Research Article
Conformal Super-Biderivations on Lie Conformal Superalgebras

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In this paper, the conformal super-biderivations of two classes of Lie conformal superalgebras are studied. By proving some general results on conformal super-biderivations, we determine the conformal super-biderivations of the loop super-Virasoro Lie conformal superalgebra and Neveu–Schwarz Lie conformal superalgebra. Especially, any conformal super-biderivation of the Neveu–Schwarz Lie conformal superalgebra is inner.

1. Introduction

Lie conformal superalgebras, introduced by Kac in [1], encode the singular part of the operator product expansion of chiral fields in conformal field theory. The conformal super-algebras play important roles in quantum field theory, vertex algebras, integrable systems, and so on and have drawn much attention in the branches of physics and mathematics. Finite simple Lie conformal superalgebras were classified by Fattori and Kac in [2], and their representation theories were developed in [3–5]. Moreover, some infinite Lie conformal superalgebras were also studied, such as loop super-Virasoro Lie conformal superalgebra [6] and Lie conformal superalgebras of Block type [7]. Other results on Lie conformal superalgebras can be seen in [8, 9].

In recent years, biderivations have been extensively studied for various algebra structures [10–14]. The authors in [15, 16] generalized biderivations of Lie algebras to the concept of super-biderivations of superalgebras independently. The authors in [17] studied super-biderivations on the super Galilean conformal algebra. The conformal biderivations of the loop $W(a, b)$ Lie conformal algebra and loop Virasoro Lie conformal algebra are determined in [18].

As a generalization of conformal biderivations of Lie conformal algebras and a parallel concept of super-biderivations of Lie superalgebras, we introduce the concept of conformal super-biderivations on Lie conformal superalgebras. We hope that biderivations would contribute to the development of structure theories of Lie conformal superalgebras. This is our motivation to present this paper.

In this paper, we concentrate on the loop super-Virasoro Lie conformal superalgebra $\mathcal{L}_s$ (see [6]), which is defined as a $\mathbb{C}[\partial]$-module $\mathcal{L}_s = (\oplus_{i \in \mathbb{Z}} \mathbb{C}[\partial] L_i) \oplus (\oplus_{i \in \mathbb{Z}} \mathbb{C}[\partial] G_i)$ with a $\mathbb{C}[\partial]$-basis $\{L_i, G_i | i \in \mathbb{Z}\}$ and $\lambda$-brackets given by

$$
\begin{align*}
[L_i L_j] &= (\partial + 2\lambda)L_{i+j}, \\
[L_i G_j] &= \left(\partial + \frac{3}{2}\lambda\right)G_{i+j}, \\
[G_i L_j] &= \left(\frac{1}{2}\partial + \frac{3}{2}\lambda\right)G_{i+j}, \\
[G_i G_j] &= 2L_{i+j},
\end{align*}
$$

for any $i, j \in \mathbb{Z}$. In section 2, we recall definition of Lie conformal superalgebras. Some general results about conformal super-biderivations are obtained in section 3. In section 4, we determine the conformal super-biderivations of $\mathcal{L}_s$ and Neveu–Schwarz Lie conformal superalgebra $\mathcal{N}_S$, and all conformal super-biderivations of the $\mathcal{N}_S$ are inner.

Throughout this paper, all vector spaces are over the complex field $\mathbb{C}$.

2. Preliminaries

In this section, we recall definition about Lie conformal superalgebra in [1, 4, 7, 8].
A vector space \( V \) is called \( \mathbb{Z}_2 \)-graded algebra if \( V = V_0 \oplus V_1 \), and \( x \in V_i \) is called \( \mathbb{Z}_2 \)-homogenous and we write \(|x| = i\).

**Definition 1.** A Lie conformal superalgebra \( R = R_0 \oplus R_1 \) is a \( \mathbb{Z}_2 \)-graded \( C[\partial] \)-module with a \( \mathfrak{C} \)-linear map \( R \otimes R \longrightarrow C[\lambda] \otimes R \), \( a \otimes b \longrightarrow [a,b] \), called the \( \lambda \)-bracket satisfying \( ([R_0]_\lambda R_1) \subseteq R_{\lambda+1} \) and the following axioms:

1. Conformal sesquilinearity: \([ (\partial a)_b b] = -\lambda [a,b], \quad [a \partial b] = (\partial + \lambda) [a,b] \).
2. Skew-supersymmetry: \([ a,b] = -(-1)^{|a||b|}[b,-\lambda a] \).
3. Jacobi identity: \([ a_j b, c] = [[a_j,b],c] + (-1)^{|a||b|} [b, [a_j,c]] \), for all \( a_i, b, c \in R \), \( \lambda, \tau \in \mathbb{Z}_2 \), where \( \lambda \) is an indeterminate and \( \partial \) is a derivation of the \( \lambda \)-bracket.

From conformal sesquilinearity, we can conclude that
\[
[f(\partial) a] b = f(-\lambda) [a,b],
\]
\[
[a_1 f(\partial)] b = f(\partial + \lambda) [a,b], \quad \forall f(\partial) \in C[\partial].
\]  
(2)

By Definition 1, the loop super-Virasoro Lie conformal superalgebra \( \text{cls} = (\text{cls})_0 \oplus (\text{cls})_1 \) with \( (\text{cls})_0 = \oplus_{\lambda \in \mathbb{Z}} C[\partial]L_\lambda \) and \( (\text{cls})_1 = \oplus_{\lambda \in \mathbb{Z}} C[\partial]G_\lambda \) and
\[
\left( (\text{cls})_{\lambda_1} (\text{cls})_{\lambda_2} \right) \subseteq (\text{cls})_{\lambda_1 + \lambda_2}[1],
\]
for \( i, j \in \mathbb{Z}_2 \).

**Example 1** (see [9]). Neveu–Schwarz Lie conformal superalgebra \( \text{NS} \) is a free \( \mathbb{Z}_2 \)-graded \( C[\partial] \)-module \( \text{NS} = C[\partial]\mathfrak{L} \otimes C[\partial]G \) with the following conditions:
\[
[L_\lambda L] = (\partial + 2\lambda) L,
\]
\[
[L_\lambda G] = (\partial + 3\lambda) G,
\]
\[
[G_\lambda L] = \left( \frac{1}{2} \partial + \frac{3}{2} \lambda \right) G,
\]
\[
[G_\lambda G] = L,
\]
(4)

where \( (\text{NS})_0 = C[\partial]L \) and \( (\text{NS})_1 = C[\partial]G \).

### 3. Conformal Super-Biderivations

**Definition 2.** Let \( R \) be a Lie conformal superalgebra. We call a conformal bilinear map \( \varphi_{\lambda}: R \times R \longrightarrow R[\lambda] \) a conformal super-biderivation of \( R \) if it satisfies the following equations:
\[
\varphi_{\lambda}(x, y) = -(1)^{|\lambda||\mu|}[\varphi_{-\lambda-\mu}(y,x)],
\]
(5)

\[
\varphi_{\lambda}(x, y) = [\varphi_{\lambda}(x, y)]_{1+\mu} + (1)^{|\lambda||\mu|}[\varphi_{\lambda}(x, y)],
\]
(6)

\[
\varphi_{\lambda}(x, y) = [\varphi_{\lambda}(x, y)]_{1+\mu} - (1)^{|\lambda||\mu|}[\varphi_{\lambda}(x, y)],
\]
(7)

for all homogeneous \( x, y, z \in R \).

**Remark 1.** Equation (6) is equivalent to equation (7).

**Proof.** We suppose equation (7) satisfies. On the one hand, using equation (5), we have
\[
\varphi_{1+\mu}(x, y) = -(1)^{|\lambda||\mu|}[\varphi_{-\lambda-\mu}(x, y)],
\]
(8)

On the other hand, using conformal sesquilinearity, we obtain
\[
\varphi_{1+\mu}(x, y) = -1)^{|\lambda||\mu|}[\varphi_{-\lambda-\mu}(x, y)],
\]
(9)

Then

Replace \( \lambda, \mu \) by \(-\partial - \lambda - \mu, \mu\), respectively, and by conformal sesquilinearity, we have

\[
(-1)^{|\lambda||\mu|}[\varphi_{-\lambda-\mu}(x, y)] = (1)^{|\lambda||\mu|}[\varphi_{-\lambda-\mu}(x, y)].
\]
(10)

This implies that equation (8) is satisfied. The reverse conclusion follows similarly.

We call the conformal super-biderivation \( \varphi_{\lambda} \) the inner conformal super-biderivation if there exists a fixed complex number \( a \) such that \( \varphi_{\lambda}(x, y) = a[x_1 y] \).

To avoid lengthy notations, we let

\[
(-1)^{|\lambda||\mu|}[\varphi_{\lambda}(x, y)] = [\varphi_{\lambda}(x, y)]_{1+\mu} + (1)^{|\lambda||\mu|}[\varphi_{\lambda}(x, y)],
\]
(11)

\[
\Phi_{\lambda,\mu,\nu}(x, y, w, v) = (1)^{|\lambda||\mu|}[[\varphi_{\lambda}(x, y)]_{1+\mu}[\varphi_{\lambda}(w, v)].
\]
(12)

**Lemma 1.** Let \( \varphi_{\lambda} \) be a conformal super-biderivation of Lie conformal superalgebra \( R \). Then,
Lemma 2. Let $\phi$ be a conformal super-biderivation of Lie conformal superalgebra $R$. Then,

\[ \Phi_{x,y} (x, y, w, v) = (-1)^{|x||y|} \Phi_{x,y} (w, y, x, v), \]

for any homogeneous $x, y, w, v \in R$.

Proof. Firstly, from the definition of conformal super-biderivation, we have

\[ \Phi_{x,y} (x, y, w, v) = (-1)^{|x||y|} \Phi_{x,y} (w, y, x, v), \]

Using the Jacobi identity of Lie conformal superalgebras, we obtain

\[ x_{\mu} \Phi_{x,y} (w, v) = \left[ x_{\mu} \Phi_{x,y} (w, v) \right]_{\lambda + \gamma} + (-1)^{|x||y|} \Phi_{x,y} (w, y, x, v), \]

Therefore, we get

\[ (-1)^{|x||y|} \Phi_{x,y} (w, y, x, v) = \left[ (-1)^{|x||y|} \Phi_{x,y} (w, y, x, v) \right] + \left[ (-1)^{|x||y|} \Phi_{x,y} (w, y, x, v) \right]_{\lambda + \gamma}, \]

which implies

\[ (-1)^{|x||y|} \Phi_{x,y} (w, y, x, v) = \left[ (-1)^{|x||y|} \Phi_{x,y} (w, y, x, v) \right] + \left[ (-1)^{|x||y|} \Phi_{x,y} (w, y, x, v) \right]_{\lambda + \gamma}, \]

Thus,

\[ \Phi_{x,y} (x, y, w, v) = (-1)^{|x||y|} \Phi_{x,y} (w, y, x, v), \]

for any homogeneous $x, y, w, v \in R$.

Proof. By skew-supersymmetry and conformal sesquilinearity, we have

\[ (i) \ [ (\phi_{x,y})_{\mu} \Phi_{x,y} (w, v) \right] = [[x_{\mu} \Phi_{x,y} (w, v) \right]_{\lambda + \gamma}, \]

for all homogeneous $x, y, w, v \in R$

(ii) If $[x_{1}] = 0$, then $\phi_{x,y} (x, y) \in Z[R_{1} R], \text{ where } Z[R_{1} R] \text{ is the center of } R_{1} R\]
\[ \Phi_{\lambda, \mu, \gamma}(x, y, w, v) = (-1)^{[x][y][\gamma]} \left( \left[ \Phi_{\mu}(x, y) \right]_{\mu + \gamma} [w_1 v] - \left[ [x, y]_{\mu + \gamma} \phi_{\lambda}(w, v) \right] \right) \]

\[ = -(-1)^{[x][y][\gamma]} \left( \left[ \Phi_{\mu}(x, y) \right]_{\mu + \gamma} [w_1 v] - \left[ [y, x]_{\mu + \gamma} \phi_{\lambda}(w, v) \right] \right) \]

\[ = -(-1)^{[x][y][\gamma]} \left( \left[ \Phi_{\mu}(x, y) \right]_{\mu + \gamma} [w_1 v] - \left[ [y, x]_{\mu + \gamma} \phi_{\lambda}(w, v) \right] \right) \]

Note that
\[ \Phi_{\lambda, \mu, \gamma}(x, y, w, v) = (-1)^{[x][y][\gamma]} \Phi_{\mu, \lambda, \gamma}(w, y, x, v) \]

On the other hand, we have
\[ \Phi_{\lambda, \mu, \gamma}(x, y, w, v) = (-1)^{[x][y][\gamma]} \Phi_{\mu, \lambda, \gamma}(w, y, x, v) \]

Thus, \( \Phi_{\lambda, \mu, \gamma}(x, y, w, v) = 0; \) this implies that
\[ \left[ \left[ \Phi_{\mu}(x, y) \right]_{\mu + \gamma} \right] = \left[ [x, y]_{\mu + \gamma} \phi_{\lambda}(w, v) \right], \] (21)

for any homogeneous \( x, y, w, v \in \mathbb{R}. \) This implies (ii) directly follows from (i).

4. Conformal Super-Biderivations of cls

Theorem 1. Every conformal super-biderivation \( \phi_{\lambda} \) on the cls has the following forms:

\[ \phi_{\lambda}(L_i, L_j) = (\partial + 2\lambda) \sum_{t \in \mathbb{Z}} a_{t-\lambda} L_t, \] (24)

for all \( i, j \in \mathbb{Z}. \)

For any \( i, j \in \mathbb{Z}, \) we may assume that
\[ \phi_{\lambda}(L_i, L_j) = \sum_{k \in \mathbb{Z}} f_{ij}^k (\partial, \lambda) L_k + \sum_{k \in \mathbb{Z}} g_{ij}^k (\partial, \lambda) G_k, \] (25)

where \( f_{ij}^k (\partial, \lambda), g_{ij}^k (\partial, \lambda) \in \mathbb{C}[\partial, \lambda]. \) By Lemma 2(i), we have
\[ \left[ \left[ \Phi_{\mu}(L_i, L_j) \right]_{\mu + \gamma} \right] = \left[ [L_i, L_j]_{\mu + \gamma} \phi_{\lambda}(L_i, L_j) \right]. \] (26)

Furthermore, we note that

\[ \sum_{k \in \mathbb{Z}} f_{ij}^k (-\mu - \gamma, \mu) \left( [L_k]_{\mu + \gamma} (\partial + 2\lambda) L_{i+j} \right) + \sum_{k \in \mathbb{Z}} g_{ij}^k (-\mu - \gamma, \mu) \left( [G_k]_{\mu + \gamma} (\partial + 2\lambda) L_{i+j} \right) \]

That is,

\[ \sum_{k \in \mathbb{Z}} f_{ij}^k (\partial - \mu + \gamma, \lambda) \left( [L_k]_{\mu + \gamma} + \sum_{k \in \mathbb{Z}} g_{ij}^k (\partial - \mu + \gamma, \lambda) \left( [G_k]_{\mu + \gamma} G_k \right) \right]. \] (28)
Therefore, we have

\[
(\partial + \mu + \gamma + 2\lambda) \sum_{k \in \mathbb{Z}} f_{ij}^k (-\mu - \gamma, \mu) \left[ (L_k)_{\mu+\gamma}^T L_{i+j} \right] + (\partial + \mu + \gamma + 2\lambda) \sum_{k \in \mathbb{Z}} g_{ij}^k (-\mu - \gamma, \mu) \left[ (G_k)_{\mu+\gamma}^T L_{i+j} \right] \\
= (\mu - \gamma) \sum_{k \in \mathbb{Z}} f_{ij} (\partial + \mu + \gamma, \lambda) \left[ (L_{i+j})_{\mu+\gamma}^T L_k \right] + (\mu - \gamma) \sum_{k \in \mathbb{Z}} g_{ij} (\partial + \mu + \gamma, \lambda) \left[ (L_{i+j})_{\mu+\gamma}^T G_k \right].
\]

(29)

It follows that

\[
(\partial + \mu + \gamma + 2\lambda) (\partial + 2\mu + 2\gamma) \sum_{k \in \mathbb{Z}} f_{ij}^k (-\mu - \gamma, \mu) L_{i+j+k} + (\partial + \mu + \gamma + 2\lambda) \left( \frac{1}{2} \partial + \frac{3}{2} \mu + \frac{3}{2} \gamma \right) \sum_{k \in \mathbb{Z}} g_{ij}^k (-\mu - \gamma, \mu) G_{i+j+k} \\
= (\mu - \gamma) (\partial + 2\mu + 2\gamma) \sum_{k \in \mathbb{Z}} f_{ij}^k (\partial + \mu + \gamma, \lambda) L_{i+j+k} + (\mu - \gamma) \left( \partial + \frac{3}{2} \mu + \frac{3}{2} \gamma \right) \sum_{k \in \mathbb{Z}} g_{ij}^k (\partial + \mu + \gamma, \lambda) G_{i+j+k},
\]

which implies

\[
(\partial + \mu + \gamma + 2\lambda) \sum_{k \in \mathbb{Z}} f_{ij}^k (-\mu - \gamma, \mu) L_{i+j+k} = (\mu - \gamma) \sum_{k \in \mathbb{Z}} f_{ij}^k (\partial + \mu + \gamma, \lambda) L_{i+j+k},
\]

(31)

\[
(\partial + \mu + \gamma + 2\lambda) \left( \frac{1}{2} \partial + \frac{3}{2} \mu + \frac{3}{2} \gamma \right) \sum_{k \in \mathbb{Z}} g_{ij}^k (-\mu - \gamma, \mu) G_{i+j+k} = (\mu - \gamma) \left( \partial + \frac{3}{2} \mu + \frac{3}{2} \gamma \right) \sum_{k \in \mathbb{Z}} g_{ij}^k (\partial + \mu + \gamma, \lambda) G_{i+j+k}.
\]

(32)

From (32), we have

\[
(\partial + \mu + \gamma + 2\lambda) \left( \frac{1}{2} \partial + \frac{3}{2} \mu + \frac{3}{2} \gamma \right) g_{ij}^k (-\mu - \gamma, \mu) \\
= (\mu - \gamma) \left( \partial + \frac{3}{2} \mu + \frac{3}{2} \gamma \right) g_{ij}^k (\partial + \mu + \gamma, \lambda).
\]

(33)

By comparing the degrees of \( \partial \) on both sides of (33), we can suppose that

\[
g_{ij}^k (\partial, \lambda) = c_0 (\lambda) + c_1 (\lambda) \partial,
\]

(34)

for some \( c_0 (\lambda), c_1 (\lambda) \in \mathbb{C}[\lambda] \). Substituting this formula to (33), we can obtain

\[
(\partial + \mu + \gamma + 2\lambda) \left( \frac{1}{2} \partial + \frac{3}{2} \mu + \frac{3}{2} \gamma \right) \left( c_0 (\mu) + c_1 (\mu) (-\mu - \gamma) \right) \\
= (\mu - \gamma) \left( \partial + \frac{3}{2} \mu + \frac{3}{2} \gamma \right) \left( c_0 (\lambda) + c_1 (\lambda) (\partial + \mu + \gamma) \right).
\]

(35)

Comparing the coefficients of \( \gamma^3 \), one can deduce \( c_1 (\mu) - c_1 (\lambda) = 0 \). Thus, \( c_1 (\lambda) \in \mathbb{C} \) which we denote by \( c_1 \). Therefore, (35) can be written as

\[
(\partial + \mu + \gamma + 2\lambda) \left( \frac{1}{2} \partial + \frac{3}{2} \mu + \frac{3}{2} \gamma \right) \left( c_0 (\mu) + c_1 (-\mu - \gamma) \right) \\
= (\mu - \gamma) \left( \partial + \frac{3}{2} \mu + \frac{3}{2} \gamma \right) \left( c_0 (\lambda) + c_1 (\partial + \mu + \gamma) \right).
\]

(36)

We get \( c_0 (\mu) = 2c_1 (\mu) \). Considering the coefficients of \( \partial \), we have

\[
\frac{1}{2} \partial^2 c_1 (\mu - \gamma) = c_1 (\mu - \gamma) \partial^2,
\]

(37)

which implies \( c_1 = 0 \) and \( g_{ij}^k (\partial, \lambda) = 0 \).

By (31), we obtain

\[
(\partial + \mu + \gamma + 2\lambda) f_{ij}^k (-\mu - \gamma, \mu) = (\mu - \gamma) f_{ij}^k (\partial + \mu + \gamma, \lambda).
\]

(38)

We suppose that

\[
f_{ij}^k (\partial, \lambda) = a_0 (\lambda) + a_1 (\lambda) \partial,
\]

(39)

for some \( a_0 (\lambda), a_1 (\lambda) \in \mathbb{C}[\lambda] \). Thus,

\[
(\partial + \mu + \gamma + 2\lambda) (a_0 (\mu) + a_1 (\mu) (-\mu - \gamma)) \\
= (\mu - \gamma) (a_0 (\lambda) + a_1 (\lambda) (\partial + \mu + \gamma)).
\]

(40)

By comparing the coefficients of \( \partial \) on both sides of (40), one can deduce

\[
a_0 (\mu) + a_1 (\mu) (-\mu - \gamma) = a_1 (\mu) (\lambda - \mu).
\]

(41)

Hence, \( a_1 (\mu) \in \mathbb{C} \), and we denote it by \( a_1 \). We also get \( a_0 (\lambda) = 2\lambda a_1 \). Therefore, \( f_{ij}^k (\partial, \lambda) = a_1 (\partial + 2\lambda) \). We denote \( a_1 \) by \( a_{ij}^k \). Thus,
\[ \varphi_\lambda(L_i, L_j) = (\partial + 2\lambda) \sum_{k \in Z} a^{i,j}_k L_k, \]  

where \( a^{i,j}_k \in \mathbb{C} \).

Furthermore, by Lemma 2(i), we have

\[ \left[ \varphi_\mu(L_i, L_j) \right]_{\mu=\gamma} = \left[ L_{\mu} L_j \right]_{\mu=\gamma} \varphi_\lambda(L_k, L_i), \]

Therefore,

\[ (\mu - \gamma) \sum_{i, j} a^{i,j}_k [L_i]_{\mu=\gamma} [L_k L_j]_{\mu=\gamma} = \left[ L_{\mu} L_j \right]_{\mu=\gamma} \varphi_\lambda(L_k, L_i). \]

That is,

\[ (\mu - \gamma) \sum_{i, j} a^{i,j}_k [L_i]_{\mu=\gamma} [L_k L_j]_{\mu=\gamma} = (\partial + \mu + \gamma + 2\lambda) \sum_{k \in Z} a^{i,j}_k \left[ L_{\mu} L_j \right]_{\mu=\gamma} L_j. \]

We get

\[ \sum_{i, j} a^{i,j}_k L_k = \sum_{i, j} a^{i,j}_k L_k L_j \]

which shows that \( a^{i,j}_k = \mu^{i+k+1-i-j} \). Hence, \( a^{i,j}_k = a^{i+k+1-i-j}_k \), denoted by \( a_i \). Thus,

\[ \varphi_\lambda(L_0, L_0) = (\partial + 2\lambda) \sum_{i \in Z} a_i L_i, \]

Then, we conclude that

\[ \varphi_\lambda(L_i, L_j) = (\partial + 2\lambda) \sum_{r \in Z} a_{r-i,j} L_r, \]

for all \( i, j \in Z \), where \( a_k \) for any \( k \in Z \) are complex numbers.

Claim 2. \( \varphi_\lambda(L_i, G_j) = (\partial + (3/2)\lambda) \sum_{k \in Z} a_{k-i,j} G_k \) for all \( i, j \in Z \).

For any \( i, j \in Z \), we suppose that

\[ \varphi_\lambda(L_i, G_j) = \sum_{k \in Z} d_{i,j}^k (\partial, \lambda) L_k + \sum_{k \in Z} h_{i,j}^k (\partial, \lambda) G_k, \]

where \( d_{i,j}^k (\partial, \lambda), h_{i,j}^k (\partial, \lambda) \in \mathbb{C}[\partial, \lambda] \). By Lemma 2(i), we have

\[ \left[ \varphi_\lambda(L_i, L_j) \right]_{\mu=\gamma} = \left[ L_{\mu} L_j \right]_{\mu=\gamma} \varphi_\lambda(L_k, G_j). \]

Therefore,

\[ \left[ (\partial + 2\mu) \sum_{k \in Z} a_{k-i,j} L_k \right]_{\mu=\gamma} \left[ L_{\mu} L_j \right]_{\mu=\gamma} \]

\[ = \left[ L_{\mu} L_j \right]_{\mu=\gamma} \left( \sum_{k \in Z} d_{i,j}^k (\partial, \lambda) L_k + \sum_{k \in Z} h_{i,j}^k (\partial, \lambda) G_k \right). \]

That is,

\[ (\mu - \gamma) \sum_{k \in Z} a_{k-i,j} [L_k]_{\mu=\gamma} [L_\lambda G_j]_{\mu=\gamma} = \sum_{k \in Z} d_{i,j}^k (\partial + \mu + \gamma, \lambda) \left[ L_{\mu} L_j \right]_{\mu=\gamma} L_k + \sum_{k \in Z} h_{i,j}^k (\partial + \mu + \gamma, \lambda) \left[ L_{\mu} L_j \right]_{\mu=\gamma} G_k. \]

We get

\[ (\mu - \gamma) \sum_{k \in Z} a_{k-i,j} (L_k)_{\mu=\gamma} (L_\lambda G_j)_{\mu=\gamma} = \sum_{k \in Z} d_{i,j}^k (\partial + \mu + \gamma, \lambda) \left[ (\partial + 2\mu) L_{i+j} \right]_{\mu=\gamma} L_k \]

\[ + \sum_{k \in Z} h_{i,j}^k (\partial + \mu + \gamma, \lambda) \left[ (\partial + 2\mu) L_{i+j} \right]_{\mu=\gamma} G_k. \]

Then

\[ (\partial + \mu + \gamma + 3/2 \lambda) \sum_{k \in Z} a_{k-i,j} [L_k]_{\mu=\gamma} [G_{i+j}]_{\mu=\gamma} = \sum_{k \in Z} d_{i,j}^k (\partial + \mu + \gamma, \lambda) \left[ L_{i+j} \right]_{\mu=\gamma} L_k + \sum_{k \in Z} h_{i,j}^k (\partial + \mu + \gamma, \lambda) \left[ L_{i+j} \right]_{\mu=\gamma} G_k. \]
Hence,

\[(\partial + \mu + \gamma + \frac{3}{2} \lambda) \sum_{k \in \mathbb{Z}} a_{k-i-j} \left( \partial + \frac{3}{2} \mu + \frac{3}{2} \gamma \right) G_{i+j+k} = (\partial + 2 \mu + 2 \gamma) \sum_{k \in \mathbb{Z}} d_{ij}^k (\partial + \mu + \gamma, \lambda) L_{i+j+k} + \left( \partial + \frac{3}{2} \mu + \frac{3}{2} \gamma \right) \sum_{k \in \mathbb{Z}} h_{ij}^k (\partial + \mu + \gamma, \lambda) G_{i+j+k},\]

which implies \(d_{ij}^k (\partial, \lambda) = 0\) and

\[(\partial + \mu + \gamma + \frac{3}{2} \lambda) \sum_{k \in \mathbb{Z}} a_{k-i-j} G_{i+j+k} = \sum_{k \in \mathbb{Z}} h_{ij}^k (\partial + \mu + \gamma, \lambda) G_{i+j+k}.\]

We get

\[(\partial + \mu + \gamma + \frac{3}{2} \lambda) a_{k-i-j} = h_{ij}^k (\partial + \mu + \gamma, \lambda).\]

Then,

\[h_{ij}^k (\partial, \lambda) = \left( \partial + \frac{3}{2} \lambda \right) a_{k-i-j}.\]

We obtain

\[
\left[ \left( \partial + 2 \mu \right) \sum_{k \in \mathbb{Z}} a_{k-i-j} L_k \right] [G_{i+j} L_j] = \left[ \left( \partial + \mu + \gamma, \lambda \right) \sum_{k \in \mathbb{Z}} p_{ij}^k (\partial, \lambda) L_k + \sum_{k \in \mathbb{Z}} q_{ij}^k (\partial, \lambda) G_k \right].
\]

That is,

\[(\mu - \gamma) \sum_{k \in \mathbb{Z}} a_{k-i-j} \left[ \left( L_k \right)_{\mu+\gamma} [G_{i+j} L_j] \right] = \sum_{k \in \mathbb{Z}} p_{ij}^k (\partial + \mu + \gamma, \lambda) \left[ \left( L_{i+j} L_j \right)_{\mu+\gamma} L_k \right] + \sum_{k \in \mathbb{Z}} q_{ij}^k (\partial + \mu + \gamma, \lambda) \left[ \left( L_{i+j} L_j \right)_{\mu+\gamma} G_k \right].\]

Hence,

\[
\left( \frac{1}{2} \partial + \frac{1}{2} \mu + \frac{1}{2} \gamma + \frac{3}{2} \lambda \right) \sum_{k \in \mathbb{Z}} a_{k-i-j} \left[ \left( L_k \right)_{\mu+\gamma} G_{i+j} \right] = \sum_{k \in \mathbb{Z}} p_{ij}^k (\partial + \mu + \gamma, \lambda) \left[ \left( L_{i+j} \right)_{\mu+\gamma} L_k \right] + \sum_{k \in \mathbb{Z}} q_{ij}^k (\partial + \mu + \gamma, \lambda) \left[ \left( L_{i+j} \right)_{\mu+\gamma} G_k \right].
\]
Then,

\[
\left(\frac{1}{2}\partial + \frac{1}{2} \mu + \frac{1}{2} \gamma + \frac{3}{2} \lambda \right) \sum_{k \in \mathbb{Z}} a_{k-i-j} \left(\frac{1}{2}\partial + \frac{3}{2} \mu + \frac{3}{2} \gamma \right) G_{i+j+k} = (\partial + 2\mu + 2\gamma) \sum_{k \in \mathbb{Z}} p_{ij}^k (\partial + \mu + \gamma, \lambda) L_{i+j+k} \\
+ (\partial + \frac{3}{2} \mu + \frac{3}{2} \gamma) \sum_{k \in \mathbb{Z}} q_{ij}^k (\partial + \mu + \gamma, \lambda) G_{i+j+k},
\]

(66)

which implies \( p_{ij}^k (\partial, \lambda) = 0 \) and

\[
\left(\frac{1}{2}\partial + \frac{1}{2} \mu + \frac{1}{2} \gamma + \frac{3}{2} \lambda \right) \sum_{k \in \mathbb{Z}} a_{k-i-j} G_{i+j+k} = \sum_{k \in \mathbb{Z}} q_{ij}^k (\partial + \mu + \gamma, \lambda) G_{i+j+k},
\]

(67)

It follows

\[
\left(\frac{1}{2}\partial + \frac{1}{2} \mu + \frac{1}{2} \gamma + \frac{3}{2} \lambda \right) a_{k-i-j} = q_{ij}^k (\partial + \mu + \gamma, \lambda).
\]

(68)

This shows that

\[
q_{ij}^k (\partial, \lambda) = \left(\frac{1}{2}\partial + \frac{3}{2} \lambda \right) a_{k-i-j}
\]

(69)

Hence,

\[
\left[ \left(\partial + 2\mu \right) \sum_{k \in \mathbb{Z}} a_{k-i-j} L_k \right]_{\mu \gamma} \left[ G_{i} G_{j} \right] = \left[ L_{i} L_{j} \right]_{\mu \gamma} \left( \sum_{k \in \mathbb{Z}} u_{ij}^k (\partial + \mu + \gamma, \lambda) L_k + \sum_{k \in \mathbb{Z}} v_{ij}^k (\partial + \mu + \gamma, \lambda) G_k \right)
\]

(73)

That is,

\[
(\mu - \gamma) \sum_{k \in \mathbb{Z}} a_{k-i-j} \left[ (L_k)_{\mu \gamma} \left[ G_{i} G_{j} \right] \right] = \sum_{k \in \mathbb{Z}} u_{ij}^k (\partial + \mu + \gamma, \lambda) \left[ L_{i} L_{j} \right]_{\mu \gamma} L_k + \sum_{k \in \mathbb{Z}} v_{ij}^k (\partial + \mu + \gamma, \lambda) \left[ L_{i} L_{j} \right]_{\mu \gamma} G_k.
\]

(74)

Hence,

\[
(\mu - \gamma) \sum_{k \in \mathbb{Z}} a_{k-i-j} \left[ (L_k)_{\mu \gamma} (2L_{i+j}) \right] = \sum_{k \in \mathbb{Z}} u_{ij}^k (\partial + \mu + \gamma, \lambda) \left[ (\partial + 2\mu) L_{i+j} \right]_{\mu \gamma} L_k + \sum_{k \in \mathbb{Z}} v_{ij}^k (\partial + \mu + \gamma, \lambda) \left[ (\partial + 2\mu) L_{i+j} \right]_{\mu \gamma} G_k.
\]

(75)

Thus, we have

\[
2 \sum_{k \in \mathbb{Z}} a_{k-i-j} \left[ (L_k)_{\mu \gamma} L_{i+j} \right] = \sum_{k \in \mathbb{Z}} u_{ij}^k (\partial + \mu + \gamma, \lambda) \left[ L_{i+j} \right]_{\mu \gamma} L_k + \sum_{k \in \mathbb{Z}} v_{ij}^k (\partial + \mu + \gamma, \lambda) \left[ L_{i+j} \right]_{\mu \gamma} G_k.
\]

(76)
It follows that
\[
2(\partial + 2\mu + 2\gamma) \sum_{k \in \mathbb{Z}} a_{k-i-j} L_{i+j+zk} = (\partial + 2\mu + 2\gamma) \sum_{k \in \mathbb{Z}} u^k_{ij}(\partial + \mu + \gamma, \lambda)L_{i+j+zk} + \left(\partial + \frac{3}{2}\mu + \frac{3}{2}\gamma\right) \sum_{k \in \mathbb{Z}} v^k_{ij}(\partial + \mu + \gamma, \lambda)G_{i+j+zk},
\]
which implies
\[
u^k_{ij}(\partial, \lambda) = 0 \quad \text{and} \quad 2 \sum_{k \in \mathbb{Z}} a_{k-i-j} L_{i+j+zk} = \sum_{k \in \mathbb{Z}} u^k_{ij}(\partial + \mu + \gamma, \lambda)L_{i+j+zk},
\]
hence
\[
2a_{k-i-j} = u^k_{ij}(\partial + \mu + \gamma, \lambda).
\]

We can get
\[
u^k_{ij}(\partial, \lambda) = 2a_{k-i-j} \quad \text{and} \quad \varphi_1(G_i, G_j) = 2\sum_{k \in \mathbb{Z}} a_{k-i-j} L_{k},
\]
for all \(i, j \in \mathbb{Z}\), where \(a_k\) for any \(k \in \mathbb{Z}\) are complex numbers.

**Theorem 2.** Every conformal super-biderivation \(\varphi_1\) on \(NS\) has the following forms:
\[
\varphi_1(L, L) = a(\partial + 2\lambda)L,
\]
\[
\varphi_1(L, G) = a\left(\partial + \frac{3}{2}\lambda\right)G,
\]
\[
\varphi_1(G, L) = a\left(\frac{1}{2}\partial + \frac{3}{2}\lambda\right)G,
\]
\[
\varphi_1(G, G) = aL,
\]
where \(a\) are complex numbers. Therefore, any conformal super-biderivation on \(NS\) is inner.

The proof of this theorem is similar to that of Theorem 1.

**Data Availability**
No data were used to support this study.

**Conflicts of Interest**
The author declares that there are no conflicts of interest regarding the publication of this paper.

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