Suppose $A = \bigoplus_{i \in \mathbb{N}} A_i$ is a left noetherian $\mathbb{N}$-graded ring. The category of left $\mathbb{Z}$-graded $A$-modules is denoted by $\text{GrMod} A$. Recall that the objects of $\text{GrMod} A$ are the left graded $A$-modules $M = \bigoplus_{i \in \mathbb{Z}} M_i$, and the morphisms are the $A$-linear homomorphisms of degree 0. A graded-injective module is an injective object $I$ in the abelian category $\text{GrMod} A$.

The following result is stated as a “pleasant exercise in homological algebra” in [VdB]. Its proof was communicated to us privately by M. Van den Bergh many years ago, and we referred to it as “quite involved” in [YZ1, Remark 4.9]. The purpose of this note is to give a modified proof of this result.

**Theorem 1** (Van den Bergh). Assume $A$ is a left noetherian $\mathbb{N}$-graded ring. Let $I$ be a graded-injective left $A$-module. Then the injective dimension of $I$ in the ungraded sense is at most 1.

Actually we prove a slightly more general result (Theorem 12), of which Theorem 1 is a special case. As far as we recall, this was also proved by Van den Bergh.

Our proof follows the same strategy as the original proof by Van den Bergh, namely passing through the Rees ring. However our proof is somewhat more conceptual, in that it isolates the precise “reason” for the dimension jump (see Lemma 6, and compare it to Example 2 below). The downside of our proof is that it uses derived categories.

The dimension jump can already be seen in the next easy commutative example.

**Example 2.** Suppose $\mathbb{k}$ is a field and $A = \mathbb{k}[t]$, the polynomial ring in the variable $t$, that has degree 1. The injective $A$-modules (in the ungraded sense, i.e. in the category $\text{Mod} A$) are direct sums of indecomposable $A$-modules. The indecomposable injectives are the field of fractions $I(0) \cong \mathbb{k}(t)$, and the torsion modules $I(m) \cong \mathbb{k}(t)/A_m$, where $m$ is a maximal ideal of $A$.

In the category $\text{GrMod} A$ there are two indecomposable injective objects, up to Serre twist: the module $I(m_0)$, where $m_0 := (t)$ is the unique graded maximal ideal of $A$; and the graded ring of fractions $J := \mathbb{k}[t, t^{-1}]$. The latter is not injective in $\text{Mod} A$; but it has injective dimension 1.

Observe that projective modules do not have a jump in dimension:

**Proposition 3.** Let $A$ be a $\mathbb{Z}$-graded ring. If $P$ is a graded-projective left $A$-module, then $P$ is also a projective $A$-module in the ungraded sense.
We let $\tilde{Q}$ be a graded isomorphism of filtered $A$-modules. A function $\phi$ is projective as an ungraded $A$-module if for all $m \in \hat{M}$ and $a \in A$ we have $\phi(m) \subset N_{i+j}$ and $\phi(a \cdot m) = a \cdot \phi(m)$. The set of all such homomorphisms is denoted by $\text{Hom}_{A}^{\text{gr}}(M, N)$. And we let

$$\text{Hom}_{A}^{\text{gr}}(M, N) := \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{A}^{\text{gr}}(M, N)_i,$$

which is a graded abelian group. Thus

$$\text{Hom}_{\text{GrMod} A}(M, N) = \text{Hom}_{A}^{\text{gr}}(M, N)_0.$$

We can view $A$ as a filtered ring, with filtration $F = \{F_i(A)\}_{i \in \mathbb{Z}}$ defined by $F_i(A) := \bigoplus_{j \leq i} A_j$. Let $\hat{A}$ be the Rees ring associated to this filtration. Thus

$$\hat{A} := \bigoplus_{i \in \mathbb{N}} F_i(A) \cdot t^i \subset A[t] = A \otimes_{\mathbb{Z}} \mathbb{Z}[t],$$

where $t$ is a variable of degree 1. There is an isomorphism of graded rings $\hat{A}/t \cdot \hat{A} \cong A$, and an isomorphism of filtered rings $\hat{A}/(t-1) \cdot \hat{A} \cong A$. According to [ATV, Theorem 8.2] the ring $\hat{A}$ is left noetherian.

Given $\check{M} \in \text{GrMod} \hat{A}$, let $\text{sp}_0(\check{M}) := \check{M} / t \cdot \check{M}$, which is a graded $A$-module. This is an additive functor (specialization)

$$\text{sp}_0 : \text{GrMod} \hat{A} \to \text{GrMod} A.$$

Likewise the operation $\text{sp}_1(\check{M}) := \check{M} / (t-1) \cdot \check{M}$ is an additive functor

$$\text{sp}_1 : \text{GrMod} \hat{A} \to \text{Mod} A.$$
is a graded module over the ring $\tilde{A} = rs^F(A)$. There is a canonical isomorphism $sp_0(rs^F(M)) \cong gr^F(M)$ in $\text{GrMod} \tilde{A}$, and a canonical isomorphism $sp_1(rs^F(M)) \cong M$ in $\text{Mod} \tilde{A}$.

A graded $A$-module $M = \bigoplus_{i \in \mathbb{Z}} M_i$ has a filtration $F = \{F_i(M)\}_{i \in \mathbb{Z}}$ defined by $F_i(M) := \bigoplus_{j \leq i} M_j$. The corresponding Rees module is denoted by $rs(M) := rs^F(M)$. Thus we get a functor

$$rs : \text{GrMod} \tilde{A} \to \text{GrMod} \tilde{A}.$$ 

This functor is exact. There is a canonical isomorphism $sp_0(rs^F(M)) \cong M$ in $\text{GrMod} \tilde{A}$.

For an abelian category $M$ we have the (unbounded) derived category $D(M)$. In it there are the full subcategories $D^+(M)$ and $D^-(M)$, whose objects are the complexes with bounded below and bounded above cohomologies, respectively. The category of complexes of objects of $M$ is $C(M)$. There is a localization functor $C(M) \to D(M)$, which is the identity on objects, and inverts quasi-isomorphisms.

We shall consider these abelian categories: $\text{GrMod} \tilde{A}$, $\text{GrMod} A$ and $\text{Mod} A$.

The exact functors $sp_1$ and $rs$ extend to triangulated functors

$$sp_1 : D(\text{GrMod} \tilde{A}) \to D(\text{Mod} A)$$

and

$$rs : D(\text{GrMod} A) \to D(\text{GrMod} \tilde{A}).$$

There is a left derived functor

$$Lsp_0 : D^-(\text{GrMod} \tilde{A}) \to D(\text{GrMod} A).$$

We use graded-projective resolutions to construct it. The right derived functors

$$\text{RHom}_{\tilde{A}}^\mathbb{Z}(-, -) : D^-(\text{GrMod} \tilde{A})^{\text{op}} \times D^+(\text{GrMod} \tilde{A}) \to D^+(\text{GrMod} \mathbb{Z})$$

and

$$\text{RHom}_{\tilde{A}}^\mathbb{Z}(-, -) : D^-(\text{GrMod} \tilde{A})^{\text{op}} \times D^+(\text{GrMod} \tilde{A}) \to D^+(\text{GrMod} \mathbb{Z})$$

are constructed either by graded-projective resolutions of the first argument, or by graded-injective resolutions of the second argument.

The key to the dimension jump is the next lemma, that reduces the question to the graded ring $\mathbb{Z}[t]$. Observe that there is an obvious graded ring homomorphism $\mathbb{Z}[t] \to \tilde{A}$, and the element $t$ is not a zero divisor in $\tilde{A}$ (cf. formula (5)).

**Lemma 6.** The functor $Lsp_0$ has cohomological dimension $\leq 1$. More precisely, if the cohomology of $\tilde{M} \in D^{b}(\text{GrMod} \tilde{A})$ is concentrated in degrees $\{i_0, \ldots, i_1\}$, then the cohomology of $Lsp_0(\tilde{M})$ is concentrated in degrees $\{i_0 - 1, i_0, \ldots, i_1\}$.

**Proof:** Recall that we constructed $Lsp_0(\tilde{M})$ by choosing a resolution $\tilde{P} \to \tilde{M}$, where $\tilde{P}$ is a bounded above complex of graded-projective $\tilde{A}$-modules, and letting $Lsp_0(\tilde{M}) := sp_0(\tilde{P})$. Since $t$ is not a zero divisor in $\tilde{A}$, we obtain a short exact sequence

$$0 \to \tilde{P}(-1) \overset{t}{\to} \tilde{P} \to sp_0(\tilde{P}) \to 0 \quad (7)$$
in $\text{C}(\text{GrMod} \tilde{A})$.

Let $\tilde{Q} \in \text{C}(\text{GrMod} \mathbb{Z}[t])$ be the semi-free complex

$$\tilde{Q} := (\mathbb{Z}[t](-1) \xrightarrow{i} \mathbb{Z}[t])$$

concentrated in cohomological degrees $-1$ and $0$. Using equation (7) we see that there are quasi-isomorphisms

$$\tilde{Q} \otimes_{\mathbb{Z}[t]} \tilde{M} \leftrightarrow \tilde{Q} \otimes_{\mathbb{Z}[t]} \tilde{P} \to sp_0(\tilde{P})$$

in $\text{C}(\text{GrMod} \mathbb{Z}[t])$. The bounds on the cohomology of the complex $\tilde{Q} \otimes_{\mathbb{Z}[t]} \tilde{M}$ are clear.

**Lemma 8.** Take $\tilde{M} \in D_{-}^-(\text{GrMod} \tilde{A})$ and $N \in D^{+}(\text{GrMod} A)$. There is an isomorphism

$$\text{RHom}_{\tilde{A}}(\tilde{M}, \text{rs}(N)) \cong \text{rs} \left( \text{RHom}_{A}^{gr}(Lsp_0(\tilde{M}), N) \right)$$

in $D(\text{GrMod} \mathbb{Z}[t])$. It is functorial in $\tilde{M}$ and $N$.

**Proof.** Any such $\tilde{M}$ admits a quasi-isomorphism $\tilde{P} \to \tilde{M}$, where $\tilde{P}$ is a bounded above complex of finite rank graded-free $\tilde{A}$-modules. There are isomorphisms

(9) $$\text{RHom}_{\tilde{A}}(\tilde{M}, \text{rs}(N)) \cong \text{Hom}_{\tilde{A}}(\tilde{P}, \text{rs}(N))$$

and

(10) $$\text{rs} \left( \text{RHom}_{A}^{gr}(Lsp_0(\tilde{M}), N) \right) \cong \text{rs} \left( \text{Hom}_{\tilde{A}}^{gr}(sp_0(\tilde{P}), N) \right)$$

in $D(\text{GrMod} \mathbb{Z}[t])$, functorial in $\tilde{P}$. We use the fact that $sp_0(\tilde{P})$ is a bounded above complex of finite rank graded-free $A$-modules.

If $\tilde{P}$ is a single graded-free $\tilde{A}$-module of rank 1, i.e. $\tilde{P} \cong \tilde{A}(i)$ for some integer $i$, and if $N$ is a single graded $A$-module, then there are isomorphisms

$$\text{Hom}_{\tilde{A}}(\tilde{P}, \text{rs}(N)) \cong \text{rs}(N)(-i) \cong \text{rs}(N(-i))$$

and

$$\text{rs} \left( \text{Hom}_{\tilde{A}}^{gr}(sp_0(\tilde{P}), N) \right) \cong \text{rs} \left( \text{Hom}_{\tilde{A}}^{gr}(A(i), N) \right) \cong \text{rs}(N(-i))$$

in $\text{GrMod} \mathbb{Z}[t]$. Since these isomorphisms are functorial in the modules $\tilde{P}$ and $N$, we get an isomorphism

$$\text{Hom}_{\tilde{A}}^{gr}(\tilde{P}, \text{rs}(N)) \cong \text{rs} \left( \text{Hom}_{\tilde{A}}^{gr}(sp_0(\tilde{P}), N) \right)$$

in $\text{C}(\text{GrMod} \mathbb{Z}[t])$ for any bounded above complex of finite rank graded-free $\tilde{A}$-modules $\tilde{P}$, and any bounded below complex of graded $A$-modules $N$. Finally we use the isomorphisms (9) and (10).

**Lemma 11.** Let $\tilde{M} \in D_{-}^-(\text{GrMod} \tilde{A})$ and $\tilde{N} \in D^{+}(\text{GrMod} \tilde{A})$. There is an isomorphism

$$\text{sp}_1 \left( \text{RHom}_{\tilde{A}}^{gr}(\tilde{M}, \tilde{N}) \right) \cong \text{RHom}_{A} \left( \text{sp}_1(M), \text{sp}_1(N) \right)$$

in $D(\text{Mod} \mathbb{Z})$. It is functorial in $\tilde{M}$ and $\tilde{N}$.
Proof. This is [YZ1, Lemma 6.3(3)]. The statement there talks about complexes \( \tilde{M}, \tilde{N} \) in \( D^b(\text{GrMod} \tilde{A}) \); but the proof (the way-out argument) works also for the slightly weaker assumptions that we have here. \( \square \)

For a graded \( A \)-module \( N \) we denote by \( \text{gr} \text{.inj.dim}(N) \) its injective dimension in \( \text{GrMod} A \). Thus \( I \) is graded-injective iff \( \text{gr} \text{.inj.dim}(I) = 0 \). Likewise, for an \( A \)-module \( N \) we denote by \( \text{inj.dim}(N) \) its injective dimension in \( \text{Mod} A \).

**Theorem 12** (Van den Bergh). Assume \( A \) is a left noetherian \( \mathbb{N} \)-graded ring. For any graded \( A \)-module \( N \) we have

\[
\text{inj.dim}(N) \leq \text{gr.inj.dim}(N) + 1.
\]

**Proof.** Take a graded \( A \)-module \( N \), and let \( d := \text{gr.inj.dim}(N) \). We may assume that \( d < \infty \). It is enough to prove that \( \text{Ext}_A^q(M, N) = 0 \) for every \( q > d + 1 \) and every finitely generated \( A \)-module \( M \).

Let \( M \) be a finitely generated \( A \)-module. Let \( F \) be a good filtration on \( M \), namely a filtration such that the graded module \( \text{gr}^F(M) \) is finitely generated over \( A \). Such filtrations exist by [MR, Lemma 7.6.11]. Then (by the graded Nakayama Lemma) the Rees module \( \tilde{M} := \text{rs}_F(M) \) is finitely generated over \( \tilde{A} \).

Consider the graded \( \tilde{A} \)-module \( \tilde{N} := \text{rs}(N) \). Since \( M \cong \text{sp}_1(\tilde{M}) \) and \( N \cong \text{sp}_1(\tilde{N}) \) in \( \text{Mod} A \), Lemma [11] says that it is enough to prove that

\[
H^q(\text{RHom}_A^\text{gr}(\tilde{M}, \tilde{N})) = 0
\]

for all \( q > d + 1 \). By Lemma [8] this is equivalent to proving that

\[
H^q(\text{RHom}_A^\text{gr}(\text{Lsp}_0(\tilde{M}), N)) = 0
\]

for \( q > d + 1 \). Let us take a quasi-isomorphism \( N \rightarrow I \) in \( \text{GrMod} A \), such that \( I \) is a complex of graded-injective modules concentrated in degrees \( \{0, \ldots, d\} \). We have to prove that

\[
H^p(\text{Hom}_A^\text{gr}(\text{Lsp}_0(\tilde{M}), I)) = 0
\]

for \( p > d + 1 \). For that is sufficient to show that \( H^p(\text{Lsp}_0(\tilde{M})) = 0 \) for \( p < -1 \). This is true by Lemma [6]. \( \square \)

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