Global well posedness and scattering for the elliptic and non-elliptic derivative nonlinear Schrödinger equations with small data

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Abstract

We study the Cauchy problem for the generalized elliptic and non-elliptic derivative nonlinear Schrödinger equations, the existence of the scattering operators and the global well posedness of solutions with small data in Besov spaces $B^s_{2,1}(\mathbb{R}^n)$ and in modulation spaces $M^s_{2,1}(\mathbb{R}^n)$ are obtained. In one spatial dimension, we get the sharp well posedness result with small data in critical homogeneous Besov spaces $\dot{B}^s_{2,1}$. As a by-product, the existence of the scattering operators with small data is also shown. In order to show these results, the global versions of the estimates for the maximal functions on the elliptic and non-elliptic Schrödinger groups are established.

Key words: Derivative nonlinear Schrödinger equation, elliptic and non-elliptic cases, estimates for the maximal function, global well posedness, small data.

MSC: 35 Q 55, 46 E 35, 47 D 08.

1 Introduction

We consider the Cauchy problem for the generalized derivative nonlinear Schrödinger equation (gNLS)

$$iu_t + \Delta u = F(u, \bar{u}, \nabla u, \nabla \bar{u}), \quad u(0, x) = u_0(x),$$

(1.1)

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where \( u \) is a complex valued function of \((t, x) \in \mathbb{R} \times \mathbb{R}^n\),

\[
\Delta_\pm u = \sum_{i=1}^{n} \varepsilon_i \partial_x^2, \quad \varepsilon_i \in \{1, -1\}, \quad i = 1, \ldots, n,
\]

(1.2)

\[
\nabla = (\partial_{x_1}, \ldots, \partial_{x_n}), \quad F : \mathbb{C}^{2n+2} \rightarrow \mathbb{C} \text{ is a polynomial},
\]

(1.3)

\[
F(z) = P(z_1, \ldots, z_{2n+2}) = \sum_{m+1 \leq |\beta| \leq M+1} c_\beta z^\beta, \quad c_\beta \in \mathbb{C},
\]

(1.3)

\( m, M \in \mathbb{N} \) will be given below.

There is a large literature which is devoted to the study of (1.1). Roughly speaking, three kinds of methods have been developed for the local and global well posedness of (1.1). The first one is the energy method, which is mainly useful to the elliptic case \( \Delta_\pm = \Delta = \partial_x^2 + \ldots + \partial_x^n \), see Klainerman [21], Klainerman and Ponce [22], where the global classical solutions were obtained for the small Cauchy data with sufficient regularity and decay at infinity, \( F \) is assumed to satisfy an energy structure condition Re \( \partial F / \partial (\nabla u) = 0 \). Chihara [6,7] removed the condition Re \( \partial F / \partial (\nabla u) = 0 \) by using the smooth operators and the commutative estimates between the first order partial differential operators and \( i\partial_t + \Delta \), suitable decay conditions on the Cauchy data are still required in [6,7]. Recently, Ozawa and Zhang [25] removed the assumptions on the decay at infinity of the initial data. They obtained that if \( n \geq 3, s > n/2 + 2, u_0 \in H^s \) is small enough, \( F \) is a smooth function vanishing of the third order at origin with \( \text{Re} \partial F / \partial (\nabla u) = \nabla (\theta(|u|^2)), \quad \theta \in C^2, \quad \theta(0) = 0 \), then (1.1) has a unique classical global solution \( u \in (C_w \cap L^\infty)(\mathbb{R}, H^s) \cap C(\mathbb{R}, H^{s-1}) \cap L^2(\mathbb{R}; H^{s-1}_{2n/(n-2)}). \)

The main tools used in [25] are the gauge transform techniques, the energy method together with the endpoint Strichartz estimates.

The second way consists in using the \( X^{s,b} \)-like spaces, see Bourgain [3] and it has been developed by many authors (see [2,15] and references therein). This method depends on both the dispersive property of the linear equation and the structure of the nonlinearities, which is very useful for the lower regularity initial data.

The third method is to mainly use the dispersive smooth effects of the linear Schrödinger equation, see Kenig, Ponce and Vega [17,18]. The crucial point is that the Schrödinger group has the following locally smooth effects \((n \geq 2)\):

\[
\sup_{\alpha \in \mathbb{Z}^n} \| e^{it\Delta} u_0 \|_{L_{t,x}^2(\mathbb{R} \times Q_\alpha)} \lesssim \| u_0 \|_{H^{-1/2}},
\]

(1.4)

\[
\sup_{\alpha \in \mathbb{Z}^n} \left\| \nabla \int_0^t e^{i(t-s)\Delta} f(s)ds \right\|_{L_{t,x}^2(\mathbb{R} \times Q_\alpha)} \lesssim \sum_{\alpha \in \mathbb{Z}^n} \| f \|_{L_{t,x}^2(\mathbb{R} \times Q_\alpha)},
\]

(1.5)

where \( Q_\alpha \) is the unit cube with center at \( \alpha \). Estimate (1.5) contains one order smooth effect, which can be used to control the derivative terms in
Cauchy data \((m\) both elliptic and non-elliptic cases was established for sufficiently smooth large Cauchy data \((m \geq 1, u_0 \in H^s \text{ with } s > n/2 \text{ large enough})\). Moreover, they showed that the solutions are almost global if the initial data are sufficiently small, i.e., the maximal existing time of solutions tends to infinity as initial data tends to 0. Recently, the local well posedness results have been generalized to the quasi-linear (ultrahyperbolic) Schrödinger equations, see [19,20]. As far as the authors can see, the existence of the scattering operators for Eq. \((1.1)\) and the global well posedness of \((1.1)\) in the non-elliptic cases are unknown.

### 1.1 Main results

In this paper, we mainly apply the third method to study the global well posedness and the existence of the scattering operators of \((1.1)\) in both the elliptic and non-elliptic cases with small data in \(B_{2,1}^s, \ s > 3/2 + n/2\). We now state our main results, the notations used in this paper can be found in Sections 1.3 and 1.4.

**Theorem 1.1** Let \(n \geq 2\) and \(s > n/2 + 3/2\). Let \(F(z)\) be as in [13] with \(2 + 4/n \leq m \leq M < \infty\). We have the following results.

(i) If \(\|u_0\|_{B_{2,1}^s} \leq \delta\) for \(n \geq 3\), and \(\|u_0\|_{B_{2,1}^s \cap H^{-1/2}} \leq \delta\) for \(n = 2\), where \(\delta > 0\) is a suitably small number, then \((1.1)\) has a unique global solution \(u \in C(\mathbb{R}, B_{2,1}^s) \cap X_0\), where

\[
X_0 = \left\{ u : \begin{array}{l}
\|D^\beta u\|_{\ell^{s+1/2}_\alpha(L^{2m}_x(H^s(\mathbb{R} \times Q_0)))} \lesssim \delta, \ |\beta| \leq 1 \\
\|D^\beta u\|_{\ell^{s+1/2}_\alpha(L^{2+4/n}_x(L^{2m}_x \cap H^s(\mathbb{R} \times Q_0)))} \lesssim \delta, \ |\beta| \leq 1
\end{array} \right\}. \tag{1.6}
\]

Moreover, for \(n \geq 3\), the scattering operator of Eq. \((1.1)\) carries the ball \(\{u : \|u\|_{B_{2,1}^s} \leq \delta\}\) into \(B_{2,1}^s\).

(ii) If \(s + 1/2 \in \mathbb{N}\) and \(\|u_0\|_{H^s} \leq \delta\) for \(n \geq 3\), and \(\|u_0\|_{H^s \cap H^{-1/2}} \leq \delta\) for \(n = 2\), where \(\delta > 0\) is a suitably small number, then \((1.1)\) has a unique global solution \(u \in C(\mathbb{R}, H^s) \cap X\), where

\[
X = \left\{ u : \begin{array}{l}
\|D^\beta u\|_{\ell^{s+1/2}_\alpha(L^{2m}_x(\mathbb{R} \times Q_0)))} \lesssim \delta, \ |\beta| \leq s + 1/2 \\
\|D^\beta u\|_{\ell^{s+4/n}_\alpha(L^{2m}_x(L^{2m}_x \cap H^s(\mathbb{R} \times Q_0)))} \lesssim \delta, \ |\beta| \leq 1
\end{array} \right\}. \tag{1.7}
\]
Moreover, for $n \geq 3$, the scattering operator of Eq. (1.1) carries the ball $\{u : \|u\|_{H^s} \leq \delta\}$ into $H^s$.

We now illustrate the proof of (ii) in Theorem 1.1. Let us consider the equivalent integral equation

$$u(t) = S(t)u_0 - i\mathcal{A}F(u, \bar{u}, \nabla u, \nabla \bar{u}),$$  \hspace{1cm} (1.8)$$

where

$$S(t) := e^{it\Delta}$$ and $$\mathcal{A} f := \int_0^t e^{i(t-s)\Delta} f(s)ds.$$  \hspace{1cm} (1.9)$$

If one applies the local smooth effect estimate (1.3) to control the derivative terms in the nonlinearities, then the working space should contains the space $\mathcal{L}^\infty_\alpha(L^2_{t,x}(\mathbb{R} \times Q_\alpha))$. For simplicity, we consider the case $F(u, \bar{u}, \nabla u, \nabla \bar{u}) = (\partial_{x_1} u)^{\nu+1}$. By (1.4) and (1.5), we immediately have

$$\|\nabla u\|_{\mathcal{L}^\infty_\alpha(L^2_{t,x}(\mathbb{R} \times Q_\alpha))} \lesssim \|u_0\|_{H^{1/2}} + \sum_{\alpha \in \mathbb{Z}^n} \|\partial_{x_1} u\|_{L^2_{t,x}(\mathbb{R} \times Q_\alpha)}^{\nu+1} \|L^2_{t,x}(\mathbb{R} \times Q_\alpha)\|.$$  \hspace{1cm} (1.10)$$

Hence, one needs to control $\|\nabla u\|_{\mathcal{L}^\infty_\alpha(L^2_{t,x}(\mathbb{R} \times Q_\alpha))}$. In [17,18], it was shown that for $\nu = 2$,

$$\|S(t)u_0\|_{\mathcal{L}^\infty_\alpha(L^2_{t,x}([0,T] \times Q_\alpha))} \leq C(T)\|u_0\|_{H^s}, \quad s > n/2 + 2. \hspace{1cm} (1.11)$$

In the elliptic case (1.11) holds for $s > n/2$. (1.11) is a time-local version which prevents us to get the global existence of solutions. So, it is natural to ask if there is a time-global version for the estimates of the maximal function. We can get the following

$$\|S(t)u_0\|_{\mathcal{L}^\infty_\alpha(L^2_{t,x}(\mathbb{R} \times Q_\alpha))} \leq C\|u_0\|_{H^s}, \quad s > n/2, \quad \nu \geq 2 + 4/n. \hspace{1cm} (1.12)$$

Applying (1.12), we have for any $s > n/2$,

$$\|\nabla u\|_{\mathcal{L}^\infty_\alpha(L^2_{t,x}(\mathbb{R} \times Q_\alpha))} \lesssim \|\nabla u_0\|_{H^s} + \|\nabla (\partial_{x_1} u)^{1+\nu}\|_{L^1(\mathbb{R}, H^s(\mathbb{R}^n)))). \hspace{1cm} (1.13)$$

One can get, say for $s = [n/2] + 1$,

$$\|\nabla (\partial_{x_1} u)^{1+\nu}\|_{L^1(\mathbb{R}, H^s(\mathbb{R}^n))} \lesssim \sum_{|\beta| \leq s+2} \|D^\beta u\|_{\mathcal{L}^\infty_\alpha(L^2_{t,x}(\mathbb{R} \times Q_\alpha))} \|\nabla u\|_{\mathcal{L}^\infty_\alpha(L^{2\nu}_{t,x} L^\infty_{x}(\mathbb{R} \times Q_\alpha)). \hspace{1cm} (1.14)$$

Hence, we need to further estimate $\|D^\beta u\|_{\mathcal{L}^\infty_\alpha(L^2_{t,x}(\mathbb{R} \times Q_\alpha))}$ for all $|\beta| \leq s + 2$ and $\|\nabla u\|_{\mathcal{L}^\infty_\alpha(L^{2\nu}_{t,x} L^\infty_{x}(\mathbb{R} \times Q_\alpha))}$.
holds:

\[
\sum_{|\beta| \leq s+2} \|D^\beta u\|_{\ell^\infty_\alpha(L^2_t L^s_x(\mathbb{R} \times Q_\alpha))} \\
\lesssim \|u_0\|_{H^{s+3/2}} + \sum_{|\beta| \leq s+2} \|D^\beta u\|_{\ell^\infty_\alpha(L^2_t L^s_x(\mathbb{R} \times Q_\alpha))} \|\nabla u\|_{\ell^\infty_\alpha(L^\infty_t L^\infty_x(\mathbb{R} \times Q_\alpha))}.
\]  

Finally, for the estimate of \( \|\nabla u\|_{\ell^\infty_\alpha(L^2_t L^\infty_x(\mathbb{R} \times Q_\alpha))} \), one needs the following

\[
\|S(t)u_0\|_{\ell^\infty_\alpha(L^2_t L^\infty_x(\mathbb{R} \times Q_\alpha))} \leq C \|u_0\|_{H^{s-1/\nu}}, \quad s > n/2, \quad \nu \geq 2 + 4/n.
\]  

Using (1.16), the estimate of \( \|\nabla u\|_{\ell^\infty_\alpha(L^2_t L^\infty_x(\mathbb{R} \times Q_\alpha))} \) becomes easier than that of \( \|\nabla u\|_{\ell^\infty_\alpha(L^\infty_t L^\infty_x(\mathbb{R} \times Q_\alpha))} \). Hence, the solution has a self-contained behavior by using the spaces \( \ell^\infty_\alpha(L^2_t(\mathbb{R} \times Q_\alpha)), \ell^\nu_\alpha(L^\infty_t(\mathbb{R} \times Q_\alpha)) \) and \( \ell^\nu_\alpha(L^2_t L^\infty_x(\mathbb{R} \times Q_\alpha)) \). We will give the details of the estimates (1.12) and (1.16) in Section 2. The nonlinear mapping estimates as in (1.14) and (1.15) will be given in Section 4.

Next, we use the frequency-uniform decomposition method developed in [31, 32, 33] to consider the case of initial data in modulation spaces \( M_{2,1}^s \), which is the low regularity version of Besov spaces \( B_{2,1}^{n/2+s} \), i.e., \( B_{2,1}^{n/2+s} \subset M_{2,1}^s \) is a sharp embedding and \( M_{2,1}^s \) has only \( s \)-order derivative regularity (see [27, 29, 32] for the final result, see [33]). We have the following local well-posedness result with small rough initial data:

**Theorem 1.2** Let \( n \geq 2 \). Let \( F(z) \) be as in (1.3) with \( 2 \leq m \leq M < \infty \). Assume that \( \|u_0\|_{M_{2,1}^2} \leq \delta \) for \( n \geq 3 \), and \( \|u_0\|_{M_{2,1}^2 \cap H^{-1/2}} \leq \delta \) for \( n = 2 \), where \( \delta > 0 \) is sufficiently small. Then there exists a \( T := T(\delta) > 0 \) such that (1.1) has a unique local solution \( u \in C([0, T], M_{2,1}^2) \cap Y \), where

\[
Y = \left\{ u : \quad \|D^\beta u\|_{\ell^3/2_\alpha(L^2_t(0, T] \times Q_\alpha))} \lesssim \delta, \quad |\beta| \leq 1 \quad \|D^\beta u\|_{\ell^\infty_\alpha(L^\infty_t(0, T] \times Q_\alpha))} \lesssim \delta, \quad |\beta| \leq 1 \right\}.
\]  

Moreover, \( \lim_{\delta \searrow 0} T(\delta) = \infty \).

The following is a global well-posedness result with Cauchy data in modulation spaces \( M_{2,1}^s \):

**Theorem 1.3** Let \( n \geq 2 \). Let \( F(z) \) be as in (1.3) with \( 2 + 4/n \leq m \leq M < \infty \). Let \( s > 3/2 + (n + 2)/m \). Assume that \( \|u_0\|_{M_{2,1}^s} \leq \delta \) for \( n \geq 3 \), and \( \|u_0\|_{M_{2,1}^s \cap H^{-1/2}} \leq \delta \) for \( n = 2 \), where \( \delta > 0 \) is a suitably small number. Then (1.1) has a unique global solution \( u \in C(\mathbb{R}, M_{2,1}^s) \cap Z \), where

\[
Z = \left\{ u : \quad \|D^\beta u\|_{\ell^{3/2-1/2}_\alpha(L^2_t(\mathbb{R} \times Q_\alpha))} \lesssim \delta, \quad |\beta| \leq 1 \quad \|D^\beta u\|_{\ell^m_\alpha(L^\infty_t(\mathbb{R} \times Q_\alpha))} \lesssim \delta, \quad |\beta| \leq 1 \right\}.
\]


Moreover, for \( n \geq 3 \), the scattering operator of Eq. (1.1) carries the ball \( \{ u : \| u \|_{M_{2,1}^s} \leq \delta \} \) into \( M_{2,1}^s \).

Finally, we consider one spatial dimension case. Denote

\[
s_\kappa = \frac{1}{2} - \frac{2}{\kappa}, \quad \bar{s}_\nu = \frac{1}{2} - \frac{1}{\nu}.
\]

(1.19)

Theorem 1.4 Let \( n = 1, M \geq m \geq 4, u_0 \in \dot{B}_{2,1}^{1+\bar{s}_M} \cap \dot{B}_{2,1}^{s_m} \). Assume that there exists a small \( \delta > 0 \) such that \( \| u_0 \|_{\dot{B}_{1+\bar{s}_M}^{s_m}} \leq \delta \). Then (1.1) has a unique global solution \( u \in X = \{ u \in \mathcal{S}'( \mathbb{R}^{1+1} ) : \| u \|_X \lesssim \delta \} \), where

\[
\| u \|_X = \sup_{s_m \leq s \leq \bar{s}_M} \sum_{i=0}^{\infty} \sum_{j \in \mathbb{Z}} \| \partial_x^i \Delta_j u \|_s \quad \text{for } m > 4,
\]

\[
\| u \|_X = \sum_{i=0}^{\infty} \left( \| \partial_x^i u \|_{L_t^\infty L_x^2} + \sup_{s_m \leq s \leq \bar{s}_M} \sum_{j \in \mathbb{Z}} \| \partial_x^i \Delta_j u \|_s \right) \quad \text{for } m = 4,
\]

\[
\| \Delta_j v \|_s := 2^{sj} \left( \| \Delta_j v \|_{L_t^\infty L_x^2} + 2^{j/2} \| \Delta_j v \|_{L_t^2 L_x^2} \right) + 2^{(s-s_m)j} \| \Delta_j v \|_{L_t^2 L_x^\infty} + 2^{(s-\bar{s}_M)j} \| \Delta_j v \|_{L_x^M L_t^\infty}.
\]

(1.20)

Recall that the norm on homogeneous Besov spaces \( \dot{B}_{2,1}^s \) can be defined in the following way:

\[
\| f \|_{\dot{B}_{2,1}^s} = \sum_{j=-\infty}^{\infty} 2^{sj} \left( \int_{2^j}^{2^{j+1}} |\mathcal{F} f(\xi)|^2 d\xi \right)^{1/2}.
\]

(1.21)

1.2 Remarks on main results

It seems that the regularity assumptions on initial data are not optimal in Theorems 1.1–1.3, but Theorem 1.4 presents the sharp regularity condition to the initial data. To illustrate the relation between the regularity index and the nonlinear power, we consider a simple cases of (1.1):

\[
iu_t + \Delta_{\pm} u = u^\nu, \quad u(0) = \phi.
\]

(1.22)

Eq. (1.22) is invariant under the scaling \( u \rightarrow u_\lambda = \lambda^{(2-\nu)/(\nu-1)}(\lambda^2 t, \lambda x) \) and moreover,

\[
\| \phi \|_{\dot{H}^s(\mathbb{R}^n)} = \| u_\lambda(0, \cdot) \|_{\dot{H}^s(\mathbb{R}^n)}, \quad s = 1 + \bar{s}_\nu - 1 := 1 + n/2 - 1/(\nu - 1).
\]

(1.23)

From this point of view, we say that \( s = 1 + \bar{s}_\nu - 1 \) is the critical regularity index of (1.22). In [23], Molinet and Ribaud showed that (1.22) is ill-posed in one spatial dimension in the sense if \( s_1 \neq \bar{s}_\nu - 1 \), the flow map of equation (1.22)
\( \phi \rightarrow u \) (if it exists) is not of class \( C^m \) from \( \dot{B}_{2,1}^s(\mathbb{R}) \) to \( C([0, \infty), \dot{B}_{2,1}^s(\mathbb{R})) \) at the origin \( \phi = 0 \). For each term in the polynomial nonlinearity \( F(u, \bar{u}, \nabla u, \nabla \bar{u}) \) as in (1.3), we easily see that the critical index \( s \) can take any critical index between \( s_m \) and \( 1 + \tilde{s}_M \). So, our Theorem 1.4 give sharp result in the case \( m \geq 4 \). On the other hand, Christ [9] showed that in the case \( \nu = 2, n = 1 \), for any \( s \in \mathbb{R} \), there exist initial data in \( H^s \) with arbitrarily small norm, for which the solution attains arbitrarily large norm after an arbitrarily short time (see also [24]). From Christ’s result together with Theorems 1.4, we can expect that there exists \( m_0 > 1 \) (might be non-integer) so that for \( \nu - 1 \geq m_0 \), \( s = 1 + \tilde{s}_M \) is the minimal regularity index to guarantee the well posedness of (1.22), at least for the local solutions and small data global solutions in \( H^s \). However, it is not clear for us how to find the exact value of \( m_0 \) even in one spatial dimension.

However, in higher spatial dimensions, it seems that \( 1/2 + 1/M \)-order derivative regularity is lost in Theorem 1.1 and we do not know how to attain the regularity index \( s \geq 1 + \tilde{s}_M \).

In two dimensional case, if \( \Delta_\pm = \Delta \) and the initial value \( u_0 \) is a radial function, we can remove the condition \( u_0 \in \dot{H}^{-1/2}, \|u_0\|_{\dot{H}^{-1/2}} \leq \delta \) by using the endpoint Strichartz estimates as in the case \( n \geq 3 \).

Considering the nonlinearity \( F(u, \nabla u) = (1 - |u|^2)^{-1} |\nabla u|^{2k} u \), Theorem 1.2 holds for the case \( k \geq 1 \). Theorems 1.1 and 1.3 hold for the case \( k \geq 2 \). Since \( (1 - |u|^2)^{-1} = \sum_{k=0}^{\infty} |u|^{2k} \), one easily sees that we can use the same way as in the proof of our main results to handle this kind of nonlinearity.

1.3 Notations

Throughout this paper, we will always use the following notations. \( \mathcal{S}(\mathbb{R}^n) \) and \( \mathcal{S}'(\mathbb{R}^n) \) stand for the Schwartz space and its dual space, respectively. We denote by \( L^p(\mathbb{R}^n) \) the Lebesgue space, \( \| \cdot \|_p := \| \cdot \|_{L^p(\mathbb{R}^n)} \). The Bessel potential space is defined by \( H^s_p(\mathbb{R}^n) := (I - \Delta)^{-s/2} L^p(\mathbb{R}^n) \), \( H^s(\mathbb{R}^n) = H^s_2(\mathbb{R}^n) \). For any quasi-Banach space \( X \), we denote by \( X^* \) its dual space, by \( L^p(I, X) \) the Lebesgue-Bochner space, \( \| f \|_{L^p(I, X)} := \left( \int_I \| f(t) \|_X^p dt \right)^{1/p} \). If \( X = L^r(\Omega) \), then we write \( L^p(I, L^r(\Omega)) = L^p_t L^r_x(I \times \Omega) \) and \( L^p(I \times \Omega) = L^p_t L^p_x(I \times \Omega) \). Let \( Q_\alpha \) be the unit cube with center at \( \alpha \in \mathbb{Z}^n \), i.e., \( Q_\alpha = \alpha + Q_0, Q_0 = \{ x = (x_1, \ldots, x_n) : -1/2 \leq x_i < 1/2 \} \). We also needs

\(^1\) \( \mathbb{R}^n \) will be omitted in the definitions of various function spaces if there is no confusion.
the function spaces $\ell^q_\alpha(L^p_t L^r_x(I \times Q_\alpha))$, 

$$\|f\|_{\ell^q_\alpha(L^p_t L^r_x(I \times Q_\alpha))} := \left(\sum_{\alpha \in \mathbb{Z}^n} \|f\|_{L^q_t L^r_x(I \times Q_\alpha)}^q\right)^{1/q}.$$  

We denote by $\mathcal{F}$ ($\mathcal{F}^{-1}$) the (inverse) Fourier transform for the spatial variables; by $\mathcal{F}_t$ ($\mathcal{F}_t^{-1}$) the (inverse) Fourier transform for the time variable and by $\mathcal{F}_{t,x}$ ($\mathcal{F}_{t,x}^{-1}$) the (inverse) Fourier transform for both time and spatial variables, respectively. If there is no explanation, we always denote by $\varphi_k(\cdot)$ the dyadic decomposition functions as in (1.24); and by $\sigma_k(\cdot)$ the uniform decomposition functions as in (1.27). $u * v$ and $u \ast v$ will stand for the convolution on time and on spatial variables, respectively, i.e.,

$$(u * v)(t, x) = \int_{\mathbb{R}} u(t - \tau, x)v(\tau, x)d\tau, \quad (u \ast v)(t, x) = \int_{\mathbb{R}^n} u(t, x - y)v(t, y)dy.$$  

$\mathbb{R}$, $\mathbb{N}$ and $\mathbb{Z}$ will stand for the sets of reals, positive integers and integers, respectively. $c < 1$, $C > 1$ will denote positive universal constants, which can be different at different places. $a \lesssim b$ stands for $a \leq Cb$ for some constant $C > 1$, $a \sim b$ means that $a \lesssim b$ and $b \lesssim a$. We denote by $p'$ the dual number of $p \in [1, \infty]$, i.e., $1/p + 1/p' = 1$. For any $a > 0$, we denote by $[a]$ the minimal integer that is larger than or equals to $a$. $B(x, R)$ will denote the ball in $\mathbb{R}^n$ with center $x$ and radial $R$.

1.4 Besov and modulation spaces

Let us recall that Besov spaces $B^{s}_{p,q} := B^{s}_{p,q}(\mathbb{R}^n)$ are defined as follows (cf. [130]). Let $\psi : \mathbb{R}^n \to [0, 1]$ be a smooth radial bump function adapted to the ball $B(0, 2)$:

$$\psi(\xi) = \begin{cases} 
1, & |\xi| \leq 1, \\
\text{smooth, } |\xi| \in [1, 2], \\
0, & |\xi| \geq 2.
\end{cases} \quad (1.24)$$

We write $\delta(\cdot) := \psi(\cdot) - \psi(2\cdot)$ and

$$\varphi_j := \delta(2^{-j} \cdot) \quad \text{for } j \geq 1; \quad \varphi_0 := 1 - \sum_{j \geq 1} \varphi_j. \quad (1.25)$$

We say that $\triangle_j := \mathcal{F}^{-1}\varphi_j \mathcal{F}$, $j \in \mathbb{N} \cup \{0\}$ are the dyadic decomposition
operators. We write \( B_{p,q}^s = B_{p,q}^s(\mathbb{R}^n) \) are defined in the following way:

\[
B_{p,q}^s = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \| f \|_{B_{p,q}^s} = \left( \sum_{j=0}^{\infty} 2^{sjq} \| \triangle_j f \|_p^q \right)^{1/q} < \infty \right\}. \tag{1.26}
\]

Now we recall the definition of modulation spaces (see [12,13,31,32,33]). Here we adopt an equivalent norm by using the uniform decomposition to the frequency space. Let \( \rho \in \mathcal{S}(\mathbb{R}^n) \) and \( \rho : \mathbb{R}^n \to [0,1] \) be a smooth radial bump function adapted to the ball \( B(0, \sqrt{n}) \), say \( \rho(\xi) = 1 \) as \( |\xi| \leq \sqrt{n}/2 \), and \( \rho(\xi) = 0 \) as \( |\xi| \geq \sqrt{n} \). Let \( \rho_k \) be a translation of \( \rho \): \( \rho_k(\xi) = \rho(\xi - k) \), \( k \in \mathbb{Z}^n \). We write

\[
\sigma_k(\xi) = \rho_k(\xi) \left( \sum_{k \in \mathbb{Z}^n} \rho_k(\xi) \right)^{-1}, \quad k \in \mathbb{Z}^n. \tag{1.27}
\]

Denote

\[
\Box_k := \mathcal{F}^{-1} \sigma_k \mathcal{F}, \quad k \in \mathbb{Z}^n, \tag{1.28}
\]

which are said to be the frequency-uniform decomposition operators. For any \( k \in \mathbb{Z}^n \), we write \( \langle k \rangle = \sqrt{1 + |k|^2} \). Let \( s \in \mathbb{R} \), \( 0 < p, q \leq \infty \). Modulation spaces \( M_{p,q}^s = M_{p,q}^s(\mathbb{R}^n) \) are defined as:

\[
M_{p,q}^s = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \| f \|_{M_{p,q}^s} = \left( \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{sq} \| \Box_k f \|_p^q \right)^{1/q} < \infty \right\}. \tag{1.29}
\]

We will use the function space \( \ell_\Box^{1,s} \ell_\alpha^q(L_t^p L_x^r(I \times Q_\alpha)) \) which contains all of the functions \( f(t,x) \) so that the following norm is finite:

\[
\| f \|_{\ell_\Box^{1,s} \ell_\alpha^q(L_t^p L_x^r(I \times Q_\alpha))} := \sum_{k \in \mathbb{Z}^n} \langle k \rangle^s \left( \sum_{\alpha \in \mathbb{Z}^n} \| \Box_k f \|_{L_t^p L_x^r(I \times Q_\alpha)}^q \right)^{1/q}. \tag{1.30}
\]

Similarly, we can define the space \( \ell_{\alpha}^{1,s} \ell_\alpha^q(L_t^p L_x^r(I \times Q_\alpha)) \) with the following norm:

\[
\| f \|_{\ell_{\alpha}^{1,s} \ell_\alpha^q(L_t^p L_x^r(I \times Q_\alpha))} := \sum_{j=0}^{\infty} 2^{sj} \left( \sum_{\alpha \in \mathbb{Z}^n} \| \triangle_j f \|_{L_t^p L_x^r(I \times Q_\alpha)}^q \right)^{1/q}. \tag{1.31}
\]

A special case is \( s = 0 \), we write \( \ell_{\alpha}^{1,0} \ell_\alpha^q(L_t^p L_x^r(I \times Q_\alpha)) = \ell_{\alpha}^{1} \ell_\alpha^q(L_t^p L_x^r(I \times Q_\alpha)) \) and \( \ell_{\Box}^{1,0} \ell_\alpha^q(L_t^p L_x^r(I \times Q_\alpha)) = \ell_{\Box}^{1} \ell_\alpha^q(L_t^p L_x^r(I \times Q_\alpha)) \).

The rest of this paper is organized as follows. In Section 2 we give the details of the estimates for the maximal function in certain function spaces. Section 3 is devoted to considering the spatial local versions for the Strichartz
estimates and giving some remarks on the estimates of the local smooth effects.
In Sections 4–7 we prove our main Theorems 1.1–1.4, respectively.

2 Estimates for the maximal function

2.1 Time-local version

Recall that $S(t) = e^{-it\triangle} = \mathcal{F}^{-1}e^{it|\xi|^2} \mathcal{F}$, where

$$|\xi|^2 = \sum_{j=1}^n \varepsilon_j \xi_j^2, \quad \varepsilon_j = \pm 1. \quad (2.1)$$

Kenig, Ponce and Vega \[17\] showed the following maximal function estimate:

$$\left( \sum_{\alpha \in \mathbb{Z}^n} \|S(t)u_0\|_{L^\infty_{t,x}([0,T] \times Q_\alpha)}^2 \right)^{1/2} \lesssim C(T) \|u_0\|_{H^s}, \quad (2.2)$$

where $s \geq 2 + n/2$. If $S(t) = e^{-it\triangle}$, then (2.2) holds for $s > n/2$, $C(T) = (1 + T)^s$. Using the frequency-uniform decomposition method, we can get the following

**Proposition 2.1** There exists a constant $C(T) > 1$ which depends only on $T$ and $n$ such that

$$\sum_{\alpha \in \mathbb{Z}^n} \left( \sum_{\alpha \in \mathbb{Z}^n} \|\Box_k S(t)u_0\|_{L^\infty_{t,x}([0,T] \times Q_\alpha)}^2 \right)^{1/2} \leq C(T) \|u_0\|_{M_{2,1}^{1/2}}, \quad (2.3)$$

In particular, for any $s > (n + 1)/2$,

$$\left( \sum_{\alpha \in \mathbb{Z}^n} \|S(t)u_0\|_{L^\infty_{t,x}([0,T] \times Q_\alpha)}^2 \right)^{1/2} \leq C(T) \|u_0\|_{H^s}. \quad (2.4)$$

**Proof.** By the duality, it suffices to prove that

$$\int_0^T (S(t)u_0, \psi(t)) dt \lesssim \|u_0\|_{M_{2,1}^{1/2}} \sup_{k \in \mathbb{Z}^n} \left( \sum_{\alpha \in \mathbb{Z}^n} \|\Box_k \psi(t)\|_{L^2_{t,x}([0,T] \times Q_\alpha)}^2 \right)^{1/2}. \quad (2.5)$$
Since \((M_{2,1}^{1/2})^* = M_{2,\infty}^{1/2}\), we have
\[
\int_0^T (S(t)u_0, \psi(t)) dt \leq \|u_0\|_{M_{2,1}^{1/2}} \left\| \int_0^T S(-t)\psi(t) dt \right\|_{M_{2,\infty}^{-1/2}}. \tag{2.6}
\]
Recalling that \(\|f\|_{M_{2,\infty}^{-1/2}} = \sup_{k \in \mathbb{Z}^n} \langle k \rangle^{-1/2} \|\square_k f\|_2\), we need to estimate
\[
\left\| \square_k \int_0^T S(-t)\psi(t) dt \right\|_2^2
= \int_0^T \left( \square_k \psi(t), \int_0^T S(t-\tau)\square_k \psi(\tau) d\tau \right) dt
\leq \sum_{\alpha \in \mathbb{Z}^n} \|\square_k \psi\|_{L^1_{t,x}([0,T] \times Q_\alpha)} \left\| \int_0^T S(t-\tau)\square_k \psi(\tau) d\tau \right\|_{L^\infty_{t,x}([0,T] \times Q_\alpha)}
\leq \|\square_k \psi\|_{L^2_{t,x}([0,T] \times Q_\alpha)} \left\| \int_0^T S(t-\tau)\square_k \psi(\tau) d\tau \right\|_{L^2_{t,x}([0,T] \times Q_\alpha)}. \tag{2.7}
\]
If one can show that
\[
\left\| \int_0^T S(t-\tau)\square_k \psi(\tau) d\tau \right\|_{L^2_{t,x}([0,T] \times Q_\alpha)} \lesssim C(T)\|\square_k \psi\|_{L^2_{t,x}([0,T] \times Q_\alpha)},
\]
then from (2.6)–(2.8) we obtain that (2.5) holds. Denote
\[
\Lambda := \{\ell \in \mathbb{Z}^n : \text{supp } \sigma_\ell \cap \text{supp } \sigma_0 \neq \emptyset\}. \tag{2.9}
\]
In the following we show (2.8). In view of Young’s inequality, we have
\[
\left\| \int_0^T S(t-\tau)\square_k \psi(\tau) d\tau \right\|_{L^\infty_{t,x}([0,T] \times Q_\alpha)}
\leq \sum_{\ell \in \Lambda} \left\| \int_0^T S(t-\tau)\square_{k+\ell} \square_k \psi(\tau) d\tau \right\|_{L^\infty_{t,x}([0,T] \times Q_\alpha)}
= \sum_{\ell \in \Lambda} \left\| \int_0^T [\mathcal{F}^{-1}(e^{i(t-\tau)|\ell|_2^2} \sigma_{k+\ell}^2)] * \square_k \psi(\tau) d\tau \right\|_{L^\infty_{t,x}([0,T] \times Q_\alpha)}
\leq \sum_{\ell \in \Lambda} \sum_{\beta \in \mathbb{Z}^n} \left\| \mathcal{F}^{-1}(e^{i|\ell|_2^2} \sigma_{k+\ell}) \right\|_{L^\infty_{t,x}([-T,T] \times Q_\beta)} \left\| \square_k \psi \right\|_{L^1_{t,x}([0,T] \times (Q_\alpha-Q_\beta))}. \tag{2.10}
\]
From (2.11) and Minkowski’s inequality that
\[
\left\| \int_0^T S(t-\tau)\square_k \psi(\tau) d\tau \right\|_{L^2_{t,x}([0,T] \times Q_\alpha)}
\leq \sum_{\ell \in \Lambda} \sum_{\beta \in \mathbb{Z}^n} \left\| \mathcal{F}^{-1}(e^{i|\ell|_2^2} \sigma_{k+\ell}) \right\|_{L^\infty_{t,x}([-T,T] \times Q_\beta)} \left\| \square_k \psi \right\|_{L^2_{t,x}([0,T] \times (Q_\alpha-Q_\beta))}, \tag{2.11}
\]
It is easy to see that
\[
\|k\psi\|_{L^2_{t,x}(\mathbb{R}^n)} \lesssim \|k\psi\|_{L^2_{t,x}(\mathbb{R}^n)}.
\]
(2.12)

Hence, in order to prove (2.8), it suffices to prove that
\[
\sum_{\beta \in \mathbb{Z}^n} \left\| \mathcal{F}^{-1}(e^{it|\xi|^2} \sigma_k) \right\|_{L^\infty_{t,x}(\mathbb{R}^n)} \lesssim C(T) \langle k \rangle.
\]
(2.13)

In fact, observing the following identity,
\[
|\mathcal{F}^{-1}(e^{it|\xi|^2} \sigma_k)| = |\mathcal{F}^{-1}(e^{it|\xi|^2} \sigma_0)(\cdot + 2tk_\pm)|,
\]
where \(k_\pm = (\varepsilon_1 k_1, ..., \varepsilon_n k_n)\), we have
\[
\left\| \mathcal{F}^{-1}(e^{it|\xi|^2} \sigma_k) \right\|_{L^\infty_{t,x}(\mathbb{R}^n)} \leq \left\| \mathcal{F}^{-1}(e^{it|\xi|^2} \sigma_0) \right\|_{L^\infty_{t,x}(\mathbb{R}^n)}.
\]
(2.14)

where
\[
Q^*_{\beta,k} = \{ x : x \in 2tk_\pm + Q_{\beta} \text{ for some } t \in [0, T] \}.
\]

Denote \(\Lambda_{\beta,k} = \{ \beta' : Q_{\beta'} \cap Q^*_{\beta,k} \neq \emptyset \}\). It follows from (2.13) that
\[
\sum_{\beta \in \mathbb{Z}^n} \left\| \mathcal{F}^{-1}(e^{it|\xi|^2} \sigma_k) \right\|_{L^\infty_{t,x}(\mathbb{R}^n)} \leq \sum_{\beta \in \mathbb{Z}^n} \sum_{\beta' \in \Lambda_{\beta,k}} \left\| \mathcal{F}^{-1}(e^{it|\xi|^2} \sigma_0) \right\|_{L^\infty_{t,x}(\mathbb{R}^n)}.
\]
(2.15)

Since each \(E_{\beta,k}\) overlaps at most \(O(T\langle k \rangle)\) many \(Q_{\beta'}\), \(\beta' \in \mathbb{Z}^n\), one can easily verify that in the sums of the right hand side of (2.16), each \(\left\| \mathcal{F}^{-1}(e^{it|\xi|^2} \sigma_0) \right\|_{L^\infty_{t,x}(\mathbb{R}^n)}\) repeats at most \(O(T\langle k \rangle)\) times. Hence, we have
\[
\sum_{\beta \in \mathbb{Z}^n} \left\| \mathcal{F}^{-1}(e^{it|\xi|^2} \sigma_k) \right\|_{L^\infty_{t,x}(\mathbb{R}^n)} \lesssim \langle k \rangle \sum_{\beta \in \mathbb{Z}^n} \left\| \mathcal{F}^{-1}(e^{it|\xi|^2} \sigma_0) \right\|_{L^\infty_{t,x}(\mathbb{R}^n)}.
\]
(2.17)

Finally, it suffices to show that
\[
\sum_{\beta \in \mathbb{Z}^n} \left\| \mathcal{F}^{-1}(e^{it|\xi|^2} \sigma_0) \right\|_{L^\infty_{t,x}(\mathbb{R}^n)} \leq C(T).
\]
(2.18)

Denote \(\nabla_{t,x} = (\partial_t, \partial_{x_1}, ..., \partial_{x_n})\). By the Sobolev inequality,
\[
\sum_{\beta \in \mathbb{Z}^n} \left\| \mathcal{F}^{-1}(e^{it|\xi|^2} \sigma_0) \right\|_{L^{2n}_{t,x}(\mathbb{R}^n)} \lesssim \sum_{\beta \in \mathbb{Z}^n} \left\| \nabla_{t,x} \mathcal{F}^{-1}(e^{it|\xi|^2} \sigma_0) \right\|_{L^{2n}_{t,x}(\mathbb{R}^n)}
\]
\[+ \sum_{\beta \in \mathbb{Z}^n} \left\| \nabla_{t,x} \mathcal{F}^{-1}(e^{it|\xi|^2} \sigma_0) \right\|_{L^{2n}_{t,x}(\mathbb{R}^n)} = I + II.
\]
(2.19)
By Hölder’s inequality, we have

\[
II \lesssim \left( \sum_{\beta \in \mathbb{Z}^n} \left\| (1 + |x|^2)^{\beta} \nabla_{t,x} \mathcal{F}^{-1} (e^{it|\xi|^2} \xi_{\sigma_0}) \right\|_{L^2_{t,x}([0,T] \times \mathbb{R}^n)}^{2n} \right)^{1/2n}
\]

\[
\lesssim \sum_{i=1}^n \left\| \mathcal{F}^{-1} ((I - \Delta)^n (e^{it|\xi|^2} \xi_{\sigma_0})) \right\|_{L^2_{t,x}([0,T] \times \mathbb{R}^n)} + \left\| \mathcal{F}^{-1} ((I - \Delta)^n (e^{it|\xi|^2} \xi^2_{\pm} \sigma_0)) \right\|_{L^2_{t,x}([0,T] \times \mathbb{R}^n)}
\]

\[
\lesssim C(T).
\]

(2.20)

One easily sees that \( I \) has the same bound as that of \( II \). The proof of (2.3) is finished. Noticing that \( H^s \subset M_{2,1}^{1/2} \) if \( s > (n + 1)/2 \) (cf. [29,32,33]), we immediately have (2.4). \( \square \)

### 2.2 Time-global version

Recall that we have the following equivalent norm on Besov spaces ([1,30]):

**Lemma 2.2** Let \( 1 \leq p, q \leq \infty, \sigma > 0, \sigma \notin \mathbb{N} \). Then we have

\[
\|f\|_{B^\sigma_{p,q}} \sim \sum_{|\beta| \leq [\sigma]} \|D^\beta f\|_{L^p(\mathbb{R}^n)} + \sum_{|\beta| \leq [\sigma]} \left( \int_{\mathbb{R}^n} |h|^{-n-q}|\sigma| \|\Delta_h D^\beta f\|_{L^q(\mathbb{R}^n)} dh \right)^{1/q},
\]

(2.21)

where \( \Delta_h f = f(\cdot + h) - f(\cdot) \), \([\sigma]\) denotes the minimal integer that is larger than or equals to \( \sigma \), \( \{\sigma\} = \sigma - [\sigma] \).

Taking \( p = q \) in Lemma 2.2, one has that

\[
\|f\|_{B^\sigma_{p,p}} \sim \sum_{|\beta| \leq [\sigma]} \|D^\beta f\|_{L^p(\mathbb{R}^n)} + \sum_{|\beta| \leq [\sigma]} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\Delta_h D^\beta f(x)|^p}{|h|^{n+p|\sigma|}} dx dh.
\]

(2.22)

**Lemma 2.3** Let \( 1 < p < \infty, s > 1/p \). Then we have

\[
\left( \sum_{\alpha \in \mathbb{Z}^n} \|u\|_{L^\infty_{t,x}([0,T] \times \mathbb{R}^n)}^p \right)^{1/p} \lesssim \|(I - \partial_t^2)^{s/2} u\|_{L^p_{t,x}([0,T] \times \mathbb{R}^n)}.
\]

(2.23)

**Proof.** We divide the proof into the following two cases.

**Case 1.** \( ns \notin \mathbb{N} \). Due to \( H^s_p(\mathbb{R}) \subset L^\infty(\mathbb{R}) \), we have

\[
\|u\|_{L^\infty_{t,x}([0,T] \times \mathbb{R}^n)} \lesssim \|(I - \partial_t^2)^{s/2} u\|_{L^p_{t,x}([0,T] \times \mathbb{R}^n)}.
\]
\[
\leq \|(I - \partial^2_t)^{s/2} u\|_{L^p_t L^\infty_x(\mathbb{R}^n)}.
\]  
(2.24)

Recalling that \(\sigma_\alpha(x) \geq 1\) for all \(x \in Q_\alpha\) and \(\alpha \in \mathbb{Z}^n\), we have from (2.24) that

\[
\|u\|_{L^\infty_{t,x}(\mathbb{R}^n \times Q_\alpha)} \leq \|(I - \partial^2_t)^{s/2} \sigma_\alpha u\|_{L^p_t L^\infty_x(\mathbb{R}^n \times \mathbb{R}^n)}.
\]  
(2.25)

Since \(B_{p,p}^{ns}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)\), in view of (2.25), one has that

\[
\|u\|_{L^\infty_{t,x}(\mathbb{R}^n \times Q_\alpha)} \leq \|(I - \partial^2_t)^{s/2} \sigma_\alpha u\|_{L^p(\mathbb{R} \times B_{p,p}^{ns}(\mathbb{R}^n))}.
\]  
(2.26)

For simplicity, we denote \(v = (I - \partial^2_t)^{s/2} u\). By (2.22) and (2.26) we have

\[
\sum_{\alpha \in \mathbb{Z}^n} \|u\|^p_{L^\infty_{t,x}(\mathbb{R}^n \times Q_\alpha)} \lesssim \sum_{|\alpha| \leq [ns]} \sum_{\alpha \in \mathbb{Z}^n} \int_{\mathbb{R}} \|D^3(\sigma_\alpha v)(t)\|^p_{L^p(Q_\alpha)} dt \\
+ \sum_{|\alpha| \leq [ns]} \sum_{\alpha \in \mathbb{Z}^n} \int_{\mathbb{R}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\|\Delta_h D^3(\sigma_\alpha v)(t,x)\|^p}{|h|^{n+p(ns)}} dx dh dt \\
:= I + II.
\]  
(2.27)

We now estimate \(II\). It is easy to see that

\[
|\Delta_h D^3(\sigma_\alpha v)| \lesssim \sum_{\beta_1 + \beta_2 = \beta} |\Delta_h (D^3 \sigma_\alpha D^3 v)| \\
\lesssim \sum_{\beta_1 + \beta_2 = \beta} (|D^3 \sigma_\alpha (\cdot + h) \Delta_h D^3 v| + |\Delta_h D^3 \sigma_\alpha D^3 v|).
\]  
(2.28)

Since \(\text{supp} \sigma_\alpha\) overlaps at most finitely many \(\text{supp} \sigma_\beta\) and \(\sigma_\beta = \sigma_0(\cdot - \beta), \beta \in \mathbb{Z}^n\), it follows from (2.28), \(|D^3 \sigma_\alpha| \lesssim 1\) and Hölder’s inequality that

\[
II \lesssim \sum_{|\alpha| \leq [ns]} \int_{\mathbb{R}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sum_{\alpha \in \mathbb{Z}^n} |D^3 \sigma_\alpha(x + h)| \frac{\|\Delta_h D^3 v(x, t + h)\|^p}{|h|^{n+p(ns)}} dx dh dt \\
+ \sum_{|\alpha| \leq [ns]} \sum_{\beta_1 + \beta_2 = \beta} \int_{\mathbb{R}^n} \frac{\|\Delta_h D^3 \sigma_\alpha\|^p_{L^\infty(\mathbb{R}^n)}}{|h|^{n+p(ns)}} dh \\
\times \sup_{h} \sum_{\alpha \in \mathbb{Z}^n} \int_{\mathbb{R}^n} \int_{B(0,\sqrt{n}) \cup B(-h,\sqrt{n})} |D^3 v(t, x + \alpha)|^p dx dt \\
\lesssim \sum_{|\beta| \leq [ns]} \int_{\mathbb{R}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\|\Delta_h D^3 v(t, x)|^p}{|h|^{n+p(ns)}} dx dh dt \\
+ \|\sigma_\alpha\|^p_{B_{p,p}^{ns}(\mathbb{R}^n)} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |D^3 v(x)|^p dx dt \\
\lesssim \|v\|^p_{L^p(\mathbb{R} \times B_{p,p}^{ns}(\mathbb{R}^n))}.
\]  
(2.29)

Clearly, one has that

\[
I \lesssim \|v\|^p_{L^p(\mathbb{R} \times B_{p,p}^{ns}(\mathbb{R}^n))}.
\]  
(2.30)
Collecting (2.27), (2.29) and (2.30), we have (2.23).

Case 2. \(ns \in \mathbb{N}\). One can take an \(s_1 < s\) such that \(s_1 > 1/p\) and \(ns_1 \notin \mathbb{N}\). Applying the conclusion as in Case 1, we get the result, as desired. \(\square\)

For the semi-group \(S(t)\), we have the following Strichartz estimate (cf. [14]):

\[\|S(t)u_0\|_{L^{\gamma}(\mathbb{R}, L^p(\mathbb{R}^n))} \lesssim \|u_0\|_{L^2(\mathbb{R}^n)}, \quad (2.31)\]
\[\|\mathcal{A}F\|_{L^{\gamma}(\mathbb{R}, L^p(\mathbb{R}^n))} \lesssim \|F\|_{L^{\gamma}(\mathbb{R}, L^{p'}(\mathbb{R}^n))}. \quad (2.32)\]

If \(p\) and \(\rho\) equal to \(2n/(n-2)\), then (2.31) and (2.32) are said to be the endpoint Strichartz estimates. Using Proposition 2.4, we have

**Proposition 2.5** Let \(p \geq 2 + 4/n := 2^*\). For any \(s > n/2\), we have

\[\left( \sum_{\alpha \in \mathbb{Z}^n} \|S(t)u_0\|^p_{L^{\infty}_t(\mathbb{R} \times Q_\alpha)} \right)^{1/p} \lesssim \|u_0\|_{H^s}. \quad (2.33)\]

**Proof.** For short, we write \(\langle \partial_t \rangle = (I - \partial_t^2)^{1/2}\). By Lemma 2.3, for any \(s_0 > 1/2^*\),

\[\left( \sum_{\alpha \in \mathbb{Z}^n} \|S(t)u_0\|^p_{L^{\infty}_t(\mathbb{R} \times Q_\alpha)} \right)^{1/p} \lesssim \left( \sum_{\alpha \in \mathbb{Z}^n} \|S(t)u_0\|^{2s_0}_{L^{\infty}_t(\mathbb{R} \times Q_\alpha)} \right)^{1/2^*}\]
\[\lesssim \|\langle \partial_t \rangle^{s_0} S(t)u_0\|_{L^{2^*}_t(\mathbb{R}, B^{s_0}_{2^*, 2^*}(\mathbb{R}^n))}. \quad (2.34)\]

We have

\[\|\langle \partial_t \rangle^{s_0} S(t)u_0\|_{L^{2^*}_t(\mathbb{R}, B^{s_0}_{2^*, 2^*}(\mathbb{R}^n))} = \sum_{k=0}^{\infty} 2^{ns_0 k^2} \|\langle \partial_t \rangle^{s_0} \Delta_k S(t)u_0\|_{L^{2^*}_t(\mathbb{R}^{1+n})}. \quad (2.35)\]

Using the dyadic decomposition to the time-frequency, we obtain that

\[\|\langle \partial_t \rangle^{s_0} \Delta_k S(t)u_0\|_{L^{2^*}_t} \lesssim \sum_{j=0}^{\infty} \|\mathcal{T}_{t,x}^{-1} \langle \tau \rangle^{s_0} \varphi_j(\tau) \mathcal{T}_{t} e^{it|\xi|^2} \varphi_k(\xi) \mathcal{T}_{\xi} u_0\|_{L^{2^*}_t}. \quad (2.36)\]

Noticing the fact that

\[\langle \mathcal{T}_{t}^{-1} \langle \tau \rangle^{s_0} \varphi_j(\tau) \rangle \star e^{it|\xi|^2} = c e^{it|\xi|^2} \varphi_j(|\xi|^2) \langle |\xi|^2 \rangle^{s_0}, \quad (2.37)\]
and using the Strichartz inequality and Plancherel’s identity, one has that
\[
\|\langle \partial_t \rangle^{s_0} \Delta_k S(t) u_0 \|_{L^2_t L^2_x} \lesssim \sum_{j=0}^{\infty} \| S(t) \mathcal{F}^{-1} (|\xi|^2 \varphi_j (|\xi|^2) \varphi_k (\xi)) \mathcal{F}_x u_0 \|_{L^2_t L^2_x} \\
\lesssim \sum_{j=0}^{\infty} \| \mathcal{F}^{-1} (|\xi|^2 \varphi_j (|\xi|^2) \varphi_k (\xi)) \mathcal{F}_x u_0 \|_{L^2_x (\mathbb{R}^n)} \\
\lesssim 2^{2s_k} \sum_{j=0}^{\infty} \| \mathcal{F}^{-1} \varphi_j (|\xi|^2) \varphi_k (\xi) \mathcal{F}_x u_0 \|_{L^2_x (\mathbb{R}^n)}.
\] (2.38)

Combining (2.35) and (2.38), together with Minkowski’s inequality, we have
\[
\|\langle \partial_t \rangle^{s_0} S(t) u_0 \|_{L^2_t (\mathbb{R}, B^{s_0}_{2,2} (\mathbb{R}^n))} \\
\lesssim \sum_{j=0}^{\infty} \left( \sum_{k=0}^{\infty} 2^{(n+2)s_0 k^2} \| \mathcal{F}^{-1} \varphi_j (|\xi|^2) \varphi_k (\xi) \mathcal{F}_x u_0 \|_{L^2_x (\mathbb{R}^n)} \right)^{1/2} \\
\lesssim \sum_{j=0}^{\infty} \| \mathcal{F}^{-1} \varphi_j (|\xi|^2) \mathcal{F}_x u_0 \|_{B^{s_0}_{2,2}}.
\] (2.39)

In view of \( H^{(n+2)s_0} \subset B^{(n+2)s_0}_{2,2} \) and Hölder’s inequality, we have for any \( \varepsilon > 0 \),
\[
\sum_{j=0}^{\infty} \| \mathcal{F}^{-1} \varphi_j (|\xi|^2) \mathcal{F}_x u_0 \|_{B^{(n+2)s_0}_{2,2}} \lesssim \sum_{j=0}^{\infty} \| \mathcal{F}^{-1} \varphi_j (|\xi|^2) \mathcal{F}_x u_0 \|_{H^{(n+2)s_0}} \\
\lesssim \left( \sum_{j=0}^{\infty} 2^{2j\varepsilon} \| \mathcal{F}^{-1} \varphi_j (|\xi|^2) \mathcal{F}_x u_0 \|_{H^{(n+2)s_0}}^2 \right)^{1/2}.
\] (2.40)

By Plancherel’s identity, and \( \text{supp} \varphi_j (|\xi|^2) \subset \{ \xi : |\xi|^2 \in [2^{j-1}, 2^{j+1}] \} \), we easily see that
\[
\left( \sum_{j=0}^{\infty} 2^{2j\varepsilon} \| \mathcal{F}^{-1} \varphi_j (|\xi|^2) \mathcal{F}_x u_0 \|_{H^{(n+2)s_0}}^2 \right)^{1/2} \lesssim \left( \sum_{j=0}^{\infty} \| \varphi_j (|\xi|^2) \mathcal{F}_x u_0 \|_{H^{(n+2)s_0 + 2\varepsilon}}^2 \right)^{1/2} \\
\lesssim \| u_0 \|_{H^{(n+2)s_0 + 2\varepsilon}}.
\] (2.41)

Taking \( s_0 \) such that \( (n+2)s_0 + 2\varepsilon < s \), from (2.39)–(2.41) we have the result, as desired.

Next, we consider the estimates for the maximal function based on the frequency-uniform decomposition method. This issue has some relations with the Strichartz estimates in modulation spaces. Recently, the Strichartz estimates have been generalized to various function spaces, for instance, in the
Wiener amalgam spaces [10][11]. Recall that in [32], we obtained the following Strichartz estimate for a class of dispersive semi-groups in modulation spaces:

\[ U(t) = \mathcal{F}^{-1} e^{itP(\xi)} \mathcal{F}, \]  

(2.42)

\[ P(\cdot) : \mathbb{R}^n \to \mathbb{R} \] is a real valued function, which satisfies the following decay estimate

\[ \|U(t)f\|_{M^\alpha_{p,q}} \lesssim (1 + |t|)^{-\delta} \|f\|_{M^\alpha_{p',q'}}, \]  

(2.43)

where \( 2 \leq p < \infty \), \( \alpha = \alpha(p) \in \mathbb{R} \), \( \delta = \delta(p) > 0 \), \( \alpha, \delta \) are independent of \( t \in \mathbb{R} \).

**Proposition 2.6** Let \( U(t) \) satisfy (2.43) and (2.44). We have for any \( \gamma \geq 2 \vee (2/\delta) \),

\[ \|U(t)f\|_{L^\gamma(\mathbb{R},M^\alpha_{p/2,1})} \lesssim \|f\|_{M^\alpha_{2,1}}. \]  

(2.44)

Recall that the hyperbolic Schrödinger semi-group \( S(t) = e^{it\Delta} \) has the same decay estimate as that of the elliptic Schrödinger semi-group \( e^{it\Delta} \):

\[ \|S(t)u_0\|_{L^\infty(\mathbb{R}^n)} \lesssim |t|^{-n/2}\|u_0\|_{L^1(\mathbb{R}^n)}. \]

It follows that

\[ \|S(t)u_0\|_{M^\infty_{\infty,1}} \lesssim |t|^{-n/2}\|u_0\|_{M^1_{1,1}}. \]  

(2.45)

On the other hand, by Hausdorff-Young’s and Hölder’s inequalities we easily calculate that

\[ \|\Box_k S(t)u_0\|_{L^\infty(\mathbb{R}^n)} \lesssim \sum_{\ell \in \Lambda} \|\mathcal{F}^{-1} \sigma_{k+\ell} \mathcal{F} \Box_k S(t)u_0\|_{L^\infty(\mathbb{R}^n)} \]

\[ \lesssim \sum_{\ell \in \Lambda} \|\sigma_{k+\ell} \mathcal{F} \Box_k u_0\|_{L^1(\mathbb{R}^n)} \lesssim \|\Box_k u_0\|_{L^1(\mathbb{R}^n)}, \]

where \( \Lambda \) is as in (2.9). It follows that

\[ \|S(t)u_0\|_{M^\infty_{\infty,1}} \lesssim \|u_0\|_{M^1_{1,1}}. \]  

(2.46)

Hence, in view of (2.45) and (2.46), we have

\[ \|S(t)u_0\|_{M^\infty_{\infty,1}} \lesssim (1 + |t|)^{-n/2}\|u_0\|_{M^1_{1,1}}. \]  

(2.47)

By Plancherel’s identity, one has that

\[ \|S(t)u_0\|_{M^1_{2,1}} = \|u_0\|_{M^1_{2,1}}. \]  

(2.48)

Hence, an interpolation between (2.47) and (2.48) yields (cf. [33]),

\[ \|S(t)u_0\|_{M^1_{p,1}} \lesssim (1 + |t|)^{-n(1/2-1/p)}\|u_0\|_{M^1_{p',1}}. \]
Applying Proposition 2.6, we immediately obtain that

**Proposition 2.7** Let $2 \leq p < \infty$, $2/\gamma(p) = n(1/2 - 1/p)$. We have for any $\gamma \geq 2 \vee \gamma(p)$,

$$
\|S(t)u_0\|_{L^{2}(\mathbb{R}, M_{p,1}^{\alpha/2})} \lesssim \|u_0\|_{M_{2,1}}. 
$$

In particular, if $p \geq 2 + 4/n := 2^*$, then

$$
\|S(t)u_0\|_{L^{p}(\mathbb{R}, M_{p,1}^{\alpha/2})} \lesssim \|u_0\|_{M_{2,1}}. 
$$

Let $\Lambda = \{ \ell \in \mathbb{Z}^n : \text{supp} \sigma_\ell \cap \text{supp} \sigma_0 \neq \emptyset \}$ be as in (2.3). Using the fact that $\Box k \Box_{k+\ell} = 0$ if $\ell \not\in \Lambda$, it is easy to see that (2.50) implies the following frequency-uniform estimates:

$$
\|\Box k S(t)u_0\|_{L^{p}_{t,x}(\mathbb{R} \times \mathbb{R}^n)} \lesssim \|k u_0\|_{2}, \quad k \in \mathbb{Z}^n. 
$$

Applying this estimate, we can get the following

**Proposition 2.8** Let $p \geq 2 + 4/n := 2^*$. For any $s > (n + 2)/p$, we have

$$
\sum_{k \in \mathbb{Z}^n} \left( \sum_{\alpha \in \mathbb{Z}^n} \|\Box k S(t)u_0\|_{L^{p}_{t,x}(\mathbb{R} \times \mathbb{R}^n)} \right)^{1/p} \lesssim \|u_0\|_{M_{2,1}^s}. 
$$

**Proof.** Let us follow the proof of Proposition 2.5. Denote $\langle \partial_\ell \rangle = (I - \partial_\ell^2)^{1/2}$. By Lemma 2.3, for any $s_0 > 1/p$,

$$
\sum_{k \in \mathbb{Z}^n} \left( \sum_{\alpha \in \mathbb{Z}^n} \|\Box k S(t)u_0\|_{L^{p}_{t,x}(\mathbb{R} \times \mathbb{R}^n)} \right)^{1/p} \lesssim \sum_{k \in \mathbb{Z}^n} \|\langle \partial_\ell \rangle^{s_0} S(t)\Box k u_0\|_{L^{p}_{t,x}(\mathbb{R} \times \mathbb{R}^n)} \lesssim \sum_{k \in \mathbb{Z}^n} \|\langle \partial_\ell \rangle^{s_0} S(t)\Box k u_0\|_{L^{p}_{t,x}(\mathbb{R} \times \mathbb{R}^n)}; 
$$

where we have used the fact that $H_{p,s_0}^{n} (\mathbb{R}^n) \subset B_{p,s_0}^{n}(\mathbb{R}^n)$. Since $\text{supp}\sigma_k \subset B(k, \sqrt{n/2})$, applying Bernstein’s multiplier estimate, we get that

$$
\sum_{k \in \mathbb{Z}^n} \|\langle \partial_\ell \rangle^{s_0} S(t)\Box k u_0\|_{L^{p}_{t,x}(\mathbb{R} \times \mathbb{R}^n)} \lesssim \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{n s_0} \|\langle \partial_\ell \rangle^{s_0} S(t)\Box k u_0\|_{L^{p}_{t,x}(\mathbb{R}^{1+n})}. 
$$

Similarly as in (2.38), using (2.51), we have

$$
\|\langle \partial_\ell \rangle^{s_0} S(t)\Box k u_0\|_{L^{p}_{t,x}(\mathbb{R}^{1+n})} \lesssim \sum_{j=0}^{\infty} \|\mathcal{F}^{-1} \langle |\xi|^{2} \rangle^{s_0} \varphi_j (|\xi|^{2}) e^{i|\xi| \cdot k} \sigma_k (\xi) \mathcal{F} u_0\|_{L^{p}_{t,x}(\mathbb{R}^{1+n})}
$$
\begin{equation}
\lesssim \sum_{j=0}^{\infty} (k)^{2s_0} \| \mathcal{F}_x^{-1} \varphi_j(|\xi|^2_{\pm}) \sigma_k(\xi) \mathcal{F}_x u_0 \|_{L^2_x(\mathbb{R}^n)}.
\end{equation}

In an analogous way as in (2.40) and (2.41), we obtain that
\begin{equation}
\sum_{j=0}^{\infty} (k)^{2s_0} \| \mathcal{F}_x^{-1} \varphi_j(|\xi|^2_{\pm}) \sigma_k(\xi) \mathcal{F}_x u_0 \|_{L^2_x(\mathbb{R}^n)} \lesssim (k)^{2s_0+2\varepsilon} \| \Box_k u_0 \|_{L^2_x(\mathbb{R}^n)}.
\end{equation}

Collecting (2.53)–(2.56), we have
\begin{equation}
\sum_{k \in \mathbb{Z}^n} \left( \sum_{\alpha \in \mathbb{Z}^n} \| \Box_k S(t) u_0 \|_{L^p(\mathbb{R}, L^r(Q_\alpha))} \right)^{1/p} \lesssim \sum_{k \in \mathbb{Z}^n} (k)^{(n+2)s_0+2\varepsilon} \| \Box_k u_0 \|_{L^2_x(\mathbb{R}^n)}.
\end{equation}

Hence, by (2.57) we have (2.52).

Using the ideas as in Lemma 2.3 and Proposition 2.5, we can show the following

Proposition 2.9 Let \( p \geq 2 + 4/n := 2^* \). Let \( 2^* \leq r, q \leq \infty, s_0 > 1/2^* - 1/q \), \( s_1 > n(1/2^* - 1/r) \). Then we have
\begin{equation}
\left( \sum_{\alpha \in \mathbb{Z}^n} \| S(t) u_0 \|_{L^p(\mathbb{R}, L^r(Q_\alpha))} \right)^{1/p} \lesssim \| u_0 \|_{H_{s+2s_0}^*}.
\end{equation}

In particular, for any \( q, p \geq 2^* \), \( s > n/2 - 2/q \),
\begin{equation}
\left( \sum_{\alpha \in \mathbb{Z}^n} \| S(t) u_0 \|_{L^p(\mathbb{R}, L^r(Q_\alpha))} \right)^{1/p} \lesssim \| u_0 \|_{H^s}.
\end{equation}

Sketch of Proof. In view of \( \ell^{2^*} \subset \ell^{p} \), it suffices to consider the case \( p = 2^* \). Using the inclusions \( H^{s_0}_p(\mathbb{R}) \subset L^p(\mathbb{R}) \) and \( B^{s_1}_{p,p}(\mathbb{R}^n) \subset L^r(\mathbb{R}^n) \), we have
\begin{equation}
\| u \|_{L^q(\mathbb{R}, L^r(Q_\alpha))} \lesssim \|(I - \partial_t^2)^{s_0/2} \sigma_\alpha u \|_{L^p(\mathbb{R}, B^{s_1}_{p,p}(\mathbb{R}^n))}.
\end{equation}

Using the same way as in Lemma 2.3, we can show that
\begin{equation}
\left( \sum_{\alpha \in \mathbb{Z}^n} \| u \|_{L^p(\mathbb{R}, L^r(Q_\alpha))} \right)^{1/p} \lesssim \|(I - \partial_t^2)^{s_0/2} u \|_{L^p(\mathbb{R}, B^{s_1}_{p,p}(\mathbb{R}^n))}.
\end{equation}

One can repeat the procedures as in the proof of Lemma 2.3 to conclude that
\begin{equation}
\sum_{\alpha \in \mathbb{Z}^n} \|(I - \partial_t^2)^{s_0/2} \sigma_\alpha S(t) u_0 \|_{L^p(\mathbb{R}, B^{s_1}_{p,p}(\mathbb{R}^n))} \lesssim \sum_{j=0}^{\infty} \| \mathcal{F}_x^{-1} \varphi_j(|\xi|^2_{\pm}) \mathcal{F}_x u_0 \|_{H^{s+2s_0}(\mathbb{R}^n)}.
\end{equation}
Applying an analogous way as in the proof of Proposition 2.5
\[ \sum_{j=0}^{\infty} \| \mathcal{F}^{-1} \varphi_j | \xi_j^2 \rangle \mathcal{F} u_0 \|_{H^{s_1+2s_0}(\mathbb{R}^n)} \lesssim \| u_0 \|_{H^{s_1+2s_0+2 \varepsilon}}. \]  (2.63)

Collecting (2.61) and (2.63), we immediately get (2.58). \( \Box \)

**Proposition 2.10** For any \( q \geq p \geq 2^* \), \( s > (n+2)/p - 2/q \),
\[ \sum_{k \in \mathbb{Z}^n} \left( \sum_{\alpha \in \mathbb{Z}^n} \| \Box_k S(t) u_0 \|_{L^p(\mathbb{R}, L^\infty(Q_\alpha))}^p \right)^{1/p} \lesssim \| u_0 \|_{M^2_{s,1}}. \]  (2.64)

### 3 Global-local estimates on time-space

#### 3.1 Time-global and space-local Strichartz estimates

We need some modifications to the Strichartz estimates, which are global on time variable and local on spatial variable. We always denote by \( S(t) \) and \( \mathcal{A} \) the generalized Schrödinger semi-group and the integral operator as in (1.9).

**Proposition 3.1** Let \( n \geq 3 \). Then we have
\[ \sup_{\alpha \in \mathbb{Z}^n} \| S(t) u_0 \|_{L^2_{t,x}(\mathbb{R} \times Q_\alpha)} \lesssim \| u_0 \|_2, \]  (3.1)
\[ \sup_{\alpha \in \mathbb{Z}^n} \| \mathcal{A} F \|_{L^2_{t,x}(\mathbb{R} \times Q_\alpha)} \lesssim \sum_{\alpha \in \mathbb{Z}^n} \| F \|_{L^1_t L^2_x(\mathbb{R} \times Q_\alpha)}. \]  (3.2)
\[ \sup_{\alpha \in \mathbb{Z}^n} \| \mathcal{A} F \|_{L^2_{t,x}(\mathbb{R} \times Q_\alpha)} \lesssim \sum_{\alpha \in \mathbb{Z}^n} \| F \|_{L^2_{t,x}(\mathbb{R} \times Q_\alpha)}. \]  (3.3)

**Proof.** In view of Hölder’s inequality and the endpoint Strichartz estimate,
\[ \| S(t) u_0 \|_{L^2_{t,x}(\mathbb{R} \times Q_\alpha)} \lesssim \| S(t) u_0 \|_{L^2_t L^{2n/(n-2)}_{x}(\mathbb{R} \times Q_\alpha)} \leq \| S(t) u_0 \|_{L^2_t L^{2n/(n-2)}_{x}(\mathbb{R} \times \mathbb{R}^n)} \lesssim \| u_0 \|_{L^2_\alpha(\mathbb{R}^n)} \].  (3.4)

Using the above ideas and the following Strichartz estimate
\[ \| \mathcal{A} F \|_{L^2_t L^{2n/(n-2)}_{x}(\mathbb{R} \times \mathbb{R}^n)} \lesssim \| F \|_{L^1_t L^2_x(\mathbb{R} \times \mathbb{R}^n)}, \]  (3.5)
\[ \| \mathcal{A} F \|_{L^2_t L^{2n/(n-2)}_{x}(\mathbb{R} \times \mathbb{R}^n)} \lesssim \| F \|_{L^2_t L^{2n/(n+2)}_{x}(\mathbb{R} \times \mathbb{R}^n)}, \]  (3.6)

one can easily get (3.2) and (3.3). \( \Box \)
Since the endpoint Strichartz estimates used in the proof of Proposition 3.1 only holds for \( n \geq 3 \), it is not clear for us if (3.1) still hold for \( n = 2 \). This is why we have an additional condition that \( u_0 \in H^{-1/2} \) is small in 2D. However, we have the following (see [17])

**Proposition 3.2** Let \( n = 2 \). Then we have for any \( 1 \leq r < 4/3 \),

\[
\sup_{\alpha \in \mathbb{Z}^n} \| S(t)u_0 \|_{L^2_t(x;Q_\alpha)} \lesssim \min \left( \| (-\Delta)^{-1/4}u_0 \|_2, \| u_0 \|_{L^2 \cap L^r(R^n)} \right). \tag{3.7}
\]

In the low frequency case, one easily sees that (3.7) is strictly weaker than (3.1).

**Proof.** By Lemma 3.4, it suffices to show

\[
\sup_{\alpha \in \mathbb{Z}^n} \| S(t)u_0 \|_{L^2_t(x;Q_\alpha)} \lesssim \| u_0 \|_{L^2 \cap L^r(R^n)}. \tag{3.8}
\]

Using the unitary property in \( L^2 \) and the \( L^p - L^{p'} \) decay estimates of \( S(t) \), we have

\[
\| S(t)u_0 \|_{L^2_t(Q_\alpha)} \lesssim (1 + |t|)^{-1/2} \| u_0 \|_{L^2 \cap L^r(R^n)}. \tag{3.9}
\]

Taking the \( L^2 \) norm in both sides of (3.9), we immediately get (3.8). Hence, the result follows.

**Proposition 3.3** Let \( n = 2 \). Then we have

\[
\sup_{\alpha \in \mathbb{Z}^n} \| \mathcal{A}F \|_{L^2_t(x;Q_\alpha)} \lesssim \sum_{\alpha \in \mathbb{Z}^2} \| F \|_{L^1_tL^2_x(R \times \mathbb{R}^n)}. \tag{3.10}
\]

**Proof.** We notice that

\[
\| S(t)f \|_{L^2_t(Q_\alpha)} \lesssim (1 + |t|)^{-1/2} \| f \|_{L^1_tL^2_x(R \times \mathbb{R}^n)} \tag{3.11}
\]

It follows that

\[
\| \mathcal{A}F \|_{L^2_t(Q_\alpha)} \lesssim \int_{\mathbb{R}} (1 + |t - \tau|)^{-1} \| F(\tau) \|_{L^1_tL^2_x(R \times \mathbb{R}^n)} d\tau. \tag{3.12}
\]

Using Young’s inequality, one has that

\[
\| \mathcal{A}F \|_{L^2_t(Q_\alpha)} \lesssim \| F \|_{L^1(\mathbb{R}, L^1_tL^2_x(R \times \mathbb{R}^n))}. \tag{3.13}
\]

In view of Hölder’s inequality, (3.13) yields the result, as desired.
3.2 Note on the time-global and space-local smooth effects

Kenig, Ponce and Vega [16,17] obtained the local smooth effect estimates for the Schrödinger group $e^{it\Delta}$, and their results can also be developed to the non-elliptical Schrödinger group $e^{it\Delta\pm}$ [18]. On the basis of their results and Proposition 3.1, we can obtain a time-global version of the local smooth effect estimates with the nonhomogeneous derivative $(I - \Delta)^{1/2}$ instead of homogeneous derivative $\nabla$, which is useful to control the low frequency parts of the nonlinearity.

**Lemma 3.4 (16)** Let $\Omega$ be an open set in $\mathbb{R}^n$, $\phi$ be a $C^1(\Omega)$ function such that $\nabla \phi(\xi) \neq 0$ for any $\xi \in \Omega$. Assume that there is $N \in \mathbb{N}$ such that for any $\bar{\xi} := (\xi_1, \ldots, \xi_{n-1}) \in \mathbb{R}^{n-1}$ and $r \in \mathbb{R}$, the equation $\phi(\xi_1, \ldots, \xi_k, x, \xi_{k+1}, \ldots, \xi_{n-1}) = r$ has at most $N$ solutions. For $a(x, s) \in L^\infty(\mathbb{R}^n \times \mathbb{R})$ and $f \in \mathcal{S}(\mathbb{R}^n)$, we denote

$$W(t)f(x) = \int_\Omega e^{i(t\phi(\xi)+x\xi)}a(x, \phi(\xi))\hat{f}(\xi)d\xi.$$  \hfill (3.14)

Then for $n \geq 2$, we have

$$\|W(t)f\|_{L^2_tL^2_x(\mathbb{R} \times \mathbb{R}(0,R))} \leq CNR^{1/2}\|\nabla \phi|^{-1/2}\hat{f}\|_{L^2(\Omega)}.  \hfill (3.15)$$

**Corollary 3.5** Let $n \geq 3$, $S(t) = e^{it\Delta\pm}$. We have

$$\sup_{t \in \mathbb{Z}^n} \|S(t)u_0\|_{L^2_tL^2_x(\mathbb{R} \times \mathbb{R}(0,R))} \lesssim \|u_0\|_{H^{-1/2}},  \hfill (3.16)$$

$$\|\mathcal{A}f\|_{L^\infty(\mathbb{R}, H^{1/2})} \lesssim \sum_{t \in \mathbb{Z}^n} \|f\|_{L^2_tL^2_x(\mathbb{R} \times \mathbb{R}(0,R))}.  \hfill (3.17)$$

For $n = 2$, (3.17) also holds if one substitutes $H^{1/2}$ by $\dot{H}^{1/2}$.

**Proof.** Let $\Omega = \mathbb{R}^n \setminus B(0,1)$, $\phi(\xi) = |\xi|_+^2$ and $\psi$ be as in (3.24), $a(x, s) = 1 - \psi(s)$ in Lemma 3.3. Taking $W(t) := S(t)\mathcal{F}^{-1}(1 - \psi)\mathcal{F}$, from (3.15) we have

$$\sup_{t \in \mathbb{Z}^n} \|S(t)\mathcal{F}^{-1}(1 - \psi)\mathcal{F}u_0\|_{L^2_tL^2_x(\mathbb{R} \times \mathbb{R}(0,R))} \lesssim \|\xi|^{-1/2}\hat{u}_0\|_{L^2_\xi(\mathbb{R}^n \setminus B(0,1))}.  \hfill (3.18)$$

It follows from Proposition 3.1 that

$$\|S(t)\mathcal{F}^{-1}\psi\mathcal{F}u_0\|_{L^2_tL^2_x(\mathbb{R} \times \mathbb{R}(0,R))} \lesssim \|\mathcal{F}^{-1}\psi\mathcal{F}u_0\|_{L^2_\xi(\mathbb{R}^n)} \lesssim \|\hat{u}_0\|_{L^2(\mathbb{R}^n)}.  \hfill (3.19)$$

From (3.18) and (3.19) we have (3.16), as desired. (3.17) is the dual version of (3.16). \hfill $\square$
When \( n = 2 \), it is known that for the elliptic case, the endpoint Strichartz estimate holds for the radial function (cf. [28]). So, Corollary 3.5 also holds for the radial function \( u_0 \) in the elliptic case. The following local smooth effect estimates for the nonhomogeneous part of the solutions of the Schrödinger equation is also due to Kenig, Ponce and Vega [17,2].

**Proposition 3.6** Let \( n \geq 2 \), \( S(t) = e^{it\Delta} \). We have

\[
\sup_{\alpha \in \mathbb{Z}^n} \|\nabla \mathcal{A} f\|_{L^2_t L^\infty_x(\mathbb{R} \times Q_\alpha)} \lesssim \sum_{\alpha \in \mathbb{Z}^n} \|f\|_{L^2_t L^\infty_x(\mathbb{R} \times Q_\alpha)}.
\]

(3.20)

4 Proof of Theorem 1.1

**Lemma 4.1** (Sobolev Inequality). Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain with \( \partial \Omega \in C^m \), \( m, \ell \in \mathbb{N} \cup \{0\} \), \( 1 \leq r, p, q \leq \infty \). Assume that

\[
\frac{\ell}{m} \leq \theta \leq 1, \quad \frac{1}{p} - \frac{\ell}{n} = \theta \left(\frac{1}{r} - \frac{m}{n}\right) + \frac{1 - \theta}{q}.
\]

Then we have

\[
\sum_{|\beta| = \ell} \|D^\beta u\|_{L^p(\Omega)} \lesssim \|u\|_{L^\infty(\Omega)}^{1 - \theta} \|u\|_{W^{m,\infty}(\Omega)}^\theta ,
\]

(4.1)

where \( \|u\|_{W^{m,\infty}(\Omega)} = \sum_{|\beta| \leq m} \|u\|_{L^r(\Omega)} \).

**Proof of Theorem 1.1** In order to illustrate our ideas in an exact way, we first consider a simple case \( s = \lfloor n/2 \rfloor + 5/2 \) and there is no difficulty to generalize the proof to the case \( s > n/2 + 3/2, s + 1/2 \in \mathbb{N} \). We assume without loss of generality that

\[
F(u, \bar{u}, \nabla u, \nabla \bar{u}) := F(u, \nabla u) = \sum_{\Lambda_{\kappa,\nu}} c_{\kappa \nu_1...\nu_n} u^\kappa u^{\nu_1}_x...u^{\nu_n}_x,
\]

(4.2)

where

\[
\Lambda_{\kappa,\nu} = \{(\kappa, \nu_1, ..., \nu_n) : m + 1 \leq \kappa + \nu_1 + ... + \nu_n \leq M + 1\}.
\]

Since we only use the Sobolev norm to control the nonlinear terms, \( \bar{u} \) and \( u \) have the same norm, whence, the general cases can be handled in the same way. Denote

\[
\lambda_1(v) := \|v\|_{L^\infty_t L^2_x(\mathbb{R} \times Q_\alpha)},
\]

2 In [17], the result was stated for the elliptic case, however, their result is also adapted to the non-elliptic cases.
\[\lambda_2(v) := \|v\|e_x^* (L^\infty_t (\mathbb{R} \times Q_o)),\]
\[\lambda_3(v) := \|v\|e_x^* (L^\infty_t L^\infty (\mathbb{R} \times Q_o)).\]

Put

\[\mathcal{D}_n = \left\{ u : \sum_{|\beta| \leq [n/2] + 3} \lambda_1(D^\beta u) + \sum_{|\beta| \leq 1} \sum_{i=2,3} \lambda_i(D^\beta u) \leq \theta \right\}. \tag{4.3}\]

We consider the mapping

\[\mathcal{T} : u(t) \rightarrow S(t)u_0 - i\mathcal{A}F(u, \nabla u), \tag{4.4}\]

and we show that \(\mathcal{T} : \mathcal{D}_n \rightarrow \mathcal{D}_n\) is a contraction mapping for any \(n \geq 2\).

**Step 1.** For any \(u \in \mathcal{D}_n\), we estimate \(\lambda_1(D^\beta \mathcal{T} u), |\beta| \leq 3 + [n/2]\). We consider the following three cases.

**Case 1.** \(n \geq 3\) and \(1 \leq |\beta| \leq 3 + [n/2]\). In view of Corollary 3.3 and Proposition 3.6, we have for any \(\beta, 1 \leq |\beta| \leq 3 + [n/2]\),

\[\lambda_1(D^\beta \mathcal{T} u) \lesssim \|S(t)D^\beta u_0\|e_x^*(L^2_t (\mathbb{R} \times Q_o)) + \sum_{\Lambda_{\kappa,\nu}} \|\mathcal{A}D^\beta (u^\kappa u_{x_1}^{\nu_1} \ldots u_{x_n}^{\nu_n})\|e_x(L^\infty_t (\mathbb{R} \times Q_o)) \lesssim \|u_0\|_{H^s} + \sum_{|\beta| \leq 2+[n/2]} \sum_{\Lambda_{\kappa,\nu}} \sum_{\alpha \in \mathbb{Z}^n} \|D^\beta (u^\kappa u_{x_1}^{\nu_1} \ldots u_{x_n}^{\nu_n})\|L^2_t (\mathbb{R} \times Q_o). \tag{4.5}\]

For simplicity, we can further assume that \(u^\kappa u_{x_1}^{\nu_1} \ldots u_{x_n}^{\nu_n} = u^\kappa u_{x_1}^\nu\) in (4.5) and the general case can be treated in an analogous way. So, one can rewrite (4.5) as

\[\sum_{1 \leq |\beta| \leq 3+[n/2]} \lambda_1(D^\beta \mathcal{T} u) \lesssim \|u_0\|_{H^s} + \sum_{|\beta| \leq 2+[n/2]} \sum_{\Lambda_{\kappa,\nu}} \sum_{\alpha \in \mathbb{Z}^n} \|D^\beta (u^\kappa u_{x_1}^\nu)\|L^2_t (\mathbb{R} \times Q_o). \tag{4.6}\]

It is easy to see that

\[|D^\beta (u^\kappa u_{x_1}^\nu)| \lesssim \sum_{\beta_1 + \ldots + \beta_{\kappa+\nu} = \beta} |D^{\beta_1} u_\ldots D^{\beta_{\kappa+\nu}} u_{x_1}^{\nu}|. \tag{4.7}\]

By Hölder’s inequality,

\[\|D^\beta (u^\kappa u_{x_1}^\nu)\|_{L^2_t (Q_o)} \lesssim \sum_{\beta_1 + \ldots + \beta_{\kappa+\nu} = \beta} \prod_{i=1}^\kappa \|D^{\beta_i} u\|_{L^\infty_t (Q_o)} \prod_{i=\kappa+1}^{\kappa+\nu} \|D^{\beta_i} u_{x_1}\|_{L^\infty_t (Q_o)}, \tag{4.8}\]

\[\text{3 One can see below for a general treating.}\]
where
\[
p_i = \begin{cases} 
\frac{2|\beta_i|}{|\beta|}, & |\beta_i| \geq 1, \\
\infty, & |\beta_i| = 0. 
\end{cases}
\]

It is easy to see that for \( \theta_i = |\beta_i|/|\beta| \),
\[
\frac{1}{p_i} - \frac{|\beta_i|}{n} = \theta_i \left( \frac{1}{2} - \frac{|\beta_i|}{n} \right) + \frac{1 - \theta_i}{\infty}.
\]

Using Sobolev’s inequality, one has that for \( B_\alpha := \{ x : |x - \alpha| \leq \sqrt{n} \} \),
\[
\| D^\alpha u \|_{L^p_x(Q_\alpha)} \lesssim \| D^\alpha u \|_{L^p_x(B_\alpha)} \lesssim \| u \|_{L^2_x(B_\alpha)}^\theta \| u \|_{W^{1,2}_x(B_\alpha)}^{1-\theta}, \quad i = 1, \ldots, \kappa; \tag{4.9}
\]
\[
\| D^\alpha u_{x_1} \|_{L^p_x(Q_\alpha)} \lesssim \| u_{x_1} \|_{L^2_x(B_\alpha)}^\theta \| u_{x_1} \|_{W^{1,2}_x(B_\alpha)}^{1-\theta}, \quad i = \kappa + 1, \ldots, \kappa + \nu. \tag{4.10}
\]

Since
\[
\sum_{i=1}^{\kappa+\nu} \theta_i = 1, \quad \sum_{i=1}^{\kappa+\nu} (1 - \theta_i) = \kappa + \nu - 1,
\]
by \(4.8\)–\(4.10\) we have
\[
\| D^\alpha (u^* u_{x_1}) \|_{L^p_x(B_\alpha)} \lesssim \sum_{|\beta| \leq 2+[n/2]} \left( \| u \|_{L^p_x(B_\alpha)} + \| u_{x_1} \|_{L^p_x(B_\alpha)}^{\theta} \right) \times \left( \| u \|_{L^p_x(B_\alpha)} + \| u_{x_1} \|_{L^p_x(B_\alpha)}^{\theta} \right) \lesssim \sum_{|\gamma| \leq 3+[n/2]} \| D^\gamma u \|_{L^p_x(B_\alpha)} \sum_{|\beta| \leq 1} \| D^\beta u \|_{L^p_x(B_\alpha)}. \tag{4.11}
\]

It follows from \(4.11\) and \(L^2 \subset L^{\kappa+\nu-1} \) that
\[
\sum_{|\beta| \leq 2+[n/2]} \sum_{\alpha \in \mathbb{Z}^n} \| D^\alpha (u^* u_{x_1}) \|_{L^2_t_x^{1,2}((\mathbb{R} \times Q_\alpha))} \lesssim \sum_{\alpha \in \mathbb{Z}^n} \sum_{|\gamma| \leq 3+[n/2]} \| D^\gamma u \|_{L^p_t_x^{1,2}(\mathbb{R} \times B_\alpha)} \sum_{|\beta| \leq 1} \| D^\beta u \|_{L^p_t_x^{1,2}(\mathbb{R} \times B_\alpha)} \lesssim \sum_{|\gamma| \leq 3+[n/2]} \lambda_1(D^\gamma u) \sum_{|\beta| \leq 1} \lambda_2(D^\beta u)^{\kappa+\nu-1} \lesssim \theta^{\kappa+\nu}. \tag{4.12}
\]

Hence, in view of \(4.6\) and \(4.12\) we have
\[
\sum_{1 \leq |\beta| \leq 3+[n/2]} \lambda_1(D^\beta u) \lesssim \| u_0 \|_{H^\nu} + \sum_{\kappa+\nu=m+1}^{M+1} \theta^{\kappa+\nu}. \tag{4.13}
\]

**Case 2.** \( n \geq 3 \) and \( |\beta| = 0 \). By Corollary 3.5 the local Strichartz estimate 3.2 and Hölder’s inequality,
\[
\lambda_1(\mathcal{F} u) \lesssim \| S(t)u_0 \|_{L^\infty_t L^2_x(\mathbb{R} \times Q_\alpha)} + \| \mathcal{A} F(u, \nabla u) \|_{L^\infty_t L^2_x(\mathbb{R} \times Q_\alpha)} \lesssim \| u_0 \|_2 + \sum_{\alpha \in \mathbb{Z}^n} \| F(u, \nabla u) \|_{L^1_t L^2_x(\mathbb{R} \times Q_\alpha)}
\]

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\[ \lesssim \| u_0 \|_2 + \sum_{\lambda_{n,\alpha} \in \mathbb{Z}^n} \sum_{\alpha} \| u^\kappa u_{x_1}^{\nu_1} \ldots u_{x_n}^{\nu_n} \|_{L^1_t L^2_x(\mathbb{R} \times Q_\alpha)} \]

\[ \lesssim \| u_0 \|_2 + \sum_{\kappa+\nu=m+1} \sum_{\alpha} \sup_{|t| \leq 1} \| D^\gamma u \|_{L^2_{t,x}(\mathbb{R} \times Q_\alpha)} \]

\[ \times \sum_{|\beta| \leq 1} \sum_{\alpha} \| D^\beta u \|_{L^{2\nu+1}(\mathbb{R} \times Q_\alpha)} \]

\[ \lesssim \| u_0 \|_2 + \sum_{\kappa+\nu=m+1} \sum_{\alpha} \lambda_1(D^\gamma u) \sum_{|\beta| \leq 1} \lambda_i(D^\beta u)^{\kappa+\nu-1} \]

\[ \lesssim \| u_0 \|_2 + \sum_{\kappa+\nu=m+1} \varrho^{\kappa+\nu}. \] (4.14)

Case 3. \( n = 2, \| \beta \| = 0 \). By Propositions 3.2 and 3.3, we have

\[ \lambda_1(\mathcal{F} u) \lesssim \| u_0 \|_{\dot{H}^{-1/2}} + \sum_{\alpha} \| F(u, \nabla u) \|_{L^1_t L^2_x(\mathbb{R} \times Q_\alpha)}. \] (4.15)

Using the same way as in Case 2, we have

\[ \lambda_1(\mathcal{F} u) \lesssim \| u_0 \|_{\dot{H}^{-1/2}} + \sum_{\kappa+\nu=m+1} \varrho^{\kappa+\nu}. \] (4.16)

**Step 2.** We consider the estimates of \( \lambda_2(D^\beta \mathcal{F} u), \| \beta \| \leq 1 \). Using the estimates of the maximal function as in Proposition 2.5, we have for \( |\beta| \leq 1, 0 < \varepsilon \ll 1 \),

\[ \lambda_2(D^\beta \mathcal{F} u) \lesssim \| S(t)D^\beta u_0 \|_{L^\infty_t(L^\infty_x(\mathbb{R} \times Q_\alpha))} + \| \mathcal{A} D^\beta F(u, \nabla u) \|_{L^\infty_t(L^\infty_x(\mathbb{R} \times Q_\alpha))} \]

\[ \lesssim \| D^\beta u_0 \|_{H^{n/2+\epsilon}} + \sum_{|\beta| \leq 1} \| D^\beta F(u, \nabla u) \|_{L^1(\mathbb{R}, H^{n/2+\epsilon}(\mathbb{R}))} \]

\[ \lesssim \| u_0 \|_{H^{n/2+1+\epsilon}} + \sum_{|\beta| \leq \lfloor n/2 \rfloor + 2} \sum_{\alpha} \| D^\beta F(u, \nabla u) \|_{L^1_t L^2_x(\mathbb{R} \times Q_\alpha)}. \] (4.17)

Applying the same way as in Step 1, for any \( |\beta| \leq \lfloor n/2 \rfloor + 2 \),

\[ \| D^\beta F(u, \nabla u) \|_{L^2_x(Q_\alpha)} \lesssim \sum_{\kappa+\nu=m+1} \sum_{|\beta| \leq 1} \| D^\beta u \|_{L^\infty_x(B_\alpha)} \sum_{|\gamma| \leq 3+[n/2]} \| D^\gamma u \|_{L^2_t(B_\alpha)}. \] (4.18)

By Hölder’s inequality, we have from (4.18) that

\[ \| D^\beta F(u, \nabla u) \|_{L^1_t L^2_x(\mathbb{R} \times Q_\alpha)} \lesssim \sum_{\kappa+\nu=m+1} \sum_{|\gamma| \leq 3+[n/2]} \| D^\gamma u \|_{L^2_{t,x}(\mathbb{R} \times B_\alpha)} \]

\[ \times \sum_{|\beta| \leq 1} \| D^\beta u \|_{L^{2\nu+1}(\mathbb{R} \times B_\alpha)}. \] (4.19)
Combining (4.17) with (4.20), we obtain that

$$\sum_{\alpha \in \mathbb{Z}^n} \| D^\beta F(u, \nabla u) \|_{L_t^1 L_x^2(\mathbb{R} \times Q_\alpha)}$$

$$\lesssim \sum_{\kappa + \nu = m + 1} \sum_{|\gamma| \leq 3 + [n/2]} \lambda_1(D^\gamma u) \sum_{|\beta| \leq 1} \sum_{\alpha \in \mathbb{Z}^n} \| D^\beta u \|_{L_t^{(\kappa + \nu - 1)} L_x^\infty(\mathbb{R} \times B_\alpha)}}$$

$$\lesssim \sum_{\kappa + \nu = m + 1} \sum_{|\gamma| \leq 3 + [n/2]} \lambda_1(D^\gamma u) \sum_{|\beta| \leq 1} \sum_{\alpha \in \mathbb{Z}^n} \| D^\beta u \|_{(L_t^{2m} L_x^\infty) \cap L_t^\infty(\mathbb{R} \times B_\alpha)}}$$

$$\lesssim \sum_{\kappa + \nu = m + 1} \sum_{|\gamma| \leq 3 + [n/2]} \lambda_1(D^\gamma u) (\lambda_2(D^\beta u)^{\kappa + \nu - 1} + \lambda_3(D^\beta u)^{\kappa + \nu - 1})$$

$$\lesssim \sum_{\kappa + \nu = m + 1} \varrho^{\kappa + \nu}.$$ (4.20)

Combining (4.17) with (4.20), we obtain that

$$\sum_{|\beta| \leq 1} \lambda_2(D^\beta \mathcal{T} u) \lesssim \| u_0 \|_{H^{n/2+1+\epsilon}} + \sum_{\kappa + \nu = m + 1} \varrho^{\kappa + \nu}.$$ (4.21)

**Step 3.** We estimate $\lambda_3(D^\beta \mathcal{T} u), |\beta| \leq 1$. In view of Proposition 2.7, one has that

$$\lambda_3(D^\beta \mathcal{T} u) \lesssim \| S(t) D^\beta u_0 \|_{L_t^\infty L_x^\infty(\mathbb{R} \times Q_\alpha)} + \| D^\beta F(u, \nabla u) \|_{L_t^\infty L_x^\infty(\mathbb{R} \times Q_\alpha)}$$

$$\lesssim \| D^\beta u_0 \|_{H^{n/2-1/m+\epsilon}} + \sum_{|\beta| \leq 1} \| D^\beta F(u, \nabla u) \|_{L_t^1(\mathbb{R}, H^{n/2-1/m+\epsilon}(\mathbb{R}^n))}$$

$$\lesssim \| u_0 \|_{H^{n/2+1}} + \sum_{|\beta| \leq [n/2]+2} \sum_{\alpha \in \mathbb{Z}^n} \| D^\beta F(u, \nabla u) \|_{L_t^1 L_x^2(\mathbb{R} \times Q_\alpha)}.$$ (4.22)

which reduces to the case as in (4.17).

Therefore, collecting the estimates as in Steps 1–3, we have for $n \geq 3$,

$$\sum_{|\beta| \leq 3 + [n/2]} \lambda_1(D^\beta \mathcal{T} u) + \sum_{i=2,3} \sum_{|\beta| \leq 1} \lambda_i(D^\beta \mathcal{T} u) \lesssim \| u_0 \|_{H^s} + \sum_{\kappa + \nu = m + 1} \varrho^{\kappa + \nu},$$ (4.23)

and for $n \geq 2$,

$$\sum_{|\beta| \leq 4} \lambda_1(D^\beta \mathcal{T} u) + \sum_{i=2,3} \sum_{|\beta| \leq 1} \lambda_i(D^\beta \mathcal{T} u) \lesssim \| u_0 \|_{H^{7/2} \cap H^{-1/2}} + \sum_{\kappa + \nu = m + 1} \varrho^{\kappa + \nu}.$$ (4.24)

It follows that for $n \geq 3$, $\mathcal{T} : \mathcal{R}_n \to \mathcal{R}_n$ is a contraction mapping if $\varrho$ and $\| u_0 \|_{H^s}$ are small enough (similarly for $n = 2$).
Before considering the case $s > n/2 + 3/2$, we first establish a nonlinear mapping estimate:

**Lemma 4.2** Let $n \geq 2$, $s > 0$, $K \in \mathbb{N}$. Let $1 \leq p, p_1, q, q_i \leq \infty$ satisfy $1/p = 1/p_1 + (K-1)/p_2$ and $1/q = 1/q_1 + (K-1)/q_2$. We have

$$
\|v_1 \ldots v_K\|_{L_{\infty}^{1,p}(L_1^p L_2^q(\mathbb{R} \times Q_\alpha))} \lesssim \sum_{k=1}^{K} \|v_k\|_{L_{\infty}^{1,p}(L_1^p L_2^q(\mathbb{R} \times Q_\alpha))} \prod_{i \neq k, i=1, \ldots, K} \|v_i\|_{L_{\infty}^{1,p}(L_1^p L_2^q(\mathbb{R} \times Q_\alpha))}. \quad (4.25)
$$

**Proof.** Denote $S_r u = \sum_{j \leq r} \Delta u$. We have

$$
v_1 \ldots v_K = \sum_{r=-1}^{\infty} (S_{r+1} v_1 \ldots S_{r+1} v_k - S_r v_1 \ldots S_r v_K), \quad \text{ (4.26)}
$$

where we assume that $S_{-1} v \equiv 0$. Recall the identity,

$$
\prod_{k=1}^{K} a_k - \prod_{k=1}^{K} b_k = \sum_{k=1}^{K} (a_k - b_k) \prod_{i \leq k-1} b_i \prod_{i \geq k+1} a_i, \quad \text{ (4.27)}
$$

where we assume that $\prod_{i \leq 0} a_i = \prod_{i \geq K+1} 1 = 1$. We have

$$
v_1 \ldots v_K = \sum_{r=-1}^{\infty} \sum_{k=1}^{K} \Delta_{r+1} v_k \prod_{i=1}^{k-1} S_r v_i \prod_{i=k+1}^{K} S_{r+1} v_i. \quad \text{ (4.28)}
$$

Hence, it follows that

$$
\|v_1 \ldots v_K\|_{L_{\infty}^{1,p}(L_1^p L_2^q(\mathbb{R} \times Q_\alpha))} = \sum_{j=0}^{\infty} 2^{sj} \sum_{a \in \mathbb{Z}^n} \|\Delta_j (v_1 \ldots v_K)\|_{L_1^q L_2^p(\mathbb{R} \times Q_\alpha)}
$$

$$
\lesssim \sum_{k=1}^{K} \sum_{j=0}^{\infty} 2^{sj} \sum_{a \in \mathbb{Z}^n} \sum_{r=-1}^{\infty} \left\|\Delta_j (\Delta_{r+1} v_k \prod_{i=1}^{k-1} S_r v_i \prod_{i=k+1}^{K} S_{r+1} v_i)\right\|_{L_1^q L_2^p(\mathbb{R} \times Q_\alpha)}. \quad (4.29)
$$

Using the support property of $\widehat{\Delta_r v}$ and $\widehat{S_r v}$, we see that

$$
\Delta_j (\Delta_{r+1} v_k \prod_{i=1}^{k-1} S_r v_i \prod_{i=k+1}^{K} S_{r+1} v_i) \equiv 0, \quad j > r + C. \quad \text{ (4.30)}
$$

Using the fact $\|f d\mu\|_X \leq \|f\|_X d\mu$, one has that

$$
\sum_{a \in \mathbb{Z}^n} \|\Delta_j f\|_{L_1^q L_2^p(\mathbb{R} \times Q_\alpha)} \leq \sum_{a \in \mathbb{Z}^n} \int_{\mathbb{R}^n} |\mathcal{F}^{-1} \varphi_j(y)| \|f(t, x-y)\|_{L_1^q L_2^p(\mathbb{R} \times Q_\alpha)} dy
$$

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the result follows.

$2$

Let

Collecting (4.29)–(4.31) and using Fubini’s Theorem, we have

Lemma 4.4

$2$

In view of Corollary 3.5 and Propositions 3.1 and 3.6, we have the results, as desired.

Proof. For short, we write

Collecting (4.29)–(4.31) and using Fubini’s Theorem, we have

\[
\|v_1 \ldots v_K\|_{L^p_t L^p_x (\mathbb{R} \times Q_\alpha)}^s \lesssim \sum_{k=1}^K \sum_{r=-1}^{\infty} \sum_{j \leq r + C} 2^{sj} \sum_{\alpha \in \mathbb{Z}^n} \left\| \Delta_j (\Delta_{r+1} v_k) \prod_{i=1}^{k-1} S_r v_i \prod_{i=k+1}^K S_r v_i \right\|_{L^p_t L^p_x (\mathbb{R} \times Q_\alpha)}
\]

the result follows.

For short, we write $\|\nabla u\|_X = \|\partial_{x_1} u\|_X + \ldots + \|\partial_{x_n} u\|_X$.

Lemma 4.3 Let $n \geq 3$. We have for any $s > 0$

\[
\sum_{k=1}^K \|S(t) \nabla^k u_0\|_{L^p_t L^p_x (\mathbb{R} \times Q_\alpha)} \lesssim \|u_0\|_{B^s_{2,1}},
\]

\[
\sum_{k=1}^K \|{\mathcal F} \nabla^k \mathcal{F} \|_{L^p_t L^p_x (\mathbb{R} \times Q_\alpha)} \lesssim \|\mathcal{F}\|_{L^p_t L^p_x (\mathbb{R} \times Q_\alpha)}.
\]

Proof. In view of Corollary 3.5 and Propositions 3.1 and 3.6 we have the results, as desired.

```

Lemma 4.4 Let $n = 2$. We have for any $s > 0$

\[
\sum_{k=1}^K \|S(t) \nabla^k u_0\|_{L^p_t L^p_x (\mathbb{R} \times Q_\alpha)} \lesssim \|u_0\|_{B^s_{2,1} \cap (-1, 1)^s},
\]

\[
\|{\mathcal F} \nabla^k \mathcal{F} \|_{L^p_t L^p_x (\mathbb{R} \times Q_\alpha)} \lesssim \|\mathcal{F}\|_{L^p_t L^p_x (\mathbb{R} \times Q_\alpha)}.
\]

Proof. By Propositions 3.2, 3.3 and 3.6 we have the results, as desired.
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We now continue the proof of Theorem 1.1 and now we consider the general case $s > n/2 + 3/2$. We write
\[
\lambda_1(v) := \sum_{i=0,1} \| \nabla^i v \|_{L^2_{\triangle} L^{\infty}_t L^{s}((0,T) \times \mathbb{R}^n))},
\]
\[
\lambda_2(v) := \sum_{i=0,1} \| \nabla^i v \|_{L^2_{\triangle} L^{\infty}_t (0,T) \times \mathbb{R}^n)}},
\]
\[
\lambda_3(v) := \sum_{i=0,1} \| \nabla^i v \|_{L^2_{\triangle} L^{\infty}_t (0,T) \times \mathbb{R}^n)}},
\]
\[
\mathcal{D} = \{ u : \sum_{i=1,2,3} \lambda_i(v) \leq \theta \}. \tag{4.38}
\]

Note $\lambda$ and $\mathcal{D}$ defined here are different from those in the above. We only give the details of the proof in the case $n \geq 3$ and the case $n = 2$ can be shown in a slightly different way. Let $\mathcal{F}$ be defined as in (4.1). Using Lemma 4.3, we have
\[
\lambda_1(\mathcal{F} u) \lesssim \| u_0 \|_{B^{s,1}_{2,1}} + \| F \|_{L^{1,s-1/2} L^{s}((0,T) \times \mathbb{R}^n)}}, \tag{4.39}
\]
For simplicity, we write
\[
(u)^{\kappa} (\nabla u)^{\nu} = u^{\kappa_1} \bar{u}^{\kappa_2} u^{\nu_1} \bar{u}^{\nu_2} u^{\nu_{2n-1}} \bar{u}^{\nu_{2n}}, \tag{4.40}
\]
$|\kappa| = \kappa_1 + \kappa_2$, $|\nu| = \nu_1 + ... + \nu_{2n}$. By Lemma 4.2, we have
\[
\| (u)^{\kappa} (\nabla u)^{\nu} \|_{L^{1,s-1/2} L^{s}((0,T) \times \mathbb{R}^n))} \lesssim \sum_{i=0,1} \| \nabla^i u \|_{L^{2} L^{\infty}((0,T) \times \mathbb{R}^n)} \sum_{k=0,1} \| \nabla^k u \|_{L^{\infty} L^{\infty}((0,T) \times \mathbb{R}^n)} \| \alpha \| + |\nu| - 1]. \tag{4.41}
\]
Hence, if $u \in \mathcal{D}$, in view of (4.39) and (4.41), we have
\[
\lambda_1(\mathcal{F} u) \lesssim \| u_0 \|_{B^{s,1}_{2,1}} + \sum_{m+1 \leq |\kappa| + |\nu| \leq M+1} \theta^{\kappa + |\nu|}. \tag{4.42}
\]
In view of the estimate for the maximal function as in Proposition 2.5, one has that
\[
\lambda_2(S(t)u_0) \lesssim \| u_0 \|_{B^{s,1}_{2,1}}, \tag{4.43}
\]
and for $i = 0, 1,$
\[
\| \mathcal{A} \nabla^i F \|_{L^{\infty}_t L^{s}((0,T) \times \mathbb{R}^n))} \leq \sum_{j=0}^{\infty} \int_{\mathbb{R}} \| S(t-\tau) (\Delta_j \nabla^i F)(\tau) \|_{L^{s}((0,T) \times \mathbb{R}^n))} \leq \sum_{j=0}^{\infty} \int_{\mathbb{R}} \| (\Delta_j \nabla^i F)(\tau) \|_{L^{s-3/2}((\mathbb{R}^n))} d\tau \leq \sum_{j=0}^{\infty} 2^{(s-1/2)j} \int_{\mathbb{R}} \| (\Delta_j F)(\tau) \|_{L^{2}((\mathbb{R}^n))} d\tau. \tag{4.44}
\]
Hence, by (4.43) and (4.44),
\begin{equation}
\lambda_2(\mathcal{F}u) \lesssim \|u_0\|_{B^{2,1}_2} + \|F\|_{\ell^{1,2}_{\Delta}L^2_1L^2_2([0,T]\times Q_\alpha)},
\end{equation}
(4.45)

Similar to (4.45), in view of Proposition 2.9, we have
\begin{equation}
\lambda_3(\mathcal{F}u) \lesssim \|u_0\|_{B^{2,1}_2} + \|F\|_{\ell^{1,2}_{\Delta}L^2_1L^2_2([0,T]\times Q_\alpha)},
\end{equation}
(4.46)

In view of Lemma 4.2, we have
\begin{equation}
\|\mathcal{A}^\alpha(\nabla u)^\nu\|_{\ell^{1,2}_{\Delta}L^2_1L^2_2([0,T]\times Q_\alpha))} \lesssim \sum_{k=0,1} \|\nabla^k u\|_{\ell^{1,2}_{\Delta}L^2_1L^2_2([0,T]\times Q_\alpha))} \times \sum_{i=0,1} \|\nabla^i u\|_{\ell^{1,2}_{\Delta}L^2_1L^2_2([0,T]\times Q_\alpha))},
\end{equation}
(4.47)

Hence, if \(u \in \mathcal{D}\), we have
\begin{equation}
\lambda_2(\mathcal{F}u) + \lambda_3(\mathcal{F}u) \lesssim \|u_0\|_{B^{2,1}_2} + \sum_{m+1 \leq |\alpha| + |\nu| \leq M+1} \theta^{|\alpha|+|\nu|}.
\end{equation}
(4.48)

Repeating the procedures as in the above, we obtain that there exists \(u \in \mathcal{D}\) satisfying the integral equation \(\mathcal{F}u = u\), which finishes the proof of Theorem 1.1.

\section{5 Proof of Theorem 1.2}

We begin with the following

\textbf{Lemma 5.1} Let \(\mathcal{A}\) be as in (1.19). There exists a constant \(C(T) > 1\) which depends only on \(T\) and \(n\) such that
\begin{equation}
\sum_{i=0,1} \|\mathcal{A}^\alpha(\nabla u)^\nu\|_{\ell^{1,2}_{\Delta}L^2_1L^2_2([0,T]\times Q_\alpha))} \leq C(T) \|F\|_{\ell^{1,2}_{\Delta}L^2_1L^2_2([0,T]\times Q_\alpha))},
\end{equation}
(5.1)

\textbf{Proof.} Using Minkowski’s inequality and Proposition 2.1,
\begin{align}
\|\mathcal{A}^\alpha(\nabla u)^\nu\|_{\ell^{1,2}_{\Delta}L^2_1L^2_2([0,T]\times Q_\alpha))} &\leq \sum_{k \in \mathbb{Z}^n} \left( \sum_{\alpha \in \mathbb{Z}^n} \left( \int_0^T \|S(t-\tau)\Box_k(\nabla^i u)F(\tau)\|_{L^2_1L^2_2([0,T]\times Q_\alpha))} d\tau \right)^2 \right)^{1/2} \\
&\leq \sum_{k \in \mathbb{Z}^n} \int_0^T \left( \sum_{\alpha \in \mathbb{Z}^n} \|S(t-\tau)\Box_k(\nabla^i u)F(\tau)\|_{L^2_1L^2_2([0,T]\times Q_\alpha))}^2 \right)^{1/2} d\tau \\
&\leq \sum_{k \in \mathbb{Z}^n} \int_0^T \|\Box_k(\nabla^i u)F(\tau)\|_{M^{1/2}_{2,1}} d\tau.
\end{align}
(5.2)
It is easy to see that for $i = 0, 1$,
\[
\|\Box_k \nabla^i F\|_{M_{2,1}^{1/2}} \lesssim \langle k \rangle^{3/2} \|\Box_k F\|_{L^2(\mathbb{R}^n)} \leq \langle k \rangle^{3/2} \sum_{\alpha \in \mathbb{Z}^n} \|\Box_k F\|_{L^2(Q_\alpha)}. \tag{5.3}
\]
By (5.2) and (5.3), we immediately have (5.1).

Proof. In view of Proposition 3.6, we have
\[
\|\nabla \mathcal{A} F\|_{\ell^1_x t^s \ell^\infty_\alpha(t^2 L^2_{t,x}([0,T]\times Q_\alpha))} \lesssim \|F\|_{\ell^1_x t^s \ell^1_\alpha(t^1 L^1_{t,x}([0,T]\times Q_\alpha))}. \tag{5.4}
\]

Proof. In view of Proposition 3.6, we have
\[
\|\nabla \mathcal{A} F\|_{\ell^1_x t^s \ell^\infty_\alpha(t^2 L^2_{t,x}([0,T]\times Q_\alpha))} \lesssim \|F\|_{\ell^1_x t^s \ell^1_\alpha(t^1 L^1_{t,x}([0,T]\times Q_\alpha))}. \tag{5.5}
\]
By Propositions 3.1 and 3.3,
\[
\|\mathcal{A} F\|_{\ell^1_x t^s \ell^\infty_\alpha(t^2 L^2_{t,x}([0,T]\times Q_\alpha))} \lesssim \|F\|_{\ell^1_x t^s \ell^1_\alpha(t^1 L^1_{t,x}([0,T]\times Q_\alpha))} 
\leq T^{1/2} \|F\|_{\ell^1_x t^s \ell^1_\alpha(t^1 L^1_{t,x}([0,T]\times Q_\alpha))}. \tag{5.6}
\]
By (5.5) and (5.6), we immediately have (5.4).

Lemma 5.3 Let $\mathcal{A}$ be as in (1.9). Then we have for $i = 0, 1$,
\[
\|\nabla^i S(t) u_0\|_{\ell^1_x t^s \ell^\infty_\alpha(t^2 L^2_{t,x}([0,T]\times Q_\alpha))} \lesssim \|u_0\|_{M_{2,1}^{i+1/2}}, \quad n \geq 3, \tag{5.7}
\]
\[
\|\nabla^i S(t) u_0\|_{\ell^1_x t^s \ell^\infty_\alpha(t^2 L^2_{t,x}([0,T]\times Q_\alpha))} \lesssim \|u_0\|_{M_{2,1}^{i+1/2} \cap H^{-1/2}}, \quad n = 2. \tag{5.8}
\]

Proof. (5.7) follows from Corollary 3.5. For $n = 2$, by Proposition 3.2, we have the result, as desired.

Lemma 5.4 Let $n \geq 2$, $s > 0$, $L \in \mathbb{N}$, $L \geq 3$. Let $1 \leq p, p_1, q, q_i \leq \infty$ satisfy $1/p = 1/p_1 + (L - 1)/p_2$ and $1/q = 1/q_1 + (L - 1)/q_2$. We have
\[
\|v_1 \ldots v_L\|_{\ell^1_x t^s \ell^\infty_\alpha(t^L L^p_{t,x}([0,T]\times Q_\alpha))} \lesssim \sum_{i=1}^{L} \|v_i\|_{\ell^1_x t^s \ell^\infty_\alpha(t^L L^p_{t,x}([0,T]\times Q_\alpha))} \times \prod_{i \neq L, i=1,\ldots,L} \|v_i\|_{\ell^1_x t^{L-1}_\alpha(t^L L^q_{t,x}([0,T]\times Q_\alpha))}. \tag{5.9}
\]

Proof. Using the identity
\[
v_1 \ldots v_L = \sum_{k_1,\ldots,k_L \in \mathbb{Z}^n} \Box_{k_1} v_1 \ldots \Box_{k_L} v_L \tag{5.10}
\]
and noticing the fact that
\[
\Box_k(\Box_{k_1}v_1\ldots\Box_{k_L}v_L) = 0, \quad |k - k_1 - \ldots - k_L| \geq C(L, n), \quad (5.11)
\]
we have
\[
\begin{align*}
\|v_1\ldots v_L\|_{L^s_tL^q_x(I \times Q_0)} &\leq \sum_{k \in \mathbb{Z}^n} (\langle k \rangle)^s \sum_{\alpha \in \mathbb{Z}^n} \|\Box_k(\Box_{k_1}v_1\ldots\Box_{k_L}v_L)\|_{L^q_tL^s_x(I \times Q_0)} \\
&\leq \sum_{k_1,\ldots,k_n \in \mathbb{Z}^n} \sum_{|k - k_1 - \ldots - k_L| \leq C} (\langle k \rangle)^s \sum_{\alpha \in \mathbb{Z}^n} \|\Box_k(\Box_{k_1}v_1\ldots\Box_{k_L}v_L)\|_{L^q_tL^s_x(I \times Q_0)}. \quad (5.12)
\end{align*}
\]
Similar to (1.31) and noticing the fact that \(\|\mathcal{F}^{-1}\sigma_k\|_{L^1(\mathbb{R}^n)} \lesssim 1\), we have
\[
\begin{align*}
\sum_{\alpha \in \mathbb{Z}^n} \|\Box f\|_{L^q_tL^s_x(I \times Q_0)} &= \sum_{\alpha \in \mathbb{Z}^n} \| (\mathcal{F}^{-1}\sigma_k) \ast f \|_{L^q_tL^s_x(I \times Q_0)} \\
&\leq \int_{\mathbb{R}^n} |(\mathcal{F}^{-1}\sigma_k)(y)| \left( \sum_{\alpha \in \mathbb{Z}^n} \| f(t, x - y) \|_{L^q_tL^s_x(I \times Q_0)} \right) dy \\
&\leq \sup_{y \in \mathbb{R}^n} \sum_{\alpha \in \mathbb{Z}^n} \| f(t, x - y) \|_{L^q_tL^s_x(I \times Q_0)} \| \mathcal{F}^{-1}\sigma_k \|_{L^1(\mathbb{R}^n)} \\
&\lesssim \sum_{\alpha \in \mathbb{Z}^n} \| f \|_{L^q_tL^s_x(I \times Q_0)}. \quad (5.13)
\end{align*}
\]
By (5.12) and (5.13), we have
\[
\begin{align*}
\|v_1\ldots v_L\|_{L^s_tL^q_x(I \times Q_0)} &\leq \sum_{k_1,\ldots,k_n \in \mathbb{Z}^n} \sum_{|k - k_1 - \ldots - k_L| \leq C} (\langle k \rangle)^s \sum_{\alpha \in \mathbb{Z}^n} \|\Box_k(\Box_{k_1}v_1\ldots\Box_{k_L}v_L)\|_{L^q_tL^s_x(I \times Q_0)} \\
&\lesssim \sum_{k_1,\ldots,k_n \in \mathbb{Z}^n} (\langle k_1 \rangle)^s + \ldots + (\langle k_L \rangle)^s \sum_{\alpha \in \mathbb{Z}^n} \|\Box_k(\Box_{k_1}v_1\ldots\Box_{k_L}v_L)\|_{L^q_tL^s_x(I \times Q_0)}. \quad (5.14)
\end{align*}
\]
By Hölder’s inequality,
\[
\begin{align*}
&\sum_{k_1,\ldots,k_n \in \mathbb{Z}^n} \langle k \rangle^s \sum_{\alpha \in \mathbb{Z}^n} \|\Box_k(\Box_{k_1}v_1\ldots\Box_{k_L}v_L)\|_{L^q_tL^s_x(I \times Q_0)} \\
&\leq \sum_{k_1,\ldots,k_n \in \mathbb{Z}^n} \langle k_1 \rangle^s \sum_{\alpha \in \mathbb{Z}^n} \|\Box_k v_1\|_{L^q_tL^p_x(I \times Q_0)} \prod_{i=2}^L \|\Box_k v_i\|_{L^q_tL^p_x(I \times Q_0)} \\
&\leq \|v_1\|_{L^q_tL^p_x(I \times Q_0)} \sum_{k_2,\ldots,k_n \in \mathbb{Z}^n} \sum_{\alpha \in \mathbb{Z}^n} \prod_{i=2}^L \|\Box_k v_i\|_{L^q_tL^p_x(I \times Q_0)} \\
&\leq \|v_1\|_{L^q_tL^p_x(I \times Q_0)} \sum_{k_2,\ldots,k_n \in \mathbb{Z}^n} \prod_{i=2}^L \|\Box_k v_i\|_{L^{q-1}_tL^p_x(I \times Q_0)} \\
&\leq \|v_1\|_{L^q_tL^p_x(I \times Q_0)} \prod_{i=2}^L \|v_i\|_{L^{q-1}_tL^p_x(I \times Q_0)}. \quad (5.15)
\end{align*}
\]
The result follows. \(\square\)
Proof of Theorem 1.2. Denote

\[
\lambda_1(v) = \sum_{i=0,1} \|\nabla^i v\|_{\ell_2^{1.3/2}\ell_2^\infty(L_{t,x}^2([0,T] \times \Omega))},
\]

\[
\lambda_2(v) = \sum_{i=0,1} \|\nabla^i v\|_{\ell_2^3(L_t^\infty([0,T] \times \Omega))}.
\]

Put

\[
\mathcal{D} = \{ u : \lambda_1(u) + \lambda_2(u) \leq \rho \}.
\]

Let \( \mathcal{S} \) be as in (4.4). We will show that \( \mathcal{S} : \mathcal{D} \to \mathcal{D} \) is a contraction mapping. First, we consider the case \( n \geq 3 \). Let \( u \in \mathcal{D} \). By Lemmas 5.2 and 5.3, we have

\[
\lambda_1(\mathcal{S}u) \lesssim \|u_0\|_{M_{2,1}^2} + (T)^{1/2} \|F\|_{\ell_2^{1.3/2}\ell_2^\infty(L_{t,x}^2([0,T] \times \Omega))}.
\]

(5.17)

We use the same notation as in (4.40). We have from Lemma 5.4 that

\[
\|(u)\kappa(\nabla u)\|_{\ell_2^{1.3/2}\ell_2^\infty(L_{t,x}^2([0,T] \times \Omega))} \lesssim \sum_{i=0,1} \|\nabla^i u\|_{\ell_2^{1.3/2}\ell_2^\infty(L_{t,x}^2([0,T] \times \Omega))} \times \sum_{k=0,1} \|\nabla^k u\|_{\ell_2^{1.3/2}\ell_2^\infty(L_{t,x}^2([0,T] \times \Omega))} \lesssim \lambda_1(u)\lambda_2(u)|\kappa|+|\nu|-1 \lesssim \rho^{1.3/2}.
\]

Hence, for \( n \geq 3 \),

\[
\lambda_1(\mathcal{S}u) \lesssim \|u_0\|_{M_{2,1}^2} + \sum_{|\kappa|+|\nu|=m+1} \rho^{1.3/2}.
\]

(5.19)

Next, we consider the estimate of \( \lambda_2(\mathcal{S}u) \). By Lemma 5.1 and Proposition 2.1

\[
\lambda_2(\mathcal{S}u) \lesssim \|u_0\|_{M_{2,1}^2} + C(T) \|F\|_{\ell_2^{1.3/2}\ell_2^\infty(L_{t,x}^2([0,T] \times \Omega))},
\]

(5.20)

which reduces to the estimates of \( \lambda_1(\cdot) \) as in (5.17). Similarly, for \( n = 2 \),

\[
\lambda_1(\mathcal{S}u) + \lambda_2(\mathcal{S}u) \lesssim \|u_0\|_{M_{2,1}^2} + \sum_{|\kappa|+|\nu|=m+1} \rho^{1.3/2}.
\]

(5.21)

Repeating the procedures as in the proof of Theorem 1.1, we can show our results, as desired.

\( \square \)
6 Proof of Theorem 1.3

The proof of Theorem 1.3 follows an analogous way as that in Theorems 1.1 and 1.2 and will be sketched. Put

$$\lambda_1(v) = \sum_{i=0,1} \|\nabla^i v\|_{\ell_{\alpha}^1 (L_t^s, R \times Q_{\alpha})}^{s-1/2} \alpha(\mathbb{L}^2_t, x(R \times Q_{\alpha})),$$

$$\lambda_2(v) = \sum_{i=0,1} \|\nabla^i v\|_{\ell_{\alpha}^m (L_t^\infty, R \times Q_{\alpha})},$$

$$\lambda_3(v) = \sum_{i=0} \|\nabla^i v\|_{\ell_{\alpha}^m (L_t^{2m} L_x^\infty, R \times Q_{\alpha})}.$$

Put

$$\mathcal{D} = \{ u : \lambda_1(u) + \lambda_2(u) + \lambda_3(u) \leq \varrho \}. \quad (6.1)$$

Let \( \mathcal{T} \) be as in (4.4). We show that \( \mathcal{T} : \mathcal{D} \to \mathcal{D} \). We only consider the case \( n \geq 3 \). It follows from Lemma 5.3 and 4.3 that

$$\lambda_1(\mathcal{T} u) \lesssim \|u_0\|_{M_{t,1}^2} + \|F\|_{G^{s-1/2} \ell_{\alpha}^1 (L_t^s, R \times Q_{\alpha})}, \quad (6.2)$$

Using Lemma 5.4 and similar to (5.18), one sees that if \( u \in \mathcal{D} \), then

$$\lambda_1(\mathcal{T} u) \lesssim \|u_0\|_{M_{t,1}^2} + \sum_{m+1 \leq \kappa + |\nu| \leq M+1} \varrho^{\kappa + |\nu|}. \quad (6.3)$$

Using Proposition 2.10 and combining the proof of (4.4)–(4.6), we see that

$$\lambda_2(\mathcal{T} u) + \lambda_3(\mathcal{T} u) \lesssim \|u_0\|_{M_{t,1}^2} + \sum_{m+1 \leq \kappa + |\nu| \leq M+1} \varrho^{\kappa + |\nu|}. \quad (6.4)$$

The left part of the proof is analogous to that of Theorems 1.1 and 1.2 and the details are omitted. \qed

7 Proof of Theorem 1.4

We prove Theorem 1.4 by following some ideas as in Molinet and Ribaud [23] and Wang and Huang [33]. The following is the estimates for the solutions of the linear Schrödinger equation, see [16 23 33]. Recall that \( \triangle_j := \mathcal{F}^{-1} \delta(2^{-j} \cdot) \mathcal{F}, j \in \mathbb{Z} \) and \( \delta(\cdot) \) is as in Section 1.4.

Lemma 7.1 Let \( g \in \mathcal{S}(\mathbb{R}), f \in \mathcal{S}(\mathbb{R}^2), 4 \leq p < \infty \). Then we have

$$\|\triangle_j S(t)g\|_{L_t^p L_x^q} \lesssim \|\triangle_j g\|_{L^2}, \quad (7.1)$$
Then
\[ \| \Delta_j S(t) g \|_{L^p_t L^q_x} \leq 2^{j\left(\frac{1}{2}-\frac{1}{p}\right)} \| \Delta_j g \|_{L^2}, \] (7.2)
\[ \| \Delta_j S(t) g \|_{L^\infty_t L^p_x} \leq 2^{-j/2} \| \Delta_j g \|_{L^2}, \] (7.3)

\[ \| \Delta_j \mathcal{A} f \|_{L^\infty_t L^p_x \cap L^6_{x,t}} \lesssim \| \Delta_j f \|_{L^{6/5}_{x,t}}, \] (7.4)
\[ \| \Delta_j \mathcal{A} f \|_{L^p_t L^\infty_x} \lesssim 2^{j\left(1-\frac{1}{p}\right)} \| \Delta_j f \|_{L^{6/5}_{x,t}}, \] (7.5)
\[ \| \Delta_j \mathcal{A} f \|_{L^\infty_t L^2_x} \lesssim 2^{-j/2} \| \Delta_j f \|_{L^{6/5}_{x,t}}, \] (7.6)

and

\[ \| \Delta_j \mathcal{A} (\partial_x f) \|_{L^\infty_t L^2_x \cap L^6_{x,t}} \lesssim 2^{j/2} \| \Delta_j f \|_{L^1_x L^2_t}, \] (7.7)
\[ \| \Delta_j \mathcal{A} (\partial_x f) \|_{L^p_t L^\infty_x} \lesssim 2^{j/2} 2^{j\left(\frac{1}{2}-\frac{1}{p}\right)} \| \Delta_j f \|_{L^1_x L^2_t}, \] (7.8)
\[ \| \Delta_j \mathcal{A} (\partial_x f) \|_{L^\infty_t L^2_x} \lesssim \| \Delta_j f \|_{L^1_x L^2_t}. \] (7.9)

For convenience, we write for any Banach function space \( X \),
\[ \| f \|_{\ell^s(X)} = \sum_{j \in \mathbb{Z}} 2^{js} \| \Delta_j f \|_X, \quad \| f \|_{\ell^1(X)} := \| f \|_{\ell^1_\Delta(X)}. \]

**Lemma 7.2** Let \( s > 0 \), \( 1 \leq p, p_i, \gamma, \gamma_i \leq \infty \) satisfy
\[ \frac{1}{p} = \frac{1}{p_1} + ... + \frac{1}{p_N}, \quad \frac{1}{\gamma} = \frac{1}{\gamma_1} + ... + \frac{1}{\gamma_N}. \] (7.10)

Then
\[ \| u_1 \ldots u_N \|_{\ell^s_\Delta(L^p_x L^q_t)} \lesssim \| u_1 \|_{\ell^s_\Delta(L^{p_1}_x L^{q_1}_t)} \prod_{i=2}^N \| u_i \|_{\ell^s_\Delta(L^{p_i}_x L^{q_i}_t)} \]
\[ + \| u_2 \|_{\ell^s_\Delta(L^{p_2}_x L^{q_2}_t)} \prod_{i=2, i\neq 1}^{N-1} \| u_i \|_{\ell^s_\Delta(L^{p_i}_x L^{q_i}_t)} \]
\[ + \ldots + \prod_{i=1}^{N-1} \| u_i \|_{\ell^s_\Delta(L^{p_i}_x L^{q_i}_t)} \| u_N \|_{\ell^s_\Delta(L^{p_N}_x L^{q_N}_t)}, \] (7.11)

and in particular, if \( u_1 = \ldots = u_N = u \), then
\[ \| u^N \|_{\ell^s_\Delta(L^p_x L^q_t)} \lesssim \| u \|_{\ell^s_\Delta(L^{p_1}_x L^{q_1}_t)} \prod_{i=2}^N \| u \|_{\ell^s_\Delta(L^{p_i}_x L^{q_i}_t)}. \] (7.12)

Substituting the spaces \( L^p_x L^q_t \) and \( L^{p_i}_x L^{q_i}_t \), respectively, the spaces \( L^{p_0}_x L^{q_0}_t \) and \( L^{q_0}_x L^{p_0}_x \), respectively, (7.11) and (7.12) also holds.
Proof. We only consider the case $N = 2$ and the case $N > 2$ can be handled in a similar way. We have

\[
    u_1 u_2 = \sum_{r = -\infty}^{\infty} \left[ (S_{r+1} u_1)(S_{r+1} u_2) - (S_r u_1)(S_r u_2) \right]
    = \sum_{r = -\infty}^{\infty} \left[ (\Delta_{r+1} u_1)(S_{r+1} u_2) + (S_r u_1)(\Delta_{r+1} u_2) \right],
\]

and

\[
    \Delta_j(u_1 u_2) = \Delta_j \left( \sum_{r \geq j-10} \left[ (\Delta_{r+1} u_1)(S_{r+1} u_2) + (S_r u_1)(\Delta_{r+1} u_2) \right] \right). \tag{7.13}
\]

We may assume, without loss of generality that there is only the first term in the right hand side of (7.14) and the second term can be handled in the same way. It follows from Bernstein’s estimate, Hölder’s and Young’s inequalities that

\[
    \sum_{j \in \mathbb{Z}} 2^{sj} \| \Delta_j(u_1 u_2) \|_{L^p_x L^\gamma_t} \lesssim \sum_{j \in \mathbb{Z}} 2^{sj} \sum_{r \geq j-10} \| (\Delta_{r+1} u_1)(S_{r+1} u_2) \|_{L^p_x L^\gamma_t}
    \lesssim \sum_{j \in \mathbb{Z}} 2^{sj} \sum_{r \geq j-10} \| \Delta_{r+1} u_1 \|_{L^p_x L^\gamma_1} \| S_{r+1} u_2 \|_{L^{p_2}_x L^\gamma_2}
    \lesssim \sum_{j \in \mathbb{Z}} 2^{s(j-r)} \sum_{r \geq j-10} 2^{r s} \| \Delta_{r+1} u_1 \|_{L^p_x L^\gamma_1} \| S_{r+1} u_2 \|_{L^{p_2}_x L^\gamma_2}
    \lesssim \| u_1 \|_{L^p_x (L^\gamma_1)} \| u_2 \|_{L^{p_2}_x (L^\gamma_2)}, \tag{7.15}
\]

which implies the result, as desired. \qed

Remark 7.3 One easily sees that (7.12) can be slightly improved by

\[
    \left\| u^N \right\|_{L^p_x (L^\gamma_1)} \lesssim \left\| u \right\|_{L^p_x (L^\gamma_1)} \prod_{i=2}^{N} \left\| u \right\|_{L^p_x L^\gamma_i}. \tag{7.16}
\]

In fact, from Minkowski’s inequality it follows that

\[
    \| S_r u \|_{L^p_x L^\gamma_1} \lesssim \| u \|_{L^p_x L^\gamma}. \tag{7.17}
\]

From (7.15) and (7.17) we get (7.16).

Proof of Theorem 1.4. We can assume, without loss of generality that

\[
    F(u, \bar{u}, u_x, \bar{u}_x) = \sum_{m+1 \leq n \leq M+1} \lambda_{\kappa \nu} u^\kappa \bar{u}^\nu \tag{7.18}
\]

and the general case can be handled in the same way.
**Step 1.** We consider the case \( m > 4 \). Recall that

\[
\|u\|_X = \sup_{s_m \leq s \leq s_M} \sum_{i=0}^{\infty} \sum_{j \in \mathbb{Z}} \|\partial_x^i \triangle_j u\|_s, \tag{7.19}
\]

\[
\|\triangle_j v\|_s := 2^sj(\|\triangle_j v\|_{L_t^\infty L_x^2} + 2^{j/2}\|\triangle_j v\|_{L_t^\infty L_x^2} + 2^{(s-s_M)j}\|\triangle_j v\|_{L_t^{s_M} L_x^\infty} + 2^{(s-s_M)j}\|\triangle_j v\|_{L_t^s L_x^\infty}). \tag{7.20}
\]

Considering the mapping

\[
\mathcal{T} : u(t) \to S(t)u_0 - i\mathcal{A}F(u, \bar{u}, u_x, \bar{u}_x), \tag{7.21}
\]

we will show that \( \mathcal{T} : X \to X \) is a contraction mapping. We have

\[
\|\mathcal{T}u(t)\|_X \lesssim \|S(t)u_0\|_X + \|\mathcal{A}F(u, \bar{u}, u_x, \bar{u}_x)\|_X. \tag{7.22}
\]

In view of (7.3), (7.4) and (7.6) we have,

\[
\|\partial_x^i \triangle_j S(t)u_0\|_s \lesssim 2^sj\|\partial_x^i \triangle_j u_0\|_2. \tag{7.23}
\]

It follows that

\[
\|S(t)u_0\|_X \lesssim \sup_{s_m \leq s \leq s_M} \sum_{i=0}^{\infty} \sum_{j \in \mathbb{Z}} 2^sj\|\partial_x^i \triangle_j u_0\|_2 \lesssim \|u_0\|_{B_{2,1}^{s_m} \cap \dot{B}_{2,1}^{1+s_M}}. \tag{7.24}
\]

We now estimate \( \|\mathcal{A}F(u, \bar{u}, u_x, \bar{u}_x)\|_X \). We have from (7.3), (7.4) and (7.6) that

\[
\|\triangle_j (\mathcal{A}F(u, \bar{u}, u_x, \bar{u}_x))\|_s \lesssim 2^sj\|\triangle_j F(u, \bar{u}, u_x, \bar{u}_x)\|_{L_t^{s_j/2}}. \tag{7.25}
\]

From (7.7), (7.8) and (7.9) it follows that

\[
\|\triangle_j (\mathcal{A} \partial_x F(u, \bar{u}, u_x, \bar{u}_x))\|_s \lesssim 2^sj2^{j/2}\|\triangle_j F(u, \bar{u}, u_x, \bar{u}_x)\|_{L_t^1 L_x^2}. \tag{7.26}
\]

Hence, from (7.19), (7.25) and (7.26) we have

\[
\|\mathcal{A}F(u, \bar{u}, u_x, \bar{u}_x)\|_X \lesssim \sum_{j \in \mathbb{Z}} 2^{sj}\|\triangle_j F(u, \bar{u}, u_x, \bar{u}_x)\|_{L_t^{s_j/2}} + \sum_{j \in \mathbb{Z}} 2^{sj}2^{j/2}\|\triangle_j F(u, \bar{u}, u_x, \bar{u}_x)\|_{L_t^1 L_x^2} = I + II. \tag{7.27}
\]

Now we perform the nonlinear estimates. By Lemma 7.2.

\[
I \lesssim \sum_{m+1 \leq \kappa + \nu \leq M + 1} \left( \|u\|_{L_t^{1, \kappa} (L_x^\infty)} \|u\|_{L_t^{1, \nu} (L_x^{\kappa + \nu - 1})} \|u_x\|_{L_t^{1, \kappa} (L_x^{\kappa + \nu - 1})} \right)
\]

\[
+ \sum_{m+1 \leq \kappa + \nu \leq M + 1} \left( \|u_x\|_{L_t^{1, \kappa} (L_x^\infty)} \|u\|_{L_t^{1, \nu} (L_x^{\kappa + \nu - 1})} \right)
\]

\[
\lesssim \sum_{m+1 \leq \kappa + \nu \leq M + 1} \left( \sum_{i=0, 1} \|\partial_x^i u\|_{L_t^{1, \kappa} (L_x^\infty)} \right) \left( \sum_{i=0, 1} \|\partial_x^i u\|_{L_t^{1, \nu} (L_x^{\kappa + \nu - 1})} \right). \tag{7.28}
\]
For any $m \leq \lambda \leq M$, we let $\frac{1}{\rho} = \frac{1}{2} - \frac{4}{3\lambda}$. It is easy to see that the following inclusions hold:

$$L_t^{\infty}(\mathbb{R}, \dot{H}^{s_0}) \cap L_t^6(\mathbb{R}, \dot{H}^{s_0}_6) \subset L_t^{3\lambda/2}(\mathbb{R}, \dot{H}^{s_0}_\rho) \subset L_t^{3\lambda/2}(\mathbb{R}, \dot{H}^{s_0}_\rho).$$

(7.29)

More precisely, we have

$$\sum_{j \in \mathbb{Z}} \| \nabla_j u \|_{L_t^{3\lambda/2}(\mathbb{R}, \dot{H}^{s_0}_\rho)} \lesssim \sum_{j \in \mathbb{Z}} \| \nabla_j u \|_{L_t^{3\lambda/2}(\mathbb{R}, \dot{H}^{s_0}_\rho)} \lesssim \sum_{j \in \mathbb{Z}} \| \nabla_j u \|_{L_t^{4/\lambda}(\mathbb{R}, \dot{H}^{s_0}_\rho)} \| \nabla_j u \|_{L_t^{4/\lambda}(\mathbb{R}, \dot{H}^{s_0}_\rho)} \lesssim \| u \|_{\ell^{1,s_0}(\mathbb{R}, \dot{H}^{s_0}_\rho)}^2.$$  

(7.30)

Using (7.30) and noticing that $s_m \leq s_{\kappa + \nu - 1} \leq s_M < \bar{s}_M$, we have

$$\| \partial^i_x u \|_{\ell^1_x(L_t^{s_{\kappa + \nu - 1} + 1/2}(\mathbb{R}, \dot{H}^{s_0}_\rho))} \lesssim \| \partial^i_x u \|_{\ell^1_x(L_t^{s_{\kappa + \nu - 1} + 1/2}(\mathbb{R}, \dot{H}^{s_0}_\rho))} \| \partial^i_x u \|_{\ell^1_x(L_t^{s_{\kappa + \nu - 1} + 1/2}(\mathbb{R}, \dot{H}^{s_0}_\rho))} \lesssim \| u \|_{X}^{\kappa + \nu - 1}.$$  

(7.31)

Combining (7.28) with (7.31), we have

$$I \lesssim \sum_{m + 1 \leq \kappa + \nu \leq M + 1} \| u \|^{\kappa + \nu}_{X}.$$  

(7.32)

Now we estimate $II$. By Lemma 7.2,

$$II \lesssim \sum_{m + 1 \leq \kappa + \nu \leq M + 1} \left( \| u \|_{\ell^1_x(L_t^{s_{\kappa + \nu - 1} + 1/2}(\mathbb{R}, \dot{H}^{s_0}_\rho))} \| u \|_{\ell^1_x(L_t^{s_{\kappa + \nu - 1} + 1/2}(\mathbb{R}, \dot{H}^{s_0}_\rho))} \| u \|_{\ell^1_x(L_t^{s_{\kappa + \nu - 1} + 1/2}(\mathbb{R}, \dot{H}^{s_0}_\rho))} \right)$$

$$\lesssim \sum_{m + 1 \leq \kappa + \nu \leq M + 1} \left( \sum_{i = 0} \| \partial^i_x u \|_{\ell^1_x(L_t^{s_{\kappa + \nu - 1} + 1/2}(\mathbb{R}, \dot{H}^{s_0}_\rho))} \right) \leq \sum_{m + 1 \leq \kappa + \nu \leq M + 1} \| u \|_{X}^{\kappa + \nu}.$$  

(7.33)

Collecting (7.27), (7.28), (7.32) and (7.33), we have

$$\| \mathcal{A} F(u, \tilde{u}, u_x, \tilde{u}_x) \|_{X} \lesssim \sum_{m + 1 \leq \kappa + \nu \leq M + 1} \| u \|^{\kappa + \nu}_{X}.$$  

(7.34)

By (7.22), (7.24) and (7.34),

$$\| \mathcal{T} u(t) \|_{X} \lesssim \| u_0 \|_{\ell^1_{\rho_m} \cap L_t^{s_{\kappa + \nu}}} + \sum_{m + 1 \leq \kappa + \nu \leq M + 1} \| u \|_{X}^{\kappa + \nu}.$$  

(7.35)
Step 2. We consider the case \( m = 4 \). Recall that

\[
\| u \|_X = \sum_{i=0,1} \left( \| \partial_x^i u \|_{L^\infty_t L^2_x \cap L^6_x, t} + \sup_{s_5 \leq s \leq s_M} \sum_{j \in \mathbb{Z}} \| \partial_x^j \Delta_j u \|_s \right).
\]

By (7.1), (7.2) and (7.3),

\[
\| S(t) u_0 \|_X \lesssim \| u_0 \|_2 + \sup_{s_5 \leq s \leq s_M} \sum_{i=0,1} \sum_{j \in \mathbb{Z}} 2^{s j} \| \partial_x^j \Delta_j u_0 \|_2 \lesssim \| u_0 \|_{B^{1+\delta,M}_{2,1}}. \tag{7.36}
\]

We now estimate \( \mathcal{A} F(u, \bar{u}, u_x, \bar{u}_x) \|_X \). By Strichartz’ and Hölder’s inequality, we have

\[
\| \mathcal{A} F(u, \bar{u}, u_x, \bar{u}_x) \|_{L^\infty_t L^2_x \cap L^6_x, t} \lesssim \sum_{5 \leq \kappa + \nu \leq M+1} \| (|u| + |u_x|)^{\kappa+\nu} \|_{L^{6/5}_x, t}^{\kappa+\nu}.
\]

\[
\lesssim \sum_{5 \leq \kappa + \nu \leq M+1} \| (|u| + |u_x|) \|_{L^6_x, t} \| (|u| + |u_x|) \|_{L^{6/(\kappa+\nu-1)}_x, t}^{\kappa+\nu-1}.
\]

Applying (7.30), we see that (7.37) implies that

\[
\| \mathcal{A} F(u, \bar{u}, u_x, \bar{u}_x) \|_{L^\infty_t L^2_x \cap L^6_x, t} \lesssim \sum_{5 \leq \kappa + \nu \leq M+1} \| u \|^{\kappa+\nu}_X. \tag{7.38}
\]

From Bernstein’s estimate and (7.7) it follows that

\[
\| \partial_x \mathcal{A} F(u, \bar{u}, u_x, \bar{u}_x) \|_{L^\infty_t L^2_x \cap L^6_x, t} \lesssim \| \mathcal{A} F(u, \bar{u}, u_x, \bar{u}_x) \|_{L^\infty_t L^2_x \cap L^6_x, t} \lesssim \| \mathcal{A} F(u, \bar{u}, u_x, \bar{u}_x) \|_{L^\infty_t L^2_x \cap L^6_x, t} + \sum_{j \geq 1} 2^{j/2} \| \Delta_j F(u, \bar{u}, u_x, \bar{u}_x) \|_{L^1_x L^2_t} \lesssim \| \mathcal{A} F(u, \bar{u}, u_x, \bar{u}_x) \|_{L^\infty_t L^2_x \cap L^6_x, t} + \sum_{j \geq 2} 2^{j/2} \| \Delta_j F(u, \bar{u}, u_x, \bar{u}_x) \|_{L^1_x L^2_t} = \text{III} + \text{IV}. \tag{7.39}
\]

The estimates of \( \text{III} \) and \( \text{IV} \) have been given in (7.38) and (7.33), respectively. We have

\[
\| \partial_x \mathcal{A} F(u, \bar{u}, u_x, \bar{u}_x) \|_{L^\infty_t L^2_x \cap L^6_x, t} \lesssim \sum_{5 \leq \kappa + \nu \leq M+1} \| u \|^{\kappa+\nu}_X. \tag{7.40}
\]

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We have from (7.4)–(7.6), (7.7)–(7.9) that
\[
\sum_{j \in \mathbb{Z}} \| \triangle_j (\mathcal{A} F(u, \bar{u}, u_x, \bar{u}_x)) \|_s \lesssim \sum_{j \in \mathbb{Z}} 2^{sj} \| \triangle_j F(u, \bar{u}, u_x, \bar{u}_x) \|_{L^{6/5}_{x,t}} \tag{7.41}
\]
\[
\sum_{j \in \mathbb{Z}} \| \Delta_j (\mathcal{A} \partial_x F(u, \bar{u}, u_x, \bar{u}_x)) \|_s \lesssim \sum_{j \in \mathbb{Z}} 2^{sj} 2^{j/2} \| \Delta_j F(u, \bar{u}, u_x, \bar{u}_x) \|_{L^1_x L^2_t} \tag{7.42}
\]
hold for all \(s > 0\). The right hand side in (7.42) has been estimated by (7.33).

So, it suffices to consider the estimate of the right hand side in (7.41). Let us observe the equality
\[
F(u, \bar{u}, u_x, \bar{u}_x) = \sum_{\kappa + \nu = 5} \lambda_{\kappa \nu} u^\kappa u_x^\nu + \sum_{5 < \kappa + \nu \leq M + 1} \lambda_{\kappa \nu} u^\kappa u_x^\nu := V + VI. \tag{7.43}
\]
For any \(s_5 \leq s \leq \tilde{s}_M\), \(VI\) has been handled in (7.28)–(7.32):
\[
\sum_{5 < \kappa + \nu \leq M + 1} \sum_{j \in \mathbb{Z}} 2^{sj} \| \Delta_j (u^\kappa u_x^\nu) \|_{L^{5/3}_{x,t}} \lesssim \sum_{5 < \kappa + \nu \leq M + 1} \| u \|_{X}^{\kappa + \nu}. \tag{7.44}
\]
For the estimate of \(V\), we use Remark 7.3 for any \(s_5 \leq s \leq \tilde{s}_M\),
\[
\sum_{\kappa + \nu = 5} \sum_{j \in \mathbb{Z}} 2^{sj} \| \Delta_j (u^\kappa u_x^\nu) \|_{L^{6/5}_{x,t}} \lesssim \left( \sum_{i = 0, 1} \| \partial_x^i u_x \|_{L^6_{x,t}}^{4} \right) \left( \sum_{i = 0, 1} \| \partial_x^i u_x \|_{L^1_{x,t}} \right) \lesssim \| u \|_{X}^5. \tag{7.45}
\]
Summarizing the estimate above,
\[
\| \mathcal{T} u(t) \|_X \lesssim \| u_0 \|_{B^{1+\delta}_{2,1}} + \sum_{5 \leq \kappa + \nu \leq M + 1} \| u \|_{X}^{\kappa + \nu}, \tag{7.46}
\]
whence, we have the results, as desired. \(\Box\)

**Acknowledgment.** This work is supported in part by the National Science Foundation of China, grants 10571004 and 10621061; and the 973 Project Foundation of China, grant 2006CB805902.

**References**

[1] J. Bergh and J. Löfström, Interpolation Spaces, Springer–Verlag, 1976.

[2] I. Bejenaru and D. Tataru, Large data local solutions for the derivative NLS equation, arXiv:math.AP/0610092 v1.

[3] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations, GAFA, 3 (1993), 107 - 156 and 209 - 262.
[4] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, A refined global well-posedness result for the Schrödinger equation with derivative, SIAM J. Math. Anal., 34 (2002), 64–86.

[5] P. Constantin and J. C. Saut, Local smoothing properties of dispersive equations, J. Amer. Math. Soc., 1 (1988), 413–446.

[6] H. Chihara, Global existence of small solutions to semilinear Schrödinger equations with gauge invariance, Publ. RIMS, 31 (1995), 731–753.

[7] H. Chihara, The initial value problem for cubic semilinear Schrödinger equations with gauge invariance, Publ. RIMS, 32 (1996), 445–471.

[8] H. Chihara, Gain of regularity for semilinear Schrödinger equations, Math. Ann. 315 (1999), 529-567.

[9] M. Christ, Illposedness of a Schrödinger equation with derivative regularity, Preprint.

[10] E. Cordero and F. Nicola. Strichartz estimates in Wiener amalgam spaces for the Schrödinger equation. Math. Nachr., 281 (2008), 25–41.

[11] E. Cordero, F. Nicola, Metaplectic representation on Wiener amalgam spaces and applications to the Schrödinger equation, J. Funct. Anal., 254 (2008), 506-534.

[12] H. G. Feichtinger, Modulation spaces on locally compact Abelian group, Technical Report, University of Vienna, 1983. Published in: “Proc. Internat. Conf. on Wavelet and Applications”, 99–140. New Delhi Allied Publishers, India, 2003. http://www.unive.at/nuhag-php/bibtex/open_files/fe03-1_modspa03.pdf.

[13] K. Gröchenig, Foundations of Time-Frequency Analysis, Birkhäuser, Boston, MA, 2001.

[14] M. Keel and T. Tao, Endpoint Strichartz estimates, Amer. J. Math., 120 (1998), 955-980.

[15] A. Grünrock On the Cauchy- and periodic boundary value problem for a certain class of derivative nonlinear Schrödinger equations, arXiv:math/0006195v1.

[16] C. E. Kenig, G. Ponce and L. Vega, Oscillatory integrals and regularity of dispersive equations, Indiana Univ. Math. J., 40 (1991), 253–288.

[17] C. E. Kenig, G. Ponce, L. Vega, Small solutions to nonlinear Schrodinger equation, Ann. Inst. Henri Poincaré, Sect C, 10 (1993), 255-288.

[18] C. E. Kenig, G. Ponce and L. Vega, Smoothing effects and local existence theory for the generalized nonlinear Schrödinger equations, Invent. Math., 134 (1998), 489–545.

[19] C. E. Kenig, G. Ponce, L. Vega, The Cauchy problem for quasi-linear Schrödinger equations, Invent. Math. 158 (2004), 343–388.
[20] C. E. Kenig, G. Ponce, C. Rolvent, L. Vega, The general quasilinear
untrahyperbolic Schrodinger equation, Advances in Mathematics 206 (2006),
402–433.

[21] S. Klainerman, Long-time behavior of solutions to nonlinear evolution
equations, Arch. Rational Mech. Anal., 78 (1982), 73–98.

[22] S. Klainerman, G. Ponce, Global small amplitude solutions to nonlinear
evolution equations, Commun. Pure Appl. Math., 36 (1983), 133–141.

[23] L. Molinet and F. Ribaud, Well posedness results for the generalized Benjamin-
Ono equation with small initial data, J. Math. Pures Appl., 83 (2004), 277–311.

[24] L. Molinet, J.D.Saut and N.Tzvetkov, Ill-posedness issues for the Benjamin-Ono
equation and related equations, SIAM J.Math. Analysis, 33 (2001), 982-988.

[25] T. Ozawa and J. Zhang, Global existence of small classical solutions to nonlinear
Schrödinger equations, Ann. I. H. Poincaré, AN, to appear.

[26] P. Sjölin, Regularity of solutions to the Schrödinger equations, Duke Math. J.,
55 (1987), 699–715.

[27] M. Sugimoto and N. Tomita, The dilation property of modulation spaces and
their inclusion relation with Besov spaces, Preprint.

[28] T. Tao, Spherically averaged endpoint Strichartz estimates for the two-
dimensional Schrödinger equation, Commun. PDE, 25 (2000), 1471–1485.

[29] J. Toft, Continuity properties for modulation spaces, with applications to
pseudo-differential calculus, I. J. Funct. Anal., 207 (2004), 399–429.

[30] H. Triebel, Theory of Function Spaces, Birkhäuser–Verlag, 1983.

[31] Baoxiang Wang, Lifeng Zhao and Boling Guo, Isometric decomposition
operators, function spaces $E^{\lambda}_{p,q}$ and applications to nonlinear evolution
equations, J. Funct. Anal., 233 (2006), 1–39.

[32] Baoxiang Wang and Henryk Hudzik, The global Cauchy problem for the NLS
and NLKG with small rough data, J. Differential Equations, 231 (2007), 36–73.

[33] Baoxiang Wang and Chunyan Huang, Frequency-uniform decomposition
method for the generalized BO, KdV and NLS equations, J. Differential
Equations, 239 (2007), 213–250.

[34] L. Vega, The Schrödinger equation: pointwise convergence to the initial data,
Proc. Amer. Math. Soc., 102 (1988), 874–878.