Equilibrium configurations of nematic liquid crystals on a torus

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The topology and the geometry of a surface play a fundamental role in determining the equilibrium configurations of thin films of liquid crystals. We propose here a theoretical analysis of a recently introduced surface Frank energy, in the case of two-dimensional nematic liquid crystals coating a toroidal particle. Our aim is to show how a different modeling of the effect of extrinsic curvature acts as a selection principle among equilibria of the classical energy, and how new configurations emerge. In particular, our analysis predicts the existence of new stable equilibria with complex windings.

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Due to their special optical properties and their controllability through electric and magnetic fields, liquid crystals have proven to be fundamental in many scientific and technological applications. Their properties have been deeply investigated for over a century and nowadays increasing emphasis is being placed on so-called nematic shells. These are microscopic particles coated by a thin film of liquid crystals, which develop defects with a topological charge, and thus have a tendency to self-assemble into metamaterials which may have new optical properties and a high potential for technological applications (see, e.g., [3–5]). In contrast, there is no universal agreement on the two-dimensional free energy to model nematic shells. Different ways to take into account the distortion effect of the substrate were proposed in [6–8] and recently by Napoli and Vergori ([9, 10]). Indeed, as observed in [11] and [12], the liquid crystal ground state (and all its stable configurations, in general) is the result of the competition between two driving principles: on one hand the minimization of the “curvature of the texture” penalized by the elastic energy, and on the other the frustration due to constraints of geometrical and topological nature, imposed by anchoring the nematic to the surface of the underlying particle. The new energy model ([9, 10]) affects these two aspects, focusing on the effects of the extrinsic geometry of the substrate on the elastic energy of the nematics. With the present Letter we aim at exploring the full consequences of the new model so that a detailed comparison with the classical one can be established. More precisely, we study the two-dimensional Napoli-Vergori director theory for nematic shells on a genus one surface: a) we analyze the dependence of the new energy on the mechanical parameters (splay, twist and bend moduli) and on the geometrical parameters (the radii of the torus); b) we highlight in which cases the new energy acts as a selection principle among the minimizers of the classical one, and in which cases new different states emerge: c) we predict the existence of stable equilibrium states carrying a higher energy than the ground state, in correspondence with the homotopy classes of the torus. Our analysis, in particular, agrees and makes more precise the statement of [10], according to which the new energy “promotes the alignment of the flux lines of the nematic director towards geodesics and/ or lines of curvature of the surface”. The aspect of high energy equilibrium states is present in the classical energy as well, but it was neglected in previous research on genus one surfaces ([8]).

Our observations are based on a rigorous mathematical analysis of the models, which combines methods from differential geometry and topology, calculus of variations, functional analysis and numerical simulations. Topology enters our work, first of all, in the choice of the torus as base substrate. Indeed the nematics would necessarily present defects on any surface with genus different than one, due to Poincaré-Hopf Theorem ([5, 13, 14]); as a consequence, when dealing with the Frank’s director theory, the space of functions in which one looks for minimizers would be empty, even in a weak sense (see [15]), requiring thus further expedients or approximations (see, e.g., [11, 16]). In order to focus on the influence of the geometry, we choose a surface where defectless ground states can be found. The case of a cylinder, the simplest surface where different results between classical and new energy can be predicted, was presented in [10] (on a sphere, the two energies differ by a constant). A related energy on hyperbolic surfaces was studied in [17] and in [18], which described the different effects of the intrinsic and of the extrinsic geometry on defects. Although the experimental generation of toroidal nematics is a challenge, recent techniques ([19]) allow for droplets of genus one or higher, and further motivate investigation of more complex surfaces.

In the classical director theory of nematics, the local orientation of the liquid crystal molecules in a sample \( \Omega \subset \mathbb{R}^3 \) is described by the unit vector field \( \mathbf{n} : \Omega \to S^2 \), where \( S^2 \) is the unit sphere. Stable configurations are minimizers of the classical elastic energy, which according
to Frank’s formula reads
\[
W(n) := \frac{1}{2} \int_{\Omega} \left[ K_1 (\text{div} n)^2 + K_2 (n \cdot \text{curl} n)^2 + K_3 |n \times \text{curl} n|^2 + (K_2 + K_{24}) \text{div} \left[ (\nabla n - (\text{div} n)) n \right] \right] dx,
\]
where \( K_1, K_2, K_3 \) and \( K_{24} \) are positive constants called the splay, twist, bend and splay-splay moduli, respectively. The last term is a null Lagrangian, hence it depends only on the behavior of \( n \) on the boundary. While the Frank’s energy above is, within the director theories, well accepted, there is not such agreement when dealing with surface energies. In the literature, one can find different proposals for such an energy \([6, 8, 11, 10]\). The main difference between the classical energy proposed in \([6, 8, 11]\) and the most recent one \([10]\) essentially lies in the choice of the differential operators on the surface \( S \). More precisely, the energy in \([6, 8, 11]\) is a functional of the covariant derivative \( Dn \) of the vector field \( n \), while the energy in \([10]\) depends on the surface gradient \( \nabla_s n \), which is defined as \( \nabla_s := \nabla n P \) (see, e.g., \([3]\) ), with \( P \) being the orthogonal projection onto the tangent plane of \( S \). In other words, \( \nabla_s \) is the restriction of the usual derivative of \( \mathbb{R}^3 \) to directions lying in the tangent plane and takes thus into account also the extrinsic curvature of \( S \). In order to describe the elastic energy of a thin film, approximated by a surface \( S \), we resort to the Darboux frame \( \{ n, t, \nu \} \), where \( \nu \) is the unit normal vector to \( S \) and \( t := \nu \times n \). Let \( \kappa_t, \kappa_n \) be the geodesic curvatures of the flux lines of \( t \) and \( n \), respectively. Let \( c_n \) be the normal curvature and let \( \tau_n \) be the geodesic torsion (see, e.g., \([20]\) for all the definitions and examples related to differential geometry). The surface divergence acting on the field \( n \) is defined as \( \text{div} n := \text{tr} Dn = \text{tr} \nabla_s n \) and can be expressed as \( \text{div} n = \kappa_s \) \([10]\). The classical form of surface free energy, for a thin film of nematics of constant thickness \( h \) around \( S \), is (\([6, 8, 11]\) )
\[
W_{C3}(n) := \frac{1}{2} \int_S \left[ k_1 (\text{div} n)^2 + k_3 (\text{curl} n)^2 \right] dA,
= \frac{1}{2} \int_S \left[ k_1 \kappa_t^2 + k_3 \kappa_n^2 \right] dA,
\]
where \( k_i = \hbar K_i \) and \( \tau_3 \) is the vector of constant curvature of \( S \) comes to play a role through \( c_3 \) and \( \tau_3 \)
\[
W_{NV}(n) := \frac{1}{2} \int_S \left[ k_1 (\text{div} n)^2 + k_2 (n \cdot \text{curl} n)^2 + k_3 |n \times \text{curl} n|^2 \right] dA,
= \frac{1}{2} \int_S \left[ k_1 \kappa_t^2 + k_2 \kappa_n^2 + k_3 \kappa_n^2 + c_n^2 \kappa_n^2 \right] dA,
\]
where \( \text{curl} n := -\epsilon \nabla_s n \) (\( \epsilon \) is the Ricci alternator). Note that \( \text{curl} n = -\tau_n n + c_n t + \kappa_n \nu \) and that, unless we restrict to flat surfaces, the vector \( \text{curl} n \) has non vanishing in-plane components. Note also that the saddle-splay term is not present in the surface energy \( W_{NV} \) (see \([10]\) for a justification). In order to study the minimizers of \( W_{C3} \) and \( W_{NV} \), it is convenient to introduce a parametrization \( X \) of \( S \). We use \( (\theta, \phi) \) as a set of local coordinates and we assume that \( \mathbf{e}_\theta := \partial_\theta X/\partial_\theta X \) and \( \mathbf{e}_\phi := \partial_\phi X/\partial_\phi X \) give a local orthonormal basis for the tangent plane to \( S \). We can then describe \( n \) through the angle \( \alpha \) defined by \( n = e_\theta \cos \alpha + e_\phi \sin \alpha \). With respect to \( \alpha \), the surface energy \([2]\) takes the form
\[
W_{NV}(\alpha) = \frac{1}{2} \int_S \left[ k_1 ((\nabla_s \alpha - \Omega) \cdot t)^2 + k_3 ((\nabla_s \alpha - \Omega) \cdot n)^2 + k_2 (c_1 - c_2)^2 \sin^2 \alpha \cos^2 \alpha \right. \\
+ k_3 (c_1 \cos^2 \alpha + c_2 \sin^2 \alpha))^2 \right] dA,
\]
where \( c_1 \) and \( c_2 \) are the principal curvatures of \( S \) and \( \Omega \) is the spin connection (see \([22]\) ). The first two terms coincide with \( W_{C3} \).

![FIG. 1. Schematic representation of the torus T.](image)

We now focus on the case of a torus \( T \), with radii \( R > r > 0 \) (see Figure 1), parametrized by
\[
X(\theta, \phi) = \begin{pmatrix} (R + r \cos \theta) \cos \phi \\ (R + r \cos \theta) \sin \phi \\ r \sin \theta \end{pmatrix},
\]
where \( (\theta, \phi) \in Q := [0, 2\pi] \times [0, 2\pi] \). In order to study the dependence of the energy \([2]\) on the mechanical and geometrical parameters, we first restrict to the case of constant angle \( \alpha \). In this case, the integral in \([2]\) can be computed explicitly as a function of five real parameters (see Appendix B):
\[
W_{NV} = W_{NV}(\alpha, 1, k_2, k_3, \mu),
\]
where \( \mu := R/r \). Since \( W_{NV} \) is \( \pi \)-periodic, we restrict to \( \alpha \in [-\pi/2, \pi/2] \). Studying the equilibrium equation associated with \( W_{NV} \), one finds the equilibrium configurations (see Figure 2)
\[
\alpha_0 = 0, \quad \alpha_p = \frac{\pi}{2}, \quad \alpha_h = \pm \frac{1}{2} \arccos \left( \frac{Bk_3 + Ck_1}{\mu^2(k_2 - k_3)} \right),
\]
where
constant term in the energy \( \mu \), the minimality depends also on the ratio \( \frac{R}{r} \). To conclude this part with a remark. Unlike the case of the one-constant approximation of \( W \), in this regard, see Figure 3, where \( W \) is plotted for fixed \( k_3 = 1 \) and different values of \( k_2 \). Studying the second derivative of \( W \), one finds critical values \( \xi_1 = 2\sqrt{\mu^2 - 1}/\mu \) and \( \xi_2 = 2 - \xi_1 \), which characterize the cases: \( \alpha_m \) is local minimum and \( \alpha_p \) is global minimum (\( k_2 > \xi_1 \), continuous line); \( \alpha_m \) is local maximum and \( \alpha_p \) is global minimum (\( \xi_1 \geq k_2 \geq \xi_2 \), between dashed and dashed-dotted lines); \( \pm \alpha_p \) are global minima, \( \alpha_p \) is local maximum, \( \alpha_m \) is global maximum (\( k_2 < \xi_2 \), dotted line).

Let \( \nu \) be the unit normal vector to the torus \( T \); the role of the extrinsic curvature of \( T \) on the free energy can clearly be seen from the orthogonal decomposition \( \nabla_s n = Dn - \nu \otimes (\nabla_s \nu)n \), recalling that \(-\nabla_s \nu \) is the extrinsic curvature tensor (see [20]). The one-constant approximation of \( W \) is then

\[
W_{NV}(n) = \frac{k}{2} \int_T |\nabla_s n|^2\,dA,
\]

or, in terms of the angle \( \alpha \) with the local coordinates,

\[
W_{NV}(\alpha) = f(\mu) + \frac{k}{2} \int_T \left[ |\nabla_s \alpha|^2 + \frac{c_1^2 - c_2^2}{2} \cos(2\alpha) \right] \,dA,
\]

where \( f(\mu) = \kappa\pi^2(2\mu + (2 - \mu^2)/\sqrt{\mu^2 - 1}) \) can be computed using the orthogonality of \( \nabla_s \alpha \) and \( \Omega \). The corresponding Euler-Lagrange equation is

\[
k\Delta_s \alpha + \frac{k}{2}(c_1^2 - c_2^2) \sin(2\alpha) = 0,
\]

where \( \Delta_s = \text{div}_s \nabla_s \) is the Laplace-Beltrami operator on the torus \( T \). Equation (5) is a novel kind of elliptic sine-Gordon equation, the only explicit solutions to which, to our knowledge, are the constants \( \alpha_m = 0 \) and \( \alpha_p = \pi/2 \). We resort thus to studying the gradient flow of \( W_{NV} \), i.e., the solutions \( \alpha = \alpha(x,t) \) defined on \( T \times [0, +\infty) \), to the evolution equation

\[
\partial_t \alpha = k\Delta_s \alpha + \frac{k}{2}(c_1^2 - c_2^2) \sin(2\alpha),
\]

equipped with an initial datum \( \alpha(x,0) = \alpha_0(x) \) on \( T \). We remark that this evolution problem is not a physical
flow of the nematics, but it constitutes an efficient mathematical artifice to approximate solutions of the stationary equation [3]. Indeed, owing to the ellipticity of the Laplace-Beltrami operator and to the regularity of the nonlinear term, for any regular initial datum $\alpha_0$ there exists a unique solution $\alpha(t)$ to [4]. Moreover, by construction, at any time $t > 0$ this solution satisfies the energy balance

$$\int_0^t \int_T |\partial_t \alpha(s)|^2 dA ds + W_{NV}(\alpha(t)) = W_{NV}(\alpha_0),$$

which implies that $t \mapsto W_{NV}(\alpha(t))$ is monotone decreasing. Finally, as $t \to +\infty$, $\alpha(t)$ converges (possibly up to a subsequence) to a function $\alpha_\infty$ which solves [5]. If $\alpha_0$ is a critical point, i.e. $\alpha_0 = \alpha_m$ or $\alpha_0 = \alpha_p$, then the evolution is clearly constant: $\alpha(t) \equiv \alpha_m$, or $\alpha(t) \equiv \alpha_p$, respectively. For different initial data, we find as a limit solution either $\alpha_p$, or a nonconstant $\alpha_\infty$ belonging to the family illustrated in Figure [6]. The only distinguishing factor between these two behaviors is the ratio of radii $R/r$. From the explicit form of $W_{NV}(\alpha)$, we know that if $\mu \geq 2$, then $c_1^2 - c_2^2 \geq 0$ and thus $\alpha_p \equiv \pi/2$ is the unique (up to rotations of $m\pi, m \in \mathbb{Z}$) global minimizer of $W_{NV}$. On the other hand, from the previous discussion on constant $\alpha$, we know that if $\mu < 2/\sqrt{3}$, then $\alpha_p$ cannot be the global minimizer, as for this ratio $W_{NV}(\alpha_m) < W_{NV}(\alpha_p)$. We conjecture that there exists a unique critical ratio $\mu^* \in (2/\sqrt{3}, 2)$ above which $\alpha_p$ is the point of minimum, and below which the nonconstant solution appears. Numerically, we found $\mu^* \sim 1.52$.

It is interesting to compare these configurations with the equilibrium ones of the classical Frank energy [8]. In the one-constant approximation, the energy on a torus is

$$W_{Cl}(\alpha) = \frac{k}{2} \int_T |\nabla_\alpha - \Omega|^2 dA$$

and the corresponding equilibrium equation is $\Delta_\alpha \alpha = 0$. (Note that in [8] $\mu = r/R$.) Therefore, in the classical case, every field $\mathbf{n} = e_\theta \cos \alpha + e_\phi \sin \alpha$, for constant $\alpha$, is an equilibrium state, with the same energy independently of $\bar{\alpha}$. For $\mu > \mu^*$, the new energy $W_{NV}$ selects $\alpha_p$, among all constants, as unique equilibrium configuration. For $\mu < \mu^*$, instead, the new lower-energy configuration shown in Figure [6] appears.

In order to describe more complex equilibrium states, we need to introduce the winding number of the director field $\mathbf{n}$ on the torus. Let $\mathbf{n}$ be given and, referring to [3], let $\alpha : Q \to \mathbb{R}$ be such that

$$\mathbf{n}(X) = e_\theta \cos \alpha + e_\phi \sin \alpha \quad \text{on } Q.$$

Though $\alpha$ needs not be $Q$-periodic, there certainly exist integers $m, n \in \mathbb{Z}$ such that

$$\alpha(2\pi, 0) = \alpha(0, 0) + m\pi, \quad \alpha(0, 2\pi) = \alpha(0, 0) + n\pi.$$

We define the \textit{winding number} of $\pm \mathbf{n}$ as the couple of indices $h = (h_\theta, h_\phi) \in \mathbb{Z} \times \mathbb{Z}$, given by

$$h_\theta := \frac{\alpha(2\pi, 0) - \alpha(0, 0)}{\pi}, \quad h_\phi := \frac{\alpha(0, 2\pi) - \alpha(0, 0)}{\pi}.$$

Note that, by allowing for a difference of an odd multiple of $\pi$, we are taking into account the symmetry $\mathbf{n} = -\mathbf{n}$ of the nematic represented by the vector field $\mathbf{n}$. Geometrically, $h_\theta$ indicates how many turns of $180^\circ$ are completed by $\mathbf{n}$ along the meridian line parametrized by $\theta \mapsto X(\theta, 0)$; similarly, the number of turns along the parallel line $\phi \mapsto X(0, \phi)$ is given by $h_\phi$. A crucial property of the winding number is its invariance under continuous transformations of $\alpha$ (so that $h$ could be equivalently computed on any pair of curves which are homotopically equivalent to the two that we chose). The

![Figure 5](image_url)

**FIG. 5.** (Color online) Plots of the vector field $\mathbf{n}_\infty$, corresponding to the limit solution $\alpha_\infty$ to [6] for $\alpha_0 = \pi/4$ and different ratios $\mu$. (The color code represents $\alpha_\infty$.) For $\mu < 2$, the configuration $\alpha_m$ is preferable in a strip close to the central hole, while for every choice of $\mu$, $\alpha_p$ is more convenient around the external equator. As the term $|\nabla \alpha|^2$ in the energy penalizes the transition from $\alpha_m$ to $\alpha_\infty$, minimizers exhibit a smooth rotation of the director field along the meridians. The amplitude of the rotation increases as $\mu$ decreases to 1.
relevant consequence is the following: for any choice of \( h = (h_\theta, h_\phi) \in \mathbb{Z} \times \mathbb{Z} \), there is an initial datum \( \alpha_0 \) (e.g., \( \alpha_0(\theta, \phi) = h_\theta \theta + h_\phi \phi \)) such that the corresponding vector field \( \pm n \) has winding number \( h \); the evolution of \( \alpha_0 \) according to (6) provides then, in the limit as \( t \to +\infty \), a function \( \alpha_\infty \) whose associated line field \( n_\infty \) is a local minimizer of \( W_{NY} \), and has winding number \( h \). There exists therefore at least one equilibrium configuration for each choice of winding number, and this configuration is stable, in the sense that it is not possible to lower its energy without breaking the pattern on a line, i.e., without creating a line defect. Nonetheless, as the elastic energy increases monotonically with \( |h| \), we expect the existence of an upper bound on the observable winding number of the nematics field, depending on the maximum elastic energy that the specific liquid crystal can undergo before breaking.

In conclusion, we analyzed nematic liquid crystals on a toroidal particle, finding new equilibrium configurations for the surface energy recently proposed by Napoli and Vergori in [9, 10] and comparing them to the equilibria of classical energies. We identified a range of parameters where the new energy selects a unique equilibrium among the classical ones, and we showed the emergence of new equilibria configurations, in accordance with the new penalization of the normal curvature and geodesic torsion of the director field. We hope that experiments could be carried out in order to confirm our analysis.

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**Appendix A. Geometric quantities on the torus**

Let \( Q := [0, 2\pi] \times [0, 2\pi] \subset \mathbb{R}^2 \), and let \( X : Q \to \mathbb{R}^3 \) be the following parametrization of an embedded torus \( T \)

\[
X(\theta, \phi) = \begin{pmatrix}
R + r \cos \theta \\
R + r \cos \theta \sin \phi \\
r \sin \theta
\end{pmatrix},
\]

(7)

Using parametrization (7), in the next paragraph we derive the main geometrical quantities, like tangent and normal vectors, first and second fundamental form, in order to obtain an explicit expression for the metric and the curvatures of \( T \) and for \( \nabla, n \).

Let \( X_\theta := \partial_\theta X, X_\phi := \partial_\phi X, \nu := \frac{X_\theta \times X_\phi}{|X_\theta \times X_\phi|} \), we have

\[
X_\theta = \begin{pmatrix}
-R \sin \theta \cos \phi \\
-R \sin \theta \sin \phi \\
r \cos \theta
\end{pmatrix}, \quad X_\phi = \begin{pmatrix}
-(R + r \cos \theta) \sin \phi \\
(R + r \cos \theta) \cos \phi \\
0
\end{pmatrix},
\]
\[ X_{\theta \theta} = \begin{pmatrix} -r \cos \theta \cos \phi \\ -r \cos \theta \sin \phi \\ -r \sin \theta \end{pmatrix}, \quad X_{\theta \phi} = \begin{pmatrix} r \sin \theta \sin \phi \\ r \sin \theta \cos \phi \\ 0 \end{pmatrix}, \quad X_{\phi \phi} = \begin{pmatrix} -(R + r \cos \theta) \cos \phi \\ -(R + r \cos \theta) \sin \phi \\ 0 \end{pmatrix}. \]

The unit tangent vectors are

\[ e_{\theta} := \frac{X_{\theta}}{|X_{\theta}|} = -\begin{pmatrix} \sin \theta \cos \phi \\ \sin \theta \sin \phi \\ \cos \theta \end{pmatrix}, \quad e_{\phi} := \frac{X_{\phi}}{|X_{\phi}|} = -\begin{pmatrix} -\sin \phi \\ \cos \phi \\ 0 \end{pmatrix}. \]

The geodesic curvatures \( \kappa_1 \) and \( \kappa_2 \) of the principal lines of curvature can thus be obtained by

\[ \kappa_1 = e_{\phi}(\nabla e_{\theta})e_{\theta} = \frac{1}{r} \frac{r \phi X_{\theta \phi} - X_{\theta \theta}}{r^2}, \quad \kappa_2 = e_{\phi}(\nabla e_{\theta})e_{\phi} = \frac{X_{\phi \phi} \cdot X_{\theta \theta}}{r^2} = -\frac{\sin \theta}{R + r \cos \theta}. \]

The spin connection \( \Omega \) is given by

\[ \Omega^1 = (e_{\theta}, D_{e_{\theta}}e_{\phi})_{R^3} = -\kappa_1 = 0, \]

\[ \Omega^2 = (e_{\theta}, D_{e_{\phi}}e_{\phi})_{R^3} = -\kappa_2 = \frac{\sin \theta}{R + r \cos \theta}. \]

The explicit forms of the surface differential operators on the torus are

\[ \nabla_{\alpha} n = g^{ij} \partial_i \alpha e_j + \frac{\partial \alpha}{r} e_j, \quad \Delta_{\alpha} = \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} \partial_j), \]

\[ = \frac{1}{r^2} \partial_{\theta \theta}^2 \frac{\sin \theta}{\cos \theta} \partial_\theta + \frac{1}{(R + r \cos \theta)^2} \partial_{\phi \phi}^2. \]

For \( n = \cos \alpha e_\theta + \sin \alpha e_\phi \), the explicit expression of the surface gradient \( \nabla_{\alpha} n \) in terms of the deviation angle \( \alpha \), with respect to the Darboux frame \( (n, t, \nu) \) is

\[ \int_Q c_1 c_2 dA = 0. \]

Ordering the terms according to the frequency in \( \alpha \), we get

\[ W(\alpha) = \frac{k_1 + k_3}{4} I_1 + \frac{k_2 + k_3}{8} (I_2 + I_3) \]

\[ + \cos(2\alpha) \left[ \frac{k_1 - k_3}{4} I_1 + \frac{k_3}{4} (I_2 - I_3) \right] \]

\[ + \cos^2(2\alpha) \left[ \frac{k_3 - k_2}{8} (I_2 + I_3) \right]. \]

and substituting the values of \( I_i \) yields \( \Omega \). We also note that the recurring value of \( (I_2 + I_3)/4 \) corresponds to

**Appendix B. Derivation of \[I \]**

The proof relies on algebraic manipulations and integration of trigonometric functions. Let \( \mu := R/r \), substituting the expressions for \( c_1, c_2, k_2, \sqrt{g} \) derived in Appendix A, we have

\[ I_1 := \int_Q (\kappa_2)^2 dA = \int_0^{2\pi} \int_0^{2\pi} \sin^2 \theta \frac{d\theta d\phi}{\mu + \cos \theta} = 4\pi^2 \left( \mu - \sqrt{\mu^2 - 1} \right), \]

\[ I_2 := \int_Q (c_1)^2 dA = \int_0^{2\pi} \int_0^{2\pi} \{ \mu + \cos \theta \} d\theta d\phi = 4\pi^2 \mu, \]

\[ I_3 := \int_Q (c_2)^2 dA = \int_0^{2\pi} \int_0^{2\pi} \cos^2 \theta \frac{d\theta d\phi}{\mu + \cos \theta} = 4\pi^2 \mu \left( \frac{\mu}{\sqrt{\mu^2 - 1}} - 1 \right), \]
Willmore’s functional on a torus, i.e.
\[ W(T) := \int_T H^2 \, dS = \int_0^{2\pi} \int_0^{2\pi} \left( \frac{c_1 + c_2}{2} \right)^2 \, dS = \pi^2 \left( \frac{\mu^2}{\sqrt{\mu^2 - 1}} \right). \]

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