MILD SOLUTIONS TO THE TIME FRACTIONAL NAVIER-STOKES DELAY DIFFERENTIAL INCLUSIONS

YEJUAN WANG* AND TONTONG LIANG

School of Mathematics and Statistics
Gansu Key Laboratory of Applied Mathematics and Complex Systems
Lanzhou University
Lanzhou 730000, China

(Communicated by Tomas Caraballo)

Abstract. In this paper, we study a Navier-Stokes delay differential inclusion with time fractional derivative of order $\alpha \in (0, 1)$. We first prove the local and global existence, decay and regularity properties of mild solutions when the initial data belongs to $C([-h, 0]; D(A^\varepsilon))$. The fractional resolvent operator theory and some techniques of measure of noncompactness are successfully applied to obtain the results.

1. Introduction. In this paper we consider the following functional Navier-Stokes differential inclusion with time fractional derivative

$$
\begin{cases}
\epsilon D_t^\alpha u - \Delta u + (u \cdot \nabla)u + \nabla p \in g(t) + f(t, u_t) \quad \text{in } (0, \infty) \times \Omega, \\
\nabla \cdot u = 0 \quad \text{in } (0, \infty) \times \Omega, \\
u = 0 \quad \text{on } (0, \infty) \times \partial \Omega, \\
u(t, x) = \varphi(t, x), \quad t \in [-h, 0], \ x \in \Omega,
\end{cases}
$$

(1.1)

where $\Omega$ is a bounded domain in $\mathbb{R}^N$, $N \geq 2$, $\partial \Omega \in C^\infty$, $\epsilon D_t^\alpha u$ is the Caputo fractional derivative (see Definition 1), $u$ is the velocity field of the fluid, $p$ the pressure, $g$ a nondelayed external force field, $f$ another external force with some hereditary characteristics and $\varphi$ the initial datum in the interval of time $[-h, 0]$, where $h$ is a fixed positive number.

This research is motivated by control problems for fluid flows, these situations may appear, for instance, when we want to control the system by applying a force which takes into account not only the present state of the system but the history of the solutions. In recent years, there is an increasing interest in the study of Navier-Stokes models in which the forcing term contains some hereditary features; see, e.g., [3, 4, 5, 6, 20, 21, 22, 23, 33, 40] and the references cited therein.

2010 Mathematics Subject Classification. Primary: 33E12, 34K37, 35Q30, 35R11, 35B65.

Key words and phrases. Fractional delay differential inclusions, Navier-Stokes equations, mild solution, singular initial data, measure of noncompactness, upper semi-continuity.

This work was supported by NSF of China (Grants No. 41875084, 11571153), the Fundamental Research Funds for the Central Universities under Grant Nos. lzujbky-2018-ot03 and lzujbky-2018-it58.

* Corresponding author.
Fractional Navier-Stokes equations have received increased attention in the last years due to their physical applications in many fields such as turbulence, heterogeneous flows and materials, viscoelasticity and electromagnetic theory. Navier-Stokes equations with time fractional derivative were studied in [8, 17, 39, 41, 45, 49, 50, 51], for instance. Also, Navier-Stokes equations with fractional dissipation have been considered in [9, 13, 16, 18, 24, 26, 44, 48] (see [14] for fractional stochastic Navier-Stokes equations as well). Numerical analysis of the fractional Navier-Stokes equations has been studied in, for example, [34, 35, 43, 45] and the references therein.

Recently, there are many works on functional Navier-Stokes equations with classical derivative. For the existence and regularity of pullback attractors for 2D Navier-Stokes equations with delay, we refer the reader to [3, 5, 6, 20, 21, 22, 38] and the references therein. The exponential stability of the stationary solution to such equations has been investigated in [4] and [23]. Globally modified three dimensional Navier-Stokes equations with delays have been studied in [25, 31, 32]. Existence and decay of solutions in full space to Navier-Stokes equations with delays have been considered in [40]. There has, however, been little mention of functional Navier-Stokes differential inclusions even in the case of classical derivative.

In this paper, we are interested in the existence, decay and regularity of mild solutions to the functional Navier-Stokes differential inclusions with time fractional derivative. The novelty and the difficulties of this work are in three aspects: (i) $f$ is a multi-valued mapping containing some hereditary features. In order to obtain the existence of mild solutions we use some techniques of measure of noncompactness and the general theory of multi-valued mappings developed in [1, 27, 29]. Some new ideas for checking the upper semi-continuity of the multi-valued operator $L$ are developed here to circumvent the difficulty caused by the multi-valued mapping $f$. (ii) The vector-valued Lebesgue space $L^r(\Omega)$ ($N < r < \infty$) will be considered. We shall use some estimates for the Mittag-Leffler families proposed in [8] and conduct new and complicated estimates for the linear and nonlinear parts of mild solutions to the functional Navier-Stokes differential inclusions with time fractional derivative. (iii) The nonlinear part $B(u) = B(u, u)$ where $B(u, v) = -P(u \cdot \nabla)v$. Here we need to compute the measure of noncompactness for the nonlinear part $B(u)$.

This paper is organized as follows. In Section 2, we recall some basic concepts and results related to fractional calculus in Banach spaces, measure of noncompactness and multi-valued mappings. In Section 3, we state and prove the main theorems of the paper, concerning the existence of local and global mild solutions. The decay of the solutions is also obtained. Finally, an illustrative example is presented.

2. Preliminaries. In this section we will recall some definitions and preliminary facts related to fractional calculus in Banach spaces, measure of noncompactness and multi-valued mappings.

2.1. Fractional setting. To set our problem (1.1) in the abstract framework, we consider the following usual abstract space:

$$L^r_\sigma = \{ u \in L^r(\Omega) : \nabla \cdot u = 0 \}$$

with norm $\| \cdot \|_{L^r}$, where $L^r$ denotes the vector-valued Lebesgue space and for $u \in L^r(\Omega)$,

$$\| u \|_{L^r}^r = \sum_{j=1}^N \int_\Omega |u_j(x)|^r \, dx.$$
Besides, let $S \subset \mathbb{R}$ and $X$ be a Banach space. We denote the space of the continuous functions from $S$ to $X$ by $C(S; X)$ and the space of the continuous and bounded functions from $S$ to $X$ by $CB(S; X)$ (equipped with its usual norm). For $1 \leq p \leq \infty$, $L^p(S; X)$ denotes the Banach space of $L^p$ integrable functions $u : S \to X$ if $p < \infty$ and the essential bounded functions when $p = \infty$. $W^{1,p}(S; X)$ is the subspace of $L^p(S; X)$ consisting of functions such that the weak derivative $\frac{\partial u}{\partial t}$ belongs to $L^p(S; X)$. Both spaces $L^p(S; X)$ and $W^{1,p}(S; X)$ are endowed with their standard norms. Given $T > 0$ and $u : [-h, T] \to L^r$, for each $t \in [0, T]$ we denote by $u_t$ the function on $[-h, 0]$ by the relation $u_t(s) = u(t + s)$, $s \in [-h, 0]$. We also denote $C_h = C([-h, 0]; L^r)$. In the sequel, $C$ denotes an arbitrary positive constant, which may be different from line to line and even in the same line.

We now recall some facts about the theory of fractional calculus.

For $\beta > 0$, define the function $g_\beta : \mathbb{R} \to \mathbb{R}$ by

$$g_\beta(t) := \begin{cases} \frac{1}{\Gamma(\beta)} t^{\beta - 1}, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

where $\Gamma(\beta)$ is Euler’s Gamma function. For a function $u \in L^1([0, T]; X)$, the Riemann-Liouville fractional integral of order $\beta$ of $u$ is given by

$$J_t^\beta u(t) := g_\beta \ast u(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} u(s) \, ds, \quad t \in [0, T].$$

Based on the definition of Riemann-Liouville fractional integral operator, we present the Caputo fractional differential operator; for more details, we refer to the books [30, 42].

**Definition 1.** Let $\beta \in (0, 1)$ and $T > 0$. Consider $u \in C([0, T]; X)$ such that the convolution $g_{1-\beta} \ast u \in W^{1,1}([0, T]; X)$. The expression

$$D_t^\beta u(t) := \frac{d}{dt} \left\{ J_t^{-\beta} [u(t) - u(0)] \right\} = \frac{d}{dt} \left\{ \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} [u(s) - u(0)] \, ds \right\}$$

is called the Caputo fractional derivative of order $\beta$ of the function $u$.

For $\beta \in (0, 1)$, the entire function $M_\beta : \mathbb{C} \to \mathbb{C}$ given by

$$M_\beta(z) := \sum_{n=0}^{\infty} \frac{z^n}{n! \Gamma(1 - \beta(1 + n))}$$

is called Mainardi function. This function is a particular case of the Wright type function introduced by Mainardi in [36].

The following classical result shows some properties of the Mainardi function [7, 37, 46].

**Proposition 2.** For $\beta \in (0, 1)$ and $-1 < r < \infty$, when we restrict $M_\beta$ to the positive real line, it holds that

$$M_\beta(t) \geq 0 \text{ for all } t \geq 0, \quad \text{and} \quad \int_0^\infty t^r M_\beta(t) \, dt = \frac{\Gamma(r+1)}{\Gamma(\beta r + 1)}.$$

Let $X$ be a Banach space and $-A : D(A) \subset X \to X$ be the infinitesimal generator of an analytic semigroup $\{T(t) : t \geq 0\}$. Then, for each $\beta \in (0, 1)$, the Mittag-Leffler families $\{E_\beta(-t^\beta A) : t \geq 0\}$ and $\{E_{\beta, h}(-t^\beta A) : t \geq 0\}$ are defined by

$$E_\beta(-t^\beta A) = \int_0^\infty M_\beta(s) T(st^\beta) \, ds,$$
Lemma 3. The operators $E_{\beta}(\cdot^{\beta}A)$ and $E_{\alpha,\beta}(\cdot^{\beta}A)$ are well defined from $X$ to $X$. Moreover, for $x \in X$ it holds

(i) $E_{\beta}(\cdot^{\beta}A)x|_{t=0} = x$ and $E_{\alpha,\beta}(\cdot^{\beta}A)x|_{t=0} = x$;

(ii) the vectorial functions $t \to E_{\beta}(\cdot^{\beta}A)x$ and $t \to E_{\alpha,\beta}(\cdot^{\beta}A)x$ are analytic from $[0, \infty)$ to $X$, and satisfy

$$cD_t^\beta E_{\beta}(\cdot^{\beta}A)x = -AE_{\beta}(\cdot^{\beta}A)x, \quad t > 0.$$ 

We rewrite the time fractional Navier-Stokes delay differential inclusions in the abstract form

$$\begin{cases}
  cD_t^\beta u \in -A_t u + B(u) + G(t, u_t), & t > 0, \\
  u(t) = \varphi(t), & t \in [-h, 0],
\end{cases}$$

(2.1)

where $A_t = -P\Delta = -\Delta P$, $B(u) = B(u, u)$, $B(u, v) = -P(u \cdot \nabla)v$, $G = P\eta$ and $F(t, u_t) = Pf(t, u_t)$. Here $P : L^r \to L^r_\sigma$ is the Helmholtz-Leray projector and $A_r : D(A_r) \subset L^r_\sigma \to L^r_\sigma$ is the Stokes operator.

For $\gamma > 0$, let $X^{\gamma,r} = D(A_r^\gamma)$ with norm

$$\|u\|_{X^{\gamma,r}} = \|A_r^\gamma u\|_{L^r}.$$ 

We recall the following well-known estimates for the semigroup generated by the Stokes operator (see, for instance, [19, 47]).

Proposition 4. Let $1 < r < \infty$, $\gamma_1 < \gamma_2$ and $v \in X^{\gamma_1,r}$. Then there exists a constant $C_1 = C_1(\gamma_1, \gamma_2)$ such that

$$\|e^{-tA_r}v\|_{X^{\gamma_2,r}} \leq C_1 t^{-\gamma_2 - \gamma_1} \|v\|_{X^{\gamma_1,r}}, \quad t > 0.$$ 

Furthermore,

$$\lim_{t \to 0} t^{\gamma_2 - \gamma_1} \|e^{-tA_r}v\|_{X^{\gamma_2,r}} = 0$$

for all $v \in X^{\gamma_1,r}$.

Similar estimates for both families of Mittag-Leffler operators were proved in [51].

Lemma 5. Let $\beta \in (0, 1)$, $1 < r < \infty$ and $\gamma_1 < \gamma_2$. Then for any $v \in X^{\gamma_1,r}$, there exists a constant $C_2 = C_2(r, \gamma_1, \gamma_2, \beta) > 0$ such that

$$\|E_{\beta}(\cdot^{\beta}A_r)v\|_{X^{\gamma_2,r}} \leq C_2 t^{-\beta(\gamma_2 - \gamma_1)} \|v\|_{X^{\gamma_1,r}}, \quad t > 0$$

and

$$\|E_{\alpha,\beta}(\cdot^{\beta}A_r)v\|_{X^{\gamma_2,r}} \leq C_2 t^{-\beta(\gamma_2 - \gamma_1)} \|v\|_{X^{\gamma_1,r}}, \quad t > 0.$$ 

Furthermore,

$$\lim_{t \to 0} t^{\beta(\gamma_2 - \gamma_1)} \|E_{\beta}(\cdot^{\beta}A_r)v\|_{X^{\gamma_2,r}} = 0$$

for all $v \in X^{\gamma_1,r}$.

We next show the continuity of the Mittag-Leffler families, since they also play an important role in the proof of the existence of solutions.
Lemma 6. Let $\beta \in (0,1)$, $1 < r < \infty$ and $0 \leq \gamma_2 - \gamma_1 < 1$. Then $E_\beta(-t^\beta A_r) : X^{\gamma_1,r} \to X^{\gamma_2,r}$ and $E_{\beta,\beta}(-t^\beta A_r) : X^{\gamma_1,r} \to X^{\gamma_2,r}$ are continuous in $(0,\infty)$ by the operator norm.

Proof. Fix $t_1 > 0$ and consider $t_2 > t_1$, since the case $0 < t_2 < t_1$ follows similarly. By Proposition 4, we have

$$\left\| e^{-t_2 A_r} v - e^{-t_1 A_r} v \right\|_{X^{\gamma_2,r}} = \left\| \int_{t_1}^{t_2} A_r e^{-s A_r} v ds \right\|_{X^{\gamma_2,r}} \leq C_1 \int_{t_1}^{t_2} s^{-(\gamma_2+1-\gamma_1)} \| v \|_{X^{\gamma_1,r}} ds \leq C \left( t_1^{-(\gamma_2-\gamma_1)} - t_2^{-(\gamma_2-\gamma_1)} \right) \| v \|_{X^{\gamma_1,r}},$$

which implies that $e^{-t^\beta A_r} : X^{\gamma_1,r} \to X^{\gamma_2,r}$ is continuous in $(0,\infty)$ by the operator norm.

Now it is sufficient to prove that $E_{\beta}(-t^\beta A_r) : X^{\gamma_1,r} \to X^{\gamma_2,r}$ is continuous in $(0,\infty)$ by the operator norm, since the other case can be proved similarly. Let $\varepsilon > 0$ be given arbitrarily. By the properties of the Mainardi function, we can choose $0 < \delta_1 < \delta_2$ such that

$$2C_1 t_1^{-\beta(\gamma_2-\gamma_1)} \int_0^{\delta_1} s^{-(\gamma_2-\gamma_1)} M_\beta(s) ds < \frac{\varepsilon}{3},$$

(2.3)

and

$$2C_1 t_1^{-\beta(\gamma_2-\gamma_1)} \int_{\delta_2}^{\infty} s^{-(\gamma_2-\gamma_1)} M_\beta(s) ds < \frac{\varepsilon}{3},$$

(2.4)

where $C_1$ is the constant in Proposition 4. Observe that $e^{-t^\beta A_r} : X^{\gamma_1,r} \to X^{\gamma_2,r}$ is continuous in $(0,\infty)$ by the operator norm, hence it follows from Lebesgue’s dominated convergence theorem that there exists $\delta' > 0$ such that

$$\int_{\delta_1}^{\delta_2} M_\beta(s) \left\| e^{-s t_1^\beta A_r} v - e^{-s t_2^\beta A_r} v \right\|_{X^{\gamma_2,r}} ds < \frac{\varepsilon}{3} \| v \|_{X^{\gamma_1,r}},$$

(2.5)

for any $t_2 > t_1 > 0$ with $|t_1 - t_2| < \delta'$ and $v \in X^{\gamma_1,r}$. Then, by using the definition of the Mittag-Leffler operators, Proposition 4 and (2.3)-(2.5), we deduce that

$$\left\| E_{\beta}(-t^\beta A_r) v - E_{\beta}(-t_2^\beta A_r) v \right\|_{X^{\gamma_2,r}} \leq \int_0^{\infty} M_\beta(s) \left\| e^{-s t_1^\beta A_r} v - e^{-s t_2^\beta A_r} v \right\|_{X^{\gamma_2,r}} ds \leq 2C_1 t_1^{-\beta(\gamma_2-\gamma_1)} \| v \|_{X^{\gamma_1,r}} \left( \int_0^{\delta_1} s^{-(\gamma_2-\gamma_1)} M_\beta(s) ds + \int_{\delta_2}^{\infty} s^{-(\gamma_2-\gamma_1)} M_\beta(s) ds \right) + \int_{\delta_1}^{\delta_2} M_\beta(s) \left\| e^{-s t_1^\beta A_r} v - e^{-s t_2^\beta A_r} v \right\|_{X^{\gamma_2,r}} ds < \frac{2\varepsilon}{3} \| v \|_{X^{\gamma_1,r}} + \frac{\varepsilon}{3} \| v \|_{X^{\gamma_1,r}} = \varepsilon \| v \|_{X^{\gamma_1,r}}$$

for any $t_2 > t_1 > 0$ with $|t_1 - t_2| < \delta'$ and $v \in X^{\gamma_1,r}$. \hfill \Box

The following estimates for the nonlinear term $B(u,v)$ can be found in [47].

Proposition 7. Let $N < r < \infty$. Then there exists a constant $C_3 = C_3(r, N) > 0$ such that for any $\nabla u, \nabla v \in L^r(\Omega)$,

$$\| B(u,v) \|_{L^r} \leq C_3 \| \nabla u \|_{L^r} \| \nabla v \|_{L^r},$$

(2.6)
\[ \|B(u, u) - B(v, v)\|_{L^r} \leq C_3 (\|\nabla u\|_{L^r} + \|\nabla v\|_{L^r}) \|\nabla u - \nabla v\|_{L^r}. \] (2.7)

2.2. Hausdorff measure of noncompactness and multi-valued mappings.

We recapitulate the standard definition of the Hausdorff measure of noncompactness and its basic properties; see [1, 27, 29] for more details.

Let \( X \) be a Banach space with norm \( \| \cdot \|_X \), and \( \mathcal{B}(X) \) be the collection of all nonempty and bounded subsets of \( X \).

**Definition 8.** A function \( \chi : \mathcal{B}(X) \to \mathbb{R}^+ \) is called the Hausdorff measure of noncompactness (MNC) on \( X \) if for every \( B \in \mathcal{B}(X) \),

\[
\chi(B) = \inf \{ \varepsilon : B \text{ has a finite } \varepsilon\text{-net} \}.
\]

**Lemma 9.** Let \( X \) be a Banach space, and \( \chi \) be the Hausdorff MNC on \( X \). Then

1. if for each \( B_1, B_2 \in \mathcal{B}(X) \) such that \( B_1 \subset B_2 \), then \( \chi(B_1) \leq \chi(B_2) \);
2. \( \chi((a) \cup B) = \chi(B) \) for any \( a \in X \), \( B \in \mathcal{B}(X) \);
3. \( \chi(K \cup B) = \chi(B) \) for every relatively compact set \( K \subset X \), \( B \in \mathcal{B}(X) \);
4. \( \chi(B_1 + B_2) \leq \chi(B_1) + \chi(B_2) \) for any \( B_1, B_2 \in \mathcal{B}(X) \);
5. \( \chi(B) = \chi(\overline{B}) \), where \( \overline{B} \) is the closure of \( B \);
6. \( \chi(B) = \chi(\text{conv } B) \), where \( \text{conv } B \) is the closed convex hull of \( B \);
7. \( \chi(B) = 0 \) is equivalent to the relative compactness of \( B \);
8. If \( B_t \) is a family of nonempty, closed and bounded sets defined for \( t > r \) that satisfy \( B_t \subset B_s \) whenever \( s \leq t \), and \( \chi(B_t) \to 0 \) as \( t \to \infty \), then \( \bigcap_{t > r} B_t \) is a nonempty, compact set in \( X \).

The following result for the Hausdorff measure of noncompactness can be found in [29].

**Lemma 10.** Let \( B \in L^1([0, T]; X) \) be such that

1. \( \|\xi(t)\|_X \leq \vartheta(t) \), for all \( \xi \in B \) and for a.e. \( t \in [0, T] \),
2. \( \chi(B(t)) \leq q(t) \) for a.e. \( t \in [0, T] \),

where \( \vartheta, q \in L^1([0, T]) \). Then for all \( t \in [0, T] \),

\[
\chi \left( \int_0^t B(s)ds \right) \leq 4 \int_0^t q(s)ds,
\]

where \( \int_0^t B(s)ds = \{ \int_0^t \xi(s)ds : \xi \in B \} \).

**Definition 11.** A sequence of functions \( \{f_n\}_{n=1}^\infty \subset L^1([0, T]; X) \) is said to be integrable bounded if

\[
\|f_n(t)\|_X \leq \vartheta(t), \quad \text{for all } n = 1, 2, \ldots \text{ and a.e. } t \in [0, T],
\]

where \( \vartheta(t) \in L^1([0, T]) \).

Now let us recall some notions of set-valued analysis and a fixed point theorem for multi-valued mappings.

Let \( Y, E \) be two metric spaces, and \( \mathcal{P}(E) \) be the collection of all nonempty subsets of \( E \).

**Definition 12.** A multi-valued mapping \( F : Y \to \mathcal{P}(E) \) is said to be

1. upper semi-continuous (u.s.c.) if \( F^{-1}(V) := \{ y \in Y : F(y) \cap V \neq \emptyset \} \) is a closed subset of \( Y \) for every closed set \( V \subset E \);
2. closed if its graph \( \Gamma_F := \{ (y, z) : z \in F(y) \} \) is a closed subset of \( Y \times E \);
3. compact if its range \( F(Y) \) is relatively compact in \( E \).
Definition 13. A sequence \( \{f_n\}_{n=1}^{\infty} \subset L^1([0,T]; E) \) is said to be semicompact if it is integrable bounded and \( \{f_n(t)\} \subset \mathcal{K}(t) \), for a.e. \( t \in [0,T] \), where \( \mathcal{K}(t) \subset E \), \( t \in [0,T] \), is a family of compact sets.

In other words, the sequence \( \{f_n\}_{n=1}^{\infty} \subset L^1([0,T]; E) \) is semicompact if it is integrable bounded and \( \{f_n(t)\}_{n=1}^{\infty} \subset K(t) \) is relatively compact for a.e. \( t \in [0,T] \); see [27] for more details.

The following result can be found in [15, Proposition 1.1].

Proposition 14. Let \( E \) be a Banach space and \( D \) be a nonempty subset of another Banach space. Assume that \( \mathcal{F} : D \rightarrow \mathcal{P}(E) \) is a multi-valued mapping with compact values. Then \( \mathcal{F} \) is u.s.c. if and only if \( \{x_n\}_{n=1}^{\infty} \subset D \) with \( x_n \rightarrow x_0 \in D \) and \( y_n \in \mathcal{F}(x_n) \) implies \( y_n \rightarrow y_0 \in \mathcal{F}(x_0) \) up to a subsequence.

In the next section, we will prove that problem (2.1) has at least one local mild solution, by using the following fixed point theorem for multi-valued mappings due to [29].

Theorem 15. Let \( E \) be a Banach space and \( M \subset E \) be a nonempty compact convex subset. If the multi-valued operator \( \mathcal{F} : M \rightarrow \mathcal{P}(M) \) is upper semi-continuous with closed convex values, then \( \mathcal{F} \) has a fixed point.

3. Existence and decay of solutions. For \( \varphi \in C_h \), we define the space

\[
C_\varphi = \left\{ u : [0,T] \rightarrow \overline{D(A_r)} \mid u \in C([0,T]; L^r_\sigma), u(0) = \varphi(0) \right\}.
\]

Then for any \( u \in C_\varphi \), we denote the function \( u[\varphi] \in C([-h,T]; L^r_\sigma) \) as follows

\[
u[\varphi](t) = \begin{cases} 
 u(t), & t \in [0,T], \\
 \varphi(t), & t \in [-h,0]. 
\end{cases}
\]

Thus,

\[
u[\varphi](\theta) = \begin{cases} 
 u(t + \theta), & \theta \in [-t,0], \\
 \varphi(t + \theta), & \theta \in [-h - t,-t]. 
\end{cases}
\]

For \( u \in C_\varphi \), let us define \( \mathcal{P}_\varphi(u) \) by setting

\[
\mathcal{P}_\varphi(u) = \left\{ f \in L^1([0,T]; L^r_\sigma) : f(t) \in F(t,u[\varphi]_t), \text{ for a.e. } t \in [0,T] \right\}.
\]

By the arguments in [2, 7, 8, 10, 11, 12, 46] and references therein, the notion of mild solutions to problem (2.1) is given by a fractional variation of constants formula which involves the Mittag-Leffler families.

Definition 16. Consider real values \( \alpha \in (0,1) \) and \( 2 \leq N < r < \infty \). A continuous function \( u : [0,T] \rightarrow X^{\varepsilon,r} \) with \( 0 < \varepsilon < \frac{1}{2} \) is called a mild solution of (2.1) if \( u(t) = \varphi(t) \) for \( t \in [-h,0] \) with \( A_r^\varepsilon \varphi \in C_h \), and for \( t \in [0,T] \), there exists \( f \in \mathcal{P}_\varphi(u) \) such that the following integral equation holds

\[
u(t) = E_\alpha(-t^\alpha A_r)\varphi(0) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-(t-s)^\alpha A_r)B(u(s))ds \\
+ \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-(t-s)^\alpha A_r)G(s)ds \\
+ \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-(t-s)^\alpha A_r)f(s)ds.
\]
Then we define the multi-valued operator $L : \mathcal{P}(L^\infty_r)$ has compact convex values; for each $t$, the multi-valued function $F(t, \cdot) : C_h \to \mathcal{P}(L^\infty_r)$ is u.s.c., and for each $\psi \in C_h$, $F(\cdot, \psi)$ has a strongly measurable selection.

Let us now introduce the following conditions:

1. The multi-valued function $F : [0, \infty) \times C_h \to \mathcal{P}(L^\infty_r)$ satisfies

   \[ \|F(t, y)\|_{L^r} := \sup \{ \|\xi\|_{L^r} : \xi \in F(t, y) \} \leq m_1(t)\|y\|_{C_h} + m_0(t) \]

   for all $t \geq 0, y \in C_h$.

2. There exist nonnegative functions $m_0 \in L^q([\mathbb{R}^+])$ with $q \in (\frac{2}{\alpha}, \infty)$ and $m_1 \in L^\infty([\mathbb{R}^+])$ such that the multi-valued mapping $F : [0, \infty) \times C_h \to \mathcal{P}(L^\infty_r)$ satisfies

   \[ \left( \int_0^\infty \|G(s)\|_{L^r}^q ds \right)^{\frac{1}{q}} = \Lambda < \infty. \]

3. There exists a constant $q \in (\frac{2}{\alpha}, \infty)$ such that the function $G : [0, \infty) \to L^\infty_r$ satisfies

   \[ \chi(F(t, B)) \leq l(t) \sup_{s \in [-h, 0]} \chi(B(s)), \]

   where $\chi$ is the Hausdorff measure of noncompactness on $L^r$.

In order to apply Theorem 15, we now introduce suitable Banach spaces. For any $\beta \in (0, 1)$, $2 \leq N < r < \infty$ and $0 < \varepsilon < \frac{1}{2}$, consider the Banach space $\mathcal{X}^\beta_r[0, T]$ of functions $v$ satisfying

\[ t^{\beta(\frac{1}{2} - \varepsilon)} \nabla v \in C_h([0, T]; L^\infty_r) \quad \text{and} \quad \lim_{t \to 0} t^{\beta(\frac{1}{2} - \varepsilon)} \nabla v(t) = 0, \]

with its natural norm

\[ \|v\|_{\mathcal{X}^\beta_r[0, T]} := \sup_{t \in [0, T]} \|A_r^\beta v(t)\|_{L^r} + \sup_{t \in [0, T]} t^{\beta(\frac{1}{2} - \varepsilon)}\|\nabla v(t)\|_{L^r}. \]

Then we define the multi-valued operator $L : \mathcal{X}^\alpha_r[0, T] \to \mathcal{X}^\alpha_r[0, T]$ as follows

\[ L(v)(t) = \left\{ E_\alpha(-t^\alpha A_r)\varphi(0) + \int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}(-(t-s)^\alpha A_r)(B(v(s)) + G(s))ds \right. \]

\[ + W(f)(t) : f \in \mathcal{P}_F(v) \}, \]

(3.2)

where

\[ W(f)(t) = \int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}(-(t-s)^\alpha A_r)f(s)ds. \]

(3.3)

Thanks to the formulation of the operator $W$, $L$ can be rewritten as

\[ L(v)(t) = E_\alpha(-t^\alpha A_r)\varphi(0) + \int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}(-(t-s)^\alpha A_r)(B(v(s)) \]

\[ + G(s))ds + W \circ \mathcal{P}_F(v). \]

(3.4)

Now we give some important properties which will be used later. First, we need the following useful result from [27, Theorem 4.2.1].
Theorem 17. Let $X$ be a Banach space, and let the sequence of functions $\{f_n\}_{n=1}^\infty \subset L^1([0,T];X)$ be integrable bounded:
\[ \|f_n(t)\|_X \leq \vartheta(t), \quad \text{for all } n = 1, 2, \ldots \text{ and a.e. } t \in [0,T], \]
where $\vartheta(t) \in L^1([0,T])$. Assume that
\[ \chi(\{f_n(t)\}_{n=1}^\infty) \leq \nu(t), \quad \text{for a.e. } t \in [0,T], \]
where $\nu \in L^1([0,T])$. Then for every $\delta > 0$, there exist a compact set $K_\delta \subset X$, a set $m_\delta \subset [0,T]$, $\text{meas}(m_\delta) < \delta$ and a set of functions $G_\delta \subset L^1([0,T];X)$ with values in $K_\delta$ such that for every $n \geq 1$ there exists $x_n \in G_\delta$ for which
\[ \|f_n(t) - x_n(t)\|_X \leq 2\nu(t) + \delta, \quad t \in [0,T] \setminus m_\delta. \]

Proposition 18. If $\mathcal{X} \subset L^q([0,T];L^r)$ is such that
\[ \|x(t)\|_{L^r} \leq p(t), \quad \text{for all } x \in \mathcal{X} \text{ and a.e. } t \in [0,T], \]
where $p \in L^q([0,T])$ with $q \in (\frac{2}{3}, \infty)$, and $\{f_n\}_{n=1}^\infty \subset \mathcal{X}$ are semicompact sequences, then $\{\nabla W(f_n(t))\}_{n=1}^\infty$ are relatively compact in $L^r$ for each $t \in [0,T]$.

Proof. Noticing that
\[ \|f_n(t)\|_{L^r} < p(t), \quad \text{for all } n = 1, 2, \ldots \text{ and a.e. } t \in [0,T], \tag{3.5} \]
and by $p \in L^q([0,T])$, we conclude that for any $\eta > 0$ there exists $\delta_1 \in (0, \frac{\eta}{4C_2T^{\frac{2}{\alpha}}})$ such that for every set $m \subset [0,T]$ with $\text{meas}(m) < \delta_1$, we get
\[ \int_m (p(s))^q ds < \left( \frac{\left( \frac{\alpha}{2} - 1 \right) \frac{q}{q-1} + 1}{4C_2T^{\frac{2}{\alpha}-1}} \right)^{\eta}, \tag{3.6} \]
Invoking Theorem 17, in view of (3.5) and the semicompactness of $\{f_n\}_{n=1}^\infty$, hence there exist a set $m_{\delta_1} \subset [0,T]$ with $\text{meas}(m_{\delta_1}) < \delta_1$, a compact set $K_{\delta_1} \subset L^r$ and a set of functions $G_{\delta_1} \subset \mathcal{X}$ with values in $K_{\delta_1}$ such that for every $n \geq 1$ there exists $y_n \in G_{\delta_1}$, for which
\[ \|f_n(t) - y_n(t)\|_{L^r} \leq \delta_1, \quad t \in [0,T] \setminus m_{\delta_1}. \tag{3.7} \]
Take $\beta = \alpha$, $\gamma_1 = 0$, $\gamma_2 = \frac{1}{2}$ in Lemma 5, in view of (3.6)-(3.7), the properties of the Beta function and Hölder’s inequality, we obtain that
\[
\|\nabla(W(f_n(t)) - W(y_n(t)))\|_{L^r} \\
= \left\| \nabla \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-(t-s)^\alpha A_r) (f_n(s) - y_n(s)) \, ds \right\|_{L^r} \\
\leq C_2 \int_0^t (t-s)^{\frac{\alpha}{2}-1} \|f_n(s) - y_n(s)\|_{L^r} ds \\
\leq 2C_2 \int_{[0,t]\setminus m_{\delta_1}} (t-s)^{\frac{\alpha}{2}-1} \|f_n(s) - y_n(s)\|_{L^r} ds \\
+ C_2 \int_{m_{\delta_1}} (t-s)^{\frac{\alpha}{2}-1} \|f_n(s) - y_n(s)\|_{L^r} ds \\
\leq 2C_2T^{\frac{\alpha}{2}} \delta_1 + 2C_2 \left( \int_0^t (t-s)^{\frac{(\alpha-1)q}{2}} ds \right)^{\frac{2}{q}} \left( \int_0^t (p(s))^q ds \right)^{\frac{1}{q}}.
\]
\[
\leq \frac{2C_2 T^{\frac{2}{\alpha}}}{\delta_1} + \frac{2C_2}{\left(\left(\frac{2}{\alpha} - 1\right)\frac{q}{q-1} + 1\right)} T^{\frac{2}{\alpha} - \frac{1}{4}} \left(\int_0^t (p(s))^{q/d} ds\right)^{\frac{1}{q}} < \eta. \tag{3.8}
\]

Therefore, \(\{\nabla W(f_n(t))\}_{n=1}^{\infty}\) belongs to a \(\eta\)-net of the set
\[
\nabla W(K_{\delta_1}) := \nabla \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^{\alpha} A_r) K_{\delta_1} ds.
\]

In order to prove the relative compactness of \(\{\nabla W(f_n(t))\}_{n=1}^{\infty}\) in \(L^r\), let us consider the relative compactness of \(\nabla W(K_{\delta_1})(t)\) in \(L^r\). Let \(v_n \in W(K_{\delta_1})(t)\) be given arbitrarily. Then there exists \(z_n \in K_{\delta_1}\) such that
\[
v_n = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^{\alpha} A_r) z_n ds.
\]

Since \(K_{\delta_1}\) is compact in \(L^r\), there exist a subsequence \(z_{n_k}\) and \(z \in K_{\delta_1}\) which is the limit of \(z_{n_k}\). Applying Lemma 5 again, we deduce that
\[
\|\nabla (W(z_{n_k})(t) - W(z)(t))\|_{L^r}
\]
\[
= \left\| \nabla \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^{\alpha} A_r) (z_{n_k} - z) ds \right\|_{L^r}
\]
\[
\leq C_2 \int_0^t (t-s)^{\frac{2}{\alpha} - 1} \|z_{n_k} - z\|_{L^r} ds
\]
\[
\leq CT^{\frac{2}{\alpha}} \|z_{n_k} - z\|_{L^r} \to 0 \tag{3.9}
\]
as \(k \to \infty\). Therefore \(\nabla W(K_{\delta_1})(t)\) is relatively compact in \(L^r\). This implies that \(\{\nabla W(f_n(t))\}_{n=1}^{\infty}\) is relatively compact in \(L^r\).

The first result is concerned with the existence of the local mild solution to problem (2.1).

**Theorem 19.** Assume that (H1)-(H4) hold. Then for \(\alpha \in (0,1)\), \(2 \leq N < r < \infty\) and \(A^2 \varphi_0 \in C(I)\) with \(0 < \varepsilon < 2\), there exists \(T_0 > 0\) such that problem (2.1) has at least one local mild solution \(u : [0, T_0] \to X^{r-3}\). Moreover,
\[
t^{\alpha-(\frac{2}{\alpha})} \nabla u \in C([0, T_0]; L^r_{x_{\alpha}})
\]
with value zero at \(t = 0\), in which \(u(0) = \varphi(0)\).

**Proof.** We divide the proof into four steps.

**Step 1.** There exist \(T_0 > 0\) and a closed convex set \(\mathcal{M} \subset X^\alpha_{\alpha}[0, T]\) satisfying that \(\mathcal{L}(\mathcal{M}) \subset \mathcal{M}\).

Let \(w \in \mathcal{L}(u)\) with \(u \in X^\alpha_{\alpha}[0, T]\) and \(T > 0\). Then it follows from (3.2) that
\[
t^{\alpha-(\frac{2}{\alpha})} \|\nabla w(t)\|_{L^r} \leq t^{\alpha-(\frac{2}{\alpha})} \|\nabla E_{\alpha,\alpha}(-t^\alpha A_r) \varphi(0)\|_{L^r}
\]
\[
+ t^{\alpha-(\frac{2}{\alpha})} \left\| \nabla \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^{\alpha} A_r) B(u(s)) ds \right\|_{L^r}
\]
\[
+ t^{\alpha-(\frac{2}{\alpha})} \left\| \nabla \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^{\alpha} A_r) G(s) ds \right\|_{L^r}
\]
\[
+ t^{\alpha-(\frac{2}{\alpha})} \left\| \nabla \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^{\alpha} A_r) f(s) ds \right\|_{L^r}
\]
\[
:= J_1 + J_2 + J_3 + J_4. \tag{3.10}
\]
Choose $\beta = \alpha$, $\gamma_1 = \varepsilon$, and $\gamma_2 = \frac{1}{2}$ in Lemma 5, we obtain

$$J_1 \leq C_2 \|\varphi(0)\|_{X^{\alpha, r}}, \quad t \geq 0.$$  \hfill (3.11)

For $J_2$, $J_3$ and $J_4$, take $\beta = \alpha$, $\gamma_1 = 0$ and $\gamma_2 = \frac{1}{2}$ in Lemma 5, in view of (H2)-(H3), Proposition 7, the properties of the Beta function and Hölder’s inequality, we deduce that for $t \geq 0$,

$$J_2 \leq C_2 t^{\alpha(\frac{1}{2}-\varepsilon)} \int_0^t (t-s)^{\frac{\alpha}{2}-1} \|B(u(s))\|_{L^r} \, ds$$

$$\leq C_2 C_4 t^{\alpha(\frac{1}{2}-\varepsilon)} \int_0^t (t-s)^{\frac{\alpha}{2}-1} \|\nabla u(s)\|_{L^r}^2 \, ds$$

$$\leq C_2 C_4 t^{\alpha(\frac{1}{2}-\varepsilon)} \int_0^t (t-s)^{\frac{\alpha}{2}-1} s^{-\alpha(1-2\varepsilon)} \left(s^{\alpha(\frac{1}{2}-\varepsilon)} \|\nabla u(s)\|_{L^r}\right)^2 \, ds$$

$$\leq C \|u\|^2_{X^\alpha[0,T]} t^{\alpha \varepsilon},$$  \hfill (3.12)

$$J_3 \leq C_2 t^{\alpha(\frac{1}{2}-\varepsilon)} \int_0^t (t-s)^{\frac{\alpha}{2}-1} \|G(s)\|_{L^r} \, ds \leq C t^{\alpha-\alpha \varepsilon-\frac{1}{2}},$$  \hfill (3.13)

and

$$J_4 \leq C_2 t^{\alpha(\frac{1}{2}-\varepsilon)} \int_0^t (t-s)^{\frac{\alpha}{2}-1} \|f(s)\|_{L^r} \, ds$$

$$\leq C_2 t^{\alpha(\frac{1}{2}-\varepsilon)} \int_0^t (t-s)^{\frac{\alpha}{2}-1} m_0(s) \, ds$$

$$+ C_2 t^{\alpha(\frac{1}{2}-\varepsilon)} \int_0^t (t-s)^{\frac{\alpha}{2}-1} m_1(s) \|u\|_{X^\alpha} \, ds$$

$$\leq C_2 t^{\alpha(\frac{1}{2}-\varepsilon)} \left(\int_0^t (t-s)^{\frac{\alpha}{2}-1} \frac{ds}{s} \int_0^t \left(\int_0^t (m_0(s))^q \, ds\right)^{\frac{1}{q}} \right)$$

$$+ C_2 t^{\alpha(\frac{1}{2}-\varepsilon)} \int_0^t (t-s)^{\frac{\alpha}{2}-1} \|m_1\|_{L^\infty} \left(\|A_{\alpha}^\varepsilon \varphi\|_{C_h} + s^{-\alpha(\frac{1}{2}-\varepsilon)} \|u\|_{X^\alpha[0,T]}\right) \, ds$$

$$\leq C t^{\alpha-\alpha \varepsilon-\frac{1}{4}} + C \|A_{\alpha}^\varepsilon \varphi\|_{C_h} t^{\alpha-\alpha \varepsilon} + C \|u\|^2_{X^\alpha[0,T]} t^{\frac{\alpha}{2}}.$$  \hfill (3.14)

Arguing as in the proof of Step 2, we see that

$$t^{\alpha(\frac{1}{2}-\varepsilon)} \nabla w \in C([0,T]; L^r_\alpha).$$

Then it follows from Lemma 5 and (3.10)-(3.14) that $t^{\alpha(\frac{1}{2}-\varepsilon)} \nabla w$ vanishes at $t = 0$.

In the similar way, we can conclude that

$$\|u\|_{X^\alpha[0,T]} = \sup_{t \in [0,T]} \|A_{\alpha}^\varepsilon w(t)\|_{L^r} + t^{\alpha(\frac{1}{2}-\varepsilon)} \|\nabla w(t)\|_{L^r}$$

$$\leq 2 C_2 \|A_{\alpha}^\varepsilon \varphi(0)\|_{L^r} + C \|A_{\alpha}^\varepsilon \varphi\|_{C_h} T^{\alpha-\alpha \varepsilon} + C T^{\alpha-\alpha \varepsilon-\frac{1}{2}}$$

$$+ C \|u\|^2_{X^\alpha[0,T]} T^{\alpha \varepsilon} + C \|u\|_{X^\alpha[0,T]} T^{\frac{\alpha}{2}}.$$  \hfill (3.15)

Denote

$$\mathcal{M}_0 = \{ u \in X^\alpha[0,T_*] : \|u\|_{X^\alpha[0,T_*]} \leq 2 C_2 \|A_{\alpha}^\varepsilon \varphi(0)\|_{L^r} + 1 \},$$

where $T_*$ is sufficiently small such that

$$C \|A_{\alpha}^\varepsilon \varphi\|_{C_h} T^{\alpha-\alpha \varepsilon} + C T^{\alpha-\alpha \varepsilon-\frac{1}{2}} + C T^{\alpha \varepsilon} (2 C_2 \|A_{\alpha}^\varepsilon \varphi(0)\|_{L^r} + 1)^2$$

$$+ C T^{\frac{\alpha}{2}} (2 C_2 \|A_{\alpha}^\varepsilon \varphi(0)\|_{L^r} + 1) \leq 1.$$
It is clear that $\mathcal{M}_0$ is a closed convex subset of $\mathcal{X}_r^\alpha [0, T_*]$ and (3.15) ensures that $\mathcal{L}(\mathcal{M}_0) \subset \mathcal{M}_0$.

Set

$$\mathcal{M}_{k+1} = \text{conv} \mathcal{L}(\mathcal{M}_k), \quad k = 0, 1, 2, \ldots,$$

here the $\text{conv}$ stands for the closure of convex hull of a subset in $\mathcal{X}_r^\alpha [0, T_*]$. Note that $\mathcal{M}_k$ is a closed, convex and $\mathcal{M}_{k+1} \subset \mathcal{M}_k$ for all $k \in \mathbb{N}$. Let $\mathcal{M} = \bigcap_{k=0}^{\infty} \mathcal{M}_k$. Then $\mathcal{M}$ is a closed convex subset of $\mathcal{X}_r^\alpha [0, T_*]$ and $\mathcal{L}(\mathcal{M}) \subset \mathcal{M}$.

**Step 2.** $\mathcal{M}$ is equicontinuous.

Indeed, for each $k \geq 0$, if we can prove that $\mathcal{L}(\mathcal{M}_k)$ is equicontinuous, then it follows that $\mathcal{M}_{k+1}$ is equicontinuous. Therefore, $\mathcal{M}$ is equicontinuous as well.

Now it only remains to show that for each $k \geq 0$, $\mathcal{L}(\mathcal{M}_k)$ is equicontinuous, that is, for any $\eta > 0$, there exists $\delta > 0$ such that if $w \in \mathcal{L}(u)$ with $u \in \mathcal{M}_k$, $t_1, t_2 \in [0, T_*]$ and $|t_1 - t_2| < \delta$, then

$$\|A^*_{\alpha}(w(t_2) - w(t_1))\|_{L_r} + \left\| t_2^{\alpha(\frac{1}{r} - \epsilon)} \nabla w(t_2) - t_1^{\alpha(\frac{1}{r} - \epsilon)} \nabla w(t_1) \right\|_{L_r} < \eta. \quad (3.16)$$

Without loss of generality, we fix $t_1 \geq 0$ and consider $t_2 > t_1$, since the case $t_2 < t_1$ follows similarly. By (3.2), we have

$$\left\| t_2^{\alpha(\frac{1}{r} - \epsilon)} \nabla w(t_2) - t_1^{\alpha(\frac{1}{r} - \epsilon)} \nabla w(t_1) \right\|_{L_r}$$

$$\leq \left\| t_2^{\alpha(\frac{1}{r} - \epsilon)} E_\alpha(-t_2^\alpha A_r) \varphi(0) - t_1^{\alpha(\frac{1}{r} - \epsilon)} E_\alpha(-t_1^\alpha A_r) \varphi(0) \right\|_{L_r}$$

$$+ \left\| t_2^{\alpha(\frac{1}{r} - \epsilon)} \nabla \int_0^{t_2} (t_2 - s)^{\alpha - 1} E_\alpha(-s^\alpha A_r) B(u(s)) ds \right\|_{L_r}$$

$$+ \left\| t_1^{\alpha(\frac{1}{r} - \epsilon)} \nabla \int_0^{t_1} (t_1 - s)^{\alpha - 1} E_\alpha(-s^\alpha A_r) B(u(s)) ds \right\|_{L_r}$$

$$+ \left\| t_2^{\alpha(\frac{1}{r} - \epsilon)} \nabla \int_0^{t_2} (t_2 - s)^{\alpha - 1} E_\alpha(-s^\alpha A_r) G(s) ds \right\|_{L_r}$$

$$+ \left\| t_1^{\alpha(\frac{1}{r} - \epsilon)} \nabla \int_0^{t_1} (t_1 - s)^{\alpha - 1} E_\alpha(-s^\alpha A_r) G(s) ds \right\|_{L_r}$$

$$+ \left\| t_2^{\alpha(\frac{1}{r} - \epsilon)} \nabla \int_0^{t_2} (t_2 - s)^{\alpha - 1} E_\alpha(-s^\alpha A_r) f(s) ds \right\|_{L_r}$$

$$+ \left\| t_1^{\alpha(\frac{1}{r} - \epsilon)} \nabla \int_0^{t_1} (t_1 - s)^{\alpha - 1} E_\alpha(-s^\alpha A_r) f(s) ds \right\|_{L_r}$$

$$:= I_1 + I_2 + I_3 + I_4. \quad (3.17)$$

For $I_1$, by Lemmas 5 and 6 we have

$$I_1 = \left\| t_2^{\alpha(\frac{1}{r} - \epsilon)} E_\alpha(-t_2^\alpha A_r) \varphi(0) - t_1^{\alpha(\frac{1}{r} - \epsilon)} E_\alpha(-t_1^\alpha A_r) \varphi(0) \right\|_{L_r} \to 0 \quad (3.18)$$

as $t_2 \to t_1$.

For $I_2$, take $\beta = \alpha$, $\gamma_1 = 0$ and $\gamma_2 = \frac{1}{2}$ in Lemma 5, in view of Proposition 7, $w \in \mathcal{L}(u)$ with $u \in \mathcal{M}_k$, and $\mathcal{M}_k \subset \mathcal{M}_0$ for each $k \in \mathbb{N}$, we find that

$$I_2 \leq \left\| t_2^{\alpha(\frac{1}{r} - \epsilon)} \nabla \int_0^{t_2} (t_2 - s)^{\alpha - 1} E_\alpha(-s^\alpha A_r) B(u(s)) ds \right\|_{L_r}$$

$$- \left\| t_1^{\alpha(\frac{1}{r} - \epsilon)} \nabla \int_0^{t_1} (t_1 - s)^{\alpha - 1} E_\alpha(-s^\alpha A_r) B(u(s)) ds \right\|_{L_r}$$

$$

\text{...}$$
where we have used the notation \( \| \cdot \| \) to denote the norm of the Mittag-Leffler operator \( E_{\alpha,\alpha}(-(t-s)\alpha A_r) \). For \( t_1 = 0 \), we can see that \( I_{21} = I_{22} = 0 \). By the properties of the Beta function,

\[
I_{23} = C t_1^{\alpha(\frac{1}{\alpha}-\varepsilon)} \int_0^{t_2} (t_2 - s)^{\frac{2}{\alpha} - 1 - \alpha} (t_1 - s)^{\alpha - 1} (t_2 - s)^{-\frac{2}{\alpha}} \int_0^{t_1} E_{\alpha,\alpha}(-(t-s)\alpha A_r) ds
\]

\[
= C t_2^{\alpha(\frac{1}{\alpha}-\varepsilon)} \int_0^{t_2} (t_2 - s)^{\frac{2}{\alpha} - 1 - \alpha} (t_2 - s)^{-\alpha(1-2\varepsilon)} ds \leq C t_2^{\alpha(\frac{1}{\alpha}-\varepsilon)} \rightarrow 0 \quad \text{as} \quad t_2 \rightarrow 0. \quad (3.20)
\]

For \( 0 < t_1 < T \) and arbitrary \( 0 < \eta < t_1 \), by Lemmas 5 and 6, the properties of the Beta function, and the arbitrariness of \( \eta \), we deduce from Lebesgue’s dominated convergence theorem that

\[
I_{22} \leq C t_1^{\alpha(\frac{1}{\alpha}-\varepsilon)} \int_0^{t_1 - \eta} (t_1 - s)^{\alpha - 1} s^{\alpha(1-2\varepsilon)} \left| E_{\alpha,\alpha}(-(t_2-s)\alpha A_r) - E_{\alpha,\alpha}(-(t_1-s)\alpha A_r) \right| ds
\]

\[
+ C t_1^{\alpha(\frac{1}{\alpha}-\varepsilon)} \int_{t_1 - \eta}^{t_1} (t_1 - s)^{\alpha - 1} s^{\alpha(1-2\varepsilon)} (t_2 - s)^{-\frac{2}{\alpha}} ds
\]

\[
+ C t_1^{\alpha(\frac{1}{\alpha}-\varepsilon)} \int_{t_1 - \eta}^{t_1} (t_1 - s)^{\alpha - 1} s^{\alpha(1-2\varepsilon)} ds
\]
conclude from Lebesgue’s dominated convergence theorem and Hölder’s inequality, we have

\[
\int_{0}^{1-t_1} (t_1-s)^{\alpha-1} s^{-\alpha(1-2\varepsilon)} |E_{\alpha,\alpha}(-(t_2-s)^{\alpha} A_r)| ds + C t_1^{\alpha \varepsilon} \int_{1-t_1}^{1} (1-s)^{\alpha-1} s^{-\alpha(1-2\varepsilon)} ds \to 0
\]

as \( t_2 \to t_1 \), where we have used the assumption \( t_2 > t_1 \). For \( I_{21} \) and \( I_{23} \), by the properties of the Beta function and Lebesgue’s dominated convergence theorem, we get

\[
I_{21} = C \int_{0}^{t_1} \left| t_2^{\alpha(1-\varepsilon)} (t_2-s)^{\alpha-1} - t_1^{\alpha(1-\varepsilon)} (t_1-s)^{\alpha-1} (t_2-s)^{-\frac{\alpha}{2}} \right| s^{-\alpha(1-2\varepsilon)} ds \to 0
\]

as \( t_2 \to t_1 \), and

\[
I_{23} \leq C t_2^{\alpha \varepsilon} \int_{1-t_1}^{1} (1-s)^{\frac{\alpha}{2}-1} s^{-\alpha(1-2\varepsilon)} ds \to 0 \quad \text{as} \quad t_2 \to t_1.
\]

For \( I_3 \), take \( \beta = \alpha, \gamma_1 = 0 \) and \( \gamma_2 = \frac{1}{2} \) in Lemma 5, we obtain

\[
I_3 \leq \left\| t_2^{\alpha(1-\varepsilon)} \nabla \int_{0}^{t_1} (t_2-s)^{\alpha-1} E_{\alpha,\alpha}(-(t_2-s)^{\alpha} A_r)G(s) ds \right\|_{L^r}
\]

\[
- t_1^{\alpha(1-\varepsilon)} \nabla \int_{0}^{t_1} (t_1-s)^{\alpha-1} E_{\alpha,\alpha}(-(t_2-s)^{\alpha} A_r)G(s) ds \right\|_{L^r}
\]

\[
+ \left\| t_1^{\alpha(1-\varepsilon)} \nabla \int_{0}^{t_1} (t_1-s)^{\alpha-1} \left( E_{\alpha,\alpha}(-(t_2-s)^{\alpha} A_r) - E_{\alpha,\alpha}(-(t_1-s)^{\alpha} A_r) \right) G(s) ds \right\|_{L^r}
\]

\[
\leq C \int_{0}^{t_1} \left| t_2^{\alpha(1-\varepsilon)} (t_2-s)^{\frac{\alpha}{2}-1} - t_1^{\alpha(1-\varepsilon)} (t_1-s)^{\alpha-1} (t_2-s)^{-\frac{\alpha}{2}} \right| \|G(s)\|_{L^r} ds
\]

\[
+ t_1^{\alpha(1-\varepsilon)} \int_{0}^{t_1} (t_1-s)^{\alpha-1} \left\| E_{\alpha,\alpha}(-(t_2-s)^{\alpha} A_r) - E_{\alpha,\alpha}(-(t_1-s)^{\alpha} A_r) \right\| ds
\]

\[
\times \|G(s)\|_{L^r} ds + C t_2^{\alpha \varepsilon} \int_{t_1}^{t_2} (t_2-s)^{\frac{\alpha}{2}-1} \|G(s)\|_{L^r} ds
\]

\[
:= I_{31} + I_{32} + I_{33}.
\]

For \( t_1 = 0 \), we see that \( I_{31} = I_{32} = 0 \). By the assumption \((H3)\) and Hölder’s inequality, we have

\[
I_{33} \leq C t_2^{\alpha \varepsilon} \left( \int_{0}^{t_2} (t_2-s)^{\frac{\alpha}{2}-1} ds \right)^{\frac{q-1}{q}} \left( \int_{0}^{t_2} \|G(s)\|_{L^q}^q ds \right)^{\frac{1}{q}}
\]

\[
\leq C t_2^{\alpha - \alpha \varepsilon - \frac{1}{q}} \to 0 \quad \text{as} \quad t_2 \to 0.
\]

For \( 0 < t_1 < T_\ast \) and any \( \varepsilon' > 0 \), by Lemmas 5 and 6, the assumption \((H3)\), we conclude from Lebesgue’s dominated convergence theorem and Hölder’s inequality
that there exist $\delta'>0$ and $\eta$ with $0<\eta<t_1$ such that if $|t_1 - t_2| < \delta'$,
\[
I_{32} \leq t_1^{\alpha(\frac{1}{2} - \varepsilon)} \int_0^{t_1-\eta} (t_1 - s)^{\alpha-1} \left\| E_{\alpha,\alpha}(-(t_2 - s)^{\alpha} A_r) - E_{\alpha,\alpha}(-(t_1 - s)^{\alpha} A_r) \right\| \\
\times \|G(s)\|_{L^r} ds + Ct_1^{\alpha(\frac{1}{2} - \varepsilon)} \int_0^{t_1} (t_1 - s)^{\alpha-1} (t_2 - s)^{-\frac{\alpha}{2}} \|G(s)\|_{L^r} ds \\
\leq t_1^{\alpha(\frac{1}{2} - \varepsilon)} \int_0^{t_1-\eta} (t_1 - s)^{\alpha-1} \left\| E_{\alpha,\alpha}(-(t_2 - s)^{\alpha} A_r) - E_{\alpha,\alpha}(-(t_1 - s)^{\alpha} A_r) \right\| \\
- E_{\alpha,\alpha}(-(t_1 - s)^{\alpha} A_r) \left\| G(s) \right\|_{L^r} ds \\
+ Ct_1^{\alpha(\frac{1}{2} - \varepsilon)} \left( \int_0^{t_1} \|G(s)\|_{L^r}^q ds \right)^{\frac{1}{q}} \left( \int_0^{t_1} (t_1 - s)^{\frac{\alpha}{2} - 1} \frac{s}{t_1} ds \right)^{\frac{q-1}{q}} \\
\leq t_1^{\alpha(\frac{1}{2} - \varepsilon)} \int_0^{t_1-\eta} (t_1 - s)^{\alpha-1} \left\| E_{\alpha,\alpha}(-(t_2 - s)^{\alpha} A_r) - E_{\alpha,\alpha}(-(t_1 - s)^{\alpha} A_r) \right\| \\
\times \|G(s)\|_{L^r} ds + Ct_1^{\alpha(\frac{1}{2} - \varepsilon)} \eta^{\frac{\alpha}{2} - \frac{1}{4}} < \varepsilon',
\] (3.25)
where we have used the assumption $t_2 > t_1$. Therefore, $I_{32}$ tends to zero independently of $u \in \mathcal{M}_k$ as $t_2 \to t_1$.

For $I_{31}$ and $I_{33}$, by the assumption (H3), Hölder’s inequality and Lebesgue's dominated convergence theorem, we obtain that
\[
I_{31} = C \int_0^{t_1} \left| t_2^{\alpha(\frac{1}{2} - \varepsilon)} (t_2 - s)^{\frac{\alpha}{2} - 1} - t_1^{\alpha(\frac{1}{2} - \varepsilon)} (t_1 - s)^{\alpha-1} (t_2 - s)^{-\frac{\alpha}{2}} \right| \|G(s)\|_{L^r} ds \to 0
\] (3.26)
as $t_2 \to t_1$, and
\[
I_{33} \leq Ct_1^{\alpha(\frac{1}{2} - \varepsilon)} \left( \int_0^{t_1} \|G(s)\|_{L^r}^q ds \right)^{\frac{1}{q}} \left( \int_0^{t_1} (t_2 - s)^{\frac{\alpha}{2} - 1} \frac{s}{t_1} ds \right)^{\frac{q-1}{q}} \\
\leq Ct_1^{\alpha(\frac{1}{2} - \varepsilon)} (t_2 - t_1)^{\frac{\alpha}{2} - \frac{1}{4}} \to 0 \text{ as } t_2 \to t_1.
\] (3.27)

For $I_4$, analogous to the arguments of (3.19), (3.21)-(3.23) and (3.25)-(3.27), in view of (H2), we get that
\[
I_4 \leq \left| t_2^{\alpha(\frac{1}{2} - \varepsilon)} \nabla \int_0^{t_1} (t_2 - s)^{\alpha-1} E_{\alpha,\alpha}(-(t_2 - s)^{\alpha} A_r) f(s) ds \right| \\
- t_1^{\alpha(\frac{1}{2} - \varepsilon)} \nabla \int_0^{t_1} (t_1 - s)^{\alpha-1} E_{\alpha,\alpha}(-(t_2 - s)^{\alpha} A_r) f(s) ds \left\| L^r \right\| + \left| t_1^{\alpha(\frac{1}{2} - \varepsilon)} \nabla \int_0^{t_1} (t_1 - s)^{\alpha-1} \left( E_{\alpha,\alpha}(-(t_2 - s)^{\alpha} A_r) - E_{\alpha,\alpha}(-(t_1 - s)^{\alpha} A_r) \right) f(s) ds \right| \\
\times f(s) ds \left\| L^r \right\| + \left| t_2^{\alpha(\frac{1}{2} - \varepsilon)} \nabla \int_0^{t_1} (t_2 - s)^{\alpha-1} E_{\alpha,\alpha}(-(t_2 - s)^{\alpha} A_r) f(s) ds \right| \left\| L^r \right\| \\
\leq C \int_0^{t_1} \left| t_2^{\alpha(\frac{1}{2} - \varepsilon)} (t_2 - s)^{\frac{\alpha}{2} - 1} - t_1^{\alpha(\frac{1}{2} - \varepsilon)} (t_1 - s)^{\alpha-1} (t_2 - s)^{-\frac{\alpha}{2}} \right| m_0(s) ds.
\]
Then it follows from (3.17)-(3.28) that
\[ \| M \|_L^\infty \| u_\alpha \|_{C_\alpha} ds + C t_2^{\alpha(\frac{1}{2}-\epsilon)} \int_{t_1}^{t_2} (t_2-s)^{\frac{3}{2}-1} m_0(s) ds \]
\[ + C \int_{t_1}^{t_2} m_1 \| u_\alpha \|_{C_\alpha} ds \]
\[ \leq C \int_{t_1}^{t_2} t_2^{\alpha(\frac{1}{2}-\epsilon)} (t_2-s)^{\frac{3}{2}-1} - t_1^{\alpha(\frac{1}{2}-\epsilon)} (t_1-s)^{\alpha-1} (t_2-s)^{-\frac{3}{2}} m_0(s) ds \]
\[ + C \int_{t_1}^{t_2} t_2^{\alpha(\frac{1}{2}-\epsilon)} (t_2-s)^{\frac{3}{2}-1} - t_1^{\alpha(\frac{1}{2}-\epsilon)} (t_1-s)^{\alpha-1} (t_2-s)^{-\frac{3}{2}} ds \]
\[ + t_1^{\alpha(\frac{1}{2}-\epsilon)} \int_{0}^{t_1} (t_1-s)^{\alpha-1} \| E_{\alpha,a}(-(t_2-s)^{\alpha} A_r) - E_{\alpha,a}(-(t_1-s)^{\alpha} A_r) \| m_0(s) ds \]
\[ + t_1^{\alpha(\frac{1}{2}-\epsilon)} \int_{0}^{t_1} (t_1-s)^{\alpha-1} \| E_{\alpha,a}(-(t_2-s)^{\alpha} A_r) - E_{\alpha,a}(-(t_1-s)^{\alpha} A_r) \| ds \]
\[ + C t_2^{\alpha(\frac{1}{2}-\epsilon)} \int_{t_1}^{t_2} (t_2-s)^{\frac{3}{2}-1} m_0(s) ds \]
\[ + C t_2^{\alpha(\frac{1}{2}-\epsilon)} \int_{t_1}^{t_2} (t_2-s)^{-\frac{3}{2}} ds \to 0 \quad \text{as} \quad t_2 \to t_1. \]

(3.28)

Then it follows from (3.17)-(3.28) that \[ \left\| t_2^{\alpha(\frac{1}{2}-\epsilon)} \nabla w(t_2) - t_1^{\alpha(\frac{1}{2}-\epsilon)} \nabla w(t_1) \right\|_{L^r} \] tends to zero independently of \( u \in \mathcal{M}_k \) as \( t_2 \to t_1 \). By the similar arguments, we can also prove that \( \| A^\gamma_k(w(t_2) - w(t_1)) \|_{L^r} \) tends to zero independently of \( u \in \mathcal{M}_k \) as \( t_2 \to t_1 \).

**Step 3.** \( \mathcal{M} \) is a compact set.

To prove this, we first show that for any fixed \( t \in [0, T_1] \), \( \mathcal{M}(t) \) is compact. This will be done if we have \( \chi(\nabla \mathcal{M}_k(t)) \to 0 \) as \( k \to \infty \), where \( \chi \) is the Hausdorff MNC on \( L^r \) and the set \( \nabla \mathcal{M}_k(t) \) is defined by

\[ \nabla \mathcal{M}_k(t) = \{ \nabla u(t) : u(t) \in \mathcal{M}_k(t) \}. \]

By (H4), (3.2), Lemmas 5, 9 and 10, we deduce that
\[
\nu_{k+1}(t) = \chi(\nabla \mathcal{M}_{k+1}(t)) = \chi(\nabla \mathcal{M}_k(t)) = \chi(\nabla \mathcal{M}_k(t))
\]
\[
\leq \chi \left( \nabla E_{\alpha,\alpha}(-t_2^\alpha A_r) \varphi(0) + \nabla \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^{\alpha} A_r) G(s) ds \right)
\]
\[
+ \chi \left( \nabla \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^{\alpha} A_r) B(M_k(s)) ds \right)
\]
\[
+ \chi \left( \nabla \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^{\alpha} A_r) \mathcal{P}_F(M_k(s)) ds \right)
\]
\[
\leq \chi \left( \nabla \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^{\alpha} A_r) B(M_k(s)) ds \right)
\]
From this and Hölder’s inequality, we can derive

\[
+ 4C_2 \int_0^t (t-s)^{\frac{\gamma}{2}-1} l(s) \sup_{\tau \in [-h,0]} \chi(\mathcal{M}_k[\varphi](s+\tau)) \, ds \\
\leq \chi \left( \nabla \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^{\alpha} A_r) B(\mathcal{M}_k(s)) \, ds \right) \\
+ 4C_2 \int_0^t (t-s)^{\frac{\gamma}{2}-1} l(s) \sup_{\tau \in [0,s]} \chi(\mathcal{M}_k[\varphi](\tau)) \, ds \\
\leq \chi \left( \nabla \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^{\alpha} A_r) B(\mathcal{M}_k(s)) \, ds \right) \\
+ C \int_0^t (t-s)^{\frac{\gamma}{2}-1} \sup_{\tau \in [0,s]} \nu_k(\tau) \, ds, \\
(3.29)
\]

where \( B(\mathcal{M}_k(s)) = \{B(u(s)) : u(s) \in \mathcal{M}_k(s)\} \) and \( \mathcal{P}_F(\mathcal{M}_k)(s) = \{\mathcal{P}_F(u(s) : u \in \mathcal{M}_k\} \). Observe that for any \( u, v \in \mathcal{M}_k \), by Proposition 7 and choosing \( \beta = \alpha \), \( \gamma_1 = 0 \) and \( \gamma_2 = \frac{1}{\beta} \) in Lemma 5, we find that

\[
\left\| \nabla \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^{\alpha} A_r) (B(u(s), u(s)) - B(v(s), v(s))) \, ds \right\|_{L^r} \\
\leq C_2 \int_0^t (t-s)^{\frac{\gamma}{2}-1} \left\| B(u(s), u(s)) - B(v(s), v(s)) \right\|_{L^r} \, ds \\
\leq C_2 C_3 \int_0^t (t-s)^{\frac{\gamma}{2}-1} \left( \|\nabla u(s)\|_{L^r} + \|\nabla v(s)\|_{L^r} \right) \|\nabla u(s) - \nabla v(s)\|_{L^r} \, ds \\
\leq C_2 C_3 \int_0^t (t-s)^{\frac{\gamma}{2}-1} s^{-\alpha(\frac{1}{2}-\epsilon)} \left( \|u\|_{X_{\alpha}^p[0,T_r]} + \|v\|_{X_{\alpha}^p[0,T_r]} \right) \|\nabla u(s) - \nabla v(s)\|_{L^r} \, ds \\
\leq C \int_0^t (t-s)^{\frac{\gamma}{2}-1} s^{-\alpha(\frac{1}{2}-\epsilon)} \|\nabla u(s) - \nabla v(s)\|_{L^r} \, ds. \\
(3.30)
\]

Hence,

\[
\nu_{k+1}(t) \leq C \int_0^t (t-s)^{\frac{\gamma}{2}-1} s^{-\alpha(\frac{1}{2}-\epsilon)} \sup_{\tau \in [0,s]} \nu_k(\tau) \, ds + C \int_0^t (t-s)^{\frac{\gamma}{2}-1} \sup_{\tau \in [0,s]} \nu_k(\tau) \, ds. \\
(3.31)
\]

From this and Hölder’s inequality, we can derive

\[
\nu_{k+1}(t) \leq C \left( \int_0^t (t-s)^{\alpha-1} s^{-\alpha(\frac{1}{2}-\epsilon)} \frac{1}{p} \, ds \right)^{\frac{1}{p}} \left( \int_0^t \left( \sup_{\tau \in [0,s]} \nu_k(\tau) \right)^p \, ds \right)^{\frac{1}{p}} \\
+ C \left( \int_0^t (t-s)^{\alpha-1} s^{-\alpha(\frac{1}{2}-\epsilon)} \frac{1}{p} \, ds \right)^{\frac{1}{p}} \left( \int_0^t \left( \sup_{\tau \in [0,s]} \nu_k(\tau) \right)^p \, ds \right)^{\frac{1}{p}} \\
\leq C t^{\alpha-\frac{1}{p}} \left( \int_0^t \left( \sup_{\tau \in [0,s]} \nu_k(\tau) \right)^p \, ds \right)^{\frac{1}{p}} + C t^{\frac{\gamma}{2}-\frac{1}{p}} \left( \int_0^t \left( \sup_{\tau \in [0,s]} \nu_k(\tau) \right)^p \, ds \right)^{\frac{1}{p}}, \\
(3.32)
\]

which implies that

\[
\left( \sup_{\tau \in [0,t]} \nu_{k+1}(\tau) \right)^p \leq C \int_0^t \left( \sup_{\tau \in [0,s]} \nu_k(\tau) \right)^p \, ds,
\]
where $p > \frac{1}{\alpha}$ and the constant $C$ in (3.32) is dependent on $T_*$. Defining $\eta_k(t) = \left(\sup_{\tau \in [0, t]} \eta_k(\tau)\right)^p$, then we have

$$\eta_{k+1}(t) \leq C \int_0^t \eta_k(s) ds.$$  

Let $\eta_\infty(t) = \lim_{k \to \infty} \eta_k(t)$. Then by Lebesgue’s dominated convergence theorem, we obtain

$$\eta_\infty(t) \leq C \int_0^t \eta_\infty(s) ds.$$  

Using Gronwall’s lemma we have $\eta_\infty(t) = 0$ for all $t \in [0, T_*]$. Noticing that $0 \leq (\chi(\nabla M_k(t)))^p \leq \eta_k(t) \to 0$ as $k \to \infty$. Thus, we have $\chi(\nabla M_k(t)) \to 0$ as $k \to \infty$ as desired, and consequently $M(t)$ is compact for each $t \in [0, T_*]$.

Recall that $M$ is equicontinuous, hence the Arzelá-Ascoli theorem gives the compactness of $M$.

**Step 4.** $L : M \to \mathcal{P}(M)$ is u.s.c. with closed convex values.

The assumption (H1) implies that $\mathcal{P}_F$ has convex values, so $L$ does. We then show that $L$ is u.s.c. with closed values. Thanks to Proposition 14, now it suffices to show that $\{u_n\}_{n=1}^\infty \subset M$ with $u_n \to u^* \in M$ and $w_n \in L(u_n)$ implies $w_n \to w^* \in L(u^*)$ up to a subsequence. Suppose not. Then there exists a neighborhood $\mathcal{O}$ of $L(u^*)$ such that

$$w_n \notin \mathcal{O}, \quad \forall \ n \in \mathbb{N}. \quad (3.33)$$

By the definition of the multi-valued operator $L$, we obtain that for any $t \in [0, T_*]$,

$$w_n(t) \in E_\alpha(-t^\alpha A_r)\varphi(0) + \int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}(-(t-s)^\alpha A_r)(B(u_n(s))) + G(s) ds + \mathcal{W} \circ \mathcal{P}_F(u_n(t)). \quad (3.34)$$

Recall that $u_n \to u^* \in M$, hence for any fixed $t \in [0, T_*]$, $\chi(\{\nabla u_n(t)\}_{n=1}^\infty) = 0$. Similar to the arguments of (3.29)-(3.31), we deduce that

$$\chi(\{\nabla w_n(t)\}_{n=1}^\infty) \leq \chi(\nabla E_\alpha(-t^\alpha A_r)\varphi(0))$$

$$+ \chi\left(\nabla \int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}(-(t-s)^\alpha A_r)G(s) ds\right)$$

$$+ \chi\left(\nabla \left\{\int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}(-(t-s)^\alpha A_r)B(u_n(s)) ds\right\}_{n=1}^\infty\right)$$

$$+ \chi\left(\nabla \left\{\int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}(-(t-s)^\alpha A_r)\mathcal{W} \circ \mathcal{P}_F(u_n(s)) ds\right\}_{n=1}^\infty\right)$$

$$\leq C \int_0^t (t-s)^{\frac{\alpha}{2}-1} s^{-\alpha(\frac{1}{2}-\epsilon)} \chi(\{\nabla w_n(s)\}_{n=1}^\infty) ds$$

$$+ C \int_0^t (t-s)^{\frac{\alpha}{2}-1} \sup_{\tau \in [0, s]} \chi(\{\nabla u_n(\tau)\}_{n=1}^\infty) ds = 0. \quad (3.35)$$

On the other hand, arguing as in the proof of Step 2, we see that $\{w_n\}_{n=1}^\infty$ is equicontinuous, i.e., for any $\eta^* > 0$, there exists $\delta^* > 0$ such that if $t_1, t_2 \in [0, T_*]$
and $|t_1 - t_2| < \delta'$, then
\[
\|A^\epsilon_t(w_n(t_2) - w_n(t_1))\|_{L^r} + \|a^{(1/2-\epsilon)}_t \nabla w_n(t_2) - a^{(1/2-\epsilon)}_t \nabla w_n(t_1)\|_{L^r} < \eta', \quad \forall n \in \mathbb{N}.
\] (3.36)

Then it follows from the Arzelà-Ascoli theorem that $\{w_n\}_{n=1}^\infty$ is relatively compact in $X^\alpha\{0, T_*\}$. Without loss of generality, we assume that there exists $w^* \in X^\alpha\{0, T_*\}$ such that
\[
\sup_{t \in [0, T^*]} \|A^\epsilon_t(w_n(t) - w^*(t))\|_{L^r} + \sup_{t \in [0, T^*]} \|a^{(1/2-\epsilon)}_t (\nabla w_n(t) - \nabla w^*(t))\|_{L^r} \to 0
\] (3.37)
as $n \to \infty$. Let $f_n \in \mathcal{P}_F(u_n)$. Then it follows from (3.34) that
\[
w_n(t) = E_\alpha(-t^\alpha A_r)\varphi(0) + \int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}(-(t-s)^\alpha A_r) G(s)ds
\]
\[+ \int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}(-(t-s)^\alpha A_r) B(u_n(s))ds
\]
\[+ \int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}(-(t-s)^\alpha A_r) f_n(s)ds.
\] (3.38)

By (3.30), the properties of the Beta function and $u_n \to u^*$ in $X^\alpha\{0, T_*\}$, we obtain that for any fixed $t \in [0, T_*],$
\[
\left\| \nabla \int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}(-(t-s)^\alpha A_r) (B(u_n(s), u_n(s)) - B(u^*(s), u^*(s)))ds \right\|_{L^r}
\]
\[\leq C \int_0^t (t-s)^{\alpha-1} \left( \|\nabla u_n(s)\|_{L^r} + \|\nabla u^*(s)\|_{L^r} \right) \|\nabla u_n(s) - \nabla u^*(s)\|_{L^r} ds
\]
\[\leq C\|u_n - u^*\|_{X^\alpha\{0, T_*\}} \int_0^t (t-s)^{\alpha-1} s^{-(1-2\epsilon)}ds \to 0 \quad \text{as} \quad n \to \infty.
\] (3.39)

By the assumptions (H2) and (H4), in view of $u_n \to u^*$ in $X^\alpha\{0, T_*\}$, we deduce from Proposition 18 that $\{\nabla W(f_n(t))\}_{n=1}^\infty$ is relatively compact in $L^r$ for each $t \in [0, T_*].$ Without loss of generality, we assume that for any fixed $t \in [0, T_*], \xi \in L^{1/2-\epsilon}$ such that
\[
\|\nabla W(f_n(t)) - \nabla \xi(t)\|_{L^r} \to 0 \quad \text{as} \quad n \to \infty.
\] (4.0)

Noticing that (H2) and $u_n \to u^*$ in $X^\alpha\{0, T_*\}$ ensure that $\{f_n\}_{n=1}^\infty$ is integrably bounded in $L^q([0, T_*]; L^r)$, hence $\{f_n\}_{n=1}^\infty$ is weakly compact in $L^q([0, T_*]; L^r)$. Then, by Mazur’s lemma, there exists a sequence $\bar{f}_n \in \text{conv}\{f_m : m \geq n\}$ such that $\bar{f}_n \to f^*$ in $L^q([0, T_*]; L^r)$, and consequently, for a.e. $t \in [0, T_*],$
\[
\|\bar{f}_n(t) - f^*(t)\|_{L^r} \to 0 \quad \text{as} \quad n \to \infty.
\] (4.1)

By the upper semi-continuity of $F(t, \cdot)$, we conclude from $u_n \to u^*$ in $X^\alpha\{0, T_*\}$ that for $\eta > 0$, $F(t, u_n[\varphi]_1) \subset F(t, u^*[\varphi]_1) + B_\eta$ for all large $n$, here $B_\eta$ denotes the closed ball in $L^r$ at origin with radius $\eta$. Therefore, $f_n(t) \in F(t, u^*[\varphi]_1) + B_\eta$ for a.e. $t \in [0, T_*].$ We observe that $F(t, u^*[\varphi]_1) + B_\eta$ is convex, so $f_n(t) \in F(t, u^*[\varphi]_1) + B_\eta$ for a.e. $t \in [0, T_*].$ This implies that $f^* \in F(t, u^*[\varphi]_1) + B_\eta$ for a.e. $t \in [0, T_*].$ Since $\eta$ is arbitrary, we have
\[
f^* \in F(t, u^*[\varphi]_1) \quad \text{for a.e.} \quad t \in [0, T_*].
\] (4.2)
By \((H2)\), \(u_n \to u^*\) in \(X^\alpha_r[0,T_s]\), and choosing \(\beta = \alpha, \gamma_1 = 0, \gamma_2 = \frac{1}{2}\) in Lemma 5, we deduce from Lebesgue’s dominated convergence theorem that
\[
\left\| \nabla \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-t-s)^{\alpha}A_r \left( \tilde{f}_n(s) - f^*(s) \right) ds \right\|_{L^r}
\leq C \int_0^t (t-s)^{\frac{\alpha}{2}-1} \left\| \tilde{f}_n(s) - f^*(s) \right\|_{L^r} ds \to 0 \quad \text{as} \quad n \to \infty. \tag{3.43}
\]
Noticing that \(\xi = W(f^*)\) and \(f^* \in \mathcal{P}_F(u^*)\). Then it follows from (3.38) that
\[
w^*(t) = E_\alpha(-t^\alpha A_r)\varphi(0) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-t-s)^{\alpha}A_r G(s) ds
+ \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-t-s)^{\alpha}A_r B(u^*(s)) ds
+ \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-t-s)^{\alpha}A_r f^*(s) ds,
\]
where \(f^* \in \mathcal{P}_F(u^*)\), and consequently \(w^* \in \mathcal{L}(u^*)\). This contradicts (3.33).

Therefore, Theorem 15 gives the existence of a fixed point of \(\mathcal{L}\), which is a local mild solution of (2.1).

Now we establish the continuation of local mild solutions of (2.1).

**Theorem 20.** Suppose that \((H1)-(H4)\) hold. Let \(\alpha \in (0,1), 2 \leq N < r < \infty\) and \(A^\nu_r \varphi \in C_h\) with \(0 < \varepsilon < \frac{1}{2}\). If \(u : [0,T_s] \to X^{r-\varepsilon}\) is a local mild solution to (2.1) in \([0,T_s]\), then there exists a continuation \(u^*\) of \(u\) in some interval \([0,T_s + \tau]\) with \(\tau > 0\). Moreover,
\[
t^{\alpha(\frac{1}{2} - \varepsilon)}\nabla u^* \in C([0,T_s + \tau]; L^r_\sigma)
\]
with value zero at \(t = 0\), in which \(u(0) = \varphi(0)\).

**Proof.** Let \(u : [0,T_s] \to X^{r-\varepsilon}\) be the local mild solution to (2.1) in \([0,T_s]\). Then there exists \(f \in \mathcal{P}_F(u)\) such that
\[
u(t) = E_\alpha(-t^\alpha A_r)\varphi(0) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-t-s)^{\alpha}A_r (B(u(s)) + G(s)) ds
+ \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-t-s)^{\alpha}A_r f(s) ds. \tag{3.44}
\]
Fix \(R^* > 0\) and consider
\[
\mathcal{M}_0^* := \left\{ \omega \in X^\alpha_r[0,T_s + \tau] : \omega(t) = u(t) \text{ for all } t \in [0,T_s] \right\}
\]
and
\[
\sup_{t \in [T_s, T_s + \tau]} \left\| A^\nu_r(\omega(t) - u(T_s)) \right\|_{L^r} + \sup_{t \in [T_s, T_s + \tau]} \left\| t^{\alpha(\frac{1}{2} - \varepsilon)}\nabla \omega(t) - T^\alpha(\frac{1}{2} - \varepsilon) \nabla u(T_s) \right\|_{L^r} \leq R^*
\]
where \(\tau > 0\) will be chosen later.

Then we define the multi-valued operator \(\tilde{\mathcal{L}} : \mathcal{M}_0^* \to X^\alpha_r[0,T_s + \tau]\) as follows
\[
\tilde{\mathcal{L}}(\omega)(t) = \left\{ E_\alpha(-t^\alpha A_r)\varphi(0) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-t-s)^{\alpha}A_r (B(\omega(s)) + G(s)) ds
+ \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-t-s)^{\alpha}A_r f(s) ds \right\}
\]
\[ + W(f)(t) : f \in P_F(\omega) \text{ and } \tilde{f}(s) = f(s) \text{ for all } s \in [0, T_1] \} \tag{3.45} \]

We check that \( \tilde{L}(M_0^\alpha) \subset M_0^\alpha \).

(a) If \( \omega \in M_0^\alpha \), then \( \omega(t) = u(t) \) in \( [0, T_*] \) with \( u \) the local mild solution to (2.1) in \( [0, T_*] \). So, if \( t \in [0, T_*] \),

\[ \tilde{L}(\omega)(t) = E_\alpha(-t^\alpha A_r)\varphi(0) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^\alpha A_r)(B(\omega(s)) + G(s))ds \]
\[ + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^\alpha A_r)\tilde{f}(s)ds \]
\[ = E_\alpha(-t^\alpha A_r)\varphi(0) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^\alpha A_r)(B(u(s)) + G(s))ds \]
\[ + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^\alpha A_r)f(s)ds = u(t). \]

(b) If \( t \in [T_*, T_\star + \tau] \), then for any \( \omega \in \tilde{L}(\omega) \) with \( \omega \in M_0^\alpha \), we have

\[
\left\| t^{\alpha(\frac{1}{2}-\varepsilon)} \nabla \omega(t) - T_*^{\alpha(\frac{1}{2}-\varepsilon)} \nabla u(T_*) \right\|_{L^r} \\
\leq \left\| t^{\alpha(\frac{1}{2}-\varepsilon)} \nabla E_\alpha(-t^\alpha A_r)\varphi(0) - T_*^{\alpha(\frac{1}{2}-\varepsilon)} \nabla E_\alpha(-T_*^\alpha A_r)\varphi(0) \right\|_{L^r} \\
+ t^{\alpha(\frac{1}{2}-\varepsilon)} \left\| \nabla \int_{T_*}^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^\alpha A_r) \left( B(\omega(s)) + G(s) + \tilde{f}(s) \right) ds \right\|_{L^r} \\
+ \left\| \nabla \int_0^{T_*} \left( t^{\alpha(\frac{1}{2}-\varepsilon)}(t-s)^{\alpha-1} - T_*^{\alpha(\frac{1}{2}-\varepsilon)}(T_* - s)^{\alpha-1} \right) \times E_{\alpha,\alpha}(-(t-s)^\alpha A_r)(B(u(s)) + G(s) + f(s))ds \right\|_{L^r} \\
+ \left\| T_*^{\alpha(\frac{1}{2}-\varepsilon)} \nabla \int_0^{T_*} (T_* - s)^{\alpha-1}(\nabla E_{\alpha,\alpha}(-(t-s)^\alpha A_r)) - \nabla E_{\alpha,\alpha}(-(T_* - s)^\alpha A_r))(B(u(s)) + G(s) + f(s))ds \right\|_{L^r} \\
:= \bar{J}_1 + \bar{J}_2 + \bar{J}_3 + \bar{J}_4. \tag{3.46} \]

For \( \bar{J}_1 \), take \( \beta = \alpha, \gamma_1 = \varepsilon \) and \( \gamma_2 = \frac{1}{2} \) in Lemmas 5 and 6, we can choose \( \tau > 0 \) sufficiently small such that

\[
\bar{J}_1 \leq \left\| t^{\alpha(\frac{1}{2}-\varepsilon)} - T_*^{\alpha(\frac{1}{2}-\varepsilon)} \right\|_{L^r} \nabla E_\alpha(-t^\alpha A_r)\varphi(0) \]
\[ + \left\| T_*^{\alpha(\frac{1}{2}-\varepsilon)} \nabla E_\alpha(-t^\alpha A_r) - \nabla E_\alpha(-T_*^\alpha A_r) \right\|_{L^r} \leq \frac{R_*}{8}. \tag{3.47} \]

For \( \bar{J}_2 \), arguing as in (3.12)-(3.14), in view of \( \omega \in M_0^\alpha \) and the properties of the Beta function, we can choose \( \tau > 0 \) sufficiently small such that

\[
\bar{J}_2 \leq C t^{\alpha(\frac{1}{2}-\varepsilon)} \int_{T_*}^t (t-s)^{\frac{3}{2} - \alpha - (1-2\varepsilon)} \sup_{s \in [T_*, T_\star + \tau]} s^{\alpha(\frac{1}{2}-\varepsilon)} \nabla \omega(s) \|_{L^r} \]
\[ + C t^\alpha (\frac{\alpha}{2} - \varepsilon) \left( \int_{T_*}^t (t-s)\left(\frac{\alpha}{2} - 1\right) s^{-\frac{2\alpha}{\alpha + 1}} ds \right)^{\frac{\alpha - 1}{\alpha}} \left( \int_{T_*}^t (m_0(s)) q ds \right)^{\frac{1}{q}} \]
\[ + C t^\alpha (\frac{\alpha}{2} - \varepsilon) \int_{T_*}^t (t-s)\left(\frac{2\alpha}{\alpha + 1}\right) m_0(s) ds \]
\[ + C t^\alpha (\frac{\alpha}{2} - \varepsilon) \int_{T_*}^t (t-s)^{\frac{2\alpha}{\alpha + 1}} \sup_{s \in [T_*, T_* + T]} s^{-\alpha (\frac{\alpha}{2} - \varepsilon)} \|
abla \omega(s)\|_{L^\infty} ds \]
\[ \leq C(R^* + C) t^\alpha (\frac{\alpha}{2} - \varepsilon) \int_{T_*}^t (t-s)^{\frac{2\alpha}{\alpha + 1}} s^{-\alpha (\frac{\alpha}{2} - \varepsilon)} ds \leq \frac{R^*}{8}. \quad (3.48) \]

For \( \tilde{J}_3 \), arguing as in (3.12)-(3.14), by Lebesgue's dominated convergence theorem, we can choose \( \tau \) sufficiently small such that

\[ \tilde{J}_3 \leq C \int_0^{T_*} \left( t^\alpha (\frac{\alpha}{2} - \varepsilon) (t-s)^{\alpha - 1} - T_*^{\alpha (\frac{\alpha}{2} - \varepsilon)} (T_* - s)^{\alpha - 1} \right) (t-s)^{-\frac{2\alpha}{\alpha + 1}} s^{-\alpha (1-2\varepsilon)} ds \]
\[ + C \int_0^{T_*} \left( t^\alpha (\frac{\alpha}{2} - \varepsilon) (t-s)^{\alpha - 1} - T_*^{\alpha (\frac{\alpha}{2} - \varepsilon)} (T_* - s)^{\alpha - 1} \right) (t-s)^{-\frac{2\alpha}{\alpha + 1}} \|G(s)\|_{L^\infty} ds \]
\[ + C \int_0^{T_*} \left( t^\alpha (\frac{\alpha}{2} - \varepsilon) (t-s)^{\alpha - 1} - T_*^{\alpha (\frac{\alpha}{2} - \varepsilon)} (T_* - s)^{\alpha - 1} \right) (t-s)^{-\frac{2\alpha}{\alpha + 1}} m_0(s) ds \]
\[ + C \int_0^{T_*} \left( t^\alpha (\frac{\alpha}{2} - \varepsilon) (t-s)^{\alpha - 1} - T_*^{\alpha (\frac{\alpha}{2} - \varepsilon)} (T_* - s)^{\alpha - 1} \right) (t-s)^{-\frac{2\alpha}{\alpha + 1}} \times \|m_1\|_{L^\infty} \left( \|A^*_\varphi\|_{C_h} + s^{-\alpha (\frac{\alpha}{2} - \varepsilon)} \|u\|_{X_\varphi[0,T_*]} \right) \]
\[ \leq C \int_0^{T_*} \left( t^\alpha (\frac{\alpha}{2} - \varepsilon) (t-s)^{\alpha - 1} - T_*^{\alpha (\frac{\alpha}{2} - \varepsilon)} (T_* - s)^{\alpha - 1} \right) (t-s)^{-\frac{2\alpha}{\alpha + 1}} s^{-\alpha (1-2\varepsilon)} ds \]
\[ + C \left( \int_0^{T_*} \left( t^\alpha (\frac{\alpha}{2} - \varepsilon) (t-s)^{\frac{2\alpha}{\alpha + 1}} - T_*^{\alpha (\frac{\alpha}{2} - \varepsilon)} (T_* - s)^{\alpha - 1} (t-s)^{-\frac{2\alpha}{\alpha + 1}} \right)^{\frac{\alpha - 1}{\alpha}} ds \right)^{\frac{\alpha}{\alpha - 1}} \]
\[ + C \int_0^{T_*} t^\alpha (\frac{\alpha}{2} - \varepsilon) (t-s)^{-\frac{2\alpha}{\alpha + 1}} - T_*^{\alpha (\frac{\alpha}{2} - \varepsilon)} (T_* - s)^{\alpha - 1} (t-s)^{-\frac{2\alpha}{\alpha + 1}} ds \]
\[ + C \int_0^{T_*} t^\alpha (\frac{\alpha}{2} - \varepsilon) (t-s)^{\alpha - 1} - T_*^{\alpha (\frac{\alpha}{2} - \varepsilon)} (T_* - s)^{\alpha - 1} (t-s)^{-\frac{2\alpha}{\alpha + 1}} s^{-\alpha (\frac{\alpha}{2} - \varepsilon)} ds \]
\[ \leq \frac{R^*}{8}. \quad (3.49) \]

Let \( 0 < \eta < T_* \) be given arbitrarily. For \( \tilde{J}_4 \), take \( \beta = \alpha, \gamma_1 = 0 \) and \( \gamma_2 = \frac{1}{2} \) in Lemma 6, by the similar way as above and the arbitrariness of \( \eta \), we can choose
\( \tau > 0 \) sufficiently small such that

\[
\mathcal{J}_4 \leq T_s^{\alpha(\frac{1}{2}-\varepsilon)} \int_0^T (T_s - s)^{\alpha - 1} \left\| E_{\alpha,a}(-(t - s)^\alpha A_r) - E_{\alpha,a}(-(T_s - s)^\alpha A_r) \right\|
\times s^{-\alpha(1-2\varepsilon)} \|u\|^2_{L^2([0,T_s])} ds
\]
\[
+ T_s^{\alpha(\frac{1}{2}-\varepsilon)} \int_0^T (T_s - s)^{\alpha - 1} \left\| E_{\alpha,a}(-(t - s)^\alpha A_r) - E_{\alpha,a}(-(T_s - s)^\alpha A_r) \right\|
\times \|G(s)\|_{L^r} ds + T_s^{\alpha(\frac{1}{2}-\varepsilon)} \int_0^T (T_s - s)^{\alpha - 1}
\times \|m_1\|_L^\infty \left( \|A_r^\varepsilon\|_{C_h} + s^{-\alpha(\frac{1}{2}-\varepsilon)} \|u\|_{L^r([0,T_s])} \right) ds
\]
\[
\leq C T_s^{\alpha(\frac{1}{2}-\varepsilon)} \int_0^T (T_s - s)^{\alpha - 1} \left\| E_{\alpha,a}(-(t - s)^\alpha A_r) - E_{\alpha,a}(-(T_s - s)^\alpha A_r) \right\|
\times s^{-\alpha(1-2\varepsilon)} ds + C T_s^{\alpha(\frac{1}{2}-\varepsilon)} \left( \int_0^T (T_s - s)^{\alpha - 1}
\times \left\| E_{\alpha,a}(-(t - s)^\alpha A_r) - E_{\alpha,a}(-(T_s - s)^\alpha A_r) \right\| \right)^{\frac{\alpha}{\alpha + 1}} ds \frac{2^{\frac{3}{\alpha}}}{\alpha}
\]
\[
+ C T_s^{\alpha(\frac{1}{2}-\varepsilon)} \int_0^T (T_s - s)^{\alpha - 1} \left\| E_{\alpha,a}(-(t - s)^\alpha A_r) - E_{\alpha,a}(-(T_s - s)^\alpha A_r) \right\| ds
\]
\[
+ C T_s^{\alpha(\frac{1}{2}-\varepsilon)} \int_0^T (T_s - s)^{\alpha - 1} \left\| E_{\alpha,a}(-(t - s)^\alpha A_r) - E_{\alpha,a}(-(T_s - s)^\alpha A_r) \right\| ds
\]
\[
\times s^{-\alpha(\frac{1}{2}-\varepsilon)} ds
\]
\[
\leq C T_s^{\alpha(\frac{1}{2}-\varepsilon)} \int_0^{T - \eta} (T_s - s)^{\alpha - 1} \left\| E_{\alpha,a}(-(t - s)^\alpha A_r) - E_{\alpha,a}(-(T_s - s)^\alpha A_r) \right\|
\times s^{-\alpha(1-2\varepsilon)} ds
\]
\[
+ C T_s^{\alpha(\frac{1}{2}-\varepsilon)} \int_0^{T - \eta} (T_s - s)^{\alpha - 1} \left( (t - s)^{-\eta} + (T_s - s)^{-\eta} \right) s^{-\alpha(1-2\varepsilon)} ds
\]
\[
+ C T_s^{\alpha(\frac{1}{2}-\varepsilon)} \left( \int_0^{T - \eta} (T_s - s)^{\alpha - 1} \left\| E_{\alpha,a}(-(t - s)^\alpha A_r) - E_{\alpha,a}(-(T_s - s)^\alpha A_r) \right\| \right)^{\frac{\alpha}{\alpha + 1}} ds \frac{2^{\frac{3}{\alpha}}}{\alpha}
\]
\[
+ C T_s^{\alpha(\frac{1}{2}-\varepsilon)} \left( \int_0^{T - \eta} (T_s - s)^{\alpha - 1} \left[ (t - s)^{-\eta} + (T_s - s)^{-\eta} \right] \left( (t - s)^{\alpha - 1} + (T_s - s)^{\alpha - 1} \right) \right)^{\frac{\alpha}{\alpha + 1}} ds \frac{2^{\frac{3}{\alpha}}}{\alpha}
\]
\[
+ C T_s^{\alpha(\frac{1}{2}-\varepsilon)} \int_0^{T - \eta} (T_s - s)^{\alpha - 1} \left\| E_{\alpha,a}(-(t - s)^\alpha A_r) - E_{\alpha,a}(-(T_s - s)^\alpha A_r) \right\| ds
\]
where we have used the notation \( \| \cdot \| \) to denote the norm of the Mittag-Leffler operator \( E_{\alpha,\alpha}(-(t-s)^\alpha A_r) : X^{0,r} \to X^{\frac{1}{2},r} \). Arguing as in (3.46)-(3.50), we can choose \( \tau > 0 \) sufficiently small such that for any \( t \in [T_*, T_* + \tau] \) and every \( \omega \in \tilde{L}(\omega) \) with \( \omega \in \mathcal{M}_0^* \),

\[
\| A_t^\alpha(\omega(t) - u(T_*)) \|_{L^r} \leq \frac{R^*}{2}, \tag{3.51}
\]

Hence, it follows from (3.46)-(3.51) that for any \( \omega \in \tilde{L}(\omega) \) with \( \omega \in \mathcal{M}_0^* \),

\[
\sup_{t \in [T_*, T_* + \tau]} \| A_t^\alpha(\omega(t) - u(T_*)) \|_{L^r} + \sup_{t \in [T_*, T_* + \tau]} \left\| \frac{\alpha(\frac{1}{2} - \varepsilon)}{r} T_*^\alpha(\frac{1}{2} - \varepsilon) \nabla u(t) \right\|_{L^r} \leq R^*,
\]

and consequently, \( \tilde{L}(\mathcal{M}_0^*) \subset \mathcal{M}_0^* \).

It is obvious that \( \mathcal{M}_0^* \) is a closed convex subset of \( X^\alpha_r[0, T_* + \tau] \). Set

\[
\mathcal{M}^*_{k+1} = \overline{\text{conv}} \tilde{L}(\mathcal{M}_0^*), \quad k = 0, 1, 2, \ldots,
\]

here the \( \overline{\text{conv}} \) stands for the closure of convex hull of a subset in \( X^\alpha_r[0, T_* + \tau] \). Let \( \mathcal{M}^* = \bigcap_{k=0}^{\infty} \mathcal{M}^*_{k+1} \). Arguing as in the proof of Theorem 19, we obtain that \( \mathcal{M}^* \) is a compact convex subset of \( X^\alpha_r[0, T_* + \tau] \) and \( \tilde{L} : \mathcal{M}^* \to \mathcal{P}(\mathcal{M}^*) \) is upper semi-continuous with closed convex values. Therefore, by Theorem 15 we conclude that there exists a fixed point \( u^* \) of \( \tilde{L} \). As in the proof of the Theorem 19, we see that \( u^* \) is the continuation of \( u \) in \([0, T_* + \tau] \).

We finish Section 3 with a result on global existence or non-continuation by blow up.

**Theorem 21.** Assume the conditions of Theorem 19. If a mild solution \( u(t) \) of problem (2.1) has a maximal interval of existence \([0, T')\), then \( u(t) \) is a global mild solution in \([0, \infty)\) or \( \lim_{t \to T'} \sup \left( \| A_t^\alpha u(t) \|_{L^r} + t^\alpha(\frac{1}{2} - \varepsilon) \| \nabla u(t) \|_{L^r} \right) = \infty \).

**Proof.** If \( T' = \infty \), then \( u \) is a global mild solution in \([0, \infty)\). Otherwise, if \( T' < \infty \), we will prove that \( \lim_{t \to T'} \sup \left( \| A_t^\alpha u(t) \|_{L^r} + t^\alpha(\frac{1}{2} - \varepsilon) \| \nabla u(t) \|_{L^r} \right) = \infty \). Assume on the contrary that this is not the case. Then there exists \( K^* \) such that \( \| u(t) \|_{X^\alpha_r[0, T']} \leq K^* \). Arguing as in the proof of Theorem 19, we deduce that \( \{ t_n \} \subset [0, T'] \) is a sequence that converges to \( T' \), then given \( \eta > 0 \), there exists \( N \in \mathbb{N} \) such that if \( m, n \geq N \), we have

\[
\| A_t^\alpha(u(t_n) - u(t_m)) \|_{L^r} + t_n^\alpha(\frac{1}{2} - \varepsilon) \| \nabla u(t_n) - t_m^\alpha(\frac{1}{2} - \varepsilon) \nabla u(t_m) \|_{L^r} < \eta.
\]
This implies that \( \{u(t_n)\}_{n=1}^{\infty} \) is a Cauchy sequence, and therefore it has a limit \( u^* \in X^{\omega_T} \). Let \( u(T') = u^* \). Then \( u \) belongs to \( X^{\alpha,\omega}_{\alpha,\omega}([0,T']) \), and for all \( t \in [0,T'] \),

\[
u(t) = E_{\alpha}(\varphi(0) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\omega}(-(t-s)^\omega A_r)(B(u(s)) + G(s) + f(s))ds,\]

where \( f \in \mathcal{P}_F(u) \). Combining this together with Theorem 20, we can extend the mild solution to some bigger interval, which is a contradiction with definition of \( T' \).

4. Application. Consider that multi-valued operator \( F \) in \( (2.1) \) is given by

\[
F(t,u_t) = F_1(t,u(t-\rho(t))) + \int_{-h}^0 F_2(t,s,u(t+s))ds + \sum_{i=1}^m b_i(t,x)v_i(t),
\]

where

\[
v_i(t) \in \left[ \int_\Omega J_{1,i}(y)u(t-h,y)dy, \int_\Omega J_{2,i}(y)u(t-h,y)dy \right], \quad 1 \leq i \leq m.
\]

Assume the following conditions:

(B1) There exist a positive constant \( k_2 \) and a function \( k_1 \in L^q([0,\infty)) \) such that the functions \( F_1 \in C([0,\infty) \times \mathbb{R}^N) \) and \( \rho \in C([0,\infty)) \) satisfy

\[
|F_i(t,u)|_{\mathbb{R}^N} \leq |k_1(t)| + k_2|u|_{\mathbb{R}^N}, \quad \forall t \in \mathbb{R}, u \in \mathbb{R}^N.
\]

(B2) There exists a nonnegative function \( l_1 \in L^{\infty}(\mathbb{R}^+) \) such that

\[
|F_1(t,u) - F_2(t,u)|_{\mathbb{R}^N} \leq l_1(t)|u|_{\mathbb{R}^N}, \quad \forall t \in \mathbb{R}, u, v \in \mathbb{R}^N.
\]

(B3) There exist positive scalar functions \( m_0 \in L^1([-h,0]), m_1 \in L^{\omega'}([-h,0]) \) with \( \frac{1}{\omega} + \frac{1}{\rho} = 1 \) such that the function \( F_2 \in C([0,\infty) \times \mathbb{R}^N) \) satisfies

\[
|F_2(t,s,u)|_{\mathbb{R}^N} \leq m_0(s) + m_1(s)|u|_{\mathbb{R}^N}, \quad \forall t \in \mathbb{R}, s \in [-h,0], u \in \mathbb{R}^N.
\]

(B4) There exists a nonnegative function \( m_3 \in L^{\omega'}([-h,0]) \) with \( \frac{1}{\omega} + \frac{1}{\rho} = 1 \) such that

\[
|F_2(t,s,u) - F_2(t,s,v)|_{\mathbb{R}^N} \leq m_3(s)|u - v|_{\mathbb{R}^N}, \quad \forall t \in \mathbb{R}, s \in [-h,0], u, v \in \mathbb{R}^N.
\]

(B5) \( b_i \in L^{\infty}(\mathbb{R}^+; L^{\omega'}(\Omega)) \) with \( q \in (\frac{2}{\alpha}, \infty) \), \( J_{1,i}, J_{2,i} \in L^{\omega'}(\Omega) \) with \( \frac{1}{\omega} + \frac{1}{\rho} = 1 \) for \( i \in \{1, \ldots, m\} \), \( F_1(t,u_t) = F_1(t,u(t-\rho(t))) \), \( F_2(t,u_t) = \int_{-h}^0 F_2(t,s,u(t+s))ds \), \( F_3(t,u_t) : \mathbb{R} \times \mathcal{C}_h \rightarrow \mathcal{P}(E_{\omega,\omega}^+) \) be such that

\[
F_3(t,u_t) = \sum_{i=1}^m b_i(t,x) \left[ \int_\Omega J_{1,i}(y)u(t-h,y)dy, \int_\Omega J_{2,i}(y)u(t-h,y)dy \right].
\]

Then we obtain that

\[
\|F_1(t,u_t)\|_{L^\omega(\Omega)} \leq \int_\Omega |F_1(t,u(t-\rho(t)))|_{E_{\omega,\omega}^+} dx
\]

\[
\leq C \int_\Omega |k_1(t)|_{\mathbb{R}^N} dx + C \int_\Omega |u(t-\rho(t))|_{E_{\omega,\omega}^+} dx
\]

\[
\leq C|k_1(t)|_{\mathbb{R}^N} + C\|u_t\|_{\mathcal{C}_h},
\]

\[\text{(4.1)}\]

\[
\|F_2(t,u_t)\|_{L^\omega(\Omega)} \leq \int_\Omega \left( \int_{-h}^0 |F_2(t,s,u(t+s))|_{\mathbb{R}^N} ds \right)^{\frac{\rho}{\omega}} dx
\]

\[
\leq \int_\Omega \left( \int_{-h}^0 m_0(s) ds + \int_{-h}^0 m_1(s)|u(t+s)|_{\mathbb{R}^N} ds \right)^{\frac{\rho}{\omega}} dx
\]
\[
\int_{\Omega} \left( \|m_0\|_{L^1([-h,0])} + \|m_1\|_{L^{r'}([-h,0])} \|u(t+s)\|_{L^r([-h,0])} \right)^r \ dx
\leq C + \int_{\Omega} \int_{-h}^0 |u(t+s)|_{L^r} \ ds \ dx
\leq C + C \|u_t\|_{C^{-1}}. \tag{4.2}
\]
and
\[
\|F_3(t,u_t)\|_{L^r} \leq \sum_{i=1}^m \|b_i(t,x)\|_{L^r} \max \{ \|J_1,i\|_{L^{r'}}, \|J_2,i\|_{L^{r'}} \} \cdot \|u_t\|_{C_h}. \tag{4.3}
\]
Thus the assumption \((H2)\) is satisfied.

For each bounded set \(B \subset C_h\), \(F_3(t,B) \subset \text{span}\{b_1(t,\cdot), \ldots, b_m(t,\cdot)\}\), which is a finite dimensional subspace of \(L^r(\Omega)\). In addition, by (4.3), \(F_3(t,B)\) is also bounded and then relatively compact. Hence
\[
\chi(F_3(t,B)) = 0, \tag{4.4}
\]
where \(\chi\) is the Hausdorff measure of noncompactness on \(L^r\). Arguing as in the proof of (4.1)-(4.2), in view of \((B2)\) and \((B4)\), we get that
\[
\chi(F_1(t,B)) \leq l_1(t) \sup_{s \in [-h,0]} \chi(B(s)),
\]
and
\[
\chi(F_2(t,B)) \leq C \sup_{s \in [-h,0]} \chi(B(s)).
\]
Thus the assumption \((H4)\) is verified.

Since \(F_1(t,\cdot)\) and \(F_2(t,\cdot)\) are continuous single-valued mappings. By a simple argument we see that \(F_3\) has closed convex values. Thus \(F\) has compact convex values.

For each \(t\), note that \(F_3(t,\cdot)\) has a closed graph, hence \(F(t,\cdot)\) is u.s.c. due to (4.4) and Proposition 14. Then the assumption \((H1)\) is satisfied.

**Acknowledgements.** The authors wish to express their thanks to the reviewer for his/her very careful reading of the paper, giving valuable comments and suggestions. The author also thanks the editors for their kind help.

**REFERENCES**

[1] R. R. Akhmerov, M. I. Kamenskii, A. S. Potapov, A. E. Rodkina and B. N. Sadovskii, *Measures of Noncompactness and Condensing Operators*, Birkhäuser, Verlag, Basel, 1992.

[2] B. De Andrade, A. N. Carvalho, P. M. Carvalho-Neto and P. Marín-Rubio, *Semilinear fractional differential equations: Global solutions, critical nonlinearities and comparison results*, *Topol. Methods Nonlinear Anal.*, 45 (2015), 439–467.

[3] T. Caraballo and X. Y. Han, *A survey on Navier-Stokes models with delays: Existence, uniqueness and asymptotic behavior of solutions*, *Discrete Contin. Dyn. Syst. Ser. S*, 8 (2015), 1079–1101.

[4] T. Caraballo and X. Y. Han, *Stability of stationary solutions to 2D-Navier-Stokes models with delays*, *Dyn. Partial Differ. Equ.*, 11 (2014), 345–359.

[5] T. Caraballo and J. Real, *Attractors for 2D-Navier-Stokes models with delays*, *J. Differential Equations*, 205 (2004), 271–297.

[6] T. Caraballo and J. Real, *Asymptotic behaviour of two-dimensional Navier-Stokes equations with delays*, *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.*, 459 (2003), 3181–3194.

[7] P. M. Carvalho-Neto, *Fractional Differential Equations: A Novel Study of Local and Global Solutions in Banach Spaces*, PhD thesis, Universidade de São Paulo, São Carlos, 2013.
[8] P. M. Carvalho-Neto and G. Planas, Mild solutions to the time fractional Navier-Stokes equations in $\mathbb{R}^N$, *J. Differential Equations*, 259 (2015), 2948–2980.

[9] Y. K. Chen and C. H. Wei, Partial regularity of solutions to the fractional Navier-Stokes equations, *Discrete Contin. Dyn. Syst.*, 36 (2016), 5309–5322.

[10] P. Y. Chen and Y. X. Li, Existence of mild solutions for fractional evolution equations with mixed monotone nonlocal conditions, *Z. Angew. Math. Phys.*, 65 (2014), 711–728.

[11] P. Y. Chen, X. P. Zhang and Y. X. Li, Study on fractional non-autonomous evolution equations with delay, *Comput. Math. Appl.*, 73 (2017), 794–803.

[12] P. Y. Chen, X. P. Zhang and Y. X. Li, A blowup alternative result for fractional non-autonomous evolution equation of Volterra type, *Commun. Pure Appl. Anal.*, 17 (2018), 1975–1992.

[13] J. W. Cholewa and T. Dlotko, Fractional Navier-Stokes equations, *Discrete Contin. Dyn. Syst. Ser. B*, 23 (2018), 2967–2988.

[14] L. Debbi, Well-posedness of the multidimensional fractional stochastic Navier-Stokes equations on the torus and on bounded domains, *J. Math. Fluid Mech.*, 18 (2016), 25–69.

[15] K. Deimling, *Multivalued Differential Equations*, De Gruyter, Berlin, 1992.

[16] T. Dlotko, Navier-Stokes equation and its fractional approximations, *Appl. Math. Optim.*, 77 (2018), 99–128.

[17] M. El-Shahed and A. Salem, On the generalized Navier-Stokes equations, *Appl. Math. Comput.*, 156 (2004), 287–293.

[18] L. Ferreira and E. Villamizar-Roa, Fractional Navier-Stokes equations and a Hölder-type inequality in a sum of singular spaces, *Nonlinear Anal.*, 74 (2011), 5618–5630.

[19] G. P. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations: Nonlinear Steady Problems*, Springer Tracts in Natural Philosophy, 38, Springer-Verlag, New York, 1994.

[20] J. García-Luengo, P. Marín-Rubio and J. Real, Some new regularity results of pullback attractors for 2D Navier-Stokes equations with delays, *Commun. Pure Appl. Anal.*, 14 (2015), 1603–1621.

[21] J. García-Luengo, P. Marín-Rubio and J. Real, Regularity of pullback attractors and attraction in $H^1$ in arbitrarily large finite intervals for 2D Navier-Stokes equations with infinite delay, *Discrete Contin. Dyn. Syst.*, 34 (2014), 181–201.

[22] J. García-Luengo, P. Marín-Rubio and J. Real, Pullback attractors for 2D Navier-Stokes equations with delays and their regularity, *Adv. Nonlinear Stud.*, 13 (2013), 331–357.

[23] M. J. Garrido-Atienza and P. Marín-Rubio, Navier-Stokes equations with delays on unbounded domains, *Nonlinear Anal.*, 64 (2006), 1100–1118.

[24] X. L. Guo and Y. Y. Men, On partial regularity of suitable weak solutions to the stationary fractional Navier-Stokes equations in dimension four and five, *Acta Math. Scie. (Engl. Ser.)*, 33 (2017), 1632–1646.

[25] S. M. Guzzo and G. Planas, On a class of three dimensional Navier-Stokes equations with bounded delay, *Discrete Contin. Dyn. Syst. Ser. B*, 16 (2011), 225–238.

[26] Q. S. Jiu and Y. Q. Wang, On possible time singular points and eventual regularity of weak solutions to the fractional Navier-Stokes equations, *Dyn. Partial Differ. Equ.*, 11 (2014), 321–343.

[27] M. Kamenskii, V. Obukhovskii and P. Zecca, *Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces*, de Gruyter Series in Nonlinear Analysis and Applications, vol. 7, Walter de Gruyter, Berlin, 2001.

[28] T. Kato, Strong $L^p$-solutions of the Navier-Stokes equation in $\mathbb{R}^m$, with applications to weak solution, *Math. Z.*, 187 (1984), 471–480.

[29] T. D. Ke and D. Lan, Global attractor for a class of functional differential inclusions with Hille-Yosida operators, *Nonlinear Anal.*, 103 (2014), 72–86.

[30] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, Vol. 204, Elsevier Science B.V., Amsterdam, 2006.

[31] P. E. Kloeden, J. A. Langa and J. Real, Pullback V-attractors of the 3-dimensional globally modified Navier-Stokes equations, *Commun. Pure Appl. Anal.*, 6 (2007), 937–955.

[32] P. E. Kloeden, P. Marín-Rubio and J. Real, Equivalence of invariant measures and stationary statistical solutions for the autonomous globally modified Navier-Stokes equations, *Commun. Pure Appl. Anal.*, 8 (2009), 785–802.
[33] P. E. Kloeden and J. Valero, The Knese property of the weak solutions of the three-dimensional Navier-Stokes equations, *Discrete Contin. Dyn. Syst.*, 28 (2010), 161–179.

[34] M. Li, C. M. Huang and F. Z. Jiang, Galerkin finite element method for higher dimensional multi-term fractional diffusion equation on non-uniform meshes, *Appl. Anal.*, 96 (2017), 1269–1284.

[35] X. C. Li, X. Y. Yang and Y. H. Zhang, Error estimates of mixed finite element methods for time-fractional Navier-Stokes equations, *J. Sci. Comput.*, 70 (2017), 500–515.

[36] F. Mainardi, On the initial value problem for the fractional diffusion-wave equation, *Ser. Adv. Math. Appl. Sci.*, 23 (1994), 246–251.

[37] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity*, Imperial College Press, London, 2010.

[38] P. Marín-Rubio, J. Real and J. Valero, Pullback attractors for a two-dimensional Navier-Stokes model in an infinite delay case, *Nonlinear Anal.*, 74 (2011), 2012–2030.

[39] S. Momani and Z. Odibat, Analytical solution of a time-fractional Navier-Stokes equation by Adomian decomposition method, *Appl. Math. Comput.*, 177 (2006), 488–494.

[40] C. J. Niche and G. Planas, Existence and decay of solutions in full space to Navier-Stokes equations with delays, *Nonlinear Anal.*, 74 (2011), 244–256.

[41] L. Peng, Y. Zhou, B. Ahmad and A. Alsaedi, The Cauchy problem for fractional Navier-Stokes equations in Sobolev spaces, *Chaos Solitons Fractals*, 102 (2017), 218–228.

[42] I. Podlubny, *Fractional Differential Equations*, vol. 198 of Mathematics in Science and Engineering, Academic Press, San Diego, California, USA, 1999.

[43] H. Singh, A new stable algorithm for fractional Navier-Stokes equation in polar coordinate, *Int. J. Appl. Comput. Math.*, 3 (2017), 3705–3722.

[44] L. Tang and Y. Yu, Partial Hölder regularity of the steady fractional Navier-Stokes equations, *Calc. Var. Partial Differential Equations*, 55 (2016), Art. 31, 18 pp.

[45] H. Y. Xu, X. Y. Jiang and B. Yu, Numerical analysis of the space fractional Navier-Stokes equations, *Appl. Math. Lett.*, 69 (2017), 94–100.

[46] R. N. Wang, D. H. Chen and T. J. Xiao, Abstract fractional Cauchy problems with almost sectorial operators, *J. Differential Equations*, 252 (2012), 202–235.

[47] F. B. Weissler, The Navier-Stokes initial value problem in $L^p$, *Arch. Ratton. Mech. Anal.*, 74 (1980), 219–230.

[48] Z. C. Zhai, *Some Regularity Estimates for Mild Solutions to Fractional Heat-Type and Navier-Stokes Equations*, Thesis (Ph.D.)-Memorial University of Newfoundland (Canada), 2009.

[49] Y. Zhou, L. Peng, B. Ahmad and A. Alsaedi, Energy methods for fractional Navier-Stokes equations, *Chaos Solitons Fractals*, 102 (2017), 78–85.

[50] Y. Zhou and L. Peng, Weak solutions of the time-fractional Navier-Stokes equations and optimal control, *Comput. Math. Appl.*, 73 (2017), 1016–1027.

[51] Y. Zhou and L. Peng, On the time-fractional Navier-Stokes equations, *Comput. Math. Appl.*, 73 (2017), 874–891.

Received March 2018; revised July 2018.

E-mail address: wangyj@lzu.edu.cn
E-mail address: liangtt15@lzu.edu.cn