TRANSVERSE ELECTRIC CONDUCTIVITY OF
QUANTUM COLLISIONAL PLASMAS

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Formulas for calculation of transverse dielectric function and transverse electric conductivity in quantum collisional plasmas under arbitrary degree of degeneracy of the electron gas are received. The Wigner – Vlasov – Boltzmann kinetic equation with collision integral in BGK (Bhatnagar, Gross and Krook) form in coordinate space is used. Various special cases are investigated. The case of fully degenerate quantum plasma was considered separately. Comparison with Lindhard’s formula has been realized.

Key words: collisional plasma, Schrödinger equation, electric conductivity, dielectric function, Lindhard formula.

PACS numbers: 52.25.Dg Plasma kinetic equations, 52.25.-b Plasma properties, 05.30 Fk Fermion systems and electron gas.

1. INTRODUCTION

In the present work formulas for calculation of electric conductivity and dielectric function in quantum collisional plasma under arbitrary temperature, i.e. under arbitrary degree of degeneration of the electron gas are deduced.

During the derivation of the kinetic equation we generalize the approach, developed by Klimontovich and Silin \cite{1}.
Dielectric function in the collisionless quantum gaseous plasma was studied by many authors (see, for example, [1] – [10]).

In the work [6], where the one-dimensional case of the quantum plasma is investigated, the importance of derivation of dielectric function with use of the quantum kinetic equation with collision integral in the form of BGK – model (Bhatnagar, Gross, Krook) [11], [12] was noted.

The present work is devoted to the performance of this problem.

A dielectric function is one of the most significant characteristics of a plasma. This quantity is necessary for description of the skin effect [13], for analysis of surface plasmons [14], for description of the process of propagation and damping of the transverse plasma oscillations [10], the mechanism of electromagnetic waves penetration in plasma [9], and for analysis of other problems of plasma physics [15], [16], [17], [18] and [19].

Kliewer and Fuchs were the first who have noticed [4], that the dielectric function for quantum plasma deduced by Lindhard in collisional case does not pass into dielectric function for classical plasma in the limit when Planck’s constant $\hbar$ converges to zero. This means, that dielectric Lindhard’s function does not take into account electron collisions correctly. Kliewer and Fuchs have corrected Lindhard’s dielectric function ”by hands” so that it passed into classical one under condition $\hbar \to 0$.

In the works [14], [15] the dielectric function received by them was applied to consideration of various questions of metal optics.

In the work [5] the correct account of collisions in framework of the relaxation model in electron momentum space for the case of longitudinal dielectric function has been carried out. At the same time the correct account of influence of collisions for transverse dielectric function has not been implemented till now.

The aim of the present work is the elimination of this lacuna.

2. KINETIC EQUATION FOR THE WIGNER FUNCTION
Let’s consider the Schrödinger equation written for a particle in an electromagnetic field in terms of density matrix $\rho$

\[ i\hbar \frac{\partial \rho}{\partial t} = H \rho - H^*\rho. \quad (1.1) \]

Here $H$ is the Hamilton operator, $H^*$ is the complex conjugate operator to $H$, $H^*$ is the complex conjugate operator to the $H$, which forces on primed spatial variables $r'$.

Hamilton operator for the free particle which is in the field of the scalar potential $U$ and in the field of vector potential $A$, has the following form

\[ H = \frac{(p - eA)}{2m} + eU = \]

\[ = \frac{p^2}{2m} - \frac{e}{2mc}(pA + Ap) + \frac{e^2}{2mc^2}A^2 + eU. \quad (1.2) \]

Here $p$ is the momentum operator, $p = -i\hbar\nabla$, $e$ is the electron charge, $m$ is the electron mass, $c$ is the light velocity.

Let’s rewrite the Hamilton operator (1.2) in the explicit form

\[ H = -\frac{\hbar^2}{2m}\Delta + \frac{ie\hbar}{2mc}(2A\nabla + \nabla A) + \frac{e^2}{2mc^2}A^2 + eU. \quad (1.3) \]

Complex conjugate to the $H$ operator $H^*$ according to (1.3) has the form

\[ H^* = -\frac{\hbar^2}{2m}\Delta - \frac{ie\hbar}{2mc}(2A\nabla + \nabla A) + \frac{e^2}{2mc^2}A^2 + eU. \]

Hence we can write down for $H\rho$

\[ H\rho = -\frac{\hbar^2}{2m}\Delta\rho + \frac{ie\hbar}{2mc}(2A\nabla\rho + \rho\nabla A) + \frac{e^2}{2mc^2}A^2\rho + eU\rho \quad (1.4) \]

and for $H^*\rho$

\[ H^*\rho = -\frac{\hbar^2}{2m}\Delta'\rho - \frac{ie\hbar}{2mc}(2A'\nabla'\rho + \rho\nabla' A) + \frac{e^2}{2mc^2}A'^2\rho + eU'\rho. \quad (1.5) \]
Operators $\nabla$ and $\Delta$ from Eqs (1.4) and (1.5) force on unprimed spatial variables of the density matrix, i.e. $\nabla = \nabla_R$, $\Delta = \Delta_R$. In the operator $H^{*'}$ is necessary to replace the operators $\nabla = \nabla_R$ and $\Delta = \Delta_R$ by operators $\nabla' \equiv \nabla_{R'}$ and $\Delta' \equiv \Delta_{R'}$, in addition we introduce the following designations

$$A' \equiv A(R', t), \quad U' \equiv U(R', t).$$

Let’s find the right-hand member of the equation (1.1), i.e. difference between relations (1.4) and (1.5): $H\rho - H^{*'}\rho$. According to (1.4) and (1.5) we have

$$H\rho - H^{*'}\rho = -\frac{\hbar}{2m}(\Delta \rho - \Delta' \rho) +$$

$$+ \frac{i\hbar}{2mc}[2(A\nabla \rho + A'\nabla' \rho) + \rho(\nabla A + \nabla' A)] +$$

$$+ \frac{e^2}{2mc^2}[A^2(R, t) - A^2(R', t)] + e[U(R, t) - U(R', t)]\rho.$$

The connection between density matrix $\rho(r, r', t)$ and Wigner function $f(r, p, t)$ is given by the inversion and direct Fourier conversions

$$f(r, p, t) = \int \rho(r + \frac{a}{2}, r - \frac{a}{2}, t)e^{-ipa/\hbar}d^3a,$$

$$\rho(R, R', t) = \frac{1}{(2\pi\hbar)^3} \int f\left(\frac{R + R'}{2}, p, t\right) e^{ip(R-R')/\hbar}d^3p.$$

The Wigner function is analogue of distribution function for quantum systems. It is widely used in the diversified physics questions. Wigner’s function was investigated, for example, in works [23] and [24].

Substituting the representation of the density matrix in terms of the Wigner function (1.2) into the equation for the density matrix (1.1), we
obtain

\[ i\hbar \frac{\partial \rho}{\partial t} = H \left\{ \frac{1}{(2\pi\hbar)^3} \int f\left( \frac{R + R'}{2}, p', t \right) e^{i p'(R - R')/\hbar} d^3p' \right\} - \]

\[-H^* \left\{ \frac{1}{(2\pi\hbar)^3} \int f\left( \frac{R + R'}{2}, p', t \right) e^{i p'(R - R')/\hbar} d^3p' \right\}. \]

Let’s use the equalities written above. Thus the right–hand member of the previous equation we may present in explicit form. As a result we receive the following equation

\[ i\hbar \frac{\partial \rho}{\partial t} = 1 \left\{ \frac{1}{(2\pi\hbar)^3} \int \left\{ -\frac{i\hbar}{m} p' \nabla f + \frac{ie\hbar}{2mc} \left[ \text{div} A(R, t) + \text{div} A(R', t) \right] f + \right. \]

\[ + \frac{ie\hbar}{2mc} \left[ A(R, t) + A(R', t) \right] \nabla f - \frac{e}{mc} \left[ A(R, t) - A(R', t) \right] p' f + \]

\[ + \frac{e^2}{2mc^2} \left[ A^2(R, t) - A^2(R', t) \right] f + \]

\[ + e \left[ U(R, t) - U(R', t) \right] f \right\} e^{i p'(R - R')/\hbar} d^3p'. \] (1.6)

In the equation (1.6) we will put

\[ R = r + \frac{a}{2}, \quad R' = r - \frac{a}{2}. \]

Then in this equation we obtain

\[ f\left( \frac{R + R'}{2}, p', t \right) e^{i p'(R - R')/\hbar} = f(r, p', t) e^{i p' a/\hbar}. \]

Let’s multiply the equation (1.6) by \( e^{-i p a/\hbar} \) and let’s integrate it by \( a \). Then we will divide both parts of the equation by \( i\hbar \). As a result we receive

\[ \frac{\partial f}{\partial t} = \iint \left\{ -\frac{p'}{m} \nabla f + \frac{e}{2mc} \left[ A(r + \frac{a}{2}, t) + A(r - \frac{a}{2}, t) \right] \nabla f + \right. \]

\[ + \frac{ie\hbar}{2mc} \left[ A(r + \frac{a}{2}, t) + A(r - \frac{a}{2}, t) \right] \nabla f - \frac{e}{mc} \left[ A(r + \frac{a}{2}, t) - A(r - \frac{a}{2}, t) \right] p' f + \]

\[ + \frac{e^2}{2mc^2} \left[ A^2(r + \frac{a}{2}, t) - A^2(r - \frac{a}{2}, t) \right] f + \]

\[ + e \left[ U(r + \frac{a}{2}, t) - U(r - \frac{a}{2}, t) \right] f \right\} e^{i p' a/\hbar} d^3p'. \]
\[
\frac{ie}{mch} \left[ \mathbf{A}(\mathbf{r} + \frac{\mathbf{a}}{2}, t) - \mathbf{A}(\mathbf{r} - \frac{\mathbf{a}}{2}, t) \right] \mathbf{p}' f + \\
\frac{e}{2mc} \left[ \text{div} \mathbf{A}(\mathbf{r} + \frac{\mathbf{a}}{2}, t) + \text{div} \mathbf{A}(\mathbf{r} - \frac{\mathbf{a}}{2}, t) \right] f - \\
\frac{-ie^2}{2mc^2\hbar} \left[ \mathbf{A}^2(\mathbf{r} + \frac{\mathbf{a}}{2}, t) - \mathbf{A}^2(\mathbf{r} - \frac{\mathbf{a}}{2}, t) \right] f - \\
\frac{i\epsilon}{\hbar} \left[ \mathbf{U}(\mathbf{r} + \frac{\mathbf{a}}{2}, t) - \mathbf{U}(\mathbf{r} - \frac{\mathbf{a}}{2}, t) \right] f \right\} e^{i(p' - p)\mathbf{a}/\hbar} \frac{d^3a d^3p'}{(2\pi\hbar)^3}. \tag{1.7}
\]

On the left-hand side of the equation (1.7) we have \( f = f(\mathbf{r}, \mathbf{p}, t) \), in the integral we have \( f = f(\mathbf{r}, \mathbf{p}', t) \).

We consider the integral
\[
\int\int \mathbf{p}'(\nabla f)e^{i(p' - p)\mathbf{a}/\hbar} \frac{d^3a d^3p'}{(2\pi\hbar)^3} = \nabla \int\int \mathbf{p}' f e^{i(p' - p)\mathbf{a}/\hbar} \frac{d^3a d^3p'}{(2\pi\hbar)^3} = \\
\nabla \int\int \mathbf{p}' f \delta(\mathbf{p}' - \mathbf{p})d\mathbf{p}' = \mathbf{p} \nabla f(\mathbf{r}, \mathbf{p}, t).
\]

Two following equalities can be verified similarly
\[
\int\int \frac{e}{mc} \mathbf{A}(\mathbf{r}, t)[\nabla f(\mathbf{r}, \mathbf{p}', t)e^{i(p' - p)\mathbf{a}/\hbar} \frac{d^3a d^3p'}{(2\pi\hbar)^3} = \\
= \frac{e}{mc} \mathbf{A}(\mathbf{r}, t) \nabla f(\mathbf{r}, \mathbf{p}, t),
\]
and
\[
\int\int \frac{e}{mc} [\text{div} \mathbf{A}(\mathbf{r}, t)] f(\mathbf{r}, \mathbf{p}', t)e^{i(p' - p)\mathbf{a}/\hbar} \frac{d^3a d^3p'}{(2\pi\hbar)^3} = \\
= \frac{e}{mc} [\text{div} \mathbf{A}(\mathbf{r}, t)] f(\mathbf{r}, \mathbf{p}, t).
\]

Then the equation (1.6) can be rewritten as following
\[
\frac{\partial f}{\partial t} + \frac{1}{m} \left( \mathbf{p} - \frac{e}{c} \mathbf{A} \right) \nabla f - \frac{e}{mc} [\text{div} \mathbf{A}(\mathbf{r}, t)] f(\mathbf{r}, \mathbf{p}, t) = W[f]. \tag{1.8}
\]
In the equation (1.8) the symbol \( W[f] \) is the Wigner — Vlasov integral, defined by the equality

\[
W[f] = \int \int \left\{ \frac{e}{2mc} \left[ A(r + \frac{a}{2}, t) + A(r - \frac{a}{2}, t) - 2A(r, t) \right] \nabla f + \right.
\]

\[
+ \frac{ie}{mch} \left[ A(r + \frac{a}{2}, t) - A(r - \frac{a}{2}, t) \right] p'f +
\]

\[
+ \frac{e}{2mc} \left[ \text{div} A(r + \frac{a}{2}, t) + \text{div} A(r - \frac{a}{2}, t) - 2 \text{div} A(r, t) \right] f -
\]

\[
- \frac{ie^2}{2mc^2\hbar} \left[ A^2(r + \frac{a}{2}, t) - A^2(r - \frac{a}{2}, t) \right] f -
\]

\[
- \frac{ie}{\hbar} \left[ U(r + \frac{a}{2}, t) - U(r - \frac{a}{2}, t) \right] f \right\} e^{i(p' - p) a / \hbar} \frac{d^3a d^3p'}{(2\pi\hbar)^3}. \tag{1.9}
\]

The energy of the particle is equal to

\[
\mathcal{E} = \mathcal{E}(r, p, t) = \frac{1}{2m} \left( p - \frac{e}{c} A \right)^2 + eU.
\]

Then the velocity of the particle \( v \) is equal to

\[
v = v(r, p, t) = \frac{\partial \mathcal{E}}{\partial p} = \frac{1}{m} \left( p - \frac{e}{c} A \right),
\]

besides,

\[
\nabla v = -\frac{e}{mc} \text{div} A.
\]

Hence, the left–hand part of the equation (1.9) equals to

\[
\frac{\partial f}{\partial t} + \frac{1}{m} \left( p - \frac{e}{c} A \right) \nabla f - f \frac{e}{mc} \text{div} A = \frac{\partial f}{\partial t} + v \nabla f + f \nabla A =
\]

\[
= \frac{\partial f}{\partial t} + \nabla(vf).
\]

Therefore the equation (1.9) can be rewritten in standard for transport theory form

\[
\frac{\partial f}{\partial t} + \nabla(vf) = W[f]. \tag{1.10}
\]
In the case of collisional plasma we may write the kinetic equation (1.10) as following

\[ \frac{\partial f}{\partial t} + \nabla (vf) = B[f, f] + W[f]. \]  

(1.11)

In the equation (1.11) the symbol \( B[f, f] \) represents the collision integral.

Under the electron scattering on impurity we will consider the equation (1.11) with collision integral in the form of relaxation \( \tau \)-model [11], [12]

\[ \frac{\partial f}{\partial t} + \nabla (vf) = \frac{f^{(0)} - f}{\tau} + W[f]. \]  

(1.12)

In the equation (1.12) \( \tau \) is the mean time between two consecutive collisions, \( \tau = 1/\nu \), \( \nu \) is the effective collision electron frequency with plasma particles, \( f^{(0)} \) is the local equilibrium Fermi — Dirac distribution function,

\[ f^{(0)} = f^{(0)}(r, p, t) = \left[ 1 + \exp\left(\frac{\mathcal{E} - \mu}{k_BT}\right)\right]^{-1}. \]

Here \( k_B \) is the Boltzmann constant, \( T \) is the plasma temperature, \( \mathcal{E} \) is the electron energy, \( \mu \) is the chemical potential of electron gas.

In an explicit form the local equilibrium distribution function has the following form

\[ f^{(0)} = f^{(0)}(r, p, t) = \left[ 1 + \exp\left(\frac{p - (e/c) A(r, t)}{2 mk_BT} + \frac{eU(r, t) - \mu}{k_BT}\right)\right]^{-1}. \]

We introduce the dimensionless electron velocity \( \mathbf{C}(r, p, t) \), scalar potential \( \phi(r, t) \) and chemical potential \( \alpha \)

\[ \mathbf{C}(r, p, t) = \frac{\mathbf{v}(r, p, t)}{v_T}, \quad \phi(r, t) = \frac{eU(r, t)}{k_BT}, \quad \alpha = \frac{\mu}{k_BT}, \]

where \( v_T = \frac{1}{\sqrt{\beta}} \) is the thermal electron velocity, \( \beta = \frac{m}{2k_BT} \).

Now local equilibrium function can be presented in terms of the electron velocity as follows

\[ f^{(0)}(r, p, t) = \left[ 1 + \exp\left(\frac{mv^2(r, p, t)}{2k_BT} + \frac{eU(r, t) - \mu}{k_BT}\right)\right]^{-1}, \]
or, in dimensionless parameters,

\[ f^{(0)}(r, p, t) = \frac{1}{1 + \exp \left[ C^2(r, p, t) + \phi(r, t) - \alpha \right]} \]  \hspace{1cm} (1.13)

We designate \( \chi = \alpha - \phi \). Then we have

\[ f^{(0)} = \frac{1}{1 + e^{C^2-\chi}}. \]

The quantity \( \chi \) is defined from the conservation law of number of particles

\[ \int f d\Omega_F = \int f^{(0)} d\Omega_F. \]

Here \( d\Omega_F \) is the quantum measure for electrons,

\[ d\Omega_F = \frac{2d^3p}{(2\pi\hbar)^3}. \]

Let’s note, that in the case of constant potentials \( U = \text{const}, \, A = \text{const} \) the equilibrium distribution function (1.13) is the solution of the equation (1.12).

Let’s find the electron concentration (numerical density) \( N \) and mean electron velocity \( u \) in an equilibrium state. These macroparameters are defined as follows

\[ N(r, t) = \int f(r, p, t)d\Omega_F, \]

\[ u(r, t) = \frac{1}{N(r, t)} \int v(r, p, t)f(r, p, t)d\Omega_F. \]

For calculation of these macroparameters in equilibrium condition it is necessary to put \( f = f^{(0)} \), where \( f^{(0)} \) is defined by equality (1.13). We designate these macroparameters in equilibrium condition through \( N^{(0)}(r, t) \) and \( u^{(0)}(r, t) \).

Let’s carry out the replacement of the integration variable

\[ p - \frac{e}{c}A(r, t) = p' \]
in the previous equalities. Then, passing to integration in spherical coordinates, for numerical density in an equilibrium state we get

\[ N^{(0)} = \frac{m^3 v^3 T}{\pi^2 \hbar^3} f_2(\alpha - \phi), \quad (1.14) \]

where

\[
 f_2(\alpha - \phi) = \int_0^\infty \frac{x^2 \, dx}{1 + \exp(x^2 + \phi - \alpha)} = \int_0^\infty x^2 f_F(\alpha - \phi) \, dx.
\]

In the same way, as for numerical density, for mean velocity in equilibrium state we derive

\[
 u^{(0)}(r, t) = \frac{1}{N^{(0)}} \int v(r, p, t) f^{(0)}(r, p, t) \, d\Omega_F,
\]

or, in explicit form,

\[
 u^{(0)}(r, t) = \frac{2}{N^{(0)}(2\pi \hbar)^3} \int \frac{\mathbf{p} - (e/c) \mathbf{A}}{1 + \exp \left[ \frac{(\mathbf{p} - (e/c) \mathbf{A})^2}{2k_B T m} + \frac{eU - \mu}{k_B T m} \right]} d^3 p.
\]

After the same change of variables \( \mathbf{p} - (e/c) \mathbf{A}(r, t) = \mathbf{p}' \) we receive

\[
 u^{(0)}(r, t) = \frac{2}{N^{(0)}(2\pi \hbar)^3} \int \frac{\mathbf{p}' \, d^3 p'}{1 + \exp \left[ \frac{\mathbf{p}'^2}{2k_B T m} + \frac{eU - \mu}{k_B T m} \right]} = 0. \quad (1.15)
\]

So, the electron velocity in an equilibrium state according to (1.15) is equal to zero.

Let’s note, that numerical electron density and their mean velocity satisfy the usual continuity equation

\[
 \frac{\partial N}{\partial t} + \text{div}(N \mathbf{u}) = 0. \quad (1.16)
\]

For the derivation of the continuity equation (1.16) is necessary to integrate the kinetic equation (1.12) by quantum measure for electrons \( d\Omega_F \) and to use the definition of numerical density and mean velocity. Then is
necessary to use the conservation law of number of particles and to check up, if the integral by quantum measure \(d\Omega_F\) of Wigner — Vlasov integral is equal to zero. Indeed, we have

\[
\int\! W[f] \frac{2 \, d^3p}{(2\pi\hbar)^3} = 2 \int\! \{ \cdots \} e^{i\mathbf{p}'a/\hbar} \delta(a) \, d^3a \, d^3p' =
\]

\[
= 2 \int\! \{ \cdots \} \bigg|_{a=0} d^3p \equiv 0,
\]
as after some algebra,

\[
\{ \cdots \} \bigg|_{a=0} \equiv 0.
\]

Here the symbol \(\{ \cdots \}\) means the same expression, as in the right-hand side of the equation (1.9).

Let’s note, that the left-hand side of the kinetic equation (1.11) or (1.12) takes standard form for transport theory under the following gauge condition:

\[
\text{div} \, \mathbf{A}(\mathbf{r}, t) = 0.
\] (1.17)

Thus, i.e. in case of gauge (1.17), the kinetic equation has the following form

\[
\frac{\partial f}{\partial t} + \mathbf{v} \nabla f = B[f, f] + W[f].
\] (1.18)

Here the Wigner — Vlasov integral equals to

\[
W[f] = \int\! \{ \frac{e}{2mc} \left[ \mathbf{A}(\mathbf{r} + \frac{\mathbf{a}}{2}, t) + \mathbf{A}(\mathbf{r} - \frac{\mathbf{a}}{2}, t) - 2\mathbf{A}(\mathbf{r}, t) \right] \nabla f +
\]

\[
+ \frac{ie}{m\hbar} \left[ \mathbf{A}(\mathbf{r} + \frac{\mathbf{a}}{2}, t) - \mathbf{A}(\mathbf{r} - \frac{\mathbf{a}}{2}, t) \right] \mathbf{p}' f -
\]

\[
- \frac{ie^2}{2mc^2\hbar} \left[ \mathbf{A}^2(\mathbf{r} + \frac{\mathbf{a}}{2}, t) - \mathbf{A}^2(\mathbf{r} - \frac{\mathbf{a}}{2}, t) \right] f -
\]

\[
- \frac{ie}{\hbar} \left[ U(\mathbf{r} + \frac{\mathbf{a}}{2}, t) - U(\mathbf{r} - \frac{\mathbf{a}}{2}, t) \right] f \} e^{i(p' - p)a/\hbar} \frac{d^3a \, d^3p'}{(2\pi\hbar)^3}.
\] (1.19)
3. LINEARIZATION OF THE KINETIC EQUATION AND ITS SOLUTION

Let’s consider the kinetic equation with collision integral in the form of \( \tau \)-model and suppose, that the scalar potential is equal to zero \( U(r, t) \equiv 0 \).

We take vector potential which is orthogonal to the direction of the wave vector \( k : kA = 0 \) in the form of a running harmonic wave

\[
A(r, t) = A_0 e^{i(kr - \omega t)}.
\]

We suppose that the vector potential is small enough. This assumption allows us to linearize the equation and to neglect terms quadratic in electric field.

Then the equation (1.18) can be reduced to

\[
\frac{\partial f}{\partial t} + v \nabla f = f^{(0)} - f \tau + W[f].
\]  

(2.1)

In this case chemical potential is equal to a constant.

In the equation (2.1) local equilibrium Fermi — Dirac distribution is simplified as following

\[
f^{(0)} = f^{(0)}(r, p, t) = \left[ 1 + \exp \left( C^2(r, p, t) - \alpha \right) \right]^{-1}.
\]  

(2.2)

The Wigner — Vlasov integral (1.19) also can be simplified essentially and has the following form

\[
W[f] = \frac{ie}{mch} \int \int \left[ A(r + \frac{a}{2}, t) - A(r - \frac{a}{2}, t) \right] p' f e^{i(p' - p)a/h} \frac{d^3a d^3p'}{(2\pi h)^3}.
\]  

(2.3)

We notice, that

\[
A(r + \frac{a}{2}, t) - A(r - \frac{a}{2}, t) = A(r, t) \left[ e^{ika/2} - e^{-ika/2} \right] .
\]

Calculating the integral in (2.3), we find that

\[
W[f] = \frac{ie}{mch} A(r, t) \int \int \left[ e^{ika/2} - e^{-ika/2} \right] e^{i(p' - p)a/h} \frac{d^3a d^3p'}{(2\pi h)^3}.
\]
The internal integral is equal to
\[
\frac{1}{(2\pi\hbar)^3} \int \left\{ \exp \left( i \left[ p' - p + \frac{k\hbar}{2} \right] \right) a \right\} - \exp \left( i \left[ p' - p + \frac{k\hbar}{2} \right] \right) \right\} d^3a = \\
= \delta \left( p' - p + \frac{\hbar k}{2} \right) - \delta \left( p' - p - \frac{\hbar k}{2} \right).
\]

We calculate the Wigner — Vlasov integral
\[
W[f] = \\
= A(r, t) \frac{ie}{mch} \int \left[ \delta(p' - p + \frac{\hbar k}{2}) - \delta(p' - p - \frac{\hbar k}{2}) \right] p' f(r, p', t) d^3p' = \\
= A(r, t) \frac{ie}{mch} \left[ (p - \frac{\hbar k}{2}) f(r, p - \frac{\hbar k}{2}, t) - \left( p + \frac{\hbar k}{2} \right) f(r, p + \frac{\hbar k}{2}, t) \right] = \\
= A(r, t) \frac{ie}{mch} \left\{ p \left[ f(r, p - \frac{\hbar k}{2}, t) - f(r, p + \frac{\hbar k}{2}, t) \right] - \\
\left( p - \frac{\hbar k}{2} \right) \left[ f(r, p - \frac{\hbar k}{2}, t) + f(r, p + \frac{\hbar k}{2}, t) \right] \right\} = \\
= A(r, t) \frac{ie}{mch} p \left( f_+ - f_- \right),
\]
where
\[
f_\pm \equiv f(r, p \mp \frac{\hbar k}{2}, t).
\]

Consequently, the Wigner — Vlasov integral is equal to
\[
W[f] = \frac{ie}{mch} p A(r, t) \left[ f_+ - f_- \right] = \\
+ \frac{ie\nu_T}{mch} p A(r, t) \left[ f_+ - f_- \right] = \\
= \frac{ie\nu_T}{ch} p A(r, t) \left[ f_+ - f_- \right].
\]

Here and below the expression $PA$ means scalar production.
Further we will use dimensionless velocity $C$ in the form

$$C = \frac{v}{v_T} = \frac{P}{p_T} - \frac{e}{c p_T} A(r, t) \equiv P - \frac{e}{c p_T} A(r, t),$$

where $P = \frac{P}{p_T}$ is the dimensionless momentum.

Then

$$v = v(r, P, t) = v_T \left( P - \frac{e}{c p_T} A(r, t) \right).$$

In linear approximation is possible to replace the function $f$ in Wigner—Vlasov integral by the absolute Fermi—Dirac distribution, i.e. we put $f = f_F(P)$, where

$$f_F(P) = \frac{1}{1 + \exp(P^2 - \alpha)}, \quad \alpha = \text{const}.$$  

Here Wigner—Vlasov integral (2.4) has the following form

$$W[f_F] = \frac{i e v_T}{\hbar} P A(r, t) \left[ f_F^+ - f_F^- \right],$$

where

$$f_F^\pm \equiv f_F^\pm(P) = \frac{1}{1 + \exp \left[ \left( P \mp \frac{\hbar k}{2 p_T} \right)^2 - \alpha \right]},$$

and $p_T = m v_T$ is the thermal electron momentum, or,

$$f_F^\pm = \frac{1}{1 + e^{P^2 - \alpha}}.$$  

Here

$$P^2_\pm = \left( P \mp \frac{\hbar k}{2 p_T} \right)^2 = \left( P_x \mp \frac{\hbar k_x}{2 p_T} \right)^2 + \left( P_y \mp \frac{\hbar k_y}{2 p_T} \right)^2 + \left( P_z \mp \frac{\hbar k_z}{2 p_T} \right)^2,$$

or

$$p^2_\pm = \frac{\left( p_x \mp \frac{\hbar k_x}{2} \right)^2 + \left( p_y \mp \frac{\hbar k_y}{2} \right)^2 + \left( p_z \mp \frac{\hbar k_z}{2} \right)^2}{p_T^2}.$$

The linearization of the Wigner equilibrium function (2.2) we will carry out in terms of vector potential $A(r, t)$

$$f^{(0)} = f^{(0)} \bigg|_{A=0} + \frac{\partial f^{(0)}}{\partial A} \bigg|_{A=0} A(r, t),$$
or, in explicit form

\[ f^{(0)} = f_{F}(P) + g(P) \frac{2e}{cPT} PA(r, t), \quad (2.5) \]

\[ g(P) = \frac{e^{P^2 - \alpha}}{(1 + e^{P^2 - \alpha})^2}. \]

Considering decomposition (2.5), we will search for Wigner’s function in the form

\[ f = f_{F}(P) + g(P) \frac{2e}{cPT} PA(r, t) + g(P)(PA(r, t))h(P). \quad (2.6) \]

We receive the following equation

\[ [PA(r, t)] g(P)(\nu - i\omega + ikv)h(P) = \]

\[ = \frac{iev_T}{\hbar}[PA(r, t)] (f_{F}^+ - f_{F}^-) + \frac{2ie}{cPT} g(P)(\nu - v_TkP)[PA(r, t)]. \]

From this equation we find

\[ [PA(r, t)] g(P)h(P) = [PA(r, t)] \frac{2ie}{cPT} \left[ \frac{\omega - v_TkP}{\nu - i\omega + iv_TkP} g(P) + \right. \]

\[ \left. + \frac{mv_T^2}{2\hbar} \frac{f_{F}^+(P) - f_{F}^-(P)}{\nu - i\omega + iv_TkP} \right]. \quad (2.7) \]

With the help of (2.6) and (2.7) we construct the full distribution function

\[ f = f^{(0)} + g(P)h(P)PA = \]

\[ = f^{(0)} + \frac{2ie}{cPT} PA \left[ \frac{\omega - v_TkP}{\nu - i\omega + iv_TkP} g(P) + \frac{mv_T^2}{2\hbar} \frac{f_{F}^+(P) - f_{F}^-(P)}{1 - i\omega + ik_P} \right], \]

or

\[ f = f^{(0)} + \frac{2ie}{cPT} PA \left[ \frac{\omega - k_P}{1 - i\omega + ik_P} g(P) + \frac{mv_T^2}{2\hbar} \frac{f_{F}^+(P) - f_{F}^-(P)}{1 - i\omega + ik_P} \right]. \quad (2.8) \]
Here $k_1 = k l$, $l$ is the electron mean free path, $l = v_T \tau$, $k_1$ is the dimensionless wave vector.

We consider the connection between electric field and potentials

$$E(r, t) = -\frac{1}{c} \frac{\partial A(r, t)}{\partial t} - \frac{\partial U(r, t)}{\partial r},$$

or

$$E(r, t) = \frac{i \omega}{c} A(r, t).$$

Hence, the current is connected with vector potential as

$$j(r, t) = \sigma_t r \frac{i \omega}{c} A(r, t).$$

By definition, the current is equal to

$$j(r, t) = e \int v(r, p, t) f^2 \frac{2 d^3 p}{(2\pi \hbar)^3}.$$

Let’s note, that the current in the equilibrium state is equal to zero

$$j^{(0)}(r, t) = e \int v(r, P, t) f^{(0)} \frac{2p_T^3 d^3 P}{(2\pi \hbar)^3} = 0.$$

Indeed, considering that mean electron velocity in the equilibrium state is equal to zero, according to (1.15) we have

$$j^{(0)}(r, t) = eN^{(0)} u^{(0)}(r, t) \equiv 0.$$

Hence, with the use of equality (2.8) we have the following equality

$$j(r, t) = i \frac{4e^2 p_T^2}{(2\pi \hbar)^3 c} \int \left( PA \right) v(r, P, t) \times$$

$$\times \left[ \frac{\omega \tau - k_1 P}{1 - i \omega \tau + i k_1 P} g(P) + \frac{\mathcal{E}_T f_F^+(P) - f_F^-(P)}{\hbar \nu} \right] d^3 P,$$

where $\mathcal{E}_T$ is the thermal kinetic energy of electrons,

$$\mathcal{E}_T = \frac{mv^2_T}{2}.$$
Substituting obvious expression for the velocity into this equality

\[ v(r, P, t) = \frac{p}{m} - \frac{eA(r, t)}{mc} = \frac{p_T P}{m} - \frac{eA(r, t)}{mc}, \]

and, after linearization of it by vector field, we receive

\[ j(r, t) = i \frac{4e^2 p^3_T}{(2\pi\hbar)^3 mc} \int \left( P A(r, t) \right) P \times \]

\[ \times \left[ \frac{\omega\tau - k_1 P}{1 - i\omega\tau + ik_1 P} g(P) + \frac{\varepsilon_T f_F^+(P) - f_F^-(P)}{\hbar \nu 1 - i\omega\tau + ik_1 P} \right] d^3P. \]

It is seen easily, that all the components of the vector \( j \), which are orthogonal to the vector \( A \) are equal to zero. Therefore

\[ j(r, t) = i \frac{4e^2 p^3_T A(r, t)}{(2\pi\hbar)^3 mc} \int \left( e_1 P \right)^2 \left[ \frac{\omega\tau - k_1 P}{1 - i\omega\tau + ik_1 P} g(P) + \right. \]

\[ \left. + \frac{\varepsilon_T f_F^+(P) - f_F^-(P)}{\hbar \nu 1 - i\omega\tau + ik_1 P} \right] d^3P. \]

Here \( e_1 = A/A \) is the unit vector directed lengthwise \( A \). In view of the symmetry the value of integral will not change, if the vector \( e_1 \) is replaced by any other unit vector \( e_2 \), perpendicular to the vector \( k_1 \). Therefore

\[ \int \left( e_1 P \right)^2 [s] d^3P = \int \left( e_2 P \right)^2 [s] d^3P = \]

\[ = \frac{1}{2} \int \left[ \left( e_1 P \right)^2 + \left( e_2 P \right)^2 \right] [s] d^3P, \]

where

\[ e_2 = \frac{A \times k_1}{|A \times k_1|} = \frac{A \times k_1}{Ak_1}, \]

and \( A \times k_1 \) is the vector product.

But we have further

\[ \left( e_1 P \right)^2 + \left( e_2 P \right)^2 = P^2 - \frac{(P k_1)^2}{k_1^2} = \]
\[ = P^2 - (Pn)^2 \equiv P_{\perp}^2, \]

where \( n \) is the unit vector directed along the vector \( k_1 \), \( n = \frac{k_1}{k_1} \).

Hence for the current density we receive the following expression

\[
j(r, t) = i \frac{2e^2p_T^3}{(2\pi \hbar)^3mc} \int \left[ \frac{\omega - k_1P}{1 - i\omega + ik_1P} g(P) + \frac{\mathcal{E}_T f_F^+(P) - f_F^-(P)}{\hbar \nu 1 - i\omega + ik_1P} \right] P_{\perp}^2 d^3P.
\]

Replacing the current in the left–hand side of this equality by the expression in terms of field, we receive:

\[
\sigma_{tr} = \frac{i\omega}{c} A(r, t) = i \frac{2e^2p_T^3}{(2\pi \hbar)^3mc} \int \times
\[
\times \left[ \frac{\omega - k_1P}{1 - i\omega + ik_1P} g(P) + \frac{\mathcal{E}_T f_F^+(P) - f_F^-(P)}{\hbar \nu 1 - i\omega + ik_1P} \right] P_{\perp}^2 d^3P.
\]

4. ELECTRIC CONDUCTIVITY AND DIELECTRIC FUNCTION

From the last formula we receive the following expression for the transverse dielectric function in quantum plasma

\[
\sigma_{tr} = \frac{2e^2p_T^3}{(2\pi \hbar)^3m\omega} \int \left[ P^2 - \left( \frac{P k_1}{k_1^2} \right)^2 \right] \left[ \frac{\omega - k_1P}{1 - i\omega + ik_1P} g(P) + \frac{\mathcal{E}_T f_F^+(P) - f_F^-(P)}{\hbar \nu 1 - i\omega + ik_1P} \right] d^3P.
\]

We will transform expression for transverse conductivity and we will bring it to the form

\[
\sigma_{tr} = \frac{2e^2p_T^3}{(2\pi \hbar)^3m\omega} \int \left[ (\omega - k_1P) g(P) + \frac{\mathcal{E}_T}{\hbar \nu} (f_F^+ - f_F^-) \right] \frac{P_{\perp}^2 d^3P}{1 - i\omega + ik_1P}.
\]
With the use of the equality (1.14) we will present the previous formula in the form

\[
\sigma_{tr} = \frac{\sigma_0}{4\pi f_2(\alpha)} \int \left[ 1 - (\omega \tau)^{-1} k_1 P \right] g(P) +
\]
\[+ \frac{\mathcal{E}_T}{\hbar \omega} \left[ f_F^+(P) - f_F^-(P) \right] \right]\frac{P^2_1 d^3 P}{1 - i\omega \tau + ik_1 P}.
\]

(3.1)

Here the function \(f_2(\alpha)\) has been entered above and in the absence of the scalar potential it is defined by equality

\[
f_2(\alpha) = \int_0^\infty x^2 f_F(x) dx = \int_0^\infty \frac{x^2 dx}{1 + e^{x^2-\alpha}} = \frac{1}{2} \int_0^\infty \ln(1 + e^{\alpha-x^2}) dx.
\]

The quantity \(\sigma_0\) is defined by classical expression for the static electric conductivity

\[
\sigma_0 = \frac{e^2 N^{(0)}}{m \nu}.
\]

Dielectric function we will find according to the formula

\[
\varepsilon_{tr} = 1 + \frac{4\pi i}{\omega} \sigma_{tr}.
\]

Substituting electric conductivity (3.1) into this equality, we receive the expression for dielectric permittivity in quantum collision plasma

\[
\varepsilon_{tr} = 1 + \frac{\omega_p^2}{\omega^2} \frac{i}{4\pi f_2(\alpha)} \int \left\{ \left[ \omega \tau - k_1 P \right] g(P) +
\]
\[+ \frac{\mathcal{E}_T}{\hbar \nu} \left[ f_F^+(P) - f_F^-(P) \right] \right\} \frac{P^2_1 d^3 P}{1 - i\omega \tau + ik_1 P}.
\]

We investigate some special cases of electroconductivity. In the long-wave limit (when \(k \to 0\)) from (3.1) we receive the well known classical expression

\[
\sigma_{tr}(k = 0) = \sigma_0 \frac{\nu}{\nu - i\omega} = \frac{\sigma_0}{1 - i\omega \tau}.
\]
Let’s consider the quantum mechanical limit of the conductivity in the case of arbitrary values of wave number, i.e. conductivity limit in the case, when Planck’s constant $\hbar \to 0$, and the quantity $k$ is arbitrary.

Now we consider the case, when values of the wave number are arbitrary, but Planck’s constant converges to zero: $\hbar \to 0$.

When the values of $\hbar$ are small we have

$$f_0^\pm(P) = f_F(P) \pm g(P)\frac{\hbar k}{2m v_T},$$

hence

$$f_F^+(P) - f_F^-(P) = 2g(P)\frac{\hbar k}{2m v_T}.$$

Therefore

$$(\omega - v_T kP)g(P) + \frac{P^2}{2m \hbar} [f_F^+(P) - f_F^-(P)] = \omega g(P).$$

Thus, in linear approximation at small $\hbar$ (independently of the quantity $k$) for transverse conductivity we receive

$$\sigma_{tr} = \sigma_{tr}^{classic},$$

where

$$\sigma_{tr}^{classic} = \frac{\sigma_0}{4\pi f_2(\alpha)} \int \frac{g(P) P^2 d^3 P}{1 - i\omega \tau + i k_1 P}.$$ (3.2)

The expression (3.2) accurately coincides with the expression of the transverse conductivity for classical plasma with arbitrary temperature.

Let’s return to the expression (3.1). We present it in the form of the sum of two components

$$\sigma_{tr} = \sigma_{tr}^{classic} + \sigma_{tr}^{quant},$$ (3.3)

where $\sigma^{classic}$ is defined by the equality (3.2), and second component $\sigma_{tr}^{quant}$ corresponds to quantum properties of the plasma under consideration

$$\sigma_{tr}^{quant} = \frac{\sigma_0}{4\pi f_2(\alpha)} \int \left[ - \frac{k_1 P}{\omega \tau} g(P) + \right.$$
\[ + \frac{\mathcal{E}_T}{\hbar \omega} [f_F^+(\mathbf{P}) - f_F^-(\mathbf{P})] \left( \frac{P^2 \, d^3P}{1 - i\omega\tau + i{k}_1 \mathbf{P}} \right). \quad (3.4) \]

The quantum summand \( \sigma_{tr}^{\text{quant}} \) we will present in the form, proportional to a square of the Planck’s constant \( \hbar \).

For this aim we use cubic expansion of \( \sigma_{tr}^{\text{quant}} \) by powers of \( \hbar \). We will remind, that in linear approximation by \( \hbar \), as it was already specified, the quantity \( \sigma_{tr}^{\text{quant}} \) disappears. We will direct an axis \( x \) along the wave vector \( \mathbf{k} \).

Let’s expand the Fermi — Dirac distribution by degrees of dimensionless wave number \( q = \frac{k}{k_T} = \frac{k_1 \hbar \nu}{mv_T^2} \), where \( k_T = \frac{p_T}{\hbar} \) is the thermal wave number.

We receive
\[
f_F^\pm(\mathbf{P}) = f_F(\mathbf{P}) \pm g(P)P_xq - \left[ g'_{P^2}(P)P^2_x + \frac{1}{2} g(P) \right] q^2 \pm \left[ g''_{P^2}(P^2)P^2_x + \frac{3}{2} g'_P(P) \right] P_xq^3 + \cdots.
\]

Here
\[
g'_{P^2}(P) = g'(P^2), \quad g''_{P^2}(P^2) = g''(P^2),
\]
\[
g'(P^2) = g(P) \frac{1 - e^{P^2 - \alpha}}{1 + e^{P^2 - \alpha}},
\]
\[
g''(P^2) = \left[ \frac{1 - e^{P^2 - \alpha}}{1 + e^{P^2 - \alpha}} \right]^2 - 2g(P).
\]

Now we will find the difference
\[
f_F^+(\mathbf{P}) - f_F^-(\mathbf{P}) = 2g(P)P_xq + \left[ g''(P^2)P^2_x + \frac{3}{2} g'(P^2) \right] P_xq^3 + \cdots.
\]

By means of this expression we find, that
\[
- \frac{k_1 P_x}{\omega \tau} g(P) + \frac{\mathcal{E}_T}{\hbar \omega} [f_F^+(\mathbf{P}) - f_F^-(\mathbf{P})] = G(\mathbf{P}) \frac{k_1^3 \hbar^2 \nu^3}{6\omega m^2 \nu_T^4} + \cdots,
\]

where
\[
G(\mathbf{P}) = P_x \left[ g''(P^2)P^2_x + \frac{3}{2} g'(P^2) \right] .
\]
Substituting this expression into (3.4), we obtain, that the quantum summand is proportional to the square of Planck's constant and it is defined by expression

\[ \sigma^\text{quant}_{tr} = \hbar^2 \sigma_0 \frac{k_1^3 \nu^3}{24\pi \omega m^2 v_T f_2(\alpha)} \int \frac{G(P)(P^2 - P_x^2) d^3 P}{1 - i\omega \tau + ik_1 P_x}. \] (3.5)

In the expressions for classical and quantum components of the conductivity we can simplify several integrals.

We break the triple integral to external one–dimensional integration by the variable \( P_x \) from \(-\infty\) to \( +\infty \) and internal double integration by plane orthogonal to the axis \( P_x \) in the expression (3.3). The internal integration we carry out in polar coordinates. Here we obtain that

\[ P^2 = P_x + P_{\perp}^2, \quad d^3 P = dP_x dP_{\perp}, \quad dP_{\perp} = P_{\perp} dP_{\perp} d\chi, \]

where \( P_{\perp} \) is the polar radius, and \( \chi \) is the polar angle.

Thus we receive, that

\[ \sigma_{tr}^\text{classic} = \frac{\sigma_0}{4\pi f_2(\alpha)} \int_{-\infty}^{\infty} dP_x \int_{0}^{2\pi} d\chi \int_0^\infty \frac{g(P) P_{\perp}^3 dP_{\perp} d\chi}{1 - i\omega \tau + ik_1 P_x}, \]

where

\[ g(P) = \frac{e^{P_x^2 + P_{\perp} - \alpha}}{(1 + e^{P_x^2 + P_{\perp} - \alpha})^2}. \]

Internal double integral we calculate in polar coordinates

\[ \int_0^{2\pi} \int_0^\infty g(P) P_{\perp}^3 dP_{\perp} d\chi = 2\pi \int_0^\infty \frac{P_{\perp} dP_{\perp}}{1 + e^{P_x^2 + P_{\perp}^2 - \alpha}} = \]

\[ \equiv 2\pi \int_0^\infty f_F(P) P_{\perp} dP_{\perp} = 2\pi \int_0^\infty \frac{e^{\alpha - P_x^2 - P_{\perp}^2} P_{\perp} dP_{\perp}}{1 + e^{\alpha - P_x^2 - P_{\perp}^2}} = \]

\[ = \pi \ln(1 + e^{\alpha - P_x^2}). \] (3.6)
Hence, the expression for the classical component is simplified to one-dimensional integral

\[
\sigma_{\text{tr}}^{\text{classic}} = \sigma_0 \frac{f_2(\alpha)}{4} \int_{-\infty}^{\infty} \frac{\ln(1 + e^{\alpha - P_x^2})}{1 - i\omega\tau + ik_1P_x} dP_x,
\]

or

\[
\sigma_{\text{tr}}^{\text{classic}} = -\sigma_0 y \frac{f_2(\alpha)q}{4} \int_{-\infty}^{\infty} \frac{\ln(1 + e^{\alpha - \tau^2})}{\tau - z/q} d\tau,
\]

where

\[
z = \frac{\omega + i\nu}{kTv_T}, \quad q = \frac{k}{k_T}.
\]

The quantum item (3.4) we present in the form of the sum of two items

\[
\sigma_{\text{tr}}^{\text{quant}} = \sigma_1 + \sigma_2.
\]

Here

\[
\sigma_1 = -\sigma_0 k_1 \frac{f_2(\alpha)}{4\pi} \int_{-\infty}^{\infty} \frac{P_x(P^2 - P_x^2)g(P)}{1 - i\omega\tau + ik_1P_x} d^3P,
\]

and

\[
\sigma_2 = \frac{\sigma_0 \mathcal{E}_T}{4\pi f_2(\alpha)\hbar\omega} \int \left[ f_+^1(P) - f_-^1(P) \right] (P^2 - P_x^2) d^3P.
\]

With the help of the equality (3.6) the expression for \(\sigma_1\) can be rewritten in the following form

\[
\sigma_1 = -\sigma_0 k_1 \frac{f_2(\alpha)}{4\pi} \int_{-\infty}^{\infty} \frac{P_x \ln(1 + e^{\alpha - P_x^2})}{1 - i\omega\tau + ik_1P_x} dP_x,
\]

or

\[
\sigma_1 = -\sigma_0 y \frac{f_2(\alpha)q}{4\pi} \int_{-\infty}^{\infty} \frac{\tau \ln(1 + e^{\alpha - \tau^2})}{\tau - z/q} d\tau.
\]

After change of variable

\[
P_x \mp \frac{\hbar k}{2p_T} \equiv P_x \mp \frac{k_1\hbar
\nu}{2mw_T^2} \equiv P_x \mp \frac{k_1\hbar
\nu}{4\mathcal{E}_T} \rightarrow P_x
\]
the difference of integrals from (3.9) will be transformed to one integral and we receive

\[ \sigma_2 = -\frac{i\sigma_0 k_1^2}{8\pi f_2(\alpha) \omega \tau} \int \frac{f_F(P)(P^2 - \frac{1}{2}P_x^2)}{(1 - i\omega \tau + i k_1 P_x)^2 + (k_1^2 \hbar \nu / 4 E_T)^2} \, d^3P. \] (3.11)

In the same way, as well as during the derivation of the formula (3.6), double internal integral in (3.11) we reduce to the one-dimensional integral

\[ \int \int f_F(P)[P^2 - \frac{1}{2}P_x^2] \, dP \perp = \int \int f_F(P) P_\perp^3 \, dP \perp d\chi = \\
= 2\pi \int P_\perp \ln(1 + e^{\alpha - \frac{1}{2}P_x^2}) \, dP_\perp. \]

Now the expression (3.11) can be written in the following form (replacing a variable of integration \( P_\perp = \rho \))

\[ \sigma_2 = -\frac{i\sigma_0 k_1^2}{4 f_2(\alpha) \omega \tau} \int \int \frac{\rho \ln(1 + e^{\alpha - \rho^2 - P_x^2}) d\rho dP_x}{(1 - i\omega \tau + i k_1 P_x)^2 + (k_1^2 \hbar \nu / 4 E_T)^2}. \] (3.12)

Hence we can to present the expression for transverse conductivity in the form of the sum of one-dimensional (3.11) and two-dimensional (3.12) integrals

\[ \sigma_{tr} = \frac{\sigma_0}{4 f_2(\alpha)} \int \int \frac{[1 - (k_1 / \omega \tau) P_x] \ln(1 + e^{\alpha - P_x^2}) dP_x}{1 - i\omega \tau + i k_1 P_x} - \\
- \frac{i\sigma_0 k_1^2}{4 f_2(\alpha) \omega \tau} \int \int \frac{\rho \ln(1 + e^{\alpha - \rho^2 - P_x^2}) d\rho dP_x}{(1 - i\omega \tau + i k_1 P_x)^2 + (k_1^2 \hbar \nu / 4 E_T)^2}. \]

In the expression (3.11) for \( \sigma_2 \) the triple integral can be reduced to one-dimensional integral. For this purpose in (3.11) we pass to integration
in spherical coordinates and present this expression in the form

\[ \sigma_2 = -\frac{i\sigma_0 k_1^2}{4f_2(\alpha)\omega\tau} \int_{0}^{\infty} f_F(P) P^{4} J(P) dP, \]

where

\[ J(P) = \int_{-1}^{1} \frac{(1 - \mu^2) d\mu}{(1 - i\omega\tau + ik_1 P \mu)^2 + (k_1^2 \hbar \nu / 4\mathcal{E}_T)^2}. \]

Let’s designate temporarily

\[ a = 1 - i\omega\tau, \quad b = ik_1 P, \quad d = \frac{\hbar \nu k_1^2}{4\mathcal{E}_T}, \]

and rewrite the integral \( J(P) \) in the form:

\[ J = \int_{-1}^{1} \frac{(1 - \mu^2) d\mu}{(a + b\mu)^2 + d^2}. \]

After change of variable \( a + b\mu = t \) this integral will be rewritten in the form

\[ J = \frac{1}{b^3} \int_{a-b}^{a+b} \frac{b^2 - (t-a)^2}{t^2 + d^2} dt. \]

This integral equals to

\[ J = -\frac{2}{b^2} + \frac{d^2 + b^2 - a^2}{b^3} \ln \frac{(a + b - id)(a - b + id)}{(a + b + id)(a - b - id)} + \]

\[ + \frac{a}{b^3} \ln \frac{(a + b - id)(a + b + id)}{(a - b - id)(a - b + id)}, \]

or

\[ J = -\frac{2}{b^2} + \frac{d^2 + b^2 - a^2}{2idb^3} \ln \frac{a^2 - (b - id)^2}{a^2 - (d + id)^2} + \frac{a}{b^3} \ln \frac{(a + b)^2 + d^2}{(a - d)^2 + d^2}. \]

Considering designations for \( a, b, d \), we receive

\[ J(P) \equiv J(P; \omega\tau, k_1) = \]
Thus, the expression of quantum transverse conductivity is defined by one-dimensional integral

\[
\sigma_{tr} = \frac{\sigma_0}{4 f_2(\alpha)} \int_{-\infty}^{\infty} \frac{1 - (k_1/\omega \tau) P_x}{1 - i \omega \tau + i k_1 P_x} \ln(1 + e^{\alpha - P_x^2}) dP_x - \frac{i \sigma_0 k_1^2}{4 f_2(\alpha) \omega \tau} \int_0^\infty f_F(P) P^4 J(P) dP,
\]

where the function \( J(P) \) is defined by expression (3.13).

Let's consider the case of degenerate plasma separately.

5. DEGENERATE QUANTUM PLASMA

Let's return to the formula (3.1) for transverse conductivity. With the help of (1.14) we will reduce it to the form

\[
\sigma_{tr} = \frac{2 e^2 m^3 v_T^3}{\omega m (2\pi \hbar)^3} \int \frac{\omega \tau - k_1 P}{1 - i \omega \tau + i k_1 P} g(P) P^2 d^3P + \frac{e^2 m^3 v_F^5}{\omega (2\pi \hbar)^3 \hbar} \int f_F^+(P) - f_F^-(P) \frac{P_\perp d^3P}{1 - i \omega \tau + i k_1 P}.
\] (4.1)

In the formula (4.1) we will pass to a new dimensionless variable \( P = \frac{P}{p_F} \), where \( p_F = mv_F \), \( v_F \) is the electron velocity on Fermi’s surface which is supposed to be spherical. Then we receive for \( \sigma_{tr} \) the following expression

\[
\sigma_{tr} = \frac{e^2 m^3 v_F^3}{(2\pi \hbar)^2 \omega k_B T} \int \frac{\omega \tau - k_1 P}{1 - i \omega \tau + i k_1 P} g(P) \frac{P^2 d^3P}{1 - i \omega \tau + i k_1 P} + \frac{e^2 m^3 v_F^3}{(2\pi \hbar)^2 \omega k_B T} \int f_F^+(P) - f_F^-(P) \frac{P_\perp d^3P}{1 - i \omega \tau + i k_1 P}.
\]
\[
+ \frac{e^2 m^3 v_F^5}{(2\pi \hbar)^3 \omega \nu \hbar} \int \left[ f'_F - f_F \right] \frac{P^2 d^3 P}{1 - i\omega + i k_1 \mathbf{P}}. \tag{4.1'}
\]

In this expression \( l = v_F \tau \) is the mean free path of electrons in degenerate plasma, \( k_1 = k l \),

\[
g(P) = \frac{\exp \left( \frac{\mathcal{E} - \mathcal{E}_F}{k_B T} \right)}{\left[ 1 + \exp \left( \frac{\mathcal{E} - \mathcal{E}_F}{k_B T} \right) \right]^2} = \frac{\exp \left( \frac{\mathcal{E}_F (P^2 - 1)}{k_B T} \right)}{\left[ 1 + \exp \left( \frac{\mathcal{E}_F (P^2 - 1)}{k_B T} \right) \right]^2},
\]

\[
f^\pm_F = f_F (P^\pm) = \frac{1}{1 + \exp \left[ \frac{\mathcal{E}_F (P^2 - \mu)}{k_B T} \right]} = \frac{1}{1 + \exp \left( \frac{\mathcal{E}^\pm - \mu}{k_B T} \right)}.
\]

Here following designations are entered

\[
\mathcal{E}^\pm = \frac{1}{2m} \left( \mathbf{p} \mp \frac{\hbar \mathbf{k}}{2} \right)^2, \quad P^\pm = \left( \mathbf{P} \mp \frac{\hbar \mathbf{k}}{2 p_F} \right)^2, \quad \mathcal{E}_F = \frac{mv_F^2}{2}.
\]

Let’s pass in (4.1') to a limit at \( T \to 0 \). Thus chemical potential passes to Fermi energy of electrons on Fermi’s surfaces, i.e. \( \mu \to \mathcal{E}_F \). We easily will show that

\[
\lim_{T \to 0} f^\pm_F = \Theta(\mathcal{E}_F - \mathcal{E}^\pm) \equiv \Theta(1 - P^2) \equiv \Theta^\pm,
\]

\[
\lim_{T \to 0} \frac{g(P)}{k_B T} = \frac{-\partial}{\partial \mathcal{E}} \left[ \lim_{T \to 0} \frac{1}{1 + \exp \left( \frac{\mathcal{E} - \mathcal{E}_F}{k_B T} \right)} \right] = -\frac{\partial}{\partial \mathcal{E}} \Theta(\mathcal{E}_F - \mathcal{E}) = \delta(\mathcal{E}_F - \mathcal{E}).
\]

Here \( \delta(x) \) is the Dirac delta-function, \( \Theta(x) \) is the Heaviside function,

\[
\Theta(x) = \begin{cases} 
1, & x > 0, \\
0, & x < 0,
\end{cases}
\]

\[
\mathcal{E}_F = \frac{mv_F^2}{2} = \frac{p_F^2}{2m} \quad \text{is the electron kinetic energy on the Fermi surface,}
\]

\[
\mathcal{E} = \frac{mv^2}{2} = \frac{p^2}{2m} = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m}.
\]
is the kinetic electron energy,

\[ \Theta^\pm(\mathcal{E}_F - \mathcal{E}^\pm) = \begin{cases} 1, & \mathcal{E}^\pm < \mathcal{E}_F, \\ 0, & \mathcal{E}^\pm > \mathcal{E}_F, \end{cases} \]

\[ \Theta^\pm(1 - P^2_{\pm}) = \begin{cases} 1, & P_{\pm} < 1, \\ 0, & P_{\pm} > 1, \end{cases} \]

though

\[ \mathcal{E}^\pm = \frac{1}{2m} \left( p_x \mp \frac{\hbar k}{2} \right)^2 + \frac{p_y^2}{2m} + \frac{p_z^2}{2m}. \]

Hence, for transverse conductivity of degenerate quantum plasma we have the following form

\[ \sigma_{tr} = \frac{e^2 m^3 v_F^5}{(2\pi \hbar)^3 \omega} \int \left[ (\omega \tau - k_1 P) \delta(\mathcal{E}_F - \mathcal{E}) + \Theta(\mathcal{E}_F - \mathcal{E}^+) - \Theta(\mathcal{E}_F - \mathcal{E}^-) \right] \frac{P_+ d^3 P}{\hbar \nu} \left( \frac{1}{1 - i\omega \tau + i k_1 P} \right). \] (4.2)

Now with the help of the equation of state for degenerate plasma

\[ \left( \frac{mv_F}{\hbar} \right)^3 = 3\pi^2 N(0), \]

we transform the formula (4.2) to the form

\[ \sigma_{tr} = \sigma_0 \frac{3mv_F^2}{8\pi} \int \left[ \left( 1 - \frac{k_1 P}{\omega \tau} \right) \delta(\mathcal{E}_F - \mathcal{E}) + \frac{1}{\hbar \omega} \left[ \Theta(\mathcal{E}_F - \mathcal{E}^+) - \Theta(\mathcal{E}_F - \mathcal{E}^-) \right] \right] \frac{P_+ d^3 P}{1 - i\omega \tau + i k_1 P}. \]

Let’s note, that

\[ \delta(\mathcal{E}_F - \mathcal{E}) = \delta\left( \frac{mv_F^2}{2}(1 - P^2) \right) = \frac{2}{mv_F^2} \delta(1 - P^2) = \frac{1}{mv_F^2} \delta(1 - P), \]

\[ \Theta(\mathcal{E}_F - \mathcal{E}^+) = \Theta(\mathcal{E}_F(1 - P^2_{\pm})) \equiv \Theta(1 - P^2_{\pm}). \]
By means of this equality the expression for $\sigma_2$ can be written in the following form

$$\sigma_{tr} = \sigma_0 \frac{3}{8\pi} \left[ \int \frac{(1 - (\omega \tau)^{-1}k_1P)\delta(1 - P)}{1 - i\omega \tau + i k_1\mathbf{P}} P_\perp^2 \, d^3\mathbf{P} + \right.$$

$$+ \frac{mv_F^2}{\hbar \omega} \int \frac{\Theta(1 - P_\perp^2) - \Theta(1 - P_\perp^2)}{1 - i\omega \tau + i k_1\mathbf{P}} P_\perp^2 \, d^3\mathbf{P} \right].$$

(4.3)

Here

$$P_\perp^2 = \left( \mathbf{P} \mp \frac{\hbar \mathbf{k}}{2p_F} \right)^2,$$

$$\Theta(1 - P_\perp^2) = \Theta \left[ 1 - \left( \mathbf{P} \mp \frac{\hbar \mathbf{k}}{2p_F} \right)^2 \right],$$

or,

$$\Theta^\pm(\mathbf{P}) \equiv \Theta(1 - P_\perp^2) =$$

$$= \begin{cases} 1, & \text{if} \quad \left( P_x \mp \frac{\hbar k_x}{2p_F} \right)^2 + \left( P_y \mp \frac{\hbar k_y}{2p_F} \right)^2 + \left( P_z \mp \frac{\hbar k_z}{2p_F} \right)^2 < 1, \\ 0, & \text{if} \quad \left( P_x \mp \frac{\hbar k_x}{2p_F} \right)^2 + \left( P_y \mp \frac{\hbar k_y}{2p_F} \right)^2 + \left( P_z \mp \frac{\hbar k_z}{2p_F} \right)^2 > 1. \end{cases}$$

Let’s consider the special case of transverse conductivity, when the wave number $k$ is equal to zero. Then in the formula (4.3) the second (quantum) item drops out and we obtain

$$\sigma_{tr}(k = 0) = \sigma_{tr}^{\text{classic}}(k = 0) =$$

$$= \sigma_0 \frac{3}{8\pi} \frac{\nu}{\nu - i\omega} \int \delta(1 - P)[P^2 - P_y^2] \, d^3\mathbf{P},$$

whence we receive the well known formula for classical plasma

$$\sigma_{tr}(k = 0) = \sigma_{tr}^{\text{classic}}(k = 0) = \sigma_0 \frac{\nu}{\nu - i\omega}.$$
Further everywhere we will direct the axis \(x\) along the vector \(k_1\). Let’s consider a case of small values of the product \(\hbar k\). We will note that

\[
\Theta^\pm \equiv \Theta(\mathcal{E}_F - \mathcal{E}^\pm) = \begin{cases} 
1, & \text{if } \left( p_x \mp \frac{\hbar k}{2} \right)^2 + p_y^2 + p_z^2 < p_F^2, \\
0, & \text{if } \left( p_x \mp \frac{\hbar k}{2} \right)^2 + p_y^2 + p_z^2 > p_F^2
\end{cases}
\]

Here

\[
\mathcal{E}^\pm = \frac{1}{2m} \left( p_x \mp \frac{\hbar k}{2} \right)^2 + \frac{1}{2m} p_y^2 + \frac{1}{2m} p_z^2.
\]

Let’s expand \(\Theta^\pm\) by powers of \(\hbar k\) to the second order inclusive

\[
\Theta^\pm = \Theta(\mathcal{E}_F - \mathcal{E}^\pm) = \Theta(1 - P_x^2) = \\
= \Theta(1 - P^2) \pm \delta(1 - P^2)P_x \frac{\hbar k}{p_F} + \\
+ \left[ \delta'(1 - P^2)P_x^2 - \frac{1}{2} \delta(1 - P^2) \right] \frac{\hbar^2 k^2}{2p_F^2} \pm \\
\pm \left[ \delta''(1 - P^2)P_x^3 - \frac{3}{2} \delta'(1 - P^2)P_x \right] \frac{\hbar^3 k^3}{6p_F^3}.
\]

From here follows that

\[
\Theta^+(1 - P_x^2) - \Theta^-(1 - P_x^2) = 2\delta(1 - P^2)P_x \frac{\hbar k}{p_F} + \\
+ \left[ \delta''(1 - P^2)P_x^3 - \frac{3}{2} \delta'(1 - P^2)P_x \right] \frac{\hbar^3 k^3}{3p_F^3}.
\]

We will present the formula for calculation of transverse conductivity in the following form

\[
\sigma_{tr} = \sigma_0 f_{tr},
\]

where

\[
f_{tr} = \frac{3}{8\pi\omega_T} \int \left[ (\omega_T - k_1 P_x)\delta(1 - P^2) + \\
+ \frac{mv_F^2}{\hbar \nu} (\Theta^+ - \Theta^-) \right] \frac{(P^2 - P_x^2) d^3 P}{1 - i\omega_T + ik_1 P_x},
\]

\[
\Theta^\pm \equiv \Theta(\mathcal{E}_F - \mathcal{E}^\pm) = \begin{cases} 
1, & \text{if } \left( p_x \mp \frac{\hbar k}{2} \right)^2 + p_y^2 + p_z^2 < p_F^2, \\
0, & \text{if } \left( p_x \mp \frac{\hbar k}{2} \right)^2 + p_y^2 + p_z^2 > p_F^2
\end{cases}
\]

Here

\[
\mathcal{E}^\pm = \frac{1}{2m} \left( p_x \mp \frac{\hbar k}{2} \right)^2 + \frac{1}{2m} p_y^2 + \frac{1}{2m} p_z^2.
\]

Let’s expand \(\Theta^\pm\) by powers of \(\hbar k\) to the second order inclusive

\[
\Theta^\pm = \Theta(\mathcal{E}_F - \mathcal{E}^\pm) = \Theta(1 - P_x^2) = \\
= \Theta(1 - P^2) \pm \delta(1 - P^2)P_x \frac{\hbar k}{p_F} + \\
+ \left[ \delta'(1 - P^2)P_x^2 - \frac{1}{2} \delta(1 - P^2) \right] \frac{\hbar^2 k^2}{2p_F^2} \pm \\
\pm \left[ \delta''(1 - P^2)P_x^3 - \frac{3}{2} \delta'(1 - P^2)P_x \right] \frac{\hbar^3 k^3}{6p_F^3}.
\]

From here follows that

\[
\Theta^+(1 - P_x^2) - \Theta^-(1 - P_x^2) = 2\delta(1 - P^2)P_x \frac{\hbar k}{p_F} + \\
+ \left[ \delta''(1 - P^2)P_x^3 - \frac{3}{2} \delta'(1 - P^2)P_x \right] \frac{\hbar^3 k^3}{3p_F^3}.
\]

We will present the formula for calculation of transverse conductivity in the following form

\[
\sigma_{tr} = \sigma_0 f_{tr},
\]

where

\[
f_{tr} = \frac{3}{8\pi\omega_T} \int \left[ (\omega_T - k_1 P_x)\delta(1 - P^2) + \\
+ \frac{mv_F^2}{\hbar \nu} (\Theta^+ - \Theta^-) \right] \frac{(P^2 - P_x^2) d^3 P}{1 - i\omega_T + ik_1 P_x},
\]
or

\[
f_{tr} = \frac{3}{8\pi} \int \left[ \left( 1 - \frac{k_1 P_x}{\omega \tau} \right) \delta(1 - P) + \frac{mv^2_F}{\hbar \omega} (\Theta^+ - \Theta^-) \right] \frac{(P^2 - P_x^2)}{1 - i\omega \tau + ik_1 P_x} d^3 P.
\]

Let’s consider the integrand from the equality for \( f_{tr} \)

\[
\left( 1 - \frac{k_1 P_x}{\omega \tau} \right) \delta(1 - P) + \frac{mv^2_F}{\hbar \omega} (\Theta^+ - \Theta^-) =
\]

\[
= \delta(1 - P) - k_1 P_x \frac{\delta(1 - P) + 2\delta(1 - P^2)P_x \frac{mv^2_F k}{\omega p_F}}{\hbar \omega} + \frac{mv^2_F}{\hbar \omega} \left[ \delta''(1 - P^2)P_x^3 - \frac{3}{2} \delta'(1 - P^2)P_x \right] \frac{\hbar k^3}{3p_F^3} =
\]

\[
= \delta(1 - P) + \left[ \delta''(1 - P^2)P_x^3 - \frac{3}{2} \delta'(1 - P^2)P_x \right] \left( \frac{\hbar \nu}{mv^2_F} \right)^2 \frac{k_1^3}{3 \omega \tau}.
\]

Therefore the transverse conductivity at small \( \hbar k \) is equal to

\[
\sigma_{tr} = \sigma_{tr}^{\text{classic}} +
\]

\[
+ \frac{\hbar^2 \nu^2 k_1^3}{8\pi (mv^2_F)^2 \omega \tau} \int \frac{\delta''(1 - P^2)P_x^2 - \frac{3}{2} \delta'(1 - P^2)}{1 - i\omega \tau + ik_1 P_x} P_x (P^2 - P_x^2) d^3 P.
\]

Here \( k_1 = kl, l = v_F \tau \) is the electron mean free path,

\[
\sigma_{tr}^{\text{classic}} = \sigma_0 \frac{3}{8\pi} \int \frac{\delta(1 - P)(P^2 - P_x^2)}{1 - i\omega \tau + ik_1 P_x} d^3 P.
\]

From the first formula follows that under \( \hbar \to 0 \) \( \sigma_{tr} \to \sigma_{tr}^{\text{classic}} \), i.e. under tendency of Planck’s constant to zero the transverse conductivity passes into the classical.

Let’s pass to decomposition on degrees of wave number of the quantum component the transverse conductivity. For this purpose we will spread out by degrees of wave number of Fermi — Dirac distributions \( \Theta^\pm = \Theta(1 - P^2_\pm) \).

We obtain that

\[
\Theta^\pm = \Theta(1 - P^2_\pm) = \Theta(1 - P^2) \pm \delta(1 - P^2)P_x \frac{\hbar k}{2p_F} +
\]
\[
+ \left[ \delta'(1 - P^2)P_x^2 - \frac{1}{2}\delta(1 - P^2) \right] \frac{\hbar^2 k^2}{2p_F^2} \pm \\
\pm \left[ \delta''(1 - P^2)P_x^3 - \frac{3}{2}\delta'(1 - P^2)P_x \right] \frac{h^3 k^3}{6p_F^3} + \\
+ \left[ \delta'''(1 - P^2)P_x^4 - 3\delta''(1 - P^2)P_x^2 + \frac{3}{4}\delta'(1 - P^2) \right] \frac{h^4 k^4}{24p_F^4} \pm \\
\pm \left[ \delta^{(4)}(1 - P^2)P_x^5 - 5\delta'''(1 - P^2)P_x^3 + \frac{15}{4}\delta''(1 - P^2)P_x \right] \frac{h^5 k^5}{120p_F^5} + \cdots.
\]

From these decomposition we receive their difference

\[
\Theta^+ - \Theta^- = 2\delta(1 - P^2)P_x \frac{\hbar k}{p_F} + \\
+ \left[ \delta''(1 - P^2)P_x^3 - \frac{3}{2}\delta'(1 - P^2)P_x \right] \frac{h^3 k^3}{3p_F^3} + \\
+ \left[ \delta^{(4)}(1 - P^2)P_x^5 - 5\delta'''(1 - P^2)P_x^3 + \frac{15}{4}\delta''(1 - P^2)P_x \right] \frac{h^5 k^5}{60p_F^5} + \cdots.
\]

Now decomposition of quantum component transverse conductivity has following expansion by degrees of wave number

\[
\sigma_{\text{tr}}^{\text{quant}} = i\sigma_0 \frac{7(\omega\tau + i)^2 k_1^6 + 10k_1^8}{140\omega\tau(\omega\tau + i)^6} \left( \frac{\hbar\nu}{2E_F} \right)^2 + \cdots,
\]

or

\[
\sigma_{\text{tr}}^{\text{quant}} = i\sigma_0 \left[ \frac{\nu v_F^4}{20\omega m^2(\omega + i\nu)^4}(\hbar^2 k^6) + \frac{\nu v_F^4}{14\omega m^2(\omega + i\nu)^6}(\hbar^2 k^8) + \cdots \right].
\]

The expression (4.3) for transverse conductivity we will present as the sum of two terms

\[
\sigma_{\text{tr}} = \sigma_{\text{tr}}^{\text{classic}} + \sigma_{\text{tr}}^{\text{quant}}.
\]

In the equality (4.4) the following designations are entered

\[
\sigma_{\text{tr}}^{\text{classic}} = \sigma_0 \frac{3}{8\pi} \int \frac{\delta(1 - P)(P^2 - P_x^2) d^3 P}{1 - i\omega\tau + ik_1 P_x},
\]

\[
\sigma_{\text{tr}}^{\text{quant}} = \sigma_0 \frac{3\nu}{8\pi\omega} \left[ - k_1 \int \frac{P_x \delta(1 - P)(P^2 - P_x^2) d^3 P}{1 - i\omega\tau + ik_1 P_x} + \right.
\]

[...]

\[
\cdot \cdot \cdot
\]
\[ + \frac{mv_F^2}{\hbar \nu} \int \frac{\Theta^+(P) - \Theta^-(P)}{1 - i \omega \tau + ik_1 P_x} (P^2 - P_x^2) d^3P. \] 

(4.6)

The expression (4.5) for \( \sigma_{\text{tr}}^\text{classic} \) is easily calculated in the explicit form

\[ \sigma_{\text{tr}}^\text{classic} = -\sigma_0 \frac{3}{4} \left[ 2\nu\frac{\nu - i \omega}{k^2 v_F^2} - i \nu \frac{(\nu - i \omega)^2 + k^2 v_F^2}{k^3 v_F^3} \ln \frac{\nu - i \omega + ik v_F}{\nu - i \omega - ik v_F} \right], \]

(4.7)

or, in dimensionless parameters,

\[ \sigma_{\text{tr}}^\text{classic} = -\sigma_0 \frac{3}{4} \left[ 2\frac{1 - i \omega \tau}{k_1^2} + i \frac{(1 - i \omega \tau)^2 + k_1^2}{k_1^3} \ln \frac{1 - i \omega \tau + ik_1}{1 - i \omega \tau - ik_1} \right]. \]

For the previous formula (4.7) we can give also such forms

\[ \sigma_{\text{tr}}^\text{classic} = \sigma_0 \frac{3i}{4} \left[ 2\frac{\omega \tau + i}{k_1^2} + \frac{(\omega \tau + i)^2 - k_1^2}{k_1^3} \ln \frac{\omega \tau + i - k_1}{\omega \tau + i + k_1} \right], \]

and

\[ \sigma_{\text{tr}}^\text{classic} = \sigma_0 \frac{3i}{4} \left[ 2\nu \frac{\nu + i \nu}{k^2 v_F^2} + \nu \frac{(\nu + i \nu)^2 - k^2 v_F^2}{k^3 v_F^3} \ln \frac{\nu + i \nu - k v_F}{\nu + i \nu + ik v_F} \right], \]

(4.8)

We introduce the dimensionless variables

\[ z = \frac{\omega + iv}{k_F v_F} = x + iy, \quad x = \frac{\omega}{k_F v_F}, \quad y = \frac{\nu}{k_F v_F}, \quad q = \frac{k}{k_F}, \]

where \( k_F \) is the Fermi wave number, \( k_F = \frac{mv_F}{\hbar} = \frac{p_F}{\hbar} \).

Then we can rewrite the formula (4.7) in the form

\[ \sigma_{\text{tr}}^\text{classic} = \sigma_0 \frac{3i}{4} \left[ 2y \frac{yz}{q^2} + y \frac{z^2 - q^2}{q^3} \ln \frac{z - q}{z + q} \right]. \]

(4.8)

We present the formula (4.6) in the form of the sum of two components

\[ \sigma_{\text{tr}}^\text{quant} = \sigma_1 + \sigma_2. \]

(4.9)

Here

\[ \sigma_1 = -\sigma_0 \frac{3 \nu v_F k}{4 \omega} J_1 = -\sigma_0 \frac{3 \nu}{4 \omega} k_1 J_1, \]

where

\[ J_1 = \frac{1}{2\pi} \int \frac{P_x \delta(1 - P)(P^2 - P_x^2)}{1 - i \omega \tau + ik_1 P_x} d^3P, \]

(4.10)
and

\[ \sigma_2 = \sigma_0 \frac{3}{4\pi} \frac{\nu mv_F^2}{\hbar} J_2, \]

where

\[ J_2 = \frac{1}{2} \int \frac{\Theta^+(\mathbf{P}) - \Theta^-(\mathbf{P})}{\nu - i\omega + ikv_F P_x} (P_x^2 - P_x^2) d^3P = \]

\[ = \frac{1}{2} \int \frac{\Theta^+(\mathbf{P}) - \Theta^-(\mathbf{P})}{\nu - i\omega + ikv_F P_x} (P_y^2 + P_z^2) d^3P = \]

\[ = \int \frac{\Theta^+(\mathbf{P}) - \Theta^-(\mathbf{P})}{\nu - i\omega + ikv_F P_x} P_y^2 d^3P. \]  \hspace{1cm} (4.11)

Calculating the integral \( J_1 \) in spherical coordinates, we receive, that

\[ J_1 = \int_1^1 \frac{\mu(1 - \mu^2) d\mu}{1 - i\omega \tau + ik_1 \mu} = \]

\[ = -\frac{4i}{3k_1} \left[ 1 + \frac{3(1 - i\omega \tau)^2}{2k_1^2} + \frac{3i(1 - i\omega \tau)}{4k_1^3} [(1 - i\omega \tau)^2 + k_1^2] \ln \frac{1 - i\omega \tau + ik_1}{1 - i\omega \tau - ik_1} \right]. \]

Therefore, the quantity \( \sigma_1 \) is equal to

\[ \sigma_1 = \frac{i\sigma_0}{\omega \tau} \left[ 1 + \frac{3(1 - i\omega \tau)^2}{2k_1^2} \right] + \]

\[ + \frac{3i(1 - i\omega \tau)}{4k_1^3} [(1 - i\omega \tau)^2 + k_1^2] \ln \frac{1 - i\omega \tau + ik_1}{1 - i\omega \tau - ik_1}. \]  \hspace{1cm} (4.12)

or

\[ \sigma_1 = i\sigma_0 \frac{\nu}{\omega} \left[ 1 - \frac{3(\omega + i\nu)^2}{2k_F v_F^2} - \frac{3(\omega + i\nu)}{k_F^3 v_F^2} [(\omega + i\nu)^2 - \right. \]

\[ \left. - k_F^2 v_F^2 \right] \ln \frac{\omega + i\nu - kl}{\omega + i\nu + kl} \right]. \]

Let’s pass to calculation of the summand \( \sigma_2 \) which we will present in the form

\[ \sigma_2 = \sigma_0 \frac{3\nu mv_F^2}{4\omega \pi \hbar \nu} \int \frac{\Theta^+(\mathbf{P}) - \Theta^-(\mathbf{P})}{1 - i\omega \tau + ik_1 P_x} P_y^2 d^3P. \]  \hspace{1cm} (4.13)
Let’s present the formula (4.13) in the form of the difference

\[ \sigma_2 = \sigma_0 \frac{3\nu \, m v_F^2}{4\omega \, \pi \hbar \nu} \left[ \int \frac{\Theta^+(P) \, P_y^2 \, d^3 P}{1 - i\omega \tau + ik_1 P_x} - \int \frac{\Theta^-(P) \, P_y^2 \, d^3 P}{1 - i\omega \tau + ik_1 P_x} \right], \]

or

\[ \sigma_2 = \sigma_0 \frac{3\nu \, m v_F^2}{4\omega \, \pi \hbar \nu} (J^+ - J^-), \]

where

\[ J^\pm = \int \frac{\Theta^\pm(P) \, P_y^2 \, d^3 P}{1 - i\omega \tau + ik_1 P_x}. \]

It is obvious that these integrals are equal to:

\[ J^\pm = \int_{S^\pm_3} \frac{P_y^2 \, d^3 P}{1 - i\omega \tau + ik_1 P_x}. \]

Here \( S^\pm_3 = S_3 \left( \pm \frac{\hbar k}{2p_F}, 0, 0 \right) \) is the three-dimensional sphere with unitary radius with the centre in the point \( \left( \pm \frac{\hbar k}{2p_F}, 0, 0 \right) \)

\[ S^\pm_3 \left( \pm \frac{\hbar k}{2p_F}, 0, 0 \right) \equiv \left\{ (P_x, P_y, P_z) : \left( P_x \mp \frac{\hbar k}{2p_F} \right)^2 + P_y^2 + P_z^2 < 1 \right\}. \]

After obvious replacement of variables we receive, that

\[ J^+ = \int_{S^+_3} \frac{P_y^2 \, d^3 P}{1 - i\omega \tau + ik_1 P_x} = \int_{S_3(0)} \frac{P_y^2 \, d^3 P}{1 - i\omega \tau + ik_1 \left( P_x \pm \frac{\hbar \nu k_1}{2mv_F^2} \right)}, \]

where \( S_3(0) \) is the sphere with the centre in zero with unitary radius,

\[ S_3(0) = \left\{ (P_x, P_y, P_z) : P_x^2 + P_y^2 + P_z^2 < 1 \right\}. \]

Now it is easy to find, that the item \( \sigma_2 \) is calculated by the formula

\[ \sigma_2 = -i\sigma_0 \frac{3k_1^2}{8\pi \omega \tau} \int_{S_3(0)} \frac{(P_y^2 + P_z^2) \, d^3 P}{(1 - i\omega \tau + ik_1 P_x)^2 + \left( \frac{\hbar \nu k_1}{2mv_F^2} \right)^2}. \]

(4.14)
We present the sphere $S_3(0)$ in the form of joining:

$$S_3(0) = \bigcup_{P_x=1}^{P_x=1} S_{1-P_x}^2(0,0).$$

Here $S_{1-P_x}^2(0,0)$ is the circle of the form:

$$S_{1-P_x}^2(0,0) = \{(P_y, P_z) : P_y^2 + P_z^2 < 1 - P_x^2\}.$$

Now we will calculate the integrals $J^\pm$ as repeated

$$J^\pm = \frac{1}{2} \int_{-1}^{1} \frac{dP_x}{1 - i\omega \tau + ik_1 \left(P_x \pm \frac{\hbar \nu k_1}{2mv_F}\right)} \int_{S_{1-P_x}^2(0,0)}^{1-P_x^2} \int_{0}^{2\pi} \int_{0}^{P_\perp^3} P_\perp dP_\perp d\chi =$$

$$= \frac{\pi}{4} \int_{-1}^{1} \frac{(1 - t^2)^2 dt}{1 - i\omega \tau + ik_1 \left(t \pm \frac{\hbar \nu k_1}{2mv_F}\right)}, \quad t = P_x.$$

We note that

$$ik_1 \frac{\hbar \nu k_1}{2mv_F^2} = i \frac{\hbar \nu k_1^2}{2mv_F^2} = \frac{\hbar \nu k_1^2}{4E_F} = ic_0,$$

moreover

$$c_0 = \frac{\hbar \nu k_1^2}{2mv_F^2} = \frac{k_1^2}{2k_F l} = \frac{k_1^2}{2k_{01}}.$$

Here $k_{01}$ is the dimensionless Fermi wave number, $k_{01} = k_F l$.

Now the summand $\sigma_2$ is equal to

$$\sigma_2 = \sigma_0 \frac{3\nu}{4\omega \pi \hbar \nu} [J^+ - J^-] =$$

$$= \sigma_0 \frac{3\nu}{16\omega \hbar \nu} \int_{-1}^{1} \left[ \frac{(1 - t^2)^2}{1 - i\omega \tau + ik_1 t + ic_0} - \frac{(1 - t^2)^2}{1 - i\omega \tau + ik_1 t - ic_0} \right] dt =$$
\[ = -i\sigma_0 \frac{3k_1^2}{16\omega\tau} \int_{-1}^{1} \frac{(1-t^2)^2 \, dt}{(1-i\omega\tau + ik_1 t)^2 + c_0^2}. \]

We consider the denominator
\[ (1-i\omega\tau + ik_1 t)^2 + c_0^2 = -\left[ k_1^2 \left( t - \frac{\omega\tau + i}{k_1} \right)^2 - c_0^2 \right] = \]
\[ = -k_1^2 \left[ \left( t - \frac{\omega\tau + i}{k_1} \right)^2 - \left( \frac{k_1\hbar}{2p_Fl} \right)^2 \right] = -k_1^2 \left[ (t-a)^2 - c^2 \right], \]
where
\[ a = \frac{\omega\tau + i}{k_1} = \frac{z}{q}, \quad c = \frac{k_1\hbar}{2p_Fl} = \frac{k_1}{2k_Fl} = \frac{k_1}{2k_0} = \frac{q}{2}. \]

Thus, the expression (4.14) can be written in the form
\[ \sigma_2 = i\sigma_0 \frac{3}{16\omega\tau} J, \quad (4.15) \]
where
\[ J = \int_{-1}^{1} \frac{(1-t^2)^2}{(t-a)^2 - c^2}. \]

After variable replacement \( t - a = x, \quad x_0 = -1 - a, \quad x_1 = -1 + a, \) we receive, that
\[ J = \int_{x_0}^{x_1} \frac{[1-(x-a)^2]^2}{x^2 - c^2} \, dx = \]
\[ = \frac{(a^2 + c^2 - 1)^2 + 4a^2c^2}{2c} \ln \frac{a^2 - (c-1)^2}{a^2 - (c+1)^2} + 6a^2 + 2c^2 - \frac{10}{3} + \]
\[ + 2a(a^2 + c^2 - 1) \ln \frac{(a-1)^2 - c^2}{(a+1)^2 - c^2}. \]

The value of this integral in parameters \( z \) and \( q \) is expressed by the equality
\[ J = \frac{(z^2 - q^2 + q^4/4)^2 + z^2q^4}{q^5} \ln \frac{z^2 - (q-q^2/2)^2}{z^2 - (q+q^2/2)^2} + 6z^2 + \frac{q^2}{2} - \frac{10}{3} + \]
\[ + 2zq(2z^2 - q^2 + q^4/4) \ln \frac{z-q^2-q^4/4}{(z+q)^2 - q^4/4}. \]
The expression (4.15) for $\sigma_2$ with the help of the previous expression for $J$ we present in the form

$$\sigma_2 = i\sigma_0 \frac{3y}{8x} \left[ -\frac{5}{3} + \frac{3z^2}{q^2} + \frac{q^2}{4} + \frac{1}{2q^5} \left( \frac{z^2 - q^2 + \frac{q^4}{4}}{2} + z^2q^4 \right) \times \right.$$

$$\times \ln \frac{z^2 - (q - q^2/2)^2}{z^2 - (q + q^2/2)} + \frac{z}{q^2} \left( \frac{z^2 - q^2 + \frac{q^4}{4}}{4} \right) \ln \frac{(z - q)^2 - q^4/4}{(z + q)^2 - q^4/4} \left. \right]. \quad (4.16)$$

We will present the formula (4.12) for $\sigma_1$ in terms of $z$ and $q$

$$\sigma_1 = i\sigma_0 \frac{y}{x} \left[ 1 - \frac{3z^2}{2q^2} - \frac{3z}{4q^3} (z^2 - q^2) \ln \frac{z - q}{z + q} \right], \quad (4.17)$$

With the help of the equalities (4.16) and (4.17) we get the quantum part of transverse permittivity

$$\sigma_{tr}^{\text{quant}} = i\sigma_0 \frac{3y}{8x} \left[ 1 - \frac{z^2}{q^2} + \frac{q^2}{4} - \frac{2z}{q^3} (z^2 - q^2) \ln \frac{z - q}{z + q} + \right.$$

$$+ \frac{1}{2q^5} \left( \frac{z^2 - q^2 + \frac{q^4}{4}}{2} + z^2q^4 \right) \ln \frac{z^2 - (q - q^2/2)^2}{z^2 - (q + q^2/2)^2} + \right.$$  

$$+ \frac{z}{q^3} \left( \frac{z^2 - q^2 + \frac{q^4}{4}}{4} \right) \ln \frac{(z - q)^2 - q^4/4}{(z + q)^2 - q^4/4} \left. \right]. \quad (4.18)$$

Let’s note, that the expressions (4.17) and (4.18) contain Kohn singularities in the form $x \ln x$, where $x = z - q$, or $x = z + q$.

Now it is necessary to sum the quantum term (4.18) and classical term (4.8), for this purpose we present the expression (4.8) in the similar to (4.18) form

$$\sigma_{tr}^{\text{classic}} = i\sigma_0 \frac{3y}{8x} \left[ \frac{4xz}{q^2} + \frac{2x}{q^3} (z^2 - q^2) \ln \frac{z - q}{z + q} \right]. \quad (4.19)$$

Adding (4.18) and (4.19), we receive the final expression for the transverse permittivity in quantum plasma

$$\sigma_{tr}(x, y, q) = i\sigma_0 \frac{3y}{8x} \left[ 1 + \frac{z(3x - iy)}{q^2} + \frac{q^2}{4} - \frac{2iy}{q^3} (z^2 - q^2) \ln \frac{z - q}{z + q} + \right.$$

$$+ \frac{z}{q^3} \left( \frac{z^2 - q^2 + \frac{q^4}{4}}{4} \right) \ln \frac{(z - q)^2 - q^4/4}{(z + q)^2 - q^4/4} \left. \right].$$
\[
+ \frac{1}{2q^5} \left[ \left( z^2 - q^2 + q^4/4 \right)^2 + z^2q^4 \right] \ln \frac{z^2 - (q - q^2/2)^2}{z^2 - (q + q^2/2)^2} + \\
+ \frac{z}{q^3} \left( z^2 - q^2 + \frac{q^4}{4} \right) \ln \frac{(z - q)^2 - q^4/4}{(z + q)^2 - q^4/4} \right].
\] \quad (4.20)

It is more convenient for graphic research of conductivity instead of formulas (4.20) to use the equivalent form

\[
\frac{\sigma_{tr}}{\sigma_0} = -iy \frac{3}{4} \int_{-1}^{1} \frac{(1 - t^2)dt}{qt - z} + i \frac{3yq}{4x} \int_{-1}^{1} \frac{t(1 - t^2)dt}{qt - z} + \\
+ i \frac{3yq^2}{16x} \int_{-1}^{1} \frac{(1 - t^2)^2dt}{(qt - z)^2 - q^4/4}.
\] \quad (4.21)

6. COMPARISON WITH LINDHARD’S FORMULAS

Let’s consider Lindhard formula (5.3.4) from \[16\] for transverse conductivity and let’s transform it to our designations. After limiting transition from this formula (5.3.4) we receive

\[
\hat{\sigma}(q, \omega) = \frac{i\sigma_0}{\omega\tau} - i \frac{e^2}{\Omega \omega m^2} \sum_k \left[ \frac{f^0(\mathcal{E}_k)}{\mathcal{E}_k - \mathcal{E}_{k-q} - \hbar(\omega - i\nu)} - \\
- \frac{f^0(\mathcal{E}_k)}{\mathcal{E}_{k+q} - \mathcal{E}_k - \hbar(\omega - i\nu)} \right] |\langle k + q | p | k \rangle|^2.
\] \quad (5.1)

Here \(f^0(\mathcal{E}_k)\) is the absolute Fermi — Dirac distribution, \(\mathcal{E}_k = \frac{\hbar^2 k^2}{2m}\); besides, the sum from (5.1) is to be understood as integral

\[
\frac{1}{\Omega} \sum_k = \int \frac{2dk}{(2\pi)^3}.
\]

Let’s present the formula (5.1) in the form

\[
\hat{\sigma}(q, \omega) = \frac{i\sigma_0}{\omega\tau} + \hat{\sigma}_2,
\] \quad (5.2)
where

\[ \hat{\sigma}_2 = -\frac{i}{\Omega \omega m^2} \sum_k \left[ \frac{f^0(\mathcal{E}_k)}{\mathcal{E}_k - \mathcal{E}_{k-q} - \hbar(\omega - i\nu)} - \frac{f^0(\mathcal{E}_k)}{\mathcal{E}_{k+q} - \mathcal{E}_k - \hbar(\omega - i\nu)} \right] |\langle k + q | p | k \rangle|^2. \tag{5.3} \]

Let’s present the formula (5.3) in the integrated form

\[ \hat{\sigma}_2 = -\frac{ie^2}{m\omega} \int \Theta(\mathcal{E}_F - \mathcal{E}) \frac{2d\mathbf{k}}{(2\pi)^3} \left[ k^2 - \left( \frac{kq}{q} \right)^2 \right] \times \]

\[ \times \left[ \frac{1}{q^2 + 2kq - \frac{2m}{\hbar}(\omega + i/\tau)} + \frac{1}{q^2 - 2kq + \frac{2m}{\hbar}(\omega + i/\tau)} \right]. \tag{5.4} \]

Let’s transform the formula (5.4). We will direct the wave vector \( \mathbf{q} \) along the \( x \)-component of momentum, i.e. we take \( \mathbf{q} = \{k, 0, 0\} \), and instead of the vector \( \mathbf{k} \) we enter the dimensionless vector \( \mathbf{P} \) with unit length by the following equality

\[ \mathbf{k} = \frac{\mathbf{P}}{\hbar} = \frac{p_F}{\hbar} \mathbf{P}, \quad p_F = mv_F. \]

Therefore

\[ k^2 - \left( \frac{kq}{q} \right)^2 = \frac{p_F^2}{\hbar^2} \left( P^2 - P_x^2 \right) = \frac{p_F^2}{\hbar^2} \left( P_y^2 + P_z^2 \right), \]

\[ \frac{2d\mathbf{k}}{(2\pi)^3} = \frac{2d^3p}{(2\pi\hbar)^3} = \frac{2p_F^3}{(2\pi\hbar)^3} d\Omega_F. \]

The absolute Fermi — Dirac distribution in our designations has the following form

\[ f^0(\mathcal{E}_k) = \Theta(\mathcal{E}_F - \mathcal{E}), \quad \mathcal{E} = \frac{p^2}{2m} = \frac{p_F^2}{2m} P^2 = \mathcal{E}_F P^2, \quad \mathcal{E}_F = \frac{p_F^2}{2m}. \]

Noticing, that the absolute Fermi — Dirac distribution is normalized in terms of numerical density, i.e.

\[ \int \Theta(\mathcal{E}_F - \mathcal{E}) d\Omega_F = N, \]
we transform the second square brackets from (5.4). We have

\[
\frac{1}{q^2 + 2kq - \frac{2m}{\hbar}(\omega + i/\tau)} + \frac{1}{q^2 - 2kq + \frac{2m}{\hbar}(\omega + i/\tau)} = \frac{i\hbar}{2m} \left[ \frac{1}{\nu - i\omega + ikv_F P_x + \frac{\hbar k^2}{2m}} - \frac{1}{\nu - i\omega + ikv_F P_x - \frac{\hbar k^2}{2m}} \right] = \frac{\hbar^2 k^2}{2m^2} \left( \nu - i\omega + ikv_F P_x \right)^2 + \left( \frac{\hbar k^2}{2m} \right)^2
\]

\[
= \frac{\hbar^2 k^2}{2p_F^2} \left( 1 - i\omega \tau + i k_1 P_x \right)^2 + \left( \frac{\hbar \nu k^2}{2mv_F^2} \right)^2.
\]

Now we receive the integrated summand of Lindhard in the form

\[
\hat{\sigma}_2 = -\frac{i\sigma_0 3 k_1^2}{8\pi \omega \tau} \int \Theta(\mathcal{E}_F - \mathcal{E})(P^2 - P_y^2) d^3 P \frac{\Theta(\mathcal{E}_F - \mathcal{E})(P^2 - P_z^2) d^3 P}{(1 - i\omega \tau + i k_1 P_x)^2 + \left( \frac{\hbar \nu k^2}{2mv_F^2} \right)^2},
\]

or, that is the same that

\[
\hat{\sigma}_2 = -\frac{i\sigma_0 3 k_1^2}{8\pi \omega \tau} \int_{S^3(0)} \frac{(P_y^2 + P_z^2) d^3 P}{(1 - i\omega \tau + i k_1 P_x)^2 + \left( \frac{\hbar \nu k^2}{2mv_F^2} \right)^2}, \quad (5.5)
\]

The formula (5.5) precisely coincides with the formula (4.14) for \(\sigma_2\). The \(\sigma_2\) is calculated according to (4.16)

\[
\sigma_2 = i\sigma_0 \frac{3y}{8x} \left[ -\frac{5}{3} + 3 \frac{z^2}{q^2} + \frac{q^2}{4} + \frac{1}{2q^5} \left( z^2 - q^2 + \frac{q^4}{4} \right)^2 + z^2 q^4 \right] \times \\
\times \ln \frac{z^2 - (q - q^2/2)^2}{z^2 - (q + q^2/2)} + \frac{z}{q^3} \left( z^2 - q^2 + \frac{q^4}{4} \right) \ln \frac{(z - q)^2 - q^4/4}{(z + q)^2 - q^4/4}, \quad (5.6)
\]

In the monograph [16] the following formula (the formula (5.3.6) from [16]) for calculation of conductivity is presented

\[
\hat{\sigma}(q, \omega) = \frac{iNe^2}{\omega m} \left( \frac{3}{8} \left[ \left( \frac{q}{2k_F} \right)^2 + 3 \left( \frac{\omega + i/\tau}{qv_F} \right)^2 + 1 \right] - \\
\right.
\]
\[-\frac{3k_F}{16q} \left[ 1 - \left( \frac{q}{2k_F} - \frac{\omega + i/\tau}{qv_F} \right)^2 \right]^2 \ln \left\{ \frac{q}{2k_F} - \frac{\omega + i/\tau}{qv_F} + 1 \right\} - \frac{3k_F}{16q} \left[ 1 - \left( \frac{q}{2k_F} + \frac{\omega + i/\tau}{qv_F} \right)^2 \right]^2 \ln \left\{ \frac{q}{2k_F} + \frac{\omega + i/\tau}{qv_F} + 1 \right\} \right]. \tag{5.7}\]

Now we can rewrite the formula (5.7) with the use of our notations

\[
\sigma_{tr}^L(x, y, q) = i\sigma_0 \frac{3y}{16x} \left\{ 2 \left[ 1 + 3 \left( \frac{x + iy}{q^2} \right)^2 + \frac{q^2}{4} \right] - \frac{1}{q^5} \left[ q^2 - \left( \frac{q^2}{2} - x - iy \right)^2 \right]^2 \ln \frac{q^2/2 - x - iy + q}{q^2/2 - x - iy - q} - \frac{1}{q^5} \left[ q^2 - \left( \frac{q^2}{2} + x + iy \right)^2 \right]^2 \ln \frac{q^2/2 + x + iy + q}{q^2/2 + x + iy - q} \right\}. \tag{5.8}\]

Subtracting from (5.8) gauge summand, for an integrated part of conductivity \(\hat{\sigma}_2\) we receive expression

\[
\hat{\sigma}_2^L(x, y, q) = i\sigma_0 \frac{3y}{16x} \left\{ 2 \left[ -\frac{5}{3} + 3 \left( \frac{x + iy}{q^2} \right)^2 + \frac{q^2}{4} \right] - \frac{1}{q^5} \left[ q^2 - \left( \frac{q^2}{2} - x - iy \right)^2 \right]^2 \ln \frac{q^2/2 - x - iy + q}{q^2/2 - x - iy - q} - \frac{1}{q^5} \left[ q^2 - \left( \frac{q^2}{2} + x + iy \right)^2 \right]^2 \ln \frac{q^2/2 + x + iy + q}{q^2/2 + x + iy - q} \right\}. \tag{5.9}\]

Let’s demonstrate on Figs. 1 and 2 graphic comparison of expressions (5.6) and (5.9).

From formulas (5.6) and (5.9), and Figs. 1 and 2 it’s clear, that these expressions differ not only analytically, but also numerically. Hence, the expression for conductivity from [16], is incorrect.
Let’s return to equality (5.2) and we will present it in the form

\[ \sigma_{tr}^{(1)}(x, y, q) = i\sigma_0 \frac{y}{x} + \sigma_2(x, y, q), \]  

(5.10)

where \( \sigma_2(x, y, q) \) is defined by the equality (5.6).

Let’s rewrite the formula (5.10) by means of (5.6) in the explicit form:

\[
\sigma_{tr}^{(1)}(x, y, q) = i\sigma_0 \frac{3y}{8x} \left[ \frac{z^2}{q^2} + \frac{q^2}{4} + \frac{1}{2q^5} \left( z^2 - q^2 + \frac{q^4}{4} \right)^2 + z^2 q^4 \right] \times \\
\times \ln \frac{z^2 - (q - q^2/2)^2}{z^2 - (q + q^2/2)^2} + \frac{z}{q^3} \left( z^2 - q^2 + \frac{q^4}{4} \right) \ln \left( \frac{z - q}{z + q} \right) \frac{z - q}{z + q}.
\]

(5.11)

Now the transverse Lindhard electric conductivity is defined by expression (5.9).

The difference of conductivities (4.20) and (5.9) is equal to:

\[ \sigma_{tr} - \sigma_{tr}^{(1)} = \sigma_0 \frac{3y}{8x} \left[ \frac{2z}{q^2} + \frac{1}{q^3} (z^2 - q^2) \ln \frac{z - q}{z + q} \right]. \]

This equality shows, that at increase \( q \) the difference \( \sigma_{tr} - \sigma_{tr}^{(1)} \) tends to zero.

Let’s show comparison of the obtained expressions of conductivity. For this purpose let’s take advantage of equalities (4.20), (5.11) and (4.8). On following three plots curves of 1, 2 and 3 answer accordingly electric conductivity, constructed according to (4.20), (5.11) and (4.8).

From Figs. 3 and 4, and also from Figs. 5 and 6 one can see, that at small values \( q \) curves of 1, answering (4.21), coincide with curves of 3, answering (4.8), and at large \( q \) curves of 1 coincide with curves of 2, answering to Lindhard’s expression (5.10).

From Figs. 7 and 8 it is clear, that at large values of dimensionless frequency \( x \) and at large values \( q \) the curves 1, 2, 3 coincide among themselves.

On Figs. 9 and 10 dependences of the real and imaginary parts of transverse conductivity on dimensionless frequency \( x \) at the various values
of parameter $q$ are presented. The curves 1, 2, 3 correspond to values $q = 0.1, 1, 2$ accordingly.

7. CONCLUSION

In the present work the correct formula for calculation of transverse electric conductivity in the quantum collisional plasma is deduced. For this purpose the Wigner — Vlasov — Boltzmann kinetic equation with collisional integral in the form of BGK–model (Bhatnagar, Gross and Krook) in coordinate space is used. The case of degenerate plasma is considered separately. Comparison with Lindhard’s formula has been realized.

![Graph](image)

Figure 1: The case: $q = 2, y = 0.01$. Dependence $|\sigma_y/\sigma_0|$ on the dimensionless frequency $x$. 
Figure 2: The case: $x = 1, y = 0.01$. Dependence $|\sigma_2/\sigma_0|$ on the dimensionless wave number $q$.

Figure 3: The case: $x = 0.001, y = 0.01$. Dependence $\text{Re}(\sigma_{tr}/\sigma_0)$ on the dimensionless wave number $q$. 
Figure 4: The case $x = 0.001, y = 0.01$. Dependence $\text{Im}(\sigma_{tr}/\sigma_0)$ on the dimensionless wave number $q$.

Figure 5: The case: $x = 0.1, y = 0.01$. Dependence $\text{Re}(\sigma_{tr}/\sigma_0)$ on the dimensionless wave number $q$. 
Figure 6: The case: $x = 0.1, y = 0.01$. Dependence $\text{Im}(\sigma_{tr}/\sigma_0)$ on the dimensionless wave number $q$.

Figure 7: The case: $y = 0.01, q = 0.5$. Dependence $\text{Re}(\sigma_{tr}/\sigma_0)$ on dimensionless frequency $x$. 
Figure 8: The case $y = 0.01, q = 0.5$. Dependence $\text{Im} \left( \frac{\sigma_{tr}}{\sigma_0} \right)$ on dimensionless frequency $x$.

Figure 9: The case: $y = 0.01$. Dependence $\text{Re} \left( \frac{\sigma_{tr}}{\sigma_0} \right)$ on dimensionless frequency $x$. 
Figure 10: The case: $y = 0.01$. Dependence $\text{Im} (\sigma_t/\sigma_0)$ on dimensionless frequency $x$. 

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