STRICTLY UNITAL $A_\infty$-ALGEBRAS

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ABSTRACT. Given a graded module over a commutative ring, we define a dg-Lie algebra whose Maurer-Cartan elements are the strictly unital $A_\infty$-algebra structures on that module. We use this to generalize Positselski’s result that a curvature term on the bar construction compensates for a lack of augmentation, from a field to arbitrary commutative base ring. We also use this to show that the reduced Hochschild cochains control the strictly unital deformation functor. We motivate these results by giving a full development of the deformation theory of a nonunital $A_\infty$-algebra.

0. INTRODUCTION

The bar and cobar constructions are an adjoint pair of functors between the categories of augmented differential graded (dg) algebras and coaugmented dg coalgebras, both defined over a fixed commutative ring $k$,

$$
\begin{align*}
dgAlg^a_k & \xrightarrow{\text{Cob}} \text{dgCoalg}^a_k. \\
\text{Bar} & \xleftarrow{\text{Cob}} dgAlg^a_k.
\end{align*}
$$

This adjoint pair is the algebraic analogue of the classifying space and Moore loop space adjoint pair between topological monoids and based topological spaces. Analogous to the situation in topology, the bar and cobar functors are decidedly non-trivial: the unit of the adjunction is a homotopy equivalence. This non-triviality is at the root of their usefulness in algebra. In particular, the bar construction gives canonical resolutions, of both modules and bimodules, and plays a large role in infinitesimal deformation theory.

Stasheff, in [Sta63a], relaxed the assumption of associativity in a topological monoid to define an $A_\infty$-space, using a generalized notion of classifying space. He then showed that a connected topological space has the homotopy type of a loop space exactly when it is an $A_\infty$-space (see also [Ada78, Chapter 2]). The algebraic analogue of a connected $A_\infty$-space is an augmented $A_\infty$-algebra [Sta63b], generalizing an augmented dg-algebra. An augmented $A_\infty$-algebra is an augmented complex $(A, m^1)$ and a sequence of augmented maps $m^n : A^\otimes n \to A$, where $m^2$ satisfies the Leibniz rule with respect to $m^1$, $m^3$ is a nullhomotopy for the associator of $m^2$, and more generally, the $m^n$ satisfy the quadratic equations necessary for a bar construction, giving the following diagram:

$$
\begin{align*}
A_\infty \text{Alg}^a_k & \xrightarrow{\text{Cob}} \text{dgCoalg}^a_k. \\
\text{Bar} & \xleftarrow{\text{Cob}} \text{dgCoalg}^a_k.
\end{align*}
$$

Every $A_\infty$-algebra is homotopy equivalent to a dg-algebra, so nothing is gained at the homotopy level by enlarging the category of dg-algebras, but often there are
dramatically smaller $\mathbb{A}_\infty$-versions smaller than the dg-models, e.g., the cochains of the classifying space of a finite group; see [Pro11 §6].

The augmentation assumption plays a vital but subtle role in the nontriviality of the above functors. Indeed, by bar construction of an augmented dg-algebra $\epsilon : A \to k$, we mean the bar construction applied to the nonunital algebra $\ker \epsilon$. The bar construction of a unital dg-algebra is homotopy equivalent to the trivial coalgebra, destroying the “homotopy type” of the unital dg-algebra. In particular, the resolutions traditionally constructed using the bar construction, will not necessarily be resolutions if one doesn’t kill the unit. Augmented algebras are exactly those we can do this to, without losing information. All of this remains true for augmented versus strictly unital $\mathbb{A}_\infty$-algebras, summarized in the following diagram:

$$\mathbb{A}_\infty \text{Alg}^\text{su}_k \searrow \bigcup \quad \mathbb{A}_\infty \text{Alg}^\text{aug}_k \rightarrow \mathbb{A}_\infty \text{Alg}^\text{aug}_k \rightarrow \text{dgCoalg}^\text{aug}_k.$$  

Positselski had the insight that the right side of the diagram can be extended to curved dg-coalgebras. He showed how to construct, for a strictly unital, but not necessarily augmented, $\mathbb{A}_\infty$-algebra, a curved bar construction, killing the unit and transferring the potentially lost information to a curvature term [Pos93, Pos11], giving the following diagram:

$$\mathbb{A}_\infty \text{Alg}^\text{su}_k \rightarrow \mathbb{A}_\infty \text{Alg}^\text{aug}_k \rightarrow \text{curv-dgCoalg}^\text{aug}_k.$$  

He proved analogous results for $\mathbb{A}_\infty$-morphisms and $\mathbb{A}_\infty$-modules, and also stated a strong converse: the curved bar construction characterizes strictly unital $\mathbb{A}_\infty$-algebras (and morphisms, and representations).

The fundamental idea that a curvature term compensates for lack of augmentation is not particularly emphasized in the long paper [Pos11] (a paper that contains many new and powerful ideas), full details of the proofs are not given, and, most importantly for us, the ground ring is assumed to be a field. In this paper, we give careful proofs of Positselski’s results, valid for an arbitrary commutative ground ring (in fact with a few small adjustments, noted in remarks, the results hold when replacing modules over a commutative ring with any symmetric monoidal category with countable coproducts, where finite coproducts are also finite products). Positselski also showed that the bar construction is homotopically non-trivial, and this opens the door to using it for the construction of resolutions. We do not pursue the generalization from a field to arbitrary base ring here, but hope to return to it in the future.

The proofs in this paper use a characterization of strictly unital $\mathbb{A}_\infty$-algebra structures as Maurer-Cartan elements of a certain dg-Lie algebra (the coassociative analogue of a construction used by Schlessinger and Stasheff for Lie coalgebras [SSS85]). We also use this characterization to show the dg-Lie algebra of reduced
Hochschild cochains controls the strictly unital infinitesimal deformations of the corresponding \( A_\infty \)-algebra. As motivation and context for using Maurer-Cartan elements of a dg-Lie algebra, we include a detailed discussion of the deformation theory of nonunital \( A_\infty \)-algebras via the dg-Lie algebra of Hochschild cochains. We also prove linear analogues of all of the above results. In particular, we recover, and generalize to arbitrary commutative base ring, Positselski’s result that strictly unital modules correspond functorially to cofree curved dg-comodules over the curved bar construction.

There has been considerable further work developing Positselski’s ideas, especially for operads \cite{HM12, FK16, Lyu14, Gri16}, see also \cite{CLM16}. There has also been much work on homotopy, or weak, units in an \( A_\infty \)-algebra; see \cite{KS09, Lyu11, MT14} and the references contained there. An \( A_\infty \)-automorphism does not necessarily preserve a strict unit, but it does preserve a homotopy unit (one can take for the definition of homotopy unit that there is an automorphism that takes it to a strict unit). Positselski’s idea on curvature gives a way of maintaining a strict unit through certain processes, e.g., transfer of \( A_\infty \)-structure, rather than working in the larger category of homotopy unital \( A_\infty \)-algebras.

Finally, let us mention one motivation for this paper. In \cite{Bur15} we study projective resolutions of modules over a commutative ring \( R = Q/I \) by putting \( Q \)-linear strictly unital \( A_\infty \)-structures on \( Q \)-projective resolutions of \( R \) and its modules. (This example emphasizes the importance of working with an arbitrary commutative base ring.) In particular, we show that minimality of \( A_\infty \)-structures characterizes Golod singularities, and the bar construction can then be used to construct the minimal free resolution of every module over a Golod ring. To work effectively with different classes of singularities, e.g., complete intersections, a relative Koszul duality (relative to \( Q \)) is needed. We hope to develop this in future work. Throughout this paper we give a sequence of running examples illustrating the elementary, but interesting example of the Koszul complex on a single element \( f \) of the ground ring \( k \), where e.g., if \( k = C[x_1, \ldots, x_n] \), then we are studying the zero set of \( f \) relative to \( C^n \).

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1. Notation and conventions

(1) Throughout, \( k \) is a fixed commutative ring. By module, complex, map, etc. we mean \( k \)-module, complex of \( k \)-modules, \( k \)-linear map, etc. We place no boundedness or connectedness assumptions on complexes. For graded modules \( M, N \), define graded modules \( \text{Hom}(M, N) \) and \( M \otimes N \) by

\[
\text{Hom}(M, N)_n = \prod_{i \in \mathbb{Z}} \text{Hom}(M_i, N_{i+n}) \quad (M \otimes N)_n = \bigoplus_{i \in \mathbb{Z}} M_i \otimes N_{n-i}.
\]

If \((M, \delta_M)\) and \((N, \delta_N)\) are complexes, then \( \text{Hom}(M, N) \) and \( M \otimes N \) are complexes with differentials

\[
\delta_{\text{Hom}}(f) = \delta_N f - (-1)^{|f|} f \delta_M \quad \delta_{\otimes} = \delta_M \otimes 1 + 1 \otimes \delta_N.
\]

A morphism of complexes is a degree 0 cycle of the complex \( \text{Hom}(M, N) \), \( \delta_{\text{Hom}} \).

(2) All elements of graded objects are assumed to be homogeneous. We write \( |x| \) for the degree of an element \( x \). If \( M \) is a graded module, \( PM \) is the graded module with \((PM)_n = M_{n-1} \). Set \( s \in \text{Hom}(M, PM)_1 \) to be the
identity map. For \( x \in M \), we set \([x] = s(x) \in \Pi M\) and more generally
\([x_1] \cdots [x_n] = sx_1 \otimes \cdots \otimes sx_n\). If \((M, \delta_M)\) is a complex, set \(\delta_{IM} = -s\delta_M s^{-1}\).
Then \(s : (M, \delta_M) \to (\Pi M, \delta_{IM})\) is a cycle in \((\Hom(\Pi M, M), \delta_{\Hom})\).

(3) We use the sign conventions that when \(x, y\) are permuted, a factor of \((-1)^{|x||y|}\) is introduced, and when applying a tensor product of morphisms, we have \((f \otimes g)(x \otimes y) = (-1)^{|g||x|} f(x) \otimes g(y)\).

(4) For complexes \(M\) and \(N\), the maps \((\Pi M) \otimes N \to \Pi(M \otimes N), [m] \otimes n \mapsto [m \otimes n]\) and \(M \otimes (\Pi N) \to \Pi(M \otimes N), m \otimes [n] \mapsto (-1)^{|m||n|} [m \otimes n]\) are isomorphisms of complexes, as are the maps \(\Pi \Hom(M, N) \to \Hom(\Pi^{-1} M, N), [f] \mapsto (-1)^{|f|} fs \Pi \) and \(\Pi \Hom(M, N) \to \Hom(M, \Pi N), [f] \mapsto sf\).

(5) String diagrams are used to represent morphisms between tensor products of graded modules. A line represents a graded module, parallel lines represent a tensor product of graded modules, and a rectangle represents a morphism. Lines may be decorated to distinguish graded modules, for instance, let \(\text{represented by} A\) represent a graded module \(A\) and \(\text{represented by} B\) represent a graded module \(B\). Then, e.g., \(\text{represents a morphism} f : A^\otimes 3 \to B\). The utility of the diagrams becomes apparent when composing such morphisms.

(6) For unexplained conventions or definitions related to differential graded Lie algebras, see [NR66], and for graded coalgebras, see Chapters 1 and 5 of [Mon93].

2. Nonunital \(A_\infty\)-algebras

In this section we recall the definitions of nonunital \(A_\infty\)-algebra, \(A_\infty\)-morphism, bar construction, and dg-Lie algebra of Hochschild cochains. We use the approach of Prouté [Pro11] and Getzler [Get93], and the string diagram notation of Hinich [Hin03]. Fix graded modules \(A, B\) throughout the section. In string diagrams \(\text{will denote} \Pi A\) and \(\text{will denote} \Pi B\).

**Definition 2.1.** Set \(CC^n(A, B) = \Hom((\Pi A)^\otimes n, \Pi B)\), and
\[
CC^\bullet(A, B) = \prod_{n \geq 1} \Hom((\Pi A)^\otimes n, \Pi B).
\]
We write \(f = (f^n) \in CC^\bullet(A, B)\) with \(f^n \in CC^n(A, B)\) and call \(f^n\) the \(n\)th tensor homogeneous component.

Since \(A\) and \(B\) are graded, so is \(CC^\bullet(A, B)\), using \((1)\). The \(i\)th homogeneous component of this grading is denoted \(CC^\bullet(A, B)_i\).

The module \(CC^\bullet(A, A)\) has a very intricate algebraic structure. In particular it is a graded Lie algebra under the commutator of the following.

**Definition 2.2.** The Gerstenhaber product of \(g = (g^n) \in CC^\bullet(A, B)_i\) and \(f = (f^n) \in CC^\bullet(A, A)_j\), denoted \(g \circ f \in CC^\bullet(A, B)_{i+j}\), has \(n\)th tensor homogeneous
component given by the following:

\[(g \circ f)^n = \sum_{i=1}^{n} \sum_{j=0}^{i-1} g^i (1 \otimes f^{n-i+1} \otimes 1 \otimes f^{i-j-1}).\]

This was defined in [Ger63] where it was shown to be a pre-Lie algebra structure (Corollary to Theorem 2, applied to Example 5.5) and by [Ger63, Theorem 1], the commutator of any pre-Lie algebra, defined to be \([x, y] = x \circ y - (-1)^{|x||y|} y \circ x\), is a graded Lie algebra.

Using string diagrams, it is a relatively easy exercise to show \(CC^\bullet(A, A)\) is a pre-Lie algebra (see [Kel, p. 20, Figure 1] for details). Performing this exercise, one will see that care must be taken with signs and string diagrams. The conventions we use for signs and string diagrams are formalized below (we encourage the reader to skip ahead and return when a sign issue occurs).

**Remark 2.3.**

1. Morphisms will always be grouped into horizontal lines, i.e., the projections of any two boxes onto the left side of the page are either disjoint or equal.
2. If all morphisms are on the same line, we visualize inputs feeding into the diagram from the right, along the front, and use sign convention 1, e.g.,

\[ (x \otimes y \otimes z) = (-1)^{|x||y|+|y||z|} f^n(x) \otimes y \otimes g^m(z), \]

where \(x = x_1 \otimes \ldots \otimes x_n\), \(y = y_1 \otimes \ldots \otimes y_l\), \(z = z_1 \otimes \ldots \otimes z_m\).
3. If there are multiple lines of morphisms, we visualize the output from each line coming out behind the diagram, needing to be twisted around to the front, where the next line of morphisms is applied as in Step 2; e.g.,

\[ (x \otimes y \otimes z) = (-1)^{|x||y|+|y||z|+(|x|+|f|)|h|} f^n(x) \otimes h^l(y) \otimes g^m(z). \]

Moving the line a morphism is on only changes the diagram by a sign. Sign rules for vertical moves of a morphism are:

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1A pre-Lie algebra structure on a graded module \(G\) is a degree zero morphism \(\circ : G \otimes G \to G\) such that for all \(f, g, h \in G\), the following equation holds, \((f \circ g) \circ h - f \circ (g \circ h) = (-1)^{|g||h|} (f \circ h) \circ g - f \circ (h \circ g))\), i.e., the associator is symmetric in the last two values. In particular, every associative algebra is a pre-Lie algebra, since the associator is zero. See e.g., [LYP §1.4] for more information.
In particular, we have

\[
\begin{array}{c|c|c}
 f & | & g \\
\hline
 f & | & g \\
\hline
 f & | & g \\
\end{array}
\]

\[
\begin{array}{c|c|c}
 f & | & g \\
\hline
 f & | & g \\
\hline
 f & | & g \\
\end{array}
\]

\[
\begin{array}{c|c|c}
 f & | & g \\
\hline
 f & | & g \\
\hline
 f & | & g \\
\end{array}
\]

Definition 2.4. The nonunital tensor coalgebra on a graded module \( V \) is \( T_\text{co}(V) = \bigoplus_{n \geq 1} V^{\otimes n} \) with comultiplication the linear extension of

\[
\Delta(v_1 \otimes \ldots \otimes v_n) = \sum_{i=1}^{n-1} (v_1 \otimes \ldots \otimes v_i) \otimes (v_{i+1} \otimes \ldots \otimes v_n).
\]

Note that \( CC^* (A,B) = \text{Hom}(T_\text{co}(\Pi A), \Pi B) \).

A graded coderivation of a graded coalgebra \( C \) with comultiplication \( \Delta \) is a homogeneous endomorphism \( d \) of \( C \) such that \( (d \otimes 1 + 1 \otimes d)\Delta = \Delta d \). We write \( \text{Coder}(C,C) \) for the set of coderivations. This is a graded Lie subalgebra of the commutator bracket on \( \text{Hom}(C,C) \).

Lemma 2.5. The tensor coalgebra satisfies the following universal properties.

1. The canonical projection \( \pi_1 : T_\text{co}(\Pi A) \to \Pi A \) induces an isomorphism,

\[
\Phi = (\pi_1)_* : \text{Coder}(T_\text{co}(\Pi A), T_\text{co}(\Pi A)) \xrightarrow{\cong} \text{Hom}(T_\text{co}(\Pi A), \Pi A) = CC^*(A,A).
\]

This is an isomorphism of graded Lie algebras, where the bracket on the source is the commutator, and the bracket on the target is the Gerstenhaber bracket. The inverse applied to \( f = (f^n) \in CC^*(A,A) \) is given by

\[
\pi_{n-i+1} \Phi^{-1}(f)|_{(\Pi A)^{\otimes n}} = \sum_{j=1}^n \begin{array}{c|c|c}
 f^j & | & g \\
\hline
 f^j & | & g \\
\hline
 f^j & | & g \\
\end{array}
\]

2. The canonical projection \( \pi_1 : T_\text{co}(\Pi B) \to \Pi B \) induces an isomorphism,

\[
\Psi = (\pi_1)_* : \text{Hom}_{\text{Coalg}_{k}}(T_\text{co}(\Pi A), T_\text{co}(\Pi B)) \xrightarrow{\cong} CC^*(A,B)_0.
\]

The inverse applied to \( g = (g^n) \in CC^*(A,B)_0 \) is given by:

\[
\pi_k \Psi^{-1}(g)|_{(\Pi A)^{\otimes n}} = \sum_{i_1 + \ldots + i_k = n} \begin{array}{c|c|c}
 g^{i_1} & | & \ldots & | & g^{i_k} \\
\hline
 g^{i_1} & | & \ldots & | & g^{i_k} \\
\hline
 g^{i_1} & | & \ldots & | & g^{i_k} \\
\end{array}
\]

For \( g \in CC^*(A,B) \) and \( f \in CC^*(A,A) \), it follows from 2.5(1) that \( g \circ f = g \Phi^{-1}(f) \). We define an analogous product using \( \Psi^{-1} \).

Definition 2.6. For \( g \in CC^*(A,B)_0 \) and \( h \in CC^*(B,B)_1 \), set

\[
h \ast g = h \Psi^{-1}(g) \in CC^*(A,B)_1.
\]
Remark 2.7. If there are no superscripts on the morphisms of a string diagram, the diagram represents an element \( \xi \) of \( CC^\bullet(A, B) \) with \( \xi^n \) given by the summing over all diagrams of the given shape that have \( n \) inputs. For example, if \( h \in CC^\bullet(B, B) \) and \( g \in CC^\bullet(A, B) \), we write

\[
(h \ast g)^n = \sum_{i_1 + \ldots + i_k = n} g^{i_1} \cdot \ldots \cdot g^{i_k}
\]

and analogously,

\[
(g \circ f) = \sum_j g_j \cdot f
\]

(We also extend this notation to tensor products of elements of \( CC^\bullet(A, A) \) in the proof below.)

Proof of Lemma 2.5. For proofs that \( \Phi \) and \( \Psi \) are isomorphisms of modules see e.g., [Pro11, 2.16, 2.19]. We will show that \( \Phi^{-1} \) is a morphism of graded Lie algebras (this is also presumably well known, but string diagrams give an easy proof). Let \( f, g \in CC^\bullet(A, A) \), and set \( d = \Phi^{-1}(f), e = \Phi^{-1}(g) \). We then have

\[
de = \Phi^{-1}(f)\Phi^{-1}(g) = \left( \sum_j \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
f_{j_1} \\
\vdots \\
\vdots \\
\vdots \\
\vdots
\end{array} \right) \left( \sum_k \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
g_{k_1} \\
\vdots \\
\vdots \\
\vdots \\
\vdots
\end{array} \right).
\]

When composing terms, \( g \) is inserted to the left of, into, or to the right of, \( f \), so

\[
de = \sum_{j,k} \left( \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
f_{j_1} \\
\vdots \\
\vdots \\
\vdots \\
\vdots
\end{array} \right) + \left( \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
g_{k_1} \\
\vdots \\
\vdots \\
\vdots \\
\vdots
\end{array} \right) = \sum_{j,k} \left( \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
f_{j_1} \\
\vdots \\
\vdots \\
\vdots \\
\vdots
\end{array} \right) + \left( \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
g_{k_1} \\
\vdots \\
\vdots \\
\vdots \\
\vdots
\end{array} \right).
\]

We then have, see 2.3 for signs,

\[
[d, e] = de - (-1)^{|d||e|} ed = \sum_{j,k} \left( \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
f_{j_1} \\
\vdots \\
\vdots \\
\vdots \\
\vdots
\end{array} \right) - \left( (-1)^{|d||e|} \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
g_{k_1} \\
\vdots \\
\vdots \\
\vdots \\
\vdots
\end{array} \right)
\]

\[
= \Phi^{-1}(f \circ g - (-1)^{|f||g|} f \circ g) = \Phi^{-1}[f, g]. \quad \square
\]

We will need the following in a later section. It follows from the explicit formulas for \( \Phi^{-1} \) and \( \Psi^{-1} \) given in Lemma 2.5.
Corollary 2.8. A graded coalgebra morphism $\gamma : T_{\co}(\Pi A) \to T_{\co}(\Pi B)$ commutes with coderivations $d_A$ and $d_B$, of $T_{\co}(\Pi A)$ and $T_{\co}(\Pi B)$, respectively, if and only if $\pi_1 \gamma d_A = \pi_1 d_B \gamma$.

Definition 2.9. Let $A, B$ be graded modules.

1. A nonunital $A_\infty$-algebra structure on $A$ is an element $\nu \in CC^*(A, A)_{-1}$ such that $\nu \circ \nu = 0$. For $\nu = (\nu^n)$, this is equivalent to

$$\sum_{1 \leq i \leq n \atop 0 \leq j \leq i-1} \nu^{i-j} \otimes \nu^j = 0$$

for all $n \geq 1$.

2. An $A_\infty$-morphism $(A, \nu_A) \to (B, \nu_B)$ between nonunital $A_\infty$-algebras is an element $g \in CC^*(A, B)_0$ such that $\nu_B \circ g = g \circ \nu_A$, where $*$ is defined in 2.6. In diagrams this means,

$$\sum_{i_1 + \ldots + i_k = n} \nu^{i_1}_B \otimes g^{i_2} \otimes \ldots \otimes g^{i_k} = \sum_{1 \leq i \leq n \atop 0 \leq j \leq i-1} \nu^{i-j} \otimes g^j$$

for all $n \geq 1$.

3. The bar construction of a nonunital $A_\infty$-algebra $(A, \nu)$ is the dg-coalgebra $\text{Bar} A = (T_{\co}(\Pi A), \Phi^{-1}(\nu))$. Since $[\nu, \nu] = 0$, and $\Phi^{-1}$ is a morphism of Lie algebras, it follows that $[\Phi^{-1}(\nu), \Phi^{-1}(\nu)] = 0$ and so $\Phi^{-1}(\nu)^2 = 0$ (assuming $1/2 \in k$; or one can modify the proof of 2.6). This is functorial with respect to $A_\infty$-morphisms, using 2.3.(2).

4. The Hochschild cochains of a nonunital $A_\infty$-algebra $(A, \nu)$ is the dg-Lie algebra $(CC^*(A, A), [\nu, -])$.

Remark 2.10. It is often convenient to pass from a family of degree -1 maps $\nu^n : (\Pi A)^{\otimes n} \to \Pi A$ to a family of degree $n-2$ maps $m^n : A^{\otimes n} \to A$, and vice versa. We use the convention that $m^n = s^{-1} \nu^n s^{\otimes n}$, and since $(s^{\otimes n})^{-1} = (-1)^{\frac{n(n-1)}{2}} (s^{-1})^{\otimes n}$, it follows that $\nu^n = (-1)^{\frac{n(n-1)}{2}} s m^n (s^{-1})^{\otimes n}$.

(Prout [Pro11] uses the convention that $\nu^n = -sm^n(s^{-1})^{\otimes n}$.) If $(A, (\nu^n))$ is an $A_{\infty}$-algebra, then, in low tensor degrees, the corresponding maps $m^n$ satisfy:

- $n = 1 \quad m^1 m^1 = 0$
- $n = 2 \quad m^1 m^2 = m^2 (m^1 \otimes 1 + 1 \otimes m^1)$
- $n = 3 \quad m^2 (1 \otimes m^2 - m^2 \otimes 1) = m^3 + m^3 \circ m^1$.

This is a slightly non-standard version of the Hochschild cochains; the standard definition is $\Sigma CC^*(A, A) \otimes A$. To see the Lie algebra structure and differential agree in the classical case when $A$ is a $k$-algebra, see equation 23 on page 280 of [Ger92] and [Sta92].
Thus, \((A, m^1)\) is a complex, \(m^2\) satisfies the Leibniz rule with respect to \(m^1\), and the associator of \(m^2\) is a boundary in the Hom-complex \((\text{Hom}(A^\otimes A), \delta_{\text{Hom}})\) between the complexes \((A^\otimes A, \delta_{\otimes})\) and \((A, m^1)\).

It follows easily from the above that a dg-algebra, i.e., a complex \((A, m^1)\) with a compatible associative multiplication \(m^2\), satisfies the Leibniz rule with respect to \(m^1\), and the associator of \(m^2\) is a boundary in the Hom-complex \((\text{Hom}(A^\otimes A), \delta_{\text{Hom}})\) between the complexes \((A^\otimes A, \delta_{\otimes})\) and \((A, m^1)\).

Remark. In this section we could replace the category of graded \(k\)-modules with the category of graded objects in an arbitrary symmetric monoidal category with coproducts, such that a finite coproduct is also a product, and such that the coproduct behaves as expected with respect to the tensor product. Indeed, given an object \(V\) in such a category, set \(T_{\text{co}}(V) = \bigoplus_{n \geq 1} V^\otimes n\), and define a comultiplication \(T_{\text{co}}(V) \to T_{\text{co}}(V) \otimes T_{\text{co}}(V)\) on the component \(V^\otimes n\) to be the map \(V^\otimes n \to \bigoplus_{i=1}^{n-1} (V^\otimes i \otimes V^\otimes (n-i)) \to T_{\text{co}}(V) \otimes T_{\text{co}}(V)\), which has components \(V^\otimes n \cong V^\otimes i \otimes V^\otimes (n-i)\). Then \(T_{\text{co}}(V)\) is a coalgebra object in the category, and satisfies the formal properties of 2.5, and so the definitions of 2.9 make sense in this context.

3. Deformation theory of \(A_\infty\)-algebras

In this section we recall how the Hochschild cochains control the infinitesimal deformation theory of an \(A_\infty\)-algebra. A goal is to give context and motivation for the definition and use of Maurer-Cartan elements of dg-Lie algebras. The reader uninterested in deformation theory only needs Definition 3.5. There are no new results, and the approach follows [GS88, KS, Kel]; see also [PS95, FP02, Sta93]. We assume that \(\frac{1}{2} \in k\).

Definition 3.1. Let \(l\) be a commutative \(k\)-algebra.

1. An \(l\)-family of \(A_\infty\)-algebra structures on \(A\) is an \(l\)-linear \(A_\infty\)-algebra structure on \(A \otimes l\). We set

\[
\text{CC}^*_l(A \otimes l, A \otimes l) = \prod_{n \geq 1} \text{Hom}_l((\Pi(A \otimes l))^\otimes n, \Pi(A \otimes l)).
\]

Using the isomorphism \(\text{CC}^*_l(A \otimes l, A \otimes l) \cong \text{CC}^*(A, A \otimes l)\), and denoting \(l\) in string diagrams as \(\xi\), we can write an \(l\)-family as \(\xi^\nu\).

2. If \(\alpha: l \to l'\) is a morphism of commutative algebras, represented by a string diagram \(\xi\), then an \(l\)-family \(\nu\) gives rise to the \(l'\)-family \(\xi^\nu\). Thus there is a functor,

\[
\text{Fam}_A : \text{ComAlg}_{k} \to \text{Set}
\]

\[
l \mapsto \{l\text{-families of } A_\infty\text{-algebra structures on } A\}.
\]

Lemma 3.2 (Yoneda). If the functor \(\text{Fam}_A\) is representable by a \(k\)-algebra \(l_u\) and an isomorphism of functors \(\zeta : \text{Hom}_{\text{ComAlg}_{k}}(l_u, -) \cong \text{Fam}_A\), then \(\text{Spec } l_u\) is the

\[3\]This assumption can be removed by treating \(0\) as a quadratic squaring map, as in [NR66, §2].
moduli space of $A_\infty$-algebra structures on $A$ and $\nu_{\text{univ}} = \zeta(1_\nu)$ is the universal family of $A_\infty$-algebra structures.

Indeed, for any commutative $k$-algebra $l$ and any $l$-family $\nu \in CC^*_k(A \otimes l, A \otimes l)$, there exists a unique morphism $f : l \to l$ such that $\nu = \text{Fam}_A(f)(\nu_{\text{univ}})$ (one can take this as the definition of moduli space and universal family). In particular, the set of $A_\infty$-algebra structures on $A$ corresponds to the set of $k$-morphisms $l \to k$.

If $A$ is a finitely generated graded projective $k$-module, and is concentrated in non-negative degrees, or in degrees at most $-2$, then $\text{Fam}_A$ is representable. Indeed, set $L^1 = CC^*(A, A)_{-1}$ and $b = [-,-] : L^1 \otimes L^1 \to L^2$. By the assumptions on $A$, $L^1$ is a finitely generated projective $k$-module. Thus, writing $(-)^*$ for the $k$-dual, the natural map $\iota : L^1_1 \otimes L^1_1 \to (L_1 \otimes L_1)^*$ is an isomorphism. If we denote by $sq^* : L_2^* \to \text{Sym}^2(L_1^*)$ the map $L^*_2 \xrightarrow{b^*} (L_1 \otimes L_1)^* \xrightarrow{-1} L^*_1 \otimes L^*_1 \to \text{Sym}^2(L_1^*)$, then the algebra $l_1 := \text{Sym}^*(L_1^*)/(sq^*(L_2^*))$ represents $\text{Fam}_A$. If $k$ is an algebraically closed field, then the closed points of $\text{Spec} l_1$ correspond to the Maurer-Cartan elements of $L^1 = CC^*(A, A)_{-1}$, i.e., $A_\infty$-algebra structures on $A$.

Regardless of whether the functor $\text{Fam}_A$ is representable, we can view the functor as a generalized scheme. By Yoneda’s Lemma\footnote{A more general (and standard) version than the one quoted in Lemma \ref{lem:yoneda}.} an $l$-family $\nu$ corresponds to the natural transformation $\nu_* : h^l \to \text{Fam}_A$ that sends $\beta \in h^l(l') = \text{Hom}_{\text{ComAlg}}(l, l')$ to $\text{Fam}_A(\beta)(\nu) \in \text{Fam}_A(l')$. Given an $A_\infty$-structure $\nu$ on $A$, we say the $l$-family $\nu$ contains $\nu$ if there is a natural transformation $\epsilon^* : h^k \to h^l$ such that the following diagram is commutative:

\[
\begin{array}{ccc}
 h^k & \xrightarrow{\nu_*} & \text{Fam}_A \\
 (\epsilon^*)_* & \downarrow & \\
 h^l & \xrightarrow{\nu} & \\
\end{array}
\]

By Yoneda again, the transformation $\epsilon^*$ is determined by a $k$-algebra morphism $\epsilon : l \to k$. Unraveling this gives an algebraic definition of $l$-family that contains a marked $k$-point $\nu$.

**Definition 3.3.** Let $(A, \nu)$ be a nonunital $A_\infty$-algebra and $\epsilon : l \to k$ a morphism of commutative $k$-algebras. An $(l, \epsilon)$-deformation of $(A, \nu)$ is an $l$-family $\nu$ such that $\text{Fam}_A(\epsilon)(\nu) = \nu$. In diagrams, this means:

\[
\begin{array}{ccc}
 \nu & \xrightarrow{\epsilon} & \bullet \\
 \downarrow & \downarrow & \downarrow \\
 1_{\nu} \otimes \epsilon & \xrightarrow{\nu} & \\
\end{array}
\]

We denote by $\text{ComAlg}_{k}^{\text{aug}}$ the category with objects pairs $(l, \epsilon)$ as above, and morphisms the algebra morphisms commuting with the augmentations. Set $\text{Def}_{(A, \nu)} : \text{ComAlg}_{k}^{\text{aug}} \to \text{Set}$ to be the functor that sends $(l, \epsilon)$ to the set of $(l, \epsilon)$-deformations of $(A, \nu)$.

If $\text{Fam}_A$ is represented by $l_u$, and $\alpha : l_u \to k$ is the morphism corresponding to $\nu$, then one checks the augmented $k$-algebra $(l_u, \alpha)$ represents $\text{Def}_{(A, \nu)}$. Regardless of the representability of $\text{Def}_{(A, \nu)}$, we can view the functor as describing the generalized scheme $\text{Fam}_A$ near the $k$-point corresponding to $\nu$. We can focus attention on...
the (generalized) infinitesimal neighborhoods of the $k$-point by restricting the domain of $\text{Def}(A,\nu)$ to $\text{finComAlg}_{u}^{\text{aug}}$, the full subcategory of $\text{ComAlg}_{u}^{\text{aug}}$ with objects $(l,\epsilon)$, such that $l$ is a finitely generated projective $k$-module and $(\ker \epsilon)^{N} = 0$ for some $N \geq 1$ (the last condition follows from the first if $k$ is a field, and $l$ is local).

**Definition 3.4.** An infinitesimal deformation of a nonunital $A_{\infty}$-algebra $(A,\nu)$ is an $(l,\epsilon)$-deformation with $(l,\epsilon)$ an object of $\text{finComAlg}_{u}^{\text{aug}}$. The corresponding functor is denoted $\text{infDef}_{(A,\nu)} = \text{Def}_{(A,\nu)} |_{\text{finComAlg}_{u}^{\text{aug}}} : \text{finComAlg}_{u}^{\text{aug}} \to \text{Set}$.

If $\text{Def}_{(A,\nu)}$ is represented by an augmented $k$-algebra $(l_{u},\epsilon)$, such that $l_{u}/(\ker \epsilon)^{n}$ is in $\text{finComAlg}_{u}^{\text{aug}}$ for all $n$ (this holds when $k$ is a field and $l_{u}$ is noetherian), then $\text{infDef}_{(A,\nu)}$ is pro-represented by the completion $\varprojlim_{n \geq 0} l_{u}/(\ker \epsilon)^{n}$. Indeed, the canonical morphism of functors,

$$\text{colim}_{n} \text{Hom}_{\text{ComAlg}}(l_{u}/(\ker \epsilon)^{n},-) \to \text{Hom}_{\text{ComAlg}}(l_{u},-),$$

is easily checked to be an isomorphism on $\text{finComAlg}_{u}^{\text{aug}}$, and this is the definition of pro-representability (see e.g., [Gro95, §2]). If $l_{u}$ is Noetherian, then $\text{Spf}$ of the completion is the formal completion of $\text{Def}_{(A,\nu)}$ along the $k$-point of $\nu$.

Since $\text{infDef}_{(A,\nu)}$ preserves limits, a result of Grothendieck, [Gro95, Corollary to 3.1], shows that it is pro-representable, but the result does not describe the pro-representing object. Drinfeld showed [Dri], in case $A$ is concentrated in non-negative degrees, or in degrees at most $-2$, and degreewise finitely generated, that the degree zero Lie algebra cohomology of the Hochschild cochains pro-represents $\text{infDef}_{(A,\mu)}$. He put the answer in the following more general context, which shows where the finiteness assumptions on $A$ enter. First note that $(\text{finComAlg}_{u}^{\text{aug}})^{\text{op}} = \text{finCocomCoalg}^{\text{aug}}$, the category of cocomplete cocommutative coalgebras that are finitely generated projective $k$-modules. The ind-completion of this category is equivalent to $\text{CocomCoalg}^{\text{aug}}$, the category of all cocommutative cocomplete coalgebras that are projective $k$-modules. It follows that $(\text{CocomCoalg}^{\text{aug}})^{\text{op}}$ is equivalent to the pro-completion of $\text{finComAlg}_{u}^{\text{aug}}$. Any functor $\text{finComAlg}_{u}^{\text{aug}} \to \text{Set}$ that preserves limits extends uniquely to a limit preserving functor on the pro-completion, and thus such a functor is pro-representable exactly when the corresponding functor on coalgebras is representable. The dual of the coalgebra is then a pro-representing object. Kontsevich and Soibelman develop this point of view extensively in [KS].

The functor $\text{infDef}_{(A,\mu)}$ extends to a functor on the category of dg-coalgebras (whose underlying graded coalgebra is in $\text{CocomCoalg}^{\text{aug}}$). We denote this category by $\text{dgCocomCoalg}_{u}^{\text{aug}}$. When $k$ is a field of characteristic zero, Quillen [Qui69], assuming certain boundedness conditions later removed by Hinich [Hin01], defined a model category structure on $\text{dgCocomCoalg}_{u}^{\text{aug}}$, and showed there is an equivalence,

$$\text{Ho}(\text{dgCocomCoalg}_{u}^{\text{aug}}) \xrightarrow{\text{Bar}} \text{Ho}(\text{CocomCoalg}_{u}^{\text{aug}}) \xrightarrow{\cong} \text{Ho}(\text{dgLie}_{k}),$$

between the homotopy category of this model category and the homotopy category of dg-Lie algebras. The equivalence is given by the commutative versions of the bar and cobar constructions. The functor $\text{infDef}_{(A,\mu)}$ induces a functor

\footnotetext{This holds since the cocomplete coalgebras are closed under colimits, and every element in a cocomplete coalgebra is contained in a sub-coalgebra that is a finitely generated projective $k$-module; see e.g., [Dem72] Chapter 1, §6. [Swe99]}
Ho(dgCocomCoalg\textsuperscript{aug}\textsubscript{k})^{op} \to \text{Set}. Such a functor is representable exactly when its representable by \text{Bar} \ L, for some dg-Lie algebra \ L, by the above equivalence. Unwinding definitions, the functor represented by \text{Bar} \ L, restricted to finComAlg\textsuperscript{aug}\textsubscript{k}, is the following (see [Ker] \S 2.7 for details of the unwinding).

**Definition 3.5.** Let \( (L, \delta) \) be a dg-Lie algebra.

1. The **Maurer-Cartan elements** are
   \[
   \text{MC}(L, \delta) := \{ v \in L_{-1} \mid \delta(v) + \frac{1}{2} [v, v] = 0 \}.
   \]

2. The **Maurer-Cartan functor** is
   \[
   \text{MC}(L, \delta) : \text{finComAlg}\textsuperscript{aug}\textsubscript{k} \to \text{Set}
   \]
   \[
   (l, \epsilon) \mapsto \text{MC}(L \otimes \overline{l}, \delta \otimes 1) := \{ v \in (L \otimes \overline{l})_{-1} \mid (\delta \otimes 1)(v) + \frac{1}{2} [v, v] = 0 \},
   \]
   where \( \overline{l} = \ker \epsilon \) and \( L \otimes \overline{l} \) has the induced bracket \( [v \otimes x, v' \otimes x'] = [v, v'] \otimes xx' \).

3. A functor \( F : \text{finComAlg}\textsuperscript{aug}\textsubscript{k} \to \text{Set} \) is controlled by the dg-Lie algebra \( (L, \delta) \) if there is an equivalence \( \text{MC}(L, \delta) \xrightarrow{\sim} F \).

We assumed that \( k \) was a characteristic zero field in the paragraph above, but Definition 3.5 makes sense over any commutative ring (with \( 1/2 \in k \)). The most natural context for this story is derived algebraic geometry, see [Lur10, Toe14]. Staying at a more concrete level, Schectmann shows [Sch98, Theorem 2.5] that if \( L \) is concentrated in strictly positive cohomological degrees, and \( L^1 \) is finite dimensional, then the zeroth cohomology of the bar construction of \( L \) pro-represents the Maurer-Cartan functor.

One motivation behind the Maurer-Cartan approach to deformation theory is that often there is an apparent dg-Lie algebra controlling a given functor, for instance \( \text{infDef}_{(A, \nu)} \). We now show that the dg-Lie algebra of Hochschild cochains controls it (this is classical, but we give details for lack of a reference at this level of generality). Paired with [Sch98 Theorem 2.5], it recovers Drinfeld’s description of the pro-representing object of \( \text{infDef}_{(A, \nu)} \), assuming certain finiteness conditions.

**Proposition 3.6.** Let \( (A, \nu) \) be a nonunital \( A\infty \)-algebra and \( (CC^\bullet(A, A), [\nu, -]) \) the Hochschild cochains. The following is an equivalence:

\[
\text{MC}(CC^\bullet(A, A), [\nu, -]) \to \text{infDef}_{(A, \nu)}
\]

where \( \theta_l \) is the canonical morphism of graded Lie algebras,

\[
\theta_l : CC^\bullet(A, A) \otimes l \to CC^\bullet(A \otimes l, A \otimes l)
\]

\[
(\nu^n) \otimes y \mapsto ([a_1 \otimes x_1] \ldots [a_n \otimes x_n] \mapsto \nu^n[a_1] \ldots [a_n] \otimes yx_1 \ldots x_n),
\]

with the induced bracket on the source and the Gerstenhaber bracket on the target.

**Proof.** Let \( (l, \epsilon) \) be an object of \( \text{finComAlg}\textsuperscript{aug}\textsubscript{k} \) and set \( \overline{l} = \ker \epsilon \). By definition,

\[
\text{MC}(CC^\bullet(A, A), [\nu, -])(l, \epsilon) = \text{MC}(CC^\bullet(A, A) \otimes \overline{l}, [\nu \otimes 1, -]),
\]

and one checks the following is a bijection,

\[
\text{MC}(CC^\bullet(A, A) \otimes \overline{l}, [\nu \otimes 1, -]) \xrightarrow{\sim} \text{MC}(1 \otimes \epsilon)^{-1}([\nu] \subseteq \text{MC}(CC^\bullet(A, A) \otimes l, 0)
\]

\[
\nu \mapsto \overline{\nu} + \nu \otimes 1.
\]
Since $l$ is a finite rank projective $k$-module, $\theta_l$ is an isomorphism, and thus induces a bijection $MC(CC^\bullet(A, A) \otimes l, 0) \xrightarrow{\sim} MC(CC^\bullet_*(A \otimes l, A \otimes l), 0)$. This restricts to a bijection $MC((1 \otimes \epsilon)^{-1}(\nu) \cong \text{Fam}_A(\epsilon)^{-1}(\nu) = \infDef_{(A, \nu)}(l)$.

One is most often interested in families and deformations modulo the following.

**Definition 3.7.** An isomorphism between $l$-families is an $l$-linear $A_\infty$-isomorphism. An equivalence of deformations is an isomorphism of families that reduces to the identity on $A$.

One can consider isomorphism and equivalence classes using the following group functors. For $l$ a commutative $k$-algebra, set $H_A(l) = \{ \overline{g} = (\overline{g}^n) \in CC^\bullet(A \otimes l, A \otimes l) \mid \overline{g}^1 \text{ is an isomorphism} \}$. There is an equality $\text{Aut}_{\text{Coalg}}(T_{co}(IA \otimes l)) = \Psi^{-1}(H_A(l))$, where $\Psi^{-1}$ is defined in \cite{25}; see \cite{Laz03} Proposition 2.5 for a proof. Thus $H_A$ is a group functor $H_A : \text{ComAlg} \to \text{Group}$. Using this, set

$$G_A : \text{ComAlg}^{aug} \to \text{Group}$$

$$G_A(l, \epsilon) = \{ \overline{g} \in H_A(l) \mid (1 \otimes \epsilon)_*(\overline{g}) = 1 \}.$$

There is an action $H_A \times \text{Fam}_A \to \text{Fam}_A$, defined using the isomorphisms of \cite{25} whose quotient functor sends $l$ to the set of isomorphism classes of $l$-families of $A_\infty$-structures on $A$. If $(A, \nu)$ is an $A_\infty$-structure, the action of $H_A$ restricts to an action $G_A \times \text{Def}_{(A, \nu)} \to \text{Def}_{(A, \nu)}$ whose quotient functor sends $(l, \epsilon)$ to the set of equivalence classes of $(l, \epsilon)$-deformations of $(A, \nu)$.

**Corollary 3.8.** Let $(A, \nu)$ be a nonunital $A_\infty$-algebra. The following is a bijection,

$$H^1(CC^\bullet(A, A), [\nu, -]) \to \infDef_{(A, \nu)}(k[t]/(t^2))/\sim, \nu \mapsto \theta(\nu \otimes t + \nu \otimes 1),$$

where the right side is the set of equivalence classes of $k[t]/(t^2)$-deformations.

**Proof.** Let $Z^1$ be the cohomological degree 1 cycles of the complex $(CC^\bullet(A, A), [\nu, -])$. The assignment $\nu \mapsto \nu \otimes t$ is a bijection $Z^1 \to MC((1 \otimes \epsilon)^{-1}(\nu)) = MC(CC^\bullet_*(A, A) \otimes k t, [\nu \otimes 1, -])$. Thus by \cite{3.0}, the assignment $\nu \mapsto \theta(\nu \otimes t + \nu \otimes 1)$ is a bijection $Z^1 \xrightarrow{\sim} \infDef_{(A, \nu)}(k[t]/(t^2))$.

We now claim that for $\nu, \nu' \in Z^1$, the deformations $\theta(\nu \otimes t + \nu \otimes 1)$ and $\theta(\nu' \otimes t + \nu \otimes 1)$ are equivalent if and only if $\theta((\nu' - \nu) \otimes t) = \theta([\mu, \alpha] \otimes t)$, for some $\alpha \in CC^\bullet_*(A, A)$. The claim finishes the proof, since then $\theta(\nu \otimes t + \nu \otimes 1)$ and $\theta(\nu' \otimes t + \nu \otimes 1)$ are equivalent if and only if $\nu' - \nu = [\mu, \alpha]$ for some $\alpha$, using that $\theta$ is a bijection.

**Remark.** Let $(L, \delta)$ be a dg-Lie algebra, and assume that $k$ contains $\mathbb{Q}$. For any $(l, \epsilon) \in \text{finComAlg}^{aug}_k$, the graded Lie algebra $L \otimes l$ is nilpotent, and thus we can define its exponential, which makes it a group. This gives a functor $\text{finComAlg}^{aug}_k \to \text{Group}$. This functor acts on the Maurer-Cartan functor of $(L, \delta)$, and is usually the group functor one hopes to quotient by (in case the dg-Lie algebra is the Hochschild cochains, the group functor agrees with $G_A|_{\text{finComAlg}^{aug}_k}$). This is another advantage of the Maurer-Cartan formalism (in characteristic zero): the group we hope to quotient by is built into the Lie algebra. This point of view is due to Deligne, see \cite{GMSS, KS}. 
4. STRICTLY UNITAL $A_{\infty}$-ALGEBRAS

In this section, given a graded module $A$, we construct a dg-Lie algebra whose Maurer-Cartan elements are the strictly unital $A_{\infty}$-structures on $A$. We first use this to recover Positselski’s construction of a functorial curved bar construction from a strictly unital $A_{\infty}$-algebra, and then use it to show that the reduced Hochschild cochains control infinitesimal strictly unital deformations.

4.1. Characterization of strictly unital structures.

**Definition 4.1.** Let $A, B$ be graded modules with fixed elements $1 \in A_0, 1 \in B_0$.

1. An element $\nu = (\nu^n) \in CC^\bullet(A, A)_{-1}$ is strictly unital (with respect to $1 \in A_0$) if

   \[
   \nu^2[1|a| = a = (-1)^{|a|}\nu^2[a|1]
   \]

   and $\nu^n[a_1|\ldots|a_i|1|a_{i+1}|\ldots|a_{n-1}|1] = 0$ for all $a, a_1, \ldots, a_{n-1} \in A$, where $n \neq 2$ and $0 \leq i \leq n$. If $\nu$ is also an $A_{\infty}$-algebra structure, we say $(A, \nu)$ is a strictly unital $A_{\infty}$-algebra.

2. An element $f = (f^n) \in CC^\bullet(A, B)_0$ is strictly unital if $f^1[1] = [1]$ and $f^n[a_1|\ldots|a_i|1|a_{i+1}|\ldots|a_{n-1}|1] = 0$ for all $a_1, \ldots, a_{n-1} \in A$, $n \geq 2$. If $f$ is also an $A_{\infty}$-morphism, we say it is a strictly unital $A_{\infty}$-morphism.

For our main results we need to place a further assumption on the pair $(A, 1)$ (that is automatically satisfied when $k$ is a field).

**Definition 4.2.** A split element of a graded module $A$ is an element that generates a rank one free module. A graded module with split element is a pair $(A, 1)$ with 1 a split element in $A$, and a fixed (unlabeled) splitting $A \to k$ of the inclusion $k \to A$. An $A_{\infty}$-algebra with split unit is a triple $(A, 1, \nu)$, such that $(A, 1)$ is a graded module with split element, and $(A, \nu)$ is a strictly unital $A_{\infty}$-algebra (with respect to 1). If $(A, 1)$ is a graded module with split element, we set $\overline{A} = A/k \cdot 1$.

We consider this as a submodule $\overline{A} \subseteq A$ via the fixed splitting of 1.

If $(A, 1)$ and $(B, 1)$ are modules with split elements, then strictly unital morphisms $f \in CC^\bullet(A, B)_0$ are assumed to preserve the fixed splittings. In string diagrams, $\vdash$ represents $\Pi\overline{A}$ (previously it denoted $\Pi A$), $\dashv$ represents $\Pi\overline{B}$, and $\dashv$ represents $\Pi k$.

**Definition 4.3.** An $A_{\infty}$-algebra with split unit is a triple $(A, 1, \nu)$ with $(A, 1)$ a module with split element and $(A, \nu)$ a strictly unital $A_{\infty}$-algebra with respect to 1. The trivial $A_{\infty}$-algebra with split unit, denoted $(A, 1, \mu_{su})$, is defined by $\mu_{su}^n = 0$ for $n \neq 2$ and

\[
\mu_{su}^2 = \mu_{su}^1 - \mu_{su}^1 + \mu_{su}^1 + \mu_{su}^1 \in CC^2(A, A)_{-1},
\]

where $(\cong)$ denotes the following canonical isomorphisms, respectively: $\Pi k \otimes \Pi\overline{A} \cong \Pi(k \otimes \Pi\overline{A}) = \Pi^2\overline{A}$. $\Pi\overline{A} \otimes \Pi k \cong \Pi(k \otimes \Pi\overline{A}) = \Pi^2\overline{A}$, $\Pi k \otimes \Pi k \cong \Pi(k \otimes \Pi k) = \Pi^2k$ (see (4) for signs). One checks (carefully, evaluating on elements) that $\mu_{su} \circ \mu_{su} = 0$, and that $\mu_{su}$ is strictly unital, thus $(A, 1, \mu_{su})$ is an $A_{\infty}$-algebra with split unit. If $B$ is a graded module with fixed element $1 \in B_0$, the trivial strictly unital morphism $g_{su}: A \to B$ is $g_{su}^1 = \Pi A \otimes \Pi k \to \Pi B$ and $g_{su}^n = 0$ for $n \geq 2$. 


Lemma 4.4. Let \((A, 1)\) be a module with split element. Every strictly unital element in \(CC^\bullet(A, A)\) is of the form \(\mu + \mu_{su}\) for a unique \(\mu \in CC^\bullet(\overline{A}, A)\). If \(B\) is another graded module with a fixed element 1 \(\in B_0\), every strictly unital element in \(CC^\bullet(A, B)_0\) is of the form \(g + g_{su}\) for a unique \(g \in CC^\bullet(\overline{A}, B)_0\).

Proof. For a strictly unital element \(\nu \in CC^\bullet(A, A)\), set \(\mu = \nu - \mu_{su} \in CC^\bullet(A, A)\). By definition, \(\mu\) is zero on any term containing a 1, and thus \(\mu \in CC^\bullet(\overline{A}, A)\). The proof for morphisms is similar (and easier).

Remark. If we replace the category of \(k\)-modules by a symmetric monoidal category, we can define \(\mu_{su}\) using the diagrams above (where \(k\) is the unit of the category), and use the lemma to define strictly unital elements of \(CC^\bullet(A, A)\) and \(CC^\bullet(A, B)_0\), when \(A, B\) are objects in the category.

We will use without remark that if \((A, 1)\) is a graded module with split element, the splitting \(A = \overline{A} \oplus k\) induces a splitting \(CC^\bullet(\overline{A}, A) = CC^\bullet(\overline{A}, \overline{A}) \oplus CC^\bullet(\overline{A}, k)\).

Definition 4.5. A strictly unital element \(\mu + \mu_{su} \in CC^\bullet(A, A)\), with \(\mu = \overline{\mu} + h \in CC^\bullet(\overline{A}, \overline{A})\), is augmented if \(h = 0\), i.e., if \(\mu\) is in \(CC^\bullet(\overline{A}, \overline{A})\). (In this case, if \(\mu + \mu_{su}\) is an \(A_\infty\)-algebra structure, the fixed splitting \(A \rightarrow k\) is a strict \(A_\infty\)-morphism, called the augmentation.)

We note the term \(h\) measuring the lack of augmentation is in \(CC^\bullet(\overline{A}, k)\).

Example 4.6. Let \((A, 1)\) be a graded module with split element such that \(A_i = 0\) for \(i < 0\), \(A_0 = k\), and 1 \(\in A_0\) is the unit in \(k\). Let \(\mu = \overline{\mu} + h \in CC^\bullet(\overline{A}, A)\) be an element such that \((A, 1, \nu = \mu + \mu_{su}\) is an \(A_\infty\)-algebra with split unit. Since \(\overline{A} = A_{\geq 1}\), it follows that \((\overline{\Pi A})_2 = 0\) for \(n \geq 2\). Thus \(h^n = 0\) for all \(n \geq 2\); the map \(h^1\) makes the following diagram commutative:

\[
\begin{array}{ccc}
(\Pi A)_2 & \xrightarrow{h^1} & (\Pi k)_1 \\
\downarrow & & \downarrow \\
A_1 & \xrightarrow{(m^1)_1} & A_0.
\end{array}
\]

Here \(m^1 = s^{-1} \nu^1 s\), see \(2.10\). Note that the image of \((m_1)_{1}\) is an ideal \(I\) in \(k = A_0\). The \(A_\infty\)-algebra \((A, 1, \mu + \mu_{su}\) is augmented exactly when \(h^1 = 0\), i.e., \(I = 0\).

By Lemma 4.8 below, \([\overline{\mu}, \mu_{su}] = 0\) for all \(\overline{\mu} \in CC^\bullet(\overline{A}, \overline{A})\), and it follows that Maurer-Cartan elements of \(CC^\bullet(\overline{A}, \overline{A})\) (i.e., nonunital \(A_\infty\)-algebra structures on \(\overline{A}\)) correspond to augmented \(A_\infty\)-algebra structures on \(A\), via the map \(\overline{\mu} \mapsto \overline{\mu} + \mu_{su}\). The following generalizes this to all strictly unital \(A_\infty\)-algebras.

Theorem 4.7. Let \((A, 1)\) be a module with split element, and \(\mu_{su} \in CC^\bullet(A, A)\) the trivial strictly unital \(A_\infty\)-algebra structure defined in \(4.3\). The submodule \(CC^\bullet(\overline{A}, A)\) is a graded Lie subalgebra of \(CC^\bullet(A, A)\), and the derivation \([\mu_{su}, -]\) of \(CC^\bullet(A, A)\) preserves \(CC^\bullet(\overline{A}, A)\). The Maurer-Cartan elements of the resulting dg-Lie algebra \((CC^\bullet(\overline{A}, A), [\mu_{su}, -])\) correspond to the strictly unital \(A_\infty\)-structures on \((A, 1)\) via \(\overline{\mu} + h \mapsto \overline{\mu} + h + \mu_{su}\).
We need the following lemma for the proof of Theorem 4.7.

**Lemma 4.8.** Let $\mu, \mu' \in CC^\bullet(\overline{A}, \overline{A})_{-1}$ and $h, h' \in CC^\bullet(\overline{A}, k)_{-1}$ be arbitrary elements. The following hold:

1. $[\mu, \mu_{\text{su}}] = 0 = [h, h']$.
2. $[\mu_{\text{su}}, h] = \mu_{\text{su}}^2 h + \mu_{\text{su}}^2 h \in CC^\bullet(\overline{A}, \overline{A})$.
3. $[\mu, \mu'] = \sum_j [\pi_j, \pi'] \in CC^\bullet(\overline{A}, \overline{A})$.
4. $[\mu, h] = \sum_j [\pi_j, h] \in CC^\bullet(\overline{A}, k)$.

**Proof.** All of the equalities are automatic except for the first half of (1), $[\mu, \mu_{\text{su}}] = 0$, and (2). To show (1), one can first check that for all $j \geq 1$, the following holds:

$$j \mu_{\text{su}}^2 h + j^{-1} \mu_{\text{su}}^2 h = 0.$$ 

(To check this one can evaluate both diagrams on the element $\left[a_1 \ldots a_j | 1 | a_{j+1} \ldots | a_n\right]$, using the sign conventions of 2.3.) The above implies that $\mu_{\text{su}} \circ \mu_{\text{su}} = \mu_{\text{su}}^2 h + \mu_{\text{su}}^2 h \in CC^\bullet(\overline{A}, \overline{A})$, and one checks this is $-\mu_{\text{su}} \circ \mu_{\text{su}}$ (by evaluating on elements as above). The proof of (2) is similar. \hfill \square

**Proof of Theorem 4.7.** For $\pi + h, \pi' + h' \in CC^\bullet(\overline{A}, A)$, we have $[\pi + h, \pi' + h'] = [\pi, \pi'] + [\pi', h] + [\pi, h'] \in CC^\bullet(\overline{A}, A)$, using the previous lemma. Thus $CC^\bullet(\overline{A}, A)$ is a graded subalgebra of $CC^\bullet(\overline{A}, A)$. Again using the lemma, we have $[\mu_{\text{su}}, \pi + h] = [\mu_{\text{su}}, h] \in CC^\bullet(\overline{A}, A)$, and thus $CC^\bullet(\overline{A}, A)$ is preserved by $[\mu_{\text{su}}, -]$.

A strictly unital element $\pi + h + \mu_{\text{su}}$ in $CC^\bullet(\overline{A}, A)_{-1}$ is an $A_\infty$-algebra structure exactly when $[\pi + h + \mu_{\text{su}}, \pi + h + \mu_{\text{su}}] = [\pi, h_{\text{su}} + h + 2[\mu_{\text{su}}, \pi + h]]$ is zero, i.e., $\frac{1}{2} [\pi + h, \pi + h + [\mu_{\text{su}}, \pi + h]] = 0$. And this is the definition of $\pi + h$ being a Maurer-Cartan element of $(CC^\bullet(\overline{A}, A), [\mu_{\text{su}}, -])$. \hfill \square
Remark. The dg-Lie algebra \((\mathbb{C}^*\mathcal{A}, A), [\mu_{su}, -]\) is an adaptation of a construction of Schlessinger and Stasheff [SSS5 §2], who use the cofree Lie coalgebra where we use the cofree coassociative coalgebra \(T_{\mathcal{A}}\). To match the definitions, one can check that the graded subalgebra \(\mathbb{C}^*(\mathcal{A}, \mathcal{A})\) of \(\mathbb{C}^*(A, A)\) acts on the \(k\)-module \(\mathbb{C}^*(\mathcal{A}, k)\), via Lemma 4.8(4), and the resulting semi-direct product \(\mathbb{C}^*(\mathcal{A}, A) \oplus \mathbb{C}^*(\mathcal{A}, k)\) is isomorphic as a graded Lie algebra to \(\mathbb{C}^*(\mathcal{A}, A)\); one then checks the derivations agree.

4.2. Curved bar construction.

Definition 4.9. Let \(C\) be a graded coalgebra and \(\xi \in \text{Hom}(C, k)\) a homogeneous linear map. Define \(\text{ad}(\xi) \in \text{Hom}(C, C)\) by \(\text{ad}(\xi) := (C \xrightarrow{\Delta} C \otimes C \xrightarrow{\xi \otimes 1} k \otimes C \cong C) - (C \xrightarrow{\Delta} C \otimes C \xrightarrow{1 \otimes \xi} C \otimes k \cong C).\)

(One checks this is a coderivation of \(C\).) A curved dg-coalgebra is a triple \((C, d, \xi)\), with \(C\) a graded coalgebra, \(d : C \rightarrow C\) a coderivation of degree \(-1\), and \(\xi : C \rightarrow k\) a degree \(-2\) linear map, such that \(d^2 = \text{ad}(\xi)\) and \(\xi d = 0\). A dg-coalgebra is a curved dg-coalgebra with \(h = 0\) (so \(d^2 = 0\)).

When \(C = T_{\mathcal{A}}(\mathcal{A})\), we can calculate \(\text{ad}(\xi)\) using the Gerstenhaber bracket and the trivial strictly unital \(A_\infty\)-structure.

Lemma 4.10. If \(\xi \in \text{Hom}(T_{\mathcal{A}}(\mathcal{A}), k)\), then \(\text{ad}(\xi) = \Phi^{-1}(\mu_{su}, s\xi)\).

Proof. Since \(\text{ad}(\xi)\) is a coderivation, it is equal to \(\Phi^{-1}(\pi_1 \text{ad}(\xi))\), using Lemma 2.5(1) (where \(\Phi^{-1}\) is defined, also). Thus it is enough to show that \(\pi_1 \text{ad}(\xi)|_{\Pi \mathcal{A} \otimes n} = [\mu_{su}, s\xi]|_{\Pi \mathcal{A} \otimes n}\) for all \(n \geq 1\). If \(\xi = (\xi^n)\), then

\[
\pi_1 \text{ad}(\xi)|_{\Pi \mathcal{A} \otimes n} = (\Pi \mathcal{A} \otimes n \xrightarrow{\xi^n \otimes 1} k \otimes \Pi \mathcal{A} \cong \Pi \mathcal{A}) - (\Pi \mathcal{A} \otimes n \xrightarrow{1 \otimes \xi^n} \Pi \mathcal{A} \otimes k \cong \Pi \mathcal{A}).
\]

By Lemma 4.8(2) we have \([\mu_{su}, s\xi]|_{\Pi \mathcal{A} \otimes n} = \mu_{su}(s\xi^{n-1} \otimes 1 + 1 \otimes s\xi^{n-1})\). Using the definition of \(\mu_{su}\), one checks these agree.

Corollary 4.11. Let \((A, 1)\) be a graded module with split element. A strictly unital element \(\mu + h + \mu_{su}\) in \(\mathbb{C}^*(A, A)\) is an \(A_\infty\)-algebra structure if and only if the triple \((T_{\mathcal{A}}(\mathcal{A}), \Phi^{-1}(\mu), -s^{-1}h)\) is a curved dg-coalgebra (\(\Phi^{-1}\) is defined in 2.5).

In diagrams, this is equivalent to:

\[
\sum_j \begin{array}{c}
\begin{array}{c}
\mu_{su} \downarrow \\
\pi \downarrow \\
\mu_{su} \downarrow \\
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
h \downarrow \\
\mu_{su} \downarrow \\
\downarrow \\
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\mu_{su} \downarrow \\
\mu_{su} \downarrow \\
\end{array}
\end{array} = 0
\]

\[
(4.11.1)
\]

\[
\sum_j \begin{array}{c}
\begin{array}{c}
\pi \downarrow \\
\mu_{su} \downarrow \\
\downarrow \\
\end{array}
\end{array} = 0.
\]

Proof. Let \(\mu + h + \mu_{su}\) be a strictly unital element with \(\mu + h \in \mathbb{C}^*(\mathcal{A}, \mathcal{A}) \oplus \mathbb{C}^*(\mathcal{A}, k) = \mathbb{C}^*(\mathcal{A}, A)\). By Theorem 4.7 this is an \(A_\infty\)-algebra structure if and
only if it is a Maurer-Cartan element of \((CC^\bullet(\overline{A}, A), [\mu_{su}, -])\). By Lemma 4.8, this is equivalent to

\[ [\mu_{su}, h] + \frac{1}{2}[\overline{\mu}, \overline{\mu}] = 0 \quad \text{and} \quad [\overline{\mu}, h] = 0, \]

and these are equivalent to the first and second equations of (4.11.1), respectively.

Set \( \overline{d} = \Phi^{-1}(\overline{\mu}) \). The triple \((T_{co}(\Pi \overline{A}), \overline{d}, -s^{-1}h)\) is a curved dg-coalgebra if and only if \( \overline{d}^2 + ad(s^{-1}h) = 0 \) and \( s^{-1}h\overline{d} = 0 \). We have \( h\overline{d} = h\Phi^{-1}(\overline{\mu}) = h \circ \overline{\mu} \), so \( s^{-1}h\overline{d} \)

is zero exactly when the second equation of (4.11.1) holds. Since \( \overline{d}^2 \) and \( ad(s^{-1}h) \)

are both coderivations of \( T_{co}(\Pi \overline{A}) \), \( \overline{d}^2 = -ad(s^{-1}h) \) holds if and only if \( \pi_1\overline{d}^2 = -\pi_1 ad(s^{-1}h) \) holds, by Lemma 4.10. We have \( \pi_1\overline{d}^2 = \overline{\mu} \circ \overline{\mu} \), and \( \pi_1 ad(s^{-1}h) = [\mu_{su}, h] \)

by Lemma 4.10. Thus \( \overline{d}^2 + ad(s^{-1}h) = 0 \) holds if and only if \( \overline{\mu} \circ \overline{\mu} + [\mu_{su}, h] = 0 \)

holds, and this is exactly the first equation of (4.11.1).

**Definition 4.12.** If \((A, 1, \overline{\mu} + h + \mu_{su})\) is an \( A_\infty \)-algebra with split unit, the *curved bar construction*, denoted \( \text{Bar} \overline{A} \), is the curved dg-coalgebra \((T_{co}(\Pi \overline{A}), \Phi^{-1}(\overline{\mu}), -s^{-1}h)\).

**Remark.** Note that \( \text{Bar} \overline{A} \) is a dg-coalgebra if and only if \( h = 0 \) if and only if \((A, 1, \mu)\) is augmented.

**Example 4.13.** Let \((A, 1)\) be a graded module with split element as in Example 4.6 and let \( \nu = \overline{\mu} + h + \mu_{su} \) be a strictly unital \( A_\infty \)-algebra structure on \((A, 1)\). Set \( \overline{d} = \Phi^{-1}(\overline{\mu}) \in \text{Coder}(T_{co}(\Pi \overline{A}), T_{co}(\Pi \overline{A})) \). By Example 4.10, \( h^n = 0 \) for \( n \geq 2 \), thus \( h \circ \overline{\mu} = h\overline{\mu} = 0 \) and \( \mu_{su} \circ h \) is concentrated in tensor degree two. It follows from Corollary 4.11 that \( \overline{d}^2 [a_1] \ldots [a_n] = 0 \) for \( n \neq 2 \) and

\[
\overline{d}^2 [a_1|a_2] = \begin{cases} 
0 & |a_1| \neq 1 \text{ and } |a_2| \neq 1 \\
 m^1(a_1) a_2 & |a_1| = 1 \text{ and } |a_2| \neq 1 \\
- m^1(a_2) a_1 & |a_1| \neq 1 \text{ and } |a_2| = 1 \\
 m^1(a_1) a_2 - m^1(a_2) a_1 & |a_1| = 1 \text{ and } |a_2| = 1.
\end{cases}
\]

The \( A_\infty \)-algebra \((A, 1, \nu)\) is augmented exactly when \( h^1 = 0 \), which is equivalent to \( (m^1)^1 = 0 \). Thus we see directly in this case that \( \overline{d}^2 = 0 \) if and only if \((A, 1, \nu)\) is augmented.

The smallest nontrivial case of the above is the following.

**Example 4.14.** Let \((A, 1, \mu)\) be the Koszul complex on \( f \in k \), so \( \mu^n = 0 \) for \( n \geq 3 \), \( \mu^2 = \mu_{su} \) and \( \mu^1 = (k \cdot [e] \xrightarrow{f} k \cdot [1]) \in CC^\bullet(\overline{A}, k)_{-1} \). Thus \( \overline{\mu} = 0 \) and \( h = \mu^1 \), so \( A \)

is augmented if and only if \( f = 0 \). We also have:

\[
T_{co}(\Pi \overline{A}) = \begin{array}{cccccccc}
0 & 0 & k[e]^0n & 0 & \cdots & 0 & k[e] & 0 & k[e] & 0 \\
2n + 1 & 2n & 2n - 1 & 5 & 4 & 3 & 2 & 1 & 0
\end{array}
\]

and \( h_1([e]) = -f \) and \( h_2 = 0 \) for \( n \geq 2 \). If we set \( T = [e] \in T_{co}(\Pi \overline{A})_{2} \), then \( T_{co}(\overline{A}) = k[T] \), the divided powers coalgebra on the 1-dimensional free module generated by \( T \). The \( k \)-dual is the symmetric algebra \( k[T^*] \), with curvature \( -fT^* \in k[T^*]_{-2} \).

We now show the curved bar construction is functorial.

**Definition 4.15.** A *morphism of curved dg-coalgebras*, \((C, d_C, h_C) \rightarrow (D, d_D, h_D)\), is a pair \((\gamma, \alpha)\), with \( \gamma : C \rightarrow D \) a graded coalgebra morphism and \( \alpha : C \rightarrow k \) a
degree $-1$ linear map, such that the following equations hold,
\[
d_B \gamma = \gamma d_C + \gamma \text{ad}(\alpha) \in \text{Hom}(C, D)
\]
\[
h_B \gamma - \alpha^2 = \alpha d_C + h_C \in \text{Hom}(C, k),
\]
where $\text{ad}(\alpha)$ is defined in \[4.9\] and $\alpha^2 = (C \xrightarrow{\Delta} C \otimes C \xrightarrow{\alpha \otimes \alpha} k \otimes k \cong k)$.

**Corollary 4.16.** Let $(A, 1, \overline{\mu}_A + h_A + \mu_{su})$ and $(B, 1, \overline{\mu}_B + h_B + \mu_{su})$ be $A_\infty$-algebras with split units. A strictly unital element $g + g_{su} \in CC^\bullet(A, B)_0$, with $g = \bar{g} + a \in CC^\bullet(\overline{A}, B) \oplus CC^\bullet(\overline{A}, k) = CC^\bullet(\overline{A}, B)$,

is a morphism of $A_\infty$-algebras if and only if
\[
(\Psi^{-1}(\bar{g}), -s^{-1}a) : \text{Bar}\overline{A} \rightarrow \text{Bar}\overline{B}
\]
is a morphism of the corresponding curved dg-coalgebras, where $\Psi^{-1}$ is defined in \[2.5\](2). In diagrams this is equivalent to:

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\text{\[4.17.1\]} \\
\end{array}
\end{array}
\end{align*}
\]

We need the following lemma for the proof. For later use, we assume that $B$ has a strict, but not necessarily split, unit; e.g., $B = k/I$ for some ideal $I$.

**Lemma 4.17.** Let $(A, 1, \overline{\mu}_A + h_A + \mu_{su})$ be an $A_\infty$-algebra with split unit and $(B, \nu_B)$ an $A_\infty$-algebra with strict unit $1 \in B_0$. A strictly unital element $g + g_{su}$, with $g \in CC^\bullet(\overline{A}, B)_0$ is an $A_\infty$-morphism if and only if $\nu_B * g$ and $g \circ \overline{\mu}_A + g_{su} \circ h_A$ are equal, where $*$ is defined in \[2.5\] and $\circ$ is the Gerstenhaber product.

**Proof.** By definition, $g + g_{su}$ is an $A_\infty$-morphism exactly when
\[
\nu_B * (g + g_{su}) = (g + g_{su}) \circ (\overline{\mu}_A + h_A + \mu_{su}).
\]
We claim this equation always holds for elements of $T_\infty(\Pi A) \setminus T_\infty(\Pi \overline{A})$. Assuming the claim, we now note that the above equation holds on elements of $T_\infty(\Pi \overline{A})$ if and only if $\nu_B * g = (g + g_{su}) \circ (\overline{\mu}_A + h_A)$, since $g_{su}$ and $\mu_{su}$ are zero on $T_\infty(\Pi \overline{A})$. Also, clearly $g \circ h_A = 0$ and $g_{su} \circ \overline{\mu}_A = 0$. Thus \[4.17.1\] holds on $T_\infty(\Pi \overline{A})$ if and only if $\nu_B * g = g \circ \overline{\mu}_A + g_{su} \circ h_A$.

We are left to prove the claim, i.e., that \[4.17.1\] holds on any element of the form $a = [a_1] \ldots [a_1][a_1-1] \ldots [a_n]$. We first compute the left side. Since $g + g_{su}$ is strictly unital, we have
\[
\nu_B * (g + g_{su})(a) = \nu_B (\Psi^{-1}(g + g_{su})[a_1] \ldots [a_1-1] \otimes [1] \otimes \Psi^{-1}(g + g_{su})[a_1+1] \ldots [a_n])
\]
Using that $\nu_B$ is strictly unital, we have: if $l = 1 = n$, the result is $[1]$; if $1 < l = 2 = n$ and $a_2 = 1$, or if $l = 2 = n$ and $a_2 = 1$, the result is $[1]$; if $1 < n$, the result is
Proof of 4.16. By the previous lemma, $g + g_{su}$ is an $A_\infty$-morphism if and only if 
\[ (\overline{\pi}_B + \overline{h}_B + \mu_{su} ) \ast g = g \circ \overline{\pi}_A + g_{su} \circ h_A. \]
Substituting $g = \overline{g} + a$, and using the equalities $\overline{\pi}_B \ast g = \overline{\pi}_B \ast \overline{g}$ and $h_B \ast g = h_B \ast \overline{g}$, this is equivalent to:
\begin{equation}
\overline{\pi}_B \ast \overline{g} + h_B \ast \overline{g} + \mu_{su} \ast (\overline{g} + a) - \overline{g} \circ \overline{\pi}_A - a \circ \overline{\pi}_A - g_{su} \circ h_A = 0.
\end{equation}
We can match each term of (4.17.2) with a diagram in (4.16.1): $\overline{\pi}_B \ast \overline{g}$ is the first diagram and $-\overline{g} \circ \overline{\pi}_A$ is the second diagram, both in the first line; $h_B \ast \overline{g}$, $-a \circ \overline{\pi}_A$ and $-g_{su} \circ h_A$ are the first, second, and third diagrams of the second line; $\mu_{su} \ast (\overline{g} + a)$ is the sum of the third and fourth diagrams of the first line, and the fourth diagram on the second line. It follows that $g + g_{su}$ is an $A_\infty$-morphism if and only if the equations (4.16.1) hold.

We now claim the equations (4.16.1) hold if and only if $(\Psi^{-1}(\overline{g}), -s^{-1}a)$ is a morphism of curved dg-coalgebras, i.e.,
\begin{align*}
\Phi^{-1}(\overline{\pi}_B) \Psi^{-1}(\overline{g}) &= \Psi^{-1}(\overline{g}) \Phi^{-1}(\overline{\pi}_A) - \psi^{-1}(\overline{g}) \operatorname{ad}(s^{-1}a) \\
-s^{-1}h_B \Psi^{-1}(\overline{g}) - (-s^{-1}a)^2 &= -s^{-1}a \Phi^{-1}(\overline{\pi}_B) - s^{-1}h_A.
\end{align*}
Using 2.8 to reduce the first equation, and applying $-s$ to the second, we have
\begin{align*}
\overline{\pi}_B \Psi^{-1}(\overline{g}) &= \overline{g} \Phi^{-1}(\overline{\pi}_A) - \overline{g} \operatorname{ad}(s^{-1}a) \\
h_B \Psi^{-1}(\overline{g}) + s(s^{-1}a)^2 &= a \Phi^{-1}(\overline{\pi}_B) + h_A.
\end{align*}
Using 4.10 one calculates that $\overline{g} \operatorname{ad}(s^{-1}a)$ is the third and fourth terms of the first equation, and one checks $s(s^{-1}a)^2$ is the last diagram in the second equation. The other terms are easily matched to their counterparts in the equations (4.16.1), which completes the proof. □

4.3. Strictly unital deformation theory. We will use without comment that if $(A, 1)$ is a graded $k$-module with split element and $l$ is a $k$-algebra, then $(A \otimes l, 1 \otimes 1)$ is a graded $l$-module with split element, and $\overline{A} \otimes l = \overline{A} \otimes l$.

Definition 4.18. A strictly unital $(l, \epsilon)$-deformation of an $A_\infty$-algebra with split unit $(A, 1, \mu)$, where $(l, \epsilon)$ is an augmented algebra, is an $l$-linear $A_\infty$-algebra with split unit of the form $(A \otimes l, 1 \otimes 1, \mu_{su})$, such that $(A \otimes l, \mu_{su})$ is a nonunital $(l, \epsilon)$-deformation of $(A, \mu)$. We denote the resulting functor
\[
\inf\text{Def}_{(A, 1, \mu)}^{su} : \text{finComAlg}_{k}^{aug} \to \text{Set}.
\]
If $(A, 1, \mu)$ is an augmented $A_\infty$-algebra, an augmented $(l, \epsilon)$ deformation is a strictly unital deformation $(A \otimes l, 1 \otimes 1, \mu_{su})$ that is augmented; the corresponding functor is denoted $\inf\text{Def}_{(A, 1, \mu)}^{aug}$.

We denote by $\mu_{su}^l \in CC_{l}^{*}(A \otimes l, A \otimes l)$ the $l$-linear trivial strictly unital algebra structure (see 4.3 for the definition). It follows from 4.4 that strictly unital elements of $CC_{l}^{*}(A \otimes l, A \otimes l)$ are of the form $\mu + \mu_{su}^l$, where
\[
\mu \in CC_{l}^{*}(\overline{A} \otimes l, A \otimes l) = CC_{l}^{*}(\overline{A} \otimes l, A \otimes l) \cong CC_{l}^{*}(\overline{A}, A \otimes l).
\]
Using the decomposition $CC_{*}^{*}(\overline{A}, A \otimes l) = CC_{*}^{*}(\overline{A}, A \otimes l) \oplus CC_{*}^{*}(\overline{A}, A \otimes l)$, the strictly unital element $\mu$ is a deformation of an $A_\infty$-algebra structure with split unit $(A, 1, \mu + \mu_{su})$ if and only if $\mu = \overline{\mu} + \theta_{l}(\mu \otimes 1)$ for some $\overline{\mu} \in CC_{*}^{*}(\overline{A}, A \otimes l)$. 
Using a different decomposition, we can write
\[
\mu = \mathfrak{p} + h \in CC^*_{\infty}(\overline{A}, \overline{A} \otimes I) \oplus CC^*_{\infty}(\overline{A}, I) = CC^*_{\infty}(\overline{A}, A \otimes I).
\]
The element \(\mu + h_{\text{su}}\) is augmented if and only if \(h = 0\). If \((A,1,\mathfrak{p} + h_{\text{su}})\) is an augmented \(A_\infty\)-algebra (so \((\overline{A},\mathfrak{p})\) is a nonunital \(A_\infty\)-algebra) it now follows easily that there is a natural equivalence of functors,
\[
\infDef_{(\pi,\mathfrak{p})} \cong \infDef_{(A,1,\mathfrak{p} + h_{\text{su}})}^{\text{aug}}.
\]
Thus by \(3.6\) the dg-Lie algebra \((CC^*_{\infty}(\overline{A},\overline{A}),[\pi,\mu])\) controls the infinitesimal augmented deformations of the augmented \(A_\infty\)-algebra \((A,1,\mathfrak{p} + h_{\text{su}})\).

**Definition 4.19.** The reduced Hochschild cochains of an \(A_\infty\)-algebra with split unit \((A,1,\mu + h_{\text{su}})\) is the dg-Lie algebra \((CC^*_{\infty}(\overline{A}, A),[\mu + h_{\text{su}},-])\) (it follows from \(4.8\) that \([\mu + h_{\text{su}},-]\) preserves the subalgebra \(CC^*_{\infty}(\overline{A}, A)\) of \(CC^*_{\infty}(A, A)\)).

**Corollary 4.20.** Let \((A,1,\mu + h_{\text{su}})\) be \(A_\infty\)-algebra with split unit. The reduced Hochschild cochains control the infinitesimal strictly unital deformation functor via the natural transformation
\[
MC_{(CC^*_{\infty}(\overline{A}, A),[\mu + h_{\text{su}},-])}(l,\mu) \xrightarrow{\theta_1} \infDef_{(A,1,\mu + h_{\text{su}})}^{\text{aug}}(\mu \otimes 1 + h_{\text{su}} \otimes 1) = \mu + h_{\text{su}}.\]

**Proof.** Let \((l,\epsilon)\) be an object of \(\text{finComAlg}_{k}^{\text{aug}}\) and set \(\tilde{l} = \ker \epsilon\). By definition,
\[
MC_{(CC^*_{\infty}(\overline{A}, A),[\mu + h_{\text{su}},-])}(l,\mu) = MC(CC^*_{\infty}(\overline{A}, A) \otimes \tilde{l},[\mu \otimes 1 + h_{\text{su}} \otimes 1,\epsilon]),
\]
and the following is seen to be a bijection,
\[
MC(CC^*_{\infty}(\overline{A}, A) \otimes \tilde{l},[\mu \otimes 1 + h_{\text{su}} \otimes 1,\epsilon]) \cong MC(CC^*_{\infty}(\overline{A}, A) \otimes l,[\mu_{\text{su}} \otimes 1,\epsilon])
\]
\[
\theta_1(\mu) \mapsto \mu \otimes 1.
\]
One checks that \(\theta_1(\mu_{\text{su}} \otimes 1) = h_{\text{su}}\), and thus \(\theta_1\) is a morphism of dg-Lie algebras \((CC^*_{\infty}(\overline{A}, A) \otimes l,[\mu_{\text{su}} \otimes 1,\epsilon]) \rightarrow (CC^*_{\infty}(\overline{A} \otimes \tilde{l}, A \otimes l),[\mu_{\text{su}},\epsilon])\). Since \(l\) is a finitely generated projective \(k\)-module, \(\theta_1\) is an isomorphism, and thus induces a bijection between MC elements. The target is the set of \(A_\infty\)-algebra structures on \(A \otimes l\) such that \(1 \otimes 1\) is a split unit by Theorem \(4.7\). Finally, one checks the bijection restricts to a bijection \(MC(1 \otimes \epsilon)^{-1}(\mu) \cong \text{Fam}_{A}(\epsilon)^{-1}(\mu) = \infDef_{(A,\mu)}(l)\).

**Remark.** The reduced Hochschild cochains are quasi-isomorphic to the standard Hochschild complex, see \cite[Theorem 4.4]{Laz03}, but not as dg-Lie algebras. Indeed, the functors they control, infinitesimal strictly unital deformations and infinitesimal nonunital deformations, are different.

### 5. Representations of \(A_\infty\)-algebras

In this section we treat strictly unital \(A_\infty\)-modules. In particular, we give a proof of Positselski’s result that strictly unital modules over a strictly unital \(A_\infty\)-algebra correspond to cofree curved dg-comodules over the curved bar construction.
5.1. Representations of nonunital $A_{\infty}$-algebras. If $(M, \delta_M)$ is a complex of modules, $\text{Hom}(M, M)$ is a dg algebra with multiplication equal to composition and differential $\delta_{\text{Hom}} = [\delta_M, -]$. We denote by $(\text{End} M, \mu_{\text{End}})$ the corresponding $A_{\infty}$-algebra, see 2.10.

Definition 5.1. A representation of a nonunital $A_{\infty}$-algebra $(A, \mu)$ on a complex $(M, \delta_M)$ is an $A_{\infty}$-morphism $p = (p^n) \in CC^*(A, \text{End} M)_0$ from $(A, \mu)$ to $(\text{End} M, \mu_{\text{End}})$.

Definition 5.2. Let $M, N$ be graded modules. The adjoint of an element $p^n$ in $CC^n(A, \text{Hom}(M, N))$ is $\lambda^{n+1} = \text{ev}(s^{-1}p^n \otimes 1) : (\Pi A)^{\otimes n} \otimes M \to N$, where $\text{ev}(f \otimes m) = f(m)$. Thus $\lambda^{n+1}$ is the image of $p^n$ under the following isomorphisms:

$$CC^n(A, \text{Hom}(M, N)) \cong \Pi \text{Hom}((\Pi A)^{\otimes n}, \text{Hom}(M, N))$$

where the first isomorphism is from (4). In string diagrams, $\Pi$ denotes $\Pi A$, $\otimes$ denotes $M$, $\otimes M$ denotes $N$, and $\otimes$ represents $\Pi \text{Hom}(M, N)$. We then have:

$$\lambda^{n+1} = \text{ev}(s^{-1}p^n \otimes 1) : (\Pi A)^{\otimes n} \otimes M \to N.$$

Lemma 5.3. An element $p = (p^n) \in CC^*(A, \text{End} M)_0$ is a representation of $(A, \mu = (\mu^n))$ on $(M, \delta_M)$ if and only if the adjoint family $(\lambda^{n+1})$, with $\lambda^1 = \delta_M$, satisfies:

$$\sum_{i=2}^{n+1} \sum_{j=0}^{i-2} \lambda^{i-2} + \sum_{i=1}^{n+1} \lambda^{i-2} = 0.$$

Proof. By the definition of $A_{\infty}$-morphism, $p$ is a representation if and only if $p \circ \mu_A - \mu_{\text{End}} * p = 0$. This equation holds if and only if it holds in every tensor degree. Applying the isomorphism $CC^n(A, \text{End} M)_0 \cong \text{Hom}((\Pi A)^{\otimes n} \otimes M, M)_{-1}$, one checks the equation in tensor degree $n$ is equivalent to the diagrams above, with $p \circ \mu_A$ corresponding to the left diagram and $-\mu_{\text{End}} * p$ to the right diagram.

To define a morphism of representations, we need to add a counit to the tensor coalgebra (else we would have to fix a morphism of complexes, and talk about morphisms of representations over that fixed morphism of complexes). Set

$$T_{\text{co}, u}(\Pi A) = k \times T_{\text{co}}(\Pi A) = \bigoplus_{n \geq 0} (\Pi A)^{\otimes n}$$

$$CC^*_u(A, B) = \text{Hom}(T_{\text{co}, u}(\Pi A), \Pi B) \cong CC^*(A, B) \oplus \Pi B.$$
Using this isomorphism, given a representation \( p \) on a complex \((M, \delta_M)\), we set \( p_M = p + \delta_M \in CC^\bullet_u(A, \text{End } M)_0 \). Conversely, we can view representations as elements \( p_M \in CC^\bullet_u(A, \text{End } M)_0 \) such that \( p_M^0 \in \text{End } M \) is a differential and \( p_M^\geq 1 \) is an \( A_\infty \)-morphism from \( A \) to the endomorphism \( A_\infty \)-algebra of the complex \((M, p_M^0)\).

**Definition 5.4.** Let \( M, N, P \) be graded modules. We consider the action

\[
\star : CC^\bullet_u(A, \text{Hom}(N, P))_k \otimes CC^\bullet_u(A, \text{Hom}(M, N))_l \to CC^\bullet_u(A, \text{Hom}(M, P))_{k+l-1}
\]

\[
\alpha \otimes \beta = (\alpha^n) \otimes (\beta^n) \mapsto \left( \gamma \sum_{j=0}^{n} \alpha^j \otimes \beta^{n-j} \right) = \alpha \star \beta,
\]

where \( \gamma = sc(s^{-1} \otimes s^{-1}) \), with \( c \) the composition map. If \( a^{n+1} \) and \( b^{n+1} \) are the adjoints of \( \alpha^n \) and \( \beta^n \), and \( 0 \) represents \( P \), then the adjoint of \( (\alpha \star \beta)^n \) is

\[
(-1)^{|a|-1} \sum_{i=1}^{n+1} \left[ \begin{array}{c}
\vdots \\
\vdots \\
\alpha^i \\
\vdots \\
\end{array} \right] = 0.
\]

**Definition 5.5.** A morphism of representations \((M, p_M) \to (N, p_N)\) of a nonunital \( A_\infty \)-algebra \((A, \mu)\) is an element \( f \in CC^\bullet_u(A, \text{Hom}(M, N))_1 \) such that \( p_N \star f + (f^\geq 1) \circ \mu + f \star p_M = 0 \). The composition with a second morphism, \( \tilde{f} \in CC^\bullet_u(A, \text{Hom}(N, P))_1 \) is \( \tilde{f} \star f \in CC^\bullet_u(A, \text{Hom}(M, N))_1 \).

**Lemma 5.6.** Let \((M, p_M)\) and \((N, p_N)\) be representations of a nonunital \( A_\infty \)-algebra \((A, \mu)\), with adjoints \( \lambda_M \) and \( \lambda_N \), respectively. An element \( f = (f^n) \in CC^\bullet_u(A, \text{Hom}(M, N))_1 \) is a morphism of representations if and only if the adjoint family \((g^{n+1})\) satisfies the equations

\[
\begin{align*}
\sum_{i=1}^{n+1} \left[ \begin{array}{c}
\vdots \\
\vdots \\
\lambda_M^i \\
\vdots \\
\end{array} \right] &- \sum_{i=2}^{n+1} \sum_{j=0}^{i-2} \left[ \begin{array}{c}
\vdots \\
\vdots \\
\mu_M^{i-j-2} \\
\vdots \\
\end{array} \right] \quad \left[ \begin{array}{c}
\vdots \\
\vdots \\
\lambda_N^{i-j} \\
\vdots \\
\end{array} \right] = 0.
\end{align*}
\]

Indeed, each of the three terms above is the adjoint of \((-1)\) times the corresponding term in the definition of morphism.

5.2. Representations of strictly unital \( A_\infty \)-algebras.

**Definition 5.7.** Let \((A, 1, \mu)\) be a strictly unital \( A_\infty \)-algebra. A strictly unital representation on a complex \((M, \delta_M)\) is a strictly unital \( A_\infty \)-morphism \( p \in CC^\bullet_u(A, \text{End } M)_0 \). A morphism of strictly unital representations \((M, p_M) \to (N, p_N)\) is a morphism of representations \( f \) such that \( f \in CC^\bullet_u(\overline{A}, \text{Hom}(M, N))_1 \), i.e.,

\[
f^n([a_1] \ldots |a_{n-1}|a_n]) = 0 \quad \text{for all } n \geq 1.
\]

In string diagrams, \( \| \) will now denote \( \Pi \overline{A} \) (previously it denoted \( \Pi A \)), while \( \langle \rangle \) continues to represent \( M \), \( \langle \rangle \) represents \( N \), and \( \| \) represents \( Hk \).

**Lemma 5.8.** Let \((A, 1, \overline{\mu} + h + \mu_{su})\) be an \( A_\infty \)-algebra with split unit.
(1) A strictly unital element \( p = \overline{p} + g_{su} \in CC^\bullet (A, \text{End} M)_0 \), with \( \overline{p} \in CC^\bullet (A, \text{End} M)_0 \), is a representation if and only if the adjoint family \( \overline{X} = (\overline{X}^{n+1}) \) of \( \overline{p} \), where \( \overline{X}^1 = \delta_M \), satisfies

\[
\sum_{i=2}^{n+1} \sum_{j=0}^{i-1} \overline{X}^{n+1} = 0.
\]

(2) An element \( f \in CC^\bullet (A, \text{Hom}(M, N))_1 \) is a morphism of strictly unital representations \( (M, \overline{P}_M) \to (N, \overline{P}_N) \) if and only if the following holds, where \( g, \overline{X}_M, \overline{X}_N \) are the adjoint families of \( f, \overline{P}_M, \overline{P}_N \):

\[
\sum_{i=1}^{n+1} \overline{X}^i = 0.
\]

Proof. By Lemma 4.17, \( \overline{p} + g_{su} \) is an \( A_\infty \)-morphism if and only if \( \overline{p} \circ \overline{p} - \mu_{\text{End}} \ast \overline{p} + g_{su} \circ h_A = 0 \). One checks that the adjoints of the three terms of this equation agree with the three families of displayed diagrams, and this proves part 1.

For part 2, the element \( f \) is a morphism of representation if and only if \( p_N \ast f + f \geq 1 \circ \mu + f \ast p_M = 0 \). Using the decompositions \( \mu = h + \mu_{su}, p_M = p_M + g_{su}, \) and \( p_N = p_N + g_{su}, \) together with the equality \( g_{su} \ast f + f \geq 1 \circ \mu_{su} + f \ast g_{su} = 0 \), we see \( f \) is a morphism if and only if \( p_N \ast f + f \geq 1 \circ \overline{p} + f \ast p_M = 0 \). Each of the three terms of this equation is the adjoint of \((-1)^{n+1}\) times the corresponding term in the displayed equation.

Example 5.9. Let \( (A, 1, \mu) \) be the Koszul complex on \( f \in k \), see 4.14 and \( M \) a graded module. Let \( \overline{P}_M \in CC^\bullet (A, \text{End} M)_0 \) be an arbitrary element with adjoint family \( (\overline{X}^n) \) and set \( \sigma^\ast : M \xrightarrow{\sigma} k[e] \otimes^\Sigma M \xrightarrow{\overline{X}^n} M \), a degree \( 2n - 1 \) endomorphism of \( M \). Since \( \overline{p} = 0 \), we see that \( \overline{p}_M \) is a representation if and only if \( \sigma^1 \sigma^0 + \sigma^0 \sigma^1 = -f \cdot 1_M \) and \( \sum_{n=0}^\infty \sigma^{n+1} \sigma^1 = 0 \) for \( n \geq 2 \). Such a system of maps was first considered by Shamash [Sha09], who assumed that \( M \) was the \( k \)-free resolution of a \( k/(f) \)-module, and has since been important in the construction of free resolutions in commutative algebra; see e.g., [Avr18, §3.1] and the references contained there.

5.3. Comodules. If \( (A, \mu) \) is a nonunital \( A_\infty \)-algebra, set \( \text{Bar}_u A \) to be the coaugmented bar construction \( (T_{\text{co}, u}(\Pi A), d) \) (here \( d|_{T_u(\Pi A)} = \Phi_{-1} (\mu) \) and \( d(1) = 0 \)).

Definition 5.10. Let \( C \) be a graded coalgebra. The cofree \( C \)-comodule on a graded module \( M \) has underlying graded module \( C \otimes M \) and comultiplication \( \Delta_C \otimes 1 \). If \( d \) is a graded coderivation of \( C \) and \( P \) is a graded \( C \)-comodule, a coderivation of \( P \) (with respect to \( d \)) is a homogeneous map \( d_P : P \to P \), with \( |d_P| = |d| \), that satisfies \( (d \otimes 1 + 1 \otimes d_P) \Delta_P = \Delta_P d_P \). We denote by \( \text{Coder}^d(P, P) \) the set of coderivations of \( P \). If \( (C, d) \) is a dg-coalgebra, a dg-comodule is a pair \((P, d_P)\) with \( P \) a graded comodule and \( d_P \) an element of \( \text{Coder}^d(P, P) \) such that \( d_P^2 = 0 \). A morphism of dg-comodules is a morphism of comodules that commutes with the given coderivations. A dg-comodule is cofree if the underlying comodule is cofree.
Cofree comodules satisfy the linear analogue of \([2.3]\).

**Lemma 5.11.** Let \((C, \epsilon)\) be a graded coalgebra with counit \(\epsilon : C \to k\), and \(M\) a graded module. The following hold.

1. For any degree \(n\) coderivation \(d\) of \(C\), the following is an isomorphism,
   \[
   \phi : \text{Coder}^d(C \otimes M, C \otimes M) \xrightarrow{\cong} \text{Hom}(C \otimes M, M)_n
   \]
   with \(\phi^{-1}(m) = d \otimes 1 + (1 \otimes m)(\Delta_C \otimes 1)\). A coderivation \(\phi^{-1}(m)\) is a differential, i.e., squares to zero, if and only if \(m \phi^{-1}(m) = 0\).

2. For any graded \(C\)-comodule \(P\), the following is an isomorphism,
   \[
   \psi : \text{Hom}_C(P, C \otimes M) \xrightarrow{\cong} \text{Hom}(P, M),
   \]
   \[
   \beta \mapsto (\epsilon \otimes 1)\beta,
   \]
   where \(\text{Hom}_C(-,-)\) denotes morphisms of graded \(C\)-comodules. The inverse is given by \(\psi^{-1}(\alpha) = (1 \otimes \alpha)\Delta_P\). A morphism \(\psi^{-1}(g)\) commutes with coderivations \(d_P\) of \(P\) and \(\phi^{-1}(m)\) of \(C \otimes M\) if and only if \(gd_P = m_N \psi^{-1}(g)\).

Note the above properties emphasize the need to adjoin a counit to \(\text{Bar}_A\).

**Proposition 5.12.** Let \((A, \mu)\) be a nonunital \(A_\infty\)-algebra with counital bar construction \(\text{Bar}_u A\), and \(M, N\) graded modules.

1. An element \(p_{MN} \in \text{CC}_0^u(A, \text{End} M)_0\) is a representation of \((A, \mu)\) if and only if, for \(\lambda_M\) the adjoint family, the pair \((\text{Bar}_u A \otimes M, \phi^{-1}(\lambda_M))\) is a \(dg\)-\(\text{Bar}_u A\) comodule.

2. An element \(f \in \text{CC}_u^u(A, \text{Hom}(M, N))_1\) is a morphism of representations \((M, p_M) \to (N, p_N)\) if and only if \(\psi^{-1}(g) : (\text{Bar}_u A \otimes M, \phi^{-1}(\lambda_M)) \to (\text{Bar}_u A \otimes N, \phi^{-1}(\lambda_N))\) is a morphism of \(dg\)-\(\text{Bar}_u A\) comodules, where \(g, \lambda_M, \lambda_N\) are the adjoint families of \(f, p_M, p_N\).

**Proof.** The pair \((\text{Bar}_u A \otimes M, \phi^{-1}(\lambda_M))\) is a \(dg\) \(\text{Bar}_u A\)-comodule if and only if \((\phi^{-1}(\lambda_M))^2 = 0\). By \([5.11](1)\), this is equivalent to the equation \(\lambda_M \phi^{-1}(\lambda_M) = 0\), and by the definition of \(\phi^{-1}\), we see this is equivalent to the equations of \([5.3]\).

Analogously, \(\psi^{-1}(g)\) is a morphism of \(dg\)-comodules if and only if it commutes with the coderivations \(\phi^{-1}(\lambda_M)\) and \(\phi^{-1}(\lambda_N)\). By \([5.11](2)\) this is equivalent to \(g \phi^{-1}(\lambda_M) = \lambda_N \psi^{-1}(g)\), and from the definitions of \(\phi^{-1}\) and \(\psi^{-1}\), this is equivalent to the equations of \([5.6]\). \(\square\)

**Corollary 5.13.** Let \((A, \mu)\) be a nonunital \(A_\infty\)-algebra. There is a functor from the category of representations of \(A\) to the category of \(dg\) \(\text{Bar}_u A\) comodules, that sends \((M, p_M)\) to \((\text{Bar}_u A \otimes M, \phi^{-1}(\lambda_M))\). This is fully faithful with image the full subcategory of cofree \(dg\) comodules.

We now assume that \(A\) has a split unit, and construct the analogue of the above for strictly unital representations of \(A\).

**Definition 5.14.** A curved \(dg\)-comodule over a curved \(dg\)-coalgebra \((C, d, \xi)\) is a pair \((P, d_P)\), with \(P\) a graded \(C\) comodule and \(d_P \in \text{Coder}^d(P, P)_{-1}\), that satisfies
\[
d^2_P = \left( P \xrightarrow{\Delta} C \otimes P \xrightarrow{\xi \otimes 1} k \otimes P \cong P \right) =: L_\xi.
\]
A morphism of curved \(dg\)-comodules \((P, d_P) \to (N, d_N)\) is a degree zero morphism of graded \(C\)-comodules \(f : P \to N\) that satisfies \(fd_P = d_N f\).
If \((A, 1, \bar{p} + h + \mu_{sa})\) is an \(A_\infty\)-algebra with split unit, we denote by \(\text{Bar}_u \overline{A}\) the counital curved bar construction \((T_{co,u}(\Pi \overline{A}), \Phi^{-1}(\bar{p}), -s^{-1}h)\), where \(\Phi^{-1}(\bar{p})\) and \(-s^{-1}h\) are extended by zero from \(T_{co}(\Pi \overline{A})\) to \(T_{co,u}(\Pi \overline{A})\).

**Theorem 5.15.** Let \((A, 1, \bar{p} + h + \mu_{sa})\) be an \(A_\infty\)-algebra with split unit and counital curved bar construction \(\text{Bar}_u \overline{A}\). Let \(M, N\) be graded modules.

1. A strictly unital element \(p = \bar{p} + g_{sa}\), with \(\bar{p} \in CC^e_u(\overline{A}, \text{End } M)\), is a representation if and only if, for \(\lambda\), the adjoint family of \(\bar{p}\), the pair \((\text{Bar}_u \overline{A} \otimes M, \phi^{-1}(\overline{A}))\) is a curved dg-Bar\(_u\) \(\overline{A}\) comodule.

2. An element \(f \in CC^e_u(\overline{A}, \text{Hom}(M, N))\) is a morphism of strictly unital representations \((M, \overline{p}_M) \rightarrow (N, \overline{p}_N)\) if and only if

\[
\psi^{-1}(g) : (\text{Bar}_u \overline{A} \otimes M, \phi^{-1}(\overline{A})) \rightarrow (\text{Bar}_u \overline{A} \otimes N, \phi^{-1}(\overline{A}))
\]

is a morphism of curved dg-Bar\(_u\) \(\overline{A}\) comodules, where \(g, \overline{A}_M, \overline{A}_N\) are the adjoint families of \(f, \overline{p}_M, \overline{p}_N\).

**Proof.** The pair \((\text{Bar}_u \overline{A} \otimes M, \phi^{-1}(\overline{A}))\) is a curved dg-Bar\(_u\) \(\overline{A}\)-comodule if and only if \(\phi^{-1}(\overline{A})^2 = L_{s^{-1}h}L_{\lambda} = 0\). Since \(\phi^{-1}(\overline{A})\) is a coderivation with respect to \(\lambda = \Phi^{-1}(\bar{p})\), \(\phi^{-1}(\overline{A})\) is also a coderivation with respect to \(\lambda = \Phi^{-1}(\bar{p})\), and one checks \(L_{s^{-1}h}L_{\lambda} = 0\) exactly when \(\overline{A}\phi^{-1}(\overline{A}) = \overline{A}(\lambda \otimes 1 + (1 \otimes \lambda)(\Delta \otimes 1))\) is equal to the first two terms of the equation of 5.8(1), while the adjoint of \((\overline{A} \otimes M, \phi^{-1}(\overline{A}))\) is the third of 5.8(1). Now by 5.11(2), \(\psi^{-1}(g)\) is a morphism of curved dg comodules if and only if \(\overline{A}_N\phi^{-1}(g) - g\phi^{-1}(\overline{A}_M) = 0\). The first term is the adjoint of the first term of 5.8(2), and the second term is the adjoint of the second and third terms of 5.8(2).

**Example 5.16.** Let \((A, 1, \mu)\) be the Koszul complex on \(f \in k\) with curved bar construction \(\text{Bar}_u \overline{A} = (k[T], fT^*)\), see 4.13 and let \((M, \overline{A})\) be a strictly unital representation, described in 5.9. Set \(d_M = \phi^{-1}(\overline{A}) : k[T] \otimes M \rightarrow k[T] \otimes M\). For \(x \in M\), \(d_M(T^j \otimes x) = \sum_{k=0}^j T^k \otimes \sigma^{-k}(x)\), where \(\sigma^{-k}\) is the composition

\[
M \xrightarrow{\cong} k[e] \xrightarrow{\phi^{-1}(-k)} M \xrightarrow{\Delta^{-k}} M.
\]

Dualizing gives a graded module over the polynomial ring \(k[T]\) and a degree \(-1\) map on \(k[T]^* \otimes M^*\) whose square is multiplication by \(-fT^*\). Sheafifying this, we get two \(k\)-modules and maps, \(M^{ev} \rightarrow M^{odd} \rightarrow \Pi^{-1} M^{ev}\), whose composition is multiplication by \(f \in k\). This is exactly a matrix factorization in the sense of Eisenbud [Eis80].

**Corollary 5.17.** Let \((A, 1, \bar{p} + h + \mu_{sa})\) be an \(A_\infty\)-algebra with split unit. There is a functor from the category of strictly unital representations of \(A\) to the category of curved dg-Bar\(_u\) \(\overline{A}\) comodules, that sends \((M, \overline{p}_M)\) to \((\text{Bar}_u \overline{A} \otimes M, \phi^{-1}(\overline{A}_M))\). This is fully faithful with image the full subcategory of cofree curved dg comodules.

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