Some asymptotic properties of SEIRS models with nonlinear incidence and random delays

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Abstract

This paper presents the dynamics of mosquitoes and humans, with general nonlinear incidence rate and multiple distributed delays for the disease. The model is a SEIRS system of delay differential equations. The normalized dimensionless version is derived; analytical techniques are applied to find conditions for deterministic extinction and permanence of disease. The BRN $R_0^*$ and ESPR $E(e^{-\mu \tau_1 + \mu \tau_2})$ are computed. Conditions for deterministic extinction and permanence are expressed in terms of $R_0^*$ and $E(e^{-\mu \tau_1 + \mu \tau_2})$, and applied to a P.vivax malaria scenario. Numerical results are given.

Keywords: Endemic equilibrium, basic reproduction number, permanence in the mean, Lyapunov functionals techniques, extinction rate.

Note 0.1 This arxiv paper published by Nonlinear Analysis: Modelling and Control is the elaborate version with added biological insights which were removed to meet space restrictions of the journal. Thanks for reading. D.W.

1 Introduction

Malaria has exhibited an increasing alarming high mortality rate between 2015 and 2016. In fact, the latest WHO-World Malaria Report 2017[12] estimates a total of 216 million cases of malaria from 91 countries in 2016, which constitutes a 5 million increase in the total malaria cases from the malaria statistics obtained previously in 2015. Moreover, the total death count was 445000, and sub-Saharan Africa accounts for 90% of the total estimated malaria cases. This rising trend in the malaria data, signals a need for more learning about the disease, improvement of the existing control strategies and equipment, and also a need for more advanced resources etc. to fight and eradicate, or ameliorate the burdens of malaria.

Malaria and other mosquito-borne diseases such as dengue fever, yellow fever, zika fever, lymphatic filariasis etc. exhibit some unique biological features. For instance, the incubation of the disease requires two hosts - the mosquito vector and human hosts, which may be either directly involved in a full life cycle of the infectious agent consisting of two separate and independent segments of sub-life cycles, which are completed separately inside the two hosts, or directly involved in two separate and independent half-life cycles of the infectious agent in the hosts. Therefore, there is a total latent time lapse of disease incubation which extends over the two segments of delay incubation times namely: (1) the incubation period of the infectious agent (or the half-life cycle) inside the vector, and (2) the incubation period of the infectious agent (or the other half-life cycle) inside the human being (cf.[11, 4]). In fact, the malaria plasmodium undergoes the first developmental half-life cycle called the sporogonic cycle inside the female Anopheles mosquito lasting approximately 10 – 18 days, following a successful blood meal from an infectious human being through a mosquito bite. Moreover, the mosquito becomes infectious.

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The parasite completes the second developmental half-life cycle called the exo-erythrocytic cycle lasting about 7-30 days inside the exposed human being\[11\] \[4\], whenever the parasite is transferred to human being in the process of the infectious mosquito foraging for another blood meal.

The exposure and successful recovery from a malaria parasite, for example, *falciparum vivae* induces natural immunity against the disease which can protect against subsequent severe outbreaks of the disease. Moreover, the effectiveness and duration of the naturally acquired immunity against malaria is determined by several factors such as the species and the frequency of exposure to the parasites (cf. [4, 11]).

Compartmental mathematical epidemic dynamic models have been used to investigate the dynamics of several different types of infectious diseases including malaria\[5\] \[6\]. In general, these models are classified as SIS, SIR, SIRS, SEIRS, and SEIR etc.\[13\] \[2\] \[17\] \[6\] \[6\] epidemic dynamic models depending on the compartments of the disease classes directly involved in the general disease dynamics. Many compartmental mathematical models with delays have been studied\[17\] \[8\] \[6\].

Some important investigations in the study of population dynamic models expressed as systems of differential equations are the permanence, and extinction of disease in the population, and also stability of the equilibria over sufficiently long time. Several papers in the literature\[10\] \[16\] \[15\] have addressed these topics. The extinction of disease seeks to find conditions that are sufficient for the disease related classes in the population such as, the exposed and infectious classes, to become extinct over sufficiently long time. The permanence of disease also answers the question about whether a significant number of people in the disease related classes will remain over sufficiently long time. Disease eradication or persistence of disease in the steady state population seeks to find conditions sufficient for the equilibria to be stable asymptotically.

The primary objectives of this paper include, to investigate (1) the extinction, and (2) the permanence of disease in a family of SEIRS epidemic models. In other words, we find conditions that are sufficient for a disease such as malaria, to become extinct from the population over time, and also conditions that cause the disease to be permanent in the population over time.

The rest of this paper is presented as follows:- in Section 2 the mosquito-human models are derived. In Section 3 some model validation and preliminary results are presented. In Section 4 the results for the permanence of the disease are presented. Moreover, simulation results for the permanence of the disease in the population are presented in Section 6. In Section 4 the results for the extinction of the disease are presented. Moreover, the numerical simulation results for the extinction of disease are presented in Section 7.

## 2 Derivation of the mosquito-host dynamics

The following assumptions are made to derive the epidemic model. Ideas from\[21\] will be used to derive the model for the mosquito-human dynamics.

(A) There are delays in the disease dynamics, and the delays represent the incubation period of the infectious agents (plasmodium or dengue fever virus etc.) in the vector $T_1$, and in the human host $T_2$. The third delay represents the natural immunity period $T_3$, where the delays are random variables with densities $f_{T_1}, t_0 \leq T_1 \leq h_1, h_1 > 0$, and $f_{T_2}, t_0 \leq T_2 \leq h_2, h_2 > 0$ and $f_{T_3}, t_0 \leq T_3 < \infty$ (cf.\[18\]).

(B) The vector (e.g. mosquito) population consists of two main classes namely: the susceptible vectors $V_s$ and the infectious vectors $V_i$. Moreover, it is assumed that the total vector population denoted $V_0$ is constant at any time, that is, $V_s(t) + V_i(t) = V_0, \forall t \geq t_0$, where $V_0 > 0$ is a positive constant. The susceptible vectors $V_s$ are infected by infectious humans $I$, and after the incubation period $T_1$, the exposed vector becomes infectious $V_i$. Moreover, there is homogenous mixing between the vector-host populations. Therefore, the birth rate and death rate of the vectors are equal, and denoted $\mu_v$. It is assumed that the turnover of the vector population is very high, and the total number of vectors $V_0$ at any time $t$, is very large, and as a result, $\mu_v$ is sufficiently large number. In addition, it is assumed that the total vectors $V_0$ is exceedingly larger than the total humans present at any time $t$, denoted $N(t), t \geq t_0$. That is, $V_0 >> N(t), t \geq t_0$.

(C) The humans consists of susceptible ($S$), Exposed ($E$), Infectious ($I$) and removed ($R$) classes. The susceptibles are infected by the infectious vectors $V_i$, and become exposed ($E$). The infectious agent
incubates for $T_2$ time units, and the exposed individuals become infectious $\hat{I}$. The infectious class recovers from the disease with temporary or sufficiently long natural immunity and become $(\hat{N})$. Therefore, the total population present at time $t$, $\hat{N}(t) = \hat{S}(t) + \hat{E}(t) + \hat{I}(t) + \hat{R}(t), \forall t \geq t_0$.

Furthermore, it is assumed that the interaction between the infectious vectors $V_i$ and susceptible humans $\hat{S}$ exhibits nonlinear behavior, due to the overcrowding of the vectors as described in (B), and resulting in psychological effects on the susceptible individuals which lead to change of behavior that limits the disease transmission rate, and consequently in a nonlinear character for the incidence rate characterized by the nonlinear incidence function $G$. $G$ satisfies the conditions of Assumption 2.1.

**Assumption 2.1**

$A1$ $G(0) = 0$; $A2$: $G(I)$ is strictly monotonic on $[0, \infty)$; $A3$: $G \in C^2([0, \infty), [0, \infty))$, and $G''(I) < 0$; $A4$. $\lim_{I \to \infty} G(I) = C, 0 \leq C < \infty$; $A5$: $G(I) \leq I, \forall I > 0$; $A6$

$$\left(\frac{G(x)}{x} - \frac{G(y)}{y}\right)(G(x) - G(y)) \leq 0, \forall x, y \geq 0. \quad (2.1)$$

These assumptions form an extension of the assumptions in [13, 19, 18]. Some examples of incidence functions include $G(x) = \frac{T}{1 + e^{\mu T}}$, $\theta > 0$ etc.

(D) There is constant birthrate of humans $\hat{B}$ in the population, and all births are susceptible individuals. It is also assumed that the natural deathrate of human beings in the population is $\hat{\mu}$ and individuals die additionally due to disease related causes at the rate $\hat{d}$. From a biological point of view, the average lifespan of vectors $\frac{1}{\hat{\mu}_v}$, is much less than the average lifespan of a human being in the absence of disease $\frac{1}{\hat{\mu}}$. It follows that assuming exponential lifetime for all individuals (both vector and host) in the population, then the survival probabilities over the time intervals of length $T_1 = s \in [t_0, h_1]$, and $T_2 = s \in [t_0, h_2]$, satisfy

$$e^{-\hat{\mu}_v T_1} < e^{-\hat{\mu}_v T_1} \quad \text{and} \quad e^{-\hat{\mu}_v T_1 - \hat{\mu}_v T_2} < e^{-\hat{\mu}_v (T_1 + T_2)}. \quad (2.2)$$

Applying similar ideas in [21], the vector dynamics from (A)-(D) follows the system

$$
\begin{align*}
\frac{dV_s(t)}{dt} &= [-\Lambda e^{-\hat{\mu}_v T_1} \hat{I}(t - T_1)V_s(t - T_1) - \hat{\mu}_v V_s(t) + \hat{\mu}_v (V_s(t) + V_i(t))]dt, \quad (2.3) \\
\frac{dV_i(t)}{dt} &= [\Lambda e^{-\hat{\mu}_v T_1} \hat{I}(t - T_1)V_s(t - T_1) - \hat{\mu}_v V_i(t)]dt, \quad (2.4) \\
V_0 &= V_s(t) + V_i(t), \forall t \geq t_0, t_0 \geq 0, \quad (2.5)
\end{align*}
$$

where $\hat{\Lambda}$ is the effective disease transmission rate from an infectious human being to a susceptible vector. Observe that the incidence rate of the disease into the vector population $\Lambda e^{-\hat{\mu}_v T_1} \hat{I}(t - T_1)V_s(t - T_1)$ represents new infectious vectors occurring at time $t$, which became exposed at earlier time $t - T_1$, and surviving natural death over the incubation period $T_1$, with survival probability rate $e^{-\hat{\mu}_v T_1}$, and are infectious at time $t$. The detailed host population dynamics is derived as follows.

At time $t$, it follows from (C) that when susceptible humans $\hat{S}$ and infectious vectors $V_i$ interact with $\hat{\beta}$ effective contacts per vector, per unit time, then under the assumption of homogenous mixing, the incidence rate of the disease into the human population is given by the term $\hat{\beta} \hat{S}(t)V_i(t)$. With the assumption of crowding effects of the vector population, it follows from (C) that the incidence rate of the disease can be written as

$$\hat{\beta} \hat{S}(t)G(V_i(t)), \quad (2.6)$$

where $G$ is the nonlinear incidence function satisfying the conditions in Assumption 2.1.

It follows easily (cf. [18]) from the assumptions (A)-(D), and (2.6) that for $T_j, j = 1, 2, 3$ fixed in the
population, the dynamics of malaria in the human population is given by the system

\[
\begin{align*}
d\hat{S}(t) &= \left[\hat{B} - \hat{\beta} \hat{S}(t)G(V_i(t)) - \hat{\mu} \hat{S}(t) + \hat{\alpha} \hat{I}(t - T) e^{-\hat{\mu} T}\right] dt, \\
d\hat{E}(t) &= \left[\hat{\beta} \hat{S}(t)G(V_i(t)) - \hat{\mu} \hat{E}(t) - \hat{\beta} \hat{S}(t - T_2) e^{-\hat{\mu} T_2} G(V_i(t - T_2))\right] dt, \\
d\hat{I}(t) &= \left[\hat{\beta} \hat{S}(t - T_2) e^{-\hat{\mu} T_2} G(V_i(t - T_2)) - (\hat{\mu} + \hat{\alpha}) \hat{I}(t)\right] dt, \\
d\hat{R}(t) &= \left[\hat{\alpha} \hat{I}(t) - \hat{\mu} \hat{R}(t) - \hat{\alpha} \hat{I}(t - T_3) e^{-\hat{\mu} T_3}\right] dt.
\end{align*}
\]  

Furthermore, the incidence function \(G\) satisfies the conditions in Assumption 2.1. And the initial conditions are given in the following:

\[
\begin{align*}
\hat{S}(t_0) &= \hat{S}_0, \\
\hat{E}(t_0) &= \hat{E}_0, \\
\hat{I}(t_0) &= \hat{I}_0, \\
\hat{R}(t_0) &= \hat{R}_0,
\end{align*}
\]

where \(\mathcal{C}([-T_{\text{max}}, t_0], \mathbb{R}^+)\) is the space of continuous functions with the supremum norm

\[
||\varphi||_{\infty} = \sup_{t \leq t_0} |\varphi(t)|.
\]

It is shown in the following that the vector-host dynamics in \((2.7)-(2.10)\) and \((2.11)-(2.14)\) lead to the malaria model in [18], which omits the dynamics of the vector population, under the assumptions (A)-(D).

Firstly, observe that the system \((2.7)-(2.11)\) satisfies [Theorem 3.1, [18]], and the total human population \(\hat{N}(t) = \hat{S}(t) + \hat{E}(t) + \hat{I}(t) + \hat{R}(t), \forall t \geq t_0\) obtained from system \((2.7)-(2.11)\) with initially condition that satisfies \(N(t_0) \leq \frac{\hat{S}_0}{\hat{\mu}}\), must satisfy

\[
\lim_{t \to \infty} \sup_{t \leq t_0} \hat{N}(t) = \frac{\hat{B}}{\hat{\mu}}.
\]

Therefore, the assumption (B) above, interpreted as \(\frac{\hat{N}(t)}{\hat{V}_0} << 1, \forall t \geq t_0\) implies that

\[
\lim_{t \to \infty} \sup_{t \leq t_0} \hat{N}(t) = \frac{\hat{B}}{\hat{\mu}}, \text{ and } \frac{\left(\frac{\hat{B}}{\hat{\mu}}\right)}{\hat{V}_0} << 1.
\]

Define

\[
\epsilon = \frac{\left(\frac{\hat{B}}{\hat{\mu}}\right)}{\hat{V}_0},
\]

then from \((2.13)-(2.15)\), it follows that \(\epsilon = \frac{\left(\frac{\hat{B}}{\hat{\mu}}\right)}{\hat{V}_0} << 1\).

Employing similar reason in [21], define two natural dimensionless time scales \(\eta\) and \(\varrho\) for the joint vector-host dynamics \((2.3)-(2.5)\) and \((2.7)-(2.11)\) in the following.

\[
\eta = \left(\frac{\hat{B}}{\hat{\mu}}\right) \Lambda t,
\]

\[
\varrho = \hat{V}_0 \Lambda t.
\]
Note that since the total vector population $V_0$ from (B) above is constant, that is, $V_s(t) + V_i(t) = V_0, \forall t \geq t_0$, and from (2.13) and [Theorem 3.1, [18]] the total human $0 < \hat{N}(t) \leq \frac{\mu}{\beta}, \forall t \geq t_0$, whenever $\hat{N}(t_0) \leq \frac{\mu}{\beta}$, then the time scales $\eta$ and $\varrho$ arise naturally to rescale the total vector and maximum total human populations $V_0$ and $\left(\frac{\mu}{\beta}\right)$, respectively, at any time. The time scale $\varrho$ is "fast", and $\eta$ is "slow" (cf. [21]).

Therefore, from above, let
\[
\hat{V}_i(t) = \frac{V_i(t)}{V_0}, \quad \text{and} \quad \hat{V}_s(t) = \frac{V_s(t)}{V_0},
\]
be the dimensionless vector variables, and
\[
S(t) = \frac{\hat{S}(t)}{\left(\frac{\beta}{\mu}\right)}, \quad I(t) = \frac{\hat{I}(t)}{\left(\frac{\beta}{\mu}\right)}, \quad E(t) = \frac{\hat{E}(t)}{\left(\frac{\beta}{\mu}\right)}, \quad R(t) = \frac{\hat{R}(t)}{\left(\frac{\beta}{\mu}\right)} \quad \text{and} \quad N(t) = \frac{\hat{N}(t)}{\left(\frac{\beta}{\mu}\right)},
\]
be the dimensionless human variables. And since $0 < \hat{N}(t) \leq \frac{\mu}{\beta}, \forall t \geq t_0$, whenever $\hat{N}(t_0) \leq \frac{\mu}{\beta}$, it follows from (2.19) that
\[
0 < S(t) + E(t) + I(t) + R(t) = N(t) \leq 1, \forall t \geq t_0.
\]

Applying (2.18)-(2.19) to (2.3)-(2.5) leads to the following
\[
d\hat{V}_i(t) = \epsilon \left[ e^{-\mu \tau_i} I(t-T_1) \hat{V}_s(t-T_1) - \frac{\hat{\mu}}{\Lambda \left(\frac{\beta}{\mu}\right)} \hat{V}_i(t) \right] d\varrho, \quad \text{(2.21)}
\]
\[
d\hat{V}_s(t) = -d\hat{V}_i(t), \quad \text{(2.22)}
\]
\[
1 = \hat{V}_s(t) + \hat{V}_i(t), \forall t \geq t_0, t_0 \geq 0. \quad \text{(2.23)}
\]

Observe from (2.20)-(2.22) that for nonnegative values for the vector variables $\hat{V}_i(t) \geq 0, \hat{V}_s(t) \geq 0, \forall t \geq t_0$, and positive values for the human variables $S(t), E(t), I(t), R(t) > 0, \forall t \geq t_0$, it is follows that
\[
-\epsilon \frac{\hat{\mu}}{\Lambda \left(\frac{\beta}{\mu}\right)} \leq d\hat{V}_i(t) d\varrho \leq \epsilon e^{-\mu \tau_i}, \quad \text{(2.24)}
\]

Thus, on the time scale $\varrho$ which is "fast", it is easy to see from (2.21)-(2.24), that under the assumption that $\epsilon$ from (2.19) is infinitesimally small, that is $\epsilon \to 0$, then
\[
\frac{d\hat{V}_i(t)}{d\varrho} = -\frac{d\hat{V}_s(t)}{d\varrho} = 0, \quad \text{(2.25)}
\]
which implies that the dynamics of $\hat{V}_i$ and $\hat{V}_s$ behaves as in steady state. And thus, it follows from (2.21)-(2.25) that
\[
\hat{V}_i(t) = \frac{e^{-\mu \tau_i}}{\hat{\mu}} \Lambda \left(\frac{\beta}{\mu}\right) I(t-T_1) \hat{V}_s(t-T_1),
\]
\[
1 = \hat{V}_s(t) + \hat{V}_i(t). \quad \text{(2.26)}
\]

It follows further from (2.26) that
\[
\hat{V}_s(t) = \frac{1}{1 + e^{-\mu \tau_i} \Lambda \left(\frac{\beta}{\mu}\right) I(t-T_1) \hat{V}_s(t-T_1)}. \quad \text{(2.27)}
\]
For sufficiently large value of the birth-death rate $\hat{\mu}_v$ (see assumption (B)), such that $\hat{\mu}_v e^{\hat{\mu}_v T_1} >> \Lambda \left( \frac{\hat{\theta}}{\mu} \right)$, then it follows from (2.27) that $\tilde{V}_i(t) \approx 1$, and consequently from (2.28) and (2.18), $V_i(t) \approx V_0$. Moreover, it follows further from (2.27) that 

$$ V_i(t) \approx e^{-\hat{\mu}_v T_1} \frac{\Lambda\left( \frac{\hat{B}}{\mu} \right)}{\hat{\mu}_v} I(t - T_1), $$

(2.28)

and equivalently from (2.18)-(2.19) that (2.28) can be rewritten as follows 

$$ V_i(t) \approx e^{-\hat{\mu}_v T_1} \Lambda V_0 I(t - T_1). $$

(2.29)

While on the fast scale $\eta$ the term $\tilde{I}(t - T_1)$ behaves as the steady state, on the slow scale $\eta$, it is expected to still be evolving. In the following, using (2.18)-(2.19), the dynamics for the human population in (2.7)-(2.11) is nondimensionalized with respect to the slow time scale $\eta$ in (2.16).

Without loss of generality (as it is usually the case e.g. $G(x) = \frac{x}{1+\alpha x}$, $G(x) = \frac{x}{1+\alpha x^2}$), it is assumed that on the $\eta$ timescale, the nonlinear term $G(V_i(\eta))$ expressed as $G(V_0 \tilde{V}_i(\eta))$, can be rewritten from (2.29) as 

$$ G(V_0 \tilde{V}_i(\eta)) = \frac{\Lambda V_0}{\hat{\mu}_v} \tilde{G}(\tilde{V}_i(\eta)) e^{-\hat{\mu}_v T_1}, $$

(2.30)

by factoring a constant term $\frac{\Lambda V_0}{\hat{\mu}_v} \tilde{G}(\tilde{V}_i(\eta))$, and the function $\tilde{G}$ carries all the properties of Assumption 2.1. Thus, from the above and (2.29), the system (2.7)-(2.11) is rewritten in dimensionless form as follows:

$$ dS(\eta) = \left[ B - \beta S(\eta) \tilde{G}(I(\eta - T_1)) e^{-\mu T_1} - \mu S(\eta) + \alpha I(\eta - T_3) e^{-\mu T_3} \right] d\eta, $$

(2.31)

$$ dE(\eta) = \left[ \beta S(\eta) \tilde{G}(I(\eta - T_1)) e^{-\mu T_1} - \mu E(\eta) - \beta S(\eta - T_2) \tilde{G}(I(\eta - T_1 - T_2) e^{-\mu T_1 - \mu T_2} \right] d\eta, $$

(2.32)

$$ dI(\eta) = \left[ \beta S(\eta - T_2) \tilde{G}(I(\eta - T_1 - T_2)) e^{-\mu T_1 - \mu T_2} - \mu I(\eta) - (\mu + d + \alpha) I(\eta) \right] d\eta, $$

(2.33)

$$ dR(\eta) = \left[ \alpha I(\eta) - \mu R(\eta) - \alpha I(\eta - T_3) e^{-\mu T_3} \right] d\eta, $$

(2.34)

where

$$ B = \frac{\hat{B}}{\left( \frac{\hat{\mu}_v}{\mu} \right)^2} \Lambda, \quad \beta = \frac{\hat{\beta} V_0}{\mu} \Lambda, \quad \mu = \frac{\hat{\mu}}{\left( \frac{\hat{\theta}}{\mu} \right) \Lambda}, \quad \alpha = \frac{\hat{\alpha}}{\left( \frac{\hat{\theta}}{\mu} \right) \Lambda} \Lambda $$

$$ \mu_v = \frac{\hat{\mu}_v}{\left( \frac{\hat{\theta}}{\mu} \right) \Lambda}, \quad d = \frac{\hat{d}}{\left( \frac{\hat{\theta}}{\mu} \right) \Lambda}, \quad T_j \eta = \left( \frac{\hat{B}}{\mu} \right) \Lambda T_j, \forall j = 1, 2, 3. $$

(2.35)

The system (2.31)-(2.34) describes the dynamics of malaria on the slow scale $\eta$. Furthermore, moving forward, the analysis of the model (2.31)-(2.34) is considered only on the $\eta$ timescale. To reduce heavy notation, the following substitutions are made. Substitute $t$ for $\eta$, and the delays $T_j, \forall j = 1, 2, 3$ will substitute $T_j \eta, \forall j = 1, 2, 3$. Moreover, since the delays are distributed with density functions $f_j, \forall j = 1, 2, 3$, it follows from (A)-(D), (2.31)-(2.34) and (2.11) that the expected SEIRS model for malaria is
given as follows:

\[
\begin{align*}
    dS(t) &= \left[ B - \beta S(t) \int_{t_0}^{h_1} f_{T_1}(s)e^{-\mu_s G(I(t-s))}ds - \mu S(t) + \alpha \int_{t_0}^{\infty} f_{T_3}(r)I(t-r)e^{-\mu_r dr} \right] dt, \\
    dE(t) &= \left[ \beta S(t) \int_{t_0}^{h_1} f_{T_1}(s)e^{-\mu_s G(I(t-s))}ds - \mu E(t) \\
    &\quad - \beta \int_{t_0}^{h_2} f_{T_2}(u)S(t-u) \int_{t_0}^{h_1} f_{T_1}(s)e^{-\mu_s G(I(t-s-u))}ds du \right] dt, \\
    dI(t) &= \beta \int_{t_0}^{h_2} f_{T_2}(u)S(t-u) \int_{t_0}^{h_1} f_{T_1}(s)e^{-\mu_s G(I(t-s-u))}ds du - (\mu + \alpha)I(t) \right] dt, \\
    dR(t) &= \left[ \alpha I(t) - \mu R(t) - \alpha \int_{t_0}^{\infty} f_{T_3}(r)I(t-r)e^{-\mu_r dr} \right] dt, 
\end{align*}
\]  

(2.36)  

(2.37)  

(2.38)  

(2.39)

where the initial conditions are given in the following: let \( h = h_1 + h_2 \) and define

\[
(S(t), E(t), I(t), R(t)) = (\varphi_1(t), \varphi_2(t), \varphi_3(t), \varphi_4(t)), t \in (-\infty, t_0], \\
\varphi_k \in UC_g^{C}((-\infty, t_0], \mathbb{R}^+), \forall k = 1, 2, 3, 4, \quad \varphi_k(t_0) > 0, \forall k = 1, 2, 3, 4, 
\]

(2.40)

where \( UC_g \) is some fading memory sub Banach space of the Banach space \( C((-\infty, t_0], \mathbb{R}^+) \) endowed with the norm

\[
\|\varphi\|_g = \sup_{t \leq t_0} \frac{|\varphi(t)|}{g(t)},
\]

(2.41)

and \( g \) is some continuous function with the following properties: (P1.) \( g((-\infty, t_0]) \subseteq [1, \infty) \), non-increasing, and \( g(t_0) = 1 \); (P2.) \( \lim_{u \to t_0} \frac{g(t+u)}{g(t)} = 1 \), uniformly on \([t_0, \infty)\); \( \lim_{t \to -\infty} g(t) = \infty \). An example of such a function is \( g(t) = e^{-at}, a > 0 \) (cf. [7]). Note that for any \( g \) satisfying (P1.)-(P2.), the Banach space \( C((-\infty, t_0], \mathbb{R}^+) \) is continuously embedded in \( UC_g \) which allows structural properties for \( C((-\infty, t_0], \mathbb{R}^+) \) with the uniform norm to hold in \( UC_g \) with \( \|\cdot\|_g \) norm. Moreover, \( \varphi \in UC_g, \exists g \) if and only if \( \|\varphi\|_g < \infty \) and \( \frac{|\varphi(t)|}{g(t)} \) is uniformly continuous on \((-\infty, t_0]\). Also, the function \( G \) in (2.36)-(2.39) satisfies the conditions of Assumption 2.1.

Observe (2.36)-(2.39) is similarly structured exactly as [(2.8)-(2.11), [15]]. Furthermore, the equations for \( E \) and \( R \) decouple from (2.36)-(2.39). Therefore, the results are exhibited for the decoupled system (2.36) and (2.38) containing equations for \( S \) and \( I \).

\[
Y(t) = (S(t), E(t), I(t), R(t))^T, X(t) = (S(t), E(t), I(t))^T, \quad \text{and} \quad N(t) = S(t) + E(t) + I(t) + R(t).
\]

(2.42)

Whilst permanence or extinction has been investigated in some delay type systems (cf. [10]-[13], the permanence and extinction in the sense of [13] in systems with multiple random delays is underdeveloped in the literature. Furthermore, as far as we know no other paper has addressed extinction and persistence of malaria in a mosquito-human population dynamics involving delay differential equations in the line of thinking of [10]-[13]. We recall the following definition from [10]-[20].

**Definition 2.1**

1. A population \( x(t) \) is called strongly permanent if \( \lim\inf_{t \to +\infty} x(t) > 0 \);
2. \( x(t) \) is said to go extinct if \( \lim_{t \to +\infty} x(t) = 0 \).
3. $x(t)$ is said to be weakly permanent in the mean if $\limsup_{t \to +\infty} \frac{1}{t} \int_0^t x(s) ds > 0$.
4. $x(t)$ is said to be strongly permanent in the mean if $\liminf_{t \to +\infty} \frac{1}{t} \int_0^t x(s) ds > 0$.
5. $x(t)$ is said to be stable in the mean if $\lim_{t \to \infty} \frac{1}{t} \int_0^t x(s) ds = c > 0$.

3 Model validation results

The consistency results for the system \((2.36)-(2.39)\) are given. Some ideas from [13] using the dimensionless parameters \((2.35)\), are applied to the new model \((2.36)-(2.39)\). Observe from \((2.35)\) that expression $\frac{B}{p}$ simplifies to 1, and this is emphasized as $\frac{B}{p} \equiv 1$.

**Theorem 3.1** For the given initial conditions \((2.40)-(2.41)\), the system \((2.36)-(2.39)\) has a unique positive solution $Y(t) \in \mathbb{R}_+^4$. Moreover,

$$
\limsup_{t \to +\infty} N(t) \leq S^*_0 = \frac{B}{\mu} \equiv 1. \tag{3.1}
$$

Furthermore, there is a positive self invariant space for the system denoted $D(\infty) = B_{\mathbb{R}_+^4}^{(-\infty,\infty)} \left(0, \frac{B}{\mu} \equiv 1\right)$, where $D(\infty)$ is the closed unit ball in $\mathbb{R}_+^4$ centered at the origin with radius $\frac{B}{\mu} \equiv 1$ containing all positive solutions defined over $(-\infty, \infty)$.

**Proof:**

The proof of this result is standard and easy to follow applying the notations \((2.42)\) to the system \((2.36)-(2.39)\).

**Theorem 3.2** The feasible region for the unique positive solutions $Y(t), t \geq t_0$ of the system \((2.36)-(2.39)\) in the phase plane that lie in the self-invariant unit ball $D(\infty) = B_{\mathbb{R}_+^4}^{(-\infty,\infty)} \left(0, \frac{B}{\mu} \equiv 1\right)$ for the system, also lie in a much smaller space $D_{\text{expl}}(\infty) \subset D(\infty)$, where

$$
D_{\text{expl}}(\infty) = \left\{ Y(t) \in \mathbb{R}_+^4 : \frac{B}{\mu + d} \leq N(t) = S(t) + E(t) + I(t) + R(t) \leq \frac{B}{\mu} \forall t \in (-\infty, \infty) \right\}. \tag{3.2}
$$

Moreover, the space $D_{\text{exp}}(\infty)$ is also self-invariant with respect to the system \((2.36)-(2.39)\).

**Proof:**

Suppose $Y(t) \in D(\infty)$, then it follows from \((2.36)-(2.39)\) and \((2.42)\) that the total population $N(t) = S(t) + E(t) + I(t) + R(t)$ satisfies the following inequality

$$
[B - (\mu + d)N(t)]dt \leq dN(t) \leq [B - (\mu)N(t)]dt. \tag{3.3}
$$

It is easy to see from \((3.3)\) that

$$
\frac{B}{\mu + d} \leq \liminf_{t \to +\infty} N(t) \leq \limsup_{t \to +\infty} N(t) \leq \frac{B}{\mu}, \tag{3.4}
$$

and \((3.2)\) follows immediately.

**Remark 3.1** Theorem 3.2 signifies that every solution for \((2.36)-(2.39)\) that starts in the unit ball $D(\infty)$ in the phase plane, oscillates continuously inside $D(\infty)$. Moreover, if the solution oscillates and enters the space $D_{\text{expl}}(\infty) = B_{\mathbb{R}_+^4}^{(-\infty,\infty)} \left(0, \frac{B}{\mu + d} \right) \cap \left( B_{\mathbb{R}_+^4}^{(-\infty,\infty)} \left(0, \frac{B}{\mu} \right) \right) \subseteq D_{\text{expl}}(\infty)$ for all time.

Biologically, observe that $\frac{B}{p}$ and $\frac{B}{\mu + d}$ represent the total births that occur over the average lifespans $\frac{1}{\mu}$ and $\frac{1}{\mu + d}$ of a human being in a malaria-free population and in a malaria- epidemic population, respectively. Thus, Theorem 3.2 signifies that when the population grows and enters a state for the total population $N(t) \in \left[ \frac{B}{\mu + d}, \frac{B}{\mu} \right]$, it stays within that range for all time.
Also, it is easy to see that the system (2.36)-(2.39) has a DFE \( E_0 = (S^*_0, 0, 0) = (\frac{R}{\mu} \equiv 1, 0, 0) = (1, 0, 0) \). The basic reproduction number (BRN) for the disease when the delays in the system \( T_1, T_2 \) and \( T_3 \) are constant, is given by

\[
\hat{R}_0^* = \frac{\beta}{(\mu + d + \alpha)}.
\]

(3.5)

Furthermore, when \( \hat{R}_0^* < 1 \), then \( E_0 = X_0^* = (S^*_0, 0, 0) = (1, 0, 0) \) is asymptotically stable, and the disease can be eradicated from the population. Also, when the delays in the system \( T_i, i = 1, 2, 3 \) are random, and arbitrarily distributed, the BRN is proportional to

\[
R_0 \propto \frac{\beta}{(\mu + d + \alpha)} + \frac{\alpha}{(\mu + d + \alpha)},
\]

(3.6)

And malaria is eradicated from the system, whenever \( R_0 \leq 1 \).

The following result can be made about the nonzero steady state of the dimensionless system (2.36)-(2.39), when Assumption 2.1 is satisfied.

**Theorem 3.3** Let the conditions of Assumption 2.1 be satisfied. Suppose \( R_0 > 1 \) (or \( \hat{R}_0^* > 1 \)) and the expected survival probability rate of the plasmodium satisfies

\[
E(e^{-\mu_i T_i - \mu T_2}) \geq \frac{R_0}{(R_0 - \frac{\alpha}{\mu + d + \alpha}) G'(0)}.
\]

(3.7)

then there exists a nonzero endemic equilibrium \( E_1 = (S^*_1, E^*_1, I^*_1) \) for the the dimensionless system (2.36)-(2.39), where

\[
E(e^{-\mu_i T_i - \mu T_2}) = \int_{t_0}^{t_1} \int_{t_0}^{h_1} e^{-\mu_i s - \mu u} f_{T_2}(u) f_{T_1}(s) ds du.
\]

(3.8)

Proof:
The dimensionless endemic equilibrium \( E_1 = (S^*_1, I^*_1) \) of the decoupled (2.36)-(2.39) is a solution to the following system:

\[
B - \beta E(e^{-\mu_i T_i}) S G(I) - \mu S + \alpha E(e^{-\mu T_2}) I = 0,
\]

(3.9)

\[
\beta E(e^{-\mu_i T_i - \mu T_2}) S G(I) - (\mu + d + \alpha) I = 0.
\]

(3.10)

Solving for \( S \) from (3.10) and substituting the result into (3.9), gives the following equation:

\[
H(I) = 0
\]

(3.11)

where,

\[
H(I) = B - \frac{1}{E(e^{-\mu_i T_i + \mu T_2})} \left[ \frac{(\mu + d + \alpha) \mu}{\beta G(I)} + (\mu + d) E(e^{-\mu_i T_i}) + \alpha E(e^{-\mu_i T_i}) (1 - E(e^{-\mu T_2}) E(e^{-\mu T_3})) \right].
\]

(3.12)

Note that \( 0 < E(e^{-\mu T_i}) \leq 1, i = 1, 2, 3 \), and \( \lim_{I \to \infty} G(I) = C < \infty \), hence for sufficiently large positive value of \( I, H(I) < 0 \). Furthermore, the derivative of \( H(I) \) is given by

\[
H'(I) = - \frac{(\mu + d + \alpha) \mu (G(I) - IG'(I))}{\beta E(e^{-\mu_i T_i - \mu T_2}) G^2(I)} - \frac{1}{E(e^{-\mu_i T_i - \mu T_2})} \left( (\mu + d) E(e^{-\mu_i T_i}) + \alpha E(e^{-\mu_i T_i}) (1 - E(e^{-\mu T_2}) E(e^{-\mu T_3})) \right).
\]

(3.13)

Assume without loss of generality that \( G'(I) > 0 \). It follows from the other properties of \( G \) in Assumption 2.1 that is, \( G(0) = 0, G''(I) < 0, \) that \( (G(I) - IG'(I)) > 0 \) and this further implies
that $H'(I) < 0$ for all $I > 0$. That is, $H(I)$ is a decreasing function over all $I > 0$. Therefore, a positive root of the equation (3.11) requires that $H(0) > 0$. Observe from (3.12) and the dimensionless expressions in (2.35),

$$H(0) = B \left( 1 - \frac{\mu + d + \alpha}{\beta G'(0) E(e^{-\mu T_1 - \mu T_2})} \right) = B \left( 1 - \frac{1}{R_0 - \frac{\alpha}{\mu + d + \alpha}} \right) \frac{G'(0) E(e^{-\mu T_1 - \mu T_2})}{R_0}$$

$$\geq B \left( 1 - \frac{1}{R_0} \right).$$

(3.14)

For $R_0 > 1$, it is easy to see that $H(0) > 0$.

The extinction of disease will be investigated in the neighborhood of the zero steady state $E_0$, and the permanence of disease will be investigated in the neighborhood of $E_1$.

4 Extinction of disease

In this section, the extinction of malaria from the system (2.36)-(2.39) is investigated. Note, the decoupled system (2.36) and (2.38) is used. The following lemma will be used to establish the extinction results.

Lemma 4.1 Let the assumptions of Theorem 3.2 hold, and define the following Lyapunov functional in $D^{exp}(\infty)$,

$$\tilde{V}(t) = V(t) + \beta \left[ \int_{t_0}^{h_2} \int_{t_0}^{h_1} f_{T_2}(u) f_{T_1}(s) e^{-(\mu + d + \alpha)} \int_{t-u}^{t} S(\theta) \frac{G(I(\theta - s))}{I(t)} d\theta ds du + \right.$$

$$\left. + \int_{t_0}^{h_2} \int_{t_0}^{h_1} f_{T_2}(u) f_{T_1}(s) e^{-(\mu + d + \alpha)} \int_{t-s}^{t} S(\theta) \frac{G(I(\theta))}{I(t)} d\theta ds du \right].$$

(4.1)

where $V(t) = \log I(t)$. It follows that

$$\limsup_{t \to \infty} \frac{1}{t} \log (I(t)) \leq \beta \frac{B}{\mu} E(e^{-(\mu T_1 + \mu T_2)}) - (\mu + d + \alpha).$$

(4.2)

Proof:

The differential operator $\dot{V}$ applied to the Lyapunov functional $\tilde{V}(t)$ with respect to the system (2.36) leads to the following

$$\dot{V}(t) = \beta \int_{t_0}^{h_2} f_{T_2}(u) \int_{t_0}^{h_1} f_{T_1}(s) e^{-(\mu + d + \alpha)} S(t) \frac{G(I(t))}{I(t)} ds du - (\mu + d + \alpha)$$

(4.3)

Since $S(t), I(t) \in D^{exp}(\infty)$, and $G$ satisfies the conditions of Assumption 2.1 it follows easily from (4.3) that

$$\dot{V}(t) \leq \beta \frac{B}{\mu} E(e^{-(\mu T_1 + \mu T_2)}) - (\mu + d + \alpha).$$

(4.4)

Now, integrating both sides of (4.4) over the interval $[t_0, t]$, it follows from (4.4) and (4.1) that

$$\log I(t) \leq \tilde{V}(t) \leq \tilde{V}(t_0) + \left[ \beta \frac{B}{\mu} E(e^{-(\mu T_1 + \mu T_2)}) - (\mu + d + \alpha) \right] (t - t_0).$$

(4.5)

Diving both sides of (4.5) by $t$ and taking the limit supremum as $t \to \infty$, it is easy to see that (4.5) reduces to

$$\limsup_{t \to \infty} \frac{1}{t} \log I(t) \leq \left[ \beta \frac{B}{\mu} E(e^{-(\mu T_1 + \mu T_2)}) - (\mu + d + \alpha) \right].$$

(4.6)
And the result (4.12) follows immediately from (4.10).

The extinction conditions for the infectious population over time are expressed in terms - (1) the BRN \( R_0^∗ \) in (3.5), and (2) the expected survival probability rate (ESPR) of the parasites \( E(e^{-(\mu \tau_1 + \mu T_2)}) \), also defined in [Theorem 5.1, [18]].

**Theorem 4.1** Suppose Lemma 4.1 is satisfied, and let the BRN \( R_0^∗ \) be defined as in (3.3). In addition, let one of the following conditions hold

1. \( R_0^∗ \geq 1 \) and \( E(e^{-(\mu \tau_1 + \mu T_2)}) < \frac{1}{R_0^*} \), or
2. \( R_0^∗ < 1 \).

Then

\[
\limsup_{t \to \infty} \frac{1}{t} \log (I(t)) < -\lambda.
\]

where \( \lambda > 0 \) is some positive constant. In other words, \( I(t) \) converges to zero exponentially.

**Proof:** Suppose Theorem 4.1[1.] holds, then from (4.2),

\[
\limsup_{t \to \infty} \frac{1}{t} \log (I(t)) < \beta B \left( E(e^{-(\mu \tau_1 + \mu T_2)}) - \frac{1}{R_0^*} \right) \equiv -\lambda, \tag{4.8}
\]

where the positive constant \( \lambda > 0 \) is taken to be as follows

\[
\lambda \equiv (\mu + d + \alpha) - \beta B \mu E(e^{-(\mu \tau_1 + \mu T_2)}) = \beta B \mu \left( \frac{1}{R_0^*} - E(e^{-(\mu \tau_1 + \mu T_2)}) \right) > 0. \tag{4.9}
\]

Also, suppose Theorem 4.1[2.] holds, then from (4.2),

\[
\limsup_{t \to \infty} \frac{1}{t} \log (I(t)) \leq \beta B \mu E(e^{-(\mu \tau_1 + \mu T_2)}) - (\mu + d + \alpha) < \beta B \mu - (\mu + d + \alpha) = -(1 - R_0^*)(\mu + d + \alpha) \equiv -\lambda, \tag{4.10}
\]

where the positive constant \( \lambda > 0 \) is taken to be as follows

\[
\lambda \equiv (1 - R_0^*)(\mu + d + \alpha) > 0. \tag{4.11}
\]

**Remark 4.1** Theorem 4.1 and Theorem 3.2 signify that all trajectories of \((S(t), I(t))\) of the decoupled system (2.3.10) and (2.3.12), that start in \( D(\infty) \) and grow into \( D^{exp}(\infty) \subset D(\infty) \) remain in \( D^{exp}(\infty) \). Moreover, on the phase plane of \((S(t), I(t))\), the trajectory of the infectious state \(I(t), t \geq t_0\) ultimately turn to zero exponentially, whenever either the ESPR \( E(e^{-(\mu \tau_1 + \mu T_2)}) < \frac{1}{R_0^*} \) for \( R_0^* \geq 1 \), or whenever the BRN \( R_0^∗ < 1 \). Furthermore, the Lyapunov exponent (LE) from (4.7) is estimated by the term \( \lambda \), defined in (4.9) and (4.11).

It follows from (4.7) that when either of the conditions in Theorem 4.1[1.-2.] hold, then the \( I(t) \) state dies out exponentially, whenever \( \lambda \) in (4.9) and (4.11) is positive, that is, \( \lambda > 0 \). In addition, the rate of the exponential decay of each trajectories of \( I(t) \) in each scenario of Theorem 4.1[1.-2.] is given by the estimate \( \lambda > 0 \) of the LE in (4.9) and (4.11).

The conditions in Theorem 4.1[1.-2.] can also be interpreted as follows. Recall, the BRN \( R_0^∗ \) in (3.3) (similarly in (3.6)) represents the expected number of secondary malaria cases that result from one infective placed in the disease free state \( S_0^0 = \frac{B}{\mu} \equiv 1 \). Thus, \( \frac{1}{R_0^*} = \frac{(\mu + \alpha + \gamma)}{\beta S_0^*} \), for \( R_0^* \geq 1 \), represents the probability rate of infectious persons in the secondary infectious population \( \beta S_0^* \) leaving the infectious state, either through natural death \( \mu \), diseases related death \( d \), or recovery and acquiring natural immunity.

\(^{1}\text{Lyapunov exponent}\)
at the rate $\alpha$. Thus, $\frac{1}{R_0}$ is the effective probability rate of surviving infectiousness until recovery with acquisition of natural immunity. Moreover, $\frac{1}{R_0}$ is a probability measure provided $R_0 > 1$.

In addition, recall Theorem 4.1 asserts that when $R_0 > 1$, and the ESPR $E(e^{-(\mu_rT_1+\mu_vT_2)})$ is significantly large, then the outbreak of malaria establishes a malaria endemic steady state population $E_1$. The conditions for extinction of disease in Theorem 4.1, that is $R_0 < 1$ and $E(e^{-(\mu_rT_1+\mu_vT_2)}) < \frac{1}{R_0}$ suggest that in the event where $R_0 > 1$, and the disease is aggressive, and likely to establish an endemic steady state population, if the expected survival probability rate $E(e^{-(\mu_rT_1+\mu_vT_2)})$ of the malaria parasites over their complete life cycle of length $T_1 + T_2$, is less than $\frac{1}{R_0}$, the effective probability rate of surviving infectiousness until recovery with natural immunity, then the malaria epidemic fails to establish an endemic steady state, and as a result, the disease ultimately dies out at an exponential rate $\lambda$ in (5.9).

In the event where $R_0 < 1$ in Theorem 4.1, extinction of disease occurs exponentially over sufficiently long time, regardless of the survival of the parasites. Moreover, the rate of extinction is $\lambda$ in (5.9).

5 Persistence of susceptibility and stability of zero equilibrium

Theorem 4.1 characterizes the behavior of the trajectories of the $I(t)$ coordinate of the solution $(S(t), I(t))$ of the decoupled system (2.36) and (2.38) in the phase plane. The question remains about how the trajectories for the $S(t)$ behave asymptotically in the phase plane.

Using Definition 2.1, the following result describes the average behavior of the trajectories of $S(t)$ over sufficiently long time, and also states conditions for the stability of the disease-free equilibrium (DFE) of the decoupled system $E_0 = (S_0, 0) = (1, 0)$, whenever Theorem 4.1 holds.

**Theorem 5.1** Suppose any of the conditions in the hypothesis of Theorem 4.1 are satisfied. It follows that in $D^{\text{expl}}(\infty)$, the trajectories of the susceptible state $S(t)$ of the decoupled system (2.36) and (2.38), satisfy

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t S(\xi) d\xi = \frac{B}{\mu} \equiv 1.$$  

(5.1)

That is, the susceptible state is strongly persistent over long-time in the mean (see definition of persistence in the mean Definition 2.1). Moreover, it is stable in the mean, and the average value of the susceptible state over sufficiently long time is equal to $S(t) = S_0 = \frac{B}{\mu}$, obtained when the system is in steady state.

**Proof:**

Suppose either of the conditions in Theorem 4.1 hold, then it follows clearly from Theorem 4.1 that for every $\epsilon > 0$, there is a positive constant $K_1(\epsilon) \equiv K_1 > 0$, such that

$$I(t) < \epsilon, \quad \text{whenever } t > K_1.$$  

(5.2)

It follows from (5.2) that

$$I(t - s) < \epsilon, \quad \text{whenever } t > K_1 + h_1, \forall s \in [t_0, h_1].$$  

(5.3)

In $D^{\text{expl}}(\infty)$, define

$$V_1(t) = S(t) + \alpha \int_t^\infty f_{T_1}(r)e^{\mu_r r} \int_{t-r}^t I(\theta) d\theta dr.$$  

(5.4)

The differential operator $\dot{V}_1$ applied to the Lyapunov functional $V_1(t)$ in (5.4) leads to the following

$$\dot{V}_1(t) = g(S, I) - \mu S(t),$$  

(5.5)

where

$$g(S, I) = B - \beta S(t) \int_{t_0}^{h_1} f_{T_1}(s)e^{-\mu_r s}G(I(t - s)) ds + \alpha e^{-\mu T_3} I(t).$$  

(5.6)
Estimating the right-hand-side of (5.5) in $D^{\exp}(\infty)$, and integrating over $[t_0, t]$, it follows from (5.5) - (5.13) that
\[
V_1(t) \leq V_1(t_0) + B(t - t_0) + \int_{t_0}^{t} \alpha I(\xi)d\xi + \int_{t_0}^{t} \alpha I(\xi)d\xi - \mu \int_{t_0}^{t} S(\xi)d\xi, \\
\leq V_1(t_0) + B(t - t_0) + \frac{B}{\mu}(K_1 - t_0) + \alpha(t - K_1)t - \mu \int_{t_0}^{t} S(\xi)d\xi. \tag{5.7}
\]
Thus, dividing both sides of (5.7) by $t$ and taking the limit supremum as $t \to \infty$, it follows that
\[
\limsup_{t \to \infty} \frac{1}{t} \int_{t_0}^{t} S(\xi)d\xi \leq \frac{B}{\mu} + \frac{\alpha}{\mu}. \tag{5.8}
\]
On the other hand, estimating $g(S, I)$ in (5.6) from below and using the conditions of Assumption 2.1 and (5.3), it is easy to see that in $D^{\exp}(\infty)$,
\[
g(S, I) \geq B - \beta S(t) \int_{0}^{h_1} f_T(s)e^{-\mu\alpha(s)}(I(t - s))ds \geq B - \beta \frac{B}{\mu} E(e^{-\mu T_1})u, \forall t > K_1 + h_1, \\
\geq B - \beta \frac{B}{\mu} \epsilon. \tag{5.9}
\]
Moreover, for $t \in [t_0, K_1 + h_1]$, then
\[
g(S, I) \geq B - \beta \left(\frac{B}{\mu}\right)^2. \tag{5.10}
\]
Therefore, applying (5.9) - (5.10) into (5.5), then integrating both sides of (5.5) over $[t_0, t]$, and diving the result by $t$, it is easy to see from (5.5) that
\[
\frac{1}{t}V_1(t) \geq \frac{1}{t}V_1(t_0) + B(1 - \frac{t_0}{t}) - \frac{1}{t} \beta \left(\frac{B}{\mu}\right)^2 (K_1 + h_1 - t_0) - \beta \frac{B}{\mu} \epsilon[1 - \frac{K_1 + h_1}{t}] - \frac{1}{t} \mu \int_{t_0}^{t} S(\xi)d\xi. \tag{5.11}
\]
Observe that in $D^{\exp}(\infty)$, $\lim_{t \to \infty} \frac{1}{t}V_1(t) = 0$, and $\lim_{t \to \infty} \frac{1}{t}V_1(t_0) = 0$. Therefore, rearranging (5.11), and taking the limit infimum of both sides as $t \to \infty$, it is easy to see that
\[
\liminf_{t \to \infty} \frac{1}{t} \int_{t_0}^{t} S(\xi)d\xi \geq \frac{B}{\mu} - \frac{1}{\mu} \beta \frac{B}{\mu} \epsilon. \tag{5.12}
\]
It follows from (5.8) and (5.12) that
\[
\frac{B}{\mu} - \frac{1}{\mu} \beta \frac{B}{\mu} \epsilon \leq \liminf_{t \to \infty} \frac{1}{t} \int_{t_0}^{t} S(\xi)d\xi \leq \limsup_{t \to \infty} \frac{1}{t} \int_{t_0}^{t} S(\xi)d\xi \leq \frac{B}{\mu} + \frac{\alpha}{\mu} \epsilon. \tag{5.13}
\]
Hence, for $\epsilon$ arbitrarily small, the result in (5.1) follows immediately from (5.13).

**Remark 5.1** Theorem 5.1 signifies that the DFE $E_0$ is strongly persistent and stable in the mean by Definition 2.1[3-5]. That is, over sufficiently long time, on average the human population will be in the DFE $E_0$. Thus, the conditions in Theorem 5.1 are sufficient for malaria to be eradicated from the population, when the population is in a steady state.

The next result confirms that not only is the zero equilibrium state of the decoupled system (2.36) and (2.35) $E_0 = (S^*_0, 0) = (1, 0)$ stable and persistent on average over time, but also stable in the sense of Lyapunov.

**Theorem 5.2** Suppose any of the conditions in the hypothesis of Theorem 4.1[1.-2.] are satisfied. Also, suppose the conditions of Theorem 5.1 hold. It follows that in $D^{\exp}(\infty)$, the DFE $E_0 = (S^*_0, 0) = (1, 0)$ is stable in the sense of Lyapunov.
Proof:
It is left to show that every trajectory that starts near $E_0$ remains near $E_0$ asymptotically. Indeed, if the hypothesis of Theorem 4.1 holds, then all trajectories in the phase-plane for the infectious state $I(t)$ converge asymptotically and exponentially to $I_0 = 0$. It is left to show that if the trajectories of the susceptible state $S(t)$ from Theorem 5.1, converge asymptotically in the mean to $S_0^* = \frac{B}{p}$, then they must remain asymptotically near $S_0^*$.

Indeed, if the contrary, there exist a trajectory for $S(t)$ starting near $S_0 = \frac{B}{p} = \equiv 1$ that does not stay near $S_0^* = \frac{B}{p}$ asymptotically, that is, suppose there exists some $\epsilon_0 > 0$ and $\delta(t_0, \epsilon_0) > 0$, such that $\|S(t_0) - S_0^*\| < \delta$, but $\|S(t) - S_0^*\| > \epsilon_0$, $\forall t \geq t_0$, then clearly from (5.1), either

$$S_0^* = \lim_{t \to \infty} \frac{1}{t} \int_{t_0}^{t} S(\xi)d\xi \geq S_0^* + \epsilon_0 \quad \text{or} \quad S_0^* = \lim_{t \to \infty} \frac{1}{t} \int_{t_0}^{t} S(\xi)d\xi \leq S_0^* - \epsilon_0. \quad (5.14)$$

Thus, $\epsilon_0$ must be zero, otherwise (5.14) is a contradiction. Hence, $E_0 = (S_0^*, 0)$ is stable in the sense of Lyapunov.

Remark 5.2: Theorem 5.1, Theorem 4.1 and Theorem 5.1 signify that all trajectories of $S(t), I(t), t \geq t_0$ of the decoupled system (2.36) and (2.38) that start in $D^{exp}(\infty) \subset D(\infty)$ remain bounded in $D^{exp}(\infty)$. Moreover, the trajectories of $I(t), t \geq t_0$ of the solution $S(t), I(t), t \geq t_0$ in phase plane, ultimately turn to zero exponentially, while trajectories of the susceptible state $S(t)$ persist strongly, and converge in the mean to the DFE $S_0^* = \frac{B}{p}$, whenever $S(t) - S_0^* \geq \epsilon_0$, $\forall t \geq t_0$, for $R_0^* \geq 1$, or whenever the basic production number satisfy $R_0 < 1$.

Moreover, from Theorem 5.1, the DFE $S(t), I(t) = E_0 = (S_0^*, 0) = (\frac{B}{p}, 0) = (1, 0)$ is uniformly globally asymptotically stable. Thus, the conditions in Theorem 4.1 are strong disease eradication conditions.

The above observations suggest that over sufficiently long time, the population that remains will be all susceptible malaria-free people, and the population size will be averagely equal to the DFE $S_0^* = \frac{B}{p}$ of (2.36) and (2.38).

6 Permanence of infectivity near nonzero equilibrium

As remarked in Theorem 6.1, when $R_0^*$ in (3.31) satisfies $R_0^* > 1$, the endemic equilibrium of the decoupled system (2.36) and (2.38) exists and is denoted $E_1 = (S_1^*, I_1^*)$. In this section, conditions for $I(t)$ to be strongly persistent (Definition 2.11) in the neighborhood of $E_1$ are given.

Lemma 6.1: Suppose the conditions of Theorem 5.1 and Theorem 3.3 are satisfied, and let the nonlinear incidence function $G$ satisfy the assumptions of Assumption 2.1.

Then every positive solution $(S(t), I(t)) \in D(\infty)$ of the decoupled system (2.36) and (2.38) with initial conditions (2.4) and (2.4) satisfies the following conditions:

$$\liminf_{t \to \infty} S(t) \geq v_1 \equiv \frac{B}{\mu + \beta G(S_0)} \quad \text{and} \quad \liminf_{t \to \infty} I(t) \geq v_2 \equiv qI_1^* e^{-(\mu + \alpha)(\rho + 1)h}, \quad (6.1)$$

where $h = h_1 + h_2$, and $\rho > 0$ is a suitable positive constant, $S_1^* < \min\{S_0, S^\lambda\}$ and $0 < q < \tilde{q} < 1$, given that,

$$\tilde{q} = \frac{B\beta E(e^{-\mu T^1})G(I_1^*) - \mu \alpha E(e^{-\mu T^1})I_1^*}{(B + \alpha E(e^{-\mu T^1})I_1^*) \beta I_1^*}, \quad S^\lambda = \frac{B}{k} (1 - e^{-kh}), \quad (6.2)$$

Proof: Recall (3.3) asserts that for $N(t) = S(t) + E(t) + I(t) + R(t)$, $\limsup_{t \to \infty} N(t) \leq S_0^* = \frac{B}{p}$. This implies that $\limsup_{t \to \infty} S(t) \leq S_0^* \equiv 1$. This further implies that for any arbitrarily small $\epsilon > 0$, there exists a sufficiently large $\Lambda > 0$, such that

$$I(t) \leq S_0^* + \epsilon, \quad \text{whenever,} \quad t \geq \Lambda. \quad (6.3)$$
Without loss of generality, let $\Lambda_1 > 0$ be sufficiently large such that 

\[ t \geq \Lambda \geq \max_{(s, r) \in [t_0, h_1] \times [t_0, \infty)} (\Lambda_1 + s, \Lambda_1 + r). \]

It follows from Assumption 2.1, (2.2) and (2.36) that

\[
\frac{dS(t)}{dt} \geq B - \beta S(t) \int_{h_0}^{h_1} f_{r_1}(s)e^{-\mu s}G(S_0^* + \epsilon)ds - \mu S(t) \geq B - [\mu + \beta G(S_0^* + \epsilon)] S(t). \quad (6.4)
\]

From (6.4) it follows that

\[
S(t) \geq \frac{B}{k_1} - \frac{B}{k_1}e^{-k_1(t-t_0)} + S(t_0)e^{-k_1(t-t_0)},
\]

where $k_1 = \mu + \beta G(S_0^* + \epsilon)$.

It is easy to see from (6.5) that

\[
\liminf_{t \to \infty} S(t) \geq \frac{B}{\mu + \beta G(S_0^* + \epsilon)}. \quad (6.6)
\]

Since $\epsilon > 0$ is arbitrarily small, then the first part of (5.1) follows immediately.

In the following it is shown that $\liminf_{t \to \infty} I(t) \geq \gamma_2$. In order to establish this result, it is first proved that it is impossible that $I(t) \leq qI^*_1$ for sufficiently large $t \geq t_0$, where $q \in (0, 1)$ is defined in the hypothesis. Suppose on the contrary there exists some sufficiently large $\Lambda_0 > t_0 > 0$, such that $I(t) \leq qI^*_1$ for all $t \geq \Lambda_0$. It follows from (2.30) that

\[
S^*_1 = \frac{B + \alpha E(e^{-\mu T_1})I_1^*}{\mu + \beta E(e^{-\mu T_1})G(I_1^*)} = \frac{B}{\mu + \beta E(e^{-\mu T_1})G(I_1^*)} = \frac{B}{\mu + \beta e^{q I^*_1} \frac{B}{\mu + \beta G(qI^*_1)}} \leq \frac{B}{\mu + \beta G(qI^*_1)}.
\]

But, it can be easily seen from (2.30) and (2.38) that

\[
B\beta E(e^{-\mu T_1})G(I_1^*) - \mu \alpha E(e^{-\mu T_1})I_1^* = \frac{\mu (\mu + d + \alpha)}{E(e^{-\mu T_1})S^*_1} \left[ S_0^* - \frac{S^*_1}{(\mu + d + \alpha)} \right] I^*_1.
\]

Therefore, from (6.7), it follows that

\[
S^*_1 < \frac{B}{\mu + \beta I^*_1 q} \leq \frac{B}{\mu + \beta G(qI^*_1)}.
\]

where $0 < q < \bar{q}$, and $\bar{q}$ is defined in (3.2).

For all vector values $(s, r) \in [t_0, h_1] \times [t_0, \infty)$ define

\[
\Lambda_{0,\text{max}} = \max_{(s, r) \in [t_0, h_1] \times [t_0, \infty)} (\Lambda_0 + s, \Lambda_0 + r), \quad (6.10)
\]

It follows from Assumption 2.1 and (2.36) that for all $t \geq \Lambda_{0,\text{max}}$,

\[
S(t) \geq \frac{B}{k} - \frac{B}{k}e^{-k(t-\Lambda_{0,\text{max}})} + S(\Lambda_{0,\text{max}})e^{-k(t-\Lambda_{0,\text{max}})}, \quad (6.11)
\]

where $k$ is defined in (3.2). For $t \geq \Lambda_{0,\text{max}} + \rho h$, where $h = h_1 + h_2$, and $\rho > 0$ is sufficiently large, it follows from (6.11) that

\[
S(t) \geq \frac{B}{k} \left[ 1 - e^{-k(t-\Lambda_{0,\text{max}})} \right] \geq \frac{B}{k} \left[ 1 - e^{-k\rho h} \right] = S^0. \quad (6.12)
\]
Hence, from (6.9) and (6.12), it follows that for some suitable choice of $\rho > 0$ sufficiently large, then
\[ S^\vartriangle > S_1^\vartriangle, \forall t \geq \Lambda_{0,max} + \rho h. \] (6.13)

For $t \geq \Lambda_{0,max} + \rho h$, define
\[
V(t) = I(t) + \beta S_1^* \int_{t_0}^{t} \int_{t_0}^{h_1} f_T(u) f_T_1(s) e^{-\mu(s+u)} \int_{t-s}^{t} G(I(v-u))dvdsdu \\
+ \beta S_1^* \int_{t_0}^{h_2} \int_{t_0}^{h_1} f_T(u) f_T_1(s) e^{-\mu(s+u)} \int_{t-s}^{h_1} G(I(v))dvdsdu.
\] (6.14)

It is easy to see from system (2.36)-(2.38), and (6.14) that differentiating $V(t)$ with respect to the system (2.36) and (2.38), leads to the following
\[
\dot{V}(t) = \beta \int_{t_0}^{t} \int_{t_0}^{h_2} f_T(u) f_T_1(s) e^{-\mu(s+u)} G(I(t-s-u))[S(t-s) - S_1^*]dsdu \\
+ \left[ \beta S_1^* E(e^{-\mu(T_1+T_2)}) \frac{G(I(t))}{I(t)} - (\mu + d + \alpha) \right] I(t).
\] (6.15)

For all $t \geq \Lambda_{0,max} + \rho h + h > \Lambda_{0,max} + \rho h + h_2$, it follows from (2.1), (6.13) and (2.36)- (2.38) that
\[
\dot{V}(t) \geq \beta \int_{t_0}^{t} \int_{t_0}^{h_2} f_T(u) f_T_1(s) e^{-\mu(s+u)} G(I(t-s-u))[S^\vartriangle - S_1^*]dsdu \\
+ \left[ \beta S_1^* E(e^{-\mu(T_1+T_2)}) \frac{G(I(t))}{I(t)} - (\mu + d + \alpha) \right] I(t) \\
= \beta \int_{t_0}^{t} \int_{t_0}^{h_1} f_T(u) f_T_1(s) e^{-\mu(s+u)} G(I(t-s-u))[S^\vartriangle - S_1^*]dsdu.
\] (6.16)

Observe that the union of the subintervals $\bigcup_{(s,u) \in [t_0,h_1] \times [t_0,h_2]} [t_0 - (s+u),t_0] = [t_0 - h_1,t_0]$, where $h = h_1 + h_2$. Denote the following
\[ i_{min} = \min_{\theta \in [t_0-h_1,t_0], (s,u) \in [t_0,h_1] \times [t_0,h_2]} I(\Lambda_{0,max} + \rho h + h + s + u + \theta). \] (6.17)

Note that (6.17) is equivalent to
\[ i_{min} = \min_{\theta \in [0,t_0-h_1]} I(\Lambda_{0,max} + \rho h + h + \theta). \] (6.18)

It is shown in the following that $I(t) \geq i_{min}, \forall t \geq \Lambda_{0,max} + \rho h + h \geq \Lambda_{0,max} + \rho h + u$, $\forall u \in [t_0, h_2]$. Suppose on the contrary there exists $\tau_1 \geq 0$ such that $I(t) \leq i_{min}$ for all $t \in [\Lambda_{0,max} + \rho h + h, \Lambda_{0,max} + \rho h + h + h_1] \supset [\Lambda_{0,max} + \rho h + h, \Lambda_{0,max} + \rho h + h + s + u + \tau_1], \forall (s,u) \in [t_0, h_1] \times [t_0, h_2]$
\[ I(\Lambda_{0,max} + \rho h + h + h + \tau_1) = i_{min}, \text{ and } \dot{I}(\Lambda_{0,max} + \rho h + h + h + \tau_1) \leq 0. \] (6.19)

For the value of $t = \Lambda_{0,max} + \rho h + h + h_1 + \tau_1$, it follows that $S(t-u) > S^\vartriangle > S_1^\vartriangle$, and $t - s - u \in [\Lambda_{0,max} + \rho h + h, \Lambda_{0,max} + \rho h + h + h + \tau_1], \forall (s,u) \in [t_0, h_1] \times [t_0, h_2]$, and it can be further seen from (2.36) and (2.38), (6.18) and (2.1) that
\[
\dot{I}(t) \geq \beta E(e^{-\mu(T_1+T_2)}) \frac{G(i_{min})}{I(t)} S^\vartriangle - (\mu + d + \alpha) i_{min} = \left[ \beta E(e^{-\mu(T_1+T_2)}) \frac{G(i_{min})}{i_{min}} S^\vartriangle - (\mu + d + \alpha) \right] i_{min} \\
> \left[ \beta E(e^{-\mu(T_1+T_2)}) \frac{G(i_{min})}{I(t)} S_1^* - (\mu + d + \alpha) \right] i_{min} = 0.
\] (6.20)

But (6.20) contradicts (6.19). Therefore, $I(t) \geq i_{min}, \forall t \geq \Lambda_{0,max} + \rho h + h \geq \Lambda_{0,max} + \rho h + u + s, \forall (s,u) \in [t_0, h_1] \times [t_0, h_2]$. 

It follows further from (6.15)-(6.17), and the Assumption 2.1 that for all \( t \geq \Lambda_{0,\max} + \rho h + h \geq \Lambda_{0,\max} + \rho h + h + s, \forall (s, u) \in [t_0, h_1] \times [t_0, h_2]. \)

\[
V(t) \geq \beta \int_{t_0}^{h_1} \int_{t_0}^{b_1} f_{T_2}(u) f_{T_1}(s) e^{-\mu(s+u)} G(I(t-s-u))[S^\Delta - S^*_1] ds du
> \beta E(e^{-\mu(T_1+T_2)}) G(i_{\text{min}})(S^\Delta - S^*_1) > 0. \quad (6.21)
\]

From (6.21), it implies that \( \limsup_{t \to \infty} V(t) = +\infty. \)

On the contrary, it can be seen from (6.1) that \( \limsup_{t \to \infty} N(t) \leq S_0^* = \frac{B}{\mu}, \) which implies that \( \limsup_{t \to \infty} I(t) \leq S_0^* = \frac{B}{\mu}. \)

This further implies that for every \( \epsilon > 0 \) infinitesimally small, there exists \( \tau_2 > 0 \) sufficiently large such that \( I(t) \leq S_0^* + \epsilon, \forall t \geq \tau_2. \)

It follows that from Assumption 2.1 that

\[
G(I(t-s-u)) \leq G(I(v-u)) \leq G(I(t-u)) \leq G(I(t)) \leq G(S_0^* + \epsilon), \forall v \in [t-s,t], (s, u) \in [t_0, h_1] \times [t_0, h_2].
\]

From (6.22), it follows that

\[
\limsup_{t \to \infty} G(I(t-s-u)) \leq \limsup_{t \to \infty} G(I(t)) \leq G(S_0^*). \quad (6.23)
\]

It is easy to see from (6.14) and (6.23) that

\[
\limsup_{t \to \infty} V(t) \leq S_0^* + \beta S_0^* G(S_0^*) E\left((T_1 + T_2)e^{-\mu(T_1+T_2)}\right) < \infty. \quad (6.24)
\]

Therefore, it is impossible that \( I(t) \leq q I^*_1 \) for sufficiently large \( t \geq t_0 \), where \( q \in (0,1). \)

Hence, the following are possible, (Case(i.)) \( I(t) \geq q I^*_1 \) for all \( t \) sufficiently large, and (Case(ii.)) \( I(t) \) oscillates about \( q I^*_1 \) for sufficiently large \( t. \)

Obviously, we need show only Case(ii.). Suppose \( t_1 \) and \( t_2 \) are sufficiently large values such that

\[
I(t_1) = I(t_2) = q I^*_1, \quad \text{and} \quad I(t) < q I^*_1, \forall (t_1, t_2). \quad (6.25)
\]

If for all \( (s, u) \in [t_0, h_1] \times [t_0, h_2], t_2 - t_1 \leq \rho h + h, \) where \( h = h_1 + h_2, \) observe that \( [t_1, t_1 + \rho h + s + u] \subseteq [t_1, t_1 + \rho h + h], \) and it is easy to see from (2.36) by integration that

\[
I(t) \geq I(t_1) e^{-(\mu+d+\alpha)(t-t_1)} \geq q I^*_1 e^{-(\mu+d+\alpha)(t+1)h} = v_1. \quad (6.26)
\]

If for all \( (s, u) \in [t_0, h_1] \times [t_0, h_2], t_2 - t_1 > \rho h + h, \) then it can be seen easily that \( I(t) \geq v_2, \)

for all \( t \in [t_1, t_1 + \rho h + s + u] \subseteq [t_1, t_1 + \rho h + h]. \)

Now, for each \( t \in (\rho h + h, t_2) \supseteq (\rho h + s + u, t_2), \) \( \forall (s, u) \in [t_0, h_1] \times [t_0, h_2], \) one can also claim that \( I(t) \geq v_2. \)

Indeed, as similarly shown above, suppose on the contrary for all \( (s, u) \in [t_0, h_1] \times [t_0, h_2], \exists T^* > 0 \) such that \( I(t) \geq v_2, \forall t \in [t_1, t_1 + \rho h + h + T^*] \supseteq [t_1, t_1 + \rho h + s + u + T^*] \)

\[
I(t_1 + \rho h + h + T^*) = v_2, \quad \text{but} \quad I(t_1 + \rho h + h + T^*) \leq 0. \quad (6.27)
\]

It follows from (2.36)-(2.38) and (2.4) that for the value of \( t = t_1 + \rho h + h + T^*, \)

\[
I(t) \geq \beta E(e^{-\mu(T_1+T_2)}) G(v_2) [S^\Delta - (\mu + d + \alpha)] v_2 > \beta E(e^{-\mu(T_1+T_2)}) \frac{G(v_2)}{v_2} S^*_1 - (\mu + d + \alpha) v_2 = 0. \quad (6.28)
\]

Observe that (6.28) contradicts (6.27). Therefore, \( I(t) \geq v_2, \) for \( t \in [t_1, t_2]. \) And since \( [t_1, t_2] \) is arbitrary, it implies that \( I(t) \geq v_2 \) for all sufficiently large \( t. \) Therefore (5.1) is satisfied.

Theorem 6.1 If the conditions of Lemma 6.1 are satisfied, then the system (2.36)-(2.38) is strongly permanent for any total delay time \( h = h_1 + h_2 \) according to Definition 2.1.
Remark 6.1

1. It can be seen from Lemma [6.7] that when \( \beta = 0 \), then \( v_1 = \frac{B}{\rho} \). That is, when disease transmission stops, then asymptotically, the smallest total susceptible that remains are new births over the average lifespan \( \frac{1}{\rho} \) of the population, equivalent to the DFE \( S_0^* = \frac{B}{\rho} \equiv 1 \). Also, as \( \beta \to \infty \), then the total susceptible that remains \( v_1 \to 0^+ \). That is, as disease transmission rises, even the new births are either infected, or die from natural or disease related causes over time.

2. From (6.1), observe that \( e^{-\beta} \) stops, then asymptotically, the smallest total susceptible that remains are new births over the average lifespan \( \frac{1}{\rho} \) of the population, equivalent to the DFE \( S_0^* = \frac{B}{\rho} \equiv 1 \). Also, as \( \beta \to \infty \), then the total susceptible that remains \( v_1 \to 0^+ \). That is, as disease transmission rises, even the new births are either infected, or die from natural or disease related causes over time.

\[ \text{Remark 6.1} \]

Remark 6.1

The question of what conditions the population ever gets extinct in time is answered from [1.] & [2.]. From (6.1), observe that \( e^{-\beta} \) stops, then asymptotically, the smallest total susceptible that remains are new births over the average lifespan \( \frac{1}{\rho} \) of the population, equivalent to the DFE \( S_0^* = \frac{B}{\rho} \equiv 1 \). Also, as \( \beta \to \infty \), then the total susceptible that remains \( v_1 \to 0^+ \). That is, as disease transmission rises, even the new births are either infected, or die from natural or disease related causes over time.

3. From (6.1), observe that \( e^{-\beta} \) stops, then asymptotically, the smallest total susceptible that remains are new births over the average lifespan \( \frac{1}{\rho} \) of the population, equivalent to the DFE \( S_0^* = \frac{B}{\rho} \equiv 1 \). Also, as \( \beta \to \infty \), then the total susceptible that remains \( v_1 \to 0^+ \). That is, as disease transmission rises, even the new births are either infected, or die from natural or disease related causes over time.

### 7 Example: Application to P. vivax malaria

In this section, the extinction and persistence results are exhibited for the P. vivax malaria example in Wanduku [19]. This is accomplished by examining the trajectories of the decoupled system (2.30) and (2.32), relative to the zero and endemic equilibria. To conserve space, we recall the dimensionless parameters in [Table 1, [19], page 3793] given in Table 1 and the reader is referred to [19] for detailed description of the P. vivax malaria scenario.

The dimensional estimates for the parameters of the malaria model given in [(a.)-(e.), [19], page 3792] are applied to (2.30) to find the dimensionless parameters for the model (2.30)-(2.32) given in Table 1.

| Disease transmission rate | \( B \) | Subsection [2.1] (0.2146383), Subsection [7.2] (0.2146383) |
|---------------------------|-------|---------------------------------|
| Constant Birth rate      | \( B \) | 8.476678e-06                     |
| Recovery rate             | \( \alpha \) | 0.08571429                      |
| Disease death rate        | \( d \) | 0.0001761252                     |
| Natural death rate        | \( \mu, \mu_e \) | 8.476678e-06,42.85714          |
| Incubation delay in vector | \( T_1 \) | 0.105                           |
| Incubation delay in host   | \( T_2 \) | 0.175                           |
| Immunity delay time       | \( T_3 \) | 2.129167                        |

Moreover, the Euler approximation scheme is used to generate trajectories for the different states \( S(t), E(t), I(t), R(t) \) over the time interval \([0,1000]\) days. The special nonlinear incidence functions \( G(I) = \frac{\beta}{1+I}, \alpha_1 = 0.05 \) in [13] is utilized. Furthermore, the following initial fractions of susceptible, exposed, infectious and removed individuals in the initial population size \( \hat{N}(t_0) = 65000 \) are used:

\[ S(t) = 10/23 \approx 28261/65000, \quad E(t) = 5/23 \approx 14131/65000, \quad I(t) = 6/23 \approx 16957/65000, \quad R(t) = 2/23 \approx 5653/65000, \forall t \in [-T, 0], T = \max(T_1 + T_2, T_3) = 2.129167. \]
Recall Section 3 asserts that the endemic equilibrium $E_1$ exists, whenever the BRN $R^*_0 > 1$, where $R^*_0$ is defined in (5.3). Thus, it follows that when $R^*_0 > 1$, the endemic equilibrium $E_1 = (S_1^*, E_1^*, I_1^*)$ satisfies the following system

$$
B - \beta Se^{-\mu T_1} G(I) - \mu S + \alpha I e^{-\mu T_3} = 0, \beta Se^{-\mu T_1} G(I) - \mu E - \beta S e^{-(\mu + T_1 + T_2)} G(I) = 0,
\beta S e^{-(\mu + T_1 + T_2)} G(I) - (\mu + d + \alpha)I = 0, \alpha I - \mu R - \alpha I e^{-\mu T_3} = 0.
$$

(7.2)

For the given set of dimensionless parameter estimates in Table 1, the BRN in (3.5) in this scenario is

$$
R_0^* = \frac{\beta S_0}{\mu} = 0.2498732 < 1.\text{ Therefore, } E_0 \text{ is stable, and the endemic equilibrium } E_1 = (S_1^*, E_1^*, I_1^*) \text{ fails to exist.}
$$

7.1 Example for extinction of disease

For the given set of dimensionless parameter estimates in Table 1, where $\beta = 0.02146383$, from (5.3) the BRN is $R^*_0 = 0.2498732 < 1$. Therefore, $E_0$ is stable, and the endemic equilibrium $E_1 = (S_1^*, E_1^*, I_1^*)$ fails to exist.

Figure 1: (a-1), and (b-1), show the trajectories of the states $(S, I)$, respectively, over sufficiently long time $t \in [0, 1000]$, whenever the intensity of the incidence of malaria is $\alpha = 0.05$. The BRN in (3.5) in this case is $R^*_0 = 0.2498732 < 1$, the estimate of the LE, or rate of extinction of the disease in (4.11) is $\lambda = 0.06443506 > 0$.

Figure 1 verifies the results about the extinction of the $I(t)$ state over time in Theorem 4.1 and the persistence of the $S(t)$ state over time in Theorem 5.1. Indeed, it is observed that for the given parameter values in Table 1 and the initial conditions in (7.2), the BRN in (3.5) in this scenario is $R^*_0 = 0.2498732 < 1$. Therefore, the condition of Theorem 4.1(a.) and Theorem 5.1 are satisfied, and from (4.11), the estimate of the rate of extinction of the malaria population $I(t)$ is $\lambda = 0.06443506 > 0$. That is,

$$
\limsup_{t \to \infty} \frac{1}{t} \log (I(t)) \leq -\lambda = -0.06443506.
$$

(7.3)

The Figure 1(b-1) confirms that over sufficiently large time, when $\lambda > 0$, then the infectious state approaches zero, that is, $\lim_{t \to \infty} I(t) = 0$. Furthermore, the BRN $R^*_0 = 0.2498732 < 1$, signifies that the disease is getting eradicated from the population over time. This is confirmed by Figure 1(a-1), where $S(t)$ appears to be rising over time, and approaching the DFE state $S_0^* = \frac{B}{\mu} = 1$, that is, $\lim_{t \to \infty} S(t) = 1$.

7.2 Persistence of malaria

For the given set of dimensionless parameter estimates in Table 1, when $\beta = 7.941616$, from (5.5) the BRN becomes $R^*_0 = 92.45307 > 1$. Therefore, the DFE $E_0 = (S_0^*, 0, 0) = (1, 0, 0)$ becomes unstable, and the endemic equilibrium exists, and given as $E_1 = (S_1^*, E_1^*, I_1^*) = (6.281296e - 06, 0.000640553, 0.04550565)$. It can be shown from (4.11) that for some suitable choice of $q \in (0, 1)$ and $\rho > 0$, for $t \in [0, 1000]$,

$$
\liminf_{t \to \infty} S(t) = 0.4806943 >> v_1 \equiv \frac{B}{\mu + \beta G(S_0)} = 4.269316e - 05,
\liminf_{t \to \infty} I(t) = 0.2360496 >> v_2 \equiv qI_1^* e^{-(\mu + d + \alpha)(\rho + 1)h} = 0.04550565qe^{-0.02405169(1+\rho)},
$$

(7.4)
Hence, from Theorem 6.1 there is a significant number of infectious people present over time \([0, 1000]\), and as a result malaria persists in the population over time. These facts are further illustrated by Figure 2 over \([0, 1000]\).

Figure 2: (a-2), and (b-2), show the trajectories of the states \((S, I)\), respectively, over sufficiently long time \(t \in [0, 1000]\), whenever the intensity of the incidence of malaria is \(a = 0.05\). The BRN in (3.5) is \(R_0^* = 92.45307 < 1\). \(E(e^{-(\mu T_1 + \mu T_2)}) = 0.01110898 > 0.0108163 = \frac{1}{R_0^*}\). Hence, Theorem 4.1 and Theorem 5.1 fail. The endemic equilibrium \(E_1 = (S_1^*, E_1^*, I_1^*)\) exists and Theorem 6.1 holds from (7.4).

8 conclusion

The vector-human population dynamic models are derived. The models have a general nonlinear incidence rate. The extinction and persistence of the vector-borne disease in the SEIRS epidemic models are studied. Numerical simulation results are given to confirm the results.

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