Wetzel’s sector covers unit arcs

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Abstract We settle J. Wetzel’s 1970’s conjecture and show that a 30° circular sector of unit radius can accommodate every planar arc of unit length. Leo Moser asked in 1966 for the smallest (convex) region in the plane that can accommodate each arc of unit length. With area $\pi/12$, this sector is the smallest such set presently known. Moser’s question has prompted a multitude of papers on related problems over the past 50 years, most remaining unanswered.

Keywords covering of unit arcs · covering by convex sets · worm problem · sectorial covers · planar arc · support line

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1 Introduction

On a list [3, Problem 9] of 50 combinatorial geometry problems circulated in 1966, L. Moser asked for the smallest set in the plane that is large enough to accommodate a congruent copy of each arc of unit length. (The list and much related material and commentary are included in W. O. J. Moser [4, p. 211, pp. 218-19].) This difficult question, still unanswered, has become the model for a family of arc-covering questions known collectively as “worm problems.” Many articles have been published on such questions, but very few specific problems have been completely solved.

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Let $\mathcal{F}$ be the set of all arcs of unit length in the plane. A *cover* of $\mathcal{F}$ is a set that contains a congruent copy of every arc in $\mathcal{F}$. Soon after Moser asked his question, A. Meir showed that a semidisk of unit diameter is such a cover (reported in [10]); and in 1973 Wetzel [10] showed more generally that for $0^\circ < \vartheta \leq 90^\circ$ a circular sector of angle $\vartheta$ and radius $\frac{1}{2} \csc \frac{1}{2} \vartheta$ is also a cover. At about this same time, Wetzel guessed that the $30^\circ$ sector $\Pi$ of unit radius might also be a cover, but although he mentioned this hunch to colleagues, he inexplicably neglected to mention it in [10]. It was first reported in [7] and more recently in [9, p. 358].

The relevant literature on related covering problems is sizeable, but not much has been published about Moser's specific question. An argument is given in [13] to show that the sector with central angle $\vartheta$ and radius $\csc 2\vartheta$ is a cover, so in particular, a $30^\circ$ sector of radius $2/\sqrt{3} \approx 1.155$ is large enough to cover $\mathcal{F}$. It is known [8] that at least every *convex* (and consequently every *drapeable*) unit arc fits in $\Pi$.

The best bounds in the literature for the least area $\alpha$ of a convex cover for $\mathcal{F}$ are $0.2322 < \alpha < 0.2709$, as reported in [11]. With area $\pi/12 \approx 0.2618$, the $30^\circ$ unit sector $\Pi$ reduces the upper bound by a little more than 3 percent.

Here we give a geometric proof of Wetzel's conjecture. The shape of $\Pi$ suggests that reflections might be helpful, and our proof relies heavily on this insight. But in one of those strange coincidences that sometimes occur, Y. Movshovich has recently found an altogether different analytic proof of the result using calculus-level tools. Her argument has not yet been published.

## 2 Preliminaries

In [12, Cor. 5] we showed in detail that a compact set in the plane is a cover for $\mathcal{F}$ if it contains a congruent copy of each simple polygonal arc of unit length. We first mention the key lemma on how a pair of support lines touches a simple arc. The parallel case is discussed in [2] and the general case is in [13]. Here we give another proof for the general case. This is called the *lambda property*. Our proof of the following lemma is related to reasoning given by J. Ralph Alexander in our unpublished paper, Alexander, Wetzel, and Wichiramala, “The $\Lambda$-property of a simple arc.” [1]. The proof of the next lemma will appear after the main theorem.

**Lemma 1** Suppose $\gamma$ is a parametrized, simple, polygonal arc and $0 < \theta < \pi$. Let $X_1, X_2, ..., X_n$ be the corner points of the convex hull of $\gamma$ and appear on $\gamma$ in this parametric order. Then $\gamma$ can be placed within two angles of size $\theta$ so that

1) $X_1$ touches an angle’s vertex or $\gamma$ touches the rays at 3 points with parameters $t_1 < t_2 < t_3$ where $\gamma(t_2)$ touches one ray and the other two points touch the other ray.

2) $X_n$ touches the other angle’s vertex or $\gamma$ touches the rays at another set of 3 alternating points.
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In total, there are at most 2 pairs of support lines with sets of such triple points. Furthermore when there are 2 pairs of support lines, they interlace as illustrated in Figure 2 or Figure 3.

The lemma originates in [14] with the same result except the obvious interlace property. We note that 2 lines make angle 0 if they are parallel. From the previous lemma, when $\theta = 0$ the 2 results 1) and 2) become existing of a unique parallel support lines with such 3 alternating points [14]. When $\theta \geq \pi$, the result is trivial and thus not interesting. In addition, we may place some $\gamma$ within an angle so that some middle $X_i$, $1 < i < n$, touches the angle’s vertex. When $\theta$ is equal to the angle of the hull at $X_1$, we can both place $\gamma$ within an angle of size $\theta$ so that $X_1$ touches the angle’s vertex and there is a set of 3 alternating points having $X_1$ and $X_2$ (on opposite rays). Recently, Movshovich extends the result in [5].

The next lemma shows how to compare the length of an arbitrary arc with the chord of a related sector.

**Lemma 2** Let $AVB$ be a sector of angle $\theta$ with chord $AB$. Suppose that the arc $\gamma$ with endpoints $P$ and $Q$ are on the rays $VA$ and $VB$, respectively (see Figure 2).

1) If $P$ is not in the sector and $\theta < 90^\circ$, then $PQ > AA'$.
2) If $P$ and $Q$ are not in the sector (one of them is allowed to be on the circular arc) and $\theta < 180^\circ$, then $PQ > AB$.
3) If a point $X$ on $\gamma$ is not in the sector and $\theta < 90^\circ$, then $\gamma$ is longer than $AA'$. In addition, if $X$ is on the angle bisector, then $\gamma$ is longer than $AB$.
4) If points $X$ and $Y$ on $\gamma$ are not in the sector and they are on opposite side of the angle bisector of $\theta < 180^\circ$, then $\gamma$ is longer than $AB$.

Proof This is clear by simple comparison and the inequality $\sin \alpha + \sin \beta > \sin \alpha \cos \beta + \cos \alpha \sin \beta = \sin(\alpha + \beta)$ for $\alpha$ and $\beta$ in $(0, 90^\circ)$. $\square$

3 The main result

We prove the conjecture on the sector $\Pi$ here.

**Theorem 3** A $30^\circ$ sector of unit radius contains a congruent copy of every unit arc.

Proof It suffices to show that every simple, polygonal unit arc fits in $\Pi$. Suppose to the contrary that $\gamma$ is a simple, polygonal unit arc that cannot be covered by $\Pi$ (see Corollary 5 in [12]). Hence $\gamma$ cannot be placed within $\Pi$ so that it touches the angle’s vertex of the sector of unit radius. By Lemma 1 for angle $\theta = 30^\circ$, there are 2 different pairs of support lines with angle in between $30^\circ$ together with 2 sets of 3 alternating points on each pair of such lines. Some of these 6 points may coincide. Let $U$ and $V$ be the intersections of the 2 pairs of support lines (see Figure 2). Hence none of the 6 points touches $U$ or $V$, and there are points $X$ and $Y$ on $\gamma$ that $XV > 1$ and $YU > 1$.

We will divide into cases according to relations between the 2 pairs of lines and the 2 sets of points.

Case 1. The 2 pairs of lines share a common line with 2 sets of points $ABD$ and $ACD$ as in Figure 3. Note that $B$ and $C$ may coincide as a degenerate subcase. This case already includes the last 3 possibilities in Figure 3 as we may ignore some points.

For the remaining cases, the 4 lines are different. The arrangement of those 6 points will determine the remaining cases.

Case 2. The 2 sets of points are $ABD$ and $CEF$ as in Figure 4(a). Note that $BC$ or $DE$ may degenerate.

Case 3. The 2 sets of points are $ABC$ and $DEF$ as in Figure 4(b). Note that $CD$ may degenerate.

Case 4. The 2 sets of points are $ABC$ and $DEF$ as in Figure 4(c).

Let $P$ be the endpoint of $\gamma$ near $A$ and $Q$ be the other endpoint. Now we will show in each case that $\gamma$ is longer than 1.
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Fig. 3 Case 1 with all possibilities

Fig. 4 (a) Case 2. (b) Case 3. (c) Case 4.

Case 1. Here we have that \( X \) is on the subarc \( PB \) and \( Y \) is on the subarc \( CQ \). First suppose that \( X \) is on the subarc \( AB \) and \( Y \) is on the subarc \( CD \) (see Figure 6). Note that \( X \) may coincide with \( A \) or \( B \) and \( Y \) may coincide with \( C \) or \( D \). We will show that the polysegment \( AXBCYD \) is longer than 1. First we reflect \( A, X \) and \( V \) across \( UB \) to \( A', X' \) and \( V' \). Then we reflect \( Y, D \) and \( U \) across \( VC \) to \( Y', D' \) and \( V' \). Since \( X'V' = XV > 1 \) and \( V'Y' = UY > 1 \), by Lemma 2, the length \( AXBCYD = A'X'BCY'D' \) is greater than the chord of a unit sector of angle 60° at \( V' \). Therefore \( \gamma \) is longer than 1.

Now suppose that \( X \) is on the subarc \( PA \) (and \( Y \) is on the subarc \( EF \)). We will compare the polygonal arc \( XABCDEYF \) with a shorter polygonal arc that satisfies the previous assumption. First let \( \bar{X} \) be the point on the left of \( A \) with \( \bar{X}A =XA \) (see Figure 6). Since \( A \) is on the left of \( V \), the center of the circular arc, \( \bar{X} \) is also on the left of the circular arc. Thus \( \bar{X}V > 1 \). Now let \( \bar{A} = \bar{X} \). Thus the polygonal arc \( \bar{A}XBCDEYF \) is shorter than \( XABCDEYF \). By the first part of case 1, we have \( \bar{A}XBCDEYF > 1 \). When \( Y \) is on the subarc \( FQ \), the argument is similar. Hence \( \gamma \) is longer than 1.

For the remaining cases, with the same argument on \( \bar{X} \) and \( \bar{A} \), we may suppose that \( X \) is on the subarc \( AB \) and \( Y \) is on the subarc \( EF \). Then we will show that the polysegment \( AXBCDEYF \) is longer than 1.

Case 2. Let \( \alpha = \angle AVU \) and \( \beta = \angle FUV \) (see Figure 7). Possibly after a rotation, we may assume \( \alpha \geq \beta \). Note that the product of 2 reflections in the sides of a 30° angle is a 60° rotation about their intersection. We create new points and segments as follows. First let \( C'D'E'Y'F' \) be the reflection of \( CDEYF \) across \( UB \). Next let \( E''Y''F'' \) be the reflection of \( E'Y'F' \) across \( UD' \). Now let \( V' \) be the rotation of \( V \) for 60° around \( U \). Then let \( Y'''F''' \) be
Fig. 5 Case 1.

Fig. 6 $X$ is on the subarc $PA$. ($\bar{X}A =XA$)

Fig. 7 Case 2
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Fig. 8 Case 3

the reflection of $Y''F''$ across $V'E''$ and $Y^*$ be the reflection of $Y'''$ across $V'F''$. Note that $V$ is also the rotation of $U$ for 60° around $V'$.

Since $\angle F''V'V = \angle FVU$, it is in $(0, 30^\circ)$. We then have $\angle Y'''V'V = \angle Y'''V'F'' + \angle F''V'V > \angle Y'''V'F'' - \angle F''V'V = \angle Y^*V'F'' - \angle Y'''V'V = \angle Y^*V'V$. Hence $V$ and $Y^*$ are on the same side of $V'F''$. Thus $Y'''V > Y^*V = YU > 1$. Together with $XY > 1$ and $\angle F'''VV' = \angle FUV = \beta$. By Lemma 2, the length of $AXBC'DE''Y'''F'''$ is at least $AF'''$ which is longer than a chord of a unit sector around $V$ of angle at least $60^\circ$. Therefore the length of $AXBCDEYF$ is greater than 1.

Case 3. We create new points and segments as follows (see Figure 8). First let $X'B'$ be the reflection of $XB$ across $UA$ and $A'X''$ be the reflection of $AX$ across $UB'$. Next let $V'$ be the rotation of $V$ around $U$ for $-60^\circ$. Let $E'Y'$ be the reflection of $EY$ across $VF$ and $V'''F''$ be the reflection of $V'F$ across $V'E'$. Note that $V'$ is also the rotation of $U$ around $V$ for $60^\circ$.

Note that $X''V' = XV > 1$ and $Y'''V' = YU > 1$. By Lemma 2, the length of $A'X''B'CDE'Y'''F'''$ is at least $AF''$ which is longer than a chord of a unit sector around $V'$ of angle at least $60^\circ$. Therefore the length of $AXBCDEYF$ is greater than 1.

Case 4. Similar to case 2, let $\alpha = \angle AVU$ and $\beta = \angle FUV$ (see Figure 9). Possibly after a rotation, we may assume $\alpha \geq \beta$. We create new points and segments as follows. First let $C'D'E'Y'F'$ be the reflection of $CDEYF$ across $UB$. Next let $D''E''Y'''F''$ be the reflection of $D'E'Y'F'$ across $UC'$. Now let
$V'$ be the rotation of $V$ around $U$ for 60°. Then let $Y'''F'''$ be the reflection of $Y''E''$ across $V'E''$ and $Y^*$ be the reflection of $Y'''$ across $V'F'''$. Note that $V$ is also the rotation of $U$ around $V'$ for 60°.

Since $\angle Y'''V'V > \angle Y^*V'V$, we have $Y'''V > Y^*V = YU > 1$. Together with $XV > 1$ and $\angle F'''V'V = \angle FUV = \beta$, by Lemma 2 the length of $AXBC'D''E''Y'''F'''$ is at least $AF'''$ which is longer than a chord of a unit sector around $V$ of angle $60° + \alpha - \beta \geq 60°$. Therefore the length of $AXBCDEYF$ is greater than 1.

In each case, we arrive at a contradiction. Therefore $\Pi$ can cover every unit arc. \qed

Now we provide the proof of Lemma 1.

Proof We first consider the angle of the convex hull of $\gamma$ at $X_1$. If the angle is not bigger than $\theta$, then we can place $\gamma$ within it so that $X_1$ touches the angle’s vertex. So we suppose the angle is bigger than $\theta$. Note that for each $i$, the segment $X_iX_{i+1}$ is a part of the boundary of the hull or cross segment (inside the interior of) the hull. Without the interior of all cross segments, the remaining of the polygonal arc $X_1X_2 \ldots X_n$ is composed of connected, external pieces on the boundary. Note that an external piece could possibly be just a single point. However the beginning and ending pieces are not a single point. Next we place 2 rays $u$ and $v$ starting at $X_1$ and on the 2 sides of the hull as in Figure 10(a). As $X_1$ is the first point on the boundary, we may assume that $v$ touches $X_2$ and $u$ touches 2 external pieces or 2 ends of the single external
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Fig. 10 (a) The 2 rays $u$ and $v$ start at $X_1$. (b) The ray $v$ rotates around $\gamma$. (c) The ray $v$ touches 2 external pieces. (d) The ray $u$ rotates around $\gamma$. (e) The ray $u$ touches 2 external pieces.

piece. We now start to rotate $v$ around and keep touching $\gamma$ as in Figure 10(b). We continue to rotate until $v$ touches 2 external pieces as in Figure 10(c). Next we start to rotate $u$ around $\gamma$ as in Figure 10(d). We continue to rotate until $u$ touches 2 external pieces as in Figure 10(e). Along this rotation, we keep the property of having 3 points alternating between $u$ and $v$ and reduce the angle between $u$ and $v$. The process stops when the angle between $u$ and $v$ is $\theta$.

For result 2), we work in the same fashion at $X_n$ and get another pair of support lines with alternating 3 points. Since $\theta > 0$, if there are 2 pairs of support lines, they are different as the 2 orientated angles between $u$’s and $v$’s are different. In the other way, we may think that we continue moving $u$ and $v$ starting from $X_1$ and decreasing the angle in between until $u$ and $v$ meet at $X_n$. When the signed angle in between is $-\theta$, we have a pair of support line for 2). We have that the number of external pieces is greater than the number of cross segments by 1 and each external piece alternately contribute to opposite side of the hull with respect to $X_1$ and $X_n$. Hence the numbers of external pieces on both sides differ by at most 1. Thus the process must stop at $X_n$ and if there are 2 such pairs of support lines, they must interlace as in Figure 3 or Figure 4.

For a pair of support lines with this property, one line must touch 2 consecutive external pieces while the other line touches the external piece ordered in between the 2 previous and opposite pieces. Hence the process has found all possible such pairs. This completes the proof of the lemma. □
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References

1. Alexander, J.R., Wetzel, J.E., and Wichiramala, W.: The A-property of a simple arc. (2014). (Unpublished)
2. Coulton, P., Movshovich, Y.: Besicovitch triangles cover unit arcs. Geom. Dedicata. 123, 79–88 (2006), doi: 10.1007/s10711-006-9107-7.
3. Moser, L.: Poorly formulated unsolved problems in combinatorial geometry. (1966) mimeographed.
4. Moser, W.O.: Problems, problems, problems. Discrete Appl. Math. 31(2), 201–225 (1991), doi: 10.1016/0166-218X(91)90071-4.
5. Movshovich, Y.: A-configurations and embeddings, submitted.
6. Movshovich, Y., Wetzel, J.E.: Drapeable unit arcs fit in the unit 30° sector. to appear in Adv. Geom, doi: 10.1515/advgem-2017-0011.
7. Norwood, R., Poole, G., Laidacker, M.: The worm problem of Leo Moser. Discrete Comput. Geom. 7(2), 153–162 (1992), doi: 10.1007/BF02187832.
8. Wang, W.: An improved upper bound for worm problem. Acta Math. Sin., Chin. Ser. 49(4), 835–846 (2006), doi: cnki:ISSN:0583-1431.0.2006-04-013.
9. Wetzel, J.E.: Fits and Covers. Math. Mag. 76(3), 349–363 (2003)
10. Wetzel, J.E.: Sectorial covers for curves of constant length. Can. Math. Bull. 16, 367-375 (1973), doi: 10.4153/CMB-1973-058-8.
11. Wetzel, J.E.: Bounds for covers of unit arcs. Geombinatorics. XXII(3), 116-122 (2013)
12. Wetzel, J.E., Wichiramala, W.: A covering theorem for families of sets in Rd. J. Comb. I(1), 69-75 (2010), doi: 10.4310/JOC.2010.v1.n1.a5.
13. Wetzel, J.E., Wichiramala, W.: Sectorial covers for unit arcs. to appear in Math. Mag, Mathematica Magazine 92(1), 42-46 (2019), doi: 10.1080/0025570X.2019.1523648.
14. Wichiramala, W.: How support lines touch an arc. (2013). (Unpublished)