P and T Odd Asymmetries in Lepton Flavor Violating $\tau$ Decays

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Abstract

We calculated the differential cross sections of the processes in which one of the pair created $\tau$ particles at an $e^+e^-$ collider decays into lepton flavor violating final states e.g. $\tau \rightarrow \mu\gamma$, $\tau \rightarrow 3\mu$, $\tau \rightarrow \mu ee$. Using the correlations between angular distributions of both sides of $\tau$ decays, we can obtain information on parity and CP violations of lepton flavor non-conserving interactions. The formulae derived here are useful in distinguishing different models, since each model of physics beyond the standard model predicts different angular correlations. We also calculate angular distributions of the major background process to $\tau \rightarrow l\gamma$ search, namely $\tau \rightarrow l\nu\bar{\nu}\gamma$, and discuss usefulness of the angular correlation for background suppression.

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1 Introduction

Recent results from neutrino experiments such as the Super-Kamiokande experiment strongly suggest neutrino oscillation so that there are flavor mixings in the lepton sector [1]. This implies that the charged lepton flavor violating processes such as $\mu \rightarrow e \gamma$, $\tau \rightarrow \mu \gamma$, $\tau \rightarrow e \gamma$, etc. also occur at some level. It is, therefore, important to search for lepton flavor violation (LFV) in rare decay processes of muon and $\tau$.

Prediction of branching fraction of LFV processes depends on models of physics beyond the standard model. In the minimal extension of the standard model which takes into account neutrino oscillations by seesaw mechanism of neutrino mass generation [2], the expected branching fraction is too small to be observable in near future [3]. On the other hand, in supersymmetric (SUSY) models, the prediction can be close to the current experimental upper bound. In this case, the flavor mixing in the slepton mass matrix becomes a new source of LFV. Even in the minimal supergravity scenario [4], in which the slepton mass matrix is proportional to a unit matrix at the Planck scale, the renormalization effects due to LFV interactions can induce sizable slepton mixings [5]. For example, such LFV Yukawa interactions exist in SUSY Grand Unified Theory (GUT) [5], SUSY model with right-handed neutrinos [6], and SUSY models with exotic vectorlike leptons [7]. Another interesting possibility is models with extra dimensions, where the neutrino masses and mixings are obtained from the Yukawa interaction between the ordinary left-handed leptons and the gauge-singlet neutrinos which propagate in the bulk of extra dimensions [8]. This Yukawa interaction breaks the lepton flavor conservation and the Kaluza-Klein modes of the bulk neutrinos can enhance $\mu \rightarrow e \gamma$ decay, $\tau \rightarrow \mu \gamma$ decay, etc. through the loop diagrams [9].

In the muon decay, the polarized muon experiments provide useful information on the nature of LFV interactions [10, 11]. We can define a parity (P) odd asymmetry for $\mu \rightarrow e \gamma$ process and P and time reversal (T) odd asymmetries for $\mu \rightarrow 3e$ process. These asymmetries are useful to distinguish different models. For example, in the SU(5) SUSY GUT model with small and intermediate value of $\tan \beta$ (a ratio of two vacuum expectation values of Higgs fields), only $\mu^+ \rightarrow e^+_L \gamma$ (or $\mu^- \rightarrow e^-_R \gamma$) occurs because LFV is induced through the right-handed slepton sector. On the other hand, the SUSY models with right-handed neutrinos predict $\mu^+ \rightarrow e^+_R \gamma$ (or $\mu^- \rightarrow e^-_L \gamma$), and the SUSY models
with vectorlike leptons can induce both $\mu^+ \rightarrow e^+_L \gamma$ and $\mu^+ \rightarrow e^+_R \gamma$ depending on how interaction breaks the lepton flavor conservation. In the models with extra dimensions, only $\mu^+ \rightarrow e^+_R \gamma$ can occur. As for the T odd asymmetry in the $\mu \rightarrow 3e$ process, it was shown that asymmetry could be sizable in the SU(5) SUSY GUT [11, 12].

In this paper, we discuss the LFV processes of $\tau$ decays such as $\tau \rightarrow \mu \gamma$, $\tau \rightarrow 3\mu$, $\tau \rightarrow \mu ee$, taking into account P and T odd asymmetries. In the $\tau^+\tau^-$ pair production at $e^+e^-$ collision, we can extract information on the spin of the decaying $\tau$ particle from the angular distribution of the $\tau$ decay products in the opposite side. Using this technique, we can obtain the P and T odd asymmetry defined in the rest frame of $\tau$. The method of the spin correlation has been developed since the days before the discovery of $\tau$ particle [13]. There have been many works on spin correlation method in search for anomalous coupling involving $\tau$ [14]. We have applied the formalism in order to obtain the information on LFV interactions under P and T symmetries. We also calculate angular correlation of the process where one of the $\tau$’s decays through $\tau \rightarrow l\nu\gamma$ mode. This mode is a background process to the $\tau \rightarrow l\gamma$ search if the neutrinos carry out little energy. As in the muon case [15], the angular correlation is useful to identify the background process and the background suppression is effective for $\tau^- \rightarrow \mu^- \gamma$ ($\tau^+ \rightarrow \mu^+_L \gamma$) search.

This paper is organized as follows. In section 2, we introduce a formalism to calculate the spin correlation. In section 3, we present a differential cross section of the production and decays of $\tau^+\tau^-$ at $e^+e^-$ colliders where one of $\tau$’s decays in $\tau \rightarrow \mu \gamma$ or $\tau \rightarrow e\gamma$ modes, and show how to extract P-odd asymmetry of $\tau$ decay. In section 4, P and T odd asymmetries in three body LFV decays ($\tau \rightarrow 3\mu$, $\tau \rightarrow \mu ee$, etc.) are considered. In section 5, we consider $\tau \rightarrow l\nu\bar{\nu}\gamma$ mode and show that the analysis of the angular distribution is useful for background suppression of the $\tau \rightarrow \mu \gamma$ and $\tau \rightarrow e\gamma$ searches. A summary is given in section 6. Appendices contain the derivation of basic formulae and a list of kinematical functions.

2 General formula for spin correlation

In this section, we present general formulae used in the calculation of differential cross sections and spin correlations.

We calculate differential cross sections of $e^+e^- \rightarrow \tau^+\tau^- \rightarrow f_B f_A$, where $f_B$ ($f_A$) repre-
sents the decay products of \( \tau^+ (\tau^-) \). If the intermediate states were spinless particles, the cross section is simply a product of a production cross section and decay branching ratios. However, in the case of spin 1/2 particles, we have to take into account spin correlation between two intermediate particles. If we take \( \tau^+ \rightarrow f_B \) to be a LFV decay mode, we can measure P and T violation of LFV interactions by using the angular correlations of decay products of \( \tau^+ \) and \( \tau^- \).

The differential cross section of \( e^+e^- \rightarrow \tau^+\tau^- \rightarrow f_Bf_A \) is given by

\[
d\sigma = d\sigma^P \ dB^{\tau^+\rightarrow f_B} \ dB^{\tau^+\rightarrow f_B} + \sum_{a, b=1}^3 d\Sigma_{ab}^P \ dB^{\tau^+\rightarrow f_A} \ dB^{f_B^{-}\rightarrow f_B},
\]

and

\[
d\sigma^P = \frac{d^3 p_A}{(2\pi)^3 2p_A^0} \ \frac{d^3 p_B}{(2\pi)^3 2p_B^0} \ \frac{1}{2s} \ (2\pi)^4 \ \delta^4(p_A + p_B - p_{e^+} - p_{e^-}) \ \alpha^P,
\]

\[
dB^{\tau^+\rightarrow f_A} = \frac{1}{\Gamma} \ \frac{d^3 q_1}{(2\pi)^3 2q_1^0} \ \cdots \ \frac{d^3 q_n}{(2\pi)^3 2q_n^0} \ \frac{1}{2m_\tau} \ (2\pi)^4 \ \delta^4 \left( \sum_{i=1}^n q_i - p_A \right) \ \alpha^{P_-},
\]

\[
dB^{\tau^+\rightarrow f_B} = \frac{1}{\Gamma} \ \frac{d^3 q_{n+1}}{(2\pi)^3 2q_{n+1}^0} \ \cdots \ \frac{d^3 q_{n+m}}{(2\pi)^3 2q_{n+m}^0} \ \frac{1}{2m_\tau} \ (2\pi)^4 \ \delta^4 \left( \sum_{i=n+1}^{n+m} q_i - p_B \right) \ \alpha^{P_+},
\]

\[
d\Sigma_{ab}^P = \frac{d^3 p_A}{(2\pi)^3 2p_A^0} \ \frac{d^3 p_B}{(2\pi)^3 2p_B^0} \ \frac{1}{2s} \ (2\pi)^4 \ \delta^4(p_A + p_B - p_{e^+} - p_{e^-}) \ \rho^P,
\]

\[
dR_{a}^{\tau^-\rightarrow f_A} = \frac{1}{\Gamma} \ \frac{d^3 q_1}{(2\pi)^3 2q_1^0} \ \cdots \ \frac{d^3 q_n}{(2\pi)^3 2q_n^0} \ \frac{1}{2m_\tau} \ (2\pi)^4 \ \delta^4 \left( \sum_{i=1}^n q_i - p_A \right) \ \rho_{a}^{P_-},
\]

\[
dR_{b}^{\tau^-\rightarrow f_B} = \frac{1}{\Gamma} \ \frac{d^3 q_{n+1}}{(2\pi)^3 2q_{n+1}^0} \ \cdots \ \frac{d^3 q_{n+m}}{(2\pi)^3 2q_{n+m}^0} \ \frac{1}{2m_\tau} \ (2\pi)^4 \ \delta^4 \left( \sum_{i=n+1}^{n+m} q_i - p_B \right) \ \rho_{b}^{P_+},
\]

where we assume that \( f_A \) is a \( n \) body system and \( f_B \) is a \( m \) body system. \( p_{e^+} (p_{e^-}) \) is \( e^+ (e^-) \) four momentum, \( p_B (p_A) \) is \( \tau^+ (\tau^-) \) four momentum, and \( q_i \)'s are momenta of final state particles. \( s \) is determined as \( s = (p_{e^+} + p_{e^-})^2 \). \( \Gamma \) and \( m_\tau \) are the width and the mass.
of the \( \tau \), respectively. In order to define \( \alpha^P, \alpha^{D-}, \alpha^{D+}, \rho_{ab}^P, \rho_a^{D-}, \) and \( \rho_b^{D+} \), we first write down the invariant amplitude of \( e^+e^- \rightarrow \tau^+\tau^- \rightarrow f_Bf_A \) as follows:

\[
M = \frac{e^2}{s} \bar{A}(p_A + m_{\tau}) \gamma^\mu (p_B - m_{\tau}) B \bar{v}_e \gamma_\mu u_e - 1 \frac{1}{p_A^2 - (m_{\tau} - \frac{\Gamma}{2})^2} - \frac{1}{p_B^2 - (m_{\tau} - \frac{\Gamma}{2})^2},
\]

where \( v_e^+ (u_e^-) \) is the wave function of positron (electron) and \( A \) and \( B \) are spinors which include wave functions of final states and interaction vertices. By using the Bouchiat-Michel formulae \[16\] and the narrow width approximation, \( \alpha^P, \alpha^{D-}, \alpha^{D+}, \rho_{ab}^P, \rho_a^{D-}, \) and \( \rho_b^{D+} \) are given by

\[
\alpha^P = \frac{1}{4} \frac{e^4}{s^2} \text{Tr} [\bar{p}_A + m_{\tau}) \gamma^\mu (p_B - m_{\tau}) \gamma^\nu] \text{Tr} [\bar{p}_e \gamma_\mu p_e \gamma_\nu] ,
\]

\[
\alpha^{D-} = \frac{1}{2} \{ \bar{A}(p_A + m_{\tau}) A \} ,
\]

\[
\alpha^{D+} = \frac{1}{2} \{ \bar{B}(p_B - m_{\tau}) B \} ,
\]

\[
\rho_{ab}^P = \frac{1}{4} \frac{e^4}{s^2} \text{Tr} [\gamma_5 \bar{p}_a A (p_A + m_{\tau}) \gamma_5 \gamma_5 \bar{p}_b (p_B - m_{\tau}) \gamma^\nu] \text{Tr} [\bar{p}_e \gamma_\mu p_e \gamma_\nu] ,
\]

\[
\rho_a^{D-} = \frac{1}{2} \{ \bar{A} \gamma_5 \bar{p}_a (p_A + m_{\tau}) A \} ,
\]

\[
\rho_b^{D+} = \frac{1}{2} \{ \bar{B} \gamma_5 \bar{p}_b (p_B - m_{\tau}) B \} ,
\]

where the spins of the final state fermions are summed over, and four vectors \( (s_A^a)^\mu \) and \( (s_B^b)^\nu \) \((a, b = 1, 2, 3)\) are a set of vectors which satisfy following equations.

\[
p_A \cdot s_A^a = p_B \cdot s_B^b = 0,
\]

\[
s_A^a \cdot s_A^b = s_B^a \cdot s_B^b = -\delta_{ab} ,
\]

\[
\sum_{a=1}^3 (s_A^a)_{\mu}(s_A^a)^{\nu} = -g_{\mu\nu} + \frac{p_A\mu p_A\nu}{m_{\tau}^2}, \quad \sum_{b=1}^3 (s_B^b)_{\mu}(s_B^b)^{\nu} = -g_{\mu\nu} + \frac{p_B\mu p_B\nu}{m_{\tau}^2} .
\]

The derivation of the above result is shown in Appendix \[A\]. Notice that \( d\sigma^P, dB^{\tau^-\rightarrow f_A}, \) and \( dB^{\tau^-\rightarrow f_B} \) in eq.(1) are the \( \tau^+\tau^- \) production cross section and \( \tau \) decay branching
ratios, in which spins of $\tau$'s are averaged, and $d\Sigma^P, dR^\tau \to fA$, and $dR^\tau \to fB$ represent the spin correlation effects of this process.

In above formulae, it is assumed that the $\tau$ pair production occurs through the photon exchange. It is straightforward to include the contribution from the $Z$ boson exchange and the $\gamma - Z$ interference. If we consider the $e^+e^-$ center of mass energy to be in the range of the $\tau^+\tau^-$ threshold energy considered in the $\tau$-charm factory or the $\Upsilon(4S)$ resonance energy where $e^+e^-$ B-factories are operated, these effects only contribute to the production cross section at the level of $O(10^{-4})$ of the photon exchanging diagram.

3 Parity asymmetry in \( \tau \to \mu \gamma \) decay

Let us calculate the cross section of $e^+e^- \to \tau^+\tau^- \to \mu^+\gamma + f_A$ processes. For $f_A$, we consider hadronic and leptonic modes such as ($\pi \nu$, $\rho \nu$, $a_1 \nu$, and $l\bar{\nu}\nu$). Below we neglect the muon mass compared to $\tau$ mass, and therefore all formulae can be applied also to the $\tau \to e\gamma$ process. The effective Lagrangian for $\tau^+ \to \mu^+\gamma$ decay is given by

$$\mathcal{L} = -\frac{4G_F}{\sqrt{2}} \left\{ m_\tau A_R \bar{\tau} \sigma^{\mu\nu} P_L \mu F_{\mu\nu} + m_\tau A_L \bar{\tau} \sigma^{\mu\nu} P_R \mu F_{\mu\nu} + \text{h.c.} \right\},$$

(18)

where $G_F$ is the Fermi coupling constant, $P_L = (1 - \gamma_5)/2$, and $P_R = (1 + \gamma_5)/2$. The operator with the coupling constant $A_R$ ($A_L$) induces the $\tau^+ \to \mu_R^+\gamma$ ($\tau^+ \to \mu_L^+\gamma$) decay. As mentioned in Introduction, each model of the physics beyond the standard model predicts a different ratio of $A_L$ and $A_R$. For example, the SU(5) SUSY GUT in the minimal supergravity scenario predicts that only $A_L$ has a non-vanishing value for small and intermediate values of $\tan \beta$. Therefore the separate determination of $A_L$ and $A_R$ provides us important information on the origin of LFV. For this purpose, we need information about the $\tau$ polarization. This can be done by observing angular distributions of final state of $\tau$ decay in the opposite side in the modes of $\tau \to \pi \nu$, $\tau \to \rho \nu$, $\tau \to a_1 \nu$, and $\tau \to l\bar{\nu}\nu$, because these processes proceed due to the $V - A$ interaction and therefore have a specific angular distribution with respect to polarization of $\tau$. Using $\tau^+ - \tau^-$ spin correlation, we can determine $|A_L|^2$ and $|A_R|^2$, separately.

We first define three coordinate systems (Fig.1). The first coordinate system (Frame 1) is the center of mass frame of the $e^+e^-$ collision in which the $z$ axis is taken to be the $e^+$ momentum direction. The second one (Frame 2) is the rest frame of the $\tau^+$, and the
third one (Frame 3) is the rest frame of the $\tau^-$. More explicitly, the relation of a four vector in the three systems are given as follows:

$$\xi_1^\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta_{\tau} & 0 & \sin \theta_{\tau} \\ 0 & 0 & 1 & 0 \\ 0 & -\sin \theta_{\tau} & 0 & \cos \theta_{\tau} \end{pmatrix} \begin{pmatrix} \gamma & 0 & 0 & \gamma_{\beta_{\tau}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma_{\beta_{\tau}} & 0 & 0 & \gamma \end{pmatrix} \xi_2^\mu$$

$$\xi_3^\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta_{\tau} & 0 & -\sin \theta_{\tau} \\ 0 & 0 & 1 & 0 \\ 0 & \sin \theta_{\tau} & 0 & \cos \theta_{\tau} \end{pmatrix} \begin{pmatrix} \gamma & 0 & 0 & \gamma_{\beta_{\tau}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma_{\beta_{\tau}} & 0 & 0 & \gamma \end{pmatrix} \xi_3^\mu ,$$

where $\gamma = \sqrt{s/(2m_{\tau})}$ and $\beta_{\tau} = \sqrt{1 - 4m_{\tau}^2/s}$, and the four vectors $\xi_{1-3}$ are defined in Frame 1-3, respectively. We calculate the production process in Frame 1, and the $\tau^+ (\tau^-)$ decay in Frame 2 (Frame 3). In the calculations, we choose the spin vectors $(s_A^a)^\mu$, $(s_B^b)^\mu$ as follows:

$$(s_A^a)^\mu = \begin{pmatrix} 0 \\ \delta_{a\mu} \end{pmatrix} \text{ (in Frame 3)} ,$$

$$(s_B^b)^\mu = \begin{pmatrix} 0 \\ \delta_{b\mu} \end{pmatrix} \text{ (in Frame 2)} .$$

The production cross section and spin dependence term are obtained from eq.(8) and
eq. (12) as follows:

\[ d\sigma^P = \frac{d\Omega}{4\pi} \frac{\pi\alpha^2}{s} \sqrt{1 - \frac{4m^2}{s}} \left\{ \left( 1 + \frac{4m^2}{s} \right) + \left( 1 - \frac{4m^2}{s} \right) \cos^2 \theta_\tau \right\}, \]  

(22)

\[ d\Sigma^P_{ab} = \frac{d\Omega}{4\pi} \frac{\pi\alpha^2}{s} \sqrt{1 - \frac{4m^2}{s}} \times \begin{pmatrix} (1 + \frac{4m^2}{s}) \sin^2 \theta_\tau & 0 & -\frac{2m_\tau}{\sqrt{s}} \sin 2\theta_\tau \\ 0 & (1 - \frac{4m^2}{s}) \sin^2 \theta_\tau & 0 \\ \frac{2m_\tau}{\sqrt{s}} \sin 2\theta_\tau & 0 & \left( 1 - \frac{4m^2}{s} \right) - \left( 1 + \frac{4m^2}{s} \right) \cos^2 \theta_\tau \end{pmatrix}. \]  

(23)

where \( \theta_\tau \) is the angle between the \( e^+ \) and \( \tau^+ \) directions in the Frame 1, and \( d\Omega_\tau \) is a solid angle element of \( \tau^+ \), \( d\Omega_\tau = d\cos \theta_\tau \, d\phi_\tau \).

For decay processes, we take \( \tau^+ \to \mu^+\gamma \) for the \( \tau^+ \) side and hadronic (\( \tau^- \to \pi^-\nu \), \( \tau^- \to \rho^-\nu \), and \( \tau^- \to a_1\nu \)) and leptonic (\( \tau^- \to l^-\bar{\nu} \)) decays for the \( \tau^- \) side. \( dB^{\tau^+\to\mu^+\gamma} \) and \( dR_b^{\tau^+\to\mu^+\gamma} \) (see eq. (1)) can be calculated from eq. (11) and eq. (14) in which the spinor \( B \) is given by

\[ B = \frac{8i}{\sqrt{2}} G_F m_\tau \sigma^{\mu\nu}(q_\gamma)_{\mu}(A_R P_L + A_L P_R)\epsilon^*_\nu(q_\mu), \]  

(24)

where \( \epsilon_\nu \) is the polarization vector of the photon and \((q_\gamma)_\mu\) is the momentum of the photon and \( v(q_\mu) \) is the wave function of the muon. These quantities are given as follows:

\[ dB^{\tau^+\to\mu^+\gamma} = \frac{d\Omega_\mu}{4\pi} \frac{1}{\Gamma} \frac{2}{\pi} G_F^2 m_\tau^5 \left( |A_L|^2 + |A_R|^2 \right), \]  

(25)

\[ dR_b^{\tau^+\to\mu^+\gamma} = \frac{d\Omega_\mu}{4\pi} \frac{1}{\Gamma} \frac{2}{\pi} G_F^2 m_\tau^5 \left( |A_L|^2 - |A_R|^2 \right) \begin{pmatrix} \sin \theta_\mu \cos \phi_\mu \\ \sin \theta_\mu \sin \phi_\mu \\ \cos \theta_\mu \end{pmatrix}, \]  

(26)

where \((\theta_\mu, \phi_\mu)\) are angles in the polar coordinate for a unit vector of the muon momentum direction in Frame 2. The three components in eq. (26) corresponds to \( b = 1, 2, \) and \( 3 \).

Next, we list \( dB \) and \( dR \) for the \( \tau^- \) decay in each mode of \( \tau^- \to \pi^-\nu \), \( \tau^- \to \rho^-\nu \), \( \tau^- \to a_1\nu \), and \( \tau^- \to l^-\bar{\nu} \). For \( \tau^- \to \pi^-\nu \) decay, the spinor \( A \) in eqs. (11) and (13) is given by

\[ A = 2iV_{ud} f_\pi G_F \hat{g}_\pi P_L u(q_\nu), \]  

(27)
where $f_\pi$ is the pion decay constant, $q_\pi$ is the momentum of the pion, and $u(q_\nu)$ is the neutrino wave function. Then, $dB^{\tau^-\rightarrow\pi^-\nu}$ and $dR_{\alpha}^{\tau^-\rightarrow\pi^-\nu}$ are given by

$$
\frac{dB^{\tau^-\rightarrow\pi^-\nu}}{d\Omega} = \frac{d\Omega_\pi}{4\pi} \frac{1}{8\pi} |V_{ud}|^2 f_\pi^2 G_F^2 m_\pi^3 \gamma_1 \frac{m_\tau}{m_\pi}, \quad (28)
$$

$$
\frac{dR_{\alpha}^{\tau^-\rightarrow\pi^-\nu}}{d\Omega} = dB^{\tau^-\rightarrow\pi^-\nu} \left( \begin{array}{c} \sin \theta_\pi \cos \phi_\pi \\ \sin \theta_\pi \sin \phi_\pi \\ \cos \theta_\pi \end{array} \right), \quad (29)
$$

where $(\theta_\pi, \phi_\pi)$ are the polar angles of $\pi^-$ momentum in Frame 3 and $d\Omega_\pi = d\cos \theta_\pi \, d\phi_\pi$ (we use a similar notation in the following expressions). Here we neglect the mass of the pion. As before three elements in eq.(29) corresponds to $a = 1, 2,$ and 3. Similar results can be obtained for the vector mesons. The spinor $A$ for $\tau \rightarrow \rho \nu$, $\tau \rightarrow a_1 \nu$ is given by

$$
A = -2V_{ud} g_V G_F f_V P_L u(q_\nu). \quad (30)
$$

where $g_V$ and $\epsilon_V$ are the decay constant and polarization vector of the corresponding vector mesons, respectively. From this expression, we can obtain $dB$ and $dR$ for the longitudinally polarized vector mesons e.g. $\tau^- \rightarrow \rho^-(L)\nu$ and $\tau^- \rightarrow a_1^-(L)\nu$ as follows:

$$
\frac{dB^{\tau^-\rightarrow V(L)^-\nu}}{d\Omega} = \frac{d\Omega_V}{4\pi} \frac{1}{8\pi} |V_{ud}|^2 \left( \frac{g_V}{m_V^2} \right)^2 G_F^2 m_\tau^4 m_V^2 (1 - \frac{m_V^2}{m_\tau^2}), \quad (31)
$$

$$
\frac{dR_{\alpha}^{\tau^-\rightarrow V(L)^-\nu}}{d\Omega} = dB^{\tau^-\rightarrow V(L)^-\nu} \left( \begin{array}{c} \sin \theta_V \cos \phi_V \\ \sin \theta_V \sin \phi_V \\ \cos \theta_V \end{array} \right), \quad (32)
$$

where $m_V$ and $(\theta_V, \phi_V)$ are the mass, and polar angles of the corresponding vector meson, respectively. For the transversely polarized vector mesons, the spin dependence terms have a minus sign contrary to the case of the pion and longitudinally polarized vector mesons.

$$
\frac{dB^{\tau^-\rightarrow V(T)^-\nu}}{d\Omega} = \frac{d\Omega_V}{4\pi} \frac{1}{8\pi} |V_{ud}|^2 \left( \frac{g_V}{m_V^2} \right)^2 G_F^2 m_\tau^4 m_V^2 \left(1 - \frac{m_V^2}{m_\tau^2} \right) \frac{2m_V^2}{m_\tau^2}, \quad (33)
$$

$$
\frac{dR_{\alpha}^{\tau^-\rightarrow V(T)^-\nu}}{d\Omega} = dB^{\tau^-\rightarrow V(T)^-\nu} \left( \begin{array}{c} -\sin \theta_V \cos \phi_V \\ -\sin \theta_V \sin \phi_V \\ -\cos \theta_V \end{array} \right). \quad (34)
$$
For leptonic decays, after integrating over the phase space of the neutrinos, the branching ratio and the spin dependence term are given by

\[
\begin{align*}
 dB_{\tau^- \rightarrow l^- \bar{\nu}\nu} & = \frac{d\Omega_l}{4\pi} \frac{d\Omega_{\nu\bar{\nu}}}{192\pi^3} \frac{G^2_F m_\tau^5}{2x^2(3 - 2x)}, \\
 dR_{\tau^- \rightarrow l^- \bar{\nu}\nu} & = \frac{d\Omega_l}{4\pi} \frac{d\Omega_{\nu\bar{\nu}}}{192\pi^3} \frac{G^2_F m_\tau^5}{2x^2(1 - 2x)} \left( \begin{array}{c}
 \sin \theta_l \cos \phi_l \\
 \sin \theta_l \sin \phi_l \\
 \cos \theta_l 
\end{array} \right),
\end{align*}
\]

where we neglect the masses of the leptons (e and \(\mu\)) and \(x\) is the lepton energy normalized by the maximum energy \(m_\tau/2\) i.e. \(x = 2E_l/m_\tau\), and \((\theta_l, \phi_l)\) are the polar angles of the lepton in Frame 3.

Substituting these results into the formula in eq.(31), we obtain the differential cross sections of each processes. For example, the differential cross section of \(e^+e^- \rightarrow \tau^+\tau^- \rightarrow \mu^+\gamma + \pi^-\nu\) process is given by

\[
\begin{align*}
 d\sigma(e^+e^- \rightarrow \tau^+\tau^- \rightarrow \mu^+\gamma + \pi^-\nu) & = \frac{\sigma(e^+e^- \rightarrow \tau^+\tau^-)}{4\pi} B(\tau^- \rightarrow \pi^-\nu) B(\tau^+ \rightarrow \mu^+\gamma) \frac{d\Omega_\tau}{4\pi} \frac{d\Omega_\gamma}{4\pi} \frac{d\Omega_\pi}{4\pi} \\
 & \times \left[ \begin{array}{c}
 \left( 1 + \frac{4m^2_\tau}{s} \right) + \left( 1 - \frac{4m^2_\tau}{s} \right) \cos^2 \theta_\tau \\
 - \frac{2m_\tau}{\sqrt{s}} \sin 2\theta_\tau \\
 \left( 1 + \frac{4m^2_\tau}{s} \right) \sin^2 \theta_\tau \\
 0 \\
 \frac{2m_\tau}{\sqrt{s}} \sin 2\theta_\tau \\
 \left( 1 - \frac{4m^2_\tau}{s} \right) - \left( 1 + \frac{4m^2_\tau}{s} \right) \cos^2 \theta_\tau \\
 \frac{\sin \theta_\mu \cos \phi_\mu}{\cos \theta_\mu} \\
 \frac{\sin \theta_\mu \sin \phi_\mu}{\cos \theta_\mu}
\end{array} \right],
\end{align*}
\]

where

\[
\sigma(e^+e^- \rightarrow \tau^+\tau^-) = \frac{4\pi\alpha^2}{3s} \sqrt{1 - \frac{4m^2_\tau}{s}} \left( 1 + \frac{2m^2_\tau}{s} \right)
\]

is the \(\tau^+\tau^-\) production cross section. The branching ratio of \(\tau^- \rightarrow \pi^-\nu\) and \(\tau^+ \rightarrow \mu^+\gamma\) is given by

\[
B(\tau^- \rightarrow \pi^-\nu) = \frac{1}{\Gamma} \frac{1}{8\pi} |V_{ud}|^2 f^2 G^2_F m_\tau^3
\]

(39)
\[ B(\tau^+ \rightarrow \mu^+\gamma) = \frac{1}{\Gamma} \frac{2}{\pi} G_F^2 m_\tau^5 (|A_L|^2 + |A_R|^2) , \]  
(40)
and the asymmetry parameter \( A_P \) is defined as follows:
\[ A_P \equiv \frac{|A_L|^2 - |A_R|^2}{|A_L|^2 + |A_R|^2} . \]  
(41)

We can see that the measurement of angular correlation of the pion and muon momentum enables us to determine the parameter \( A_P \), so that we can obtain \( |A_L|^2 \) and \( |A_R|^2 \) separately.

A simpler expressions can be obtained if we integrate over the angle \( \theta_\tau, \phi_\tau, \phi_\pi, \) and \( \phi_\gamma \) in eq.(37). The differential cross section is given by
\[ d\sigma(e^+e^- \rightarrow \tau^+\tau^- \rightarrow \mu^+\gamma + \pi^-\nu) = \sigma(e^+e^- \rightarrow \tau^+\tau^-)B(\tau^+ \rightarrow \mu^+\gamma)B(\tau^- \rightarrow \pi^-\nu) \frac{d\cos\theta_\mu}{2} \frac{d\cos\theta_\pi}{2} \times \left( 1 - \frac{s - 2m_\tau^2}{s + 2m_\tau^2} A_P \cos\theta_\mu \cos\theta_\pi \right) . \]  
(42)

Notice that angular distribution in the rest frames of \( \tau^+ \) and \( \tau^- \) can be easily converted to the energy distribution in the center of mass frame of the \( e^+e^- \) collision. We obtain
\[ d\sigma(e^+e^- \rightarrow \tau^+\tau^- \rightarrow \mu^+\gamma + \pi^-\nu) = \sigma(e^+e^- \rightarrow \tau^+\tau^-)B(\tau^+ \rightarrow \mu^+\gamma)B(\tau^- \rightarrow \pi^-\nu) \frac{s}{s - 4m_\tau^2} dz_\mu dz_\pi \times \left( 1 - \frac{s(s - 2m_\tau^2)}{(s - 4m_\tau^2)(s + 2m_\tau^2)} A_P (2z_\mu - 1)(2z_\pi - 1) \right) , \]  
(43)
where \( z_\mu = E_\mu/E_\tau \) \( (z_\pi = E_\pi/E_\tau) \), and \( E_\mu \), \( E_\pi \), and \( E_\tau = \sqrt{s}/2 \) are the energies of the muon, pion, and \( \tau \) in the center of mass frame, respectively.

The angular (or energy) distributions in eq.(42) (eq.(43)) can be understood as follows. Because of the helicity conservation of the \( \tau^+\tau^- \) production process, the helicities of \( \tau^+ \) and \( \tau^- \) are correlated, namely \( \tau^+_L \tau^-_R \) or \( \tau^+_R \tau^-_L \) is produced. This means that two \( \tau \) spins are parallel in the limit of \( \sqrt{s} \gg m_\tau \). In the decay process, the \( \pi^- \) tends to be emitted to the spin direction of \( \tau^- \) for \( \tau^- \rightarrow \pi^-\nu \), because of the \( V-A \) interaction. On the other hand, for \( \tau^+ \rightarrow \mu^+\gamma \) decay, the muon tends to be emitted to the same direction of the \( \tau^+ \) spin if \( A_P > 0 \). Therefore the differential branching ratio is enhanced (suppressed) if the sign of \( \cos\theta_\mu \cos\theta_\pi \) is negative (positive). In other words, pion and muon energies in
the center of mass frame of the $e^+e^-$ collision have a negative correlation if $A_P > 0$. If $A_P < 0$, we have an opposite correlation.

We can define an asymmetry $A_{\mu^+\gamma,\pi^-}\nu$ by the following asymmetric integrations

$$A_{\mu^+\gamma,\pi^-}\nu = \frac{\int d\cos\theta_\mu \, d\cos\theta_\pi \, w(\cos\theta_\mu, \cos\theta_\pi) \, \frac{d^2\sigma}{d\cos\theta_\mu d\cos\theta_\pi}}{\sigma(e^+e^- \rightarrow \tau^+\tau^-)B(\tau^+ \rightarrow \mu^+\gamma)B(\tau^- \rightarrow \pi^-\nu)}$$

$$= \frac{N^{++} + N^{--} - N^{+-} - N^{-+}}{N^{++} + N^{--} + N^{+-} + N^{-+}},$$

(44)

where the weight function $w(u,v)$ is defined by

$$w(u,v) = \frac{uv}{|uv|},$$

(45)

and shown in Fig.3. In the second line, $N^{\pm\pm}$ are the event numbers where the first $\pm$ represent the sign of $\cos\theta_\mu$ and the second one is that of $\cos\theta_\pi$, respectively. $A_{\mu^+\gamma,\pi^-}\nu$ is related to the parameter $A_P$ by

$$A_{\mu^+\gamma,\pi^-}\nu = -\frac{s - 2m^2_\mu}{4(s + 2m^2_\mu)} A_P.$$  

(46)

In Fig.3, $\sqrt{s}$ dependence of $A_{\mu^+\gamma,\pi^-}\gamma$ is shown for $A_P = -1$. We can see that the asymmetry is already close to the maximal value at the B-factory energy.
It is straightforward to extend the above formula to other cases. We only present here formulae corresponding to eq.(42) for different decay modes of $\tau^-$. 

\begin{align}
\frac{d\sigma}{dt}(e^+e^- & \to \tau^+\tau^- \to \mu^+\gamma + V^-\nu)
= \sigma(e^+e^- \to \tau^+\tau^-)B(\tau^+ \to \mu^+\gamma)B(\tau^- \to V^-\nu)\frac{d\cos \theta_\mu}{2} \frac{d\cos \theta_V}{2} \\
& \times \left( 1 \pm \frac{s - 2m_{\tau}^2}{s + 2m_{\tau}^2} A_P \cos \theta_\mu \cos \theta_V \right), \\
\end{align}

(47)

where $+$ corresponds to the vector mesons with transverse polarization $V = \rho(T), a_1(T)$ and $-$ corresponds to those with longitudinal polarization $V = \rho(L), a_1(L)$. For leptonic decay, we obtain

\begin{align}
\frac{d\sigma}{dt}(e^+e^- & \to \tau^+\tau^- \to \mu^+\gamma + l^-\bar{\nu}\nu)
= \sigma(e^+e^- \to \tau^+\tau^-)B(\tau^+ \to \mu^+\gamma)B(\tau^- \to l^-\bar{\nu}\nu)\frac{d\cos \theta_\mu}{2} \frac{d\cos \theta_l}{2} dx \frac{2x^2}{2} \\
& \times \left\{ 3 - 2x - \frac{s - 2m_{\tau}^2}{s + 2m_{\tau}^2} (1 - 2x) A_P \cos \theta_\mu \cos \theta_l \right\}.
\end{align}

(48)

The measurement of the polarization of the vector mesons can be done by the analysis of the distribution of the two (or three) pions from the $\rho \ (a_1)$ meson decay [17].
In the case of $\tau^-$ decays into $\mu^-\gamma$ and $\tau^+$ decays via $V-A$ interaction, the $dR_a^{\tau^-\rightarrow fA}$ and $dR_b^{\tau^+\rightarrow fB}$ acquire extra minus signs. For example,

$$dR_a^{\tau^-\rightarrow \mu^-\gamma} = -\frac{dQ^\mu}{4\pi} \frac{1}{\Gamma} \frac{2}{8\pi} G_F^2 m_{\tau}^5 (|A_L|^2 - |A_R|^2) \begin{pmatrix} \sin \theta'_\mu \cos \phi'_\mu \\ \sin \theta'_\mu \sin \phi'_\mu \\ \cos \theta'_\mu \end{pmatrix},$$

$$dR_b^{\tau^+\rightarrow \pi^+\nu} = -\frac{dQ^\pi}{4\pi} \frac{1}{\Gamma} \frac{1}{8\pi} |V_{ud}|^2 f^2_F G_F^2 m_{\tau}^3 \begin{pmatrix} \sin \theta'_\mu \cos \phi'_\mu \\ \sin \theta'_\mu \sin \phi'_\mu \\ \cos \theta'_\mu \end{pmatrix},$$

where $(\theta'_\mu, \phi'_\mu)$ are the polar angle of the muon (pion) momentum in Frame 3 (Frame 2). The formula in eq.(42) can be applied to the $\tau^- \rightarrow \mu^-\gamma$ case by the replacement of $(\theta_\mu, \theta_\pi)$ by $(\theta'_\mu, \theta'_\pi)$, and therefore same angular and energy correlation holds as in the $\tau^+ \rightarrow \mu^+\gamma$ case. In a similar way, we can obtain the formulae corresponds to eqs.(47) and (48) for the $\tau^- \rightarrow \mu^-\gamma$ case by replacement of $(\theta_V, \theta_\tau)$ by $(\theta'_V, \theta'_\tau)$, where $\theta'_V$ and $\theta'_\tau$ are the angle between the vector meson (lepton) momentum and $\tau^+$ direction in Frame 2.

4 P and T asymmetries in LFV three body $\tau$ decays

In this section, we consider LFV three body decays i.e. $\tau \rightarrow 3\mu$, $\tau \rightarrow 3e$, $\tau \rightarrow \mu ee$, and $\tau \rightarrow e\mu\mu$. Within the approximation that the muon and electron masses are neglected, $\tau \rightarrow 3\mu$ and $\tau \rightarrow 3e$ (or $\tau \rightarrow \mu ee$ and $\tau \rightarrow e\mu\mu$) give the same formula, so that we only consider $\tau \rightarrow 3\mu$ and $\tau \rightarrow \mu ee$ processes. In these processes, we can define the P odd as well as T odd asymmetries of $\tau$ decays.

For $\tau^+ \rightarrow \mu^+\mu^+\mu^-$ decay, the effective Lagrangian is given by

$$\mathcal{L} = -\frac{4G_F}{\sqrt{2}} \left\{ m_\tau A_R \bar{\tau} \sigma^{\mu\nu} P_L \mu F_{\mu\nu} + m_\tau A_L \bar{\tau} \sigma^{\mu\nu} P_R \mu F_{\mu\nu} + g_1 (\bar{\tau} P_L \mu)(\bar{\mu} P_L \mu) + g_2 (\bar{\tau} P_R \mu)(\bar{\mu} P_R \mu) + g_3 (\bar{\tau} \gamma^\mu P_R \mu)(\bar{\mu} \gamma^\mu P_R \mu) + g_4 (\bar{\tau} \gamma^\mu P_L \mu)(\bar{\mu} \gamma^\mu P_L \mu) + g_5 (\bar{\tau} \gamma^\mu P_L \mu)(\bar{\mu} \gamma^\mu P_L \mu) + g_6 (\bar{\tau} \gamma^\mu P_R \mu)(\bar{\mu} \gamma^\mu P_R \mu) + \text{h.c.} \right\}.$$

With this Lagrangian in eq.(51), we can calculate the differential branching ratio $dB^{\tau^+\rightarrow 3\mu}$ and the spin dependence term $dR_b^{\tau^+\rightarrow 3\mu}$ in eq.(51). In order to calculate these quantities we first define the Lorentz frame (Frame 4) for the three body decays [12]. Frame 4 is
Figure 4: The coordinate system in the $\tau \to 3\mu$ calculation.

Figure 5: The relation between Frame 2 and Frame 4.
the rest frame of $\tau^+$ and we take $z$-direction to be $\mu^-$ momentum direction, and $xz$-plane to be the decay plane. The $x$-direction is determined so that the $x$-component of the momentum for the $\mu^+$ with larger energy is positive. The coordinate system are shown in Fig.4. Any four vector in Frame 4 is related to that in Frame 2 by the Euler rotation with three angles $(\theta, \phi, \psi)$ as follows (Fig.5):

\[
\xi_{\mu}^{\tau,4} = \left( \begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & \cos \phi & -\sin \phi & 0 \\
0 & \sin \phi & \cos \phi & 0 \\
0 & 0 & 0 & 1
\end{array} \right) \left( \begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & \cos \theta & 0 & \sin \theta \\
0 & 0 & 1 & 0 \\
0 & \sin \theta & 0 & \cos \theta
\end{array} \right) \left( \begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & \cos \psi & -\sin \psi & 0 \\
0 & \sin \psi & \cos \psi & 0 \\
0 & 0 & 0 & 1
\end{array} \right) \xi_{\mu}^{\tau,2},
\]  

(52)

where

\[
0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi, \quad 0 \leq \psi \leq 2\pi.
\]  

(53)

We also define the energy variables $x_1 = 2E_1/m_\tau$ and $x_2 = 2E_2/m_\tau$ where $E_1$ ($E_2$) is the energy of $\mu^+$ with a larger (smaller) energy in the rest frame of $\tau^+$.

With these angles and energy variables, the branching ratio and spin dependence term can be expressed as follows:

\[
\frac{d B^{\tau^+ \rightarrow 3 \mu}}{\Gamma} \propto \frac{m^5 G_F^2}{256\pi^5} \frac{1}{x_1 x_2} \frac{d \cos \theta}{d \phi} \frac{d \psi}{d \phi} X,
\]  

(54)

\[
\frac{d R_b^{\tau^+ \rightarrow 3 \mu}}{\Gamma} \propto \frac{m^5 G_F^2}{256\pi^5} \frac{1}{x_1 x_2} \frac{d \cos \theta}{d \phi} \frac{d \psi}{d \phi}
\]

\[
\times \left( -Y s_\theta c_\psi + Z (c_\theta c_\phi c_\psi - s_\phi s_\psi) + W (-c_\theta s_\phi s_\psi + c_\phi c_\psi) \right) + Y c_\theta c_\phi + Z s_\theta c_\phi + W s_\theta s_\phi,
\]  

(55)

where $s_\theta$ ($s_\phi$, $s_\psi$) and $c_\theta$ ($c_\phi$, $c_\psi$) represent $\sin \theta$ ($\sin \phi$, $\sin \psi$) and $\cos \theta$ ($\cos \phi$, $\cos \psi$), respectively. The functions $X$, $Y$, $Z$, and $W$ are defined as follows:

\[
X = \left( \frac{|g_1|^2}{16} + \frac{|g_2|^2}{16} + |g_3|^2 + |g_4|^2 \right) \alpha_1(x_1, x_2) + \left( |g_5|^2 + |g_6|^2 \right) \alpha_2(x_1, x_2)
\]

\[
+ \left( |eA_R|^2 + |eA_L|^2 \right) \alpha_3(x_1, x_2) - \text{Re}(eA_R g_4^* + eA_L g_3^*) \alpha_4(x_1, x_2)
\]

\[
- \text{Re}(eA_R g_6^* + eA_L g_5^*) \alpha_5(x_1, x_2),
\]  

(56)
\[ Y = \left( \frac{|g_1|^2}{16} - \frac{|g_2|^2}{16} + |g_3|^2 - |g_4|^2 \right) \alpha_1(x_1, x_2) + \text{Re}(eA_R g_4^* - eA_L g_3^*) \alpha_4(x_1, x_2) \\
- \text{Re}(eA_R g_4^* - eA_L g_3^*) \alpha_3(x_1, x_2) + \left( |g_5|^2 - |g_6|^2 \right) \beta_1(x_1, x_2) \\
+ \left( |eA_R|^2 - |eA_L|^2 \right) \beta_2(x_1, x_2), \quad (57) \]

\[ Z = \left( |g_5|^2 - |g_6|^2 \right) \gamma_1(x_1, x_2) + \left( |eA_R|^2 - |eA_L|^2 \right) \gamma_2(x_1, x_2) \\
- \text{Re}(eA_R g_4^* - eA_L g_3^*) \gamma_3(x_1, x_2) + \text{Re}(eA_R g_6^* - eA_L g_5^*) \gamma_4(x_1, x_2), \quad (58) \]

\[ W = -\text{Im}(eA_R g_4^* + eA_L g_3^*) \gamma_3(x_1, x_2) + \text{Im}(eA_R g_6^* + eA_L g_5^*) \gamma_4(x_1, x_2), \quad (59) \]

where \( e > 0 \) is the positron charge and functions \( \alpha_{1-5}, \beta_{1-2}, \) and \( \gamma_{1-4} \) are given in Appendix [3]. Notice that the \( Y \) and \( Z \) terms represent P odd quantities with respect to the \( \tau^+ \) spin in the rest frame of \( \tau^+ \) and the \( W \) term represents a T odd quantity. These are the same as P and T odd terms considered in the differential decay width of \( \mu^+ \rightarrow e^+e^- \) \[72,131\].

The differential cross section is obtained by substituting this into eq.(11). In the case that the opposite side \( \tau \) decays into \( \pi^-\nu \), we obtain after integrating over \( \phi_\pi, \psi, \theta_\tau, \) and \( \phi_\tau \)

\[
d\sigma(e^+e^- \rightarrow \tau^+\tau^- \rightarrow \mu^+\mu^- + \pi^-\nu) = \frac{\sigma(e^+e^- \rightarrow \tau^+\tau^-)B(\tau^- \rightarrow \pi^-\nu) \left( \frac{m^5_G}{128\pi^4} / \Gamma \right)}{2} \frac{d\cos \theta_\tau}{dx_1 \ dx_2 \ d\cos \theta \ d\phi} \times \left[ X - \frac{s - 2m^2}{s + 2m^2} \{ Y \cos \theta + Z \sin \theta \cos \phi + W \sin \theta \sin \phi \} \cos \theta_\pi \right]. \quad (60)\]

The terms \( X, Y, Z, \) and \( W \) can be extracted by the following (asymmetric) integrations.

\[
\int d\cos \theta \ d\phi \ d\cos \theta_\pi \frac{d^5 \sigma}{dx_1 dx_2 d\cos \theta \ d\phi \ d\cos \theta_\pi} \times \left\{ \sigma(e^+e^- \rightarrow \tau^+\tau^-)B(\tau^- \rightarrow \pi^-\nu) \left( \frac{m^5_G}{32\pi^3} / \Gamma \right) \right\}^{-1} = X, \quad (61)\]

\[
\int d\cos \theta \ d\cos \theta_\pi \ w(\cos \theta, \cos \theta_\pi) \frac{d^4 \sigma}{dx_1 dx_2 d\cos \theta \ d\cos \theta_\pi} \times \left\{ \sigma(e^+e^- \rightarrow \tau^+\tau^-)B(\tau^- \rightarrow \pi^-\nu) \left( \frac{m^5_G}{32\pi^3} / \Gamma \right) \right\}^{-1} = -\frac{(s - 2m^2)}{4(s + 2m^2)} Y, \quad (62)\]

17
\[
\int d\phi \, d\cos \theta_c \, w(\cos \phi, \cos \theta_c) \, \frac{d^4 \sigma}{dx_1 dx_2 \, d\phi \, d\cos \theta_c} \times \left\{ \sigma(e^+e^- \rightarrow \tau^+\tau^-)B(\tau^- \rightarrow \pi^-\nu) \left( \frac{m\mu G_F^2}{32\pi^3} / \Gamma \right) \right\}^{-1} = - \frac{(s - 2m^2)}{4(s + 2m^2)} Z , \quad (63)
\]
\[
\int d\phi \, d\cos \theta_c \, w(\sin \phi, \cos \theta_c) \, \frac{d^4 \sigma}{dx_1 dx_2 \, d\phi \, d\cos \theta_c} \times \left\{ \sigma(e^+e^- \rightarrow \tau^+\tau^-)B(\tau^- \rightarrow \pi^-\nu) \left( \frac{m\mu G_F^2}{32\pi^3} / \Gamma \right) \right\}^{-1} = - \frac{(s - 2m^2)}{4(s + 2m^2)} W . \quad (64)
\]

Notice that the function \( W \) represents CP violating LFV interaction. We can see that this is induced by the relative phase between the photon-penguin coupling constants \( A_L \) and \( A_R \) and the four-fermion coupling constants \((g_3 - g_6)\).

A similar formula can be obtained for the \( \tau^+ \rightarrow \mu^+e^+e^- \) decay. The effective Lagrangian for the \( \tau^+ \rightarrow \mu^+e^+e^- \) is given by
\[
\mathcal{L} = - \frac{4G_F}{\sqrt{2}} \left\{ m_\tau A_R \bar{\tau} \sigma^{\mu\nu} P_L \mu F_{\mu\nu} + m_\tau A_L \bar{\tau} \sigma^{\mu\nu} P_R \mu F_{\mu\nu} + \lambda_1(\bar{\tau} P_L \mu)(\bar{e} P e) + \lambda_2(\bar{\tau} P_L \mu)(\bar{e} P R e) + \lambda_3(\bar{\tau} P_R \mu)(\bar{e} P L e) + \lambda_4(\bar{\tau} P_R \mu)(\bar{e} P R e) + \lambda_5(\tau \gamma^\mu P_L \mu)(\bar{e} \gamma e P L e) + \lambda_6(\tau \gamma^\mu P_L \mu)(\bar{e} \gamma e P R e) + \lambda_7(\tau \gamma^\mu P_R \mu)(\bar{e} \gamma e P L e) + \lambda_8(\tau \gamma^\mu P_R \mu)(\bar{e} \gamma e P R e) + \lambda_9(\bar{\tau} \sigma^{\mu\nu} P L \mu)(\bar{e} \sigma_{\mu\nu} e) + \lambda_{10}(\bar{\tau} \sigma^{\mu\nu} P R \mu)(\bar{e} \sigma_{\mu\nu} e) + \text{h.c.} \right\} . \quad (65)
\]

In this calculation, we define Frame 4' which is almost the same as Frame 4 in the \( \tau^+ \rightarrow \mu^+\mu^-\mu^- \) case. The definition is obtained by the replacement of \( \mu^- \) of \( \tau^+ \rightarrow \mu^+\mu^-\mu^- \) by \( e^- \) of \( \tau^+ \rightarrow \mu^+e^-e^- \), the \( \mu^+ \) with a larger energy of \( \tau^+ \rightarrow \mu^+\mu^- \) by \( \mu^+ \) of \( \tau^+ \rightarrow \mu^+e^-e^- \), and \( \mu^- \) with a smaller energy of \( \tau^+ \rightarrow \mu^+\mu^- \) by \( e^- \) of \( \tau^+ \rightarrow \mu^+e^-e^- \). If we take the definition of \((\theta, \phi, \psi)\) in such a way that the same relation is satisfied as in eq.(62), the branching ratio and the spin dependence term are given by
\[
dB_{\tau^+ \rightarrow \mu^+e^-e^-} = \frac{1}{\Gamma} \frac{m\mu G_F^2}{256\pi^3} \, dx_1 \, dx_2 \, d\cos \theta \, d\phi \, d\psi \, X' , \quad (66)
\]
\[
dR_{\tau^+ \rightarrow \mu^+e^-e^-} = \frac{1}{\Gamma} \frac{m\mu G_F^2}{256\pi^3} \, dx_1 \, dx_2 \, d\cos \theta \, d\phi \, d\psi \times \left( Y' s_\theta c_\psi + Z'(c_\theta s_\phi c_\psi - s_\theta s_\psi) + W'(c_\theta s_\phi c_\psi + c_\theta s_\phi) \right) \left( Y' s_\theta s_\psi + Z'(c_\theta s_\phi s_\psi - s_\theta c_\phi) + W'(c_\theta s_\phi s_\psi + c_\theta c_\phi) \right) , \quad (67)
\]
where the functions $X'$, $Y'$, $Z'$, and $W'$ are given by

$$
X' = (|e_{AR}|^2 + |e_{AL}|^2) A_1(x_1, x_2) + Re (e_{AR} \lambda_6^* + e_{AL} \lambda_8^*) A_2(x_1, x_2) \\
+ Re (e_{AR} \lambda_6^* + e_{AL} \lambda_7^*) A_3(x_2) + (|\lambda_1|^2 + |\lambda_2|^2 + |\lambda_3|^2 + |\lambda_4|^2) A_4(x_1) \\
+ (|\lambda_5|^2 + |\lambda_8|^2) A_5(x_1, x_2) + (|\lambda_6|^2 + |\lambda_7|^2) A_6(x_2) \\
+ (|\lambda_9|^2 + |\lambda_{10}|^2) A_7(x_1, x_2) + Re (\lambda_1 \lambda_6^* + \lambda_4 \lambda_{10}^*) A_8(x_1, x_2),
$$

(68)

$$
Y' = -Re (e_{AR} \lambda_5^* - e_{AL} \lambda_8^*) A_2(x_1, x_2) + Re (e_{AR} \lambda_6^* - e_{AL} \lambda_7^*) A_3(x_1, x_2) \\
- (|\lambda_5|^2 - |\lambda_8|^2) A_5(x_1, x_2) + (|e_{AR}|^2 - |e_{AL}|^2) B_1(x_1, x_2) \\
+ (|\lambda_1|^2 + |\lambda_2|^2 - |\lambda_3|^2 - |\lambda_4|^2) B_2(x_1, x_2) + (|\lambda_6|^2 - |\lambda_7|^2) B_3(x_1, x_2) \\
+ (|\lambda_9|^2 - |\lambda_{10}|^2) B_4(x_1, x_2) + Re (\lambda_1 \lambda_6^* - \lambda_4 \lambda_{10}^*) B_5(x_1, x_2),
$$

(69)

$$
Z' = (|e_{AR}|^2 - |e_{AL}|^2) C_1(x_1, x_2) + Re (e_{AR} \lambda_5^* - e_{AL} \lambda_8^*) C_2(x_1, x_2) \\
+ Re (e_{AR} \lambda_6^* - e_{AL} \lambda_7^*) C_3(x_1, x_2) + (|\lambda_1|^2 + |\lambda_2|^2 - |\lambda_3|^2 - |\lambda_4|^2) C_4(x_1, x_2) \\
+ \{ |\lambda_6|^2 - |\lambda_7|^2 + Re (-2 \lambda_1 \lambda_6^* + 2 \lambda_4 \lambda_{10}^*) \} C_5(x_1, x_2) \\
+ (|\lambda_9|^2 - |\lambda_{10}|^2) C_6(x_1, x_2),
$$

(70)

$$
W' = \text{Im} (e_{AR} \lambda_5^* + e_{AL} \lambda_8^*) C_2(x_1, x_2) + \text{Im} (e_{AR} \lambda_6^* + e_{AL} \lambda_7^*) C_3(x_1, x_2) \\
+ \text{Im} (\lambda_1 \lambda_6^* + \lambda_4 \lambda_{10}^*) C_7(x_1, x_2).
$$

(71)

The functions $A_{1-7}, B_{1-4}, C_{1-6}$ are given in Appendix 3. The $X'$, $Y'$, $Z'$, and $W'$ can be extracted in the same way as in eqs. (61)–(64).

Next we consider the decay mode of $\tau^+ \rightarrow \mu^- e^+ e^+$. This case is different from above in the point that both $\tau \rightarrow e$ and $\mu \rightarrow e$ transitions are necessary. The effective Lagrangian for this process is given by

$$
\mathcal{L} = -\frac{4G_F}{\sqrt{2}} \{ g_1'(\bar{\tau}P_L e)(\bar{\mu}P_L e) + g_2'(\bar{\tau}P_R e)(\bar{\mu}P_R e) \\
+ g_3'(\bar{\tau}\gamma^\mu P_R e)(\bar{\mu}\gamma_\mu P_L e) + g_4'(\bar{\tau}\gamma^\mu P_L e)(\bar{\mu}\gamma_\mu P_R e) \\
+ g_5'(\bar{\tau}\gamma^\mu P_R e)(\bar{\mu}\gamma_\mu P_L e) + g_6'(\bar{\tau}\gamma^\mu P_L e)(\bar{\mu}\gamma_\mu P_R e) + \text{h.c.} \}.
$$

(72)

If we take a coordinate system similar to Frame 4, in which the larger (smaller) energy $\mu^+$ is replaced by the larger (smaller) energy $e^+$, $dB_{\tau^+ \rightarrow \mu^- e^+ e^+}$ and $dR_{\tau^+ \rightarrow \mu^- e^+ e^+}$ are given
by

\[ dB^{\tau^+ \rightarrow \mu^- e^+ e^+} = \frac{1}{\Gamma} \frac{m_\tau^2 G_F^2}{256\pi^5} dx_1 \, dx_2 \, d\cos \theta \, d\phi \, d\psi \, X'', \]

\[ dR^b_{\tau^+ \rightarrow \mu^+ e^+ e^-} = \frac{1}{\Gamma} \frac{m_\tau^2 G_F^2}{256\pi^5} dx_1 \, dx_2 \, d\cos \theta \, d\phi \, d\psi \times \left( \begin{array}{c} -Y'' s_\theta c_\psi + Z''(c_\phi c_\phi c_\psi - s_\phi s_\psi) \\ Y'' s_\theta s_\psi + Z''(-c_\theta c_\phi s_\psi - s_\phi c_\psi) \\ Y'' c_\theta + Z'' s_\theta c_\phi \end{array} \right), \]

where functions \( X'', Y'', \) and \( Z'' \) are given by

\[ X'' = \left( \frac{|g'_1|^2}{16} + \frac{|g'_2|^2}{16} + |g'_3|^2 + |g'_4|^2 \right) \alpha_1(x_1, x_2) + \left( |g'_5|^2 + |g'_6|^2 \right) \alpha_2(x_1, x_2), \]

\[ Y'' = \left( \frac{|g'_1|^2}{16} - \frac{|g'_2|^2}{16} + |g'_3|^2 - |g'_4|^2 \right) \alpha_1(x_1, x_2) + \left( |g'_5|^2 - |g'_6|^2 \right) \beta_1(x_1, x_2), \]

\[ Z'' = \left( |g'_5|^2 - |g'_6|^2 \right) \gamma_1(x_1, x_2), \]

where \( \alpha_{1-2}, \beta_1, \) and \( \gamma_1 \) are the same functions that we defined in \( \tau^+ \rightarrow \mu^+ \mu^+ \mu^- \) calculation. \( X'', Y'', \) and \( Z'' \) can be extracted by asymmetric integrations as before, but we cannot obtain information on CP violation in this case.

Notice that the above three cases exhaust all possibilities in the three body decay of \( \tau \) to \( e \) and/or \( \mu \) as long as we neglect the electron and muon masses compared to the \( \tau \) mass. Namely, the formula for other cases can be obtained by appropriate replacements of \( e \) and/or \( \mu \).

The formulae for LFV decays with \( \tau^- \) can be obtained in a similar substitution as the \( \tau \rightarrow \mu \gamma \) case. Using appropriate angles of \( \tau^- \) decay in Frame 3 and \( \tau^+ \) decay in Frame 2, \( dR_b \) gets an extra minus sign in eqs.\( (55), (67), \) and \( (74) \).

5 \( \tau \rightarrow \mu \nu \bar{\nu} \gamma \) process and background suppressions

In this section, we consider the background processes for the \( \tau \rightarrow \mu \gamma \) search, and we show that the measurement of angular distributions is useful in identifying the background process. In the muon decay, the physical background can be suppressed if we use polarized
muons [13]. In the following, we show a similar suppression mechanism holds for \( \tau \) decay if we use the spin correlation.

One of the main background for the \( \tau \to \mu \gamma \) search comes from the kinematical endpoint region of the \( \tau \to \mu \nu \bar{\nu} \gamma \) process where two neutrinos carry out a little energy at the rest frame of \( \tau \). In the following, we assume that \( \tau^+ \) decays into \( \mu^+ \nu \bar{\nu} \gamma \) and \( \tau^- \) decays through one of hadronic and leptonic decay processes. For the \( \tau^- \) decay, the differential branching ratio and the spin dependence term are given in eqs.(28)–(36). For \( \tau^+ \to \mu^+ \nu \bar{\nu} \gamma \), these quantities are given by

\[
\begin{align*}
\frac{d}{d\Omega} B_{\pm^2 G} &= \frac{1}{\Gamma} \frac{G_F^2 m_\tau^5 \alpha}{3 \times 2^{11} \pi^5} \, dx \, dy \, d\Omega \sin z \frac{\beta_\mu}{y} \, F, \\
\frac{d}{d\Omega} R_{b^2 G} &= \frac{1}{\Gamma} \frac{G_F^2 m_\tau^5 \alpha}{3 \times 2^{11} \pi^5} \, dx \, dy \, d\Omega \sin z \frac{\beta_\mu}{y} \\
&\times (-\beta_\mu G + H \cos z) \left( \begin{array}{c}
\sin \theta_\mu \cos \phi_\mu \\
\sin \theta_\mu \sin \phi_\mu \\
\cos \theta_\mu
\end{array} \right),
\end{align*}
\]

where \( x \) and \( y \) are the muon and photon energies normalized by \( m_\tau/2 \), respectively, and \( (\theta_\mu, \phi_\mu) \) is the polar coordinate of the unit vector of the muon momentum direction, all defined in the rest frame of \( \tau^+ \) (Frame 2). \( \beta_\mu = \sqrt{1 - 4r/x^2} \) with \( r \equiv m_\mu^2/m_\tau^2 \). The angle \( z \) is defined by \( z \equiv \pi - \theta_{\mu \gamma} \), where \( \theta_{\mu \gamma} \) is the angle between the muon and photon momentum in the same frame. These quantities can be obtained by a simple replacement from the formula of the differential decay width for the radiative muon decay presented in Ref.[18]. For completeness, the functions \( F, G, \) and \( H \) are given in Appendix B.

The background comes from the kinematical region near \( x = 1 + r \) and \( y = 1 - r \), at which the branching fraction vanishes. However, with finite detector resolutions, this kinematical region gives physical backgrounds. If we take the signal region as \( 1 + r - \delta x \leq x \leq 1 + r \) and \( 1 - r - \delta y \leq y \leq 1 - r \), the leading terms of the branching ratio and spin dependence term expanded in terms of \( r, \delta x, \) and \( \delta y \), after integrating over \( z \), are given by

\[
\begin{align*}
\frac{d}{d\Omega} B_{\pm^2 G} &\sim \frac{1}{\Gamma} \frac{G_F^2 m_\tau^5 \alpha}{3 \times 2^{11} \pi^5} \, d\Omega \mu \left( \delta x^4 \delta y^2 + \frac{8}{3} \delta x^3 \delta y^3 \right), \\
\frac{d}{d\Omega} R_{b^2 G} &\sim \frac{1}{\Gamma} \frac{G_F^2 m_\tau^5 \alpha}{3 \times 2^{11} \pi^5} \, d\Omega \mu \left( -\delta x^4 \delta y^2 + \frac{8}{3} \delta x^3 \delta y^3 \right) \left( \begin{array}{c}
\sin \theta_\mu \cos \phi_\mu \\
\sin \theta_\mu \sin \phi_\mu \\
\cos \theta_\mu
\end{array} \right). \hspace{1cm} (81)
\end{align*}
\]
Then after integrating over $\phi_\mu$, $\phi_\pi$, $\phi_\tau$, and $\theta_\tau$, the differential cross section for $e^+e^- \rightarrow \tau^+\tau^- \rightarrow \mu^+\nu\bar{\nu}\gamma + \pi^-\nu$ is given by

$$d\sigma(e^+e^- \rightarrow \tau^+\tau^- \rightarrow \mu^+\nu\bar{\nu}\gamma + \pi^-\nu) = \sigma(e^+e^- \rightarrow \tau^+\tau^-) B(\tau^- \rightarrow \pi^-\nu) \left( \frac{G_F^2 m_\tau^5 \alpha}{3 \times 2^9 \pi^4 / \Gamma} \right) \frac{d \cos \theta_\mu}{2} \frac{d \cos \theta_\pi}{2} \times \left\{ \left( \delta x^4 \delta y^2 + \frac{8}{3} \delta x^3 \delta y^3 \right) - \frac{s - 2m_\tau^2}{s + 2m_\tau^2} \left( -\delta x^4 \delta y^2 + \frac{8}{3} \delta x^3 \delta y^3 \right) \cos \theta_\mu \cos \theta_\pi \right\}. \quad (82)$$

If the photon energy resolution is worse than the muon energy resolution, the term $\delta x^4 \delta y^2$ is small compared to $(8/3)\delta x^3 \delta y^3$. In such a case, the angular distribution is similar to the $A_R = 0$, $A_L \neq 0$ case of the $\tau \rightarrow \mu \gamma$ angular distribution. See eqs.(25), (29), and (42). This feature is useful for the background suppressions for $\tau^+ \rightarrow \mu_R^+\gamma$ search because signal and background processes have different angular correlation. For $\tau^+ \rightarrow \mu_L^+\gamma$ search, the signal to background ratio is almost the same even if we take into account angular correlation.

A similar background suppression works for $\tau \rightarrow e\gamma$ case because eqs.(80) and (81) do not include the mass of the muon explicitly.

### 6 Summary

In this paper, we have calculated the differential cross sections of $e^+e^- \rightarrow \tau^+\tau^- \rightarrow f_B f_A$ processes, where one of $\tau$’s decays through LFV processes. Using spin correlations of $\tau^+\tau^-$, we show that the P odd asymmetry of $\tau \rightarrow \mu \gamma$ and $\tau \rightarrow e \gamma$ and P and T asymmetries of three body LFV decays of $\tau$ can be obtained by angular correlations. These P and T odd quantities are important to identify a model of new physics responsible for LFV processes.

We have also considered the background suppression of the $\tau \rightarrow \mu \gamma$ and $\tau \rightarrow e \gamma$ search by the angular distributions. We see that the analysis of the angular distributions are useful for the $\tau^+ \rightarrow \mu_R^+\gamma$ ($\tau^- \rightarrow \mu_L^-\gamma$) and $\tau^+ \rightarrow e_R^+\gamma$ ($\tau^- \rightarrow e_L^-\gamma$) searches.

We can obtain similar information in muon decay experiments if initial muons are polarized. Although highly polarized muons are available experimentally, a special setup for production and transportation of a muon beam is necessary for actual experiment.
The advantage of the $\tau$ case is that we can extract the information on $\tau$ spins by looking at the decay distribution of the other side of $\tau$ decay so that we do not need a special requirement for experimental setup.

Acknowledgments

The authors would like to thank A. I. Sanda for suggesting us to consider spin correlation in the LFV decay of $\tau$. They also would like to thank J. Hisano for useful discussions and comments.

A The derivation of the general formulae

In this section, we derive the eq.(1) from the amplitude in eq.(8).

By using the completeness relation of the fermion spinors, the amplitude squared is deformed to

$$
|\bar{A}(p_A + m_\tau)\gamma^\mu(p_B - m_\tau)B|^2
= \left| \sum_{\lambda_1=\pm} \sum_{\lambda_2=\pm} \bar{A}(p_A, \lambda_1)\bar{u}(p_A, \lambda_1)\gamma^\mu v(p_B, \lambda_2)\bar{v}(p_B, \lambda_2)B \right|^2
= \sum_{\lambda_1=\pm} \sum_{\lambda_2=\pm} \sum_{\lambda_1'=\pm} \sum_{\lambda_2'=\pm} \left( \bar{A}(p_A, \lambda_1)\bar{u}(p_A, \lambda_1)A \right)
\times (\bar{u}(p_A, \lambda_1)\gamma^\mu v(p_B, \lambda_2)\bar{v}(p_B, \lambda_2')\gamma^\nu u(p_A, \lambda_2')) (B v(p_B, \lambda_2')\bar{v}(p_B, \lambda_2)B) ,
$$

where $\lambda$'s are the spin eigenvalues. The spin summation can be performed by using the Bouchiat-Michel formulae as follows $[16, 19]$:

$$
|\bar{A}(p_A + m_\tau)\gamma^\mu(p_B - m_\tau)B|^2
= \alpha D^- \text{ Tr } [\bar{\gamma}(p_A + m_\tau)\gamma^\mu(p_B - m_\tau)\gamma^\nu] \alpha D^+
+ \alpha D^- \text{ Tr } [\bar{\gamma}(p_A + m_\tau)\gamma^\mu\gamma_5 p_B(p_B - m_\tau)\gamma^\nu] \rho_b^{D^-}
+ \rho_a^{D^-} \text{ Tr } [\gamma_5 p_A(p_A + m_\tau)\gamma^\mu(p_B - m_\tau)\gamma^\nu] \alpha D^+
+ \rho_a^{D^-} \text{ Tr } [\gamma_5 p_A(p_A + m_\tau)\gamma^\mu\gamma_5 p_B(p_B - m_\tau)\gamma^\nu] \rho_b^{D^-} ,
$$

where

$$
\alpha D^- = \frac{1}{2} \{ \bar{A}(p_A + m_\tau)A \} \quad \alpha D^+ = \frac{1}{2} \{ \bar{B}(p_B - m_\tau)B \} ,
$$

23
\[ \rho_a^{D_1} = \frac{1}{2} \{ \tilde{A} \gamma_5 \not{F}^a_A (p_A + m_\tau) A \} \quad \rho_b^{D_1} = \frac{1}{2} \{ \tilde{B} \gamma_5 \not{F}^b_B (p_B - m_\tau) B \} , \]

where \((s_A^a)^\mu\) and \((s_B^b)^\nu\) are four vectors which satisfy the following equations:

\[
\begin{align*}
    p_A \cdot s_A^a &= p_B \cdot s_B^b = 0, \\
    s_A^a \cdot s_A^b &= s_B^a \cdot s_B^b = -\delta^{ab}, \\
    \sum_{a=1}^{3} (s_A^a)_{\mu}(s_A^a)_{\nu} &= -g_{\mu\nu} + \frac{P_{\mu\nu}}{m^2}, \\
    \sum_{b=1}^{3} (s_B^b)_{\mu}(s_B^b)_{\nu} &= -g_{\mu\nu} + \frac{P_{\mu\nu}}{m^2}.
\end{align*}
\]

The second and third terms in eq. (84) vanish because the production parts are antisymmetric on \(\mu\) and \(\nu\) indices while the square of the electromagnetic current from \(e^+e^-\) collision is symmetric on \(\mu\) and \(\nu\) indices. Explicit calculation gives

\[
\begin{align*}
    \text{Tr} \left( (p_A + m_\tau) \gamma^\mu \gamma_5 \not{F}^b_B (p_B - m_\tau) \gamma^\nu \right) &= 4im_\tau \epsilon_{\mu\nu\rho\sigma} P_{\rho\sigma} (s_B^b)_{\sigma}, \\
    \text{Tr} \left[ \gamma_5 \not{F}^a_A (p_A + m_\tau) \gamma^\mu (p_B - m_\tau) \gamma^\nu \right] &= 4im_\tau \epsilon_{\mu\nu\rho\sigma} P_{\rho\sigma} (s_A^a)_{\sigma}.
\end{align*}
\]

\[
\sum_{\text{spin}} |\bar{v}_{e^+} \gamma_{\mu} u_{e^-}|^2 = \text{Tr} [\gamma \gamma^\mu \gamma^\nu] = 4p_{e^+\mu}p_{e^-\nu} + 4p_{e^+\nu}p_{e^-\mu} - 4g_{\mu\nu}p_{e^+} \cdot p_{e^-}.
\]

Using the narrow width approximation,

\[
\left| \frac{1}{q^2 - (m - \gamma)^2} \right|^2 \sim \frac{\pi}{m \Gamma} \delta(q^2 - m^2),
\]

the first and last terms in eq. (84) give formula (1) after the phase space integral.

**B The kinematical functions**

In this section, we list the kinematical functions used in the formulae of branching ratios.

The functions \(\alpha_{1-5}, \beta_{1-2}\), and \(\gamma_{1-4}\) in the \(\tau^+ \rightarrow \mu^+\mu^-\mu^-\) and \(\tau^+ \rightarrow \mu^-e^+e^-\) decay calculations are given as follows. These functions are the same as those used in \(\mu^+ \rightarrow e^+e^-\) decay \([12]\). \(x_1\) and \(x_2\) are given by \(x_1 = 2E_1/m_\tau\) and \(x_2 = 2E_2/m_\tau\).

\[
\begin{align*}
    \alpha_1(x_1, x_2) &= 8(2 - x_1 - x_2)(x_1 + x_2 - 1), \\
    \alpha_2(x_1, x_2) &= 2 \{x_1(1 - x_1) + x_2(1 - x_2)\},
\end{align*}
\]

24
\[ \alpha_3(x_1, x_2) = 8 \left\{ \frac{2x_2^2 - 2x_2 + 1}{1 - x_1} + \frac{2x_1^2 - 2x_1 + 1}{1 - x_2} \right\}, \] 
\[ \alpha_4(x_1, x_2) = 32(x_1 + x_2 - 1), \] 
\[ \alpha_5(x_1, x_2) = 8(2 - x_1 - x_2), \] 
\[ \beta_1(x_1, x_2) = \frac{2(x_1 + x_2)(x_1^2 + x_2^2) - 6(x_1 + x_2)^2 + 12(x_1 + x_2) - 8}{2 - x_1 - x_2}, \] 
\[ \beta_2(x_1, x_2) = \frac{8}{(1 - x_1)(1 - x_2)(2 - x_1 - x_2)} \times \left\{ 2(x_1 + x_2)(x_1^3 + x_2^3) - 4(x_1 + x_2)(2x_1^2 + x_1x_2 + 2x_2^2) + (19x_1^2 + 30x_1x_2 + 19x_2^2) - 12(2x_1 + 2x_2 - 1) \right\}, \] 
\[ \gamma_1(x_1, x_2) = \frac{4\sqrt{(1 - x_1)(1 - x_2)(x_1 + x_2 - 1)(x_2 - x_1)}}{2 - x_1 - x_2}, \] 
\[ \gamma_2(x_1, x_2) = 32\sqrt{\frac{x_1 + x_2 - 1}{(1 - x_1)(1 - x_2)}} \frac{(x_1 + x_2 - 1)(x_2 - x_1)}{2 - x_1 - x_2}, \] 
\[ \gamma_3(x_1, x_2) = 16\sqrt{\frac{x_1 + x_2 - 1}{(1 - x_1)(1 - x_2)}} \frac{(x_1 + x_2 - 1)(x_2 - x_1)}{2 - x_1 - x_2}, \] 
\[ \gamma_4(x_1, x_2) = 8\sqrt{\frac{x_1 + x_2 - 1}{(1 - x_1)(1 - x_2)}} \frac{(2 - x_1 - x_2)(x_2 - x_1)}{2 - x_1 - x_2}. \]

The functions \( A_{1-7}, \) \( B_{1-4}, \) and \( C_{1-6} \) in the \( \tau^+ \to \mu^+e^+e^- \) decay calculation are given by
\[ A_1(x_1, x_2) = \frac{8 \left( 2 - x_1 - 4x_2 + 2x_1x_2 + 2x_2^2 \right)}{1 - x_1}, \] 
\[ A_2(x_1, x_2) = -8 \left( x_1 + x_2 - 1 \right). \]
\[ A_3(x_2) = -8 \left(1 - x_2\right), \quad (107) \]

\[ A_4(x_1) = \frac{x_1 (1 - x_1)}{2}, \quad (108) \]

\[ A_5(x_1, x_2) = 2 \left(2 - x_1 - x_2\right) \left(x_1 + x_2 - 1\right), \quad (109) \]

\[ A_6(x_2) = 2 x_2 \left(1 - x_2\right), \quad (110) \]

\[ A_7(x_1, x_2) = -8 \left(4 - 5 x_1 + x_1^2 - 8 x_2 + 4 x_1 x_2 + 4 x_2^2\right), \quad (111) \]

\[ A_8(x_1, x_2) = -4 \left(1 - x_1\right) \left(x_1 + 2 x_2 - 2\right), \quad (112) \]

\[ B_1(x_1, x_2) = \frac{-8}{(1 - x_1)(2 - x_1 - x_2)} \times \left(-6 + 8 x_1 - 3 x_1^2 + 12 x_2 - 11 x_1 x_2 + 2 x_1^2 x_2 - 8 x_2^2 + 4 x_1 x_2^2 + 2 x_2^3\right), \quad (113) \]

\[ B_2(x_1, x_2) = \frac{-(1 - x_1)(2 - 2 x_1 + x_1^2 - 2 x_2 + x_1 x_2)}{2 \left(2 - x_1 - x_2\right)}, \quad (114) \]

\[ B_3(x_1, x_2) = \frac{2 \left(1 - x_2\right)(2 - 2 x_1 - 2 x_2 + x_1 x_2 + x_2^2)}{2 - x_1 - x_2}, \quad (115) \]

\[ B_4(x_1, x_2) = \frac{8}{2 - x_1 - x_2} \times \left(-10 + 16 x_1 - 7 x_1^2 + x_1^3 + 22 x_2 - 23 x_1 x_2 + 5 x_1^2 x_2 - 16 x_2^2 + 8 x_1 x_2^2 + 4 x_2^3\right), \quad (116) \]

\[ B_5(x_1, x_2) = \frac{4 \left(1 - x_1\right)(2 - 4 x_1 + x_1^2 - 4 x_2 + 3 x_1 x_2 + 2 x_2^2)}{2 - x_1 - x_2}, \quad (117) \]

\[ C_1(x_1, x_2) = \frac{-16 \left(x_1 + x_2 - 1\right) \sqrt{(1 - x_1)(1 - x_2)(x_1 + x_2 - 1)}}{(1 - x_1)(2 - x_1 - x_2)}, \quad (118) \]

\[ C_2(x_1, x_2) = \frac{8 \left(x_1 + x_2 - 1\right) \sqrt{(1 - x_1)(1 - x_2)(x_1 + x_2 - 1)}}{1 - x_1}, \quad (119) \]
\[ C_3(x_1, x_2) = -8 \frac{(1 - x_2) \sqrt{(1 - x_1)(1 - x_2)(x_1 + x_2 - 1)}}{1 - x_1}, \]  

(120)

\[ C_4(x_1, x_2) = \frac{(1 - x_1) \sqrt{(1 - x_1)(1 - x_2)(x_1 + x_2 - 1)}}{2 - x_1 - x_2}, \]  

(121)

\[ C_5(x_1, x_2) = \frac{4(1 - x_2) \sqrt{(1 - x_1)(1 - x_2)(x_1 + x_2 - 1)}}{2 - x_1 - x_2}, \]  

(122)

\[ C_6(x_1, x_2) = \frac{-16 \sqrt{(1 - x_1)(1 - x_2)(x_1 + x_2 - 1)(3 - x_1 - 2x_2)}}{2 - x_1 - x_2}, \]  

(123)

\[ C_7(x_1, x_2) = 8 \sqrt{(1 - x_1)(1 - x_2)(x_1 + x_2 - 1)}. \]  

(124)

Finally, the functions \( F, G, \) and \( H \) in the \( \tau \rightarrow \mu \nu \bar{\nu} \gamma \) decay calculation are given by

\[ F = F^{(0)} + rF^{(1)} + r^2F^{(2)}, \]  

(125)

\[ G = G^{(0)} + rG^{(1)} + r^2G^{(2)}, \]  

(126)

\[ H = H^{(0)} + rH^{(1)} + r^2H^{(2)}, \]  

(127)

where \( F^{(0)}-F^{(2)}, G^{(0)}-G^{(2)}, \) and \( H^{(0)}-H^{(2)} \) are the functions of \( x(\equiv 2E_\mu/m_\tau), y(\equiv 2E_\gamma/m_\tau), \) \( d(\equiv 1 + \beta_\mu \cos z) \) with \( \beta_\mu = \sqrt{1 - 4r/x^2} \) \( (r \equiv m_\mu^2/m_\tau^2) \) and \( z = \pi - \theta_{\mu\gamma}. \) These functions are given by

\[ F^{(0)}(x, y, d) = \frac{d}{-8 \left(-3 + 2x + 2y\right) \left(2x^2 + 2xy + y^2\right)} \]  

\[ + \frac{8}{x^2} \left\{ x^2 (2 + 4y) + y (-3 + y + 2) + x (-3 + y + 4y^2) \right\} \]  

\[ - 2x^2 y \left\{ -6 + (5 + 2y) + 2x (4 + 3y) \right\} \]  

\[ + 2x^3 y^2 (2 + y)^2 , \]  

(128)

\[ F^{(1)}(x, y, d) = \frac{d}{32 \left(x + y\right) \left(-3 + 2x + 2y\right)} + \frac{8}{x^2} \left\{ 6x^2 + (6 - 5y) y - 2x (4 + y) \right\} \]  

\[ - 8x \left\{ -4 - (-3 + y) y + 3x (1 + y) \right\} + 6x^2 y (2 + y) , \]  

(129)

\[ F^{(2)}(x, y, d) = \frac{d}{-32 \left(-4 + 3x + 3y\right)} + \frac{48 y}{x^2} , \]  

(130)
\[ G^{(0)}(x, y, d) = \frac{-8 x \{4 x^2 + y (-1 + 2 y) + x (-2 + 6 y)\}}{d} + \frac{4 x^2 \{-2 + 3 y + 4 y^2 + x (4 + 6 y)\}}{} - 4 x^3 y (2 + y) d , \quad (131) \]

\[ G^{(1)}(x, y, d) = \frac{32 (-1 + 2 x + 2 y)}{d^2} + \frac{8 x (6 x - y)}{d} - 12 x^2 (2 + y) , \quad (132) \]

\[ G^{(2)}(x, y, d) = \frac{-96}{d^2} , \quad (133) \]

\[ H^{(0)}(x, y, d) = \frac{-8 y (x + y) (-1 + 2 x + 2 y)}{d} + \frac{4 x y \{2 x^2 + 2 y (1 + y) + x (-1 + 4 y)\}}{} - 2 x^2 y^2 (-1 + 4 x + 2 y) d + 2 x^3 y^3 d^2 , \quad (134) \]

\[ H^{(1)}(x, y, d) = \frac{32 y (-1 + 2 x + 2 y)}{x d^2} - \frac{8 y (-2 + x + 5 y)}{d} - 4 x (3 x - 2 y) y + 6 x^2 y^2 d , \quad (135) \]

\[ H^{(2)}(x, y, d) = \frac{-96 y}{x d^2} + \frac{48 y}{d} . \quad (136) \]
References

[1] Y. Fukuda et al. [Super-Kamiokande Collaboration], Phys. Rev. Lett. 81, 1562 (1998);
E. Kearns, a talk given in International Conference on High Energy Physics, Osaka, Japan, (2000).

[2] T. Yanagida, in Proceedings of the Workshop on Unified Theory and Baryon Number of the Universe, edited by O. Sawada and A. Sugamoto (KEK, 1979) p.95;
M. Gell-Mann, P. Ramond, and R. Slansky, in Supergravity, edited by P. van Nieuwenhuizen and D. Freedman (North Holland, Amsterdam, 1979).

[3] T. P. Cheng and L. Li, Phys. Rev. Lett. 45, 1908 (1980).

[4] A. H. Chamseddine, R. Arnowitt and P. Nath, Phys. Rev. Lett. 49, 970 (1982);
R. Barbieri, S. Ferrara and C. A. Savoy, Phys. Lett. B119, 343 (1982);
L. Hall, J. Lykken and S. Weinberg, Phys. Rev. D27, 2359 (1983).

[5] L. J. Hall, V. A. Kostelecky and S. Raby, Nucl. Phys. B267, 415 (1986).

[6] R. Barbieri and L. J. Hall, Phys. Lett. B338, 212 (1994);
R. Barbieri, L. Hall and A. Strumia, Nucl. Phys. B445, 219 (1995);
N. Arkani-Hamed, H. Cheng and L. J. Hall, Phys. Rev. D53, 413 (1996);
J. Hisano, T. Moroi, K. Tobe and M. Yamaguchi, Phys. Lett. B391, 341 (1997).

[7] F. Borzumati and A. Masiero, Phys. Rev. Lett. 57, 961 (1986);
J. Hisano, T. Moroi, K. Tobe, M. Yamaguchi and T. Yanagida, Phys. Lett. B357, 579 (1995);
J. Hisano, T. Moroi, K. Tobe and M. Yamaguchi, Phys. Rev. D53, 2442 (1996);
J. Hisano, D. Nomura and T. Yanagida, Phys. Lett. B437, 351 (1998);
J. Hisano and D. Nomura, Phys. Rev. D59, 116005 (1999);
J. Ellis, M. E. Gomez, G. K. Leontaris, S. Lola and D. V. Nanopoulos, Eur. Phys. J. C14, 319 (2000);
J. Sato, K. Tobe and T. Yanagida, hep-ph/0010348.

[8] R. Kitano and K. Yamamoto, Phys. Rev. D62, 073007 (2000).
[9] N. Arkani-Hamed, S. Dimopoulos, G. Dvali and J. March-Russell, [hep-ph/9811448](https://arxiv.org/abs/hep-ph/9811448); Y. Grossman and M. Neubert, Phys. Lett. **B474**, 361 (2000).

[10] A. E. Faraggi and M. Pospelov, Phys. Lett. **B458**, 237 (1999); R. Kitano, Phys. Lett. **B481**, 39 (2000).

[11] Y. Okada, K. Okumura and Y. Shimizu, Phys. Rev. **D58**, 051901 (1998).

[12] Y. Okada, K. Okumura and Y. Shimizu, Phys. Rev. **D61**, 094001 (2000).

[13] Y. Tsai, Phys. Rev. **D4**, 2821 (1971); S. Kawasaki, T. Shirafuji, and S. Y. Tsai, Prog. Theor. Phys. **49**, 1656 (1973); S. Pi and A. I. Sanda, Phys. Rev. Lett. **36**, 1 (1976); K. Fujikawa and N. Kawamoto, Phys. Rev. **D14**, 59 (1976); A. Pais and S. B. Treiman, Phys. Rev. **D14**, 293 (1976); T. Hagiwara, S. Pi and A. I. Sanda, Annals Phys. **106**, 134 (1977); S. Pi and A. I. Sanda, Annals Phys. **106**, 171 (1977).

[14] H. Kuhn and F. Wagner, Nucl. Phys. **B236**, 16 (1984); C. A. Nelson, Phys. Rev. **D40**, 123 (1989); Phys. Rev. **D41**, 2805 (1990); Phys. Rev. **D53**, 5001 (1996); W. Fetscher, Phys. Rev. **D42**, 1544 (1990); W. Bernreuther, G. W. Botz, O. Nachtmann and P. Overmann, Z. Phys. **C52**, 567 (1991); J. Bernabeu, N. Rius and A. Pich, Phys. Lett. **B257**, 219 (1991); S. Goozovat and C. A. Nelson, Phys. Rev. **D44**, 2818 (1991); G. Couture, Phys. Lett. **B272**, 404 (1991); Phys. Lett. **B305**, 306 (1993); B. K. Bullock, K. Hagiwara and A. D. Martin, Phys. Lett. **B273**, 501 (1991); R. Alemany, N. Rius, J. Bernabeu, J. J. Gomez-Cadenas and A. Pich, Nucl. Phys. **B379**, 3 (1992); A. Aeppli and A. Soni, Phys. Rev. **D46**, 315 (1992); P. Privitera, Phys. Lett. **B288**, 227 (1992); W. Bernreuther, O. Nachtmann and P. Overmann, Phys. Rev. **D48**, 78 (1993); B. Ananthanarayan and S. D. Rindani, Phys. Rev. Lett. **73**, 1215 (1994); Phys. Rev.
D51, 5996 (1995);
C. A. Nelson, H. S. Friedman, S. Goozovat, J. A. Klein, L. R. Kneller, W. J. Perry and S. A. Ustin, Phys. Rev. D50, 4544 (1994);
Y. S. Tsai, Phys. Rev. D51, 3172 (1995);
J. Bernabeu, G. A. Gonzalez-Sprinberg, M. Tung and J. Vidal, Nucl. Phys. B436, 474 (1995).

[15] Y. Kuno and Y. Okada, Phys. Rev. Lett. 77, 434 (1996).

[16] C. Bouchiat and L. Michel, Nucl. Phys. 5, 416 (1958);
L. Michel, Suppl. Nuovo Cim. 14, 95 (1959).

[17] B. K. Bullock, K. Hagiwara and A. D. Martin, Nucl. Phys. B395, 499 (1993).

[18] Y. Kuno and Y. Okada, hep-ph/9909265, to appear in Rev. Mod. Phys.

[19] For a review of spin formalism, see for instance,
H. E. Haber, hep-ph/9405376.