Multiplicative properties of quantum channels

Mizanur Rahaman

Department of Mathematics and Statistics, University of Regina, Regina, Saskatchewan S4S 0A2, Canada

E-mail: rahamami@uregina.ca

Received 4 February 2017, revised 15 June 2017
Accepted for publication 23 June 2017
Published 26 July 2017

Abstract

In this paper, we study the multiplicative behaviour of quantum channels, mathematically described by trace preserving, completely positive maps on matrix algebras. It turns out that the multiplicative domain of a unital quantum channel has a close connection to its spectral properties. A structure theorem (theorem 2.5), which reveals the automorphic property of an arbitrary unital quantum channel on a subalgebra, is presented. Various classes of quantum channels (irreducible, primitive, etc) are then analysed in terms of this stabilising subalgebra. The notion of the multiplicative index of a unital quantum channel is introduced, which measures the number of times a unital channel needs to be composed with itself for the multiplicative algebra to stabilise. We show that the maps that have trivial multiplicative domains are dense in completely bounded norm topology in the set of all unital completely positive maps. Some applications in quantum information theory are discussed.

Keywords: quantum channel, multiplicative domain, fixed points, irreducible, primitive, peripheral eigenvalues, multiplicative index

Introduction

Quantum channels are the most general input–output transformations allowed by quantum mechanics. Physically, they play a central role in quantum information theory, where they represent the communication from a sender to a receiver [41] in quantum information processing [12, 15, 42], and in the theory of quantum open systems (see the monograph [20]). The steps of a quantum computation, and also the effects of errors and noise on quantum registers, are modeled as quantum channels. In quantum statistical mechanics involving finitely many particles, the typical domain on which a quantum channel acts is the set of $d \times d$ complex matrices which we denote by $\mathcal{M}_d$. 
Although the maps in discussion are linear, their domain, the set of $d \times d$ complex matrices $\mathcal{M}_d$, has an algebraic structure. Thus, it is of interest to investigate the multiplicative nature of such linear maps. The domain on which the quantum channel is multiplicative is known as the multiplicative domain and the main theme of this paper is to study this domain in detail. The scheme of quantum error correction, one of the major themes of current research in information theory, was successfully analysed from the point of view of the multiplicative nature of channels (see [14, 24, 28]). Also, the multiplicative domain appears to be a useful area to explore in the study of private algebras and complementary quantum channels (see [31]).

From a purely operator algebraic perspective, the multiplicative domains of positive and completely positive maps have been studied by many authors for independent interests [9, 13, 37]. In this context, it is essential to mention the work of Bulinski in [8] where the author considers the dynamical system $(\mathcal{M}, \omega, \tau)$ and investigates the asymptotic automorphic behaviour of the dynamical system, where $\mathcal{M}$ is a von Neumann algebra, $\tau = (\tau_t)_{t \geq 0}$ is a family of unital normal completely positive maps on $\mathcal{M}$ parametrised over the positive real numbers and $\omega$ is a faithful normal state on $\mathcal{M}$ such that $\omega(x) = \omega \circ \tau_t(x)$ for every $t \geq 0$ and $x \in \mathcal{M}$. The main result of this work asserts that there exists a subalgebra (named ‘automorphy subalgebra’) on which each $\tau_t$ is an automorphism. Later, Størmer in [37] studied multiplicative properties of a positive map on a von Neumann algebra and obtained more general results concerning the automorphic behaviour of the given map.

Our aim in this paper is to study multiplicative properties of quantum channels acting on a matrix algebra $\mathcal{M}_d$. It is well known that a quantum channel $\mathcal{E} : \mathcal{M}_d \to \mathcal{M}_d$ is represented by a set of (non-unique) Kraus operators $\{k_i\}_{i=1}^n$ such that

$$\mathcal{E}(x) = \sum_{j=1}^n k_j x k_j^*, \forall x \in \mathcal{M}_d \text{ and } \sum_{j=1}^n k_j^* k_j = 1.$$ 

The operators $\{k_i\}_{i=1}^n$ are known as Kraus operators. Our first result is: for a unital quantum channel, we have the equality of two sets

$$\mathcal{M}_\mathcal{E} = \mathcal{F}_{\mathcal{E}^* \circ \mathcal{E}},$$

where $\mathcal{M}_\mathcal{E}$ is the multiplicative domain of $\mathcal{E}$ and $\mathcal{F}_{\mathcal{E}^* \circ \mathcal{E}}$ denotes the fixed point set of $\mathcal{E}^* \circ \mathcal{E}$. Here $\mathcal{E}^*$ is the adjoint of $\mathcal{E}$ when $\mathcal{M}_d$ is thought of as a Hilbert space with the Hilbert–Schmidt inner product $(a, b) = \text{tr}(ab^*)$ for all $a, b \in \mathcal{M}_d$. The adjoint $\mathcal{E}^*$ satisfies the relation $\text{tr}(\mathcal{E}(a)b) = \text{tr}(a \mathcal{E}^*(b))$ for all $a, b \in \mathcal{M}_d$. This result seems to be known before (see theorems 10, 11 in [14] and also [28]) but we present a different proof here. Exploiting the same relationship for powers of the channel, that is for $\mathcal{E}^n, n \geq 1$, we arrive at the chain of subalgebras with the following inclusion

$$\mathcal{M}_\mathcal{E} \supseteq \mathcal{M}_\mathcal{E}^2 \supseteq \cdots \supseteq \mathcal{M}_\mathcal{E}^n \supseteq \cdots .$$

Since the underlying space is of finite dimension, this finding motivates us to predict the existence of a stabilising subalgebra which we denote by

$$\mathcal{M}_\mathcal{E}^\infty := \bigcap_{n=1}^\infty \mathcal{M}_\mathcal{E}^n .$$

This subalgebra captures precisely the automorphic behaviour of $\mathcal{E}$ which is the content of the main theorem (2.5) of this paper. It turns out that $\mathcal{E}$ acts as a bijective homomorphism on $\mathcal{M}_\mathcal{E}^\infty$ and also this set is the algebra generated by the eigen operators of $\mathcal{E}$ corresponding to the peripheral eigenvalues. The sub algebra $\mathcal{M}_\mathcal{E}^\infty$ carries the intrinsic automorphic attribute of a unital channel which also manifests the asymptotic behaviour of $\mathcal{E}$. Theorem 2.5 then sets the foundation for introducing the notion of a multiplicative index of a unital channel.
is the smallest \( n \in \mathbb{N} \) such that \( \mathcal{M}_{E_E} = \mathcal{M}_{E_{\infty}} \). It turns out that the multiplicative index has an important connection in quantum error correction, which is described in section 5.

Much of the work presented in this paper is based on viewing a channel in the Schrödinger picture where trace preservation of the map is assumed and exploited heavily. In a non-unital case, such a channel is realised as a unital completely positive (ucp) map in the Heisenberg picture (that is, in the dual picture). In section 4, we prove that the set of unital completely positive (ucp) maps that have trivial multiplicative domains is cb dense in the class of all unital completely positive maps. Also a new result is obtained (theorem 4.5) which can be viewed as an extension to Arveson’s boundary theorem (4.3) on matrix algebras.

The paper is organised as follows. For a unital channel \( \mathcal{E} : \mathcal{M}_d \to \mathcal{M}_d \) we start with section 1 that develops the techniques needed to prove the relationship between the fixed point set of \( \mathcal{E}^* \circ \mathcal{E} \) and the multiplicative domain of \( \mathcal{E} \). Some related corollaries are noted down. In section 2 we introduce the notion of multiplicative index. The main theorem (theorem 2.5) characterises the stabilising subalgebra \( \mathcal{M}_{\mathcal{E}^*} \) in terms of the peripheral eigenvectors of \( \mathcal{E} \).

Also the multiplicative index is calculated for some quantum channels. Section 3 is concerned with the multiplicative properties of irreducible and primitive quantum channels—types of channels that appear to be very important in information theory. In section 4 the multiplicative nature of ucp (not necessarily trace preserving) maps is explored. The tools and techniques developed throughout the paper are exploited in section 5 to demonstrate some applications in information theory, specifically in quantum error correction. Section 6 contains a summary and some discussions on the topological aspects of sets with a fixed multiplicative index.

1. Fixed point and multiplicative domain

We begin with some terminology and the general theory of positive and completely positive maps on C*-algebras which will be required for further discussion. References [35] and [39] are amongst the many good resources on this topic.

Let \( \mathcal{E} : \mathcal{A} \to \mathcal{B} \) be a ucp map of C*-algebras \( \mathcal{A} \) and \( \mathcal{B} \). The following sets are called the set of fixed points and the multiplicative domain respectively:

\[
\mathcal{F}_{\mathcal{E}} = \{ a \in \mathcal{A} : \mathcal{E}(a) = a \},
\]

\[
\mathcal{M}_{\mathcal{E}} = \{ a \in \mathcal{A} : \mathcal{E}(ab) = \mathcal{E}(a)\mathcal{E}(b), \mathcal{E}(ba) = \mathcal{E}(b)\mathcal{E}(a) \forall b \in \mathcal{A} \}.
\]

Recall that a completely positive unital map \( \mathcal{E} \) satisfies the Schwarz inequality \( \mathcal{E}(aa^*) \geq \mathcal{E}(a)\mathcal{E}(a^*) \), for every \( a \in \mathcal{A} \). Choi ([13]) showed, for a ucp map, the set \( \mathcal{M}_{\mathcal{E}} \) is the same as the following set

\[
\mathcal{S} = \{ x \in \mathcal{A} : \mathcal{E}(xx^*) = \mathcal{E}(x)\mathcal{E}(x^*), \mathcal{E}(x^*x) = \mathcal{E}(x^*)\mathcal{E}(x) \}.
\]

Recall that for a ucp map \( \Phi : \mathcal{A} \to \mathcal{B}(\mathcal{H}) \), the Stinespring dilation theorem says that there exists a Hilbert space \( \mathcal{K} \), a bounded linear operator \( V : \mathcal{H} \to \mathcal{K} \) and a \* homomorphism \( \pi : \mathcal{A} \to \mathcal{B}(\mathcal{K}) \) such that \( \Phi(x) = V^*\pi(x)V \), for all \( x \in \mathcal{A} \). Furthermore, \( \| V \|^2 \leq\| \Phi(1) \| = 1 \).

We first state a theorem regarding the multiplicative domain of a ucp map defined on a C*-algebra which can be found as an exercise in [35] (see also [9]).

**Theorem 1.1.** Let \( \mathcal{A} \) be a unital C*-algebra, and let \( \mathcal{E} : \mathcal{A} \to \mathcal{B}(\mathcal{H}) \) be a ucp map with the minimal Stinespring representation \( (\pi, V, \mathcal{K}) \). An element \( a \in \mathcal{A} \) satisfies \( \mathcal{E}(aa^*) = \mathcal{E}(a)\mathcal{E}(a)^* \) and \( \mathcal{E}(a^*a) = \mathcal{E}(a)^*\mathcal{E}(a) \) if and only if \( VH \) is a reducing subspace for \( \pi(a) \). Moreover, the collection of such elements is a C*-sub-algebra of \( \mathcal{A} \) and is equal to the multiplicative domain of \( \mathcal{E} \).

The following theorem provides useful characterisations for projections and unitaries to belong to the multiplicative domain of a ucp map.
Theorem 1.2. Let $E : A \to B$ be a ucp map between unital $C^*$ algebras. Then
1. for a projection $p \in A$, $p \in M_E$ if and only if $E(p)$ is a projection;
2. for a unitary element $u \in A$, $u \in M_E$ if and only if $E(u)$ is a unitary element.

Proof.
1. If a projection $p \in M_E$, then by theorem 1.1 $E(p) = E(p^2) = E(p)E(p)$.
   Conversely, for a projection $p \in A$, if $E(p)^2 = E(p)$, then $E(p) = E(p)E(p)$ and hence $p$ gives equality in the Schwarz inequality and by the theorem $p \in M_E$.
2. If $u \in M_E$ and $u$ is a unitary, then $E(1) = 1 = E(uu^*) = E(u)E(u)^*$ and similarly the other direction.
   Conversely, if $E(u)$ is unitary for a unitary $u \in A$, then it is easy to see that $u$ satisfies the equality in the Schwarz inequality as well and hence we get $u \in M_E$.

Assumptions.
1. With the exception of example 1.6, all quantum channels considered henceforth in this paper are assumed to act on the matrix algebra $M_d$.
2. Given a quantum channel $E : M_d \to M_d$, we can identify its dual map or adjoint map $E^*$ via the relation $\text{tr}(E(ab)) = \text{tr}(aE(b)) \forall a, b \in M_d$ where $a, b = \text{tr}(ab^*)$ for all $a, b \in M_d$, defines an inner product on $M_d$ which makes $M_d$ a Hilbert space. This is known as the Hilbert–Schmidt inner product. We will frequently denote the norm of an element $x \in M_d$ arising from this inner product as $\|x\|_{HS}^2 := \langle x, x \rangle = \text{tr}(xx^*)$.

We are now ready to state the following theorem. The result has been obtained before (see [14, 28]) but we present a different proof here. The technique used in this proof will be used extensively throughout the rest of the paper.

Theorem 1.3. Let $E : M_d \to M_d$ be a unital quantum channel. Then
$$M_E = F(E \circ E),$$
that is, the multiplicative domain of $E$ is equal to the fixed point set of $(E^* \circ E)$.

Proof. If $x \in M_E$, then $E(xy) = E(x)E(y)$ and $E(yx) = E(y)E(x)$ for all $y \in M_d$.

We then have, for any $z \in M_d$,
$$\langle x, z \rangle = \text{tr}(z^*x) = \text{tr}(E(z^*x)) = \text{tr}(E(z)^*E(x)) = \langle E(x), E(z) \rangle.$$ Invoking the adjoint relation, we have
$$\langle x, z \rangle = \langle E^* \circ E(x), z \rangle.$$ Since this happens for all $z \in M_d$, by the non-degeneracy of the pairing we have $E^* \circ E(x) = x$.

Conversely, if $x \in M_d$ is such that $E^* \circ E(x) = x$, then
$$\langle x, x \rangle = \text{tr}(x^*x) = \text{tr}(E(x^*x)) \geq \text{tr}(E(x)^*E(x)) = \langle E(x), E(x) \rangle = \langle E^* \circ E(x), x \rangle = \langle x, x \rangle.$$
where the inequality arises from the Schwarz inequality for the ucp map $E$. Since the extreme ends of the above equations are the same, the inequalities become an equality. So we have $\text{tr}(E(x^*x)) = \text{tr}(E(x)^*E(x))$, and hence by the faithfulness of the trace we have $E(x^*x) = E(x)^*E(x)$. Now by theorem 1.1, we conclude $x \in \mathcal{M}_E$.

Now we relate the multiplicative domain with the commutant of the product of Kraus operators of a unital channel.

**Corollary 1.4 (Commutant of the Kraus operators).** Let $E : \mathcal{M}_d \to \mathcal{M}_d$ be a unital quantum channel with the Kraus representation: $E(x) = \sum_{j=1}^n a_j x a_j^*$, for all $x \in \mathcal{M}_d$. Then an element $\rho \in \mathcal{M}_E$ if and only if $\rho a_i^* a_j = a_i^* a_j \rho$ for all $i, j = 1, 2, \cdots, n$.

**Proof.** Let us note that if $E$ is a unital channel, then $E^* \circ E$ is a unital channel as well. Since $E$ is completely positive and trace preserving, $E^*$ is unital because $\text{tr}(x) = \text{tr}(E(x) \cdot 1) = \text{tr}(x E^*(1))$ for all $x \in \mathcal{B}(\mathcal{H})$ and hence $E^*(1) = 1$. If $E$ is completely positive, then so is $E^*$ (see [39]). Also since $E$ is unital, $E^*$ is trace preserving because $\text{tr}(E^*(x)) = \text{tr}(E^*(x)1) = \text{tr}(xE(1)) = \text{tr}(x)$, and hence $E^*$ is a ucp and trace preserving map and hence the composition $E^* \circ E$ is a unital channel. Now if $E(x) = \sum_{j=1}^n a_j x a_j^*$, for all $x \in \mathcal{M}_d$ is a Kraus representation, a small calculation shows that $E^* \circ E$ is represented by $E^* \circ E(x) = \sum_{i,j=1}^n a_i^* a_j x a_j^* a_i$, and so the Kraus operators for $E^* \circ E$ are $\{a_i^* a_j\}_{i,j}$. Now by the fixed point and commutant theorem ([27], theorem 2.1) and by the theorem 1.3, we get $\rho \in \mathcal{M}_E$ if and only if $\rho a_i^* a_j = a_i^* a_j \rho$, $\forall i, j$.

**Remark 1.5.** Trace preservation of $E$ (equivalently, the unitality of $E^*$) is the key factor of the above theorem. Below we give an example where an element lies in the multiplicative domain of a ucp (but not trace preserving) map with a given Kraus representation that does not satisfy the commutant condition. We use the example of a ucp map that arose in [7].

Let $\Phi : \mathcal{M}_3 \to \mathcal{M}_3$ be a ucp map given by

$$
\Phi \left( \begin{bmatrix} x_{11} & x_{12} & x_{13} \\
 x_{21} & x_{22} & x_{23} \\
 x_{31} & x_{32} & x_{33} \end{bmatrix} \right) = \begin{bmatrix} x_{11} & 0 & 0 \\
 0 & x_{22} & 0 \\
 0 & 0 & \frac{x_{11} + x_{22}}{2} \end{bmatrix}
$$

Let us choose a set of Kraus operators for $\Phi$ as follows

$$
k_1 = \begin{bmatrix} 1 & 0 & 0 \\
 0 & 0 & 0 \\
 0 & 0 & 0 \end{bmatrix},
k_2 = \begin{bmatrix} 0 & 0 & 0 \\
 0 & 1 & 0 \\
 0 & 0 & 0 \end{bmatrix},
k_3 = \begin{bmatrix} 0 & 0 & 0 \\
 0 & 0 & 0 \\
 \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix},
k_4 = \begin{bmatrix} 0 & 0 & 0 \\
 0 & 0 & 0 \\
 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}
$$
Now note that \( a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathcal{M}_\Phi \) because \( \Phi(a) = 1 \) where 1 denotes the \( 3 \times 3 \) identity matrix and so \( \Phi(a^*) = 1 \). Moreover \( \Phi(aa^*) = \Phi(a) = 1 \) and \( \Phi(a)\Phi(a^*) = 1 \), and hence \( a \in \mathcal{M}_\Phi \).

But \( a \) does not commute with \( k_3 k_3^* = k_3 \).

In an infinite dimension, corollary 1.4 is not true, as the following example suggests. This example was first given in [2] within the context of proving that the fixed point set is not necessarily equal to the commutant of the algebra generated by the Kraus operators of a channel. Since our context is similar, we use this example and expand it accordingly. Recall, in an infinite dimension, the issue of trace preserving needs to be addressed as the algebra in context might not have a finite trace. To this end, we follow the notion of quantum operation given in [2]. For a Hilbert space \( \mathcal{H} \), if \( \mathcal{B}(\mathcal{H}) \) is a set of bounded linear operators, then a linear map \( \mathcal{E} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \) which is induced by a set of operators \( \{a_i\}_{i=1}^\infty \) and defined as \( \mathcal{E}(x) = \sum_{i=1}^\infty a_i x a_i^* \) is called trace preserving if \( \text{tr}(\mathcal{E}(b)) = \text{tr}(b) \), for every trace class operator \( b \). It turns out that if \( \sum_{i=1}^\infty a_i^* a_i = 1 \), where the convergence is in the ultra-weak topology, then \( \mathcal{E} \) is trace preserving. With these notations in hand, a quantum operation is a completely positive trace preserving map.

**Example 1.6.** Let \( \mathbb{F}_2 \) be the free group of two generators \( g_1, g_2 \), with the identity element \( e \). So \( \mathbb{F}_2 \) is a countable group. Let \( \mathcal{H} \) be the complex separable Hilbert space

\[
\mathcal{H} = \{ f : \mathbb{F}_2 \to \mathbb{C} : \sum_x |f(x)|^2 \leq \infty \}.
\]

Now define the following function for \( x \in \mathbb{F}_2 \),

\[
\delta_x(y) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{otherwise}. \end{cases}
\]

Then \( \{\delta_x : x \in \mathbb{F}_2 \} \) is an orthonormal basis for \( \mathcal{H} \). It is well known that the group \( C^* \)-algebra \( C^*(\mathbb{F}_2) \) corresponding to the left regular representation \( \Gamma : \mathbb{F}_2 \to \mathcal{B}(\mathcal{H}) \) has a faithful trace \( \tau \), defined by \( \tau(x) = \langle x\delta_e, \delta_e \rangle \).

Define two unitary operators on \( \mathcal{H} \) by \( u, v \) in the following way:

\[
u(\delta_x) = \delta_{g_1x} \text{ and } v(\delta_x) = \delta_{g_2x} \text{ for all } x \in \mathbb{F}_2.
\]

Now define \( \mathcal{E} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \) by

\[
\mathcal{E}(a) = \frac{1}{2} u a u^* + \frac{1}{2} v a^* v \text{ for all } a \in \mathcal{B}(\mathcal{H}).
\]

Then \( \mathcal{E} \) defines a ucp and trace preserving map. Let us call \( \mathcal{M} \) the von Neumann algebra generated by \( \{u^*v, v^*u, 1\} \).

Suppose an operator \( b \) is defined as \( b(\delta_x) = \lambda_x \delta_x \) for all \( x \in \mathbb{F}_2 \) and \( \lambda_x \in [0, 1] \). The operator \( b \) is positive and in the multiplicative domain \( \mathcal{M}_\mathcal{E} \) of \( \mathcal{E} \) if and only if we have \( \mathcal{E}(bb^*) = \mathcal{E}(b)\mathcal{E}(b)^* \) which yields for all \( x \in \mathbb{F}_2 \), \( \mathcal{E}(bb^*)(\delta_x) = \mathcal{E}(b)\mathcal{E}(b)^*(\delta_x) \). Unwinding the definition of \( \mathcal{E} \), we get

\[
\frac{1}{2} (ubb^* u^* + vbb^* v^*)(\delta_x) = \frac{1}{2} (ubu^* + vbu^*)\frac{1}{2} (ub^* u^* + vb^* v^*)(\delta_x).
\]
Now applying the definition of \( u, v \) and \( b \), we get

\[
\left( \frac{1}{2} \lambda_2 g_1^{-1} + \frac{1}{2} \lambda_2 g_2^{-1} \right) = \left( \frac{1}{2} \lambda_1 + \frac{1}{2} \lambda_1 \right)^2 \quad \text{for all } x \in \mathbb{F}_2.
\] (1)

Now, if we assume \( b \in \mathcal{M}' \), and if \( x = g_1^{-1} g_2 y \) for some \( y \in \mathbb{F}_2 \), then we have,

\[
\lambda_0 \delta_x = b \delta_x = b \delta_1^{-1} g_1^{-1} g_2 y = bu^* v \delta_y = \lambda_1 u^* v \delta_y = \lambda_1 \delta_y,
\]

and hence

\[
\lambda_1^{-1} g_1^{-1} g_2 y = \lambda_1, \quad \forall y \in \mathbb{F}_2.
\] (2)

If \( b \) is defined as \( b(\delta_x) = \lambda_0 \delta_x \), where

\[
\lambda_x = \begin{cases} 
0 & \text{if } x \text{ ends with } g_2^{-1} \\
1 & \text{if } x \text{ ends with } g_1^{-1} \\
\frac{1}{2} & \text{otherwise}
\end{cases}
\]

then one can check that \( b \) satisfies equation (1) and hence \( b \in \mathcal{M}_E \). But \( b \notin \mathcal{M}' \) because otherwise, we saw from equation (2), \( \lambda_1^{-1} g_2 y = \lambda_1 \) for all \( y \in \mathbb{F}_2 \). Now putting \( y = g_2^{-1} \) we have \( 1 = \lambda_1 g_2^{-1} = \lambda_1 g_2^{-1} = 0 \), a contradiction.

## 2. Multiplicative index of a unital quantum channel

In this section we discover more intrinsic properties of a unital channel concerning its multiplicative behaviour. The relationship between spectral properties and multiplicative nature will be explored but first we start with the following lemma which will be useful in the subsequent discussion.

**Lemma 2.1.** For a unital quantum channel \( \mathcal{E} \), we have

\[
\mathcal{M}_{\mathcal{E}^* \circ \mathcal{E}} = \mathcal{F}_{\mathcal{E}^* \circ \mathcal{E}} = \mathcal{M}_E.
\]

**Proof.** The last equality is from theorem 1.3. We will establish the first equality of sets. Since the fixed point set is a subalgebra of the multiplicative domain, we automatically have \( \mathcal{F}_{\mathcal{E}^* \circ \mathcal{E}} \subseteq \mathcal{M}_{\mathcal{E}^* \circ \mathcal{E}} \). For the converse part, let \( a \in \mathcal{M}_{\mathcal{E}^* \circ \mathcal{E}} \). So it gives equality in the Schwarz inequality for the map \( \mathcal{E}^* \circ \mathcal{E} \) and we get

\[
\mathcal{E}^* \circ \mathcal{E}(aa^*) = \mathcal{E}^* \circ \mathcal{E}(a)\mathcal{E}^* \circ \mathcal{E}(a^*).
\] (3)

Now we have

\[
\text{tr}(aa^*) = \text{tr}(\mathcal{E}^* \circ \mathcal{E}(aa^*)) \\
\geq \text{tr}(\mathcal{E}^* \mathcal{E}(a)\mathcal{E}(a^*)) \\
\geq \text{tr}(\mathcal{E}^* \mathcal{E}(a)\mathcal{E}^* \mathcal{E}(a^*)) \\
= \text{tr}(\mathcal{E}^* \circ \mathcal{E}(aa^*)) \\
= \text{tr}(aa^*)
\]
where the first two inequalities follow from the Schwarz inequality for \( E \) and \( E^* \) and then we have used the relation in equation (3). Since the extreme ends of the above equation are the same, we have equalities in all the inequalities. This gives \( \text{tr}(E^* \circ E(aa^*)) = \text{tr}(E^*(a)E(a^*)) \).

Since \( E^* \) is trace preserving, we get \( \text{tr}(E(aa^*)) = \text{tr}(E(a)E(a^*)) \). Hence by the faithfulness of the trace, we get \( E(aa^*) = E(a)E(a^*) \) and \( a \in M_E = F_{E^* \circ E} \).

Given a linear map \( \Phi : M_d \rightarrow M_d \), the spectrum of \( \Phi \), which is denoted by \( \text{Spec}(\Phi) \), is defined as

\[
\text{Spec}(\Phi) = \{ \lambda \in \mathbb{C} : (\lambda 1 - \Phi) \text{ is not invertible on } M_d \},
\]

where \( 1 \) denotes the identity operator on \( M_d \). Recall that the spectral radius of \( \Phi \), which is denoted as \( r(\Phi) \), is defined as

\[
r(\Phi) = \sup\{ |\lambda| : \lambda \in \text{Spec}(\Phi) \}.
\]

It follows that if \( \Phi \) is a unital positive map, then \( r(\Phi) \leq 1 \) and hence all eigenvalues lie in the unit disc of the complex plane (see proposition 6.1 in [43]). For a unital channel \( \Phi \) the set \( \text{Spec}(\Phi) \cap \mathbb{T} \) is called the set of peripheral eigenvalues, where \( \mathbb{T} \) is the unit circle in the complex plane. If an element \( a \in M_d \) satisfies the relation \( \Phi(a) = \lambda a \) for \( |\lambda| = 1 \), then \( a \) is called a peripheral eigenvector of \( \Phi \). Note that the fixed point set of \( \Phi \), that is \( F_\Phi \), is the set of all peripheral eigenvectors corresponding to the eigenvalue 1. With this terminology in hand, we are ready to note down the corollary to lemma 2.1.

**Corollary 2.2.** For any unital channel \( E \), the channel \( E^* \circ E \) does not have any peripheral eigenvalue other than 1.

**Proof.** We will show that any peripheral eigenvector corresponding to a peripheral eigenvalue of a unital channel \( \Phi \) is in the multiplicative domain \( M_\Phi \). To this end, let \( \lambda (\neq 1) \) be a peripheral eigenvalue of \( \Phi \), that is \( |\lambda| = 1 \) and suppose \( a \in M_d \) is such that \( \Phi(a) = \lambda a \). Now it follows from the positivity of \( \Phi \) that \( \Phi \) preserves the \( * \)-operation, that is, \( \Phi(x^*) = \Phi(x)^* \), for every \( x \in M_d \). Hence \( \Phi(a^*) = \lambda a^* \). We get

\[
\Phi(aa^*) \geq \Phi(a)\Phi(a^*) = \lambda a\lambda a^* = aa^*.
\]

Using the trace preservation and the faithfulness of the trace, we get \( \Phi(aa^*) = aa^* \) and hence \( a \in M_\Phi \).

Now for a unital channel \( E \), if \( a \) is a peripheral eigenvector of \( E^* \circ E \) corresponding to a peripheral eigenvalue \( \lambda (\neq 1) \), then \( a \in M_{E^* \circ E} \). But lemma 2.1 asserts that \( M_{E^* \circ E} = F_{E^* \circ E} \), which implies that \( a \) is an eigenvector of \( E^* \circ E \) corresponding to the eigenvalue 1. Hence we get a contradiction.

The next lemma sets the foundation of the concept of the multiplicative index of a quantum channel. It gives the description of the multiplicative domain of a composition of two quantum channels.

**Lemma 2.3.** Let \( E = \Phi \circ \Psi \) where \( \Phi, \Psi \) are two unital quantum channels. Then

\[
M_E = \{ a \in M_\Psi : \Psi(a) \in M_\Phi \}.
\]

**Proof.** If \( a \in M_\Psi \) such that \( \Psi(a) \in M_\Phi \), then

\[
E(aa^*) = \Phi(\Psi(aa^*)) = \Phi(\Psi(a)\Psi(a^*)) = \Phi \circ \Psi(a)\Phi \circ \Psi(a^*) = E(a)E(a^*)
\]
and hence $a \in \mathcal{M}_E$.

Conversely, let $a \in \mathcal{M}_E$. Then
\[
\mathcal{E}(a)\mathcal{E}(a^*) = \Phi \circ \Psi(aa^*) \\
\geq \Phi(\Psi(a))\Psi(a^*) \\
\geq \Phi(\Psi(a))\Phi(\Psi(a^*)) \\
= \mathcal{E}(a)\mathcal{E}(a^*).
\]
Hence all the inequalities must be equalities and we first get $\Phi \circ \Psi(aa^*) = \Phi(\Psi(a))\Psi(a^*)$. Now the trace preservation property of $\Phi$ would imply $\Psi(aa^*) = \Psi(a)\Psi(a^*)$ and we get $a \in \mathcal{M}_\Phi$. Using the second inequality, it is immediately apparent that $\Psi(aa^*) = \Psi(a)\Psi(a^*)$ and we get $a \in \mathcal{M}_\Psi$. □

Now we can proceed in exploring the multiplicative domain of the compositions of a channel $\mathcal{E}$ with itself.

**Corollary 2.4.** For $\mathcal{E}^n = \mathcal{E} \circ \cdots \circ \mathcal{E}$ (n-times, $n \in \mathbb{N}$), we have
\[
\mathcal{M}_{\mathcal{E}^n} = \{a \in \mathcal{M}_{\mathcal{E}^{n-1}} : \mathcal{E}(a) \in \mathcal{M}_{\mathcal{E}^{n-1}}\}.
\]
In particular, we have the following inclusion
\[
\mathcal{M}_E \supseteq \mathcal{M}_{\mathcal{E}^2} \supseteq \cdots \supseteq \mathcal{M}_{\mathcal{E}^n} \supseteq \cdots.
\]
Now since the multiplicative domain of $\mathcal{E}^n$ is a $C^*$-algebra for each $n \in \mathbb{N}$ and the underlying subspace is of finite dimension, the above decreasing chain of subalgebras will stabilise at a fixed subalgebra. Let us denote this subalgebra as $\mathcal{M}_E^\infty$, i.e.
\[
\mathcal{M}_E^\infty = \bigcap_{n=1}^\infty \mathcal{M}_{\mathcal{E}^n}.
\]
We will see that on this subalgebra $\mathcal{M}_E^\infty$, $\mathcal{E}$ acts as an automorphism. Also, it is not necessarily true that if $a \in \mathcal{M}_E$, then $\mathcal{E}(a) \in \mathcal{M}_E$. However, it will be evident that if $a \in \mathcal{M}_E^\infty$, then $\mathcal{E}(a) \in \mathcal{M}_E^\infty$. Note that Stormer in [37] deals with a set related to a positive map, which he calls the multiplicative core and proves that the positive linear map, when restricted to this set, is a Jordan automorphism. In our context, the underlying space is of finite dimension and the linear map is completely positive which is stronger than positivity. The indispensable effect of the adjoint of a quantum channel in this whole discussion about multiplicative property might be traced back to the work of Kümmnerer in [29] (section 3.2), where the author deals with dilations of asymptotic automorphic dynamical systems. Now we state and prove the main theorem of this section.

**Theorem 2.5.** Let $\mathcal{E} : \mathcal{M}_d \to \mathcal{M}_d$ be a unital quantum channel. Then

1. There exists a subalgebra of $\mathcal{M}_d$, namely $\mathcal{M}_E^\infty$, upon which $\mathcal{E}$ acts as a bijective homomorphism with the inverse being the adjoint $\mathcal{E}^*$.
2. $\mathcal{M}_d$ decomposes into two orthogonal subspaces as $\mathcal{M}_d = \mathcal{M}_E^\infty \bigoplus \mathcal{M}_{\mathcal{E}^\infty}$, where the orthogonality is with respect to the Hilbert–Schmidt inner product. Moreover, we have a precise description for the set $\mathcal{M}_{\mathcal{E}^\infty}$ given by
\[
\mathcal{M}_{\mathcal{E}^\infty} = \{x \in \mathcal{M}_d : \lim_{n \to \infty} \|\mathcal{E}^n(x)\| = 0\}.
\]
3. The spectrum of this automorphism is equal to the peripheral spectrum of $\mathcal{E}$, that is $\text{Spec}(\mathcal{E}|_{\mathcal{M}_E^\infty}) = \text{Spec}(\mathcal{E}) \cap \mathbb{T}$. Moreover, if $\mathcal{N}_E = \{a \in \mathcal{M}_d : \mathcal{E}(a) = \lambda a, |\lambda| = 1\}$, then $\mathcal{M}_E^\infty$ is the algebra generated by $\mathcal{N}_E$. 

M Rahaman

J. Phys. A: Math. Theor. 50 (2017) 345302
Proof.

1. We will first show that \( \mathcal{E}(M_{E^\infty}) \subseteq M_{E^\infty} \). To see this, let \( a \in M_{E^\infty} \). For any \( k \in \mathbb{N} \), we have

\[
E^{k+1}(aa^*) = E^k(E(aa^*)) = E^k(E(a)E(a^*)) \\
\geq E^kE(a)E^kE(a^*) \\
= E^{k+1}(a)E^{k+1}(a^*) \\
= E^{k+1}(aa^*)
\]

where we have just used the Schwarz inequality for the map \( E^k \) and the equality follows because of the fact that \( a \in M_{E^\infty} = \bigcap_{k=1}^{\infty} M_{E^k} \). Clearly, we have equality in all the inequalities and we obtain \( E^k(E(a)E(a^*)) = E^k(E(a))E^k(E(a^*)) \). Hence \( E(a) \in M_{E^k} \) for every \( k \in \mathbb{N} \) and we get \( E(a) \in M_{E^\infty} \).

Injectivity follows easily; let \( a \in M_{E^\infty} \) such that \( E(a) = 0 \), then

\[
\text{tr}(aa^*) = \text{tr}(E(aa^*)) = \text{tr}(E(a)E(a^*)) = 0,
\]

which forces \( a = 0 \). So \( E : M_{E^\infty} \to M_{E^\infty} \) is an injective linear map. Since \( M_{E^\infty} \) is finite dimensional vector space, the rank-nullity theorem asserts that \( E \) is surjective.

Now we prove that the inverse of \( E \big|_{M_{E^\infty}} \) is \( E^* \). From theorem 1.3, it is evident that on \( M_{E^\infty} \subseteq M_E \), \( E^* \circ E = 1 \) where 1 is the identity operator. As \( E \) is bijective on \( M_{E^\infty} \), for any \( a \in M_{E^\infty} \), there exists an element \( b \in M_{E^\infty} \) such that \( E(b) = a \). Applying the adjoint on both sides we get \( E^*E(b) = E^*(a) \). But \( M_{E^\infty} \) is a subset of the fixed point of \( E^* \circ E \) and hence we get \( b = E^*(a) \). As \( a \) was arbitrary, we have proved \( E^*(M_{E^\infty}) \subseteq M_{E^\infty} \).

One can show similarly, as was shown for \( E \), that \( E^* \) is a bijective homomorphism on \( M_{E^\infty} \). Now to show that \( E^* \) is right inverse, we let \( a \in M_{E^\infty} \) and find a \( b \in M_{E^\infty} \) such that \( E(b) = a \) and hence \( E \circ E^*(a) = E \circ E^*(E(b)) = E(b) = a \). So on \( M_{E^\infty} \), we get \( E \circ E^* = E^* \circ E = 1 \).

2. The decomposition of \( M_d \) is now clear since \( M_{E^\infty} \) is an invariant subspace for both \( E \) and \( E^* \), as was shown in part 1 of the proof. Now to obtain the characterisation of \( M_{E^\infty} \), we follow the method taken by Størmer in [37]. Let \( a \in M_{E^\infty} \). Then we compute for any \( n \in \mathbb{N} \),

\[
\|E^{n+1}(a)\|_{HS}^2 = \text{tr}(E^{n+1}(a)E^{n+1}(a^*)) \\
\leq \text{tr}(E(E^n(a))E^n(a^*)) \\
= \text{tr}(E^n(a)E^n(a^*)) \\
= \|E^n(a)\|_{HS}^2
\]

so \( \{\|E^n(a)\|_{HS}^2\}_{n=1}^{\infty} \) is a decreasing sequence and suppose \( E^n(a) \to a_0 \) in the H.S sense.

We will show that \( a_0 \in M_{E^\infty} \). Since \( \{\|E^n(a)\|_{HS}^2\}_{n=1}^{\infty} \) is decreasing, we have

\[
\|E^n(a)\|_{HS}^2 - \|E^{n+1}(a)\|_{HS}^2 \to 0 \text{ as } n \to \infty.
\]  

(4)

By Schwarz inequality we see that

\[
E(E^n(a)E^n(a^*)) - E^{n+1}(a)E^{n+1}(a^*) \geq 0.
\]
Taking the trace and using equation (4), we obtain $\mathcal{E}(\mathcal{E}^n(a)\mathcal{E}^n(a^*) ) = \mathcal{E}^{n+1}(a)\mathcal{E}^{n+1}(a^*)$ as $n \to \infty$, i.e. $\lim_{n \to \infty} \mathcal{E}^n(a) \in \mathcal{M}_E$. Using this argument repeatedly we can show $\lim_{n \to \infty} \mathcal{E}^n(a) \in \mathcal{M}_E^n$ for every $k \geq 1$, and we obtain $\lim_{n \to \infty} \mathcal{E}^n(a) \in \mathcal{M}_E^\infty$. Since the underlying space is a finite dimensional $C^*$-algebra, the weak limit is the norm limit and because $\mathcal{M}_E^\infty$ is a $C^*$-algebra, we find $a_0 \in \mathcal{M}_E^\infty$.

Lastly, if $a \in \mathcal{M}_E^\infty$, then $\mathcal{E}^k(a) \in \mathcal{M}_E^\infty$ for any $k \geq 1$. Indeed, note that for every such $k$, $\mathcal{E}^k$ is bijective on $\mathcal{M}_E^\infty$ and for any $b \in \mathcal{M}_E^\infty$, there exists an element $c \in \mathcal{M}_E^\infty$ such that $\mathcal{E}^k(c) = b$. So for any $b \in \mathcal{M}_E^\infty$, we have

$$\text{tr}(\mathcal{E}^k(a)b) = \text{tr}(\mathcal{E}^k(a)\mathcal{E}^k(c)) = \text{tr}(\mathcal{E}^k(ac)) = \text{tr}(ac) = 0.$$ 

Hence $\mathcal{E}^k(a) \in \mathcal{M}_E^\infty$ for all $k \geq 1$. So we have $a_0 \in \mathcal{M}_E^\infty \cap \mathcal{M}_E^\infty$, which forces $a_0 = 0$. Hence we get

$$\mathcal{M}_E^\infty = \{ x \in \mathcal{M}_E : \lim_{n \to \infty} \| \mathcal{E}^n(x) \|_{HS} = 0 \}.$$ 

As $\| \cdot \| \leq \| \cdot \|_{HS}$, we have the desired description of the set.

3. We first prove that the eigen operators corresponding to the peripheral eigenvalues algebraically span the set $\mathcal{M}_E^\infty$. It is not hard to see that $\mathcal{N}_E \subseteq \mathcal{M}_E^\infty$. Indeed if $\mathcal{E}(a) = \lambda a$ where $|\lambda| = 1$, then we get $\mathcal{E}^k(a) = \lambda^k a$, for any $k \geq 1$. Hence $\mathcal{E}^k(aa^*) \geq \mathcal{E}^k(a)\mathcal{E}^k(a^*) = \lambda^k \bar{\lambda} a a^* = aa^*$. Taking trace and using the faithfulness of the trace, we get $\mathcal{E}^k(aa^*) = \mathcal{E}^k(a)\mathcal{E}^k(a^*)$. Hence it follows that $a \in \mathcal{M}_E^\infty$ for all $k \geq 1$ and subsequently, $a \in \mathcal{M}_E^\infty$. Hence the algebra generated by $\mathcal{N}_E$ is contained in $\mathcal{M}_E^\infty$. Conversely, the map $\mathcal{E} : \mathcal{M}_E^\infty \to \mathcal{M}_E^\infty$ is a bijection which satisfies $\mathcal{E} \circ \mathcal{E}^* = \mathcal{E}^* \circ \mathcal{E} = 1$, that is a unitary operator on the Hilbert subspace $\mathcal{M}_E^\infty$ equipped with the Hilbert–Schmidt inner product. In particular, $\mathcal{E}$ is a normal operator and hence diagonalisable, that is, there exists a basis of eigenvectors of $\mathcal{E}$ which span the entire space $\mathcal{M}_E^\infty$. Now if $a \in \mathcal{M}_E^\infty$ such that $\mathcal{E}(a) = \lambda a$, then $\mathcal{E}(aa^*) = \mathcal{E}(a)\mathcal{E}(a^*) = \lambda \bar{\lambda} a a^*$. Taking the trace in both sides we see that $\lambda \bar{\lambda} = |\lambda|^2 = 1$, that is, $a$ is one of the peripheral eigen operators of $\mathcal{E}$. So, $\mathcal{M}_E^\infty$ is spanned by the eigen operators corresponding to the peripheral eigenvalues of $\mathcal{E}$ showing $\mathcal{M}_E^\infty$ is contained in the algebra spanned by $\mathcal{N}_E$ and hence we get the required equality. It is now obvious that $\text{Spec}(\mathcal{E}|_{\mathcal{M}_E^\infty}) = \text{Spec}(\mathcal{E}) \cap T$.  

**Remark 2.6.** Theorem 2.5 displays the stable behaviour of the channel on the algebra generated by the peripheral eigen-operators, and on its complementary part, $\mathcal{E}$ asymptotically approaches zero. In [4], similar results were obtained for a uc map (not necessarily trace preserving) where the given map acts as an isometry on the operator system spanned by $\mathcal{N}_E$. (See also [11] for a discussion relating to quantum Markov semigroups.) Our attention has been focused on finite dimensional $C^*$-algebra and the maps $\mathcal{E}$ were uc and trace preserving. The existence of the adjoint $\mathcal{E}^*$ and subsequently the identity $\mathcal{M}_E = \mathcal{F}_{\mathcal{E}^*\mathcal{E}}$ helps to provide a very different approach to this topic from the above-mentioned cases.

**Remark 2.7.** The decreasing chain of subalgebras $\mathcal{M}_E \supseteq \mathcal{M}_E^2 \supseteq \cdots \supseteq \mathcal{M}_E^n \supseteq \cdots$ must stabilise at a finite stage because the underlying space is of finite dimension. Call $\kappa$ the smallest number for which we have $\mathcal{M}_E^n = \mathcal{M}_E^{n+1}$, $n \geq \kappa$. It is evident that $\mathcal{M}_E^\kappa = \mathcal{M}_E^\infty$. This $\kappa$ is uniquely determined for every quantum channel $\mathcal{E}$ and can be used to differentiate between two quantum channels. We call this $\kappa$ the *multiplicative index* of $\mathcal{E}$.
Now we note down the following corollary which will be useful in future reference. Note that a channel is called normal or diagonalisable if $E^* \circ E = E \circ E^*$. 

**Corollary 2.8.** If a unital channel $E$ satisfies $E^* \circ E = E \circ E^*$, then $M_{E^\infty} = M_E$. Hence the multiplicative index of such channels is 1.

**Proof.** If $a \in M_E = F_{E^* \circ E}$, then $E^* \circ E(a) = a$ and applying $E^* \circ E$ once again we get $E^* \circ E(E^* \circ E(a)) = a$. Now using the commutativity, we get $(E^*)^2 \circ E^2(a) = a$. We will show $M_{E^2} = M_E$.

For $a \in M_E$, we have

$$\text{tr}(aa^*) = \text{tr}(E^2(aa^*)) = \text{tr}(E(E(a)E(a^*)))$$

$$\geq \text{tr}(E^2(a)E^2(a^*))$$

$$= \text{tr}((E^*)^2E^2(a)a^*)$$

$$= \text{tr}(aa^*)$$

so the inequality must be an equality and hence we get $E(a) \in M_E$ which yields $a \in M_{E^2}$. Now by corollary 2.4, we get $M_{E^2} = M_E$. This process can be repeated for every $n \in \mathbb{N}$ to get $a \in M_{E^n}$, which forces $a \in M_{E^\infty}$. $\square$

Recall that the qubit Pauli operators are described by the following $2 \times 2$ matrices

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$ 

Channels whose Kraus decompositions consist of Pauli operators are called Pauli Channels. The generalised Pauli channels in dimension $d$ consist of random mixtures of unitaries in the discrete Weyl–Heisenberg representation. The Pauli or generalised Pauli channels are diagonalisable ([10]) and hence they all have a multiplicative index of 1.

**Remark 2.9.** By theorem 2.5, we get a decomposition of the unital channel $E$ as follows

$$E = \begin{pmatrix} \mathcal{E}_0 & 0 \\ 0 & \mathcal{E}_1 \end{pmatrix}$$

where $\mathcal{E}_0$ is the automorphism on $M_{E^\infty}$ and $\mathcal{E}_1$ is another quantum channel.

This decomposition of $E$ gives more information than the Jordan decomposition for $E$. Recall that a linear operator $\Phi : M_d \rightarrow M_d$, regarded as an element of $M_{d^2}$, admits a Jordan decomposition of the form

$$\Phi = P \left( \bigoplus J_k(\lambda_k) \right) P^{-1}, \quad J_k(\lambda) = \begin{pmatrix} \lambda & 1 \\ \vdots & \ddots & 1 \\ & & \lambda \end{pmatrix},$$

where the $J_k$s are Jordan blocks of size $d_k$ and $\sum_k d_k = d^2$. From the Jordan decomposition it follows that each of the peripheral eigenvalues for a trace preserving or unital positive map $\Phi$, has one-dimensional Jordan blocks (see for example [43], proposition 6.2). Although the Jordan decomposition is very useful in studying eigen values and their locations, the multiplicative nature of a channel cannot be investigated by just looking at the Jordan decomposition.

Now we enlist some examples showing the different values of the multiplicative index $\kappa$ of unital channels and the corresponding algebras $M_{E^\infty}$. In what follows $e_1, \ldots, e_d$ denote the standard basis of $\mathbb{C}^d$. For any two vectors $\xi, \eta \in \mathbb{C}^d$, the rank one operator $\xi \eta^* : \mathbb{C}^d \rightarrow \mathbb{C}^d$ is defined by $\xi \eta^*(x) = \langle x, \eta \rangle \xi$, for all $x \in \mathbb{C}^d$. 

J. Phys. A: Math. Theor. 50 (2017) 345302
Example 2.10. If $\mathcal{E}$ is a unitary channel that is $\mathcal{E}(x) = u x u^*$ for some unitary $u$ and for all $x \in \mathcal{M}_d$, then $\mathcal{E}$ is multiplicative in the whole domain and $\mathcal{M}_{E^\infty} = \mathcal{M}_d$ and $\kappa = 1$.

Example 2.11. Let $\omega \in \mathbb{C}$ be such that $\omega^3 = 1$. Define a quantum channel $\mathcal{E} : \mathcal{M}_3 \to \mathcal{M}_3$ in the following way:

$$\mathcal{E}(x) = \sum_{j=1}^{3} s_j x s_j^*,$$

for all $x \in \mathcal{M}_3$, where

$$s_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad s_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 0 & 1 & 0 \\ 0 & \omega & 0 \\ 0 & \omega^2 & 0 \end{bmatrix}, \quad s_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & \omega \\ 0 & 0 & \omega^2 \end{bmatrix}.$$

The calculation shows that

$$\mathcal{E}^* \circ \mathcal{E}(x) = \sum_{j=1}^{3} a_j x a_j^*,$$

where $a_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $a_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $a_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Clearly $\mathcal{M}_\mathcal{E} = \mathcal{F}_{\mathcal{E}^* \circ \mathcal{E}} = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} : a, b, c \in \mathbb{C} \right\}$, which is the algebra of diagonal matrices, a commutative $\mathbb{C}^*$-algebra.

Now $\mathcal{E}^2(x) = \sum_{j=1}^{9} b_j x b_j^*$, where

$$b_1 = \frac{1}{3} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad b_2 = \frac{1}{3} \begin{bmatrix} 0 & 0 & 0 \\ \omega & 0 & 0 \\ \omega^2 & 0 & 0 \end{bmatrix}, \quad b_3 = \frac{1}{3} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad b_4 = \frac{1}{3} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & \omega^2 & 0 \end{bmatrix}, \quad b_5 = \frac{1}{3} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & \omega & 0 \end{bmatrix}, \quad b_6 = \frac{1}{3} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Actually $\mathcal{E}^2(x) = \frac{\text{tr}(x)}{3}$ for all $x \in \mathcal{M}_3$. We have $\mathcal{M}_{\mathcal{E}^2} = \mathcal{A}$, where $\mathcal{A}$ is the algebra generated by the set $\{ b_i b_j : i, j = 1, \cdots, 9 \}$. Now since $\mathcal{A}$ generates the full matrix algebra $\mathcal{M}_3$, we get $\mathcal{A} = \mathbb{C} 1$. So $\mathcal{M}_{\mathcal{E}^2} = \mathbb{C} 1$. Hence for every $n \geq 2$, $\mathcal{M}_{\mathcal{E}^n} = \mathbb{C} 1$, resulting in $\mathcal{M}_{\mathcal{E}^\infty} = \mathcal{M}_{\mathcal{E}^2} = \mathbb{C} 1$. Hence the multiplicative index $\kappa = 2$ and also we have found $\mathcal{M}_{\mathcal{E}^\infty} \subset \mathcal{M}_{\mathcal{E}}$. 

13
**Example 2.12.** The above example is a particular case \((d = 3)\) of a more general example that can be constructed on \(\mathcal{M}_d\) (see the example after theorem 13 in [10]), where \(\mathcal{E} : \mathcal{M}_d \to \mathcal{M}_d\) is defined by
\[
\mathcal{E}(x) = \sum_{j=1}^{d} s_j x s_j^*,
\]
where \(s_k = f_k e_k^*\), where \(e_k\)s are the standard basis of \(\mathbb{C}^d\) and \(f_k\)s are the Fourier basis \(f_k = \frac{1}{\sqrt{d}} \sum_{j=1}^{d} e^{2\pi i kj/d} e_j\). It follows that \(\mathcal{E}^2(x) = \text{tr}(x) \mathbb{1}\) and hence has a trivial multiplicative domain and hence \(\mathcal{M}_{\mathcal{E}^\infty} = \mathcal{M}_{\mathcal{E}^2} = \mathbb{C}1\). However the multiplicative domain of \(\mathcal{E}\) is generated by the rank one projections \(e_1 e_1^*, \ldots, e_d e_d^*\). So \(\mathcal{M}_{\mathcal{E}^\infty} \subset \mathcal{M}_{\mathcal{E}^2}\) and \(\kappa(\mathcal{E}) = 2\) for every dimension \(d\).

**Example 2.13.** Now we construct a channel on \(\mathcal{M}_3\) with \(\kappa = 3\). Let \(\mathcal{E} : \mathcal{M}_3 \to \mathcal{M}_3\) be given by
\[
\mathcal{E}(x) = \sum_{j=1}^{3} s_j x s_j,
\]
where \(s_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\), \(s_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\) and \(s_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\).

It can be seen that \(\mathcal{E}^* \circ \mathcal{E}(x) = \sum_{j=1}^{3} a_j x a_j^*, \) where \(a_j = e_j e_j^*\), for \(j = 1, 2, 3\), that is the rank one projections on standard basis \(\{e_1, e_2, e_3\}\). So 
\[
\mathcal{M}_{\mathcal{E}^2} = \mathcal{F}_{\mathcal{E}^* \circ \mathcal{E}} = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & c \end{bmatrix} : a, b, c \in \mathbb{C} \right\}.
\]

Now, take \(p = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\), then it can be checked that \(\mathcal{E}(p) = \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\), and hence \(\mathcal{E}(p) \notin \mathcal{M}_{\mathcal{E}^2}\). Since \(p \in \mathcal{M}_{\mathcal{E}^2}\) and \(\mathcal{E}(p) \notin \mathcal{M}_{\mathcal{E}}\), we get \(p \notin \mathcal{M}_{\mathcal{E}^2}\). Indeed one checks that 
\[
\mathcal{M}_{\mathcal{E}^2} = \text{span} \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}.
\]

Now for \(q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\), it turns out that \(\mathcal{E}^2(q) \notin \mathcal{M}_{\mathcal{E}^2}\) and hence \(q \notin \mathcal{M}_{\mathcal{E}^2}\). Indeed we get \(\mathcal{M}_{\mathcal{E}^2} = \mathbb{C}1 \subset \mathcal{M}_{\mathcal{E}^2}\), which yields \(\mathcal{M}_{\mathcal{E}^\infty} = \mathcal{M}_{\mathcal{E}^2} = \mathbb{C}1\) and hence \(\kappa(\mathcal{E}) = 3\).

It is worth mentioning at this point that the value of the multiplicative index \(\kappa\) of a unital channel \(\mathcal{E} : \mathcal{M}_d \to \mathcal{M}_d\) cannot range from 1 to \(d^2\). The reason is the reduction of the dimension of maximal proper 
\(+\)-subalgebras of \(\mathcal{M}_d\). The maximum possible value of \(\kappa\) is the longest chain of the following subalgebras
Now note that the subalgebra $\mathcal{M}_\mathcal{E}$ cannot be taken as $\mathcal{M}_d$ to begin with. This would mean that the multiplicative domain of $\mathcal{E}$ is $\mathcal{M}_d$ and hence $\mathcal{E}$ is a homomorphism and the kernel of $\mathcal{E}$, $\text{Ker}(\mathcal{E})$ is an ideal of $\mathcal{M}_d$. Since $\mathcal{M}_d$ is simple as an algebra, that is, it cannot contain a non-trivial two-sided ideal, $\mathcal{E}$ has to have a trivial kernel which makes $\mathcal{E}$ an isomorphism. So there exists another unital homomorphism $\Phi : \mathcal{M}_d :\rightarrow \mathcal{M}_d$ such that $\mathcal{E} \circ \Phi(x) = \Phi \circ \mathcal{E}(x) = x$, for all $x \in \mathcal{M}_d$. Hence we get for every $x$,

$$||x|| = ||\Phi \circ \mathcal{E}(x)|| \leq ||\mathcal{E}(x)|| \leq ||x||.$$  

This makes $\mathcal{E}$ an isometric isomorphism and hence, following Kadison’s work in [25], we conclude that the maximal dimension of a proper unital subalgebra of $\mathcal{M}_\mathcal{E}$ is a partial isometry. We note down this fact as the following proposition.

To get a possible value of $\kappa$ which is bigger than 1, one starts with a channel $\mathcal{E}$ which is a unitary channel. Now by example 2.10 we get $\mathcal{E}(x) = uax^*a^*$ for some unitary $u$ in $\mathcal{M}_d$, that is, $\mathcal{E}$ is a unitary channel. Now by Theorem 2.14 ([10]), it was shown that the maximal dimension of a proper unital subalgebra of $\mathcal{M}_d$ is $d^2 - d + 1$. Since we are concerned with proper subalgebras which are also $\ast$-closed, the dimension of a maximal proper unital $\ast$-subalgebra of $\mathcal{M}_d$ will be bounded by $d^2 - d + 1$. With this choice of $\mathcal{M}_\mathcal{E}$, the algebra $\mathcal{M}_\mathcal{E}$ will also have lesser dimension than that of $\mathcal{M}_d$ and at the end, even if $\mathcal{M}_\mathcal{E} = \mathcal{M}_\mathcal{E} = \mathcal{C}1$, we will get $\kappa < d^2$.

In the case of $d = 3$, it is more easily seen. By Wedderburn’s theorem, the $\ast$-subalgebras of $\mathcal{M}_3$ are described below up to isomorphism

$$\mathcal{M}_2 \oplus \mathcal{M}_1, \mathcal{M}_1 \oplus \mathcal{M}_1 \oplus \mathcal{M}_1, (\mathcal{M}_1 \otimes 1_2) \oplus \mathcal{M}_1, \mathcal{M}_1 \otimes 1_3,$$

where $\mathcal{M}_1 = \mathcal{C}1$ and $1_2, 1_3$ are identity matrices in $\mathcal{M}_2$ and $\mathcal{M}_3$ respectively. Example 2.13 shows the chain in 5 starting with $\mathcal{M}_1 \oplus \mathcal{M}_1 \oplus \mathcal{M}_1$ and going down all the way to $\mathcal{M}_1 \otimes 1_3$ yielding $\kappa = 3$.

We end this section with the following proposition which asserts that $\mathcal{M}_\mathcal{E} = \mathcal{M}_d$. The choice significantly reduces the path of the chain in equation (5) because of the dimension criteria for any proper $\ast$-subalgebras. In a very recent article [1], it was shown that the maximal dimension of a proper unital subalgebra of $\mathcal{M}_d$ is $d^2 - d + 1$. Since we are concerned with proper subalgebras which are also $\ast$-closed, the dimension of a maximal proper unital $\ast$-subalgebra of $\mathcal{M}_d$ will be bounded by $d^2 - d + 1$. Example 2.10 we get $\mathcal{E}(x) = uax^*a^*$ for some unitary $u$ in $\mathcal{M}_d$, that is, $\mathcal{E}$ is a unitary channel. Now by Theorem 2.14 ([10]), it was shown that the maximal dimension of a proper unital subalgebra of $\mathcal{M}_d$ is $d^2 - d + 1$. Since we are concerned with proper subalgebras which are also $\ast$-closed, the dimension of a maximal proper unital $\ast$-subalgebra of $\mathcal{M}_d$ will be bounded by $d^2 - d + 1$. With this choice of $\mathcal{M}_\mathcal{E}$, the algebra $\mathcal{M}_\mathcal{E}$ will also have lesser dimension than that of $\mathcal{M}_d$ and at the end, even if $\mathcal{M}_\mathcal{E} = \mathcal{M}_\mathcal{E} = \mathcal{C}1$, we will get $\kappa < d^2$.

In the case of $d = 3$, it is more easily seen. By Wedderburn’s theorem, the $\ast$-subalgebras of $\mathcal{M}_3$ are described below up to isomorphism

$$\mathcal{M}_2 \oplus \mathcal{M}_1, \mathcal{M}_1 \oplus \mathcal{M}_1 \oplus \mathcal{M}_1, (\mathcal{M}_1 \otimes 1_2) \oplus \mathcal{M}_1, \mathcal{M}_1 \otimes 1_3,$$

where $\mathcal{M}_1 = \mathcal{C}1$ and $1_2, 1_3$ are identity matrices in $\mathcal{M}_2$ and $\mathcal{M}_3$ respectively. Example 2.13 shows the chain in 5 starting with $\mathcal{M}_1 \oplus \mathcal{M}_1 \oplus \mathcal{M}_1$ and going down all the way to $\mathcal{M}_1 \otimes 1_3$ yielding $\kappa = 3$.

We end this section with the following proposition which asserts that $\mathcal{M}_\mathcal{E}$ is generated by partial isometries. Recall that an element $v \in \mathcal{M}_d$ is a partial isometry if $vv^* = p$ and $v^*v = q$ where $p, q$ are projections. Before we state our proposition, we state the following theorem.

**Theorem 2.14 ([10]).** Let $\Phi : \mathcal{M}_d \rightarrow \mathcal{M}_d$ be a unital channel given by $\Phi(x) = \sum_{j=1}^{n} a_j \alpha_j^*$. Then an element $a \in \mathcal{M}_d$ satisfies $\Phi(a) = \lambda a$ with $\lambda = 1$ if and only if $a_j a_j^* = \lambda a_j$, $\forall j = 1, \ldots, n$.

Moreover, the eigenspace of $\Phi$ corresponding to $\lambda$ is (linearly) spanned by partial isometries.

Now in part 3 of theorem 2.5, $\mathcal{M}_\mathcal{E}$ is generated by the eigenvectors corresponding to the peripheral eigenvalues. Since the above theorem asserts that each peripheral eigenvector is a linear combination of partial isometries, we conclude that $\mathcal{M}_\mathcal{E}$ is generated by partial isometries. We note down this fact as the following proposition.

**Proposition 2.15.** For a unital channel $\mathcal{E} : \mathcal{M}_d \rightarrow \mathcal{M}_d$, the algebra $\mathcal{M}_\mathcal{E}$ is generated by partial isometries.

3. Irreducibility and primitivity

In this section we study the fixed points and multiplicative properties of irreducible positive linear maps on $\mathcal{M}_d$. We recall some definitions and mention some well known facts below.
Definition 3.1. A positive linear map $\Phi : \mathcal{M}_d \to \mathcal{M}_d$ is called irreducible if there exists no non trivial projection $p \in \mathcal{M}_d$ such that

$$\Phi(p) \leq \lambda p \text{ for } \lambda > 0.$$ 

We note some basic facts about irreducible positive linear maps on finite dimensional $C^*$ algebras (see also [16, 19]). In the literature of quantum information theory, such maps are also known as ergodic linear maps (see [10]). In what follows $\mathcal{M}_d^+$ denotes the set of all positive semidefinite elements of $\mathcal{M}_d$. We start with the spectral properties of an irreducible positive linear map.

Theorem 3.2 (See [16]). Let $\Phi$ be a positive linear map on $\mathcal{M}_d$ and let $r$ be its spectral radius. Then

1. There is a non zero $x \in \mathcal{M}^+_d$ such that $\Phi(x) = rx$.
2. If $\Phi$ is irreducible and if $y \in \mathcal{M}^+_d$ is an eigenvector of $\Phi$ corresponding to some eigenvalue $s$ of $\Phi$, then $s = r$ and $y$ is a positive scalar multiple of $x$.
3. If $\Phi$ is unital, irreducible and satisfies the Schwarz inequality for positive linear maps then

- $r = 1$ and $\mathcal{F}_\Phi = \mathbb{C}1$.
- Every peripheral eigenvalue $\lambda \in \text{Spec}(\Phi) \cap \mathbb{T}$ is simple and the corresponding eigenspace is spanned by a unitary $u_\lambda$ which satisfies $\Phi(u_\lambda x) = \lambda u_\lambda \Phi(x)$ for all $x \in \mathcal{M}_d$.
- The set $\Gamma = \text{Spec}(\Phi) \cap \mathbb{T}$ is a cyclic subgroup of the group $\mathbb{T}$ and the corresponding eigenvectors form a cyclic group which is isomorphic to $\Gamma$ under the isomorphism $\lambda \to u_\lambda$.

With these set up we are ready to note down the following definition.

Definition 3.3. Let $E : \mathcal{M}_d \to \mathcal{M}_d$ be an irreducible positive linear map. If $E$ has a trivial peripheral spectrum, then $E$ is called primitive.

We refer to [43] for some properties of primitive maps. We begin by describing the set $\mathcal{M}_E^\infty$ for an irreducible and primitive channel $E$.

Lemma 3.4. Let $E$ be a unital irreducible channel. Then $\mathcal{M}_E^\infty$ is a commutative $C^*$-algebra.

Proof. Since the channel $E$ is unital and satisfies the Schwarz inequality, the peripheral spectrum $\Gamma = \text{Spec}(E) \cap \mathbb{T}$ is a cyclic subgroup. So $\Gamma = \exp(2\pi i \mathbb{Z}_m)$ for some $m \leq d^2$. If $u$ is the eigen vector for the eigenvalue $\lambda = \exp(2\pi i/m)$, then it is easy to see that $E(u^k) = \lambda^k u^k$ for all $k \in \mathbb{N}$. This shows that the peripheral eigen operators are generated by the powers of $u$ and hence $\mathcal{N}_E$ is spanned by the single unitary $u$. Since $\mathcal{M}_E^\infty$ is then algebraically generated by a unitary, we get that $\mathcal{M}_E^\infty$ is a commutative $C^*$-algebra. \hfill $\Box$

Note that from the work of Størmer in [38], it can be deduced that if $E$ is a unital entanglement breaking channel, then $\mathcal{M}_E$ (and hence $\mathcal{M}_E^\infty$) is abelian $C^*$-algebra where an entanglement breaking channel is one whose all Kraus operators are of rank one [23]. Lemma 3.4 reflects upon a similar characterisation of $\mathcal{M}_E^\infty$ in the case of irreducible channels which are found in abundance in the set of quantum channels.

Corollary 3.5. A unital quantum channel is primitive if and only if $\mathcal{M}_E^\infty = \mathbb{C}1$.

The following proposition captures the description of the stabilising algebra of a composition of two channels. The commutativity of the channels is a necessary condition.
Proposition 3.6. If two unital channels $\Phi, \Psi$ commute that is $\Phi \circ \Psi = \Psi \circ \Phi$, then $\mathcal{M}_{(\Phi \circ \Psi)\sim} = \{a \in \mathcal{M}_{\Psi\sim} : \Psi^k(a) \in \mathcal{M}_{\Phi\sim} \forall k \in \mathbb{N}\}$.

Proof. The proof is based on the same idea which was used in lemma 2.3. One side is straightforward. For the other side, let $a \in \mathcal{M}_{(\Phi \circ \Psi)\sim}$. Then for any $k \in \mathbb{N}$, we have

$$
(\Phi \circ \Psi)^k(aa^*) = \Phi^k(\Psi^k(aa^*))
$$

$$
\geq \Phi^k(\{\Psi^k(a)\}\Psi^k(a^*))
$$

$$
\geq \Phi^k(\Psi^k(a))\Psi^k(\Psi^k(a^*))
$$

$$
= \Psi^k\Psi^k(aa^*)
$$

$$
= (\Phi \circ \Psi)^k(aa^*).
$$

Since the two extreme ends are the same, the inequalities become an equality and using the trace preservation of $\Phi, \Psi$ and the faithfulness of the trace, we obtain the result.

Note that starting from the Perron–Frobenius theory of positive and non-negative matrices, the notions of the irreducibility and primitivity of matrices have attracted wide attention in matrix theory [22]. These concepts have been applied in algebraic graph theory, finite Markov chains and other related fields. In [36], the author is concerned with the irreducibility and primitivity of sums and products of non-negative matrices. Since unital quantum channels can be thought of as non-commutative generalisations of bistochastic matrices (matrices with non-negative real entries where each row and column sums to 1), the remaining part of this section addresses similar notions of the irreducibility and primitivity of the products of unital channels.

Note that a product of primitive channels need not be primitive. To get a glimpse of the importance of the primitivity of a product of channels, we observe that to quantify the increment of the entropy of a quantum system under a primitive channel, the logarithmic–Sobolev (LS) constant plays an important role [34]. The discrete LS constant in [34] is defined assuming that $E^* \circ E$ is primitive. So the primitivity of the composition of two channels seems to be a useful assertion. We derive a necessary condition for the composition of two channels to be primitive.

Theorem 3.7. Let two unital channels $\Phi, \Psi$ be such that $\Phi \circ \Psi = \Psi \circ \Phi$. If one of the two channels is primitive, then $\Phi \circ \Psi$ is primitive.

Proof. Proposition 3.6 implies that $\mathcal{M}_{(\Phi \circ \Psi)\sim} \subseteq \mathcal{M}_{\Phi\sim} \cap \mathcal{M}_{\Psi\sim}$.

Now if one of them (say $\Phi$) is primitive, then $\mathcal{M}_{\Phi\sim} = \mathbb{C}1$, forcing $\mathcal{M}_{(\Phi \circ \Psi)\sim} = \mathbb{C}1$, making $\Phi \circ \Psi$ primitive.

Now we give a new proof to the following theorem concerning the primitivity of $E^* \circ E$, which was first given in [10], theorem 13.

Theorem 3.8. Let $E$ be a unital channel. If $E^* \circ E$ is irreducible, then $E$ is primitive. Furthermore, if $E^* \circ E = E \circ E^*$ and $E$ is primitive, then $E^* \circ E$ is primitive and hence irreducible.

Proof. Let $E^* \circ E$ be irreducible. Then we have $F_{E^* \circ E} = \mathbb{C}1$. It is easy to see that if for any $a$, $E(a) = \lambda a$ with $|\lambda| = 1$, then $a \in \mathcal{M}_E$. Now by lemma 2.1 we get $\mathcal{M}_E = F_{E^* \circ E} = \mathbb{C}1$. Hence the peripheral spectrum has to be trivial. So $E$ is primitive.

The other assertion is a direct application of theorem 3.7.
The following result appeared in [40], lemma 2, which was used to show that the set of primitive channels with a fixed Kraus rank is path connected. We give an alternate proof of the result using the techniques we have developed.

**Theorem 3.9.** Let $\mathcal{E}$ be a unital quantum channel. Then $\mathcal{E}$ is irreducible if and only if $\mathcal{E} + \mathcal{E}^2$ is primitive.

**Proof.** First we observe that $\mathcal{E} + \mathcal{E}^2 = \mathcal{E}(1 + \mathcal{E}) = (1 + \mathcal{E})\mathcal{E}$ and hence $\mathcal{E}$ and $1 + \mathcal{E}$ commute. Now it is well known (see [43], theorems 6.2, 6.7) that if $\mathcal{E}$ is irreducible, then $1 + \mathcal{E}$ is primitive. Now by the proposition 3.6, $\mathcal{M}_{(\mathcal{E} + \mathcal{E}1)}^n = \mathcal{M}_{(\mathcal{E}(1+\mathcal{E}))}^n \subseteq \mathcal{M}_{\mathcal{E}^n} \cap \mathcal{M}_{(1+\mathcal{E})}^n$. As $\mathcal{M}_{(1+\mathcal{E})} = \mathcal{C}1$, we have $\mathcal{M}_{(\mathcal{E} + \mathcal{E}1)} = \mathcal{C}1$, forcing $\mathcal{E} + \mathcal{E}^2$ to be primitive.

Conversely, let $\mathcal{E} + \mathcal{E}^2$ be primitive. If $\mathcal{E}$ is not irreducible, then there exists a non trivial projection $p$ such that $\mathcal{E}(p) \leq \lambda p$. Furthermore, using the Kraus operators, it can be shown that $\mathcal{E}(p) \leq p$ ([7], lemma 3.1). Since $\mathcal{E}$ is trace preserving, we get by the faithfulness of the trace, $\mathcal{E}(p) = p$. This yields $(\mathcal{E} + \mathcal{E}^2)(p) = 2p$ and subsequently $(\mathcal{E} + \mathcal{E}^2)^n(p) = 2^n p$. Now following [43], theorem 6.7, if $\mathcal{E} + \mathcal{E}^2$ is primitive, then $\lim_{n \to \infty} (\mathcal{E} + \mathcal{E}^2)^n(p)$ is a positive definite matrix. But if $(\mathcal{E} + \mathcal{E}^2)^n(p) = 2^n p$, then $\lim_{n \to \infty} (\mathcal{E} + \mathcal{E}^2)^n(p)$ does not exist, contradicting the primitivity of $(\mathcal{E} + \mathcal{E}^2)$.

Exploiting the relation $\mathcal{F}_{\mathcal{E}+\mathcal{E}^2} = \mathcal{M}_\mathcal{E}$, we give a different proof of the following proposition given in [10], proposition 2.

**Proposition 3.10.** Let $\mathcal{E}$ be a unital channel. Then $\mathcal{E}^\ast \circ \mathcal{E}$ is irreducible if and only if there is no projection $p < 1$ and no unitary $u$ such that $\mathcal{E}(p) = upu^\ast$.

**Proof.** Let $\mathcal{E}^\ast \circ \mathcal{E}$ be irreducible. Then $\mathcal{C}1 = \mathcal{F}_{\mathcal{E}^\ast \circ \mathcal{E}} = \mathcal{M}_\mathcal{E}$. If there is a projection $p$ and a unitary $u$ such that $\mathcal{E}(p) = upu^\ast$, then $\mathcal{E}(p)^2 = upu^\ast = upu^* = \mathcal{E}(p) = \mathcal{E}(p^2)$ and hence $p \in \mathcal{M}_\mathcal{E}$, contradicting the hypothesis.

Conversely, assume the contrary, that is, $\mathcal{F}_{\mathcal{E}^\ast \circ \mathcal{E}} \neq \mathcal{C}1$, then $\mathcal{M}_\mathcal{E} \neq \mathcal{C}1$ and hence assume $a \in \mathcal{M}_\mathcal{E}$. As $\mathcal{M}_\mathcal{E}$ is a $\mathbb{C}$-algebra, replacing $a$ by the real part $\Re a$ and the imaginary $\Im a$ part of $a$, we can assume $a$ is self adjoint. Furthermore, replacing $a$ by $a + \|a\|1$ and noting that $\mathcal{E}$ is unital, we can assume $a$ is positive. By the spectral decomposition theorem, let $a = \lambda_1 p_1 + \cdots + \lambda_k p_k$ where $\lambda_i$’s are the eigenvalues and $p_i$’s are the corresponding eigen projections. Since $a \in \mathcal{M}_\mathcal{E}$, it is easy to see that $p_i \in \mathcal{M}_\mathcal{E}$ for every $i = 1, 2, \cdots, k$. Hence for any such $j$, using theorem 1.2 we get $\mathcal{E}(p_j)$ is a projection and because $\mathcal{E}$ is trace preserving, $\mathcal{E}(p_j)$ is a projection of the same rank as $p_j$. This means there is a unitary $u$ such that $\mathcal{E}(p_j) = up_j u^\ast$. This violates the assumption.

4. In the Heisenberg picture

In a general setting, a quantum channel ($\mathcal{E}$) that is a completely positive trace preserving linear map is defined on the trace class operators $\mathcal{T}(\mathcal{H})$ where $\mathcal{H}$ can be an infinite dimensional Hilbert space. The positivity of a map on $\mathcal{T}(\mathcal{H})$ is realised by considering $\mathcal{T}(\mathcal{H})$ as a matrix ordered space. This map $\mathcal{E} : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$ is seen as a linear operator in the Schrödinger picture. Since the dual of $\mathcal{T}(\mathcal{H})$ is $\mathcal{B}(\mathcal{H})$, that is $\mathcal{B}(\mathcal{H})^\ast = \mathcal{T}(\mathcal{H})$, the map $\mathcal{E}$ induces a unital, normal and completely positive map on $\mathcal{B}(\mathcal{H})$. The dual picture where a completely positive and trace preserving map on trace class operators induces a unital completely positive map $\mathcal{E}^\ast : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is also an important association with a channel. The two maps are related via the relation $\text{tr}(a\mathcal{E}(b)) = \text{tr}(\mathcal{E}^\ast(a)b)$ for all $a \in \mathcal{B}(\mathcal{H}), b \in \mathcal{T}(\mathcal{H})$. This scenario where the
map $\mathcal{E}^*: \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ is unital normal and completely positive is known as the Heisenberg picture. In finite dimensional Hilbert spaces, both the spaces are the same but thought of as different matrix ordered spaces. That is, although the matrix ordered spaces $\mathcal{M}_d$, and $\mathcal{M}_d$ are different, a linear map, if it is completely positive on $\mathcal{M}_d$, is equivalent to being completely positive on $\mathcal{M}_d$. The discussion on positivity and the quantum channel is mentioned in detail in [18], section III.A. We discuss the multiplicative property of a linear map in the Heisenberg picture and hence we will consider ucp maps on $\mathcal{M}_d$.

4.1. Density of trivial multiplicative domain maps

**Theorem 4.1.** The set of ucp maps on $\mathcal{M}_d$ that have trivial multiplicative domains is dense in the completely bounded norm topology inside the set of ucp maps.

**Proof.** Let us start with an arbitrary ucp map $\Phi$ and define a new map $\mathcal{E} : \mathcal{M}_d \rightarrow \mathcal{M}_d$ by $\mathcal{E}(x) = (1 - \frac{1}{n})\Phi(x) + \frac{1}{n}\text{tr}(x)\frac{1}{M}$ for all $x \in \mathcal{M}_d$ where $n(>1) \in \mathbb{N}$ and 1 is the identity operator in $\mathcal{M}_d$.

One can check $\mathcal{E}$ is ucp. Note also that $\mathcal{E}$ is strictly positive which means it sends positive semi definite operators to positive definite, i.e. invertible and positive elements. Hence it is clear that $\mathcal{E}$ is irreducible. Now we see $\mathcal{E}$ approximates $\Phi$. Since the map $\mathcal{E}(x) = \text{tr}(x)\frac{1}{M}$ is unital and positive and the cb norm is attained at 1, we have

$$||\Phi - \mathcal{E}||_{cb} = \frac{1}{n}||\Phi - \mathcal{E}||_{cb} \leq \frac{1}{n}||\Phi||_{cb} + ||\mathcal{E}||_{cb} = \frac{2}{n}.$$  

As we can take $n$ large enough, it shows that $\mathcal{E}$ approximates an arbitrary ucp map $\Phi$.

Suppose $\mathcal{E}$ has a non trivial multiplicative domain and let $a \in \mathcal{M}_E$. Since $\mathcal{M}_E$ is a $C^*$ algebra, $a^*$ is also in $\mathcal{M}_E$. So we can assume $a$ to be self adjoint. Also since $\mathcal{E}$ is unital by replacing $a$ by $a + ||a||1$ we can assume $a$ is a positive operator. Now let us assume $a$ has the following spectral decomposition: $a = \lambda_1 p_1 + \cdots + \lambda_k p_k$ where each $p_i$ is a projection. Since $\mathcal{E}(a^2) = \mathcal{E}(a,a) = \mathcal{E}(a)\mathcal{E}(a)$ we have $\mathcal{E}(a^n) = (\mathcal{E}(a))^n$ for all $m > 0$. So for any polynomial $f$, we have $\mathcal{E}(f(a)) = f(\mathcal{E}(a))$, and $f(a)$ is in the multiplicative domain of $\mathcal{E}$, and since it is a $C^*$-algebra, all the spectral projections $p_i \in \mathcal{M}_E$ for every $i$. By theorem 1.2 $\mathcal{E}(p_i)$ is also a projection. Since $\mathcal{E}$ is strictly positive, the only possibility of $\mathcal{E}(p_i)$ is 1. Now, if $\mathcal{E}(p_i) = 1$, then $\mathcal{E}(1 - p_i) = 0$ which violates the irreducibility of $\mathcal{E}$. This shows that the multiplicative domain of $\mathcal{E}$ cannot contain any non-trivial elements. □

From the standpoint of $C^*$-algebra theory on finite dimensional Hilbert spaces, theorem 4.1 reveals an interesting property—that the majority of ucp maps on $\mathcal{M}_d$ do not display a multiplicative nature, even when restricted to a subdomain. From the viewpoint of quantum information theory, especially in quantum error correction where a tacit relationship between the multiplicative domain of a unital channel and error correcting codes have been discovered (see [14, 24]), theorem 4.1 rules out the possibility of perfect error correction for the majority of channels. Section 5 contains further discussion on the effect of a multiplicative domain of a unital channel on error correction.
We end this section with another observation which reinforces the effect of the theorem 4.1 in the study of the convex set of all ucp maps on $\mathcal{M}_d$. We first state the following theorem.

**Theorem 4.2 (Choi, [13]).** If $\mathcal{E}, \Phi, \Psi$ are ucp maps on a C*-algebra $\mathcal{A}$ such that $\mathcal{E} = \frac{1}{2}(\Phi + \Psi)$, then

$$\mathcal{M}_\mathcal{E} = \mathcal{M}_\Phi \cap \mathcal{M}_\Psi \cap \{x \in \mathcal{A} : \mathcal{E}(x) = \Phi(x) = \Psi(x)\}.$$ 

Now if a ucp map $\Phi$ is such that $\mathcal{M}_\Phi = \mathbb{C}1$, then Choi’s theorem implies that any ucp map lying in the line-segment passing through $\Phi$ must have a trivial multiplicative domain. Since by theorem 4.1 such $\Phi$s are dense, it follows that every line-segment passing through the elements of this dense set contains ucp maps with a trivial multiplicative domain. Hence ucp maps which do not show a multiplicative nature are quite generic in a general sense.

4.2. On Arveson’s boundary type theorem on matrix algebras

Recall that an operator $a$ on a Hilbert space $\mathcal{H}$ is called irreducible if there exists no non trivial projection $p$ such that $ap = pa$. The celebrated boundary theorem ([3]) of Arveson in finite dimension asserts:

**Theorem 4.3 ([17]).** If $a \in \mathcal{M}_d$ is irreducible and if $\Phi : \mathcal{M}_d \rightarrow \mathcal{M}_d$ is a ucp linear transformation such that $a \in \mathcal{F}_\Phi$, then $\Phi(x) = x$ for all $x \in \mathcal{M}_d$.

We state a simple lemma, the proof of which can be easily derived using the fact that the algebra generated by $\{1, a, a^*\}$ is $\mathcal{M}_d$, where $a$ is an irreducible operator.

**Lemma 4.4.** Let $\mathcal{E} : \mathcal{M}_d \rightarrow \mathcal{M}_d$ be a ucp map and let $a \in \mathcal{M}_d$ be irreducible such that $a \in \mathcal{M}_\mathcal{E}$. Then $\mathcal{E}$ is an automorphism.

**Proof.** Easy.

Now we state and prove the main theorem of the subsection. Since, in the discussion of the multiplicative index, the eigen operators play an important role, the following theorem provides some more information extending the theorem 4.3. For a ucp map $\mathcal{E} : \mathcal{M}_d \rightarrow \mathcal{M}_d$, we recall that $\mathcal{N}_\mathcal{E} = \{x \in \mathcal{M}_d : \mathcal{E}(x) = \lambda x, |\lambda| = 1\}$. Clearly the set $\mathcal{N}_\mathcal{E}$ contains the fixed point set $\mathcal{F}_\mathcal{E}$. Then we have the following theorem.

**Theorem 4.5.** Let $\mathcal{E} : \mathcal{M}_d \rightarrow \mathcal{M}_d$ be a ucp map. Suppose an irreducible operator $a \in \mathcal{M}_d$ is inside $\mathcal{N}_\mathcal{E}$. Then $\mathcal{N}_\mathcal{E}$, the peripheral eigen operators of $\mathcal{E}$, span the entire $\mathcal{M}_d$.

**Proof.** For a ucp $\mathcal{E}$, Kuperberg in [30] proved that there is a sequence of integers $n_1 < n_2 < \cdots$ such that $\mathcal{E}^n$ converges to a unique idempotent and completely positive map $P : \mathcal{M}_d \rightarrow \mathcal{M}_d$ and $\text{span-cone}(P)$ spans $\mathcal{N}_\mathcal{E}$. Now we proceed similarly as in [17] and using the Choi–Effrose product on span-$\mathcal{N}_\mathcal{E}$ we can make it a C* algebra. Let us define $a \circ b = P(ab)$ where $a, b \in \text{span-$\mathcal{N}_\mathcal{E}$}$ and it is well known that with this multiplication span-$\mathcal{N}_\mathcal{E} = \text{ran$(P)$}$ is a C* algebra. Since $P$ is a conditional expectation, we have $P(yz) = P(yP(z))$ for $y \in \text{span-$\mathcal{N}_\mathcal{E}$}$, $z \in \mathcal{M}_d$. Now, we see for any $a \in \text{span-$\mathcal{N}_\mathcal{E}$}$, $P(a) = a$ and $P(xy) = P(xP(y)) = x \circ P(y) = P(x) \circ P(y)$, $x \in \text{span-$\mathcal{N}_\mathcal{E}$}$, $y \in \mathcal{M}_d$. Similarly if $x_1, x_2 \in \text{span-$\mathcal{N}_\mathcal{E}$}$, $y \in \mathcal{M}_d$ we get $P(x_1x_2y) = P(x_1P(x_2y)) = x_1 \circ P(x_2y) = (P(x_1) \circ P(x_2)) \circ P(y)$. Hence by induction, if $\gamma$ is any term with $2n$ non commutative variables and if $x_1, \cdots, x_n \in \text{span-$\mathcal{N}_\mathcal{E}$}$ and $y \in \mathcal{M}_d$ we have
ρH. An important class of subsystems is the ρ···P(...)=((subsystems of any unital channel

theory of error correction.

appropriate set up for exhibiting such applications

— see [14, 26] for more detailed discussion on the

quantum error correction is one of the areas where we find an appro-

In this section we note down some applications of the techniques and results developed

rem 4.3) on finite dimensional spaces from a more spectral theoretic viewpoint.

theorem 4.5 also suggests a different prospective of the

of Arveson (theo-

boundary theorem

is an isomorphism and therefore the set span-

M ∈M
d

B

an irreducible operator as a peripheral eigenvector. Since

in [4]. Theorem 4.5 exhibit similar behaviour of a ucp map if it contains

isometric invertible operators acting

on a general Banach space

where the maps in discussion satisfy certain properties. Isometric invertible operators

acting on a general Banach space X, whose peripheral eigenvectors span the domain X, seem to have

played a significant role in studying the asymptotic stability of ucp maps. Such maps were

named diagonalisable in [4]. Theorem 4.5 exhibit similar behaviour of a ucp map if it contains

an irreducible operator as a peripheral eigenvector. Since F_E ⊆N_E for a unital linear map, theorem 4.5 also suggests a different prospective of the boundary theorem of Arveson (theo-

rem 4.3) on finite dimensional spaces from a more spectral theoretic viewpoint.

5. Application

In this section we note down some applications of the techniques and results developed throughout this paper. Quantum error correction is one of the areas where we find an appropriate set up for exhibiting such applications—see [14, 26] for more detailed discussion on the theory of error correction.

The standard model of error correction can be described by a triple (E, R, C). Here E : B(H) → B(H) is a quantum channel, where H is a finite dimensional Hilbert space. C is a subspace of H known as the code and R is another quantum channel on B(H) known as recovery operation. Denote PC by the projection onto C. The triple should satisfy the condition

R(E(ρ)) = ρ, where P_CP_C = ρ.

With this set up, the ‘noiseless subsystem’ protocol (see [26, 21]) seeks subsystems H^B (with dim H^B > 1) of the full system H such that E = (H^A ⊗ H^B) ⊕ K, where K is another sub-

space of H such that ∀ρ^A, ∀ρ^B there exists a γ^A satisfying

E(ρ^A ⊗ ρ^B) = γ^A ⊗ ρ^B.

Here we write ρ^A (respectively ρ^B) for operators on B(H^A) (respectively B(H^B)). Noiseless subsystems of any unital channel E are obtained precisely from the fixed point set F_E of E.

A subsystem B is called correctable for E via a recovery operation R if it is a noiseless subsystem for the quantum operation R ◦ E. An important class of subsystems is the unitarily correctable subsystem (UCS) which is also known as unitarily correctable codes where the recovery operation R can be chosen to be the unitary channel x → u*x, for a unitary operator u and for all x ∈ B(H). The following theorem relates the UCS of a unital channel E to the noiseless subsystems of E* ◦ E.
Theorem 5.1 ([28]). Let $\mathcal{E}$ be a unital quantum channel. Then the following statements are equivalent:

1. $B$ is a unitarily correctable subsystem for $\mathcal{E}$.
2. $B$ is a noiseless subsystem for $\mathcal{E}^* \circ \mathcal{E}$.

In [14], the UCC algebra for a unital channel is defined to be the $F_{\mathcal{E}^* \circ \mathcal{E}}$ and in theorem 11 it was shown that this is precisely the multiplicative domain.

With this background we are ready to exhibit the application of the techniques developed in this paper. We will prove that for a unital channel $\mathcal{E}$, even if we require the recovery operation $R$ to be a unital channel (not necessarily a unitary channel), we still get the multiplicative domain of $\mathcal{E}$ to be the exact correctable code. This means we do not get any extra correctable codes other than those arising from the multiplicative domain as in theorem 5.1, even if we allow our recovery map to be any unital channel. We formulate the following proposition:

Proposition 5.2. Given a unital channel $\mathcal{E} : \mathcal{M}_d \to \mathcal{M}_d$, define the following set

$$C_\mathcal{E} = \{ F_{R \circ \mathcal{E}} : \text{for unital channels } R \text{ on } \mathcal{M}_d \}.$$ 

Then we have

$$M_\mathcal{E} = C_\mathcal{E}.$$ 

Proof. First we see that $M_\mathcal{E} \subseteq C_\mathcal{E}$ by the aid of theorem 1.3 that asserts $M_\mathcal{E} = F_{\mathcal{E}^* \circ \mathcal{E}}$. Hence $\mathcal{E}^*$ is one of the choices of $R$.

Conversely, let $a \in M_d$ be such that $a \in F_{R \circ \mathcal{E}}$ for a unital channel $R$. Since $a^* \in F_{R \circ \mathcal{E}}$ as well, we have

$$\text{tr}(aa^*) = \text{tr}(R \circ \mathcal{E}(aa^*)) = \text{tr}(R(\mathcal{E}(a)\mathcal{E}(a^*))) \geq \text{tr}(\mathcal{E}(\mathcal{E}(a)\mathcal{E}(a^*))) = \text{tr}(aa^*)$$

where we have used the Schwarz inequality for the unital maps $\mathcal{E}$ and $R$. So the inequalities above are all equalities. Subsequently, by the trace preservation property of $R$, we get $\text{tr}(\mathcal{E}(aa^*)) = \text{tr}(\mathcal{E}(a)\mathcal{E}(a^*))$ and hence by the faithfulness of the trace and the Schwarz inequality for $\mathcal{E}$ we get $\mathcal{E}(aa^*) = \mathcal{E}(a)\mathcal{E}(a^*)$, which shows $a \in M_\mathcal{E}$. Hence we have the desired equality of sets. 

Since the unitary channels are a particular case for arbitrary unital channels, it can be noted that theorem 5.1 appears as a special case of proposition 5.2. It is now evident from proposition 5.2 that even if we collect the unitarily correctable codes of a unital channel via a unital recovery operation, we do not get anything other than the unitarily correctable codes.

A UCS code $\mathcal{B}$ of $\Phi$ is called a unitarily noiseless subsystem (UNS) for $\Phi$ (see [33, 6]) if $\mathcal{B}$ is UCS code for $\Phi^n$ for every $n \geq 1$. Now by the aid of theorem 5.1, the UCS codes of $\Phi^n$ are exactly the noiseless subsystems for $\Phi^n \circ \Phi^n$ for $n \geq 1$, which can be obtained precisely from the algebra $F_{\Phi^n \circ \Phi^n}$. Since for each $n \in \mathbb{N}$, $\Phi^n$ is a unital channel, theorem 1.3 asserts that

$$M_{\Phi^n} = F_{\Phi^n \circ \Phi^n}.$$ 

So if $\mathcal{B}$ is a UNS code, then it can be obtained from the set $\bigcap_{n \geq 1} F_{\Phi^n \circ \Phi^n}$ which is essentially the stabilising algebra $M_{\Phi^\infty}$. Hence the UNS codes for a unital channel $\Phi$ are exactly those that arise from the stabilising subalgebra $M_{\Phi^\infty}$.
In this connection the following proposition relates to the notion of the multiplicative index of a unital channel and the UNS codes.

**Proposition 5.3.** Let \( E : \mathcal{M}_d \to \mathcal{M}_d \) be a unital channel with multiplicative index 1, that is \( \kappa(E) = 1 \). Then every UCS code for \( E \) is a UNS code. Moreover, if \( \kappa(E) > 1 \), then there exist UCS codes which are not UNS.

**Proof.** As \( \kappa(E) = 1 \), \( \mathcal{M}_{E^n} = \mathcal{M}_{E^\infty} \) is obtained after applying the channel only once. Clearly if \( B \) is UCS then \( B \) is obtained from the set \( \mathcal{M}_E = \mathcal{M}_{E^\infty} \) which is the same as \( \mathcal{M}_{E^n} \) for all \( n \geq 1 \) and the first assertion follows.

For the other assertion, note that if \( \kappa(E) > 1 \), then we have proper containment \( \mathcal{M}_{E^n} \subset \mathcal{M}_{E^\infty} \). Hence there is a UCS code \( B \) corresponding to \( \mathcal{M}_E \) such that it is not UCS for \( E^n \), and hence not UCS for \( E^n \) if \( n \geq 2 \). So \( B \) cannot be UNS. \( \square \)

Note that by the corollary 2.8, all the channels that commute with the adjoints have a multiplicative index of 1 and hence have UCS codes as UNS codes. As a corollary to the above result we capture the following well known result for Pauli channels which constitute an important class of quantum channels.

**Corollary 5.4 ([32], section 3.2.1).** A UCS code for Pauli channels or generalised Pauli channels is also a UNS code.

**Proof.** If \( E \) is a Pauli channel or a generalised Pauli channel, then \( E^* \circ E = E \circ E^* \) ([10]—see discussion after definition 6). Hence by the corollary 2.8, we get \( \mathcal{M}_E = \mathcal{M}_{E^\infty} \) and hence \( \kappa(E) = 1 \) and proposition 5.3 applies. \( \square \)

6. Summary and discussion

We have put forward a structure theorem for a unital quantum channel that depicts the asymptotic automorphic behaviour of the channel on a stabilising subalgebra. This subalgebra is generated by the peripheral eigen operators of the channel. Based on the finite dimensionality of the system, the notion of a multiplicative index has been introduced which measures the number of times the channel needs to be composed with itself for the multiplicative domain to stabilise. Some applications of the results obtained in the paper have been shown in quantum error correction.

It is interesting to note that any unital channel \( \Phi : \mathcal{M}_d \to \mathcal{M}_d \) can be approximated by a channel \( E \) with \( \kappa(E) = 1 \). Indeed the set

\[ \mathcal{S}_1 = \{ E : \mathcal{M}_d \to \mathcal{M}_d : \kappa(E) = 1 \} \]

is dense inside the convex set of unital channels. For given a channel \( \Phi \), proceeding similarly as in the proof of the theorem 4.1, for each \( n \in \mathbb{N} \), there exists a channel \( E \) such that \( \| \Phi - E \|_{cb} < \frac{1}{n} \) and \( \mathcal{M}_E = \mathbb{C} I \). Each of these \( E \)'s has \( \kappa(E) = 1 \) as \( \mathcal{M}_{E^n} = \mathcal{M}_E = \mathbb{C} I \).

Even though \( \mathcal{S}_1 \) is dense, it follows that \( \mathcal{S}_1 \) is not a relatively open set inside the set of unital channels. To see this, define a channel \( \Phi : \mathcal{M}_2 \to \mathcal{M}_2 \) as follows

\[ \Phi(x) = pxp + qxq, \quad \forall x \in \mathcal{M}_2, \]

where \( p = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \).
Then it is easily verified that

\[ \mathcal{M}_\Phi = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b \in \mathbb{C} \right\}. \]

Also since \( \Phi^* = \Phi \), we get from corollary 2.8 that \( \mathcal{M}_\Phi = \mathcal{M}_{\Phi^*} \) and hence \( \kappa = 1 \) and we have \( \Phi \in \mathcal{S}_1 \).

Now for each \( t \in [0, 1] \), define \( \Phi_t : \mathcal{M}_2 \to \mathcal{M}_2 \) by

\[ \Phi_t(x) = p(t)x p(t)^* + q(t)x q(t)^*, \]

where \( p(t) = \frac{1}{t} \begin{bmatrix} 1 + t & 0 \\ t & 0 \end{bmatrix}, \quad q(t) = \frac{1}{t} \begin{bmatrix} 0 & -t \\ 0 & 1 + t \end{bmatrix}, \quad \text{and} \quad c = \sqrt{(1 + t)^2 + t^2}. \]

Then \( \Phi_t \) is a unital quantum channel for each \( t \in [0, 1] \) such that \( \Phi_0 = \Phi \). So it follows that \( t \mapsto \Phi_t \) is a continuous path starting from \( \Phi \) and ending at \( \Phi_1 \). We will show that for each \( t > 0 \), we will have \( \kappa(\Phi_t) > 1 \). Since any open neighbourhood of \( \Phi \) will intersect this path, it will be evident then that \( \Phi \) cannot admit a neighbourhood consisting of channels with \( \kappa = 1 \) only, implying that \( \mathcal{S}_1 \) is not a relatively open set. To this end, we note that for any \( x \in \mathcal{M}_2 \),

\[ \Phi_t^* \circ \Phi_t(x) = pxp + qxq, \]

where \( p = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \) and \( q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \). Hence we have

\[ \mathcal{M}_{\Phi_t} = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b \in \mathbb{C} \right\}. \]

Now for \( p = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{M}_{\Phi_t} \), we compute \( \Phi_t(p) = \frac{1}{t} \begin{bmatrix} (1 + t)^2 & (1 + t)t \\ (1 + t)t & t^2 \end{bmatrix} \). Evidently \( \Phi_t(p) \notin \mathcal{M}_{\Phi_t} \) if \( t \neq 0 \), and by definition we get \( p \notin \mathcal{M}_{\Phi_t} \). Hence \( \mathcal{M}_{\Phi_t} \subset \mathcal{M}_{\Phi} \) and subsequently \( \kappa > 1 \) as desired.

It can also be seen that for any value \( n > 1 \), the set

\[ \mathcal{S}_n = \{ \mathcal{E} : \mathcal{M}_d \to \mathcal{M}_d : \kappa(\mathcal{E}) = n \} \]

cannot be relatively open as \( \mathcal{S}_1 \) is a dense set.

Notwithstanding the discussion on channels with a fixed multiplicative index, it will be interesting to know: given a unital channel \( \mathcal{E} \) on \( \mathcal{M}_d \), how to get the exact value of the multiplicative index \( \kappa(\mathcal{E}) \) of \( \mathcal{E} \). The exact upper bound for \( \kappa \) is an interesting problem itself. The discussion after the example 2.13 suggests that it is strictly less than \( d^2 \). The question may have a connection with the exact dimension of proper maximal unital \( * \)-subalgebras of \( \mathcal{M}_d \).

**Acknowledgments**

This work is supported by the Graduate Research Fellowship at the University of Regina. The author would like to thank Dr Douglas Farenick for various stimulating discussions, Dr Sarah Plosker for reading the manuscript and providing feedback and Sam Jaques for many insightful discussions on this topic. The author would also like to thank the anonymous referees for various comments and suggestions which have improved the quality and exposition of this work.
References

[1] Agore A L 2017 The maximal dimension of unital subalgebras of the matrix algebra Forum Math. 29 1–5
[2] Arias A, Gheondea A and Gudder S 2002 Fixed points of quantum operations J. Math. Phys. 43 5872–81
[3] Arveson W 1972 Subalgebras of C*-algebras. II Acta Math. 128 271–308
[4] Arveson W 2004 Asymptotic stability. I. Completely positive maps Int. J. Math. 15 289–312
[5] Arveson W 2007 The asymptotic lift of a completely positive map J. Funct. Anal. 248 202–24
[6] Blume-Kohout R, Ng H K, Poulin D and Viola L 2008 Characterizing the structure of preserved information in quantum processes Phys. Rev. Lett. 100 030501
[7] Bratteli O, Jorgensen P E T, Kishimoto A and Werner R F 2000 Pure states on C4 J. Operator Theory 43 97–143
[8] Bulinski A V 1995 Some asymptotic properties of W*-dynamical systems Funktsional. Anal. Prilozhen. 29 64–7
[9] Bunce J and Slinian N 1976 Completely positive maps on C*-algebras and the left matricial spectra of an operator Duke Math. J. 43 747–74
[10] Burgarth D, Chiribella G, Giovannetti V, Perinotti P and Yuasa K 2013 Ergodic and mixing quantum channels in finite dimensions New J. Phys. 15 073045
[11] Carbone R, Sasso E and Umanita V 2013 Decoherence for quantum Markov semi-groups on matrix algebras Ann. Henri Poincare 14 681–97
[12] Chiribella G and Xie J 2013 Optimal design and quantum benchmarks for coherent state amplifiers Phys. Rev. Lett. 110 213602
[13] Choi M D 1974 A Schwarz inequality for positive linear maps on C*-algebras Illinois J. Math. 18 565–74
[14] Choi M D, Johnston N and Kribs D W 2009 The multiplicative domain in quantum error correction J. Phys. A: Math. Theor. 42 245303
[15] D’Ariano G M, Macchiavello C and Perinotti P 2005 Superbroadcasting of mixed states Phys. Rev. Lett. 95 060503
[16] Evans D E and Høegh-Krohn R 1978 Spectral properties of positive maps on C*-algebras J. Lond. Math. Soc. 17 345–55
[17] Farenick D 2011 Arveson’s criterion for unitary similarity Linear Algebra Appl. 435 769–77
[18] Farenick D, Jaques S and Rahaman M 2016 The fidelity of density operators in an operator-algebraic framework J. Math. Phys. 57
[19] Farenick D R 1996 Irreducible positive linear maps on operator algebras Proc. Am. Math. Soc. 124 3381–90
[20] Heinosaari T and Ziman M 2012 The Mathematical Language of Quantum Theory: From Uncertainty to Entanglement (Cambridge: Cambridge University Press)
[21] Holbrook J A, Kribs D W and Laflamme R 2003 Noiseless subsystems and the structure of the commutant in quantum error correction Quantum Inf. Process. 2 381–419
[22] Horn R A and Johnson C R 1990 Matrix Analysis (Cambridge: Cambridge University Press) (corrected reprint of the 1985 original)
[23] Horodecki M, Shor P W and Ruskai M B 2003 Entanglement breaking channels Rev. Math. Phys. 15 629–41
[24] Johnston N and Kribs D W 2011 Generalized multiplicative domains and quantum error correction Proc. Am. Math. Soc. 139 627–39
[25] Kadison R V 1951 Isometries of operator algebras Ann. Math. 54 325–38
[26] Knill E and Laflamme R 1997 Theory of quantum error-correcting codes Phys. Rev. A 55 900–11
[27] Kribs D W 2003 Quantum channels, wavelets, dilations and representations of Cn Proc. Edinburgh Math. Soc. 46 421–33
[28] Kribs D W and Spekkens R W 2006 Quantum error-correcting subsystems are unitarily recoverable subsystems Phys. Rev. A 74 042329
[29] Kümmerer B 1985 Markov dilations on W*-algebras J. Funct. Anal. 63 139–77
[30] Kuperberg G 2003 The capacity of hybrid quantum memory IEEE Trans. Inf. Theory 49 1465–73
[31] Levick J 2016 An uncertainty principle for completely positive maps in preparation (arXiv e-prints)
[32] Magesan E 2008 Gaining information about a quantum channel via twirling PhD Thesis University of Waterloo
[33] Magesan E 2012 Characterizing noise in quantum systems PhD Thesis University of Waterloo
[34] Müller-Hermes A, Stilck França D and Wolf M M 2016 Entropy production of doubly stochastic quantum channels J. Math. Phys. 57 022203
[35] Paulsen V 2002 Completely Bounded Maps and Operator Algebras (Cambridge Studies in Advanced Mathematics vol 78) (Cambridge: Cambridge University Press)
[36] Schwarz v 1966 New kinds of theorems on non-negative matrices Czech. Math. J. 16 285–95
[37] Størmer E 2007 Multiplicative properties of positive maps Math. Scand. 100 184–92
[38] Størmer E 2008 Separable states and positive maps J. Funct. Anal. 254 2303–12
[39] Størmer E 2013 Positive Linear Maps of Operator Algebras (Springer Monographs in Mathematics) (Heidelberg: Springer)
[40] Szehr O and Wolf M M 2016 Connected components of irreducible maps and 1D quantum phases J. Math. Phys. 57 081901
[41] Watrous J 2015 Theory of Quantum Information (draft manuscript)
[42] Werner R F 1998 Optimal cloning of pure states Phys. Rev. A 58 1827–32
[43] Wolf M M 2012 Quantum Channels and Operations—Guided Tour (Online Lecture Notes) www-m5.ma.tum.de/foswiki/pub/M5/Allgemeines/MichaelWolf/QChannelLecture.pdf (Accessed: 10 July 2017)