Distributional Behavior of Time Averages of Non-$L^1$ Observables in One-dimensional Intermittent Maps with Infinite Invariant Measures

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Abstract In infinite ergodic theory, two distributional limit theorems are well-known. One is characterized by the Mittag-Leffler distribution for time averages of $L^1(m)$ functions, i.e., integrable functions with respect to an infinite invariant measure. The other is characterized by the generalized arc-sine distribution for time averages of non-$L^1(m)$ functions. Here, we provide another distributional behavior of time averages of non-$L^1(m)$ functions in one-dimensional intermittent maps where each has an indifferent fixed point and an infinite invariant measure. Observation functions considered here are non-$L^1(m)$ functions which vanish at the indifferent fixed point. We call this class of observation functions weak non-$L^1(m)$ function. Our main result represents a first step toward a third distributional limit theorem, i.e., a distributional limit theorem for this class of observables, in infinite ergodic theory. To prove our proposition, we propose a stochastic process induced by a renewal process to mimic a Birkhoff sum of a weak non-$L^1(m)$ function in the one-dimensional intermittent maps.

Keywords Infinite ergodic theory · Anomalous diffusion · Distributional limit theorem

1 Introduction

In statistical physics, many observables result from time averages of the microscopic observation functions. Ergodic theory plays an important role in providing the asymptotic behavior
Distributional Behavior of Time Averages of Non-$L^1$ of time-averaged observables in dynamical systems. Trajectories in chaotic dynamical systems cannot be predicted due to the sensitivity dependence of initial conditions. However, with the aid of the unpredictability, trajectories can be regarded as a stochastic process. Then, one can introduce a measure in dynamical systems. In fact, an invariant measure characterizes chaotic orbits. Birkhoff’s ergodic theorem tells us that time averages of an observation function converge to a constant for almost all initial conditions if the observation function is integrable with respect to an absolutely continuous invariant measure [14]. On the other hand, when an invariant measure cannot be normalized (infinite measure), the asymptotic behavior of time-averaged observables is completely different from that stated by the Birkhoff’s ergodic theorem. In infinite measure systems (infinite ergodic theory), one of the most striking points is that a time-averaged observable does not converge to a constant but converges in distribution [1,2,4,35,37].

In infinite ergodic theory, two different distributional limit theorems for time averages have been known. Distribution of time averages of an $L^1(m)$ function, which is an integrable function with respect to an invariant measure $m$, follows the Mittag-Leffler distribution [1,2]. This distributional limit theorem is based on Darling–Kac theorem in stochastic processes [17]. The other distributional limit theorem states that time averages of a non-$L^1(m)$ function converges to the generalized arc-sin distribution [3,4,35,37,38], which is based on Dynkin–Lamperti’s generalized arc-sine law [18,26]. In infinite ergodic theory, it is important to determine the distribution of time averages for arbitrary observation functions as well as arbitrary ensembles of initial points. Recently, one of the authors has shown that the distribution of time averages of $L_1(m)$ functions depends also on the ratio of a measurement time and the time at which system started, i.e., aging distributional behavior [8]. Here, we provide another distributional behavior that is in-between the above two distributional limit theorems.

Infinite ergodic theory has attracted the interest from not only mathematics but also physics community [4–7,10,13,19,23]. This is because distributional behaviors of time-averaged observables are ubiquitous in phenomena ranging from fluorescence in nano material [15] to biological transports [21,33,39]. Theoretical studies on distributional behaviors of time-averaged observables have been extensively conducted using stochastic models with divergent mean trapping-time distributions such as continuous-time random walks [22,30], random walk with static disorder [29], and dichotomous processes [28]. The distribution function of time-averaged observables depends on the type of observation function. In particular, the distribution of time-averaged mean square displacement follows the Mittag-Leffler distribution [22,30], while that of the ratio of occupation time of on state in dichotomous processes follows the generalized arc-sine distribution [28]. Although distributional limit theorems in stochastic processes have been elucidated, it will be possible to construct another distributional limit theorem of time-averaged observables by introducing another type of the observation function in stochastic models with divergent mean trapping-time distributions. In fact, one of the authors has shown a novel distributional behavior for time-averaged mean square displacements in stored-energy-driven Lévy flight [11,12].

In this paper, we provide a novel distribution for time averages of a class of non-$L^1(m)$ functions in one-dimensional maps with indifferent fixed points having infinite invariant measures. The value of the observation function at the indifferent fixed point is zero. Because the observation function is non-$L^1(m)$, the generalized arc-sine distribution can be applied to those observation functions. However, it only gives a trivial result that time averages converge to zero. Our distributional limit theorem gives a non-trivial broad distribution of normalized time averages. In other words, we refine the distribution of normalized time averages of such observation functions by introducing a normalizing sequence. The proof is based on
a stochastic process induced by a renewal process proposed here, which mimics a Birkhoff sum of a non-$L^1(m)$ function.

2 From Dynamical System to Stochastic Process: Partial Sums of Non-$L^1(m)$ Functions

A dynamical system considered here is a transformation $T : [0, 1] \to [0, 1]$ which satisfies the following conditions for some $c \in (0, 1)$: (i) the restrictions $T : (0, c) \to (0, 1)$ and $T : (c, 1) \to (0, 1)$ are $C^2$ and onto, and have $C^2$-extensions to the respective closed intervals; (ii) $T'(z) > 1$ on $(0, c] \cup [c, 1]$; $T'(0) = 1$; (iii) $T(z) - z$ is regularly varying at zero with index $1 + 1/\alpha$, $T(z) - z \sim az^{1+1/\alpha}$ ($\alpha > 0$). For example, a transformation,

$$T_\alpha(z) = z\left(1 + \left(\frac{z}{1+z}\right)^{\frac{1-\alpha}{\alpha}} - z^{\frac{1-\alpha}{\alpha}}\right)^{-\frac{\alpha}{1-\alpha}} \pmod{1},$$

satisfies the conditions ($\alpha = 1$). It is known that an invariant measure $m$ of the map is given by $dm/dz \propto z^{-1/\alpha}$ ($z \to 0$) [34]. Thus, the invariant measure cannot be normalized for $\alpha \leq 1$. While this dynamical system has zero-Lyapunov exponent, the dynamical instability can be characterized as a sub-exponential instability [7,19,23].

For $z_t \equiv T^t(z_0) \to 0$, the following ordinary differential equation can be used to describe the dynamics [20,27]:

$$\frac{d\tilde{z}_t}{dt} = a\tilde{z}_t^{1+1/\alpha},$$

where we use $\tilde{z}_t$ as the solution of the ordinary differential equation (2) with an initial condition $\tilde{z}_0 = z_0$. The solution is given by

$$\tilde{z}_t = z_0 \left(1 - \frac{t}{\tau}\right)^{-\alpha},$$

where

$$\tau = \alpha/(z_0^{1/\alpha} a)$$

is a characteristic time scale that a trajectory with initial point $z_0$ escapes from $[0, c]$. In fact, a time when $\tilde{z}_t$ becomes unity denoted by $t_c$, i.e., $\tilde{z}_{t_c} = 1$, is given by $t_c = \tau - \alpha/a$. In what follows, we use a sequence $c_n$ defined by $c_n = T(c_{n+1})$ with $c_n < c$ ($n = 1, 2, \cdots$) and $c_0 = c$ (see Fig. 1). Trajectory is reinjected to $[0, c]$ from $(c, 1]$. Because this dynamical system has a sub-exponential dynamical instability, the reinjection points, $z_0$, can be regarded as a random variable, and it is known that the reinjection points are almost uniformly distributed on $[0, c]$. Because the distribution of $\tau$ is determined by that of $z_0$, by assuming the probability density function (PDF) of $z_0$ is uniform on $[0, c]$, we have the PDF of residence times on $[0, c]$:

$$w(\tau) \sim \frac{A_c}{\Gamma(-\alpha)}\tau^{-1-\alpha} \text{ as } \tau \to \infty (z_0 \to 0),$$

where $A_c$ depends on not only $\alpha$ and $c$ but also details of the map $T(x)$. We note that the mean residence time $\langle \tau \rangle$ diverges when an invariant measure cannot be normalized ($\alpha \leq 1$). Here, we give a rigorous result that a normalized Birkhoff’s sum can be represented by the trajectory generated by Eq. (2).

**Lemma 1** For $t \ll N$, there exists $N > 0$ such that $z_t < \tilde{z}_t < z_{t+1}$ where $z_0 = \tilde{z}_0 = c_N$. 

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Fig. 1 Transformation (1) with $\alpha = 0.5$. The origin, $z = 0$, is the indifferent fixed point, i.e., $T'(0) = 1$. The sequence $c_n$ is plotted for $n = 0, 1, $ and 2.

**Proof** By Eqs. (3) and (4), we have

$$\tilde{z}_1 - z_1 = c_N \left(1 - \frac{c_N^{1/a} d}{\alpha}\right)^{-\alpha} - c_{N-1} > c_N + ac_N^{1+1/a} - c_{N-1} \simeq 0,$$

and

$$z_2 - \tilde{z}_1 \simeq c_{N-1} + ac_{N-1}^{1+1/a} - c_N \left(1 - \frac{c_N^{1/a} d}{\alpha}\right)^{-\alpha} = c_{N-1} - c_N + a(c_{N-1}^{1+1/a} - c_N^{1+1/a}) + o(c_N^{1+1/a}).$$

Because the sequence $c_n$ is given by $c_n \sim \alpha^n (an)^{-\alpha}$ for $n \to \infty$, we have

$$c_{N-1} - c_N \sim \left(\frac{\alpha}{aN}\right)\left[\left(1 - \frac{1}{N}\right)^{-\alpha} - 1\right] \sim a\left(\frac{\alpha}{aN}\right)^{\alpha+1} = O(N^{-1-\alpha}),$$

and

$$c_N^{1+1/a} \sim \left(\frac{\alpha}{aN}\right)^{\alpha+1}.$$  

It follows that $\tilde{z}_1 > z_1$, $z_2 > \tilde{z}_1$ and $z_2 - \tilde{z}_1 = O(N^{-1-\alpha})$. We assume $z_t < \tilde{z}_t < z_{t+1}$ and $z_{t+1} - \tilde{z}_t = O(N^{-1-\alpha})$ for $N \to \infty$. Then, we have

$$\tilde{z}_{t+1} - z_{t+1} \simeq \tilde{z}_t \left(1 - \frac{\tilde{z}_t^{1/a} d}{\alpha}\right)^{-\alpha} - z_t - az_t^{1+1/a}$$

$$> \tilde{z}_t + az_t^{1+1/a} - z_t - az_t^{1+1/a},$$

and

$$z_{t+2} - \tilde{z}_{t+1} \simeq z_{t+1} + az_{t+1}^{1+1/a} - \tilde{z}_t \left(1 - \frac{\tilde{z}_t^{1/a} d}{\alpha}\right)^{-\alpha}$$

$$> \tilde{z}_t + az_t^{1+1/a} - z_t - az_t^{1+1/a},$$

$$z_t + az_t^{1+1/a} - \tilde{z}_t \left(1 - \frac{\tilde{z}_t^{1/a} d}{\alpha}\right)^{-\alpha}.$$
By Lemma 1, 
\[ \sum_{k=0}^{t-1} f(z_k) = z_{t+1} - \tilde{z}_t + a(z_{t+1}^{1+1/\alpha} - \tilde{z}_t^{1+1/\alpha}) + o(N^{-1-\alpha}). \] (14)

Because we assume \( z_t < \tilde{z}_t < z_{t+1} \) and \( \tilde{z}_{t+1} - z_t = O(N^{-1-\alpha}) \), we have \( \tilde{z}_{t+1} - z_{t+1} > 0 \), \( z_{t+2} - \tilde{z}_{t+1} > 0 \) and \( z_{t+2} - \tilde{z}_{t+1} = O(N^{-1-\alpha}) \). It follows by mathematical induction that there exists \( N \) such that \( z_t < \tilde{z}_t < z_{t+1} \) for \( t \ll N \). \( \square \)

Here, we consider the following bounded continuous observation function, \( f(z) \sim Cz^{-\alpha} (z \to 0) \), which is not an \( L^1(m) \) function for \( \alpha \leq \gamma < 1 \); we call this type of functions as weak non-\( L^1(m) \) functions. In particular, we study statistical properties of partial sums of this type of observables,

\[ S_l = \sum_{k=0}^{t-1} f(z_k) \] (15)
to elucidate the ergodic properties (Note that \( S_l/t \) is the time average).

**Lemma 2** For \( z_0 \in [c_{N+1}, c_N) \) and \( l(\ll N) \), there exists \( N \) such that

\[ \left| \sum_{k=0}^{l} f(z_k) - \int_{0}^{l} f(\tilde{z}_t)dt \right| < \Delta I, \] (16)

where \( \Delta I \equiv f(c_{N-l}) - f(c_{N+1}) \).

**Proof** First, we define \( I_{\max} \) and \( I_{\min} \) as

\[ I_{\max} = \sum_{k=N-l}^{N} f(c_k) \quad \text{and} \quad I_{\min} = \sum_{k=N-l}^{N} f(c_{k+1}), \] (17)

and

\[ \Delta I \equiv I_{\max} - I_{\min} = f(c_{N-l}) - f(c_{N+1}). \] (18)

By Lemma 1,

\[ I_{\min} < \sum_{k=0}^{l} f(z_k) < I_{\max} \quad \text{and} \quad I_{\min} < \int_{0}^{l} f(\tilde{z}_t)dt < I_{\max}. \] (19)

It follows

\[ \left| \sum_{k=0}^{l} f(z_k) - \int_{0}^{l} f(\tilde{z}_t)dt \right| < \Delta I. \] (20)

\( \square \)

Here, we decompose the function \( f(z) \) into an \( L^1(m) \) part and a non-\( L^1(m) \) part, where an \( L^1(m) \) part, \( f^R_\delta (z) \), is defined by \( f^R_\delta (z) \equiv 0 \) on \( [0, \delta] \) and \( f^R_\delta (z) \equiv f(z) \) on \( (\delta, 1] \), and a non-\( L^1(m) \) part, \( f^L_\delta (z) \), is defined by \( f^L_\delta (z) \equiv f(z) \) on \( [0, \delta] \) and \( f^L_\delta (z) \equiv 0 \) on \( (\delta, 1] \). By the Aaronson’s distributional limit theorem, \( \sum_{k=0}^{-1} f^R(z_k)/n^\alpha \) converges in distribution for all \( \delta > 0 \) because \( f^R_\delta (z) \) is an \( L^1(m) \) function for all \( \delta > 0 \). It follows that for a sequence \( a_n \) such that \( a_n/n^\alpha \to \infty \) as \( n \to \infty \), the normalized time averages, \( \sum_{k=0}^{-1} f^R(z_k)/a_n \), converge to zero: \( \sum_{k=0}^{-1} f^R(z_k)/a_n \to 0 \) as \( n \to \infty \).

For the dynamical systems defined above, a trajectory is trapped in the interval \( [0, \delta] \) for a long time and then escapes to the other interval \( [\delta, 1] \) for small \( \delta \). Let us consider \( k \)-th such
trapping state. We note that the k-th trapping time denoted by \( \tau_k \) is approximately given by 
\[
\alpha/\alpha(z_{0,k}^{-\alpha} - \delta^{-\alpha})/\alpha, \quad \text{where } z_{0,k} \text{ is the k-th re-injection point.}
\]
We will show that a partial sum during the k-th trap in \([0, \delta]\),
\[
I(\tau_e, \tau_k) = \sum_{i=t_{k-1}}^{t_k-1} f(z_i),
\]
can be replaced as \( \int_0^{\tau_e} f(\tilde{z}_t)dt \) with \( \tilde{z}_0 = z_{t_{k-1}} \), where
\[
\int_0^{\tau_e} f(\tilde{z}_t)dt \approx B \tau_k^\gamma \left[ 1 - \left( 1 - \frac{\tau_e}{\tau_k} \right)^\gamma \right],
\]
for \( \tilde{z}_0 \to 0, t_{k-1} = \tau_1 + \cdots + \tau_{k-1}, \tau_e \in [0, \tau_k] \) is the elapsed time since the beginning of the trapping, and \( B \) is a constant given by \( B = (\alpha/\alpha)^{-\gamma}/C/\gamma \). Because we assume that \( z_{0,k} \) is uniformly distributed on \([0, \delta]\) or equivalently assume Eq. (5), the PDF of \( I(\tau_k) \equiv I(\tau_k, \tau_k) = B \tau_k^\gamma \) is given by
\[
I(x) \sim \frac{A_3 \beta^\gamma}{\gamma|\Gamma(-\alpha)|} x^{1-\frac{\alpha}{\gamma}} (x \to \infty).
\]

We note that the constant \( A_3 \) depend on \( \delta \).

**Lemma 3** For \( a_t \propto t^\gamma \), the asymptotic behavior of the normalized time average, \( S_t/a_t \), is given by
\[
\frac{S_t}{a_t} \sim \frac{1}{a_t} \sum_{k=1}^{N_t} \sum_{i=0}^{t_k-1} f(z_i) + \int_0^{\tau_e} f(\tilde{z}_t)dt/a_t,
\]
where \( \tilde{z}_0 = z_{t_{N_t}}, N_t \) is the number of re-injections to \([0, \delta]\) until time \( t \) and \( \tau_k \) is the k-th trapping time on \([0, \delta]\) and \( \delta \ll 1 \).

**Proof** A partial sum is given by
\[
S_t = \sum_{k=0}^{t_k-1} f^L_\delta(z_i) + \sum_{k=0}^{t_k-1} f^R_\delta(z_i),
\]
where the second term contributes to a Mittag-Leffler distribution but it can be ignored when we consider a normalized time averages of weak non-L\(^1\)(m) functions, because the order of the normalizing sequence is greater than that of the return sequence. In fact, the normalizing sequences for \( f^R_\delta \) and \( f^L_\delta \) are given by \( \langle \sum_{k=0}^{t_k-1} f^R_\delta(z_k) \rangle \propto t^\alpha \) and \( \langle \sum_{k=0}^{t_k-1} f^L_\delta(z_k) \rangle \propto t^\gamma \), respectively (\( \gamma > \alpha \)). Therefore, it is sufficient to consider the first term only. By Lemma 2, for \( \delta \ll 1 \) and \( z_i \leq \delta (i = 0, \cdots, \tau_e, \cdots, \tau_k) \), there exists a constant \( \varepsilon \) such that
\[
\left| \sum_{i=0}^{t_k} f^L_\delta(z_i) - I(\tau_k) \right| < \varepsilon \quad \text{and} \quad \left| \sum_{i=0}^{t_e} f^L_\delta(z_i) - \int_0^{\tau_e} f(\tilde{z}_t)dt \right| < \varepsilon,
\]
where the constant \( \varepsilon \) does not depend on \( \tau_k \) but depend on \( \delta \). For \( a_t = O(t^\gamma) \), we have
\[
\frac{1}{a_t} \sum_{k=0}^{t_k-1} f^L_\delta(z_k) - \sum_{k=1}^{N_t} I(\tau_k) - \int_0^{\tau_e} f(\tilde{z}_t)dt \ll \frac{\varepsilon(N_t + 1)}{a_t}.
\]
Because \( \langle N_t \rangle \propto t^\alpha \) [10], the left-hand-side goes to zero as \( t \to \infty \). \( \square \)

In the following sections, we will show that there exists a sequence \( a_t \) such that the normalized time average, \( S_t/a_t \), converges in distribution:
\[
\frac{1}{a_t} \sum_{k=0}^{t_k-1} f(z_k) \Rightarrow Y_{\alpha, \gamma} \quad \text{as} \quad n \to \infty,
\]
where the Laplace transform of the random variable $Y_{\alpha,\gamma}$ is given by Eq. (45). We note that the sequence $a_t$ is given by $a_t \equiv \langle \sum_{k=0}^{t-1} f(z_k) \rangle \propto t^\gamma$, which is not the so-called return sequence in infinite ergodic theory [2]. In particular, the order of the return sequence is given by $t^\alpha$, which is smaller than that of $a_t$, i.e., $t^\alpha / a_t \to 0$ as $t \to \infty$.

### 3 Continuous Accumulation Process

To analyze the partial sum (Eq. 15), we generalize a renewal process. Renewal process is a point process where the time intervals between point events are independent and identically distributed (i.i.d) random variables [16]. Because residence times near the indifferent fixed point in intermittent maps are considered to be almost i.i.d. random variables, one can apply renewal processes to study dynamical systems [9].

Here, we consider a **cumulative process** by introducing an intensity of each renewal event [16], where intensity is correlated with the time interval between successive renewals. This process can be characterized by the total intensity $X_t$ until time $t$, whereas renewal processes are characterized by the number of renewals in the time interval $[0, t]$, denoted by $N_t$. Let $\tau_1, \ldots, \tau_k$ be the time intervals between successive renewals, which are i.i.d. random variables with PDF $w(\tau)$. We assume that the $k$-th intensity is determined by the $k$-th interevent time (IET) $\tau_k$ as $I(\tau_k) \equiv B \tau_k^\gamma$, where $\gamma \in [0, 1)$ and $B > 0$. Thus, the longer the IET between renewals becomes, the larger the intensity is. Furthermore, we propose a **continuous accumulation process** induced by a renewal process to consider a Birkhoff sum. In the continuous accumulation process, the intensity is gradually accumulated according to a function $I(\tau_e, \tau_k)$ in between the two successive renewals, where $\tau_e \in [0, \tau_k]$ is the elapsed time after the $(k - 1)$-th renewal (see Fig. 2). Here, we use the following intensity function:

$$I(\tau_e, \tau_k) = I(\tau_k) \left[ 1 - \left( 1 - \frac{\tau_e}{\tau_k} \right)^\gamma \right]. \tag{28}$$

This intensity function $I(\tau_e, \tau)$ mimics an increase of the Birkhoff sum in dynamical systems [see Eq. (21)]. As shown in the previous section, the following stochastic variable $X_t$ (the integrated intensity up to time $t$),

$$X_t = \sum_{k=1}^{N_t} I(\tau_k) + I(t - t_{N_t}, \tau_{N_t+1}), \tag{29}$$

where $t_k = \tau_1 + \ldots \tau_k$, is related to the Birkhoff sum of the non-integrable function. The case in which $\gamma = 0$ and $B = 1$ is exactly equivalent to the usual renewal process, because $X_t = N_t$.

Here, we consider the case that the mean interevent times of renewals diverges ($\alpha \leq 1$). In particular, we use Eq. (5) as the PDF of IETs. Thus, the survival probability $W(\tau)$ is given by

$$W(\tau) \equiv 1 - \int_0^\tau w(\tau')d\tau' \sim \frac{A}{\Gamma(1-\alpha)} \tau^{-\alpha} \quad (\tau \to \infty), \tag{30}$$

and the PDF of $I(\tau_k)$, denoted by $l(x)$, is given by Eq. (22). Because the mean intensity $\langle I \rangle$ diverges for $\alpha \leq \gamma$, the renewal theory cannot be straightforwardly applied.
4 Theory of a Continuous Accumulation Process

4.1 Generalized Renewal Equation

Distribution of $X_t$ can be derived by a generalized renewal equation, which is similar to a generalized master equation for the continuous-time random walk (CTRW) [32]. First, we define a joint PDF of the IET $\tau$ and the intensity increment $x$ as $\psi(x, \tau) = w(\tau)\delta(x - I(\tau))$, and we also use $\Psi(x, \tau; \tau) = \delta(x - I(\tau_e, \tau))\theta(\tau - \tau_e)$, where $\theta(x) = 0$ for $x < 0$ and 1 otherwise. Let $Q(x, t)$ be the PDF of $X_t$ at time $t$ when a renewal occurs, then we have

$$Q(x, t) = \int_0^\infty dx'\psi(x', t)\delta(x - x')$$
$$+ \int_0^x dx'\int_0^t dt'\psi(x', t')Q(x - x', t - t').$$

(31)

The conditional PDF of $X_t$ at time $t$ on the condition of $\tau_{N_t + 1} = \tau$, denoted by $P(x, t; \tau)$ is given by

$$P(x, t; \tau) = \int_0^x dx'\int_0^t dt'\Psi(x', t'; \tau)Q(x - x', t - t') + \Psi(x, t; \tau).$$

(32)

It follows that the PDF of $X_t$ at time $t$ reads

$$P(x, t) = \int_0^\infty w(\tau)P(x, t; \tau)d\tau.$$

(33)

Here, we assume that a renewal occurs at time $t = 0$, i.e., ordinary renewal process [16]. Using the double Laplace transform with respect to time ($t \to s$) and $X_t$ ($x \to k$), defined by

$$\hat{P}(k, s) = \int_0^\infty dt\int_0^\infty dx e^{-st-kx}P(x, t),$$

(34)
we have

\[ \hat{P}(k, s) = \int_0^\infty w(\tau) \hat{\psi}(k, s; \tau) \frac{d\tau}{1 - \hat{\psi}(k, s)}, \]  

(35)

where

\[ \hat{\psi}(k, s) \equiv \int_0^\infty d\tau \int_0^\infty dx e^{-s\tau - kx} \psi(x, \tau) = \int_0^\infty e^{-s\tau} e^{-kI(\tau)} w(\tau) d\tau, \]

(36)

and

\[ \hat{\Psi}(k, s; \tau) \equiv \int_0^\infty dt \int_0^\infty dx e^{-st - kx} \Psi(x, t; \tau) \int_0^\tau e^{-st - kI(t, \tau)} dt. \]

(37)

In what follows, we use the asymptotic behaviors of the Laplace transforms of \( w(\tau) \) and \( W(\tau) \), i.e., \( 1 - \hat{w}(s) \sim As^\alpha \) and \( \hat{W}(s) \sim As^\alpha - 1 \) for \( s \to 0 \).

4.2 Moments of \( X_t \)

4.2.1 First Moment

The Laplace transform of \( \langle X_t \rangle \), denoted by \( \langle X_s \rangle \), is given by \( \langle X_s \rangle = -\frac{\partial \hat{P}(k, s)}{\partial k} \bigg|_{k=0} \). As shown in the Appendix 1, the leading order of \( \langle X_s \rangle \) is given by

\[ \langle X_s \rangle \sim \begin{cases} 
BM_1(\alpha, \gamma) \frac{1}{\Gamma(1+\gamma)} & (\gamma > \alpha) \\
\frac{B}{\Gamma(-\alpha)} \frac{1}{s^{1+\gamma}} \log \left( \frac{1}{s} \right) & (\gamma = \alpha) \\
\langle I \rangle \frac{1}{A} s^{1+\alpha} & (\gamma < \alpha)
\end{cases} \]

(38)

where \( M_1(\alpha, \gamma) = \Gamma(\gamma - \alpha) \left[ 1 + \gamma(\gamma - \alpha) \right] \). The inverse Laplace transform reads

\[ \langle X_t \rangle \sim \begin{cases} 
BM_1(\alpha, \gamma) t^\gamma & (\gamma > \alpha) \\
\frac{B}{\Gamma(-\alpha)} \frac{1}{\Gamma(1+\gamma)} t^\alpha \log t & (\gamma = \alpha) \\
\langle I \rangle \frac{1}{A \Gamma(1+\alpha)} t^\alpha & (\gamma < \alpha)
\end{cases} \]

(39)

We note that the asymptotic behavior of \( \langle X_t \rangle \) is determined by \( B, \alpha, \) and \( \gamma \). In other words, it does not depend on \( A \).
4.2.2 Second Moment

The Laplace transform for the second moment of $X_t$, denoted by $\langle X_t^2 \rangle$ is given by $\langle X_t^2 \rangle = \frac{\partial^2 \hat{P}(k,s)}{\partial k^2} \bigg|_{k=0}$. As shown in the Appendix 2, the leading order of $\langle X_t^2 \rangle$ is given by

$$\langle X_t^2 \rangle \sim \begin{cases} \frac{B^2 M_2(\alpha, \gamma)}{|\Gamma(-\alpha)|} \frac{1}{s^{1+2\gamma}}, & (\gamma > \alpha) \\ \frac{2B^2}{|\Gamma(-\alpha)|^{n+1} s^{1+2\alpha}} \log^2 \left( \frac{1}{s} \right), & (\gamma = \alpha) \\ \frac{2(I)^2}{A^2} \frac{1}{s^{1+2\alpha}}, & (\gamma < \alpha) \end{cases} \quad (40)$$

where $M_2(\alpha, \gamma) = \left\{ 1 + \frac{\gamma(\gamma-\alpha)}{2+\alpha-\gamma} \right\} \Gamma(2\gamma - \alpha) + \left\{ 1 + \frac{\gamma(\gamma-\alpha)}{2+\alpha-\gamma} \right\} \frac{2\Gamma(\gamma-\alpha)^2}{\Gamma(-\alpha)}$. The inverse Laplace transform reads

$$\langle X_t^2 \rangle \sim \begin{cases} \frac{B^2 M_2(\alpha, \gamma)}{|\Gamma(-\alpha)|\Gamma(1+2\gamma)} t^{2\gamma}, & (\gamma > \alpha) \\ \frac{2B^2}{|\Gamma(-\alpha)|^{n+1} (\log t)^2}, & (\gamma = \alpha) \\ \frac{2(I)^2}{A^2} t^{2\alpha}, & (\gamma < \alpha) \end{cases} \quad (41)$$

4.2.3 nth Moment

As shown in the Appendix 3, the leading order of the Laplace transform of $\langle X_t^n \rangle$ with $n > 0$ for $s \to 0$ is given by

$$\langle X_t^n \rangle = \hat{P}^{(n)}(0, s) \sim \begin{cases} \frac{B^n M_n(\alpha, \gamma)}{|\Gamma(-\alpha)|} \frac{1}{s^{1+n\gamma}}, & (\gamma > \alpha) \\ \frac{(-1)^n n! [1 - \hat{\psi}(s)]}{s [1 - \hat{\psi}(0, s)]^{n+1}} \{\hat{\psi}'(0, s)\}^n, & (\gamma \leq \alpha) \end{cases} \quad (42)$$

$$\sim \begin{cases} \frac{B^n M_n(\alpha, \gamma)}{|\Gamma(-\alpha)|} \frac{1}{s^{1+n\gamma}}, & (\gamma > \alpha) \\ \frac{n! B^n}{|\Gamma(-\alpha)|^{n+1} s^{1+n\alpha}} \left\{ \log \left( \frac{1}{s} \right) \right\}^n, & (\gamma = \alpha) \\ \frac{n! (I)^n}{A^n} \frac{1}{s^{1+n\alpha}}, & (\gamma < \alpha) \end{cases} \quad (43)$$
where \( M_n(\alpha, \gamma) \) is given by (61). The inverse Laplace transform reads

\[
\langle X^n_t \rangle \sim \begin{cases} 
\frac{B^n M_n(\alpha, \gamma)}{|\Gamma(-\alpha)| \Gamma(1+n\gamma)} t^{n\gamma} & (\gamma > \alpha) \\
\frac{n! B^n}{|\Gamma(-\alpha)| n \Gamma(1+n\alpha)} t^{na} (\log t)^n & (\gamma = \alpha) \\
\frac{n!}{A^n \Gamma(1+n\alpha)} t^{na} & (\gamma < \alpha).
\end{cases}
\]

(44)

It follows that \( X_t/\langle X_t \rangle \) converges in distribution to \( Y_{\alpha,\gamma} \), where

\[
\langle e^{zY_{\alpha,\gamma}} \rangle = \begin{cases} 
\sum_{k=0}^{\infty} \frac{M_k(\alpha, \gamma) z^k}{k! M_1(\alpha, \gamma) k \Gamma(1+k\gamma)} & (\gamma > \alpha) \\
\sum_{k=0}^{\infty} \frac{\Gamma(1+\alpha) z^k}{\Gamma(1+k\alpha)} & (\gamma \leq \alpha),
\end{cases}
\]

(45)

and \( M_0(\alpha, \gamma) = 1 \). We note that the distribution of the normalized random variable \( X_t/\langle X_t \rangle \) does not depend on \( \gamma \) for \( \gamma \leq \alpha \) (in this case the observation function is \( L^1(m) \) in the dynamical system) and the distribution is called the Mittag-Leffler distribution of order \( \alpha \), whereas the distribution of a scaled sum \( X_t/\langle X_t \rangle \) for \( \gamma > \alpha \) converges to a time-independent non-trivial distribution, which is not the Mittag-Leffler distribution. Figure 3 shows the PDFs of \( X_t/\langle X_t \rangle \) for different \( \alpha \) and \( \gamma \).

5 Distribution of Time Averages of Weak Non-\( L^1(m) \) Functions

In the previous section, we have shown that the normalized random variable \( X_t/\langle X_t \rangle \) converges to \( Y_{\alpha,\gamma} \) in distribution. Because \( S_t = \sum_{k=0}^{t-1} f(z_k) \) can be represented by \( X_t \) for \( \gamma > \alpha \) (Lemma 3), we have the following proposition for the distribution of time average of a weak non-\( L^1(m) \) function.

Proposition 1  For a transformation satisfying the conditions (i), (ii), and (iii), and a bounded continuous observation function \( f(z) \sim \alpha^{1-\gamma} C z^{\frac{1}{\alpha}} (z \to 0) \) with \( \alpha \leq 1 \) and \( \alpha < \gamma < 1 \), there exists sequence \( a_n \) such that the normalized time average, \( S_n/a_n \), converges in distribution:

\[
\frac{1}{a_n} \sum_{k=0}^{n-1} f(z_k) \Rightarrow Y_{\alpha,\gamma} \quad \text{as} \quad n \to \infty,
\]

(46)

where the Laplace transform of the random variable \( Y_{\alpha,\gamma} \) is given by Eq. (45).

Remark 1  The sequence \( a_n \) is given by \( a_n = \langle S_n \rangle \propto n^{\gamma} \) for \( \gamma > \alpha \), which is not the return sequence, where \( \langle . \rangle \) means an average with respect to the initial point \( z_0 \).

Proof  By Lemma 3, the distribution of time averages of \( f(z) \) in the dynamical system considered here can be regarded as that in a continuous accumulation process. The result in Sect. 4 implies the proposition.

To demonstrate our proposition, we use the map \( T_\alpha : [0, 1] \to [0, 1] \) with \( \alpha \leq 1 \) [36] defined by Eq. (1). The asymptotic behavior of \( T(x) \) for \( x \to 0 \) is given by \( T_\alpha(x) - x \sim x^{1+1/\alpha} \). Thus, \( a = 1 \). The invariant density \( \rho_\alpha(x) \) of this map is exactly known as [36]
**Fig. 3** Probability density functions of $y = X_t / \langle X_t \rangle$ for a $\alpha = 0.25$, b $\alpha = 0.5$ and c $\alpha = 0.75$. The PDFs are obtained by numerical simulations of continuous accumulation processes. Total simulation time $t$ is $10^7$, $10^6$ and $10^5$ for $\alpha = 0.25$, 0.5 and 0.75, respectively. PDFs crucially depend on $\gamma$ for $\gamma > \alpha$. Because of finite simulation time, PDFs for $\gamma < \alpha$ slightly depend on $\gamma$. In the numerical simulations, we used the PDF of $\tau$ as $w(\tau) = \alpha \tau^{-1 - \alpha}$ ($\tau \geq 1$) and $B = 1$.

\begin{align*}
\rho_\alpha(x) &= \frac{C_\alpha}{x^\frac{1}{\alpha}} + \frac{C_\alpha}{(1 + x)^\frac{1}{\alpha}},
\end{align*} 

(47)
Fig. 4 Moments of $S_t$ in the map (1), where $\alpha = 0.5$, $\gamma = 0.9$, and $f(z) = z^{1-\gamma}/\alpha$. Symbols are the results of numerical simulations. Solid lines are the theoretical ones (39), (41), and (44). The asymptotic behaviors are well described by the theory without fitting.

Fig. 5 Probability density functions of $S_t/\langle S_t \rangle$ with $f(z) = z^{1-\gamma}/\alpha$ in the map (1) with $\alpha = 0.5$ for $\gamma = 0.6$ and 0.9. Histograms are the results of numerical simulations with a $\gamma = 0.6$ and b $\gamma = 0.9$. The solid lines are the PDFs obtained by numerical simulation of continuous accumulation processes. Total simulation time is $t = 10^7$. Histograms are in good agreement with the PDFs obtained in the corresponding continuous accumulation processes.

where $C_\alpha$ is a multiplicative constant. By numerical simulations, we have confirmed that the asymptotic behaviors of moments of $S_t$ are well described by the theory (44) as shown in Fig. 4. Figure 5 shows that PDF of $S_t/\langle S_t \rangle$ is in good agreement with the PDF of $X_t/\langle X_t \rangle$ in the corresponding continuous accumulation process.

Because the ensemble average of $f(z)$ with respect to an infinite measure diverges, one cannot obtain a relation between the time average and the ensemble average with respect to the infinite measure. However, we have shown that $\langle S_t/t^\gamma \rangle$ converges to a constant, and the constant is determined by $\alpha$, $\gamma$, and $B$. Because these constants are determined by the asymptotic behaviors of $T(z)$ and $f(z)$, the normalizing sequence, $a_n$, in the proposition can be determined by the asymptotic behaviors of $T(z)$ and $f(z)$. In other words, the sequence does not depend on the details of a map $T(z)$ and $f(z)$ except for a small $z$ behavior. We have numerically confirmed them (not shown).

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6 Conclusion

For one-dimensional intermittent maps with infinite invariant measures, we have shown a novel distributional behavior for time averages of weak non-$L^1$ functions. The distribution refines the generalized arc-sine distribution of time average for weak non-$L^1$ functions because the normalizing sequence is not $n$ but is proportional to $n^\gamma$ ($\gamma < 1$). Therefore, the distribution is not the generalized arc-sine distribution nor the Mittag-Leffler distribution. In other words, we have made an important first step for a foundation of the third distributional limit theorem in infinite ergodic theory. Recently, distributional behaviors in intermittent maps with more than two indifferent fixed points has been studied [24, 25, 31]. This kind of extension will be interesting for a future work. The proof of our proposition is based on the theory of the continuous accumulation process proposed here. Our result is summarized in Fig. 6. This novel distributional limit theorem is related to a distributional behavior of time-averaged diffusion coefficients in a model of anomalous diffusion like stored-energy-driven Lévy flight [11].

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Appendix 1: First Moment

The Laplace transform of $\langle X_t \rangle$ is given by

$$
\langle X_t \rangle = -\left. \frac{\partial \hat{P}(k, s)}{\partial k} \right|_{k=0} = -\frac{\hat{w}'(0, s) \int_0^\infty W(\tau) e^{-st} d\tau}{[1 - \hat{\psi}(0, s)]^2} + \int_0^\infty \frac{w(\tau) [\int_0^\tau I(t, \tau) e^{-st} dt] d\tau}{1 - \hat{\psi}(0, s)}
$$

Because the asymptotic behavior of $w(\tau) I(t, \tau)$ is given by $w(\tau) I(t, \tau) \sim AB \gamma t^{-2+\gamma-\alpha}/|\Gamma(-\alpha)|$ for $t/\tau \ll 1$, the integral in the second term can be calculated...
as follows:

\[
\int_0^\infty dt \left[ \int_t^\infty d\tau w(\tau) I(t, \tau) \right] e^{-st} \sim \frac{AB\gamma}{|\Gamma(-\alpha)|} \int_0^\infty t \left[ \int_t^\infty \tau^{-2-\alpha+\gamma} d\tau \right] e^{-st} dt \\
= \frac{AB\gamma}{|\Gamma(-\alpha)|(1+\alpha-\gamma)} \int_0^\infty t^{\gamma-\alpha} e^{-st} dt \\
\sim \frac{AB\gamma \Gamma(\gamma - \alpha + 1)}{|\Gamma(-\alpha)|(1+\alpha-\gamma)s^{\gamma-\alpha+1}}.
\]  

(49)

Using the asymptotic behavior of \( \hat{\psi}'(0, s) \),

\[
\hat{\psi}'(0, s) \sim -\frac{AB\Gamma(\gamma - \alpha)}{|\Gamma(-\alpha)|} \frac{1}{s^{\gamma-\alpha}} \ (\gamma > \alpha),
\]

(50)

we have

\[
\langle X_s \rangle = \frac{B\Gamma(\gamma - \alpha)}{|\Gamma(-\alpha)|} \frac{1}{s^{1+\gamma}} + \frac{B\gamma(\gamma - \alpha)\Gamma(\gamma - \alpha)}{|\Gamma(-\alpha)|(1+\alpha-\gamma)s^{1+\gamma}} \\
= \frac{B\Gamma(\gamma - \alpha)}{|\Gamma(-\alpha)|} \left[ 1 + \frac{\gamma(\gamma - \alpha)}{1+\alpha-\gamma} \right] \frac{1}{s^{1+\gamma}}.
\]

(51)

**Appendix 2: Second Moment**

The Laplace transform for the second moment of \( X_t \) is given by

\[
\langle X_s^2 \rangle = \left. \frac{\partial^2 \hat{P}(k, s)}{\partial k^2} \right|_{k=0} \\
= \int_0^\infty w(\tau) \int_0^\infty I(t, \tau)^2 e^{-st} dt d\tau - \frac{2\hat{\psi}'(0, s)f_0^\infty w(\tau)f_0^\tau I(t, \tau)e^{-st} dt d\tau}{(1-\hat{w}(s))^2} \\
+ \frac{2[\hat{\psi}'(0, s)]^2}{s[1-\hat{w}(s)]^2} + \frac{\hat{\psi}''(0, s)}{s[1-\hat{w}(s)]} \\
= \int_0^\infty dt \left[ \int_t^\infty d\tau w(\tau)I(t, \tau)^2 \right] e^{-st} \\
- \frac{2\hat{\psi}'(0, s)f_0^\infty dt \left[ \int_t^\infty d\tau w(\tau)I(t, \tau) \right] e^{-st}}{(1-\hat{w}(s))^2} \\
+ \frac{2[\hat{\psi}'(0, s)]^2}{s[1-\hat{w}(s)]^2} + \frac{\hat{\psi}''(0, s)}{s[1-\hat{w}(s)]}.
\]

(52)

Because the asymptotic behavior of \( w(\tau)I(t, \tau)^2 \) is given by \( w(\tau)I(t, \tau)^2 \sim A(\beta\gamma t^2)^{-3+2\gamma-\alpha}/|\Gamma(-\alpha)| \) for \( t/\tau \ll 1 \), the integral in the first term can be calculated as follows:

\[
\int_0^\infty dt \left[ \int_t^\infty d\tau w(\tau)I(t, \tau)^2 \right] e^{-st} \sim \frac{A(\beta\gamma)^2}{|\Gamma(-\alpha)|} \int_0^\infty t^2 \left[ \int_t^\infty \tau^{-3-\alpha+2\gamma} d\tau \right] e^{-st} dt \\
= \frac{A(\beta\gamma)^2}{|\Gamma(-\alpha)|(2+\alpha-2\gamma)} \int_0^\infty t^{2\gamma-\alpha} e^{-st} dt \\
\sim \frac{A(\beta\gamma)^2 \Gamma(2\gamma - \alpha + 1)}{|\Gamma(-\alpha)|(2+\alpha-2\gamma)s^{2\gamma-\alpha+1}}.
\]

(53)
Using Eq. (49) and

\[
\hat{\psi}''(0, s) \sim \frac{AB^2 \Gamma(2\gamma - \alpha)}{|\Gamma(-\alpha)|} \frac{1}{s^{2\gamma - \alpha}}, \quad (\gamma > \alpha)
\]

we have

\[
\langle X^2 \rangle_s \sim \left[ \frac{(B\gamma)^2 \Gamma(2\gamma - \alpha + 1)}{|\Gamma(-\alpha)|(2 + \alpha - 2\gamma)} + \frac{2B\Gamma(\gamma - \alpha)}{|\Gamma(-\alpha)|} \frac{B\gamma \Gamma(\gamma - \alpha + 1)}{|\Gamma(-\alpha)|(1 + \alpha - \gamma)} \right. \\
+ \left. \frac{2B^2 \Gamma(\gamma - \alpha)^2}{|\Gamma(-\alpha)|^2} + \frac{B^2 \Gamma(2\gamma - \alpha)}{|\Gamma(-\alpha)|} \right] \frac{1}{s^{2\gamma + 1}}
\]

\[
= \frac{B^2}{|\Gamma(-\alpha)|} \left[ \gamma^2 (2\gamma - \alpha) \frac{\Gamma(2\gamma - \alpha)}{2 + \alpha - 2\gamma} + \frac{2\gamma (\gamma - \alpha) \Gamma(\gamma - \alpha)^2}{|\Gamma(-\alpha)|(1 + \alpha - \gamma)} \right. \\
+ \left. \left\{ 1 + \gamma (\gamma - \alpha) \right\} \frac{2\gamma (\gamma - \alpha)^2}{|\Gamma(-\alpha)|} \right] \frac{1}{s^{2\gamma + 1}}.
\]

The inverse Laplace transform reads

\[
\langle X^2 \rangle_t \sim \frac{B^2}{|\Gamma(-\alpha)||\Gamma(1 + 2\gamma)} \left[ \left\{ 1 + \gamma^2 (2\gamma - \alpha) \right\} \frac{\Gamma(2\gamma - \alpha)}{2 + \alpha - 2\gamma} \right. \\
+ \left. \left\{ 1 + \gamma (\gamma - \alpha) \right\} \frac{2\gamma (\gamma - \alpha)^2}{|\Gamma(-\alpha)|} \right] t^{2\gamma}.
\]

**Appendix 3: nth (n > 1) Moment and Its Coefficient M_n(\alpha, \gamma)**

The n-th (n > 1) differentiation of \( \hat{P}(k, s) \) is given by the recursion relation:

\[
\hat{p}^{(n)}(k, s) = \frac{1}{1 - \hat{\psi}(k, s)} \sum_{i=1}^{n-1} c_{n,i} \hat{p}^{(i)}(k, s) \hat{\psi}^{(n-i)}(k, s) + \hat{P}(k, s) \hat{\psi}^{(n)}(k, s)
\]

where \( c_{n,i} = c_{n-1,i} + c_{n-1,i-1} \) (\( i = 2, \ldots, n - 2 \)) and \( c_{n,n-1} = c_{n,1} = n \).

Here,

\[
\int_0^\infty d\tau w(\tau) \hat{\psi}^{(n)}(0, s; \tau) = \int_0^\infty d\tau w(\tau) \int_0^\tau I(t, \tau)^n e^{-st} dt
\]

\[
= \int_0^\infty d\tau \int_t^\infty d\tau w(\tau) I(t, \tau)^n e^{-st}
\]

\[
\sim \frac{A(B\gamma)^n}{|\Gamma(-\alpha)|} \int_0^\infty dt \int_t^\infty d\tau w(\tau) \tau^{-(n+1)-\alpha+n\gamma} e^{-st}
\]

\[
\sim \frac{AB^n \gamma^n \Gamma(n\gamma - \alpha + 1)}{|\Gamma(-\alpha)|(n + \alpha - n\gamma)} s^{1-\alpha+n\gamma}.
\]
We assume
\[
\hat{P}^{(i)}(0, s) \sim (-1)^i \frac{B^i M_i(\alpha, \gamma)}{|\Gamma(-\alpha)|} \frac{1}{s^{1+i\gamma}},
\]
for \(i < n\). It follows that
\[
\hat{P}^{(n)}(0, s) = \left[ \sum_{i=1}^{n-1} c_{n,i} M_i(\alpha, \gamma) \frac{\Gamma((n-i)\gamma - \alpha)}{|\Gamma(-\alpha)|} \right]
+ \frac{\Gamma(n\gamma - \alpha)}{|\Gamma(-\alpha)|} \frac{(-B)^n}{s^{1+n\gamma}}
\]
\[
= \left[ \sum_{i=1}^{n-1} c_{n,i} M_i(\alpha, \gamma) \frac{\Gamma((n-i)\gamma - \alpha)}{|\Gamma(-\alpha)|} \right]
+ \left( 1 + \frac{\gamma^n(n\gamma - \alpha)}{n + \alpha - n\gamma} \right) \frac{\Gamma(n\gamma - \alpha)}{|\Gamma(-\alpha)|} \frac{(-B)^n}{s^{1+n\gamma}}.
\]
Therefore,
\[
M_n(\alpha, \gamma) = \sum_{i=1}^{n-1} c_{n,i} M_i(\alpha, \gamma) \frac{\Gamma((n-i)\gamma - \alpha)}{|\Gamma(-\alpha)|} + \left( 1 + \frac{\gamma^n(n\gamma - \alpha)}{n + \alpha - n\gamma} \right) \frac{\Gamma(n\gamma - \alpha)}{|\Gamma(-\alpha)|}.
\]

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