A characterization of the central shell-focusing singularity in spherical gravitational collapse

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Abstract
We give a characterization of the central shell-focusing curvature singularity that can form in the spherical gravitational collapse of a bounded matter distribution obeying the dominant energy condition. This characterization is based on the limiting behaviour of the mass function in the neighbourhood of the singularity. Depending on the rate of growth of the mass as a function of the area radius $R$, the singularity may be either covered or naked. The singularity is naked if this growth rate is slower than $R$, covered if it is faster than $R$, and either naked or covered if the growth rate is same as $R$.

1 Introduction

An outstanding open problem in classical general relativity and relativistic astrophysics is to determine the final fate of the continual gravitational collapse of a star. The singularity theorems show that, given a few physically reasonable assumptions, such a collapse will terminate in a gravitational singularity. However, the theorems do not by themselves imply that the star ultimately becomes a black hole. In order for a black hole to form, the singularity must be invisible to far away observers. The theorems however allow for the possibility that the singularity may be naked, i.e. visible to a far away observer. Naked singularities, if they were to form in gravitational collapse, are expected to have theoretical and observational properties very different from those of black holes. Because of the difficulty in obtaining a general solution of Einstein equations, it is not known whether classical general relativity generically admits the formation of naked singularities in gravitational collapse.
Over the years, various examples of the formation of covered (i.e. not naked) and naked singularities have been found in general relativity, largely in studies of spherical gravitational collapse. Typically, it has been observed that for any given equation of state, both naked and covered singularities arise in these examples, depending on the choice of the initial density and velocity distribution of the star. While on the one hand there is a similarity amongst these various examples, suggesting an underlying pattern, on the other hand the issue of the genericity of these examples is an open one.

In the present paper we give a characterization of covered and naked singularities in spherical collapse, on the basis of the behaviour of the mass function in the neighbourhood of the singularity.

Our characterization is related to an elementary observation regarding light propagation in Newtonian gravity. The divergence of the density at the center of a collapsing spherical Newtonian star can be taken to mean that the ratio $M/R^3$ diverges as $R \to 0$, where $M(R)$ is the mass to the interior of the radius $R$. However, according to the usual escape velocity argument, the escaping of light (moving at speed $c$) from this density singularity is governed by the ratio $2M(R)/R$, not by the ratio $M/R^3$. If in the approach to the singularity, $2M(R)/R$ goes to a limiting value less than unity, light will be able to escape, but not otherwise. We may write

$$\frac{2M}{R} = \frac{2M}{R^3} R^2$$

and it is evident that if $M/R^3$ diverges then $2M/R$ may either diverge or go to zero or be finite and non-zero, depending on whether $M/R^3$ diverges faster than $1/R^2$, slower than $1/R^2$, or as fast as $1/R^2$, respectively. On the other hand, if $2M/R$ diverges, then $M/R^3$ also diverges.

Continuing with this Newtonian argument, let us consider the dependence of the mass function $M(R)$ on the radius $R$, near the center, at any fixed time $t$. Clearly, $M(R)$ will grow slower than $R^0$, and faster than or equal to $R^3$, if we assume that the density at the center is non-zero, and if $M(R = 0) = 0$. If the evolution is non-singular, then $M$ goes as
$R^3$. A density singularity will form if there comes a stage during the evolution when $M$ goes faster than $R^3$. We know on the basis of theoretical and observational evidence that for a typical equation of state there are initial conditions (distribution of density and velocity) for which the collapsing matter cloud does not become singular, and either rebounces or attains equilibrium. These initial conditions correspond to selecting $M$ behaving as $R^3$ throughout the evolution. Let us now imagine choosing different classes of initial conditions, for the same equation of state, so as to move away from such non-singular evolutions and towards possible singular evolutions. In so far as $M(R)$ is concerned, this is equivalent to changing the final behavior of $M(R)$ from $M$ going as $R^3$ to $M$ going as faster than $R^3$, and approaching $R^0$, while possibly realizing growth rates between $R^3$ and $R^0$. So long as we are considering the class of solutions for which the final growth rate is between $R^3$ and $R$, the ratio $2M/R$ will go to zero and light will be able to escape the density singularity - this is an analog of the naked singularity in general relativity. If the growth rate lies between $R$ and $R^0$, the ratio $2M/R$ will go to infinity and light will not be able to escape from the density singularity. This is an analog of the covered singularity. If $M$ grows as $R$, either case may be realized, depending on the actual value the ratio takes.

Again on the basis of theoretical and observational evidence, we expect that the black hole type (i.e. covered) singular solutions, which correspond to $M$ going as faster than $R$, will be realized in gravitational collapse, from some initial conditions. As the initial conditions for a given equation of state are varied from the ‘weak’ to the ‘strong’, one goes from the non-singular class $M \sim R^3$ to the black hole type singular class ($M$ grows faster than $R$). In doing so, one may or may not encounter solutions of the intermediate ‘naked singularity’ class ($M$ goes slower than $R$ but faster than $R^3$). In this sense naked singularities, if they occur, lie between non-singular solutions and covered singular solutions.

Considering the similarity of spherical Newtonian collapse to the corresponding general relativistic case, one expects to be able to formulate such a characterization in spherical general relativity. We show below how this can be done, by considering the growth of a curvature invariant like the Kretschmann scalar, and by assuming the dominant energy
condition for the collapsing matter distribution. According to the dominant energy condition, pressure components cannot exceed the value of the density at any point in the spacetime.

We also point out that we deal here only with the characterization of local nakedness of singularities, i.e. the issue of whether outgoing light rays at all terminate on the singularities in the past. We do not concern ourselves with the question of whether or not these light rays escape to infinity (i.e. whether or not the singularities are globally naked).

2 A classification of shell-focusing singularities

We will assume the matter fields to have an energy-momentum tensor of Type I [2]. With respect to an orthonormal basis \( (E_0, E_1, E_2, E_3) \), and with \( E_0 \) being timelike, the energy-momentum tensor can be expressed in the form \( T_{ik} = diag(\rho, p_1, p_2, p_3) \). All known matter fields, including those of zero rest mass, have an energy-momentum tensor of this form, with the exception of directed radiation. \( \rho \) is the energy density as measured by an observer whose world line has a unit tangent vector \( E_0 \), while the components \( p_\alpha \) are the principal pressures in the three spacelike directions. The dominant energy condition implies that \( \rho \geq 0, |p_\alpha| \leq \rho \).

In comoving coordinates \((t, r, \theta, \phi)\) the spherically symmetric line element in general relativity is given by

\[
ds^2 = e^\sigma dt^2 - e^\omega dr^2 - R^2 d\Omega^2
\]  

(2)

where \( \sigma, \omega \) and \( R \) are functions of \( t \) and \( r \). The components of the energy-momentum tensor of a collapsing matter distribution are given, in these coordinates, by \( T_{ik}^i = diag(\rho, -p_r, -p_T, -p_T) \). The functions \( \rho, p_r \) and \( p_T \) have the interpretation of being the density, radial pressure and tangential pressure of the matter field, respectively.

If the matter content of the collapsing object is being described as a fluid, then one must supplement the above description of the energy-momentum tensor with two equations of state, one each for the radial and the tangential pressure. If on the other hand the matter content is a fundamental field, say a massless scalar field, then the components \( \rho, p_r \) and \( p_T \)
of $T^i_k$ follow from the Lagrangian density of the scalar field.

The Einstein field equations for this system are

$$\rho = \frac{F'}{R^2 R'},$$  \hspace{1cm} (3)

$$p_r = -\frac{\dot{F}}{R^2 \dot{R}},$$  \hspace{1cm} (4)

$$\sigma' = -\frac{2p'_r}{\rho + p_r} + \frac{4R'}{R(\rho + p_r)}(p_T - p_r),$$  \hspace{1cm} (5)

$$\dot{\omega} = -\frac{2\dot{\rho}}{\rho + p_r} - \frac{4\dot{R}(\rho + p_T)}{R(\rho + p_r)},$$  \hspace{1cm} (6)

and

$$e^{-\sigma} \dot{R}^2 = \frac{F}{R} + f.$$  \hspace{1cm} (7)

Here, prime and dot denote derivatives w.r.t. $r$ and $t$, respectively. In equations (3) and (4) we have set $8\pi G/c^4 = 1$. The function $F(t, r)$ results from the integration of Einstein equations, and has the interpretation of being twice the mass to the interior of the shell labeled by the coordinate $r$. The function $f(t, r)$ is defined by the relation $e^{\omega} = \dot{R}^2/(1 + f)$ and satisfies the condition $f \geq -1$.

We assume that at time $t = 0$ the spherical object starts undergoing gravitational collapse, and we choose the initial scaling $R(0, r) = r$. Next, we assume that a shell-focusing curvature singularity, given by $R(t_s(r), r) = 0$, forms during the evolution. At this singularity, the Kretschmann scalar $R_{abcd}R^{abcd}$ diverges. The shell labeled by the comoving coordinate $r$ becomes singular at the time $t_s(r)$. In particular, the ‘central’ curvature singularity, i.e. the one at $r = 0$, forms at the time $t_s(0)$. It is the central singularity that is
of primary interest from the point of view of nakedness, and is the object of study in the present paper. We will assume that \( t_\text{s}(0) < t_\text{s}(r \neq 0) \) and that the central shell-focusing singularity is not preceded by any shell-crossing singularity. Since \( F(t, r) \) is twice the mass, it is physically reasonable to take \( F(t, 0) = 0 \), prior to singularity formation. Furthermore, since the center is the first point to become singular, according to time \( t \), we assume that \( F(t_\text{s}(0), 0) = 0 \).

Let us consider the propagation of an outgoing radial null geodesic congruence \( \zeta^i \). As is known, the occurrence of a covered or locally naked singularity is determined by the behaviour of the geodesic expansion \( \theta = \zeta^i \), in the approach to the singularity. If the expansion of the outgoing null congruence continues to remain non-negative in the approach to the singularity, the singularity will be naked. On the other hand, the singularity will be covered if \( \theta \) becomes negative in the approach to the singularity. It can be shown [3] that in a spherically symmetric spacetime \( \theta \) can be written, using the form (2) of the metric, as

\[
\theta = \frac{2R'}{R} \left( 1 - \sqrt{\frac{f + F/R}{1 + f}} \right) \zeta^r.
\]  

(8)

Here, \( R' \) and \( \zeta^r \) are non-negative quantities. We will assume that the ratio \( 2R'\zeta^r/R \) remains non-zero in the limit. While this is an untested assumption, it is seen to hold at least in those cases when the dependence of \( R(t, r) \) on \( r \) is a power law - since that gives \( R'/R \sim 1/r \), and also, \( \zeta^r = dr/dk \sim r/k \). It then follows that the necessary and sufficient condition for the singularity to be naked is that \( F/R \) approach a value less than or equal to unity along the outgoing geodesic \( \zeta^i \), as it meets the singularity in the past. (The only exceptions to this condition arise when \( f \) takes the value \(-1\) or \(+\infty\) in the limit. In the former case, the singularity could be covered even if \( F/R \) goes to unity in the limit, and in the latter case it could be naked even if \( F/R \) takes a limiting value greater than unity.)

The above condition on \( F/R \) is nothing but the usual condition of no trapping, and is analogous to the condition on \( 2M/R \) in the Newtonian case.

Consider next the rate of divergence of the Kretschmann scalar at the central curvature
singularity, as the singularity is approached in the past along an outgoing null geodesic. Because of the dominant energy condition, the leading divergence in the Kretschmann scalar will be due to the energy density $\rho$. Also, if the density is non-singular, this scalar will be finite.

We first transform from the comoving coordinates $(t, r, \theta, \phi)$ to coordinates $(t, R, \theta, \phi)$, thus using the area radius as one of the coordinates. From equation (3) it follows that we can write, in these new coordinates

$$\rho(t, R) = \left(\frac{\partial F}{\partial R}\right)_t R^2.$$  

(9)

In order to evaluate the growth of $\rho$ along an outgoing null geodesic, we use the trajectory of the outgoing null geodesic itself as one of the coordinates. We label the trajectory by a parameter $X$, which will be a function of $t$ and $r$. We eliminate the coordinate $t$ in favour of $X$, and in the $(X, R, \theta, \phi)$ coordinates we get the evolution of the density to be

$$\rho(X, R) = \left(\frac{\partial F}{\partial R}\right)_X + \left(\frac{\partial X}{\partial R}\right)\left(\frac{\partial F}{\partial X}\right)_R R^2.$$  

(10)

Now, if the density evolution is to be non-singular, $F(X, R)$ must grow as $R^3$ or slower, as one approaches $R = 0$, keeping $X$ fixed. (A faster approach rate will cause the density to diverge, as is evident from (10)). On the other hand, if the evolution results in a singularity, we cannot conclude anything about the growth rate of $F$ as a function of $R$, at a fixed $X$. This is because the dominant divergence may come from the second term on the right hand side in Eqn. (10), rather than the first term. However, we know from (8) that in order for the singularity to be covered, it is necessary that this growth rate be at least as fast as $F(X, R) \sim R$, or faster.

We can now give a classification of the nature of the evolution (i.e. whether it is non-singular, or if singular whether it is naked or covered), depending on the behaviour of the mass function $F(X, R)$ as a function of $R$, for a fixed $X$, in the approach to the singularity. For a given form of matter, (say a fluid with a known equation of state, or a massless scalar
field) first consider initial data for which the evolution is non-singular. This means that along an outgoing null geodesic, $F$ grows as $R^3$ or slower.

Consider next changing the initial data so that we come to the class of data that lead to singular evolution. We divide such data into two possible subclasses, one for which the limiting growth of $F$, for a fixed $X$, is $R^3$ or slower, and the other for which it is faster than $R^3$. The singularities in the former class would be naked, since the ratio $F/R$ will go to zero in the limit. Hence, if the growth of $F(R)$ is as $R^3$ or slower, the evolution is either non-singular or singular and naked.

Those singular evolutions for which the limiting growth rate of $F$ is faster than $R^3$ can be further subdivided into three classes. If $F$ grows faster than $R^3$ but slower than $R$, the singularity is naked. If $F$ grows as $R$ the singularity is covered if the limiting value of $F/R$ exceeds unity, and naked if this limiting value is less than or equal to one. If $F$ grows faster than $R$ the singularity will be covered.

In view of the above characterization one can understand better the conditions that are required for naked singularity formation in spherical inhomogeneous dust collapse. For simplicity, we recall the case of marginally bound dust collapse, for which the initial density profile near the center, written in a series as

$$\rho(R) = \rho_0 + \rho_1 R + \frac{1}{2} \rho_2 R^2 + \frac{1}{6} \rho_3 R^3 + ...$$

(11)

determines the outcome of the collapse. It turns out that the singularity is naked if $\rho_1 < 0$ or if $\rho_1 = 0, \rho_2 < 0$. It is covered if $\rho_1 = \rho_2 = \rho_3 = 0$. If $\rho_1 = \rho_2 = 0$ and $\rho_3 < 0$ the singularity is naked if the dimensionless quantity $\zeta = \sqrt{3} \rho_3 / 4 \rho_0^{5/2}$ is less than or equal to $-25.9904$, and covered when $\zeta$ exceeds this value [4].

One could ask why the transition from the naked to the covered case takes place at the level of the third derivative. A physical answer is the following. It can be shown that in the approach to the central dust singularity along an outgoing null geodesic, $R \sim r^\alpha$, with $\alpha$ taking the value $(1 + 2p/3)$ where $p$ is defined such that the first non-vanishing derivative in
the above expansion for the density is the $p$th one. According to our naturalness argument above, if the singularity has to be naked, $F/R^3$ must not vary faster than $1/R^2$. Noting that $F \sim r^3$, we can write $F/R^3 \sim R^{3(1-\alpha)/\alpha}$ which implies that $\alpha \leq 3$, i.e. $p \leq 3$, is necessary for nakedness ($p < 3$ is sufficient for nakedness). This physically explains the significance of the transition taking place at the level of the third derivative. We also note that the condition that $2R'\zeta r/R$ does not vanish in the limit is satisfied in the dust model.

3 Discussion and Conclusions

We summarize here the assumptions that we have made: (i) We have considered Type I matter fields obeying the dominant energy condition; (ii) we have assumed that the chosen initial conditions are such that any central shell-focusing singularity that might form in collapse is not preceded by a shell-crossing singularity; (iii) in the expression (8) for $\theta$ we have assumed that the quantity $2R'\zeta r/R$ remains non-zero in the limit.

Subject to these assumptions, and subject to choosing a specific form of matter, we have given a partial classification of the nature of evolution (as to whether it is non-singular, and if singular whether it is covered or naked) in terms of the behaviour of the mass function in the neighbourhood of the origin. For a given set of initial conditions, the evolution is non-singular or naked singular if this growth rate is $R^3$ or slower. If at some epoch during the evolution the growth rate becomes faster than $R^3$, this shows the formation of a curvature singularity at that epoch. This singularity is naked if the growth rate is slower than $R$, it is covered if the growth rate is faster than $R$, and it could be either covered or naked if the growth rate is same as $R$.

We make the following observation about the nature of the shell-focusing singularity. Suppose it is the case that for a given form of matter, there are generic initial data leading to non-singular evolution, and other generic initial data leading to covered singularities, but the initial data leading to naked singularities is non-generic. This implies the following regarding the behaviour of the mass function. There are generic initial conditions for which the mass
function grows as $R^3$ and other generic initial conditions for which, near the singularity, it grows as $R$ or faster. However, the initial conditions for which it grows faster than $R^3$ but slower than $R$ near the singularity are non-generic.

A priori, one cannot draw conclusions as to how the dependence of the mass function on $R$ changes, when the initial data is changed so as to go from the non-singular class of solutions to the singular class. If this change is a continuous one, then there will necessarily be generic data for which $M(R)$ grows faster then $R^3$ but slower than $R$, and naked singularity formation will be generic. On the other hand, if the change is a discontinuous one, and there is a jump from $M(R) \sim R^3$ to $M(R) \sim$ faster than $R$, naked singularities will be non-generic (provided also that evolutions of the type $M \sim R^3$ are generically non-singular). Whether or not this change is a continuous one cannot be decided from the present study, and further investigation is necessary.

Although we cannot conclude here about the genericity or non-genericity of locally naked singularities, we compare our results briefly with other recent developments relating to naked singularities in spherical gravitational collapse. Christodoulou [5] has shown that for the spherical collapse of a massless scalar field, the initial data leading to globally naked singularities is non-generic. To our understanding, Christodoulou’s result does not have a bearing on whether or not the locally naked singularities in his model are non-generic. Since the arguments we have given above are only for local nakedness, they cannot be directly compared with Christodoulou’s conclusions. Joshi and Dwivedi [6] have demonstrated that the formation of locally naked singularities is generic in the spherical collapse of Type I matter fields obeying the weak energy condition. Since the weak energy condition does not necessarily imply the dominant energy condition, their result does not necessarily imply that $M(R)$ grows faster than $R^3$ and slower than $R$ near the singularity, for generic initial data.

Finally, we comment briefly on numerical studies of spherical collapse. In these studies (including those on scalar fields and fluids) it has been found that dispersive evolutions and black hole formation both result from generic initial data. In many of the the black hole class
of solutions, the amount of mass that collapses to form the black hole behaves as $(p - p_*)^\gamma$, where $p$ is a parameter labeling a family of solutions, and $p_*$ is the critical value below which the evolution is dispersive. The limiting solution for which $p \to p_*$ is a naked singularity, and numerical studies suggest that such a naked singularity is non-generic, because it appears to occur only at one value of $p$, i.e. $p = p_*$.

To our understanding however, the following important issue should be taken note of. In most numerical studies to date, the identification of a black hole in a simulation is carried out via the detection of an apparent horizon. Such a detection does not by itself rule out the development of a Cauchy horizon and a naked singularity which is not covered by the apparent horizon. Hence, in principle, the solutions that are classified as black holes could include, as a subset, naked singular solutions. The few simulations that have been carried out using double null coordinates that can be extended up to the center of spherical symmetry do not appear to offer a completely clear picture in this regard. Hence it appears to us that further numerical investigation is necessary in order to examine the issue of genericity of naked singularities in spherical collapse.

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References

[1] for recent reviews of the status of naked singularities in classical general relativity, and for references to the literature, see for instance, R. Penrose in Black Holes and Relativistic Stars ed. R. M. Wald (Chicago University Press, 1998); R. M. Wald, gr-qc/9710068. P. S. Joshi, Global Aspects in Gravitation and Cosmology (Oxford, 1993), gr-qc/9702030. T. P. Singh, gr-qc/9606016, gr-qc/9805060.

[2] S. W. Hawking and G. F. R. Ellis, The large scale structure of space-time, Cambridge University Press (1973).
[3] T. P. Singh, Phys. Rev. D58 (1998) 024004.

[4] T. P. Singh and P. S. Joshi, Class. Quantum Gravity 13, 559 (1996).

[5] D. Christodoulou, Annals of Mathematics, 149 (1999) 183.

[6] P.S. Joshi and I.H. Dwivedi, Class. Quantum Grav. 16, 41 (1999).