Explicitly solvable algebraic equations of degree 8 and 9

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Abstract

The generic monic polynomial of degree \(N\) features \(N\) a priori arbitrary coefficients \(c_m\) and \(N\) zeros \(z_n\). In this paper we limit consideration to \(N=8\) and \(N=9\). We show that if the \(N\)—a priori arbitrary—coefficients \(c_m\) of these polynomials are appropriately defined—as it were, a posteriori—in terms of 6 arbitrary parameters, then the \(N\) roots of these polynomials can be explicitly computed in terms of radicals of these 6 parameters. We also report the constraints on the \(N\) coefficients \(c_m\) implied by the fact that they are so defined in terms of 6 arbitrary parameters; as well as the explicit determination of these 6 parameters in terms of the \(N\) coefficients \(c_m\).

1 Introduction

This paper is a follow-up to the paper \([1]\), and we refer to that paper for the motivation of this kind of investigations.

Notation 1-1. Hereafter the \(N\) zeros \(z_n\) and the \(N\) coefficients \(c_m\) of the monic polynomial \(P_N(z)\) of degree \(N\) are defined as follows:

\[
P_N(z) = \prod_{n=1}^{N} (z - z_n) = z^N + \sum_{m=0}^{N-1} (c_m z^m) ; \tag{1}
\]

and in this paper we limit consideration to \(N=8\) and \(N=9\).

In this paper we identify special cases of the polynomial (1) with \(N=8\) and \(N=9\)—with its \(N\) coefficients \(c_m\) appropriately defined in terms of 6 arbitrary parameters—so that its \(N\) zeros \(z_n\) can then be themselves explicitly determined in terms of radicals of these 6 parameters; and we also report below both the expressions of the 6 parameters in terms of the \(N\) coefficients \(c_m\) and the constraints on the coefficients \(c_m\) implied by this approach.

Of course the results reported below do not contradict the implications of the findings about the zeros of polynomials implied by the Theory of Galois Groups, and are certainly contained as special cases within that framework; but they are obtained by much more elementary means.

In the following two Sections we report separately our findings for the two cases with \(N=8\) and \(N=9\); the proofs of these findings are provided in Appendix A. A terse final Section outlines possible future developments.

2 Results for \(N=8\)

Proposition 2-1. Assume that the 8 coefficients \(c_m\) of the polynomial (1) with \(N=8\) may be expressed as follows in terms of the 6 arbitrary parameters \(\alpha_0, \alpha_1, \beta_0, \beta_1, \gamma_0, \gamma_1\):

\[
c_7 = 4\alpha_1 , \tag{2a}
\]
c_6 = 4\alpha_0 + 6(\alpha_1)^2 + 2\beta_1 , \hspace{1cm} (2b)
c_5 = 12\alpha_0\alpha_1 + 4(\alpha_1)^3 + 6\alpha_1\beta_1 , \hspace{1cm} (2c)
c_4 = 6\alpha_0 \left[ \alpha_0 + 2(\alpha_1)^2 + \beta_1 \right] + (\alpha_1)^2 \left[ (\alpha_1)^2 + 6\beta_1 \right] + 2\beta_0 + (\beta_1)^2 + \gamma_1 , \hspace{1cm} (2d)
c_3 = 4\alpha_0\alpha_1 \left[ 3\alpha_0 + (\alpha_1)^2 + 3\beta_1 \right] + 2\alpha_1 \left[ 2\beta_0 + (\alpha_1)^2 \beta_1 + (\beta_1)^2 + \gamma_1 \right] , \hspace{1cm} (2e)
c_2 = 2\alpha_0 \left[ 2(\alpha_0)^2 + 3\alpha_0(\alpha_1)^2 + 2\beta_0 + 3 \left[ \alpha_0 + (\alpha_1)^2 \right] \beta_1 + (\beta_1)^2 \right] + 2(\alpha_1)^2 \beta_0 + 2\beta_0 + (\alpha_1)^2 \beta_1 \beta_1 + 2\beta_0 + (2\alpha_0 + \beta_1)\gamma_1 , \hspace{1cm} (2f)
c_1 = \alpha_1 \left\{ 2\alpha_0 \left[ 2(\alpha_0)^2 + 2\beta_0 + (3\alpha_0 + \beta_1)\beta_1 \right] + 2\beta_0 + (2\alpha_0 + \beta_1)\gamma_1 \right\} , \hspace{1cm} (2g)
c_0 = \alpha_0 \left\{ (\alpha_0)^3 + 2(\alpha_0)^2 + (\alpha_1)^2 + (\beta_1)^2 \right\} + 2\beta_0 + (\alpha_0 + \beta_1)\gamma_1 \right\} + \beta_0 (\beta_0 + \gamma_1) + \gamma_0 . \hspace{1cm} (2h)

Then the 8 roots \( z_n \equiv z_{\lambda\mu\nu} \ (n = 1, 2, 3, 4, 5, 6, 7, 8; \ \lambda = 0, 1, \mu = 0, 1, \nu = 0, 1) \) of the polynomial \((11)\) with \( N = 8 \) are explicitly given, in terms of the 6 parameters \( \alpha_0, \alpha_1, \beta_0, \beta_1, \gamma_0, \gamma_1 \), by the following definitions: the 8 numbers \( z_{\lambda\mu\nu} \) are the 2 roots (with \( \lambda = 1, 2, \mu = 1, 2, \nu = 1, 2 \)) of the following 4 quadratic equation in \( z \),

\[ z^2 + \alpha_1 z + \alpha_0 = y_{\mu\nu}, \quad \mu = 1, 2, \quad \nu = 1, 2, \]

where the 4 numbers \( y_{\mu\nu} \) (with \( \mu = 1, 2, \nu = 1, 2 \)) are the 2 roots of the following 2 quadratic equations in \( y \),

\[ y^2 + \beta_1 y + \beta_0 = x_\nu, \quad \nu = 1, 2, \]

where the 2 numbers \( x_\nu \) (with \( \nu = 1, 2 \)) are the 2 roots of the following quadratic equation in \( x \),

\[ x^2 + \gamma_1 x + \gamma_0 = 0 . \] \hspace{1cm} (3a)

Remark 2-1. The 3 quadratic eqs. \((3)\) can of course be solved explicitly. The resulting explicit formula expressing the 8 zeros \( z_n \equiv z_{\lambda\mu\nu} \) in terms of the 6 parameters \( \alpha_0, \alpha_1, a_2, b_0, b_1 \) involves—in a nested way—only square roots; anybody interested can obtain it easily, the only tool needed to that end is the formula giving the 2 roots \( x_{\pm} \) of a monic polynomial of second degree:

\[ x^2 + a_1 x + a_0 = (x - x_+) (x - x_-) , \quad x_{\pm} = \left( -a_1 \pm \sqrt{(a_1)^2 - 4a_0} \right) / 2 . \]

Next, let us face the "inverse" task to assign—as it were, a priori—the 8 coefficients \( c_m \) of a monic polynomial of degree 8 (see \((1)\) with \( N = 8 \)) and to then find the 6 parameters \( \alpha_0, \alpha_1, \beta_0, \beta_1, \gamma_0, \gamma_1 \) which determine—as it were, a posteriori—via explicit formulas both these 8 coefficients \( c_m \) and the 8 zeros \( z_n \) of that polynomial \((1)\), as well as explicit formulas displaying the corresponding constraints implied by these assignments on the 8 coefficients \( c_m \).

The following proposition provides these findings, which complement those reported in Proposition 2-1.

Proposition 2-2. If the 8 parameters \( c_m \) are expressed in terms of the 6 parameters \( \alpha_0, \alpha_1, \beta_0, \beta_1, \gamma_0, \gamma_1 \) by the 8 formulas \((2)\), then the 4 parameters \( \alpha_0, \alpha_1, \beta_0, \beta_1 \) are themselves expressed as follows in terms of the 8 coefficients \( c_m \) and of the "free" parameters \( \alpha_0 \) and \( \beta_0 \):

\[ \alpha_1 = c_7 / 4 , \] \hspace{1cm} (5a)

\[ \beta_1 = - \left[ 32\alpha_0 - 8c_6 + 3(c_7)^2 \right] / 16 , \] \hspace{1cm} (5b)

\[ \gamma_0 = c_0 - \left\{ \alpha_0 \left[ 8c_6 - 3(c_7)^2 \right] - 16 \left[ (\alpha_0)^2 - \beta_0 \right] \right\} \cdot \left\{ (c_7)^4 - 8(c_6)^2 + 32c_4 - 2\alpha_0 \left[ 8c_6 - 3(c_7)^2 \right] + 32 \left[ (\alpha_0)^2 - \beta_0 \right] \right\} / 512 , \] \hspace{1cm} (5c)
\[
\gamma_1 = -\alpha_6 + 2 \left[ (\alpha_0)^2 - \beta_0 \right] + c_4 + \left[ -8 (\alpha_6)^2 + 12\alpha_0 (c_7)^2 + (c_7)^4 \right] / 32 ,
\]
while the 8 coefficients \(c_m\) satisfy the following 4 constraints:

\[
c_5 = c_7 \left[ 24\alpha_6 - 7 (c_7)^2 \right] / 32 ,
\]
\[
c_3 = c_7 \left[ 128\alpha_4 - 20\alpha_6 (c_7)^2 + 7 (c_7)^4 \right] / 256 ,
\]
\[
c_2 = \left\{ 512\alpha_4 \left[ 4\alpha_6 - (c_7)^2 \right] - 64 \left[ 8\alpha_6 - 3 (c_7)^2 \right] (\alpha_6)^2 \\
+ \left[ 16\alpha_6 - 7 (c_7)^2 \right] (c_7)^4 \right\} / 4096 ,
\]
\[
c_1 = c_7 \left[ 8\alpha_6 - 3 (c_7)^2 \right] \left[ 32\alpha_4 - 8 (\alpha_6)^2 + (c_7)^4 \right] / 2048 .
\]

3 Results for \(N=9\)

The findings reported in this Section 3 are analogous, but different, from those reported in Section 2; accordingly, the variables and parameters used in this Section 3 are different from those having the same names in Section 2, although they play analogous roles.

**Proposition 3-1.** Assume that the 9 coefficients \(c_m\) of the polynomial \(P\) with \(N = 9\) may be expressed as follows in terms of the 6 arbitrary parameters \(\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2\):

\[
c_0 = \alpha_0 \left[ \beta_1 + \alpha_0 (\alpha_0 + \beta_2) \right] + \beta_0 ,
\]
\[
c_1 = \alpha_0 \alpha_1 (3\alpha_0 + 2\beta_2) + \alpha_1 \beta_1 ,
\]
\[
c_2 = 3\alpha_0 \left[ (\alpha_1)^2 + \alpha_0 \alpha_2 \right] + \alpha_2 \left[ (\alpha_1)^2 + 2\alpha_0 \alpha_2 \right] + \alpha_2 \beta_1 ,
\]
\[
c_3 = 3\alpha_0 (\alpha_0 + 3\alpha_1 \alpha_2) + 2\beta_2 (\alpha_0 + \alpha_1 \alpha_2) + (\alpha_1)^3 + \beta_1 ,
\]
\[
c_4 = 3\alpha_0 \left[ 2\alpha_1 + (\alpha_2)^2 \right] + 3 (\alpha_1)^2 \alpha_2 + \left[ 2\alpha_1 + (\alpha_2)^2 \right] \beta_2 ,
\]
\[
c_5 = 3\alpha_1 \left[ (\alpha_1 + (\alpha_2)^2 \right] + 2\alpha_2 (3\alpha_0 + \beta_2) ,
\]
\[
c_6 = 3\alpha_0 + \alpha_2 \left[ 6\alpha_1 + (\alpha_2)^2 \right] + \beta_2 ,
\]
\[
c_7 = 3 \left[ (\alpha_1 + (\alpha_2)^2 \right] ,
\]
\[
c_8 = 3\alpha_2 .
\]

Then the 9 roots \(z_n \equiv z_{\lambda\mu}\) (with \(n = 1, ..., 9\) and \(\lambda = 1, 2, 3, \mu = 1, 2, 3\)) of the polynomial \(P\) with \(N = 9\) are explicitly given, in terms of the 6 parameters \(\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2\) by the following definitions: \(z_n \equiv z_{\lambda\mu}\) is one of the 3 roots of the following 3 cubic equations in \(z\):

\[
z^3 + \alpha_2 z^2 + \alpha_1 z + \alpha_0 = y_\mu , \quad \mu = 1, 2, 3 ,
\]

where the 3 numbers \(y_\mu\) are the 3 roots of the following cubic equation in \(y\):

\[
y^3 + \beta_2 y^2 + \beta_1 y + \beta_0 = 0 .
\]

**Proposition 3-2.** If the 9 parameters \(c_m\) of the polynomial \(P\) with \(N = 9\) are expressed in terms of the 6 parameters \(\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2\) by the 9 formulas \(7\), then the 6 parameters \(\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2\) are themselves expressed as follows in terms of the 9 coefficients \(c_m\):

\[
\alpha_2 = c_8 / 3 ,
\]
coefficients of the resulting polynomial of degree 9 with the coefficients—which allow their zeros to be identified the 8 coefficients of this equation—after having expanded it in powers of $z$ with the expression $P$ the polynomial scalar analogous manner polynomial equations the arguments and coefficients of which are more complicated objects than $c_m$ are themselves required to satisfy as many as 5 constraints, which can be expressed in the form of explicit expressions of any 5 of the 6 coefficients $c_\ell$ with $\ell = 1, 2, ..., 6$ in terms of the remaining coefficients $c_m$ with $m = 1, 2, ..., 8$: as explained in Appendix A.

4 Outlook

It is possible—but not necessarily useful—to try and extend the simple-minded approach utilized in this paper in order to identify other special polynomials—of higher order than 9, and characterized by appropriate restrictions on their coefficients—which allow their zeros to be explicitly computed in terms of their coefficients; or even to treat in analogous manner polynomial equations the arguments and coefficients of which are more complicated objects than scalar quantities (for instance, matrices).

5 Appendix A

In this Appendix we justify the results reported in Sections 2 and 3.

To derive the results reported in Propositions 2-1 one should firstly replace $x$ in the quadratic eq. (6) with the expression $y^2 + \beta y + \beta_0$ (see eq. (3b)) and secondly replace, in the resulting quartic equation for $y$, this variable with the expression $z^2 + \alpha_1 z + \alpha_0$ (see eq. (5)), obtaining thereby an equation of degree 8 for the variable $z$; then identify the 8 coefficients of this equation—after having expanded it in powers of $z$—with the 8 coefficients $c_m$ of the polynomial $P_8(z)$, see eq. (1). Likewise, to obtain the results reported in Proposition 3-1, replace $y$ in the cubic eq. (5) with $z^3 + \alpha_2 z^2 + \alpha_1 z + \alpha_0$ (see eq. (4)), expand the outcome in powers of $z$ and identify the 9 coefficients of the resulting polynomial of degree 9 with the coefficients $c_m$ of the polynomial $P_9(z)$, see (1).

The results reported in Propositions 2-2 are consequences of the inversion of the formulas (5), performed in order to obtain the parameters (identified by Greek letters) they feature in terms of the 8 coefficients $c_m$; they have been obtained via a judicious choice of the order in which these equations are solved, and they have then been checked via the algebraic manipulation package Mathematica.

The procedure to obtain the results reported in Proposition 3-2 is a bit more tricky. The issue is of course to invert the algebraic eqs. (6). It is convenient to consider them in the opposite order in which they are listed. Then from the last 2 (i.e. eqs. (7i) and (7h)) we immediately get

$$\alpha_2 = c_8 / 3 , \quad \alpha_1 = \left[ 3c_7 - (c_8)^2 \right] / 9 .$$

\hspace{1cm} (10)

Hence hereafter we shall use—for simplicity—the variables $\alpha_1$ and $\alpha_2$ in place of the variables $c_7$ and $c_8$.

It is then convenient to rewrite the 3 eqs. (6a), (7i) and (7h) as follows:

$$3\alpha_0 + \beta_2 = c_6 - \alpha_2 \left[ 6\alpha_1 + (\alpha_2)^2 \right] , \quad \text{(11a)}$$

$$3\alpha_0 + \beta_2 = \left\{ c_5 - 3\alpha_1 \left[ \alpha_1 + (\alpha_2)^2 \right] \right\} / (2\alpha_2) , \quad \text{(11b)}$$

$$\alpha_1 = \left[ 3c_7 - (c_8)^2 \right] / 9 , \quad \alpha_0 = \frac{c_4 - 3(\alpha_1)^2 \alpha_2}{3 \left[ 2\alpha_1 + (\alpha_2)^2 \right]} - \beta_2 / 3 , \quad \beta_2 = \frac{c_3 - 3\alpha_0 (\alpha_0 - 2\alpha_1 \alpha_2) - (\alpha_1)^3 - \beta_1}{2(\alpha_0 + \alpha_1 \alpha_2)} , \quad \beta_1 = \frac{[c_1 - \alpha_0 \alpha_1 (3\alpha_0 + 2\beta_2)] / \alpha_1}{1} , \quad \beta_0 = c_0 - \alpha_0 \left[ (\alpha_0)^2 + \alpha_0 \beta_2 + \beta_1 \right] ; \quad \text{(9)}$$

note that these formulas (9), via their sequentially nested character, express explicitly the 6 quantities $\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2$ in terms of the 6 coefficients $c_8, c_7, c_4, c_3, c_1, c_0$.

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It is possible—but not necessarily useful—to try and extend the simple-minded approach utilized in this paper in order to identify other special polynomials—of higher order than 9, and characterized by appropriate restrictions on their coefficients—which allow their zeros to be explicitly computed in terms of their coefficients; or even to treat in analogous manner polynomial equations the arguments and coefficients of which are more complicated objects than scalar quantities (for instance, matrices).

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The results reported in Propositions 2-2 are consequences of the inversion of the formulas (5), performed in order to obtain the parameters (identified by Greek letters) they feature in terms of the 8 coefficients $c_m$; they have been obtained via a judicious choice of the order in which these equations are solved, and they have then been checked via the algebraic manipulation package Mathematica.

The procedure to obtain the results reported in Proposition 3-2 is a bit more tricky. The issue is of course to invert the algebraic eqs. (6). It is convenient to consider them in the opposite order in which they are listed. Then from the last 2 (i.e. eqs. (7i) and (7h)) we immediately get

$$\alpha_2 = c_8 / 3 , \quad \alpha_1 = \left[ 3c_7 - (c_8)^2 \right] / 9 .$$

\hspace{1cm} (10)

Hence hereafter we shall use—for simplicity—the variables $\alpha_1$ and $\alpha_2$ in place of the variables $c_7$ and $c_8$.

It is then convenient to rewrite the 3 eqs. (6a), (7i) and (7h) as follows:

$$3\alpha_0 + \beta_2 = c_6 - \alpha_2 \left[ 6\alpha_1 + (\alpha_2)^2 \right] , \quad \text{(11a)}$$

$$3\alpha_0 + \beta_2 = \left\{ c_5 - 3\alpha_1 \left[ \alpha_1 + (\alpha_2)^2 \right] \right\} / (2\alpha_2) , \quad \text{(11b)}$$

$$\alpha_1 = \left[ 3c_7 - (c_8)^2 \right] / 9 , \quad \alpha_0 = \frac{c_4 - 3(\alpha_1)^2 \alpha_2}{3 \left[ 2\alpha_1 + (\alpha_2)^2 \right]} - \beta_2 / 3 , \quad \beta_2 = \frac{c_3 - 3\alpha_0 (\alpha_0 - 2\alpha_1 \alpha_2) - (\alpha_1)^3 - \beta_1}{2(\alpha_0 + \alpha_1 \alpha_2)} , \quad \beta_1 = \frac{[c_1 - \alpha_0 \alpha_1 (3\alpha_0 + 2\beta_2)] / \alpha_1}{1} , \quad \beta_0 = c_0 - \alpha_0 \left[ (\alpha_0)^2 + \alpha_0 \beta_2 + \beta_1 \right] ; \quad \text{(9)}$$

note that these formulas (9), via their sequentially nested character, express explicitly the 6 quantities $\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2$ in terms of the 6 coefficients $c_8, c_7, c_4, c_3, c_1, c_0$.
$3\alpha_0 + \beta_2 = \left[ c_4 - 3(\alpha_1)^2 \alpha_2 \right] / \left[ 2\alpha_1 + (\alpha_2)^2 \right]$ ; (11c)

and it is also easily seen that the 3 eqs. (11d), (11e) and (11f) can also be combined to yield the following 3 relations:

$3\alpha_0 + \beta_2 = (c_2\alpha_1 - c_1\alpha_2) / (\alpha_1)^3$ , (11d)

$3\alpha_0 + \beta_2 = \left[ c_3\alpha_1 - c_1 - (\alpha_1)^4 \right] / \left[ 2(\alpha_1)^2 \alpha_2 \right]$ , (11e)

$3\alpha_0 + \beta_2 = \left[ c_3\alpha_2 - c_2 - (\alpha_1)^3 \alpha_2 \right] / \left[ \alpha_1 \left[ 2(\alpha_2)^2 - \alpha_1 \right] \right]$ . (11f)

Since the left-hand sides of these 6 equations (11) are all equal, and the right-hand sides involve—via (10)—only the 8 parameters $c_m$ (with $m = 1, 2, ..., 8$), this implies that these 8 coefficients $c_m$ must satisfy, among themselves, 5 constraints; and it is also clear that it is quite easy to reformulate these 5 constraints in the form of 5 explicit expressions of any 5 of the 6 coefficients $c_\ell$ with $\ell = 1, 2, ..., 6$ in terms of the remaining coefficients $c_m$ (with $m = 1, 2, ..., 8$)—by just solving the system of 5 algebraic linear equations for these coefficients implied by the 6 eqs. (11).

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References

[1] F. Calogero and F. Payandeh, "Two classes of explicitly solvable sextic equations", arXiv:2104.03072 [math.DS] April 8, 2021.