij} + \ddot{\xi}_{ikl} \left( A_{k,l}^{k, l} + \ddot{\xi}_{kl} \right) \right) (u^0)^2
\]

\[
H_{i0} = \left( \dddot{g}_{ijk} \dot{C}_{ik} - \dddot{g}_{ijk} \dddot{C}_{ik} \right) u^0 u^j - \dddot{g}_{ijk} \dddot{C}_{ik} + \dddot{g}_{ijk} + 2\dot{A}_{(l)j} + \ddot{\xi}_{ij} \right) u^k u^j.
\]

(20)

(21)

\footnote{Here we have re-inserted the velocity of light ‘c’, despite the fact that in this subsection we are putting \( c = 1 \). The \( c \) dependence will, however, be of use in section 3.}

9
\[
H_{0i} = \left( \dot{\epsilon}_{ijk} \dot{C}^{jk} - \epsilon_{ijk} \left( A^{k;l}_{;i} - \dot{\epsilon}_{i}^{k;l} \right) \right) u^{0} u^{j} - \dot{\epsilon}_{jl}^{m} \left( \delta^{l}_{[k} C_{i];m} + \xi^{l}_{[k;i]m} + \xi_{m[j;k]} \right) u^{j} u^{k}.
\] (22)

As can be now easily verified, the electromagnetic tidal tensors are, generically, very different from their gravitational counterparts. But if one takes time independent electromagnetic potentials/gravitational perturbations, and a "static observer" \( U^{\mu} = \delta^{\mu}_{0} \), then

\[
E_{ij} = -\phi_{;ij} = E_{ij}, \quad B_{ij} = \epsilon_{ikl} A^{k;l}_{;j} = H_{ij}.
\] (23)

One may regard this matching between the tidal tensors of the two theories as an analogy between the electromagnetic potential \( A^{\mu} \) and some components of the metric tensor, and hence define the "gravito-electromagnetic fields":

\[
E_{G}^{i} = -\phi^{i}, \quad B_{G}^{i} = \epsilon^{ikl} A_{k;l}.
\] (24)

These fields help us visualise geodesic motion and frame dragging in analogy with the more familiar picture of a charged particle subject to a electromagnetic Lorentz force. Indeed, the geodesic equation \( DU^{\alpha}/D\tau = 0 \) yields, to linear order in the velocity of the test particle, the space components:

\[
\frac{d^{2}\vec{x}}{dt^{2}} = -\vec{E}_{G} - 2\vec{v} \times \vec{B}_{G}.
\] (25)

The "gravito-magnetic" field in (24) also leads directly to the Lense and Thirring precession for test gyroscopes \([52, 2, 3]\): as for a magnetic dipole placed in a magnetic field, a "torque" \( \tau = -\vec{S} \times \vec{B}_{G} \) acts on a gyroscope of angular momentum \( \vec{S} \) (since in gravity \( \vec{S} \) plays the role of the magnetic moment; see discussion in section 2.2.1 below), causing it to precess with angular frequency \( \vec{\omega} = \vec{B}_{G} \), which is accurate to linear order.

We must however stress the fact that this construction is restricted to a mapping of static electromagnetic fields to stationary gravitational setups, and is clearly non-covariant: as readily seen by comparing equations (15)-(16) with (18)-(22), tidal tensors will not match for any observer other than the above considered static observer \( U^{\mu} = \delta^{\mu}_{0} \).

### 2.2.1 Spinning charge versus spinning mass

Let us choose an elementary example of analogue physical systems, namely a rotating spherical mass in General Relativity and a rotating spherical charge in Electromagnetism.

Consider a sphere of charge \( q \), mass \( m \), rotating with constant angular momentum \( J e_{z} \) in Minkowski spacetime. Taking the leading electric monopole and magnetic dipole contributions, the potential one form becomes:

\[
A = -\frac{q}{r} dt + \frac{\mu}{r} \sin^{2} \theta d\phi,
\]
where \( \mu \equiv Jq/(2m) \) is the magnetic dipole moment of the rotating sphere. Then the Maxwell tensor and its dual are:

\[
F = -\frac{q}{r^{2}} dt \wedge dr - \frac{\mu}{r^{2}} \sin^{2} \theta dr \wedge d\phi + \frac{\mu}{r} \sin 2\theta d\theta \wedge d\phi,
\]

\[^{4}\text{We define the "static observer" as being an observer for which the setup is stationary. An observer at rest relative to the centre of mass of the spinning spheres considered in 2.2.1 is an example of such an observer.}\]
This metric is asymptotically flat. Thus it allows us to compare its asymptotic gravitational tidal

tensors with the previous electromagnetic tidal tensors. For a static observer, with four velocity
$U^\mu = \delta^\mu_0$, the electric and magnetic tidal tensors are, asymptotically

$$
E_{\alpha\beta} dx^\alpha dx^\beta = \frac{2q}{r^3} dr^2 + \frac{q}{r} d\Omega_2 ,
$$

$$
B_{\alpha\beta} dx^\alpha dx^\beta = \frac{3\mu}{r^2} \left( - \frac{2\cos \theta}{r^2} dr^2 + \cos \theta d\Omega_2 - \frac{2\sin \theta}{r} d\theta d\phi \right) .
$$

(26)

Now consider the metric outside a rotating spherical mass, which is asymptotically described

by the Kerr solution \[50\]. In Boyer-Linquist coordinates the line element takes the form

$$
ds^2 = \frac{\Delta}{\Sigma} \left( dt - a \sin^2 \theta d\phi \right)^2 + \frac{\sin^2 \theta}{\Sigma} \left( a dt - (r^2 + a^2) d\phi \right)^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 ,
$$

where

$$
\Delta \equiv r^2 - 2mr + a^2 , \quad \Sigma = r^2 + a^2 \cos^2 \theta
$$

This metric is asymptotically flat. Thus it allows us to compare its asymptotic gravitational tidal
tensors with the previous electromagnetic tidal tensors. For a static observer, with four velocity
$U^\mu \simeq \delta^\mu_0$, the electric and magnetic gravitational tidal tensors are, asymptotically

$$
E_{\alpha\beta} dx^\alpha dx^\beta \simeq - \frac{2m}{r^3} dr^2 + \frac{m}{r} d\Omega_2 + \mathcal{O} \left( \frac{1}{r^4} \right) d\theta d\phi ,
$$

$$
H_{\alpha\beta} dx^\alpha dx^\beta \simeq \frac{3J}{r^2} \left( - \frac{2\cos \theta}{r^2} dr^2 + \cos \theta d\Omega_2 - \frac{2\sin \theta}{r} d\theta d\phi \right) ,
$$

(27)

where $J = ma$ is the physical angular momentum of the spacetime. The electric gravitational and

electromagnetic tidal tensors \[27\] and \[26\] match approximately identifying $q \leftrightarrow m$; the magnetic

tensors match up to a factor of 2; this factor, already manifest in \[12iv\], is related to the fact that

gravity is a spin 2 interaction, while electromagnetism is spin 1. It can be interpreted by drawing

the analogy $\mu \leftrightarrow J$ (i.e., assigning to gravity a gyromagnetic ratio equal to 1).

Now let’s see what happens when the observer moves relative to the spheres of mass/charge; for simplicity we will consider an observer moving radially with 3-velocity $\vec{v} = v \vec{e}_r$, such that its four velocity is, in the Minkowski spacetime $U^\alpha = \gamma (\delta^\alpha_0 + v^\alpha_r)$, and, in the Kerr spacetime, asymptotically, $U^\alpha \simeq \gamma (\delta^\alpha_0 + v^\alpha_r)$, where $\gamma \equiv \sqrt{1 - v^2}$. For this observer, the electromagnetic tidal
tensors are not symmetric: $E_{\alpha\beta} = E_{(\alpha\beta)} + E_{[\alpha\beta]}$ and $B_{\alpha\beta} = B_{(\alpha\beta)} + B_{[\alpha\beta]}$, with:

$$
E_{(\alpha\beta)} dx^\alpha dx^\beta = \gamma \left( \frac{2q}{r^3} (v dt - dr) dr + \frac{q}{r} d\Omega_2 - \frac{3\mu v}{r^3} \sin^2 \theta dr d\phi \right)
$$

$$
E_{[\alpha\beta]} = - \frac{q\gamma v}{r^2} dr \wedge dt + \frac{3\mu \gamma v}{2r^2} \left( \frac{\sin^2 \theta}{r} dr \wedge d\phi - \sin 2\theta d\theta \wedge d\phi \right)
$$

$$
B_{(\alpha\beta)} dx^\alpha dx^\beta = \frac{3\mu \gamma}{r^2} \left( \frac{2\cos \theta}{r^2} (v dt - dr) dr + \cos \theta d\Omega_2 + \frac{\sin \theta}{r} (v dt - 2dr) d\theta \right)
$$

$$
B_{[\alpha\beta]} = \frac{q\gamma v \sin \theta}{r} d\theta \wedge d\phi - \frac{3\mu \gamma v}{r^3} \left( \frac{\cos \theta}{r} dr \wedge dt + \frac{\sin \theta}{2} d\theta \wedge dt \right)
$$
while their gravitational counterparts are the symmetric tensors:

\[
\mathbb{E}_{\alpha\beta} dx^\alpha dx^\beta \approx - \frac{2m\gamma^2}{r^3} (dr - vdt)^2 + \frac{m}{r} d\Omega_2 + \frac{6J\gamma^2 v}{r^3} \sin^2 \theta (vdt - dr) d\phi \\
+ O(r^{-4}) (dr + d\phi + dt) d\theta
\]

\[
\mathbb{H}_{\alpha\beta} dx^\alpha dx^\beta \approx \frac{3J}{r^2} \left( \frac{2\gamma^2 \cos \theta}{r^2} (dr - vdt)^2 + \cos \theta d\Omega_2 + \frac{2\gamma^2 \sin \theta}{r} (vdt - dr) d\theta \right) \\
+ (O(r^{-4}) dr + O(r^{-3}) d\theta + O(r^{-4}) dt) d\phi
\]

Thus, while the static observer finds a similarity between gravitational and electromagnetic tidal forces, for the observer moving relative to the central body these forces are very different. Both observers will however agree on the similarity between the invariants constructed from gravitational and electromagnetic tidal tensors (see 5.1.1).

### 2.3 Ultra-stationary spacetimes

From the analysis of the previous section, one might conclude that a matching between electromagnetic and gravitational tidal tensor components is only possible in linearised theory. Indeed, the non-linear nature of the curvature tensor would seem to preclude that an identification of metric components with the components of the electromagnetic potential could yield analogous tidal tensors, when expressed in terms of these components. There is, however, a class of spacetimes where the matching between electromagnetic and gravitational tidal tensors is exact. This class corresponds to *ultra-stationary spacetimes* defined as stationary spacetimes whose metric has a constant \( g_{00} \) component in the chart where it is explicitly time independent. The most general metric for such spacetimes is

\[
ds^2 = - \left( dt + A_i(x^k)dx^i \right)^2 + \tilde{g}_{ij}(x^k)dx^i dx^j . \tag{28}\]

Consider now, in these spacetimes, the Klein-Gordon equation for a particle of mass \( m \): \( \Box \Phi = m^2 \Phi \); it reduces (with the ansatz \( \Phi = e^{-iEt} \Psi(x^j) \)) to a time independent Schrödinger equation \( \hat{H} \Psi = \epsilon \Psi \), where

\[
\hat{H} = \left( \frac{\tilde{P} + E \tilde{A}}{2m} \right)^2 , \quad \epsilon = \frac{E^2 - m^2}{2m}; \tag{29}\]

this is the non-relativistic problem of a “charged” particle with charge \(-E\) and mass \( m \) under the influence of a magnetic field \( \tilde{B} = \nabla \times \tilde{A} \) in a curved space with metric \( \tilde{g}_{ij} \). This has been used to map the Landau levels due to constant magnetic fields in three-spaces of constant curvature to energy quantisation in spacetimes with Closed Timelike Curves (like the Gödel space) [38, 39].

Computing now the magnetic tidal tensor \( \mathbb{H} \) for the magnetic field \( \tilde{B} = \nabla \times \tilde{A} \) in a three space with metric \( \tilde{g}_{ij} \):

\[
B_{ij} = \tilde{D}_j B_i = \hat{\epsilon}_{ikl} \hat{D}_j \hat{D}_l A^k . \tag{30}\]

where \( \hat{D} \) denotes covariant differentiation carried out in the three space with metric \( \tilde{g}_{ij} \). This construction holds for the static observer of 4-velocity \( U^\mu = \delta_0^\mu \); for the same observer, the magnetic gravitational tidal tensor is exactly given by

\[
\mathbb{H}_{ij} = \frac{1}{2} \hat{\epsilon}_{ikl} \hat{D}_j \hat{D}_l A^k . \tag{31}\]

Up to a factor of 2, $H_{ij}$ and $B_{ij}$ are the same. This shows that our interpretation of the magnetic part of the Riemann tensor as a magnetic tidal tensor is indeed correct even outside the scope of linearised theory. Turning the argument around, one can see this matching of tidal tensors as a justification for the very close analogy between the physics in these apparently very different setups. In our approach this is to be understood by the similarity of the tidal forces.

It is somewhat surprising that, identifying some metric components with electromagnetic potentials, the magnetic gravitational tidal tensor can have exactly the same form as the magnetic tidal tensor, since the former would normally be non-linear in the metric. Indeed, this is what happens to the electric gravitational tidal tensor of (28): all its components are non-linear in $A_i$, preventing an interpretation in terms of an electromagnetic field. Since there is no electric tidal tensor analogue, this analogy is not perfect. But it is quite suggestive from the above mapping between Klein-Gordon and Schrödinger equation that the magnetic part of the problem (together with the spacial curvature) captures all the physics. Let us now exhibit three explicit examples of the analogy between the gravitational magnetic tidal tensor of ultra-stationary spacetimes and the magnetic tidal tensor of the analogous magnetic configuration.

### 2.3.1 Som-Raychaudhuri, Van-Stockum and Heisenberg spacetimes

The Som-Raychaudhuri metrics \[51\] are a family of solutions to the Einstein-Maxwell equations with a source term representing charged incoherent matter. They are characterised by two parameters, $a$ and $b$. The metric can be written in the form (28) with

$$A_i dx^i = -ar^2 d\phi, \quad \hat{g}_{ij} dx^i dx^j = e^{-b^2 r^2} dr^2 + r^2 d\phi^2 + e^{-b^2 r^2} dz^2.$$ (32)

The special case with $a = b$ corresponds to the solution rediscovered in 1937 by Van Stockum \[51\] (first found by Lanczos \[48\]) describing the metric in the interior of an infinitely long rotating cylinder of dust (i.e., $p = 0$) with energy density given by $\mathcal{E}_\alpha = 4\pi \rho = 2a^2 \exp a^2 r^2$.

The magnetic gravitational tidal tensor has non-vanishing components, for an observer with 4-velocity $U^\mu = \delta^\mu_0$:

$$\mathbb{H}_{rz} = \mathbb{H}_{zr} = -ab^2 r.$$ (33)

According to the Klein-Gordon equation, the analogous magnetic configuration is the magnetic field:

$$\vec{B} = \nabla \times \vec{A} = -2ae^{-b^2} e_z.$$ (34)

living in a three space with metric $\hat{g}_{ij}$. The tidal tensor of this field indeed has non-vanishing components $B_{rz} = B_{zr} = 2\mathbb{H}_{rz}$.

If we put $b = 0$, but keep $a \neq 0$, the transverse space metric $\hat{g}_{ij}$ becomes flat. The four dimensional space becomes homogeneous, and the three dimensional part $(t, r, \phi)$ is indeed a group manifold, the $Nil$ or Heisenberg group manifold, which is the group associated to the Bianchi II Lie algebra. The absence of tidal forces, in such case reflects the homogeneity of the spacetime and of the analogous magnetic field. This group manifold is a three dimensional version of the five dimensional maximally supersymmetric Gödel type universes found in $\mathcal{N} = 2$ minimal ungauged five dimensional supergravity in \[49\].

One other interesting point about this example is that the tidal tensors $B_{\alpha\gamma}$ and $\mathbb{H}_{\alpha\gamma}$ are symmetric. According to (12v) this shows there are no mass currents. This is counter-intuitive
for a rotating cylinder of fluid, unless the spacetime is described in co-moving coordinates. Such coordinates correspond, precisely, to the coordinate system above [55].

2.3.2 The Gödel Universe

In 1949 Kurt Gödel [53] found an exact solution of Einstein’s field equations which is usually portrayed as a homogeneous rotating universe. His solution arose much discussion over the years due to the existence of Closed Timelike Curves passing through any point which led Gödel to recognise, and explicitly discuss for the first time, that “an observer travelling in this universe could be able to travel to his own past”.

This description is conceptually hard. If the Gödel universe is rotating and is also homogeneous, then it must be rotating around any point! What does this mean? Of course one can argue that this solution should be discarded as being unphysical, due to the CTC’s. But even if such solution is unphysical (which is not clear) building an intuition about it could be useful in understanding General Relativity. We argue that the gravito-electromagnetic analogy proposed herein helps us to construct such an intuition. The metric can be written in the form (28) with

\[ A_i dx^i = e^{\sqrt{2}\omega x} dy \]  \quad \hat{g}_{ij} dx^i dx^j = dx^2 + e^{2\sqrt{2}\omega x} dy^2 + dz^2. \]  \quad \quad (35)

The magnetic gravitational tidal tensor vanishes, for an observer with 4-velocity \( U^\mu = (1, \vec{0}) \), which has led to conceptual difficulties in literature (see section 4.3). The Klein-Gordon equation maps this metric into a magnetic field:

\[ \vec{B} \equiv \nabla \times \vec{A} = 2\omega \vec{e}_z, \]  on the three space with metric \( \hat{g}_{ij} \). This field is uniform, since the magnetic tidal tensor \( B_{\alpha\beta} \) vanishes. Thus, the physical interpretation for the vanishing of \( H_{\alpha\beta} \) is that the Gödel universe, like the Heisenberg group manifold, has a uniform gravitomagnetic field. The concept of homogeneous rotation can now be easily assimilated by an analogy with the more familiar picture of a gas of charged particles (e.g. the conduction electrons in a metal) subject to a uniform magnetic field: there are Larmor orbits around any point.

2.3.3 The metric of Lense and Thirring

In 1918, Lense and Thirring [52] solved the linearised Einstein equations for a rotating thin spherical shell of matter, and found that rotation can ‘drag inertial frames’. Since then, this Lense-Thirring effect has become the archetype for magnetic effects in relativistic gravity. To linear order in \( \omega \), the Lense-Thirring line element inside the spherical shell is

\[ ds^2 = -dt^2 - 2\omega r^2 dt d\phi + dr^2 + r^2 d\phi^2 + dz^2. \]  \quad \quad (36)

The metric can be written in the form (28) with

\[ A_i dx^i = \omega r^2 d\phi \]  \quad \hat{g}_{ij} dx^i dx^j = dr^2 + (r^2 + \omega^2 r^4) d\phi^2 + dz^2. \]  \quad \quad (37)

It follows that the only non-trivial component of the magnetic gravitational tidal tensor, for an observer with 4-velocity \( U^\mu = (1, \vec{0}) \), is

\[ H_{tr} = -\frac{\omega^3 r}{(1 + r^2 \omega^2)^{3/2}}. \]  \quad \quad (38)
The tidal tensor is not zero. Thus, the usual interpretation of this spacetime as the gravitational analogue of a constant magnetic field (see e.g. [7]), must not be an exact assumption. To clear this matter, we note, from (29), that the magnetic field configuration analogous to the metric is:

$$\vec{A} = \frac{\omega}{1 + \omega^2 r^2} \vec{e}_\phi \quad \Rightarrow \quad \vec{B} = \frac{2\omega}{\sqrt{1 + \omega^2 r^2}} \vec{e}_z,$$

which is not uniform (in the curved space with metric $\hat{g}_{ij}$), since its tidal tensor has the (only) non-vanishing component $B_{zr} = 2H z r$.

This is the correct interpretation for the magnetic analogue of the Lense-Thirring metric if one takes it as an exact metric. However, for small $r$ (and indeed the line element (36) relates to the neighbourhood of the centre of the hollow sphere, cf. [52]), the uniform magnetic field picture is justified in the linear approach, since $\hat{g}_{zz} \sim \mathcal{O}(\omega^3)$. Under these conditions this spacetime is equivalent to the Heisenberg group manifold, which is exactly homogeneous and analogous to a constant magnetic field in flat space.

### 2.4 Force acting on a gyroscope

The force acting on a dipole is a purely tidal effect; it is therefore the most obvious physical application for a tidal tensor based analogy. There is no gravitational analogue to the electric dipole, since there are no “negative masses”; for the same reason, there can be no gravitational analogue to a magnetic dipole “alone”. But there is a clear gravitational analogue to the electric pole - magnetic dipole particle, which is the ideal gyroscope (i.e., a pole-dipole spinning test particle, as defined in [56]). In gravity no force arises from the monopole term, since a spinless particle moves along a geodesic; hence, the force exerted on a gyroscope should indeed, in the spirit of our approach, be the gravitational version of the electromagnetic force exerted on a magnetic dipole.

In electromagnetism, the force acting on a magnetic dipole when placed in a magnetic field $\vec{B}$ is usually given in literature (see, for instance, [3], p. 318) by:

$$\vec{F}_{EM} = \nabla (\vec{\mu} \cdot \vec{B}) \quad \text{(39)}$$

where $\vec{\mu} = \frac{q}{2m} \vec{S}$ denotes the magnetic dipole moment, and $\vec{S}$ the classical angular momentum. A covariant form of this equation is:

$$F^\beta_{EM} = \frac{q}{2m} B^{\alpha\beta} S_\alpha, \quad \text{(40)}$$

where $B_{\alpha\beta} = F_{\alpha\gamma\beta} U^\gamma$ is the magnetic tidal tensor “seen” by the dipole of 4-velocity $U^\gamma$, and $S^\alpha$ is the “intrinsic angular momentum” (e.g. [43], p. 158), defined as being the 4-vector with components $(0, \vec{S})$ in the dipole rest frame. Equation (40) is valid for a dipole moving with arbitrary velocity; hence, from the definition of the magnetic tidal tensor (41), and a simple covariantization of the non-relativistic expression (42), we have just obtained the important equation of the electromagnetic force exerted on a moving magnetic dipole, avoiding an otherwise more demanding computation (e.g. [42], pp. 80-84).

In the light of our approach, the analog force in gravity should then be:

$$F^\beta_G = -\hat{g}^{\alpha\beta} S_\alpha, \quad \text{(41)}$$

---

The existence of such force causes the gyroscope to deviate from geodesic motion; it well known, since the works of Mathisson and Papapetrou [56], that a spinning particle violates the weak equivalence principle.
where $H_{\alpha\beta} = R_{\alpha\gamma\beta\delta} U^\gamma U^\delta$ is the gravitational magnetic tidal tensor “seen” by the gyroscope of 4-velocity $U^\gamma$ and intrinsic angular momentum $S^\alpha$. The minus sign reflects the fact that two masses “of the same sign” attract each other, by contrast with electromagnetism, where two charges of same sign repel each other; this implies that mass currents in the same direction repel each other, while charge currents in the same direction attract each other (see [15] p. 246 and [16] 13-6 for an excellent explanation of these analogue effects). The factor $q/(2m)$ in (40) also drops out since the gravitational analog of $\vec{\mu}$ is $\vec{S}$, as can be seen by comparing (12iv) to (6iv) (see also discussion in section 2.2.1).

Equation (41) turns out to be the exact result derived by Papapetrou [56]-[58][11, 12, 13]:

$$\frac{DP^\alpha}{D\tau} = -\frac{1}{2} R^\alpha_{\beta\mu\nu} U^\beta S^{\mu\nu}$$

(42)

with Pirani [57] supplementary condition $S^{\mu\nu} U_\nu = 0$, where $S^{\mu\nu}$ denotes the spin tensor (e.g. [43], p. 158). This result may therefore be seen as a definite confirmation that our interpretation of $H_{\alpha\beta}$ as a magnetic tidal tensor is indeed correct.

This gravito-electromagnetic analogy based derivation of the Papapetrou equation has two major strengths. The first, its obvious simplicity: (41) was obtained in a two step derivation, avoiding the lengthy original computation [56]. The second is that, when written in a form explicitly analogue to its electromagnetic counterpart (40), it makes possible, by comparison with the more familiar electromagnetic ones, to visualise effects which are not transparent at all in the form (42) presented in literature. At the same time, it also reveals in a clear fashion significant differences between the electromagnetic and gravitational forces, which arise from the different symmetries of the tidal tensors. In particular, due to these symmetries, as follows from equations (6iv) and (12iv), the two forces can only be similar when the fields are static (besides weak) in the dipole rest frame. This will be discussed in detail in [60]. Herein, we will just focus on the temporal components of these forces. Let us start by the electromagnetic force (40); its temporal component is not (generically) zero in the dipole rest frame, as one might naively expect. In the dipole rest frame we have:

$$F^0_{EM} = B^0 S_i q/2m = B^0 \mu_i = -\frac{\partial B^i}{\partial t} \mu_i$$

The magnetic dipole may be seen as a small current loop; denoting the area of the loop by $a$, and its current by $I$, the magnetic dipole moment is then given by $\vec{\mu} = IA\vec{n}$, where $\vec{n}$ is the vector normal to the plane of the loop. Thus:

$$F^0_{EM} = -\frac{\partial B}{\partial t} \vec{n}AI = -\frac{\partial \Phi}{\partial t} I = I \oint_{\text{loop}} \vec{E} \cdot d\vec{s} = P$$

where $\Phi \equiv \{$magnetic flux through the loop$\}$, $\vec{E} \equiv \{$induced electric field$\}$ and $P \equiv \{$net work done per unit time by the induced electric forces$\}$. This way, in the dipole rest frame, the time component of (40) gets a simple physical interpretation: it is the power transferred to the dipole by electromagnetic Maxwell-Faraday induction, due to a time varying magnetic field.

---

6Consider, as an example, the tensorial form of the Lorentz Force: $DP^\alpha/D\tau = qF^{\alpha\beta} U_\beta$. It also exhibits a time component, given by $DP^0/D\tau = \vec{v}.\vec{E}/\sqrt{1-v^2}$, which represents the work done, per unit time, by the electromagnetic field on the particle of charge $q$. It is zero in the particle rest frame, since the particle (obviously) does not move relative to itself. But that is not the case of the time component of force (40), which depends on quantities that are all measurable in the dipole rest frame.
We turn now to gravity. Unlike its electromagnetic counterpart, $H_{\alpha\beta}$ is a spacial tensor; this means that if $H_{\alpha\beta}$ is the gravitational magnetic tidal tensor as measured by a given observer of 4-velocity $U^\gamma$, then $H_{\alpha\beta} U^\alpha = H_{\alpha\beta} U^\beta = 0$. Or, equivalently, that, in the coordinates of the observer’s rest frame, $H_{\alpha\beta}$ has only space components. Therefore, in the gyroscope rest frame:

$$F^0_G = -H^0_i S_i = 0$$

and one may thus regard the spatial character of the gravitational tidal tensors as evidence for the nonexistence of electromagnetic-like induction effects in gravity, which is in accordance with discussion in section 2.1.3.

We close this section with a remark on how this application emphasises the universality of the gravito-electromagnetic analogy based on tidal tensors: gravitational and electromagnetic tidal tensors will not be similar except under very special conditions; generically, $B_{\alpha\beta}$ and $H_{\alpha\beta}$ will not even exhibit the same symmetries - but they will always play analogous roles in dynamics, as shown by equations (40) and (41).

3 Linearised theory approach

We started our analogy by looking under which circumstances tidal effects could have a similar tensorial description in gravity and electromagnetism. We have seen (section 2.2) that the similarity between tidal tensors we found on certain special conditions could be regarded as an analogy between some components of the metric tensor and the electromagnetic gauge potentials.

Such an analogy is also suggested by the formal similarity between the linearised Einstein equations in the harmonic gauge, and the Maxwell equations in the Lorentz Gauge. In the linearised theory approach to gravitoelectromagnetism (GEM), this analogy is worked out to define the gravitational analogues to the electromagnetic fields $\vec{E}$ and $\vec{B}$, with which one aims to describe gravity. Of course, since gravity is pure geometry, such forces or fields have no place in it; in this sense, we may see this approach as an analogy between physical quantities which are present in one theory, but do not exist in the other. This marks an important conceptual difference from the approach based on tidal tensors, where we compare quantities common to both theories.

Nevertheless, we know that in the Newtonian limit (i.e., when the field is weak, and the relative motion of the sources is negligible), the linearised Einstein equations reduce to the Poisson equation, and gravity is well described by the Newtonian gravitational field. We also know that when the motion of the sources is taken into account, magnetic-type phenomena occur in gravity; and given the aforementioned similarity between the linearised Einstein equations in the harmonic gauge and the Maxwell equations in the Lorentz gauge, it is legitimate to suppose that gravity can be described, in some appropriate limit, by fields analogous to the electromagnetic ones.

There is no general agreement about the limit of validity of this analogy (see, for instance, [1] and [12]), apart from the fact that it is defined in the weak field and slow motion approximation. While some authors limit the analogy to stationary configurations [1]-[4], for others it can be extended to time dependent setups [5]-[12].

We argue that our analysis of the tidal tensors sheds light into this debate: gravity can be described, in an approximate way, by fields analogue to those from electromagnetism only when the gravitational tidal tensors are similar to their electromagnetic counterparts (i.e., when the tidal...
forces, which are the only forces present in gravity, are correctly determined by the variations of those fields). In particular, this requires the fields to be static in the observer’s rest frame, as is clear from the comparison of equations (10) with (12) (see also section 2.2). Otherwise, gravitational tidal tensors do not even exhibit the same symmetries as the covariant derivative of an electromagnetic type field, signalling that such fields are no longer appropriate to describe gravity.

We shall now briefly review and discuss these two versions of linear approach to GEM.

3.1 Time independent approach

One starts by considering small perturbations $h_{\mu\nu}$ around Minkowski spacetime, such that the metric is given by:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1 \quad (43)$$

It is more convenient to work with the quantity $\tilde{h}_{\mu\nu} \equiv h_{\mu\nu} - \eta_{\mu\nu} h^\alpha_{\alpha}/2$; imposing a gauge condition called de Donder or harmonic gauge, given by $\tilde{h}^\alpha_{\alpha \beta} = 0$, the linearised Einstein equations take the form:

$$\Box \tilde{h}^\alpha_{\alpha \beta} = -\frac{16\pi}{c^4} T^\alpha_{\beta} \quad (44)$$

We will consider a metric with general perturbations of the form (17), but with $g_{ij} = \delta_{ij}$, i.e., using nearly Lorentz coordinates, so that conditions (43) are satisfied. For such metric the harmonic gauge condition yields:

$$\frac{1}{c^3} \frac{\partial}{\partial t} (\phi + 3C) + \frac{2}{c^2} \nabla . \vec{A} = 0, \quad \frac{2}{c^5} \frac{\partial A_i}{\partial t} + \frac{1}{c^2} \left[(\phi - C),_i + 2 \xi_{ij}^j\right] = 0 \quad (45)$$

In the special case $C(t, x^k) = \phi(t, x^i)$, $\xi_{ij}(t, x^k) = 0$ one gets $\tilde{h}_{ij} = 0$ and (44) reduces to the set of four equations for $\tilde{h}^{0\beta}$ which, apart from the different tensorial structure, exhibit a clear analogy with the Maxwell equations in the Lorentz gauge:

$$\Box A^\beta = -\frac{4\pi}{c} j^\beta \quad (46)$$

Furthermore, the harmonic gauge condition (45) reduces to:

$$\frac{1}{c^3} \frac{\partial \phi}{\partial t} + \frac{\nabla . \vec{A}}{2c^2} = 0, \quad \frac{1}{c^3} \frac{\partial \vec{A}}{\partial t} = 0 \quad (46)$$

The second equation imposes a static potential $\vec{A}$, while the first one is similar (up to a factor of 2 in the second term) to the Lorentz gauge condition in electromagnetism; thus, defining the gravitoelectric and gravitomagnetic fields in direct analogy with electromagnetism:

$$\vec{E}_G \equiv -\nabla \phi, \quad \vec{B}_G \equiv \nabla \times \vec{A} \quad (47)$$

these two definitions, together with equations (44) and (46) lead to the following set of equations:

$$\begin{align*}
  &i) \nabla . \vec{E}_G = 4\pi \rho_m \\
  &ii) \nabla \times \vec{E}_G = 0 \\
  &iii) \nabla . \vec{B}_G = 0 \\
  &iv) \frac{1}{2} \nabla \times \vec{B}_G = \frac{4\pi}{c} \vec{J} + \frac{1}{c} \frac{\partial \vec{E}_G}{\partial t}
\end{align*} \quad (48)$$

\footnote{In this section, following the standard procedure in the literature, we reinsert the speed of light $c$, while still making $G = 1$.}
The last equation would generally imply \( \nabla \times \vec{B}_G \neq 0 \) in empty space, what seems to contradict our exact equation (12iv), which states that the magnetic gravitational tidal tensor is always symmetric in empty space; to clear this matter we will check the geodesics. To linear order in the potentials and velocity of the test particle, the geodesic equation yields:

\[
\frac{d^2 \vec{x}}{dt^2} = -E_G - \frac{2}{c} \vec{v} \times \vec{B}_G - \frac{2}{c^2} \frac{\partial \phi}{\partial t} \vec{v}
\]

which reduces to the form (25) analogue to the electromagnetic Lorentz force only if \( \partial \phi / \partial t = 0 \); thus, this analogy between electromagnetism and linearised gravity holds only if the potentials are static, as is asserted in [2] p. 163, and in accordance with our discussion in section 2.2. In this case, the last term of (48iv) vanishes, and equations (48) become but an approximation, in the limit of static weak fields and stressless sources, to our exact equations (12).

The non-covariance of this analogy (already manifest in its formalism) is obvious: if one considers a Lorentz boosted observer, the analogy will no longer hold since, for that observer, the potentials will be time dependent.

Nevertheless, despite its limitations, this approach still proves extremely useful whenever it applies. The gravitomagnetic fields (47) are the same as (24) for the special case \( g_{ij} = \delta_{ij} \); thus, cf. discussion in section 2.2 they help us visualise frame dragging, and lead directly to the Lense and Thirring precession of (for instance) a test gyroscope. The torque applied on the gyroscope is \( \tau = -\vec{S} \times \vec{B}_G / c \), which yields the precession frequency \( \vec{\omega} = \vec{B}_G / c \). The gravitomagnetic field \( \vec{B}_G \) has also been used [1, 3, 58] to obtain a first order estimate to the Papapetrou force applied on the gyroscope (11); in direct analogy with the electromagnetic expression (39), that first order estimate is \( \vec{F}_G = -\nabla (\vec{S} \cdot \vec{B}_G) / c \). However, as is asserted in [58], and made clear by the discussion in section 2.4, this expression is valid only when the gyroscope is at rest in a stationary spacetime; otherwise, the gravitational and electromagnetic forces will differ significantly, due to the different symmetries of the tidal tensors “seen” by the gyroscope.

### 3.2 “Maxwellian” gravity

On some of the literature covering the linear perturbation approach to GEM, a set of “gravitational Maxwell equations” is derived [5]-[8][10]-[12], which include time dependent terms; these equations differ from the set (48) by the following: the last term of (48iv) is now kept, and (48ii) is replaced by:

\[
\nabla \times \vec{E}_G = -\frac{1}{2c} \frac{\partial \vec{B}_G}{\partial t}
\]

Based on this kind of equations, some exotic gravitational phenomena have been predicted, such as the existence of a gravitational analogue of Faraday’s law of induction [7, 10, 11], deduced from (50). Experiments to detect it have been proposed [11] and, recently, such an experiment has actually been performed [13].

---

8The source of the gravito-electric field is here \( \rho_m \) instead of \( 2\rho_m + T_{\alpha}^\alpha \) which appears in our equation (12). This physical content is lost here since, by choosing \( \phi = C \) and \( \xi_{ij} = 0 \) above, we imposed \( \bar{h}_{ij} = 0 \); to be valid everywhere, this condition indeed demands, by virtue of equations (44), \( T_{ij} = 0 \), so that \( T_{\alpha}^\alpha = -\rho_m \). This is consistent with the slow-source assumption, where the pressure and all material stresses are taken to be negligible compared to the energy density term.
These equations have been under debate in literature, and there is no consensus around them; they are also in contradiction with the results from our approach, since in dynamical situations they would imply tidal tensors having symmetries that, by virtue of our exact, covariant equations (12i) and (12v), gravitational tidal tensors can never have.

Note that the physical content of the equations depends crucially on these symmetries. For instance, in the framework of the approach based on tidal tensors, it is clear that such phenomena as an analogue of Faraday’s induction cannot take place in gravity: our exact equation (12ii) shows that the electric tidal tensor is always symmetric, i.e., the “gravito-electric field” is (always) irrotational.

In order to clarify these contradictions, let us review the derivation of these Maxwell-like equations. One starts by making $C(t, x^k) = \phi(t, x^i)$ in (17), then one neglects all terms of order $c^{-4}$ or lower; that amounts to neglect (see eg. [8]) the tensor perturbations term in (17) since in the slow motion approximation $\xi_{ij}(t, x^k) \sim c^{-2}$. The next step is to define the gravitational electric $\vec{E}_G$ and magnetic $\vec{B}_G$ fields; the latter is defined as in (47), while the former is here:

$$\vec{E}_G \equiv -\nabla \phi - \frac{1}{2c} \frac{\partial A}{\partial t}(51)$$

This definition leads directly to equation (50). The other equations are obtained using these definitions together with equations (44) and (45).

In a consistent approximation, however, the last term of (51) should have been dropped as in (17), since by virtue of the second equation in (45), neglecting tensor perturbations amounts to neglect the time dependence of $\vec{A}$ (cf. [4], p. 78). This condition is however ignored, again with the argument (see eg. [8]) that it involves terms of order $c^{-4}$; and this procedure is indeed the origin of the controversy surrounding this particular approach (see eg. [1] and [12]). In what follows we will show that indeed this analogy breaks down precisely when time dependent terms are present.

An analysis of the tidal effects reveals quite clearly that definition (51) cannot be appropriate for a gravitational field. According to (51) an observer at rest would be subject to tidal forces described by a tensor of the form:

$$(E_G)_{i,j} = -\phi_{,ij} - \frac{1}{2c} \frac{\partial A_{i,j}}{\partial t}(52)$$

which is not symmetric when $\partial \vec{A}/\partial t \neq 0$. But we know, from the geodesic deviation equation, that the tidal tensor, for this observer, is actually given, to linear order, by:

$$E_{ij} = R_{i\beta j\gamma} \delta_{0}^{\beta} \delta_{0}^{\gamma} c^2 \approx -\phi_{,ij} - \frac{2}{c} \frac{\partial A_{(i,j)}}{\partial t} - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} \delta_{ij}(53)$$

which is symmetric as expected; note that the second term in (52), as well as both the second and third terms in (53) are of same order $c^{-2}$, and cannot be consistently neglected in a approximation where terms down to $c^{-3}$ are kept. Therefore, within this approximation, the only way to make expressions (52) and (53) match is to consider static potentials (where both reduce to the Newtonian tidal tensor).

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9To follow this debate, see [11], [1]. [2] p. 163, and [12].

10Note, from eqs. [14], that in the slow motion approximation, $|\vec{A}| \sim c^{-1}$. 

20
An inspection of the geodesics leads to a similar conclusion. To linear order in the potentials and velocity of the test particle, the geodesic equation yields:

\[
\frac{d^2 \vec{x}}{dt^2} = -\vec{E}_G + \frac{1}{c} \left( \frac{3}{2} \frac{\partial \vec{A}}{\partial t} - 2 \vec{v} \times \vec{B}_G \right) - \frac{2}{c^2} \frac{\partial \phi}{\partial t} \vec{v}
\]

in this equation all terms apart from $\vec{E}_G$ are of same order $c^{-2}$; therefore, in this approximation, the only way to obtain a Lorentz like expression (25), is to consider static potentials [2]-[6][58].

As a third evidence that gravity cannot be described by electromagnetic-like fields when the spacetime is not stationary, we point the fact that, on those conditions, the gravito-magnetic field $\vec{B}_G$ does not yield the correct force applied on a gyroscope. According to the literature where this “Maxwellian” analogy is pursued, that force would be given by (see eg. [5, 6, 8]):

\[
\vec{F} = -\frac{1}{c} \nabla (\vec{S} \cdot \vec{B}_G) \quad \Leftrightarrow \quad F^j = -\frac{1}{c} (B^i_G)^{i,j} S_i
\]

But, as already discussed in previous section, the validity of this equation requires the field to be static [1, 3, 58]. And it is easy to show it leads to an incorrect result when applied to time dependent fields. The correct force is given by the Papapetrou equation (41), which, when applied to a gyroscope at rest in the above considered spacetime reads, to linear order:

\[
F^j_G = -\frac{1}{c} \mathbb{H}^{ij} S_i \approx -\frac{1}{c} \left[ (B_G)^{i,j} - \frac{1}{c} \frac{\partial}{\partial t} \left( \epsilon^{ijk} \phi_k \right) \right] S_i
\]

This equation only matches (54) if the potentials are static (again, both terms in (55) are of same order in powers of $c^{-1}$; hence, none can be neglected). This way we see that, even within the scope of a linear perturbation approach, one must use the analogy based on tidal tensors instead of gravito-electromagnetic fields in order to obtain the correct result when the spacetime is not stationary.

Thus, to conclude and summarise: we argue that the physical analogy \{\vec{E}, \vec{B}\} $\leftrightarrow$ \{\vec{E}_G, \vec{B}_G\} must be restricted to time independent setups, as is done in most textbooks covering the subject (e.g. [2, 3, 4]), and in accordance with the results from our approach based on tidal tensors.

4 The analogy between gravitational tidal tensors and electromagnetic fields

There is a different analogy [14]-[37] between Maxwell’s equations of electromagnetism and the so called “higher order” gravitational field equations which has led to the definition of an electric and a magnetic part of the curvature tensor, originally introduced in [14] (see also [15]). These electric and magnetic parts of the curvature tensor form, moreover, invariants in a similar fashion to the relativistic invariants formed by the electric and magnetic fields [14, 17, 18, 22]. In this section we will review this analogy and dissect its physical content in the light of our approach based on tidal tensors.
4.1 General arguments

Maxwell’s field equations in vacuum are

\[ F_{\nu;\mu} = 0 , \quad F_{[\mu\nu;\alpha]} = 0 , \quad (56) \]

where the second set of equations are, in fact, Bianchi identities. Given a congruence of observers with 4-velocity field \( U^\alpha \), we can split the Maxwell tensor into the two spatial vector fields:

\[ E^\alpha = F^\alpha_\beta U^\beta , \quad B^\alpha = \star F^\alpha_\beta U^\beta , \quad (57) \]

which are, of course, the electric and magnetic fields as measured by the observer of 4-velocity \( U^\alpha \), but in this context dubbed electric and magnetic parts of the Maxwell tensor. They completely characterise the latter, as can be seen by writing

\[ F^\alpha_\beta = 2 U_{[\alpha} E^\beta] + \frac{1}{2} \epsilon^\alpha_\beta\mu\nu B^\mu U^\nu , \quad (58) \]

Thus, all 6 independent components of \( F_{\mu\nu} \) are encoded in the 3+3 independent components of \( E^\alpha \) and \( B^\alpha \) and there is an equivalence between the vanishing of \( F^\alpha_\beta \) and the simultaneous vanishing of \( E_\alpha \) and \( B_\alpha \). In spite of their dependence on \( U^\alpha \), one can use \( E^\alpha \) and \( B^\alpha \) to define two tensorial quantities which are \( U^\alpha \) independent, namely

\[ E^\alpha E_\alpha - B^\alpha B_\alpha = -\frac{F^\alpha_\beta F^\alpha_\beta}{2} , \quad E^\alpha B_\alpha = -\frac{F^\alpha_\beta \star F^\alpha_\beta}{4} ; \quad (59) \]

these are the two independent relativistic invariants in four spacetime dimensions.

Let us now turn to gravity. The curvature tensor obeys the second Bianchi identity

\[ R_{\sigma\tau;\mu}^\mu = 0 . \quad (60) \]

by virtue of the field equations \( R_{\mu\nu} = 0 \). Let us change the perspective by observing the following result (originally due to Lichnerowicz, see \[15\]): Let \( \Sigma \) be a spacelike hypersurface and \( V \) a neighbourhood of \( \Sigma \); then, if \( R_{\mu\nu;\sigma}^\sigma = 0 \) in \( V \), it follows from \( R_{\sigma\tau;[\mu\nu]} = 0 \) that \( R_{\mu\nu} = 0 \) in \( V \) iff \( R_{\mu\nu} = 0 \) in \( \Sigma \). Thus, \( R_{\mu\nu} = 0 \) becomes an initial condition in \( \Sigma \) which is propagated to \( V \) by virtue of the higher order field equations \( (60) \) and the Bianchi identities. Taking this perspective the gravitational analogue of \( (56) \) are

\[ R_{\nu\sigma\tau;\mu}^\mu = 0 , \quad R_{\sigma\tau;[\mu\nu]} = 0 . \quad (61) \]

From decomposition \[8\] we see that in vacuum: \( R_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} \). Again, given a congruence of observers with 4-velocity field \( U^\alpha \), we can split the curvature tensor into its electric and magnetic parts given in \( (11) \). These two spatial tensors, both of which are symmetric and traceless, completely characterise the Weyl tensor, as can be seen by writing

\[ C_{\alpha\beta}^{\gamma\delta} = 4 \left\{ 2 U_{[\alpha} U^{[\gamma} + g_{[\alpha}^{[\gamma} \right\} E_{\beta]}^{\delta]} + 2 \left\{ \epsilon_{\alpha\beta\mu\nu} U^{[\gamma} H^{\delta]\mu U^{\nu} + \epsilon^{\gamma\delta\mu\nu} U_{[\alpha} H_{\beta]}^{\mu U^{\nu}} \right\} . \quad (62) \]

Thus, all 10 independent components (in vacuum) of \( R_{\mu\nu\alpha\beta} \) are encoded in the 5+5 independent components of \( E_{\alpha\beta} \) and \( H_{\alpha\beta} \). Again, in spite of their dependence on \( U^\alpha \), one can use \( E_{\alpha\beta} \) and \( H_{\alpha\beta} \) to define two tensorial quantities which are \( U^\alpha \) independent, namely
\[
\mathcal{E}^{\alpha\beta} \mathcal{E}_{\alpha\beta} - \mathcal{H}^{\alpha\beta} \mathcal{H}_{\alpha\beta} = \frac{C_{\alpha\beta\mu\nu} C^{\alpha\beta\mu\nu}}{8}, \quad \mathcal{E}^{\alpha\beta} \mathcal{H}_{\alpha\beta} = \frac{C_{\alpha\beta\mu\nu} (\ast C)^{\alpha\beta\mu\nu}}{16}. \tag{63}
\]

The construction (61)-(63) is, clearly, formally analogous to (56)-(59). Let us now consider electromagnetic/gravitational sources. Maxwell’s field equations (56) get replaced by

\[
F^{\mu\nu} = J_{\nu}, \quad F_{[\mu\nu;\alpha]} = 0, \tag{64}
\]

where \( J_{\nu} \) is the 4-current. Since including sources does not change the number of independent components of the Maxwell tensor, we can still split it as in (57), so that the remaining construction (57)-(59) is unchanged. The same does not apply, however, to the Riemann tensor: in the presence of sources it cannot be completely characterised by its electric and magnetic parts. The reason can be seen as follows. Including sources endows the Riemann tensor with non-vanishing trace and thus with its maximal number of independent components: 20 in four dimensions. On the other hand, from equations (9)-(10) we see that the sources endow \( E_{\alpha\beta} \) with a trace and \( H_{\alpha\beta} \) with an anti-symmetric part; then, these tensors combined possess at most 6+8 components, which is insufficient to encode all the information in the Riemann tensor.

Decomposition (62), in its turn, remains valid in the presence of sources, since the Weyl tensor is by definition traceless and has generically only 10 independent components in four dimensions, which are completely encoded into the 5+5 components of \( \mathcal{E}_{\alpha\beta} \) and \( \mathcal{H}_{\alpha\beta} \). For this reason, generically, in this construction one uses \( C_{\alpha\beta\gamma\delta} \) instead of \( R_{\alpha\beta\gamma\delta} \). Hence, we replace (61) by

\[
C^{\mu}_{\nu\sigma\tau;\mu} = T_{\nu}[\tau;\sigma] - \frac{1}{3} g_{\nu[\tau} T_{\sigma],} \quad R_{\sigma\tau}[\mu\nu;\alpha] = 0, \tag{65}
\]

where \( T_{\mu\nu} \) is the energy momentum tensor, with trace \( T \), and we have used the Einstein equations in the form \( G_{\mu\nu} = T_{\mu\nu} \). Equations (65), together with (62)-(63) form, in the spirit of this approach, the general gravitational analogue to construction (57)-(59).

Generically we can say that the electric and magnetic parts of the Weyl tensor describe the free gravitational field, since they correspond to the parts of the curvature that do not couple directly to sources, but rather just via the integrability conditions (i.e. the Bianchi identities). In 3 spacetime dimensions, the curvature is completely determined by the sources; thus there is no free gravitational field. That is why the Weyl tensor is identically zero in 3 spacetime dimensions.

### 4.2 The analogy in the 1 + 3 covariant formalism

This analogy can be further worked out by making “spatial” and “temporal” projections of both the Maxwell equations (64) and the higher order gravitational equations (65). Using the 1+3 covariant formalism (see e.g. [22] and references therein), such projections can be made and still keep covariance. To explain this formalism, keeping this paper self-contained, we need to introduce the notation given in [22].

Consider a congruence of observers with 4-velocity \( U^\mu \), and the projector into local rest frames \( h_{\mu\nu} \equiv g_{\mu\nu} + U_\mu U_\nu \). Note that

\[
h_{\mu\nu} U^\mu = 0, \quad h^\nu_{\mu} h^\alpha_{\nu} = h^\alpha_{\mu}, \quad h_{\mu\nu} g^{\nu\alpha} = h^\alpha_{\mu}. \tag{66}
\]

Note that the two equations in (66) are equivalent, using the usual Einstein equations, unlike the two equations in (65). But the perspective in this analogy is that these are the fundamental equations, and the usual Einstein equations are initial conditions propagated to the whole spacetime by virtue of these equations.
The fundamental idea is that $U^\alpha$ provides a covariant time projection whereas $h_{\alpha\beta}$ provides a covariant spatial projection of tensorial quantities. Thus, denote the spatially projected part of a vector as $V_{(\mu)} \equiv h^{\mu}_\nu V_\nu$ and the spatially projected, symmetric and trace free part of a rank two tensor as

$$A_{(\mu\nu)} \equiv h^{\mu}_\nu h_\alpha^\beta A_{\alpha\beta} - \frac{1}{3} h_{\mu\nu} h_{\alpha\beta} A^{\alpha\beta} .$$

(67)

The covariant spatial vector product is denoted

$$[V, W]_\mu \equiv \epsilon^{\mu\nu\alpha\beta} V_\nu W_\alpha U_\beta ,$$

(68)

where $\epsilon_{\mu\nu\alpha\beta}$ is the Levi-Civita tensor. The covariant vector product for spatial tensors is

$$[A, B]_\mu \equiv \epsilon^{\mu\nu\alpha\beta} A_\nu B_\alpha U_\beta .$$

(69)

The covariant time derivative for an arbitrary tensor is

$$\dot{A}^{\mu\ldots\nu\ldots} = U^\alpha \nabla_\alpha A^{\mu\ldots\nu\ldots} ,$$

(70)

where $\nabla$ denotes (in this section) the standard spacetime covariant derivative. The covariant spatial derivative is

$$D_\alpha A^{\mu\ldots\nu\ldots} = h^{\sigma}_\alpha h^\mu_\tau \ldots h^\rho_\nu \ldots \nabla_\sigma A^{\tau\ldots\rho\ldots} .$$

(71)

The covariant spatial divergence and curl of vectors is

$$\text{div}V = D^\mu V_\mu , \quad \text{curl}V_\mu = \epsilon_{\mu\nu\alpha\beta} D^\nu V^\alpha U^\beta ,$$

(72)

whereas the spatial divergence and curl of rank two tensors is

$$(\text{div}A)_\alpha = D^\mu A_{\alpha\mu} , \quad \text{curl}A_{\alpha\beta} = -\epsilon_{\mu\nu\gamma\alpha} D^\mu A_\beta U^\gamma .$$

(73)

Finally, the kinematics of the $U^\mu$-congruence is described by the following quantities: expansion, $\Theta \equiv D^\mu U_\mu$; shear, $\sigma_{\mu\nu} \equiv D_{(\mu} U_{\nu)}$; vorticity, $w_\mu \equiv -\text{curl}U_\mu/2$; acceleration, $\dot{U}_\mu \equiv \dot{U}_{(\mu)}$.

We can now express the Maxwell equations in this formalism. The first equation in (64) yields, as its covariant time and space components, respectively

$$D_\mu E^\mu = \rho - 2\omega_\mu B^\mu ,$$

$$\dot{E}_{(\mu)} - \text{curl}B_\mu = -\frac{2}{3} E_\mu \Theta + \sigma_{\mu\nu} E^\nu - [\omega, E]_\mu + [\dot{U}, B]_\mu - j_\mu ,$$

(74)

where the charge density is given by $\rho = -J_\mu U^\mu$ and the current density vector is $j_\mu \equiv J_\mu h^\alpha_\mu$. The Bianchi identities in (64), yield, as their covariant time and space components, respectively

$$D_\mu B^\mu = 2\omega_\mu E^\mu ,$$

$$\dot{B}_{(\mu)} + \text{curl}E_\mu = -\frac{2}{3} B_\mu \Theta + \sigma_{\mu\nu} B^\nu - [\omega, B]_\mu - [\dot{U}, E]_\mu .$$

(75)

Note that that the vector equations in (74) and (75) are spatial and therefore there are $3+1+3+1=8$ equations in accordance to Maxwell’s theory.
Now we turn to the gravitational equations (65) taking the energy momentum tensor of a perfect fluid \( T_{\mu\nu} = p g_{\mu\nu} + (p + \rho) U_\mu U_\nu \). Taking the trace of the first equation in (65) implies energy-momentum conservation. The trace free part gives, taking spacial and temporal projections, a set of equations [22] which resemble (74) and (75):

\[
D^\mu \mathcal{E}_{\nu\mu} = \frac{1}{3} D_\nu \rho - 3 \omega^\mu \mathcal{H}_{\nu\mu} + [\sigma, \mathcal{H}]_\nu ,
\]

(76)

\[
\dot{\mathcal{E}}_{(\mu\nu)} - \text{curl} \mathcal{H}_{\mu\nu} = - \mathcal{E}_{\nu\mu} \Theta + 3 \sigma_{\tau(\mu} \mathcal{E}_{\nu)}^\tau - \omega^\tau \epsilon_{\tau\rho(\mu} \mathcal{E}_\nu^\rho + 2 \dot{U}^\rho \epsilon_{\rho\tau(\mu} \mathcal{H}_\nu^\tau - \frac{1}{2} (\rho + p) \sigma_{\mu\nu} ,
\]

where \( \epsilon_{\mu\nu\rho\tau} \equiv \epsilon_{\mu\nu\rho\tau} U^\tau \) and

\[
D^\mu \mathcal{H}_{\nu\mu} = (\rho + p) \omega_{\nu} + 3 \omega^\mu \mathcal{E}_{\nu\mu} - [\sigma, \mathcal{E}]_\nu ,
\]

(77)

\[
\dot{\mathcal{H}}_{(\mu\nu)} + \text{curl} \mathcal{E}_{\mu\nu} = - \mathcal{H}_{\nu\mu} \Theta + 3 \sigma_{\tau(\mu} \mathcal{H}_{\nu)}^\tau - \omega^\tau \epsilon_{\tau\rho(\mu} \mathcal{H}_\nu^\rho - 2 \dot{U}^\rho \epsilon_{\rho\tau(\mu} \mathcal{E}_\nu^\tau .
\]

Equations (74)-(75) and (76)-(77) exhibit a clear analogy. The obvious question is the physical content of this analogy.

4.3 What physical conclusions can we extract from this analogy?

The physical content of the analogy \( \{ E^\alpha, B^\alpha \} \leftrightarrow \{ \mathcal{E}^{\mu\nu}, \mathcal{H}^{\mu\nu} \} \) is an unanswered question in the literature, and that is mainly due to the fact that (especially) the magnetic part of the Weyl tensor is not well understood [23, 32, 33]. On some of the literature where this analogy is pursued, it has been suggested (see e.g. [20, 21, 22, 24, 27]) that rotation should source the magnetic part of the Weyl tensor; but immediately contradictions arise. Whereas a number of examples, like the Kerr metric or the Van-Stockum solution [20], are known to support the idea that rotation sources \( \mathcal{H}_{\mu\nu} \), there are also well known counterexamples, like the Gödel universe [20, 21, 22]. Should we be surprised?

In the framework of our analogy based on tidal tensors, both the questions of 1) the physical interpretation of \( \mathcal{E}_{\mu\nu} \) and \( \mathcal{H}_{\mu\nu} \) and 2) the physical content of this analogy based on the splitting of the Weyl tensor are readily answered.

1. \( \mathcal{E}_{\mu\nu} \) and \( \mathcal{H}_{\mu\nu} \) are tidal tensors. As equations (9)-(10) point out, they are, respectively, the trace-free and symmetric parts of \( E_{\alpha\beta} \) and \( \mathbb{H}_{\alpha\beta} \), which, in their turn, are the gravitational counterparts of the electromagnetic tidal tensors \( E_{\mu\nu} \) and \( B_{\mu\nu} \).

2. It follows that the analogy \( \{ E^\alpha, B^\alpha \} \leftrightarrow \{ \mathcal{E}^{\mu\nu}, \mathcal{H}^{\mu\nu} \} \) is purely formal, since it compares electromagnetic fields with gravitational tidal tensors.

From 1) it is then quite clear why \( \mathcal{H}_{\mu\nu} \) is not zero for Kerr (which is rotating) but should vanish for Gödel, which is also interpreted as being rotating. The electromagnetic analogue for the (asymptotic) Kerr metric is a rotating charge (see 2.2.1), whose magnetic tidal tensor is not zero. But the electromagnetic analogue for Gödel is a uniform magnetic field (see section 2.3.2), whose magnetic tidal tensor \( \mathbb{H}_{\alpha\beta} \) is obviously vanishing; this is the reason for the vanishing of the magnetic part of the Weyl tensor since, by virtue of equation (10), \( \mathcal{H}_{\alpha\beta} = \mathbb{H}_{(\alpha\beta)} \). One can also check that \( \mathcal{H}_{\mu\nu} = 0 \) for the Heisenberg spacetime, which is again analogous to a uniform magnetic field.
Evidence for 2) is the simple observation that the gravitational equation analogous to Gauss’s law (74), \( \nabla \cdot \vec{E} = \rho + \ldots \) is (76),

\[
D^\mu \mathcal{E}_{\nu \mu} = \frac{1}{3} D^\nu \rho + \ldots
\]

clearly revealing that these gravitational objects are one order higher in differentiation than their electromagnetic counterparts. Another signature of the unphysical character of this analogy is that the invariants (63) are not similar, in simple gravitational backgrounds with an obvious electromagnetic analogue, to the invariants (69) of the electromagnetic analogue (see section 5.1.1).

Under debate in literature has been the question of whether it is possible to neglect \( H_{\mu \nu} \) in the Newtonian limit; we believe our interpretation of \( H_{\mu \nu} \) may also shed light into this debate: \( H_{\mu \nu} \) is a magnetic tidal tensor, thus, motion of mass/energy is its source; Newtonian gravity, like electrostatics, is obtained by neglecting all relativistic effects; that includes magnetism. Hence, from the viewpoint of our analogy based on tidal tensors, \( H_{\mu \nu} \) must vanish in the Newtonian limit for the same reason that \( B_{\alpha \beta} \) (or the magnetic field \( \vec{B} \)) has no place in electrostatics. This supports what is asserted by most authors (see [23]-[26] [28] [30]-[33] [36, 37]), and the claim by Ellis and Dunsby [36] that the “Newtonian” equations for \( H_{\mu \nu} \) derived in [34, 35] were in fact obtained by going beyond Newtonian theory.

5 Invariants and a new proof of a set of tensor identities

The fact that both the contractions of \( E^\alpha \) and \( B^\alpha \) given in (59) and the contractions of \( \mathcal{E}^{\alpha \beta} \) and \( \mathcal{H}^{\alpha \beta} \) given in (63) form \( U^\mu \) independent tensor quantities, suggests that there might be some general underlying structure common to both. To see that this is indeed the case we prove the following lemma:

**Lemma:** Let \( A_{\alpha \beta} \) and \( B_{\alpha \beta} \) be tensors which are antisymmetric in two of their indices; these are the only indices displayed. Let \( \star A_{\alpha \beta} \) and \( \star B_{\alpha \beta} \) be the four dimensional Hodge duals of \( A \) and \( B \) with respect to these two indices, i.e

\[
\star A^{\alpha \beta} = \frac{1}{2} \epsilon^{\alpha \beta \mu \nu} A_{\mu \nu}
\]

and similarly for \( B \). Then, in four dimensions

\[
\star A^{\alpha \nu} B_{\alpha \mu} + \star B^{\alpha \nu} A_{\alpha \mu} = \frac{1}{4} \left[ \star A^{\sigma \tau} B_{\sigma \tau} + \star B^{\sigma \tau} A_{\sigma \tau} \right] \delta^\nu_\mu.
\]

**Proof:** Take the following four dimensional identity:

\[
\epsilon^{\alpha \beta \sigma \tau} A_{[\alpha \beta} B_{\sigma \tau]} \delta^\nu_\mu \equiv 0.
\]

The anti-symmetrisation in five indices guarantees the identity in four dimensions; \( \epsilon \) is the Levi-Civita tensor (not the tensor density). A tedious, but straightforward, computation shows that

\[
\epsilon^{\alpha \beta \sigma \tau} A_{[\alpha \beta} B_{\sigma \tau]} \delta^\nu_\mu = \frac{6}{30} \left( [\star A^{\sigma \tau} B_{\sigma \tau} + \star B^{\sigma \tau} A_{\sigma \tau}] \delta^\nu_\mu - 4 \star A^{\alpha \nu} B_{\alpha \mu} - 4 \star B^{\alpha \nu} A_{\alpha \mu} \right).
\]

---

12To follow this debate, see [25] pp. 124-135, [34]-[36], [32], and [31].

13See [45] p. 246 and [46] 13-6 for insightful discussions on how magnetism arises from relativistic effects in gravity and in electromagnetism.
which proves the Lemma.

We can now apply this lemma to different tensors $A_{\alpha\beta}$ and $B_{\alpha\beta}$. In doing so we will understand why the contractions (59) and (63) are $U^\alpha$ independent, as well as build others.

**Corollary 1:** Take $A_{\alpha\beta} = B_{\alpha\beta} = F_{\alpha\beta}$, the Maxwell tensor. One gets the identity

$$\star F^\alpha\beta F_{\alpha\beta} = \frac{\star F^\alpha\beta F_{\alpha\beta}}{4} \delta^\nu_\mu .$$  \hfill (82)

**Corollary 2:** Take $A_{\alpha\beta} = F_{\alpha\beta}$ and $B_{\alpha\beta} = \star F_{\alpha\beta}$. One gets the identity

$$F^\alpha\beta F_{\alpha\beta} - \star F^\alpha\beta F_{\alpha\beta} = \frac{F^\alpha\beta F_{\alpha\beta}}{2} \delta^\nu_\mu .$$  \hfill (83)

Contracting (82) and (83) with a normalised four velocities $U^\mu$ and $U_\mu$ one finds (59). In particular, one realises that the important point in getting a scalar invariant which is independent on the four velocity is the antisymmetric structure of the tensors $A_{\alpha\beta}$ and $B_{\alpha\beta}$ as well as the normalisation of the four velocity. Likewise we can understand the construction of the invariants (63), but now one uses a two step process:

**Corollary 3a:** Take $A_{\alpha\beta} = C_{\alpha\beta\gamma\eta}$ and $B_{\alpha\beta} = \tilde{C}_{\alpha\beta\gamma\eta}$, the Weyl tensor. Contracting $\gamma$ with $\tilde{\gamma}$ and $\eta$ with $\tilde{\eta}$ one gets the identity

$$\star C^\alpha\beta\gamma\eta C_{\alpha\beta\gamma\eta} = \frac{\star C^\alpha\beta\gamma\eta C_{\alpha\beta\gamma\eta}}{4} \delta^\nu_\mu .$$  \hfill (84)

**Corollary 3b:** Take $A_{\alpha\beta} = C_{\alpha\beta\gamma\eta}$ and $B_{\alpha\beta} = \tilde{C}_{\alpha\beta\gamma\eta}$. Contracting $\gamma$ with $\tilde{\gamma}$ (but not $\eta$ with $\tilde{\eta}$) one gets the identity

$$\star C^\alpha\beta\gamma\eta C_{\alpha\beta\gamma\eta} + \star C^\alpha\beta\gamma\eta C_{\alpha\beta\gamma\eta} = \frac{\star C^\alpha\beta\gamma\eta C_{\alpha\beta\gamma\eta} + \star C^\alpha\beta\gamma\eta C_{\alpha\beta\gamma\eta}}{2} \delta^\nu_\mu .$$  \hfill (85)

Using (84), one obtains the identity

$$\star C^\alpha\beta\gamma\eta C_{\alpha\beta\gamma\eta} - \star C^\alpha\beta\gamma\eta C_{\alpha\beta\gamma\eta} = \frac{\star C^\alpha\beta\gamma\eta C_{\alpha\beta\gamma\eta}}{2} \delta^\nu_\mu .$$  \hfill (86)

**Corollary 4a:** Take $A_{\alpha\beta} = C_{\alpha\beta\gamma\eta}$ and $B_{\alpha\beta} = \star C_{\alpha\beta\gamma\eta}$. Contracting $\gamma$ with $\tilde{\gamma}$ and $\eta$ with $\tilde{\eta}$ one gets the identity

$$C^\alpha\beta\gamma\eta C_{\alpha\beta\gamma\eta} - \star C^\alpha\beta\gamma\eta (\star C)_{\alpha\beta\gamma\eta} = \frac{C^\alpha\beta\gamma\eta C_{\alpha\beta\gamma\eta}}{2} \delta^\nu_\mu .$$  \hfill (87)

**Corollary 4b:** Take $A_{\alpha\beta} = C_{\alpha\beta\gamma\eta}$ and $B_{\alpha\beta} = \star C_{\alpha\beta\gamma\eta}$. Contracting $\gamma$ with $\tilde{\gamma}$ (but not $\eta$ with $\tilde{\eta}$) one gets the identity

$$\star C^\alpha\beta\gamma\eta C_{\alpha\beta\gamma\eta} - \star C^\alpha\beta\gamma\eta C_{\alpha\beta\gamma\eta} = \frac{C^\alpha\beta\gamma\eta C_{\alpha\beta\gamma\eta}}{8} \delta^\nu_\mu \delta^\eta_\eta .$$  \hfill (88)
The invariants (63) follow, now, by contracting (86) and (89) with normalised four velocities $U^\mu$, $U^\nu$, $U^\eta$ and $\tilde{U}^\eta$. Again, the important underlying structure is the antisymmetry and the normalisation of the four velocities.

Having understood the structure in the construction of the above invariants, we can in an obvious fashion build another two invariants:

**Corollary 5:** Take $A_{\alpha\beta} = F_{\alpha\beta;\eta}$ and $B_{\alpha\beta} = F_{\alpha\beta;\tilde{\eta}}$. Contracting $\eta$ with $\tilde{\eta}$ one gets the identity

$$\star F_{\sigma\nu;\eta} F_{\sigma\mu;\eta} = \frac{\star F_{\sigma\nu;\eta} F_{\sigma\mu;\eta}}{4} \delta_{\mu}.$$

**Corollary 6:** Take $A_{\alpha\beta} = F_{\alpha\beta;\eta}$ and $B_{\alpha\beta} = \star F_{\alpha\beta;\tilde{\eta}}$. Contracting $\eta$ with $\tilde{\eta}$ one gets the identity

$$F_{\sigma\nu;\eta} F_{\sigma\mu;\eta} = \frac{F_{\sigma\nu;\eta} F_{\sigma\mu;\eta}}{2} \delta_{\mu}.$$

Contracting with the normalised four velocities $U^\mu$, $U^\nu$ we find the invariants:

- From corollary 5 and 6:

\[
L \equiv E_{\alpha\beta}^\alpha - B_{\alpha\beta} B^{\alpha\beta} = - \frac{F_{\sigma\tau;\eta} F_{\sigma\mu;\eta}}{2}, \quad M \equiv E_{\alpha\beta}^\alpha B^{\alpha\beta} = - \frac{\star F_{\sigma\tau;\eta} F_{\sigma\mu;\eta}}{4};
\]

In the same way that the other invariants were:

- From corollary 1 and 2 the invariants based on the electromagnetic fields:

\[
L_F \equiv E_\mu E^\mu - B_\mu B^\mu = - \frac{F_{\sigma\tau} F_{\sigma\tau}}{2}, \quad M_F \equiv E_\mu B^\mu = - \frac{\star F_{\sigma\tau} F_{\sigma\tau}}{4};
\]

- From corollary 3 and 4:

\[
\mathcal{L} \equiv \mathcal{E}_{\alpha\beta} \mathcal{E}^{\alpha\beta} - \mathcal{H}_{\alpha\beta} \mathcal{H}^{\alpha\beta} = \frac{C^{\tau\gamma\eta} C_{\sigma\tau\gamma\eta}}{8}, \quad \mathcal{M} \equiv \mathcal{E}_{\alpha\beta} \mathcal{H}^{\alpha\beta} = \frac{\star C^{\tau\gamma\eta} C_{\sigma\tau\gamma\eta}}{16}.
\]

All these quantities are independent of the observer $O$ whose four velocity is $U^a$ despite the fact that the definition of electric and magnetic tidal tensors (3)-(4), electric and magnetic parts of the Maxwell tensor (57) and electric and magnetic parts of the Weyl tensor (11) depend on $O$.

### 5.1 Invariants for the gravito-electromagnetic analogy based on tidal tensors

The previous section established that the observer independence of the invariants (93) and (94) stems from the underlying tensor structure rather than from some physical similarity between \{$E_\alpha$, $B_\alpha$\} and \{$\mathcal{E}_{\alpha\beta}$, $\mathcal{H}_{\alpha\beta}$\}; in fact, the example 5.1.1 below, shows that they do not match.

A natural question is if it is possible to define, in electromagnetism, invariants physically analogue to $L$ and $M$. In vacuum, $\mathcal{E}_{\alpha\beta}$ and $\mathcal{H}_{\alpha\beta}$ are the gravitational tidal tensors. Since we have shown in (92) that the electromagnetic tidal tensors also define the invariants $L$ and $M$, then these should be the physical analogues of the invariants $L$ and $M$. The example 5.1.1 supports this claim. Thus, we establish the analogy:

\[
L \equiv E^{\alpha\beta} E_{\alpha\beta} - E^{\alpha\beta} H_{\alpha\beta} \leftrightarrow \mathcal{L} \equiv \mathcal{E}^{\alpha\beta} \mathcal{E}_{\alpha\beta} - \mathcal{H}^{\alpha\beta} \mathcal{H}_{\alpha\beta},
\]

\[
M \equiv E^{\alpha\beta} H_{\alpha\beta} \leftrightarrow \mathcal{M} \equiv \mathcal{E}^{\alpha\beta} \mathcal{H}_{\alpha\beta}.
\]
which holds in vacuum. One then might guess that, in the presence of sources, replacing \{E_{\alpha\beta}, H_{\alpha\beta}\} by \{E_{\alpha\beta}, H_{\alpha\beta}\} would lead to a natural extension of the analogy with the electromagnetic invariants \(L\) and \(M\). This turns out not to be the case: in gravity it is not possible, except in vacuum, to construct scalar invariants using only the electric and magnetic tidal tensors. As already discussed in section 4.1, equations (9)-(10) show that the sources endow \(E_{\alpha\beta}\) with a trace and \(H_{\alpha\beta}\) with an anti-symmetric part; thus these tensors combined possess 6+8 independent components which is insufficient to encode all the information in the Riemann tensor (20 independent components in four dimensions). A third spatial, symmetric tensor, defined by (see [18] and [43] pp. 360-361):

\[
F_{\alpha\gamma} \equiv \star R \star_{\alpha\beta\gamma\delta} U_{\beta} U_{\delta},
\]

(95)

where Hodge duality is taken both with respect to the first and second pair of indices, is needed to account for the remaining 6 components. It is then straightforward to show that the invariants formed by the Riemann tensor are:

\[
\mathcal{L} = \frac{R_{\sigma\tau\gamma\eta} R_{\sigma\tau\gamma\eta}}{8} = \frac{E^{\alpha\gamma} E_{\alpha\gamma} + F^{\alpha\gamma} F_{\alpha\gamma}}{2} - \frac{H^{\alpha\gamma} H_{\alpha\gamma}}{2}, \quad \mathcal{M} = \frac{R_{\sigma\tau\gamma\eta} R_{\sigma\tau\gamma\eta}}{16} = \frac{E^{\alpha\gamma} H_{\alpha\gamma} - F^{\alpha\gamma} H_{\alpha\gamma}}{2}.
\]

(96)

Noting that:

\[
E^{\alpha\gamma} H_{\alpha\gamma} = E^{\alpha\gamma} H_{\alpha\gamma} - \frac{1}{2} H^{\alpha\gamma} R_{\alpha\gamma}, \quad F^{\alpha\gamma} H_{\alpha\gamma} = -E^{\alpha\gamma} H_{\alpha\gamma} - \frac{1}{2} H^{\alpha\gamma} R_{\alpha\gamma}.
\]

it follows that \(\mathcal{M} = M\) generically (i.e. not only in vacuum). Note also that in vacuum:

\[
R_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} = -\star C_{\alpha\beta\gamma\delta} \Rightarrow F_{\alpha\gamma} = -E_{\alpha\gamma} = -E_{\alpha\gamma}
\]

so that (96) reduces to (94), as expected.

It is a curious fact that the most relevant energy conditions can be expressed in a simple fashion in terms of the three tensors in which the Riemann tensor is generically decomposed (\(E_{\alpha\beta}, H_{\alpha\beta}, F_{\alpha\beta}\)). Indeed we have seen from (12i) that \(E^{\alpha}_{\alpha} > 0\) is the strong energy condition. A simple computation reveals that \(F^{\alpha}_{\alpha} = 8\pi T_{\mu\nu} U_{\mu} U_{\nu}\), which shows that \(F^{\alpha}_{\alpha} > 0\) is the weak energy condition. Finally, the dominant energy condition is the statement that the vector

\[
J_{\nu} = \frac{1}{8\pi} \left( H_{\alpha\beta} U_{\mu} e^{\alpha\beta\mu\nu} + F^{\alpha}_{\alpha} U_{\nu} \right)
\]

is timelike and future directed, where we have used (12v).

5.1.1 A simple example

Having exhibited all the above invariants we now return to the example of section 2.2.1. In the electromagnetic case, we can compute both the usual relativistic invariants

\[
L_F = \frac{q^2}{r^4} - \frac{\mu^2 (5 + 3 \cos 2\theta)}{2r^6}, \quad M_F = \frac{2\mu q \cos \theta}{r^5},
\]

(97)

as well as the invariants based on the tidal tensors

\[
L = \frac{6q^2}{r^6} - \frac{\mu^2 (6 + 3 \cos 2\theta)}{r^8}, \quad M = \frac{18\mu q \cos \theta}{r^7}.
\]

(98)
On the gravitational side, since there are no sources, we have the asymptotic expressions:

\[ L = \mathcal{L} \approx \frac{6m^2}{r^6}, \quad M = \mathcal{M} \approx \frac{18Jm\cos \theta}{r^7}. \]  

(99)

Identifying the black hole mass \( m \) and angular momentum \( J \) with, respectively, the electromagnetic charge \( q \) and the magnetic dipole moment \( \mu \), the asymptotic matching between \( L \) and \( \mathcal{L} \) (not \( L_F \)), and between \( M \) and \( \mathcal{M} \) (not \( M_F \)), is perfect. It is a curious fact that the matching of the invariants is exact between Schwarzschild and a point charge; the non-vanishing invariants are

\[ L = \frac{6q^2}{r^6}, \quad \mathcal{L} = \frac{6m^2}{r^6}. \]  

(100)

By contrast \( L_F = q^2/r^4 \). Thus, it is clear from this example that the physical analogues of the gravitational invariants \( \{ \mathcal{L}, \mathcal{M} \} \) (in vacuum) are the invariants \( \{ L, M \} \) built from tidal tensors, and not the invariants \( \{ L_F, M_F \} \) built from electromagnetic fields.

6 Discussion

In this paper we have proposed a tensorial description of the physical gravitoelectromagnetism (GEM) - a new analogy between general relativity and electromagnetism, based on tidal tensors. Physical gravitoelectromagnetism has been studied in literature in the framework of linearised gravity (section 3), where the analogy \( \{ \vec{E}, \vec{B} \} \leftrightarrow \{ \vec{E}_G, \vec{B}_G \} \) is drawn. However, this analogy is non-covariant, and valid only for weak static fields and stressless sources. There is another well known analogy between gravity and electromagnetism: \( \{ E^\alpha, B^\alpha \} \leftrightarrow \{ E^{\mu\nu}, H^{\mu\nu} \} \), which has also been dubbed gravitoelectromagnetism (section 4); this analogy is covariant and general, but (as we have discussed in section 4.3) it is purely formal. Hence, there is a void in literature concerning a general, physical, gravito-electromagnetic analogy. Our proposal, the analogy \( \{ E^{\alpha\beta}, B^{\alpha\beta} \} \leftrightarrow \{ E^{\alpha\beta}, H^{\alpha\beta} \} \), fills that void, since it is a physical analogy, and relies on covariant, universal equations.

The analogy based in tidal tensors naturally embodies all the correct predictions from the linearised theory approach, but, since it is exact and general, allows for an understanding of gravitomagnetism beyond the limits of the latter. The Som-Raychaudhuri/Van-Stockum and Gödel solutions are examples of universes which are beyond the scope of the linear approach, and that we were able to study under our approach (sections 2.3.1 and 2.3.2). And even in the linear limit, if the configurations are time dependent, one must use the analogy based on tidal tensors, and not the analogy \( \{ \vec{E}, \vec{B} \} \leftrightarrow \{ \vec{E}_G, \vec{B}_G \} \), to obtain correct results (see section 3.2).

We have shown that the Maxwell equations can be regarded as tidal tensor equations, and, in this framework, we have derived their gravitational analogues (section 2.1). It follows that the (non-covariant) equations derived form the analogy \( \{ \vec{E}, \vec{B} \} \leftrightarrow \{ \vec{E}_G, \vec{B}_G \} \) turn out to be a special case of our exact equations in the regime of static, weak fields, and stressless sources - which is, therefore, the regime of validity of the latter approach. This regime of validity is hence revealed in an unambiguous way, thus shedding light into an ongoing debate (cf. section 3).

In the framework of this analogy, suggestive similarities between gravity and electromagnetism have been revealed:

- For stationary configurations (in the observer rest frame), the gravitational and electromagnetic tidal tensors obey strikingly similar equations (section 2.1.3);
• In vacuum, the gravitational tidal tensors form invariants in the same way the electromagnetic ones do (section 5.1).

• As an illustration of the previous points, the gravitational tidal tensors of a Kerr black hole asymptotically match the electromagnetic tidal tensors from a spinning charge (identifying the appropriate parameters, cf. section 2.2.1). And there is also a matching between observer independent quantities: the invariants built on those tidal tensors (section 5.1.1).

• The Klein Gordon equation reduces, in ultra-stationary spacetimes, to the non-relativistic Schrödinger equation for a particle subject to a certain magnetic field, living in a curved three-space; the tidal tensor of that magnetic field turns out to match exactly (up to the usual factor of 2) the magnetic part of the Riemann tensor (section 2.3).

But our approach also unveiled deep differences between the two interactions. These differences are revealed, in a very clear fashion, by the symmetries of the tidal tensors. Unlike its electromagnetic analogue, the gravitational electric tidal tensor is always symmetric; in vacuum, the same applies to the gravitational magnetic tidal tensor. This means that in gravity there can be no electromagnetic-like induction effects: the “gravito-electromagnetic” induced fields which have been predicted in the literature (see section 3.2), and whose detection was recently experimentally attempted, do not exist.

One should not be surprised: the fact that in generic dynamics gravity cannot be described by electromagnetic like fields, just reminds us that since gravity is pure geometry, such fields or forces have no place in it. This does not preclude, of course, a unification of these two interactions, as in Kaluza-Klein theory; but it gives us a strong hint that a possible geometrisation of electromagnetism (at least in four dimensions) would have to take a very different character from the geometrisation of gravity.

Nevertheless, despite the intrinsic differences between the two interactions, a physical gravito-electromagnetic analogy is still of great value for the understanding of both theories. The non-geodesic motion of a gyroscope is an example of an effect which we showed in section 2.4 that can not only be easily understood, but also exactly described, in an analogy with the more familiar electromagnetic force exerted on a magnetic dipole. The latter is generated by non-homogeneities in the electromagnetic field - it is therefore a purely tidal effect, and hence the most obvious application for our tidal tensor based analogy. Indeed, a simple application of our analogy leads to the Papapetrou equation for the force applied on a gyroscope. There have been previous attempts to describe this force in an analogy with electromagnetism; a first order estimate has been derived in the framework of the standard linearised theory approach to GEM (cf. section 3.1); that expression, however, is valid only when the gyroscope is at rest in a static, weak gravitational field (by contrast, our result is exact, thus valid for fields arbitrary large, and with an arbitrary time dependence) and therefore not suited to describe motion. Moreover that force accounts only for the coupling between the intrinsic spin of the source and the spin of the gyroscope (that is why the force on a gyroscope, and the violation of the weak equivalence principle, are often referred in the literature as arising from a spin-spin interaction) hiding the fact that the gyroscope will indeed deviate from geodesic motion even in the absence of rotating sources; for example, in the Schwarzschild spacetime. The underlying reason is readily understood in the framework of the approach based on tidal tensors. We derived two fundamental results, \[(41)\] and \[(40)\], which have a very important physical interpretation:
it is the magnetic tidal tensor as seen by the dipole/gyroscope what completely determines the force exerted upon it. Therefore, the gyroscope deviates from geodesic motion in Schwarzschild spacetime by the very same reason that a magnetic dipole will suffer a Stern-Gerlach type deviation even in the coulomb field of a point charge: in its rest frame, there is a non-vanishing magnetic tidal tensor.

Another important physical content unveiled by the analogy based on tidal tensors (and which is, again, lost in the three-dimensional expression derived the linear theory approach) concerns the temporal component of these forces. In the dipole rest frame, the time component of (40) is the power transferred to the dipole by Faraday’s induction, and the fact that it is zero in the gravitational case (41), may be regarded as another evidence for the absence of electromagnetic-like induction effects in gravity.

Both this example of the force exerted on a dipole/gyroscope, and the worldline deviation equations (1)-(2) exhibit one of the strongest aspects of our analogy: the electromagnetic and gravitational tidal tensors always play analogous roles in dynamics, despite being in general very different (they do not even exhibit generically the same symmetries).

There are many other effects which can be easily assimilated with the help of a physical gravito-electromagnetic analogy (indeed, the variety is far too wide to be herein discussed at length). Amongst the best known ones are the dragging of inertial frames and gyroscope precession in the vicinity of rotating bodies, by analogy with Larmor orbits of charged particles and precession of magnetic dipole in magnetic fields (section 2.2). Another example, explored in this paper, is the Gödel universe (section 2.3.2). It is commonly found in the literature that the Gödel universe should be interpreted as a rotating non-expanding universe. Since it is homogeneous one is inevitably led to the conclusion that it must be rotating around every point! Herein, we have suggested that a more insightful interpretation is as follows. The magnetic gravitational tidal tensor is zero for the Gödel Universe. Thus, one may think on the Gödel Universe as a gravitational version of a constant magnetic field (in a curved space). This gives, for the rotation of test particles around any point, a more intuitive picture than the aforementioned one. As a final example consider the similarity between the Coulomb and the Newtonian gravitational potential and also the analogy between the Biot-Savart law and the force between two mass currents. Note that in both cases the electromagnetic force (for charges of equal sign) has the opposite sign to its gravitational analog. Thus, two parallel mass currents will attract one another if they move with opposite velocity. This builds a physical intuition to explain why, in rotating black holes, test particles with counter rotating angular momentum can generically reach closer to the black hole (see [40] for an explicit discussion of this point).

The approach proposed herein clarifies several issues concerning the gravito-electromagnetic approaches found in the literature. As mentioned above, it reveals in an unambiguous way the range of validity of the analogy $\{\vec{E}, \vec{B}\} \leftrightarrow \{\vec{E}_G, \vec{B}_G\}$, for which there is no consensus in the literature. It also sheds light on some conceptual difficulties concerning the analogy $\{E^\alpha, B^\alpha\} \leftrightarrow \{\mathcal{E}^{\mu\nu}, \mathcal{H}^{\mu\nu}\}$, by giving simple answers to three longstanding questions debated in the literature (see section 4.3), namely: 1) the source of the magnetic part of the Weyl tensor, 2) its vanishing in homogeneous rotating universes and 3) its Newtonian limit: 1) it is a tidal magnetic tensor; thus, motion (of masses or transfer of momentum) is, generically, its source, 2) it vanishes in such spacetimes because they are analogous to uniform magnetic fields and 3) it has no place in Newtonian gravity by the same reason that $B_{\alpha\beta}$ has no place in electrostatics, i.e, both limits are obtained neglecting all
relativistic effects, which includes magnetism and (of course) the magnetic tidal tensors. In the framework of our approach, it is obvious that the analogy \( \{ E^\alpha, B^\alpha \} \leftrightarrow \{ \mathcal{E}^{\mu\nu}, \mathcal{H}^{\mu\nu} \} \) is purely formal, since it compares electromagnetic fields to gravitational tidal tensors. In section 5 we dissected the common underlying tensorial structure which allows \( \{ \mathcal{E}^{\mu\nu}, \mathcal{H}^{\mu\nu} \} \) and \( \{ E^\alpha, B^\alpha \} \) to form observer independent scalars in a formally similar way, and showed that indeed, by the same mathematical reasons, it is also possible to construct formally similar invariants from the electromagnetic tidal tensors \( \{ E^{\mu\nu}, B^{\mu\nu} \} \); these invariants are (in vacuum) the physical analogues of the invariants formed by the electric and magnetic parts of the Weyl tensor \( \{ \mathcal{E}^{\mu\nu}, \mathcal{H}^{\mu\nu} \} \).

The relation between the two analogies found in literature is now clear. Take the example of the Heisenberg spacetime (section 2.3.1). According to the linearised theory approach, it has a uniform gravito-magnetic field \( \vec{B}_G \) and a vanishing gravito-electric field \( \vec{E}_G \); however, the analogy \( \{ E^\alpha, B^\alpha \} \leftrightarrow \{ \mathcal{E}^{\mu\nu}, \mathcal{H}^{\mu\nu} \} \) apparently leads to an opposite conclusion: the magnetic part of the Weyl tensor, which is therein taken to be the gravitational analogue of the magnetic field, vanishes, while its electric counterpart is non-zero. But there is indeed no contradiction. Firstly, the former is a physical analogy, while the latter is purely formal. Then, the two approaches refer (on the gravitational side) to different things: the first, to (fictitious) gravitational fields; the second to tidal tensors, which are one order higher in differentiation of the metric potentials than the former. Then, the vanishing of \( \mathcal{H}_{\alpha\beta} \) in the second approach is in accordance with the fact that the field is uniform in the first approach. \( \mathcal{E}_{\alpha\beta} \) does not vanish, unlike one might expect from the vanishing of \( \vec{E}_G \) in the linear theory approach; again there is no contradiction: it is simply due to the fact that unlike its magnetic counterpart (in ultra-stationary spacetimes), the electric tidal tensor is not linear in the metric potentials.

Our approach has also achieved a unification within gravito-electromagnetism. Indeed, the analogy based on linearised theory (within its range of validity) was seen to originate from the same fundamental principle as the (exact) connection between ultra-stationary spacetimes and magnetic fields in some curved manifolds: the similarity between tidal tensors.

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**A Worldline deviation**

**A.1 Gravity (geodesic deviation)**

Consider two particles on infinitesimally close geodesics \( x^\alpha(\tau) \) and \( x^\alpha_2(\tau) = x^\alpha(\tau) + \delta x^\alpha(\tau) \). The geodesic equation \( DU^\alpha/D\tau = 0 \) yields, for the first particle:

\[
0 = \frac{d^2 x^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\beta}(x)U^\mu U^\beta
\]  

(101)
while for the second particle, by making a first order Taylor expansion \( \Gamma_{\mu\beta}^\alpha(x + \delta x) \approx \Gamma_{\mu\beta}^\alpha(x) + \Gamma_{\mu\beta,\sigma}^\alpha(x) \delta x^\sigma \), we obtain:

\[
0 = \frac{d^2 x^\alpha}{d\tau^2} + \frac{d^2 \delta x^\alpha}{d\tau^2} + \left[ \Gamma_{\mu\beta}^\alpha(x) + \Gamma_{\mu\beta,\sigma}^\alpha(x) \delta x^\sigma \right] \left[ U^\mu U^\beta + 2U^\mu \frac{d\delta x^\beta}{d\tau} + \frac{d\delta x^\mu}{d\tau} \frac{d\delta x^\beta}{d\tau} \right]
\] (102)

subtracting (101) to (102):

\[
-\frac{d^2 \delta x^\alpha}{d\tau^2} = \Gamma_{\mu\beta}^\alpha(x) \left[ 2U^\mu \frac{d\delta x^\beta}{d\tau} + \frac{d\delta x^\mu}{d\tau} \frac{d\delta x^\beta}{d\tau} \right] + \Gamma_{\mu\beta,\sigma}^\alpha(x) \delta x^\sigma \left[ U^\mu U^\beta + 2U^\mu \frac{d\delta x^\beta}{d\tau} + \frac{d\delta x^\mu}{d\tau} \frac{d\delta x^\beta}{d\tau} \right]
\] (103)

The second covariant derivative of the connecting vector along the geodesic curve \( x^\alpha(\tau) \) is:

\[
\frac{D^2 \delta x^\alpha}{D\tau^2} = \frac{d^2 \delta x^\alpha}{d\tau^2} - R_{\mu\beta,\sigma}^\alpha U^\mu U^\sigma \delta x^\beta + \Gamma_{\mu\beta,\sigma}^\alpha(U^\mu U^\beta \delta x^\sigma + 2\Gamma_{\mu\beta}^\alpha \frac{d\delta x^\mu}{d\tau} \frac{d\delta x^\beta}{d\tau})
\] (104)

Although it may not be manifest, this equation is indeed covariant, as shown in [59]. It gives the “relative acceleration” of two neighbouring geodesics with arbitrary tangent vectors. If we consider the geodesics to be (at some instant) parallel, i.e., the two particles to have the same velocity, then \( \frac{d\delta x^\mu}{d\tau} = 0 \) and this equation reduces to the traditional “geodesic deviation equation”:

\[
\frac{D^2 \delta x^\alpha}{D\tau^2} = -R_{\mu\beta,\sigma}^\alpha U^\mu U^\sigma \delta x^\beta
\] (105)

### A.2 Electromagnetism

We will now consider the analogue electromagnetic problem: two particles with the same ratio \( q/m \) placed in an electromagnetic field on Minkowski spacetime, following the infinitesimally close worldlines \( x^\alpha(\tau) \) and \( x_2^\alpha(\tau) = x^\alpha(\tau) + \delta x^\alpha(\tau) \). We choose global Cartesian coordinates to perform this analysis. The equation of motion of the first particle is:

\[
\frac{d^2 x^\alpha}{d\tau^2} = \frac{q}{m} F_{\beta}^\alpha(x) U^\beta
\] (106)

Using a first order Taylor expansion, \( F_{\beta}^\alpha(x + \delta x) \approx F_{\beta}^\alpha(x) + F_{\beta,\sigma}^\alpha(x) \delta x^\sigma \); thus, in this coordinate system, the equation of motion for the second particle is:

\[
\frac{d^2 (x^\alpha + \delta x^\alpha)}{d\tau^2} = \frac{q}{m} \left[ F_{\beta}^\alpha(x) + F_{\beta,\sigma}^\alpha(x) \delta x^\sigma \right] \left( U^\beta + \frac{d\delta x^\beta}{d\tau} \right)
\] (107)

Subtracting (106) to (107) we obtain:

\[
\frac{d^2 \delta x^\alpha}{d\tau^2} = \frac{q}{m} \left[ F_{\beta}^\alpha \delta x^\sigma + F_{\beta,\sigma}^\alpha \delta x^\sigma \frac{d\delta x^\beta}{d\tau} + F_{\beta}^\alpha \frac{d\delta x^\beta}{d\tau} \right]
\] (108)
which is the electromagnetic analogue of (104). Since we are using Cartesian coordinates we can now replace partial derivatives by covariant derivatives, hence obtaining a manifestly covariant equation. It gives the acceleration of the vector connecting two infinitesimally close particles with arbitrary velocities. If the particles have the same velocity, then \( \frac{d\delta x^\mu}{d\tau} = 0 \) and the deviation equation reduces to:

\[
\frac{D^2 \delta x^\alpha}{D\tau^2} = \frac{q}{m} F^\alpha_{\beta\gamma} U^\beta \delta x^\sigma
\]

(109)
in a clear analogy with (105).

References

[1] Edward G. Harris “Analogy between general relativity and electromagnetism for slowly moving particles in weak gravitational fields,” Am J. Phys. 59 (5) (1991) 421

[2] Hans C. Ohanian, Remo Ruffini, “Gravitation and Spacetime,” W.W. Norton & Company, Second Edition (1994)

[3] I. Ciufolini, J. A. Wheeler, “Gravitation and Inertia,” Princeton Series in Physics (1995)

[4] Robert M. Wald, “General Relativity,” The University of Chicago Press (1984)

[5] B. Mashhoon, “Gravitoelectromagnetism”, in: “Reference Frames and Gravitomagnetism”, edited by J.-F. Pascual-Sanchez, L. Floria, A. San Miguel, World Scientific, Singapore (2001), pp. 121 [arXiv:gr-qc/0011014]; B. Mashhoon “Gravitoelectromagnetism: A Brief Review,” [arXiv:gr-qc/0311030].

[6] M. L. Ruggiero and A. Tartaglia, “Gravitomagnetic effects,” Nuovo Cim. 117B (2002) 743 [arXiv:gr-qc/0207065].

[7] A. Tartaglia and M. L. Ruggiero, “Gravitoelectromagnetism versus electromagnetism,” Eur. J. Phys. 25 (2004) 203 [arXiv:gr-qc/0311024].

[8] B. Mashhoon, F. Gronwald and H. I. M. Lichtenegger, “Gravitomagnetism and the Clock Effect,” Lect. Notes Phys. 562 (2001) 83 [arXiv:gr-qc/9912027].

[9] R. L. Forward, “General Relativity for the Experimentalist,” Proceedings of the IRE, 49 (1961) 892

[10] R.L. Forward, “Guidelines to Antigravity,” Am. J. Phys. 31 (1963); R. L. Forward, “Antigravity,” Proc. IRE, 49 (1961) 1442

[11] V. B. Braginsky, C. M. Caves, Kip S. Thorne, “Laboratory experiments to test relativistic gravity,” Phys. Rev. D 15 (1977) 2047

[12] J. -F. Pascual Sánchez, “The harmonic gauge condition in the gravitomagnetic equations”, Il Nuovo Cimento B, 115, p. 725 (2000) [arXiv:gr-qc/0010075]. 0010075;
[13] M. Tajmar, F. Plesescu, K. Marhold, C. J. de Matos, “Experimental detection of the Gravitomagnetic London Moment,” arXiv:gr-qc/0603033 (2006) www.esa.int/specials/GSP/SEM0L6OVGE_0.html

[14] A. Matte, “Sur de nouvelles solutions des equations de la gravitation,” Canadian J. Math. 5 (1953) 1.

[15] L. Bel, “Radiation states and the problem of energy in General Relativity,” Cahiers de Physique, 16 (1962) 59; english translation: Gen. Rel. Grav., 32 (2000) 2047.

[16] G. F. R. Ellis, P. A. Hogan, “The Electromagnetic Analogue of Some Gravitational Perturbations in Cosmology,” Gen. Rel. Grav. 29 (1997) 235.

[17] W. B. Bonnor, “The Electric and Magnetic Weyl tensors,” Class. Quant. Grav. 12 (1995) 499.

[18] C. Cherubini, D. Bini, S. Capozziello, R. Ruffini, “Second Order Scalar Invariants of the Riemann Tensor: Applications to Black Hole Spacetimes,” Int. J. Mod. Phys. D 11 (6) (2002) 827 arXiv:gr-qc/0302095.

[19] Naresh Dadhich, “Electromagnetic duality in general relativity,” Gen. Rel. Grav. 32 (2000) 1009 arXiv:gr-qc/9909067.

[20] W. B. Bonnor, “The magnetic Weyl tensor and the van Stockum solution,” Class. Quant. Grav. 12 (1995) 1483.

[21] C. Lozanovski and C. B. G. McIntosh, “Perfect Fluid Spacetimes with a Purely Magnetic Weyl Tensor,” Gen. Rel. Grav. 31 (1999) 1355.

[22] R. Maartens and B. A. Bassett, “Gravito-electromagnetism,” Class. Quant. Grav. 15 (1998) 705 arXiv:gr-qc/9704059.

[23] C. Lozanovski and M. Aarons (1999) “Irrotational perfect fluid spacetimes with a purely magnetic Weyl tensor” Class. Quantum Grav. 16 (1999) 4075.

[24] G. Fodor, M. Marklund and Z. Perjés, “Axistationary perfect fluids: A tetrad approach,” Class. Quant. Grav. 16 (1999) 453 arXiv:gr-qc/9805017.

[25] G. F. R. Ellis, “General relativity and Cosmology,” Proceedings of the International School of Physics ’Enrico Fermi, Course XLVII, Edited by B. K. Sachs (1971).

[26] S. Matarrese, O. Pantano and D. Saez, “A General relativistic approach to the nonlinear evolution of collisionless matter,” Phys. Rev. D 47 (1993) 1311.

[27] S. Matarrese, O. Pantano and D. Saez, “General relativistic dynamics of irrotational dust: Cosmological implications,” Phys. Rev. Lett. 72 (1994) 320 arXiv:astro-ph/9310036.

[28] M. Bruni, S. Matarrese and O. Pantano, “Dynamics of silent universes,” Astrophys. J. 445 (1995) 958 arXiv:astro-ph/9406068.

[29] W. M. Lesame, G. F. R. Ellis and P. K. S. Dunsby, “Irrotational dust with div H = 0,” Phys. Rev. D 53 (1996) 738 arXiv:gr-qc/9508049.
[30] R. Maartens, W. M. Lesame and G. F. R. Ellis, “Consistency of dust solutions with $\text{div } H = 0$,” Phys. Rev. D 55 (1997) 5219 [arXiv:gr-qc/9703080].

[31] R. Maartens, W. M. Lesame and G. F. R. Ellis, “Newtonian-like and anti-Newtonian universes,” Class. Quant. Grav. 15 (1998) 1005 [arXiv:gr-qc/9802014].

[32] P. K. S. Dunsby, B. A. C. Bassett and G. F. R. Ellis, “Covariant analysis of gravitational waves in a cosmological context,” Class. Quant. Grav. 14 (1997) 1215 [arXiv:gr-qc/9811092].

[33] N. Van den Bergh, “Purely gravito-magnetic vacuum space-times,” Class. Quant. Grav. 20 (2003) L1 [arXiv:gr-qc/0211025].

[34] E. Bertschinger and A. J. S. Hamilton, “Lagrangian evolution of the Weyl tensor,” Astrophys. J. 435 (1994) 1 [arXiv:astro-ph/9403016].

[35] Lam Hui and E. Bertschinger, “Local approximations to the gravitational collapse of cold matter,” Astrophys. J. 471 (1994) 1.

[36] G. F. R. Ellis and P. K. S. Dunsby, “Newtonian evolution of the Weyl tensor,” Astrophys. J. 479 (1997) 97 [arXiv:astro-ph/9410001].

[37] J. J. Ferrando, J. A. Sáez, “Gravito-magnetic vacuum spacetimes: kinematic restrictions,” Class. Quantum Grav. 20 (2003) 2835.

[38] G.W.Gibbons, C. A. R. Herdeiro, Unpublished; C. A. R. Herdeiro, “Aspects of Causality and Flux-branes in superstring theory,” Ph.D. thesis, University of Cambridge, 2001.

[39] N. Drukker, B. Fiol and J. Simon, “Goedel-type universes and the Landau problem,” JCAP 0410 (2004) 012 [arXiv:hep-th/0309199].

[40] G. W. Gibbons and C. A. R. Herdeiro, “Supersymmetric rotating black holes and causality violation,” Class. Quant. Grav. 16 (1999) 3619 [arXiv:hep-th/9906098].

[41] R. T. Jantzen, p. Carnini, D. Bini, “The many faces of Gravitoelectromagnetism,” Ann. Phys. 215 (1992) 1 [arXiv:gr-qc/0106043].

[42] Hans Stephani, “Relativity” Cambridge Univ. Press, third ed. (2004).

[43] Charles W. Misner, Kip. S. Thorne, John A. Wheeler “Gravitation,” W. H Freeman and Company, San Francisco (1973)

[44] J. M. Stewart, “Perturbations of Friedmann-Robertson-Walker,” Class. Quant. Grav. 7 (1990) 1169.

[45] B. Schutz, “Gravity from the ground up,” Cambridge Univ. Press (2003).

[46] “The Feynman Lectures on Physics” Vol. II, Addison-Wesley Publishing Company (1964)

[47] B. Mashhoon, “On the gravitational analogue of Larmor’s theorem,” Phys. Lett. A173 (1993) 347.
[48] K. Lanczos, “Über eine stationäre Kosmologie im Sinne der Einsteinschen Gravitationstheorie,” Z. Phys. 21 (1924) 73.

[49] J. P. Gauntlett, J. B. Gutowski, C. M. Hull, S. Pakis and H. S. Reall, “All supersymmetric solutions of minimal supergravity in five dimensions,” Class. Quant. Grav. 20 (2003) 4587 [arXiv:hep-th/0209114].

[50] Roy P. Kerr, “Gravitational field of a spinning mass as an example of algebraically special metrics,” Phys. Rev. Lett. 11 (1963) 237

[51] W. van Stockum, “The gravitational field of a distribution of particles rotating about an axis of symmetry,” Proc. R. Soc. Edin. 57 (1937) 135.

[52] J. Lense, H. Thirring, “Über den Einflus der Eigenrotation der Zentralkörper auf die Bewegung der Planeten und Monde nach der Einsteinschen Gravitationstheorie,” Physikalishe Zeitschrift, 19 (1918) 156.

B. Mashhoon, F. W. Hehl, D. S. Theiss, “On the gravitational effects of rotating masses: the Thirring-Lense papers”, Gen. Rel. Grav. 16 (1984) 711

[53] K. Gödel, “An example of a new type of cosmological solutions of Einstein’s field equations of gravitation,” Rev. Mod. Phys. 21 (1949) 447.

[54] M. Som, A. Raychaudhuri, “Cylindrically symmetric charged dust distribution in rigid rotation in general relativity,” Proc. Roy. Soc. A304 (1968) 81.

[55] B. Mashhoon and N. O. Santos, “Rotating cylindrical systems and gravitomagnetism,” Annalen Phys. 9 (2000) 49 [arXiv:gr-qc/9807063].

[56] A. Papapetrou, “Spinning test particles in general relativity. I,” Proc. R. Soc. London Ser. A 209 (1951) 248

[57] F.A.E. Pirani, “On the physical significance of the Riemann tensor,” Acta Phys. Pol. 15 (1956) 389

[58] Robert M. Wald, “Gravitational Spin Interaction,” Phys. Rev. D 6 (1972) 406

[59] I. Ciufolini, “Generalized geodesic deviation equation”, Phys. Rev D 34 (1986) 1014

[60] L. Filipe Costa, C. A. R. Herdeiro, in preparation