Geometric construction of the classical $R$-matrices for the elliptic and trigonometric Calogero-Moser systems.

G.E.Arutyunov *
and
P.B.Medvedev †

Abstract

By applying the Hamiltonian reduction scheme we recover the R-matrix of the trigonometric and elliptic Calogero-Moser system.
1 Introduction

Recently the theory of integrable many-body systems has a number of interesting developments. In particular, classical dynamical R-matrices were found for rational, trigonometric [1], elliptic [2] Calogero systems and also for trigonometric Ruijsenaars-Schneider model [3].

About twenty years ago it was found [4]-[7] that finite-dimensional dynamical systems of Toda- and Calogero-types can be obtained by the Hamiltonian reduction of geodesic motions on the cotangent bundles of semi-simple Lie groups. The L-operator of a system arises as a point on the reduced phase space $P$, whereas the Lax representation $\frac{d}{dt} = [M, L]$ - as the equation of motion on $P$. Later on it was realized [8] that this construction provides an effective tool to deduce the classical R-matrix and to prove thereby the integrability of a system. In the recent paper by Avan, Babelon and Talon [9] a nice computation scheme along the lines described above was worked out in detail and applied to recover the known [1] dynamical R-matrices of the rational and trigonometric Calogero-Moser models.

The dynamical R-matrix for the elliptic Calogero model was found in [2] by the direct computation. In that paper a task to find a geometric interpretation of this R-matrix was adduced (see also [9]). The essential ingredient to solve this task, the Hamiltonian reduction procedure, was elaborated in the recent paper by Gorsky and Nekrasov [10]. They proved that the phase space of the elliptic Calogero model is the cotangent bundle to the moduli space of holomorphic connections on a torus with a marked point. We employ this reduction and following the computation scheme of [9] deduce the Sklyanin R-matrix. It appears to be a solution of the first order differential equation:

$$X = [R(X), D] - k\bar{\partial}R(X) + Q,$$

where $D$ and $Q$ are constant diagonal matrices, and $X$ is an $sl(n, \mathbb{C})$ -valued function on a torus.

The paper is organized as follows. In section 2 we show how the classical R-matrix is related to the factorization problem for $su(n)$ connection on a circle (holomorphic $sl(n, \mathbb{C})$ connection on a torus). Sections 3 and 4 are devoted to the explicit solution of the corresponding factorization problems. As the result we recover the trigonometric and elliptic R-matrices.

2 R-matrix from the Hamiltonian reduction

We start with the brief review of some basic facts about the Hamiltonian reduction of cotangent bundles over affine algebras.

Denote by $G$ a finite-dimensional Lie group with a Lie algebra $\mathcal{G}$. Let $\mathcal{L}\mathcal{G} = \{ \phi : S^1 \rightarrow \mathcal{G} \}$ be a current algebra and $\tilde{\mathcal{L}}\mathcal{G}$ be its central extension. The commutation relations in $\tilde{\mathcal{L}}\mathcal{G}$ are

$$[(\phi, c), (\phi', c')] = ([\phi, \phi'], \int_{S^1} d\varphi \text{tr}(\phi \partial \phi')), \quad c \in \mathbb{R}.$$
The dual $\hat{\mathcal{LG}}^*$ to $\hat{\mathcal{LG}}$ consists of pairs $(A, k)$, where $A: S^1 \to G$ and $k \in \mathbb{R}$. The nondegenerate pairing is

$$< (A, k), (\phi, c) > = \int_{S^1} \text{tr}(A\phi) + kc,$$

(2.1)

The sum $\hat{\mathcal{LG}} \oplus \hat{\mathcal{LG}}^*$ can be identified with the cotangent bundle $T^*\hat{\mathcal{LG}}$ over $\hat{\mathcal{LG}}$ supplied with the standard Poisson (symplectic) structure:

$$\{f, h\} = \int_{S^1} d\phi \text{tr} \left( \frac{\delta f}{\delta \phi} \frac{\delta h}{\delta A} - \frac{\delta f}{\delta A} \frac{\delta h}{\delta \phi} \right) + \frac{\delta f}{\delta c} \frac{\delta h}{\delta k} - \frac{\delta f}{\delta k} \frac{\delta h}{\delta c}. $$

(2.2)

Sometimes we will denote by $p$ the whole set $(\phi, c; A, k)$ being the point of the phase space $\mathcal{P}$.

A current group $\mathcal{LG}$ acts on $\hat{\mathcal{LG}}$ and on $\hat{\mathcal{LG}}^*$ by the adjoint and coadjoint actions respectively:

$$(\phi, c) \to (g\phi g^{-1}, c + \int_{S^1} d\phi (\phi g^{-1} \partial g)), $$

(2.3)

$$(A, k) \to (gAg^{-1} - k\partial gg^{-1}, k). $$

(2.4)

This action is Hamiltonian and it gives rise to a moment map $\mu: T^*\hat{\mathcal{LG}} \to \mathcal{LG}$

$$\mu(p) = k\partial \phi + [A, \phi]. $$

(2.5)

Let $J \in \hat{\mathcal{LG}}^*$. Then a quotient $\mu^{-1}(J)/G_J$ by the action of the isotropy group $G_J \subset G$ of $J$ admits under some natural assumptions a symplectic structure [13].

Let $A$ be a smooth function on $S^1$ with values in some real semi-simple Lie algebra. A differential equation

$$A = -k\partial f f^{-1} $$

(2.6)

has a unique solution if one fixes $f(0) = 1$. Since $A$ is periodic, then $f(\varphi + 2\pi) = f(\varphi)M(A)$, where $M(A)$ is a constant matrix called the monodromy of $A$. Thus we have a mapping $M: A \to M(A)$. Performing the gauge transformation

$$A^g = gAg^{-1} - k\partial gg^{-1} $$

(2.7)

with an element $g$ of the current group $(g(0) = g(2\pi))$, one finds that $M(A^g) = g(0)M(A)g(0)^{-1}$. The inverse assertion is also true, i.e. if we have two fields $A$ and $\tilde{A}$ with monodromies related as $M(\tilde{A}) = gM(A)g^{-1}$, then the function

$$g(\varphi) = \tilde{f}(\varphi)gf^{-1}(\varphi) $$

is a gauge transform from $A$ to $\tilde{A}$. Thus, the spectral invariants of $M(A)$ completely parametrize the orbits of (2.7).

Let us diagonalize $M(A)$: $M(A) \to \mathcal{D}(A)$ by some gauge transformation of $A$ and define $D$ as $\exp D = \mathcal{D}(A)$. Now consider the function $f(\varphi) = e^{i\frac{2\pi}{2n}D}$. On the one hand, it is the solution of (2.6) for the constant diagonal field $\tilde{A} = -\frac{k}{2n}D$ and on the other hand, it defines the monodromy conjugated to $M(A)$. Thus every field
A is gauge equivalent to a constant matrix with values in the Cartan subalgebra. Moreover, if we fix the order of eigenvalues of $D$ (the fundamental Weyl chamber), then $D$ is uniquely defined. Consequently we have a decomposition

$$A = g(A, k)D(A, k)g(A, k)^{-1} - k\delta g(A, k)g(A, k)^{-1},$$

(2.8)

where $D(A, k)$ is the constant diagonal matrix with a fixed order of eigenvalues.

Recall that in the approach of [11, 12] the moment map is fixed by the condition

$$k\partial\phi + [A, \phi] = \nu \sum_{\alpha \in \Delta_+} (e_{\alpha} + e_{-\alpha})\delta(\phi),$$

(2.9)

where $\nu$ is a coupling constant. The isotropy group of this moment is also large enough to reduce $A$ to a constant diagonal matrix $D = iX$. Then solving (2.9) with respect to $\phi$, one gets [11] the $L$-operator of the trigonometric Calogero model [11]:

$$L(\varphi) = iP - \frac{i\nu}{2k} \sum_{\alpha \in \Delta_+} \left( \frac{e^{-i\alpha(X)}\varphi}{e^{-\frac{\pi\alpha(X)}{k}} - 1} e_{\alpha} + \frac{e^{i\alpha(X)}\varphi}{e^{\frac{\pi\alpha(X)}{k}} - 1} e_{-\alpha} \right).$$

(2.10)

A pair $(L, D)$ is a point of the reduced phase space $P_\mu$. For the $su(n)$ case (2.10) reduces to

$$L(\varphi) = i \sum_i p_i E_{ii} + \frac{\nu}{2k} \sum_{i \neq j} e^{i\alpha_{ij}(X)(\pi - \varphi)} \sin \frac{\pi\alpha_{ij}(X)}{k} E_{ij},$$

(2.11)

where $E_{ij}$ is the basis of matrix unities. The description of the reduction completes by mentioning that the entries of the constant diagonal matrices $P$ and $X$ form a set of the canonically conjugated variables being the coordinates on $P_\mu$.

Now we proceed further with our analysis of (2.8). In contrast to $D(A, k)$, the factor $g(A, k)$ is not uniquely defined. In the case of a simple real Lie group $G$ this ambiguity is described in the following

**Proposition 1** Factor $g(A, k)$ in (2.8) is defined up to the right multiplication by elements of the maximal torus $T$ of $G$.

The proof is as follows. Suppose that we have two elements $g$ and $g'$ of $L\mathcal{G}$ such that $A = g \circ D = g' \circ D$, where $\circ$ is a shorthand for (2.8). Then $g'^{-1} g \circ D = D$, i.e. $\gamma = g'^{-1} g$ belongs to the stabilizer of $D$, so that $\gamma(0)$ lies in the centralizer of $M(D)$. Since $M(D)$ is a generic diagonal matrix, we have $\gamma(0) \in T$. It is readily seen that $\gamma(\varphi)$ is given by $\gamma(\varphi) = f(\varphi)\gamma(0)f(\varphi)^{-1}$, where $f(\varphi)$ is the solution of (2.6) for $A = D$. Recalling that $f(\varphi) = e^{-\frac{\pi}{k}D}$, we get $\gamma(\varphi) = \gamma(0) \in T$, i.e. $g'^{-1} g \in T$.

Let us associate with any element $X \in L\mathcal{G}$ a function

$$F_X((\phi, c), (A, k)) = \langle \phi, g(A, k)Xg(A, k)^{-1} \rangle,$$

(2.12)

where $g(A, k)$ is defined by (2.8). The isotropy group $G_J$ acting on the surface $\mu^{-1}(J)$ coincides with a group of smooth mappings $G_J = \{g : S^1 \rightarrow G, \ g(0) \in H \}$, where $H$ is the isotropy group of $J = \sum_{\alpha \in \Delta_+} (e_{\alpha} + e_{-\alpha})$. Since $H \cap T = 0$, there is no ambiguity in the choice of $g(A, k)$ for $A$ lying on the surface $\mu^{-1}(J)$, so that $F_X$ is
well defined. Moreover, $F_X$ is invariant with respect to (2.7) with $g \in G_J$, i.e. it is a function on the reduced phase space.

Denote by $\xi_X$ the hamiltonian vector field corresponding to $X \in \mathcal{L}G$. In the sequel, we need to know the action of $\xi_X$ on $F_Z$. The calculation is straightforward

$$\xi_X F_Z = \frac{d}{dt} F_Z(p(t))_{|t=0} = \frac{d}{dt} \langle e^{tX} \phi e^{-tX}, g(e^{tX} \circ (A, k)) Z g(e^{tX} \circ (A, k))^{-1} \rangle_{|t=0}$$

$$= \langle \phi, g(A, k) \left( g(A, k)^{-1} \nabla_{(A,k)} g(X) - g(A, k)^{-1} X g(A, k, Z) \right) g(A, k)^{-1} \rangle,$$

where the derivative along an orbit of gauge transformations

$$\nabla_{(A,k)} g(X) = \frac{d}{dt} g(e^{tX} \circ (A, k))_{|t=0}$$

was introduced. On $P_\mu$ the previous formula takes the form

$$\xi_X F_Z = \langle L, \left[ \nabla_{(D,k)} g(X) - X, Z \right] \rangle.$$

(2.14)

As it was explained in [9], the Poisson bracket on the reduced phase space can be presented in the following convenient form

$$\{f, h\}_r = \{f, h\} - \langle J, [V_f, V_h] \rangle,$$

(2.15)

where $f, h$ are functions on $P$ whose restrictions on $\mu^{-1}(J)$ are invariant with respect to the action of the isotropy group and $V_f$ is the solution of $\langle J, [X, V_f] \rangle = \xi_X f$ (see [9] for details). Since $F_X, F_Y$ are invariant on $\mu^{-1}(J)$, we get

$$\{F_X, F_Y\}_r = \{F_X, F_Y\} - \langle J, [V_{F_X}, V_{F_Y}] \rangle,$$

(2.16)

The second term in (2.16) follows from (2.14):

$$\langle J, [V_{F_X}, V_{F_Y}] \rangle = \langle L, \left[ \nabla_{(D,k)} g(V_{F_X}) - V_{F_X}, Y \right] \rangle.$$

To obtain the bracket $\{F_X, F_Y\}$ we first calculate

$$\frac{\delta F_X}{\delta A_{ij}(\phi')} = \langle \phi, g(A) \left( g(A)^{-1} \frac{\delta g(A)}{\delta A_{ij}(\phi')} X \right) g(A)^{-1} \rangle,$$

$$\frac{\delta F_X}{\delta \phi_{ij}(\phi')} = \left( g(A) X g(A)^{-1} \right)_{ij}(\phi')$$

and then substitute the result in (2.2):

$$\{F_X, F_Y\} = \langle \phi, g(A) \left[ g(A)^{-1} \int_{S^1} (g(A) X g(A)^{-1})_{ij}(\phi') \frac{\delta g(A)}{\delta A_{ij}(\phi')} Y \right] g(A)^{-1} \rangle$$

$$- \langle \phi, g(A) \left[ g(A)^{-1} \int_{S^1} (g(A) Y g(A)^{-1})_{ij}(\phi') \frac{\delta g(A)}{\delta A_{ij}(\phi')} X \right] g(A)^{-1} \rangle.$$

(2.17)

Thus we have
Proposition 2 There exists a linear operator $R: \mathcal{G} \to \mathcal{G}$ given by
\[
R(X)(\phi) = \sum_{ij} \int_{S^1} d\phi' X_{ij}(\phi') \frac{\delta g(D,k)}{\delta A_{ij}(\phi')}(\phi) - \frac{1}{2} (\nabla_{(D,k)} g(V_{F_X}) - V_{F_X})
\] (2.18)
such that the Poisson bracket on the reduced phase space is of the form
\[
\{F_X, F_Y\}_r = \langle L, [R(X), Y] + [X, R(Y)] \rangle
\] (2.19)

Remark. Proposition 2 is a generalization of Theorem 4.1 of \cite{9} to the reduction procedure described above.

$R$-matrix (2.18) depends on the extension of $F_X$ in the vicinity of $\mu^{-1}(J)$. We extend $F_X$ in a specific way.

For the sake of being definite, assume that $\mathcal{G} = su(n)$. Then $\mathcal{G}$ has the decomposition in the direct sum $\mathcal{G} = \mathcal{H} \oplus \mathcal{B} \oplus \mathcal{C}$, where $\mathcal{H}$ is a Lie algebra of the isotropy group $H$, $\mathcal{B}$ is the Lie algebra of $T$ and $\mathcal{C}$ is defined as an orthogonal to $\mathcal{H} \oplus \mathcal{B}$ with respect to the Killing metric. Consider the factor $g(A,k) = \exp X$ in (2.8), where $X \equiv X(\phi)$ is an element of the current algebra. Using the ambiguity in the definition of $g(A,k)$ described by Proposition 1, we arrive at

Proposition 3 For any pair $(A,k)$ with $k \neq 0$ decomposition (2.8) defines a unique element $g(A,k) = e^X(\phi)$ of the current group such that $X(\phi)$ is an element of the current algebra obeying the boundary condition $X(0) \in \mathcal{H} \oplus \mathcal{C}$. In addition, if $(A,k) \in \mu^{-1}(J)$, then $X(0) \in \mathcal{H}$.

Now return to (2.18) and choose the extension of $g(A,k)$ as described in Proposition 3. Then $\nabla_{(D,k)} g(X) = P_{\mathcal{H} \oplus \mathcal{C}}$, where $P_{\mathcal{H} \oplus \mathcal{C}}$ is a projector on $\mathcal{H} \oplus \mathcal{C}$ parallel to $\mathcal{B}$. Now if $X(\phi)$ is such that $X(0) \in \mathcal{H} \oplus \mathcal{C}$, then
\[
0 =\langle L, \nabla_{(D,k)} g(X) - X \rangle = \langle J, [X, V_{F_Y}] \rangle.
\]

Since $\mathcal{H} \oplus \mathcal{C}$ is an isotropic space of the form $\Xi(X,Y) = \langle J, [X, Y] \rangle$, we conclude that $V_{F_Y} \in \mathcal{H} \oplus \mathcal{C}$ for any $Y$. Hence under the choice of $g(A,k)$ given by Proposition 3 $R$-matrix (2.18) reduces to
\[
R(X)(\phi) = \sum_{ij} \int_{S^1} d\phi' X_{ij}(\phi') \frac{\delta g(D,k)}{\delta A_{ij}(\phi')}(\phi).
\] (2.20)

The last formula has a transparent geometric meaning. Defining the time evolution of the field $A(t)$ such that $A(0) = D$ and $\frac{dA}{dt}|_{t=0} = X$, one has $R(X)(\phi) = \frac{d}{dt} g(A(t))(\phi)|_{t=0}$. Since eq. (2.8) is valid for any $t$
\[
A(t) = g(A,k)(t)D(A,k)(t)g(A,k)(t)^{-1} - k \partial g(A,k)(t)g(A,k)(t)^{-1},
\]
we differentiate it with respect to $t$ and put $t = 0$. The result is
\[
X = [R(X), D] - k \partial R(X) + Q,
\] (2.21)
where $Q = \left. \frac{\text{d}}{\text{d}t} D \right|_{t=0}$. Eq. (2.21) is a differential equation of the first order. For any smooth function $X(\varphi)$ on a circle it has a unique solution $R(X)$ obeying the boundary condition $X(0) \in \mathcal{H} \oplus \mathcal{C}$. From (2.21) we can also read off that the $R$-matrix is dynamical since it depends on $D$ that accumulates the coordinates on the reduced phase space. In the sequel, we refer to (2.21) as to the factorization problem for $su(n)$ ($\mathfrak{sl}(n, \mathbb{C})$) connection. Hence, by construction the $R$-matrix of the trigonometric Calogero model is defined as a unique solution of the factorization problem for $su(n)$ connection obeying some specific boundary condition.

3 Trigonometric case

In this section we solve explicitly the factorization problem for $su(n)$ connection and thereby recover the $R$-matrix of the trigonometric Calogero model first found in [1].

We start with equation

$$V = [\Lambda, D] - k\Lambda' + t, \quad V, \Lambda \in \mathcal{L}su(n)$$

(3.22)

and $D, t$ are constant diagonal matrices. Writing down the root decomposition of $su(n)$ elements

$$V = \sum_{\alpha \in \Delta_+} v_\alpha e_\alpha - \bar{v}_\alpha e_{-\alpha} + \sum_i v_i h_i, \quad \Lambda = \sum_{\alpha \in \Delta_+} x_\alpha e_\alpha - \bar{x}_\alpha e_{-\alpha} + \sum_i x_i h_i,$$

and introducing $v_\alpha = \langle V, e_\alpha \rangle$, $v_i = \langle V, h_i \rangle$, etc., we get from (3.22) two equations on diagonal and nondiagonal parts of $\Lambda$ respectively. Imposing on $\Lambda$ the periodicity condition: $\Lambda(0) = \Lambda(2\pi)$, we reconstruct $\Lambda$ up to a constant diagonal matrix $h$ with pure imaginary entries:

$$\Lambda(\varphi) = \frac{\varphi}{2\pi k} \int_0^{2\pi} v_i h_i - \frac{1}{k} \int_0^\varphi v_i h_i + h$$

(3.23)

$$\frac{i}{2k} \sum_{\alpha \in \Delta_+} \left( \frac{e^{-i\pi \alpha_\alpha(X)}}{\sin \frac{\pi}{k} \alpha(X)} \int_0^{2\pi} d\theta e^{-i\frac{\pi}{k} \alpha(X)(\varphi - \theta)} v_\alpha(\theta) e_\alpha + \frac{e^{i\pi \alpha_\alpha(X)}}{\sin \frac{\pi}{k} \alpha(X)} \int_0^{2\pi} d\theta e^{i\frac{\pi}{k} \alpha(X)(\varphi - \theta)} \bar{v}_\alpha(\theta) e_{-\alpha} \right)$$

$$- \frac{1}{k} \sum_{\alpha \in \Delta_+} \left( \int_0^\varphi d\theta e^{-i\frac{\pi}{k} \alpha(X)(\varphi - \theta)} v_\alpha(\theta) e_\alpha - \int_0^\varphi d\theta e^{i\frac{\pi}{k} \alpha(X)(\varphi - \theta)} \bar{v}_\alpha(\theta) e_{-\alpha} \right),$$

where we put $D = iX$.

Now we fix $h$ by requiring $\Lambda(0)$ to be an element of $\mathcal{H} \oplus \mathcal{C}$. To this end we choose the explicit realization of the root basis by the matrix unities and evaluate $\Lambda(\varphi)$ in (3.23) at zero point:

$$\Lambda(0) = \frac{i}{2k} \sum_{i \neq j} \frac{e^{i\pi \alpha_{ij}(X)}}{\sin \frac{\pi}{k} \alpha_{ij}(X)} \int_0^{2\pi} e^{i\frac{\pi}{k} \alpha_{ij}(\varphi - \theta)} v_{ij} E_{ij} + h.$$  

(3.24)

In (3.24) the convenient notation $v_{ij} = -\bar{v}_{ij}$ was used. Recall that any element $\Lambda(0) \in \mathcal{H} \oplus \mathcal{C}$ should satisfy the relation (see [1] for the proof)

$$\text{Im} \sum_j (\Lambda((0)))_{ij} = \frac{1}{n} \sum_{i \neq j} \text{Im}(\Lambda(0))_{ij}.$$  

(3.25)
Substituting (3.24) in (3.25), one finds $h$ that makes (3.25) true:

$$h = \frac{i}{4k} \sum_{i \neq j} e^{-\frac{i\alpha_{ij}(X)}{k}} \int_0^{2\pi} e^{\frac{\alpha_{ij}}{k} \varphi''} v_{ij} \left( \frac{1}{n} - E_{ii} \right) + \left( \frac{1}{n} - E_{ii} \right).$$ (3.26)

Combining (3.23) and (3.26), we finally get

**Proposition 4** The solution of the factorization problem (3.22) obeying the constraint $\Lambda(0) \in \mathcal{H} \oplus \mathcal{C}$ has the form:

$$\Lambda(\varphi) = \frac{\varphi - \pi}{2\pi k} \int_0^{2\pi} v_i E_{ii} - \frac{1}{k} \int_0^{\varphi} v_i E_{ii} +$$

$$\frac{i}{2k} \sum_{i \neq j} \cos \frac{\pi \alpha_{ij}(X)}{k} \int_0^{2\pi} e^{-\frac{i\alpha_{ij}(X)}{k} (\varphi'' - \varphi')} v_{ij} E_{ij} - \frac{1}{k} \sum_{i \neq j} \int_0^{\varphi} e^{-\frac{i\alpha_{ij}(X)}{k} (\varphi'' - \varphi')} v_{ij} E_{ij} +$$

$$\frac{i}{4k} \sum_{i \neq j} \left( e^{-\frac{i\alpha_{ij}(X)}{k}} \int_0^{2\pi} e^{\frac{i\alpha_{ij}(X)}{k} \varphi''} v_{ij} \right) \left( \frac{1}{n} - E_{ii} \right) + \left( \frac{1}{n} - E_{jj} \right).$$

Clearly, we can rewrite it as

$$\Lambda(\varphi) = \frac{\varphi - \pi}{2\pi k} \int_0^{2\pi} v_i E_{ii} + \frac{i}{2k} \sum_{i \neq j} \cos \frac{\pi \alpha_{ij}(X)}{k} \int_0^{2\pi} e^{-\frac{i\alpha_{ij}(X)}{k} (\varphi'' - \varphi')} v_{ij} E_{ij} +$$

$$\frac{i}{4k} \sum_{i \neq j} \left( e^{-\frac{i\alpha_{ij}(X)}{k}} \int_0^{2\pi} e^{\frac{i\alpha_{ij}(X)}{k} \varphi''} v_{ij} \right) \left( \frac{1}{n} - E_{ii} \right) + \frac{1}{2k} \int_0^{2\pi} \left( \sum_{i \neq j} e^{-\frac{i\alpha_{ij}(X)}{k} (\varphi'' - \varphi')} v_{ij} E_{ij} + \sum_i v_i E_{ii} \right) \epsilon(\varphi - \varphi'),$$

where the function

$$\epsilon(\varphi - \varphi') = \begin{cases} 
1, & \text{if } \varphi \geq \varphi', \\
-1, & \text{otherwise}
\end{cases}$$

was used.

From this form of the solution we can read off that $R$-matrix of the trigonometric Calogero system is a function on $S^1 \times S^1$ having the following explicit form

$$R(\varphi, \varphi') = \frac{\varphi - \pi}{2\pi k} \sum_i E_{ii} \otimes E_{ii} + \frac{i}{2k} \sum_{i \neq j} \cos \frac{\pi \alpha_{ij}(X)}{k} e^{-\frac{i\alpha_{ij}(X)}{k} (\varphi'' - \varphi')} E_{ij} \otimes E_{ij}$$

$$- \frac{1}{2k} \sum_{i \neq j} \left( E_{ii} - \frac{1}{n} \right) \otimes \left( \frac{e^{-\frac{i\alpha_{ij}(X)}{k}}}{1 - e^{-\frac{2\pi \alpha_{ij}(X)}{k}}} E_{ij} - \frac{e^{\frac{i\alpha_{ij}(X)}{k}}}{e^{\frac{2\pi \alpha_{ij}(X)}{k}}} E_{ji} \right) + \frac{1}{2k} r(\varphi, \varphi'),$$

where we have introduced a matrix $r$:

$$r(\varphi, \varphi') = \left( \sum_{i \neq j} e^{-\frac{i\alpha_{ij}(X)}{k} (\varphi'' - \varphi')} E_{ij} \otimes E_{ji} + \sum_i E_{ii} \otimes E_{ii} \right) \epsilon(\varphi - \varphi'),$$ (3.29)

It holds the following...
Proposition 5. Matrix $r$ leads to the trivial Poisson bracket on the reduced phase space, i.e. the following relation is satisfied

$$[r_{12}(\varphi, \varphi'), L(\varphi) \otimes I] - [r_{21}(\varphi', \varphi), I \otimes L(\varphi')] = 0. \quad (3.30)$$

See Appendix A for the proof. Consequently,

Proposition 6. R-matrix of the trigonometric Calogero model is given by (3.28) with $r(\varphi, \varphi') = 0$.

Remark. On the reduced phase space the variables $(P, X)$ are canonically conjugated. In standard tensor notation,

$$\{P_1, X_2\} = -\frac{1}{2\pi} \sum_i E_{ii} \otimes E_{ii}. \quad (3.31)$$

$L$-operator (2.11) as well as $R$-matrix (3.28) depend on the phase $\varphi$. However, this dependence may be removed by the similarity transformation $L \rightarrow \tilde{L} = Q(\varphi) L(\varphi) Q(\varphi)^{-1}$, where $Q(\varphi) = e^{-\frac{i}{2<k} \partial g}$. The modified $L$-operator:

$$\tilde{L} = i \sum_i p_i E_{ii} + \frac{\nu}{2k} \sum_{i \neq j} \frac{E_{ij}}{\sin \frac{\pi \alpha_{ij}(X)}{k}}$$

with $(P, X)$ subjected to (3.31) has also an $R$-matrix bracket with

$$R = i \left( \sum_i \cos \frac{\pi \alpha_{ii}(X)}{k} E_{ii} \otimes E_{ii} + \frac{1}{2} \sum_{i \neq j} \frac{1}{\sin \frac{\pi \alpha_{ij}(X)}{k}} \left( E_{ii} - \frac{1}{n} \right) \otimes (E_{ij} - E_{ji}) \right).$$

This $R$-matrix was first found in [1] and then recovered in [9] by the Hamiltonian reduction applied to the cotangent bundle $T^*G$ over a finite-dimensional simple Lie group $G$.

4 Elliptic case

As it was first shown in [10] the phase space of the elliptic Calogero model coincides with the moduli space of holomorphic connections on a torus $\Sigma_\tau$ with a marked point. We employ this construction to deduce the classical $R$-matrix of the elliptic Calogero model [2].

This time the phase space is characterized by the set $\mathcal{P} = (\phi, c; A, k)$, where $\phi, A$ are functions on $\Sigma_\tau$ with values in $sl(n, \mathbb{C})$, $c, k \in \mathbb{C}$. $\mathcal{P}$ can be identified with the cotangent bundle over the centrally extended current algebra $(\phi, c)$ of $sl(n, \mathbb{C})$-valued functions on $\Sigma_\tau$. On $\mathcal{P}$ there is an action of the current group $\mathcal{L}SL(n, \mathbb{C})$

$$(\phi(z, \bar{z}), c) \rightarrow (g(z, \bar{z}) \phi(z, \bar{z}) g^{-1}(z, \bar{z}), c + \int_{\Sigma_\tau} d\eta \text{tr} \phi A), \quad (4.32)$$

$$(A(z, \bar{z}), k) \rightarrow (g(z, \bar{z}) A(z, \bar{z}) g^{-1}(z, \bar{z}) - k \partial g(z, \bar{z}) g^{-1}(z, \bar{z}), k) \quad (4.33)$$
that preserves the standard symplectic structure \((2.2)\). The moment map of this action is fixed to be

\[ k\tilde{\partial}\phi + [A, \phi] = \nu J\delta(z, \bar{z}) \] (4.34)

that defines the phase space and the \(L\)-operator of the elliptic Calogero model \([10, 13]\). In (4.34) \(J\) denotes some element on the coadjoint \(\text{sl}(n, \mathbb{C})\) orbit.

Retracing the same steps as in the trigonometric case we can prove without problems that the \(R\)-matrix corresponding to the \(L\)-operator arising from (4.34) is given by the similar formula

\[ R(V)(z, \bar{z}) = \sum_{ij} \int_{\Sigma_\tau} d\eta d\bar{\eta} \frac{\delta g(A,k)}{\delta A_{ij}(z, \bar{z})} \delta(A,k) \] (4.35)

where \(X(z, \bar{z})\) is a solution of the factorization problem

\[ V(z, \bar{z}) = [X(z, \bar{z}), D] - k\tilde{\partial}X(z, \bar{z}) + t, \] (4.36)

for \(\text{sl}(n, \mathbb{C})\) connection \(V\). Here \(D\) and \(t\) are constant diagonal matrices, and \(V(z, \bar{z})\) and \(X(z, \bar{z})\) are \(\text{sl}(n, \mathbb{C})\)-valued functions on \(\Sigma_\tau\). As usual we regard functions on a torus as twice periodic functions on \(\mathbb{C}\). From (4.36) it is easy to see that \(X(z, \bar{z})\) is defined only up to a constant diagonal matrix \(h\). As it was shown above \(h\) must be fixed in a specific way to make (4.35) correct.

We start the study of (4.36) with solving the equation

\[ \bar{\partial}x(z, \bar{z}) = f(z, \bar{z}), \] (4.37)

where \(f(z, \bar{z})\) is a twice periodic function on \(\mathbb{C}\) with periods \(\tau\) and \(\tau' = 1\). Suppose that we have a solution \(x(z, \bar{z})\) of (4.37), then \(x(z + \tau, \bar{z} + \tau)\) is also a solution. Indeed, substituting \(z = w + \tau\) in (4.37), we get

\[ \frac{\partial}{\partial w}x(w + \tau, \bar{w} + \bar{\tau}) = f(w + \tau, \bar{w} + \bar{\tau}) = f(w, \bar{w}). \]

Consider the difference \(\psi = x(w + \tau, \bar{w} + \bar{\tau}) - x(w, \bar{w})\). It satisfies the homogeneous equation \(\bar{\partial}\psi = 0\), i.e. \(\psi\) is an entire function. In this way we found the monodromy property of a general solution of (4.37):\n
\[ x(w + \tau, \bar{w} + \bar{\tau}) = x(w, \bar{w}) + \psi_\tau(w). \] (4.38)

Suppose we have an equation on a torus

\[ \bar{\partial}\mathcal{E}(z, \bar{z}) = \delta(z, \bar{z}). \] (4.39)

In the vicinity of the origin eq.(4.39) defines a meromorphic function with a first order pole with the residue \(1/2\pi i\). Define a solution of (4.39) as a meromorphic function on \(\mathbb{C}\) having simple poles at the points \(Z\tau + Z\tau'\) with the residues \(1/2\pi i\) and satisfying the quasiperiodicity conditions

\[ \mathcal{E}(z + \tau, \bar{z} + \bar{\tau}) = \mathcal{E}(z, \bar{z}) + C_\tau, \] (4.40)
\[ \mathcal{E}(z + \tau', \bar{z} + \bar{\tau}') = \mathcal{E}(z, \bar{z}) + C_{\tau'}, \]

where \( C_{\tau}, C_{\tau'} \) are complex numbers. As it will be seen only these fundamental solutions are relevant to define a solution of (4.37).

**Remark.** Note that \( \mathcal{E}_{C_{\tau}, C_{\tau'}}(z) \) can not be twice periodic since there is no elliptic functions of the first order. Here the subscript marks the monodromy property of the solution.

Suppose that we have two solutions \( \mathcal{E}_{C_{\tau_1}, C_{\tau_1'}}(z) \) and \( \mathcal{E}_{C_{\tau_2}, C_{\tau_2'}}(z) \). Their difference is an entire but nonperiodic function \( \psi \) (poles and residues of \( \mathcal{E} \)'s coincide) with

\[
\psi(z + \tau) = \psi(z) + C_{\tau_1} - C_{\tau_2} = \psi(z) + \delta,
\]

\[
\psi(z + \tau') = \psi(z) + C_{\tau'_1} - C_{\tau'_2} = \psi(z) + \delta'.
\]

Recall that numbers \( C_{\tau} \) and \( C_{\tau'} \) are not arbitrary. They obey the relation (Legendre's identity [16])

\[ C_{\tau} \tau' - C_{\tau'} \tau = 1 \]

that originates from integrating \( \mathcal{E}_{C_{\tau}, C_{\tau'}}(z) \) around the pole at zero point. Therefore, we get

\[ \delta \tau' - \delta' \tau = 0. \quad (4.42) \]

The only entire function with (4.41) is \( \psi = \alpha z + \beta, \alpha, \beta \in \mathbb{C} \). We can always choose \( \alpha (\alpha = -\delta/\tau) \) to put \( \delta = 0 \), then from (4.42) it follows that \( \delta' = 0 \) also. Hence, any two solutions of (4.39) are related as

\[ \mathcal{E}(z) = \tilde{\mathcal{E}}(z) + \alpha z + \beta. \]

The Weierstrass \( \zeta \)-function

\[
\zeta(z) = \frac{1}{z} + \sum_{\mathbf{Z}} \left( \frac{1}{z - \omega_{nm}} + \frac{1}{\omega_{nm}} + \frac{z}{\omega_{nm}^2} \right), \quad \omega_{nm} = n \tau + m \tau',
\]

satisfies the properties listed above and therefore gives a peculiar solution of (4.39). Thus, we have proved the following

**Proposition 7** Any meromorphic function \( \mathcal{E}(z) \) with only simple poles at the points \( \mathbb{Z} \tau + \mathbb{Z} \tau' \) with the residues \( 1/2\pi i \) and satisfying (4.40) is of the form

\[ \mathcal{E}(z) = \frac{1}{2\pi i} \zeta(z) + \alpha z + \beta \] (4.44)

Clearly, when \( \beta = 0 \) these functions are odd \( \mathcal{E}(-z) = -\mathcal{E}(z) \).

Taking into account this proposition we can write down a general solution of \( \tilde{\partial}x_i(z, \bar{z}) = \frac{1}{k}(t_i - v_i(z, \bar{z})) \)

\[ x_{\text{diag}}(z, \bar{z}) = \frac{1}{k} \int_{\Sigma_{\tau}} d\eta d\bar{\eta} \mathcal{E}(z - \eta)(t_i - v_i(\eta, \bar{\eta}))E_{ii} + h, \quad (4.45) \]
where \( h \) is a diagonal matrix in \( sl(n, \mathbb{C}) \). Let us require this solution to be periodic, i.e. \( \psi_\tau = \psi_{\tau r} = 0 \). This determines the unknown diagonal part \( t \):

\[
    t = \frac{1}{2i \Sigma r} \int \Sigma_r \, d\tilde{\eta} d\eta \, v_i(\eta, \tilde{\eta}) E_{ii}.
\]  

(4.46)

Now we turn to the equation

\[
\bar{\partial}x_\alpha(z, \bar{z}) + \frac{\alpha(D)}{k} x_\alpha(z, \bar{z}) = -\frac{1}{k} v_\alpha(z, \bar{z}).
\]

(4.47)

First we find the fundamental solution \( \mathcal{E}_\alpha \)

\[
\bar{\partial}\mathcal{E}_\alpha(z) + \frac{\alpha(D)}{k} \mathcal{E}_\alpha(z) = \delta(z, \bar{z}).
\]

(4.48)

Writing \( \mathcal{E}_\alpha(z) = e^{\frac{\alpha(D)}{k}(z-\bar{z})} Q_\alpha(z, \bar{z}) \), we get for \( Q_\alpha(z, \bar{z}) \) the following equation

\[
\bar{\partial}Q_\alpha(z, \bar{z}) = e^{-\frac{\alpha(D)}{k}(z-\bar{z})} \delta(z, \bar{z}) = \delta(z, \bar{z})
\]

that tells us that \( Q_\alpha(z, \bar{z}) \) is a meromorphic function having the simple pole at \( z = 0 \) with the residue equal to \( 1/2\pi i \). Assuming the fundamental solution \( \mathcal{E}_\alpha(z) \) to be twice periodic we find immediately the monodromy properties of \( Q_\alpha(z, \bar{z}) \):

\[
Q_\alpha(z + 1) = Q_\alpha(z), \\
Q_\alpha(z + \tau) = e^{-\frac{\alpha(D)}{k}(\tau-\bar{\tau})} Q_\alpha(z).
\]

(4.49)

The solution of the last problem is unique and is given by

\[
\mathcal{E}_\alpha(z) = \frac{e^{\frac{\alpha(D)}{k}(z-\bar{z})}}{2\pi i} \frac{\theta_{11}(z + \frac{\alpha(D)}{\pi k} \text{Im}\tau) \theta'_{11}(0)}{\theta_{11}(z) \theta_{11}(\frac{\alpha(D)}{\pi k} \text{Im}\tau)},
\]

where

\[
\theta_{11}(z, \tau) = \sum_{n \in \mathbb{Z}} e^{i\pi\tau(n+\frac{1}{2})^2 + 2\pi i(n+\frac{1}{2})(z+\frac{1}{2})}
\]

is the Jacobi \( \theta \)-function with monodromy properties

\[
\theta_{11}(z + 1, \tau) = -\theta_{11}(z, \tau), \quad \theta_{11}(z + \tau, \tau) = -e^{-i\pi\tau - 2\pi iz} \theta_{11}(z, \tau).
\]

Combining all pieces of our analysis together we can write a general solution of the factorization problem for \( sl(n, \mathbb{C}) \) connection

\[
X(z) = \frac{1}{k} \int J_{\Sigma r} \, d\tilde{\eta} d\eta \, E(z - \eta)(t_i - v_i(\eta, \tilde{\eta})) E_{ii} + h
\]

(4.50)

\[
-\frac{1}{k} \sum_{i \neq j} \int J_{\Sigma r} \, d\tilde{\eta} d\eta \frac{e^{\frac{\alpha(D)}{k}(z-\eta)-(\bar{z}-\bar{\eta})}}{2\pi i} \frac{\theta_{11}(z - \eta + \frac{\alpha(D)}{\pi k} \text{Im}\tau) \theta'_{11}(0)}{\theta_{11}(z - \eta) \theta_{11}(\frac{\alpha(D)}{\pi k} \text{Im}\tau)} v_{ij}(\eta, \tilde{\eta}) E_{ij}.
\]

(4.51)

Now the problem is to fix the undetermined matrix \( h \) by using the boundary condition for \( X(0) \).
Recall [10] that in the elliptic case one should choose the following representative $J$ on the coadjoint $\mathfrak{sl}(n, \mathbb{C})$ orbit

$$ J = 1 - u \otimes s^\dagger, \quad (4.52) $$

where $u, s$ are some vectors in $\mathbb{C}^n$. The requirement of vanishing the diagonal entries of $J$ fixes the choice of components $s_i$: $s_i^* = 1/u_i$. Every solution $(\phi, A)$ of (4.34) can be brought to the form $(L, D)$, where $D$ is a constant diagonal matrix and $L$ is the $L$-operator

$$ L = \sum_i p_i E_{ii} - \frac{\nu}{2\pi i} \sum_{i \neq j} e^{\frac{\alpha_{ij}(D)}{2}(z - z)} \frac{u_i}{u_j} \frac{\theta_{11}(z + \frac{\alpha_{ij}(D)}{\pi k} \text{Im} \tau)}{\theta_{11}(\frac{\alpha_{ij}(D)}{\pi k} \text{Im} \tau)} E_{ij} \quad (4.53) $$

providing the Lax representation for the elliptic Calogero system [10]. Below we point out the connection of (4.53) with the $L$-operator found by Krichever [14].

The Lie algebra $\mathcal{H}$ of the isotropy group of $J$ in $\mathfrak{sl}(n, \mathbb{C})$ is determined by the equation $[X, J] = 0$, $X \in \mathcal{H}$ that is equivalent to

$$ u_i (s^\dagger X)_j - (Xu)_i s_j^* = 0. \quad (4.54) $$

Choosing in (4.54) $i = j$ one gets $(s^\dagger X)_i = \frac{s_i^*}{u_i} (Xu)_i$ and thereby (4.54) reduces to $(Xu)_i = (Xu)_i$, $s_i^* = \lambda$, where $\lambda \in \mathbb{C}$. One also has

$$ (s^\dagger X)_i = \frac{s_i^*}{u_i} \lambda u_i = \lambda s_i^* = \lambda (s^\dagger)_i. $$

Thus, we find $\mathcal{H}$:

$$ \mathcal{H} = \{ X \in \mathfrak{sl}(n, \mathbb{C}) : \ X u = \lambda u, \ s^\dagger X = \lambda s^\dagger, \ \lambda \in \mathbb{C} \}. \quad (4.55) $$

From (4.53) we can read off that $\mathcal{H} \cap B = 0$, where $B$ is a maximal torus of $\mathfrak{sl}(n, \mathbb{C})$ (diagonal matrices). Since the real dimension of $\mathfrak{sl}(n, \mathbb{C})$ is $2(n^2 - 1)$ and $\mathcal{H}$ is defined by $4n - 4$ equations, we get $\dim \mathcal{H} = 2(n^2 - 1) - (4n - 4) = 2(n - 1)^2$.

Now consider the decomposition of $\mathcal{G} = \mathfrak{sl}(n, \mathbb{C})$ in the direct sum

$$ \mathcal{G} = \mathcal{H} \oplus B \oplus \mathcal{C}, $$

where $\mathcal{C}$ is defined as an orthogonal subspace to $\mathcal{H} \oplus B$ with respect to the Killing metric $(X, Y) = \text{Tr}(XY)$. To describe $\mathcal{C}$ explicitly we introduce a matrix $C$

$$ C = Z \otimes s^\dagger - u \otimes Y^\dagger $$

depending on two vectors $Z, Y \in \mathbb{C}^n$. Let $X \in \mathcal{H}$ and $X u = s^\dagger X = 0$, then $\text{Tr}(XC) = \text{Tr}(Z \otimes s^\dagger X - X u \otimes Y^\dagger) = 0$. On the other hand for $X$ arbitrary we have

$$ \text{Tr}(XC) = \sum_{ik} (z_i s_k^* - u_i y_k^*) x_{ki} $$

13
and therefore for $C$ orthogonal to any element $X = (x_i \delta_{ij}) \in \mathcal{B}$ we have $z_i s_i^* - u_i y_i^* = \beta$ for any $i$, where $\beta$ is an arbitrary complex number. Orthogonality of $\mathcal{H}$ and $C$ also implies the fulfillment of

$$\text{Tr}(JC) = \beta n(1 - <s^\dagger, u>) = \beta n(1 - n) = 0$$

that gives $\beta = 0$. Thus, $C$ is an element of $\mathcal{C}$ if $y_i^* = \frac{1}{u_i^*} z_i$. We also put $\sum_i \frac{z_i^*}{u_i^*} = 0$ to have the correct dimension of $\mathcal{C}$: $\dim \mathcal{C} = 2(n - 1)$. This completes the description of $\mathcal{C}$.

Remark. Just as for $su(n)$ case \[9\] one can prove that $\mathcal{B}$ and $\mathcal{C}$ form a pair of complementary Lagrangian subspaces with respect to the symplectic form $\Xi(X, Y) = <J, [X, Y]>$ defined on $\mathcal{B} \oplus \mathcal{C}$.

Now one can easily find that $(Cu)_i = nz_i$ and $(s^\dagger C)_i = -\frac{n}{u_i^2} z_i$. This allows us to describe the action of a generic element $X \in \mathcal{H} \oplus \mathcal{C}$ on the vectors $u$ and $s$:

$$\begin{align*}
(Xu)_i &= \lambda u_i + nz_i, \\
(s^\dagger X)_i &= \frac{1}{u_i} - \frac{n}{u_i} z_i.
\end{align*}$$

(4.56)

Summing up the second lines in (4.56) and taking into account $\sum \frac{z_i^*}{u_i} = 0$, we find $\lambda$:

$$\lambda = \frac{1}{n} \sum_i \frac{(Xu)_i}{u_i}.$$ (4.57)

Solving (4.57) for $z_i$, we arrive at

**Proposition 8** Let $X$ be an arbitrary element of $\mathcal{H}_J \oplus \mathcal{C}$. Then the following relation

$$u_i (s^\dagger X)_i + (Xu)_i \frac{1}{u_i} = \frac{2}{n} \sum_j \frac{(Xu)_j}{u_j}$$

is valid for any $i$.

Note that we can rewrite (4.56) as follows:

$$(s^\dagger X)_i u_i + \sum_j \left( \delta_{ij} - \frac{2}{n} \frac{(Xu)_j}{u_j} \right) = 0.$$ (4.58)

We use Proposition 2 to fix the element $h$ in (4.51). To this end we put $X(0) \in \mathcal{H} \oplus \mathcal{C}$. Let us show that this requirement determines $h$ completely. To simplify the calculations we introduce

$$w_{ij}(\eta, \bar{\eta}) = \frac{e^{-\frac{\alpha_{ij}(D)}{\pi k}(\eta - \bar{\eta})}}{2\pi i} \frac{\theta_{11}(\eta - \alpha_{ij}(D) \pi k \text{Im} \tau) \theta_{11}'(0)}{\theta_{11}(\eta) \theta_{11}(\alpha_{ij}(D) \pi k \text{Im} \tau)}$$, (4.59)

and

$$q_i(\eta, \bar{\eta}) = t_i - v_i(\eta, \bar{\eta}),$$ (4.60)
so that

\[ X \equiv X(0) = \frac{1}{k} \int_{\Sigma_r} d\bar{\eta} d\eta \, E(-\eta)q_i(\eta, \bar{\eta})E_{ii} + h \quad (4.61) \]

\[ -\frac{1}{k} \sum_{i \neq j} \int_{\Sigma_r} d\bar{\eta} d\eta \, w_{ij}(\eta, \bar{\eta})v_{ij}(\eta, \bar{\eta})E_{ij}. \]

Using the explicit form of \( X \) we calculate

\[ (Xu)_i = \frac{1}{k} \int_{\Sigma_r} d\bar{\eta} d\eta \, E(-\eta)q_iu_i - \frac{1}{k} \sum_{i \neq j} \int_{\Sigma_r} d\bar{\eta} d\eta \, w_{ij}v_{ij}u_j + h_iu_i, \quad (4.62) \]

\[ (s^\dagger X)_i = \frac{1}{k} \int_{\Sigma_r} d\bar{\eta} d\eta \, E(-\eta)q_i \frac{1}{u_i} - \frac{1}{k} \sum_{i \neq j} \int_{\Sigma_r} d\bar{\eta} d\eta \, \frac{1}{u_j}w_{ji}v_{ji}u_i + h_i \frac{1}{u_i}. \quad (4.63) \]

With the help of (4.62) we also find

\[ \sum_i \frac{(Xu)_i}{u_i} = -\frac{1}{k} \sum_{i \neq j} \int_{\Sigma_r} d\bar{\eta} d\eta \, \frac{u_j}{u_i}w_{ij}v_{ij}. \quad (4.65) \]

Substitution of (4.62)-(4.65) in (4.57) results in

\[ \frac{2}{k} \int_{\Sigma_r} d\bar{\eta} d\eta \, E(-\eta)q_i - \frac{1}{k} \sum_{i \neq j} \int_{\Sigma_r} d\bar{\eta} d\eta \, \left( \frac{u_j}{u_i}w_{ij}v_{ij} + \frac{u_i}{u_j}w_{ji}v_{ji} \right) + \frac{1}{2} \sum_{i \neq j} \int_{\Sigma_r} d\bar{\eta} d\eta \, \left( \frac{u_i}{u_i}w_{ij}v_{ij} + \frac{u_i}{u_j}w_{ji}v_{ji} \right) \left( E_{ii} - \frac{1}{n}I \right). \quad (4.66) \]

for any \( i \). This equation allows us to find \( h \):

\[ h = -\frac{1}{k} \int_{\Sigma_r} d\bar{\eta} d\eta \, E(-\eta)q_i + \frac{1}{2} \sum_{i \neq j} \int_{\Sigma_r} d\bar{\eta} d\eta \, \left( \frac{u_i}{u_i}w_{ij}v_{ij} + \frac{u_i}{u_j}w_{ji}v_{ji} \right) \left( E_{ii} - \frac{1}{n}I \right), \quad (4.67) \]

Thus, we arrive at

**Proposition 9** A general solution \( X(z, \bar{z}) \) of the factorization problem for \( \text{sl}(n, \mathbb{C}) \) connection satisfying \( X(0) \in \mathcal{H} \oplus \mathbb{C} \) has the form

\[ X(z, \bar{z}) = \frac{1}{k} \sum_i \int_{\Sigma_r} d\eta d\bar{\eta} \, (\mathcal{E}(z - \eta) - \mathcal{E}(-\eta))(t_i - v_i(\eta, \bar{\eta}))E_{ii} \quad (4.68) \]

\[-\frac{1}{k} \sum_{i \neq j} \int_{\Sigma_r} d\eta d\bar{\eta} \, w_{ij}(\eta - z, \bar{\eta} - \bar{z})v_{ij}(\eta, \bar{\eta})E_{ij} \]

\[ + \frac{1}{2k} \sum_{i \neq j} \int_{\Sigma_r} d\eta d\bar{\eta} \, \left( \frac{u_j}{u_i}w_{ij}(\eta, \bar{\eta})v_{ij}(\eta, \bar{\eta}) + \frac{u_i}{u_j}w_{ji}(\eta, \bar{\eta})v_{ji}(\eta, \bar{\eta}) \right) \left( E_{ii} - \frac{1}{n}I \right), \]

where \( t \) and \( w_{ij} \) are given by (4.4a) and (4.5a) respectively.
Using the explicit form (4.44) of $E(z)$ and taking into account (4.46) it is easy to find that the first line in (4.68) reduces to

$$
\Phi(z, \bar{z}) \sum_i \int_{\Sigma_\tau} d\eta d\bar{\eta} \ v_i(\eta, \bar{\eta}) E_{ii} - \frac{1}{2\pi i k} \sum_i \int_{\Sigma_\tau} d\eta d\bar{\eta} \ (\zeta(z - \eta) + \zeta(\eta)) v_i(\eta, \bar{\eta}) E_{ii},
$$

where we have introduced a function

$$
\Phi(z, \bar{z}) = \int_{\Sigma_\tau} d\eta d\bar{\eta} \ (\zeta(z - \eta) + \zeta(\eta)).
$$

Hence, despite the function $\alpha z + \beta$ enters the fundamental solution (4.44) solution (4.36) of the factorization problem does not depend on it. Just as in the trigonometric case we get from (4.68) the following

**Theorem 1** The $R$-matrix corresponding to $L$-operator (4.53) is the following matrix function on $\Sigma_\tau \times \Sigma_\tau$

$$
R(z, \eta) = \frac{\Phi(z, \bar{z})}{2i k \Sigma_\tau} E_{ii} \otimes E_{ii} - \frac{1}{2\pi i k} (\zeta(z - \eta) + \zeta(\eta)) \sum_i E_{ii} \otimes E_{ii}
$$

$$
- \frac{1}{k} \sum_{i \neq j} w_{ij}(\eta - z, \bar{\eta} - \bar{z}) E_{ij} \otimes E_{ji}
$$

$$
+ \frac{1}{2k} \sum_{i \neq j} \left( E_{ii} - \frac{1}{n} I \right) \otimes \left( \frac{u_j}{u_i} w_{ij}(\eta, \bar{\eta}) E_{ji} + \frac{u_i}{u_j} w_{ji}(\eta, \bar{\eta}) E_{ij} \right).
$$

(4.70)

**Remark.** $L$-operator as well as $R$-matrix (4.70) depends on vector $u \in L^n$. However, by conjugating $L$ with a matrix $Q = e^U$, $U_{ij} = u_i \delta_{ij}$ this dependence may be removed. The corresponding $R$-matrix is given by (4.70) with all $u_i = 1$.

Now we are going to make a connection with the Sklyanin result [2]. Without loss of generality we can assume that the integration domain $\Sigma_\tau$ has vertexes at points $\pm \frac{1}{2} \pm \frac{\tau}{2}$. Then by the oddness of $\zeta$-function one has $\int_{\Sigma_\tau} d\eta d\bar{\eta} \ z(\eta) = 0$ and therefore $\Phi(z, \bar{z})$ reduces to

$$
\Phi(z, \bar{z}) = \int_{\Sigma_\tau} d\eta d\bar{\eta} \ z(z - \eta).
$$

(4.71)

Eq. (4.71) means that $\Phi(z, \bar{z})$ is a solution of the equation $\partial \Phi(z, \bar{z}) = 1$, i.e. $\Phi(z, \bar{z}) = \bar{z} + f(z)$, where $f(z)$ is an entire function. The monodromy properties of $\zeta$ define the ones for $\Phi(z, \bar{z})$:

$$
\Phi(z + \tau, \bar{z} + \bar{\tau}) = \Phi(z, \bar{z}) + \frac{\Sigma_\tau}{\pi} C_\tau,
$$

where $\tau$ denotes any of two periods 1 and $\tau$. For $f(z)$ the equation above implies that

$$
f(z + \tau) - f(z) = \frac{\Sigma_\tau}{\pi} C_\tau - \bar{\tau}.
$$

(4.72)
The only entire function obeying (1.72) is \( f(z) = az + \beta \) with \( \alpha = \frac{C_{\tau}/\pi - \bar{\tau}}{\tau} \). However, we have to prove that \( \alpha \) is defined by the same formula for both periods 1 and \( \tau \). In other words, we have to prove the relation

\[
\frac{C_{\tau}/\pi - \bar{\tau}}{\tau} = \frac{C_1}{\pi - 1},
\]

(4.73)

where \( C_1 \) corresponds to the shift of \( \zeta \)-function along the period 1. Fortunately, (4.73) is satisfied since it reduces to the identity \( \text{Im} \tau = \frac{\tau - \bar{\tau}}{2\pi} = \frac{C_1}{\pi} - \frac{C_{\tau}}{\pi} \) that follows in its turn from Legendre’s identity on the numbers \( C_\tau, C_{\tau'} \). The constant \( \beta \) is equal to zero by the oddness of \( \Phi(z) \). Thus, we get for \( \Phi(z, \bar{z}) \) the following explicit answer

\[
\Phi(z, \bar{z}) = \bar{z} - z + \frac{C_1}{\pi} \text{Im} \tau \; z.
\]

(4.74)

In [2] Krichever’s \( L \)-operator [14]:

\[
L^{Kr} = \sum_i p_i E_{ii} - \frac{\nu}{2\pi i} \sum_{i \neq j} \frac{\sigma(z + \frac{\alpha_{ij}(D)}{\pi k} \text{Im} \tau)}{\sigma(z) \sigma(\frac{\alpha_{ij}(D)}{\pi k} \text{Im} \tau)} E_{ij}
\]

(4.75)

was used to find the corresponding \( R \)-matrix. Due to the identity

\[
G_{ij}(z) = \frac{\sigma(z + \frac{\alpha_{ij}(D)}{\pi k} \text{Im} \tau)}{\sigma(z) \sigma(\frac{\alpha_{ij}(D)}{\pi k} \text{Im} \tau)} = \exp \left( C_1 \alpha_{ij}(D) \text{Im} \tau \right) \frac{\theta_{11}(z + \frac{\alpha_{ij}(D)}{\pi k} \text{Im} \tau) \theta'_{11}(0)}{\theta_{11}(z) \theta'_{11}(\frac{\alpha_{ij}(D)}{\pi k} \text{Im} \tau)}
\]

it is easy to see that \( L^{Kr} \) is related to (4.53) by the similarity transformation

\[
L^{Kr}(z) = Q(z, \bar{z}) L(z, \bar{z}) Q(z, \bar{z})^{-1},
\]

(4.76)

where \( Q(z, \bar{z}) = e^{\frac{D}{\pi k} \Phi(z, \bar{z})} \). Note also, that \( L^{Kr}(z) \) does not depend on \( \bar{z} \). The Poisson bracket for \( L^{Kr} \) is obtained in the usual way

\[
\{ L_1^{Kr}, L_2^{Kr} \} = [Q_1 \{ L_1, Q_2 \} Q_1^{-1} Q_2^{-1}, L_2^{Kr}] + [Q_2 \{ Q_1, L_2 \} Q_1^{-1} Q_2^{-1}, L_1^{Kr}]
\]

where the omitted spectral parameters \( z \) and \( \eta \) can be easily restored. Calculating \( \{ L_1, Q_2 \} \) with the help of the canonically conjugated variables \( \{ P, D \} = \frac{1}{2\pi i k} \sum_i E_{ii} \otimes E_{ii}, P = \sum_i p_i E_{ii} \) on the reduced phase space, we recover the \( R \)-matrix for \( L^{Kr} \):

\[
R(z, \eta) = -\frac{1}{2\pi i k} (\zeta(z - \eta) + \zeta(\eta)) \sum_i E_{ii} \otimes E_{ii}
\]

(4.77)

\[-\frac{1}{2\pi i k} \sum_{i \neq j} G_{ij}(z - \eta) E_{ij} \otimes E_{ji} - \frac{1}{4\pi i k} \sum_{i \neq j} G_{ij}(\eta) \left( E_{ii} + E_{jj} - \frac{1}{n} I \right) \otimes E_{ij}
\]

that is precisely the result of [2].
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APPENDIX

A Proof of Proposition 5.

Recalling the explicit form of the $L$-operator we first compute

$$
\sum_{i\neq j} e^{-i\alpha_{ij}(X)/(\varphi-\varphi')} e^{i\alpha_{kl}(X)/(\pi-\varphi)} \sin \frac{\pi\alpha_{ijl}(X)}{k} [E_{ij}; E_{kl}] \otimes E_{ji} =
$$

$$
\sum_{i\neq j} e^{-i\alpha_{ij}(X)/(\varphi-\varphi')} e^{i\alpha_{kl}(X)/(\pi-\varphi)} \sin \frac{\pi\alpha_{ijl}(X)}{k} E_{il} \otimes E_{ji} = \sum_{i\neq j} e^{-i\alpha_{ij}(X)/(\varphi-\varphi')} e^{i\alpha_{kl}(X)/(\pi-\varphi)} \sin \frac{\pi\alpha_{kl}(X)}{k} E_{kj} \otimes E_{ji}.
$$

The preceding sums can be divided in four parts

$$
\sum_{i\neq j} e^{-i\alpha_{ij}(X)/(\varphi-\varphi')} e^{i\alpha_{kl}(X)/(\pi-\varphi)} \sin \frac{\pi\alpha_{ijl}(X)}{k} E_{ii} \otimes E_{ji} = \quad (A.1)
$$

$$
\sum_{i\neq j} e^{-i\alpha_{ij}(X)/(\varphi-\varphi')} e^{i\alpha_{kl}(X)/(\pi-\varphi)} \sin \frac{\pi\alpha_{ijl}(X)}{k} E_{jj} \otimes E_{ji} = \quad (A.2)
$$

$$
\sum_{i\neq j\neq l} e^{-i\alpha_{ij}(X)/(\varphi-\varphi')} e^{i\alpha_{kl}(X)/(\pi-\varphi)} \sin \frac{\pi\alpha_{ijl}(X)}{k} E_{il} \otimes E_{ji} = \quad (A.3)
$$

$$
\sum_{i\neq j\neq k} e^{-i\alpha_{ij}(X)/(\varphi-\varphi')} e^{i\alpha_{kl}(X)/(\pi-\varphi)} \sin \frac{\pi\alpha_{kl}(X)}{k} E_{kj} \otimes E_{ji}. \quad (A.4)
$$

Taking into account the relations

$$
-\alpha_{ij}(X)(\varphi-\varphi') + \alpha_{jl}(X)(\pi - \varphi) = \pi\alpha_{jl}(X) + \alpha_{ij}(X)\varphi' + \alpha_{ii}(X)\varphi,
$$

$$
-\alpha_{ij}(X)(\varphi-\varphi') + \alpha_{kl}(X)(\pi - \varphi) = \pi\alpha_{kl}(X) + \alpha_{ij}(X)\varphi' + \alpha_{jk}(X)\varphi,
$$

and combining (A.1) and (A.2), we finally get

$$
- \sum_{i\neq j} e^{-i\alpha_{ij}(X)/(\pi-\varphi')} \sin \frac{\pi\alpha_{ijl}(X)}{k} (E_{ii} - E_{jj}) \otimes E_{ji} \quad (A.5)
$$

$$
+ \sum_{i\neq j\neq l} e^{i\alpha_{jl}(X)\varphi' + \alpha_{ii}(X)\varphi} \sin \frac{\pi\alpha_{ijl}(X)}{k} E_{il} \otimes E_{ji} \quad (A) \quad (A.6)
$$
Now we calculate the commutator

$$\left[ \sum_k E_{kk} \otimes E_{kk}, I \otimes \sum_{i\neq j} e^{i \alpha_{ij}(X)} \frac{\sin \pi \alpha_{ij}(X)}{\sin \frac{\pi \alpha_{ij}(X)}{k}} E_{ij} \right] =$$

$$\sum_{i\neq j} e^{i \alpha_{ij}(X)} \frac{\sin \pi \alpha_{ij}(X)}{\sin \frac{\pi \alpha_{ij}(X)}{k}} (E_{ii} - E_{jj}) \otimes E_{ij}.$$

Combining the previous expression with (A.5), we obtain

$$-\left( \sum_{i\neq j} e^{-i \alpha_{ij}(X)} \frac{\sin \pi \alpha_{ij}(X)}{\sin \frac{\pi \alpha_{ij}(X)}{k}} (E_{ii} - E_{jj}) \otimes E_{ji} \right) (\epsilon(\varphi - \varphi') + \epsilon(\varphi' - \varphi)) = 0.$$

As to (A.6) and (A.7), they have the counterparts coming from the second commutator in (3.30). These counterparts can be immediately derived by twisting the factors of the tensor product in (A.6) and (A.7) and changing \( \varphi \) for \( \varphi' \). We denote them by \( C \) and \( D \):

$$- \sum_{i\neq j} e^{i \alpha_{ij}(X)} \frac{\sin \pi \alpha_{ij}(X)}{\sin \frac{\pi \alpha_{ij}(X)}{k}} E_{ji} \otimes E_{il}. \quad \text{(C)} \quad (A.9)$$

$$- \sum_{i\neq j} e^{i \alpha_{ij}(X)} \frac{\sin \pi \alpha_{ij}(X)}{\sin \frac{\pi \alpha_{ij}(X)}{k}} E_{ji} \otimes E_{lj}. \quad \text{(D)} \quad (A.10)$$

The change of indices in (A.9) and (A.10) leads to \( C = -B \) and \( D = -A \). Now subtracting \((C + D)\epsilon(\varphi' - \varphi)\) from \((A + B)\epsilon(\varphi - \varphi')\), we get zero. Continuing this line of reasoning, it is not hard to prove that the moment part of the \( L \)-operator also satisfies (3.30).

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