Strong nd-M-fuzzy lattice

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Abstract

Tepavčević and Goran Trajakovski [1, 2] introduced the concept of L-fuzzy lattices where a bounded lattice is fuzzified by using a complete lattice [3]. Based on this concept strong L-fuzzy Lattices were introduced. In this paper we generalize the concept of Strong L-fuzzy lattices to Strong Nd-M-fuzzy lattices using a complete consistent distributive multilattice M with respect to the Egli-Milner ordering of subsets.

Keywords

Nd-M fuzzy subset, Nd-M-fuzzy lattice, Strong nd-M-fuzzy lattice.

AMS Subject Classification

03G10, 06B99, 06D72, 03E72.

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1. Introduction

The paper deals with the current trend of fuzzification of crisp concepts. [0, 1] valued fuzzy sets was introduced by Zadeh and Later the concept of L-fuzzy sets [8, 15] was generalized and developed by Goguen. But in these concepts, the membership function gives unique value in a complete lattice L [4, 9] for each element of its domain. Now we have introduced the concept of Nd-M-fuzzy subsets [11, 12] whose membership function takes one or more values to each member of its domain, where M has the structure of a multilattice.

Recently, an alternative definition of Multilattices was developed by Cordero [6] which simplify structure of lattices. My intention is how to bring these kind of structures for the generalization of fuzzy related concepts.

The L-fuzzy lattice was developed by Tepavčević and Goran Trajakovski, where a bounded lattice is fuzzified by using a complete lattice L [1, 2]. Following this definition the Strong nd-M-fuzzy lattice were introduced [16]. In this paper we generalize the concept of Strong L-fuzzy lattice to Strong nd-M-fuzzy lattice using a complete consistent distributive multilattice M with respect to the Egli-Milner ordering of subsets.

The organization of the paper is as follows. In section 2, the definition and preliminary (theoretical) results about multilattice, Nd-M-fuzzy subsets and L-fuzzy lattice are introduced; Later, in section 3 the concept of Strong nd-M-fuzzy lattice are presented.

2. Preliminary Results

Let (M, ≤) be a partially ordered set and U ⊆ M. Then multisupremum of U is a minimal element of the set of upper bounds of U and multiinfimum of U is a maximal element of the set of lower bounds of U. The multisuprema of U is denoted by Multi sup(U) and multiinfima of U is denoted by Multi inf(U).

Definition 2.1. [5, 6] The necessary and sufficient condition for a poset (M, ≤) to be an ordered multilattice is that for any p, q, x with p ≤ x and q ≤ x, there exist z ∈ Multi sup \{p, q\} such that z ≤ x and its dual.

When we comparing with lattices, we can see that least upper bound (which is a unique element) is replaced by the non empty set of all minimal (instead of least) upper bounds and dually.

Definition 2.2. [13, 14] A multilattice M is distributive if for each p, q, r ∈ M, the conditions (p ∨ q) ∨ (p ∨ q) ≠ ∅ and
\((p \land q) \cap (p \land r) \neq \emptyset \Rightarrow q = r\) (where \(\cap\) is the usual set intersection and \(\cup\) is the usual set unions)

**Definition 2.3.** A multilattice is said to be complete if for any subset \(U\) of \(M\), \(\text{Multi}(U)\) and \(\text{Multi}(U)\) exists and non empty.

**Definition 2.4.** A multisemilattice is a poset \((M, \leq)\) which satisfies the condition that for any \(p, q, x \in M\) with \(p \leq x, q \leq x\), there exist \(z \in \text{muti sup}\{p, q\}\) such that \(z \leq x\) and dually.

**Definition 2.5.** Let \((M, \leq)\) be a poset. The element \(p \in M\) is called a greatest element of \(M\) if all other element are smaller. That is if \(p \geq x\) for every \(x \in M\). Similarly \(q \in M\) is said to be a smallest element of \(M\) if \(q \leq x\) for every \(x \in M\). If a multilattice has a greatest element and smallest element, then \((M, \leq)\) is said to be bounded. Normally greatest element is taken as 1 and smallest element is taken as 0.

**Definition 2.6.** A multilattice \(M\) with 0 and 1 is called complemented if for each \(p \in M\), there is at least one element \(q\) such that \(p \land q = \{0\}\) and \(p \lor q = \{1\}\). Also if \(M\) is a complete distributive multilattice, then every element in \(M\) has exactly one complement in \(M\).

Now we introduce some ordering among the subsets of posets which are the Hoare ordering, the Smyth ordering and the Egli -Milner ordering respectively.

**Definition 2.7.** \([4]\) consider \(U, V \in 2^M\), then
- \(U \subseteq V\) if and only if for all \(a \in U\) there exists \(b \in V\) such that \(a \leq b\)
- \(U \subseteq V\) if and only if for every \(b \in V\) there exists \(a \in U\) such that \(a \leq b\)
- \(U \subseteq V\) if and only if \(U \subseteq V\) and \(U \subseteq V\).

**Definition 2.8.** \([9]\) \((M, \land, \lor)\) - be a algebraic multilattice . Let \(p\) and \(U\) and \(V\) be subsets of \(M\), then
\[
p \land U = \{x \land y \mid x \in U\}
p \lor U = \{x \lor y \mid x \in U\}
\]
Also
\[
U \land V = \{x \land y \mid x \in U, y \in V\}
U \lor V = \{x \lor y \mid x \in U, y \in V\}
\]

**Definition 2.9.** \([4]\) A multilattice \(M\) is said to be consistent if it satisfies the inequalities \(\text{LB}(U) \subseteq \text{EM} \text{Multi}(U)\text{ and} \text{Multi}(U) \subseteq \text{UB}(U)\) for all subset \(U\) of \(M\). Where \(\text{LB}(U)\) and \(\text{UB}(U)\) are the lower bound of \(U\) and upper bound of \(U\) respectively.

**Definition 2.10.** \([11, 12]\) A Non deterministic \(M\)-Fuzzy subset of \(X\) is a function from \(X\) to \(2^M\), where \(M\) is a complete distributive multilattice. Then the collection of all the \(Nd-M\)-fuzzy subsets of \(X\) is called \(Nd-M\)-fuzzy space and is denoted by \((2^M)^X\).

**Definition 2.11.** A completely distributive multilattice \(M\) is called a \(F\)-multilattice if \(M\) has an order reversing involution \(\alpha: M \rightarrow M\).

Let \(X\) be a non empty ordinary set and \(M\) a \(F\)-multilattice. Let \(U \in (2^M)^X\). Then \(U' = (|U(a)|) = \{p' \mid p \in U(a)\}\)

If \(M\) is a completely distributive multilattice, then \(U' = (|U(a)|) = \{p' \mid p \in U(a)\}\), then \(U': \rightarrow (2^M)^X\), the pseudo complementary operation on \((2^M)^X\) is the pseudo complementary set of \(U\) in \((2^M)^X\).

**Definition 2.12.** \([11, 12]\) Rules of set relations on \((2^M)^X\).

1. \(U = V\) if \(U(a) = V(a)\) for every \(a \in X\)
2. \(U \leq V\) if \(U(a) \subseteq V(a)\) for every \(a \in X\)
3. \(C = U \lor V\) if \(C(a) = \text{Multi}(U(a), V(a))\) for every \(a \in X\) \(= \{p \lor q \mid p \in U(a), q \in V(a)\}\) for every \(a \in X\)
4. \(D = U \land V\) if \(D(a) = \text{Multi}(U(a), V(a))\) for every \(a \in X\) \(= \{p \land q \mid p \in U(a), q \in V(a)\}\) for every \(a \in X\)
5. \(E = X\) and \(E(a) = \{p' \mid p \in U(a)\}\) for every \(a \in X\)

**Definition 2.13.** \([1, 2]\) Let \(X\) be a lattice and \((L, \land, \lor)\) is a complete lattice with \(0\) and \(1\). Let \(\mu\) be a \(L\)-fuzzy set defined on \(X\). The \(p\) cut \((p \in L)\) of \(\mu\) is defined by \(\mu_p = \{a \in X : \mu(a) \geq p\}\). A fuzzy set \(\mu\) defined on \(L\) is a fuzzy sub lattice of \(L\) if
\[
\mu(a \land b) \land \mu(a \lor b) \leq \min(\mu(a), \mu(b))\quad a, b \in X
\]
\[
\mu(a \land b) \land \mu(a \lor b) \leq \mu(a) \land \mu(b)
\]

**Note 2.14.** \(\mu \in L^X\) is a \(L\)-fuzzy sub lattice of \(X\) if and only if \(\mu_p\) is a sub lattice of \(X\) for each \(p \in L\).

**Definition 2.15.** \([16]\)

Let \((M, \land)\) be a join- semilattice and \((L, \land, \lor)\) be a complete lattice with \(0\) and \(1\). Let \(\mu\) be a \(L\)-fuzzy set defined on \(X\). Then \(\mu\) is called an \(L\)-fuzzy join- semilattice of of \(M\), if all the \(p \in L\) level sets of \(\mu\) are sub join-semilattices of \(M\).

**Definition 2.16.** \([16]\)

Let \((M, \land)\) be a meet semilattice and \((L, \land, \lor)\) be a complete lattice with \(0\) and \(1\). Let \(\mu\) be a \(L\)-fuzzy set defined on \(X\). Then \(\mu\) is called an \(L\)-fuzzy meet semilattice of of \(M\), if all the \(p \in L\) level sets of \(\mu\) are sub meet semilattices of \(M\).

**Definition 2.17.** \([11, 12]\)

Let \((L, \land)\) be a join – semilattice and \((M, \land, \lor)\) be a complete and consistent multilattice. A mapping \(U : L \rightarrow 2^M\) is called a \(Nd-M\) Fuzzy join-semilattice of \(L\), if each \(\alpha\)–level set satisfies \(\alpha \subseteq U(a) \land M(U(b)\) for every \(a, b \in U\) and are sub join-semilattices of \(L\).
Proposition 2.18. [11, 12] Let \((L, \lor_L)\) be a join-semilattice and \((M, \land_M, \lor_M)\) be a complete and consistent multilattice. Assume that \(U_\alpha\) satisfying \(\alpha \subseteq EM \ U(a) \land_M U(b)\). Then a mapping \(U : L \rightarrow 2^M\) is an \(N\!d-M\!-fuzzy\) join-semilattice of \(L\) if and only if mult\((f(U(a), U(b)) \subseteq EM \ U(a \lor_L b) \land \forall a, b \in L\). That is

\[
U(a) \land_M U(b) \subseteq EM \ U(a \lor_L b) \land \forall a, b \in L
\]

Definition 2.19. [11, 12] Let \((L, \land_L)\) be a meet-semilattice and \((M, \land_M, \lor_M)\) be a complete and consistent multilattice. A mapping \(U : L \rightarrow 2^M\) is called an \(N\!d-M\!-fuzzy\) meet-semilattice of \(L\) if for each \(\alpha -\) level sets satisfies \(, \alpha \subseteq EM U(a) \land_M U(b)\) for every \(a, b \in U_\alpha\), and are sub meet-semilattices of \(L\).

Proposition 2.20. [11, 12] Let \((L, \land_L)\) be a meet-semilattice and \((M, \land_M, \lor_M)\) be a complete and consistent multilattice. Assume that \(U_\alpha\) satisfying \(\alpha \subseteq EM U(a) \land_M U(b)\) for every \(a, b \in U_\alpha\). Then mapping \(U : L \rightarrow 2^M\) is an \(n\!d-M\!-fuzzy\) meet-semilattice of \(L\) if and only if mult\((f(U(a), U(b)) \subseteq EM \ U(a \land_L b) \land \forall a, b \in L\). That is

\[
U(a) \land_M U(b) \subseteq EM \ U(a \land_L b) \land \forall a, b \in L
\]

Note 2.21. Let \((L, \land_L, \lor_L)\) be a lattice and \((M, \land_M, \lor_M)\) be a complete and consistent multilattice. A mapping \(U : L \rightarrow 2^M\) is called an \(N\!d-M\!-fuzzy\) lattice if and only if \(U\) is both an \(N\!d-M\!-fuzzy\) meet-semilattice and an \(N\!d-M\!-fuzzy\) join-semilattice.

### 3. Strong \(N\!d-M\!-fuzzy\) lattice

Definition 3.1. Let \((L, \land_L, \lor_L)\) be a lattice and \((M, \land_M, \lor_M)\) be a multilattice with the least element \(0_M\) and the greatest element \(1_M\). The mapping \(U : L \rightarrow 2^M\) is called a strong \(N\!d-M\!-fuzzy\) lattice if for each \(\alpha, \beta \in 2^M\), the set \(U_\alpha^\beta = \{a \in L |\alpha \subseteq EM a \subseteq EM \beta\}\) satisfies,

1. \(\alpha \subseteq EM U(a) \land_M U(b)\) for every \(a, b \in U_\alpha\)
2. \(U(a) \lor_M U(b) \subseteq EM \beta\) for every \(a, b \in U_\alpha\)
3. \(U_\alpha^\beta\) is a sub-lattice of \(L\), for all \(\alpha, \beta \in 2^M\).

Theorem 3.2. Let \((L, \land_L, \lor_L)\) be a lattice and \((M, \land_M, \lor_M)\) be a complete and consistent multilattice with \(0_M\) and \(1_M\). Then the mapping \(U : L \rightarrow 2^M\) is a strong \(N\!d-M\!-fuzzy\) lattice if and only if \(U\) satisfies the following conditions, for all \(a, b \in L\):

1. \(U(a) \land_M U(b) \subseteq EM U(a \land_L b) \land U(a) \land_M U(b)\)
2. \(U(a) \land_M U(b) \subseteq EM U(a \lor_L b) \land U(a) \land_M U(b)\)

Proof. Assume that \(U : L \rightarrow 2^M\) is a strong \(N\!d-M\!-fuzzy\) lattice of \(L\). Then for each \(\alpha, \beta \in 2^M\), \(U\) satisfies

1. \(\alpha \subseteq EM U(a) \land_M U(b)\) for every \(a, b \in U_\alpha\)
2. \(U(a) \lor_M U(b) \subseteq EM \beta\) for every \(a, b \in U_\alpha\)
3. \(U_\alpha^\beta\) is a sub-lattice of \(L\), for all \(\alpha, \beta \in 2^M\).

Let \(a, b \in L\), \(P = U(a) \land_M U(b)\) and \(Q = U(a) \lor_M U(b)\). Then \(P \subseteq EM U(a) \lor_M Q\) and \(P \subseteq EM U(b) \lor_M Q\). Hence \(a, b \in U_\alpha\). That is, \(P \subseteq EM U(a \land_L b) \lor_M Q\) and \(P \subseteq EM U(a \lor_L b) \lor_M Q\).

Hence \(U(a) \land_M U(b) \subseteq EM U(a \land_L b) \land U(a) \land_M U(b)\) and \(U(a) \land_M U(b) \subseteq EM U(a \lor_L b) \land U(a) \land_M U(b)\). Conversely assume that \(U : L \rightarrow 2^M\) satisfies the conditions \(U(a) \land_M U(b) \subseteq EM U(a \land_L b) \land U(a) \land_M U(b)\) and \(U(a) \land_M U(b) \subseteq EM U(a \lor_L b) \land U(a) \land_M U(b)\). Then \(P \subseteq EM U(a) \lor_M Q\) and \(P \subseteq EM U(a) \lor_M Q\).

Hence \(P \subseteq EM U(a \land_L b) \lor_M Q\) and \(P \subseteq EM U(a \lor_L b) \lor_M Q\). That is, from our assumption, we have

- \(P \subseteq EM U(a) \land_M U(b)\)
- \(P \subseteq EM U(a) \land_M U(b)\)
- \(P \subseteq EM U(a) \land_M U(b)\)
- \(P \subseteq EM U(a) \land_M U(b)\)

That is \(a, b \in U_\alpha\) and \(a \land_L b \in U_\alpha^\beta\). Hence \(U_\alpha^\beta\) is a sub-lattice of \(L\) and so \(L\) is a strong sub lattice of \(L\).

Theorem 3.3. Let \(L\) be a lattice, \(M\) be a complete and consistent multilattice and \(U : L \rightarrow 2^M\) be a strong \(N\!d-M\!-fuzzy\) lattice, for each \(j \in J\), then \(\land_{j \in J}\) and \(\lor_{j \in J}\) are \(N\!d-M\!-fuzzy\) lattices.

Proof. Assume that \(U : L \rightarrow 2^M\) is a strong \(N\!d-M\!-fuzzy\) lattice of \(L\). Then for each \(\alpha, \beta \in 2^M\), \(U\) satisfies

\[
((\land_{j \in J} U_{j}(a)) \land_M ((\lor_{j \in J} U_{j}(b)))\]
\[
= (\land_{j \in J} U_{j}(a)) \land_M (\lor_{j \in J} U_{j}(b))\]
\[
= (\lor_{j \in J} U_{j}(a \land_L b))\]
\[
\subseteq EM (\lor_{j \in J} U_{j}(a \land_L b))\]
\[
= (\land_{j \in J} U_{j}(a \land_L b))\]
\[
\subseteq EM (\land_{j \in J} U_{j}(a \land_L b))\]
\[
E_{EM} (\land_{j \in J} U_{j}(a \land_L b))\]
\[
E_{EM} (\lor_{j \in J} U_{j}(b))\]

Therefore \(((\land_{j \in J} U_{j}(a)) \land_M ((\lor_{j \in J} U_{j}(b))) \subseteq EM (\land_{j \in J} U_{j}(a \land_L b))\)

Therefore \((\land_{j \in J} U_{j}(a)) \land_M ((\lor_{j \in J} U_{j}(b))) \subseteq EM (\land_{j \in J} U_{j}(a \land_L b))\).
Similarly

\[
((\bigvee U_j(a)) \land_M (\bigvee U_j(b)) \\
= (\bigvee U_j(a)) \land_M (\bigvee U_j(b)) \\
= \bigvee (U_j(a) \land_M U_j(b)) \\
\subseteq EM (\bigvee U_j(a) \lor_M U_j(b)) \\
= \bigvee (U_j(a) \lor M b) \\
= \bigvee (U_j(a) \lor b) \\
\subseteq EM (U_j(a) \lor M U_j(b)) \\
= (\bigvee (U_j(a)) \lor_M (\bigvee (U_j(b))) \\
= ((\bigvee U_j(a)) \lor_M ((\bigvee U_j(b)))
\]

Hence the expressions from (1) and (2) together implies \( (\bigvee U_j) \) is a \( Nd-M \)-fuzzy lattices, Similarly,

\[
((\bigwedge U_j(a)) \land_M ((\bigwedge U_j(b)) \\
= (\bigwedge U_j(a)) \land_M (\bigwedge U_j(b)) \\
= \bigwedge (U_j(a) \land_M U_j(b)) \\
\subseteq EM \bigwedge (U_j(a) \land L b) \\
= \bigwedge (U_j(a) \land L b) \\
= \bigwedge (U_j(a) \lor M U_j(b)) \\
\subseteq EM \bigwedge (U_j(a) \lor M U_j(b)) \\
= \bigwedge (U_j(a) \land (\bigwedge U_j(b))) \\
= ((\bigwedge U_j(a)) \land_M ((\bigwedge U_j(b)))
\]

Therefore

\[
((\bigwedge U_j(a)) \land_M ((\bigwedge U_j(b))) \subseteq EM (\bigwedge U_j(a) \land L b) \\
\subseteq EM ((\bigwedge U_j(a)) \land L ((\bigwedge U_j(b)))
\]

Hence from the expression above we have \( (\bigwedge U_j) \) is an \( Nd-M \)-fuzzy lattices.

4. Conclusion

We have generalized the concept of Strong L-fuzzy lattice to Strong nd-M-fuzzy lattice using a complete consistent distributive multilattice \( M \). Finally, we observe that Non-deterministic fuzzification of lattices is valid using a complete consistent distributive multilattice \( M \) with respect to the Egli Milner ordering of subsets.

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