NEUMANN LI-YAU GRADIENT ESTIMATE UNDER INTEGRAL RICCI CURVATURE BOUNDS

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Abstract. We prove a Li-Yau gradient estimate for positive solutions to the heat equation, with Neumann boundary conditions, on a compact Riemannian submanifold with boundary $M^n \subseteq N^n$, satisfying the integral Ricci curvature assumption:

$D^2 \sup_{x \in N} \left( \frac{1}{\gamma} \int_{B(x,D)} |Ric|^p dy \right)^{\frac{1}{p}} < K$

for $K(n, p)$ small enough, $p > n/2$, where $\text{diam}(M) \leq D$. The boundary of $M$ is not necessarily convex, but it needs to satisfy the interior rolling $R-$ball condition.

1. Introduction

Let $(M^n, g)$ be a Riemannian manifold with boundary $\partial M$. In [LY], P. Li and S.T. Yau proved a series of Li-Yau gradient estimates for positive solutions to the heat equation on $M$. In particular, they proved that if $M$ is a compact manifold with $\text{Ric} \geq -K$, for some $K \geq 0$, and the boundary of $M$ is convex (i.e. its second fundamental form is nonnegative $II \geq 0$), then any positive solution $u(x,t)$ to the Neumann problem:

$\begin{cases}
\partial_t u - \Delta u = 0 & \text{in } \overset{o}{M} \times (0, \infty) \\
\partial_{\nu} u = 0 & \text{on } \partial M \times (0, \infty)
\end{cases}$

where $\overset{o}{M} = M \setminus \partial M$ and $\nu$ denotes the outer unit normal vector to $\partial M$, satisfies:

$\frac{|
abla u|^2}{u^2} - \alpha \frac{\partial_t u}{u} \leq C_1 + C_2 \frac{1}{t}$

for all $\alpha > 1$, where:

$C_1 = \frac{n}{\sqrt{2}} \alpha^2 (\alpha - 1)^{-1} K$

$C_2 = \frac{n}{2} \alpha^2$

Later, in [W], J. Wang generalized this result to a case where $\partial M$ is not necessarily convex; more precisely, he considers the case where $II \geq -H$ for some $H \geq 0$, adding the necessary “interior rolling $R-$ball” condition, inspired by the work of R. Chen in [C].

Definition 1.1. A Riemannian manifold with boundary $M$ is said to satisfy the interior rolling $R-$ball condition if for any $p \in \partial M$ there exists $q \in M$ such that $B(q, R) \subseteq M$, and $B(q, R) \cap \partial M = \{p\}$, where $B(q, R)$ denotes the geodesic ball centered at $q$ with radius $R$.

Key words and phrases. Geometric analysis, Differential geometry, Neumann heat kernel, Integral Ricci curvature, Li-Yau gradient estimate.
Under these assumptions, J. Wang proved that $u(x, t)$ satisfies the Li-Yau gradient estimate (1.2), with constants:

$$C_1 = \frac{6n\alpha(\alpha - 1)(1 + H)^7K}{(\alpha - (1 + H)^2)^2} + \frac{309n^2\alpha^3(\alpha - 1)(1 + H)^{10}H}{(\alpha - (1 + H)^2)^3R^2\beta}$$

$$C_2 = \frac{n\alpha^2(\alpha - 1)^2(1 + H)^4}{(2 - \beta)(1 - \beta)(\alpha - (1 + H)^2)^2}$$

for any $\alpha > (1 + H)^2$ and any $0 < \beta < \frac{1}{2}$.

Notice that in both of the examples above, the authors were assuming a pointwise lower bound on the Ricci curvature. This pointwise assumption has been recently weakened by Q.S. Zhang and M. Zhu, in [ZZ1] and [ZZ2]. One of their results is a Li-Yau type gradient estimate under the integral Ricci curvature bounds introduced by P. Petersen and G. Wei in [PW1] and [PW2].

In particular, in [ZZ2] they prove the Li-Yau type gradient estimate:

$$\alpha J \frac{\lvert \nabla u \rvert^2}{u^2} - \frac{\partial_t u}{u} \leq \frac{C_1}{J} \left[ 1 + \frac{C_2}{J} \right] + \frac{C_3}{J} \frac{1}{t}$$

for the heat kernel on a manifold $N$, where $C_1$, $C_2$ and $C_3$ are constants depending on $n$, $p$ and $\alpha$, and $J = J(t)$ is a decreasing exponential function (see [ZZ2] for more details). Their curvature assumption is that, for $\kappa$ small enough, $p > n/2$:

$$\sup_{x \in N} \left( \frac{1}{\int_{B(x, 1)} |Ric^-|^p dV} \right)^{1/p} \leq \kappa$$

where $|Ric^-|$ is the negative part of the Ricci curvature, i.e. $|Ric^-| = \max\{0, -\rho(x)\}$ where $\rho(x)$ denotes the lowest eigenvalue of the Ricci curvature. Here $\int$ denotes the average integral over the domain. The smallness of the integral Ricci curvature is a necessary condition in general, as shown by X. Dai, G. Wei, and Z. Zhang in [DWZ]. Note that this result is for manifolds without boundary.

To deal with a manifold with boundary with Ricci curvature bounds, we use the technique developed in [W] together with the technique developed in [ZZ1] and [ZZ2]. We consider a manifold $N^n$ satisfying the scaling invariant curvature condition:

$$D^2 \sup_{x \in N} \left( \frac{1}{\int_{B(x, D)} |Ric^-|^p dy} \right)^{\frac{1}{p}} < K$$

with $p > n/2$ and $K(n, p) > 0$ small enough to have the volume doubling property proved in [PW2]. Let $M^n \subseteq N^n$ be a compact Riemannian submanifold with boundary, $\text{diam}(M) \leq D$, whose boundary is not necessarily convex, but that satisfies the interior rolling $R-$ball condition. Then we derive a Li-Yau gradient estimate for the positive solutions $u(x, t) > 0$ to the problem (1.1).

More precisely, our main theorem is:
Theorem 1.2. Given $H > 0$, $n > 0$, $p > \frac{n}{2}$, and $R > 0$ small enough, there exists $K(n, p) > 0$ such that if $M^n$ is a compact Riemannian submanifold with boundary of a Riemannian manifold $N^n$ with the properties:

1. $D^2 \sup_{x \in N} \left( \int_{B(x, D)} |Ric \cdot |^p \, dy \right)^{\frac{1}{p}} < K$, where $\text{diam}(M) \leq D$

2. $H \geq -H$, where $H$ is the second fundamental form of $\partial M$

3. $M$ satisfies the interior rolling $R$-ball condition

then any positive solution $u(x, t)$ to:

\begin{align*}
(1.6) & \left\{ \begin{array}{l}
\partial_t u - \Delta u = 0 \quad \text{in } \overset{\circ}{M} \times (0, \infty) \\
\partial_\nu u = 0 \quad \text{on } \partial M \times (0, \infty)
\end{array} \right.
\end{align*}

satisfies the Li-Yau type gradient estimate:

\begin{align*}
(1.7) \quad \alpha J \left( \left| \nabla u \right|^2 - \frac{\partial_t u}{u} \right) \leq C_1 + \frac{C_2}{J(t)}
\end{align*}

where, given any $0 < \xi < 1$, we can choose any $0 < \alpha \leq \frac{1-\xi}{(1+H)^2}$ and any $0 < \beta \leq \frac{\xi^2(1-\xi)}{2\xi^2+n^2(1+H)^2}$, and where:

\begin{align*}
C_1 &= \frac{n^2}{\alpha \sqrt{2\xi^3(1-2\beta)}} \left( \frac{32n^2H^2(1+H)^2}{\xi^3R^2} + 2\alpha(1+H) \left[ \frac{H}{R^2} + \frac{2(n-1)H(3H+1)}{R} \right] + \right. \\
& \quad \left. + (\beta + 4\alpha^{-1}) \left[ \frac{4\alpha H(1+H)}{R} \right]^2 \right)
\end{align*}

(1.8)

\begin{align*}
C_2 &= \frac{n^2}{\alpha(1-2\beta)}
\end{align*}

(1.9)

\begin{align*}
J(t) := 2^{n} \frac{1}{\alpha} e^{-\frac{c}{\alpha} t}
\end{align*}

(1.10)

where $c = (3 + \frac{1}{n})\frac{1}{\beta}$ and

\begin{align*}
(1.11) \quad \tilde{C}_3 = C_3(\alpha, \beta, n, p) \left[ \frac{K}{D^2 R^p} + \frac{K^p}{D R^{\frac{p}{p-1}} R^{\frac{1}{p-1}}} \right] > 0
\end{align*}

Remark 1.3. Notice that the range of values of $\alpha$ for which this theorem holds is consistent with the range of values of $\alpha$ in [W].

One of the key tools that we used are the Gaussian upper bounds of the Neumann heat kernel proved by M. Choulli, L. Kayser, and E.M. Ouhabaz in [CKO]. To apply their result, we use the volume doubling results of [PW2], and we show in Lemma 3.3 that the interior rolling $R$-ball condition ensures that the volume doubling property in $M$ holds up to the boundary. These two results, together with the Sobolev inequality of [DWZ], allow us to use the Gaussian upper bounds to find $J(t)$, as in [ZZ2].
2. Proof of main theorem

As in the proof of [W] and [C], consider a nonnegative $C^2$ function $\psi(r)$ defined on $[0, \infty)$ such that

\[
\psi(r) \leq H \quad \text{if } r \in [0, 1/2) \\
\psi(r) = H \quad \text{if } r \in [1, \infty)
\]

with $\psi(0) = 0$, $0 \leq \psi'(r) \leq 2H$, $\psi'(0) = H$ and $\psi''(r) \geq -H$. Define $\phi(x) := \psi\left(\frac{r(x)}{R}\right)$, where $r(x)$ denotes the distance from $x \in M$ to $\partial M$. Let $\varphi(x) := (1 + \phi(x))^2$, and $\tilde{\varphi}(x) := \alpha \varphi(x)$, where $\alpha > 0$ will be determined below.

**Lemma 2.1.** The function $\tilde{\varphi}$ satisfies:

\[
(2.1) \quad \alpha \leq \tilde{\varphi} \leq \alpha(1 + H)^2 \\
(2.2) \quad |\nabla \tilde{\varphi}| \leq \frac{4\alpha H(1 + H)}{R} \\
(2.3) \quad \Delta \tilde{\varphi} \geq -2\alpha(1 + H) \left[ \frac{H}{R^2} + \frac{2(n-1)H(3H+1)}{R} \right]
\]

**Proof of Lemma 2.1.** The inequalities (2.1) and (2.2) are immediate from the definitions. To prove (2.3) we follow the same argument as in [W] and [C]: we need to use that

\[
(2.4) \quad \Delta r \geq -(n-1)(3H+1)
\]

For a detailed argument on how to derive (2.4) see [C]. Then:

\[
(2.5) \quad \Delta \phi = \frac{\psi''|\nabla r|^2}{R^2} + \frac{\psi' \Delta r}{R} \geq -\frac{H}{R^2} - \frac{2H(n-1)(3H+1)}{R}
\]

So:

\[
(2.6) \quad \Delta \tilde{\varphi} = 2\alpha|\nabla \phi|^2 + 2\alpha(1 + \phi)\Delta \phi \geq -2\alpha(1 + H) \left[ \frac{H}{R^2} + \frac{2H(n-1)(3H+1)}{R} \right]
\]

\[\square\]

**Remark 2.2.** The inequality (2.4) holds as long as $R < 1$ is chosen small enough so that:

\[
(2.7) \quad \sqrt{K_R \tan(R \sqrt{K_R})} \leq \frac{1 + H}{2}
\]

and

\[
(2.8) \quad \frac{H}{\sqrt{K_R} \tan(R \sqrt{K_R})} \leq \frac{1}{2}
\]

where $K_R$ is the supremum of the sectional curvature at distance $R$ from the boundary.

Before we start the proof of the main theorem, we will need the following technical lemma which will be proved in the following section.
Lemma 2.3. There exists a unique smooth solution $J(x,t)$ to the problem:

\[
\begin{cases}
\Delta J - \partial_t J - c|\nabla J|^2 - 2J|\operatorname{Ric}| = 0 & \text{in } \bar{M} \times (0, \infty) \\
\partial_t J = 0 & \text{on } \partial M \times (0, \infty) \\
J = 1 & \text{on } M \times \{0\}
\end{cases}
\]

for $c > 1$ constant, and it satisfies

\[(2.10) \ 0 < \underline{J} \leq J \leq \overline{J} \leq 1\]

where $\underline{J} = \underline{J}(t)$ is given by

\[(2.11) \quad \underline{J}(t) := 2 - \frac{1}{c+1} e^{-\frac{C_3}{\alpha} t}\]

and $C_3 = C_3(c, n, p) \left[ \frac{K}{D^2 - R_{\mathbb{P}}} \right] > 0$.

Remark 2.4. The function $\underline{J}(t)$ is decreasing in $t$.

The proof of this lemma is provided in section 3. With it, we can start the proof of our main theorem.

Proof of Theorem 1.2. Consider the function:

\[(2.12) \quad G(x, t) := t \left[ \tilde{\varphi}_J(|\nabla f|^2 + \epsilon) - \partial_t f \right] \]

where $f = \ln u$, $\epsilon > 0$, and $J(x,t)$ is the function from Lemma 2.3 corresponding to $c = (3 + \frac{1}{\alpha}) \frac{1}{\beta}$, where $\alpha, \beta > 0$ are constants.

Let $(p, t_0)$ be the maximum of $G$ in $M \times [0, T]$, for $T > 0$. Notice that we can assume w.l.o.g. that $t_0 > 0$, since otherwise $t_0 = 0$ implies that $G \leq 0$ in $M \times [0, T]$, which is stronger than what we want to prove.

Case 1: $p \in \partial M$

In that case, $\partial_{\nu} G(p, t_0) \geq 0$, and choosing an orthonormal frame at $p$ so that $\nu_n = \nu$, we get:

\[(2.13) \quad 0 \leq t_0 \left[ \partial_{\nu} \tilde{\varphi}_J (|\nabla f|^2 + \epsilon) + \tilde{\varphi}_J \left( \partial_{\nu} J (|\nabla f|^2 + \epsilon) + 2J \sum_{i=1}^n \partial_i f \partial_{\nu} \partial_i f \right) \right] - \partial_{\nu} \partial_t f \]

Using that $\partial_{\nu} f = 0$ and $\partial_{\nu} J = 0$ on $\partial M \times (0, \infty)$, and dividing by $t_0 \tilde{\varphi}_J (|\nabla f|^2 + \epsilon)$, we get that at $(p, t_0)$:

\[(2.14) \quad 0 \leq \frac{1}{\tilde{\varphi}_J} \partial_{\nu} \tilde{\varphi} + 2 \frac{\sum_{i=1}^{n-1} \partial_i f \partial_{\nu} \partial_i f}{|\nabla f|^2 + \epsilon} \]

But now, following the argument of [W], this leads to a contradiction. By direct computation:

\[(2.15) \quad \sum_{i=1}^{n-1} \partial_i f \partial_{\nu} \partial_i f = -2II(\nabla f, \nabla f) \leq H|\nabla f|^2 \]
Also, since $p \in \partial \mathbf{M}$ we have that $r(p) = 0$, and so $\phi(p) = \psi(0) = 0$; so using that $\psi'(0) = H$ and that $\nabla r \cdot \nu(p) = -1$ we get:

\[ \partial_u \tilde{\varphi}(p) = \nabla \tilde{\varphi} \cdot \nu(p) = 2\alpha(1 + \phi(p))\psi' \left( \frac{r(p)}{R} \right) \frac{1}{R} \nabla r \cdot \nu = -\alpha \frac{2H}{R} \]

Now, from the two expressions above, assuming w.l.o.g. that $R < 1$ and using that $\tilde{\varphi}(p) = \alpha$, for any $\epsilon > 0$ we get:

\[ \frac{1}{\varphi} \partial_u \tilde{\varphi} + 2 \sum_{i=1}^{n-1} \partial_i f \partial_u \partial_i f = \leq -\frac{2H}{R} + 2 \frac{H|\nabla f|^2}{|\nabla f|^2 + \epsilon} < 0 \]

which is a contradiction. Thus, Case 1 can not occur.

Case 2: $p \in \mathbf{M} = \mathbf{M} \setminus \partial \mathbf{M}$

In this case, $(p, t_0)$ is a local maximum, thus $\nabla G(p, t_0) = 0$, $\partial_t G(p, t_0) \geq 0$, and $\Delta G(p, t_0) \leq 0$, which implies that $\Delta G - \partial_t G \leq 0$ at $(p, t_0)$. We can assume w.l.o.g. that $G(p, t_0) > 0$, since otherwise we get a stronger statement than what we are going to prove.

Now we proceed similarly as in [ZZ2]. By direct computation, using Bochner’s formula:

\[ (\Delta - \partial_t) \left( \frac{\nabla u^2}{u} + \epsilon u \right) = 2 \left( \partial_i \partial_j u - \frac{\partial_i u \partial_j u}{u} \right)^2 + 2R_{ij} \frac{\partial_i u \partial_j u}{u} \]

Let’s call $g = \frac{\nabla u^2}{u} + \epsilon$ and $\tilde{g} = ug = \frac{\nabla u^2}{u} + \epsilon u$. Then:

\[ (\Delta - \partial_t) (\tilde{g} - \partial_t u) = (\Delta - \partial_t) (\tilde{g} - \partial_t u) + 2\nabla (\tilde{g}) \nabla \tilde{g} + \tilde{g} [ (\Delta - \partial_t) \tilde{g}] \]

\[ = (\Delta - \partial_t) (\tilde{g} - \partial_t u) + 2\nabla (\tilde{g}) \nabla \tilde{g} + \tilde{g} \left[ \frac{2}{u} \left( \partial_i \partial_j u - \frac{\partial_i u \partial_j u}{u} \right)^2 + 2R_{ij} \frac{\partial_i u \partial_j u}{u} \right] \]

Using the quotient formula for the operator $\mathcal{L} = \Delta - \partial_t$, which is:

\[ \mathcal{L} \left( \frac{A}{B} \right) + 2\nabla \ln B \nabla \frac{A}{B} = \frac{\Delta A}{B} - \frac{A \mathcal{L} B}{B^2} \]

for $A = \tilde{g} - \partial_t u$ and $B = u$, and defining $Q = G/t = \tilde{g} - \frac{2u^2}{u}$, we get:

\[ (\Delta - \partial_t) Q + 2 \frac{u}{u} \nabla u \nabla Q = \frac{\Delta - \partial_t}{u} (\tilde{g} - \partial_t u) \]

And from (2.19) we get:

\[ (\Delta - \partial_t) Q + 2 \frac{u}{u} \nabla u \nabla Q = (\Delta - \partial_t) (\tilde{g} - \partial_t u) + 2 \nabla (\tilde{g}) \nabla \tilde{g} + \tilde{g} \left[ \frac{2}{u^2} \left( \partial_i \partial_j u - \frac{\partial_i u \partial_j u}{u} \right)^2 + 2R_{ij} \frac{\partial_i u \partial_j u}{u^2} \right] \]
Also, using that \( \nabla \left( \frac{\left| \nabla u \right|^2}{u^2} \right) \frac{1}{u} = \nabla \left( \frac{\left| \nabla u \right|^2}{u^2} \right) + \frac{\nabla u}{u} \nabla u \), and the notation \( f = \ln u \), we observe that:

\[
\frac{\nabla g}{u} = \nabla \left( \frac{\left| \nabla u \right|^2}{u^2} \right) + \frac{\nabla u}{u} \nabla u + \epsilon \frac{\nabla u}{u} = \nabla \left( ||\nabla f||^2 \right) + g\nabla f
\]

Hence (2.23) becomes:

\[
(\Delta - \partial_t) Q + 2\nabla f \nabla Q = (\Delta - \partial_t) (\bar{\varphi} J) \cdot g + 2\nabla (\bar{\varphi} J) \left[ \nabla \left( ||\nabla f||^2 \right) + g\nabla f \right] + 2\bar{\varphi} J \left[ \partial_t \partial_j f^2 + \text{Ric}(\nabla f, \nabla f) \right]
\]

Now notice that, for \( \beta > 0 \):

\[
\nabla J \nabla \left( ||\nabla f||^2 \right) \geq -\frac{1}{\beta J} ||\nabla J||^2 ||\nabla f||^2 - \beta J \partial_t \partial_j f^2
\]

So (2.25) gives us:

\[
(\Delta - \partial_t) Q + 2\nabla f \nabla Q \geq (\Delta - \partial_t) (\bar{\varphi} J) \cdot g + 2\nabla (\bar{\varphi} J) \left[ \nabla \left( ||\nabla f||^2 \right) + g\nabla f \right] + 2\bar{\varphi} J \left[ ||\nabla f||^2 - g\text{Ric} \right]
\]

Using that \( -2||\nabla f|| \nabla f \geq -\frac{\left| \nabla J \right|^2}{\beta J} - \beta J ||\nabla f||^2 \) and Cauchy-Schwarz, we get:

\[
(\Delta - \partial_t) Q + 2\nabla f \nabla Q \geq (\Delta - \partial_t) (\bar{\varphi} J) \cdot g + 2\nabla (\bar{\varphi} J) \left[ \nabla \left( ||\nabla f||^2 \right) + g\nabla f \right] + 2\bar{\varphi} J \left[ ||\nabla f||^2 - g\text{Ric} \right]
\]

Now expanding the first term on the right:

\[
(\Delta - \partial_t) Q + 2\nabla f \nabla Q \geq g [\Delta \bar{\varphi} J + 2\nabla \bar{\varphi} \nabla J] + 2(1 - \beta)\bar{\varphi} J \left[ \partial_t \partial_j f^2 - \beta Jg \nabla \bar{\varphi} \nabla f \right]^2
\]

Using (2.21) we get \( \nabla \bar{\varphi} \nabla J \geq -\frac{1}{\alpha g} \frac{||\nabla f||^2}{\beta J} - \beta J ||\nabla \bar{\varphi}||^2 \), hence:

\[
(\Delta - \partial_t) Q + 2\nabla f \nabla Q \geq g [\Delta \bar{\varphi} J + 2\nabla \bar{\varphi} \nabla J] + 2(1 - \beta)\bar{\varphi} J \left[ \partial_t \partial_j f^2 - \beta Jg \nabla \bar{\varphi} \nabla f - \beta \nabla \bar{\varphi} \nabla f \right]^2
\]
Since $J$ solves the problem of Lemma 2.3 for $c = (3 + \frac{1}{\alpha}) \frac{1}{\beta}$, we see that (2.31) becomes:

\[(\Delta - \partial_t) Q + 2\nabla f \nabla Q \geq J \left[ g \Delta \bar{\varphi} + 2\nabla \bar{\varphi} \nabla (|\nabla f|^2) + 2g \nabla \bar{\varphi} \nabla f - \beta |\nabla \bar{\varphi}|^2 g\right.
\begin{equation}
+2(1 - \beta)\bar{\varphi} |\partial_t \partial_j f|^2 - \beta g \bar{\varphi} |\nabla f|^2 \right]
\end{equation}

(2.32)

Note that:

\[(\Delta - \partial_t) G + 2\nabla f \nabla G = (\Delta - \partial_t) (tQ) + 2\nabla f \nabla (tQ) = t[(\Delta - \partial_t) Q + 2\nabla f \nabla Q] - Q\]

Since we know that $(p, t_0)$ is a local maximum, $(\Delta - \partial_t) G(p, t_0) + 2\nabla f \nabla G(p, t_0) = (\Delta - \partial_t) G(p, t_0) \leq 0$, so at $(p, t_0)$ we have:

\[(2.34) 0 \geq t_0 J \left[ g \Delta \bar{\varphi} + 2\nabla \bar{\varphi} \nabla (|\nabla f|^2) + 2g \nabla \bar{\varphi} \nabla f - \beta |\nabla \bar{\varphi}|^2 g + 2(1 - \beta)\bar{\varphi} |\partial_t \partial_j f|^2 - \beta g \bar{\varphi} |\nabla f|^2 \right] - Q\]

Expanding the second term on the right, notice that:

\[(2.35) 2\nabla \bar{\varphi} \nabla (|\nabla f|^2) \geq -4\alpha^{-1} |\nabla \bar{\varphi}|^2 |\nabla f|^2 - \alpha |\partial_t \partial_j f|^2\]

Hence:

\[(2.36) 0 \geq t_0 J \left[ g \Delta \bar{\varphi} + 2g \nabla \bar{\varphi} \nabla f - \beta |\nabla \bar{\varphi}|^2 g + (2(1 - \beta)\bar{\varphi} - \alpha) |\partial_t \partial_j f|^2 - 4\alpha^{-1} |\nabla \bar{\varphi}|^2 |\nabla f|^2 - \beta g \bar{\varphi} |\nabla f|^2 \right] - Q\]

Now using that

\[(2.37) \sum_{i,j=1}^{n} |\partial_t \partial_j f|^2 \geq \frac{1}{n^2} \left( \sum_{i,j=1}^{n} |\partial_t \partial_j f| \right)^2 \geq \frac{1}{n^2} (\Delta f)^2 = \frac{1}{n^2} (|\nabla f|^2 - \partial_t f)^2\]

we get:

\[(2.38) 0 \geq t_0 J \left[ g \Delta \bar{\varphi} + 2g \nabla \bar{\varphi} \nabla f - \beta |\nabla \bar{\varphi}|^2 g + \frac{2(1 - \beta)\bar{\varphi} - \alpha}{n^2} (|\nabla f|^2 - \partial_t f)^2 - 4\alpha^{-1} |\nabla \bar{\varphi}|^2 |\nabla f|^2 - 4\alpha^{-1} |\nabla \bar{\varphi}|^2 |\nabla f|^2 - \beta g \bar{\varphi} |\nabla f|^2 \right] - Q\]

In the discussion below, $O(\epsilon)$ denotes a function that goes to zero as $\epsilon$ goes to zero. Expanding the terms containing $g = |\nabla f|^2 + \epsilon$, by Lemma 2.1 and the elementary inequality $2\nabla \bar{\varphi} \nabla f \geq -(|\nabla \bar{\varphi}|^2 + |\nabla f|^2)$, we get:

\[(2.39) 0 \geq t_0 J \left[ |\nabla f|^2 \Delta \bar{\varphi} + 2|\nabla f|^2 \nabla \bar{\varphi} \nabla f - \epsilon |\nabla f|^2 - \beta |\nabla \bar{\varphi}|^2 |\nabla f|^2 \right.
\begin{equation}
+ \frac{2(1 - \beta)\bar{\varphi} - \alpha}{n^2} (|\nabla f|^2 - \partial_t f)^2 - 4\alpha^{-1} |\nabla \bar{\varphi}|^2 |\nabla f|^2 - \beta \epsilon |\nabla f|^2 - \beta g \bar{\varphi} |\nabla f|^2 \right] - Q + O(\epsilon)
\end{equation}

Using Cauchy-Schwarz and rearranging terms:
\[ 0 \geq t_0 \left[ (\Delta \varphi - (\beta + 4\alpha^{-1})|\nabla \varphi|^2 + O(\varepsilon))|\nabla f|^2 + 2|\nabla \varphi||\nabla f|^3 - \beta \varphi|\nabla f|^4 \right. \]
\[ + \frac{2(1 - \beta)\varphi - \alpha}{n^2} (|\nabla f|^2 - \partial_t f)^2 \left] - Q + O(\varepsilon) \right. \]

Now we will relate the term \((|\nabla f|^2 - \partial_t f)^2\) to \(Q^2\). Notice that, by direct computation:

\[ Q^2 = (|\nabla f|^2 - \partial_t f)^2 + (\varphi^2 J^2 - 1)|\nabla f|^4 + 2(1 - \varphi J)|\nabla f|^2 \partial_t f + O(\varepsilon)|\nabla f|^2 + O(\varepsilon)\partial_t f + O(\varepsilon) \]

Using that \(\partial_t f = \varphi J (|\nabla f|^2 + \varepsilon) - Q\), we get:

\[ Q^2 = (|\nabla f|^2 - \partial_t f)^2 - (1 - \varphi J)^2|\nabla f|^4 - 2(1 - \varphi J)Q|\nabla f|^2 + O(\varepsilon)|\nabla f|^2 + O(\varepsilon)Q + O(\varepsilon) \]

Hence:

\[ Q^2 = (|\nabla f|^2 - \partial_t f)^2 - (1 - \varphi J)^2|\nabla f|^4 - 2(1 - \varphi J)Q|\nabla f|^2 + O(\varepsilon)|\nabla f|^2 + O(\varepsilon)Q + O(\varepsilon) \]

Notice that if we choose \(\alpha \leq \frac{1 - \xi}{4 + \varepsilon}\) for some \(0 < \xi < 1\), from (2.11) and (2.10), we get \(1 - \varphi J \geq \xi\), so \(2(1 - \varphi J)Q|\nabla f|^2 \geq 0\), hence:

\[ (|\nabla f|^2 - \partial_t f)^2 \geq Q^2 + (1 - \varphi J)^2|\nabla f|^4 + O(\varepsilon)|\nabla f|^2 + O(\varepsilon)Q + O(\varepsilon) \]

Making sure that our later choice of \(\beta\) is so that \(\frac{2(1 - \beta)\varphi - \alpha}{n^2} > 0\), (2.40) becomes:

\[ 0 \geq t_0 \left[ (\Delta \varphi - (\beta + 4\alpha^{-1})|\nabla \varphi|^2 + O(\varepsilon))|\nabla f|^2 + 2|\nabla \varphi||\nabla f|^3 + \\
+ \left(\frac{2(1 - \beta)\varphi - \alpha}{n^2}\right) (1 - \varphi J)^2 - \beta \varphi \right] |\nabla f|^4 + \frac{2(1 - \beta)\varphi - \alpha}{n^2} Q^2 \right] - (1 + O(\varepsilon)) Q + O(\varepsilon) \]

Now we are going to choose \(\beta > 0\) so that the coefficient of \(|\nabla f|^4\) is positive; namely, for some \(A > 0\), we want to have:

\[ \left(\frac{2(1 - \beta)\varphi - \alpha}{n^2}\right) (1 - \varphi J)^2 - \beta \varphi \geq A \]

Using Lemmas 2.1 and 2.3 and the choice of \(\alpha\), we know that:

\[ \left(\frac{2(1 - \beta)\varphi - \alpha}{n^2}\right) (1 - \varphi J)^2 - \beta \varphi \geq \left[\frac{2(1 - \beta)\alpha - \alpha}{n^2}\right] \xi^2 - \beta \alpha (1 + H)^2 \]

Now setting the right hand side to be greater or equal than \(A\), we get the condition:

\[ \beta \leq \frac{\alpha \xi^2 - An^2}{2\alpha \xi^2 + \alpha n^2 (1 + H)^2} \]

To ensure that there are positive values of \(\beta\), we choose \(A = \frac{a \xi^3}{n^2}\), so that the condition above becomes:
Choosing $\beta$ in this way and using Lemmas 2.1 and 2.3, the coefficient of $|\nabla f|^2$ satisfies:

$$
\Delta \tilde{\varphi} - (\beta + 4 \alpha^{-1})|\nabla \tilde{\varphi}|^2 + O(\epsilon) \geq -C(\alpha, \beta, n, H) + O(\epsilon) =: -C
$$

where:

$$
C := 2\alpha (1 + H) \left[ \frac{H}{R^2} + \frac{2(n - 1)H(3H + 1)}{R} \right] + (\beta + 4 \alpha^{-1}) \left[ \frac{4 \alpha H(1 + H)}{R} \right]^2
$$

The one of $|\nabla f|^3$ satisfies:

$$
-2|\nabla \tilde{\varphi}| \geq -\frac{8 \alpha H(1 + H)}{R} =: -B(\alpha, H, R)
$$

Finally, the one of $Q^2$ satisfies:

$$
\frac{2(1 - \beta)}{n^2} \varphi - \alpha \frac{n^2}{n^2} \geq \frac{\alpha(1 - 2\beta)}{n^2} =: E(\alpha, \beta, n)
$$

So (2.49) becomes:

$$
0 \geq t_0 J \left[ -C_\epsilon |\nabla f|^2 - B|\nabla f|^3 + A|\nabla f|^4 + EQ^2 \right] - (1 + O(\epsilon)) Q + O(\epsilon)
$$

Now following the same argument as in [W], calling $y = |\nabla f|^2$, notice that:

$$
Ay^2 - By^{3/2} - C_\epsilon y = \
= A y^2 + \left( \sqrt{\frac{A}{2}} y - \frac{B}{\sqrt{2A}} y^{1/2} \right)^2 - \frac{B^2}{2A} y - C_\epsilon y \geq \
\geq \left( \sqrt{\frac{A}{2}} y - \frac{1}{\sqrt{2A}} \left( \frac{B^2}{2A} + C_\epsilon \right) \right)^2 - \frac{1}{2A} \left( \frac{B^2}{2A} + C_\epsilon \right)^2 \geq \
\geq - \frac{1}{2A} \left( \frac{B^2}{2A} + C_\epsilon \right)^2 = - \frac{1}{2A} \left( \frac{B^2}{2A} + C \right)^2 + O(\epsilon) =: -\tilde{D} + O(\epsilon)
$$

Thus (2.54) becomes:

$$
0 \geq t_0 JQ^2 - (1 + O(\epsilon))Q - \tilde{D}t_0 J + O(\epsilon)
$$

Or equivalently, multiplying by $t_0$:

$$
0 \geq EJQ^2 - (1 + O(\epsilon))G - \tilde{D}t_0^2 J + O(\epsilon)
$$

Notice that the right hand side is quadratic in $G$, with the leading coefficient being positive. So if it is nonpositive, we must have:
\begin{equation}
G(p,t_0) \leq \frac{1}{2EJ(t_0)} + \sqrt{\frac{1}{4E^2J_2(t_0)}} + \frac{\bar{D}t_0^2}{E} + O(\epsilon) + O(\epsilon)
\end{equation}

Since \((p,t_0)\) is the maximum of \(G\) in \(M \times [0,T]\) and using Lemma 2.3 and that \(J\) is decreasing in \(t\), we get that for any \(x \in M\):

\begin{equation}
G(x,T) \leq \frac{1}{2EJ(T)} + \sqrt{\frac{1}{4E^2J_2(T)}} + \frac{\bar{D}T^2}{E} + O(\epsilon) + O(\epsilon)
\end{equation}

So using again Lemma 2.3:

\begin{equation}
T \left[ \tilde{\varphi}_J \left( |\nabla f|^2 + \epsilon \right) - \partial_t f \right] \leq G \leq \frac{1}{2EJ} + \sqrt{\frac{1}{4E^2J_2^2}} + \frac{\bar{D}T^2}{E} + O(\epsilon) + O(\epsilon)
\end{equation}

where all the functions in the previous inequality are being evaluated at \((x,T)\). At this point, the inequality does not depend on the point \((p,t_0)\) (which could change when we vary \(\epsilon\)), and since the inequality holds for any \(\epsilon > 0\), we get:

\begin{equation}
T \left[ \tilde{\varphi}_J \left( |\nabla f|^2 + \epsilon \right) - \partial_t f \right] \leq \frac{1}{2EJ} + \sqrt{\frac{1}{4E^2J_2^2}} + \frac{\bar{D}T^2}{E}
\end{equation}

The argument above works for any value of \(T > 0\), so at any point \((x,t) \in M \times (0,\infty)\) we have:

\begin{equation}
t \left[ \tilde{\varphi}_J |\nabla f|^2 - \partial_t f \right] \leq \frac{1}{2EJ} + \sqrt{\frac{1}{4E^2J_2^2}} + \frac{\bar{D}t^2}{E} \leq \frac{1}{EJ} + t\sqrt{\frac{\bar{D}}{E}}
\end{equation}

Hence, using Lemma 2.1 \(f = \ln(u)\), and dividing by \(t\), we get what we wanted:

\begin{equation}
\alpha J |\nabla u|^2 - \frac{\partial_t u}{u} \leq C_1 + \frac{C_2}{J} \frac{1}{t}
\end{equation}

where \(C_1 = \sqrt{\frac{\bar{D}}{E}}\) and \(C_2 = \frac{1}{E}\). \(\square\)

3. Proof of Lemma 2.3

Using the transformation \(w = J^{-(c-1)}\), the problem from Lemma 2.3 becomes:

\begin{equation}
\begin{cases}
\Delta w - \partial_t w + Vw = 0 & \text{in } M \times (0,\infty) \\
\partial_n w = 0 & \text{on } \partial M \times (0,\infty) \\
w = 1 & \text{on } M \times \{0\}
\end{cases}
\end{equation}

where \(V(x) := 2(c-1)|\text{Ric}^{-1}(x)| \geq 0\). Note that this is a linear parabolic PDE with Neumann boundary conditions, which has a unique smooth solution given by Duhamel’s formula:
where $h(t, x, y) \geq 0$ is the Neumann heat kernel on $M$.

**Claim 1:** $w(x, t) > 0$

**Proof.** Argue by contradiction. First, define:

$$w(t) = \min_{x \in M} w(x, t)$$

If the claim is false, there exists some $t \geq 0$ such that $w(t) \leq 0$. Since $w$ is continuous and $w(0) = 1$, there exists $t_0 > 0$ such that $w(t_0) = 0$ and $w(t) > 0$ for any $0 \leq t < t_0$. Let $x_0 \in M$ be a point that realizes the minimum $w(x_0, t_0) = w(t_0) = 0$. Then:

$$\int_0^{t_0} \int_M h(t_0 - s, x_0, y)V(y)w(y, s)dyds \geq 0$$

So:

$$w(x_0, t_0) \geq 1$$

But that’s a contradiction. □

By the previous claim, we know that $J = w^{-\frac{1}{c-1}}$ is well defined and smooth. Moreover:

**Claim 2:** $J(x, t) \leq 1$

**Proof.** This statement is equivalent to $w(x, t) \geq 1$. By a similar argument as above, since $w > 0$:

$$\int_0^t \int_M h(t - s, x, y)V(y)w(y, s)dyds \geq 0$$

hence:

$$w(x, t) \geq 1$$

□

**Claim 3:** $J(x, t) \geq J(t)$ where $J(t) := 2^{-\frac{1}{c-1}}e^{-\frac{C_3}{c}t}$ and $\overline{C_3} = C_3(c, n, p) \left[ \frac{K}{D^2} + \frac{K}{R \eta} \right] + \frac{K}{D^2} + \frac{K}{R \eta}$.

**Proof.** To prove this we will follow a similar argument as in [ZZ2]: we will find Gaussian upper bounds for $w(x, t)$. To do so, we use the following result from Choulli, Kayser, and Ouhabaz in [CKO] (Theorem 1.1):
Lemma 3.1 (Choulli, Kayser, Ouhabaz [CKO]). Suppose that $\mathbb{N}^n$ is a smooth Riemannian manifold that satisfies the volume doubling property and whose heat kernel $p(t, x, y)$ satisfies:

\begin{equation}
(3.9) \quad p(t, x, y) \leq \frac{C}{[V(x, \sqrt{t})V(y, \sqrt{t})]^{1/2}} e^{-c\frac{d(x,y)^2}{t}}
\end{equation}

where $C, c > 0$ are constants, $d(x,y)$ denotes the geodesic distance between $x, y \in \mathbb{N}^n$, and $V(x, r) = \text{Vol}(B(x, r))$ is the volume of a geodesic ball of radius $r > 0$. Suppose also that $M^a \subseteq \mathbb{N}^n$ is a Riemannian submanifold with boundary such that $\text{diam}(M) < \infty$, satisfying the volume doubling property:

\begin{equation}
(3.10) \quad V_M(x, s) \leq \tilde{C}\left(\frac{s}{r}\right)^\gamma V_M(x, r)
\end{equation}

where $V_M(x, r) = \text{Vol}(B(x, r) \cap M)$ is the volume in $M^a$ of a geodesic ball $B(x, r)$ of $\mathbb{N}^n$, and $\tilde{C}, \gamma > 0$ are positive constants. Then, the Neumann heat kernel on $M^a$ satisfies:

\begin{equation}
(3.11) \quad h(t, x, y) \leq \frac{C}{[V_M(x, \sqrt{t})V_M(y, \sqrt{t})]^{1/2}} \left(1 + \frac{d(x, y)^2}{4t}\right)^\gamma e^{-\frac{d(x, y)^2}{4t}}
\end{equation}

All the hypothesis of this lemma are known to be satisfied, except maybe for property (3.10) near the boundary. More precisely, $\text{diam}(M) < \infty$ since $M$ is compact, and the volume doubling property for $\mathbb{N}$ was proved in [PW2] and can be stated as:

Lemma 3.2 (Petersen, Wei [PW2]). Given $p > n/2$ and $D > 0$, there exists $K = K(n, p) > 0$ such that if

\begin{equation}
(3.12) \quad D^2 \sup_{x \in \mathbb{N}} \left(\int_{B(x, D)} |Ric^{-1}|^p \right)^{1/p} < K
\end{equation}

then for all $x \in \mathbb{N}$ and $r < s < D$ we have:

\begin{equation}
(3.13) \quad V(x, s) \leq 2 \left(\frac{s}{r}\right)^n V(x, r)
\end{equation}

Choosing $D$ so that $\text{diam}(M) \leq D$, we get the volume doubling property for all the balls completely contained in $M$. Also, (3.14) follows from the volume doubling property and a Sobolev inequality proved in [DWZ], as explained in [ZZ2]. Now we will show that property (3.10) holds as well:

Lemma 3.3. Given $D, \gamma > 0$, if a manifold $\mathbb{N}$ satisfies the volume doubling property

\begin{equation}
(3.14) \quad V(x, s) \leq C\left(\frac{s}{r}\right)^\gamma V(x, r)
\end{equation}

for $r \leq s \leq D$ and $x \in \mathbb{N}$, then a compact submanifold with boundary $M \subseteq \mathbb{N}$, with $\text{diam}(M) \leq D$, whose boundary satisfies the interior rolling $R-$ball condition, satisfies:

\begin{equation}
(3.15) \quad V_M(x, s) \leq \tilde{C}\left(\frac{s}{r}\right)^\gamma V_M(x, r)
\end{equation}

for $0 < r \leq s$, $x \in M$, $\tilde{C} = \max\{3\gamma C, \left(\frac{2D}{R}\right)^\gamma C\}$. 
Proof. We only need to consider the situation where \( B(x, s) \cap \partial M \neq \emptyset \). If that’s the case, then let \( p \in \partial M \) denote the closest point to \( x \) in \( \partial M \). Using the interior rolling \( R \)-ball condition, we know that there exists \( q \in M \) such that \( B(q, R) \subseteq M \) and \( B(q, R) \cap \partial M = \{p\} \).

Notice that, by definition, \( p \) minimizes the distance to the boundary from \( x \) and from \( q \), hence the geodesics joining \( p \) and \( x \), \( \gamma_{px} \), and \( p \) and \( q \), \( \gamma_{pq} \), must both be perpendicular to \( \partial M \) at \( p \). Hence, since geodesics do not branch, \( x \), \( q \) and \( p \) are on the same geodesic.

**Case 1:** \( 0 < r \leq R \)

Consider a point \( q' \) on the geodesic joining \( x \) and \( q \), such that \( d(q', x) = r/2 \) and \( d(q', q) = d(x, q) - r/2 \). Notice that \( p \) also minimizes the distance from \( q' \) to \( \partial M \). It’s easy to see that \( B(q', r/2) \subseteq B(x, r) \cap M \), and also \( B(x, s) \subseteq B(q', 3s/2) \).

If \( \frac{3s}{2} \leq \gamma \) we get:

\[
V_M(x, s) \leq V(q', \gamma) \leq C \left( \frac{3s}{2} \right) \gamma V_M(x, r)
\]

If \( \frac{3s}{2} \geq \gamma \) we get:

\[
V_M(x, s) \leq V(q', D) \leq C \left( \frac{2D}{r} \right) \gamma V(q', r/2) \leq \bar{C} \left( \frac{3s}{2} \right) \gamma V_M(x, r)
\]

**Case 2:** \( R \leq r \)

In this case, consider also a point \( q'' \) on the geodesic joining \( x \) and \( q \), such that \( d(q'', x) = R/2 \) and \( d(q'', q) = d(x, q) - R/2 \). As before, it’s easy to see that \( B(q'', R/2) \subseteq B(x, r) \cap M \). Thus, we can see that:

\[
V_M(x, s) \leq V(q'', D) \leq C \left( \frac{2D}{R} \right) \gamma V(q'', R/2) \leq \tilde{C} V_M(x, r)
\]

which is stronger than what we want to prove, so in particular:

\[
V_M(x, s) \leq \tilde{C} \left( \frac{3s}{2} \right) \gamma V_M(x, r)
\]

In our case, in particular, we can get:

\[
V_M(x, s) \leq \left( \frac{2^{n+1} \cdot 3^n D^n}{R^n} \right)^n V_M(x, r)
\]

Notice that, as a consequence of the volume doubling in \( N \) and the curvature condition \( (3.12) \), we can derive:

\[
\left( \int_M |\text{Ric}^-|^p \right)^{1/p} < 2^{\frac{n}{p}} \left( \frac{1}{D^{\frac{n}{p}} R^{\frac{n}{p}}} \right) K
\]

Now, using Lemma \[3.1\] and following the same argument as in \[ZZ2\], we can finish the proof of the claim. Let
\[ w(t) = \sup_{(x,s) \in M \times [0,t]} w(x,s) \]

Then:

\[ w(x,t) \leq 1 + \int_0^t w(s) \int_M h(t-s, x, y)V(y)dy ds \]

**Case 1: \( t - s \geq D^2 \)**

In this case \( V_M(z, \sqrt{t-s}) = |M| \), so:

\[ \int_M h(t-s, x, y)V(y)dy \leq C(c,n,|M|) \int_M |Ric^-| |y|dy \leq C(c,n) \left( \int_M |Ric^-|^p \right)^{\frac{1}{p}} \leq C_4 \frac{K}{D^{2-n} R^p} =: \bar{C}_4 \]

where \( C_4 = C_4(c,n,p) \).

**Case 2: \( t - s \leq D^2 \)**

Using Lemma 3.3:

\[ V_M(z, \sqrt{t-s}) \geq \frac{R^n}{2^{n+1} \cdot 3^n D^{2n}} (t-s)^n/2 V_M(z, D) = C(n) \frac{R^n}{D^{2n}} (t-s)^n/2 |M| \]

Thus, by Hölder’s inequality:

\[ \int_M h(t-s, x, y)V(y)dy \leq ||V||_{L^p(M)} \left( \int_M h |h|^{-\frac{1}{p}} \right)^{\frac{1}{p-1}} \leq \frac{C_5 K}{D^{2-n} R^p} (t-s)^{-\frac{n}{p}} =: \tilde{C}_5 (t-s)^{-\frac{n}{p}} \]

where \( C_5 = C_5(c,n,p) \), and where we have used that:

\[ h(t,x,y) \leq C(\frac{R^n}{D^{2n}}) \left( \int_M h |h|^{-\frac{1}{p}} \right)^{\frac{1}{p-1}} \leq \frac{C_5 K}{D^{2-n} R^p} (t-s)^{-\frac{n}{p}} =: \tilde{C}_5 (t-s)^{-\frac{n}{p}} \]

With these estimates, \( \text{3.28} \) becomes:

\[ w(x,t) \leq 1 + \tilde{C}_4 \int_0^{t-D^2} \bar{w}(s)ds + \tilde{C}_5 \int_{t-D^2}^t (t-s)^{-\frac{n}{p}} \bar{w}(s)ds \]

The second term on the right can be written as:
\[ \int_{t-D^2}^{t} (t-s)^{-\frac{2p}{n}} \overline{w}(s) ds = \int_{t-D^2}^{t-\epsilon} (t-s)^{-\frac{2p}{n}} \overline{w}(s) ds + \int_{t-\epsilon}^{t} (t-s)^{-\frac{2p}{n}} \overline{w}(s) ds \leq \epsilon^\frac{2p}{n} \int_{t-D^2}^{t-\epsilon} \overline{w}(s) ds + \frac{2p}{2p-n} \overline{w}(t) \]

where we have used that \( p > \frac{n}{2} \). Then, taking supremum over \((x,t) \in M \times [0,t]\), \(3.28\) becomes:

\[ \left[ 1 - \frac{2p\overline{C}_5 \epsilon^{\frac{2p-n}{2p-n}}}{} \right] \overline{w}(t) \leq 1 + \max\{\overline{C}_4, \overline{C}_5 \epsilon^{-\frac{2p}{n}} \} \int_0^t \overline{w}(s) ds \]

Now, choosing \( \epsilon = \left( \frac{4p\overline{C}_3}{2p-n} \right)^{-\frac{2p-n}{2p-n}} \), we get:

\[ \overline{w}(t) \leq 2 + \overline{C}_3 \int_0^t \overline{w}(s) ds \]

where \( \overline{C}_3 \) is chosen so that \( \overline{C}_3 \geq 2 \max\{\overline{C}_4, \overline{C}_5 \epsilon^{\frac{2p}{n}} \} \), for example:

\[ \overline{C}_3 = C_3 \left[ \frac{K}{D^{2-p} R^{\frac{2p-n}{n}}} + \frac{K \frac{2p}{n}}{D^{4p-n} R^{2p-n}} \right] \]

for some constant \( C_3 = C_3(c,n,p) \). So by Grönwall’s inequality:

\[ w(x,t) \leq \overline{w}(t) \leq 2e^{\overline{C}_3 t} \]

Thus, using \( J = w^{-\frac{1}{n}} \), we get

\[ J(t) := 2^{-\frac{1}{n}} e^{-\frac{\overline{C}_3}{n}} t \leq J(x,t) \]

\[ \square \]

**Acknowledgements**

The author would like to thank Professors Qi S. Zhang, Meng Zhu, and Lihan Wang for their valuable advise and comments.
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