NEW EXTREMAL BINARY SELF-DUAL CODES OF LENGTHS 64 AND 66 FROM BICUBIC PLANAR GRAPHS

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Abstract. In this work, connected cubic planar bipartite graphs and related binary self-dual codes are studied. Binary self-dual codes of length 16 are obtained by face-vertex incidence matrices of these graphs. By considering their lifts to the ring \( R_2 \) new extremal binary self-dual codes of lengths 64 are constructed as Gray images. More precisely, we construct 15 new codes of length 64. Moreover, 10 new codes of length 66 were obtained by applying a building-up construction to the binary codes. Codes with these weight enumerators are constructed for the first time in the literature. The results are tabulated.

1. Introduction

Self-dual codes are connected to areas such as design theory, graph theory and lattices. Therefore they form an interesting family of linear codes. Such codes have been studied extensively especially the ones over the binary alphabet. \([24, 12, 8]\) extended binary Golay code and \([48, 24, 12]\) extended quadratic residue code are celebrated examples of this type. An upper bound for the minimum distance of a binary self-dual code was given in [2]. That was finalized in [19] as follows; the minimum distance \( d \) of a binary self-dual code of length \( n \) satisfies \( d \leq 4 \lfloor n/24 \rfloor + 6 \) if \( n \equiv 22 \pmod{24} \) and \( d \leq 4 \lfloor n/24 \rfloor + 4 \), otherwise. A self-dual code is said to be extremal if it meets the bound. Since the appearance of [2, 5, 19] construction and classification of extremal binary self-dual codes have been a captivating research area.

The existence of extremal binary self-dual codes are open problems for various lengths. The most famous of these is the existence of doubly-even \([72, 36, 16]\) self-dual code. The possible weight enumerators of extremal self-dual binary codes of lengths up to 64 and 72 were determined in [2]. The weight enumerators for the remaining lengths up to 100 were characterized in [5]. Different techniques such as circulant constructions, Hadamard matrices, automorphism groups and extensions are used to obtain new extremal binary self-dual codes. [8] is a survey on self-dual codes over different alphabets. We refer the reader to [4, 5, 15, 12] for more information in this direction.

In recent years, new binary self-dual codes have been constructed as Gray images of self-dual codes over some rings of characteristic 2. Four circulant construction was applied to the ring \( F_2 + uF_2 \) in [9]. \( F_4 + uF_4 \)-lifts of self-dual quaternary codes were considered in [13]. The ring \( F_2 + uF_2 + vF_2 + uvF_2 \) were used in [11, 15]. In [10], Karadeniz and Yildiz obtained self-dual codes as lifts of \([8, 4, 4]\) extended binary Hamming code to the ring \( R_3 \) which is of size \( 2^8 \).

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On the other hand, codes can be obtained from some special graphs by considering their incidence or adjacency matrices. See [3, 17, 18].

The possible weight enumerators of extremal singly-even [64, 32, 12] codes were determined in [2] as:

\[
W_{64,1} = 1 + (1312 + 16\beta) y^{12} + (22016 - 64\beta) y^{14} + \cdots ; 14 \leq \beta \leq 284,
\]

\[
W_{64,2} = 1 + (1312 + 16\beta) y^{12} + (23040 - 64\beta) y^{14} + \cdots ; 0 \leq \beta \leq 277.
\]

The existence of the codes is unknown for the most of the \(\beta\) values. Recently, codes with \(\beta = 25, 59\) and 74 in \(W_{64,1}\) are constructed in [15] by a bordered four circulant construction. Ten new codes with weight enumerators in \(W_{64,2}\) were obtained in [10]. Together with these, codes exist with weight enumerators for \(\beta = 14, 18, 22, 25, 29, 32, 36, 39, 44, 46, 53, 59, 60, 64\) and 74 in \(W_{64,1}\) and for \(\beta = 0, 1, 2, 4, 5, 6, 8, 9, 10, 12, 13, 14, 16, 17, 18, 20, 21, 22, 23, 24, 25, 28, 29, 30, 32, 33, 36, 37, 38, 40, 41, 44, 48, 51, 52, 56, 58, 64, 72, 80, 88, 96, 104, 108, 112, 114, 118, 120\) and 184 in \(W_{64,2}\).

In this work, we obtain fifteen new extremal binary self-dual codes of length 64. The codes with these weight enumerators are constructed for the first time in the literature. More precisely, the codes with weight enumerators for \(\beta = 16, 20, 24, 26, 28, 30, 34\) and 38 in \(W_{64,1}\); \(\beta = 3, 7, 11, 15, 26, 27\) and 35 in \(W_{64,2}\) are discovered.

The rest of the work is organized as follows; Section 2 is devoted to the preliminaries on codes and graphs, bicubic planar graphs and related binary codes have been studied in Section 3. In Section 4, we consider \(R_2\)-lifts of the codes generated by the face-vertex incidence matrices of connected bicubic planar graphs. As Gray images of these codes extremal binary self-dual codes of length 64 are constructed. Section 5 is devoted to the extensions of the new codes of length 64. We were able to obtain ten new extremal binary self-dual codes of length 66. MAGMA computational algebra system [11] have been used for computational results. Section 6 concludes the paper with potential lines of research.

2. Preliminaries

The ring \(R_2 = \mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2\) was introduced in [20] as a generalization of \(\mathbb{F}_2 + u\mathbb{F}_2\). The ring is a commutative local Frobenius ring of characteristic 2 and size 16 that is defined via the restrictions \(u^2 = 0 = v^2\) and \(uv = vu\). The following isomorphism follows by the definition

\[
R_2 \cong \mathbb{F}_2[u, v]/\langle u^2, v^2, uv - vu \rangle.
\]

Let \(R\) denote a commutative Frobenius ring. A linear code \(C\) of length \(n\) over \(R\) is an-\(R\) submodule of \(R^n\). The Euclidean dual \(C^\perp\) of the code \(C\) is defined with respect to the standard inner product as

\[
C^\perp := \left\{ (b_1, b_2, \ldots b_n) \in R^n \mid \sum_{i=1}^{n} a_i b_i = 0, \forall (a_1, a_2, \ldots a_n) \in C \right\}.
\]

A code \(C\) is said to be self-orthogonal if \(C \subseteq C^\perp\), and self-dual if \(C = C^\perp\). A binary self-dual code is called Type II (doubly-even) if the weight of any codeword is divisible by 4 and Type I (singly-even) otherwise.

In [20] an orthogonality preserving Gray map from \(R_2^n\) into \(\mathbb{F}_2^{4n}\) was given as follows:

\[
\phi(\bar{a} + u\bar{b} + v\bar{c} + uv\bar{d}) = (\bar{d}, \bar{c} + \bar{d}, \bar{b} + \bar{d}, \bar{a} + \bar{b} + \bar{c} + \bar{d}),
\]
where \( \bar{a}, \bar{b}, \bar{c}, \bar{d} \in \mathbb{F}_2 \).

Generator matrices in a special form could be used for lifts. For more details about the ring \( R_2 \) and lifting a binary code to \( R_2 \) we refer to [6, 11, 20, 11, 15].

**Definition 2.1.** [11] A matrix \([I_n | A]\) which generates a self-dual code is called an LRM (lift-ready-matrix) if each upper left \( k \times k \) square submatrix of \( A \) is invertible.

Consider the projection \( \pi : R_2^n \rightarrow \mathbb{F}_2^n \) defined by \( \pi (\overline{a + u b + v c + uv d}) = \overline{a} \), where \( \overline{a}, \overline{b}, \overline{c}, \overline{d} \in \mathbb{F}_2^n \). Let \( D \) be a binary self-dual code, a code \( C \) over \( R_2 \) is said to be a lift of \( D \) if \( \pi (C) = D \).

A graph \( G = (V, E) \) is an incidence structure where \( V \) and \( E \) are vertices and edges, respectively. A graph that can be drawn without crossings in the plane is called planar. A graph is said to be bipartite if a partition of the vertex set \( V \) into \( \{V_1, V_2\} \) such that there is no edge with both endpoints in \( V_1 \) or \( V_2 \).

The graph \( G \) is called simple if it is free of loops and multi-edges. In a simple graph, the degree of a vertex \( w \) denoted by \( \deg (w) \) is the number of edges incident with \( w \). A graph is called \( k \)-regular when every vertex has degree \( k \). A 3-regular graph is said to be cubic. A bicubic graph is both bipartite and cubic. If there is a path between every pair of vertices than it is said to be connected.

3. Bicubic planar graphs and binary self-dual codes

In this section, we consider connected bicubic planar graphs and related self-dual codes. With respect to [17] any cubic planar graph could be used to construct self-orthogonal codes but we focus on the following result for bipartite graphs which gives self-dual codes.

**Theorem 3.1.** [17] Let \( G \) be a connected cubic planar bipartite graph with vertex set \( \{1, 2, \ldots, n\} \) and face-vertex incidence matrix \( D \). Let \( f_1, f_2 \) be any two faces of \( G \) of different colours in a 3-face coloring of \( G \). If we delete the rows corresponding to \( f_1 \) and \( f_2 \) from \( D \), the resulting matrix is a generator matrix for a self-dual code of length \( n \). Moreover, this code is independent of the choice of faces \( f_1, f_2 \).

The extended binary Hamming code of length 8 is obtained from the face-vertex incidence matrix of the cube in the following example:

**Example 3.2.** Consider the cube which is a connected bicubic planar graph.

![Cube Diagram](image)

Obviously, the faces \( f_5 \) and \( f_6 \) will have different colors in a 3-face coloring of the cube. Hence, by Theorem 3.1 the following submatrix of the face-vertex incidence
matrix of the graph generates a self-dual code.

\[
\begin{array}{cccccccc}
\hline
 & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 \\
\hline
f_1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
f_2 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
f_3 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
f_4 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
\hline
\end{array}
\]

The code is the extended binary Hamming code of parameters \([8, 4, 4]_2\). When we express the matrix in standard form, we observe that it is a lift-ready matrix in the form:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
\end{bmatrix}
\]

In [10], lifts of the extended binary Hamming code to the ring \(R_3\) have been studied in detail, where \(R_3 := \mathbb{F}_2[u_1, u_2, u_3]/\langle u_i^2, u_i u_j - u_j u_i \rangle, 1 \leq i, j \leq 3\). They were able to obtain ten new extremal binary self-dual codes with weight enumerators in \(W_{64,2}\).

In the following we give an example that is obtained by combining two cycles of length 8. That is the graph \(E_{16}\) in [18].

![Figure 1. The graph \(G_1\)](image)

**Example 3.3.** Let us consider the graph \(G_1\) in Figure 1. Let \(C\) be the code generated by the face-vertex incidence matrix of the graph. Then, \(C\) is a self-dual binary code of length 16 by Theorem 5.1. A generator matrix in standard form is \([I_8 \mid A_1]\), where

\[
A_1 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\
\end{bmatrix}
\]
The code $C$ is a self-dual Type II code of parameters $[16,8,4]_2$ with an automorphism group of order $2^{14}3^25\times 7$ and weight distribution $1+28z^4+198z^8+28z^{12}+z^{16}$.

4. Lifts of self-dual graph codes

Lifts of the binary self-dual codes related to some bicubic planar graphs on 16 vertices have been considered in this section. $\mathbb{F}_2+u\mathbb{F}_2$-lifts of four circulant binary self-dual codes of length 32 have been studied in [9]. Lifting quaternary codes to $\mathbb{F}_4+u\mathbb{F}_4$ have been used in [13]. For more research in this direction we refer the reader to [9, 10, 11, 13, 15, 16]. The binary self-dual codes obtained from face-vertex incidence matrix of connected bicubic planar graphs is in good shape which is suitable for lifting. In this section, by considering binary images of $R_2$-lifts of graph codes we obtain 15 new extremal binary self-dual Type I codes of length 64.

We need a brief representation for the elements of the ring $R_2$ in order to tabulate the results. We prefer to use hexadecimals. The correspondence between the binary 4 tuples and the hexadecimals is as follows:

0 $\leftrightarrow$ 0000, 1 $\leftrightarrow$ 0001, 2 $\leftrightarrow$ 0010, 3 $\leftrightarrow$ 0011, 4 $\leftrightarrow$ 0100, 5 $\leftrightarrow$ 0101, 6 $\leftrightarrow$ 0110, 7 $\leftrightarrow$ 0111, 8 $\leftrightarrow$ 1000, 9 $\leftrightarrow$ 1001, A $\leftrightarrow$ 1010, B $\leftrightarrow$ 1011, C $\leftrightarrow$ 1100, D $\leftrightarrow$ 1101, E $\leftrightarrow$ 1110, F $\leftrightarrow$ 1111.

The ordered basis $\{uv, v, u, 1\}$ is used to express elements of $R_2$. For instance, $1+u+v$ is represented as 0111 which corresponds to hexadecimal 7.

**Example 4.1.** The generator matrix $[I_8|A_1]$ of the code $C$ in Example 3.3 that is obtained from the graph $G_1$ is in lift-ready form. $A_1$ is lifted to $K_1$ where

$$K_1 = \begin{bmatrix}
9 & C & 0 & 8 & E & 4 & D & 7 \\
4 & 5 & 4 & E & 8 & 8 & B & 1 \\
E & 2 & 1 & 6 & 2 & C & F & B \\
E & 8 & 6 & 9 & 6 & A & F & 7 \\
4 & A & E & A & 1 & A & F & 7 \\
8 & C & 6 & 0 & C & B & 3 & 5 \\
7 & 5 & F & 7 & 3 & B & 5 & 8 \\
B & 9 & D & 5 & D & B & E & 5
\end{bmatrix}.$$  

Let $K_1$ be the code over $R_2$ generated by $[I_8|K_1]$ then its Gray image $\phi(K_1)$ is a self-dual Type I $[64,32,12]_2$-code with weight enumerator for $\beta = 20$ in $W_{64,1}$. This the first example of an extremal Type I code of length 64 with this weight enumerator.

As lifts of $C$ we were able to obtain five new extremal binary self-dual Type I codes. We were able to construct codes with weight enumerators $\beta = 20, 24, 26$ and 30 in $W_{64,1}$; $\beta = 26$ in $W_{64,2}$. The codes are given below in Table I. In order to save space only the upper triangular parts of the matrices are given since the rest is determined by orthogonality relations.

**Remark 4.2.** The extremal binary self-dual codes in Table I are constructed by considering the binary image $\phi(K_i)$ where $K_i$ is the code of length 16 over $R_2$ that is generated by $[I_8|K_i]$. 

Table 1. New extremal self-dual Type I $[64, 32, 12]_2$ codes as binary images of $R_2$-lifts of $C$

| $K_i$ | Upper triangular part of the matrix $K_i$ | $\beta$ in $W_{64,1}$ |
|-------|------------------------------------------|---------------------|
| $K_1$ | 9C08E4D754E88B1162CFBF96AF71AF7B35585 | 20 in $W_{64,1}$ |
| $K_2$ | 9A8C663FF2A4855D2463516CD943F95BB8B | 24 in $W_{64,1}$ |
| $K_3$ | D24022373664C1D1AC671126799C7759FB0F | 26 in $W_{64,1}$ |
| $K_4$ | 9E4CEEF332A0C55D6CE755A83D5C3FD9F78B | 30 in $W_{64,1}$ |
| $K_5$ | 9A8C6273FEECC119A8E75D6CF51CF3513F87 | 26 in $W_{64,2}$ |

Let us recall the graph $F_{16}$ from [18] which is a connected bicubic planar graph. Its face-vertex incidence matrix leads to a Type I self-dual code of length 16: The graph $G_2$ in Figure 2 is a connected bicubic planar graph. So, by Theorem [31], the face-vertex incidence matrix of $G_2$ generates a self-dual binary code $D$. The code is Type I with parameters $[16, 8, 4]_2$. The generator matrix in standard form is $[I_8 | A_2]$ where:

$$A_2 = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Weight distribution of $D$ is $1 + 12z^4 + 64z^6 + 102z^8 + 64z^{10} + 12z^{12} + z^{16}$ and $|Aut(D)| = 2^{13}3^2$. The matrix is an $LRM$ and as binary images of the $R_2$-lifts of $D$ we obtain fifteen Type I $[64, 32, 12]_2$-codes with previously unknown weight enumerators. More precisely, the codes with weight enumerators $\beta = 16, 20, 24, 26, 28, 30, 34$ and $38$ in $W_{64,1}$; $\beta = 3, 7, 11, 15, 26, 27$ and $35$ in $W_{64,2}$. Those are listed in Table 2.
Table 2. New extremal self-dual Type I $[64,32,12]_2$ codes as binary images of $R_2$-lifts of $D$ (15 new codes)

| $\mathcal{L}_i$ | Upper triangular part of the matrix $L_i$ | $\beta$ in $W_{64,1}$ |
|-----------------|------------------------------------------|------------------------|
| $\mathcal{L}_1$ | 5EASBBDE945739ADBDB3436ABF7237B5105B | 16 in $W_{64,1}$       |
| $\mathcal{L}_2$ | F4ED97582719FA159532748135383F97DCTF | 20 in $W_{64,1}$       |
| $\mathcal{L}_3$ | DAE7379AD85FBDE1D3BB0BA33F6FF5D0DF   | 24 in $W_{64,1}$       |
| $\mathcal{L}_4$ | 5A2B7796DC53F1E99FBBCEFAB77B37F1D907F | 26 in $W_{64,1}$       |
| $\mathcal{L}_5$ | 5E2F335610D7FDA957F0BAEF3FD234F59813 | 28 in $W_{64,1}$       |
| $\mathcal{L}_6$ | D6A3BBDED05739AB34FAABF323F3D1C13    | 30 in $W_{64,1}$       |
| $\mathcal{L}_7$ | 3C211FDB23D1FAD15F70CBB83F94F3       | 34 in $W_{64,1}$       |
| $\mathcal{L}_8$ | 5205997E1A7750739D92A92817F01535B9   | 38 in $W_{64,1}$       |
| $\mathcal{L}_9$ | 742D979876F15F2599936F041BFBCBF135437 | 3 in $W_{64,2}$ |
| $\mathcal{L}_{10}$ | 16055DB29A3B990771952D6013B09F33D0B5 | 7 in $W_{64,2}$       |
| $\mathcal{L}_{11}$ | 3C295F50FA399B619D96F481BF3B57D88BF | 11 in $W_{64,2}$      |
| $\mathcal{L}_{12}$ | 56491536127F014755DE5241F0D75587D    | 15 in $W_{64,2}$       |
| $\mathcal{L}_{13}$ | 5A23B31ADB8177AD5B7BC7A6BBB7AFF719B1B | 26 in $W_{64,2}$ |
| $\mathcal{L}_{14}$ | DEC1D9F2D63FD94B9D5A12CD330D35B1C3D | 27 in $W_{64,2}$       |
| $\mathcal{L}_{15}$ | 1023DB74723537D99955ED458DBD2371457  | 35 in $W_{64,2}$       |

Remark 4.3. The binary codes in Table 2 are constructed as the Gray image $\phi(\mathcal{L}_i)$ where $\mathcal{L}_i$ is the code of length 16 over $R_2$ that is generated by $[8_i|L_i]$.

The $\beta$ parameters of the codes in Table 1 reoccurs in Table 2. We see that the corresponding codes are not equivalent when we check the invariants. If two extremal binary self-dual Type I codes of length 64 have the same weight enumerator, then for each code let $c_1,c_2,\ldots,c_N$ be the codewords of weight 12. Let $A_j = |\{(c_k,c_l) \mid d(c_k,c_l) = j, k < l\}|$ where $d$ is the Hamming distance. $A_{12}$ is invariant under a permutation of the coordinates. Hence, two codes are inequivalent if their $A_{12}$-values are not equal. Those are given in Table 3 which indicates that the corresponding codes are not equivalent.

Table 3. The inequivalence of the codes in Table 1 and Table 2

| $\mathcal{K}_i$ | $\mathcal{L}_i$ | $\mathcal{K}_i$ | $\mathcal{L}_i$ | $\beta$ in $W_{64,1}$ |
|-----------------|----------------|----------------|----------------|------------------------|
| $\mathcal{K}_1$ | $\mathcal{L}_2$ | 15732          | $\mathcal{L}_3$ | 14964          | 20 in $W_{64,1}$ |
| $\mathcal{K}_2$ | $\mathcal{L}_3$ | 16488          | $\mathcal{L}_4$ | 17264          | 24 in $W_{64,1}$ |
| $\mathcal{K}_3$ | $\mathcal{L}_4$ | 17676          | $\mathcal{L}_5$ | 17898          | 26 in $W_{64,1}$ |
| $\mathcal{K}_4$ | $\mathcal{L}_5$ | 20544          | $\mathcal{L}_6$ | 19890          | 30 in $W_{64,1}$ |
| $\mathcal{K}_5$ | $\mathcal{L}_{13}$ | 18876          | $\mathcal{L}_{13}$ | 19680          | 26 in $W_{64,2}$ |

Theorem 4.4. The existence of extremal binary self-dual codes of length 64 is known for 23 parameters in $W_{64,1}$; 56 parameters in $W_{64,2}$.

Remark 4.5. The codes in Table 1 and Table 2 have an automorphism group of order $2^3$. The $R_2$ and binary generator matrices of these are available online at [14]. We also note that the symmetry in the graphs $G_1$ and $G_2$ emerge in the standard forms of the generator matrices of the corresponding binary self-dual codes.
5. New binary self-dual codes of length 66 as extensions

Building-up construction which is also known as extension in the literature is an efficient method to construct self-dual codes from shorter ones. We refer to [12, 7] for different versions of the construction. Such methods have been effectively used recently in [9, 11, 13, 15, 16] to obtain new self-dual codes of lengths 58, 66 and 68.

In this section, we apply the following extension method to the codes in Section 4. As a result of this, 10 new extremal binary self-dual codes of length 66 are obtained.

Theorem 5.1. ([7]) Let $C$ be a self-dual code over $R$ of length $n$ and $G = (r_i)$ be a $k \times n$ generator matrix for $C$, where $r_i$ is the $i$-th row of $G$, $1 \leq i \leq k$. Let $c$ be a unit in $R$ such that $c^2 = 1$ and $X$ be a vector in $R^n$ with $\langle X, X \rangle = 1$. Let $y_i = \langle r_i, X \rangle$ for $1 \leq i \leq k$. Then the following matrix

$$\begin{pmatrix}
1 & 0 & X \\
y_1 & cy_1 & r_1 \\
\vdots & \vdots & \vdots \\
y_k & cy_k & r_k
\end{pmatrix},$$

generates a self-dual code $C'$ over $R$ of length $n + 2$.

A self-dual $[66, 33, 12]_2$-code has a weight enumerator in one of the following forms ([5])

$$W_{66,1} = 1 + (858 + 8\beta)y^{12} + (18678 - 24\beta)y^{14} + \cdots \text{ where } 0 \leq \beta \leq 778,$$

$$W_{66,2} = 1 + 1690y^{12} + 7990y^{14} + \cdots$$

and

$$W_{66,3} = 1 + (858 + 8\beta)y^{12} + (18166 - 24\beta)y^{14} + \cdots \text{ where } 14 \leq \beta \leq 756,$$

Recently, new codes with weight enumerators in $W_{66,1}$ are constructed in [9, 13, 16]. More precisely, 5 new codes are obtained in [9], 24 new codes in [13] and 11 new codes in [16]. Together with these, the existence of such codes is known for $\beta = 0, 1, 2, 3, 5, 6, 8, \ldots, 11, 14, \ldots, 56, 59, \ldots, 69, 71, \ldots, 90, 92, 94$ and 100 in $W_{66,1}$.

We construct the codes with weight enumerators $\beta = 13$ and 57 in $W_{66,1}$.

Most recently, 14 codes were discovered in [15] by applying the building-up construction to the binary images of modified four-circulant codes of length 16 over $R_2$. Together with these the existence of codes in $W_{66,3}$ is known for $\beta = 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 66, 67, 70, 71, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 90, 92.

In this work, eight new codes with new weight enumerators are constructed. More precisely, the codes with weight enumerator for $\beta = 24, 25, 26, 27, 39, 40, 41$ and 42 in $W_{66,3}$ which are listed in Table 4.

**Remark 5.2.** The extremal codes of length 66 in Table 4 are obtained by considering the binary code generated by the matrix

$$\begin{pmatrix}
1 & 0 & X \\
y_1 & cy_1 & r_1 \\
\vdots & \vdots & \vdots \\
y_k & cy_k & r_k
\end{pmatrix}$$
Table 4. 10 new extremal self-dual binary codes of length 66 by Theorem 5.1.

| $C_i$ | Code $X$ | $\beta$ in $W_{66,1}$ |
|-------|------------|------------------|
| $C_1$ | $L_9$ 001001101001100001010011101100 | 13 in $W_{66,1}$ |
| $C_2$ | $L_{15}$ 011011001101100100110111010001111 | 57 in $W_{66,1}$ |
| $C_3$ | $L_1$ 000100001110101111111101000011 | 24 in $W_{66,3}$ |
| $C_4$ | $L_1$ 01111101101111111110100011110001 | 25 in $W_{66,3}$ |
| $C_5$ | $K_1$ 0010001100110101010011 | 26 in $W_{66,3}$ |
| $C_6$ | $L_1$ 110110001001010011111111011101 | 27 in $W_{66,3}$ |
| $C_7$ | $K_3$ 010110111011111111110011011101 | 25 in $W_{66,3}$ |
| $C_8$ | $L_4$ 00101101010101101000011011011101 | 29 in $W_{66,3}$ |
| $C_9$ | $K_3$ 00100011011111000110101111011101 | 40 in $W_{66,3}$ |
| $C_{10}$ | $K_3$ 10101100110111101110110111011011 | 41 in $W_{66,3}$ |

where $G = (r_i)$ is the matrix determined by the Gray images of $[I_8|L_i]$, $u[I_8|L_i]$, $v[I_8|L_i]$ and $uv[I_8|L_i]$. The binary generator matrices of the codes in Table 4 are available online at [14]. The codes all have an automorphism group of order 2.

**Theorem 5.3.** The existence of extremal binary self-dual codes of length 66 is known for 89 parameters in $W_{66,1}$; 64 parameters in $W_{66,3}$.

6. Conclusion

The codes obtained by considering face-vertex incidence matrices of bicubic planar graphs have a nice structure. In this work, we considered two such graphs on 16 vertices. New extremal binary self-dual codes of length 64 obtained as the Gray images of $R_2$-lifts of face-vertex incidence matrices of the graphs. Such methods can be applied to different bicubic planar graphs and lifts can be considered over various rings. Another possible research area is to generalize Oral’s work to a larger family of planar graphs.

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