(BOUNDDED) CONTINUOUS COHOMOLOGY
AND GROMOV'S PROPORTIONALITY PRINCIPLE

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ABSTRACT. Let $X$ be a topological space, and let $C^*(X)$ be the complex of singular cochains on $X$ with coefficients in $\mathbb{R}$. We denote by $C^*_c(X) \subseteq C^*(X)$ the subcomplex given by continuous cochains, i.e. by such cochains whose restriction to the space of simplices (endowed with the compact–open topology) defines a continuous real function. We prove that at least for “reasonable” spaces the inclusion $C^*_c(X) \hookrightarrow C^*(X)$ induces an isomorphism in cohomology, thus answering a question posed by Mostow. We also prove that such isomorphism is isometric with respect to the $L^\infty$–norm on cochains defined by Gromov.

As an application, we discuss a cohomological proof of Gromov’s proportionality principle for the simplicial volume of Riemannian manifolds.

1. Preliminaries and statements

Let $X$ be a topological space. We denote by $C_*(X)$ (resp. by $C^*(X)$) the usual complex of singular chains (resp. cochains) on $X$ with coefficients in $\mathbb{R}$. For $i \in \mathbb{N}$, we let $S_i(X)$ be the set of singular $i$–simplices in $X$, and we endow $S_i(X)$ with the compact–open topology (see Appendix A for basic definitions and results about the compact–open topology). We also regard $S_i(X)$ as a subset of $C_i(X)$, so that for any cochain $\varphi \in C^i(X)$ it makes sense to consider its restriction $\varphi|_{S_i(X)}$. For every $\varphi \in C^i(X)$, we set

$$||\varphi|| = ||\varphi||_{\infty} = \sup \{|\varphi(s)|, s \in S_i(X)\} \in [0, \infty].$$

We denote by $C^*_b(X)$ the submodule of bounded cochains, i.e. we set $C^*_b(X) = \{\varphi \in C^*(X) | ||\varphi|| < \infty\}$. Since the differential takes bounded cochains into bounded cochains, $C^*_b(X)$ is a subcomplex of $C^*(X)$. We introduce the following submodules of $C^*(X)$, which in fact are easily seen to be subcomplexes of $C^*(X)$:

$$C^*_c(X) = \{\varphi \in C^*(X) | \varphi|_{S_i(X)} \text{ is continuous}\},$$

$$C^*_{b,c}(X) = C^*_c(X) \cap C^*_b(X),$$

We denote by $H^*(X)$, $H^*_c(X)$, $H^*_b(X)$, $H^*_{b,c}(X)$ respectively the homology of the complexes $C^*(X)$, $C^*_c(X)$, $C^*_b(X)$, $C^*_{b,c}(X)$. Of course, $H^*(X)$ is the usual singular

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cohomology module of $X$, while $H^*_b(X)$ is the usual bounded cohomology module of $X$. Also note that the norm on $C^i(X)$ descends (after the suitable restrictions) to a seminorm on each of the modules $H^*(X)$, $H^*_b(X)$, $H^*_b(X)$, $H^*_b,c(X)$. More precisely, if $\varphi \in H$ is a class in one of these modules, which is obtained as a quotient of the corresponding module of cocycles $Z$, then we set

$$||\varphi|| = \inf \{ ||\psi||, \psi \in Z, [\psi] = \varphi \text{ in } H \}.$$  

This seminorm may take infinite values on elements in $H^*(X)$, $H^*_b(X)$, and may be a priori null on non-zero elements in $H^*_b(X)$, $H^*_b(X)$, $H^*_b,c(X)$ (but not on non-zero elements in $H^*(X)$: it is easy to see that a cohomology class with norm equal to zero has to be null on any cycle, whence null in $H^*(X) \cong (H_*(X))^*$).

A map $\alpha : E \to F$ between seminormed spaces is norm-decreasing if $||\alpha(v)|| \leq ||v||$ for every $v \in E$ (this definition makes sense even when the seminorms considered take $+\infty$ as value). The natural inclusions

$$C^*_b(X) \hookrightarrow C^*(X), \quad C^*_b,c(X) \hookrightarrow C^*_b(X),$$

induce norm-decreasing maps in cohomology

$$i^* : H^*_b(X) \to H^*(X), \quad i^*_b : H^*_b,c(X) \to H^*_b(X).$$

Moreover, the natural inclusions

$$C^*_b(X) \hookrightarrow C^*(X), \quad C^*_b,c(X) \hookrightarrow C^*_b(X),$$

induce maps in cohomology

$$c^* : H^*_b(X) \to H^*(X), \quad c^*_b : H^*_b,c(X) \to H^*_b(X).$$

The map $c^*$ (resp. $c^*_b$) is called comparison map (resp. continuous comparison map). By the very definitions, for $\varphi \in H^*(X)$, $\psi \in H^*_b(X)$ we have

$$||\varphi|| = \inf \{ ||\varphi_b||, \varphi_b \in H^*_b(X), c^*(\varphi_b) = \varphi \},$$

$$||\psi|| = \inf \{ ||\psi_b||, \psi_b \in H^*_b,c(X), c^*_b(\psi_b) = \psi \},$$

where such infima are intended to be equal to $\infty$ when taken over the empty set. Moreover, we obviously have $i^* \circ c^*_b = c^* \circ i^*_b$.

1.1. Continuous vs. usual cohomology. It is well-known that bounded cohomology and standard cohomology are very different from each other, while Bott stated in [Bot75] that, at least for “reasonable spaces”, the map $i^*$ is an isomorphism. However, Mostow asserted in [Mos76, Remark 2 at p. 27] that the natural proof of this fact seems to raise some difficulties.

More precisely, it is quite natural to ask whether continuous cohomology satisfies Eilenberg-Steenrod axioms for cohomology. In fact, it is not difficult to show that continuous cohomology verifies the so-called “dimension axiom” (see Subsection 2.2) and “homotopy axiom” (see Subsection 2.3). However, if $Y$ is a subspace of $X$ it is in general not possible to extend cochains in $C^*_c(Y)$ to cochains in $C^*_c(X)$, so
that it is not clear if a natural long exact sequence for pairs actually exists in the realm of continuous cohomology. This difficulty can be overcome either by considering only pairs $(X,Y)$ where $X$ is metrizable and $Y$ is closed in $X$, or by exploiting a cone construction, as described in \[Mdz09\]. A still harder issue arises about excision: even if the barycentric subdivision operator consists of a finite sum (with signs) of continuous self-maps of $S_s(X)$, the number of times a simplex should be subdivided in order to become “small” with respect to a given open cover depends in a decisive way on the simplex itself. Thus the dual map of (suitable iterations) of the barycentric subdivision does not necessarily carry continuous cochains to continuous cochains. However, in Proposition 2.6 we show that such dual map takes \textit{locally zero} continuous cochains into \textit{locally zero} continuous cochains. Together with some sheaf–theoretic arguments, this turns out to be sufficient in order to prove the following:

**Theorem 1.1.** Suppose $X$ has the homotopy type of a metrizable and locally contractible topological space. Then the map $i^*$ is an isomorphism.

A similar result has recently been obtained in \[Mdz09\] under stronger assumptions on $X$ (but considering a larger class of groups of coefficients) in \[Mdz09\]. Mdzinarishvili has shown that continuous cohomology satisfies in fact the Eilenberg-Steenrod axioms for cohomology (at least when considering continuous cohomology as a functor defined on the category of \textit{metric} spaces). As expected, Mdzinarishvili’s argument for showing that continuous cohomology satisfies the axiom of excision is quite subtle (see also Remark 2.5 below).

In order to show that the isomorphism $i^*$ is isometric, at least for a large class of spaces, in Section \ref{section:iso} we describe how the continuous bounded cohomology and the ordinary bounded cohomology of $X$ are related to the bounded cohomology of the fundamental group of of $X$. Building on results and techniques developed by Ivanov \[Iva87\] and Monod \[Mon01\], we are then able to prove the following result:

**Theorem 1.2.** Suppose $X$ is path connected and has the homotopy type of a countable CW–complex. Then the map $i^*_b$ is surjective and norm–decreasing. Moreover, $i^*_b$ admits a right inverse which is an isometric embedding.

If $X$ satisfies the hypothesis of Theorem \ref{thm:iso}, then it is metrizable, so by Theorem \ref{thm:iso} the map $i^*: H^*_c(X) \rightarrow H^*(X)$ is an isomorphism. Moreover, for every $\varphi \in H^*(X)$ we have

$$||\varphi|| = \inf \{ ||\varphi_b|| \mid \varphi_b \in H^*_b(X), c^*(\varphi_b) = \varphi \} = \inf \{ ||\varphi_{b,c}|| \mid \varphi_{b,c} \in H^*_b(X), c^*(i^*_b(\varphi_{b,c})) = \varphi \} = \inf \{ ||\varphi_{b,c}|| \mid \varphi_{b,c} \in H^*_b(X), i^*(c^*_c(\varphi_{b,c})) = \varphi \} = \inf \{ ||\varphi_{b,c}|| \mid \varphi_{b,c} \in H^*_b(X), c^*_c(\varphi_{b,c}) = (i^*)^{-1}(\varphi) \} = ||(i^*)^{-1}(\varphi)||,$$
Theorem 1.3. Suppose $X$ is path connected and has the homotopy type of a countable CW–complex. Then the map $i^*$ is an isometric isomorphism.

1.2. The case of aspherical spaces. When $X$ has contractible universal covering, the relation between the (bounded) cohomology of $X$ and the continuous (bounded) cohomology of $X$ is more explicit than in the general case. More precisely, in Section 5 we prove the following:

Theorem 1.4. Suppose $X$ is path connected and paracompact with contractible universal covering. Then the maps $i^*$ and $i_b^*$ are isometric isomorphism. Moreover, the inverse maps $(i^*)^{-1} : H^*(X) \to H^c(X)$, $(i_b^*)^{-1} : H_b^*(X) \to H_{b,c}(X)$ can be described by explicit formulae.

The techniques developed for the proof of Theorem 1.4 can be adapted to provide, in dimension one, a more explicit description of the inverse map $(i^1)^{-1} : H^1(X) \to H^1_c(X)$, even when $\tilde{X}$ is not contractible. More precisely, in Subsection 5.3 we prove the following:

Theorem 1.5. Suppose that $X$ is paracompact, locally path connected and semilocally simply connected. Then the map $i^1 : H^1_c(X) \to H^1(X)$ is an isomorphism, whose inverse map can be described by an explicit formula. Moreover, we have $H^1_{b,c}(X) = 0$ ($= H^1_b(X)$).

In Section 6 we give examples of path connected (and in one case even simply connected!) “pathological” spaces whose first continuous cohomology module is not isomorphic (through $i^*$) to the standard first cohomology module. Such spaces are suitable variations of the Hawaiian earring space and of the comb space.

1.3. Simplicial volume. The $L^\infty$–seminorm on cohomology introduced above naturally arises as dual to the $L^1$–seminorm on homology we are going to describe. If $X$ is a topological space and $\alpha \in C_i(X)$, we set

$$||\alpha|| = ||\alpha||_1 = \sum_{\sigma \in S_i(X)} |a_\sigma|,$$

where $\alpha = \sum_{\sigma \in S_i(X)} a_\sigma \sigma$.

(Note that the sums in the formula above are indeed finite, due to the definition of singular chain). This norm descends to a seminorm on $H^*_i(X)$, which is defined as follows: if $[\alpha] \in H_i(X)$, then

$$||[\alpha]|| = \inf\{|||\beta||, \beta \in C_i(X), d\beta = 0, [\beta] = [\alpha]\}.$$
This seminorm can be null on non-zero elements of $H_*(X)$.

If $X$ is a $n$-dimensional connected closed orientable manifold, then we denote by $[X]_\mathbb{R}$ the image of a generator of $H_n(X;\mathbb{Z}) \cong \mathbb{Z}$ under the change of coefficients map $H_n(X;\mathbb{Z}) \to H_n(X;\mathbb{R}) = H_n(X)$. The simplicial volume of $X$ is $||X|| = ||[X]_\mathbb{R}||$. It is easily seen that if $Y$ is the total space of a $d$-sheeted covering of $X$, then $||Y|| = d \cdot ||X||$. Thus, if $X$ is non-orientable, it is reasonable to set $||X|| = ||X'||/2$, where $X'$ is the double covering of $X$ with orientable total space.

1.4. The proportionality principle. Suppose now $X,Y$ are closed connected Riemannian manifolds. If $X,Y$ admit a common finite Riemannian covering, then multiplicativity of Riemannian and simplicial volumes under finite coverings implies that $||X||/\text{Vol}(X) = ||Y||/\text{Vol}(Y)$. Gromov’s proportionality principle [Gro82] ensures that this equality still holds even when $X,Y$ only share the universal covering:

**Theorem 1.6** ([Gro82, Löh06, BK08]). Let $X,Y$ be closed Riemannian manifolds with isometric Riemannian universal covering. Then

$$\frac{||X||}{\text{Vol}(X)} = \frac{||Y||}{\text{Vol}(Y)}.$$ 

A detailed proof of the proportionality principle has been provided by Löh [Löh06] following the “measure homology” approach due to Thurston [Thu79], while the strategy described by Bucher-Karlsson in [BK08] makes explicit use of bounded cohomology, more in the spirit of the original argument by Gromov (however, it may be worth mentioning that, in [Löh06], the proof of the fact that measure homology is isometric to the standard singular homology, which is the key step towards the proportionality principle, still relies on results about bounded cohomology).

However, Gromov’s approach to the proportionality principle exploits an averaging process which can be defined only on sufficiently regular (e.g. bounded Borel measurable) cochains. Moreover, one needs a regularity result for those cochains that are defined by integrating differential forms. As a consequence, singular (continuous or Borel) cohomology has to be replaced by smooth singular (continuous or Borel) cohomology, i.e. by the homology of the complex of (continuous or Borel) cochains defined on the set of smooth simplices, endowed with the $C^1$-topology (rather than with the compact–open topology). These technical details seem to raise some difficulties in Bucher-Karlsson’s proof of the proportionality principle, which slightly relies on the expected isomorphism between ordinary and continuous cohomology (attributed in [BK08] to Bott [Bot75]), and does not deal with the fact that the integration of a volume form does not define a continuous cochain with respect to the compact–open topology (see Remark 8.2). Building on the results about smooth continuous cohomology described in Section 7 in Section 8 we closely follow the strategy described in [BK08] for filling in the details in Gromov’s original proof of the proportionality principle.
1.5. **Plan of the paper.** In Section 2 we deal with the basic properties of continuous cohomology. In particular, we prove functoriality and homotopy invariance of continuous cohomology, we compute the continuous cohomology of the point, and we discuss barycentric (co)subdivisions of continuous cochains. The results obtained are then used in Section 3 for showing, via sheaf–theoretic arguments, that continuous cohomology is canonically isomorphic to standard singular cohomology for a large class of spaces. In Section 4 we first introduce the needed algebraic notions for dealing with the seminorms on continuous cohomology and bounded continuous cohomology introduced above: in particular, we describe the machinery of injective and relative injective strong resolutions developed in [Iva87, Mon01]. This machinery is then used in Section 5 for proving Theorems 1.2, 1.4 and 1.5. Examples of spaces whose continuous cohomology is not isomorphic to the singular one are given in Section 6. As explained above, when dealing with smooth manifolds, it is useful to consider the space of smooth simplices, endowed with the $C^1$–topology. In Section 7 we describe how definitions and results for continuous cohomology can be adapted to this setting, proving results that are used in the final section, which is devoted to Gromov’s proportionality principle for the simplicial volume of Riemannian manifolds.

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2. **Basic properties of continuous cohomology**

We begin by showing that all the theories we have introduced indeed provide homotopy functors from the category of topological spaces to the category of graded real vector spaces.

2.1. **Functoriality.** By Lemma A.1 if $f: X \to Y$ is a continuous map, then $f_*|S_i(X): S_i(X) \to S_i(Y)$ is continuous, so its dual map sends continuous cochains to continuous cochains, thus defining a map $f^*_c: H^*_c(Y) \to H^*_c(X)$. Of course, if $f = \text{Id}_X$, then $f^*_c = \text{Id}_{H^*_c(X)}$, and if $g: Y \to Z$ is continuous, then $(g \circ f)^*_c = f^*_c \circ g^*_c$. The same results hold true also for continuous bounded cohomology.

**Remark 2.1.** Let us denote by $i^*_X: H^*_c(X) \to H^*(X)$, $i^*_Y: H^*_c(Y) \to H^*(Y)$ the maps induced by the inclusion of continuous cochains into the space of singular cochains. With the above notations, it is readily seen that $i^*_Y \circ f^*_c = f^* \circ i^*_X$. This shows that $i^*$ provides a *natural transformation* from the functor $H^*_c(\cdot)$ to the functor $H^*(\cdot)$. The analogous result also holds in the bounded case for the tranformation $i^*_b$. 
2.2. The dimension axiom. Suppose $X$ consists of only one point. Then the space $S_n(X)$ consists of only one point (the constant simplex), so any cochain is automatically continuous, and continuous cohomology coincides with the usual singular cohomology theory. We have thus proved the following:

**Proposition 2.2.** Suppose $X$ consists of only one point. Then

$$H^0_c(X) \cong H^0_{b,c}(X) \cong \mathbb{R},$$

$$H^i_c(X) = H^i_{b,c}(X) = 0 \quad \text{for } i \geq 1.$$  

□

Let now $X = \bigsqcup_{i \in I} X_i$ be the disjoint union of the topological spaces $X_i$, $i \in I$, and endow $X$ with the disjoint union topology (so $\Omega \subseteq X$ is open in $X$ if and only if $\Omega \cap X_i$ is open in $X_i$ for every $i \in I$). Since each $X_i$ is open in $X$ and the standard simplex is connected, we have $C^*_c(X) = \bigoplus_{i \in I} C^*_c(X_i)$, whence $C^*_c(X) = \prod_{i \in I} C^*_c(X_i)$. Moreover, we have the natural inclusion $l_c: C^*_c(X) \hookrightarrow \prod_{i \in I} C^*_c(X_i)$, which restrict to the inclusion $l_{b,c}: C^*_c(X) \hookrightarrow \prod_{i \in I} C^*_c(X_i)$.

Now, since each $X_i$ is open in $X$, each $S_n(X_i)$ is open in $S_n(X)$. This readily implies that $\varphi: S_n(X) \to \mathbb{R}$ is continuous if and only if $\varphi|_{S_n(X_i)}$ is continuous for every $i \in I$. Therefore $l_c$ is an isomorphism. Moreover, if $I$ is finite, then $\varphi: S_n(X) \to \mathbb{R}$ is bounded if and only if $\varphi|_{S_n(X_i)}$ is bounded, so $l_{b,c}$ is also an isomorphism. We can summarize the above discussion in the following:

**Proposition 2.3.** Suppose $X = \bigsqcup_{i \in I} X_i$ is as above. Then $H^*_c(X) \cong \prod_{i \in I} H^*_c(X_i)$. Moreover, if $I$ is finite then $H^*_c(X) \cong \prod_{i \in I} H^*_c(X_i)$.

□

2.3. Homotopy invariance. We now show that all the cohomology theories we have introduced indeed provide homotopy invariants.

**Proposition 2.4.** Let $f, g: X \to Y$ be continuous maps with induced morphisms

$$f^*_c, g^*_c: H^*_c(Y) \to H^*_c(X), \quad f^*_b,c, g^*_b,c: H^*_c(Y) \to H^*_b,c(X).$$

If $f, g$ are homotopic, then $f^*_c = g^*_c$, and $f^*_b,c = g^*_b,c$.

**Proof:** If $H: X \times [0, 1] \to Y$ is a homotopy between $f$ and $g$, then there exists an algebraic homotopy $T_s: C^*_c(X) \to C^*_c(Y)$ between $f_*$ and $g_*$ which is constructed as follows: for any $s \in S_t(X)$, the chain $T(s) \in C_{i+1}(Y)$ is obtained as the image through $H \circ (s \times 1)$ of a suitable fixed subdivision of the prism $\Delta_i \times [0, 1]$. Therefore $T_s$ takes any simplex $s \in S_t(X)$ to a fixed number of simplices in $S_{i+1}(Y)$, each of which continuously depends on $s$ (see Lemma A.1). Thus the dual homotopy $T^*$ takes continuous cochains into continuous cochains, and bounded cochains into bounded cochains, whence the conclusion. □
2.4. **Barycentric subdivisions.** The aim of this subsection is to show that the barycentric subdivision operator on singular chains can be suitably dualized in order to provide a well-defined operator on the space of locally zero continuous cochains. This fact will play a crucial rôle in the proof of Theorem 1.1.

Suppose \( \mathcal{U} = \{ U_i \}_{i \in I} \) is an open cover of \( X \). We say that a simplex \( s \in C_s(X) \) is \( \mathcal{U} \)-small if its image lies in \( U_i \) for some \( i \in I \), we denote by \( S_s(X)^{\mathcal{U}} \subseteq S_s(X) \) the space of \( \mathcal{U} \)-small simplices, and by \( C_s(X)^{\mathcal{U}} \) the subspace of \( C_s(X) \) generated by \( S_s(X)^{\mathcal{U}} \).

The usual construction of the barycentric subdivision operator (see e.g. [Hu68] page 56) provides operators

\[
\text{sd}_n : C_n(X) \to C_n(X), \quad D_n : C_n(X) \to C_{n+1}(X)
\]
such that the following conditions hold:

- for every \( s \in S_n(X) \), there exists \( k \in \mathbb{N} \) such that \( \text{sd}^k_n(s) \in C_n(X)^{\mathcal{U}} \);
- \( \text{sd}_n \circ d_{n+1} = d_n \circ \text{sd}_n \) and \( d_{n+1} \circ D_n + D_{n-1} \circ d_n = \text{sd}_n - \text{Id}_{C_n(X)} \) for every \( n \geq 0 \), where \( d_n : C_n(X) \to C_{n-1}(X) \) is the usual differential (and \( D_{-1} \) is intended to be the null operator);
- \( \text{sd}_n(C_s(Y)) \subseteq C_s(Y) \), \( D_n(C_s(Y)) \subseteq C_{s+1}(Y) \) for every subset \( Y \) of \( X \).

Moreover, for any given singular simplex \( s : \Delta_n \to X \), the value of \( \text{sd}_n(s) \) and of \( D_n(s) \) only involves sums (with signs) of compositions of suitable restrictions of \( s \) with affine parameterizations of convex subsets of \( \Delta_n \). Therefore, by Lemma [A.1], the restrictions of \( \text{sd}_n \) and \( D_n \) to \( S_n(X) \) can both be expressed as algebraic sums of a finite number of continuous functions. In particular,

- if \( \varphi \in C^n_c(X) \), then \( \varphi \circ \text{sd}_n \in C^n_c(X) \), \( \varphi \circ D_{n-1} \in C^{n-1}_c(X) \).

If \( s \in C_s(X) \) is any simplex, we set \( \xi^{\mathcal{U}}(s) = \min \{ n \in \mathbb{N} \mid \text{sd}^n(s) \in C_s(X)^{\mathcal{U}} \} \). For every \( s \in C_n(X) \), let us denote by \( s^0, \ldots, s^n \) the faces of \( s \) (i.e. the maps obtained by composing with \( s \) the affine inclusions of \( \Delta_{n-1} \) onto the faces of \( \Delta_n \)).

Following [Hu68], for every \( n \in \mathbb{N} \) we define the homomorphisms

\[
\tau_n^{\mathcal{U}} : C_n(X) \to C_n(X)^{\mathcal{U}}, \quad \Omega_n^{\mathcal{U}} : C_n(X) \to C_{n+1}(X)
\]
as the unique linear maps such that for every \( s \in S_n(X) \)

\[
\tau_n^{\mathcal{U}}(s) = \text{sd}^n(s) + \sum_{i=0}^n (-1)^i \left( \sum_{j=\xi^{\mathcal{U}}(s')} D_{n-1}(\text{sd}_{n-1}(s')) \right),
\]

\[
\Omega_n^{\mathcal{U}}(s) = \sum_{j=0}^{\xi^{\mathcal{U}}(s)-1} D_n(\text{sd}_n(s)).
\]

It is easily seen (and shown in [Hu68]) that \( \tau_n^{\mathcal{U}} \) is a chain map, and that, if \( j_n^{\mathcal{U}} : C_s(X)^{\mathcal{U}} \to C_s(X) \) is the natural inclusion, then for every \( n \geq 0 \) we have

\[
j_n^{\mathcal{U}} \circ \tau_n^{\mathcal{U}} - \text{Id}_{C_n(X)} = d_{n+1} \circ \Omega_n^{\mathcal{U}} + \Omega_{n-1}^{\mathcal{U}} \circ d_n,
\]
i.e. \( j_n^{\mathcal{U}} \circ \tau_n^{\mathcal{U}} \) is chain homotopic to the identity of \( C_s(X) \). Since \( j_s^{\mathcal{U}} \circ j_s^{\mathcal{U}} = \text{Id}_{C_s(X)^{\mathcal{U}}} \), this implies that \( \tau_s^{\mathcal{U}}, j_s^{\mathcal{U}} \) are chain homotopy equivalences.
Let now $C^n(X)^U$ be the dual space of $C^n(X)^U$, and endow $C^*(X)^U$ with the usual differential. We denote by $\tau^*_U: C^n(X)^U \to C^n(X)$, $\Omega^*_n: C^n(X) \to C^{n-1}(X)$, $\tilde{j}^*_U: C^n(X) \to C^n(X)^U$ the dual maps of $\tau^*_U$, $\Omega^*_n$, $\tilde{j}^*_n$ respectively.

**Remark 2.5.** We now would like to prove that the maps just introduced take continuous cochains into continuous cochains. However, this is in general not true, due to the fact that the map $\xi^*: S_n(X) \to \mathbb{N}$, taking values in a discrete set, has no hope to be continuous. Therefore, even if $s d^*$ and $D^*$ preserve continuity of cochains, the operators $\tau^*_U$ and $\Omega^*_n$ may not enjoy this property. Therefore, in order to prove an excision theorem for continuous cohomology, it seems that one cannot carry out the naive strategy of trying to dualize the barycentric subdivision operator in order to obtain a homotopy equivalence between the complex of continuous cochains and the complex given by the restriction of continuous cochains to $C_*(X)^U$.

Let us now set $\mu C^*_c(X) = C^*_c(X) \cap \ker \tilde{j}^*_U$, so that $\mu C^*_c(X)$ is the complex of continuous cochains which vanish on $U$–small chains. Even if the above remark shows that the operators just introduced cannot preserve continuity of cochains, for our purposes it will be sufficient to prove that $\Omega^*_n$ takes $\mu C^*_c(X)$ into $\mu C^*_c(X)$:

**Proposition 2.6.** For every $n \in \mathbb{N}$ we have

$$\Omega^*_n(\mu C^*_c(X)) \subseteq \mu C^{n-1}_c(X).$$

**Proof:** Take $\varphi \in \mu C^*_c(X)$. We begin by showing that $\Omega^*_n(\varphi)$ is continuous at every simplex $\varpi \in S_{n-1}(X)$. If $\overline{F} = \xi^*(\varpi)$, then by the very definitions the chain $sd\overline{F}(\varpi)$ is $U$–small. Moreover, as mentioned above, for every $s \in S_{n-1}(X)$ the chain $sd_{n-1}(s)$ is the sum (with signs) of a fixed number of singular simplices, each of which continuously depends on $s$. This easily implies that a neighbourhood $W$ of $\varpi$ in $S_n(X)$ exists such that for every $s \in W$ the chain $sd\overline{F}_{n-1}(s)$ is $U$–small, or equivalently $\xi^*(s) \leq \overline{F}$. More in general, for every $j \in \mathbb{N}$ the restrictions of $sd_{n-1}$ and $D_n$ to $S_{n-1}(X)$ can both be expressed as algebraic sums of a finite number of continuous functions, and this implies that since $\varphi \in C^*_c(X)$, then $\varphi \circ D_n \circ sd_{n-1} \in C^*_c(X)$.

Therefore, in order to prove that $\varphi$ is continuous on $W$, whence at $\varpi$, it is sufficient to show that for every $s \in W$ we have

$$\Omega^*_n(\varphi)(s) = \left( \sum_{j=0}^{k-1} \varphi \circ D_{n-1} \circ sd_{n-1}^j \right)(s),$$

or, equivalently, that

$$0 = \left( \sum_{j=0}^{k-1} \varphi(D_{n-1}(sd_{n-1}^j(s))) \right) - \Omega^*_n(\varphi)(s) = \varphi \left( \sum_{j=\xi^*(s)}^{k-1} D_{n-1}(sd_{n-1}^j(s)) \right)$$
However, for every \( s \in W \) and \( j \geq \xi \) the chain \( \sum_{j=\xi}^{n-1} D_{n-1}(s_d) \) is \( \mathcal{U} \)-small, and since \( \mathcal{D}^{n-1}(C_{n-1}(Y)) \subseteq C_n(Y) \) for every subset \( Y \) of \( X \), this implies that the chain \( \sum_{j=\xi}^{n-1} D_{n-1}(s_d) \) is \( \mathcal{U} \)-small. Since \( \phi \) is null on \( \mathcal{U} \)-small chains, this readily gives equality (2).

We have thus proved that \( \Omega^{n-1}_{\mathcal{U}}(\phi) \) is continuous. In order to conclude we are left to show that \( \Omega^{n-1}_{\mathcal{U}}(\phi) \) is locally zero. However, if \( c \in C^{n-1}(X) \) is \( \mathcal{U} \)-small, then \( \Omega^{n-1}_{\mathcal{U}}(\phi)(c) = \phi(\delta(\Omega^{n-1}_{\mathcal{U}}(\phi))) = 0 \), whence the conclusion.

We now denote by \( \mathcal{O} C^*_e(X) \) the complex of continuous locally zero cochains, where we say that a continuous cochain is locally zero if it belongs to \( \mathcal{U} C^*_e(X) \) for some open covering \( \mathcal{U} \) of \( X \). As a consequence of Proposition 2.6 we get the following result, which will play a crucial rôle in the proof of Theorem 1.1.

**Proposition 2.7.** We have \( H^* (\mathcal{O} C^*_e(X)) = 0 \).

**Proof:** Take \( [\phi] \in H^n (\mathcal{O} C^*_e(X)) \), and let \( \mathcal{U} \) be an open cover of \( X \) such that \( \phi \in \mathcal{U} C^*_e(X) \). Since \( j^n_{\mathcal{U}}(\phi) = 0 \), we have

\[
(3) \quad -\phi = \tau^n_{\mathcal{U}}(j^n_{\mathcal{U}}(\phi)) - \phi = \Omega^{n+1}(\delta(\phi)) + \delta(\Omega^n_{\mathcal{U}}(\phi)) = \delta(\Omega^n_{\mathcal{U}}(\phi)),
\]

where we also used that \( \delta(\phi) = 0 \). By Proposition 2.6 therefore, \( \phi \) is the coboundary of a continuous cochain which belongs to \( \mathcal{U} C^*_e(X) \), whence to \( \mathcal{O} C^*_e(X) \), and this readily implies the conclusion. \( \square \)

### 3. Unbounded continuous cohomology of topological spaces

This section is entirely devoted to the proof of Theorem 1.1. After defining sheafified versions of ordinary cohomology and continuous cohomology, we will prove in Theorem 3.2 that these cohomology theories are isomorphic to each other (this fact was already mentioned in \[Mos76\]). Since ordinary cohomology is isomorphic to its sheafified version (see Theorem 3.1), in order to conclude we will exploit Proposition 2.7 for proving Theorem 3.3 which asserts that also continuous cohomology is isomorphic to its sheafified version. This last result provides the missing step in Mostow’s approach to the proof of Theorem 1.1.

#### 3.1. Sheaves and presheaves: preliminaries and notations.

We now introduce some notations and recall some basic results about sheaves and presheaves. For further reference see e.g. \[Bre97\]. Let \( X \) be a topological space. If \( F \) is a presheaf of real vector spaces on \( X \), and \( U \subseteq X \) is open, we denote by \( F(U) \) the sections of \( F \) over \( U \). As usual, if \( V \subseteq U \) are open subsets of \( X \) and \( \phi \in F(U) \), we denote by \( \phi|_V \) the restriction of \( \phi \) to \( V \).

A presheaf \( F \) is a *sheaf* if the following conditions hold:
• if $U = \bigcup_{i \in I} U_i$, where each $U_i$ is open, and $\varphi \in F(U)$ is such that $\varphi|_{U_i} = 0$ for every $U_i$, then $\varphi = 0$.

• if $U = \bigcup_{i \in I} U_i$, where each $U_i$ is open, and $\varphi_i \in F(U_i)$ are such that $\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$ for every $i, j \in I$, then there exists $\varphi \in F(U)$ such that $\varphi|_{U_i} = \varphi_i$ for every $i \in I$.

Let now $F$ be a presheaf of vector spaces on $X$. We will now describe the classical sheafification procedure, which canonically associates to $F$ a sheaf $\mathcal{F}$. If $x \in X$, we denote by $F_x$ the stalk of $F$ at $x$; if $x \notin U$ and $\varphi \in F(U)$, we denote by $[\varphi]|_U$ the class of $\varphi$ in $F_x$. If $U \subseteq X$ is open, we say that a map $\psi: U \to \bigsqcup_{x \in U} F_x$ is a continuous section over $U$ if for every $x \in U$ there exist an open set $V_x$ with $x \in V_x \subseteq U$ and an element $\varphi_x \in F(V_x)$ such that $\psi(y) = [\varphi_x]|_y$ for every $y \in V_x$. We denote by $\mathcal{F}(U)$ the set of continuous sections over $U$. It is well-known that the association $U \mapsto \mathcal{F}(U)$ indeed defines a sheaf (restrictions are easily induced by those of $F$).

Moreover, there exists an obvious morphism of presheaves $\rho_F: F \to \mathcal{F}$, which is natural in the following sense: if $f: F \to G$ is a morphism of presheaves, then a unique morphism of sheaves $\hat{f}: \mathcal{F} \to \mathcal{G}$ exists such that $\rho_G \circ f = \hat{f} \circ \rho_F$. Moreover, if $F$ is already a sheaf, then $\rho_F$ is an isomorphism.

3.2. The continuous singular cohomology sheaf. Remark 2.4 and Proposition 2.4 ensure that, in order to prove Theorem 1.1, we may assume without loss of generality that $X$ is a metrizable locally contractible topological space. Therefore, this assumption will be taken for granted from now until the end of this section.

The presheaf of continuous singular $n$–cochains on $X$ associates to each open set $U \subseteq X$ the real vector space $\mathcal{C}^n_c(U)$, and to every inclusion of open sets the obvious restriction of cochains. We will denote such presheaf by $\mathcal{C}^n_c[X]$, and $\mathcal{C}^n_c[X]$ will be the sheaf associated to $\mathcal{C}^n_c[X]$. If $U \subseteq X$ is open, we will denote simply by $\mathcal{C}^n_c(U)$ (and not by $\mathcal{C}^n_c[X](U)$) the space of sections of $\mathcal{C}^n_c[X]$ over $U$. The morphism of presheaves $\delta: \mathcal{C}^n_c[X] \to C^{n+1}_c[X]$ induces a morphism between $\mathcal{C}^n_c[X]$ and $C^{n+1}_c[X]$, which we will still denote by $\delta$. We will denote by $\mathcal{H}^*_c(X)$ the homology of the complex

$$0 \to \mathcal{C}^0_c(X) \xrightarrow{\partial^0} \mathcal{C}^1_c(X) \xrightarrow{\partial^1} \cdots \xrightarrow{\partial^n} \mathcal{C}^{n+1}_c(X) \xrightarrow{\delta^{n+1}} \cdots$$

i.e. we will set $\mathcal{H}^*_c(X) = H^*(\mathcal{C}^*_c(X))$. We denote simply by $\rho^*_c$ the map $\rho_{C^*_c[X]}: C^*_c[X] \to \mathcal{C}^*_c[X]$ described above, and by $\overline{\rho}^*_c: H^*_c(X) \to \mathcal{H}^*_c(X)$ the map induced in homology by the restriction of $\rho^*_c$ to global sections.

The same procedure just described can be applied to standard (i.e. not necessarily continuous) cochains: to the presheaf $\mathcal{C}^n[X]$ there is associated the sheaf $\mathcal{C}^n[X]$, and the usual differential on cochains induces a differential on the complex $\mathcal{C}^*(X)$, whose homology will be denoted by $\mathcal{H}^*(X)$. As in the case of continuous cochains, we have a natural map $\overline{\rho}^*: H^*(X) \to \mathcal{H}^*(X)$. Moreover, the mentioned naturality of the sheafification process provides a chain map of complexes of sheaves $\mathcal{C}^*_c(X) \to \mathcal{C}^*_c(X)$, and the restriction of this map to global sections induces in homology a map
\( i^*_\text{sh} : \mathcal{H}_c^*(X) \to \mathcal{H}^*(X) \) which makes the following diagram commute:

\[
\begin{array}{ccc}
H_c^*(X) & \xrightarrow{\varphi} & \mathcal{H}_c^*(X) \\
\downarrow i^* & & \downarrow i^*_\text{sh} \\
H^*(X) & \xrightarrow{\varphi} & \mathcal{H}^*(X).
\end{array}
\]

Observe now that, since \( X \) is metrizable, the family of all closed subsets of \( X \) is paracompactifying in the sense of [Bre97, page 21]. As a consequence we get the following classical result (see e.g. [Bre97, page 26]):

**Theorem 3.1.** The map \( \varphi^* : H^*(X) \to \mathcal{H}^*(X) \) is an isomorphism.

\[ \square \]

In order to conclude the proof of Theorem 1.1, it is now sufficient to prove the following results:

**Theorem 3.2.** The map \( i^*_\text{sh} : \mathcal{H}_c^*(X) \to \mathcal{H}^*(X) \) is an isomorphism.

**Theorem 3.3.** The map \( \varphi_c^* : H_c^*(X) \to \mathcal{H}_c^*(X) \) is an isomorphism.

3.3. **Proof of Theorem 3.2** We refer to [Bre97, Section II.9] for the definition of soft and fine sheaf. Let \( R[X] \) be the sheaf of real continuous functions over \( X \). For every open subset \( U \subseteq X \), the space \( C_c^0(U) \) is naturally a module over the ring \( R(U) \), with the operation defined by \( (f \cdot \varphi)(s) = f(s(e_0)) \cdot \varphi(s) \) for every \( f \in R(U), \varphi \in C_c^0(U), s \in S_n(U) \), where \( e_0 \) is the first vertex of the standard \( n \)-simplex \( \Delta_n \). So \( C_c^n[X] \) is a presheaf of modules over the sheaf of rings with unit \( R[X] \), and this readily implies that \( C_c^n[X] \) is a sheaf of modules over \( R[X] \). Now, since \( X \) is paracompact, the sheaf \( R[X] \) is soft (see [Bre97, Example II.9.4]), so \( C_c^n[X] \) is fine [Bre97, Theorem II.9.16], whence acyclic. The very same argument also applies to \( \mathcal{C}_c^n[X] \), which therefore is also acyclic.

Let now \( \tilde{\mathbb{R}} \) be the constant sheaf on \( X \) with stalks isomorphic to \( \mathbb{R} \). We recall from Subsection 3.2 that the inclusion induces a chain map of complexes of sheaves:

\[
\begin{array}{cccccccc}
0 & \xrightarrow{\delta^{-1}} & \tilde{\mathbb{R}}^0[X] & \xrightarrow{\delta^0} & \mathcal{C}_b^0[X] & \xrightarrow{\delta^1} & \mathcal{C}_b^1[X] & \xrightarrow{\delta^2} & \cdots \\
0 & \xrightarrow{1d} & \tilde{\mathbb{R}}^0[X] & \xrightarrow{\delta^0} & \mathcal{C}_b^0[X] & \xrightarrow{\delta^1} & \mathcal{C}_b^1[X] & \xrightarrow{\delta^2} & \cdots \\
\end{array}
\]

where \( \delta^{-1} \) is induced by the usual augmentation map on presheaves which sends \( t \in \mathbb{R} \) to the \( 0 \)-cochain which takes the value \( t \) on every \( 0 \)-simplex.

Recall now that both standard singular cohomology and continuous cohomology are homotopy invariant: since \( X \) is locally contractible, this implies that the rows in the diagram above provide exact sequences of sheaves. Moreover, we have seen...
that both \( C^n_c[X] \) and \( C^n[X] \) are acyclic for every \( n \geq 0 \). A classical result of sheaf theory (see e.g. [Bre97, Theorem II.4.1]) now ensures that \( i^*_\text{sh} : \mathcal{H}^*_c(X) \to \mathcal{H}^*(X) \) is an isomorphism. \( \square \)

3.4. Proof of Theorem 3.3. We begin with the following:

**Lemma 3.4.** Let \( Z \) be a locally finite closed cover of \( X \), and suppose we are given an element \( \varphi_Z \in C^*_c(Z) \) for every \( Z \in \mathcal{Z} \), in such a way that

\[
\varphi_Z|_{Z \cap Z'} = \varphi_{Z'}|_{Z \cap Z'} \quad \text{for every } Z, Z' \in \mathcal{Z}.
\]

Then there exists \( \varphi \in C^*_c(X) \) such that \( \varphi|_Z = \varphi_Z \) for every \( Z \in \mathcal{Z} \).

**Proof:** We prove the following equivalent statement: let \( \{ f_Z : S_n(Z) \to \mathbb{R}, Z \in \mathcal{Z} \} \) be a family of continuous functions such that \( f_Z|_{S_n(Z \cap Z')} = f_{Z'}|_{S_n(Z \cap Z')} \) for every \( Z, Z' \in \mathcal{Z} \); then a continuous function \( f : S_n(X) \to \mathbb{R} \) exists such that \( f|_{S_n(Z)} = f_Z \) for every \( Z \in \mathcal{Z} \).

Now, since \( \mathcal{Z} \) is a locally finite closed cover of \( X \), the family \( \{ S_n(Z) \mid Z \in \mathcal{Z} \} \) provides a locally finite collection of closed subsets of \( S_n(X) \). As a consequence, the set \( W = \bigcup_{Z \in \mathcal{Z}} S_n(Z) \) is closed in \( S_n(X) \), and there exists a well-defined continuous map \( g : W \to \mathbb{R} \) such that \( g|_{S_n(Z)} = f_Z \) for every \( Z \in \mathcal{Z} \). Since \( X \) is metrizable and the standard \( n \)-simplex is compact, the space \( S_n(X) \) is metrizable, so Tietze’s extension theorem [Dug66, p. 149] ensures that \( g \) can be extended to a continuous function defined on the whole of \( S_n(X) \). \( \square \)

**Lemma 3.5.** The map \( \rho^*_c : C^*_c(X) \to C^*_c(X) \) is surjective.

**Proof:** Take \( \varphi \in C^n_c(X) \). By the very definitions, for every \( x \in X \) an open set \( U_x \ni x \) in \( X \) and a section \( \psi_x \in C^n_c(U_x) \) exist such that \( \varphi|_{U_x} = \rho^*_c(\psi_x) \). Since \( X \) is metrizable, it is paracompact, so there exist a locally finite open covering \( \{ V_i \}_{i \in I} \) of \( X \) and a function \( x : I \to X \) such that \( V_i \subseteq U_{x(i)} \). For every \( y \in X \) we set \( I(y) = \{ i \in I \mid y \in V_i \} \). Since \( \{ V_i \}_{i \in I} \) is locally finite also the family \( \{ V_i \}_{i \in I} \) is locally finite, so the set \( I(y) \) is finite. Moreover the set \( \bigcup_{i \in I(y)} V_i \) is closed, so an open neighbourhood \( W_y \) of \( y \) exists such that \( W_y \cap V_i = \emptyset \) for every \( i \notin I(y) \), and this readily implies that \( I(y') \subseteq I(y) \) for every \( y' \in W_y \). Since \( V_i \subseteq U_{x(i)} \) for every \( i \in I, y \) up to shrinking \( W_y \) we may also assume that \( W_y \subseteq U_{x(i)} \) for every \( i \in I(y) \) and \( \psi_{x(i)}|_{W_y} = \psi_{x(j)}|_{W_y} \) for every \( i, j \in I(y) \). For every \( y \in X \) we now set \( \psi'_y = \psi_{x(i)}|_{W_y} \in C^n_c(W_y) \) for some \( i \in I(y) \) (by construction, this definition does not depend on \( i \in I(y) \)). We now claim that the collection of sections \( \psi'_y \in C^n_c(W_y), y \in X \) satisfies the following properties:

1. \( \rho^*_c(\psi'_y) = \varphi|_{W_y} \) for every \( y \in X \);
2. \( \psi'_y|_{W_y \cap W_{y'}} = \psi'_{y'}|_{W_y \cap W_{y'}} \) for every \( y, y' \in X \) such that \( W_y \cap W_{y'} \neq \emptyset \).
Property (1) is obvious, while if \( \emptyset \neq W_y \cap W_{y'} \ni z \), then \( I(y) \cap I(y') \supseteq I(z) \neq \emptyset \): in particular, if \( \mathcal{I}_0 \in I(y) \cap I(y') \), then \( \psi_{y'}|_{W_y \cap W_{y'}} = \psi_{x(\mathcal{I}_0)}|_{W_y \cap W_{y'}} = \psi_{y'}|_{W_y \cap W_{y'}} \), whence property (2).

Let now \( \mathcal{Z} = \{ Z_j \}_{j \in J} \) be a locally finite open cover of \( X \) such that for every \( j \in J \) there exists \( y(j) \in X \) such that \( \overline{Z_j} \subseteq W_{y(j)} \). By property (2) above and Lemma 3.4, a global cochain \( \psi'' \in C_c^n(X) \) exists such that \( \psi''|_{Z_j} = \psi_{y(j)}|_{Z_j} \) for every \( j \in J \). Now, by property (1) above, for every \( j \in J \) we have \( \rho_c^n(\psi'')|_{Z_j} = \rho_c^n(\psi''|_{Z_j}) = \varphi|_{Z_j} \). Since \( C_c^n[X] \) is a sheaf, this readily implies that \( \rho_c^n(\psi'') = \varphi \), whence the conclusion. \( \square \)

We can now conclude the proof of Theorem 3.3. We defined in Subsection 2.4 the subcomplex \( C_c^e(X) \) of locally zero continuous cochains. By its very definition, \( C_c^e(X) \) is equal to the kernel of the map \( \rho_c^e : C_c^e(X) \to C_c^e(X) \). By Lemma 3.5 the short sequence of complexes

\[
0 \longrightarrow C_c^e(X) \longrightarrow C_c^e(X) \longrightarrow C_c^e(X) \longrightarrow 0
\]

is therefore exact. This gives a long exact sequence

\[
\cdots \longrightarrow H^n(0C_c^e(X)) \longrightarrow H^n_b(X) \overset{\underline{\rho}^e}{\longrightarrow} H^n_c(X) \longrightarrow H^{n+1}(0C_c^e(X)) \longrightarrow \cdots
\]

By Proposition 2.7 we now have \( H^n(0C_c^e(X)) = H^{n+1}(0C_c^e(X)) = 0 \), so \( \underline{\rho}^e \) is an isomorphism. \( \square \)

**Remark 3.6.** The hypothesis that \( X \) is metrizable came into play only in the proof of Lemma 3.4 where, in order to extend continuous cocycles defined on closed subspaces of \( S_n(X) \), we exploited the fact that the space \( S_n(X) \) is normal. One could wonder if normality of \( S_n(X) \) could be proved under somewhat weaker hypotheses on \( X \). However, it is exhibited in [Sto63] an example of a paracompact locally contractible space \( X \) such that \( S_1(X) \) is not normal. This seems to suggest that metrizability is the most reasonable assumption on \( X \) which ensures that \( S_n(X) \) is normal for every \( n \in \mathbb{N} \).

**4. Bounded continuous cohomology of topological spaces**

This section is mainly devoted to the proofs of Theorems 122 and 124. We begin by reviewing some definitions introduced in [Iva87, Mon01].

Let \( G \) be a group (which should be thought as endowed with the discrete topology). In what follows, a \( G \)-module (resp. a Banach \( G \)-module) is a real vector space (resp. a real Banach space) endowed with an action of \( G \) (by isometries, in the Banach case) on the left. Sometimes, we will stress the fact that a \( G \)-module \( E \) is not endowed with a Banach structure by saying that \( E \) is an unbounded \( G \)-module.

If \( E \) is a (Banach) \( G \)-module, we denote by \( E^G \subseteq E \) the submodule of \( G \)-invariant elements in \( E \), i.e. we set

\[
E^G = \{ v \in E \mid g \cdot v = v \text{ for every } g \in G \}.
\]
A $G$–map between (Banach) $G$–modules is a (bounded) $G$–equivariant linear map.

4.1. **Relative injectivity.** A $G$–map $\nu : A \to B$ between Banach $G$–modules is strongly injective if there exists a linear map $\sigma : B \to A$ with $\|\sigma\| \leq 1$ such that $\sigma \circ \nu = \text{Id}_A$ (we do not require $\sigma$ to be a $G$–map; note also that strongly injective obviously implies injective). We now define the important notion of relative injectivity for Banach (resp. for unbounded) $G$–modules.

**Definition 4.1.** A Banach $G$–module $U$ is relatively injective if the following holds: whenever $A, B$ are Banach $G$–modules, $\nu : A \to B$ is a strongly injective $G$–map and $\alpha : A \to U$ is a $G$–map, there exists a $G$–map $\beta : B \to U$ such that $\beta \circ \nu = \alpha$ and $\|\beta\| \leq ||\alpha||$.

\[
\begin{array}{ccc}
0 & \longrightarrow & A \\
& \sigma \downarrow & \searrow \nu \\
& \alpha & \searrow \beta \\
& U & \searrow \end{array}
\]

**Definition 4.2.** An unbounded $G$–module $U$ is injective if the following holds: whenever $A, B$ are unbounded $G$–modules, $\nu : A \to B$ is an injective $G$–map and $\alpha : A \to U$ is a $G$–map, there exists a $G$–map $\beta : B \to U$ such that $\beta \circ \nu = \alpha$.

Note that any injective map between unbounded $G$–modules admits a (maybe not $G$–equivariant) left inverse, so the notion of relative injectivity can be considered an extension to the Banach setting of the notion of injectivity for unbounded modules.

4.2. **Resolutions.** A (Banach) $G$–complex (or simply a (Banach) complex) is a sequence of (Banach) $G$–modules $E^i$ and $G$–maps $\delta^i : E^i \to E^{i+1}$ such that $\delta^{i+1} \circ \delta^i = 0$ for every $i$, where $i$ runs over $\mathbb{N} \cup \{-1\}$:

\[
0 \longrightarrow E^{-1} \overset{\delta^{-1}}{\longrightarrow} E^0 \overset{\delta^0}{\longrightarrow} E^1 \overset{\delta^1}{\longrightarrow} \ldots \overset{\delta^n}{\longrightarrow} E^{n+1} \overset{\delta^{n+1}}{\longrightarrow} \ldots
\]

Such a sequence will be often denoted by $(E^*, \delta^*)$.

A chain map between (Banach) $G$–complexes $(E^*, \delta^*_E)$ and $(F^*, \delta^*_F)$ is a sequence of $G$–maps $\{\alpha^i : E^i \to F^i, i \geq -1\}$ such that $\delta^i_E \circ \alpha^i = \alpha^{i+1} \circ \delta^i_F$ for every $i \geq -1$. If $\alpha^*, \beta^*$ are chain maps between $(E^*, \delta^*_E)$ and $(F^*, \delta^*_F)$ which coincide in degree $-1$, a $G$–homotopy between $\alpha^*$ and $\beta^*$ is a sequence of $G$–maps $\{T^i : E^i \to F^{i-1}, i \geq 0\}$ such that $\delta^i_E \circ T^i + T^{i+1} \circ \delta^i_F = \alpha^i - \beta^i$ for every $i \geq 0$, and $T_0 \circ \delta^*_E = 0$. We recall that, according to our definition of $G$–maps for Banach modules, both chain maps between Banach $G$–complexes and $G$–homotopies between such chain maps have to be bounded (more precisely, such maps have to be bounded in every degree, while there does not need to be a uniform bound on their norms as maps from $\bigoplus_{i \geq -1} E^i$ to $\bigoplus_{i \geq -1} F^i$).
A complex is exact if $\delta^{-1}$ is injective and $\ker \delta^{i+1} = \text{Im} \delta^i$ for every $i \geq -1$. Let $E$ be a (Banach) $G$–module. A resolution of $E$ as a (Banach) $G$–module is an exact (Banach) $G$–complex $(E^*, \delta^*)$ with $E^{-1} = E$.

A resolution $(E^*, \delta^*)$ is relatively injective (resp. injective) if $E^n$ is relatively injective (resp. injective) for every $n \geq 0$.

A contracting homotopy for a resolution $(E^*, \delta^*)$ is a sequence of linear maps $k^i : E^i \to E^{i-1}$ such that $\delta^{i-1} \circ k^i + k^{i+1} \circ \delta^i = \text{Id}_{E^i}$ if $i \geq 0$, and $k_0 \circ \delta^{-1} = \text{Id}_{E}$. If $(E^*, \delta^*)$ is a resolution of Banach modules, the condition $||k^i|| \leq 1$ is also required.

\[
\begin{array}{cccccccccccc}
0 & \longrightarrow & E^{-1} & \xrightarrow{k_0} & E^0 & \xrightarrow{k_1} & E^1 & \xrightarrow{k_2} & \cdots & \xrightarrow{k_n} & E^n & \xrightarrow{k_{n+1}} & \cdots \\
& & \delta^{-1} & & \delta^0 & & \delta^1 & & \cdots & & \delta^n & & \cdots
\end{array}
\]

Note however that it is not required that $k_i$ is $G$–equivariant. A resolution of a (Banach) $G$–module is strong if it admits a contracting homotopy.

The following results can be proved by means of standard homological algebra arguments (see [Iva87], [Mon01, Lemmas 7.2.4 and 7.2.6] for the details in the case of Banach resolutions).

**Proposition 4.3.** Let $\alpha : E \to F$ be a $G$–map between Banach (resp. unbounded) $G$–modules, let $(E^*, \delta^*_F)$ be a strong resolution of $E$, and suppose $(F^*, \delta^*_E)$ is a $G$–complex with $F^{-1} = F$ and $F^i$ relatively injective (resp. injective) for every $i \geq 0$. Then $\alpha$ extends to a chain map $\alpha^*$, and any two extensions of $\alpha$ to chain maps are $G$–homotopic.

### 4.3. (Bounded) group cohomology.
We recall that if $E$ is a (Banach) $G$–module, we denote by $E^G \subseteq E$ the submodule of $G$–invariant elements in $E$.

Let $(E^*, \delta^*)$ be a relatively injective strong resolution of the trivial Banach $G$–module $\mathbb{R}$ (such a resolution exists, see Subsection 4.4). Since coboundary maps are $G$–maps, they restrict to the $G$–invariant submodules of the $E^i$’s. Thus $((E^*)^G, \delta^*)$ is a subcomplex of $(E^*, \delta^*)$. A standard application of Proposition 4.3 now shows that the isomorphism type of the homology of $((E^*)^G, \delta^*)$ does not depend on the chosen resolution (while the seminorm induced on such homology module by the norms on the $E^i$’s could depend on it, see Proposition 4.5 below). For every $i \geq 0$, we now define the $i$–dimensional bounded cohomology module $H^i_b(G)$ of $G$ (with real coefficients) as follows: if $i \geq 1$, then $H^i_b(G)$ is the $i$–th homology module of the complex $((E^*)^G, \delta^*)$, while if $i = 0$ then $H^i_b(G) = \ker \delta^0 \cong \mathbb{R}$.

The same construction applies verbatim when considering an injective strong resolution $(E^*, \delta^*)$ of $\mathbb{R}$ as an unbounded $G$–module. In this case, the homology of $((E^*)^G, \delta^*)$ is the standard cohomology of $G$, and will be denoted by $H^*(G)$.

### 4.4. The standard $G$–resolutions.
For every $n \in \mathbb{N}$, let $F^n(G) = \{f : G^{n+1} \to \mathbb{R} \}$ and $F^n_b(G) = \{f \in F^n(G) \mid f \text{ is bounded} \}$, and endow $F^n_b(G)$ with the supremum norm, thus obtaining a real Banach space. Let $G$ act on $F^n(G)$ in such a way that
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$(g : f)(g_0,\ldots, g_n) = f(g^{-1}g_0,\ldots, g^{-1}g_n)$. It is easily seen that this action leaves $F^n_b(G)$ invariant, and endows $F^n(G)$ (resp. $F^n_b(G)$) with a structure of $G$–module (resp. of Banach $G$–module). For $n \geq 0$, define $\delta^n : F^n(G) \to F^{n+1}(G)$ by setting:

$$
\delta^n(f)(g_0, g_1, \ldots, g_{n+1}) = \sum_{i=0}^{n+1} (-1)^i f(g_0, \ldots, \widehat{g_i}, \ldots, g_{n+1}).
$$

It is easily seen that $\delta(F^n_b(G)) \subseteq F^{n+1}_b(G)$, so it makes sense to define $\delta^n_b : F^n_b(G) \to F^{n+1}_b(G)$ by restricting $\delta^n$. Moreover, we let $F^{-1}(G) = \mathbb{R}$ be a trivial unbounded $G$–module and $F_0^{-1}(G) = \mathbb{R}$ be a trivial Banach $G$–module, and we define $\delta^{-1}_b : \mathbb{R} \to F^0_b(G)$ by setting $\delta^{-1}_b(t)(g) = t$ for every $g \in G$, and $\delta^{-1}$ by composing $\delta^{-1}_b$ with the inclusion of $F^0_b(G)$ in $F^0(G)$.

**Remark 4.4.** With slightly different conventions and notations, Ivanov proved in [Iva87] that the complex $(F^*_b(G), \delta^*_b)$ provides a relatively injective strong resolution of $\mathbb{R}$ as a bounded $G$–module. His argument can be easily adapted for showing that the complex $(F^*(G), \delta^*)$ is an injective strong resolution of $\mathbb{R}$ as an unbounded $G$–module. However, these results will not be necessary for our purposes.

The resolution $(F^*(G), \delta^*)$ (resp. $(F^*_b(G), \delta^*_b)$) is usually known as the standard $G$–resolution of $\mathbb{R}$ as an unbounded $G$–module (resp. as a Banach $G$–module). The seminorm induced on $H^*_b(G)$ by the standard bounded resolution is known as the canonical seminorm. The following result [Iva87, Mon01] gives a useful characterization of the canonical seminorm, and plays a decisive rôle in our proof of Theorem 1.2.

**Proposition 4.5.** Let $(E^*, \delta^*)$ be any strong resolution of $\mathbb{R}$ as a Banach $G$–module. Then the identity of $\mathbb{R}$ can be extended to a chain map $\alpha^*_b$ between $E^*$ and the standard resolution of $\mathbb{R}$ as Banach $G$–module, in such a way that $|\alpha^*_b| \leq 1$ for every $n \geq 0$. In particular, the canonical seminorm is not bigger than the seminorm induced on $H^*_b(G)$ by any relatively injective strong resolutions.

**Proof:** One can define $\alpha^*_b$ by induction setting, for every $v \in E^n$ and $g_j \in G$:

$$
\alpha^*_b(v)(g_0,\ldots, g_n) = \alpha^{n-1}_b(g_0(k^n(g_0^{-1}(v))))(g_1, \ldots, g_n),
$$

where $\{k^n\}_{n \in \mathbb{N}}$ is a contracting homotopy for $E^*$. It is not difficult to prove by induction that $\alpha^*$ is indeed a norm–decreasing chain $G$–map (see [Iva87, Mon01] Theorem 7.3.1) for the details).

**Remark 4.6.** It is readily seen that, if $(E^*, \delta^*)$ is any strong resolution of $\mathbb{R}$ as an unbounded $G$–module, then the formula described in the proof of Proposition 4.5 also provides an extension of the identity of $\mathbb{R}$ to a chain map $\alpha^*$ between $E^*$ and the standard resolution of $\mathbb{R}$ as unbounded $G$–module. Moreover, $\alpha^n$ is norm–decreasing for every $n \geq 0$.  

4.5. Some notations and a useful lemma. From now until the end of the section, we assume that \( X \) is a path connected paracompact topological space with universal covering \( p: \tilde{X} \to X \), and we fix an identification between the fundamental group of \( X \) and the group \( \Gamma \) of covering automorphisms of \( \tilde{X} \). Thus every \( g \in \Gamma \) defines a chain map \( g_*: C_*(\tilde{X}) \to C_*(\tilde{X}) \). It is a standard fact of algebraic topology that the action of \( \Gamma \) on \( \tilde{X} \) is wandering, i.e. any \( x \in \tilde{X} \) admits a neighbourhood \( U_x \) such that \( g(U_x) \cap U_x = \emptyset \) for every \( g \in \Gamma \setminus \{1\} \) (if \( X \), whence \( \tilde{X} \), is locally compact, then an action on \( \tilde{X} \) is wandering if and only if it is free and proper). In the following lemma we describe a particular instance of generalized Bruhat function (see [Mon01] Lemma 4.5.4) for a more general result based on [Bou63, Proposition 8 in VII §2 N° 4]).

**Lemma 4.7.** There exists a continuous map \( h_{\tilde{X}}: \tilde{X} \to [0,1] \) with the following properties:

1. For every \( x \in \tilde{X} \) there exists a neighbourhood \( W_x \) of \( x \) in \( \tilde{X} \) such that the set \( \{ g \in \Gamma \mid g(W_x) \cap \text{supp} h_{\tilde{X}} \neq \emptyset \} \) is finite.
2. For every \( x \in \tilde{X} \), we have \( \sum_{g \in \Gamma} h_{\tilde{X}}(g \cdot x) = 1 \) (note that the sum on the left-hand side is finite by (1)).

**Proof:** Let us take a locally finite open cover \( \{U_i\}_{i \in I} \) of \( X \) such that for every \( i \in I \) there exists \( V_i \subseteq \tilde{X} \) with \( p^{-1}(U_i) = \bigcup_{g \in \Gamma} g(V_i) \) and \( g(V_i) \cap g'(V_i) = \emptyset \) whenever \( g \neq g' \). Let \( \{\varphi_i\}_{i \in I} \) be a partition of unity adapted to \( \{U_i\}_{i \in I} \). It is easily seen that the map \( \psi_i: \tilde{X} \to \mathbb{R} \) which coincides with \( \varphi_i \circ p \) on \( V_i \) and is null elsewhere is continuous. We can now set \( h_{\tilde{X}} = \sum_{i \in I} \psi_i \). Since \( \{U_i\}_{i \in I} \) is locally finite, also \( \{V_i\}_{i \in I} \), whence \( \{\text{supp} \psi_i\}_{i \in I} \), is locally finite, so \( h_{\tilde{X}} \) is indeed well-defined and continuous.

In order to show that \( h_{\tilde{X}} \) satisfies (1), let \( x \in \tilde{X} \), and suppose \( p(x) \in U_{i_0} \). Then there exists \( g_0 \in \Gamma \) such that \( x \in g_0(V_{i_0}) \). We set \( W_x = g_0(V_{i_0}) \), and let \( J = \{j \in I \mid U_j \cap U_{i_0} \neq \emptyset \} \). By construction, \( J \) is finite. Now, if \( i \in I \setminus J \) then for every \( g \in \Gamma \) we have \( p(g(W_x) \cap V_i) \subseteq U_{i_0} \cap U_i = \emptyset \), so if \( g(W_x) \cap \text{supp} h_{\tilde{X}} \neq \emptyset \) then \( g(W_x) \cap V_j \neq \emptyset \) for some \( j \in J \). However, since \( g(W_x) \cap g'(W_x) = \emptyset \) for every \( g \neq g' \), for every \( j \in J \) there is at most one \( g \in \Gamma \) such that \( g(W_x) \cap V_j \neq \emptyset \), so \( \{ g \in \Gamma \mid g(W_x) \cap \text{supp} h_{\tilde{X}} \neq \emptyset \} \) is finite.

Finally, for every \( x \in \tilde{X} \), \( i \in I \) we have by construction \( \sum_{g \in \Gamma} \psi_i(g(x)) = \varphi_i(p(x)) \), so

\[
\sum_{g \in \Gamma} h_{\tilde{X}}(g(x)) = \sum_{g \in \Gamma} \left( \sum_{i \in I} \psi_i(g(x)) \right) = \sum_{i \in I} \left( \sum_{g \in \Gamma} \psi_i(g(x)) \right) = \sum_{i \in I} \varphi_i(p(x)) = 1,
\]

whence (2). \(\square\)
4.6. **Singular cochains as (relatively) injective modules.** For every $n \geq 0$, we define an action of $\Gamma$ on $C^n(\widetilde{X})$ by setting $g \cdot \varphi = (g^{-1})^*(\varphi)$ for any $g \in \Gamma$ and $\varphi \in C^n(\widetilde{X})$, where $g^* = g_*^\beta$ is the usual map induced by $g$ on cochains. This action leaves $C^*_c(\widetilde{X})$, $C^*_b(\widetilde{X})$ and $C^*_{b,c}(\widetilde{X})$ invariant and commutes with the differential, thus endowing these modules (and $C^*(\widetilde{X})$, of course) with a $\Gamma$–complex structure.

It is proved in [1va87] that for every $n \geq 0$ the $\Gamma$–module $C^n(\widetilde{X})$ is relatively injective. We show here that the same is true for $C^n_{b,c}(\widetilde{X})$, and that the modules $C^n_c(\widetilde{X})$, $C^n(\widetilde{X})$ are injective.

**Proposition 4.8.** Let $n \geq 0$. The $\Gamma$–modules $C^n(\widetilde{X})$ and $C^n_c(\widetilde{X})$ are injective. The Banach $\Gamma$–modules $C^n_b(\widetilde{X})$ and $C^n_{b,c}(\widetilde{X})$ are relatively injective.

**Proof:** Let $\iota : A \rightarrow B$ be an injective map between unbounded $\Gamma$–modules, with left inverse $\sigma : B \rightarrow A$, and suppose we are given a $\Gamma$–map $\alpha : A \rightarrow C^n(\widetilde{X})$. We denote by $e_0, \ldots, e_n$ the vertices of the standard $n$–simplex, and define $\beta : B \rightarrow C^n(\widetilde{X})$ as follows: given $b \in B$, the cochain $\beta(b)$ is the unique linear extension of the map that on the singular simplex $s$ takes the following value:

$$\beta(b)(s) = \sum_{g \in \Gamma} h_X(g^{-1}(s(e_0))) \cdot (\alpha(g(\sigma(g^{-1}(b))))(s)),$$

where $h_X$ is the map provided by Lemma 4.7. By Lemma 4.7–(1), the sum involved is in fact finite, so $\beta$ is well-defined. Moreover, for every $b \in B$, $g_0 \in \Gamma$ and $s \in S_n(\widetilde{X})$ we have

$$\beta(g_0 \cdot b)(s) = \sum_{g \in \Gamma} h_X(g^{-1}(s(e_0))) \cdot (\alpha(g(\sigma(g^{-1}(g_0 b))))(s)) = \sum_{g \in \Gamma} h_X(g^{-1}(g_0 \cdot s)(e_0)) \cdot (\alpha(g g_0 g(\sigma(g^{-1} g_0 b)))(s)) = \sum_{k \in \Gamma} h_X(k^{-1}(g_0^{-1} \cdot s)(e_0)) \cdot (\alpha(g_0 k(\sigma(k^{-1} b)))(s)) = \sum_{k \in \Gamma} h_X(k^{-1}(g_0^{-1} \cdot s)(e_0)) \cdot (\alpha(k(\sigma(k^{-1} b)))(g_0^{-1} \cdot s)) = \beta(b)(g_0^{-1} \cdot s) = (g_0 \cdot \beta(b))(s),$$

so $\beta$ is a $\Gamma$–map. Finally,

$$\beta(\iota(b))(s) = \sum_{g \in \Gamma} h_X(g^{-1}(s(e_0))) \cdot (\alpha(g(\sigma(g^{-1}(\iota b))))(s)) = \sum_{g \in \Gamma} h_X(g^{-1}(s(e_0))) \cdot (\alpha(g(\sigma(g^{-1} \cdot b)))(s)) = \sum_{g \in \Gamma} h_X(g^{-1}(s(e_0))) \cdot (\alpha(b)(s)) = \left(\sum_{g \in \Gamma} h_X(g^{-1}(s(e_0)))\right) \cdot (\alpha(b)(s)) = \alpha(b)(s),$$

so $\beta \circ \iota = \alpha$. Thus $C^n(\widetilde{X})$ is an injective $\Gamma$–module.

Let now $s \in S_n(\widetilde{X})$ be any singular $n$–simplex. By Lemma 4.7–(1) there exists a neighbourhood $U$ of $s$ in $S_n(\widetilde{X})$ such that the set \{ $g \in \Gamma$ | $h_X(g^{-1}(s'(e_0))) \neq 0$ for some $s' \in U$ \} is finite. This readily implies that if $\alpha(A) \subseteq C^n_c(\widetilde{X})$, then also $\beta(B) \subseteq C^n_c(\widetilde{X})$. Thus also $C^n_c(\widetilde{X})$ is an injective $\Gamma$–module.
The same argument applies verbatim if $C^n(\tilde{X})$ is replaced by $C^n_b(\tilde{X})$, and $A,B$ are Banach modules: moreover, it is easily seen that if $\alpha$ is bounded and $||\sigma|| \leq 1$, then also $\beta$ is bounded, and $||\beta|| \leq ||\alpha||$. This, together with the argument above about continuity, implies that $C^n_b(\tilde{X})$ and $C^n_{b,c}(\tilde{X})$ are relatively injective Banach $\Gamma$–modules.

4.7. Singular cochains as strong resolutions of $\mathbb{R}$. If $E^*$ is one of the complexes $C^*(\tilde{X}), C^*_c(\tilde{X}), C^*_b(\tilde{X}), C^*_{b,c}(\tilde{X})$, endowed with the usual structures of $\Gamma$–complexes, then a natural map $\delta^{-1} : E^{-1} := \mathbb{R} \to E^0$ is defined, such that for any $t \in \mathbb{R}$ and $x_0 \in S_0(\tilde{X})$ we have $\delta^{-1}(t)(x_0) = t$. It is readily seen that $\delta^0 \circ \delta^{-1} = 0$, so the augmented sequence of modules thus obtained, which will still be denoted by $(E^*, \delta^*)$ from now on, is a complex. We would now like to show that in some cases such an augmented complex is exact (for example, this is obviously true if $E^* = C^*(\tilde{X})$ and $\tilde{X}$ is contractible), and moreover admits a contracting homotopy.

**Proposition 4.9.** Suppose $\tilde{X}$ is contractible. Then the complexes $C^*(\tilde{X})$ and $C^*_c(\tilde{X})$ are strong resolutions of $\mathbb{R}$ as an unbounded $\Gamma$–module. Moreover, the complexes $C^*_b(\tilde{X})$ and $C^*_{b,c}(\tilde{X})$ are strong resolutions of $\mathbb{R}$ as a Banach $\Gamma$–module.

**Proof:** Since $\tilde{X}$ is contractible, there exist $x_0 \in \tilde{X}$ and a continuous map $H : \tilde{X} \times [0,1] \to \tilde{X}$ such that $H(x,0) = x$ and $H(x,1) = x_0$ for every $x \in \tilde{X}$. For $n \geq 0$, let $e_0^n, \ldots, e_n^n$ be the vertices of the standard simplex $\Delta_n \subset \mathbb{R}^{n+1}$, and let $Q_0^n$ be the face of $\Delta_n$ opposite to $e_0^n$. Let also $r_n : Q_0^{n+1} \to \Delta_n$ be defined by $r_n(t_1 e_1^{n+1} + \ldots + t_{n+1} e_{n+1}^{n+1}) = t_1 e_0^n + \ldots + t_{n+1} e_n^n$. For $n \geq 0$, we define $T_n : C_n(\tilde{X}) \to C_{n+1}(\tilde{X})$ as the unique linear map such that if $s \in S_n(\tilde{X})$, then the following holds: if $p = t e_0^{n+1} + (1-t)q \in \Delta_{n+1}$, where $q \in Q_0^{n+1}$, then $(T_n(s))(p) = H(s(r_n(q)), t)$. $(T_n(s))$ is just the “cone” over $s$ with vertex $x_0$, contructed by using the contracting homotopy $H$). By Lemma A.3, $T_n(s)$ is well-defined and continuous. Moreover, we define $T_{-1} : \mathbb{R} \to C_0(\tilde{X})$ by $T_{-1}(t) = tx_0$. It is readily seen that, if $d_n$ is the usual (augmented) differential on singular chains, then $d_n T_{-1} = \text{Id}_{\mathbb{R}}$, and for every $n \geq 0$ we have $T_{n-1} \circ d_n + d_{n+1} \circ T_n = \text{Id}_{C_n(\tilde{X})}$.

For every $n \geq 0$, let now $k^n : C^n(\tilde{X}) \to C^{n-1}(\tilde{X})$ be defined by $k^n(\varphi(c)) = \varphi(T_{n-1}(c))$. It is readily seen that $(k^n)_{n \in \mathbb{N}}$ is a contracting homotopy for the complex $C^*(\tilde{X})$, which is therefore a strong resolution of $\mathbb{R}$. By Lemma A.3, the map $T_n|_{S_n(\tilde{X})} : S_n(\tilde{X}) \to S_{n+1}(\tilde{X})$ is continuous, so the contracting homotopy $(k^n)_{n \in \mathbb{N}}$ restricts to a contracting homotopy for the augmented complex of continuous cochains $C^*_c(\tilde{X})$, which therefore also gives a strong resolution of $\mathbb{R}$. Moreover, since $T_n$ sends a simplex to a simplex, if $\alpha \in C^n_b(\tilde{X})$ then $||k^n(\alpha)|| \leq ||\alpha||$. Thus a suitable restriction of $k^*$ provide contracting homotopies for the complexes of Banach
\[ \Gamma \text{-modules } C^*_b(\tilde{X}) \text{ and } C^*_{b,c}(\tilde{X}). \] These complexes give therefore strong resolutions of \( \mathbb{R} \) as a Banach \( \Gamma \)–module. \( \square \)

The following result is very deep, and plays a fundamental rôle in the study of bounded cohomology of topological spaces. Together with a separate argument providing the required control on seminorms, it implies, for example, that the bounded cohomology of a countable CW–complex is canonically isomorphic to the bounded cohomology of its fundamental group \([\text{Gro}82\text{, Section 3.1}]\).

**Theorem 4.10** \([\text{Iva}87]\). Suppose \( X \) has the homotopy type of a path connected countable CW–complex. Then \( C^*_b(\tilde{X}) \) is a relatively injective strong resolution of \( \mathbb{R} \) as a Banach \( \Gamma \)–module.

We now come to the following important:

**Proposition 4.11.** Let \( F^*(\Gamma) \) (resp. \( F^*_b(\Gamma) \)) be the standard resolution of \( \mathbb{R} \) as an unbounded (resp. Banach) \( \Gamma \)–module. There exists a chain map \( \beta^*: F^*(\Gamma) \to C^*_b(\tilde{X}) \) which extends the identity of \( \mathbb{R} \) and is such that \( \beta^n \) is norm–decreasing for every \( n \in \mathbb{N} \). In particular, \( \beta^* \) restricts to a chain map \( \beta^*_b: F^*_b(\Gamma) \to C^*_{b,c}(\tilde{X}) \) which extends the identity of \( \mathbb{R} \) and is such that \( \|\beta^n\| \leq 1 \) for every \( n \in \mathbb{N} \).

**Proof:** Let \( n \geq 0 \). For every \( f \in F^n(\Gamma) \), we define \( \beta^n(f) \) to be the unique singular cochain such that for every \( s \in S^n(\tilde{X}) \) we have

\[ \beta^n(f)(s) = \sum_{(g_0, \ldots, g_n) \in \Gamma^{n+1}} h_{\tilde{X}}(g_0^{-1}(s(e_0))) \cdot \ldots \cdot h_{\tilde{X}}(g_n^{-1}(s(e_n))) \cdot f(g_0, \ldots, g_n), \]

where \( h_{\tilde{X}}: \tilde{X} \to \mathbb{R} \) is the continuous map provided by Lemma 4.7. It is readily seen that the sum involved in the definition above is finite, so \( \beta^n(f)(s) \) is well-defined. Moreover, Lemma 4.7–(1) ensures that for every \( s \in S_n(\tilde{X}) \) a neighbourhood \( U \) of \( s \) exists such that

\[ \{(g_0, \ldots, g_n) \in \Gamma^{n+1} \mid \exists s' \in U \text{ s.t. } h_{\tilde{X}}(g_i^{-1}(s'(e_i))) \neq 0 \forall i = 0, \ldots, n\} \]

is finite. This easily implies that \( \beta^n(f) \) is indeed continuous.

The fact that \( \beta^n \) is norm–decreasing for every \( n \in \mathbb{N} \) is immediate, and it is straightforward to check that \( \beta^* \) is indeed a \( \Gamma \)–equivariant chain map. \( \square \)

5. **Proofs of Theorems 1.2, 1.4, 1.5**

We are now ready to deduce Theorems 1.2, 1.4 from the results about resolutions proved in the preceding section. Looking closely at the formula involved in the proof that \( C^*_b(\tilde{X}) \) is relatively injective, and at the explicit description for the contracting homotopy for \( C^*(\tilde{X}) \) when \( \tilde{X} \) is contractible, we will be able to write down the explicit formulae required in the statement of Theorem 1.4. Moreover,
in Theorem 1.5 similar formulae will be obtained in the one-dimensional case even without the assumption that \( \bar{X} \) is contractible.

We begin with the following:

**Lemma 5.1.** For any topological space \( X \), the chain map \( p^*: C^*(X) \to C^*(\bar{X}) \) restricts to the following isometric isomorphisms of complexes:

\[
p^*: C^*(X) \to C^*(\bar{X})^{\Gamma}, \\
p^*|_{C^k(X)}: C^k_b(X) \to C^k_b(\bar{X})^{\Gamma}, \\
p^*|_{C^k_{b,c}(X)}: C^k_{b,c}(X) \to C^k_{b,c}(\bar{X})^{\Gamma},
\]

which induce therefore isometric isomorphisms

\[
H^*(X) \cong H^*(C^*(\bar{X})^{\Gamma}), \\
H^*_b(X) \cong H^*(C^*_b(\bar{X})^{\Gamma}), \\
H^*_b(X) \cong H^*(C^*_b(\bar{X})^{\Gamma}).
\]

**Proof:** The fact that \( p^* \) is an isometric embedding on the space of \( \Gamma \)-invariant cochains on \( \bar{X} \) is obvious, thus the only non-trivial issue to prove is the fact that \( p^*(\varphi) \) is continuous if and only if \( \varphi \) is continuous. By Lemma 5.1 the map \( p_\ast: S_n(\bar{X}) \to S_n(X) \) is a covering. In particular, it is continuous, open and surjective, and this readily implies that if \( \varphi: S_n(X) \to \mathbb{R} \) is any map, then \( \varphi \) is continuous if and only if \( \varphi \circ p_\ast: S_n(\bar{X}) \to \mathbb{R} \) is continuous, whence the conclusion. \( \square \)

### 5.1. Proof of Theorem 1.2

Suppose now that \( X \) is a path connected countable CW–complex. The inclusion \( i^*: C^*_b(\bar{X}) \to C^*(\bar{X}) \) is a norm–decreasing chain \( \Gamma \)-map, and induces therefore a norm–decreasing chain \( \Gamma \)-map \( \bar{i}_b^*: C^*_b(\bar{X}) \to C^*_b(\bar{X}) \).

We would like to show that \( \bar{i}_b^* \) admits a norm–decreasing right homotopy inverse.

By Theorem 1.10 the complex \( C^*_b(\bar{X}) \) provides a relatively injective strong resolution of \( \mathbb{R} \) as a Banach \( \Gamma \)-module. Thus Proposition 4.5 shows that there exists a norm–decreasing chain map \( \alpha_b^*: C^*_b(\bar{X}) \to F^*_b(\Gamma) \) which extends the identity of \( \mathbb{R} \).

Let now \( \beta_b^*: F^*_b(\Gamma) \to C^*_b(\bar{X}) \) be the chain map provided by Proposition 4.11. By Propositions 4.8 and 4.3 the map \( \bar{i}_b^* \circ (\beta_b^* \circ \alpha_b^*) \) is \( \Gamma \)-homotopic to the identity of \( C^*_b(\bar{X}) \). Since both \( \bar{i}_b^* \) and \( \beta_b^* \circ \alpha_b^* \) are norm–decreasing, the map \( \bar{i}_b^* \) restricts to a map

\[
\bar{i}_b^*|_{C^*_b,c(\bar{X})^{\Gamma}}: C^*_b,c(\bar{X})^{\Gamma} \to C^*_b(\bar{X})^{\Gamma}
\]

which induces in homology a norm–decreasing map \( \bar{i}_b^* \) admitting a norm–decreasing right inverse (such an inverse is therefore an isometric embedding). Moreover, under the isometric identifications \( H^*_b(X) \cong H^*(C^*_b(\bar{X})^{\Gamma}), H^*_b(X) \cong H^*(C^*_b(\bar{X})^{\Gamma}) \) provided by Lemma 5.1 the map \( \bar{i}_b^* \) corresponds to \( i_b^*: H^*_b(X) \to H^*_b(X) \), whence the conclusion. \( \square \)

**Remark 5.2.** If we were able to show that the complex \( C^*_b,c(\bar{X}) \) provides a strong resolution of \( \mathbb{R} \), we could prove that, under the hypothesis of Theorem 1.2 the map
\(i^*_\theta\) is an isometric isomorphism. However, it is not clear to us why the contracting homotopy for \(C_b^*(\tilde{X})\) constructed in [Iva87, Theorem 2.4] should take continuous cochains to continuous cochains, thus restricting to a contracting homotopy for \(C_{b,\theta}^*(\tilde{X})\).

5.2. **Proof of Theorem 1.3.** Let \(\tilde{\theta}^*: C^*(\tilde{X}) \to C^*_c(\tilde{X})\) be defined as the composition \(\beta^* \circ \alpha^*\), where \(\alpha^*, \beta^*\) are the maps described in Remark 4.6 and Proposition 4.11. Since \(\tilde{X}\) is contractible, by Propositions 4.8 and 4.9 both \(C^*(\tilde{X})\) and \(C_c^*(\tilde{X})\) provide injective strong resolutions of \(\mathbb{R}\). Therefore, by Proposition 4.3 the compositions \(\tilde{i}^* \circ \tilde{\theta}^*\) and \(\theta^* \circ \tilde{i}^*\) are \(\Gamma\)-homotopic to the identity respectively of \(C^*(\tilde{X})\) and of \(C_c^*(\tilde{X})\). Therefore, \(\tilde{i}^*, \tilde{\theta}^*\) restrict to homotopy equivalences between \(C^*_c(\tilde{X})\) and \(C^*(\tilde{X})\), which in turn define isomorphisms

\[
\tilde{\tau}^*: H^*(C^*_c(\tilde{X})) \to H^*(C^*(\tilde{X})), \quad \tilde{\theta}^*: H^*(C^*(\tilde{X})) \to H^*(C^*_c(\tilde{X})),
\]

that are one the inverse of the other. Finally, under the identifications provided by Lemma 5.1 the map \(\tilde{i}^*\) corresponds to \(i^*: H^*_c(X) \to H^*(X)\), while \(\tilde{\theta}^*\) corresponds to the inverse \(\theta^*: H^*(X) \to H^*_c(X)\) of \(i^*\). This proves in particular that \(i^*\) is an isomorphism. Moreover, since \(\alpha^*, \beta^*\) are norm-decreasing, so is \(\theta^*\), and this readily implies that \(i^*\) is isometric.

In order to write down an explicit formula for \((i^*)^{-1} = \theta^*\) it is sufficient to exhibit an explicit construction of \(\tilde{\theta}\). To this aim, let \(H: \tilde{X} \times [0, 1] \to \tilde{X}\) be a homotopy such that \(H(x, 0) = x\) and \(H(x, 1) = x_0\) for every \(x \in \tilde{X}\), and let \(T_*: C_*(\tilde{X}) \to C_{\theta,1}(\tilde{X})\) be the induced contracting homotopy, as described in the proof of Proposition 4.9. Given \(n \geq 0\) and \((g_0, g_1, \ldots, g_n) \in \Gamma^{n+1}\), we will now construct a simplex \(\overline{s}(g_0, g_1, \ldots, g_n) \in S_n(\tilde{X})\) such that \(\overline{s}(g_0, g_1, \ldots, g_n)(e_i) = g_i(x_0)\) for every \(i = 0, 1, \ldots, n\). If \(n = 0\), we set \(\overline{s}(g_0) = g_0(x_0)\), while if \(\overline{s}(g_0, g_1, \ldots, g_n) \in S_n(\tilde{X})\) has been defined for every \((g_0, g_1, \ldots, g_n) \in \Gamma^{n+1}\), then for every \((g_0, g_1, \ldots, g_{n+1}) \in \Gamma^{n+2}\) we set

\[
\overline{s}(g_0, g_1, \ldots, g_{n+1}) = g_0 \cdot (T_n(g_0^{-1} \cdot \overline{s}(g_1, g_2, \ldots, g_{n+1})))
\]

(so \(\overline{s}(g_0, g_1, \ldots, g_{n+1})\) is the image under \(g_0\) of the “cone” of vertex \(x_0\) based on \(\overline{s}(g_1, g_2, \ldots, g_{n+1})\)).

It is not difficult to check that, according to Propositions 4.9, 4.11 and Remark 4.6, the map \(\tilde{\theta}^* = \beta^* \circ \alpha^*: C^*(\tilde{X}) \to C_c^*(\tilde{X})\) is defined as follows: if \(\overline{\varphi} \in C^n(\tilde{X})\), then \(\tilde{\theta}^n(\overline{\varphi}) \in C^n_c(\tilde{X})\) is the unique cochain such that for every \(\overline{s} \in S_n(\tilde{X})\) we have

\[
\tilde{\theta}^n(\overline{\varphi})(\overline{s}) = \sum_{(g_0, g_1, \ldots, g_n) \in \Gamma^{n+1}} h_{\tilde{X}}(g_0^{-1}(\overline{s}(e_0))) \cdots h_{\tilde{X}}(g_n^{-1}(\overline{s}(e_n))) \cdot \varphi(\overline{s}(g_0, \ldots, g_n)).
\]

As a consequence, for every \([\varphi] \in H^n(X)\), the coclass \((i^*)^{-1}([\varphi]) = \theta^n(p^*(\varphi))(\overline{s})\) is represented by the cocycle that maps every \(s \in S_n(X)\) to the real number \(\tilde{\theta}^n(p^*(\varphi))(\overline{s})\), where \(\overline{s}\) is any lift of \(s\) to \(\tilde{X}\).
Being norm-decreasing in every degree, the map \( \partial^* : C^*_c(\tilde{X}) \to C^*_c(\tilde{X}) \) restricts to a chain map \( \partial^*_b : C^*_c(\tilde{X}) \to C^*_b(X) \) extending the identity of \( \mathbb{R} \). By Propositions 4.9 and 4.8 this map provides a \( \Gamma \)-homotopy inverse to \( \tilde{i}^*_b \), and induces as above a norm-decreasing inverse \( \theta^*_b : H^*_b(X) \to H^*_b(X) \) of \( i^*_b \). This proves that \( i^*_b \) is an isometric isomorphism. Moreover, for every \( [\varphi] \in H^n_b(X) \), the coclass \( (i^*_b)^{-1}([\varphi]) = \theta^*_b([\varphi]) \) is represented by the cocycle that maps every \( s \in S_\eta(X) \) to the real number \( \theta^*_b(p^*(\varphi))(s) \), where \( s \) is any lift of \( s \) to \( \tilde{X} \).

### 5.3. Proof of Theorem 1.5

In this subsection we concentrate on the relations between continuous and ordinary cohomology in dimension one. We begin by dealing with the map \( i^1 : H^1_b(X) \to H^1(X) \) between unbounded cohomology modules. Since \( X \) is locally path connected, by Proposition 2.3 we can suppose that \( X \) is path connected.

We first show that \( i^1 \) is surjective, i.e. that any cocycle \( \varphi \in C^1(X) \) is cohomologous to a continuous cocycle. So, let \( \tilde{\varphi} = p^*(\varphi) \in C^1(\tilde{X}) \). Since \( \varphi \) is a cocycle and \( \tilde{X} \) is simply connected, \( \tilde{\varphi} \) is a coboundary, so \( \tilde{\varphi} = \delta f \) for some \( f \in C^0(\tilde{X}) \). Moreover, for every \( g \in \Gamma \) we have \( g \cdot \tilde{\varphi} = \tilde{\varphi} \), and this readily implies that

\[
\tag{4} f(g(y)) - f(g(x)) = f(y) - f(x) \quad \text{for every } x, y \in \tilde{X}, \quad g \in \Gamma.
\]

We now replace \( f \) with the map \( f_c : \tilde{X} \to \mathbb{R} \) defined by

\[
f_c(x) = \sum_{g \in \Gamma} h_{\tilde{X}}(g^{-1}(x)) \cdot f(g(x_0)),
\]

where \( x_0 \in \tilde{X} \) is a fixed basepoint, and \( h_{\tilde{X}} \) is the function provided by Lemma 4.7.

By Lemma 4.7(1), if \( x \in \tilde{X} \) there exists a neighbourhood \( U \) of \( x \) in \( \tilde{X} \) such that the set \( \{ g \in \Gamma | h_{\tilde{X}}(g^{-1}(x')) \neq 0 \text{ for some } x' \in U \} \) is finite. This readily implies that \( f_c \) is well-defined and continuous, and determines therefore a continuous 0-cocycle, which we will still denote by \( f_c \).

Let us consider the difference \( \tilde{k} = (f - f_c) : \tilde{X} \to \mathbb{R} \). For every \( g_0 \in \Gamma, \) \( x \in \tilde{X} \) we have

\[
\tilde{k}(g_0^{-1}(x)) - \tilde{k}(x) = f(g_0^{-1}(x)) - f(x) - \sum_{g \in \Gamma} h_{\tilde{X}}(g^{-1}(x))(f(gg_0^{-1}(x)) - f(g(x)))
\]

\[
= f(g_0^{-1}(x)) - f(x) - \sum_{g \in \Gamma} h_{\tilde{X}}(g^{-1}(x))(f(g_0^{-1}(x)) - f(x))
\]

\[
= f(g_0^{-1}(x)) - f(x) - \left( \sum_{g \in \Gamma} h_{\tilde{X}}(g^{-1}(x)) \right) (f(g_0^{-1}(x)) - f(x))
\]

\[
= 0,
\]

where the second equality is obtained by specializing equation (4) to the case \( y = g_0^{-1}(x) \). Being \( \Gamma \)-equivariant, the map \( \tilde{k} \) defines therefore a unique map \( k : X \to \mathbb{R} \) such that \( \tilde{k}(x) = k(p(x)) \) for every \( x \in \tilde{X} \). We now set

\[
\varphi_c = \varphi - \delta k \in C^1(X).
\]
We have by construction $|\varphi_c| = |\varphi|$ in $H^1(X)$, so in order to show that $i^1$ is surjective it is sufficient to show that $\varphi_c$ is continuous, or, equivalently, that $\tilde{\varphi}_c = p^*(\varphi_c) \in C^1(\tilde{X})$ is continuous (see Lemma \ref{lem:cohomology}). However, we have
\[\tilde{\varphi}_c = p^*(\varphi - \delta k) = p^*(\varphi) - \delta p^*(k) = \tilde{\varphi} - \delta k = \delta f - \delta(f - f_c) = \delta f_c,\]
which is continuous since $f_c \in C^0_c(\tilde{X})$. We have thus proved that $i^1$ is surjective, also providing a somewhat explicit procedure for replacing a singular 1–cocycle with a cohomologous continuous 1–cocycle.

Let now $\varphi \in C^1_c(X)$ be a continuous cocycle with $i^1(|\varphi|) = 0$, fix a basepoint $x_0 \in X$, and, for every $q \in X$, let $s_q: [0,1] \to X$ be a fixed continuous path with $s_q(0) = x_0$, $s_q(1) = q$. We define a real function $f: X \to \mathbb{R}$ by setting $f(q) = \varphi(s_q)$. This function defines a 0–cocycle which will still be denoted by $f$. We will now show that $\delta(f) = \varphi$, and that $f$ is continuous, thus proving that $\varphi$ is the coboundary of a continuous 0–cocycle, and that $i^1$ is injective.

So, let $s \in S_1(X)$ be a simplex with endpoints $q_0, q_1$. Since $s + s_{q_0} - s_{q_1}$ is a cycle and $\varphi$ is a coboundary, we have
\[\varphi(s) = \varphi(s_{q_1}) - \varphi(s_{q_0}) = f(q_1) - f(q_0) = \delta(f)(s),\]
so $\varphi = \delta(f)$. We now show that $f$ is continuous. Let $q \in X$ and $\varepsilon > 0$ be given. If $c_q$ is the constant 1–simplex with $c_q(t) = q$ for every $t \in [0,1]$, then $\varphi(c_q) = f(q) - f(q) = 0$. Thus it is not difficult to show that since $\varphi$ is continuous there exists a path connected neighbourhood $U$ of $q$ such that $|\varphi(s)| < \varepsilon$ for every simplex $s$ with values in $U$. In particular, for every $r \in U$ there exists a simplex $s_{q,r}$ such that $s_{q,r}(0) = q$, $s_{q,r}(1) = r$, and $|\varphi(s_{q,r})| < \varepsilon$, so $|f(r) - f(q)| = |\varphi(s_{q,r})| < \varepsilon$.

We have thus shown that $f$ is continuous, so $i^1$ is injective. Also note that if $\varphi$ is supposed to be bounded, then $||f|| \leq ||\varphi||$, so $i^1_b: H^1_{b,c}(X) \to H^1_b(X)$ is injective too. Moreover, the last statement is obviously still true even when $X$ is not path connected (but still locally path connected).

The fact that $H^1_b(X) = 0$ is well-known and easy: a bounded 1–cocycle $\omega$ defines a bounded homomorphism between $\pi_1(X)$ and $\mathbb{R}$, but $\mathbb{R}$ does not contain non-trivial bounded subgroups, so $\omega$ has to be the coboundary of a 0–cocycle $\psi$; moreover, the same argument showing above that $||f|| \leq ||\varphi||$ applies here ensuring that $\psi$ can be chosen in such a way that $||\psi|| \leq ||\omega||$, and this implies that $[\omega] = 0$ in $H^1_b(X)$, so $H^1_b(X) = 0$. Therefore also $H^1_{b,c}(X) = 0$, since $i^1_b: H^1_{b,c}(X) \to H^1_b(X)$ is injective as noted above. \hfill $\square$

6. Two (counter)examples

It is not difficult to construct disconnected spaces whose continuous cohomology is not isomorphic to standard cohomology, even in dimension 0. For example, if $Y$ is the Cantor set, then any real function on $Y$ defines a 0–cocycle, while a 0–cocycle
Figure 1. The spaces\( X_1 \) (on the left) and\( X_2 \) (on the right).

is continuous if and only if it is defined by a continuous real function on\( Y \), and this readily shows that in this case the map \( i^0 : H^0_c(Y) \to H^0(Y) \) is not surjective.

The following examples provide path connected spaces whose continuous cohomology is not isomorphic (through \( i^* \)) to standard cohomology, even in dimension one.

Let \( X_1 = \left( \left\{ 0 \right\} \cup \left( \bigcup_{n \geq 1} \left\{ \frac{1}{n} \right\} \right) \right) \times [0, 1] \cup ([0, 1] \times \{0, 1\}) \subset \mathbb{R}^2 \)

be endowed with the Euclidean topology (see Figure 1–left). For \( i \in \mathbb{N} \), let \( \alpha_i : [0, 1] \to X \) be the constant–speed parameterization of the polygonal path with vertices \((0, 0), (1/i, 0), (1/i, 1) \) and \((0, 1) \) if \( i \neq 0 \), and \( \alpha_i(t) = (0, 1-t) \) if \( i = 0 \). It is not difficult to prove that \( H_1(X_1) \) is freely generated (as a vector space) by the classes represented by the loops \( \alpha_i \ast \alpha_0 \), \( i \geq 1 \). Thus, if \( i^1 \) were surjective, by the Universal Coefficient Theorem, for any real sequence \( \{\epsilon_i\}_{i \geq 1} \) there should exist a continuous cocycle \( \psi \in C^1_c(X_1) \) such that \( \psi(\alpha_0) + \psi(\alpha_i) = \epsilon_i \). Now choosing \( \epsilon_i = (-1)^i \) we would get

\[
|\psi(\alpha_i) - \psi(\alpha_{i+1})| = |\psi(\alpha_i) + \psi(\alpha_0) - (\psi(\alpha_{i+1}) + \psi(\alpha_0))| = 2,
\]

and this would contradict the continuity of \( \psi \), since \( \lim_{i \to \infty} \alpha_i = \alpha_0^{-1} \) in the compact–open topology, where \( \alpha_0^{-1} : [0, 1] \to X_1 \) is defined by \( \alpha_0^{-1}(t) = \alpha_0(1-t) \) for every \( t \in [0, 1] \).

It is maybe worth mentioning that as a byproduct of our results we obtain that \( X_1 \) does not admit a universal covering: in fact, \( X_1 \) is metrizable, whence paracompact, while the surjectivity of \( i^1 \) was established in the proof of Theorem 1.5 without using any local path connectedness assumption.

Let now \( X_2 \subset \mathbb{R}^2 \) be defined as follows (see Figure 1–right):

\[
X_2 = \left( \left( \bigcup_{n \geq 1} \left\{ \frac{1}{n} \right\} \right) \times [0, 1] \right) \cup ([0, 1] \times \{-1, 1\}) \cup (\{0\} \times [-1, 0]) \cup (\{1\} \times [-1, 1]).
\]
We let \( f: X_2 \to \mathbb{R} \) be the 0–cochain such that for every \((x, y) \in X_2\)

\[
f(x, y) = \begin{cases} 
0 & \text{if } x = 0, x = 1, \text{ or } y = -1 \\
1 - y & \text{otherwise}
\end{cases}
\]

and set \( \varphi = \delta f \in C^1(X_2) \). The map \( f \) is continuous at every point of \( X_2 \) except \((0, 0)\), and we now sketch a proof of the fact that \( \varphi \) is indeed continuous. Since \( X_2 \) is metrizable and \([0, 1]\) is compact, it is sufficient to show that \( \varphi \) is sequentially continuous. So, let us suppose by contradiction that \( \lim_{n \to \infty} s_n = s \in S_1(X_2) \), while \( \varphi(s_n) = f(s_n(1)) - f(s_n(0)) \) does not tend to \( \varphi(s) = f(s(1)) - f(s(0)) \). Then \( \lim_{n \to \infty} f(s_n(t_0)) = f(s(t_0)) \) for some \( t_0 \in \{0, 1\} \). Since \( f \) is everywhere continuous except that at \((0, 0)\) we must have \( s(t_0) = (0, 0) \), and \( x(s_n(t_0)) > 0 \) for an infinite number of indices. But it is easily seen that a sequence of continuous paths starting (resp. ending) in points of \( X_2 \) with positive \( x \)-coordinate can converge to a path starting (resp. ending) in \((0, 0)\) if and only if it converges to the constant path. However, if \( s \) is the constant path in \((0, 0)\) and \( \lim_{n \to \infty} s_n = s \), then it is easily seen that definitively \( x(s_n(0)) = x(s_n(1)) \), so \( \lim_{n \to \infty} s_n = 0 = \varphi(s) \), a contradiction.

Thus \( \varphi \in C^1_c(X_2) \), and \( \delta \varphi = \delta^2 f = 0 \). We now claim that \([\varphi] \neq 0 \in H^1_c(X_2) \), thus showing that \( \iota^1: H^1_c(X_2) \to H^1(X_2) \) is not injective, since \([\varphi] = [\delta f] = 0 \) in \( H^1(X_2) \). In fact, if \( \varphi \) were the coboundary of a continuous 0–cochain \( f_c \), then we would have \( \delta(f - f_c) = 0 \). Since \( X_2 \) is path connected, this would imply that \( k \in \mathbb{R} \) should exist such that \( f(x) = f_c(x) + k \) for every \( x \in X_2 \), a contradiction since \( f_c \) is continuous while \( f \) is not.

Note that this example does not contradict Theorem 1.4. In fact, even if it is simply connected and 1–dimensional, \( X_2 \) is not contractible, since by Proposition 2.4 any contractible space has trivial first continuous cohomology group.

7. Smooth cohomology

Suppose now \( X \) is a smooth manifold. As mentioned in the introduction, we would like to concentrate our attention only on smooth simplices, and on cochains which are continuous with respect to the \( C^1 \)–topology on smooth simplices. More precisely, we set \( sS_q(X) = \{ s \in S_q(X) \mid s \text{ is smooth} \} \) and we denote by \( sC_q(X) \) the subcomplex of \( C_q(X) \) generated by \( sS_q(X) \); moreover, we let \( sC^q(X) \) be the dual space of \( sC_q(X) \). Of course, \( sC^*(X) \) is a differential complex, whose elements will be called smooth cochains. The homology of \( sC^*(X) \) will be called smooth cohomology of \( X \) and denoted by \( sH^*(X) \). We will also denote by \( sC^*_b(X) \subseteq sC^*(X) \) the subcomplex of bounded cochains, and by \( sH^*_b(X) \) the corresponding cohomology.

The inclusion \( j_*: sC_*(X) \to C_*(X) \) induces restrictions \( j^*: C^*(X) \to sC^*(X) \), \( j_b^*: C^*_b(X) \to sC^*_b(X) \), which induce in turn maps \( r^*: H^*(X) \to sH^*(X) \), \( r^*_b: H^*_b(X) \to sH^*_b(X) \). The following well-known result shows that smooth cohomology is isometric to the usual one:
Proposition 7.1. The maps \( r^*: H^*(X) \to sH^*(X) \), \( r_b^*: H_b^*(X) \to sH_b^*(X) \) are isometric isomorphisms.

**Proof:** It is well-known (see e.g. [Lee03, Theorem 16.6]) that there exist maps \( l_s: C_s(X) \to sC_s(X) \), \( T_s: C_s(X) \to C_{s+1}(X) \) with the following properties: for every \( s \in S_n(X) \), \( l(s) \) is a simplex in \( sS_n(X) \), while \( T_n(s) \) is the sum (with signs) of a fixed number (depending only on \( n \)) of simplices in \( S_{n+1}(X) \); \( l_s \circ j_s = \text{Id}_{sC_s(X)} \); \( j_s \circ l_s - \text{Id}_{C_s(X)} = d \circ T_s + T_s \circ d \). The dual maps \( l^* \), \( j^* \) are therefore norm-decreasing, and one the \( \Gamma \)-homotopy inverse of the other. Since \( T_n \) is bounded for every \( n \), the same is true for the restrictions of \( l^* \), \( j^* \) to bounded cochains, whence the conclusion. □

For every \( n \in \mathbb{N} \), we now endow \( sS_n(X) \) with the \( C^1 \)-topology (see Appendix A for the definition and the needed properties of \( C^1 \)-topology). We say that a cochain \( \varphi \in sC^n(X) \) is continuous if it restricts to a continuous map on \( sS_n(X) \), and we denote by \( sC^*_c(X) \) the subcomplex of continuous cochains in \( sC^*(X) \). We also denote by \( sC_{b,c}^*(X) = sC^*_c(X) \cap sC^*_b(X) \) the complex of bounded continuous cochains. The corresponding cohomology modules will be denoted by \( sH^*_c(X) \) and \( sH^*_b(X) \). The natural inclusions of cochains induce maps

\[
s_i^*: sH^*_c(X) \to sH^*_c(X), \quad s_i^*_b: sH^*_b(X) \to sH^*_b(X).
\]

Basically, all the results proved in the preceding sections for continuous and singular cohomology of sufficiently nice topological space extend to the cohomology theories just introduced for smooth manifolds. We state here the facts we will need in Section 8, also giving an outline of their proofs.

Lemma 7.2. The map \( s_i^*_b \) admits a norm–decreasing right inverse.

**Proof:** Since the \( C^1 \)-topology is finer than the compact–open topology, the restriction map \( j^*: C^*(X) \to sC^*(X) \) introduced above takes continuous cochains into continuous smooth cochains. Therefore \( j^* \) restricts to a chain map \( C_{b,c}^*(X) \to sC_{b,c}^*(X) \), which induces in turn a map \( r_{b,c}^*: H_{b,c}^*(X) \to sH_{b,c}^*(X) \). We have therefore the sequence of maps

\[
H_{b,c}^*(X) \xrightarrow{r_{b,c}^*} sH_{b,c}^*(X) \xrightarrow{s_i^*_b} sH_b^*(X) \xrightarrow{(r_b^*)^{-1}} H_b^*(X)
\]

where \( \beta^* \) is the right inverse of \( i_b^* \) provided by Theorem 1.2. By the very definitions we have \( s_i^*_b \circ r_{b,c}^* = r_b^* \circ i_b^* \), so

\[
s_i^*_b \circ \left( r_{b,c}^* \circ \beta^* \circ (r_b^*)^{-1} \right) = \left( s_i^*_b \circ r_{b,c}^* \right) \circ \beta^* \circ (r_b^*)^{-1} = (r_b^* \circ i_b^* \circ \beta^* \circ (r_b^*)^{-1}) = \text{Id}_{H_b^*(X)}.
\]

Therefore \( s_i^*_b \) admits a norm–decreasing right inverse. □
Theorem 7.3. The map $s^* : sH^*_c(X) \to sH^*(X)$ is an isometric isomorphism.

Proof: All the arguments developed in Section 3 in order to prove Theorem 1.1 apply verbatim to smooth cohomology, thus proving that $s^*$ is an isomorphism. More precisely, since every smooth manifold is locally smoothly contractible, by Lemma A.6 the graded sheaves of smooth cochains and continuous smooth cochains both provide resolutions of the constant sheaf $\mathbb{R}$ on $X$. Moreover, such sheaves admit a structure of modules over the sheaf of continuous functions, and are therefore acyclic. Therefore, sheafified smooth cohomology is canonically isomorphic to sheafified continuous smooth cohomology. Now, in order to prove that singular cohomology is isomorphic to sheafified singular cohomology, in Section 3 we only needed the following facts:

1. the existence of a barycentric subdivision (co)operator taking continuous locally zero cochains to continuous locally zero cochains (see Proposition 2.6);
2. the fact that a locally defined continuous cochain could be extended to a global one (seeLemma 3.4).

The proof of fact (2) given in Section 2 applies verbatim when restricting our attention to smooth simplices, endowed with the $C^1$–topology. Moreover, using Lemma A.5 and the fact that the $C^1$–topology is finer than the compact–open topology, it is readily seen that the barycentric subdivision (co)operators defined in Section 2 take continuous locally zero smooth cochains to continuous locally zero smooth cochains, so (1) also holds.

Since $s^*$ is obviously norm–decreasing, the fact that $s^*$ is an isometry is now a consequence of Lemma 7.2. □

Let now $p : \tilde{X} \to X$ be the smooth universal covering of $X$ (see the Appendix for the definition and basic properties of smooth coverings). By Lemma A.5, the covering $p$ induces a well-defined map $p^* : sC^*_c(X) \to sC^*_c(\tilde{X})$. Moreover, if $\Gamma \cong \pi_1(X)$ is the group of the covering automomorphisms of $p$, then $\Gamma$ acts on $\tilde{X}$ as a group of diffeomorphisms. Therefore, as noted in the Appendix, $\Gamma$ also acts on $sC^*_c(\tilde{X})$, and we will denote by $sC^*_c(\tilde{X})^\Gamma \subseteq sC^*_c(\tilde{X})$ the subcomplex of $\Gamma$–invariant continuous smooth cochains.

The following result easily follows from Lemma A.7 (see also the proof of Lemma 5.1):

Lemma 7.4. The chain map $p^* : sC^*_c(X) \to sC^*_c(\tilde{X})$ induces the isometric isomorphism of complexes

$$p^* |_{sC^*_c(X)} : sC^*_c(X) \to sC^*_c(\tilde{X})^\Gamma,$$

which induces in turn an isometric isomorphism $sH^*_c(X) \cong H^*(sC^*_c(\tilde{X})^\Gamma)$.

8. Gromov’s proportionality principle

Before going into the proof of the proportionality principle, we briefly describe Gromov’s original approach to the issue.
As mentioned in Subsection 1.3, bounded cohomology provides the natural “dual” theory to $L^1$-homology, and is therefore deeply related to the simplicial volume. More precisely, it is not difficult to show that if $X$ is a compact connected Riemannian manifold, then $\text{Vol}(X)/||X||$ equals the seminorm of the coclass of $H^n(X)$ represented by the Riemannian volume form of $X$ (see Corollary 8.4 below). Keeping notations from the preceding section, if $\Gamma \cong \pi_1(X)$ is the group of the covering automorphisms of $\tilde{X}$, then the volume form of $X$ lifts to the volume form of $\tilde{X}$, which is $G$-invariant, where $G$ is the group of orientation–preserving isometries of $\tilde{X}$. Moreover, the seminorm of the volume form of $X$ is equal to the seminorm of the volume form of $\tilde{X}$ in the homology of the appropriate complex of $\Gamma$–invariant cochains. An averaging process on cochains now allows to show that this seminorm is equal to the seminorm of the volume form of $\tilde{X}$ in the homology of $G$–invariant cochains. As a consequence, the role of $\Gamma$ turns out to be immaterial, and $\text{vol}(X)/||X||$ only depends on the geometry of $\tilde{X}$.

The argument just outlined is basically Gromov’s original approach to the proportionality principle [Gro82, Section 2.3]. However, as Gromov himself points out, in order to formally define the above mentioned averaging process, one should restrict only to continuous (or at least Borel measurable) cochains. The fact that this assumption is harmless is a consequence of the (isometric) isomorphism Theorem 1.3.

Our exposition closely follows (and is in fact inspired by) Bucher-Karlsson’s argument (see [BK08]). However, since the volume form is not continuous with respect to the compact–open topology (see Remark 8.2 below), we work here in the slightly different setting of continuous smooth cohomology, endowing the space of smooth simplices with the $C^1$–topology, rather than with the compact–open topology.

8.1. The duality principle. From now on, we denote by $X$ a $n$–dimensional compact connected oriented Riemannian manifold with real fundamental class $[X]_R \in H_n(X)$. Recall that there exists a well-defined product $(\cdot, \cdot): H^n(X) \times H_n(X) \to \mathbb{R}$, called Kronecker product, such that $(\varphi, [z]) = \varphi(z)$ for every cocycle $\varphi \in C^n(X)$ and every cycle $z \in C_n(X)$. We denote by $[X]_R \in H^n(\mathbb{R}) \cong \mathbb{R}$ the fundamental coclass of $X$, i.e. the unique coclass in $H^n(X)$ such that $(\langle [X]_R, [X]_R \rangle = 1$. Also recall that we denote by $c^n: H^n_b(X) \to H^n(X)$ the comparison map induced by the inclusion of cochains. The following result, due to Gromov [Gro82], is based on Hahn-Banach Theorem, and is proved e.g. in [BP92, Proposition F.2.2]:

**Proposition 8.1.** Let $||X|| = ||[X_R]||$ be the simplicial volume of $X$. Then

$$||X|| = \frac{1}{||[X]_R||} = \sup \left\{ \frac{1}{||\varphi||}, \varphi \in H^n_b(X), c^n(\varphi) = [X]_R \right\}.$$ 

In particular, $||X|| = 0$ if and only if $[X]_R \notin \text{Im} c^n$, i.e. if and only if $\text{Im} c^n = 0$. 

8.2. The volume form. We define a map \( \text{Vol}_X : \text{sS}_n(X) \to \mathbb{R} \) by setting

\[
\text{Vol}_X(s) = \int_s \omega_X,
\]

where \( \omega_X \in \Omega^n(X) \) is the volume form of \( X \). Of course, \( \text{Vol}_X \) is continuous with respect to the \( C^1 \)-topology on \( \text{sS}_n(X) \), so its linear extension to smooth \( n \)-chains, which will still be denoted by \( \text{Vol}_X \), defines an element in \( \text{sC}^n_c(X) \). By Stokes’ Theorem, such an element is closed, and defines therefore elements \([\text{Vol}_X] \in \text{sH}^n(X), [\text{Vol}_X]_c \in \text{sH}^n_c(X)\).

**Remark 8.2.** The following example shows that the cochain \( \text{Vol}_X \) is in general not continuous with respect to the compact–open topology. In fact, let us consider the Euclidean plane \( \mathbb{R}^2 \), endowed with the usual volume form \( dx_1 \wedge dx_2 \). Let \( Y = \mathbb{R}^2 \setminus \{(0,0)\} \), and for every \( n \geq 1 \) let \( f_n : Y \to Y \) be the map which corresponds to \( z \mapsto z^n/(n|z|^{n-1}) \) under the identification \( Y \cong \mathbb{C} \setminus \{0\} \) (this map is the composition of the rescaling of ratio \( 1/n \) with a map that “wraps” \( Y \) around the origin \( n \) times). An easy computation shows that \( f_n \) is an area-preserving local diffeomorphism of \( Y \) onto itself, so if \( s \in \text{sS}_2(\mathbb{R}^2) \) is any smooth simplex with \( \text{Im}(s) \subseteq Y \) and \( \text{Vol}_{\mathbb{R}^2}(s) = \alpha \neq 0 \), then \( s_n = f_n \circ s \) is a smooth simplex such that \( \text{Vol}_{\mathbb{R}^2}(s_n) = \alpha \neq 0 \). On the other hand, if \( \text{Im}(s) \) is contained in the ball \( B(0,R) \subseteq \mathbb{R}^2 \) of radius \( R \) centered at the origin, then \( \text{Im}(s_n) \) is contained in the ball \( B(0,R/n) \). This readily implies that \( \lim_{n \to \infty} s_n = s_0 \) in the compact–open topology, where \( s_0 \) is the constant simplex with values in \( \{0\} \subseteq \mathbb{R}^2 \). Since \( \text{Vol}_{\mathbb{R}^2}(s_0) = 0 \), this shows that \( \text{Vol}_{\mathbb{R}^2} \) is not continuous if we endow \( \text{sS}_2(\mathbb{R}^2) \) with the compact–open topology.

Let \( \tau^n : \text{sH}^n(X) \to \text{sH}^n(X) \) be the map introduced at the beginning of Section 7.

**Lemma 8.3.** We have \((\tau^n)^{-1}([\text{Vol}_X]) = \text{Vol}(X) \cdot [X]^\mathbb{R}\).

**Proof:** Since \( \text{sH}^n(X) \cong \mathbb{R} \), we have \((\tau^n)^{-1}([\text{Vol}_X]) = \langle(\tau^n)^{-1}([\text{Vol}_X]), [X]_\mathbb{R}\rangle \cdot [X]^\mathbb{R}\). Moreover, it is well-known that the fundamental class of \( X \) can be represented by the sum of the simplices in a positively oriented smooth triangulation of \( X \). Evaluating the cohomology class \((\tau^n)^{-1}([\text{Vol}_X])\) on such a sum we get the sum of the volumes of the simplices of the triangulation, i.e. the volume of \( X \). \( \square \)

**Corollary 8.4.** We have

\[
\frac{||X||}{\text{Vol}(X)} = \frac{1}{||[\text{Vol}_X]_c||}.
\]

**Proof:** Since \( \text{s}^n([\text{Vol}_X]_c) = [\text{Vol}_X] \), by Proposition 7.1, Theorem 7.3 and Lemma 8.3 we have

\[
||[\text{Vol}_X]_c|| = ||(\tau^n)^{-1}(s^n([\text{Vol}_X]_c))|| = \text{Vol}(X) \cdot ||[X]^\mathbb{R}||.
\]

Therefore, by Proposition 8.1 we get \( ||X|| = 1/||[X]^\mathbb{R}|| = \text{Vol}(X)/||[\text{Vol}_X]_c||\). \( \square \)

From now on, we denote by \( \tilde{X} \) the Riemannian universal covering of \( X \). By Corollary 8.4, the proportionality principle can be restated as follows:
Theorem 8.5. The value $|||\text{Vol}_X|||$ only depends on the isometry type of $\tilde{X}$.

Thus our efforts will be henceforth devoted to proving Theorem 8.5.

8.3. The transfer map. From now on, we denote by $G$ the group of orientation-preserving isometries of $\tilde{X}$, and by $\Gamma \cong \pi_1(X) < G$ the group of covering automorphisms of $\tilde{X}$. It is well-known that $G$ admits a Lie group structure inducing the compact–open topology. Moreover, there exists on $G$ a left-invariant regular Borel measure $\mu_G$, which is called Haar measure of $G$ and is unique up to scalars. Since $G$ contains a cocompact subgroup, its Haar measure is in fact also right-invariant [Sau02 Lemma 2.32]. Since $X \cong \tilde{X}/\Gamma$ is compact, there exists a Borel subset $F \subseteq G$ with the following properties: $F$ contains exactly one representative for each class in $\Gamma \backslash G$ and $F$ is relatively compact in $G$. We will call such an $F$ a fundamental region for $\Gamma$ in $G$. From now on, we normalize the Haar measure $\mu_G$ in such a way that $\mu_G(F) = 1$.

In order to avoid too heavy notations, if $H$ is a subgroup of $G$ we set $sH^*(\tilde{X})^H = H^*(sC_c^*(\tilde{X})^H)$. We also endow $sH^*_e(\tilde{X})^H$ with the seminorm induced by $sC_c^*(\tilde{X})^H$.

Recall that by Lemma [Sau04] we have an isometric isomorphism $sH^*_e(\tilde{X})^\Gamma \cong sH^*_e(X)$. The chain inclusion $sC_c^*(\tilde{X})^G \rightarrow sC_c^*(\tilde{X})^\Gamma$ induces a norm–decreasing map 

$$\text{res}^*: sH^*_b(\tilde{X})^G \rightarrow sH^*_b(\tilde{X})^\Gamma \cong sH^*_e(X).$$

Following [BK08], we will now construct a norm–decreasing left inverse of $\text{res}$. We begin with the following:

Lemma 8.6. Let $s_0 \in sS_s(\tilde{X})$ be fixed. Then the map $G \rightarrow sS_s(\tilde{X})$ defined by $g \mapsto g \cdot s_0 = g \circ s_0$ is continuous.

Proof: Let us consider $G$ as a subset of the space $F_s(\tilde{X}, \tilde{X})$ of smooth functions from $\tilde{X}$ to itself. Since the elements of $G$ are isometries, the compact–open and the $C^1$–topology coincide on $G$ (see e.g. [Loh04 Theorem 5.12]). Therefore the conclusion follows from Lemma [BK08] \qed

Take now $\varphi \in sC_c^*(\tilde{X})$ and $s \in sC_c(\tilde{X})$, and consider the function $f^s_\varphi: G \rightarrow \mathbb{R}$ defined by $f^s_\varphi(g) = \varphi(g \cdot s)$. By Lemma 8.6, $f^s_\varphi$ is continuous, whence bounded on the relatively compact subset $F \subseteq G$. Therefore a well-defined cochain $\text{trans}^s(\varphi) \in sC^s(\tilde{X})$ exists such that for every $s \in sS_s(\tilde{X})$ we have

$$\text{trans}^s(\varphi)(s) = \int_{\tilde{X}} f^s_\varphi(g) \, d\mu_G(g) = \int_{\tilde{X}} \varphi(g \cdot s) \, d\mu_G(g).$$

Proposition 8.7. The cochain $\text{trans}^s(\varphi)$ is continuous. Moreover, if $\varphi$ is $\Gamma$–invariant, then $\text{trans}^s(\varphi)$ is $G$–invariant, while if $\varphi$ is $G$–invariant, then $\text{trans}^s(\varphi) = \varphi$.

Proof: Let us define a distance $d_S$ on $sS_s(\tilde{X})$ as follows. It is well-known that the Riemannian structure on $\tilde{X}$ induces a distance $d_{T\tilde{X}}$ on the tangent bundle $T\tilde{X}$. 

\begin{align*}
\text{trans}^s(\varphi)(s) &= \int_{\tilde{X}} f^s_\varphi(g) \, d\mu_G(g) \\
&= \int_{\tilde{X}} \varphi(g \cdot s) \, d\mu_G(g) \\
&= \int_{\tilde{X}} \varphi(g) \, d\mu_G(g) \\
&= \varphi.
\end{align*}
Moreover, \( d_{\tilde{T}X} \) is \( G \)-invariant, in the sense that for every \( g \in G \) the differential \( dg: T\tilde{X} \to T\tilde{X} \) acts as an isometry of \((T\tilde{X}, d_{\tilde{T}X})\). For every \( s, s' \in sS_t(\tilde{X}) \) we now set
\[
d_S(s, s') = \sup_{x \in T\Delta_t} d_{\tilde{T}X}(ds(x), ds'(x)).
\]
It is well-known that \( d_S \) induces on \( sS_t(\tilde{X}) \) the \( C^1 \)-topology.

Let now \( s_0 \in sS_t(\tilde{X}) \) and \( \varepsilon > 0 \) be fixed. By Lemma 8.6 the set \( F \cdot s_0 \subseteq sS_t(\tilde{X}) \) is compact. Since \( \varphi \) is continuous, this easily implies that there exists \( \eta > 0 \) such that \( |\varphi(s_1) - \varphi(s_2)| \leq \varepsilon \) for every \( s_1 \in F \cdot s_0, s_2 \in B_{d_S}(s_1, \eta) \), where \( B_{d_S}(s_1, \eta) \) is the open ball of radius \( \eta \) centered at \( s_1 \). Take now \( s \in B_{d_S}(s_0, \eta) \). Since \( G \) acts isometrically on \( sS_t(X) \), for every \( g \in F \) we have \( d_S(g \cdot s_0, g \cdot s) = d_S(s_0, s) < \eta \), so \( |\varphi(g \cdot s) - \varphi(g \cdot s_0)| \leq \varepsilon \). Together with the fact that \( \mu_G(F) = 1 \), this readily implies
\[
|\text{trans}^i(\varphi)(s) - \text{trans}^i(\varphi)(s_0)| \leq \int_{\tilde{X}} |\varphi(g \cdot s) - \varphi(g \cdot s_0)| \, d\mu_G(g) \leq \varepsilon.
\]
We have thus proved that \( \text{trans}^i(\varphi) \) is continuous.

Now, if \( \varphi \) is \( G \)-invariant then by the very definition we have \( \text{trans}^i(\varphi) = \varphi \). The fact that \( \text{trans}^i(\varphi) \) is \( G \)-invariant if \( \varphi \) is \( \Gamma \)-invariant follows from the very same computations described in [BK08, Subsection 6.3].

Proposition 8.7 provides a well-defined map \( \text{trans}^*: sC_c^\ast(\tilde{X})^\Gamma \to sC_c^\ast(\tilde{X})^G \). It is readily seen that \( \text{trans}^* \) is a chain map, and we still denote by \( \text{trans}^*: sH_c^\ast(\tilde{X})^\Gamma \to sH_c^\ast(\tilde{X})^G \) the resulting map in cohomology. Since \( \text{trans}^* \) restricts to the identity on \( G \)-invariant cochains, we have the following commutative diagram:

\[
\begin{array}{ccc}
\text{Id} & \longrightarrow & sH_c^\ast(\tilde{X})^\Gamma \\
\text{res}^* \downarrow & & \downarrow \text{trans}^* \\
\text{trans}^* \downarrow & & \downarrow \\
sH_c^\ast(X) & \longrightarrow & sH_c^\ast(\tilde{X})^G
\end{array}
\]

where the vertical row describes the isomorphism provided by Lemma 7.4. Since \( \text{trans}^* \) is obviously norm-decreasing, we get the following:

**Proposition 8.8.** The map \( \text{res}^*: sH_c^\ast(\tilde{X})^G \to sH_c^\ast(\tilde{X})^\Gamma \) is an isometric embedding.

8.4. **Proof of Theorem 8.5.** Since \( G \) acts on \( \tilde{X} \) via orientation-preserving isometries, the cochain \( \text{Vol}_{\tilde{X}} \in sC_c^\ast(\tilde{X}) \) is of course \( G \)-invariant, whence \( \Gamma \)-invariant. Let us denote by \( [\text{Vol}_{\tilde{X}}]^G \) (resp. [\( [\text{Vol}_{\tilde{X}}]^\Gamma \]) the cohomology class in \( sH_c^\ast(\tilde{X})^\Gamma \) (resp. in \( sH_c^\ast(\tilde{X})^G \)) represented by \( \text{Vol}_{\tilde{X}} \). Of course \( \text{res}^n([\text{Vol}_{\tilde{X}}]^G) = [\text{Vol}_{\tilde{X}}]^\Gamma \), while since
p: \tilde{X} \to X is a local isometry we have \(p^*(\text{Vol}_X) = \text{Vol}_{\tilde{X}}\). Therefore Proposition 8.8 and Lemma 7.4 imply
\[
\|[\text{Vol}_X]_c|| = \|[\text{Vol}_{\tilde{X}}]_c|| = \|[\text{res}^n([\text{Vol}_{\tilde{X}}]_c^G)]|| = \|[\text{Vol}_{\tilde{X}}]_c^G||,
\]
whence Theorem 8.5, since \(\|[\text{Vol}_{\tilde{X}}]_c^G||\) only depends on the isometry type of \(\tilde{X}\).

**APPENDIX A. COMPACT–OPEN AND C\(^1\)–TOPOLOGY**

**A.1. Compact–open topology.** Recall that if \(X, Y\) are topological spaces, the compact–open topology on the space \(F(X, Y) = \{f: X \to Y, \ f \text{ continuous}\}\) admits as a subbasis the set
\[
\{\Omega(K, U), \ K \subset X \text{ compact}, \ U \subset Y \text{ open}\},
\]
where
\[
\Omega(K, U) = \{f \in F(X, Y) \mid f(K) \subseteq U\}.
\]
In this subsection, all the function spaces involved are endowed with the compact–open topology. The following result is proved in [Dug66, page 259]:

**Lemma A.1.** Let \(X, Y, Z\) be topological spaces, and \(f: Y \to Z, g: X \to Y\) be continuous. The maps \(f_*: F(X, Y) \to F(X, Z), g^*: F(Y, Z) \to F(X, Z)\) defined by \(f_*(h) = f \circ h, g^*(h) = h \circ g\) are continuous.

**Lemma A.2.** Suppose \(X\) is compact and Hausdorff, let \(C \subseteq X\) be closed and set \(F_C(X, Y) = \{h \in F(X, Y) \mid h|_C\ \text{is constant}\}\). Let \(\pi: X \to X/C\) be the canonical projection, and for \(h \in F_C(X, Y)\), let \(\psi(h) \in F(X/C, Y)\) be the unique map such that \(\psi(h) \circ \pi = h\). Then \(\psi: F_C(X, Y) \to F(X/C, Y)\) is well-defined and continuous.

**Proof:** The fact that \(\psi\) is well-defined is an immediate consequence of the definition of quotient topology. Moreover, if \(K \subseteq X/C\) is compact and \(U \subseteq Y\) is open, then \(\psi^{-1}(\Omega(K, U)) = \Omega(\pi^{-1}(K), U)\), which is open: in fact, since \(X\) is compact Hausdorff and \(C\) is closed, \(X/C\) is Hausdorff, so \(K\) is closed, and \(\pi^{-1}(K)\) is closed in a compact space, whence compact.

As in the proof of Proposition 1.9 let now \(e_0^n, \ldots, e_n^n\) be the vertices of the standard simplex \(\Delta_n\), let \(Q^n_0\) be the face of \(\Delta_n\) opposite to \(e_0^n\), and let \(r_n: Q^n_0 \to \Delta_n\) be defined by \(r_n(t_1 e_1^{n+1} + \ldots + t_{n+1} e_{n+1}^{n+1}) = t_1 e_0^n + \ldots t_{n+1} e_{n+1}^n\).

**Lemma A.3.** Let \(X, Y, Z\) be topological spaces, let \(x_0 \in X\) be fixed and suppose \(H: X \times [0, 1] \to X\) is a continuous map such that \(H(x, 1) = x_0\) for every \(x \in X\).

1. The map \((h \times \text{Id}): F(X, Y) \to F(X \times [0, 1], Y \times [0, 1])\) is continuous (where products are endowed with the product topology).
2. Let \(T_n: F(\Delta_n, X) \to F(\Delta_{n+1}, X)\) be defined as follows: if \(s \in F(\Delta_n, X)\) and \(p = t e_0^{n+1} + (1 - t) q \in \Delta_{n+1}\), then
\[
(T_n(s))(p) = H(s(r_n(q)), t).
\]
Then \(T_n\) is well-defined and continuous.
PROOF: (1): By [Dug66, page 264], the compact–open topology of $F(X \times [0,1], Y \times [0,1])$ admits as a subbasis the set
\[
B = \{ \Omega(K, U \times U') | K \subseteq X \times [0,1] \text{ compact}, U \subseteq Y \text{ open}, U' \subseteq [0,1] \text{ open} \}. 
\]
Now if $\pi_1: X \times [0,1] \to X$, $\pi_2: X \times [0,1] \to [0,1]$ are the natural projections and $\Omega(K, U \times U') \in B$, then $(h \times Id)^{-1}(\Omega(K, U \times U')) = \Omega(\pi_1(K), U)$ if $\pi_2(K) \subseteq U'$, and $(h \times Id)^{-1}(\Omega(K, U \times U')) = 0$ otherwise. In any case, $(h \times Id)^{-1}(\Omega(K, U \times U'))$ is open in $F(X,Y)$.

(2): Since $r^{-1}_n: \Delta_n \to Q_0^{n+1} \subseteq \Delta_{n+1}$ is a homeomorphism, the map $\varphi: \Delta_n \times [0,1] \to \Delta_{n+1}$ defined by $\varphi(q,t) = (1-t)r^{-1}_n(q)+te^{n+1}_0$ is well-defined and continuous. Moreover, we have $\varphi(q,1) = \varphi(q',1)$ for every $q, q' \in \Delta_n$, so $\varphi$ induces a map $\overline{\varphi}: (\Delta_n \times [0,1])/(\Delta_n \times \{1\}) \to \Delta_{n+1}$ which is easily shown to be bijective. Moreover, $\overline{\varphi}$ is continuous by the very definition of quotient topology, and is closed since it is defined on a compact space with values in a Hausdorff space. Thus, $\overline{\varphi}$ is a homeomorphism. Now, if $s \in F(\Delta_n, X)$, since $H(q,1) = x_0$ for every $q \in X$, the map $H \circ (s \times Id): \Delta_n \times [0,1] \to X$ defines a continuous map $\overline{H} \circ (s \times Id): (\Delta_n \times [0,1])/(\Delta_n \times \{1\}) \to X$, and by construction we have $T_n(s) = \overline{H} \circ (s \times Id) \circ \overline{\varphi}^{-1}$. This shows that $T_n(s)$ is indeed well-defined and continuous.

Let us show now that $T_n(s)$ continuously depends on $s$. Since $\overline{\varphi}^{-1}$ is a homeomorphism, it is sufficient to show that the map $\overline{H} \circ (s \times Id)$ continuously depends on $s$. But this is a consequence of Lemma [A.1] Lemma [A.2] and point (1). □

Let now $p: \tilde{X} \to X$ be a covering, and denote by $p_*: C_n(\tilde{X}) \to C_n(X)$ the induced map on singular chains. The following result was proved in [Loh06] under the hypothesis that $\tilde{X}$ is metrizable.

Lemma A.4. The restriction $p_*|_{S_n(\tilde{X})}: S_n(\tilde{X}) \to S_n(X)$ is a covering map.

PROOF: Let $s_0 \in S_n(\tilde{X})$ be a simplex, and set $s_0(e_0) = x_0$. Since $p$ is a covering, there exists an open neighbourhood $U_0$ of $x_0 \in X$ such that $p^{-1}(U_0) = \bigsqcup_{j \in J} \tilde{U}_j^0$, $\tilde{U}_j^0$ is open in $\tilde{X}$ and $p|_{\tilde{U}_j^0}: \tilde{U}_j^0 \to U_0$ is a homeomorphism for every $j \in J$.

We set $V_0 = \{ s \in S_n(X) | s(e_0) \in U_0 \}$, $\tilde{V}_j^0 = \{ \tilde{s} \in S_n(\tilde{X}) | \tilde{s}(e_0) \in \tilde{U}_j^0 \}$. Of course, $V_0$ and $\tilde{V}_j^0$ are open subsets of $S_n(X)$ and $S_n(\tilde{X})$ respectively. Moreover, since the standard simplex is path connected and simply connected, for every $j \in J$ and every simplex in $s \in V_0$ there exists a unique lift $\tilde{s}_j^0 \in \tilde{V}_j^0$. This readily implies that $p^{-1}_s(V_0) = \bigsqcup_{j \in J} \tilde{V}_j^0$, and that $p_*|_{\tilde{V}_j^0}: \tilde{V}_j^0 \to V_0$ is bijective for every $j \in J$. Moreover, by Lemma [A.1] the map $p_*|_{S_n(\tilde{X})}$ is continuous, so in order to conclude we are only left to show that $p_*|_{\tilde{V}_j^0}: \tilde{V}_j^0 \to V_0$ is open for every $j \in J$.

We fix $j \in J$, and denote $\tilde{V}_j^0$ (resp. $\tilde{U}_j^0$) simply by $\tilde{V}_j^0$ (resp. $\tilde{U}_j^0$). Since $p_*|_{\tilde{V}_j^0}$ is injective, it preserves unions and intersections, thus it is sufficient to prove that if
\( s \in \tilde{V}_0 \) is a simplex and \( s \in \Omega(K, Y) \), where \( K \subseteq \Delta_n \) is compact and \( Y \subseteq \tilde{X} \) is open, then \( p_* (\Omega(K, Y) \cap \tilde{V}_0) \) is a neighbourhood of \( s = p_*(\tilde{s}) \). Since \( \tilde{s}(\Delta_n) \) is compact, there exists a finite open cover \( \{ \tilde{Z}_i \}_{i=0}^l \) of \( \tilde{s}(\Delta_n) \) such that \( \tilde{Z}_i \) homeomorphically projects onto an open subset \( Z_i \subseteq X \) for every \( i \). Moreover, it is easily seen that there exists a decomposition \( \Delta_n = \bigcup_{i=0}^N \Delta^i \) with the following properties:

1. each \( \Delta^i \) is closed in \( \Delta_n \);
2. \( e_0 \in \Delta^i \) if and only if \( i = 0 \);
3. \( \tilde{s}(\Delta^0) \subseteq \tilde{U}_0 \);
4. for every \( i = 0, \ldots, N \) there exists \( j_i \in \{0, \ldots, l \} \) such that \( \tilde{s}(\Delta^i) \subseteq \tilde{Z}_{j_i} \).

Let \( I = \{0, \ldots, N\} \). If \( L \subseteq I \), we set \( \Delta^L = \bigcap_{i \in L} \Delta^i \) and \( \tilde{Z}_L = \bigcap_{i \in L} \tilde{Z}_{j_i} \). We also set

\[
\tilde{H} = \bigcap_{L \subseteq I} \Omega(\Delta^L, \tilde{Z}_L), \quad \tilde{R} = \bigcup_{L \subseteq I} \Omega(K \cap \Delta^L, Y \cap \tilde{Z}_L), \quad \tilde{W}_0 = \Omega(\Delta^0, \tilde{U}_0 \cap \tilde{Z}_{j_0}),
\]

and

\[
H = \bigcap_{L \subseteq I} \Omega(\Delta^L, p(\tilde{Z}_L)), \quad R = \bigcup_{L \subseteq I} \Omega(K \cap \Delta^L, p(Y \cap \tilde{Z}_L)), \quad W_0 = \Omega(\Delta^0, p(\tilde{U}_0 \cap \tilde{Z}_{j_0})).
\]

Note that since \( p \) is open, the sets \( \tilde{H}, \tilde{R}, \tilde{W}_0, H, R, W_0 \) are open. Moreover, by construction we have \( \tilde{s} \in \tilde{W}_0 \cap \tilde{H} \cap \tilde{R} \subseteq \tilde{V}_0 \cap \Omega(K, Y) \) and \( s \in W_0 \cap H \cap R \). Thus in order to conclude we only need to prove that

\[
p_*(\tilde{W}_0 \cap \tilde{H} \cap \tilde{R}) = W_0 \cap H \cap R.
\]

The inclusion \( \subseteq \) is obvious. Let \( s_1 \in W_0 \cap H \cap R \), and let \( \tilde{s}_1 \in \tilde{V}_0 \) be the unique lift of \( s_1 \) whose first vertex lies in \( \tilde{U}_0 \).

We now try to reconstruct \( \tilde{s}_1 \) as explicitly as possible. For every \( i \in I \), we denote by \( r_i : Z_{j_i} \to \tilde{Z}_{j_i} \) the local inverse of \( p \). For every \( q \in \Delta_n \), let \( I(q) = \{i \in I \mid q \in \Delta^i\} \). We claim that for every \( q \in \Delta_n \) and \( i, i' \in I(q) \) we have \( r_i(s_1(q)) = r_{i'}(s_1(q)) \): in fact, since \( s_1 \in H \) we have \( s_1(q) \in p(\tilde{Z}_{I(q)}) \), so a point \( a \in \tilde{Z}_{j_i} \cap \tilde{Z}_{j_{i'}} \) exists such that \( p(a) = s_1(q) \). But \( p(r_i(s_1(q))) = p(r_{i'}(s_1(q))) = s_1(q) \), and \( p|_{\tilde{Z}_{j_i}} : p|_{\tilde{Z}_{j_{i'}}} \) are injective, so \( r_i(s_1(q)) = a = r_{i'}(s_1(q)) \).

Therefore, a well defined map \( \tilde{s}_1 : \Delta_n \to \tilde{X} \) exists such that \( \tilde{s}_1|_{\Delta^i} = r_i \circ s_1|_{\Delta^i} \) for every \( i \in I \). Since the \( \Delta^i \)'s provide a finite closed cover of \( \Delta_n \), this map is continuous. Moreover, \( p \circ \tilde{s}_1 = s_1 \), and since \( s_1 \in W_0 \) we have \( \tilde{s}_1(e_0) \in p^{-1}(p(\tilde{U}_0 \cap \tilde{Z}_{j_0})) \). Since by construction \( \tilde{s}_1(e_0) \in \tilde{Z}_{j_0} \) and \( p|_{\tilde{Z}_{j_0}} \) is injective, this implies \( \tilde{s}_1(e_0) \in \tilde{U}_0 \), so \( \tilde{s}_1(e_0) = \tilde{s}_1(e_0) \). This, together with the fact that \( p \circ \tilde{s}_1 = p \circ \tilde{s}_1 \) and the fact that \( \Delta_n \) is path connected, implies that \( \tilde{s}_1 = \tilde{s}_1 \).

We are then left to show that \( \tilde{s}_1 \) indeed belongs to \( \tilde{W}_0 \cap \tilde{H} \cap \tilde{R} \). The fact that \( \tilde{s}_1 \) belongs to \( \tilde{W}_0 \cap \tilde{H} \) is an immediate consequence of our construction. Now, if \( q \in \Delta^L \cap K \), since \( s_1 \in R \) there exists \( a \in Y \cap \tilde{Z}_L \) such that \( p(a) = s_1(q) \). Moreover,
by construction we have \( p(a) = p(\overline{s}_1(q)) \) and \( \overline{s}_1(q) \in \overline{Z}_L \), so \( a = s_1(q) \) since \( p|_{\overline{Z}_L} \) is injective. Therefore we have \( \overline{s}_1(q) = a \in Y \cap \overline{Z}_L \), and this proves that \( \overline{s}_1 \) belongs to \( \overline{R} \), whence the conclusion. \( \square \)

A.2. \( C^1 \)-topology on maps between smooth manifolds. If \( X,Y \) are smooth manifolds, we denote by \( F_s(X,Y) \) the set of smooth maps from \( X \) to \( Y \). If \( TX,TY \) are the tangent bundles of \( X,Y \) respectively, the usual differential defines a map \( d: F_s(X,Y) \to F_s(TX,TY) \), which is obviously injective. The \( C^1 \)-topology on \( F_s(X,Y) \) is the pull-back via \( d \) of the compact–open topology of \( F_s(TX,TY) \).

Putting together Lemma A.1 and the fact that the differential of the composition of smooth maps is the composition of their differentials it is not difficult to get the following:

**Lemma A.5.** Let \( X,Y,Z \) be smooth manifolds, let \( f: Y \to Z \), \( g: X \to Y \) be smooth, and endow \( F_s(X,Y), F_t(Y,Z), F_s(X,Z) \) with the \( C^1 \)-topology. The maps \( f_*: F(X,Y) \to F(X,Z), g^*: F(Y,Z) \to F(X,Z) \) defined by \( f_*(h) = f \circ h, g^*(h) = h \circ g \) are continuous.

In particular, if \( h: X \to X \) is a smooth map and \( \varphi \in \_sC^*_c(X) \), then \( h^*(\varphi) \in \_sC^*_c(X) \). Therefore, continuous smooth cohomology is a functor from the category of manifolds and smooth maps to the category of graded \( \mathbb{R} \)-vector spaces and linear maps. Moreover, using Lemma A.5 and arguing just as in the proof of Proposition 2.4 it is not difficult to prove the following:

**Lemma A.6.** Let \( f,g: X \to Y \) be smooth maps between smooth manifolds, and let \( f^*\_c, g^*\_c: \_sH^*_c(Y) \to \_sH^*_c(X) \) be the induced maps in smooth cohomology. If \( f \) is smoothly homotopic to \( g \), then \( f^*\_c = g^*\_c \).

Moreover, if \( G \) acts on \( X \) as a group of diffeomorphisms, then it makes sense to define \( G \)-invariant continuous smooth cochains on \( X \).

Let now \( X,\overline{X} \) be smooth manifolds, and suppose \( p: \overline{X} \to X \) is a smooth covering (i.e. a covering which is also a local diffeomorphism). We endow \( \_sS_n(X), \_sS_n(\overline{X}) \) with the \( C^1 \)-topology.

**Lemma A.7.** The map \( p_+: \_sS_n(\overline{X}) \to \_sS_n(X) \) is a covering map.

**PROOF:** If \( X,Y \) are smooth manifolds, we say that \( g: TX \to TY \) is integrable if there exists a smooth \( f: X \to Y \) such that \( g = df \). Of course, a smooth map is integrable if and only if it is locally integrable. Let us denote by \( (dp)_*: F_s(T\Delta_n,TX) \to F_s(T\Delta_n,T\overline{X}) \) the composition with \( dp \). We have the commutative diagram

\[
\begin{array}{ccc}
F_s(\Delta_n,\overline{X}) & \xrightarrow{d} & F_s(T\Delta_n,T\overline{X}) \\
\downarrow p_* & & \downarrow (dp)_* \\
F_s(\Delta_n,X) & \xrightarrow{d} & F_s(T\Delta_n,TX)
\end{array}
\]
It is easily seen that, since \( p \) is a smooth covering, \( dp: T\tilde{X} \to TX \) is also a smooth covering. Since \( T\Delta_n \) is simply connected, a slight modification of the proof of Lemma [Δ.4] shows that \((dp)_*\) is a covering, where we are endowing \( F_s(T\Delta_n,T\tilde{X}) \), \( F_s(T\Delta_n, TX) \) with the compact–open topology. Moreover, it is readily seen that \( f \in F_s(T\Delta_n,T\tilde{X}) \) is integrable if and only if \((dp)_*(f)\) is integrable. Since the subset of integrable maps in \( F_s(T\Delta_n,T\tilde{X}) \) (resp. in \( F_s(T\Delta_n, TX) \)) coincide by definition with \( d(F_s(\Delta_n,\tilde{X})) \) (resp. with \( d(F_s(\Delta_n, X)) \)), \((dp)_*\) restricts to a covering \( d(F_s(\Delta_n,\tilde{X})) \to d(F_s(\Delta_n, X)) \). The conclusion follows since the horizontal rows of the diagram are by definition homeomorphisms on their images. \( \square \)

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