The iterative solution of wave propagation in transverse magnetic mode for graded positive-negative

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Abstract. The iterative solution was used to obtain the electromagnetic wave propagation in transverse magnetic (TM) mode for a graded positive-negative refractive index. The graded graphs of negative permittivity and negative permeability were obtained in hyperbolic functions. By using hyperbolic function for permittivity and permeability in Maxwell equation and by separation variable, we obtained the electromagnetic differential equation. From the differential equation, we used the approachment using MacLaurin series to obtain the wave vector and magnetic fields equation. The distribution of the magnetic fields were given in graph visualization using Matlab software.

1. Introduction
A new class of artificial materials, namely a negative refractive index material (NIM) or so-called left-handed materials (LHM) has attracted the interest of many scientists. NIM or LHM is better known as a metamaterial. [1-2] The theoretical concept of metamaterial was brought in the first time, in 1968 by Vasilago. [3] And then, Pendry, et al brought the concept of metamaterial into the experiment in 1996 and in 1999 by research which regarding to negative permeability in material.[4-5] A year later, Pendry investigated about the perfect lens with negative refraction index material .[6] In the same year, Smith et al.[7] studied about the simultaneously of negative permeability and negative permittivity.

Metamaterial has unique properties with negative refractive index value in a certain wavelength range. The refraction index of a material depends on the permittivity and permeability. LHM has the negative value of permittivity and negative value of permeability ($\varepsilon < 0$ & $\mu < 0$), while for the common material in nature with the positive value of permittivity and positive value of permeability ($\varepsilon > 0$ & $\mu > 0$) is called right-handed material (RHM) [1]. Permittivity and permeability are the basic properties to know the material's response due to the presence of electromagnetic waves. [3] The electromagnetics wave propagation was mostly done through modeling by software. [8-9]

The electromagnetic waves equation in various permeability and permittivity profile can be reduced into the second order differential equation, like the solution of Schrodinger equation or Dirac equation. The second order differential equation of electromagnetic wave can be solved to obtain the fields distribution and wave vector using the various methods. Some methods to solve the second order differential equation, such us Nikiforov-Uvarov method [14-15], Supersymmetry Quantum Mechanics (SUSY QM) [16], the polynomial Romanovski method [17] and Asymptotic Iteration Methods (AIM) [18-21]. The solution for the case where the refraction index profile is a hyperbolic tangent function has been investigated by Dalarsson and Tassin [10]. The gradation profile are mostly
used in hyperbolic tangent [11-12]. In recently, Husein et al. investigated the solution of wave propagation in a graded interface positive-negative for the hyperbolic cosecant and cotangent functions of permittivity and permeability using AIM. [2-13]

In this research, we considered the wave propagation in transverse magnetic mode for a graded negative-positive refraction index. By using iterative solution which is called AIM, we obtained the fields equation and wave vector.

2. The fields equation
Maxwell equation for dielectric, are given as

\[- \nabla \times \vec{E} = i \omega \mu (\omega, y) \vec{H} , \quad \nabla \times \vec{H} = i \omega \varepsilon (\omega, y) \vec{E} \quad (1)\]

where \( \varepsilon (\omega, y) \) and \( \mu (\omega, y) \) are frequency and spatial (as functions of material thickness, \( y \)) dependent dielectric permittivity and magnetic permeability, respectively. [2,13] We let the \( yz \) plane as the plane of ray incident, so \( \frac{d}{dx} = 0 \). By considering the geometry of the material, we obtain a second order differential equation of electromagnetic waves in inhomogeneous materials on mode transverse magnetic (TM). Actually, in two-dimensional study, there are two independent modes i.e. TM mode, the group of fields: \( \{ E_\parallel, E_y, H_z \} \), \( \{ E_y, E_z, H_x \} \), \( \{ E_z, E_x, H_y \} \) and transverse electric (TE) mode, the group of fields: \( \{ H_x, H_y, E_z \} \), \( \{ H_y, H_z, E_x \} \), \( \{ H_z, H_x, E_y \} \). In principle, these two modes (TM and TE) are similar to each other, so we only need to do one calculation, e.g. the TM mode. By using the plane wave solution which is harmonic in time, is given

\[ H_x = H(y,z) e^{i\omega t} \quad (2) \]

For the group fields in TM mode which we use i.e. \( \{ E_y, E_z, H_x \} \). From equation (1) and equation (2) we obtain

\[ \frac{\partial^2 H_x}{\partial y^2} - \frac{\varepsilon'}{\varepsilon} \frac{\partial H_x}{\partial y} + \omega^2 \mu \varepsilon H_x + \frac{\partial^2 H_x}{\partial z^2} = 0 \quad (3) \]

and by using \( H_x = H(y,z) = H(y)H(z) \) in equation (3), we obtained

\[ \frac{\partial^2 H(z)}{\partial z^2} + k_z^2 H(z) = 0 \quad (4) \]

\[ \frac{\partial^2 H(y)}{\partial y^2} - \frac{\varepsilon'}{\varepsilon} \frac{\partial H(y)}{\partial y} + \left( \omega^2 \mu \varepsilon - k_z^2 \right) H(y) = 0 \quad (5) \]

where \( k_z \) is the wave vector in \( z \) direction. Equation (4) is the simple differential equation which it has the solution,

\[ H(z) = e^{i k_z z} \quad (6) \]

and, by using AIM we solved equation (5) with the graded interface permittivity and permeability.

3. Iterative method using Asymptotic Iteration Method
AIM is used to solve second order differential equation in terms: [18-21]

\[ f_n(o) - \lambda_n(o) f_n(o) - s_n(o) f_n(o) = 0 \quad (7) \]

where \( \lambda_n(o) \neq 0 \) and \( s_n(o) \) are coefficients of the differential equation and are well-defined functions as well as sufficiently differentiable. The solution of equation (7) can be obtained by using iteration of \( \lambda_j \) and \( s_j \):

\[ \lambda_j(o) = \lambda_{j-1} + \lambda_{j-1} \lambda_0 + s_{j-1} + s_{j-1} \lambda_{j-1}, j = 1,2,3, ... \quad (8) \]
Eigenvalues can be obtained using equation (9) below,
\[ \lambda_j(o) s_{j-1}(o) - \lambda_{j-1}(o) s_j(o) = 0 = \Delta_j, j = 1,2,3 \ldots \]  
(9)

For the eigenfunctions, following in [22] we use the Hermite differential equation. Hermite differential equation is given as,
\[ f''(o) - 2of'(o) + 2jf(o) = 0 \]  
(10)

Equation (10) has the general solution in polynomial Hermite:
\[ f_j(o) = (-1)^j e^{o^2} \frac{d^j}{ds^j}(e^{-o^2}); j = 0,1,2,3... \]  
(11)

From equation (11) we can get some the first polynomial Hermite functions are given as
\[ f_j(o) = \cdots \]  
(12)

4. Solution of the field equations
The permittivity and permeability equation are expressed as
\[ \mu(\omega, y) = -\mu_0 \mu_r \left( \sinh(\rho y) + \tanh(\rho y) \right) \]  
(13)

\[ \varepsilon(\omega, y) = -\varepsilon_0 \varepsilon_r \tanh(\rho y) \]  
(14)

with \( \rho \) is abruptness’s parameter describing the transition of wave propagation from RHM to LHM.. The graphs of permeability (Figure 1.a) and permittivity (Figure 1.b) vary according to the material thickness \( y \) where the interface between RHM and LHM is located at \( y=0 \).
By inserting equations (13-14) into equation (5) we obtained the electromagnetic wave equation as follows:

\[
\frac{\partial^2 H(y)}{\partial y^2} - \frac{\rho \text{sech}^2(\rho y)}{\text{tanh}(\rho y)} \frac{\partial H(y)}{\partial y} + \left( \omega^2 \mu_0 \mu \varepsilon_0 \varepsilon_r \right) \text{tanh}(\rho y) \left( \sinh(\rho y) + \text{tanh}(\rho y) \right) - k_0^2 \right) H(y) = 0
\]

(15)

by using \( c = \left( \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \right) \); \( \mu \varepsilon_r = n_0^2 \); \( k_0 = (\omega/c) \) where \( c \) is light velocity in vacuum, \( n_0 \) is the refraction index and \( k_0 \) is wave vector in vacuum in equation (15), we get

\[
\frac{\partial^2 H(y)}{\partial y^2} - \frac{\rho \text{sech}^2(\rho y)}{\text{tanh}(\rho y)} \frac{\partial H(y)}{\partial y} + \left( k_0^2 n_0^2 \text{tanh}(\rho y) \left( \sinh(\rho y) + \text{tanh}(\rho y) \right) - k_0^2 \right) H(y) = 0
\]

(16)

If we set \( q = \rho y \) in equation (16),

\[
\frac{d^2 H(y)}{dq^2} - \frac{\text{sech}^2(q)}{\text{tanh}(q)} \frac{dH(y)}{dq} + \frac{1}{\rho^2} \left( k_0^2 n_0^2 \text{tanh}(q) \left( \sinh(q) + \text{tanh}(q) \right) - k_0^2 \right) H(y) = 0
\]

(17)

and in the solution domain close to zero, equation (17) can be approached using the MacLaurin series, as follows

\[
\text{tanh}(q) \approx q; \quad \text{sech}(q) \approx 1; \quad \sinh(q) + \text{tanh}(q) \approx 2q
\]

(18)

so we have

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**Figure 1.** The graded graphs of (a) permeability equation (b) permittivity equation.
\[
\frac{\text{sech}^2(q)}{\tanh(q)} \approx \frac{1}{q}
\]  
(19)

But, at \( q = 0 \) equation (19) is not defined, so it needs to make an assumption. If \( \beta \) is the value of \( q \) that close to zero then using equation (19) can be obtained intervals \( -\beta \leq q \leq \beta \). Using this interval, equation (19) is considered to have the form of a linear function connecting the points \((-\beta, -1/\beta)\) and \((\beta, 1/\beta)\), ie [2, 13]

\[
b(q) = \frac{1}{\beta^2} q
\]  
(20)

Using that assumption in equation (17), we obtained

\[
\frac{d^2 H(y)}{dq^2} - \frac{1}{q} \frac{dH(y)}{dq} + \frac{1}{\rho^2} \left(2k_0^2 \frac{n_o^2 q^2}{\rho^2} - k_z^2\right) H(y) = 0
\]  
(21)

By setting \( s = \left(2k_0^2 \frac{n_o^2}{\rho^2}\right)^{1/4} q \), into equation (21), we got

\[
\frac{d^2 H(y)}{ds^2} - \frac{s}{\beta^2 \sqrt{2k_0 n_o}} \frac{dH(y)}{ds} + \left(s^2 - \frac{k_z^2}{\sqrt{2k_0 n_o \rho}}\right) H(y) = 0
\]  
(22)

Equation (22) is AIM-type differential equation, like in equation (7) with

\[
\lambda_0 = \frac{s}{\beta^2 \sqrt{2k_0 n_o}}; s_0 = \frac{k_z^2}{\sqrt{2k_0 n_o \rho}} - s^2
\]  
(23)

for \( s \) is close to zero, we approached equation (23) using Maclaurin series, given as

\[
\lambda_0(s) \approx \frac{s}{\beta^2 \frac{n_o}{2k_0}}; s_0(s) \approx \frac{k_z^2}{\sqrt{2k_0 n_o \rho}}
\]  
(24)

in equation (22), so we obtained

\[
\frac{d^2 H(y)}{ds^2} - \frac{s}{\beta^2 \sqrt{2k_0 n_o}} \frac{dH(y)}{ds} - \frac{k_z^2}{\sqrt{2k_0 n_o \rho}} H(y) = 0
\]  
(25)

By setting \( r = s \sqrt{\frac{\rho}{2\sqrt{2k_0 n_o \beta^2}}} \) in equation (25), we got

\[
\frac{d^2 H(y)}{dr^2} - 2r \frac{dH(y)}{dr} - \frac{2k_z^2 \beta^2}{\rho^2} H(y) = 0
\]  
(26)

Equation (26) is AIM-type differential equation, and by using equation (9) we obtained,

\[
\Delta_j = \prod_{j=0}^{m} \left[-\frac{2k_z^2 \beta^2}{\rho^2} - 2j\right]; m = 1, 2, 3, ...
\]  
(27)

according to equation (27), the eigenvalues can be obtained as follows

\[
-\frac{k_z^2 \beta^2}{\rho^2} = j \Rightarrow k_z^2 = -\frac{\rho^2 j}{\beta^2} \Rightarrow k_z = i \frac{\rho}{\beta} \sqrt{j}; j = 0, 1, 2, 3, ...
\]  
(28)

so equation (26) become

\[
\frac{d^2 H(y)}{dr^2} - 2r \frac{dH(y)}{dr} + 2jH(y) = 0
\]  
(29)

which known as Hermite differential equation. The approximate solution of equation (29) is
\[
H_j(y) = (-1)^j e^{i\gamma} \frac{d^j}{dr^j} \left(e^{-i\gamma}\right)
\]  \hspace{1cm} (30)

where
\[
r = \frac{\rho}{\beta \sqrt{2}} y, \quad -\frac{1}{\sqrt{2}} \leq r \leq \frac{1}{\sqrt{2}}, \quad \beta \ll 1
\]  \hspace{1cm} (31)

Then for the fields distribution, we plotted from the total eigenfunctions using equations (6,12,30-31) for \( j=5 \) and \( j=8 \), and we take the positive sign for \( H(z) \), we obtained the magnetic fields distribution which is shown in Figure 2.

\[\text{Figure 2. Distribution of the magnetic fields (a) } j=5, \text{ (b) } j=8\]
Figure 2.a is the five level $j=5$ while Figure 2.b for $j=8$. The magnetic field has the different distribution in various the material thickness $y$ which in higher level, the amplitude of magnetic fields increase. While for the wave vector $k_z$ in equation (28) depend on the abruptness’s parameter, the material thickness and the level $j$. From equation (28), we can show that the level $j$ is proportional to the wave vector, which the value of wave vector increase in the higher level.

5. Conclusion

Iterative method was used to investigated the wave propagation for TM mode in a graded interface negative-positive index. We obtained the graded graphs of permittivity and permeability as a function of material thickness in hyperbolic function. The wave vector equation was obtained using AIM, while the magnetic fields equation in Hermite polynomial were expressed in two-dimensional graph fields versus $y$.

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