On the construction of $A_\infty$-structures

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To Tornike Kadeishvili

Abstract

We relate a construction of Kadeishvili’s establishing an $A_\infty$-structure on the homology of a differential graded algebra or more generally of an $A_\infty$ algebra with certain constructions of Chen and Gugenheim. Thereafter we establish the links of these constructions with subsequent developments.

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1 Introduction

We will relate a construction of Kadeishvili’s establishing an $A_\infty$-structure on the homology of a differential graded algebra or more generally of an $A_\infty$ algebra with certain constructions of Chen and Gugenheim. We will then establish the links of these constructions with subsequent developments.

Let $R$ be a commutative ring and $A$ a differential graded algebra over $R$. Suppose that, as a graded $R$-module, the homology $H(A)$ of $A$ is free. Then $H(A)$ acquires an $A_\infty$-algebra structure that is equivalent to $A$. Over a field, so that the freeness hypothesis relative to $H(A)$ is automatically satisfied, this fact is nowadays quoted as the “minimality theorem” for differential graded algebras— we will discuss the issue of “minimality theorem” below. Such a result was published by Kadeishvili in 1980 [41]. Over a general ground ring $R$, a related result involving HPT was published by V. Gugenheim in 1982 [13]. More precisely, starting from a simply connected coaugmented differential graded coalgebra $C$ over the ground ring $R$ that is homology split (e. g. free as a module over the ground ring), the homology $H(C)$ acquires a coaugmented graded coalgebra structure, and a perturbation of the ordinary cobar construction relative to the coalgebra structure on $H(C)$ yields an $A_\infty$-coalgebra structure on $H(C)$ that is equivalent to the original coalgebra $C$. This is a version of the “minimality theorem” in the realm of coalgebras. Gugenheim’s approach relies on a perturbation argument developed over the reals by Chen [7], published in 1977 and, furthermore, in a sense, Theorem 3.1.1 in Chen’s paper [6] establishes a version of the “minimality theorem”. In 1982, Kadeishvili published a result which extends his original approach to a more general “minimality theorem” saying that, over a field, the homology $H(A)$ of a general $A_\infty$-algebra $A$ acquires an $A_\infty$-algebra structure that is equivalent to $A$ [42].

Because of the present renewed interest in the “minimality theorem”, and to help the presently young avoid loss of contact with the past, it seems worthwhile explaining the original insight into the “minimality theorem”. This will place the original results properly in the literature. The contributions of Gugenheim and in particular those of Chen seem to have been largely forgotten. Also there has been a debate in the literature
to what extent the various constructions of $A_\infty$-structures were explicit; the constructions by Chen, Gugenheim and Kadeishvili are perfectly explicit. In fact, we hope to convince the reader that these constructions essentially boil down to the very same basic idea. We will then relate the old approaches to subsequent ones and show that the recent ones \[48\] (6.4), \[50\] essentially still come down to the same basic idea. In particular we will illustrate how the constructions in terms of labelled oriented rooted trees \[48\] (6.4) are instances of ordinary HPT constructions. This will, perhaps, demystify the labelled oriented rooted trees method and make it accessible to a wider audience. We will also explain the Lie algebra case. It turns out that the various constructions establishing the corresponding statement of the “minimality theorem” in the algebra, coalgebra, Lie algebra situation, etc. all boil down to essentially the same kind of construction, as comparison of (6.1)–(6.7), (7.1)–(7.7), and (12.1)–(12.7) below shows.

It is a pleasure to dedicate this paper to Tornike Kadeishvili. I am indebted to him for collaboration and for discussion much beyond that collaboration. Our collaboration \[37\] led to a number of results related with the perturbation lemma; in particular we have elaborated on the compatibility of the perturbation lemma with suitable algebraic structure. The perturbation lemma is lurking behind the formulas in Chapter II of Section 1 of \[59\] and seems to have first been made explicit by M. Barrat (unpublished). The first instance known to us where it appeared in print is \[3\]. By means of that lemma, in \[12\], V. Gugenheim developed a lucid proof of the twisted Eilenberg-Zilber theorem which, in turn, was established by E. H. Brown \[2\] originally via acyclic models. We have already mentioned Chen’s construction of a perturbation given in \[6\], extended and clarified by V. Gugenheim in \[13\]. These constructions of Chen’s and Gugenheim’s are somewhat by hand and do not involve the perturbation lemma. In \[19\], Gugenheim and Stasheff extended that construction of a perturbation to the case where the contracted object is admitted to have non-zero differential. In \[24\], I had developed the tensor trick, see Section 9 below, and the idea of iterative perturbation. My collaboration with T. Kadeishvili involved the tensor trick and iterative perturbations and produced in particular the (co)algebra perturbation lemma. This lemma then enabled us to recover the perturbations of the kind constructed by Chen, Gugenheim, and Gugenheim-Stasheff in a conceptual way. It led as well to a lucid proof of the minimality theorem. We also developed a perturbation theory for a general homotopy equivalence, not necessarily a contraction. It is, furthermore, worthwhile noting that the labelled rooted trees are lurking behind the (co)algebra perturbation lemma but at the time there was no need to spell them out explicitly in \[37\]. We will explain all these and more issues in this paper. At these days the perturbation lemma and variants thereof are, perhaps, more vivid than ever, see e.g. \[1\], \[31\], \[34\], \[49\] and the references there; in particular, the perturbation theory for a general homotopy equivalence developed in \[37\] has been taken up again and pushed further in \[49\].

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2 Preliminaries

The ground ring is a commutative ring with 1 and will be denoted by $R$. Later some condition has, perhaps, to be imposed upon $R$ so that the symmetric coalgebra on the $R$-module under discussion exists but $R$ is not necessarily a field.

Indeed, to avoid confusion, recall that, given the graded $R$-module $Y$, for $j \geq 0$, the notation $S'_j[Y] \subseteq T'_j[Y]$ refers to the submodule of invariants in the $j$'th tensor power $T'_j[Y]$ relative to the obvious action on $T'_j[Y]$ of the symmetric group $S_j$ on $j$ letters, and $S^c[Y]$ refers to the direct sum

$$S^c[Y] = \oplus_{j=0}^{\infty} S'_j[Y]$$

of graded $R$-modules. Some hypothesis is, in general, necessary in order for the homogeneous constituents

$$T_{j+k}[Y] \to T_j[Y] \otimes T_k[Y] (j, k \geq 0)$$

of the diagonal map $\Delta : T^c[Y] \to T^c[Y] \otimes T^c[Y]$ of the graded tensor coalgebra $T^c[Y]$ to induce a graded diagonal map on $S^c[Y]$, so that $S^c[Y]$ is then the symmetric coalgebra on $Y$. See Section 3 of [31]. In particular, let $V$ be a projective graded $R$-module, concentrated in odd degrees, and consider the graded exterior algebra $\Lambda[V]$ on $V$. The diagonal map $V \to V \oplus V$ is well known to induce a diagonal map for $\Lambda[V]$ turning the latter into a graded Hopf algebra. We then denote the resulting graded coalgebra by $\Lambda'[V]$ and, as usual, refer to it as the exterior coalgebra. Whenever a graded exterior coalgebra of the kind $\Lambda'[V]$ is under discussion, we will suppose throughout that the resulting coalgebra is the graded symmetric coalgebra $S^c[V]$ on $V$, that is, that the canonical morphism of coalgebras from $\Lambda'[V]$ to $S^c[V]$ (induced by the canonical projection from $\Lambda'[V]$ to $V$) is an isomorphism of graded coalgebras. This excludes the prime 2 being a zero divisor in the ground ring $R$. In particular, a field of characteristic 2 is not admitted as ground ring. Indeed in characteristic 2 the entire theory requires special treatment.

We will take chain complex to mean differential graded $R$-module. A chain complex will not necessarily be concentrated in non-negative or non-positive degrees. The differential of a chain complex will always be supposed to be of degree $-1$. For a filtered chain complex $X$, a perturbation of the differential $d$ of $X$ is a (homogeneous) morphism $\partial$ of the same degree as $d$ such that $\partial$ lowers the filtration and $(d + \partial)^2 = 0$ or, equivalently,

$$[d, \partial] + \partial d = 0. \quad (2.1)$$

Thus, when $\partial$ is a perturbation on $X$, the sum $d + \partial$, referred to as the perturbed differential, endows $X$ with a new differential. When $X$ has a graded coalgebra structure such that $(X, d)$ is a differential graded coalgebra, and when the perturbed differential $d + \partial$ is compatible with the graded coalgebra structure, we refer to $\partial$ as a coalgebra perturbation; the notion of algebra perturbation is defined similarly. Given a differential graded coalgebra $C$ and a coalgebra perturbation $\partial$ of the differential $d$ on $C$, we will occasionally denote the new or perturbed differential graded coalgebra by $C_\partial$. Likewise given a differential graded algebra $A$ and an algebra perturbation $\partial$ of the differential $d$ on $A$, we will occasionally denote the new or perturbed differential graded algebra by $A_\partial$.

A contraction

$$\left( N \xrightarrow{\nabla} \pi M, h \right) \quad (2.2)$$
of chain complexes \([\mathcal{C}]\) consists of
- chain complexes \(N\) and \(M\),
- chain maps \(\pi: N \to M\) and \(\nabla: M \to N\),
- a morphism \(h: N \to N\) of the underlying graded modules of degree 1;

these data are required to satisfy

\[
\begin{align*}
\pi \nabla &= \text{Id}, \\
Dh &= \text{Id} - \nabla \pi, \\
\pi h &= 0, \quad h \nabla &= 0, \quad hh &= 0.
\end{align*}
\]

The requirements \((2.5)\) are referred to as \textit{annihilation properties} or \textit{side conditions}.

Let \(C\) be a \textit{coaugmented} differential graded coalgebra with coaugmentation map \(\eta: R \to C\) and \textit{coaugmentation} coideal \(JC = \text{coker}(\eta)\), the diagonal map being written as \(\Delta: C \to C \otimes C\) as usual. Recall that the counit \(\varepsilon: C \to R\) and the coaugmentation map determine a direct sum decomposition \(C = R \oplus JC\). The \textit{coaugmentation} filtration \(\{F_nC\}_{n \geq 0}\) is as usual given by

\[
F_nC = \ker(C \longrightarrow (JC)^{\otimes (n+1)}) \quad (n \geq 0)
\]

where the unlabelled arrow is induced by some iterate of the diagonal \(\Delta\) of \(C\). This filtration is well known to turn \(C\) into a \textit{filtered} coaugmented differential graded coalgebra; thus, in particular, \(F_0C = R\). We recall that \(C\) is said to be \textit{cocomplete} when \(C = \bigcup F_nC\).

Write \(s\) for the \textit{suspension} operator as usual and accordingly \(s^{-1}\) for the \textit{desuspension} operator. Thus, given the chain complex \(X\), \((sX)_j = X_{j-1}\), etc., and the differential \(d: sX \to sX\) on the suspended object \(sX\) is defined in the standard manner so that \(ds + sd = 0\).

Given two chain complexes \(X\) and \(Y\), recall that \(\text{Hom}(X, Y)\) inherits the structure of a chain complex by the operator \(D\) defined by

\[
D\phi = d\phi - (-1)^{\mid \phi\mid} \phi d
\]

where \(\phi\) is a homogeneous homomorphism from \(X\) to \(Y\) and where \(\mid \phi\mid\) refers to the degree of \(\phi\).

Let \(g\) be a chain complex having the property that the cofree coaugmented differential graded cocommutative coalgebra \(S^c[s\mathfrak{g}]\) on the suspension \(s\mathfrak{g}\) of \(\mathfrak{g}\) exists. This happens to be the case, e.g., when \(\mathfrak{g}\) is projective as a graded \(R\)-module. Let

\[
\tau_\mathfrak{g}: S^c[s\mathfrak{g}] \longrightarrow \mathfrak{g}
\]

be the composite of the canonical projection to \(S^1[s\mathfrak{g}] = s\mathfrak{g}\) with the desuspension map. Suppose that \(\mathfrak{g}\) is endowed with a graded skew-symmetric bracket \([\cdot, \cdot]\) that is compatible with the differential but not necessarily a graded Lie bracket, i.e. does not necessarily satisfy the graded Jacobi identity. Let \(C\) be a coaugmented differential graded cocommutative coalgebra. Given homogeneous morphisms \(a, b: C \to \mathfrak{g}\), with a slight abuse of the bracket notation \([\cdot, \cdot]\), the \textit{cup bracket} \([a, b]\) is given by the composite

\[
C \xrightarrow{\Delta} C \otimes C \xrightarrow{a \otimes b} \mathfrak{g} \otimes \mathfrak{g} \xrightarrow{[\cdot, \cdot]} \mathfrak{g}.
\]
The cup bracket \([\cdot, \cdot]\) is well known to be a graded skew-symmetric bracket on \(\text{Hom}(C, \mathfrak{g})\) which is compatible with the differential on \(\text{Hom}(C, \mathfrak{g})\). Define the coderivation

\[
\partial : S^c[\mathfrak{s}\mathfrak{g}] \longrightarrow S^c[\mathfrak{s}\mathfrak{g}]
\]
on \(S^c[\mathfrak{s}\mathfrak{g}]\) by the requirement

\[
\tau_\mathfrak{g}\partial = \frac{1}{2}[\tau_\mathfrak{g}, \tau_\mathfrak{g}] : S^2[\mathfrak{s}\mathfrak{g}] \rightarrow \mathfrak{g}.
\] (2.7)

Then \(D\partial = d\partial + \partial d = 0\) since the bracket on \(\mathfrak{g}\) is supposed to be compatible with the differential \(d\). Moreover, the bracket on \(\mathfrak{g}\) satisfies the graded Jacobi identity if and only if \(\partial\partial = 0\), that is, if and only if \(\partial\) is a coalgebra perturbation of the differential \(d\) on \(S^c[\mathfrak{s}\mathfrak{g}]\), cf. e. g. [38].

We now suppose that the graded bracket \([\cdot, \cdot]\) on \(\mathfrak{g}\) turns \(\mathfrak{g}\) into a differential graded Lie algebra and continue to denote the resulting coalgebra perturbation by \(\partial\), so that \(S^0[\mathfrak{s}\mathfrak{g}]\) is a coaugmented differential graded cocommutative coalgebra; in fact, \(S^0[\mathfrak{s}\mathfrak{g}]\) is then precisely the ordinary \(C(\text{ARTAN-})C(\text{HEVALLEY-})E(\text{ILENBERG})\) or classifying coalgebra for \(\mathfrak{g}\) and, following [57] (p. 291), we denote it by \(C[\mathfrak{g}]\) (but the construction given above is different from that in [57] which, in turn, is carried out only over a field of characteristic zero). Furthermore, given a coaugmented differential graded cocommutative coalgebra \(C\), the cup bracket turns \(\text{Hom}(C, \mathfrak{g})\) into a differential graded Lie algebra. In particular, \(\text{Hom}(S^c, \mathfrak{g})\) and \(\text{Hom}(F_nS^c, \mathfrak{g}) (n \geq 0)\) acquire differential graded Lie algebra structures.

Given a coaugmented differential graded cocommutative coalgebra \(C\) and a differential graded Lie algebra \(\mathfrak{h}\), a Lie algebra twisting cochain \(t : C \rightarrow \mathfrak{h}\) is a homogeneous morphism of degree \(-1\) whose composite with the coaugmentation map is zero and which satisfies

\[
Dt = \frac{1}{2}[t, t],
\] (2.8)
cf. [53], [57]. In particular, relative to the graded Lie bracket \(\mathcal{D}\) on \(\mathfrak{g}\), the morphism \(\tau_\mathfrak{g} : C[\mathfrak{g}] \rightarrow \mathfrak{g}\) is a Lie algebra twisting cochain, the \(C(\text{ARTAN-})C(\text{HEVALLEY-})E(\text{ILENBERG})\) or universal Lie algebra twisting cochain for \(\mathfrak{g}\). It is, perhaps, worth noting that, when \(\mathfrak{g}\) is viewed as an abelian differential graded Lie algebra relative to the zero bracket, \(S^c[\mathfrak{s}\mathfrak{g}]\) is the corresponding CCE or classifying coalgebra and \(\tau_\mathfrak{g} : S^c[\mathfrak{s}\mathfrak{g}] \rightarrow \mathfrak{g}\) is still the universal Lie algebra twisting cochain.

**Remark 2.1.** The terms master equation, Maurer-Cartan equation, Lie algebra twisting cochain, and integrability condition all refer to the same mathematical object. Historically the Maurer-Cartan equation came first.

At the risk of making a mountain out of a molehill, we note that, in (2.7) and (2.8) above, the factor \(\frac{1}{2}\) is a mere matter of convenience. The correct way of phrasing graded Lie algebras when the prime 2 is not invertible in the ground ring is in terms of an additional operation, the *squaring* operation \(\text{Sq} : \mathfrak{g}_{\text{odd}} \rightarrow \mathfrak{g}_{\text{even}}\) and, by means of this operation, the factor \(\frac{1}{2}\) can be avoided. Indeed, in terms of this operation, the equation (2.8) takes the form

\[Dt = \text{Sq}(t)\].
For intelligibility, we will follow the standard convention, avoid spelling out the squaring operation explicitly, and keep the factor $\frac{1}{2}$. A detailed description of the requisite modifications when the prime 2 is not invertible in the ground ring is given in [34].

Finally we comment on the usage of the terminology “minimal”: Given an augmented differential graded algebra $A$ over a field $k$, with augmentation map $\varepsilon: A \to k$ and augmentation ideal $IA$, its graded vector space of indecomposables $Q(A)$ is the cokernel of the canonical map $IA \otimes IA \to IA$ induced by the multiplication map of $A$; since $A$ is a differential graded algebra, this cokernel inherits a differential, and the augmented differential graded algebra $A$ is said to be minimal when it is cofibrant and when the differential on the indecomposables $Q(A)$ is zero. Any connected differential graded algebra has a canonical augmentation. In rational homotopy theory, a minimal model of a connected differential graded commutative algebra $A$ is a minimal differential graded commutative algebra $mA$ together with a morphism $mA \to A$ of differential graded algebras which is an isomorphism on homology. Likewise, the differential graded Lie algebra $L$ is said to be minimal when it is cofibrant and when the differential on the abelianization $Q(L)$ is zero. A minimal model of a connected differential graded Lie algebra $L$ is a minimal differential graded Lie algebra $mL$ together with a morphism $mL \to L$ of differential graded Lie algebras which is an isomorphism on homology. There are also corresponding notions of minimal differential graded coalgebra and of minimal model for a differential graded coalgebra. See e. g. [55] (Section 5) for details and more references. Over a local ring $R$, with maximal ideal $m \subseteq R$, a free resolution

$$
0 \leftarrow M \xleftarrow{\varepsilon} F_0 \xleftarrow{d} F_1 \xleftarrow{d} \cdots
$$

is said to be minimal when $d(F_j) \subseteq mF_{j-1}$ for $j \geq 1$. The meaning of the term “minimal” in the present paper refers to $A_\infty$-algebras, see Section 3 below. While in rational homotopy theory, a homology algebra is not necessarily a Sullivan algebra, suitably interpreted, the notion of minimality of $A_\infty$-algebras is consistent with the usage of the concept of minimality in rational homotopy theory. See Remarks 4.2 and 5.2 below.

3 $A_\infty$-algebras

To introduce language and notation we reproduce a precise definition of an $A_\infty$-algebra and of a morphism of $A_\infty$-algebras, cf. [60]. Our convention is that a differential lowers degree by 1.

An $A_\infty$-algebra over the ground ring $R$ is a graded $R$-module $A$ equipped with a family $\{m_n\}_{n=1}^\infty$ of $R$-multilinear maps

$$m_n: A^\otimes n \to A$$

of degree $n - 2$ that satisfy the identities

$$
\sum_{r+s+t=n} (-1)^{r+s+t} m_{r+1+t}(\text{Id}^r \otimes m_s \otimes \text{Id}^t) = 0
$$

(3.1)

A morphism $f: A \to B$ of $A_\infty$-algebras is a family $\{f_n\}_{n=1}^\infty$ of $R$-multilinear maps

$$f_n: A^\otimes n \to B$$
of degree $n-1$ that satisfy the identities

$$
\sum_{r+s+t=n} (-1)^{r+st} f_{r+1+t}(\text{Id}^r \otimes m_s \otimes \text{Id}^t) = \sum (-1)^w m(q_{i_1} \otimes \cdots \otimes f_{i_q})
$$

(3.2)

where $i_1 + \cdots + i_q = n$ and $w = (q-1)(i_1-1) + \cdots + 2(i_{q-2}-1) + (i_{q-1}-1)$.

We now reproduce the familiar equivalent description of an $A_\infty$-algebra structure as a coalgebra perturbation. To save trouble, we will do this only for the supplemented case. Thus let $M$ be a graded $R$-module which comes with a direct sum decomposition $M = IM \oplus R$ of graded $R$-modules. We view $M$ as a graded algebra with zero multiplication on $IM$, so that $IM$ can then be interpreted as the augmentation ideal of $M$. The graded tensor coalgebra $T \circ [sIM]$ (with zero differential) is then the corresponding reduced bar construction for $M$; let $\tau_M : T \circ [sIM] \to M$ be the universal bar construction twisting cochain, that is, the canonical projection to $sIM$, followed by the desuspension mapping.

For $j \geq 1$, let

$$
m_j : (IM)^{\otimes j} \to IM
$$

be a homogeneous degree $j-2$ operation and define the coderivation

$$
D^{j-1} : T \circ [sIM] \to T \circ [sIM]
$$

by the identity

$$
m_j = s^{-1} \circ D^{j-1} \circ s^{\otimes j} : (IM)^{\otimes j} \to IM.
$$

For convenience, we write $d = D^0$. Then

$$
D = D^1 + D^2 + \ldots : T \circ [sIM] \to T \circ [sIM]
$$

is a coderivation, and so is the sum

$$
d + D = d + D^1 + D^2 + \ldots : T \circ [sIM] \to T \circ [sIM].
$$

**Proposition 3.1.** (i) The family $\{m_j\}_j$ turns $M$ into an (augmented) $A_\infty$-algebra if and only if $d + D$ is a differential, that is, if and only if

$$
dD + Dd + DD = 0.
$$

(ii) Given the augmented $A_\infty$-algebras $A$ and $B$, a family $\{f_j\}_j$ of $R$-multilinear maps

$$
f_j : (IA)^{\otimes n} \to IB
$$

of degree $j-1$ is a morphism of (augmented) $A_\infty$-algebras if and only if the constituents of the family combine to a morphism

$$
(T \circ [sIA], d + D) \to (T \circ [sIB], d + D)
$$

of differential graded coalgebras.
Thus, when \( \{m_j\}_j \) turns \( M \) into an (augmented) \( A_\infty \)-algebra, in particular, \( m_1 \) is a differential on \( M \) and on \( IM \), and \( d \) is a differential on \( T^c[\text{sIM}] \), in fact, the differential induced by that on \( IM \). The special case where only \( m_1 \) and \( m_2 \) are non-zero is that of an ordinary differential graded algebra structure, and \( (T^c[\text{sIM}], d + D) \) is then the ordinary reduced bar construction \( BM \).

Kadeishvili introduced the terminology minimal for an \( A_\infty \)-algebra \( (M, \{m_i\}) \) having \( m_1 \) zero, i.e. trivial differential. Minimal \( A_\infty \)-algebras behave similar to Sullivan’s minimal differential graded algebras: each weak equivalence of minimal \( A_\infty \)-algebras is an isomorphism. See also Remark 5.2 below.

We will henceforth consider \( IM \) as a chain complex, use the notation \( B^D M = (T^c[\text{sIM}], d + D) \), \( (3.3) \) and occasionally refer to \( B^D M \) as the standard construction. It is also customary to write \( \tilde{BM} \) and to refer to the bar tilde construction. Apart from Section 4 below, we will henceforth exclusively use the description of an \( A_\infty \)-algebra structures on the chain complex \( M \) in terms of the coalgebra perturbation \( D \) on the associated differential graded coalgebra \( (T^c[\text{sIM}], d) \). Likewise we will exclusively use the description of an \( A_\infty \)-coalgebra structure in terms of the corresponding algebra perturbation on the associated differential graded algebra and we will use the description of an \( L_\infty \)-algebra structure merely in terms of the corresponding coalgebra perturbation on the associated differential graded coalgebra.

### 4 Kadeishvili’s minimality theorem for algebras

In [40], Kadeishvili studied the homology of a fiber bundle with structure group \( G \) and fiber \( F \). He noticed that the Pontrjagin ring structure of the homology \( H_*(G) \) and the action of \( H_*(G) \) on \( H_*(F) \) in general fail to recover the geometry of the action. To fix this failure, he then constructed certain higher operations

\[
f^i : H_*(G) \otimes \ldots (i \text{ times}) \ldots \otimes H_*(G) \to H_*(G), \quad i = 3, 4, \ldots,
\]

and homomorphisms

\[
A^i : H_*(G) \otimes \ldots (i \text{ times}) \ldots \otimes H_*(G) \to C_*(G), \quad i = 1, 2, 3, 4, \ldots.
\]

which are solutions of certain equations. For example, \( f = f^3 + f^4 + \ldots \), interpreted as a Hochschild cochain in \( C^*(H^*(G), H^*(G)) \), satisfies the condition

\[
\delta f = f \cup_1 f
\]

where the operation

\[
\cup_1 : (a, b) \mapsto a \cup_1 b
\]

refers to the operation in the Hochschild (cochain) complex introduced by Gerstenhaber. Kadeishvili referred to this operation as a “cup-one” product since it has the same properties as Steenrod’s \( \cup_1 \)-product, and he called such an \( f \) Hochschild twisting cochain (page 3 of [40]).
With hindsight we see that Kadeishvili’s construction just explained yields an $A_\infty$-algebra structure $(H_\ast(G), \{f^i\})$ and a morphism (weak equivalence) of $A_\infty$-algebras $\{A\} : (H_\ast(G), \{f^i\}) \to C_\ast(G)$. While, at the time of writing [40], Kadeishvili did not know about Stasheff’s notion of $A_\infty$-algebra he realized thereafter that the condition $\delta f = f \cup_1 f$ is exactly Stasheff’s defining condition for an $A_\infty$-algebra $(M, \{m_i\})$ with $m_1 = 0$. This led to the paper [41]. Here is the main result thereof, valid for a general differential graded algebra, not necessarily of the kind $C_\ast(G)$ for a group $G$.

**Theorem 4.1** (Minimality theorem). Let $A$ be a differential graded algebra over a field. There is an $A_\infty$-algebra structure on $H(A)$ and an $A_\infty$-algebra quasi-isomorphism $f : H(A) \to A$ such that $f_1$ is a cycle-choosing quasi-isomorphism of chain complexes where the differential $m_1$ on $H(A)$ is zero and $m_2$ is a strictly associative multiplication induced by the multiplication in $A$. The resulting structure is unique up to quasi-isomorphism. When $A$ has a unit, then the structure and quasi-isomorphism can be chosen to be strictly unital.

In particular, the $A_\infty$-algebra structure on homology resulting from the minimality theorem is minimal and this structure, which is of course not uniquely determined, is unique up to isomorphism in the category of $A_\infty$-algebras.

To compare the original approach with other developments, we partly reproduce the proof in [41].

**Proof.** Since $A$ is a differential graded algebra, its associated $A_\infty$-structure is encapsulated in the operations $m_1$ and $m_2$, the higher operations being zero. We shall denote $m_1$ by $d$ and refer to $m_2$ by the notation $\cdot$ or simply by juxtaposition.

To start an inductive construction of an $A_\infty$-structure on $H(A)$, we pick $m_1 = 0$ and take $m_2$ to be the induced strictly associative multiplication on $H(A)$. Furthermore, we take $f_1$ to be some linear map $H(A) \to A$ that picks a cycle in each homology class.

Given $a_1, a_2 \in A$, let

$$\Psi_2(a_1, a_2) = f_1(a_1a_2) - f_1(a_1)f_1(a_2).$$

This yields a boundary, since $f_1(a_1a_2)$ is defined to be a representative cycle of the homology class containing $f_1(a_1)f_1(a_2)$. Hence, $\Psi_2(a_1, a_2) = dw$ for some $w$. Abstracting from the particular elements $a_1$ and $a_2$, since we are over a field, we find a morphism $f_2$ such that $df_2 = \Psi_2$.

Now, let $n > 2$. Given $a_1, \ldots, a_n \in A$, let

$$\Psi_n(a_1, \ldots, a_n) = \sum_{s=1}^{n-1} (-1)^{\varepsilon_1(a_1, \ldots, a_n, s)} f_s(a_1, \ldots, a_s) \cdot f_{n-s}(a_{s+1}, \ldots, a_n) +
\sum_{j=2}^{n-1} \sum_{k=0}^{n-j} (-1)^{\varepsilon_2(a_1, \ldots, a_n, k, j)} f_{n-j+1}(a_1, \ldots, a_k, m_j(a_{k+1}, \ldots, a_{k+j}), \ldots, a_n)$$

where the expressions

$$\varepsilon_1(a_1, \ldots, a_n, s) = s + (n-s+1)(|a_1| + \cdots + |a_s|)$$
$$\varepsilon_2(a_1, \ldots, a_n, k, j) = k + j(n-k-j + |a_1| + \cdots + |a_k|)$$
are the signs in (3.2), adjusted according to the Eilenberg-Koszul convention. The term $\Psi_n$ arises from the identity (3.2), with the two terms $f_1 m_n$ and $m_1 f_n$ removed. To complete the inductive step we must exhibit suitable terms $m_n$ and $f_n$ in such a way that the identity (3.2) holds.

Tedious but straightforward calculation shows that the element $\Psi_n(a_1, \ldots, a_n)$ is a $d$-cycle, and we take

$$m_n(a_1, \ldots, a_n) = [\Psi_n(a_1, \ldots, a_n)] \in H(A).$$

This yields the operation $m_n$ on $H(A)$. Since now $f_1(m_n(a_1, \ldots, a_n))$ and $\Psi_n(a_1, \ldots, a_n)$ are in the same class, there is some $w \in A$ such that

$$f_1(m_n(a_1, \ldots, a_n)) - \Psi_n(a_1, \ldots, a_n) = dw.$$ 

Abstracting from the particular elements $a_1, \ldots, a_n$, since we are over a field, we find a morphism $f_n$ such that

$$df_n = \Psi_n.$$ 

Thus $m_n$ and $f_n$ match the definitions of the corresponding constituents of an $A_\infty$-algebra and of a morphism of $A_\infty$-algebras, respectively.

M. Vejdemo-Johansson has observed that this proof can be translated into an algorithm for the computation of the $A_\infty$-structure maps [62]. This justifies the claim made earlier that Kadeishvili’s construction can be made explicit (by means of a choice of contracting homotopy, see Remark 7.2 below).

Remark 4.2. Let $A$ be a connected minimal differential graded commutative algebra over the rationals. Theorem 4.1 applies to it and yields a “minimal” $A_\infty$-structure on $H(A)$, associated with $A$. However the two notions of minimality clearly differ unless $A$ is formal.

In 1982, Kadeishvili extended this construction and arrived at a more general “minimality theorem” saying that, over a field, the homology $H(A)$ of a general $A_\infty$-algebra $A$ acquires an $A_\infty$-algebra structure that is equivalent to $A$ [42].

Remark 4.3. Kadeishvili’s original problem, that is, that of constructing a small model for the chains on the total space of a fiber bundle, has received much attention in the literature, as has the problem, given a group $G$, to isolate, under suitable circumstances, a suitable structure on the homology $H(X)$ of a $G$-space $X$ such that $X$ and $H(X)$ are equivalent in the $A_\infty$-sense. In the de Rham setting, this problem was studied, e.g., in [11], where the corresponding $A_\infty$-structure on de Rham cohomology is encoded in terms of what are referred to there as cohomology operations. In [36], we have explained how equivariant de Rham theory can be subsumed under relative homological algebra. This includes an explanation of those $A_\infty$-structures on de Rham cohomology.

In the situation of the minimality theorem, Theorem 4.1 above, when the differential graded algebra $A$ is graded commutative, the resulting $A_\infty$-algebra structure on $H(A)$ has peculiar features encoded by Kadeishvili in the notion of $CA_\infty$-algebra, meaning that, in this case, the bar tilde construction is not only a differential graded coalgebra but also
an algebra with respect to the shuffle product, and the two structures combine to that of a differential graded bialgebra. Later this kind of structure has been christened \( C_\infty \)-structure. In the special situation where \( A \) is the algebra of rational cochains on a space \( X \), the resulting \( C_\infty \)-algebra structure on \( \text{H}^*(X,Q) \) determines the rational homotopy type of \( X \). Kadeishvili has worked this out in \([44]\); an extended version can be found in \([46]\).

The general situation is this: Given an augmented \( C_\infty \)-algebra \( A \), the structure being given in terms of its standard construction \( B_\partial A \), by the very definition of \( C_\infty \)-structure, the shuffle multiplication turns \( B_\partial A \) into a graded commutative differential graded Hopf algebra, the space of indecomposables relative to the algebra structure is a differential graded Lie coalgebra \( L \), in fact, a perturbation of the cofree differential graded Lie coalgebra \( L_\partial(sIA) \) cogenerated by \( sIA \), and the projection \( B_\partial A \rightarrow L_\partial \) actually spells out the differential graded coalgebra \( B_\partial A \) as the universal coalgebra \( U_\partial[L_\partial] \) cogenerated by \( L_\partial \), the space \( L_\partial \) necessarily being that of indecomposables relative to the graded commutative algebra structure. More formally, the standard construction for the augmented \( C_\infty \)-algebra \( A \) boils down a perturbation of the cofree differential graded Lie coalgebra \( L_\partial(sIA) \) cogenerated by \( sIA \). In other words, the \( C_\infty \)-structure on \( A \) is given by a perturbation of the differential on the cofree differential graded Lie coalgebra \( L_\partial(sIA) \) cogenerated by \( sIA \).

In the situation of Kadeishvili’s observation explained above, the differential graded Lie coalgebra \( L_\partial \) is the dual of the familiar minimal Lie algebra model in rational homotopy theory. Indeed, the structure dual to that of a \( C_\infty \)-structure has been explored in the literature in the context of rational homotopy theory. We will explain this in Remark 5.2 below.

Lie coalgebras are still not widely known objects. See e. g. \([51]\) for a thorough approach to ordinary (ungraded) Lie coalgebras.

5 The Chen perturbation

Let \( X \) be a (connected) smooth manifold, and let \( \mathcal{A}(X) \) be its ordinary de Rham algebra. Our convention is that \( \mathcal{A}(X) \) is graded by negative degrees, that is, \( \mathcal{A}(X)_j = \mathcal{A}^{-j}(X) \) for \( j \leq 0 \). Let \( V \) be a graded vector space which, to avoid unnecessary complications, we suppose to be of finite type (that is, finite-dimensional in each degree), and let \( \mathring{T}[V] \) denote the graded completion of the graded tensor algebra \( T[V] \) relative to the augmentation filtration. Let \( \mathring{T}_{\mathcal{A}(X)}[V] \) be the de algebra of formal power series in a basis of \( V \) with coefficients in the de Rham algebra \( \mathcal{A}(X) \) of \( X \). More formally, let \( V^* \) be the graded dual of \( V \)—still of finite type—, consider the graded tensor coalgebra \( T^c[V^*] \), and let

\[
\mathring{T}_{\mathcal{A}(X)}[V] = \text{Hom}(T^c[V^*], \mathcal{A}(X)).
\]

Thus, the algebra \( \mathring{T}_{\mathcal{A}(X)}[V] \) is the appropriate completion

\[
\mathcal{A}(X) \otimes T[V]
\]

of the tensor product \( \mathcal{A}(X) \otimes T[V] \). In his paper \([7]\), Chen referred to \( \mathring{T}_{\mathcal{A}(X)}[V] \) as the algebra of \( \mathring{T}[V] \)-valued differential forms on \( X \) and defined a \( \mathring{T}[V] \)-valued formal power series connection on \( X \) to be a degree \(-1\) element of \( \mathring{T}_{\mathcal{A}(X)}[V] \).
Let \( H \) be the de Rham cohomology of \( X \), and let

\[
\begin{array}{c}
\nabla \\
\pi \\
\end{array}
\Rightarrow
\begin{array}{c}
A(X) \\
h \end{array}
\]

be a contraction of chain complexes. Thus, any \( \alpha \in A(X) \) can be written as

\[
\alpha = (dh + h\pi + \nabla\pi)\alpha = dh\alpha + \nabla\pi\alpha + h\alpha
\]

in a unique fashion. The resulting decomposition

\[
A(X) = dA(X) \oplus \ker(h) = dA(X) \oplus H \oplus hA(X)
\]

where \( H = \nabla H(A(X)) \) plays the role of the Hodge decomposition in Kodaira-Spencer deformation theory. On p. 19 of [56], a decomposition of the kind (5.2) (not phrased in the language of contractions) is indeed referred to as a “Hodge decomposition”, and on p. 187 of [7] it is remarked that a Hodge decomposition of the de Rham complex of a Riemannian manifold is a special case of a decomposition of the kind (5.2), \( H \) being the space of harmonic forms.

Let \( \tilde{H} \) be the reduced real homology of \( X \), let \( V = s^{-1}\tilde{H} \), and consider the algebra \( \hat{T}_A(X)[s^{-1}\tilde{H}] \) of \( \hat{T}[s^{-1}\tilde{H}] \)-valued differential forms on \( X \). We now recall, in the language of the present paper, Chen’s Theorem 1.3.1 in [7].

**Theorem 5.1.** The contraction (5.1) determines a \( \hat{T}[s^{-1}\tilde{H}] \)-valued formal power series connection \( \tau \in \hat{T}_A(X)[s^{-1}\tilde{H}] \) and an algebra differential \( \partial \) on \( \hat{T}[s^{-1}\tilde{H}] \) such that, when \( d \) refers to the ordinary de Rham differential, relative to the total differential

\[
d \otimes \partial = d \otimes \text{Id} + \text{Id} \otimes \partial
\]

on \( \hat{T}_A(X)[s^{-1}\tilde{H}] = A(X) \otimes T[s^{-1}\tilde{H}] \), the following master equation is satisfied:

\[
d \otimes \tau = \tau \tau.
\] (5.3)

We will show below how this theorem is a consequence of a somewhat more general result. For intelligibility, at the present stage, we note the following: Suppose that \( X \) is simply connected. Then the algebra \( T[s^{-1}\tilde{H}] \) is already complete, that is, coincides with \( \hat{T}[s^{-1}\tilde{H}] \), and the algebra \( \hat{T}_A(X)[s^{-1}\tilde{H}] \) comes down to the ordinary tensor product \( A(X) \otimes T[s^{-1}\tilde{H}] \). The differential \( \partial \) can be written as an infinite series

\[
\partial = \partial^1 + \partial^2 + \ldots
\]

where \( \partial^1 \) is the ordinary cobar construction differential relative to the coalgebra structure on the real homology \( H_*(X) \) of \( X \), the resulting differential graded algebra \( (T[s^{-1}\tilde{H}], \partial) \) is then a kind of perturbed reduced cobar construction, and this perturbed reduced cobar construction is a model for the real chain algebra of the based loop space of \( X \); we therefore write this differential graded algebra as \( \Omega_{\partial}|H_*(X)| \). In particular, when the higher terms \( \partial^j \) for \( j \geq 2 \) are zero, the manifold \( X \) is formal over the reals.
Chen already established a version of the “minimality theorem”, though not in the language of $A_{\infty}$-structures. Indeed, Theorem 5.1 together with Theorem 3.1.1 in [6], include the statement that, for simply connected $X$, the real homology $H_*(X)$ of the manifold $X$ acquires an $A_{\infty}$-coalgebra structure such that the chains on $X$ and $H_*(X)$, endowed with the $A_{\infty}$-coalgebra structure, are equivalent. Dualized, this is precisely the statement of the “minimality theorem” over the reals.

**Remark 5.2.** Suitably interpreted, Chen’s construction makes perfect sense over the rationals. Let $X$ be a simply connected space and take $\mathcal{A}(X)$ to be the graded commutative algebra of rational forms on $X$. We continue to denote the resulting differential graded algebra, now over the rationals, $\mathbb{Q}$, by $\Omega_0[H_*(X)]$. The rational homology $H_*(X)$ acquires a graded cocommutative coalgebra structure whence the shuffle diagonal map turns $\Omega_0[H_*(X)]$ into a graded cocommutative Hopf algebra. Thus, by the Milnor-Moore theorem, $\Omega_0[H_*(X)]$ is the universal enveloping algebra $U[L]$ of a differential graded Lie algebra $L$. This differential graded Lie algebra $L$ is in fact the familiar minimal Lie algebra model for the based loop space on $X$ nowadays widely used in rational homotopy theory. This has been worked out in [20], [21], [61]; indeed, in the latter reference, the differential graded Lie algebra $L$, together with the formal power series connection, is referred to as the Chen model for (the rational homotopy type of) the space $X$. It is in this sense that the usage of the term “minimal” in the present paper is compatible with its usage in rational homotopy theory.

The notion of $C_{\infty}$-coalgebra is lurking behind these observations. We shall elucidate this kind of structure in Remark 6.3 below.

### 6 Perturbations for coalgebras

Abstracting from the perturbation argument for Chen’s theorem quoted above, V. Gugenheim developed a general perturbation theory for differential graded coalgebras which includes and explains Chen’s theorem [13]. Gugenheim made it entirely clear that his perturbation argument is formally exactly the same as that of Chen, but placed in a much more general context. We will now explain a version of Gugenheim’s result, somewhat more general than the original one in [13]. The proofs of all the claims in this section will be given in Section 9 below.

Let $C$ be a coaugmented differential graded coalgebra and let

$$\begin{align*}
(M \xrightarrow{\nabla} C, h)
\end{align*}$$

be a contraction of chain complexes. The situation considered by Gugenheim in [13] is the special case where $M$ has zero differential, so that $M$ then amounts to the homology of $C$. In the general case, the counit $\varepsilon: C \to R$ and coaugmentation $\eta: R \to C$ induce a “counit” $\varepsilon: M \to R$ and “coaugmentation” $\eta: R \to M$ for $M$ in such a way that (6.1) is a contraction of augmented and coaugmented chain complexes. Thus $M$ admits a direct sum decomposition $M = R \oplus JM$; indeed we can view $M$ as a differential graded coalgebra with zero diagonal on $JM$, and $JM$ can then be interpreted as the coaugmentation coideal of $M$. The differential graded tensor algebra $T[s^{-1}JM]$ is then the corresponding reduced
coar{}bar construction for $M$; let $\tau_M: M \to T[s^{-1}JM]$ be the universal coar{}bar construction twisting cochain, that is, the desuspension mapping, followed by the canonical injection.

Let

$$\tau^1 = \tau_M \pi: C \to s^{-1}JM \subseteq T[s^{-1}JM]$$

(6.2)

and, for $j \geq 2$, let

$$\tau^j: C \to (s^{-1}JM)^{\otimes j} \subseteq T[s^{-1}JM]$$

be the degree $-1$ morphism defined recursively by

$$\tau^j = (\tau^1 \cup \tau^{j-1} + \cdots + \tau^{j-1} \cup \tau^1) h: C \to (s^{-1}JM)^{\otimes j}.$$  

(6.3)

Thereafter, for $j \geq 1$, define the degree $-1$ derivation $D_j$ on $T[s^{-1}JM]$ by

$$D_j \tau_M = (\tau^1 \cup \tau^j + \cdots + \tau^j \cup \tau^1) \nabla: M \to (s^{-1}JM)^{\otimes (j+1)} \subseteq T[s^{-1}JM].$$

(6.4)

Let $\hat{T}[s^{-1}JM]$ denote the graded completion of the graded tensor algebra $T[s^{-1}JM]$, the term completion being taken relative to the augmentation filtration.

**Theorem 6.1.** The infinite series

$$D = D_1 + D_2 + \ldots : \hat{T}[s^{-1}JM] \to \hat{T}[s^{-1}JM]$$

(6.5)

is an algebra perturbation of the differential $d$ on $\hat{T}[s^{-1}JM]$ induced from the differential on $M$, and the infinite series

$$\tau = \tau^1 + \tau^2 + \ldots : C \to \hat{T}[s^{-1}JM]$$

(6.6)

is a twisting cochain

$$C \to (\hat{T}[s^{-1}JM], d + D).$$

For the special case where $M$ has zero differential so that $M$ then coincides with the homology $H(C)$ of $C$, this theorem is essentially Proposition 2.1 in [13]. We will henceforth write

$$\hat{\Omega}_D M = (\hat{T}[s^{-1}JM], d + D).$$

(6.7)

**Complement 1.** Suppose that, in addition, $M$ is a coaugmented differential graded coalgebra and that $\nabla$ is a morphism of differential graded coalgebras. Then $D^j$ is zero for $j \geq 2$ and $D^1$ is the algebra perturbation determined by the coalgebra structure of $M$.

Under suitable circumstances, the graded tensor algebra $T[s^{-1}JM]$ is already complete. This happens to be the case when $M$ is simply connected in the sense that it is concentrated in non-negative degrees and that $JM_1$ is zero, or when $JM$ is concentrated in non-positive degrees. In this case, the series

$$\tau = \tau^1 + \tau^2 + \ldots : C \to T[s^{-1}JM],$$

(6.8)

$$D = D^1 + D^2 + \ldots : T[s^{-1}JM] \to T[s^{-1}JM]$$

(6.9)

converge naively in the sense that, applied to a specific element, only finitely many terms are non-zero. Furthermore, $D$ is then an algebra perturbation of the differential on $T[s^{-1}JM]$, and

$$\tau = \tau^1 + \tau^2 + \ldots : C \to \Omega_D M$$
is a twisting cochain where we use the notation
\[
\Omega_D M = (T[s^{-1} J M], d + D).
\] (6.10)

At the risk of being, perhaps, repetitive, we point out explicitly that the piece of structure \(D\) in \(\Omega_D M\) is precisely an \(A_\infty\)-coalgebra structure on \(M\).

**Complement 2 to Theorem 6.1.** When \(C\) is simply connected or when \(C\) is concentrated in non-positive degrees, the adjoint
\[
\tau : \Omega C \longrightarrow \Omega_D M
\]
of the twisting cochain \(\tau\), necessarily a morphism of differential graded algebras, is a chain equivalence. If, furthermore, \(M\) is a coaugmented differential graded coalgebra and \(\nabla\) is a morphism of differential graded coalgebras, \(\Omega_D M\) is the ordinary reduced cobar construction on \(M\).

Indeed, in the situation of the “Furthermore” statement of Complement 2, the vanishing of the higher terms \(D^j\) for \(j \geq 2\) is a consequence of the annihilation property \(h \nabla = 0\) and the construction of the twisting cochain \(\tau\) comes essentially down to \([18]\) (4.1)*. A result somewhat weaker than the above Complement 2 is Theorem 3.2 in \([13]\) which says that \(\tau\) is a homology isomorphism.

Complement 2 to Theorem 6.1 includes the statement that, in the simply connected case, \(M\), endowed with the \(A_\infty\)-coalgebra structure \(D\), and \(C\), endowed with the \(A_\infty\)-coalgebra structure associated with the differential graded coalgebra structure, are, via \(\tau\), equivalent as \(A_\infty\)-coalgebras.

**Remark 6.2** (Lemma 2.2.1 in \([13]\)). Suppose that the differential of \(M\) is zero. Then \(M\) amounts to the homology \(H(C)\) of \(C\) and acquires the structure of a coaugmented graded coalgebra. In this case, even though neither the morphism \(\nabla\) nor the morphism \(\pi\) in the contraction (6.1) are supposed to be compatible with the coalgebra structures, the operator \(D^1\) is the ordinary cobar construction differential on the graded tensor algebra \(T[s^{-1} J H(C)]\) determined by the diagonal map \(\Delta_{H(C)}\) of \(H(C)\). Indeed, write the diagonal map of \(C\) as \(\Delta\) as usual. By construction, the diagonal map \(\Delta_M\) of \(M\) coincides with the composite
\[
M \xrightarrow{\nabla} C \xrightarrow{\Delta} C \otimes C \xrightarrow{\pi \otimes \pi} M \otimes M.
\]
Consequently
\[
D^1 \tau_M = (\tau^1 \cup \tau^1) \nabla = ((\tau_M \pi) \cup (\tau_M \pi)) \nabla = (\tau_M \otimes \tau_M)(\pi \otimes \pi) \Delta \nabla = (\tau_M \otimes \tau_M) \Delta_M.
\]

However the identity
\[
D^1 \tau_M = (\tau_M \otimes \tau_M) \Delta_M
\]
says that \(D^1\) is the ordinary cobar construction differential. Thus, in this case, \(\Omega_D H(C)\) is a perturbation of the ordinary reduced cobar construction \(\Omega H(C)\) over \(H(C)\) and thence endows \(H(C)\) with an \(A_\infty\)-coalgebra structure equivalent to \(C\).
Remark 6.3. The concept dual to that of a $C_{\infty}$-algebra is that of a $C_{\infty}$-coalgebra: Let $C$ be a coaugmented differential graded $A_{\infty}$-coalgebra, with standard construction $\Omega_\partial(C)$. Then this $A_{\infty}$-coalgebra structure on $C$ is a $C_{\infty}$-coalgebra structure provided the shuffle diagonal turns $\Omega_\partial(C)$ into a differential graded Hopf algebra, necessarily graded cocommutative. Exactly the same reasoning as that in Remark 5.2 reveals that the standard construction $\Omega_\partial(C)$ of a coaugmented $C_{\infty}$-coalgebra $C$ is the universal enveloping algebra $U[L]$ of a differential graded Lie algebra $L$ which, in turn, is a perturbation of the free differential graded Lie algebra generated by $s^{-1}(JC)$. Thus the standard construction of a (coaugmented) $C_{\infty}$-coalgebra comes down to a perturbation of the free differential Lie algebra generated by $s^{-1}(JC)$.

In the special case where, as a differential graded coalgebra, $C$ is an ordinary (coaugmented) cocommutative differential graded coalgebra, the shuffle diagonal plainly turns the ordinary cobar construction $\Omega(C)$ into a differential graded Hopf algebra; this situation has been explored by J. Moore in [52] and [53].

7 Perturbations for algebras

We now spell out the situation dual to that in the previous section. Again the proofs of all the claims in this section will be given in Section 9 below.

Thus, let $A$ be an augmented differential graded algebra and let

$$(M \xleftarrow{\nabla} \xrightarrow{\pi} A, h)$$

be a contraction of chain complexes. The unit $\eta: R \to A$ and augmentation $\varepsilon: A \to R$ induce a “unit” $\eta: R \to M$ and “augmentation” $\varepsilon: M \to R$ for $M$ in such a way that (7.1) is a contraction of augmented and coaugmented chain complexes. Thus $M$ admits a direct sum decomposition $M = R \oplus IM$; indeed we can view $M$ as a differential graded algebra with zero multiplication on $IM$, and $IM$ can then be interpreted as the augmentation ideal of $M$. The differential graded tensor coalgebra $T^c[\mathfrak{sIM}]$ is then the corresponding reduced bar construction for $M$; let $\tau_M: T^c[\mathfrak{sIM}] \to M$ be the universal bar construction twisting cochain, that is, the canonical projection to $\mathfrak{sIM}$, followed by the desuspension mapping.

Let

$$\tau^1 = \nabla \tau_M: T^c[\mathfrak{sIM}] \to \mathfrak{sIM} \to A$$

and, for $j \geq 2$, let

$$\tau^j: T^c[\mathfrak{sIM}] \to (\mathfrak{sIM})^{\otimes j} \to A$$

be the degree $-1$ morphism defined recursively by

$$\tau^j = h(\tau^1 \cup \tau^{j-1} + \cdots + \tau^1 \cup \tau^j): (\mathfrak{sIM})^{\otimes j} \to A.$$  

Thereafter, for $j \geq 1$, define the degree $-1$ coderivation $D^j$ on $T^c[\mathfrak{sIM}]$ by

$$\tau_M D^j = \pi(\tau^1 \cup \tau^j + \cdots + \tau^{j-1} \cup \tau^1): T^c[\mathfrak{sIM}]_{j+1} = (\mathfrak{sIM})^{\otimes (j+1)} \to M.$$  

In particular, for $j \geq 1$, the coderivation $D^j$ is zero on $F_j T^c[\mathfrak{sIM}]$ and lowers coaugmentation filtration by $j$. 

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Theorem 7.1. The infinite series
\[ \mathcal{D} = D^1 + D^2 + \ldots : T^c[sIM] \to T^c[sIM] \]  
(7.5)
is a coalgebra perturbation of the differential \( d \) on \( T^c[sIM] \) induced from the differential on \( M \), and the infinite series
\[ \tau = \tau^1 + \tau^2 + \ldots : T^c[sIM] \to A \]  
(7.6)
is a twisting cochain
\[ (T^c[sIM], d + \mathcal{D}) \to A. \]
Furthermore, the adjoint
\[ \varpi : B_\mathcal{D}M \to BA \]
of the twisting cochain \( \tau \), necessarily a morphism of differential graded coalgebras, is a chain equivalence.

To our knowledge, such a result for a general chain complex \( M \) appears in the literature for the first time in [19]—the special case where the ground ring is a field and where \( M \) has zero homology is due to Chen, as noted earlier. The sums (7.5) and (7.6) are in general infinite. However, applied to a specific element which, since \( T^c[sIM] \) is cocomplete, necessarily lies in some finite filtration degree subspace, since the operators \( D^j \) \((j \geq 1)\) lower coaugmentation filtration by \( j \), only finitely many terms will be non-zero, whence the convergence is naive. We will henceforth write
\[ B_\mathcal{D}M = (T^c[sIM], d + \mathcal{D}). \]  
(7.7)
The piece of structure \( \mathcal{D} \) in \( B_\mathcal{D}M \) is precisely an \( A_\infty \)-algebra structure on \( M \) and Theorem 7.1 includes the statement that \( M \), endowed with the \( A_\infty \)-algebra structure \( \mathcal{D} \), and \( A \), endowed with the \( A_\infty \)-algebra structure associated with the differential graded algebra structure, are, via \( \tau \), equivalent as \( A_\infty \)-algebras. This recovers, in particular, Kadeishvili’s result, Theorem 4.1.

Complement to Theorem 7.1. Suppose that, in addition, \( M \) is an augmented differential graded algebra and that \( \pi \) is a morphism of differential graded algebras. Then \( \mathcal{D}^j \) is zero for \( j \geq 2 \), the operator \( D^1 \) is the coalgebra perturbation determined by the algebra structure of \( M \), and \( B_\mathcal{D}M \) is the ordinary reduced bar construction for \( M \).

Indeed, in the situation of the Complement, the vanishing of the higher terms \( \mathcal{D}^j \) for \( j \geq 2 \) is a consequence of the annihilation property \( \pi h = 0 \) and the construction of the twisting cochain comes essentially down to [18] (4.1).

In the special case where \( A \) has zero differential and the original contraction (7.1) is the trivial contraction of the kind
\[ (A \xrightarrow{\text{id}} A, 0), \]  
(7.8)
\( M \) and \( A \) coincide, the perturbation \( \mathcal{D} \) coincides with the perturbation \( \partial \) determined by the algebra structure on \( A \), and \( T^c[sM] \) coincides with the ordinary reduced bar
construction for $A$; the twisting cochain $\tau$ then comes down to the bar construction twisting cochain and in fact coincides with $\tau^1$. In this case, the higher terms $\tau^j$ and $D^j$ $(j \geq 2)$ are obviously zero, and the operator $D^1$ manifestly coincides with the bar construction operator.

Likewise, suppose that the differential of $M$ is zero. Then $M$ amounts to the homology $H(A)$ of $A$ and acquires the structure of an augmented graded algebra. In this case, even though neither the morphism $\nabla$ nor the morphism $\pi$ in the contraction (7.1) are supposed to be compatible with the algebra structures, the operator $D^1$ is the ordinary bar construction differential on the graded tensor coalgebra $T^e[sIH(A)]$ determined by the multiplication map of $H(A)$, and $B_DH(A)$ is a perturbation of the ordinary reduced bar construction $BH(A)$.

**Remark 7.2.** [Relationship with Kadeishvili’s minimality theorem for algebras] Comparison of the proof of Theorem 4.1 with that of Theorem 7.1 reveals the following: The terms $\tau^j$ in the proof of Theorem 7.1 exhibit precisely terms of the kind $f_j$ in the proof of Theorem 4.1, and the operators $D^j$ in the proof of Theorem 7.1 yield terms of the kind $m_j$ in the proof of Theorem 4.1. The coalgebra structure of $T^e[sM]$ exploited in the proof of Theorem 7.1 organizes the otherwise tedious calculations in the original proof of Theorem 4.1 and the usage, in the proof of Theorem 7.1 of the chain homotopy $h$ in the contraction (7.1) removes the ambiguities related with the choices of the bounding chains in the proof of Theorem 4.1 and thus leads to an algorithm, cf. [62].

### 8 A proof of Chen’s theorem

We will now briefly explain how Theorem 7.1 includes Theorem 5.1; this will make it clear that the basic perturbation argument goes back to Chen: Given the smooth manifold $X$, let $A = \mathcal{A}(X)$ and pick a contraction of the kind (5.1). Thus $H$ is then the de Rham cohomology algebra of $X$. The recursive construction (7.3) yields the twisting cochain

$$\tau = \tau^1 + \tau^2 + \ldots : B_DH \to \mathcal{A}(X)$$

and the formulas (7.4) yield the coalgebra differential

$$D = \mathcal{D}^1 + \mathcal{D}^2 + \ldots : T^e[sIH] \to T^e[sIH].$$

To simplify the exposition we will suppose that the homology of $X$ is of finite type. Then the differential graded algebra which is the real dual of $B_DH$ is precisely a differential graded algebra of the kind

$$\hat{\Omega}_\partial[H_*(X)]$$

where $\partial$ is the algebra differential dual to the coalgebra differential $D$. Thus the twisting cochain $\tau$ then appears as an element of the differential graded algebra

$$\text{Hom}(B_DH, \mathcal{A}(X)) \cong \tilde{T}_{\mathcal{A}(X)}[s^{-1}\tilde{H}] = \mathcal{A}(X) \otimes T[s^{-1}\tilde{H}],$$

endowed with the total differential

$$d^\otimes = d \otimes \text{Id} + \text{Id} \otimes \partial.$$
The twisting cochain property says that $\tau$ satisfies the master equation (5.3). The formulas (7.3) and (7.4) are then essentially the same as those used by Chen to establish the existence of the formal power series connection and of the differential $\partial$ in the proof of his Theorem 1.3.1 in [7].

9 Homological perturbations and algebraic structure

In the 1980’s, I noticed that various standard HPT-constructions are compatible with algebraic structure, and I used this observation to exploit $A_\infty$-structures arising in group cohomology via HPT-constructions of suitable small free resolutions. These small free resolutions enabled me to do explicit numerical calculations in group cohomology which until today are still not doable by other methods. In particular, spectral sequences show up which do not collapse from $E_2$. This illustrates a typical phenomenon: Whenever a spectral sequence arises from a certain mathematical structure, a certain strong homotopy structure is lurking behind and the spectral sequence is an invariant thereof. The higher homotopy structure is actually finer than the spectral sequence itself. The results have been published in the papers [25]–[30].

The observation that compatibility with algebraic structure is hidden in various standard HPT-constructions led to an alternate approach to Theorem 6.1 and Theorem 7.1 and, indeed, leads to considerable generalization, cf. Remark 10.3 below. In the academic year 1987/88, lifting of the restrictions in the USSR enabled T. Kadeishvili to accept an invitation to the mathematics department of the University of Heidelberg. During that period, in collaboration, T. Kadeishvili and I developed HPT for general chain equivalences and, within this research collaboration, we worked out in particular the alternate approach. This kind of approach was also worked out in [14]–[16].

We will now explain the outcome of this alternate approach and how it actually leads to proofs of these theorems and to additional insight. To this end, let

$$\left( M \xleftarrow{\nabla} \xrightarrow{\pi} N, h \right)$$

be a filtered contraction. For intelligibility, we recall the following.

**Lemma 9.1 (Ordinary perturbation lemma).** Let $\partial$ be a perturbation of the differential on $N$, and let

$$\mathcal{D} = \sum_{n \geq 0} \pi \partial (-h \partial)^n \nabla = \sum_{n \geq 0} \pi (-\partial h)^n \partial \nabla$$

(9.2)

$$\nabla_{\partial} = \sum_{n \geq 0} (-h \partial)^n \nabla$$

(9.3)

$$\pi_{\partial} = \sum_{n \geq 0} \pi (-\partial h)^n$$

(9.4)

$$h_{\partial} = - \sum_{n \geq 0} (-h \partial)^n h = - \sum_{n \geq 0} h (-\partial h)^n.$$  

(9.5)
When the filtrations on $M$ and $N$ are complete, these infinite series converge, $D$ is a perturbation of the differential on $M$ and, when $N$ and $M$ refer to the new chain complexes,

$$
M \xrightarrow{\nabla_{\partial}} N_{\partial}, h_{\partial}
$$

constitute a new filtered contraction that is natural in terms of the given data.

**Proof.** See [3] or [12].

The issue addressed in the perturbation lemma has been described in the literature as a *transference* problem. The ordinary perturbation lemma solves this transference problem relative merely to a perturbation of the differential on the larger object $N$. We will now explore the transference problem relative to additional structure.

Let $T[N]$ and $T[M]$ be the differential graded tensor algebras on $N$ and $M$ respectively, denote the multiplication map of $T[N]$ by $m$, let $T\pi$ and $T\nabla$ be the induced morphisms of differential graded algebras, and define an operator $Th: T[N] \to T[N]$ by means of

$$(Th)_{N^\otimes k} = h \otimes (\nabla\pi)^{\otimes(k-1)} + \text{Id} \otimes h \otimes (\nabla\pi)^{\otimes(k-2)} + \cdots + \text{Id}^{\otimes(k-1)} \otimes h, \quad k \geq 1.$$  

This is an instance of what is referred to as the *tensor trick*, developed in [24] and exploited in [16], [24], [25], [26], [37] and elsewhere; cf. also §3 of [15]. We will come back to the tensor trick in Section 11 below.

With the above preparations out of the way, the morphisms $m$, $T\pi$, $Th$, etc. are related by the identities

$$m(\text{Id} \otimes Th + Th \otimes (T\nabla)(T\pi)) = (Th) m,$$

and

$$D(Th) = d(Th) + (Th)d = (T\nabla)(T\pi) - \text{Id},$$

that is, $Th$ is a homotopy $\text{Id} \simeq (T\nabla)(T\pi)$ of morphisms of differential graded algebras, whence the data

$$
T[M] \xrightarrow{T\pi} T[N], Th
$$

constitute a contraction of augmented algebras. Further, with respect to the augmentation filtrations on $T[N]$ and $T[M]$, these data constitute in fact a contraction of filtered algebras. This idea goes back to Theorem 12.1 in *Eilenberg and Mac Lane I* [9] where it is spelled out in the dual situation as a *contraction of bar constructions*. The contraction of coalgebras that corresponds to (9.7) is also spelled out in [14] (3.2), in [15] (§3), and in [16] (2.2).

Given a chain complex $X$ and a multiplicative perturbation $\partial$ of the algebra differential on $T[X]$, we shall denote the new differential graded algebra by $T_{\partial}[X]$. Maintaining terminology introduced in [37], we shall refer to a chain complex $X$ (without any additional structure) as being *connected in the reduced sense* provided it is zero in degree zero and, furthermore, either non-negative or non-positive; the reader will note that “connected in the reduced sense” does not coincide with the standard usage of the term “connected”. On the other hand, an augmented differential graded algebra is *connected* in the usual
sense if and only its augmentation ideal is, as a chain complex, connected in the reduced sense.

Henceforth a tensor algebra $T[W]$ on a graded $R$-module $W$ will always be viewed as a filtered algebra with respect to the augmentation filtration. Here is Theorem 2.2* of [37].

**Theorem 9.2.** Suppose that $T[N]$ and $T[M]$ are connected and let $\partial$ be a multiplicative perturbation of the differential on $T[N]$ with respect to the augmentation filtration. Then the perturbation $\mathcal{D}$ given by (9.2) together with the morphisms given by (9.3) – (9.5) yield a contraction

$$
\left( T_\mathcal{D}[M] \xrightarrow{\frac{T_\partial \nabla}{T_\partial \pi}} T_\partial[N], T_\partial h \right)
$$

of filtered differential graded algebras which is natural in terms of the given data.

Plainly the perturbation $\mathcal{D}$ on $T[M]$ and the morphisms $T_\partial \nabla$, $T_\partial \pi$, $T_\partial h$ are determined by their restrictions to $N \subseteq T[N]$ and $M \subseteq T[M]$ as appropriate.

The proof of Theorem 9.2 given in [37] relies on the Algebra Perturbation Lemma ([37] Lemma 2.1*) and involves the “tensor trick”.

We will now sketch a proof of Theorem 6.1. Let $C$ be a coaugmented differential graded coalgebra, with structure maps $\Delta$ and $\eta$, and let

$$
\Omega C = T_\partial[s^{-1}(JC)]
$$

be its cobar construction, $\partial$ being the derivation on the differential graded tensor algebra $T[s^{-1}(JC)]$ induced by the diagonal map of $C$. Thus with respect to the augmentation filtration, $T[s^{-1}(JC)]$ is a filtered differential graded algebra, $\partial$ is a multiplicative perturbation, and the cobar construction $\Omega C$ appears as a “perturbation” of $T[s^{-1}(JC)]$.

Suppose momentarily that $C$ is simply connected or concentrated in non-positive degrees. Then $N = s^{-1}(JC)$ is connected in the reduced sense, and the algebra $T[s^{-1}(JC)]$ is complete. The contraction (6.1) determines a contraction

$$
(s^{-1}JM \xrightarrow{\pi} N, h)
$$

of the kind used in Theorem 9.2 where now $s^{-1}JM$ plays the role of $M$ in Theorem 9.2. Theorem 9.2 then yields a multiplicative perturbation $\mathcal{D}$ of the differential on $T[s^{-1}JM]$ and a contraction

$$
\left( T_\mathcal{D}[s^{-1}JM] \xrightarrow{T_\partial \nabla}{T_\partial \pi} \Omega C, T_\partial h \right)
$$

(9.9)

of filtered differential graded algebras. The naturality of the constructions implies that $T_\partial \pi$ is the adjoint of a twisting cochain

$$
\tau : C \longrightarrow T_\mathcal{D}[s^{-1}JM].
$$

A closer look reveals that $\tau$ and $\mathcal{D}$ actually coincide with (6.8) and (6.9), respectively. This establishes the statement of Complement 2 to Theorem 6.1 and yields, furthermore,
explicit morphisms which then can be extended, by suitable HPT-constructions, to $A_{\infty}$-morphisms between $C$ and $M$ (with its $A_{\infty}$-coalgebra structure) and thus establish an $A_{\infty}$-equivalence between $C$ and $M$.

To establish Theorem 6.1 for a general coaugmented differential graded coalgebra $C$, we note that the contraction (9.7) of filtered algebras induces a contraction

$$
\left( \hat{T}[M] \xrightarrow{\hat{T} \nu} \hat{T}[N], \hat{T}h \right)
$$

(9.10)

of complete algebras. The statement of Theorem 9.2 extends to that situation and a proof of Theorem 6.1 for a general coaugmented differential graded coalgebra $C$ can then be concocted, just as for the particular case handled first.

Finally we will indicate the necessary modifications to arrive at a proof of Theorem 7.1. Instead of the contraction (9.7), we now consider the corresponding contraction

$$
\left( T_c[M] \xrightarrow{T_c \nu} T_c[N], T_c h \right)
$$

(9.11)

of coaugmented coalgebras. Relative to the coaugmentation filtrations, the coalgebra version of Theorem 9.2 is true without any connectivity assumption and takes the following form; actually this is Theorem 2.2* of [37], not spelled out explicitly there.

**Theorem 9.3.** Let $\partial$ be a coalgebra perturbation of the differential on $T_c[N]$ with respect to the coaugmentation filtration. Then the perturbation $D$ given by (9.2) together with the morphisms given by (9.3) – (9.5) yield a contraction

$$
\left( T_\partial[M] \xrightarrow{T_\partial \nu} T_\partial[N], T_\partial h \right)
$$

(9.12)

of filtered differential graded coalgebras which is natural in terms of the given data.

The proof of Theorem 9.3 relies on the Coalgebra Perturbation Lemma ([37] Lemma 2.1) and involves likewise the “tensor trick”.

Dualizing the above reasoning which leads to a proof of Theorem 6.1, the reader is now invited to concoct a proof of Theorem 7.1. We refrain from spelling out details.

## 10 General $A_{\infty}$-algebras and $A_{\infty}$-coalgebras

The reasoning in the previous section extends immediately to general $A_{\infty}$-algebras and $A_{\infty}$-coalgebras and thus yields solutions of the corresponding transference problems:

Let $A$ be an augmented $A_{\infty}$-algebra, the $A_{\infty}$-algebra structure being given by a coalgebra perturbation $\partial$ of the differential on $T^c[\text{sIA}]$ relative to the coaugmentation filtration and, as before, write $B_\partial A$ for the perturbed differential graded coalgebra. Moreover, let

$$
(\text{sIM} \xrightarrow{\nu} \text{sIA}, h)
$$

(10.1)

be a contraction of chain complexes. Such a contraction arises plainly from a contraction of augmented chain complexes from $A$ onto $M$. 

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Theorem 10.1. The perturbation $D$ given by (9.2) yields an augmented $A_\infty$-algebra structure on $M$, and the morphisms given by (9.3) – (9.5) yield a contraction

\[
\begin{align*}
BDM & \xrightarrow{T_D \nabla} BDA, T_\partial h \\
\end{align*}
\] (10.2)

of filtered differential graded coalgebras which is natural in terms of the given data.

Proof. This is an immediate consequence of Theorem 9.3. □

For the special case where $M$ is the homology of $A$, Theorem 10.1 yields the “minimality theorem” for general augmented $A_\infty$-algebras.

Likewise let $C$ be a coaugmented $A_\infty$-coalgebra that is simply connected or concentrated in non-positive degrees, the $A_\infty$-coalgebra structure being given by an algebra perturbation $\partial$ of the differential on $T[s^{-1}]JC$ relative to the augmentation filtration and, as before, write $\Omega_\partial C$ for the perturbed differential graded coalgebra. Moreover, let

\[
(sJM \xleftarrow{\nabla} \pi sJC, h)
\] (10.3)

be a contraction of chain complexes. Such a contraction arises plainly from a contraction of coaugmented chain complexes from $C$ onto $M$.

Theorem 10.2. The perturbation $D$ given by (9.2) yields a coaugmented $A_\infty$-coalgebra structure on $M$, and the morphisms given by (9.3) – (9.5) yield a contraction

\[
\begin{align*}
\Omega DM & \xrightarrow{T_D \nabla} \Omega C, T_\partial h \\
\end{align*}
\] (10.4)

of filtered differential graded algebras which is natural in terms of the given data.

Proof. This is an immediate consequence of Theorem 9.2. □

For the special case where $M$ is the homology of $C$, Theorem 10.2 yields the “minimality theorem” for general coaugmented $A_\infty$-coalgebras.

Remark 10.3. At the risk of making a mountain out of a molehill we point out that Theorem 10.1 is more general than Theorem 7.1 since, in Theorem 10.1, $A$ is a general (augmented) $A_\infty$-algebra; likewise, Theorem 10.2 is more general than Theorem 6.1 since, in Theorem 10.2, $C$ is a general (coaugmented) $A_\infty$-coalgebra. In other words, Theorem 10.1 provides a solution of the transference problem for $A_\infty$-algebra structures and Theorem 10.2 provides a solution of the transference problem for $A_\infty$-coalgebra structures.

11 Summation over oriented rooted planar trees

In [48] it has been observed that $A_\infty$-algebra structures of the kind reproduced in previous sections can be described in terms of sums over oriented rooted planar trees endowed with suitable labels. Indeed the authors of [48] bravely acknowledged that the oriented rooted
planar trees description is essentially a reworking of the earlier HPT-constructions. We will now explain how these sums over oriented rooted planar trees come out of the HPT-constructions.

We return to the circumstances of Theorem 7.1 above. We denote the multiplication map of $A$ by $\mu: A \otimes A \to A$.

By construction, the operation $m_2: M \otimes M \to M$ is the composite

$$M \otimes M \xrightarrow{\nabla \otimes \nabla} A \otimes A \xrightarrow{\mu} A \xrightarrow{\pi} M.$$  

This is interpreted as an oriented rooted planar tree with three edges and four vertices, one vertex where the three edges meet and an end point vertex for each edge. The vertex where three edges meet is labelled $\mu$, the root vertex is labelled $\pi$, and the two remaining vertices are labelled $\nabla$. The two edges having a vertex labelled $\nabla$ are oriented from $\nabla$ to $\mu$ whereas the edge having $\mu$ and $\pi$ as vertices is oriented from $\mu$ to $\pi$.

For $j \geq 1$, simply by construction, the homogeneous degree $j-1$ operation

$$m_{j+1} = s^{-1} \circ D^j \circ s^{\otimes (j+1)}: (IM)^{\otimes (j+1)} \to IM$$

is given by

$$\pi \circ (\tau^1 \cup \tau^j + \cdots + \tau^j \cup \tau^1) \circ s^{\otimes (j+1)}$$

and is thus the sum of the $j$ terms $\pi \circ (\tau^\ell \cup \tau^{j+1-\ell}) \circ s^{\otimes (j+1)}$ where $1 \leq \ell \leq j$. Each of these $j$ terms can be represented by an oriented rooted planar tree with suitable labels, in the following way:

The operation $m_2$ has already been dealt with. The operation $m_3: M^{\otimes 3} \to M$ is the sum of the two composite morphisms

$$M \otimes M \otimes M \xrightarrow{\nabla^{\otimes 3}} A \otimes A \otimes A \xrightarrow{A \otimes \mu} A \otimes A \xrightarrow{A \otimes h} A \otimes A \xrightarrow{\mu} A \xrightarrow{\pi} M$$

and

$$M \otimes M \otimes M \xrightarrow{\nabla^{\otimes 3}} A \otimes A \otimes A \xrightarrow{\mu \otimes A} A \otimes A \xrightarrow{h \otimes A} A \otimes A \xrightarrow{\mu} A \xrightarrow{\pi} M.$$  

Each of them is interpreted as an oriented rooted planar tree with four external edges, one internal edge, four external vertices, and two internal vertices in the following manner:

(i) Three external vertices are labelled $\nabla$; these correspond to the three tensor factors $\nabla$ in the above constituent

$$\nabla^{\otimes 3}: M \otimes M \otimes M \to A \otimes A \otimes A;$$

(ii) one external vertex—the root vertex—is labelled $\pi$; this corresponds to the right-most arrow $\pi: A \to M$;

(iii) the two internal vertices are labelled $\mu$; in the upper morphism, these correspond to the arrows labelled $\mu: A \otimes A \to A$ and $A \otimes \mu: A \otimes A \otimes A \to A \otimes A$; in the lower morphism, they correspond to the arrows labelled $\mu: A \otimes A \to A$ and $\mu \otimes A: A \otimes A \otimes A \to A \otimes A$;

(iv) the single internal edge is labelled $h$; this corresponds to the arrow labelled

$$A \otimes h: A \otimes A \to A \otimes A.$$
in the upper morphism and labelled 

\[ h \otimes A : A \otimes A \rightarrow A \otimes A \]

in the lower morphism;
(v) the three external edges having \( \nabla \) as a vertex are oriented from the vertices labelled \( \nabla \) to the vertices labelled \( \mu \);
(vi) the remaining external edge is oriented from a vertex labelled \( \mu \) to the root vertex labelled \( \pi \);
(vii) two edges having a vertex labelled \( \nabla \) as starting point meet at a vertex labelled \( \mu \), this vertex is joined to the other vertex labelled \( \mu \), oriented in that manner, and it meets the third edge having a vertex labelled \( \nabla \) as starting point at its end point.

There are two such oriented rooted planar trees, one being the mirror image of the other, and the two composite morphisms spelled out above correspond to these two labelled oriented rooted planar trees.

Likewise the operation \( m_4 : M \otimes^4 \rightarrow M \) is the sum of three composite morphisms of a similar nature which, in the language of twisting cochains, arise from the three constituents \( \tau_1 \cup \tau_3, \tau_2 \cup \tau_2, \) and \( \tau_3 \cup \tau_1 \), cf. (7.4) above; each such composite morphisms can be encoded in the appropriate labelled oriented rooted planar tree.

Formalizing this procedure one arrives at the description of the operations \( m_j \) in terms of sums over labelled oriented rooted planar trees worked out in detail in [48].

The requisite combinatorics for the construction in Theorem 7.1 is provided by the concept of cofree coalgebra; likewise in Theorem 6.1 the necessary combinatorics is provided by the concept of free algebra. The machinery of labelled oriented rooted planar trees yield an alternate description of the requisite combinatorial tool.

This discussion so far refers to ordinary strict algebra (or coalgebra) structures on the larger object coming into play in the corresponding contraction. Instead of the original contraction (7.1) where the constituent \( A \) is an ordinary augmented differential graded algebra, consider now a contraction of the kind (10.1) where \( A \) is merely an \( A_\infty \)-algebra, with structure maps

\[ \mu_j : A \otimes^j \rightarrow A \ (j \geq 1). \]

The construction in [48] in terms of labelled oriented rooted planar trees extends to that situation. Indeed, define the \textit{arity} of a vertex to be the number of incoming edges. To the trees described above having only vertices of arity 2, one simply adds trees where internal vertices \( v \) have general arities \( j > 2 \), a vertex of arity \( j > 2 \) being labelled by the operation \( \mu_j \); one then takes the sum over all labelled oriented rooted planar trees. This kind of construction yields an \( A_\infty \)-algebra structure on \( M \) and, extended suitably, it also yields an \( A_\infty \)-equivalence between \( A \) and \( M \).

However, unravelling the perturbation \( D \) on \( T^c[sIM] \) spelled out in Theorem 10.1 we find precisely that very same \( A_\infty \)-structure as that given by the labelled oriented rooted planar trees. Indeed, the infinite series (9.2), evaluated relative to the contraction (9.11), takes the form

\[ D = (T^c \pi) \partial(T^c \nabla) - (T^c \pi) \partial(T^c h) \partial(T^c \nabla) + (T^c \pi) \partial(T^c h) \partial(T^c h) \partial(T^c \nabla) + \ldots \]  (11.1)

Now, when \( \partial \) arises from an ordinary associative differential graded algebra structure on \( A \), the first term in the development (11.1) yields, on \( M \), the \( A_\infty \)-constituent \( m_2 \), the
second term yields the $A_\infty$-constituent $m_3$, and so forth, precisely in the form given by the labelled oriented rooted planar trees construction. More general, when $\partial$ arises from a general $A_\infty$-algebra structure $\{\mu_j\}_j$ on $A$, the perturbation $\partial$ on $T^c[slA]$ has the form

$$\partial = \partial^1 + \partial^2 + \ldots$$

in such a way that, for $j \geq 1$, the constituent $\partial^j \mu$ corresponds to $\mu_{j+1}$. Consequently each term in the development (11.1) involves all the constituents $\mu_j$, and reordering the terms that show up in (11.1), we obtain precisely the $A_\infty$-constituent $m_j$ in the form given by the labelled oriented rooted planar trees construction for the transference of a general $A_\infty$-algebra structure. Likewise, exploiting the series (9.3), (9.4), and (9.5) to unravel the other terms, respectively, $T^c_\partial \nabla$, $T^c_\partial \pi$, and $T^c_\partial h$ that are spelled out in Theorem 10.1, we obtain the requisite remaining data that establish the necessary chain equivalence, precisely in the form given by the corresponding labelled oriented rooted planar trees constructions for the transference of a general $A_\infty$-algebra structure.

The same kind of remark applies to the dual situation encapsulated in Theorem 10.2

The construction of these perturbations $D$ on the tensor algebra or tensor coalgebra relies on the tensor trick mentioned above, which we developed in [24].

Thus we see that the more recent constructions of an $A_\infty$-structure in [48] (6.4) and [50] still come down to the earlier constructions.

**Remark 11.1.** The construction in [50] is slightly more general in the sense that the initial data considered there are required to satisfy requirements somewhat weaker than those which characterize an ordinary contraction. Indeed, in [50], a system

$$(N \overset{\nabla}{\underset{\pi}{\longrightarrow}} M, h)$$

(11.2)

of chain complexes is explored satisfying the requirements (2.1) and (2.2) but not necessarily (2.3), that is, it is not required that $\pi \nabla = \text{Id}$; indeed, no condition is imposed upon $\pi \nabla$. We will now show that application of the constructions in the perturbation lemma without the requirement that $\pi \nabla = \text{Id}$ does not lead to a more general theory.

Indeed, a system of the kind (11.2) can arise by the following specific construction from an ordinary contraction and in fact every system of the kind (11.2) arises in this way: Consider an ordinary contraction

$$(N_1 \overset{\nabla_1}{\underset{\pi_1}{\longrightarrow}} M, h)$$

(11.3)

of chain complexes, let $N_2$ be an arbitrary chain complex, let $N = N_1 \oplus N_2$, let $\pi : M \to N$ be the composite of $\pi_1$ with the canonical injection into $N$, let

$$\nabla_2 : N_2 \to \ker(\pi) = \ker(\pi_1) \subseteq M$$

be a chain map, and let $\nabla = (\nabla_1, \nabla_2) : N_1 \oplus N_2 \to M$. Then

$$(N \overset{\nabla}{\underset{\pi}{\longrightarrow}} M, h)$$

(11.4)
is a system of the kind (11.2). We will now show that every system of the kind (11.2) arises in this way. Given a perturbation of the differential on $M$, application of the perturbation lemma involves only the summand $N_1$ of $N$ and leaves $N_2$ unchanged in the sense that this application yields a new system of the kind (11.4) where the new contraction of the kind (11.3) arises by an application of the perturbation lemma to the contraction of the kind (11.3) before application of the perturbation and where the summand $N_2$ remains unchanged. Hence application of the constructions in the perturbation lemma without the requirement that $\pi \nabla = \text{Id}$ does not lead to a more general theory.

Thus consider a system of the kind (11.2). The requirement (2.4), viz. $Dh = \text{Id} - \nabla \pi$, implies

$$D(\pi h \nabla) = \pi \nabla - \pi \nabla \pi \nabla$$

which, in view of the annihilation properties (2.5), comes down to $\pi \nabla = \pi \nabla \pi \nabla$. Hence the endomorphism $\pi \nabla$ of $N$ is a projector. Let $N_1 = \pi \nabla(N)$ and $N_2 = (\text{Id} - \pi \nabla)(N)$. Then $N = N_1 \oplus N_2$. Let $\pi_1 : M \to N_1$ be the composite of $\pi$ with the canonical projection to $N_1$, and let $\nabla_1 : N_1 \to M$ be the injection of $N_1$ into $N$, followed by $\nabla$. The resulting data

$$\left( N_1 \xrightarrow{\nabla_1} M, h \right)$$

constitute a contraction of chain complexes.

Indeed, to understand the situation, suppose momentarily that $N_1$ is zero, that is, the composite $\pi \nabla$ is zero. Then $\nabla_1 = \nabla_1 \nabla = 0$ whence $Dh = \text{Id}$, that is, $M$ is contractible, the homotopy $h$ being a conical contracting homotopy. In the general case where $N_1$ is not necessarily zero, let $M_1 = \nabla(N_1)$ and $M_2 = \ker(\pi)$. The requirement (2.4), viz. $Dh = \text{Id} - \nabla \pi$, together with the annihilation properties (2.5), implies that

$$M = \ker(\pi) + \nabla \pi(M) = \ker(\pi) + \nabla(N) = \ker(\pi) + \nabla(N_1) + \nabla(N_2).$$

But $\pi \nabla(N_2)$ is zero whence $\nabla(N_2) \subseteq \ker(\pi)$ and hence

$$M = \ker(\pi) + \nabla(N_1) = M_1 + \ker(\pi) = M_1 + M_2.$$ 

By construction, $\pi$ restricted to $M_2$ is zero and $\pi$ restricted to $M_1$ amounts to the restriction of $\pi_1$ to $M_1$ which, in turn is an isomorphism having $\nabla_1$ as its inverse. Moreover, exploiting once more the fact that $\pi \nabla(N_2)$ is zero we conclude that $\nabla$ restricted to $N_2$ is a morphism of the kind $\nabla : N_2 \to M_2$. Hence the decomposition $M = M_1 + M_2$ is a direct sum decomposition; the obvious inclusion $\ker(\pi_1) \subseteq \ker(\pi)$ is the identity; the original system (11.2) can be written as

$$\left( N_1 \oplus N_2 \xrightarrow{\left( \nabla_1, \nabla_2 \right) \atop \left( \pi_1, 0 \right)} M_1 \oplus M_2, h \right);$$

and the annihilation properties (2.5) imply that $h(M_1)$ is zero and that $h(M_2) \subseteq M_2$. Indeed, since $h \nabla$ is zero, $h$ vanishes on $M_1$ and, likewise, since $\pi h$ is zero, $h(M_2) \subseteq M_2$. 

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Hence (11.6) actually takes the form
\[
\left( N_1 \oplus N_2 \xrightarrow{(\nabla_1, \nabla_2)} M_1 \oplus M_2, (0, h_2) \right)
\] (11.7)
and the system
\[
\left( N_2 \xrightarrow{\nabla_2} M_2, (0, h_2) \right) \quad (11.8)
\]
is of the kind of the special case where \( N_1 \) is zero—indeed, the corresponding morphism \( \pi_1 \) is now even zero—whence, in particular, \( h_2 \) is a conical contracting homotopy for \( M_2 \). Consequently
\[
\left( N_1 \xrightarrow{(\nabla_1, 0)} M_1 \oplus M_2, (0, h_2) \right), \quad (11.9)
\]
is an ordinary contraction as asserted and the original system (11.2) is indeed of the special kind (11.4).

12 Perturbations for Lie algebras and \( L_\infty \)-algebras, and the master equation

Let \( \mathfrak{g} \) be a differential graded \( R \)-Lie algebra that is projective as a graded \( R \)-module and let
\[
(M \xleftarrow{\nabla} \mathfrak{g}, h) \quad (12.1)
\]
be a contraction of chain complexes. Suppose that the cofree coaugmented differential graded cocommutative coalgebra \( S^c[\mathfrak{s}\mathfrak{g}] \) on the suspension \( s\mathfrak{g} \) of \( \mathfrak{g} \) and, likewise, the cofree coaugmented differential graded cocommutative coalgebra \( S^c = S^c[sM] \) on the suspension \( sM \) of \( M \) exist. This kind of coalgebra is well known to be cocomplete. Further, let \( d^0: S^c \rightarrow S^c \) denote the coalgebra differential on \( S^c = S^c[sM] \) induced by the differential on \( M \). For \( b \geq 0 \), we will henceforth denote the homogeneous degree \( b \) component of \( S^c[sM] \) by \( S^c_b \); thus, as a chain complex, \( F_b S^c = R \oplus S^c_1 \oplus \cdots \oplus S^c_b \). Likewise, as a chain complex, \( S^c = \bigoplus_{j=0}^\infty S^c_j \). We denote by
\[
\tau_M: S^c \rightarrow M
\]
the composite of the canonical projection \( \text{proj}: S^c \rightarrow sM \) from \( S^c = S^c[sM] \) to its homogeneous degree 1 constituent \( sM \) with the desuspension map \( s^{-1} \) from \( sM \) to \( M \). When \( M \) is viewed as an abelian differential graded Lie algebra, \( S^c = S^c[sM] \) may be viewed as the CCE or classifying coalgebra \( C[M] \) for \( M \), and \( \tau_M: S^c \rightarrow M \) is then the universal differential graded Lie algebra twisting cochain for \( M \).

Let
\[
\tau^1 = \nabla \tau_M: S^c \rightarrow \mathfrak{g} \quad (12.2)
\]
and, for \( j \geq 2 \), let
\[
\tau^j: S^c \rightarrow \mathfrak{g}
\]
be the degree \(-1\) morphism defined recursively by
\[
\tau^j = \frac{1}{2} h([\tau^1, \tau^{j-1}] + \cdots + [\tau^{j-1}, \tau^1]): S^c \to g. \tag{12.3}
\]
Thereafter, for \(j \geq 1\), define the degree \(-1\) coderivation \(D^j\) on \(S^c\) by
\[
\tau_M D^j = \frac{1}{2} \pi ([\tau^1, \tau^j] + \cdots + [\tau^j, \tau^1]): S^c_{j+1} \to M. \tag{12.4}
\]
In particular, for \(j \geq 1\), the coderivation \(D^j\) is zero on \(F_j S^c\) and lowers coaugmentation filtration by \(j\).

We now spell out the main result of [31].

**Theorem 12.1.** The infinite series
\[
D = D^1 + D^2 + \ldots : S^c \to S^c \tag{12.5}
\]
is a coalgebra perturbation of the differential \(d\) on \(S^c\) induced from the differential on \(M\), and the infinite series
\[
\tau = \tau^1 + \tau^2 + \ldots : S^c \to g \tag{12.6}
\]
is a Lie algebra twisting cochain
\[
(S^c, d + D) \longrightarrow g.
\]
Furthermore, the adjoint
\[
\tau: (S^c, d + D) \longrightarrow C[g]
\]
of the twisting cochain \(\tau\), necessarily a morphism of differential graded coalgebras, is a chain equivalence. More precisely, the data determine a contraction
\[
\left( (S^c[sM], d + D) \overset{\tau}{\longrightarrow} C[g], H \right)
\]
of chain complexes which is natural in terms of the data.

This is precisely Theorem 2.1 in [31] where also a complete proof can be found. For the special case where \(M\) is the homology of \(g\), this result yields the “minimality theorem” for ordinary differential graded Lie algebras.

**Remark 12.2.** The attempt to treat, as for the cases explained in Section 9 above, the requisite higher homotopies by means of a suitable version of the perturbation lemma relative to the additional algebraic structure, that is, to develop a version of the perturbation lemma compatible with Lie brackets or more generally with sh-Lie structures, led to the paper [38], but technical complications arise since the tensor trick breaks down for cocommutative coalgebras; indeed, the notion of homotopy of morphisms of cocommutative coalgebras is a subtle concept [58], and only a special case was handled in [38], with some of the technical details merely sketched. The article [31] provides a complete solution with all the necessary details and handles the case of a general contraction whereas in [38] only the case of a contraction of a differential graded Lie algebra onto its homology was treated. Also in [38], the proof is only sketched, and a detailed proof is given in [31]. The twisting cochain \(\tau\) is the most general solution of the master equation, under the circumstances of [31].
We will write
\[ C_D M = (S^c, d + D). \] (12.7)
The piece of structure \( D \) in \( C_D M \) is precisely an \( L_\infty \)-algebra structure on \( M \), Theorem 12.1 includes the statement that \( M \), endowed with the \( L_\infty \)-algebra structure \( D \), and \( g \), endowed with the \( L_\infty \)-algebra structure associated with the differential graded Lie algebra structure, are, via \( \tau \), equivalent as \( L_\infty \)-algebras.

The general sh-Lie algebra perturbation lemma, Theorem 2.5 in [34], extends Theorem 12.1 to the more general case where the constituent \( g \) in the contraction (12.1) is merely an sh-Lie algebra. This sh-Lie algebra perturbation lemma yields, in particular, the “minimality theorem” for general \( L_\infty \)-algebras.

13 More about the relationship with deformations

We have pointed out above that the idea of combining the Gerstenhaber operation in the Hochschild complex mentioned in Section 4.1 with Stasheff’s notion of \( A_\infty \)-algebra let Kadeishvili to the minimality theorem. There is an obvious formal relationship between homological perturbations and deformation theory but the relationship is actually much more profound: In [23], S. Halperin and J. Stasheff developed a procedure by means of which the classification of rational homotopy equivalences inducing a fixed cohomology algebra isomorphism can be achieved. Moreover, one can explore the rational homotopy types with a fixed cohomology algebra by studying perturbations of a free differential graded commutative model by means of techniques from deformation theory. This was initiated by M. Schlessinger and J. Stasheff [58]. A related and independent development, phrased in terms of what is called the functor \( D \), is due to N. Berikashvili and his students at Georgia, notably T. Kadeishvili and S. Saneblidze; see [29] for some details and references. A third approach in which only the underlying graded vector space was fixed is due to Y. Felix [10]. More remarks about the relationship between homological perturbations and deformation theory can be found in [32].

Kadeishvili contributed once more to the relationship between deformations and higher homotopies: He observed an interpretation of \( A_\infty \)-operations in terms of a suitable notion of twisting cochain, with respect to the Gerstenhaber operation in the Hochschild (cochain) complex [43]. An expanded version of that approach will appear as [45].

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