Notes on the Feynman path integral for the
Dirac equation

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This paper is a continuation of the author’s preceding one. In the preceding paper the author has rigorously constructed the Feynman path integral for the Dirac equation in the form of the sum-over-histories, satisfying the superposition principle, over all paths of one electron in space-time that goes in any direction at any speed, forward and backward in time with a finite number of turns. In the present paper, first we will generalize the results in the preceding paper and secondly prove in a direct way that our Feynman path integral satisfies the unitarity principle and the causality one.

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1 Introduction

This paper is a continuation of the author’s preceding one [8]. Let $T > 0$ be an arbitrary constant. We will study the Dirac equation

$$i\hbar \frac{\partial u}{\partial t}(t) = H(t)u(t)$$

$$:= \left[ c \sum_{j=1}^{d} \hat{\alpha}^{(j)} \left( \frac{\hbar}{i} \frac{\partial}{\partial x_j} - eA_j(t, x) \right) + \hat{\beta}mc^2 + eV(t, x)I_N \right] u(t), \quad (1.1)$$

where $t \in I_T := [-T, T]$, $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, $u(t) = ^t(u_1(t), \ldots, u_N(t)) \in \mathbb{C}^N$, $(V(t, x), A(t, x)) = (V, A_1, \ldots, A_d) \in \mathbb{R}^{d+1}$ is an electromagnetic potential, $\hat{\alpha}^{(j)}(j = 1, 2, \ldots, d)$ and $\hat{\beta}$ are constant $N \times N$ Hermitian matrices, $I_N$ is the $N \times N$ identity matrix, $c$ is the velocity of light, $\hbar$ is the Planck constant and $e$ is the charge of an electron. Though the relations

$$\hat{\alpha}^{(j)}\hat{\alpha}^{(k)} + \hat{\alpha}^{(k)}\hat{\alpha}^{(j)} = 2\delta^{jk}I_N, \quad \hat{\alpha}^{(0)} = \hat{\beta} \quad (1.2)$$

for $j, k = 0, 1, \ldots, d$ are assumed for the genuine Dirac equation (cf. (8) of §67 in [1]), in the present paper $\hat{\alpha}^{(j)}$ and $\hat{\beta}$ are assumed to be only Hermitian as in [8], where $\delta^{jk}$ denotes the Kronecker delta. For the sake of simplicity we suppose $\hbar = 1$ and $e = 1$ hereafter, and will sometimes omit $I_N$.

In the preceding paper [8] the author has rigorously constructed the Feynman path integral for the Dirac equation (1.1) in the form of the sum-over-histories, satisfying the superposition principle, over all possible paths of one electron in space-time that goes in any direction at any speed, forward and backward in time with a finite number of turns. In addition, the author has proved that the Feynman path integral constructed above satisfies the Dirac equation (1.1). It should be noted that Feynman had said for the application of his path integral to quantum electrodynamics that the electron goes in any
direction at any speed forward and backward in time, as seen on p.376 of \[2\], in \[3\] and on p.388 of \[13\].

In the present paper, first we will generalize the results in \[8\] and secondly prove in a direct way that our Feynman path integral satisfies the unitarity principle and the causality one. We basically owe our arguments in their proofs to the theory of pseudo-differential operators.

First, we will prove that the assumptions about a magnetic strength tensor can be generalized for the Feynman path integral to be determined. The assumptions about this haven’t been able to be generalized for a long time since \[7\] in 1999. Our proof will be obtained by returning to the original idea of Theorem 3.7 in \[6\].

The second generalization is in the $L^2$ space. In the present paper we will determine the Feynman path integral in the form of the sum-over-histories, satisfying the superposition principle, over all possible paths of one electron that goes in any direction at any speed, forward and backward in time particularly with a countably infinite number of turns. Here, $L^2 = L^2(\mathbb{R}^d)$ denotes the space of all square integrable functions on $\mathbb{R}^d$ with inner product $(f, g) := \int f(x)\overline{g(x)}dx$ and norm $\|f\|$, where $\overline{g(x)}$ is the complex conjugate of $g(x)$. Our proof will be obtained as in the proof of Theorem 2.1 in \[8\] by using the estimate (3.10) in the present paper.

Next, we will study the properties of our Feynman path integral for the Dirac equation. First, we will prove that the Feynman path integral makes a unitary operator on the $(L^2)^N$ space. This result gives another proof of the unitarity on $(L^2)^N$ of the fundamental solution to the Dirac equation (1.1), which is well known in the theory of partial differential equations. Our proof is based on the estimate (3.10) in the present paper too.
Secondly, we will prove that our Feynman path integral satisfies the causality principle, i.e. has the speed not exceeding the velocity of light of propagation of disturbances. This result gives another proof that every solution to the Dirac equation has the same property, which is also well known in the theory of partial differential equations. Our proof is based on the Paley-Wiener theorem (cf. Theorem IX.11 in [12]), which theorem characterizes the size of the support of functions by their Fourier transforms. As seen above, to construct the Feynman path integral we use the paths, of one electron in space-time, violating causality. Consequently our result, that the Feynman path integral satisfies causality, implies that the probability amplitudes for such paths are completely canceled out by the effect of interference among themselves and other probability ones, as argued in §1-3 of [4].

Our proof that the Feynman path integral satisfies unitarity and causality is more direct than the proof in the theory of partial differential equations that every solution to the Dirac equation has the same properties. Our results are yielded from (4.1) and (4.6), and (2.7) and the Paley-Wiener theorem, respectively.

The plan of the present paper is as follows. In §2 we will state the results on the Feynman path integral. In §3 we will prove them. In §4 we will state the results on unitarity and causality for the Feynman path integral and prove them.

2 Results on the Feynman path integral

For an $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and a multi-index $\alpha = (\alpha_1, \ldots, \alpha_d)$, we write $|x| = \sqrt{\sum_{j=1}^{d} x_j^2}$, $|\alpha| = \sum_{j=1}^{d} \alpha_j$, $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$, $\partial_{x_j} = \partial/\partial x_j$ and $\partial^\alpha = \partial^{\alpha_1} \cdots \partial^{\alpha_d}$. 
\[ \partial_{x_1} \cdots \partial_{x_d}. \] In the present paper we often use symbols \( C, C_\alpha, C_{\alpha,\beta} \) and \( C_a \) to write down constants, although these values are different in general.

Let us write the classical Hamiltonian function

\[ H(t,x,p) = c \sum_{j=1}^{d} \hat{\alpha}^{(j)}(p_j - A_j(t,x)) + \hat{\beta} mc^2 + V(t,x)I_N \]  

(2.1)

for \( H(t) \) defined by (1.1) as in (23) on p.261 of [1], where \( p \in \mathbb{R}^d \) is the canonical momentum. We write the kinetic momentum as \( \xi := p - A(t,x) \in \mathbb{R}^d \). Then the classical Lagrangian function is given by

\[ L(t,x,\dot{x},\xi) = p \cdot \dot{x} - H(t,x,p) \]

\[ = \xi \cdot \dot{x} + \dot{x} \cdot A(t,x) - V(t,x)I_N - (c\hat{\alpha} \cdot \xi + \hat{\beta} mc^2), \]  

(2.2)

where \( p \cdot \dot{x} = \sum_{j=1}^{d} p_j \dot{x}_j, \hat{\alpha} = (\hat{\alpha}^{(1)}), \ldots, \hat{\alpha}^{(d)} \) and \( \hat{\alpha} \cdot \xi = \sum_{j=1}^{d} \hat{\alpha}^{(j)} \xi_j \).

Let \( t \) and \( s \) be in \( I_T \) such that \( t \neq s \). For \( x \) and \( y \) in \( \mathbb{R}^d \) we define

\[ q_{t,s}^{l,s}(\theta) := y + \frac{\theta - s}{t - s}(x - y) \]  

(2.3)

in \( s \leq \theta \leq t \) or \( t \leq \theta \leq s \). Let \( \xi \in R^d \) and consider a path \((q_{x,y}^{l,s}(\theta), \xi) \in \mathbb{R}^{2d} \) in phase space. The classical action for this path is given by

\[ S(t,s;x,\xi,y) := \int_{t}^{s} L(\theta, q_{x,y}^{l,s}(\theta), q_{x,y}^{l,s}(\theta), \xi)d\theta = (x - y) \cdot \xi \]

\[ + \int_{t}^{s} \left\{ q_{x,y}^{l,s}(\theta) \cdot A(\theta, q_{x,y}^{l,s}(\theta)) - V(\theta, q_{x,y}^{l,s}(\theta)) \right\} d\theta - (t - s)(c\hat{\alpha} \cdot \xi + \hat{\beta} mc^2) \]

\[ = (x - y) \cdot \xi + (x - y) \cdot \int_{0}^{1} A(t - \theta \rho, x - \theta(x - y))d\theta \]

\[ - \rho \int_{0}^{1} V(t - \theta \rho, x - \theta(x - y))d\theta - \rho(c\hat{\alpha} \cdot \xi + \hat{\beta} mc^2), \quad \rho = t - s \]  

(2.4)

from (2.2), where \( q_{x,y}^{l,s}(\theta) = dq_{x,y}^{l,s}(\theta)/d\theta \). The matrices \( \hat{\alpha}^{(j)} \) and \( \hat{\beta} \) are assumed to be Hermitian and so is \( S(t,s;x,\xi,y) \). Noting (2.4), we will define

\[ S(s,s;x,\xi,y) := (x - y) \cdot \xi + (x - y) \cdot \int_{0}^{1} A(s, x - \theta(x - y))d\theta, \]  

(2.5)
which we write \( \int_{s}^{s} \mathcal{L}(\theta, q^{\text{s}, s}_{x,y}(\theta), \dot{q}^{\text{s}, s}_{x,y}(\theta), \xi) d\theta \) formally.

Let \( t_{i} \in I_{T} \) and \( t_{f} \in I_{T} \) be an initial time and a final one respectively, where \( t_{i} \leq t_{f} \) or \( t_{i} > t_{f} \). Take \( \tau_{j} \in I_{T} \) \((j = 1, 2, \ldots, \nu-1)\) and consider a time-division \( \Delta := \{ \tau_{j} \}_{j=1}^{\nu-1} \), where \( \tau_{j} \leq \tau_{j+1} \) or \( \tau_{j} > \tau_{j+1} \). We set \( \tau_{0} = t_{i} \) and \( \tau_{\nu} = t_{f} \). We take a point \( x \in \mathbb{R}^{d} \) and fix it. Taking points \( x^{(j)} \in \mathbb{R}^{d} \) \((j = 0, 1, \ldots, \nu-1)\) arbitrarily, we define a piecewise linear path \((\Theta_{\Delta}, q_{\Delta}(x^{(0)}, \ldots, x^{(\nu-1)}, x))\) in space-time \( I_{T} \times \mathbb{R}^{d} \) by joining \((\tau_{j}, x^{(j)}) \) \((j = 0, 1, \ldots, \nu)\) in order. Next, taking points \( \xi^{(j)} \in \mathbb{R}^{d} \) \((j = 0, 1, \ldots, \nu-1)\) arbitrarily, we also define a piecewise constant path \((\Theta_{\Delta}, \xi_{\Delta}(\xi^{(0)}, \ldots, \xi^{(\nu-1)}))\) in \( I_{T} \times \mathbb{R}^{d} \) by using \( \xi_{\Delta} \) that has the value \( \xi^{(j)} \) \((j = 0, 1, \ldots, \nu-1)\) for \( \theta \in [\tau_{j}, \tau_{j+1}] \) if \( \tau_{j} \leq \tau_{j+1} \) or \( \theta \in [\tau_{j+1}, \tau_{j}] \) if \( \tau_{j+1} < \tau_{j} \). We note that the paths \((\Theta_{\Delta}, q_{\Delta})\) and \((\Theta_{\Delta}, \xi_{\Delta})\) go in any direction forward and backward in time and that \( q_{\Delta} \) has any speed, even the infinite speed.

Let us consider the path \((\Theta_{\Delta}, q_{\Delta}(x^{(0)}, \ldots, x^{(\nu-1)}, x), \xi_{\Delta}(\xi^{(0)}, \ldots, \xi^{(\nu-1)}))\) in \( I_{T} \times \mathbb{R}^{2d} \) connecting \((t_{i}, x^{(0)}, \xi^{(0)})\) with \((t_{f}, x, \xi^{(\nu-1)})\). We define the probability amplitude \( \exp \ast iS(t_{f}, t_{i}, q_{\Delta}, \xi_{\Delta}) \) for this path in terms of the classical action \( \mathcal{L}_{\text{1}} \) and \( \mathcal{L}_{\text{2}} \) by the product of unitary matrices

\[
\exp i \int_{t_{i}}^{t_{f}} \mathcal{L}(\theta, q^{t_{i}, t_{f}, \nu-1}_{x,x^{(\nu-1)}}(\theta), \dot{q}^{t_{i}, t_{f}, \nu-1}_{x,x^{(\nu-1)}}(\theta), \xi^{(\nu-1)})d\theta \cdot \exp i \int_{t_{i}}^{t_{f}} \mathcal{L}(\theta, q^{t_{i}, t_{f}, \nu-1, \nu-2}_{x^{(\nu-1)}, x^{(\nu-2)}}(\theta), \dot{q}^{t_{i}, t_{f}, \nu-1, \nu-2}_{x^{(\nu-1)}, x^{(\nu-2)}}(\theta), \xi^{(\nu-2)})d\theta \cdots \exp i \int_{t_{i}}^{t_{f}} \mathcal{L}(\theta, q^{t_{i}, t_{f}, \nu-1}_{x,x^{(0)}}(\theta), \dot{q}^{t_{i}, t_{f}, \nu-1}_{x,x^{(0)}}(\theta), \xi^{(0)})d\theta.
\]

(2.6)

Let \( \mathcal{S} = \mathcal{S}(\mathbb{R}^{d}) \) be the Schwartz space of all rapidly decreasing functions on \( \mathbb{R}^{d} \) with the well-known topology. We take a function \( \chi \in \mathcal{S}(\mathbb{R}^{d}) \) such that \( \chi(0) = 1 \). Let \( f = (f_{1}, \ldots, f_{N}) \in \mathcal{S}(\mathbb{R}^{d})^{N} \) and define an approximation
\(K_{\Delta}(t_f, t_i) f\) of the Feynman path integral for the Dirac equation (1.1) by

\[
K_{\Delta}(t_f, t_i) f = \int \cdots \int e^{iS(t_f, t_i, \Delta)} f(q^{(0)}) \prod_{j=0}^{\nu-1} \{ \chi(\epsilon x^{(j)}) \chi(\epsilon \xi^{(j)}) \} \, dx^{(0)} \cdots dx^{(\nu-1)}
\]

\[= \lim_{\epsilon \to 0^+} \int \cdots \int e^{iS(t_f, t_i, \Delta)} f(x^{(0)}) \prod_{j=0}^{\nu-1} \{ \chi(\epsilon x^{(j)}) \chi(\epsilon \xi^{(j)}) \} \, dx^{(0)} \cdots dx^{(\nu-1)}
\cdot d\xi^{(0)} \cdots d\xi^{(\nu-1)},
\]

(2.7)

where \(d\xi^{(j)} = (2\pi)^{-d} d\xi^{(j)}\). As stated in Theorem 2.A below, \(K_{\Delta}(t_f, t_i) f\) is determined independently of the choice of \(\chi\). Hence the integral (2.7) is often called the oscillatory integral and written as

\[
\text{Os} - \int \cdots \int e^{iS(t_f, t_i, \Delta)} f(x^{(0)}) \, dx^{(0)} \cdots dx^{(\nu-1)} d\xi^{(0)} \cdots d\xi^{(\nu-1)}
\]

(cf. p. 45 of [10]).

For \(f = (f_1, \ldots, f_N) \in L^2(\mathbb{R}^d)^N\) we write its norm \(\sqrt{\sum_{j=1}^{N} \|f_j\|^2}\) as \(\|f\|\).

Let \(E(t, x) = (E_1, \ldots, E_d) \in \mathbb{R}^d\) and \((B_{jk}(t, x))_{1 \leq j < k \leq d} \in \mathbb{R}^{d(d-1)}\) be electric strength and a magnetic strength tensor, respectively. In Theorems 2.1 and 2.2 of [8] we have proved the following.

**Theorem 2.A.** Let \(\partial_x^\alpha E_j(t, x) (j = 1, 2, \ldots, d), \partial_x^\alpha B_{jk}(t, x) (1 \leq j < k \leq d)\) and \(\partial_t B_{jk}(t, x)\) be continuous in \(I_T \times \mathbb{R}^d\) for all \(\alpha\). We assume

\[
|\partial_x^\alpha E_j(t, x)| \leq C_\alpha, \quad |\alpha| \geq 1,
\]

(2.8)

\[
|\partial_x^\alpha B_{jk}(t, x)| \leq C_\alpha < x >^{-(1+\delta_\alpha)}, \quad |\alpha| \geq 1
\]

(2.9)

in \(I_T \times \mathbb{R}^d\) with constants \(\delta_\alpha > 0\) for \(j = 1, 2, \ldots, d\) and \(1 \leq j < k \leq d\), where \(< x > = \sqrt{1 + |x|^2}\). Let \((V, A)\) be an electromagnetic potential inducing \(E(t, x)\) and \((B_{jk}(t, x))_{1 \leq j < k \leq d}\) via equations

\[
E = -\frac{\partial A}{\partial t} - \frac{\partial V}{\partial x},
\]

\[
B_{jk} = \frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k} \quad (1 \leq j < k \leq d)
\]

(2.10)
such that \(V, \partial_x V, \partial_t A_k\) and \(\partial_x A_k\) \((j, k = 1, 2, \ldots, d)\) are continuous in \(I_T \times \mathbb{R}^d\), where \(\partial V/\partial x = (\partial V/\partial x_1, \ldots, \partial V/\partial x_d)\). We take \(t_i\) and \(tf\) in \(I_T\). Let \(\tau_j \in I_T\) \((j = 1, 2, \ldots, \nu - 1)\), determine \(\Delta = \{\tau_j\}_{j=1}^{\nu-1}\) and define \(K_{D\Delta}(tf, ti) f\) for \(f \in S^N\) by (2.7).

Then we have: (1) \(K_{D\Delta}(tf, ti)\) on \(S^N\) is determined independently of the choice of \(\chi \in S\) and can be extended to a bounded operator on \((L^2)^N\). (2) Let \(f \in (L^2)^N\). Let \(L_0 \geq 0\) be an arbitrary constant and consider only time-divisions \(\Delta\) satisfying
\[
\sum_{j=0}^{\nu-1} |\tau_{j+1} - \tau_j| \leq L_0. \tag{2.11}
\]
Then, as \(|\Delta| := \max_{j=0,1,\ldots,\nu-1} |\tau_{j+1} - \tau_j| \to 0\), \(K_{D\Delta}(tf, ti) f\) converges in \((L^2)^N\) uniformly with respect to \(tf\) and \(tf\) in \(I_T\), and this limit \(K_D(tf, ti) f\), called the Feynman path integral, is determined independently of the choice of \(L_0\). (3) \(K_D(tf, ti) f\) for \(f \in (L^2)^N\) belongs to \(\mathcal{E}^0_k(I_T; (L^2)^N)\) and satisfies the Dirac equation (1.1) in the distribution sense with \(u(ti) = f\), where \(\mathcal{E}^0_k(I_T; (L^2)^N)\) \((j = 0, 1, \ldots)\) denotes the space of all \((L^2)^N\)-valued \(j\)-times continuously differentiable functions on \(I_T\). (4) Let \(\psi(t, x)\) be a real-valued function such that \(\partial_x \partial_x \psi(t, x)\) and \(\partial t \partial_x \psi(t, x)\) \((j, k = 1, 2, \ldots, d)\) are continuous in \(I_T \times \mathbb{R}^d\). We consider the gauge transformation
\[
V' = V - \frac{\partial \psi}{\partial t}, \quad A'_j = A_j + \frac{\partial \psi}{\partial x_j} \quad (j = 1, 2, \ldots, d) \tag{2.12}
\]
and write (2.7) for this \((V', A')\) as \(K'_{D\Delta}(tf, ti) f\). Then we have a formula
\[
k'_{D\Delta}(tf, ti) f = e^{i\psi(tf, \cdot)} K_{D\Delta}(tf, ti) (e^{-i\psi(ti, \cdot)} f) \tag{2.13}
\]
for all \(f \in (L^2)^N\) and so have the same formula for \(K_{D}(tf, ti) f\).

Let \(M\) and \(a\) be positive integers. We introduce the weighted Sobolev spaces \(B^a_M(\mathbb{R}^d)^N := \{f \in L^2(\mathbb{R}^d)^N; \|f\|_{B^a_M} := \|f\| + \sum_{|\alpha|=a} \|\partial^\alpha f\| + \sum_{|\alpha|=a} \|\partial^\alpha_x f\| < \infty\}\). Let \(B^a_M(\mathbb{R}^d)^N\) denote their dual spaces and set \(B^a_M(\mathbb{R}^d)^N := L^2(\mathbb{R}^d)^N\).
Theorem 2.B. Besides the assumptions of Theorem 2.A we assume the following: (1) We have
\[ |\partial^\alpha_x A_j(t,x)| \leq C_\alpha, \quad |\alpha| \geq 1 \]  \tag{2.14}
in \mathcal{I}_T \times \mathbb{R}^d \text{ for } j = 1, 2, \ldots, d. (2) There exists an integer \( M \geq 1 \) such that
\[ |\partial^\alpha_x \partial_t A_j(t,x)| \leq C_\alpha x^M \]  \tag{2.15}
for all \( \alpha \) in \( \mathcal{I}_T \times \mathbb{R}^d \). Then we have: (1) \( K_{D\Delta}(t_f,t_i) \) on \( S^N \) can be extended to a bounded operator on \( (B^a_{M+1})^N \) \( (a = 0, 1, \ldots) \). (2) Let \( f \in (B^a_{M+1})^N \) and \( L_0 \geq 0 \) an arbitrary constant. Then, as \( |\Delta| \to 0 \) under the assumption (2.11), \( K_{D\Delta}(t_f,t_i)f \) converges to \( K_D(t_f,t_i)f \) in \( (B^a_{M+1})^N \) uniformly with respect to \( t_f \) and \( t_i \) in \( \mathcal{I}_T \).

Remark 2.1. In [8] we used \( \chi \in C^\infty_0(\mathbb{R}^d) \), i.e. an infinitely differentiable function in \( \mathbb{R}^d \) with compact support, to define \( K_{D\Delta}(t_f,t_i) \) by (2.7) in place of \( \chi \in \mathcal{S}(\mathbb{R}^d) \). However, the proof of Proposition 3.2 in [8] or Proposition 3.2 in the present paper assures us that Theorems 2.A and 2.B above remain true.

Remark 2.2. In Theorem 2.2 in [8] we assumed
\[ |\partial^\alpha_x V(t,x)| \leq C_\alpha x^M, \quad |\alpha| \geq 1 \]  \tag{2.16}
besides (2.13) and (2.15). We note that (2.16) are derived from (2.8), (2.10) and (2.15).

We will state Theorems 2.1 and 2.2 as the main results on the Feynman path integral.

Theorem 2.1. In Theorems 2.A and 2.B we replace the assumption (2.9) with (2.14), (2.15) and
\[ |\partial^\alpha_x \partial_t B_{jk}(t,x)| \leq C_\alpha x^{-(1+\delta_\alpha)}, \quad |\alpha| \geq 1 \]  \tag{2.17}
for $1 \leq j < k \leq d$ in $I_T \times \mathbb{R}^d$ where $\delta_\alpha > 0$ are constants. Then the same assertions as in Theorems 2.A and 2.B hold respectively.

Remark 2.3. If $A_j$ ($j = 1, 2, \ldots, d$) satisfying (2.14) are independent of $t \in I_T$, (2.15) holds and (2.17) follows from (2.10). Hence we can see that Theorem 2.1 gives new results. The assumption (2.9) hasn’t been able to be generalized for a long time since [7] in 1999.

We will consider the Feynman path integral in the $L^2$ space.

**Theorem 2.2.** We suppose the assumptions of Theorem 2.A or make in Theorem 2.A the same replacement of the assumption (2.9) as in Theorem 2.1. For time-divisions $\Delta = \{\tau_j\}_{j=1}^{\nu-1}$ we set

$$\sigma(\Delta) := \sum_{j=0}^{\nu-1} (\tau_{j+1} - \tau_j)^2. \quad (2.18)$$

Then we obtain: (1) Under the assumption $\sigma(\Delta) \leq 1$ we have

$$\|K_{D\Delta}(t_f, t_i)f\| \leq e^{K_0\sigma(\Delta)}\|f\| \quad (2.19)$$

for all $t_i, t_f$ in $I_T$ with a constant $K_0 \geq 0$. (2) Let $f \in (L^2)^N$. Then, as $\sigma(\Delta) \to 0$, $K_{D\Delta}(t_f, t_i)f$ converges to the Feynman path integral $K_D(t_f, t_i)f$ in $(L^2)^N$ uniformly with respect to $t_f$ and $t_i$ in $I_T$.

Remark 2.4. Theorem 2.2 gives a generalization of Theorem 2.A and a part of Theorem 2.1 because of

$$\sigma(\Delta) = \sum_{j=0}^{\nu-1} (\tau_{j+1} - \tau_j)^2 \leq |\Delta| \sum_{j=0}^{\nu-1} |\tau_{j+1} - \tau_j|. $$

The corollary below follows from (2) of Theorem 2.2.
Corollary 2.3. Consider time-divisions \( \Delta(n) := \{\tau_j^{(n)}\}_{j=1}^{\nu(n)-1} \) \((n = 1, 2, \ldots)\) such that \(\lim_{n \to \infty} \sigma(\Delta(n)) = 0\) and for each \(n\) there exist \(j_k\) and \(j'_k\) \((k = 1, 2, \ldots, n)\) satisfying \(\tau_{j_k}^{(n)} = T\) and \(\tau_{j'_k}^{(n)} = -T\). Let \(f \in (L^2)^N\). Then, under the assumptions of Theorem 2.2 we have

\[
\lim_{n \to \infty} K_{D\Delta(n)}(t_f, t_i) f = K_D(t_f, t_i) f
\]  

(2.20)
in \((L^2)^N\) uniformly with respect to \(t_f\) and \(t_i\) in \(I_T\).

Example 2.1. We can easily construct time-divisions \(\Delta(n) \) \((n = 1, 2, \ldots)\) satisfying the properties stated in Corollary 2.3. In fact, let \(t_i < t_f\) and take \(\nu(n) = (2n + 1)n^2\). We can easily determine \(\Delta(n) = \{\tau_j\}_{j=1}^{\nu(n)-1}\) such that

\[
t_i < \tau_1 < \tau_2 < \ldots < \tau_{j_1} = T > \tau_{j_1+1} > \ldots > \tau_{j'_1} = -T < \tau'_{j'_1+1} < \ldots < \tau_{j_2} = T > \tau_{j_2+1} > \ldots > \tau_{j'_n} = -T < \tau'_{j'_n+1} < \ldots < \tau_{\nu(n)-1} < t_f
\]

and \(|\tau_{j+1} - \tau_j| \leq 2T/n^2\). For example, we have only to take \(j_k = (2k - 1)n^2\) and \(j'_k = 2kn^2\) for \(k = 1, 2, \ldots, n\). Then we have

\[
\sigma(\Delta(n)) = \sum_{j=0}^{\nu(n)-1} (\tau_{j+1} - \tau_j)^2 \leq (2n + 1)n^2 \left(\frac{2T}{n^2}\right)^2,
\]

which tends to zero as \(n \to \infty\).

Remark 2.5. The left-hand side of (2.20) gives the Feynman path integral in the form of sum-over-histories over all paths of one electron that goes forward and backward in time with a countably infinite number of turns.
3 Proofs of Theorems 2.1 and 2.2

Let $t$ and $s$ be in $I_T$. We set

$$
\Psi_j(t, s; x, y, z) := -\int_0^1 A_j(s, z + \theta(x - z))d\theta \\
+ (t-s) \int_0^1 \int_0^1 \sigma_1 E_j(t - \sigma_1(t-s), y + \sigma_1(z-y) + \sigma_1\sigma_2(x-z))d\sigma_1d\sigma_2 \\
+ \sum_{k=1}^d (y_k - z_k) \int_0^1 \int_0^1 \sigma_1 B_{jk}(t - \sigma_1(t-s), y + \sigma_1(z-y) + \sigma_1\sigma_2(x-z))d\sigma_1d\sigma_2
$$

(3.1)

as in (3.7) of [8] and

$$
\Psi_j'(t, s; x, y, z) := -\int_0^1 A_j(s, z + \theta(x - z))d\theta \\
+ (t-s) \int_0^1 \int_0^1 \sigma_1 E_j(t - \sigma_1(t-s), y + \sigma_1(z-y) + \sigma_1\sigma_2(x-z))d\sigma_1d\sigma_2 \\
+ (t-s) \int_0^1 d\theta \sum_{k=1}^d (y_k - z_k) \int_0^1 \int_0^1 \sigma_1(1 - \sigma_1) \frac{\partial B_{jk}}{\partial t}(s + \theta(t-s)(1 - \sigma_1), \\
y + \sigma_1(z-y) + \sigma_1\sigma_2(x-z))d\sigma_1d\sigma_2
$$

(3.2)

for $j = 1, 2, \ldots, d$.

Lemma 3.1. We have

$$(x-z) \cdot \Psi(t, s; x, y, z) = (x-z) \cdot \Psi'(t, s; x, y, z).$$

(3.3)

Under the assumptions (2.8), (2.14) and (2.17) we have

$$
|\partial_x^\alpha \partial_y^\beta \partial_z^\gamma \Psi_j'(t, s; x, y, z)| \leq C_{\alpha, \beta, \gamma}, \quad |\alpha + \beta + \gamma| \geq 1
$$

(3.4)

in $I_T^2 \times \mathbb{R}^{3d}$ for $j = 1, 2, \ldots, d$. 

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Proof. Let us return to the proof of Lemma 3.4 in [8]. Let \( \Lambda \) be the 2-dimensional plane in \( I_T \times \mathbb{R}^d \) determined in (3.8) of [8]. Then we have

\[
\lim_{t \to s \pm 0} \int \int_{\Lambda} d(A \cdot dx - V dt) = 0.
\]

Hence from the proof of Lemma 3.2 in [6] we can see

\[
\sum_{j=1}^{d} (x_j - z_j) \sum_{k=1}^{d} (y_k - z_k) \int_{0}^{1} \int_{0}^{1} \sigma_1 B_{jk}(s, y + \sigma_1(z - y))
\]

\[
+ \sigma_1 \sigma_2(x - z))d\sigma_1 d\sigma_2 = 0
\]

for all \((x, y, z) \in \mathbb{R}^{3d}\). Consequently, subtracting the coefficient of \(x_j - z_j\) in the above from \(\Psi_j(t, s; x, y, z)\), we get (3.2) and (3.3).

It follows from (2.8) and (2.14) that the first term and the second one on the right-hand side of (3.2) satisfy (3.4). Applying Lemma 3.5 in [6] to the third term on the right-hand side of (3.2), we can see from (2.17) that the third term satisfies (3.4) as well. Thus, the proof is complete.

Now we will prove Theorem 2.1. Let us define an operator on \( S^{N} \) by

\[
(G_\epsilon(t, s)f)(x) = \int \int e^{iS(t, s; x, \xi, y)} f(y) \chi(\epsilon \xi) dy d\xi \tag{3.5}
\]

for \( \epsilon > 0 \) in terms of (2.4) and (2.5) as in (1.12) of [8], where \( \chi \in S(\mathbb{R}^d) \) such that \( \chi(0) = 1 \). Let \( G_\epsilon(t, s)^* \) denote the formally adjoint operator of \( G_\epsilon(t, s) \).

We will do use Lemma 3.1. Then, noting Lemma 3.4 in [8], as in the proof of Proposition 3.5 in [8] we can prove

\[
(G_\epsilon(t, s)^*G_\epsilon(t, s)f)(x) = \int \int e^{i(x-z) \cdot \xi} d\xi d\xi \int \int e^{-i\eta \cdot w} e^{i(t-s)(c\tilde{\alpha} \cdot \xi + c\tilde{\alpha} \cdot \Psi' + \tilde{\beta} m^2)}
\]

\[
\times e^{-i(t-s)(c\tilde{\alpha} \cdot \xi + c\tilde{\alpha} \cdot \Psi' + \tilde{\beta} m^2 - c\tilde{\alpha} \cdot \eta)} \chi(e(\xi + \Psi')) \chi(e(\xi + \Psi' - \eta)) f(z)dud\eta \tag{3.6}
\]
for \( f \in \mathcal{S}^N \) with \( \Psi' = \Psi'(t, s; x, w + z, z) \), where \( \eta \in \mathbb{R}^d \) and \( w \in \mathbb{R}^d \). Hence, taking account of (3.4), we can prove Theorem 2.1 as in the proofs of Theorems 2.1 and 2.2 in [8].

Next we will prove Theorem 2.2. As proved in Lemma 6.1 of [7], under the assumptions of Theorem 2.2 there exists an integer \( M \geq 1 \) such that we have (2.14)-(2.16). Therefore, we can see from Proposition 3.6 in [8] that under the assumptions of Theorem 2.2 we have (2.14)-(2.16), and (3.4) or

\[
|\partial_x^\alpha \partial_y^\beta \partial_z^\gamma \Psi_j(t, s; x, y, z)| \leq C_{\alpha, \beta, \gamma}, \quad |\alpha + \beta + \gamma| \geq 1
\] (3.7)

in \( I_T \times \mathbb{R}^d \) for \( j = 1, 2, \ldots, d \).

Let \( G_\epsilon(t, s) \) be the operator defined by (3.5). The following proposition has already been shown in the proof of Proposition 3.2 of [8].

**Proposition 3.2.** Assume (2.16) and

\[
|\partial_x^\alpha A_j(t, x)| \leq C_{\alpha} < x >^M, \quad |\alpha| \geq 1
\]

in \( I_T \times \mathbb{R}^d \) for \( j = 1, 2, \ldots, d \). Then, \( \{G_\epsilon(t, s)\}_{0 < \epsilon \leq 1} \) is a bounded family of operators from \( \mathcal{S}^N \) into itself and there exists an operator \( G(t, s) \) on \( \mathcal{S}^N \) independent of the choice of \( \chi \) such that we have

\[
G(t, s)f = \lim_{\epsilon \to 0} G_\epsilon(t, s)f
\] (3.8)

in \( \mathcal{S}^N \) for all \( f \in \mathcal{S}^N \) uniformly with respect to \( t \) and \( s \) in \( I_T \). In particular, we have \( G(s, s)f = f \) for all \( f \in \mathcal{S}^N \).

The following proposition has been stated as Theorem 5.2 of [8], that had been proved in [5].

**Proposition 3.3.** Under the assumptions of Proposition 3.2 consider the Dirac equation (1.1) with \( u(s) = f \in B^3_{3M+1} (a = 0, 1, 2, \ldots) \) for \( s \in I_T \). Then
there exists a unique solution $U(t, s)f \in \mathcal{E}_t^0(I_T; B_{M+1}^a) \cap \mathcal{E}_t^1(I_T; B_{M+1}^{a-1})$, which satisfies

$$\|U(t, s)f\| = \|f\|, \quad \|U(t, s)f\|_{B_{M+1}^a} \leq C_a(T)\|f\|_{B_{M+1}^a} \quad (a = 1, 2, \ldots) \quad (3.9)$$

for $t$ and $s$ in $I_T$.

We have proved (3.4) or (3.7) under the assumptions of Theorem 2.2. Hence we can prove the following as in the proof of Theorem 3.3 in [8].

**Proposition 3.4.** Under the assumptions of Theorem 2.2 we have: (1) $G(t, s)$ defined in Proposition 3.2 can be extended to a bounded operator on $(L^2)^N$. (2) There exists a constant $K_0 \geq 0$ such that

$$\|G(t, s)f\| \leq e^{K_0(t-s)^2}\|f\| \quad (3.10)$$

for all $f \in L^2$ and $t, s \in I_T$ with $|t - s| \leq 1$.

**Remark 3.1.** The inequality (3.10) above has been yielded directly from (3.16) in the proof of Theorem 3.3 of [8]. As in the completely same way, we can prove

$$\|G(t, s)f\|^2 = (f, f) + (t - s)^2(P(t, s; X, D_X, X')f, f) \geq \|f\|^2 - 2K_0(t - s)^2\|f\|^2$$

for $t$ and $s$ in $I_T$ with $|t - s| \leq 1$. This shows

$$\|G(t, s)f\|^2 \geq e^{-4K_0(t-s)^2}\|f\|^2 \quad (3.11)$$

for $t$ and $s$ in $I_T$ with $|t - s| \leq 1$ and $4K_0(t - s)^2 \leq \log 2$, because $1 - \theta \geq e^{-2\theta}$ holds for $0 \leq \theta \leq \log 2/2$. 

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Noting Lemma 3.1, we can prove the following as well as Proposition 3.4, as in the proof of Proposition 5.3 of [8].

**Proposition 3.5.** Under the assumptions of Theorem 2.2 we have

\[ \|G(t, s)f - U(t, s)f\|_{B_{M+1}^a} \leq C_a(t - s)^2 \|f\|_{B_{M+1}^{a+2}}, \quad -T \leq s, t \leq T \quad (3.12) \]

for \( a = 0, 1, 2, \ldots \) and \( f \in S^N \).

Now, let us prove Theorem 2.2. We take an electromagnetic potential \((V, A)\) satisfying (2.14)-(2.16) that induces \( E(t, x) \) and \((B_{jk}(t, x))_{1 \leq j < k \leq d}\), as stated in the early part of this section. For this \((V, A)\) we will prove the assertions (1) and (2) in Theorem 2.2. The general case can be proved from these results by the use of the gauge transformation (2.12), as in the proof of Theorem 2.1 of [8] on pp. 506-507.

For \( f \in S^N \) we can write (2.7) by using (3.5) and Proposition 3.2 as

\[ K_{D\Delta}(t_f, t_i)f = \lim_{\epsilon \to 0} G_\epsilon(t_f, \tau_{\nu-1})\chi(\epsilon)G_\epsilon(\tau_{\nu-1}, \tau_{\nu-2})\chi(\epsilon)\cdots\chi(\epsilon)G_\epsilon(\tau_1, t_i)f \]

\[ = G(t_f, \tau_{\nu-1})G(\tau_{\nu-1}, \tau_{\nu-2})\cdots G(\tau_1, t_i)f \]

in \( S^N \), which proves

\[ K_{D\Delta}(t_f, t_i)f = G(t_f, \tau_{\nu-1})G(\tau_{\nu-1}, \tau_{\nu-2})\cdots G(\tau_1, t_i)f \quad (3.13) \]

in \((L^2)^N\) for \( f \in (L^2)^N\) from Proposition 3.4. Hence, applying (3.10) to (3.13), we can easily prove (2.19) in Theorem 2.2 from (2.18). Consequently, (1) in Theorem 2.2 has been proved.

Let \( f \in (B_{M+1}^2)^N \). From (3.13) we can write

\[ K_{D\Delta}(t_f, t_i)\|f - U(t_f, t_i)f = G(t_f, \tau_{\nu-1})\cdots G(\tau_1, t_i)f - U(t_f, \tau_{\nu-1})\cdots U(\tau_1, t_i)f \]

\[ = \sum_{j=0}^{\nu-1} G(t_f, \tau_{\nu-1})\cdots G(\tau_{j+2}, \tau_{j+1})\{G(\tau_{j+1}, \tau_j) - U(\tau_{j+1}, \tau_j)\}U(\tau_j, t_i)f. \]

\[ (3.14) \]
Let $\sigma(\Delta) \leq 1$ and apply Propositions 3.3-3.5 to the last equation in (3.14). Then we have

$$\| K_D \Delta (t_f, t_i) f - U(t_f, t_i) f \| \leq \sum_{j=0}^{\nu-1} e^{K_0 \sigma(\Delta)} C_0 (\tau_{j+1} - \tau_j)^2 \| U(\tau_j, t_i) f \|_{B_{M+1}^2}$$

$$\leq C_0^\nu \sigma(\Delta) e^{K_0 \sigma(\Delta)} \| f \|_{B_{M+1}^2}. \quad (3.15)$$

Let $f \in (L^2)^N$ and $\sigma(\Delta) \leq 1$. For an arbitrary constant $\epsilon > 0$ take a function $g \in (B_{M+1}^2)^N$ such that $\| g - f \| < \epsilon$. Then, using (2.19), (3.9) and (3.15), we can prove

$$\| K_D \Delta (t_f, t_i) f - U(t_f, t_i) f \| \leq \| K_D \Delta (t_f, t_i) g - U(t_f, t_i) g \| + \| K_D \Delta (t_f, t_i) (f - g) \| + \| U(t_f, t_i) (f - g) \| \leq C_0^\nu \sigma(\Delta) e^{K_0 \sigma(\Delta)} \| g \|_{B_{M+1}^2} + e^{K_0 \sigma(\Delta)} \| g - f \| + \| g - f \|, \quad (3.16)$$

which shows

$$\lim_{\sigma(\Delta) \to 0} \| K_D \Delta (t_f, t_i) f - U(t_f, t_i) f \| \leq 2\epsilon.$$

Consequently we have been able to prove (2) of Theorem 2.2. Therefore, the proof of Theorem 2.2 has been completed.

### 4 Unitarity and Causality

In this section we will study the properties of the Feynman path integral $K_D(t_f, t_i)$ determined in Theorems 2.1 and 2.2.

First we will prove the unitarity of $K_D(t_f, t_i)$ on $(L^2)^N$. This result gives another proof of the unitarity of the fundamental solution $U(t_f, t_i)$ to (1.1) on $(L^2)^N$ because of $U(t_f, t_i) = K_D(t_f, t_i)$ in Theorem 2.2, which is well known.
in the theory of partial differential equations. We note that we can prove Theorem 2.2 without the use of the unitarity of \(U(t_f, t_i)\).

**Theorem 4.1.** Under the assumptions of Theorem 2.2 \(K_D(t_f, t_i)\) is unitary on \((L^2)^N\).

**Proof.** We have proved (2.19) in Theorem 2.2. In the same way we can prove

\[
\|K_D\Delta(t_f, t_i)f\| \geq e^{-2K_0\sigma(\Delta)}\|f\|
\]

for small \(\sigma(\Delta)\) from (3.11) and (3.13), which shows

\[
e^{-2K_0\sigma(\Delta)}\|f\| \leq \|K_D\Delta(t_f, t_i)f\| \leq e^{K_0\sigma(\Delta)}\|f\|.
\]  

(4.1)

Letting \(\sigma(\Delta)\) tend to zero, we obtain

\[
\|K_D(t_f, t_i)f\| = \|f\|  
\]

(4.2)

for \(f \in (L^2)^N\) from (2) of Theorem 2.2.

From (3.5) we can easily have

\[
(G_\varepsilon(t, s)^*f)(x) = \int\int e^{-iS(t,s;y,\xi,x)}f(y)\chi(\varepsilon \xi)dyd\xi
\]

(4.3)

for \(f \in \mathcal{S}^N\). From (2.4) and (2.5) we can write

\[
S(t, s; x, \xi, y)
= (x - y) \cdot \xi + \int_{q_{t,s}^{y}}^{q_{t,s}^{x}} (A \cdot dx - Vdt) - (t - s)(c\hat{\alpha} \cdot \xi + \hat{\beta}mc^2)
\]

(4.4)

as in the proof of (2.3) in [6], where

\[
q_{t,s}^{x,y}(\theta) = (\theta, q_{t,s}^{x,y}(\theta)) \in I_T \times \mathbb{R}^d  
\]

\((s \leq \theta \leq t \text{ or } t \leq \theta \leq s)\).
This gives
\[
-S(t, s; y, \xi, x) = -(y - x) \cdot \xi - \int_{q_{t,y}^{t,s}} (A \cdot dx - V dt)
\]
\[
+ (t - s)(c\hat{\alpha} \cdot \xi + \hat{\beta}mc^2) = (x - y) \cdot \xi + \int_{q_{t,y}^{t,s}} (A \cdot dx - V dt)
\]
\[
- (s - t)(c\hat{\alpha} \cdot \xi + \hat{\beta}mc^2) = S(s, t; x, \xi, y),
\]
which shows
\[
(G\epsilon(t, s)^* f)(x) = (G\epsilon(s, t) f)(x) \quad (4.5)
\]
together with (4.3). Consequently the expression (3.13) indicates
\[
K_D \Delta^*(t_f, t_i) f = K_D \Delta^*(t_i, t_f) f \quad (4.6)
\]
for \(f \in (L^2)^N\) with the time-division \(\Delta^*\) corresponding to \(\Delta\), which proves
\[
\|K_D(t_f, t_i)^* f\| = \|K_D(t_i, t_f) f\| = \|f\| \quad (4.7)
\]
from (2) of Theorem 2.2 and (4.2).

The equalities (4.2) and (4.7) imply that \(K_D(t_f, t_i)\) is unitary on \((L^2)^N\), as well known. In fact, it is easily seen from the polarization identity (cf. p.63 of [12]) that if and only if \(F := K_D(t_f, t_i)\) is isometric on \((L^2)^N\), \((F f, F g) = (f, g)\) are true for all \(f\) and \(g\) in \((L^2)^N\), which is equivalent to \(F^* F = \text{Identity on } (L^2)^N\). Since \(F^*\) is also isometric, \(FF^* = \text{Identity on } (L^2)^N\) is yielded. Thus, it has been proved that \(F = K_D(t_f, t_i)\) is unitary on \((L^2)^N\).

Secondly, we will prove that the Feynman path integral \(K_D(t_f, t_i) f\) satisfies the causality principles, i.e. has the speed not exceeding the velocity of light of propagation of disturbances. This result gives another proof that every solution to the Dirac equation (1.1) has the same property, which is also well known in the theory of partial differential equations. For example, see the 5th
problem in §5.3 on p.170 of [9], Theorem 6.10 and its Note 2 on pp.364-365 of [11] and §4 in Chapter IV on p.79 of [14]. In all of these references, the method of proving the causality principle is based on the energy inequality and the introduction of a hypersurface spacelike with respect to the operator defining the equation. Thereby, a delicate analysis is needed.

Let \( \hat{\alpha}^{(j)} \) \( (j = 1, 2, \ldots, d) \) be the \( N \times N \) Hermitian matrix in (1.1) and \( \lambda_k(\xi) \) \( (k = 1, 2, \ldots, N) \) the eigenvalue of the matrix \( \hat{\alpha} \cdot \xi \), which is continuous on \( \mathbb{R}_\xi^d \). We set

\[
\lambda_{\text{max}} := \max_{j=1,2,\ldots,N} \max_{|\xi|=1} \lambda_j(\xi),
\]

which is non-negative because of

\[
\lambda_j(s\xi) = s\lambda_j(\xi) \quad (s \in \mathbb{R}).
\]
Proof. From (1.2) we can easily have

\[(\hat{\alpha} \cdot \xi)^2 = |\xi|^2 \quad (\xi \in \mathbb{R}^d)\]  

(4.10)

by the same argument as in §67 of [1], which shows \(\lambda_j(\xi)^2 = |\xi|^2\) and so \(|\lambda_j(\xi)| = |\xi|\). It follows from the hermiticity of \(\hat{\alpha} \cdot \xi\) that \(\lambda_j(\xi)\) is real, which implies \(\lambda_{\text{max}} = 1\) from (4.8) and (4.9). Consequently we obtain Corollary 4.3 from Theorem 4.2.

Now, we will state the well-known results as the Paley-Wiener theorem (cf. Theorem IX.11 on p.333 in [12]) and Lie product formula (cf. Theorem VIII.29 on p.295 in [12]) that will be used to prove Theorem 4.2.

**Proposition 4.A** (Paley-Wiener). Let \(\zeta := \xi + i\eta \in \mathbb{C}^d\) be complex variables where \(\xi \in \mathbb{R}^d\) and \(\eta \in \mathbb{R}^d\). An entire analytic function \(g(\zeta)\) on \(\mathbb{C}^d\) is the Fourier transform \(\hat{f}(\zeta) := \int e^{-i\zeta \cdot x} f(x) dx\) of a function \(f(x) \in C_0^\infty(\mathbb{R}^d)\) with support in \(B(0; R)\), if and only if for each \(n = 1, 2, \ldots\) there is a constant \(C_n \geq 0\) so that

\[|g(\zeta)| \leq \frac{C_n e^{R|\eta|}}{(1 + |\zeta|)^n}\]

for all \(\zeta \in \mathbb{C}^d\).

Let \(A\) be an \(N \times N\) matrix. We write its norm \(\sup_{|u|=1} |Au|\) as \(\|A\|\), where \(u = (u_1, \ldots, u_N) \in \mathbb{C}^N\) and \(|u| = \sqrt{\sum_{j=1}^N |u_j|^2}\).

**Proposition 4.B** (Lie product formula). Let \(A\) and \(B\) be finite dimensional matrices. Then we have

\[
\exp(A + B) = \lim_{n \to \infty} \left[ \exp \frac{A}{n} \exp \frac{B}{n} \right]^n
\]

in the topology of the norm.

The following lemma is essential for the proof of Theorem 4.2.
**Lemma 4.4.** Let $\lambda_{\text{max}}$ be the constant defined by (4.8). Let $\rho \in \mathbb{R}$ and $u = (u_1, \ldots, u_N) \in \mathbb{C}^N$. Then we have

$$\left| \exp \left( -i \rho c \hat{\alpha} \cdot (\xi + i \eta) \right) u \right| \leq \left( \exp(|\rho|c|\eta|\lambda_{\text{max}}) \right) |u|.$$ 

**Proof.** Let $\eta = 0$. Then

$$\left| \exp \left( -i \rho c \hat{\alpha} \cdot (\xi + i \eta) \right) u \right| = |\exp(-i \rho c \hat{\alpha} \cdot \xi) u| = |u|$$

holds since $\exp(-i \rho c \hat{\alpha} \cdot \xi)$ is unitary, which shows Lemma 4.4.

Let $\eta \neq 0$. Since $\hat{\alpha} \cdot \eta$ is Hermitian, we can have a diagonal matrix

$$\mathbf{U}^{-1}(\hat{\alpha} \cdot \eta) \mathbf{U} = \begin{pmatrix} 
\lambda_1(\eta) & 0 & 0 & \cdots & 0 \\
0 & \lambda_2(\eta) & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \lambda_N(\eta) 
\end{pmatrix}$$

by using a unitary matrix $\mathbf{U}$. Consequently we get

$$\mathbf{U}^{-1} \exp(\rho c \hat{\alpha} \cdot \eta) \mathbf{U} = \exp \rho c \begin{pmatrix} 
\lambda_1(\eta) & 0 & 0 & \cdots & 0 \\
0 & \lambda_2(\eta) & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \lambda_N(\eta) 
\end{pmatrix}$$

$$= \exp \rho c |\eta| \begin{pmatrix} 
\lambda_1(\eta/|\eta|) & 0 & 0 & \cdots & 0 \\
0 & \lambda_2(\eta/|\eta|) & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \lambda_N(\eta/|\eta|) 
\end{pmatrix}$$

$$= \begin{pmatrix} 
e^{\rho c |\eta| \lambda_1(\eta/|\eta|)} & 0 & 0 & \cdots & 0 \\
0 & \ne^{\rho c |\eta| \lambda_2(\eta/|\eta|)} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \ne^{\rho c |\eta| \lambda_N(\eta/|\eta|)} 
\end{pmatrix}$$
together with (4.9), which yields

\[ |\exp(\rho c\hat{\alpha} \cdot \eta) u| = |\mathbf{U}^{-1} \exp(\rho c\hat{\alpha} \cdot \eta) \mathbf{U}^{-1} u| \]
\[ = |\mathbf{U}^{-1} \exp(\rho c\hat{\alpha} \cdot \eta) \mathbf{U}^{-1} u| \leq \left(\exp(|\rho| c |\eta| \lambda_{\text{max}})\right) |\mathbf{U}^{-1} u| \]
\[ = \left(\exp(|\rho| c |\eta| \lambda_{\text{max}})\right) |u| \]

by using the unitarity of \( \mathbf{U} \). Hence we obtain

\[ |\exp(\rho c\hat{\alpha} \cdot \eta) u| \leq \left(\exp(|\rho| c |\eta| \lambda_{\text{max}})\right) |u|. \] (4.11)

Now, Proposition 4.B indicates

\[ \exp\left(-i\rho c\hat{\alpha} \cdot (\xi + i\eta)\right) u = \exp(\rho c\hat{\alpha} \cdot \eta - i\rho c\hat{\alpha} \cdot \xi) u \]
\[ = \lim_{n \to \infty} \left[ \exp\left(\frac{\rho c\hat{\alpha} \cdot \eta}{n} \right) \exp\left(\frac{-i\rho c\hat{\alpha} \cdot \xi}{n} \right) \right]^n u \quad \text{in } \mathbb{C}^N. \] (4.12)

Noting (4.11) and the unitarity of \( \exp(-i\rho c\hat{\alpha} \cdot \xi/n) \), we can easily prove

\[ |\exp(\rho c\hat{\alpha} \cdot \eta/n) \exp(-i\rho c\hat{\alpha} \cdot \xi/n) u| \]
\[ \leq \left(\exp\left(|\rho| c |\eta| \lambda_{\text{max}}/n\right)\right) |\exp(-i\rho c\hat{\alpha} \cdot \xi/n) u| = \left(\exp\left(|\rho| c |\eta| \lambda_{\text{max}}/n\right)\right) |u|. \]

In the same way we have

\[ \left|\left[ \exp(\rho c\hat{\alpha} \cdot \eta/n) \exp(-i\rho c\hat{\alpha} \cdot \xi/n) \right]^2 u \right| \]
\[ \leq \left(\exp\left(|\rho| c |\eta| \lambda_{\text{max}}/n\right)\right) |\exp(\rho c\hat{\alpha} \cdot \eta/n) \exp(-i\rho c\hat{\alpha} \cdot \xi/n) u| \]
\[ \leq \left(\exp\left(2|\rho| c |\eta| \lambda_{\text{max}}/n\right)\right) |u|. \]

Repeating this argument, we can prove

\[ \left|\left[ \exp(\rho c\hat{\alpha} \cdot \eta/n) \exp(-i\rho c\hat{\alpha} \cdot \xi/n) \right]^n u \right| \]
\[ \leq \left(\exp\left(|\rho| c |\eta| \lambda_{\text{max}}\right)\right) |u|. \] (4.13)

which completes the proof of Lemma 4.4 together with (4.12).
Lemma 4.5. Let $\rho \in \mathbb{R}$ and $u = u(t, u_1, \ldots, u_N) \in \mathbb{C}$. Then we have

$$\left| \exp \left(-i\rho \{c\hat{\alpha} \cdot (\xi + i\eta) + \beta mc^2 \} \right) u \right| \leq \left( \exp \left| \rho |c| \eta \lambda_{\text{max}} \right| \right) |u|.$$ 

Proof. We have

$$\exp \left(-i\rho \{c\hat{\alpha} \cdot (\xi + i\eta) + \beta mc^2 \} \right) u = \lim_{n \to \infty} \left[ \exp \left(-i\rho \frac{c\hat{\alpha} \cdot (\xi + i\eta)}{n} \right) \exp \left(-i\rho \frac{\beta mc^2}{n} \right) \right]^n u \text{ in } \mathbb{C}^N \quad (4.14)$$

from Proposition 4.B. Lemma 4.4 shows

$$\left| \exp \left(-i\rho \frac{c\hat{\alpha} \cdot (\xi + i\eta)}{n} \right) \exp \left(-i\rho \frac{\beta mc^2}{n} \right) \right| |u| \leq \left( \exp \left| \rho |c| \eta \lambda_{\text{max}} / n \right| \right) |u|$$

because of the unitarity of $\exp(-i\rho \beta mc^2 / n)$. Hence we can prove

$$\left| \left[ \exp \left(-i\rho \frac{c\hat{\alpha} \cdot (\xi + i\eta)}{n} \right) \exp \left(-i\rho \frac{\beta mc^2}{n} \right) \right]^n u \right| \leq \left( \exp \left| \rho |c| \eta \lambda_{\text{max}} / n \right| \right) |u| \quad (4.15)$$

as in the proof of (4.13), which completes the proof of Lemma 4.5 together with (4.14).

Taking a function $\psi(x) \in C_0^\infty(\mathbb{R}^d)$ with support in $B(0; 1)$ and $\int \psi(x) dx = 1$, we define $\chi(\xi) \in \mathcal{S}$ by its Fourier transform $\hat{\psi}(\xi)$. Then $\chi(0) = 1$ holds. We fix this $\chi(\xi)$ hereafter. For $\epsilon > 0$ and $f \in \mathcal{S}$ let us write

$$\left( G^0_\epsilon(t, s)f \right)(x) := \int \int e^{i(x-y) \cdot \xi - i\rho (c\hat{\alpha} \cdot \xi + \beta mc^2)} f(y) \chi(\epsilon \xi) dy d\xi$$

$$= \int e^{ix \cdot \xi - i\rho (c\hat{\alpha} \cdot \xi + \beta mc^2)} f(\xi) \chi(\epsilon \xi) d\xi, \quad \rho = t - s, \quad (4.16)$$

which is equal to $G_\epsilon(t, s)f$ defined by (3.5) with $V = 0$ and $A = 0$. 24
Proposition 4.6. Let \( f \in C_0^\infty(\mathbb{R}^d)^N \) with support in \( B(0; R) \). Then we have \( \text{supp} G_0^0(t, s)f \subset B(0; c\lambda_{\max}|t - s| + R + \epsilon) \).

Proof. The expression (4.16) gives that the Fourier transform of \( G_0^0(t, s)f \) is

\[
v_{\epsilon}(t, s; \xi) := e^{-i\rho(c\hat{\alpha} \cdot \xi + \hat{\beta}mc^2)} \hat{f}(\xi) \chi(\epsilon \xi).
\]

(4.17)

Proposition 4.A indicates that \( \hat{f}_j(\xi) (j = 1, 2, \ldots, N) \) and \( \chi(\xi) \) can be extended to entire functions on \( \mathbb{C}^d \) and satisfy

\[
|\hat{f}_j(\xi)| \leq C_n e^{R|\eta|} \frac{(1 + |\xi|)^n}{(1 + |\xi|)^n}, \quad |\chi(\xi)| \leq C_n e^{R|\eta|} \frac{(1 + |\xi|)^n}{(1 + |\xi|)^n}
\]

(4.18)

for each \( n = 1, 2, \ldots \) with a constant \( C_n \geq 0 \). Hence, applying Lemma 4.5 to (4.17), we can see by (4.18) that \( v_{\epsilon}(t, s; \xi) \) can be extended to an entire function on \( \mathbb{C}^d \) and satisfies

\[
|v_{\epsilon}(t, s; \xi)| \leq e^{\rho|\eta|\lambda_{\max}} |\hat{f}(\xi)| |\chi(\epsilon \xi)| \leq C_n^2 e^{(\rho|c\lambda_{\max} + R + \epsilon)|\eta|} \frac{(1 + |\xi|)^n(1 + |\epsilon \xi|)^n}{(1 + |\xi|)^n(1 + |\xi|)^n}
\]

(4.19)

for each \( n \), which proves Proposition 4.6 from Proposition 4.A.

Corollary 4.7. Let \( f \in C_0^\infty(\mathbb{R}^d)^N \) with support in \( B(a; R) \) for \( a \in \mathbb{R}^d \). Then we have \( \text{supp} G_0^0(t, s)f \subset B(a; c\lambda_{\max}|t - s| + R + \epsilon) \).

Proof. Set \( g(x) := f(x + a) \). Then \( \hat{g}(\xi) = e^{ia\cdot \xi} \hat{f}(\xi) \) and supp \( g \subset B(0; R) \). Consequently from (4.16) we can see

\[
\left(G_0^0(t, s)f\right)(x) = \int e^{ix \cdot \xi - i\rho(c\hat{\alpha} \cdot \xi + \hat{\beta}mc^2)} e^{-ia \cdot \xi} \hat{g}(\xi) \chi(\epsilon \xi) d\xi
\]

\[
= \left(G_0^0(t, s)g\right)(x - a).
\]

Hence Corollary 4.7 is yielded since \( \text{supp} G_0^0(t, s)g \subset B(0; c\lambda_{\max}|t - s| + R + \epsilon) \) follows from Proposition 4.6.
Proposition 4.8. Let \( t \) and \( s \) be in \( I_T \). We consider the operator \( G(t, s) \) on \( S^N \) defined in Proposition 3.2. Then \( G(t, s)f \) for \( f \in C_0^\infty(\mathbb{R}^d)^N \) has the speed not exceeding \( c\lambda_{\max} \) of propagation of disturbances.

Proof. Let \( f \in C_0^\infty(\mathbb{R}^d)^N \) with support in \( B(a; R) \). Take \( \varphi_R(x) \in C_0^\infty(\mathbb{R}^d) \) such that \( \varphi_R(x) = 1 \) if \( |x| \leq R \). Noting (2.4), (2.5) and \( \text{supp} f \subset B(a; R) \), we can write \( G_\epsilon(t, s)f \) defined by (3.5) as
\[
(G_\epsilon(t, s)f)(x) = \int\int e^{i(x-y)\cdot\xi - i\rho(\gamma_\alpha \cdot \xi + \beta \dot{\gamma_\beta}c^2)} w(t, s; x, y) f(y) \chi(\epsilon \xi) dy d\xi, \tag{4.20}
\]
where
\[
w(t, s; x, y) = \varphi_R(y - a) \exp\left\{ i(x - y) \cdot \int_0^1 A(t - \theta \rho, x - \theta(x - y)) d\theta - i\rho \int_0^1 V(t - \theta \rho, x - \theta(x - y)) d\theta \right\}. \tag{4.21}
\]

Let \( \alpha \) be multi-indices such that \( |\alpha| = 2d \). It follows from the assumptions of Proposition 3.2 that we have
\[
|\partial_\gamma^\beta w(t, s; x, y)| \leq C(\langle x \rangle^{M+1} < y >^{M+1})^{2d} \sum_{|\beta| \leq 2d} |\partial_\gamma^\beta \varphi_R(y - a)|
\leq C'(\langle x \rangle^{2(M+1)} < y >^{M+1})^{2d} \sum_{|\beta| \leq 2d} |\partial_\gamma^\beta \varphi_R(y - a)|
\leq C' < R + |a| >^{2(M+1)d} < x >^{4(M+1)d} \sum_{|\beta| \leq 2d} |\partial_\gamma^\beta \varphi_R(y - a)|, \tag{4.22}
\]
where we used \( \langle x - y \rangle \leq \sqrt{2} < x > < y > \).

Using \( \text{supp} w(t, s; x, \cdot) \subset B(a; R) \), we can expand \( w(t, s; x, y) \) into a Fourier series with respect to variables \( y \in B(a; R) \)
\[
w(t, s; x, y) = \sum_{n_1 = -\infty}^{\infty} \cdots \sum_{n_d = -\infty}^{\infty} c_n(t, s; x) e^{i n\omega \cdot (y - a)}, \tag{4.23}
\]
26
\[ c_n(t, s; x) = \left( \frac{1}{2l} \right)^d \int_I w(t, s; x, y) e^{-in\omega \cdot (y-a)} dy, \]  
\( \text{(4.24)} \)

where \( n = (n_1, \ldots, n_d), \ l \geq R \) is a constant, \( \omega = \pi/l \) and \( I \) is a cube in \( \mathbb{R}^d \) with edges of length \( 2l \). From (4.22) and (4.24) we have

\[ |c_n(t, s; x)| \leq C <x>^{\frac{d(M+1)}{n_1^2 \cdots n_d^2}}. \]  
\( \text{(4.25)} \)

Hence, using (4.20) and (4.23), we can write \( (G_\epsilon(t, s)f)(x) \) as an infinite series

\[
\sum_{n_1=-\infty}^{\infty} \cdots \sum_{n_d=-\infty}^{\infty} c_n(t, s; x) \int \int e^{i(x-y) \cdot \xi - i\rho(\tilde{c} \tilde{\alpha} \cdot \xi + 3mc^2)} e^{in\omega \cdot (y-a)} \\
\times f(y) \chi(\epsilon \xi) dy d\xi,
\]  
\( \text{(4.26)} \)

which converges uniformly on compact sets in \( \mathbb{R}^d \). Consequently we see

\[
\text{supp } G_\epsilon(t, s)f \subset B(a; c\lambda_{\text{max}}|t-s| + R + \epsilon)
\]  
\( \text{(4.27)} \)

since Corollary 4.7 shows that the support of each term in (4.26) is in \( B(a; c\lambda_{\text{max}}|t-s| + R + \epsilon) \). Therefore, we can complete the proof of Proposition 4.8 from Proposition 3.2.

Now, we will prove Theorem 4.2. Let us use the gauge transformation as in the proof of Theorem 2.2. Hence we may assume (2.14)-(2.16). Suppose \( t_i < t_f \). Another case can be proved in the same way. Take a time-division \( \{\tau_j\}_{j=1}^{\nu-1} \) satisfying

\[
t_i < \tau_1 < \tau_2 < \cdots < \tau_{\nu-1} < t_f.
\]  
\( \text{(4.28)} \)

Then, applying Proposition 4.8 to each \( G(\tau_j, \tau_{j-1}) \) in (3.13), we can see that \( K_{D\Delta}(t_f, t_i)f \) for \( f \in C_0^\infty(\mathbb{R}^d)^N \) has the speed not exceeding \( c\lambda_{\text{max}} \) of propagation of disturbances and so does \( K_{D\Delta}(t_f, t_i)f \) for \( f \in (L^2)^N \), which can be proved from (2.19) by making \( f \) approximated in \( (L^2)^N \) by functions in \( C_0^\infty(\mathbb{R}^d)^N \). Therefore, we have been able to complete the proof of Theorem 4.2 from (2) in Theorem 2.2.
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