Hadronic light-by-light scattering in the muon $g-2$: a Dyson-Schwinger equation approach

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We determine the hadronic light-by-light scattering contribution to the anomalous magnetic moment of the muon using the framework of Dyson-Schwinger and Bethe-Salpeter equations of QCD. Our result for the pseudoscalar ($\pi^0, \eta, \eta'$) meson exchange diagram is commensurate with previous calculations. In our calculation of the quark loop contribution we improve upon previous approaches by explicitly implementing constraints due to gauge invariance. The impact of transverse contributions, presumably dominated by vector meson poles, are only estimated at this stage. As a consequence, our value $a_{\mu}^{\text{LBL;quarkloop}} = (136 \pm 59) \times 10^{-11}$ is significantly larger. Taken at face value, this then leads to a revised estimate of the total $a_{\mu} = 116591891.0(105.0) \times 10^{-11}$.

I. INTRODUCTION

One of the most impressive successes of the standard model of particle physics is the determination of the anomalous magnetic moment of the electron. This quantity is determined both experimentally and theoretically to such a degree of precision that the underlying physical description is vindicated. However, when it comes to the question of new physics, the anomalous magnetic moment of the muon is an even more interesting quantity, see e.g. [1–3] for reviews. This is due to the large mass of the muon as compared to the electron, which leads to an enhanced sensitivity to physics in and beyond the standard model. Experimental efforts at Brookhaven and theoretical efforts of the past ten years have pinned $a_{\mu}$ down to the $10^{-11}$ level, leading to significant deviations between theory [1] and experiment [4]:

Experiment: 116 592 089.0(63.0) × 10^{-11} , \hspace{1cm} (1)

Theory: 116 591 790.0(64.6) × 10^{-11} . \hspace{1cm} (2)

Whilst the theoretical and experimental values are determined to comparable errors, the central values give rise to a discrepancy at the 3.3 $\sigma$ confidence level. This difference has been present for a number of years and can be interpreted as a signal for the existence of physics beyond the standard model. However, to clearly distinguish between New Physics and possible shortcomings in the SM calculations the uncertainties present in both experimental and theoretical values of $a_{\mu}$ need to be further reduced.

The greatest uncertainties in the theoretical determination of $a_{\mu}$ are encountered in the hadronic contributions, i.e. those terms which involve QCD beyond perturbation theory. The most prominent of these is given by the vacuum polarisation tensor dressing of the QED vertex, see Fig. (1a). Fortunately it can be related to experimental data of $e^+e^-$-annihilation and $\tau$-decay via dispersion relations and the optical theorem, thus resulting in a precise determination with systematically improvable errors [3]. Considered individually, its (leading and subleading order) contribution to the anomalous magnetic moment of the muon is [2]

\[ [6.903.0(52.6) - 100.3(1.1)] \times 10^{-11} . \hspace{1cm} (3) \]

Although currently these uncertainties dominate the error of the theoretical result in Eq. (2) it is foreseeable that future experiments reduce this error below that of another, more problematic source. This is the hadronic light-by-light (LBL) scattering diagram, shown in Fig. (1b). This contribution cannot be directly related to experiment and must hence be calculated entirely through theory. The central object in such a calculation is the photon four-point function. It receives important contributions from the small momentum region below 2 GeV, where perturbative QCD breaks down and non-perturbative methods are imperative. Recent determinations of $a_{\mu}^{\text{LBL}}$ are provided in Table I. Although the magnitude of the LBL contribution is much smaller than the one from vacuum polarization it is significant because its error is of a comparable size. Taken together, with the errors added in quadrature, the

![FIG. 1: The two classifications of corrections to the photon-muon vertex function: (a) hadronic vacuum polarisation contribution to $a_{\mu}$. The vertex is dressed by the vacuum polarisation tensor $\Pi_{\mu\nu}$; (b) the hadronic light-by-light scattering contribution to $a_{\mu}$.](image)

## TABLE I: Recent calculations of the hadronic light-by-light scattering contribution to the anomalous magnetic moment of the muon.

| Ref. | $a_{\mu}^{\text{LBL}}$ |
|------|----------------------|
| [5]  | 116(40) × 10^{-11}   |
| [6]  | 105(26) × 10^{-11}   |
| [7]  | 110(40) × 10^{-11}   |
| [8, 10] | 89(15) × 10^{-11}   |
hadronic contributions constitute the largest uncertainty in the standard model determination of the anomalous magnetic moment of the muon.

The theoretical approaches to determine the LBL contribution are centered around two main ideas. One is chiral symmetry, its breaking pattern and the associated low energy effective descriptions of QCD [11]; the other is the large-$N_c$ expansion of the four-photon function and the associated ordering of diagrams. These ideas have been put together in [12] and led to various refined calculations of LBL within the frameworks of large-$N_c$ and vector meson dominance [3–15], the extended Nambu-Jona-Lasinio model (ENJL) [7,16], the (very similar) hidden local symmetry model [8–10], or a non-local chiral quark model [17,18], see also [6] for a summary. Although in terms of diagrams individual contribution are of varying size in these approaches, their sum leads to consistent results as can be inferred from Table I. In all these calculations the (pseudoscalar) meson exchange contributes the most and the meson loop has been found to be small. A possible explanation of the latter is given in [14–16]. As a result, we quote the recent value for LBL $\sigma_{\text{LBL}} = 105(26) \times 10^{-11}$ proposed in Ref. [6], which also agrees with the one in [5].

One of the most important goals for these and future calculations is the reduction of the model dependence and subsequently of the systematic error involved in these calculations. Since LBL is non-perturbative in nature, all estimates in Table I are plagued by systematic model dependencies. It is therefore desirable to also explore other calculational tools which have the potential to go beyond these limitations. Certainly, lattice gauge theory is one such method. However, due to the multi-scale nature of the problem no reliable estimates for LBL have been extracted on the lattice so far. This multi-scale nature also makes EFT methods less desirable as it proves more difficult to impose suitable matching conditions.

Another non-perturbative method, well suited to accommodate for largely different scales is the framework of Dyson-Schwinger and Bethe-Salpeter equations [19–22]. In the past years this approach has been used to study fundamental properties of QCD such as confinement and dynamical chiral symmetry breaking. On the other hand the approach served as a tool for hadron physics. In this work we expand upon this and apply the formalism to a calculation of the LBL contribution to the muon anomalous magnetic moment. To this end we separate different contributions to the light-by-light four-point function according to their topology of gluon exchange and their status with respect to the large-$N_c$ expansion. Diagrammatically this translates to considering resummations of planar diagrams involving gluon exchange. In this scheme we then determine the dressed quark-loop diagram and an approximation in terms of pseudoscalar meson ($\pi^0, \eta, \eta'$) exchange contributions. In principle, the off-shell meson amplitudes involved in these diagrams could be calculated from inhomogeneous Bethe-Salpeter equations. Here, due to numerical complexity, in this work we resort to a commonly used ansatz that extrapolates on-shell wave functions. Our results are then compared with the ones of previous approaches. First results of our analysis have been published in Ref. [23]. Here we discuss our method in much more detail and present new and more elaborate results for the quark-loop diagram. The paper is organized thus: in section II we recall the definition of the light-by-light scattering amplitude and focus upon the pseudoscalar pole contributions; in section III we introduce our Dyson-Schwinger approach and discuss the necessary truncation schemes; in section IV we present and discuss our results. We conclude in section V.

II. THE LBL SCATTERING AMPLITUDE

In the hadronic light-by-light scattering contribution, Fig. 1(b) the muon is coupled to an external photon source via the hadronic photon four-point function $\Pi_{\mu\nu\alpha\beta}$, defined through

$$\Pi_{\mu\nu\alpha\beta}(q_1,q_2,q_3) = \int_{xyz} e^{iq_1 \cdot x + iq_2 \cdot y + iq_3 \cdot z} \langle j_\mu(x) j_\nu(y) j_\alpha(z) \rangle ,$$  \hspace{1cm} (4)

where $\int_{xyz} = \int d^4x \int d^4y \int d^4z$ represents integration over four-dimensional space, $q_1,2,3$ are the photon momenta that are connected to the muon line, and $j_\mu$ is the electromagnetic quark current

$$j_\mu = \frac{2}{3} \bar{u} \gamma_\mu u - \frac{1}{3} i \epsilon_\mu d - \frac{1}{3} \bar{s} \gamma_\mu s + \frac{2}{3} \bar{c} \gamma_\mu c .$$  \hspace{1cm} (5)

A detailed discussion of this object can be found in the literature, see e.g. [14, 16]. Instead of working directly with the light-by-light scattering diagram given in Fig. 1(b) it is more convenient to follow the strategy employed in Ref. [24, 25]. Here gauge symmetry is exploited to construct quantities that are finite. Through use of the Ward-Takahashi-identity $k_\mu \Pi_{\mu\nu\alpha\beta} = 0$ it follows via differentiation that

$$\Pi_{\rho\nu\alpha\beta} = - k_\mu \frac{\partial}{\partial k_\rho} \Pi_{\mu\nu\alpha\beta} =: - k_\mu \tilde{\Pi}_{(\rho)\nu\alpha\beta},$$  \hspace{1cm} (6)

which serves as definition of the five-point-function $\tilde{\Pi}_{(\rho)\nu\alpha\beta}$. Here $k = q_1 + q_2 + q_3$ is the momentum of the external photon. The virtue of the derivative is that it lowers the dimensionality of the integral thus ensuring that integrals employing necessary truncation schemes; in section IV we present and discuss our results. We conclude in section V.
Here $D_{\mu\nu}(q)$ are perturbative photon propagators (we use Feynman gauge) with momenta $q_i$. The perturbative muon propagators are given by $S(p)$.

The anomalous magnetic moment can now be obtained by applying the appropriate projection operator to Eq. (7)

$$a_\mu = \frac{1}{48m_\mu} \text{tr} \left[ (iP + m_\mu)(\gamma_\sigma, \gamma_\rho) (iP + m_\mu) \Gamma_{\sigma\rho} \right]_{k \equiv 0},$$

that we write here in Euclidean convention for later convenience. Using Eqs. (6–9) we are able to evaluate the light-by-light scattering contributions for an arbitrary photon four-point function. What remains now is the specification of this four-point function within our approach.

### A. Expansion using EFT approaches

As already mentioned in the introduction, chiral and large-$N_c$ arguments have been established to expand the full LBL scattering amplitude into the diagrammatic parts shown in Fig. 2 [1]. These diagrams belong to different orders with respect to chiral and large-$N_c$ counting. Whereas the meson exchange diagrams and the quark-loop diagram are leading in large-$N_c$, it is the meson-loop diagram that is leading in the chiral counting. Thus a priori, one does not know which expansion is to be preferred. Therefore it is certainly interesting that all explicit calculations of these contributions seem to favor the $N_c$-counting scheme; meson-loop contributions have been found to be suppressed. Arguments as to why this is the case have been presented in Ref. [12].

Strictly speaking, however, one does not actually perform a large-$N_c$ expansion as this would necessitate the inclusion of an infinite number of resonances. Instead, only the lowest lying meson exchange contributions in the pseudoscalar, scalar and axialvector channel have been subsumed. Here, the pseudoscalar $\pi^0$-exchange has been identified as the leading contribution, followed by $\eta$ and $\eta'$-exchange.

Concerning the pseudoscalar (PS) exchange contribution a few remarks are in place. The photons in the exchange diagrams are coupled to the PS mesons via the PS-$\gamma\gamma$ form-factor, $F_{\text{PS}\gamma\gamma}$. It is evident that there are two limiting features of the pseudoscalar-pole approximation. The first is the actual provision of the form-factors themselves, which are in general subject to systematic errors depending on how they are modeled or calculated. The second is the procedure under which the form-factor is taken off-shell. Previous approaches mainly used vector meson dominance ideas to determine this form factor and there has been an extensive debate as to whether and how short distance constraints have to be implemented [5, 6, 13, 14, 18]. Rather than employ the principles of vector-meson dominance and construct an ansatz for the on-shell/off-shell form-factor, we wish to calculate it from first principles. This is possible within the framework of Dyson-Schwinger and Bethe-Salpeter equations using a well explored and successful truncation scheme [20].

As for the quark-loop diagram, different interpretations have been given in the literature. Whereas in [12] it has been argued that the quark loop is a separate contribution that has to be added to the other two, in many other approaches it has been treated as a complementary one, which is only added in the large spacelike momentum region, say above a typical cutoff for an effective model. In our functional approach, described below, it is clearly the first point of view that is correct. Moreover, as we will see, the quark loop is subject to large dressing effects not only for the quark propagators in the
loop but also for the quark-photon vertices. This will be the main result of our work.

B. Expansion using functional methods

From a functional integral approach to QCD, featuring quarks and gluons as the fundamental degrees of freedom, the analogous picture to what is normally considered in the literature is shown in Fig. 3 where we give an expansion in terms of non-perturbatively dressed one-particle irreducible Green’s functions. The basic idea of this expansion is not a separation of long distance and short distance scales, but rather a separation of different classes of diagrams based on their topology. Clearly, the expansion is such that no double counting of diagrams is involved. By considering a restricted subset of contributions in which only diagrams with a planar topology are resummed, we effectively adhere to the \( N_c \)-counting scheme as favored in the EFT approaches mentioned above. Though there are similarities between the two pictures, since we work with a truncated formulation of exact-QCD rather than an effective field theory there are some differences that we will comment on here to avoid confusion. First of all, our quarks are to be interpreted in the same way as those extracted via Lattice QCD; they are characterised by momentum dependent dressing functions that interpolate between the current and constituent quark limits, cf. the discussion below Fig. 11. Secondly, the quark-photon coupling is a non-perturbative form-factor and not merely a tree-level bare vertex; it can be calculated self-consistently for a given truncation scheme. Finally, the planar resummation of gluons is related to the \( T \)-matrix of quark-antiquark scattering and contains meson poles that can be associated with pseudoscalars, vectors, scalars etc. This will be exploited below, where we return to the conventional meson exchange picture to approximate these contributions.

We wish to emphasize that the expansion displayed in Fig. 3 has been used successfully in a different context already in Ref. [26]. There \( \pi - \pi \) scattering has been considered using similar quark-box and ladder exchange parts as displayed in Fig. 3. In this setup, the authors of Ref. [26] could reproduce the isospin 0 and 2 scattering lengths in exact agreement with Weinberg’s low energy results. Moreover, in Ref. [27] it has been checked, that the corresponding resonant expansion similar to the one displayed in Fig. 2 is a good approximation to the ladder exchange part of Fig. 3. Note that in both these calculations the quark-box diagram had to be added to the ladder exchange or the resonant ’meson-exchange’ part respectively. We believe that these results add further support to our approach.

1. Quark-loop contribution

Within our proposed truncation, the quark-loop is composed of dressed quark propagators and dressed quark-photon vertices. On expanding these one-particle irreducible Green’s functions, within the rainbow-ladder approximation, we find planar-like diagrams such as the ones shown in Fig. 4 (all propagators are fully dressed), where in fact infinite ladders of gluons are taken into account. Should we consider corrections beyond rainbow-ladder, such as those considered in Ref. [28, 29], one would also include diagrams in which the gluons have self-interactions as well as crossed-ladder components. Taking into account such corrections is, however, beyond the scope of the present work.

Considering this contribution to the muon-photon vertex, we obtain the diagrams as shown in Fig. 5 where we have shown permutation of the external photon legs but have omitted the topologies that merely involve reversal of the quark-spin line (these give identical contributions and hence constitute a factor of two). As is well-known, these diagrams are individually logarithmically divergent with only their sum finite and convergent; thus one employs the aforementioned trick, Eq. (6), of taking the derivative of the photon four-point function [24, 25].

Since this is now to be applied to loop integrals over non-perturbative quantities, namely the quark-propagator, it is no longer possible to reduce the integration to be five-dimensional as in the case of perturbative studies. More generally, on considering the planar nature of the diagrams, one must deal with 8-dimensional integrals which necessitate Monte-Carlo methods [30]. However, for reasons of calculational simplicity we actually integrate in nine. We did check, however, that we were able to reproduce the well-known perturbative results for the electron loop contribution to the anomalous magnetic of the electron and the muon [31–33]. Additionally, due to the somewhat involved Dirac algebra [34, 35] we will content ourselves with taking the quark-photon vertices inside the quark-loop contribution to be: (a) bare; (b) 1BC; (c) full BC. The precise meaning of these ab-
FIG. 6: Ladder-exchange contribution (upper equation) and ring-ladder contribution (lower equation) to the photon four-point amplitude.

\[
\begin{align*}
T & = 1 + \cdots \\
T & = 1 + \cdots \\
T & = P^2 \rightarrow -M^2
\end{align*}
\]

FIG. 7: The \( T \)-matrix in Rainbow-Ladder approximation. (a) shows the series expansion in terms of dressed quarks and gluon, whilst (b) represents Dyson’s equation. (c) Shows the pole-ansatz for the \( T \)-matrix on-mass shell. The abbreviations and the relation to the full quark-photon vertices will become clear in sections [11B] and [1V B]. The extension to employ the numerically calculated non-perturbative form of the vertex will be explored in a later publication. The results of our calculation are presented in section [I V].

2. Ladder-exchange and ladder-ring contribution

Two contributions that are leading and sub-leading in large-\( N_c \) respectively are the so-called ladder-exchange and ladder-ring diagrams of Fig. 3. These infinite ladder resummations are in fact related to the \( T \)-matrix of bound-state theory in a certain approximation scheme (that produces planar diagrams). Thus, another way to portray these contributions is given in Fig. 6. The \( T \)-matrix in rainbow-ladder approximation is given in Fig. 7. At this point, we make it clear that there is no conflict nor double counting between the quark-loop and ladder-exchange diagrams, as they clearly consider and resum different topologies of diagrams.

As it stands, the full \( T \)-matrix is a very complicated object to solve in its entirety though its structure admits several approximations and simplifications [36]. The one which we employ here is similar to the viewpoint taken by Effective Field Theory approaches; that is, we consider pole contributions to be dominant. Now, since it is well-known that such an infinite gluon-ladder resummation dynamically generates bound-state poles, one can expand the \( T \)-matrix in terms of meson pole contributions as shown in Fig. 7(c). On mass shell we then have a unique definition of the Bethe-Salpeter amplitude, described below in Fig. 12 that gives the form-factor describing coupling of a meson to two quarks. From this point the (on-shell) pseudoscalar-photon-photon form-factor can be defined and calculated, giving rise to the ‘leading’ pseudoscalar meson exchange part, as shown in Fig. 8 and Fig. 9.

In a similar fashion, the ring-ladder diagram contains contributions akin to the pion loop on meson mass-shell. However, since these are generally considered to be sub-leading we will not consider them further here and instead concentrate on the quark-loop and ladder-exchange diagrams. After we present our approach and formalism in the next section, results will be discussed in section [I V].

III. FRAMEWORK

The dressed quark propagator is one of the most important quantities in the covariant description of mesons. It encodes non-perturbative properties of QCD such as dynamical mass generation and the realization of a non-zero condensate. Its equation of motion, the quark Dyson-Schwinger equation (DSE) displayed in Fig. 10, also contains the dressed gluon propagator and a dressed quark-gluon vertex. Whereas the dressed gluon propagator in Landau gauge is a well-known
such functions on the renormalisation point independent quark mass function. The dependence of momentum only. While this is certainly a severe approximation in principle, in practice it turned out to be very successful as concerns the calculation of masses and electromagnetic properties of mesonic observables \[28, 29, 41\]. While the parameters of the model are tuned such that it reproduces the experimental values for the masses and decay constants of the pion, it also reproduces the pion charge radius and \(\pi\gamma\gamma\) transition form factors on the percent level. In the vector channel the agreement with experimental masses and decay constants is on the five and ten percent level. Thus, while one has to keep in mind possible systematic caveats, we nevertheless believe that such a model is an excellent starting point for a systematic evaluation of hadronic LBL.

In Euclidean momentum space, the renormalized dressed gluon and quark propagators in the Landau gauge are given by

\[
D_{\mu\nu}(p) = \left( \delta_{\mu\nu} - \frac{p_{\mu}p_{\nu}}{p^2} \right) \frac{Z(p^2; \mu^2)}{p^2},
\]

\[
S_F(p) = \frac{Z_f(p^2; \mu^2)}{ip^\mu + M(p^2)} = \frac{1}{ip^\mu A(p^2; \mu^2) + B(p^2; \mu^2)},
\]

where \(Z(p^2; \mu^2)\) is the gluon dressing function, \(Z_f(p^2; \mu^2)\) is the quark wave-function and \(M(p^2)\) is the renormalisation point independent quark mass function. The dependence of such functions on the renormalisation point \(\mu^2\) will be implicitly assumed from here on. The quark dressing functions can be recombined into the quark mass and wave-function by \(M(p^2) = B(p^2)/A(p^2)\) and \(Z_f(p^2) = 1/A(p^2)\).

These propagators may be obtained by solving their respective Dyson–Schwinger equations. The DSE for the quark propagator, shown diagrammatically in Fig.10 is written

\[
S^{-1}(p) = Z_2 S_0^{-1}(p) + \Sigma(p),
\]

\[
\Sigma(p) = g^2 C_F Z_{1F} \int \frac{d^4q}{(2\pi)^4} \Gamma_\nu(q, p) D_{\mu\nu}(k) \gamma_\mu S_F(q),
\]

where \(\Sigma(p)\) is the quark self-energy, \(k = p - q\) and the Casimir \(C_F = 4/3\) stems from the colour trace. We introduced the reduced quark-gluon vertex \(\Gamma_\nu(q, p)\) defined by \(\Gamma^\nu_\mu(q, p) = ig^2 \kappa^\nu_\mu\). The bare inverse quark propagator is \(S_0^{-1}(p) = i\not{p} + m\). The renormalisation factors are \(Z_{1F} = Z_2 / \tilde{Z}_3\) for the quark-gluon vertex, \(Z_2\) for the quark propagator and \(\tilde{Z}_3\) for the ghost dressing function.

The scalar dressing functions of the quark DSE are solved for by appropriate projections of Eq. (14). This is a coupled non-linear integral equation that is solvable provided we know the gluon dressing function and the structure of the quark-gluon vertex. In the rainbow approximation both are specified by Ansätze, with in particular the choice \(\Gamma_\nu(q, p) := \Gamma_{YM}^\nu(k^2)\gamma_\nu\), with scalar function \(\Gamma_{YM}\) representing the non-perturbative dressing of the quark-gluon vertex and \(k = q - p\) the gluon momenta. Here, the gluon dressing function \(Z(k^2)\) from Eq. (12) and the Yang-Mills part \(\Gamma_{YM}^\nu(k^2)\) of the quark-
gluon vertex are combined to form a phenomenological effective interaction. For the Maris-Tandy (MT) model \[44\] this function is given by

\[
Z(k^2)\Gamma_{YM}(k^2) = \frac{4\pi}{g^2} \left( \frac{\pi}{\omega^2} D k^4 \exp(-k^2/\omega^2) \right) + \frac{2\pi g}{\omega^2} \log \left( \frac{\tau + \left( 1 + k^2/\Lambda_{QCD}^2 \right)^2}{\tau} \right) \times \left[ 1 - \exp \left( -k^2/\left[ 4m_t^2 \right] \right) \right],
\]

with

\[
m_t = 0.5 \text{ GeV}, \quad \tau = \frac{e^2}{1}, \quad \Lambda_{QCD} = 0.234 \text{ GeV}.
\]

This interaction corresponds to a Gaussian distribution in the infrared that provides for sufficient interaction strength to generate DCSB, together with the one-loop behavior of the running coupling at large, perturbative, momenta. The latter is mandatory to provide for the correct short distance behavior of the quark propagator. The remaining parameters \( \omega \) and \( D \) essentially constitute a single one-parameter family of solutions for which pion observables remain comparable, via \( \omega D = (0.72 \text{GeV})^3 \).

For the convenience of the reader, in Fig. [11] we again show the two dressing functions \( Z_f(p^2) \) and \( M(p^2) \) that characterize the non-perturbative quark propagator, obtained by solving Eq. \([13]\) using the Maris-Tandy interaction \([44]\). Clearly, in the mass function there are three distinguished momentum regions. In the infrared, the quark propagator is essentially constant displaying the behavior of a constituent quark. Then for \( 1 \text{ GeV}^2 < p^2 < 10 \text{ GeV}^2 \) there is a region of rapid change, where the quark mass function follows the well known \( 1/p^2 \) behavior expected from the operator product expansion. For even larger momenta and non-vanishing current quark mass the quark mass function behaves logarithmically as expected for a current quark. The fully dressed quark propagator thus naturally interpolates between the constituent and current quark picture. We consider this feature of the Dyson-Schwinger approach to QCD as an advantage compared with effective models such as the ENJL model.

\section{Bethe-Salpeter equation}

The chiral symmetry preserving truncation for the Bethe-Salpeter equation, consistent with the rainbow-approximation above, is given by the ladder approximation

\[
\Gamma^{q\bar{q}}(p; P) = \int \frac{d^4 k}{(2\pi)^4} K_{tuv;rs}(p, k; P) \left[ S_F(k_+)\Gamma^{q\bar{q}}(k; P)S_F(k_-) \right]_{sr}
\]

with the kernel \( K_{tuv;rs} \) given by

\[
K_{tuv;sr}(q, p; P) = \frac{g^2}{k^2} Z(k^2) \frac{\Gamma_{YM}(k^2)}{Z_{1F}} \left( \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \left[ \gamma^a \frac{\lambda^a}{2} \gamma^\mu \right]_{ts} \left[ \gamma^a \frac{\lambda^a}{2} \gamma^\nu \right]_{ru},
\]

see Fig. [12] for a graphical representation. Here \( \Gamma^{q\bar{q}}(p; P) \) is the Bethe-Salpeter vertex function corresponding to a pseudoscalar quark anti-quark bound-state, specified below. The momenta \( k_+ = k + P/2 \) and \( k_- = k - P/2 \) are such that the total momentum \( P \) of the meson is given by \( P = k_+ - k_- \) and the relative momentum \( k = (k_+ + k_-)/2 \). The Latin indices \((t, u, r, s)\) of the kernels refer to colour, flavour and Dirac structure.

The form of the kernel Eq. \((17)\) is uniquely determined from the axial-vector Ward-Takahashi identity and ensures, that the pion is a Goldstone-boson in the chiral limit without any fine-tuning of parameters. It also ensures that important constraints from chiral symmetry such as the Gell-Mann-Oakes-Renner relation are satisfied.

In general, the covariant structure of the Bethe-Salpeter vertex function, \( \Gamma^{q\bar{q}}(p; P) \), determines the quantum numbers of the bound-state under consideration. In particular, a pseudoscalar meson is completely specified by the following form

\[
\Gamma^{q\bar{q}}(p; P) = \gamma_5 \left[ F_1^{q\bar{q}}(p; P) - iPF_2^{q\bar{q}}(p; P) \right] - i\not{p} \left[ F_3^{q\bar{q}}(p; P) - |P, p'\rangle F_4^{q\bar{q}}(p; P) \right],
\]

This amplitude is obtained through solution of Eq. \((16)\) on-mass shell: \( P^2 = -m_{q\bar{q}}^2 \) in Euclidean space. While \((18)\) rep-

![FIG. 12: The homogeneous Bethe-Salpeter equation for the meson amplitude.](image-url)
and necessary for the calculation of the PS vertex. This quantity is genuinely non-perturbative in nature.

The hat in Eq. (19) is the chiral limit value of the leptonic decay constant, and represents a quark-anti-quark bound-state, physical mesons are defined as matrices in flavour space built out of the $q\bar{q}$ amplitudes. This then leads to the same decomposition as in Eq. (13) but with flavour matrix valued quantities $\Gamma$ and $F_i$. In the following, however, we will keep the flavour index implicit in the isospin-limit considered herein the pion amplitude differs from the $\pi\bar{u}u$ amplitudes have a similar form as the quark mass function in the dynamical information are identical (up to normalization). This is different for the quark-photon vertex, shown diagrammatically in Fig. 13. Qualitatively, the amplitudes a have a similar form as the quark mass function in the chiral limit. For large momenta and up to logarithmic corrections they fall off like $1/p^2$, which is a necessary condition to correctly describe the anomalous decay of the pion and to reproduce the asymptotics of the pion form factor [48].

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\begin{equation}
\hat{F}_1^\pi(p; P) := \lambda_3 B(p^2)/f_\pi \tag{19}
\end{equation}

where $B(p^2)$ is the scalar dressing function of the quark, $f_\pi$ is the chiral limit value of the leptonic decay constant, and $\lambda_3$ is a Gell-Mann matrix that represents the flavour structure. The hat in Eq. (19) indicates that the object is matrix valued in flavour space. The expressions for the calculation of $f_\pi$ and the normalization condition of the Bethe-Salpeter amplitude together with details on the numerical procedure for dealing with the BSE are given in [45]. For the convenience of the reader we display the resulting Bethe-Salpeter amplitudes for $p^2 = -m_\pi^2$ and $p, P = 0$ in Fig. 13. Qualitatively, the amplitudes have a similar form as the quark mass function in the chiral limit. For large momenta and up to logarithmic corrections they fall off like $1/p^2$, which is a necessary condition to correctly describe the anomalous decay of the pion and to reproduce the asymptotics of the pion form factor [48].

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{FIG_13}
\caption{Normalised Bethe-Salpeter amplitudes $F_1$ and $F_2$ of the pion solved with the Maris-Tandy interaction (leading Chebyshev component) [44].}
\end{figure}

B. Quark-photon Vertex

An important quantity for the determination of the LBL contribution to the muon $g - 2$ is the dressed quark-photon vertex. This quantity is genuinely non-perturbative in nature and necessary for the calculation of the PS$\gamma\gamma$ form-factor. It describes the coupling of a fully dressed quark to a photon and is dominated by QCD corrections. As a function of one Lorentz and two spinor indices, it can be decomposed into twelve Dirac structures

\begin{equation}
\Gamma_{\mu}(P, k) = \sum_{i=1}^{12} \lambda_i(P, k) V_{\mu i}(P, k) \tag{20}
\end{equation}

where $V_{\mu i}(P, k)$ represents the basis components, and $\lambda_i(P, k)$ the non-perturbative dressing functions. A common basis is that of Ball and Chiu [47], in which $V_{\mu i}(P, k)$ is split into terms that are transverse and non-transverse with respect to the photon momentum. The Ward-Takahashi identity and regularity assumptions constrain the form of the non-transverse part in terms of quark propagator functions

\begin{equation}
\Gamma_{\mu}^{BC}(k, P) := \left[ \gamma_\mu A(k_+^2) + 2 \left( \frac{1}{2} A(k_+^2) - A(k_-^2) \right) \right. \left. + (k_+ + k_-)(k_+ + k_-) \frac{1}{2} A(k_+^2) - A(k_-^2) \right. \left. + i(k_+ + k_-) \mu B(k_+^2) - B(k_-^2) \right] \tag{21}
\end{equation}

leaving only the strictly transverse pieces undetermined. Eq. (21) can therefore be seen as an approximation of the full vertex and has been used in situations, where the full vertex cannot be determined numerically.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{FIG_14}
\caption{Inhomogeneous BS equation for the quark-photon vertex, in rainbow-ladder approximation.}
\end{figure}

A more sophisticated approach, however, is to solve the inhomogeneous Bethe-Salpeter equation of the quark-photon vertex, shown diagrammatically in Fig. 14. It is given by

\begin{equation}
\Gamma_{\mu tu}(k, P) = Z_1 \gamma_{tu}^\mu + \int_q K_{tu;rs} \left[ S_F(k_+) \Gamma_{\mu}(q, P) S_F(k_-) \right]_{sr} \tag{22}
\end{equation}

where $Z_1$ is the renormalisation factor associated with the quark-photon vertex. By using the same interaction kernel as in the BSE for mesons, Eq. (17), not only do we achieve self-consistency within the truncation scheme, but also by virtue of its symmetry preserving nature we satisfy the Ward-Takahashi identity. Consequently, the non-transverse part of the vertex, given in Eq. (21), is nicely reproduced numerically, with transverse terms additionally generated [48, 52, 54].

We note here that such a determination of the quark-photon vertex automatically contains poles in the time-like region corresponding to vector meson exchange. Thus, presupposing that vector-meson dominance is an important feature in
we plot the form factor as a function of the two photon vertices and where \( k \) dynamical, selfconsistent solution of Eq. (22) for our calculation of the PS \( \rightarrow \gamma \gamma \) form-factor and the resulting meson exchange contribution to LBL. Unfortunately, because of the numerical complexity we have to restrict ourselves to the exact longitudinal part given by Eq. (27) in the quark-loop diagram. It will be a subject of future work to overcome this limitation.

C. The PS \( \rightarrow \gamma \gamma \) form-factor

The coupling of the exchanged pseudoscalar mesons to photons is the quantity that is central to the resonant expansion of Fig. 2. In impulse approximation, consistent with the rainbow-ladder truncation scheme introduced in sections III, the Bethe-Salpeter amplitude is connected via a quark-triangle to the fully-dressed quark-photon vertex, as shown in Fig. 13. For a pseudoscalar PS we have

\[
\Lambda_{\mu\nu}^{\text{PS}\gamma\gamma} (k_1, k_2) = 2e^2 N_c \int k \left[ i \tilde{Q}_\nu \Gamma_\nu (k_2, p_{12}) S_F (p_2) \times \tilde{\Gamma}^{\text{PS}} (p_{23}, P) S_F (p_3) \tilde{\Gamma}_\mu (k_1, p_{31}) S_F (p_3) \right],
\]

(23)

where \( k_1 \) and \( k_2 \) are the outgoing photon momenta, \( p_1 = q, p_2 = q - k_2 \) and \( p_3 = q + k_1 \) are the quark momenta and \( p_{ij} = (p_i + p_j) / 2 \). The factor of two stems from exchange of the two photon vertices and \( \tilde{Q} = \text{diag} [2/3, -1/3, -1/3] \) gives the quark’s charge. The PS vertex \( \tilde{\Gamma}^{\text{PS}} \) is explicitly matrix valued in flavour space. It is defined as

\[
\pi^0: \quad \tilde{\Gamma}^{\pi^0} = \frac{1}{\sqrt{2}} \text{diag} \left[ \Gamma^{u\bar{u}}, -\Gamma^{d\bar{d}}, 0 \right]
\]

\[
\eta^8: \quad \tilde{\Gamma}^{\eta^8} = \frac{1}{\sqrt{6}} \text{diag} \left[ \Gamma^{u\bar{u}}, \Gamma^{d\bar{d}}, -2\Gamma^{s\bar{s}} \right]
\]

\[
\eta^0: \quad \tilde{\Gamma}^{\eta^0} = \frac{1}{\sqrt{3}} \text{diag} \left[ \Gamma^{u\bar{u}}, \Gamma^{d\bar{d}}, \Gamma^{s\bar{s}} \right],
\]

(24)

for the pseudoscalar mesons. The \( \Gamma^{q\bar{q}} \) are solutions of Eq. (16). In addition we work in the isospin-limit (\( \Gamma^{u\bar{u}} = \Gamma^{d\bar{d}} \)). Since the quantities in Eq. (24) are defined in the singlet-octet basis we have to rotate in order to obtain the \( \eta- \eta' \) amplitudes

\[
\tilde{\Gamma}^{\eta} = \cos \theta \tilde{\Gamma}^{\eta^8} - \sin \theta \tilde{\Gamma}^{\eta^0},
\]

\[
\tilde{\Gamma}^{\eta'} = \sin \theta \tilde{\Gamma}^{\eta^8} + \cos \theta \tilde{\Gamma}^{\eta^0},
\]

(25)

where we have taken \( \theta = -15.4^\circ \). The pseudoscalar electromagnetic form-factor can be described by a single scalar function, \( F^{\pi^0\gamma\gamma} \). For the pion this function can be given a natural normalization via the Abelian anomaly

\[
\Lambda_{\mu\nu}^{\pi\gamma\gamma} (k_1^2, k_2^2) = \frac{\alpha_{\text{em}}}{\pi f_\pi} \varepsilon_{\mu\nu\alpha\beta} k_1^\alpha k_2^\beta F^{\pi\gamma\gamma} (k_1^2, k_2^2),
\]

(26)

where \( \alpha_{\text{em}} \) is the fine structure constant and \( f_\pi \) the pion decay constant. The definition of the prefactors is such that \( F^{\pi\gamma\gamma} (0,0) = 1 \). The \( \eta- \eta' \)-mesons have the same tensor structure.

Note that the form factors determined here do not accurately reflect all effects due to the topological mass of the \( \eta_0 \), simply because the \( U_A (1) \)-anomaly is not represented correctly in the Maris-Tandy model\(^2\). In the form factors this may be a minor problem. The effect is larger, however, in the meson propagators attached to the form factors. We therefore prefer to use the experimental masses in these propagators thereby taking care of the majority of the \( U_A (1) \)-anomaly effects.

The \( \pi^0 \) electromagnetic form-factor has been explored in detail in Ref. [53], wherein it has been confirmed that the correct normalization is satisfied. In addition it has been shown analytically (and numerically) that the correct asymptotic behaviour, modulo potential logarithms is obtained [53, 57].

\[
\lim_{Q^2 \rightarrow \infty} F^{\pi^0\gamma\gamma} (0, Q^2) \propto \frac{1}{Q^2}
\]

\[
\lim_{Q^2 \rightarrow \infty} F^{\pi^0\gamma\gamma} (Q^2, Q^2) \propto \frac{1}{Q^2}. \quad (27)
\]

In Fig. 16 we plot the form factor as a function of the two photon momenta \( k_1^2 \) and \( k_2^2 \) and compare with the VMD inspired

---

2 Perspectives to improve this issue in the framework of Dyson-Schwinger equations have been reported in Refs. [49, 51]. In Ref. [51] a topological mass of the \( \eta_0 \) has been obtained which goes well with lattice results of the topological susceptibility via the Witten-Veneziano relation.
model used in Ref. [13]. One clearly sees that both form factors agree nicely on a qualitative and even quantitative level. Whereas the low-momentum behaviour is governed by the anomaly, at large momenta both form factors fall off according to Eqs. (27). There are small quantitative differences in the mid-momentum regime, which will lead to a small difference in the meson exchange contributions to LBL. discussed below. In general, however, the results of our calculation may be viewed as a confirmation of the previously used model approaches almost from first principles.

D. Off-shell prescription

It is evident from the kinematics of the diagram shown in Fig. 2 that the form-factors, thus far defined as on-shell quantities, must be evaluated for momenta of the exchanged pseudoscalar meson that would be far from the pole mass. In the approach considered here, the pseudoscalar amplitude is obtained from its homogeneous Bethe-Salpeter equation and hence is by definition an on-shell quantity. Thus, to proceed we must introduce a prescription for the continuation of this quantity to the off-shell momentum region.

Since the off-shell behaviour should be dominated by the pseudoscalar pole contribution, the introduction of any prescription that provides for a suppression at off-shell momentum should be suitable starting point. Here, we will employ a prescription that is inspired from the axial-vector Ward-Takahashi identity in the chiral limit

\[ 2P_\mu \Gamma^{5,3}_\mu (k, P) = iS^{-1}(k_+)_5 + i\gamma_5 S^{-1}(k_-) . \]  

(28)

Here the axial vector vertex is defined as the correlation function \( \Gamma^{5,3}_\mu = \langle j^{5,3}_\mu q \rangle \) that includes the axial vector current in the pion channel \( j^{5,3}_\mu = \tilde{q} \gamma_\mu \gamma_5 \gamma^3 \). It is clear that Eq. (28) relates \( \Gamma^{5,3}_\mu \) to the quark propagator. Taking explicit parameterisations for vertex and propagator (see [45]) that include the pion pole in the axial vector vertex, the following form of the dominant amplitude for the \( \pi^0 \) can be deduced:

\[ \tilde{F}^{PS}_1 (k, P) = \lambda_3 \frac{B(k_+) + B(k_-)}{2f_\pi} . \]  

(29)

Here \( k_+ = k \pm P/2 \). Note that in the chiral on-shell limit \( (P^2 = 0) \) the above equation reduces to Eq. (19). We generalize the pseudoscalar amplitude Eq. (18) by using Eq. (29) as a guideline for all four structures also away from the chiral limit. The final off-shell meson amplitude reads:

\[ \tilde{F}^{PS}_i (p; P) = f(p^2) \left\{ -iP \tilde{F}^{PS}_2 (p; P) - i\varphi (p \cdot P) \tilde{F}^{PS}_3 (p; P) - [P, \varphi] \tilde{F}^{PS}_4 (p; P) \right\} \]  

(30)

where the hat over the functions \( \tilde{F}_i^{PS} \) indicates that the flavour structure of the corresponding meson is included in the same manner as in Eqs. (24) [25]. The scalar off-shell amplitudes \( F^{\pi^0}_i (k, P) \) are defined in terms of the on-shell amplitudes \( F^{\pi^0}_i (k, P) \) through

\[ F^{\pi^0}_i (k, P) = \frac{F^{\pi^0}_i (k_+, k_+ - P) + F^{\pi^0}_i (k_-, k_- - P)}{2} \]  

(31)

for which \( i = 1, \ldots, 4 \). The on-shell amplitudes are obtained via Eq. (16). In order to account for the mass dimensions of the form-factors \( \tilde{F}^{PS}_{2,3,4} \) we attach to each the function

\[ f(p^2) = \sqrt{\frac{m_\pi^2}{P^2 + 2m_\pi^2}} . \]  

(32)

This prevents an unnatural enhancement at high meson virtuality whilst at the same time leaving the on-shell behaviour unchanged. The off-shell form factor

\[ \Lambda^{PS, \gamma^* \gamma^*}_\mu (P, k_1, k_2) = \epsilon_{\mu \nu \alpha \beta} k_1^\alpha k_2^\beta F_{PS, \gamma^* \gamma^*} (P^2, k_1^2, k_2^2) \]  

(33)

is then obtained via the generalisation of Eq. (24) by taking the Bethe-Salpeter amplitude \( F^{\gamma^0} \) to be defined via Eq. (30).

The contribution to the derivative of the four-point-function can now be written as [13]:

\[ \bar{\Pi}_{(\rho) \mu \nu \lambda \sigma} (q_1, q_2, -q_{12}) = \frac{F_{PS, \gamma^* \gamma^*} (q_1^2, q_2^2, 0)}{q_{12}^2 + m_{PS}^2} \epsilon_{\mu \nu \alpha \beta} q_1^\alpha q_2^\beta \epsilon_{\lambda \sigma \rho \tau} q_{12}^\rho q_{12}^\tau \]

\[ + \frac{F_{PS, \gamma^* \gamma^*} (q_1^2, q_1^2, 0)}{q_1^2 + m_{PS}^2} \epsilon_{\mu \sigma \rho \tau} q_1^\tau \epsilon_{\nu \lambda \beta} q_2^\alpha q_{12}^\rho \]

\[ + \frac{F_{PS, \gamma^* \gamma^*} (q_2^2, q_2^2, 0)}{q_2^2 + m_{PS}^2} \epsilon_{\mu \lambda \alpha \beta} q_1^\rho q_2^\tau \epsilon_{\nu \sigma \rho \tau} q_2^\sigma . \]  

(34)

where \( q_{12} = q_1 + q_2 \). The function \( \bar{\Pi}_{(\rho) \mu \nu \lambda \sigma} \) is now only dependent upon two momenta since the limit \( k \to 0 \) for the external photon momentum has been carried out.
IV. RESULTS

With the approach to hadronic LBL scattering within the functional approach outlined, and our truncation scheme defined we proceed to combine our propagators, vertices, amplitudes and form-factors together and calculate the respective contributions to hadronic LBL scattering in the muon $g-2$.

A. Pion-pole contribution to LBL

To demonstrate parity between our approach and others in the determination of hadronic LBL, we calculate the ladder-exchange diagram of Fig. 3 assuming pseudoscalar pole-dominance. Once more, we re-iterate that on mass-shell this is identical to the pseudoscalar exchange diagram portrayed in Fig. 2 whereas off-shell we make the common assumption that the meson-exchange picture provides a good approximation.

In order to determine the pseudoscalar exchange contribution we must numerically determine the dressed quark propagator, the quark-photon vertex, and the homogeneous Bethe-Salpeter amplitude for the pseudoscalar meson. Combining these altogether we calculate from first principles the $\pi\gamma\gamma$ form-factor. We wish to emphasize again that the resulting quark-photon vertex also contains time-like poles corresponding to vector meson exchange \(52\). Thus the main ideas of VMD are naturally included here in the form-factor due to the non-perturbative approach that we employ. We have checked that the total numerical error of our calculation is of the order of one percent. In a similar fashion we also evaluate the corresponding form factors for the $\eta$ and $\eta'$ mesons. We then use our results for the form factors to evaluate the pseudoscalar meson exchange contribution to LBL. For this, we use the off-shell prescription for the pseudoscalar Bethe-Salpeter amplitude proposed in Eq. (30) for the exchanged mesons. This prescription gives a reduction of the contribution that is similar to that found in other approaches.

The systematic error of our calculation of the pseudoscalar exchange diagrams can be attributed entirely to the validity of the rainbow-ladder approximation, by the Maris-Tandy (MT) model, Eq. (15), and the off-shell prescription for the mesons, Eq. (29).

No other approximations have been used. While in the Goldstone-Boson sector the MT model works well, there is certainly a larger error in the flavour singlet sector. We therefore guesstimate a total systematic error: ten percent for the pion contribution, and twenty percent for the $\eta$ and $\eta'$ contributions. With a numerical error of two percent we then obtain: $a_{\mu}^{\text{LBL};\eta'} = (57.5 \pm 6.9) \times 10^{-11}$, $a_{\mu}^{\text{LBL};\eta} = (13.6 \pm 3.0) \times 10^{-11}$ and $a_{\mu}^{\text{LBL};\eta'} = (9.6 \pm 2.1) \times 10^{-11}$ leading to

$$a_{\mu}^{\text{LBL};\text{ps}} = (80.7 \pm 12.0) \times 10^{-11} \quad (35)$$

for the pseudoscalar meson exchange contribution to LBL. As compared to our previous work \[23\], the values for $a_{\mu}^{\text{LBL};\eta}$ and $a_{\mu}^{\text{LBL};\eta'}$ are slightly reduced due to a more consistent off-shell prescription in these channels. Our result \[35\] is compatible with previous ones \[4, 6\] and all approaches make the assumption that pseudoscalar pole-dominance is valid far from the meson mass-shell.

B. Quark loop contribution to LBL

Having convinced ourselves that the method works we now focus on the quark loop contribution to LBL. Here we follow the strategy described around Eq. (6) where the Ward identity obeyed by the four-point function with respect to the external field is exploited to construct quantities that are explicitly finite. As for our numerical error we verified that we reproduce the well known perturbative result for the corresponding electron loop with an accuracy of better than one per mille. In the quark loop we use the fully dressed quark propagators for the up, down, strange and charm quarks, extracted from the DSE, Fig. 10. As already mentioned above, due to numerical complexity we are unfortunately not yet in a position where we can use the full, numerically determined quark-photon vertex used for the meson exchange contributions in the previous section. Instead, we use three different approximations to the full vertex and compare the results. As explained in section III B due to the Ward-identity we are in possession of exact expressions for the non-transversal, Ball-Chiu part of the vertex, Eq. (21).

We exploit this knowledge to compare results with (a) a bare vertex, (b) the first term of Eq. (21) (IBC)

$$\Gamma_{\mu}(p, q) = \frac{A(p^2) + A(q^2)}{2} \eta_{\mu} \quad (36)$$

where $p, q$ are the quark and antiquark momenta, and (c) the full Ball-Chiu (BC) expression Eq. (21). A comparison between these three approximations may serve as a guide for the systematic error due to the relevance of vertex effects. We emphasize, however, that only our most elaborate approximation, (c), satisfies the constraints of gauge invariance. Previous approximations based on purely transverse parts of the vertex \[16\] do not satisfy this constraint. We believe that the ansatz (c) provides an excellent basis for the calculation of the quark-loop diagram, which can and should be expanded in future work to also include transverse parts of the vertex.

As a result of our calculation we find

$\begin{align*}
a_{\mu}^{\text{LBL; quarkloop (bare vertex)}} & = (61 \pm 2) \times 10^{-11} \\
a_{\mu}^{\text{LBL; quarkloop (BC)}} & = (176 \pm 4) \times 10^{-11} \\
\end{align*}$

for the quark loop contribution. Clearly these are sizable contributions. Whereas the bare vertex result roughly agrees with the number $60 \times 10^{-11}$ given in \[14\], the dressing effects of the vertex lead to a drastic increase. As compared to our result for the first part of the Ball-Chiu vertex \[23\] we again find a drastic increase from $10^7 \times 10^{-11}$ to $176 \times 10^{-11}$ when the other two terms of the full Ball-Chiu vertex are included. In
As for the quark-loop contribution, analytic constraints have been used which arise from the requirement of gauge invariance: the quark-photon vertex appearing in this loop has to satisfy the vector Ward-Takahashi identity. In contrast to previous approaches, we have implemented this identity by using the Ball-Chiu ansatz for this vertex. We believe this is a systematic improvement. The consequences are drastic: we observe a dramatic increase for the quark-loop contribution to LBL. Our result of $(176 \pm 4) \times 10^{-11}$ is more than three times larger than the constituent quark result of Ref. [14].

When combining our two results, Eq. (35) and Eq. (37), we arrive at a hadronic LBL contribution of

$$d_{\mu}^{\text{LBL;PS+quarkloop}} = (257 \pm 31) \times 10^{-11}.$$  (39)

This value, however, does not yet account for transverse parts of the quark-photon vertex in the quark-loop contribution and for effects from the right hand diagrams of Fig. 2 or Fig. 3. In general, it is difficult to gauge the effects of additional transverse vertex contributions in the quark-loop diagram. In Ref. [16] part of these effects have been taken into account by using an ansatz motivated by vector-meson dominance (VMD) ideas. They found a reduction due to these effects of roughly $40 \times 10^{-11}$. Since we agree with Ref. [16] on the size of the pion exchange contribution, where VMD works very well, it may be justified to use their result as a rough estimate for the size and also for the potential error in these effects. We therefore add a contribution of $d_{\mu}^{\text{LBL;quarkloop,transverse}} = (-40 \pm 40) \times 10^{-11}$ to arrive at

$$d_{\mu}^{\text{LBL;quarkloop}} = (136 \pm 59) \times 10^{-11}.$$  

The additional contributions due to the right hand diagrams of Fig. 2 or Fig. 3 are also difficult to judge. It may help, though, to observe that these involve an additional quark-loop. Typically such contributions are negative and of the order of ten to twenty percent of the leading-Nc contributions 28-29. Since on the other hand one also expects positive contributions of a similar size from non-pseudoscalar exchange diagrams 29 we choose to subsume all these contributions to another

\footnote{Note that cross-terms between transverse and non-transverse contributions from the four vertices in the quark-box diagram could provide additional suppression. This is expressed in our error estimate.}
\(a_{\mu}^{\text{LBL\ other}} = (0 \pm 20) \times 10^{-11}\), where the error is clearly subjective. This gives us the following total hadronic LBL contribution

\[
a_{\mu}^{\text{LBL}} = (217 \pm 91) \times 10^{-11} ,
\]

in our approach. Note that the increase of the central value as compared to our previous result in Ref. [23] is due to a combination of taking the full BC-vertex instead of 1BC and in addition accounting for the transverse corrections in the quark-loop using the results of Ref. [16]. Taken at face value these numbers together with the other contributions quoted in [1] clearly reduce the discrepancy between theory and experiment. Combining our light-by-light scattering results with the quark-loop using the results of Ref. [16].

\[a_{\mu}^{\text{theor}} = 116.591 \times 10^{-11} .\]

To put this result in perspective we wish to recall the caveats that to our mind are tied to it. First, there is the contribution of transverse parts of the quark-photon vertex to the quark-loop diagram. Although the results of Ref. [16] may serve as an estimate, we definitely need to explicitly calculate these contributions in our approach. Second, there is the question whether the pseudoscalar meson exchange diagram provides for a good approximation of the gluon exchange contribution discussed around Fig. 3. Also this assumption needs to be questioned by an explicit calculation. In this sense, our results certainly do not provide final answers but still have to be seen as a further step towards a fundamental determination of \(a_{\mu}\).

Finally, we point out that the current approach will also be checked by a calculation of the hadronic vacuum polarization contribution to \(a_{\mu}\). Preliminary results in this direction are encouraging.

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