ALGEBRAIC AND TRANSCENDENTAL FORMULAS FOR THE
SMALLEST PARTS FUNCTION

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Abstract. Building on work of Hardy and Ramanujan, Rademacher proved a well-known exact formula for the values of the ordinary partition function \( p(n) \). More recently, Bruinier and Ono obtained an algebraic formula for these values. Here we study the smallest parts function introduced by Andrews; \( \text{spt}(n) \) counts the number of smallest parts in the partitions of \( n \). The generating function for \( \text{spt}(n) \) forms a component of a natural mock modular form of weight \( 3/2 \) whose shadow is the Dedekind eta function. Using automorphic methods (in particular the theta lift of Bruinier and Funke), we obtain an exact formula and an algebraic formula for its values. In contrast with the case of \( p(n) \), the convergence of our expression is non-trivial, and requires power savings estimates for weighted sums of Kloosterman sums for a multiplier in weight \( 1/2 \). These are proved with spectral methods (following an argument of Goldfeld-Sarnak) and depend on bounds for the smallest positive eigenvalue for the Laplacian on \( L^2(\Gamma_0(N) \backslash \mathbb{H}) \).

1. Introduction

Let \( p(n) \) denote the ordinary partition function. Hardy and Ramanujan [28] developed the circle method to prove the asymptotic formula

\[
p(n) \sim \frac{e^\pi \sqrt{24n}}{4\sqrt{3}n}.
\]

Building on their work, Rademacher [36, 37, 38] proved the famous exact formula

\[
p(n) = \frac{2\pi}{(24n-1)^{3/4}} \sum_{c=1}^{\infty} A_c(n) I_{\frac{3}{2}} \left( \frac{\pi \sqrt{24n-1}}{6c} \right), \tag{1.1}
\]

where \( I_\nu \) is the \( I \)-Bessel function, \( A_c(n) \) is the Kloosterman sum

\[
A_c(n) := \sum_{d \mod c \atop (d,c)=1} e^{ixs(d,c)}\left(-\frac{dn}{c}\right), \quad e(x) := e^{2\pi ix} \tag{1.2}
\]

and \( s(d,c) \) is the Dedekind sum

\[
s(d,c) := \sum_{r=1}^{c-1} \left( \frac{r}{c} - \left[ \frac{r}{c} \right] - \frac{1}{2} \right) \left( \frac{dr}{c} - \left[ \frac{dr}{c} \right] - \frac{1}{2} \right). \tag{1.3}
\]
The existence of formula (1.1) is made possible by the fact that the generating function for \( p(n) \) is a modular form, namely
\[
q^{-1/24} \sum_{n \geq 0} p(n)q^n = \frac{1}{\eta(\tau)}, \quad q := \exp(2\pi i \tau),
\]
where \( \eta(\tau) \) denotes the Dedekind eta function.

There are a number of ways to prove (1.1), which can be viewed as a formula for the Fourier coefficients of a modular form of weight \(-1/2\). For example, Pribitkin [17] obtained a proof using a modified Poincaré series which represents \( \eta^{-1}(\tau) \). A similar technique can be used to obtain general formulas for the coefficients of modular forms of negative weight (see Appendix D of [29] for a discussion of this method). The authors [1] recovered (1.1) from Poincaré series representing a weight 5/2 harmonic Maass form whose shadow is \( \eta^{-1}(\tau) \). As pointed out by Bruinier and Ono [16], the exact formula can be recovered from the algebraic formula (1.6) stated below (this was partially carried out by Dewar and Murty [18]). The equivalence of (1.1) and (1.6) (in a more general setting) is made explicit by Proposition 7 of [4]. We also mention that Bringmann and Ono [10] established an exact formula for the coefficients of the weight 1/2 mock theta function \( f(q) \); this proved conjectures of Dragonette [20] and Andrews [5].

The smallest parts function \( \text{spt}(n) \), introduced by Andrews in [6], counts the number of smallest parts in the partitions of \( n \). Andrews proved that the generating function for \( \text{spt}(n) \) is
\[
S(\tau) := \sum_{n \geq 1} \text{spt}(n)q^n = \prod_{n \geq 1} \frac{1}{1 - q^n} \left( \sum_{n \geq 1} \frac{nq^n}{1 - q^n} + \sum_{n \neq 0} \frac{(-1)^n q^{n(n+1)/2}}{(1 - q^n)^2} \right).
\]
Work of Bringmann [9] shows that \( S(\tau) \) is a component of a mock modular form of weight 3/2 whose shadow is the Dedekind eta-function; using the circle method, she obtained an asymptotic expansion for \( \text{spt}(n) \). In particular we have
\[
\text{spt}(n) \sim \frac{\sqrt{6n}}{\pi} p(n).
\]
Many authors have investigated the properties of the coefficients of this mock modular form, which is a prototype for modular forms of this type (see, e.g., [2, 3, 7, 22, 24, 25, 26, 34]).

In analogy with (1.1), we prove the following formula for \( \text{spt}(n) \).

**Theorem 1.** For all \( n \geq 1 \), we have
\[
\text{spt}(n) = \frac{\pi}{6} (24n - 1)^{1/2} \sum_{c=1}^{\infty} \frac{A_c(n)}{c} \left( I_{1/2} - I_{3/2} \right) \left( \frac{\pi \sqrt{24n - 1}}{6c} \right).
\]

The convergence of Rademacher’s formula for \( p(n) \) follows from elementary estimates on the size of the I-Bessel function and the trivial bound \( |A_c(n)| \leq c \). By contrast, the convergence of the series for \( \text{spt}(n) \) is quite subtle, and requires non-trivial estimates for weighted sums of the Kloosterman sum (1.2) (this is discussed in more detail below).

Formulas (1.1) and (1.4) express integers as infinite series involving values of transcendental functions. By contrast, Bruinier and Ono [16] (see also [11]) obtained a formula for \( p(n) \) as a finite sum of algebraic numbers. Let \( P(\tau) \) denote the \( \Gamma_0(6) \)-invariant function
\[
P(\tau) := -\frac{1}{2} \left( q \frac{d}{dq} + \frac{1}{2\pi y} \right) \frac{E_2(\tau) - 2E_2(2\tau) - 3E_2(3\tau) + 6E_2(6\tau)}{(\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau))^2}.
\]
For $n \geq 1$ define
\[ Q_{1-24n}^{(1)} := \left\{ ax^2 + bxy + cy^2 : b^2 - 4ac = 1 - 24n, 6 \mid a > 0, \text{ and } b \equiv 1 \mod 12 \right\}. \]
The group $\Gamma := \Gamma_0(6)/\left\{ \pm 1 \right\}$ acts on this set. For each $Q \in Q_{1-24n}^{(1)}$ let $\tau_Q$ denote the root of $Q(\tau, 1)$ in the upper-half plane $\mathbb{H}$. Bruinier and Ono showed that
\[
p(n) = \frac{1}{24n - 1} \sum_{Q \in \Gamma \setminus Q_{1-24n}^{(1)}} P(\tau_Q). \quad (1.6)
\]
We obtain an analogue of (1.6) for $spt(n)$. Define the weakly holomorphic modular function
\[
f(\tau) := \frac{1}{24} \frac{E_4(\tau) - 4E_4(2\tau) - 9E_4(3\tau) + 36E_4(6\tau)}{(\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau))^2}. \quad (1.7)
\]
Then we have the following algebraic formula.

**Theorem 2.** For all $n \geq 1$, we have
\[
spt(n) = \frac{1}{12} \sum_{Q \in \Gamma \setminus Q_{1-24n}^{(1)}} (f(\tau_Q) - P(\tau_Q)) \cdot (1.8)
\]
Bruinier and Ono showed that the values $P(\tau_Q)$ are algebraic numbers with bounded denominators, and the classical theory of complex multiplication implies that the values $f(\tau_Q)$ are algebraic as well (see, for instance, Section 6.1 of [44]).

**Example.** We illustrate the simplest case of Theorem 2. The class number of $Q(\sqrt{-23})$ is 3, so $\Gamma \setminus Q_{-23}^{(1)}$ consists of 3 classes. These are represented by the forms
\[
Q_1 = 6x^2 + xy + y^2, \quad Q_2 = 12x^2 + 13xy + 4y^2, \quad Q_3 = 18x^2 + 25xy + 9y^2,
\]
whose roots are
\[
\tau_1 = \frac{-1 + \sqrt{-23}}{12}, \quad \tau_2 = \frac{-13 + \sqrt{-23}}{24}, \quad \tau_3 = \frac{-25 + \sqrt{-23}}{36}.
\]
Let $g = f - P$. Since the values $\{g(\tau_k)\}$ are conjugate algebraic numbers, we compute the class polynomial
\[
\prod_{k=1,2,3} (x - g(\tau_k)) = x^3 - 12x^2 - \frac{1008}{23}x - \frac{1728}{23} \quad (1.9)
\]
by approximating each $g(\tau_k)$ using (1.5) and (1.7). This shows that $spt(1) = 1$. Computing the roots of the polynomial in (1.9) gives the values
\[
g(\tau_1) = 4 \left( 1 + \frac{2}{23} \beta + \frac{22}{\beta} \right), \quad g(\tau_2) = 4 \left( 1 + \frac{2}{23} \zeta_3 \beta + \frac{22 \zeta_3^2}{\beta} \right), \quad g(\tau_3) = f(\tau_2),
\]
where $\zeta_3 := e^{2\pi i/3}$ and
\[
\beta := \sqrt[3]{\frac{23}{2} \left( 391 + 21\sqrt{69} \right)}.
\]
We return to the problem of obtaining estimates for weighted sums of the \( A_c(n) \), which is of independent interest. Define
\[
A_n(x) := \sum_{c \leq x} \frac{A_c(n)}{c}.
\]
Selberg’s formula
\[
A_c(n) = \sqrt{\frac{c}{3}} \sum_{\ell \mod 2c, \ell^2 + \ell = -n(c)} (-1)^\ell \cos \left( \frac{6\ell + 1}{6c} \pi \right),
\]
which was proved by Whiteman in \cite{Whiteman1955}, shows that for fixed \( n \) we have
\[
A_c(n) \ll n^{\frac{1}{2} + \epsilon}
\]
for any \( \epsilon > 0 \). This is the analogue of the Weil bound (see \cite{Weil1948}) for ordinary Kloosterman sums; it implies that \( A_n(x) \ll n^{\frac{1}{2} + \epsilon} \). Since this estimate is not sufficient to prove the convergence of the series in (1.4), we require a power savings estimate for \( A_n(x) \). In Section 6 we prove a result which has the following corollary.

**Theorem 3.** Suppose that \( -n = \frac{k(3k^2 + 1)}{2} \) is a pentagonal number. Then for any \( \epsilon > 0 \) we have
\[
\sum_{c \leq x} \frac{A_c(n)}{c} = (-1)^k \frac{4\sqrt{3}}{\pi} x^{\frac{1}{2}} + O \left( x^{\frac{1}{6} + \epsilon} \right).
\]
If \( -n \) is not pentagonal then we have
\[
\sum_{c \leq x} \frac{A_c(n)}{c} = O \left( x^{\frac{1}{6} + \epsilon} \right).
\]
The implied constants depend on \( n \) and \( \epsilon \).

This can be compared with Kuznetsov’s bound \cite{Kuznetsov1986}
\[
\sum_{c \leq x} \frac{k(m, n; c)}{c} \ll_{m,n} x^{\frac{1}{6}} (\log x)^{\frac{1}{3}}
\]
where \( k(m, n, c) \) is the ordinary Kloosterman sum defined in (4.3) below.

2. Preliminaries

We briefly introduce some of the objects which we will require.

2.1. **Quadratic forms and Atkin-Lehner involutions.** We require some facts about Atkin-Lehner involutions \cite{Atkin-Lehner1975}. For each divisor \( d \) of 6, we define the Atkin-Lehner involution \( W_d \) on \( M_0(\Gamma_0(6)) \) as the map \( f \mapsto f|_0 W_d \), where
\[
W_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad W_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 2 & -1 \\ 6 & -2 \end{pmatrix}, \quad W_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 3 & 1 \\ 6 & 3 \end{pmatrix}, \quad W_6 = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & -1 \\ 6 & 0 \end{pmatrix},
\]
and \((f|_0^\gamma)(\tau) := f(\gamma \tau)\). The normalizing factors are chosen so that \( W_d \in \text{SL}_2(\mathbb{R}) \), which will be convenient later. If \( d, d' | 6 \), then
\[
W_d W_{d'} = W_{dd'} \quad \text{mod} \quad (d, d')^2.
\]
Suppose that \( r \in \{1, 5, 7, 11\} \) and that \( D > 0 \), and define
\[
Q^{(r)}_D := \left\{ ax^2 + bxy + cy^2 : b^2 - 4ac = -D, \ 6 | a > 0, \text{ and } b \equiv r \mod 12 \right\}.
\]
Let \( \Gamma^*_0(6) \subset \text{SL}_2(\mathbb{R}) \) denote the group generated by \( \Gamma_0(6) \) and the Atkin-Lehner involutions \( W_d \) for \( d | 6 \). Matrices \( g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma^*_0(6) \) act on binary quadratic forms on the left by
\[
gQ(x, y) := Q(\delta x - \beta y, -\gamma x + \alpha y).
\]
This action is compatible with the action \( g\tau := \frac{\alpha \tau + \beta}{\gamma \tau + \delta} \) on the root \( \tau_Q \in \mathbb{H} \) of \( Q(\tau, 1) \): for \( g \in \Gamma^*_0(6) \), we have
\[
g\tau_Q = \tau_{gQ}.
\]
(2.3)

Define
\[
Q^-_D := \bigcup_{r \in \{1, 5, 7, 11\}} Q^{(r)}_D.
\]
A computation involving (2.1) and (2.2) shows that
\[
W_d : Q^-_D \leftrightarrow Q^-_D
\]
is a bijection, where
\[
r' \equiv (2d\mu(d) - 1)r \mod 12.
\]
(2.5)
For each \( r \), we have
\[
Q^-_D = \bigcup_{d | 6} W_d Q^{(r)}_D.
\]
(2.6)

2.2. **Quadratic spaces of signature** \((1, 2)\). The proof of Theorem 2 uses a theta lift of Bruinier-Funke associated to an isotropic rational quadratic space of signature \((1, 2)\). To access the necessary results requires some background, which we develop briefly in the next two subsections. For further details, see [15, 13].

Let \( V \) be an isotropic rational quadratic space of signature \((1, 2)\) with non-degenerate symmetric bilinear form \((\ , \ )\). Let the positive square-free integer \( d \) denote the discriminant of the quadratic form \( q \) given by \( q(v) = \frac{1}{2}(v, v) \). We may view \( V \) as the subspace of pure quaternions with oriented basis
\[
\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}
\]
in the quaternion algebra \( M_2(\mathbb{Q}) \); in other words we have
\[
V = \left\{ X = \begin{pmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{pmatrix} : x_i \in \mathbb{Q} \right\}.
\]
With this identification we have
\[
q(X) = d \det(X), \quad (X, Y) = -d \text{Tr}(XY).
\]
We identify \( G := \text{Spin}(V) \) with \( \text{SL}_2(\mathbb{Q}) \) and \( \overline{G} \simeq \text{PSL}_2(\mathbb{Q}) \) with its image in \( \text{SO}(V) \). The group \( G \) acts on \( V \) by conjugation; we write
\[
g.X := gXg^{-1}.
\]
The group \( G(\mathbb{R}) \) acts transitively on the Grassmannian \( D \) of positive lines in \( V \):
\[
D := \{ z \subseteq V(\mathbb{R}) : \text{dim } z = 1 \text{ and } q|_z > 0 \}.
\]
Choosing the base point \( z_0 = \text{span} \left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \in \mathbb{D} \), we find that \( z_0 \) is stabilized by \( \text{SO}_2(\mathbb{R}) \), so that

\[
\mathbb{D} \simeq \text{SO}_2(\mathbb{R}) \backslash G(\mathbb{R})
\]

is a Hermitian symmetric space.

An explicit isomorphism \( \mathbb{H} \simeq \mathbb{D} \) can be described as follows. For \( \tau = x + iy \in \mathbb{H} \), let

\[
g_\tau := \frac{1}{\sqrt{y}} \begin{pmatrix} x & -y \\ 1 & 0 \end{pmatrix} \in G(\mathbb{R}),
\]

so that \( g_\tau i = \tau \). Then define

\[
X(\tau) := g_\tau \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \frac{1}{y} \begin{pmatrix} -x & x^2 + y^2 \\ -1 & x \end{pmatrix},
\]

and define an isomorphism \( \mathbb{H} \to \mathbb{D} \) by \( \tau \mapsto \text{span}(X(\tau)) \). We have

\[
X(g\tau) = g.X(\tau)
\]

for all \( g \in \text{SL}_2(\mathbb{R}) \).

Let \( L \subseteq V(\mathbb{Q}) \) be the lattice

\[
L := \left\{ \begin{pmatrix} b & c/6 \\ -a & -b \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}.
\]

The dual lattice is

\[
L' = \left\{ \begin{pmatrix} b/12 & c/6 \\ -a & -b/12 \end{pmatrix} : a, b, c \in \mathbb{Z} \right\},
\]

and we identify \( L'/L \) with \( \mathbb{Z}/12\mathbb{Z} \).

The group \( \Gamma_0(6) \subseteq \text{Spin}(L) \) fixes \( L \). Let

\[
\{ \mathbf{e}_h : h \in \mathbb{Z}/12\mathbb{Z} \}
\]
denote the standard basis of the group ring \( \mathbb{C}[L'/L] \). A computation shows that matrices \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0^*(6) \) act on \( \mathbb{C}[L'/L] \) by

\[
g.\mathbf{e}_h = \mathbf{e}_{(1 + 2bc)h}.
\]

In particular, if \( g \in \Gamma_0(6) \), then \( g \) acts trivially on \( L'/L \).

Let

\[
M := \Gamma_0(6) \backslash \mathbb{D}
\]

be the modular curve. If \( X \in V(\mathbb{Q}) \) has positive length, then we define

\[
D_X := \text{span}(X) \subseteq \mathbb{D}.
\]

For each positive rational number \( m \) and each \( h \in L'/L \), define

\[
L_{h,m} := \{ X \in L + h : q(X) = m \}.
\]

Then \( \Gamma_0(6) \) acts on \( L_{h,m} \) with finitely many orbits.

The set \( \text{Iso}(V) \) of isotropic lines in \( V(\mathbb{Q}) \) is identified with the cusps of \( G(\mathbb{Q}) \) via the map

\[
(\alpha : \beta) \mapsto \text{span} \left( \begin{pmatrix} -\alpha \beta & \alpha^2 \\ -\beta^2 & \alpha \beta \end{pmatrix} \right).
\]
The cusps of $M$ are the $\Gamma_0(6)$ classes of $\text{Iso}(V)$; these are represented by the lines $\ell_j := \text{span}(X_j)$, where

$$X_0 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X_1 := \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad X_2 := \begin{pmatrix} -2 & 1 \\ -4 & 2 \end{pmatrix}, \quad \text{and} \quad X_3 := \begin{pmatrix} -3 & 1 \\ -9 & 3 \end{pmatrix}.$$ 

For each $\ell \in \text{Iso}(V)$ choose $\sigma_\ell \in \text{SL}_2(\mathbb{Z})$ with $\sigma_\ell \ell_0 = \ell$, and let $\alpha_\ell$ be the width of the cusp $\ell$. For each $\ell$, there is a positive rational number $\beta_\ell$ such that

$$\ell_0 \cap \sigma_\ell^{-1} L = \begin{pmatrix} 0 & \beta_\ell \mathbb{Z} \\ 0 & 0 \end{pmatrix},$$

and we define $\varepsilon_\ell := \alpha_\ell / \beta_\ell$.

Suppose that $q(X) < 0$ and that $Q(X) \in -6(\mathbb{Q}^\times)^2$. Then (see [23, Lemma 3.6]) $X$ is orthogonal to two isotropic lines, $\text{span}(Y)$ and $\text{span}(\tilde{Y})$. We associate $\ell_X := \text{span}(Y)$ to $X$ if $(X, Y, \tilde{Y})$ is a positively oriented basis of $V$. We then have $\ell_{-X} = \text{span}(\tilde{Y})$. For each $\ell$, define

$$L_{h, -6m^2, \ell} := \{ X \in L_{h, -6m^2 : \ell = \ell} \}.$$ 

Then $\Gamma_0(6)$ acts on these sets, and equation (4.7) of [15] shows that

$$v_\ell(h, -6m^2) := |\Gamma_0(6) \setminus L_{h, -6m^2, \ell}| = \begin{cases} 2m\varepsilon_\ell & \text{if } L_{h, -6m^2, \ell} \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases} \quad (2.11)$$

2.3. Harmonic Maass forms and the theta lift. Let $\text{Mp}_2(\mathbb{R})$ denote the metaplectic two-fold cover of $\text{SL}_2(\mathbb{R})$. The elements of this group are pairs $(M, \phi(\tau))$, where

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R}),$$

and $\phi : \mathbb{H} \to \mathbb{C}$ is a holomorphic function satisfying $\phi(\tau)^2 = (c\tau + d)$.

Let $\tilde{\Gamma}$ denote the inverse image of $\text{SL}_2(\mathbb{Z})$ under the covering map; this group is generated by

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau}. \quad \text{We fix the lattice } L \text{ defined by } (2.9). \text{ The Weil representation (see Chapter 1 of } [12])$$

is defined by

$$\rho_L : \text{Mp}_2(\mathbb{Z}) \to \text{GL}(\mathbb{C}[L'/L])$$

is defined by

$$\rho_L(T)e_h = e\left(\frac{-h^2}{24}\right)e_h,$$

$$\rho_L(S)e_h = \frac{\sqrt{i}}{\sqrt{12}} \sum_{h' \in L'/L} e\left(\frac{hh'}{12}\right)e_{h'}.$$ \quad (2.12)

Denote by $H_{k, \rho_L}$ the space of weak harmonic Maass forms of weight $k$ for the representation $\rho_L$; these are functions $F : \mathbb{H} \to \mathbb{C}[L'/L]$ which satisfy the following conditions:

(1) For $(\gamma, \phi) \in \text{Mp}_2(\mathbb{Z})$,

$$f(\gamma \tau) = \phi(\tau)^{2k} \rho_L(\gamma, \phi) f(\tau).$$

(2) $\Delta_k f = 0$, where

$$\Delta_k = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right).$$
(3) \( f \) has at most linear exponential growth at \( \infty \).

(4) \( \xi_k f \) is holomorphic at \( \infty \), where

\[
\xi_k = 2iy^k \frac{\partial}{\partial \tau}.
\]

Let \( M'_0(\Gamma_0(6)) \) be the space of weakly holomorphic modular functions on \( \Gamma_0(6) \). The theta lift of \( f \in M'_0(\Gamma_0(6)) \) is given by

\[
I(\tau, f) = \int_M f(z) \Theta(\tau, z) = \sum_{h \in L'/L} I_h(\tau, f) e_h, \tag{2.13}
\]

with

\[
I_h(\tau, f) := \int_M f(z) \theta_h(\tau, z), \tag{2.14}
\]

where \( \theta_h(\tau, z) \) and \( \Theta(\tau, z) \) are defined in §3.2 of [15].

We have

\[
I(\tau, f) \in H_{\frac{3}{2}, \rho_L}.
\]

To see this, note that the transformation properties follow from those of the theta function (§3.2 of [15]). The other conditions follow from the explicit description of the Fourier expansion of \( I(\tau, f) \) given in Theorem 4.5 of [15].

By (3.7) and (3.9) of [15] and (2.8) we have the relation

\[
\theta_h(\tau, gz) = \theta_{g^{-1}h}(\tau, z)
\]

for any \( g \in \Gamma_0(6) \). From this and (2.14) it follows that

\[
I_{gh}(\tau, f) = I_h(\tau, f) |_{0} g. \tag{2.15}
\]

By Theorem 4.5 of [15] we have

\[
I_h(\tau, f) = \sum_{m \geq 0} t_f(h, m) q^m + \sum_{m > 0} t_f(h, -6m^2) q^{-6m^2} + N(\tau). \tag{2.16}
\]

A formula for the non-holomorphic part \( N(\tau) \) is given in [15] but we do not include it here.

Let \( \Gamma := \Gamma_0(6)/\{\pm 1\} \). The terms of positive index \( m \) in (2.16) are given by

\[
t_f(h, m) = \sum_{X \in \Gamma \setminus L_{h,m}} |\Gamma_X|^{-1} f(D_X), \tag{2.17}
\]

where \( \Gamma_X \subseteq \Gamma \) is the stabilizer of \( X \). Let

\[
f(\sigma_\ell \tau) = \sum_{n \in \frac{1}{\sigma_\ell} \mathbb{Z}} a_\ell(n) q^n
\]

denote the Fourier expansion of \( f \) at the cusp \( \ell \). By Proposition 4.7 of [15], the terms of negative index \(-6m^2\) in (2.16) are given by

\[
t_f(h, -6m^2) = - \sum_{\ell \in \Gamma \setminus \text{Is}(V)} \sum_{n \in \frac{2m}{\ell} \mathbb{N}} a_\ell(-n) \left( v_\ell(h, -6m^2) e \left( \frac{r_{h,\ell} n}{2m} \right) + v_\ell(h, -6m^2) e \left( \frac{r_{-h,\ell} n}{2m} \right) \right), \tag{2.18}
\]

where \( r_{h,\ell} \) is defined by

\[
\sigma_\ell^{-1} X = \begin{pmatrix} m & r_{h,\ell} \\ 0 & -m \end{pmatrix} \quad \text{for any } X \in L_{h,-6m^2,\ell}.
\]
Note that \( t_f(h, -6m^2) = 0 \) for \( m \) sufficiently large.

3. Proof of Theorem [2]

We now define \( f \in M_0^!(\Gamma_0(6)) \) by

\[
f(\tau) := \frac{1}{24} E_4(\tau) - 4E_4(2\tau) - 9E_4(3\tau) + 36E_4(6\tau) = \frac{1}{q} + 12 + 77q + \ldots
\]

and we define \( F(\tau) \) by

\[
F(\tau) := \sum_{n=1}^{\infty} spt(n)q^{n-\frac{1}{24}} - \frac{1}{12} \cdot \frac{E_2(\tau)}{\eta(\tau)} + \frac{\sqrt{3i}}{2\pi} \int_{-\tau}^{i\infty} \frac{\eta(w)}{(\tau + w)^{\frac{3}{2}}} dw
\]

so that

\[
s(n) = spt(n) + \frac{1}{12} (24n - 1)p(n).
\]

Work of Bringmann [9] shows that \( F(24\tau) \) is a harmonic Maass form on \( \Gamma_0(576) \) with character \( \left( \frac{12}{\tau} \right) \), and that \( F(24\tau) \) has eigenvalue \(-1\) under the Fricke involution \( W_{576} \). Using these facts we find that

\[
F(\tau + 1) = e \left( -\frac{1}{24^2} \right) F(\tau), \quad F(-1/\tau) = i^{\frac{1}{4}} \tau^{\frac{3}{2}} F(\tau).
\]

Now set

\[
\mathcal{F}(\tau) := \sum_{h \in L'/L} \left( \frac{12}{h} \right) F(\tau) \epsilon_h
\]

(we use the identification of \( L'/L \) with \( \mathbb{Z}/12\mathbb{Z} \) to define \( \left( \frac{12}{\tau} \right) \)). Using (3.3), (2.12), and the fact that

\[
\sum_{h \in L'/L} \left( \frac{12}{h} \right) e \left( \frac{hh'}{12} \right) = \left( \frac{12}{h'} \right) \sqrt{12},
\]

we find that

\[
\mathcal{F}(\tau) \in H_{3/2, \rho_L}.
\]

Proposition 4. We have \( I(\tau, f) = 24 \mathcal{F}(\tau) \).

Proof. For \( d \mid 6 \) we find that

\[
f|_0 W_d = \begin{cases} f & \text{if } d = 1, 6, \\ -f & \text{if } d = 2, 3. \end{cases}
\]

We first claim that for \( h \in L'/L \) we have

\[
I_h(\tau, f) = \left( \frac{12}{h} \right) I_1(\tau, f).
\]

When \( (h, 12) = 1 \) the claim follows from (2.10), (2.14), (2.15) and (3.5). If \( (h, 12) \neq 1 \) then by (2.10), \( h \) is fixed by either \( W_2 \) or \( W_3 \). In this case, (2.15) and (3.5) imply that \( I_h(\tau, f) = 0 \), and (3.6) follows.

Now let

\[
\mathcal{G}(\tau) \in H_{3/2, \rho_L}
\]
denote the difference of the two forms in the statement of the lemma. By (3.4) and (3.6), there is a function \( G \) such that

\[
G = \sum_{h \in \mathbb{L}/L} \left( \frac{12}{h} \right) G \varepsilon_h.
\]

Arguing as above, we find that \( G \) satisfies the transformation laws described by (3.3).

Using (2.18), we compute the principal part of \( I_1(\tau, f) \) as follows. Since

\[
\beta_{\ell_0} = \frac{1}{6}, \quad \beta_{\ell_1} = 1, \quad \beta_{\ell_2} = \frac{1}{2}, \quad \text{and} \quad \beta_{\ell_3} = \frac{1}{3},
\]

we see that \( t_f(1, -6m^2) = 0 \) for \( m > \frac{1}{12} \). A computation shows that

\[
v_{\ell}(1, -\frac{1}{24}) = \begin{cases} 1 & \text{if } \ell = \ell_0, \\ 0 & \text{otherwise}, \end{cases} \quad \text{and} \quad v_{\ell}(-1, -\frac{1}{24}) = \begin{cases} 1 & \text{if } \ell = \ell_1, \\ 0 & \text{otherwise}. \end{cases}
\]

Furthermore, \( r_{1, \ell_0} = r_{-1, \ell_3} = 0 \), so \( t_f(1, -\frac{1}{24}) = -2 \). Therefore the principal part of \( I_1(\tau, f) \) is given by \(-2q^{-1/24}\), which agrees with the principal part of \( 24F(\tau) \). It follows from (3.4) and (3.6) that \( G(\tau) \) has trivial principal part.

Let \( g = \xi_2^2 G \). Then \( g \) is holomorphic on \( \mathbb{H} \) and at \( \infty \), and we have

\[
g(\tau + 1) = e \left( \frac{1}{24} \right) g(\tau) \quad \text{and} \quad g(-1/\tau) = (-i)^{\frac{1}{2}} \tau^{\frac{1}{2}} g(\tau).
\]

It follows that \( g(\tau) \) is in fact a constant multiple of \( \eta(\tau) \). Theorem 3.6 of [14] then implies that \( G(\tau) \) is a holomorphic modular form. It follows that \( G = 0 \); otherwise the product of \( G \) with \( \eta \) would be a non-zero modular form of weight 2 for \( \text{SL}_2(\mathbb{Z}) \).

**Remark.** A more direct approach to the proof of Proposition 4 is to compute \( N(\tau) \) using the formula in Theorem 4.5 of [15] and to match it directly to the non-holomorphic part of (3.1).

**Proof of Theorem 2.** Let \( \Gamma := \Gamma_0(6)/\{\pm 1\} \). Suppose that \( n \geq 1 \), and let \( \hat{n} := n - \frac{1}{24} \). By (2.16), (2.17), (3.1), and Proposition 4 we have

\[
s(n) = \frac{1}{24} \sum_{X \in \Gamma \backslash L_{1, \hat{n}}} |\Gamma_X|^{-1} f(D_X).
\]

Note that for each \( X \), we have \( D_X = D_X \), so we restrict our attention to the subset

\[
L_{1, \hat{n}}^+ := \left\{ \left( \begin{array}{cc} b + \frac{12}{12} & \frac{c}{6} \\ -a & b - \frac{1}{12} \end{array} \right) : a, b, c \in \mathbb{Z}, \ a > 0, \text{ and } q(X) = \hat{n} \right\}.
\]

There is a natural bijection between \( L_{1, \hat{n}}^+ \) and \( Q_{1-24n}^{(1)} \) given by

\[
X = \left( \begin{array}{cc} b + \frac{12}{12} & \frac{c}{6} \\ -a & b - \frac{1}{12} \end{array} \right) \leftrightarrow Q_X := [6a, 12b + 1, c].
\]

It is easy to check that the action of \( \Gamma \) on \( L_{1, \hat{n}}^+ \) translates under this bijection to the usual action of \( \Gamma \) on \( Q_{1-24n}^{(1)} \). Since the stabilizer of \( Q \) is trivial for every \( Q \in Q_{1-24n}^{(1)} \), we have \( |\Gamma_X| = 1 \) for all \( X \in \Gamma \backslash L_{1, \hat{n}}^+ \). A computation involving (2.7) shows that \( D_X \mapsto \tau_{Q_X} \) under the isomorphism \( \mathbb{D} \cong \mathbb{H} \). Thus we have

\[
s(n) = \frac{1}{12} \sum_{Q \in \Gamma \backslash Q_{1-24n}^{(1)}} f(\tau_Q).
\]
Theorem follows from (1.6) and (3.2). □

4. Poincaré series and the function \( f(\tau) \)

In this section we construct the modular function \( f(\tau) \) in terms of weak Maass-Poincaré series. To this end, we construct an auxiliary function \( f(\tau, s) \), defined for \( \text{Re}(s) > 1 \) and compute its Fourier expansion to obtain an analytic continuation of \( f(\tau, s) \) to \( \text{Re}(s) > \frac{3}{4} \). We then show that \( f(\tau, 1) = f(\tau) \).

Throughout this section, we let
\[
\Gamma := \Gamma_0(6)/\{\pm I\},
\]
and we write \( \tau = x + iy \) and \( s = \sigma + it \). Letting \( \Gamma_\infty = \{ (1 \ 0 \ \ast \ 0) \} \) denote the stabilizer of \( \infty \), we define
\[
F(\tau, s) := \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} \phi_s(\text{Im} \gamma \tau) e(-\text{Re} \gamma \tau),
\]
where
\[
\phi_s(y) := 2\pi \sqrt{y} I_{s - \frac{1}{2}}(2\pi y).
\]

Since \( \phi_s(y) \ll y^\sigma \) as \( y \to 0 \), we have
\[
F(\tau, s) \ll y^\sigma \sum_{(c,d) \in \Gamma_\infty \setminus \Gamma} |c\tau + d|^{-2\sigma},
\]
so \( F(\tau, s) \) converges normally for \( \sigma > 1 \). A computation involving (13.14.1) of [19] shows that
\[
\Delta_0 F(\tau, s) = s(1 - s) F(\tau, s).
\]

Let \( \mu \) denote the Möbius function, and define
\[
f(\tau, s) := \sum_{r|6} \mu(r) F(W_r \tau, s).
\]
The following proposition gives the Fourier expansion of \( f(\tau, s) \) in terms of the ordinary Kloosterman sum
\[
k(m, n; c) := \sum_{d \mod c \atop (c,d) = 1} e\left(\frac{md + nd}{c}\right)
\]
and the \( I, J, \) and \( K \)-Bessel functions.

**Proposition 5.** For \( \sigma > 1 \) we have
\[
f(\tau, s) = 2\pi \sqrt{y} I_{s - \frac{1}{2}}(2\pi y)e(-x) + a_s(0)y^{1-s} + 2\sqrt{y} \sum_{n \neq 0} a_s(n) K_{s - \frac{1}{2}}(2\pi |n| y)e(nx),
\]
where
\[
a_s(0) = \frac{2\pi^{s+1}}{(s - \frac{1}{2})\Gamma(s)} \sum_{r|6} \mu(r) \sum_{0 < c \equiv 0(6/r) \atop (c,r) = 1} \frac{k(-r,0;c)}{(c\sqrt{r})^{2s}},
\]
\[
a_s(n) = 2\pi \sum_{\nu | 6} \mu(\nu) \sum_{0 < c \equiv 0(6/r) \atop (c,\nu) = 1} k(-\nu, n; c) \times \begin{cases} I_{2s-1} \left( \frac{4\pi \sqrt{n}}{c\sqrt{r}} \right) & \text{if } n > 0, \\ J_{2s-1} \left( \frac{4\pi \sqrt{|n|}}{c\sqrt{r}} \right) & \text{if } n < 0. \end{cases}
\]

**Proof.** The function \( f(\tau, s) - \phi_s(y)e(-x) \) has at most polynomial growth as \( y \to \infty \). For \( r \mid 6, r \neq 1 \), a complete set of representatives for \( \Gamma_{\infty} \setminus \Gamma W_r \) is given by

\[
\left\{ \left( \frac{ra}{rc}, \frac{rd}{r} \right) : c > 0, \gcd(c, rd) = 1, (6/r) \mid c, a \in \{1, \ldots, c-1\}, a \equiv rd \pmod{c} \right\}.
\]

So we have the Fourier expansion

\[
f(\tau, s) = \phi_s(y)e(-x) + \sum_{n \in \mathbb{Z}} \sum_{r \mid 6} \mu(\nu) A_r(n, y, s)e(nx),
\]

where

\[
A_r(n, y, s) = \sum_{\gamma \in \Gamma_{\infty} \setminus \Gamma W_r \atop c(\gamma) > 0} \int_0^1 \phi_s \left( \text{Im} \gamma \tau \right) e(-\text{Re} \gamma \tau) e(-nx) \, dx.
\]

For \( \gamma \in \Gamma_{\infty} \setminus \Gamma W_r \) with \( c(\gamma) > 0 \) we write

\[
\gamma \tau = \frac{ra}{rc} \tau + * = \frac{a}{c} - \frac{1}{rc^2(\tau + d/c)}
\]

and make the change of variable \( x \to x - d/c \) to obtain

\[
A_r(n, y, s) = \sum_{0 < c \equiv 0(6/r) \atop (c,rd) = 1} e \left( -\frac{a + nd}{c} \right) \int_{d/c}^{1+d/c} \phi_s \left( \frac{y}{rc^2|\tau|^2} \right) e \left( \frac{x}{rc^2|\tau|^2} - nx \right) \, dx.
\]

Since \( a \equiv rd \pmod{c} \) we have

\[
\sum_{d \mod{c} \atop (d, c) = 1} e \left( -\frac{a + nd}{c} \right) = k(-\tau, n; c),
\]

so that

\[
A_r(n, y, s) = 2\pi \sum_{0 < c \equiv 0(6/r) \atop (r,c) = 1} \frac{k(-\tau, n; c)}{c\sqrt{r}} y^2 \int_{-\infty}^{\infty} \left| \tau \right|^{-1} I_{s-\frac{1}{2}} \left( \frac{2\pi y}{rc^2|\tau|^2} \right) e \left( \frac{x}{rc^2|\tau|^2} - nx \right) \, dx.
\]

Let \( I \) denote the integral above. We make the substitution \( x = yu \) and set \( A = \frac{1}{rc^2y} \) and \( B = -ny \), so that

\[
I = \int_{-\infty}^{\infty} (u^2 + 1)^{-\frac{1}{2}} I_{s-\frac{1}{2}} \left( \frac{2\pi A}{u^2 + 1} \right) e \left( \frac{Au}{u^2 + 1} + Bu \right) \, du.
\]

Using Lemma 5.5 on page 357 and (xiv) and (xv) on page 345 of [29], and the fact that

\[
2^{1-2s} \sqrt{\pi} \frac{\Gamma(2s)}{\Gamma(s + \frac{1}{2}) \Gamma(s)} = 1,
\]

the result follows.
we find that
\[
I = \begin{cases} 
2K_{s-\frac{1}{2}}(2\pi B)J_{2s-1}\left(4\pi\sqrt{AB}\right) & \text{if } B > 0, \\
\frac{\pi^s A^{s-\frac{1}{2}}}{(s-\frac{1}{2})\Gamma(s)} & \text{if } B = 0, \\
2K_{s-\frac{1}{2}}(2\pi |B|)J_{2s-1}\left(4\pi\sqrt{A|B|}\right) & \text{if } B < 0.
\end{cases}
\]
The proposition follows. \[\square\]

The function \( f(\tau, s) \) has an analytic continuation to \( \text{Re}(s) > \frac{3}{4} \), as we now show. Suppose that \( \frac{3}{4} < \sigma_0 < \frac{3}{2} \), and fix \( \epsilon_0 \) with \( 0 < \epsilon_0 < 2\sigma_0 - \frac{3}{2} \). We will show that the Fourier expansion in Proposition 5 converges absolutely and uniformly for \( s \) in the region \( R \) defined by \( \sigma_0 \leq \sigma \leq \frac{3}{2} \), \( |t| \leq T \) (the estimates below are for \( s \in R \)). Using the Weil bound \[\text{(11)}\] for Kloosterman sums we have
\[
k(a, b; c) \ll \gcd(a, b, c)^{\frac{1}{2}+\epsilon_0},
\]
from which
\[
a_s(0) \ll \sum_{c > 0} c^{-2\sigma_0 + \frac{1}{2} + \epsilon_0} \ll 1.
\]
By (10.40.2) of \[\text{(19)}\] we have
\[
\sqrt{|y|} K_{s-\frac{1}{2}}(2\pi |n| y) \ll |n|^{-\frac{3}{2}} e^{-2\pi |n| y} \text{ as } n \to \infty. \tag{4.4}
\]
From (10.40.1) and (10.30.1) of \[\text{(19)}\] we have
\[
I_{2s-1}(x) \ll \frac{e^x}{\sqrt{x}} \text{ as } x \to \infty,
\]
\[
I_{2s-1}(x) \ll x^{2s-1} \text{ as } x \to 0.
\]
Suppose that \( n > 0 \). Taking absolute values in the series defining \( a_s(n) \), we find that
\[
a_s(n) \ll n^{-\frac{1}{4}} \sum_{c < \sqrt{n}} e^{\sigma_0} e^{\frac{3\pi n}{c}} + n \sum_{c \geq \sqrt{n}} c^{-2\sigma_0 + \frac{1}{2} + \epsilon_0} \ll e^{4\pi\sqrt{n}} n^{\frac{1}{4} + \epsilon_0} + n \ll e^{6\pi\sqrt{n}}. \tag{4.5}
\]
From (10.7.8) and (10.7.3) of \[\text{(19)}\] we have
\[
J_{2s-1}(x) \ll \frac{1}{\sqrt{x}} \text{ as } x \to \infty,
\]
\[
J_{2s-1}(x) \ll x^{2s-1} \text{ as } x \to 0.
\]
Arguing as above we find that for \( n < 0 \) we have \( a_s(n) \ll n \). With (4.5) and (4.4), this shows that the Fourier expansion converges absolutely and uniformly for \( s \in R \). This provides the analytic continuation of \( f(\tau, s) \) to \( \sigma > \frac{3}{4} \).

Since
\[
2\sqrt{|y|} K_{\frac{1}{2}}(2\pi |n| y) = |n|^{-\frac{1}{2}} e^{-2\pi |n| y} \quad \text{and} \quad 2\pi \sqrt{|y|} I_{\frac{1}{2}}(2\pi y) = 2 \sinh(2\pi y),
\]
the Fourier expansion of \( f(\tau, 1) \) is
\[
f(\tau, 1) = e(-\tau) + a_1(0) + \sum_{n > 0} \frac{a_1(n)}{\sqrt{|n|}} e(n\tau) - e(-\tau) + \sum_{n < 0} \frac{a_1(n)}{\sqrt{|n|}} e(n\tau). \tag{4.6}
\]
Using (4.2) and (4.6) we find that $\xi_0 f(\tau, 1)$ is an element of $S_2(\Gamma_0(6)) = \{0\}$. Therefore $f(\tau, 1)$ is holomorphic on $\mathbb{H}$. Since the principal parts of $f(\tau, 1)$ and $f(\tau)$ are equal, we conclude that
\[ f(\tau, 1) = f(\tau), \quad (4.7) \]
as desired.

5. Proof of Theorem 1

By Theorem 2 and equations (4.7), (3.2), and (2.3) we have
\[ s(n) = \frac{1}{12} \sum_{Q \in \Gamma \backslash Q^{(1)}_{1-24n}} f(\tau_Q) \]
\[ = \lim_{s \to 1^+} \frac{1}{12} \sum_{Q \in \Gamma \backslash Q^{(1)}_{1-24n}} \sum_{d \mid 6} \sum_{\gamma \in \Gamma \backslash \Gamma_\infty} \mu(d) \phi_s(\text{Im} \tau_{W_d Q}) e(-\text{Re} \tau_{W_d Q}). \]

By (2.4) and (2.6) the map $(\gamma, d, Q) \mapsto \gamma W_d Q$ is a bijection
\[ \Gamma_\infty \backslash \Gamma \times \{1, 2, 3, 6\} \times \Gamma \backslash Q^{(1)}_{1-24n} \longleftrightarrow \Gamma_\infty \backslash Q_{1-24n}. \quad (5.1) \]

If $Q \in Q^{(1)}_{1-24n}$ and $Q' = W_d Q = [a, b, *]$ then $\mu(d) = \left( \frac{12}{d} \right)$ by (2.4) and (2.5). Thus we have
\[ s(n) = \lim_{s \to 1^+} \frac{1}{12} \sum_{Q \in \Gamma \backslash Q_{1-24n}} \frac{1}{2} \sum_{b \equiv \pm 1-24n(4a)} \frac{a^{b/2}I_{s-1/2}}{a} \left( \frac{\pi \sqrt{24n-1} - 6}{a} \right) \sum_{b \equiv \pm 1-24n(4a)} \frac{12}{b} e\left( \frac{b}{2a} \right). \]

Since $\left( \frac{1}{4} \right) [a, b, *] = [a, b - 2ka, *]$, there is a bijection
\[ \Gamma_\infty \backslash Q_{1-24n} \longleftrightarrow \{(a, b) : a > 0, \ 6 \mid a, \ 0 \leq b < 2a, \ b^2 \equiv 1 - 24n \mod 4a\}, \]
which, together with (4.1), gives
\[ s(n) = \lim_{s \to 1^+} \frac{\pi}{6\sqrt{2}} (24n - 1)^{1/4} \sum_{6 \mid a > 0} a^{-1/2} I_{s-1/2} \left( \frac{\pi \sqrt{24n-1}}{6} \right) \sum_{b \equiv \pm 1-24n(4a)} \frac{12}{b} e\left( \frac{b}{2a} \right). \]

Writing $a = 6c$, we see that the inner sum is equal to
\[ \frac{1}{2} \sum_{b \equiv \pm 1-24n(24c)} \frac{12}{b} e\left( \frac{-b}{12c} \right). \]

By Proposition 6 of [4] (see also [42]), this equals
\[ \frac{2\sqrt{3}}{\sqrt{c}} A_c(n). \]

We conclude that
\[ s(n) = \lim_{s \to 1^+} \frac{\pi}{6} (24n - 1)^{1/4} \sum_{c=1}^{\infty} \frac{A_c(n)}{c} I_{s-1/2} \left( \frac{\pi \sqrt{24n-1}}{6c} \right). \quad (5.2) \]

We will use Theorem 3 to justify the interchange of the limit and the sum in formula (5.2), but first we require a straightforward lemma.
Lemma 6. For $a > 0$ and $\frac{1}{2} \leq \nu \leq \frac{3}{2}$, define $f_\nu(x) := I_\nu(a/x)$. Then

$$|f'_\nu(x)| \ll_a x^{-\nu-1} \quad \text{as } x \to \infty.$$  

Proof. By (10.29.1) of [19] we find that

$$I'_\nu(x) = \frac{1}{2} (I_{\nu-1}(x) + I_{\nu+1}(x)).$$

For fixed $x$, the function $I_\nu(x)$ is decreasing as a function of $\nu$. Therefore

$$|f'_\nu(x)| = \frac{a}{2x^2} (I_{\nu-1}(a/x) + I_{\nu+1}(a/x)) \leq \frac{a}{x^2} I_{\nu-1}(a/x). \quad (5.3)$$

From (10.30.1) of [19] we have

$$f_\nu(x) \ll_a x^{-\nu} \quad \text{as } x \to \infty \quad \text{for } \nu \in [1/2, 3/2]. \quad (5.4)$$

The lemma follows from (5.3) and (5.4). □

Set $a := \pi \sqrt{24n-1}/6$, suppose that $s \in [1, 2]$, and define

$$A_n(x) = \sum_{c \leq x} A_c(n) / c. \quad (5.5)$$

By Theorem 3 we have $A_n(x) \ll_{\epsilon,n} x^{3/4+\epsilon}$ for any $\epsilon > 0$. For large $N$, partial summation gives

$$\sum_{c > N} A_c(n) f_{s-1/2}(c) = \frac{\pi}{x} A_n(x) f_{s-1/2}(x) - A_n(N) f_{s-1/2}(N) - \int_N^\infty A_n(t) f_{s-1/2}(t) \, dt \ll_a N^{3/4-s+\epsilon} + \int_N^\infty t^{-3/4-s+\epsilon} \, dt \ll_a N^{-3/4+\epsilon}.$$  

It follows that the series (5.2) converges uniformly for $s \in [1, 2]$. Interchanging the limit and the sum gives Theorem 1. □

6. Sums of Kloosterman sums

In this section we will consider sums of Kloosterman sums $S(m, n, c, \chi)$ associated to a multiplier in weight $k$, which were studied when $m, n > 0$ by Goldfeld and Sarnak [27]. Work of Folsom-Ono [21] and Pribitkin [35] applies to the case of general $m$ and $n$. For completeness we record a general asymptotic formula here, referring the reader to [27] for details.

Let $\Gamma$ be a finite-index subgroup of $\text{SL}_2(\mathbb{Z})$ which contains $-I$. Suppose that $k \in \mathbb{R}$ and that $\chi$ is a multiplier on $\Gamma$ for the weight $k$. Suppose that $q$ is the smallest positive integer for which $\begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} \in \Gamma$ and define $\alpha \in [0, 1)$ by

$$\chi \left( \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} \right) = e(-\alpha).$$

For simplicity, we write

$$\tilde{n} := \frac{n - \alpha}{q}.$$

For $c > 0$, the generalized Kloosterman sum is given by

$$S(m, n, c, \chi) := \sum_{\substack{0 \leq a, d < qc \\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \gamma = (\begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix})}} \chi(\gamma) e \left( \frac{m a + \tilde{n} d}{c} \right),$$
and Selberg’s Kloosterman zeta function is defined as

$$Z_{m,n}(s,\chi) := \sum_{c>0} \frac{S(m,n,c,\chi)}{c^{2s}}.$$  

The space $L^2 (\Gamma \backslash \mathbb{H}, \chi, k)$ consists of functions $f : \mathbb{H} \to \mathbb{C}$ such that

$$f (\gamma \tau) = \chi (\gamma) \left( \frac{c \tau + d}{|c \tau + d|} \right)^k f (\tau)$$

for all $\gamma \in \Gamma$ and $\|f\| < \infty$, where

$$\|f\|^2 := \frac{1}{\text{Vol}(\Gamma \backslash \mathbb{H})} \int \int_{\Gamma \backslash \mathbb{H}} |f(\tau)|^2 \frac{dxdy}{y^2}.$$  

The operator

$$\tilde{\Delta}_k := y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - ik \left( y \frac{\partial}{\partial x} \right)$$

(which is not the operator $\Delta_k$ in §2.3) has a self-adjoint extension to $L^2 (\Gamma \backslash \mathbb{H}, \chi, k)$ with real spectrum. The asymptotic formula of [27] depends on the discrete spectrum, which we denote by

$$\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_\ell < \frac{1}{4}.$$  

For each $j$ let $u_j$ be the normalized Maass cusp form corresponding to $\lambda_j$ and define $s_j \in (1/2, 1)$ by

$$\lambda_j = s_j (1 - s_j).$$

Then we have the expansion

$$u_j (\tau) = \sum_{m=-\infty}^{\infty} \hat{u}_j(m,y)e(\tilde{m}x),$$

where

$$\hat{u}_j(m,y) = \begin{cases} 
\rho_j(m)W_{\frac{s_j}{2}} \text{sgn}(\tilde{m}),s_j - \frac{1}{2} (4\pi |\tilde{m}|y)e(\tilde{m}x) & \text{if } \tilde{m} \neq 0, \\
\rho_j(0)y^{s_j} + \rho_j'(0)y^{1-s_j} & \text{if } \tilde{m} = 0.
\end{cases}$$

Define

$$\beta := \limsup_{c \to \infty} \frac{\log |S(m,n,c,\chi)|}{\log c}.$$  

With this notation we have the following

**Proposition 7.** Suppose that $m > 0$ and that $n \in \mathbb{Z}$. For any $\epsilon > 0$ we have

$$\sum_{c \leq x} \frac{S(m,n,c,\chi)}{c} = \sum_{1/2 < s_j < 1} \tau_j(m,n) \frac{x^{2s_j - 1}}{2s_j - 1} + O \left( x^{\frac{3}{2} + \epsilon} \right),$$

where the sum runs over the exceptional values $s_j$ described above, and

$$\tau_j(m,n) = 2q^2 t^k \rho_j(m) \rho_j(n) \pi^{1-2s_j} (4\tilde{m} |\tilde{n}|)^{1-s_j} \frac{\Gamma(s_j + \text{sgn}(\tilde{n})\frac{k}{2}) \Gamma(2s_j - 1)}{\Gamma(s_j - \frac{k}{2})}.$$  

The implied constant depends on $k, \chi, m, n, \epsilon,$ and $\Gamma$. 
When \( n > 0 \) this is Theorem 2 of [27], but the constants in (3.2) of [27] differ from those in (6.3). Figure 1 in Section 7 gives an example which supports the accuracy of (6.3). The existence of such a formula is implicit in [21].

For the case when \( n \leq 0 \) we argue as in Lemma 2 of [27], relating \( Z_{m,n}(s, \chi) \) to the inner product of two Poincaré series. We compute

\[
\langle P_m(\tau, s, \chi, k), P_{1-n}(\tau, s + 2, \chi, -k) \rangle = \frac{(-i)^k}{4^{s+1} \pi^{-n^2}} \cdot \frac{\Gamma(2s + 1)}{\Gamma(s - \frac{k}{2}) \Gamma(s + \frac{k}{2} + 2)} Z_{m,n}(s, \chi) + R(s),
\]

where \( R(s) \) is holomorphic in \( \sigma > \frac{1}{2} \) and is \( O\left(\frac{1}{\sigma - \frac{1}{2}}\right) \) in this region. Arguing as in Section 2 of [27] we find that for \( m > 0 \) and for all \( n \) we have

\[
\text{Res}_{s = s_j} Z_{m,n}(s, \chi) = q^{2i} \frac{k}{\rho_j(m) \rho_j(n) \pi^{1-2s_j} (4m|\tilde{n}|)^{1-s_j} \Gamma(s_j + \text{sgn}(\tilde{n})\frac{k}{2}) \Gamma(2s_j - 1)} \frac{\Gamma(s_j - \frac{k}{2})}{\Gamma(s_j - \frac{k}{2})}.
\]

The proposition follows by the method of Section 3 of [27].

7. Sums of Kloosterman sums for the \( \eta \)-multiplier

We specialize the results of Section 6 to \( k = \frac{1}{2}, \Gamma = \text{SL}_2(\mathbb{Z}) \), and \( \chi \) the multiplier system attached to the \( \eta \)-function. For matrices in \( \text{SL}_2(\mathbb{Z}) \) with \( c > 0 \) we have (see, for example, §2.8 of [30])

\[
\chi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \sqrt{-i} e\left(\frac{a + d}{24c}\right) e^{-\pi i s(d,c)},
\]

where \( s(d,c) \) is the Dedekind sum defined in (1.3). In this case we have \( q = 1 \) and \( \alpha = \frac{23}{24} \).

Recall that the pentagonal numbers are those numbers of the form \( \frac{k(3k \pm 1)}{2} \) for \( k \in \mathbb{Z} \). We have the following (c.f. [39, Theorem 4.5]).

**Theorem 8.** Suppose that \( m > 0 \) and that \( n \in \mathbb{Z} \). If \( m - 1 = \frac{k_1(3k_1 \pm 1)}{2} \) and \( n - 1 = \frac{k_2(3k_2 \pm 1)}{2} \) are both pentagonal then for any \( \epsilon > 0 \) we have

\[
\sum_{c \leq x} S(m, n, c, \chi) = \sqrt{i} (-1)^{k_1 + k_2} \frac{4\sqrt{3}}{\pi} x^{\frac{1}{2}} + O\left(x^{\frac{1}{6} + \epsilon}\right).
\]

Otherwise we have

\[
\sum_{c \leq x} S(m, n, c, \chi) = O\left(x^{\frac{1}{6} + \epsilon}\right).
\]

The implied constants depend on \( m, n, \) and \( \epsilon \).

Recalling the definition (1.2), we find that

\[
\sqrt{i} A_c(n) = S(1, -n + 1, c, \chi),
\]

so Theorem 3 is an immediate corollary. Figure 1 shows values of the summatory function \( A_n(x) \) for the pentagonal number \( -n = 1 \) (along with the asymptotic curve) and the non-pentagonal number \( -n = -1 \).

In the proof we will need to know the Petersson norm of the eta function, which is given by the next lemma.
Lemma 9. We have \( \|y^{\frac{1}{4}} \eta\|^2 = \frac{1}{\sqrt{6}} \). 

Proof. For Re\( (s) > 1 \), let

\[
E(\tau, s) = \sum_{\gamma \in \Gamma_{\infty} \backslash SL_2(\mathbb{Z})} (\text{Im} \gamma \tau)^s
\]
denote the nonholomorphic Eisenstein series (see, for example, [43]), and define

\[
I(s) := \int_{SL_2(\mathbb{Z}) \backslash \mathbb{H}} E(\tau, s) y^{\frac{1}{2}} |\eta(\tau)|^2 \frac{dxdy}{y^2}.
\]

Since \( E(\tau, s) \) has a pole at \( s = 1 \) with residue \( 3/\pi \), we have

\[
\|y^{\frac{1}{2}} \eta\|^2 = \text{Res}_{s=1} I(s).
\]

On the other hand, we have

\[
I(s) = \int_{\Gamma_{\infty} \backslash \mathbb{H}} y^{s+\frac{1}{2}} |\eta(\tau)|^2 \frac{dxdy}{y^2} = \sum_{n,m \geq 1} \left( \frac{12}{nm} \right) \int_0^\infty y^{s-\frac{1}{2}} e^{-2\pi(n^2+m^2)\frac{y}{\pi}} \frac{dy}{y} \times \int_0^1 e\left( \frac{n^2-m^2}{24} x \right) dx
\]

\[
= \left( \frac{6}{\pi} \right)^{s-\frac{3}{2}} \Gamma(s-\frac{1}{2}) \sum_{n \geq 1 \atop (n,6)=1} \frac{1}{n^{2s-1}} = \left( \frac{6}{\pi} \right)^{s-\frac{1}{2}} \Gamma(s-\frac{1}{2})(1-\frac{1}{2})(1-\frac{1}{3})\zeta(2s-1).
\]

From this we obtain

\[
\text{Res}_{s=1} I(s) = \frac{1}{\sqrt{6}},
\]

as desired. \( \square \)

Proof of Theorem 8. By Proposition 1.2 of [39] and the discussion that follows, we see that the minimal eigenvalue of \( \Delta_{\frac{1}{2}} \Delta \) is \( \lambda_0 = \frac{3}{16} \), so that \( s_0 = \frac{3}{4} \). This is achieved by a unique normalized Maass cusp form in \( L^2(\SL_2(\mathbb{Z}) \backslash \mathbb{H}, \chi, \frac{1}{2}) \), namely

\[
u_0(\tau) = \frac{y^{\frac{1}{4}} \eta(\tau)}{\|y^{\frac{1}{4}} \eta\|} = (6y)^{\frac{1}{4}} \eta(\tau).
\]

(7.2)
We next show that
\[ \lambda_1 \geq \frac{1131}{4624}, \]  
(7.3)
which implies that
\[ 2s_1 - 1 \leq \frac{5}{34} < \frac{1}{6}. \]  
(7.4)
It is likely that there are no exceptional eigenvalues, but this would not improve our bound.

To prove (7.3) we note that the map \( z \mapsto 24z \) gives an injection
\[ L^2 \left( \text{SL}_2(\mathbb{Z}) \setminus \mathbb{H}, \chi, \frac{1}{2} \right) \hookrightarrow L^2 \left( \Gamma_0(576) \setminus \mathbb{H}, \chi_\theta \chi_{12}, \frac{1}{2} \right), \]
where \( \chi_\theta \) is the multiplier attached to the theta function \( \theta(\tau) = \sum_{n \in \mathbb{Z}} q^n \). This map takes eigenforms to eigenforms with the same eigenvalue. Let \( \hat{\lambda}_1 \) be the least non-residual eigenvalue in the spectrum of \( L^2 \left( \Gamma_0(576) \setminus \mathbb{H}, \chi_\theta \chi_{12}, \frac{1}{2} \right) \). In the proof of Theorem 3.6 of [39], Goldfeld and Sarnak showed that \( \hat{\lambda}_1 \geq \frac{1}{4} \nu + \frac{3}{16} \), where \( \nu \) is any lower bound for the exceptional spectrum in weight 0. Selberg [40] proved that one can take \( \nu = \frac{3}{16} \), which yields the bound \( \hat{\lambda}_1 \geq \frac{15}{64} \) stated in Theorem 3.6 of [39]. As a consequence of their proof of Langlands functoriality for the symmetric cube of an automorphic representation on \( \text{GL}_2 \), Kim and Shahidi [32] obtained \( \nu = \frac{66}{289} \), which gives \( \hat{\lambda}_1 \geq \frac{1131}{4624} \), and (7.3) follows (this was improved to \( \hat{\lambda}_1 \geq \frac{4047}{16384} \) by Kim and Sarnak [31, Appendix 2] using functoriality for the symmetric fourth power).

Recalling (1.10), we can take \( \beta = 1/2 \). Proposition 7 and (7.4) imply that
\[ \sum_{c \leq x} \frac{S(m,n,c,\chi)}{c} = 2\tau_0(m,n)x^{\frac{1}{2}} + O(x^{\frac{3}{4} + \epsilon}), \]
where (with \( \tilde{n} = n - \frac{23}{24} \) as before) we have
\[ \tau_0(m,n) = 2\sqrt{2}i\pi^{-\frac{1}{2}} \rho_0(m)\rho_0(n)|\tilde{n}|^{\frac{1}{4}} \Gamma \left( \frac{3}{4} + \frac{1}{4} \text{sgn}(\tilde{n}) \right). \]
Equations (6.1), (6.2), and (7.2) give the relation
\[ \sum_{m \in \mathbb{Z}} \rho_0(m)W_{\frac{1}{4} \text{sgn}(\tilde{m})} \left( 4\pi |\tilde{m}| y \right) e(\tilde{m}x) \]
\[ = (6y)^{\frac{1}{4}} \left( q^{\frac{1}{2\pi}} + \sum_{k=1}^{\infty} (-1)^k \left( q^{\frac{k(3k-1)}{2} + \frac{1}{2\pi}} + q^{\frac{k(3k+1)}{2} + \frac{1}{2\pi}} \right) \right). \]
Since
\[ W_{\frac{1}{4},\frac{1}{4}}(y) = y^{\frac{1}{4}} e^{-y/2}, \]
we find that
\[ \rho_0(m) = \begin{cases} (-1)^k 6^{\frac{1}{4}} (4\pi \tilde{m})^{-\frac{1}{4}} & \text{if } m - 1 = \frac{k(3k\pm 1)}{2}, \\ 0 & \text{otherwise}. \end{cases} \]

Therefore
\[ \tau_0(m,n) = \begin{cases} (-1)^{k_1+k_2} \frac{2\sqrt{3i}}{\pi} & \text{if } m - 1 = \frac{k_1(3k_1\pm 1)}{2} \text{ and } n - 1 = \frac{k_2(3k_2\pm 1)}{2}, \\ 0 & \text{otherwise}, \end{cases} \]
and the theorem follows.
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