Well-conditioned eigenvalue problems that overflow

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May 13, 2020

Abstract

In this note we present a parameterized class of lower triangular matrices. The components of the eigenvectors grow rapidly and will exceed the representational range of any finite number system. The eigenvalues and the eigenvectors are well-conditioned with respect to componentwise relative perturbations of the matrix. This class of matrices is well suited for testing software for computing eigenvectors as these routines must be able to handle overflow successfully.

1 Introduction

Given a matrix $A \in \mathbb{R}^{m \times m}$ the standard eigenvalue problem consists of finding eigenvalues $\lambda \in \mathbb{C}$ and eigenvectors $x \in \mathbb{C}^m$ such that

$$Ax = \lambda x.$$ 

If $A$ is a dense nonsymmetric matrix then the standard eigenvalue problem is often solved by first reducing $A$ to Hessenberg form

$$H = Q_1^T AQ_1$$

and then to real Schur form

$$S = Q_2^T HQ_2$$

using orthogonal similarity transformations. Here the matrix $H$ is upper Hessenberg and the matrix $S$ is upper quasi-triangular with diagonal blocks that are either 1-by-1 or 2-by-2. The eigenvalues of $A$ can be determined from the diagonal blocks of $S$. Every 1-by-1 diagonal block on the diagonal of $S$ is a real eigenvalue of $A$ and every 2-by-2 block on the diagonal of $S$ specifies a pair of complex conjugate eigenvalues of $A$. The eigenvectors of $S$ can be computed using a variant of substitution and transformed to eigenvectors of $A$. Specifically, if $y \neq 0$ satisfies $Sy = \lambda y$, then $x = Q_1 Q_2 y$ is an eigenvector of $A$ corresponding to the eigenvalue $\lambda$.

However, substitution is very vulnerable to floating point overflow and special software has been developed to handle this problem. We say that an algorithm is robust if all intermediate and final results are in the representable range, i.e., overflow is prevented. In LAPACK
the robust subroutines for computing eigenvectors are all derived from \texttt{xlatrs} \cite{1}. In Elemental \cite{2} there are parallel subroutines that can be used to compute eigenvectors of triangular matrices, but they are not fully robust. In StarNEig \cite{3} there are parallel robust subroutines for computing standard and generalized eigenvectors from matrices or matrix pairs in real Schur form.

The contribution of this note is to exhibit a class of lower triangular matrices that can be used to test robust solvers. The components of the eigenvectors grow rapidly and will exceed the representational range of any finite set of numbers sooner rather than later. Moreover, the eigenvalues and the eigenvectors are well-conditioned with respect to componentwise relative perturbations of the matrix. These results are all established in Section \ref{sec:results}. We briefly cover the transformation to upper triangular problems in Section \ref{sec:upper} and finish with some concluding remarks in Section \ref{sec:conclusion}.

Here we offer the following small example as an appetizer. Consider the matrices $A$ and $X$ given by

$$
A = \begin{bmatrix}
1 & 2 & 3 \\
-5 & -5 & -5 \\
-5 & -5 & -5 & 4 \\
-5 & -5 & -5 & -5 & 5
\end{bmatrix}, \quad X = \begin{bmatrix}
1 & 5 \\
5 & 15 \\
15 & 5 & 1 \\
35 & 15 & 5 & 1 \\
70 & 35 & 15 & 5 & 1
\end{bmatrix}.
$$

It is straightforward to verify that the $j$th column of $X$ is an eigenvector of $A$ corresponding to the eigenvalue $\lambda_j = j$ and $x_{ij} \geq 2^{i-j}$. However, by Theorem \ref{thm:main} and Lemma \ref{lem:condition} these properties are preserved when our example is generalized to any dimension $m$. Moreover, the nontrivial components of the first column of $X = [x_{ij}]$ can be obtained by solving the linear system

$$
\begin{bmatrix}
1 & 2 \\
-5 & 3 \\
-5 & 4 \\
-5 & 5
\end{bmatrix}
\begin{bmatrix}
x_{21} \\
x_{31} \\
x_{41} \\
x_{51}
\end{bmatrix} =
\begin{bmatrix}
5 \\
5 \\
5 \\
5
\end{bmatrix}.
$$

In this case, forward substitution consists of adding and dividing real numbers that are strictly positive. Hence it is not surprising that this eigenvector is well-conditioned with respect to componentwise relative perturbations of the matrix $A$. A general upper bound for the relevant condition numbers is established as Theorem \ref{thm:condition}.

\section{Auxiliary results}

In this section we derive a set of elementary results related to the solution of very special triangular linear systems. These results are used to prove the main results.

Consider the very special linear system given by

$$
Gx := \begin{bmatrix}
d_1 \\
-c & d_2 \\
\vdots & \ddots & \ddots \\
-c & \ldots & -c & d_m
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_m
\end{bmatrix} = \begin{bmatrix}
c \\
c \\
c \\
c
\end{bmatrix} =: f
$$

where $G \in \mathbb{R}^{m \times m}$ and $f \in \mathbb{R}^m$. If the diagonal elements are nonzero, then $G$ is nonsingular and the unique solution $x$ can be found using forward substitution. The familiar formula takes the form:

$$
x_k = a_k \left( 1 + \sum_{j=1}^{k-1} x_j \right), \quad a_k = \frac{c}{d_k}.
$$
The components of $x$ can also be expressed compactly using the following theorem.

**Theorem 2.1.** If $d_j \neq 0$ for all $j$, then the solution of the linear system (1) is given by

$$x_k = a_k \omega_k$$

where

$$\forall k \in \{1, 2, \ldots, m\} : \omega_k = \prod_{j=1}^{k-1} (1 + a_j), \quad a_k = \frac{c}{d_k}.$$

**Proof.** The proof is split into two steps.

1. We begin by showing that

$$\omega_k = 1 + \sum_{i=1}^{k-1} a_i \omega_i.
\tag{3}$$

Let $S_m = \{1, 2, \ldots, m\}$ and let $V \subseteq S_m$ be given by

$$V = \left\{ j \in S_m : \omega_j = 1 + \sum_{i=1}^{j-1} a_i \omega_i \right\}.$$

It is clear that $1 \in V$, because $\omega_1 = 1$. Now assume that $S_m \setminus V \neq \emptyset$. Then $S_m \setminus V$ has a smallest element $k$. We must have $k > 1$ because $1 \in V$. By definition of $\omega_k$ we have

$$\omega_k = (1 + a_{k-1}) \omega_{k-1} = \omega_{k-1} + a_{k-1} \omega_{k-1}.$$

Since $k$ is the smallest element of $V$ we must have $k - 1 \in V$ and we can therefore write

$$\omega_{k-1} = 1 + \sum_{j=1}^{k-2} a_i \omega_i.$$

It follows that

$$\omega_k = \omega_{k-1} + a_{k-1} \omega_{k-1} = \left[ 1 + \sum_{i=1}^{k-2} a_i \omega_i \right] + a_{k-1} \omega_{k-1} = 1 + \sum_{i=1}^{k-1} a_i \omega_i.$$

We conclude that $k \in V$. This is a contradiction, because $k \in S_m \setminus V$. Therefore, we must have $V = S_m$.

2. We will now show that

$$x_k = a_k \omega_k.$$

Let $W \subseteq S_m = \{1, 2, \ldots, m\}$ be given by

$$W = \{ j \in S_m : x_j = a_j \omega_j \}.$$

It is clear that $1 \in W$ because $\omega_1 = 1$ and $x_1 = a_1$ implies $x_1 = a_1 \omega_1$. Now assume that $S_m \setminus W \neq \emptyset$. Then $S_m \setminus W$ has a smallest element $k$. Equation (2) is simply the statement that

$$x_k = a_k \left( 1 + \sum_{j=1}^{k-1} x_j \right).$$

Now since $k$ is the smallest element of $S_m \setminus W$ we have $j \in W$ for $j < k$ or equivalently $x_j = a_j \omega_j$ for $j < k$. It follows that

$$x_k = a_k \left( 1 + \sum_{j=1}^{k-1} a_j \omega_j \right).$$
Equation (3) now implies that
\[ x_k = a_k \omega_k \]
and \( k \in W \). This is a contradiction because \( k \in S_m \setminus W \). We conclude that \( W = S_m \).

This completes the proof. \( \square \)

We will now state a formula for the inverse matrix \( H = G^{-1} \).

**Theorem 2.2.** If \( d_j \neq 0 \) for all \( j \), then \( G \) is nonsingular and the components of the inverse matrix \( G^{-1} = H = [h_{ij}] \) are given by

\[
h_{ij} = \begin{cases} 
0 & i < j, \\
\frac{1}{d_j} & i = j, \\
\frac{a_i \omega_i}{d_j \omega_{j+1}} & i > j.
\end{cases}
\]

**Proof.** Let \( x = [x_1 \ x_2 \ \ldots \ x_m]^T \in \mathbb{R}^m \) denote the \( j \)th column of the inverse matrix \( H \). Since \( G \) is lower triangular, we have

\[
x_i = \begin{cases} 
0 & i < j, \\
\frac{1}{d_j} & i = j.
\end{cases}
\]

The remaining components of \( x \) are obtained by solving the linear system

\[
\begin{bmatrix} d_{j+1} & -c & d_{j+2} & \cdots & -c & d_m \\
-c & \ddots & \ddots & \ddots & \ddots & \ddots \\
-c & \cdots & -c & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & \cdots & 0 & \cdots
\end{bmatrix}
\begin{bmatrix} x_{j+1} \\
x_{j+2} \\
\vdots \\
x_m
\end{bmatrix}
= \begin{bmatrix} c \\
c \\
\vdots \\
c
\end{bmatrix}.
\]

Now let \( i \in \{1, 2, \ldots, m - j\} \). Then by Theorem 2.1 we have

\[
x_{j+i} = x_j \left( a_{j+i} \prod_{k=1}^{i-1} (1 + a_{j+k}) \right) = x_j a_{j+i} \prod_{k=j+1}^{j+i-1} (1 + a_k) = x_j a_{j+i} \frac{\omega_{j+i}}{\omega_{j+1}}.
\]

This shows that

\[
x_i = \frac{a_i \omega_i}{d_j \omega_{j+1}}, \quad i > j.
\]

This completes the proof. \( \square \)

Skeel's condition number for a nonsingular linear system \( Ax = b \) where \( A \in \mathbb{R}^{m \times m} \) and \( b \in \mathbb{R}^m \) depends explicitly on the solution \( x \) and is given by

\[
\kappa_\infty(A, x) = \frac{||A^{-1}||_\infty ||A||_1 ||x||_\infty}{||x||_\infty}.
\]

The following result will be used to compute Skeel’s condition number for systems of the form given by equation (1).
Lemma 2.3. Assume $d_j > 0$ for all $j$ and $c > 0$. Let $x$ denote the solution of the linear system (1). Let $y = |G||x|$ and let $z = |G^{-1}|y$. Then

$$y_i = c(2\omega_i - 1)$$

and

$$z_i = a_i(2\omega_i - 1) + \sum_{j=1}^{i-1} a_ia_j \frac{\omega_i}{\omega_{j+1}}(2\omega_j - 1).$$

Proof. By Theorem 2.1 the solution of $Gx = f$ is given by

$$x_k = a_k\omega_k$$

where our assumptions ensure that

$$a_k = \frac{c}{d_k} > 0, \quad \omega_k = \prod_{j=1}^{k-1} (1 + a_j) > 0.$$  

This shows that $x_k > 0$ for all $k$, so $|x| = x$. Since $|G|$ is lower triangular, we have

$$y_i = (|A||x|)_i = d_ix_i + \sum_{j=1}^{i-1} cx_j = d_i a_i\omega_i + c \sum_{j=1}^{i-1} a_i\omega_i$$

$$= c\omega_i + c[\omega_i - 1] = c(2\omega_i - 1).$$

Here we have used equation (3) during the last reduction. By Theorem 2.2 the elements of $H = G^{-1}$ are nonnegative. Hence $|G^{-1}| = G^{-1}$. Since $H = G^{-1}$ is lower triangular, we have

$$z_i = (|G^{-1}|y)_i = (Hy)_i = h_{ii}y_i + \sum_{j=1}^{i-1} h_{ij}y_j$$

$$= a_i(2\omega_i - 1) + \sum_{j=1}^{i-1} a_ia_j \frac{\omega_i}{\omega_{j+1}}(2\omega_j - 1).$$

This completes the proof. 

3 Main results

In this section we present a class of lower triangular matrices parameterized using three real numbers $a$, $b$, and $c$. The components of the eigenvectors tend to infinity when $\gamma = \frac{c}{b} > 1$ and the growth is at least exponential when $\gamma \geq m$ where $m$ is the dimension of the matrix. Regardless, the eigenvectors are well-conditioned with respect to componentwise relative perturbations of the matrix $A$ when $b > 0$ and $c > 0$.

Let $a, b, c \in \mathbb{R}$ and consider the lower triangular matrix $A \in \mathbb{R}^{m \times m}$ given by

$$a_{ij} = \begin{cases} 
0 & i < j, \\
 a + jb & i = j, \\
- c & i > j.
\end{cases}$$

(4)

The case of $m = 4$ is illustrated by the matrix $A$ given by

$$A = \begin{bmatrix} 
 a + b & a + 2b & a + 3b & a + 4b \\
 - c & a + 4b & a + 3b \\
 - c & - c & a + 3b \\
 - c & - c & - c & a + 4b
\end{bmatrix}.$$
The eigenvalues of \( A \) can be read off from the diagonal of \( A \), i.e.,
\[
\lambda_j = a + jb
\]
and they are trivially well-conditioned with respect to componentwise relative perturbations of the matrix. If \( b \neq 0 \), then the eigenvalues are distinct and \( A \) is diagonalizable. The eigenvectors are determined up to a scaling by the sequence \( \{z_k\}_{k=0}^{\infty} \) given by
\[
z_k = \left( \frac{\gamma + k - 1}{k} \right), \quad \gamma = \frac{c}{b}. \tag{5}
\]
Here \( \binom{i}{j} \) denotes the binomial coefficient given by
\[
\forall x \in \mathbb{R} \forall k \in \mathbb{N}_0 : \binom{x}{k} = \frac{\prod_{i=0}^{k-1} (x - i)}{k!} = \frac{x(x - 1)(x - 2) \ldots (x - k + 1)}{k!}.
\]
Specifically, we have the following theorem.

**Theorem 3.1.** Let \( A \in \mathbb{R}^{m \times m} \) be given by equation (4) where \( a, b, c \in \mathbb{R} \) and \( b \neq 0 \). Let \( X \in \mathbb{R}^{m \times m} \) denote the lower triangular matrix given by
\[
 x_{ij} = \begin{cases} 
 0 & i < j, \\
 z_i - j & i \geq j.
\end{cases}
\]
Then the \( j \)th column of \( X \) is an eigenvector of \( A \) with respect to the eigenvalue \( \lambda_j = a + jb \).

**Proof.** Consider the problem of computing an eigenvector \( x \) of \( A \) with respect to the eigenvalue \( \lambda_l = a + lb \). We have \( x_i = 0 \) for \( i < l \) and we are free to choose \( x_l = z_0 = 1 \), provided the nontrivial components of \( x \), i.e., the vector \( [x_{l+1}, \ldots, x_m]^T \in \mathbb{R}^{m-l} \) solves the linear system
\[
\begin{bmatrix}
 b & & & \\
 -c & 2b & & \\
 & \ddots & \ddots & \\
 -c & \ldots & -c & (m-l)b \\
\end{bmatrix}
\begin{bmatrix}
 x_{l+1} \\
 x_{l+2} \\
 \vdots \\
 x_m \\
\end{bmatrix}
= \begin{bmatrix}
 c \\
 c \\
 \vdots \\
 c \\
\end{bmatrix}.
\]
This linear system has dimension \( m-l \). It is a special case of Theorem 2.1 and corresponds to the choice of \( d_j = jb \) and \( a_j = \frac{c}{b} \). Let \( y_i = x_{l+i} \) for \( i = 1, 2, \ldots, m-l \), then by Theorem 2.1 we have
\[
y_k = a_k \omega_k = a_k \prod_{j=1}^{k-1} \left( 1 + a_j \right) = \frac{\gamma}{k} \prod_{j=1}^{k-1} \left( 1 + \frac{\gamma}{j} \right) = \frac{\gamma}{k} \prod_{j=1}^{k-1} \frac{\gamma + j}{j} = \left( \frac{\gamma + k - 1}{k} \right) = z_k.
\]
We conclude that
\[
x_i = \begin{cases} 
 0 & i < l, \\
 z_{i-l} & i \geq l.
\end{cases}
\]
This completes the proof.

The behavior of the sequence given by equation (5) is entirely controlled by the ratio \( \gamma = \frac{c}{b} \). In particular, we have the following lemma.

**Lemma 3.2.** Let \( \alpha \in \mathbb{R} \) be any real number and let \( \{y_k\}_{k=0}^{\infty} \subset \mathbb{R} \) be the sequence given by
\[
y_k = \left( \frac{\alpha + k}{k} \right).
\]
Then the following statements hold:
1. If $\alpha > 0$, then $\{y_k\}_{k=0}^{\infty}$ is increasing and
   \[ y_k \to \infty, \quad k \to \infty, \quad k \in \mathbb{N}_0. \]

2. If $\alpha = 0$, then $y_k = 1$ for all $k \in \mathbb{N}_0$.

3. If $\alpha < 0$ is an integer, then $y_k = 0$ for all sufficiently large $k \in \mathbb{N}_0$.

4. If $\alpha < 0$ is not an integer, then $y_k \neq 0$, but
   \[ y_k \to 0, \quad k \to \infty, \quad k \in \mathbb{N}_0. \]

Moreover, the convergence is strictly monotone for all sufficiently large $k$ and the rate of convergence is sublinear.

**Proof.** When studying the different cases it is convenient to exploit that $y_k$ can be rewritten as
\begin{equation}
   y_k = \frac{\prod_{j=1}^{k} (\alpha + j)}{k!} = \prod_{j=1}^{k} \left(1 + \frac{\alpha}{j}\right).
\end{equation}

1. If $\alpha > 0$ then equation (6) shows that $y_k$ is strictly positive and the sequence is increasing because
   \[ y_{k+1} = \left(1 + \frac{\alpha}{k+1}\right) y_k > y_k. \]

   Now choose any $l \in \mathbb{N}$ such that $\alpha \leq l$ and consider any $k \geq l$. Then
   \[ y_k = \frac{\prod_{j=1}^{k} (\alpha + j)}{k!} = C p_k, \]
   where we have introduced
   \begin{equation}
   C = \prod_{j=1}^{l-1} \left(1 + \frac{\alpha}{j}\right), \quad p_k = \prod_{j=l}^{k} \left(1 + \frac{\alpha}{j}\right).
   \end{equation}

   By design, $C > 0$ is a constant that is independent of $k$. We now investigate the convergence of the sequence $\{p_k\}_{k=l}^{\infty}$. We have
   \[ \log p_k = \sum_{j=l}^{k} \log \left(1 + \frac{\alpha}{j}\right) \geq \sum_{j=l}^{k} \frac{1}{2} \frac{\alpha}{j}, \]
   simply because
   \[ \forall x \in [0, 1] : \log(1 + x) \geq \frac{1}{2} x. \]

   Since the harmonic series is divergent we first conclude that
   \[ p_k \to \infty, \quad k \to \infty, \quad k \geq l, \]
   and then
   \[ y_k = C p_k \to \infty, \quad k \to \infty, \quad k \in \mathbb{N}_0, \]
   because $C > 0$.

2. If $\alpha = 0$, then for all $k \in \mathbb{N}_0$:
   \[ y_k = \frac{\prod_{j=1}^{k} (\alpha + j)}{k!} = \frac{\prod_{j=1}^{k} j}{k!} = \frac{k!}{k!} = 1. \]
3. If $\alpha < 0$ is an integer, then $\alpha = -l$ for exactly one $l \in \mathbb{N}$, and

$$\forall k \geq l: \quad y_k = \frac{\prod_{j=1}^{k}(\alpha + j)}{k!} = \frac{\prod_{j=1}^{l}(j - l)}{k!} = 0$$

simply because the term corresponding to $j = l$ is zero.

4. If $\alpha < 0$ is not an integer, then equation (6) shows that $y_k \neq 0$ because no term is zero. We also have $-l < \alpha < 1 - l$ for exactly one $l \in \mathbb{N}$. Now let $k \geq l$, then we again write

$$y_k = C p_k$$

where $C$ and $p_k$ are given by equation (7). We again investigate the convergence of the sequence $\{p_k\}_{k=l}^\infty$. Since $\alpha < 0$ we have

$$p_k = \prod_{j=l}^{k} \left(1 - \frac{|\alpha|}{j}\right). \quad (8)$$

This shows that $p_k$ is a product of strictly positive terms because $|\alpha| < l$. It follows that

$$-\log p_k = \sum_{j=l}^{k} -\log \left(1 - \frac{|\alpha|}{j}\right) \geq \sum_{j=l}^{k} \frac{1}{2} \frac{|\alpha|}{j}$$

simply because

$$\forall x \in [0,1]: \quad -\log(1 - x) \geq -\frac{1}{2} x.$$ 

The divergence of the harmonic series now implies that

$$-\log(p_k) \to \infty, \quad k \to \infty, \quad k \geq l.$$ 

We can now conclude that

$$p_k \to 0, \quad k \to \infty, \quad k \geq l.$$ 

It follows that

$$y_k = C p_k \to 0, \quad k \to \infty, \quad k \in \mathbb{N}_0.$$ 

The convergence is strictly monotone for $k \geq l$ simply because

$$p_{k+1} = p_k \left(1 - \frac{|\alpha|}{k + 1}\right) < p_k. \quad (9)$$

The convergence is sublinear, because

$$\frac{y_{k+1}}{y_k} = \frac{\alpha + k + 1}{k + 1} \to 1, \quad k \to \infty, \quad k \in \mathbb{N}_0.$$ 

This completes the proof. $\square$

By Theorem 3.1 and Lemma 3.2 the eigenvectors of the matrix $A$ will eventually exceed the representational range provided $\gamma = \frac{c}{\delta} > 1$ and the dimension of the matrix is sufficiently large. However, if we increase the size of $\gamma$, then the eigenvectors grow at least exponentially. We have the following lemma.

**Lemma 3.3.** Let $X = [x_{ij}] \in \mathbb{R}^{m \times m}$ denote the lower triangular matrix given by Theorem 3.1 and let $\gamma$ denote the ratio $\gamma = \frac{c}{\delta}$. If $\gamma \geq m$, then

$$x_{ij} \geq 2^{i-j}, \quad 1 \leq j \leq i \leq m.$$
Proof. Let \( i \geq j \) be given and set \( k = i - j \geq 0 \). Then
\[
x_{ij} = z_k = \binom{\gamma + k - 1}{k} = \prod_{l=0}^{k-1} \frac{\gamma + l}{l + 1} \geq \prod_{l=0}^{k-1} \frac{m + l}{l + 1}.
\]
We now claim that
\[
\forall l \in \{0, 1, 2, \ldots, k - 1 \} : \frac{m + l}{l + 1} \geq 2. \tag{10}
\]
We have
\[
\frac{m + l}{l + 1} \geq 2 \iff m + l \geq 2(l + 1) \iff m \geq l + 2.
\]
We have \( l \leq k - 1 \) and \( k \leq m - 1 \). It follows that
\[
l + 2 \leq (k - 1) + 2 = k + 1 \leq (m - 1) + 1 = m.
\]
This shows that inequality (10) is satisfied. The immediate implication is that
\[
x_{ij} \geq 2^k = 2^{i-j}.
\]
This completes the proof. \qed

Remark 3.4. We emphasize that the growth of the components of the matrix \( X \) is independent of the location and clustering of the eigenvalues of \( A \). If \( a \neq 0 \) and if \( \frac{mb}{a} \) is small, then the eigenvalues \( \lambda_j = a + jb \) are clustered near \( a \). If \( a = 0 \) and \( b \neq 0 \), then the eigenvalues are not clustered anywhere. In any case, it is the fraction \( \gamma = \frac{c}{b} \) and not the value of \( a \) that decides the behavior of the eigenvectors.

We now study the conditioning of the eigenvectors of the matrix \( A \) given by equation (4). We consider perturbations \( |\Delta A| \) that are bounded componentwise relative to \( A \), i.e.,
\[
|\Delta A| \leq \epsilon |A|,
\]
where \( \epsilon > 0 \) is a small number. The eigenvectors of the matrix \( A \) are determined by the columns of the matrix \( X \) given in Theorem 3.1. The nontrivial components of the \( j \)th column of \( X = [x_{ij}] \) are computed as the solution of a linear system of the form
\[
Bx = \begin{bmatrix} b & 2b & \cdots & 2b \\ -c & 0 & \cdots & -c \\ \vdots & \ddots & \ddots & \vdots \\ -c & \cdots & -c & nb \end{bmatrix} \begin{bmatrix} x_{j+1,j} \\ x_{j+2,j} \\ \vdots \\ x_{m,j} \end{bmatrix} = \begin{bmatrix} c \\ c \\ \vdots \\ c \end{bmatrix} =: f \tag{11}
\]
where the dimension of the system is \( n = m - j \). It is straightforward to verify that componentwise relative perturbations of \( A \) induces componentwise relative perturbations of \( B \) and \( f \). In this situation, the relevant condition number is Skeel’s condition number given by
\[
\kappa_\infty(B, f) = \frac{\|B^{-1}\|B\|x\|\infty}{\|x\|\infty}.
\]
We have the following theorem.

Theorem 3.5. Let \( b > 0 \) and \( c > 0 \) satisfy \( \gamma = \frac{c}{b} > 1 \). Let \( B \in \mathbb{R}^{n \times n} \) and \( f \in \mathbb{R}^n \) be as in equation (11). Then Skeel’s condition number satisfies
\[
\kappa_\infty(B, f) \leq 2 \left(1 + \gamma \log \left(\frac{\gamma + n - 1}{\gamma}\right)\right).
\]
Proof. Let \( x \in \mathbb{R}^n \) denote the solution of equation (11). Let \( z = |B^{-1}|B|x| \). Our objective is to show that
\[
\frac{\|z\|_\infty}{\|x\|_\infty} \leq 2 \left( 1 + \gamma \log \left( \frac{\gamma + n - 1}{\gamma} \right) \right).
\]

By the definition of \( z \) we have \( 0 \leq z_i \). We will now bound \( z_i \) from above. By Lemma 2.3 we have
\[
z_i = a_i(2\omega_i - 1) + \sum_{j=1}^{i-1} a_i a_j \omega_i \omega_j \omega_{j+1}(2\omega_j - 1)
\]
where \( a_i = \frac{c_i}{d_i} > 0 \) and \( \omega_i = \prod_{k=1}^{i-1} (1 + a_k) > 0 \). It is clear that
\[
z_i \leq 2 \left( a_i \omega_i + \sum_{j=1}^{i-1} a_i a_j \omega_i \omega_j \omega_{j+1} \right) = 2x_i \left( 1 + \sum_{j=1}^{i-1} \frac{a_j}{1 + a_j} \right).
\]

We now utilize that \( a_j = \frac{c_j}{d_j} = \frac{\gamma}{j} \). This implies
\[
\sum_{j=1}^{i-1} \frac{a_j}{1 + a_j} = \sum_{j=1}^{i-1} \frac{\gamma}{\gamma + j} \leq \int_0^{i-1} \frac{\gamma}{\gamma + x} \, dx = \gamma \log \left( \frac{\gamma + i - 1}{\gamma} \right).
\]

Here we have used that the continuous function \( x \to \frac{\gamma}{\gamma + x} \) is decreasing on the interval \([0, i - 1] \). We conclude that
\[
z_i \leq 2x_i \left( 1 + \gamma \log \left( \frac{\gamma + i - 1}{\gamma} \right) \right) \leq 2\|x\|_\infty \left( 1 + \gamma \log \left( \frac{\gamma + n - 1}{\gamma} \right) \right).
\]

This implies
\[
\|z\|_\infty \leq 2\|x\|_\infty \left( 1 + \gamma \log \left( \frac{\gamma + n - 1}{\gamma} \right) \right)
\]
and the proof is complete. \( \square \)

By Lemma 3.3 the eigenvectors grow at least exponentially when \( \gamma = \frac{\ell}{b} = m \). However, if \( b > 0 \) and \( c > 0 \), then the eigenvectors are well-conditioned with respect to componentwise relative perturbations of the matrix because
\[
\kappa_\infty(B, f) \leq 2 \left( 1 + m \log(2) \right).
\]

This is less surprising when we consider that the eigenvectors can be computed using additions and divisions of real numbers which are strictly positive.

4 Upper triangular problems

The choice of using lower rather than upper triangular matrices was made exclusively for pedagogical reasons. We find it simpler to apply the well-ordering principle when we are moving forward. We pass from lower triangular eigenvalue problems \( Ax = \lambda x \) to equivalent upper triangular problems with ease. Simply replace the lower triangular matrices \( A \) and \( X \) with the upper triangular matrices \( A' = JAJ \) and \( X' = JXJ \) where \( J \) is the anti-diagonal identity matrix. This similarity transformation reverses the numbering of the rows and columns of the matrices \( A \) and \( X \). The specific example given in the introduction is transformed into
\[
A' = \begin{bmatrix}
5 & -5 & -5 & -5 & -5 \\
4 & -5 & -5 & -5 & -5 \\
3 & -5 & -5 & -5 & -5 \\
2 & -5 & -5 & -5 & -5 \\
1 & -5 & -5 & -5 & -5 \\
\end{bmatrix}, \quad X' = \begin{bmatrix}
1 & 5 & 15 & 35 & 70 \\
1 & 5 & 15 & 35 & 60 \\
1 & 5 & 15 & 35 & 70 \\
1 & 5 & 15 & 35 & 70 \\
1 & 5 & 15 & 35 & 70 \\
\end{bmatrix}.
\]
5  Conclusion

We have shown that there exists matrices for which the eigenvalues and eigenvectors are well-conditioned with respect to componentwise relative perturbations of the matrix. However, the eigenvectors cannot be computed using regular substitution because they exceed the representational range. These matrices can be used to test subroutines for computing eigenvectors as these subroutines must be able to deal successfully with overflow.

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