Subspace Codes based on Graph Matchings, Ferrers Diagrams and Pending Blocks

Natalia Silberstein and Anna-Lena Trautmann

Abstract

This paper provides new constructions and lower bounds for subspace codes, using Ferrers diagram rank-metric codes from matchings of the complete graph and pending blocks. We present different constructions for constant dimension codes with minimum injection distance $2$ or $k - 1$, where $k$ is the constant dimension. Furthermore, we present a construction of new codes from old codes for any minimum distance. Then we construct non-constant dimension codes from these codes. The examples of codes obtained by these constructions are the largest known codes for the given parameters.

Index Terms

Subspace codes, constant dimension codes, Grassmannian, Ferrers diagram rank-metric codes, graph matchings.

I. INTRODUCTION

Let $\mathbb{F}_q$ be the finite field of size $q$. Given two integers $k, n$, such that $0 \leq k \leq n$, the set of all $k$-dimensional subspaces of $\mathbb{F}_q^n$ forms the Grassmannian over $\mathbb{F}_q$, denoted by $G_q(k, n)$. It is well known that the cardinality of the Grassmannian is given by the $q$-ary Gaussian coefficient

$$|G_q(k, n)| = \binom{n}{k}_q = \frac{1}{k!} \prod_{i=0}^{k-1} q^{n-i} - 1.$$

The set of all subspaces of $\mathbb{F}_q^n$ is denoted by $P_q(n)$. It holds that $P_q(n) = \bigcup_{k=0}^{n} G_q(k, n)$.

Both the subspace distance, defined as

$$d_S(X,Y) \overset{\text{def}}{=} \dim X + \dim Y - 2 \dim(X \cap Y),$$

and the injection distance, defined as

$$d_I(X,Y) \overset{\text{def}}{=} \max\{\dim X, \dim Y\} - \dim(X \cap Y),$$

for any two distinct subspaces $X$ and $Y$ in $P_q(n)$, are metrics on $P_q(n)$, and hence also on $G_q(k, n)$. Note that for $X, Y \in G_q(k, n)$ it holds that $d_S(X,Y) = 2d_I(X,Y)$.

We say that $C \subseteq G_q(k, n)$ is an $(n, M, d, k)_q$ code in the Grassmannian, or constant-dimension code, if $M = |C|$ and $d_I(X,Y) \geq d$ for all distinct elements $X,Y \in C$ (or equivalently $d_S(X,Y) \geq 2d$). Furthermore, we call $C \subseteq P_q(n)$ an $(n, M, d)_q$ subspace code, or projective space code, if $M = |C|$ and $d_S(X,Y) \geq d$ for all distinct elements $X,Y \in C$. If we use the injection distance instead of the subspace distance, we denote it by $(n, M, d)_I^q$. $A_q(n, d, k)$ will denote the maximum size of an $(n, M, d, k)_q$-code. By $A_q^*(n, d, k)$ we denote the size of the largest known $(n, M, d, k)_q$-code.

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Subspace codes, and constant dimension codes in particular, have drawn significant attention in the last six years due to the work by Koetter and Kschischang [14], where they presented an application of such codes for error correction in random network coding. Constructions and bounds for constant dimension codes were given e.g. in [2]–[5], [7], [8], [11], [13], [15], [17], [18], [20], [23]. For non-constant dimension codes some results can be found in [3], [5], [10]–[12].

One notes that the codes obtained by a simple construction based on lifting of maximum rank distance (MRD) codes [19] are almost optimal, i.e., asymptotically attain the known upper bounds [5], [14]. However, it is of interest to provide constructions of constant dimension codes which are larger than the lifted MRD codes. The first step in this direction was done in [3], where the multilevel construction was presented. This construction generalizes the lifted MRD codes construction by introducing a new family of rank-metric codes having a given shape of their codewords, namely, Ferrers diagram rank-metric codes. Further, some other constructions were presented in [2], [4], [5], [7], [8], [11]. [20], [23]. Most of them provide constant dimension codes which contain a lifted MRD code as a subcode. Another type of constructions includes orbit or cyclic codes [5], [13], [22]. In [4], an upper bound on the cardinality of codes which contain a lifted MRD code was presented for some sets of parameters. For constant dimension $k = 3$ this bound was attained by using a generalization of a pending dots based construction, presented in [23].

In this paper, we continue with this direction of constructing large constant dimension codes which contain lifted MRD codes. We present new families of codes which have the largest known cardinality. The ideas for these constructions generalize the ideas presented in [3], [4], [19], [23]. First, we present new \((n, M, k − 1, k)_q\)-codes. These codes have the second largest possible injection distance $k − 1$ (codes having the largest possible injection distance $k$ are called (partial) spread codes and were considered in e.g. [1], [9], [15]). Our new codes are based on a two-dimensional generalization of pending dots, which we call pending blocks. Based on this approach we construct \((n, M, k − 1, k)_q\)-codes of cardinality

$$M \geq q^{2(n−k)} + \sum_{j=3}^{k-1} q^{2(n−\sum_{i=j}^{k} i)} + \left[ n - \frac{k^2 + k - 6}{2} \right]_q.$$

This construction requires the field size to be large enough. For smaller fields, we slightly modify the construction and the obtained codes have almost the same cardinality as in [3].

Next, we focus on codes with the smallest non-trivial injection distance $d_I = 2$ (a code with the smallest possible distance $d_I = 1$ is the trivial code which contains the whole Grassmannian). We start with the multilevel construction of [3]. The main drawback of this construction is that it depends on the choice of the underlying constant weight code, but the best choice for such a code is still unknown. As a consequence, the cardinality of constant dimension codes obtained by the multilevel construction can not be written in a general form. We consider a specific choice of a constant weight code for the multilevel construction. This constant weight code is based on an one-factorization of a complete graph. The cardinality of the proposed \((n, M, 2, k)_q\)-code can be derived and is given by

$$M \geq \sum_{i=1}^{\lfloor \frac{n-2}{2} \rfloor} \left( q^{(k-1)(n-ik)} + \frac{(q^{2(k-2)} - 1)(q^{2(n-ik-1)} - 1)}{(q^4 - 1)^2} q^{(k-3)(n-ik-2)+4} \right).$$

Then, we combine the idea of one-factorization based constant weight codes with the pending blocks construction and present a new family of \((n, M, 2, k)_q\) codes. Here, we use the one-factorization of a specific node labelling of the complete graph to provide codes with large cardinality.

In addition, we present a simple way to construct a new constant dimension code from an old one, with the same minimum distance. Surprisingly, for some parameters this construction provides the largest known codes. In particular, we derive the following recursive formula for the maximum cardinality of a constant dimension code, for any $n \geq 3k$ and $n \geq \Delta \geq k$:

$$A_q(n, d, k) \geq q^\Delta(k-d+1) A_q(n - \Delta, d, k) + A_q(\Delta, d, k).$$

Finally, we consider non-constant dimension codes. We use the constant dimension codes constructed in this paper as well as the largest codes from [3], [4] and apply the puncturing method [3] to obtain large codes for both the subspace and the injection metric.
We present tables comparing our constructions with the previously known (constant and non-constant dimension) codes and show that for some parameters our new codes have larger cardinality.

The rest of this paper is organized as follows. In Section II we introduce the necessary definitions and two known constructions which will be the starting points to our new constructions. In Section III we introduce the notation of pending blocks and present a construction for an \((n, M, k-1, k)_{q}\)-code. In Section IV we consider properties of Ferrers diagrams arising from matchings of complete graphs and discuss the constructions for \((n, M, 2, k)_{q}\)-codes. In Section V we present a construction of a new code from a given one. Section VI presents the comparison between the new codes obtained in the paper and some previously known codes. We consider constructions of non-constant dimension codes in Section VII and conclude with Section VIII.

II. PRELIMINARIES AND RELATED WORK

In this section we briefly provide the definitions and previous results used in our constructions. More details can be found in [3], [4], [23].

A. Representations of Subspaces and Multilevel Construction

Let \(X\) be a \(k\)-dimensional subspace of \(\mathbb{P}_q^n\). We represent \(X\) by the matrix \(\text{RE}(X)\) in reduced row echelon form, such that the rows of \(\text{RE}(X)\) form a basis of \(X\). The identifying vector of \(X\), denoted by \(v(X)\), is the binary vector of length \(n\) and weight \(k\), where the \(k\) ones of \(v(X)\) are exactly in the positions where \(\text{RE}(X)\) has the leading coefficients (the pivots). All the binary vectors of length \(n\) and weight \(k\) can be considered as the identifying vectors of all the subspaces in \(G_q(k, n)\). These \(\binom{n}{k}\) vectors partition \(G_q(k, n)\) into \(\binom{n}{k}\) different classes, where each class, also called a cell of \(G_q(k, n)\), consists of all subspaces in \(G_q(k, n)\) with the same identifying vector.

Recall that the Hamming metric on \(\mathbb{F}_q^n\) is defined as \(d_H(u, v) \overset{\text{def}}{=} \text{wt}(u - v)\), where \(\text{wt}(w)\) denotes the number of nonzero entries in the vector \(w\). The asymmetric metric on \(\mathbb{F}_q^n\) is defined as \(d_{\text{asym}}(u, v) \overset{\text{def}}{=} \max\{N(u, v), N(v, u)\}\), where \(N(u, v)\) denotes the number of coordinates \(i\) where \(u_i = 1\) and \(v_i = 0\) [11]. The following results are useful tools for constructions of subspace codes.

**Proposition 1** ([3], [11], [12]). For \(X, Y \in \mathcal{P}_q(n)\) we have

- \(d_S(X, Y) \geq d_H(v(X), v(Y))\)
- \(d_I(X, Y) \geq d_{\text{asym}}(v(X), v(Y))\).

The Ferrers tableaux form of a subspace \(X\), denoted by \(\mathcal{F}(X)\), is obtained from \(\text{RE}(X)\) first by removing from each row of \(\text{RE}(X)\) the zeroes to the left of the leading coefficient; and after that removing the columns which contain the leading coefficients. All the remaining entries are shifted to the right. The Ferrers diagram of \(X\), denoted by \(\mathcal{F}_X\), is obtained from \(\mathcal{F}(X)\) by replacing the entries of \(\mathcal{F}(X)\) with dots.

Given \(\mathcal{F}(X)\), the unique corresponding subspace \(X \in G_q(k, n)\) can easily be found. Also given \(v(X)\), the unique corresponding \(\mathcal{F}_X\) can be found. When we fill the dots of a Ferrers diagram by elements of \(\mathbb{F}_q\), we obtain a \(\mathcal{F}(X)\) for some \(X \in G_q(k, n)\).

**Example 2.** Let \(X\) be the subspace in \(G_2(3, 7)\) with the following generator matrix in reduced row echelon form:

\[
\text{RE}(X) = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1
\end{pmatrix}.
\]

Its identifying vector is \(v(X) = 1011000\), and its Ferrers tableaux form and Ferrers diagram are given by

\[
\begin{pmatrix}
0 & 1 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 1
\end{pmatrix},
\]

\[
\begin{pmatrix}
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet
\end{pmatrix},
\]

respectively.

In the following we will consider Ferrers diagram rank-metric codes, which are closely related to constant dimension codes. For two \(m \times \ell\) matrices \(A\) and \(B\) over \(\mathbb{F}_q\) the rank distance, \(d_R(A, B)\), is defined by

\[d_R(A, B) \overset{\text{def}}{=} \text{rank}(A - B)\].
Proposition 3 ([5], [11]). For \( X, Y \in \mathcal{P}_q(n) \) we have that if \( v(X) = v(Y) \) then

- \( d_S(X, Y) = 2d_R(RE(X), RE(Y)) \).
- \( d_I(X, Y) = d_R(RE(X), RE(Y)) \).

Let \( \mathcal{F} \) be a Ferrers diagram with \( m \) dots in the rightmost column and \( \ell \) dots in the top row. A code \( \mathcal{C}_\mathcal{F} \) is an \([\mathcal{F}, \rho, d] \) Ferrers diagram rank-metric (FDMRD) code if all codewords of \( \mathcal{C}_\mathcal{F} \) are \( m \times \ell \) matrices in which all entries not in \( \mathcal{F} \) are zeroes, they form a linear subspace of dimension \( \rho \) of \( \mathbb{F}_q^{m \times \ell} \), and for any two distinct codewords \( A \) and \( B \), \( d_R(A, B) \geq d \). If \( \mathcal{F} \) is a rectangular \( m \times \ell \) diagram with \( m\ell \) dots then the FDMRD code is a classical rank-metric code [6], [16]. The following theorem provides an upper bound on the cardinality of \( \mathcal{C}_\mathcal{F} \).

Theorem 4 ([3]). Let \( \mathcal{F} \) be a Ferrers diagram and \( \mathcal{C}_\mathcal{F} \) the corresponding \([\mathcal{F}, \rho, d] \) FDMRD code. Then \( \rho \leq \min_i \{ w_i \} \), where \( w_i \) is the number of dots in \( \mathcal{F} \) which are not contained in the first \( i \) rows and the rightmost \( d - 1 - i \) columns \((0 \leq i \leq d - 1)\).

A code which attains the bound of Theorem 4 is called a Ferrers diagram maximum rank distance (FDMRD) code. Maximum rank distance (MRD) codes are a class of \([\mathcal{F}, \ell(m - d + 1), d] \) FDMRD codes, \( \ell \geq m \), with a full \( m \times \ell \) diagram \( \mathcal{F} \), which attain the bound of Theorem 4 [6], [16].

It was proved in [3] that for general diagrams the bound of Theorem 4 is attained for \( d = 1, 2 \):

Theorem 5. For any Ferrers diagram \( \mathcal{F} \) there exists a \([\mathcal{F}, \rho, d] \) FDMRD code for \( d = 1 \) or \( d = 2 \).

Some special cases, when this bound is attained for \( d > 2 \), can also be found in [3].

For a codeword \( A \in \mathcal{C}_\mathcal{F} \subseteq \mathbb{F}_q^{k\times(n-k)} \) let \( A_\mathcal{F} \) denote the part of \( A \) related to the entries of \( \mathcal{F} \) in \( A \).

Definition 6. Given an FDMRD code \( \mathcal{C}_\mathcal{F} \), a lifted FDMRD code \( \mathcal{C}_\mathcal{F} \) is defined as follows:

\[
\mathcal{C}_\mathcal{F} = \{ X \in \mathcal{G}_q(k, n) : \mathcal{F}(X) = A_\mathcal{F}, A \in \mathcal{C}_\mathcal{F} \}.
\]

This definition is the generalization of the definition of a lifted MRD code [19]. Note, that all the codewords of a lifted MRD code have the same identifying vector of the type \( (11\ldots000\ldots00) \). The following theorem [3] is the generalization of the result given in [19].

Theorem 7. If \( \mathcal{C}_\mathcal{F} \subseteq \mathbb{F}_q^{k\times(n-k)} \) is an \([\mathcal{F}, \rho, d] \) Ferrers diagram rank-metric code, then its lifted code \( \mathcal{C}_\mathcal{F} \) is an \((n, q^d, k, d)_q \) constant dimension code.

The Multilevel Construction [3] for constant dimension codes is based on Proposition 1 and Theorem 7.

Multilevel Construction. First, a binary constant weight code of length \( n \), weight \( k \), and Hamming distance \( 2d \) is chosen to be the set of the identifying vectors for \( \mathcal{C} \). Then, for each identifying vector a corresponding lifted FDMRD code with minimum injection distance \( d \) is constructed. The union of these lifted FDMRD codes is an \((n, M, d, k)_q \) code.

B. One-Factorization of Complete Graphs and the Pending Dots Construction

In the construction provided in [4], for \( k = 3 \) and \( d = 2 \), in the stage of choosing the identifying vectors for a code \( \mathcal{C} \), a set of vectors with minimum (Hamming) distance \( 2d - 2 = 2 \) is allowed, by using a method based on pending dots in a Ferrers diagram [23], which will be explained in the following.

The pending dots of a Ferrers diagram \( \mathcal{F} \) are the leftmost dots in the first row of \( \mathcal{F} \) whose removal has no impact on the size of the corresponding Ferrers diagram rank-metric code. The following lemma follows from [23].

Lemma 8. Let \( X \) and \( Y \) be two subspaces in \( \mathcal{G}_q(k, n) \) with \( d_H(v(X), v(Y)) = 2d - 2 \), such that the leftmost one of \( v(X) \) is in the same position as the leftmost one of \( v(Y) \). Let \( P_X \) and \( P_Y \) be the sets of pending dots of \( X \) and \( Y \), respectively. If \( P_X \cap P_Y \neq \emptyset \) and the entries in \( P_X \cap P_Y \) (of their Ferrers tableaux forms) are assigned with different values in at least one position, then \( d_S(X, Y) = 2d_I(X, Y) \geq 2d \).

Example 9. Let \( X \) and \( Y \) be subspaces in \( \mathcal{G}_q(3, 6) \) which are given by the following generator matrices:

\[
\begin{pmatrix}
1 & 0 & 0 & u_1 & v_2 & 0 \\
0 & 0 & 1 & v_3 & v_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 1 & u_1 & 0 & v_2 & 0 \\
0 & 0 & 0 & 1 & u_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]
where \( v_i, u_i \in \mathbb{F}_q \), and the pending dots are emphasized by circles. Their identifying vectors are \( v(X) = 101001 \) and \( v(Y) = 100101 \). Clearly, \( d_H(v(X), v(Y)) = 2 \), while \( d_S(X, Y) \geq 4 \).

The following results from the area of graph theory will be useful in the following code constructions. We denote by \( K_m \) the complete graph with \( m \) nodes. A matching of \( K_m \) is a set of non-adjacent edges of \( K_m \). A perfect (resp. nearly perfect) matching is a matching that covers all (resp. all but one) nodes of \( K_m \). A one-factorization (OF) (resp. near one-factorization (NOF)) of \( K_m \) is a partition of all edges into perfect (resp. nearly perfect) matchings of \( K_m \). If one labels all nodes of \( K_m \) with the numbers from \( 1, \ldots, m \), then one can easily see the \( 1-1 \)-correspondence between the edges of the graph and the weight-2 vectors of \( \mathbb{F}_2^m \) by assigning the two ones of the vector in the coordinates labelled by the numbers of the two nodes in the graph which are connected by the corresponding edge.

The following lemma, which follows from a one-factorization and near-one-factorization of a complete graph [24], [25], will be used in our constructions.

**Lemma 10.** Let \( D \) be the set of all binary vectors of length \( m \) and weight 2.

- If \( m \) is even, \( D \) can be partitioned into \( m - 1 \) classes, each of \( \frac{m}{2} \) vectors with pairwise disjoint positions of ones;
- If \( m \) is odd, \( D \) can be partitioned into \( m \) classes, each of \( \frac{m-1}{2} \) vectors with pairwise disjoint positions of ones.

The following construction for \( k = 3 \) and \( d = 2 \) based on pending dots from [4] will be used as the base step of our recursive construction proposed in the sequel.

**Pending Dots Construction.** Let \( n \geq 8 \) and \( q^2 + q + 1 \geq \ell \), where \( \ell = n - 4 \) for odd \( n \) and \( \ell = n - 3 \) for even \( n \). In addition to the lifted MRD code (which has the identifying vector \( v_0 = (11100\ldots0) \)), the final code \( C \) will contain the codewords with identifying vectors of the form \( (x||y) \), where the prefix \( x \in \mathbb{F}_2^q \) is of weight 1 and the suffix \( y \in \mathbb{F}_2^{n-3} \) is of weight 2. By Lemma 10 we partition the set of suffixes into \( \ell \) classes \( P_1, P_2, \ldots, P_\ell \) and define the following three sets:

\[
A_1 = \{(001||y) : y \in P_1\},
\]

\[
A_2 = \{(010||y) : y \in P_1, 2 \leq i \leq \min\{q+1, \ell\}\},
\]

\[
A_3 = \begin{cases} 
(100||y) : y \in P_1, q + 2 \leq i \leq \ell & \text{if } \ell > q + 1 \\
\emptyset & \text{if } \ell \leq q + 1 
\end{cases}.
\]

Elements with the same prefix and distinct suffixes from the same class \( P_i \) have Hamming distance 4. When we use the same prefix for two different classes \( P_i, P_j \), we assign different values in the pending dots of the Ferrers tableaux forms. Then the corresponding lifted FDMRD codes of injection distance 2 are constructed, and their union with the lifted MRD code forms the final code \( C \) of size \( q^{2(n-3) + \left\lceil \frac{n-3}{2} \right\rceil} \).

In the following sections we will generalize this construction in various ways and obtain codes for any \( k \geq 4 \) with minimum injection distance \( d = 2 \) or with \( d = k - 1 \), or equivalently minimum subspace distance \( 2d = 4 \) or with \( 2d = 2(k - 1) \).

**III. CONSTRUCTION FOR \((n, M, k - 1, k)_q \) CODES**

In this section we provide a recursive construction for \((n, M, k - 1, k)_q \) codes, which uses the Pending Dots Construction described in Section III as an initial step. Codes obtained by this construction contain a lifted MRD code. The upper bound on the cardinality of such codes is derived in [4] and given in the following theorem.

**Theorem 11 ([4]).** If an \((n, M, k - 1, k)_q \) code \( C \), \( k \geq 3 \), contains an \((n, q^2(n-k), k - 1, k)_q \) lifted MRD code then

\[
M \leq q^{2(n-k)} + A_q(n - k, k - 2, k - 1).
\]

Note that for \( k = 3 \) this bound is given by

\[
M \leq q^{2(n-3)} + \left\lceil \frac{n-3}{2} \right\rceil.
\]
which is attained by the Pending Dots Construction. Our recursive construction provides a new lower bound on the cardinality of such codes for general \( k \).

To present the construction we first need to extend the definition of pending dots of [23] to a two-dimensional setting, which we will do in the following subsection.

### A. Pending Blocks

**Definition 12.** Let \( F \) be a Ferrers diagram with \( m \) dots in the rightmost column and \( \ell \) dots in the top row. We say that the \( \ell_1 < \ell \) leftmost columns of \( F \) form a pending block (of length \( \ell_1 \)) if the upper bound on the size of FDMRD code \( C_F \) from Theorem 4 is equal to the upper bound on the size of \( C_F \) without the \( \ell_1 \) leftmost columns.

**Example 13.** Consider the following Ferrers diagrams:

\[
F_1 = \begin{array}{cccc}
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet \\
\end{array}, \quad F_2 = \begin{array}{cccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array}.
\]

For \( d = 3 \) by Theorem 4 both codes \( C_{F_1} \) and \( C_{F_2} \) have \( |C_{F_i}| \leq q^3, i = 1, 2 \). The diagram \( F_1 \) has the pending block \( \bullet \bullet \) and the diagram \( F_2 \) has no pending block.

**Definition 14.** Let \( F \) be a Ferrers diagram with \( m \) dots in the rightmost column and \( \ell \) dots in the top row, and let \( \ell_1 < \ell \), and \( m_1 < m \). If the \( (m_1 + 1) \)st row of \( F \) has less dots than the \( m_1 \)th row of \( F \) and at most \( m - \ell_1 \) dots, then the \( \ell_1 \) leftmost columns of \( F \) are called a quasi-pending block (of size \( m_1 \times \ell_1 \)).

Note that a pending block is also a quasi-pending block.

**Theorem 15.** Let \( X, Y \in G_{q}(k,n) \), such that \( \text{RE}(X) \) and \( \text{RE}(Y) \) have a quasi-pending block of size \( m_1 \times \ell_1 \) in the same position and \( d_H(v(X), v(Y)) = 2d \). Denote the submatrices of \( F(X) \) and \( F(Y) \) corresponding to the quasi-pending blocks by \( B_X \) and \( B_Y \), respectively. Then \( d_I(X,Y) \geq d + \text{rank}(B_X - B_Y) \) or equivalently \( d_S(X,Y) \geq 2d + 2\text{rank}(B_X - B_Y) \).

**Proof:** Since the quasi-pending blocks are in the same position, the first \( h \) pivots of \( \text{RE}(X) \) and \( \text{RE}(Y) \) are in the same columns. To compute the rank of \[
\begin{bmatrix}
\text{RE}(X) \\
\text{RE}(Y)
\end{bmatrix}
\]
we permute the columns such that the \( h \) first pivot columns are to the very left, then the columns of the pending block, then the other pivot columns and then the rest:

\[
\begin{bmatrix}
1 & \ldots & 0 & B_X & \ldots \\
0 & \ldots & 1 & 0 & \ldots \\
0 & \ldots & 0 & 0 & \ldots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & 0 & \ldots \\
0 & \ldots & 0 & 0 & \ldots \\
\end{bmatrix}
\]

Now we subtract the lower half from the upper one and get
The additional pivots of \( \text{RE}(X) \) and \( \text{RE}(Y) \) (to the right in the above representation) that were in different columns in the beginning are still in different columns, hence it follows that \( \text{rank} \begin{bmatrix} \text{RE}(X) \\ \text{RE}(Y) \end{bmatrix} \geq k + \text{rank}(B_X - B_Y) + d \), which implies the statement.

This theorem implies that for the construction of an \((n, M, d, k)_q\)-code, by filling the (quasi-) pending blocks with a suitable Ferrers diagram rank metric code, one can choose a set of identifying vectors with lower minimum Hamming distance than \( 2d \).

\[ \begin{array}{c}\text{= rank} \\
\begin{bmatrix}
1 & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & 1 \\
0 & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & 0 \\
0 & \ldots & 0 \\
\end{bmatrix}
\end{array} \]

\( B_X \)

\( B_X - B_Y \)

\[ \begin{array}{c}\ldots \\
\vdots \\
\vdots \\
\vdots \\
\end{array} \]

\[ \begin{array}{c}\ldots \\
\vdots \\
\vdots \\
\vdots \\
\end{array} \]

\[ \begin{array}{c}\ldots \\
\vdots \\
\vdots \\
\vdots \\
\end{array} \]

The additional pivots of \( \text{RE}(X) \) and \( \text{RE}(Y) \) (to the right in the above representation) that were in different columns in the beginning are still in different columns, hence it follows that \( \text{rank} \begin{bmatrix} \text{RE}(X) \\ \text{RE}(Y) \end{bmatrix} \geq k + \text{rank}(B_X - B_Y) + d \), which implies the statement.

This theorem implies that for the construction of an \((n, M, d, k)_q\)-code, by filling the (quasi-) pending blocks with a suitable Ferrers diagram rank metric code, one can choose a set of identifying vectors with lower minimum Hamming distance than \( 2d \).

\( B \)

**B. The Construction.**

The following two lemmas will be useful for our construction.

**Lemma 16.** Let \( n - k - 2 \geq n_1 \geq k - 2 \) and \( v \) be an identifying vector of length \( n \) and weight \( k \), such that there are \( k - 2 \) many ones in the first \( n_1 \) positions of \( v \). Then the Ferrers diagram arising from \( v \) has more or equally many dots in any of the first \( k - 2 \) rows than in the last column, and the upper bound for the dimension of a Ferrers diagram code with minimum rank distance \( k - 1 \) is the number of dots in the lower two rows.

**Proof:** Naturally, the last column of the Ferrers diagram has at most \( k \) many dots. It holds that any column has at most as many dots as the last one. Since there are \( k - 2 \) many ones in the first \( n_1 \) positions of \( v \), it follows that there are \( n - n_1 - 2 \) zeros in the last \( n - n_1 \) positions of \( v \). Thus, there are at least \( n - n_1 - 2 \) many dots in any but the lower two rows of the Ferrers diagram arising from \( v \). Therefore, if \( n - n_1 - 2 \geq k \iff n - k - 2 \geq n_1 \) the Ferrers diagram arising from \( v \) has more than or equally many dots in any of the first \( k - 2 \) rows than in the last column, and hence than in any column.

From Theorem 4 we know that the bound on the dimension of the FDRM code is given by the minimum number of dots not contained in the first \( i \) rows and last \( k - 2 - i \) columns for \( i = 0, \ldots, k - 2 \). If we start with \( i = k - 2 \) we get that the dimension of the code is at most the number of dots in the last two rows of the diagram. Inductively, if we decrease \( i \) by one, we add a row (of the first \( k - 2 \) rows) and erase a column of the previous diagram, which results in more points, hence the minimum is attained for \( i = k - 2 \).

**Remark 17.** If an \( m \times \ell \)-Ferrers diagram has \( d - 1 \) rows with \( \ell \) dots each, then the construction of \([3]\) provides respective FDMRD codes of minimum distance \( d \) attaining the bound of Theorem 4.

We need yet another special case of Ferrers diagrams where we can attain the upper bound on the dimension of the code size.

**Lemma 18.** For an \( m \times \ell \)-Ferrers diagram where the \( j \)th row has at least \( x \) more dots than the \((j + 1)\)th row for \( 1 \leq j \leq m - 1 \) and the lowest row has \( x \) many dots, there is a FDMRD code with minimum rank distance \( m \) and cardinality \( q^x \).

**Proof:** The construction is as follows: For each codeword take a different \( w \in \mathbb{F}_q^x \) and fill the first \( x \) dots of every row with this vector, whereas all other dots are filled with zeros. The minimum distance follows easily from the fact that the positions of the \( w \)’s in each row have no column-wise intersection. Since they are all different, any difference of two codewords has a non-zero entry in each row and it is already row-reduced.
The cardinality is clear, hence it remains to show that this attains the bound of Theorem 3. Plugging in $i = k - 1$ in Theorem 3, we get that the dimension of the code is less than or equal to the number of dots in the last row, which is achieved by this construction.

We now have all the machinery to describe the new construction for $(n, M, k - 1, k)_q$-codes.

**Construction A.**

Let $k \geq 4$, $s = \sum_{i=3}^{k} i = \frac{k^2 + k - 6}{2}$, $n \geq s + 2 + k = \frac{k^2 + 3k - 2}{2}$ and $q^2 + q + 1 \geq \ell$, where $\ell = n - s = n - \frac{k^2 + k - 6}{2}$ for odd $n - s$ (or $\ell = n - s - 1 = n - \frac{k^2 + k - 4}{2}$ for even $n - s$).

**Identifying vectors:** In addition to the identifying vector $v_{00}^k = (11\ldots1100\ldots0)$ of the lifted MRD code $C^k_*$ (of size $q^{2(n-k)}$ and minimum subspace distance $2(k - 1)$), the other identifying vectors of the codewords are defined as follows. First, by Lemma 10, we partition the weight-2 vectors of $F_{q}^{n-s}$ into classes $P_1, \ldots, P_t$ of size $\frac{q}{2}$ (where $\ell = \ell - 1 = n - s - 1$ if $n - s$ even and $\ell = \ell + 1 = n - s$ if $n - s$ odd) with pairwise disjoint positions of the ones. We define the sets of identifying vectors by a recursion. Let $v_0 \in F_{q}^{n-s+3}$ and $A_1, A_2, A_3 \subseteq F_{q}^{n-s+3}$, as defined in the Pending Dots Construction (see Section II-B). Then $v_{00}^3 = v_0$.

For $k \geq 4$ we define:

$$A_k^i = \{v_{01}^k, \ldots, v_{0k-3}^k\},$$

where $v_{0j}^k = (000\, w_j^k \, || v_{0j-1}^k)$ (1 $\leq j \leq k - 3$), such that the $w_j^k$ are all different weight-1 vectors of $F_2^{k-3}$. Furthermore we define:

$$A_1^k = \{(0010\ldots00|z) : z \in A_1^{k-1}\},$$

$$A_2^k = \{(0100\ldots00|z) : z \in A_2^{k-1}\},$$

$$A_3^k = \{(1000\ldots00|z) : z \in A_3^{k-1}\},$$

such that the prefixes of the vectors in $\bigcup_{i=0}^{3} A_k^i$ are vectors of $F_2^s$ of weight 1. Note, that the suffix $y \in F_q^{n-s}$ (from the Pending Dots Construction) in all the vectors from $A_1^k$ belongs to $P_1$, the suffix $y$ in all the vectors from $A_2^k$ belongs to $\bigcup_{i=2}^{\min\{q+1, \ell\}} P_i$, and the suffix $y$ in all the vectors from $A_3^k$ belongs to $\bigcup_{i=q+2}^{\ell} P_i$ (the set $A_3^k$ is empty if $\ell \leq q + 1$).

**Pending blocks:**

- All Ferrers diagrams that correspond to the vectors in $A_1^k$ have a common pending block with $k - 3$ rows and $\sum_{i=3}^{k} i$ dots in the $j$th row, for $1 \leq j \leq k - 3$. We fill each of these pending blocks with a different element of a suitable FDMRD code with minimum rank distance $k - 3$ and size $q^3$, according to Lemma 18. Note, that the initial conditions always imply that $q^3 \geq \ell$.

- All Ferrers diagrams that correspond to the vectors in $A_2^k$ have a common pending block with $k - 2$ rows and $\sum_{i=3}^{k} i + 1$ dots in the $j$th row, $1 \leq j \leq k - 2$. Every vector which has a suffix $y$ from the same $P_i$ will have the same value $a_i \in F_q$ in the first entry in each row of the common pending block, such that the vectors with suffixes from different classes will have different values in these entries. (This corresponds to a FDMRD code of distance $k - 2$ and size $q$.) Given the filling of the first entries of every row, all the other entries of the pending blocks are filled by a FDMRD code with minimum distance $k - 3$, according to Lemma 18.

- All Ferrers diagrams that correspond to the vectors in $A_3^k$ have a common pending block with $k - 2$ rows and $\sum_{i=3}^{k} i + 2$ dots in the $j$th row, $1 \leq j \leq k - 2$. The filling of these pending blocks is analogous to the previous case, but for the suffixes from the different $P_i$-classes we fix the first two entries in each row of a pending block.

**Ferrers tableaux forms:** On the dots corresponding to the last $n - s - 2$ columns of the Ferrers diagrams for each vector $v_i$ in a given $A_k^i$, $0 \leq i \leq 3$, we construct a FDMRD code with minimum distance $k - 1$ (according to Remark 17) and lift it to obtain $C^k_{i,j}$. We define $C^k_i = \bigcup_{j=1}^{3} C^k_{i,j}$. The final code is defined as $C^k = \bigcup_{i=0}^{3} C^k_i \cup C^k$. 

**Code:** The final code is defined as $C^k = \bigcup_{i=0}^{3} C^k_i \cup C^k$. 


Theorem 19. The code $\mathbb{C}^k$ obtained by Construction A has minimum injection distance $k - 1$ and cardinality $|\mathbb{C}^k| = q^{2(n-k)} + q^{2(n-(k+(k-1)))} + \ldots + q^{2(n-k^2+k-2)} + \left[ n - \frac{k^2+k-6}{2} \right]_q$.

Proof: We will first prove the cardinality by induction on $k$. Observe that the only identifying vector that contributes additional codewords in $\mathbb{C}^k$ compared to $\mathbb{C}^{k-1}$ is $v^k_{00}$, since for all the other identifying vectors, the additional line of dots of the corresponding Ferrers diagrams does not increase the cardinality due to Lemma 16, and thus $|\mathbb{C}^k| = |\mathbb{C}^{k-1}| + q^{2(n-k)}$ for any $k \geq 4$. Solving this recursively results in the above formula.

Next we prove that the minimum injection distance of $\mathbb{C}^k$ is $k - 1$. Let $X, Y \in \mathbb{C}^k$, $X \neq Y$. If $v(X) = v(Y)$, then by Proposition 3, $d_I(X, Y) \geq 2(k - 1)$, i.e. $d_I(X, Y) \geq k - 1$. Now we assume that $v(X) \neq v(Y)$. Note, that according to the definition of identifying vectors, $d_I(X, Y) = d_H(v(X, v(Y))/2 = k - 1$ for $(X, Y) \in \mathbb{C}^k \times \mathbb{C}_i^k$, $0 \leq i \leq 3$, for $(X, Y) \in \mathbb{C}_0^k \times \mathbb{C}_0^k$, and for $(X, Y) \in \mathbb{C}_i^k \times \mathbb{C}_j^k$, $i \neq j$. Now let $X, Y \in \mathbb{C}_1^k$, for some $1 \leq i \leq 3$. If the suffixes of $X$ and $Y$ of length $n - 3$ belong to the same class $P_i$, then $d_H(v(X), v(Y)) = 4$ and $d_R(B_X, B_Y) = k - 3$, for the submatrices $B_X, B_Y$ of $\mathcal{F}(X), \mathcal{F}(Y)$ corresponding to the common pending blocks. Then by Theorem 15, $d_I(X, Y) \geq 2 + (k - 3) = k - 1$. If the suffixes of $X$ and $Y$ of length $n - 3$ belong to different classes $P_{t_1}, P_{t_2}$, then $d_H(v(X), v(Y)) = 2$ and $d_R(B_X, B_Y) = k - 2$, for the submatrices $B_X, B_Y$ of $\mathcal{F}(X), \mathcal{F}(Y)$ corresponding to the common pending blocks. Then by Theorem 15, $d_I(X, Y) \geq 1 + (k - 2) = k - 1$.

Hence, for any $X, Y \in \mathbb{C}^k$ it holds that $d_I(X, Y) \geq k - 1$.

Corollary 20. Let $n \geq \frac{k^2+3k-2}{2}$ and $q^2 + q + 1 \geq \ell$, where $\ell = n - \frac{k^2+k-6}{2}$ for odd $n - \frac{k^2+k-4}{2}$ (or $\ell = n - \frac{k^2+k-4}{2}$ for even $n - \frac{k^2+k-6}{2}$). Then

$$A_q(n, k - 1, k) \geq q^{2(n-k)} + \sum_{j=3}^{k-1} q^{2(n-k^2+i-j)} + \left[ n - \frac{k^2+k-6}{2} \right]_q$$

Example 21. Let $k = 5$, $d = 8$, $n = 19$, and $q = 2$. The code $\mathbb{C}_5^5$ obtained by Construction A has cardinality $2^{28} + 2^{20} + 2^{14} + \left[ \frac{7}{2} \right]_q = 2^{28} + 1067627$ (the largest previously known code is of cardinality $2^{28} + 1052778$ [3]).

We now illustrate the construction:

First, we partition the set of suffixes $y \in \mathbb{F}_2^7$ of weight 2 into 7 classes, $P_1, \ldots, P_7$ of size 3 each. The identifying vectors of the code are partitioned as follows:

$$v^5_{00} = (11111||00000|000|000000)$$
$$\mathcal{A}_{5,1} = \{(00001||11111||000|0000000), (00010||00011||11111|00000000)\}$$
$$\mathcal{A}_{5,2} = \{(001000|00100|001||y) : y \in P_1\}$$
$$\mathcal{A}_{5,3} = \{(010000|01000|010||y) : y \in \{P_2, P_3\}\}$$
$$\mathcal{A}_{5,4} = \{(100000|10000|100||y) : y \in \{P_4, P_5, P_6, P_7\}\}$$

To demonstrate the idea of the construction we will consider only the set $\mathcal{A}_{5,2}$. All the codewords corresponding to $\mathcal{A}_{5,2}$ have the following common pending block $B$:

$$\cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot$$

If the suffix $y \in P_2$, or $y \in P_3$ then to distinguish between these two classes we assign the following values to $B$, respectively:

$$\begin{array}{cccccc}
1 & \cdot & \cdot & \cdot & \cdot & 0 \\
1 & \cdot & \cdot & \cdot & \cdot & 0 \\
0 & \cdot & \cdot & \cdot & \cdot & 0
\end{array}$$

For the identifying vectors with the suffixes $y$ from $P_i$, $i = 2, 3$, we construct a FDMRD code of distance 2 for
the remaining dots of $B$ (here, $a = 0$ or $a = 1$), as follows:

\[
\begin{array}{cccccccccccc}
\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\alpha & 0 & 0 & 0, & a & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\alpha & a & a & a & a & a & a & a & a & a & a & a \\
\alpha & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\alpha & 0 & 1 & 0, & a & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
\alpha & a & a & a & a & a & a & a & a & a & a & a \\
\alpha & 1 & 1 & 0 & 0 & 0 & 0 & a & 1 & 0 & 1 & 0 \\
\alpha & a & 1 & 1, & a & 1 & 1 & 1, & a & a & a & a \\
\end{array}
\]

Since $P_i$ contains only three elements, we only need to use three of the above tableaux. We proceed analogously for the pending blocks of $A_0^5, A_1^5$. Then we fill the Ferrers diagrams corresponding to the last 7 columns of the identifying vectors with an FDMRD code of minimum rank distance 4 and lift these elements. Moreover, we add the lifted MRD code corresponding to $v_0^5$, which has cardinality $2^{28}$. The number of codewords which corresponds to the set $A_0^5$ is $2^{20} + 2^{14}$. The number of codewords that correspond to $A_1^5 \cup A_2^5 \cup A_3^5$ is $7^2/7$.

For small alphabets, when $q^2 + q + 1 < \ell$, we use as the initial step for the recursion the Modified Pending Dots Construction (Construction II in [4]), where the last $n-3$ coordinates of the identifying vectors are partitioned into sets of size $q^2 + q + 2$ and then the same idea for the construction of the identifying vectors is applied in each such set. This Modified Pending Dots Construction generates an $(n, M, k, 3)_q$ constant dimension code with $M = q^{2(n-3)} + \sum_{i=1}^{\alpha} \left[ q^{2(n-3-(q^2+q+2)i)} \right]$, which contains the lifted MRD code, where $\alpha = \left[ \frac{n-3}{q^2+q+2} \right]$.

Then the size of an $(n, M, k-1, k)_q$ constant dimension code $C^K$ obtained from the modified recursive construction is given by

\[
|C^K| = q^{2(n-k)} + \sum_{j=3}^{k-1} q^{2(n-\sum_{i=j}^{k} i)} + \sum_{i=1}^{\alpha_k} q^{2(n-k) - (q^2+q+1)i},
\]

where $\alpha_k = \left[ \frac{n-k^2+k-6}{q^2+q+2} \right]$. Then, we obtain the following corollary.

**Corollary 22.** Let $n \geq \frac{k^2+2k-2}{2}$ and $q^2 + q + 1 < \ell$, where $\ell = n - \frac{k^2+k-6}{2}$ for odd $n - \frac{k^2+k-6}{2}$ (or $\ell = n - \frac{k^2+k-6}{2}$ for even $n - \frac{k^2+k-6}{2}$). Then

\[
A_q(n, k-1, k) \geq q^{2(n-k)} + \sum_{j=3}^{k-1} q^{2(n-\sum_{i=j}^{k} i)} + \sum_{i=1}^{\alpha_k} q^{2(n-k) - (q^2+q+1)i},
\]

where $\alpha_k = \left[ \frac{n-k^2+k-6}{q^2+q+2} \right]$.

In the following, we compare the size of the codes obtained from Construction A (and its modification for small alphabets) to the bound in Theorem 11. In particular, we are interested in an estimation of the function $F(n, k, q)$ defined by $F(n, k, q) := \frac{\alpha_n}{\alpha_q(n-k,k-2,k-1)}$. The following bound on $A_q(n, d, k)$ was established in [5], [27], [28].
\[
A_q(n, d, k) \leq \left[ \frac{n}{k-d+1} \right]_q \left[ \frac{k}{k-d+1} \right]_q
\]

Then
\[
F(n, k, q) = \frac{C^k - q^{2(n-k)}}{A_q(n-k, k-2, k-1)} \geq \frac{C^k - q^{2(n-k)}}{\left[ \frac{n-k}{2} \right]_q / \left[ \frac{k-1}{2} \right]_q}
\]

One can show that \(F(n, k, q)\) is an increasing function in \(k\) and \(q\) and that for \(k \geq 10, n \geq \frac{k^2 + 3k - 2}{2}\), it holds that \(F(n, k, 2) \geq 0.99\). Hence, Construction A asymptotically attains the bound of Theorem 11 for any \(k\) and \(q\). In fact it gets very close to the bound already for small values of \(k\) and \(q\). In comparison, the lifted MRD construction attains the bound asymptotically as well, but is much further away from the bound for small parameters.

The comparison between the cardinality of codes obtained by Construction A and other known codes is given in Section VI, Table I.

IV. CONSTRUCTIONS FOR \((n, M, 2, k)_q\) CODES

In this section we present two constructions for \((n, M, 2, k)_q\)-codes with \(k \geq 4\) and \(n \geq 2k + 2\). These constructions will then give rise to new lower bounds on the size of constant dimension codes with minimum injection distance 2 (or equivalently subspace distance 4). The first one (Construction B), which is a modification of the Multilevel Construction from [3], is based on a specific choice of a set of identifying vectors obtained from matchings and the complement of matchings of the corresponding complete graphs and is given for general \(k \geq 4\). The second one (Construction C) combines the results on pending blocks and Ferrers diagrams arising from different (nearly) perfect matchings of the complete graph. Since it improves the first construction only for the parameters \(k = 4\) and \(k = 5\), it will only be explained for these two cases.

A. A Special Instance of the Multilevel Construction

The Multilevel Construction (see Section II) is a general code construction which usually provides large codes. However, it does not give rise to a general formula for the cardinality of the arising codes, since this construction depends on the specific choice of a related constant weight code. In the following, we will use a specific (nearly) perfect matching and the complement of matchings of the corresponding complete graphs and is given for general \(k \geq 4\).

We first need the following result, which is similar to Lemma 16.

**Lemma 23.** Let \(n \geq 2k + 2\). Let \(v\) be an identifying vector of length \(n\) and weight \(k\), such that there are \(k-2\) many ones in the first \(k\) positions of \(v\). Then the Ferrers diagram arising from \(v\) has more or equally many dots in the first row than in the last column, and the upper bound for the dimension of a Ferrers diagram code with minimum distance 2 is the number of dots that are not in the first row.

**Proof:** Analogous to the proof of Lemma 16.

From Theorems 4 and 5 the next statement follows.

**Corollary 24.** The dimension of a Ferrers diagram code with minimum distance 2 in the setting of Lemma 23 is the number of dots that are not in the first row.

Let \(n \geq 2k + 2\) and define
\[
O_{n-k} := \{(110\ldots0), (00110\ldots0), (0000110\ldots0), \ldots \} \subseteq \mathbb{F}_{2}^{n-k},
\]
which has \(\left\lfloor \frac{n-k}{2} \right\rfloor\) elements (the two ones are always shifted to the right by two positions). In other words, if we denote by \(v_i(j)\) the \(j\)th coordinate of the vector \(v_i\), the set \(O_{n-k}\) contains binary vectors \(v_i\) of length \(n-k\) and...
weight 2, such that \( v_i(j) = 1 \) if and only if \( \left\lceil \frac{j}{2} \right\rceil = i \). Note, that for odd \( n - k \) the last entry of all vectors in \( O_{n-k} \) is always zero.

Also, we define

\[ O_k := \{(11\ldots100),(11\ldots10011),(11\ldots1001111),\ldots\} \subseteq \mathbb{F}_2^k, \]

which has \( \left\lceil \frac{k}{2} \right\rceil \) elements (the two zeroes are always shifted to the left by two positions). In other words, the set \( O_k \) contains binary vectors \( u_i \) of length \( k \) and weight \( k - 2 \), such that \( u_i(j) = 0 \) if and only if \( \left\lceil \frac{k-j+1}{2} \right\rceil = i \). Note, that for odd \( k \) the first entry of all vectors in \( O_k \) is always one.

**Remark 25.** The elements of \( O_{n-k} \) and \( O_k \) form a (nearly) perfect matching of \( K_{n-k} \) and the complement of a (nearly) perfect matching of \( K_k \), respectively.

**Construction B.**

Let \( n \geq 2k + 2 \). We use the following sets of identifying vectors for the Multilevel Construction:

\[
\mathcal{A}_0^k = \{(11\ldots1111||0\ldots0)\}
\]

\[
\mathcal{A}_1^k = \{(11\ldots1110||v) \mid v \in O_{n-k}\}
\]

\[
\mathcal{A}_2^k = \{(11\ldots1001||v) \mid v \in O_{n-k}\}
\]

\[
\vdots
\]

\[
\mathcal{A}_{\frac{n-k}{2}}^k = \{(w||v) \mid v \in O_{n-k}, \text{ } w = \begin{cases} (0011\ldots1) & \text{if } k \text{ even} \\ (10011\ldots1) & \text{if } k \text{ odd} \end{cases}\}.
\]

where the prefixes are the different elements from \( O_k \) (except for \( \mathcal{A}_0^k \)). Then we construct the corresponding lifted FDMRD codes with injection distance 2. Note that the code corresponding to \( \mathcal{A}_0^k \) is the conventional lifted MRD code. Furthermore we add the largest known \((n-k,M,2,k)_{q}\)-code, with \( k \) zero columns appended in front of every codeword.

**Theorem 26.** The code from Construction B has minimum injection distance 2 and cardinality

\[
q^{(k-1)(n-k)} + \left( \sum_{i=0}^{\left\lfloor \frac{k-2}{2} \right\rfloor} q^{(k-3)(n-k)-4i} + \epsilon(k-1)q^{(k-3)(n-k)} \right) \sum_{i=0}^{\frac{n-k-1}{2}} q^{2(2i+\epsilon(n-k))} + A_q^*(n-k,2,k),
\]

where \( \epsilon(i) = 1 \) if \( i \) odd and \( \epsilon(i) = 0 \) if \( i \) even.

**Proof:** The minimum distance for elements with different identifying vectors follows by Proposition 1 from the Hamming distance of the identifying vectors, which is always at least 4. For elements with the same identifying vector it follows from the minimum rank distance of the FDMRD code, by Proposition 3.

The cardinality can be shown as follows. From Theorem 5, Lemma 23 and Corollary 24 we know that the number of dots not in the first row of the FD is the dimension of the FDMRD code. Hence, the subcode arising from \( \mathcal{A}_0^k \) has dimension \((k-1)(n-k)\). The number of matrix fillings for the height-2 Ferrers diagrams corresponding to \( O_{n-k} \) is equal to \( \sum_{i=0}^{\left\lfloor \frac{k-2}{2} \right\rfloor} q^{2i+\epsilon(n-k)} \) (where the empty matrix is also counted). The number of fillings for the Ferrers diagrams corresponding to \( O_k \) without the first rows is equal to \( \sum_{i=0}^{\left\lfloor \frac{k-2}{2} \right\rfloor} q^{(k-3)(n-k)-4i} + \epsilon(k-1)q^{(k-3)(n-k-2)} \). Hence the formula follows.

**Corollary 27.** Let \( n \geq 2k + 2 \). Then

\[
A_q(n,2,k) \geq \sum_{i=1}^{\left\lfloor \frac{n-k}{2} \right\rfloor} \left( q^{(k-1)(n-ik)} + \frac{q^{2(k-2)} - 1}{(q^2 - 1)^2} q^{(k-3)(n-ik-2)+4} \right).
\]

**Proof:** From Theorem 26 it follows that the value for \( A_q(n,2,k) - A_q^*(n-k,2,k) - q^{(k-1)(n-k)} \) is lower
bounded by
\[
\left( \sum_{i=0}^{\left\lfloor \frac{n-3}{2} \right\rfloor} q^{(k-3)(n-k)-4i} \right) \left( \sum_{i=0}^{\left\lfloor \frac{n-k-2}{2} \right\rfloor} q^{4i+2\epsilon(n-k)} \right) = q^{(k-3)(n-k)+2\epsilon(n-k)} \left( \sum_{i=0}^{\left\lfloor \frac{n-3}{2} \right\rfloor} q^{-4i} \right) \left( \sum_{i=0}^{\left\lfloor \frac{n-k-2}{2} \right\rfloor} q^{4i} \right).
\]

Solving the sums and then using the equality \(4\left\lfloor \frac{x}{2} \right\rfloor = 2x - 2\epsilon(x)\) we get that this expression is equal to
\[
q^{(k-3)(n-k)+2\epsilon(n-k)} \frac{q^{-4\left\lfloor \frac{n-3}{2} \right\rfloor}(q^{4\left\lfloor \frac{n-k-2}{2} \right\rfloor+1}) - 1}{(q^4 - 1)^2} = q^{(k-3)(n-k)+2(\epsilon(n-k)+\epsilon(k-1))} \frac{(q^{2(k-1)-\epsilon(k-1)}) - 1}{(q^4 - 1)^2}(q^{2(n-k-\epsilon(n-k))} - 1)
\]

This expression takes its minimum for \(\epsilon(n-k) = \epsilon(k-1) = 1\), hence
\[
A_q(n, 2, k) \geq A_q^*(n-k, 2, k) + q^{(k-1)(n-k)} + q^{(k-3)(n-k-2)+4} \frac{(q^{2(k-2)} - 1)(q^{2(n-k-1)} - 1)}{(q^4 - 1)^2}.
\]

Applying this bound recursively yields the desired formula.

**Remark 28.** Here we derived a closed cardinality formula for the special instance of the Multilevel Construction for \(d = 2\). Note that one can also apply this idea to obtain a bound on the cardinality for constant dimension codes with other values for the minimum injection distance.

**B. New \((n, M, 2, 4)_q^*\) and \((n, M, 2, 5)_q^*\)-Codes from One-Factorizations and Pending Dots**

The construction presented in this subsection is based on a one-factorization of a complete graph which is used to construct a set of identifying vectors for the proposed codes, by generalizing the Pending Dots Construction to \(k > 3\). However, in contrast to the Pending Dots Construction, here we use not all but specifically chosen perfect matchings which result in a large constant dimension code. First, we consider one-factorizations and the Ferrers diagrams arising from them.

1) **Ferrers Diagrams from One-Factorizations of the Complete Graph:** We will now present some results on Ferrers diagrams arising from the weight-2 vector representation of matchings of the complete graph \(K_n\). To do so we will use some graph theoretic results (see e.g. \([24], [25]\)) that will be useful for our choice of identifying vectors later on. We start by with the existence proof of (near) one-factorizations, (see also Lemma 10 in Section II), since we need the idea of this proof for our following results.

**Theorem 29 (\([24], [25]\)).**
1) If \(n\) is odd there always exists a near one-factorization (NOF) of \(K_n\).
2) If \(n\) is even there always exists a one-factorization (OF) of \(K_n\).

**Proof:**

1) If \(n\) is odd we can draw the nodes of \(K_n\) as a circle. Then we can choose one edge and all its parallels, which will give us a nearly perfect matching of \(K_n\). We can repeat this step for any edge that is not covered yet and get a NOF of \(K_n\).

2) If \(n\) is even we can use \(n - 1\) nodes of \(K_n\) as a circle, just like before, and use the remaining node as the center of the circle. Then we use again the set of parallel edges plus the edge that connects the remaining node on the circle with the center of the circle, which is a perfect matching. The set of all these different perfect matchings is an OF of \(K_n\).

Then one can easily count the number of elements in the sets of a NOF or an OF of \(K_n\) (see also Lemma 10):

**Lemma 30.**
1) For a given odd \(n\) the NOF of \(K_n\) has \(n\) many nearly perfect matchings and each one of them contains \(\frac{n-1}{2}\) elements.
2) For a given even \(n\) the OF of \(K_n\) has \(n - 1\) many perfect matchings and each one of them contains \(\frac{n}{2}\) elements.
As in Section [11], we denote the different (nearly) perfect matchings of a (near) one-factorization in the vector representation by $P_i$ and call them classes.

In the following construction we want to use the matchings which contribute the largest possible FDRM codes. So we need the following lemma, which gives the sizes of the corresponding Ferrers diagrams and, as a consequence, the cardinality of the FDRM codes. We use the construction of matchings described in the proof of Theorem 29. We denote $n' := n - k$ and label all the outside nodes counter-clock-wise from 1 to $n' - 1$ if $n'$ is even, and from 1 to $n'$ if $n'$ is odd. If $n'$ is even, the center node is labeled by $n'$ and we name $P_i$ the perfect matching that contains the edge $(n', i)$ as the center edge (i.e. all other edges are orthogonal to this one). If $n'$ is odd, there is no center node and we name $P_i$ the nearly perfect matching that corresponds to the matching that does not cover node $i$.

**Lemma 31.** For a given $P_i$, the size of the respective FDRM code with rank distance 1 (i.e. the number of different matrix fillings for the corresponding Ferrers diagrams) is given by

- $(\frac{n'}{2} - i) q^{(n'-2i)} + (i - 1)q^{(2(n'-i)-1)} + q^{(n'-i-1)}$ if $i \leq \frac{n'}{2}$ and $n'$ is even,
- $(i - \frac{n'}{2}) q^{3(n'-2i+1)} + (n' - i - 1)q^{(2(n'-i)-1)} + q^{(n'-i-1)}$ if $i > \frac{n'}{2}$ and $n'$ is even,
- $(\frac{n'+1}{2} - i) q^{(n'-2i-1)} + (i - 1)q^{(2(n'-i)-1)}$ if $i \leq \frac{n'+1}{2}$ and $n'$ is odd,
- $(i - \frac{n'+1}{2}) q^{3(n'-2i-1)} + (n' - i)q^{(2(n'-i)-1)}$ if $i > \frac{n'+1}{2}$ and $n'$ is odd.

**Proof:** Can be found in Appendix A.

2) **Code Construction:** We will now describe a construction for constant dimension codes with $k = 4$ and $k = 5$. The idea in both cases is similar to the Multilevel Construction: To construct the identifying vectors, we start with $(1 \ldots 10 \ldots 0)$ and then construct sets of identifying vectors with prefixes of length $k$ and weight $k-2$, and suffixes of length $n' := n - k$ and weight 2. The suffixes will be chosen from some of the (nearly) one-factors $P_i$ of $K_{n-k}$. We choose the prefixes and suffixes that contribute the largest FDRM codes, using Lemma 31. In addition, we use pending dots to allow for a choice of identifying vectors with a smaller Hamming distance.

**Construction C-4**

Let $n \geq 10$ and $n' = n - 4$. Hence, $n'$ is even if and only if $n$ is even. We use the following sets of identifying vectors

$$A_4^i = \{(1111||0\ldots0)\}$$
$$A_4^i = \{(1100||v), (0011||v) \mid v \in P_{\frac{n'}{2}+1}\}$$
$$A_4^i = \{(1001||v), (0110||v) \mid v \in P_2\}$$
$$\begin{align*}
A_4^i &= \{(1010||v), (0101||v) \mid v \in \bigcup_{i=2}^{\min\{\frac{n'}{2}+1,\frac{n'}{2}\}} P_{\frac{n'}{2}+1} \bigcup_{i=3}^{\min\{\frac{n'}{2}+2,\frac{n'}{2}\}} P_i\}
\end{align*}$$

and construct the corresponding lifted FDMRD codes with injection distance 2, where we use the pending dot in $A_4^i$. Note that the code corresponding to $A_0^i$ is the conventional lifted MRD code. Furthermore, we add the largest known $(n - 4, M, 2, 4)q$-code, with 4 zero columns appended in front of every codeword, to obtain a constant dimension code $C^4$.

**Theorem 32.** The code $C^4$ obtained by Construction C-4 has minimum subspace distance 4 and cardinality given by

- $q^{3(n-4)} + (q^{n-4}) [q^{2(n-6)} + (\frac{n}{2} - 4)q^{(n-7)} + q^{(\frac{n}{2} - 4)}] +$  
- $(q^{n-5} + q^{n-6}) \left[\sum_{i=2}^{\min\{\frac{n}{2}+1,\frac{n}{2}\}} (i)q^{2n-2i-10} + (\frac{n}{2} - i)q^{n-2i-5} + q^{\frac{n}{2} - i} \right] +$  
- $\sum_{i=1}^{\min\{\frac{n}{2}+1,\frac{n}{2}\}} (i)q^{2n-2i-11} + (\frac{n}{2} - i)q^{n-2i-6} + q^{n-i-6})] + A_4^i(n - 4, 2, 4)$

if $n$ is even.

- $q^{3(n-4)} + (q^{n-4}) [q^{2(n-6)} + (\frac{n}{2} - 3)q^{(n-8)}] +$  
- $(q^{n-5} + q^{n-6}) \left[\sum_{i=2}^{\min\{\frac{n}{2}+1,\frac{n}{2}\}} (i)q^{2n-2i-10} + (\frac{n}{2} - i)q^{n-2i-6}) +$
\[
\sum_{i=1}^{\min\{\lfloor \frac{n}{2} \rfloor+1,\lfloor \frac{n}{2} \rfloor\}} (iq^{2n-2i+1} + \left(\frac{n-5}{2} - i\right)q^{n-2i-7}) + A_q^*(n-4, 2, 4)
\]
if \( n \) is odd.

**Proof:** The minimum distance for elements with different identifying vectors follows from the Hamming distance of the identifying vectors, together with the pending dots, i.e., from Proposition \([\text{I}]\) and Lemma \([\text{II}]\). For elements with the same identifying vector it follows from the minimum rank distance of the FDMRD code, by Proposition \([\text{III}]\).

The proof for the cardinality can be found in Appendix B. \(\blacksquare\)

**Example 33.** Let \( q = 2, \ n = 10 \). Then we have \( A_2(10, 2, 4) \geq 2^{18} + 37456 + 21 \), where \( A_2(6, 2, 4) = 21 \). The largest previously known code obtained by the Multilevel Construction \([\text{III}]\) has cardinality \( 2^{18} + 34768 \).

**Example 34.** Let \( q = 2, \ n = 12 \). Then we have \( A_2(12, 2, 4) \geq 2^{24} + 2333568 + 701 + 2^{12} = 2^{24} + 2333568 \), where \( A_2(8, 2, 4) \geq 701 + 2^{12} \). The largest previously known code obtained by the Multilevel Construction \([\text{III}]\) has cardinality \( 2^{24} + 2290845 \).

**Construction C-5**

Let \( n \geq 12 \) and \( n' = n - 5 \). Hence, \( n' \) is even if and only if \( n \) is odd. We use the following sets of identifying vectors

\[
A_1^5 = \{(11100||v), (10011||v) \mid v \in P_{\lfloor \frac{n'}{2} \rfloor+1}^q\}
\]

\[
A_2^5 = \{(11010||v), (01101||v) \mid v \in P_3^q\}
\]

\[
A_3^5 = \{(01110||v), (10101||v) \mid v \in P_{\lfloor \frac{n'}{2} \rfloor+2}^q\}
\]

\[
A_4^5 = \{(00111||v), (11001||v) \mid v \in P_3^q\}
\]

\[
A_5^5 = \{(10110||v), (01011||v) \mid v \in \bigcup_{i=3}^{\min\{\lfloor \frac{n}{2} \rfloor+2,\lfloor \frac{n}{2} \rfloor\}} P_{\lfloor \frac{n}{2} \rfloor+i}^q \cup \bigcup_{i=4}^{\min\{\lfloor \frac{n}{2} \rfloor+3,\lfloor \frac{n}{2} \rfloor\}} P_i^q\}
\]

and construct the corresponding lifted FDMRD codes with injection distance 2, where we use the pending dot in \( A_5^5 \). Note that the code corresponding to \( A_0^5 \) is the conventional lifted MRD code. Furthermore, we add the largest known \( (n-5, M, 2, 5)_q \)-code, with 5 zero columns appended in front of every codeword to obtain a constant dimension code \( C^5 \).

**Theorem 35.** The code \( C^5 \) obtained by Construction C-5 has minimum subspace distance 4 and cardinality given by

- \( q^{4(n-5)} + (q^{2n-10} + q^{2n-14})(q^{2(n-7)} + (\frac{n-8}{2})q^{(n-9)}) + (q^{2n-11} + q^{2n-13})(q^{2n-10} + q^{(n-13)}) + (q^{2n-12} + q^{2n-13})(2q^{2(n-8)} + \frac{n-10}{2}q^{(n-11)}) + (q^{2n-12} + q^{2n-14})\sum_{i=3}^{\min\{\lfloor \frac{n}{2} \rfloor+2,\lfloor \frac{n}{2} \rfloor\}} (iq^{2n-2i-13} + (\frac{n-6}{2} - i)q^{n-2i-8}) + A_q^*(n-5, 2, 5)\)

if \( n \) is even.

- \( q^{4(n-5)} + (q^{2n-10} + q^{2n-14})(q^{2n-14} + (\frac{n-9}{2})q^{(n-8)} + \frac{n-9}{2}) + (q^{2n-11} + q^{2n-13})(q^{2n-9} + q^{(n-15)} + q^{n-8}) + (q^{2n-12} + q^{2n-13})(q^{2n-16} + (\frac{n-11}{2})q^{(n-10)} + \frac{n-11}{2}) + (q^{2n-12} + q^{2n-14})\sum_{i=3}^{\min\{\lfloor \frac{n}{2} \rfloor+2,\lfloor \frac{n}{2} \rfloor\}} (iq^{2n-2i-13} + (\frac{n-7}{2} - i)q^{n-2i-6} + q^{n-2i-6} + \frac{n-7}{2} - i)q^{n-7} + q^{n-7} - q^{n-7}) + A_q^*(n-5, 2, 5)\)

if \( n \) is odd.

**Proof:** The minimum distance for elements with different identifying vectors follows from the Hamming distance of the identifying vectors, together with the pending dots, by Proposition \([\text{IV}]\) and Lemma \([\text{V}]\). For elements
with the same identifying vector it follows from the minimum rank distance of the FDMRD code, by Proposition 3.

The proof for the cardinality can be found in Appendix B.

**Remark 36.** One can easily generalize Constructions C-4 and C-5 to larger values of $k$ by choosing the prefixes for the sets $A^k_i$ as follows: Choose an OF (or NOF) of $K_k$, look at its vector representation and add the all-one vector to all these vectors (i.e. bitflip all coordinates). Thus, the prefixes in a given set $A^k_i$ form a code with constant weight $k - 2$ and minimum Hamming distance $4$ in $\mathbb{F}_2^k$. But one can then prove that there is no such set with pending dots in all its elements. Hence, this generalization would not improve the Multilevel Construction from [3]. This is why we only describe the construction for $k = 4$ or $k = 5$ in this work.

The comparison between the Multilevel Construction and the codes obtained by Constructions B and C can be found in Section VII Table II. One can see that Constructions C-4 and C-5 improve Construction B, but remember that Construction B works for general $k$ as opposed to the Multilevel Construction. The advantage still is that we have a closed formula for all constructions explained in some parameter sets. On the other hand, Construction C-5 does not improve the cardinality of the codes arising from Construction C-4 yields larger codes than the Multilevel Construction and hence results the largest known codes for some parameter sets. On the other hand, Construction C-5 does not improve the cardinality of the codes arising from the Multilevel Construction. The advantage still is that we have a closed formula for all constructions explained in this section, in contrast to the Multilevel Construction.

**V. CONSTRUCTION FOR A NEW $(n, M, d, k)_q$ CODE FROM AN OLD CODE**

In the following we discuss a way for constructing a new constant dimension code with minimum injection distance $d$ (or subspace distance $2d$) from a given one. This approach is fairly simple, but surprisingly, for some families of parameters it provides the largest known codes.

**Construction D.**

Let $C \in \mathcal{G}_q(k, n)$ be an $(n, M, d, k)_q$-code, let $\Delta$ be an integer such that $\Delta \geq k$, and let $C$ be an $[F, \Delta(k-d+1), d]$ FDMRD code with a full $k \times \Delta$ rectangular Ferrers diagram. Define

$$C' = \{X' \in \mathcal{G}_q(k, n') : \text{RE}(X') = [\text{RE}(X)A], X \in C, A \in C\}.$$

**Theorem 37.** The code $C'$ obtained by Construction D is an $(n' = n + \Delta, M', d, k)_q$-code in $\mathcal{G}_q(k, n')$, such that

$$M' = Mq^{\Delta(k-d+1)}.$$

**Proof:** Since $|C| = q^{\Delta(k-d+1)}$, it follows from Theorem 3 that $M' = Mq^{\Delta(k-d+1)}$. To prove the minimum distance we distinguish between two cases:

1. Let $X', Y' \in C'$, such that $\text{RE}(X') = [\text{RE}(X)A], \text{RE}(Y') = [\text{RE}(X)B]$, for $X \in C$ and $A, B \in C$, $A \neq B$. Then $v(X') = v(Y')$ since all the ones of the identifying vectors of the codewords from $C'$ appear in the first $k$ coordinates. Hence, by Proposition 3 $d_f(X', Y') = d_R(\text{RE}(X'), \text{RE}(Y'))$. Since $\text{RE}(X') - \text{RE}(Y') = [0A - B]$, where 0 is a $k \times n$ zeroes matrix, we have $d_f(X', Y') = d_R(A, B) \geq d$, since $A, B \in C$.

2. Let $X', Y' \in C'$, such that $\text{RE}(X') = [\text{RE}(X)A], \text{RE}(Y') = [\text{RE}(Y)B]$, for $X \in C, Y \neq Y$, and $A, B \in C$. Then $d_f(X', Y') = k - \dim(X' \cap Y') \geq k - \dim(X \cap Y) \geq d$, since $X, Y \in C$.

**Example 38.** We take the $(8, 2^{12} + 701, 2, 4)_2$ code $C$ constructed in [4] and apply on it Construction D with $\Delta = 4$. Then the new code $C'$ has cardinality $|C'| = 2^{24} + 701 \cdot 2^{12} = 2^{24} + 2871296$ and has parameters $(12, |C'|, 2, 4)_2$. The largest previously known code of these parameters of size $2^{24} + 2290845$ was obtained in [3].

Like in the constructions before we can then also add codes of shorter length with zeroes appended in front to these codes. Hence we get a new lower bound as follows.

**Corollary 39.** Let $n \geq 3k$. Then for any positive integer $\Delta$, such that $n \geq \Delta \geq k$, it holds that

$$A_q(n, d, k) \geq q^{\Delta(k-d+1)}A_q(n - \Delta, d, k) + A_q(\Delta, d, k).$$

In particular, for $\Delta = k$, we get

$$A_q(n, d, k) \geq q^{k(k-d+1)}A_q(n - k, d, k) + 1.$$
and, for $\Delta = n - k$, we get
\[ A_q(n, d, k) \geq q^{(n-k)(k-d+1)} + A_q(n - k, d, k) \]
which, if recursively solved, corresponds exactly to the formula of the multi-component lifted MRD codes from [27].

**Remark 40.** Note that Construction D is related to the interleaved rank-metric codes (see e.g. [26]). In particular, the code obtained in Construction D can be considered as a lifted Ferrers diagram interleaved code, where to the FDRM code raised from the first $n$ coordinates is appended another FDRM code with the same minimum rank distance and with a full rectangular $k \times \Delta$ Ferrers diagram. Then, this construction can be considered as a generalization of an interleaved construction, since every code can be used as the initial step of construction.

### VI. Tables of Constant Dimension Code Sizes

Tables [I – IV] show the examples of code cardinalities of the different constructions from this paper compared to the Multilevel Construction of [3]. The bold value for each line shows the largest cardinality for the given parameters.

For Construction A we use the cardinality formula of Theorem 19 for Construction B the formula of Theorem 26 without adding $A_q^*(n - k, 2, k)$. For the values of Construction C we use the formulas of Theorems 32 and 35 for $k = 4$ and $k = 5$ respectively, without adding $A_q^*(n - k, 2, k)$. For Construction D we use the respective multilevel codes (see [3]) of length $2k$ (i.e., $\Delta = n - 2k$), and the $(8, 4797, 2, 4)$ code from [4], as the old code from which we construct a new code. The cardinality formula for Construction D can be found in Theorem 37.

All the $(n, M, d, k)_q$-codes presented in these tables contain a lifted MRD code of size $q^{(n-k)(k-d+1)}$, so the cardinalities of the constructed codes are written in the form $q^{(n-k)(k-d+1)} + (M - q^{(n-k)(k-d+1)})$.

**TABLE I**: Comparison of cardinalities of codes constructed according to Constructions A and D with the Multilevel Construction.

| $(n, d, k)_q$ | A | D | Multilevel |
|--------------|---|---|------------|
| $(13, 3, 4)_2$ | $2^{18} + 4747$ | $2^{18} + 4096$ | $2^{18} + 4357$ |
| $(14, 3, 4)_2$ | $2^{20} + 19051$ | $2^{20} + 16384$ | $2^{20} + 17204$ |
| $(15, 3, 4)_2$ | $2^{22} + 76331$ | $2^{22} + 65536$ | $2^{22} + 68378$ |
| $(19, 4, 5)_2$ | $2^{28} + 1067627$ | $2^{28} + 1048576$ | $2^{28} + 1052778$ |
| $(20, 4, 5)_2$ | $2^{30} + 4270635$ | $2^{30} + 4194304$ | $2^{30} + 4211044$ |
| $(19, 4, 5)_3$ | $3^{28} + 3491666833$ | $3^{28} + 3486784401$ | $3^{28} + 3487316403$ |
| $(20, 4, 5)_3$ | $3^{30} + 31425002590$ | $3^{30} + 3138105639$ | $3^{30} + 31385846853$ |

**TABLE II**: Comparison of cardinalities of codes constructed according to Constructions B, C-4, C-5, and D with the Multilevel Construction.

| $(n, d, k)_q$ | B | C | D | Multilevel |
|--------------|---|---|---|------------|
| $(10, 2, 4)_2$ | $2^{18} + 21840$ | $2^{18} + 37456$ | $2^{21} + 292896$ | $2^{18} + 35685$ |
| $(11, 2, 4)_2$ | $2^{21} + 174720$ | $2^{21} + 2333568$ | $2^{24} + 2871296$ | $2^{21} + 285889$ |
| $(12, 2, 4)_2$ | $2^{21} + 1398080$ | $2^{24} + 8480128$ | $2^{27} + 22970368$ | $2^{21} + 2290845$ |
| $(13, 2, 4)_2$ | $2^{27} + 11184640$ | $2^{27} + 29377536$ | $2^{32} + 447025152$ | $2^{27} + 18328921$ |
| $(12, 2, 5)_2$ | $2^{28} + 19009536$ | $2^{28} + 29377536$ | $2^{32} + 113059954688$ | $2^{28} + 30877839$ |
| $(13, 2, 5)_2$ | $2^{32} + 304222208$ | $2^{32} + 447025152$ | $2^{40} + 124519448576$ | $2^{32} + 494999563$ |
| $(15, 2, 5)_2$ | $2^{40} + 7788199360$ | $2^{40} + 7788199360$ | $2^{40} + 124519448576$ | $2^{40} + 126773908793$ |
| $(16, 2, 5)_2$ | $2^{44} + 1246111989760$ | $2^{44} + 1903742156800$ | $2^{44} + 1992311177216$ | $2^{44} + 2028469279328$ |

One can see that Construction A always results in the largest cardinality for a valid set of parameters (remember that Construction A is only defined for $d = 2$). Furthermore, Construction C-4 beats the Multilevel Construction, whereas Construction C-5 does not for the parameter sets we used. Moreover, Construction D yields the largest known codes e.g. for $(15, 2, 5)_2$- and $(16, 2, 5)_2$-codes.
Overall, our new constructions presented in this paper beat the known constructions for many sets of parameters. Note that, by construction, we cannot expect Construction B to improve on the cardinality of the Multilevel Construction. We still wanted to describe this construction to derive a closed cardinality formula, in contrast to the Multilevel Construction, for which no such formula exists.

VII. NON-CONSTANT DIMENSION CODES

In this section we consider codes in \( \mathcal{P}_q(n) \) which are not constant dimension codes. Constructions of such codes were considered for the subspace metric in \([3], [12]\) and for the injection metric in \([12]\). A code in the projective space can be considered as a union of constant dimension codes with different dimensions. Moreover, a construction of a code in \( \mathcal{P}_q(n) \) can be done in a multilevel manner, i.e., first, the identifying vectors of the subspaces are chosen and then the corresponding lifted Ferrers diagrams rank-metric codes are constructed \([3], [12]\). For this recall Proposition 1, which states that for any subspaces are chosen and then the corresponding lifted Ferrers diagrams rank-metric codes are constructed \([3]\), \([12]\).

The lower bound on the cardinality of the punctured code is given in the following theorem \([3]\):

First, we briefly describe the puncturing method presented in \([3]\). Let \( C \) be an \( (n, M, d) \)-code in \( \mathcal{P}_q(n) \) of subspace distance \( d \), such that there exist codewords \( X_1, X_2 \in \mathbb{C} \) with \( X_1 \subseteq Q \) and \( v \in X_2 \). Then the \textit{punctured} code \( C'_{Q,v} \), defined by

\[
C'_{Q,v} = \{ \Gamma(X) : X \in \mathbb{C}, X \subseteq Q \} \cup \{ \Gamma(X \cap Q) : X \in \mathbb{C}, v \in X \},
\]

is a code in \( \mathcal{P}_q(n-1) \) with minimum subspace distance \( d-1 \), i.e., an \( (n-1, M', d-1)^q \)-code. If \( \mathbb{C} \) is a constant dimension code in \( \mathcal{G}_q(k,n) \), then the punctured code contains subspaces of dimensions \( k \) and \( k-1 \).

The following lemma considers the minimum \textit{injection} distance of a punctured code of a constant dimension code.

**Lemma 41.** Let \( \mathbb{C} \in \mathcal{G}_q(k,n) \) be a code with minimum injection distance \( d \), i.e., an \( (n, M, d, k)_q \)-constant dimension code. Let \( Q \in \mathcal{G}_q(n-1, n) \) and \( v \in \mathbb{F}_q^{n}, v \notin Q \), such that there exist two codewords \( X_1, X_2 \in \mathbb{C} \) with \( X_1 \subseteq Q \) and \( v \in X_2 \). Then the punctured code \( C'_{Q,v} \) has minimum injection distance \( d \), i.e., it is an \( (n-1, M', d)_q \)-code.

**Proof:** Since for any two subspaces \( X, Y \) of the same dimension it holds that \( d_I(X, Y) = d_S(X, Y)/2 \), it is sufficient to check two subspaces \( X, Y \in C'_{Q,v} \) of different dimensions \( k \) and \( k-1 \):

\[
d_I(X, Y) = k - \dim(X \cap Y) = \frac{2k - 2 \dim(X \cap Y)}{2} = \frac{d_S(X, Y) + 1}{2} \geq \frac{(2d-1) + 1}{2} = d.
\]

The lower bound on the cardinality of the punctured code is given in the following theorem \([3]\):
**Theorem 42.** If \( C \) is an \((n, M, d, k)_q\) constant dimension code then there exists an \((n - 1)\)-dimensional subspace \( Q \) and a vector \( v \notin Q \), such that

\[
|C'_{Q,v}| \geq M \frac{q^{n-k} + q^k - 2}{q^n - 1}.
\]

Now we present a construction for codes in the projective space. This construction generalizes the constructions for non-constant dimension codes from [3], [12].

**Construction of codes in projective space.** Let \( C \in G_q(\lfloor \frac{n+1}{2} \rfloor, n+1) \) be a constant dimension code of minimum injection distance \( d_I = d \). Let \( C' \) be the code obtained by puncturing \( C \). \( C' \) contains subspaces of \( F^d_q \) of dimensions \( \lfloor \frac{n+1}{2} \rfloor \) and \( \lfloor \frac{n+1}{2} \rfloor - 1 \) and has minimum subspace distance \( 2d - 1 \) and minimum injection distance \( d \), by Lemma 41.

1. For the injection metric, we add to \( C' \) the codewords of the largest known constant dimension codes with minimum injection distance \( d \) from \( G_q(\lfloor \frac{n+1}{2} \rfloor - 1-id, n) \), for \( i = 1, \ldots, \lfloor \frac{n+1}{2d} \rfloor - 1 \) and from \( G_q(\lfloor \frac{n+1}{2} \rfloor + id, n) \), for \( i = 1, \ldots, \lfloor \frac{n-\frac{n+1}{d}}{2} \rfloor \). The resulting code \( \tilde{C}_I \) is a code in \( P_q(n) \) with minimum injection distance \( d \).
2. For the subspace metric, we add to \( C' \) the codewords of the largest known constant dimension codes with minimum subspace distance \( 2d \) from \( G_q(\lfloor \frac{n+1}{2} \rfloor - 1 - i(2d - 1), n) \), for \( i = 1, \ldots, \lfloor \frac{n+1}{2d} \rfloor - 1 \) and from \( G_q(\lfloor \frac{n+1}{2} \rfloor + i(2d - 1), n) \), for \( i = 1, \ldots, \lfloor \frac{n-\frac{n+1}{d}}{2} \rfloor \). The resulting code \( \tilde{C}_S \) is a code in \( P_q(n) \) with minimum subspace distance \( 2d - 1 \).

**Remark 43.** The cardinality of the code obtained by the above construction is lower bounded by using the results from the previous sections and by Theorem 42.

We illustrate the idea of the construction for projective space codes based on the puncturing method, for both the subspace and the injection metric, in the following example.

**Example 44.** Let \( q = 2 \) and \( n = 11 \). First, let \( C \in G_2(6, 12) \) be a constant dimension code with minimum injection distance \( d_I = 2 \) and size 1196288829, obtained by the Multilevel Construction [3]. By puncturing it, we can obtain a code in \( P_2(11) \) of size at least 36808900 (by Theorem 42), which includes subspaces of dimensions 5 and 6 of \( F^{11}_2 \), and has minimum subspace distance \( d_S = 3 \) and minimum injection distance \( d_I = 2 \).

1. We add the codewords of constant dimension codes with minimum injection distance 2 from \( G_2(1, 11), G_2(3, 11), G_2(8, 11), G_2(10, 11) \) of sizes 1, 76331, 76331, 1, respectively. The final code \( \tilde{C}_I \) has minimum injection distance \( d_I = 2 \) (and subspace distance \( d_S = 2 \)) and size \( |\tilde{C}_I| = 36961564 \), such that \( \log(|\tilde{C}_I|) = 25.1395 \) (compare to 24.63210 in [10]).
2. We add the codewords of constant dimension codes with minimum injection distance 2 from \( G_2(2, 11), G_2(9, 11) \) of size 681 each. The final code \( \tilde{C}_S \) has minimum subspace distance \( d_S = 3 \) (and injection distance \( d_I = 2 \)) and size \( |\tilde{C}_S| = 36810200 \), such that \( \log(|\tilde{C}_S|) = 25.1336 \).

Table III shows some examples of cardinalities of our codes based on puncturing (for both the subspace and the injection metric) in \( P_q(n) \) compared to the codes of [10] (for the injection metric), for \( q = 2 \). To make the comparison easier we present the cardinalities in the logarithmic form.

![Table III](image-url)
VIII. CONCLUSION AND OPEN PROBLEMS

In this work we presented new constructions for constant dimension codes, and based on these also new constructions for non-constant dimension codes. To do so we used the known techniques of the Multilevel Construction and pending dots, as well as new results on Ferrers diagrams arising from matchings of the complete graph. Moreover, we derived a way of constructing new codes from old codes. The new constructions give rise to the largest known codes for most sets of parameters, as shown in the tables of Section VI and VII. This means that these codes have the best known transmission rate for a given error-correction capability.

For future research it would be interesting to derive bounds analogous to the one of Theorem 11 for other values of \(d\), and see if any of our constructions attain such a bound (asymptotically). Furthermore, we would like to develop results of Ferrers diagrams rank metric codes related to the complete graph for codes of minimum rank distance \(d \neq 2\), and investigate if we could use such results for constant dimension code constructions with minimum injection distance \(d\) (and respective non-constant dimension codes).

Another open question is how these codes can be decoded efficiently. Due to their similarity to the Multilevel Construction the codes constructed to our new constructions can be decoded with an analogous decoding algorithm but the structure of the identifying vectors might be useful and could be exploited for a more efficient algorithm.

APPENDIX

A. Proof of Lemma 31

Proof: We will prove the first statement for \(n'\) even and \(i \leq \frac{n'}{2}\). The other statements can be proven analogously.

Let us look at the graph of the proof of Theorem 29 again, labeled as mentioned before. Choose some center \(A\).

Now we look at all edges whose smaller entry \(i'\) satisfies \(1 \leq i' < i\). Such an edge will always be of the form \((i - j, i + j)\) for \(1 \leq j < i\), thus there are \((i - 1)\) of these edges. One can see by induction that all of these edges give rise to Ferrers diagrams of the same size, since a FD corresponding to \((x, y)\) can be obtained from the FD corresponding to \((x', y)\) by adding a point in the first row and deleting a point in the second row. We can count the dots e.g. in the FD corresponding to \(1, 2i - 1\): There are \(n' - 2\) dots in the first row and \(n' - (2i - 1)\) in the second, hence a sum of \(2n' - 2i - 1\) dots for the whole FD.

The edges that are left are of the form \((\frac{n'}{2} + i - 1 - j, \frac{n'}{2} + i + j)\) for \(0 \leq j < \frac{n'}{2} - i\). With the same argument as in the paragraph before, all of these FD have the same number of dots and there are \(\frac{n'}{2} - i\) many of them. We can count the dots in the FD arising from \((i, n' - 1 + i)\): There are \(n' - 1 - i\) dots in the first row and \(n' - (n' - 1 + i)\) in the second, hence a sum of \(n' - 2i\) dots for the whole FD.

B. Proof of the cardinalities in Theorems 32 and 33

Proof: We derive the cardinalities of each component of the set of identifying vectors from Theorems 32 and 33.

Let \(n\) be even. The FDRM code with rank distance \(d = 2\) arising from the identifying vectors of

- \(A_1^1\) has cardinality \(q^{(n-4)}(q^{2(n-6)} + (\frac{n}{2} - 4)q^{(n-7)} + q^{(n-4)})\).
- \(A_2^1\) has cardinality \(q^{(n-5)}(q^{2(n-6)} + (\frac{n}{2} - 4)q^{(n-8)} + q^{(2n-13)} + q^{(n-7)})\).
- \(A_3^1\) has cardinality \(q^{(n-5)}(q^{2(n-6)} + (\frac{n}{2} - 4)q^{(n-8)} + q^{(2n-13)} + q^{(n-7)})\).
- \(A_4^1\) has cardinality \(q^{2n-11} + q^{2n-10} + (n-6 - i)q^{n-2i-6} + q^{n-6-i}\).
- \(A_5^1\) has cardinality \(q^{2n-10} + q^{2n-14} + q^{(n-7)} + q^{(n-9)}\).
- \(A_6^1\) has cardinality \(q^{2n-11} + q^{2n-13} + q^{(n-10)} + q^{(2n-15)}\).
- \(A_7^1\) has cardinality \(q^{2n-12} + q^{2n-13} + q^{(n-10)} + q^{(2n-15)}\).
- \(A_8^1\) has cardinality \(q^{2n-12} + q^{2n-14} + q^{(n-10)} + q^{(2n-15)}\).
- \(A_9^1\) has cardinality \(q^{2n-12} + q^{2n-14} + q^{(n-10)} + q^{(2n-15)}\).

Let \(n\) be odd. The FDRM code with rank distance \(d = 2\) arising from the identifying vectors of

- \(A_1^2\) has cardinality \(q^{(n-4)}(q^{2(n-6)} + (\frac{n}{2} - 4)q^{(n-7)} + q^{(n-4)})\).
- \(A_2^2\) has cardinality \(q^{(n-5)}(q^{2(n-6)} + (\frac{n}{2} - 4)q^{(n-8)} + q^{(2n-13)} + q^{(n-7)})\).
- \(A_3^2\) has cardinality \(q^{(n-5)}(q^{2(n-6)} + (\frac{n}{2} - 4)q^{(n-8)} + q^{(2n-13)} + q^{(n-7)})\).
- \(A_4^2\) has cardinality \(q^{2n-11} + q^{2n-10} + (n-6 - i)q^{n-2i-6} + q^{n-6-i}\).
- \(A_5^2\) has cardinality \(q^{2n-10} + q^{2n-14} + q^{(n-7)} + q^{(n-9)}\).
- \(A_6^2\) has cardinality \(q^{2n-11} + q^{2n-13} + q^{(n-10)} + q^{(2n-15)}\).
- \(A_7^2\) has cardinality \(q^{2n-12} + q^{2n-13} + q^{(n-10)} + q^{(2n-15)}\).
- \(A_8^2\) has cardinality \(q^{2n-12} + q^{2n-14} + q^{(n-10)} + q^{(2n-15)}\).
- \(A_9^2\) has cardinality \(q^{2n-12} + q^{2n-14} + q^{(n-10)} + q^{(2n-15)}\).
Let $n$ be odd. The FDRM code with rank distance $d = 2$ arising from the identifying vectors of

- $A_1^4$ has cardinality $(q^{(n-4)} + q^{(n-6)}) (q^{(2n-6)} + (\frac{n-3}{2})^2 q^{(n-8)})$
- $A_2^4$ has cardinality $(q^{(n-5)} + q^{(n-6)}) (\frac{n-3}{2} q^{n-9} + q^{2n-13})$
- $A_3^4$ has cardinality $(q^{(n-5)} + q^{(n-6)}) \left[ \sum_{i=2}^{\lfloor \frac{n}{2} \rfloor} (iq^{2n-2i-10} + (\frac{n-5}{2} - i)q^{n-2i-6}) + \sum_{i=2}^{\lfloor \frac{n}{2} \rfloor + 1} (iq^{2n-2i-11} + (\frac{n-5}{2} - i)q^{n-2i-7}) \right]$
- $A_4^5$ has cardinality $(q^{2n-10} + q^{2n-14}) (q^{2n-14} + (\frac{n-9}{2})q^{(n-8)} + q^{\frac{n-7}{2}})$
- $A_5^4$ has cardinality $(q^{2n-11} + q^{2n-13}) (\frac{n-9}{2} q^{(n-9)} + q^{2n-15}) + q^{n-8}$
- $A_6^5$ has cardinality $(q^{2n-12} + q^{2n-13}) (q^{2n-16} + (\frac{n-11}{2})q^{(n-10)} + q^{\frac{n-8}{2}})$
- $A_7^4$ has cardinality $(q^{2n-12} + q^{2n-14}) (\frac{n-11}{2} q^{(n-11)} + 2q^{(2n-17)} + q^{n-9})$
- $A_8^5$ has cardinality $(q^{2n-14} + q^{2n-12}) \left[ \sum_{i=3}^{\lfloor \frac{n}{2} \rfloor + 2} (iq^{2n-12} + (\frac{n-7}{2} - i)q^{n-2i-6} + q^{\frac{n-7}{2} - i}) + \sum_{i=3}^{\lfloor \frac{n}{2} \rfloor + 2} (iq^{2n-2i-13} + (\frac{n-7}{2} - i)q^{n-2i-7} + q^{n-7-i}) \right]$

These formulas imply the cardinality formulas of Theorems 32 and 35 by summing them up and adding the largest known code of length $n - k$ with zeros appended in front. Note that when summing them up we can merge the cardinalities of $A_2^4$ and $A_3^4$, as well as $A_5^4$ and $A_6^5$, respectively. An index shift in the second sums results in the formulas of Theorems 32 and 35.

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