The J1-J2 model: First order phase transition versus deconfinement of spinons

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We revisit the phase transition from the Néel ordered to a valence bond solid (VBS) state in the two-dimensional $J_1-J_2$ antiferromagnetic Heisenberg model. In the first part we address the question whether or not this transition could be an example of a second order phase transition due to a deconfinement of spinons. We give arguments based on series expansion and spin-wave theory that this is not the case and the transition is most likely first order. The method proposed here to detect first order phase transitions seems to be very sensitive and might be useful in other models as well. In the second part we analyze possible VBS patterns in the magnetically disordered phase based on numerical data for different susceptibilities, obtained in the ordered phase, which test the breaking of lattice symmetries. We conclude that a columnar dimerization pattern is the most likely candidate.

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I. INTRODUCTION

It is well known that microscopically different systems can show similar behavior near a critical point, a phenomenon termed universality. The universal behavior is caused by the fact that only a small number of long-wave length degrees of freedom are relevant for most physical quantities at the critical point. It is indeed often sufficient to consider an effective theory in which all modes other than the order parameter have been eliminated. The concept of the order parameter and effective theories based on this quantity has been developed by Landau and Ginzburg.

Recently, it has been argued that there is a second order phase transition in the $S = 1/2$ square lattice antiferromagnet between the Néel state and a paramagnetic valence bond solid state which is not described by a Ginzburg-Landau (GL) type critical theory. Instead the proposed theory involves fractional degrees of freedom (spinons) interacting with an emergent gauge field. One of the best studied models where this scenario could possibly be realized is the spin-1/2 $J_1-J_2$ Heisenberg antiferromagnet in two dimensions

$$H = J_1 \sum_{i,j} S_i S_j + J_2 \sum_{i,j} S_i S_j.$$  

(1)

Here $J_1 > 0$ is the nearest-neighbor interaction and $J_2 \geq 0$ a frustrating next-nearest neighbor exchange. There are two well understood limits: For $J_1 \neq 0$, $J_2 = 0$ the model is just the usual two-dimensional Heisenberg antiferromagnet which is known to possess Néel order although with an order parameter $M \approx 0.3$ which is reduced compared to its classical value $M = 1/2$. For $J_1 \to 0$ and $J_2 \neq 0$ and fixed, on the other hand, both sub-lattices are Néel ordered and $J_1$ then induces a so called collinear order. For general couplings $J_1$, $J_2$, both limiting ground states become frustrated. Therefore a parameter region might be expected where the magnetic order vanishes and a spin liquid or VBS ground state is formed. Numerical studies including exact diagonalization, variational Quantum-Monte-Carlo, as well as series expansion, indeed indicate that for $0.4 \lesssim g \lesssim 0.6$, with $g = J_2/J_1$, no magnetic order exists.

To address the question of whether or not the ground state in the non-magnetic region breaks a lattice symmetry, the response to a field

$$F_i = \delta \sum_{i,j} (-1)^i S_{i,j} S_{i+1,j},$$

(2)

has been calculated. The series expansion studies show that the corresponding susceptibility becomes very large or even diverges when $g \approx 0.4$ is approached from the Néel phase, indicating that translational symmetry by one site is broken and a VBS state is formed. The results obtained in Ref. by a variational quantum Monte Carlo (QMC) technique seem to support this scenario. Furthermore, the QMC data have been shown to be in good agreement with an exact diagonalization of clusters with $N = 16,32$ spins. In a later exact diagonalization study, however, it has been shown that the susceptibility decreases when going from the $4 \times 4$ to the $6 \times 6$-cluster. Further evidence in favour of a breaking of translational invariance has been obtained by a dimer series expansion showing directly that the corresponding order parameter is indeed nonzero in this phase. On the basis of the series expansion data we therefore believe that the existence of a homogeneous spin liquid phase in this parameter region, as proposed in Ref., is highly unlikely.

A very important result of the series calculations in Refs. is that the point $g_{c1}$, where magnetic order vanishes, and the point $g_{c2}$, where dimer order becomes established, are very close or even coincident. Further-
more, a crossing of energies between an Ising expansion and a dimer expansion, which would be the indication of a first order transition, could not be detected. This lead to the assumption that the transition is second order. For a second order transition it is, however, difficult to understand within GL theory why $g_{cv}$ and $g_{cφ}$ should be equal. Each phase has a different broken symmetry (spin rotational symmetry versus lattice symmetry) so that one would naively expect that each transition is described by its own effective theory containing only the staggered magnetization (order parameter in the Néel phase) or the dimer order parameter, respectively. In this case the transitions should be independent from each other. That this seems to be not the case tells us that the two order parameters must be related. A GL-type theory should therefore at least contain also terms describing the interaction between the two order parameters. Such an effective theory has been proposed in Ref. 11. However, it has been found that within this theory $g_{cv}$ and $g_{cφ}$ will only be identical if the non-magnetic phase has massless excitations. This would correspond to a transition to a translationally invariant spin liquid phase which we believe can be ruled out based on the numerical data obtained in Ref. 6.

The new effective theory for the second order phase transition between the Néel and a VBS state proposed in Refs. 23 follows an entirely different route. Here the order parameters in the two phases are represented in terms of fractional degrees of freedom (spinons) which become deconfined exactly at the critical point. As the spinons are the constituents of both order parameters this would offer a natural explanation for a direct second order phase transition between these at first sight very different phases. Among the models proposed to show such a transition is the two-dimensional spin-1/2 model with a four-spin exchange.12 Quite recently, however, it has been argued that the transition in this model is more likely to be first order.13 In the present work we will address the same question about the order of the phase transition for the $J_1 - J_2$ model.

Our paper is organized as follows: In section II we give arguments based on series expansion and spin-wave theory why the phase transition is most likely a weak first order instead of a second order transition. In section III we will discuss numerical data for three different susceptibilities probing the VBS order in the non-magnetic phase. These susceptibilities are obtained in the Néel phase where the ground state is known and the series is therefore not biased. We will discuss why these data provide additional evidence against the deconfinement scenario and will conclude that the VBS order is most likely of the columnar dimer type. The last section presents a summary and conclusions.

II. ORDER OF THE PHASE TRANSITION

Usually a first order phase transition is detected in series calculations by looking for the crossing of energies obtained by expansions starting from different states. For the $J_1 - J_2$ model such an energy-crossing has been detected between an Ising expansion in the collinear regime and a dimer expansion in the VBS phase. This shows that the transition at $g \sim 0.62$ from the VBS to the collinear state is first order (see also Fig. 7). For the transition from the Néel to the VBS state at $g \sim 0.4$, on the other hand, no crossing has been found. More precisely, the energies for an Ising expansion and various dimer expansions are so close over a relatively large parameter regime around the transition point that it is not possible to decide within the accuracy of the series if there is a crossing or not (see Ref. 9 and Fig. 7). In all previous series studies it has been implicitly assumed that the transition is second order. Here we propose a more sensitive method to distinguish between first and second order transitions and conclude that the transition is most likely weak first order.

Let us consider the ground-state energy $e(δ)$ for the Hamiltonian 1 with the field in Eq. (2) included for $|δ| \ll 1$ and $g \leq 0.4$. We have calculated $e(δ)$ for different $g$ by Ising series expansion. Using an Ising expansion means that we start with a state which breaks spin-rotational symmetry whereas the lattice symmetries are intact. Obviously, it is then impossible to restore spin-rotational symmetry and break lattice symmetries - as would be required when going from the Néel to the VBS state - in any finite order in the expansion. However, we can expect that an instability of the state we are starting with is signaled by a susceptibility, with respect to the corresponding symmetry breaking field, which is divergent.

In Fig. 1 we present our numerical data. For a fixed $g$

![FIG. 1: Ising series data for the ground state energy $e(δ)$ with the field $F_1$ included and $g = 0.0, 0.1, 0.2, 0.3$. The lines are a guide to the eye.](image-url)
we have fitted \( e(\delta) \) by a polynomial of the form

\[
e(\delta) - e(0) = \frac{a}{2} \delta^2 + \frac{b}{4} \delta^4 + \frac{c}{6} \delta^6 .
\]

(3)

The susceptibility is then given by

\[
\chi_1 = -\frac{\partial^2 e}{\partial \delta^2} \bigg|_{\delta=0} = -a .
\]

(4)

As the series for the ground state energy shows better convergence than the series for the susceptibility itself (which was calculated in Ref. 11), we were able to obtain \( \chi_1 \) with much smaller error bars than before as shown in Fig. 2. The new data are nevertheless consistent with the old data within the given error bars. A strong response \( \gamma_e \) is the critical exponent for the staggered magnetic susceptibility with \( \gamma_e = 1 \) in mean-field theory. At \( g < g_{cv} \) the vector field \( v \) will then show a nonzero ground state expectation value \( \langle v \rangle = \sqrt{a_v(g_{cv} - g)/u_v} \).

Consider now the case that we are in the magnetically ordered phase and add the field \( F_1 \) as given in Eq. (2) with \( |\delta| \ll 1 \). The Néel order will then coexist with a small dimerization described by a scalar field

\[
H_\phi = \frac{1}{2} \left\{ (\partial_\ell \phi)^2 + c_\phi^2 (\nabla \phi)^2 + m_\phi^2 \phi^2 \right\} + \frac{u_\phi}{4} \phi^4 + \frac{r_\phi}{6} \phi^6 - \delta \phi .
\]

If a second order phase transition with respect to \( \phi \) at a critical point \( g_{cv} \) would occur, we would have \( m_\phi^2 = a_\phi(g_{cv} - g)^\gamma_\phi \) and \( u_\phi > 0 \). We also want to include an interaction between the vector and the scalar field. The lowest order coupling term allowed by symmetry is

\[
H_{\text{int}} = \frac{u_{\phi v}}{2} \phi^2 v^2 .
\]

(6)

The effective field theory in the ordered phase for \( \delta \neq 0 \) is then given by \( H = H_e + H_\phi + H_{\text{int}} \) and we will have a nonzero ground state expectation value

\[
\langle \phi \rangle = \frac{\delta}{A} \frac{u_\phi}{A^2} \delta^3 + \frac{3u_\phi^2}{A^3} - \frac{Ar_\phi}{A^4} \delta^5 + O(\delta^7) .
\]

(8)

This leads to a ground state energy given by

\[
e(\delta) - e(\delta = 0) = -\frac{1}{2A} \delta^2 + \frac{u_\phi}{4A^4} \delta^4 + \frac{Ar_\phi - 3u_\phi^2}{6A^7} \delta^6 + O(\delta^8) .
\]

(9)

FIG. 2: The susceptibility \( \chi_1 \) calculated as described in the text (solid line) compared to the old data from Ref. 11 (dashed line). The dotted line represents a linear fit of the new data \( 1/\chi_1 = 6.58 \cdot (0.43 - g) \).

to the field \( F_1 \) is visible indicating that translational symmetry is broken in the non-magnetic phase. If the phase transition with respect to the corresponding order parameter would be second order we expect \( 1/\chi_1 \propto (g_{cv} - g)^\gamma_\phi \) where \( g_{cv} \) is the critical point and \( \gamma_\phi \) the critical exponent. In mean-field theory \( \gamma_\phi = 1 \) and we can indeed obtain a nice linear fit of \( 1/\chi_1 \) as shown in Fig. 2. Note, that \( \gamma_\phi \) is not expected to change dramatically even if fluctuations are taken into account (an O(1)-model, for example, would have \( \gamma_\phi \approx 1.2 \)) so that changes to the value of \( g_{cv} \) would be minor.

To study different possible scenarios for this phase transition we consider an effective field theory for the magnetically ordered phase. In the effective field theory for a two-dimensional antiferromagnet in the ordered phase no topological term (Berry phase) is present. One can therefore describe the system by the following \( O(3) \)-model

\[
H_v = \frac{1}{2} \left\{ (\partial_\ell v)^2 + c_\phi v^2 + m_v^2 v^2 \right\} + \frac{u_v}{4} (v^2)^2 .
\]

(5)

Assuming a second order phase transition at a critical point \( g_{cv} \) we have \( m_v^2 = a_v(g - g_{cv})^\gamma_v \) and \( u_v > 0 \). Here

(10,17) Surprisingly, LSWT

A check of our series data for \( b \) is provided by calculating this coefficient in spin-wave theory. We know that linear spin-wave theory (LSWT) yields accurate results for quantities like ground-state energy or sub-lattice magnetization for the two-dimensional Heisenberg antiferromagnet without frustration.10,17 Surprisingly, LSWT
The coefficient of the $\delta$ decreases with increasing $g$ the series data. Again we find that relations is shown in the inset of Fig. 3 in comparison to MFSWT for the $J$ calculated the ground-state energy $e$ spin-wave theory results are reasonable. $g$ theory in the strongly frustrated regime. If we consider $g$ close to zero, on the other hand, we might expect that spin-wave theory results are reasonable.

In complete analogy to the series calculations we have calculated the ground-state energy $e(\delta)$ in LSWT and MFSWT for the $J_1 - J_2$ model which is very similar to the one found by numerical calculations. In particular, a non-magnetic ground state for $g \in [0.38, 0.51]$ is found. In a spin-wave theory where the quartic terms are treated self-consistently in one-loop approximation (MFSWT), however, the phase diagram changes dramatically and Néel order remains stable up to $g \sim 0.6^{19}$. This tells us that we cannot trust spin-wave theory in the strongly frustrated regime. If we consider $g$ close to zero, on the other hand, we might expect that spin-wave theory results are reasonable.

In this section we provide additional, independent arguments against the deconfinement scenario and in favor of a first order phase transition. We also address the question which kind of VBS order is actually realized. In Refs. 2,3 dealing with a possible deconfined critical point separating a Néel ordered from a VBS phase in a two-dimensional antiferromagnet it has been implicitly assumed that the VBS order is either of columnar dimer type as shown in Fig. 4(a) or of plaquette type as shown in Fig. 4(b). The order parameter for columnar

more exotic scenario, where $\langle v \rangle$ jumps to a smaller but nonzero value and then decreases further before vanishing at a critical point, is also possible. This would imply that there is a region in the phase diagram where dimerization and Néel order coexist. Based on the currently available numerical data it is impossible to decide which scenario is actually realized. We also want to mention that our mean-field treatment of the order parameters and the use of the expansion are a posteriori justified. The finding of a first order phase transition means that we never get to the assumed critical point $g_{c\phi}$ where the length scale for fluctuations would diverge and a mean-field treatment would therefore be very questionable. It also means that the expansion exists and its radius of convergence will be finite everywhere in the ordered phase.

III. VBS ORDER IN THE NON-MAGNETIC PHASE

In this section we provide additional, independent arguments against the deconfinement scenario and in favor of a first order phase transition. We also address the question which kind of VBS order is actually realized. In Refs. 2,3 dealing with a possible deconfined critical point separating a Néel ordered from a VBS phase in a two-dimensional antiferromagnet it has been implicitly assumed that the VBS order is either of columnar dimer type as shown in Fig. 4(a) or of plaquette type as shown in Fig. 4(b). The order parameter for columnar...
dimerization is given by \( O_c = (-1)^i S_{i,j} S_{i+1,j} \) whereas the order parameter for the plaquette phase can be represented as \( O_p = (-1)^i S_{i,j} S_{i+1,j} + (-1)^i S_{i,j} S_{i,j+1} \). As discussed in Ref. 4, they can also be interpreted as a single complex order parameter where only the phase is different for the two patterns. At a deconfined critical point, however, this phase will only appear as an irrelevant operator. Therefore both order parameters are expected to show power law correlations at such a point and susceptibilities testing the breaking of lattice symmetries with respect to columnar or plaquette order should diverge when the deconfined critical point is approached from the magnetically ordered phase.

First, consider the field

\[
F_2 = \delta \sum_{i,j} (S_{i,j} S_{i+1,j} - S_{i,j} S_{i,j+1})
\]

(10)

which tests if rotational symmetry is broken, which would be the case for the columnar dimer state but not for the plaquette state. The corresponding susceptibility \( \chi_2 \) shows only a very moderate increase with \( g \) and certainly no sign of divergence when the possible critical point \( g_{c,0} \approx 0.43 \) is approached (see Fig. 5). We mention here that our results for the susceptibilities \( \chi_1 \) and \( \chi_2 \) are in qualitative agreement with Ref. 5 where the finite size scaling of these susceptibilities has been investigated by exact diagonalization and variational QMC methods. Note that it does not come as a surprise that there is no quantitative agreement: the variational QMC method explores only a small part of the Hilbert space so that a convergence to the true value for the susceptibility cannot be expected even if arbitrary large clusters could be treated. If a good variational wave-function is chosen, however, one might expect that the finite size scaling is at least qualitatively similar to the true finite size scaling.

Next, we consider the susceptibility \( \chi_3 \) corresponding to the field

\[
F_3 = \delta \sum_{i,j} (-1)^{i+j} (S_{i,j}^x S_{i+1,j+1}^x + S_{i,j}^y S_{i+1,j+1}^y)
\]

(11)

which has already been calculated in Ref. 11. We reproduce the result in Fig. 6. \( \chi_3 \) even decreases when the non-magnetic phase is approached which seems to indicate that the corresponding lattice symmetry, which would be broken for the plaquette but not for the columnar dimer state, is intact.

The fact that \( \chi_2 \) and \( \chi_3 \) do not diverge when \( g \to g_{c,0} \approx 0.43 \) as would be expected for a deconfined critical point provides an argument independent of the considerations in section 11 that the phase transition in the \( J_1 - J_2 \) model is not an example for the scenario proposed in Refs. 2, 3. Putting the arguments given in section 11 aside, this alone, however, does not exclude the possibility of another kind of second order phase transition from the Néel ordered to a state with VBS order. Nevertheless, in any second order scenario we should take the fact that \( \chi_2 \) and \( \chi_3 \) do not diverge seriously. This means that in this case the VBS order cannot be of columnar dimer type because then \( \chi_2 \) should diverge, and not of plaquette type because this would lead to a diverging \( \chi_3 \). So in such a scenario the quest is to find an ordering pattern with lattice symmetries which agree with our findings for all three susceptibilities.

A possible pattern is shown in Fig. 4(c). Translational symmetry by one site along the x-axis is broken leading to a divergent \( \chi_1 \). \( \chi_3 \) will be finite because of the 180° rotational symmetry around the axis marked by “1” in Fig. 4(c) or identical positions. \( \chi_2 \) will be finite for this pattern because of the following symmetry: A rotation by 90° around the axis “2” or identical positions, followed by a shift of every second row of plaquettes by one lattice site, followed by a shift of every second column of plaquettes by one lattice site. Although other patterns which have the correct symmetries might be possible, the

![FIG. 5: Susceptibility \( \chi_2 \) as a function of \( g \) calculated by Ising series expansion. The line is a guide to the eye.](image)

![FIG. 6: Susceptibility \( \chi_3 \) as a function of \( g \) calculated by Ising series expansion. The line is a guide to the eye.](image)
pattern in Fig. 4(c) seems to be the one with the smallest unit cell.

We have calculated the ground-state energy as well as the singlet and triplet dispersion for different $g$ for this state by series expansion starting from the decoupled plaquettes shown in Fig. 4(c). The expansion parameter $x$ describes the couplings between the plaquettes. For $x = 0$ the plaquettes are decoupled whereas $x = 1$ corresponds to the case we are finally interested in where the couplings within and between the plaquettes are of equal strength.

In Fig. 4 we show the ground-state energy obtained from this expansion compared to other series data.

![Graph showing ground state energies](image)

**FIG. 7:** Ground state energies calculated by different series expansions starting from the nonmagnetic states shown in Fig. 4 as well as from the magnetic Néel and collinear states. The curve 4(a) (open squares) corresponds to the columnar dimer state shown in Fig. 4(a), the curve 4(b) (black dots) to the plaquette state shown in Fig. 4(b), and the curve 4(c) (red dots) to the plaquette state shown in Fig. 4(c). The black triangles show results of Ising series expansions starting from the Néel and collinear states.

The obtained energies for $0.4 \leq g \leq 0.5$ are similar or even slightly lower than the energies obtained from the columnar dimer expansion. This shows that the state in Fig. 4(c) is indeed a possible candidate for the ground state in this parameter region. In series calculations it is, however, important to make sure that the state which is the starting point for the expansion remains stable when extrapolating to the isotropic limit $x \rightarrow 1$. We have therefore calculated also the singlet and triplet dispersions. The plaquettes for this VBS state are arranged on a triangular lattice. With respect to this lattice both singlet and triplet dispersion show a minimum at momentum $\mathbf{k} = (0, 2\pi/\sqrt{3})$. The series coefficients for the singlet and triplet gaps at this point and $g = 0.25, 0.45$ are given in Table I. Dlog Padé approximants for these series show that the singlet gap always vanishes before the triplet gap and at a value $x_c < 1$. This indicates that the state in Fig. 4(c) is unstable.

For the columnar dimer state, on the other hand, it is very difficult to obtain the singlet dispersion by series expansion because the state we are starting with consisting of decoupled dimers does not contain a singlet excitation. We have, however, calculated the triplet gap in this case which shows a minimum at $\mathbf{k} = (0, \pi)$ and Dlog Padé approximants suggest that in this case the gap remains nonzero for $x \rightarrow 1$. In addition, the instability of the state in Fig. 4(c) which manifests itself by a vanishing of the singlet gap at $\mathbf{k} = (0, 2\pi/\sqrt{3})$ for $x < 1$ is an instability towards a columnar dimerization. Although this does not prove that the columnar state is finally stable, it makes the columnar dimerization pattern the most likely candidate for the VBS order in the non-magnetic region of the phase diagram.

**IV. CONCLUSIONS**

We have calculated the ground-state energy $e$ for the $J_1 - J_2$ model including a small field $F_1$ with strength $\delta$, which induces a columnar dimerization in the Néel ordered phase, by Ising series expansion. We have argued that everywhere except directly at a critical point it is possible to expand $e(\delta)$ in a regular series in $\delta^2$ and that this series has a finite radius of convergence which goes to zero when the tentative critical point is approached. The prefactor of the $\delta^2$-term in that series gives the susceptibility $\chi_1$ with respect to $F_1$. The data for this susceptibility obtained by an Ising series expansion indicate that the Néel state becomes unstable for $g > g_c \approx 0.43$ and that the ground state for $g > g_c$ breaks translational symmetry by one site and therefore seems to be of the VBS type and not a spin liquid. Based on a mean-field treatment of an effective field theory describing the Néel state coexisting with the small dimerization induced by $F_1$ we have argued that the sign of the $\delta^4$-term in the expansion of $e(\delta)$ determines whether the transition with respect to the VBS order parameter is first or second order. We believe that this is an in general more sensitive and less biased method to distinguish between a first and a second order transition than looking for a crossing of energies obtained by different expansions. From the series data we found that the $\delta^4$-term has a negative sign and we showed that the same is true in spin-wave theory. Within the presented GL-type theory this means that the transition is expected to be first order. Our mean-field treatment of the order parameter is a posteriori justified because a critical point where such a treatment would break down is never reached.
In the second part we gave arguments in favor of a first order transition which are independent of any effective field theory by analyzing two additional susceptibilities testing different lattice symmetries. These susceptibilities were calculated based on an Ising expansion in the Néel phase so that the series is not biased by any assumed dimerization pattern. We argued that at a deconfined critical point all three susceptibilities considered here are expected to diverge and that the fact that $\chi_2$ and $\chi_3$ do not diverge excludes this scenario. We further argued that in any second order scenario the non-divergence of $\chi_2$ would mean that the VBS state is not of the columnar dimer type and the non-divergence of $\chi_3$ would mean that the VBS state is not of plaquette type either. For an assumed second order transition we have been able to find a VBS pattern which does have the correct lattice symmetries to explain our data for all three susceptibilities. Series expansion data starting from this pattern, however, have proven that this state is unstable with respect to the columnar dimerization pattern.

Taking the arguments given in the two parts together shows that the transition from the Néel state to a VBS state with columnar dimerization is most likely first order. For a first order phase transition we still expect that the susceptibility $\chi_1$ when calculated by Ising series expansion diverges because the field $F_1$ directly tests the instability of the Néel state with respect to columnar dimer order irrespective of the order of the phase transition. For the susceptibility $\chi_2$, on the other hand, we can only expect that it diverges if the transition is second order. The fact that $\chi_2$ does increase with $g$ without diverging at $g \approx 0.4$ therefore further supports that the transition is most likely first order and the VBS state most likely of the columnar dimer type.

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**TABLE I:** Series coefficients $x^n$ for the minimum singlet gap $\Delta_s$ and triplet gap $\Delta_t$ for the plaquette state from Fig. 1(c) and $g = 0.25$ ($g = 0.45$), respectively.

| $n$ | $\Delta_s / J_1 (g = 0.25)$ | $\Delta_s / J_1 (g = 0.25)$ | $\Delta_s / J_1 (g = 0.45)$ | $\Delta_s / J_1 (g = 0.45)$ |
|-----|-----------------------------|-----------------------------|-----------------------------|-----------------------------|
| 0   | 1.5000000000               | 1.0000000000               | 1.1000000000               | 1.000000000                   |
| 1   | 0.0000000000               | -8.33333334 $\times 10^{-1}$ | 0.0000000000               | -4.33333334 $\times 10^{-1}$ |
| 2   | -4.356195887 $\times 10^{-1}$ | -5.72808442 $\times 10^{-1}$ | -4.15103339 $\times 10^{-1}$ | -5.00596637 $\times 10^{-1}$ |
| 3   | -3.880918282 $\times 10^{-1}$ | 1.4032850965               | -1.832689336 $\times 10^{-1}$ | 8.2113320127 $\times 10^{-2}$ |
| 4   | -4.828163654 $\times 10^{-1}$ | -2.265506840 $\times 10^{-1}$ | -1.537390367 $\times 10^{-1}$ | -3.080545443 $\times 10^{-2}$ |
| 5   | -6.632418702 $\times 10^{-1}$ | 7.5790435073 $\times 10^{-3}$ | -1.592497102 $\times 10^{-1}$ | -1.108096117 $\times 10^{-1}$ |
| 6   | -7.988310579 $\times 10^{-1}$ | -4.517676083 $\times 10^{-2}$ | -1.653261118 $\times 10^{-1}$ | 1.6084176410 $\times 10^{-2}$ |

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