Indexing and querying color sets of images

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Abstract

We aim to study the set of color sets of continuous regions of an image given as a matrix of $m$ rows over $n \geq m$ columns where each element in the matrix is an integer from $[1, \sigma]$ named a color. The set of distinct colors in a region is called fingerprint. We aim to compute, index and query the fingerprints of all rectangular regions named rectangles. The set of all such fingerprints is denoted by $F$. A rectangle is maximal if it is not contained in a greater rectangle with the same fingerprint. The set of all locations of maximal rectangles is denoted by $L$. We first explain how to determine all the $|L|$ maximal locations with their fingerprints in expected time $O(nm^2\sigma)$ using a Monte Carlo algorithm (with polynomially small probability of error) or within deterministic $O(nm^2\sigma \log \frac{|L|}{nm^2} + 2)$ time. We then show how to build a data structure which occupies $O(nm \log n + |L|)$ space such that a query which asks for all the maximal locations with a given fingerprint $f$ can be answered in time $O(|f| + \log \log n + k)$, where $k$ is the number of maximal locations with fingerprint $f$. If the query asks only for the presence of the fingerprint, then the space usage becomes $O(nm \log n + |F|)$ while the query time becomes $O(|f| + \log \log n)$. We eventually consider the special case of squared regions (squares).

1 Introduction

In this paper, we are interested in studying and indexing the set of all sets of distinct colors of continuous regions of a given image in order to quickly answer to several queries on this set, for instance, does there exist at least one region with a given set of colors and if so, what are the positions of all such regions in the image?

Also the considered indexing structures and algorithms are a good approach towards efficient image comparisons and clustering based on color sets. To our knowledge, this is the first time such a problem is formalized and analyzed, and to build a general well founded algorithmic framework we begin by formalizing our notions.

We consider an image as a matrix $M$ of $m$ rows over $n$ columns (see Fig. 1) where each $a_{i,j}$ is an integer from $[1, \sigma]$ [1]. This matrix is called an image

1 We assume that $\sigma \leq nm$. If this is not the case, then we can build a dictionary data structure that occupies $O(nm)$ space and that can remap in constant time distinct characters from the original alphabet to distinct integers in $[1, nm]$. 

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and henceforth, we will refer to the integers stored in the matrix by colors or characters.

To design our indexing structures and algorithms, we extend the definition of fingerprints (or sets of distinct characters), initially defined on sequences \([2, 6, 7, 12]\), to images. While classical pattern matching approaches on images have already been studied \([3, 1, 4]\), this paper is the first (to our knowledge) to focus on the character sets. We assume below that \(m \leq n\). We denote by \(\langle i_0, i_1; j_0, j_1 \rangle\) where \(i_0 \leq i_1\) and \(j_0 \leq j_1\), the rectangle in \(M\) bounded by the \(i_0\)-th and \(i_1\)-th rows and \(j_0\)-th and \(j_1\)-th columns including these rows and columns. We also denote by \(f(i_0, i_1; j_0, j_1)\) the set of distinct colors contained in the rectangle \(\langle i_0, i_1; j_0, j_1 \rangle\). This set is called the fingerprint of \(\langle i_0, i_1; j_0, j_1 \rangle\).

**Definition 1** A rectangle in an image is maximal if it is not contained in a greater rectangle with the same fingerprint.

In other words a rectangle is maximal if any extension of the rectangle in one of the four directions will add at least one color not present in the rectangle. Figure 2 shows an example of an image containing many maximal rectangles.

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**Theorem 1** Given an image of \(m\) rows by \(n \geq m\) columns, we can determine all the \(|\mathcal{L}|\) maximal locations with their fingerprints in expected time \(O(nm^2\sigma)\).
with a Monte Carlo algorithm (with polynomially small probability of error) or within deterministic $O(\frac{nm^2 \log(|L| \cdot n m^2 + 2)}{nm^2})$ time.

Note that the total deterministic time is $O(nm^2 \log \sigma)$ in the worst case (when we have $|L| = \Theta(nm^2 \sigma)$), but is only $O(nm^2 \sigma)$ (which is as good as the Monte Carlo time) as long as $|L| \leq O(nm^2)$.

Theorem 2 Given an image of $m$ rows by $n \geq m$ columns, whose set $L$ of all maximal locations and set $F$ of associated fingerprints have been already determined, we can build a data structure which occupies space $O(nm^2 \log n + |L|)$ such that a query which asks for all the maximal locations with a given fingerprint $f$ can be answered in time $O(|f| + \log \log n + k)$, where $k$ is the number of maximal locations with fingerprint $f$. If the query asks only for the presence of the fingerprint, then the space usage becomes $O(nm \log n + |F|)$ while the query time becomes $O(|f| + \log \log n)$.

The construction times of the data structures mentioned in Theorem 2 are respectively $O(nm \log n \log \log n + |L|)$ and $O(nm \log n \log \log n + |F|)$.

We eventually consider the special case in which only squared regions called squares are considered instead of rectangles. We develop a specialized faster algorithm for this case.

1.1 Notations and tools

Let $\langle i_0, i_1; j_0, j_1 \rangle$ be a rectangle. For this rectangle, the $i_0$-th row is called the bottom row, the $i_1$-th row is called the top row, the $j_0$-th column is called the left column, and the $j_1$-th column is called the right column. We also define the following notions derived from Definition 1: a rectangle $\langle i_0, i_1; j_0, j_1 \rangle$ is maximal to the left (resp. to the right) if rectangle $\langle i_0, i_1; j_0-1, j_1 \rangle$ (resp. $\langle i_0, i_1; j_0, j_1+1 \rangle$) doesn’t have the same fingerprint, and is maximal to the bottom (resp. to the top) if rectangle $\langle i_0-1, i_1; j_0, j_1 \rangle$ (resp. $\langle i_0, i_1+1; j_0, j_1 \rangle$) doesn’t have the same fingerprint. It is obvious that a rectangle is maximal if and only if it is maximal in all the directions. One of our solutions uses the following lemma by Muthukrishnan [14]:

**Lemma 1** Given a sequence of colors $T[1,n]$ each chosen from the same alphabet of colors $[1,\sigma]$, in time $O(n)$ we can preprocess the sequence into a data structure which occupies $O(n)$ space so that given any range $[i,j]$ we can find the set of all distinct colors occurring in $T[i..j]$ in time $O(k)$ where $k$ is the number of reported colors. Moreover the data structure reports the colors ordered by their last occurrence, returning the last occurrence of each color.

And also the following lemma from [13]:

**Lemma 2** Given a set of $n$ colored points (with colors from $[1,\sigma]$) stored in a grid of $U$ columns by $U$ rows, we can build a data structure which occupies
We note that a (highly) naive algorithm which would try all possible rectangles and explicitly compute the fingerprint of each rectangle would take time \(\Omega(n^2m^3)\). In the following, we show that this cost can be reduced to just about \(O(nm^2\sigma)\).

Let, for some fixed \(i_0\) and \(i_1\), \(R(i_0, i_1)\) be the set of all rectangles with \(i_0\)-th bottom row and \(i_1\)-th upper row. Let, for some fixed \(j_0\) and \(j_1\), \(R(i_0, i_1; j_0, \ast)\) be the set of all rectangles from \(R(i_0, i_1)\) with \(j_0\)-th left column, and \(R(i_0, i_1; \ast, j_1)\) be the set of all rectangles from \(R(i_0, i_1)\) with \(j_1\)-th right column. For each \(i_0, i_1, j_0\), we construct the sequence \(\varphi(i_0, i_1; j_0)\) consisting of distinct colors and end markers as follows.

Let \(r\) be the longest rectangle from \(R(i_0, i_1; j_0, \ast)\) which is maximal to the left and to the right (if it exists). Then the subsequence formed by all distinct colors of \(\varphi(i_0, i_1; j_0)\) in left-to-right order is actually the sequence of all distinct colors from the fingerprint of \(r\) satisfying the following property: if in this sequence a color \(c\) is before a color \(b\) then the leftmost occurrence of \(c\) in \(\langle i_0, i_1; j_0, n \rangle\) is not to the right of the leftmost occurrence of \(b\) in this rectangle.

By definition, each maximal rectangle from \(R(i_0, i_1; j_0, \ast)\) is a subrectangle of \(r\), so the set of colors in the fingerprint of such maximal rectangle is a prefix of the sequence \(\varphi(i_0, i_1; j_0)\). Then we include in \(\varphi(i_0, i_1; j_0)\) after the last color of this prefix the end marker for this maximal rectangle. It is obvious that distinct maximal rectangles from \(R(i_0, i_1; j_0, \ast)\) have distinct fingerprints, and thus distinct maximal rectangles correspond to distinct end markers.

2.1 Computing all \(\varphi(i_0, i_1; j_0)\)

2.1.1 Data structures

For computing \(\varphi(i_0, i_1; j_0)\) the following data structures are used.

**SLL.** The sequence of last colors (SLL) for a string \(p\) of colors is the sequence \((p_{i_1}, i_1), (p_{i_2}, i_2), \ldots, (p_{i_k}, i_k)\) where \(p_{i_1}, p_{i_2}, \ldots, p_{i_k}\) are all colors contained in \(p\),

\[^2^2\text{We have } O(m^2n^2) \text{ rectangles which we can process in a certain order so that we can determine the set of colors contained in every rectangle as the union of the } O(m) \text{ colors of the last column of the rectangle with the set of colors of another smaller rectangle that was previously processed.}

2 Determination of all fingerprints and maximal locations

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and \(i_j\) are the rightmost position of color \(p_{i_j}\) in \(p\), and \(i_1 < i_2 < \ldots < i_k\). We denote the SLL for the string \(a_{i_1}a_{i_2}\ldots a_{i_j}\) (the segment of the line number \(i\) that spans columns 1 to \(j\)) by \(\text{SLL}(i; j)\).

For any rectangle \(r\) we also consider two following related data structures.

**SLC.** The sequence of last columns (SLC) for a rectangle \(r\) is a sequence \((l_1, k_1), (l_2, k_2), \ldots, (l_s, k_s)\) with \(l_1 < l_2 < \ldots < l_s\) and where \(l_j\) is the number of a column containing at least one rightmost occurrence of some color in \(r\), and \(k_j\) is the number of distinct colors whose rightmost occurrences are contained in column \(l_j\) (for convenience we will assume that colors of the input alphabet which have no occurrences in the rectangle are contained in 0-th column, i.e. if there exist such colors then SLC has first item \((0, k)\) where \(k\) in the number of such colors).

**LLP.** The array of last color pointers (LLP) for a rectangle \(r\), is an array of size \(\sigma\) which for each color \(c\) contains the pointer to the item of SLC corresponding to the column containing the rightmost occurrence(s) of \(c\) in \(r\).

The SLC sequence and LLP array for \(\langle i_0, i_1; 1, j \rangle\) are denoted by \(\text{SLC}\langle i_0, i_1; j \rangle\) and \(\text{LLP}\langle i_0, i_1; j \rangle\). To support efficient updates, the sequences SLC and SLL are implemented using doubly linked lists.

\[
\begin{align*}
1 &: 2: 3: 4: 5: 6: 7: 8: 9: 10: \\
6 &: b\ g\ d\ i\ f\ f\ e\ e\ c\ g \\
5 &: e\ d\ e\ i\ f\ h\ e\ e\ a\ i \\
4 &: i\ f\ d\ b\ b\ i\ i\ i\ e\ e \\
3 &: a\ e\ j\ d\ b\ j\ i\ a\ h\ e \\
2 &: e\ f\ h\ a\ i\ e\ b\ b\ e\ f \\
1 &: c\ a\ b\ e\ g\ f\ i\ b\ g\ i
\end{align*}
\]

Figure 3: An instance of a rectangle image.

Figure 3 shows an instance of a rectangular image. In this example:

- \(\text{SLL}(1; 10) = (c, 1), (a, 2), (e, 4), (f, 6), (b, 8), (g, 9), (i, 10)\)
- \(\text{SLL}(6; 10) = (b, 1), (d, 3), (i, 4), (f, 6), (e, 8), (c, 9), (g, 10)\)
- \(\text{SLC}(2, 5; 10) = (0, 2), (4, 1), (6, 1), (8, 1), (9, 2), (10, 3)\)
- \(\text{LLP}(2, 5; 10)\) contains for colors \(a, b, c, d, e, f, g, h, i, j\) the pointers to \((9, 2), (8, 1), (0, 2), (4, 1), (10, 3), (10, 3), (0, 2), (9, 2), (10, 3), (6, 1)\) respectively.

**2.1.2 Algorithm: main frame**

We present the following algorithm for computing all the sequences \(\varphi(i_0, i_1; j_0)\) for fixed \(i_0\) and \(i_1\). This algorithm processes sequentially all columns of the
rectangle \( \langle i_0, i_1; 1, n \rangle \) from first to \( n \)-th. Before processing the \( j \)-th column we assume that we have computed the sequences \( \text{SLL}(i_0-1; j-1) \), \( \text{SLL}(i_1+1; j-1) \), \( \text{SLC}(i_0, i_1; j-1) \) and the array \( \text{LLP}(i_0, i_1; j-1) \). We compute also the fingerprint \( f(i_0, i_1; j, j) \) for the \( j \)-th column of \( \langle i_0, i_1; 1, n \rangle \). Let \( \text{SLL}(i_0-1; j-1) = \{(a'_1, i'_1), (a'_2, i'_2), \ldots, (a'_k, i'_k)\} \), \( \text{SLL}(i_1+1; j-1) = \{(a''_1, i''_1), (a''_2, i''_2), \ldots, (a''_k, i''_k)\} \), \( \text{SLC}(i_0, i_1; j-1) = \{(l_1, k_1), (l_2, k_2), \ldots, (l_s, k_s)\} \) and \( f(i_0, i_1; j, j) = f \). The processing of the \( j \)-th column has two stages.

### 2.1.3 First stage

At the first stage we append to the sequences \( \varphi(i_0, i_1; j_0) \) the end markers for all maximal rectangles from \( R(i_0, i_1; *, j-1) \).

#### Left and right borders

To that end, we initially find the leftmost among the columns of \( \langle i_0, i_1; 1, j-1 \rangle \) that contain the rightmost occurrences of colors from \( f \) and the item of \( \text{SLC}(i_0, i_1; j-1) \) corresponding to this column (this can be done in \( O(\sigma) \) time, using \( \text{LLP}(i_0, i_1; j-1) \)). Let \( l_t \) be the number of this column. Note that \( l_s = j-1 \). It is easy to see that, if \( t < s \), in the set \( R(i_0, i_1; *, j-1) \) the rectangles with left columns in positions \( l_t + 1, l_{t+1} + 1, \ldots, l_{s-1} + 1 \) are all the rectangles which are maximal to the left and to the right (if \( t = s \), there are no such rectangles).

#### Bottom and top borders

However, these rectangles may not be maximal since they may not be maximal to the bottom or to the top. Thus we need to check additionally for these rectangles the maximality to the bottom and to the top. For any color \( c \) denote by \( l(c) \) the number of the column in \( \langle i_0, i_1; 1, j-1 \rangle \) which contains the rightmost occurrences of the color \( c \) in \( \langle i_0, i_1; 1, j-1 \rangle \). Using \( \text{SLC}(i_0, i_1; j-1) \) and \( \text{LLP}(i_0, i_1; j-1) \), the value \( l(c) \) can be computed in constant time. Note that a rectangle \( \langle i_0, i_1; j_0, j-1 \rangle \) from \( R(i_0, i_1; *, j-1) \) is maximal to the bottom if and only if in \( \text{SLL}(i_0-1; j-1) \) there exists a color \( a'_p \) such that \( l(a'_p) < j_0 \leq i'_p \). Thus, if \( l(a'_p) < i'_p \), the color \( a'_p \) yields the interval \([l(a'_p) + 1, i'_p]\) of numbers of left columns for rectangles from \( R(i_0, i_1; *, j-1) \) which are maximal to the bottom, and this interval can be computed in constant time. Therefore, in \( O(\sigma) \) time we can construct the ordered sequence \( I' \) of all the intervals yielded by the colors from \( \text{SLL}(i_0-1; j-1) \) (if some of these intervals are adjacent or overlapped we merge them in one interval in \( I' \)), and have that \( I' \) represents the set of all numbers of left columns for rectangles from \( R(i_0, i_1; *, j-1) \) which are maximal to the bottom. Using \( \text{SLL}(i_1+1; j-1) \), \( \text{SLC}(i_0, i_1; j-1) \) and \( \text{LLP}(i_0, i_1; j-1) \), we construct analogously the ordered sequence \( I'' \) of intervals presenting the set of all numbers of left columns for rectangles from \( R(i_0, i_1; *, j-1) \) which are maximal to the top.

We then construct the ordered sequence \( I \) of intervals which are intersections of intervals from \( I' \) and \( I'' \). Since both \( I' \) and \( I'' \) contain no more than \( \sigma \) intervals, \( I \) contains no more than \( 2\sigma \) intervals, and therefore can be computed in \( O(\sigma) \) time. The sequence \( I \) represents the set of all numbers of left columns
for rectangles from \( R(i_0, i_1; *, j-1) \) which are maximal to the bottom and to the top. Thus, to find all maximal rectangles from \( R(i_0, i_1; *, j-1) \), for each \( q = t, t+1, \ldots, s-1 \) we have to check if the possible position \( l_q + 1 \) for a left column of such rectangle is contained in an interval of \( I \) (this checking can be done in total \( O(\sigma) \) time). If it is the case, we insert in \( \varphi(i_0, i_1; l_q + 1) \) the end marker for the maximal rectangle with \((j-1)\)-th right column.

2.1.4 Second stage

At the second stage of the processing of the \( j \)-th column we add to the sequences \( \varphi(i_0, i_1; j_0) \) new colors from \( f \) which can be contained in maximal rectangles from \( R(i_0, i_1) \) which contain \( j \)-th column. To that end, we firstly compute for each \( l_q\)-th column from \( \text{SLC}(i_0, i_1; j-1) \) the number \( k'_q \) of distinct colors from \( f \) whose rightmost occurrences are contained in this column (it can be done in \( O(\sigma) \) time, using \( \text{SLC}(i_0, i_1; j-1) \) and \( \text{LLP}(i_0, i_1; j-1) \)).

We call the \( l_q \)-th column from \( \text{SLC}(i_0, i_1; j-1) \) feasible if \( k'_q < k_q \). Using the numbers \( k'_q \), we can compute in \( O(\sigma) \) time the subsequence of feasible columns from \( \text{SLC}(i_0, i_1; j-1) \). Using \( \text{SLC}(i_0, i_1; j-1) \) and \( \text{LLP}(i_0, i_1; j-1) \), we can also compute in \( O(\sigma) \) time for each color \( c \) from \( f \) the index \( t(a) \) such that the rightmost occurrences of \( c \) are contained in \( l_{t(a)} \)-th column of \( (i_0, i_1; j-1) \). After computing these indexes for each color \( c \) from \( f \) we add the color \( c \) to \( \varphi(i_0, i_1; l_q + 1) \) for each feasible \( l_q \)-th column from \( \text{SLC}(i_0, i_1; j-1) \) such that \( t(a) \leq q \).

2.1.5 Complexity.

The time complexity of the processing of the \( j \)-th column is \( O(\sigma + S) \) where \( S \) is the number of inserted colors. Thus the total time complexity of this procedure for all \( j = 1, 2, \ldots, n \) is \( O(n\sigma + \hat{S}) \) where \( \hat{S} \) is the sum of values \( S \) of all sequences \( \varphi(i_0, i_1; j_0) \). Since any sequence \( \varphi(i_0, i_1; j_0) \) contains no more than \( \sigma \) colors, we have \( \hat{S} \leq n\sigma \) i.e. the total time complexity of this procedure for all \( j \) is \( O(n\sigma) \).

2.1.6 Algorithm: data structures update from column \( j \) to \( j+1 \).

After processing the \( j \)-th column we compute the structures \( \text{SLL}(i_0 - 1; j) \), \( \text{SLL}(i_1 + 1; j) \), \( \text{SLC}(i_0, i_1; j) \) and \( \text{LLP}(i_0, i_1; j) \), required for processing the \( j+1 \)-th column, from \( \text{SLL}(i_0 - 1; j-1) \), \( \text{SLL}(i_1 + 1; j-1) \), \( \text{SLC}(i_0, i_1; j-1) \) and \( \text{LLP}(i_0, i_1; j-1) \). It can be done obviously in \( O(\sigma) \) time.

2.1.7 Last step

Once we have processed all the columns, we include in the sequences \( \varphi(i_0, i_1; j_0) \) the end markers for maximal rectangles from \( R(i_0, i_1; *, n) \), using the structures \( \text{SLL}(i_0 - 1; n) \), \( \text{SLL}(i_1 + 1; n) \), \( \text{SLC}(i_0, i_1; n) \) and \( \text{LLP}(i_0, i_1; n) \). This procedure is similar to the procedure described above except that in this case we consider for possible positions of left columns the position 1 and positions \( l_q + 1 \) for each \( l_q \)-th column from \( \text{SLC}(i_0, i_1; n) \).
The time complexity for computing all sequences $\varphi(i_0, i_1; j_0)$ is clearly $O(nm^2\sigma)$ excluding the time for the computation of the sets $f$ which is shown next.

2.2 Computing the all sets of colors: two variants

We present below two variants for resolving the problem of effective computation of the sets $f$. The first variant relies on Lemma 1, the second is more complex but does not rely on any external data structure. Both variants run in $O(nm^2\sigma)$ time.

First variant. The first variant of resolving this problem uses the result stated in Lemma 1 which allows to compute $f$ in $O(|f|)$ time, which is bounded by $O(\sigma)$. Thus, given $i_0$ and $i_1$, computing $\varphi(i_0, i_1; j_0)$ for all $j_0$ is bounded by $O(n\sigma)$ and repeating $O(m^2)$ such iterations for all pairs $i_0, i_1$ takes $O(nm^2\sigma)$ time.

Second variant. The second variant avoids the direct computation of the set $f$ by computing in parallel for each fixed $i_0$ all the sequences $\varphi(i_0, i_1; j_0)$ for any $j_0$ and any $i_1 \geq i_0$. In this case at the $j$-th step we process at once the $j$-th columns sequentially in each rectangle $\langle i_0, i_1; 1, n \rangle$ for $i_1 = i_0, i_0 + 1, \ldots, m$.

We assume that before processing this step we have computed all sequences $\text{SLL}(i_0 - 1, j - 1), \text{SLL}(i_0; j - 1), \text{SLL}(i_0 + 1; j - 1), \ldots, \text{SLL}(m; j - 1)$. Note that the data structures $\text{SLC}(i_0, i_1; j - 1)$ and $\text{LLP}(i_0, i_1; j - 1)$, required for processing the $j$-th column in $\langle i_0, i_1; 1, n \rangle$, can be computed from the disposed sequence $\text{SLL}(i_1; j - 1)$ and the structures $\text{SLC}(i_0, i_1 - 1; j - 1)$ and $\text{LLP}(i_0, i_1 - 1; j - 1)$ used for processing the $j$-th column in $\langle i_0, i_1 - 1; 1, n \rangle$ in $O(\sigma)$ time. Moreover, the set $f(i_0, i_1; j, j)$, required for processing the $j$-th column in $\langle i_0, i_1; 1, n \rangle$, can be computed in constant time from the set $f(i_0, i_1 - 1; j, j)$, used for processing the $j$-th column in $\langle i_0, i_1 - 1; 1, n \rangle$. Thus, after processing the $j$-th column in $\langle i_0, i_1 - 1; 1, n \rangle$ we can compute in $O(\sigma)$ time all the data structures, required for processing the $j$-th column in $\langle i_0, i_1; 1, n \rangle$. Moreover, after processing the $j$-th step each sequence $\text{SLL}(i; j)$, required for $(j + 1)$-th step, can be computed in constant time from the sequence $\text{SLL}(i; j - 1)$. So all sequences $\text{SLL}(i; j)$, required for $(j + 1)$-th step, can be computed in $O(m)$ time. Thus, the time required for computing all data structures at each step is $O(m\sigma)$, i.e. the total time for computing all data structures in this parallel procedure is $O(mn\sigma)$.

Since the whole algorithm requires $m$ such iterations, the whole complexity is also $O(nm^2\sigma)$ time.

2.3 Naming

The naming technique is used to give a unique name to the fingerprint (color set) of each maximal rectangle of an image. The technique which originated
in [10] was first adapted to the fingerprinting problem in [2] and later its speed was improved in [8].

We assume for simplicity, but without loss of generality, that $\sigma$ is a power of two. We consider a stack of $\log \sigma + 1$ arrays $B_0 \ldots B_{\log \sigma}$ on top of each other. The levels are numbered bottom-up starting from 0. The lowest $B_0$, called the fingerprint table, contains $\sigma$ names that are only $[0]_0$ or $[1]_0$, where a $[1]_0$ indicates the presence of the character. A name is indexed by its level number. The array of any level $i > 0$ contains half the number of names of the array of level $i - 1$ it is placed upon. The highest array only contains a single name that will be the name of the whole array. Such a name is called a fingerprint name. Figure 4 shows a simple example with $\sigma = 8$, where the left shows the stack for an empty character set and the right shows the stack for a non-empty one.

![Figure 4: Naming initialization and instance.](image)

The names in the fingerprint table are only $[0]_0$ or $[1]_0$ and are given. Each cell, $c$, of an upper array represents two disjoint and consecutive cells of the array it is placed upon, and thus a pair of two names. Conceptually, the naming is done in the following way: for each level going from the lowest to the highest, if the cell represents a new pair of names, give this pair a new name and assign it to the cell. If the pair has already been named, place this name into the cell. In the example in Figure 4, the name $[1]_1$ is associated to $([1]_0, [0]_0)$ the first time this pair is encountered. The second time, this name is directly retrieved. The approach above can be directly implemented by using a dictionary for level $i$ that stores all the pairs of names from level $i$ associated with a name from level $i + 1$. The dictionary is either kept using a binary search tree [2] or a hash table [5]. In [8] the naming is only done at the end of the computation of the maximal locations. Here we show a slightly modified version of that naming adapted for our case.

**Naming algorithm.** Before doing the naming, we first start by preprocessing the sequences $\varphi(i_0, i_1; j_0)$ by sorting (using radix sort) the subsequences of characters that lie between two end markers in the sequences (or before the first end marker). If $t$ is the number of characters to sort, then this phase will take time $O(t(\frac{\log \sigma}{\log t} + 1))$. 
Our naming will proceed in \( \log \sigma \) phases. At phase \( i \in [1..\log \sigma] \), we will determine all the cells at level \( i \). Before the first phase, the stack of arrays \( B_0 \ldots B_\sigma \) is initialized at zero.

At phase 1, we build a unique list \( L_1 \) from all the sequences \( \varphi(i_0, i_1; j_0) \) processed in sequential order and using two global counters \( C_0 \) and \( C_1 \) (initially at 0) as follows: in every list \( \varphi(i_0, i_1; j_0) \), and for every subsequence of characters that lies between two consecutive end markers (or before the first one), we do the following: we set to one all the positions \( B_0[\alpha] \) for every character \( \alpha \) in the subsequence. We then scan the subsequence again and replace every character \( \alpha \) by \( \lceil \alpha/2 \rceil \) (and potentially merge any two consecutive resulting characters if they are equal), generate a pair of pairs \( ((C_1, C_0), (B_0[2\alpha - 1], B_0[2\alpha])) \) and increment \( C_0 \). After processing the subsequence we increment the counter \( C_1 \) and reinitialize \( C_0 \) at 0.

At the end of processing a sequence \( \varphi(i_0, i_1; j_0) \), we reinitialize \( B_0 \) to zero by undoing all the operations done on \( B_0 \). After we have processed all the sequences \( \varphi \), we sort the elements in the generated pairs of pairs by their last components (which are actually pairs of names from level 0) and replace all equal last components with a unique sequential number (a name for level 1). We then again sort the resulting pairs, this time by their first components (the value of \((C_1, C_0)) \) resulting in a list \( L_1 \), which we partition into sublists according to the value of \( C_1 \). At this time, we have already built all the names at level 1 and we can proceed to the phase number 2 which builds all the names for level 2. We first reset \( C_1 \) to 0. We then scan all the sequences \( \varphi(i_0, i_1; j_0) \) and the list \( L_1 \) in parallel and for a subsequence number \( k \) (as before the subsequences are separated by end markers and their number is indicated by \( C_1 \)) do the following two steps. In the first step, we apply the changes of \( B_1 \) indicated by the corresponding sublist in \( L_1 \). That is, if the character number \( j \) in the subsequence is \( \alpha \), we write in \( B_1[\alpha] \) the last component (the name for level 1) coming from the pair number \( j \) in the sublist number \( k \) of \( L_1 \).

In the second step, we rescan again the subsequence \( k \) and replace every character \( \alpha \) by \( \lceil \alpha/2 \rceil \) (and removing duplicates), generate a pair \( ((C_1, C_0), (B_1[2\alpha - 1], B_1[2\alpha])) \) and increment \( C_0 \). After processing the subsequence \( k \) we increment the counter \( C_1 \) and reinitialize \( C_0 \) at 0. At the end of processing a sequence \( \varphi(i_0, i_1; j_0) \), we reinitialize \( B_1 \) to zero by undoing all the operations done on \( B_1 \). Everything else is exactly the same as in level number 1. The sorting step will result in an array \( L_2 \) that allows to replace every unique pair of names from level 1 by a name for level 2.

At any phase \( i > 2 \), the algorithm is exactly the same as in level 2, except that we now use \( B_{i-1} \), \( L_i \) and \( L_{i-1} \) instead of \( B_1, L_2 \) and \( L_1 \).

At the end of the phase number \( \sigma \), every sublist of \( L_\sigma \) will contain a single pair whose last component indicates the name of a maximal rectangle.

**Complexity.** We first bound the overhead due to the sorting phases. Sorting is first used on the \( t \) characters in a subsequence and takes (recall that a subsequence stores all the \( t \) characters that are added to produce the fingerprint of a
maximal rectangle) \( O(t(\log \sigma \log t + 1)) \leq O(t(\log \sigma + 1)) \) time. It is also used twice for each phase \( i \), in order to produce the list \( L_i \).

We can prove that the naming of each maximal location involving the addition of \( t \) new colors is done in total time \( O(t(\log \sigma + 1)) \). In this case all \(|L|\) maximal locations can be named in total time \( O(m^2n\sigma \max\{1, \log(|L|/nm^2)\}) \) which will be faster than a naive naming when \(|L| = o(nm^2\sigma)\).

The trick is to view the stack as a full binary tree of height \( \log(\sigma) \), where the leaves (the \( t \) positions in the fingerprint table) are marked and the internal nodes in the root-to-leaves paths leading to those \( t \) leaves are also marked (those nodes correspond to generated names). Then the total number of marked nodes will be less than \( O(t(\log \sigma + 1)) \). This is obvious: consider the upper \( \lceil \log t \rceil \) levels of the perfect binary tree. The total number of nodes into these upper levels can not exceed \( 2^{\lceil \log t \rceil} \leq 2^t \). The total number of internal nodes in the lower \( \log \sigma - \lceil \log t \rceil \) levels will be clearly no more than \( t(\log \sigma - \lceil \log t \rceil) = O(t \log \sigma / t) \).

Then, the naming will be done level by level starting from the bottom level and taking in total \( O(t(\log \sigma + 1)) \) time for a maximal location whose fingerprint adds \( t \) colors to the previous one.

We prove now a \( O(m^2n\sigma \max\{1, \log |L| / nm^2\}) \) bound for the naming of all maximal locations. For convenience, the \log function will refer to the base \( e \) logarithm instead of base 2 (as is the case in the rest of the paper).

We denote the number of color(s) added by the maximal location number \( i \) by \( t_i \). We let \( \ell = |L| \) and \( \sum_{i=1}^{\ell} t_i = T \). We have:

\[
\sum_{i=1}^{\ell} O \left( t_i (\log \left( \frac{\sigma}{t_i} \right) + 1) \right) \\
= \sum_{i=1}^{\ell} O \left( t_i \log \left( \frac{\sigma}{t_i} \right) \right) + \sum_{i=1}^{\ell} O \left( t_i \right) \\
= \sum_{i=1}^{\ell} O \left( t_i \log \left( \frac{\sigma}{t_i} \right) \right) + O(T).
\]

Since \( T = O(m^2n) \), in order to bound the naming time we only need to prove that \( P = \sum_{i=1}^{\ell} O \left( t_i \log \left( \frac{\sigma}{t_i} \right) \right) \leq O(m^2n\sigma \max\{1, \log |L| / nm^2\}) \).

We now apply Jensen’s inequality to the convex function \( f(x) = x \log x \). Recall that the Jensen inequality for a convex function \( f(x) \) states that:

\[
f \left( \frac{\sum_{i=1}^{n} x_i}{n} \right) \leq \frac{\sum_{i=1}^{n} f(x_i)}{n}
\]

By replacing \( x_i \) by \( t_i \), \( f(x) \) by \( x \log x \) and \( n \) by \( \ell \) we obtain:

\[
\left( \frac{\sum_{i=1}^{\ell} t_i \ell}{\ell} \right) \log \left( \frac{\sum_{i=1}^{\ell} t_i}{\ell} \right) \leq \frac{\sum_{i=1}^{\ell} t_i \log t_i}{\ell}
\]

which simplifies to the following inequality: \( T \log \frac{T}{\ell} \leq \sum_{i=1}^{\ell} t_i \log t_i \).
By replacing in the term $P$ we get:

$$P = \sum_{i=1}^{t} O \left( t_i \log \left( \frac{\sigma}{t_i} \right) \right) = O \left( \left( \sum_{i=1}^{t} t_i \log \sigma \right) - \left( \sum_{i=1}^{t} t_i \log t_i \right) \right) \leq O \left( T \log \sigma - T \log \frac{\sigma}{T} \right) = O \left( T \left( \log \frac{\sigma}{T} \right) \right) = O \left( \sigma g \left( \frac{T}{\sigma} \right) \right)$$

where $g(x) = x \log(\ell/x)$. Note that $g$ is monotonically increasing for $x \leq \ell/e$ and monotonically decreasing for $x \geq \ell/e$ and is maximal at $x = \ell/e$. So we consider two cases:

1. $\ell/e \leq nm^2$, in this case

$$\sigma g(T/\sigma) \leq \sigma g(\ell/e) = \sigma (\ell/e) \log e = O(\sigma \ell) = O(nm^2 \sigma)$$

2. $\ell/e > nm^2$. Since $T \leq nm^2 \sigma$ we have that $T/\sigma \leq nm^2 < \ell/e$, and

$$\sigma g(T/\sigma) \leq \sigma g(nm^2) = nm^2 \sigma \log(\ell/nm^2)$$

So in the both cases $P = \sigma g(T/\sigma) \leq O(nm^2 \sigma \max\{1, \log \frac{|L|}{nm^2} \})$ which concludes the analysis.

### 2.4 Storing fingerprints

We now show how to store the fingerprints efficiently in $O(|L| + nm \log n)$ space such that a query for a fingerprint $f$ will take only time $O(\log \log n + |f| + k)$, where $k$ is the number of maximal locations that correspond to the fingerprint. If a query asks just for the presence of a fingerprint, without requiring to return its corresponding maximal locations, then we can do with just query time $O(\log \log n + |f|)$ with space usage reduced to just $O(|F| + nm \log n)$. We note that storing all the fingerprints could take much more space than just $O(|F|)$. This is unlike the one dimensional case in which every fingerprint adds only one color to another fingerprint and hence all the fingerprints can be coded with just $O(1)$ space overhead per fingerprint (actually the overhead can even be reduced to just $2 \log \sigma + O(1)$ bits per fingerprint) \[5\]. In our case of two dimensional fingerprints, a fingerprint can add an arbitrary number of colors to another fingerprint. Hence coding a fingerprint by just storing the differing colors with other fingerprints would require too much space. Our solution to use just $O(1)$ overhead per fingerprint is to index the matrix using the data structure of Lemma 2. This would use space $O(nm \log(nm)) = O(nm \log n)$. We also store in a hash table all the hash values associated with all distinct fingerprints (perfect hash function). To each value, we associate a pointer to one of the rectangles having the corresponding fingerprint. More details on the hash function implementation are given in the next subsection. For now we assume that the perfect hash function can be evaluated in time $O(|f|)$ on a
fingerprint $f$. Now, given a fingerprint $f$, a query will proceed in four steps: (1) First compute the hash value $h$ associated with $f$. This is done in time $O(|f|)$. (2) Probe the hash table for the value $h$ and if found, retrieve the pointer to the rectangle associated with that hash value. Otherwise declare a failure and stop the query. (3) Retrieve all the colors occurring inside that rectangle but stop if there are more than $c|f|$ distinct colors (for some fixed and known constant $c$). If that was the case then the query is stopped with a failure. (4) Match the reported colors with the set $|f|$.

We give now more details on the four steps. The first step can obviously be done in time $O(|f|)$. The second step is done in $O(1)$ time. Then third step can be done in time $O(|f| + \log \log (n + m))$ if the data structure of Lemma 2 is used (with the constant $c$ as defined in the lemma). Eventually the last step can be done in $O(|f|)$ time as follows. We only use a bit-vector of length $\sigma$ bits which will be reserved specifically for queries. Initially, all the bits in the bit-vector are set to zero. For a given query consisting in a fingerprint $f$, we first set to one all the bits in the bit-vector whose positions correspond to colors in the fingerprint. To check that the colors reported by the data structure of Lemma 2 are the same as those of the fingerprint, we start with a counter set to zero and then do the following for each retrieved color. First check if the corresponding bit in the bitvector is set to one. If this is the case, we reset the bit to zero and increment the counter. After we have processed all the reported colors, we can report a success if and only if the value of the counter is $|f|$. Eventually at the end of the query, and regardless of the result of the query, we reset all the bits that were set to one during the query, in such a way that the bitvector is made only of zeros (by setting to zero all the positions of the bits which were set to one during the query). We have thus proved that existential queries can be answered in time $O(|f| + \log \log n)$ using space $O(|F| + nm \log n)$. In order to support reporting queries, we will additionally store the list of maximal locations (maximal rectangles) that correspond to each fingerprint. This adds $O(|L|)$ space to the data structure. At query time, we only need to check that the colors in the first maximal location correspond to the given query fingerprint $f$ in time $O(|f| + \log \log n)$. Afterward, we can report the remaining maximal $k - 1$ rectangles by traversing the list in time $O(k - 1)$.

2.5 Hash table and probabilistic naming

In order to implement the hash table described in previous subsection, we can make use of a polynomial hash function (Rabin-Karp signatures [11]) so as toinjectively reduce the fingerprints to integers of length $O(\log(|F|))$. This approach was used and described in detail in [5]. We only give a sketch here. The used polynomial hash function $H$ is parametrized by a prime number $r$ from the range $[1..nc]$ for some suitable constant $c$. For a randomly chosen $r$ from the range, the hash functions maps injectively the fingerprints to integers in $[1,|F|^{O(1)}]$ with high probability (probability $1 - \frac{1}{|F|^{|F|^{O(1)}}}$). If it fails to do so, then we choose a different $r$ from the range until we find a hash function that maps all fingerprints to distinct integers. Then we can use any minimal
perfect hashing scheme [9] to map those integers to the final range $[1..|F|]$. Note that evaluating a perfect hash function on any given fingerprint $f$ takes $O(|f|)$ time as it consists in first computing the Rabin-Karp signature followed by the computation of the perfect hash function applied on the obtained integer.

The use of hashing provides an alternative naming that will work with high probability (Monte Carlo). Simply omit the phase described in section 2.3 and simply rely on the polynomial hash function $H$ to give distinct names to different fingerprints. The polynomial hash function will give distinct names to all fingerprints with high probability. The advantage of such an approach is that the total complexity of the naming is $O(nm^2\sigma)$. This is because computing the hash value of a fingerprint that adds $t$ characters to another fingerprint given the hash value of the latter can be done in time $O(t)$.

3 Matching squares

We can obtain a faster algorithm in case we are only aiming at matching square maximal locations. A square is maximal if and only if it is not included in a greater square having the same set of colors.

3.1 Notations

We denote by $[i, j; k]$ the square $\langle i, i + k - 1; j, j + k - 1 \rangle$ where $a_{i,j}$ is the left bottom corner, $a_{i+k-1,j}$ is the left upper corner, $a_{i,j+k-1}$ is the right bottom corner, and $a_{i+k-1,j+k-1}$ is the right upper corner. By $f[i, j; k]$ we denote the fingerprint of $\langle i, i + k - 1; j, j + k - 1 \rangle$ and we call the parameter $k$ the size of this square. Let the $r$-th diagonal be the set of all points $a_{i,j}$ such that $i - j = r$. A square is on a diagonal if its left bottom and right upper corners are in this diagonal.

We also denote by $S[i, j; \ast]$ the set of all squares with the left bottom corner $a_{i,j}$, and by $S[\ast; i, j]$ the set of all squares with the right upper corner $a_{i,j}$. For any two colors $a_{i',j'}$ and $a_{i'',j''}$ the distance between $a_{i',j'}$ and $a_{i'',j''}$ is $\min(|i' - i''|, |j' - j''|)$. By $[i, j; k]$ we denote the triangle with corners $a_{i,j}$, $a_{i+k,j}$ and $a_{i+k,j+k}$, and by $[i;j;k]$ we denote the triangle with corners $a_{i,j}$, $a_{i,j+k}$ and $a_{i+k,j+k}$. The $i$-th line of a triangle is the intersection of the $i$-th line of the input image with this triangle, and the $j$-th column of a triangle is the intersection of the $j$-th column of the input image with this triangle.

3.2 Maximality conditions

A square $[i, j; k]$ is maximal if it fulfills the square maximality conditions $L$, $R$, $U$, $D$, $LU$, $LD$, $RU$, $RD$ defined as follows:

- $L$ is true if $f[i, i + k - 1; j - 1, j + k - 1] \neq f[i, j; k]$  
- $R$ is true if $f[i, i + k - 1; j, j + k] \neq f[i, j; k]$  
- $U$ is true if $f[i, i + k; j, j + k - 1] \neq f[i, j; k]$  

\begin{itemize}
    \item $D$ is true if $f(i - 1, i + k - 1; j, j + k - 1) \neq f[i, j; k]$
    \item $LU$ is true if $a_{i+k,j-1} \notin f[i, j; k]$ and $f(i, i + k; j, j + k - 1) = f[i, j; k]$
    \item $LD$ is true if $a_{i-1,j-1} \notin f[i, j; k]$
    \item $RU$ is true if $a_{i+k,j+k} \notin f[i, j; k]$
    \item $RD$ is true if $a_{i-1,j+k} \notin f[i, j; k]$ and if $f(i, i + k - 1; j, j + k) = f[i, j; k]$

Note that a square is maximal if and only if the condition
\[
(L \lor U \lor LU) \land (L \lor D \lor LD) \land (R \lor U \lor RU) \land (R \lor D \lor RD)
\]
holds for the square.

For each $i$ and $j$ we construct the following sequence $\hat{\phi}(i; j)$ consisting of distinct colors and end markers. Let $s_{i,j}$ be the greatest square in $S[i, j; \ast]$. For any color $c$ contained in $s_{i,j}$ the distance of $c$ in $s_{i,j}$ is the minimal distance between $a_{i,j}$ and occurrences of $c$ in $s_{i,j}$. The color subsequence of $\hat{\phi}(i; j)$ is a sequence of all distinct colors contained in $s_{i,j}$ such that if in this sequence a color $c$ is before a color $b$ then the distance of $c$ in $s_{i,j}$ is not greater than the distance of $b$ in $s_{i,j}$. Note that for any square from $S[i, j; \ast]$ the fingerprint of this square forms a starting segment of the considered color subsequence. For any maximal square from $S[i, j; \ast]$ we insert in $\hat{\phi}(i; j)$ after the last color of the corresponding segment the end marker for this maximal square (it is obvious that distinct maximal squares have distinct end markers). Thus, to find all maximal squares on a fixed diagonal we have to compute all sequences $\hat{\phi}(i; j)$ such that $a_{i,j}$ is in this diagonal.

### 3.3 Computing $\hat{\phi}(i; j)$

#### 3.3.1 Data structures

For computing $\hat{\phi}(i; j)$ the following data structures are used.

**SCT.** For the triangle $[i, j; k]$ we consider the two-way queue $\text{SCT}[i, j; k]$. This is the sequence $(c_1, q_1), (c_2, q_2), \ldots, (c_s, q_s)$ such that $c_1 < c_2 < \ldots < c_s$ where $c_t$ is the number of a column of the triangle $[i, j; k]$ which contains occurrences of colors such that the square $[i + c_t - j, c_t + 1; j + k + 1 - c_t]$ has no occurrences of these colors. We will call such colors respective to $c_t$-th column of the triangle $[i, j; k]$. $q_t$ is the number of distinct colors respective to $c_t$-th column of the triangle $[i, j; k]$.

**PCT.** We will also consider the array $\text{PCT}[i, j; k]$ which for each color $c$ contains the pointer to the item of $\text{SCT}[i, j; k]$ for the column of $[i, j; k]$ respective to $c$ if such column exists; otherwise this pointer is undefined.

**SLT.** For the triangle $[i, j; k]$ we consider the two-way queue $\text{SLT}[i, j; k]$. This is the sequence $(l_1, p_1), (l_2, p_2), \ldots, (l_s, p_s)$ such that $l_1 < l_2 < \ldots < l_s$ where $l_t$
is the number of a line of the triangle \([i, j; k]\) which contains occurrences of colors such that the square \([l_t + 1, j + l_t - i; i + k + 1 - l_t]\) has no occurrences of these colors. We will also call such colors respective to \(l_t\)-th line of the triangle \([i, j; k]\).

**PLT.** Similarly to the array \(PCT[i, j; k]\) for \(SCT[i, j; k]\), we consider also the array \(PLT[i, j; k]\) for \(SLT[i, j; k]\).

**SLD.** Moreover, for each \(i\) and \(j\) we consider the two-way queue \(SLD(i, j)\). This is the sequence \((a_{i_1, i_1 + r, i_1}), (a_{i_2, i_2 + r, i_2}), \ldots, (a_{i_s, i_s + r, i_s})\) where \(r = j - i, i_1 < i_2 < i_3 < \ldots < i_s = i\), and \(a_{i_t, i_t + r}\) has no occurrences in the square with the left bottom corner \(a_{i_t + 1, i_t + r + 1}\) and the right upper corner \(a_{i, j}\).

**PLD.** For \(SLD(i, j)\) we also consider array \(PLD(i, j)\) which for each color \(c\) contains the pointer to the item of \(SLD(i, j)\) with this color if such item exists; otherwise this pointer is undefined.

**SLL and SLL\(^T\).** Besides the sequences \(SLL(i, j)\) we also use the sequences \(SLL^T(i, j)\) which are the SLL sequences for the string \(a_{1, j}a_{2, j} \ldots a_{i, j}\).

\[
\begin{aligned}
1 & : \ 2 & : \ 3 & : \ 4 & : \ 5 & : \ 6 & : \ 7 & : \ 8 : \\
8 & : \ a & f & d & a & f & f & i & c \\
7 & : \ h & f & d & g & i & j & a & i \\
6 & : \ j & d & i & b & g & g & a & c \\
5 & : \ i & f & i & i & a & h & i & f \\
4 & : \ f & j & d & b & b & g & j & h \\
3 & : \ h & d & a & i & f & h & c & b \\
2 & : \ a & g & i & g & i & a & h & b \\
1 & : \ h & f & e & e & b & d & c & b \\
\end{aligned}
\]

Figure 5: An instance of a square.

Figure 5 shows an instance of a square. In this example:

- \(SCT[2, 1; 6] = (3, 1), (4, 1), (6, 2), (7, 1)\)
- \(PCT[2, 1; 6]\) contains for letters \(b, d, f, i, j\) the pointers to \((4, 1), (3, 1), (6, 2), (7, 1), (6, 2)\) respectively
- \(SLT[1, 2; 6] = (1, 1), (4, 1), (5, 1), (7, 1)\)
- \(PLT[1, 2; 6]\) contains for letters \(b, e, h, i\) the pointers to \((4, 1), (1, 1), (5, 1), (7, 1)\) respectively
- \(SLD(8, 8) = (b, 4), (g, 6), (a, 7), (c, 8)\)
- \(PLD(8, 8)\) contains for letters \(a, b, c, g\) the pointers to \((a, 7), (b, 4), (c, 8), (g, 6)\) respectively
3.3.2 Algorithm

Consider the $r$-th diagonal of the input image where $r \geq 0$ (the case $r < 0$ is processed in the same way). To find all maximal squares on this diagonal, we need to compute all sequences $\phi(1; r + 1), \phi(2; r + 2), \ldots$. For computing these sequences we process sequentially all colors $a_{i,i+r}$ where $i = 1, 2, \ldots$. Before processing a color $a_{i,j}$, where $i = j + r$, on this diagonal, we assume that we have computed the following structures: $\text{SCT}[2, 1 + r; i - 2], \text{PCT}[2, r + 1; i - 2], \text{SLT}[1, r + 2; i - 2], \text{PLT}[1, r + 2; i - 2], \text{SLD}(i - 1, j - 1), \text{PLD}(i - 1, j - 1), \text{SLL}(i; j - 1)$, and $\text{SLL}^T(i - 1; j)$. The processing of $a_{i,j}$ is done in two stages.

First stage. At the first stage we insert in the computed sequences $\phi(l; r + l)$ the end markers for all maximal squares from $S[*; i - 1, j - 1]$. Note that each square $[i - k, j - k; k]$ from $S[*; i - 1, j - 1]$ is uniquely defined by its size $k$. Thus, for each maximality condition $P \in \{L, R, D, U, LD, RU, LU, RD\}$ we can compute the characteristic set $\chi_P$ of sizes of all squares from $S[*; i - 1, j - 1]$ which satisfy the condition $P$. First of all, for any color $c$, using the structures $\text{SCT}[2, 1 + r; i - 2], \text{PCT}[2, r + 1; i - 2], \text{SLT}[1, r + 2; i - 2], \text{PLT}[1, r + 2; i - 2], \text{SLD}(i - 1, j - 1)$, and $\text{PLD}(i - 1, j - 1)$, we can compute in constant time the maximal size $K_c$ of a square from $S[*; i - 1, j - 1]$ which has no occurrences of color $c$. Thus we can compute in constant time the set $\chi_{RU} = [1..K_{a_{i,j}}]$. Further, if

$$\text{SLD}(i - 1, j - 1) = (a_{i_1,i_1+r,i_1}, a_{i_2,i_2+r,i_2}, \ldots, a_{i_s,i_s+r,i_s})$$

then we have obviously (taking into account that $i_s = i - 1$)

$$\chi_{LD} = \{i - i_{s-1} - 1, i - i_{s-2} - 1, \ldots, i - i_1 - 1\}$$

Now let

$$\text{SCT}[2, 1 + r; i - 2] = (c_1, q_1), (c_2, q_2), \ldots, (c_s, q_s)$$

Then obviously

$$\chi_L = \{j - c_s - 1, j - c_{s-1} - 1, \ldots, j - c_1 - 1\}$$

In the symmetrical way we can compute $\chi_D$ from $\text{SLT}[1, r + 2; i - 2]$. To compute $\chi_U$, note that a square $[i - k, j - k; k]$ from $S[*; i - 1, j - 1]$ satisfies the condition $U$ if and only if in $\text{SLL}(i; j - 1)$ there exists an item $(c_t, j_t)$ such that $j - j_t \leq k \leq K_c$. Therefore, each item $(c_t, j_t)$ of $\text{SLL}(i; j - 1)$ such that $j - j_t \leq K_c$ yields the interval $[j - j_t, K_c]$ to $\chi_U$, and $\chi_U$ has no other intervals. Moreover, each of these intervals $[j - j_t, K_c]$ can be computed in constant time. Thus, $\chi_U$ consists of no more than $\sigma$ intervals and can be computed in $O(\sigma)$ time. In the symmetrical way, $\chi_R$ also consists of no more than $\sigma$ intervals and can be computed in $O(\sigma)$ time from $\text{SLL}^T(i - 1; j)$. Computing $\chi_{LU}$ is based on the following considerations. Let $[i - k, j - k; k]$ be a square from $S[*; i - 1, j - 1]$ satisfying the condition $LU$. So $a_{i,j-k-1} \notin f[i - k, j - k; k] = f(i - k, i; j - k, j - 1)$, i.e. the color $a_{i,j-k-1}$ is not contained in the string $a_{i,j-k}a_{i,j-k+1} \ldots a_{i,j-1}$. Thus
$a_{i,j-k-1}$ is the rightmost occurrence of this color in the string $a_{i,1}a_{i,2}\ldots a_{i,j-1}$, i.e. $a_{i,j-k-1}$ is contained in $SLL(i, j-1)$. Hence, to compute $\chi_{LU}$, we can check only values $k = j-j_t-1$ for each item $(c_t, j_t)$ from $SLL(i, j-1)$. More precisely, for each item $(c_t, j_t)$ from $SLL(i, j-1)$ we check if the color $c_t$ has occurrences in $[i-k, j-k; k]$ where $k = j-j_t-1$ (since $K_{c_t}$ can be computed in constant time, it can be checked in constant time). If it is the case, we include values $k$ in $\chi_{LU}$. Thus, $\chi_{LU}$ contains no more than $\sigma$ values and can be computed in $O(\sigma)$ time. In the symmetrical way $\chi_{RD}$ contains no more than $\sigma$ values and can be computed in $O(\sigma)$ time from $SLL^T(i-1; j)$. Thus all characteristic sets for the considered maximality conditions $L, R, D, U, LD, RU, LU, RD$ contain no more than $\sigma$ values or intervals and can be computed in $O(\sigma)$ time. Note that each value $k$ can be considered as interval $[k..k]$, so without loss of generality we can assume that all characteristic sets for the maximality conditions consist of no more than $\sigma$ intervals. Thus the characteristic set

$$\chi_M = (\chi_L \cup \chi_U \cup \chi_{LU}) \cap (\chi_L \cup \chi_D \cup \chi_{LD}) \cap (\chi_R \cup \chi_U \cup \chi_{RU}) \cap (\chi_R \cup \chi_D \cup \chi_{RD})$$

of sizes $k$ of all maximal squares $[i-k, j-k; k]$ from $S[\ast; i-1, j-1]$ consists of $O(\sigma)$ intervals and can be computed in $O(\sigma)$ time. After computing $\chi_M$, for each $k$ from $\chi_M$ we insert in $\hat{\varphi}(i-k; j-k)$ the end marker for the maximal square $[i-k, j-k; k]$. Thus, the time complexity of the first stage computations is $O(\sigma + S)$ where $S$ is the number of maximal squares in $S[\ast; i-1, j-1]$. So the overall time complexity of the second stage computations is $O(nm\sigma + S)$ where $S$ is the total number of maximal squares. Since $S \leq n\sigma$, we have that the overall time complexity of the second stage computations is $O(nm\sigma)$.

**Second stage.** At the second stage of processing $a_{i,j}$ we add to the computed sequences $\hat{\varphi}(i-k; j-k)$ new colors which are not contained in squares $[i-k, j-k; k]$ but are contained in squares $[i-k, j-k; k+1]$. We make the following computations. Note that for each color $c$, using $SLL(i; j-1)$, $SLL^T(i-1; j)$, $a_{i,j}$ and $K_c$, we can compute in constant time the maximal size $K_{c}$ of a square from $S[\ast; i, j]$ which has no occurrences of $c$. Then, if $K_c \leq K_{c}$, we add the color $c$ to each of the sequences $\hat{\varphi}(i-K_c; j-K_c), \hat{\varphi}(i-K_c+1; j-K_c+1), \ldots, \hat{\varphi}(i-K_{c}; j-K_{c})$. The time complexity of these computations is $O(\sigma + S)$ where $S$ is the number of added colors. Thus the overall time complexity of the second stage computations is $O(nm\sigma + S)$ where $S$ is the total number of colors in all sequences $\hat{\varphi}(i; j)$. So, since $S \leq n\sigma$, the overall time complexity of the second stage computations is $O(nm\sigma)$.

Note also that the data structures $SCT[2, 1+r; i-2]$, $PCT[2, r+1; i-2]$, $SLT[1, r+2; i-2]$, $PLT[1, r+2; i-2]$, $SLD(i-1, j-1)$, $PLD(i-1, j-1)$, required for processing the color $a_{i,j}$ can be computed in $O(\sigma)$ time from the data structures $SCT[2, 1+r; i-3]$, $PCT[2, r+1; i-3]$, $SLT[1, r+2; i-3]$, $PLT[1, r+2; i-3]$, $SLD(i-2, j-2)$, $PLD(i-2, j-2)$, $SLD(i-1, j-2)$, and $SLL^T(i-2; j-1)$, used for processing the previous color $a_{i-1,j-1}$. Thus the overall time complexity of computing the structures $SCT[2, 1+r; i-2]$, $PCT[2, r+1; i-2]$, $SLT[1, r+2; i-2]$, $PLT[1, r+2; i-2]$, $SLD(i-1, j-1)$,
and PLD\langle i - 1, j - 1 \rangle is $O(nm\sigma)$. The ”bottleneck” of this algorithm is the computation of the remaining structures SLL\langle i; j - 1 \rangle and SLL^T\langle i - 1; j \rangle. To resolve this problem, we can use the result stated in Lemma 1 which allows to compute these structures in $O(\sigma)$ time. Another way of resolving this problem is to process all diagonals in parallel. More precisely, the algorithm makes $n$ steps for $j = 1, 2, \ldots, n$, and at $j$-th step the colors $a_{1,j}, a_{2,j}, \ldots, a_{m,j}$ are sequentially processed. We assume that before $j$-th step, all data structures required for processing these colors, except the structures SLL^T\langle 1; j \rangle, SLL^T\langle 2; j \rangle, \ldots, SLL^T\langle m; j \rangle are already computed. In this case the sequence SLL^T\langle i - 1; j \rangle, required for processing $a_{i,j}$, can be computed in constant time from the sequence SLL^T\langle i - 2; j \rangle, used for processing the previous color $a_{i-1,j}$. Thus the overall time complexity of computing all sequences SLL^T\langle i - 1; j \rangle is $O(mn)$. Moreover, each sequence SLL\langle i; j - 1 \rangle, required at $j$-th step, can be computed in constant time from the sequence SLL\langle i; j - 2 \rangle, used at $(j - 1)$-th step. So the overall time complexity of computing all sequences SLL\langle i; j - 1 \rangle is also $O(mn)$. Thus the total time complexity of the algorithm for finding all maximal squares is $O(nm\sigma)$.

3.4 Complexity

By combining the previous algorithm with the naming scheme of section 2.3, we obtain the following theorem.

**Theorem 3** Given an image of $m$ rows by $n \geq m$ columns, we can compute all the $|S|$ maximal squares with the set $F$ of all distinct fingerprints of these squares in deterministic time $O(mn\sigma \log(\frac{|S|}{nm} + 2))$ or in Monte Carlo algorithm in time $O(mn\sigma)$ such that a query which asks for all the maximal squares with a given fingerprint $f$ can be answered in time $O(|f| + \log \log n + k)$, where $k$ is the number of maximal squares. If the query asks only for the presence of squares with a given fingerprint $f$, then the query time becomes $O(|f| + \log \log n)$ while the space usage becomes $O(nm \log n + |F|)$.

The construction times of the data structures mentioned in the theorem are respectively $O(nm \log n \log \log n + |S|)$ and $O(nm \log n \log \log n + |F|)$.

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References

[1] A. Amir, G. M. Landau, and D. Sokol. Inplace 2d matching in compressed images. *J. Algorithms*, 49(2):240–261, 2003.

[2] Amihood Amir, Alberto Apostolico, Gad M. Landau, and Giorgio Satta. Efficient text fingerprinting via parikh mapping. *J. Discrete Algorithms*, 1(5-6):409–421, 2003.

[3] Amihood Amir and Martin Farach. Two-dimensional dictionary matching. *Information Processing Letters*, 44(5):233–239, 1992.

[4] Theodore P Baker. A technique for extending rapid exact-match string matching to arrays of more than one dimension. *SIAM Journal on Computing*, 7(4):533–541, 1978.

[5] Djamal Belazzougui, Roman Kolpakov, and Mathieu Raffinot. Various improvements to text fingerprinting. *J. Discrete Algorithms*, 22:1–18, 2013.

[6] C.-Y. Chan, H.-I Yu, W.-K. Hon, and B.-F. Wang. A faster query algorithm for the text fingerprinting problem. In ESA, pages 123–135, 2007.

[7] C.-Y. Chan, H.-I Yu, W.-K. Hon, and B.-F. Wang. Faster query algorithms for the text fingerprinting problem. *Inf. Comput.*, 209(7):1057–1069, 2011.

[8] Gilles Didier, Thomas Schmidt, Jens Stoye, and Dekel Tsur. Character sets of strings. *Journal of Discrete Algorithms*, 5(2):330–340, 2007.

[9] Michael L. Fredman, János Komlós, and Endre Szemerédi. Storing a sparse table with O(1) worst case access time. In FOCS, pages 165–169, 1982.

[10] Richard M Karp, Raymond E Miller, and Arnold L Rosenberg. Rapid identification of repeated patterns in strings, trees and arrays. In *Proceedings of the fourth annual ACM symposium on Theory of computing*, pages 125–136. ACM, 1972.

[11] Richard M. Karp and Michael O. Rabin. Efficient randomized pattern-matching algorithms. *IBM Journal of Research and Development*, 31(2):249–260, 1987.

[12] Roman Kolpakov and Mathieu Raffinot. New algorithms for text fingerprinting. *J. Discrete Algorithms*, 6(2):243–255, 2008.

[13] Kasper Green Larsen and Freek van Walderveen. Near-optimal range reporting structures for categorical data. In *SODA*, pages 265–276, 2013.

[14] S. Mukherjee. Efficient algorithms for document retrieval problems. In *SODA*, pages 657–666, 2002.