Mean-Field Neural ODEs via Relaxed Optimal Control

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Abstract

We develop a framework for the analysis of Bayesian neural ODE models that are trained with stochastic gradient algorithms. We do that by identifying the connections between control theory, deep learning and theory of statistical sampling. We derive Pontryagin’s optimality principle and study the corresponding gradient flow in the form of Mean-Field Langevin dynamics (MFLD) for solving relaxed data-driven control problems. Subsequently, we study uniform-in-time propagation of chaos of time-discretised MFLD. We derive explicit convergence rate in terms of the learning rate, the number of particles/model parameters and the number of iterations of the gradient algorithm. In addition, we study the error arising when using a finite training data set and thus provide quantitative bounds on the generalisation error. Crucially, the obtained rates are dimension-independent. This is possible by exploiting the regularity of the model with respect to the measure over the parameter space.

Keywords: Neural ODE, Relaxed Control, Gradient Flow, Generalisation Error

1. Introduction

There is overwhelming empirical evidence that deep neural networks trained with stochastic gradient descent perform (extremely) well in high dimensional setting LeCun et al. (2015); Silver et al. (2016); Mallat (2016). Nonetheless, a complete mathematical theory that would provide theoretical guarantees why and when these methods work so well has been elusive.

In this work, we establish connections between high dimensional data-driven control problems, deep neural network models and statistical sampling. We demonstrate how all of them are fundamentally intertwined. Optimal (relaxed) control perspective on deep learning tasks provides new insights, with a solid theoretical foundation. In particular, the powerful idea of relaxed control, that dates back to the work of L.C. Young on generalised solutions of problems of calculus of variations Young (2000), paves the way for efficient algorithms used in the theory of statistical sampling Majka et al. (2018); Durmus and Moulines (2017); Eberle (2016); Cheng et al. (2019). Indeed, in a recent series of works, see Hu et al. (2019b); Mei et al. (2018); Rotskoff and Vanden-Eijnden (2018); Chizat and Bach (2018);
Sirignano and Spiliopoulos (2018), the task of learning the optimal weights in deep neural networks is viewed as a sampling problem. The picture that emerges is that the aim of the learning algorithm is to find optimal distribution over the parameter space (rather than optimal values of the parameters). As a consequence, individual values of the parameters are not important in the sense that different sets of weights sampled from the correct (optimal) distribution are equally good. To learn optimal weights, one needs to find an algorithm that samples from the correct distribution. It has been shown recently in Hu et al. (2019b); Mei et al. (2018) that in the case of Bayesian one-hidden layer network the noisy gradient algorithm does precisely that. The key mathematical tools to these results turn out to be the theory of gradient flows and differential calculus on the measure space.

Until recently the depth of neural networks used in practice has been limited. Practitioners reported that as number of layers in traditional DNN increases the training becomes harder. However, the arrival of the so-called residual neural networks (ResNet) in 2016 which outperformed traditional networks across a variety of tasks dramatically changed this situation, He et al. (2016). To extend Hu et al. (2019b); Mei et al. (2018) to multi-layer setup we build upon the connection between deep learning and controlled ODEs that has been explored in the pioneering works Weinan (2017); Li et al. (2017); E et al. (2018); Hu et al. (2019a).

1.1 Overview of the main results

The key motivation behind this work is to demonstrate that continuous-time and space analysis of gradient flow on the space of probability measures combined with probabilistic numerical analysis, that yield quantitative convergence bounds in terms of the learning rate, the number of iterations of the gradient algorithm and the size of the training set, offers a general framework for studying of recurrent neural networks models trained with stochastic gradient algorithm. This perspective has been recently reinforced by Weinan et al. (2019).

This section provides a high level overview of key findings of this work. Precise definitions of appropriate spaces, assumptions and theorems are presented in Section 2. As such this section provides a roadmap for the rest of the paper.

**Bayesian Neural ODEs via Relaxed Control**

Let \((\xi, \zeta) \in D\) represent training data (possibly e.g. continuous paths), distributed according to a distribution \(M \in \mathcal{P}(D)\) (typically unknown), and let \(\varphi : \mathbb{R} \rightarrow \mathbb{R}\) be an activation function. Each layer, indexed by \(t\), of the deep network takes input \(z \in \mathbb{R}^d\) and outputs

\[
\frac{1}{n} \sum_{i=1}^{n} \beta_{t,i} \varphi(\alpha_{t,i} \cdot z + \rho_{t,i} \cdot \zeta_t) = \int_{\mathbb{R}^d} \beta \varphi(\alpha \cdot z + \rho \cdot \zeta_t) \nu^\beta_t(d\beta, d\alpha, d\rho), \quad \nu^\beta_t = \frac{1}{n} \sum \delta_{\{\beta_{t,i}, \alpha_{t,i}, \rho_{t,i}\}}.
\]

In other words, applying the mean-field scaling at each layer of the neural networks, allows one to shift focus from specific values of the weights to the probability distribution over the weights. This perspective is widely adapted in deep learning community under the banner probabilistic or Bayesian neural networks and lends itself to the uncertainly quantification for deep learning, see e.g. MacKay (1995); Neal (2012); Gal and Ghahramani (2015).
Let \( a := (\beta, \alpha, \rho) \) and \( \phi(z, a) := \beta \varphi(\alpha \cdot z + \rho \cdot \zeta) \). An example of deep neural network architecture studied in this paper is given by

\[
X_{t_{k+1}}^{\nu, \xi, \zeta} = X_{t_k}^{\nu, \xi, \zeta} + \frac{\Delta t}{n} \sum_{i=1}^{n} \phi(X_{t_k}^{\nu, \xi, \zeta}, a_i^k), \quad \text{where } \Delta t = (t_{k+1} - t_k).
\]

Taking \( \Delta t \to 0 \) corresponds to sending the number of layers to infinity Weinan (2017); Liu and Markowich (2020). Similarly note that as \( n \to \infty \) the empirical distribution over the weights of neural networks \( \nu^n \) is expected to converge to continuous distribution over the weights Hu et al. (2019b). Analysis of a model with infinite number of layers and neurons has the following advantages.

a) The continuous limit enjoys number of different numerical approximations. Each approximation corresponds to a different network architecture. Different approximations / architectures then lead to different mean-field approximation error, see Chassagneux et al. (2019) and different time discretization errors, see Hairer et al. (1987).

b) Theoretical guarantees established for the performance of the continuous limit model will translate to networks with an arbitrary number of parameters and layers as long as the architecture after discretization preserves relevant properties of the continuous model. This provides a strategy for identifying good neural network architectures.

Motivated by these observations we consider probabilistic neural ODE model given by

\[
X_t^{\nu, \xi, \zeta} = \xi + \int_0^t \int \phi(X_r^{\nu, \xi, \zeta}, a) \nu_r(da) \, dr
\]  

(1)

Training of such model, can be recast as optimisation over the space of probability measures

\[
\min_{\nu} J^{\sigma, \mathcal{M}}(\nu), \quad \text{with } J^{\sigma, \mathcal{M}}(\nu) := \int_D \left( g(\xi, X_T^{\nu, \xi, \zeta}) + R^\sigma(\xi, \zeta, \nu) \right) \mathcal{M}(d\xi, \zeta),
\]

(2)

where \( g \) is an unregularised loss and \( R^\sigma \) is a regulariser. Motivated by the theory of regularised control we take

\[
R^\sigma(\xi, \zeta, \nu) := \int_0^T \left( \int_{\mathbb{R}^p} f(X_s^{\nu, \xi, \zeta}, a) \nu_s(da) + \frac{\sigma^2}{2} \text{Ent}(\nu_s) \right) \, ds,
\]

where \( f \) in theory of control is called a running cost function, and \( \text{Ent}(\mu) \) is relative entropy with respect to prior distribution \( \gamma(x) \approx e^{-U(x)} \) for some potential \( U \). The entropy term allows one to incorporate prior knowledge in the form on distribution over the weights into the training. Note that unlike in mean-field modes for one hidden layer studied in Hu et al. (2019b); Mei et al. (2018); Rotskoff and Vanden-Eijnden (2018); Chizat and Bach (2018); Sirignano and Spiliopoulos (2018) the unregularised loss \( J^{0, \mathcal{M}} \) is not convex even if \( g \) is. Consequently, it is not clear a priori that the entropy term turns the problem strictly convex as it was the case in Hu et al. (2019b); Mei et al. (2018). It is a common practice to consider the cost function \( J^{0, \mathcal{M}} \), i.e. \( \sigma = 0 \) corresponding to no entropic regularisation, but to train the network with stochastic gradient algorithm, including randomly sampling a mini
batch at each step of training or training a random subset of the network’s weights at each step i.e. dropout. The randomness introduced by the stochastic gradient algorithm leads to so called implicit regularisation, see Neyshabur et al. (2017, 2018); Heiss et al. (2019) and is modelled here by taking $\sigma > 0$. To put it differently, the noise introduced during training means that some bias is introduced and one should not expect that $J^{0,\mathcal{M}}$ will decrease along the gradient flow. The variational perspective, that we take in this paper, makes the connection between randomness introduced during the training and the exact form of the (implicit) regularisation at the level of cost function $J^{\sigma,\mathcal{M}}$.

We tackle the optimisation problem (2) using tools from relaxed control theory. One of the contributions of this work is a derivation of Pontryagin maximum principle for measure valued controls using variational calculus. Indeed we show (see Theorem 5) that if $\nu$ is (locally) optimal then it must solve the forward-backward system given by

$$\nu^* = \arg\min_{\nu \in \mathcal{P}(\mathbb{R}^p)} \int_D H_t^\sigma (X_t^{\xi,\zeta}, P_t^{\xi,\zeta}, m, \xi) \, \mathcal{M}(d\xi, d\zeta),$$

where the regularised Hamiltonian is given by

$$H_t^\sigma (x, p, m, \zeta) := \int h_t(x, p, a, \zeta) \, m(da) + \frac{\sigma^2}{2} \text{Ent}(m),$$

$$h_t(x, p, a, \zeta) := \phi_t(x, a, \zeta)p + f_t(x, a, \zeta).$$

Note that the necessary condition described in (3) involves “layer-by-layer” optimisation of the Hamiltonian with $(X_t^{\xi,\zeta}, P_t^{\xi,\zeta})$ fixed. It is instructive to note the differences and similarities between “training” using (3) and usual algorithm for training neural network by gradient descent. We see that in (3), just as is common, we “run the network” for each data input $(\xi, \zeta)$ (the equation for $X_t^{\xi,\zeta}$). We do the “back-propagation” again for each data input (the equation for $P_t^{\xi,\zeta}$). But thanks to Pontryagin optimality criteria (see Theorem 5 later) we can “update the weights” by minimizing the Hamiltonian “layer-by-layer” instead of the cost (2). We show (see Theorem 7) that the optimal distribution over the space of parameters of the neural network, at least for sufficiently large $\sigma$ exists and is unique. Moreover $\nu^{*,\sigma} = \arg\min_{\nu} J^{\sigma,\mathcal{M}}(\nu)$ is for each $t$ is given by coupling the forward-backward system for $X_t^{\xi,\zeta}$ and $P_t^{\xi,\zeta}$ in (3) with the functional equation

$$\nu^*_t(a) = \frac{1}{Z_t(\nu^*)} e^{\int_D h_t(X_t^{\nu^*,\xi,\zeta}, P_t^{\nu^*,\xi,\zeta}, a, \zeta) \, \mathcal{M}(d\xi, d\zeta) \gamma(a)},$$

where $Z_t(\nu^*)$ is the normalising factor constant in $a$. We observe that $\nu^*_t(a)$, the solution to (4), enjoys Bayesian interpretation with $\gamma(a)$ being the prior and $\nu^{*,\mathcal{M}}_t$ posterior distributions of weights of the neural ODE.

**Mean-Field Langevin sampling**

Thus, we showed that the training of neural ODEs in mean-field regime can be recast as a relaxed control problem which we solve using Pontryagin principle on the space of probability measures. The next step is to demonstrate that mean-field Langevin sampling algorithm,
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also known in literature as Wasserstein gradient flow, studied previously in literature in the context of training one hidden layer neural network Mei et al. (2018); Hu et al. (2019b), corresponds to a noisy stochastic gradient algorithm used for training deep neural networks. Let $\nabla \cdot$ denote the divergence operator and let $\delta H_0^{\sigma}$ be the linear functional derivative (first variation), see Appendix A for the definition. From (3) and the theory of gradient flows, cf. Villani (2008), one would hope that the gradient flow equation, for each “layer” $t \in [0, T]$, is given by

$$
\frac{d}{ds} \nu_{s,t} = \nabla \cdot \left( \left( \nabla_a \delta H_0^{\sigma} \right)(X_{s,t}, P_{s,t}, \nu_{s,t}, a, \mathcal{M}) \nu_{s,t} \right), \quad \nu_{0,t} = \mu, \quad s \geq 0, \quad (5)
$$

where $X_{s,t} = (X_{s,t}^{\xi,\zeta})_{\xi,\zeta}$ and $P_{s,t} = (P_{s,t}^{\xi,\zeta})_{\xi,\zeta}$. The gradient flow is coupled with, for each “gradient flow time” $s \geq 0$, with the forward backward system

$$
\begin{align*}
X_{s,t}^{\xi,\zeta} &= \xi + \int_0^t \Phi_r(X_{s,r}^{\xi,\zeta}, \nu_{s,r}, \zeta) \, dr, \quad t \in [0, T], \\
P_{s,t}^{\xi,\zeta} &= (\nabla_x g)(X_{s,T}^{\xi,\zeta}) + \int_t^T (\nabla_x H_r^0)(X_{s,r}^{\xi,\zeta}, P_{s,r}^{\xi,\zeta}, \nu_{s,r}, \zeta) \, dr, \quad t \in [0, T].
\end{align*} \quad (6)
$$

We stress again that in this setting the forward process $(X_{s,t}^{\xi,\zeta})_t$ plays the role of running the neural network with input $(\xi, \zeta)$ while the backward / adjoint process $(P_{s,t}^{\xi,\zeta})_t$ plays the role of back propagation. One of the main results of this work is to show that (5) is indeed a “correct” gradient flow in a sense that $\frac{d}{ds} J_{\sigma, \mathcal{M}}(\nu_{s,t}) \leq 0$ i.e the loss function is a decreasing function along the gradient flow (5). Furthermore, for sufficiently large $\sigma$ we show in Theorem 7 that

$$
W_2^T(\mathcal{L}(\theta_s, \cdot), \nu_{s\cdot}, \mathcal{M})^2 \leq e^{-\lambda_s} W_2^T(\mathcal{L}(\theta_0, \cdot), \nu_{s\cdot}, \mathcal{M})^2,
$$

where $W_2^T(\mu, \nu) := \left( \int_0^T W_2(\mu_t, \nu_t)^2 \, dt \right)^{1/2}$ with $W_q$ being the usual Wasserstein distance. We remark that the rate of convergence does not dependent on the dimension of the state and parameter spaces and the results hold for either $\mathcal{M}$ or its empirical approximation. It is useful to contrast this result with recent work on the linearizations of neural networks around initialisation known as neural tangent kernel, see Jacot et al. (2018); Arora et al. (2019), or lazy training regime Chizat et al. (2018). For linearised model the loss function can be shown to go to zero exponentially fast. Furthermore, Chizat et al. (2018); Mei et al. (2019) showed that by appropriately rescaling one-hidden layer neural network model one can show that, at least asymptotically, the distributions over the weights does not change from its initialisation. Subsequent works Ghorbani et al. (2019, 2020) studied the limitation of lazy regime. Here, we prove exponential convergence in mean-field regime and go beyond one hidden-layer model, but require strong regularisation. In fact, one can see that by taking large $\sigma$ we are imposing “strong” prior on the posterior distribution over weights $\nu_{s\cdot}^{\sigma, \mathcal{M}}(a)$. That way, our work provides an alternative link between mean-field and lazy regimes. Similar observation for one hidden layer has recently been made in Tzen and Raginsky (2020).

Let

$$
h_t(a, \mu, \mathcal{M}) := \frac{\delta H_0^0}{\delta m}(X_t(\mu), P_t(\mu), a, \mathcal{M}) = \int_D h_t(X_t^{\xi,\zeta}(\mu), P_t^{\xi,\zeta}(\mu), a, \zeta, \mathcal{M}(d\xi, d\zeta)).
$$

5
To simulate $(\nu_{s,t})_{s>0}$ we rely on the probabilistic representation of (5) given by the mean-field equation

$$d\theta_{s,t} = -\left((\nabla_\theta h_t)(\theta_{s,t},\nu_{s,t},\mathcal{M}) + \frac{\sigma^2}{2}(\nabla_\theta U)(\theta_{s,t})\right) \, ds + \sigma dB_s \quad s \geq 0,$$

(7)

where $(\theta^0_{0,t})_{t \in [0,T]}$ is a given initial condition and where $\nu_{s,t}$ is the law of $\theta_{s,t}$. We must again couple (7) with (6). Mean-field Langevin dynamics (7) can be viewed as continuous time noisy gradient descent. We remark that perturbing weights during training with Gaussian noise is common practice when designing differentially private models, see Dziugaite and Rov (2018).

Particle approximation to (7) leads to familiar (noisy) stochastic gradient algorithms used to train neural networks. Indeed, consider a sequence $(\xi^i,\zeta^i)_{i=1}^{N_1}$ of i.i.d copies of $(\xi,\zeta)$ and let $\mathcal{M}^{N_1} := \frac{1}{N_1} \sum_{j=1}^{N_1} \delta_{(\xi^j,\zeta^j)}$ be the empirical measure representing the available training sample. Further, we fix an increasing sequence $0 = s_0 < s_1 < s_2 \cdots$ and define the $(\tilde{\theta}^i_{s,t})_{t \in \mathbb{N}, 0 \leq t \leq T}$ satisfying, for $i = 1, \ldots, N_2$ and $l \in \mathbb{N},$

$$\tilde{\theta}^i_{s+1,t} = \tilde{\theta}^i_{s,t} - \left((\nabla_\theta h_t)(\tilde{\theta}^i_{s,t},\tilde{\nu}^0_{s,t},\mathcal{M}^{N_1}) + \frac{\sigma^2}{2}(\nabla_\theta U)(\tilde{\theta}^i_{s,t})\right) (s_{l+1} - s_l) + \sigma (B^i_{s_{l+1}} - B^i_{s_l}),$$

(8)

where $\tilde{\nu}^0_{s,t} = \frac{1}{N_2} \sum_{j=1}^{N_2} \delta_{\tilde{\theta}^j_{s,t}}$. In this work we provide dimension independent and uniform-in-time bounds for convergence of (8) to (7), see Theorems 8 and 9, and use them to study generalisation error. In practice one also considers a numerical approximation of the process $(X_t)_{t \in [0,T]}$ on a finite time partition $0 = t_0 < \cdots < t_n = T$ of the interval $[0,T]$. Different numerical approximations of process $(X_t)_{t \in [0,T]}$ can be interpreted as different neural network architectures, see again Chen et al. (2018). We omit numerical error in the introduction and remark that numerical analysis of ODEs is a mature field of study, see Hairer et al. (1987).

**Generalisation Error**

Let $\nu^{*,\sigma,N_1} := \arg\min_\nu J^{\nu^*,\sigma,M^{N_1}}(\nu)$ be the optimal distribution over the parameter space when minimising empirical loss with $N_1$ data points, with noisy gradient descent (7). Let $\nu^{\sigma,N_1,N_2,\Delta s}$ denote the distribution over the parameter space induced by the gradient algorithm when training with $N_1$ data samples, finite number of model parameters (the number of which is $\approx N_2 \times p \times n$, where $n$ is the number of grid points of $[0,T]$), the learning rate $\Delta s = \max_{0 \leq s < T} (s_{l+1} - s_l)$, and training time $S$. The generalisation error $J^{0,M}(\nu^{\sigma,N_1,N_2,\Delta s})$ is the the value of the loss function under population measure $\mathcal{M}$ evaluated at $\nu^{\sigma,N_1,N_2,\Delta s}$. Note that we can write

$$J^{0,M}(\nu^{\sigma,N_1,N_2,\Delta s}) = J^{0,M}(\nu^{\sigma,N_1,N_2,\Delta s}) - J^{0,M}(\nu^{*,\sigma}) - \frac{\sigma^2}{2} \int_0^T \text{Ent}(\nu^{*,\sigma}_t) \, dt + \min_\mu \int \mathcal{J}^{\nu^{*,\sigma}}(\mu),$$

since $\min_\nu J^{\nu,M}(\mu) = J^{0,M}(\nu^{*,\sigma}) + \frac{\sigma^2}{2} \int_0^T \text{Ent}(\nu^{*,\sigma}_t) \, dt$.

Thus we see that the generalisation error consist of three errors: (a) the numerical error of approximating an invariant measure with discrete time particle system, b) the relative
entropy between the Gibbs measure $\gamma$ (a prior) and the $\nu^{*,\sigma}$, c) the minimum value of the cost function under population measure $J^{\sigma,M}$. Under appropriate smoothness of the loss function with respect to measure over the parameter space obtain (in Theorem 10)

$$\mathbb{E} \left[ \left| J^{0,M}(\nu^{*,\sigma}) - J^{0,M}(\nu^{\sigma,N_1,N_2,\Delta s}_S) \right|^2 \right] \leq c \left( e^{-\lambda S} + \frac{1}{N_1} + \frac{1}{N_2} + h \right),$$

where $h := \max_{0<s_l<s<s_l-1} (s_l - s_{l-1})$. We stress that the rates that we obtained are dimension independent, which is not common in the literature. Indeed, such bound wouldn’t hold in a dimension-free way for functions that are only Lipschitz continuous w.r.t. the Wasserstein distance, see Fournier and Guillin (2015) or Dereich et al. (2013). For more regular functions it is possible to obtain such dimension-free estimates (Delarue et al., 2019, Lem. 5.10), (Szpruch and Tse, 2019, Lem.2.2) and Jabir (2019). In this paper we also exploit this additional regularity to obtain such estimates.

It is a common practice to terminate the training when $J^{0,M,N_1}$ is negligible Zhang et al. (2016); Belkin et al. (2019); Montanari et al. (2019); Hastie et al. (2019); Mei and Montanari (2019) and such models have been observed to generalise well. In a situation when $J^{0,M,N_1}$ is not negligible after training for sufficiently long time $S$, then the model is considered untrained and one would not expect it to generalise well. If one assumes that for fixed $\varepsilon > 0$ and $N_1 > 0$, $J^{0,M_1}(\nu^{*,\sigma,N_1}) \leq \varepsilon$ then we show (see Theorem 12) that

$$\mathbb{E} \left[ \left| J^{0,M}(\nu^{\sigma,N_1,N_2,\Delta s}_S) \right|^2 \right] \leq \varepsilon^2 + c \left( e^{-\lambda S} + \frac{1}{N_1} + \frac{1}{N_2} + h \right).$$

The assumption that $J^{0,M_1}(\nu^{*,\sigma,N_1}) \leq \varepsilon$ could be verified by establishing universal approximation theorem for the neural ODEs in a spirit of Sontag and Sussmann (1997); Cuchiero et al. (2019) and combining it with the analysis presented in this work. We postpone this direction of research to future work.

While this work is motivated by the desire to put deep learning on a solid mathematical foundation, as a byproduct, ideas emerging from machine learning provide new perspective on classical dynamic optimal control problems. Indeed, high dimensional control problems are ubiquitous in technology and science Bertsekas (1995); Bensoussan (2004); Bensoussan and Lions (2011); Fleming and Soner (2006); Carmona and Delarue (2018). There are many computational methods designed to find, or approximate, the optimal control functional, see e.g. Kushner and Dupuis (2001) or Gyöngy and Šiska (2009) and references therein. These, typically, rely on dynamic programming, discrete space-time Markov chains, finite-difference methods or Pontryagin’s maximum principle and, in general, do not scale well with the dimensions. Indeed the term “curse of dimensionality” (computational effort grows exponentially with the dimension) has been coined by R. E. Bellman when considering problems in dynamic optimisation Bellman (1966). This work advances the study of a new class of algorithm for control problems that is particularly well adapted to high dimensional setting.

1.2 Related work

In Weinan (2017); Li et al. (2017); E et al. (2018), Pontryagin’s optimality principle is leveraged and the convergence of the method of successive approximations for training the neural
ODEs is studied. The authors suggested the possibility of combining their algorithm with gradient descent but have not studied this connection in full detail. The term Neural ODEs has been coined in Chen et al. (2018) where the authors exploited the computational advantage of Pontryagin’s principle approach (with its connections to automatic differentiation Baydin et al. (2018)). Finally, Hu et al. (2019a) (see also Bo et al. (2019)) formulated the relaxed control problem and showed that the recently methodology developed in Hu et al. (2019b) can be successfully applied in this setup, to prove convergence of flows of measures induced by the mean-field Langevin dynamics to invariant measure that minimised relaxed control problem. The starting point of Hu et al. (2019a) is a relaxed Pontryagin’s representation for the control problem and the authors prove convergence of the continuous time dynamics in a fixed data regime and, in the case of the neural ODE application, infinite number of parameters. Our work complements Hu et al. (2019a): first we derive the Pontryagin’s optimality principle for the data-driven relaxed control problem. From there we identify the gradient flow on the space of probability measures along which corresponding energy function is decreasing. Next, we study the complete algorithm and provide quantitative convergence bounds in terms of the learning rate, the number of iterations of the gradient algorithm and the size of the training set. Finally we derive quantitative bounds on the generalisation error by exploiting smoothness of the energy function.

We remark that control problem perspective is fruitful when studying universal approximation results for (neural) ODEs, Sontag and Sussmann (1997); Cuchiero et al. (2019); Ma et al. (2019).

1.3 Examples of application to machine learning problems

The setting in this paper is relevant to many types of machine learning problems. Below we give two examples of how the results of this paper can be applied to gain insight into machine learning tasks.

**Example 1 (Nonlinear regression and function approximation)** Consider a function $f: \mathbb{R}^d \to \mathbb{R}^d$ one wishes to approximate. This is to be done by sampling $(\xi, \zeta)$ from $\mathcal{M}$ where $\xi = f(\zeta)$. We see that here $\mathcal{S} = \mathbb{R}^d$.

The objective can be taken as

$$ J^\sigma(\nu) = \int_{\mathbb{R}^d \times \mathbb{R}^d} |X^\xi_\tau - f(\zeta)|^2 \mathcal{M}(d\xi, d\zeta) + \frac{\sigma^2}{2} \int_0^T \text{Ent}(\nu_t) \, dt. $$

We now fix a nonlinear activation function $\varphi: \mathbb{R}^d \to \mathbb{R}^d$ and take $\phi$ to be

$$ \phi(x, a) := a_1 \varphi(a_2 x), \quad a = (a_1, a_2) \in \mathbb{R}^p, $$

with $a_1 \in \mathbb{R}^{d \times d}$, $a_2 \in \mathbb{R}^{d \times d}$ so $p := 2d^2$. The neural network will then be given by discretizing (1).

In the following example we consider a time-series application. In this setting the input will be one or more “time-series” each a continuous a path on $[0, T]$. The reason for considering paths on $[0, T]$ is that this covers the case of unevenly spaced observations.
Example 2 (Missing data interpolation) The learning data set consists of the true path \( \zeta(2) \in C([0,T];\mathbb{R}^d) \) and a set of observations \((\zeta_i^{(1)})_{i=1}^{N_{\text{obs}}} \) on \( 0 \leq t_1 \leq \cdots \leq t_{N_{\text{obs}}} \leq T \). We will extend this to entire \([0,T]\) by piecewise linear interpolation denoted \( \zeta^1 \). The learning data is then \((\zeta^{(1)},\zeta^{(2)}) =: \zeta \in (C([0,T];\mathbb{R}^d))^2 =: S \) distributed according to the data measure \( M \in \mathcal{P}(S) \). As before we fix a nonlinear activation function \( \varphi : \mathbb{R}^d \to \mathbb{R}^d \). Let us take \( \phi \) to be
\[
\phi_t(x,a,\zeta) := a_1 \varphi(a_2 x + a_3 \zeta^{(1)}), \quad a = (a_1,a_2,a_3) \in \mathbb{R}^p,
\]
with \( a_1 \in \mathbb{R}^{d \times d} \), \( a_2 \in \mathbb{R}^{d \times d} \), \( a_3 \in \mathbb{R}^{d \times d} \), so that \( d^2 + d^2 + d^3 = p \). Let \( L \in \mathbb{R}^{d \times d} \) be a matrix. The learning task is then to minimize
\[
J^\sigma(\nu) = \int_S \int_0^T |LX_t^\nu \zeta^{(1)} - \zeta^{(2)}|^2 dt \mathcal{M}(d\zeta^1,d\zeta^2) + \sigma^2 \int_0^T \text{Ent}(\nu_t) dt
\]
over all measures \( \nu \in \mathcal{V}_2 \) subject to the dynamics of the forward ODE given by (1).

2. Main results

In this section we state precisely the assumptions and the results discussed in Section 1. Some notions from Section 1 are repeated here so that this section can be read on its own.

2.1 Statement of problem

Given some metric space \( E \) and \( 0 < q < \infty \), let \( \mathcal{P}_q(E) \) denote the set of probability measures defined on \( E \) with finite \( q \)-th moment. Let \( \mathcal{P}_0(E) = \mathcal{P}(E) \) be the set of probability measures on \( E \). Let \( \mathcal{V}_2 \) denote the set of positive Borel measures on \([0,T] \times \mathbb{R}^p \) with the first marginal equal to the Lebesgue measure, the second marginal being a probability measure with finite second moment. That is
\[
\mathcal{V}_2 := \left\{ \nu \in \mathcal{M}([0,T] \times \mathbb{R}^p) : \nu(dt,da) = \nu_t(da)dt, \nu_t \in \mathcal{P}_2(\mathbb{R}^p), \int_0^T \int |a|^2 \nu_t(da)dt < \infty \right\}, \tag{9}
\]
where here and elsewhere any integral without an explicitly stated domain of integration is over \( \mathbb{R}^p \). We consider the following controlled ordinary differential equation (ODE):
\[
X_t^{\xi,\zeta}(\nu) = \xi + \int_0^t \int \nu_t(\phi_r(X_r^{\xi,\zeta}(\nu),a,\zeta) \nu_r(da)dr, \quad t \in [0,T], \tag{10}
\]
where \( \nu = (\nu_t)_{t \in [0,T]} \in \mathcal{V}_2 \) is the control, where \((\xi,\zeta) \in \mathbb{R}^d \times \mathcal{S} \) denotes some external data, distributed according to \( \mathcal{M} \in \mathcal{P}_2(\mathbb{R}^d \times \mathcal{S}) \) and where \((\mathcal{S},\|\cdot\|_\mathcal{S})\) is a normed space.

For \( \nu \in \mathcal{P}(\mathbb{R}^p) \) let \( \text{Ent}(\nu) := \infty \) if \( \nu \) is not absolutely continuous with respect to Lebesgue measure. Otherwise let
\[
\text{Ent}(\nu) := \int [\log \nu(a) - \log \gamma(a)] \nu(a) da, \quad \text{where} \quad \gamma(a) = e^{-U(a)} \quad \text{with} \quad U \text{ s.t.} \quad \int e^{-U(a)} da = 1. \tag{11}
\]

Given \( f \) and \( g \) we define objective functionals as
\[
J^\sigma(\nu,\xi,\zeta) := \int_0^T \int f_t(X_t^{\xi,\zeta}(\nu),a,\zeta) \nu_t(da) dt + g(X_T^{\xi,\zeta}(\nu),\zeta) + \sigma^2 \int_0^T \text{Ent}(\nu_t) dt \tag{12}
\]
as well as
\[ J^\sigma,\mathcal{M}(\nu) := \int_{\mathbb{R}^d \times \mathcal{S}} \tilde{J}^\sigma(\nu, \xi, \zeta) \mathcal{M}(d\xi, d\zeta). \] (13)

Note that when \( \mathcal{M} \) is fixed in (13) then we don’t emphasise the dependence of \( J^\sigma,\mathcal{M} \) on \( \mathcal{M} \) in our notation and write only \( J^\sigma \) and we write \( J^0 = J \) and \( J^{0,\mathcal{M}} = J^\mathcal{M} \). Once \( \mathcal{M} \) is fixed, the aim is to minimize \( J^\sigma \) over all controls \( \nu \in \mathcal{V}_2 \), subject to the controlled process \( X^{\xi,\zeta}(\nu) \) satisfying (10). In case \( \sigma \neq 0 \) we additionally require that \( \nu_t \) is absolutely continuous with respect to the Lebesgue measure for almost all \( t \in [0, T] \). For convenience define
\[ \Phi_t(x, m, \zeta) := \int \phi_t(x, a, \zeta) \, m(da) \] and \( F_t(x, m, \zeta) := \int f_t(x, a, \zeta) \, m(da) \).

### 2.2 Assumptions and theorems

We start by briefly introducing some terminology and notation. We will use \( \frac{\partial}{\partial t} \) to denote the flat derivative on \( \mathcal{V}_2 \) and \( \frac{\delta}{\delta m} \) to denote the flat derivative on \( \mathcal{P}_2(\mathbb{R}^p) \), see Section A for definitions of these objects. We will say that some function \( \psi = \psi_t(x, a, \zeta) \) is Lipschitz continuous in \( (x, a) \), uniformly in \( (t, \zeta) \in [0, T] \times \mathcal{S} \) if we have that
\[ \sup_{t \in [0, T]} \sup_{\zeta \in \mathcal{S}} \sup_{x, x' \in \mathbb{R}^d, a, a' \in \mathbb{R}^p, (x, a) \neq (x', a')} \frac{|\psi_t(x, a, \zeta) - \psi_t(x', a', \zeta)|}{|(x, a) - (x', a')|} < \infty. \]

For \( \psi = \psi(w) \) we will use \( \|\psi\|_{\infty} := \sup_w |\psi(w)| \) to denote the supremum norm and we will use \( \|\psi\|_{\text{Lip}} := \sup_{w \neq w'} \frac{|\psi(w) - \psi(w')|}{|w - w'|} \) to denote the Lipschitz norm. We will use \( c \) to denote a generic constant that may change from line to line but must be independent of \( p, d \) and all other parameters that appear explicitly in the same expression.

**Assumption 1**

i) \( \int_{\mathbb{R}^d \times \mathcal{S}} [||\xi||^2 + ||\zeta||^2] \mathcal{M}(d\xi, d\zeta) < \infty. \)

ii) \( \phi, \nabla_a \phi, \nabla_x \phi, f, \nabla_a f, \nabla_x f \) and \( \nabla_x g \) are all Lipschitz continuous in \( (x, a) \), uniformly in \( (t, \zeta) \in [0, T] \times \mathcal{S} \). Moreover, \( x \mapsto \nabla_x \phi, \nabla_x f \) and \( \nabla_x g \) are all continuously differentiable.

iii) \( \sup_{t \in [0, T]} \int_{\mathbb{R}^d \times \mathcal{S}} \left[ |g(0, \zeta)|^2 + |f_t(0, 0, \zeta)|^2 + |\phi_t(0, 0, \zeta)|^2 \\
+ |\nabla_x g(0, \zeta)|^2 + |\nabla_x f_t(0, 0, \zeta)|^2 + |\nabla_a f_t(0, 0, \zeta)|^2 \right] \mathcal{M}(d\xi, d\zeta) < \infty. \)

**Definition 2 (Permissible flows)** We will call \((b_{s,t}(a))_{t \in [0, T]} \) a permissible flow if for all \( s \geq 0, t \in [0, T] \) the map \( a \mapsto b_{s,t}(a) \) has at most linear growth.

**Lemma 3** If \( b = b_{s,t}(a) \) is a permissible flow (c.f. Definition 2) then, for all \( t \in [0, T] \) the equation
\[ \partial_s \nu_{s,t} = \nabla_a \cdot \left( b_{s,t} \nu_{s,t} + \frac{\sigma^2}{2} \nabla a \nu_{s,t} \right), \quad s \in [0, \infty), \nu_{0,t} \in \mathcal{P}_2(\mathbb{R}^p), \]
has a unique solution \( \nu_{s,t} \in C^{1,2}((0, \infty) \times \mathbb{R}^p) \) and moreover \( \nu_{s,t} > 0 \) for all \( s > 0, t \in [0, T] \).
The existence and uniqueness stated in Lemma 3 are proved in e.g. (Ladyzenskaja et al., 1968, Chapter IV) and the fact that the solution is strictly positive can be obtained from its stochastic representation and via a Girsanov transform.

We now introduce the relaxed Hamiltonian:

\[ H^0_t(x, p, m, \zeta) := \int h_t(x, p, a, \zeta) m(da), \]

where \( h_t(x, p, a, \zeta) := \phi_t(x, a, \zeta)p + f_t(x, a, \zeta), \]

\[ H^\sigma_t(x, p, m, \zeta) := H^0_t(x, p, m, \zeta) + \frac{\sigma^2}{2} \text{Ent}(m). \]

(14)

We will also use the adjoint process

\[ dP_{\xi,\zeta}^\nu_t(\mu) = -\nabla_x H^0_\sigma(X^{\xi,\zeta}_t, P^{\xi,\zeta}_t, \mu, \nu_t) dt, \quad t \in [0, T], \quad P_{\xi,\zeta}^\nu_t(\mu) = (\nabla_x g)(X^{\xi,\zeta}_t(\nu), \zeta), \]

(15)

and note that trivially \( \nabla_x H^0 = \nabla_x H^\sigma \). In Lemma 20 we will prove that, under Assumption 1, the system (10), (15) has a unique solution. For \( \mu \in V_2, M \in \mathcal{P}_2(\mathbb{R}^d \times \mathcal{S}) \) let

\[ h_t(a, \mu, M) := \int_{\mathbb{R}^d \times \mathcal{S}} h_t(X^{\xi,\zeta}_t(\mu), P^{\xi,\zeta}_t(\mu), a, \zeta) M(d\xi, d\zeta), \]

(16)

where \((X^{\xi,\zeta}_t(\mu), P^{\xi,\zeta}_t(\mu))\) is the unique solution to (10) and (15).

We now state a key result on how to choose the gradient flow to solve the control problem.

**Theorem 4** Fix \( \sigma > 0 \), let Assumption 1 hold, let \( b \) be a permissible flow (c.f. Definition 2) and \((\nu_s, \cdot)_{s \geq 0}\) the corresponding measure flow resulting from Lemma 3. Let \( X^{\xi,\zeta}_s, P^{\xi,\zeta}_s \) be the forward and backward processes arising from data \((\xi, \zeta)\) with control \( \nu_s, \in V_2 \). Then

\[
\frac{d}{ds} J^\sigma(\nu_s, \cdot) = -\int_0^T \left[ \int_{\mathbb{R}^d \times \mathcal{S}} \left( \nabla_a \frac{\delta H^0}{\delta m} \right)(X^{\xi,\zeta}_{s,t}, P^{\xi,\zeta}_{s,t}, \nu_{s,t}, a, \zeta) \right] \cdot \left[ b_{s,t}(a) + \frac{\sigma^2}{2} \nabla_a \log \nu_{s,t}(a) \right] \nu_{s,t}(da) dt.
\]

(17)

The complete proof of Theorem 4 will come in Section 3.2 but a sketch is given in Section 3.1.

**Theorem 5 (Necessary condition for optimality)** Fix \( \sigma \geq 0 \). Let Assumption 1 hold. If \( \nu \in V_2 \) is (locally) optimal for \( J^{\sigma, M} \) given by (13), \( X^{\xi,\zeta} \) and \( P^{\xi,\zeta} \) are the associated optimally controlled state and adjoint processes for data \((\xi, \zeta)\), given by (10) and (15) respectively, then for any other \( \mu \in V_2 \) we have

i) For a.a. \( t \in (0, T) \)

\[
\int \left( \int_{\mathbb{R}^d \times \mathcal{S}} \frac{\delta H^0}{\delta m}(X^{\xi,\zeta}_t, P^{\xi,\zeta}_t, \nu_t, a, \zeta) \right) M(d\xi, d\zeta) + \frac{\sigma^2}{2} \log \nu_t(a) - \frac{\sigma^2}{2} U(a) \right) (\mu_t - \nu_t)(da) \geq 0.
\]
\textbf{Assumption 6 (For existence, uniqueness and invariant measure)}

For a.a. $t \in (0, T)$ there exists $\varepsilon > 0$ (small and depending on $\mu_t$) such that

\[
\int_{\mathbb{R}^d \times S} H^\sigma_t(X_t^{\xi,\zeta}, P_t^{\xi,\zeta}, \nu_t + \varepsilon(\mu_t - \nu_t)) \mathcal{M}(d\xi, d\zeta) \geq \int_{\mathbb{R}^d \times S} H^\sigma_t(X_t^{\xi,\zeta}, P_t^{\xi,\zeta}, Z_t^{\xi,\zeta}, \nu_t) \mathcal{M}(d\xi, d\zeta).
\]

In other words, the optimal relaxed control $\nu \in \mathcal{V}_2$ locally minimizes the Hamiltonian $H^\sigma$.

We remark that we also prove a sufficient optimality condition, but do not state it here.

The proof of Theorem 5 is postponed until Section 3.2. However it tells us that if $\nu \in \mathcal{V}_2$ is (locally) optimal then it must solve (together with the forward and adjoint processes) the following system:

\[
\begin{align*}
\nu_t &= \arg \min_{\mu \in \mathcal{P}_2(\mathbb{R})} \int_{\mathbb{R}^d \times S} H^\sigma_t(X_t^{\xi,\zeta}, P_t^{\xi,\zeta}, \mu, \zeta) \mathcal{M}(d\xi, d\zeta), \\
\kappa_t &= \Phi_t(X_t^{\xi,\zeta}, \nu_t, \zeta) dt, \quad t \in [0, T],
\end{align*}
\]

\[
\begin{align*}
\kappa_t &= \xi \in \mathbb{R}^d, \zeta \in S, \\
\kappa_t &= \frac{\sigma^2}{2}(\nabla_a U)(\theta_{s,t}) ds + \sigma dB_s, \quad s \in I,
\end{align*}
\]

Let $\{\theta^0_t\}_{t \in [0, T]}$ be an $\mathbb{R}^p$-valued stochastic process on this space. Let $\Omega := \Omega^B \times \Omega^\theta \times \mathbb{R}^d \times \mathcal{S}$, $\mathcal{F} := \mathcal{F}^B \otimes \mathcal{F}^\theta \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathcal{S})$ and $\mathbb{P} := \mathbb{P}^B \otimes \mathbb{P}^\theta \otimes \mathcal{M}$. Let $I := [0, \infty)$. Consider the mean-field system given by:

\[
d\theta_{s,t} = - \left( \int_{\mathbb{R}^d \times S} (\nabla_a h_t)(X_{s,t}^{\xi,\zeta}, P_{s,t}^{\xi,\zeta}, \theta_{s,t}, \zeta) \mathcal{M}(d\xi, d\zeta) + \frac{\sigma^2}{2}(\nabla_a U)(\theta_{s,t}) \right) ds + \sigma dB_s, \quad s \in I,
\]

where $\{\theta^0_t\}_{t \in [0, T]}$ is a given initial condition and where for each $s \geq 0$

\[
\begin{align*}
\nu_{s,t} &= L(\theta_{s,t}), \quad t \in [0, T], \\
X_{s,t}^{\xi,\zeta} &= \xi + \int_0^t \Phi_r(X_{s,r}^{\xi,\zeta}, \nu_{s,r}, \zeta) dr, \quad t \in [0, T], \\
P_{s,t}^{\xi,\zeta} &= (\nabla_a g)(X_{T}^{\xi,\zeta}, \zeta) + \int_0^T (\nabla_a H_r)(X_{s,r}^{\xi,\zeta}, P_{s,r}^{\xi,\zeta}, \nu_{s,r}, \zeta) dr, \quad t \in [0, T].
\end{align*}
\]

\textbf{Assumption 6 (For existence, uniqueness and invariant measure)}

\begin{enumerate}
\item[i)] Let $\int_0^T \mathbb{E}[|\theta^0_t|^2] dt < \infty$.
\item[ii)] Let $\nabla_a U$ be Lipschitz continuous in $a$ and moreover let there $\kappa > 0$ such that:

\[
(\nabla_a U(a') - \nabla_a U(a)) \cdot (a' - a) \geq \kappa |a' - a|^2, \quad a, a' \in \mathbb{R}^p.
\]
\item[iii)] One of the two following conditions holds:
\end{enumerate}
Theorem 7 Let Assumptions 1 and 6 hold. Then

\[ \text{Let } \mathcal{M} \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{S}) \text{ and } \mathcal{M} \text{ has compact support.} \]

To introduce topology on the space \( \mathcal{V}_2 \) we define the integrated Wasserstein metric as follows. Let \( q = 1, 2 \) and let \( \mu, \nu \in \mathcal{V}_2 \). Then

\[
W_q^T(\mu, \nu) := \left( \int_0^T W_q(\mu_t, \nu_t)^q \, dt \right)^{1/q},
\]

where \( W_q \) is the usual Wasserstein metric on \( \mathcal{P}_q(\mathbb{R}^p) \).

In Lemma 21 we show that Assumptions 1 and 6 imply that for any \( \mathcal{M} \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{S}) \) there exists \( L > 0 \) such that for all \( a, a' \in \mathbb{R}^p \) and \( \mu, \mu' \in \mathcal{V}_2 \)

\[
|(\nabla_a h_t)(a, \mu, \mathcal{M}) - (\nabla_a h_t)(a', \mu', \mathcal{M})| \leq L |a - a'| + W_1^T(\mu, \mu').
\]

**Theorem 7** Let Assumptions 1 and 6 hold. Then (19)-(20) has a unique solution. Moreover, assume that \( J \) defined in (13) is bounded from below and that there exists \( \nu_0 \) such that \( J^{\sigma}(\nu) < \infty \) and that \( \sigma > 0 \). Then

\[ i) \text{ arg min}_{\nu \in \mathcal{V}_2} J^{\sigma}(\nu) \neq \emptyset, \]

\[ ii) \text{ if } \nu^* \in \text{ arg min}_{\nu \in \mathcal{V}_2} J^{\sigma}(\nu) \text{ then for a.a. } t \in (0, T) \text{ we have that} \]

\[ h_t(a, \nu^*, \mathcal{M}) + \frac{\sigma^2}{2} \log(\nu^*(a)) + \frac{\sigma^2}{2} U(a) \text{ is constant for a.a. } a \in \mathbb{R}^p \]

and \( \nu^* \) is an invariant measure for (19)-(20). Moreover,

\[ iii) \text{ if } \sigma^2 \kappa - 4L > 0 \text{ then } \nu^* \text{ is unique and for any solution } \theta_{s,t} \text{ to (19)-(20) we have that} \]

\[
W_2^T(\mathcal{L}(\theta_{s,\cdot}), \nu^*)^2 \leq e^{-\lambda s} W_2^T(\mathcal{L}(\theta_{0,\cdot}), \nu^*)^2.
\]

Theorem 7 is proved in Section 3.3. Let us point out that parts i) and ii) are proved in Hu et al. (2019a) and are included for completeness. Part iii) is proved in Hu et al. (2019a) under different assumptions.

Let us now consider the particle approximation and propagation of chaos property. Consider a sequence \((\xi^i, \zeta^i)_{i=1}^{N_2}\) of i.i.d copies of \((\xi, \zeta)\) and let \( \mathcal{M}_i^{N_1} := \frac{1}{N_1} \sum_{j=1}^{N_1} \delta_{\xi^i_j, \zeta^i_j} \). Furthermore, we assume that initial distribution of weights \((\theta^i_0)_{i=1}^{N_2}\) are i.i.d copies of \( (\theta^0) \) and that \((B^i_{t \in [0,T]}\) are independent Brownian motions and we extend our probability space to accommodate these. For \( s \in [0, S], t \in [0, T] \) and \( 1 \leq i \leq N_2 \) define

\[
\theta^i_{s,t} = \theta^i_{0,t} - \int_0^s \left( (\nabla_a h_t)(\theta^i_{v,t}, \nu^i_{v,t}, \mathcal{M}^{N_1}) + \frac{\sigma^2}{2} (\nabla_a U)(\theta^i_{v,t}) \right) \, dv + \sigma B^i_s,
\]

where \( \nu^i_{v,t} \in \mathcal{V}_2 \) is the empirical measures defined as \( \nu^i_{v,t} = \frac{1}{N_2} \sum_{j=1}^{N_2} \delta_{\theta^i_{v,t}} \), and where \( h_t \) is defined in (16).
Theorem 8 Let Assumptions 1 and 6 hold. Fix \( \lambda = \frac{\sigma^2 \kappa}{2} - \frac{L}{2}(3 + T) + \frac{1}{2} \). Then, there exists a unique solution to (23) with \( \mathcal{L}(\theta^{i}_{s,t}) \in \mathcal{V}_2 \). Moreover, for \( (\theta^{i,\infty}_{s,t}) \) solution to

\[
\theta^{i,\infty}_{s,t} = \theta^{0}_{0,t} - \int^{s}_0 \left( \nabla_a h_t(\theta^{i,\infty}_{v,t}, \mathcal{L}(\theta^{i,\infty}_{v,t}), \mathcal{M}) + \frac{\sigma^2}{2} \nabla_a U(\theta^{i,\infty}_{v,t}) \right) \, dv + \sigma B^i_s, \quad 0 \leq s \leq S, \quad 0 \leq t \leq T,
\]

there exists \( c \), independent of \( s, N_1, N_2, p, d \), such that, for all \( i \)

\[
\int_{0}^{T} \mathbb{E} \left[ \left| \theta^{i,\infty}_{s,t} - \theta^{i,\infty}_{s,t} \right|^2 \right] \, dt \leq \frac{c}{\lambda} \left( 1 - e^{-\lambda s} \right) \left( \frac{1}{N_1} + \frac{1}{N_2} \right).
\]

The proof of Theorem 8 is given in Section 3.4 (see Lemma 24 for the well-posedness and Theorem 25 for propagation of chaos).

Euler–Maruyama approximations with non-homogeneous time steps can be used to obtain an algorithm for the gradient descent (23). Fix an increasing sequence \( 0 = s_0 < s_1 < s_2 \cdots \) and define the family of processes \( (\tilde{\theta}^{i}_{t})_{t \in \mathbb{N}, 0 \leq t \leq T} \) satisfying, for \( i = 1, \ldots, N_2 \) and \( l \in \mathbb{N} \),

\[
\tilde{\theta}^{i}_{t+1, t} = \tilde{\theta}^{i}_{t, t} - \left( \nabla_a h_t \left( \tilde{\theta}^{i}_{t, t}, \mathcal{M}^{N_i} \right) + \frac{\sigma^2}{2} \nabla_a U(\tilde{\theta}^{i}_{t, t}) \right) (s_{l+1} - s_l) + \sigma (B^{i}_{s_{l+1}} - B^{i}_{s_l}), \quad (24)
\]

where \( \mathcal{M}^{N_i} = \frac{1}{N_2} \sum_{j \geq 2} 1/\delta^2 \). The error estimate for this discretisation is given in the theorem below.

Theorem 9 Let Assumptions 1 and 6 hold. Assume also that \( (s_{l})_{l \geq 1} \) is a non-decreasing sequence of times, starting from 0, such that the increments \( (s_{l} - s_{l-1})_{l \geq 1} \) are positive and non-increasing, \( \sum_{l \geq 1} (s_{l} - s_{l-1})^2 < \infty \) and that, \( \kappa \) is large enough so that

\[
\max_{l \geq 1} (s_{l} - s_{l-1}) < \frac{\sigma^2 \kappa - L}{2L \left( 1 + \frac{\sigma^2}{2} \left\| \nabla_a U \right\|_{Lip}^2 \right)}.
\]

Then, for all \( i, l \),

\[
\mathbb{E} \left[ \int_{0}^{T} |\tilde{\theta}^{i}_{s,t} - \theta^{i}_{s,t}|^2 \, dt \right] \leq c \max_{1 \leq \nu \leq l} (s_{l, \nu} - s_{\nu-1}) \left( 1 + \max_{0 \leq s \leq s_l} \int_{0}^{T} \mathbb{E} \left[ |\theta^{i}_{s,t}|^2 \right] \, dt \right).
\]

The proof of this theorem is in Appendix D, where we will also briefly discuss the additional discretization along the time variable \( t \).

2.3 Generalisation error

Recall \( \nu^{\sigma, N_1, N_2, \Delta s}_{s, t} \) denote distribution over parameter space induced by gradient algorithm when training with \( N_1 \) data samples, finite number of model parameters (the number of which is \( \approx N_2 \times p \times n \), where \( n \) is the number of grid points of \([0, T])\), the learning rate \( \Delta s = \max_{0 \leq s \leq S} (s_{l} - s_{l-1}) \), and training time \( S \). In the next theorem we establish a bound for the generalisation error \( J^M(\nu^{\sigma, N_1, N_2, \Delta s}_{s, t}) \).
Theorem 10 Let Assumptions 1 and 6 hold. Assume that \( \sigma^2 \) is sufficiently large relative to \( L \) and \( T \). Then there is \( c > 0 \) independent of \( \lambda, S, N_1, N_2, d, p \) and the time partition used in Theorem 9 such that
\[
\mathbb{E} \left[ \left| J^{0,\mathcal{M}}(\nu^* \sigma) - J^{0,\mathcal{M}}(\nu^* \sigma, N_1, N_2, \Delta s) \right|^2 \right] \leq c \left( e^{-\lambda S} + \frac{1}{N_1} + \frac{1}{N_2} + h \right),
\]
where \( h := \max_{0 < s_i < S} (s_t - s_{t-1}) \). The generalisation error is given by
\[
J^{0,\mathcal{M}}(\nu^* \sigma, N_1, N_2, \Delta s) = J^{0,\mathcal{M}}(\nu^* \sigma, N_1, N_2, \Delta s) - J^{0,\mathcal{M}}(\nu^* \sigma) - \frac{\sigma^2}{2} \int_0^T \text{Ent}(\nu^*_\mu) \, dt + \min_{\mu \in \mathcal{M}} J^{\sigma,\mathcal{M}}(\mu),
\]
since \( \min_{\mu \in \mathcal{M}} J^{\sigma,\mathcal{M}}(\mu) = J^{\sigma,\mathcal{M}}(\nu^* \sigma) = J^{0,\mathcal{M}}(\nu^* \sigma) + \frac{\sigma^2}{2} \int_0^T \text{Ent}(\nu^*_\mu) \, dt. \)

Theorem 10 tells us that the generalisation error consist of three errors: a) the numerical error of approximating an invariant measure with discrete time particle system, b) the relative entropy between the Gibbs measure \( \gamma \) (a prior) and the \( \nu^* \sigma \), c) the minimum value of the cost function under population measure \( J^{\sigma,\mathcal{M}}. \)

The proof of Theorem 10 is postponed until Section 3.6.

2.4 Conditional generalisation error

It is a common practice to terminate the training when \( J^{0,N_1} \) is negligible and such models have been observed to generalise well. To link our results to such regime we postulate the following assumption.

Assumption 11 Fix \( \varepsilon > 0 \) and \( N_1 > 0 \). Assume that \( J^{N_1}(\nu^* \sigma, N_1) \leq \varepsilon. \)

Theorem 12 Let Assumptions 1, 6 and 11 hold. Then there is \( c > 0 \) independent of \( \lambda, S, N_1, N_2, d, p \) and the time partition used in Theorem 9 such that
\[
\mathbb{E} \left[ \left| J^{\mathcal{M}}(\nu^* \sigma, N_1, N_2, \Delta s) \right|^2 \right] \leq \varepsilon^2 + c \left( e^{-\lambda S} + \frac{1}{N_1} + \frac{1}{N_2} + h \right),
\]

Proof We decompose the error as follows
\[
J^{0,\mathcal{M}}(\nu^* \sigma, N_1, N_2, \Delta s)
= \left( J^{0,\mathcal{M}}(\nu^* \sigma, N_1, N_2, \Delta s) - J^{0,\mathcal{M}}(\nu^* \sigma) \right) + \left( J^{0,\mathcal{M}}(\nu^* \sigma) - J^{0,\mathcal{M}}(\nu^* \sigma) \right) + J^{0,\mathcal{M}}(\nu^* \sigma).
\]

The bound on first term follows from Theorem 10. Next there exists \( c > 0 \) such that
\[
\mathbb{E} \left[ \left| J^{\mathcal{M}}(\nu^* \sigma, N_1) - J^{\mathcal{M}}(\nu^* \sigma, N_1) \right|^2 \right] = \mathbb{E} \left[ \left| \int J(\nu^* \sigma, \xi, \zeta)(\mathcal{M}^{N_1} - \mathcal{M})(d\xi, d\zeta) \right|^2 \right] \leq \frac{c}{N_1}.
\]

The proof is complete.
3. Proofs

3.1 Outline of proof of Theorem 4

Before we proceed to proofs of the main result in full generality we present a sketch the the proof of Theorem 4. For brevity we take $f = 0$. Our aim is to solve this control problem using a (stochastic) gradient descent algorithm. The goal is to find, for each $t \in [0, T]$ a vector field flow $(b_{s,t})_{s \geq 0}$ such that the measure flow $(\nu_{s,t})_{s \geq 0}$ given by

$$
\partial_s \nu_{s,t} = \text{div} \left( \nu_{s,t} b_{s,t} + \frac{\sigma^2}{2} \nabla_a \nu_{s,t} \right), \quad s \geq 0, \quad \nu_{0,t} = \nu_t^0 \in \mathcal{P}_2(\mathbb{R}^p),
$$

(26)
satisfies that $s \mapsto J^\sigma(\nu_{s,\cdot})$ is decreasing. The aim is to compute $\frac{d}{ds} J(\nu_{s,\cdot})$, in terms of $b_{s,\cdot}$, and use this expression to choose $b_{s,\cdot}$ such that the derivative is negative. Let us keep $(\xi, \zeta)$ fixed and use $X_{s,t}$ for the solution of (10) when the control is given by $\nu_{s,\cdot}$. Let $V_{s,t} := \frac{d}{ds} X_{s,t}$. Let $B_{s,t} := b_{s,t} + \frac{\sigma^2}{2} \nabla_a \nu_{s,t}$. We will show (see Lemma 14) that

$$
dV_{s,t} = \left[ (\nabla \Phi)(X_{s,t}, \nu_{s,t}, \xi, \zeta) V_{s,t} - \int (\nabla \phi_t)(X_{s,t}, \nu_{s,t}, a, \xi, \zeta) B_{s,t}(a) \nu_{s,t}(da) \right] dt .
$$

Since the equation is affine we can write its solution using an integrating factor

$$
V_{s,t} = - \int_0^t \int I(r, t; \nu_{s,\cdot})(\nabla \phi_r)(X_{s,r}, a, \xi, \zeta) B_{s,r}(a) \nu_{s,r}(da) dr .
$$

Further (see Lemma 17) we can show that

$$
\frac{d}{ds} J^\sigma(\nu_{s,\cdot}, \xi, \zeta) = - \int_0^T \int \left[ (\nabla \phi_r)(X_{s,r}, a, \xi, \zeta) I(r, T; \xi, \nu_{s,\cdot})(\nabla \phi_r)(X_{s,r}, a, \xi, \zeta) + \frac{\sigma^2}{2} \frac{\nabla_a \nu_{s,t}(a)}{\nu_{s,t}(a)} + \nabla_a U(a) \right] B_{s,r}(a) \nu_{s,r}(da) dr .
$$

Now we define

$$
P_{s,r}^{\xi, \zeta} := I^{\xi, \zeta}(r, T; \nu_{s,\cdot})(\nabla \phi_r)(X_{s,r}, \xi, \zeta)
$$

so that

$$
\frac{d}{ds} J^\sigma(\nu_{s,\cdot}, \xi, \zeta) = - \int_0^T \int \left[ (\nabla \phi_r)(X_{s,r}, \xi, \zeta) P_{s,r}^{\xi, \zeta} \nabla a U(a) \right] B_{s,r}(a) \nu_{s,r}(da) dr .
$$

After integrating over $\xi, \zeta$ w.r.t. $\mathcal{M}$ we get

$$
\frac{d}{ds} J^\sigma(\nu_{s,\cdot}) = - \int_0^T \int \left[ \int_{\mathbb{R}^d \times \mathcal{S}} (\nabla \phi_r)(X_{s,r}^{\xi, \zeta}, a, \xi, \zeta) P_{s,r}^{\xi, \zeta} \mathcal{M}(d\xi, d\zeta) + \frac{\sigma^2}{2} \frac{\nabla_a \nu_{s,t}(a)}{\nu_{s,t}(a)} + \nabla_a U(a) \right] B_{s,r}(a) \nu_{s,r}(da) dr .
$$

At this point it is clear how to choose the flow to make this negative: we must take

$$
b_{s,r}(a) := \int_{\mathbb{R}^d \times \mathcal{S}} (\nabla \phi_r)(X_{s,r}^{\xi, \zeta}, a, \xi, \zeta) P_{s,r}^{\xi, \zeta} \mathcal{M}(d\xi, d\zeta) + \frac{\sigma^2}{2} \nabla_a U(a)
$$
so that
\[
\frac{d}{ds} J^\sigma(\nu_{s,}) = -\int_0^T \int_{\mathbb{R}^d \times S} (\nabla_a \phi_t)(X^\xi,\zeta_{s,r},a,\zeta) P^\xi,\zeta_{s,r} M(d\xi, d\zeta) + \frac{\sigma^2}{2} \left( \nabla_a \nu_{s,t}(a) + \nabla_a U(a) \right)^2 \nu_{s,r}(da) dr \leq 0.
\]

Moreover, Theorem 7. says that for \( \sigma > 0 \) the \( \nu^* \) minimizing \( J^\sigma \) exists and satisfies the following first order condition: for a.a. \( t \in (0, T) \) we have
\[
\int_{\mathbb{R}^d \times S} \phi_t(X^\xi,\zeta_t(a),\zeta) P^\xi,\zeta_t M(d\xi, d\zeta) + \frac{\sigma^2}{2} \log(\nu_t^*(a)) + \frac{\sigma^2}{2} U(a) \text{ is constant for a.a. } a \in \mathbb{R}^p,
\]
where \( P^\xi,\zeta_t := I^{\xi,\zeta}(t, T; \nu^*)((\nabla_a g)(X^{\xi,\zeta}_t(\nu^*), \zeta) \). Such first order condition is essentially another way of stating the necessary condition from Pontryagin optimality principle. Here we will provide a derivation from first principles for measure-valued control processes with entropy regularization. We note that a simple calculation using Itô’s formula shows that the law in (26) with the choice of \( b_{s,r} \) made above is the law of
\[
d\theta_{s,t} = -\left( \int_{\mathbb{R}^d \times S} (\nabla_a \phi_t)(X^t_{s,t}(\mathcal{L}(\nu_{s,}), \theta_{s,t}, \zeta) P^\xi,\zeta_t(\mathcal{L}(\nu_{s,})) M(d\xi, d\zeta) - \frac{\sigma^2}{2} U(\theta_{s,t}) \right) ds + \sigma dB_s
\]
i.e. \( \mathcal{L}(\theta_{s,}) = \nu_{s,} \).

### 3.2 Proof of Theorem 4 and related optimality conditions

In this section we derive Pontryagin’s optimality principle for the relaxed control problem by proving Theorems 4 and 5. Note that later we also prove the Pontryagin sufficient condition for optimality in Theorem 19. This is done purely for completeness as Theorem 19 is not used anywhere in the analysis carried out in this work.

We will work with an additional control \( (\mu_t)_{t \in [0, T]} \) and define \( \nu^\varepsilon := \nu_t + \varepsilon (\mu_t - \nu_t) \). In case \( \sigma \neq 0 \) assume that \( \mu_t \) are absolutely continuous w.r.t. the Lebesgue measure for all \( t \in [0, T] \). We will write \( (X^\xi,\zeta_t)_{t \in [0, T]} \) for the solution of (10) driven by \( \nu \) and \( (X^{\xi,\zeta^\varepsilon}_t)_{t \in [0, T]} \) for the solution of (10) driven by \( \nu^\varepsilon \) both with the data \( (\xi, \zeta) \). Moreover, let \( V_0 = 0 \) be fixed and
\[
\frac{dV^\xi,\zeta_t}{\varepsilon} = \left[ (\nabla_a \Phi_t)(X^\xi,\zeta_t, \nu_t, \zeta) V_t + \int \frac{\delta \Phi_t}{\delta m}(X^\xi,\zeta_t, \nu_t, a, \zeta)(\mu_t - \nu_t)(da) \right] dt. \quad (27)
\]
We observe that this is a linear equation. Let
\[
V^\xi,\zeta^\varepsilon := \frac{X^\xi,\zeta^\varepsilon - X^\xi,\zeta}{\varepsilon} - V^\xi,\zeta \text{ i.e. } X^\xi,\zeta^\varepsilon = X^\xi,\zeta + \varepsilon (V^\xi,\zeta^\varepsilon + V^\xi,\zeta).
\]

**Lemma 13** Under Assumption 1 we have
\[
\limsup_{\varepsilon \to 0, t \leq T} \left| \frac{X^\xi,\zeta^\varepsilon - X^\xi,\zeta}{\varepsilon} - V^\xi,\zeta \right|^2 = 0.
\]
Proof Since we will be working with $\xi, \zeta$ fixed we will omit them from the notation in the proof. We note that

$$
\Phi_t(X_t^\varepsilon, \nu_t^\varepsilon) - \Phi_t(X_t, \nu_t) = \Phi_t(X_t^\varepsilon, \nu_t^\varepsilon) - \Phi_t(X_t^\varepsilon, \nu_t) + \Phi_t(X_t^\varepsilon, \nu_t) - \Phi_t(X_t, \nu_t)
$$

$$
= \varepsilon \int_0^1 (\nabla_x \Phi_t)(X_t + \lambda \varepsilon (V_t^\varepsilon + V_t), \nu_t)(V_t^\varepsilon + V_t) d\lambda + \varepsilon \int_0^1 \frac{d\Phi_t}{dm}(X_t^\varepsilon, (1 - \lambda)\nu_t^\varepsilon + \nu_t, a)(\mu_t - \nu_t)(da) d\lambda.
$$

Hence

$$
\frac{1}{\varepsilon} \left[ \Phi_t(X_t^\varepsilon, \nu_t^\varepsilon) - \Phi_t(X_t, \nu_t) - \varepsilon(\nabla_x \Phi_t)(X_t, \nu_t)V_t - \varepsilon \int \frac{d\Phi_t}{dm}(X_t, \nu_t, a)(\mu_t - \nu_t)(da) \right]
$$

$$
= \int_0^1 (\nabla_x \Phi_t)(X_t + \lambda \varepsilon (V_t^\varepsilon + V_t), \nu_t)V_t^\varepsilon d\lambda + \int_0^1 [(\nabla_x \Phi_t)(X_t + \lambda \varepsilon (V_t^\varepsilon + V_t), \nu_t) - (\nabla_x \Phi_t)(X_t, \nu_t)] V_t d\lambda
$$

$$
+ \int_0^1 \int \left[ \frac{d\Phi_t}{dm}(X_t^\varepsilon, (1 - \lambda)\nu_t^\varepsilon + \nu_t, a) - \frac{d\Phi_t}{dm}(X_t, \nu_t, a) \right] (\mu_t - \nu_t)(da) d\lambda =: I_t^{(0)} + I_t^{(1)} + I_t^{(2)} =: I_t.
$$

Note that

$$
dV_t^\varepsilon = \frac{1}{\varepsilon}[dX_t^\varepsilon - dX_t] - dV_t
$$

and so we then see that $d|V_t^\varepsilon|^2 = 2V_t^\varepsilon I_t dt$. Hence we have that

$$
\sup_{s \leq t} |V_s^\varepsilon|^2 \leq c \int_0^t |I_s|^2 ds.
$$

Let us now consider

$$
\int_0^T |I_t^{(2)}|^2 dt \leq 2 \int_0^T \left| \int_0^1 \left[ \frac{d\Phi_t}{dm}(X_t^\varepsilon, (1 - \lambda)\nu_t^\varepsilon + \nu_t, a) - \frac{d\Phi_t}{dm}(X_t, \nu_t, a) \right] \mu_t(da) d\lambda \right|^2 dt
$$

$$
+ 2 \int_0^T \left| \int_0^1 \left[ \frac{d\Phi_t}{dm}(X_t^\varepsilon, (1 - \lambda)\nu_t^\varepsilon + \nu_t, a) - \frac{d\Phi_t}{dm}(X_t, \nu_t, a) \right] \nu_t(da) d\lambda \right|^2 dt.
$$

Taking the terms separately and using the fact that we are working with probability measures we see that

$$
\int_0^1 \left| \int_0^1 \left[ \frac{d\Phi_t}{dm}(X_t^\varepsilon, (1 - \lambda)\nu_t^\varepsilon + \nu_t, a) - \frac{d\Phi_t}{dm}(X_t, \nu_t, a) \right] \mu_t(da) d\lambda \right|
$$

$$
\leq \int_0^1 \left| \int_0^1 \left[ \frac{d\Phi_t}{dm}(X_t^\varepsilon, (1 - \lambda)\nu_t^\varepsilon + \nu_t, a) - \frac{d\Phi_t}{dm}(X_t, \nu_t, a) \right] \mu_t(da) d\lambda \right|
$$

$$
= \int_0^1 \left| \phi_t(X_t^\varepsilon, a) - \phi_t(X_t, a) + \int [\phi_t(X_t, a') - \phi_t(X_t^\varepsilon, a')] \mu_t(da') \right| \mu_t(da) d\lambda
$$

$$
\leq \int_0^1 \left| \phi_t(X_t^\varepsilon, a) - \phi_t(X_t, a) \right| + \int \left| \phi_t(X_t, a') - \phi_t(X_t^\varepsilon, a') \right| \mu_t(da') \right| \mu_t(da) d\lambda \leq L |X_t^\varepsilon - X_t|.
$$

Hence

$$
\int_0^T |I_t^{(2)}|^2 dt \leq 4L^2 \int_0^T |X_t^\varepsilon - X_t|^2 dt.
$$
Similarly
\[ \int_0^T |t^{(1)}_t|^2 \, dt \leq 4L^2 \int_0^T |X^\xi_t - X_t|^2 \, dt \, . \]
This and the assumption that the derivatives w.r.t. the spatial variable are bounded lead to
\[ \sup_{s \leq t} |V^\xi_s|^2 \leq c \int_0^t |V^\xi_s|^2 \, ds + \int_0^T |X^\xi_s - X_s|^2 \, ds \, . \]
Finally note that
\[ \delta_\varepsilon := \int_0^T |X^\xi_t - X_t|^2 \, dt \to 0 \quad \text{as} \quad \varepsilon \to \infty \, . \]
So
\[ \sup_{s \leq t} |V^\xi_s|^2 \leq c \int_0^t \sup_{s \leq \tau} |V^\xi_{\tau}|^2 \, d\tau + \delta_\varepsilon \]
and by Gronwall’s lemma $\sup_{s \leq t} |V^\xi_s|^2 \leq \delta_\varepsilon e^{CT} \to 0 \quad \text{as} \quad \varepsilon \to \infty$.

Note that we effectively have $\frac{d}{d\varepsilon}X^{\xi,\nu+\varepsilon(\mu-\nu)}|_{\varepsilon=0} = V^{\xi,\nu}$ and moreover due to the affine structure of (27) the solution can be expressed as
\[ \frac{d}{d\varepsilon}X^{\xi,\nu+\varepsilon(\mu-\nu)}|_{\varepsilon=0} = \int_0^t \int \xi^{\nu}(r,t;\nu) \frac{\partial \Phi_r}{\partial \nu}(X^{\xi,\nu},\nu_r,a,\zeta) \big( b_{s,r}(a) + \frac{\sigma^2}{2} \nabla_a \log \nu_{s,r}(a) \big) \nu_{s,r}(da) \, dr \, . \]  
(28)

**Lemma 14** Let $\sigma > 0$ be fixed and let Assumption 1 hold. Let $b$ be a permissible flow (c.f. Definition 2) with $\nu_{s,\cdot} \in \mathcal{V}_2$ the corresponding solution from Lemma 3. Let $X^{\xi,\nu}_{s,t}$ be the solution to (10) from data $(\xi,\zeta)$ with control $\nu_{s,\cdot} \in \mathcal{V}_2$. Let $V^{\xi,\nu}_{s,t} := \frac{d}{d\varepsilon}X^{\xi,\nu+\varepsilon(\mu-\nu)}|_{\varepsilon=0}$. Then
\[ V^{\xi,\nu}_{s,t} = - \int_0^t \int \xi^{\nu}(r,t;\nu) \left( \nabla_a \frac{\partial \Phi_r}{\partial \nu}(X^{\xi,\nu},\nu_{s,r},a,\zeta) \cdot \left( b_{s,r}(a) + \frac{\sigma^2}{2} \nabla_a \log \nu_{s,r}(a) \right) \nu_{s,r}(da) \right) \, dr \, . \]  
(29)

and this can be written in differential form (w.r.t time $t$) as
\[ dV^{\xi,\nu}_{s,t} = \left[ (\nabla_a \Phi_r)(X^{\xi,\nu}_{s,t},\nu_{s,t},\zeta) V^{\xi,\nu}_{s,t} \right. \]
\[ - \int \left( \nabla_a \frac{\partial \Phi_r}{\partial \nu}(X^{\xi,\nu}_{s,t},\nu_{s,t},a,\zeta) \cdot \left( b_{s,t}(a) + \frac{\sigma^2}{2} \nabla_a \log \nu_{s,t}(a) \right) \nu_{s,t}(da) \right) \, dt \, . \]  
(30)

**Proof** We will keep $(\xi,\zeta)$ fixed for the moment and hence omit it from the notation. Let us now fix as $\nu_t = \nu_{s_0,t}$ and $\mu_t = \nu_{s_1,t}$ for all $t \in [0,T]$. Define $\nu^\varepsilon_t := \nu_t + \varepsilon(\mu_t - \nu_t)$ and $\mu^\varepsilon_t := \mu_t + \varepsilon(\mu_t - \mu_t)$, so that $\mu^\varepsilon_t - \nu^\varepsilon_t = \mu_t - \nu_t$. From the Fundamental Theorem of Calculus
\[ X_t(\mu) - X_t(\nu) = \int_0^1 \lim_{\delta \to 0} \frac{1}{\delta} \left( X_t(\nu^\varepsilon + \varepsilon(\mu_t - \nu_t)) - X_t(\nu + \varepsilon(\mu_t - \nu_t)) \right) \, d\varepsilon \]
\[ = \int_0^1 \lim_{\delta \to 0} \frac{1}{\delta} \left( X_t(\nu^\varepsilon + \delta(\mu^\varepsilon - \nu^\varepsilon)) - X_t(\nu^\varepsilon) \right) \, d\varepsilon \, . \]  
19
Due to Lemma 13 and (28) we see that
\[
\lim_{\delta \to 0} \frac{1}{\delta} \left( X_t(\nu^\varepsilon + \delta(\mu^\varepsilon - \nu^\varepsilon)) - X_t(\nu^\varepsilon) \right) = \int_0^t \int I^{\xi,\zeta}(r, t; \nu^\varepsilon) \frac{\delta \Phi_r}{\delta m}(X_r(\nu^\varepsilon), \nu_r^\varepsilon, a) (\mu_r - \nu_r)(da) \, dr .
\]
Hence
\[
X_t(\mu) - X_t(\nu) = \int_0^1 \int_0^t \int I^{\xi,\zeta}(r, t; \nu^\varepsilon) \frac{\delta \Phi_r}{\delta m}(X_r(\nu^\varepsilon), \nu_r^\varepsilon, a) (\mu_r - \nu_r)(da) \, dr \, \, \, d\varepsilon .
\]
Now fix \( s \geq 0 \) and take \( \mu_t = \nu_{s+h,t}, \nu_t = \nu_{s,t} \) and \( \nu_{s,t}^\varepsilon = \nu_{s,t} + \varepsilon(\nu_{s+h,t} - \nu_{s,t}) \). Note that \( \nu_{s,t}^\varepsilon \to \nu_{s,t} \) as \( h \searrow 0 \) and hence
\[
I^{\xi,\zeta}(r, t; \nu_{s,t}^\varepsilon) \frac{\delta \Phi_r}{\delta m}(X_r(\nu_{s,t}^\varepsilon), \nu_{s,t}^\varepsilon, a) \to I^{\xi,\zeta}(r, t; \nu_{s,t}, \nu_{s,t}, a) \frac{\delta \Phi_r}{\delta m}(X_r(\nu_{s,t}, \nu_{s,t}, a))
\]
as \( h \searrow 0 \). Moreover
\[
\frac{d}{ds} X_t(\nu_{s,t}) = \lim_{h \searrow 0} \frac{1}{h} (X_t(\nu_{s+h,t}) - X_t(\nu_{s,t}))
\]
\[
= \lim_{h \searrow 0} \frac{1}{h} \int_0^1 \int_0^t \int I^{\xi,\zeta}(r, t; \nu_{s,t}^\varepsilon) \frac{\delta \Phi_r}{\delta m}(X_r(\nu_{s,t}^\varepsilon), \nu_{s,t}^\varepsilon, a) (\nu_{s+h,t} - \nu_{s,t})(da) \, dr \, \, \, d\varepsilon
\]
\[
= \int_0^1 \int_0^t \lim_{h \searrow 0} \left[ I^{\xi,\zeta}(r, t; \nu_{s,t}^\varepsilon) \frac{\delta \Phi_r}{\delta m}(X_r(\nu_{s,t}^\varepsilon), \nu_{s,t}^\varepsilon, a) \frac{1}{h} (\nu_{s+h,t} - \nu_{s,t})(a) \right] \, da \, dr \, \, \, d\varepsilon .
\]
Thus from Lemma 3 which states that the control laws evolve according to a gradient flow we get
\[
\frac{d}{ds} X_t(\nu_{s,t})
\]
\[
= \int_0^t \int I^{\xi,\zeta}(r, t; \nu_{s,t}) \frac{\delta \Phi_r}{\delta m}(X_r(\nu_{s,t}, \nu_{s,t}, a)) \nabla_a \cdot \left( b_{s,t}(a) + \frac{\sigma^2}{2} \nabla_a \log \nu_{s,t}(a) \right) \nu_{s,t}(a) \, da \, dr
\]
\[
= - \int_0^t \int I^{\xi,\zeta}(r, t; \nu_{s,t}) \left( \nabla_a \frac{\delta \Phi_r}{\delta m}(X_r(\nu_{s,t}, \nu_{s,t}, a)) \cdot \left( b_{s,t}(a) + \frac{\sigma^2}{2} \nabla_a \log \nu_{s,t}(a) \right) \nu_{s,t}(a) \right) \, da \, dr ,
\]
where the last equality is due to integration by parts. 

\begin{Lemma} \label{lemma15}
Let \( \sigma > 0 \) be fixed and let Assumption 1 hold. Let \( b \) be a permissible flow (c.f. Definition 2) with \( \nu_{s,t} \in V_2 \) the corresponding solution from Lemma 3. Then
\[
d\text{Ent}(\nu_{s,t}) = - \int \left( \nabla_a \log \nu_{s,t} + \nabla_a U \right) \cdot \left( b_{s,t} + \frac{\sigma^2}{2} \nabla_a \log \nu_{s,t} \right) \nu_{s,t}(da) \, ds .
\]
\end{Lemma}

\begin{proof}
The key part of the proof is done in (Hu et al., 2019b, Proof of Proposition 2.4).
\end{proof}
Lemma 16 Under Assumption 1 the mapping $\nu \mapsto \bar{J}^0(\nu, \xi, \zeta)$ defined by (12) satisfies
\[
\frac{d}{d\varepsilon} \bar{J}^0((\nu_t + \varepsilon(\mu_t - \nu_t))_{t \in [0,T]}, \xi, \zeta) \bigg|_{\varepsilon = 0} = \int_0^T \left[ \int f_t(X_t^\xi, a, \zeta)(\mu_t - \nu_t)(da) + \int (\nabla_x f_t)(X_t^\xi, a, \zeta)V_t^\xi \nu_t(da) \right] dt + (\nabla_x g)(X_T^\xi, \zeta)V_T^\xi.
\]

Proof We need to consider the difference quotient for $\bar{J}^0$ and to that end we consider
\[
I_\varepsilon := \frac{1}{\varepsilon} \int_0^T \left[ F(X_t^\xi, \nu_t^\varepsilon, \zeta) - F(X_t^\xi, \nu_t, \zeta) + F(X_t^\xi, \nu_t^\varepsilon, \zeta) - F(X_t^\xi, \nu_t, \zeta) \right] dt
\]
\[
= \int_0^T \int_0^1 \int \left[ f(X_t^\xi, \nu_t^\varepsilon, \zeta) - f(X_t^\xi, \nu_t, \zeta) \right] \nu_t^\varepsilon(da) d\lambda dt
\]
\[
+ \int_0^T \int_0^1 \int (\nabla_x f)(X_t^\xi, \zeta) \nu_t^\varepsilon(da) d\lambda dt
\]
\[
= \int_0^T \int f(X_t^\xi, \nu_t^\varepsilon, \zeta)(\mu_t - \nu_t)(da) dt
\]
\[
+ \int_0^T \int (\nabla_x f)(X_t^\xi, \zeta)(\mu_t^\varepsilon - \mu_t) \nu_t(da) d\lambda dt.
\]
Using Lebesgue’s dominated convergence theorem and Lemma 13 we get
\[
\lim_{\varepsilon \to 0} I_\varepsilon = \int_0^T \int f_t(X_t^\xi, a, \zeta)(\mu_t - \nu_t)(da) dt + \int_0^T \int (\nabla_x f_t)(X_t^\xi, a, \zeta)V_t^\xi \nu_t(da) dt.
\]
The term involving $g$ can be treated using the differentiability assumption and Lemma 13. 

Lemma 17 Let $\sigma > 0$ be fixed and let Assumption 1 hold. Let $b$ be a permissible flow (c.f. Definition 2) with $\nu_{s_\cdot} \in \mathcal{V}_2$ the corresponding solution from Lemma 3. Let $X_{s,t}^\xi$ be the solution to (10) from data $(\xi, \zeta)$ with control $\nu_{s_\cdot} \in \mathcal{V}_2$. Let $B_{s,t} := b_{s,t} + \frac{\sigma}{2} \nabla_a \log \nu_{s,t}$. Then
\[
\frac{d}{ds} \bar{J}^0(\nu_{s_\cdot}, \xi, \zeta) = -\int_0^T \int (\nabla_a f_t)(X_{s,t}, a, \zeta)B_{s,t}(a) \nu_{s,t}(da) dt
\]
\[
- \int_0^T (\nabla_x f)(X_{s,t}, \nu_{s,t}, \zeta) \int_0^t \int \left( \nabla_a \frac{\delta \Phi_r}{\delta m} \right)(X_r(\nu_{s_\cdot}), \nu_{s,r}, a)B_{s,r}(a) \nu_{s,r}(da) dr dt
\]
\[
- (\nabla_x g)(X_{s,T}, \zeta) \int_0^T \int \left( \nabla_a \frac{\delta \Phi_r}{\delta m} \right)(X_r(\nu_{s_\cdot}), \nu_{s,r}, a)B_{s,t}(a) \nu_{s,r}(da) dt.
\]

Proof We will keep $(\xi, \zeta)$ fixed for the moment and hence omit it from the notation. Let us now fix $\nu_t = \nu_{s_{0,t}}$ and $\mu_t = \nu_{s_{1,t}}$ for all $t \in [0,T]$. Define $\nu_t^\varepsilon := \nu_t + \varepsilon(\mu_t - \nu_t)$ and $\mu_t^\varepsilon := \mu_t + \varepsilon(\mu_t - \mu_t)$, so that $\mu_t^\varepsilon - \nu_t^\varepsilon = \mu_t - \nu_t$. From the Fundamental Theorem of Calculus
\[
\bar{J}^0(\mu) - \bar{J}^0(\nu) = \int_0^1 \lim_{\delta \downarrow 0} \frac{1}{\delta} \bar{J}^0(\nu + \varepsilon(\mu - \nu)) - \bar{J}^0(\nu + \varepsilon(\mu - \nu)) d\varepsilon
\]
\[
= \int_0^1 \lim_{\delta \downarrow 0} \frac{1}{\delta} \left( \bar{J}^0(\nu^\varepsilon + \delta(\varepsilon - \nu^\varepsilon)) - \bar{J}^0(\nu^\varepsilon) \right) d\varepsilon.
\]
Due to Lemma 16 we see that
\[
\mathcal{J}^0(\mu) - \mathcal{J}^0(\nu) = \int_0^1 \left( \int_0^T \left[ f_t(X_t(\nu^s)) \left( \mu_t - \nu_t \right)(a) + \langle \nabla_x F_t(X_t(\nu^s)) \rangle V_t^a \right] dt + \langle \nabla_x g(X_T(\nu^s)) \rangle V_T^a \right) d\nu.
\]

Now fix \( s \geq 0 \) and take \( \mu_t = \nu_{s+h,t}, \nu_t = \nu_{s,t} \) and \( \nu_{s,t}^h = \nu_{s,t} + \varepsilon(\nu_{s+h,t} - \nu_{s,t}) \) and note that \( \nu_{s,t}^h \to \nu_{s,t} \) as \( h \to 0 \). Then
\[
\frac{d}{ds} \mathcal{J}^0(\nu_{s,\cdot}) = \lim_{h \to 0} \frac{1}{h} \left( \mathcal{J}^0(\nu_{s+h,\cdot}) - \mathcal{J}^0(\nu_{s,\cdot}) \right)
\]
\[
= \int_0^1 \left( \int_0^T \lim_{h \to 0} \left[ f_t(X_t(\nu_{s,t}^h), a) \frac{1}{h} (\nu_{s+h,t} - \nu_{s,t}) \right] (da) dt 
+ \int_0^T \int_0^t \lim_{h \to 0} \left[ \langle \nabla_x F_t(X_t(\nu_{s,t}^h)) \rangle I(r, t; \nu_{s,r}^h, a) \right] \frac{\delta \Phi_r}{\delta m}(X_r(\nu_{s,r}^h), \nu_{s,r}^h, a) \frac{1}{h} (\nu_{s+h,r} - \nu_{s,r}) (da) dr dt 
+ \int_0^T \lim_{h \to 0} \left[ \langle \nabla_x g(X_T(\nu_{s,r}^h)) \rangle I(r, T; \nu_{s,r}^h, a) \right] \frac{\delta \Phi_T}{\delta m}(X_T(\nu_{s,r}^h), \nu_{s,r}^h, a) \frac{1}{h} (\nu_{s+h,r} - \nu_{s,r}) (da) dr \right) d\nu.
\]

Thus from Lemma 3 which states that the control laws evolve according to a gradient flow we get
\[
\int \lim_{h \to 0} \left[ f_t(X_t(\nu_{s,t}^h), a) \frac{1}{h} (\nu_{s+h,t} - \nu_{s,t}) \right] (da)
= \int f_t(X_t(\nu_{s,\cdot}, a) \nabla_a \cdot \left( b_{s,r}(a) + \frac{\sigma^2}{2} \frac{\nabla_a \nu_{s,r}(a)}{\nu_{s,r}(a)} \right) \nu_{s,r}(a) da
= - \int (\nabla_a f_t)(X_t(\nu_{s,\cdot}, a) \cdot B_{s,t} \nu_{s,r}(a) da,
\]
where the last equality is due to integration by parts. The integrands in the other two integrals are treated similarly and so the result follows.

**Proof** [Proof of Theorem 4] Recall that for each \( (\xi, \zeta) \) and each \( s \geq 0 \) we have that \( X_{\xi,\zeta}^s \) is the forward process arising in (10) with control \( \nu_{s,\cdot} \in \mathcal{V}_2 \). From Lemmas 15 and 17 we have that
\[
\frac{d}{ds} \mathcal{J}^0(\nu_{s,\cdot}, \xi, \zeta) = - \int_0^T \int \left[ (\nabla_a f)(X_{s,t}, a, \xi, \zeta) + \frac{\sigma^2}{2} \frac{\nabla_a \nu_{s,t}(a)}{\nu_{s,t}(a)} + \nabla_a U(a) \right] B_{s,t}(a) \nu_{s,t}(a) (da) dt
- \int_0^T (\nabla_x F)(X_{s,t}, \nu_{s,t}) \int_0^t I(r, t, \nu_{s,\cdot}) \left( \nabla_a \frac{\delta \Phi_r}{\delta m} \right) (X_{s,r}, a) B_{s,r}(a) \nu_{s,r}(a) (da) dr dt
- (\nabla_x g)(X_{s,T}, \zeta) \int_0^T I(t, T, \nu_{s,\cdot}) \left( \nabla_a \frac{\delta \Phi_t}{\delta m} \right) (X_{s,t}, a) B_{s,t}(a) \nu_{s,t}(a) (da) dt.
\]

We now perform a change of order of integration in the triangular region
\[
\{(r, t) \in \mathbb{R}^2 : 0 \leq t \leq T, \ 0 \leq r \leq t\} = \{(r, t) \in \mathbb{R}^2 : 0 \leq r \leq T, \ r \leq t \leq T\}
\]

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which transforms (31) into
\[
\frac{d}{ds} J^\sigma (v_{s,}, \xi, \zeta) = - \int_0^T \int \left[ (\nabla_a f_t)(X_{s,t}, a, \zeta) + \frac{\sigma^2}{2} \nabla_a v_{s,t}(a) \right] B_{s,t}(a) v_{s,t}(da) dt \\
- \int_0^T \int \int I(t, t', v_{s,-}) \left( \nabla_a \frac{\delta F_{t'}}{\delta m} \right) \left( (X_{s,t}, a)(\nabla_x F)(X_{s,t}, v_{s,t}, \zeta) B_{s,t}(a) v_{s,t}(da) dt dr \\
- \int_0^T \int I(t, T, v_{s,-}) \left( \nabla_a \frac{\delta F_T}{\delta m} \right) \left( (X_{s,t}, a)(\nabla_x g)(X_{s,T}, \zeta) B_{s,t}(a) v_{s,t}(da) dt .
\right.
\]

Now we recall that for each \((\xi, \zeta)\) and each \(s \geq 0\) we have that \(P_{s,-}^{\xi, \zeta}\) is the backward process arising in (15) from the forward process \(X_{s,-}^{\xi, \zeta}\) and the control \(v_{s,-} \in \mathcal{V}_2\). We can see that since (15) is affine
\[
P_{s,t} = (\nabla_x g)(X_{s,T}, \zeta) I(r, T, v_{s,-}) + \int_r^T (\nabla_x F)(X_{s,t}, v_{s,t}, \zeta) I(r, T, v_{s,-}) dt
\]
so that (32) can be written as
\[
\frac{d}{ds} J^\sigma (v_{s,}, \xi, \zeta) = - \int_0^T \int \left[ (\nabla_a f_t)(X_{s,t}, a, \zeta) + \frac{\sigma^2}{2} \nabla_a v_{s,t}(a) \right] B_{s,t}(a) v_{s,t}(da) dt \\
- \int_0^T \int (\nabla_a \phi_t)(X_{s,t}, a, \zeta) P_{s,t} B_{s,t}(a) v_{s,t}(da) dr .
\]

Recalling the Hamiltonian defined in (14) completes the proof.

\[\]
From this and Lemma 16 and noting that $(\nabla_x g)(X_T, \zeta)V_T = P_T V_T$ we see that
\[
\frac{d}{d\epsilon} J^0((\nu_t + \epsilon(\mu_t - \nu_t))_{t\in[0,T]}, \xi, \zeta) \bigg|_{\epsilon=0} = \int_0^T \left[ \int f_t(X_t, a, \zeta) (\mu_t - \nu_t)(da) + \int (\nabla_x f_t)(X_t, a, \zeta) V_t \nu_t(da) \right] dt \\
+ \int_0^T \left[ \int P_t \phi_t(X_t, a, \zeta) (\mu_t - \nu_t)(da) - \int V_t (\nabla_x f_t)(X_t, a, \zeta) \nu_t(da) \right] dt \\
= \int_0^T \int h_t(X_t, P_t, a, \zeta)(\mu_t - \nu_t)(da) dt.
\]

We can conclude the proof by properties of the linear derivative. \hfill \blacksquare

**Proof** [Proof of Theorem 5.] Let $(\mu_t)_{t\in[0,T]}$ be an arbitrary relaxed control. Since $(\nu_t)_{t\in[0,T]}$ is optimal we know that
\[ J^\sigma((\nu_t + \epsilon(\mu_t - \nu_t))_{t\in[0,T]}) \geq J^\sigma(\nu) \quad \text{for any } \epsilon > 0. \]

From this and Lemma 18 and Lemma 34 we get, after integrating over $(\xi, \zeta) \in \mathbb{R}^d \times \mathcal{S}$, that
\[
0 \leq \limsup_{\epsilon \to 0} \frac{1}{\epsilon} \left( J^\sigma((\nu_t + \epsilon(\mu_t - \nu_t))_{t\in[0,T]} - J^\sigma(\nu) \right) \\
\leq \int_0^T \int \left( \int_{\mathbb{R}^d \times \mathcal{S}} \frac{\delta H^0}{\delta m}(X_t^{\xi,\zeta}, P_t^{\xi,\zeta}, \nu_t, a, \zeta) \mathcal{M}(d\xi, d\zeta) + \frac{\sigma^2}{2} (\log \nu_t(a) - U(a)) \right) (\mu_t - \nu_t)(da) dt.
\]

Now assume there is $S \in \mathcal{B}([0,T])$, with strictly positive Lebesgue measure, such that
\[
\int_0^T \mathbf{1}_S \int \left( \int_{\mathbb{R}^d \times \mathcal{S}} \frac{\delta H^0}{\delta m}(X_t^{\xi,\zeta}, P_t^{\xi,\zeta}, \nu_t, a, \zeta) \mathcal{M}(d\xi, d\zeta) + \frac{\sigma^2}{2} (\log \nu_t(a) - U(a)) \right) (\mu_t - \nu_t)(da) dt < 0
\]

Define $\tilde{\mu}_t := \mu_t \mathbf{1}_S + \nu_t \mathbf{1}_{S^c}$. Then by the same argument as above
\[
0 \leq \int_0^T \int \left( \int_{\mathbb{R}^d \times \mathcal{S}} \frac{\delta H^0}{\delta m}(X_t^{\xi,\zeta}, P_t^{\xi,\zeta}, \nu_t, a, \zeta) \mathcal{M}(d\xi, d\zeta) + \frac{\sigma^2}{2} (\log \nu_t(a) - U(a)) \right) (\tilde{\mu}_t - \nu_t)(da) dt \\
= \mathbb{E} \int_0^T \mathbf{1}_S \left( \int_{\mathbb{R}^d \times \mathcal{S}} \frac{\delta H^0}{\delta m}(X_t^{\xi,\zeta}, P_t^{\xi,\zeta}, \nu_t, a, \zeta) \mathcal{M}(d\xi, d\zeta) + \frac{\sigma^2}{2} (\log \nu_t(a) - U(a)) \right) (\mu_t - \nu_t)(da) dt < 0
\]

leading to a contradiction. This proves i).

From i), properties of linear derivatives and Lemma 34 we have
\[
0 \leq \int \left( \int_{\mathbb{R}^d \times \mathcal{S}} \frac{\delta H^0}{\delta m}(X_t^{\xi,\zeta}, P_t^{\xi,\zeta}, \nu_t, a, \zeta) \mathcal{M}(d\xi, d\zeta) + \frac{\sigma^2}{2} (\log \nu_t(a) - U(a)) \right) (\mu_t - \nu_t)(da) \\
\leq \limsup_{\epsilon \to 0} \int_{\mathbb{R}^d \times \mathcal{S}} \left[ H^\sigma(X_t^{\xi,\zeta}, P_t^{\xi,\zeta}, \nu_t, \zeta) - H^\sigma(X_t^{\xi,\zeta}, P_t^{\xi,\zeta}, \nu_t, \zeta) \right] \mathcal{M}(d\xi, d\zeta).
\]

From this ii) follows. \hfill \blacksquare
Theorem 19 (Sufficient condition for optimality) Fix $\sigma \geq 0$. Assume that $g$ and $h$ are continuously differentiable in the $x$ variable. Assume that $(\nu_t)_{t \in [0,T]}$, $X^{\xi,\zeta}$, $P^{\xi,\zeta}$ are a solution to (18). Finally assume that

i) the map $x \mapsto g(x, \zeta)$ is convex for every $\zeta \in \mathcal{S}$ and

ii) the map $(x,m) \mapsto H^t_{\sigma}(x, P^{\xi,\zeta}, m, \zeta)$ satisfies that for all $t \in [0,T]$, $\xi \in \mathbb{R}^d$, $\zeta \in \mathcal{S}$, $x, x' \in \mathbb{R}^d$ and all $m, m' \in P_2(\mathbb{R}^p)$ (absolutely continuous w.r.t. the Lebesgue measure if $\sigma > 0$) it holds that

$$H^t_{\sigma}(x, P^{\xi,\zeta}, m, \zeta) - H^t_{\sigma}(x', P^{\xi,\zeta}, m', \zeta) \leq (\nabla_x H^0_0)(x, P^{\xi,\zeta}, m, \zeta) (x - x') + \int \left( \frac{\delta H^0}{\delta m}(x, P^{\xi,\zeta}, m, a, \zeta) + \frac{\sigma^2}{2} \log m(a) - \frac{\sigma^2}{2} U(a) \right) (m - m')(da).$$

Then the relaxed control $(\nu_t)_{t \in [0,T]}$ is an optimal control.

Proof [Proof of Theorem 19.] Let $(\tilde{\nu}_t)_{t \in [0,T]}$ be another control with the associated family of forward and backward processes $\tilde{X}^{\xi,\zeta}$, $\tilde{P}^{\xi,\zeta}$, $(\xi, \zeta) \in \mathbb{R}^d \times \mathcal{S}$. Of course $X_0^{\xi,\zeta} = \tilde{X}_0^{\xi,\zeta}$. For now, we have $(\xi, \zeta)$ fixed and so write $X = X^{\xi,\zeta}$, $P = P^{\xi,\zeta}$.

First, we note that due to convexity of $x \mapsto g(x, \zeta)$ for every $\zeta \in \mathcal{S}$, we have

$$g(X_T, \zeta) - g(\tilde{X}_T, \zeta) \leq (\nabla_x g)(X_T, \zeta)(X_T - \tilde{X}_T) = P_T(X_T - \tilde{X}_T)$$

$$= \int_0^T (X_t - \tilde{X}_t) dP_t + \int_0^T P_t(dX_t - d\tilde{X}_t)$$

$$= -\int_0^T (X_t - \tilde{X}_t)(\nabla_x H^0_0)(X_t, P_t, \nu_t, \zeta) dt + \int_0^T P_t \left( \Phi_t(X_t, \nu_t, \zeta) - \Phi_t(\tilde{X}_t, \tilde{\nu}_t, \zeta) \right) dt.$$

Moreover, since $F(x, \nu, \zeta) + \frac{\sigma^2}{2} \text{Ent}(\nu) = H^0_{\sigma}(x, p, \nu, \zeta) - \Phi(x, \nu) p$ we have

$$\int_0^T \left[ F(X_t, \nu_t, \zeta) - F(\tilde{X}_t, \tilde{\nu}_t, \zeta) + \frac{\sigma^2}{2} \text{Ent}(\nu_t) - \frac{\sigma^2}{2} \text{Ent}(\tilde{\nu}_t) \right] dt$$

$$= \int_0^T \left[ H^0_{\sigma}(X_t, P_t, \nu_t, \zeta) - \Phi_t(X_t, \nu_t, \zeta)Y_t - H^0_{\sigma}(\tilde{X}_t, P_t, \tilde{\nu}_t, \zeta) + \Phi_t(\tilde{X}_t, \tilde{\nu}_t, \zeta) P_t \right] dt.$$

Hence

$$\tilde{J}^\sigma(\nu, \xi, \zeta) - J^\sigma(\tilde{\nu}, \xi, \zeta)$$

$$\leq -\int_0^T (X_t - \tilde{X}_t)(\nabla_x H^\sigma)(X_t, P_t, \nu_t, \zeta) dt + \int_0^T \left[ H^\sigma(X_t, P_t, \nu_t, \zeta) - H^\sigma(\tilde{X}_t, P_t, \tilde{\nu}_t, \zeta) \right] dt.$$
Integrating over $(\xi, \zeta)$ w.r.t. $\mathcal{M}$ we thus have
\[ J'(\nu) - J'(\bar{\nu}) \]
\[ \leq \int_0^T \left( \int_{\mathbb{R}^d \times S} \frac{\delta H^0_t}{\delta m}(X^\xi_\theta, P^\xi_\theta) \mathcal{M}(d\xi, d\zeta) + \frac{\sigma^2}{2} \log \nu_t(a) - \frac{\sigma^2}{2} U(a) \right) (\nu_t - \bar{\nu}_t)(da). \]

Moreover $\nu_t = \arg \min_{\mu} \int_{\mathbb{R}^d \times S} H^0_t(X^\xi_\theta, P^\xi_\theta, \mu, \zeta) \mathcal{M}(d\xi, d\zeta)$ implies that
\[ \int \left( \int_{\mathbb{R}^d \times S} \frac{\delta H^0_t}{\delta m}(X^\xi_\theta, P^\xi_\theta, \nu_t, a) \mathcal{M}(d\xi, d\zeta) + \frac{\sigma^2}{2} \log \nu_t(a) - \frac{\sigma^2}{2} U(a) \right) (\nu_t - \bar{\nu}_t)(da) \leq 0. \]

Hence $J(\nu) - J(\bar{\nu}) \leq 0$ and $\nu$ is an optimal control. We note that if either $g$ or $H^0$ are strictly convex then $\nu$ is the optimal control. A particular case is whenever $\sigma > 0$. \[ \blacksquare \]

### 3.3 Existence, uniqueness, convergence to the invariant measure

The reader will recall the space of relaxed controls $\mathcal{V}_2$ given by (9) and the integrated Wasserstein metric given by (21). The Mean-field Langevin system induces an s-time marginal law in the space $\mathcal{V}_2$ i.e. for each $s \geq 0$ we have $\mathcal{L}(\theta_s) \in \mathcal{V}_2$ and moreover the map $s \mapsto \mathcal{L}(\theta_{s, \cdot})$ is continuous in the $W^2_T$ metric on $\mathcal{V}_2$. Hence, as in Hu et al. (2019a), we have that the map $I \ni s \mapsto \mathcal{L}(\theta_{s, \cdot}) \in \mathcal{V}_2$ belongs to the space
\[ \mathcal{C}(I, \mathcal{V}_2) := \left\{ \nu = (\nu_{s, \cdot})_{s \in I} : \nu_{s, \cdot} \in \mathcal{V}_2 \text{ and } \lim_{s' \to s} W^2_T(\nu_{s', \cdot}, \nu_{s, \cdot}) = 0 \forall s \in I \right\}. \]

The existence of a unique solution to the system (19)-(20) will be established by a fixed point argument in $\mathcal{C}(I, \mathcal{V}_2)$.

**Lemma 20** Let Assumption 1 hold and let $\mu \in \mathcal{V}_2$.

i) The system
\[
\begin{align*}
X^\xi_\theta(\mu) &= \xi + \int_0^t \Phi_r(X^\xi_\theta(\mu), \mu_r, \zeta) \, dr, \quad t \in [0, T], \\
\int_{\mathbb{R}^d \times S} \, dt, \quad P^\xi_\theta(\mu) &= -\langle \nabla_x H_t \rangle(X^\xi_\theta(\mu), P^\xi_\theta(\mu), \mu_t, \zeta) \, dt, \quad P^\xi_\theta(\mu) = \langle \nabla g \rangle(X^\xi_\theta(\mu), \zeta).
\end{align*}
\]

has a unique solution.

ii) Assume further point iii) of Assumption 6 holds. Then
\[ \sup_{t \in [0, T]} \sup_{\mathcal{P}(\mathbb{R}^d \times S)} \sup_{\nu \in \mathcal{V}_2} \int_{\mathbb{R}^d \times S} \left[ |X^\xi_\theta(\nu)|^2 + |P^\xi_\theta(\nu)|^2 \right] \mathcal{M}(d\xi, d\zeta) < \infty. \]

**Proof** Assumption 1 implies that the functions $x \mapsto \Phi(x, \mu_t, \zeta)$ and $(x, p) \mapsto \langle \nabla_x H_t \rangle(x, p, \mu_t, \zeta)$ are Lipschitz continuous and hence unique solution $(P^\xi_\theta, X^\xi_\theta)_{t \in [0, T]}$ exists for all $\xi, \zeta$. This proves point i). Point ii) follows from Lemma 35 under our assumptions. \[ \blacksquare \]
Recall that (16) defines
\[
\mathbf{h}_t(a, \mu, \mathcal{M}) = \int_{\mathbb{R}^d \times S} h_t(X^\xi_t(\mu), P^\xi_t(\mu), a, \zeta) \mathcal{M}(d\xi, d\zeta),
\]
for \( \mu \in \mathcal{V}_2, \mathcal{M} \in \mathcal{P}_2(\mathbb{R}^d \times S) \), where \((X^\xi_t(\mu), P^\xi_t(\mu))\) is the unique solution to (34) given by Lemma 20.

**Lemma 21** Let Assumptions 1 and 6 hold. Then for any \( \mathcal{M} \in \mathcal{P}_2(\mathbb{R}^d \times S) \) there exists \( L > 0 \) such that for all \( a, a' \in \mathbb{R}^p \) and \( \mu, \mu' \in \mathcal{V}_2 \)
\[
|\left( \mathbf{h}_t \right)(a, \mu, \mathcal{M}) - \left( \mathbf{h}_t \right)(a', \mu', \mathcal{M})| \leq L \left( |a - a'| + W_1^T(\mu, \mu') \right). \tag{35}
\]

**Proof** Due to Theorem 38 we know that there is \( L' > 0 \) such that
\[
|\left( \mathbf{h}_t \right)(a, \mu, \mathcal{M}) - \left( \mathbf{h}_t \right)(a', \mu', \mathcal{M})| \leq L' \left( 1 + \sup_{t \in [0, T]} \sup_{\mu \in \mathcal{V}_2} \int_{\mathbb{R}^d \times S} |P^\xi_t(\mu)| \mathcal{M}(d\xi, d\zeta) \right) \left( |a - a'| + W_1^T(\mu, \mu') \right).
\]
This, together with point ii) of Lemma 20 provides the desired \( L \). \( \blacksquare \)

**Lemma 22 (Existence and uniqueness)** Let Assumptions 1 and 6 hold. Then there is a unique solution to (19)-(20) for any \( s \in I \). Moreover
\[
\int_0^T \mathbb{E}[|\theta_{s,t}|^2] dt \leq e^{(K - \frac{a^2}{2}\kappa)s} \int_0^T \mathbb{E}[|\theta|^2] dt + \int_0^s e^{(K - \frac{a^2}{2}\kappa)(s-v)} \left( \frac{\sigma^2 T}{4\kappa} + \frac{\sigma^2 T}{2} |(\nabla_s U)(0)|^2 + K \right) dv. \tag{36}
\]
where \( K \) is a finite constant depending on \( \phi, f, \nabla_s g, \) and \( \mathcal{M} \).

**Proof** Consider \( \mu \in C(I, \mathcal{V}_2) \). For each \( \mu_s \in \mathcal{V}_2 \), \( s \geq 0 \) we obtain unique solution to (34) which we denote \((X^\xi_{s,t}(\mu_s), (P^\xi_{s,t}(\mu_s))\). Moreover, for each \( t \in [0, T] \) the SDE
\[
d\theta_{s,t}(\mu) = -\left( (\nabla_s \mathbf{h}_t)(\theta_{s,t}(\mu), \mu_s, \mu, \mathcal{M}) + \frac{\sigma^2}{2}(\nabla_s U)(\theta_{s,t}(\mu)) \right) ds + \sigma dB_s
\]
has unique strong solution and for each \( s \in I \) we denote the measure in \( \mathcal{V}_2 \) induced by \( \theta_{s,t} \) as \( \mathcal{L}(\theta_{s,t}) \). Consider now the map \( \Psi \) given by \( \mathcal{L}(I, \mathcal{V}_2) \ni \mu \mapsto \{ \mathcal{L}(\theta_{s}(\mu)) : s \in I \} \).

**Step 1.** We need to show that \( \{ \mathcal{L}(\theta_{s}(\mu)) : s \in I \} \in C(I, \mathcal{V}_2) \). This amounts to showing that we have the appropriate integrability and continuity i.e. that there is \( K > 0, \lambda > 0 \) such that
\[
\int_0^T \mathbb{E}[|\theta_{s,t}(\mu)|^2] dt \leq e^{(K - \sigma^2 \kappa)s} \left( \int_0^T \mathbb{E}[|\theta|^2] dt + \int_0^s e^{\lambda v} \frac{\sigma^2 T}{4\kappa} |(\nabla_s U)(0)|^2 + K \right) dv.
\]
and that
\[
\lim_{s' \to s} \int_0^T \mathbb{E}[|\theta_{s',t}(\mu) - \theta_{s,t}(\mu)|^2] dt = 0.
\]
To show the integrability observe that Assumption 6 point ii) together with the Young’s inequality: \( \forall a, y \in \mathbb{R}^p \ |ay| \leq \frac{3}{2} |a|^2 + (2\kappa)^{-1} |y|^2 \) imply that

\[
(\nabla_a U)(a) a \geq \frac{\kappa}{2} |a|^2 - \frac{1}{2\kappa} |(\nabla_a U)(0)|^2.
\]

Hence, for any \( \lambda > 0 \), we have

\[
\int_0^T \mathbb{E}[e^{\lambda s}|\theta_{s,t}(\mu)|^2] \, dt = \int_0^T \mathbb{E}[|\theta_{0,t}|^2] \, dt + \lambda \int_0^T \int_0^s \mathbb{E}[e^{\lambda v}|\theta_{v,t}|^2] \, dv \, dt + \frac{\lambda^2}{2} T \int_0^s \mathbb{E}[|\theta_{v,t}|^2] \, dv \, dt
\]

\[
- \frac{\lambda^2}{2} \int_0^s \mathbb{E}[e^{\lambda v} (\sigma^2 \nabla_a U(\theta_{v,t}) + (\nabla_a h_t)(\theta_{v,t}(\mu), \mu_{s, \ldots, M})) \theta_{v,t}(\mu)] \, dv \, dt.
\]

From the definition of the Hamiltonian and Lemma 20, we have

\[
|((\nabla_a h_t)(\theta_{v,t}, \mu_{s, \ldots, M}))| \leq c \left( 1 + |\theta_{v,t}| + \int_0^T \mathbb{E}[|\theta_{v,t}|] \, dt \right).
\]

Hence

\[
\int_0^T \mathbb{E} \left[ |((\nabla_a h_t)(\theta_{v,t}(\mu), \mu_{s, \ldots, M})) \theta_{v,t}| \right] \, dt \leq c \left( 1 + \int_0^T \mathbb{E}[|\theta_{v,t}(\mu)|^2] \, dt \right).
\]

Therefore,

\[
\int_0^T \mathbb{E}[e^{\lambda s}|\theta_{s,t}(\mu)|^2] \, dt \leq \int_0^T \mathbb{E}[|\theta_{0,t}|^2] \, dt + \left( \frac{\lambda}{2} - \frac{\sigma^2}{2 \kappa} \right) \int_0^s \mathbb{E}[e^{\lambda v}|\theta_{v,t}(\mu)|^2] \, dv \, dt
\]

\[
+ \int_0^s \mathbb{E}[e^{\lambda v} (\sigma^2 T + \frac{\sigma^2 T}{4\kappa}) |(\nabla_a U)(0)|^2 + c)] \, dv.
\]

Take \( \lambda = \frac{\sigma^2}{2 \kappa} - c \) to conclude the required integrability property. To establish the continuity property note that for \( s' \geq s \) we have

\[
|\theta_{s',t}(\mu) - \theta_{s,t}(\mu)| \leq L \int_s^{s'} \left( 1 + \int_0^T |a| \mu_{r,t}(da) \, dr \right) \, dr + \sigma |B_s - B_{s'}|.
\]

This, together with Lebesgue’s theorem on dominated convergence enables us to establish the continuity property.

**Step 2.** Take \( \lambda = \frac{\sigma^2}{2 \kappa} - \frac{3}{2} L. \) Fix \( t \in [0, T] \). From Itô’s formula we get

\[
d(e^{2\lambda s}|\theta_{s,t}(\mu) - \theta_{s,t}(\mu')|^2) = 2e^{2\lambda s} \left( \lambda |\theta_{s,t}(\mu) - \theta_{s,t}(\mu')|^2
\right.
\]

\[
- \int e^{2\lambda s} \frac{\sigma^2}{2} (\theta_{s,t}(\mu) - \theta_{s,t}(\mu'))((\nabla_a U)(\theta_{s,t}(\mu)) - ((\nabla_a U)(\theta_{s,t}(\mu')))) \mathcal{M}(d\xi, d\zeta)
\]

\[
- (\theta_{s,t}(\mu) - \theta_{s,t}(\mu'))(\nabla_a h_t)(\theta_{s,t}(\mu), \mu_{s, \ldots, M}) - (\nabla_a h_t)(\theta_{s,t}(\mu'), \mu_{s, \ldots, M})) ds.
\]
Assumption 6 point ii), along with Young’s inequality: \( \forall x, y, L > 0, |xy| \leq L|x|^2 + (4L)^{-1}|y|^2 \) yields that

\[
\begin{align*}
\frac{d}{ds}(e^{2\lambda s}|\theta_{s,t}(\mu) - \theta_{s,t}(\mu')|^2) &
\leq 2 \left( \lambda - \frac{\sigma^2}{2} \kappa \right) e^{2\lambda s}|\theta_{s,t}(\mu) - \theta_{s,t}(\mu')|^2 ds \\
&+ 2 e^{2\lambda s} \left( L|\theta_{s,t}(\mu) - \theta_{s,t}(\mu')|^2 + \frac{L}{2} \left( |\theta_{s,t}(\mu) - \theta_{s,t}(\mu')|^2 + \frac{L}{2} \mathcal{W}_2^T(\mu, \nu, \mu', \nu')^2 \right) \right) ds \\
&\leq 2 e^{2\lambda s} \left( \lambda - \left( \frac{\sigma^2}{2} \kappa - \frac{3}{2} L \right) \right) |\theta_{s,t}(\mu) - \theta_{s,t}(\mu')|^2 + \frac{L}{2} \mathcal{W}_2^T(\mu, \nu, \mu', \nu')^2 \right) ds.
\end{align*}
\]

Recall that for any \( s \in I \) we have

\[
\mathcal{W}_2^T(\mathcal{L}(\theta_{s,}(.)), \mathcal{L}(\theta_{s,}(.)))^2 \leq \int_0^T \mathbb{E} \left[ |\theta_{s,t}(\mu) - \theta_{s,t}(\nu)|^2 \right] dt.
\tag{37}
\]

Recall that for all \( t \in [0, T] \), \( \theta_{0,t}(\mu) = \theta_{0,t}(\nu) \). Fix \( S > 0 \) and note that

\[
de^{2\lambda S} \mathcal{W}_2^T(\mathcal{L}(\theta_{S,}(.)), \mathcal{L}(\theta_{S,}(.)))^2 \leq \frac{LT}{2} \int_0^S e^{2\lambda s} \mathcal{W}_2^T(\mu_s, \nu_s)^2 ds.
\tag{38}
\]

**Step 3.** Let \( \Psi^k \) denote the \( k \)-th composition of the mapping \( \Psi \) with itself. Then, for any integer \( k > 1 \),

\[
\sup_{s \in [0, S]} \mathcal{W}_2^T(\Phi^k(\mu_s), \Phi^k(\nu_s))^2 \leq e^{-2\lambda S} \left( \frac{LT}{2} \right)^k \frac{S^k}{k!} \sup_{s \in [0, S]} \mathcal{W}_2^T(\mu_s, \nu_s)^2.
\]

Hence, for any \( S \in I \) there is \( k \), such that \( \Phi^k \) is a contraction and then Banach fixed point theorem gives existence of the unique solution on \([0, S]\). The estimate (36) follows simply from Step 1. The proof is complete.

**Remark 23** As a by-product of the proof of the above result we obtained the rate of convergence of the Picard iteration algorithm on \( C(I, \mathcal{V}_2) \) for solving (19)-(20). Such an algorithm can be combined with particle approximations in the spirit of Szpruch et al. (2015).

**Proof** [Proof of Theorem 7] Lemma 22 provides the unique solution to (19)-(20). The existence of the invariant measures follows by the similar argument as in (Hu et al., 2019b, Prop 2.4) and (Hu et al., 2019a, Prop 3.9). We present the proof for completeness. Since \( \mathcal{M} \) is fixed here, we omit it from the notation in \( \mathbf{h}_t \) defined in (16). Moreover, the required integrability and regularity for solutions to the Kolmogorov–Fokker–Planck equations are exactly those proved in (Hu et al., 2019b, Proposition 5.2 and Lemmas 6.1-6.5) and we do not state them here.
Step 1. First we show that \( \arg \min_{\nu \in \mathcal{V}_2} J^\sigma(\nu) \neq \emptyset \). Denote
\[
\mathcal{K} := \left\{ \nu \in \mathcal{V} : \frac{\sigma^2}{2} \operatorname{Ent}(\nu) \leq J^\sigma(\nu) - \inf_{\nu} J^\sigma(\nu) \right\}.
\]
Note that since \( \operatorname{Ent}(\nu) \) is weakly lower-semicontinuous so is \( J^\sigma \). Further, as sub-level set of relative entropy, \( \mathcal{K} \) is weakly compact, (Dupuis and Ellis, 2011, Lem 1.4.3). Further note that for \( \nu \notin \mathcal{K} \)
\[
J^\sigma(\nu) \geq \frac{\sigma^2}{2} \operatorname{Ent}(\nu) + \inf_{\nu} J^\sigma(\nu) > J^\sigma(\tilde{\nu}).
\]
Hence \( \{\nu \in \mathcal{V}_2 : J^\sigma(\nu) \leq J^\sigma(\tilde{\nu})\} \subset \mathcal{K} \), and so \( \inf_{\nu \in \mathcal{V}_2} J^\sigma(\nu) = \inf_{\nu \in \mathcal{K}} J^\sigma(\nu) \). A weakly lower continuous function achieves a global minimum on a weakly compact set and hence \( \arg \min_{\nu \in \mathcal{V}_2} J^\sigma(\nu) \neq \emptyset \). This proves point i).

Step 2. Let \( \nu^* \in \arg \min_{\nu \in \mathcal{V}_2} J(\nu) \). Necessarily \( J(\nu^*) \leq J(\tilde{\nu}) < \infty \). This together with the fact that \( J \) is bounded from below means that \( \int_0^T \operatorname{Ent}(\mu_t) dt < \infty \). In particular, \( \nu^* \) is absolutely continuous with respect to the Lebesgue measure. Theorem 5 i) tells us that for any \( \mu \in \mathcal{V}_2 \),
\[
\int \int_{\mathbb{R}^d \times \mathcal{S}} \frac{\delta H^\sigma}{\delta m}(X_t^{\xi,\zeta}(\nu^*), P_t^{\xi,\zeta}(\nu^*), \nu_t^*, a, \zeta) \mathcal{M}(d\xi, d\zeta) (\mu_t - \nu_t^*)(da) \geq 0 \text{ for a.a. } t \in (0, T).
\]
From this and Lemma 33 we deduce that for a.a. \( t \in [0, T] \) we have that
\[
a \mapsto \Gamma_t(a) := \int_{\mathbb{R}^d \times \mathcal{S}} \frac{\delta H^\sigma}{\delta m}(X_t^{\xi,\zeta}, P_t^{\xi,\zeta}, \nu_t^*, a, \zeta) \mathcal{M}(d\xi, d\zeta)
\]
stays constant in \( a \) and \( \Gamma_t(a) = \int \Gamma(a') \nu_t^*(da') := \Gamma_t \) for \( \nu_t^* \)-a.a. \( a \in \mathbb{R}^p \). From the fact that \( \nu^* \) is absolutely continuous w.r.t. Lebesgue measure and from (14) we see that
\[
a \mapsto \Gamma_t(a) = h_t(a, \nu^*) + \frac{\sigma^2}{2} \log(\nu^*(a)) + \frac{\sigma^2}{2} U(a)
\]
is constant in \( a \), possibly \( -\infty \). We know \( \nu^*(a) \geq 0 \) and on \( S := \{(t, a) \in \mathbb{R}^p \times [0, T] : \nu_t^*(a) > 0\} \) the probability measure \( \nu_t^* \) satisfies the equation
\[
\nu_t^*(a) = e^{-\frac{a^2}{2\sigma^2} \Gamma_t} e^{-U(a) - \frac{a^2}{2\sigma^2} h_t(a, \nu^*)}.
\]
Since \( \nu^* \) is a probability measure
\[
1 = \int \nu^*(a) da = e^{-\frac{a^2}{2\sigma^2} \Gamma_t} \int e^{-U(a) - \frac{a^2}{2\sigma^2} h_t(a, \nu^*)} da.
\]
Due Lemma 20. part ii) we see that \( a \mapsto h_t(a, \nu^*) \) has at most linear growth. This and Assumption 6 point ii) implies that \( 0 < \int e^{-U(a) - \frac{a^2}{2\sigma^2} h_t(a, \nu^*)} da < \infty \). This implies that for a.a. \( t \in [0, T] \) we have \( \Gamma_t \neq -\infty \). Hence \( \nu_t^*(a) > 0 \) for Lebesgue a.a \( a \in (0, T) \times \mathbb{R}^p \) and \( \nu^* \) is equivalent to the Lebesgue measure on \( (0, T) \times \mathbb{R}^p \).
**Step 3.** Let $\mathcal{L}(\theta_0^t) := \nu^*$ and consider $\theta_{s,t}$ given by (19)-(20). Since $\nu^* \in \arg \min_{\nu \in \mathcal{V}_2} J(\nu)$ we know that

$$0 \leq \lim_{s \to 0} \frac{J(\mathcal{L}(\theta_{s,t})) - J(\nu^*)}{s} = \frac{d}{ds} J(\mathcal{L}(\theta_{s,t})) \bigg|_{s=0}.$$  

Moreover, from Theorem 4 we have that

$$\frac{d}{ds} J(\mathcal{L}(\theta_{s,t})) \bigg|_{s=0} = - \int_0^T \left( (\nabla_a h_t)(a, \nu^*) + \frac{\sigma^2}{2} \nabla_t \nu^*(a) + \frac{\sigma^2}{2} \nabla U(a) \right) \nu^*(a) \, da \, dt \leq 0.$$  

Hence for Lebesgue a.a. $(t, a) \in (0, T) \times \mathbb{R}^p$ we have

$$0 = (\nabla_a h_t)(a, \nu^*) + \frac{\sigma^2}{2} \nabla_t \nu^*(a) + \frac{\sigma^2}{2} \nabla U(a).$$  

Multiplying by $\nu^*(a) > 0$ and applying the divergence operator we see that for a.a. $t \in [0, T]$ the function $a \mapsto \nu_t^*(a)$ solves the stationary Fokker–Planck equation

$$\nabla \cdot \left[ \left( (\nabla_a h_t)(\cdot, \nu_t^*) + \frac{\sigma^2}{2} (\nabla_a U) \right) \nu_t^* \right] + \frac{\sigma^2}{2} \Delta a \nu_t^* = 0 \text{ on } \mathbb{R}^p.$$  

From this we can conclude that $\nu_t^*$ is an invariant measure for (19)-(20). This concludes the proof of point ii).

**Step 4.** Take two solutions to (19)-(20) denoted by $(\theta, X, P)$ and $(\theta', X', P')$ with initial distributions $\mathcal{L}((\theta_{0,t})_{t \in [0,T]})$ and $\mathcal{L}((\theta'_{0,t})_{t \in [0,T]})$, respectively. Take $\lambda = \sigma^2 \kappa - 4L > 0$. Due to (37), Lemma 21 and Assumption 6 point ii) we have

$$d \left( e^{\lambda s} \mathcal{W}_2^T(\mathcal{L}(\theta_{s,t}), \mathcal{L}(\theta'_{s,t})) \right)^2 \leq d \left( e^{\lambda s} \mathbb{E} \int_0^T |\theta_{s,t} - \theta'_{s,t}|^2 dt \right) = e^{\lambda s} \mathbb{E} \int_0^T \left[ \lambda |\theta_{s,t} - \theta'_{s,t}|^2 - 2(\theta_{s,t} - \theta'_{s,t}) \left( (\nabla_a h_t)(\theta_{s,t}, \mathcal{L}(\theta_{s,t})) - (\nabla_a h_t)(\theta'_{s,t}, \mathcal{L}(\theta'_{s,t})) \right) + \frac{\sigma^2}{2} \left( (\nabla_a U)(\theta_{s,t}) - (\nabla_a U)(\theta'_{s,t}) \right) \right] dt \, ds \leq e^{\lambda s} \left( -L \int_0^T \mathbb{E} |\theta_{s,t} - \theta'_{s,t}|^2 dt + L \mathcal{W}_2^T(\mathcal{L}(\theta_{s,t}), \mathcal{L}(\theta'_{s,t})) \right) ds \leq 0,$$

where we used (37) again to obtain the last inequality. Integrating this leads to

$$\mathcal{W}_2^T(\mathcal{L}(\theta_{s,t}), \mathcal{L}(\theta'_{s,t}))^2 \leq e^{-\lambda s} \mathcal{W}_2^T(\mathcal{L}(\theta_{0,s}), \mathcal{L}(\theta'_{0,s}))^2.$$  

Take $\theta'$ to be solution to (19)-(20) with initial condition $\mathcal{L}(\theta'_{0,s}) = \nu^*$ and $\theta$ to be solution to (19)-(20) with an arbitrary initial condition $(\theta^t_{0,s})_{t \in [0,T]}$. We see that we have the convergence claimed in (22). Moreover if $\mu^*$ is another invariant measure and we can start (19)-(20) with $\mathcal{L}(\theta_{0,s}) = \nu^*$ and $\mathcal{L}(\theta'_{0,s}) = \mu^*$ to see that for any $s \geq 0$ we have $\mathcal{W}_2^T(\mu^*, \nu^*)^2 \leq e^{-\lambda s} \mathcal{W}_2^T(\mu^*, \nu^*)^2$. This is a contradiction unless $\mu^* = \nu^*$. This proves point iii).
3.4 Particle approximation and propagation of chaos

In this section, we study particle system that corresponds to (19)-(20). Define $\mathcal{M}^{N_1} = \frac{1}{N_1} \sum_{j=1}^{N_1} \delta_{(\xi^j, \zeta^j)}$. For convenience, we introduce $h^\sigma_i$ defined on $\mathbb{R}^p \times \mathcal{V}_2 \times \mathcal{P}(\mathbb{R}^d)$ by

$$\nabla_a h^\sigma_i (a, \nu, \mathcal{M}) = \frac{\sigma^2}{2} \nabla_a U(a) + \nabla_a h(a, \nu, \mathcal{M}).$$

Lemma 24 Let Assumptions 1 and 6 hold. Then, for any finite time horizon $0 < S < \infty$, there exists a unique solution $(\theta_{s,t}^1, \ldots, \theta_{s,t}^{N_2})$ to the $\mathbb{R}^{N_2}$ dimensional system of SDEs (23). Moreover, for all $0 \leq s \leq S$, $\int_0^T \mathbb{E}[|\theta_{s,t}^i|^2] \, dt < \infty$.

Proof Under Assumptions 1 and 6 the coefficients of $\mathbb{R}^{N_2}$-dimensional system of SDEs (23) are globally Lipschitz continuous and therefore existence and uniqueness and integrability results can be derived by adapted classical techniques for stochastic differential equations, see e.g. (Karatzas and Shreve, 2012, Chapter 5).

The rate of convergence between (19) and (23) is given by the following theorem.

Theorem 25 Let Assumptions 1 and 6 hold. Fix $\lambda = \frac{\sigma^2}{2} - \frac{1}{2}(3 + T) + \frac{1}{2}$. Define $(\theta_{s,t}^{i,\infty})_{i=1}^{N_2}$ consisting of $N_2$ independent copies of (19), given by

$$\theta_{s,t}^{i,\infty} = \theta_{0,t}^i - \int_0^S \nabla_a h^\sigma_i (\theta_{s,t}^{i,\infty}, \mathcal{L}^{(\theta_{s,t}^{i,\infty})}, \mathcal{M}) \, dv + \sigma B_{s,t}^i, 0 \leq s \leq S, 0 \leq t \leq T. \quad (39)$$

Then there exists $c$, independent of $S$, $N_1, N_2, d, p$, such that, for all $i = 1, \ldots, N_2$ we have

$$\int_0^T \mathbb{E} \left[ \left| \theta_{s,t}^i - \theta_{s,t}^{i,\infty} \right|^2 \right] \, dt \leq \frac{c}{\lambda} (1 - e^{-\lambda S}) \left( \frac{1}{N_1} + \frac{1}{N_2} \right).$$

Proof Step 1. Noticing that the particle system (23) is exchangeable, it is sufficient to prove our claim for $i = 1$. By uniqueness of solutions to (19), $\mathcal{L}(\theta_{v,.}) = \mathcal{L}(\theta_{v,.}^{1,\infty})$. We also define $\nu_{s,t}^{N_2,\infty} = \frac{1}{N_2} \sum_{j=1}^{N_2} \delta_{\theta_{s,t}^{j,\infty}}$. Furthermore, for any $i$,

$$\theta_{s,t}^i - \theta_{s,t}^{i,\infty} = - \left( \int_0^S \nabla_a h^\sigma_i (\theta_{s,t}^{i,v,.}, \nu_{s,t}^{N_2,\infty}, \mathcal{M}^{N_1}) \, dv - \int_0^S \nabla_a h^\sigma_i (\theta_{s,t}^{i,\infty}, \mathcal{L}(\theta_{s,t}^{i,.}), \mathcal{M}) \, dv \right).$$

For $\lambda > 0$, to be chosen later on, we have

$$\int_0^T \mathbb{E} \left[ e^{\lambda S} \left| \theta_{s,t}^1 - \theta_{s,t}^{1,\infty} \right|^2 \right] \, dt = \lambda \int_0^T \int_0^S \mathbb{E} \left[ e^{\lambda v} \left| \theta_{s,t}^1 - \theta_{s,t}^{1,\infty} \right|^2 \right] \, dv \, dt$$

$$- 2 \left( \int_0^T \int_0^S e^{\lambda v} \mathbb{E} \left[ \nabla_a h^\sigma_i (\theta_{s,t}^{1,v,.}, \nu_{s,t}^{N_2,\infty}, \mathcal{M}^{N_1}) - \nabla_a h^\sigma_i (\theta_{s,t}^{1,\infty}, \mathcal{L}(\theta_{s,t}^{i,.}), \mathcal{M}) \right] \, dv \, dt \right). \quad (40)$$
Observe that
\[\mathbb{E} \left[ \left( \nabla_a h^\vartheta (t^{1,\infty}, L(t,v), \mathcal{M}) - \nabla_a h^\vartheta (t^{1,\infty}, \nu^{N_2}, \mathcal{M}^{N_1}) \right) \left( \theta_{v,t}^{1,\infty} - \theta_{v,t}^{1,\infty} \right) \right] \]
\[= \mathbb{E} \left[ \frac{\sigma^2}{2} \left( U(t,v) - U(t,v^{1,\infty}) \right) \left( \theta_{v,t}^{1,\infty} - \theta_{v,t}^{1,\infty} \right) \right] \]
\[+ \mathbb{E} \left[ \left( \nabla_a h^\vartheta (t^{1,\infty}, L(t,v), \mathcal{M}) - \nabla_a h^\vartheta (t^{1,\infty}, \nu^{N_2,\infty}, \mathcal{M}^{N_1}) \right) \left( \theta_{v,t}^{1,\infty} - \theta_{v,t}^{1,\infty} \right) \right] \]
\[+ \mathbb{E} \left[ \left( \nabla_a h^\vartheta (t^{1,\infty}, \nu^{N_2,\infty}, \mathcal{M}^{N_1}) - \nabla_a h^\vartheta (t^{1,\infty}, \nu^{N_2}, \mathcal{M}^{N_1}) \right) \left( \theta_{v,t}^{1,\infty} - \theta_{v,t}^{1,\infty} \right) \right] . \]

By Lemma (21)
\[\left| \nabla_a h_t(t^{1,\infty}, \nu^{N_2,\infty}, \mathcal{M}^{N_1}) - \nabla_a h_t(t^{1,\infty}, \nu^{N_2}, \mathcal{M}^{N_1}) \right| \leq L \left( \left| \theta_{v,t}^{1,\infty} - \theta_{v,t}^{1,\infty} \right| + \frac{1}{N_2} \sum_{j=1}^{N_2} \mathbb{E} \left[ \int_0^T \left| \theta_{v,r}^{j,\infty} - \theta_{v,r}^{j,\infty} \right| dr \right] . \]

Therefore, by Young’s inequality and the exchangeability of \( (\theta_{s,r}^{j,\infty})_{j=1}^{N_2} \),
\[\mathbb{E} \left[ \left( \nabla_a h_t(t^{1,\infty}, \nu^{N_2,\infty}, \mathcal{M}^{N_1}) - \nabla_a h_t(t^{1,\infty}, \nu^{N_2}, \mathcal{M}^{N_1}) \right) \left( \theta_{v,t}^{1,\infty} - \theta_{v,t}^{1,\infty} \right) \right] \]
\[\leq L(1 + \frac{T}{2}) \mathbb{E} \left[ \left| \theta_{v,t}^{1,\infty} - \theta_{v,t}^{1,\infty} \right|^2 \right] + \frac{L}{2} \int_0^T \mathbb{E} \left[ \left| \theta_{v,r}^{1,\infty} - \theta_{v,r}^{1,\infty} \right|^2 \right] dt .

Using Young’s inequality again, we have
\[\mathbb{E} \left[ \left( \nabla_a h_t(t^{1,\infty}, \nu^{N_2,\infty}, \mathcal{M}^{N_1}) - \nabla_a h_t(t^{1,\infty}, \nu^{N_2}, \mathcal{M}^{N_1}) \right) \left( \theta_{v,t}^{1,\infty} - \theta_{v,t}^{1,\infty} \right) \right] \]
\[\leq \frac{1}{2} \mathbb{E} \left[ \left| \nabla_a h_t(t^{1,\infty}, \nu^{N_2,\infty}, \mathcal{M}^{N_1}) - \nabla_a h_t(t^{1,\infty}, \nu^{N_2}, \mathcal{M}^{N_1}) \right|^2 \right] + \frac{1}{2} \mathbb{E} \left[ \left| \theta_{v,t}^{1,\infty} - \theta_{v,t}^{1,\infty} \right|^2 \right] .

Coming back to (40) and employing Assumption 6-ii), we get
\[\int_0^T \mathbb{E} \left[ e^{\lambda s} \left| \theta_{s,t}^{1,\infty} - \theta_{s,t}^{1,\infty} \right|^2 \right] dt \leq \int_0^T \int_0^s \mathbb{E} \left[ 2(\lambda - \frac{\sigma^2 \kappa}{2} - \frac{L}{2}(3 + T) + \frac{1}{2}) \left| \theta_{v,t}^{1,\infty} - \theta_{v,t}^{1,\infty} \right|^2 \right] dv dt \]
\[+ 2 \int_0^s \mathbb{E} \left[ \left| \nabla_a h_t(t^{1,\infty}, \nu^{N_2,\infty}, \mathcal{M}^{N_1}) - \nabla_a h_t(t^{1,\infty}, \nu^{N_2,\infty}, \mathcal{M}^{N_1}) \right|^2 \right] dv dt .

(41)

**Step 2.** The statistical errors will be derived from
\[I := \int_0^T \mathbb{E} \left[ \left| \nabla_a h_t(t^{1,\infty}, \nu^{N_2,\infty}, \mathcal{M}^{N_1}) \right|^2 \right] dt \]
\[\leq 2 \int_0^T \mathbb{E} \left[ \left| \nabla_a h_t(t^{1,\infty}, \nu^{N_2,\infty}, \mathcal{M}^{N_1}) \right|^2 \right] dt \]
\[+ 2 \int_0^T \mathbb{E} \left[ \left| \nabla_a h_t(t^{1,\infty}, \nu^{N_2,\infty}, \mathcal{M}) \right|^2 \right] dt := I_1 + I_2 .

Recall that
\[\nabla_a h_t(t^{1,\infty}, \nu^{N_2,\infty}, \mathcal{M}^{N_1}) = \frac{1}{N_1} \sum_{j=1}^{N_1} \nabla_a h_t(X_t^{j,\zeta}(\nu^{N_2}), \theta_{v,t}, P_t^{j,\zeta}(\nu^{N_2}, \zeta)) .\]
Note that $E \left[ \nabla_a h_t(\theta_{v,t}^{1,\infty}, \nu_{v,t}^{N_1,\infty}, M^{N_1}) \big| \theta_{v,t}^{1,\infty}, \nu_{v,t}^{N_1,\infty} \right] = \nabla_a h_t(\theta_{v,t}^{1,\infty}, \nu_{v,t}^{N_1,\infty}, M^{N_1})$. Hence, from Assumption 1 and Lemmas 22 and 35

$$I_1 = \frac{2}{N_1} \int_0^T E \left[ \text{Var} \left[ \nabla_a h_t(\theta_{v,t}^{1,\infty}, \nu_{v,t}^{N_1,\infty}, M^{N_1}) \big| \theta_{v,t}^{1,\infty}, \nu_{v,t}^{N_1,\infty} \right] \right] \, dt \, dv \leq \frac{2}{N_1} \int_0^T E \left[ \| f_t(x, \cdot, \zeta^1) \|_{\text{Lip}}^2 + \| \phi_t(x, \cdot, \zeta^1) \|_{\text{Lip}}^2 P^t \xi, \zeta^1 \nu^{N_1,\infty} \right] \, dt \leq \frac{c}{N_1}.$$ 

Next, we aim to show that there exists constant $c$ independent of $S, d, p$, such that

$$I_2 := 2 E \left[ \nabla_a h_t(\theta_{v,t}^{1,\infty}, \nu_{v,t}, M) - \nabla_a h_t(\theta_{v,t}^{1,\infty}, \nu_{v,t}^{N_1,\infty}, M) \right]^2 \leq \frac{c}{N_2}. \quad (42)$$

By Assumption 1, we have

$$\left| \nabla_a h_t(\theta_{v,t}^{1,\infty}, \nu_{v,t}, M) - \nabla_a h_t(\theta_{v,t}^{1,\infty}, \nu_{v,t}^{N_1,\infty}, M) \right| \leq \int_{R^d \times S} \left( \| \nabla_a f_t(\cdot, \theta_{v,t}^{1,\infty}, \zeta) \|_{\text{Lip}} + \| \nabla_a \phi_t(\cdot, \theta_{v,t}^{1,\infty}, \zeta) \|_{\text{Lip}} P^t \xi, \zeta(\nu_{v,t}) \right) |X^\xi, \zeta_t(\nu_{v,t}) - X^\xi, \zeta^{N_1,\infty}_t(\nu_{v,t})| M(d\xi, d\zeta)$$

$$+ \| \nabla_a \phi_t \|_{\infty} \int_{R^d \times S} P^t \xi, \zeta(\nu_{v,t}) - P^t \xi, \zeta^{N_1,\infty}(\nu_{v,t}) |M(d\xi, d\zeta)| \leq c \left( \int_{R^d \times S} \left( 1 + |P^t \xi, \zeta(\nu_{v,t})| \right)^2 M(d\xi, d\zeta) \right)^{1/2} \left( \int_{R^d \times S} \left| X^\xi, \zeta_t(\nu_{v,t}) - X^\xi, \zeta^{N_1,\infty}_t(\nu_{v,t}) \right|^2 M(d\xi, d\zeta) \right)^{1/2}$$

$$+ c \left( \int_{R^d \times S} \left| P^t \xi, \zeta(\nu_{v,t}) - P^t \xi, \zeta^{N_1,\infty}(\nu_{v,t}) \right|^2 M(d\xi, d\zeta) \right)^{1/2}.$$

Lemma 37 tells us that there exists (an explicit) constant $c$ such that

$$\left| X^\xi, \zeta_t(\nu_{v,t}) - X^\xi, \zeta^{N_1,\infty}_t(\nu_{v,t}) \right| \leq c \int_0^T \left| \frac{1}{N_2} \sum_{j=1}^{N_2} \phi_r X^\xi, \zeta_t(\nu_{v,t}), \theta_{v,r}^{j,\infty}, \zeta \right| - \frac{1}{N_2} \sum_{j=1}^{N_2} \phi_r X^\xi, \zeta_t(\nu_{v,t}), \theta_{v,r}^{j,\infty}, \zeta \nu_{v,r}(da) \, dr$$

and

$$\left| P^t \xi, \zeta(\nu_{v,t}) - P^t \xi, \zeta^{N_1,\infty}(\nu_{v,t}) \right| \leq c \int_0^T \left| \frac{1}{N_2} \sum_{j=1}^{N_2} \phi_r X^\xi, \zeta_t(\nu_{v,t}), \theta_{v,r}^{j,\infty}, \zeta \right| - \frac{1}{N_2} \sum_{j=1}^{N_2} \phi_r X^\xi, \zeta_t(\nu_{v,t}), \theta_{v,r}^{j,\infty}, \zeta \nu_{v,r}(da) \, dr.$$

Consequently, and due to Lemma 35

$$E \left[ \nabla_a h_t(\theta_{v,t}^{1,\infty}, \nu_{v,t}, M) - \nabla_a h_t(\theta_{v,t}^{1,\infty}, \nu_{v,t}^{N_1,\infty}, M) \right]^2 \leq c \int_0^T \int_{R^d \times S} \left| \frac{1}{N_2} \sum_{j=1}^{N_2} \phi_r X^\xi, \zeta_t(\nu_{v,t}), \theta_{v,r}^{j,\infty}, \zeta \right| - \frac{1}{N_2} \sum_{j=1}^{N_2} \phi_r X^\xi, \zeta_t(\nu_{v,t}), \theta_{v,r}^{j,\infty}, \zeta \nu_{v,r}(da) \, dM(d\xi, d\zeta).$$

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Note that \( \mathbb{E} \left[ \phi_r(X^\xi_\tau^{\infty}(\nu_\tau), \theta_1^{\infty}(\nu_\tau), \zeta) \right] X^\xi_\tau^{\infty}(\nu_\tau), \zeta = \int \phi_r(X^\xi_\tau^{\infty}(\nu_\tau), a, \zeta) \nu_{\tau}(da) \). Hence
\[
\mathbb{E} \left[ \nabla_a h_t(\theta_1^{\infty}(\nu_{\tau}, a, \zeta), \theta_{\tau}^{\infty}(\nu_{\tau}, a, \zeta), \zeta) \right]^2 \right] \leq \frac{c}{N_2} \int_0^T \int_{\mathbb{R}^d \times \mathcal{S}} \mathbb{E} \left[ \nabla \phi_r(X^\xi_\tau^{\infty}(\nu_\tau), \theta_1^{\infty}(\nu_\tau), \zeta) \right] \mathcal{M}(d\xi, d\zeta) \ dt.
\]
(43)
By Assumption 1 and Lemmas 20 and 22 we know that
\[
\int_{\mathbb{R}^d \times \mathcal{S}} \mathbb{E} \left[ \nabla \phi_r(X^\xi_\tau^{\infty}(\nu_\tau), \theta_1^{\infty}(\nu_\tau), \zeta) \right] \mathcal{M}(d\xi, d\zeta) \leq c \int_{\mathbb{R}^d \times \mathcal{S}} (1 + \mathbb{E}|X^\xi_\tau^{\infty}(\nu_\tau)|^2 + \mathbb{E}|\theta_1^{\infty}(\nu_\tau)|^2) \mathcal{M}(d\xi, d\zeta) < \infty.
\]
From this and (43) we conclude that the estimate (42) holds.

**Step 3.** Coming back to (40), with \( \lambda = \frac{\sigma^2 \kappa}{2} - \frac{3}{2}(3 + T) + \frac{1}{2} \) we get that
\[
\int_0^T \mathbb{E} \left[ e^{\lambda s} |\theta_{s,t} - \theta_{t}^{\infty}|^2 \right] dt \leq c \left( \frac{1}{N_1} + \frac{1}{N_2} \right) \int_0^T e^{\lambda s} \ dt.
\]
Calculating the integral on the right hand side concludes the proof.

### 3.5 Time discretisation of the gradient descent

#### 3.5.1 Proof of Theorem 9

First, we consider only discretisation of the overdamped Langevin dynamics (23). A simple, explicit numerical scheme for the numerical approximation of (23) can be introduced through Euler–Maruyama approximations with non-homogeneous time steps. Fix an increasing sequence of times \( 0 = s_0 < s_1 < \cdots < s_l < \cdots \) and set
\[
\Lambda(s) = \sup \{ s_l : s_l \leq s \}.
\]
Now define the family of processes \( \{ \bar{\theta}_{s,t}^l \} \) satisfying, for any \( i \),
\[
\bar{\theta}_{s,t}^l = \theta_{t}^{\infty} - \int_0^s \nabla_a h_t \left( \bar{\theta}_{l}(\nu_{i}, t), \nu_{i}^{N_l} \right) \ dt + \sigma B_{s,t}^i\bar{\nu}_{v_{i}, t} = \frac{1}{N_2} \sum_{j=1}^{N_2} \delta_{(\bar{\theta}_{v_{j}, t}^l)}.
\]
(44)
For this approximation, the rate of convergence is given by the following lemma.

**Lemma 26** Let Assumptions 1 and 6 hold. Assume also that \( \{ s_l \} \) is a non-decreasing sequence times, starting from 0, such that \( \{ s_l - s_{l-1} \} \) is non-increasing, \( \sum_l (s_l - s_{l-1})^2 < \infty \) and, for \( \kappa \) large enough, (25) holds. Then, for all \( i, 1 \leq l \leq n \),
\[
\mathbb{E} \left[ \int_0^T \left| \theta_{s,t}^i - \bar{\theta}_{s,t}^i \right|^2 dt \right] \leq c \max_l \left( s_{l+1} - s_{l} \right) \left( 1 + \max_{0 \leq s \leq s_l} \mathbb{E} \left[ \left| \theta_{s,t}^i \right|^2 \right] \right).
\]

**Proof** As the components \( (\xi^i, \zeta^i) \), will be fixed throughout the proof, for the sake of simplicity, \( \mathcal{M}^{N_l} \) will be omitted from now on in the notation of \( h^\sigma \), and will be re-introduced only when needed.
Step 1. As, for all \(s_{t-1} \leq s \leq s_t, 0 \leq t \leq T\), the difference \(\Delta_{s,t} \theta^i := \theta^i_{s,t} - \tilde{\theta}^i_{s,t}\) is given by

\[
\Delta_{s,t} \theta^i = \Delta_{s_{t-1},t} \theta^i + \int_{s_{t-1}}^s \left( \nabla_a h_t^i \left( \theta^i_{t,v,t}, \nabla_{N_2}^t \nu_{v,t} \right) - \nabla_a h_t^i \left( \tilde{\theta}^i_{s_{t-1},t}, \tilde{\nu}^i_{s_{t-1},t} \right) \right) dv. \tag{45}
\]

In particular, for \(s = s_t\), and by integration by parts,

\[
|\Delta_{s_t,t} \theta^i|^2 = |\Delta_{s_{t-1},t} \theta^i|^2
- 2 \int_{s_{t-1}}^{s_t} (\Delta_{v,t} \theta^i) \cdot \left( \nabla_a h_t^i \left( \theta^i_{t,v,t}, \nabla_{N_2}^t \nu_{v,t} \right) - \nabla_a h_t^i \left( \tilde{\theta}^i_{s_{t-1},t}, \tilde{\nu}^i_{s_{t-1},t} \right) \right) dv
- 2 \int_{s_{t-1}}^{s_t} (\Delta_{v,t} \theta^i) \cdot \left( \nabla_a h_t^i \left( \theta^i_{s_{t-1},t}, \nabla_{N_2}^t \nu_{s_{t-1},t} \right) - \nabla_a h_t^i \left( \tilde{\theta}^i_{s_{t-1},t}, \tilde{\nu}^i_{s_{t-1},t} \right) \right) dv
= (s_t - s_{t-1})^2 \left( \nabla_a h_t^i \left( \theta^i_{s_{t-1},t}, \nabla_{N_2}^t \nu_{s_{t-1},t} \right) - \nabla_a h_t^i \left( \tilde{\theta}^i_{s_{t-1},t}, \tilde{\nu}^i_{s_{t-1},t} \right) \right)^2.
\tag{46}
\]

Using again (45) and, by the Young inequality: \(2a \cdot b \leq |a|^2 + |b|^2\), observe then that

\[
- 2 \int_{s_{t-1}}^{s_t} (\Delta_{v,t} \theta^i) \cdot \left( \nabla_a h_t^i \left( \theta^i_{s_{t-1},t}, \nabla_{N_2}^t \nu_{s_{t-1},t} \right) - \nabla_a h_t^i \left( \tilde{\theta}^i_{s_{t-1},t}, \tilde{\nu}^i_{s_{t-1},t} \right) \right) dv
\leq (1 - \sigma^2 \kappa)(s_t - s_{t-1}) |\Delta_{s_{t-1},t} \theta^i|^2
+ \int_{s_{t-1}}^{s_t} (1 + (v - s_{t-1})) \left| \nabla_a h_t^i \left( \theta^i_{s_{t-1},t}, \nabla_{N_2}^t \nu_{s_{t-1},t} \right) - \nabla_a h_t^i \left( \tilde{\theta}^i_{s_{t-1},t}, \tilde{\nu}^i_{s_{t-1},t} \right) \right|^2 dv.
\]

In the same way, we have

\[
- 2 \int_{s_{t-1}}^{s_t} (\Delta_{v,t} \theta^i) \cdot \left( \nabla_a h_t^i \left( \theta^i_{s_{t-1},t}, \nabla_{N_2}^t \nu_{s_{t-1},t} \right) - \nabla_a h_t^i \left( \tilde{\theta}^i_{s_{t-1},t}, \tilde{\nu}^i_{s_{t-1},t} \right) \right) dv
\leq (s_t - s_{t-1}) |\Delta_{v,t} \theta^i|^2
+ \int_{s_{t-1}}^{s_t} (1 + (v - s_{t-1})) \left| \nabla_a h_t^i \left( \theta^i_{s_{t-1},t}, \nabla_{N_2}^t \nu_{s_{t-1},t} \right) - \nabla_a h_t^i \left( \tilde{\theta}^i_{s_{t-1},t}, \tilde{\nu}^i_{s_{t-1},t} \right) \right|^2 dv.
\]

Coming back to (46), we obtain

\[
|\Delta_{s,t} \theta^i|^2 \leq |\Delta_{s_{t-1},t} \theta^i|^2 \left( 1 - (\sigma^2 \kappa - 3)(s_t - s_{t-1}) \right)
+ \int_{s_{t-1}}^{s_t} (1 + (v - s_{t-1})) \left| \nabla_a h_t^i \left( \theta^i_{s_{t-1},t}, \nabla_{N_2}^t \nu_{s_{t-1},t} \right) - \nabla_a h_t^i \left( \tilde{\theta}^i_{s_{t-1},t}, \tilde{\nu}^i_{s_{t-1},t} \right) \right|^2 dv.
\tag{47}
\]
Taking the expectation of the above and integrating the resulting expression over \( [0, T] \), we get
\[
\int_0^T \mathbb{E}[|\triangle_{s,t} \theta|^2] dt \leq (1 - (1 + \sigma^2 \kappa)(s_l - s_{l-1})) \int_0^T \mathbb{E}[|\triangle_{s_l-1,t} \theta^i|^2] dt \\
+ (s_l - s_{l-1}) \int_0^T \mathbb{E}[\nabla_a h_t \left( \theta^i_{s_l-1,t}, v_{s_l-1} \right) - \nabla_a h_t \left( \hat{\theta}^i_{s_l-1,t}, \bar{v}_{s_l-1} \right) |^2] dt \\
+ 2(s_l - s_{l-1})^2 \int_0^T \mathbb{E}[\nabla_a h_t^\sigma \left( \theta^i_{s_l-1,t}, v_{s_l-1} \right) - \nabla_a h_t^\sigma \left( \hat{\theta}^i_{s_l-1,t}, \bar{v}_{s_l-1} \right) |^2] dt \\
+ \int_0^{s_l} (1 + (v - s_{l-1})) \int_0^T \mathbb{E}[\nabla_a h_t^\sigma \left( \hat{\theta}^i_{v,t}, \bar{v}_{v,t} \right) - \nabla_a h_t^\sigma \left( \hat{\theta}^i_{s_l-1,t}, \bar{v}_{s_l-1} \right) |^2] dt dv.
\]

Lemma 21 then yields
\[
\int_0^T \mathbb{E}[\nabla_a h_t \left( \theta^i_{s_l-1,t}, v_{s_l-1} \right) - \nabla_a h_t \left( \hat{\theta}^i_{s_l-1,t}, \bar{v}_{s_l-1} \right) |^2] dt \leq L \int_0^T \mathbb{E}[|\triangle_{s_l-1,t} \theta^i|^2] dt,
\]
and
\[
\int_0^T \mathbb{E}[\nabla_a h_t^\sigma \left( \theta^i_{s_l-1,t}, v_{s_l-1} \right) - \nabla_a h_t^\sigma \left( \hat{\theta}^i_{s_l-1,t}, \bar{v}_{s_l-1} \right) |^2] dt \leq L (1 + \frac{\sigma^2}{4} \|\nabla_a U\|_{Lip}^2) \int_0^T \mathbb{E}[|\triangle_{s_l-1,t} \theta^i|^2] dt.
\]

Plugged into (48), using the exchangeability of \( \theta^i \) and \( \hat{\theta}^i \), we get
\[
\int_0^T \mathbb{E}[|\triangle_{s,t} \theta|^2] dt \\
\leq \left( 1 + (s_l - s_{l-1}) \left\{ (L - \sigma^2 \kappa) + 2(s_l - s_{l-1}) L \left( 1 + \frac{\sigma^4}{2} \|\nabla_a U\|_{Lip}^2 \right) \right\} \right) \times \int_0^T \mathbb{E}[|\triangle_{s_l-1,t} \theta^i|^2] dt \\
+ \int_0^{s_l} (1 + (v - s_{l-1})) \int_0^T \mathbb{E}[\nabla_a h_t^\sigma \left( \hat{\theta}^i_{v,t}, \bar{v}_{v,t} \right) - \nabla_a h_t^\sigma \left( \hat{\theta}^i_{s_l-1,t}, \bar{v}_{s_l-1} \right) |^2] dt dv.
\]

Recalling the recurrence estimate:
\[
u_{t+1} \leq c_{t+1} \nu_t + b_{t+1}, \quad \forall l \quad \Rightarrow \quad \nu_t \leq \sum_{l=1}^t \left( \Pi_{l=1}^l c_{l+1} \right) b_{l-1} + u_0 \left( \Pi_{l=1}^l c_{l+1} \right), \quad \forall l,
\]
we get, since \( \triangle_{0,t} \theta = 0 \),
\[
\int_0^T \mathbb{E}[|\triangle_{s,t} \theta|^2] dt \\
\leq \sum_{l=1}^t \left( \Pi_{l=1}^l \left( 1 + (s_l - s_{l-1}) \left( L - \sigma^2 \kappa \right) + (s_l - s_{l-1})^2 L \left( 1 + \frac{\sigma^4}{2} \|\nabla_a U\|_{Lip}^2 \right) \right) \right) \times \int_0^{s_l} (1 + (v - s_{l-1})) \int_0^T \mathbb{E}[\nabla_a h_t^\sigma \left( \hat{\theta}^i_{v,t}, \bar{v}_{v,t} \right) - \nabla_a h_t^\sigma \left( \hat{\theta}^i_{s_l-1,t}, \bar{v}_{s_l-1} \right) |^2] dt dv
\]

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Step 2. Observing that

\[
\mathbb{E} \left[ \left| \nabla_a h^0_t \left( \theta^i_{v,t}, \varphi^N_{v,t} \right) - \nabla_a h^0_t \left( \theta^i_{s_{t-1},t}, \varphi^N_{s_{t-1},t} \right) \right|^2 \right] \leq L \left( 1 + \frac{\sigma^4}{2} \| \nabla_a U \|_{lip}^2 \right) \int_0^T \mathbb{E} \left[ \left| \theta^i_{v,t} - \theta^i_{s_{t-1},t} \right|^2 \right] dt,
\]

and since, for all \( s_{t-1} \leq v \leq s_t \),

\[
\mathbb{E} \left[ \left| \theta^i_{v,t} - \theta^i_{s_{t-1},t} \right|^2 \right] \leq L(s_t - s_{t-1}) \left( 1 + 2\sigma^2 + \frac{\sigma^4}{2} (s_t - s_{t-1}) \| \nabla_a U \|_{lip}^2 \right) \int_{s_{t-1}}^{s_t} \mathbb{E} \left[ \left| \theta^i_{s,t} \right|^2 \right] dv,
\]

we have

\[
\int_{s_{t-1}}^{s_t} (1 + (v - s_{t-1})) \int_0^T \mathbb{E} \left[ \left| \nabla_a h^0_t \left( \theta^i_{v,t}, \varphi^N_{v,t} \right) - \nabla_a h^0_t \left( \theta^i_{s_{t-1},t}, \varphi^N_{s_{t-1},t} \right) \right|^2 \right] dt dv
\]

\[
\leq c(s_t - s_{t-1})^2 \left( 1 + \max_{0 \leq s \leq s_t} \int_0^T \mathbb{E} \left[ \left| \theta^i_{s,t} \right|^2 \right] dt \right),
\]

from which we get

\[
\int_0^T \mathbb{E} \left[ |\triangle_{s_{t-1}} \theta|^2 \right] dt \leq c \left( 1 + \max_{0 \leq s \leq s_t} \int_0^T \mathbb{E} \left[ \left| \theta^i_{s,t} \right|^2 \right] dt \right)
\]

\[
\times \sum_{l_1=1}^l \left( \Pi_{l_2=l_1}^l \left( 1 + (s_{l_1} - s_{l_1-1}) \left( (L - \sigma^2 \kappa) + (s_{l_1} - s_{l_1-1}) L \left( 1 + \frac{\sigma^4}{2} \| \nabla_a U \|_{lip}^2 \right) \right) \right) \right) (s_{l_1} - s_{l_1-1})^2.
\]

By the assumption (25), any time step \( s_{l} - s_{l-1} \) is small enough so that the coefficient

\[
(L - \sigma^2 \kappa) + (s_{l_1} - s_{l_1-1}) L \left( 1 + \frac{\sigma^4}{2} \| \nabla_a U \|_{lip}^2 \right) := \kappa,
\]

is negative. It now remains to prove that the sum

\[
\sum_{l=1}^l \left( \Pi_{l_2=l_1}^l \left( 1 + (s_{l_1} - s_{l_1-1}) \left( (L - \sigma^2 \kappa) + (s_{l_1} - s_{l_1-1}) L \left( 1 + \frac{\sigma^4}{2} \| \nabla_a U \|_{lip}^2 \right) \right) \right) \right) (s_{l_1} - s_{l_1-1})
\]

\[
= \sum_{l=1}^l \left( \Pi_{l_2=l_1}^l (1 - |\kappa|(s_{l_2} - s_{l_2-1})) \right) (s_{l_1} - s_{l_1-1})
\]

is finite. Since \( 1 - x \leq e^{-x} \) for all \( x \geq 0 \), we have

\[
\sum_{l=1}^l \left( \Pi_{l_2=l_1}^l (1 - |\kappa|(s_{l_2} - s_{l_2-1})) \right) (s_{l_1} - s_{l_1-1}) \leq \sum_{l=1}^l \exp \{-|\kappa|(s_{l_1} - s_{l_1})\} (s_{l_1} - s_{l_1-1}).
\]
Comparing this upper-bound with the integral \( \int_0^{s_i} \exp\{-|\mathcal{R}|(s_t-v)\} \, dv \), the assumption \( \sum_l(s_l-s_{l-1})^2 < \infty \) is enough to ensure the finiteness of the sum as:

\[
\sum_{l=1}^{l-1} \exp\{-|\mathcal{R}|(s_{l}-s_{l-1})\}(s_{l}-s_{l-1}) - \int_0^{s_l} \exp\{-|\mathcal{R}|(s_t-v)\} \, dv \\
= \sum_{l=1}^{l-1} \int_{s_{l-1}}^{s_l} \left( \exp\{-|\mathcal{R}|(s_{l}-s_{l-1})\}(s_{l}-s_{l-1}) - \exp\{-|\mathcal{R}|(s_t-v)\} \right) \, dv - \int_0^{s_l} \exp\{-|\mathcal{R}|(s_t-v)\} \, dv \\
\leq \exp\{-|\mathcal{R}|s_1\} \sum_{l=1}^{l-1} \int_{s_{l-1}}^{s_l} (v-s_{l-1}) \exp\{|\mathcal{R}|v\} \, dv \leq \sum_{l=1}^{l-1} (s_{l}-s_{l-1})^2 < \infty.
\]

This ends the proof.

### 3.6 Generalisation estimates

**Lemma 27** Let Assumptions 1 and 6 hold. Then there exist constants \( L_{1,M} \) and \( L_{2,M} \), such that for any \( \mu, \nu \in \mathcal{V}_2 \) we have

i) For all \( M \in \mathcal{P}(\mathbb{R}^d \times \mathcal{S}) \),

\[
|J^M(\mu) - J^M(\nu)| \leq L_{1,M} W^1_T(\mu, \nu),
\]

ii) For any stochastic processes \( \eta, \eta' \) such that \( \mathbb{E} \int_0^T |\eta_t|^2 + |\eta'_t|^2 \, dt < \infty \) we have

\[
\mathbb{E} \left[ \sup_{\nu \in \mathcal{V}_2} \left| \int_0^T \frac{\delta J^M}{\delta \nu} (\nu, t, \eta_t) \, dt \right|^2 \right] + \mathbb{E} \left[ \sup_{\nu \in \mathcal{V}_2} \left| \int_0^T \int_0^T \frac{\delta^2 J^M}{\delta \nu^2} (\nu, t, \eta_t, \eta'_t) \, dt \, dt' \right|^2 \right] \leq L_{2,M}.
\]

The expression \( \frac{\delta J^M}{\delta \nu}(\nu, t, a) \) here is to the derivative of \( \nu \in \mathcal{V}_2 \mapsto J^M(\nu) \) (see Definition 30) and \( \frac{\delta^2 J^M}{\delta \nu^2}(\nu, t, a, t', a') \) is the derivative of \( \nu \in \mathcal{V}_2 \mapsto \frac{\delta J^M}{\delta \nu}(\nu, t, a) \) (Definition 31).

**Proof**

Let \( \nu^\lambda := \nu + \lambda(\mu - \nu) \) for \( \lambda \in [0,1] \) and recall definition of \( \bar{J} \) from (12). From Lemma 18 (with \( \sigma = 0 \)) we have

\[
\frac{d}{d\varepsilon} J^M((\nu + (\lambda + \varepsilon)(\mu - \nu)), \xi, \zeta) \bigg|_{\varepsilon = 0} = \int_0^T \int h_t(X_t^{\xi,\zeta}(\nu^\lambda), P_t^{\xi,\zeta}(\nu^\lambda), a, \zeta)(\mu_t - \nu_t)(da) \, dt.
\]

Due to this and the fundamental theorem of calculus we have

\[
J^M(\mu, \xi, \zeta) - J^M(\nu, \xi, \zeta) = \int_0^1 \frac{d}{d\varepsilon} J^M((\nu + (\lambda + \varepsilon)(\mu - \nu)), \xi, \zeta) \bigg|_{\varepsilon = 0} \, d\lambda \]

\[
= \int_0^1 \int_0^T \int h_t(X_t^{\xi,\zeta}(\nu^\lambda), P_t^{\xi,\zeta}(\nu^\lambda), a, \zeta)(\mu_t - \nu_t)(da) \, dt \, d\lambda.
\]

Assumptions 1, 6 point iii) and Lemma 35 allow us to conclude that

\[
a \mapsto h_t(X_t^{\xi,\zeta}(\nu^\lambda), P_t^{\xi,\zeta}(\nu^\lambda), a, \zeta) = \phi_t(X_t^{\xi,\zeta}(\nu^\lambda), a, \zeta) P_t^{\xi,\zeta}(\nu^\lambda) + f(X_t^{\xi,\zeta}(\nu^\lambda), a, \zeta),
\]

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is uniformly Lipschitz in $a$. From Fubini’s Theorem and Kantorovich (dual) representation of the Wasserstein distance (Villani, 2008, Th 5.10) we conclude that

$$\|f_t(X_t^{\xi,\zeta}(\nu), P_t^{\xi,\zeta}(\nu))\|_{L_{tp}} = L_1(\xi, \zeta) \sup_{t \in [0, T], \nu \in V_2} \|h_t(X_t^{\xi,\zeta}(\nu), P_t^{\xi,\zeta}(\nu), \cdot, \zeta)\|_{L_{tp}},$$

(52)

and we see, by Lemma 35, that

$$|J^M(\mu, \xi, \zeta) - J^M(\nu, \xi, \zeta)| \leq L_1(\xi, \zeta) W_1(\mu, \nu), \quad L_1(\xi, \zeta) := \sup_{t \in [0, T], \nu \in V_2} \|h_t(X_t^{\xi,\zeta}(\nu), P_t^{\xi,\zeta}(\nu), \cdot, \zeta)\|_{L_{tp}},$$

This completes the proof of part i).

From (51), we are able to identify the derivative of $\nu \in V_2 \mapsto J(\nu, \xi, \zeta)$ and see that

$$\frac{\delta J}{\delta \nu}(\mu, t, a, \xi, \zeta) = \phi_t(X_t^{\xi,\zeta}(\mu), a, \zeta) P_t^{\xi,\zeta}(\mu) + f_t(X_t^{\xi,\zeta}(\mu), a, \zeta).$$

Hence due to Definition 30 and a following similar computation as in Lemma 36 we have

$$\frac{\delta^2 J}{\delta \nu^2}(\mu, t, a, t', a', \xi, \zeta) = (\nabla_x \phi_t)(X_t^{\xi,\zeta}(\mu), a, \zeta) \frac{\delta X_t^{\xi,\zeta}}{\delta \nu}(\mu, t', a') P_t^{\xi,\zeta}(\mu) + \phi_t(X_t^{\xi,\zeta}(\mu), a, \zeta) \frac{\delta P_t^{\xi,\zeta}}{\delta \nu}(\mu, t', a')$$

+ $$(\nabla_x f_t)(X_t^{\xi,\zeta}(\mu), a, \zeta) \frac{\delta X_t^{\xi,\zeta}}{\delta \nu}(\mu, t', a').$$

Note that

$$\frac{\delta J^M}{\delta \nu}(\mu, t, a) = \int_{\mathbb{R}^d \times S} \frac{\delta J}{\delta \nu}(\mu, t, a, \xi, \zeta) M(d\xi, d\zeta),$$

$$\frac{\delta^2 J^M}{\delta \nu^2}(\mu, t, a, t', a') = \int_{\mathbb{R}^d \times S} \frac{\delta^2 J}{\delta \nu^2}(\mu, t, a, t', a', \xi, \zeta) M(d\xi, d\zeta).$$

From Lemma 36 we see that for $\eta$ and $\eta'$ such that $\int_0^T E[|\eta|^2 + |\eta'|^2]dt < \infty$ we have

$$E \left[ \sup_{\nu \in V_2} \left( \int_0^T \frac{\delta J^M}{\delta \nu}(\nu, t, \eta)dt \right)^2 \right] + E \left[ \sup_{\nu \in V_2} \left( \int_0^T \int_0^T \frac{\delta^2 J^M}{\delta \nu^2}(\nu, t, \eta, t', \eta')dt'dt \right)^2 \right] \leq L_{2,M}.$$  

Lemma 28 We assume that the 2nd order linear functional derivative, in a sense of definition in 30, of $J$ exists, and that there is $L > 0$ such that for any random variables $\eta$, $\eta'$ such that $E[|\eta|^2 + |\eta'|^2] < \infty$, it holds that

$$E \left[ \sup_{\nu \in V_2} \left( \int_0^T \frac{\delta J^M}{\delta \nu}(\nu, t, \eta)dt \right)^2 \right] + E \left[ \sup_{\nu \in V_2} \left( \int_0^T \int_0^T \frac{\delta^2 J^M}{\delta \nu^2}(\nu, t, \eta, t', \eta')dt'dt \right)^2 \right] \leq L.$$  

(53)

Let $(\theta_i)^N_{i=1}$ be i.i.d such that $\theta_i \sim \mu$, $i = 1, \ldots, N$. Let $\mu^N := \frac{1}{N} \sum_{i=1}^N \delta_{\theta_i}$. Then there is $c$ (independent of $N$, $p$ and $d$) such that

$$E \left[ |J^M(\mu^N) - J^M(\mu)| \right] \leq \frac{c}{N}.$$
By the definition of the second order functional derivative of $J^\mathcal{M}(\mu^N) - J^\mathcal{M}(\mu)$ for $\lambda \in [0, 1]$ and let $(\tilde{\theta}^i)^N_{i=1}$ be i.i.d., independent of $(\theta^i)^N_{i=1}$ and with law $\mu$. By the definition of linear functional derivatives, we have

\[
J^\mathcal{M}(\mu^N) - J^\mathcal{M}(\mu) = \int_0^1 \int_0^T \int \frac{\delta J^\mathcal{M}}{\delta \nu}(\mu^N, t, a)(\mu^N - \mu_t)(da) dt d\lambda \\
= \int_0^1 \int_0^T \int \frac{\delta J^\mathcal{M}}{\delta \nu}(\mu^N, t, \tilde{\theta}^i_t) - E\left[\frac{\delta J^\mathcal{M}}{\delta \nu}(\mu^N, t, \tilde{\theta}^i_t)\right] dt d\lambda \\
= \int_0^1 \int_0^T \frac{1}{N} \sum_{i=1}^N (\frac{\delta J^\mathcal{M}}{\delta \nu}(\mu^N, t, \tilde{\theta}^i_t) - E\left[\frac{\delta J^\mathcal{M}}{\delta \nu}(\mu^N, t, \tilde{\theta}^i_t)\right]) dt d\lambda
\]

where, for $i \in \{1, \ldots, N\}$ and $\lambda \in [0, 1]$,

\[
\varphi^i = \frac{\delta J^\mathcal{M}}{\delta \nu}(\mu^N, t, \theta^i) - E\left[\frac{\delta J^\mathcal{M}}{\delta \nu}(\mu^N, t, \tilde{\theta}^i_t)\right]. \quad (54)
\]

Note that the expectation only applies to $\tilde{\theta}^i$ and that $\varphi^i$ is zero mean random variable. We have the estimate

\[
E\left[|J^\mathcal{M}(\mu^N) - J^\mathcal{M}(\mu)|^2\right] \leq \frac{T}{N^2} \int_0^1 \int_0^T E\left[\sum_{i=1}^N \varphi^i + \sum_{i_1 \neq i_2} \varphi^i \varphi^{i_2}\right] dt d\lambda. \quad (55)
\]

**Step 2.** By our assumption (53) we have

\[
E\left[\sum_{i=1}^N \varphi^i\right] \leq 2 \sum_{i=1}^N E\left[\frac{\delta J^\mathcal{M}}{\delta \nu}(\mu^N, t, \theta^i)\right] \leq 2LN.
\]

**Step 3.** For any $(i_1, i_2) \in \{1, \ldots, N\}^2$, we introduce the (random) measures

\[
\mu^N_{\lambda, -(i_1, i_2)} := \mu^N + \frac{\lambda}{N} \sum_{k \in \{i_1, i_2\}} (\delta_{\tilde{\theta}^k} - \delta_{\theta^k}) \text{ and } \mu^N_{\lambda, \lambda_1} := (\mu^N_{\lambda, -(i_1, i_2)} - \mu^N)\lambda_1 + \mu^N, \lambda, \lambda_1 \in [0, 1].
\]

By the definition of the second order functional derivative

\[
\frac{\delta^2 J^\mathcal{M}}{\delta \nu^2}(\mu^N_{\lambda, -(i_1, i_2)}, t, \theta^i) - \frac{\delta^2 J^\mathcal{M}}{\delta \nu^2}(\mu^N, t, \theta^i) \\
= \int_0^1 \int_0^T \int \frac{\delta^2 J^\mathcal{M}}{\delta \nu^2}(\mu^N_{\lambda, t, \lambda_1}, t, \tilde{\theta}^i_t, t', y_1)(\mu^N_{\lambda, t, \lambda_1} - \mu^N_{\lambda, t, \lambda_1})(dy_1) d\lambda_1 dt' \\
= \frac{\lambda}{N} \int_0^1 \int_0^T \int \sum_{k \in \{i_1, i_2\}} \frac{\delta^2 J^\mathcal{M}}{\delta \nu^2}(\mu^N_{\lambda, t, \lambda_1}, t, \tilde{\theta}^i_t, t', y_1)(\delta_{\tilde{\theta}^k} - \delta_{\theta^k})(dy_1) d\lambda_1 dt'. \quad (56)
\]

By our assumption (53) we have

\[
E\left[\left|\frac{\delta^2 J^\mathcal{M}}{\delta \nu^2}(\mu^N_{\lambda, -(i_1, i_2)}, t, \theta^i) - \frac{\delta^2 J^\mathcal{M}}{\delta \nu^2}(\mu^N, t, \theta^i)\right|^2\right] \leq \frac{4TL}{N^2}.
\]
In the same way we can show that
\[
E \left[ E \left[ \frac{\delta J^M}{\delta \nu} (\mu_\lambda^{N,-(1,i_2)}, t, \tilde{\theta}^i) \right] - E \left[ \frac{\delta J^M}{\delta \nu} (\mu_\lambda^{N,-(1,i_2)}, t, \tilde{\theta}^i) \right] \right]^2 \leq \frac{4TL}{N^2}.
\]
Hence
\[
E[|\varphi^i - \varphi^i, -(i_1,i_2)|^2] \leq \frac{8TL}{N^2}, \text{ where } \varphi^i, -(i_1,i_2) = \frac{\delta J^M}{\delta \nu} (\mu_\lambda^{N,-(1,i_2)}, t, \tilde{\theta}^i) - E \left[ \frac{\delta J^M}{\delta \nu} (\mu_\lambda^{N,-(1,i_2)}, t, \tilde{\theta}^i) \right].
\]
Finally, by writing \( \varphi^i = (\varphi^i - \varphi^i, -(i_1,i_2)) + \varphi^i, -(i_1,i_2) \), applying Cauchy-Schwarz inequality and using (53) we have
\[
E \left[ \sum_{i_1 \neq i_2} \varphi^i_1 \varphi^i_2 \right] \leq \sum_{i_1 \neq i_2} \left( \frac{1}{N} + E[\varphi^i, -(i_1,i_2) \varphi^i, -(i_1,i_2)] \right) = N + \sum_{i_1 \neq i_2} E[\varphi^i, -(i_1,i_2) \varphi^i, -(i_1,i_2)].
\]
By conditional independence argument the last term above is zero. Combining this, (55), Conclusions of Step 1 and 2 concludes the proof.

**Proof** [Proof of Theorem 10] Throughout the proof we write \( J = J^M \). We decompose the error as follows:
\[
E \left[ J^M(\nu^{*,\sigma,N_1}) - J^M(\nu^{\sigma,N_1,N_2}, \Delta^s) \right]^2 \leq 4 \left( E|\mathcal{E}_1|^2 + E|\mathcal{E}_2|^2 + E|\mathcal{E}_3|^2 \right),
\]
where
\[
\mathcal{E}_1 := J^M(\nu^{*,\sigma,N_1}) - J^M(\nu^{\sigma,N_1}), \quad \mathcal{E}_2 := J^M(\nu^{\sigma,N_1}) - J^M(\nu^{\sigma,N_1,N_2}), \quad \mathcal{E}_3 := J^M(\nu^{\sigma,N_1,N_2}) - J^M(\nu^{\sigma,N_1,N_2}, \Delta^s).
\]
Here \( \mathcal{E}_1 \) is the error arising from running the mean-field gradient descent only for finite time \( S \). The error arising from replacing the mean-field gradient descent by a particle approximation is \( \mathcal{E}_2 \) and finally \( \mathcal{E}_3 \) arises from doing a time discretisation of the particle gradient descent.

**Step 1.** From Lemma 27-\( i) \) and Theorem 7 we conclude that
\[
E|\mathcal{E}_1|^2 = E \left[ J^M(\nu^{*,\sigma,N_1}) - J^M(\nu^{\sigma,N_1,N_2}, \Delta^s) \right]^2 \leq e^{-\lambda S} L^2_{1,M} \mathbb{E} \left[ \mathcal{W}_2^2 \left( \mathcal{L}(\theta^0), \mathcal{L}(\nu^{*,\sigma,N_1}) \right) \right].
\]

**Step 2.** Consider i.i.d copies of the mean-field Langevin dynamic \( \theta^i\infty_{t=1} \)
\[
\theta_{s,t}^i = \theta_{0,t}^i - \int_0^s \left( \nabla h_t(\theta_{v,t}^i, \mathcal{L}(\theta_{v,t}^i), \mathcal{M}^{N_1}) + \frac{\sigma^2}{2} \nabla U(\theta_{v,t}^i) \right) dv + \sigma dB_t^i.
\]
The associated empirical measure is defined as \( \bar{\nu}^{\sigma,N_1,N_2} = 1/N_2 \sum_{i=1}^{N_2} \delta_{\theta^i\infty} \). We have
\[
E|\mathcal{E}_2|^2 = E \left[ J^M(\nu^{\sigma,N_1}) - J^M(\nu^{\sigma,N_1,N_2}) \right]^2 \leq 2E \left[ J^M(\bar{\nu}^{\sigma,N_1}) - J^M(\bar{\nu}^{\sigma,N_1,N_2}) \right]^2 + 2E \left[ J^M(\nu^{\sigma,N_1,N_2}) - J^M(\nu^{\sigma,N_1,N_2}) \right]^2 =: 2E|\mathcal{E}_{2,1}|^2 + 2E|\mathcal{E}_{2,2}|^2.
\]
From Lemmas 27-\(ii\) and 28 we see that \(\mathbb{E}|\mathcal{E}_{2,1}|^2 \leq \frac{c}{N_2}\). Next we observe that by Lemma 27-\(i\) and by the definition of Wasserstein distance
\[
\mathbb{E}|\mathcal{E}_{2,2}|^2 = \mathbb{E}\left|J^\mathcal{M}(\nu_{S,1}^{\sigma,N_1,N_2}) - J^\mathcal{M}(\nu_{S,1}^{\sigma,N_1,N_2})\right|^2 \leq (L_{1,M})^2 \mathbb{E}\left[\mathcal{W}_2^T\left(\nu_{S,1}^{\sigma,N_1,N_2}, \nu_{S,1}^{\sigma,N_1,N_2}\right)^2\right]
\leq \frac{(L_{1,M})^2}{N_2} \sum_{i=1}^{N_2} \int_0^T \mathbb{E}|\theta_{S,t}^{i,\infty} - \theta_{S,t}^{i}|^2 dt.
\]
Finally, due to Theorem 8 we see that \(\int_0^T \mathbb{E}|\theta_{S,t}^{i,\infty} - \theta_{S,t}^{i}|^2 dt \leq c\left(\frac{1}{N_1} + \frac{1}{N_2}\right)\) and so
\[
\mathbb{E}|\mathcal{E}_{2}|^2 \leq c\left(\frac{1}{N_1} + \frac{1}{N_2}\right).
\]
**Step 3.** By Lemma 27, by Theorem 9 and by the definition of Wasserstein distance
\[
\mathbb{E}|\mathcal{E}_{3}|^2 \leq \mathbb{E}\left|J^\mathcal{M}(\nu_{S,1}^{\sigma,N_1,N_2}) - J^\mathcal{M}(\nu_{S,1}^{\sigma,N_1,N_2,\Delta s})\right|^2 \leq (L_{1,J})^2 \mathbb{E}\left[\mathcal{W}_2^T\left(\nu_{S,1}^{\sigma,N_1,N_2}, \nu_{S,1}^{\sigma,N_1,N_2,\Delta s}\right)^2\right]
\leq \frac{(L_{1,J})^2}{N_2} \sum_{i=1}^{N_2} \int_0^T \mathbb{E}|\theta_{S,t}^{i,\infty} - \theta_{S,t}^{i}|^2 dt \leq (L_{1,J})^2 c\Delta s,
\]
where \(\Delta s := \max_{0 \leq t < S}(s_i - s_{i-1})\). Collecting conclusions of Steps 1, 2 and 3 we obtain
\[
\mathbb{E}\left|J^\mathcal{M}(\nu^{*,\sigma,N_1}) - J^\mathcal{M}(\nu_{S,1}^{\sigma,N_1,N_2,\Delta s})\right|^2 \leq c\left(e^{-\lambda S} + \frac{1}{N_1} + \frac{1}{N_2} + \Delta s\right),
\]
where \(c\) is independent of \(\lambda, S, N_1, N_2, d, p\) and the time partition used in Theorem 9.

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**Appendix A. Measure derivatives**

We first define flat derivative on \(\mathcal{P}_2(\mathbb{R}^p)\). See e.g. (Carmona and Delarue, 2018, Section 5.4.1) for more details.

**Definition 29** A functional \(U : \mathcal{P}_2(\mathbb{R}^p) \to \mathbb{R}\) is said to admit a linear derivative if there is a (continuous on \(\mathcal{P}_2(\mathbb{R}^p)\)) map \(\delta U/\delta m : \mathcal{P}(\mathbb{R}^p) \times \mathbb{R}^d \to \mathbb{R}\), such that \(|\delta U/\delta m(a, \mu)| \leq C(1 + |a|^2)\) and, for all \(m, m' \in \mathcal{P}_2(\mathbb{R}^p)\), it holds that
\[
U(m) - U(m') = \int_0^1 \int \frac{\delta U}{\delta m}(m + \lambda(m' - m), a) (m' - m)(da) d\lambda.
\]
Since $\frac{\delta U}{\delta m}$ is only defined up to a constant we make a choice by demanding $\int \frac{\delta U}{\delta m}(m,a) m(da) = 0$.

We will also need the linear functional derivative on $\mathcal{V}_2$, which provides a slight extension of the one introduced in the above Definition 29.

**Definition 30** A functional $F : \mathcal{V}_2 \rightarrow \mathbb{R}^d$, is said to admit a first order linear derivative, if there exists a functional $\frac{\delta F}{\delta \nu} : \mathcal{V}_2 \times (0, T) \times \mathbb{R}^p \rightarrow \mathbb{R}^d$, such that

i) For all $(t, a) \in (0, T) \times \mathbb{R}^p$, $\nu \in \mathcal{V}_2 \mapsto \frac{\delta F}{\delta \nu}(\nu, t, a)$ is continuous (for $\mathcal{V}_2$ endowed with the weak topology of $\mathcal{M}_b^+(((0, T) \times \mathbb{R}^p))$).

ii) For any $\nu \in \mathcal{V}_2$ there exists $C = C_{\nu,T,d,p} > 0$ such that for all $a \in \mathbb{R}^p$ we have that

$$\left| \frac{\delta F}{\delta \nu}(\nu, t, a) \right| \leq C(1 + |a|^q).$$

iii) For all $\nu, \rho \in \mathcal{V}_2$,

$$F(\rho) - F(\nu) = \int_0^1 \int_0^T \int \frac{\delta F}{\delta \nu}(1 - \lambda)\nu + \lambda \rho, t, a \right) (\rho_t - \nu_t) (da) dt d\lambda. \quad (59)$$

The functional $\frac{\delta F}{\delta \nu}$ is then called the linear (functional) derivative of $F$ on $\mathcal{V}_2$.

The linear derivative $\frac{\delta F}{\delta \nu}$ is here also defined up to the additive constant $\int_0^T \int \frac{\delta F}{\delta \nu}(\nu, t, a)\nu_t (da) dt$.

By a centering argument, $\frac{\delta F}{\delta \nu}$ can be generically defined under the assumption that $\int_0^T \int \frac{\delta F}{\delta \nu}(\nu, t, a)\nu_t (da) dt = 0$. Note that if $\frac{\delta F}{\delta \nu}$ exists according to Definition 30 then

$$\forall \nu, \rho \in \mathcal{V}_2, \lim_{\epsilon \to 0^+} \frac{F(\nu + \epsilon(\rho - \nu)) - F(\nu)}{\epsilon} = \int_0^T \int \frac{\delta F}{\delta \nu}(\nu, t, a) (\rho_t - \nu_t) (da) dt. \quad (60)$$

Indeed (59) immediately implies (60). To see the implication in the other direction take $\nu^\lambda := \nu + \lambda(\rho - \nu)$ and $\rho^\lambda := \rho - \nu + \nu^\lambda$ and notice that (60) ensures for all $\lambda \in [0, 1]$ that

$$\lim_{\epsilon \to 0^+} \frac{F(\nu^\lambda + \epsilon(\rho - \nu)) - F(\nu^\lambda)}{\epsilon} = \lim_{\epsilon \to 0^+} \frac{F(\nu^\lambda + \epsilon(\rho^\lambda - \nu^\lambda)) - F(\nu^\lambda)}{\epsilon} = \int_0^T \int \frac{\delta F}{\delta \nu}(\nu^\lambda, t, a) (\rho_t^\lambda - \nu_t^\lambda) (da) dt = \int_0^T \int \frac{\delta F}{\delta \nu}(\nu^\lambda, t, a) (\rho_t - \nu_t) (da) dt.$$

By the fundamental theorem of calculus

$$F(\rho) - F(\nu) = \int_0^1 \lim_{\epsilon \to 0^+} \frac{F(\nu^{\lambda + \epsilon}) - F(\nu^{\lambda})}{\epsilon} d\lambda = \int_0^1 \int_0^T \int \frac{\delta F}{\delta \nu}(\nu^\lambda, t, a) (\rho_t - \nu_t) (da) dt d\lambda.$$

For the estimate on the generalization error (Section 3.6), we will also need to use a second order variation of $\nu \in \mathcal{V}_2 \mapsto F(\nu)$ which is given by:
Definition 31 We will say that \( F : \mathcal{V}_2 \rightarrow \mathbb{R}^d \) admits a second order linear functional derivative if, for all \( t, a, \nu \mapsto \frac{\partial^2 F}{\partial \nu^2}(\nu, t, a) \) itself admits a linear functional derivative in the sense of Definition 30. We will denote this second order linear derivative by \( \frac{\partial^2 F}{\partial \nu^2} \). In particular

\[
\frac{\partial F}{\partial \nu}(\nu, t, a) - \frac{\partial F}{\partial \nu}(\nu, t, a) = \int_0^1 \int_0^T \int \frac{\partial^2 F}{\partial \nu^2}((1-\lambda)\nu + \lambda\nu', t, a, t', a') (\nu' - \nu') (da') dt' d\lambda,
\]

where \( (t', a', \nu) \mapsto \frac{\partial F}{\partial \nu}(\nu, t, a, t', a') \) satisfies the properties i) and ii) of Definition 30.

Let us finally point out the following chain rule:

Lemma 32 Assume that \( F : \mathcal{V}_2 \rightarrow \mathbb{R}^d \) admits a linear functional derivative, in the sense of Definition 30, and \( J : \mathbb{R}^d \times \mathcal{V}_2 \rightarrow \mathbb{R}^d \) is such that, for all \( \nu \in \mathcal{V}_2, x \mapsto J(x, \nu) \) admits a continuous differential \( \nabla_x J(x, \nu) \) such that \( \nu \mapsto \nabla_x J(x, \nu) \) is continuous on \( \mathcal{V}_2 \), and for all \( x, \nu \mapsto J(x, \nu) \) admits a continuous differential \( \frac{\partial J}{\partial \nu}(x, \nu, t, a) \) on \( \mathcal{V}_2 \). Then \( \nu \mapsto J(\nu, F(\nu)) \) admits a linear functional derivative on \( \mathcal{V}_2 \) given by

\[
\frac{\partial J}{\partial \nu}(F(\nu), \nu, t, a) + \nabla_x J(F(\nu), \nu) \frac{\partial F}{\partial \nu}(\nu, t, a).
\]

Proof For all \( \nu, \rho, \) we have

\[
J(F(\nu + \epsilon(\rho - \nu)), \nu + \epsilon(\rho - \nu)) - J(F(\nu), \nu)
= J(F(\nu + \epsilon(\rho - \nu)), \nu + \epsilon(\rho - \nu)) - J(F(\nu), \nu) + J(F(\nu), \nu + \epsilon(\rho - \nu)) - J(F(\nu), \nu)
= \epsilon \int_0^1 \nabla_x J(F(\nu + (\lambda + \epsilon)(\rho - \nu)), \nu + \epsilon(\rho - \nu)) (F(\nu + \epsilon(\rho - \nu)) - F(\nu)) d\lambda
+ \epsilon \int_0^1 \int_0^T \int \frac{\partial J}{\partial \nu}(F(\nu), \nu + (\lambda + \epsilon)(\rho - \nu), t, a)(\rho_t(da) - \nu_t(da)) dt d\lambda.
\]

Dividing this expression by \( \epsilon \), the limit \( \epsilon \to 0 \), which grants the derivative of \( \nu \mapsto J(\nu, F(\nu)) \) (using (60)), follows by dominated convergence. \( \blacksquare \)

The connection between the linear functional derivative \( \frac{\partial F}{\partial \nu} \) introduced in Definition 30 and \( \frac{\partial m}{\partial \nu} \) introduced in Definition 29 is the following one: Let \( \pi^t, 0 \leq t \leq T \) be the family of operators, which, for any \( 0 \leq t \leq T \) and \( \nu \in \mathcal{V}_2 \) assigns the measure \( \pi^t(\nu) = \nu_t \) of \( \mathcal{P}_2(\mathbb{R}^p) \). For any functional \( U : \mathcal{P}_2(\mathbb{R}^p) \rightarrow \mathbb{R} \), define its extension on \( \mathcal{V}_2 \) as \( U^t(\nu) = U(\pi^t(\nu)) \). Whenever the functional \( U^t \) admits a linear functional derivative on \( \mathcal{V}_2 \), then

\[
U^t(\nu') - U^t(\nu) = \int_0^1 \int_0^T \int \frac{\partial U^t}{\partial \nu}(\nu + \lambda(\nu' - \nu), r, a)(\nu' - \nu) dr d\lambda.
\]

For any \( m \in \mathcal{P}_2(\mathbb{R}^p) \), define for the measure \( \nu^m \) of \( \mathcal{V}_2 \) constant in the sense \( \nu_t^m(da) = m(da) \), for a.e. \( t \). Therefore we have \( U^t(\nu^m) = U(\nu) \), and for all \( m, m' \in \mathcal{P}_2(\mathbb{R}^p) \)

\[
U(m') - U(m) = U^t(\nu^m) - U^t(\nu^m)
= \int_0^1 \int_0^T \int \frac{\partial U^t}{\partial \nu}(\nu^m + \lambda(\nu^m - \nu^m), r, a)(\nu^m - \nu^m(da) - \nu^m(da)) dr d\lambda
= \int_0^1 \left( \int_0^T \int \frac{\partial U^t}{\partial \nu}(\nu^m + \lambda(\nu^m - \nu^m), r, a) dr \right) (m'(da) - m(da)) d\lambda.
\]

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Lemma 33 Fix $\nu \in \mathcal{P}(\mathbb{R}^m)$. Let $u : \mathbb{R}^m \to \mathbb{R}$ be such that for all $\mu \in \mathcal{P}(\mathbb{R}^m)$ we have that

$$0 \leq \int u(a) (\mu - \nu)(da).$$

Then $u$ is a constant function: for all $a \in \mathbb{R}^m$ we have $u(a) = \int u(a') \nu(da')$.

Proof Let $M := \int u(a) \nu(da)$. Fix $\varepsilon > 0$. Assume that $\nu(u - M \leq -\varepsilon) > 0$. Indeed take

$$d\mu := \frac{1}{\nu(u - M \leq -\varepsilon)} \mathbb{1}_{\{u - M \leq -\varepsilon\}} \, d\nu.$$ Then

$$0 \leq \int u(a) (\mu - \nu)(da) = \int [u(a) - M] \mu(da)$$

$$= \int \mathbb{1}_{\{u - M \leq -\varepsilon\}} [u(a) - M] \mu(da) + \int \mathbb{1}_{\{u - M > -\varepsilon\}} [u(a) - M] \mu(da)$$

$$= \int \mathbb{1}_{\{u - M \leq -\varepsilon\}} [u(a) - M] \frac{1}{\nu(u - M \leq -\varepsilon)} \nu(da) \leq -\varepsilon.$$ As this is a contradiction we get $\nu(u - M \leq -\varepsilon) = 0$ and taking $\varepsilon \to 0$ we get $\nu(u - M < 0) = 0$. On the other hand assume that $\nu(u - M \geq \varepsilon) > 0$. Then, since $u - M \geq 0$ holds $\nu$-a.s., we have

$$0 = \int [u(a) - M] \nu(da) \geq \int_{\{u - M \geq \varepsilon\}} [u(a) - M] \nu(da) \geq \varepsilon \nu(u - M \geq \varepsilon) > 0$$

which is again a contradiction meaning that for all $\varepsilon > 0$ we have $\nu(u - M \geq \varepsilon) = 0$ i.e. $u = M$ $\nu$-a.s..

Lemma 34 Let $\nu_t, \mu_t \in \mathcal{V}_2$ and let $\nu^\varepsilon = \nu + \varepsilon (\mu - \nu)$. Then

i) for any $\varepsilon \in (0, 1)$ we have

$$\frac{1}{\varepsilon} \int_0^T \left[\text{Ent}(\nu_t^\varepsilon) - \text{Ent}(\nu_t)\right] dt \geq \int_0^T \left[\log \nu_t(a) - \log \gamma(a)\right] (\mu_t - \nu_t)(da) dt,$$

ii)

$$\limsup_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^T \left[\text{Ent}(\nu_t^\varepsilon) - \text{Ent}(\nu_t)\right] dt \leq \int_0^T \left[\log \nu_t(a) - \log \gamma(a)\right] (\mu_t - \nu_t)(da) dt,$$

where $m \mapsto \text{Ent}(m)$ is defined by (11) for $m \in \mathcal{P}(\mathbb{R}^p)$.

Proof This follows the steps of (Hu et al., 2019b, Proof of Proposition 2.4). For i) we begin by observing that

$$\frac{1}{\varepsilon} \left(\text{Ent}(\nu_t^\varepsilon) - \text{Ent}(\nu_t)\right) = \frac{1}{\varepsilon} \int \left[\frac{\nu_t^\varepsilon(a)}{\gamma(a)} \nu_t^\varepsilon(a) - \frac{\nu_t(a)}{\gamma(a)} \nu_t(a)\right] da$$

$$= \frac{1}{\varepsilon} \int \left(\nu_t^\varepsilon(a) - \nu_t(a)\right) \log \frac{\nu_t(a)}{\gamma(a)} da + \frac{1}{\varepsilon} \int \nu_t^\varepsilon(a) \left[\frac{\nu_t^\varepsilon(a)}{\gamma(a)} - \frac{\nu_t(a)}{\gamma(a)}\right] da$$

$$= \int \left(\mu_t(a) - \nu_t(a)\right) \log \frac{\nu_t(a)}{\gamma(a)} da + \frac{1}{\varepsilon} \int \nu_t^\varepsilon(a) \log \frac{\nu_t^\varepsilon(a)}{\nu_t(a)} da$$

$$= \int \left[\log \nu_t(a) - \log \gamma(a)\right] (\mu_t - \nu_t)(da) + \frac{1}{\varepsilon} \int \nu_t^\varepsilon(a) \log \frac{\nu_t^\varepsilon(a)}{\nu_t(a)} \nu_t(a) da.$$
Since $x \log x \geq x - 1$ for $x \in (0, \infty)$ we get
\[
\frac{1}{\varepsilon} \int \frac{\nu^\varepsilon_t(t)}{\nu_t(t)} \log \frac{\nu^\varepsilon_t(t)}{\nu_t(t)} \nu_t(t) \, da \geq \frac{1}{\varepsilon} \int \left[ \frac{\nu^\varepsilon_t(t)}{\nu_t(t)} - 1 \right] \nu_t(t) \, da = \frac{1}{\varepsilon} \int \left[ \nu^\varepsilon_t(t) - \nu_t(t) \right] \, da = 0.
\]
Hence
\[
\frac{1}{\varepsilon} \left( \text{Ent}(\nu^\varepsilon_t) - \text{Ent}(\nu_t) \right) \geq \int \left[ \log \nu_t(t) - \log \gamma(a) \right] (\mu_t - \nu_t)(da).
\]
From this i) follows.

To prove ii) start by noting that
\[
\frac{1}{\varepsilon} \left( \text{Ent}(\nu^\varepsilon_t) - \text{Ent}(\nu_t) \right) = \frac{1}{\varepsilon} \int \left[ \left( \log \nu^\varepsilon_t(t) - \log \gamma(a) \right) \nu^\varepsilon_t(t) - \left( \log \nu_t(t) - \log \gamma(a) \right) \nu_t(t) \right] \, da
\]
\[
= \int \frac{1}{\varepsilon} \left[ \nu^\varepsilon_t(t) \log \nu^\varepsilon_t(t) - \nu_t(t) \log \nu_t(t) - \log \gamma(a) \left( \nu^\varepsilon_t(t) - \nu_t(t) \right) \right] \, da
\]
Now
\[
-\frac{1}{\varepsilon} \log \gamma(a) \left( \nu^\varepsilon_t(t) - \nu_t(t) \right) = -\log \gamma(a) \left( \mu_t(a) - \nu_t(a) \right).
\]
Moreover, since the map $x \mapsto x \log x$ is convex for $x > 0$ we have
\[
\frac{1}{\varepsilon} \left[ \nu^\varepsilon_t(t) \log \nu^\varepsilon_t(t) - \nu_t(t) \log \nu_t(t) \right] \leq \mu(a) \log \mu(a) - \nu(a) \log \nu(a).
\]
Hence
\[
\frac{1}{\varepsilon} \left( \text{Ent}(\nu^\varepsilon_t) - \text{Ent}(\nu_t) \right) \leq \text{Ent}(\mu_t) - \text{Ent}(\nu_t).
\]
Since $\mu, \nu \in \mathcal{W}_2^W$, the right hand side is finite. Finally, by the reverse Fatou’s lemma,
\[
\limsup_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[ \text{Ent}(\nu^\varepsilon_t) - \text{Ent}(\nu_t) \right] \leq \int \limsup_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[ \nu^\varepsilon_t(t) \log \nu^\varepsilon_t(t) - \nu_t(t) \log \nu_t(t) - \log \gamma(a) \left( \nu^\varepsilon_t(t) - \nu_t(t) \right) \right] \, da.
\]
Calculating the derivative of $x \mapsto x \log x$ for $x > 0$ leads to
\[
\limsup_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[ \nu^\varepsilon_t(t) \log \nu^\varepsilon_t(t) - \nu_t(t) \log \nu_t(t) - \log \gamma(a) \left( \nu^\varepsilon_t(t) - \nu_t(t) \right) \right]
\]
\[
= (1 + \log \nu_t(t))(\mu_t(a) - \nu_t(a)) - \log \gamma(a) \left( \mu_t(a) - \nu_t(a) \right).
\]
Hence
\[
\limsup_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[ \text{Ent}(\nu^\varepsilon_t) - \text{Ent}(\nu_t) \right] \leq \int \left[ \log \nu_t(t) - \log \gamma(a) \right] (\mu_t - \nu_t)(da).
\]
This completes the proof. \[\blacksquare\]
Appendix B. Bounds and regularity for the forward-backward system (19)

In this section, we establish the boundedness and regularity of the mapping assigning to each $\nu \in \mathcal{V}_2$ the solution to

$$
\begin{cases}
X^\xi_\nu(t) = \xi + \int_0^t \int \phi_r(X^\xi_\nu(r), a, \zeta) \nu_r(da) dr,

P^\xi_\nu(t) = \nabla_x g(X^\xi_\nu(t), \zeta) + \int_t^T \int \left( \nabla_x f_r(X^\xi_\nu(r), a, \zeta) + \nabla_x \phi_r(X^\xi_\nu(r), a, \zeta) \cdot P^\xi_\nu(r) \right) \nu_r(da) dr.
\end{cases}
$$

(61)

Hereafter, we will work mostly under the sole assumption 1, and, for a fixed couple of data $\xi, \zeta$ chosen according to (1)-iii), so that for a.e. $0 \leq t \leq T$,

$$
|\nabla_x g(0, \zeta)| + |\phi_t(0, 0, \zeta)| + |\nabla_a f_t(0, 0, \zeta)| + |\nabla_x f_t(0, 0, \zeta)| < \infty.
$$

Let us also recall the function $h$ given by:

$$
h_t(x, a, p, \zeta) = f_t(x, a, \zeta) + \phi_t(x, a, \zeta)p.
$$

Let us point out that the process $(P^\xi_\nu(t))_{0 \leq t \leq T}$ can be written as:

$$
P^\xi_\nu(t) = \Xi^\xi_\nu(T, t) \nabla_x g(X^\xi_\nu(T, t), \zeta) - \int_t^T \Xi^\xi_\nu(r, t) \left( \int \nabla_x f_r(X^\xi_\nu(r), a, \zeta) \nu_r(da) \right) dr,
$$

(62)

where $\Xi^\xi_\nu$ denote $\nu \in \mathcal{V}_2$, the continuous $\mathbb{R}^{d \times d}$-valued function, solution to:

$$
\frac{d\Xi^\xi_\nu(t, t_0)}{dt_0} = \left( \int \nabla_x \phi_r(X^\xi_\nu(t, a, \zeta) \nu_r(da) \right) \Xi^\xi_\nu(t, t_0), \ t_0 \leq t \leq T, \ \Xi^\xi_\nu(t, t) = I_d,
$$

(63)

for $I_d$ denoting the identity matrix of size $d$. As a first estimate, let us show the following lemma:

**Lemma 35 (Uniforms bounds and continuity)** Under Assumption 1, for any $\nu \in \mathcal{V}_2$, $(X^\xi_\nu(t), P^\xi_\nu(t))_{0 \leq t \leq T}$ given by (61) satisfies:

$$
|X^\xi_\nu(t)| \leq \left( |\xi| + \int_0^t |\phi_r(0, a, \zeta)| \nu_r(da) dr \right) \times \exp\{T \sup_{t,a,\zeta} \|\phi_t(\cdot, a, \zeta)\|_{Lip}\},
$$

$$
|P^\xi_\nu(t)| \leq \|\nabla_x g(\cdot, \zeta)\|_{\infty} \wedge \left( \|\nabla_x g(\cdot, \zeta)\|_{Lip} |X^\xi_\nu(t)| + |\nabla_x g(0, \zeta)| \right) \times \exp\{T \sup_{t,a,\zeta} \|\phi_t(\cdot, a, \zeta)\|_{Lip}\}
$$

$$
+ \|\nabla_x f_t\|_{\infty} \times \exp\{T \sup_{t,a,\zeta} \|\phi_t(\cdot, a, \zeta)\|_{Lip}\}.
$$

Additionally, for all $0 \leq t \leq T$, $\nu \mapsto (X^\xi_\nu(t), P^\xi_\nu(t))$ is continuous on $\mathcal{V}_2$ equipped with the topology related to the metric $W^T$. 

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Proof Owing to the Lipschitz and differentiability properties of \((x,a) \mapsto \phi_t(x,a)\),

\[
|X^\xi,\zeta_t(\nu)| \leq |\xi| + \sup_{t,\alpha,\zeta} \|\phi_t(\cdot,a,\zeta)\|_{\text{Lip}} \int_0^t |X^\xi,\zeta_r(\nu)| \, dr + \int_0^t \int |\phi_t(0,a,\zeta)|_{\nu_r}\,(da) \, dr.
\]

Applying Gronwall’s inequality: For non-negative continuous functions \(u,w,\alpha\)

\[
u(t) \leq w(t) + \alpha \int_0^t u(s) \, ds, \forall 0 \leq t \leq T \Rightarrow \nu(t) \leq w(t) \exp\{\alpha T\}, \forall 0 \leq t \leq T,
\]

yields to the estimate of \(X^\xi,\zeta_t(\nu)\). The estimate for \(P^\xi,\zeta_t(\nu)\) follows directly from (62).

For the continuity of \(\nu \mapsto (X^\xi,\zeta_t(\nu), P^\xi,\zeta_t(\nu))\), let \(\{\nu^\epsilon\}_{\epsilon > 0}\) be a family of elements of \(\mathcal{V}_2\) such that \(\lim_{\epsilon \to 0^+} \mathcal{W}_2(T, \nu^\epsilon, \nu) = 0\). Observe that

\[
\left| X^\xi,\zeta_t(\nu^\epsilon) - X^\xi,\zeta_t(\nu) \right| \leq \int_0^t \int \left\{ |\phi_r(X^\xi,\zeta_r(\nu^\epsilon), a, \zeta) - \phi_r(X_r(\nu), a, \zeta)| \right\} \nu^\epsilon_r(\,da) \, dr
\]

and

\[
\left| X^\xi,\zeta_t(\nu^\epsilon) - X^\xi,\zeta_t(\nu) \right| \leq \sup_{t,\alpha,\zeta} \left\{ |\phi_r(X^\xi,\zeta_r(\nu^\epsilon), a, \zeta) - \phi_r(X_r(\nu), a, \zeta)| \right\} \nu_r(\,da) \, dr.
\]

Applying Gronwall’s inequality (64) and since \(\mathcal{W}_1^T \leq \mathcal{W}_2^T\), it follows that \(\lim_{\epsilon \to 0^+} X^\xi,\zeta_t(\nu^\epsilon) = X^\xi,\zeta_t(\nu)\).

Since \((x,a) \mapsto \nabla_x g(x,\zeta), \nabla_x \phi_t(x,a,\zeta), \nabla_x f_t(x,a,\zeta)\) are bounded continuous (uniformly in \(t\)), the continuity of \((\nu) \mapsto X^\xi,\zeta_t(\nu)\) also ensure, by dominated convergence that

\[
\lim_{\epsilon \to 0} \nabla_x g(X^\xi,\zeta_t(\nu^\epsilon), \zeta) = \nabla_x g(X_T(\nu), \zeta),
\]

\[
\lim_{\epsilon \to 0} \int_0^t \int \left\{ |\nabla_x f_r(X^\xi,\zeta_r(\nu^\epsilon), a, \zeta)| \right\} \nu^\epsilon_r(\,da) \, dr = \int_0^t \int \left\{ |\nabla_x f_r(X^\xi,\zeta_r(\nu), a, \zeta)| \right\} \nu_r(\,da) \, dr,
\]

\[
\lim_{\epsilon \to 0} \int_0^t \int \left\{ |\nabla_x \phi_r(X^\xi,\zeta_r(\nu^\epsilon), a, \zeta)| \right\} \nu^\epsilon_r(\,da) \, dr = \int_0^t \int \left\{ |\nabla_x \phi_r(X^\xi,\zeta_r(\nu), a, \zeta)| \right\} \nu_r(\,da) \, dr.
\]

This ensures that \(\lim_{\epsilon \to 0} P^\xi,\zeta_t(\nu^\epsilon) = P^\xi,\zeta_t(\nu)\). \(\blacksquare\)

Let us prove the differentiability, in the sense of Definition 30, of the mapping \(\nu \in \mathcal{V}_2 \mapsto (X^\xi,\zeta_t(\nu), Y^\xi,\zeta_t(\nu))\):

Lemma 36 Let \(\Xi^\xi,\zeta,\nu\) be as in (63) and let \((t,t_0) \mapsto \Xi^\xi,\zeta,\nu(t,t_0)\) be the solution to

\[
d\Xi^\xi,\zeta,\nu(t,t_0) = (\int \nabla_x \phi_t(X^\xi,\zeta(\nu), a, \zeta)\nu_r(\,da))\Xi^\xi,\zeta,\nu(t,t_0), \, 0 \leq t \leq T, \Xi^\xi,\zeta,\nu(t_0,t_0) = I_d.
\]

For all \(0 \leq t \leq T\), \(\nu \mapsto (X^\xi,\zeta_t(\nu), Y^\xi,\zeta_t(\nu))\) admits a linear functional derivative given as the solution to the ODEs

\[
\frac{\delta X^\xi,\zeta_t(\nu,r,a)}{\delta \nu}(\nu,r,a) = \mathbf{1}_{\{r \leq t\}} \Xi^\xi,\zeta,\nu(t,r)\phi_r(X^\xi,\zeta(\nu), a, \eta),
\]

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and
\[
\frac{\delta P_{t}^{\xi,\zeta}}{\delta \nu}(\nu, r, a) = \Xi_{t}^{\xi,\zeta}(T, t) \nabla_{x}^{2}g(X_{T}^{\xi,\zeta}(\nu, \zeta)) \times \frac{\delta X_{t}^{\xi,\zeta}}{\delta \nu}(\nu, r, a)
\]
\[
+ \int_{t}^{T} \Xi_{r}^{\xi,\zeta}(r, t) \nabla_{x} h_{r}(X_{r}^{\xi,\zeta}(\nu), P_{r}^{\xi,\zeta}(\nu), a, \zeta) \, dr
\]
\[
+ \int_{t}^{T} \Xi_{r}^{\xi,\zeta}(r, t) \nabla_{x}^{2} h_{r}(X_{r}^{\xi,\zeta}(\nu), P_{r}^{\xi,\zeta}(\nu), a', \zeta) \nu_{r}(da') \, dr'
\]
\[
+ \left\{ \int_{t}^{T} \Xi_{r}^{\xi,\zeta}(r, t) \int \nabla_{x}^{2} h_{r}(X_{r}^{\xi,\zeta}(\nu), P_{r}^{\xi,\zeta}(\nu), a', \zeta) \nu_{r}(da') \, dr' \right\} \frac{\delta X_{T}^{\xi,\zeta}}{\delta \nu}(\nu, r, a),
\]
where \( h \) is defined as in (14).

**Proof** The derivative of \( \nu \mapsto X_{t}^{\xi,\zeta}(\nu) \) follows directly Lemma 13, which grants
\[
\int_{0}^{T} \frac{\delta X_{t}^{\xi,\zeta}}{\delta \nu}(r, a, \nu + \epsilon(\rho - \nu)) (\rho(\nu) - \nu(\nu)) \, dr = \lim_{\epsilon \to 0} \frac{X_{t}^{\xi,\zeta}(\nu + \epsilon(\rho - \nu)) - X_{t}^{\xi,\zeta}(\nu)}{\epsilon} = V_{t}^{\xi,\zeta}
\]
for \( (V_{t}^{\xi,\zeta})_{0 \leq t \leq T} \) satisfying:
\[
V_{t}^{\xi,\zeta} = \int_{0}^{t} \left( \int \nabla_{x} \phi_{r}(X_{r}^{\xi,\zeta}(\nu), a, \zeta) \nu_{r}(da) \right) V_{r}^{\xi,\zeta} \, dr + \int_{0}^{t} \int \phi_{r}(X_{r}^{\xi,\zeta}(\nu), a, \zeta) (\rho_{r}(da) - \nu_{r}(da)) \, dr, \quad 0 \leq t \leq T.
\]
Recall that any solution to the ODE \( du(t)/dt = b(t)u(t) + \alpha(t) \) on \([0, T]\), \( u(0) = u_{0} \), admits the representation:
\[
u(t) = \Psi(t, 0)u_{0} + \int_{0}^{t} \Psi(t, s)\alpha(s) \, ds, \quad \frac{d\Psi(t, t_{0})}{dt} = b(t)\Psi(t, t_{0}), \quad \Psi(t_{0}, t_{0}) = I_{d},
\]
we get, using the expression of \( V_{t}^{\xi,\zeta} \),
\[
\int_{0}^{T} \frac{\delta X_{t}^{\xi,\zeta}}{\delta \nu}(\nu, r, a) (\rho_{r}(da) - \nu_{r}(da)) \, dr
\]
\[
= \int_{0}^{t} \Xi_{r}^{\xi,\zeta}(r, t) \phi_{r}(X_{r}^{\xi,\zeta}(\nu), a, \zeta) (\rho_{r}(da) - \nu_{r}(da)) \, dr.
\]
The Lipschitz properties of \( (x, a) \mapsto \phi_{t}(x, a, \zeta) \) and \( (x, a) \mapsto \nabla_{x} \phi_{t}(x, a, \zeta) \) ensure, with Lemma 35 that \( \nu \mapsto X_{t}^{\xi,\zeta}(\nu) \), and by extension \( \nu \mapsto \phi_{r}(X_{r}^{\xi,\zeta}(\nu), a, \zeta), \int_{\nu}^{t} \int \nabla_{x} \phi_{r}(X_{r}^{\xi,\zeta}(\nu), a', \zeta) \nu_{r}(da') \, dr' \) are continuous. In particular, \( \nabla_{x} \phi_{t}(x, a, \zeta) \) is uniformly bounded and
\[
|\phi_{r}(X_{r}^{\xi,\zeta}(\nu), a, \zeta)| \leq C(1 + |X_{r}^{\xi,\zeta}(\nu)| + |a| + |\phi_{t}(0, 0, \zeta)|) \leq C(1 + |\xi| + |a| + |\phi_{t}(0, 0, \zeta)|),
\]
for some finite constant \( C \) depending only on \( T, d, p, \|\nabla_{x} \phi\|_{\infty} \) and \( \|\nabla_{a} \phi\|_{\infty} \). Therefore, as
\[
\|\Xi_{t}^{\xi,\zeta}(t, t_{0})\| := \sup_{|v|=1} \|\Xi_{t}^{\xi,\zeta}(t, r)v\| \leq 1 + \|\nabla_{x} \phi\|_{\infty} \int_{t_{0}}^{t} \|\Xi_{t}^{\xi,\zeta}(r, t_{0})\| \, dr
\]
Gronwall’s inequality yields

$$\|\Xi^{\xi,\nu}(t, t_0)\| \leq \exp\{(t - t_0)\|\nabla_x\phi\|_\infty\}$$

and so

$$\left|\Xi^{\xi,\nu}(t, r)\phi_r(X^{\xi,\nu}_r(\nu), a, \zeta)\right| \leq C(1 + |a| + |\xi| + |\phi_t(0, 0, \zeta)|).$$

This enables us to conclude that

$$\mathbb{1}_{\{r \leq t\}} \Xi^{\xi,\nu}(t, r)\phi_r(X^{\xi,\nu}_r(\nu), a, \zeta),$$

is the linear derivative functional of $\nu \mapsto X^{\xi,\nu}_t(\nu)$.

In the same way, for $\nu_\epsilon = \nu + \epsilon(\rho - \nu)$, we have

$$P^{\xi,\nu}_t(\nu_\epsilon) - P^{\xi,\nu}_t(\nu) = \nabla_x g(X^{\xi,\nu}_T(\nu_\epsilon), \zeta) - \nabla_x g(X^{\xi,\nu}_T(\nu), \zeta)
+ \int_t^T \left( \int \nabla_x f_r(X^{\xi,\nu}_r(\nu_\epsilon), a, \zeta)\nu_r^\epsilon(da) - \int \nabla_x f_r(X^{\xi,\nu}_r(\nu), a, \zeta)\nu_r(da) \right) dr
+ \int_t^T \left( \int \nabla_x \phi_r(X^{\xi,\nu}_r(\nu_\epsilon), a, \zeta)\nu_r^\epsilon(da) - \int \nabla_x \phi_r(X^{\xi,\nu}_r(\nu), a, \zeta)\nu_r(da) \right) P^{\xi,\nu}_r(\nu) dr
+ \int_t^T \int \nabla_x \phi_r(X^{\xi,\nu}_r(\nu_\epsilon), a) \left( P^{\xi,\nu}_r(\nu_\epsilon) - P^{\xi,\nu}_r(\nu) \right) \nu_r(da) dr
= \nabla_x g(X^{\xi,\nu}_T(\nu_\epsilon), \zeta) - \nabla_x g(X^{\xi,\nu}_T(\nu), \zeta)
+ \int_t^T \int \left( \nabla_x h_r(X^{\xi,\nu}_r(\nu_\epsilon), P^{\xi,\nu}_r(\nu), a)\nu_r^\epsilon(da) - \nabla_x h_r(X^{\xi,\nu}_r(\nu), P^{\xi,\nu}_r(\nu), a, \zeta)\nu_r(da) \right) dr
+ \int_t^T \left( \int \nabla_x \phi_r(X^{\xi,\nu}_r(\nu_\epsilon), a, \zeta)\nu_r(da) \right) \left( P^{\xi,\nu}_r(\nu_\epsilon) - P^{\xi,\nu}_r(\nu) \right) dr.$$

Using (63), we deduce the formulation:

$$P^{\xi,\nu}_t(\nu_\epsilon) - P^{\xi,\nu}_t(\nu) = \Xi^{\xi,\nu_\epsilon}(T, t) \times \left( \nabla_x g(X^{\xi,\nu}_T(\nu_\epsilon), \zeta) - \nabla_x g(X^{\xi,\nu}_T(\nu), \zeta) \right)
+ \int_t^T \Xi^{\xi,\nu_\epsilon}(r, t)
\times \left( \int \nabla_x h_r(X^{\xi,\nu}_r(\nu_\epsilon), P^{\xi,\nu}_r(\nu), a, \zeta)\nu_r^\epsilon(da) - \int \nabla_x h_r(X^{\xi,\nu}_r(\nu), P^{\xi,\nu}_r(\nu), a, \zeta)\nu_r(da) \right) dr.$$
As \( \lim_{\epsilon \to 0} \frac{\partial \xi, \nu^\epsilon(T, t)}{\partial \epsilon} = \frac{\partial \xi, \nu(T, t)}{\partial \epsilon} \), using Lemma 32 and Lemma 35 (observing that \( \mathcal{W}_{2, \nu}^\epsilon(\nu, \nu) \leq \epsilon \mathcal{W}_{2, \nu}(\nu, \nu) \to 0 \) as \( \epsilon \to 0^+ \)), we obtain

\[
\lim_{\epsilon \to 0} \frac{P^\xi, \nu^\epsilon(\nu + \epsilon(\rho - \nu)) - P^\xi, \nu(\nu)}{\epsilon} = \int_0^T \frac{\partial \xi, \nu}{\partial \nu}(T, t) \int \left( \nabla^2 g(X^\xi, \nu, \eta, \zeta) \frac{\partial X^\xi}{\partial \nu}(\nu, r, a) \right) (\rho_r(da) - \nu_r(da)) \, dr
\]

\[
+ \int_0^T \frac{\partial \xi, \nu}{\partial \nu}(r, t) \left( \int_0^T \nabla_h(X^\xi(\nu, \eta, \zeta), P^\xi(\nu, \eta, \zeta), a, \zeta) (\rho_r(da) - \nu_r(da)) \, dr \right)
\]

\[
+ \int_0^T \frac{\partial \xi, \nu}{\partial \nu}(r, t) \times \left( \int_0^T \nabla^2 h(X^\xi(\nu, \eta, \zeta), P^\xi(\nu, \eta, \zeta), a, \zeta) \frac{\partial X^\xi}{\partial \nu}(\nu, r, a') (\rho_r(da') - \nu_r(da')) \, dr \right) \nu_r(da) \, dr,
\]

from which we identify the value of \( \frac{\partial P^\xi, \nu}{\partial \nu}(\nu, r, a) \).

Lemma 36 together with the definition of linear derivative allows to compute \( X(\nu) - X(\mu) \) and \( P(\nu) - P(\mu) \). However, to establish propagation of chaos an alternative representation is more convenient.

**Lemma 37** Let Assumptions 1 and 6 hold. Then

\[
|X^\xi(T, \nu) - X^\xi(T, \mu)| \leq \exp(||\nabla \phi||_{\infty}(T - t)) \int_0^t \left| \int \phi_t(X^\xi(\mu, a, \zeta)(\nu_r - \mu_r)(da) \right| \, dr,
\]

and

\[
|P^\xi(T, \nu) - P^\xi(T, \mu)| \leq c_1 |X^\xi(T, \nu) - X^\xi(T, \mu)| + c_2 \int_0^T |X^\xi(T, \nu) - X^\xi(T, \mu)| \, dr,
\]

where \( c_1 \) and \( c_2 \) are given in (67).

**Proof** Let \( X^\lambda_r := X^\xi(\nu) + \lambda(X^\xi(\mu) - X^\xi(\nu)) \) and write

\[
X^\xi(T, \nu) - X^\xi(T, \mu) = \int_0^T \left( \int \phi_r(X^\xi(\nu), a, \zeta) \nu_r(da) - \int \phi_r(X^\xi(\mu), a, \zeta) \nu_r(da) \right) \, dr
\]

\[
- \int_0^T \left( \int \phi_r(X^\xi(\nu), a, \zeta)(\nu_r - \mu_r)(da) \right) \, dr
\]

\[
+ \int_0^T \left( \int \phi_r(X^\xi(\mu), a, \zeta)(\nu_r - \mu_r)(da) \right) \, dr
\]

\[
- \int_0^T \left( \int \int_0^1 (\nabla_x \phi_r)(X^\lambda_r, a, \zeta) d\lambda \nu_r(da) \right) \left( X^\xi(T, \nu) - X^\xi(T, \mu) \right) \, dr
\]

\[
+ \int_0^T \left( \int \phi_r(X^\xi(\mu), a, \zeta)(\nu_r - \mu_r)(da) \right) \, dr.
\]
Let \((r, t) \mapsto \Gamma_{r,t}\) be the solution to
\[
\frac{d\Gamma_{r,t}}{dr} = \left( \int \int_0^1 (\nabla_X \phi_r)(X_r^\lambda, a, \zeta) \nu_r(da) d\lambda \right), \quad \Gamma_{r,0} = \Gamma_d.
\]
We then have
\[
X_t^{\xi, \zeta}(\nu) - X_t^{\xi, \zeta}(\mu) = \int_0^t \Gamma_{r,t} \left( \int \phi_r(X_r^{\xi, \zeta}(\mu), a, \zeta)(\nu_r - \mu_r)(da) \right) dr.
\]
Assumption 1 implies that \(|\Gamma_{r,t}| \leq \exp\left((t-r)\|\nabla_X \phi\|_\infty\right)\) and leads immediately to the estimate for \(|X_t^{\xi, \zeta}(\nu) - X_t^{\xi, \zeta}(\mu)|\).

Let us fix \(\xi, \zeta\). Let us write \(f_t(x, a) := f_t(x, a, \zeta)\) and \(G(x) := g(x, \zeta)\). From (62) we have
\[
P_t^{\xi, \zeta}(\nu) - P_t^{\xi, \zeta}(\mu) = \widehat{\Xi}^{\xi, \zeta, \nu}(T, t) (\nabla_X G(X_T(\nu)) - \nabla_X G(X_T(\mu)))
+ \left( \widehat{\Xi}^{\xi, \zeta, \mu}(T, t) - \widehat{\Xi}^{\xi, \zeta, \nu}(T, t) \right) \nabla_X G(X_T(\mu))
- \int_t^T \Xi^{\xi, \zeta, \nu}(r, t) \left( \int \nabla_x f_r(X_r(\nu), a) \nu_r(da) - \int \nabla_x f_r(X_r(\mu), a) \nu_r(da) \right) dr
- \int_t^T \left( \widehat{\Xi}^{\xi, \zeta, \mu}(r, t) - \widehat{\Xi}^{\xi, \zeta, \nu}(r, t) \right) \left( \int \nabla_x f_r(X_r(\mu), a) \nu_r(da) \right) dr.
\]
Applying the mean-value theorem in \(X\) and using Assumption 1 implies that
\[
|P_t^{\xi, \zeta}(\nu) - P_t^{\xi, \zeta}(\mu)| \leq c_1|X_T^{\xi, \zeta}(\nu) - X_T^{\xi, \zeta}(\mu)| + c_2 \int_t^T |X_t^{\xi, \zeta}(\nu) - X_t^{\xi, \zeta}(\mu)| dr,
\]
where
\[
c_1 = \|\nabla^2 g\|_\infty e^T \|\nabla_x \phi\|_\infty
\]
\[
c_2 = \|\nabla^2 g\|_\infty e^T \|\nabla_x \phi\|_\infty + \|\nabla^2 f\|_\infty e^T \|\nabla_x \phi\|_\infty + T \|\nabla_x f\|_\infty e^T \|\nabla_x \phi\|_\infty \|\nabla^2 f\|_\infty.
\]

**Appendix C. Regularity estimates on the Hamiltonian**

In this section, we prove the following result:

**Theorem 38** Let Assumption 1 hold. Let \(\nabla_a h\) be the function defined on \([0, T] \times \mathbb{R}^p \times \mathcal{V}_2 \times \mathcal{P}(\mathbb{R}^d \times S)\) by
\[
\nabla_a h_t(a, \nu, \mathcal{M}) = \int_{\mathbb{R}^d \times S} \nabla_a f_t(a, X_t^{\xi, \zeta}(\nu)) + \nabla_a \phi_t(a, X_t^{\xi, \zeta}(\nu)) \cdot P_t^{\xi, \zeta}(\nu) \mathcal{M}(d\xi, d\zeta),
\]
for \((X_t^{\xi, \zeta}(\nu), P_t^{\xi, \zeta}(\nu))\) satisfying
\[
\begin{align*}
X_t^{\xi, \zeta}(\nu) &= \xi + \int_0^t \int_{\mathbb{R}^d} \phi_r(X_r^{\xi, \zeta}(\nu), a, \zeta) \nu_r(da) dr, \\
P_t^{\xi, \zeta}(\nu) &= \nabla_x g(X_T^{\xi, \zeta}(\nu), \zeta) + \int_t^T \int_{\mathbb{R}^d} \nabla_x h_r(X_r^{\xi, \zeta}(\nu), a, P_r^{\xi, \zeta}(\nu), \zeta) \nu_r(da) dr.
\end{align*}
\]
Then there exists $L > 0$ such that for all $\mathcal{M} \in \mathcal{P}_2(\mathbb{R}^d \times S)$, for all $a, a' \in \mathbb{R}^p$ and $\mu, \mu' \in \mathcal{V}_2$

$$\left| \nabla_a h_t(a, \mu, \mathcal{M}) - (\nabla_a h_t)(a', \mu', \mathcal{M}) \right|$$

$$\leq L \left( 1 + \max_t \left( \frac{\max_{\mu \in \text{Lin}(\nu, \nu')} \int_{\mathbb{R}^d \times S} |P_t^{\xi, \zeta}(\mu)| \mathcal{M}(d\xi, d\zeta) \right) \right) (|a - a'| + W_1^T(\mu, \mu')) . \quad (69)$$

for $\text{Lin}(\nu, \nu') := \{ \mu \in \mathcal{V}_2 : \mu = (1 - \lambda)\nu' + \lambda\nu, \text{ for some } 0 \leq \lambda \leq 1 \}$.

**Proof** From the definition of $\nabla_a h$, we have

$$(\nabla_a h_t)(a, \nu, \mathcal{M}) = \int_{\mathbb{R}^d \times S} \left[ \nabla_a \phi_t(X_t^{\xi, \zeta}(\nu), a, \zeta) \cdot P_t^{\xi, \zeta}(\mu) + \nabla_a f_t(X_t^{\xi, \zeta}(\mu), a, \zeta) \right] \mathcal{M}(d\xi, d\zeta) .$$

This, Lemma 20 and Assumption 1 ii) allow us to conclude that for any $\mathcal{M} \in \mathcal{P}_2(\mathbb{R}^d \times S)$ there exists $L$ such that for all $\mu \in \mathcal{V}_2$ we have

$$\left| \nabla_a h_t(a, \nu, \mathcal{M}) - (\nabla_a h_t)(a', \nu, \mathcal{M}) \right| \leq L \left( 1 + \int_{\mathbb{R}^d \times S} |P_t^{\xi, \zeta}(\nu)| \mathcal{M}(d\xi, d\zeta) \right) |a - a'| , \quad (70)$$

for some constant $L$ depending only on the Lipschitz coefficients of $\nabla_a f$ and $\nabla_a \phi$.

On the other hand, for all $a \in \mathbb{R}^p$, we have

$$\left| \nabla_a h_t(a, \nu, \mathcal{M}) - (\nabla_a h_t)(a, \nu', \mathcal{M}) \right|$$

$$\leq \int_{\mathbb{R}^d \times S} \left[ \|\nabla_a f_t(a, \cdot, \zeta)\|_{\text{Lip}} |X_t^{\xi, \zeta}(\nu) - X_t^{\xi, \zeta}(\nu')| \right] \mathcal{M}(d\xi, d\zeta)$$

$$+ \int_{\mathbb{R}^d \times S} \left[ \|\phi_t(x, \cdot, \cdot)\|_{\text{Lip}} |P_t^{\xi, \zeta}(\nu) - P_t^{\xi, \zeta}(\nu')| + \|\nabla_a \phi(t, \cdot, \zeta)\|_{\text{Lip}} |P_t^{\xi, \zeta}(\nu)| \right] |X_t^{\xi, \zeta}(\nu) - X_t(\nu)| \mathcal{M}(d\xi, d\zeta) .$$

Recalling Lemma 36, $\nu \in \mathcal{V}_2 \mapsto X_t^{\xi, \zeta}(\nu)$ and $P_t^{\xi, \zeta}(\nu)$ both admit a linear functional derivative $\frac{\delta X_t^{\xi, \zeta}}{\delta \nu}(r, a, \nu)$ and $\frac{\delta P_t^{\xi, \zeta}}{\delta \nu}(r, a, \nu)$ (see Definition 30), which are given by

$$\frac{\delta X_t^{\xi, \zeta}}{\delta \nu}(\nu, r, a) = 1_{\{0 \leq r \leq t\}} \Xi^{\xi, \zeta, \nu}(t, r) \phi_r(X_r^{\xi, \zeta}(\nu), a, \zeta), \quad (71)$$

$$\frac{\delta P_t^{\xi, \zeta}}{\delta \nu}(\nu, r, a) = \Xi^{\xi, \zeta, \nu}(T, t) \nabla_x g(X_T^{\xi, \zeta}(\nu), \zeta) \times \frac{\delta X_t^{\xi, \zeta}}{\delta \nu}(\nu, r, a)$$

$$+ \int_t^T \Xi^{\xi, \zeta, \nu}(r, t) \nabla_x h_r(X_r^{\xi, \zeta}(\nu), P_r^{\xi, \zeta}(\nu), a, \zeta) \, dr$$

$$+ \int_t^T \Xi^{\xi, \zeta, \nu}(r, t) \nabla_x h_r^2(X_r^{\xi, \zeta}(\nu), P_r^{\xi, \zeta}(\nu), a', \zeta) \nu_r(da') \, dr'$$

$$+ \left\{ \int_t^T \Xi^{\xi, \zeta, \nu}(r, t) \int \nabla_x h_r^2(X_r^{\xi, \zeta}(\nu), P_r^{\xi, \zeta}(\nu), a', \zeta) \nu_r(da') \, dr' \right\} \frac{\delta X_t^{\xi, \zeta}}{\delta \nu}(\nu, r, a) . \quad (72)$$
Since $a \mapsto \phi_t(x, a, \zeta)$ is uniformly Lipschitz continuous, $a \mapsto \frac{\delta X_t^{\xi, \zeta}}{\delta \nu}(\nu, r, a)$ is also uniformly Lipschitz continuous, uniformly in $r, \nu, \xi$ and $\zeta$ with

$$\|\frac{\delta X_t^{\xi, \zeta}}{\delta \nu}(\nu, r, \cdot)\|_{Lip} := \sup_{a \neq a'} \frac{\left| \frac{\delta X_t^{\xi, \zeta}}{\delta \nu}(\nu, r, a') - \frac{\delta X_t^{\xi, \zeta}}{\delta \nu}(\nu, r, a) \right|}{|a - a'|},$$

(73)

In the same way, $a \mapsto \frac{\delta P_t^{\xi, \zeta}}{\delta \nu}(\nu, r, a)$ is also uniformly Lipschitz continuous with

$$\|\frac{\delta P_t^{\xi, \zeta}}{\delta \nu}(\nu, r, \cdot)\|_{Lip} := \sup_{a \neq a'} \frac{\left| \frac{\delta P_t^{\xi, \zeta}}{\delta \nu}(\nu, r, a') - \frac{\delta P_t^{\xi, \zeta}}{\delta \nu}(\nu, r, a) \right|}{|a - a'|},$$

(74)

For $\lambda \in (0, 1)$, define $\mu^\lambda = (1 - \lambda)\mu' + \lambda \mu$. Let $Lip(1)$ denote the class of Lipschitz functions with Lipschitz constant bounded by 1. From the definition of the functional derivative 30,

$$\left| X_t^{\xi, \zeta}(\nu) - X_t^{\xi, \zeta}(\nu') \right| = \left| \int_0^1 \int_0^T \frac{\delta X_t^{\xi, \zeta}}{\delta \nu}(\nu^\lambda, r, a) (\nu_t(da) - \nu'_t(da)) \, dr \, d\lambda \right|,$$

$$\leq \sup_{t, r \in (0, T), \nu \in \mathcal{V}_2} \|\frac{\delta X_t^{\xi, \zeta}}{\delta \nu}(\nu, r, \cdot)\|_{Lip} \left| \sup_{c \in Lip(1)} \int_0^T c(a) (\nu_t(da) - \nu'_t(da)) \, dr \right|,$$

$$\leq L \left| \sup_{c \in Lip(1)} \int_0^T c(a) (\nu_t(da) - \nu'_t(da)) \, dr \right|.$$

We can estimate $P_t^{\xi, \zeta}(\mu) - P_t^{\xi, \zeta}(\nu)$ in the same way, obtaining here:

$$\left| P_t^{\xi, \zeta}(\nu) - P_t^{\xi, \zeta}(\nu') \right| = \left| \int_0^1 \int_0^T \frac{\delta P_t^{\xi, \zeta}}{\delta \nu}(r, a, \nu^\lambda) (\nu_t(da) - \nu'_t(da)) \, dr \, d\lambda \right|,$$

$$\leq L \left( \sup_{t \in (0, T), \mu \in Lin(\nu, \nu')} \left| \int_{\mathbb{R}^d \times S} P_t^{\xi, \zeta}(\mu) \mathcal{M}(d\xi, d\zeta) \right| \right) \left| \sup_{c \in Lip(1)} \int_0^T c(a) (\nu_t(da) - \nu'_t(da)) \, dr \right|,$$

for $Lin(\nu, \nu') := \{ \mu \in \mathcal{V}_2 : \mu = (1 - \lambda)\nu' + \lambda \nu, \text{ for some } 0 \leq \lambda \leq 1 \}$. By Kantorovich (dual) representation of the Wasserstein distance (Villani, 2008, Th 5.10) we conclude that there is $L > 0$ such that

$$\left| (\nabla \mathbf{h}_t)(a, \nu, \mathcal{M}) - (\nabla \mathbf{h}_t)(a, \nu', \mathcal{M}) \right| \leq L \mathcal{W}_{1}^T(\nu, \nu').$$

(75)

Combining (70) and (75), we then conclude.
Appendix D. Full discretisation scheme

In this section, we illustrate a simple example of a full-time discretization of the particle (23), complementing the time-discretization (44) by adding a discretization in the \( t \) variable.

Consider a finite partition \( 0 = t_0 < t_1 < \ldots < t_n = T \) of the interval \([0, T]\). Let \( \{\tilde{X}^\xi_k(\nu)\}_k \) and \( \{\tilde{P}^\xi_k(\nu)\}_k \) be a uniform \( \beta \)-order approximation (\( \beta > 0 \)) of \( (X^\xi_t(\nu)) \) and \( (P^\xi_t(\nu)) \), in the sense that

\[
\sup_{\xi, \nu} |X^\xi_{t_k}(\nu) - \tilde{X}^\xi_k(\nu)| + |P^\xi_{t_k}(\nu) - \tilde{P}^\xi_k(\nu)| \leq C \max_{k_1 \leq k} |t_{k_1} - t_{k_1-1}|^\beta.
\]  

(76)

We again refer to Hairer et al. (1987) for an exhaustive presentation of numerical approximation scheme for ODEs.

On the other hand, from the partial discretization \( (\bar{\theta}^i_j) \) defined in (44), we introduce the discretization of the time variable \( t \) with from the frozen dynamic:

\[
\bar{\theta}_{i,s} = \theta^0_{\eta_n(t)} - \int_0^s \nabla_a \bar{H}^n_{\eta_n(t)} \left( \bar{\theta}^i_{\lambda M(\nu, \eta_n(t)), \nu}, \bar{\nu}^N_{\lambda M(\nu, \eta_n(t)), \nu, \eta_n(t)} \right) dv + \sigma B^i_s, \bar{\nu}^{N_2}_s = \frac{1}{N_2} \sum_{j_2=1}^{N_2} \delta(\bar{\gamma}^i_{j_2}),
\]  

(77)

where \( \eta_n(t) = \inf\{t_k : t_k \leq t\} \), and \( \nabla \bar{H}^n_{\eta} \) is defined on \([0, T] \times \mathcal{V}_2 \times \mathcal{P}(\mathbb{R}^d \times S)\) by

\[
\nabla_a \bar{H}^n_{\eta_i}(a, \nu, M^{N_1}) = \nabla_a U(a) + \nabla_a \bar{H}^n_{\eta_i}(a, \nu, M^{N_1}),
\]

\[
\bar{H}^n_{\eta_i}(a, \nu, M^{N_1}) = \int_{\mathbb{R}^d \times S} h_{\eta_i} (\tilde{X}^\xi_k(\nu^{\eta_i}(\cdot), \cdot), \tilde{P}^\xi_k(\nu^{\eta_i}(\cdot), \cdot), a, \zeta, M^{N_1}(d\xi, d\zeta)).
\]

where \( \nu^{\eta_i}(\cdot) \) is the discretized version of \( \nu \) at times \( t_0, t_1, \ldots \).

The rate of convergence between (44) and (77) is given by:

**Proposition 39** Assume that the assumptions of Lemma 26 hold. Assume also that the following properties hold:

(D1) The discrete schemes \( \tilde{X} \) and \( \tilde{P} \) satisfy (76) as well as the properties:

i) \( \sup_{\nu, \nu'} \int |\tilde{P}^\xi_k(\nu)| M^{N_1}(d\xi, d\zeta) < \infty, \)

ii) there exists some constant \( L' \) such that for all \( \nu, \nu' \) in \( \mathcal{V}_2 \),

\[
|\tilde{X}^\xi_k(\nu) - \tilde{X}^\xi_k(\nu')| \leq L'T \sup_{0 \leq t \leq T} W_1(\nu_t, \nu'_t),
\]

and

\[
|\tilde{P}^\xi_k(\nu) - \tilde{P}^\xi_k(\nu')| \leq L'T \sup_{0 \leq t \leq T} W_1(\nu_t, \nu'_t).
\]

(D2) The functions

\[
t \mapsto \phi_t(x, a, \zeta), \nabla_x \phi_t(x, a, \zeta), \nabla_x f_t(x, a, \zeta), \nabla_a \phi_t(x, a, \zeta), \nabla_a f_t(x, a, \zeta)
\]

are all (uniformly in \( x, a, \zeta \)) of class \( C^\alpha \) (for \( 0 < \alpha \leq 1 \)), that is, for some \( 0 < L'' < \infty \),

\[
|\phi_t(x, a, \zeta) - \phi_t(x, a, \zeta')| + |\nabla_x \phi_t(x, a, \zeta) - \nabla_x \phi_t(x, a, \zeta')| + |\nabla_a \phi_t(x, a, \zeta) - \nabla_a \phi_t(x, a, \zeta')| + |\nabla_x f_t(x, a, \zeta) - \nabla_x f_t(x, a, \zeta')| + |\nabla_a f_t(x, a, \zeta) - \nabla_a f_t(x, a, \zeta')| \leq L''|t - t'|^\alpha,
\]

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for all $t, t' \in [0, T]$.

(D3) The initial flow $t \mapsto \theta_t^{0,i}$ satisfies the properties: $\sup_{0 \leq t \leq T} \mathbb{E}[|\theta_t^{0,i}|^2] < \infty$ and

$$\mathbb{E}\left[ |\theta_t^{0,i} - \theta_{t'}^{0,i}|^2 \right] \leq L |t - t'|^{2\alpha}, \forall t, t' \in [0, T].$$

Then, there exists $0 < c < \infty$ independent of $(s_t)_t$, $N_1$ and $N_2$, such that, for all integers $i, L$,

$$\sup_{k,1 \leq l \leq L} \mathbb{E}\left[ |\tilde{\theta}_{i,l} - \bar{\theta}_{i,l}^{s_t,tk}|^2 \right] \leq c(1 + \max_{l \leq L} (s_l - s_{l-1})) \left( \max_{t} |t - \eta(t)|^{2(\alpha \wedge \beta \wedge 1)} + \int_0^T \mathbb{E}\left[ |\tilde{\theta}_{i,t'}^{s_l,t}|^2 \right] dt' \right).$$

**Proof**

**Step 1.** The assumption (D1) immediately ensures that: for all $a, a', \nu, \nu' \in \mathcal{V}_2$,

$$|\nabla \tilde{a}_l^i(a, \nu, \mathcal{M}^{N_1}) - \nabla \tilde{a}_l^i(a', \nu', \mathcal{M}^{N_1})| \leq L \left( |a - a'| + \sup_{0 \leq t \leq T} W_1(\nu, \eta(t), \nu', \eta(t)) \right). \quad (78)$$

For simplicity, we will again omit from now on the explicit notation of the component $\mathcal{M}^{N_1}$ in most of the calculations below.

Setting $\triangle_{s_l,tk}^{\theta^i} := \tilde{\theta}_{s_l,tk}^i - \bar{\theta}_{s_l,tk}^i$, observe that, from all $1 \leq l \leq M, 1 \leq k \leq n$,

$$\triangle_{s_l,tk}^{\theta^i} = \triangle_{s_{l-1},tk}^{\theta^i} - \frac{\sigma^2}{2} \left( s_l - s_{l-1} \right) \left( \nabla_a U(\tilde{\theta}_{s_{l-1},tk}^i) - \nabla_a U(\tilde{\theta}_{s_{l-1},tk}^i) \right)$$

$$\quad - \left( s_l - s_{l-1} \right) \left( \nabla_a \tilde{h}_{tk}^i \left( \tilde{\theta}_{s_{l-1},tk}^i, \tilde{\nu}_{s_{l-1},\ldots,t}^{N_2}, \mathcal{M}^{N_1} \right) - \nabla_a \tilde{h}_{tk}^i \left( \tilde{\theta}_{s_{l-1},tk}^i, \tilde{\nu}_{s_{l-1},\ldots,t}^{N_2}, \mathcal{M}^{N_1} \right) \right)$$

$$\quad - \left( s_l - s_{l-1} \right) \left( \nabla_a \tilde{h}_{tk}^n \left( \tilde{\theta}_{s_{l-1},tk}^i, \tilde{\nu}_{s_{l-1},\ldots,t}^{N_2}, \mathcal{M}^{N_1} \right) - \nabla_a \tilde{h}_{tk}^n \left( \tilde{\theta}_{s_{l-1},tk}^i, \tilde{\nu}_{s_{l-1},\ldots,t}^{N_2}, \mathcal{M}^{N_1} \right) \right).$$

Proceeding as in (47),

$$|\triangle_{s_l,tk}^{\theta^i}|^2 \leq |\triangle_{s_{l-1},tk}^{\theta^i}|^2 \left( 1 - (\sigma^2 \kappa - 3)(s_l - s_{l-1}) \right)$$

$$\quad + \left( s_l - s_{l-1} \right) \nabla_a \tilde{h}_{tk}^i \left( \tilde{\theta}_{s_{l-1},tk}^i, \tilde{\nu}_{s_{l-1},\ldots,t}^{N_2}, \mathcal{M}^{N_1} \right) - \nabla_a \tilde{h}_{tk}^n \left( \tilde{\theta}_{s_{l-1},tk}^i, \tilde{\nu}_{s_{l-1},\ldots,t}^{N_2}, \mathcal{M}^{N_1} \right) \right|^2$$

$$\quad + 2(s_l - s_{l-1})^2 \left( \nabla_a \tilde{h}_{tk}^i \left( \tilde{\theta}_{s_{l-1},tk}^i, \tilde{\nu}_{s_{l-1},\ldots,t}^{N_2}, \mathcal{M}^{N_1} \right) - \nabla_a \tilde{h}_{tk}^n \left( \tilde{\theta}_{s_{l-1},tk}^i, \tilde{\nu}_{s_{l-1},\ldots,t}^{N_2}, \mathcal{M}^{N_1} \right) \right|^2$$

$$\quad + \left( s_l - s_{l-1} \right)(1 + (s_l - s_{l-1})/2) \nabla_a \tilde{h}_{tk}^i \left( \tilde{\theta}_{s_{l-1},tk}^i, \tilde{\nu}_{s_{l-1},\ldots,t}^{N_2}, \mathcal{M}^{N_1} \right) - \nabla_a \tilde{h}_{tk}^n \left( \tilde{\theta}_{s_{l-1},tk}^i, \tilde{\nu}_{s_{l-1},\ldots,t}^{N_2}, \mathcal{M}^{N_1} \right) \right|^2.$$

From (78), we deduce

$$\mathbb{E}\left[ \left| \nabla_a \tilde{h}_{tk}^i \left( \tilde{\theta}_{s_{l-1},tk}^i, \tilde{\nu}_{s_{l-1},\ldots,t}^{N_2}, \mathcal{M}^{N_1} \right) - \nabla_a \tilde{h}_{tk}^n \left( \tilde{\theta}_{s_{l-1},tk}^i, \tilde{\nu}_{s_{l-1},\ldots,t}^{N_2}, \mathcal{M}^{N_1} \right) \right|^2 \right] \leq L \max_{k} \mathbb{E}\left[ |\triangle_{s_{l-1},tk}^{\theta^i}|^2 \right],$$

and

$$\mathbb{E}\left[ \left| \nabla_a \tilde{h}_{tk}^i \left( \tilde{\theta}_{s_{l-1},tk}^i, \tilde{\nu}_{s_{l-1},\ldots,t}^{N_2}, \mathcal{M}^{N_1} \right) - \nabla_a \tilde{h}_{tk}^n \left( \tilde{\theta}_{s_{l-1},tk}^i, \tilde{\nu}_{s_{l-1},\ldots,t}^{N_2}, \mathcal{M}^{N_1} \right) \right|^2 \right] \leq L(1 + \frac{\sigma^4}{2} \|\nabla_a U(\cdot)\|_L^2) \max_{k} \mathbb{E}\left[ |\triangle_{s_{l-1},tk}^{\theta^i}|^2 \right].$$
Then it follows that
\[
\mathbb{E} \left[ |\Delta_{s_l,t_0} \theta|^{2} \right] \leq \mathbb{E} \left[ |\Delta_{s_{l-1},t_0} \theta|^{2} \right] (1 - (\sigma^2 \kappa - 3)(s_l - s_{l-1})) \\
+ L(s_l - s_{l-1}) \left( 1 + (s_l - s_{l-1}) \left( 1 + \frac{\sigma^4}{2} \left\| \nabla a U (\cdot) \right\|_{L_{lip}} \right) \right) \max_k \mathbb{E} \left[ |\Delta_{s_{l-1},t_0} \theta|^{2} \right] \\
+ (s_l - s_{l-1}) (1 + (s_l - s_{l-1})/2) \mathbb{E} \left[ \left| \nabla a h_{k} \left( \bar{\theta}_{s_{l-1},t_0}, \bar{\nu}_{s_{l-1},t_0}, \mathcal{M}^{N_l} \right) - \nabla a \bar{h}_{k} \left( \bar{\theta}_{s_{l-1},t_0}, \bar{\nu}_{s_{l-1},t_0}, \mathcal{M}^{N_l} \right) \right|^2 \right].
\]

(79)

**Step 2.** Owing to the regularity of \( f, \phi \) and \( \nabla a U \), we have, for all \( 0 \leq t \leq T \),
\[
\mathbb{E} \left[ \left| \nabla a h_{t} \left( \bar{\theta}_{s_{l-1},t}, \bar{\nu}_{s_{l-1},t}, \mathcal{M}^{N_l} \right) - \nabla a \bar{h}_{t} \left( \bar{\theta}_{s_{l-1},t}, \bar{\nu}_{s_{l-1},t}, \mathcal{M}^{N_l} \right) \right|^2 \right] \\
\leq \int \mathbb{E} \left[ \left| h_{t_k} \left( X_{t}^{L,K} \left( \bar{\nu}_{s_{l-1},t_{k}}, \mathcal{M}^{N_l} \right), P_{t_k}^{L,K} \left( \bar{\nu}_{s_{l-1},t_{k}}, \mathcal{M}^{N_l} \right) \right), a, \zeta \right) - h_{t_k} \left( \bar{X}_{t}^{L,K} \left( \bar{\nu}_{s_{l-1},t_{k}}, \mathcal{M}^{N_l} \right), \bar{P}_{t_k}^{L,K} \left( \bar{\nu}_{s_{l-1},t_{k}}, \mathcal{M}^{N_l} \right) \right), a, \zeta \right) \right|^2 \mathcal{M}^{N_l}(d\xi, d\zeta) \\
+ \int \mathbb{E} \left[ \left| h_{t_k} \left( X_{t}^{L,K} \left( \bar{\nu}_{s_{l-1},t_{k}}, \mathcal{M}^{N_l} \right), P_{t_k}^{L,K} \left( \bar{\nu}_{s_{l-1},t_{k}}, \mathcal{M}^{N_l} \right) \right), a, \zeta \right) - h_{t_k} \left( \bar{X}_{t}^{L,K} \left( \bar{\nu}_{s_{l-1},t_{k}}, \mathcal{M}^{N_l} \right), \bar{P}_{t_k}^{L,K} \left( \bar{\nu}_{s_{l-1},t_{k}}, \mathcal{M}^{N_l} \right) \right), a, \zeta \right) \right|^2 \mathcal{M}^{N_l}(d\xi, d\zeta).
\]

From (76) and (78), we deduce directly that
\[
\mathbb{E} \left[ \left| \nabla a h_{t} \left( \bar{\theta}_{s_{l-1},t}, \bar{\nu}_{s_{l-1},t}, \mathcal{M}^{N_l} \right) - \nabla a \bar{h}_{t} \left( \bar{\theta}_{s_{l-1},t}, \bar{\nu}_{s_{l-1},t}, \mathcal{M}^{N_l} \right) \right|^2 \right] \\
\leq c \sup_k |t_k - t_{k-1}|^{2\beta} + c \int_{0}^{T} \mathbb{E} \left[ |\bar{\theta}_{s_{l-1},t} - \bar{\theta}_{s_{l-1},t_{k}}|^{2} \right] dt.
\]

(80)

It now remains to estimate \( \mathbb{E}[|\bar{\theta}_{s_{l},t} - \bar{\theta}_{s_{l},\eta_{0}(t)}|^{2}] \). When \( l = 0 \), \( \mathbb{E}[|\bar{\theta}_{s_{l},t} - \bar{\theta}_{s_{l},\eta_{0}(t)}|^{2}] \) is an immediate consequence of \( (D_3) \). When \( l > 1 \), we have, by \( (D_2), (1) \) and \( 6 \):
\[
\mathbb{E} \left[ \left| \bar{\theta}_{s_{l},t} - \bar{\theta}_{s_{l},\eta_{0}(t)} \right|^{2} \right] \\
\leq \mathbb{E} \left[ \left| \bar{\theta}_{s_{l-1},t} - \bar{\theta}_{s_{l-1},\eta_{0}(t)} \right| + (s_l - s_{l-1}) \left( \nabla a h^{\sigma}_{t} \left( \bar{\theta}_{s_{l-1},t}, \bar{\nu}_{s_{l-1},t}, \mathcal{M}^{N_l} \right) - \nabla a h^{\sigma}_{t_{k}} \left( \bar{\theta}_{s_{l-1},t_{k}}, \bar{\nu}_{s_{l-1},t_{k}}, \mathcal{M}^{N_l} \right) \right) \right] \\
\leq \mathbb{E} \left[ \left| \bar{\theta}_{s_{l-1},t} - \bar{\theta}_{s_{l-1},\eta_{0}(t)} \right|^{2} \right] (1 - (\sigma^2 \kappa - 3)(s_l - s_{l-1})) \\
+ E \left[ \left| \bar{\theta}_{s_{l-1},t} - \bar{\theta}_{s_{l-1},\eta_{0}(t)} \right|^{2} \right] L(s_l - s_{l-1}) \left( 1 + (s_l - s_{l-1}) \left( 1 + \frac{\sigma^4}{2} \left\| \nabla a U (\cdot) \right\|_{L_{lip}} \right) \right) \\
+ 2c(s_l - s_{l-1}) (1 + (s_l - s_{l-1}) \left( (t - \eta_{0}(t))^{2\alpha} \right) + \int \mathbb{E} \left[ \left| X_{t}^{\xi,\zeta} \left( \bar{\nu}_{s_{l-1},t}, \mathcal{M}^{N_l} \right) - X_{t_{k}}^{\xi,\zeta} \left( \bar{\nu}_{s_{l-1},t_{k}}, \mathcal{M}^{N_l} \right) \right|^{2} + \left| P_{t}^{\xi,\zeta} \left( \bar{\nu}_{s_{l-1},t}, \mathcal{M}^{N_l} \right) - P_{t_{k}}^{\xi,\zeta} \left( \bar{\nu}_{s_{l-1},t_{k}}, \mathcal{M}^{N_l} \right) \right|^{2} \right] \mathcal{M}^{N_l}(d\xi, d\zeta).\]

With the assumptions \( 1 \) and \( 6 \) one can check that
\[
\mathbb{E} \left[ \left| X_{t}^{\xi,\zeta} \left( \bar{\nu}_{s_{l-1},t}, \mathcal{M}^{N_l} \right) - X_{t_{k}}^{\xi,\zeta} \left( \bar{\nu}_{s_{l-1},t_{k}}, \mathcal{M}^{N_l} \right) \right|^{2} + \mathbb{E} \left[ \left| P_{t}^{\xi,\zeta} \left( \bar{\nu}_{s_{l-1},t}, \mathcal{M}^{N_l} \right) - P_{t_{k}}^{\xi,\zeta} \left( \bar{\nu}_{s_{l-1},t_{k}}, \mathcal{M}^{N_l} \right) \right|^{2} \right] \\
\leq c \max_{t} |t - \eta_{0}(t)|^{2} \left( 1 + \mathbb{E} \left[ \int_{0}^{T} |\bar{\theta}_{s_{l-1},t}|^{2} dt \right] \right)
\]
so that

\[
\mathbb{E} \left[ \left| \tilde{\theta}_{s_l,t}^i - \tilde{\theta}_{s_l,\eta_n(t)}^i \right|^2 \right] \leq c(s_l - s_{l-1})(1 + (s_l - s_{l-1}))(t - \eta_n(t))^{2(\alpha \wedge 1)} \\
\quad + \mathbb{E} \left[ \left| \tilde{\theta}_{s_{l-1},t}^i - \tilde{\theta}_{s_{l-1},\eta_n(t)}^i \right|^2 \right] \left( 1 - (\sigma^2 \kappa - 3)(s_l - s_{l-1}) + L(s_l - s_{l-1}) \left( 1 + (s_l - s_{l-1})\left( 1 + \frac{\sigma^4}{2}\|\nabla u\|_{\text{Lip}} \right) \right) \right).
\]

Proceeding as in Step 2 of the proof of Lemma 26, we get

\[
\mathbb{E} \left[ \left| \tilde{\theta}_{s_l,t}^i - \tilde{\theta}_{s_l,\eta_n(t)}^i \right|^2 \right] \leq c(1 + \max_{l'} (s_{l'} - s_{l'-1}))(t - \eta_n(t))^{2(\alpha \wedge 1)}
\]

(81)

**Step 3.** Plugging (81) into (80), coming back to (79), and then following again Step 2 from the proof of Lemma 26, we conclude.

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**References**

S. Arora, S. Du, W. Hu, Z. Li, and R. Wang. Fine-grained analysis of optimization and generalization for overparameterized two-layer neural networks. In *International Conference on Machine Learning*, pages 322–332. PMLR, 2019.

A. G. Baydin, B. A. Pearlmutter, A. A. Radul, and J. M. Siskind. Automatic differentiation in machine learning: a survey. *Journal of machine learning research*, 18(153), 2018.

M. Belkin, D. Hsu, S. Ma, and S. Mandal. Reconciling modern machine-learning practice and the classical bias–variance trade-off. *Proceedings of the National Academy of Sciences*, 116(32):15849–15854, 2019.

R. Bellman. Dynamic programming. *Science*, 153(3731):34–37, 1966.

A. Bensoussan. *Stochastic control of partially observable systems*. Cambridge University Press, 2004.

A. Bensoussan and J.-L. Lions. *Applications of variational inequalities in stochastic control*. Elsevier, 2011.

D. P. Bertsekas. *Dynamic programming and optimal control*. Athena scientific Belmont, MA, 1995.

L. Bo, A. Capponi, and H. Liao. Relaxed control and gamma-convergence of stochastic optimization problems with mean field. *arXiv preprint arXiv:1906.08894*, 2019.

R. Carmona and F. Delarue. *Probabilistic Theory of Mean Field Games with Applications I-II*. Springer, 2018.

J.-F. Chassagneux, L. Szpruch, and A. Tse. Weak quantitative propagation of chaos via differential calculus on the space of measures. *arXiv:1901.02556*, 2019.
R. T. Q. Chen, Y. Rubanova, J. Bettencourt, and D. Duvenaud. Neural ordinary differential equations. In *Advances in neural information processing systems*, pages 6571–6583, 2018.

X. Cheng, P. L. Bartlett, and M. I. Jordan. Quantitative $w_1$ convergence of Langevin-like stochastic processes with non-convex potential state-dependent noise. *arXiv:1907.03215*, 2019.

L. Chizat and F. Bach. On the global convergence of gradient descent for over-parameterized models using optimal transport. In *Advances in neural information processing systems*, pages 3040–3050, 2018.

L. Chizat, E. Oyallon, and F. Bach. On lazy training in differentiable programming. *arXiv preprint arXiv:1812.07956*, 2018.

C. Cuchiero, M. Larsson, and J. Teichmann. Deep neural networks, generic universal interpolation, and controlled odes. *arXiv preprint arXiv:1908.07838*, 2019.

F. Delarue, D. Lacker, and K. Ramanan. From the master equation to mean field game limit theory: A central limit theorem. *Electronic Journal of Probability*, 24, 2019.

S. Dereich, M. Scheutzow, and R. Schottstedt. Constructive quantization: approximation by empirical measures. *Ann. Inst. Henri Poincaré Probab. Stat.*, 49(4):1183–1203, 2013.

P. Dupuis and R. S. Ellis. *A weak convergence approach to the theory of large deviations*. John Wiley & Sons, 2011.

A. Durmus and E. Moulines. Nonasymptotic convergence analysis for the unadjusted Langevin algorithm. *The Annals of Applied Probability*, 27(3):1551–1587, 2017.

G. K. Dziugaite and D. M. Roy. Data-dependent pac-bayes priors via differential privacy. *arXiv preprint arXiv:1802.09583*, 2018.

W. E, J. Han, and Q. Li. A mean-field optimal control formulation of deep learning. *arXiv:1807.01083*, 2018.

A. Eberle. Reflection couplings and contraction rates for diffusions. *Probability theory and related fields*, 166(3-4):851–886, 2016.

W. H. Fleming and H. M. Soner. *Controlled Markov processes and viscosity solutions*. Springer, 2006.

N. Fournier and A. Guillin. On the rate of convergence in Wasserstein distance of the empirical measure. *Probability Theory and Related Fields*, 162(3-4):707–738, 2015.

Y. Gal and Z. Ghahramani. Bayesian convolutional neural networks with bernoulli approximate variational inference. *arXiv preprint arXiv:1506.02158*, 2015.

B. Ghorbani, S. Mei, T. Misiakiewicz, and A. Montanari. Linearized two-layers neural networks in high dimension. *arXiv preprint arXiv:1904.12191*, 2019.
B. Ghorbani, S. Mei, T. Misiakiewicz, and A. Montanari. When do neural networks outperform kernel methods? *arXiv preprint arXiv:2006.13409*, 2020.

I. Gyöngy and D. Šiška. On finite-difference approximations for normalized Bellman equations. *Appl. Math. Optim.*, 60(3):297–339, 2009. doi: 10.1007/s00245-009-9082-0. URL https://doi.org/10.1007/s00245-009-9082-0.

E. Hairer, S. P. Nørsett, and G. Wanner. *Solving ordinary differential equations. I*. Springer, Berlin, 1987. ISBN 3-540-17145-2. doi: 10.1007/978-3-662-12607-3. URL https://doi.org/10.1007/978-3-662-12607-3.

T. Hastie, A. Montanari, S. Rosset, and R. J. Tibshirani. Surprises in high-dimensional ridgeless least squares interpolation. *arXiv preprint arXiv:1903.08560*, 2019.

K. He, X. Zhang, S. Ren, and J. Sun. Deep residual learning for image recognition. In *Proceedings of the IEEE conference on computer vision and pattern recognition*, pages 770–778, 2016.

J. Heiss, J. Teichmann, and H. Wutte. How implicit regularization of neural networks affects the learned function—part i. *arXiv preprint arXiv:1911.02903*, 2019.

K. Hu, A. Kazeykina, and Z. Ren. Mean-field Langevin system, optimal control and deep neural networks. *arXiv:1909.07278*, 2019a.

K. Hu, Z. Ren, D. Šiška, and L. Szpruch. Mean-field Langevin dynamics and energy landscape of neural networks. *arXiv:1905.07769*, 2019b.

J.-F. Jabir. Rate of propagation of chaos for diffusive stochastic particle systems via Girsanov transformation. *arXiv:1907.09096*, 2019.

A. Jacot, F. Gabriel, and C. Hongler. Neural tangent kernel: Convergence and generalization in neural networks. *arXiv preprint arXiv:1806.07572*, 2018.

I. Karatzas and S. Shreve. *Brownian motion and stochastic calculus*. Springer, 2012.

H. J. Kushner and P. Dupuis. *Numerical methods for stochastic control problems in continuous time*. Springer, 2001. doi: 10.1007/978-1-4613-0007-6.

O. A. Ladyzenskaja, V. A. Solonnikov, and N. N. Ural’ceva. *Linear and quasi-linear equations of parabolic type*. Translations of Mathematical Monographs. AMS, 1968.

Y. LeCun, Y. Bengio, and G. Hinton. Deep learning. *Nature*, 521(7553):436–444, 2015.

Q. Li, L. Chen, C. Tai, and W. E. Maximum principle based algorithms for deep learning. *The Journal of Machine Learning Research*, 18(1):5998–6026, 2017.

H. Liu and P. Markowich. Selection dynamics for deep neural networks. *Journal of Differential Equations*, 269(12):11540–11574, 2020.

C. Ma, L. Wu, et al. Barron spaces and the compositional function spaces for neural network models. *arXiv preprint arXiv:1906.08039*, 2019.
D. J. MacKay. Probable networks and plausible predictions—a review of practical bayesian methods for supervised neural networks. *Network: computation in neural systems*, 6(3): 469–505, 1995.

M. B. Majka, A. Mijatović, and L. Szpruch. Non-asymptotic bounds for sampling algorithms without log-concavity. *arXiv:1808.07105*, 2018.

S. Mallat. Understanding deep convolutional networks. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 374(2065), 2016.

S. Mei and A. Montanari. The generalization error of random features regression: Precise asymptotics and double descent curve. *arXiv:1908.05355*, 2019.

S. Mei, A. Montanari, and P.-M. Nguyen. A mean field view of the landscape of two-layer neural networks. *Proceedings of the National Academy of Sciences*, 115(33):E7665–E7671, 2018.

S. Mei, T. Misiakiewicz, and A. Montanari. Mean-field theory of two-layers neural networks: dimension-free bounds and kernel limit. In *Conference on Learning Theory*, pages 2388–2464. PMLR, 2019.

A. Montanari, F. Ruan, Y. Sohn, and J. Yan. The generalization error of max-margin linear classifiers: High-dimensional asymptotics in the overparametrized regime. *arXiv:1911.01544*, 2019.

R. M. Neal. *Bayesian learning for neural networks*, volume 118. Springer Science & Business Media, 2012.

B. Neyshabur, R. Tomioka, R. Salakhutdinov, and N. Srebro. Geometry of optimization and implicit regularization in deep learning. *arXiv:1705.03071*, 2017.

B. Neyshabur, Z. Li, S. Bhojanapalli, Y. LeCun, and N. Srebro. Towards understanding the role of over-parametrization in generalization of neural networks. *arXiv:1805.12076*, 2018.

G. M. Rotskoff and E. Vanden-Eijnden. Neural networks as interacting particle systems: Asymptotic convexity of the loss landscape and universal scaling of the approximation error. *arXiv:1805.00915*, 2018.

D. Silver, A. Huang, C. J. Maddison, A. Guez, L. Sifre, G. Van Den Driessche, J. Schrittwieser, I. Antonoglou, V. Panneershelvam, M. Lanctot, et al. Mastering the game of Go with deep neural networks and tree search. *Nature*, 529(7587):484–489, 2016.

J. Sirignano and K. Spiliopoulos. Mean field analysis of neural networks. *arXiv:1805.01053*, 2018.

E. Sontag and H. Sussmann. Complete controllability of continuous-time recurrent neural networks. *Systems & control letters*, 30(4):177–183, 1997.

L. Szpruch and A. Tse. Antithetic multilevel particle system sampling method for McKean–Vlasov SDEs. *arXiv:1903.07063*, 2019.
L. Szpruch, S. Tan, and A. Tse. Iterative particle approximation for McKean-Vlasov SDEs with application to multilevel Monte Carlo estimation. *To appear in Annals of Applied Probability*, 2019.

B. Tzen and M. Raginsky. A mean-field theory of lazy training in two-layer neural nets: entropic regularization and controlled mckean-vlasov dynamics. *arXiv preprint arXiv:2002.01987*, 2020.

C. Villani. *Optimal transport: old and new*. Springer, 2008.

E. Weinan. A proposal on machine learning via dynamical systems. *Communications in Mathematics and Statistics*, 5(1):1–11, 2017.

E. Weinan, C. Ma, and L. Wu. Machine learning from a continuous viewpoint. *arXiv preprint arXiv:1912.12777*, 2019.

L. C. Young. *Lectures on the calculus of variations and optimal control theory*, volume 304. American Mathematical Soc., 2000.

C. Zhang, S. Bengio, M. Hardt, B. Recht, and O. Vinyals. Understanding deep learning requires rethinking generalization. *arXiv:1611.03530*, 2016.