Holographic thermalization with Lifshitz scaling and hyperscaling violation

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Abstract

A Vaidya type geometry describing gravitation collapse in asymptotically Lifshitz spacetime with hyperscaling violation provides a simple holographic model for thermalization near a quantum critical point with non-trivial dynamic and hyperscaling violation exponents. The allowed parameter regions are constrained by requiring that the matter energy momentum tensor satisfies the null energy condition. We present a combination of analytic and numerical results on the time evolution of holographic entanglement entropy in such backgrounds for different shaped boundary regions and study various scaling regimes, generalizing previous work by Liu and Suh.

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1 Introduction

One of the interesting questions regarding quantum information is how fast quantum correlations can propagate in a physical system. In a groundbreaking study in 1972, Lieb and Robinson [1] derived an upper bound for the speed of propagation of correlations in an interacting lattice system and in recent years there has been growing interest in this and related questions in connection with a number of new advances. The study of ultracold atom systems has developed to the level where experiments on the time evolution of quantum correlations are possible (see e.g. [2]), new techniques have been developed for the theoretical study of time evolution of observables in perturbed quantum lattices (see e.g. [3]), analytical results have been obtained for the time evolution of observables after quenches in conformal field theory [4–6], and entanglement entropy has been given a geometric interpretation [7–10] in the context of the holographic duality of strongly interacting conformal field theory [11]. The present paper follows up on this last direction.

In the context of holographic duality, different ways of introducing quenches in a conformal theory have been studied. One line of work focuses on constructing holographic duals for quenches in strongly coupled theories [12–16] in the spirit of similar work in weakly coupled quantum field theory involving a sudden change in the parameters of the Hamiltonian [4–6,17–20]. In another approach, the focus has instead been on perturbing the state of the system by turning on homogeneous sources for a short period of time. By a slight abuse of terminology, this process has also been called a "quench", although perhaps a "homogenous explosion" would be a closer term to describe the sudden change in the state of the boundary theory. There are two good reasons to study this model. One of them is that there is an elegant and tractable gravitational dual description of such a process in terms of the gravitational collapse of a thin shell of null matter to a black hole, the AdS-Vaidya geometry. The other good reason is that the time evolution of quantum correlations manifest in the holographic entanglement entropy following such an explosion was found to behave in the same manner as in the 1+1 dimensional conformal field theory work [4–6] – in a relativistic case quantum correlations were found to propagate at the speed of light [21–29]. The interesting lesson there is that even a strongly coupled conformal theory with no quasiparticle excitations may behave as if the correlations were carried by free-streaming particles. The model also allows for an easy extrapolation of the results to higher dimensional field theory at strong coupling. In generic dimensions, it turns out that the time evolution of holographic entanglement entropy has a more refined structure, characterized by different scaling regimes [30,31]: (I) a pre-local equilibrium power law growth in time, (II) a post-local equilibration linear growth in time, (III) a saturation regime. For entanglement surfaces of more general shape, one can also identify late-time memory loss, meaning that near saturation the time-evolution becomes universal with no memory on the detailed shape of the surface.

Many condensed matter and ultracold atom systems feature more complicated critical behavior with anisotropic (Lifshitz) scaling [32], characterized by the dynamic critical exponent $\zeta > 1$, or hyperscaling violation characterized by a non-zero hyperscaling violation exponent $\theta$ [33–35]. Hyperscaling violation leads to an effective dimension $d_\theta = d - \theta$. It was found that for a critical value $d_\theta = 1$ the entanglement entropy exhibits a logarithmic violation from the usual area law [36], which is also generic for compressible states with hidden Fermi surfaces [37].

By now there exist various holographic dual models for critical points involving Lifshitz scaling and hyperscaling violation [33–36,38–54]. In the light of the rich scaling structure in the time evolution of entanglement entropy, it is interesting to see how it carries over to systems with Lifshitz scaling and hyperscaling violation. In [55] a Lifshitz scaling generalization of the AdS-Vaidya geometry was constructed, and it was found that time evolution of entanglement entropy still contains a linear regime, where entanglement behaves as if it was carried by free streaming particles at finite velocity. This is non-trivial, since in the non-relativistic case $\zeta > 1$ there is no obvious characteristic scale like the speed of light in relativistic theories. The authors of [30,31], on the other hand, considered a relativistic system with hyperscaling violation, and found that their previous analysis easily carries over to that case, with the spatial dimension $d$ replaced by the effective dimension $d_\theta$. In this paper we extend the analysis to systems that exhibit both Lifshitz scaling and hyperscaling violation. We do this by first constructing the extension of the Lifshitz-AdS-Vaidya geometry to the hyperscaling violating case, and then analyzing the time evolution of the entanglement entropy for various boundary regions. We compute numerically the evolution of the holographic entanglement entropy.
for the strip and the sphere in backgrounds with non-trivial $\zeta$ and $\theta$. We then extract some analytic behavior in the thin shell limit for the temporal regimes (I), (II) and (III), generalizing the results of [30,31] to the case of $\zeta \neq 1$ and $\theta \neq 0$. In an appendix, we also consider briefly quench geometries where the critical exponents themselves are allowed to vary. This can be motivated from a quasiparticle picture and one could, for instance, consider a system where the dispersion relation is suddenly altered from $\omega \sim k^2 + \cdots$ to $\omega \sim k + \cdots$ or vice versa, by rapidly adjusting the chemical potential. We take some steps in this direction by considering holographic geometries where the dynamical critical exponent and the hyperscaling violation parameter are allowed to vary with time and show that such solutions can be supported by matter satisfying the null energy condition, at least in some simple cases. We leave a more detailed study for future work.

This paper is organized as follows. Hyperscaling violating Lifshitz-AdS-Vaidya solutions are introduced in Section 2 and parameter regions allowed by the null energy condition determined. In Section 3 the holographic entanglement entropy for a strip and for a sphere is analyzed in static backgrounds and Vaidya-type backgrounds are considered in Section 4. In Section 5 scaling regions in the time evolution of the entanglement entropy are studied for differently shaped surfaces. The details of some of the computations are presented in appendices along with a brief description of holographic quench geometries where the hyperscaling violation parameter and the dynamical critical exponent are allowed to vary with time.

**Note added.** As we were preparing this manuscript, [56] appeared with significant overlap with some of our results. A preliminary check finds that where overlap exists, the results are compatible.

### 2 Backgrounds with Lifshitz and hyperscaling exponents

The starting point of our analysis is the following gravitational action [55]

$$S = \frac{1}{16\pi G_N} \int \left( R - \frac{1}{2} (\partial \phi)^2 - V(\phi) - \frac{1}{4} \sum_{i=1}^{N_F} e^{\lambda_i \phi} F_i^2 \right) \sqrt{-g} \, d^{d+2}x,$$

(2.1)

which describes the interaction between the metric $g_{\mu\nu}$, $N_F$ gauge fields and a dilaton $\phi$. The simplest $d+2$ dimensional time independent background including the Lifshitz scaling $\zeta$ and the hyperscaling violation exponent $\theta$ is given by [33–35]

$$ds^2 = z^{-2d_\theta/d} (-z^{2-2\zeta} dt^2 + dz^2 + dx^2),$$

(2.2)

where $z > 0$ is the holographic direction and the cartesian coordinates $x$ parameterize $\mathbb{R}^d$ (we denote a vectorial quantity through a bold symbol). Hereafter the metric (2.2) will be referred as hvLif. In (2.2) we have introduced the convenient combination

$$d_\theta \equiv d - \theta.$$

(2.3)

When $\theta = 0$ and $\zeta = 1$, (2.2) reduces to $AdS_{d+2}$ in Poincaré coordinates.

In the following, we will consider geometries that are asymptotic to the hyperscaling violating Lifshitz (hvLif) spacetime (2.2). In particular, static black hole solutions with Lifshitz scaling and hyperscaling violation have been studied in [35,51,52]. The black hole metric is

$$ds^2 = z^{-2d_\theta/d} \left( -z^{2-2\zeta} \frac{dz^2}{F(z)} + \frac{dz^2}{F(z)} + dx^2 \right),$$

(2.4)

where the emblackening factor $F(z)$, which contains the mass $M$ of the black hole, is given by

$$F(z) = 1 - M z^{d_\theta + \zeta}.$$

(2.5)

The position $z_h$ of the horizon is defined as $F(z_h) = 0$ and the standard near horizon analysis of (2.4) provides the temperature of the black hole $T = z_h^{1-\zeta} |F'(z_h)|/(4\pi)$. In order to have $F(z) \to 1$ when $z \to 0$, we need to require

$$d_\theta + \zeta \geq 0.$$

(2.6)
The Einstein equations are $G_{\mu\nu} = T_{\mu\nu}$, where $G_{\mu\nu}$ is the Einstein tensor and $T_{\mu\nu}$ the energy-momentum tensor of the matter fields, i.e. the dilaton and gauge fields in (2.1). The Null Energy Condition (NEC) prescribes that $T_{\mu\nu}N^\mu N^\nu \geq 0$ for any null vector $N^\mu$. On shell, the NEC becomes $G_{\mu\nu}N^\mu N^\nu \geq 0$ and, through an astute choice of $N^\mu$, one finds

$$d_\theta(\zeta - 1 - \theta/d) \geq 0, \quad (2.7)$$

$$d_\theta(\zeta - 1 + \zeta) \geq 0. \quad (2.8)$$

In the critical case $\theta = d - 1$, they reduce to $\zeta \geq 2 - 1/d$. In Fig. 1 we show the region identified by (2.7) and (2.8) in the ($\zeta, \theta$) plane.

In order to construct an infalling shell solution, it is convenient to write the static metric (2.4) in an Eddington-Finkelstein-like coordinate system, by introducing a new time coordinate $v$ through the relation

$$dv = dt - \frac{dz}{z^{1-\zeta}F(z)}, \quad (2.9)$$

and rewriting (2.4) as

$$ds^2 = z^{-2d_\theta/d}( - z^{2(1-\zeta)}F(z)dv^2 - 2z^{1-\zeta}dv dz + dx^2). \quad (2.10)$$

The dynamical background that we are going to consider is of Vaidya type [57,58] and it is obtained by promoting the mass $M$ in (2.10) to a time dependent function $M(v)$, namely

$$ds^2 = z^{-2d_\theta/d}( - z^{2(1-\zeta)}F(v,z)dv^2 - 2z^{1-\zeta}dv dz + dx^2), \quad (2.11)$$

where

$$F(v,z) = 1 - M(v)z^{d_{\theta}+\zeta}. \quad (2.12)$$

The metric (2.11) with the emblackening factor (2.12) is a solution of the equation of motion $G_{\mu\nu} = T_{\mu\nu}$, where the energy-momentum tensor is given by the one of the static case with $M$ replaced by $M(v)$, except for the component $T_{vv}$, which now contains the following additional term

$$\tilde{T}_{vv} = \frac{d_\theta}{2}z^{d_{\theta}}M'(v). \quad (2.13)$$
Now consider the null vectors $N^\mu = (N^v, N^z, N^w)$ given by

$$
N^\mu_I = (0, 1, 0), \quad N^\mu_I = \left( -\frac{2z^{\zeta-1}}{F(v,z)}, 1, 0 \right), \quad N^\mu_{III} = \left( \pm \frac{z^{\zeta-1}}{\sqrt{F(v,z)}}, 0, n_1 \right),
$$

(2.14)

where $n_1$ is a $d-1$ dimensional vector with unit norm. The NEC for the vectors (2.14) leads to the following inequalities

$$
d_0(\zeta - 1 - \theta/d) \geq 0, \quad d_0 \left[(\zeta - 1 - \theta/d)F^2 - 2z^\zeta F_v \right] \geq 0, \quad 2(\zeta - 1)(d_0 + \zeta)F^2 + [zF_{zz} - (d_0 + 3(\zeta - 1))F_z]zF - z^\zeta d_0 F_v \geq 0,
$$

(2.15-2.17)

where the notation $F_z \equiv \partial_z F$, $F_v \equiv \partial_v F$ and $F_{zz} \equiv \partial^2_z F$ has been adopted. When $F(v,z) = 1$ identically, (2.16) and (2.17) simplify to (2.7) and (2.8) respectively. Plugging (2.12) into (2.16) and (2.17), we get

$$
d_0 \left[(\zeta - 1 - \theta/d)(1 - M(v)z^{d_0 + \zeta})^2 + 2z^{d_0 + 2\zeta}M'(v) \right] \geq 0, \quad 2(\zeta - 1)(d_0 + \zeta)(1 - M(v)z^{d_0 + \zeta}) + z^{d_0 + 2\zeta}d_0 M'(v) \geq 0.
$$

(2.18-2.19)

In the special case of $\theta = 0$ and $\zeta = 1$ we recover the condition $M'(v) \geq 0$, as expected. In the following we will choose the following profile for $M(v)$

$$
M(v) = \frac{M}{2} \left( 1 + \tanh(v/a) \right),
$$

(2.20)

which is always positive and increasing with $v$. It goes to 0 when $v \to -\infty$ and to $M$ when $v \to +\infty$. The parameter $a > 0$ encodes the rapidity of the transition between the two regimes of $M(v) \sim 0$ and $M(v) \sim M$. In the limit $a \to 0$ the mass function becomes a step function $M(v) = M\theta(v)$. This is the thin shell regime and it applies to many of the calculations presented below. We have checked numerically that the profiles (2.20) that we employ satisfy the inequalities (2.18) and (2.19) for all $v$ and $z$.

### 3 Holographic entanglement entropy for static backgrounds

#### 3.1 Strip

Let us briefly review the simple case when the region $A$ in the boundary theory is a thin long strip, which has two sizes $\ell \ll \ell_\perp$. Denoting by $x$ the direction along the short length and by $y_i$ the remaining ones, the domain in the boundary is defined by $-\ell/2 \leq x \leq \ell/2$ and $0 \leq y_i \leq \ell_\perp$, for $i = 1, \ldots, d$. Since $\ell \ll \ell_\perp$, we can assume translation invariance along the $y_i$ directions and this implies that the minimal surface is completely specified by its profile $z = z(x)$, where $z(\pm \ell/2) = 0$. We can also assume that $z(x)$ is even. Computing from (2.4) the induced metric on such a surface, the area functional reads

$$
\mathcal{A}[z(x)] = 2\ell_\perp^{d-2} \int_0^{\ell/2} \frac{1}{z^{d_0}} \sqrt{1 + \frac{z'^2}{F(z)}} \, dx.
$$

(3.1)

Since the integrand has no explicit $x$ dependence of $x$, the corresponding integral of motion is constant giving a first order equation for the profile

$$
z' = -\sqrt{F(z)[(z_s/z) z^{2d_0} - 1]}. \quad (3.2)
$$

Here we have introduced $z(0) = z_s$ and we have used that $z'(0) = 0$ and $z(x) < 0$. Plugging (3.2) into (3.1), it is straightforward to find that the area of the extremal surface is

$$
\mathcal{A} = 2\ell_\perp^{d-1} z_s^{d_0} \int_0^{\ell/2-\eta} z(x)^{-2d_0} \, dx, \quad = 2\ell_\perp^{d-1} \int_{z_s}^{z_s} z^{d_0} \sqrt{F(z)[(z_0 z_\infty) z^{2d_0} - z^{2d_0}]} \, dz, \quad (3.3)
$$

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with $z(x)$ a solution of (3.2). A cutoff $z \geq \epsilon > 0$ has been introduced to render the integral (3.3) finite, and a corresponding one along the $x$ direction

$$z(\ell/2 - \eta) = \epsilon. \quad (3.4)$$

The relation between $z_*$ and $\ell$ reads

$$\frac{\ell}{2} = \int_0^{z_*} \frac{dz}{\sqrt{F(z)[(z_*/z)^{2d_\theta} - 1]}}. \quad (3.5)$$

The vacuum case of $F(z) = 1$ can be solved analytically. Indeed, one can then integrate (3.2), obtaining

$$x(z) = \frac{\ell}{2} - \frac{z_*}{1 + d_\theta} \left( \frac{z}{z_*} \right)^{d_\theta + 1} _2 F_1 \left( \frac{1}{2}, \frac{1}{2} + \frac{3}{2} \frac{1}{2d_\theta} ; \frac{1}{2d_\theta} ; (z/z_*)^{2d_\theta} \right). \quad (3.6)$$

Imposing $x(z_*) = 0$ in (3.6) one finds

$$\frac{\ell}{2} = \frac{\sqrt{\pi} \Gamma\left( \frac{1}{2} + \frac{1}{2d_\theta} \right)}{\Gamma\left( \frac{1}{2d_\theta} \right)} z_* \quad (3.7)$$

The area (3.3) with $F(z) = 1$ is then [35]

$$d_\theta \neq 1 \quad A = \frac{2\ell^{d-1}}{d_\theta - 1} \left[ \frac{1}{\epsilon^{d_\theta - 1}} - \frac{1}{\ell^{d_\theta - 1}} \left( \frac{\sqrt{\pi} \Gamma\left( \frac{1}{2} + \frac{1}{2d_\theta} \right)}{\Gamma\left( \frac{1}{2d_\theta} \right)} \right)^{d_\theta} \right] + O \left( \epsilon^{1 + d_\theta} \right), \quad (3.8)$$

$$d_\theta = 1 \quad A = \frac{2\ell^{d-1}}{d_\theta} \log(\ell/\epsilon) + O \left( \epsilon^2 \right). \quad (3.9)$$

The critical value $d_\theta = 1$ is characterized by this divergence, which is logarithmic instead of power-like.

### 3.2 Sphere

If the perimeter between the two regions in the boundary theory is a $d - 1$ dimensional sphere of radius $R$ it is convenient to adopt spherical coordinates in the bulk (we denote by $\rho$ the radial coordinate) for $\mathbb{R}^d$ in (2.2) and (2.4), namely $d\mathbf{x}^2 = d\rho^2 + \rho^2 d\Omega_{d-1}^2$. In this case, the problem reduces to computing $z = z(\rho)$. The area functional reads

$$A[z(\rho)] = \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^R \rho^{d-1} \sqrt{1 + \frac{z^\prime^2}{F(z)}} d\rho, \quad (3.10)$$

where the factor in front of the integral is the volume of the $d - 1$ dimensional unit sphere. The key difference compared to the strip (see (3.1)) is that now the integrand of (3.10) depends explicitly on $\rho$ and one has to solve a second order ODE to find the $z(\rho)$ profile,

$$2d_\theta \rho F^2 + z\left[ \rho z - 2(d - 1) z^\prime \right] z^\prime^2 - 2 F \left[ \rho z z^\prime + (d - 1) z z^\prime + d_\theta \rho z^2 \right] = 0, \quad (3.11)$$

subject to the boundary conditions $z(R) = 0$ and $z^\prime(0) = 0$. For a trivial emblackening factor $F(z) = 1$ the equation of motion (3.11) simplifies to

$$\rho z z^\prime + \left[ d_\theta \rho + (d - 1) z z^\prime \right] (1 + z^\prime^2) = 0. \quad (3.12)$$

In the absence of hyperscaling violation ($\theta = 0$) it is well known that $z(\rho) = \sqrt{R^2 - \rho^2}$ describes an extremal surface for any dimension $d$ [8]. Since the extremal surface is computed for $t = \text{const.}$, the Lifshitz exponent $\zeta$ does not enter in the computation but equation (3.12) does involve the hyperscaling exponent through the effective dimension $d_\theta$. The extremal surface cannot be found in closed form for general values of $d_\theta \neq 0$ but the leading behavior of the extremal surface area, including the UV divergent part, can be obtained from
Figure 2: The profiles $z(x)$ of the extremal surfaces for a strip with $\ell = 8$ for different boundary times:
$t = 0$ (hvLif regime, red curve), $t = 3.6$ (intermediate regime, when the shell is crossed, blue curve) and $t = 5$
(black hole regime, black curve). The final horizon is $z_h = 1$. These plots have $d = 2$, $\theta = 2/3$ and $\zeta = 1.5$.
The left panel shows the situation in the thin shell limit ($a = 0.01$), while in the right panel $a = 0.5$.

the small $z$ asymptotics when $\rho = R$ is approached from below. For this it is convenient to rewrite (3.12) in
terms of a dimensionless variables $z = R \tilde{z}(x)$, $\rho = R(1 - x)$,

$$(1 - x)\tilde{z}^{\frac{d}{2}} + \left[d_\theta(1 - x) - (d - 1)\tilde{z}^{\frac{d}{2}}\right](1 + \tilde{z}^2) = 0,$$ (3.13)

where $\tilde{z}$ denotes $d\tilde{z}/dx$. As long as the effective dimension is not an odd integer $d_\theta \notin \{1, 3, 5, \ldots\}$ the equation
can be solved order by order in small $x$ by inserting an expansion of the form

$$\tilde{z} = z_0 \sqrt{x}(1 + z_1 x + z_2 x^2 + \ldots),$$ (3.14)

and solving for the coefficients,

$$z_0 = \sqrt{\frac{2(d_\theta - 1)}{d - 1}}, \quad z_1 = \frac{2d - 2 - (d_\theta - 1)^2}{4(d_\theta - 1)(d_\theta - 3)}, \quad \ldots.$$ (3.15)

The leading behavior of the area functional (3.10) is then

$$\mathcal{A}[z(\rho)] = \frac{2\pi^{d/2}}{\Gamma(d/2)} \left[ \frac{1}{(d_\theta - 1)} \frac{R^{d-1}}{\epsilon^{d_\theta-1}} + \ldots \right].$$ (3.16)

A more detailed analysis is required when $d_\theta \in \{1, 3, 5, \ldots\}$, along the lines of the procedure given in [59] for
$d = 2$, $d_\theta = 1$. In particular this is needed to see the logarithmic enhancement of the area law when $d_\theta = 1$.

4 Holographic entanglement entropy in Vaidya backgrounds

4.1 Strip

In this section we consider the strip introduced in §3.1 as the region in the boundary and compute holographically its entanglement entropy in the background given by the Vaidya metric (2.11), employing the prescription of [9]. The problem is more complicated than in the static case considered in §3.1 because
the profile is now specified by two functions $z(x)$ and $v(x)$ which must satisfy $v(-\ell/2) = v(\ell/2) = t$ and
\( z(-\ell/2) = z(\ell/2) = 0 \), with \( t \) the time coordinate in the boundary. Since in our problem \( v(x) \) and \( z(x) \) are even, the area functional reads
\[
A[v(x), z(x)] = 2^{d_\theta - 1} \int_0^{\ell/2} \sqrt{B} \frac{dz}{z^{d_\theta}} \, dx, \quad B = 1 - F(v, z)z^{2(1-\zeta)}v'^2 - 2z^{1-\zeta}z'v', \tag{4.1}
\]
and the boundary conditions for \( v(x) \) and \( z(x) \) are given by
\[
z'(0) = v'(0) = 0, \quad v(\ell/2) = t, \quad z(\ell/2) = 0. \tag{4.2}
\]
Since the integrand in (4.1) does not depend explicitly on \( x \), the corresponding integral of motion is constant, namely \( z^{d_\theta} B = \text{const} \). By recalling that \( z(0) = z_* \), this constancy condition can be written as
\[
\left( \frac{z_*}{z} \right)^{2d_\theta} = B. \tag{4.3}
\]

The equations of motion obtained extremizing the functional (4.1) are
\[
\begin{align*}
\partial_x \left[ z^{1-\zeta}(z^{1-\zeta} F v' + z') \right] &= z^{2(1-\zeta)} F_v v'^2 / 2, \tag{4.4} \\
\partial_x \left[ z^{1-\zeta} z' \right] &= d_\theta B/z + z^{2(1-\zeta)} F_v v'^2 / 2 + (1 - \zeta) z^{-\zeta} (z' + z^{1-\zeta} F v') v'. & \tag{4.5}
\end{align*}
\]

In Fig. 2 the typical profiles \( z(x) \) obtained by solving these equations numerically are depicted. For \( t \leq 0 \) the extremal surface is entirely in the hvLif part of the geometry. As time evolves and the black hole is forming, part of the surface enters into the shell and for large times, when the black hole is formed, the extremal surface stabilizes to its thermal result. In the special case of \( \theta = 0 \) and \( \zeta = 1 \), (4.4) and (4.5) simplify to
\[
\begin{align*}
F_v v'^2 &= 2 \left[ F_v v' + (F_v v' + F_z z') v' + z'' \right], \tag{4.6} \\
2z v'' &= z F_z v'^2 + 2d(1 - F v'^2 - 2z'v'). \tag{4.7}
\end{align*}
\]

Once a solution of (4.4) and (4.5) satisfying the boundary conditions (4.2) has been found, the surface area is obtained by plugging the solution into (4.1). By employing (4.3), one finds that the area of the extremal surface reads
\[
A = 2^{d_\theta - 1} \int_0^{\ell/2} \frac{z^{d_\theta}}{z^{2d_\theta}} \, dx. \tag{4.8}
\]
The integral is divergent and we want to consider its finite part. As in the static case, one introduces a cutoff \( \epsilon \) along the holographic direction and a corresponding one \( \eta \) along the \( x \) direction, as defined in (3.4).

One way to obtain a finite quantity is to subtract the leading divergence, which, for the strip, is the only one (see (3.8) for the static case),
\[
d_\theta \neq 1 \quad A^{(1)}_{\text{reg}} = \int_0^{\ell/2-\eta} z^{d_\theta} \frac{dz}{z^{2d_\theta}} \, dx - \frac{1}{(d_\theta - 1) \epsilon^{d_\theta - 1}}, \tag{4.9}
\]
\[
d_\theta = 1 \quad A^{(1)}_{\text{reg}} = \int_0^{\ell/2-\eta} z^{d_\theta} \frac{dz}{z^{2d_\theta}} \, dx - \log(\ell/\epsilon).
\]

Another way to get a finite result is by subtracting the area of the extremal surface at late time, after the black hole has formed
\[
A^{(2)}_{\text{reg}} = \int_0^{\ell/2-\eta} z^{d_\theta} \frac{dz}{z^{2d_\theta}} \, dx - \int_0^{\ell/2-\eta} z^{d_\theta} \frac{dz}{z^{2d_\theta}} \, dx, \tag{4.10}
\]
or by subtracting the area of the extremal surface at early time, when the background is hvLif, namely
\[
A^{(3)}_{\text{reg}} = \int_0^{\ell/2-\eta} z^{d_\theta} \frac{dz}{z^{2d_\theta}} \, dx - \int_0^{\ell/2-\eta} z^{d_\theta} \frac{dz}{z^{2d_\theta}} \, dx. \tag{4.11}
\]

The quantities corresponding to the the black hole are tilded, while the ones associated to hvLif are hatted. In particular, \( \tilde{z}(\ell/2 - \eta) = \epsilon \) and \( \tilde{z}(\ell/2 - \eta) = \epsilon \). In Fig. 3 we compare the regularizations (4.9), (4.10) and (4.11) as functions of \( \ell \) and of the boundary time \( t \) at the critical value \( \theta = d - 1 \).
Figure 3: Strip and $a = 0.01$ (thin shell). Regularizations (4.9), (4.10) and (4.11) of the area for $d = 1$ (dashed red), $d = 2$ (blue) and $d = 3$ (green) with $\theta = d - 1$ and $\zeta = 2 - 1/d$. Left panels: areas as functions of $\ell/2$ for fixed $t = 1.5$ (bottom curves) and $t = 2.5$ (upper curves). Right: area as functions of the boundary time $t$ with fixed $\ell = 3$ and $\ell = 5$. The latter ones are characterized by larger variations.

4.1.1 Thin shell regime

Let us consider the limit $a \to 0$ in (2.20), which leads to a step function

$$M(v) = M\theta(v).$$  \hspace{1cm} (4.12)

The holographic entanglement entropy in this background has been studied analytically for $\theta = 0$, $\zeta = 1$ and $d = 1$ in \cite{23,24}. For more general values of $\theta$ and $\zeta$ the thin shell regime is obtained by solving the differential equations (4.4) and (4.5) in the vacuum (hvLif) for $v < 0$ and in the background of a black hole of mass $M$ for $v > 0$. The solutions are then matched across the shell. Thus, the metric is (2.11) with

$$F(v, z) = \begin{cases} 
1 & v < 0 \text{ hvLif,} \\
F(z) & v > 0 \text{ black hole,}
\end{cases} \hspace{1cm} (4.13)$$

where $F(z)$ is given by (2.5). Recall that the symmetry of the problem allows us to work with $0 \leq x \leq \ell/2$. From Fig. 2 and by comparing Fig. 8 with Fig. 5 one can appreciate the difference between the thin shell
regime and the one where \( M(v) \) is not a step function. Denoting by \( x_c \) the position where the two solutions match, we have
\[
v(x_c) = 0, \quad z(x_c) \equiv z_c. \tag{4.14}
\]
Thus, when the extremal surface crosses the shell, the part having \( 0 \leq x < x_c \) is inside the shell (hvLif geometry) and the part with \( x_c < x \leq \ell/2 \) is outside the shell (black hole geometry).

The matching conditions can be obtained in a straightforward way by integrating the differential equations (4.4) and (4.5) in a small interval which properly includes \( x_c \), and then sending to zero the size of the interval. In this procedure, since both \( v(x) \) and \( z(x) \) are continuous functions with discontinuous derivatives, only a few terms contribute \( \delta \). In particular, \( F_v = -M z^{d_0+\zeta} \delta(v) \) is the only term on the r.h.s.’s of (4.4) and (4.5) that provides a non vanishing contribution. Thus, considering (4.5) first, we find the following matching condition
\[
v'_+ = v'_- = v'_c, \quad \text{at} \quad x = x_c. \tag{4.15}
\]
Then, integrating across the shell (4.4) and employing (4.15) (we have also used that \( \delta(v) = \delta(x - x_c)/|v'_c| \), where \( v'_c > 0 \), as discussed below), we find (notice that the term containing \( v' \) on the l.h.s. provides a non vanishing contribution)
\[
z'_+ - z'_- = \frac{z_c^{1-\zeta} v'_c}{2} (1 - F(z_c)), \quad \text{at} \quad x = x_c. \tag{4.16}
\]
Since \( F_v \) vanishes for \( v \neq 0 \), the differential equation (4.4) tells us that
\[
z^{1-\zeta} (v' z^{1-\zeta} F + z') = \text{const} \equiv \begin{cases} E_- & 0 \leq x < x_c \quad \text{hvLif}, \\ E_+ & x_c < x \leq \ell/2 \quad \text{black hole}. \end{cases} \tag{4.17}
\]
Let us consider the hvLif part \( (v < 0) \) first, where \( F = 1 \). Since \( v'(0) = 0 \) and \( z'(0) = 0 \), (4.17) tells us that \( E_- = 0 \). Thus, (4.17) implies that
\[
v' = -z^{-1} z', \quad 0 \leq x < x_c. \tag{4.18}
\]
Plugging this result into (4.3) with \( F = 1 \), it reduces to the square of (3.2) with \( F = 1 \), as expected. Taking the limit \( x \to x_c^- \) of (4.18), one finds a relation between the constant value \( v'_c \) defined in (4.15) and \( z'_c \), i.e.
\[
v'_c = -z_c^{1-\zeta} z'_c > 0, \tag{4.19}
\]
Figure 5: Regularized area (4.11) for the strip with \( a = 0.5 \). These plots should be compared with Fig. 4 because the parameters \( d, \theta \) and \( \zeta \) and the color code are the same.

where we have used that \( z'_c < 0 \). Integrating (4.18) from \( x = 0 \) to \( x = x_c \), we obtain that

\[
z'_c + \zeta v_c = z'_c + \zeta v_c .
\] (4.20)

Now we can consider the region outside the shell (\( v > 0 \)), where the geometry is given by the black hole. From (4.17) with \( F = F(z) \) given in (2.5) we have that

\[
v' = \frac{1}{z^{1-\zeta}F(z)} \left( \frac{E_+}{z^{1-\zeta}} - z' \right), \quad x_c < x \leq \ell/2 .
\] (4.21)

Then, plugging this result into (4.3), one gets

\[
z'^2 = F(z) \left[ \left( \frac{z}{z_c} \right)^{2d_0} - 1 \right] + \frac{E_+^2}{z^{2(1-\zeta)}}, \quad x_c < x \leq \ell/2 .
\] (4.22)

We remark that (4.22) becomes (3.2) when \( E_+ = 0 \). The constant \( E_+ \) can be related to \( z'_c \) by taking the difference between the equations in (4.17) across the shell. By employing (4.15), the result reads

\[
E_+ - E_- = z_c^{1-\zeta} \left[ z'_c - z'_c + z_c^{-1-\zeta} v'_c (F(z_c) - 1) \right] .
\] (4.23)

Then, with \( E_- = 0 \), the matching conditions (4.16) and (4.19) lead to

\[
E_+ = \frac{z_c^{1-\zeta}}{2} \left( 1 - F(z_c) \right) z'_c ,
\] (4.24)

where \( E_+ < 0 \) because of (4.18). Moreover, from (4.3), one finds that

\[
B_+ = B_- = \left( \frac{z_c}{z_c} \right)^{2d_0} , \quad \text{at} \ x = x_c .
\] (4.25)

Finally, the size \( \ell \) can be expressed in terms of the profile function \( z(x) \) (we recall that \( z' < 0 \)) by summing the contribution inside the shell (from (4.22) with \( F(z) = 1 \)) and the one outside the shell (from (4.22))

\[
\frac{\ell}{2} = \int_{z_c}^{z_c} \left( \frac{z_c^{2d_0} - z^{2d_0}}{z^{2d_0}} \right)^{-1/2} dz + \int_{z_c}^{z_c} \left( \frac{F(z)}{z^{2(1-\zeta)}} \right)^{-1/2} dz .
\] (4.26)
Let us consider a circle of radius $R$ in the boundary of the asymptotically $hvLif$ spacetime. As discussed in §3.2 for the static case, it is more convenient to adopt spherical coordinates in the Vaidya metric (2.11) for
\( \mathbb{R}^d \). The area functional is given by
\[
A[v(\rho), z(\rho)] = \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^R \frac{\rho^{d-1}}{z^d} \sqrt{B} \, d\rho, \quad B \equiv 1 - F(v, z)z^{2(1-\zeta)}v'^2 - 2z^{1-\zeta}z'v',
\]
where now the prime denotes the derivative w.r.t. \( \rho \). An important difference compared to the strip, as already emphasized for the static case, is that the Lagrangian of (4.30) depends explicitly on \( \rho \). This implies that we cannot find an integral of motion which allows to get a first order differential equation to describe the extremal surface. Thus, we have to deal with the equations of motion, which read
\[
\frac{z^d \sqrt{B}}{\rho^{d-1}} \partial_\rho \left[ \frac{\rho^{d-1}z^{1-\zeta} - d}{\sqrt{B}} \left( v'z^{1-\zeta}F + z' \right) \right] = \frac{z^{2(1-\zeta)}}{2} F_v v'^2, \tag{4.31}
\]
\[
\frac{z^d \sqrt{B}}{\rho^{d-1}} \partial_\rho \left[ \frac{\rho^{d-1}z^{2(1-\zeta)} - d}{\sqrt{B}} v' \right] = \frac{d_\rho}{z} B + \frac{z^{2(1-\zeta)}}{2} F_v v'^2 + \frac{1-\zeta}{z^\zeta} (z' + z^{1-\zeta}F_v v') v'. \tag{4.32}
\]
These equations have to be supplemented by the following boundary conditions
\[
v(R) = t, \quad v'(0) = 0, \quad \text{and} \quad z(R) = 0, \quad z'(0) = 0. \tag{4.33}
\]
We are again mainly interested in the limiting case of a thin shell \((4.12)\).

### 4.2.1 Thin shell regime

Considering the thin shell regime, defined by \((4.12)\), we can adopt to the sphere some of the observations made in §4.1.1 for the strip. Again, there is a value \( \rho_c \) such that for \( 0 \leq \rho < \rho_c \) the extremal surface is inside the shell (hvLif geometry), while for \( \rho_c < \rho \leq R \) it is outside the shell (black hole geometry).

The matching conditions can be found by integrating (4.31) and (4.32) across the shell, as was done in §4.1.1 for the strip. Introducing
\[
\tilde{v}' \equiv \frac{v'}{\sqrt{B}}, \quad \tilde{z}' \equiv \frac{z'}{\sqrt{B}}, \tag{4.34}
\]
we can use (4.32), whose r.h.s. does not contain \( F_v \), to obtain
\[
\tilde{v}'_+ = \tilde{v}'_-, \quad \text{at} \quad \rho = \rho_c \tag{4.35}
\]
while from (4.31) and employing (4.35) as well, we get
\[
\tilde{z}'_+ - \tilde{z}'_- = \frac{z^{1-\zeta} \tilde{v}'_+}{2} \left( 1 - F(\rho_c) \right), \quad \text{at} \quad x = x_c. \tag{4.36}
\]

Considering (4.31), since \( F_v = 0 \) for \( v \neq 0 \), we have
\[
\frac{\rho^{d-1}z^{1-\zeta} - d}{\sqrt{B}} (v'z^{1-\zeta}F + z') = \text{const} \equiv \begin{cases} E_- & \text{hvLif}, \\ E_+ & \rho_c < \rho \leq R \text{ black hole}, \end{cases} \tag{4.37}
\]
where \( E_- = 0 \) because \( v'(0) = 0 \) and \( z'(0) = 0 \). By using (4.34), one can write
\[
\frac{1}{B_+} = 1 + \tilde{v}'_+ \tilde{z}_c^{(1-\zeta)} \left[ \tilde{v}'_+ F(\rho_c) + 2 \tilde{z}'_+ \right], \tag{4.38}
\]
\[
\frac{1}{B_-} = 1 + \tilde{v}'_+ \tilde{z}_c^{(1-\zeta)} \left[ \tilde{v}'_+ F(\rho_c) + 2 \tilde{z}'_+ \right]. \tag{4.39}
\]
Taking the difference of these expressions and using (4.35) and (4.36), one finds
\[
B_+ = B_- . \tag{4.40}
\]
Figure 7: Holographic entanglement entropy for the sphere in the thin shell regime with $a = 0.01$ (see §4.2). The parameters $d$, $\theta$ and $\zeta$ are the same of Fig. 4 (same color coding). Left panel: fixed $t = 1.5$ (lower curve) and $t = 3$ (upper curve). Right panel: fixed $R = 2$ and $R = 4$ (larger spheres thermalize later).

By using (4.35), (4.36) and (4.37), we get

$$E_+ = \frac{\rho c^{d-1} z_c^{2(1-\zeta)} - \rho}{2\sqrt{B_+}} (F(z_c) - 1) v_c'.$$  \hfill (4.41)

Then, from (4.37) in the black hole region, one obtains

$$v' = \frac{z^{\zeta-1}}{F(z)} \left( A E_+ \sqrt{1 + z'^2/F(z)} - z' \right), \quad A \equiv \frac{z^{d_0+\zeta-1}}{\rho^{d-1}}. \hfill (4.42)$$

Plugging this expression into (4.32) leads to

$$2d_0 \rho F^2 + z[z F_z - 2(d-1)z'] z'^2 - 2F \left[ \rho z z'' + (d-1)z' + d_0 \rho z'^2 \right] + E_+^2 A^2 \rho [z(F_z + 2z'') - 2(\zeta - 1)(F + z'^2)] = 0,$$ \hfill (4.43)

which reduces to (3.11) when $E_+ = 0$, as expected. The boundary time $t$ is obtained by integrating (4.42) outside the shell $\rho_c \leq \rho < R$ (see e.g. (4.27) for the strip)

$$t = \int_{\rho_c}^{R} \frac{z^{\zeta-1}}{F(z)} \left( A E_+ \sqrt{1 + z'^2/F(z)} - z' \right) d\rho.$$ \hfill (4.44)

Notice that we cannot provide a similar expression for $R$, like we did for the strip in (4.26). Finally, the area of the extremal surface at time $t$ is the sum of two contributions, one inside (finite) and one outside (infinite) the shell, and is given by

$$\mathcal{A} = \frac{2\pi^{d/2}}{\Gamma(d/2)} \left( \int_0^{\rho_c} \rho^{d-1} \sqrt{1 + z'^2} \frac{d\rho}{z^{d_0}} + \int_{\rho_c}^{R} \rho^{d-1} \sqrt{1 + A^2 F_+^2/F(z)} \frac{d\rho}{z^{d_0}} \right).$$ \hfill (4.45)

Numerical results for the regularized extremal area $A_{\text{reg}}^{(3)}$ for a sphere (defined via an appropriate adaptation of (4.11)) in the thin shell regime are shown in Fig. 7.
Figure 8: Initial growth of the holographic entanglement entropy for \( d = 2 \) (see §5.1). The points come from the numerical solution of (4.4)-(4.5) for the strip (left) and (4.31)-(4.32) for the sphere (right) in the thin shell regime. The black dashed lines are obtained through the formula (5.3), which is independent of \( \theta \) and of the shape of the region in the boundary. Left panel: strip with \( \ell = 4 \). Right panel: sphere with \( R = 4 \).

5 Regimes in the growth of the holographic entanglement entropy

In this section we extend the analysis performed in [30] to \( \theta \neq 0 \) and \( \zeta \neq 1 \). For \( t < 0 \) we have \( A_{\text{reg}}^{(3)} = 0 \) because the background is hVLif. When \( t > 0 \), it is possible to identify three regimes: an initial one, when the growth is characterized by a power law, an intermediate regime where the growth is linear and a final regime, when \( A_{\text{reg}}^{(3)}(t) \) saturates to the thermal value. We report our results for the different regimes in the main text while the details of the computation are described in Appendix §A.

5.1 Initial growth

The initial regime is characterized by times that are short compared to the horizon scale

\[
0 < t \ll z_h.
\]  

(5.1)

In Appendix §A.1 following [31], we expand \( A_{\text{reg}}^{(3)} \) around \( t = 0 \) and consider the first non trivial order for an \( n \) dimensional spatial region whose boundary \( \Sigma \) has a generic shape. Given the metric (2.11) with (4.13), the final result for this regime is (see (A.14))

\[
A_{\text{reg}}^{(3)}(t) = \frac{M A_\Sigma (\zeta t)^{d(1-n/d)+\zeta+1}/\zeta}{2^{d(1-n/d)+\zeta+1}},
\]  

(5.2)

where \( A_\Sigma \) is the area of \( \Sigma \). Notice that for the holographic entanglement entropy \( n = d \), for the holographic counterpart of the Wilson loop \( n = 2 \) and for the holographic two point function \( n = 1 \). Explicitly, for the holographic entanglement entropy, (5.2) becomes

\[
A_{\text{reg}}^{(3)}(t) = \frac{M A_\Sigma \zeta^{1+1/\zeta}}{2(\zeta+1)} t^{1+1/\zeta},
\]  

(5.3)

which is independent of \( d \) and \( \theta \). This generalizes the result of [31] (see [61] for \( d = 1 \)). In Fig. 8 we show some numerical checks of (5.3) both for the strip and for the sphere.

5.2 Linear growth

When \( z_* \) is large enough, the holographic entanglement entropy displays a linear growth in time. The computational details for the strip are explained in Appendix §A.2. The result for (4.13) is that, in the
regime given by

\[ z_h \ll t \ll \ell, \]  

and if the following condition is satisfied

\[ d_\theta \geq 2 - \zeta, \]  

we find a linear growth in time for the holographic entanglement entropy, namely

\[ A_{\text{reg}}^{(3)}(t) \equiv 2t_{\perp}^{d-1}v_{\text{linear}} t. \]

The method of \cite{31} for the thin shell regime, extended to \( \theta \neq 0 \) and \( \zeta \neq 1 \), tells us that

\[ A_{\text{reg}}^{(3)}(t) = 2t_{\perp}^{d-1}A_{\text{reg}}^{(3)}(t), \quad A_{\text{reg}}^{(3)}(t) = \frac{\sqrt{-F(z_m)}}{z_m^{d_\theta + \zeta - 1}} t = \frac{v_E}{z_h^{d_\theta + \zeta - 1}} t, \]  

where, for \( F(z) \) given by \eqref{2.5}, \( v_E \) reads

\[ v_E = \frac{(\eta - 1) \frac{\eta - 1}{\eta^2}}{\eta^2}, \quad \eta = \frac{2(d_\theta + \zeta - 1)}{d_\theta + \zeta}. \]

It can be easily seen that \( v_E = 1 \) when \( \eta = 1 \) and \( v_E \to 0 \) as \( \eta \to +\infty \) monotonically. Notice that the linear regime depends only on the combination \( d_\theta + \zeta \). In Fig. \ref{fig:9}, where the points are computed using the

\[ \text{FIG. 9: Typical example of linear growth for the holographic entanglement entropy in the thin shell regime. Here } d = 2, \ z_h = 1, \ \theta = 1 \ \text{and } \zeta = 2 \text{ for two large strips: } \ell \sim 16 \text{ (green squares) and } \ell \sim 20 \text{ (blue squares). In the bottom panel, the dashed line is obtained through (5.7) and (5.8).} \]
Figure 10: Linear regime for the strip: the colored squares are values of the slope (see (5.6)) found from the numerical data as in the bottom panel of Fig. 9. The black empty circles denote the corresponding results of \( v_E \) from (5.8). In this plot \( z_h = 1 \).

Numerical solutions of (4.4) and (4.5), we see a typical linear behavior in time for two strips with large \( \ell \). The agreement between the slope of the numerical data and the value computed from (5.8) is quite good. In Fig. 10 we compare the slopes of the numerical curves with the values obtained from (5.8) for other values of \( \theta \) and \( \zeta \). We consider the linear growth regime for more generic backgrounds in Appendix B. In order to get a better understanding of the origin of the \( \zeta \) dependence in (5.7).

5.3 Saturation

We define the saturation time \( t_s \) as the boundary time such that, for \( t > t_s \), the extremal surface probes only the black hole part of the geometry. It is possible to estimate \( t_s \) as a function of \( z_* \) for sufficiently large regions with generic shapes. The relevant computations for this regime are explained in Appendix B.3. To leading order, \( t_s \) is given by

\[ t_s = -\frac{z_h^{-1}}{F_h'} \log(z_h - z_*) , \tag{5.9} \]

where \( F_h' = -\partial_z F(z) \big|_{z = z_h} \). Since the relation between \( z_* \) and the characteristic length of the boundary region depends on its shape, we have to consider the strip and the sphere separately. For a strip, if \( \partial_t A_{reg}(t) \) is continuous at \( t = t_s \), we find the following linear relation

\[ t_s = z_h^{-1} \sqrt{\frac{d\theta}{2z_h F_h'}} \ell + \ldots , \tag{5.10} \]

where the dots denote subleading orders at large \( \ell \). Notice that (5.10) can be generalized to \( n \) dimensional spatial surfaces in the boundary according to the observation made in the end of §4.1.1, namely \( d\theta \) should be replaced by \( nd\theta/d \) while \( F(z) \) kept equal to (2.5). This gives

\[ t_s = z_h^{-1} \sqrt{\frac{nd\theta}{2z_h F_h'}} \ell + \ldots . \tag{5.11} \]

It can also be shown that, whenever \( \partial_t A_{reg}(t) \) is continuous at saturation, we have

\[ A_{reg}^{(2)}(t) \propto (t - t_s)^2 + o((t - t_s)^2) , \tag{5.12} \]
Figure 11: Saturation time as a function of the transverse length scale $\ell$ for geodesic correlators. The dashed black line is a reference line with slope equal to 1, while the colored ones are obtained through (5.11) with $n = 1$, $\zeta = 2$ and the corresponding values of $\theta$ indicated in the legend. The agreement improves for large $\ell$.

for a strip for any values of $\zeta$ and $\theta$ (see Appendix A.3.3).

When the boundary region is a sphere and in the regime of large $R$, the transition to the saturated value is always smooth. In Appendix A.3.2 we show that

$$t_s = z_h^{\zeta - 1} \frac{2d_\theta}{z_h F_h} R - z_h^{\zeta - 1} \frac{(d-1)}{F(1)(z_h)} \log R + \ldots.$$  \hspace{1cm} (5.13)

The saturation time has also been evaluated numerically for the geodesic correlator, with the following procedure from [55]. The action for the geodesics has solutions with turning points either inside or outside the horizon. We first choose turning points $z_\ast$ inside the horizon, generate the corresponding geodesic and find the coordinates of the endpoints at the boundary and the length of the geodesic. The results are regulated by subtracting the vacuum value. For sufficiently large $\ell$, at early times the bulk geodesics will all have turning points inside the horizon, and also pass through the infalling shell extending into the part of the spacetime with vacuum geometry. In this case the corresponding observable will not be thermal. At later times the turning point will be outside the horizon and the observable takes a thermal value. The conversion between these two types of behavior is sharp and defines the saturation time. Following [55], the saturation times can be calculated by fitting surfaces to the data of the above solution. The intersection of the surfaces then defines the curve for the saturation time as a function of the transverse length scale. In Fig. 11 the numerical results for the saturation time of the geodesics are compared with the corresponding results from (5.11). Notice that the agreement improves for large $\ell$, as expected.

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A Computational details for the entanglement growth

A.1 Initial growth: generic shape

Let us consider a \( n \) dimensional region embedded into \( \mathbb{R}^d \), which is the spatial part of the boundary (i.e. \( z = 0 \)) of the Vaidya background (2.11). The boundary of such region will be denoted by \( \Sigma \) and it has a generic shape. The submanifold \( \Sigma \) is \( n - 1 \) dimensional and therefore it can be parameterized through a \( n - 1 \) dimensional vector of intrinsic coordinates \( \xi^\alpha \). Thus, being \( x_\alpha \) the cartesian coordinates of \( \mathbb{R}^d \), the submanifold \( \Sigma \) is specified by

\[
x_\alpha (\xi^\alpha), \quad a \in \{1, \ldots, d\}, \quad \alpha \in \{1, \ldots, n-1\}.
\]

The surface \( \Gamma_\Sigma \) we are looking for is also \( n \) dimensional and it extends into the bulk, arriving to the boundary along \( \Sigma \), i.e. \( \partial F_{\Sigma} = \Sigma \) at certain time \( t \). It is described by the functions

\[
v(\xi^\alpha, z), \quad X_\alpha (\xi^\alpha, z),
\]

satisfying the following boundary conditions

\[
v(\xi^\alpha, 0) = t, \quad X_\alpha (\xi^\alpha, 0) = x_\alpha (\xi^\alpha).
\]

We remark that for the holographic entanglement entropy \( n = d \), for the holographic counterpart of the Wilson loop \( n = 2 \) and for the holographic two point function \( n = 1 \) (\( \Gamma_\Sigma \) is a geodesic and \( \Sigma \) is made by two points spacelike separated).

The area \( A_\Sigma \) of \( \Gamma_\Sigma \) is given by

\[
A_\Sigma = \int_0^{z_*} dx \int d\xi^\alpha \frac{\sqrt{\text{det} \gamma}}{z^{nd_{a}/d}}, \quad \text{(A.4)}
\]

where \( z^{-2nd_{a}/d} \gamma_{ab} \) is the induced metric on \( \Gamma_\Sigma \) and \( \text{det} \gamma \) denotes the determinant of \( \gamma_{ab} \). Differentiating (A.2) and plugging the results into (2.11), we find that

\[
\gamma_{zz} = -z^{2(1-\zeta)} F v_z^2 - 2z^{1-\zeta} v_z + X_z \cdot X_z, \quad \text{(A.5)}
\]

\[
\gamma_{\alpha z} = -z^{2(1-\zeta)} F v_{\alpha} v_z - z^{1-\zeta} v_{\alpha} + X_\alpha \cdot X_z, \quad \text{(A.6)}
\]

\[
\gamma_{\alpha \beta} = -z^{2(1-\zeta)} F v_{\alpha} v_\beta + X_\alpha \cdot X_\beta, \quad \text{(A.7)}
\]

where \( X \) denotes the vector whose components are \( X_\alpha \), the dots stand for the scalar product and the subindices indicate the corresponding partial derivatives.

Here we consider the analogue of \( A^{(3)}_{\Sigma, \text{reg}} \) defined in (4.11), namely the area of \( \Gamma_\Sigma \) regularized through the area of \( \hat{\Gamma}_\Sigma \) computed in hvLif, when \( F = 1 \). Given that the hatted quantities refer to hvLif, it reads

\[
A^{(3)}_{\Sigma, \text{reg}}(t) = \int_0^{z_*} \frac{\sqrt{\text{det} \gamma}}{z^{nd_{a}/d}} dx - \int_0^{z_*} \frac{\sqrt{\text{det} \gamma}}{z^{nd_{a}/d}} dz \right] d^{n-1} \xi, \quad \text{(A.8)}
\]

The initial regime is characterized by \( 0 < t \ll z_0 \) and we want to compute \( A^{(3)}_{\Sigma, \text{reg}}(t) \) for small \( t \). Keeping the first order in (A.8) and repeating the same arguments discussed in [31], we find

\[
A^{(3)}_{\Sigma, \text{reg}}(t) = \int_0^{z_*} \frac{\partial F}{\frac{\sqrt{\text{det} \gamma}}{z^{nd_{a}/d}}} \delta F dz + \frac{\sqrt{\text{det} \gamma}}{z^{nd_{a}/d}} \delta z_* + \sum_{A=0,a} \frac{\partial}{\partial X_{A,z}} \left( \frac{\sqrt{\text{det} \gamma}}{z^{nd_{a}/d}} \right) \delta X_A \bigg|_0 d^{n-1} \xi, \quad \text{(A.9)}
\]
where \(X_0 \equiv v\), \(X_{A,z} \equiv \partial_z X_A\) and only the first term within the square brackets provides a non vanishing contribution. In order to find it, we employ the well known formula for the variation of the determinant

\[
\partial_F \left( \sqrt{\det \gamma} \right) = \frac{\sqrt{\det \gamma}}{2} \text{Tr} \left( \gamma^{-1} \partial_F \gamma \right).
\]  

(A.10)

From (A.5), (A.6) and (A.7), we get respectively

\[
\partial_F (\gamma_{zz})|_{F=1} = -\frac{v^2}{z^2(1-\zeta)} , \quad \partial_F (\gamma_{\alpha z})|_{F=1} = -\frac{v_\alpha v_z}{z^2(1-\zeta)} , \quad \partial_F (\gamma_{\alpha \beta})|_{F=1} = -\frac{v_\alpha v_\beta}{z^2(1-\zeta)}.
\]

(A.11)

Now, from (2.9) with \(F = 1\) we find that \(v = t - z^\zeta/\zeta\). Since \(t\) is a constant in terms of \(\xi^\alpha\), in (A.11) we have that \(v_\alpha = o(t)\) and \(v_z = -z^{\zeta-1} + o(t)\). Plugging these behaviors into (A.11), only the first expression is non vanishing and equal to \(-1\). Then, by using that \(X_\alpha (\xi^\alpha, z) = x_\alpha (\xi^\alpha) + o(z)\), where \(o(z)\) vanishes fast enough when \(z \to 0\), we have

\[
\gamma_{\alpha \beta} = h_{\alpha \beta} + o(z), \quad \gamma_{\alpha z} = o(z), \quad \gamma_{zz} = 1 + o(z),
\]

(A.12)

where \(h_{\alpha \beta} \equiv \partial_\alpha x_\beta \partial_\beta x_\alpha\) is the induced metric on \(\Sigma\). Notice that (A.12) tells us that the contribution of the term \(\text{Tr}(\gamma^{-1} \partial_F \gamma)\) to \(\partial_F (\sqrt{\det \gamma})|_{F=1}\) is equal to \(-1\). Collecting all these informations, we find

\[
\partial_F (\sqrt{\det \gamma})|_{F=1} = -\frac{\sqrt{\det h}}{2}.
\]

(A.13)

Finally, since in our case \(\delta F = F(z) - 1 = -M z^{(\delta z + \zeta)}\) is non vanishing only for \(0 < z < z_c\), the first term in (A.14) becomes

\[
\mathcal{A}^{(3)}_{\text{reg}} (t) = \frac{MA_\Sigma}{2} \int_0^{z_c} z^{d(z) - n/d + \zeta} dz = \frac{MA_\Sigma}{2} \int_0^{z_c} z^{d(z) - n/d + \zeta + 1} dz = \frac{MA_\Sigma}{2} \int_0^{z_c} z^{d(z) - n/d + \zeta + 1} dz.
\]

(A.14)

In the last step we have used that \(z_c = (\zeta t)^{1/\zeta}\) to the first order, which is obtained from \(v = t - z^\zeta/\zeta\) and the condition \(v = 0\) at the shell.

### A.2 Linear growth

In order to study this regime, we consider the strip (see §4.1). Following [31], let us start from (4.22) for the black hole regime. By employing (4.24) and (4.22) with \(F(z) = 1\), we can write it as follows

\[
z^2 = F(z) \left[ \left( \frac{z}{z} \right)^{2d} - 1 \right] + g(z) \left[ \left( \frac{z}{z_c} \right)^{2d} - 1 \right] = H(z), \quad x_c < x < \ell/2,
\]

(A.15)

where

\[
g(z) \equiv \frac{(F(z_c) - 1)^2}{4} \left( \frac{z_c}{z} \right)^{2(1-\zeta)}.
\]

(A.16)

Notice that the dependence on \(z\) of \(g(z)\) disappears when \(\zeta = 1\). Assuming that \(H(z)\) has a minimum at \(z = z_m\) with \(z_m < z^*_c\), its defining equation \(\partial_z H(z)|_{z = z_m} = 0\) gives

\[
z^{2d} \equiv z_m^{2d} F'(z_m) + 2(\zeta - 1)g(z_m)\]

\[
\frac{z_m^{2d} F'(z_m) + 2(\zeta - 1)g(z_m)}{z_m F'(z_m) - 2d z_m F(z_m) + 2(\zeta - 1)(z_m/z_c)^{2d} g(z_m)} = z_m^{2d}.
\]

(A.17)

Assuming also that at \(z = z_m\), it is possible to find \(z_c = z^*_c\) such that \(H(z_m) = 0\) (thus \(z^*_c = z^*_c(z_m)\)). Then, \(z^*_c\) is given by

\[
\frac{2d z_m F'(z_m) [F(z_m) + g(z_m)]_{z = z^*_c} + [(z_m/z_c)^{2d} - 1] [2(1 - \zeta) F(z_m) + z_m F'(z_m)] g(z_m)]_{z = z^*_c} = 0.
\]

(A.18)
When \( F(z) \) is given by (2.5), (A.17) and (A.18) become respectively

\[
z_m^{2d_h} = \frac{(d_h + \zeta)(z_m / z_h)^{d_h + \zeta} + (1 - \zeta)(z_c / z_h)^{d_h + \zeta}(z_c / z_m)^{2(1 - \zeta)}}{4d_h - 2(d_h - \zeta)(z_m / z_h)^{d_h + \zeta} + (1 - \zeta)(z_m / z_h)^{2(d_h + \zeta)}(z_c / z_m)^{2(1 - \zeta)}} z_m^{2d_h}, \tag{A.19}
\]

and

\[
2d_h \left[ 1 - \frac{(z_m / z_h)^{d_h + \zeta}}{(z_c / z_m)^{2(1 - \zeta)}} \right]^2 = \frac{\left(\frac{z_m}{z_c}\right)^{2d_h + \zeta}}{\left(\frac{z_c}{z_m}\right)^{2(1 - \zeta)}} \left[ 1 - \left(\frac{z_m}{z_h}\right)^{d_h + \zeta} \right] + \left(\frac{z_m}{z_c}\right)^{2d_h} \left(\frac{z_c}{z_h}\right)^{2d_h + \zeta} \left(1 - \eta \right) - \left(\frac{z_m}{z_c}\right)^{2d_h + \zeta} \left(\frac{z_c}{z_h}\right)^{2d_h + \zeta} \left(2(1 - \zeta) - (d_h + 2 - \zeta) \left(\frac{z_m}{z_h}\right)^{d_h + \zeta}\right). \tag{A.20}
\]

At this point, let us consider the limit \( z_s \to \infty \) with both \( z_m \) and \( z_c \) kept fixed. For the moment we just assume to be in a regime where this is allowed. The equations (A.17) and (A.18) become respectively

\[
z_m F'(z_m) - 2d_h F(z_m) = 2(1 - \zeta) \left(\frac{z_m}{z_c}\right)^{2d_h} g(z_m), \quad F(z_m) = - \left(\frac{z_m}{z_c}\right)^{2d_h} g(z_m) |_{2d_h}. \tag{A.21}
\]

Plugging the second equation in (A.21) into the first one, one finds

\[
z_m F'(z_m) + 2(1 - \zeta - d_h) F(z_m) = 0, \quad \text{at} \quad z_c = z_c^*, \tag{A.22}
\]

which can be written also in the following form

\[
\partial_{z_m} \left( \frac{F'(z_m)}{z_m^{2d_h + \zeta - 1}} \right) = 0. \tag{A.23}
\]

For \( F(z) \) given by (2.5) this equation tells us that

\[
z_m = \left(\frac{2(d_h + \zeta - 1)}{d_h + \zeta - 2}\right) z_h^{\frac{1}{d_h + \zeta}} \left(\frac{\eta}{\eta - 1}\right)^{\frac{1}{2}(2 - \eta)} \eta, \quad \eta = \frac{2(d_h + \zeta - 1)}{d_h + \zeta}. \tag{A.24}
\]

Notice that in this expression, the dimensionality, the Lifshitz and the hyperscaling exponents occur only through the combination \( d_h + \zeta \). In order to have a positive expression within the brackets of the first equation in (A.24), we need to require \( \eta > 1 \), i.e.

\[
d_h + \zeta > 2. \tag{A.25}
\]

Plugging (A.24) into the second equation of (A.21) computed for (2.5), we find that

\[
z_c^* = \frac{z_c}{z_h} = \frac{2(\eta - 1)\eta^{(\eta - 1)}}{\eta^{2\eta}}. \tag{A.26}
\]

It is useful to plot curves \( C_\ell \) with constant \( \ell \) in the plane \((z_s, z_c)\) or \((v_s, z_c)\) as done in Fig. 12. As \( t \) evolves, \( z_s \) decreases along each curve. After some time (which changes with \( \ell \)), all the curves lie on a limiting one \( C^* \). For any fixed \( \ell \), it will be shown that \( A_{reg}(t) \) is linear when the curve \( C_\ell \) coincides with \( C^* \). From Fig. 12 it is clear that, as \( \ell \) increases, also the linear regime increases. Thus, now we are considering

\[
z_s \to \infty, \quad \eta > 1, \quad z_c = (1 - \varepsilon)z_c^*, \tag{A.27}
\]

where \( 0 < \varepsilon \ll 1 \). When \( z_s \) is large, for \( F(z) \) given by (2.5), from (A.24) and (A.26) we have that

\[
\frac{z_m}{z_c^*} = \frac{\eta}{2\sqrt{\eta - 1}} > 1. \tag{A.28}
\]
This tells us that the solutions $z(x)$ are not injective for $0 \leq x \leq \ell/2$, which implies that we cannot employ (4.26), (4.27) and (4.28) because they have been derived assuming that $z(x)$ is invertible. In this case we have to use the following ones (see [31] for a detailed discussion)

\[
\frac{\ell}{2} = \int_{z_c}^{z_*} \frac{z^d}{\sqrt{z^{2d} - z^{2d_2}}} dz + \int_{z_*}^{z_m} \frac{dz}{\sqrt{H(z)}} + \int_{z_m}^{z_c} \frac{dz}{\sqrt{H(z)}},
\]

\[
t = \int_{z_c}^{z_*} \frac{1}{z^{1-\xi}F(z)} \left( \frac{E_+(z)}{z^{1-\xi}\sqrt{H(z)}} + 1 \right) dz + \int_{z_*}^{z_m} \frac{dz}{z^{1-\xi}F(z)} \left( \frac{E_+}{z^{1-\xi}\sqrt{H(z)}} + 1 \right) dz,
\]

\[
A = 2^{d-1} z_{d_2} \left( \int_{z_c}^{z_*} \frac{dz}{z^d \sqrt{z^{2d} - z^{2d_2}}} \int_{z_*}^{z_m} \frac{dz}{z^{2d} \sqrt{H(z)}} + \int_{z_m}^{z_c} \frac{dz}{z^{2d} \sqrt{H(z)}} \right).
\]

Comparing these equations with (4.26), (4.27) and (4.28), notice that only the part outside the shell is different.

Since the point $z = z_m$ and $z_c = z_*$ is a quadratic zero of $H(z)$, it provides a leading contribution to the integrals in (A.29), (A.30) and (A.31). Thus, expanding $H(z)$ around $z = z_m$, we find

\[
H(z) = H_2(z - z_m)^2 + b\varepsilon.
\]

By employing that for a smooth function $f(z)$ we have

\[
\int \frac{f(z)}{\sqrt{H_2(z - z_m)^2 + b\varepsilon}} \, dz = \frac{f(z_m)}{\sqrt{H_2}} \arcsinh(H_2(z - z_m)/(b\varepsilon)) + \cdots = -\frac{f(z_m)}{\sqrt{H_2}} \log \varepsilon + \cdots,
\]

we conclude that

\[
\frac{\ell}{2} = 2^{d-1} \frac{z_{d_2}^d}{\sqrt{H_2}} \log \varepsilon,
\]

\[
t = -\frac{E_+}{z_m^{1-\xi}F(z_m)\sqrt{H_2}} \log \varepsilon = -\frac{z_{d_2}^d}{z_m^{1-\xi}\sqrt{-H_2F(z_m)}} \log \varepsilon,
\]

\[
A_{\text{reg}}^{(3)} = -2^{d-1} \frac{z_{d_2}^d}{z_m^{2d_2} \sqrt{H_2}} \log \varepsilon,
\]
where in the second equality of (A.35) we used the second equation of (A.21). Combining (A.35) and (A.36), we also obtain that

\[ A_{reg}^{(3)} = 2 \ell_1^{d-1} \sqrt{-F(z_m)} z_m^{d_a + \zeta - 1} t = 2 \ell_1^{d-1} \frac{v_E}{z_h^{d_a + \zeta - 1}} t. \]  

(A.37)

For a \( F(z) \) given by (2.5), the linear growth velocity reads

\[ v_E = \left( \frac{z_h}{z_m} \right)^{d_a + \zeta - 1} \sqrt{-F(z_m)} = \frac{(\eta - 1)^{\frac{d-1}{2}}}{\eta^2}, \]  

(A.38)

where \( \eta \) has been defined in (A.24).

### A.3 Saturation

#### A.3.1 Large regions in static backgrounds

In order to understand the regime of saturation, when the holographic entanglement entropy approaches the thermal value, let us consider the static case when the size of the boundary region is large with respect to \( z_h \). In this case a large part of the extremal surface is very close to the horizon. Starting with the strip, when \( \ell \gg z_h \), we have that (we recall that tilded values of \( z \) refer to the static black hole case, following the notation introduced in §4.1)

\[ \tilde{z}_* = (1 - \epsilon)z_h, \]  

(A.39)

where \( \epsilon \) is a positive infinitesimal parameter. Expanding (3.5), we find

\[ \frac{\ell}{2} = -\frac{z_h \log \epsilon}{\sqrt{2d_0 z_h F'_h}} + \ldots, \quad F'_h \equiv -\partial_z F(z) \bigg|_{z = z_h}. \]  

(A.40)

In a similar way, plugging (A.39) into (3.1) and keeping the first divergent term as \( \epsilon \to 0 \), we get

\[ \mathcal{A} = -\frac{\ell}{2} \sqrt{2d_0 z_h F'_h} \log \epsilon / z_h + \ldots = \frac{\ell^{d-1}}{z_h^{d_a}} + \ldots. \]  

(A.41)

For a sphere, the analysis is slightly more complicated because we have to expand the differential equation for the minimal surface [62]. Setting

\[ z(\rho) = z_h - \epsilon a(\rho) + O(\epsilon^2), \]  

(A.42)

and expanding (3.11), the first order reads

\[ 2z_h [(d - 1)a' + \rho a''] a - z_h \rho a'^2 - 2d_0 F'_h a^2 = 0. \]  

(A.43)

This equation cannot be solved exactly, but, at large \( \rho \), we can find that the solution behaves as

\[ a(\rho) = C e^{\rho \sqrt{2d_0 F'_h / z_h}} / \rho^{d-1} + \ldots, \]  

(A.44)

where \( C \) is an arbitrary constant. Keeping only the first order in \( \epsilon \) in (A.42) and imposing \( z(R) = 0 \), one finds \( a(R) = z_h / \epsilon \), whose logarithm gives

\[ -\log \epsilon = R \sqrt{-2d_0 F'(z_h) / z_h} - (d - 1) \log R + \ldots. \]  

(A.45)

As for the area, plugging (A.42) into (3.10) and keeping the first divergent term as \( \epsilon \to 0 \), (A.45) allows us to conclude that

\[ \mathcal{A} = \frac{2\pi^{d/2} R^d}{d \Gamma(d/2) z_h^{d_a}} + \ldots. \]  

(A.46)
A.3.2 Saturation time

In the thin shell regime and whenever the saturation to the thermal value of the holographic entanglement entropy is smooth (the derivative does not jump), we can define the saturation time $t_s$ as the time such that $\tilde{v}_s = 0$. For $t > t_s$, the extremal surface is entirely within the black hole region. Thus, the equation for $t_s$ reads

$$0 = \tilde{v}_s(t_s) = t_s - \int_0^{\tilde{z}^*_s} \frac{dz}{z^{1-\zeta}F(z)}.$$  \hfill (A.47)

For $F(z)$ given by (2.3) the integral can be solved explicitly, finding

$$t_s = \left(\frac{\tilde{z}^*_s}{\zeta}\right)^2 F_1\left(1, \frac{\zeta}{d\theta + \zeta}; 1 + \frac{\zeta}{(d\theta + \zeta)(\tilde{z}_s/z_h)^d + \zeta}\right).$$  \hfill (A.48)

For very large regions, $\tilde{z}_s = z_h(1 - \varepsilon)$ and therefore (A.48) expanded to the first order in $\varepsilon$ gives

$$t_s = -z_h^{\zeta-1}\log \varepsilon = \frac{z_h^{\zeta-1}}{F_h} \sqrt{\frac{d\theta}{2(\zeta)}} \ell + \ldots$$  \hfill (A.49)

where in the second step we have employed (2.3). If the region on the boundary is a strip, we can use (A.40) to obtain

$$t_s = z_h^{\zeta-1} \sqrt{\frac{2d\theta}{z_h F_h^2}} \ell + \ldots = z_h^{\zeta-d\theta-1} \sqrt{\frac{d\theta}{2(\zeta)}} \ell + \ldots$$  \hfill (A.50)

For a sphere, (A.45) gives us

$$t_s = z_h^{\zeta-1} \sqrt{\frac{2d\theta}{z_h F_h^2}} R - \frac{(d-1)z_h^{\zeta-1}}{F_h} \log R + \ldots.$$  \hfill (A.51)

A.3.3 Saturation of the holographic entanglement entropy

In this section we try to estimate $A^{(2)}_{\text{ent}}(t)$ as a function of $t - t_s$, being $t_s$ the saturation time computed above. As the holographic entanglement entropy approaches its thermal value, the extremal surface is almost entirely within the black hole region. This means that the point $z_c$, where the extremal surface crosses the shell, is very close to $\tilde{z}_s$.

Let us consider the strip first and introduce a positive infinitesimal parameter $\varepsilon$ as follows

$$z_c = z_s \left(1 - \frac{\varepsilon^2}{2d\theta}\right).$$  \hfill (A.52)

Plugging this expansion into (4.24), at first order we get

$$E_+ = \frac{z_s^{1-\zeta}(F(z_s) - 1)}{2} \sqrt{\left(\frac{z_s}{z_c}\right)^{2d\theta} - 1} = \frac{z_s^{1-\zeta}(F(z_s) - 1)}{2} \varepsilon + O(\varepsilon^2).$$  \hfill (A.53)

Since we are approaching the extremal surface corresponding to the one of the static black hole, $z_s$ is close to its thermal value $\tilde{z}_s$, namely we are allowed to introduce another positive infinitesimal parameter $\delta$ as

$$z_s = \tilde{z}_s \left(1 - \frac{\delta}{2d\theta}\right).$$  \hfill (A.54)
We want to estimate $t - t_s$ in terms of the infinitesimal parameters $\varepsilon$ and $\delta$. Using (4.27) and (A.48), we find that

$$t - t_s = \int_0^{z_c} z^{\zeta-1} \left( \frac{E_+}{z^{1-\zeta}\sqrt{H(z)}} + 1 \right) - \int_0^{z_c} z^{\zeta-1} \frac{E_+}{F(z)} dz,$$

(A.55)

$$= \int_{z_c}^{z_\infty} z^{\zeta-1} \frac{E_+}{F(z)} dz + \int_0^{z_c} \frac{E_+}{F(z)\sqrt{H(z)}} dz - \int_{z_c}^{z_\infty} \frac{E_+ z^{2(\zeta-1)}}{F(z)\sqrt{H(z)}} dz,$$

(A.56)

$$= -\frac{z^{\zeta}}{2d\bar{z}F(\bar{z})} \frac{z \sqrt{F(\bar{z})} \left( F(\bar{z}) - 1 \right) Q_1(\bar{z})}{2} \varepsilon + \ldots,$$

(A.57)

where $H(z)$ is defined as the r.h.s. of (4.22) (see also (A.15)), $Q_1(z_s)$ is defined as follows

$$Q_1(z_s) \equiv \int_0^{z_c} \frac{z^{2(\zeta-1)}}{F(z)\sqrt{F(z)} \left( (z_s/z)^{2d\bar{z}} - 1 \right)} dz,$$

(A.58)

and the dots denote higher orders in $\varepsilon$ and $\delta$. Following [31], one can find a relation between $\delta$ or $\varepsilon$ from the expansion of (4.26). The presence of $\zeta$ does not modify the result, which reads

$$\delta = \frac{1 - F(\bar{z})}{F(\bar{z})Q_2(\bar{z})} \varepsilon + O(\varepsilon^2),$$

(A.59)

where (see [31] for further details)

$$Q_2(z_s) \equiv \int_0^{z_c} \frac{dz}{\sqrt{F(z) \left( (z_s/z)^{2d\bar{z}} - 1 \right)}}.$$

(A.60)

Thus, plugging this result into (A.57), one finds

$$t - t_s \propto \varepsilon + O(\varepsilon^2),$$

(A.61)

where the coefficient in front of $\varepsilon$ depends on $\zeta$ and $\theta$, as can be clearly seen from (A.57), but the power of $\varepsilon$ does not. Given this result, one can repeat precisely the computation of [31] and show that in this regime

$$A_{\text{reg}}^{(2)}(t) \propto \varepsilon^2 + O(\varepsilon^3),$$

(A.62)

i.e.

$$A_{\text{reg}}^{(2)}(t) \propto (t - t_s)^2 + O((t - t_s)^3).$$

(A.63)

Notice that the exponent is independent of $\theta$ and $\zeta$.

**B Strip in more generic backgrounds**

In order to understand the terms of the metric determining the linear regime, let us consider the following static background

$$ds^2 = \frac{1}{z^{2d\bar{z}/d}} \left( -Q(z)dt^2 - \frac{P(z)^2}{Q(z)} dz^2 + dx^2 \right),$$

(B.1)

which reduces to the black hole (2.4) when $Q(z) = z^{2(1-\zeta)}F(z)$ and $P(z) = z^{1-\zeta}$. By introducing the time coordinate $v$ as

$$dv = dt - \frac{P(z)}{Q(z)} dz,$$

(B.2)
we have but we recall that it takes two different values while for obtained by finding the extremal surface of the following functional area

\[ \mathcal{A}[v(x), z(x)] = 2\ell_\perp^{d-1} \int_0^{\ell/2} \frac{\sqrt{B}}{z_{\text{ds}}} \, dx, \quad B = 1 - Q(v, z)v'^2 - 2P(z)z'v', \]  

(B.5)

and the boundary conditions for \( v(x) \) and \( z(x) \) are given by (4.2). We only have to adapt the analysis performed in §4.1 to the background (B.4). The equations of motion of (B.5) read

\[ \partial_x [Qv' + Pz'] = Qv'^2/2, \]  

(B.6)

\[ \partial_x [Pv'] = dB/z + Qzv'^2/2 + Pz'v'. \]  

(B.7)

Choosing the thin shell profile

\[ Q(v, z) = P(z)^2 + \theta(v) [Q(z) - P(z)^2], \]  

(B.8)

we have that for \( v < 0 \) the backgrounds is

\[ ds^2 = \frac{1}{z_{\text{ds}}/d} \left( - P(z)^2 dt^2 + dz^2 + dx^2 \right), \]  

(B.9)

while for \( v > 0 \) the metric becomes (B.1). The equation (B.6) tells us that \( Qv' + Pz' \) is constant for \( v \neq 0 \) but we recall that it takes two different values \( E_- \) (for \( v < 0 \)) and \( E_+ \) (for \( v > 0 \)). Since \( v'(0) = z'(0) = 0 \), we have that \( E_- = 0 \). Integrating across the shell as in [4.1] (B.7) implies again that

\[ v_+ = v'_+ = v'_-, \quad \text{at} \quad x = x_c. \]  

(B.10)

Then (B.6) leads to

\[ z'_+ - z'_- = -\frac{1}{2P(z)}(Q(z) - P^2(z))v'_c. \]  

(B.11)

From these equations, we get

\[ E_+ = \frac{(Q_c - P^2_c)v'_c}{2} = -\frac{(Q_c - P^2_c)z'_c}{2P_c}, \]  

(B.12)

where \( P_c \equiv P(z_c), \) \( Q_c \equiv Q(z_c) \) and again \( z'_c = -\sqrt{(z_c/z_c)^{2d_\perp} - 1}. \) Thus, in the black hole part \( x_c < x \leq \ell/2 \) we have

\[ v' = \frac{E_+ - Q(z)z'}{P(z)}, \]  

(B.13)

\[ z'^2 = \frac{Q(z)}{P(z)^2} \left[ \left( \frac{z_c}{z} \right)^{2d_\perp} - 1 \right] + \frac{(Q_c - P^2_c)^2}{4P_c^2 P(z)^2} \left[ \left( \frac{z_c}{z_c} \right)^{2d_\perp} - 1 \right] \equiv H(z). \]  

(B.14)

Repeating the steps explained to get (4.27) and (4.28), in this case we find

\[ t = \int_0^{z_c} \frac{P(z)}{Q(z)} \left( \frac{E_+}{P(z)\sqrt{H(z)}} + 1 \right) \, dz, \quad A = 2\ell_\perp^{d-1} z_{\text{ds}}^{d_\perp} \int_0^{z_c} \frac{dz}{z_{\text{ds}} \sqrt{H(z)}}. \]  

(B.15)
B.1 Linear growth

At this point we take the limit of large $z$, keeping $z_m$ and $z_c$ finite. In this limit, (B.14) becomes

\[ z'^2 = \left( \frac{Q(z)}{z^{2d_o}} + \frac{(Q_c - P_c^2)}{4z^{2d_o}P_c^2} \right) \frac{z^{2d_o}}{P(z)^2} = H(z). \]  

(B.16)

The equation $\partial_{z_m} H(z_m) = 0$, which defines $z_m$, reads

\[ (Q'_m P_m - 2Q_m P'_m) z_m - 2d_\theta P_m Q_m - 2P'_m G_c z_m^{2d_o + 1} = 0, \]  

(B.17)

where the subindex $m$ denotes that the corresponding quantity is computed at $z = z_m$ and we defined

\[ \gamma_c \equiv \frac{(Q_c - P_c^2)^2}{4z_c^{2d_o}P_c^2}. \]  

(B.18)

Introducing $\gamma^*_c \equiv \gamma_c |_{z_c = z^*_c}$, the equation for $z^*_c$ reads

\[ \gamma^*_c = -\frac{Q_m}{z^*_m}, \]  

(B.19)

which reduces to the second equation of (A.21) for the case considered in the Appendix A. Then, plugging (B.19) into (B.17) we find

\[ Q'_m z_m - 2d_\theta Q_m = 0, \]  

at $z_c = z^*_c$,  

(B.20)

which can also be written as

\[ \partial_{z_m} \left( \frac{Q_m}{z^*_m} \right) = 0. \]  

(B.21)

Repeating the steps done to get (A.34), (A.35) and (A.36), in this case we obtain

\[ \ell/2 = \sqrt{\pi} \Gamma(1/(2d_o) + 1/2) \frac{z^*_m \log \epsilon}{\sqrt{H_2}} \]  

(B.22)

\[ t = -\frac{E_+}{Q_m \sqrt{H_2}} \log \epsilon = -\frac{z^*_m}{z^*_m \sqrt{-H_2} Q_m} \log \epsilon, \]  

(B.23)

\[ A^{(3)}_{\text{reg}} = -2d_o \frac{z^*_m}{z^*_m \sqrt{-H_2 Q_m} \log \epsilon}. \]  

(B.24)

Thus, (B.23) and (B.24) allow us to find that

\[ A^{(3)}_{\text{reg}} = 2d_o \frac{\sqrt{-Q_m}}{z^*_m} t. \]  

(B.25)

We conclude that $P(z)$ does not affect the linear growth regime.

C Vaidya backgrounds with time dependent exponents

In this appendix we consider the following generalization of (2.11)

\[ ds^2 = z^{2\theta(v)/d - 2} \left( -z^{2(1 - \zeta(v))} F(v, z) dv^2 - 2z^{1 - \zeta(v)} dv dz + dx^2 \right), \]  

(C.1)

where we have introduced a temporal dependence in the Lifshitz and hyperscaling exponents. Let us discuss the energy-momentum tensor when the metric (C.1) is on shell. For simplicity, we consider only the backgrounds (C.1) with $F(v, z) = 1$ identically.

The first case we consider is given by $\theta(v) = \text{const}$. The associated energy-momentum tensor reads

\[ T_{\mu\nu} = T_{\mu\nu}^{(\theta)} + T_{\mu\nu}^{(C)}, \]  

(C.2)
where $T^{(\mu \nu)}_{\mu \nu}$ is the part containing the hyperscaling exponent, which occurs also when $\zeta(v)$ is constant, namely

$$
T^{(\mu \nu)}_{\mu \nu} = \begin{pmatrix} 
-z^{-2\zeta}(d_0 + 1 + \theta/d)d_0/2 & -z^{1-\zeta}(d_0 + 1 + \theta/d)d_0/2 & 0 \\
-z^{1-\zeta}(d_0 + 1 + \theta/d)d_0/2 & z^{-2d_0}(\theta/d - \zeta + 1) & 0 \\
0 & 0 & z^{-2}(d_0^2(d-1)/d + 2\zeta(\zeta - 1 + d_0)) I_d/2 
\end{pmatrix},
$$

(C.3)

(we have denoted by $I_d$ the $d$ dimensional identity matrix), while $T^{(\nu \zeta)}_{\mu \nu}$ is the term due to $\zeta' \neq 0$

$$
T^{(\nu \zeta)}_{\mu \nu} = \begin{pmatrix} 
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & z^\zeta \zeta' I_d 
\end{pmatrix}.
$$

(C.4)

Similarly, we can consider the situation where $\zeta(v) = \text{const}$. It leads to

$$
T_{\mu \nu} = T^{(\nu \zeta)}_{\mu \nu} + T^{(\theta)}_{\mu \nu},
$$

(C.5)

where $T^{(\mu \nu)}_{\mu \nu}$ is (C.3) and

$$
T^{(\theta)}_{\mu \nu} = \frac{\theta'}{z} \begin{pmatrix} 
z^{-1\zeta}[2 + \log z(\zeta - d_0 - \theta/d + (\theta'/d) \log z)] & (1 - d_0 \log z) & 0 \\
(1 - d_0 \log z) & 0 & 0 \\
0 & 0 & z^{\zeta - 1}[2 + (d-1)(d_0/d) \log z] I_d 
\end{pmatrix},
$$

(C.6)

which vanishes when $\theta(v)$ is constant, as expected. When both $\theta'(v) \neq 0$ and $\zeta'(v) \neq 0$, we find that

$$
T_{\mu \nu} = T^{(\nu \zeta)}_{\mu \nu} + T^{(\zeta)}_{\mu \nu} + T^{(\theta)}_{\mu \nu} + T^{(\theta \zeta)}_{\mu \nu},
$$

(C.7)

where $T^{(\nu \zeta)}_{\mu \nu}$, $T^{(\zeta)}_{\mu \nu}$ and $T^{(\theta)}_{\mu \nu}$ have been defined respectively in (C.3), (C.4) and (C.6), while $T^{(\theta \zeta)}_{\mu \nu}$ is given by

$$
T^{(\theta \zeta)}_{\mu \nu} = \begin{pmatrix} 
-\zeta \theta' \log^2(z) & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0_d 
\end{pmatrix},
$$

(C.8)

being $0_d$ is the $d \times d$ matrix whose elements are zero.

It could be interesting to analyze the Null Energy Condition for these kind of backgrounds. Unfortunately, since the inequalities turn out to be lengthy and not very illuminating, we will consider here only the case of $\theta(v) = \text{const}$. First, since a null vector with respect to the metric (C.1) is null also with respect to (2.1), we can employ the vectors (2.14). Secondly, given the additive structure of $T_{\mu \nu}$ in (C.2), we can consider the results of (2.14) and add to them the contribution of $T^{(\nu \zeta)}_{\mu \nu} N^\mu N^\nu$. The resulting inequalities read

$$
d_0 [\zeta(v) - 1 - \theta/d] \geq 0, \quad (C.9)
$$

$$
[\zeta(v) - 1] [d_0 + \zeta(v)] + z^\zeta \zeta'(v) \geq 0, \quad (C.10)
$$

which reduce respectively to (2.7) and (2.8) when $\zeta(v) = \text{const}$, as expected. When $\theta = 0$, the inequality (C.9) tells us that $\zeta(v) \geq 1$. As for (C.10), it allows, for instance, a profile with $\zeta'(v) \geq 0$. In the critical case $\theta = d - 1$, (C.9) becomes $\zeta(v) \geq 2 - 1/d \geq 1$ while (C.9) becomes $[\zeta(v)^2 - 1] + z^\zeta \zeta'(v) \geq 0$. Thus, for instance, profiles having $\zeta'(v) \geq 0$ are again allowed.

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