Ordered Orthogonal Array Construction Using LFSR Sequences

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Abstract—We present a new construction of ordered orthogonal arrays (OOA) of strength \( t \) with \((q + 1)t\) columns over a finite field \( \mathbb{F}_q \) using linear feedback shift register sequences (LFSRs). OOAs are naturally related to \((t, m, s)\)-nets, linear codes, and MDS codes. Our construction selects suitable columns from the array formed by all subintervals of length \( \frac{t}{q-1} \) of an LFSR sequence generated by a primitive polynomial of degree \( t \) over \( \mathbb{F}_q \). We prove properties about the relative positions of runs in an LFSR which guarantee that the constructed OOA has strength \( t \). The set of parameters of our OOAs are the same as the ones given by Rosenbloom and Tsfasman (1997) and Skriganov (2002), but the constructed arrays are different. We experimentally verify that our OOAs are stronger than the Rosenbloom-Skriganov OOA in the sense that ours are "closer" to being a "full" orthogonal array. We also discuss how our OOA construction relates to previous techniques to build OOAs from a set of linearly independent vectors over \( \mathbb{F}_q \), as well as to hypergraph homomorphisms.

Index Terms—Ordered orthogonal arrays, linear feedback shift registers, runs in LFSR sequences, hypergraph homomorphisms.

I. INTRODUCTION

ORDERED orthogonal arrays (OOA) are a generalization of orthogonal arrays introduced independently by Lawrence [8] and Mullen and Schmid [14]. A survey of constructions of ordered orthogonal arrays is in [9, Chapter 3]; we also refer to [2, Section VI.59.3].

Let \( t, m, s, v, \lambda \) be positive integers such that \( 2 \leq t \leq ms \), and let \( N = \lambda^t \). Let \( A \) be an \( N \times ms \) array over an alphabet \( V \) of size \( v \). An \( N \times t \) subarray of \( A \) is \( \lambda \)-covered if it has each \( t \)-tuple over \( V \) as a row exactly \( \lambda \) times. A set of \( t \) columns is \( \lambda \)-covered if the \( N \times t \) array formed by them is \( \lambda \)-covered. If \( \lambda = 1 \), we simply say an \( N \times t \) subarray or a set of \( t \) columns is covered. Being \( \lambda \)-covered is often referred to as having the OA property.

Let \( A \) be an array with \( ms \) columns labeled by \( [m] \times [s] = \{(i, j) : 1 \leq i \leq m, 1 \leq j \leq s \} \). A subset \( L \) of columns is left-justified if \( (i, j) \in L \) with \( j > 1 \) implies \( (i, j-1) \in L \). An ordered orthogonal array \( OOA_{\lambda}(N; t, m, s, v) \) is an \( N \times ms \) array \( A \) with columns labeled by ordered pairs \((i, j) \in [m] \times [s] \) and with elements from an alphabet \( V \) of size \( v \), with the property that every left-justified set \( L \) of \( t \) columns of \( A \) is \( \lambda \)-covered. The parameter \( t \) is known as the strength of the OOA. Since the parameter \( N \) is determined by the other parameters, we sometimes write \( OOA_{\lambda}(t, m, s, v) \). If \( \lambda = 1 \), we just write \( OOA(t, m, s, v) \). When \( s = 1 \), an \( OOA_{\lambda}(N; t, m, 1) \) is the well-known orthogonal array \( OA_{\lambda}(N; t, m) \).

For example, Fig. 1 shows a binary ordered orthogonal array of strength 3. The columns of the \( OOA(3, 3, 2, 2) \) in Fig. 1 are labeled by \([3] \times [2] = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (3, 2)\} \). There are seven left-justified sets of size 3, namely\[\{(1, 1), (1, 2), (2, 1)\}, \{(1, 1), (1, 2), (3, 1)\}, \{(1, 1), (2, 1), (2, 2)\}, \{(1, 1), (2, 1), (3, 1)\}, \{(1, 1), (3, 1), (3, 2)\}, \{(2, 1), (2, 2), (3, 1)\}, \{(2, 1), (3, 1), (3, 2)\}.\]

The \( 8 \times 3 \) subarray given by each of them is covered.

Ordered orthogonal arrays are related to the Niederreiter-Rosenbloom-Tsfasman metric and \((t, m, s)\)-nets in base \( b \) (see the definition of \((t, m, s)\)-nets in Section VII). Rosenbloom and Tsfasman [23] introduced a metric on linear spaces over finite fields and discussed possible applications of this metric to interference in parallel channels of communication systems. This metric is commonly known as the Niederreiter-Rosenbloom-Tsfasman (NRT) metric. Rosenbloom and Tsfasman [24] and Skriganov [26] constructed a class of maximum distance separable (MDS) codes over this metric. For \( q \) prime power and \( s \leq t \), they show that there exists an MDS code with respect to the NRT metric with length \((q+1)s\), dimension \( t \), and minimum distance \((q+1)s-t+1\). This class of MDS codes is known as Reed-Solomon \( s \)-codes and they are equivalent to an \( OOA(t, q+1, t, q) \). In this paper we provide a new construction of OOAs with these parameters.

Niederreiter [17] introduced \((t, m, s)\)-nets in base \( b \) which we define in Section VII; several applications of these objects to numerical integration (quasi-Monte Carlo methods) can be found in [12]. Ordered orthogonal arrays are a combinatorial characterization of \((t, m, s)\)-nets. Lawrence [8] and Mullen and Schmid [14] show that there exists a \((t, m, s)\)-net in base \( b \) if and only if there exists an \( OOA_{b^t}(m-t, s, m-t, b) \). An \( OOA(t, q+1, t, q) \) corresponds to a \((0, t, q+1)\)-net in base \( q \). The OOA constructed in this paper is a linear OOA, that is, the rows of the \( OOA(t, q+1, t, q) \) form a subspace of \( \mathbb{F}_q^{t+1} \). The \((t, m, s)\)-nets corresponding...
to linear OOAs are known as digital nets, see for instance [2, Section VI.59].

Our OOAs differ from all previous constructions of ordered orthogonal arrays since they are derived from linear feedback shift register (LFSR) sequences. We review LFSR concepts in Section II. Munemasa [16] first used LFSR sequences to construct classical orthogonal arrays; see also [3], [20]. This was later extended to covering arrays in [22], [26]. A survey of finite field constructions of combinatorial arrays is in [12].

In order to construct OOAs, we need some properties of maximum period LFSR sequences that to the best of our knowledge are identified for the first time in this paper; see Section II. Our main result in this section is Theorem 10 which relates runs of zeroes in the sequence to roots of a certain polynomial over \( F_q \). In Section VI we give further properties of maximum period LFSR sequences. Theorem 10 is crucial in our construction which is given in Section VI. The main result of this paper is Theorem 16 where we construct an OOA\((t, q + 1, t, q)\) using LFSR sequences. In Section VI we show experimentally that our method to construct OOAs covers substantially more \( t \)-sets of columns than both Rosenbloom and Tsfasman [24] and Skriagov [23] methods. This also shows that our construction is new and intrinsically different from their construction. In Section VII we relate our construction to other combinatorial structures. In Subsection VII-A we discuss sets of linearly independent vectors over \( F_q \), and OOA constructions that utilize linear independence and coding theory. In Subsection VII-B we consider hypergraphs and homomorphisms to construct OOAs, showing that this is a framework that includes many previous OOA constructions and non-existence proofs. In this framework, ours is the first construction for general strength \( t \) that satisfies a non-triviality condition asked for by Martin [11].

II. Preliminaries on LFSRs

We present concepts and results on finite fields and linear feedback shift register sequences that are needed to construct ordered orthogonal arrays of strength \( t \) in the subsequent sections.

Let \( q \) be a prime power, \( t \geq 1 \), and let \( F_q \) be the finite field of \( q \) elements. Let \( F_q^* \) be the multiplicative group of \( F_q \). If \( \alpha \in F_q^* \) generates \( F_q^* \), \( \alpha \) is a primitive element of \( F_q^* \). A polynomial \( f \in F_q[x] \) of degree \( t \geq 1 \) is a primitive polynomial over \( F_q \) if \( f \) is the minimal polynomial over \( F_q \) of a primitive element of \( F_q^* \).

| (1,1) | (1,2) | (2,1) | (2,2) | (3,1) | (3,2) |
|-------|-------|-------|-------|-------|-------|
| 0     | 0     | 1     | 1     | 1     | 1     |
| 1     | 0     | 1     | 1     | 0     | 0     |
| 1     | 0     | 1     | 0     | 0     | 1     |
| 0     | 1     | 0     | 0     | 0     | 1     |
| 1     | 0     | 0     | 0     | 0     | 0     |

Fig. 1. A binary ordered orthogonal array of strength 3.

A linear feedback shift register sequence, henceforth LFSR sequence, with characteristic polynomial \( f(x) = c_0 + c_1x + \cdots + c_{t-1}x^{t-1} + x^t \in F_q[x] \) and initial values \( T = (b_0, \ldots, b_{t-1}) \in F_q^t \) is a sequence \( S(f, T) = (a_i)_{i \geq 0} \) over \( F_q \) defined as

\[
a_i = \begin{cases} 
  b_t & \text{if } 0 \leq i < t, \\
  -\sum_{j=0}^{t-1} c_j a_{i-t+j} & \text{if } i \geq t.
\end{cases}
\]

A sequence \((a_i)_{i \geq 0}\) is periodic if there exists an integer \( r > 0 \) such that \( a_{i+r} = a_i \) for all \( i \geq 0 \); the smallest such \( r \) is the least period (or simply period of the sequence). It is well known (see for example [10, Theorem 8.33]) that if \( f \) is a primitive polynomial of degree \( t \) over \( F_q \), then the LFSR sequence generated by \( f \) and nonzero initial values \( T = (b_0, \ldots, b_{t-1}) \) has maximum period \( q^t - 1 \). An LFSR sequence with maximum period is an \( m \)-sequence.

An important tool when working with LFSRs is the trace function from the extension field \( F_{q^t} \) to the field \( F_q \). The trace function is defined by

\[
\begin{align*}
\text{Tr} : & \ F_{q^t} \to F_q \\
& x \mapsto x + x^q + x^{q^2} + \cdots + x^{q^{t-1}}.
\end{align*}
\]

This function is \( F_q \)-linear; see for example [10, Theorem 2.23].

The trace function provides a one-to-one correspondence between initial values \( T \in F_q^t \) of an LFSR sequence and elements of \( F_{q^t} \), as shown next.

**Proposition 1:** [10, Theorem 8.21] Let \( f \) be a primitive polynomial of degree \( t \) over \( F_q \) and \( \alpha \in F_{q^t} \) a root of \( f \). For any initial values \( T = (b_0, \ldots, b_{t-1}) \), there exists a unique element \( \gamma \in F_{q^t} \) such that \( b_t = \text{Tr}(\gamma \alpha^i) \) for all \( 0 \leq i < t \). Moreover, the LFSR \((a_i)_{i \geq 0}\) generated by \( f \) and \( T \) has the property that for all \( i \geq 0 \), \( a_i = \text{Tr}(\gamma \alpha^i) \).

For a positive integer \( l \) and a sequence \( S = (a_i)_{i \geq 0} \), we define

\[
C_l^1(S) = (a_i, a_{i+1}, \ldots, a_{i+l-1})
\]

to be the subinterval of \( S \) of length \( l \) beginning at position \( i \).

Let \( \delta \in F_q \). The subinterval \( C_l^1(S) \) is a run of \( \delta \)'s of length \( l \) if \( a_{i+j} = \delta \), \( 0 \leq j < l \), and \( a_{i-1} = \delta \).

A sequence \((a_i)_{i \geq 0}\) is a Golomb's second randomness postulate. Property (5) below is Golomb's fourth randomness postulate.
Proposition 2: \cite{section5.2} Let $f$ be a primitive polynomial of degree $t$ over $\mathbb{F}_q$ and nonzero initial values $T = (b_0, \ldots, b_{t-1}) \in \mathbb{F}_q^t$. In a period of the LFSR sequence $S(f,T)$ the following properties hold:

1. For $1 \leq l \leq t-2$, the runs of every element in $\mathbb{F}_q$ of length $l$ occur $(q-1)^2q^{l-2}$ times.
2. The runs of every nonzero element in $\mathbb{F}_q$ of length $t-l$ occur $q-2$ times.
3. The runs of the zero element in $\mathbb{F}_q$ of length $t-1$ occur $q-1$ times.
4. The run of any nonzero element in $\mathbb{F}_q$ of length $t$ occurs once, and there is no run of zeroes of length $t$.
5. Each nonzero $t$-tuple in $\mathbb{F}_q^t$ appears exactly once as consecutive elements.

The behaviour of the positions of zeroes in any two subintervals of length $k = \frac{q^t-1}{q-1}$ beginning in positions that differ by a multiple of $k$ is presented in the following result.

Proposition 3: \cite{corollary1} Let $k = \frac{q^t-1}{q-1}$. If $f$ is a primitive polynomial of degree $t$ over $\mathbb{F}_q$, then the LFSR sequence generated by $f$ and $T \in \mathbb{F}_q^t$, $T \neq (0, \ldots, 0)$, has the following properties:

1. For any $i \geq 0$, $C_{ik}^l(S(f,T))$ contains exactly $\frac{q^{2t}-1}{q-1}$ zeroes.
2. For any $i \geq 0$, $j \geq 0$, the positions of zeroes in $C_{ik}^l(S(f,T))$ and $C_{ik+jk}^l(S(f,T))$ are identical.

III. NEW PROPERTIES OF LFSRS OF MAXIMUM PERIOD

In this section, we study properties of linear feedback shift register sequences of maximum period. They are used to investigate the relationship between runs of elements in $\mathbb{F}_q$ in the LFSR sequence generated by a primitive polynomial and nonzero initial values. In Section \cite{section5} we apply these results to prove that the arrays constructed in that section are ordered orthogonal arrays of strength $t$.

Proposition 4: Let $S(f,T) = (a_i)_{i \geq 0}$ be the LFSR sequence generated by a primitive polynomial $f$ of degree $t$ over $\mathbb{F}_q$ and nonzero initial values $T \in \mathbb{F}_q^t$. Consider $\alpha \in \mathbb{F}_q^t$, a root of $f$. For each $\beta \in \mathbb{F}_q^t$, let $k_\beta \in \mathbb{Z}_{q^t-1}$ such that $\alpha^{k_\beta}(\alpha - \beta) = 1$. Then, in a period of $S(f,T)$

$$a_{i+1} - \beta a_i = a_{i-k_\beta}$$

for all $i \geq 0$, where the subscripts of $a$’s are taken modulo $q^t-1$.

Proof: By Proposition \cite{section5.2} there exists a unique $\gamma \in \mathbb{F}_q^t$ such that $a_i = \text{Tr}(\gamma \alpha^i)$ for all $i \geq 0$. By the hypotheses, $\alpha - \beta = \alpha^{-k_\beta}$. Since the trace function is $\mathbb{F}_q$-linear it follows, for all $i \geq 0$, that

$$a_{i+1} - \beta a_i = \text{Tr}(\gamma \alpha^{i+1}) - \beta \text{Tr}(\gamma \alpha^i) = \text{Tr}(\gamma \alpha^i(\alpha - \beta)) = \text{Tr}(\gamma \alpha^{-k_\beta}) = a_{i-k_\beta}.$$

The previous result means that the difference $a_{i+1} - \beta a_i$ is determined by counting back $k_\beta$ positions from position $i$ in a period of $S(f,T)$.

Example 5: Let $f(x) = 1 + x + x^4$ be a primitive polynomial over $\mathbb{F}_2$ and $\alpha \in \mathbb{F}_2^t$ be a root of $f$. Consider the LFSR sequence $S(f,0001) = (a_i)_{i \geq 0}$. We take $k_1 = 11 \in \mathbb{Z}_{15}$, since $\alpha^{11}(\alpha - 1) = 1$. By Proposition \cite{section5.2} $a_{i+1} - a_i = a_{i+11}$ which means that the difference between consecutive elements of $S(f,0001)$ is determined by counting back 11 positions from position $i$ in a period of $S(f,0001)$ as shown below

$$0 \quad 0100110101110$$

This property allows us to obtain a run of zeroes in $\mathbb{F}_2$ of length $l$ from any run of length $l+1$ by counting back 11 positions as illustrated below

$$10001011010111100$$

In the same way a run of zeroes of length $l$ is obtained from a run of length $l+1$, the process can be reversed and a run of length $l+1$ is reached by counting forward 11 positions from a run of zeroes of length $l$.

The next proposition shows that this process can be done for any LFSR sequence generated by a primitive polynomial.

Proposition 6: Let $f$ be a primitive polynomial of degree $t$ over $\mathbb{F}_q$ with $\alpha \in \mathbb{F}_q^t$, a root of $f$. Let $S(f,T) = (a_i)_{i \geq 0}$ be the LFSR sequence generated by $f$ and nonzero initial values $T \in \mathbb{F}_q^t$. Consider $\beta \in \mathbb{F}_q^t$ and $k_\beta \in \mathbb{Z}_{q^t-1}$ such that $\alpha^{k_\beta}(\alpha - \beta) = 1$. For $l \in \{1, \ldots, t\}$ and $\delta \in \mathbb{F}_q$, if $C_{ik}^l(S(f,T))$ is a run of $\delta$’s of length $l$, then $C_{ik-k_\beta}^l(S(f,T))$ is a run of $\delta(1-\beta)$’s of length $l-1$.

Proof: Since $C_{ik}^l(S(f,T))$ is a run of $\delta$’s of length $l$ then $a_{n-1} \neq \delta$ and $a_{n+1} \neq \delta$. Proposition \cite{section5.2} yields $a_i = \delta(1-\beta)$ for all $i = n-k_\beta, \ldots, n-k_\beta+1-2$. We claim that $a_{n-1-k_\beta} \neq \delta(1-\beta)$ and $a_{n-1-k_\beta+1} \neq \delta(1-\beta)$. Suppose by contradiction that $a_{n-1-k_\beta} = \delta(1-\beta)$. By Proposition \cite{section5.2} $a_{n-k_\beta} = a_n - \beta a_{n-1}$. If $a_{n-1-k_\beta} = \delta(1-\beta)$ and $a_n = \delta$, we have $a_{n-1} = \delta$, which is a contradiction. A similar argument shows that $a_{n-k_\beta+1} \neq \delta(1-\beta)$.

In other words, in a period of $S(f,T)$, a run of $\delta$’s of length $l$ is turned into a run of $\delta(1-\beta)$’s of length $l-1$ by making multiple scalar differences between consecutive elements. Another way of interpreting the above result is: in a period of an LFSR sequence when we count back $k_\beta$ positions from a run of $\delta$’s of length $l$, we find a run of $\delta(1-\beta)$’s of length $l-1$.

Proposition 7: Let $f$ be a primitive polynomial of degree $t$ over $\mathbb{F}_q$ and $\alpha \in \mathbb{F}_q^t$, a root of $f$. Let $S(f,T) = (a_i)_{i \geq 0}$ be the LFSR sequence generated by $f$ and $T \in \mathbb{F}_q^t$, $T \neq (0, \ldots, 0)$. For each $\beta \in \mathbb{F}_q^t$, let $k_\beta \in \mathbb{Z}_{q^t-1}$ such that $\alpha^{k_\beta}(\alpha - \beta) = 1$. If $C_{ik}^{l-1}(S(f,T))$ is a run of zeroes of length $l - 1$, then $C_{ik}^l(S(f,T))$ is a run of zeroes of length $l - 1$. Moreover, (1) if $l = t$, then $a_n \neq 0$; (2) if $l \in \{2, \ldots, t\}$ and $a_n$ is nonzero (zero) then $a_{n+1}, \ldots, a_{n+l-1}$ are nonzero (zero); (3) if $\beta = 1$, then $C_{ik}^l(S(f,T))$ is a run of $a_n$’s of length $l$.

Proof: Since $C_{ik-k_\beta}^{l-1}(S(f,T))$ is a run of zeroes of length $l-1$, by Proposition \cite{section5.2} we have that $a_{n+1} = \beta a_{n+1}$ for all
\[ i = 1, \ldots, l - 1. \] Therefore, \( a_{n+i} = \beta^i a_n \) for all \( i = 1, \ldots, l - 1 \) and \( C^l_n(S(f, T)) = (a_n, \beta a_n, \beta^2 a_n, \ldots, \beta^{l-1} a_n) \).

(1) Suppose by contradiction that \( a_n = 0 \). The subinterval \( C^l_n(S(f, T)) \) is a run of zeroes of length \( l \), which is a contradiction as such run does not exist by item (4) of Proposition 2.

(2) If \( a_n \neq 0 \) it follows that \( a_{n+i} \neq 0 \) for all \( i = 1, \ldots, l - 1 \).

(3) If \( \beta = 1 \), then \( a_{n+i} = a_n \) for all \( i = 1, \ldots, l - 1 \). We claim that \( a_{n-1} \neq a_n \) and \( a_{n+t} \neq a_n \). Suppose by contradiction that \( a_{n-1} = a_n \). Proposition 4 implies that \( a_{n-k_1-1} = a_n - a_{n-1} = 0 \), which contradicts the hypotheses that \( C^l_{n-k_1}(S(f, T)) \) is a run of zeroes of length \( l = 1 \). Analogously one can prove that \( a_{n+t} \neq a_n \).

Remark 8: Propositions 6 and 7 hold for a period of an LFSR sequence. However, \( k_3 \in \mathbb{Z}_{q'-1} \) satisfying \( \alpha k_3 (\alpha - \beta) = 1 \) can be considered modulo \( k = q^q - 1 \) and Propositions 6 and 7 still hold in a subinterval of \( S(f, T) \) of length \( k \). This is possible because the constant position of the zeroes in subintervals of \( S(f, T) \) of length \( k \), given in Proposition 5.

Let \( f(x) = c_0 + c_1 x + \cdots + c_{l-1} x^{l-1} + x^l \) be a primitive polynomial of degree \( t \) over \( \mathbb{F}_q \) and \( \alpha \in \mathbb{F}_q^* \) be a root of \( f \). Let \( S(f, T) = (a_i)_{i \geq 0} \) be the LFSR sequence generated by \( f \) and \( T = (b_0, \ldots, b_{t-1}) \in \mathbb{F}_q^t \). For each \( \beta \in \mathbb{F}_q^* \), let \( \beta \) be the LFSR sequence generated by \( \beta f, T \).

For each \( \beta \in \mathbb{F}_q^* \), let \( \beta \in \mathbb{Z}_{q'-1} \) satisfying \( \alpha k_3 (\alpha - \beta) = 1 \). Let \( l = \{0, \ldots, t - 3\} \) and \( C^l_n(S(f, T)) \) a run of zeroes of length \( l \). Then

\[
\begin{align*}
a_{n+k_3+j} &= \begin{cases} 
\beta a_{n+k_3} & \text{if } j = 1, \ldots, l, \\
\sum_{i=0}^{j-1} \beta^i a_{n+i} + \beta^j a_{n+k_3} & \text{if } j = 1, \ldots, l.
\end{cases}
\end{align*}
\]

**Proof:** We show first that

\[
a_{n+k_3+j} = \sum_{i=0}^{j-1} \beta^i a_{n+i} + \beta^j a_{n+k_3}.
\]

Proposition 4 implies that \( a_{n+k_3+1} = a_{n+k_3} \), and the result holds for \( j = 1 \). For the induction step, suppose that

\[
a_{n+k_3+j-1} = \sum_{i=0}^{j-2} \beta^i a_{n+i} + \beta^{j-1} a_{n+k_3}.
\]

Applying Proposition 4 again, we can write \( a_{n+k_3+j} = a_{n+k_3+1} + a_{n+k_3} \). Therefore

\[
a_{n+k_3+j} = a_{n+k_3+1} + a_{n+k_3} = a_{n+k_3+1} + a_{n+k_3} + a_{n+k_3} = \sum_{i=0}^{j-1} \beta^i a_{n+i} + \beta^j a_{n+k_3}.
\]

Since \( a_{n+i} = 0 \) for all \( i = 0, \ldots, l - 1 \) the result follows.

Combining Eq. (2) and Lemma 9 the following equation arises

\[
\sum_{j=0}^{l-1} c_j \beta^j a_{n+k_3} + \sum_{j=1}^{l} c_j \left( \sum_{i=0}^{j-1} \beta^i a_{n+i} + \beta^j a_{n+k_3} \right) + \sum_{i=l}^{t-1} \beta^{i-1} a_{n+i} + \beta^l a_{n+k_3} = 0.
\]

Rearranging the coefficients of \( a_{n+k_3} \) in Eq. (3), we get

\[
\begin{align*}
a_{n+k_3} &\left( \sum_{j=0}^{t-1} c_j \beta^j + \beta^l \right) + \sum_{j=1}^{t-1} c_j \sum_{i=0}^{j-1} \beta^i a_{n+i} + \sum_{i=l}^{t-1} \beta^{i-1} a_{n+i} = 0.
\end{align*}
\]

Let \( c_l = 1 \). Eq. (4) can be rewritten as

\[
\begin{align*}
a_{n+k_3} f(\beta) + \sum_{j=1}^{t-1} c_j \sum_{i=0}^{j-1} \beta^{i-1} a_{n+i} + \sum_{i=l}^{t-1} \beta^{i-1} a_{n+i} = 0.
\end{align*}
\]

and, therefore,

\[
\begin{align*}
a_{n+k_3} f(\beta) + \sum_{j=1}^{t} c_j \sum_{i=0}^{j-1} \beta^{i-1} a_{n+i} = 0.
\end{align*}
\]

Let \( P \) be the following polynomial of degree \((t - l - 1)\) over
\[
P(x) = \sum_{j=0}^{t-1} c_j \sum_{i=0}^{j-1} a_{n+i} x^{j-i-1} = \sum_{j=0}^{t-1} c_{j+t} \sum_{i=0}^{j} a_{n+t+j-i} x^i.
\]

Therefore, \( \beta \) is a root of the polynomial \( a_{n+k_{\beta}} f(x) + P(x) \) over \( \mathbb{F}_q \).

Consider \( C_{n+k_{\beta}}^l (S(f, T)) \) a run of zeroes of length \( l \). We recall that \( z = z (C_{n_{\beta}}^l, \beta) \in \mathbb{Z} \) is defined as the positive integer such that, for \( j = 1, \ldots, z \), \( a_{n+jk_{\beta}} = 0 \) and \( a_{n+(z+1)k_{\beta}} \neq 0 \).

**Theorem 10:** Let \( f \) be a primitive polynomial of degree \( t \) over \( \mathbb{F}_q \) and \( \alpha \in \mathbb{F}_{q^t} \) be a root of \( f \). Let \( S(f, T) = (a_i)_{i \geq 0} \) be the LFSR sequence generated by \( f \) and \( T \in \mathbb{F}_q^t, T \neq (0, \ldots, 0) \). For each \( \beta \in \mathbb{F}_{q^t} \), let \( k_{\beta} \in \mathbb{Z}_{q^t} \) satisfying \( \alpha^{k_{\beta}} (\alpha - \beta) = 1 \). Let \( l \in \{0, \ldots, t-3 \} \) and \( C_{n}^{l} (S(f, T)) \) a run of zeroes of length \( l \).

1. The subinterval \( C_{n+k_{\beta}}^{l+1} (S(f, T)) \) is a run of zeroes of length \( l+1 \) if and only if \( \beta \) is a root of the polynomial \( P \in \mathbb{F}_q[x] \) of degree \( (t-l-1) \) given by

\[
P(x) = \sum_{j=0}^{t-1} c_{j+t+1} \sum_{i=0}^{j} a_{n+t+j-i} x^i.
\]

2. The number \( z(C_{n_{\beta}}^l, \beta) \) is equal to the multiplicity of \( \beta \) as a root of \( P(x) \).

**Proof:** (1) By the previous arguments, we know that \( \beta \) is a root of the polynomial \( a_{n+k_{\beta}} f(x) + P(x) \). If \( C_{n+k_{\beta}}^{l+1} (S(f, T)) \) is a run of zeroes of length \( l+1 \), then \( a_{n+k_{\beta}} = 0 \) and therefore \( \beta \) is a root of \( P(x) \). Conversely, if \( \beta \) is a root of \( P(x) \), then \( a_{n+k_{\beta}} f(\beta) = 0 \). Since \( \beta \in \mathbb{F}_q^t \) and \( f \) is a primitive polynomial over \( \mathbb{F}_q \), we conclude that \( f(\beta) \neq 0 \). Thus \( a_{n+k_{\beta}} = 0 \), and by Proposition 4, it follows that \( C_{n+k_{\beta}}^{l+1} (S(f, T)) \) is a run of zeroes of length \( l+1 \).

(2) Suppose that \( C_{n+k_{\beta}}^{l+1} (S(f, T)) \) is a run of zeroes of length \( l+1 \). By item (1), we have that \( \beta \) is a root of the polynomial \( P(x) \) given in Eq. \( \text{(5)} \). Now, suppose that \( C_{n+k_{\beta}}^{l+2} (S(f, T)) \) is a run of zeroes of length \( l+2 \). Applying the previous arguments for \( C_{n+k_{\beta}}^{l+1} (S(f, T)) \) and \( C_{n+k_{\beta}}^{l+2} (S(f, T)) \), and item (1), we obtain that \( \beta \) is a root of the polynomial

\[
P_{\beta}(x) = \sum_{j=0}^{t-1} c_{j+t+2} \sum_{i=0}^{j} a_{n+k_{\beta}+t+l+j-i} x^i.
\]

We claim that \( P(x) = \sum_{j=0}^{t-1} c_{j+t+2} \sum_{i=0}^{j} a_{n+k_{\beta}+t+l+j-i} x^i \).

\[
\sum_{j=0}^{t-1} c_{j+t+2} \sum_{i=0}^{j} a_{n+k_{\beta}+t+l+j-i} x^i.
\]

Since \( a_{n+k_{\beta}+t} = 0 \), Proposition 4 implies \( a_{n+k_{\beta}+t+1} = a_{n+t} \). Applying Proposition 4 several times, we obtain

\[
\sum_{j=0}^{t-1} c_{j+t+2} \sum_{i=0}^{j} a_{n+k_{\beta}+t+l+j-i} x^i.
\]

\[
\sum_{j=0}^{t-1} c_{j+t+2} \sum_{i=0}^{j} a_{n+k_{\beta}+t+l+j-i} x^i.
\]

Now, we get

\[
(x - \beta) P_{\beta}(x) = -\beta \sum_{j=0}^{t-1} c_{j+t+2} a_{n+k_{\beta}+t+l+j} + \sum_{j=0}^{t-1} c_{j+t+2} a_{n+k_{\beta}+t+l+1-j} x^i.
\]

Since \( C_{n+k_{\beta}}^{l+1} (S(f, T)) \) is a run of zeroes of length \( l+1 \), Eq. \( \text{(1)} \) implies

\[
a_{n+k_{\beta}+t+1} = -c_0 a_{n+k_{\beta}-1} - \sum_{j=0}^{t-1} c_{j+t+2} a_{n+k_{\beta}+t+l+1+j}.
\]

Since \( a_{n+k_{\beta}+t} - \beta a_{n+k_{\beta}+t-1} = a_{n-1} \) and \( a_{n+k_{\beta}} = 0 \),

\[
-\beta \sum_{j=0}^{t-1} c_{j+t+2} a_{n+k_{\beta}+t+l+1+j} = \beta (c_0 a_{n+k_{\beta}-1} - c_0 a_{n+k_{\beta}-1})
\]

\[
= \beta \left( c_0 a_{n+k_{\beta}-1} - c_0 a_{n+k_{\beta}-1} \right)
\]

\[
= c_0 a_{n+k_{\beta}-1} = -c_0 a_{n-1}.
\]

Eq. \( \text{(1)} \) and the fact that \( C_{n+k_{\beta}}^{l} (S(f, T)) \) is a run of zeroes of length \( l \) yield

\[
-c_0 a_{n-1} = \sum_{j=0}^{t-1} c_{j} a_{n+j+1} + a_{n+t+1}
\]

\[
= \sum_{j=0}^{t-1} c_{j+t+1} a_{n+t+j}.
\]

Eq. \( \text{(7)} \) is written as

\[
(x - \beta) P_{\beta}(x)
\]

\[
= \sum_{j=0}^{t-1} c_{j+t+1} a_{n+t+j} + \sum_{j=0}^{t-2} c_{j+t+2} a_{n+t+j+1-i} x^i
\]

\[
= \sum_{j=0}^{t-1} c_{j+t+1} a_{n+t+j} + \sum_{j=1}^{t-1} c_{j+t+1} a_{n+t+j-i} x^i.
\]

\[
= \sum_{j=0}^{t-1} c_{j+t+1} a_{n+t+j-i} x^i.
\]

Therefore, \( P(x) = (x - \beta) P_{\beta}(x) \). By item (1), we conclude that this process can be repeated as long as \( \beta \) is a root of \( P(x) \). Hence, the number \( z(C_{n_{\beta}}^l, \beta) \) is equal to the multiplicity of \( \beta \) as root of \( P(x) \).

**Example 11:** Let \( f(x) = 2 + 2x + x^4 \) be a primitive polynomial over \( \mathbb{F}_3 \) and \( \alpha \in \mathbb{F}_{3^t} \) a root of \( f \). The parameters
k_1 = 27 and k_2 = 76 satisfy \( \alpha^{k_1}(\alpha - 1) = 1 \) and \( \alpha^{k_2}(\alpha - 2) = 1 \), respectively. A period of \( S(f,1000) \) is 1000101101210021020221220111122201121 200020202122001201021120220221111122112220.

Consider the run \( C_{18}^1(S(f,1000)) \). Since \( c_0 = c_1 = 2, c_2 = c_0 = 0 \) and \( c_4 = 1 \), the polynomial described by Eq. [5] is \( P(x) = 1 + 2x + 2x^2 \). The polynomial \( P_3 \) given in Eq. [6] is \( P_1(x) = 2(x-1)(x-2) \) for \( \beta = 1 \) and \( P_2(x) = 2(x-1) \) for \( \beta = 2 \). Then \( P(x) = (x-2)P_2(x) \) for both \( \beta = 1 \) and \( \beta = 2 \).

By Theorem 10, \( C_{18}^1(f,1000) \) and \( C_{18}^2(f,1000) \) are runs of zeroes of length 2 as illustrated in Fig. 2.

Fig. 2. Runs of zeroes obtained from the run \( C_{18}^1(S(f,1000)) \).

Now, choose the run \( C_{27}^1(S(f,1000)) \). The polynomial \( P \) in this case is \( P(x) = 1 + x + x^2 \). The polynomial \( P_3 \) given in Eq. [6] is \( P_1(x) = x \) for \( \beta = 1 \) and \( P_1(x) = x \) for \( \beta = 2 \). Thus, \( P(x) = (x-1)P_1(x) \) and \( P(x) = (x-2)P_2(x) \). By Theorem 10, \( C_{27}^1(S(f,1000)) \) is a run of zeroes of length 2 and \( C_{27}^2(S(f,1000)) \) is not a run of zeroes. Moreover, \( C_{27}^1(S(f,1000)) \) is a run of zeroes of length 3. Fig. 3 shows the runs of zeroes obtained from the run \( C_{27}^1(S(f,1000)) \).

Fig. 3. Runs of zeroes obtained from the run \( C_{27}^1(S(f,1000)) \).

IV. FURTHER RESULTS OF LFSR SEQUENCES

In this section, we show that there exists a bijection between the runs of nonzero elements of length greater than \( l \) and the runs of zeroes of length exactly \( l \). Although we do not use this result in this paper, we think it may be of independent interest as a new combinatorial property of LFSRs.

Proposition 2 states, for \( l \in \{1, \ldots, t-2\} \), there exists exactly \( (q-1)^2q^{t-l-2} \) runs of zeroes of length \( l \). Let \( R(l) \) be the total number of runs of nonzero elements of \( \mathbb{F}_q \) of length greater than \( l \). Proposition 2 implies

\[
R(l) = (q - 1) \sum_{i=l+1}^{t-2} (q - 1)^2 q^{t-i-2} + (q - 1)^2 \\
= (q - 1)^2 (1 + (q - 1) \sum_{i=l+1}^{t-2} q^{t-i-2}) \\
= (q - 1)^2 q^{t-l-2}.
\]

Thus \( R(l) \) is equal to the number of runs of zeroes of length \( l \). For \( l = t - 1 \) the same property holds. The application of Propositions 6 and 7 and the counting argument on \( R(l) \) suggest a bijection between the runs of nonzero elements of length greater than \( l \) and the runs of zeroes of length exactly \( l \). We give this bijection next.

**Proposition 12:** Let \( 1 \leq l \leq t - 1 \). In a period of an LFSR sequence \( S(f,T) \), there is a bijection between runs of zeroes of length \( l \) and runs of nonzero elements of \( \mathbb{F}_q \) with length larger than \( l \). Moreover, this bijection is such that the difference of indices between the start of the runs is a multiple of \( k_1 \), where \( k_1 \in \mathbb{Z}_{q^t-1} \) satisfies \( \alpha^{k_1}(\alpha - 1) = 1 \).

**Proof:** Let \( C_{n,l}^{(l)}(S(f,T)) \) be a run of zeroes of length \( l \) and \( k_1 \in \mathbb{Z}_{q^t-1} \) such that \( \alpha^{k_1}(\alpha - 1) = 1 \). By iterating Proposition 7 there exists an integer \( j \) such that \( C_{n+l+jk_1}^{(l)}(S(f,T)) \) is a run of a nonzero element in \( \mathbb{F}_q \) beginning at position \( n + jk_1 \), and \( C_{n+l+jk_1}^{(l)}(S(f,T)) \) is a run of zeroes of length \( l + i \) for \( i = 1, \ldots, j - 1 \). Furthermore, \( j \leq t - l \) and let \( l' \in \mathbb{Z} \) such that \( j = l' - l \). Let \( S(l) \) be the set formed by the starting positions of the runs of zeroes of length \( l \). Define the map \( g(i) = i + jk_1 = i + (l' - l)k_1 \) where \( i \in S(l) \).

For the inverse map, let \( C_{n,l}^{(l)}(S(f,T)) \) be a run of a nonzero element in \( \mathbb{F}_q \) of length \( l' > l \). We get \( C_{n,(l'-l)k_1}^{(l'-l)}(S(f,T)) \), a run of zeroes of length \( l \) by iterating Proposition 6 exactly \( l' - l \) times. Let \( T(l) \) be the set formed by the starting positions of the runs of a nonzero element in \( \mathbb{F}_q \) of length \( l' > l \). Consider the map \( h(i) = i - (l' - l)k_1 \) where \( i \in T(l) \).

Propositions 6 and 7 describe inverse processes and so the maps \( g \) and \( h \) are inverse maps.

**Example 13:** Consider the primitive polynomial \( f(x) = 1 + 2x + x^3 \) over \( \mathbb{F}_3 \). A period of \( S(f,100) \) is given by:

100202121022001201011211120112112220

Then \( k_1 = 23 \) satisfies \( \alpha^{23}(\alpha - 1) = 1 \), where \( \alpha \in \mathbb{F}_3 \) is a root of \( f \). In Fig. 4 the period of the sequence \( S(f,100) \) is represented twice and clockwise ordered. The runs highlighted by two arcs represent how the runs of zeroes of length 1 and runs of nonzero elements of \( \mathbb{F}_3 \) of length larger than 1.
one-to-one correspondence between runs of zeroes of length 1 and runs of nonzero elements of \( F_3 \) of length larger than 1, established in Proposition 12.

V. ORDERED ORTHOGONAL ARRAYS FROM LFSRS

Linear feedback shift register sequences of maximum period are used to construct orthogonal arrays \( [3, 10, 20] \) and covering arrays \( [21, 23, 26] \). A subinterval array of the sequence is the key for building such arrays. We construct an ordered orthogonal array by choosing suitable columns of this subinterval array.

Let \( f \) be a primitive polynomial of degree \( t \geq 3 \) and \( \alpha \in F_q^t \), a root of \( f \). Let \( T \in \mathbb{F}_q^t \) be nonzero initial values for the sequence \( S(f, T) \) generated by \( f \). Let \( k = \frac{q^t-1}{q-1} \) and consider the following \( q^t \times k \) array

\[
M = M(f, T) = \begin{bmatrix}
C_0^k(S(f, T)) \\
C_1^k(S(f, T)) \\
\vdots \\
C_{q^t-2}^k(S(f, T)) \\
0, 0, \ldots, 0
\end{bmatrix},
\]

where \( C_i^k(S) \) is the subinterval of \( S \) of length \( k \) beginning at position \( i \). Matrix \( M \) is the subinterval array of \( f \). Label the columns of \( M \) by \( \mathbb{Z}_k \). A characterization of which sets of \( t \) columns of \( M \) are covered is given in [22].

Theorem 14: [22] Theorem 2] Let \( f \) be a primitive polynomial of degree \( t \geq 3 \) over \( F_q^t \) and \( \alpha \in F_q^t \) a root of \( f \). Let \( k = \frac{q^t-1}{q-1} \), and let \( M \) be the \( q^t \times k \) subinterval array of \( f \). The following are equivalent:

1. A set of \( t \) columns \( \{i_1, \ldots, i_t\} \) is covered by \( M \).
2. There is no row \( r \) other than the all-zero row of \( M \) such that \( r_{i_1} = \cdots = r_{i_t} = 0 \).
3. The set \( \{\alpha^{i_1}, \ldots, \alpha^{i_t}\} \) is linearly independent over \( F_q \).

Proposition 15: Any subarray of \( M \) formed by \( t \) consecutive columns of \( M \) is covered.

Proof: It is an immediate consequence of Proposition 2, item (5), and the definition of \( M(f, T) \).

We are ready to construct ordered orthogonal arrays from the subinterval array of a primitive polynomial.

Theorem 16: Let \( f \) be a primitive polynomial of degree \( t \geq 3 \) over \( F_q^t \) and \( \alpha \in F_q^t \) be a root of \( f \). Consider an array \( A \) with columns labeled by \( [q + 1] \times [t] = \{(i, j) : 1 \leq i \leq q + 1, 1 \leq j \leq t\} \) where

1. columns labeled by \( (1, 1), \ldots, (1, t) \) correspond to the columns of \( M \) labeled by \( t-1, \ldots, 0 \), respectively;
2. columns labeled by \( (2, 1), \ldots, (2, t) \) correspond to the columns of \( M \) labeled by \( t, \ldots, 2t-1 \), respectively; and
3. columns labeled by \( (i, 1), (i, 2), \ldots, (i, t) \), for \( i \in \{3, \ldots, q + 1\} \), correspond to the columns of \( M \) labeled by \( t + k_\beta, t + 2k_\beta, \ldots, t + tk_\beta \), for each \( \beta \in \mathbb{F}_q^* \), respectively.

Then, the array \( A \) is a \( OOA(t, q + 1, t, q) \).

Proof: Let \( L \) be a set of \( t \) columns in \( M \) corresponding to a left-justified set of \( t \) columns of \( A \). Given any row \( r = (r_0 \ldots r_{k-1}) \) distinct from the all zero row, we prove that for some \( i \in L \), and by Theorem 14 the subarray of \( M \) labeled by \( L \) is covered. Four cases need to be considered:

Case 1: \( L \subset \{0, \ldots, 2t-1\} \). Since \( L \) is a left-justified set, the elements of \( L \) are consecutive elements of \( \{0, \ldots, 2t-1\} \). Then, \( L \) is a left-justified set that labels consecutive \( t \) columns of \( M \), and by Proposition 15 these \( t \) columns of \( M \) are covered.

Case 2: \( |L \cap \{0, \ldots, 2t-1\}| = t-1 \). In this case, the left-justified set \( L \) has the form \( L = \{n, n+ t-2\} \cup \{t+k_\beta\} \) for some \( n \in \{1, \ldots, t\} \) and \( \beta \in \mathbb{F}_q^* \). If \( r_i = 0 \) for some \( i \in \{n, n+ t-2\} \) we have nothing to prove. Otherwise, \( r_i = 0 \) for all \( i \in \{n, \ldots, n+ t-2\} \). Since an LFSR sequence does not contain a run of zeroes of length \( t \), the consecutive zeroes in the positions \( n_1, n_2, \ldots, n_2+t-2 \) form a run of zeroes of length \( t-1 \). Proposition 7 items (1) and (2), yield \( r_{n+k_\beta+j} \neq 0 \) for all \( j = 0, \ldots, t-1 \). In particular, for \( j = t-n \) we conclude that \( r_{t+k_\beta} \neq 0 \) as desired.

Case 3: \( 1 \leq |L \cap \{0, \ldots, 2t-1\}| < t-1 \). We can write \( L = L_1 \cup L_2 \), where \( L_1 \subset \{2, \ldots, 2t-3\} \) and \( L_2 \subset \{t+ jk_\beta : \beta \in \mathbb{F}_q^*, 1 \leq j \leq t\} \). The set \( L_1 \) is formed by consecutive elements of \( \{2, \ldots, 2t-3\} \). Furthermore, since \( L \) is a left-justified set, \( t-1 \) or \( t \) belongs to \( L_1 \). Let \( l = |L_1| \). Without loss of generality we assume that the consecutive elements in \( L_1 \) form a run of zeroes of length \( l \)

\[
r_{n-1 \ldots 0 \ldots 0_{r_{n+t-1}}} = 0_{r_{n+1+t-1}} \ldots r_{n+t-2r_{n+t-1}},
\]

where \( n \in \{2, \ldots, t\} \). If \( r_{n_1}, \ldots, r_{n+t-1} \) were not all zero, then the proof of this case would be complete. By Theorem 10 item (2), the number \( z(C_n^l, \beta) \) is equal to the multiplicity of \( \beta \in \mathbb{F}_q^* \) as a root of the polynomial

\[
P(x) = \sum_{j=0}^{t-1} c_{j+t+1} \sum_{i=0}^j r_{n+t+j-i} x^i,
\]

where \( c_0, \ldots, c_{t-1} \) are the coefficients of \( f \). By Proposition 7 items (1) and (2), if \( \beta \) is a root of \( P(x) \) with multiplicity \( z \), then \( r_{t+jk_\beta} = 0 \) for \( 1 \leq j \leq z \), and \( r_{t+(z+1)k_\beta} \neq 0 \). Since the number of roots of \( P(x) \) in \( \mathbb{F}_q \) counting with multiplicity is at most \( t-1 \), the number of elements \( i \in L_2 \) such that \( r_i = 0 \) is at most \( t-1 \). Therefore, the number of zeroes in row \( r \) in positions \( i \in L \) is at most \( t-1 \).

Case 4: \( L \cap \{0, \ldots, 2t-1\} = \emptyset \). We consider two subcases.

If \( r_i = 0 \), then we have a run of zeroes of length 0 beginning at position \( t \). By Theorem 10 item (2), the number \( z(C_n^l, \beta) \) is equal to the multiplicity of \( \beta \in \mathbb{F}_q^* \) as a root of the polynomial

\[
P(x) = \sum_{j=0}^{t-1} c_{j+t+1} \sum_{i=0}^j r_{t+j-i} x^i.
\]

Since the number of roots of \( P(x) \) in \( \mathbb{F}_q \) counting with multiplicity is at most \( t-1 \), the number of elements \( i \) in \( L \) such that \( r_i = 0 \) is at most \( t-1 \).

If \( r_i = 0 \), then there exists \( n \in \{2, \ldots, t\} \) such that \( C_n^l = (r_{n+1}, r_{n+2}, \ldots, r_{n+t-1}) \) is a run of zeroes of length \( l \). Now, we can apply the same argument as we applied in Case 3 and conclude that the number of zeroes in row \( r \) in positions
$i \in L$ is at most $t - 1$.

The previous theorem constructs an OOA with the same parameter sets as Skriganov [25] which generalizes and also covers the range given by Rosenbloom and Tsfasman [24]. However, our construction yields different OOAs than the ones in [25]. In the next section, we compare characteristics of both constructions, showing that ours cover a larger number of $t$-sets of columns beyond the coverage required for the left-justified sets of the set $[q + 1] \times [t]$. In this sense, the array $A$ in Theorem 16 is “closer” to being a full orthogonal array than the one given in [25].

VI. A COMPARATIVE ANALYSIS OF THE OOA CONSTRUCTIONS

In this section, we compare our construction of ordered orthogonal arrays given in Theorem 16 with the construction of Skriganov [25]. Since the construction of Skriganov [25] is broader than Rosenbloom and Tsfasman [24] construction, we use the former for this comparison. We refer to the construction given in Theorem 16 as the RUNS construction and to the one by Rosenbloom and Tsfasman [24] and Skriganov [25] as the RTS construction.

One criteria we use for comparing different OOA $(t, q + 1, t, q)$ is their extent of coverage, that is, the total number of $t$-sets of columns that are covered. In both constructions, we know that the columns labeled by left-justified sets of size $t$ are covered, but there may be many other $t$-sets of columns that are covered. We were not able to determine this quantity in general for RUNS and RTS, but for some values of $t$ and $q$, we computed the total number of $t$-sets of columns covered by RUNS using all distinct primitive polynomials and by RTS, as shown in Table I. These experiments show that RUNS covers many more $t$-sets of columns than RTS.

For every $q$ and $t$ considered in Table I we show statistics on the ratio of $t$-sets covered by RUNS over all possible $t$-sets of columns, for all primitive polynomials of degree $t$ over $\mathbb{F}_q$. In this table, $\# f$ is the number of primitive polynomials of degree $t$ over $\mathbb{F}_q$, $\text{RUNS}_{\text{min}}, \text{RUNS}_{\text{max}}$ and $\text{RUNS}_{\text{avg}}$ give the minimum, maximum and average ratio of coverage of RUNS for all primitive polynomial for $t$ and $q$; RTS is the ratio of $t$-sets of columns covered by the RTS construction.

Table I shows that in both constructions many $t$-sets of columns are covered in addition to the left-justified ones. For all pairs of parameters $(t, q)$ the experiments show that the number of $t$-sets of columns covered by RUNS is always greater than this number for RTS. For RUNS, Table I shows that the minimum ratio and the maximum ratio of $t$-sets of columns covered are relatively close, and we conjecture that, for $t$ and $q$ fixed, there exists a nontrivial lower bound for the number of $t$-sets of columns covered in all ordered orthogonal arrays obtained with the RUNS construction. Finally, when we fix $t$ and vary $q$, the average ratio of $t$-sets of columns covered by the RUNS construction grows faster than the ratio of $t$-sets of columns covered by the RTS construction. Moreover, when we fix $q$ and increase $t$, the ratio of $t$-sets of columns covered by the RTS construction decreases faster than the average ratio of $t$-sets of columns covered by the RUNS construction.

VII. ORDERED ORTHOGONAL ARRAYS AND OTHER COMBINATORIAL STRUCTURES

In this section, we relate our methods to other combinatorial structures. In the first subsection, we look at other constructions that depend on linear independence including coding theory constructions. In the second subsection, we look at hypergraphs and homomorphisms.

Ordered orthogonal arrays are a combinatorial characterization of $(t, m, s)$-nets [8]. [14, Lawrence 8 Theorem 4.1] and Mullen and Schmid [14, Theorem 7] independently show the equivalence between $(t, m, s)$-nets and OOAs. To state this equivalence, we define the concept of $(t, m, s)$-net in base $b$.

Let $[0, 1]^s$ be the half-open unit cube of dimension $s$ and suppose numerical computation is to be done in base $b \geq 2$. An elementary interval in base $b$ in $[0, 1]^s$ is an euclidean set of the form

$$E = \prod_{i=1}^{s} \left[ \alpha_i \frac{d_i}{b^{d_i}}, \alpha_{i+1} \frac{d_i}{b^{d_i}} \right]$$

where, for each $i$, $d_i \geq 0$ and $0 \leq \alpha_i < b^{d_i}$. The volume of $E$ is $b^{-\sum d_i}$. Let $s \geq 1$, $b \geq 2$, and $m \geq t \geq 0$ be integers. A $(t, m, s)$-net in base $b$ is a multiset $N$ of $b^m$ points in $[0, 1]^s$ with the property that every elementary interval in base $b$ of volume $b^{m-t}$ contains precisely $b^t$ points from $N$.

Theorem 17: [8, 14] Let $s \geq 1$, $b \geq 2$, $t \geq 0$ and $m$ be integers, and assume that $m \geq t + 1$ to avoid degeneracy. Then there exists a $(t, m, s)$-net in base $b$ if and only if there exists an OOA $\text{A}_{\text{OOA}}(b^m; m-t, s, m-t, b)$.

An ordered orthogonal array $\text{OOA}(t, m, s, q)$ over $\mathbb{F}_q$ is linear if the rows of the OOA form a subspace of $\mathbb{F}_q^m$. Linear OOAs correspond to digital nets, see for instance [2] Section VI.59]. Thus our OOA $\text{OOA}(t, q + 1, t, q)$ is a linear OOA and corresponds to a $(0, t, q + 1)$-net in base $q$, which is consequently a digital net.

A. Ordered orthogonal arrays and sets of independent vectors over $\mathbb{F}_q$

Ordered orthogonal arrays can be constructed from a set of vectors in $\mathbb{F}_q^t$ such that any subset of $t$ vectors is linearly independent over $\mathbb{F}_q$. We show, in this subsection, that the RUNS construction of OOAs uses a set of vectors in $\mathbb{F}_q^t$ such that not all choices of $t$ vectors form a linearly independent set over $\mathbb{F}_q$.

A method to construct digital nets in prime power bases is introduced in [9]. This construction makes use of sets of independent vectors over finite fields. An $(n, t)$-set in $\mathbb{F}_q^{m+t}$ is a set of $n$ vectors in $\mathbb{F}_q^{m+t}$ such that any $t$ of them are linearly independent over $\mathbb{F}_q$. For our purpose, we state [9, Theorem 1] in terms of ordered orthogonal arrays.

Theorem 18: [9, Theorem 1] Let $q$ be a prime power, and let $n, u \geq 0$ and $t \geq 2$ be integers. Given an $(n, t)$-set in $\mathbb{F}_q^{m+t}$, a linear $\text{OOA}_{\text{OOA}}(q^{u+m}; t, m, t, q)$ can be constructed over $\mathbb{F}_q$ with
TABLE I

| $t$ | $q$ | # $f$ | $\text{RUNS}_{\text{min}}$ | $\text{RUNS}_{\text{max}}$ | $\text{RUNS}_{\text{avg}}$ | $\text{RTS}$ |
|-----|-----|-----|-----------------|-----------------|-----------------|--------|
| 3   | 2   | 2   | 0.595238        | 0.595238        | 0.595238        | 0.464286 |
| 3   | 3   | 4   | 0.709091        | 0.740909        | 0.723964        | 0.545455 |
| 3   | 5   | 20  | 0.810049        | 0.839461        | 0.824387        | 0.573529 |
| 3   | 7   | 36  | 0.853261        | 0.889822        | 0.867054        | 0.583004 |
| 4   | 2   | 2   | 0.484848        | 0.523232        | 0.50404        | 0.345455 |
| 4   | 3   | 8   | 0.588462        | 0.702747        | 0.632143        | 0.325824 |
| 4   | 5   | 48  | 0.776774        | 0.801525        | 0.78791        | 0.449558 |
| 5   | 2   | 6   | 0.38628         | 0.46953         | 0.444388       | 0.196803 |
| 5   | 3   | 22  | 0.602941        | 0.660733        | 0.635038       | 0.243292 |
| 6   | 2   | 6   | 0.38914         | 0.446509        | 0.410032       | 0.135693 |
| 6   | 3   | 48  | 0.453089        | 0.633845        | 0.59164        | 0.205296 |
| 7   | 2   | 18  | 0.308763        | 0.423719        | 0.363138       | 0.0897059 |

In Corollary 21 we give an upper bound for the parameter $m$ of the OOA($t, m, t, q$) constructed in Theorem 18 depending on $q$ and $t$. We first state a classical bound on orthogonal arrays, before we show the mentioned upper bound.

**Theorem 19:** [6, Theorem 2.19] In an OOA($q^t; t, n, q$), the following inequalities hold

- $n \leq t + 1$ if $q \leq t$,
- $n \leq q + t - 2$ if $3 \leq t < q$ and $q$ is odd,
- $n \leq q + t - 1$ otherwise.

The concept of $(n, t)$-set in $F_q^{n+t}$ is connected to the theory of error-correcting codes. The existence of an $(n, t)$-set in $F_q^{n+t}$ is equivalent to the existence of a linear $[n, n - (u + t), t + 1]$-code over $F_q$, see for example [23, Theorem 5.3.7]. Using this connection of $(n, t)$-sets and coding theory, and Theorem 19 we derive the following upper bound on $n$.

**Proposition 20:** Let $n \geq t \geq 1$, and let $q$ be a prime power. If there exists an $(n, t)$-set in $F_q^t$, then there exists an OOA($q^t; t, n, q$). In particular, $n \leq q + t - 1$.

**Proof:** We first show that the existence of an $(n, t)$-set in $F_q^t$ implies the existence of an OOA($q^t; t, n, q$). The existence of an $(n, t)$-set in $F_q^t$ is equivalent to the existence of a linear $[n, n - t, t + 1]$-code over $F_q$, which is consequently an MDS (maximum distance separable) code. It is known that if a linear code over $F_q$ is an MDS code, then its dual code is also a linear MDS code. Thus if there exists an MDS $[n, n - t, t + 1]$-code over $F_q$, then there exists an MDS $[n, n - t, t + 1]$-code over $F_q$. The existence of a linear MDS $[n, n - t, t + 1]$-code over $F_q$ implies the existence of an OOA($q^t; t, n, q$), see for instance [6, Theorem 4.6]. Now, Theorem 12 implies that $n \leq q + t - 1$.

**Corollary 21:** For the OOA($q^t; t, m, t, q$) given in Theorem

	ext{rem}_{18} \text{using } u = 0,

$$m \leq \left\lfloor \frac{q}{t} \right\rfloor + 1.$$  

**Proof:** From Proposition 20, we get $n \leq q + t - 1$, which combined with the relationship of $m$ and $n$ given in Theorem 18 gives the desired bound on $m$.

In the next subsection, we look at Theorem 18 again from a related but more abstract point of view.

A well-known upper bound for the parameter $m$ from the theory of digital nets over $F_q$ is $m \leq \frac{q^{t+1} - 1}{q^t - 1}$. Thus, for $u = 0$, this upper bound is $m \leq q + 1$. Therefore, for $t = 3$ and $q$ odd, from the bound on $m$ given by Corollary 21, we conclude the construction given in Theorem 18 does not achieve the $q + 1$ bound for digital nets. In other words, using $(n, t)$-sets all we can hope for are linear OOAs bounded by $m \leq \left\lfloor \frac{q}{t} \right\rfloor + 1$, which is generally much weaker than $q + 1$.

On the other hand, our OOA construction in Theorem 16 (RUNS) and the OOA construction in [24, 25] (RTS) build linear OOAs with $m = q + 1$, which is the best possible. In the rest of this section, we show that RUNS is implicitly derived from a set of $n$ vectors in $F_q^t$ such that not all choices of $t$ vectors are linearly independent over $F_q$. To do this, we describe it as follows.

Let $f$ be a primitive polynomial of degree $t \geq 3$ over $F_q$ and $\alpha \in F_q$ be a root of $f$. Let $T(j, i) = (a_i)_{i \geq 0}$ be the LFSR sequence generated by $f$ and $T = \{b_0, \ldots, b_{t-1}\} \in F_q^t$, $T \neq \{0, \ldots, 0\}$. By Proposition 1, there exists a unique $\gamma \in F_q^t$ such that $T = (\text{Tr}(\gamma \alpha^0), \ldots, \text{Tr}(\gamma \alpha^{t-1}))$. Since $\alpha$ is a primitive element in $F_q$, there exists $v \in \{0, \ldots, q^t - 2\}$ such that $\gamma = \alpha^v$, and so $T = (\text{Tr}(\alpha^v), \ldots, \text{Tr}(\alpha^{v^{t-1}}))$. Let $\mathcal{X} = \{T_j \in F_q^t : j \in \Omega[q + 1, t]\}$.
The next result gives a criterion to know which subsets of $\mathcal{X}$ of size $t$ are linearly independent over $\mathbb{F}_q$.

**Proposition 22:** Under the conditions above, a subset $\{T_{j_0}, \ldots, T_{j_{t-1}}\}$ of $\mathcal{X}$ is linearly independent over $\mathbb{F}_q$ if and only if $\{\alpha^{j_0}, \ldots, \alpha^{j_{t-1}}\}$ is linearly independent over $\mathbb{F}_q$.

**Proof:** By the conditions above, $T_{j_i} = (\text{Tr}(\alpha^{j_i+1} \alpha^0), \ldots, \text{Tr}(\alpha^{j_{t-1}+1} \alpha^0))$ for $0 \leq i \leq t-1$. Let $y_0, \ldots, y_{t-1} \in \mathbb{F}_q$ not all of which are zero. The following relation holds

$$\sum_{i=0}^{t-1} y_i T_{j_i} = \left( \text{Tr} \left( \alpha^0 \sum_{i=0}^{t-1} y_i \alpha^{j_i} \right), \ldots, \text{Tr} \left( \alpha^{t-1} \sum_{i=0}^{t-1} y_i \alpha^{j_i} \right) \right).$$

If $\{\alpha^{j_0}, \ldots, \alpha^{j_{t-1}}\}$ is linearly dependent over $\mathbb{F}_q$, then the set $\{T_{j_0}, \ldots, T_{j_{t-1}}\}$ is linearly dependent over $\mathbb{F}_q$ by Eq. (8). Conversely, if $\{T_{j_0}, \ldots, T_{j_{t-1}}\}$ is linearly dependent over $\mathbb{F}_q$ by Eq. (8), for $0 \leq i \leq t-1$, we have

$$\text{Tr} \left( \alpha^{j_i+1} (y_0 \alpha^{j_0} + \cdots + y_{t-1} \alpha^{j_{t-1}}) \right) = 0.$$

Since $\{\alpha^{j_0}, \ldots, \alpha^{j_{t-1}}\}$ is linearly independent over $\mathbb{F}_q$, we conclude that

$$0 = \text{Tr} (\gamma (y_0 \alpha^{j_0} + \cdots + y_{t-1} \alpha^{j_{t-1}})).$$

for all $\gamma \in \mathbb{F}_q$. If $b = y_0 \alpha^{j_0} + \cdots + y_{t-1} \alpha^{j_{t-1}}$ is nonzero, then $b$ has an inverse $b^{-1}$ in $\mathbb{F}_q$. For any $\omega \in \mathbb{F}_q$, we can write $\omega = (\omega b^t) b$. By choosing $\gamma = \omega b^t$ we have

$$0 = \text{Tr} (\gamma b) = \text{Tr} \left( (\omega b^t) b \right) = \text{Tr} (\omega).$$

This implies that $\text{Tr} (\omega) = 0$ for all $\omega \in \mathbb{F}_q$, which is not possible. Therefore $b = 0$, implying that $\{\alpha^{j_0}, \ldots, \alpha^{j_{t-1}}\}$ is linearly dependent over $\mathbb{F}_q$. \hfill $\blacksquare$

Now we see how the previous result relates to our OOA construction. For each $T_j \in \mathcal{X}$, let $S(f, T_j)$ be the LFSR sequence generated by $f$ and $T_j$. Consider the array whose columns are $C^q_{j_0-1}((S(f, T_j))$ for $j \in \Omega[q+1, t]$. Add to this array the all-zero row. This array is exactly the OOA constructed in Theorem 16. Thus these OOAs are constructed from the set $\mathcal{X}$ of vectors in $\mathbb{F}_q^t$. The set $\mathcal{X}$ is not an $(|\mathcal{X}|, t)$-set in $\mathbb{F}_q^t$, since $|\mathcal{X}| \geq q + t$. Indeed, for every $K \subseteq \mathcal{X}$ with $|K| \geq q + t$, we conclude $K$ is not a $(|K|, t)$-set in $\mathbb{F}_q^t$, otherwise by Proposition 20 we could construct an OA($q^t$, $t$, $|K|$, $q$).

In the next subsection, we generalize the notion of linear independence using hypergraphs and see that Theorem 16 and our Theorem 16 belong to a family of constructions which are instances of a general principle given in Theorem 25.

**B. Ordered orthogonal arrays, hypergraphs and homorphisms**

The construction in this paper can be viewed through the lens of hypergraph homorphisms, which has the potential for further fruitful use in the construction of ordered orthogonal arrays and of other generalizations of orthogonal arrays. In some sense, we are abstracting the notion of linear independence with hypergraphs and abstracting the notion of the choices of columns with homorphisms. Informally to start, suppose we have an $N \times k$ v-ary array that has a collection $H$ of $t$-subsets of the columns that is $\lambda$-covered. In the previous subsection, the array was either an OA and $H$ the set of all $t$-sets of columns or the array was a subinterval array from an LFSR sequence and $H$ the set of all linearly independent $t$-sets of columns. Considering $H$ as a hypergraph, suppose we additionally have a homomorphism from another hypergraph $G \rightarrow H$. Then we can construct an $N \times |V(G)|$ v-ary that is $\lambda$-covered on the hyperedges of $G$. This map simply records the choices of columns from one array we are using to build a new array as described by Theorems 18 and 16 respectively. We now make this construction formal and review existing results on $(t, m, s)$-net and ordered orthogonal array literature, including our construction, via this framework.

A t-uniform hypergraph, $G$, consists of a finite vertex set $V(G)$ and a collection, $E(G)$, of $t$-subsets of $V(G)$ called hyperedges. Let $H_{t,m,s}$ be the hypergraph whose hyperedges are precisely the left-justified subsets of size $t$ of $[m] \times [s]$. A complete t-uniform hypergraph of order $n$, $K_n^t$, is the hypergraph $(X, E)$ where $|X| = n$ and $E = \binom{X}{t}$, where $\binom{X}{t}$ is the set of all subsets of $X$ of cardinality $t$. Given a finite vector space, $\mathbb{F}_q^t$, the linear independence hypergraph, $LI_{d,q}$ is the hypergraph with $V(\{L_{d,q}\}) = \mathbb{F}_q^d$ and $e = \{v_0, v_1, \ldots, v_{d-1}\} \in E(LI_{d,q})$ if and only if $e$ is a linearly independent set over $\mathbb{F}_q$ (equivalently $e$ is not contained in any dimension $d-1$ subspace). Similarly given a projective space $PG(d,q)$, the projective independence hypergraph, $PI_{d,q}$ is the hypergraph with the $(q^{d+1} - 1)/(q-1)$ points of the space as vertices and $e = \{v_0, v_1, \ldots, v_d\} \in E(PI_{d,q})$ if and only if $e$ is not contained in any dimension $d-1$ subspace. A reference for the geometry concepts used here is [1].

**Definition 23:** For two $t$-uniform hypergraphs $G, H$, a hypergraph homomorphism $f: G \rightarrow H$ is a map $f: V(G) \rightarrow V(H)$ such that $e = \{v_0, \ldots, v_{m-1}\} \in E(G)$, then $f(e) = \{f(v_0), \ldots, f(v_{m-1})\} \in E(H)$, and $|e| = |f(e)|$.

The fact that there exists an injective homomorphism [1]

$$PI_{d,q} \rightarrow LI_{d+1,q}$$

gives a geometrical motivation for the truncation of subintervals of the LFSR at length $(q^t - 1)/(q-1)$ used in Section 16.

**Definition 24:** Let $G$ be a $t$-uniform hypergraph on $k$ vertices. A variable strength orthogonal array, $VOA(N; G, v)$, is an $N \times k$ array over $\{0, \ldots, v-1\}$ with columns labeled by $V(G)$ such that if $B = \{b_0, \ldots, b_{m-1}\} \in E(G)$, then $N \times t$ subarray labeled by $B$ is $\lambda$-covered, where $\lambda = N/v^t$.

In this language, an $OOA_{\lambda}(N; t; m, s, v)$ is equivalent to a $VOA(N; H_{t,m,s}, v)$, and an $OA_{\lambda}(N; t; n, v)$ is a $VOA(N; K_n^t, v)$.

The following theorem is a general purpose construction linking homorphisms and variable strength orthogonal arrays.

**Theorem 25:** [21] Let $G$ and $H$ be $t$-uniform hypergraphs. Suppose that there exists a $VOA(N; H, v)$ and a hypergraph homomorphism $f: G \rightarrow H$. Then there exists a $VOA(N; G, v)$. 

For this construction we need two things: an already existing array where the edges of the hypergraph $H$ describes the sets of columns that are $\lambda$-covered, and a homomorphism from hypergraph $G$ to $H$. In most of the uses of this construction that we are aware of, the hypergraph $H$ is well known and the work is establishing the homomorphism. This framework also applies to more general objects called covering arrays \cite{21}.

We now demonstrate how previous results on $(t,m,s)$-nets and OOAs can be viewed with the hypergraph homomorphism language. Theorem 2 of \cite{15}, Theorem 5.4 of \cite{19}, and Theorem 1 of \cite{19} all use the existence of an $OA_3(N;2,m,v)$, equivalently a $VOA(N;K_m^2,v)$, to construct an $OOA_\lambda(N;2,m,2,v)$ by establishing a homomorphism from $H_{2,m,2} \rightarrow H_{2,m,1} = K_m^2$. In homomorphism terms, they also use the fact that there is a more obvious homomorphism $K_m^2 \rightarrow H_{2,m,2}$ to show that the existence of an $OA_3(N;2,m,v) = OOA_\lambda(N;2,m,1,v)$ is equivalent to the existence of an $OOA_\lambda(N;2,m,2,v)$. Theorem 3 in \cite{14} extends this to higher strength showing that the existence of an $OOA_\lambda(N;t,m,t−1,v)$ is equivalent to an $OOA_\lambda(N;t,m,t,v)$ by establishing the homomorphisms

$$H_{t,m,t−1} \rightarrow H_{t,m,t} \rightarrow H_{t,m,t−1}.$$  

They point out that this yields a construction when the $OOA_\lambda(N;t,m,t−1,v)$ is known to exist which is powerful in the $t = 2$ case, since there are many known $OA_3(N;2,m,v) = OOA_\lambda(N;2,m,1,v) = VOA(N;K_m^2,v)$. In some of these cases the OOAs are linear and the $VOA(N;K_m^2,q)$ is derived from $(n,t)$-sets and thus linear independence plays a role in the constructions.

The Hammersley net, an $OOA(t,2,t,v)$ given in Example 14.8.5 in \cite{15}, is constructed by giving a homomorphism from $H_{t,2,t} \rightarrow K_1^t = H_{t,1,t}$ and using the trivial $OOA(t,t,1,v) = OA(t,t,v) = VOA(e^t;K_1^t,v)$. Example 14.8.6 in \cite{13} constructs an $OOA_\lambda(N;1,m,1,v)$ by essentially using a homomorphism from $K_m^n \rightarrow K_1^t$.

Theorem 5.1 in \cite{8} and Theorem 1 in \cite{9} use the existence of an $OA_3(N;t,n,v)$ to construct an $OOA_\lambda(N;t,m,t,v)$ where if $t = 2h + \delta, \delta \in \{0,1\}$, we have $n = mh + \delta$. From the hypergraph homomorphism point of view, both articles can be seen as establishing a homomorphism from $H_{t,m,t} \rightarrow K_n^t$. This is the equivalent of Theorem 18 from the previous subsection.

Fuji-Hara and Miao \cite{4} determine the geometric structures in $PG(d,q)$ that are equivalent to linear $OOA_{d−3}(q^{d+1};3,m,3,q)$ and $OOA_{d−3}(q^{d+1};4,m,4,q)$. This can be viewed as determining the existence of homomorphisms

$$H_{3,m,3} \rightarrow PI_{d,q}$$

for prime powers $q$, when $q = 2$ and $m \leq 2^d − 1$ or when $q > 2$ and there exists a set of $m$ points in $PG(d,q)$, no three of which are co-linear; and

$$H_{4,m,4} \rightarrow PI_{d,q}$$

for prime powers $q$, when a particular point configuration exists in $PG(d,q)$. They then show that this configuration always exists when $d = 3$ and $m = q + 1$. In some of their methods, there are non-constructive proofs for the point configurations.

In the hypergraph homomorphism language, our Theorem 16 is proved by using the existence of the subinterval array from the LFSR which is a $VOA(q^d;PI_{t−1,q},q)$ and then building a homomorphism

$$H_{t,q+1,t} \rightarrow PI_{t−1,q}.$$  

All these examples show the power of the hypergraph homomorphism technique to construct OOAs. As pointed out by Martin \cite{11} in discussions with the fourth author, the existing OOA constructions prior to 2002 (all the work surveyed above prior to our own and that of Fuji-Hara and Miao) essentially repeated columns of existing orthogonal arrays in clever ways so that the resulting arrays satisfied the required column coverage for the OOA definition. Martin \cite{11} conjectures that there are similar homomorphic techniques that construct OOAs, but which do not simply repeat columns. The work of Fuji-Hara and Miao for $t = 3,4$ and our Theorem 16 for arbitrary $t$ are the first such homomorphic constructions of OOAs which meet this goal. Our work is fully constructive.

Another use of hypergraph homomorphisms in the OOA and $(t,m,s)$-net literature is to prove non-existence results. Because there is a simple homomorphism

$$K_m^t \rightarrow H_{t,m,t},$$

Theorem 25 shows that if an $OA_3(N;t,m,v)$ does not exist then an $OOA_\lambda(N;t,m,t,v)$ cannot exist. This homomorphism technique is the essence of several non-existence results for OOAs in the literature \cite{17}, \cite{9}. Fuji-Hara and Miao \cite{4}, in homomorphism terms, use this to prove that an $OOA_{d−3}(q^{d+1};4,m,4,q)$ cannot exist unless

$$(q + 1)m + (q − 1)\left(\frac{m}{2}\right) \leq \frac{q^d − 1}{q − 1}$$

and

$$m \leq \frac{q^d − 2}{q − 1}. $$

Since this method is so general, we believe that it has the potential to yield new non-existence results. The goal should be to find a hypergraph homomorphism $G \rightarrow H_{t,m,s}$ and also show that a $VOA(N;G,v)$ does not exist. This would establish a non-existence result for $OOA_\lambda(N;t,m,s,v)$.

We also believe that the homomorphism construction has great potential to yield new constructions of ordered orthogonal arrays, variable strength orthogonal arrays and variable strength covering arrays. In particular, in Section VI for $3 \leq t \leq 5$, we experimentally counted the number of $t$-sets of columns that were covered in our construction of Theorem 16 and also in the Rosenbloom-Tsfasman-Skriganov construction \cite{24}, \cite{25}. It would be interesting to characterize the full set of hyperedges that are covered in the arrays from these constructions.

ACKNOWLEDGMENT

Brett Stevens would like to thank William J. Martin for his many discussions over the years about the construction of or-
orthogonal array-like objects using hypergraph homomorphisms.

The authors would like to thank the anonymous referees for their suggestions that greatly improved this paper.

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