NUMERICAL APPROXIMATION OF REGULARIZED NON–CONVEX ELLIPTIC OPTIMAL CONTROL PROBLEMS BY THE FINITE ELEMENT METHOD

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Abstract. We investigate the numerical approximation of an elliptic optimal control problem which involves a nonconvex local regularization of the $L^q$-quasinorm penalization (with $q \in (0,1)$) in the cost function. Our approach is based on the difference–of–convex function formulation, which leads to first-order necessary optimality conditions, which can be regarded as the optimality system of an auxiliar convex $L^1$–penalized optimal control problem. We consider piecewise–constant finite element approximation for the controls, whereas the state equation is approximated using piecewise–linear basis functions. Then, convergence results are obtained for the proposed approximation. Under certain conditions on the support’s boundary of the optimal control, we deduce an order of $h^{\frac{1}{2}}$ approximation rate of convergence where $h$ is the associated discretization parameter. We illustrate our theoretical findings with numerical experiments that show the convergence behavior of the numerical approximation.

1. Introduction

Optimal control problems with nonconvex $L^q$-quasinorms with $q \in (0,1)$ can be regarded as an approximation of the so-called $L^0$ sparse penalized optimal control problems, see [23] and [28]. This kind of penalization induces attractive sparsification effects on the control, which might be useful in applications requiring a localized action of the control within the domain e.g., inverse sparse reconstruction problems.

Optimal controls induced by $L^q$-quasinorm penalizers share similarities with those promoted by $L^1$-norm. It is well known that these controls tend to have small supports, depending on the sparse regularization cost, see [32]. However, a simple but essential difference in considering $L^q$-quasinorms is that optimal controls might jump at the boundary of its support. This feature is potentially advantageous for specific applications. In particular, in those applications where sparse solutions are crucial in achieving the desired state, see [12]. Other examples arise in image-processing-related applications where jumps can be used to achieve certain graphical features.

One recognizable difficulty in considering the $L^q$-quasinorm in the cost function is the lack of convexity of the penalizer. Therefore, $L^p$-spaces ($p > 1$) are not well suited for this class of problems. In [24], a penalization in the gradient is added to the cost, which allows using compactness arguments to guarantee the existence of solutions in $H^1$. However, the gradient cost term eliminates the jumps of the solutions, which is an essential feature of the quasinorm penalization. Therefore, a more convenient space to promote piecewise
smooth controls is the space of functions of bounded variation denoted by $BV$. In this case, the existence of solutions can also be argued by compactness arguments.

Several contributions published in the last decade are devoted to the numerical approximation of optimal control problems related to sparse controls. A linear order of convergence was derived in [37] for linear elliptic optimal control problems with an $L^1$ cost term, where a general discretization scheme including piecewise constant approximation was studied. The same order of convergence was obtained for sparse optimal control problems governed by linear elliptic equations by considering regular Borel measures as control space, where controls were approximated using linear combinations of Dirac measures, see [9], and [27]. The semilinear case with $L^1$–norm penalization in the cost was studied in [10] and [11] for piecewise constant and piecewise linear approximations of the control, respectively. These papers also reported a linear order of error for both approximations.

On the other hand, the distributed optimal control problem with a $L^q$–quasinorm ($q \in (0,1)$) penalization has the form:

$$
(P) \quad \begin{cases}
\min_{(y,u)} & \frac{1}{2}\|y - y_d\|_{L^2(\Omega)}^2 + \frac{\alpha}{2}\|u\|_{L^2(\Omega)}^2 + \beta\int_\Omega |u|^q \, dx \\
\text{subject to:} & Ay = u + f, \quad \text{in } \Omega, \\
& y = 0, \quad \text{on } \Gamma, \\
& u \in U_{ad} \cap BV(\Omega),
\end{cases}
$$

in our setting, $\Omega \subset \mathbb{R}^2$ is star-shaped domain with boundary $\Gamma$. $y_d$ is given in $L^p(\Omega)$, with $p > n$ and $f$ is given in $L^2(\Omega)$. The admissible control set $U_{ad}$ is a closed convex set in $L^2(\Omega)$. Let $A$ be a uniformly elliptic second–order differential operator of the form:

$$
(Ay)(x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial y(x)}{\partial x_j} \right) + c_0 y(x),
$$

with coefficients $a_{ij}$ in $C(\bar{\Omega})$, and $c_0 \geq 0$ in $L^\infty(\Omega)$. Moreover, the matrix $(a_{ij})$ is symmetric and fulfills the uniform ellipticity condition:

$$
\exists \sigma > 0 : \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \sigma |\xi|^2, \quad \forall \xi \in \mathbb{R}^n, \text{for almost all } x \in \Omega.
$$

A related problem was studied in [23] with a quadratic penalizer on the gradient. There, the authors established the existence of solutions of controls in $H^1_0(\Omega)$ using compactness arguments. Moreover, by using a suitable regularization of the $L^q$-quasinorm, a primal-dual scheme for its numerical solution was also proposed. On the other hand, the optimal control problem with an additional quadratic $L^2$ penalizer on the control was discussed in [28] through a difference–of–convex function formulation of a regularization of the nonconvex problem. A regularizing assumption on the control space in $H^1$ was also considered due to the nonconvex nature of the problem and its intrinsic difficulties in proving the solution’s existence for $(P)$. 
**Our contribution.** This paper aims to analyze the numerical approximation by the finite element method of the regularized nonconvex optimal control problem (P). Although several related papers investigated optimality conditions and numerical methods (see [22],[36]), we cannot track literature for a priori error estimates for the discretization by the finite element method for this type of problem.

One of the difficulties in studying the numerical approximation of nonconvex problems is related to the expected nonuniqueness of solutions. For differentiable nonconvex problems, the associated analysis is based on second-order optimality conditions. See for example [1].

Moreover, we consider the space $BV(\Omega)$ of bounded variation functions as a control space that allows the presence of jumps in the solutions. We rely on the interpolation operator introduced in [2]. It is known that piecewise constant functions are not a suitable approximation for problems involving functions of bounded variation due to the norm in $BV(\Omega)$. However, the total variation term is not present in our problem.

It turns out that the difference-of-convex function formulation, together with its associated optimality system derived in [28], are useful to carry out an approximation analysis for the finite element approximation and help us to establish second-order sufficient conditions for local solutions of our problem. In fact, the DC formulation can be expressed as a linearly perturbed $L^1$ sparse optimal control, for which we can carry out a similar analysis as in [37]; then, we end up expressing the error in terms of the corresponding linear perturbations. However, the results of [37] cannot be applied to our problem. We still require to analyze this perturbation (which depends on the control), taking into account the numerical approximation for the control space. Therefore, for our analysis, we use the pointwise characterization of the solution resulting from the maximum principle established in [22]. Following a similar assumption on the active sets as in [37], we can obtain an error estimate of order $h^{\frac{1}{2}}$ provided that the boundary of the solution support is contained in a subset of the mesh whose measure is proportional to the discretization parameter.

**Organization of the paper.** In Section 2, we state the optimal control problem with nonconvex penalties and briefly discuss its main properties. The discretization technique is presented in Section 3. Section 4 is devoted to estimating the convergence rate of the FEM approximation. Finally, we confirm the theoretical findings with numerical measures of the convergence rate with respect to $h$.

### 2. Regularized optimal control problem

For $q \in (0, 1)$ and $\gamma >> 1$, we introduce the regularization mapping

$$u \mapsto \Upsilon_{q,\gamma}(u) := \int_{\Omega} h_{q,\gamma}(u(x))^q dx,$$

where $h_{q,\gamma}$ is Huber-like local smoothing of the absolute value introduced in [28], and defined by

$$h_{q,\gamma}(t) := \begin{cases} 
q \gamma^\frac{1-q}{q} |t|^\frac{1}{q}, & \text{if } t \in [-\frac{1}{\gamma}, \frac{1}{\gamma}], \\
|t| - \frac{1-q}{\gamma}, & \text{otherwise}.
\end{cases}$$
The mapping $\Upsilon_{p,\gamma}$ allows us to define the following family of (non-convex and non-differentiable) optimal control problems that approximate problem $(P)$:

$$
(P_\gamma) \begin{cases}
\min_{(y,u)} \mathcal{J}_\gamma(u, y) := \frac{1}{2} \|y - y_d\|_{L^2(\Omega)}^2 + \frac{\alpha}{2} \|u\|_{L^2(\Omega)}^2 + \beta \Upsilon_{p,\gamma}(u)
\text{subject to:} \\
u \in U_{ad} \cap BV(\Omega) \quad \text{and} \quad Ay = u + f, \quad \text{in } \Omega,
\end{cases}
$$

Here, the admissible of controls $U_{ad}$ corresponds to the box–constraint type, i.e.

$$U_{ad} = \{u \in L^2(\Omega) : u_a \leq u(x) \leq u_b, \text{ a.a. } x \in \Omega\},$$

for reals $u_a$ and $u_b$, such that $u_a < 0 < u_b$.

By the classical theory of elliptic partial differential equations, the state equation is formulated in the weak sense by introducing the bilinear form associated with the elliptic operator $(\mathcal{L})$, denoted by $\alpha$.

Thanks to the Lax–Milgram theorem and elliptic regularity [4], we know that for every $w \in L^2(\Omega)$ there exist $y \in H^1_0(\Omega)$ and a positive constant $c_a > 0$, satisfying:

$$a(y, v) = (w, v)_{L^2(\Omega)}, \quad \forall v \in H^1_0(\Omega),$$

$$\|y\|_{H^1_0(\Omega)} \leq c_a \|w\|_{L^2(\Omega)}.$$ 

Let us introduce the solution operator $S : L^2(\Omega) \to H^1_0(\Omega)$ corresponding to the linear and continuous operator which assigns to each $u \in L^2(\Omega)$ the corresponding solution $y = y(u) \in H^1_0(\Omega)$ satisfying the state equation in $(P_\gamma)$. Moreover, as in [37], the operator $S$ and its adjoint $S^*$ are continuous from $L^2(\Omega)$ to $L^\infty(\Omega)$. Thus, the state $y$ associated to the control $u$ has the representation $y = S(u + f) = Su + y_f$, with $y_f = Sf$.

**Assumption 1.** There exists a positive constant $M$ such that $\|\nabla u\|_{\mathcal{M}(\Omega)} \leq M$ for all $u \in U_{ad}$. Here $\| \cdot \|_{\mathcal{M}(\Omega)}$ is the norm on the space of regular Borel measures.

**Theorem 1.** Under Assumption 1, there exists a solution for problem $(P_\gamma)$.

**Proof.** Let $\{(y_k, u_k)\}_{k \in \mathbb{N}} \subset H^1_0(\Omega) \times U_{ad}$ a minimizing sequence for problem $(P_\gamma)$ such that $y_k \in H^1_0(\Omega)$ is the corresponding state associated to $u_k$. Let us check the boundedness of $\{(y_k, u_k)\}_{k \in \mathbb{N}}$. Indeed, $u_k \in U_{ad}$ hence $\|u_k\|_{L^\infty(\Omega)} \leq \max\{-u_a, u_b\}$. Furthermore, by Assumption 1 the sequence $\{u_k\}_{k \in \mathbb{N}}$ is bounded in $BV(\Omega)$.

On the other hand, since $y_k$ satisfies the state equation

$$Ay_k = u_k + f, \quad \text{in } \Omega,
\quad y = 0, \quad \text{on } \Gamma
$$

and, since $BV(\Omega) \hookrightarrow L^2(\Omega)$ in two dimensions (see [35, Theorem 10.1.3]), the Lax-Milgram Theorem guarantees the existence of a positive constant $c$, such that

$$\|y_k\|_{H^1_0(\Omega)} \leq c \|u_k\|_{L^2(\Omega)} \leq c \max\{-u_a, u_b\},$$

which shows that the sequence $\{y_k\}_{k \in \mathbb{N}}$ is bounded in $H^1_0(\Omega)$. Thus, there is a subsequence (without renaming) $\{(y_k, u_k)\}_{k \in \mathbb{N}}$, such that $u_k \overset{\ast}{\rightharpoonup} \bar{u}$ in $BV(\Omega)$ and $y_k \rightharpoonup \bar{y}$ in
We have that \( u_k \to \bar{u} \) in \( L^1(\Omega) \) and \( y_k \to \bar{y} \) in \( L^2(\Omega) \). Since \( U_{ad} \) is closed, it also follows that \( u \in U_{ad} \).

By continuity of \( \Upsilon_{q,\gamma} \) from \( L^1(\Omega) \) into \( \mathbb{R} \), see [28, Lemma 2] we can pass to the limit to obtain that

\[
\lim_{k \to \infty} \Upsilon_{q,\gamma}(u_k) = \Upsilon_{q,\gamma}(\bar{u}) \quad \text{and} \quad \lim_{k \to \infty} \frac{1}{2} \|y_k - y_d\|^2_{L^2(\Omega)} = \frac{1}{2} \|\bar{y} - y_d\|^2_{L^2(\Omega)}.
\]

Moreover, \( u \mapsto \int_{\Omega} u^2 \, dx \) is lower semi-continuous. Since \( \{u_k\}_{k \in \mathbb{N}} \subset L^2(\Omega) \), then \( u_k \to \bar{u} \) in \( L^1(\Omega) \) implies

\[
\liminf_{k \to \infty} \frac{\alpha}{2} \|u_k\|^2_{L^2(\Omega)} = \liminf_{k \to \infty} \frac{\alpha}{2} \int_{\Omega} u_k(x)^2 \, dx \geq \frac{\alpha}{2} \|\bar{u}\|^2_{L^2(\Omega)}.
\]

Altogether, we infer that \( J_\gamma(\bar{y}, \bar{u}) \geq \lim_{k \to \infty} J_\gamma(y_k, u_k) = \inf J_\gamma(y, u) \). Thus \( (\bar{y}, \bar{u}) \) is a solution for \( (P^\gamma) \).

### 2.1. Optimality conditions via DC–programming

DC–functions consist of those functions represented by the difference of two convex functions. This class of functions originated the DC–programming theory, which is well known in nonconvex optimization cf. [20]. A difference–of–convex functions formulation for elliptic optimal control problems involving \( L^q \)–quasinorms was applied in [28]. This formulation can be conveniently analyzed in the framework of DC programming in order to derive optimality conditions in the form of a KKT system. By replacing this expression in the objective function and incorporating the indicator function \( I_{U_{ad}} \) for the admissible control set, we get the following reduced problem:

\[
(P') \quad \min_{u \in BV(\Omega)} J_\gamma(u) := \frac{1}{2} \|Su + y_f - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2} \|u\|^2_{L^2(\Omega)} + \beta \Upsilon_{p,\gamma}(u) + I_{U_{ad}}.
\]

Let,

\[
F(u) := \frac{1}{2} \|Su + Sf - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2} \|u\|^2_{L^2(\Omega)}.
\]

Then, taking into account that \( U_{ad} \subset BV(\Omega) \cap L^\infty(\Omega) \), the DC–formulation of \( J_\gamma \) is defined by expressing \( J_\gamma = G - H \), where

\[
\begin{align*}
G : \ & L^2(\Omega) \to \mathbb{R} \\
& u \mapsto G(u) := F(u) + \beta \delta_\gamma \|u\|_{L^1(\Omega)} + I_{U_{ad}}, \quad \text{with} \quad \delta_\gamma := q^\gamma 1 - q \\
H : \ & L^2(\Omega) \to \mathbb{R} \\
& u \mapsto H(u) := \beta \left( \delta_\gamma \|u\|_{L^1(\Omega)} - \Upsilon_{p,\gamma}(u) \right),
\end{align*}
\]

therefore, \( (P') \) admits the formulation

\[
(D) \quad \min_u J_\gamma(u) = G(u) - H(u).
\]

This representation permits the application of DC programming theory, see [20]. In fact, a local solution of the problem \( (DC) \) must satisfy the so-called DC–criticality:

\[
\partial H(\bar{u}) \subset \partial G(\bar{u}),
\]

from which the associated optimality system is derived, see [28, Theorem 6]. We summarize these conditions in the following proposition.
Proposition 1. Let \( \bar{u} \in U_{ad} \) be a solution of (P), then there exist: \( \bar{y} = S \bar{u} + y_f \) in \( H^1_0(\Omega) \), with \( y_f = Sf \) an adjoint state \( \phi \in H^1_0(\Omega) \cap L^\infty(\Omega) \), a multiplier \( \bar{\zeta} \) and \( \bar{w} \) in \( \in L^\infty(\Omega) \) such that the following optimality system is satisfied:

\[
\begin{align*}
(7a) & \quad A\bar{y} = \bar{u} + f, \quad \text{in } \Omega, \\
(7b) & \quad \bar{y} = 0, \quad \text{on } \Gamma, \\
(7c) & \quad A^*\bar{\phi} = \bar{y} - y_d, \quad \text{in } \Omega, \\
(7d) & \quad \bar{\phi} = 0, \quad \text{on } \Gamma, \\
(7e) & \quad \bar{\zeta}(x) = 1, \quad \text{if } \bar{u}(x) > 0, \\
(7f) & \quad \bar{\zeta}(x) = -1, \quad \text{if } \bar{u}(x) < 0, \quad \text{and} \\
(7g) & \quad \left| \bar{\zeta}(x) \right| \leq 1, \quad \text{if } \bar{u}(x) = 0,
\end{align*}
\]

\( \bar{w}(x) := \begin{cases} \\
\delta_\gamma - q \left( |\bar{u}(x)| + \frac{q-1}{\gamma} \right)^{q-1} \text{sign}(\bar{u}(x)), & \text{if } |\bar{u}(x)| > \frac{1}{\gamma}, \\
0, & \text{otherwise},
\end{cases} \]

for almost all \( x \in \Omega \).

Moreover, there exist \( \lambda_a \) and \( \lambda_b \) in \( L^2(\Omega) \) such that the last optimality system can be written as a KKT optimality system:

\[
\begin{align*}
(8a) & \quad A\bar{y} = \bar{u} + f \quad \text{in } \Omega, \\
(8b) & \quad \bar{y} = 0 \quad \text{on } \Gamma, \\
(8c) & \quad A^*\bar{\phi} = \bar{y} - y_d \quad \text{in } \Omega, \\
(8d) & \quad \bar{\phi} = 0 \quad \text{on } \Gamma, \\
(8e) & \quad \bar{\phi} + \alpha\bar{u} + \beta(\delta_\gamma \bar{\zeta} - \bar{w}) + \lambda_b - \lambda_a = 0
\end{align*}
\]

\[\lambda_a \geq 0, \quad \lambda_b \geq 0, \quad \lambda_a(\bar{u} - u_a) = 0, \quad \lambda_b(\bar{u}_b - \bar{u}) = 0, \quad \bar{\zeta}(x) = 1 \quad \text{if } \bar{u}(x) > 0, \quad \bar{\zeta}(x) = -1 \quad \text{if } \bar{u}(x) < 0, \quad \left| \bar{\zeta}(x) \right| \leq 1 \quad \text{if } \bar{u}(x) = 0, \]

with \( \bar{w} \) given by (7e).

The presence of \( \bar{w} \) in this optimality system characterizes the jumps occurring in the solutions. Analyzing this term is crucial for the numerical approximation of (P). Note that \( w \) represents the superposition operator \( u \mapsto w \), defined by \( w(x) = j(u(x)) \) for almost all \( x \in \Omega \), where \( j : \mathbb{R} \to \mathbb{R} \), is defined by

\[
(9) \quad j(t) = \left[ \delta_\gamma - q \left( |t| + \frac{q-1}{\gamma} \right)^{q-1} \right] \text{sign}(t), \quad \text{if } |t| > \frac{1}{\gamma},
\]

and by 0, otherwise. Also, it satisfies \( |j(t)| \leq \delta_\gamma \) for all \( t \).

Lemma 1. The adjoint state \( \bar{\phi} \), solution of the adjoint equation (8b), satisfies

\( i) \ |\bar{\phi}|_{L^\infty(\Omega)} + |\bar{\phi}|_{H^1_0(\Omega)} \leq c|\bar{\bar{y}} - y_d|_{L^2(\Omega)} \), for some positive constant \( c \).
(ii) There exists $p > 2$, such that $\bar{\phi} \in W^{1,p}_0(\Omega)$.

Proof. (i) is a consequence of standard elliptic regularity since $\bar{y} - y_d \in L^p(\Omega)$, see [KK]. On the other hand, notice that by the embedding $H^1_0(\Omega) \hookrightarrow L^p(\Omega)$ with $p > 2$ implies that $\bar{y} - y_d \in W^{-1,p}(\Omega)$. Therefore, owing to [26, Theorem 2.1.2] the adjoint state $\bar{\phi}$ belongs to $W^{1,p}_0(\Omega)$ for some $p > 2$. Hence, (ii) follows.

Lemma 2. Let $j$ be the function defined in (9). The following properties are satisfied:

(a) $|j(t)| \leq \delta_\gamma$, for all $t \in \mathbb{R}$.

(b) $j(\cdot)$ is Lipschitz continuous with Lipschitz constant $L_j = 2\gamma \delta_\gamma \frac{1}{q}$, i.e.

$$|j(t_1) - j(t_2)| \leq L_j |t_1 - t_2|,$$

for all $t_1, t_2 \in \mathbb{R}$.

Proof. The first property is a consequence of the definition of $j$ for the case $|t| > \frac{1}{\gamma}$. Regarding the second property, we analyze the following cases:

i) $|t_1| \leq \frac{1}{\gamma}$ and $|t_2| \leq \frac{1}{\gamma}$, the result follows without difficulties by the definition of $j$.

ii) $t_1 > \frac{1}{\gamma}$ and $t_2 > \frac{1}{\gamma}$ (similarly for $t_1 < -\frac{1}{\gamma}$ and $t_2 < -\frac{1}{\gamma}$), by using the mean value theorem, we get

$$|j(t) - j(t_2)| = q \left| \left( t_1 + \frac{q - 1}{\gamma} \right)^{q-1} - \left( t_2 + \frac{q - 1}{\gamma} \right)^{q-1} \right|$$

$$\leq (1 - q) q \sup_{r > \frac{1}{\gamma}} \left( r + \frac{q - 1}{\gamma} \right)^{q-1} |t_1 - t_2|$$

$$\leq q^{-1} q^{q-1} |t_1 - t_2|$$

$$= q^{-1} q^{1-q} |t_1 - t_2|.$$

iii) $|t_1| \leq \frac{1}{\gamma}$ and $|t_2| > \frac{1}{\gamma}$ (similarly for $|t_2| \leq \frac{1}{\gamma}$ and $|t_1| > \frac{1}{\gamma}$), we have that $j(t_1) = 0$, which implies

$$|j(t_1) - j(t_2)| = q \left| \left( \frac{q}{\gamma} \right)^{q-1} - \left( t_2 + \frac{q - 1}{\gamma} \right)^{q-1} \right|,$$

by applying the mean value theorem, we get $|j(t_1) - j(t_2)| \leq \frac{1}{q} \gamma \delta_\gamma |t_2 - t_1|$. Then, using our assumption $0 < t_2 - \frac{1}{\lambda} \leq t_2 - t_1$ we obtain

$$|j(t_1) - j(t_2)| \leq \frac{1}{q} \gamma \delta_\gamma |t_2 - t_1|.$$
Thus, we conclude that

\[ (10) \]

For a given function \( \alpha \), we have

\[ \| j(t_1) - j(t_2) \| = \| \delta - q \left( -t_1 + \frac{q-1}{\gamma} \right) q^{-1} + \delta - q \left( t_2 + \frac{q-1}{\gamma} \right) q^{-1} \| \]

\[ \leq | \delta - q \left( -t_1 + \frac{q-1}{\gamma} \right) q^{-1} | + | \delta - q \left( t_2 + \frac{q-1}{\gamma} \right) q^{-1} | \]

\[ = q \left( \frac{q}{\gamma} \right)^{-1} \left( -t_1 + \frac{q-1}{\gamma} \right) q^{-1} + q \left( \frac{q}{\gamma} \right)^{-1} \left( t_2 + \frac{q-1}{\gamma} \right) q^{-1} \]

analogously to the previous cases, the mean value theorem implies that

\[ | j(t_1) - j(t_2) | \leq q(1-q) \sup_{|r|>\frac{1}{q} \left( r + \frac{q-1}{\gamma} \right)^{-1} \left( t_1 + \frac{1}{\lambda} \right) + \left( t_2 - \frac{1}{\lambda} \right) \leq \gamma \delta \gamma \frac{1}{q} | t_2 - t_1 | . \]

On another hand, since \(-1/\lambda < t_2 \) then \(-1/\lambda - t_1 < t_2 - t \). Moreover, since \( t_1 < 1/\lambda \), it follows that \( t_2 - 1/\lambda < t_2 - t \). Therefore,

\[ | j(t_1) - j(t_2) | \leq 2 \gamma \delta \gamma \frac{1}{q} | t_2 - t_1 | . \]

Thus, we conclude that \( j \) is Lipschitz continuous with constant \( 2 \gamma \delta \gamma \frac{1}{q} \).

**Remark 1.** For a given function \( g \) in \( L^2(\Omega) \), let us consider the following auxiliary \( L^1 \)-sparse optimal control problem:

\[
\min_{(y,u)} J_g(y,u) := \frac{1}{2} \| y - y_d \|^2_{L^2(\Omega)} + \frac{\alpha}{2} \| u \|^2_{L^2(\Omega)} - \beta \int_{\Omega} g u \, dx + \beta \delta \| u \|_{L^1(\Omega)}
\]

subject to:

\[
u \in U_{ad} \quad \text{and} \quad Ay = u + f, \quad \text{in} \, \Omega, \quad y = 0, \quad \text{on} \, \Gamma.
\]

Let \( \bar{u} \) satisfying the optimality system \( (7) \) and fix \( g = \bar{w} = j(\bar{u}) \) in \( (10) \). Interestingly, \( (7) \) corresponds to the optimality system of the optimal control problem \( (10) \), taking \( g = \bar{w} \) defined in \( (7a) \), see \( (32) \). Therefore, a local optimal control \( \bar{u} \) for \( (P_{\alpha}) \) is also a solution of the problem \( (10) \). In other words, for a control \( \bar{u} \) satisfying first-order optimality conditions, an auxiliary convex \( L^1 \) sparse optimal control problem can be associated for which, \( \bar{u} \) is its unique solution if \( \alpha > 0 \), see \( (32) \).

This observation provides several tools, developed for \( L^1 \)-sparse optimal control problems, that will serve in the numerical approximation of problem \( (P_{\alpha}) \). Notice, however, the presence of the function \( \bar{w} \) in the optimality system, c.f. \( (7c) \). This function depends on \( \bar{u} \), which directly affects the overall numerical approximation. The analysis of this element will be crucial for obtaining error estimates in the forthcoming sections.

Besides the characterization provided by optimality system \( (7) \) for an optimal control \( \bar{u} \), additional valuable information about the optimal control’s structure regarding its
sparsity properties can be obtained. The lower bound, estimated in the following Proposition, was also proved in [28, Remark III] for the unconstrained case. Its extension to the constrained case is direct.

We apply the maximum principle established in [22] to problem \( P' \). This leads to the following result.

**Theorem 2.** Let \( \bar{u} \) be the optimal control of the problem \( (P_\gamma) \) and let \( \bar{\phi} \in H^1_0(\Omega) \cap L^\infty(\Omega) \) be the associated adjoint state. Then,

(i) For almost all \( x \in \Omega \), \( \bar{u}(x) \) satisfies

\[
\bar{u}(x) = \arg\min_{u \in \{u_a, u_b\}} \{ (\bar{\phi}(x) - \beta \bar{w}(x))u + \frac{\alpha}{2} u^2 + \beta \delta_\gamma |u| \}.
\]

(ii) For almost all \( x \in \Omega \), the value \( \bar{u}(x) \) is characterized by

\[
\bar{u}(x) = \begin{cases} 0, & \text{if } |\beta \bar{w}(x) - \bar{\phi}(x)| \leq \beta \delta_\gamma, \\ P_{[u_a, u_b]} \left( \frac{\beta \bar{w}(x) - \bar{\phi}(x)}{\alpha} - \frac{\beta \gamma}{\alpha} \text{sign}(\beta \bar{w}(x) - \bar{\phi}(x)) \right), & \text{otherwise}. \end{cases}
\]

(iii) Let \( s^* = \left( \frac{\beta}{2} q(1-q) \right)^{\frac{1}{2-q}} \), we have for almost all \( x \), \( \bar{u}(x) \neq 0 \) such that \( u_a < \bar{u}(x) < u_b \), there exists \( \gamma_0 \) such that the inequality:

\[
s^* \left( 1 + \frac{1}{1-q} \right) + \frac{1-q}{\gamma} \leq |\bar{u}(x)| + \frac{\beta q}{\alpha} \left( |\bar{u}(x)| + \frac{q-1}{\gamma} \right)^{q-1} \leq \frac{1}{\alpha} \| \bar{\phi} \|_{L^\infty(\Omega)},
\]

holds for all \( \gamma > \gamma_0 \).

(iv) There exist \( \rho > 0 \) and \( \gamma_0 > 0 \) such that for all \( \gamma > \gamma_0 \) we have that the set

\[
\Omega_\rho := \{ x \in \Omega : 0 < |\bar{u}(x)| \leq \rho + \frac{1}{\gamma} \},
\]

has zero measure.

Proof. The proof follows from [23, Theorem 2.2]. Indeed, by choosing \( \ell(x, y(x)) = \frac{1}{2} (y(x) - y_d)^2 \), \( h(v) = \frac{\alpha}{2} |v|^2 + h^\alpha v^2(v), \ f(\cdot, y, u(\cdot)) = u(\cdot), \ (Ey, \cdot, \gamma^*, \gamma^* X, X^*, X) = a(y, \cdot) \) and \( X = H^1_0(\Omega) \), we can verify the hypothesis of [23, Theorem 2.2] and conclude that for almost all \( x \in \Omega \):

\[
\bar{u}(x) \in \arg\min_{u \in \{u_a, u_b\}} \{ \bar{\phi}(x) u + \frac{\alpha}{2} |u|^2 + \beta h^\alpha_q \gamma(u) \}
\]

\[
= \arg\min_{u \in \{u_a, u_b\}} \{ \bar{\phi}(x) u + \frac{\alpha}{2} |u|^2 + \beta h^\alpha_q \gamma(u) + I_{[u_a, u_b]} \}.
\]

Here, \( I_{[u_a, u_b]} \) denotes the indicator function of the interval \( [u_a, u_b] \). In terms of DC-programming [20], it follows that

\[
\bar{u}(x) \in \arg\min_{u \in \mathbb{R}} \{ g(u) - h(u) \},
\]

where \( g(u) := \bar{\phi}(x) u + \frac{\alpha}{2} u^2 + \beta \delta_\gamma |u| + I_{[u_a, u_b]} \) and \( h(u) := \beta h^\alpha_q \gamma(u) - \beta \delta_\gamma |u| \) are convex functions. Thus, by considering that \( h'(\bar{u}(x)) = j(\bar{u}(x)) = \bar{w}(x) \) and noticing that the optimality condition for the problem \( (12) \) is given by \( \partial h(\bar{u}(x)) \subset \partial g(\bar{u}(x)) \), we can find that \( \bar{u}(x) \) fulfills:

\[
0 \in \bar{\phi}(x) - \beta \bar{w}(x) + \alpha \bar{u}(x) + \beta \delta_\gamma \partial |(\bar{u}(x)) + \partial I_{[u_a, u_b]}(\bar{u}(x)),
\]
which corresponds to the optimality condition of problem formulated in (i). Characterization (ii) follows by considering the soft–thresholding operator ([14] Example 6.22) of the convex function $\beta \delta_{\gamma} \cdot | \cdot |$, which is given by

$$\text{prox}_{\beta \delta_{\gamma}}(y) = \arg\min_u \left\{ \frac{1}{2} (y - u)^2 + \beta \delta_{\gamma} |u| \right\} = \begin{cases} 0, & \text{if } |y| \leq \beta \delta_{\gamma}, \\ y - \beta \delta_{\gamma} \text{sign}(y), & \text{otherwise}. \end{cases}$$

(14) Therefore, for the unconstrained case, we may express $\bar{u}(x)$ with the implicit formula

$$\bar{u}(x) = \begin{cases} 0, & \text{if } |\beta \bar{x}(x) - \bar{\phi}(x)| \leq \beta \delta_{\gamma}, \\ \frac{\beta \bar{x}(x) - \bar{\phi}(x)}{\alpha} - \frac{\beta \delta_{\gamma}}{\alpha} \text{sign}(\beta \bar{x}(x) - \bar{\phi}(x)), & \text{if } |\beta \bar{x}(x) - \bar{\phi}(x)| > \beta \delta_{\gamma}. \end{cases}$$

(15) By considering the lower and upper bounds and complementarity conditions (8d), we arrive to the formula in (ii).

Finally, the part (iii) was proved in [28, Remark 3 (ii)].

Remark 2. The bound given in Theorem 3 (iv) is analogous to the bound obtained in [36] for the $L^0$ penalizer. In contrast to the $L^0$ penalizer, we can not use the Hard–thresholding in our analysis, which gives an explicit value for the jump of the optimal control when changing from zero to nonzero values. In our case, the soft–thresholding operator was used to derive part (iii).
Note that nonzero values of the optimal control necessarily satisfy the critical point condition (17), which might be satisfied by two values, see Figure 1. Recall that a unique solution is not guaranteed for problem \( P_\gamma \). If \( \bar{u}(x) > \frac{1}{\gamma} \), we look for the value that minimizes \( \frac{\alpha}{2} u^2 + \bar{\varphi}(x)u + \beta(u + \frac{q-1}{\gamma})^q \). Furthermore, its second-order derivative at \( \bar{u}(x) \) is given by

\[
\alpha + \beta q(q-1) \left( \bar{u}(x) + \frac{q-1}{\gamma} \right)^{-2},
\]

which is positive, provided that \( \bar{u}(x) \geq s^* + \frac{1-q}{\gamma} \). By symmetry, the case \( \bar{u}(x) < -\frac{1}{\gamma} \) is analogous. Therefore, we choose the solution of (17), such that

\[
|\bar{u}(x)| > s^* + \frac{1-q}{\gamma} \quad \text{with} \quad s^* = \left( \frac{\beta}{\alpha q(1-q)} \right)^{\frac{1}{q-1}},
\]

for almost all \( x \in \Omega \).

![Figure 1. Solutions of the critical point equation (17)](image)

In view of Remark 2 we state the following assumption.

**Assumption 2.** Let \( \bar{u} \in U_{ad} \) satisfying optimality system (8). Then, there exists a constant \( C > 1 \) such that:

\[
|\bar{u}(x)| \geq Cs^* + \frac{1-q}{\gamma},
\]

for almost all \( x \in \Omega \), such that \( \bar{u}(x) \neq 0 \).

It is clear that (local) optimality for nonsmooth DC–problems can not be established using second-order derivatives due to the lack of differentiability of the objective functional. Instead, we will rely on the underlying convexity given by the DC-representation of the cost functional.

**Theorem 3.** Let \( \bar{u} \in U_{ad} \) satisfying optimality conditions (8). Then, under Assumption 2 there exist constants \( \varrho > 0 \) and \( \sigma > 0 \), such that the following relation holds:

\[
\sigma \| u - \bar{u} \|^2_{L^2(\Omega)} \leq J_\gamma(y, u) - J_\gamma(\bar{y}, \bar{u}), \quad \forall u \in B_\infty(\varrho, \bar{u}) \cap U_{ad},
\]

for \( \gamma \) sufficiently large.
Proof. Aiming to prove the local optimality of $\bar{u}$, we estimate the difference of the cost function at a point $u$ close to $\bar{u}$. Let $\bar{y}$ and $y$ be the corresponding states associated with $\bar{u}$ and $u$, respectively. We consider

$$J_\gamma(y, u) - J_\gamma(\bar{y}, \bar{u}) = J_\gamma(u) - J_\gamma(\bar{u}) = G(u) - H(u) - G(\bar{u}) + H(\bar{u}).$$

Recalling that $G(u) = F(u) + \beta \delta \|u\|^2_1$, with $F$ the quadratic functional given by $F(u) = \frac{1}{2} \|Su + Sf - y\|^2_{L^2(\Omega)} + \frac{\rho}{2} \|u\|^2_{L^2(\Omega)}$, we have:

$$G(u) - G(\bar{u}) = (S^*(\bar{y} - y_\delta) + \alpha \bar{u}, u - \bar{u})_{L^2(\Omega)} + (S(\bar{u} - u), S(\bar{u} - u))_{L^2(\Omega)} + \beta \delta \|u\|^2_{L^1(\Omega)} - \beta \delta \|\bar{u}\|^2_{L^1(\Omega)}$$

$$\geq (\bar{\delta} + \alpha \bar{u}, u - \bar{u})_{L^2(\Omega)} + \|\bar{y} - y\|^2_{L^2(\Omega)} + \alpha \|u - \bar{u}\|^2_{L^2(\Omega)},$$

where the last inequality is obtained from the fact that $\bar{\delta} \in \partial \|\cdot\|_{L^1(\Omega)}(\bar{u})$. Taking into account that $\bar{u}$ satisfies the variational inequality \cite{72}, we deduce

$$G(u) - G(\bar{u}) \geq (\beta \bar{w}, u - \bar{u})_{L^2(\Omega)} + \|\bar{y} - y\|^2_{L^2(\Omega)} + \alpha \|u - \bar{u}\|^2_{L^2(\Omega)}.$$

On the other hand, by using the Gâteaux differentiability and the convexity of function $H$, see \cite{28}, we get

$$H(\bar{u}) - H(u) \geq H'(\bar{u})(u - \bar{u}) = (\beta \bar{w}, u - \bar{u})_{L^2(\Omega)}, \quad \forall u \in L^2(\Omega).$$

Replacing (21) and (22) in (20) we get

$$J_\gamma(y, u) - J_\gamma(\bar{y}, \bar{u}) \geq \|\bar{y} - y\|^2_{L^2(\Omega)} + \alpha \|\bar{u} - u\|^2_{L^2(\Omega)} - (\bar{\delta} + \alpha \bar{u}, u - \bar{u})_{L^2(\Omega)}.$$

Notice that the last term in the above inequality is nonnegative due to the monotonicity of $J'$. In view of Assumption 2, we are allowed to chose a $\varrho$, such that for all $u \in B_\infty(\varrho, \bar{u})$ they also satisfy

$$|u(x)| \geq C s^* + \frac{1 - q}{\gamma}, \quad \text{for almost all} \ x \in \Omega \setminus \Omega_\varrho.$$

Thus, we estimate

$$\int_\Omega \beta(\bar{w}(x) - w(x))(\bar{u}(x) - u(x)) \, dx$$

$$\leq \int_{\Omega \setminus \Omega_\varrho} \beta |\bar{w}(x) - w(x)||\bar{u}(x) - u(x)| \, dx + \beta \int_{\Omega_\varrho} |\bar{u}(x) - w(x)||\bar{u}(x) - u(x)| \, dx$$

$$\leq \beta q(1 - q) \int_{\Omega \setminus \Omega_\varrho} \left(\bar{u}(x) + \frac{q - 1}{\gamma}\right)\gamma^{-2}(\bar{u}(x) - u(x))||\bar{u}(x) - u(x)|| \, dx + \beta \int_{\Omega_\varrho} |w(x)||u(x)| \, dx,$$

where $\bar{u}(x)$ lies between $\bar{u}(x)$ and $u(x)$. Observing that $|\bar{u}(x)| \geq C s^* + \frac{1 - q}{\gamma}$ and, using Theorem 2(iv), we have that $\Omega_\varrho$ has zero measure for $\gamma$ sufficiently large. Therefore, it follows that

$$\int_\Omega \beta(\bar{w}(x) - w(x))(\bar{u}(x) - u(x)) \, dx < \alpha \|\bar{u} - u\|^2_{L^2(\Omega)}.$$

Replacing (24) in (23) and taking into account that the bilinear form $(S\cdot, S\cdot)_{L^2(\Omega)}$ is coercive, there exists $\sigma > 0$ satisfying (19). \qed
3. Finite element approximation

On $\bar{\Omega}$, we consider a family of meshes $(T_h)_{h>0}$ which consist of triangles $T \in T_h$, such that the following conditions are satisfied:

**Assumption 3.** (i) $\bigcup_{T \in T_h} T = \bar{\Omega}$ and

(ii) For two triangular elements $T_i$ and $T_j$, $i \neq j$ they share a vertex, a side or are disjoints.

For each triangle $T \in T_h$, we denote $\rho(T)$ the diameter of $T$, and $\sigma(T)$ the diameter of the largest ball contained in $T$. The mesh size $h$ associated to the mesh is defined by

$h = \max_{T \in T_h} \rho(T)$. 

Throughout this paper, we impose the following regularity assumption on the grid:

**Assumption 4.** There exist two positive constants $\rho$ and $\sigma$ such that

$\frac{\rho(T)}{\sigma(T)} \leq \sigma$ and $\frac{h}{\rho(T)} \leq \rho$, \quad \forall T \in T_h,$

for all $h > 0$.

Associated with the triangulation $T_h$, we define the following approximation spaces:

\begin{align*}
Y_h &= \{ y_h \in C(\bar{\Omega}) : y_h|_T \in P_1(T), \forall T \in T_h, \; y_h = 0 \; \text{on} \; \Gamma \}, \\
U_h &= \{ u_h \in L^2(\Omega) : u_h|_T = P_0(T), \forall T \in T_h \},
\end{align*}

where $P_0(T)$ and $P_1(T)$ denote the set of real valued constant functions and linear-affine continuous real-valued functions defined on $T$, respectively.

### 3.1. Discretization of the state equation

The discrete state equation is defined as the following variational problem formulated in $Y_h \subset H^1_0(\Omega)$: for every $w \in L^2(\Omega)$, we seek a function $y_h \in Y_h$ satisfying the equation

\begin{equation}
\begin{aligned}
\langle a(y_h, v_h) \rangle &= \langle (w, v_h) \rangle, \\
\forall v_h &\in Y_h.
\end{aligned}
\end{equation}

This problem has a unique solution $y_h \in Y_h$ which depends continuously on the data $w$, see [34]. Analogously to the continuous counterpart, we introduce the discrete solution operator $S_h : U_h \rightarrow Y_h$ which assigns to each $w$, the corresponding solution $y_h = y_h(w)$ satisfying (27). Moreover, the following estimate holds: there exists a positive constant $c$ (independent of $h$ and $w$) such that $\|y_h\|_{H^1_0(\Omega)} \leq c\|w\|_{L^2(\Omega)}$.

Similarly, we have that the state $y_h$ associated to a control $u \in U_h$ can be written as $y_h = S_h u + y_{h,f}$, with $y_{h,f} = S_h f$. However, for simplicity and without loss of generality, we will assume that $y_{h,f}$ is computed exactly, i.e. $y_{h,f} = y_f$.

The approximation for linear-elliptic problems of the form (4) is well known, see [13, Section 17] for the proof of the following rate of approximation.

**Proposition 2.** Let $y$ and $y_h$ be the solutions, associated to a control $u \in L^2(\Omega)$, of equations (27) and (27), respectively. There exists a constant $c_A > 0$, independent of $h$, such that

$\|y - y_h\|_{L^2(\Omega)} + h\|y - y_h\|_{H^1(\Omega)} \leq c_A h^2$. 

3.2. **Numerical approximation of the optimal control problem.** The approximation of the control functions by piecewise constant functions space $U_h$ is motivated by the discontinuous nature of the solution, as discussed in Theorem 2. Therefore, a discrete control $u \in U_h$ can be written as

$$ u = \sum_{T \in \mathcal{T}_h} u_T \chi_T, $$

where $\chi_T$ is the characteristic function of $T$ and $u_T \in \mathbb{R}$, for all $T \in \mathcal{T}_h$. As a result of this type of approximation, the set of discrete admissible controls is given by:

$$ (28) \quad U^h_{ad} = U_{ad} \cap U_h. $$

In view of (27) and (28) we formulate the associated discrete optimal control problem:

$$ (P^h) \quad \begin{cases} 
\min_u & J^h(u) := \frac{1}{2} \| S_h u + S_h f - y_d \|_{L^2(\Omega)}^2 + \frac{\alpha}{2} \| u \|_{L^2(\Omega)}^2 + \beta \Upsilon_{q,\gamma}(u) \\
\text{subject to:} & u \in U^h_{ad}. 
\end{cases} $$

It is clear that for $\alpha > 0$ this problem has at least one solution $\bar{u}_h \in U_h$. Similarly to the continuous problem $(P_\gamma)$, we use the indicator function of the discrete admissible set to formulate the discrete reduced problem, as in (7), which also admits a DC–formulation of the form $J^h_\gamma(u) = G^h(u) - H(u)$. Again, the DC-splitting is given by:

$$ \begin{align*}
G^h : & \quad U^h \rightarrow \mathbb{R} \\
& \quad u \mapsto G^h(u) := F^h(u) + \beta \delta_{\gamma} \| u \|_{L^1(\Omega)} + I_{U^h_{ad}}, \quad \text{and} \\
H : & \quad U^h \rightarrow \mathbb{R} \\
& \quad u \mapsto H(u) := \beta \left( \delta_{\gamma} \| u \|_{L^1(\Omega)} - \Upsilon_{q,\gamma}(u) \right),
\end{align*} $$

where $F^h(u) := \frac{1}{2} \| S_h u + S_h f - y_d \|_{L^2(\Omega)}^2 + \frac{\alpha}{2} \| u \|_{L^2(\Omega)}^2$.

Optimality conditions for the discrete problem $(P^h)$ can also be obtained following the DC–programming; therefore, we omit the proof.

**Proposition 3.** Let $\bar{u}_h \in U_h$ be a local solution of $(P^h)$, then there exist: $\bar{y}_h = S_h \bar{u}_h + y_{h,f}$ in $Y_h$ (with $y_{h,f} = S_h f$), an adjoint state $\phi_h \in Y_h$, a multiplier $\zeta_h$ and $\bar{w}_h$ in $L^\infty(\Omega)$...
such that the following optimality system is satisfied:

\begin{align}
(29a) \quad a(\bar{y}_h, v) &= (\bar{u}_h + f, v), \quad \forall v \in Y_h, \\
(29b) \quad a^*(\bar{\phi}_h, v) &= (\bar{y}_h - y_d, v), \quad \forall v \in Y_h, \\
(29c) \quad (\bar{\phi}_h + \beta (\delta_{\gamma} \zeta_h - \bar{w}_h) + \alpha \bar{u}_h, u - \bar{u}_h)_{L^2(\Omega)} &\geq 0, \quad \forall u \in U^h \\
(29d) \quad \zeta_h(x) &= 1, \quad \text{if } \bar{u}_h(x) > 0, \\
|\zeta_h(x)| &= 1, \quad \text{if } \bar{u}_h(x) = 0,
\end{align}

Moreover, there exist \( \lambda_{ah} \) and \( \lambda_{bh} \) in \( U_h \) such that the last optimality system can be written as a KKT system by replacing (29c) with:

\begin{align}
(30a) \quad \bar{\phi}_h + \alpha \bar{u}_h + \beta (\delta_{\gamma} \zeta_h - \bar{w}_h) + \lambda_{bh} - \lambda_{ah} &= 0 \\
(30b) \quad \lambda_{ah} &\geq 0, \quad \lambda_{bh} \geq 0, \\
\zeta_h(x) &= 1 \quad \text{if } \bar{u}_h(x) > 0, \\
|\zeta_h(x)| &\leq 1 \quad \text{if } \bar{u}_h(x) = 0,
\end{align}

with \( \bar{w}_h \) given by (29e).

Proceeding as in the continuous case, we have a result on the structure of the discrete optimal control.

**Proposition 4.** There exist \( \gamma_0 > 0 \) and \( \rho_h > 0 \) such that for all \( \gamma > \gamma_0 \) the set

\begin{equation}
\Omega_{\rho_h} := \{ x \in \Omega : 0 < |\bar{u}_h(x)| \leq \rho_h + \frac{1}{\gamma} \},
\end{equation}

has zero Lebesgue measure. In addition, in view of the maximum principle, the support’s bound (18) also holds for \( \bar{u}_h \).

**Proof.** We follow the arguments from [28, Remark 3, (ii)] with slight modifications. First observe that \( \|\bar{\phi}_h\|_{L^\infty(\Omega)} \) is bounded. Indeed, since the discrete adjoint state satisfies equation (29b), there exists a constant \( c > 0 \) such that \( c \|\bar{\phi}_h\|_{L^\infty(\Omega)} \leq \|\bar{y}_h - y_d\|_{L^2(\Omega)} \leq (2J_h^0(0))^{\frac{1}{2}} = \|y_h, f - y_d\|_{L^2(\Omega)} \).

Now, we claim that the measure of the set \( \{ x \in \Omega : 0 < |\bar{u}_h(x)| \leq 1/\gamma \} \) vanishes for sufficiently large \( \gamma \). We prove this statement by assuming that the corresponding measure is positive. Without loss of generality, let us also assume that \( u_a < \bar{u}_h(x) < u_b \) for almost all \( x \) in \( \Omega_{\rho_h} \). From equation (30a), we infer that for almost all \( x \) in \( \Omega_{\rho_h} \) it holds:

\[ |\bar{\phi}_h(x)| \geq \beta \delta_{\gamma} - \frac{\alpha}{\gamma} \.]
This relation contradicts the fact that $|\tilde{\phi}_h(x)|$ essentially bounded whenever $\gamma$ is taken sufficiently large. Thus, our claim is true.

In the same fashion, we assume that for all $\rho_h > 0$ the set $\{x \in \Omega : 1/\gamma < |\tilde{u}_h(x)| \leq \rho_h + 1/\gamma\}$ has zero Lebesgue measure. Further, we assume that $u_a < \tilde{u}_h(x) < u_b$ and for every $\rho_h > 0$ we take $\gamma$ such that $1/\gamma < \rho_h$. Therefore, (29e) and (30a) imply

$$|\tilde{\phi}_h(x)| = \alpha u_h(x) + \beta q \left( |\tilde{u}_h(x)| + \frac{q-1}{\gamma} \right)^{q-1} \frac{\text{sign}(\tilde{u}_h(x))}{\gamma} \geq \alpha \gamma + \beta q \left( \rho_h + \frac{q}{\gamma} \right)^{q-1},$$

which leads us to a contradiction by taking $\rho_h$ sufficiently small. Therefore, $\rho_h$ satisfying property (31) exists.

For the second part of the proposition, we apply the maximum principle, tacitly assuming that the integrals of the cost function are computed exactly. By following the lines of the Remark 2, we have that for almost all $x \in \Omega$, where $\tilde{u}_h(x) > 0$, the discrete optimal control $\tilde{u}_h(x)$ also satisfies:

$$\alpha \tilde{u}_h(x) + \beta q \left( \tilde{u}_h(x) + \frac{q-1}{\gamma} \right)^{q-1} = -\bar{\phi}_h(x).$$

Using the same arguments as in the continuous case, we get that $|\tilde{u}_h(x)| \geq s^* + \frac{1-q}{\gamma}$ for almost all $x$ where $|\tilde{u}_h(x)| \neq 0$. Moreover, we may interpret the solution $\tilde{u}_h$ of $\tilde{P}_h^0$ as the solution of the following auxiliary discrete $L^1$–sparse optimal control problem by choosing $\hat{g} = \tilde{w}_h$. The auxiliary discrete problem reads:

$$\begin{align*}
\min_{u \in U_h} & \frac{1}{2} \|S_h u + y - y_d\|_{L^2(\Omega)}^2 + \alpha \|u\|_{L^2(\Omega)}^2 + \beta (\hat{g}, u)_{L^2(\Omega)} + \beta \delta_u \|u\|_{L^1(\Omega)} \\
\text{subject to:} & \quad u \in U_{ad}^h.
\end{align*}$$

Analogously, the former discrete $L^1$–sparse optimal control problem has a unique solution in $U_h$, namely $u_h$ depending of the function $\hat{g} = \tilde{w}_h$.

Now, we introduce the following interpolation operator to represent functions in $U_h$.

**Definition 1.** The quasi–interpolant operator $\Pi_h : L^1(\Omega) \to U_h$ is defined as follows (see [16]):

$$\Pi_h u := \sum_{T \in T_h} u_T \chi_T, \quad \text{with} \quad u_T = \frac{1}{|T|} \int_T u.$$

The next orthogonality result is also known [37]. For $u \in L^2(\Omega)$, it holds that

$$\int_{\Omega} (u - u_T) \chi_T = 0, \quad \forall T \in T_h.$$

In case of functions belonging to $BV(\Omega)$, following [2], we consider the interpolation operator defined on the smooth approximation $u_\varepsilon$ of $BV$ functions (see [2, Proposition 2.1]), as follows

$$C_h u := \Pi_h u_\varepsilon.$$
For convenience in the forthcoming analysis, we fix \( \varepsilon = h \). The interpolation error of \( \Pi_h \) is known for \( L^2 \) and \( H^{-1} \) norms in \[16\]. Updating the arguments also \( L^1 \)-norm interpolation errors are obtained. See Lemma \[8\] in the Appendix.

**Lemma 3.** There exists a positive constant \( c \) such that
\[
\|u - C_h u\|_{L^2(\Omega)} \leq ch^{\frac{3}{2}}|Du(\Omega)|^{\frac{1}{2}},
\]
holds for all \( u \in BV(\Omega) \cap L^2(\Omega) \).

Proof. Let \( u \) be in \( BV(\Omega) \) and let \( c \) be a positive generic constant independent of \( h \).
We apply Lemma \[8\] with \( s = 1 \) to \( u_e \). Next, since \( C_h u \) is uniformly bounded in \( L^\infty \) by its construction \[2\] and, using the \[2, Proposition 2.1\], we get
\[
\|u - C_h u\|_{L^2(\Omega)} \leq \|u - C_h u\|_{L^\infty(\Omega)} \|u - C_h u\|_{L^1(\Omega)}^{\frac{1}{2}}
\]
\[
\leq c \|u\|_{L^\infty(\Omega)} \|u - \Pi_h u_e\|_{L^1(\Omega)}^{\frac{1}{2}}
\]
\[
\leq c \left( \|u - u_e\|_{L^1(\Omega)} + \|u_e - \Pi_h u_e\|_{L^1(\Omega)} \right)^{\frac{1}{2}}
\]
\[
\leq c \left( \|Du(\Omega)\| + c \|\nabla u_e\|_{L^1(\Omega)} \right)^{\frac{1}{2}},
\]
\[
\leq c(\varepsilon + h)^{\frac{1}{2}}|Du(\Omega)|^{\frac{1}{2}}.
\]
Recalling that \( \varepsilon = h \) in the Definition \[1\] of the interpolant \( C_h \) the result follows from assumption \[1\].

Similarly, we obtain an interpolation error for \( \Pi_h \) in the \( L^1 \)-norm.

**Corollary 1.** There exists a positive constant \( c \) such that
\[
\|u - \Pi_h u\|_{L^1(\Omega)} \leq ch,
\]
holds for all \( u \in BV(\Omega) \).

Proof. As in the Lemma \[3\] according to \[2, Proposition 2.1\] and Lemma \[8\] we obtain the following estimation
\[
\|u - \Pi_h u\|_{L^1(\Omega)} \leq \|u - C_h u\|_{L^1(\Omega)} + \|C_h u - \Pi_h u\|_{L^1(\Omega)}
\]
\[
\leq \|u - u_e\|_{L^1(\Omega)} + \|u_e - \Pi_h u_e\|_{L^1(\Omega)} + \|\Pi_h (u_e - u)\|_{L^1(\Omega)}
\]
\[
\leq \varepsilon |Du(\Omega)| + \|u_e - \Pi_h u_e\|_{L^1(\Omega)} + C\|u - u_e\|_{L^1(\Omega)}
\]
\[
\leq c(\varepsilon + h)|Du(\Omega)|,
\]
The result follows by considering that \( \varepsilon = h \) and Assumption \[1\].

The following convergence result will be crucial for our error analysis. In fact, the second-order optimality conditions established in Theorem \[3\] require the approximation of the local solution in the \( L^\infty \) topology.

**Proposition 5.** Let \( \bar{\phi} \) and \( \bar{\phi}_h \) be the adjoint and the discrete adjoint states satisfying equations \((7a)\) and \((29b)\) with controls \( \bar{u} \) and \( \bar{u}_h \), respectively. There exist positive constants \( c_A \) and \( c \) (independent of \( h \)) such that the following relations hold:
\[
\begin{align*}
(34a) & \quad \|\bar{\phi} - \bar{\phi}_h\|_{H^1_0(\Omega)} \leq c_A (h + \|\bar{u} - \bar{u}_h\|_{L^2(\Omega)}), \text{ and} \\
(34b) & \quad \lim_{h \to 0} \|\bar{\phi} - \bar{\phi}_h\|_{L^\infty(\Omega)} \leq c \lim_{h \to 0} \|\bar{u} - \bar{u}_h\|_{L^2(\Omega)}.
\end{align*}
\]
assuming that the last limit exists.

Proof. The proof is rather standard since the adjoint equation is linear. Indeed, let \( \tilde{\phi} \) be the solution of the equation

\[
a^*(\phi, v) = (S\tilde{u}_h + y_f - y_d, v)_{L^2(\Omega)}, \quad \forall v \in H^1_0(\Omega),
\]

by linearity, \( \hat{\phi} := \tilde{\phi} - \tilde{\phi} \) satisfies

\[
a^*(\hat{\phi}, v) = (S\tilde{u} - \tilde{u}_h, v)_{L^2(\Omega)} = (S(\bar{u} - \bar{u}_h), v)_{L^2(\Omega)}, \quad \forall v \in H^1_0(\Omega).
\]

From this equation and standard elliptic estimates, we have that \( \|\tilde{\phi} - \hat{\phi}\|_{H^1_0(\Omega)} \leq c\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)}. \) On the other hand, the error estimates for linear elliptic PDE from Proposition 2 also applies to the finite element approximation of the adjoint equation. Hence, there exists a constant \( c > 0, \) such that

\[
\|\tilde{\phi} - \hat{\phi}\|_{H^1_0(\Omega)} \leq \|\tilde{\phi} - \hat{\phi}\|_{H^1_0(\Omega)} + \|\tilde{\phi} - \tilde{\phi}h\|_{H^1_0(\Omega)}
\]

\[
= \|\tilde{\phi} - \hat{\phi}\|_{H^1_0(\Omega)} + \|S^*(S\tilde{u}_h + y_f - y_d) - S^*(S\bar{u}_h + y_f - y_d)\|_{H^1_0(\Omega)}
\]

\[
\leq \|\tilde{\phi} - \hat{\phi}\|_{H^1_0(\Omega)} + \|S^*(S\tilde{u}_h + y_f - y_d) - S^*(S\bar{u}_h + y_f - y_d)\|_{H^1_0(\Omega)}
\]

\[
+ \|S^*(S\bar{u}_h + y_f - S\bar{u}_h - y_n.f)\|_{H^1_0(\Omega)}
\]

\[
\leq c(\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)} + h).
\]

Next, we prove relation (34b). Let us define \( \tilde{\phi}_h \) as the solution to equation

\[a^*(\phi, v) = (\bar{u} - y_d, v)_{L^2(\Omega)}, \quad \forall v \in Y_h.\]

Note that the last equation, defining \( \tilde{\phi}_h, \) has the same right-hand of the adjoint equation (7b). We get \( \|\tilde{\phi}_h - \bar{y}\|_{L^\infty(\Omega)} \leq c\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)}. \) Then, by Lemma 1 (ii) we know that \( \tilde{\phi}_h \in W^{1,p}_0(\Omega), \) for some \( p > n. \) Thus, using [13, Theorem 21.5] we have that \( \tilde{\phi}_h \to \tilde{\phi} \) in \( L^\infty(\Omega). \) Therefore

\[
\lim_{h \to 0} \|\bar{u} - \bar{u}_h\|_{L^\infty(\Omega)} = \lim_{h \to 0} \big( \|\tilde{\phi} - \tilde{\phi}_h\|_{L^\infty(\Omega)} + \|\tilde{\phi}_h - \bar{y}_h\|_{L^\infty(\Omega)} \big)
\]

\[
\leq \lim_{h \to 0} \|\tilde{\phi} - \tilde{\phi}_h\|_{L^\infty(\Omega)} + c \lim_{h \to 0} \|\bar{y} - \bar{y}_h\|_{L^2(\Omega)}
\]

\[
\leq c \lim_{h \to 0} \|\bar{u} - \bar{u}_h\|_{L^2(\Omega)}.
\]

According to Theorem 2, a jump may occur when the optimal control for the continuous problem is not zero. Intuitively, the elements containing this jump should converge to a set of zero measure, provided that the boundary of the function’s support is sufficiently regular. Although expected, we do not prove this property, and we will assume that the support’s boundary crosses a set of elements with an area proportional to \( h. \) In this regard, let us introduce the following sets:

\[
T^0_h = \{ T \in T_h : \bar{u}_T = \Pi_h \bar{u}(x)|_{T} = 0 \}, \quad \text{and}
\]

\[
T^\#_h = \{ T \in T_h : |\bar{u}(x)| > 0, \text{a.a. } x \in T \}.
\]

The following assumption is analogous to [37, Assumption 3.3], imposed on the structure of the active sets in the context of our discretization.
Assumption 5. Let $T^*_h := T_h \setminus \left( T^0_h \cup T^#_h \right)$. There exists $c^* > 0$ such that

$$|T^*_h| \leq c^* h, \quad \text{for all } h > 0.$$  

Theorem 4. Let $\bar{u}_h$ a global solution of problem $\{P^h\}$ defined in a mesh $T_h$ of size $h > 0$. Then, the sequence $\{\bar{u}_h\}$ contains a subsequence $\{\bar{u}_h\}$ (without renaming) in $U_h$, such that

(i) $\lim_{h \to 0} J_h(\bar{u}_h) = \inf \{ J(u) : u \in U_{ad} \cap BV(\Omega) \}$,

(ii) $\bar{u}_h$ converges to a solution $\bar{u}$ of $\{P\}$ in $L^2(\Omega)$.

(iii) Under Assumptions 2 and 3, $\lim_{h \to 0} \|\bar{u}_h - \bar{u}\|_{L^\infty(\Omega)} = 0$.

Proof. (i) Let $\{\bar{u}_h\}_h \subset U^0_{ad} \subset BV(\Omega)$ be a sequence of solutions $\{P^h\}$ (with discretization parameter $h > 0$) and associated sequence of corresponding states $\{\bar{y}_h\}_h$ in $H^1_0(\Omega)$. By the definition of $U_{ad}$ and the Tikhonov term in the cost function, it follows that $\{\bar{y}_h\}_h$ is a bounded sequence in $BV(\Omega) \cap L^2(\Omega)$. Also, we confirm that $\{\bar{y}_h\}_h$ is a bounded sequence in $H^1_0(\Omega)$ given our assumption and the inequality:

$$\|\bar{y}_h\|_{H^1_0(\Omega)} = \|S_h \bar{u}_h\|_{H^1_0(\Omega)} \leq \|(S_h - S) \bar{u}_h\|_{H^1_0(\Omega)} + \|S \bar{u}_h\|_{H^1_0(\Omega)}$$

$$\leq c_4 h + c \|\bar{u}_h\|_{L^2(\Omega)}.$$

Therefore, we subtract (without renaming) weakly* convergent subsequence $\{\bar{u}_h\}_h$ in $BV(\Omega)$, such that $\bar{u}_h \rightharpoonup \bar{u}$ in $BV(\Omega)$ and $\bar{u}_h \rightarrow \bar{u}$ in $L^2(\Omega)$. Here $\bar{u} \in U_{ad}$ because the admissible set $U_{ad}$ is weakly closed in $L^2(\Omega)$. The sequence of corresponding states: $\{\bar{y}_h\}_h$ is such that $\bar{y}_h \rightarrow \bar{y}$ in $H^1_0(\Omega)$, as $h \to 0$. Moreover, the compact embedding of $H^1_0(\Omega)$ into $L^2(\Omega)$ implies that $\bar{y}_h \rightarrow \bar{y}$ in $L^2(\Omega)$ as $h \to 0$.

Using the fact that $\bar{u}_h \rightharpoonup \bar{u}$ in $BV(\Omega)$ implies that $\bar{u}_h \rightarrow \bar{u}$ in $L^1(\Omega)$ and by Lemma 2] we have also that $\Upsilon_{q,\gamma}(\bar{u}_h) \rightarrow \Upsilon_{q,\gamma}(\bar{u})$. Then, by using the convexity of $F^h$ and the fact that $\bar{u}_h$ is a solution for $\{P^h\}$, we estimate:

$$J_s(\bar{u}) = \frac{1}{2} \|\bar{y} + yf - yd\|^2_{L^2(\Omega)} + \frac{\alpha}{2} \|\bar{u}\|^2_{L^2(\Omega)} + \beta \Upsilon_{q,\gamma}(\bar{u})$$

$$\leq \liminf_{h \to 0} \left( \frac{1}{2} \|\bar{y}_h + yf - yd\|^2_{L^2(\Omega)} + \frac{\alpha}{2} \|\bar{u}_h\|^2_{L^2(\Omega)} \right) + \beta \Upsilon_{q,\gamma}(\bar{u})$$

$$\leq \liminf_{h \to 0} \left( F^h(\bar{u}_h) + \beta \Upsilon_{q,\gamma}(\bar{u}_h) \right) + \beta \Upsilon_{q,\gamma}(\bar{u}) - \liminf_{h \to 0} \beta \Upsilon_{q,\gamma}(\bar{u}_h)$$

$$= \liminf_{h \to 0} J^h(\bar{u}_h)$$

$$\leq \liminf_{h \to 0} \beta c^*(\bar{u}_h),$$

Moreover, by continuity of $S$ we have that $\bar{S} = \bar{y}$, since $\bar{S}_h \rightarrow \bar{S}$ and $\bar{u}_h \rightarrow \bar{u}$ in $L^1(\Omega)$, respectively. Taking into account that $c_h u \rightarrow u$ in $L^2(\Omega)$ and $\bar{S}^{\prime} \bar{u} \rightarrow \bar{S}$ in $L^2(\Omega)$ as $h \to 0$, it follows that $F^{\prime}(\bar{S}^\prime \bar{u}) \rightarrow F(\bar{u})$. Hence, from (39), we obtain

$$J_s(\bar{\bar{u}}) \leq J_s(\bar{\bar{u}}), \quad \forall \bar{u} \in U_{ad} \cap BV(\Omega).$$

Since $\bar{\bar{u}} \in U_{ad} \cap BV(\Omega)$ then $\bar{\bar{u}}$ is an optimal control for $\{P\}$. Therefore, we use the notation $\bar{u} = \bar{\bar{u}}$ and $\bar{y} = \bar{\bar{y}}$. 

Recalling that $\bar{u}_h \to \bar{u}$ in $L^2(\Omega)$ and $\bar{u}_h \to \bar{u}$ in $L^1(\Omega)$, we have by (i) that $J_\alpha(\bar{u}_h) \to J_\alpha(\bar{u})$ and $(\alpha \bar{u} + \bar{\phi}, \bar{u}_h - \bar{u})_{L^2(\Omega)} \to 0$. Furthermore, we have that $\beta \mathcal{T} \bar{u}_h \to \beta \mathcal{T} \bar{u}_h$ by Lemma 2) and $\bar{y}_h \to \bar{y}$ in $L^2(\Omega)$. Passing to the limit altogether gives the result.

In order to prove (ii) we use Assumption 2] let us first consider $T \in \mathcal{T}_h^0 = \{T \in \mathcal{T}_h : \bar{u}_T = \Pi_h \bar{u}(x) | T = 0 \}$, therefore

$$\left| \bar{u}_{h,T} \right| = \frac{1}{|T|} \int_T \bar{u}_h(x)dx = \frac{1}{|T|} \left| \int_T \bar{u}_h(x) - \bar{u}_T dx \right|$$

$$\leq \frac{1}{|T|} \left| \int_T \bar{u}_h(x) - \Pi_h \bar{u}(x) dx \right|$$

$$\leq \frac{1}{|T|} \left| \int_T \bar{u}_h(x) - \bar{u}(x) dx + \left| \int_T \bar{u}(x) - \Pi_h \bar{u}(x) dx \right| \right|$$

by using Lemma 3 we get

$$\left| \bar{u}_{h,T} \right| \leq \frac{1}{|T|} \int_T |\bar{u}_h(x) - \bar{u}(x)|dx + \|\bar{u} - \mathcal{C}_h \bar{u}\|_{L^2(\Omega)} + \|\Pi_h (\bar{u}_x - \bar{u})\|_{L^2(\Omega)}$$

$$\leq \frac{1}{|T|} \int_T |\bar{u}_h(x) - \bar{u}(x)|dx + ch^\frac{1}{2} |Du(\Omega)|^{\frac{1}{2}} + ch.$$

From this relation and since $\bar{u}_h \to \bar{u}$ in $L^1(\Omega)$ as $h \to 0$ it follows that $|\bar{u}_{h,T}| < \frac{1}{\epsilon}$ for $h$ sufficiently small, we have that $\bar{u}_{h,T} = 0$.

Now, let us analyze the case $T \in \mathcal{T}_h^\#$; since $\bar{u}_h \to \bar{u}$ in $L^1(\Omega)$, there exists $h_1 > 0$ such that $|\bar{u}_{h,T} - \bar{u}_T| = |\bar{u}_{h,T} - \frac{1}{|T|} \int_T \bar{u}(x)dx| < \epsilon$, for all $h < h_1$. Moreover, by noticing that $|\bar{u}_T| = \frac{1}{|T|} \int_T \bar{u}(x)dx > \rho + \frac{1}{\gamma}$, we have that sign$(\bar{u}_{h,T}) = \text{sign}(\bar{u}_T) \neq 0$. Consequently,
sign(\(\bar{w}(x) - \bar{\phi}(x)\)) = sign(\(\bar{w}_h(x) - \bar{\phi}_h(x)\)). Therefore, using Theorem 2 it follows that

\[
|\bar{u}_h(x) - \bar{u}(x)| = \left| P_{u_a, u_h} \left( \frac{\beta \bar{w}(x) - \bar{\phi}(x)}{\alpha} - \frac{\beta \delta}{\alpha} \text{sign}(\bar{w}(x) - \bar{\phi}(x)) \right) \right|
\]

\[
- P_{u_a, u_h} \left( \frac{\beta \bar{w}_h(x) - \bar{\phi}_h(x)}{\alpha} - \frac{\beta \delta}{\alpha} \text{sign}(\bar{w}_h(x) - \bar{\phi}_h(x)) \right)
\]

\[
\leq \left| \frac{\beta \bar{w}(x) - \bar{\phi}(x)}{\alpha} - \frac{\beta \bar{w}_h(x) - \bar{\phi}_h(x)}{\alpha} \right|
\]

\[
\leq \frac{\beta}{\alpha} |\bar{w}(x) - \bar{w}_h(x)| + \frac{1}{\alpha} |\bar{\phi}(x) - \bar{\phi}_h(x)|.
\]

(40)

By the definition of \(\bar{w}_h\), we have that \(\bar{w}_h = j(\bar{u}_h) = j(\bar{u}_{h_T})\), for almost all \(x \in T\). Then, denoting \(u_\tau = \bar{u}(x) + \tau(\bar{u}_{h_T} - \bar{u}(x))\), with \(\tau \in (0, 1)\), and considering that \(\text{sign}(\bar{u}_{h_T}) = \text{sign}(\bar{u}_T) \neq 0\) the following estimate holds:

\[
|\bar{w}(x) - \bar{w}_h(x)| \leq q \left( |\bar{u}(x)| + \frac{q - 1}{\gamma} \right)^{q-1} - \left( |\bar{u}_{h_T}| + \frac{q - 1}{\gamma} \right)^{q-1}
\]

\[
= q(1 - q)|\bar{u}(x) - \bar{u}_{h_T}| \int_0^1 \left( |u_\tau| + \frac{q - 1}{\gamma} \right)^{q-2} d\tau
\]

\[
= c_T(x)|\bar{u}(x) - \bar{u}_{h_T}|,
\]

where \(c_T : T \rightarrow \mathbb{R}\) is given by \(c_T(x) := q(1 - q) \int_0^1 \left( |u_\tau(x)| + \frac{q - 1}{\gamma} \right)^{q-2} d\tau\). In addition, by Assumption 2 and construction of \(u_\tau\), there exist \(C_0 > 1\) such that \(|u_\tau(x)| > c_0 s^* + \frac{1-q}{\gamma}\). Then, there exists \(c_0 < 1\) such that \(\|c_T\|_{L^\infty(T)} < c_0 S^*\) for all \(T \in T^\#\). Therefore

\[
|\bar{w}(x) - \bar{w}_h(x)| < c_0 \frac{\alpha}{\beta} |\bar{u}(x) - \bar{u}_{h_T}| = c_0 \frac{\alpha}{\beta} |\bar{u}(x) - \bar{u}_h(x)|.
\]

Since \(x\) is fixed, the last relation can be replaced in (40), implying that

\[
|\bar{u}_h(x) - \bar{u}(x)| \leq c_1 |\bar{\phi}(x) - \bar{\phi}_h(x)|,
\]

for some constant \(c_1 > 0\) independent of \(h\). Moreover, we can estimate

\[
|\bar{u}_h(x) - \bar{u}(x)| \leq c_1 |\bar{\phi}(x) - \bar{\phi}_h(x)|
\]

\[
\leq c_1 \|\bar{\phi} - \bar{\phi}_h\|_{L^\infty(T)}
\]

\[
\leq c_1 \|\bar{\phi} - \bar{\phi}_h\|_{L^\infty(\Omega)}.
\]

Finally, in view of Proposition 5 and (ii), taking the limit \(h \rightarrow \infty\) we obtain

\[
\lim_{h \rightarrow 0} |\bar{u}_h(x) - \bar{u}(x)| \leq c_1 \lim_{h \rightarrow 0} \|\bar{\phi} - \bar{\phi}_h\|_{L^\infty(\Omega)} \leq c \lim_{h \rightarrow 0} \|\bar{u} - \bar{u}_h\|_{L^2(\Omega)} = 0.
\]

4. Error estimates

In what follows, let us denote by \(\{\bar{u}_h\}\), the sequence of global solutions of problems \(\{P_h\}\) such that, under Assumptions 2 and 3, \(\lim_{h \rightarrow 0} \|\bar{u}_h - \bar{u}\|_{L^\infty(\Omega)} = 0\), where \(\bar{u}\) is a solution of \(\{P\}\), see Theorem 2. We will obtain the estimate of \(\bar{u} - \bar{u}_h\) in the \(L^2\) norm.
using optimality conditions. We start this section with the following lemma, which invokes necessary optimality conditions \(^{(7c)}\) and \(^{(29c)}\).

**Lemma 4.** Let \(g\) and \(\hat{g}\) be functions in \(L^2(\Omega)\), and let \(u\) and \(\hat{u}_h\) be optimal controls, with corresponding states \(y\) and \(\hat{y}_h\), the solutions of \(^{(10)}\) and \(^{(33)}\) associated to \(g\) and \(\hat{g}\), respectively. Then, for all \(v \in U_{ad}\) and all \(v_h \in U_{ad}^h\) the following estimate holds:

\[
\alpha \|u - \hat{u}_h\|_{L^2(\Omega)}^2 + \|y - \hat{y}_h\|_{L^2}^2 \\
\leq (\phi, v - \hat{u}_h + v_h - u)_{L^2(\Omega)} + \alpha(u, v - \hat{u}_h + v_h - u)_{L^2(\Omega)} \\
+ \alpha(\hat{u}_h, v_h - u)_{L^2(\Omega)} \\
+ (\hat{y}_h - y, (S_h - S)v_h + S(v_h - u))_{L^2(\Omega)} + (y - y_d, (S_h - S)(v_h - \hat{u}_h)_{L^2(\Omega)} \\
+ \beta \delta_{\gamma}(\|v\|_{L^1(\Omega)} + \|u\|_{L^1(\Omega)} - \|v_h\|_{L^1(\Omega)} - \|\hat{u}_h\|_{L^1(\Omega)}) \\
- \beta(g, v - u)_{L^2(\Omega)} - \beta(\hat{g}, v_h - \hat{u}_h)_{L^2(\Omega)}.
\]

Proof. Let \(u\) and \(\hat{u}_h\) satisfying their corresponding first optimality conditions, with states \(y = Su + y_f\) and \(\hat{y}_h = S_h\hat{u}_h + y_f\), adjoint states \(\hat{\phi} \in H_0^1(\Omega)\) and \(\hat{\phi}_h \in Y_h\), multipliers \(\zeta\) and \(\hat{\zeta}\), respectively. Then, we have

\[
(\phi + \beta(\delta_\gamma \zeta - g), v - u)_{L^2(\Omega)} + \alpha(u, v - u)_{L^2(\Omega)} \geq 0, \quad \forall v \in U_{ad}\] and 

\[
(\hat{\phi}_h + \beta(\delta_\gamma \hat{\zeta} - \hat{g}), v_h - \hat{u}_h)_{L^2(\Omega)} + \alpha(\hat{u}_h, v_h - \hat{u}_h)_{L^2(\Omega)} \geq 0, \quad \forall v_h \in U_{ad}^h.
\]

by adding these both inequalities we get that

\[
0 \leq (\phi, v - \hat{u}_h)_{L^2(\Omega)} + \alpha(u, v - u)_{L^2(\Omega)} + (\phi, \hat{u}_h - u)_{L^2(\Omega)} + \alpha(\hat{u}_h, \hat{u}_h - u)_{L^2(\Omega)} \\
+ (\hat{\phi}_h, v_h - u)_{L^2(\Omega)} + \alpha(\hat{u}_h, v_h - u)_{L^2(\Omega)} + (\hat{\phi}_h, u - \hat{u}_h)_{L^2(\Omega)} + \alpha(\hat{u}_h, u - \hat{u}_h)_{L^2(\Omega)} \\
+ \beta(\delta_\gamma \zeta - g, v - u)_{L^2(\Omega)} + \beta(\delta_\gamma \hat{\zeta} - \hat{g}, v_h - \hat{u}_h)_{L^2(\Omega)}.
\]

Taking the right terms conveniently to the left–hand side, it follows that

\[
\alpha(u - \hat{u}_h, u - \hat{u}_h)_{L^2(\Omega)} \leq (\phi, v - \hat{u}_h)_{L^2(\Omega)} + \alpha(u, v - u)_{L^2(\Omega)} \\
+ (\hat{\phi}_h, v_h - u)_{L^2(\Omega)} + \alpha(\hat{u}_h, v_h - u)_{L^2(\Omega)} + (\phi - \hat{\phi}_h, \hat{u}_h - u)_{L^2(\Omega)} \\
+ \beta(\delta_\gamma \zeta - g, v - u)_{L^2(\Omega)} + \beta(\delta_\gamma \hat{\zeta} - \hat{g}, v_h - \hat{u}_h)_{L^2(\Omega)}.
\]

Let us focus on the last two terms. Since \(\zeta \in \partial \|\cdot\|_{L^1(\Omega)}(u)\) and \(\hat{\zeta} \in \partial \|\cdot\|_{L^1(\Omega)}(\hat{u}_h)\), we obtain:

\[
\beta(\delta_\gamma \zeta - g, v - u)_{L^2(\Omega)} + \beta(\delta_\gamma \hat{\zeta} - \hat{g}, v_h - \hat{u}_h)_{L^2(\Omega)} = \beta\delta_{\gamma}(\zeta, v - u)_{L^2(\Omega)} \\
- \beta(g, v - u)_{L^2(\Omega)} + \beta(\hat{\gamma}, v_h - \hat{u}_h)_{L^2(\Omega)} - \beta(\hat{\gamma}, v_h - \hat{u}_h)_{L^2(\Omega)} \\
\leq \beta\delta_{\gamma}(\|v\|_{L^1(\Omega)} - \|u\|_{L^1(\Omega)} + \|v_h\|_{L^1(\Omega)} - \|\hat{u}_h\|_{L^1(\Omega)}) \\
- \beta(g, v - u)_{L^2(\Omega)} - \beta(\hat{g}, v_h - \hat{u}_h)_{L^2(\Omega)}.
\]

Now, by considering the adjoint–state terms in \(^{(41)}\), it follows that

\[
(\phi, v - \hat{u}_h)_{L^2(\Omega)} + (\hat{\phi}_h, v_h - u)_{L^2(\Omega)} + (\phi - \hat{\phi}_h, \hat{u}_h - u)_{L^2(\Omega)} \\
= (\phi, v - \hat{u}_h + v_h - u)_{L^2(\Omega)} + (\phi - \hat{\phi}_h, u - v_h)_{L^2(\Omega)} + (\phi - \hat{\phi}_h, \hat{u}_h - u)_{L^2(\Omega)} \\
= (\phi, v - \hat{u}_h + v_h - u)_{L^2(\Omega)} - (\phi, v_h - \hat{u}_h)_{L^2(\Omega)} + (\hat{\phi}_h, v_h - \hat{u}_h)_{L^2(\Omega)}.
\]
Furthermore, we have that
\[(\phi, v - \hat{u}_h + v_h - u)_{L^2(\Omega)}\]
\[- (y - y_d, S v_h - S \hat{u}_h)_{L^2(\Omega)} + (\hat{y}_h - y_d, S h v_h - S h \hat{u}_h)_{L^2(\Omega)}\]
\[= (\phi, v - \hat{u}_h + v_h - u)_{L^2(\Omega)} - (y - y_d, S v_h - S \hat{u}_h)_{L^2(\Omega)} + (\hat{y}_h - y_d, S h v_h - \hat{y}_h)_{L^2(\Omega)}\]
\[= (\phi, v - \hat{u}_h + v_h - u)_{L^2(\Omega)} - (\hat{y}_h - y, y - \hat{y}_h)_{L^2(\Omega)} + (\hat{y}_h - y, (S_h - S)(v_h - \hat{u}_h))_{L^2(\Omega)}\]
\[(43) \quad - (\hat{y}_h - y, y - \hat{y}_h)_{L^2(\Omega)} + (\hat{y}_h - y, (S_h - S)v_h + S(v_h - u))_{L^2(\Omega)}\]

Furthermore, we have that
\[\alpha(u, v - \hat{u}_h)_{L^2(\Omega)} + \alpha(\hat{u}_h, v_h - u)_{L^2(\Omega)} = \alpha(u, v - \hat{u}_h + v_h - u)_{L^2(\Omega)}\]
\[+ \alpha(\hat{u}_h - u, v_h - u)_{L^2(\Omega)}\]
\[(44) \quad \text{Replacing relations (43), (42) and (44) in (41) we conclude the result.} \]

**Lemma 5.** Let \(g\) and \(\hat{g}\) be functions in \(L^\infty(\Omega)\), and let \(u\) and \(\check{u}_h\), with corresponding states \(y\) and \(\check{y}_h\), the solutions of (10) and (33) associated to \(g\) and \(\hat{g}\), respectively. Then, there exists a constant \(c(\epsilon)\) independent of \(h\), such that
\[\frac{1}{2} \alpha\|y - \hat{y}_h\|_{L^2}^2 + \frac{1}{2} \|y - \hat{y}_h\|_{L^2}^2 \leq \alpha(h + \beta(g - \hat{g}, u - \check{u}_h))_{L^2(\Omega)},\]
for some \(\epsilon \in (0, 1)\).

**Proof.** Following the argument from [37] Lemma 4.2], it follows that
\[\|\Pi_h u\|_{L^1} = \|\sum_{T \in \mathcal{T}_h} u_T \chi_T\|_{L^1} \leq \sum_{T \in \mathcal{T}_h} \int_{\Omega} |u_T| \chi_T = \sum_{T \in \mathcal{T}_h} \int_T |u| \, dx = |T| = \|u\|_{L^1}.\]
In addition, by the inclusion \(U^h_{ad} \subset U_{ad}\), let us consider Lemma [3] with \(v = \check{u}_h\) and \(v_h = \Pi_h u\). Thus, we get the relation
\[\alpha(u - \check{u}_h)_{L^2(\Omega)} + \|y - \check{y}_h\|_{L^2}^2 \leq (\phi, \Pi_h u - u)_{L^2(\Omega)} + \alpha(u, \Pi_h u - u)_{L^2(\Omega)} + \alpha(\check{u}_h - u, \Pi_h u - u)_{L^2(\Omega)}\]
\[+ (\hat{y}_h - y, (S_h - S)(\Pi_h u - u))_{L^2(\Omega)} + \beta(g, \hat{y}_h - y, (S_h - S)(\Pi_h u - u))_{L^2(\Omega)}\]
\[\leq \|\phi\|_{L^\infty(\Omega)} \|\Pi_h u - u\|_{L^1(\Omega)} + \alpha\|u\|_{L^\infty(\Omega)} \|\Pi_h u - u\|_{L^1(\Omega)}\]
\[+ \alpha(\check{u}_h - u, \Pi_h u - u)_{L^2(\Omega)} + (\hat{y}_h - y, (S_h - S)(\Pi_h u - u))_{L^2(\Omega)}\]
\[+ \|y - y_d\|_{L^2(\Omega)} (S_h - S)(\Pi_h u - u)_{L^2(\Omega)} \]
\[- \beta(g, \check{u}_h - u, \Pi_h u - u)_{L^2(\Omega)} - \beta(g, \hat{u}_h - y, (S_h - S)(\Pi_h u - u))_{L^2(\Omega)}\]
\[(45) \quad - \beta(g, \hat{u}_h - u, \Pi_h u - u)_{L^2(\Omega)} - \beta(g, \check{u}_h - u, \Pi_h u - u)_{L^2(\Omega)}\]

Notice that the \(L^1\)-norm terms have vanished. Moreover by using the Corollary [1] for \(\|\Pi_h u - u\|_{L^1(\Omega)}\), we are able to get a estimation for the first and second term given by
As for third term, due to Young’s inequality and the Corollary 1 we have that
\[
\alpha(\hat{u}_h - u, \Pi_h u - u)_{L^2(\Omega)} \leq \epsilon \alpha \|\hat{u}_h - u\|_{L^2(\Omega)}^2 + \frac{\alpha}{4\epsilon} \|\Pi_h u - u\|_{L^2(\Omega)}^2
\]
\[
\leq \epsilon \alpha \|\hat{u}_h - u\|_{L^2(\Omega)}^2 + \frac{\alpha}{4\epsilon} \|\Pi_h u - u\|_{L^2(\Omega)}^2
\]
\[
\leq \epsilon \alpha \|\hat{u}_h - u\|_{L^2(\Omega)}^2 + \frac{\alpha}{4\epsilon} h,
\]
for any \(\epsilon \in (0,1)\). Again, by using Young’s inequality, it follows that
\[
(y_h - y, (S_h - S)\Pi_h u + S\hat{u}_h - u)) \leq \frac{1}{2} \|y - \hat{y}_h\|_{L^2(\Omega)}^2 + \frac{1}{2} \|(S_h - S)\Pi_h u + S(\Pi_h u - u)\|_{L^2(\Omega)}^2
\]
\[
= \frac{1}{2} \|y - \hat{y}_h\|_{L^2(\Omega)}^2 + \|(S_h - S)\Pi_h u\|_{L^2(\Omega)}^2 + \|S(\Pi_h u - u)\|_{L^2(\Omega)}^2.
\]
In view of Proposition 2 (applied with control \(\Pi_h u\)) and the continuity of control–to–state operator; using a generic constant \(c\), we get that
\[
(y_h - y, (S_h - S)\Pi_h u + S\hat{u}_h - u)) \leq \frac{1}{2} \|y - \hat{y}_h\|_{L^2(\Omega)}^2 + c^2 h^4 + c\|\Pi_h u - u\|_{L^2(\Omega)}^2
\]
\[
\leq \frac{1}{2} \|y - \hat{y}_h\|_{L^2(\Omega)}^2 + ch^4 + ch.
\]
The last two terms in (45) can be majorized as follows:
\[
-\beta(\hat{u}_h - u)_{L^2(\Omega)} - \beta(\hat{g}, \Pi_h u - \hat{u}_h)_{L^2(\Omega)} = \beta(\hat{g} - g, \hat{u}_h - u)_{L^2(\Omega)} - \beta(\hat{g}, \Pi_h u - u)_{L^2(\Omega)}
\]
\[
\leq \beta(\hat{g} - g, \hat{u}_h - u)_{L^2(\Omega)} + \beta \|\hat{g}\|_{L^\infty(\Omega)} \|u - \Pi_h u\|_{L^2(\Omega)}.
\]
By applying Proposition 2 and Corollary 1 and inserting (46)–(48) in (45); using a generic constant \(c\) independent of \(h\), we obtain
\[
\alpha \|u - \hat{u}_h\|_{L^2(\Omega)}^2 + \|y - \hat{y}_h\|_{L^2(\Omega)}^2
\]
\[
\leq c(\|\phi\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\Omega)}) h + \epsilon \alpha \|\hat{u}_h - u\|_{L^2(\Omega)}^2
\]
\[
+ c \frac{\alpha}{4\epsilon} h + \frac{1}{2} \|y - \hat{y}_h\|_{L^2(\Omega)}^2 + ch^4 + ch + \|y - y_d\|_{L^2(\Omega)} c\|\hat{g}\|_{L^\infty(\Omega)}^2 + h^2
\]
\[
+ \beta c \|\hat{g}\|_{L^\infty(\Omega)} h + \beta(\hat{g} - g, \hat{u}_h)_{L^2(\Omega)},
\]
by taking similar terms to the left–hand side and using \(c\) as a generic constant (independent of \(h\)) in the higher order terms, we get
\[
(1 - \epsilon) \alpha \|u - \hat{u}_h\|_{L^2(\Omega)}^2 + \frac{1}{2} \|y - \hat{y}_h\|_{L^2(\Omega)}^2 \leq ch + c \frac{\alpha}{4\epsilon} h + ch^4 + ch + \|y - y_d\|_{L^2(\Omega)} c\|\hat{g}\|_{L^\infty(\Omega)}^2
\]
\[
+ \beta c \|\hat{g}\|_{L^\infty(\Omega)} h + \beta(\hat{g} - g, \hat{u}_h)_{L^2(\Omega)}
\]
\[
= \hat{c}(\epsilon) h + \beta(\hat{g} - g, u - \hat{u}_h)_{L^2(\Omega)}
\]
Therefore, by considering the constant \(\hat{c}(\epsilon) = \frac{\alpha}{4\epsilon} c + c\|\hat{g}\|_{L^\infty(\Omega)}^2\), independent of \(h\), we obtain the desired result.
4.1. Order of convergence.

**Remark 3.** We emphasize that the estimates obtained in the previous results depend on the difference \( g - \hat{g} \). For problem \( (P_2) \), we may apply these results taking \( g = w(\bar{u}) \) and \( \hat{g} = w(\bar{u}_h) \) and estimate the terms involving the difference \( w(\bar{u}) - w(\bar{u}_h) \) taking advantage of the Lipschitz continuity of the mapping \( u \mapsto w(u) \). However, more information can be acquired using Pontryagin’s maximum principle.

**Theorem 5.** Let \( \bar{u} \in U_{ad} \) solution problem \( (P_2) \). Under Assumptions 2 and 3, we have the following error estimate for sufficiently small mesh–size parameter \( h \):

\[
\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)} + \|\bar{y} - \bar{y}_h\|_{L^2} \leq c h^{\frac{1}{2}},
\]

for small enough and some constant \( c > 0 \) independent of \( h \).

Proof. By choosing \( u = \bar{u} \) and \( \hat{u}_h = \bar{u}_h \) as well as \( g = \bar{w} = j(\bar{u}) \) and \( \hat{g} = \bar{w}_h = j(\bar{u}_h) \) in Lemma 5 it follows that

\[
(1 - \epsilon)\alpha\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)}^2 + \frac{1}{2}\|\bar{y} - \bar{y}_h\|_{L^2}^2 \leq \hat{c}(\epsilon)h + \beta(\bar{w} - \bar{w}_h, \bar{u} - \bar{u}_h)_{L^2(\Omega)}.
\]

Notice that \( \|\bar{y}\|_{L^\infty(\Omega)} \leq \|w(\bar{u}_h)\|_{L^\infty(\Omega)} \leq \delta \gamma \) due to the Lemma 2, thus, \( \hat{c}(\epsilon) \) does not depend on \( h \).

Now, let us estimate the nonnegative term \( (\bar{w} - \bar{w}_h, \bar{u} - \bar{u}_h)_{L^2(\Omega)} \) in a manner that can be taken to the left–hand side of (49). Since the Corollary 1 (applied with control \( \|w(x) - \Pi_h w(x)\|_{L^1(\Omega)} \)), there is a positive constant \( \hat{c} \) such that

\[
\int_{\Omega} (\bar{w}(x) - \bar{w}_h(x))(\bar{u}(x) - \bar{u}_h(x)) \, dx \leq \hat{c} h + \int_{\Omega} (\Pi_h \bar{w}(x) - \bar{w}_h(x))(\bar{u}(x) - \bar{u}_h(x)) \, dx.
\]

Next, observe that \( \Pi_h \bar{w} \) and \( \bar{w}_h \) belong to \( U_h \). Therefore, by expressing both quantities in terms of the basis of \( U_h \), we have

\[
\int_{\Omega} (\Pi_h \bar{w}(x) - \bar{w}_h(x))(\bar{u}(x) - \bar{u}_h(x)) \, dx = \int_{\Omega} \sum_{T \in T_h} (\bar{w}_T - \bar{w}_{h,T}) \chi_T(x)(\bar{u}(x) - \bar{u}_h(x)) \, dx,
\]

We continue our analysis of this expression by taking into account the sets of elements \( T_h^0 \), \( T_h^\# \) and \( T_h^* \). First, using the same arguments of Theorem 4, we infer that \( \bar{w}_{h,T} = 0 \) for all \( T \in T_h^0 \) and that sign(\( u_{h,T} \)) = sign(\( \bar{w}_{h,T} \)) \( \neq 0 \), for all \( T \in T_h^\# \) if \( h \) is sufficiently small. Hence, using the Assumption 3, we obtain

\[
\int_{\Omega} (\Pi_h \bar{w}(x) - \bar{w}_h(x))(\bar{u}(x) - \bar{u}_h(x)) \, dx \leq \int_{\Omega} \sum_{T \in T_h^\#} \left( \frac{1}{|T|} \int_T \bar{w}(\xi) \, d\xi - \bar{w}_{h,T} \right) \chi_T(x) \left| \bar{u}(x) - \bar{u}_h(x) \right| \, dx + \hat{c} h.
\]
Since $j$ is differentiable for $T \in T_h^\#$, this gives

$$\left| \sum_{T \in T_h^\#} \left( \frac{1}{|T|} \int_T \bar{w}(\xi) \, d\xi - \bar{w}_{ht} \right) \chi_T(x) \right| \leq \sum_{T \in T_h^\#} \left| \left( \frac{1}{|T|} \int_T \bar{w}(\xi) \, d\xi - \bar{w}_{ht} \right) \chi_T(x) \right|
$$

$$\leq \sum_{T \in T_h^\#} \left( \frac{1}{|T|} \int_T \left| \bar{w}(\xi) - \bar{w}_{ht} \right| \, d\xi \right) \chi_T(x)
$$

$$\leq \sum_{T \in T_h^\#} \left( \frac{q}{|T|} \int_T \left( \left| \bar{u}_{ht} \right| + \frac{q-1}{\gamma} \right)^{q-1} \left( \left| \bar{u}(\xi) \right| + \frac{q-1}{\gamma} \right)^{q-1} \left| d\xi \right| \chi_T(x) \right)
$$

$$\leq \sum_{T \in T_h^\#} q(1-q) \left( \frac{1}{|T|} \int_T \left( \bar{u}(\xi) \right) + \frac{q-1}{\gamma} \right)^{q-3} \left( \bar{u}(\xi) - \bar{u}_{ht} \right)^2 d\xi) \chi_T(x)
$$

(52)

$$+ \sum_{T \in T_h^\#} \frac{q(1-q)(2-q)}{|T|} \left( \int_T \left( \bar{u}(\xi) \right) + \frac{q-1}{\gamma} \right)^{q-3} \left( \bar{u}(\xi) - \bar{u}_{ht} \right)^2 d\xi) \chi_T(x),
$$

where $\bar{u}(\xi)$ lies between $\bar{u}(\xi)$ and $\bar{u}_{ht}$. Moreover, since $|\bar{u}(\xi)| > \rho + \frac{1}{\gamma}$, it follows that

$$\left| \sum_{T \in T_h^\#} \left( \frac{1}{|T|} \int_T \bar{w}(\xi) \, d\xi - \bar{w}_{ht} \right) \chi_T(x) \right|
$$

$$\leq \sum_{T \in T_h^\#} q(1-q) \left( \frac{1}{|T|} \int_T \left( \bar{u}(\xi) \right) + \frac{q-1}{\gamma} \right)^{q-2} \left| \bar{u}(\xi) - \bar{u}_{ht} \right| d\xi \chi_T(x)
$$

$$+ \sum_{T \in T_h^\#} \frac{q(1-q)(2-q)}{|T|} \left( \int_T \left( \frac{q}{\gamma} \right)^{q-3} \left( \bar{u}(\xi) - \bar{u}_{ht} \right)^2 d\xi \right) \chi_T(x)
$$

$$= \sum_{T \in T_h^\#} q(1-q) \left( \frac{1}{|T|} \int_T \left( \bar{u}(\xi) \right) + \frac{q-1}{\gamma} \right)^{q-2} \left| \bar{u}(\xi) - \bar{u}_{ht} \right| d\xi \chi_T(x)
$$

(53)

$$+ c_q \sum_{T \in T_h^\#} \left( \frac{1}{|T|} \int_T \left( \bar{u}(\xi) - \bar{u}_{ht} \right)^2 d\xi \right) \chi_T(x),
$$

with $c_q = q(1-q)(2-q) \left( \frac{q}{\gamma} \right)^{q-3}$. Therefore, Theorem 2 and Assumption 2 imply that

(54)

$$\beta(1-q)q \left( \bar{u}(x) + \frac{q-1}{\gamma} \right)^{q-2} < c_\# \alpha,
$$

where $c_\#$ is a positive constant in $(0, 1)$. Then, because of monotonicity it follows that if $0 < \bar{u}(\xi) \leq u_b$, therefore $\beta(1-q)q \left( \bar{u}(\xi) + \frac{q-1}{\gamma} \right)^{q-2} < \alpha$. By similar arguments, we arrive to the same conclusion in the case $u_a \leq \bar{u}(\xi) < -\rho - \frac{1}{\gamma} < 0$. Hence, using this
relation in (53) we obtain
\[
\beta \left| \sum_{T \in \mathcal{T}_h} \left( \frac{1}{|T|} \int_T \bar{w}(\xi) \, d\xi - \bar{w}_h \right) \chi_T(x) \right| \leq \alpha c_h \sum_{T \in \mathcal{T}_h} \left( \frac{1}{|T|} \int_T |\bar{u}(\xi) - \bar{u}_h| \, d\xi \right) \chi_T(x)
+ \beta c_q \sum_{T \in \mathcal{T}_h} \left( \frac{1}{|T|} \int_T (\bar{u}(\xi) - \bar{u}_h)^2 \, d\xi \right) \chi_T(x)
\leq \alpha c_h \Pi_h(|\bar{u} - \bar{u}_h|) + \beta c_q \Pi_h(\bar{u} - \bar{u}_h)^2,
\]
which we insert into (51). Also, applying Young’s inequality we get
\[
\beta \int_{\Omega} (\Pi_h \bar{w}(x) - \bar{w}_h(x))(\bar{u}(x) - \bar{u}_h(x)) \, dx \leq \alpha c_h \int_{\Omega} \Pi_h(|\bar{u} - \bar{u}_h|)|\bar{u}(x) - \bar{u}_h(x)| \, dx
+ \beta c_q \int_{\Omega} \Pi_h(\bar{u} - \bar{u}_h)^2|\bar{u}(x) - \bar{u}_h(x)| \, dx + ch
\leq \alpha c_h \|\bar{u} - \bar{u}_h\|_{L^2(\Omega)}^2 + c\|\bar{u} - \bar{u}_h\|_{L^\infty(\Omega)}^2 \|\bar{u} - \bar{u}_h\|_{L^2(\Omega)}^2 + ch,
\]
where \(c\) and \(C\) are positive constants. In addition, by Theorem 4 (iii), there exists \(h_2 > 0\) such that \(\|\bar{u} - \bar{u}_h\|_{L^\infty(\Omega)} \leq c\alpha/c\), for all \(h \leq h_2\). Therefore
\[
\beta \int_{\Omega} (\Pi_h \bar{w}(x) - \bar{w}_h(x))(\bar{u}(x) - \bar{u}_h(x)) \, dx \leq \alpha c_h \|\bar{u} - \bar{u}_h\|_{L^2(\Omega)}^2 + c\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)}^2 + ch.
\]
Using (49), (50) and (56) we estimate
\[
(1 - 2c)\alpha \|\bar{u} - \bar{u}_h\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\bar{y} - \bar{y}_h\|_{L^2}^2 \leq \hat{c}(\epsilon) h + \alpha c_h \|\bar{u} - \bar{u}_h\|_{L^2(\Omega)}^2,
\]
where the positive constants \(\hat{c}(\epsilon)\) and \(c\) were redefined. Finally, choosing \(\epsilon = \frac{1 - \epsilon_h}{4}\) and taking the term \(c_h \alpha \|\bar{u} - \bar{u}_h\|_{L^2(\Omega)}^2\) to the left-hand side and obtaining the square root, we conclude the assertion.

\[\Box\]

5. Numerical experiments

In this final section, we verify the previous error estimate through an example implemented in Matlab. The numerical solution is computed by discretizing the state and adjoint equation in the corresponding approximation spaces. Then we solve the optimality system using a semi-smooth Newton method in the spirit of [29].

To confirm our theoretical convergence rate, we compute the experimental order of convergence by using a family of decreasing values of \(h\) meshes. We consider the exact solution as the corresponding numerical solution computed for a thin mesh, with which we use a reference to compare the approximated solutions for each mesh. The experimental order of convergence is calculated by:

\[
EOC := \frac{\log(||\bar{u}_h - \bar{u}_{h_1}||)}{\log(h_1) - \log(h_2)}.
\]
where \( h_1 \) and \( h_2 \) are two consecutive mesh sizes, and \( \bar{u}_h^* \) the approximate reference solution of the problem. In on the unit square domain \( \Omega = (0, 1) \times (0, 1) \).

**Example 1.** Our first example is defined on the unit square domain \( \Omega = (0, 1) \times (0, 1) \) and consider \( A = -\Delta \) and \( y_d = 10e^{-5(x^2+y^2)} \) together with fixed parameters \( \alpha = 0.24, \beta = 0.0002 \). The set of admissible controls is

\[
U_{ad} = \{ u \in L^2 : -0.8 \leq u \leq 0.55, \text{ a.a. } x \in \Omega \}.
\]

The regularization parameter is set to \( \gamma = 16000 \). The exact solution to this problem is not known. Therefore, we compute a reference solution \( \bar{u}_h^* = \bar{u}_h \) for \( h = 0.0006 \). The results are presented for different values of \( q \) in Table 1.

| \( q \) | \( h_1 \) | \( h_2 \) | \( h_3 \) | \( h_4 \) | \( h_5 \) | \( h_6 \) |
|---|---|---|---|---|---|---|
| 0.5 | 0.0366 | 0.0183 | 0.0092 | 0.0046 | 0.0023 | 0.0011 |
| EOC | 1.784 | 0.0796 | 0.0318 | 0.0155 | 0.0083 | 0.0047 |
| 0.48 | 0.1819 | 0.0818 | 0.0333 | 0.0165 | 0.0104 | 0.0063 |
| EOC | 1.2 | 1.3 | 1.0 | 0.7 | 0.7 |
| 0.45 | 0.1859 | 0.0862 | 0.0365 | 0.0170 | 0.0117 | 0.0067 |
| EOC | 1.1 | 1.2 | 1.1 | 0.6 | 0.8 |
| 0.41 | 0.1876 | 0.0912 | 0.0369 | 0.0193 | 0.0128 | 0.0083 |
| EOC | 1.0 | 1.3 | 0.9 | 0.6 | 0.6 |
| 0.40 | 0.2019 | 0.0975 | 0.0417 | 0.0259 | 0.0167 | 0.0112 |
| EOC | 1.1 | 1.2 | 0.7 | 0.6 | 0.6 |
| 0.37 | 0.2112 | 0.1104 | 0.0467 | 0.0276 | 0.0185 | 0.0125 |
| EOC | 0.9 | 1.2 | 0.8 | 0.6 | 0.6 |
| 0.36 | 0.2340 | 0.1084 | 0.0465 | 0.0301 | 0.0210 | 0.0137 |
| EOC | 1.0 | 1.2 | 0.6 | 0.5 | 0.6 |
| 0.34 | 0.2382 | 0.1118 | 0.0494 | 0.0349 | 0.0235 | 0.0151 |
| EOC | 1.1 | 1.1 | 0.5 | 0.6 | 0.6 |
| 0.33 | 0.2382 | 0.1118 | 0.0493 | 0.0349 | 0.0234 | 0.0150 |
| EOC | 1.1 | 1.1 | 0.5 | 0.6 | 0.6 |
| 0.32 | 0.2382 | 0.1118 | 0.0493 | 0.0349 | 0.0234 | 0.0150 |
| EOC | 1.1 | 1.1 | 0.5 | 0.6 | 0.6 |
| 0.31 | 0.2382 | 0.1118 | 0.0493 | 0.0349 | 0.0234 | 0.0151 |
| EOC | 1.1 | 1.2 | 0.5 | 0.6 | 0.6 |

**Table 1.** Computed error \( \| \bar{u}_h - \bar{u}_h^* \|_{L^2(\Omega)} \) and experimental order of convergence for different values of the exponent \( q \).

We notice the tendency of an order \( \approx \frac{1}{2} \) in each for different values of \( q \) as \( h \) decreases, see Figure 2. The plots of approximated optimal control for \( h = 0.0046 \) are shown in the Figure 3 for different values of \( q \).
(a) \( q = 0.5 \).
(b) \( q = 0.38 \).
(c) \( q = 0.31 \).

Figure 2. \(- \log(h)\) versus \(- \log(\|\bar{u}_h^* - \bar{u}_h\|)\) (solid line) compared with \(- \frac{1}{2} \log(h)\) (dashed line) varying the fractional exponent \( q \).

(\(A\)) \( q = 0.5 \).
(\(B\)) \( q = 0.38 \).
(\(C\)) \( q = 0.31 \).

Figure 3. \( \bar{u}_h \) computed at \( h = 0.0046 \) for different values of the fractional exponent \( q \).

6. Appendix

In this appendix, we present an extension of the results obtained in [16, Section 4] for a quasi-interpolation operator given by

\[
P_B u := \sum \pi_i(u) \phi_i, \quad \text{with} \quad \pi_i(u) = \frac{\int_{\Omega} u \phi_i}{\int_{\Omega} \phi_i},
\]

(59)

where \( \phi_i \), for \( i = 1, \ldots, n \), denote the ansatz functions, and \( \omega_i = \text{supp} \phi_i \).

We can notice the quasi-interpolation operator given in the Definition in [1] is a particular case of (59). Therefore, the results obtained in this appendix are used to prove the Lemma [3] and Corollary [1].

Lemma 6. For each \( i \in \{1, \ldots, n\} \), there is a constant \( c \) which may depend on \( \text{diam} \ \omega_i \) such that

\[
\|u - \pi_i(u)\|_{L^s(\omega_i)} \leq c \|\nabla u\|_{L^s(\omega_i)}, \quad \forall u \in W^{1,s}(\omega_i),
\]

for all \( \frac{2d}{d+2} \leq s < \infty \).

Proof. Since \( L^2(\Omega) \hookrightarrow L^1(\Omega) \), the result directly follows by using [16, Lemma 4.1].
Lemma 7. There is a constant $c$ which is independent of $h$ such that

$$
\|u - \pi_i(u)\|_{L^1(\omega_i)} \leq ch^d(1-\frac{1}{s}) + \|\nabla u\|_{L^s(\omega_i)}, \quad \forall u \in W^{1,s}(\omega_i),
$$

for all $i \in \{1, \ldots, n\}$ and all $\frac{2d}{d+2} \leq s < \infty$.

Proof. The proof of this theorem follows the same ideas of [16, Lemma 4.2]. We consider an arbitrary patch $\omega_i$ consisting of the cells $T_j^{(i)}$, for $j = 1, \ldots, M_i$. Then, for each $\omega_i$ we associate a surface $\hat{\omega}_i$ whose vertices lie on the unit ball in $\mathbb{R}^d$, and every $\hat{\omega}_i$ consists of $M_i$ congruent cells $\hat{T}_j^{(i)}$. Thus, we can define the function

$$
\hat{\pi}_i(v) = \frac{\int_{\hat{\omega}_i} \hat{\phi}_i v \, d\hat{x}}{\int_{\hat{\omega}_i} \hat{\phi}_i \, d\hat{x}} = \frac{\int_{\hat{\omega}_i} (\phi_i \circ F_i) v \, d\hat{x}}{\int_{\hat{\omega}_i} \phi_i \circ F_i \, d\hat{x}},
$$

where $F_i$ denotes the bi-Lipschitz transformation from $\hat{\omega}_i$ to $\omega_i$. Therefore, we get the next estimation

$$
\|u - \pi_i(u)\|_{L^1(\omega_i)} = \sum_{j=1}^{M_i} \frac{|T_j^{(i)}|}{|T_j^{(i)}|} \int_{T_j^{(i)}} |u(F_j^{(i)} \hat{x}) - \pi_i(u)| \, d\hat{x}
\leq c h^d \int_{\hat{\omega}_i} |u \circ F_i - \hat{\pi}_i(u \circ F_i)| \, d\hat{x}
\leq c h^d \int_{\hat{\omega}_i} \left| \int_{\hat{\omega}_i} (\hat{\phi}_i \circ F_i)(\hat{y})(u \circ F_i(\hat{x}) - u \circ F_i(\hat{y})) \, d\hat{y} \right| \, d\hat{x}
\leq c h^d \int_{\hat{\omega}_i} \|\hat{\phi}_i \circ F_i\|_{L^s(\hat{\omega}_i)} \left( \int_{\hat{\omega}_i} |\nabla \hat{x}(u \circ F_i)(\hat{x} - \hat{y})|^s \, d\hat{y} \right)^{\frac{1}{s}} \, d\hat{x}
\leq c h^d \|\hat{\phi}_i \circ F_i\|_{L^s(\omega_i)} \left( \int_{\hat{\omega}_i} |\nabla \hat{x}(u \circ F_i)|^s \, d\hat{x} \right)^{\frac{1}{s}}
\leq c h^d \sum_{j=1}^{M_i} \frac{|T_j^{(i)}|}{|T_j^{(i)}|} \left( \int_{T_j} |\nabla u|^s |\frac{\partial x}{\partial \hat{x}}|^s \, dx \right)^{\frac{1}{s}}
\leq c h^d (1-\frac{1}{s}) + \|\nabla u\|_{L^s(\omega_i)}.
$$

Lemma 8. There is a constant $c$ independent of $h$, such that

$$
\|u - \Pi_h(u)\|_{L^1(\Omega)} \leq ch^d(1-\frac{1}{s}) + \|\nabla u\|_{L^s(\Omega)}, \quad \forall u \in W^{1,s}(\Omega),
$$

with $\frac{2d}{d+2} \leq s \leq 2$. 
Proof. Following the proof of [16, Lemma 4.3] we can estimate
\[\|u - \pi_i(u)\|_{L^1(\Omega)} \leq c h^{d(1-\frac{1}{s})+1} \sum_{i=1}^n \|\nabla u\|_{L^s(\omega_i)}^{\frac{1}{s}} \leq c h^{d(1-\frac{1}{s})+1} \left(\sum_{i=1}^n \|\nabla u\|_{L^s(\omega_i)}^s\right)^{\frac{1}{s}} \leq c h^{d(1-\frac{1}{s})+1} \|\nabla u\|_{L^s(\Omega)}\].

\[\|u - \pi_i(u)\|_{L^1(\Omega)} \leq c h^{d(1-\frac{1}{s})+1} \sum_{i=1}^n \|\nabla u\|_{L^s(\omega_i)}^{\frac{1}{s}} \leq c h^{d(1-\frac{1}{s})+1} \left(\sum_{i=1}^n \|\nabla u\|_{L^s(\omega_i)}^s\right)^{\frac{1}{s}} \leq c h^{d(1-\frac{1}{s})+1} \|\nabla u\|_{L^s(\Omega)}\].

\[\|u - \pi_i(u)\|_{L^1(\Omega)} = \int_{\Omega} \left|u - \sum_{i=1}^n \pi_i(u) \phi_i\right| dx \leq \sum_{i=1}^n \int_{\omega_i} |u - \pi_i(u)| \phi_i d\hat{x} \leq c h^{d(1-\frac{1}{s})+1} \sum_{i=1}^n \|\nabla u\|_{L^s(\omega_i)}\]

\[\|u - \pi_i(u)\|_{L^1(\Omega)} = \int_{\Omega} \left|u - \sum_{i=1}^n \pi_i(u) \phi_i\right| dx \leq \sum_{i=1}^n \int_{\omega_i} |u - \pi_i(u)| \phi_i d\hat{x} \leq c h^{d(1-\frac{1}{s})+1} \sum_{i=1}^n \|\nabla u\|_{L^s(\omega_i)}\]

References

[1] Nadir Arada, Eduardo Casas and Fredi Tröltzsch. Error Estimates for the Numerical Approximation of a Semilinear Elliptic Control Problem, Computational Optimization and Applications, Kluwer Academic Publishers, volume 23: 201-229, 2002.
[2] Sören Bartels, Ricardo Nochetto, and Abner Salgado A total variation diminishing interpolation operator and applications, Mathematics of Computation, 84(296), 2569-2587, 2015.
[3] Frédéric Bonnas and Eduardo Casas. Contrôle de systems elliptiques semilinéaires comportant des contraintes sur l’état, Nonlinear partial differential equations and their applications, Longman, volume 8: 69–86, 1988.
[4] Haim Brezis. Functional analysis, Sobolev spaces and partial differential equations Springer Science & Business Media, 2010.
[5] Eduardo Casas, Christian Clason, and Karl Kunisch. Parabolic control problems in measure spaces with sparse solutions. SIAM Journal on Control and Optimization, 51(1):28–63, 2013.
[6] Eduardo Casas and Karl Kunisch. Parabolic control problems in space-time measure spaces. ESAIM: Control, Optimisation and Calculus of Variations, 22(2):355–370, 2016.
[7] Eduardo Casas. A review on sparse solutions in optimal control of partial differential equations. SeMA Journal, volume 74(3):319–344, 2017.
[8] Eduardo Casas, Mariano Mateos and Arnd Rosch. Finite element approximation of sparse parabolic control problems American Institute of Mathematical Sciences, volume 7(3): 393–417 doi: 10.3934/mcrf.2017014, 2017.
[9] Eduardo Casas, Cristhian Clason and Karl Kunisch. Approximation of elliptic control problems in measure spaces with sparse solutions SIAM Journal on Control and Optimization, volume 50(4): 1735-1752, 2012.
[10] Eduardo Casas, Roland Herzog and Gerd Wachsmuth. Approximation of sparse controls in semilinear equations by piecewise linear functions Numerische Mathematik, volume 122:645-669, 2012.
[11] Eduardo Casas, Roland Herzog and Gerd Wachsmuth. Optimality Conditions and Error Analysis of Semilinear Elliptic Control Problems with L1 Cost Functional SIAM Journal on Optimization, volume 22(3):795-820, 2012.
[12] Eduardo Casas, Christopher Ryll and Fredi Tröltzsch. Sparse optimal control of the Schröd and FitzHugh–Nagumo systems Computational Methods in Applied Mathematics, volume 13(4): 415–442, 2013.
[13] Philippe G. Ciarlet and Jacques Luis Lions. Handbook of Numerical Analysis II, North-Holland, volume 2. 1990.
[14] Philippe G. Ciarlet. *Linear and nonlinear functional analysis with applications*, SIAM, volume 130. 2013.

[15] Christian Clason and Tuomo Valkonen. *Introduction to nonsmooth analysis and optimization* arxiv: 2001.00216v2, 2020.

[16] Juan Carlos De los Reyes, Christian Meyer and Boris Vexler. Finite element error analysis for state-constrained optimal control of the stokes equations. *Control & Cybernetics*, 2007.

[17] Tao Pham Dinh and Hoai An Le Thi. Recent advances in dc programming and DCA. In *Transactions on Computational Intelligence XIII*, pages 1–37. Springer, 2014.

[18] Hinze, Michael. A Variational Discretization Concept in Control Constrained Optimization: The Linear–Quadratic Case. *Computational Optimization and Applications*, 30: 45–61, 2005.

[19] Michael Hinze, René Pinnau, Michael Ulbrich, and Stefan Ulbrich. *Optimization with PDE constraints*, volume 23. Springer Science & Business Media, 2008.

[20] Jean-Baptiste Hiriart-Urruty. From convex optimization to nonconvex optimization. Necessary and sufficient conditions for global optimality. In *Nonsmooth optimization and related topics*, pages 219–239. Springer, 1989.

[21] Jean-Baptiste Hiriart-Urruty and Claude Lemaréchal. *Fundamentals of convex analysis*. Springer Science & Business Media, 2012.

[22] Kazufumi Ito and Karl Kunisch. Lagrange multiplier approach to variational problems and applications. *SIAM Advances in Design and Control*, 52(2):1251–1275, 2014.

[23] Kazufumi Ito and Karl Kunisch. Optimal control with $L^p(\Omega)$, $p \in [0, 1)$, control cost. *SIAM Journal on Control and Optimization*, Philadelphia, 2008.

[24] Johannes Jahn. *Introduction to the Theory of Nonlinear Optimization*. Springer-Verlag Berlin Heidelberg, 3 edition, 2007.

[25] Youfeng Liu, Hao Helen Zhang, Cheolwoo Park and Jeongyoun Ahn Support vector machines with adaptive $L_q$ penalty. *Computational Statistics & Data Analysis*, 51(12):6380–6394, 2007.

[26] Mariano Mateos Problemas de Control Óptimo Gobernados por Ecuaciones Semilineales con Restricciones de Tipo Integral sobre el Gradiente del Estado. Ph.D. Thesis 2000.

[27] Konstantin Pieper and Boris Vexler. A priori error analysis for discretization of sparse elliptic optimal control problems in measure space *SIAM Journal on Control and Optimization*, volume 51(4): 2788-2808, 2013.

[28] Pedro Merino. A difference-of-convex functions approach for sparse PDE optimal control problems with nonconvex costs. Springer *Computational Optimization and Applications*, 74:225-258, 2019.

[29] Pedro Merino. A Semismooth Newton Method for Regularized $L^q$-quasinorm Sparse Optimal Control Problems. Springer *Numerical Mathematics and Advanced Applications ENUMATH 2019*, 723-731, 2021.

[30] Ronny Ramlau and Clemens A Zarzer. On the minimization of a Tikhonov functional with a nonconvex sparsity constraint. *Comput. Optim. Appl.*, 44(2):159–181, 2006.

[31] Tomáš Roubíček. *Nonlinear partial differential equations with applications* Springer Science & Business Media, Vol.153 , 2013.

[32] G. Stadler. Elliptic optimal control problems with $L^1$-control cost and applications for the placement of control devices. *Comput. Optim. Appl.*, 44(2):159–181, 2006.

[33] Guido Stampacchia. *Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus*. Annales de l’institut Fourier, 15:189–257, 1965.

[34] Fredi Tröltzsch. *Optimal control of partial differential equations: theory, methods, and applications* American Mathematical Society, Vol.112, 2010.

[35] Hedy Attouch and Giuseppe Buttazzo and Gérard Michaille. Variational analysis in Sobolev and BV spaces: applications to PDEs and optimization SIAM, 2014.

[36] Daniel Wachsmuth. *Iterative Hard-Thresholding Applied to Optimal Control Problems with $L^0(\Omega)$ Control Cost*. *SIAM Journal on Control and Optimization*, 57:854–879, 2019.

[37] Gerd Wachsmuth and Daniel Wachsmuth. *Convergence and regularization results for optimal control problems with sparsity functional*. ESAIM COCV, 17:858–886, 2011.
