DELAY-INDUCED SWITCHED STATES IN A SLOW-FAST SYSTEM

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Abstract. We consider the two-component delay system
\[
\varepsilon x'(t) = -x(t) - y(t) + f(x(t - 1)),
\]
\[
y'(t) = \eta x(t)
\]
with small parameters \( \varepsilon, \eta \), and positive feedback function \( f \). Previously, such systems have been reported to model switching in optoelectronic experiments, where each switching induces another one after approximately one delay time, related to one round trip of the signal. In this paper, we study these delay-induced switched states. We provide conditions for their existence and show how the formal limits \( \varepsilon \to 0 \) and/or \( \eta \to 0 \) facilitate our understanding of this phenomenon.

1. Introduction

Many nonlinear dynamical systems exhibit oscillations on different time-scales characterized by interchanged phases of slow and fast motion. Their analysis relies on the study of reduced systems, the properties of which are then recombined using geometric singular perturbation theory [1]. Analogous tools are increasingly available for functional differential equations [2, 3], yet the effects of time-delay on multiple time-scale systems are largely uncharted territory. It is very natural to observe relaxation oscillations [4], canard trajectories [5, 6, 7] and mixed mode oscillations [8, 9]; phenomena which are not delay-specific and extensively studied in the non-delayed case [10, 11]. Here, we study delay-induced switched states as one further type of solution that naturally arises in this set-up, yet cannot be found in the non-delayed case [12, 13, 14, 15, 16].

Let us consider the system
\[
\begin{align*}
\varepsilon x'(t) &= -x(t) - y(t) + f(x(t - 1)), \\
y'(t) &= \eta x(t),
\end{align*}
\]
where the dot denotes the derivative with respect to time. The parameters \( \varepsilon, \eta \) are assumed positive and sufficiently small so that \( (1) - (2) \) can be regarded as a singular perturbation problem acting on at least three different timescales \( \varepsilon, 1, \) and \( 1/\eta \). Figure 1(a) shows an example solution where the \( x \)-component switches back and forth between negative and positive values after approximately one delay time. The switches are sharp, on short intervals of order \( \varepsilon \). Between the switches, the solution changes slowly on time scale \( 1/\eta \). Note that \( (1) - (2) \) can be transformed to an equivalent system with short time delay \( \eta \), or long time delay \( 1/\varepsilon \).

This work is motivated by recent optoelectronic experiments [12, 13, 14, 15], where delay-induced switched states were reported. Interestingly, this set-up allows for switching between regular and irregular phases reoccurring after approximately one delay time, see Fig. 1(b) for a numerically computed example trajectory showing this behavior, together with the corresponding feedback function. In analogy to systems of coupled oscillators [17] and spatially extended systems [18], the authors refer to them as chimera states [19, 20] in Delay Differential Equations. For numerical purposes, throughout the paper, we choose Mackey-Glass like [21] feedback functions \( f_1(\xi) = 1.2\xi/(1 + \xi^4) \).
and \( f_2(\xi) = 2\xi/(1+ (\xi - 0.8)^4) \), which are capable to reproduce the qualitative dynamical behavior of the physical set-up, yet are easier to manipulate.

The paper is organized as follows: Sec. 2 collects preliminary results on scalar positive delayed feedback equations and establishes the necessary vocabulary. After having these premises set up, Sec. 3 lays out the conditions for the existence of delay induced states. Sections 4–7 contain more technical results. In particular, Sec. 4 introduces the mathematical framework for delay differential equations and establish certain basic properties of (1)–(2). In Sec. 3, we establish the concept of a balance point of delay-induced switched states. Section 5 provides more insight into the mechanisms behind the emergence of switched states close to the balance point. In Secs. 6–5, we show that the formal limit system for \( \varepsilon = 0 \) and \( \eta = 0 \) explains the profile of delay-induced switched states and another class of solutions that coexist. Our rigorous results are given in the form of propositions and lemmas: the proofs are included in the supplementary material.

2. Positive Delayed Feedback

To set the stage, allow us to back up a little and build some intuition. Let us consider the simpler case of a scalar delayed feedback equation

\[
\varepsilon x'(t) = -x(t) + f(x(t-1)),
\]

obtained from (1)–(2) by setting \( \eta \) and \( y(0) \) to zero. Equations of type (3) have been studied extensively in the literature. They are known to cause a wide range of dynamical phenomena from oscillatory, to excitable and chaotic behavior [22]; in a variety of processes such as physiological control systems [21], nonlinear optics [23], population dynamics [24] and neuroscience [25].

For the purpose of this paper, we commit to a specific case, where \( f \) is continuously differentiable and

- \( f(\xi) = \xi \) possesses exactly three solutions: \( \xi^- < 0, \xi = 0, \xi^+ > 0 \), and \( f(\xi) \neq \xi \) otherwise.
- \( f \) satisfies the positive feedback condition in some bounded but sufficiently large interval: \( \xi f(\xi) > 0 \) for all \( \xi \in [\chi^-, \chi^+] \backslash \{0\} \), where \( \chi^+ = \max_{\xi \in I} f(\xi), \chi^- = \min_{\xi \in I} f(\xi) \) and \( I = [\xi^-, \xi^+] \).

Figure 1 shows examples of functions satisfying properties (H1) – (H2).
Let us consider an initial function $x_0(t) \in [\chi^-, \chi^+]$, for $t \in [-1,0]$, taking positive as well as negative values. For example, choose any slice of length one of the $x$-component of the switching solution shown in Fig. 1(b). It can be shown that (H1) and (H2) imply $x(t) \in [\chi^-, \chi^+]$ for all $t \geq 0$ [20]. Note that each switching between two different phases contains a point, at which this function changes sign. Under conditions (H1) and (H2), the solution to (3) has a remarkable property: The number of sign changes counted on intervals of an appropriate length (that may be slightly larger than one) cannot increase [27, 28]. As a direct consequence, the number of phases counted in this way cannot increase either. Their width however, is subject to change; it might very well decrease until a point at which its enclosing sign changes merge and the phase disappears. We refer to this process as the coarsening of a phase [29]. It is a spatiotemporal phenomenon best visualized in a spatiotemporal plot [18], see Fig. 2(a). Here, the solution is sliced into subsequent segments of appropriate length (about one), and the slice’s values are then plotted with respect to their counting number on the vertical axis, encoded in a color map, see Ref. [18] for details. Fig. 2(a) shows the temporal evolution of each phase in color/gray-scale, separated by their sign changes in black. Interestingly, the amount with which the position of a sign change alters from one segment to the next seems independent of the phase width; as long as it is sufficiently large [30]. We refer to this constant trend as the drift of this sign change. If this sign change has positive slope, we refer to the corresponding drift as an up-drift $\delta_{\text{up}}$; analogously we use the term down-drift $\delta_{\text{down}}$.

Generally, up-drift and down-drift along a solution do not coincide, leading to the coarsening of phases. This behavior is very well studied when $f$ is monotone [31]. In this case, the long-term behavior of solutions to (3) consists of only equilibrium solutions, unstable periodic orbits, and homoclinic/heteroclinic connections between them [32] [33]. Generically, solutions converge to equilibrium, yet with possibly complicated transient behavior as the number of (unstable) periodic solutions is large for small $\varepsilon$ [34]. The dynamics is largely determined by the equilibria tugging at the solution until the “stronger” wins. When $f$ is monotone, the up- and down-drift along the solution can be computed explicitly, see Ref. [35, 36] and references therein. There, it is shown that two neighboring sign changes interact weakly over exponentially small tails, and therefore, they may appear independent numerically. The drift of the sign changes itself goes to zero as $\varepsilon \to 0$, whereas $\delta_{\text{up}} = \delta_{\text{up}}/\varepsilon$ and $\delta_{\text{down}} = \delta_{\text{down}}/\varepsilon$ converge to a bounded constant. If additionally $f$ is an odd function, the situation is so well-adapted that up- and down-drift coincide to leading order in $\varepsilon$, allowing for super-transient solutions with transient times scaling as $e^{-c/\varepsilon}$, for some $c > 0$. We call any function $f$ with the property that up- and down-drift coincide balanced (not necessarily monotone or odd).

We want to remark that the case of negative delayed feedback has received equal attention [37], and a lot of techniques for positive delayed feedback are motivated by the techniques developed for the negative feedback case [38, 39, 40, 41, 42].

Note that our example $f = f_1$ in Fig. 1(a) is odd and restriction of $f_1$ to the values of the solution is monotone. As a direct consequence, $f_1$ is balanced. It is easy to show that continuous differentiability of $f$ implies that $f$, when shifted along its graph by a small amount $f \mapsto f^b = f(\cdot + b) - f(b)$, still satisfies (H1)-(H2). As a result $f_1$ has the following property:

\((H3)\) There exist $\beta \in \mathbb{R}$, and $r > 0$ such that $f^b(\xi) = f(\xi + b) - f(b)$ satisfies (H1)-(H2) for all $b \in [\beta - r, \beta + r]$ and $f^b$ is balanced. We call $\beta$ the balance point.
Figure 2. (color online) Numerical computation of the drift $d_{\text{up}} = \delta_{\text{up}} / \varepsilon$ of up (blue) and $d_{\text{down}} = \delta_{\text{down}} / \varepsilon$ of down (green) transitions layers for fixed $\varepsilon = 0.01$, $f_2(x) = 2x/(1 + (0.8 - x)^4)$, and initial function $x(t) = x_0(t)$, $t \in [-1, 0]$ with two sign changes (specified in the text). Panel (a)-(d) show the spatiotemporal plot of the solution to (4) for $T = 1 + 0.755\varepsilon$ and different values of $y(0)$. Panel (e) shows $d_{\text{up}}$ and $d_{\text{down}}$ with varying $y(0) \in [0, 1]$.

In particular, $\beta = 0$ for $f = f_1$. Now obviously, our second example $f_2$ is neither odd or monotone, nor is it balanced, as the corresponding solution displays coarsening, see Fig. 2(a). Nevertheless, it is straightforward to check that $f_2$ satisfies (H1)-(H2). In addition, we show numerically that $f_2$ satisfies (H3).

Let us now vary $y(0)$ as a parameter and consider the equation

\begin{equation}
\varepsilon x'(t) = -x(t) - y(0) + f(x(t-1)),
\end{equation}

with $y(0)$ not necessarily zero. For sufficiently small $y(0)$, the transformation $x^b(t) = x(t) - b$, $f^b(\xi) = f(\xi + b) - f(b)$, where $b$ is the solution of $\xi = f(\xi) - y(0)$ with the smallest modulus, takes Eq. (4) into Eq. (3) and our analysis above applies.

We investigate numerically how the spatiotemporal behavior of (4) changes as we increase $y(0)$. In the example in Fig. 2(a), we observe that for $y(0) = 0.2$ the difference of up- and down-drift of the solution is smaller, and coarsening occurs at a later time. This suggest that as we increase $y(0)$ the difference between up- and down- drift further decreases. We show this numerically in Fig. 2(c) and (d). In fact, the values of the up- and down- drift change continuously until they coincide at the balance point $y(0) = \beta \approx 0.514$, see Fig. 2(e). For larger values, the order of the drifts is reversed and coarsening leads to another 'winning' phase of the solution Fig. 2(d).

Note that throughout this process of the change of $y(0)$, the function $f^{y(0)}$ satisfies (H1)-(H2), and as a result, $f_2$ satisfies (H3). It is now intuitively clear, that close to the balance point system (1)-(2), as a (small) perturbation of (4) exhibits a long transient solution with initial conditions close to the balance point. In fact, in the next section we provide strong arguments in favor of these solution not being mere transients anymore, but delay-induced switched states. We conjecture that conditions (H1)-(H3) are necessary conditions on $f$ to allow for such states.
3. Main result: balance point and delay-induced switched states

As we will show in Sec. 3, all solutions of (1)-(2) for \( \eta > 0 \) that are not converging to or identically zero are oscillating, i.e. its first component possesses infinitely many sign changes for \( t > 0 \) (Propositions 2 and 3). We are interested in oscillatory solutions that satisfy the following definition motivated by the spatiotemporal plot in Fig. 2(c).

**Definition** (delay induced switched state). We refer to a solution \( x \in C([-1, \infty), \mathbb{R}) \) of (1)-(2) as a *delay-induced switched state*, if there exist \( 1 \leq T \leq 2 \) and \( 0 \leq s \leq 1 \), such that \( x(t) \) has a constant even number of sign changes on each interval \( [nT + s, (n + 1)T + s] \) for all \( n \geq 0 \).

Additionally, we refer to \( \delta = T - 1 \), as the *drift* of a delay induced switched state. It is easy to show that \( \delta \to 0 \) as \( \varepsilon \to 0 \). For this reason, in our numerical exploration, we show the rescaled value \( d = \delta/\varepsilon \) which converges to a constant as \( \varepsilon \to 0 \). See more details on the drift in [29, 30, 18].

In a neighborhood of the balance point \( \beta \) of system (1), it is possible to obtain delay-induced switched states for (1)-(2). This can be explained as follows: If \( y(t) < \beta \), the coarsening of the solution leads to an increase of the average value \( \frac{1}{T} \int_{-T}^{0} x(t + s)ds \) of \( x(t) \) and therefore, to an increase of \( y(t) = y(t - T) + \eta \int_{-T}^{0} x(t + s)ds \) towards the value \( \beta \). Analogously, for \( y(t) > \beta \), the average \( \frac{1}{T} \int_{-T}^{0} x(t + s)ds \) decreases as well as \( y(t) \). This allows for the “dynamic stabilization” of the delay-induced switched states.

Figure 3 shows numerical study of the up- and down-drifts for the \( \eta \)-component of (1)-(2) for different values of \( \eta \) and \( y(0) \). As the spatiotemporal plots in Figs. 3a,b,c1,c2,c3,d show, the coarsening may still occur for certain parameters as in Figs. 3a,h,d, while switched states are observed for the other parameters as in Figs. 3c1,c2,c3). The conditions, where switched states exist, i.e. where the up- and down-drifts coincide \( \delta^+ = \delta^- = \delta \), are shown as the gray region in Figs. 3e,f). The existence region has the form of a cone attached to a balance point \( y(0) = \beta \). That is, the transient switched state for \( \eta = 0 \) exists only in the balance point, and any deviation of the initial condition \( y(0) \) from \( \beta \) would lead to the coarsening. With increasing \( \eta \), the allowed range of \( y(0) \), where delay-induced switched states exist, increases. This indicates that such states become stable for \( \eta > 0 \), and in order to reach them, one should initiate \( y(0) \) in the neighborhood of the balance point \( \beta \), while the size of this neighborhood grows with \( \eta \). We want to remark that this mechanism resembles a locking cone with respect to the delayed self-feedback, and indeed, similar solutions have been observed in forced systems [43].

Our rigorous analysis in the following sections shows that the solution of (1)-(2) can be obtained as a perturbation of (1) (Lemma 3) and that this perturbation contracts the distance of subsequent sign changes to a critical distance (implied by Lemma 5). The “largeness” of this correcting effect strongly depends on \( \eta \). Too far from the balance point, it is too weak to prevent the coarsening of a phase, see Fig. 3a). Similarly, the time scale of this process is of importance. Closer to the balance point the influence of \( \eta \) is strong enough to prevent the coarsening of one phase, but then it can lead to the coarsening towards the other, see Fig. 3b). We also show that the profile of the switched state can be obtained from a reduced system (Proposition 6).

Supported by our rigorous analysis of the “stabilization” mechanism described above as well as numerical evidences in Fig. 3 we make the following conjecture.
Conjecture 1. Let \( f \in C^1(\mathbb{R}) \) satisfy (H1)–(H3) with balance point \( \beta \). Then, for all \( \varepsilon > 0 \) and \( \eta > 0 \) sufficiently small, all initial conditions with \( y(0) \) sufficiently close to \( \beta \), and \( x(\theta) \in [\chi^-, \chi^+] \), for \( -1 \leq \theta \leq 0 \), with at least one sign changes lead to a delay-induced switched state.

An initial condition without sign changes leads to oscillatory solutions of relaxation type, similar to that described in [8, 9]. We have included a numerical exploration and discussion of this type of oscillatory behavior in Sec. 7.

4. Basic properties of system (1)–(2)

Equations (1)–(2) define a (semi-)dynamical system with phase space \( C := C([-1, 0], \mathbb{R}^2) \), that is the Banach space of continuous functions taking values in \( \mathbb{R}^2 \), with norm \( \|\phi\| = \sup_{\theta \in [-1, 0]} |\phi(\theta)| \).
Let $|\cdot|$ be the Euclidean norm in $\mathbb{R}^2$. Given an initial function $\phi \in C$, the solution $z(t, \phi) = (x(t, \phi), y(t, \phi))$, $t \geq 0$, of the initial value problem

$$
\frac{dz}{dt}(t) = A^{\varepsilon, \eta}z(t) + F(z(t-1)) , \quad z_0 = \phi,
$$

where $A^{\varepsilon, \eta} = \begin{pmatrix} -\frac{\varepsilon}{\eta} & -\frac{\varepsilon}{\eta} \\ 0 & 0 \end{pmatrix}$, $F(\zeta) = \begin{pmatrix} f(\zeta) \\ 0 \end{pmatrix}$, exists and is unique [44]. Here, we use the standard notation $z_t(\phi) = z(t + \theta; \phi)$, for $\theta \in [-1, 0]$, respectively for $x$ and $y$, to refer to the state of the system $z_t \in C$ at time $t \in [0, \infty)$, whenever it is convenient. Note that in order to have full invariant manifolds theory available, we would need to extend this concept of solution to the state space $\mathbb{R}^2 \times L^\infty([-1, 0], \mathbb{R}^2)$ [2], which is not strictly necessary in our case. The solution defines a $C^1$ semi-flow $\Phi_t : \phi \mapsto x_t(\phi)$ on $C$ [44]. Recall that the $\omega$-limit set $\omega(\phi)$ of $\phi \in C$ under the semi-flow $\Phi_t$ is defined to be

$$
\omega(\phi) = \{ \psi \in C \mid \Phi_{t_n} \phi \to \psi \text{ for some sequence } t_n \to \infty \}.
$$

If $z_t(\phi)$ exists for all $t$, $z_t(\phi)$ is $C^1$ with a uniform bound on its derivatives for all $t \geq t_0 + 1$. By the Arzela-Ascoli Theorem then, $\omega(\phi)$ is nonempty and compact in $C$ (with its $C^0$ norm). We discuss some of the objects in $\omega(\phi)$ of [1]–[2]. As $\Phi_t$ is $C^1$ we rely on the principle of linearized stability to determine the asymptotic behavior of initial conditions close to $\omega(\phi)$.

For our further analysis, it is convenient to write the solution in terms of the solution operator that takes solution segments that have length $T = 1 + \delta$ slightly larger than one from one to the next. Consider the solution segment $\phi_n(\theta) := z(nT + \theta) \in C^1_{\delta} := C^1([-T, 0], \mathbb{R}^2)$, $n \in \mathbb{N}$ and $\theta \in [-T, 0]$, $1 \leq T < 2$. Then, (5) is equivalent to

$$
\frac{d}{d\theta}\phi_{n+1}(\theta) = A^{\varepsilon, \eta}\phi_{n+1}(\theta) + \begin{cases} F(\phi_n(\theta + \delta)), & -T < \theta < -\delta, \\
F(\phi_n(\theta - 1)), & -\delta < \theta < 0,
\end{cases}
$$

$$
\phi_{n+1}(-T) = \phi_n(0) \quad \text{for all } n \in \mathbb{N}, \phi_0|_{[-1, 0]} = z_0 \in C([-1, 0], \mathbb{R}^n).
$$

Eq. (6) defines a map given by $M^{\varepsilon, \eta}_T : C_T \to C_T$, $\psi \mapsto M^{\varepsilon, \eta}_T \psi$ with

$$
[M^{\varepsilon, \eta}_T \psi](\theta) = e^{A^{\varepsilon, \eta}(\theta + T)}\psi(-T) + \int_{-T}^{\min\{\theta, -T\}} e^{A^{\varepsilon, \eta}(\theta - s)}F(\psi(s + \delta)) \, ds
$$

$$
+ \int_{\min\{\theta - 1, -T\}}^{\theta - 1} e^{A^{\varepsilon, \eta}(\theta - 1 - s)}F\left(e^{A^{\varepsilon, \eta}(s + T)}\psi(-T) + \int_{-T}^{s} e^{A^{\varepsilon, \eta}(\theta - 3)}\psi(s + \delta) \, ds \right) \, ds,
$$

which can be easily seen using the variations of constants formula. The following two propositions collect basic properties of [1]–[2].

**Proposition 2** (zero solution). Let $\varepsilon, \eta > 0$ and $f$ satisfy (H1)-(H2). Then,

(i) [7]–[8] has the unique equilibrium solution $z \equiv 0$.

(ii) there exist $\bar{\varepsilon} > 0$, such that $z \equiv 0$ is unstable for all $\varepsilon < \bar{\varepsilon}$. At $\varepsilon = \bar{\varepsilon}$, the equilibrium undergoes a Hopf bifurcation. More specifically, there exists $\varepsilon_k \to 0$ as $k \to \infty$, such that for every $k$ there is a complex conjugated pair of eigenvalues crossing the imaginary axis with nonzero speed.

**Proof.** This is a direct result of the asymptotic spectral properties of steady states of delay equations with one single discrete delay [45]. We will not repeat the full analysis here. It is easy to see that
system \([1] - [2]\) has the unique equilibrium solution \(z \equiv 0\). The behavior of solutions in its neighborhood is determined by the characteristic growth rates, which are solutions to the corresponding characteristic equation \(0 = \lambda (\varepsilon \lambda + 1 - f'(0)e^{-\lambda}) + \eta\). Using the change \(\mu = \varepsilon \lambda\), this equation reads as \(0 = \mu (\mu + 1 - f'(0)Y(\mu/\varepsilon)) + \varepsilon \eta\), where \(Y = e^{-\mu/\varepsilon}\), and the analysis of \([15]\) directly applies. In particular, the spectrum for small \(\varepsilon\) can be characterized by its asymptotic properties. The strong asymptotic spectrum is absent. The weak asymptotic spectrum is given by \(\mu_{\text{weak}}(\varphi) = \varepsilon \gamma(\varphi) + i\varphi\), where \(\gamma(\varphi) = -\frac{1}{2} \log |Y(i\varphi)|\) and
\[
|Y(i\varphi)| = \frac{(\varepsilon \eta - \varphi^2)^2 + \varphi^2}{f'(0)^2 \varphi^2}.
\]
(H1)-(H2) imply \(|f'(0)| > 1\), such that \(\max_{\varphi} \gamma(\varphi) > 0\), i.e. the weak spectrum is unstable. Our assertion follows directly from \([15]\).

We have established that the zero equilibrium is unstable, with an arbitrary large dimension of the unstable manifold. As a result, we expect the deviation from it to grow exponentially for a generic small perturbation. This destabilization is due to a cascade of Hopf bifurcations as the unstable manifold. As a result, we expect the deviation from it to grow exponentially for a generic small perturbation.

Proposition 3 (oscillatory solutions). Let \(\varepsilon, \eta > 0\) and \(f\) satisfy (H1)-(H2). Then any solution \(z(t, \phi)\) which does not converge to zero, possesses the following properties:

(i) There exists \(0 < t < \infty\), such that \(z^1(t, \phi) = 0\).
(ii) The first component \(z^1\) of \(z\) has infinitely many sign changes.
(iii) For all \(t_1 < t_2\), such that \(z^2(t_1, \phi) = z^2(t_2, \phi)\), it holds that \(\int_{t_1}^{t_2} z^1(s, \phi) ds = 0\).

Proof. (i) Suppose that the assertion is false, i.e. \(z^1(t, \phi) > 0\) for all \(t \in [t_0, \infty)\); the other case of different sign is completely analogous. Since \(z^1(t, \phi)\) is strictly bounded away from zero, \(z^2(t, \phi)\) is strictly monotonously increasing with \(z^2(t, \phi) = \eta \int_0^t z^1(s, \phi) ds\). Thus, there exists \(t_1\) such that for all \(t > t_1\), \((z^1)'(t, \phi) < -z^1(t, \phi) - z^2(t, \phi) + M < -c\) for some \(c > 0\). This implies that \(z^1(t, \phi)\) changes sign leading to a contradiction. (ii) is a direct consequence of (i). (iii) Direct integration of \([2]\) gives the result, \(0 = z^2(t_2, \phi) - z^2(t_1, \phi) = \eta \int_{t_1}^{t_2} z^1(s, \phi) ds\).

As the zero solution is unstable, we have thus established that every solution, except the zero solution and the solutions that lie on the stable manifold of the zero solution are oscillatory for all \(t \geq 0\).

5. Stabilization mechanism of delay-induced switched states

As we have indicated in Sec. 3, initial conditions 'close' the balance point lead to delay-induced switched states, whereas those too far from the balance point, may appear as delay-induced switched states at first, yet coarsen after some possibly long transient time, and as a result no delay-induced switching can be observed anymore. This can be explained by a competition of the fast \(\varepsilon\)-dynamics and slow \(\eta\)-dynamics. In order to investigate these processes independently, we 'split off' the \(\eta\)-part of the dynamics. In doing so, we provide the conceptional mechanism for the stabilization of
Delay-induced switched states; the more involved arguments being given as self-contained lemmas. Recall from Sec. 4 that the solution to (1)–(2) can be written in integral form
\[ x(t) = x(t_0)e^{-(t-t_0)/\varepsilon} + \int_{t_0}^{t} e^{-(t-s)/\varepsilon} (f(x(s-1)) - y(t_0)) \, ds - \eta \int_{t_0}^{t} e^{-(t-s)/\varepsilon} \left( \int_{t_0}^{s} x(\tilde{s}) \, d\tilde{s} \right) \, ds, \]
which can be readily checked using the variation of constants formula. Our motivation is simple, find a transformation such that the resulting solution is independent of \( \eta \). A closer look at system (8)–(9) suggests to subtract the respective last terms \(-\eta \int_{t_0}^{t} e^{-(t-s)/\varepsilon} \left( \int_{t_0}^{s} x(\tilde{s}) \, d\tilde{s} \right) \, ds \) and \( \eta \int_{t_0}^{t} x(s) \, ds \) in (8)–(9). If we assume a delay-induced switched state \( \bar{\psi} \) with length \( T \), then this transformation takes the form of a linear operator \( L^{\varepsilon,\eta}_{T} : C^{1}_{T} \rightarrow C^{1}_{T}, \psi \mapsto L^{\varepsilon,\eta}_{T}\psi \), with
\[ [L^{\varepsilon,\eta}_{T}\phi](\theta) = \phi(\theta) + \eta \left( \frac{1}{\varepsilon} \int_{0}^{\theta} e^{-\frac{(s-\theta)}{\varepsilon}} \left( \int_{0}^{s} \phi^1(\tilde{s}) \, d\tilde{s} \right) \, ds \right). \]

The next Lemma collects several properties of \( L^{\varepsilon,\eta}_{T} \); the proof of which is contained in the supplementary material. Recall that \( M^{\varepsilon,\eta}_{T} \) is the solution operator that maps segments of delay-induced switched states of length \( T \) to the next.

**Lemma 4.** Let \( \varepsilon, \eta, T > 0, \phi \in C^{1}_{T} \) be fixed, and \( L^{\varepsilon,\eta}_{T} \) be defined as in (10). Then,
\begin{enumerate}[(i)]  
  
  \item \( L^{\varepsilon,\eta}_{T} \phi \) is a \( \eta \)-small perturbation of \( \phi \).
  
  \item \( L^{\varepsilon,\eta}_{T} \) is one-to-one. \( K^{\varepsilon,\eta}_{T} = (L^{\varepsilon,\eta}_{T})^{-1} \) has the form
    \[ (K^{\varepsilon,\eta}_{T}\phi)(\theta) = \phi(\theta) + \int_{-\theta}^{\theta} \left( A^{\varepsilon,\eta}(\theta-s) - e^{A^{\varepsilon,\eta}(\theta-s)} A^{\varepsilon,\eta}(s) \right) \phi(s) \, ds. \]
  
  \item \( L^{\varepsilon,\eta}_{T} M^{\varepsilon,\eta}_{T} = M^{\varepsilon,\eta}_{T} \), and \( M^{\varepsilon,\eta}_{T} = K^{\varepsilon,\eta}_{T} M^{\varepsilon,\eta}_{T} \).
\end{enumerate}

**Proof.** Let \( \phi \in C^{1}_{T} \) and denote \( \bar{\phi} = L^{\varepsilon,\eta}_{T}\phi \). (i) One readily checks the estimates \( |\bar{\phi}^2(\theta) - \phi^2(\theta)| = \eta |\int_{-\theta}^{\theta} \phi^1(s) \, ds| \leq \eta |T| \phi^1| \leq 2\eta \| \phi \| \) and
\[ |\bar{\phi}^1(\theta) - \phi^1(\theta)| = \eta \frac{1}{\varepsilon} \int_{-\theta}^{\theta} e^{-\frac{(s-\theta)}{\varepsilon}} \left( \int_{-\theta}^{s} \phi^1(\tilde{s}) \, d\tilde{s} \right) \, ds \leq \eta \frac{2\varepsilon}{\varepsilon} \| \phi \| \int_{-\theta}^{\theta} e^{-\frac{(s-\theta)}{\varepsilon}} \, ds \leq 2\eta \| \phi \|, \]
such that \( ||L^{\varepsilon,\eta}_{T}\phi - \phi|| \leq 3\eta \| \phi \| \) where we have used \( 1 \leq T < 2 \). Note that \( (\bar{\phi}^1)'(\theta) = (\phi^1)'(\theta) - (\phi^1(\theta) - \phi^1(s) + \phi^2(\theta) - \phi^2(\theta)), (\bar{\phi}^2)'(\theta) = (\phi^2)'(\theta) + \eta \phi^1(\theta) \) such that \( ||\bar{\phi}' - \phi'|| \leq 2 ||\phi' - \phi|| + \eta T ||\phi|| \leq 8\eta \| \phi \|. \) Thus, \( L^{\varepsilon,\eta}_{T} - \text{id} \) is uniformly bounded by \( 8\eta \). (ii) Taking the derivative of \( L^{\varepsilon,\eta}_{T}\phi \) one observes that \( \bar{\phi} \in C^{1}_{T} \), with
\[ \frac{d}{d\theta} \bar{\phi}(\theta) = A^{\varepsilon,\eta} \bar{\phi}(\theta) + \frac{d}{d\theta} \phi(\theta) - A^{\varepsilon,\eta} \phi(\theta), \]
and \( A^{\varepsilon,n} \) is defined as in Eq. (5). This is ODE that can be solved explicitly to recover Eq. (10). On the other hand, it can be rearranged as an IVP problem for \( \phi \), i.e.

\[
\begin{align*}
\frac{d}{d\theta} \phi(\theta) &= A^{\varepsilon,n}(\theta) + \frac{d}{d\theta} \tilde{\phi}(\theta) - A^{\varepsilon,0} \tilde{\phi}(\theta), \\
\phi(-T) &= \tilde{\phi}(-T).
\end{align*}
\]

In this way \( \phi \) is uniquely determined given \( \tilde{\phi} \in C_T \) and the explicit form of \( K_T^{\varepsilon,n} \) is readily obtained by the variation of constants formula. (iii) Let \( \phi_0 \in C_T \), and consider the sequence \( (\tilde{\phi}_n)_n \) where \( \tilde{\phi}_n = L_T^{\varepsilon,n} \phi_n \) and \( \phi_{n+1} = M_T^{\varepsilon,n} \phi_n \) for all \( n \geq 0 \). Then, it is easy to check that \( \tilde{\phi}_n \) satisfies

\[
\frac{d}{d\theta} \tilde{\phi}_n(\theta) = A^{\varepsilon,0}(\theta) + \begin{cases} F(\phi_{n-1}(\theta + \delta)), & -T < \theta < -\delta, \\
F(\tilde{\phi}_n(\theta - 1)), & -\delta < \theta < 0,
\end{cases}
\]

such that \( L_T^{\varepsilon,n} M_T^{\varepsilon,n} \phi_{n-1} = L_T^{\varepsilon,n} \phi_n = M_T^{\varepsilon,0} \phi_{n-1} \). This is independent of \( \phi_{n-1} \) and therefore \( L_T^{\varepsilon,n} M_T^{\varepsilon,n} = M_T^{\varepsilon,0} \). The second assertions follows immediately using (ii).

As a result, we obtain a decomposition of the solution operator \( M_T^{\varepsilon,n} = K_T^{\varepsilon,n} M_T^{\varepsilon,0} \) into two parts.

We use now that we have qualitative information about \( M_T^{\varepsilon,0} \), see Sec. 3. In particular, the number of sign changes of a given function \( \phi \in C_T \) does not increase, and the amount of which sign changes drift can be read of by our numerical studies for \( \eta = 0 \). In addition, we can show how \( K_T^{\varepsilon,n} \) effects the position of sign changes of a function \( \phi \in C_T \) along the interval \([-T,0]\). We do so indirectly, via studying its inverse \( L_T^{\varepsilon,n} \). The results are collected in the following Lemma. For brevity we say that a function \( \phi \) has the property

\((SC_N)\) The first component \( \phi_1 \) of \( \phi \) has an even number of sign changes and no other zeros. More specifically, there exist \( N \in \mathbb{N} \), and a series of sign changes \( (\theta^i) \), \( 0 \leq i \leq 2N \), of \( \phi_1 \) such that \( \phi_1(\theta^i) = 0 \), \( (\phi_1)'(\theta^i) \neq 0 \) and \( \phi_1 \) has no other zeros.

**Lemma 5** (dynamics of sign changes under \( L_T^{\varepsilon,n} \)). Let \( \varepsilon, \eta, T > 0 \) be fixed. Let \( \phi \in C_T \) satisfy \((SC_N)\) and consider \( \tilde{\phi} = L_T^{\varepsilon,n} \phi \). Then the following hold true.

(i) There exists \( \tilde{\eta} > 0 \), depending on \( \phi \), such that for all \( \eta < \tilde{\eta} \), \( \tilde{\phi} \) also satisfies \((SC_N)\). In particular, for each sign change \( \theta \) of \( \phi_1 \), there exists exactly one sign change \( \tilde{\theta} \) of \( \tilde{\phi}_1 \) with \( |\theta - \tilde{\theta}| = \mathcal{O}(\eta) \) and \( \tilde{\phi}_1 \) has no other zeros.

(ii) If \( \phi \in C_T^2 \), there exists \( \tilde{\eta}' > 0 \), depending on \( \phi \), such that for all \( \eta < \tilde{\eta}' \), the sign changes \( \tilde{\theta} \) of \( \tilde{\phi}_1 \) are given by

\[
\tilde{\theta} = \theta - \frac{\eta}{\varepsilon(\phi_1)'(\theta)} \left[ \int_{-T}^{\theta} e^{-(\eta^2 - \varepsilon \eta^2)} \left| \int_{-T}^{s} \phi_1(\tilde{s}) d\tilde{s} \right| ds + \mathcal{O}(\eta^2) \right],
\]

where \( \theta \) is a sign change of \( \phi \).

(iii) Assume as in (ii). There exist \( \tilde{\varepsilon} > 0 \) such that if \( \varepsilon < \tilde{\varepsilon} \), then for each subsequent pair of sign changes \( \theta^i, \theta^{i+1} \), \( i \leq 2N \) there exist unique (up to order \( \eta^2 \)) \( \theta^i \geq 0 \), depending on \( \phi \), such that if \( |\theta^i - \theta^{i+1}| < \tilde{\eta}' + \mathcal{O}(\eta^2) \) (respectively \( |\theta^i - \theta^{i+1}| > \tilde{\eta}' + \mathcal{O}(\eta^2) \)), then \( |\theta^i - \theta^{i+1}| < |\theta^i - \theta^{i+1}| \) (respectively \( |\theta^i - \theta^{i+1}| > |\theta^i - \theta^{i+1}| \)). In particular, \( \theta^i > 0 \) (up to
order \( \eta^2 \), if and only if

\[
\frac{1}{(\phi^1)'(\bar{\theta})} \left( \int_{-T}^{0} e^{-\frac{(\phi^1(s))}{\epsilon}} \left[ \int_{-T}^{s} \phi^1(\tilde{s}) ds \right] ds \right) < 0.
\]

(iv) assume as in (iii). If \( \phi \) is \((SC_2)\) with \(-T < \theta^1 < \theta^2 < 0\), \( \phi^1 > 0 \) (up to an error of order \( \eta^2 \)), and \( \phi^1 \) is implicitly given (up to an error of order \( \eta^2 \)) by

\[
\int_{\theta^i}^{\theta^i + \phi^1} \phi^1(s) ds = \left( \frac{(\phi^1)'(\theta^2)}{(\phi^1)'(\theta^1)} - 1 \right) \int_{-T}^{\theta^i} \phi^1(s) ds.
\]

Proof. The proof relies on Newton’s method to approximate the sign changes of \( \phi^{*} \) or \( \phi^1 \).

(i) We show that for sufficiently small \( \eta \), \( \phi^1 \) also has property \((SC_N)\). From \((SC_N)\) of \( \phi \), there exist \( \theta^i \), \( 0 < i \leq 2N \), \( \phi^1(\theta^i) = 0 \) and intervals \( (\alpha_i, \beta_i) \supset \theta^i \), such that \( -(1)^{i-1} \phi^1(\alpha_i) < 0 < -(1)^{i-1} \phi^1(\beta_i) \), and \( \phi^1 \mid_{\alpha_i, \beta_i} \) is strictly monotone. Let \( I = \bigcup_{i} (\alpha_i, \beta_i) \). Define \( \epsilon' = \min_{\theta \in I} \| (\phi^1)'(\theta) \| \). Using Lemma 4, \( \phi^1 \) is a continuously differentiable, \( \eta \)-small perturbation of \( \phi \) with \( C^1 \)-uniform bound \( \eta \| \phi \| \). Thus, we may choose \( \eta = \min \{ c, c', \min_i \{ \| \phi^1(\alpha_i) \|, \min_i \{ \| \phi^1(\beta_i) \| \} \} \}/(\min_{\theta \in I} \| (\phi^1)'(\theta) \|) > 0 \), such that \( -(1)^{i-1} \phi^1(\alpha_i) < 0 < -(1)^{i-1} \phi^1(\beta_i) \), \( \phi^1 \mid_{\alpha_i, \beta_i} \) is strictly monotone with \( c' = \min_{\theta \in I} \| (\phi^1)'(\theta) \| \) > 0, and \( \phi^1 \) is bounded away from zero outside of \( I \) for all \( \eta < \eta \). Then, by the Intermediate Value

Theorem, there exist unique (since \( \phi^1 \) is strictly monotone there) \( \bar{\theta} \in (\alpha_i, \beta_i) \) such that \( \bar{\theta} \) satisfies the

\[
\int_{\theta^i}^{\bar{\theta}} \phi^1(s) ds = \left( \frac{(\phi^1)'(\theta^2)}{(\phi^1)'(\theta^1)} - 1 \right) \int_{-T}^{\theta^i} \phi^1(s) ds.
\]

for all \( i \), where \( b = 3T \| \phi \| \| c'/\eta \| \). Here, we have used \( \bar{\theta} \) for all \( i \).

(ii) Let \( \eta < \eta \). Fix a sign change \( \theta \) of \( \phi^1 \) and the corresponding sign change \( \bar{\theta} \) of \( \phi^1 \). We show that under certain conditions to be specified \( \theta \) is determined up to a bounded error of order \( \eta^2 \) by only one step of Newton’s method with the initial guess \( \theta \). Recall from (i) that there exists an interval \( (\alpha, \beta) \subseteq (\theta, \bar{\theta}) \) such that the derivatives of \( \phi^1 \) and \( \phi^1 \) are bounded away from 0 by a constant \( b' \). For example, \( b' = \min \{ c', c'' \} \), where \( c', c'' \) defined as in (i). Let us denote \( \theta' \) the first iterate of the Newton approximation step \( \theta \to \theta' \), then

\[
\theta' = \theta - \frac{\eta}{(\phi^1)'(\theta)} \int_{-T}^{0} e^{-\frac{(\phi^1(s))}{\epsilon}} \left[ \int_{-T}^{s} \phi^1(\tilde{s}) ds \right] ds
\]

Here, we used the definition of \( \bar{\theta} \), the fact \( \phi^1(\theta) = 0 \), and multiplied the numerator and the denominator by \( 1/(\phi^1)'(\theta) \). For sufficiently small \( \eta < \eta'/(2T \| \phi \|) \), we can expand the denominator into a geometric series such that \( \theta' \) satisfies Eq. (13) (with \( \theta \) replaced by \( \theta' \)). \( \phi \in C^2_T \) satisfies the
estimate $\|\hat{\phi}'' - \phi''\| < 10\eta \|\phi\|$ such that $\|\hat{\phi}''\| < \|\phi''\| + 10\eta \|\phi\| =: b''$. Then, the error $|\bar{\theta} - \theta'|$ of the Newton approximation step $\theta \mapsto \theta'$ is bounded as

$$ |\bar{\theta} - \theta'| < \frac{b''}{2b} |\bar{\theta} - \theta'|^2 \left( < \eta \frac{b''}{2b} |\bar{\theta} - \theta'| \right),$$

where for the last inequality, we have used $|\bar{\theta} - \theta'| < \eta b$ from (i). It is a well known fact that if $\eta \frac{b''}{2b} < 1$ here, Newton’s method converges, and quadratically so, such that $|\bar{\theta} - \theta'| = O((|\bar{\theta} - \theta'|)^2) = O(\eta^2)$.

As a result, Eq. (13) holds for $\eta < \eta = \min \{\bar{\eta}, b/(2T \|\phi\|), 2b'/b''\}.$

(iii) Consider two subsequent sign changes $\theta^i, \theta^{i+1}$ as discussed in (i) and assume as in (ii).

We show that as we apply $\hat{L}^\varepsilon_T$, for sufficiently small $\varepsilon$, the distance between the sign changes $\theta^{i+1} - \theta^i$ decreases (increases), if it is smaller (larger) than a given “critical” distance $\bar{\theta}$, depending on $\phi$ and $\theta^i$, such that $|\bar{\theta}^{i+1} - \bar{\theta}^i| < |\theta^{i+1} - \theta^i|$ (respectively $|\bar{\theta}^{i+1} - \bar{\theta}^i| > |\theta^{i+1} - \theta^i|$).

Let $(\phi^1)'(\theta^i) > 0$, $(\phi^1)'(\theta^{i+1}) < 0$ be fixed such that $\phi^1(\theta) > 0$ for all $\theta \in (\theta^i, \theta^{i+1})$ and vary $\bar{\theta} = \theta^{i+1} - \theta^i > 0$ as an independent variable; the proof of the case with signs exchanged is completely analogous. To increase readability, we introduce new variables $Y(-T, \theta) = \int_0^\theta \phi^1(s)ds$ and $X(-T, \theta) = \frac{1}{a_i} \int_T e^{\frac{-a_i}{\theta^i}} Y(-T, \theta) ds$ such that at $\theta^{i+1}$, we can express the value of $X(-T, \theta^{i+1})$ as

$$ X(-T, \theta^i + \bar{\theta}^i) = X(\theta^i, \theta^i + \bar{\theta}^i) + Y(-T, \theta^i) + e^{-\frac{\bar{\theta}^i}{\theta^i}}(X(-T, \theta^i) - Y(-T, \theta^i)).$$

We note that $|X(-T, \theta) - Y(-T, \theta)| \leq \varepsilon T \|\phi\|$ for all $\theta \in [-T, 0]$. Using the above expression and Eq. (13) the difference $\bar{\theta}^{i+1} - \bar{\theta}^i$ can be written as

$$ \bar{\theta}^{i+1} - \bar{\theta}^i = \theta + \frac{a_i \eta}{(\phi^1)'(\theta^i)} Z^i(\theta) + O(\eta^2),$$

where $a_i = -(\phi^1)'(\theta^i)/(\phi^1)'(\theta^{i+1}) > 0$ and

$Z^i(\theta) = X(-T, \theta^i + \bar{\theta}^i) + \frac{1}{a_i} X(-T, \theta^i) = X(\theta^i, \theta^i + \bar{\theta}^i) + \frac{1}{a_i} X(-T, \theta^i) + e^{-\frac{\bar{\theta}^i}{\theta^i}}(X(-T, \theta^i) - Y(-T, \theta^i)).$

Thus, up to an error of order $\eta^2$, $\bar{\theta}^{i+1} - \bar{\theta}^i$ is smaller (larger) than $\theta$, if $Z^i(\theta) < 0$ (respectively $Z^i(\theta) > 0$). Clearly, $Y(\theta^i, \theta^i + \bar{\theta}^i) = \int_0^{\theta^i + \bar{\theta}^i} \phi^1(s)ds$ and therefore $X(\theta^i, \theta^i + \bar{\theta}^i) = \frac{1}{a_i} \int_0^{\theta^i + \bar{\theta}^i} e^{-\frac{a_i}{\theta^i + \bar{\theta}^i}} Y(\theta^i, \theta^i + s)ds$ are monotonously increasing with $Y(\theta^i, \theta^i) = 0$ and $X(\theta^i, \theta^i) = 0$. We remark that in the limit $\varepsilon \to 0$, statement (iii) is trivial now: Consider

$$ \lim_{\varepsilon \to 0} Z^i(\theta) = Y(\theta^i, \theta^i + \theta) + (1 + \frac{1}{a_i}) Y(-T, \theta^i),$$

and observe that the second term is constant with $1 + 1/a_i > 0$. Thus, if $Y(-T, \theta^i) < 0$, there exist a unique $\bar{\theta}^i > 0$ such that $Z^i(\bar{\theta}^i) = 0$ and $\bar{\theta}^{i+1} - \bar{\theta}^i = \bar{\theta}^i + O(\eta^2)$. If $Y(\theta^i) \geq 0$, then $\lim_{\varepsilon \to 0} Z^i(\bar{\theta}^i) > 0$ for all $\bar{\theta} > 0$ and we may choose $\bar{\theta}^i = 0$.

For $0 < \varepsilon < \tilde{\varepsilon} = \eta/(T \|\phi\|)$, we have that $|X(-T, \theta) - Y(-T, \theta)| < \eta$ and thus,

$$ Z^i(\theta) = X(\theta^i, \theta^i + \bar{\theta}^i) + (1 + \frac{1}{a_i}) X(-T, \theta^i) + O(\eta).$$
We can apply the same reasoning as before. The remaining terms of order $\eta$ affect the difference $	heta^{i+1} - \theta^i$ only of order $\eta^2$ and can be neglected. In particular, $\theta^i > 0$ (up to an error of order $\eta^2$), if and only $X(-T, \theta^i) < 0$.

(iv) Let $X(-T, \theta)$ and $Y(-T, \theta)$ be defined as in (iii). By assumption $(\phi_1')'(\theta_1) > 0$ and $\phi_1'(\theta) < 0$ for all $\theta \in [-T, \theta_1]$ such that $X(-T, \theta_1) < 0$. Then, $Z^i(\theta^i) = 0$ implies $\theta^i > 0$. The implicit expression for $\theta^i$ follows directly from the condition $Z^i(\theta^i) = 0$ using that $||X - Y|| < \eta$.

As an immediate consequence, for sufficiently small $\eta$, $K_{\varepsilon, \eta}^T$ preserves number of zeros and contracts the distance between two zeros to some positive distance. Thus, if for a given $\varepsilon$, we are sufficiently close to the balance point, the individual drift of the sign changes can be mitigated by the action of $K_{\varepsilon, \eta}^T$. Naturally, as $K_{\varepsilon, \eta}^T$ influences the profile of delay induced states, we expect small variations in the 'shape' of delay-induced switched states as opposed to the solution of (14)–(15) for $\eta = 0$. The following section discusses this effect in more detail.

6. Profiles of delay-induced switched states

We reason that for a given function $f$ the “shape” of a delay-induced switched state of (14)–(15) can be qualitatively determined from the formal limit $\varepsilon = 0$. The motivation is not unlike the scalar case of delayed feedback: If $\varepsilon x'(t)$ acts as a small perturbation, then the solution to (14)–(15) can be sufficiently well approximated by the reduced system

\begin{align}
  x(t) &= -y(t) + f(x(t-1)), \\
  y'(t) &= \eta x(t).
\end{align}

Only when the derivative is large with respect to $1/\varepsilon$ this viewpoint becomes inadequate. Using the variations of constants formula, (14)–(15) satisfies

\begin{align}
  x(t) &= -y(t) + f(x(t-1)) + (x(t_0) + y(t_0) - f(x(t_0) - 1)) e^{-(t-t_0)/\varepsilon} \\
  &\quad + \int_{t_0}^{t} e^{-(t-s)/\varepsilon} [f'(x(s-1))x'(s-1) - \eta x(s)] \, ds \\
  y'(t) &= \eta x(t),
\end{align}

The integral representation (16)–(17) of (14)–(15) allows for the following insight: As long as $|x'(t)|$ is bounded and independent of $\varepsilon$, i.e. $|x'(t)| \ll C/\varepsilon$ for some $C < \infty$, it immediately follows that $x(t) = -y(t) + f(x(t-1)) + O(\varepsilon)$ for all $t$. On the other hand, if the derivative is large, yet only on a short interval $(a, b) \subset [-1, 0]$, with $|b - a| \to 0$ as $\varepsilon \to 0$, and $c/\varepsilon < |x'(\theta)| < C/\varepsilon$ for some $c < C$; then $x(t) = -y(t) + f(x(t-1)) + O(1)$ for $\theta \in (a, b)$. So the derivative may act as a large perturbation within $(a, b)$. However then, for $\theta \in (b, 1]$ one can easily show that this perturbation decays fast again as $x(t) = -y(t) + f(x(t-1)) + O(e^{-(t-b)/\varepsilon}) + O(\varepsilon)$. This results in the solution segment on $[t, t+1]$ to appear as closely related to the solution segment on $[t-1, t]$, but possibly shifted with respect to that segment.

Note that although $\varepsilon$ is supposed to be small, (14)–(15) cannot (or rather should not) be considered a ‘small’ perturbation of (16)–(17) as (i) $\varepsilon x'(t)$ ought not be small, and (ii) the nature of our problem changes from a Delay Differential Equation to a Delay Difference Equation. This singular perturbation point of view comes with several such technical difficulties.

Here, we focus on the analysis of the reduced system (14)–(15). The following proposition states that (14)–(15) exhibits a period-1 solution, and sufficient conditions for it to be unstable.
Proposition 6 (Period-1 solutions). Let \( \eta > 0 \) be sufficiently small. Then,

(i) There exists a period-1 solution \((x, y)\) of (14)–(15).

(ii) \( x \) satisfies \( \int_{-1}^{0} x(t + s)ds = 0 \) for all \( t \).

(iii) If the derivative \( x'(t) \) is defined, then it satisfies

\[
x'(t) = \eta x(t)/(f'(x(t)) - 1).
\]

(iv) If |\( f'(x(t)) \)| > 1 for some \( t \in [-1, 0] \), then \( x \) is unstable.

Proof. We refer to functions \( x \) (of bounded variation) and \( y \) (differentiable) that satisfy (14)–(15), and \( x(t) = x(t-1), \ y(t) = y(t-1) \) for all \( t \), as period-1 solutions of (14)–(15). A solution to (14)–(15) satisfies the functional equation

\[
x(t) = -y(t-1) - \eta \int_{t-1}^{t} x(s)ds + f(x(t-1)),
\]

\[
y(t) = y(t-1) + \eta \int_{t-1}^{t} x(s)ds.
\]

(14)–(20) follow from straightforward integration of (15). We construct period-1 solutions of (14)–(15) using (19)–(20). If such a solution exists, it immediately satisfies

\[
0 = y(t) - y(t-1) = \eta \int_{-1}^{0} x(s)ds
\]

for all \( t \) and the values of \( x \) and \( y \) are fixed as \( x_k(\theta) = x_{k-1}(\theta) = x(\theta) \) and \( y_k(\theta) = y_{k-1}(\theta) = y(\theta) \) for all \( k \in \mathbb{Z} \) and \( \theta \in [-1, 0] \). The resulting system

\[
x_k(\theta) = -y(\theta) + f(x_k(\theta)),
\]

\[
y_k(\theta) = y(\theta),
\]

is decoupled in each “point in space” \( \theta \) and \( y(\theta) = y(0) + \eta \int_{-1}^{0} x_0(s)ds \) is to be considered as a parameter. We prescribe a fixed number of sign changes \(-1 < \theta^1 < \theta^2 < \cdots < \theta^{2N} < 0\), \( 2N \in \mathbb{N} \), and in addition, choose initial values \( \chi^i, 1 \leq i \leq 2N, \chi^i \in \mathbb{R} \), with \( \text{sgn}(\chi^i) = -\text{sgn}(\chi^{i+1}) \), \( \chi^{2N+1} = \chi^1 \), and \( f(\chi^i) \neq 1 \). For \( \theta \in [\theta^1, \theta^{i+1}] \) (with \( \theta^{2N+1} = \theta^1 \)), \( x_0(\theta) \) can be defined as the solution of the Ordinary Differential Equation

\[
x'_0(\theta) = \eta x_0(\theta)/(f'(x_0(\theta)) - 1),
\]

with \( x_0(\theta^i) = \chi^i \). Here \( \eta \) has to be small enough that \( f'(x_0(\theta)) \neq 1 \) for all \( \theta \in [-1, 0] \). As a result, \( x_0 \) is piecewise differentiable (continuously differentiable from the right) with discontinuities at \( \theta^i \), and \( x_0 \) satisfies \( x_0(0) = x_0(-1) \). Eq. (24) can be integrated such that \( x_0(\theta) = y_0(\theta) + f(x_0(\theta)) \), where \( y_0(\theta) := y_0(-1) + \eta \int_{-1}^{0} x_0(s)ds \) and \( y_0(-1) := -x_0(-1) + f(x_0(-1)) \). As \( (\theta^i), (\chi^i) \) were arbitrary, we may choose them such that \( \int_{-1}^{0} x_0(s)ds = 0 \) and \( y_0(0) = y_0(-1) \). As a result, the function \((x, y)\) defined as \( x_k(\theta) = x_0(\theta), y_k(\theta) = y_0(\theta) \), for all \( k \in \mathbb{Z} \) and \( \theta \in [-1, 0] \) is a 1-periodic solution of (14)–(15). By construction, \( x(t) = x(t-1) \) and \( y'(t) = \eta x(t) \) for all \( t \), such that \((x, y)\) satisfies (14)–(15). Thus follows (i) and (ii) is obvious from Eq. (24); (iii) from Eq. (24).

For (iv), we consider the time evolution \((\xi^s, v^s)\) of a point perturbation \((\xi^s_0, 0)\) along the the period-1 solution \((x_0, y_0)\), where \( \xi^0_0(s) = 1 \) and \( \xi^0_0(\theta) = 0 \) for all \( \theta \in [-1, 0]/\{s\} \). One can easily
show that $\xi^s$ satisfies
\begin{equation}
\xi^s_n(s) = f'(x_0(s))\xi^s_{n-1}(s),
\end{equation}
and $\xi^s_n(\theta) = 0$ for all $n \in \mathbb{N}$ and $\theta \in [-1,0]/\{s\}$. Here, we have used that $\eta \int_0^\theta \xi^s_n(s)ds = 0$ and $v^s_n(\theta) = v^s_{n-1}(\theta) = 0$ for all $n \in \mathbb{N}$ and $\theta \in [-1,0]$. As a result, the perturbation $\xi^s_0$ grows if and only if $|f'(x_0(s))| > 1$. This condition is sufficient for $x_0$ to be unstable.

Our numerics shows that the solutions from the Proposition 6 are related to the delay-induced switched states in [1]–[2]. Figure 4 displays the solution to (14)–(15) for two example functions $f$ and suitable initial conditions. Fig. 4(a) shows a period-$\eta$ switched states in (1)–(2). Figure 4 displays the solution to (14)–(15) for two example functions and suitable initial conditions. Fig. 4(a) shows a period-$\eta$ switched states in (1)–(2). In fact, it changes exponentially with a small rate, and Prop. 6(iii) determines this rate to be 0.

Remark that for small values of $\xi$ and $\eta$, this trend can be approximated as
\begin{equation}
x(t + \Delta t) = x(t)e^{\int_0^{t+\Delta t} (f(x(t))-1)ds} \approx x(t) + O(\eta\Delta t),
\end{equation}
so that along such regular “plateaus” it appears to be linear in $\eta$. Here, we assumed that $f'(x(t))-1$ is uniformly bounded away from zero. In Fig. 4(a), the parameters are chosen such that $f'(x(t)) < 1$ along the solution. If this condition is not met for some values of $x$, the corresponding period-$\eta$ solution is unstable, and the numerically computed solution appears irregular for those values, see Fig. 4(b) and compare Prop. 6(iii). This qualitatively resembles delay-induced switched states with incoherent part in (1)–(2), although the solution is more erratic than for positive $\varepsilon$, compare Fig. 1. We attribute this to the “smoothing” effect of the term $\varepsilon x'$ in (1)–(2), which is not present in (14)–(15).

The solutions shown in Fig. 4 correspond to well chosen initial conditions satisfying the integral property in Prop. 6(ii). For strictly positive (or analogously strictly negative) initial conditions the solution to (14)–(15) does not resemble a delay-induced switched state. As a result, for such initial conditions, we do not expect delay-induced states in (1)–(2). The next section discusses this fact in more detail.

### 7. Coexistence with Fold-induced Slow-fast Oscillations

Section 3 revealed that for a given function $f$ and carefully chosen initial conditions close to the balance point, (1)–(2) displays delay-induced switched states. Here, we show that this class of solutions coexists with another type of solutions oscillating on a much longer timescale of order $1/\eta$, which can be selected via the choice of initial conditions.

To begin with, let us choose $f(x) = 1.2x/(1 + x^4)$. The balance point is given by $y(0) = 0$ as we have argued in Sec. 2. Fig. 5(a) shows the $\omega$-limit set of initial conditions with $y(0) = 0$ and two sign changes for various $\eta$. For $\eta = 0.001$, we observe a delay-induced switch state with almost constant plateaus. The orange dashed curve in Fig. 5(a) shows the fixed points of Eq. (14) parameterized by constant $y = v$, i.e., the fixed points of the iteration $(\chi, v) \mapsto (f(\chi) - v, v)$ in $\mathbb{R}^2$. We refer to this curve as the critical manifold. We observe that each sign change in the past, causes the solution to switch to the opposite branch of the critical manifold (with opposite sign) after some time close to one. Along the critical manifold, $x$ satisfies Eq. (18) up to an error of order $\varepsilon$, and as a result, $x$ and $y$ change with rate proportional to $\eta$ on timescale 1. Therefore, as we increase $\eta$, the variations along the plateaus become more pronounced, and the range of $y$ approximately scales proportional to $\eta$; compare Fig. 5(a).
Section 2, we defined the balance point $y(0) = v$ such that the solution to
\begin{equation}
\epsilon x'(t) = -x(t) - v + f(x(t - 1)),
\end{equation}
displays exponentially long transients for initial conditions that have sign changes. Let us put $v = 0$ for now. Then, starting with an initial condition without sign changes, the solutions preserves its sign [26]. The $\omega$-limit set in this case may consist of the nontrivial equilibrium point, a periodic orbit or a chaotic attractor depending on $f$ [26]. If the equilibrium is linearly stable, then the solution converges to it exponentially fast. The value of the equilibrium is given by the intersection of the critical manifold with the line $\{y = 0, x \leq 0\}$ in Fig. 5(b). For sufficiently small $\eta$, we can then think of $v = y(t)$ as changing adiabatically, that is the convergence to the equilibrium for $v \neq 0$ is fast, compared to the flow along the critical manifold. This mechanism gives rise to slow-fast relaxation oscillations with period of order $1/\eta$, see Fig. 5(b).

Let now $f(x) = 2x/(1 + (x - 0.8)^4)$ and $y(0) \approx 0.514$ as numerically determined in Sec. 3. The stability of an equilibrium $x(t) = \xi = \text{const}$ of (26) is easily determined. If $|f'(\xi)| > 1$, there exists $\bar{\epsilon} > 0$, such that it is unstable for all $\epsilon < \bar{\epsilon}$; and it is stable for all $\epsilon > 0$, otherwise. At $\epsilon = \bar{\epsilon}$ the equilibrium undergoes a Hopf bifurcation with respect to $\epsilon$; the proof is analogous to the zero equilibrium in Sec. 4. This condition is very much related to Prop. 6(iv), for which we obtain delay-induced switched states with incoherent parts. The corresponding solution to initial
Figure 5. (color online) Delay-induced switched states and “fold-induced” slow-fast oscillations coexist in (1)–(2). Panels (a–d) show the projection to the \((x, y)\)-plane of the \(\omega\)-limit set for different initial conditions, choices of the function \(f\) and parameters \(\eta_1 = 0.001, \eta_2 = 0.01, \eta_3 = 0.1\) and \(\eta_4 = 0.3\). The dashed (orange) curve corresponds to fixed points of the map \((x, y) \mapsto (f(x) - y, y)\), dotted (black) lines correspond to \(x = 0\) and \(y = 0\). Panel (a-b): \(\varepsilon = 0.01, f(x) = 1.2x/(1 + x^4), y(0) = 0\); (a): \(x_0(t) = -1\) for \(-1 \leq t \leq -1/2, x_0(t) = 1\) for \(-1/2 < t \leq 0\) (two sign changes), and (b): \(x_0(t) = -1\) for \(-1 < t \leq 0\) (no sign changes). Panel (c-d): \(\varepsilon = 0.003, f(x) = 2x/(1 + (0.8 - x)^4)\) and \(y(0) = 0.514\); (c): \(x_0\) as (a), and (d): \(x_0\) as (b).

Conditions without sign changes is shown in Fig. 5(d). This type of solution has been coined chaotic breathers [8]; we will not discuss this type of solution in detail. However, we want to emphasize that they are likely candidates for the omega-limit set of initial condition with sign changes in (1)–(2) when is \(y(0)\) chosen too far from the balance point.
8. OUTLOOK AND DISCUSSION

We have investigated delay-induced switched states in systems of the form (1)-(2) and how it arises from the formal limits $\varepsilon \to 0$ and/or $\eta \to 0$. The phenomenon itself is more general and has been observed in various slow-fast systems including time delay, such as laser systems \cite{12, 13} and in a model of neuronal activity \cite{16}. In our examples, we restricted ourselves to the case where switching occurred once per delayed interval, yet multiple switched solution can be observed as well \cite{14, 15}. Likewise, we did not investigate whether delayed-induced switched states can be reached from non-switched initial conditions. Such a transition could for example be induced by interaction of the solutions with a fold, similar to canard solutions \cite{5, 6}, that may lead to a fast increase of the number of sign changes along a delay interval. We have introduced the concept of a balance point of (1)-(2), and alongside our rigorous results on the “stabilization” mechanism close to the balance point, we included a detailed numerical exploration. A rigorous proof of the existence of delay-induced states, is beyond the expository character of this article, and we will address this question in future works. A promising direction poses the customization of the asymptotic methods used to study the drift in scalar delayed feedback equations \cite{39, 38, 40, 41, 36, 35, 42}.

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REFERENCES

[1] Kuehn C. Multiple Time Scale Dynamics. Springer, Switzerland, 1 ed, 2015.
[2] Diekmann O, van Gils SA, Verduyn Lunel SM, and Walther H-O. Delay Equations, Functional-, Complex-, and Nonlinear Analysis. Springer-Verlag, New York, 1995.
[3] Bates P, Lu K, and Zeng Z. Existence and persistence of invariant manifolds for semiflows in Banach spaces. Mem. Amer. Math. Soc., 135, 1998.
[4] Fowler AC, Mackey MC. Relaxation Oscillations in a Class of Delay Differential Equations. SIAM J. Appl. Math., 63(1):299–323, 2002.
[5] Campbell SA, Stone E, Erneux T. Delay induced canards in a model of high speed machining. Dyn. Syst., 24(3):373–392, 2009.
[6] Krupa M, Touboul JD. Canard explosion in delayed equations with multiple timescales. J. Dyn. Differ. Equations, 28(2):471–491, 2016.
[7] De Souza DC, Humphries AR. Dynamics of a Mathematical Hematopoietic Stem-Cell Population Model. arXiv:1712.08308, 2017.
[8] Chembo Kouomou Y, Colet P, Larger L, Gastaun N. Chaotic breathers in delayed electro-optical systems. Phys. Rev. Lett., 95(20):203903, 2005.
[9] Tallu Mbé JH, Tallu AF, Goune Chengui GR, Coillet A, Larger L, Woafa P, Chembo YK. Mixed-mode oscillations in slow-fast delayed optoelectronic systems. Phys. Rev. E, 91(1):012902, 2015.
[10] Krupa M, Szmolyan P. Relaxation oscillation and canard explosion. J. Differ. Equ., 174(2):312–368, 2001.
[11] Desroches M, Guckenheimer J, Krauskopf B, Kuehn C, Osinga HM, Wechselberger M. Mixed-Mode Oscillations with Multiple Time Scales. SIAM Rev., 54(2):211–288, 2012.
[12] Weicker L, Erneux T, D’Huys O, Danckaert J, Jacquot M, Chembo Y, Larger L. Strongly asymmetric square waves in a time-delayed system. Phys. Rev. E, 86(5):055201, 2012.
[13] Weicker L, Erneux T, D’Huys O, Danckaert J, Jacquot M, Chembo Y, Larger L. Slow-fast dynamics of a time- delayed electro-optic oscillator. Phil. Trans. R. Soc. A, 371:20120459, 2013.
[14] Larger L, Penkovsky B, Maistrenko Y. Virtual Chimera States for Delayed-Feedback Systems. Phys. Rev. Lett., 111(5):54103, 2013.
[15] Larger L, Penkovsky B, Maistrenko Y. Laser chimeras as a paradigm for multistable patterns in complex systems. Nat Commun, 6:7752, 2015.
[16] Erneux T, Weicker L, Bauer L, Hövel P. Short-time-delay limit of the self-coupled FitzHugh-Nagumo system. Phys. Rev. E, 93(2):022208, 2016.
[17] Klinshov V, Shchapin D, Yanchuk S, Wolfrum M, Nekorkin V. Embedding the dynamics of a single delay system into a feed-forward ring. Phys. Rev. E, 96:042217, 2017.
[18] Yanchuk S, Giacomelli G. Spatio-temporal phenomena in complex systems with time delays. J. Phys. A Math. Theor., 50(10):103001, 2017.
[19] Abrams DM, Strogatz SH. Chimera states for coupled oscillators. Phys. Rev. Lett., 93(17):174102, 2004.
[20] Kuramoto Y, Battogtokh D. Coexistence of Coherence and Incoherence in Nonlocally Coupled Phase Oscillators. Nonlinear Phenom. Complex Syst., 4:380–385, 2002.
[21] Mackey MC, Glass L. Oscillation and Chaos in Physiological Control Systems. Sci. New Ser., 197(4300):287–289, 1977.
[22] an der Heiden U, Walther H-O. Existence of Chaos in Control Systems with Delayed Feedback. J. Differ. Equ., 47:273–295, 1983.
[23] Ikeda K. Multiple-valued stationary state and its stability of the transmitted light by a ring cavity system. Opt. Commun., 30(2):257–261, 1979.
[24] Gurney WSC, Blythe SP, Nisbet RM. Nicholson’s blowflies revisited. Nature, 287:17–21, 1980.
[25] Marcus CM, Westervelt M. Stability of analog neural networks with delay. Phys. Rev. A, 39(1):347–359, 1989.
[26] Röst G, Wu J. Domain-decomposition method for the global dynamics of delay differential equations with unimodal feedback. Proc. R. Soc. A, 463(2086):2655–2669, 2007.
[27] Cao Y. The discrete Lyapunov function for scalar differential delay equations. J. Differ. Equ., 87(2):365–390, 1990.
[28] Mallet-Paret J, Sell GR. Systems of Differential Delay Equations : Floquet Multipliers and Discrete Lyapunov Functions. J. Differ. Equ., 125(0036):385–440, 1996.
[29] Giacomelli G, Marino M, Zaks MA, Yanchuk S. Coarsening in a bistable system with long-delayed feedback. EPL, 99(5):58005, 2012.
[30] Giacomelli G, Marino F, Zaks MA, Yanchuk S. Nucleation in bistable dynamical systems with long delay. Phys. Rev. E, 88(6):062920, 2013.
[31] Krisztin T. Global dynamics of delay differential equations. Period. Math. Hungarica, 56(1):83–95, 2008.
[32] Smith H. Monotone semiflows generated by functional differential equations. J. Differ. Equ., 66(3):420–442, 1987.
[33] Mallet-Paret J, Sell GR. The Poincare Bendixson Theorem for Monotone Cyclic Feedback Systems with Delay. J. Differ. Equ., 125:0037, 1996.
[34] Yanchuk S, Perlikowski P. Delay and Periodicity. Phys. Rev. E, 79:046221, 2009.
[35] Grotta-Ragazzo C, Pakdaman K, Malta CP. Metastability for delayed differential equations. Phys. Rev. E, 60(5):6230–6233, 1999.
[36] Grotta-Ragazzo C, Malta CP, Pakdaman K. Metastable Periodic Patterns in Singularly Perturbed Delayed Equations. J. Dyn. Differ. Equations, 22:203–252, 2010.
[37] Fiedler B, Mallet-Paret J. Connection between Morse sets for delay-differential equations. J. reine angew. Math., 397:23–41, 1989.
[38] Chow SN, Lin XB, Mallet-Paret J. Transition layers for singularly perturbed delay differential equations with monotone nonlinearities. J. Dyn. Differ. Equations, 1(1):3–43, 1989.
[39] Lin XB. Exponential dichotomies and homoclinic orbits in functional differential equations. J. Differ. Equ., 63(2):227–254, 1986.
[40] Nizette M. Front dynamics in a delayed-feedback system with external forcing. Phys. D, 183(3):220–244, 2003.
[41] Nizette M. Stability of square oscillations in a delayed-feedback system. Phys. Rev. E, 70(5):6, 2004.
[42] Wattis JAD. Shape of transition layers in a differential-delay equation. IMA J. Appl. Math., 82(3):681–696, 2017.
[43] Lefebvre J, Hutt A, LeBlanc VG, Longtin A. Reduced dynamics for delayed systems with harmonic or stochastic forcing Chaos, 22:043121, 2012.
[44] Hale JK, Verduyn Lunel SM. Introduction to Functional Differential Equations, volume 99 Applied Mathematical Sciences. Springer New York, New York, NY, 1993.
[45] Lichtner M, Wolfrum M, Yanchuk S. The spectrum of delay differential equations with large delay. SIAM J. Math. Anal., 43:788–802, 2011.