Wave functions of a particle with polarizability in the Coulomb potential

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Abstract

Quantum mechanical scalar particle with polarizability is considered in the presence of the Coulomb field. Separation of variables is performed with the use of Wigner D-functions, the radial system of 15 equations is reduced to a single second order differential equation, which among the Coulomb term includes an additional interaction term of the form $\sigma\alpha^2/M^2r^4$. Various physical regimes exist that is demonstrated by examining the behavior of the curves of generalized squared radial momentum $P_2(r)$. Eigenstates of the equations can be constructed in terms of double confluent Heun functions. Numerical analysis proves the existence of the bound states in the system; the lowest energy level and corresponding solution are calculated based on generalization of Ritz variational procedure.

1 Separation of the variables

A particle with spin 0 and polarizability can be described with the use of tetrad formalism by the following equation (the main references to original papers concerning this model are given in [1])

$$\left[ \Gamma^\alpha(x) (\partial_\alpha + B_\alpha(x)) - m \right] \Psi(x) = 0,$$

$$\Gamma^\alpha(x) = \Gamma^\alpha e^{\alpha}_0(x), \quad B_\alpha(x) = \frac{1}{2} J^{ab} e^{\beta}_{(a)} \nabla_{e^{(b)\beta}}.$$

(1.1a)

In Minkowski space and in spherical coordinates and tetrad, eq. (1.1a) takes the form

$$\left[ \Gamma^0 \partial_0 + \Gamma^3 \partial_r + \frac{\Gamma^1 J^{31} + \Gamma^2 J^{32}}{r} + \frac{1}{r} \Sigma_{\theta,\phi} - m \right] \Psi(x) = 0,$$

(1.1b)

$$\Sigma_{\theta,\phi} = \Gamma^1 \partial_\theta + \Gamma^2 \frac{\partial_\phi + \cos \theta J^{12}}{\sin \theta}.$$

(1.1c)

General form for 15-component spherical wave function $\epsilon, j, m$ is (more details see in [2]; the notation for Wigner functions is used; $D_\sigma = D_{j-m,\sigma}(\phi, \theta, 0)$):

$$C(x) = e^{-i\epsilon t} C(r) D_0, \quad C_0(x) = e^{-i\epsilon t} C_0(r) D_0, \quad \Phi_0(x) = e^{-i\epsilon t} \Phi_0(x) D_0,$$

$$\tilde{C}(x) = e^{-i\epsilon t} \begin{vmatrix} C_1(r) & \Phi_1(r) \\ C_2(3) & \Phi_2(r) \\ C_3(r) & \Phi_3(r) \end{vmatrix} D_{-1}, \quad \tilde{\Phi}(x) = e^{-i\epsilon t} \begin{vmatrix} \Phi_1(r) \\ \Phi_2(r) \\ \Phi_3(r) \end{vmatrix} D_{+1}. $$
\[
\vec{E}(x) = e^{-i \epsilon t} \begin{bmatrix}
E_1(r) D_{-1} \\
E_2(r) D_0 \\
E_3(r) D_{+1}
\end{bmatrix}, \quad \vec{H}(x) = e^{-i \epsilon t} \begin{bmatrix}
H_1(r) D_{-1} \\
H_2(r) D_0 \\
H_3(r) D_{+1}
\end{bmatrix}.
\]

After separation of variables we obtain the radial system [2]

\begin{align}
-i (\epsilon + \frac{\alpha}{r}) C_0 - \left( \frac{d}{dr} + \frac{2}{r} \right) C_2 - \frac{\nu}{r} (C_1 + C_3) &= m C, \quad (1.2a) \\
-i (\epsilon + \frac{\alpha}{r}) C - \sigma \left( \frac{d}{dr} + \frac{2}{r} \right) E_2 - \frac{\nu \sigma}{r} (E_1 + E_3) &= m C_0, \\
i (\epsilon + \frac{\alpha}{r}) \sigma E_1 + \sigma \left( \frac{d}{dr} + \frac{1}{r} \right) H_1 - \frac{\nu}{r} C + \frac{i \nu \sigma}{r} H_2 &= m C_1, \\
i (\epsilon + \frac{\alpha}{r}) \sigma E_2 + \frac{d}{dr} C - \frac{i \nu \sigma}{r} (H_1 - H_3) &= m C_2, \\
i (\epsilon + \frac{\alpha}{r}) \sigma E_3 - \sigma \left( \frac{d}{dr} + \frac{1}{r} \right) H_3 - \frac{\nu}{r} C - \frac{i \nu \sigma}{r} H_2 &= m C_3, \quad (1.2b) \\
-i (\epsilon + \frac{\alpha}{r}) C = m \Phi_0, \quad -\frac{\nu}{r} C = m \Phi_1, \\
\frac{d}{dr} C = m \Phi_2, \quad -\frac{\nu}{r} C = m \Phi_3, \quad (1.2c)
\end{align}

\begin{align}
(\pm) \left[ -i (\epsilon + \frac{\alpha}{r}) \Phi_1 + \frac{\nu}{r} \Phi_0 \right] &= m E_1, \\
(\pm) \left[ -i (\epsilon + \frac{\alpha}{r}) \Phi_2 - \frac{d}{dr} \Phi_0 \right] &= m E_2, \\
(\pm) \left[ -i (\epsilon + \frac{\alpha}{r}) \Phi_3 + \frac{\nu}{r} \Phi_0 \right] &= m E_3, \\
(\pm) \left[ -i \left( \frac{d}{dr} + \frac{1}{r} \right) \Phi_1 + \frac{i \nu}{r} \Phi_2 \right] &= m H_1, \\
(\pm) \left[ \frac{i \nu}{r} (\Phi_1 - \Phi_3) \right] &= m H_2, \\
(\pm) \left[ +i \left( \frac{d}{dr} + \frac{1}{r} \right) \Phi_3 + \frac{i \nu}{r} \Phi_2 \right] &= m H_3. \quad (1.2d)
\end{align}

From (1.2c) it follows that \( \Phi_3 = + \Phi_1 \). Then from (1.2d) we get \( E_3 = + E_1, \ H_3 = - H_1, \ H_2 = 0 \). Finally, from (1.2b) it follows \( C_3 = + C_1 \). So, the collected restrictions are

\[
\Phi_3 = + \Phi_1, \quad C_3 = + C_1, \quad E_3 = + E_1, \quad H_3 = - H_1, \quad H_2 = 0. \quad (1.3)
\]

Taking into account (1.3), the radial system reads

\begin{align}
-i (\epsilon + \frac{\alpha}{r}) C_0 - \left( \frac{d}{dr} + \frac{2}{r} \right) C_2 - \frac{\nu}{r} 2C_1 &= m C, \quad (1.4a) \\
-i (\epsilon + \frac{\alpha}{r}) C - \sigma \left( \frac{d}{dr} + \frac{2}{r} \right) E_2 - \frac{\nu \sigma}{r} 2E_1 &= m C_0, \\
i (\epsilon + \frac{\alpha}{r}) \sigma E_1 + \sigma \left( \frac{d}{dr} + \frac{1}{r} \right) H_1 - \frac{\nu}{r} C &= m C_1,
\end{align}

2
\[ i (\epsilon + \frac{\alpha}{r}) \sigma E_2 + \frac{d}{dr} C - \frac{i \nu \sigma}{r} 2H_1 = m C_2 , \quad (1.4b) \]

\[ -i (\epsilon + \frac{\alpha}{r}) C = m \Phi_0 , \quad - \frac{\nu}{r} C = m \Phi_1 , \]

\[ \frac{d}{dr} C = m \Phi_2 , \quad (1.4c) \]

\[ (\pm) \left[ -i (\epsilon + \frac{\alpha}{r}) \Phi_1 + \frac{\nu}{r} \Phi_0 \right] = m E_1 , \]

\[ (\pm) \left[ -i (\epsilon + \frac{\alpha}{r}) \Phi_2 - \frac{d}{dr} \Phi_0 \right] = m E_2 , \]

\[ (\pm) \left[ -i \left( \frac{d}{dr} + \frac{1}{r} \right) \Phi_1 - \frac{i \nu}{r} \Phi_2 \right] = m H_1 . \quad (1.4d) \]

Taking expressions for \( C_i \) according to (1.4c), from (1.4d) one gets

\[ E_1 = 0 , \quad E_2 = (\pm) \left( - \frac{i \alpha}{m^2 r^2} \right) C , \quad H_1 = 0 . \quad (1.5) \]

Now, from (1.4b) with the use of (1.5), one obtains

\[ mC_0 = -i(\epsilon + \frac{\alpha}{r}) \pm \frac{i\alpha \sigma}{m^2 r^2} \frac{dC}{dr} , \quad mC_1 = - \frac{\nu}{r} C , \quad mC_2 = \frac{dC}{dr} \pm \frac{\alpha \sigma}{m^2 r^2} C . \quad (1.6) \]

Finally, we arrive at a second order differential equation \( C(r) \) (changing \( m^2 \) to \( -M^2 \), and \( 2\nu^2 \) to \( j(j+1) \)):

\[ \left( \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} + (\epsilon + \frac{\alpha}{r})^2 - M^2 - \frac{j(j+1)}{r^2} \pm \frac{\sigma \alpha^2}{M^2 r^4} \right) C = 0 . \quad (1.7) \]

2 Qualitative analysis of the radial equation

With the use of notation (the signs \( \pm \) can be included into the parameter \( \sigma \))

\[ \epsilon^2 - M^2 = -K^2 , \quad j(j+1) - \alpha^2 = J^2 , \quad \sigma \frac{\alpha^2}{M^2} = \Sigma^2 \quad (2.1) \]

the main equation (1.7) reads

\[ C(r) = \frac{1}{r} f(r) , \quad \frac{d^2 f}{dr^2} + \left( -K^2 + \frac{2 \epsilon \alpha}{r} - \frac{J^2}{r^2} - \frac{\Sigma^2}{r^4} \right) f(r) = 0 . \quad (2.2) \]

Let us examine the behavior of the squared radial momentum. Near the origin and at infinity we have

\[ P^2(r \to 0) \sim \frac{\Sigma^2}{r^4} , \quad P^2(r \to \infty) \sim (\epsilon^2 - M^2) . \]

To describe classical turning points it is convenient to factorize the expression for \( P^2(r) \)

\[ P^2(r) = \frac{(\epsilon^2 - M^2)r^4 + 2 \epsilon \alpha r^3 - J^2 r^2 + \Sigma^2}{r^4} = \]

\[ = \frac{(\epsilon^2 - M^2)(r - r_1)(r - r_2)(r - r_3)(r - r_4)}{r^4} = 0 . \quad (2.3) \]
The roots of the 4-th order polynomial obey relations below

\[- \frac{2e\alpha}{\epsilon^2 - M^2} = r_1 + r_2 + r_3 + r_4, \quad (2.4a)\]

\[- \frac{J^2}{\epsilon^2 - M^2} = r_1 r_2 + r_1 r_3 + r_1 r_4 + r_2 r_3 + r_2 r_4 + r_3 r_4, \quad (2.4b)\]

\[0 = r_1 r_3 r_4 + r_2 r_3 r_4 + r_3 r_1 r_2 + r_4 r_1 r_2, \quad (2.4c)\]

\[\frac{\Sigma^2}{\epsilon^2 - M^2} = r_1 r_2 r_3 r_4. \quad (2.4d)\]

First, let us consider the case of the \textbf{bound states}. There exist two different possibilities depending on the sign of \(\Sigma^2\).

\[I \quad \epsilon^2 - M^2 < 0, \quad \Sigma^2 = \sigma(\alpha^2/M^2) < 0; \quad (2.5a)\]

so, from (2.4) we conclude that two roots can be positive and two negative (or complex and conjugate to each other)

\[r_1 < 0, \quad r_2 < 0, \quad r_3 > 0, \quad r_4 > 0. \quad (2.5b)\]

that can be illustrated by Fig. 1.

With the use of (2.4a) and (2.4b) one expresses the roots \(r_1, r_2\) through two classical turning points \(r_3, r_4\)

\[r_1 + r_2 = \frac{2e\alpha}{M^2 - \epsilon^2} - r_3 - r_4, \quad r_1 r_2 = \frac{\Sigma^2}{\epsilon^2 - M^2} \frac{1}{r_3 r_4} \implies\]

\[r_1 = \frac{1}{2} \left[ - \left( r_3 + r_4 - \frac{2e\alpha}{M^2 - \epsilon^2} \right) - \sqrt{\left( r_3 + r_4 - \frac{2e\alpha}{M^2 - \epsilon^2} \right)^2 + \frac{4\Sigma^2}{(M^2 - \epsilon^2)r_3 r_4} } \right],\]

\[r_2 = \frac{1}{2} \left[ - \left( r_3 + r_4 - \frac{2e\alpha}{M^2 - \epsilon^2} \right) + \sqrt{\left( r_3 + r_4 - \frac{2e\alpha}{M^2 - \epsilon^2} \right)^2 + \frac{4\Sigma^2}{(M^2 - \epsilon^2)r_3 r_4} } \right]; \quad (2.5c)\]
To obtain two positive and two negative roots we require two conditions

\[-(r_1 + r_2) = r_3 + r_4 - \frac{2\epsilon\alpha}{M^2 - \epsilon^2} > 0,\]

\[\left(r_3 + r_4 - \frac{2\epsilon\alpha}{M^2 - \epsilon^2}\right)^2 + \frac{4\Sigma^2}{(M^2 - \epsilon^2)r_3r_4} > 0.\]  

(2.5d)

The case with two positive and two complex roots (with two negative real parts) is realized if two following conditions are imposed

\[-(r_1 + r_2) = r_3 + r_4 - \frac{2\epsilon\alpha}{M^2 - \epsilon^2} > 0,\]

\[\left(r_3 + r_4 - \frac{2\epsilon\alpha}{M^2 - \epsilon^2}\right)^2 + \frac{4\Sigma^2}{(M^2 - \epsilon^2)r_3r_4} < 0.\]  

(2.5e)

It should be noted that another possibility exists for the bound states at positive values for \(\Sigma^2\):

\[II \quad \epsilon^2 - M^2 < 0, \quad \Sigma^2 = C \frac{\alpha^2}{\sigma M^2} > 0;\]  

(2.6a)

from (2.4) we conclude that three roots can be positive and one negative

\[r_1 < 0, \quad r_2 > 0, \quad r_3 > 0, \quad r_4 > 0,\]

\[r_1 = -\frac{r_2r_3r_4}{r_3r_4 + r_3r_2 + r_4r_2}, \quad r_1 = (r_2 + r_3 + r_4 - \frac{2\epsilon\alpha}{M^2 - \epsilon^2}) < 0.\]  

(2.6b)

This case is illustrated in Fig. 2.

![Figure 2: Finite classical motion: \(r \in [0, r_2]\), \(r \in [r_3, r_4]\)](image)

The formulas for two roots \(r_1\) and \(r_2\) in terms of \(r_3, r_4\) are

\[r_1 = \frac{1}{2} \left[-\left(r_3 + r_4 - \frac{2\epsilon\alpha}{M^2 - \epsilon^2}\right) - \sqrt{\left(r_3 + r_4 - \frac{2\epsilon\alpha}{M^2 - \epsilon^2}\right)^2 + \frac{4\Sigma^2}{(M^2 - \epsilon^2)r_3r_4}}\right],\]

\[r_2 = \frac{1}{2} \left[-\left(r_3 + r_4 - \frac{2\epsilon\alpha}{M^2 - \epsilon^2}\right) + \sqrt{\left(r_3 + r_4 - \frac{2\epsilon\alpha}{M^2 - \epsilon^2}\right)^2 + \frac{4\Sigma^2}{(M^2 - \epsilon^2)r_3r_4}}\right];\]
but now the parameter is negative $\Sigma^2 > 0$, and correspondingly the root $r_2$ is positive.

The situation when one root is positive and three are negative is also possible

$$r_1 < 0, \quad r_2 > 0, \quad r_3 < 0, \quad r_4 > 0,$$

$$r_4 = -\frac{r_1 r_2 r_3}{r_1 r_2 + r_1 r_3 + r_2 r_3}; \quad (2.6d)$$

that is illustrated in Fig. 3.

Figure 3: Finite classical motion: $r \in [0, r_4]$

Now let us consider possible infinite motions. The first possibility is

$$III \quad \epsilon^2 - M^2 > 0, \quad \Sigma^2 = \sigma(\alpha^2/M^2) < 0; \quad (2.7a)$$

$$r_1 < 0, \quad r_2 > 0, \quad r_3 > 0, \quad r_4 > 0. \quad (2.7b)$$

that is illustrated in Fig. 4.

Figure 4: Infinite classical motion: $r \in [r_2, r_3], r \in [r_4, +\infty)$

Besides, another case can be realized if three roots are negative and one is positive

$$r_1 < 0, \quad r_2 < 0, \quad r_3 < 0, \quad r_4 > 0; \quad (2.7c)$$

that is illustrated in Fig. 5.
Finally, there exists one more case

$$IV \quad \epsilon^2 - M^2 > 0 , \quad \Sigma^2 = \sigma (\alpha^2 / M^2) > 0 ;$$

(2.8a)

when two root are positive and two roots are negative

$$r_1 < 0 , \quad r_2 < 0 , \quad r_3 > 0 , \quad r_4 > 0 ;$$

(2.8b)

that is illustrated in Fig. 6.

3 Analytical treatment of the problem

Let us turn back to eq. (2.2) and introduce a new variable

$$x = \frac{i(-K^2 \Sigma^2)^{1/4} r + \Sigma}{i(-K^2 \Sigma^2)^{1/4} r - \Sigma} , \quad r = \frac{-i \Sigma}{-(-K^2 \Sigma^2)^{1/4}} \frac{(x + 1)}{(x - 1)} .$$

(3.1a)

it is readily verified the main physical singularities $r = 0$ and $r = \infty$ in the new variable look as

$$r = \infty \implies x = +1 , \quad r = 0 \implies x = -1 .$$

(3.1b)

In general, the variable $x$ is complex-valued

$$x(r) = \frac{i Ar + \Sigma}{i Ar - \Sigma} , \quad (-K^2 \Sigma^2)^{1/4} = [(\epsilon^2 - M^2) \Sigma^2]^{1/4} = A .$$

(3.1c)
There exist four different combinations of signs

\[ (e^2 - M^2, \Sigma^2) \implies \begin{cases} (+, +) & (+, -) \\ (-, +) & (-, -) \end{cases} \]  

(3.2a)

Below we will consider only the case of bound states when \( e^2 - M^2 < 0 \), which corresponds to variants \((-+, +)\) and \((--, -)\). If \( \Sigma^2 \) (the case \((-+, +))\), the variable \( x \) is complex-valued

\[ x(r) = \frac{Ar + \Sigma}{iAr - \Sigma} = \frac{\Sigma^2 - A^2 r^2}{\Sigma^2 + A^2 r^2} - i \frac{2A \Sigma r}{\Sigma^2 + A^2 r^2} = e^{i\varphi(r)} , \quad |x(r)| = 1 ; \]  

(3.2b)

that is illustrated in Fig. 7.

![Figure 7: Variable x on complex plane](image)

If \( \Sigma^2 = -b^2 < 0 \) (the case \((-+, -))\), the variable \( x \) is real-valued

\[ x(r) = \frac{Ar + b}{Ar - b} , \quad x \in [-1, +1] . \]  

(3.2c)

With the variable \( x \), the differential equation (2.2) takes the form

\[
\begin{align*}
\frac{d^2}{dx^2} \Omega + \frac{2}{x-1} \frac{d}{dx} \Omega + \left( \frac{4K^2 \Sigma^2}{\sqrt{-K^2 \Sigma^2} (x+1)^4} + \frac{4K^2 \Sigma^2}{\sqrt{-K^2 \Sigma^2} (x-1)^4} - \frac{4i \epsilon \alpha \Sigma}{(-K^2 \Sigma^2)^{1/4}(x-1)^3} - \frac{J^2}{(x+1)^2} + \frac{2i \epsilon \alpha \Sigma - J^2(-K^2 \Sigma^2)^{1/4}}{(-K^2 \Sigma^2)^{1/4}(x-1)^2} + \frac{i \epsilon \alpha \Sigma - J^2(-K^2 \Sigma^2)^{1/4}}{(-K^2 \Sigma^2)^{1/4}(x+1)} + \frac{-i \epsilon \alpha \Sigma + J^2(-K^2 \Sigma^2)^{1/4}}{(-K^2 \Sigma^2)^{1/4}(x-1)} \right) f(x) &= 0. 
\end{align*}
\]

(3.3)

With the use of the notation

\[ (-K^2 \Sigma^2)^{1/4} = A , \quad -K^2 \Sigma^2 = A^4 , \quad A^2 = \sqrt{-K^2 \Sigma^2} = \pm iK \Sigma , \]

eq. (3.3) reads

\[
\begin{align*}
\frac{d^2}{dx^2} f + \frac{2}{x-1} \frac{d}{dx} f + \left( \frac{-4A^2}{(x+1)^4} - \frac{4A^2}{(x-1)^4} - \frac{4i \epsilon \alpha \Sigma}{A(x-1)^3} \right) f(x) &= 0.
\end{align*}
\]
\[- \frac{J^2}{(x+1)^2} + \frac{2ie\alpha\Sigma - J^2A}{A(x-1)^2} + \frac{ie\alpha\Sigma - J^2A}{A(x+1)} + \frac{-ie\alpha\Sigma + J^2A}{A(x-1)} \] 
\( f(x) = 0. \)  
(3.4)

Applying the substitution

\[ f(x) = (x + 1)^B(x - 1)^C \exp \left( \frac{Dx}{(x+1)(x-1)} \right) F(x) \]  
(3.5)

we arrive at the equation for \( F \)

\[
\frac{d^2F}{dx^2} + \left( \frac{2B}{x+1} + \frac{2C + 2}{x-1} - \frac{D}{(x+1)^2} - \frac{D}{(x-1)^2} \right) \frac{df}{dx} + 
\] 
\[
\frac{-D(B-1)}{(x+1)^3} - \frac{CDA + 4ie\alpha\Sigma}{A(x-1)^3} + \frac{D^2 + 8B^2 - 8B + 4CD + 4D - 8J^2}{8(x+1)^2} + 
\] 
\[
+ \frac{D^2A + 8CA - 4BD A + 8C^2A + 16ie\alpha\Sigma - 8J^2A}{8A(x-1)^2} + 
\] 
\[
+ \frac{D^2A + 2CD A - 2BD A - 8BC A + 2DA - 8B A + 8ie\alpha\Sigma - 8J^2A}{8A(x+1)} + 
\] 
\[
+ \frac{-D^2A - 2CD A + 2BD A + 8BC A - 2DA + 8B A - 8ie\alpha\Sigma + 8J^2A}{8A(x-1)} \] 
\( F = 0. \)  
(3.6)

When \( B, C, D \) are given as (see (3.5))

\[ B = \frac{1}{2}, \quad C = -\frac{1}{2}, \quad D = \pm \frac{4iK\Sigma}{A} = \pm 4A, \]

\[ f = \sqrt{\frac{x+1}{x-1}} \exp \left( \frac{Dx}{(x+1)(x-1)} \right) F(x) \]  
(3.7)

eq. (3.6) becomes more simple

\[
\frac{d^2F}{dx^2} + \left[ \frac{1}{x+1} + \frac{1}{x-1} - \frac{D}{(x+1)^2} - \frac{D}{(x-1)^2} \right] \frac{dF}{dx} + 
\] 
\[
+ \frac{1}{2A(x+1)^3(x-1)^3} \left[ (D^2A - 8J^2A - 2A + 16ie\alpha\Sigma)x^2 + 
\right. 
\] 
\[
\left. + (8DA - 32ie\alpha\Sigma)x + 2A - 16ie\alpha\Sigma - D^2A + 8J^2A \right] F = 0. \]  
(3.8)

It coincides with the double confluent Heun equation \[3, 4\] for \( H(\mu, \beta, \gamma, \delta, z) \):

\[
\frac{d^2H}{dx^2} + \left( \frac{1}{x+1} + \frac{1}{x-1} - \frac{\mu}{2(x+1)^2} - \frac{\mu}{2(x-1)^2} \right) \frac{dH}{dx} + 
\]
\[
\frac{\beta x^2 + (\gamma + 2\mu)x + \delta}{(x+1)^3(x-1)^3}H = 0
\]

with parameters
\[
\mu = 2D = \pm 8A, \quad \gamma = -\frac{16i\alpha \Sigma}{A},
\]
\[
\beta = -1 - 4J^2 + 8A^2 - 8i\alpha \Sigma, \\
\delta = +1 + 4J^2 - 8A^2 - 8i\alpha \Sigma;
\]

with additional constrain \(\beta + \delta = \gamma\).

### 4 Numerical simulations

To possibility for quantum mechanical bound states with energy \(\epsilon [\pm M]\) there must corresponds in classical description a finite region for classical motion. Let us examine the condition at which the 4-th order polynomial (this analysis will enable us to find left limiting boundary \(V\) for possible quantum energy levels, \(\epsilon \geq \epsilon_0\)):

\[
\Pi(r) = \frac{(r^2 - M^2)r^4 + 2\epsilon\alpha r^3 - j(j+1)r^2 + \sigma\alpha^2/M^2}{\epsilon^2 - M^2} = 0
\]

just starts to have a double root \(r_0\) at positive real axis

\[
\Pi(r) = (r - r_0)^2(r - a + ib)(r - a - ib) = 0.
\]

This is just the bifurcation value of \(e\).

It is convenient to measure \(\epsilon\) in unit of \(M\) introducing \(\epsilon = eM\). From comparing (4.1) and (4.2) we get the system of algebraic equations for \(r_0, a, b, e\)

\[
-(a^2 + b^2)r_0^2 + \frac{\alpha^2\sigma}{(-1 + e^2)M^4} = 0,
\]
\[
2r_0(a^2 + b^2 + ar_0) = 0,
\]
\[
-a^2 - b^2 - \frac{j(1 + j)}{(-1 + e^2)M^2} - 4ar_0 - r_0^2 = 0,
\]
\[
2(a + r_0) - \frac{2\epsilon\alpha}{M - e^2M} = 0.
\]

From whence it follows
\[
a = \frac{M(r_0 - e^2r_0) - e\alpha}{(-1 + e^2)M},
\]
\[
b^2 = \frac{(e\alpha((-1 + e^2)Mr_0 + e\alpha)}{(-1 + e^2)^2M^2},
\]
\[
r_0 = -\frac{3e\alpha + \sqrt{8(-1+e^2)j(1+j) + 9e^2\alpha^2}}{4(-1 + e^2)M},
\]
and we arrive at a rather complicated equation for $e$

$$-16(-1 + e^2)^2 j^3 - 8(-1 + e^2)^2 j^4 - 27e^4 \alpha^4 - 9e^3 \alpha^3 \sqrt{8(-1 + e^2)j(1 + j) + 9e^2 \alpha^2} - 4e(-1 + e^2)j\alpha(9e\alpha + 2\sqrt{8(-1 + e^2)j(1 + j) + 9e^2 \alpha^2}) - 4(-1 + e^2)j^2(-2 + e^2(2 + 9\alpha^2)) + 2e\alpha \sqrt{8(-1 + e^2)j(1 + j) + 9e^2 \alpha^2} + 32(-1 + e^2)^3 \alpha^2 \sigma = 0 \quad (4.4)$$

There exists additional constraint (condition for real-valuedness of $r_0$)

$$|e| > \frac{\sqrt{8j + 8j^2}}{\sqrt{8j + 8j^2 + 9\alpha^2}}. \quad (4.5)$$

Numerically one easily finds the value of $e_{\text{min}}$ which is a lower boundary for the existence of bound states.

At $j = 0, \sigma = -1, \alpha = 1, M = 1$ we obtain $e_{\text{min}} = 0.614659$, the plot of curve is illustrated in Fig. 8.

![Figure 8: $\Pi(r)$ at $j = 0, \sigma = -1, \alpha = 1, M = 1$](image)

As one can find by numerical simulations at $\sigma >$ possible values for $e$ lay outside the interval $[-1,1]$ and lead to $r_0 < 0$.

After obtaining information of regions of possible bound state energy $e$ for the system with small quantum number $j, \ldots$ we can start to construct numerical solution for the lowest eigenstate. To this goal we extend the known Ritz variational approach [5], performing simulations for simplicity at fixed values $M = 1, \sigma = -1, j = 0$. 
First we find the asymptotic behavior of the eq. (1.7) at the origin. Selecting the most singular terms we get
\[ f''(r) - \alpha^2 f(r)/r^4 = 0 \]
with the leading term in asymptotic solution of the form \( r \ll 1, \) \( f(r) = 1/(2\alpha) \exp(-\alpha/r). \) In a similar way at infinity we introduce the variable \( u = 1/r, \) rewriting the equation and selecting most singular terms only we get
\[ y''(u) - y(u)/u^4(1 - \epsilon^2) = 0 \quad (4.6a) \]
with leading term in asymptotic solution at \( r \to \infty \) as
\[ f(r) \sim r^{-1}(\exp(-\sqrt{1 - \epsilon^2}r)/(2\sqrt{1 - \epsilon^2})). \quad (4.6b) \]

Then the trial lowest eigenstate can be chosen as a product of these leading terms with yet unknown energy. Introducing notion \( \kappa = \sqrt{1 - \epsilon^2} \) first we find the normalization condition
\[ \int (\exp(-(\alpha/r - r\kappa))/(4\kappa))^2 r^2 dr = \frac{(\frac{\alpha}{\kappa})^{3/2} K_3(4\sqrt{\alpha\kappa})}{8\alpha^2\kappa^2} \quad (4.7) \]
where \( K_i \) is the Bessel function of imaginary argument of order \( i. \)

Multiplying from the left the left-hand side of eq. on normalized trial function \( f(r) \) and integrate over radial coordinate. As a result we get the expression of the form
\[ \sqrt{\frac{\alpha}{\kappa}} K_3(4\sqrt{\alpha\kappa}) (\alpha\kappa - \kappa^2 + \epsilon^2 - 1) + \alpha(2\epsilon - 1) K_2(4\sqrt{\alpha\kappa}) = 0 \quad (4.8) \]
which is a second order algebraic equation on \( \epsilon \) with elementary solution for roots.

Now, in accord with usual Ritz variational method we have to minimize one of the root on respect to \( \kappa \) that leads to the approximate value of the lowest eigenstate energy. The appropriate dependence is shown in Fig. 9 for \( \alpha = 1. \)

As one can see from Fig. 9 we indeed have a minimum for the second root, its value and location are \( \epsilon = 0.749279 \) at \( \kappa = 0.625342. \) The appropriate eigenfunction and squared radial momentum \( P^2(r) \) are shown in Fig. 10.
Figure 10: Plots of the lowest energy eigenfunction and $P^2(r)$ at $M = 1, \sigma = -1, j = 0, \alpha = 1$

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