Absence of a dissipative quantum phase transition in Josephson junctions

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Half a century after its discovery, the Josephson junction has become the most important nonlinear quantum electronic component at our disposal. It has helped reshaping the SI system around quantum effects and is used in scores of quantum devices. By itself, the use of Josephson junctions in the Volt metrology seems to imply an exquisite understanding of the component in every aspects. Yet, surprisingly, there have been long-standing subtle issues regarding the modeling of the interaction of a junction with its electromagnetic environment which has generated broadly accepted misconceptions and paradoxical predictions. Here, we invalidate experimentally one such prediction, namely that a Josephson junction connected to a resistor becomes insulating beyond a given value of the resistance, due to a dissipative quantum phase transition. Our work clarifies how this key quantum component should be modeled and resolves contradictions in the theory.

Introduction In 1983, Schmid [1] predicted that a dissipation-driven Quantum Phase Transition (dQPT) should occur for any Josephson junction connected to a resistance $R$ : when $R > R_Q = \frac{h}{4e^2} \approx 6.5k\Omega$ the junction should be insulating at zero temperature, while if $R < R_Q$ the junction should be superconducting (see Fig. 1). The prediction was precised shortly after by Bulgadaev [2], and since then, many theoretical works using different techniques [3–11] have further confirmed it. Attempts to investigate this prediction experimentally are scarce [12–14] and these early experiments were all affected by technical limitations [15] that made their interpretation debatable. In this work we revisit this prediction using well-controlled linear response measurements on the insulating side of the phase diagram, and we find no sign of the junctions becoming insulating. By meticulously reinterpreting the theory, we show that no transition is expected, actually. This clarifies long-standing issues on the applicability of the compact and extended phase descriptions.

Let us first motivate our work by explaining why the predicted phase diagram is problematic. The left axis in the Schmid-Bulgadaev (SB) phase diagram (Fig. 1b) corresponds to $R \to \infty$, where we are left with a junction in parallel with its geometric capacitor $C$ defining the charging energy $E_C = (2e)^2/2C$. This system is known as a Cooper Pair Box (CPB) in the domain of quantum circuits; it behaves as a non-linear oscillator and has been extensively investigated theoretically and experimentally [16–18]. In particular, for any junction with a non-zero Josephson coupling $E_J$, it has finite charge fluctuations through the junction, in contradiction with being on the insulating side of the phase transition. Furthermore, since the anharmonicity of the CPB vanishes upon increasing the ratio $E_J/E_C$, one expects (at least in the large $E_J/E_C$ range) the effect of a finite parallel resistance $R$ on this non-harmonic oscillator to be similar to that on an harmonic oscillator [19–20]: as $R$ is reduced, the phase and charge fluctuations vary continuously, with $R = R_Q$ playing no particular role. Approaches that go beyond considering the junction as a pure inductor [21–22] confirm this intuition down to the moderately large $E_J/E_C$ range: they predict a supercon-

\[ E_J \]

\[ \frac{E_J}{E_C} \]

\[ S \]

\[ I \]

\[ R \]

\[ R_Q \]

\[ S_1 \]

\[ S_2 \]

Figure 1. (a) A Josephson junction with a Josephson coupling $E_J$ connected to a resistor $R$. The junction’s capacitance $C$ determines the charging energy $E_C = (2e)^2/2C$. (b) The Schmid-Bulgadaev phase diagram for the circuit in (a). In the phase $I(S)$, the junction is predicted to be insulating (superconducting) at zero temperature. The insulating phase is paradoxical because the left axis (red line, where $R = \infty$) is the location of the Cooper pair box family of superconducting qubits for which it is well known that the junction is superconducting. Similarly, our samples $S_1$ and $S_2$ are found to remain superconducting when lowering the temperature, even though they are supposed to be well inside the insulating phase.

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ductive junction that smoothly retrieves the “bare” (with no resistor) CPB behavior as the environment impedance gets large and cold. More generally, any Josephson junction connected to a large impedance $Z$ is intuitively expected to smoothly recover the (superconducting) behavior of the CPB in the $Z \to \infty$ limit. This was confirmed theoretically in the specific case of a purely inductive environment in Ref. [23]. In summary, several known theoretical results, many experimental results and intuitive expectations on electrical circuits are consistent among themselves and all conflict with the prediction of the insulating phase shown in Fig. 1b, making it paradoxical.

Experiment In order to test the SB prediction, we have designed an experiment that closely implements the circuit of Fig. 1a while allowing to probe the linear response of Josephson junctions in ac. A schematics of the experiment and a micrograph of a sample are shown in Fig. 2 and the main sample parameters are given in Table I. Instead of a single junction, we use a SQUID behaving as an effective tunable Josephson junction : by applying a magnetic flux $\Phi$ in the SQUID loop, its Josephson coupling energy is tuned as $E_c \simeq E_c^{\text{max}} \cos(\pi \Phi/\Phi_0)$ with $\Phi_0 = h/2e$ the flux quantum. The input capacitor $C_c$ is chosen small enough that, at the measurement frequency, it essentially converts the input ac signal into a current source for the parallel Junction-Capacitance-Resistance system. This current is split between these components according to their admittance. The fraction of the current flowing through the resistor is routed off-chip to a microwave bias tee. The dc port of the bias tee is shorted to ground, closing the circuit in dc and ensuring there is no dc bias applied on the junction. At the high frequency port of the bias tee, the ac signal coming from the resistor is sent through circulators and filters to a chain of microwave amplifiers with an overall gain of 106 dB. We used microwave simulations of the circuit to check that in this design, the actual impedance seen by the junction is close to $R/C$ up to frequencies well above $(RC)^{-1}$ (note that the impedance to ground of the circuit following the resistor is negligible compared to $R$ at all frequencies). We use a Vector Network Analyzer to perform continuous wave homodyne measurements of the transmission $S_{21}$ through the sample. Although in this setup we measure variations of the fraction of the dc current flowing through the resistor, they are directly related to the variations of the junction admittance.

The operating conditions of the experiment were subject to constraints that we now detail. First, in order to improve our sensitivity to the junction’s admittance [24], the measurements need to be performed at a frequency well below the “plasma frequency” $\omega_p = (CL_c^{\text{eff}})^{-1/2}$ of the junction so that, as seen from the input capacitance, the ac current through $C$ is negligible. The current is then essentially divided between the resistor and the junction’s effective inductance $L_c^{\text{eff}}$, should it exist, in proportion of their respective admittance $1/R$ and $1/L_c^{\text{eff}} \omega$. We selected an operating frequency of order of 1 GHz in order to simultaneously fulfill this constraint (except in the vicinity of the maximal frustration of the SQUID) and have a reasonably good noise temperature for our microwave amplifier. Second, since we aim to probe the linear response of the junction at equilibrium, the ac phase excursion must be $\delta \varphi \ll 2\pi$, so that the junction is properly described by an admittance $1/iL_c^{\text{eff}} \omega$. Assuming the worst case where all the current flows through the resistor, this restricts the ac amplitude at the sample input $V_{\text{in}} \ll \Phi_0/RC_c$, (that is, $P_{\text{in}} \ll -50 \text{dBm}$ for the values used in the experiment, see below). Correspondingly, all the measurements shown here were taken in the low power limit where $S_{21}$ no longer depends on the input power [15]. The last constraint also restricts the admissible input power : the Joule power dissipated by ac current flowing through the resistor should not raise significantly its temperature. We used the results of Ref. [25] to estimate the electronic heating. Neglecting electron-phonon cooling in the resistor, for the maximum $S_{21}$ value of $-50 \text{dB}$, and at the input power of $-70 \text{dBm}$ used for the Sample 2 data at the lowest temperature ($T_{\text{ph}} = 13 \text{mK}$) in Fig. 3, one predicts an upper bound for the electronic temperature rise of $\sim 1.0 \text{mK}$ (0.5 mK for Sample 1) close to the junction [15]. Note that for such low power level, the signal-to-noise ratio at

![Figure 2. Top: simplified schematics of the experimental setup. Bottom: one of the samples measured. Two SEM micrographs were stitched to show the entire central part and colorized to evidence the different metals used [see 15 for fabrication details].](image)

| Sample | $E_{c}(k_B K)$ | $E_{J}^{\text{max}}(k_B K)$ | $R$ (k$\Omega$) | $C_c$ (F) |
|--------|----------------|-----------------|--------------|--------|
| 1      | 2.6            | 0.12            | 12           | 0.3    |
| 2      | 0.64           | 0.39            | 8            | 0.3    |

Table I. Main sample parameters. See [15] for details on their determination.
the input of the first cryogenic HEMT amplifier was such that each data point necessitated averaging for about 20 minutes. Above about 50 mK, electron-phonon cooling becomes effective [15]; it was then possible to speed up the measurement by increasing the excitation amplitude (still remaining in the linear regime) without raising the temperature. Above about 50 mK, electron-phonon cooling becomes effective [15]; it was then possible to speed up the measurement by increasing the excitation amplitude (still remaining in the linear regime) without raising the temperature.

In Fig. 3 we show the transmission $S_{21}$ for the two samples we have measured, for different flux through the SQUIDs, at the lowest temperature. We observe that when the flux is zero in the SQUID the junction has the highest admittance ($S_{21}$ minimum), whereas its admittance is minimum when the SQUID is frustrated with half a flux quantum in the loop. On the right panels, we show the temperature dependence of $S_{21}$ for several values of the flux in the SQUIDs. We observe that in the low temperature range, for any fixed value of the flux, $S_{21}$ reaches plateaus indicating that the junction admittance saturates to a finite value. In other words, at low temperature the modulation of $S_{21}$ with the flux proves that the SQUID still carries supercurrent, and it shows no tendency to become insulating at lower temperatures.

Discussion If the predicted insulating phase existed, the junctions would be in the quantum critical regime where one expects the junction admittance to follow a power law of the temperature. This is clearly not the case in our experiments. In a totally independent experiment with a different objective, Grimm and co-workers [27] have recently observed that a Josephson junction with $E_J/E_C \simeq 0.3$ in series with a 32 kΩ resistance ($R_Q/R \simeq 0.2$) allows a supercurrent flow. We consider their observation backs our results.

Together with the known $R \to \infty$ limit of qubits and the observed superconducting junctions at $E_J/E_C \gtrsim 7$ and $R_Q/R \sim 0.6$ in Ref. [13, 25] (see [15]), we conclude experimental observations are consistent with a complete absence of the predicted insulating phase.

We now explain the failure of the SB prediction. In a first step we introduce the framework in which the prediction was made. In a second step we explain the exact nature of the phase transitions found by Schmid, showing that they have been misinterpreted.

The SB prediction was cast using the model introduced by Caldeira and Leggett [29], which describes a Josephson junction and its capacitor (forming a CPB) analogously to a massive particle in a washboard potential, coupling the particle position (the junction phase) to a bath of harmonic oscillators that provide viscous damping. The corresponding Hamiltonian is

$$H = E_C N^2 - E_J \cos \phi + \sum_n 4e^2 N_n^2 2C_n + \frac{\hbar^2}{4e^2} \frac{(\varphi_n - \varphi)^2}{2L_n},$$

where $\varphi$ (resp. $N$) denotes the junction’s phase (resp. number of transmitted Cooper pairs) which are conjugate $[\varphi, N] = i$, and the $\varphi_n$ (resp. $N_n$) denote the phase (resp. dimensionless charge) of the harmonic oscillators. $H$ is not invariant upon $\varphi \to \varphi + 2\pi$, so that values of $\varphi$ differing by $2\pi$ are naturally regarded as distinguishable states of the junction and $\varphi$ is said to be an “extended phase”. Correspondingly, $N$ has its spectrum in $\mathbb{R}$ and we call it an extended charge too.

A unitary transformation $H' = U^\dagger H U$ with $U = \exp(iN_R \phi)$, where $N_R = \sum_n N_n$ (the charge passed through the resistor), yields another Hamiltonian of interest

$$H' = E_C (N - N_R)^2 - E_J \cos \phi + \sum_n 4e^2 N_n^2 2C_n + \frac{\hbar^2}{4e^2} \varphi_n^2 2L_n,$$

where the CPB now couples to the environment through $N$. Unlike $H$, $H'$ is invariant upon the discrete translation $\varphi \to \varphi + 2\pi$, so that the values of $\varphi$ differing by $2\pi$ can be regarded as indistinguishable (wavefunctions in $\varphi$ are $2\pi$-periodic), and the usual terminology is that $\varphi$ is a “compact phase”. In principle, $\varphi$ can still be described as an extended variable, in which case the periodicity of the potential implies that wavefunctions in $\varphi$ are Bloch functions $\Psi_{q}(\varphi) = \sum_n a_n(q) e^{i(n+q)\varphi}$. However the “quasicharge” $q$ is a conserved quantity fixed by initial conditions and any non-zero value of $q$ can be “gauged away” by shifting the initial environment charges. Hence, for simplicity, one can always choose to use a compact phase ($q = 0$). In this description, $N$ has a discrete spectrum in $\mathbb{Z}$ (even though there is no “island” in the circuit) and the Josephson coupling term can be written as $E_J \cos \phi = \frac{1}{2}E_J \sum_{N \in \mathbb{Z}} |N(N+1) + \text{H.c.}|$ as customary for CPBs which we expect to recover in the $R \to \infty$ limit.
As $H$ and $H'$ apparently operate on wavefunctions with different symmetries, they seem to describe different physical systems. This issue was known from the start and several theory papers considered the suitability of either phase description for the system considered here, but no clear-cut answer emerged (for an overview see [30]). However, a unitary transformation cannot break a symmetry of the system, and the contradiction resolves when one properly transforms the boundary and initial conditions together with the Hamiltonian [30, 31]. We thus argue that when describing a Josephson junction using the effective Josephson Hamiltonian $-E_J \cos \varphi$ (assuming to be within its validity domain, of course – see [15]), only a compact phase which possesses the highest symmetry should be considered, unless a spontaneous symmetry breaking of the discrete phase translation invariance occurs, a phenomenon also known as the “decompactification” [6, 52] of the phase (and which goes along an “undiscretization” of the charge).

The SB theory is precisely all about dissipation causing spontaneous symmetry breaking; we now expose the core ideas of this theory. Close to the bottom axis of the phase diagram, in the so-called scaling limit where $E_C \to \infty$ (which constrains $N = N_R$), $H'$ becomes equivalent to the tight-binding model used in Refs [3, 4]. In this model, at low friction (low $R$), the zero-temperature reduced density matrix $\rho$ is completely delocalized in the discrete charge basis, and thus corresponds to a perfectly localized compact phase. For such states, an extended description for both charge and phase, the diagonal of $\rho$ is a Dirac comb in both charge and phase representation (Fig. 4b, bottom right). For $R > R_Q$, however, the discrete charge translational invariance symmetry is broken and the charge localizes at a given value of $\langle N \rangle = \text{Tr} \rho N$. Using renormalization flow arguments, the charge localization behavior can be extended to the region $R_Q / R > \sqrt{E_J / E_C}$, where the cutoff frequency of the Ohmic damping is the fastest dynamics in the system (see part CL in Fig. 4a - Note that our experimental parameters are in this zone). In $\rho$, the result of this charge localization can be seen as multiplying the charge Dirac comb by a bell-shaped function $b$ and broadening each peak of the charge Dirac comb by convolving it with the Fourier transform of $b$ (Fig. 4b, bottom left). Thus, these localized charge states closely resemble those of the bare CPB, and they very naturally coincide with them in the $R \to \infty$ limit. The difference between the resistively shunted junction and the CPB with an island is that in the first case there is a degenerate continuum of localized charge states at all values of $\langle N \rangle$, while in the second case where no dc current can flow, $\langle N \rangle$ is pinned and the ground state is unique. Across the transition, both the charge fluctuations (the width of $b$) [3] and the coherence $\langle \cos \varphi \rangle$ vary continuously, showing that the junction remains superconducting throughout. Yet, this partial charge localization transition has been recurrently (over-)interpreted as a transition to an insulating state so far in the literature, and their conclusions questionable.

Close to the top axis of the phase diagram, one follows similar reasoning in the “dual” picture [33], where charge and phase are interchanged. One then starts from a tight-binding description of phase states located in the different wells of the cosine potential (and where the strength of the friction is inverted). Mirroring what occurs on the bottom axis, this predicts that the diagonal of $\rho$ is again a Dirac comb in both charge and phase representations (upper left of Fig. 4b) at low friction (large $R$) and that a smooth spontaneous symmetry breaking transition to partial “phase localization” occurs for $R < R_Q$ (part PL in Fig. 4a). We thus identify this transition as a progressive decompactification of $\varphi$. This shows that a generic decompactified phase state is the dual of a CPB state, i.e. a superposition of classical phase states differing by $2\pi$ in several adjacent wells of the cosine. To our knowledge, this is the first time the decompactification process is clearly exposed and it is a key result as it shows this spontaneous symmetry breaking does not yield generic extended phase states, contrarily to what was generally assumed so far. In particular, Schmid and subsequent authors treated $\varphi$ as extended, which lead them to attribute an insulating character to the “delocalized phase” in all the wells of the cosine (for $R > R_Q$). This, however, is generally incorrect: when considering a compact phase, the junction is insulating only when the phase is completely delocalized within one period, meaning that the diagonal of $\rho$ is completely flat in the phase representation.

In Fig. 4a we show our reinterpretation of the SB phase diagram, where the junction is superconducting everywhere, except at $E_J = 0$. This is in agreement with experiments and resolves all the paradoxes.

At this point, what remains of the SB prediction are smooth transitions from fluctuation-less phase states to states having finite zero-point phase fluctuations, i.e. classical-to-quantum transitions. However, a classical state can only result from some approximation. In the Supplemental Material [15] we show that this transition is actually an artifact resulting from the use of the effective Josephson Hamiltonian in the model, and that a proper microscopic treatment of the tunnel junction coupled to its environment would actually suppress such classical-to-quantum dQPT (restoring small phase fluctuations in the phases $S$), and hence suppressing all phase boundaries in the diagram.

Our final understanding of this system is represented pictorially in Fig. 4b: the junction is superconducting everywhere and its reduced density matrix evolves continuously as a function of the parameters, interpolating between the limit cases depicted. From this diagram one sees that when the effective Josephson Hamiltonian is deemed adequate to model a Josephson junction, the junction phase can be essentially regarded as compact (and one can use the discrete charge basis of a CPB), except in the upper right part of the phase diagram where one expects a partial decompactification of the phase. As mentioned above, the usual extended phase approach...
considers states that do not have the appropriate symmetries within this understanding. Consequently, using an extended phase to describe the low-energy states in such system is at best approximate or it appeals to (perhaps unspoken) ingredients external to the model. Yet, many predictions (besides the dQPT) were made assuming an extended phase to describe the low-energy states. This raises the question of when one can safely use such description? A non-operative answer is that such description is fine as long as interference effects that would appear in a proper treatment of the phase (more or less complete) translation invariance play no significant role.

Before concluding, let us comment the striking dips observed in the temperature dependence of the transmitted power near $T \sim 100 \text{ mK}$, corresponding to a maximum of the junction admittance. They can be understood at a qualitative level using the usual charge description of the CPB (consistently with the above discussion), assuming the resistance is large enough. In the regime $E_J \ll E_C$ and at very low temperature, the state of the CPB is nearly a classical state at the minimum of a charging energy parabola with a given $N$. This state nevertheless has quantum fluctuations that can be computed by 2nd order perturbation theory, with virtual transitions through the neighboring charge states. This results in an effective Josephson coupling for the ground state $E_{J_{\text{eff}}} = E_J/E_C$, the energy denominator $E_C$ being the energy of the virtual states. At finite temperatures $k_B T \lesssim E_C$, low energy modes of the resistance are thermally populated; they can lend their energy to the virtual state, lowering the energy denominator and thus increasing the effective Josephson coupling. At higher temperatures, thermal fluctuations eventually reduce the gap of Al, reducing the Josephson coupling.

In conclusion, our experimental results show that the prediction of a superconducting-to-insulating dQPT as formulated by Schmid and Bulgadaev is invalid, contrary to present wide-spread belief. We point out the misunderstandings at the origin of this erroneous prediction and reach a global and consistent qualitative description of the behavior of a junction connected to an environment. As an important by-product, our analysis for the first times clearly exposes how phase decompactification occurs in Josephson junctions. This shows that generic extended phase states are not rigorous solutions for the system, hopefully settling decades of controversies. Our work also highlights that there are presently no comprehensive and quantitative predictions for the effect of dissipation on the CPB able to reproduce our results. Finally, our invalidation of the Schmid-Bulgadaev prediction prompts for a critical reexamination of the works where it was used to obtain predictions in other systems such as superconducting nanowires proposed to implement quantum phase slip junctions.

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I. FORMER EXPERIMENTAL TESTS

Former experimental tests of the dQPT \cite{12,14,28} measured the zero-bias differential conductance of dc-biased resistively shunted junctions, typically using a Lock-in technique at frequencies $f_{LI} \sim 100\text{Hz}$ or below. In these experiments, the junction and its Ohmic shunt resistance $R$ were typically “current-biased” using a voltage source in series with a large resistor $R_{bias} > R$. Could such setup properly measure the linear response of the junction?

For junctions with small critical current, it is well known that spurious noise in the setup rapidly reduces the apparent maximum supercurrent \cite{35,37}, and particularly so for under-damped junctions, \textit{i.e.} when $E_J/E_C \gg (R_Q/R)^2$. However, even when the technical noises are completely eliminated, a lockin measurement has intrinsic limitations when the junction’s admittance becomes smaller than $1/R$. In that case, keeping small phase excursion in these setups obviously requires an ac voltage excitation at the junction $V_{ac} \ll \Phi_0 f_{LI} < 1\text{pV}$ which, even taking into account the resistive bridge division $R/(R+R_{bias})$, is several orders of magnitude smaller than required to have a sufficient signal-to-noise ratio in lockin measurements. Thus, the former experiments aiming to test the dQPT could not properly measure the linear response of junctions with very low admittances: several periods of the cosine were explored, rapidly averaging any small supercurrent to zero. In contrast, in our setup measuring at much higher frequencies enable to use larger excitation voltages while remaining in the linear phase response regime, even when the admittance of the junction becomes very low.

On the other hand, it is easy to observe the supercurrent branch of junctions having a large critical current, even with an imperfect setup, because the junction very effectively shunts noise. Indeed, the authors of Ref. \cite{13,28} found that a superconducting branch was observed for all junctions supposed to be in the insulating phase, provided that $E_J/E_C \gtrsim 7$. At the time of this result, the discrepancy with the dQPT prediction was resolved by arguing that the observed superconducting state was a transient, and that the true equilibrium insulating state would only be reached after a possibly cosmologically long time \cite{6,13,28}. The argument given was that when the junction’s (extended) phase starts localized in one well of the cosine potential, it will eventually delocalize in all other wells of the cosine by tunneling (and this delocalized state was assumed insulating), but the tunneling rate become immeasurably small for large $E_J/E_C$. However, when timescales become very long and energies very small one should reconsider seriously all other approximations made in the modeling, such as, for instance, neglecting the level separation in the electrodes. When considering a compact phase such slow phenomenon simply does not exist: the phase is always instantly delocalized in all wells of the cosine and moreover that state is superconducting. The superconducting state observed in these experiments was then the genuine equilibrium state.

II. FABRICATION DETAILS

The fabrication of the sample starts from a gold 50 $\Omega$ coplanar waveguide (CPW) defined by optical lithography and providing the input and output ports for the microwave signals. The central conductor of the transmission line is interrupted on a length of 38 $\mu$m, creating a cavity in which the resistor and junctions are fabricated in two subsequent steps, using e-beam lithography and evaporation through suspended masks. The resistor consists of a 8.5 nm-thick, ~100 nm-wide and 16$\mu$m-long Cr wire, periodically overlapped with 45 nm-thick, $1 \times 1\text{(\mu m)}^2$ Cr cooling pads. One end of the resistor connects to the output transmission line. The junctions were produced by standard double-angle interruption on a length of 38 $\mu$m.

A. Determination of the sample parameters

Since the values of $E_J$ and $R$ cannot be independently measured directly on the sample, the values reported in Table 1 of the main text come from the room temperature measurements of the resistance of several other junctions and resistors having the same dimensions and fabricated at the same time on the sample. From the scatter of these measurements, the values reported are believed accurate within $\pm 15\%$. The value of $E_C$ is estimated from the area of the junction, using the commonly used value $100 \text{fF(\mu m)}^{-2}$ for the capacitance per unit area of aluminum-aluminum oxide junctions. The value of the coupling capacitance was obtained from microwave simulations.
III. JOULE HEATING IN THE RESISTOR

We here show that for the measurements shown in Fig. 4, the Joule power dissipated in the chromium resistor did not substantially raise the electronic temperature. For this we rely on the analysis of heating in diffusive wires detailed in Ref. [25] where it is assumed that the electron temperature can be well defined locally, i.e. that the thermalization between electrons occurs faster than their diffusion through the wire, and that we can neglect the radiative cooling of the wire. In this reference, the diffusive wire is supposed to be connected to two normal-metal reservoirs at both ends, and these reservoirs are supposed to be large enough so that their electronic temperature is equal to the phonon temperature. In our case, on the junction side the Cr wire is connected to superconducting Al which blocks any heat exchange at very low temperatures. We can nevertheless obtain the electronic temperature at this point by considering the results of Ref. [25] in the middle of a wire with twice the length, twice the resistance and twice the dissipated power.

We first evaluate the maximum Joule power \( P_R \) dissipated in the Cr resistor for the measurements performed at the lowest temperature (13 mK) in Fig. 4. This power is proportional to the power \( P_{\text{out}} \) at the output of the sample by

\[
P_R = \frac{R}{Z_0} P_{\text{out}},
\]

where \( Z_0 = 50 \Omega \) is the impedance of the microwave circuitry and

\[
P_{\text{out}} = P_{\text{VNA}} 10^{\left(\frac{S_{\text{21}}-G}{10}\right)}.
\]

where \( P_{\text{VNA}} \) is the power at the Vector Network Analyzer (VNA) output, \( S_{\text{21}} \) is the measured transmission of the setup (in dB) and \( G = +106 \text{ dB} \) the overall gain (in dB) of the microwave chain from the sample output to the VNA input. For Sample 2, using the maximum value \( \operatorname{Max}(|S_{\text{21}}|) = -50 \text{ dB} \), \( P_{\text{VNA}} = +3 \text{ dBm} \) and \( R = 8k\Omega \), this leads to a maximum \( P_R \simeq 80 \text{ aW} \) (for Sample 1: \( \operatorname{Max}(|S_{\text{21}}|) = -50 \text{ dB} \), \( P_{\text{VNA}} = -4 \text{ dBm} \) and \( R = 12k\Omega \) give a maximum \( P_R \simeq 25 \text{ aW} \)).

Looking for an upper bound for the electronic temperature, we consider the simple “interacting hot-electron” limit, where electron-phonon interaction in the wire are neglected, so that cooling occurs only through diffusive electronic exchange with the reservoir (here the gold central conductor of the CPW). In this limit, the maximum temperature (reached in the middle of the wire in [25], and at the Cr-Al interface in our case) is

\[
T_{\text{max}} = \sqrt{T_{\text{ph}}^2 + \frac{3}{4\pi^2} \left(\frac{e}{k_B}\right)^2 2R^2P_R}
\]

where \( T_{\text{ph}} \) is the phonon temperature in the reservoir. At the lowest temperature \( T_{\text{ph}} = 13 \text{ mK} \), for the above values this yields:

\[
T_{\text{max}} \simeq T_{\text{ph}} + \begin{cases} 1 \text{ mK for sample 2} \\ 0.5 \text{ mK for sample 1} \end{cases}
\]

which sets an upper bound for the electronic temperature of the electromagnetic environment in our experiments. This shows that in the entire experimental range, Joule heating of the resistor was negligible.

In the above analysis, the thick intermediate pads incorporated in the wire design (see Fig. 2) play absolutely no role. They are meant to increase electron-phonon coupling, but they are effective only at higher temperature as we now discuss. At the maximum power dissipated in the resistor, we can estimate the electronic temperature \( T_{\Sigma} = (P_R^2/\Sigma \Omega)^{1/5} \) [25] that would be reached if only electron-phonon cooling was taking place. Taking the entire volume of the resistive wire and of the intermediate cooling pads \( \Omega \simeq 0.20 (\mu\text{m})^3 \) and assuming the standard electron-phonon coupling constant \( \Sigma \simeq 2 \text{ nW} \ (\mu\text{m})^{-3} \text{ K}^{-5} \) gives \( T_{\Sigma} \simeq 36 \text{ mK} \) (for Sample 2). This justifies we could increase measurement power at temperatures above 50 mK in order to speed up the measurements while still not heating the electrons.

IV. CHECKING THE LINEARITY OF THE RESPONSE

In order to ascertain that we measure properly the linear response of the junction, we checked that \( S_{\text{21}} \) did no longer depend on applied power at low power. In Fig. S1 we show the variations of \( |S_{\text{21}}| \) as a function of the applied
Figure S1. Variations of the transmission through the samples, as a function of the power at the sample input, for different values of the flux through the SQUIDs, at the lowest temperature (variations are taken with respect to the value at $P_{\text{in}} = -80\, \text{dBm}$). The size of the error bars does not vary monotonically because the averaging time was increased when reducing the power. The dashed lines indicate the power level that were chosen to take the data shown in Fig. 3 of the main text.

Measurement power for various fluxes in the two samples, at the lowest temperature (13 mK). We indeed observe that in the low power range $|S_{21}|$ no longer changes, confirming that we measure the linear response and that we are not heating the resistor. We used such measurement to choose the operating power for the data presented in Fig. 4 of the main text, selecting the value at the end of the horizontal plateau (shown as the dashed vertical line in Fig. S1), i.e. $-77\, \text{dBm}$ for sample 1 and $-70\, \text{dBm}$ for sample 2.

V. EFFECTIVE JOSEPHSON HAMILTONIAN AND ZERO POINT FLUCTUATIONS ACROSS THE JUNCTION

In the main text we show that, when correctly reinterpreted, the Schmid-Bulgadaev phase diagram has smooth transitions from fluctuation-less classical phase states to states having finite zero-point phase fluctuations. The absence of quantum phase fluctuations can only be the result from some approximation, and one expects that a better theoretical treatment should remove such artificial transitions.

We argue here that these artifacts stem from modeling the junction using the effective Josephson Hamiltonian which only describes Cooper pair tunneling. This effective washboard potential emerges from the tunneling of quasiparticles at 2nd order in perturbation theory in absence of an environment [38, 39] and it is commonly admitted it describes well a junction at energies much lower than the superconducting gap $\Delta$ and in absence of quasiparticles (which is expected at $k_B T \ll 2\Delta$). A more rigorous and consistent way of considering the effect of the environment on the junction consists in going back to the tunneling of quasiparticles [21, 40, 41]. Doing so, one however finds that at 2nd order in tunneling (corresponding to order of the effective Josephson Hamiltonian used in $H$) the junction sees the bare zero point fluctuations of the $RC$ circuit. In this circuit, the voltage fluctuations can exceed $2\Delta/e$ for large $E_C$ [see below], which is clearly inconsistent with considering only Cooper pair tunneling in the model. Moreover, even if voltage fluctuations are not large, phase fluctuations are divergent for any Ohmic environment impedance and this would lead to a complete suppression of the supercurrent at all temperatures [see VB below], even for $R < R_Q$, in the predicted classical compact phase phase. This shows that the description of the system using $H$ is inconsistent when considering an Ohmic environment. These inconsistencies resolve at higher orders in the tunneling Hamiltonian (or using a self-consistent approximation [21]), when the inductive-like shunting back-action of the junction on the environment is taken into account: voltage and phase fluctuations are reduced and they acquire an effective super-Ohmic spectral density for which no dQPT is expected. We thus think that the unphysical classical-to-quantum phase transition described in the main text would be removed by a proper microscopic treatment.

More generally, this shows the modeling of the interaction of a junction with its environment using the standard effective Hamiltonian is inconsistent when considering an Ohmic environment. Thus, any result critically depending
on the Ohmic character of the environment impedance (like the dQPT) is to be considered with skepticism.

**A. Voltage fluctuations**

At 2nd order in tunneling (corresponding to the effective Josephson Hamiltonian used in $H$ and $H'$ in the main text) the junction sees the fluctuations of the impedance $Z(\omega)$ constituted of the Ohmic environment (the “resistor”) in parallel with the junction capacitance $C$.

At zero temperature voltage fluctuations are given by

$$S_{vv}(t) = \langle V(t)V(0) \rangle = \int_0^{\infty} \frac{\hbar \omega}{\pi} \text{Re}Z(\omega) e^{-i\omega t} d\omega.$$

Considering the “resistor” is frequency-independent up to a sharp UV cutoff at $\omega = \Omega$

$$\text{Re}Z(\omega) = \frac{R}{1 + \left(\frac{\omega}{\omega_c}\right)^2} \theta(|\Omega - \omega|)$$

where $\omega_c = (RC)^{-1}$ which yields

$$S_{vv}(0) = \frac{1}{2\pi} \frac{R \hbar \omega_c^2}{\hbar} \ln \left(1 + \frac{\Omega^2}{\omega_c^2}\right).$$

The rms value of this voltage can be written as

$$V_{\text{rms}} = \sqrt{S_{vv}(0)} = \frac{E_C}{e} \sqrt{\frac{R Q}{\pi}} \frac{1}{\sqrt{\ln \left(1 + \frac{\Omega^2}{\omega_c^2}\right)}}.$$

For large $E_C$ one sees that this voltage becomes comparable to or even greater than the gap voltage $2\Delta/e$ of the junction at which point direct quasiparticle tunneling occurs.

For a given junction technology (superconducting material and tunnel barrier thickness), $E_C$ (resp. $E_J$, or $E_{\text{max}}$ for a SQUID) scales as the junction inverse area (resp. area) so that $E_C \propto 1/\sqrt{E_J/E_C}$, this enables to express the parameter region where the model is invalid as an approximate straight line (neglecting the variations of the logarithm):

$$\frac{E_J}{E_C} < \text{constant} \times \frac{R Q}{R}.$$

For our experimental parameters we estimate $\epsilon V_{\text{rms}}/2\Delta_{A1} \sim 0.3(0.1)$ for Sample 1 (2). While these rms values are not above the direct quasiparticle tunneling threshold, more or less rare events in the tails of the distribution would be. The effective Josephson coupling in $H$ and $H'$ is clearly beyond its domain of validity on these occurrences, while they are intrinsically properly handled in the tunneling Hamiltonian approach.

**B. Phase fluctuations**

Phase fluctuations across the environment are given by

$$S_{\phi\phi}(t) = \langle \varphi(t)\varphi(0) \rangle = \int_{-\infty}^{t+\infty} \frac{d\omega}{\omega} \frac{2\text{Re}Z(\omega)}{R} \frac{e^{-i\omega t}}{1 - e^{-\hbar \omega/k_B T}}.$$

For an Ohmic environment $\text{Re}Z(\omega = 0) > 0$ so that the integrand has an IR divergence yielding infinite zero point fluctuations $S_{\phi\phi}(0) = \infty$.

Whether one uses the effective Josephson Hamiltonian or the Tunneling Hamiltonian at second order, in presence of an environment the supercurrent is reduced by the factor $e^{-\frac{1}{2}S_{\phi\phi}(0)}$ [6 21 42]. Such reduction of the Josephson coupling was quantitatively observed recently for resonant environments (assuming $\text{Re}Z(\omega \approx 0) = \mathcal{O}(\omega^2)$) [43 44].

However, for an Ohmic environment ($\text{Re}Z(\omega = 0) = R > 0$), whatever the value of $R$, this predicts a complete suppression of the supercurrent so that the junction should be insulating (for all values of $E_J$ and $E_C$). While this does not invalidate the insulating phase by itself, it contradicts the predicted superconducting phase, pointing differently to the inconsistency of the model. At higher order in tunneling this complete suppression of the supercurrent is removed.
VI. COMPACT VS EXTENDED PHASE

The analysis of the phase diagram conducted in the main text is based solely on symmetry considerations on the Hamiltonian and shows that phase decompactification is expected to be progressive above the anti-diagonal of the phase diagram.

In the past, this decompactification process was not at all clearly understood and resulted in a lot of ambiguities and confusion. This incomplete understanding played a role in the misidentification of the phases in the Schmid-Bulgadaev phase diagram. Here, we try to put into perspective why the situation was so confuse.

An extended phase description contains the compact phase solutions as solutions of higher symmetry (periodic solutions in phase representation), so that, in principle, it should be the only description ever needed. However, when starting from a Hamiltonian such as $H$ in the main text, for which an extended phase is the “natural” point of view, one needs to consider highly non-trivial initial and boundary conditions in order to obtain the compact phase solutions. In the existing literature based on using $H$ this was not done and, as a consequence, compact phase solutions were not found (or not recognized as a superconducting state). Accordingly, compact vs extended phase descriptions were regarded as a dichotomic choice, and an extended phase was very generally assumed to be a decompactified phase. The reasons were that an explicit decompactification process had never been worked out (or not convincingly) and thus the partial decompactification we put forward in the main text was never envisioned. Schematically, for a very long time it was broadly considered that the symmetry of the phase and the Hamiltonian used were somehow tied: ($H \Leftrightarrow$ extended phase, assumed to be a decompactified phase) XOR ($H' \Leftrightarrow$ compact phase).

To support this dichotomic view, several arguments or criteria were used to favor using a compact or an extended phase description, depending on the problem considered. For instance it was frequently argued that a compact junction phase is suitable only in circuits having an “island” connected to the junction as it would be a manifestation of the charge quantization in the island or of the tunneling of individual Cooper pairs through the junction. In other words, a compact phase should not be appropriate in a circuit where the charge can flow continuously. Although the general discussion of the main text only relies on symmetries and already shows such arguments are not relevant, in the following we nevertheless specifically discuss why these arguments are incorrect.

A. Phase compactness is not due to the tunneling of individual Cooper pairs through the junction

If instead of a Josephson junction one considers a superconducting ballistic (or nearly ballistic) weak link, then the current-phase relation is still periodic with the phase, so that one can again use a discrete charge basis to describe the state of the weak link. In that case this apparent “charge discretization” obviously cannot be directly linked to an underlying charge quantization due to the tunneling of charge carriers.

B. Is “charge quantization” due to the presence of “islands”?

As discussed in the main text, using the discrete charge basis of the CPB (equivalent to considering a compact phase) arises from the symmetries of the system. It does not require the presence of “an island” in which the charge is “naturally quantized”. The simplest argument against this is that in a CPB the mere presence of the Josephson junction destroys this charge quantization (the ground state of the CPB consists of a coherent superposition of charge states). This “charge quantization”, is not observable, it is only a mathematical illusion, actually.

Our statement is further supported by the fact that the form of the Caldeira-Leggett Hamiltonian is independent of whether the circuit has an island or not. This can be shown using the explicit decomposition of the total circuit impedance into oscillators according to the rules in Ref. [20].

Finally one can show that the Hamiltonian of a circuit with an island has a smooth limit to the island-less case by taking the limit where the capacitance defining the island becomes infinite. Correspondingly, all the finite-frequency linear response functions of the system have smooth limits too. However, as the system is non-linear, the linearity range may vanish at low frequency (Sec. e.g. Section [4], depending on the type of response probed. This agrees with the obvious expectation that at strictly zero frequency no dc current can flow when there is an island, while it can if there is no island. As explained in the main text, the absence of dc current in a circuit with island results from having a single ground state, while there is a continuum of them in the island-less case permitting a dc current flow.

As a conclusion, whether one considers a CPB with an island or a galvanically shunted junction does not radically change the way the system is modeled.
C. Issues associated with the use of an extended phase

The progressive decompactification process presented in the main text naturally resolves many of the tricky issues that previously arose when considering an extended phase:

• Previous to that, no precise symmetry-breaking mechanism was invoked to justify the use of the extended phase picture. Only qualitative or vague arguments were presented.

• On the contrary, an arbitrary wavefunction $\Psi(\varphi)$ in the extended phase description can be represented as a superposition of (compact phase) Bloch wavefunctions $\Psi(\varphi) = \int dq f(q) \Psi_q(\varphi)$ with different quasicharges and such superposition states are expected to be very rapidly projected on a single $q$ value by the decoherence due to the Ohmic environment. This makes it complicated to justify the use of an extended phase.

• If one insists on using an extended phase picture, there is no consistent way of viewing the dQPT predicted by Schmid and Bulgadaev on the top axis of the phase diagram. This resulted in a dramatic misinterpretation.

• When using an extended phase, one needs to invoke a “recompactification” of the phase in the $R \to \infty$ limit in order to recover Qubit results. It is not clear at which point this recompactification should occur when $R$ grows. Furthermore this recompactification would likely occur on an $R-$dependent timescale, requiring different descriptions for a single setup depending on the type of measurement performed...

D. Junction’s phase in the fluxonium

It is frequently argued that one must use an extended phase description for describing the fluxonium circuit \[45\] where a Josephson junction is connected in parallel with a inductor (instead of a resistor in this paper).

Indeed, for the fluxonium, the Hamiltonian proposed in Refs. \[23, 45\] is

$$H_{f1} = \frac{q^2}{2C} - E_J \cos \varphi + \frac{(\Phi_{\text{ext}} - \frac{\hbar}{2e} \varphi)^2}{2L}$$  \hspace{1cm} (S1)

where $\frac{\hbar}{2e} \varphi$ and $q$ denote the branch flux and charge of the junction and $\Phi_{\text{ext}}$ is the magnetic flux enclosed by the loop formed between the junction and the inductor, considered as an external control parameter, i.e. a fixed real number. In this model, obviously not invariant upon $\varphi \to \varphi + 2\pi$, the junction’s phase clearly appears as extended. However, the eigenstates of the system have current fluctuations that, in addition to vacuum flux fluctuations, cause fluctuations of $\Phi_{\text{ext}}$... Thus, the model is not fully consistent.

Another fluxonium Hamiltonian is derived in Ref. \[46\]. It reads:

$$H_{f2} = \frac{(Q + q)^2}{2C} - E_J \cos \varphi + \frac{\Phi^2}{2L}$$  \hspace{1cm} (S2)

In this writing $\Phi$ and $Q$ denote the branch flux and charge of the inductor while $\frac{\hbar}{2e} \varphi$ and $q$ still denote the branch flux and charge of the junction. This Hamiltonian thus has two quantum degrees of freedom (with fluctuations), and the flux in the loop is given by the Kirchhoff’s law

$$\frac{\hbar}{2e} \varphi - \Phi = \Phi_{\text{loop}}$$

so that $\Phi_{\text{loop}}$ fluctuates too (as expected) and has an expectation value related to the externally applied flux $\Phi_{\text{ext}}$. It is only by suppressing one of the quantum degree of freedom, turning it into a classical one, that \[S2\] becomes \[S1\] (and, strictly, $\varphi$ can no longer be considered as a degree of freedom describing the sole junction). The junction’s phase appearing as extended in \[S1\] thus results from an approximation (perhaps a very good one); it is not an obligation.

The inconsistency pointed above is a general problem of the circuit quantization scheme proposed in \[20\], where loop fluxes are always assumed constant. It can be easily fixed, though. Other quantization schemes have also been proposed \[46, 48\] which do not necessarily force this approximation.
The fluxonium is not in the phase diagram. In the fluxonium circuit, the impedance seen by the junction has $\text{Re} \ Z(\omega = 0) = 0$, which would naively locate it on the right axis of the phase diagram. However, in that limit, the system considered in the main text is ill-defined as neither the loop inductance $L$ (which defines a new energy scale $E_L = \hbar^2 / 8e^2L$ in the problem) nor the external flux $\Phi_{\text{ext}}$ threading the loop are specified. Thus the phase diagram would need to be refined with extra parameters close to the right axis.

Nevertheless, depending on its parameters, the fluxonium’s junction phase will evolve between fully decompactified (in a single well of the cosine) when $E_L \gg E_J$ and $\Phi_{\text{ext}} \mod \Phi_0 \neq \frac{1}{2}$, partially decompactified (in several wells) when $E_L \sim E_J$ and essentially compact (populating many wells nearly equally) [23] when $E_L \to 0$.

E. Phase in current-biased junctions

When considering the case of a current-biased junction, where the current source “tilts the washboard potential”, the different wells of the cosine appear as non-equivalent. Here again, the obligation to use an extended phase, is only apparent.

First, the current source can be modeled by considering a very large inductor loaded with an initial flux. So we are back to considering the fluxonium case for which we argued above that there is no obligation to use an extended phase.

One can arrive to a similar conclusion by performing a time-dependent unitary transformation [31] that removes the tilt of the washboard, restoring the periodicity of the cosine potential. In this case, however, the states of the system will be time-dependent.

In such a current-biased junction, the final degree of phase decompactification will depend on the dissipation in the system and on the ratio $E_J / E_C$ (as in the unbiased case), but certainly also on the current bias $I_b$ which sets an extra energy scale $I_b \Phi_0$ in the system, with an associated dynamics.