Research Article

Embedding of Besov Spaces and the Volterra Integral Operator

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The boundedness and compactness of the inclusion mapping from Besov spaces to tent spaces are studied in this paper. Meanwhile, the boundedness, compactness, and essential norm of the Volterra integral operator $T_g$ from Besov spaces to a class of general function spaces are also investigated.

1. Introduction

Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the class of functions analytic in $\mathbb{D}$. For $1 < p < \infty$, the Besov space, denoted by $B_p$, is the space of all functions $f \in H(\mathbb{D})$ satisfying

$$
\|f\|_{B_p} = \|f(0)\|^p + \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2} \, dA(z) < \infty.
$$

(1)

Let $0 < p < \infty$, $-2 < q < \infty$, and $0 \leq s < \infty$. The space $F(p,q,s)$ is the space consisting of all $f \in H(\mathbb{D})$ such that

$$
\|f\|_{F(p,q,s)} = \|f(0)\|^p + \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^q (1-|z|^2)^{s} \, dA(z) < \infty,
$$

(2)

where $a_s(z) = ((a - z)/(1-\overline{az}))$. This space was first introduced by Zhao in [1]. $F(2,0,0)$ is the BMO space, $F(p,a,0)$ is called the Dirichlet space, denoted by $D_p$. In particular, $F(p,p-2,0)$ is the Besov space $B_p$. $F(p,p,0)$ is just the classical Bergman space $A_p$. When $s > 1$, from [1], we see that $F(p,p-2,s)$ is equivalent to the Bergman space, denoted by $A_s$, which consists of all $f \in H(\mathbb{D})$ such that

$$
\|f\|_{A_s} = \|f(0)\|^p + \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^q (1-|z|^2)^{s} \, dA(z) < \infty.
$$

(3)

For $1 < q < \infty$ and $0 < s, t < \infty$, let $\mathcal{L}(q,q-2,s,t)$ denote the space of all $f \in H(\mathbb{D})$ such that

$$
\|f\|_{\mathcal{L}(q,q-2,s,t)} = \left( \frac{1}{\log(2/1-|a|^2)} \right)^s \int_{\mathbb{D}} |f'(z)|^q (1-|z|^2)^{s-t} \, dA(z) < \infty.
$$

(4)

The norm for $f \in \mathcal{L}(q,q-2,s,t)$ is given by

$$
\|f\|_{\mathcal{L}(q,q-2,s,t)} = \|f(0)\| + \|f\|_{\mathcal{L}(q,q-2,s,t)}.
$$

(5)

The Volterra integral operator $T_g$ was introduced by Pommerenke in [3]. Here,

$$
T_g f(z) = \int_0^z f(w) g'(w) \, dw, \quad f \in H(\mathbb{D}).
$$

In [3], Pommerenke showed that $T_g$ is bounded on $H^2$ if and only if $g \in BMO$. Aleman and Siskakis showed that $T_g$ is bounded on $H^p(p \geq 1)$ if and only if $g \in BMO$ in [4]. In [5], Aleman and Siskakis proved that $T_g: A^p \rightarrow A^p$ is bounded (compact) if and only if $g \in \mathcal{B}$ ($g \in \mathcal{B}_b$). Recently, the operator $T_g$ has been receiving much attention. See [4–18] and the references therein for more study of the operator $T_g$. 


For any arc $I \subseteq \mathbb{D}$, the boundary of $\mathbb{D}$, let $|I| = (1/2\pi) \int_{|z| = 1} d|z|$ denote the normalized length of $I$ and $S(I)$ be the Carleson box defined by

$$S(I) = \left\{ z \in \mathbb{D} : 1 - |I| \leq |z| < 1, \frac{z}{|z|} \in I \right\}.$$  \hspace{1cm} (6)

Let $0 < s < \infty$ and $\mu$ be a positive Borel measure on $\mathbb{D}$. We say that $\mu$ is an $\alpha$-Carleson measure if

$$\|\mu\|_{CM_{\alpha}} = \sup_{I \subseteq \mathbb{D}} \frac{\mu(S(I))}{|I|^\alpha} < \infty.$$  \hspace{1cm} (7)

When $\alpha = 1$, it gives the classical Carleson measure, $\mu$ is said to be a vanishing $\alpha$-Carleson measure if $\lim_{|I| \to 0} (\mu(S(I))/|I|^\alpha) = 0$. The Carleson measure is very useful in the theory of function spaces and operator theory. The famous embedding theorem says that the inclusion mapping $I_d^1 : H^1 \rightarrow L^2(\mu)$ is bounded if and only if $\mu$ is a Carleson measure (see [19]). See [7, 20] for the study of the inclusion mapping $I_d^1 : B_p \rightarrow L^p(\mu)$.

Let $0 < s, q < \infty, 0 < t < \infty$, and $\mu$ be a positive Borel measure on $\mathbb{D}$. Let $\mathcal{S}^q_s(\mu)$ denote the space of all $\mu$-measurable functions $f$ such that (see, e.g., [21])

$$\sup_{I \subseteq \mathbb{D}} \frac{1}{|I|^{1/2} \log(2/|I|)} \int_{S(I)} |f(z)|^q d\mu(z) < \infty.$$  \hspace{1cm} (8)

The tent space $\mathcal{S}^q_s(\mu)$ was introduced by Liu et al. in [21]. When $t = 0$, $\mathcal{S}^q_s(\mu)$ will be denoted by $\mathcal{S}^q_s(\mu)$ for the simplicity. In [21], Liu et al. studied the embedding of some Möbius invariant spaces, such as the Bloch space and the $Q_p$ space, into $\mathcal{S}^q_s(\mu)$.

In [12], Pau and Zhao showed that the inclusion mapping $I_d^1 : F(p, p - 2, s) \rightarrow \mathcal{S}^q_s(\mu)$ is bounded if and only if $\mu$ is a $p$-logarithmic $s$-Carleson measure. In [9], Li et al. proved that the inclusion mapping $I_d^1 : \mathcal{S}^p_{s-1} \rightarrow \mathcal{S}^q_s(\mu)$ is bounded if and only if $\mu$ is a $(s - 1)$-Carleson measure. In [14], Qian and Li proved that the inclusion mapping $I_d^1 : B_p \rightarrow \mathcal{S}^q_s(\mu)$ is bounded (resp. compact) if and only if $\mu$ is a $q(1 - (1/p))$-logarithmic $s$-Carleson measure (resp. vanishing $q(1 - (1/p))$-logarithmic $s$-Carleson measure) under the assumption that $1 < p < q < \infty$ and $0 < s < \infty$.

Motivated by [14, 21], in this paper, we study the boundedness and compactness of the inclusion mapping $I_d^1 : B_p \rightarrow \mathcal{S}^q_s(\mu)$. More precisely, we show that $I_d^1 : B_p \rightarrow \mathcal{S}^q_s(\mu)$ is bounded (resp. compact) if and only if $\mu$ is an $s$-Carleson measure (resp. vanishing $s$-Carleson measure) under the assumption that $1 < p < q < \infty$ and $0 < s < \infty$. As an application, we study the boundedness of the operator $T_q^1 : B_p \rightarrow \mathcal{L}F(q, q - 2, s, q - (q/p))$. Moreover, the compactness and essential norm of the operator $T_q^2 : B_p \rightarrow \mathcal{L}F(q, q - 2, s, q - (q/p))$ are also investigated.

In this paper, the symbol $f = g$ means that $f \leq g \leq f$. We say that $f \leq g$ if there exists a constant $C$ such that $f \leq Cg$.

2. Embedding the Besov Space $B_p$ into $\mathcal{S}^q_s(\mu)$

We need the following equivalent description of $\alpha$-Carleson measure (see Lemma 2.2 in [12]).

**Lemma 1.** Let $0 < \alpha, t < \infty$ and $\mu$ be a positive Borel measure on $\mathbb{D}$. Then, $\mu$ is an $\alpha$-Carleson measure if and only if

$$\sup_{\mathbb{D}} \int_{|I|} \left( \frac{1 - |e|}{1 - \bar{w}e} \right)^t d\mu(z) < \infty.$$  \hspace{1cm} (9)

Moreover,

$$\|\mu\|_{CM_{\alpha}} \approx \sup_{\mathbb{D}} \int_{|I|} \left( \frac{1 - |e|}{1 - \bar{w}e} \right)^\alpha d\mu(z).$$  \hspace{1cm} (10)

Using Lemma 3.10 in [22], we can easily obtain the following result.

**Lemma 2.** Let $1 < p < \infty$ and $\omega \in \mathbb{D}$. Set

$$f_\omega(z) = \left( \frac{1}{\log(2/|\omega|)} \right)^{1/p} \log \frac{2}{1 - \bar{\omega}z}, \quad z \in \mathbb{D}.$$  \hspace{1cm} (11)

Then, $f_\omega \in B_p$.

**Theorem 1.** Let $1 < p < q < \infty$, $0 < s < \infty$, and $\mu$ be a positive Borel measure on $\mathbb{D}$. Then, the inclusion mapping $I_d^1 : B_p \rightarrow \mathcal{S}^q_s(1 - (1/p))$ is bounded if and only if $\mu$ is an $s$-Carleson measure.

**Proof.** First, we assume that $I_d^1 : B_p \rightarrow \mathcal{S}^q_s(1 - (1/p))$ is bounded. For any given arc $I \subseteq \mathbb{D}$, set $a = (1 - |I|)\eta$, and $\eta$ is the center point of $I$. It is easy to see that

$$|1 - \bar{\eta}z| \approx 1 - |\eta|^2 = |I|, \quad z \in S(I).$$  \hspace{1cm} (12)

Let

$$f_a(z) = \left( \frac{1}{\log(2/|a|)} \right)^{1/p} \log \frac{2}{1 - \bar{a}z}.$$  \hspace{1cm} (13)

By Lemma 2, we see that $f_a \in B_p$. From the boundedness of $I_d^1 : B_p \rightarrow \mathcal{S}^q_s(1 - (1/p))$, we have

$$\|f_a\|_{\mathcal{S}^q_s(\mu)} = \sup_{I \subseteq \mathbb{D}} \frac{1}{|I|^{1/2} \log(2/|I|)} \int_{S(I)} |f_a(z)|^q d\mu(z) < \infty.$$  \hspace{1cm} (14)

By the fact that $|f_a(z)| \approx (\log(2/|I|))^{1/(1/p)}$ when $z \in S(I)$, we get

$$\sup_{I \subseteq \mathbb{D}} \frac{\mu(S(I))}{|I|^\alpha} < \infty.$$  \hspace{1cm} (15)

Hence, $\mu$ is an $s$-Carleson measure.
Conversely, assume that $\mu$ is an $s$-Carleson measure. Let $f \in B_p$. For any given arc $I \subset \partial D$, set $w = (1 - |I|)\eta$, and $\eta$ is the center point of $I$. Then,

\[
\frac{1}{|I|^q} \int_{S(I)} |f(z)|^q \, d\mu(z) \leq \frac{1}{|I|^{q(1 - (1/p)\beta)}} \int_{S(I)} |f(z) - f(w)|^q \, d\mu(z) + \frac{1}{|I|^{q(1 - (1/p)\beta)}} \int_{S(I)} |f(w)|^q \, d\mu(z)
\]

where

\[
A = \frac{1}{|I|^{q(1 - (1/p)\beta)}} \int_{S(I)} |f(z) - f(w)|^q \, d\mu(z),
\]

\[
B = \frac{1}{|I|^{q(1 - (1/p)\beta)}} \int_{S(I)} |f(w)|^q \, d\mu(z).
\]

Since

\[
|f(w)| \leq \left( \log \frac{2}{1 - |w|^2} \right)^{1 - (1/p)} \|f\|_{L_p^q} \leq \left( \log \frac{2}{|I|} \right)^{1 - (1/p)} \|f\|_{L_p^q},
\]

we get

\[
B \leq \frac{\mu(S(I))}{|I|^q} \|f\|_{L_p^q}^q \leq \|f\|_{L_p^q}^q.
\]

Now, we turn to estimate $A$. By Theorem 1 in [7], we see that $\mu$ is an $s$-Carleson measure if and only if $I_d: D^p_{p-2s(p/q)} \rightarrow L^q(\mu)$ is bounded. Note that

\[
f \in B_p \subset D^p_{p-2s(p/q)}.
\]

Then,

\[
A \leq \left( 1 - |w|^2 \right)^{s} \int_{S(I)} \left| f(z) - f(w) \right|^q \, d\mu(z)
\]

\[
\leq \left( 1 - |w|^2 \right)^{pq/q} \int_{\partial |z|^{2(p/q)}} \left( f(z) - f(w) \right)^p \left( 1 - \frac{w}{z} \right)^{(2s/q) - 1} \, dA(z).
\]

Since

\[
\left( \frac{f(z) - f(w)}{1 - w/z} \right)^q = \frac{f^q(z)(1 - w/z)^{(2s/q)} + w \left( 2s/q \right) f(z) - f(w)(1 - w/z)^{(2s/q) - 1}}{(1 - w/z)^{(4s/q)}},
\]

we deduce that $A \leq (W_1 + W_2)^{(pq/p)}$, where

\[
W_1 = \left( 1 - |w|^2 \right)^{(ps/q)} \int_{|z|^{2(p/q)}} \left| f(z) \right|^p \left( 1 - |z|^2 \right)^{p - 2s(p/q)} \, dA(z),
\]

\[
W_2 = \left( 1 - |w|^2 \right)^{(ps/q)} \int_{|z|^{2(p/q)}} \left| f(z) - f(w) \right|^p \left( 1 - |z|^2 \right)^{p - 2s(p/q)} \, dA(z).
\]

Since

\[
\frac{1 - |w|^2(1 - |z|^2)}{|1 - w/z|^2} = \left( 1 - |\sigma_w(z)|^2 \right)^2 \leq 1,
\]

we get that

\[
W_1 \leq \|f\|_{L^p}^p.
\]

Making the change of variable $\eta = \sigma_w(z)$ and combining with Proposition 4.2 in [22], we have
\[ W_2 = (1 - |w|^2)^{(p/q)} \int \frac{|(f^* \sigma_w)(\eta) - (f^* \sigma_w)(0)|^p}{|1 - w \sigma_w(\eta)|^{2(p/q)} + p} (1 - |\sigma_w(\eta)|^2)^{p - 2(p/q)} \frac{(1 - |\eta|^2)^{p - 2 + (p/q)}}{|1 - \overline{\eta}|^p} dA(\eta) \]

\[ = \int \frac{|(f^* \sigma_w)(\eta) - (f^* \sigma_w)(0)|^p}{|1 - w \sigma_w(\eta)|^{2(p/q)} + p} (1 - |\eta|^2)^{p - 2 + (p/q)} dA(\eta) \]

\[ \leq \int |f^* (\sigma_w(\eta))|^p (1 - |\sigma_w(\eta)|^2)^{p - 2 + (p/q)} \frac{(1 - |\eta|^2)^{p - 2 + (p/q)}}{|1 - \overline{\eta}|^p} dA(\eta) \]

\[ \leq \int |f^* (z)|^p (1 - |z|^2)^{p - 2 + (p/q)} \frac{(1 - |\sigma_w(z)|^2)^{p - 2 + (p/q)}}{|1 - \overline{w}\sigma_w(z)|^p} |1 - \overline{z}|^p dA(z) \]

\[ \leq \|f\|_{B^p}^p. \]

Therefore,

\[ \sup_{I \in \mathbb{D}} \frac{1}{|I|^s (\log(2/|I|))^{s(q1-(1/p))}} \int_{S(I)} |f(z)|^q d\mu(z) \leq \|f\|_{B^p}^q, \]

which implies the desired result. The proof is completed.

We say that the inclusion mapping \( I_d: B_p \rightarrow \mathcal{T}_{s,q}^{q1-(1/p)}(\mu) \) is compact if

\[ \lim_{n \to \infty} \frac{1}{|I|^s (\log(2/|I|))^{s(q1-(1/p))}} \int_{S(I)} |f_n(z)|^q d\mu(z) = 0 \]

whenever \( I \subset \mathbb{D} \) and \( \{f_n\} \) is a bounded sequence in \( B_p \), that converges to 0 uniformly on compact subsets of \( \mathbb{D} \).

**Theorem 2.** Let \( 1 < p < q < \infty \) and \( 0 < s < \infty \). Let \( \mu \) be a nonnegative Borel measure on \( \mathbb{D} \) such that point evaluation is a bounded functional on \( \mathcal{T}_{s,q}^{q1-(1/p)}(\mu) \). Then, the inclusion mapping \( I_d: B_p \rightarrow \mathcal{T}_{s,q}^{q1-(1/p)}(\mu) \) is compact if and only if \( \mu \) is a vanishing \( s \)-Carleson measure.

**Proof.** First, we assume that \( I_d: B_p \rightarrow \mathcal{T}_{s,q}^{q1-(1/p)}(\mu) \) is compact. Let \( \{f_k\} \) be a sequence with \( \lim_{k \to \infty} |I_k| = 0 \). Set \( \alpha_k = (1 - |I_k|) \eta_k \), where \( \eta_k \) is the midpoint of arc \( I_k \). Take

\[ f_k(z) = \left( \frac{1}{\log(2/|\alpha_k|^2)} \right)^{1/p} \log \frac{2}{1 - \alpha_k z}. \]

We see that \( f_k \in B_p \), and \( \{f_k\} \) converges to 0 uniformly on compact subsets of \( \mathbb{D} \) when \( k \to \infty \). Then, we get

\[ \mathcal{M}(S(I_k)) \leq \frac{1}{|I_k|^r (\log(2/|I_k|))^{r(q1-(1/p))}} \int_{S(I_k)} |f_k(z)|^q d\mu(z), \]

as \( k \to \infty \), which implies that \( \mu \) is a vanishing \( s \)-Carleson measure.

Conversely, assume that \( \mu \) is a vanishing \( s \)-Carleson measure. From [12], we see that

\[ \|\mu - \mu_r\|_{CM, r} \to 0, \quad r \to 1. \]

Here, \( \mu_r(z) = \mu(z) \) for \( |z| < r \) and \( \mu_r(z) = 0 \) for \( r \leq |z| < 1 \). Let \( \|f_k\|_{B_p} \leq 1 \) and \( \{f_k\} \) converge to 0 uniformly on compact subsets of \( \mathbb{D} \). Then,

\[ \frac{1}{|I|^s (\log(2/|I|))^{s(q1-(1/p))}} \int_{S(I)} |f_k(z)|^q d\mu(z) \]

\[ \leq \frac{1}{|I|^s (\log(2/|I|))^{s(q1-(1/p))}} \int_{S(I)} |f_k(z)|^q d\mu_r(z) \]

\[ + \frac{1}{|I|^s (\log(2/|I|))^{s(q1-(1/p))}} \int_{S(I)} |f_k(z)|^q d(\mu - \mu_r)(z) \]

\[ \leq \frac{1}{|I|^s (\log(2/|I|))^{s(q1-(1/p))}} \int_{S(I)} |f_k(z)|^q d\mu_r(z) + \|\mu - \mu_r\|_{CM} \|f_k\|_{B_p}^q \]

\[ \leq \frac{1}{|I|^s (\log(2/|I|))^{s(q1-(1/p))}} \int_{S(I)} |f_k(z)|^q d\mu_r(z) + \|\mu - \mu_r\|_{CM} \|f_k\|_{B_p}^q. \]

Letting \( k \to \infty \) and then \( r \to 1 \), we have

\[ \lim_{k \to \infty} \|f_k\|_{\mathcal{T}_{s,q}^{q1-(1/p)}(\mu)} = 0. \]
Therefore, \( I_d; B_p \rightarrow \mathcal{F}_{q,(1-(1/p))}^q (u) \) is compact. The proof is completed. \( \square \)

3. Volterra Integral Operator \( T_g; B_p \rightarrow \mathcal{L} \mathcal{F} \) \((q, q - 2, s, q - (q/p))\)

**Lemma 3.** Let \( 1 < q < \infty \) and \( 0 < s, t < \infty \). Then, \( f \in \mathcal{L} \mathcal{F}(q, q - 2, s, q - (q/p)) \) if and only if
\[
\sup_{t \in \mathcal{D} \mathcal{D}} \frac{1}{|t|^q (\log (2/|t|))} \int_{S(I)} \left| (T_g f) (z) \right|^q (1 - |z|^2)^{q-2s} dA(z) < \infty.
\]

**Proof.** The proof is similar to that of Proposition 1 in [15]. Thus, we omit the details of the proof. \( \square \)

**Theorem 3.** Let \( 1 < p < q < \infty \) and \( 0 < s < \infty \). Then, \( T_g; B_p \rightarrow \mathcal{L} \mathcal{F}(q, q - 2, s, q - (q/p)) \) is bounded if and only if \( g \in F(q, q - 2, s) \). For any fixed arc \( I \subset \mathcal{D} \mathcal{D} \), let \( e^{i\theta} \) denote the center of \( I \) and \( a = (1 - |I|) e^{i\theta} \). Set
\[
f_a(z) = \left( \frac{1}{\log (2/|a|^2)} \right)^{1/p} \log \frac{2}{1 - az}.
\]

By Lemma 2, we have \( f_a \in B_p \) for \( 1 < p < \infty \). In addition, it is easy to see that
\[
|1 - az| = 1 - |a| = |I|,
\]
and
\[
\left| f_a(z) \right| = \left( \frac{2}{|I|} \right)^{1 - 1/p}.
\]

Therefore, \( T_g; B_p \rightarrow \mathcal{L} \mathcal{F}(q, q - 2, s, q - (q/p)) \) is bounded.

Next, we give an estimation for the essential norm of \( T_g \). First, we recall some definitions. The essential norm of \( T: X \rightarrow Y \), denoted by \( \|T\|_{e,X \rightarrow Y} \), is defined by
\[
\|T\|_{e,X \rightarrow Y} = \inf_{K} \|T - K\|_{X \rightarrow Y}: K \text{ is compact from } X \text{ to } Y.
\]

By Lemma 3, we get that
\[
\sup_{t \in \mathcal{D} \mathcal{D}} \frac{1}{|t|^q (\log (2/|t|))} \int_{S(I)} \left| (T_g f) (z) \right|^q (1 - |z|^2)^{q-2s} dA(z) < \infty.
\]

Therefore, \( T_g; B_p \rightarrow \mathcal{L} \mathcal{F}(q, q - 2, s, q - (q/p)) \) is bounded.

Here, \( X \) and \( Y \) are Banach spaces, and \( T: X \rightarrow Y \) is a bounded linear operator. It is easy to see that \( T: X \rightarrow Y \) is compact if and only if \( \|T\|_{e,X \rightarrow Y} = 0 \). Let \( A \) be a closed subspace of \( X \). Given \( f \in X \), the distance from \( f \) to \( A \), denoted by \( \text{dist}_X (f, A) \), is defined by
\[
\text{dist}_X (f, A) = \inf_{g \in A} \|f - g\|_X.
\]
Lemma 4. (see [14]). Let $1 < q < \infty$ and $0 < s < \infty$. If $g \in F(q, q - 2, s)$, then
\[
\text{dist}_{F(q,q-2,s)}(g, F_0(q, q - 2, s)) = \limsup_{\alpha \to 1} \|g - g_\alpha\|_{F(q,q-2,s)} = \limsup_{\alpha \to 1} \int_{\mathbb{D}} |g'(z)|^q (1 - |z|^2)^{(q-2)(1 - |\sigma_a(z)|^2)}^s \, dA(z).
\]

Here, $g_\alpha(z) = g(rz)$, $0 < r < 1$, $z \in \mathbb{D}$.

\begin{equation}
\|T_g f_n\|_{L^q_{\mathbb{D}}(q,q-2,s,(q/p))} = \sup_{a \in \mathbb{D}} \left( \log \left( \frac{2}{1 - |a|^2} \right) \right)^q \int_{\mathbb{D}} |f_n(z)|^q |g'(z)|^q (1 - |z|^2)^{(q-2)(1 - |\sigma_a(z)|^2)}^s \, dA(z)
\end{equation}

\begin{equation}
\leq \frac{\|g\|_{L^q_{\mathbb{D}}(q,q-2,s)}^q}{(1 - r^2)^q} \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f_n(z)|^p (1 - |z|^2)^{q-2}(1 - |\sigma_a(z)|^2) \, dA(z)
\end{equation}

By the dominated convergence theorem, we get the desired result. The proof is completed.

Lemma 5. Let $1 < p < q < \infty$ and $0 < s < \infty$. If $0 < r < 1$ and $g \in F(q, q - 2, s)$, then
\[
T_g : B_p \to L^q_{\mathbb{D}}(q,q-2,s,(q/p))
\]

Proof: Given $\{f_n\} \subset B_p$ such that $\{f_n\}$ converges to zero uniformly on any compact subset of $\mathbb{D}$ and $\sup_{\alpha \in \mathbb{D}} \|f_n\|_{L^p_{\mathbb{D}}(s,(q/p))} \leq 1$. Since $g \in F(q, q - 2, s) \subset \mathbb{D}$, we get that
\[
\|g_n'(z)\| \leq \frac{\|g\|_{L^q_{\mathbb{D}}(q,q-2,s)}}{1 - r^2}.
\]

Hence,
\[
\|T_g f_n\|_{L^q_{\mathbb{D}}(q,q-2,s,(q/p))} = \text{dist}_{F(q,q-2,s)}(g, F_0(q, q - 2, s)).
\]

\begin{equation}
\|T_g\|_{L^p_{\mathbb{D}}(s,(q/p))} = \text{dist}_{F(q,q-2,s)}(g, F_0(q, q - 2, s)).
\end{equation}

Proof. Let $|I_n| \subset \partial \mathbb{D}$ and $|I_n| \to 0$ as $n \to \infty$. Suppose $e^{i\theta_n}$ is the center of $I_n$ and $\omega_n = (1 - |I_n|) e^{i\theta_n}$. For each $n$, let
\[
f_{w_n}(z) = \left( \frac{1}{\log \left( 2/1 - |w_n|^2 \right)} \right)^{1/p} \log \frac{2}{1 - |w_n|^2} \frac{2}{z - \omega_n}.
\]

Then, $|f_{w_n}(z)| = (\log (2/|w_n|^2))^{1-1/p}$ when $z \in S(I_n)$, and $\{f_{w_n}\}$ is bounded in $B_p$. Furthermore, $\{f_{w_n}\}$ converges to zero uniformly on every compact subset of $\mathbb{D}$. Given a compact operator $K : B_p \to L^q_{\mathbb{D}}(q,q-2,s,(q/p))$, by Lemma 6, we have
\[
\lim_{n \to \infty} \|K f_{w_n}\|_{L^q_{\mathbb{D}}(q,q-2,s,(q/p))} = 0.
\]

So,
\[
\left\| T^*_g - K \right\| > \limsup_{n \to \infty} \left\| (T^*_g - K) f_{w_n} \right\|_{\mathcal{L}^p((q,q-2,s,q-(q/p))})
\]
\[
> \limsup_{n \to \infty} \left( \left\| T^*_g f_{w_n} \right\|_{\mathcal{L}^p((q,q-2,s,q-(q/p))}) - \left\| K f_{w_n} \right\|_{\mathcal{L}^p((q,q-2,s,q-(q/p))}) \right)
\]
\[
= \limsup_{n \to \infty} \left\| T^*_g f_{w_n} \right\|_{\mathcal{L}^p((q,q-2,s,q-(q/p))})
\]
\[
\geq \limsup_{n \to \infty} \left( \frac{1}{(\log (2/1 - |w_n|^2))^{q(1-1/p)}} \int_{\mathbb{S}} \left| f_{w_n}(z) \right|^q \left| g^*(z) \right|^q \left( 1 - |z|^2 \right)^{q-2} \left( 1 - \left| \sigma_{w_n}(z) \right|^2 \right) dA(z) \right)^{(1/q)} \tag{48}
\]
\[
\geq \limsup_{n \to \infty} \left( \frac{1}{(2 - |w_n|^2)^{p(1-1/p)}} \int_{\mathbb{S}} \left| f_{w_n}(z) \right|^q \left| g^*(z) \right|^q \left( 1 - |z|^2 \right)^{q-2} \left( 1 - \left| \sigma_{w_n}(z) \right|^2 \right) dA(z) \right)^{(1/q)}
\]
\[
> \limsup_{n \to \infty} \left( \frac{1}{|T_n|^q} \int_{\mathbb{S}} \left| g^*(z) \right|^q \left( 1 - |z|^2 \right)^{q-2} \left( 1 - \left| \sigma_{w_n}(z) \right|^2 \right) dA(z) \right)^{(1/q)} ,
\]
which implies that
\[
\left\| T^*_g e_{B_p} \to \mathcal{L}^p((q,q-2,s,q-(q/p))} \right\| > \limsup_{n \to \infty} \left( \int_{\mathbb{S}} \left| g^*(z) \right|^q \left( 1 - |z|^2 \right)^{q-2} \left( 1 - \left| \sigma_{w_n}(z) \right|^2 \right) dA(z) \right)^{(1/q)} . \tag{49}
\]

It follows from Lemma 4 that
\[
\left\| T^*_g e_{B_p} \to \mathcal{L}^p((q,q-2,s,q-(q/p))} \right\| \rightarrow \text{dist}_{\mathcal{F}(q,q-2,s)} \left( g, F_0(q,q-2,s) \right).
\tag{50}
\]

On the contrary, by Lemma 5, \( T^*_g e_{B_p} \to \mathcal{L}^p((q,q-2,s,q-(q/p))} \) is compact. Then,
\[
\left\| T^*_g e_{B_p} \to \mathcal{L}^p((q,q-2,s,q-(q/p))} \right\| \leq \left\| T^*_g - T^*_g \right\| = \left\| T^*_g - g^*_g \right\|
\tag{51}
\]
\[
\approx \left\| g - g^*_g \right\|_{\mathcal{F}(q,q-2,s)} .
\]

Using Lemma 4 again, we have
\[
\left\| T^*_g e_{B_p} \to \mathcal{L}^p((q,q-2,s,q-(q/p))} \right\| \leq \limsup_{r \to 1} \left\| g - g^*_g \right\|_{\mathcal{F}(q,q-2,s)}
\approx \text{dist}_{\mathcal{F}(q,q-2,s)} \left( g, F_0(q,q-2,s) \right).
\tag{52}
\]

The proof is completed.

The following result can be deduced by Theorem 4 directly. □

**Corollary 1.** Let \( 1 < p < q < \infty \) and \( 0 < s < \infty \). If \( T^*_g e_{B_p} \to \mathcal{L}^p((q,q-2,s,q-(q/p))} \) is bounded, then \( T^*_g e_{B_p} \to \mathcal{L}^p((q,q-2,s,q-(q/p))} \) is compact if and only if \( g \in F_0(q,q-2,s) \). \tag{53}

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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