Special Solutions of the Sixth Painlevé Equation with Solvable Monodromy

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Abstract

We will study two types of special solutions of the sixth Painlevé equation, which are invariant under the symmetries obtained from the Bäcklund transformations. In most cases, the fixed points of the Bäcklund transformations are classical solutions, but our solutions are not classical for generic parameters. We will calculate the linear monodromy of these solutions exactly, and we will characterize them on Fricke’s cubic surface of monodromy.

1 Introduction

R. Fuchs showed that the sixth Painlevé equation is represented as an isomonodromic deformation condition of the Fuchs type linear equation with four regular singular points [3]. Garnier showed that every type of the Painlevé equation is also obtained from monodromy preserving deformation of linear equation [5]. Nevertheless it is difficult to calculate the monodromy itself for the generic Painlevé functions.

We call the monodromy data of the linear equation as a linear monodromy of the Painlevé function. Our interests is in the Painlevé function whose corresponding linear monodromy can be determined exactly. In this paper, we call such Painlevé functions monodromy solvable.

One example of monodromy solvable Painlevé functions is Umemura’s classical solutions [16], which are fixed points of the Bäcklund transformations. For any classical solutions, we can calculate the linear monodromy. But conversely, there exist some non-classical Painlevé functions whose linear monodromy can be calculated. We are interested in non-classical monodromy solvable solutions.

R. Fuchs is the first to find a non-classical monodromy solvable solution [4]. He calculates the linear monodromy of Picard’s solution [15], which satisfies the sixth Painlevé equation with a special parameter. His work is found again in [11] recently.

Another example of monodromy solvable solution is a symmetric solution. The first, second and fourth Painlevé equations

\begin{align*}
(P1) \quad \frac{d^2 y}{dt^2} &= 6y^2 + t, \\
(P2) \quad \frac{d^2 y}{dt^2} &= 2y^3 + ty + \alpha, \\
(P4) \quad \frac{d^2 y}{dt^2} &= \frac{1}{2y} \left( \frac{dy}{dt} \right)^2 + \frac{3}{2} y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y}
\end{align*}

are invariant under the transformations

\begin{align*}
(P1) \quad t \to \rho t, \quad y \to \rho^3 y \quad (\rho^5 = 1), \\
(P2) \quad t \to \omega t, \quad y \to \omega^3 y \quad (\omega^3 = 1), \\
(P4) \quad t \to -t, \quad y \to -y.
\end{align*}
There exist symmetric solutions which are invariant under the action of the cyclic groups. The symmetric solutions are studied by Kitaev [3] for (P1) and (P2), and by Kaneko [8] for (P4). For these cases, \((y, t) = (0, 0)\) is a fixed point of the transformations, and there exist symmetric solutions with initial values at the fixed point \(y(0) = 0\). Kitaev calculated the linear monodromy of the symmetric solutions of (P1) and (P2) by adopting the fixed point as the initial condition. In the same way, Kaneko calculated the linear monodromy of the symmetric solution of (P4). We remark that these symmetric solutions are not classical for generic parameters.

In this paper, we will construct monodromy solvable solutions of the sixth Painlevé equation.

For generic parameter, we cannot find a simple symmetry as above. However, for some restricted parameters, there exist two types of symmetries obtained from the Bäcklund transformations. One is

\[
\sigma_1 : t \to 1 - t, \quad y \to 1 - y, \quad \beta \to -\gamma, \quad \gamma \to -\beta,
\]

and the other is

\[
\sigma_2 \circ \sigma_1 : t \to 1/(1 - t), \quad y \to 1/(1 - y), \quad \alpha \to -\beta, \quad \beta \to -\gamma, \quad \gamma \to \alpha.
\]

The sixth Painlevé equation is invariant under the action \(\sigma_1\) if \(-\beta = \gamma\), and is invariant under the action \(\sigma_2 \circ \sigma_1\) if \(\alpha = -\beta = \gamma\).

In the section 3, we will show there exist symmetric solutions, which are fixed points of these actions. In many cases, the fixed points of the Bäcklund transformations are classical solutions. However, our symmetric solutions are not classical for generic parameters.

In the section 4, we will show that the linearizations of these symmetric solutions are reduced to the Gauss hypergeometric equations. In the section 5, we will calculate the linear monodromy of our symmetric solutions explicitly.

If the monodromy matrices \(\{M_0, M_1, M_1, M_\infty\}\) are elements of \(SL(2, \mathbb{C})\), then

\[
p_j = \text{tr} M_j, \quad p_{jk} = \text{tr} M_j M_k,
\]

satisfies the following relation:

\[
\begin{align*}
p_{01}p_{10}p_{02} + p_{01}^2 + p_{10}^2 + p_{02}^2 - (p_0 p_1 + p_1 p_\infty)p_{01} - (p_1 p_0 + p_0 p_\infty)p_{10} - (p_0 p_1 + p_1 p_\infty)p_{00} \\
+ p_0^2 + p_1^2 + p_\infty^2 + p_0 p_1 p_\infty - 4 = 0.
\end{align*}
\]

It is known that the monodromy matrices \(\{M_0, M_1, M_1, M_\infty\}\) are determined by \(\{p_0, p_1, p_\infty; p_{01}, p_{10}, p_{00}\}\) up to gauge transformations ([2], [3], [4], [10]). We call \(1\) as Fricke’s cubic surface of monodromy. In the section 6, we will characterize the monodromy of our symmetric solutions on Fricke’s cubic surface of monodromy.

Andreev and Kitaev constructed many algebraic solutions of the sixth Painlevé equation, by rational transformations of the hypergeometric equations [1]. By the rational transformations of confluent hypergeometric equation, Ohyama and Okumura constructed algebraic solutions and symmetric solutions of the first to the fifth Painlevé equations [13]. We remark that Boach also constructed many algebraic solutions of the sixth Painlevé equation, whose linear monodromy is isomorphic to the complex reflection group [2].

The solutions of this paper essentially appeared in [1]. But, since their interest are in algebraic solutions, they did not go to detail of these solutions. In this paper, we will calculate the linear monodromy explicitly, and we will show that they are symmetric solutions.

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## 2 Linear Problem

The sixth Painlevé equation [14] is obtained from the isomonodromic deformation equation of linear equation with four regular singular points. We will use Garnier-Okamoto’s linearization [14].

\[
\frac{\partial^2 \psi}{\partial x^2} + p(x, t) \frac{\partial \psi}{\partial x} + q(x, t) \psi = 0, \tag{2}
\]

\[
\frac{\partial \psi}{\partial t} = a(x, t) \frac{\partial \psi}{\partial x} + b(x, t) \psi, \tag{3}
\]
where
\begin{align*}
p(x, t) &= \frac{1 - \alpha_4}{x} + \frac{1 - \alpha_3}{x - 1} + \frac{1 - \alpha_0}{x - t} - \frac{1}{x - y}, \\
q(x, t) &= \frac{\alpha_3(\alpha_1 + \alpha_2)}{x(x - 1)} - \frac{t(t - 1)H_{VI}}{x(x - 1)(x - t)} + \frac{y(y - 1)z}{x(x - 1)(x - y)}, \\
a(x, t) &= \frac{y - t}{t(t - 1)}x(x - 1), \\
b(x, t) &= \frac{(1 - \alpha_4 - \alpha_3 - \alpha_0)(y - t)}{2t(t - 1)} - \frac{y(y - 1)(y - t)z}{t(t - 1)(x - y)},
\end{align*}
and
\[
H_{VI} = \frac{1}{t(t - 1)} \left[ y(y - 1)(y - t)z^2 - \left\{ \alpha_4(y - 1)(y - t) + \alpha_3y(y - t) + (\alpha_0 - 1)y(y - 1) \right\}z + \alpha_2(\alpha_1 + \alpha_2)(y - t) \right].
\]

The Riemann scheme of (2) is
\[
\left\{ \begin{array}{c}
0 \\
1 \\
t \\
y \\
\infty
\end{array} \right\}.
\]

From the compatibility condition of (2) and (3), we have the following Hamiltonian system
\[
\frac{dy}{dt} = \frac{\partial H_{VI}}{\partial z}, \quad \frac{dz}{dt} = -\frac{\partial H_{VI}}{\partial y},
\]
that is
\[
t(t - 1)\frac{dy}{dt} = 2y(y - 1)(y - t)z - (\alpha_0 - 1)y(y - 1)y_3y(y - t) - \alpha_4(y - 1)(y - t),
\]
\[
t(t - 1)\frac{dz}{dt} = -\left( y(y - 1) + (y - 1)(y - t) + y(y - t) \right)z^2
+ \left( 2y - 1)(\alpha_0 - 1) + (2y - t)\alpha_3 + (2y - t - 1)\alpha_4 \right)z - (\alpha_1 + \alpha_2)\alpha_2.
\]

Let us eliminate \(z\) from this system, and we have the sixth Painlevé equation
\[
\frac{d^2y}{dt^2} = \frac{1}{2} \left( \frac{1}{y} + \frac{1}{y - 1} + \frac{1}{y - t} \right) \left( \frac{dy}{dt} \right)^2 - \left( \frac{1}{t} + \frac{1}{t - 1} + \frac{1}{y - t} \right) \frac{dy}{dt}
+ \frac{y(y - 1)(y - t)}{t^2(t - 1)^2} \left\{ \alpha + \beta \frac{t}{y^2} + \gamma \frac{t - 1}{(y - 1)^2} + \delta \frac{t(t - 1)}{(y - t)^2} \right\},
\]
where
\[
\alpha = \frac{\alpha_1^2}{2}, \quad \beta = -\frac{\alpha_2^2}{2}, \quad \gamma = \frac{\alpha_3^2}{2}, \quad \delta = \frac{1 - \alpha_0^2}{2}.
\]

We will calculate the linear monodromy of some special Painlevé functions.

\section{Bäcklund Transformations of the Sixth Painlevé Equation}

For the sixth Painlevé equation, we cannot find a simple symmetry for general parameters. However, for some restricted parameters, there exist some symmetries obtained from the Bäcklund transformations. And there exist symmetric solutions of the Bäcklund transformations.

The sixth Painlevé equation (9) is invariant under the following transformations (12):
We remark that
\[
\sigma \text{ is an extended affine Weyl group of type } D_4 \text{, where }
\]
\[
\alpha = \alpha_3 \text{ is the sixth Painlevé equation is invariant with the condition } \alpha = \alpha_4, \text{ and one of the fixed point is }
\]
\[
t = 1/2, \quad y = 1/2, \quad z = 0.
\]
Since this is not a critical point of \( s \), there exist a unique holomorphic solution of \( s \) with the initial condition
\[
y(1/2) = 1/2, \quad z(1/2) = 0.
\]
And if \( \alpha = \alpha_4 \), this solution is expanded as follows:
\[
y(t) = \sum_{k=0}^{\infty} a_k \left( t - \frac{1}{2} \right)^k, \quad z(t) = \sum_{k=0}^{\infty} b_k \left( t - \frac{1}{2} \right)^k,
\]
where
\[
a_0 = \frac{1}{2}, \quad a_1 = 1 - a_0, \quad a_2 = 0,
\]
\[
a_3 = -\frac{4}{3} a_0 \{ 2a_2(\alpha_1 + \alpha_2) - (1 - a_0)(2a_2 + \alpha_1 + 1) \}, \quad a_4 = 0, \ldots,
\]
\[
b_0 = 0, \quad b_1 = 4a_2(\alpha_1 + \alpha_2), \quad b_2 = 0,
\]
\[
b_3 = -\frac{16}{3} a_2(\alpha_1 + \alpha_2) \{ \alpha_2(\alpha_1 + \alpha_2) - (\alpha_1 + 2a_2)(1 - 2\alpha_0) + \alpha_0 - 2 \}, \quad b_4 = 0, \ldots.
\]
Proposition 1. The Painlevé function $[\text{11}]$ is a symmetric solution of $\sigma_1$:
\[ y(1-t) = 1 - y(t), \quad z(1-t) = -z(t). \]

Proof. Let us set
\[ \tau = t - 1/2, \quad \lambda = y - 1/2, \quad \mu = z, \]
then the action of $\sigma_1$ maps $\tau \to -\tau$, $\lambda \to -\lambda$, and $\mu \to -\mu$. $[\text{11}]$ becomes the following Hamiltonian system $\frac{d\lambda}{d\tau} = \frac{\partial K}{\partial \mu}$, $\frac{d\mu}{d\tau} = -\frac{\partial K}{\partial \lambda}$:
\begin{align*}
\tau^2 - \frac{1}{4} \frac{d\lambda}{d\tau} &= 2 \left( \lambda^2 - \frac{1}{4} \right) \mu (\lambda - \tau) - \left( \lambda^2 - \frac{1}{4} \right) (\alpha_0 - 1) - 2\lambda (\lambda - \tau) \alpha_3, \\
\tau^2 - \frac{1}{4} \frac{d\mu}{d\tau} &= - \left( \lambda^2 - \frac{1}{4} \right) + 2\lambda (\lambda - \tau) \mu^2 + 2 \left( \lambda(\alpha_0 - 1) + (2\lambda - \tau)\alpha_3 \right) \mu - 2(\alpha_1 + \alpha_2).
\end{align*}

The Hamiltonian $K$ is
\[ (\tau^2 - \frac{1}{4}) K = \mu^2 (\lambda - \tau) (\lambda^2 - \frac{1}{4}) - \mu \left\{ \left( \lambda^2 - \frac{1}{4} \right) (\alpha_0 - 1) + 2\lambda (\lambda - \tau) \alpha_3 \right\} + (\lambda - \tau) \alpha_2 (\alpha_1 + \alpha_2). \]
Then $[\text{11}]$ corresponds to the following solution of $[\text{12}]$:
\[ \lambda = (1 - \alpha_0) \tau + O(\tau)^3, \quad \mu = 4\alpha_2 (\alpha_1 + \alpha_2) \tau + O(\tau)^3. \]

Higher order expansions are determined from $[\text{12}]$ inductively, and it has only odd powers of $\tau$. Therefore it determines symmetric solution. Thus $[\text{11}]$ is a symmetric solution.

By the Bäcklund transformations $s_0$, $s_1$, $s_2$ and $\pi_2$, which are commutative with $\sigma_1$, we have four symmetric solutions around $t = 1/2$.

**Proposition 2.** If $\alpha_3 = \alpha_4$, all of the symmetric solutions on $\sigma_1$ around $t = 1/2$ are
\begin{align*}
\begin{cases}
 y - \frac{1}{2} = (1 - \alpha_0) \left( t - \frac{1}{2} \right) + O \left( t - \frac{1}{2} \right)^3, \\
 z = 4\alpha_2 (\alpha_1 + \alpha_2) \left( t - \frac{1}{2} \right) + O \left( t - \frac{1}{2} \right)^3,
\end{cases} & \quad (\text{S2-1}) \\
\begin{cases}
 y - \frac{1}{2} = (1 + \alpha_0) \left( t - \frac{1}{2} \right) + O \left( t - \frac{1}{2} \right)^3, \\
 z = \left( t - \frac{1}{2} \right)^{-1} + O \left( t - \frac{1}{2} \right),
\end{cases} & \quad (\text{S2-2}) \\
\begin{cases}
 y - \frac{1}{2} = \frac{1}{4\alpha_1} \left( t - \frac{1}{2} \right)^{-1} + O \left( t - \frac{1}{2} \right), \\
 z = -4\alpha_1 \alpha_2 \left( t - \frac{1}{2} \right) + O \left( t - \frac{1}{2} \right)^3,
\end{cases} & \quad (\text{S2-3}) \\
\begin{cases}
 y - \frac{1}{2} = \frac{1}{4\alpha_1} \left( t - \frac{1}{2} \right)^{-1} + O \left( t - \frac{1}{2} \right), \\
 z = 4\alpha_1 (\alpha_1 + \alpha_3) \left( t - \frac{1}{2} \right) + O \left( t - \frac{1}{2} \right)^3,
\end{cases} & \quad (\text{S2-4})
\end{align*}
where $[\text{S2-1}]$ is $[\text{11}]$. The actions of $s_0$, $s_1$, $s_2$ and $\pi_2$ interchange the symmetric solutions as follows:

In section 4.3 we will show these symmetric solutions are monodromy solvable.

5
3.2 Symmetric Solution of $\sigma_2 \circ \sigma_1$

For the Bäcklund transformation $\sigma_2 \circ \sigma_1$:
\[
y \to \frac{1}{1 - y}, \quad z \to -(1 - y)(-z(1 - y) + \alpha_2), \quad t \to \frac{1}{1 - t},
\]
the sixth Painlevé equation is invariant with the condition $\alpha_1 = \alpha_3 = \alpha_4$, and one of the fixed point is
\[
y = -\omega^2, \quad z = \frac{2\omega + 1}{3}\alpha_2, \quad t = -\omega^2.
\]
Since this is not a critical point of $\sigma_2$, there exist a unique holomorphic solution of $\sigma_2$ with the initial condition
\[
y(-\omega^2) = -\omega^2, \quad z(-\omega^2) = \frac{2\omega + 1}{3}\alpha_2.
\]
And if $\alpha_1 = \alpha_3 = \alpha_4$, this solution is expanded as follows:
\[
y(t) = -\omega^2 + (1 - \alpha_0)(t + \omega^2) + \frac{1 + 2\omega}{3}\alpha_0(1 - \alpha_0)(t + \omega^2)^2 + O(t + \omega^2)^3, \\
z(t) = \frac{2\omega + 1}{3}\alpha_2 - \frac{\alpha_2}{3}(1 - \alpha_0)(t + \omega^2) + O(t + \omega^2)^2.
\]

Proposition 3. The Painlevé function $[13]$ is a symmetric solution of $\sigma_2 \circ \sigma_1$:
\[
y \left( \frac{1}{1 - t} \right) = \frac{1}{1 - y(t)}, \quad z \left( \frac{1}{1 - t} \right) = -(1 - y(t)) \{ -z(t)(1 - y(t)) + \alpha_2 \}.
\]

Proof. To prove the symmetry of $\sigma_2 \circ \sigma_1$, we set
\[
\tau = -\omega t - 1 \frac{t + \omega}{t + \omega}, \quad \lambda = -\omega y - 1 \frac{y + \omega}{y + \omega},
\]
which diagonalize the action of $\sigma_2 \circ \sigma_1$: $\tau \to \omega \tau$, $\lambda \to \omega \lambda$. Then the sixth Painlevé equation (10) becomes
\[
(\lambda - \tau) \frac{d^2 \lambda}{d\tau^2} = \frac{4\lambda^3 - 3\tau \lambda^2 + 1}{2(\lambda^3 + 1)} \left( \frac{d\lambda}{d\tau} \right)^2 + \frac{-3\tau^2 \lambda + 2\tau^3 - 1 \frac{d\lambda}{d\tau}}{\tau^3 + 1} + \frac{(\lambda^3 + 1)^2(\lambda + 1) (1 - \alpha_0)}{2(\lambda^3 + 1)(\tau^3 + 1)^2},
\]
which is equivalent to the following Hamiltonian system $\frac{d\lambda}{d\tau} = \frac{\partial K}{\partial \mu}$, $\frac{d\mu}{d\tau} = -\frac{\partial K}{\partial \lambda}$:
\[
(\tau^3 + 1) \frac{d\lambda}{d\tau} = 2(1 + \lambda^3) \mu (\lambda - \tau) + 3\lambda^2 (\lambda - \tau) \alpha_1 + (1 + \lambda^3)(1 + \alpha_0), \\
(\tau^3 + 1) \frac{d\mu}{d\tau} = \mu^2 (-4\lambda^3 + 3\lambda^2 \tau - 1) - 3\lambda \mu (1 + \alpha_0 + (3\lambda - 2\tau) \alpha_1) - \lambda(\alpha_2 - 1)^2 + \frac{\alpha_2 - 1}{2} ((\lambda + \tau)(1 + \alpha_0) + 3(\lambda - \tau) \alpha_1).
\]
The Hamiltonian $K$ is
\[
(\tau^3 + 1)K = \mu^2 (\lambda^3 + 1)(\lambda - \tau) + \mu \left\{ (\lambda^3 + 1)(1 + \alpha_0) + 3\lambda^2 (\lambda - \tau) \alpha_1 \right\} + \frac{1 + \alpha_0 + 3\alpha_1}{4} \lambda \left\{ (\lambda + \tau)(1 + \alpha_0) + 3(\lambda - \tau) \alpha_1 \right\},
\]
and $\mu$ is determined by
\[
\mu = \frac{t(t - 1)(y + \omega)^2}{2y(y - 1)(y - t)(1 - \omega^2)} \frac{dy}{dt} + \frac{(\omega - 1)(t + \omega)^2(\alpha_0 + 1)}{6(y - t)} + \frac{(1 - \omega^2)\alpha_1}{6y} + \frac{\omega^2(1 - \omega^2)\alpha_1}{6(y - 1)} + \frac{(1 - \omega)\alpha_1 y}{6} + \frac{1}{6} \left\{ (\omega - 1)(t + \omega)(1 + \alpha_0) + (\omega^2 - 1) \alpha_1 \right\}.
\]
By the action of $\sigma_2 \circ \sigma_1$, we have
\[
\tau \to \omega \tau, \quad \lambda \to \omega \lambda, \quad \mu \to \omega^2 \mu, \quad K \to \omega^2 K.
\]
There exists a solution of (13):
\[
\lambda = (1 - \alpha_0) \tau + O(\tau)^4, \\
\mu = \tau^{-1} + O(\tau)^2,
\]
which are corresponding to (13). And inductively, we have (16) in the following forms
\[
\lambda = \sum a_n \tau^{3n+1}, \quad \mu = \sum b_n \tau^{3n-1}.
\]
Therefore it is a symmetric solution. And thus (13) is a symmetric solution.

By the Backlund transformations $s_0$ and $s_2$, which are commutative with $\sigma_2 \circ \sigma_1$, we have three symmetric solutions around $t = -\omega^2$.

**Proposition 4.** If $\alpha_1 = \alpha_3 = \alpha_4$, all of symmetric solutions on $\sigma_2 \circ \sigma_1$ around $t = -\omega^2$ are
\[
\begin{align*}
\{ y &= -\omega + O(t + \omega^2)^2, \\
z &= -\frac{1}{3}(1 + 2\omega)\alpha_2 - \frac{1}{6}(\alpha_0 - 3\alpha_1 - 1)\alpha_2(t + \omega^2) + O(t + \omega^2)^2, \\
y &= -\omega^2 + (1 - \alpha_0)(t + \omega^2) + O(t + \omega^2)^2, \\
z &= \frac{1}{3}(1 + 2\omega)\alpha_2 + \frac{1}{6}(\alpha_0 - 1)\alpha_2(t + \omega^2) + O(t + \omega^2)^2, \\
y &= -\omega^2 + (1 + \alpha_0)(t + \omega^2) + O(t + \omega^2)^2, \\
z &= (t + \omega^2)^{-1} + \frac{1}{2}(1 + 2\omega)(\alpha_0 - \alpha_1 + 1) + O(t + \omega^2),
\end{align*}
\]
where (S3-2) is (13). The actions of $s_0$ and $s_2$ interchange the symmetric solutions as follows:

\[
s_0 \xrightarrow{(S3-1)} s_2 \xrightarrow{(S3-2)} s_0 \xrightarrow{(S3-3)} s_2
\]

There are more three symmetric solutions around $t = -\omega$:
\[
\begin{align*}
\{ y &= -\omega^2 + O(t + \omega)^2, \\
z &= -\frac{1}{3}(1 + 2\omega)\alpha_2 - \frac{1}{6}(\alpha_0 - 3\alpha_1 - 1)\alpha_2(t + \omega) + O(t + \omega)^2, \\
y &= -\omega + (1 - \alpha_0)(t + \omega) + O(t + \omega)^2, \\
z &= \frac{1}{3}(1 + 2\omega)\alpha_2 + \frac{1}{6}(\alpha_0 - 1)\alpha_2(t + \omega) + O(t + \omega)^2, \\
y &= -\omega + (1 + \alpha_0)(t + \omega) + O(t + \omega)^2, \\
z &= (t + \omega)^{-1} + \frac{1}{2}(1 + 2\omega)(\alpha_0 - \alpha_1 + 1) + O(t + \omega),
\end{align*}
\]
which correspond to (S3-1), (S3-2) and (S3-3) by $\omega \to \omega^2$. The actions of $s_0$ and $s_2$ interchange the symmetric solutions as follows:

\[
s_0 \xrightarrow{(S3-4)} s_2 \xrightarrow{(S3-5)} s_0 \xrightarrow{(S3-6)} s_2
\]

In section 4.2, we will show these symmetric solutions are monodromy solvable.
3.3 Comparison with classical solutions

In the case of \( \alpha_0 = 0 \) \((\delta = 0)\), the sixth Painlevé equation \((9)\) admits the Riccati type solution
\[
y(t) = t,
\]
\[
t(t-1) \frac{dz}{dt} = -t(t-1)z^2 + \left\{ 1 - 2t + \alpha_3 t + \alpha_4(t-1) \right\} z - (\alpha_1 + \alpha_2)\alpha_2.
\]
(17)

If \( \alpha_3 = \alpha_4 \), \((17)\) admits a solution
\[
y = 1/2 + (t - 1/2),
\]
\[
z = \frac{d}{dt} \log \left[ {}_2F_1 \left( \frac{\alpha_2}{2}, \frac{\alpha_1 + \alpha_2}{2}, \frac{1}{2}, \frac{4(t-1/2)^2}{1} \right) \right],
\]
which is a special case of \((S2-1)\). If \( \alpha_1 = \alpha_3 = \alpha_4 \), \((17)\) admits a solution
\[
y = -\omega^2 + (t + \omega^2),
\]
\[
z = \frac{d}{dt} \log \left[ {}_2F_1 \left( \frac{1 + \alpha_1}{2}, \frac{1 + 3\alpha_1}{6}, \frac{2}{3}, -\left( \frac{-\omega t - 1}{t + \omega} \right)^3 \right) \right] + \frac{t(1-t)(t + \omega^2) + (t^3 + 3\omega t^2 + 3(1 - \omega)t - 2)\alpha_1}{2(t-1)(t^2 - t + 1)},
\]
which is a special case of \((S2-2)\).

4 Transformation of the Linearization

Since the Painlevé equation is the isomonodromic deformation condition, to show the monodromy solvability of a solution of Painlevé equation, it is sufficient to show the monodromy solvability at a special \( t = t_0 \). In the following, we will set such special \( t = t_0 \) at the fixed points of \( \sigma_1 \) and \( \sigma_2 \circ \sigma_1 \).

4.1 Monodromy Solvability of the Symmetric Solution of \( \sigma_1 \)

We substitute the symmetric solution \((S2-1)\) into \((2)\) and take the limit \( t \to 1/2 \), then
\[
\frac{d^2 \psi}{dx^2} + \left( \frac{1 - \alpha_3}{x} + \frac{1 - \alpha_3}{x - 1} + \frac{-\alpha_0}{x - 1/2} \right) \frac{d\psi}{dx} + \frac{(\alpha_1 + \alpha_2)\alpha_2}{x(x - 1)} \psi = 0.
\]
(18)

This is Heun’s equation, whose Riemann scheme is
\[
P \left\{ \begin{array}{c c c c c}
0 & 1 & 1/2 & \infty \\
0 & 0 & 0 & \alpha_2 \\
\alpha_3 & \alpha_3 & \alpha_0 + 1 & \alpha_1 + \alpha_2
\end{array} \right.; x
\]

Now we set \( (2x - 1)^2 = 1 - \xi \), then \((18)\) becomes
\[
\frac{d^2 \psi}{d\xi^2} + \left( \frac{1 - \alpha_3}{\xi} + \frac{1 - \alpha_0}{2(\xi - 1)} \right) \frac{d\psi}{d\xi} + \frac{(\alpha_1 + \alpha_2)\alpha_2}{4\xi(\xi - 1)} \psi = 0,
\]
which is the hypergeometric equation.

**Theorem 5.** The symmetric solution \((S2-1)\) is monodromy solvable. Its linearization \((2)\) at \( t = 1/2 \) is reduced to the hypergeometric equation:
\[
P \left\{ \begin{array}{c c c c c}
0 & 1 & \infty \\
0 & 0 & \frac{\alpha_2}{2} \\
\alpha_3 & \frac{\alpha_0 + 1}{2} & \frac{\alpha_1 + \alpha_2}{2}
\end{array} \right.; \xi
\]

A fundamental solution of \((18)\) is given by
\[
\left( {}_2F_1 \left( \frac{\alpha_2}{2}, \frac{\alpha_1 + \alpha_2}{2}, 1 - \alpha_3; 4x(1 - x) \right), (4x(1 - x))^\alpha_3 {}_2F_1 \left( \frac{\alpha_2}{2}, \frac{\alpha_1 + \alpha_2}{2} + \alpha_3, 1 + \alpha_3; 4x(1 - x) \right) \right).
\]
We substitute the symmetric solution (S3-2) into (2) and take the limit \( \sigma \). Then we have the following.

Let us denote the standard paths around \( x \). Then we make a cut from \( \xi \). In the same way, the linearizations of (S2-2), (S2-3) and (S2-4) can be reduced to the hypergeometric equation.

In section 5.1, we will calculate the linear monodromy of the symmetric solutions of \( \sigma_1 \) explicitly.

### 4.2 Monodromy Solvability of the Symmetric Solution of \( \sigma_2 \circ \sigma_1 \)

We substitute the symmetric solution (S3-2) into (2) and take the limit \( t \to -\omega^2 \), then we have

\[
\frac{d^2\psi}{dx^2} + \left( \frac{1 - \alpha_1}{x} + \frac{1 - \alpha_1}{x - 1} + \frac{-\alpha_0}{x + \omega^2} \right) \frac{d\psi}{dx} + \left( \frac{\alpha_2(\alpha_1 + \alpha_2)}{x(x - 1)} - \frac{2\omega + 1}{3} \frac{\alpha_0\alpha_2}{x(x - 1)(x + \omega^2)} \right) \psi = 0. \tag{19}
\]

This is Heun’s equation, whose Riemann scheme is

\[
P \begin{pmatrix} 0 & 1 & -\omega^2 & \infty \\ 0 & 0 & 0 & \alpha_2 \\ \alpha_1 & \alpha_1 & 1 + \alpha_0 & \alpha_1 + \alpha_2 \end{pmatrix} = (x + \omega^2)^{-\alpha_2} P \begin{pmatrix} 0 & 1 & -\omega^2 & \infty \\ 0 & 0 & \alpha_2 & 0 \\ \alpha_1 & \alpha_1 & 1 + \alpha_0 + \alpha_2 & \alpha_1 \end{pmatrix}.
\]

By the transformation

\[\psi = (x + \omega^2)^{-\alpha_2} \tilde{\psi}, \quad \xi = \omega \frac{x + \omega}{x + \omega^2}, \quad \eta = \xi^3, \]

(19) is reduced to the hypergeometric equation

\[\eta(\eta - 1) \frac{d^2\tilde{\psi}}{d\eta^2} + \left( \frac{2}{3} - \left( 1 + \frac{1 + \alpha_0 + 2\alpha_2}{3} \right) \right) \frac{d\tilde{\psi}}{d\eta} - \frac{\alpha_2}{9} (1 + \alpha_0 + \alpha_2) \tilde{\psi} = 0. \]

Then we have the following.

**Theorem 6.** The symmetric solution (S3-2) on \( \sigma_2 \circ \sigma_1 \) is monodromy solvable. Its linearization (2) at \( t = -\omega^2 \) is reduced to the hypergeometric equation

\[
P \begin{pmatrix} 0 & 1 & \infty \\ 0 & 0 & \alpha_2/3 \\ 1/3 & \alpha_1 & (1 + \alpha_0 + \alpha_2)/3 \end{pmatrix}.
\]

A fundamental solution is

\[
\left( \begin{array}{c} 2F_1 \left( \frac{\alpha_2}{3}, \frac{1}{3}(\alpha_0 + \alpha_2 + 1), \frac{2}{3}; \eta \right), \\ \eta^{1/3} 2F_1 \left( \frac{\alpha_2 + 1}{3}, \frac{1}{3}(\alpha_0 + \alpha_2 + 2), \frac{4}{3}; \eta \right) \end{array} \right).
\]

In the same way, (S3-1) and (S3-3) can be reduced to the hypergeometric equation.

In section 5.2, we will calculate the linear monodromy of the symmetric solution of \( \sigma_2 \circ \sigma_1 \) explicitly.

### 5 Linear Monodromy

#### 5.1 Linear Monodromy of the Symmetric Solution of \( \sigma_1 \)

By \((2x - 1)^2 = 1 - \xi \), \( \xi \)-surface \( \mathbb{P}_1 \setminus \{0, 1, \infty\} \) is double covered by the \( x \)-surface \( \mathbb{P}_1 \setminus \{0, 1/2, 1, \infty\} \). Let us make a cut from \( \xi = 1 \) to \( \infty \) \((x = 1/2 \to \infty)\), and suppose that \( x = 1 \) is on the second \( \xi \)-plane, and the base point and \( x = 0 \) are on the first \( \xi \)-plane.

Let us denote the standard paths around \( x = 0, 1/2, 1, \infty \) by \( \gamma_0, \gamma_{1/2}, \gamma_1, \gamma_\infty \), and denote the standard paths around \( \xi = 0, 1, \infty \) by \( L_0, L_1, L_\infty \).
Then we have
\[ \gamma_0 = L_0, \quad \gamma_{1/2} = L_1^2, \quad \gamma_1 = L_1^{-1}L_0L_1, \quad \gamma_\infty = L_2^2, \]
which satisfy
\[ \gamma_0 \gamma_{1/2} \gamma_\infty = L_0(L_1^2)(L_1^{-1}L_0L_1)L_2^2 = 1. \]

Then we obtain monodromy matrix \( M_j \) along path \( \gamma_j \) (\( j = 0, 1/2, 1, \infty \)).

**Theorem 7.** The linear monodromy of the symmetric solution \((11)\) of \( \sigma_1 \) is represented as

\[
M_0 = \Gamma_0 \Lambda_0 \Gamma_0^{-1}, \quad M_{1/2} = \Gamma_1 \Lambda_{1/2} \Gamma_{1/2}^{-1}, \\
M_1 = (\Gamma_1 \Lambda_1 \Gamma_1^{-1})(\Gamma_0 \Lambda_0 \Gamma_0^{-1})(\Gamma_{1/2} \Lambda_{1/2} \Gamma_{1/2}^{-1}), \quad M_\infty = (e^{2\pi i T_\infty})^2,
\]

where
\[
\begin{align*}
\Lambda_0 = \Lambda_1 &= \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i \alpha_2} \end{pmatrix}, \\
\Lambda_{1/2} &= \begin{pmatrix} 1 & 0 \\ 0 & e^{\pi i \alpha_0} \end{pmatrix}, \\
e^{2\pi i T_\infty} &= \begin{pmatrix} e^{\pi i \alpha_2} & 0 \\ 0 & e^{\pi i (\alpha_1 + \alpha_2)} \end{pmatrix},
\end{align*}
\]

\[
\begin{align*}
\Gamma_0 & = \Gamma_1 = \Gamma_{1/2} = \Gamma_{1/2}^{-1} = \Gamma_\infty = \Gamma_{1/2} \Gamma_\infty^{-1}, \\
\Gamma_0 \Gamma_1 \Gamma_{1/2} &= (\Gamma_1 \Gamma_0 \Gamma_{1/2})^{-1}, \\
e^{-2\pi i \frac{\alpha_2}{2}} &= \begin{pmatrix} e^{-\alpha_2 \pi i / 2} & \Gamma(1-\alpha_3) \Gamma(\alpha_1/2) \\ \Gamma(1+\alpha_2) \Gamma(-1-\alpha_3/2) & e^{-(\alpha_2 + \alpha_1) \pi i / 2} \end{pmatrix}, \\
e^{-\alpha_2 \pi i / 2} &= \begin{pmatrix} e^{-(\alpha_2 + \alpha_1) \pi i / 2} & \Gamma(1-\alpha_3) \Gamma(\alpha_1/2) \\ \Gamma(1+\alpha_2) \Gamma(-1-\alpha_3/2) & e^{\alpha_2 + \alpha_1 \pi i / 2} \end{pmatrix}.
\end{align*}
\]
5.2 Linear Monodromy of the Symmetric Solution of $\sigma_2 \circ \sigma_1$

Let us denote the standard paths around $\xi = \omega, \omega^2, 1, \infty$ by $\gamma_\omega$, $\gamma_{\omega^2}$, $\gamma_1$, $\gamma_\infty$, and the standard paths around $\eta = 0, 1, \infty$ by $L_0$, $L_1$, $L_\infty$.

The $\xi$-space and the $\eta$-space is connected by $\xi^3 = \eta$.

Then we have

$$
\begin{align*}
\gamma_\omega &= L_0L_1L_0^{-1}, & \gamma_{\omega^2} &= L_0^{-1}L_1L_0, & \gamma_1 &= L_1, & \gamma_\infty &= L_\infty^3,
\end{align*}
$$

where

$$
\gamma_\omega \gamma_{\omega^2} \gamma_\infty = (L_0L_1L_0^{-1})(L_0^{-1}L_1L_0)L_1(L_\infty^3) = L_0L_1L_0^{-3}L_\infty.
$$

Since the monodromy along the path $L_0^3$ is 1, the monodromy along the path $\gamma_\omega \gamma_{\omega^2} \gamma_\infty$ is 1.

Then we obtain monodromy matrix $M_j$ along path $\gamma_j$ ($j = \omega, \omega^2, 1, \infty$).

**Theorem 8.** The linear monodromy of the symmetric solution [82] of $\sigma_2 \circ \sigma_1$ is represented as follows:

$$
\begin{align*}
M_{\omega^2} &= e^{-2\pi i \alpha_2}(e^{2\pi i T_\infty})^3, & M_0 &= (\Gamma_1^\infty \Lambda_1 \Gamma_1^{-1}^\infty), \\
M_1 &= (\Gamma_0^\infty \Lambda_0 \Gamma_0^{-1}^\infty)^3(\Gamma_1^\infty \Lambda_1 \Gamma_1^{-1}^\infty)(\Gamma_0^\infty \Lambda_0^{-1} \Gamma_0^{-1}^\infty), \\
M_\infty &= e^{2\pi i \alpha_2}(\Gamma_0^\infty \Lambda_0^{-1} \Gamma_0^{-1}^\infty)(\Gamma_1^\infty \Lambda_1 \Gamma_1^{-1}^\infty)(\Gamma_0^\infty \Lambda_0 \Gamma_0^{-1}^\infty),
\end{align*}
$$
\[ \Gamma_0 = \begin{pmatrix} e^{\alpha_2 \pi i/3} & e^{(1+\alpha_0)\pi i/3} \\ e^{(1+\alpha_0+\alpha_2)\pi i/3} & e^{(2+\alpha_0+\alpha_2)\pi i/3} \end{pmatrix}, \quad \Gamma_1 = \begin{pmatrix} e^{\alpha_2 \pi i/3} & e^{(1+\alpha_0)\pi i/3} \\ e^{(1+\alpha_0+\alpha_2)\pi i/3} & e^{(2+\alpha_0+\alpha_2)\pi i/3} \end{pmatrix}, \]

\[ \lambda_0 = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i (1-\alpha_0-2\alpha_2)/3} \end{pmatrix}, \quad \lambda_1 = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i (1+\alpha_0+\alpha_2)/3} \end{pmatrix}, \quad e^{2\pi i \Gamma_\infty} = \begin{pmatrix} e^{2\pi i \alpha_2/3} & 0 \\ 0 & e^{2\pi i (1+\alpha_0+\alpha_2)/3} \end{pmatrix}, \]

where

\[ \Gamma_0 = \frac{\Gamma(2/3)\Gamma((1+\alpha_0)/3)}{\Gamma((1+\alpha_0+\alpha_2)/3)\Gamma((2+\alpha_0+\alpha_2)/3)} \frac{\Gamma(4/3)\Gamma((1+\alpha_0)/3)}{\Gamma((1+\alpha_0+\alpha_2)/3)\Gamma((2+\alpha_0+\alpha_2)/3)}, \]

\[ \Gamma_1 = \frac{\Gamma((2+\alpha_0+2\alpha_2)/3)\Gamma((1+\alpha_0)/3)}{\Gamma((1+\alpha_0+\alpha_2)/3)\Gamma((2+\alpha_0+\alpha_2)/3)\Gamma((1+\alpha_0-\alpha_2)/3)} \frac{\Gamma((4-\alpha_0-2\alpha_2)/3)\Gamma((1+\alpha_0)/3)}{\Gamma((2-\alpha_0-2\alpha_2)/3)\Gamma((1+\alpha_0-\alpha_2)/3)\Gamma((2-\alpha_0-2\alpha_2)/3)}. \]

6 Characterizations of the monodromy

In this section, we will characterize the linear monodromy of our symmetric solutions on Fricke’s cubic surface of monodromy.

To normalize the monodromy matrices in \( SL(2, \mathbb{C}) \), we replace \( \psi \) of the linearization \( \mathfrak{L} \) as

\[ \psi \to x^{\alpha_2/2}(x-1)^{\alpha_3/2}(x-t)^{(\alpha_0-1)/2}\psi, \]

then the Riemann scheme of \( \mathfrak{L} \) becomes

\[ P \begin{pmatrix} x: \quad 0 & 1 & t & y & \infty \\ \frac{\alpha_2}{2} & -\frac{\alpha_3}{2} -\frac{\alpha_0-1}{2} & 0 & -\frac{\alpha_1}{2} & \frac{\alpha_1+1}{2} \end{pmatrix}, \]

and then the monodromy matrices \( \{M_0, M_t, M_1, M_\infty\} \) become elements of \( SL(2, \mathbb{C}) \). Hereafter we use this linearization.

6.1 Linear monodromy of the Symmetric Solutions of \( \sigma_1 \)

From the theorem \( \mathfrak{T} \) we can calculate \( \{p_0, p_1, p_t, p_\infty; p_{01}, p_{1t}, p_{0t}\} \) for the symmetric solution \( \mathfrak{S}_2-1 \). We have

\[ p_0 = p_1 = 2 \cos \pi \alpha_3, \quad p_t = 2 \cos \pi (\alpha_0 + 1), \quad p_\infty = 2 \cos \pi \alpha_1, \]

and

\[ p_{01} = -2 \left(1 + \cos \pi (\alpha_0 + 1) + \cos \pi \alpha_1 + 4 \cos \frac{\pi}{2} \alpha_1 \cos \pi \alpha_3 \cos \frac{\pi}{2} (\alpha_0 + 1)\right). \]

In the same way, for the solution \( \mathfrak{S}_2-2 \), we have

\[ p_0 = p_1 = 2 \cos \pi \alpha_3, \quad p_t = 2 \cos \pi (\alpha_0 + 1), \quad p_\infty = 2 \cos \pi \alpha_1, \]

\[ p_{1t} = p_{00} = 2 \left( \cos \pi (\alpha_3 + 1) + \cos \frac{\pi}{2} (\alpha_0 - \alpha_1 - 1) + \cos \frac{\pi}{2} (\alpha_0 + \alpha_1 - 1)\right), \]

\[ p_{01} = -2 \left(1 + \cos \pi (\alpha_0 - 1) + \cos \pi \alpha_3 + 4 \cos \frac{\pi}{2} \alpha_1 \cos \pi \alpha_3 \cos \frac{\pi}{2} (\alpha_0 - 1)\right), \]

for \( \mathfrak{S}_2-3 \), we have

\[ p_0 = p_1 = 2 \cos \pi \alpha_3, \quad p_t = 2 \cos \pi (\alpha_0 + 1), \quad p_\infty = 2 \cos \pi \alpha_1, \]

\[ p_{1t} = p_{00} = 2 \left( \cos \pi \alpha_3 + \cos \frac{\pi}{2} (\alpha_0 - \alpha_1 - 1) + \cos \frac{\pi}{2} (\alpha_0 + \alpha_1 + 1)\right), \]

\[ p_{01} = -2 \left(1 + \cos \pi \alpha_0 + \cos \pi (\alpha_1 + 1) + 4 \cos \frac{\pi}{2} (\alpha_1 + 1) \cos \pi \alpha_3 \cos \frac{\pi}{2} \alpha_0\right), \]

and

\[ p_{01} = -2 \left(1 + \cos \pi \alpha_0 + \cos \pi (\alpha_1 + 1) + 4 \cos \frac{\pi}{2} (\alpha_1 + 1) \cos \pi \alpha_3 \cos \frac{\pi}{2} \alpha_0\right), \]
Since \( M_0 \) is an element of \( \text{SL}(2, \mathbb{C}) \), we have
\[
\begin{align*}
p_0 &= p_1 = 2 \cos \pi \alpha_0, & p_t &= 2 \cos \pi (\alpha_0 + 1), & p_\infty &= 2 \cos \pi \alpha_1, \\
p_{0t} &= p_{1t} = p_\infty = 2 \left( \cos \pi \alpha_0 + \cos \frac{\pi}{2} (\alpha_0 - \alpha_1 + 1) \right), \\
p_{01} &= -2 \left( 1 + \cos \pi \alpha_0 + \cos \pi (\alpha_1 - 1) + 4 \cos \frac{\pi}{2} (\alpha_1 - 1) \cos \pi \alpha_3 \cos \frac{\pi}{2} \alpha_0 \right). 
\end{align*}
\] (26)

Certainly, they satisfy (1).

Theorem 9. Under the constraints \( p_0 = p_1 \) and \( p_{0t} = p_{1t} \), the relation of monodromy (11) admits a double root as an equation of \( p_{01} \), if and only if \( \{M_0, M_t, M_1, M_\infty\} \) is the linear monodromy of a symmetric solution of \( \sigma_1 \).

6.2 Linear monodromy of the Symmetric Solutions of \( \sigma_2 \circ \sigma_1 \)

From the theorem 8 we can calculate \( \{p_0, p_1, p_t, p_\infty; p_{01}, p_{1t}, p_{0t}\} \) for the symmetric solution (S3-2). We have
\[
\begin{align*}
p_0 &= p_1 = p_\infty = 2 \cos \pi \alpha_1, & p_t &= 2 \cos \pi (\alpha_0 - 1), \\
p_{0t} &= p_{1t} = p_{1t} = -1 - 2 \cos \left( \frac{\pi}{2} \alpha_0 \right) + 4 \cos \left( \frac{\pi}{2} \alpha_0 \right) \cos \pi \alpha_1.
\end{align*}
\] (28) (29)

In the same way, for the solution (S3-1), we have
\[
\begin{align*}
p_0 &= p_1 = p_\infty = 2 \cos \pi \alpha_1, & p_t &= 2 \cos \pi (\alpha_0 - 1), \\
p_{0t} &= p_{1t} = p_{1t} = -1 - 2 \cos \left( \frac{\pi}{2} \alpha_0 \right) + 4 \cos \left( \frac{\pi}{2} \alpha_0 \right) \cos \pi \alpha_1.
\end{align*}
\] (30) (31)

and, for (S3-3), we have
\[
\begin{align*}
p_0 &= p_1 = p_\infty = 2 \cos \pi \alpha_1, & p_t &= 2 \cos \pi (\alpha_0 - 1), \\
p_{0t} &= p_{1t} = p_{1t} = -1 - 2 \cos \left( \frac{\pi}{2} \alpha_0 \right) + 4 \cos \left( \frac{\pi}{2} \alpha_0 \right) \cos \pi \alpha_1.
\end{align*}
\] (32) (33)

Now, \( \{p_0, p_1, p_t, p_\infty; p_{01}, p_{1t}, p_{0t}\} \) also parameterize the monodromy, and they satisfy the following relation
\[
p_{\infty} p_{1t} p_{t\infty} + p_{\infty}^2 + p_{\infty}^2 + p_{t\infty}^2 - (p_{\infty} p_1 + p_t p_0) p_{0\infty} - (p_1 p_t + p_\infty p_0) p_{1t} - (p_t p_\infty + p_0 p_0) p_{1t} \\
+p_{0\infty}^2 + p_{t\infty}^2 + p_{\infty}^2 + p_{t\infty}^2 + p_0 p_{t1} p_t p_\infty - 4 = 0.
\] (34)

Since \( M_0, M_t, M_1, M_\infty \) are elements of \( \text{SL}(2, \mathbb{C}) \), we have
\[
p_{0t} = \text{tr} M_0 M_t = \text{tr} (M_1 M_\infty)^{-1} = \text{tr} M_\infty M_1 = p_{0\infty}.
\]
And thus, \( \{p_0, p_1, p_t, p_{t\infty}; p_{t0}, p_{t1}, p_{t\infty}\} \) also parameterize the monodromy.

Conversely, if we set

\[
X = p_{t0} = p_{t1} = p_{t\infty}, \quad A = p_0 = p_1 = p_{\infty} = 2\cos(\pi\alpha_1), \quad B = p_t = 2\cos\pi(\alpha_0 - 1),
\]

then (34) becomes

\[
X^3 + 3X^2 - 3(A^2 + AB)X + 3A^2 + B^2 + A^3B - 4 = 0.
\]

We can factorize the left hand side of this equation as follows

\[
(X - X_1)(X - X_2)(X - X_3),
\]

where \(X_1, X_2, X_3\) are the right hand sides of (29), (31), (33).

Now we have the following theorem.

**Theorem 10.** Under the constraints \( p_0 = p_1 = p_{\infty} \) and \( p_{t0} = p_{t1} = p_{t\infty} \), the relation (34) becomes a third order equation of \( p_{t0} \), whose solutions correspond to the linear monodromy of the symmetric solutions of \( \sigma_2 \circ \sigma_1 \).

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