BOUNDDED OPERATORS OF STOCHASTIC DIFFERENTIATION ON SPACES OF NONREGULAR GENERALIZED FUNCTIONS IN THE LÉVY WHITE NOISE ANALYSIS

Background. Operators of stochastic differentiation play an important role in the Gaussian white noise analysis. In particular, they can be used in order to study properties of the extended stochastic integral and of solutions of normally ordered stochastic equations. Although the Gaussian analysis is a developed theory with numerous applications, in problems of mathematics not only Gaussian random processes arise. In particular, an important role in modern research belongs to Lévy processes. So, it is necessary to develop a Lévy analysis, including the theory of operators of stochastic differentiation.

Objective. During recent years the operators of stochastic differentiation were introduced and studied, in particular, on spaces of regular test and generalized functions and on spaces of nonregular test functions of the Lévy analysis. In this paper, we make the next step: introduce and study such operators on spaces of nonregular generalized functions.

Methods. We use, in particular, the theory of Hilbert equipments and Lytvynov’s generalization of the chaotic representation property.

Results. The main result is a theorem about properties of operators of stochastic differentiation.

Conclusions. The operators of stochastic differentiation are considered on the spaces of nonregular generalized functions of the Lévy white noise analysis. This can be interpreted as a contribution in a further development of the Lévy analysis. Applications of the introduced operators are quite analogous to the applications of the corresponding operators in the Gaussian analysis.

Keywords: operator of stochastic differentiation; extended stochastic integral; Hida stochastic derivative; Lévy process.

Introduction

Denote \( R_+ = [0, +\infty) \). Let \( L = (L_t)_{t \in R_+} \) be a Lévy process, i.e., a random process on \( R_+ \) with stationary independent increments and such that \( L_0 = 0 \) (see, e.g., [1] for details), without Gaussian part and drift. In [2] the extended Skorohod stochastic integral with respect to \( L \) and the corresponding Hida stochastic derivative on the space of square integrable random variables \( (L^2) \) were constructed in terms of Lytvynov’s generalization of the chaotic representation property (CRP) [3], some properties of these operators were established; and it was shown that the above-mentioned integral coincides with the well-known extended stochastic integral with respect to a Lévy process, constructed in terms of Itô’s generalization of the CRP [4] (see, e.g., [5, 6]). In [7, 8] the stochastic integral and derivative were extended to spaces of test and generalized functions that belong to riggings of \( (L^2) \), this gives a possibility to extend an area of their possible applications. Together with the mentioned operators, it is natural to introduce and to study so-called operators of stochastic differentiation in the Lévy white noise analysis, by analogy with the Gaussian analysis [9, 10]. Such operators are closely related with the extended stochastic integral with respect to a Lévy process and with the corresponding Hida stochastic derivative and, by analogy with the “classical case”, can be used, in particular, in order to study properties of the extended stochastic integral and properties of solutions of so-called normally ordered stochastic equations. In [11, 12] the operators of stochastic differentiation on spaces belonging to a so-called regular parametrized rigging of \( (L^2) \) [7] were introduced and studied. But, in connection with some problems of the stochastic analysis, sometimes it can be necessary to consider another, a so-called nonregular rigging of \( (L^2) \) [7] and various operators on spaces (of nonregular test and generalized functions) belonging to this rigging. In [13] operators of stochastic differentiation were introduced and studied on the spaces of nonregular test functions of the Lévy white noise analysis. In particular, it was shown that these operators are the restrictions to the above-mentioned spaces of the corresponding operators on \( (L^2) \). The next natural step consists in introduction and study of operators of stochastic differentiation on the spaces of nonregular generalized functions. But, unfortunately, the operators of stochastic differentiation on \( (L^2) \) (in the same way as the Hida stochastic derivative)
cannot be naturally continued to the above-
mentioned spaces. Nevertheless, one can introduce
on these spaces linear operators with properties
quite analogous to properties of the operators of
stochastic differentiation. These linear operators
will be called the operators of stochastic differentia-
tion on the spaces of nonregular generalized functions.
In the present paper we introduce these operators
and establish some their properties.

**Problem definition**

The aim of this paper is to introduce the opera-
tors of stochastic differentiation on the spaces of
nonregular generalized functions of the Lévy white
noise analysis; and to establish some properties of
these operators.

**Preliminaries**

In this paper we deal with a real-valued locally
square integrable Lévy process $L$ on $R$, 
without Gaussian part and drift. As is well known,
the characteristic function of such a process is

$$ E[e^{iux}] = \exp[i\int K(e^{iaz} - 1 - iax)v(\text{d}x)], \quad (1)$$

where $v$ is the Lévy measure of $L$, $E$ denotes
the expectation. We assume that $v$ is a Radon measure
whose support contains an infinite number of points,
$v([0]) = 0$, there exists $\varepsilon > 0$ such that
$$ \int K x^2 e^{\varepsilon x}v(\text{d}x) < \infty,$$
and $\int K x^2 v(\text{d}x) = 1$.

Let us define a measure of the white noise of $L$.
Let $D$ denote the set of all real-valued infinite-
differentiable functions on $R$ with compact sup-
ports. As is well known, $D$ can be endowed by the
projective limit topology generated by some Sobol-
lev spaces (more details are given below, a detailed
presentation is given in, e.g., [14]). Let $D'$ be the
set of linear continuous functionals on $D$. For
$\varphi \in D'$ and $\varphi \in D$ denote $\langle \varphi, \varphi \rangle := \varphi(\varphi)$; note that
one can understand $\langle \cdot, \cdot \rangle$ as the dual pairing
generated by the scalar product in the space $L^2(R)$
of square integrable with respect to the Lebesgue
measure real-valued functions on $R$. The notation
$\langle \cdot, \cdot \rangle$ will be preserved for dual pairings in tensor
powers of spaces.

**Definition** ([3]). A probability measure $\mu$ on
$(D', C(D'))$, where $C$ denotes the cylindrical $\sigma$-
algebra, with the Fourier transform
$$ \int_D e^{i\langle \omega, \varphi \rangle} \mu(\text{d}\omega)$$

$$ = \exp[\int_{K \times K} \{e^{i\varphi(u)x} - 1 - i\varphi(u)x\}v(\text{d}x)], \quad \varphi \in D, \quad (2)$$
is called the Lévy white noise measure.

Denote $(\mathcal{L}^2) := L^2(D', C(D'), \mu)$ the space of
real-valued square integrable with respect to $\mu$
functions on $D'$; let also $H := L^2(R)$. Let $f \in H$
and a sequence $(\varphi_k \in D_k)_{k \in N}$ converge to $f$
in $H$ as $k \to \infty$. One can show [2, 3, 5, 6] that $\langle \varphi, f \rangle :=
= (\mathcal{L}^2) - \lim_{k \to c} \langle \varphi_k \rangle$ is well defined as an element of
$(\mathcal{L}^2)$. Let us consider $(\varphi, I_{1(0,1)}) \in (\mathcal{L}^2)$, $t \in R$, (here
and below $I_A$ denotes the indicator of a set $A$). It
follows from (1) and (2) that $(\langle \varphi, I_{1(t,t)} \rangle)_{t \in R}$
can be identified with a Lévy process $L$.

Consider Lytvynov’s generalization of the
CRP (see [3] for details). Denote by $\otimes$ the sym-
metric tensor product. For $m \in Z_+$, $m \subseteq N \cup \{0\}$ set

$$ P_m := \{\varphi(\omega) = \sum_{k=0}^m (\varphi_{\varepsilon_k}, \varphi^{(k)}) | \omega \in D', \varphi^{(k)} \in D^{\varepsilon_k}, l \leq m\}.$$ 

Denote by $\overline{P}_m$ the closure of $P_m$ in $(\mathcal{L}^2)$. Let for
$m \in N$ $P_m$ be defined from the condition $\overline{P}_m =
= P_m \otimes \overline{P}_m$, $P_0 := \overline{P}_0$. Let $f^{(m)} \in D^{\otimes_m}$, $m \in Z_+$. Denote by $\langle \cdot, \cdot \rangle_{\otimes m}$ the orthogonal projection of
a monomial $\langle \varphi_{\varepsilon_m}, f^{(m)} \rangle$ onto $P_m$. Let us introduce
scalar products $(\cdot, \cdot)_{\otimes m}$ on $D^{\otimes_m}$, $m \in Z_+(D^{\otimes_0} = R)$,
by setting for $f^{(m)}, g^{(m)} \in D^{\otimes_m}$

$$ \langle f^{(m)}, g^{(m)} \rangle_{\otimes m} := \frac{1}{m!} \int D \langle \varphi_{\varepsilon_m}, f^{(m)} \rangle \langle \varphi_{\varepsilon_m}, g^{(m)} \rangle \mu(\text{d}\omega),$$

and let $\| \cdot \|_{\otimes m}$ be the corresponding norms. Denote by
$H^{(m)}_{\otimes}$, $m \in Z_+$, the completions of $D^{\otimes_m}$ with
respect to the norms $\| \cdot \|_{\otimes m}$. For $F^{(m)} \in H^{(m)}_{\otimes}$
we define a Wick monomial $\langle \varphi_{\varepsilon_m}, f^{(m)} \rangle_{\otimes m} :=
(\mathcal{L}^2) - \lim_{k \to \infty} \langle \varphi_{\varepsilon_k}, f^{(m)} \rangle_{\otimes m}$, here for each $k \in N$
$f^{(m)} \in D^{\otimes_m}$, and $f^{(m)} \to f^{(m)}$ as $k \to \infty$ in $H^{(m)}_{\otimes}$
(well-posedness of this definition can be proved by
the method of “mixed sequences”). One can show [3]
that $F \in (\mathcal{L}^2)$ if and only if there is a unique se-
sequence of kernels \( F^{(m)} \in H^{(m)}_{\text{ext}} \), \( m \in \mathbb{Z}_+ \), such that
\[
F = \sum_{m=0}^{\infty} \langle \delta^{(m)}, F^{(m)} \rangle
\]
with
\[
E[F]^2 = \| F \|^2_{(E)} = \int_{D} |F(\omega)|^2 \mu(d\omega) = \sum_{m=0}^{\infty} m! |F^{(m)}|^2_{\text{ext}} < \infty.
\]

Note that \( H^{(1)}_{\text{ext}} = H \) and for \( m \in \mathbb{N} \setminus \{1\} \) one can identify \( H^{\otimes m} \) with a proper subspace of \( H^{(m)}_{\text{ext}} \) that consists of “vanishing on diagonals” elements [2, 3]. In this sense the space \( H^{(m)}_{\text{ext}} \) is an extension of \( H^{\otimes m} \).

Denote by \( T \) the set of indexes \( \tau = (\tau_1, \tau_2) \), where \( \tau_1 \in \mathbb{N} \), \( \tau_2 : R \to [1, \infty) \) is an infinite differentiable function. Let \( H_\tau \), \( \tau \in T \), be the Sobolev space on \( R \) of order \( \tau \) weighted by the function \( \tau_2 \) (e.g., [14]). It is well known that
\[
D = \text{pr} \lim_{\tau \to T} H_\tau \quad (\text{moreover, } D^{\otimes m} = \text{pr} \lim_{\tau \to T} H^{\otimes m}_{\text{ext}}, \ m \in \mathbb{N})
\]
and for each \( \tau \in T \) \( H_\tau \) is densely and continuously embedded into \( H \). Therefore one can consider a chain
\[
D' \ni H_{-\tau} \ni H \ni H_\tau \ni D,
\]
where \( H_{-\tau} \), \( \tau \in T \), are the spaces dual of \( H_\tau \) with respect to \( H \). Note that by the Schwartz theorem (e.g., [14]) \( D' = \bigcup_{\tau \in T} H_{-\tau} \).

By analogy with [15] (see also [3]) one can show that the measure \( \mu \) is concentrated on \( H_{-\tau} \) with some \( \tau \in T \), i.e., \( \mu(H_{-\tau}) = 1 \). So, excepting from \( T \) some indexes, we will assume, in what follows, that for each \( \tau \in T \) \( \mu(H_{-\tau}) = 1 \). Further, denote the norms in \( H_\tau \) and its tensor powers by \( | \cdot |_\tau \). It is shown in [7] that, again excepting from \( T \) some indexes, we obtain the next statement.

Proposition. For each \( \tau \in T \) and for each \( m \in \mathbb{N} \) the space \( H^{\otimes m}_{\text{ext}} \) is densely and continuously embedded into the space \( H^{(m)}_{\text{ext}} \), and there exists \( c(\tau) > 0 \) such that for all \( f^{(m)} \in H^{\otimes m}_{\text{ext}} \) we have
\[
| f^{(m)} |_{\text{ext}}^2 \leq m! c(\tau)^m | f^{(m)} |_\tau^2.
\]

Denote
\[
P_{W} := \left\{ f = \sum_{m=0}^{N_f} \langle \delta^{(m)}, f^{(m)} \rangle : f^{(m)} \in D^{\otimes m}, N_f \in \mathbb{Z}_+ \right\} \subset (L^2).
\]
Accept on default \( q \in Z_+ \), \( \tau \in T \); set \( H^{(0)}_{\tau} = R \); and define scalar products \( \langle \cdot, \cdot \rangle_{\tau, q} \) on \( P_{W} \) by setting for \( f, g \in P_{W} \)
\[
\langle f, g \rangle_{\tau, q} = \min(N_f, N_g) \sum_{m=0}^{\min(N_f, N_g)} (m!)^2 2^{\text{pm}} \langle f^{(m)}, g^{(m)} \rangle_{H^{\otimes m}_{\text{ext}}}
\]
(see [13] for details). Let \( \| \cdot \|_{\tau, q} \) be the corresponding norms. Now we define Kondratiev spaces of nonregular test functions \((H_\tau)_q\) as completions of \( P_{W} \) with respect to the norms \( \| \cdot \|_{\tau, q} \). As is easy to see, \( f \in (H_\tau)_q \) if and only if \( f \) can be presented in the form
\[
f = \sum_{m=0}^{\infty} \langle \delta^{(m)}, f^{(m)} \rangle : f^{(m)} \in H^{\otimes m}_{\tau}, \quad (3)
\]
with \( \| f \|^2_{(H_\tau)_q} = \sum_{m=0}^{\infty} (m!)^2 2^{\text{pm}} | f^{(m)} |_{\tau}^2 < \infty \).

Further, it is proved in [7] that for each \( \tau \in T \) there exists \( q_\theta(\tau) \in \mathbb{Z}_+ \) such that for each \( q \in \mathbb{Z}_+ \), \( q \geq q_\theta(\tau) \), the space \((H_\tau)_q\) is densely and continuously embedded into \( (L^2) \). In view of this statement for \( \tau \in T \) and \( q \geq q_\theta(\tau) \) one can consider a chain
\[
(H_{-\tau})_q \ni (L^2) \ni (H_\tau)_q, \quad (4)
\]
where \((H_{-\tau})_q\) is the space dual of \((H_\tau)_q\) with respect to \((L^2)\). Chain (4) is called a nonregular rigging of the space \((L^2)\). The negative spaces \((H_{-\tau})_q\) of such chains (with various \( \tau \) and \( q \)) are called Kondratiev spaces of nonregular generalized functions.

Now let us describe natural orthogonal bases in the spaces \((H_{-\tau})_q\). In view of the Proposition formulated above let us consider chains
\[
D^{(m)} \ni H^{(m)}_{-\tau} \ni H^{(m)}_{\text{ext}} \ni H^{\otimes m}_{\text{ext}} \ni D^{\otimes m}, \ m \in \mathbb{N} \quad (5)
\]
(for \( m = 0 \) set \( D^{(0)} = H^{(0)}_{-\tau} = H^{(0)}_{\text{ext}} = H^{\otimes 0} = D^{\otimes 0} = R \)), where \( H^{(m)}_{-\tau} \) and \( D^{(m)} \) are the spaces dual of \( H^{\otimes m}_{\text{ext}} \) and \( D^{\otimes m} \) with respect to \( H^{(m)}_{\text{ext}} \).

Proposition ([7]). There exists a system of generalized functions
\[
\{ \langle \delta^{(m)}, f^{(m)} \rangle : \langle f^{(m)} \rangle \in (H_{-\tau})_q, f^{(m)} \in H^{\otimes m}_{\text{ext}}, m \in \mathbb{Z}_+ \}
\]
such that
1) for \( F^{(m)}_{\text{ext}} \in H^{(m)}_{\text{ext}} \subset H^{(m)}_{-\tau} : \langle \cdot^m, F^{(m)}_{\text{ext}} \rangle \) is a Wick monomial;
2) any generalized function \( F \in (H_{-\tau})_q \) can be presented as a formal series

\[
F = \sum_{m=0}^{\infty} \langle \cdot^m, F^{(m)}_{\text{ext}} \rangle ; F^{(m)}_{\text{ext}} \in H^{(m)}_{-\tau},
\]

that converges in \((H_{-\tau})_q\), i.e.,

\[
\|F\|_{(H_{-\tau})_q}^2 = \sum_{m=0}^{\infty} 2^{-qm} \|F^{(m)}_{\text{ext}}\|_{H^{(m)}_{-\tau}}^2 < \infty;
\]

and any formal series (6) with finite norm (7) is a generalized function from \((H_{-\tau})_q\);
3) the dual pairing between \( F \in (H_{-\tau})_q \) and \( f \in (H_{\tau})_q \) that is generated by the scalar product in \((L^2)\), has the form

\[
\langle (F, f) \rangle_{(L^2)} = \sum_{m=0}^{\infty} m! \langle F^{(m)}_{\text{ext}}, f^{(m)} \rangle_{\text{ext}},
\]

where \( F^{(m)}_{\text{ext}} \in H^{(m)}_{\text{ext}} \) and \( f^{(m)} \in H^{(m)}_{\tau} \) are the kernels from decompositions (6) and (3) for \( F \) and \( f \) respectively, \( \langle \cdot, \cdot \rangle_{\text{ext}} \) denotes the dual pairings between elements of negative and positive spaces from chains (5), these pairings are generated by the scalar products in \( H^{(m)}_\tau, m \in Z_+\).

Following [13], we recall now a notion of the extended stochastic integral on \((H_{-\tau})_q \otimes H_{-\tau}\). First we note that there exists a system of orthogonal in \((H_{-\tau})_q \otimes H_{-\tau}\) generalized functions

\[
\{ \langle \cdot^m, F^{(m)}_{\text{ext}} \rangle : \in (H_{-\tau})_q \otimes H_{-\tau} \mid F^{(m)}_{\text{ext}} \in H^{(m)}_{\text{ext}} \otimes H_{\tau}, m \in Z_+ \}
\]

such that any \( F \in (H_{-\tau})_q \otimes H_{-\tau} \) can be presented as a convergent in this space series

\[
F() = \sum_{m=0}^{\infty} \langle \cdot^m, F^{(m)}_{\text{ext}} \rangle ; F^{(m)}_{\text{ext}} \in (H^{(m)}_{\text{ext}}) \otimes H_{\tau},
\]

with \( \|F\|^2_{(H_{-\tau})_q \otimes H_{\tau}} = \sum_{m=0}^{\infty} 2^{-qm} \|F^{(m)}_{\text{ext}}\|_{H^{(m)}_{\text{ext}}}^2 \|H^{(m)}_{\tau} < \infty \).

Consider a family of chains

\[
D^{(m)} \supseteq H^{(m)}_{\tau} \supseteq H^{(m)}_{\tau} \supseteq H^{(m)}_{-\tau} \supseteq D^{(m)}, m \in N
\]

(as is well known (e.g., [14]), \( H^{(m)}_{\tau} \) and \( D^{(m)} \)

\[
\supseteq \bigcup_{m \in N} H^{(m)}_{\tau}
\]

are the spaces dual of \( H^{(m)}_{\tau} \) and \( D^{(m)} \) with respect to \( H^{(m)}_{-\tau} \); for \( m = 0 \) all spaces from (10) are equal to \( R \) by definition). Since the spaces of test functions in chains (10) and (5) coincide, there exists a family of natural isomorphisms \( U_m : D^{(m)} \rightarrow D^{(m)}_{\tau} \) such that for all \( F^{(m)}_{\text{ext}} \in D^{(m)}_{\text{ext}} \) and \( f^{(m)} \in D^{(m)}_{\tau} \),

\[
\langle F^{(m)}_{\text{ext}}, f^{(m)} \rangle_{\text{ext}} = (U_m F^{(m)}_{\text{ext}}, f^{(m)}).
\]

It is easy to see that the restrictions of \( U_m \) to \( H^{(m)}_{\tau} \) are isometric isomorphisms between the spaces \( H^{(m)}_{\tau} \) and \( H^{(m)}_{-\tau} \). Now we introduce an extended stochastic integral \( \int \langle (u) dL_{\tau}; (H_{-\tau})_q \otimes H_{-\tau} \rightarrow (H_{-\tau})_q \) as a linear continuous operator that is defined for \( F \in (H_{-\tau})_q \otimes H_{-\tau} \) of form (9) as

\[
\int F(u) dL_{\tau} = \sum_{m=0}^{\infty} \langle \cdot^m, F^{(m)}_{\text{ext}} \rangle ;
\]

here \( F^{(m)}_{\text{ext}} := U^{-1}_m \{ \langle \cdot^m, F^{(m)}_{\text{ext}} \rangle \} \in H^{(m)}_{-\tau} \), \( \langle \cdot, \cdot \rangle_{\text{ext}} \) is the orthoprojector acting for each \( m \in Z_+ \) from \( H^{(m)}_{\tau} \otimes H_{-\tau} \) to \( H^{(m+1)}_{\tau} \) (the symmetrization operator), \( 1 \) is the identity operator. It is shown in [13] that this integral is an extension of the extended Skorohod stochastic integral with respect to a Lévy process \( L \).

Unfortunately, the extended stochastic integral with respect to a Lévy process cannot be naturally restricted to the spaces of nonregular test functions. More precisely, for \( f \in (H_{\tau})_q \otimes H_{\tau} \subset (H_{-\tau})_q \otimes H_{-\tau} \)

\[
\int f(u) dL_{\tau} \text{ is not necessary a nonregular test function.}
\]

Nevertheless, one can introduce on each space of nonregular test functions a linear operator that has properties quite analogous to the properties of the above-mentioned integral. First we note that for \( f^{(m)} \in H^{(m)}_{\tau} \otimes H_{\tau} \subset H^{(m)}_{-\tau} \) the above-introduced generalized functions \( \langle \cdot^m, f^{(m)} \rangle \)

belong to the spaces \( (H_{\tau})_q \otimes H_{\tau} \) and form orthogonal bases in these spaces [13]. So, any \( f \in (H_{\tau})_q \otimes H_{\tau} \) can be presented as

\[
f() = \sum_{m=0}^{\infty} \langle \cdot^m, f^{(m)} \rangle ; f^{(m)} \in H^{(m)}_{\tau} \otimes H_{\tau},
\]

with

\[
\|f\|^2_{(H_{\tau})_q \otimes H_{\tau}} = \sum_{m=0}^{\infty} \langle \cdot^m, f^{(m)} \rangle ; f^{(m)} \in H^{(m)}_{\tau} \otimes H_{\tau},
\]

Now we define a linear continuous operator \( I : (H_{\tau})_q \otimes H_{\tau} \rightarrow (H_{\tau})_q \) by setting for \( f \in (H_{\tau})_q \otimes H_{\tau} \)
of form (11)

\[
I(f) := \sum_{m=0}^{\infty} \langle \cdot^m, f^{(m)} \rangle ;
\]

here

\[
f^{(m)} := \langle \cdot^m, f^{(m)} \rangle ; \text{ Pr is the orthoprojector}
\]
acting for each \( m \in \mathbb{Z}^+ \) from \( H^\otimes m \odot H^\tau \) to \( H^\otimes m+1 \) (the symmetrization operator). The well-posedness of this definition is proved in [13].

Finally, as is well known, an important role in the Lévy white noise analysis belongs to the Hida stochastic derivative, which is the adjoint operator of the extended stochastic integral. In terms of Lytvynov’s generalization of the CRP this derivative is considered on \((L^2)\) [2], on the spaces of regular test and generalized functions [11,12] and on the spaces of nonregular test functions [7,13].

But, unfortunately, this operator has no a natural extension to the spaces of nonregular generalized functions. Nevertheless, one can define natural analogs of the Hida stochastic derivative on these spaces as operators adjoint to \( I \). More exactly, we define a linear continuous operator \( \hat{\partial} : (H^-)_{-q} \to (H^-)_{-q-1} \otimes H^- \) as the adjoint operator to \( I (\hat{\partial} = I^\dagger) \), i.e., for all \( F \in (H^-)_{-q} \) and \( f \in (H_{q+1}) \otimes H^- \), 
\[
\langle \hat{\partial} F, f \rangle_{(L^2)\otimes H} = \langle (F, I(f)) \rangle_{(L^2)},
\]
here \( \langle \cdot, \cdot \rangle_{(L^2)\otimes H} \) denotes the dual pairing generated by the scalar product in \((L^2)\otimes H\).

A simple calculation shows that \( \hat{\partial} F = \sum_{m=0}^\infty (m+1) \langle \langle \circ_{(m)}^\otimes, F \rangle \rangle^{(m+1)} \rangle \),
where
\[
F^{(m+1)} = (U_{m+1} \otimes \mathbb{1}) [U_{m+1} \circ_{(m+1)} X] \in H^- \otimes H^- \, ,
\]
and the kernels from decomposition \((6)\) for \( F \).

**Operators of stochastic differentiation**

As we said above, the operators of stochastic differentiation on \((L^2)\) [11,12] cannot be naturally continued to the spaces of nonregular generalized functions (because the kernels from decompositions \((6)\) for elements of \((H^-)_{-q} \) belong to the spaces wider than \( H^{(m)} \)). Nevertheless, one can introduce on these spaces natural analogs of the above-mentioned operators. We begin from a preparation. Let \( F^{(m)} \in H^{(m)} \), \( f^{(n)} \in H_{\tau} \odot H_{\tau} \, , \, n, m \in \mathbb{N} \). We define a generalized partial pairing \( \langle F^{(m)}, f^{(n)} \rangle_{\otimes \otimes} \) by setting for any \( g^{(m-n)} \in H^\otimes m-n \)
\[
\langle F^{(m)} \circ_{(m-n)} g^{(m-n)} \rangle_{\otimes \otimes} = \langle F^{(m)} \circ_{(m)} f^{(n)} \rangle_{\otimes \otimes} \cdot \tag{12}
\]
As is easy to verify by the generalized Cauchy–Bunyakovsky inequality, this definition is well-posed and
\[
|\langle F^{(m)} \circ_{(m)} f^{(n)} \rangle_{\otimes \otimes} |_{H_{\tau} \odot H_{\tau}} \leq |F^{(m)}|_{H^\otimes m} |f^{(n)}|_{\tau} \cdot \tag{13}
\]

**Definition.** Let \( f^{(n)} \in H_{\tau} \odot H_{\tau} \), \( n \in \mathbb{N} \). We define the operator of stochastic differentiation (\( \hat{\partial}^{(m,n)}(f^{(n)}) \)):
\[
(H^-)_{-q} \to (H^-)_{-q-1} \, \text{by the formula}
\]
\[
(\hat{\partial}^{(m,n)}(f^{(n)}))_{(\hat{\partial}^{(m,n)}(f^{(n)}))^\dagger} \}
\]
where \( F^{(m)} \in H_{\tau} \) are the kernels from decomposition \((6)\) for \( F \in (H^-)_{-q} \). Also we denote
\[
(\hat{\partial}^{(m)}(f^{(n)}))_{(\hat{\partial}^{(m)}(f^{(n)}))^\dagger} \}
\]
By direct calculation with use \((7)\) and \((13)\) one obtains the estimate
\[
\langle (\hat{\partial}^{(m)}(f^{(n)}))^\dagger \rangle_H^2 \leq 2^m |f^{(n)}|_{\tau} \max_{m \in \mathbb{Z}^+} \left[ 2^{-m} \left( \frac{(m+n)!}{m!} \right)^2 \right] |F^{(m)}|_{H_{\tau} \odot H_{\tau}}^2 \, ,
\]
whence it follows that \((\hat{\partial}^{(m)}(f^{(n)}))\) is a well-defined linear continuous (bounded) operator.

Consider properties of \((\hat{\partial}^{(m)}(f^{(n)}))\).

**Theorem 1.** For \( k_1, k_2, k_3, k_4 \in \mathbb{N} \), \( f^{(k_1)} \in H_{\tau} \odot H_{\tau} \), \( f \in \{1, \ldots, m\} \)
\[
(\hat{\partial}^{(k_1 \cdots k_3)}(f^{(k_1)})(f^{(k_2)})(f^{(k_3)})(f^{(k_4)}) \cdots)(f^{(k_n)}) = (\hat{\partial}^{(k_1 \cdots k_n)}(f^{(k_1)})(f^{(k_2)})(f^{(k_3)})(f^{(k_4)})(f^{(k_5)})(f^{(k_6)})(f^{(k_7)})(f^{(k_8)})(f^{(k_9)}))(f^{(k_10)}).
\]

2) For each \( F \in (H^-)_{-q} \) the kernels \( F^{(m)} \in H_{\tau} \odot H_{\tau} \) from decomposition \((6)\) can be presented in a form
\[
F^{(m)} = \frac{1}{m!} E(\hat{\partial}^{(m)} F) \, ,
\]
i.e., for each \( f^{(m)} \in H_{\tau} \odot H_{\tau} \)
\[
\langle F^{(m)} \circ_{(m)} f^{(m)} \rangle_{\otimes \otimes} = \frac{1}{m!} E(\langle (\hat{\partial}^{(m)} F)(f^{(m)}) \rangle) \, ,
\]
here \( E := \langle \cdot, \cdot \rangle_{(L^2)} \) is a generalized expectation.

3) The adjoint to \( \hat{\partial}^{(m)} \) operator has a form
\[
\hat{\partial}^{(m)} g^{(m)} \odot f^{(n)} = \sum_{m=0}^\infty \langle \circ_{(m+n)}^\otimes, g^{(m)} \odot f^{(n)} \rangle \in (H_{\tau})_q \, , \tag{15}
\]
where $g \in (H_\tau)_{q+1}$, $f^{(n)} \in H_\tau^{\otimes n}$, and $g^{(m)} \in H_\tau^{\otimes m}$ are the kernels from decomposition (3) for $g$.

4) For all $g \in (H_\tau)_{q+1}$ and $f^{(1)} \in H_\tau$

$$(\mathcal{D}g)(f^{(1)})^* = I(g \otimes f^{(1)}) \in (H_\tau)_{q}.$$  \hfill (16)

5) For all $F \in (H_{-\tau})_{-q}$ and $f^{(1)} \in H_\tau$ we have

$$(\mathcal{D}F)(f^{(1)}) = (\mathcal{F}_\tau F, f^{(1)}(\cdot)) \in (H_{-\tau})_{-q-1},$$

where $\langle \mathcal{F}_\tau F, f^{(1)}(\cdot) \rangle$ is a partial pairing, i.e., the unique element of $(H_{-\tau})_{-q-1}$ such that for arbitrary $g \in (H_\tau)_{q+1}$

$$\langle \langle \mathcal{F}_\tau F, f^{(1)}(\cdot) \rangle, g \rangle_{(L_\tau)} = \langle \langle \mathcal{F}_\tau F, g \otimes f^{(1)}(\cdot) \rangle \rangle_{(L_\tau) \otimes H}.$$  

Formally $\mathcal{D} = (\mathcal{D}_\tau \otimes \mathcal{D}_\tau)(\mathcal{F}_\tau)$, where $\mathcal{F}_\tau$ is the Dirac delta-function.

6) Let $F \in (H_{-\tau})_{-q} \otimes H_{-\tau}$, $f^{(1)} \in H_\tau$. Then

$$\langle \mathcal{D} F(u) \rangle f^{(1)} \rangle \mathcal{D}_\tau L_\tau(u) < f^{(1)} > = \int (\mathcal{D} F(u)) \langle f^{(1)} \rangle dL_\tau + \langle F(\cdot), f^{(1)}(\cdot) \rangle \in (H_{-\tau})_{-q+1},$$

here $\langle F(\cdot), f^{(1)}(\cdot) \rangle$ is a partial pairing.

Proof. 1) The application of the mathematical induction method.

2) The direct calculation with use (14) and (8).

3) The direct calculation with use (14), (3), (12), (8) and (6).

4) The consequence of (15) and the definition of $I$.  

5) The direct calculation with use (16) and the definition of $\mathcal{D}$.  

6) The direct calculation with use (14), the definition of the extended stochastic integral, (12) and (8). □

Conclusions

In this paper the operators of stochastic differentiation are considered on the spaces of nonregular generalized functions of the Lévy white noise analysis; and some properties of these operators are established. This can be interpreted as a contribution in a further development of the Lévy analysis. In particular, using the introduced operators one can study some properties of the extended stochastic integral and of solutions of so-called normally ordered stochastic equations. In forthcoming papers we’ll consider elements of the Wick calculus on the spaces of nonregular test and generalized functions, the connection between the Wick calculus and the stochastic differentiation and integration, etc.

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ОГРАНИЧЕННЫЕ ОПЕРАТОРЫ СТОХАСТИЧЕСКОГО ДИФФЕРЕНЦИРОВАНИЯ НА ПРОСТРАНСТВАХ НЕРЕГУЛЯРНЫХ ОБОБЩЕННЫХ ФУНКЦИЙ В АНАЛИЗЕ БЕЛОГО ШУМА ЛЕВИ

Проблематика. Операторы стохастического дифференцирования играют важную роль в гауссовском анализе белого шума. В частности, эти операторы можно использовать для изучения свойств расширенного стохастического интеграла и решений нормально упорядоченных стохастических уравнений. Хотя гауссовский анализ – это развитая теория с многочисленными приложениями, в математических задачах появляются не только гауссовские случайные процессы. В частности, важная роль в современных исследованиях принадлежит процессам Леви. Поэтому необходимо развивать анализ Леви, включая теорию операторов стохастического дифференцирования.

Цель исследования. В последние годы операторы стохастического дифференцирования были введены и изучены, в частности, на пространствах регулярных основных и обобщенных функций и на пространствах нерегулярных основных функций анализа Леви. В этой статье мы делаем следующий шаг: вводим и изучаем такие операторы на пространствах нерегулярных обобщенных функций.

Методика реализации. Мы используем, в частности, теорию гильбертовых оснащений и литвиновское обобщение свойства хаотического разложения.

Результаты исследования. Основной результат – теорема о свойствах операторов стохастического дифференцирования.

Выводы. Операторы стохастического дифференцирования рассмотрены на пространствах нерегулярных обобщенных функций анализа белого шума Леви. Это можно понимать как вклад в дальнейшее развитие анализа Леви. Применения введенных операторов вполне аналогичны применениям соответствующих операторов в гауссовском анализе.

Ключевые слова: оператор стохастического дифференцирования; расширенный стохастический интеграл; стохастическая производная Хиды; процесс Леви.