THE INVERSE PROBLEM OF THE CALCULUS OF VARIATIONS FOR SYSTEMS OF SECOND-ORDER PARTIAL DIFFERENTIAL EQUATIONS IN THE PLANE

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Abstract. A complete solution to the multiplier version of the inverse problem of the calculus of variations is given for a class of hyperbolic systems of second-order partial differential equations in two independent variables. The necessary and sufficient algebraic and differential conditions for the existence of a variational multiplier are derived. It is shown that the number of independent variational multipliers is determined by the nullity of a completely algebraic system of equations associated to the given system of partial differential equations. An algorithm for solving the inverse problem is demonstrated on several examples. Systems of second-order partial differential equations in two independent and dependent variables are studied and systems which have more than one variational formulation are classified up to contact equivalence.

1. Introduction

In this paper we examine the inverse problem of the calculus of variations for systems of partial differential equations

\[ u^\alpha_{xy} = f^\alpha(x, y, u^\gamma, u^\gamma_x, u^\gamma_y), \quad \alpha, \gamma = 1, 2, \ldots, m, \]

in \( m \) dependent variables \( u^\alpha \) and two independent variables \( x \) and \( y \). Systems (1.1) arise in various contexts such as symmetry reduction in general relativity, non-linear \( \sigma \)-models, and generalized Toda lattices (see [11], [13], [20], [21], and [24]). For systems (1.1) we give a complete solution to the variational multiplier version of the inverse problem. In particular, we give an explicit algorithm for determining the number of possible inequivalent Lagrangians for a given system (1.1).

The variational multiplier problem is stated as follows: Given a system of differential equations \( F^\alpha(x^i, u^\gamma, u^\gamma_i, u^\gamma_{ij}, \ldots) = 0 \), does there exist a Lagrangian \( L(x^i, u^\gamma, u^\gamma_i, u^\gamma_{ij}, \ldots) \) and functions \( M^\gamma_\alpha \), with \( \det (M^\gamma_\alpha) \neq 0 \), such that

\[ E^\alpha(L) = M^\gamma_\alpha F^\gamma, \]

where \( E \) is the Euler-Lagrange operator. If such an \( M^\gamma_\alpha \) exists, then it is referred to as a variational multiplier. If (1.2) is satisfied, the equations \( F^\alpha = 0 \) are equivalent to a system of Euler-Lagrange equations in the sense that solutions of the Euler-Lagrange equations for \( L \) are solutions to \( F^\alpha = 0 \) and conversely solutions to \( F^\alpha = 0 \) are solutions for \( E^\alpha(L) = 0 \).

Also of importance is the number of Lagrangians for a particular system. It is well known that two Lagrangians \( L \) and \( L' \) have identical Euler-Lagrange expressions if and only if \( L \) and \( L' \) differ by a total divergence, at least locally. Therefore we consider two Lagrangians to be equivalent if they differ by a total divergence. We note that it is possible to have a system of
differential equations which have two or more inequivalent Lagrangians (and thus more than one multiplier).

The variational multiplier problem is of some interest in theoretical physics. It is widely accepted that fundamental physical theories can be derived from an action principle, or Lagrangian. For a given theory it is important to determine whether the action is unique. Examples with multiple Lagrangians exist in Newtonian Mechanics and the SU(2) chiral model ([14], [15]).

The simplest case of the variational multiplier problem is for a scalar second-order ordinary differential equation \( u_{xx} = F(x, u, u_x) \). It has been shown by several authors, including Darboux [10], that any scalar second-order ODE is variational and the most general multiplier (and Lagrangian) depends on an arbitrary function of two variables. The multiplier problem for systems of second-order ordinary differential equations has been studied by many authors including Douglas [9], Anderson and Thompson [4], Thompson, Crimin, Sarlet, Prince and Martinez (See [7], [8], [22], and [23].) Our solution to the multiplier problem for systems (1.1) is partially based on ideas developed in Anderson and Thompson’s paper [4]. In that paper Anderson and Thompson used the variational bicomplex to derive a system of algebraic and differential conditions, with components of the multiplier matrix as unknowns, for the existence of a multiplier. They showed that there is a one-to-one correspondence between variational multipliers for the given system and certain cohomology classes in the variational bicomplex associated to the given system of differential equations. While a significant amount of research has been done on the variational multiplier problem for second-order ODE systems, a complete solution remains elusive in the sense that there is no general closed form characterization, in terms of invariants for the system, for determining the existence and degree of uniqueness of multipliers for the given system.

The variational multiplier problem for higher order scalar ordinary differential equations has been studied by Fels [12] and Juráš [17]. Fels [12] obtained a complete solution to the variational multiplier problem for fourth-order scalar ordinary differential equations \( u_{xxxx} = F(x, u, u_x, u_{xx}, u_{xxx}) \). Using Cartan’s method of equivalence, he was able to produce two differential invariants whose vanishing completely characterizes the existence of a variational multiplier. Unlike the second-order case, the multiplier is unique up to a constant multiple. In [17], Juráš obtained a similar solution for sixth and eighth-order scalar ordinary differential equations, although the differential invariants are increasingly complicated for higher order systems.

For partial differential equations, there are fewer papers on the variational multiplier problem. Anderson and Duchamp [2] studied the variational multiplier problem for scalar second-order quasilinear partial differential equations \( F = 0 \) where \( F = A^{ij}(x^k, u, u_k)u_{ij} + B(x^k, u, u_k) \). Anderson and Duchamp [2] proved that if \( \det(A^{ij}) \neq 0 \), then \( F \) has a variational multiplier if and only if a certain 1-form \( \chi \) is closed. Moreover, \( \chi \) is expressed explicitly in terms of \( F \) and its derivatives and the multiplier, if it exists, is unique up to a constant multiple. They also showed that second-order scalar evolution equations are never variational. Juráš [16] examined the inverse problem for a scalar hyperbolic second-order partial differential equations in two independent variables. He was able to show that an equation is variational if and only if two particular differential invariants \( H \) and \( K \) are identically equal. Moreover, the multiplier is unique up to a constant. Juráš’ result is equivalent to the Anderson-Duchamp result if the equation is quasilinear. However, his results also apply to hyperbolic Monge-Ampere equations.

In this paper we study the variational multiplier problem for \( f \)-Gordon systems

\[
(1.3) \quad u_{xy}^\alpha = f^\alpha(x, y, u^\gamma, u_x^\gamma, u_y^\gamma), \quad \alpha, \gamma = 1, 2, \ldots, m.
\]

We remark that we are working under the assumption that the functions \( f^\alpha \) are \( C^\infty \) on some open set \( U \subset \mathbb{R}^2 \times \mathbb{R}^{3m} \). In the following section we outline the complete solution to the
multiplier version of the inverse problem for systems (1.3). In particular, we give an algorithm for determining the number of variation multipliers (and Lagrangians) for a given system (1.3). We show that the most general multiplier depends on finitely many constants determined by the dimension of the nullspace of a certain matrix depending on the functions $f^\alpha$ and their derivatives. In Section 3 we demonstrate our algorithm for solving the inverse problem on several examples. In Section 4 we classify all systems (1.3) in two dependent variables that admit two or more inequivalent Lagrangians. In Sections 5 and 6, we turn our attention to proving two technical propositions stated in Section 2. In particular, we define the variational bicomplex for systems (1.3) and show there is a one-to-one correspondence between the Lagrangians for systems (1.3) and special classes of 3-forms $\omega$ with $d\omega = 0$. We then derive necessary and sufficient algebraic and differential conditions for the existence of a variational multiplier. In the appendix we prove a general theorem on the existence of solutions to systems of combined differential and algebraic equations that allows us to determine the number of variational multipliers from purely algebraic data.

This work is an extension of a result established in my PhD dissertation. I wish to thank my advisor Ian Anderson for suggesting this problem and an uncountable number of helpful discussions on this subject.

2. Main Results

In this section we state our solution to the variational multiplier problem for systems (1.1). Our solution depends on two propositions which we will prove in Section 6. The first proposition gives us a general normal form for variational systems (1.1) and states that the Lagrangian associated to a particular variational multiplier is unique up to modification by a total divergence.

**Proposition 2.1.** If there exists a first-order Lagrangian $L(x, y, u^\gamma, u_x^\gamma, u_y^\gamma)$ and a non-degenerate variational multiplier $M_{\alpha\beta}(x, y, u^\gamma, u_x^\gamma, u_y^\gamma)$ such that

$$E_\alpha(L) = M_{\alpha\beta}(u_x^\beta - f^\beta(x, y, u^\gamma, u_x^\gamma)), \quad (2.1)$$

then $M_{\alpha\beta} = M_{\alpha\beta}(x, y, u^\gamma)$, $M_{\alpha\beta} = M_{\beta\alpha}$.

Moreover, if $\tilde{L}(x, y, u^\gamma, \tilde{u}_x^\gamma, \tilde{u}_y^\gamma)$ is another Lagrangian associated to the variational multiplier $M_{\alpha\beta}$ such that $E_\alpha(\tilde{L}) = M_{\alpha\beta}(u_x^\beta - f^\beta)$, then $\tilde{L} = L + \text{Div} Q$, where $Q = (Q_1(x, y, u^\gamma), Q_2(x, y, u^\gamma))$ and $\text{Div} Q = D_xQ_1 + D_yQ_2$.

It follows immediately from Proposition 2.1 that we may restrict ourselves to systems of the form

$$u_x^\alpha + C_{\beta\gamma}^\alpha(x, y, u^\gamma) u_x^\beta u_y^\gamma + A_1^\alpha(x, y, u^\gamma) u_x^\gamma + B_2^\alpha(x, y, u^\gamma) u_y^\gamma + E^\alpha(x, y, u^\gamma) = 0. \quad (2.2)$$

We remark that the condition that $M_{\alpha\beta}$ is symmetric and has no first-order derivative dependence also follows from a result established by Henneaux in [14].
Of paramount importance in our study of the inverse problem are three quantities $H^\gamma_\alpha$, $K^\gamma_\alpha$, and $S^\alpha_{\alpha\beta}$ defined by

\begin{align}
(2.3a) \quad H^\gamma_\alpha &= \frac{\partial f^\gamma}{\partial u^\sigma} + \frac{\partial f^\gamma}{\partial u^\gamma_y} \frac{\partial f^\sigma}{\partial u^\alpha} - \frac{\partial^2 f^\gamma}{\partial u^\alpha \partial x} - \frac{\partial^2 f^\gamma}{\partial u^\alpha \partial u^\sigma} u^\beta_x - \frac{\partial^2 f^\gamma}{\partial u^\alpha \partial u^\gamma_y} u^\beta_y, \\
(2.3b) \quad K^\gamma_\alpha &= \frac{\partial f^\gamma}{\partial u^\alpha} + \frac{\partial f^\gamma}{\partial u^\sigma} \frac{\partial f^\sigma}{\partial u^\alpha} - \frac{\partial^2 f^\gamma}{\partial u^\alpha \partial y} - \frac{\partial^2 f^\gamma}{\partial u^\sigma \partial u^\alpha} u^\beta_y - \frac{\partial^2 f^\gamma}{\partial u^\sigma \partial u^\gamma_y} u^\beta_y, \\
(2.3c) \quad S^\alpha_{\gamma\rho} &= \frac{\partial^2 f^\gamma}{\partial u^\beta_x \partial u^\alpha_y} - \frac{\partial^2 f^\gamma}{\partial u^\beta_x \partial u^\alpha_y}.
\end{align}

In the formulas (2.3) and throughout this paper we are adopting the Einstein summation convention where repeated indices are summed upon. We remark that the quantities $H^\gamma_\alpha$, $K^\gamma_\alpha$, and $S^\alpha_{\alpha\beta}$ are relative invariants under the pseudo-group of local contact transformations that preserve systems (1.1). The functions $H^\gamma_\alpha$ and $K^\gamma_\alpha$ are natural extensions of the generalized Laplace invariants defined in [3] for scalar hyperbolic equations. For systems (2.2), it follows from (2.3) that the functions $H^\gamma_\alpha$ and $K^\gamma_\alpha$ have no second-order derivative dependence and $S^\alpha_{\alpha\beta}$ has no first-order derivative dependence, that is $H^\gamma_\alpha = H^\gamma_\alpha(x, y, u^\tau, u^\alpha_x, u^\gamma_y)$, $K^\gamma_\alpha = K^\gamma_\alpha(x, y, u^\tau, u^\alpha_x, u^\gamma_y)$, and $S^\alpha_{\alpha\beta} = S^\alpha_{\alpha\beta}(x, y, u^\tau)$, where $H^\gamma_\alpha$ and $K^\gamma_\alpha$ are quadratic in the variables $u^\alpha_x$ and $u^\alpha_y$. Our second proposition characterizes the algebraic and differential conditions on a variational multiplier for a system (2.2).

**Proposition 2.2.** A system (2.2) is multiplier variational with a multiplier $M_{\alpha\beta}$ if and only if $M_{\alpha\beta}$ satisfies det $M_{\alpha\beta} \neq 0$ and

\begin{align}
(2.4a) \quad M_{\alpha\beta} H^\gamma_\beta &= M_{\beta\sigma} K^\sigma_\alpha, \quad M_{\alpha\sigma} S^\sigma_{\beta\gamma} = -M_{\beta\sigma} S^\sigma_{\alpha\gamma}, \\
(2.4b) \quad dM_{\alpha\beta} = M_{\alpha\sigma} \Omega^\sigma_\beta + M_{\beta\sigma} \Omega^\sigma_\alpha,
\end{align}

where $\Omega^\sigma_\alpha = C^\sigma_{\alpha\tau} du^\tau + A^\sigma_\alpha dy + B^\sigma_\alpha dx$.

**Remark 2.3.** In Section 6, we will derive the algebraic and differential conditions (2.4) and give an algorithm for constructing the first-order Lagrangian $L$ associated to a variational multiplier $M_{\alpha\beta}$ satisfying (2.4). The algebraic conditions (2.4a) include the integrability conditions for $d^2 M_{\alpha\beta} = 0$ for (2.4b) and genuine algebraic constraints not arising from the integrability conditions. We will show that the set of all solutions to (2.4) is a finite dimensional real vector space with the dimension determined by the nullity of a completely algebraic system of equations. In particular, for any system of the form (2.2), the solutions to (2.4) are completely determined. However, in some cases there are non-trivial solutions $M_{\alpha\beta}$ to (2.4) that do not satisfy det $M_{\alpha\beta} \neq 0$. In practice, we first determine a basis for the solutions to (2.4) and then determine if a non-degenerate variational multiplier can be constructed from a linear combination of the basis solutions.

We are now ready to state our solution to the inverse problem which says that the variational multipliers $M_{\alpha\beta}$ satisfying (2.4) are completely characterized by the solutions to an algebraic system $\Phi^\alpha_{\alpha\beta} M_{\alpha\beta} = 0$, where $\Phi$ depends on the given system $u^\alpha_{xy} = f^\alpha$.

**Theorem 2.4.** Let $u^\alpha_{xy} = f^\alpha(x, y, u^\gamma, u^\alpha_x, u^\gamma_y)$ be a system of $m$ partial differential equations where $f^\alpha \in C^\infty(U)$ for some open set $U \subset \mathbb{R}^2 \times \mathbb{R}^{3m}$. Then there is a matrix $\Phi$, with $\text{Rank}(\Phi) \leq m(m + 1)/2$, depending on the functions $H^\gamma_\alpha$, $K^\gamma_\alpha$, and $S^\alpha_{\alpha\beta}$ and their derivatives, whose nullspace completely determines the number of linearly independent solutions to (2.4). Specifically, if $r = \text{Rank} \Phi(z)$ is constant for all points $z$ in a open set $V \subset U$, then at every point $z_0 \in V$
there is a neighborhood \( W \subset V \) of \( z_0 \) where the set of solutions to (2.4) is a \( m(m+1)/2 - r \) dimensional vector space over \( \mathbb{R} \).

**Proof.** The algebraic system for a multiplier \( M_{\alpha \beta} \) is constructed as follows. As a consequence of proposition (2.1), the functions \( M_{\alpha \beta} \) have no 1-jet dependence and \( H_{\alpha}^0 \) and \( K_{\alpha}^0 \) are quadratic in \( u_x^i \) and \( u_y^i \). We can decompose the algebraic condition \( M_{\alpha \gamma} H_{\alpha}^0 = M_{\beta \gamma} K_{\alpha}^0 \) into several linear algebraic systems on the components of the multiplier \( M_{\alpha \beta} \), with the coefficients depending only on \( x, y, \) and \( u \). We then express all algebraic conditions (2.4a) on \( M_{\alpha \gamma} \) as a single system of linear equations

\[
(\Phi_0)^{\alpha \beta}_a (x, y, u^T) M_{\alpha \beta} = 0, \quad a = 1, 2, \ldots, k_0,
\]

where \( \Phi_0 \) may be viewed as \( k_0 \times m(m+1)/2 \) matrix. Differentiating (2.5) and substituting from (2.4b), we get the algebraic condition

\[
(\Phi_1)^{\alpha \beta}_a M_{\alpha \beta} = 0, \quad a = 1, \ldots, k_1.
\]

We can proceed inductively to define a system of equations

\[
(\Phi_{i+1})^{\alpha \beta}_a M_{\alpha \beta} = 0, \quad a = 1, \ldots, k_{i+1},
\]

where (2.7) consists of the system \( (\Phi_i)^{\alpha \beta}_a M_{\alpha \beta} = 0 \) along with the system

\[
[(d\Phi_i)^{\alpha \beta}_a + 2(\Phi_i)^{\alpha \gamma}_a \Omega^{\beta}_\sigma + (\Phi_i)^{\sigma \beta}_a \Omega^{\alpha}_\sigma] M_{\alpha \beta} = 0.
\]

Clearly, at each point \( z_0 = (x_0, y_0, u_0^\gamma) \in U \subset \mathbb{R}^2 \times \mathbb{R}^m \) and for all \( j \geq 0 \), the ranks of the matrices \( \Phi_i \) satisfy

\[
0 \leq \text{Rank} \, \Phi_0(z_0) \leq \text{Rank} \, \Phi_1(z_0) \leq \cdots \leq \text{Rank} \, \Phi_j(z_0) \leq \frac{m(m+1)}{2}.
\]

If given a point \( z_0 \in U \), we see that after \( k \leq m(m+1)/2 \) differentiations, the rank of the matrices \( \Phi_i(z_0) \) must stabilize at some \( 0 \leq l \leq m(m+1)/2 \). More precisely, at the point \( z_0 \) we have

\[
l = \text{Rank} \, \Phi_k(z_0) = \text{Rank} \, \Phi_{k+i}(z_0), \quad \forall \, i \geq 0.
\]

Let \( \{C^1_{\alpha \gamma}, C^2_{\alpha \gamma}, \ldots, C^s_{\alpha \gamma}\} \), where \( s = m(m+1)/2 - l \), be a basis for the set of solutions to the system of linear equations

\[
(\Phi_k)^{\alpha \gamma}_a M_{\alpha \gamma} = 0
\]

at the point \( z_0 \). In particular, \( \{C^i_{\alpha \gamma} \mid i = 1, \ldots \} \) is a linearly independent collection of constant bilinear forms such that \( (\Phi_k)^{\alpha \gamma}_a(z_0) C^i_{\alpha \gamma} = 0 \).

If the rank of \( \Phi_k \) is constant in a neighborhood \( V \subset U \) of \( z_0 \), then a general result on systems of algebraic-differential equations, which we prove in the appendix, states that there exists a neighborhood \( W \subset V \) of \( z_0 \) and a linearly independent collection of functions \( \{M^1_{\alpha \beta}, M^2_{\alpha \beta}, \ldots, M^s_{\alpha \beta}\} \) such that

\[
M^i_{\alpha \beta}(z_0) = C^i_{\alpha \beta}, \quad (\Phi_k)^{\alpha \beta}_a(z) M^i_{\alpha \beta}(z) = 0, \quad dM^i_{\alpha \beta} = M^i_{\alpha \sigma} \Omega^\sigma_{\beta} + M^i_{\beta \sigma} \Omega^\sigma_{\alpha},
\]
for all $z \in W$, and $i = 1, 2, \ldots, s$. Moreover, we claim that if a given collection of functions $M_{\alpha \beta}$ satisfies the algebraic and differential conditions $(2.9)$, then $M_{\alpha \beta}$ can be expressed as a linear combination

\begin{equation}
M_{\alpha \beta} = \sum_{i=1}^{s} c_i M_{\alpha \beta}^i, \quad c_i \in \mathbb{R}.
\end{equation}

Indeed, if $M_{\alpha \beta}$ satisfies the algebraic condition $(\Phi_k)^{\alpha \beta}(z) M_{\alpha \beta}(z) = 0$ for all $z \in W$, it follows that $M_{\alpha \beta}$ can be uniquely expressed as

\begin{equation}
M_{\alpha \beta}(z) = \sum c_i(z) M_{\alpha \beta}^i(z).
\end{equation}

If $M_{\alpha \beta}$ satisfies the differential condition $(2.4b)$, then it follows from $(2.9)$ and $(2.11)$ that $M_{\alpha \beta}^i \ dc_i = 0$. Since the functions $M_{\alpha \beta}^i$ are pointwise linearly independent in some neighborhood of $z_0$, we deduce that $dc_i = 0$, which in turn implies that $c_i \in \mathbb{R}$ for $i = 1, 2, \ldots, s$. This establishes $(2.10)$ and completes the proof of the theorem. \hfill \Box

3. Examples

In this section we demonstrate our algorithm for solving the variational multiplier problem on several examples. For each example we calculate the invariants $H^\gamma_{\alpha \beta}$, $K^\gamma_{\beta \alpha}$, and $S^\gamma_{\alpha \beta \gamma}$ using $(2.3)$ and we explicitly list the initial algebraic conditions $(2.4a)$ on a multiplier $M_{\alpha \beta}$. We then determine the differential condition $(2.4b)$ and differentiate the algebraic conditions to uncover any additional algebraic constraints. In each case we find the most general Lagrangian and multiplier for the given system.

Example 3.1. For our first example, consider the system

\begin{equation}
(3.1) \quad u_{xy} = v, \quad v_{xy} = u.
\end{equation}

We will show that $(3.1)$ admits two Lagrangians. In this case $S^\gamma_{\beta \gamma} = 0$, so that the only nontrivial algebraic condition from $(2.4)$ is

\begin{equation}
M_{\alpha \gamma} H^\gamma_{\beta} = M_{\beta \gamma} K^\gamma_{\alpha}.
\end{equation}

The differential condition is $dM_{\alpha \beta} = 0$. Using $(2.3)$ we calculate

\[ H = K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \]

and it follows from $(3.2)$ that the only algebraic constraint is $M_{11} = M_{22}$. Since the differential condition is $dM_{\alpha \beta} = 0$, differentiating $M_{11} = M_{22}$ produces no additional algebraic conditions. The most general multiplier in this case is $M = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$, $a, b \in \mathbb{R}$, $a^2 - b^2 \neq 0$.

The Lagrangian corresponding to the multiplier $M$ is given by

\[ L = -\frac{a}{2}(u_x u_y + v_x v_y + 2uv) - \frac{b}{2}(2u_x v_y + u^2 + v^2). \]

A routine calculation shows that the Euler-Lagrange equations for $L$ are

\[ E_1(L) = a(u_{xy} - v) + b(v_{xy} - u), \quad E_2(L) = b(u_{xy} - v) + a(v_{xy} - u). \]
Example 3.2. Consider the system

\[ u_{xy} = v, \quad v_{xy} = xu. \]  

The only difference between this system and the one in the first example is the \( x \) in the second equation. We will show that (3.3) admits a unique Lagrangian. The initial algebraic conditions (2.4) on the multiplier \( M_{\alpha \beta} \) reduce to

\[ M_{11} = x M_{22}. \]  

The differential condition is again \( dM_{\alpha \beta} = 0 \). Differentiating (3.4) and substituting \( dM_{\alpha \beta} = 0 \) results in \( M_{22} dx = 0 \), implying that \( M_{22} = M_{11} = 0 \). Further differentiations add no new algebraic conditions. Consequently the multiplier \( M \), where \( M_{11} = M_{22} = 0 \) and \( M_{12} = M_{21} = 1 \), is unique up to scalar multiplication. The Lagrangian for (3.3) is \( L = -uxv - (u^2 + xuv^2)/2 \).

Example 3.3. For our third example, we again we make a slight change on the first example (3.1) to get a system that is not variational. Let

\[ u_{xy} = v, \quad v_{xy} = u_x. \]  

In this case the algebraic and differential conditions (2.4) on the multiplier \( M_{\alpha \beta} \) are given by

\[ M_{11} = 0, \quad dM_{11} = 2M_{12} dy, \quad dM_{12} = M_{22} dy, \quad dM_{22} = 0. \]  

Differentiating the algebraic condition \( M_{11} = 0 \) we find that \( M_{12} dy = 0 \), implying that \( M_{12} = 0 \). Differentiating \( M_{12} = 0 \), it follows that \( M_{22} dy = 0 \). Consequently, the only solution to (3.6) is the trivial solution \( M \equiv 0 \).

Example 3.4. Consider the system

\[ u_{\alpha \gamma} + \Gamma^\alpha_{\beta \gamma} (u^\beta) u^\beta_y = 0, \]  

where \( \Gamma^\alpha_{\beta \gamma} \) are the components of a symmetric connection on an \( m \)-dimensional manifold \( \mathcal{M} \). We will show that (3.7) is variational if and only if \( \Gamma \) is a metric connection. Since \( \Gamma \) is symmetric we have \( S^\alpha_{\beta \gamma} = 0 \) and the Laplace invariants for (3.7) are given by \( H^\alpha_{\beta} = R^\gamma_{\alpha \sigma \epsilon} u^\sigma_{\epsilon} u^\gamma_y \) and \( K^\gamma_{\alpha \beta} = R^\gamma_{\alpha \epsilon \beta} u^\epsilon_y u^\gamma_y \), where \( R^\alpha_{\gamma \beta} \) are the components of the curvature tensor associated to the connection \( \Gamma \). Using properties of the curvature tensor, we can show that the conditions (2.4) on the multiplier are

\[ M_{\alpha \gamma} R^\gamma_{\beta \epsilon \sigma} + M_{\beta \gamma} R^\gamma_{\alpha \epsilon \sigma} = 0, \quad dM_{\alpha \beta} = \left( M_{\alpha \gamma} \Gamma^\gamma_{\beta \epsilon} + M_{\beta \gamma} \Gamma^\gamma_{\alpha \epsilon} \right) du^\epsilon. \]  

We see immediately from the differential condition in (3.8) that any multiplier \( M_{\alpha \beta} \) must satisfy \( \partial M_{\alpha \beta}/\partial x = 0 \) and \( \partial M_{\alpha \beta}/\partial y = 0 \). Consequently, \( M_{\alpha \beta} = M_{\alpha \beta}(u^\epsilon) \). It follows that the differential condition on the multiplier simplifies to \( \nabla^\gamma_{\alpha \beta} = 0 \), where \( \nabla^\gamma_{\alpha \beta} \) denotes covariant differentiation with respect to \( u^\gamma \). This proves that \( \Gamma \) is the Levi-Civita connection for the metric \( M_{\alpha \beta} \). Subsequent differentiations of the algebraic condition (3.8) imply that \( M_{\alpha \beta} \) must satisfy \( M_{\alpha \gamma} \nabla^\gamma_{\beta \epsilon \sigma} = M_{\beta \gamma} \nabla^\gamma_{\alpha \epsilon \sigma}, \) where \( \nabla^\gamma_{\alpha \epsilon \sigma} = \nabla^\gamma_{\alpha \epsilon} \nabla^\gamma_{\alpha \sigma} \cdots \nabla^\gamma_{\alpha 1} \). We remark that if \( (\mathcal{M}, \Gamma) \) is a (locally) symmetric space, then the algebraic condition (3.8) involving the curvature completely determines the number of linearly independent metrics for the given connection.

Example 3.5. Consider the system of differential equations

\[ u_{\alpha \gamma} + C^\alpha_{\beta \gamma} u^\beta_y = 0, \]  

where \( C^\alpha_{\beta \gamma} \in \mathbb{R} \) are the structure constants of an \( m \)-dimensional Lie algebra \( g \). The constants \( C^\alpha_{\beta \gamma} \) are skew symmetric in the lower indices and satisfy the Jacobi identity. Using (2.3) we
calculate the invariants
\[ H^\gamma_\alpha = (C^\gamma_\epsilon C^\sigma_\alpha - C^\gamma_\sigma C^\epsilon_\alpha)u^\epsilon_x u^\gamma_y, \quad K^\gamma_\alpha = (C^\gamma_\epsilon C^\sigma_\alpha - C^\gamma_\sigma C^\epsilon_\alpha)u^\epsilon_x u^\gamma_y, \]
\[ S^\gamma_\alpha_\beta = C^\gamma_\alpha_\beta - C^\gamma_\beta_\alpha = 2C^\gamma_\alpha_\beta. \]

Consequently, the algebraic conditions (3.10a) can be summarized as
\[ M_{\alpha\gamma} (C^\epsilon_\sigma C^\gamma_\alpha - C^\gamma_\epsilon C^\sigma_\alpha) = M_{\beta\gamma} (C^\epsilon_\sigma C^\gamma_\beta - C^\gamma_\epsilon C^\sigma_\beta), \]
\[ M_{\alpha\gamma} C^\gamma_\beta + M_{\beta\gamma} C^\gamma_\alpha = 0. \]

The differential condition (2.4b) on \( M_{\alpha\beta} \) is \( dM_{\alpha\beta} = (M_{\alpha\sigma} C^\sigma_\beta + M_{\beta\sigma} C^\sigma_\alpha)du^\gamma \), so that \( dM_{\alpha\beta} = 0 \) as a result of (3.10b). It is easy to check that (3.10b) implies (3.10a). It follows that there is a Lagrangian for (3.9) with multiplier \( M_{\alpha\beta} \) if and only if \( M_{\alpha\beta} \) is constant and \( M_{\alpha\beta} \) satisfies equation (3.10b). Moreover, the Lagrangian is given by
\[ L = -\frac{1}{6} M_{\alpha\beta} (3u^\alpha_x u^\beta_y - 2C^\alpha_\epsilon u^\beta u^\gamma_y). \]

We remark that (3.10b) is exactly the same as the condition for the existence of a bi-invariant symmetric bilinear form for a Lie algebra \( \mathfrak{g} \) with structure constants \( C^\alpha_\beta \). If \( \mathfrak{g} \) is semi-simple, the Killing form provides us with a non-degenerate solution to (3.10b). Consequently, (3.9) is variational whenever \( \mathfrak{g} \) is semi-simple. Moreover, the number of solutions to (3.10b) is equal to the dimension of the Lie algebra cohomology space \( H_3(\mathfrak{g}) \). If \( \mathfrak{g} \) is simple, then \( \dim H_3(\mathfrak{g}) = 1 \) and the Killing form determines the only non-degenerate solution to (3.10b) up to a scalar multiple (See [19], Theorems 11.1, 11.2). We remark that semi-simplicity is not a necessary condition for (3.10b) to hold, as there are solvable Lie algebras which also admit bi-invariant bilinear forms. For example, consider the solvable 4-dimensional Lie algebra \( \mathfrak{g} \) of consisting of real matrices of the form
\[ A = \begin{bmatrix} 0 & a_1 & a_2 \\ 0 & a_3 & a_4 \\ 0 & 0 & 0 \end{bmatrix}. \]

It is easy to check that the bilinear map \( M : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R} \) defined by
\[ M(A, B) = \lambda b_3 + \mu(a_2 b_3 + a_3 b_2 - a_1 b_4 - a_4 b_1) \]

is non-degenerate and bi-invariant for all \( \mu \neq 0 \) and all \( \lambda \in \mathbb{R} \).

4. Classification of Variational Systems in Two Dependent Variables

In this section we establish a result characterizing the variational systems
\[ u^\alpha_{xy} = f^\alpha(x, y, u^\gamma, u^\gamma_x, u^\gamma_y), \]
of two equations and two dependent variables that admit multiple Lagrangians. In order to proceed, we need to make precise two concepts that are paramount to our discussion. We first define what it means for a system to have multiple Lagrangians. Then we review the notion of contact equivalence of two systems of differential equations (4.1).

We say that a system of differential equations (4.1) admits \( k \) Lagrangians if there exists a set of linearly independent Lagrangians \( \{L_1, L_2, \ldots, L_k\} \) and a set of linearly independent variational multipliers \( \{M^1_{\alpha\beta}, \ldots, M^k_{\alpha\beta}\} \), such that \( E_\alpha(L_i) = M^i_{\alpha\beta}(u^\beta_{xy} - f^\beta) \) for \( i = 1, \ldots, k \). We say that two \( f \)-Gordon systems (4.1) are contact equivalent if there exists a local diffeomorphism
\[ \Phi : J^2(\mathbb{R}^2, \mathbb{R}^m) \to J^2(\mathbb{R}^2, \mathbb{R}^m), \]
where \( J^2(\mathbb{R}^2, \mathbb{R}^m) \) denotes the second-order jet-bundle of local sections \( s : \mathbb{R}^2 \to \mathbb{R}^m \), such that
\[
\Phi^*(\bar{u}_x^\alpha - \bar{f}^\alpha) = Q^\gamma_\gamma(u_x^\gamma - f^\gamma),
\]
and \( \Phi^* \bar{C} \subseteq \mathcal{C} \), where \( \mathcal{C} \) is the ideal generated by the 1-forms
\[
du^\alpha - u_x^\alpha dx - u_y^\alpha dy, \quad du_x^\alpha - u_{xx}^\alpha dx - u_{xy}^\alpha dy, \quad du_y^\alpha - u_{xy}^\alpha dx - u_{yy}^\alpha dy.
\]
It was shown in [5] that any contact equivalence \( \Phi \) of two systems of the form (4.1) is the prolongation of a fiber preserving transformation
\[
\bar{x} = A(x), \quad \bar{y} = B(y), \quad \bar{u}^\alpha = C^\alpha(x, y, u^\gamma),
\]
up to an interchange \( x \leftrightarrow y \) of the independent variables.

We have the following theorem that completely characterizes the variational \( f \)-Gordon systems (4.1) in two dependent variables which admit two or more inequivalent Lagrangians.

**Theorem 4.1.** Let \( \mathcal{R} \) denote the system
\[
(4.3) \quad u_{xy} = f(x, y, u, v, u_x, u_y, v_x, v_y), \quad v_{xy} = g(x, y, u, v, u_x, v_x, u_y, v_y).
\]

1) \( \mathcal{R} \) admits three Lagrangians if and only if \( \mathcal{R} \) is contact equivalent to a system
\[
u_{xy} = \lambda(x, y)u, \quad v_{xy} = \lambda(x, y)v.
\]

2) \( \mathcal{R} \) admits two Lagrangians if and only if \( \mathcal{R} \) is contact equivalent to a system
\[
u_{xy} = W_u(x, y, u, v), \quad v_{xy} = W_u(x, y, u, v),
\]
where \( W \) satisfies one of \( W_{uu} + W_{uv} = 0 \), \( W_{uu} = W_{uv} \), or \( W_{uv} = 0 \).

**Remark 4.2.** According to Proposition (2.2) and Remark (2.3), a system (4.3) admits at most three Lagrangians. Moreover, since the dimension of the vector space of symmetric \( m \times m \) matrices is \( m(m+1)/2 \), a system (4.1) of \( m \) equations admits at most \( m(m+1)/2 \) Lagrangians. If a system of \( m \) equations (4.1) admits the maximal number of Lagrangians, then it can be shown the given system is contact equivalent to a system \( u_{xy}^\alpha = \lambda(x, y)u^\alpha \).

The proof of the Theorem (4.1) depends on the following lemma that completely characterizes the \( f \)-Gordon equations \( u_{xy}^\alpha = g^\alpha(x, y, u^\gamma) \) up to contact equivalence. The proof of Lemma (4.3) is quite tedious and is delayed until after the proof of Theorem (4.1).

**Lemma 4.3.** A system of partial differential equations \( u_{xy}^\alpha = f^\alpha(x, y, u^\gamma, u_x^\gamma, u_y^\gamma) \) is contact equivalent to a system \( u_{xy}^\alpha = g^\alpha(x, y, u^\gamma) \) if and only if \( H = K \) and \( S^\alpha_{\beta\gamma} = 0 \). If the number of dependent variables \( m > 1 \) and \( H = K = \lambda I \), then \( \lambda = \lambda(x, y) \) and the given system is contact equivalent to a system \( u_{xy}^\alpha = \lambda(x, y)u^\alpha \).

**Proof of Theorem 4.1.** We first show that if a system (4.3) admits multiple Lagrangians, then (4.3) satisfies the hypotheses of Lemma (4.3) and is contact equivalent to a system of the form
\[
u_{xy} = F(x, y, u, v), \quad v_{xy} = G(x, y, u, v).
\]
According to Proposition (2.2), if (4.3) admits a first-order Lagrangian \( L \) with a symmetric multiplier \( M_{\alpha\beta} \), then \( M_{\alpha\beta} \) satisfies the algebraic conditions
\[
M_{\alpha\gamma}H^\gamma_{\beta} = M_{\beta\gamma}K^\gamma_{\alpha}, \quad M_{\alpha\sigma}S^\sigma_{\beta\gamma} + M_{\beta\sigma}S^\sigma_{\alpha\gamma} = 0,
\]
where \( H^\gamma_{\gamma}, K^\gamma_{\alpha} \), and \( S^\alpha_{\beta\gamma} \) are given by (2.3). Multiplying the second equation of (4.5) by \( M^\alpha_{\alpha\beta} \) yields
\[
0 = M^\alpha_{\alpha\beta} M_{\alpha\sigma}S^\sigma_{\beta\gamma} + M^\alpha_{\alpha\beta} M_{\beta\sigma}S^\sigma_{\alpha\gamma} = \delta^\beta_{\alpha}S^\gamma_{\beta\gamma} + \delta^\gamma_{\alpha}S^\sigma_{\alpha\gamma} = 2S^\alpha_{\alpha\gamma}.
\]
If their are only two dependent variables, then equation (4.6) implies that $S_{12}^1 = S_{12}^2 = 0$. Since $S_{\beta \gamma}^\alpha = -S_{\beta \gamma}^\alpha$, we have $S_{\beta \gamma}^\alpha = 0$ for all $\alpha, \beta, \gamma = 1, 2$. Then the algebraic conditions (4.5) can be expressed as

$$
(4.7) \quad A \cdot \begin{bmatrix} M_{11} \\ M_{12} \\ M_{22} \end{bmatrix} = 0, \quad A = \begin{bmatrix} H_1^1 - K_1^1 & H_1^2 - K_1^2 & 0 \\ 0 & H_2^1 - K_2^1 & H_2^2 - K_2^2 \\ H_1^1 & H_2^1 & -K_1^2 \end{bmatrix}.
$$

If (4.1) admits 2 or more inequivalent Lagrangians, then there are at least two linearly independent solutions to (4.4) and it follows that the rank of $A$ is at most one. Moreover, the rank of $A$ is one or less if and only if $H = K$ and is rank zero if and only if $H = K = \lambda I$. From Lemma (4.3), we deduce that $\text{Rank}(A) = 1$ if and only if (4.3) is contact equivalent to (4.1).

We will complete the proof of the theorem by analyzing the algebraic conditions (4.7) for systems of the form (4.4). A calculation of $H$ and $K$ for (4.4) reveals that

$$
H = K = \begin{bmatrix} F_u & F_v \\ G_u & G_v \end{bmatrix}.
$$

In this case the algebraic conditions (4.7) for the existence of a multiplier $M_{\alpha \beta}$ simplify to

$$
(4.8) \quad F_v M_{11} + (G_v - F_u) M_{12} - G_u M_{22} = 0.
$$

The differential condition (2.4b) reduces to $dM_{\alpha \beta} = 0$. Consequently, solving the variational multiplier problem for (4.4) is equivalent to determining all constant solutions $M_{\alpha \beta}$ to the equation (4.8).

There are 3 linearly independent solutions to (4.8) if and only if $H = K = \lambda I$. In this case (4.4) is contact equivalent to a system

$$
(4.9) \quad u_{xy} = \lambda(x, y) u, \quad v_{xy} = \lambda(x, y) v.
$$

The most general Lagrangian for (4.9) is a linear combination of the Lagrangians $L_1 = u_x u_y + \lambda u^2$, $L_2 = u_x v_y + \lambda u v$, and $L_3 = v_x v_y + \lambda v^2$.

We analyze the case where the rank of $A$ is exactly one and we assume there are two non-degenerate, linearly independent, constant solutions $M_{1 \alpha \beta}$ and $M_{2 \alpha \beta}$ to (4.8). We claim there exists an indefinite multiplier $M = (M_{\alpha \beta})$ that satisfies (4.8). If one of $\det M_1 < 0$ or $\det M_2 < 0$, then we are done, so we assume that $\det M_1 > 0$ and $\det M_2 > 0$. If

$$
M_1 = \begin{bmatrix} a & b \\ b & c \end{bmatrix}, \quad M_2 = \begin{bmatrix} p & q \\ q & r \end{bmatrix},
$$

then it follows that $ac > 0$ and $pr > 0$. We claim there is a scalar $\mu$ such that $\det(M_1 - \mu M_2) < 0$. Indeed, if

$$
P(\mu) = \det(M_1 - \mu M_2) = (\det M_2)^2 \mu^2 + (2bp - ar - pc) \mu + (\det M_1)^2,
$$

then the discriminant $\Delta$ can be expressed as

$$
\Delta = \frac{(aq - bp)^4 + 2(p^2 \det M_1 + a^2 \det M_2)(aq - bp)^2 + (a^2 M_2 - p^2 M_1)^2}{a^2 p^2}.
$$

It follows from (4.10) and (4.12) that $\Delta \geq 0$ with equality holding if and only if $M_1 = t M_2$ for some $t \in \mathbb{R}$. Consequently, the polynomial (4.11) has two real roots and there exists $\mu \in \mathbb{R}$ such that $\det(M_1 - \mu M_2) < 0$. Now we have established that if there are two independent solutions to (4.8), with at least one of the solutions non-degenerate, then there is an indefinite multiplier $M_{\alpha \beta}$. 

If we make a linear change of variables \( u^\alpha = T_\gamma^\alpha \tilde{u}^\gamma \), then a direct calculation of the Euler-Lagrange equations for the Lagrangian \( L \) and the transformed Lagrangian \( \tilde{L} \) verifies that the corresponding variational multipliers transform according to the rule

\[
M_{\alpha\beta} = T_\sigma^\alpha T_\tau^\beta \tilde{M}_{\sigma\tau}.
\]

After a linear change of variables \( u^\alpha \to T_\gamma^\alpha u^\gamma \), we may then assume that the indefinite multiplier \( M_{\alpha\beta} \) is of the form

\[
M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

Substituting (4.14) into (4.8) implies that

\[
Gv - Fu = 0.
\]

As a consequence of the de Rham theorem, we see there exists a smooth function \( W(x, y, u, v) \) such that

\[
Wv = F, \quad Wu = G.
\]

There is a second multiplier \( N \), independent of \( M \), satisfying (4.8). We may assume, possibly after subtracting a scalar multiple of \( M \), that

\[
N = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.
\]

According to (4.13), a linear transformation \( u \to \lambda u, v \to \lambda^{-1} v \) preserves \( M \) and transforms \( N \) as

\[
N \to \begin{pmatrix} a\lambda^2 & 0 \\ 0 & b\lambda^{-2} \end{pmatrix}.
\]

We may then assume, possibly after a scaling \( N \to kN \), that \( N \) has the form

\[
N = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix},
\]

where \( \varepsilon = 0, 1, \) or \(-1\). Taking into account that (4.15) holds, substituting \( N \) into (4.8) implies that

\[
W_{vv} - \varepsilon W_{uu} = 0.
\]

If \( \varepsilon = -1 \), then \( W \) satisfies Laplace’s equation \( W_{uu} + W_{vv} = 0 \) and there exists a function \( Z(x, y, u, v) \) such that \( Z_u = W_v = F \) and \( -Z_v = W_u = G \). The most general Lagrangian in this case is given by

\[
L = c_1(u_x v_y + W) + c_2(u_x u_y - v_x v_y) + Z).
\]

If \( \varepsilon = 0 \), then \( W_{vv} = 0 \) and (4.4) can be expressed

\[
u_{xy} = a'(u), \quad v_{xy} = a''(u)v + b'(u).
\]

The most general Lagrangian for (4.16) is

\[
L = c_1[u_x u_y + 2a(u)] + c_2[u_x v_y + a'(u)v + b(u)].
\]

If \( \varepsilon = 1 \), then \( W \) satisfies the wave equation and \( W = W_1(u + v) + W_2(u - v) \). The most general Lagrangian in the case is

\[
L = c_1[u_x u_y + v_x v_y + 2W_1(u + v) - 2W_2(u - v)] + c_2[u_x v_y + W_1(u + v) + W_2(u - v)].
\]

This establishes the second statement of the theorem. \( \square \)
Proof of Lemma 4.3. If we assume that two systems $\bar{u}_{xy} = f^\alpha$ and $u_{xy} = f^\alpha$ are contact equivalent with the change of coordinates given by (4.12), then the formulas (2.3) and a tedious application of the chain rule will verify that the transformation rules for $H$, $K$, and $S$ are

$$
H^\alpha_\beta = \frac{1}{A'B'} \frac{\partial \bar{u}^\alpha}{\partial u^\gamma} \frac{\partial u^\gamma}{\partial \bar{u}^\beta} H^\gamma_{\tau}, \quad K^\alpha_\beta = \frac{1}{A'B'} \frac{\partial \bar{u}^\alpha}{\partial u^\gamma} \frac{\partial u^\gamma}{\partial \bar{u}^\beta} K^\gamma_{\tau}, \quad S^\alpha_{\gamma\tau} = \frac{\partial \bar{u}^\alpha}{\partial u^\mu} \frac{\partial u^\mu}{\partial \bar{u}^\gamma} \frac{\partial u^\tau}{\partial \bar{u}^\delta} S^\mu_{\gamma\delta},
$$

where $(\partial u^\alpha / \partial \bar{u}^\gamma) \cdot (\partial \bar{u}^\gamma / \partial u^\beta) = \delta^\alpha_\beta$. It follows from (4.17) that the conditions $H = K$ and $S = 0$ are invariant with respect to transformation (4.12).

For a system $u^\alpha_{xy} = g^\alpha(x, y, u^\gamma)$, using (2.3) we see that

$$
S^\alpha_{\gamma\tau} = 0.
$$

From (4.17) and (4.18), we deduce that any system $u^\alpha_{xy} = f^\alpha(x, y, u^\gamma)$ that is contact equivalent to $u^\alpha_{xy} = g^\alpha(x, y, u^\gamma)$ must necessarily have $H = K$ and $S^\alpha_{\gamma\tau} = 0$.

We now prove that any system

$$
u^\alpha_{xy} = f^\alpha(x, y, u^\gamma, u^\gamma_x, u^\gamma_y)
$$

with the property that $H = K$ and $S^\alpha_{\gamma\tau} = 0$ is equivalent to a system $u^\alpha_{xy} = g^\alpha(x, y, u^\gamma)$. It follows from (2.3) that if $H^\alpha_\gamma = K^\alpha_\gamma$, then

$$
\frac{\partial^2 g^\alpha}{\partial u^\beta \partial u^\gamma} = 0, \quad \frac{\partial^2 f^\alpha}{\partial u^\beta \partial u^\gamma} = 0, \quad \forall \alpha, \beta, \gamma = 1, 2, \ldots, m.
$$

Consequently, equation (4.19) simplifies to an equation of the form

$$
u^\alpha_{xy} + C^\alpha_{\gamma\tau} u^\beta u^\gamma + A^\alpha_{\gamma} u^\gamma_x + B^\alpha_{\gamma} u^\gamma_y + G^\alpha = 0,
$$

where $C^\alpha_{\gamma\tau}$, $A^\alpha_{\gamma}$, $B^\alpha_{\gamma}$, and $G^\alpha$ are functions of $x$, $y$, and $u^\gamma$. Moreover, $S^\alpha_{\gamma\tau} = 0$ if and only if $C^\alpha_{\gamma\tau} = C^\alpha_{\gamma\tau}$. With a judicious choice of coordinates, we will now eliminate the quadratic terms of (4.20). If we let $u^\alpha = g^\alpha(x, y, \bar{u}^\gamma)$, then (4.20) transforms as

$$
\frac{\partial^2 g^\alpha}{\partial \bar{u}^\beta \partial \bar{u}^\gamma} + C^\alpha_{\gamma\tau}(x, y, \bar{u}^\gamma) \frac{\partial g^\alpha}{\partial \bar{u}^\beta} \frac{\partial g^\tau}{\partial \bar{u}^\gamma} \bar{u}^\beta \bar{u}^\gamma + A^\alpha_{\gamma} \bar{u}^\gamma_x + B^\alpha_{\gamma} \bar{u}^\gamma_y + \bar{G}^\alpha.
$$

We see that $\bar{C}^\alpha_{\gamma\tau} = 0$ whenever $g^\alpha$ satisfies

$$
\frac{\partial^2 g^\alpha}{\partial \bar{u}^\beta \partial \bar{u}^\gamma} + C^\alpha_{\gamma\tau}(x, y, g^\tau) \frac{\partial g^\alpha}{\partial \bar{u}^\beta} \frac{\partial g^\tau}{\partial \bar{u}^\gamma} = 0.
$$

We differentiate (4.22) with respect to $\bar{u}^\delta$, and after substituting from (4.22) and skew-symmetrizing over $\beta$ and $\gamma$, we obtain the integrability conditions on (4.22)

$$
\left( \frac{\partial C^\alpha_{\gamma\tau}}{\partial \bar{u}^\mu} - C^\alpha_{\gamma\tau} C^\gamma_{\mu\sigma} - C^\alpha_{\tau\mu} C^\gamma_{\sigma\tau} \right) \frac{\partial g^\delta}{\partial \bar{u}^\beta} \frac{\partial g^\sigma}{\partial \bar{u}^\gamma} \frac{\partial g^\tau}{\partial \bar{u}^\delta} = 0.
$$

On the other hand a calculation of $H$ and $K$ for (4.20) yields

$$
\frac{\partial^2}{\partial u^\beta \partial u^\gamma} (H^\alpha_\gamma - K^\alpha_\gamma) = \frac{\partial C^\alpha_{\gamma\sigma}}{\partial u^\tau} - \frac{\partial C^\alpha_{\gamma\tau}}{\partial u^\sigma} + C^\alpha_{\tau\mu} C^\gamma_{\sigma\tau} - C^\alpha_{\tau\mu} C^\gamma_{\sigma\mu}.
$$

As a consequence of (4.24), the integrability conditions (4.23) are satisfied whenever $H = K$. The system of partial differential equations (4.22) then satisfies the Frobenius condition and we deduce that there exists, at least locally, a non-degenerate collection of functions $g^\alpha$ satisfying (4.22).
We may now assume that \( u^\alpha_{xy} = f^\alpha(x, y, u^\gamma, u^2_x, u^2_y) \) is of the form
\[
(4.25) \quad u^\alpha_{xy} + A^\alpha_\gamma(x, y, u^\epsilon)u^\epsilon_x + B^\alpha_\gamma(x, y, u^\epsilon)u^\epsilon_y + G^\alpha(x, y, u^\epsilon) = 0.
\]
For (4.25), we calculate
\[
(4.26) \quad H^\alpha_\gamma - K^\alpha_\gamma = \frac{\partial A^\alpha_\gamma}{\partial u^\beta} u^\beta_x + \frac{\partial B^\alpha_\gamma}{\partial u^\beta} u^\beta_y + A^\gamma_\beta B^\alpha_\gamma - B^\gamma_\beta A^\alpha_\gamma + \frac{\partial A^\alpha_\gamma}{\partial x} - \frac{\partial B^\alpha_\gamma}{\partial y}.
\]
Evidently, \( H^\alpha_\gamma = K^\alpha_\gamma \) only if \( \partial B^\gamma_\beta / \partial u^\beta = \partial A^\alpha_\gamma / \partial u^\beta = 0 \) for all \( \alpha, \beta, \) and \( \gamma \). It follows that (4.25) simplifies to
\[
(4.27) \quad u^\alpha_{xy} + A^\alpha_\gamma(x, y)u^\gamma_x + B^\alpha_\gamma(x, y)u^\gamma_y + G^\alpha(x, y, u^\epsilon) = 0.
\]
We now eliminate the functions \( A^\alpha_\gamma(x, y) \) and \( B^\alpha_\gamma(x, y) \) from (4.27) with a transformation \( u^\alpha \rightarrow N^\alpha_\gamma(x, y)u^\gamma \), where \( N^\alpha_\gamma \) satisfies the system of partial differential equations
\[
(4.28) \quad \frac{\partial N^\alpha_\gamma}{\partial y} + A^\alpha_\gamma N^\gamma_\gamma = 0, \quad \frac{\partial N^\alpha_\gamma}{\partial x} + B^\alpha_\gamma N^\gamma_\gamma = 0.
\]
The integrability conditions for (4.28) are given by
\[
(4.29) \quad \frac{\partial A^\alpha_\gamma}{\partial x} + A^\gamma_\beta B^\alpha_\gamma = \frac{\partial B^\alpha_\gamma}{\partial y} + B^\gamma_\beta A^\alpha_\gamma.
\]
As a consequence of (4.26) we see that (4.29) holds whenever \( H = K \). It follows that the system (4.28) satisfies the Frobenius condition and there exists functions \( N^\alpha_\gamma(x, y) \) such that \( \det N^\alpha_\gamma \neq 0 \) and (4.28) is satisfied. Moreover, we arrive at an equation of the desired form
\[
(4.30) \quad \frac{\partial g^\alpha}{\partial u^\beta} = \lambda \delta^\alpha_\beta, \quad \alpha = 1, 2, \ldots, m,
\]
with no summation on \( \alpha \). Differentiating equation (4.31) with respect to \( u^\beta, 1 \leq \beta \neq \alpha \leq m \), results in \( \partial \lambda / \partial u^\beta = 0 \) for all \( \beta \). We deduce that \( \lambda = \lambda(x, y) \) and it becomes apparent from (4.31) that \( g^\alpha = \lambda(x, y)u^\alpha + k^\alpha(x, y) \). After a transformation \( u^\alpha \rightarrow u^\alpha + m^\alpha(x, y), \) where \( n^\alpha_{xy} = m^\alpha + k^\alpha \), we see that (4.30) is equivalent to \( u^\alpha_{xy} = \lambda(x, y)u^\alpha \).

5. The Variational Bicomplex For Systems of PDE

In this section we introduce some basic definitions and results used in our solution to the variational multiplier problem in Section 6, including infinite jet bundles and variational bicomplexes. As we are only interested in applications to our study of the variational multiplier problem, our discussion will be of a rather brief nature. For a detailed and intrinsic construction of variational bicomplex, we refer the reader to [1], [3], [18], and [25].

Let \( \pi^k : J^k(E) \rightarrow \mathbb{R}^n \) denote the bundle of \( k \)-jets of local sections of the trivial bundle \( E = \mathbb{R}^n \times \mathbb{R}^m \). Local coordinates for \( J^k(R^n, R^m) \) are given by
\[
(x^i, u^\alpha_i, u^\alpha_{1i}, u^\alpha_{12i}, \ldots, u^\alpha_{1i2...ik}),
\]
where \( 1 \leq i_1 \leq i_2 \leq \ldots \leq i_k \leq n \) and \( 1 \leq \alpha \leq m \). There are natural projections \( \pi^l_k : J^l(E) \rightarrow J^k(E) \) for \( l \geq k \). The infinite jet bundle over \( E \), \( J^\infty(E) \), is defined as the inverse limit of the sequence
of finite jet bundles \( \{ J^k(E) \mid k = 0, 1, 2, \ldots \} \), along with the projections \( \pi_k^\infty : J^\infty(E) \to J^k(E) \) and \( \pi^\infty : J^\infty(E) \to \mathbb{R}^n \). The contact ideal \( \mathcal{C}(J^\infty(E)) \) is generated by the 1-forms
\[
\theta^\alpha_{i_1 i_2 \ldots i_k} = du^\alpha_{i_1 i_2 \ldots i_k} - u^\alpha_{i_1 i_2 \ldots i_k} dx^j, \quad \forall k = 0, 1, 2, \ldots .
\]
The full exterior algebra \( \Omega^s(J^\infty(E)) \) of differential forms on \( J^\infty(E) \) is generated by the 1-forms
\[
dx^i, \, \theta^\alpha, \, \theta^\alpha_{ij}, \, \theta^\alpha_{ij}, \ldots .
\]
There is a bi-grading of the differential forms on \( J^\infty(E) \),
\[
\Omega^p(J^\infty(E)) = \bigoplus_{r+s=p} \Omega^{r,s}(J^\infty(E)),
\]
where \( \Omega^{r,s}(J^\infty(E)) \) is the \( C^\infty(J^\infty(E)) \)-module generated by differential forms of the type
\[
dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_r} \wedge \theta^\alpha_{j_1} \wedge \cdots \wedge \theta^\alpha_{j_s}.
\]
The exterior derivative \( d : \Omega^p(J^\infty(E)) \to \Omega^{p+1}(J^\infty(E)) \) splits into the horizontal and vertical differentials \( d = d_H + d_V \), where
\[
d_H : \Omega^{r,s}(J^\infty(E)) \to \Omega^{r+1,s}(J^\infty(E)), \quad d_V : \Omega^{r,s}(J^\infty(E)) \to \Omega^{r,s+1}(J^\infty(E)).
\]
Since \( d^2 = 0 \), it follows that \( d_H^2 = 0, \, d_V^2 = 0 \), and \( d_H d_V = -d_V d_H \). The local coordinate expressions for the horizontal and vertical derivatives of a smooth function \( f \in C^\infty(J^\infty(E)) \) and 1-forms \( dx^i \) and \( \theta^\alpha_i \) are given by
\[
d_H \theta^\alpha_i = dx^i \wedge \theta^\alpha_{ii}, \quad d_V \theta^\alpha_i = 0, \quad d_H(dx^i) = 0, \quad d_V(dx^i) = 0,
\]
\[
d_H f = (D_i f) dx^i, \quad d_V f = \frac{\partial f}{\partial u^\alpha} \theta^\alpha + \frac{\partial f}{\partial u^\alpha_{ij}} \theta^\alpha_{ij} + \cdots ,
\]
where \( D_i \) denotes total differentiation with respect to \( x^i \). The free variational bicomplex is defined to be the double complex \( \{ \Omega^{r,s}, J^\infty(E), d_H, d_V \}_{s \geq 0, \, r = 0, 1, \ldots , n} \).

In our solution to the variational multiplier problem for the \( f \)-Gordon systems\footnote{\textcopyright 2023 M. Biesecker}, we begin with a trivial bundle \( \pi : \mathbb{R}^2 \times \mathbb{R}^m \to \mathbb{R}^2 \) and consider the second-order jet bundle \( J^2(\mathbb{R}^2, \mathbb{R}^m) \) with coordinates given by
\[
(x, y, u^\alpha, u^\alpha_{x}, u^\alpha_{y}, u^\alpha_{xy}, u^\alpha_{xxy}, u^\alpha_{xyy}), \quad \alpha = 1, \ldots , m.
\]
An \( f \)-Gordon system\footnote{\textcopyright 2023 M. Biesecker} defines a \( (5m + 2) \)-dimensional submanifold \( \mathcal{R}^2 \supset J^2(E) \) called the equation manifold of \( \mathcal{R}^2 \). We define the first prolongation of \( \mathcal{R}^2 \) as the \( 7m + 2 \)-dimensional submanifold \( \mathcal{R}^3 \supset J^3(E) \) defined by \footnote{\textcopyright 2023 M. Biesecker} and \( u^\alpha_{xy} = D_x f^\alpha \) and \( u^\alpha_{xxy} = D_y f^\alpha \). Further differentiations of \footnote{\textcopyright 2023 M. Biesecker} will yield submanifolds \( \mathcal{R}^k \supset J^k(E) \). For convenience we define \( \mathcal{R}^0 = E \) and \( \mathcal{R}^1 = J^1(E) \). We define the infinite prolonged equation manifold \( \mathcal{R}^\infty \) to be the inverse limit of the sequence \( \{ \mathcal{R}^k \mid k = 0, 1, 2, \ldots \} \), along with the natural projections \( \pi^\infty_M : \mathcal{R}^\infty \to M \) and \( \pi^\infty_k : \mathcal{R}^\infty \to \mathcal{R}^k \). We remark that there is a unique map \( \iota^\infty : \mathcal{R}^\infty \to J^\infty(E) \) that satisfies the commutative diagram
\[
\begin{array}{ccc}
\mathcal{R}^\infty & \xrightarrow{\iota^\infty} & J^\infty(E) \\
\pi^\infty_k \downarrow & & \downarrow \pi^\infty_k \\
\mathcal{R}^k & \xrightarrow{\iota} & J^k(E).
\end{array}
\]
For an f-Gordon system \((5.2)\), coordinates on \(\mathcal{R}^\infty\) are given by
\[
(x, y, u^\alpha, u_x^\alpha, u_y^\alpha, u_{x^2}^\alpha, u_{y^2}^\alpha, u_{x^2y}^\alpha, u_{y^2x}^\alpha, \ldots).
\]

We define the contact ideal \(\mathcal{C}(\mathcal{R}^\infty)\) on \(\mathcal{R}^\infty\) via the pullback of the contact ideal on \(J^\infty(E)\), that is \(\mathcal{C}(\mathcal{R}^\infty) = \iota^*\mathcal{C}(J^\infty(E))\). The contact ideal \(\mathcal{C}(\mathcal{R}^\infty)\) for an f-Gordon system \((5.2)\) is generated by the 1-forms \(\{\theta, \theta_x^\alpha, \theta_y^\alpha, \theta_{xx}^\alpha, \theta_{yy}^\alpha, \theta_{xxy}^\alpha, \theta_{yyx}^\alpha, \ldots\}\), where
\[
\theta^\alpha = du^\alpha - u_x^\alpha dx - u_y^\alpha dy, \quad \theta_x^\alpha = du_x^\alpha - u_{xx}^\alpha dx - f^\alpha dy, \quad \theta_y^\alpha = du_y^\alpha - f^\alpha dx - u_{yy}^\alpha dy,
\]
\[
\theta_{xx}^\alpha = du_{x^2}^\alpha - u_{x^2x}^\alpha dx - \iota^*(D_x f^\alpha) dy, \quad \theta_{yy}^\alpha = du_{y^2}^\alpha - \iota^*(D_y f^\alpha) dx - u_{y^2y}^\alpha dy, \ldots.
\]

A basis for the \(C^\infty(\mathcal{R}^\infty)\)-module of 1-forms \(\Omega^1(\mathcal{R}^\infty)\) is given by
\[
(5.3) \quad \{dx, dy, \theta^\alpha, \theta_x^\alpha, \theta_y^\alpha, \theta_{xx}^\alpha, \theta_{yy}^\alpha, \theta_{xxy}^\alpha, \theta_{yyx}^\alpha, \ldots\}
\]

For every \(p\), we have a bi-grading \(\Omega^p(\mathcal{R}^\infty) = \bigoplus \Omega^{r,s}(\mathcal{R}^\infty)\), where \(r + s = p\) and \(\Omega^{r,s}(\mathcal{R}^\infty) = \iota^*\Omega^{r,s}(J^\infty(E))\). The exterior derivative \(d : \Omega^p(\mathcal{R}^\infty) \to \Omega^{p+1}(\mathcal{R}^\infty)\) splits as \(d = d_H + d_V\), where \(d_H : \Omega^{r,s} \to \Omega^{r+1,s}\) and \(d_V : \Omega^{r,s} \to \Omega^{r,s+1}\) are the horizontal and vertical differentials, respectively. In the following section we will frequently use the \(d_H\) and \(d_V\) structures for the coframe \((5.3)\), which are given by
\[
(5.4) \quad d_H \theta^\alpha = dx \wedge \theta_x^\alpha + dy \wedge \theta_y^\alpha, \quad d_H \theta_x^\alpha = dx \wedge \theta_x^\alpha + dy \wedge f^\alpha,
\]
\[
d_H \theta_y^\alpha = dx \wedge f^\alpha + dy \wedge \theta_{xy}^\alpha, \quad d_V \theta^\alpha = d_V \theta_x^\alpha = d_V \theta_y^\alpha = 0,
\]
\[
d_H g = D_x g dx + D_y g dy, \quad d_V g = \frac{\partial g}{\partial u^\alpha} \theta^\alpha + \frac{\partial g}{\partial u_x^\alpha} \theta_x^\alpha + \frac{\partial g}{\partial u_y^\alpha} \theta_y^\alpha + \cdots,
\]
where \(g \in C^\infty(\mathcal{R}^\infty)\) and \(D_x\) and \(D_y\) denote total differentiation constrained to the equation manifold \(\mathcal{R}^\infty\).

**Definition 5.1.** The constrained variational bicomplex \(\Omega^{*,s}(\mathcal{R}^\infty, d_H, d_V)\) associated to an f-Gordon system is the pullback of the free variational bicomplex \(\Omega^{*,s}(J^\infty(E), d_H, d_V)\) by \(\iota_\infty : \mathcal{R}^\infty \to J^\infty(E)\).

\[
\begin{array}{cccc}
0 & \longrightarrow & \Omega^0(\mathcal{R}^\infty) & \xrightarrow{d_V} \Omega^0(\mathcal{R}^\infty) & \xrightarrow{d_V} \Omega^0(\mathcal{R}^\infty) \\
& & d_H & & d_H \\
0 & \longrightarrow & \Omega^1(\mathcal{R}^\infty) & \xrightarrow{d_V} \Omega^1(\mathcal{R}^\infty) & \xrightarrow{d_V} \Omega^1(\mathcal{R}^\infty) \\
& & d_H & & d_H \\
& & & & \ldots
\end{array}
\]

The columns of the variational bicomplex on \(\mathcal{R}^\infty\) are locally exact, while the rows will not be exact in general. We define the horizontal cohomology classes of the variational bicomplex by
\[
H^{r,s}(\mathcal{R}^\infty) = \frac{\{\omega \in \Omega^{r,s}(\mathcal{R}^\infty) \mid d_H \omega = 0\}}{\{d_H \eta \mid \eta \in \Omega^{r-1,s}(\mathcal{R}^\infty)\}}, \quad r, s \geq 0.
\]

It is easy to see that \(H^{r,s}(\mathcal{R}^\infty)\) is a vector space over \(\mathbb{R}\). The cohomology classes in \(H^{r,s}(\mathcal{R}^\infty)\) have interesting interpretations. For example, each class \([\omega] = H^{1,0}(\mathcal{R}^\infty)\) represents a classical conservation law. Indeed, if \(\omega = M dx + N dy\), then \(d_H \omega = 0\) if and only if \(D_x M = D_y N\) when restricted to the equation manifold \(\mathcal{R}^\infty\). We refer to a cohomology class \([\omega] \in H^{1,s}(\mathcal{R}^\infty)\) as a...
type $(1, s)$ conservation law or form-valued conservation law. In the following section, we will show that the solution to the variational multiplier problem is closely related to the existence of non-trivial classes $[\omega] \in H^{1,2}(\mathcal{R}^\infty)$. Since systems of the form (5.2) are of Cauchy-Kovaleskaya type, it follows from a general result of Vinogradov [25] that the horizontal cohomology spaces $H^{0,s}(\mathcal{R}^\infty)$ satisfy

\[ H^{0,0}(\mathcal{R}^\infty) = \mathbb{R}, \quad H^{0,s}(\mathcal{R}^\infty) = 0, \quad s > 0. \]

We remark that (5.5) was also established in [5] by constructing a coframe adapted to systems (5.2).

6. Derivation of Necessary and Sufficient Conditions for the Existence of a Variational Multiplier

Our first result states that the problem of determining all Lagrangians and variational multipliers for an $f$-Gordon system is equivalent to determining all $d$ closed forms $\omega \in \Omega^{1,2}(\mathcal{R}^\infty)$ of a certain type. We also give a description of the general form of possible Lagrangians for an $f$-Gordon system. Finally, we show that a variational multiplier has no one-jet dependence. Proposition (6.1) along with Corollaries (6.2) and (6.3) will suffice to establish Proposition (2.1), which was stated without proof in Section 2.

**Theorem 6.1.** For a system of differential equations $u^\omega_{xy} = f^\alpha(x, y, u^\gamma, u^\gamma_x, u^\gamma_y)$ the following statements are equivalent.

(i) There exists a type $(1, 2)$ form

\[ (6.1) \quad \omega = (T_{\alpha\beta} dx + S_{\alpha\beta} dy) \wedge \theta^\alpha \wedge \theta^\beta + R_{\alpha\beta}(\theta^\alpha \wedge \theta^\alpha_x \wedge dx - \theta^\beta \wedge \theta^\alpha_y \wedge dy) \]

such that $d\omega = 0$ on $\mathcal{R}^\infty$.

(ii) There exists a first-order multiplier $M_{\alpha\beta}(x, y, u^\gamma, u^\gamma_x, u^\gamma_y)$ and a first-order Lagrangian $L(x, y, u^\gamma, u^\gamma_x, u^\gamma_y)$ such that $E^\alpha(L) = M_{\alpha\beta}(u^\beta_{xy} - f^\beta)$.

(iii) There exists a multiplier $M_{\alpha\beta} = M_{\alpha\beta}(x, y, u^\gamma)$ and a Lagrangian

\[ L = -R_{\alpha\beta}(x, y, u^\gamma)u^\alpha_xu^\beta_y + Q^\alpha(x, y, u^\gamma)u^\alpha_x + P^\alpha(x, y, u^\gamma)u^\alpha_y + N(x, y, u^\gamma) \]

such that $E^\alpha(L) = M_{\alpha\beta}(u^\beta_{xy} - f^\beta)$.

**Proof.** We first show that (i) implies (iii). Suppose that $\omega$ is given by (6.1) and that $d\omega = 0$. on $\mathcal{R}^\infty$. It follows immediately that $d_\mu^\omega = d_\nu^\omega = 0$. A routine calculation using (5.4) shows that if $d_\nu^\omega = 0$, then $R_{\alpha\beta} = R_{\alpha\beta}(x, y, u^\gamma)$ and $\omega$ is of the form

\[ \omega = \left[ \frac{1}{2} \left( \frac{\partial R_{\alpha\gamma}}{\partial u^\beta} - \frac{\partial R_{\beta\gamma}}{\partial u^\alpha} \right) u^\gamma_x + T^0_{\alpha\beta}(x, y, u^\gamma) \right] \theta^\alpha \wedge \theta^\beta \wedge dx + R_{\alpha\beta}\theta^\alpha \wedge \theta^\beta_x \wedge dx \]

\[ + \left[ \frac{1}{2} \left( \frac{\partial R_{\gamma\beta}}{\partial u^\alpha} - \frac{\partial R_{\gamma\alpha}}{\partial u^\beta} \right) u^\beta_y + S^0_{\alpha\beta}(x, y, u^\gamma) \right] \theta^\alpha \wedge \theta^\beta \wedge dy - R_{\beta\alpha}\theta^\alpha \wedge \theta^\beta_y \wedge dy. \]

If we define $\rho_0 \in \Omega^{1,1}(\mathcal{R}^\infty)$ by

\[ \rho_0 = -R_{\alpha\beta}u^\alpha_x \theta^\alpha \wedge dx + R_{\beta\alpha}u^\beta_y \theta^\alpha \wedge dy, \]

then using (5.4) we see that

\[ \omega - d_\nu\rho_0 = T^0_{\alpha\beta} \theta^\alpha \wedge \theta^\beta \wedge dx + S^0_{\alpha\beta} \theta^\alpha \wedge \theta^\beta \wedge dx. \]
Since $d_V(\omega - d_V \rho_0) = 0$, there exists a form $\rho_1 \in \Omega^{1,1}(\mathcal{R}^\infty)$ such that $d_V \rho_1 = \omega - d_V \rho_0$. Since the functions $S_{\alpha\beta}^0$ and $T_{\alpha\beta}^0$ have no 1-jet dependence, we may choose $\rho_1$ to be of the form

$$\rho_1 = -P_{\alpha}(x, y, u^\epsilon) \theta^\alpha \wedge dx + Q_{\alpha}(x, y, u^\epsilon) \theta^\alpha \wedge dy.$$

We now define $\rho = \rho_0 + \rho_1$ so that $d_V \rho = \omega$. Since $d_V d_H = -d_V d_H$, we have $d_V(d_H \rho) = -d_H d_V \rho = -d_H \omega = 0$ on $\mathcal{R}^\infty$. Therefore there is a Lagrangian form $\lambda \in \Omega^{0,2}(\mathcal{R}^\infty)$ with $d_V \lambda = d_H \rho$ on $\mathcal{R}^\infty$. We will now show that $\lambda = L dx \wedge dy$, where $L$ is of the form given in statement (iii). Computing $d_H \rho$ on $J^\infty(E)$ yields

$$d_H \rho = - \left[ (R_{\alpha\beta} + R_{\beta\alpha}) u_{xy}^\beta + g_{\alpha}(x, y, u^\epsilon, u_x^\epsilon, u_y^\epsilon) \right] \theta^\alpha \wedge dx \wedge dy - (R_{\alpha\beta} u_x^\beta + P_{\alpha}) \theta^\alpha \wedge dx \wedge dy - (R_{\beta\alpha} u_y^\beta + Q_{\alpha}) \theta^\alpha \wedge dx \wedge dy,$$

where

$$(6.2) \quad g_{\alpha} = \left( \frac{\partial R_{\alpha\beta}}{\partial u^\gamma} + \frac{\partial R_{\gamma\alpha}}{\partial u^\beta} \right) u_x^\gamma u_y^\alpha + \left( \frac{\partial R_{\alpha\beta}}{\partial y} + \frac{\partial Q_{\alpha}}{\partial u^\beta} \right) u_x^\beta + \left( \frac{\partial R_{\beta\alpha}}{\partial x} + \frac{\partial P_{\alpha}}{\partial u^\beta} \right) u_y^\beta.$$

When restricted to $\mathcal{R}^\infty$,

$$-d_H \rho = \left[ (2R_{\alpha\beta} f^\beta + g_{\alpha}) \theta^\alpha + (R_{\alpha\beta} u_x^\beta + P_{\alpha}) \theta^\alpha + (R_{\beta\alpha} u_y^\beta + Q_{\alpha}) \theta^\alpha \right] \wedge dx \wedge dy,$$

where $R_{\alpha\beta} = (R_{\alpha\beta} + R_{\beta\alpha})/2$ and $g_{\alpha}$ is given by (6.2). Since $\lambda = L dx \wedge dy$ and $d_V \lambda = d_H \rho$ on $\mathcal{R}^\infty$, we see that $L$ must satisfy

$$\frac{\partial L}{\partial u^\alpha} = -2R_{\alpha\beta} f^\beta - g_{\alpha}, \quad \frac{\partial L}{\partial u_x^\alpha} = -R_{\beta\alpha} u_y^\beta - Q_{\alpha}, \quad \frac{\partial L}{\partial u_y^\alpha} = -R_{\alpha\beta} u_x^\beta - P_{\alpha}.$$

It follows that there exists a function $N(x, y, u^\epsilon)$ such that

$$(6.3) \quad L = - (R_{\alpha\beta} u_x^\beta u_y^\alpha + Q_{\alpha} u_x^\alpha + P_{\alpha} u_y^\alpha + N(x, y, u^\epsilon)).$$

If we apply the Euler-Lagrange operator $E(\lambda) = E_{\alpha}(L) \theta^\alpha \wedge dx \wedge dy$ on $J^\infty(E)$, then

$$d_H \rho + E(\lambda) = d_V(\lambda).$$

Since $d_H \rho = d_V \lambda$ when restricted to the equation manifold $\mathcal{R}^\infty$, we deduce that implies $\iota^* E(\lambda) = 0$. On the other hand, a direct computation of the Euler-Lagrange equations for (6.3) gives us

$$E_{\alpha}(L) = 2R_{\alpha\beta} f^\beta - g_{\alpha} + \frac{\partial L}{\partial u^\alpha}.$$

Since $\iota^* E_{\alpha}(L) = 0$ for all $\alpha$, we get $g_{\alpha} + \partial L/\partial u^\alpha = -2R_{\alpha\beta} f^\beta$, which implies

$$E_{\alpha}(L) = 2R_{\alpha\beta}(u_{xy}^\beta - f^\beta),$$

and (iii) is proved. Moreover, the multiplier $M_{\alpha\beta}$ is defined by $M_{\alpha\beta} = R_{\alpha\beta} + R_{\beta\alpha}$.

We now prove that statement (iii) implies (i). Assume that there is a variational multiplier $M_{\alpha\beta}(x, y, u^\epsilon)$, a first-order Lagrangian $\lambda = L dx \wedge dy$, where $L$ is of the form (6.3), and

$$(6.4) \quad E_{\alpha}(L) = M_{\alpha\beta}(u_{xy}^\beta - f^\beta).$$

Note that (6.4) explicitly determines that the multiplier is $M_{\alpha\beta} = R_{\alpha\beta} + R_{\beta\alpha}$. By the first variational formula (See [1], Corollary 5.3), we have $E(\lambda) + d_H \eta = d_V \lambda$, where

$$\eta = \frac{\partial L}{\partial u_{xy}^\alpha} \theta^\alpha \wedge dx - \frac{\partial L}{\partial u_x^\alpha} \theta^\alpha \wedge dy.$$
Since \( \iota^* E(\lambda) = 0 \), it follows that \( \iota^* d_H \eta = d_V \lambda \). Then define \( \omega = d_V \eta \) and the resulting calculation of \( d\omega \) on the equation manifold is
\[
d\omega = d_H d_V \omega + d_V \omega = d_H (d_V \eta) + d_V d_V \eta = -d_V d_V \lambda = 0.
\]

Moreover, using (5.4) to compute \( d_V \eta \) will verify that \( \omega \) is of the form \( (6.1) \).

The following corollary states that a Lagrangian \( \lambda \) corresponding to a type \((2,1)\) cohomology class \([\omega]\) is unique up to modification by a total divergence.

**Corollary 6.2.** Let \( \omega \) be given by \( (6.1) \) with \( d\omega = 0 \). If \( \lambda = L(x, y, u^\gamma, u_x^\gamma, u_y^\gamma) \) and \( \lambda' = L'(x, y, u^\gamma, u_x^\gamma, u_y^\gamma) \) are two Lagrangians and \( \eta, \eta' \in \Omega^{1,1}(\mathcal{R}^\infty) \) satisfy
\[
d_H \eta = d_V \lambda, \quad d_H \eta' = d_V \lambda', \quad d_V \eta = d_V \eta' = \omega,
\]
then \( \lambda' = \lambda + d_H \beta \) for some form \( \beta \in \Omega^{1,0}(\mathcal{R}^\infty) \).

**Proof.** Since \( d_V (\eta' - \eta) = 0 \), there exists a form \( \beta_0 \in \Omega^{1,0}(\mathcal{R}^\infty) \) such that \( \eta' = \eta + d_V \beta_0 \). It follows that
\[
d_V \lambda' = d_H \eta' = d_H (\eta + d_V \beta_0) = d_V \lambda + d_H d_V \beta_0 = d_V (\lambda - d_H \beta_0).
\]
Consequently, \( \lambda \) and \( \lambda' \) satisfy
\[
d_V (\lambda' - \lambda + d_H \beta_0) = 0,
\]
or equivalently that
\[
\lambda' - \lambda + d_H \beta_0 = a(x, y) dx \wedge dy.
\]
Defining \( \beta = \beta_0 - A(x, y) dy \), where \( A(x, y) = a(x, y) \), we have that
\[
\eta' = \eta + d_V \beta \quad \text{and} \quad \lambda' = \lambda - d_H \beta,
\]
as required.

The following corollary gives a description the general form of a \( f \)-Gordon systems that is variational. Corollary (6.3), along with Theorem (6.1) and Corollary (6.2), establishes Proposition (2.1).
Corollary 6.3. If $u^\beta_{xy} = f^\beta(x, y, u, u^\gamma, u^\delta_y, u^\delta_y)$ has a nonsingular variational multiplier $M_{\alpha\beta}$, then

$$f^\beta = C^\beta_{\epsilon\gamma} u^\gamma_x u^\gamma_y + A^\beta_x u^\epsilon_x + B^\beta_x u^\epsilon_y + G^\beta,$$

where $A^\beta_x, B^\beta_x, C^\beta_{\epsilon\gamma}, G^\beta \in C^\infty(J^0(E))$.

Proof. According to Theorem (6.1), we may assume that there is Lagrangian $L$ of the form (6.3) and a variational multiplier $M_{\alpha\beta}(x, y, u)$, with $\text{det} M_{\alpha\beta} \neq 0$ and

$$(6.8) \quad E_\alpha(L) = M_{\alpha\beta}(u^\beta_{xy} - f^\beta).$$

On the other hand, the Euler-Lagrange equations for (6.3) are explicitly given by

$$E_\alpha(L) = (R_{\alpha\beta} + R_{\beta\alpha}) u^\beta_x + \left(\frac{\partial R_{\alpha\gamma}}{\partial u^\gamma} + \frac{\partial R_{\beta\alpha}}{\partial u^\gamma} - \frac{\partial R_{\beta\gamma}}{\partial u^\alpha}\right) u^\beta u^\gamma_x +$$

$$\left(\frac{\partial P_{\alpha}}{\partial u^\beta} - \frac{\partial P_{\beta}}{\partial u^\alpha} + \frac{\partial P_{\beta}}{\partial x} + \frac{\partial Q_{\alpha}}{\partial y} - \frac{\partial Q_{\gamma}}{\partial x} + \frac{\partial P_{\beta}}{\partial y} - \frac{\partial N}{\partial u^\alpha} + \frac{\partial Q_{\alpha}}{\partial y}ight) u^\beta_x +$$

Clearly, $M_{\alpha\beta} = (R_{\alpha\beta} + R_{\beta\alpha})$ and multiplying equations (6.8) and (6.9) by $M^{\alpha\gamma}$, where $M^{\alpha\gamma}M_{\beta\gamma} = \delta^\alpha_{\beta}$, yields the desired result.

In view of Theorem (6.1), we see that solving the multiplier problem is equivalent to finding all type (1,2) forms $\omega$ of the form (6.1) with $d\omega = 0$. From Corollary (6.3), we can restrict ourselves to examining systems of partial differential equations

$$u^\alpha_{xy} + C^\alpha_{\gamma\lambda}(x, y, u^\beta) u^\gamma_x u^\delta_y + A^\alpha_{\gamma\lambda}(x, y, u^\beta) u^\gamma_x + B^\alpha_{\gamma\lambda}(x, y, u^\beta) u^\gamma_y + G^\alpha(x, y, u^\beta) = 0.$$  

For systems (6.10), the following proposition gives necessary and sufficient conditions for $d\omega = 0$.

Proposition 6.4. There exists a differential form $\omega \in \Omega^{1,2}(\mathcal{R}^\infty)$ of type (6.1) with $d\omega = 0$ if and only if $M_{\alpha\beta} = (R_{\alpha\beta} + R_{\beta\alpha})/2$ satisfies $M_{\alpha\beta} = M_{\alpha\beta}(x, y, u^\gamma)$ and

$$M_{\alpha\gamma} H_{\beta\gamma} = M_{\beta\gamma} K_{\alpha\gamma},$$

$$M_{\alpha\gamma} S_{\beta\gamma} = -M_{\beta\gamma} S_{\alpha\gamma},$$

$$dM_{\alpha\beta} = M_{\alpha\theta} \tilde{Q}_{\beta}\beta + M_{\beta\gamma} \Omega^\gamma_{\alpha};$$

where $S_{\gamma\beta} = C_{\gamma\beta}^\gamma - C_{\gamma\beta}^\beta$ and $\Omega^\gamma_{\alpha} = C_{\beta\gamma}^\gamma du^\epsilon + B_{\beta\gamma}^\epsilon dx + A_{\beta\gamma}^\epsilon dy$.

Proof. In the proof of Theorem (6.1), we showed that if $dV\omega = 0$, then $R_{\alpha\beta}$ has no 1-jet dependence. Since $M_{\alpha\beta} = (R_{\alpha\beta} + R_{\beta\alpha})/2$, it follows immediately that the functions $M_{\alpha\beta}$ depends only $x, y$ and $u$.

To simplify our calculations, we define $\omega' = \omega - \tfrac{1}{2} d_H (R_{\alpha\beta} - R_{\beta\alpha}) \theta^\alpha \wedge \theta^\beta$. A routine calculation using (5.4) yields

$$\omega' = (T_{\alpha\beta} dx + V_{\alpha\beta} dy) \wedge \theta^\alpha \wedge \theta^\beta + M_{\alpha\beta}(\theta^\alpha \wedge \theta^\beta \wedge dx - \theta^\alpha \wedge \theta^\beta \wedge dy),$$

where $M_{\alpha\beta} = (R_{\alpha\beta} + R_{\beta\alpha})/2$. Without loss of generality we may assume that $T_{\alpha\beta}$ and $V_{\alpha\beta}$ are skew-symmetric. Using (5.5) and the exactness of the columns of the variational bicomplex, it can be shown that $d\omega = 0$ if and only if $d_H \omega' = 0$ and $d_V \omega' = d_H$ exact. We will later show that if $d_H \omega' = 0$ for (6.14), then $d_V \omega'$ is necessarily $d_H$ exact.

We first calculate $d_H \omega'$ and show that $d_H \omega' = 0$ if and only if (6.11)-(6.13) hold. Using (5.4) we calculate

$$d_H \omega' = [(D_x V_{\alpha\beta} - D_y T_{\alpha\beta}) \theta^\alpha \wedge \theta^\beta - 2M_{\alpha\beta} \theta^\alpha \wedge (d_V f^\beta)] \wedge dx \wedge dy -$$

$$[(D_y M_{\alpha\beta} - 2V_{\alpha\beta}) \theta^\alpha \wedge \theta^\beta + (D_x M_{\alpha\beta} + 2T_{\alpha\beta}) \theta^\alpha \wedge \theta^\beta] \wedge dx \wedge dy.$$
It follows that \( d_H \omega' = 0 \) if and only if

\[
\begin{align*}
(6.16) & \quad D_x M_{\alpha \beta} + 2T_{\alpha \beta} + 2M_{\alpha \gamma} \frac{\partial f_\gamma}{\partial u_\gamma} = 0, \\ (6.17) & \quad D_x V_{\alpha \beta} - D_y T_{\alpha \beta} - M_{\alpha \gamma} \frac{\partial f_\gamma}{\partial u_\gamma} + M_{\beta \gamma} \frac{\partial f_\gamma}{\partial u_\gamma} = 0.
\end{align*}
\]

Decomposing the two equations (6.16) into their symmetric and skew symmetric parts produces four equations

\[
\begin{align*}
(6.18) & \quad D_x M_{\alpha \beta} = -M_{\alpha \gamma} \frac{\partial f_\gamma}{\partial u_\gamma} - M_{\beta \gamma} \frac{\partial f_\gamma}{\partial u_\gamma}, \\ (6.19) & \quad T_{\alpha \beta} = \frac{1}{2} \left( M_{\beta \gamma} \frac{\partial f_\gamma}{\partial u_\gamma} - M_{\alpha \gamma} \frac{\partial f_\gamma}{\partial u_\gamma} \right), \\ & \quad V_{\alpha \beta} = \frac{1}{2} \left( M_{\alpha \gamma} \frac{\partial f_\gamma}{\partial u_\gamma} - M_{\beta \gamma} \frac{\partial f_\gamma}{\partial u_\gamma} \right).
\end{align*}
\]

Substituting (6.19) into (6.17) and taking (6.18) into account, we arrive at

\[
(6.20) \quad M_{\alpha \gamma} (H_\alpha^\gamma + K_\alpha^\gamma) = M_{\beta \gamma} (H_\alpha^\gamma + K_\alpha^\gamma).
\]

where

\[
H_\alpha^\gamma = \frac{\partial f_\gamma}{\partial u_\alpha} + \frac{\partial f_\gamma}{\partial u_\alpha} \frac{\partial f_\gamma}{\partial u_\alpha} - D_x \left[ \frac{\partial f_\gamma}{\partial u_\alpha} \right], \quad K_\alpha^\gamma = \frac{\partial f_\gamma}{\partial u_\alpha} + \frac{\partial f_\gamma}{\partial u_\alpha} \frac{\partial f_\gamma}{\partial u_\alpha} - D_y \left[ \frac{\partial f_\gamma}{\partial u_\alpha} \right].
\]

It follows that \( d_H \omega' = 0 \) if and only if (6.18) and (6.20) hold. For a system of differential equations (6.10), we expand (6.18) to arrive at

\[
\begin{align*}
\frac{\partial M_{\alpha \beta}}{\partial x} + \frac{\partial M_{\alpha \beta}}{\partial u^\epsilon} u_x^\epsilon & = M_{\alpha \gamma} (C^\gamma_{\epsilon \beta} u_x^\epsilon + B^\gamma_{\beta}) + M_{\beta \gamma} (B^\gamma_{\alpha \epsilon} u_x^\epsilon + A^\gamma_{\epsilon}), \\
\frac{\partial M_{\alpha \beta}}{\partial y} + \frac{\partial M_{\alpha \beta}}{\partial u^\epsilon} u_y^\epsilon & = M_{\alpha \gamma} (C^\gamma_{\epsilon \beta} u_y^\epsilon + A^\gamma_{\epsilon}) + M_{\beta \gamma} (C^\gamma_{\alpha \epsilon} u_y^\epsilon + A^\gamma_{\epsilon}).
\end{align*}
\]

Since \( M_{\alpha \beta} \) has no first-order dependence, we have

\[
\begin{align*}
\frac{\partial M_{\alpha \beta}}{\partial u^\epsilon} & = M_{\alpha \gamma} C^\gamma_{\epsilon \beta} + M_{\beta \gamma} C^\gamma_{\alpha}, \\
\frac{\partial M_{\alpha \beta}}{\partial x} & = M_{\alpha \gamma} B^\gamma_{\beta} + M_{\beta \gamma} B^\gamma_{\alpha}.
\end{align*}
\]

It follows that (6.21) is equivalent to conditions (6.12) and (6.13). If (6.12) holds, then it can be shown that the integrability conditions \( d^2 M_{\alpha \beta} = 0 \) for (6.13) are

\[
(6.22) \quad M_{\alpha \gamma} (H_\beta^\gamma - K_\beta^\gamma) + M_{\beta \gamma} (H_\alpha^\gamma - K_\alpha^\gamma) = 0.
\]

Equations (6.22) and (6.20) both hold if and only if \( M_{\alpha \sigma} H_\beta^\gamma = M_{\beta \sigma} K_\alpha^\gamma \), which is precisely condition (6.11).

We now show that \( d_V \omega' \) is \( d_H \) exact whenever \( d_H \omega' = 0 \). We claim that \( d_H \zeta = d_V \omega' \) for

\[
\zeta = \frac{1}{6} M_{\alpha \sigma} S^\sigma_{\beta \gamma} \theta^\alpha \wedge \theta^\beta \wedge \theta^\gamma.
\]

From (6.10), (6.14), and (6.19), we deduce that \( \omega' \) can be expressed as

\[
\omega' = M_{\alpha \sigma} \left( C^\sigma_{\beta \gamma} u_x^\epsilon + B^\sigma_{\beta} \right) \theta^\alpha \wedge \theta^\beta \wedge dx + M_{\alpha \beta} \theta^\alpha \wedge \theta^\beta \wedge dx - M_{\alpha \sigma} \left( C^\sigma_{\beta \gamma} u_y^\epsilon + A^\sigma_{\beta} \right) \theta^\alpha \wedge \theta^\beta \wedge dy - M_{\alpha \beta} \theta^\alpha \wedge \theta^\beta \wedge dy.
\]

Consider the system of total differential equations

\[ \text{Proposition A.1.} \quad \text{existence of solutions to systems of total differential equations with an algebraic constraint.} \]

It was derived in this paper that the equation

\[ \partial_{\gamma} \alpha = A_{\gamma}^\alpha (x)z^\gamma \]

satisfies the condition

\[ Q_{\alpha \beta \gamma} \theta^\alpha \wedge \theta^\beta \wedge \theta^\gamma = 0 \]

for all \( \gamma \), where \( Q_{\alpha \beta \gamma} \) is a skew-symmetric tensor. If we skew-symmetrize over \( \alpha \) and \( \beta \), it follows from (6.12) that (6.24) simplifies to

\[ \partial_{\gamma} \alpha = A_{\gamma}^\alpha (x)z^\gamma \]

Since we are assuming that \( \det Q_{\alpha \beta \gamma} \neq 0 \) and \( \det h_{\alpha \beta} \neq 0 \), then the system (7.1) is a generalization of harmonic map equation. A partial solution of the inverse problem harmonic map equation was obtained was obtained by Henneaux [14]. It would be some interest to derive a set of invariant analogs to the invariant derived in this paper that completely characterize the existence of Lagrangians for systems (7.1).

7. Concluding Remarks

We conjecture that the methods used in this paper can be used to obtain a complete solution of the inverse problem for systems of the form

\[ g^{ij}(x, y, u^\gamma)h_{\alpha \beta}u_{ij}^\beta + f_\alpha (x^k, u^\gamma, u^\xi_k) = 0, \]

where \( \alpha, \beta = 1, \ldots, m \) and \( i, j = 1, \ldots, n \). In particular, if \( \det g^{ij} \neq 0 \) and \( \det h_{\alpha \beta} \neq 0 \), then the system (7.1) is a generalization of harmonic map equation. A partial solution of the inverse problem harmonic map equation was obtained by Henneaux [14]. It would be some interest to derive a set of invariant analogs to the invariant derived in this paper that completely characterize the existence of Lagrangians for systems (7.1).

APPENDIX

To complete our solution to the inverse problem, we have the following general result on existence of solutions to systems of total differential equations with an algebraic constraint. Proposition A.1 establishes (2.9) and (2.11) and completes the proof of Theorem (2.4).

Proposition A.1. Consider the system of total differential equations

\[ \text{(A.1)} \quad \frac{\partial z^\alpha}{\partial x^i} = A_{\gamma}^\alpha (x)z^\gamma \]

coupled with the algebraic condition

\[ \text{(A.2)} \quad B_{\gamma}^\alpha (x)z^\gamma = 0, \quad 1 \leq a \leq l. \]

Suppose the integrability conditions for (A.1) are satisfied whenever (A.2) holds and that given \( x_0 \in \mathbb{R}^n \), there exists a neighborhood \( U \) of \( x_0 \) such that \( \text{Rank} \{ B^\alpha_{\gamma} (x_0) \} = \text{Rank} \{ B^\alpha_{\gamma} (x) \} = k \leq l \), for all \( x \in U \). We also assume that the algebraic system (A.2) is complete in the sense that differentiating (A.2) and substituting from (A.1) produces no new algebraic conditions. Given \( x_0 \in U \subset \mathbb{R}^n \) and \( z_0 \in \mathbb{R}^m \) with \( B_{\gamma}^\alpha (x_0)z^\gamma_0 = 0 \), there exists unique smooth functions \( z^\alpha \) defined on a neighborhood \( V \subset U \) of \( x_0 \), such that \( z^\alpha (x) \) satisfy both (A.1) and (A.2), and \( z^\alpha (x_0) = z^\alpha_0 \).
Lemma IV.2.4.) If the integrability conditions are satisfied for (A.1) whenever (A.2) holds, then \( \phi \) is the unique maximal integral manifold. Moreover, \( \phi \) satisfies (A.5) for any point \( (x, z) \in \Sigma \). It follows immediately that \( \Sigma \) is involutive. For \( \Sigma \) to be a regular submanifold of dimension \( m + n - k \), we differentiate (A.2) with respect to \( x^i \) and substitute from (A.1). This guarantees the existence of functions \( G^a_i(x) \) such that \[
\frac{\partial B^a_i}{\partial x^i} + B^a_i A^a_{j(i)} = G^a_i \cdot B^a_j.
\]

The system of differential equations (A.1) are associated with the \( C^\infty \) distribution \( \Delta \) on \( \mathbb{R}^n \times \mathbb{R}^m \) generated by the vector fields
\[
X_i = \frac{\partial}{\partial x^i} + A^a_{j(i)} \gamma^j \frac{\partial}{\partial z^j}, \quad i = 1, 2, \ldots, n.
\]

We then define the \( C^\infty \) function \( F: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^l \) by \( F(x, z) = B^a_i(x) \gamma^i \) and let \( \Sigma = F^{-1}(0) \). As a consequence of \( \Delta \) we see that \( \text{Rank} \{ DF(x, z) \} = \text{Rank} \{ B^a_i(x) \} = k \) for all \( (x, z) \in \Sigma \). It follows immediately that \( \Sigma \subseteq \mathbb{R}^n \times \mathbb{R}^m \) is regular submanifold of dimension \( m + n - k \). In addition, we can use \( \Delta \) to show that \( X_i(B^a_i \gamma^i) \) vanishes identically for all \( (x, z) \in \Sigma \). We conclude that for any \( p \in \Sigma \) and \( i = 1, 2, \ldots, m \),
\[
X_{ip} \in \text{Ker}(F \circ T_p : \mathbb{R}^n + \mathbb{R}^m \to \mathbb{R}^l) = T_p \Sigma.
\]

It follows from \( \Delta \) that \( \Delta \) is a \( C^\infty \) distribution on the submanifold \( \Sigma \). (See Boothby [6], Lemma IV.2.4.) If the integrability conditions are satisfied for (A.1) whenever (A.2) holds, then \( \Delta \) is involutive. For any point \( (x_0, z_0) \in \Sigma \), we can invoke the Frobenius Theorem to obtain a unique maximal integral manifold \( \phi: V \to \Sigma \) with \( V \subseteq \mathbb{R}^n \), \( \phi(0) = (x_0, z_0) \), and \( T_p \phi(V) = \Delta_p \). Moreover, \( \phi \) is \( C^\infty \) viewed as map \( \phi: V \to \mathbb{R}^n + m \) and \( \phi(V) \) defines the graph of a smooth solution \( z^\alpha(x) \) to the system of differential equations (A.1). Since \( \phi(V) \subseteq \Sigma \), we see that \( z^\alpha(x) \) satisfies the algebraic condition (A.2).

\[\square\]

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