A LICHERNOWICZ ESTIMATE FOR THE SPECTRAL GAP OF THE SUB-LAPLACIAN

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Abstract. For a second order operator on a compact manifold satisfying the strong Hörmander condition, we give a bound for the spectral gap analogous to the Lichnerowicz estimate for the Laplacian of a Riemannian manifold. We consider a wide class of such operators which includes horizontal lifts of the Laplacian on Riemannian submersions with minimal leaves.

1. Introduction

Consider a second order operator $L = \sum_{i=1}^{k} X_i^2 + X_0$ of strong Hörmander type defined on a compact manifold $M$, symmetric with respect to a smooth volume density $\mu$. In other words, we assume that $X_1, \ldots, X_k$ along with their iterated brackets span the entire tangent bundle, giving us that that $L$ is hypoelliptic with discrete spectrum [9], see also e.g. [4]. The first general result regarding the spectral gap of $L$ was given in [11] using coordinate expression of the vector fields $X_1, \ldots, X_k$. Several results exist for the special case when $L$ is the sub-Laplacian of a CR-manifold, see e.g. [5, 12, 10]. More general coordinate-free results exist in [1, Section 2], [2, Section 3.5], [3] and [6, Section 4.1]. A majority of the latter mentioned group of results concern generalizations of Lichnerowicz bound for the spectral gap [13]. More precisely, if $L$ is the Laplacian of a compact, $n$-dimensional Riemannian manifold with Ricci curvature bounded from below by $\rho > 0$, then the first non-zero eigenvalue $\lambda_1$ has the lower bound

$$|\lambda_1| \geq \frac{n}{n-1} \rho.$$

In this paper, we will present a further generalization of the Lichnerowicz spectral gap. In order to put the result in perspective, consider the following special case. Let $\pi : (M, g) \rightarrow (N, g_N)$ be a surjective Riemannian submersion, meaning that the orthogonal complement $H$ of the vertical bundle $V = \ker d\pi$ satisfies

$$g|_H = \pi^* g_N|_H.$$

Consider the operator $L$ defined to be the horizontal lift with respect to $H$ of the Laplacian on $N$. Then $L$ is symmetric with respect to the volume density $\mu_g$ of $g$ if and only if the submanifolds $M_y = \pi^{-1}(y), y \in N$ are minimal, see [6, Section 2] for more details. If in addition these submanifolds are totally geodesic and $H$ satisfies the Yang-Mills condition, a spectral gap for $L$ is given in [3]. This estimate is sharp for the cases of the Hopf fibration and the quaternionic Hopf...
fibration. We will generalize this result, removing the requirement of the Yang-Mills condition and only requiring that the leaves are minimal (Condition (C), Section 2.2). Furthermore, we do not need that the complement $V$ of $H$ is an integrable subbundle, allowing for the result to be applied outside the cases of submersions or foliations.

The main result with proof is presented in Section 2. In Section 3 we give examples where the estimate on the spectral gap is computed explicitly. We emphasize the fact that the removal of the requirement of the Yang-Mills condition allows us to consider conformal changes of the metric $g_N$ in (1.1); see Section 3.2.

2. Statement of main result and proof of the spectral gap

2.1. Sub-Laplacians and notation. Let $M$ be a connected, compact manifold with a volume density $\mu$. Let $L = \sum_{i=1}^k X_i^2 + X_0$ be a smooth second order operator on $M$ that is symmetric with respect to $\mu$. Any such operator $L$ is uniquely determined by a symmetric positive semi-definite tensor $q_L \in \Gamma(\text{Sym}^2 T^* M)$ on the cotangent bundle, given by

$$L(f_1 f_2) - f_2 Lf_1 - f_1 Lf_2 = 2\langle df_1, df_2 \rangle_{q_L}.$$ 

A sub-Riemannian structure on $M$ is a pair $(H, g_H)$ where $H$ is a subbundle of the tangent bundle $T^* M$ and $g_H$ is a fiber metric defined only on $H$. We will call $H$ the horizontal bundle. Equivalently, a sub-Riemannian structure can be considered as a metric tensor $g_H^*$ on $T^* M$ that degenerates along a subbundle. These two points of view are related through the map

$$\sharp^H : T^* M \rightarrow TM,$$

$$\sharp^H \alpha = \langle \alpha, \cdot \rangle_{g_H^*}, \quad \langle \alpha, \beta \rangle_{g_H^*} = \langle \sharp^H \alpha, \sharp^H \beta \rangle_{g_H}.$$ 

Any operator $L$ where $q_L = g_H^*$ degenerates along a subbundle can be considered as the sub-Laplacian of $(M, H, g)$ with respect to $\mu$, defined by

$$Lf = \Delta_{H, \mu} f := \text{div}_\mu \sharp^H df.$$ 

For the special case when $H = T M$, $\Delta_{TM, \mu}$ is called the Witten-Laplacian or just the Laplacian if $\mu$ is the Riemannian volume density.

In what follows, we assume that $H$ is bracket-generating, meaning that the vector fields with values in $H$ and their iterated brackets span the entire tangent bundle. This is equivalent to assuming that $L = \Delta_{H, \mu}$ satisfies the strong Hörmander condition. The operator $L$ is essentially self-adjoint on smooth functions [14] and the main result will give a bound for the first non-zero eigenvalue of $\Delta_{H, \mu}$ using bounds on tensors.

For the rest of the paper, we will use the following notation. The manifold $M$ will be connected, compact and of dimension $m + n$, where $n$ denotes the rank of the horizontal bundle $H$. For any 2-tensor $\xi \in T^* M \otimes 2$, define $\text{tr}_H \xi(x, \cdot) = \xi(g_H^*)$. In other words, $\text{tr}_H \xi(x, \cdot)(x) = \sum_{i=1}^n \xi(v_i, v_i)$ where $v_1, \ldots, v_n$ is an orthonormal basis of $H_x$.

For a given volume density $\mu$ we will write $\|f\|_{L^2}$ for the corresponding $L^2$ metric. If $g$ is a possibly degenerate metric tensor on a vector bundle $E \rightarrow M$, we write $\|e\|_g = (e, e)_g^{1/2}$ for any $e \in E$, and $\|Z\|_{L^2(g)} := \|Z\|_{L_1} \|Z\|_{L^2}$ for any $Z \in \Gamma(E)$. Finally, we introduce the following operators on forms. If $T : \bigwedge^k TM \rightarrow TM$ is a
vector-valued $k$-form, define an operator $\nu_T$ such that

$$\nu_T \alpha = \alpha(T(\cdot)) \quad \text{if} \ \alpha \ \text{is a one-form,}$$

and extend its definition to arbitrary forms by requiring that it satisfies $\nu_T (\alpha \wedge \beta) = (\nu_T \alpha) \wedge \beta + (-1)^{k+\deg \alpha} \alpha \wedge \nu_T \beta$. Note that $\nu_T$ becomes the usual contraction when $T$ is a vector field, i.e. $k = 0$.

2.2. A Lichnerowicz Estimate. Let $(M, H, g_H)$ be a sub-Riemannian manifold. We introduce the following assumptions.

(A) There exists a complement $V$, i.e. a choice of subbundle satisfying $TM = H \oplus V$, such that $g_H$ and the corresponding projection $\text{pr}_H : TM \to H$ satisfy

$$(2.1) \quad (L_Z \text{pr}_H g_H)(X, X) = 0, \quad \text{for } Z \in \Gamma(V) \text{ and } X \in \Gamma(H),$$

with $L$ denoting the Lie derivative. In this case $V$ is called a metric preserving complement of $(H, g_H)$. 

(B) With respect $V$, define the curvature and the cocurvature of $H$, by

$$\mathcal{R}(X, Y) = \text{pr}_V \left[ \text{pr}_H X, \text{pr}_H Y \right] \quad \text{and} \quad \bar{\mathcal{R}}(X, Y) = \text{pr}_H \left[ \text{pr}_V X, \text{pr}_V Y \right],$$

respectively. We will assume that

$$\text{tr} \bar{\mathcal{R}}(X, \mathcal{R}(X, \cdot)) = 0.$$

This condition has appeared in [8], [9] and [7]. Note that if $V$ is integrable then $\mathcal{R} \equiv 0$, hence $(2.2)$ is always satisfied.

(C) A Riemannian metric $g$ is said to tame $g_H$ if $g|H = g_H$. Let $g$ be a taming metric for $g_H$ making $H$ and $V$ orthogonal. Define $g_V$ to be the restriction of $g$ to $V$. It will be assumed that

$$\text{tr}_{g_V}(L_X g)(\cdot, \cdot) = 0, \quad \text{whenever } X \in \Gamma(H).$$

Let $\nabla^V$ be any affine connection on $V$ compatible with $g_V$, and denote the Levi-Civita connection of $g$ by $\nabla^g$. Define the affine connection $\nabla$ on $TM$ by

$$\nabla_X Y = \text{pr}_V \nabla^g_{\text{pr}_H X} \text{pr}_H Y + \text{pr}_H \left[ \text{pr}_V X, \text{pr}_H Y \right] + \nabla^V_{\text{pr}_V Y}.$$ 

Let $T = T^V$ and $R^V$ denote the torsion and curvature tensor of $\nabla$. In what follows we introduce the tensors playing a role in the estimate of the spectral gap.

Define the vector valued one-form $B : TM \to TM$ by

$$B(v) = \text{tr}_H (\nabla^V_x T)(x, v) - \text{tr}_H T(x, T(x, v)).$$

Furthermore, we introduce the endomorphisms $\text{Ric}, \mathcal{W} : T^*M \to T^*M$ and the map $S : T^*M \to T^*M \otimes^2$ by

$$\text{Ric}(\alpha)(v) = \text{tr}_H R^V(x, v)\alpha(x),$$

$$\langle \mathcal{W}(\alpha), \beta \rangle_{g^*} = \langle \nu_T \alpha, \nu_T \beta \rangle_{g^H} + \langle \nu_B \alpha, \beta \rangle_{g_V^*},$$

$$S(\alpha)(v, w) = \langle w, T(\text{pr}_H v, T^* \alpha) \rangle_{g^H}$$

for any $\alpha, \beta \in T^*M$ and $v, w \in TM$. We will use bounds on these tensors to obtain the estimate for the spectral gap.

**Theorem 2.1.** Assume that $(M, H, g_H)$ is a compact sub-Riemannian manifold of rank $n$ with a taming metric $g$ such that the orthogonal complement $V$ is metric
The above result is a generalization of a result found \[3\]. The latter result was
proved for the case when \(V\) is a Riemannian foliation. To see this, observe that
\(\rho_1>0\) and \(\kappa_j \geq 0\). Assume that
\[
\rho_1 \rho_2 - 4 \kappa_2^2 - 3 \kappa_1 \kappa_3 - 8 \kappa_2 \sqrt{\kappa_1 \kappa_3} > 0.
\]
Then we have the following estimate on the spectral gap
\[
-\lambda_1 \geq \left( \sqrt{\frac{\rho_1 \rho_2 - 4 \kappa_2^2 + \frac{n-1}{n} \rho_2 \kappa_3}{n-1} \rho_2 + 3 \kappa_1} + \left( \frac{4 \kappa_2 \sqrt{\kappa_1}}{n-1} \rho_2 + 3 \kappa_1 \right)^2 \right)^2 - \kappa_3,
\]
where \(\lambda_1\) is the first non-zero eigenvalue of \(\Delta_{H,\mu}\).

Remark 2.2. (a) If \(H = TM\), we can choose \(\kappa_3 = 0\) and \(\rho_2 = \infty\) to obtain the
original Lichnerowicz estimate.

(b) One can interpret a complement \(V\) satisfying (2.1) as a generalization of vertical
bundles on sub-Riemannian manifolds coming from submersions. If \(\pi: M \to N\)
is a surjective submersion into a Riemannian manifold \((N, g_N)\), \(V = \ker d\pi\) and
\(H\) satisfy \(TM = H \oplus V\), then we can define a sub-Riemannian metric by
\(g_H = \pi^* g_N|H\). In this case, \(V\) is a metric preserving complement and \(\text{Ric}\) is
the pullback of the Ricci curvature on \(N\), see [6, Prop 3.4]. More generally, if
\(V\) is an integrable and metric preserving complement, then the corresponding
foliation of \(V\) is a Riemannian foliation.

(c) The assumption \(\rho_2 > 0\) is equivalent to the assumption \(H + [H, H] = TM\),
i.e. that \(\mathcal{R}\) is surjective on \(V\). To see this, observe that \(T(v, w) = -\mathcal{R}(v, w)\)
whenever \(v, w \in H\). Hence, \(|v_T df|_{g_H}^2 = |v \mathcal{R} df|_{g_H}^2\). It follows that the existence
of a covector \(\alpha\) such that \(v_T \alpha \equiv 0\) while satisfying \(|\alpha|_{g_V}^2 \neq 0\), implies
\(\rho_2 = 0\). Conversely, if \(\mathcal{R}\) is surjective, then \(g^\mathcal{R}_V(\alpha, \beta) := (v \mathcal{R} \alpha, v \mathcal{R} \beta)_{g_H}\) is a
cometric degenerate on \(H\). Hence, it induces an inner product on \(V\), and from
compactness, we get that \(|v|_{g_V}^2 \geq \rho_2 |v|_{g_V}^2\) for any \(v \in V\) and some \(\rho_2 > 0\).

(d) The above result is a generalization of a result found \[3\]. The latter result was
proved for the case when \(V\) is an integrable subbundle, and the corresponding
foliation of \(V\) is totally geodesic, which is equivalent to the requirement

\[
(\mathcal{L}_X g)(Z, Z) = 0,
\]
for any \(X \in \Gamma(H)\) and \(Z \in \Gamma(V)\). If this assumption holds, we can choose \(\nabla\)
as the Bott connection

\[
\nabla_X Y = \text{pr}_H \nabla_{\text{pr}_H X} \text{pr}_H Y + \text{pr}_H [\text{pr}_V X, \text{pr}_H Y]
+ \text{pr}_V \nabla_{\text{pr}_V X} \text{pr}_V Y + \text{pr}_V [\text{pr}_H X, \text{pr}_V Y].
\]
Under the additional assumptions of the Yang-Mills condition, i.e. that \(\text{tr}_H(\nabla_X T)(x, \cdot) = 0\), we get that \(B = 0\), \(\mathcal{F} = 0\), and the result simplifies to
\[
-\lambda_1 \geq \frac{\rho_1 \rho_2}{n-1} \rho_2 + 3 \kappa_1.
\]
(e) If the condition (2.9) holds, we can still use the connection (2.10), even without the Yang-Mills assumption or even assuming $V$ integrable. From the fact that $T(H, V) = 0$ and assumption (2.2), we get $\mathcal{H} = 0$ and hence the estimate

$$-\lambda_1 \geq \left( \frac{\rho_1\rho_2 - 4\kappa_2^2}{2\kappa_1 - \rho_2} + \frac{4\kappa_2\sqrt{\kappa_1}}{2\kappa_1 - \rho_2 + 3\kappa_1} \right)^2 - \left( \frac{4\kappa_2\sqrt{\kappa_1}}{2\kappa_1 - \rho_2 + 3\kappa_1} \right)^2.$$ 

In this case, the result is non-trivial for $\rho_1\rho_2 - 4\kappa_2^2 > 0$. The latter assumption is analogous to positive Ricci curvature in the sense that if $g$ is a complete Riemannian metric and this inequality holds, then $M$ is compact [7, Prop 5.3].

(f) Note that the result is invariant under scaling of the vertical part of the metric. Consider a variation of the metric $g_c = g_H \oplus \frac{1}{\mu} g_V$. If $g$ has volume density $\mu$, then clearly the density of $g_c$ is $\varepsilon^{-m/2} \mu$. Hence, the sub-Laplacian with respect to $g$ and $g_c$ coincides. Applying this scaled metric to Theorem 2.1, the parameters $\rho_1, \rho_2, \kappa_1, \kappa_2$ and $\kappa_3$ are respectively replaced by $\rho_1, \varepsilon^{-1} \rho_2, \varepsilon^{-1} \kappa_1, \varepsilon^{-1/2} \kappa_2$ and $\kappa_3$, leaving us with the same result for the spectral gap.

2.3. Properties of the connection. The proof of the main theorem will be divided into several intermediate steps. We begin by proving some general properties of the connections used in this setting, given assumptions (A), (B) and (C).

**Lemma 2.3.** Let $(M, H, g_H)$ be a sub-Riemannian manifold and let $g$ be a taming metric such that the vertical bundle $V$ is metric preserving. If $\nabla$ is an affine connection defined as in (2.4), then it satisfies

(i) $\nabla g_H^\ast = 0$ and $\nabla g = 0$.

Furthermore, we have that its torsion $T$ satisfies:

(ii) $\text{pr}_H T(v, w) = 0$ whenever $v \in H$.

(iii) $T(v, w) = -\mathcal{R}(v, w)$ whenever $v, w \in H$.

(iv) $\text{pr}_H T(v, w) = -\mathcal{R}(v, w)$ whenever $v, w \in V$.

(v) If $B$ is defined as in (2.5), then it takes its values in $V$.

(vi) $\text{tr} T(v, \cdot) = 0$ whenever $v \in H$.

**Proof.** Properties (i) to (iv) follows from the definition of $\nabla$ and assumption (A). The result (v) follows from (iii) and assumption (B). For the proof of (vi), let $X$ be any section of $H$. Let $X_1, \ldots, X_n$ and $Z_1, \ldots, Z_m$ be local orthonormal bases of respectively $H$ and $V$. Then we have that

$$\text{tr} T(X, \cdot) = \sum_{j=1}^n \langle T(X, X_j), X_j \rangle_g + \sum_{k=1}^m \langle T(X, Z_k), Z_k \rangle_g = \sum_{k=1}^m \langle T(X, Z_k), Z_k \rangle_g$$

$$= \sum_{k=1}^m \langle Z_k, \nabla_X Z_k - \nabla_Z X - [X, Z_k] \rangle_g$$

$$= -\sum_{k=1}^m \langle Z_k, [X, Z_k] \rangle_g = \frac{1}{2} \text{tr}_V (\mathcal{L}_X g)(\cdot, \cdot) = 0,$$

where the last equality follows from assumption (C). \hfill \Box

**Lemma 2.4.** Let $\text{Ric}$ be defined as in equation (2.6) with respect to $\nabla$. Then

$$\langle \text{Ric}(\alpha), \beta \rangle_{g^\ast} = \langle \text{Ric}(\alpha), \beta \rangle_{g_H^\ast}, \quad \alpha, \beta \in T^\ast M.$$ 

Additionally, we have that $\text{Ric}$ is symmetric.
Proof. Since $\nabla$ is compatible with $g$, we get $\langle R^\nabla(\cdot, \cdot)v, v \rangle_g = 0$, for all $v \in TM$. Using the first Bianchi identity for affine connections with torsion, when $X, Y \in \Gamma(H)$ and $Z \in \Gamma(V)$,

$$\langle R^\nabla(X, Z)Y + R^\nabla(Y, X)Z + R^\nabla(Z, Y)X, X \rangle_g = \langle R^\nabla(X, Z)Y, X \rangle_g$$

$$= \langle -T(X, T(Z, Y)) - T(Y, T(X, Z)) - T(Z, T(Y, X)), X \rangle_g$$

$$+ \langle (\nabla_X T)(Z, Y) + (\nabla_Y T)(X, Z) + (\nabla_Z T)(Y, X), X \rangle_g = -\langle X, R(Z, R(Y, X)) \rangle_g.$$ 

Hence

$$\langle \text{Ric}(\alpha), \beta \rangle_g = \text{tr}_H \langle R^\nabla(x, \sharp^H \beta \sharp^H \alpha, x) \rangle_g$$

$$= \text{tr}_H \langle R^\nabla(x, \sharp^H \beta \sharp^H \alpha, x) \rangle_g - \text{tr} R(\sharp^V \beta, R(\sharp^H \alpha, \cdot)) = \langle \text{Ric}(\alpha), \beta \rangle_{g^*_H},$$

by assumption (C). Another application of the first Bianchi identity shows that $\langle R^\nabla(X, Y)Z, X \rangle_{g^*_H} = \langle R^\nabla(X, Z)Y, X \rangle_{g^*_H}$ when $X, Y, Z \in \Gamma(H)$, giving us that Ric is symmetric.

2.4. Necessary identities. Fix a Riemannian metric $g$ taming $g_H$. For any connection $\nabla$, define the operators on tensors by $D[X, Y] = b X \otimes \nabla_Y$ and $L[X, Y] = \nabla^2_{X,Y} := \nabla_X \nabla_Y - \nabla_{\nabla_X Y}$. Extend these operators to arbitrary sections of $TM \otimes 2$ by $C^\infty(M)$-linearity, and define $L = L[g^*_H]$ and $D = D[g^*_H]$. The following results are found in [3].

**Lemma 2.5.** Assume that $\nabla$ is compatible with $g$ and $g^*_H$.

(a) We have $L = -D^* D$ on tensors if and only if

$$\text{tr} T(v, \cdot) = 0, \text{ for any } v \in H. \tag{2.11}$$

Here the dual is with respect to the $L^2$-inner product on tensors induced by $g$. Also, in this case $D^* = -\text{tr}_H \iota_X \nabla_X$. In particular, this is true for the connection $\nabla$ in (2.4) given assumption (C).

(b) If $\nabla$ satisfies (2.11), then for any $f \in C^\infty(M)$, we have $L f = \Delta_{H, \mu} f$ with $\mu$ being the volume density of $g$, and furthermore

$$\text{Ldf} - dL f = -2 \mathcal{D} df +\mathcal{A}(df), \tag{2.12}$$

$$\mathcal{A}(\alpha) := \text{Ric}(\alpha) + \iota_{2D^*T + B} \alpha,$$

$$\mathcal{D} \alpha := \text{tr}_H \nabla_X \alpha(T(x, \cdot)).$$

Notice the following identity $\mathcal{D} \alpha = -D^* \iota_T \alpha + \iota_{D^*T} \alpha$, while $\text{pr}^*_H \mathcal{D} \alpha = S^* D \alpha$.

We will use the Weitzenböck-type formula (2.12) to find the spectral gap. However, we first need the following equalities.

**Lemma 2.6.** Let $f \in C^\infty(M)$ be an arbitrary function.

(a) We have the relation

$$\langle \iota_T df, Ddf \rangle_{g^*_H} = \frac{1}{2} |\iota_T df|_{g^*_H^2}^2.$$ 

Hence, in particular, $\| (D + \iota_T) df \|_{L^2(g^*_H \otimes 2)}^2 = \| D df \|_{L^2(g^*_H \otimes 2)}^2$. Furthermore,

$$\frac{1}{2} \| \iota_T df \|_{L^2(g^*_H \otimes 2)}^2 - \langle \iota_B df, df \rangle_{L^2(g^*_H \otimes 2)} = \langle (D + \iota_T) df, S(df) \rangle_{L^2(g^*_H \otimes 2)}. \tag{2.13}$$

(b) We have the relation

$$-\langle dLf, df \rangle_{L^2(g^*_V)} = \| (D + \iota_T) df \|_{L^2(g^*_H \otimes g^*_V)}^2 - \langle \mathcal{A} df, df \rangle_{L^2(g^*_V)}.$$
(c) Let $(Df)^*$ denote the symmetrized tensor $2(Df)^*(v, w) = \nabla_v df(w) + \nabla_w df(v)$. Then

$$\|Lf\|_{L^2}^2 = \|(Df)^*\|_{L^2(g^*_{H})}^2 - \frac{1}{2} \langle \text{ric}(df), df \rangle_{L^2(g^*_{H})}$$

$$+ \langle \text{ric}(df), df \rangle_{L^2(g^*_{H})} - \frac{3}{2} \langle (D + t\tau)df, S(df) \rangle_{L^2(g^*_{H})}.$$ 

**Proof.** Observe that since $\nabla$ preserves $H$, we have that $D\text{pr}^*_H\alpha = (\text{pr}^*_H \otimes \text{pr}^*_H)D\alpha$. Hence, for any one-form $\alpha$ and two-tensor $\xi$

$$\langle \alpha, D^\ast \xi \rangle_{L^2(g^*_{H})} = \langle \text{pr}^*_H \alpha, D^\ast \xi \rangle_{L^2(g^*_{H})} = \langle D\alpha, \xi \rangle_{L^2(g^*_{H})}.$$ 

Similarly, $\langle \alpha, D^\ast \xi \rangle_{L^2(g^*_{H})} = \langle D\alpha, \xi \rangle_{L^2(g^*_{H})}.$

(a) If $X_1, \ldots, X_n$ is a local orthonormal basis of $H$, then

$$\langle Ddf, \tau Tdf \rangle_{g^*_{H}} = \sum_{r,s=1}^n df(T(X_r, X_s))\nabla_{X_r, X_s}^2 f$$

$$= \frac{1}{2} \sum_{r,s=1}^n df(T(X_r, X_s))(\nabla_{X_r}^2 f - \nabla_{X_s}^2 f)$$

$$= \frac{1}{2} \sum_{r,s=1}^n df(T(X_r, X_s))^2 = -\frac{1}{2} \|\tau Tdf\|_{g^*_{H}}^2.$$ 

We can use this identity to find

$$\frac{1}{2} \|\tau Tdf\|_{L^2(g^*_{H})}^2 = -\langle D^\ast \tau Tdf, df \rangle_{L^2(g^*_{H})}$$

$$= -\langle \tau Tdf, df \rangle_{L^2(g^*_{H})} + \langle Ddf, S(df) \rangle_{L^2(g^*_{H})}$$

$$= \langle Ddf, df \rangle_{L^2(g^*_{H})} + \langle (D + \tau T)df, S(df) \rangle_{L^2(g^*_{H})}.$$ 

(b) If we evaluate (2.12) with $D^\ast df$, we obtain

$$\langle Ldf, df \rangle_{L^2(g^*_{H})} - \langle Ddf, df \rangle_{L^2(g^*_{H})} = -\|Ddf\|_{L^2(g^*_{H})}^2 - \langle Ddf, df \rangle_{L^2(g^*_{H})}$$

$$= 2\|\tau Tdf, Ddf \rangle_{L^2(g^*_{H})} + \langle Ddf, df \rangle_{L^2(g^*_{H})}.$$ 

The result now follows from completing the square.

(c) By evaluating (2.12) with $D^\ast df$ and doing similar computations as in (b), we obtain

$$\|Lf\|_{L^2}^2 = \langle Ddf, df \rangle_{L^2(g^*_{H})}$$

$$= \|D + \tau T\rangle_{L^2(g^*_{H})}^2 - \|\tau Tdf\|_{L^2(g^*_{H})}^2$$

$$+ \langle \text{ric}(df), df \rangle_{L^2(g^*_{H})}.$$ 

From (a), we know that $\|D + \tau T\rangle_{L^2(g^*_{H})}^2 = \|Ddf\|_{L^2(g^*_{H})}^2$ and furthermore,

$$\|Ddf\|_{g^*_{H}}^2 = \langle Ddf, df \rangle_{g^*_{H}} + \frac{1}{4} \sum_{r,s=1}^n (\nabla_{X_r}^2 f - \nabla_{X_s}^2 f)^2$$

$$= \langle Ddf, df \rangle_{g^*_{H}} + \frac{1}{4} \sum_{r,s=1}^n (df(T(X_r, X_s))^2 = \langle Ddf, df \rangle_{g^*_{H}} + \frac{1}{4} \|\tau Tdf\|_{g^*_{H}}^2.$$ 

Applying (2.13) to

$$\|Lf\|_{L^2}^2 = \langle Ddf, df \rangle_{L^2(g^*_{H})} - \frac{3}{4} \|\tau Tdf\|_{L^2(g^*_{H})}^2 + \langle \text{ric}(df), df \rangle_{L^2(g^*_{H})}$$
we get the result.

\[ \Box \]

2.5. **Proof of Theorem 2.1** Introduce the eigenfunction \( f \) with eigenvalue \( \lambda < 0 \). We normalize \( f \) such that \( \| f \|_{L^2} = 1 \) and hence \( \| df \|^2_{L^2(g^*_H)} = -\lambda \). Assume the bounds of Theorem 2.1

**Lemma 2.7.** We will have the following bounds involving the vertical part of the gradient;

\[
\rho_2 \| df \|_{L^2(g^*_H)} \leq 2 \left( \sqrt{\kappa_1(\kappa_3 - \lambda) + 2\kappa_2} \right) \sqrt{-\lambda}.
\]

\[
|\langle (D + \tau)df, S(df) \rangle_{L^2(g^*_H)}| \leq -\frac{2\lambda}{\rho_2} \sqrt{\kappa_1(\kappa_3 - \lambda)} \left( \sqrt{\kappa_1(\kappa_3 - \lambda) + 2\kappa_2} \right).
\]

**Proof.** Using Lemma 2.3(b), we obtain the equality

\[ \langle \mathcal{W} df, df \rangle_{L^2(g^*_H)} = 2\lambda \| df \|^2_{L^2(g^*_H)}, \]

which implies that \( \| (D + \tau)df \|^2_{L^2(g^*_H)} \leq (\kappa_3 - \lambda) \| df \|^2_{L^2(g^*_H)} \). The bounds then follow from the estimate

\[
\frac{\rho_2}{2} \| df \|^2_{L^2(g^*_H)} - 2\kappa_2 \| df \|_{L^2(g^*_H)} \| df \|_{L^2(g^*_H)} \leq \frac{1}{2} \| (D + \tau)df \|^2_{L^2(g^*_H)} - \langle \tau df, df \rangle_{L^2(g^*_H)}
\]

\[
= \langle (D + \tau)df, S(df) \rangle_{L^2(g^*_H)} \leq \sqrt{\kappa_1(\kappa_3 - \lambda)} \| df \|_{L^2(g^*_H)} \| df \|_{L^2(g^*_H)}.
\]

\[ \Box \]

We are now ready to complete the proof of Theorem 2.1. Observe from Lemma 2.6(c) that

\[
\lambda^2 = \| Lf \|^2_{L^2} = \| (Ddf)^* \|^2_{L^2(g^*_H)} - \frac{1}{2} \langle \tau df, df \rangle_{L^2(g^*_H)}
\]

\[
+ \langle \text{Ric}(df), df \rangle_{L^2(g^*_H)} - \frac{3}{2} \langle (D + \tau)df, S(df) \rangle_{L^2(g^*_H)}
\]

\[
\geq \frac{1}{n} \| Lf \|^2_{L^2} - \kappa_2 \| df \|_{L^2(g^*_H)} \| df \|_{L^2(g^*_H)} + \rho_1 \| df \|^2_{L^2(g^*_H)}
\]

\[
+ \frac{3\lambda}{2 \rho_2} \sqrt{\kappa_1(\kappa_3 - \lambda)} \left( \sqrt{\kappa_1(\kappa_3 - \lambda) + 2\kappa_2} \right)
\]

\[
\geq \frac{\lambda^2}{n} + \frac{2\kappa_2 \lambda}{\rho_2} \left( \sqrt{\kappa_1(\kappa_3 - \lambda) + 2\kappa_2} \right) - \rho_1 \lambda
\]

\[
+ \frac{3\lambda}{\rho_2} \sqrt{\kappa_1(\kappa_3 - \lambda)} \left( \sqrt{\kappa_1(\kappa_3 - \lambda) + 2\kappa_2} \right).
\]

This gives us the inequality

\[
-\frac{n-1}{n} \rho_2 \lambda \geq \rho_1 \rho_2 - \left( 3\sqrt{\kappa_1(\kappa_3 - \lambda) + 2\kappa_2} \right) \left( \sqrt{\kappa_1(\kappa_3 - \lambda) + 2\kappa_2} \right).
\]

Define \( s = \sqrt{\kappa_1(\kappa_3 - \lambda)} \). Then

\[
\frac{n-1}{n} \left( \frac{\rho_2}{\kappa_1} \right)^2 \left( \rho_1 \rho_2 - 3s^2 - 8s\kappa_2 - 4\kappa_2^2 \right) \geq 0.
\]

or

\[
\left( \frac{n-1}{n} \frac{\rho_2}{\kappa_1} + 3 \right) s^2 + 8s\kappa_2 - \left( \rho_1 \rho_2 - 4\kappa_2^2 + \frac{n-1}{n} \rho_2 \kappa_3 \right) \geq 0.
\]
From this inequality, we have
\[ s \geq \sqrt{\frac{\rho_1 \rho_2 - 4\kappa_2^2 + \frac{2}{n} \rho_2 \kappa_3}{\frac{n-1}{n} \rho_2 + 3}} + \left( \frac{4\kappa_2}{\frac{n-1}{n} \rho_2 + 3} \right)^2 - \frac{4\kappa_2}{\frac{n-1}{n} \rho_2 + 3}. \]
The result follows.

### 3. Examples

#### 3.1. Example with non-integrable orthogonal complement.
Consider the Lie group $\text{SO}(4)$ with Lie algebra $\mathfrak{so}(4)$, the latter consisting of all $4 \times 4$ skew-symmetric matrices. This Lie algebra is spanned by matrices $B^{ij} = e_i e_j - e_j e_i$, where $e_1, e_2, e_3, e_4$ is the standard basis of $\mathbb{R}^4$. By abuse of notation, we will use the same symbol to denote an element of the Lie algebra and the vector field on $\text{SO}(4)$ obtained by left translation. Consider the horizontal bundle $H = \text{span}\{X_1, X_2, X_3, X_4\} := \text{span}\{B^{12}, B^{14}, B^{24}, B^{34}\}$, and define a metric $g_H$ such that $X_1, X_2, X_3, X_4$ is an orthonormal basis. We will show that the first eigenvalue $\lambda_1$ of the operator $L = X_1^2 + X_2^2 + X_3^2 + X_4^2$ is bounded by
\[ \lambda_1 \leq -\frac{8}{51}. \]
Define an inner product on the Lie algebra by $\langle A, B \rangle_{\mathfrak{so}(4)} = -\frac{1}{2} \text{tr} AB$ and extend this to a Riemannian metric $g$ by left translation. Observe that $g$ is a bi-invariant metric on $\text{SO}(4)$ which tames $g_H$. Furthermore, if $\mu$ is the volume form of $g$, then $L = \Delta_{H, \mu}$. The vertical bundle $V = \text{span}\{Z_1, Z_2\} := \text{span}\{B^{13}, B^{23}\}$, is not integrable, but properties [A], [B] and [C] are still satisfied. From bi-invariance, we also have that (2.9) holds, so we can use the Bott connection (2.10) from Remark 2.2. The bracket relations, connection and torsion are given by

| $[X, Y]$ | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $Z_1$ | $Z_2$ | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $Z_1$ | $Z_2$ |
|----------|------|------|------|------|------|------|------|------|------|------|------|------|
| $X_1$    | 0    | $-X_3$ | $X_2$ | 0    | $-Z_2$ | $Z_1$ | 0    | $-X_3$ | $X_2$ | 0    | $-Z_2$ | $Z_1$ |
| $X_2$    | $X_3$ | 0    | $-X_1$ | $-Z_1$ | 0    | $X_4$ | $X_3$ | 0    | $-X_1$ | $-Z_1$ | 0    | $X_4$ |
| $X_3$    | $-X_2$ | $X_4$ | 0    | $-Z_2$ | 0    | $X_1$ | $-X_2$ | $X_4$ | 0    | $-Z_2$ | 0    | $X_1$ |
| $X_4$    | $X_1$ | 0    | $-X_3$ | 0    | $-Z_1$ | $Z_2$ | 0    | $X_1$ | 0    | $-X_3$ | 0    | $-Z_1$ |
| $Z_1$    | $Z_2$ | 0    | $-X_4$ | 0    | $X_2$ | 0    | $Z_1$ | 0    | $-X_4$ | 0    | $X_2$ | 0    |
| $Z_2$    | $-Z_1$ | 0    | $-X_3$ | $X_3$ | 0    | $Z_2$ | $-Z_1$ | 0    | $-X_3$ | $X_3$ | 0    | $Z_2$ |

Additionally, Ric is given by

| Ric$(\nabla X)(Y)$ | $X_1$ | $X_2$ | $X_3$ | $X_4$ |
|-------------------|------|------|------|------|
| $X_1$             | 1/2  | 0    | 0    | 0    |
| $X_2$             | 0    | 3/2  | 0    | 0    |
| $X_3$             | 0    | 0    | 3/2  | 0    |
| $X_4$             | 0    | 0    | 0    | 2    |
hence \( \rho_1 = 1/2 \). Using the tables above we have \( B = 0 \) and \( \mathcal{W} = 0 \), hence \( \kappa_2 = \kappa_3 = 0 \). The last two constants can be set to \( \rho_2 = \kappa_1 = 2 \), since
\[
|\kappa T\alpha|_{g_H}^2 \otimes 2 = 2(\alpha(Z_1)^2 + \alpha(Z_2)^2) = 2|\alpha|_{g_H}^2,
\]
\[
|S(\alpha)|_{g_H}^2 \otimes g_{H} = \alpha(X_2)^2 + \alpha(X_3)^2 + 2\alpha(X_4)^2 \leq 2|\alpha|_{g_H}^2.
\]
Using Theorem 2.1 with the constants \( \rho_1 = 1/2 \), \( \rho_2 = \kappa_1 = 2 \) and \( \kappa_2 = \kappa_3 = 0 \) we obtain the bound.

3.2. Example with conformal change of metric and the Yang-Mills condition. For the second example consider the Lie group
\[
\text{SU}(2) = \left\{ a \in M_{2 \times 2}(\mathbb{C}) : a = \begin{pmatrix} z_1 & z_2 \\ z_2 & \bar{z}_1 \end{pmatrix}, \det(a) = 1 \right\}.
\]
Its Lie algebra \( \mathfrak{su}(2) \) is spanned by the matrices
\[
X = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad Y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad Z = \frac{1}{\sqrt{2}} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.
\]
Let \( g_H \) be a sub-Riemannian metric on the subspace spanned by \( X \) and \( Y \) making the vector fields orthonormal. For any smooth function \( f \) satisfying \( Z(f) = 0 \), define a new sub-Riemannian metric by \( g_{f,H} = e^{2f}g_H \). Then \( X_f = e^{-f}X \) and \( Y_f = e^{-f}Y \) is an orthonormal frame for \( g_{f,H} \). Let \( g_f \) be the taming metric making \( X_f, Y_f, Z \) orthonormal. Define the connection \( \nabla \) as in equation (2.10) with respect to the taming metric \( g_f \). In this setting we have the sub-Laplacian
\[
L_f = e^{-2f}(X^2 + Y^2) =: e^{-2f}L.
\]
The bracket relations, connection and torsion are given by
\[
[X_f, Y_f] = e^{-2f}Z - X_f(f)Y_f + Y_f(f)X_f, \quad [Y_f, Z] = X_f, \quad [Z, X_f] = Y_f,
\]
\[
\begin{array}{c|c|c|c|c|c|c|c}
\nabla_X Y & X_f & Y_f & Z & T(X,Y) & X_f & Y_f & Z \\
\hline
X_f & -(Y_f)Y_f & (Y_f)X_f & 0 & X_f & 0 & -e^{-2f}Z & 0 \\
Y_f & (X_f)Y_f & -(X_f)X_f & 0 & Y_f & e^{-2f}Z & 0 & 0 \\
Z & Y_f & -X_f & 0 & Z & 0 & 0 & 0 \\
\end{array}
\]
The constant \( \rho_1 \) is
\[
\rho_1 = \inf_{a \in \text{SU}(2)} e^{-2f(a)(1 - Lf(a))}
\]
since the only non-zero components of the curvature are
\[
\langle R(Y_f, X_f)X_f, Y_f \rangle_{g_f} = \langle R(X_f, Y_f)Y_f, X_f \rangle_{g_f} = e^{-2f}(1 - Lf).
\]
Observe that
\[
\mathcal{W} = 0, \quad |\kappa T\alpha|_{g_f, H}^2 = 2e^{-4f}|\alpha|_{g_H}^2, \quad \text{and} \quad |S(\alpha)|_{g_f, H}^2 \otimes g_{H} = e^{-4f}|\alpha|_{g_H}^2.
\]
Hence we can choose
\[
\rho_2 = 2 \quad \inf_{a \in \text{SU}(2)} e^{-4f(a)}, \quad \kappa_1 = \sup_{a \in \text{SU}(2)} e^{-4f(a)}.
\]
Finally, we have
\[
B : X_f \mapsto -2e^{-2f}(Y_f)Z, \quad Y_f \mapsto 2e^{-2f}(X_f)Z,
\]
so
\[
\langle \kappa_B \alpha, \alpha \rangle_{g_f, H} = 2e^{-2f} \alpha(Z)(-(Y_f)X_f + (X_f)Y_f, Y_f, H) \alpha)_{g_f, H}
\leq 2e^{-2f} \|df\|_{g_f, H}^2 |\alpha|_{g_H}^2 |\alpha|_{g_H},
\]
and we can choose
\[ \kappa_2 = \sup_{a \in SU(2)} e^{-3f} |df|_{g_H}. \]
Inserting this in the formula of Theorem 2.1, we have the spectral gap estimate.

To give a more concrete formula, define \( m = \inf_{a \in SU(2)} f(a) \) and \( m + M = \sup_{a \in SU(2)} f(a) \). Using the inequalities
\[ \rho_1 \leq e^{-2(m+M)(1 - \|Lf\|_{L^{\infty}})} \quad \text{and} \quad \kappa_2 \leq e^{-3m}\|df\|_{L^{\infty}(g_H)}, \]
we obtain
\[ -\lambda_1 \geq e^{-2m} \left( \sqrt{\frac{2e^{-6M(1-\|Lf\|_{L^{\infty}})}-4\|df\|_{L^{\infty}(g_H)}^2}{e^{-4m+3}}} + \left( \frac{4\|df\|_{L^{\infty}(g_H)}^2}{e^{-4m+3}} \right)^2 - \frac{8c}{e^{-4c+3}} \right)^2. \]

For the special case
\[ f(a) = c|z_1|^2, \quad c > 0, \]
we have \( m = 0 \) and \( M = c \). Furthermore, we have
\[ X(f) = \frac{ci}{\sqrt{2}}(\bar{z}_2 z_1 - \bar{z}_1 z_2) = \sqrt{2} \text{Im}(z_1 \bar{z}_2), \quad Y(f) = \frac{c}{\sqrt{2}}(\bar{z}_2 z_1 + \bar{z}_1 z_2) = \sqrt{2} \text{Re}(z_1 \bar{z}_2) \]
and
\[ X(X(f)) = Y(Y(f)) = c(|z_2|^2 - |z_1|^2). \]
Hence
\[ |df|_{g_H}^2 = 2c^2|z_1|^2|z_2|^2, \quad Lf = 2c(|z_2|^2 - |z_1|^2), \]
so \( \|df\|_{L^{\infty}(g_H)} = \|Lf\|_{L^{\infty}} = 2c \). In conclusion
\[ -\lambda_1 \geq \left( \sqrt{\frac{2e^{-6c(1-2c)} - 16c^2}{e^{-4c} + 3}} + \left( \frac{8c}{e^{-4c} + 3} \right)^2 - \frac{8c}{e^{-4c} + 3} \right)^2. \]
This estimate is non-trivial whenever \( e^{-6c(1-2c)} > 8c^2 \). In particular, this is true when \( c \in [0, 1.17139] \).

### 3.3. Example with a non-totally geodesic foliation.
Consider the Lie group \( SU(2) \times SU(2) \) with its Lie algebra \( su(2) \times su(2) \). Let \( X_j, Y_j, Z_j, j = 1, 2 \) be the matrices given in the previous example for each copy. Define a new frame by
\[ X^\pm = X_1 \pm X_2, \quad Y^\pm = Y_1 \pm Y_2, \quad Z^\pm = Z_1 \pm Z_2. \]
For any real number \( c \in \mathbb{R} \), define
\[ X^c = X^- + cX^+. \]
Let \( H^c \) be the span of \( X^c, Y^- \) and \( Z^- \), with the corresponding metric \( g_{H^c} \) making the vector fields \( X^c, Y^- \) and \( Z^- \) orthonormal. Define the taming metric \( g_c \) by the vector fields \( X^c, Y^-, Z^-, X^+, Y^+ \) and \( Z^+ \) forming an orthonormal frame. Denote by \( \mu \) the volume density of \( g_c \), and note that it is independent of \( c \). The sub-Laplacian then becomes
\[ L_c = \Delta_{H^c, \mu} = (X^c)^2 + (Y^-)^2 + (Z^-)^2. \]
For this basis the bracket relations become

\[
\begin{array}{c|cccccc}
[A,B] & X^c & Y^- & Z^- & X^+ & Y^+ & Z^+ \\
\hline
X^c & 0 & Z^+ + cZ^- & -Y^+ - cY^- & 0 & Z^- + cZ^+ & -Y^- - cY^+
\end{array}
\]

\[
\begin{array}{c|cccccc}
Y^- & -Z^+ - cZ^- & 0 & X^+ & -Z^- & 0 & X^+ - cX^+
\end{array}
\]

\[
\begin{array}{c|cccccc}
Z^- & Y^+ + eY^- & 0 & Y^- & eX^+ - X^c & 0 & X^+
\end{array}
\]

\[
\begin{array}{c|cccccc}
X^+ & 0 & Z^- & -Y^- & 0 & -Z^+ & -Y^+
\end{array}
\]

\[
\begin{array}{c|cccccc}
Y^+ & -Z^- - cZ^+ & 0 & X^- - cX^+ & Z^+ & 0 & X^+
\end{array}
\]

\[
\begin{array}{c|cccccc}
Z^+ & Y^- + eY^+ & cX^+ - X^c & 0 & Y^- & -X^+ & 0
\end{array}
\]

The complement \( V = \text{span}\{X^+, Y^+, Z^+\} \) is integrable and metric preserving, hence assumption \( (A) \) and \( (B) \) are satisfied. It is also straightforward to check that \( \text{tr}_{H^c}(L_A g_V)(x, x) = 0 \) whenever \( A \) is vertical. Note that we can not use the Bott connection for \( c \neq 0 \) since

\[
(L_{Y^-} \text{pr}_V^* g_V)(r X^+ + s Z^+, r X^+ + s Z^+) = -crs, \quad r, s \in \mathbb{R}^n.
\]

Define the connection \( \nabla \) by

\[
\nabla_X Y = \text{pr}_{H^c} \nabla_{\text{pr}_{H^c} X} \text{pr}_{H^c} Y + \text{pr}_{H^c} [\text{pr}_V X, \text{pr}_{H^c} X] + \text{pr}_V \nabla_{\text{pr}_V X} \text{pr}_V Y,
\]

where \( \nabla' \) is defined by making \( X^+, Y^+ \) and \( Z^+ \) into a parallel frame. The covariant derivative and torsion are

\[
\begin{array}{c|cccccc}
\nabla_A B & X^c & Y^- & Z^- & X^+ & Y^+ & Z^+ \\
\hline
X^c & 0 & cX^- - cY^- & 0 & 0 & 0 & 0 \\
Y^- & 0 & 0 & 0 & 0 & 0 & 0 \\
Z^- & 0 & 0 & 0 & 0 & 0 & 0 \\
X^+ & 0 & 0 & 0 & 0 & 0 & 0 \\
Y^+ & Z^- & cZ^- & 0 & cX^+ & 0 & -Z^+ \\
Z^+ & Y^- & -X^- & 0 & 0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{c|cccccc}
T(A,B) & X^c & Y^- & Z^- & X^+ & Y^+ & Z^+ \\
\hline
X^c & 0 & -Z^+ & Y^- & 0 & -cZ^+ & cY^+
\end{array}
\]

\[
\begin{array}{c|cccccc}
Y^- & Z^+ & 0 & X^- & 0 & 0 & 0 \\
Z^- & -Y^+ & X^+ & 0 & 0 & -cX^+ & 0 \\
X^+ & 0 & 0 & 0 & 0 & 0 & Z^+ \\
Y^+ & cX^+ & 0 & cX^+ & -Z^+ & 0 & -X^+ \\
Z^+ & -cX^+ & -cX^+ & 0 & -Y^- & X^- & 0 \\
\end{array}
\]

The only non-zero curvature terms that affect the tensor Ric are given by

\[
1 = \langle R^V(Y^-, X^c)X^c, Y^- \rangle_{g_{H^c}} = \langle R^V(Z^-, X^c)X^c, Z^- \rangle_{g_{H^c}} = \langle R^V(X^c, Y^-)Y^-, X^c \rangle_{g_{H^c}}
\]

\[
= \langle R^V(Z^-, Y^-)Y^-, Z^- \rangle_{g_{H^c}} = \langle R^V(X^c, Z^-)Z^-, X^c \rangle_{g_{H^c}} = \langle R^V(Y^-, Z^-)Z^-, Y^- \rangle_{g_{H^c}}
\]

Hence \( \rho_1 = 2 \). Furthermore, \( |\text{tr}_A|_{g_{H^c}}^2 = 2|\alpha|_{g_V}^2 \), while \( |S(\alpha)|_{g_{H^c} \otimes g_V}^2 = 2|\alpha|_{g_{H^c}}^2 \), so we can choose \( \rho_2 = \kappa_1 = 2 \). Next, we have

\[
B : X^c \mapsto -2cX^+, \quad Y^- \mapsto 0, \quad Z^- \mapsto 0,
\]

and we can set \( \kappa_2 = |c| \). To calculate \( \kappa_3 \), we see that

\[
\alpha(\text{tr}_{H^c} T(x, T(x, z_{g_{H^c}} \alpha))) = -c^2 (\alpha(Y^+)^2 + \alpha(Z^+)^2),
\]

\[
|\text{tr}_A|_{g_{H^c} \otimes g_V}^2 = c^2 (2\alpha(X^+)^2 + \alpha(Y^+)^2 + \alpha(Z^+)^2),
\]

and

\[
\alpha((\text{tr}_{H^c} \nabla_X T)(x, z_{g_{H^c}} \alpha)) = 0.
\]

In conclusion, we have \( \langle \mathcal{W}(\alpha), \alpha \rangle_{g_V} = 2c^2 |\alpha|_{g_V}^2 \), leading to \( \kappa_3 = 2c^2 \). Using Theorem 2.1 we have that

\[
-\lambda_1 \geq \frac{2}{121} \left( \sqrt{33} + 25c^2 - 6|c| \right)^2 - 2c^2,
\]

This estimate on the spectral gap is non-trivial when \( |c| < \frac{1}{4} \sqrt{\frac{11}{123}} \).
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