Equal-time hierarchies for transport descriptions of fermionic fields

R.A. Ionescu\textsuperscript{1}, H.H. Wolter
Sektion Physik, Universität München, D-85748 Garching, Germany

November 4, 2018

Abstract

A transport theory which is not restricted to the gradient and quasi-particle approximations is presented which is formulated in terms of the energy moments, or equivalently the equal-time derivatives of the one-particle Green functions. A set of infinite hierarchies of kinetic and constraint equations for equal-time quantities for the spectral and the kinetic part of the one-particle Green function is derived. The hierarchies for the spectral function truncate automatically as in the mean field approximation. The possibility of a systematic truncation of the hierarchies for the kinetic part is discussed. The effects of the quantum corrections are illustrated in a simple one-dimensional model.

\textsuperscript{1}Permanent Address: National Institute for Physics and Nuclear Engineering ”H. Hulubei”, P.O.Box MG-6, Bucharest, Romania; e-mail: amilcar@tandem.nipne.ro
1 Introduction

A transport description of many-body non-equilibrium systems is a very convenient approximation used with success in the study of heavy ion collisions in the last two decades [1]. It has also been used for more than 40 years in solid state physics to describe the transport properties like electric conductivity, thermal conductivity, etc. In this approach the one-body phase-space distribution function satisfies a semi-classical transport equation of Boltzmann type. The validity of a transport description can be established by a rigorous derivation from quantum field theory [2].

A main step in the derivation of transport equations starting from the Dyson equation for the Green function is to put particles on the mass-shell [3]. On the other hand, off-shell effects are thought to be important in heavy ion collisions for processes like subthreshold production of particles or for the inclusion of baryonic resonances in the transport description [3]. This is the case, in particular, for the $\Delta$-resonance for which the spectral function has a large width even outside the medium.

The off-shell effects are related to the dependence of one-particle Green functions on two time arguments. A systematic way to include the off-shell effects into the dynamics of a particle in an external $c$-number field (or in the mean field approximation) was developed [4] resulting in transport and constraint equations for equal time quantities which are derivatives of the Green function in the relative time at zero relative time, or equivalently energy moments of its Wigner transform. The main advantage of this equal-time description is the possibility to specify the initial conditions necessary to solve the transport equations.

More recently new derivations of semiclassical transport equations [5, 6, 7] do not use the on-shell approximation. They result in first order partial differential equation for some function defined on 8-dimensional (phase) space which is numerically solved using the test particle method by further approximating it by a sum of $\delta$ functions. It has been argued in ref. [6] that in general the function that can be represented by a test particle ansatz is not the phase space distribution function, but some more general quantity. This complicates the solution of the transport equations in 8-dimensional phase space, giving interest to alternative methods. In addition these treatments use the first-order gradient approximation which is based on the assumption of a weakly inhomogeneous
and slowly evolving system. The gradient approximation may lose validity when the range of the interactions becomes comparable with that of the space inhomogeneities of the system. Other authors therefore generalize the Boltzmann equation by including third order terms in the space gradients and obtain equations for quantities which depend on position, time, energy and two momentum components.

In the present work we propose a systematic way to derive equations for quantities which characterize the off-shell behaviour of the Green functions and also include higher order gradient terms, i.e. go beyond the mean field approximation. The idea is to formulate a theory for the energy moments of the one-particle Green functions. These energy moments measure the shape of the Green functions in energy, e.g. the second order energy moment is a measure of the width of the spectral function. Generally the \( n \)-th-order energy moment is related to the \( n \)-th-order derivative of the Green function with respect to the relative time taken at zero relative time. We use the method of real-time Green functions defined on the Schwinger-Keldysh time path, which is the appropriate formalism to describe quantum systems in non-equilibrium. Taking the Wigner transform of the Dyson equation in space coordinates only we obtain evolution and constraint equations for the energy moments of Green functions. The result is a set of four infinite hierarchies of equations. A systematic way to truncate these hierarchies is discussed.

As we are going beyond the first order gradient approximation, we can study the non-localization effects due to the quantum motion. There have been attempts to include the nonlocal terms in the scattering integral and this leads to consistency with the thermodynamic virial corrections.

In the last section we will illustrate in a simple 1-dimensional example the effects of the higher order gradient terms in the evolution equation as well as the influence of the memory effects on the velocity of thermalization.
2 Spatial Wigner transforms of Dyson equations

To keep the formalism simple we consider in this work a non-relativistic system of fermions described by the Hamiltonian

\[ H = \int d^3x \Psi_H^\dagger(x) \left( -\frac{1}{2m} \Delta \right) \Psi_H(x) + \frac{1}{2} \int d^3x \int d^3y \Psi_H^\dagger(x) \Psi_H^\dagger(y) V(x-y) \Psi_H(y) \Psi_H(x) \]  

(1)

where \( \Psi_H^\dagger \) and \( \Psi_H \) are the field operators in the Heisenberg picture and \( V \) is the two-particle interaction potential \((\hbar = 1)\). A Dirac or Klein-Gordon field can be treated along the same lines, however, here we want to avoid the technical complications related to the Dirac matrix structure of the Green function and self-energy.

For a non-equilibrium situation it is convenient to introduce a path-ordered Green function defined on the Schwinger-Keldysh double time contour \( \{1\} \). The one-particle Green function on the contour \( G \) is defined as the expectation value of a product of field operators in the Heisenberg picture

\[ iG(1,1') = Tr \left[ \rho_H P(\Psi_H(1)\Psi_H^\dagger(1')) \right], \]

(2)

where \( 1 \equiv (x_1,t_1) \) and \( \rho_H \) is the time-independent density operator which specifies the state. Depending on the position of the two time arguments on the contour \( G \) is equivalent to four different Green functions defined on the real time axis: causal and anticausal when the two time arguments are on the same branch of the contour and correlation Green functions when the time arguments are on different branches

\[ iG^{c,a}(1,1') = Tr \left[ \rho_H T^{c,a}(\Psi_H(1)\Psi_H^\dagger(1')) \right] \]

\[ iG^{>}(1,1') = Tr \left[ \rho_H (\Psi_H(1)\Psi_H^\dagger(1')) \right] \]

\[ iG^{<}(1,1') = -Tr \left[ \rho_H (\Psi_H^\dagger(1')\Psi_H(1)) \right] \]  

(3)

The causal and anticausal Green functions satisfy the following decomposition relations

\[ G^{c,a}(1,1') = \theta(t_1-t_{1'})G^{>}(1,1') + \theta(-(t_1-t_{1'}))G^{<}(1,1') \]  

(4)

It is useful to define retarded and advanced Green functions

\[ G^{+-}(1,1') = \pm \theta(\pm(t_1-t_{1'})) [G^{>}(1,1') - G^{<}(1,1')] \]  

(5)
as well as spectral and Keldysh (or kinetic) Green functions

\[ G^A(1, 1') = G^>(1, 1') - G^<(1, 1') \]  
\[ G^K(1, 1') = G^>(1, 1') + G^<(1, 1'). \]  

We note [2] that all the Green functions defined above can be expressed in terms of only two independent Green functions, for which we take \( G^A \) and \( G^K \). For future reference we note the relations

\[ (G^{K,A}(1, 1'))^* = -G^{K,A}(1', 1) \]  

and, as a result of the equal-time commutation relations, the property

\[ G^A(1, 1') \big|_{t_1 = t_1'} = \delta(x_1 - x_1') \]  

Collecting the Green functions into a 2 \( \times \) 2 matrix \( \mathcal{G} = \begin{pmatrix} G^c & G^< \\ G^> & G^a \end{pmatrix} \), and correspondingly for the other one-particle quantities, the Dyson equation can be written in the equivalent forms

\[ \mathcal{G}(1, 1') = \mathcal{G}_0(1, 1') + \int d1' \int d1'' \mathcal{G}_0(1, 1'') \tilde{\Sigma}(1'', 1'') \mathcal{G}(1'', 1') \]  
\[ \mathcal{G}(1, 1') = \mathcal{G}_0(1, 1') + \int d1' \int d1'' \mathcal{G}(1, 1'') \tilde{\Sigma}(1'', 1'') \mathcal{G}_0(1'', 1') \]  

where the time arguments take values on the Schwinger-Keldysh contour and \( \mathcal{G}_0 \) is the free Green function which satisfies the equations

\[ (i \frac{\partial}{\partial t_1} + \frac{1}{2m} \Delta_1) \mathcal{G}_0(1, 1') = \delta(1, 1') \]  
\[ (-i \frac{\partial}{\partial t_{1'}} + \frac{1}{2m} \Delta_{1'}) \mathcal{G}_0(1, 1') = \delta(1, 1') \]  

with \( \delta \) the Dirac \( \delta \)-function defined on the contour. The self-energy \( \tilde{\Sigma} \) contains all the information on the interaction. The way in which it is approximated specifies the type of approximation used. We will discuss this for a few cases below. Its causal and anti-causal components have the same decomposition relations as the Green functions, eq. (5), provided the singular, \( i.e. \) time-local, part on the contour is separated off [12]

\[ \tilde{\Sigma}(1, 1') = \Sigma_\delta(x_1, x_{1'}, t_1) \delta(t_1 - t_{1'}) + \Sigma(1, 1') \]
This singular part is given by the Hartree and Fock diagrams which are local in time, and are also discussed further in Sect. 5 below. Also for Σ we apply analogous definitions as in eqs. (6,7). In the following, to avoid writing integrals, we consider the space-time coordinates as matrix indices and use the Einstein convention to sum (integrate) over the repeated indices (coordinates).

The equations satisfied by the interacting Green functions, eqs. (10) and (11), can be put in a triangular matrix form for $\tilde{G} = \begin{pmatrix} G^- & 0 \\ G K & G^+ \end{pmatrix}$ [14]. From the non-diagonal part we obtain the equations for the kinetic part of the Green function

$$s_1 G^K(1', 1') - \Sigma_\delta(1', 1'') G^K(1'', 1') = \Sigma^+(1', 1'') G^K(1'', 1') + \Sigma^K(1', 1'') G^-(1'', 1')$$

$$s_1^\dagger G^K(1', 1') - G^K(1', 1') \Sigma_\delta(1'', 1') = G^K(1, 1'') \Sigma^-(1', 1'') + G^+(1, 1'') \Sigma^K(1', 1')$$ [14]

where $s_i = i \partial_i + \frac{1}{2m} \Delta_i$. From the diagonal part we obtain analogous equations for the spectral part of the Green function, $G^A$.

To derive an equal-time formulation we perform a Wigner transform of these equations in the space coordinates only. For the Wigner transform of a general function $F$ we use the notation

$$f(R, p, T; \tau) = \int d^3r e^{-i pr} F(R + r/2, T + \tau/2, R - r/2, T - \tau/2).$$ [15]

The transforms of $G$ and $\Sigma$ are denoted by $g$ and $\sigma$, respectively.

Introducing the differential operator (Poisson operator)

$$\hat{\diamond}(f \cdot g) = (\nabla_R f)(\nabla_R g) - (\nabla_p f)(\nabla_R g)$$ [16]

the equations for the Wigner transformed spectral and kinetic Green functions, $g^A$ and $g^K$, are

$$\left[ \pm \frac{i}{2} \frac{\partial}{\partial T} + \frac{1}{2m} \left( \frac{1}{4} \Delta_R \pm i p \nabla_R - p^2 \right) + i \frac{\partial}{\partial \tau} - e^{\pm \frac{\tau}{2}} \sigma_\delta(T \pm \tau/2) \right] \begin{pmatrix} g^A(T; \tau) \\ g^K(T; \tau) \end{pmatrix} = e^{\pm \frac{\tau}{2}} \begin{pmatrix} S^A_\pm(T; \tau) \\ S^K_\pm(T; \tau) \end{pmatrix}$$ [17]

with the collision integrals

$$S^A_\pm(T; \tau) = \int_0^\tau dt \sigma^A(T \pm t \pm \tau - t; t) \cdot g^A(T \mp \frac{\tau - t}{2}; t)$$ [18]

$$S^K_\pm(T; \tau) = \pm \int_{-\infty}^{\pm \tau} dt \sigma^A(T + \frac{t}{2}; \mp \tau \mp t) \cdot g^K(T \mp \frac{\tau + t}{2}; \pm t)$$

$$\mp \int_{-\infty}^{\pm \tau} dt \sigma^K(T + \frac{t}{2}; \mp \tau \pm t) \cdot g^A(T \pm \frac{\tau \pm t}{2}; \pm t)$$ [19]
where we did not indicate the dependence of \( g^{A,K} \) and \( \sigma^{A,K} \) on \( R \) and \( p \). The upper and lower signs in eqs. (17-19) result when starting from eqs. (10) and (11), respectively. The boundaries of the integrals in eqs. (18,19) come from the step function in eq. (5).

Setting \( \sigma^> = \sigma^< = 0 \), i.e. when the scattering is neglected, the problem is treated in the mean field approximation, the terms on the right-hand side of the above equations vanish, and eqs. for \( g^A \) and \( g^K \) are identical. Therefore one has only two equations for \( g^< = \frac{1}{2}(g^K - g^A) \) which is directly related to the particle density. In the general case it is neccessary to consider all the four equations. Taking the sum and the difference of eqs. (17) we obtain for \( g^A \) and \( g^K \) two sets of equations

\[
\begin{align*}
&\left\{ \frac{i}{\hbar} \partial_T + \frac{\hbar}{m} \nabla_R - \left[ e^{\frac{i}{2}\sigma_\delta(T + \tau/2)} - e^{-\frac{i}{2}\sigma_\delta(T - \tau/2)} \right] \\
&\quad \left\{ \frac{1}{m} \left( \frac{1}{2} \Delta_R - p^2 \right) + 2i\hbar \frac{\partial}{\partial \tau} - \left[ e^{\frac{i}{2}\sigma_\delta(T + \tau/2)} + e^{-\frac{i}{2}\sigma_\delta(T - \tau/2)} \right] \right\} \right. \\
&\quad e^{\frac{i}{2}\sigma_+} S_+^{A,K}(T;\tau) - e^{-\frac{i}{2}\sigma_+} S_-^{A,K}(T;\tau) \\
&\quad e^{\frac{i}{2}\sigma_-} S_+^{A,K}(T;\tau) + e^{-\frac{i}{2}\sigma_-} S_-^{A,K}(T;\tau) \\
&\left. \right\} \right. 
\end{align*}
\]

(20)

The equations in the upper row are called kinetic equations because they involve the time evolution in the global time \( T \). The ones in the lower row are called constraint equations \([4]\), since under certain conditions (see below) they are algebraic equations.

### 3 Hierarchies of equations for the spectral functions

The eqs. (20) are, of course, still exact. The quantities \( g^{A,K}(T;\tau), \sigma^{A,K}(T;\tau) \) depend on two time arguments, the global time \( T \) and the relative time \( \tau \). The integrations in the collision terms, (eqs. (18) and (19)), involve these functions for all relative times from \(-\infty\) to \(+\infty\). Thus these equations are extremely difficult to handle in practice. A drastic approximation is to put the relative time equal to zero, i.e. to write an equal-time formalism. This route has been followed in other approaches \([2]\). This is equivalent to the so-called quasi-particle approximation, in which the energy dependence of the spectral function is approximated by a \( \delta \) function, and thus off-shell properties of the distribution functions are neglected. As mentioned in the introduction this is a drastic approximation for many reasons. An improvement is therefore expected by expanding all quantities around \( \tau = 0 \). The off-shell information is then contained in the time derivatives with respect to \( \tau \) at \( \tau = 0 \), either inside of the convergence radius by Taylor expansion or by analytic continuation. We therefore will develop an infinite set of equations, or a
hierarchy of equations, for these equal-time derivatives originating from the kinetic and the constraint equations.

The equal-time \( \tau \)-derivatives of order \( n \) \( g^{(n)}(T, \tau) \) are related to the energy moments of order \( n \) of the Fourier transformed function \( g(T, \omega) \) since we can write

\[
g^{(n)}(T) := \left. \partial^n\tau g(T, \tau) \right|_{\tau=0} = \left. \partial^n\tau \int \frac{d\omega}{2\pi} e^{i\omega\tau} g(T, \omega) \right|_{\tau=0} = \left. \left( -i \right)^n \omega^n g(T, \omega) \right|_{\tau=0} = \frac{(-i)^n}{2\pi} \int d\omega \omega^n g(T, \omega). \tag{21}
\]

However, the energy moments may not exist in general. E.g. if \( g(T, \omega) \) behaves like \( \frac{1}{\omega^2} \) for large \( \omega \), then the second and higher energy moments of the spectral component diverge. Such a behaviour e.g. occurs in the gradient approximation of the spectral function neglecting the dependence of the imaginary part of the retarded self-energy on \( \omega \). The reason is that integration and differentiation cannot, in general, be interchanged in eq. (21). In contrast to the energy moments the time derivatives \( g^{(n)}(T) \) exist as seen from the definition of \( g \) in terms of operator products.

The problem of the divergence of the energy moments could be overcome by introducing a cut-off in the energy-integral, but we do not choose this procedure here. However, we can retain the physical intuition that, e.g. for the spectral Green function, the zeroth derivative \( g^{(0)}(T) \) is related to the normalization of the function, the first derivative \( g^{(1)}(T) \) to the mean energy of the quasi-particle, and the second derivative \( g^{(2)}(T) \) to the width of the spectral function. Indeed, neglecting the possible divergence problems, we have for the spectral function \( A \) defined e.g. by Henning [1]

\[ A(T, \omega) = -\frac{1}{2\pi} \int d\tau e^{i\omega\tau} g^A(T; \tau). \]

Then,

\[ \int d\omega \omega^n A(T, \omega) = i^{n+1} g^{A(n)} \]

and the \( n \)th \( \tau \)-derivative is exactly the \( n \)th energy moment of the four dimensional Wigner transform of the spectral function up to a factor \( i^{n+1} \).

We start by setting up the hierarchy for the spectral function \( g^A \). From the upper row of eqs. (20) for \( g^A \) we obtain the hierarchy of kinetic equations for the equal-time \( \tau \)-derivatives of the spectral component. The equal-time commutation relation, through eq.(9), results into

\[ g^{A(0)} = -i \tag{22} \]

which is the sum rule for the spectral function. Using this relation, the first three
equations of the kinetic hierarchy are
\begin{align*}
&i \frac{\partial}{\partial T} g^{A(0)} - 2i \sin \frac{1}{2} \hat{\phi}(\epsilon \cdot g^{A(0)}) = 0 \\
&i \frac{\partial}{\partial T} g^{A(1)} - 2i \sin \frac{1}{2} \hat{\phi}(\epsilon \cdot g^{A(1)}) - \cos \frac{1}{2} \hat{\phi}(\frac{\partial \epsilon}{\partial T} \cdot g^{A(0)}) = 0 \\
&i \frac{\partial}{\partial T} g^{A(2)} - 2i \sin \frac{1}{2} \hat{\phi}(\epsilon \cdot g^{A(2)}) - 2 \cos \frac{1}{2} \hat{\phi}(\frac{\partial \epsilon}{\partial T} \cdot g^{A(1)}) = 2i \sin \frac{1}{2} \hat{\phi}(\sigma^{A(0)} \cdot g^{A(1)}) - i \frac{\partial \sigma^{A(0)}}{\partial T} \\
&\text{where } \epsilon = \frac{p^2}{2m} + \sigma_0(T) \text{ and where we used the identities}
\end{align*}
\begin{align*}
\frac{p}{m} \nabla_R f &= -2 \sin \frac{1}{2} \hat{\phi}(\frac{p^2}{2m} \cdot f) \\
\frac{1}{m} (\frac{1}{4} \triangle_R - p^2) f &= -2 \cos \frac{1}{2} \hat{\phi}(\frac{p^2}{2m} \cdot f)
\end{align*}

The integrals on the rhs of eqs. (20) for \( g^A \) vanish at \( \tau = 0 \), eqs. (18). From the second row of equation (20) we obtain a constraint hierarchy from which we write down the first two members.
\begin{align*}
-2 \cos \frac{1}{2} \hat{\phi}(\epsilon \cdot g^{A(0)}) + 2ig^{A(1)} &= 0 \\
-2 \cos \frac{1}{2} \hat{\phi}(\epsilon \cdot g^{A(1)}) + 2ig^{A(2)} &= 2 \cos \frac{1}{2} \hat{\phi}(\sigma^{A(0)} \cdot g^{A(0)}) = 2 \cos \frac{1}{2} \hat{\phi}(\sigma^{A(0)} \cdot g^{A(0)})
\end{align*}

In general the \( n \)th equation of the constraint hierarchy gives the \( n \)th \( \tau \)-derivative of the spectral function, \( g^{A(n)} \), in terms of the lower order \( \tau \)-derivatives of the spectral function, \( g^{A(j)} \) and of the spectral self-energy , \( \sigma^{A(j)} \) with \( j \leq n - 1 \). On the other hand the \( (n+1) \)th equation of the kinetic hierarchy is a kinetic equation for the \( n \)th \( \tau \)-derivative of the spectral function, \( g^{A(n)} \), in which only the lower order \( \tau \)-derivatives of the spectral function and spectral self-energy, \( g^{A(j)} \) and \( \sigma^{A(j)} \) for \( j \leq n - 1 \), appear. The first kinetic equation (23) is satisfied by \( g^{A(0)} = -i \), eq. (22).

The natural question arises: are the \( n \)th equation of the constraint hierarchy and the \( (n+1) \)th equation of the kinetic hierarchy compatible? In fact, the \( (n+1) \)th kinetic equation is not a new equation, but it can be expressed as a combination of lower order constraint and kinetic equations. Since we have not made any approximation yet, this result is true beyond the mean field approximation, in which only the singular part of the self-energy, \( i.e. \) Hartree-Fock self-energy, is retained. We will show this explicitly in the following up to order \( n = 2 \). This means for the spectral part that the algebraic solutions
of the constraint hierarchy satisfy the kinetic hierarchy, and that the time evolution is
described only by the $T$-dependence of the self-energies.

With eq. (22) and eq. (24) we have from the constraint hierarchy

\begin{align}
g^{A(1)} &= -\epsilon \\
g^{A(2)} &= i \cos \frac{1}{2} \Diamond (\epsilon \cdot \epsilon) - \sigma^{A(0)}
\end{align}

To prove the compatibility of the kinetic and constraint hierarchies we have to
verify that the solutions (25,26) satisfy eqs. (23). This is done in Appendix B using
the properties of the differential operator $\Diamond$ given in Appendix A. Therefore, only the
constraint hierarchy is necessary, the kinetic hierarchy being redundant, as the solutions
of the former satisfy automatically the latter one. The $T$-derivatives of the spectral
function can be determined from the constraint hierarchy if the $\tau$-derivatives of the
spectral self-energy are known. In general, these depend on the $\tau$-derivatives of the
kinetic part of the Green function, $g^K$, as will be shown later. Thus the spectral and
kinetic Green functions are coupled.

## 4 Hierarchies of equations for the kinetic Green functions

Following the same way as for the spectral function we obtain from eqs. (20) two infinite
hierarchies of equations for the equal-time $\tau$-derivatives of the kinetic part $g^K$. Now also
the equal-time derivatives of the collision integral enter. The first three equations of the
kinetic hierarchy are

\begin{align}
i \frac{\partial}{\partial T} g^{K(0)} - 2i \sin \frac{1}{2} \Diamond (\epsilon \cdot g^{K(0)}) &= e^{\frac{1}{2} \Diamond} S^{K(0)}_+ - e^{-\frac{1}{2} \Diamond} S^{K(0)}_-\\
i \frac{\partial}{\partial T} g^{K(1)} - 2i \sin \frac{1}{2} \Diamond (\epsilon \cdot g^{K(1)}) - \cos \frac{1}{2} \Diamond (\frac{\partial \epsilon}{\partial T} \cdot g^{K(0)}) &= e^{\frac{1}{2} \Diamond} S^{K(1)}_+ - e^{-\frac{1}{2} \Diamond} S^{K(1)}_-\\
i \frac{\partial}{\partial T} g^{K(2)} - 2i \sin \frac{1}{2} \Diamond (\epsilon \cdot g^{K(2)}) - 2 \cos \frac{1}{2} \Diamond (\frac{\partial \epsilon}{\partial T} \cdot g^{K(1)}) - i \frac{1}{2} \sin \frac{1}{2} \Diamond (\frac{\partial^2 \epsilon}{\partial T^2} \cdot g^{K(0)}) &= e^{\frac{1}{2} \Diamond} S^{K(2)}_+ - e^{-\frac{1}{2} \Diamond} S^{K(2)}_-
\end{align}

where $S^{K}_\pm$ are given by eq. (19).
The first two equations of the constraint hierarchy are

\[-2\cos \frac{1}{2} \diamond (\varepsilon \cdot g^{K(0)}) + 2ig^{K(1)} = e^{\frac{i}{2} \diamond} S_{\pm}^{K(0)} + e^{-\frac{i}{2} \diamond} S_{-}^{K(0)}
- i \sin \frac{1}{2} \diamond (\frac{\partial \varepsilon}{\partial T} \cdot g^{K(0)}) - 2 \cos \frac{1}{2} \diamond (\varepsilon \cdot g^{K(1)}) + 2ig^{K(2)} = e^{\frac{i}{2} \diamond} S_{\pm}^{K(1)} + e^{-\frac{i}{2} \diamond} S_{-}^{K(1)} \tag{28}\]

Let us first consider the mean field approximation, when collisions are neglected \((S_{\pm}^{K}(\tau) \equiv 0)\). The \(n^{th}\) equation of the constraint hierarchy gives the \(n^{th}\) \(\tau\)-derivative, \(g^{K(n)}\), in terms of the lower order \(\tau\)-derivatives, \(g^{K(j)}\) for \(j \leq n - 1\). The \((n+1)^{th}\) equation of the kinetic hierarchy is an evolution equation for the \(n^{th}\) \(\tau\)-derivative, \(g^{K(n)}\). In this case we show in Appendix B that at least up to order \(n = 2\) only the first equation of the kinetic hierarchy and all the equations of the constraint hierarchy are independent; the \(n^{th}\) equation of the kinetic hierarchy, \(n > 1\), can be expressed in terms of the lower order kinetic and constraint equations. Therefore the two hierarchies are compatible in the mean field approximation (at least up to order \(n = 2\)). We note that in general the singular part of the self-energy on the contour may depend on \(p\), in contrast to the usual assumption \(\mathbb{A}\).

The inclusion of the collision contribution is more difficult because we have to approximate the integrals \(S_{\pm}^{K(0)}\) and \(S_{\pm}^{K(1)}\). The main contribution to the integrals

\[S_{\pm}^{K(0)} = \pm \int_{-\infty}^{0} dt [\sigma^{A}(T + \frac{t}{2}, \mp t)g^{K}(T + \frac{t}{2}, \pm t) - \sigma^{K}(T + \frac{t}{2}, \mp t)g^{A}(T + \frac{t}{2}, \mp t)]\]

comes from small values of the variable of integration \(t\) since large values involve Green functions at large relative time, which are expected to be small. Therefore we can approximate the integrals using as a lower limit a judiciously chosen parameter \(-\Delta\) instead of \(-\infty\) (or \(T_{0} - T\)). For this purpose we expand the functions under the integral of the type \(f(T + t/2, \pm t)\) in \(t\) around \(f(T, t = 0)\), i.e. around the equal-time value at current global time \(T\), and integrate over \(dt\). If we expand the integrand to first order in \(t\) we obtain the second order approximation in the parameter \(\Delta\) for the collision integrals

\[S_{\pm}^{K(0)} = \pm \Delta \left(1 - \frac{\Delta}{4} \frac{\partial}{\partial T}\right) [\sigma^{A(0)} g^{K(0)} - \sigma^{K(0)} g^{A(0)}] + \frac{1}{2} \Delta^{2} (\sigma^{A(1)} g^{K(0)} - \sigma^{A(0)} g^{K(1)} - \sigma^{K(1)} g^{A(0)} + \sigma^{K(0)} g^{A(1)})] \tag{29}\]

where the time argument \(T\) has been omitted everywhere. For the derivatives of \(S_{\pm}^{K}\) and \(S_{-}^{K}\), by expanding the integrand in zero order in \(t\), we obtain an expression linear in \(\Delta\)

\[S_{\pm}^{K(1)} = \frac{1}{2} (\sigma^{A(0)} g^{K(0)} + \sigma^{K(0)} g^{A(0)})\]
\[ \pm \frac{\Delta}{2} \left( \sigma^{A(1)} g^{K(0)} + \sigma^{A(0)} g^{K(1)} - \sigma^{K(1)} g^{A(0)} - \sigma^{K(0)} g^{A(1)} \right) \]
\[ \pm \frac{1}{2} \left( \frac{\partial \sigma^{A(0)}}{\partial T} g^{K(0)} - \sigma^{A(0)} \frac{\partial g^{K(0)}}{\partial T} - \frac{\partial \sigma^{K(0)}}{\partial T} g^{A(0)} + \sigma^{K(0)} \frac{\partial g^{A(0)}}{\partial T} \right) \]  

(30)

Another way to approximate the collision integrals by a convergence factor \( e^{-\alpha} \) is presented in the appendix C. There it is seen that the results are essentially the same as above, with \( \Delta \) replaced by a term proportional to \( 1/\alpha \).

Using the above expressions (29) and (30) for the collision integrals the first two equations of the kinetic and constraint hierarchies for the kinetic Green function, \( g^K \), include memory effects due to the presence of time derivatives. The eqs. (29) and (30) are second, respectively first, order polynomial approximations in \( \Delta \) for the collision terms. In general, in the \( k \)th order approximation of \( S^K_\pm(0) \) in \( \Delta \) appear \( \tau \)-derivatives of the Green functions and of the self-energy up to order \( k - 1 \). As we will see in the next section, the latter can be obtained in terms of only the same order \( \tau \)-derivatives of the Green functions. However, in the same approximation in \( \Delta \), the collision terms in the \( n \)th-order equations of the two hierarchies contain \( \tau \)-derivatives of the self-energy up to order \( n + k - 2 \). Thus the appropriate truncation of the hierarchies is not as obvious as in the mean field approximation; only the hierarchy for the spectral function truncates by itself, completely similar to the mean field case.

If we approximate the collision terms \( S^K_\pm(0) \) in order \( k \), \( k \geq 1 \), with respect to \( \Delta \) then the first equation of the kinetic hierarchy contains the \( \tau \)-derivatives of the Green functions up to order \( k - 1 \). Therefore in general we choose to approximate the collision terms which appear in the \( n \)th-order equations of the two hierarchies, \( S^K_\pm(n-1) \), in order \( k - (n-1) \). Within this approximation the first \( k - 1 \) equations of the constraint hierarchy contain also the \( \tau \)-derivatives of the Green functions up to order \( k - 1 \). Therefore, one has \( k \) equations for \( k \) \( \tau \)-derivatives. The \( \tau \)-derivatives of the self-energies appear up to order \( k - 1 \), i.e. \( N \) \( \tau \)-derivatives and they can be obtained self-consistently.

In the following we will consider the \( k = 2 \) approximation. The kinetic and constraint equations are

\[ i \frac{\partial}{\partial T} g^{K(0)} - 2i \sin \frac{1}{2} \hat{\phi} (\epsilon \cdot g^{K(0)}) = \]
\[ 2\Delta \left( 1 - \frac{\Delta}{4} \frac{\partial}{\partial T} \right) \cos \frac{1}{2} \hat{\phi} (\sigma^{A(0)} g^{K(0)} - \sigma^{K(0)} g^{A(0)}) \]
\[ + i \Delta^2 \sin \frac{1}{2} \left( \sigma^{A(1)} g^{K(0)} - \sigma^{A(0)} g^{K(1)} - \sigma^{K(1)} g^{A(0)} + \sigma^{K(0)} g^{A(1)} \right) \]  
\[ - 2 \cos \frac{1}{2} \left( \epsilon \cdot g^{K(0)} + 2 i g^{K(1)} = \right) \]
\[ 2i \Delta \left( 1 - \frac{\Delta}{4} \frac{\partial}{\partial T} \right) \sin \frac{1}{2} \left( \sigma^{A(0)} g^{K(0)} - \sigma^{K(0)} g^{A(0)} \right) \]
\[ + \Delta^2 \cos \frac{1}{2} \left( \sigma^{A(1)} g^{K(0)} - \sigma^{A(0)} g^{K(1)} - \sigma^{K(1)} g^{A(0)} + \sigma^{K(0)} g^{A(1)} \right) \]

(31)  
(32)

The familiar Vlasov and Boltzmann equations are particular cases of the above approximation. When in the collision integrals $S^K_{\pm}$ only the terms that are zeroth order in $\Delta$ are retained (the collision integrals are neglected), the equations for $g^{K(0)}$ has the same form as the corresponding equation for $g^{A(0)}$. Thus the first equation of the kinetic hierarchy is

\[ i \frac{\partial}{\partial T} g^{K(0)} - 2i \sin \frac{1}{2} \left( \epsilon \cdot g^{K(0)} \right) = 0 \]

which in the gradient approximation is the Vlasov equation with a self consistent mean-field, which is included in $\epsilon$. The validity of this approximation is not directly related to the quasi-particle approximation. Here we only use the assumption that the collision term can be approximated in zeroth order in $\Delta$.

The lowest order nontrivial approximation is $k = 1$, which results in taking only terms with $g^{A(0)}$ and $g^{K(0)}$ in eqs. (31,32). We then have the kinetic equation for $g^{K(0)}$

\[ i \frac{\partial}{\partial T} g^{K(0)} - 2i \sin \frac{1}{2} \left( \epsilon \cdot g^{K(0)} \right) = \left( \sigma^{A(0)} g^{K(0)} - \sigma^{K(0)} g^{A(0)} \right) \]

(33)

Let us note that the term $(\sigma^{A(0)} g^{K(0)} - \sigma^{K(0)} g^{A(0)})$ in the above equation can be put into the familiar form of the collision term of a Boltzmann-like equation, i.e.

\[ (\sigma^{A(0)} g^{K(0)} - \sigma^{K(0)} g^{A(0)}) = (\sigma^{>}(0) - \sigma^{<}(0))(g^{>}(0) + g^{<}(0)) - (\sigma^{>}(0) + \sigma^{<}(0))(g^{>}(0) - g^{<}(0)) \]

\[ = 2(\sigma^{>}(0) g^{<}(0) - \sigma^{<}(0) g^{>}(0)) \]

(34)

Eq. (33) goes beyond to the gradient approximation, and the higher order space derivatives contain information about quantum effects. In the gradient approximation we obtain a Boltzmann-like equation for the zeroth $\tau$-derivative of the Green function.

Consider now the $k=2$ approximation for the collision integrals, eqs.(31, 32), we need to calculate the first two $\tau$-derivatives of the kinetic Green function. In the gradient
approximation \((\cos \frac{1}{2} \nabla \rightarrow 1, \sin \frac{1}{2} \nabla \rightarrow \frac{1}{2} \{, \})\) eq. (32) is an expression for \(g^{K(1)}\) depending on \(g^{K(0)}\) and the first order \(\tau\)-derivative of the self-energy. This expression can be substituted into the kinetic equation (31) yielding a transport equation for \(g^{K(0)}\). Since \(g^{K(1)}\) appears in eq. (31) under the Poisson bracket, the Poisson bracket in eq. (32) has to be neglected consistently with the gradient approximation, because a substitution of it would result in a term in eq. (31) which involves second order derivatives. Moreover, \(g^{K(1)}\) appears in eq. (31) in the second order term in \(\Delta\). Therefore we can use in eq. (31) the solution of the algebraic eq. (32) for \(g^{K(1)}\) in zeroth order in \(\Delta\) 

\[g^{K(1)} \approx -i\epsilon g^{K(0)}\].

Thus, in a consistent approximation, substituting this expression in eq. (31), we obtain

\[
i \frac{\partial}{\partial T} g^{K(0)} - i\{\epsilon, g^{K(0)}\} - i\frac{\Delta^2}{2} \left[\{\sigma^{A(0)}, g^{K(0)}\} + \epsilon \{i\sigma^{A(0)}, g^{K(0)}\}\right] = 2\Delta \left(1 - \frac{\Delta}{4} \frac{\partial}{\partial T}\right) \left(\sigma^{A(0)} g^{K(0)} - \sigma^{K(0)} g^{A(0)}\right) + i\frac{\Delta^2}{2} \left[\{i\sigma^{A(0)}, \epsilon\} g^{K(0)} - \{i\sigma^{K(0)}, \epsilon\} g^{A(0)}\right]\]

(35)

where \(\sigma^{A(1)}\) can be calculated using the same approximation in for \(g^{K(1)}\) in terms of \(g^{K(0)}\) as above.

To go beyond the gradient approximation one has to keep the higher order gradient terms in the otherwise algebraic constraint equation for \(g^{K(1)}\).

5 **Self-energy in the Born approximation**

To solve the equations for the \(\tau\)-derivatives of the Green functions we need the corresponding \(\tau\)-derivatives of the self-energies. The self-consistent Hartee-Fock self-energy is singular in time on the contour; it appears in the equations as a potential \(\sigma_\delta\) (momentum dependent in general) and it depends only on the zeroth \(\tau\)-derivative of the Green functions. The Hartree self-energy can be written

\[\Sigma_H(1, 1') = -i\delta(t_1 - t_1')\delta(x_1 - x_1') \int d^3 x_2 V(x_1 - x_2) G^< (x_2, t_1, x_2, t_1)\]

The Wigner transformation does not depend on momentum due to the presence of the \(\delta\)-function in space coordinates and it also has no \(\tau\)-dependence. One obtains

\[\sigma_H(R, T) = -i \int d^3 \xi V(\xi) \frac{1}{(2\pi)^3} \int d^3 p g^{< (0)} (R - \xi, p, T)\]

\[= - \frac{i}{(2\pi)^3} \sum m! \frac{1}{m!} \partial_R^m \left[\int d^3 p g^{< (0)} (R, p, T)\right] \int d^3 \xi \xi^m V(\xi)\]

(36)
The Green function appearing above is

\[ g^{<}(0)(\mathbf{R}, \mathbf{p}, T) = -\frac{i}{2} \left( i g^{K(0)}(\mathbf{R}, \mathbf{p}, T) - 1 \right) \]  

(37)

using definitions (7) and the sum rule (22) for the spectral function. The gradient expansion in the second line of eq. (36) gives non-locality contributions to the Hartree self-energy due to the dependence of the Green function \( g^{K(0)} \) on \( \mathbf{R} \) which are not present in a translationally invariant system. However, the lowest non-locality contribution (proportional to the gradient of the Green function) has a factor \( \int d^3\xi V(\xi) \) which is zero for inter-particle interaction potentials with \( V(\xi) = V(-\xi) \). Therefore, the specification of the Hartree self-energy in the local approximation

\[ \sigma_H(\mathbf{R}, T) = -\frac{i}{(2\pi)^3} \left[ \int d^3p g^{<}(0)(\mathbf{R}, \mathbf{p}, T) \right] \left[ \int d^3\xi V(\xi) \right] \]  

(38)

is justified at least in the gradient approximation. The second order term in the gradient expansion in eq. (36) for spherically symmetric potentials \( V(\xi) \) is proportional to

\[ \Delta R \left[ \int d^3p g^{<}(0)(\mathbf{R}, \mathbf{p}, T) \right] \int d^3\xi \xi^4 V(\xi). \]  

(39)

For a \( \delta \)-interaction potential the local approximation is exact; as the nucleon - nucleon potential is of short range, one can expect that in nuclear processes the local approximation is reasonable even for large non-localities.

In the same manner we obtain for the Fock self-energy \[12\]

\[ \sigma_F(\mathbf{R}, \mathbf{p}, T) = \frac{i}{(2\pi)^3} \int d^3\xi e^{-i\mathbf{p}\xi} V(\xi) \int d^3p' g^{<}(0)(\mathbf{R}, \mathbf{p}', T)e^{-i\mathbf{p}'\xi} 
\]

\[ = \frac{i}{(2\pi)^3} \int d^3p' g^{<}(0)(\mathbf{R}, \mathbf{p}', T)v(\mathbf{p}' - \mathbf{p}), \]  

(40)

where \( v \) is the Fourier transform of the interaction potential. The above expression has no gradient corrections. Thus, the Fock contribution is the convolution of the Green function \( g^{<}(0) \) with the Fourier transform of the interaction potential. We note that the Hartree-Fock self-energy can be obtained from only zero order \( \tau \)-derivatives of the Green function \( g^{K(0)} \) using eq. (37).

The lowest order contribution to the non-singular part of the self-energy on the Schwinger-Keldysh contour is given by the direct and exchange Born diagrams \[12\]. We
will discuss here only the contribution for the direct term, the exchange term is given in an analogous way.

\[
\sigma_{Bd}^{\text{>}<}(\mathbf{R}, \mathbf{p}, T; \tau) = \frac{1}{(2\pi)^6} \int d^3 \xi e^{-i\mathbf{p}\mathbf{\xi}} \int d^3 \rho_1 \int d^3 \rho_2 V(\rho_1)V(\rho_2) \int d^3 p_1 e^{+i\mathbf{p}_1\mathbf{\xi}}
\]

\[
g^{\text{>}<}(\mathbf{R}, \mathbf{p}_1, T; \tau) \int d^3 p_2 e^{i\mathbf{p}_2(\xi-(\rho_1+\rho_2))} g^{\text{>}<}(\mathbf{R} + \frac{\rho_2 - \rho_1}{2}, \mathbf{p}_2, T; \tau)
\]

\[
\int d^3 p_3 e^{-i\mathbf{p}_3(\xi-(\rho_1+\rho_2))} g^{\text{<}>}(\mathbf{R} + \frac{\rho_2 - \rho_1}{2}, \mathbf{p}, T; -\tau)
\]

Expanding eq. (41) around \(\tau = 0\), it is seen that the \(n\)th \(\tau\)-derivative of the Born self-energy involves the \(\tau\)-derivatives of the Green functions up to order \(n\). This property remains true if one replaces the Born approximation for the self-energy by the T-matrix approximation if the off-shell values of the T-matrix can be approximated by the on-shell ones as it was discussed in [13]. Further we make a gradient expansion in eq. (41) of the last two Green functions around \(\mathbf{R}\). In the local approximation, \(i.e.\) in zero order, we have for the zeroth \(\tau\)-derivative of the direct Born self-energy

\[
\sigma_{Bd}^{\text{>}<}(0)(\mathbf{R}, \mathbf{p}, T) = \frac{1}{(2\pi)^6} \int d^3 p_2 \int d^3 p_3 g^{\text{>}<}(0)(\mathbf{R}, \mathbf{p}_2, T)g^{\text{<}>}(0)(\mathbf{R}, \mathbf{p}_3, T)
\]

\[
g^{\text{>}<}(0)(\mathbf{R} - \mathbf{p}_2 + \mathbf{p}_3, T)v^2(\mathbf{p}_2 - \mathbf{p}_3)
\]

In a chosen order in the gradient expansion for the constraint and kinetic equations we need the self-energy in the same order. It is straightforward to prove that the first order terms in the gradient expansion of eq. (41) for \(\tau = 0\) cancel. Thus, as for the Hartree-Fock part of the self-energy, the local approximation for the Born self-energy is consistent with the gradient approximation for the transport equations. When considering second order terms in the gradient expansion for the transport equations, also second order terms in the expansion of eq. (41) have to be retained. The contribution of second order gradient terms to the zero order \(\tau\)-derivative of the direct Born self-energy is of the form

\[
-\frac{1}{8} \frac{1}{(2\pi)^3} \int d^3 p_2 \frac{1}{(2\pi)^3} \int d^3 p_3 \partial_i^R \partial_j^R \left(g^{\text{>}<}(0)(\mathbf{R}, \mathbf{p}_2, T)g^{\text{<}>}(0)(\mathbf{R}, \mathbf{p}_3, T)\right)
\]

\[
g^{\text{>}<}(0)(\mathbf{R}, \mathbf{p} - \mathbf{p}_2 + \mathbf{p}_3, T)\partial_i^R \partial_j^R v^2(\mathbf{p}_2 - \mathbf{p}_3)
\]

In the local approximation, from eq. (42) with eqs. (7) and eq. (22) one has

\[
i\sigma_{Bd}^{A(0)}(\mathbf{R}, \mathbf{p}, T) = \frac{1}{4} \frac{1}{(2\pi)^6} \int d^3 p_2 \int d^3 p_3 \left\{ [g^K(0)(\mathbf{R}, \mathbf{p}_2, T)g^K(0)(\mathbf{R}, \mathbf{p}_3, T) + 1] \right\}
\]
\[ + g^K(0)(\mathbf{R}, \mathbf{p} - \mathbf{p}_2 + \mathbf{p}_3, T) \left[ g^K(0)(\mathbf{R}, \mathbf{p}_2, T) - g^K(0)(\mathbf{R}, \mathbf{p}_3, T) \right] \right) v^2(\mathbf{p}_2 - \mathbf{p}_3) \]
\[ = \frac{1}{4} \frac{1}{(2\pi)^6} \int d^3p_2 \int d^3p_3 \left[ 1 + g^K(0)(\mathbf{R}, \mathbf{p}_2, T) g^K(0)(\mathbf{R}, \mathbf{p}_3, T) \right] v^2(\mathbf{p}_2 - \mathbf{p}_3) \]  

(44)

In the second step we assumed that \( g^K(0)(\mathbf{p}) = g^K(0)(-\mathbf{p}) \) which is valid e.g. in the fireball region of a heavy ion collision. This step is not necessary but we use it for the following estimate. Using the non-interacting Green functions \([1]\)

\[ g_0^>(0) = -i(1 - n(\mathbf{p})) , \quad g_0^<(0) = in(\mathbf{p}) \]  

(45)

where \( n(\mathbf{p}) \) is the occupation number of the state with momentum \( \mathbf{p} \) and \( 0 < n(\mathbf{p}) < 1 \) we obtain an estimate of \( i\sigma A(0) \) in eq. (44)

\[ i\sigma_0^{A(0)} = \frac{1}{2} \frac{1}{(2\pi)^3} \int d^3p_2 \frac{1}{(2\pi)^3} \int d^3p_3 \left[ n(\mathbf{p}_2)[1 - n(\mathbf{p}_3)] + 
\quad n(\mathbf{p}_3)[1 - n(\mathbf{p}_2)] \right] v^2(\mathbf{p}_2 - \mathbf{p}_3) \]  

(46)

We note that the spectral part of the self-energy in the Born approximation is an integral of a positive function and thus positive.

Eqs. (25,26) in the gradient approximation (\( \cos(\frac{1}{2}\dot{\phi} \rightarrow 1) \) imply

\[ g^A(\tau) = -ie^{-i\tau - \frac{i\sigma A(0)}{2} \tau^2} \]  

(47)

where we approximate \( \ln g^A(\tau) \) in second order around \( \tau = 0 \). Here the spectral function decays by a gaussian in the relative time when the spectral part of the self-energy is calculated using eq. (46). Then a characteristic correlation time is

\[ \frac{1}{\tau_1^e} = \sqrt{\frac{i\sigma_0^{A(0)}}{2}} . \]  

(48)

This estimate can be used to fix the parameter \( \Delta \) (or \( \alpha \) in Appendix C) in the calculation of the collision integrals, eqs. (29,30), choosing \( \Delta = \tau_1^e \) (or \( \alpha = \sqrt{\frac{i\sigma_0^{A(0)}}{2}} \)) where the spectral self-energy, \( i\sigma_0^{A(0)} \), is determined as above using the initial occupation numbers.

6 Numerical results

We used eq.(33) (k=1 approximation of the collision integrals) to test the influence of the terms not usually retained in the gradient approximation on the time evolution of
the phase space nucleon density distribution, \( n = (1 - ig^{K(0)})/2 \). To keep the numerical problem as simple as possible we restrict ourselves to a one-dimensional configuration space (the phase space is two dimensional); physically, this could be the case of two colliding nuclear matter slabs. For simplicity, a nucleon-nucleon interaction of the shape of an attractive rectangular well potential with typical depth (40 MeV) and range (2 fm) was used.

We discretize the partial differential equation, eq. (33), and solve the resulting finite difference equation. We evolve the phase space density distribution in time steps of 0.1 \( fm/c \); the density distribution is given at mesh points between -8 \( fm \) and +8 \( fm \) in steps of 0.4 \( fm \) in space and between -8 \( fm^{-1} \) and +8 \( fm^{-1} \) in steps of 0.4 \( fm^{-1} \) in momentum. The initial density was taken as a superposition of two gaussians (the widths were 2 \( fm \) in position and 1 \( fm^{-1} \) in momentum) centered at -2 \( fm \) and +2 \( fm \) in position and at +2 \( fm^{-1} \) and -2 \( fm^{-1} \) in momentum; the mean momenta of the gaussians correspond to an energy of the colliding slabs of 80 MeV/A.

The contour plot of the results of various approximations are collected in Fig. 1. The initial phase-space density is presented in panel (a). To test the program we calculated first the phase-space density for free motion, i.e. when there is no interaction and only the first order gradient terms are retained from the drift term in the left hand side of eq. (33) and the collision terms in the right hand side of eq. (33) are neglected. The results after 10 \( fm/c \) are presented in panel (b) and they are consistent with the classical Hamiltonian flow (i.e. with the test particle method).

The next step is to include the mean field (the Hartree-Fock self-energy). The results obtained at 10 \( fm/c \) in the gradient approximation (Vlasov equation) are presented in panel (c) of Fig.1. It is seen that there is considerable distortion of the phase space distribution. Including the next order gradient correction in the Vlasov equation the results are given in panel (d). We find that the quantum corrections to the drift term result in a much smoother behaviour of the phase-space distribution function, in fact, more similar to the free evolution. The results of including the collision terms but with the first order gradient expansion is given in panel (e). The hole in the center of phase space is filled and the distribution is more extended. This calculation corresponds to the usual BUU approach. In panel (f) both the next order gradient and the collisions terms are included. The density distribution becomes still smoother.
It is seen that higher order gradient corrections result in a smoother phase space distribution. This might seem contradictory since with higher order gradient corrections one moves closer to the evolution of a quantum state, the Wigner transform of which is expected to show more oscillations. However, it has to be remembered that the initial distribution does not correspond to a pure quantum state.

In Fig. 2 we present the corresponding density distribution in space and momentum; they are obtained by integrating the phase-space distribution over momenta, respectively over space. One can see that the spreading in space of the density distribution increases by including the third order gradient term in the drift part (panel (d) in Fig. 1) relative to the Vlasov approximation (c) and it increases further when the collisions are included in the calculation (e,f). The momentum distributions are rather stable with respect to the gradient terms (d), but the collision terms result in a considerable spreading of the distribution in momenta (e,f), as expected. With a closer look one sees that in the Vlasov approximation the momentum distribution spreads in a self consistent mean-field (c), but this is no longer the case when the gradient (quantum) corrections are taken into account (d).

These calculations include nonlocal terms simultaneously in the drift and the collision part of the Boltzmann-like equation (33). The inclusion of nonlocality terms in the scattering integral given by Morawetz et al. [9] leads to consistency with the thermodynamic virial corrections, but we think that nonlocal corrections should be also included in the drift term for the case of nonequilibrium systems.

Up to now we have discussed higher order gradient terms, i.e. quantum corrections to the space evolution. It is of interest to look also at the off-shell properties of the Green functions, i.e. at the properties of the spectral function. A first look is obtained by discussing \( g^{A(2)} \), i.e. the width of the spectral function. According to eq. (26) it depends on \( \sigma^{A(0)} \) which we have already calculated. According to eq. (48) it is also proportional to the inverse of the parameter \( \Delta \), connected with memory effects which are discussed below. The width of the spectral function is shown for the slab-slab collision in Fig. 3 as a function of the one-dimensional distance variable for different times, in particular for the initial time (panel (a) of Fig. 1) and for 5 and 10 fm/c for the BUU approximation (panel (e) of Fig. 1). It is seen that the width increases in the time evolution of the initially nonequilibrated momentum configuration. It is largest during the phase of maximum
overlap of the slabs. In the later phase of the evolution it becomes more spread out and lower. This is an expected behaviour as the final configuration should evolve towards hot nuclear matter (the initial collective kinetic energy transforms in thermal energy) and the final width has a contribution from thermal effects. It is to be noted that the width goes to zero automatically outside of the system because there the spectral function has to go back on shell.

Another quantum effect on the time evolution of density distribution is due to the inclusion of higher order terms in $\Delta$ in the collision integrals. This is a memory effect as it involves time derivatives of the self energies. To illustrate numerically this quantum effect we used eq. (35) in the simplest case, namely two interpenetrating nuclear matter systems in one dimension, i.e. relative to the previous case the space extension is disregarded. In such a translationally invariant system all the quantities depend only on momentum and time and eq. (35) can be written for the the time dependent density in momentum space

$$\frac{\partial n(p; T)}{\partial T} = 2\Delta(p; T) \left( 1 - \frac{\Delta(p; T)}{4} \frac{\partial}{\partial T} \right) [(1-n(p; T))(-i\sigma^{<}(0)(p; T))-n(p; T)(i\sigma^{>}(0)(p; T))]$$

where we used the relations $g^{K(0)} = 2i(n - 1/2)$ and $g^{A(0)} = -1$ and the self-energies in the Born approximation are

$$i\sigma^{>}(0)(p; T) = \frac{1}{(2\pi)^6} \int d^3p_2 \int d^3p_3 v^2(p_2 - p_3)[1 - n(p_2)]n(p_3)[1 - n(p - p_2 + p_3)]$$

$$-i\sigma^{<}(0)(p; T) = \frac{1}{(2\pi)^6} \int d^3p_2 \int d^3p_3 v^2(p_2 - p_3)n(p_2)[1 - n(p_3)]n(p - p_2 + p_3)$$

and, using eq. (48)

$$\Delta(p; T) = \sqrt{\frac{2}{(i\sigma^{>}(0)(p; T)) + (-i\sigma^{<}(0)(p; T))}}$$

If we neglect the second order term in $\Delta$ we obtain an equation which has the structure of a Boltzmann-like equation

$$\frac{\partial n(p; T)}{\partial T} = I_{gain}(p; T) - I_{loss}(p; T)$$

with the collision integrals of the Uehling-Uhlenbeck form

$$I_{gain}(p; T) = \int d^3p_2 \int d^3p_3 s(p, p_2 - p_3; T)[1 - n(p)][1 - n(p_3)]n(p_2)n(p - p_2 + p_3)$$

$$I_{loss}(p; T) = \int d^3p_2 \int d^3p_3 s(p, p_2 - p_3; T)n(p)[1 - n(p_2)]n(p_3)[1 - n(p - p_2 + p_3)]$$
but with an effective in-medium cross-section $s(p, p_2 - p_3; T)$ that depends not only on the transferred momentum, but also on the actual distribution. Thus the above equation includes nonequilibrium effects through the in-medium cross-section

$$s = \frac{2v^2\sqrt{2}}{(2\pi)^3} \int d^3p_2' \int d^3p_3'v^2(p_2' - p_3')[n(p_3')(1 - n(p_2')) + n(p - p_2 + p_3')(n(p_2') - n(p_3'))]^{-1/2}$$

which is a complicated functional of $n(p; T)$. Thus nonequilibrium effects affect the collision integrals, in the sense that the cross section is dependent on the momentum distribution. In general it is therefore not justified to use a cross-section calculated for equilibrated configurations in momentum space in a BUU description of a nuclear collision.

When we retain the second order term in $\Delta$ in eq. (49), using eq.(50), we obtain

$$\frac{\Delta^2}{2} \left[ \frac{\partial}{\partial T}(-i\sigma^{<\,(0)}(p; T)) - n \frac{\partial}{\partial T} \left( \frac{2}{\Delta^2} \right) \right] = I_{\text{gain}}(p; T) - I_{\text{loss}}(p; T)$$

(52)

In equilibrium the rhs of eq. (49) vanishes and one obtains

$$-i\sigma^{<\,(0)}_{eq}(p; T) = n(p; T) \frac{2}{\Delta(p; T)^2}$$

$$i\sigma^{>\,(0)}_{eq}(p; T) = (1 - n(p; T)) \frac{2}{\Delta(p; T)^2}$$

(53)

(54)

Let us remark that if we assume that the condition (53) remains true also in nonequilibrium (this can be justified if the system is not too far from equilibrium) and use it in the lhs of eq.(52) we obtain exactly eq. (51). More generally we can write

$$-i\sigma^{<\,(0)}(p; T) = n(p; T) \frac{2}{\Delta(p; T)^2} + C(p; T)$$

(55)

where $C$ is a correction term of the form

$$C(p; T) = \frac{1}{(2\pi)^6} \int d^3p_2 \int d^3p_3v^2(p_2 - p_3) \left\{ n(p_2)[1 - n(p_3)]n(p - p_2 + p_3) - n(p; T)[n(p_3)(1 - n(p_2)) + n(p - p_2 + p_3)(n(p_2) - n(p_3))] \right\}$$

(56)

and eq.(52) can be rewritten as

$$\frac{\partial n(p; T)}{\partial T} = I_{\text{gain}}(p; T) - I_{\text{loss}}(p; T) - \frac{\Delta(p; T)^2}{2} \frac{\partial}{\partial T} C(p; T)$$

(57)

which differs from eq.(51) only in the last term which can be considered as a memory term.
We investigated numerically eq.(51) and (57) in a one-dimensional case starting with an initial distribution of two gaussians. In the case \( n(p) = n(-p) \) the parameter \( \Delta \) does not depend on \( p \), but it depends on time. The results are presented in Fig. 4. One can see that using \( \Delta \) calculated at the initial time instead of a dynamical \( \Delta \) in eq. (51), i.e. without non-equilibrium effects on the in medium cross-section, the high momentum distribution tails are overestimated. Further inclusion of the memory term, eq. (57), changes mainly the distribuition at small momenta.

7 Conclusions

A transport theory for equal-time quantities related to the spectral and the kinetic part of the one-particle Green function has been derived. The present description is not in principle restricted to the mean field approximation and higher order terms in the gradient expansion can be systematically retained.

We note that in the mean field approximation the above derived equations reduce to the full quantum equations of motion for the Wigner function in a self-consistently determined mean-field. We can then study the effect of higher order terms in the gradient expansion, i.e. the quantum corrections to the classical equations of motion. This could be of interest for systems which are localized in space, as is the case for heavy ion collisions, especially in the initial stage. The condition of validity for the gradient approximation, the product of the characteristic lengths at which the Green function varies in position and in momentum is much larger than 1 \([15]\), is not fulfilled for a nucleus (one is of the order of the nuclear dimension, \( \sim 1 \text{ fm} \), and the other of order of Fermi momentum, \( \sim 1 \text{ fm}^{-1} \)).

We also consider the inclusion of terms which go beyond the mean-field approximation in the self-energy and make a detailed discussion of the self-energy in the self-consistent Born approximation. The compatibility of the two hierarchies (kinetic and constraint) for the \( \tau \)-derivatives of the spectral Green function at zero relative time (equivalent to the energy moments of the spectral function provided they exist) has been proved up to the second order. This extends the similar result obtained in the mean-field approximation in \([4]\).

For the kinetic part of the Green function similar hierarchies of equations have been
The lowest order truncation is a Boltzmann-like equation (33) for the equal-time kinetic part of the Green function; all the terms in the gradient expansion are retained and this allows to study, as in the mean field case, the effect of the terms neglected in the gradient approximation. The collision term in eq. (33) has the Uehling-Uhlenbeck form (see eqs. (42) and (34)), in which the Pauli exclusion principle is automatically taken into account.

Let us mention that it is possible to prove that any truncation of the gradient expansion in eq. (33) conserves the total number of particles if the first order one, i.e. the usual gradient approximation, does. Integrating eq. (33) over position and momentum, and using integration by parts to transfer all the derivatives to the Green functions (all the selfenergies are supposed to vanish together with all their derivatives at infinity, and $\Delta$ is considered constant) the only surviving term in the rhs of eq. (33) will be $2\Delta \int d^3R \int d^3p (\sigma^A(0)g^K(0) - \sigma^K(0)g^A(0))$, i.e. the term present in the gradient approximation. In the gradient approximation, the change in the Boltzmann equation in the next order truncation of the hierarchies has been derived, eq. (35).

The methods developed in this work were tested in a simple model of one-dimensional slab-on-slab collisions. Here we went only to the lowest order of the hierarchy, i.e. we remained in the quasi-particle approximation. However, we investigated the effect of the next order gradient terms, i.e. of true quantum corrections, and found them of considerable importance. Thus these corrections could also influence the dynamical evolution of real heavy ion collisions.

The next step will be to consider higher order truncations of the hierarchies. This will result in a dependence of the time evolution of phase-space density on the higher order moments of the Green function $g^K$. As these moments are a measure of the off-shellness, this could be an alternative way to include the off-shell motion in heavy ion collision calculations. For other proposed approaches in the literature see [5, 6].

The approach of the present paper can be used to treat a system of bosons, or a coupled fermion-boson system, as quantum hadrodynamics [16] which is the modern way for describing the nuclear many-body problem in the framework of effective field theories.
Appendix A
Properties of the differential operators $\sin \frac{1}{2} \clubsuit$ and $\cos \frac{1}{2} \clubsuit$

$$e^{\frac{i}{2} \clubsuit} (f \cdot g) = f * g$$

is a noncommutative, associative product defined on the space of complex valued functions $f(R,p)$.

It is necessary to verify the associativity only for functions of the form $f_i = e^{i(\alpha_i R + \beta_i p)}$ with arbitrary $\alpha_i$, $\beta_i$ as every function can be expressed as a linear combination of the above functions (Fourier decomposition) with complex coefficients and the product $*$ is obviously distributive. Using

$$f_i * f_j = e^{\frac{i}{2}(\alpha_i \beta_j - \beta_i \alpha_j)} f_i f_j$$

it is a straightforward exercise to prove that

$$f_1 * (f_2 * f_3) = (f_1 * f_2) * f_3.$$

We have the identities

$$\cos \frac{1}{2} \clubsuit (f \cdot g) = \frac{1}{2} (f * g + g * f)$$

$$\sin \frac{1}{2} \clubsuit (f \cdot g) = \frac{1}{2i} (f * g - g * f)$$

and the following relation is obvious

$$\sin \frac{1}{2} \clubsuit (f \cdot f) = 0 \quad (A1)$$

Using the definitions of the differential operators and the associativity we obtain

$$\sin \frac{1}{2} \clubsuit (f \cdot \cos \frac{1}{2} \clubsuit (g \cdot h)) = \cos \frac{1}{2} \clubsuit (g \cdot \sin \frac{1}{2} \clubsuit (f \cdot h)) + \cos \frac{1}{2} \clubsuit (h \cdot \sin \frac{1}{2} \clubsuit (f \cdot g))$$

The particular case of the above relation $g = f$ is

$$\cos \frac{1}{2} \clubsuit (f \cdot \sin \frac{1}{2} \clubsuit (f \cdot h)) = \sin \frac{1}{2} \clubsuit (f \cdot \cos \frac{1}{2} \clubsuit (f \cdot h)) \quad (A2)$$

We have also the identity

$$\cos \frac{1}{2} \clubsuit (g \cdot \cos \frac{1}{2} \clubsuit (f \cdot h)) + \sin \frac{1}{2} \clubsuit (g \cdot \sin \frac{1}{2} \clubsuit (f \cdot h)) - \sin \frac{1}{2} \clubsuit (f \cdot \sin \frac{1}{2} \clubsuit (g \cdot h))$$

24
$$= \frac{1}{4} [g \star (f \star h + h \star f) + (f \star h + h \star f) \star g - g \star (f \star h - h \star f)$$
$$+ (f \star h - h \star f) \star g + f \star (g \star h - h \star g) - (g \star h - h \star g) \star f]$$
$$= \frac{1}{4} [f \star (g \star h + h \star g) + (g \star h + h \star g) \star f]$$
$$= \cos \frac{1}{2} \langle f \cdot \cos \frac{1}{2} \langle g \cdot h \rangle \rangle$$
\hspace{1cm} (A3)$$

**Appendix B**

*Compatibility of the constraint and kinetic hierarchies for the Green functions*

Using eq. (22), eq. (A1) for $f = \epsilon$, $\frac{\partial \epsilon}{\partial T} = \frac{\partial \sigma}{\partial T}$ and $\cos \frac{1}{2} \langle f \cdot g^{A(0)} \rangle = fg^{A(0)} = -if$ it is a straightforward algebraic exercise to prove that the second kinetic equation (23) is satisfied by the solution (25) of the first constraint equation.

To show that the solution of the second constraint equation (26) satisfies the third kinetic equation (23) we take the action of the operator $i \frac{\partial}{\partial T} - 2i \sin \frac{1}{2} \langle \epsilon \rangle$ on the eq. (26) and we obtain

$$i \frac{\partial}{\partial T} g^{A(2)} - 2i \sin \frac{1}{2} \langle \epsilon \rangle \cdot g^{A(2)} = -2 \cos \frac{1}{2} \langle \epsilon \rangle \cdot \epsilon - i \frac{\partial \sigma^{A(0)}}{\partial T}$$
$$+ 2 \sin \frac{1}{2} \langle \epsilon \cdot \cos \frac{1}{2} \langle \epsilon \cdot \epsilon \rangle \rangle + 2i \sin \frac{1}{2} \langle \epsilon \cdot \sigma^{A(0)} \rangle$$
$$= -2 \cos \frac{1}{2} \langle \epsilon \rangle \cdot \epsilon - i \frac{\partial \sigma^{A(0)}}{\partial T} - 2i \sin \frac{1}{2} \langle \sigma^{A(0)} \cdot \epsilon \rangle,$$
\hspace{1cm} (B1)

where in the last step we used the identity (A2) in the Appendix A

$$\sin \frac{1}{2} \langle \epsilon \cdot \cos \frac{1}{2} \langle \epsilon \cdot \epsilon \rangle \rangle = \cos \frac{1}{2} \langle \epsilon \cdot \sin \frac{1}{2} \langle \epsilon \cdot \epsilon \rangle \rangle = 0$$

The third equation (23) can be rewritten using the relation (25)

$$i \frac{\partial}{\partial T} g^{A(2)} - 2i \sin \frac{1}{2} \langle \epsilon \rangle \cdot g^{A(2)} = -2 \cos \frac{1}{2} \langle \epsilon \rangle \cdot \epsilon - i \frac{\partial \sigma^{A(0)}}{\partial T}$$
$$- 2i \sin \frac{1}{2} \langle \sigma^{A(0)} \cdot \epsilon \rangle$$(B2)

Comparing eqs. (B1) and (B2) we conclude that the solution (26) of the constraint hierarchy satisfies the third equation of the kinetic hierarchy.
In the same way we will show the compatibility of the constraint and kinetic hierarchies for the kinetic part of the Green function in the mean field approximation. The second kinetic equation (27) can be obtained from the first kinetic equation and the first constraint equation (28). Acting with the operator $i\partial_T - 2i\sin\frac{1}{2}\hat{\Diamond}(\epsilon^\cdot)$ on the first constraint equation we obtain ($S^K_\pm = 0$)

\[
 i\frac{\partial}{\partial T}g^{K(1)}(1) - 2i\sin\frac{1}{2}\hat{\Diamond}(\epsilon \cdot g^{K(1)}) + i(i\frac{\partial}{\partial T} - 2i\sin\frac{1}{2}\hat{\Diamond}(\epsilon^\cdot))\cos\frac{1}{2}\hat{\Diamond}(\epsilon \cdot g^{K(0)}) = 0
\]

Using the first kinetic equation we obtain

\[
 i\frac{\partial}{\partial T}g^{K(1)}(1) - 2i\sin\frac{1}{2}\hat{\Diamond}(\epsilon \cdot g^{K(1)}) = 0
\]

The last two terms cancel and we obtain exactly the second kinetic equation (27).

In the same way one can obtain the third kinetic equation. Acting with the operator $i\partial_T - 2i\sin\frac{1}{2}\hat{\Diamond}(\epsilon^\cdot)$ on the second constraint equation (28) and using the first two kinetic equations we obtain

\[
 i\frac{\partial}{\partial T}g^{K(2)}(1) - 2i\sin\frac{1}{2}\hat{\Diamond}(\epsilon \cdot g^{K(1)}) = 0
\]

The 5th and the 7th cancel and we obtain

\[
 i\frac{\partial}{\partial T}g^{K(2)}(2) - 2i\sin\frac{1}{2}\hat{\Diamond}(\epsilon \cdot g^{K(2)}) - \frac{i}{2}\sin\frac{1}{2}\hat{\Diamond}(\epsilon \cdot g^{K(0)}) - 2\sin\frac{1}{2}\hat{\Diamond}(\epsilon \cdot \sin\frac{1}{2}\hat{\Diamond}(\epsilon \cdot g^{K(0)})) = 0
\]

where we used the identity (A3) in Appendix A. Using the first constraint equation (28) we obtain

\[
 i\frac{\partial}{\partial T}g^{K(2)}(2) - 2i\sin\frac{1}{2}\hat{\Diamond}(\epsilon \cdot g^{K(2)}) - \frac{i}{2}\sin\frac{1}{2}\hat{\Diamond}(\epsilon \cdot g^{K(0)}) = 0
\]
\[
\cos \frac{1}{2} \diamond \left( \frac{\partial \epsilon}{\partial T} \cdot g^{K(1)} \right)
\]
which coincides with the third kinetic equation (27).

Appendix C

Approximation of integrals appearing in the hierarchies for \( g^K \)

In the hierarchies of equations for \( g^K \) the terms due to collisions, \( S_{\pm}^{K(k)} \) are of the form \( I = \int_{-\infty}^{0} dx f(x) \). We will approximate such an integral introducing an arbitrary real and positive parameter \( \alpha \)

\[
I = \int_{-\infty}^{0} dx \left[ e^{-\alpha x} f(x) \right] e^{\alpha x}.
\]

If we assume that \( e^{-\alpha x} f(x) \) is a function localized around zero (this is reasonable for the functions discussed here depending on the relative time) and \( \alpha \) is choosen such that the radius of convergence of the Taylor series for \( f \) around zero is greater than \( \frac{1}{\alpha} \) (\( \alpha > \frac{1}{\rho} \)) then the main contribution to the integral comes from the region in which the function is well approximated by its Taylor series in the origin and we have

\[
I = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n}{\partial x^n} \left( e^{-\alpha x} f(x) \right) \big|_{x=0} \int_{-\infty}^{0} dx x^n e^{\alpha x}
\]

Using the notation \( f^{(k)} = \frac{\partial^k}{\partial x^k} f(x) \big|_{x=0} \) we obtain

\[
I = \frac{1}{\alpha} \sum_{n=0}^{\infty} \frac{n!}{n \sum_{k=0}^{n} k!(n-k)!} \frac{f^{(k)}}{(-\alpha)^k}
\]

We consider truncations of the sum over \( n \), \( \sum_{n=0}^{N} \), and we will use \( N = 1 \) or \( N = 2 \) to approximate the integral by the derivatives of \( f \) up to order \( N \).

We note that if the radius of convergence of the Taylor series is infinity then the sum of the above series gives the exact result which does not depend on \( \alpha \). However, the result of the truncation of the series depends on the value of the parameter \( \alpha \); a low order approximation is good if \( \frac{1}{\alpha} \) is choosen as a characteristic decay time of \( f \), physically a correlation time of the system. For example, if we calculate in the \( N = 2 \) approximation the integral with \( f = e^{-\beta x^2} \) using \( \alpha = \beta \) the approximation will be \( \frac{1}{\beta} \) which differs from the exact value \( \frac{\sqrt{\pi} 1}{2 \beta} \) by 10 percent.
We can use the above results to evaluate the integrals $S_{\pm}^{K(0)}$ and $S_{\pm}^{K(1)}$. It is straightforward but tedious to write the collision terms in order $N = 2$. The results are

$$S_{\pm}^{K(0)} = \pm \frac{3}{\alpha} \left( 1 - \frac{1}{2\alpha} \frac{\partial}{\partial T} \right) (\sigma^{A(0)} g^{K(0)} - \sigma^{K(0)} g^{A(0)})$$

$$+ \frac{3}{\alpha^2} (\sigma^{A(1)} g^{K(0)} - \sigma^{A(0)} g^{K(1)} - \sigma^{K(1)} g^{A(0)} + \sigma^{K(0)} g^{A(1)})$$

and

$$S_{\pm}^{K(1)} = \frac{1}{2} (\sigma^{A(0)} g^{K(0)} + \sigma^{K(0)} g^{A(0)})$$

$$\pm \frac{1}{3\alpha} \left[ (\sigma^{A(1)} g^{K(0)} + \sigma^{A(0)} g^{K(1)} - \sigma^{K(1)} g^{A(0)} - \sigma^{K(0)} g^{A(1)})$$

$$\pm \frac{1}{2} \left( \frac{\partial \sigma^{A(0)}}{\partial T} g^{K(0)} - \sigma^{A(0)} \frac{\partial g^{K(0)}}{\partial T} - \sigma^{K(0)} \frac{\partial g^{A(0)}}{\partial T} + \sigma^{K(0)} \frac{\partial g^{A(0)}}{\partial T} \right) \right]$$

The similarity of these expansions with the $\Delta$-expansion used in Chapter 4 is obvious. It corresponds to replacing $\frac{1}{\alpha}$ by $\Delta$ with slightly different numerical factors. The reason is also clear: in the $\Delta$-expansion we cut the integral over the past history with a sharp cut-off $\Delta$, in the present method by a smooth exponential cut-off with a parameter $\alpha$.

**Acknowledgements:** We would like to thank in particular Dr. Rolf Fauser, whose discussions in the early part of this work were very important. One of the authors (R. A. I.) thanks Deutscher Akademischer Austauschdienst (DAAD) for a fellowship.
References

[1] W. Botermans, R. Malfliet - Phys. Rep. 198 (1990) 115; P.A. Henning - Phys. Rep. 253 (1995) 235; U. Mosel - Prog. Part. Nucl. Phys. 42 (1999) 163

[2] L.P. Kadanoff, G. Baym - Quantum statistical theory, Benjamin/Cummings, Menlo Park, CA, 1962

[3] J. Helgesson, J. Randrup - Phys. Lett. B439 (1998) 423; C. Fuchs, Z. Wang, L. Sehn, A. Faessler, V.S. Uma Maheswari, D.S. Kosov - Phys. Rev. C56 (1997) 606; M. Post, U. Mosel - Nucl. Phys. A688 (2001) 808

[4] P. Zhuang, U. Heinz - Phys. Rev. D53 (1996) 2096; Ann. Phys. 245 (1996) 311; Phys.Rev. bf D57 (1998) 6525; A. Abada, M.C. Birse, P.Zhuang, U. Heinz - Phys. Rev. D54 (1996)311

[5] W. Cassing, S. Juchem - Nucl. Phys. A665 (2000) 377; Nucl. Phys. A672 (2000) 417; Nucl. Phys. A677 (2000) 445

[6] S. Leupold - Nucl. Phys. A672 (2000) 475; J. Lehr, M. Effenberger, H. Lenske, S. Leupold, U. Mosel - Phys. Lett. B483 (2000) 324

[7] J. Knoll - Prog. Part. Nucl. Phys. 42 (1999) 177

[8] M. Joyce, K. Kainulanen, T. Prokopec - Phys. Lett. B474 (2000) 402

[9] K. Morawetz, P. Lipavsky, V. Spicka - Prog. Part. Nucl. Phys. 42 (1999) 147; P. Lipavsky, K. Morawetz, V. Spicka - Ann. Phys. Fr. 26 (2001) no 1

[10] A.L. Fetter, J.D. Walecka - Quantum theory of many-particle systems, McGraw-Hill, N.Y., 1971

[11] K.C.Chou, Z. Su, B. Hao, L. Yu - Phys. Rep. 118 (1985) 1; E. Calzetta, B.L.Hu - Phys. Rev. D37 (1988) 2878

[12] J.E. Davis, R.J. Perry - Phys. Rev. C43 (1991) 1893; P. Danielewicz - Ann. Phys. 152 (1984) 239

[13] C. Fuchs, A. Faessler, M. El-Shabshiry - Phys. Rev. C64 (2001) 024003
[14] J. Rammer, H. Smith - Rev. Mod. Phys. 58 (1986) 323

[15] S. Mrowczynski, U. Heinz - Ann. Phys. 229 (1994) 1

[16] B.D. Serot, J.D. Walecka - Int. J. Mod. Phys. E6 (1997) 515
Figure 1: Contour plots of the phase space distribution of slab-on-slab collisions at 80 MeV per nucleon displayed for the direction of the finite extension of the slabs (x-direction). Panel (a) shows the initial distribution. The other panels show the distributions at time 10 fm/c in the following approximations: (b) free motion; (c) Vlasov approximation, i.e. first order gradient expansion; (d) Vlasov plus next (third) order gradient terms; (e) Vlasov plus collision term (corresponds to the usual BUU approximation); (f) Vlasov plus next order gradient plus collision terms. The contour lines are give in panels (a - d) for 0.2(0.2) and in panels (e,f) for 0.1(0.1) with the notation: outermost contour line (increment).
Figure 2: Space (top) and momentum (bottom) distributions for the same slab-on-slab collisions as in Fig. 1, i.e. the projections on the coordinate and momentum axes. The curves correspond to the following approximations (with the appropriate panel of Fig.1): free motion (solid, panel b); Vlasov (longer dashed, panel c), Vlasov plus next order gradient terms (short dashed, panel d), Vlasov plus collision terms (dotted, panel e), Vlasov plus next order gradient plus collision terms (dash-dotted, panel f).
Figure 3: Width (i.e. second order energy moment) of the spectral function at different times as a function of the coordinate of the one-dimensional model.
Figure 4: Time evolution of an initial distribution of two gaussians in momentum (top) and at 10 fm/c (bottom). The curves correspond to the following approximations: Boltzmann with time dependent (nonequilibrium) cross-section $s(\mathbf{p}, \mathbf{p}_2 - \mathbf{p}_3; T)$ (solid line), Boltzmann with time independent cross-section calculated from the initial configuration $s(\mathbf{p}, \mathbf{p}_2 - \mathbf{p}_3; T = 0)$ (long dashed line), Boltzmann with time dependent cross-section plus memory term (short dashed line).