MODULI SPACES AND MULTIPLE POLYLOGARITHM MOTIVES

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Abstract. In this paper, we give a natural construction of mixed Tate motives whose periods are a class of iterated integrals which include the multiple polylogarithm functions. Given such an iterated integral, we construct two divisors \( A \) and \( B \) in the moduli spaces \( \overline{M}_{0,n} \) of \( n \)-pointed stable curves of genus 0, and prove that the cohomology of the pair \( (\overline{M}_{0,n} - A, B - B \cap A) \) is a framed mixed Tate motive whose period is that integral. It generalizes the results of A. Goncharov and Yu. Manin for multiple \( \zeta \)-values. Then we apply our construction to the dilogarithm and calculate the period matrix which turns out to be same with the canonical one of Deligne.

1. Introduction

1.1. Multiple Polylogarithms. The multiple polylogarithm functions were defined in Goncharov’s paper [11] as the following power series:

\[
Li_{n_1\ldots n_m}(x_1, \ldots, x_m) = \sum_{0<k_1<k_2< \cdots <k_m} \frac{x_1^{k_1}x_2^{k_2} \cdots x_m^{k_m}}{k_1^{n_1}k_2^{n_2} \cdots k_m^{n_m}}
\]

where the \( x_i \) are in the unit disk of the complex plane for \( i = 1, \ldots, m \) and \( n_1 \geq 1, \ldots, n_m-1 \geq 1, n_m \geq 2 \) are positive integers. For \( m = 1 \), we get the classical \( n \)-th polylogarithm which was first introduced by Leibniz [18] in 1696:

\[
Li_n(z) = \sum_{k=1}^{+\infty} \frac{z^k}{k^n}, \quad |z| \leq 1
\]

And for \( x_1 = \cdots = x_m = 1 \), we obtain the multiple \( \zeta \)-values which were first studied by Euler [3]:

\[
\zeta(n_1, \ldots, n_m) = \sum_{0<k_1<k_2< \cdots <k_m} \frac{1}{k_1^{n_1}k_2^{n_2} \cdots k_m^{n_m}}.
\]

Moreover, multiple polylogarithms can be represented as iterated integrals. Recall that iterated integrals are defined as follows. Let \( \omega_1, \ldots, \omega_n \) be smooth one-forms on a manifold \( M \) and \( \gamma : [0, 1] \rightarrow M \) be a piecewise smooth path. Then we define inductively as follows:

\[
\int_\gamma \omega_1 \circ \cdots \circ \omega_n := \int_0^1 (\int_\gamma \omega_1 \circ \cdots \circ \omega_{n-1}) \gamma^* \omega_n
\]
where $\gamma_t$ is the restriction of $\gamma$ on $[0, t]$ and $\int_{[0, t]} \omega_1 \circ \cdots \circ \omega_{n-1}$ is a function of $t$ on $[0, 1]$. More explicitly, it can be computed in the following way:

$$
\int \omega_1 \circ \cdots \circ \omega_n = \int_{0 \leq t_1 \leq \cdots \leq t_n \leq 1} f_1(t_1) dt_1 \wedge \cdots \wedge f_n(t_n) dt_n
$$

where $f_i(t) dt = \gamma^* \omega_i$ are the pullback one-forms on $[0, 1], i = 1, \ldots, n$. For example,

$$
\zeta(2) = \sum_{n=1}^{+\infty} \frac{1}{n^2} = \int_0^1 \frac{dt}{1-t} = \int_{0 \leq t_1 \leq t_2 \leq 1} \frac{dt_1}{1-t_1} \wedge \frac{dt_2}{t_2}
$$

In [12 Chap2], the following formula was proved:

$$
Li_{n_1, \ldots, n_m}(x_1, \ldots, x_m) = (-1)^m \int_0^1 \frac{dt}{t-(x_1 \ldots x_m)^{-1}} \circ \frac{dt}{t} \circ \cdots \circ \frac{dt}{t} \circ \cdots \circ \frac{dt}{t}.
$$

This formula also provides the analytic continuation of multiple polylogarithms.

1.2. **Moduli Spaces.** We denote by $\overline{\mathcal{M}}_{0,S}$ the moduli space of $S$-labeled pointed stable curves of genus 0, where $S$ is a finite set. It’s been studied by Grothendieck [3], Deligne, Mumford, Knudsen [22] and many others. It is defined over $\mathbb{Z}$. Roughly speaking, a complex point of $\overline{\mathcal{M}}_{0,S}$ is a tree of complex projective lines with $|S|$ distinct smooth points marked by the set $S$. Here $|S|$ denotes the cardinality of the set $S$. We know that it is a smooth irreducible projective variety of complex dimension $|S| - 3$. Moreover, $\overline{\mathcal{M}}_{0,S}(\mathbb{C})$ provides a natural compactification of the space $\mathcal{M}_{0,S}(\mathbb{C})$ of $|S|$ distinct points on $\mathbb{C}P^1$ modulo automorphisms of $\mathbb{C}P^1$. By cross-ratio, $\mathcal{M}_{0,S}(\mathbb{C})$ is isomorphic to

$$
\{(x_1, \ldots, x_n) \in (\mathbb{C}P^1)^n | x_i \neq x_j, i \neq j ; x_k \neq 0, 1, \infty, k = 1, \ldots, n\}, \ n = |S| - 3.
$$

The boundary $\partial \overline{\mathcal{M}}_{0,S} := \overline{\mathcal{M}}_{0,S} - \mathcal{M}_{0,S}$ is a normal crossing divisor. It can be described by the combinatorial data of the set $S$. For more detail, see section 2. Now given any subset $S_1 \subset S$, with $|S_1| \geq 3$, there is a contraction morphism:

$$
\pi_{S_1} : \overline{\mathcal{M}}_{0,S} \to \overline{\mathcal{M}}_{0,S_1}
$$

which contracts stably all sections but those marked by $S_1$. In particular, for any $s_0 \in S$, let $S' = S \setminus \{s_0\}$, the contraction morphism

$$
\pi : \overline{\mathcal{M}}_{0,S} \to \overline{\mathcal{M}}_{0,S'}
$$

is the universal $S'$-labeled curve with universal sections $\sigma_i$, for each $i \in S'$. For more information and proofs of $\overline{\mathcal{M}}_{0,S}$, we refer to [22] [20] [24].

If the set $S = \{1, 2, \ldots, n\}$, we’ll denote this space by $\overline{\mathcal{M}}_{0,n}$. And $\overline{\mathcal{M}}_{0,S}$ is non-canonically isomorphic to $\overline{\mathcal{M}}_{0,|S|}$. From now on, we’ll take $S = \{0, s_1, s_2, \ldots, s_n, 1, \infty\}$ and fix the cyclic order $\rho : 0 < s_1 < s_2 < \cdots < s_n < 1 < \infty < 0$ on $S$ unless otherwise stated.
1.3. **Main results.** In [10], for each multiple \( \zeta \)-value (1.3), the authors construct two divisors \( A \) and \( B \) of \( \overline{\mathcal{M}}_{0,S} \) and then show that

\[
H^n(\overline{\mathcal{M}}_{0,S} - A, B - B \cap A)
\]

is a framed mixed Tate motive whose period is this value. In the end of that paper, they suggest to generalize their results to the following convergent iterated integral:

\[
I_\gamma(a_1, \ldots, a_n) := \int_{\gamma} \frac{dt}{t - a_1} \circ \cdots \circ \frac{dt}{t - a_n} \quad a_1 \neq 0, a_n \neq 1
\]

where \( \gamma : [0, 1] \to \mathbb{C} \) is a piecewise smooth simple path from 0 to 1 and \( a_i \notin \gamma((0,1)), i = 1, \ldots, n \). In particular, by the formula (1.4), multiple polylogarithms are of this type. In this paper, we show that the analogous results hold for the iterated integral (1.5).

In section 2, we review the basic combinatorial facts about the boundary divisors of \( \overline{\mathcal{M}}_{0,S} \) and the stable 2-partitions of the set \( S \). Next, we briefly recall the divisor \( B_n \) in \( \overline{\mathcal{M}}_{0,S} \) which was introduced in [10], and then prove some interesting combinatorial properties of \( B_n \). In the end, we proceed to study in detail some non-boundary divisors of \( \overline{\mathcal{M}}_{0,S} \) which we’ll use later on.

In section 3, for the integral (1.5), we define a meromorphic differential form \( \Omega_S(\vec{a}) \) of \( \overline{\mathcal{M}}_{0,S}(\mathbb{C}) \). Let \( A_S(\vec{a}) \) be its divisor of singularities in \( \overline{\mathcal{M}}_{0,S}(\mathbb{C}) \). We explicitly determine the divisor \( A_S(\vec{a}) \). Then we use it to prove the key proposition that the divisor \( A_S(\vec{a}) \) does not contain any \( k \)-dimensional face of the divisor \( B_n, 0 \leq k \leq n \).

In section 4, we review the definitions of framed Hodge-Tate structure and its period, and discuss their basic properties. Finally combining with all the information of \( A_S(\vec{a}) \) and \( B_n \) in section 3 and 4, we can prove:

**Theorem 1.** Let \( \vec{a} = (a_1, \ldots, a_n) \). For the iterated integral \( I_\gamma(a_1, \ldots, a_n) \) of (1.5), \( a_i \in \mathbb{C}, a_1 \neq 0, a_n \neq 1 \), there exists two divisors \( A_S(\vec{a}) \) and \( B_n \) in \( \overline{\mathcal{M}}_{0,S} \), \( |S| = n + 3 \), such that

\[
H^n(\overline{\mathcal{M}}_{0,S} - A_S(\vec{a}), B_n - B_n \cap A_S(\vec{a})) \quad (**)
\]

carries an \( n \)-framed Hodge-Tatestructure with two canonical frames

\[
[\Omega_S(\vec{a})] \in Gr_2 \Lambda H^n(\overline{\mathcal{M}}_{0,S} - A_S(\vec{a})); \quad [\Delta_B(\gamma)] \in (Gr_0 \Lambda H^n(\overline{\mathcal{M}}_{0,S}, B_n))^\vee.
\]

and the period with respect to these frames is exactly the iterated integral \( I_\gamma(a_1, \ldots, a_n) \), where \( [\Omega_S(\vec{a})] \) is a meromorphic n-form on \( \overline{\mathcal{M}}_{0,S} \), and \( \Delta_B(\gamma) \) is a relative n-cycle.

Furthermore, if the \( a_i \) are elements of a number field \( F, i = 1, \ldots, n \), then (**) is a framed mixed Tate motive over \( F \).

In section 5, we apply our construction to the dilogarithm. Namely, we consider the following integral:

\[
Li_2(z) = -\int_{0 \leq t_1 \leq t_2 \leq 1} \frac{dt_1}{t_1 - z^{-1}} \wedge \frac{dt_2}{t_2}
\]

By the Theorem 1 above, \( H^2(\overline{\mathcal{M}}_{0.5} - A(\vec{a}), B_2 - B_2 \cap A(\vec{a})) \) carries a 2-framed Hodge-Tate structure. We calculate it and prove that:

**Theorem 2.** The mixed Hodge structure given in Theorem 1 for the dilogarithm, that is, \( H^2(\overline{\mathcal{M}}_{0.5} - A(\vec{a}), B_2 - B_2 \cap A(\vec{a})) \), is isomorphic to the one given by P. Deligne. And
for our case the period matrix is the following: 
If $z \neq 0, 1$, it equals

$$
egin{bmatrix}
1 & 0 & 0 \\
-Li_1(z) & 2\pi i & 0 \\
-Li_2(z) & 2\pi i \log z & (2\pi i)^2 \\
\end{bmatrix}
$$

If $z = 1$, it is

$$
egin{bmatrix}
1 & 0 \\
-Li_2(1) & (2\pi i)^2 \\
\end{bmatrix}
$$

With above Theorem 2, the interesting question is that for the classical $n$-th polylogarithm ($n \geq 3$), whether or not our construction is isomorphic to the canonical one given by Deligne. The general situation is more delicate than the dilogarithm case. We have some results and it seems that they are not isomorphic if $n \geq 3$. More detail will appear in [27].

**Remark 1.** Another construction of the multiple polylogarithm motives has been given by A. Goncharov in [15] where he uses another sequence of blowups. As framed mixed Tate motives, the two constructions should be equivalent. But our construction is canonical and more natural.

### 2. Geometry of the moduli space $\overline{M}_{0,S}$ and $B_n$

First recall that there is a one-to-one correspondence between the boundary divisors of $\overline{M}_{0,S}$ and the stable unordered 2-partitions of the set $S$. Let $\sigma = \sigma_1|\sigma_2$ be a 2-partition of the set $S$, then the stability condition means that $|\sigma_1| \geq 2$ and $|\sigma_2| \geq 2$. We’ll denote by $D(\sigma)$ the corresponding boundary divisor.

**Definition 1.** Let $T = \{t_1 < t_2 < \ldots, < t_k < t_1\}$ with the given cyclic order $\rho$. A subset $A$ of $T$ is called strictly ordered if there exists some $t_i \in T$ and $l$ a positive integer, such that $A = \{t_i, t_{i+1}, \ldots, t_{i+l}\}$ (the subscripts are counted mod $k$). That is, its elements are in consecutive order with respect to $\rho$. Given a 2-partition $\sigma$ of $T$, $\sigma = \sigma_1|\sigma_2$, we say that $\sigma$ is strictly ordered with respect to $\rho$ if one of the $\sigma_i's$ is a strictly ordered subset of $T$.

For example, take $k = 4$, then $A = \{s_1, s_2, s_3\}$ is strictly ordered, but $B = \{s_1, s_3, s_4\}$ is not.

Now consider the open $n$-simplex $\Delta_n = \{(t_1, \ldots, t_n) \in \mathbb{R}^n | 0 < t_1 < \ldots < t_n < 1\}$. As mentioned in the Introduction, via cross-ratio, $\mathcal{M}_{0,S}(\mathbb{C})$ is identified with the subset $\{(x_1, \ldots, x_n) \in \mathbb{C}^n | x_i \neq x_j, i \neq j ; x_k \neq 0, 1; k = 1, \ldots, n\}$ of $\mathbb{C}^n$. Under this identification, $\Delta_n$ is a subset of $\mathcal{M}_{0,S}(\mathbb{C})$. Thus we have a natural map $\Phi$ which embeds $\Delta_n$ into $\overline{M}_{0,S}(\mathbb{C})$. Let $B_n$ be the Zariski closure of the boundary of the closure of $\Phi(\Delta_n)$ in $\overline{M}_{0,S}(\mathbb{C})$, which is called algebraic Stasheff polytope (see [10]). Then we have the following:

**Proposition 3.** $B_n$ is a union of boundary divisors indexed by the stable 2-partitions of $S$ which are strictly ordered with respect to the cyclic order $\rho$. That is, they correspond to breaking a circle into two connected arcs. Furthermore, $B_n$ is an “algebraic Stasheff polytope”. I.e., there is a bijection between the irreducible components $D_i$ of $B_n$ and the codimension one faces $F_i$ of the Stasheff polytope $K_{n+2}$ such that a subset of $D_i$’s has
a non-empty intersection of expected codimension if and only if the respective subset of 
F_i’s has this property.

Proof. See [11], Proposition 2.1. □ □

For two unordered stable 2-partitions \( \sigma = \sigma_1|\sigma_2 \) and \( \tau = \tau_1|\tau_2 \) of S, we define:

\[
a(\sigma, \tau) := \text{the number of non-empty intersections of } \sigma_i \cap \tau_j; \ i, j = 1, 2.
\]

Clearly, \( a(\sigma, \tau) = 2, 3 \) or 4, and \( a(\sigma, \tau) = 2 \) if and only if \( \sigma = \tau \), which implies that the boundary divisors \( D(\sigma) = D(\tau) \).

**Lemma 1.** (1) \( a(\sigma, \tau) = 3 \) if and only if \( D(\sigma) \neq D(\tau) \) and \( D(\sigma) \cap D(\tau) \neq \emptyset \).

(2) \( a(\sigma, \tau) = 4 \) if and only if \( D(\sigma) \cap D(\tau) = \emptyset \).

**Proof.** (1) By definition, \( a(\sigma, \tau) = 3 \) if and only if one of the following holds:

\[
\sigma_1 \subseteq \tau_1, \ \sigma_1 \subseteq \tau_2, \ \sigma_2 \subseteq \tau_1, \ \sigma_2 \subseteq \tau_2.
\]

By the Fact 4 in [20 page 552], this is exactly the sufficient and necessary condition for that \( D(\sigma) \neq D(\tau) \) and \( D(\sigma) \cap D(\tau) \neq \emptyset \).

(2) By definition, \( a(\sigma, \tau) = 4 \) means that there are distinct elements \( i, j, k, l \in S \) such that

\[
i \in \sigma_1 \setminus \tau_1, \ j \in \tau_1 \setminus \sigma_1, \ k \in \sigma_1 \cap \tau_1, \ l \notin \sigma_1 \cup \tau_1.
\]

Then it follows from the proof of the Fact 4 in [20 page 552]. □ □

**Definition 2.** A family of stable 2-partitions \( \{\sigma_1, \ldots, \sigma_m\} \) of S is called good if \( a(\sigma_i, \sigma_j) = 3 \), for \( i \neq j \). A family of boundary divisors \( \{D(\sigma_1), \ldots, D(\sigma_m)\} \) is called compatible, if the corresponding partitions \( \sigma_1, \ldots, \sigma_m \) are good.

**Lemma 2.** Given \( m \) pairwise distinct boundary divisors \( D(\sigma_1), \ldots, D(\sigma_m) \), then

\[
D(\sigma_1) \cap \cdots \cap D(\sigma_m) \neq \emptyset
\]

if and only if \( \{\sigma_1, \ldots, \sigma_m\} \) is good.

**Proof.** First, if \( \{\sigma_1, \ldots, \sigma_n\} \) is not good, then \( a(\sigma_i, \sigma_j) \neq 3 \) for some \( i \neq j \). Since \( D(\sigma_i) \) and \( D(\sigma_j) \) are distinct, by Lemma [11] \( D(\sigma_i) \cap D(\sigma_j) = \emptyset \). Hence,

\[
D(\sigma_1) \cap \cdots \cap D(\sigma_m) = \emptyset.
\]

Now suppose that \( \{\sigma_1, \ldots, \sigma_n\} \) is good. The by [23 Proposition 3.5.2], there exists a stable S-tree \( \tau \) with \( m \) (internal) edges \( \{e_1, \ldots, e_m\} \) such that \( D(\sigma_i) = D(e_i), i = 1, \ldots, m \), where \( \sigma(e) \) is the stable 2-partition of \( S \) corresponding to the one edge S-tree obtained by contracting all (internal) edges but \( e_i \). Denote by \( D(\tau) \) the closure of the stratum parametrizing the stable curves of the combinatorial type \( \tau \). Then for each \( i \), we have \( D(\tau) \subset D(\sigma) \). Hence, \( D(\sigma_1) \cap \cdots \cap D(\sigma_m) \neq \emptyset \). And since the boundary divisors meet transversely, we can conclude that in this case \( D(\sigma_1) \cap \cdots \cap D(\sigma_m) = D(\tau) \). □

Now fix an \( s_i \in S \), and let \( S' = S - \{s_i\} \) with the induced cyclic order \( \rho' : 0 < s_1 < \cdots < s_{i-1} < s_{i+1} < \cdots < s_n < 1 < \infty < 0 \). Let

\[
\pi_{S'} : \overline{\mathcal{M}}_{0,s} \longrightarrow \overline{\mathcal{M}}_{0,s'}
\]

be the contraction morphism which contracts the section marked by \( s_i \). Similarly, we have the natural embedding of \( \Delta_{n-1} = \{(t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n) \in \mathbb{R}^{n-1} | 0 < t_1 < \cdots < t_{i-1} < t_{i+1} < \cdots < t_n < 1 \} \) into \( \overline{\mathcal{M}}_{0,S'}(\mathbb{C}) \). And we also have the corresponding algebraic Stasheff polytope \( B_{n-1} \). Now we have the following key property:
Proposition 4. Under the contraction morphism $\pi_{S'}$, the vertices of $B_n$ are mapped to the vertices of $B_{n-1}$.

Proof. Let $v$ be a vertex of $B_n$. Then by Proposition 3 there exists exactly $n$ compatible boundary divisors of $B_n$, say, $\{D(\sigma_1), \ldots, D(\sigma_n)\}$ such that $v \in D(\sigma_1) \cap \cdots \cap D(\sigma_n)$. Consider the images $\pi_{S'}(D(\sigma_j))$ in $\overline{\mathcal{M}}_{0,S'}(\mathbb{C})$, for $j = 1, 2, \ldots, n$. There are two possible cases for the $\sigma_j$'s:

Case 1: $\sigma_j = \{s_{i_0}, s_{i_0+1}\}|S \setminus \{s_{i_0}, s_{i_0+1}\}$; or $\sigma_j = \{s_{i_0-1}, s_{i_0}\}|S \setminus \{s_{i_0-1}, s_{i_0}\}$;

Case 2: $\sigma_j = S_1|S_2$, with $|S_1| \geq 3$ and $s_{i_0} \in S_1$.

In Case 1, we get

$$\pi_{S'}(D(\sigma_j)) = \overline{\mathcal{M}}_{0,s'}.$$ and in Case 2, we have

$$\pi_{S'}(D(\sigma_j)) = D(\sigma'_j),$$

where $\sigma'_j = S_1 \setminus \{s_{i_0}\}|S_2$ is a stable partition of $S'$, and $D(\sigma'_j)$ is an irreducible component of $B_{n-1}$. It's clear that if $\sigma_1 \neq \sigma_k$ belong to Case 2, then $\pi_{S'}(D(\sigma_1)) \neq \pi_{S'}(D(\sigma_k))$. Now we claim that among the $\sigma_1, \ldots, \sigma_m$, there exists exactly one of them belonging to Case 1. Indeed, first suppose all of them belong to Case 2. Then we have $n$ pairwise distinct boundary divisors of $\overline{\mathcal{M}}_{0,S'}$ and their intersection is empty due to the dimension consideration. But $\pi_{S'}(v)$ is contained in this intersection, contradiction. Secondly, suppose there are two of them in Case 1. Say they are $\sigma_1 = \{s_{i_0}, s_{i_0+1}\}|S \setminus \{s_{i_0}, s_{i_0+1}\}$ and $\sigma_2 = \{s_{i_0-1}, s_{i_0}\}|S \setminus \{s_{i_0-1}, s_{i_0}\}$. But then $a(\sigma_1, \sigma_2) = 4$, thus $D(\sigma_1) \cap D(\sigma_2) = \emptyset$. It contradicts that $v \in D(\sigma_1) \cap \cdots \cap D(\sigma_n)$. Hence the claim is proved.

By the claim, we see that $\pi_{S'}(v)$ is contained in the intersection of $n-1$ distinct compatible divisors of $B_{n-1}$. By Proposition 3 $\pi_{S'}(v)$ is a vertex of $B_{n-1}$.  

Using Proposition 4 we can prove the following:

Proposition 5. Let $\pi_i : \overline{\mathcal{M}}_{0,S} \longrightarrow \overline{\mathcal{M}}_{0,\{0,s_i,1,\infty\}} = \mathbb{P}^1$ be the contraction morphism which forgets all sections but the ones marked by $\{0, s_i, 1, \infty\}$. Then the images of the vertices of $B_n$ are contained in the set $\{0, 1\}$.

Proof. Under the identification of $\overline{\mathcal{M}}_{0,\{0,s_i,1,\infty\}}$ with $\mathbb{P}^1$, $B_1$ is the interval $[0,1]$. Thus the vertices are 0, 1. The map $\pi_i$ is a composition of $n-1$ contraction morphisms, each of which forgets only one section of those labeled by the subset $S \setminus \{0, s_i, 1, \infty\}$. Now we can apply Proposition 4 repeatedly to those $n-1$ contraction morphisms. Therefore, the images of the vertices of $B_n$ are either 0 or 1.

Next, we'll introduce some non-boundary divisors in $\overline{\mathcal{M}}_{0,S}$ and prove some properties which we need later on. For each $i = 1, \ldots, n$, consider the morphism

$$\pi_{(n),i} : \overline{\mathcal{M}}_{0,S} \longrightarrow \overline{\mathcal{M}}_{0,\{0,s_i,1,\infty\}} = \mathbb{P}^1$$

which contracts all sections but those labeled by $0, s_i, 1, \infty$. Now let $a \in \mathbb{P}^1 \setminus \{0,1,\infty\}$, that is, it lies in the open stratum $\mathcal{M}_{0,\{0,s_i,1,\infty\}}$. Then the inverse image $\pi_{(n),i}^{-1}(a)$ is a divisor of $\overline{\mathcal{M}}_{0,S}$. The following proposition describes how $\pi_{(n),i}^{-1}(a)$ intersects the boundary divisors of $\overline{\mathcal{M}}_{0,S}$.

Proposition 6. Let $\sigma = \sigma_1|\sigma_2$ be a stable 2-partition of $S$ and $D(\sigma)$ be the corresponding boundary divisor. Then:
(1) If \( \{0, s_1\} \) and \( \{1, \infty\} \), \( \{1, s_1\} \) and \( \{0, \infty\} \), or \( \{\infty, s_1\} \) and \( \{1, 0\} \) belong to different parts of \( \sigma \), then
\[
\pi_{(n), i}^{-1}(a) \cap D(\sigma) = \emptyset.
\]
(2) Otherwise, we can assume that \( |\sigma_1 \cap \{0, s_1, 1, \infty\}| \geq 3 \). Then:
\[
\pi_{(n), i}^{-1}(a) \cap D(\sigma) = \pi_{(\sigma_1 \cap -2), i}^{-1}(a) \times \overline{\mathcal{M}}_{0, \sigma_2 \cup \{u\}}
\]
where \( u \) is some fixed element of \( \sigma_1 \) and \( \pi_{(\sigma_1 \cap -2), i} : \overline{\mathcal{M}}_{0, \sigma_1 \cup \{t\}} \longrightarrow \overline{\mathcal{M}}_{0, \{0, s_1, 1, \infty\}} \) is the map contracting all sections but those marked by \( 0, s_1, 1, \infty \); and if \( \{0, s_1, 1, \infty\} \subset \sigma_1 \), \( t \) is some fixed element of \( \sigma_2 \); otherwise, \( |\sigma_1 \cap \{0, s_1, 1, \infty\}| = 3 \), then \( t \) is the only element of \( \{0, s_1, 1, \infty\} \) not in \( \sigma_1 \).

Proof. For the first case, it’s easy to check that the image of \( D(\sigma) \) under \( \pi_{(n), i} : \overline{\mathcal{M}}_{0, S} \longrightarrow \overline{\mathcal{M}}_{0, \{0, s_1, 1, \infty\}} \) is contained in the set \( \{0, 1, \infty\} \). But \( a \neq 0, 1, \infty \), thus \( \pi_{(n), i}^{-1}(a) \cap D(\sigma) = \emptyset \).

For the second case, consider the morphism:
\[
\beta : \overline{\mathcal{M}}_{0, S} \longrightarrow \overline{\mathcal{M}}_{0, \sigma_1 \cup \{t\}} \times \overline{\mathcal{M}}_{0, \sigma_2 \cup \{u\}}
\]
which is the product of contracting all sections labeled by \( \sigma_2 \) but \( t \) and contraction of all sections labeled by \( \sigma_1 \) but \( u \).

By the Fact 2 of [20] page 551], the restriction of \( \beta \) on \( D(\sigma) \) is an isomorphism which is independent of the choices of \( u \) and \( t \). Then we have the following commutative diagram:
\[
\begin{array}{ccc}
D(\sigma) & \xrightarrow{\beta_{D(\sigma)}} & \overline{\mathcal{M}}_{0, \sigma_1 \cup \{t\}} \times \overline{\mathcal{M}}_{0, \sigma_2 \cup \{u\}} \\
\pi_{(n), i} \downarrow & & \downarrow \text{pr}_2 \\
\overline{\mathcal{M}}_{0, \{0, s_1, 1, \infty\}} & \xleftarrow{\pi_{(\sigma_1 \cap -2), i}} & \overline{\mathcal{M}}_{0, \sigma_1 \cup \{t\}}
\end{array}
\]
where \( \beta_{D(\sigma)} \) and \( \pi_{(n), i}|_{D(\sigma)} \) are the restrictions of the maps \( \beta \) and \( \pi_{(n), i} \) on the divisor \( D(\sigma) \) respectively, and \( \text{pr}_1 \) is the projection on the first factor.

Thus we obtain that:
\[
\pi_{(n), i}^{-1}(a) \cap D(\sigma) = \pi_{(\sigma_1 \cap -2), i}^{-1}(a)
\]
\[
= \beta_{D(\sigma)}^{-1}(\text{pr}_1^{-1}(\pi_{(\sigma_1 \cap -2), i}^{-1}(a)))
\]
\[
= \beta_{D(\sigma)}^{-1}(\pi_{(\sigma_1 \cap -2), i}^{-1}(a) \times \overline{\mathcal{M}}_{0, \sigma_2 \cup \{u\}})
\]
Since \( \beta_{D(\sigma)} \) is an isomorphism, we have
\[
\pi_{(n), i}^{-1}(a) \cap D(\sigma) = \pi_{(\sigma_1 \cap -2), i}^{-1}(a) \times \overline{\mathcal{M}}_{0, \sigma_2 \cup \{u\}}
\]
via the map \( \beta \). \[\square\]

Next we have the following description of \( \pi_{(n), i}^{-1}(a) \):

Proposition 7. (1) \( \pi_{(n), i}^{-1}(a) \) is an irreducible, reduced, smooth divisor of \( \overline{\mathcal{M}}_{0, S} \).
(2) It is a Tate variety, that’s, its motive is a direct sum of pure Tate motives.

Proof. We’ll use the induction on \( n \). When \( n = 1 \), \( S = \{0, s_1, 1, \infty\} \) and
\[
\pi_{(1), 1} : \overline{\mathcal{M}}_{0, \{0, s_1, 1, \infty\}} \longrightarrow \overline{\mathcal{M}}_{0, \{0, s_1, 1, \infty\}}
\]
is the identity map. And the Proposition is obvious. Let’s suppose that it’s true for
\( \pi^{-1}_{(k),i}(a) \), \( k \leq n \), and consider \( \pi^{-1}_{(n+1),i}(a) \). Now \( S = \{0, s_1, \ldots, s_n, s_{n+1}, 1, \infty\} \). Let \( S' = S \setminus \{s_{n+1}\} \). Then we have the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{M}_{0,S} & \xrightarrow{\gamma} & \mathcal{M}_{0,S'} \times \mathcal{M}_{0,\{0,s_{n+1},1,\infty\}} \\
\pi \downarrow & & \downarrow \text{pr}_1 \\
\mathcal{M}_{0,S'} & \xrightarrow{\pi_{(n),i}} & \mathcal{M}_{0,\{0,s_{n+1},1,\infty\}}
\end{array}
\]

where \( \pi \) is the map forgetting the section \( s_{n+1} \), \( \gamma \) is the product of \( \pi \) and the morphism from \( \mathcal{M}_{0,S} \) to \( \mathcal{M}_{0,\{0,s_{n+1},1,\infty\}} \) contracting all sections but those marked by \( 0, s_{n+1}, 1, \infty \); and \( \text{pr}_1 \) is the projection on the first factor. Moreover, we have:

\( \pi_{(n+1),i} = \pi_{(n),i} \circ \pi : \mathcal{M}_{0,S} \longrightarrow \mathcal{M}_{0,\{0,s_{n+1},1,\infty\}} \).

Hence,

\( \pi^{-1}_{(n+1),i}(a) = \pi^{-1}_{(n),i}(\gamma^{-1}(\pi^{-1}_{(n),i}(a))) \).

It’s known that \( \pi : \mathcal{M}_{0,S} \longrightarrow \mathcal{M}_{0,S'} \) is the universal curve on \( \mathcal{M}_{0,S'} \), so \( \pi \) has geometrically reduced fibers, and by induction, \( \pi^{-1}_{(n),i}(a) \) is reduced, thus \( \pi^{-1}_{(n+1),i}(a) \) is reduced.

In [20], Keel proved that the map \( \gamma \) in the above diagram is isomorphic to a sequence of blowups of \( \mathcal{M}_{0,S'} \times \mathcal{M}_{0,\{0,s_{n+1},1,\infty\}} \) along all the boundary divisors of \( \mathcal{M}_{0,S'} \). From the above diagram, we see that:

\( \pi^{-1}_{(n+1),i}(a) = \gamma^{-1}(\text{pr}_1^{-1}(\pi^{-1}_{(n),i}(a))) = \gamma^{-1}(\pi^{-1}_{(n),i}(a) \times \mathcal{M}_{0,\{0,s_{n+1},1,\infty\}}) \).

By the explicit blowups given in [20], \( \pi^{-1}_{(n+1),i}(a) \) is the composition of blowups of \( \pi^{-1}_{(n),i}(a) \times \mathcal{M}_{0,\{0,s_{n+1},1,\infty\}} \) along the intersections of \( \pi^{-1}_{(n),i}(a) \) with the boundary divisors. By the Proposition 3, these intersections are either empty or of the form:

\( \pi^{-1}_{(l),i}(a) \times \mathcal{M}_{0,T} \)

where \( l < n \) and \( T \subset S \).

By the induction assumption, \( \pi^{-1}_{(l),i}(a) \times \mathcal{M}_{0,T} \) is irreducible and smooth, and \( \pi^{-1}_{(n),i}(a) \times \mathcal{M}_{0,\{0,s_{n+1},1,\infty\}} \) is an irreducible smooth divisor of \( \mathcal{M}_{0,S'} \times \mathcal{M}_{0,\{0,s_{n+1},1,\infty\}} \). Altogether we see that \( \pi^{-1}_{(n+1),i}(a) \) is the composition of blowups of an irreducible smooth variety along an irreducible smooth subvariety. Therefore, it is irreducible and smooth. And this proves the first part of the proposition.

For the second part, we need the following formula of the motive of the blowup of a variety along a subvariety. This was proved by Manin in [23, Corollary, page 463]. Let \( X' \) be the blowup of a variety \( X \) along a subvariety \( Y \) of codimension \( r \). Then we have:

\( h(X') = h(X) \oplus (l^r_{-1}h(Y) \otimes \mathbb{L}^j) \).

where \( h(X') \), \( h(X) \) and \( h(Y) \) denote the motives of \( X' \), \( X \) and \( Y \) respectively; \( \mathbb{L} \) is the Tate motive, \( h(\mathbb{P}^1) \).
Therefore, if both $X$ and $Y$ are Tate varieties, by the formula above, the blowup $X'$ is also a Tate variety. Because the product of Tate varieties is also a Tate variety, and by induction, $\pi_{(n+1),i}^{-1}(a)$ is the composition of blowups of a Tate variety along a Tate subvariety. Thus it’s a Tate variety. This proves the second part. \hfill $\Box$ \hfill $\Box$

Now we consider the intersections of the divisors $\pi_{(n),i}^{-1}(a)$. Let $a_1, a_2, \ldots, a_m \in \mathbb{C} \setminus \{0, 1\}$, and $m \leq n$. Then we have:

**Proposition 8.** The intersection $\pi_{(n),1}^{-1}(a_1) \cap \pi_{(n),2}^{-1}(a_2) \cap \cdots \cap \pi_{(n),m}^{-1}(a_m)$ is a smooth irreducible subvariety; it’s also a Tate variety.

**Proof.** First let’s look at the case $m = 2$. Notice for any irreducible boundary divisor $D(\sigma)$, where $\sigma$ is some stable 2-partition of $S$, we have

$$\pi_{(n),1}^{-1}(a_1) \cap \pi_{(n),2}^{-1}(a_2) \cap D(\sigma) = (\pi_{(n),1}^{-1}(a_1) \cap D(\sigma)) \cap (\pi_{(n),2}^{-1}(a_2) \cap D(\sigma)).$$

By the Proposition 8, $\pi_{(n),1}^{-1}(a_1) \cap D(\sigma)$ is either empty or of the form:

$$(\pi_{(\lfloor a_{-2} \rfloor),1}^{-1}(a_1) \cap \pi_{(\lfloor a_{-2} \rfloor),2}^{-1}(a_2)) \times \overline{M}_{0, \sigma_2 \cup \{a\}}.$$

Here we use the notations in Proposition 8. Now by induction on $n$, we see that $\pi_{(n),1}^{-1}(a_1) \cap \pi_{(n),2}^{-1}(a_2)$ is obtained by the blowups of smooth irreducible Tate varieties along the Tate subvarieties. Hence it’s also smooth, irreducible and Tate.

When $m > 2$, similarly for any irreducible boundary divisor $D(\sigma)$, we have

$$\left(\bigcap_{i=1}^{m} \pi_{(n),i}^{-1}(a_i)\right) \cap D(\sigma) = \bigcap_{i=1}^{m} (\pi_{(n),i}^{-1}(a_i) \cap D(\sigma)).$$

Then similarly, use Proposition 8 and induction on $n$, it’s true for any $m$. \hfill $\Box$ \hfill $\Box$

**Corollary 1.** Let $\sigma_1, \sigma_2, \ldots, \sigma_k$ be stable 2-partitions of $S$. For each $i = 1, \ldots, k$, $D(\sigma_i)$ is the corresponding boundary divisor. Then the intersection

$$\pi_{(n),1}^{-1}(a_1) \cap \pi_{(n),2}^{-1}(a_2) \cap \cdots \cap \pi_{(n),m}^{-1}(a_m) \cap D(\sigma_1) \cap D(\sigma_2) \cap \cdots \cap D(\sigma_k)$$

is either empty or a smooth irreducible Tate variety.

**Proof.** By the combinatorial description of the intersection $D(\sigma_1) \cap D(\sigma_2) \cap \cdots \cap D(\sigma_k)$, (for $k = 2$, see [20, Fact 4], and for general $k$, it can be done inductively) we see that the non-empty intersection in the corollary can be written as the product of irreducible smooth Tate varieties. Thus, it’s also an irreducible smooth Tate variety. \hfill $\Box$ \hfill $\Box$

### 3. The meromorphic form $\Omega_S(\vec{a})$ and The divisor $A_S(\vec{a})$

#### 3.1. The form $\Omega_S(\vec{a})$.

Let $\vec{a} = (a_1, \ldots, a_n) \in \mathbb{C}^n$ and consider the following $n$ contraction morphisms:

$$\beta_{s_i} : \overline{M}_{0,S} \to \overline{M}_{0,\{0,s_i,1,\infty\}} = \mathbb{P}^1 \quad i = 1, \ldots, n$$

which forget all sections but those labeled by $0, s_i, 1, \infty$. We define the meromorphic $n$-form $\Omega_S(\vec{a})$ of $\overline{M}_{0,S}$ as follows:

$$\Omega_S(\vec{a}) := \frac{d\beta_{s_1}}{\beta_{s_1} - a_{s_1}} \wedge \cdots \wedge \frac{d\beta_{s_n}}{\beta_{s_n} - a_{s_n}}$$

(3.1)
Furthermore, we have another useful description of $\Omega_S(\vec{a})$. Let

$$\beta : \overline{M}_{0,S} \longrightarrow \prod_{i=1}^{n} \overline{M}_{0,\{0,s_i,1,\infty\}} = (\mathbb{P}^1)^n$$

be the birational proper morphism of the products of $\beta_{s_i}, i = 1, \ldots, n$. And let $(t_{s_1}, \ldots, t_{s_n})$ be the affine coordinates of $(\mathbb{P}^1)^n$, then by definition, we have

$$\Omega_S(\vec{a}) = \beta^* \left( \frac{dt_{s_1}}{t_{s_1} - a_{s_1}} \wedge \ldots \wedge \frac{dt_{s_n}}{t_{s_n} - a_{s_n}} \right).$$

3.2. The divisor $A_S(\vec{a})$. We define $A_S(\vec{a})$ to be the divisor of singularities of $\Omega_S(\vec{a})$. Following [10], for $\alpha \in \{0, 1, \infty\}$, we define $S(\alpha)$ by

$$S(0) := \{s \in S | a_s = 0\}, \quad S(1) := \{s \in S | a_s = 1\}$$

and

$$S(0, 1) := S(0) \cup S(1), \quad S(\infty) := S \setminus \{0, 1, \infty\}.$$ 

we say that a 2-partition of $S$ has type $\alpha$ if one part of it is of form $\{\alpha\} \cup T$ where $T$ is a non-empty subset of $S(\alpha)$. Now we have the explicit description of the divisor $A_S(\vec{a})$:

**Proposition 9.** The divisor $A_S(\vec{a})$ of singularities of $\Omega_S(\vec{a})$ on $\overline{M}_{0,S}$ consists of the following irreducible components:

(a) The boundary divisors $D(\sigma)$ corresponding to those stable 2-partitions of $S$ which have some type $\alpha$, $\alpha \in \{0, 1, \infty\}$;

(b) The non-boundary divisors $\pi^{-1}_{(n),i}(a_{s_i})$, for each $s_i \notin S(0,1)$; where $\pi_{(n),i} : \overline{M}_{0,S} \longrightarrow \overline{M}_{0,\{0,s_i,1,\infty\}}$ is the morphism contracting all sections but those labeled by $0, s_i, 1, \infty$. Moreover, $A_S(\vec{a})$ is a normal-crossing divisor.

**Proof.** We’ll prove it by induction on $n$. For the case $n = 1$, $S = \{0, s_1, 1, \infty\}$ and $\overline{M}_{0,S} = \mathbb{P}^1$. Then $\Omega_S(\vec{a}) = \frac{dt}{t - a_{s_1}}$, where $t$ is the affine coordinate of $\mathbb{P}^1$. Thus $A_S(\vec{a}) = (a_{s_1}) + (\infty)$, and the proposition is clear in this case.

Assume that it’s true for $n$. Now $S = \{0, s_1, \ldots, s_n, s_{n+1}, 1, \infty\}$ and $\vec{a} = (a_{s_1}, \ldots, a_{s_n}, a_{s_{n+1}})$. Let $\vec{a}' = (a_{s_1}, \ldots, a_{s_n})$, $S' = S \setminus \{s_{n}\}$, and $A_{S'}(\vec{a}')$ the divisor of singularities of the meromorphic form $\Omega_{S'}(\vec{a}')$ in $\overline{M}_{0,S'}$. Consider the morphism $\beta$ of the product of two contraction maps:

$$\beta : \overline{M}_{0,S} \longrightarrow \overline{M}_{0,S'} \times \overline{M}_{0,\{0,s_{n+1},1,\infty\}}.$$ 

Then we obtain that

$$\Omega_S(\vec{a}) = \beta^* (\Omega_{S'}(\vec{a}') \wedge \frac{dt}{t - a_{s_{n+1}}}).$$

Let $A(a_{s_{n+1}})$ be the divisor of singularities of the form $\frac{dt}{t - a_{s_{n+1}}}$ in $\overline{M}_{0,\{0,s_{n+1},1,\infty\}}$. Clearly, the divisor of singularities of the meromorphic form $\Omega_{S'}(\vec{a}') \wedge \frac{dt}{t - a_{s_{n+1}}}$ in $\overline{M}_{0,S'} \times \overline{M}_{0,\{0,s_{n+1},1,\infty\}}$ is $pr_1^* (A_{S'}(\vec{a}')) + pr_2^* (A(a_{s_{n+1}}))$, where $pr_1$ and $pr_2$ are the projections on the first and second factors respectively. And it’s a normal crossing divisor. By [20], $\beta$ is isomorphic to a sequence of blowups of $\overline{M}_{0,S'} \times \overline{M}_{0,\{0,s_{n+1},1,\infty\}}$ along all the boundary divisors of $\overline{M}_{0,S'}$. By the following Lemma [3] on blowups, and Proposition [3] and [7], we can conclude that $\beta^* (pr_1^* (A_{S'}(\vec{a}'))) + pr_2^* (A(a_{s_{n+1}})))$ is a normal crossing divisor in $\overline{M}_{0,S}$. $A_S(\vec{a})$ is contained in it. Thus it’s also a normal crossing divisor.
On the other hand, the meromorphic form $\Omega_S(\vec{a})$ does not necessarily have as singularities all the exceptional divisors of $\beta$ in $\beta^*(pr_1^* (A_{S'} (\vec{a})) + pr_2^* (A(a_{s_{n+1}})))$. In fact, $A_S (\vec{a})$ equals $\beta^*(pr_1^* (A_{S'} (\vec{a})) + pr_2^* (A(a_{s_{n+1}})))$ minus those spurious divisors.

By [15, Lemma 3.8], the spurious divisors are those irreducible boundary divisors in $\mathcal{M}_{0, S}$ which get blown down by $\beta$ to a subvariety of the product which is not a stratum of the divisor $pr_1^* (A_{S'} (\vec{a})) + pr_2^* (A(a_{s_{n+1}}))$.

By [20, Lemma 1, Page 554], the exceptional divisors of $\beta$ are exactly those boundary divisors corresponding to the following 2-stable partitions:

\[(*) \quad : \quad \sigma = \sigma_1|\sigma_2, \text{ with } s_{n+1} \in \sigma_1, |\sigma_1| \geq 3, \text{ and } |\sigma_1 \cap \{0, 1, \infty\}| \leq 1.\]

By induction, the divisor $A_{S'} (\vec{a})$ consists of the following four types of irreducible divisors (we identify the boundary divisors with the corresponding 2-stable partitions here):

\[0T_0|\cdots|1T_1|\cdots|\infty T_\infty|\cdots|01; \quad \pi_{n, \lambda}^{-1}(a_{s_i}), \ a_{s_i} \neq 0, 1\]

where $T_\alpha$ is a non-empty subset of $S(\alpha)$, $\alpha \in \{0, 1, \infty\}$.

First, notice the pullback of the divisor $\pi_{n, \lambda}^{-1}(a_{s_i})$ equals $\pi_{(n+1), \lambda}^{-1}(a_{s_i})$ which is not an exceptional divisor of $\beta$, thus it’s a component of the divisor $A_S (\vec{a})$.

For each $\alpha \in \{0, 1, \infty\}$, by [20, Fact 3, page 552], the pullback of the boundary divisor $\alpha T_{\alpha}|\cdots$ in the list above is equal to the sum of the following two components:

\[\alpha T_{\alpha} s_{n+1}|\cdots + \alpha T_{\alpha} s_{n+1} |\cdots.\]

By the condition $(*)$, $\alpha T_{\alpha} s_{n+1}|\cdots$ is not an exceptional divisor of $\beta$, hence it’s a component of the divisor $A_S (\vec{a})$; but $\alpha T_{\alpha} s_{n+1} |\cdots$ is an exceptional divisor. We need to check if it’s a spurious divisor. Notice that

\[\beta(\alpha T_{\alpha} s_{n+1}|\cdots) = \alpha T_{\alpha}|\cdots \times \alpha s_{n+1}\{0, 1, \infty\} \setminus \alpha\]

Now clearly $\alpha T_{\alpha}|\cdots \times \alpha s_{n+1}\{0, 1, \infty\} \setminus \alpha$ is a stratum of the divisor $pr_1^* (A_{S'} (\vec{a})) + pr_2^* (A(a_{s_{n+1}}))$ if and only if $\alpha$ is in the divisor $A(a_{s_{n+1}})$, therefore, $\alpha T_{\alpha} s_{n+1}|\cdots$ is a component of the divisor $A_S (\vec{a})$ if and only if $\alpha$ is in the divisor $A(a_{s_{n+1}})$. For $\alpha = 0, 1$, it depends upon whether or not $a_{s_{n+1}} = \alpha$. And $\alpha = \infty$ is always in the divisor $A(a_{s_{n+1}})$, hence $\infty T_{\infty} s_{n+1}|\cdots|01$ is a component of $A_S (\vec{a})$.

Now there are only two more components of $A_S (\vec{a})$. One is $\infty s_{n+1}|\cdots|01$. It’s not an exceptional divisor and is contained in the pullback of $\infty s_{n+1}|01$. The other one is $\pi_{(n+1), (n+1)}^{-1}(a_{s_{n+1}})$, when $a_{s_{n+1}} \neq 0, 1$. And it is contained in the pullback of $A(a_{s_{n+1}})$. Thus we have all the components of $A_S (\vec{a})$ listed in the proposition. This concludes the proof.

Now let’s prove a lemma on blowups which is used in the proof of the preceding proposition.

**Lemma 3.** Let $M$ be a smooth complex variety of dimension $n$, $B$ an irreducible smooth subvariety of $M$ with codimension $m \geq 2$, and $D = D_1 \cup D_2 \cup \cdots \cup D_k$ a normal-crossing divisor of $M$ with smooth irreducible components $D_i$, $i = 1, \ldots, k$. Suppose that for each $D_i$, only one of the following cases happens: $B \subset D_i$, $B \cap D_i = \emptyset$, or: $B \subset D_i$ is an irreducible smooth subvariety with codimension $m - 1$. Let $\pi : M'_B \rightarrow M$ be the blowup of $M$ along $B$. Let $D'_i$ be the strict transform of $D_i$, $1 \leq i \leq k$, and $E = \pi^{-1}(B)$ the exceptional divisor. Then the divisor $D' = D'_1 \cup D'_2 \cup \cdots \cup D'_k \cup E$ is a normal-crossing divisor in $M'_B$. 


Proof. Let $x' \in D'$. If $x' \notin E$, the $\pi$ is locally at $x'$ an isomorphism. So we only need to consider the case $x' \in E$. Then $x = \pi(x') \in B$. By rearranging the indices of the $D_i$, if necessary, we may assume that $B \subset D_i$ for $1 \leq i \leq a$, $x \in D_j \cap B \neq \emptyset$ for $a + 1 \leq j \leq b$, and $x \notin D_j \cap B$ for $j > b$. Here $1 \leq a \leq \min\{m, k\}$, $a \leq b \leq \min\{m, k\}$ and $b - a \leq n$. Since $D$ is a normal-crossing divisor, there exists an open neighborhood $U$ of $x$ with local coordinates $(z_1, z_2, \ldots, z_n)$ such that $z_1(x) = z_2(x) = \cdots = z_n(x) = 0$ and $U \cap B = \{y \in U|z_1(y) = z_2(y) = \cdots = z_m(y) = 0\}$, for $1 \leq i \leq a, U \cap D_i = \{y \in U|z_i(y) = 0\}$; for $a + 1 \leq j \leq b, U \cap D_j = \{y \in U|z_j(y) = 0\}$. And $\pi^{-1}(U) = \{(y, l) \in U \times \mathbb{P}^{m-1}|z_i(y)l_j = z_j(y)l_i, 1 \leq i, j \leq m\}$. Here $l = [l_1 : \cdots : l_m]$ is the homogenous coordinate of $\mathbb{P}^{m-1}$. Then a point $(y, l)$ in the inverse image of $\pi^{-1}(U \cap (D_i \setminus B))$ has the property that $l_i = 0$. Now in $D'$, suppose that $x' \in D'_j$ for $1 \leq i < c$, and $x' \notin D'_j$ for $j > c$. Then $c \leq b$. Now since $x' = (y, l) \in \pi^{-1}(U)$, it can not happen that all $l_i = 0$. Then there exists some $i_0 > c$ such that $l_{i_0} \neq 0$. Consider the open neighborhood $U_{i_0} = \{(y, l) \in \pi^{-1}(U)|l_{i_0} \neq 0\}$ of $\pi^{-1}(U)$. And the restriction of the blowup $\pi$ on the $U_{i_0}$ has the following formula:
\[(s_1, \cdots, z_{i_0}, \cdots, s_m, z_{m+1}, \cdots, z_n) \mapsto (s_1z_{i_0}, \cdots, z_{i_0}, \cdots, s_mz_{i_0}, z_{m+1}, \cdots, z_n)\]
where $s_j = l_j/l_{i_0}, 1 \leq j \leq m, j \neq i_0$. Therefore for $1 \leq j \leq c, D'_j \cap U_{i_0} = \{(y, l) \in U_{i_0}|s_j = 0\}$ and $E \cap U_{i_0} = \{(y, l) \in U_{i_0}|s_{i_0}(y) = 0\}$. Thus $D'$ is a normal-crossing divisor in $M'_B$. \hfill \Box

Next we’ll prove the main theorem in this section:

**Theorem 10.** The divisor $A_S(\tilde{a})$ of singularities $\Omega_S(\tilde{a})$ does not contain any $k$-dimensional face of the algebraic stasheff polytope $B_n, 0 \leq k \leq n$. Here $\tilde{a} = (a_{s_1}, \ldots, a_{s_n})$ with $a_{s_1} \neq 0$ and $a_{s_n} \neq 1$.

**Proof.** Proof. First observe that if $A_S(\tilde{a})$ contains some $k$-dimensional face of $B_n$, then it also contains the vertices of this face. Therefore it suffices to show that $A_S(\tilde{a})$ does not contain any vertex of $B_n$.

We’ll prove that each irreducible component of $A_S(\tilde{a})$ listed in the preceding Proposition can’t contain any vertex of $B_n$.

First consider the non-boundary divisor $\pi_{(n), i}^{-1}(a_{s_i})$, for $a_{s_i} \neq 0, 1$. Clearly, its image under the contraction morphism:
\[\pi_{(n), i} : \overline{M}_{0, S} \to \overline{M}_{0, \{0, s_i, 1, \infty\}}\]
is $a_{s_i}$. On the other hand, by the Proposition in Section 2, the image of a vertex of $B_n$ under $\pi_{(n), i}$ is 0, or 1. Since $a_{s_i} \neq 0, 1$, $\pi_{(n), i}^{-1}(a_{s_i})$ doesn’t contain any vertex of $B_n$.

Let’s consider the boundary divisor in $A_S(\tilde{a})$. It has one of the following types:
\[0T_0|\cdots|1\infty; \quad 1T_1|\cdots|0\infty; \quad \infty T_\infty|\cdots|01\]
By the Proposition in Section 2, we know that a vertex of $B_n$ is contained in the intersection of exactly $n$ irreducible components of $B_n$. Notice that the intersection of any $n + 1$ irreducible boundary divisors of $\overline{M}_{0, S}$ is empty, therefore, it’s enough to show that none of the boundary divisors in $A_S(\tilde{a})$ appears as an irreducible component of $B_n$.

By the Proposition in Section 2, the irreducible components of $B_n$ correspond to the stable 2-partitions of $S$ which are strictly ordered with respect to the cyclic order $\rho : 0 < s_1 < s_2 < \cdots < s_n < 1 < \infty < 0$. The partition $0T_0|\cdots|1\infty$ is not strict with respect to $\rho$ because $a_{s_1} \neq 0$ then $s_1$ and $\infty$ separate any element of $T_0$ from 0;
similarly, $1 T_1 | \cdots | 0 \cdot \infty$ is not strict with respect to $\rho$ because $a_{s_n} \neq 1$ and $s_n$ and $\infty$ block any element of $T_1$ to 1; finally $\infty T_\infty | \cdots | 01$ is not strict with respect to $\rho$ because 1 and 0 separate $\infty$ from any element of $T_\infty$. Hence, none of the boundary divisors in $A_S(\vec{a})$ contains a vertex of $B_n$. This concludes the proof. □ □

4. Multiple polylogarithm motives

We’ll continue to use the notations in the previous sections. Now consider the convergent iterated integral:

$$I_\gamma(a_{s_1}, \ldots, a_{s_n}) = \int_{\gamma(\Delta_n)} \frac{dt_{s_1}}{t_{s_1} - a_{s_1}} \wedge \cdots \wedge \frac{dt_{s_n}}{t_{s_n} - a_{s_n}} \quad a_{s_1} \neq 0, a_{s_n} \neq 1.$$  

where $\gamma : [0, 1] \to \mathbb{C}$ is a piecewise smooth simple path from 0 to 1 and $a_{s_i} \notin \gamma((0, 1))$, $i = 1, \ldots, n$, $\Delta_n = \{(t_{s_1}, \ldots, t_{s_n}) \in \mathbb{R}^n | 0 < t_{s_1} < \cdots < t_{s_n} < 1\}$ is an open $n$-simplex in $\mathbb{R}^n$, and $\gamma(\Delta_n) = \{(\gamma(t_{s_1}), \ldots, \gamma(t_{s_n}))|(t_{s_1}, \ldots, t_{s_n}) \in \Delta_n\}$.

As mentioned in the Introduction, we can identify the open stratum $M_{0,s}(\mathbb{C})$ with the set $\{(x_1, \ldots, x_n) \in \mathbb{C}^n | x_i \neq x_j, i \neq j ; x_k \neq 0, 1, k = 1, \ldots, n\}$. Then $\gamma(\Delta_n)$ is a subset of $M_{0,s}(\mathbb{C})$, thus we have a natural map $\Phi_n$ which embeds $\gamma(\Delta_n)$ into $\overline{M}_{0,s}(\mathbb{C})$.

Clearly we have the following equality:

$$I_\gamma(a_{s_1}, \ldots, a_{s_n}) = \int_{\Phi(\gamma(\Delta_n))} \Omega_S(\vec{a})$$

where $\Omega_S(\vec{a})$ is the meromorphic $n$-form in $\overline{M}_{0,s}(\mathbb{C})$ defined in Section 2.

Let $\Phi_n(\gamma)$ be the closure of $\gamma(\Delta_n)$ in $\overline{M}_{0,s}$. Then the Zariski closure of the boundary of $\Phi_n(\gamma)$ is $B_n$. This follows from the fact that $B_n$ is the Zariski closure of the boundary of the closure of $\Delta_n$ in $\overline{M}_{0,s}$. Hence we have a relative homology class:

$$[\Phi_n(\gamma)] \in H_n(\overline{M}_{0,s}(\mathbb{C}), B_n(\mathbb{C}); \mathbb{Q})$$

The meromorphic $n$-form $\Omega_S(\vec{a})$ gives rise to a cohomology class:

$$[\Omega_S(\vec{a})] \in H^n_{DR}(\overline{M}_{0,s}(\mathbb{C}) - A_S(\vec{a})); \mathbb{C}).$$

Now given the iterated integral (4.1), we’ll define:

$$I^M(a_{s_1}, \ldots, a_{s_n}) := H^n(\overline{M}_{0,s} - A_S(\vec{a}), B_n - \bigcap A_S(\vec{a})).$$

Then we have the following main theorem:

**Theorem 11.** $I^M(a_{s_1}, \ldots, a_{s_n})$ carries an $n$-framed Hodge-Tate structure with the frames coming from $\Omega_S(\vec{a})$ and $\Phi_n(\gamma)$. The period is just the iterated integral $I_\gamma(a_{s_1}, \ldots, a_{s_n})$. Moreover, if all the $a_{s_i}$ are elements of a number field $F$, then $I^M(a_{s_1}, \ldots, a_{s_n})$ is a framed mixed Tate motive over $F$.

Before we prove the theorem, first let’s briefly recall the definitions of framed Hodge-Tate structure and its period.

**Definition 3.** A mixed Hodge $\mathbb{Q}$-structure consists of the following data:

1) a finite-dimensional $\mathbb{Q}$-vector space $H_\mathbb{Q}$ with a finite increasing filtration $W_n$ called the weight filtration;

2) a finite decreasing filtration $F^p$ of $H_\mathbb{C} := H_\mathbb{Q} \otimes_\mathbb{Q} \mathbb{C}$ called the Hodge filtration;
these data satisfy the following conditions:
For each associated graded piece $\text{Gr}_k^W(H_C) = \frac{W_nH_q \otimes C}{W_{n-1}H_q \otimes C}$, one has the decomposition:
$$\text{Gr}_k^W(H_C) = \bigoplus_{p+q=k} H^{p,q}$$
with
$$H^{p,q} = F^p\text{Gr}_k^W(H_C) \cap \overline{F^q\text{Gr}_k^W(H_C)}, \text{ and } H^{p,q} = \overline{H^{q,p}}.$$ 
Here $\overline{\cdot}$ means the complex conjugation of $H_C$ with respect to $H_R := H_Q \otimes Q \mathbb{R}$. The Hodge numbers are the integers $h^{p,q} = \dim_C H^{p,q} = h^{q,p}$.

By definition, a Hodge-Tate structure $H$ is a mixed Hodge $\mathbb{Q}$-structure $H$ with the Hodge numbers $h^{p,q} = 0$ unless $p = q$. This means that for the weight filtration, $\text{Gr}_{2n+1}^W H = 0$ and $\text{Gr}_2^W H$ is a finite direct sum of $\mathbb{Q}(-n)$. It also implies that, for each $p \in \mathbb{Z}$, the natural map
$$F^pH_C \cap W_{2p}H_C \to \text{Gr}_{2p}^WH_C$$
is an isomorphism.

**Definition 4.** An $n$-framed Hodge-Tate structure $H$ is a Hodge-Tate structure $H$ equipped with a nonzero vector in $\text{Gr}_{2n}^W H$ and a nonzero functional on $\text{Gr}_0^W H$, that is, we have two nonzero morphisms:
$$v : \mathbb{Q}(-n) \to \text{Gr}_{2n}^WH, \ f : \mathbb{Q}(0) \to \text{Gr}_0^W(H)^\vee.$$ 

To define the period of an $n$-framed Hodge-Tate structure, we need to choose a map of $\mathbb{Q}$-vector spaces $F : \mathbb{Q} \to H_Q^\vee$ which lifts $f$, that is, $\text{Gr}_0^W F = f$. Now let $f' = F(1)$. Consider the composition:
$$\mathbb{Q}(-n) \to \text{Gr}_{2n}^WH_Q \to F^nH_C \cap W_{2n}H_C,$$
where the first one is $v$ and the second is provided by (4.3). It gives rise to a vector $v' \in F^nH_C \cap W_{2n}H_C$. The period is the number $\langle v', f' \rangle$. A different choice of the lifting $F$ will change this period by $2\pi i \times \text{"weight } n - 1 \text{ period"}$. In this sense, periods are multi-valued.

For more information about periods of framed mixed Tate Motives or periods of framed Hodge-Tate structures, we refer to [13, Chapter 5] and [14, Section 3.2].

Now let’s prove some lemmas about Hodge-Tate structures.

**Lemma 4.** The category of Hodge-Tate structures is abelian. It’s closed under subquotients and extensions.

**Proof.** It’s straightforward (c.f. [3 Théorème 2.3.5]).

**Lemma 5.** Let $A$ be a complex algebraic variety. Suppose that
$$A = \bigcup_{i=1}^k A_i,$$
where $A_i$ are closed subvarieties of $A$, $i = 1, \ldots, k$. Let $d$ be a fixed positive integer. For any non-empty subset $I \subset \{1, 2, \ldots, k\}$, denote by $A_I = \cap_{i \in I} A_i$. Suppose that all the cohomology groups of $A_I$ carry the Hodge-Tate structures for all $I$. Then $H^d(A, \mathbb{Q})$ also carries a Hodge-Tate structure.
carries a Hodge-Tate structure. Now we need to show that $\Omega_{S}$ frames. First, there are natural non-zero morphisms of pure Tate structures by $\Omega_{H}$, by the Lemma 4, we can conclude that the relative cohomology knows that it’s also a long exact sequence of mixed Hodge structures. Similar to Lemma 5.\[\text{Notice that } A = A' \cup A_{n+1},\]
where $A' = \bigcup_{i=1}^{n} A_{i}$. By the Mayer-Vietoris exact sequence for cohomology, we have:
\[\cdots \longrightarrow H^{d-1}(A' \cap A_{n+1}, \mathbb{Q}) \longrightarrow \partial \longrightarrow H^{d}(A, \mathbb{Q}) \longrightarrow i \longrightarrow H^{d}(A', \mathbb{Q}) \oplus H^{d}(A_{n+1}, \mathbb{Q}) \longrightarrow j \longrightarrow \cdots\]

It’s known that the Mayer-Vietoris sequence is also an exact sequence of mixed Hodge structures. Therefore, we get the short exact sequence of mixed Hodge structures:
\[0 \longrightarrow H^{d-1}(A' \cap A_{n+1}, \mathbb{Q}) \xrightarrow{\partial} H^{d}(A, \mathbb{Q}) \xrightarrow{i} H^{d}(A', \mathbb{Q}) \oplus H^{d}(A_{n+1}, \mathbb{Q}) \xrightarrow{j} \cdots\]

Notice that $A' \cap A_{n+1} = \bigcup_{i=1}^{n} (A_{i} \cap A_{n+1})$,
therefore, by the induction assumption, both $H^{d-1}(A' \cap A_{n+1}, \mathbb{Q})$ and $H^{d}(A', \mathbb{Q})$ are Hodge-Tate structures. Now by the Lemma 6, $H^{d}(A, \mathbb{Q})$ is a Hodge-Tate structure. \[\square\]

**Lemma 6.** Let $X$ be a complex algebraic variety, and $A$ be a subvariety of codimension 1. Then if both $H^{n-1}(A; \mathbb{Q})$ and $H^{n}(X; \mathbb{Q})$ are Hodge-Tate, so are the relative cohomology $H^{n}(X, A; \mathbb{Q})$ and $H^{n}(X - A; \mathbb{Q})$.

**Proof.** Consider the long exact sequence of cohomologies for the pair $(X, A)$ and we know that it’s also a long exact sequence of mixed Hodge structures. Similar to Lemma 5 by the Lemma 6 we can conclude that the relative cohomology $H^{n}(X, A; \mathbb{Q})$ carries a Hodge-Tate structure, then by duality, so does $H^{n}(X - A; \mathbb{Q})$. \[\square\]

Now let’s prove the theorem:

**Proof.** Proof of Theorem 11. By Deligne 3,4, there is a canonical mixed Hodge structure on the relative cohomology $H^{n}(\bar{\mathcal{M}}_{0,S} - A_{S}(\bar{a}), B_{n} - B_{n} \cap A_{S}(\bar{a}))$. First we’ll show that $H^{n}(\bar{\mathcal{M}}_{0,S} - A_{S}(\bar{a}), B_{n} - B_{n} \cap A_{S}(\bar{a}))$ is a Hodge-Tate structure.

By the Proposition 9 and Corollary 11 we know that the intersections of the components of $A_{S}(\bar{a})$ are either empty or Tate varieties. By Lemma 5 and 6 $H^{d}(\bar{\mathcal{M}}_{0,S} - A_{S}(\bar{a}); \mathbb{Q})$ carries a Hodge-Tate structure. For the same reason, $H^{d}(B_{n} - B_{n} \cap A_{S}(\bar{a}); \mathbb{Q})$ also has a Hodge-Tate structure. By the Lemma 6 again, $H^{n}(\bar{\mathcal{M}}_{0,S} - A_{S}(\bar{a}), B_{n} - B_{n} \cap A_{S}(\bar{a}))$ carries a Hodge-Tate structure. Now we need to show that $\Omega_{S}(\bar{a})$ and $\Phi_{n}(\gamma)$ give the frames. First, there are natural non-zero morphisms of pure Tate structures by $\Omega_{S}(\bar{a})$ and $\Phi_{n}(\gamma)$:

\[\Omega_{S}(\bar{a}) : \mathbb{Q}(-n) \rightarrow Gr_{2n}^{W}H^{n}(\bar{\mathcal{M}}_{0,S} - A_{S}(\bar{a})),\]
\[\Phi_{n}(\gamma) : \mathbb{Q}(0) \rightarrow Gr_{0}^{W}H_{n}(\bar{\mathcal{M}}_{0,S}, B_{n})\]

Then composing with the canonical isomorphisms:

\[Gr_{2n}^{W}H^{n}(\bar{\mathcal{M}}_{0,S} - A_{S}(\bar{a}), B_{n} - B_{n} \cap A_{S}(\bar{a})) \rightarrow Gr_{2n}^{W}H^{n}(\bar{\mathcal{M}}_{0,S} - A_{S}(\bar{a}))\]
and

\[Gr_{0}^{W}H_{n}(\bar{\mathcal{M}}_{0,S} - A_{S}(\bar{a}), B_{n} - B_{n} \cap A_{S}(\bar{a})) \rightarrow Gr_{0}^{W}H_{n}(\bar{\mathcal{M}}_{0,S}, B_{n})\]
we get the frame morphisms:

\[
[\Omega(\vec{a})]': \mathbb{Q}(-n) \to Gr^W_{2n} H^n(\overline{\mathcal{M}}_{0,S} - A_S(\vec{a}), B_n - B_n \cap A_S(\vec{a})),
\]

\[
[\Phi_n(\gamma)]: \mathbb{Q}(0) \to Gr^W_0 H_n(\overline{\mathcal{M}}_{0,S} - A_S(\vec{a}), B_n - B_n \cap A_S(\vec{a})).
\]

Since \(a_s, 1 \leq i \leq n\) are elements of a number field \(F\). We'll show that \(IM(a_{s_1}, \ldots, a_{s_n})\) is a framed mixed Tate motive over \(F\). For this, we'll follow the proof of [10] Theorem 4.1 very closely. In [13], Goncharov constructed the abelian category of mixed Tate motive over any number field \(F\), where he used the theory of triangulated category of mixed motives by Voevodsky in [25]. Let's apply it to our case. First, consider the standard cosimplicial variety:

\[
S_\bullet(\overline{\mathcal{M}}_{0,S} - A_S(\vec{a}), B_n - B_n \cap A_S(\vec{a})).
\]

Here \(S_0 = \overline{\mathcal{M}}_{0,S} - A_S(\vec{a})\), and \(S_k\) is the disjoint union of the codimension \(k\) strata of the divisor \(B_n - B_n \cap A_S(\vec{a})\).

According to the standard procedure, we can get a complex \(S_\bullet(A_S(\vec{a}), B_n)\) of varieties from the above cosimplicial variety with \(S_0\) at the degree 0. Then it gives an object in Voevodsky’s triangulated category of mixed motives over \(F\). In fact it also belongs to the triangulated subcategory \(D_T(\mathbb{F})\) of mixed Tate motives over \(F\). And there is a canonical \(t\)-structure \(t\) on \(D_T(\mathbb{F})\). Then \(H^n_t(S_\bullet(A_S(\vec{a}), B_n))\) is our mixed Tate motive.

Similarly, by using the construction above for the Hodge-Tate structures and the fact that the Hodge realization is a fully faithful functor on the category of pure Tate motives, we obtain the frame morphisms for our motive coming from \([\Omega_S(\vec{a})]'\) and \([\Phi_n(\gamma)]'\) defined above. The theorem is proved.

5. Motivic Construction of Dilogarithm

In this section, we’ll apply our preceding construction to the dilogarithm and show that the corresponding period matrix is the same as the one given by P. Deligne. Hence they are isomorphic. Recall that

\[
Li_2(z) = -\int_{0 \leq t_1\leq t_2 \leq 1} \frac{dt_1}{t_1 - z^{-1}} \wedge \frac{dt_2}{t_2}
\]

Now let’s consider the moduli space \(\overline{\mathcal{M}}_{0,5}\). It’s known that \(\overline{\mathcal{M}}_{0,5}\) is the blow-up of \(\mathbb{P}^1 \times \mathbb{P}^1\) at three points \(\{0,0\}, \{1,1\}, \{\infty, \infty\}\) on the diagonal. Let \(\pi: \overline{\mathcal{M}}_{0,5} \to \mathbb{P}^1 \times \mathbb{P}^1\) denote
this blow-up. And let \((t_1, t_2)\) be the affine coordinates on \(\mathbb{P}^1 \times \mathbb{P}^1\). We list the ten boundary divisors of \(\overline{M}_{0,5}\) as follows (See Figure 1).

\[
\begin{align*}
D_{1,\infty} &= \text{the proper (strict) transform of the divisor } t_1 = \infty, \\
D_{2,\infty} &= \text{the proper transform of the divisor } t_2 = \infty, \\
D_\infty &= \pi^{-1}(\infty, \infty), \\
D_{1,1} &= \text{the proper transform of the divisor } t_1 = 1, \\
D_{1,2} &= \text{the proper transform of the divisor } t_2 = 1, \\
D_1 &= \pi^{-1}(1, 1), \\
D_{0,1} &= \text{the proper transform of the divisor } t_1 = 0, \\
D_{0,2} &= \text{the proper transform of the divisor } t_2 = 0, \\
D_0 &= \pi^{-1}(0, 0), \\
D &= \text{the proper transform of the divisor } t_1 = t_2.
\end{align*}
\]

Moreover we can describe the divisors \(A(z)\) and \(B_2\) explicitly. By definition, \(A(z)\) is the divisor of singularities of the pullback meromorphic 2-form \(\pi^*\left(\frac{dt_1}{t_1-z^{-1}} \wedge \frac{dt_2}{t_2}\right)\) of \(\overline{M}_{0,5}\). We can check the following lemma directly and leave it to the reader.

**Lemma 7.**

\[
A(z) = \begin{cases}
D_{\infty,1} \cup D_{\infty,2} \cup D_\infty \cup D_{0,2} \cup D_{1,1}, & \text{if } z = 1; \\
D_{\infty,1} \cup D_{\infty,2} \cup D_\infty \cup D_{0,2} \cup l_z, & \text{if } z \neq 0, 1.
\end{cases}
\]

where \(l_z\) is the pullback of the divisor \(t_1 = z^{-1}\). That is, \(l_z = \pi^{-1}(t_1 = z^{-1})\).

For \(B_2\), by the Proposition in Section 2, we see that \(B_2 = D_{0,1} \cup D_{1,2} \cup D_1 \cup D \cup D_0\) (see Figure 1). For \(z \neq 0\), we'll call the following the **motivic dilogarithm** :

\[
Li^M_{\infty}(z) := H^2(\overline{M}_{0,5} - A(z), B_2 - B_2 \cap A(z)).
\]

**5.1. The computations of the relative cohomology groups.** First we state the results. In the following statements, we only list the (relative) Betti homology groups
because by the comparison theorem the deRham cohomology groups are isomorphic to the corresponding Betti cohomology groups tensor with $\mathbb{C}$.

**Theorem 12.** (a) If $z \neq 0,1$, then the Betti homology groups of $\overline{M}_{0,5} - A(z)$ are:

$$H_i(\overline{M}_{0,5} - A(z); \mathbb{Q}) = \begin{cases} 0, & \text{if } i \geq 3 \\ \mathbb{Q} \oplus \mathbb{Q}, & \text{if } i = 2 \\ \mathbb{Q}, & \text{if } i = 0, 1 \end{cases}$$

(b) If $z = 1$, then the Betti homology groups of $\overline{M}_{0,5} - A(1)$ are:

$$H_i(\overline{M}_{0,5} - A(1); \mathbb{Q}) = \begin{cases} \mathbb{Q}, & \text{if } i = 0, 2 \\ 0, & \text{otherwise} \end{cases}$$

**Theorem 13.** (a) If $z \neq 0,1$ then:

$$H_2(\overline{M}_{0,5} - A(z), B_2 - B_2 \cap A(z); \mathbb{Q}) = \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q}$$

Furthermore, there are natural bases such that the corresponding period matrix is

$$\begin{bmatrix} 1 & 0 & 0 \\ -Li_1(z) & 2\pi i & 0 \\ -Li_2(z) & 2\pi i \log z & (2\pi i)^2 \end{bmatrix}.$$ 

Hence, it coincides with the one given by Deligne and they are isomorphic as Hodge-Tate structures.

(b) if $z = 1$, then:

$$H_2(\overline{M}_{0,5} - A(1), B_2 - B_2 \cap A(1); \mathbb{Q}) = \mathbb{Q} \oplus \mathbb{Q}$$

And there are natural bases such that the corresponding period matrix is

$$\begin{bmatrix} 1 & 0 \\ -Li_2(1) & (2\pi i)^2 \end{bmatrix}.$$ 

**Remark 2.** There is a dimension jump when $z$ goes to 1. This fact is predicted by the specialization theorem. For more detail about it, see [15].

Now we’ll prove these two theorems. Before we do it, let’s give a more explicit description of the affine variety $\overline{M}_{0,5} - A(z)$. We know $\overline{M}_{0,5}$ is the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at the points $\{0,0\}, \{1,1\}, \{\infty, \infty\}$, hence $\overline{M}_{0,5} - (D_{1,\infty} \cup D_{2,\infty} \cup D_{\infty}) = Bl_{\{0,1\}}(\mathbb{C}^2)$ in which $Bl_{\{0,1\}}(\mathbb{C}^2)$ denotes the blow-up of $\mathbb{C}^2$ at the points $\{0,0\}, \{1,1\}$ (see Figure 1). We’ll use the same letter $\pi : Bl_{\{0,1\}}(\mathbb{C}^2) \to \mathbb{C}^2$ for this blow-up.

**Proof.** Proof of Theorem [12] The idea is to decompose $\overline{M}_{0,5} - A(z)$ into the union of two subspaces, then use the Mayer-Vietoris exact sequences.

**Case (a):** $z \neq 0,1$. In this case we have:

$$\overline{M}_{0,5} - A(z) = (Bl_{\{0,1\}}(\mathbb{C}^2) - \pi^{-1}\{t_2 = 0\} - \pi^{-1}\{t_1 = z^{-1}\}) \cup (D_0 - \{\ast\})$$

where $\ast$ denotes the unique intersection point of $D_0$ and the proper transform of $t_2 = 0$. Let $U_1 = Bl_{\{0,1\}}(\mathbb{C}^2) - \pi^{-1}\{t_2 = 0\} - \pi^{-1}\{t_1 = z^{-1}\}$, then it’s open and isomorphic to $Bl_{\{1\}}(\mathbb{C}^2 - \{t_2 = 0\} \cup \{t_1 = z^{-1}\})$ which denotes the blow-up of $(\mathbb{C}^2 - \{t_2 =
0} ∪ \{t_1 = z^{-1}\}) at the point (1,1). Let \(B(\delta) = \{(t_1, t_2) \in \mathbb{C}^2 : |t_1|^2 + |t_2|^2 < 4\delta^2\}\) be a small open disk around (0,0) in \(\mathbb{C}^2\). Define \(U_2 = \pi^{-1}(B(\delta)) \cap (\overline{\mathcal{M}_{0,5}} - A(z))\), then it's clear that \(D_0 - \{\ast\}\) is the deformation retract of \(U_2\) by the line contraction: \((t_1, t_2) \to (at_1, at_2), 0 \leq a \leq 1\). So we have the decomposition: \(\overline{\mathcal{M}_{0,5}} - A(z) = U_1 \cup U_2\). We claim that \(U_1 \cap U_2\) deformation retracts to \(S_3^3 - S_3^1\), where \(S_3^k\) is the k-sphere of radius \(\delta\). Indeed, notice that \(U_1 \cap U_2 = \pi^{-1}(B(\delta) - \{(0,0)\}) \cap (\overline{\mathcal{M}_{0,5}} - A(z))\), hence it is isomorphic to \(B(\delta) - \{(t_1,0) : |t_1| < 2\delta\}\) which deformation retracts to \(S_3^3 - S_3^1\) under the map \(f_a(t) = (1-a)t + a \frac{\delta \cdot t}{\|t\|}, 0 \leq a \leq 1\), where \(t = (t_1, t_2)\) and \(\|t\| = \sqrt{|t_1|^2 + |t_2|^2}\).

Hence, \(H_i(U_1 \cap U_2) = H_i(S_3^3 - S_3^1)\). By the Alexander duality, we have

\[
H_i(S_3^3 - S_3^1; \mathbb{Q}) = H^i(S^1; \mathbb{Q}) = \begin{cases} \mathbb{Q}, & i = 0, 1 \\ 0, & \text{otherwise.} \end{cases}
\]

The natural generator of \(H_1(U_1 \cap U_2)\) is a simple loop around the divisor \(D_{0,2}\). Since \(U_2\) deformation retracts to \(D_0 - \{\ast\} = \mathbb{C}\), we have \(H_0(U_2) = \mathbb{Q}\) and \(H_i(U_2) = 0, i \neq 0\). Now \(U_1 \cong \text{Bl}_{\{1\}}(\mathbb{C}^2 - (\{t_2 = 0\} \cup \{t_1 = z^{-1}\}))\), so it follows from [9] Page 473-474 that:

\[
H_i(U_1) = \begin{cases} \mathbb{Q}, & i = 0 \\ H_1(\mathbb{C}^2 - (\{t_2 = 0\} \cup \{t_1 = z^{-1}\})), & i = 1 \\ H_2(\mathbb{C}^2 - (\{t_2 = 0\} \cup \{t_1 = z^{-1}\})) \oplus H_2(\mathbb{P}^1), & i = 2 \\ 0, & i \geq 3 \end{cases}
\]

Now apply Mayer-Vietoris exact sequence, we obtain:

\[
H_2(\overline{\mathcal{M}_{0,5}} - A(z)) \cong H_2(\mathbb{C}^2 - (\{t_2 = 0\} \cup \{t_1 = z^{-1}\})) \oplus H_2(\mathbb{P}^1) = \mathbb{Q} \oplus \mathbb{Q};
\]

\[
H_i(\overline{\mathcal{M}_{0,5}} - A(z)) = \mathbb{Q}; H_1(\overline{\mathcal{M}_{0,5}} - A(z)) = \mathbb{Q}; H_i(\overline{\mathcal{M}_{0,5}} - A(z)) = 0, i > 2.
\]

We also see that the generator for \(H_1(\overline{\mathcal{M}_{0,5}} - A(z))\) is a small simple loop around \(t_1 = z^{-1}\).

**Case (b):** \(z = 1\). We see in this case: \(\overline{\mathcal{M}_{0,5}} - A(1) = (\text{Bl}_{\{1\}}(\mathbb{C}^2) - \pi^{-1}(\{t_2 = 0\})) \cup (\{D_0 - \{\ast\}\} \cup (D_1 - \{\bullet\}))\), where \(\ast\) is the intersection point of \(D_0\) and \(D_{0,2}\) (the proper transform of \(t_2 = 0\)); \(\bullet\) is the intersection point of \(D_1\) and \(D_{1,1}\) (the proper transform of \(t_1 = 1\)). Let \(U_1 = \text{Bl}_{\{1\}}(\mathbb{C}^2) - \pi^{-1}(\{t_2 = 0\} \cup \{t_1 = 1\})\). It's open and isomorphic to \(\mathbb{C}^2 - (\{t_2 = 0\} \cup \{t_1 = 1\})\). As we did in Case (a), we can construct a small open neighborhood \(N_0\) of \(D_0 - \{\ast\}\) and \(N_1\) of \(D_1 - \{\bullet\}\) such that: (1) \(N_0\) deformation retracts to \(D_0 - \{\ast\}\), and \(N_1\) deformation retracts to \(D_1 - \{\bullet\}\); (2) \(N_0 \cap N_1 = \emptyset\); (3) \(N_1 \cap U_1\) deformation retracts to \(S_3^3 - S_3^1\). Now we can proceed as in Case (a) and find \(H_2(\overline{\mathcal{M}_{0,5}} - A(1)) \cong H_2(\mathbb{C}^* \times \mathbb{C}^*; \mathbb{Q}) = \mathbb{Q}, H_0(\overline{\mathcal{M}_{0,5}} - A(1); \mathbb{Q}) = \mathbb{Q}\), other homology groups vanish. Theorem [12] is proved.

**Proof.** Proof of Theorem [13]. Now we’ll use the results of Theorem [12] to compute the relative homology and cohomology groups.

**Case (a):** \(z \neq 0, 1\). First, let’s look at the divisor \(B_2 = B_2 \cap A(z)\) (See Figure 2). It consists of five irreducible components. Let’s label them as follows (See the Figure 1 and Figure 2):

\[
l_1 = D_{1,2} - \{z^{-1}, \infty\} = \mathbb{C} - \{z^{-1}\},
\]

\[
l_2 = D_1 = \mathbb{P}^1, l_3 = D - \{z^{-1}, \infty\} = \mathbb{C} - \{z^{-1}\},
\]

\[
l_4 = D_0 - \{0\} = \mathbb{C}, l_5 = D_{0,1} - \{\infty\} = \mathbb{C}.
\]
We have five intersection points of these components:
\[ a_1 = l_1 \cap l_5; \ a_2 = l_1 \cap l_2; \ a_3 = l_2 \cap l_3; \ a_4 = l_3 \cap l_4; \ a_5 = l_4 \cap l_5. \]

Let \( B_1 = \bigcup_{i=1}^{5} l_i \) be the disjoint union of the irreducible components \( l_i \); and \( B_0 = \bigcup_{i=1}^{5} a_i \) be the disjoint union of the points \( a_i; \ i = 1, 2, \ldots, 5 \). Then we can compute \( H_{i}(\mathcal{M}_{0,5} - A(z), B_2 - B_2 \cap A(z)) \) by the following bicomplex \((C_{p,q}, d, \delta)\):

\[
C_{p,q} = \begin{cases} 
  C_q(\mathcal{M}_{0,5} - A(z)) & \text{if } p = 0 \\
  C_q(B_1) & \text{if } p = 1 \\
  C_q(B_0) & \text{if } p = 2 \\
  0 & \text{otherwise}
\end{cases}
\]

where \( C_i(X) \) denote the vector space of singular \( i \)-chain on \( X \) with coefficients in \( \mathbb{Q} \). The vertical differential \( d : C_{p,q} \rightarrow C_{p,q-1} \) is the differential of the singular chain complex of \( X \) and the horizontal differential \( \delta \) is defined as following:

\[
\delta : C_q(B_1) \rightarrow C_q(\mathcal{M}_{0,5} - A(z)); \quad (c_i)_i \mapsto \sum_{i=1}^{5} (-1)^{i-1} c_i
\]

where each \( c_i \) is a \( q \)-chain of \( l_i, 1 \leq i \leq 5 \). And

\[
\delta : C_q(B_0) \rightarrow C_q(B_1); \quad \phi_{ij} \mapsto \phi_{ij}|_j - \phi_{ij}|_i
\]
where for each pair \( i < j \), \( \phi_{ij} \) is a \( q \)-chain of \( l_i \cap l_j \) and \( \phi_{ij} \mid_{l_i} \) represents the image of \( \phi_{ij} \) in \( l_i \) under the inclusion \( l_i \cap l_j \hookrightarrow l_i \). That is, we have the following diagram:

\[
\begin{array}{cccc}
\vdots & \vdots & \vdots & \\
\downarrow d & \downarrow -d & \downarrow d & \\
C_2(M_{0,5} - A(z)) & \overset{\delta}{\leftarrow} C_2(B_1) & \overset{\delta}{\leftarrow} C_2(B_0) & \\
\downarrow d & \downarrow -d & \downarrow d & \\
C_1(M_{0,5} - A(z)) & \overset{\delta}{\leftarrow} C_1(B_1) & \overset{\delta}{\leftarrow} C_1(B_0) & \\
\downarrow d & \downarrow -d & \downarrow d & \\
C_0(M_{0,5} - A(z)) & \overset{\delta}{\leftarrow} C_0(B_1) & \overset{\delta}{\leftarrow} C_0(B_0) & \\
\end{array}
\]

Consider the spectral sequence with \( E_1 \) terms given as:

\[
E_1^{p,q} = \begin{cases} 
H_q(M_{0,5} - A(z)) & \text{if } p = 0 \\
H_q(B_1) & \text{if } p = 1 \\
H_q(B_0) & \text{if } p = 2 
\end{cases}
\]

The differential \( d_1 : E_1^{p+1,q} \rightarrow E_1^{p,q} \) is induced by the horizontal differential \( \delta \). Use the explicit generators of the homology groups in the Theorem 12, we immediately get the \( E_2 \) terms and \( d_2 = 0 \):

\[
E_2^{p,q} = \begin{cases} 
\mathbb{Q} & \text{if } (p, q) = (0, 2), (1, 1), (2, 0) \\
0 & \text{otherwise} 
\end{cases}
\]

Hence the spectral sequence degenerates at the \( E_2 \) terms. So,

\[
H_2(M_{0,5} - A(z), B - B \cap A(z); \mathbb{Q}) = \begin{cases} 
\mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q}, & i = 2 \\
0, & \text{otherwise} 
\end{cases}
\]

And the filtration induced by this spectral sequence coincides with the weight filtration. Now let’s turn to the relative de Rham cohomology group \( H^i_{dR}(M_{0,5} - A(z), B - B \cap A(z)) \). It is defined as the \( i \)-th cohomology of the total complex of the following bicomplex \( (C^{p,q}, d, \delta) \):

\[
C^{p,q} = \begin{cases} 
A^q(M_{0,5} - A(z)) & \text{if } p = 0 \\
A^q(B_1) & \text{if } p = 1 \\
A^q(B_0) & \text{if } p = 2 \\
0 & \text{otherwise} 
\end{cases}
\]

where \( A^i(X) \) denote the vector space of \( C^\infty \) complex \( i \)-forms on \( X \), the vertical differential \( d : C^{p,q} \rightarrow C^{p,q+1} \) is the exterior differentiation of forms and \( \delta \) is defined as follows:

\[
\delta : A^q(M_{0,5} - A(z)) \rightarrow A^q(B_1); \quad \omega \mapsto (-1)^{q-1} \omega \mid_{l_i}
\]

where \( \omega \) is a \( q \)-form of \( M_{0,5} - A(z) \) and \( \omega \mid_{l_i} \) is the restriction of \( \omega \) on \( l_i \).
where $\theta = (\theta_i)$ is a $q$-form of $B_1$ and for each pair $i < j$, $\theta_{ij}$ is the restriction of $\theta_i$ on $l_i \cap l_j$. We have the following diagram:

\[
\begin{array}{ccc}
\vdots & \vdots & \vdots \\
\delta & \delta & \delta \\
A^2(\overline{M}_{0,5} - A(z)) & \delta & A^2(B_1) \\
\delta & \delta & \delta \\
A^1(\overline{M}_{0,5} - A(z)) & \delta & A^1(B_1) \\
\delta & \delta & \delta \\
A^0(\overline{M}_{0,5} - A(z)) & \delta & A^0(B_1) \\
\end{array}
\]

Similar to the case for homology, we see that the corresponding spectral sequence degenerates at the $E_2$ terms. So we obtain: $H^2_{dR}(\overline{M}_{0,5} - A(z), B - B \cap A(z)) = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$, and other relative cohomology groups vanish. For the case $z = 1$, the proof is similar and we leave it to the reader.

5.2. Computation of the Period Matrix. We have the following three cochains in the total complex associated to the deRham bicomplex $(\mathbb{C}^{p,q}, d, \delta)$:

\[
e_1 = (0, 0, \delta A_1); e_2 = (0, \frac{dt_1}{t_1 - z}, 0); e_3 = (\pi^*(\frac{dt_1}{t_1 - z} \wedge \frac{dt_2}{t_2}), 0, 0)
\]

where $\delta A_i$ is a function on $B_0$ satisfying $\delta A_i(A_1) = 1, \delta A_i(A_j) = 0, j \neq 1$ and $\frac{dt_i}{t_i - z}$ is a 1-form on the component $l_i$ of $B_1$. Since $\frac{dt_1}{t_1 - z}$ and $\pi^*(\frac{dt_1}{t_1 - z} \wedge \frac{dt_2}{t_2})$ are holomorphic on $l_1$ and $\overline{M}_{0,5} - A(z)$ respectively, they are closed forms. Hence $e_i$ are cocycles and represent elements in $H^2_{dR}(\overline{M}_{0,5} - A(z), B - B \cap A(z))$.

Next, let’s consider the cycles in the Betti bicomplex $(\mathbb{C}^{p,q}, d, \delta)$.

Let $b_1 = (\overline{\mathbf{F}}_2, \partial(\overline{\mathbf{F}}_2), \sum_{i=1}^{5} a_i)$, where $\partial(\overline{\mathbf{F}}_2)$ denotes the boundary of $\overline{\mathbf{F}}_2$ in $B_1$. It’s just the 5 sides of the dotted pentagon in figure 2. Let $b_2 = (C, \partial C, 0)$, where $C = \pi^{-1}(C_0), C_0 = (z^{-1} + \epsilon e^{2\pi i u}, v + (1 - v)(z^{-1} + \epsilon e^{2\pi i u})) \subset \mathbb{C} \times \mathbb{C} \subset \mathbb{P}^1 \times \mathbb{P}^1$, and $\epsilon$ is a small positive number, $0 \leq u \leq 1, 0 \leq v \leq 1$. It’s clear that the boundary $\partial C_0$ consists of 2 small cycles $(z^{-1} + \epsilon e^{2\pi i u}, 1)$ and $(z^{-1} + \epsilon e^{2\pi i u}, z^{-1} + \epsilon e^{2\pi i u})$. Since $C_0$ does not contain the blow-up points, $C$ is isomorphic to $C_0$ and the boundary $\partial C = \partial C_0$. The key point is that $\partial C$ is contained in $B_1 = \bigcup_{i=1}^{5} l_i$, which means exactly that $b_2$ is a cycle in the total complex. Finally, let $b_3 = (T, 0, 0)$, where $T = \pi^{-1}(z^{-1} + \epsilon e^{2\pi i u}, \epsilon e^{2\pi i v}), (0 \leq u \leq 1, 0 \leq v \leq 1)$ is the inverse image of a torus and since the three blow-up points are not on the torus, $T$ is isomorphic to its image. Clearly the boundary of $T$ is zero. Hence $b_3$ is a cycle.

Now we can calculate the period matrix $P = (p_{ij})$ between $(b_1, b_2, b_3)$ and $(e_1, e_2, e_3)$, here $p_{ij} = < e_i, b_j >, 1 \leq i, j \leq 3$. It’s straightforward that $< e_1, b_1 >= 1, < e_2, b_1 >= -L_i(z), < e_3, b_1 >= -L_i(z), and < e_1, b_3 >= 0, < e_2, b_3 >= 0, < e_3, b_3 >= (2\pi i)^2, < e_1, b_2 >= 0$.

Since $< e_2, b_2 >$ is equal to the integral of the 1-form $\frac{dt_1}{t_1 - z}$ over a small circle with
center $t_1 = z^{-1}$, we have $<e_2, b_2> = 2\pi i$. And

$$<e_3, b_2> = \int_C \pi^*(\frac{dt_1}{t_1 - z^{-1}} \wedge \frac{dt_2}{t_2}) = \int_{C_0} \frac{dt_1}{t_1 - z^{-1}} \wedge \frac{dt_2}{t_2}$$

$$= \int_0^1 \int_0^1 2\pi i du \cdot d(\log (v + (1 - v)(z^{-1} + e^{2\pi i u})))$$

$$= -2\pi i \int_0^1 \log (z^{-1} + e^{2\pi i u}) du = -2\pi i \log z^{-1}$$

$$= 2\pi i \log z$$

Therefore, we get the period matrix:

$$\begin{pmatrix}
1 & 0 & 0 \\
-Li_1(z) & 2\pi i & 0 \\
-Li_2(z) & 2\pi i \log z & (2\pi i)^2
\end{pmatrix}.$$

Since it’s nonsingular, it follows that \{e_1, e_2, e_3\} and \{b_1, b_2, b_3\} are bases. For part(b), it’s clear that: for the cocycles, we have $e_1 = (0, 0, \delta_{A_1}), e_3 = (\pi^*(\frac{dt_1}{t_1 - z^{-1}} \wedge \frac{dt_2}{t_2}), 0, 0)$; for the cycles, we have $b_1 = (\Phi_2, \partial(\Phi_2), \sum_{i=1}^5 a_i), b_3 = (T, 0, 0)$. Therefore, its period matrix is

$$\begin{pmatrix}
1 & 0 \\
-Li_2(1) & (2\pi i)^2
\end{pmatrix}.$$

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