Monomial Testing and Applications

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Abstract. In this paper, we devise two algorithms for the problem of testing \( q \)-monomials of degree \( k \) in any multivariate polynomial represented by a circuit, regardless of the primality of \( q \). One is an \( O^*(2^k) \) time randomized algorithm. The other is an \( O^*(12.8^k) \) time deterministic algorithm for the same \( q \)-monomial testing problem but requiring the polynomials to be represented by tree-like circuits. Several applications of \( q \)-monomial testing are also given, including a deterministic \( O^*(12.8^{mk}) \) upper bound for the \( m \)-set \( k \)-packing problem.

Keywords: Group algebra; complexity; multivariate polynomials; monomials; monomial testing; randomized algorithms; derandomization.

1 Introduction

Recent research on testing multilinear monomials and \( q \)-monomials in multivariate polynomials \cite{14,18,7,8,10,6,9} requires that \( \mathbb{Z}_q \) be a field, which is true when \( q \geq 2 \) is prime. When \( q > 2 \) is not prime, \( \mathbb{Z}_q \) is no longer a field, hence the group algebra based approaches in \cite{14,18,10,9} become inapplicable. When \( q \) is not prime, it remains open whether the problem of testing \( q \)-monomials in a multivariate polynomial can be solved in some compatible complexity, such as \( O^*(c^k) \) time for a constant \( c \geq 2 \). Our work in \cite{2} presents a randomized \( O^*(7.15^k) \) algorithm for testing \( q \)-monomials of degree \( k \) in a multivariate polynomial that is represented by a tree-like circuit. This algorithm works for any fixed integer \( q \geq 2 \), regardless of \( q \)'s primality. Moreover, for prime \( q > 7 \), it provides us with some substantial improvement on the time complexity of the previously known algorithm \cite{10,9} for testing \( q \)-monomials.

Randomized algebraic techniques have recently led to the once fastest randomized algorithms of time \( O^*(2^k) \) for the \( k \)-path problem and other problems \cite{14,18}. Another recent seminal example is the improved \( O(1.657^n) \) time randomized algorithm for the Hamiltonian path problem by Björklund \cite{3}. This algorithm provided a positive answer to the question of whether the Hamiltonian path problem can be solved in time \( O(c^n) \) for some constant \( 1 < c < 2 \), a challenging problem that had been open for half of a century. Björklund \textit{et al.} further extended the above randomized algorithm to the \( k \)-path testing problem with \( O^*(1.657^k) \) time complexity \cite{4}. Very recently, those two algorithms were simplified further by Abasi and Bshouty \cite{1}.
This paper consists of three key contributions: The first is an $O^*(2^k)$ time randomized algorithm that gives an affirmative answer to the $q$-monomial testing problem for polynomials represented by circuits, regardless of the primality of $q \geq 2$. We generalize the circuit reconstruction and variable replacements proposed in [2] to transform the $q$-monomial testing problem, for polynomials represented by a circuit, into the multilinear monomial testing problem and furthermore enabling the usage of the group algebraic approach originated by Koutis [13] to help resolve the $q$-monomial testing problem. The second is an $O^*(12.8^k)$ deterministic algorithm for testing $q$-monomials in multivariate polynomials represented by tree-like circuits. Inspired by the work in [10,9], we devise this deterministic algorithm by derandomizing the first randomized algorithm for tree-like circuits with the help of the perfect hashing functions by Chen et al. [11] and the deterministic polynomial identity testing algorithm by Raz and Shpilka [17] for non-communicative polynomials. The third is to exhibit several applications of $q$-monomial testing to designing algorithms for concrete problems. Specifically, we show how $q$-monomial testing can be applied to the non-simple $k$-path testing problem, the generalized $m$-set $k$-packing problem, and the generalized $P_2$-Packing problem. In particular, we design a deterministic algorithm for solving the $m$-set $k$-packing problem in $O^*(12.8^{mk})$, which is, to our best knowledge, the best upper bound for deterministic algorithms to solve this problem.

2 Notations and Definitions

For variables $x_1, \ldots, x_n$, for $1 \leq i_1 < \cdots < i_k \leq n$, $\pi = x_{i_1}^{s_1} \cdots x_{i_t}^{s_t}$ is called a monomial. The degree of $\pi$, denoted by $\deg(\pi)$, is $\sum_{j=1}^{t} s_j$. $\pi$ is multilinear, if $s_1 = \cdots = s_t = 1$, i.e., $\pi$ is linear in all its variables $x_{i_1}, \ldots, x_{i_t}$. For any given integer $q \geq 2$, $\pi$ is called a $q$-monomial if $1 \leq s_1, \ldots, s_t \leq q - 1$. In particular, a multilinear monomial is the same as a $2$-monomial.

An arithmetic circuit, or circuit for short, is a directed acyclic graph consisting of + gates with unbounded fan-ins, $\times$ gates with two fan-ins, and terminal nodes that correspond to variables. The size, denoted by $s(n)$, of a circuit with $n$ variables is the number of gates in that circuit. A circuit is considered a tree-like circuit if the fan-out of every gate is at most one, i.e., the underlying directed acyclic graph that excludes all the terminal nodes is a tree. In other words, in a tree-like circuit, only the terminal nodes can have more than one fan-out (or out-going edge).

Throughout this paper, the $O^*(\cdot)$ notation is used to suppress poly($n, k$) factors in time complexity bounds.

By definition, any polynomial $F(x_1, \ldots, x_n)$ can be expressed as a sum of a list of monomials, called the sum-product expansion. The degree of the polynomial is the largest degree of its monomials in the expansion. With this expanded expression, it is trivial to see whether $F(x_1, \ldots, x_n)$ has a multilinear monomial, or a monomial with any given pattern. Unfortunately, such an expanded expression is essentially problematic and infeasible due to the fact that a polynomial
may often have exponentially many monomials in its sum-product expansion. The challenge then is to test whether $F(x_1, \ldots, x_n)$ has a multilinear monomial, or any other desired monomial, efficiently but without expanding it into its sum-product representation.

For any integer $k \geq 1$, we consider the group $\mathbb{Z}_2^k$ with the multiplication $\cdot$ defined as follows. For $k$-dimensional column vectors $x, y \in \mathbb{Z}_2^k$ with $x = (x_1, \ldots, x_k)^T$ and $y = (y_1, \ldots, y_k)^T$, $x \cdot y = (x_1 + y_1, \ldots, x_k + y_k)^T$. $v_0 = (0, \ldots, 0)^T$ is the zero element in the group. For any field $\mathcal{F}$, the group algebra $\mathcal{F}[\mathbb{Z}_2^k]$ is defined as follows. Every element $u \in \mathcal{F}[\mathbb{Z}_2^k]$ is a linear sum of the form

$$u = \sum_{x_i \in \mathbb{Z}_2^k, a_i \in \mathcal{F}} a_i x_i. \quad (1)$$

For any element $v = \sum_{x_i \in \mathbb{Z}_2^k, b_i \in \mathcal{F}} b_i x_i$, we define

$$u + v = \sum_{a_i, b_i \in \mathcal{F}, x_i \in \mathbb{Z}_2^k} (a_i + b_i)x_i, \text{ and}$$

$$u \cdot v = \sum_{a_i, b_j \in \mathcal{F}, x_i, y_j \in \mathbb{Z}_2^k} (a_i b_j)(x_i \cdot y_j).$$

For any scalar $c \in \mathcal{F}$,

$$cu = c \left( \sum_{x_i \in \mathbb{Z}_2^k, a_i \in \mathcal{F}} a_i x_i \right) = \sum_{x_i \in \mathbb{Z}_2^k, a_i \in \mathcal{F}} (ca_i)x_i.$$

The zero element in the group algebra $\mathcal{F}[\mathbb{Z}_2^k]$ is $0 = \sum_v 0v$, where 0 is the zero element in $\mathcal{F}$ and $v$ is any vector in $\mathbb{Z}_2^k$. For example, $0 = 0v_0 = 0v_1 + 0v_2 + 0v_3$, for any $v_i \in \mathbb{Z}_2^k$, $1 \leq i \leq 3$. The identity element in the group algebra $\mathcal{F}[\mathbb{Z}_2^k]$ is $1 = 1v_0 = v_0$, where 1 is the identity element in $\mathcal{F}$. For any vector $v = (v_1, \ldots, v_k)^T \in \mathbb{Z}_2^k$, for $i \geq 0$, let $(v)^i = (iv_1, \ldots, iv_k)^T$. In particular, when the field $\mathcal{F}$ is $\mathbb{Z}_2$ (or in general, of characteristic 2), in the group algebra $\mathcal{F}[\mathbb{Z}_2^k]$, for any $z \in \mathbb{Z}_2^k$ we have $(v)^0 = (v)^2 = v_0$, and $z \pm z = 0$.

### 3 A New Transformation

In this section, we shall design a new method to transform any given polynomial $F$ represented by a circuit $C$ to a new polynomial $G$ represented by a new circuit $C''$ such that the $q$-monomial testing problem for $F$ is reduced to the multilinear monomial testing problem for $G$. This method is an extension of the circuit reconstruction and randomized variable replacement methods proposed by us in [2].

To simplify presentation, we assume that if any given polynomial has $q$-monomials in its sum-product expansion, then the degrees of those multilinear
Lemma 1. Let the monomials are at least $k$ and one of them has degree exactly $k$. This assumption is feasible, because when a polynomial has $q$-monomials of degree $\leq k$, e.g., the least degree of those is $\ell$ with $1 \leq \ell < k$, then we can multiply the polynomial by a list of $k - \ell$ new variables so that the resulting polynomial will have $q$-monomials with degrees satisfying the aforementioned assumption.

3.1 A New Circuit Reconstruction Method

In this section and the next, we shall extend the transformation methods designed in [2] to general circuits. The circuit reconstruction and variable replacement methods developed by us in [2] work for tree-like circuits only. In essence, the methods are as follows: Replace each original variable $x$ in the polynomial by a $+$ gate $g$; for each outgoing edge of $x$, duplicate a copy of $g$; for each $g$, allow it to receive inputs from $q - 1$ many new $y$-variables; for each edge from a $y$-variable to a duplicated gate $g$, replace it with a new $\times$ gate that receives inputs from the $y$-variable and a new $z$-variable that then feeds the output to $g$. Additionally, the methods add a new $\times$ gate $f'$ that multiplies the output of $f$ with a new $z$-variable for each $x$-gate $f$ in the original circuit.

For any given polynomial $F(x_1, x_2, \ldots, x_n)$ represented by a circuit $C$ of size $s(n)$, we first reconstruct the circuit $C$ in three steps as follows:

**Duplicating + gates.** Starting at the bottom layer of the circuit $C$, for each + gate $g$ with outgoing edges $f_1, f_2, \ldots, f_\ell$, replace $g$ with $\ell$ copies $g_1, g_2, \ldots, g_\ell$ such that each $g_i$ has the same input as $g$, but the only outgoing edge of $g_i$ is $f_i$, $1 \leq i \leq \ell$.

**Duplicating terminal nodes.** For each variable $x_i$, if $x_i$ is the input to a list of gates $g_1, g_2, \ldots, g_s$, then create $\ell$ terminal nodes $u_1, u_2, \ldots, u_\ell$ such that each of them represents a copy of the variable $x_i$ and $g_j$ receives input from $u_j$, $1 \leq j \leq \ell$.

Let $C^*$ denote the reconstructed circuit after the above two reconstruction steps. Obviously, both circuits $C$ and $C^*$ compute the same polynomial $F$.

**Adding new $\times$ gates and new variables.** Having completed the reconstruction to obtain $C^*$, we then expand it to a new circuit $C'$ as follows. For every edge $e_i$ in $C^*$ (including every edge between a gate and a terminal node) such that $e_i$ conveys the output of $u_i$ to $v_i$, add a new $\times$ gate $g_i$ that multiplies the output of $u_i$ with a new variable $z_i$ and passes the outcome to $v_i$.

Assume that a list of $h$ new $z$-variables $z_1, z_2, \ldots, z_h$ have been introduced into the circuit $C'$. Let $F'(z_1, z_2, \ldots, z_h, x_1, x_2, \ldots, x_n)$ be the new polynomial represented by $C'$.

**Example 1.** Consider $F(x_1, x_2) = 16x_1^5 + 32x_1^3x_2 + 2x_1^2x_2 + 16x_1x_2^2 + 2x_2^3$. Figure 1 shows the circuit $C$ that computes $F(x_1, x_2)$. Figures 2 and 3 show the circuit $C^*$ and the circuit $C'$, respectively.

**Lemma 1.** Let the $t$ be the length of longest path from the root gate of $C$ to its terminal nodes. $F(x_1, x_2, \ldots, x_n)$ has a monomial $\pi$ of degree $k$ in its sum-product expansion if and only if there is a monomial $\alpha \pi$ in the sum-product.
expansion of $F'(z_1, z_2, \ldots, z_h, x_1, x_2, \ldots, x_n)$ such that $\alpha$ is a multilinear monomial of $z$-variables with degree $\leq tk + 1$. Furthermore, if $\pi$ occurs more than once in the sum-product expansion of $F'$, then every occurrence of $\pi$ in $F'$ has a unique coefficient $\alpha$; and any two different monomials of $x$-variables in $F'$ will have different coefficients that are multilinear products of $z$-variables.

Proof. Recall that, by the reconstruction processes, $C^*$ computes exactly the same polynomial $F$. If $F$ has a monomial of degree $k$, then let $T$ be the sub-circuit of $C^*$ that generates the monomial $\pi$, and $T'$ be the corresponding sub-circuit in $C'$. By the way by which the new $z$-variables are introduced, the monomial generated by $T'$ is $\alpha\pi$ with $\alpha$ as the product of all the $z$-variables added to the edges of $T$ to yield $T'$. Since $\pi$ has degree $k$, $T$ has $k$ terminal nodes, corresponding to $k$ paths from the root to those terminal nodes. Thus, $T$ has at most $tk$ edges. Note that one additional $z$-variable is added to the output edge of the root gate. This implies that $\alpha$ is a multilinear monomial of $z$-variables with degree $\leq tk + 1$. 

Fig. 1. Circuit $C$ for $F(x_1, x_2)$  
Fig. 2. Circuit $C^*$ for $F(x_1, x_2)$
Fig. 3. Circuit $C'$ for $F(x_1, x_2)$. Due to space limitation, all $z$-variables for the new $\times$ gates are not shown in the figure.

If $F'$ has a monomial $\alpha \pi$ such that $\alpha$ is a product of $z$-variables and $\pi$ is a product of $x$-variables, then let $\mathcal{M}'$ be the sub-circuit of $C'$ that generates $\alpha \pi$. According to the construction of $C^*$ and $C'$, removing all the $z$-variables along with the newly added $\times$ gates from $\mathcal{M}'$ will result in a sub-circuit $\mathcal{M}$ of $C^*$ that generates $\pi$. Thereby, $\pi$ is a monomial in $F$.

Now, consider that $F'$ has two monomials $\alpha \pi$ and $\beta \phi$ such that, $\pi$ and $\phi$ are products of $x$-variables and $\alpha$ and $\beta$ are products of $z$-variables. Let $\mathcal{T}'_1$ and $\mathcal{T}'_2$ be the sub-circuits in $C'$ that generate $\alpha \pi$ and $\beta \phi$, respectively. Again, according to the construction of $C^*$ and $C'$, removing all the $z$-variables along with the newly added $\times$ gates from $\mathcal{T}'_1$ and $\mathcal{T}'_2$ will result in two sub-circuits $\mathcal{T}_1$ and $\mathcal{T}_2$ of $C^*$ that generate $\pi$ and $\phi$, respectively. When $\pi \neq \phi$, $\mathcal{T}_1$ and $\mathcal{T}_2$ are different sub-circuits, this implies that there is at least an edge $e$ that is in either $\mathcal{T}_1$ or $\mathcal{T}_2$, but not both. Since a new $\times$ gate is added for $e$ with a new $z$-variable, there is at least one $z$-variable that is in either $\mathcal{T}'_1$ or $\mathcal{T}'_2$, but not both. Hence, $\alpha$ and $\beta$ do not share the same set of $z$-variables, because $z$-variables are one to one correspondent to the edges in a sub-circuit. Hence, $\alpha \neq \beta$. Also, since the $z$-variables in $\alpha$ correspond to edges in $\mathcal{T}'_1$, $\alpha$ is multilinear. Similarly, $\beta$ is also multilinear.

Combining the above analysis completes the proof for the lemma.
3.2 Variable Replacements

Following Subsection 3.1, we continue to address how to further transform the new polynomial \( F'(z_1, z_2, \ldots, z_n, x_1, x_2, \ldots, x_n) \) computed by the circuit \( C' \). The method for this part of the transformation is similar to, but different from, the method proposed by us in [2].

**Variable replacements:** Here, we start with the new circuit \( C' \) that computes \( F'(z_1, z_2, \ldots, z_n, x_1, x_2, \ldots, x_n) \). For each variable \( x_i \), we replace it with a “weighted” linear sum of \( q - 1 \) new \( y \)-variables \( y_{i1}, y_{i2}, \ldots, y_{i(q-1)} \). The replacements work as follows: For each variable \( x_i \), we first add \( q - 1 \) new terminal nodes that represent \( q - 1 \) many \( y \)-variables \( y_{i1}, y_{i2}, \ldots, y_{i(q-1)} \). Then, for each terminal node \( u_j \) representing \( x_i \) in \( C' \), we replace \( u_j \) with a + gate. Later, for each new + gate \( g_j \) that is created for \( u_j \) of \( x_i \), let \( g_j \) receive input from \( y_{i1}, y_{i2}, \ldots, y_{i(q-1)} \). That is, we add an edge from each of such \( y \)-variables to \( g_j \). Finally, for each edge \( e_{ij} \) from \( y_{ij} \) to \( g_j \), replace \( e_{ij} \) by a new \( \times \) gate that takes inputs from \( y_{ij} \) and a new \( z \)-variable \( z_{ij} \) and sends the output to \( g_j \).

Let \( C'' \) be the circuit resulted from the above transformation, and

\[
G(z_1, \ldots, z_n, y_{11}, \ldots, y_{1(q-1)}, \ldots, y_{n1}, \ldots, y_{n(q-1)})
\]

be the polynomial computed by the circuit \( C'' \).

**Example 2.** We continue Example 1 in Subsection 3.1. The new circuit \( C'' \) for \( F(x_1, x_2) \) is given in Figure 4.

**Lemma 2.** Let \( F(x_1, x_2, \ldots, x_n) \) be any given polynomial represented by a circuit \( C \) and \( t \) be the length of the longest path of \( C \). For any fixed integer \( q \geq 2 \), \( F \) has a \( q \)-monomial of \( x \)-variables with degree \( k \), then \( G \) has a unique multilinear monomial \( \alpha \pi \) such that \( \pi \) is a degree \( k \) multilinear monomial of \( y \)-variables and \( \alpha \) is a multilinear monomial of \( z \)-variables with degree \( \leq k(t+1)+1 \). If \( F \) has no \( q \)-monomials, then \( G \) has no multilinear monomials of \( y \)-variables, i.e., \( G \) has no monomials of the format \( \beta \phi \) such that \( \beta \) is a monomial of \( z \)-variables and \( \phi \) is a multilinear monomial of \( y \)-variables.
Proof. We first show the second part of the lemma, i.e., if $F$ has no $q$-monomials, then $G$ has no multilinear monomials of $y$-variables. Suppose otherwise that $G$ has a multilinear monomial $\phi$ of $y$-variables with a coefficient $\beta$, which is a monomial of $z$-variables. Let $\phi = \phi_1 \phi_2 \cdots \phi_s$ such that $\phi_j$ is the product of all the $y$-variables in $\phi$ that are used to replace the variable $x_{i_j}$, and let $\deg(\phi_j) = d_j$, $1 \leq j \leq s$. Consider the sub-circuit $T''$ of $C''$ that generates $\beta \phi$ when the $x$-variables are replaced by a "weighted" linear sum of $y$-variables according to the aforementioned variable replacements. Derive the sub-circuit $T'$ in $C'$ that corresponds to $T''$ in $C''$. Also, derive the sub-circuit $T$ in $C^*$ that corresponds to $T'$ in $C$. Then, the sub-circuit $T$ in $C^*$ computes a monomial $\pi = x_1^{d_1} x_2^{d_2} \cdots x_s^{d_s}$ and $\phi$ is a multilinear monomial in the expansion of the replacement

$$r(\pi) = \prod_{j=1}^{s} \left( \prod_{\ell=1}^{d_j} (z_{j\ell} y_{j1} + z_{j\ell} y_{j2} + \cdots + z_{j\ell(q-1)} y_{j(q-1)}) \right).$$

Fig. 4. Circuit $C''$ for $F(x_1, x_2)$. Due to space limitation, all $z$-variables for the new $\times$ gates are not shown in the figure.
which is obtained by the variable replacements described above. If there is one
d_j such that d_j \geq q, then let us look at the replacement for \( x_{i_j}^{d_j} \), denoted as
\[
r(x_{i_j}^{d_j}) = \prod_{\ell=1}^{d_j} (z_{j(\ell-1)}y_{j(\ell-1)}).
\]

Since \( d_j \geq q \), by the pigeon hole principle, the expansion of the above \( r(x_{i_j}^{d_j}) \) has no multilinear monomials. Thereby, we must have \( 1 \leq d_j \leq q - 1, 1 \leq j \leq s \). Hence, \( \pi \) is a \( q \)-monomial in \( F \), a contradiction to our assumption at the beginning. Therefore, when \( F \) has no \( q \)-monomials, then \( G \) must not have any multilinear monomials of \( y \)-variables.

We now prove the first part of the lemma. Suppose \( F \) has a \( q \)-monomial \( \pi = x_{i_1}^{s_1}x_{i_2}^{s_2}\cdots x_{i_t}^{s_t} \) with \( 1 \leq s_j \leq q - 1, 1 \leq j \leq t \). Let \( k = \deg(\pi) \). By Lemma 4, \( F' \) has at least one monomial corresponding to \( \pi \). Moreover, each of those monomials in \( F' \) has a format \( \alpha \pi \) such that \( \alpha \) is a unique product of \( z \)-variables with \( \deg(\alpha) \leq tk + 1 \). Let \( \pi' = \alpha \pi \) be one of those monomials. Consider the sub-circuit \( T' \) of \( C' \) that generates \( \pi' \). Based on the construction of \( C' \), \( T' \) has \( k \) terminal nodes representing \( k \) occurrences of all the \( x \)-variables in \( \pi \). Following the aforementioned variable replacements, each occurrence of those \( x \)-variables is replaced by a + gate with inputs from \( q - 1 \) many \( \times \) gates. Moreover, each of such \( \times \) gates receives inputs from a \( y \)-variable and a \( z \)-variable. For each \( g \) of those \( + \) gates, we select one of the \( q - 1 \) many \( \times \) gates that are inputs to \( g \). Then, the expanded sub-circuit \( T'' \) of \( T' \) with all the selected \( \times \) gates is a sub-circuit in \( C'' \) that generates a monomial \( \beta \phi \), where \( \phi \) is a multilinear monomial of \( y \)-variables with degree \( k \), and \( \beta \) is the product of \( \alpha \) with those additional \( z \)-variables in \( T'' \) but not in \( T' \), and the degree of \( \beta \) is \( k(t + 1) + 1 \).

4 A Faster Randomized Algorithm

Recently, an \( O^*(7.15^k) \) time randomized algorithm has been devised by us in \cite{2} for testing \( q \)-monomials in any polynomial represented by a tree-like circuit. We now extend this result to general circuits with a better \( O^*(2^k) \) upper bound.

Consider any given polynomial \( F(x_1, x_2, \ldots, x_n) \) that is represented by a circuit \( C \) of size \( s(n) \). Note that the length of the longest path from the root of \( C \) to any terminal node is no more than \( s(n) \).

Let \( d = \log_2(k(s(n) + 1) + 1) \) and \( \mathcal{F} = \text{GF}(2^d) \) be a finite field of \( 2^d \) many elements. We consider the group algebra \( \mathcal{F}[Z_2^d] \). Please note that the field \( \mathcal{F} = \text{GF}(2^d) \) has characteristic 2. This implies that, for any given element \( w \in \mathcal{F} \), adding \( w \) for any even number of times yields 0. For example, \( w + w = 2w = w + w + w + w = 4w = 0 \).

The algorithm RTM for testing whether \( F(x_1, x_2, \ldots, x_n) \) has a \( q \)-monomial of degree \( k \) is given in the following.

Algorithm RTM (Randomized Testing of \( q \)-Monomials):
1. As described in Subsections 3.1 and 3.2, reconstruct the circuit $C$ to obtain $C^*$ that computes the same polynomial $F$ and then introduce new $z$-variables to $C^*$ to obtain the new circuit $C'$ that computes $F'(z_1, z_2, \ldots, z_h, x_1, x_2, \ldots, x_n)$. Finally, obtain a circuit $C''$ by variable replacements so that $F'$ is transformed to $G(z_1, \ldots, z_h, y_{11}, \ldots, y_{1(q-1)}, \ldots, y_{n1}, \ldots, y_{n(q-1)})$.

2. Select uniform random vectors $v_{ij} \in Z^k_2 - \{v_0\}$, and replace the variable $y_{ij}$ with $(v_{ij} + v_0)$, $1 \leq i \leq n$ and $1 \leq j \leq q - 1$.

3. Use $C''$ to calculate $G' = G(z_1, \ldots, z_h, (v_{11} + v_0), \ldots, (v_{1(q-1)} + v_0), \ldots, (v_{n1} + v_0), \ldots, (v_{n(q-1)} + v_0))$

\[ = \sum_{j=1}^{2^k} f_j(z_1, \ldots, z_h) \cdot v_j, \quad (2) \]

where each $f_j$ is a polynomial of degree $\leq k(s(n)+1)+1$ (see Lemma 2) over the finite field $F = GF(2^d)$, and $v_j$ with $1 \leq j \leq 2^k$ are the $2^k$ distinct vectors in $Z^k_2$.

4. Perform polynomial identity testing with the Schwartz-Zippel algorithm [15] for every $f_j$ over $F$. Return ”yes” if one of those polynomials is not identical to zero. Otherwise, return ”no”.

It should be pointed out that the actual implementation of Step 4 would be running the Schwartz-Zippel algorithm concurrently for all $f_j$, $1 \leq j \leq 2^k$, utilizing the circuit $C''$. If one of those polynomials is not identical to zero, then the output of $G'$ as computed by circuit $C''$ is not zero.

The group algebra technique established by Koutis [14] assures the following two properties:

**Lemma 3.** ([14]) Replacing all the variables $y_{ij}$ in $G$ with group algebraic elements $v_{ij} + v_0$ will make all monomials $\alpha \pi$ in $G'$ to become zero, if $\pi$ is non-multilinear with respect to $y$-variables. Here, $\alpha$ is a product of $z$-variables.

**Proof.** Recall that $F$ has characteristic 2. For any $v \in Z^k_2$, in the group algebra $F[Z^k_2],$

\[ (v + v_0)^2 = v \cdot v + 2 \cdot v \cdot v_0 + v_0 \cdot v_0 \]

\[ = v_0 + 2 \cdot v + v_0 \]

\[ = 2 \cdot v_0 + 2 \cdot v = 0. \quad (3) \]

Thus, the lemma follows directly from expression (3).

**Lemma 4.** ([14]) Replacing all the variables $y_{ij}$ in $G$ with group algebraic elements $v_{ij} + v_0$ will make any monomial $\alpha \pi$ to become zero, if and only if the
vectors \( v_{ij} \) are linearly dependent in the vector space \( \mathbb{Z}_2^k \). Here, \( \pi \) is a multilinear monomial of \( y \)-variables and \( \alpha \) is a product of \( z \)-variables. Moreover, when \( \pi \) becomes non-zero after the replacements, it will become the sum of all the vectors in the linear space spanned by those vectors.

**Proof.** The analysis below gives a proof for this lemma. Suppose \( V \) is a set of linearly dependent vectors in \( \mathbb{Z}_2^k \). Then, there exists a nonempty subset \( T \subseteq V \) such that \( \prod_{v \in T} v = v_0 \). For any \( S \subseteq T \), since \( \prod_{v \in T} v = (\prod_{v \in S} v) \cdot (\prod_{v \in T - S} v) \), we have \( \prod_{v \in S} v = \prod_{v \in T - S} v \). Thereby, we have

\[
\prod_{v \in S} (v + v_0) = \prod_{v \in T - S} (v + v_0) = 0,
\]

since every \( \prod_{v \in S} v \) is paired by the same \( \prod_{v \in T - S} v \) in the sum above and the addition of the pair is annihilated because \( F \) has characteristic 2. Therefore,

\[
\prod_{v \in V} (v + v_0) = \sum_{S \subseteq T} (\prod_{v \in S} v) = 0.
\]

Now consider that vectors in \( V \) are linearly independent. For any two distinct subsets \( S, T \subseteq V \), we must have \( \prod_{v \in T} v \neq \prod_{v \in S} v \), because otherwise vectors in \( S \cup T - (S \cap T) \) are linearly dependent, implying that vectors in \( V \) are linearly dependent. Therefore,

\[
\prod_{v \in V} (v + v_0) = \sum_{T \subseteq V} (\prod_{v \in T} v)
\]

is the sum of all the \( 2^{|V|} \) distinct vectors spanned by \( V \).

**Theorem 1.** Let \( q > 2 \) be any fixed integer and \( F(x_1, x_2, \ldots, x_n) \) be an \( n \)-variate polynomial represented by a circuit \( C \) of size \( s(n) \). Then, the randomized algorithm RTM can decide whether \( F \) has a \( q \)-monomial of degree \( k \) in its sum-product expansion in time \( O^*(2^k s^6(n)) \).

Since we are often interested in circuits with polynomial sizes in \( n \), the time complexity of algorithm RTM is \( O^*(2^k) \) for those circuits.

**Proof.** From the introduction of the new \( z \)-variables to the circuit \( C' \), it is easy to see that every monomial in \( F' \) has the format \( \alpha \pi \), where \( \pi \) is a product of \( x \)-variables and \( \alpha \) is a product of \( z \)-variables. Since only \( x \)-variables are replaced by their respective "weighted" linear sums of new \( y \)-variables as specified in Subsection 3.2, monomials in \( G \) have the format \( \beta \phi \), where \( \phi \) is a product of \( y \)-variables and \( \beta \) is a product of \( z \)-variables.

Suppose that \( F \) has no \( q \)-monomials. By Lemma 2, \( G \) has no monomials \( \beta \phi \) such that \( \phi \) is multilinear with respect to \( y \)-variables. Moreover, by Lemma 3
replacing \( y \)-variables by group algebraic elements at Step 2 will make \( \phi \) in every monomial \( \beta \phi \) in \( G \) to become zero. Hence, the group algebraic replacements will make \( G \) to become zero and so the algorithm RTM will return "no".

Assume that \( F \) has a \( q \)-monomial of degree \( k \). By Lemma \( 2 \), \( G \) has a monomial \( \beta \phi \) such that \( \phi \) is a multilinear monomial of degree \( k \) with respect to \( y \)-variables and \( \beta \) is a multilinear monomial of degree \( \leq k(s(n) + 1) + 1 \) with respect to \( z \)-variables. It follows from a lemma in \([5]\) (see also, \([4]\)) , that a list of uniform random vectors from \( Z_2^k \) will be linearly independent with probability at least 0.28. By Lemma \( 4 \) with probability at least 0.28, the multilinear monomial \( \phi \) will not be annihilated by the group algebraic replacements at Step 2. Precisely, with probability at least 0.28, \( \beta \phi \) will become

\[
\lambda(\beta \phi) = \sum_{i=1}^{2^k} \beta v_i , \tag{4}
\]

where \( v_i \) are distinct vectors in \( Z_2^k \).

Let \( S \) be the set of all those multilinear monomials \( \beta \phi \) that survive the group algebraic replacements for \( y \)-variables in \( G \). Then,

\[
G' = G(z_1, \ldots, z_h, (v_{11} + v_0), \ldots, (v_{1(q-1)} + v_0), \ldots, (v_{n1} + v_0), \ldots, (v_{n(q-1)} + v_0))
\]

\[
= \sum_{\beta \phi \in S} \lambda(\beta \phi)
\]

\[
= \sum_{\beta \phi \in S} \left( \sum_{i=1}^{2^k} \beta v_i \right)
\]

\[
= \sum_{j=1}^{2^k} \left( \sum_{\beta \phi \in S} \beta \right) v_j . \tag{5}
\]

Let

\[
f_j(z_1, \ldots, z_h) = \sum_{\beta \phi \in S} \beta .
\]

By Lemmas \( 2 \) and \( 3 \) the degree of \( \beta \) is at most \( k(s(n) + 1) + 1 \). Hence, the coefficient polynomial \( f_j \) with respect to \( v_j \) in \( G' \) after the group algebraic replacements has a degree \( \leq k(s(n) + 1) + 1 \). Also, by Lemma \( 4 \), \( \beta \) is unique with respect to every \( \phi \) for each monomial \( \beta \phi \) in \( G \). Thus, the possibility of a "zero-sum" of coefficients from different surviving monomials is completely avoided during the computation for \( f_j \). Therefore, conditioned on that \( S \) is not empty, \( G' \) must not be identical to zero, i.e., there exists at least one \( f_j \) that is not identical to zero. At Step 4, we use the randomized algorithm by Schwartz-Zippel \([15]\) to test whether \( f_j \) is identical to zero. Since the degree of each \( f_j \) is at most \( k(s(n) + 1) + 1 \), it is known that this testing can be done with
probability at least $1 - \frac{\deg(f_j)}{2^{s(n)}} \geq \frac{1}{2}$ in time polynomially in $s(n)$ and $\log_2 |\mathcal{F}| = \log_2 (k(s(n)+1)+1)+1$. Since $\mathcal{S}$ is not empty with probability at least 0.28, the success probability of testing whether $G$ has a degree $k$ multilinear monomial of $y$-variables is at least $0.28 \times \frac{1}{2} > \frac{1}{8}$.

Finally, we address the issues of how to calculate $G'$ and the time needed to do so. Naturally, every element in the group algebra $\mathcal{F}[\mathbb{Z}_2^k]$ can be represented by a vector in $\mathbb{Z}_2^k$. Adding two elements in $\mathcal{F}[\mathbb{Z}_2^k]$ is equivalent to adding the two corresponding vectors in $\mathbb{Z}_2^k$, and the latter can be done in $O(2^k \log_2 |\mathcal{F}|)$ time via component-wise sum. In addition, multiplying two elements in $\mathcal{F}[\mathbb{Z}_2^k]$ is equivalent to multiplying the two corresponding vectors in $\mathbb{Z}_2^k$, and the latter can be done in $O(k2^{k+1} \log_2 |\mathcal{F}|)$ with the help of a similar Fast Fourier Transform style algorithm as in Williams [18]. By the circuit reconstruction and variable replacements in Subsections 3.1 and 3.2, the size of the circuit $C''$ is at most $s^3(n)$. Calculating $G'$ by the circuit $C''$ consists of $n \times s^6(n)$ arithmetic operations of either adding or multiplying two elements in $\mathcal{F}[\mathbb{Z}_2^k]$ based on the circuit $C''$. Hence, the total time needed is $O(n \times s^6(n)k2^{k+1} \log_2 |\mathcal{F}|)$. At Step 4, we run the Schwartz-Zippel algorithm on $G'$ to simultaneously test whether there is one $f_j$ such that $f_j$ is not identical to zero. Recall that $\log_2 |\mathcal{F}| = \log_2 (k(s(n)+1)+1)+1$. The total time for the entire algorithm is $O^*(2^k s^6(n))$.

5 A Deterministic Algorithm via Derandomization

We shall devise a deterministic algorithm for testing $q$-monomials in a multivariate polynomial represented by a tree-like circuit. Our approach is to derandomize Steps 2 and 4 in algorithm RTM respectively with the help of two advanced techniques of perfect hashing by Chen et al. [11] (see also Naor et al. [10]) and noncommunicative multivariate polynomial identity testing by Raz and Shpilka [17]. Our approach follows the work in [10-14]. However, we are no longer require $q$ to be a prime and also obtain a better time bound.

Definition 1. (See, Chen et al. [11]) Let $n$ and $k$ be two integers such that $1 \leq k \leq n$. Let $\mathcal{A} = \{1, 2, \ldots, n\}$ and $\mathcal{K} = \{1, 2, \ldots, k\}$. A $k$-coloring of the set $\mathcal{A}$ is a function from $\mathcal{A}$ to $\mathcal{K}$. A collection $\mathcal{F}$ of $k$-colorings of $\mathcal{A}$ is a $(n, k)$-family of perfect hashing functions if for any subset $W$ of $k$ elements in $\mathcal{A}$, there is a $k$-coloring $h \in \mathcal{F}$ that is injective from $W$ to $\mathcal{K}$, i.e., for any $x, y \in W$, $h(x)$ and $h(y)$ are distinct elements in $\mathcal{K}$.

Like in the design of algorithm RTM, we assume, without loss of generality, that when a polynomial has $q$-monomials in its sum-product expansion, one of the $q$-monomials has exactly a degree of $k$ and all the rest of those will have degrees at least $k$.

Theorem 2. Let $q \geq 2$ be fixed integer. Let $F(x_1, x_2, \ldots, x_n)$ be an $n$-variate polynomial of degree $k$ represented by a tree-like circuit $\mathcal{C}$ of size $s(n)$. There is a deterministic $O^*(12.8^k s^6(n))$ time algorithm to test whether $F$ has a $q$-monomial of degree $k$ in its sum-product expansion.
Proof. Let \( d = \log_2(k(s(n) + 1) + 1) + 1 \) and \( \mathcal{F} = \text{GF}(2^d) \) be a finite field of \( 2^d \) elements. The deterministic algorithm DTM for testing whether \( F \) has a \( q \)-monomial of degree \( k \) is given as follows.

Algorithm DTM (Deterministic Testing of \( q \)-Monomials):
1. As in the Algorithm RTM, following circuit reconstruction and variable replacements in Subsections 3.1 and 3.2, reconstruct the circuit \( \mathcal{C} \) to obtain \( \mathcal{C}^* \) that computes the same polynomial \( F \) and then introduce new \( z \)-variables to \( \mathcal{C}^* \) to obtain the new circuit \( \mathcal{C}' \) that computes \( F'(z_1, z_2, \ldots, z_h, x_1, x_2, \ldots, x_n) \). Finally, perform variable replacements to obtain the circuit \( \mathcal{C}'' \) that transforms \( F' \) to

\[
G(z_1, \ldots, z_h, y_{11}, \ldots, y_{1(q-1)}, \ldots, y_n, \ldots, y_{n(q-1)}).
\]

2. Construct with the algorithm by Chen at el. \[11\] a \( ((q-1)ns(n), k) \)-family of perfect hashing functions \( \mathcal{H} \) of size \( O(6.4^k \log_2((q-1)ns(n))) \).

3. Select \( k \) linearly independent vectors \( v_1, \ldots, v_k \in \mathbb{Z}_2^k \). (No randomization is needed at this step, either.)

4. For each perfect hashing function \( \lambda \in \mathcal{H} \) do

4.1. Let \( \gamma(i, j) \) be any given one-to-one mapping from \( \{(i, j)| 1 \leq i \leq n \text{ and } 1 \leq j \leq q - 1 \} \) to \( \{1, 2, \ldots, (q-1)n\} \) to label variables \( y_{ij} \). Replace each variable \( y_{ij} \) in \( G \) with \( (v_{\lambda(\gamma(i, j))} + v_0) \), \( 1 \leq i \leq n \) and \( 1 \leq j \leq q - 1 \).

4.2. Use \( \mathcal{C}'' \) to calculate

\[
G' = G(z_1, \ldots, z_h, (v_{\lambda(\gamma(1, 1))} + v_0), \ldots, (v_{\lambda(\gamma(1, (q-1)))} + v_0),
\]

\[
\ldots, (v_{\lambda(\gamma(n, 1))} + v_0), \ldots, (v_{\lambda(\gamma(n, (q-1)))} + v_0))
\]

\[
= \sum_{j=1}^{2^k} f_j(z_1, \ldots, z_h) \cdot v_j,
\]

where each \( f_j \) is a polynomial of degree \( \leq k(s(n) + 1) + 1 \) (see, Lemma \[2\] over the finite field \( \mathcal{F} = \text{GF}(2^d) \), and \( v_j \) with \( 1 \leq j \leq 2^k \) are the \( 2^k \) distinct vectors in \( \mathbb{Z}_2^k \).

4.3. Perform polynomial identity testing with the Raz and Shpilka algorithm \[17\] for every \( f_j \) over \( \mathcal{F} \). Stop and return "yes" if one of them is not identical to zero.

5. If all perfect hashing functions \( \lambda \in \mathcal{H} \) have been tried without returning "yes", then stop and output "no".

The correctness of algorithm DTM is guaranteed by the nature of perfect hashing and the correctness of algorithm RTM. We shall now focus on analyzing the time complexity of the algorithm.

Note that \( q \) is a fixed constant. By Chen at el.\[11\], Step 2 can be done in \( O(6.4^k n \log^2((q - 1)n)) = O^*(6.4^k) \) time. Step 3 can be easily done in \( O(k^2) \) time.
It follows from Lemma 3 that all those monomials that are not \(q\)-monomials in \(F\), and hence in \(F'\), will be annihilated when variables \(y_{ij}\) are replaced by \((v_{\lambda((i,j))} + v_0)\) in \(G\) at Step 4.1.

Consider any given \(q\)-monomial \(\pi = x_{i_1}^{s_1} \cdots x_{i_t}^{s_t}\) of degree \(k\) in \(F\) with \(1 \leq s_j \leq q - 1\) and \(k = \deg(\pi)\), \(j = 1, \ldots, t\). By Lemma 2, there are monomials \(\alpha \pi\) in \(F'\) such that \(\alpha\) is a multilinear monomial of \(z\)-variables with degree \(\leq k(s(n)+1)+1\), and all such monomials are distinct. By Lemma 4 \(\pi\) (hence, \(\alpha \pi\)) will survive the replacements at Step 4.1. Let \(S\) be the set of all the surviving \(q\)-monomials \(\alpha \pi\). Following the same analysis as in the proof of Theorem 1, we have

\[
G' = G(z_1, \ldots, z_h, (v_{\lambda(\gamma(1,1))} + v_0), \ldots, (v_{\lambda(\gamma(1,(q-1)))} + v_0), \ldots, (v_{\lambda(\gamma(n,1))} + v_0), \ldots, (v_{\lambda(\gamma(n,(q-1)))} + v_0))
\]

\[
= \sum_{j=1}^{2^k} \left( \sum_{\beta \in S} \beta \right) v_j
\]

\[
= \sum_{j=1}^{2^k} f_j(z_1, \ldots, z_h) v_j
\]

\[
\neq 0
\]

since \(S\) is not empty. Here,

\[
f_j(z_1, \ldots, z_h) = \sum_{\beta \in S} \beta.
\]

This means that, conditioned on that \(S\) is not empty, there is at least one \(f_j\) that is not identical to zero. Again, as in the analysis for algorithm RTM, the time needed for calculating \(G'\) is \(O^*(2^k s^6(n))\) when the replacements are fixed for \(x\)-variables and the subsequent algebraic replacements are given for \(y\)-variables.

We now consider imposing noncommunicativity on \(z\)-variables in \(C''\). This can be done by imposing an order for \(z\)-variable inputs to any gates in \(C''\). Technically, however, we shall allow values for \(z\)-variables to communicate with those for \(y\)-variables. Finally, we use the algorithm by Raz and Shpilka [17] to test whether \(f_j(z_1, \ldots, z_h)\) is identical to zero of not. This can be done in time polynomially in \(s(n)\) and \(n\), since with the imposed order for \(z\)-variables \(f_j\) is a non-communicative polynomial represented by a tree-like circuit.

Combining the above analysis, the total time of the algorithm DTM is \(O^*(6.4^k \times 2^k s^6(n)) = O^*(12.8^k s^6(n))\).

When the circuit size \(s(n)\) is a polynomial in \(n\), the time bound becomes \(O^*(12.8^k)\).

6 Applications

We list three applications of the \(q\)-monomial testing to concrete algorithm designs. Here, we assume \(q \geq 2\) is a fixed integer. Notably, algorithm DTM can
help us to derive a deterministic algorithm for solving the $m$-set $k$-packing problem in $O^*(12.8^{mk})$, which is, to our best knowledge, the best upper bound for deterministic algorithms to solve this problem.

### 6.1 Allowing Overlapping in $m$-Set $k$-Packing

Let $S$ be a collection of sets so that each member in $S$ is a subset of an $n$-element set $X$. Additional, members in $S$ have the same size $m \geq 3$. We may like to ask whether there are $k$ members in $S$ such that those members are either pairwise disjoint or at most $q-1$ members may overlap. This problem with respect to $q$ is a generalized version of the $m$-Set $k$-packing problem.

We can view each element in $X$ as a variable. Thus, a member in $S$ is a monomial of $m$ variables. Let

$$F(S, k) = \left( \sum_{A \in S} f(A) \right)^k,$$

where $f(A)$ denotes the monomial derived from $A$. Then, the above generalized problem $m$-set $k$-packing with respect to $q$ is equivalent to ask whether $F(S, k)$ has a $q$-monomial of degree $mk$. Again, algorithm RTM solves this problem in $O^*(2^{mk})$ time. When $q = 2$, the $O^*(2^{mk})$ bound was obtained in [14].

Since $F(S, k)$ can be represented by a tree-like circuit, we can choose $q = 2$ and apply algorithm DTM to test whether $F(S, k)$ has multilinear monomial (i.e., 2-monomial) of degree $mk$. Therefore, we have a deterministic algorithm to solve the $m$-set $k$-packing problem in $O^*(12.8^{mk})$ time. Although there are many faster randomized algorithms for solving this problem, for deterministic algorithms our $O^*(12.8^{mk})$ upper bound significantly improves the best known upper bound $O^*(\exp(O(mk)))$ by Fellow et al. [12]. The upper bound in [12] has a large hidden constant in the exponent, e.g., in the case of $r = 3$, their upper bound is $O^*((12.7D)^{3m})$ for some $D \geq 10.4$.

### 6.2 Testing Non-Simple $k$-Paths

Given any undirected graph $G = (V, E)$ with $|V| = n$, we may like to know whether there is a $k$-path in $G$ such that the path may have loops but any vertex in the path can appear at most $q-1$ times. It is easy to see that this non-simple $k$-path problem with respect to $q$ is a generalized version of the simple $k$-path problem.

For each vertex $v_i \in V$, define a polynomial $F_{k,i}$ as follows:

$$F_{1,i} = x_i,$$

$$F_{k+1,i} = x_i \left( \sum_{(v_i, v_j) \in E} F_{k,j} \right), \quad k > 1.$$
We define a polynomial for $G$ as

$$F(G, k) = \sum_{i=1}^{n} F_{k,i}.$$ 

Obviously, $F(G, k)$ can be represented by an arithmetic circuit. It is easy to see that the graph $G$ has a non-simple $k$-path with respect to $q$, if and only if $F(G, k)$ has a $q$-monomial of degree $k$. Algorithm RTM can solve this problem in $O^*(2^k)$ time. When $q = 2$, the $O^*(2^k)$ bound was obtained in [14,18].

### 6.3 A Generalized $P_2$-Packing Problem

Given any undirected graph $G = (V, E)$ with $|V| = n$ and an integer $k$, we can collect $P_2$’s from $G$, i.e., simple paths of length 2 in $G$. The generalized $P_2$-packing problem with respect to $q$ asks whether there is a collection of $k$ many $P_2$’s such that either all those $P_2$’s are pairwise disjoint, or at most $q - 1$ of them may share a common vertex. The generalized $P_2$-packing problem with respect to $q$ can be easily transformed to a generalized 3-Set $k$-Packing problem with respect to $q$. Thereby, an $O^*(2^{3k})$ time randomized solution is given by algorithm RTM. When $q = 2$, the $O^*(2^{3k})$ bound was obtained in [13].

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