Abstract. In previous work Regev used part of the representation theory of Lie superalgebras to compute the values of a character of the symmetric group whose decomposition into irreducible constituents is described by semistandard \((k, \ell)\)-tableaux. In this short note we give a new proof of Regev’s result using skew characters.

1. Introduction

1.1. For any partition \(\alpha \in P(n)\) of an integer \(n \geq 0\) we have a corresponding irreducible character \(\chi_\alpha\) of the symmetric group \(S_n\), we assume this labelling is as in [Mac95, 7.4]. In [Reg13] Regev observed that the values of the character \(\Gamma_n = \sum_{\alpha} \chi_\alpha\) obtained by summing over all hook partitions were particularly simple. Specifically if \(\nu \in P_r(n)\) is a partition of length \(r\) then we have

\[
\Gamma_n(\nu) = \begin{cases} 
2^{r-1} & \text{if all parts of } \nu \text{ are odd}, \\
0 & \text{otherwise}.
\end{cases}
\]

Here we write \(\Gamma_n(\nu)\) for the value of \(\Gamma_n\) at an element of cycle type \(\nu\).

1.2. To prove this result Regev considered a more general but related problem which we now recall. For any integers \(k, \ell \geq 0\) and any partition \(\alpha \in P(n)\) we denote by \(s_{k,\ell}(\alpha)\) the number of all semistandard \((k, \ell)\)-tableaux of shape \(\alpha\), see 2.6 for the definition. Motivated by the representation theory of Lie superalgebras Regev considered the following character of \(S_n\)

\[
\Lambda_{n}^{k,\ell} = \sum_{\alpha \in P(n)} s_{k,\ell}(\alpha)\chi_\alpha.
\]

The main result of [Reg13] is the following.

Theorem 1.3 (Regev). If \(\nu = (v_1, \ldots, v_r) \in P_r(n)\) is a partition of length \(r\) then

\[
\Lambda_{n}^{k,\ell}(\nu) = \prod_{i=1}^{r} (k + (-1)^{v_i-1}\ell).
\]

Remark 1.4. Note that \(s_{k,\ell}(\alpha) \neq 0\) if and only if \(\alpha\) is contained in the \((k, \ell)\)-hook, as defined in [BR87, 2.3]. In particular, we have \(\Lambda_{n}^{k,\ell}\) is the same as the character \(\chi_{\neq (k, \ell), \alpha}\) defined in [Reg13].

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1.5. Using formulas for the coefficients $s_{k,r}(\alpha)$ obtained in [BR87] Regev deduces that $\Lambda_n^{1,1} = 2\Gamma_n$ from which the statement of 1.1 follows immediately. To prove Theorem 1.3 Regev used results of Berele–Regev on representations of Lie superalgebras [BR87]. However, Regev asks whether this result can be proven entirely in the setting of the symmetric group. The purpose of this note is to provide such a proof. Our proof is based on a description of the character $\Lambda_n^{k,l}$ as a sum of skew characters. With this we can use the Murnaghan–Nakayama formula to compute the values of $\Lambda_n^{k,l}$ and thus prove Theorem 1.3. As a closing remark we use our description in terms of skew characters to show that $\Lambda_n^{1,1} = 2\Gamma_n$ using Pieri’s rule.

Remark 1.6. The idea we use in this paper was prompted by a recent question of Marcel Novaes on MathOverflow [Nov16], which is where we also first learned of Regev’s work.

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2. The Result

2.1. Let $\mathbb{N} = \{1,2,\ldots\}$ be the natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Throughout we use the term diagram to mean a subset of $\mathbb{N}^2$. The notion of connected diagram and connected components of a diagram have their usual natural meanings, see [Mac95, I, §1] for details. A diagram $T$ will be called a horizontal line, resp., vertical line, if for any $(i,j), (i',j') \in T$ we have $i = i'$, resp., $j = j'$.

2.2. For any $k \in \mathbb{N}_0$ we denote by $C_k$ the set of all compositions $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}_0^k$ of length $k$; we call $\alpha_i$ a part of $\alpha$. For such a composition we denote by $|\alpha|$ the sum $\alpha_1 + \cdots + \alpha_k$ and by $\alpha^\circ$ the composition obtained from $\alpha$ by removing all parts equal to 0 but maintaining the original order. If $n \in \mathbb{N}_0$ then we denote by $P_k(n)$ the set of all $\alpha = (\alpha_1, \ldots, \alpha_k) \in C_k$ such that $\alpha_1 \geq \cdots \geq \alpha_k > 0$ and $|\alpha| = n$, which are the partitions of $n$ of length $k$. Moreover we denote by $P(n)$ the set $\bigcup_{k \in \mathbb{N}} P_k(n)$ of all partitions of $n$. To each partition $\alpha \in P(n)$ we have a corresponding diagram $T_\alpha = \{(i,j) \mid 1 \leq j \leq \alpha_i\}$ called the Young diagram of $\alpha$. A diagram $S$ is then called a skew diagram if $S = T_\alpha \setminus T_\beta$ for some Young diagrams $T_\beta \subseteq T_\alpha$. We recall that if $S$ is a skew diagram with $|S| = n$ then we have a corresponding character $\psi_S$ of $\mathcal{S}_n$ called a skew character, c.f., [Mac95, §7, Example 3]. The following property of these characters is well known.

Lemma 2.3 (see [Mac95, 5.7]). If $S$ and $S'$ are skew diagrams with the same connected components then $\psi_S = \psi_{S'}$.

2.4. For any $k,\ell,n \in \mathbb{N}_0$ we denote by $B_{k,\ell}(n) \subseteq C_k \times C_\ell$ the set of all pairs $(\lambda \mid \mu)$ of compositions such that $|\lambda| + |\mu| = n$; we call these bicompositions of $n$. Now for each bicomposition $(\lambda \mid \mu) \in B_{k,\ell}(n)$ we denote by $S_{(\lambda|\mu)}$ some (any) skew diagram whose connected components $H_1, \ldots, H_r, V_1, \ldots, V_s \subseteq S_{(\lambda|\mu)}$ are such that $H_i$ is a horizontal line, resp., $V_j$ is a vertical line, and $(|H_1|, \ldots, |H_r|) = \lambda^\circ$, resp., $(|V_1|, \ldots, |V_s|) = \mu^\circ$. It is easy to see that such a diagram exists. By Lemma 2.3 we then get a well defined character $\psi_{(\lambda|\mu)} := \psi_{S_{(\lambda|\mu)}}$ of $\mathcal{S}_n$.

Example 2.5. Consider the bicomposition $(4,0,5;2,3) \in B_{3,2}(14)$ then an example of a corresponding skew diagram $S_{(4,0,5;2,3)}$ is given by
2.6. We now prove Theorem 1.3 but before proceeding we recall some definitions from [BR87, 2.1]. Specifically, let \( D = \{1, \ldots, k, 1', \ldots, \ell'\} \) be a totally ordered set with \( 1 < \cdots < k < 1' < \cdots < \ell' \). If \( \alpha \in \mathcal{P}(n) \) is a partition and \( (\lambda \mid \mu) \in B_{k,\ell}(n) \) is a bicomposition then we say a function \( f : T_\alpha \rightarrow D \) is a \((k, \ell)\)-tableau of shape \( \alpha \) and weight \((\lambda \mid \mu)\) if \( \lambda_i = |\{x \in T_\alpha \mid f(x) = i\}| \) for any \( 1 \leq i \leq k \) and \( \mu_j = |\{x \in T_\alpha \mid f(x) = j'\}| \) for any \( 1 \leq j \leq \ell \). As in [BR87, 2.1] we say \( f \) is semistandard if \( T_f = f^{-1}(\{1, \ldots, k\}) \) is a Young tableau whose rows are weakly increasing and whose columns are strictly increasing and \( T_\alpha \setminus T_f \) is a skew tableau whose columns are weakly increasing and whose rows are strictly increasing. If \( s_{(\lambda \mid \mu)}(\alpha) \) is the number of semistandard \((k, \ell)\)-tableaux of shape \( \alpha \) and weight \((\lambda \mid \mu)\) then \( s_{k,\ell}(\alpha) := \sum_{(\lambda \mid \mu)} s_{(\lambda \mid \mu)}(\alpha) \) is the number of all semistandard \((k, \ell)\)-tableaux of shape \( \alpha \).

**Lemma 2.7.** For any \( k, \ell, n \in \mathbb{N}_0 \) we have

\[
\Lambda^{k,\ell}_n = \sum_{(\lambda \mid \mu) \in B_{k,\ell}(n)} \psi_{(\lambda \mid \mu)}.
\]

**Proof.** It follows from [Mac95, 3.4, 5.1, 5.4, 7.3] that for any bicomposition \((\lambda \mid \mu) \in B_{k,\ell}(n)\) we have

\[
\psi_{(\lambda \mid \mu)} = \text{Ind}^{\mathbb{S}_n}_{\mathbb{S}_\lambda \times \mathbb{S}_\mu} (\chi_{(\lambda_1)} \boxtimes \cdots \boxtimes \chi_{(\lambda_k)} \boxtimes \chi_{(1^{\mu_1})} \boxtimes \cdots \boxtimes \chi_{(1^{\mu_\ell})}),
\]

where \( \mathbb{S}_\lambda \times \mathbb{S}_\mu \) is the Young subgroup determined by the parts of \( \lambda \) and \( \mu \). By [BR87, Lemma 3.23] we have the decomposition of this character into irreducibles is given by

\[
\psi_{(\lambda \mid \mu)} = \sum_{\alpha \in \mathcal{P}(n)} \chi_{\alpha}.
\]

Note that when \( \ell = 0 \) this statement is just Young’s rule and as in [BR87] the general case can be proved easily by induction on \( \ell \) using the definition of \((k, \ell)\)-tableaux. With this we obtain the desired statement

\[
\sum_{(\lambda \mid \mu) \in B_{k,\ell}(n)} \psi_{(\lambda \mid \mu)} = \sum_{(\lambda \mid \mu) \in B_{k,\ell}(n)} \sum_{\alpha \in \mathcal{P}(n)} s_{(\lambda \mid \mu)}(\alpha) \chi_{\alpha} = \sum_{\alpha \in \mathcal{P}(n)} s_{k,\ell}(\alpha) \chi_{\alpha}.
\]

**Proof (of Theorem 1.3).** Choose a part \( a \) of \( \upsilon \) and let \( \hat{\upsilon} \in \mathcal{P}(n-a) \) be the partition obtained by removing the part \( a \) from \( \upsilon \) but maintaining the original order. If \( \lambda \in \mathcal{C}_k \) is a composition such that \( \lambda_i \geq a \) then we denote by \( \lambda \downarrow_i a \in \mathcal{C}_k \) the composition obtained by replacing \( \lambda_i \) with \( \lambda_i - a \). By the Murnaghan–Nakayama formula for skew characters, see [JK81, 2.4.15], we have that

\[
\psi_{(\lambda \mid \mu)}(\upsilon) = \sum_{\lambda_i \geq a} \psi_{(\lambda \downarrow_i a \mid \mu)}(\hat{\upsilon}) + \sum_{\mu_j \geq a} (-1)^{a-1} \psi_{(\lambda \mu \downarrow_i a)}(\hat{\upsilon}).
\]
where the first, resp., second, sum is over all \(1 \leq i \leq k\), resp., \(1 \leq j \leq \ell\), such that \(\lambda_i \geq a\), resp., \(\mu_j \geq a\). Indeed, the connected component of the skew diagram \(S_{(\lambda |\mu)}\) labelled by \(\lambda_i\), resp., \(\mu_j\), has an \(a\)-hook of leg length 0, resp., \((-1)^{a-1}\), if and only if \(\lambda_i \geq a\), resp., \(\mu_j \geq a\). Now clearly every bicomposition \((\lambda' | \mu') \in B_{k,\ell}(n-a)\) arises from exactly \(k + \ell\) bicompositions \((\lambda | \mu) \in \mathcal{B}_{k,\ell}(n)\) via the process \(\downarrow_i a\) and so by Lemma 2.7 we have

\[
\Lambda_{n,a}^{k,\ell}(\psi) = \sum_{(\lambda | \mu) \in \mathcal{B}_{k,\ell}(n)} \psi(\lambda | \mu) = (k + (-1)^{a-1}\ell)\Lambda_{n-a}^{k,\ell}(\hat{\psi}) .
\]

An easy induction argument completes the proof.

Remark 2.8. Recall that the decomposition of \(\psi(\lambda | \mu)\) into irreducible characters is given by the Littlewood–Richardson coefficients, see [Mac95, 5.3]. If \(k = \ell = 1\) then a simple application of Pieri’s rule shows that

\[
\psi(a | n-a) = \begin{cases} 
\chi(1^n) & \text{if } a = 0, \\
\chi(a,1^{n-a}) + \chi(a+1,1^{n-a-1}) & \text{if } 0 < a < n, \\
\chi(n) & \text{if } a = n.
\end{cases}
\]

This gives an alternative way to see that \(\Lambda_{n,1}^{1,1} = 2\Gamma_n\).

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