A CONFORMING DISCONTINUOUS GALERKIN FINITE ELEMENT METHOD

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Abstract. A new finite element method with discontinuous approximation is introduced for solving second order elliptic problem. Since this method combines the features of both conforming finite element method and discontinuous Galerkin (DG) method, we call it conforming DG method. While using DG finite element space, this conforming DG method maintains the features of the conforming finite element method such as simple formulation and strong enforcement of boundary condition. Therefore, this finite element method has the flexibility of using discontinuous approximation and simplicity in formulation of the conforming finite element method. Error estimates of optimal order are established for the corresponding discontinuous finite element approximation in both a discrete $H^1$ norm and the $L^2$ norm. Numerical results are presented to confirm the theory.

Key words. weak Galerkin, discontinuous Galerkin, finite element methods, second order elliptic problem

AMS subject classifications. Primary, 65N15, 65N30, 76D07; Secondary, 35B45, 35J50

1. Introduction. For the sake of clear presentation, we consider Poisson equation with Dirichlet boundary condition in two dimension as our model problem. This conforming DG method can be extended to solve other elliptic problems. The Poisson problem seeks an unknown function $u$ satisfying

\begin{align}
-\Delta u &= f, \quad \text{in } \Omega, \\
u &= g, \quad \text{on } \partial \Omega,
\end{align}

where $\Omega$ is a polytopal domain in $\mathbb{R}^2$.

Researchers started to use discontinuous approximation in finite element procedure in the early 1970s [2, 7, 12, 17]. Local discontinuous Galerkin methods were introduced in [6]. Then a paper [1] in 2002 provides a unified analysis of discontinuous Galerkin (DG) finite element methods for Poisson equation. Since then, many new finite element methods with discontinuous approximations have been developed.
such as hybridizable discontinuous Galerkin (HDG) method [5], mimetic finite differences method [10], hybrid high-order (HHO) method [11], virtual element (VE) method [13], weak Galerkin (WG) method [14] and references therein.

The weak form of the problem (1.1)-(1.2) is given as follows: find $u \in H^1(\Omega)$ such that $u = g$ on $\partial\Omega$ and

\[
(\nabla u, \nabla v) = (f, v) \quad \forall v \in H^1_0(\Omega).
\]

(1.3)

The conforming finite element method for the problem (1.1)-(1.2) keeps the same simple form as in (1.3). However, when discontinuous approximation is used, finite element formulations tend to be more complex than (1.3) to ensure connection of discontinuous function across element boundary. For example, the following is the formulation for the symmetric interior penalty discontinuous Galerkin (IPDG) method for the Poisson equation (1.1) with homogeneous boundary condition: find $u_h \in V_h$ such that for all $v_h \in V_h$,

\[
\sum_{T \in T_h} (\nabla u_h, \nabla v_h)_T - \sum_{e \in E_h} \int_e \left( \{\nabla u_h\}[v_h] + \{\nabla v_h\}[u_h] - \alpha h_e^{-1}[u_h][v_h] \right) = (f, v_h),
\]

where $\alpha$ is called a penalty parameter that needs to be tuned.

A first order weakly over-penalized symmetric interior penalty method is proposed in [3] aiming for simplifying the above IPDG formulation by eliminating the two nonsymmetric middle terms: find $u_h \in V_h$ such that for all $v_h \in V_h$,

\[
\sum_{T \in T_h} (\nabla u_h, \nabla v_h)_T + \alpha \sum_{e \in E_h} h_e^{-3}(\Pi_0[u_h], \Pi_0[v_h])_e = (f, v_h),
\]

where $\Pi_0$ is the $L^2$ projection to the constant space and $\alpha$ is a positive number. The price paid for a simpler formulation is a worse condition number for the resulting system of linear equations.

In this paper, we propose a new conforming DG method using the same finite element space used in the IPDG method for any polynomial degree $k \geq 1$ but having a simple symmetric and positive definite system: find $u_h \in V_h$ satisfying $u_h = I_h g$ on $\partial\Omega$ and

\[
(\nabla_w u_h, \nabla_w v_h) = (f, v_h) \quad \forall v_h \in V^0_h,
\]

(1.4)

where $\nabla_w$ is called weak gradient introduced in the weak Galerkin finite element method [14, 15]. It follows from (1.4) that the conforming DG method can be obtained from the conforming formulation simply by replacing $\nabla$ by $\nabla_w$ and enforcing the
boundary condition strongly. The simplicity of the conforming DG formulation will ease the complexity for implementation of DG methods. The computation of weak gradient $\nabla w v$ is totally local. Optimal convergence rates for the conforming DG approximation are obtained in a discrete $H^1$ norm and in the $L^2$ norm. This new conforming DG method is tested numerically for $k = 1, 2, 3, 4$ and $5$, and the results confirm the theory.

2. Finite Element Method. In this section, we will introduce the conforming DG method. For any given polygon $D \subseteq \Omega$, we use the standard definition of Sobolev spaces $H^s(D)$ with $s \geq 0$. The associated inner product, norm, and semi-norms in $H^s(D)$ are denoted by $(\cdot, \cdot)_{s,D}$, $\| \cdot \|_{s,D}$, and $| \cdot |_{s,D}$, respectively. When $s = 0$, $H^0(D)$ coincides with the space of square integrable functions $L^2(D)$. In this case, the subscript $s$ is suppressed from the notation of norm, semi-norm, and inner products. Furthermore, the subscript $D$ is also suppressed when $D = \Omega$.

Let $T_h$ be a triangulation of the domain $\Omega$ with mesh size $h$ that consists of triangles. Denote by $E_h$ the set of all edges in $T_h$, and let $E^0_h = E_h \setminus \partial \Omega$ be the set of all interior edges.

We define the average and the jump on edges for a scalar-valued function $v$. For an interior edge $e \in E^0_h$, let $T_1$ and $T_2$ be two triangles sharing $e$. Let $n_1$ and $n_2$ be the two unit outward normal vectors on $e$, associated with $T_1$ and $T_2$, respectively. Define the average $\{ \cdot \}$ and the jump $[\cdot]$ on $e$ by

\begin{align}
\{ v \} &= \frac{1}{2} (v|_{T_1} + v|_{T_2}) \quad \text{and} \quad [v] = v|_{T_1} n_1 + v|_{T_2} n_2,
\end{align}

respectively. If $e$ is a boundary edge, then

\begin{align}
\{ v \} &= v, \quad [v] = v n.
\end{align}

For simplicity, we adopt the following notations,

$$
(v, w) = (v, w)_{T_h} = \sum_{T \in T_h} (v, w)_T = \sum_{T \in T_h} \int_T v w \, dx,
$$

$$
\langle v, w \rangle_{\partial T_h} = \sum_{T \in T_h} \langle v, w \rangle_{\partial T} = \sum_{T \in T_h} \int_{\partial T} v w \, ds.
$$

First we define two discontinuous finite element spaces for $k \geq 1$,

\begin{align}
V_h = \{ v \in L^2(\Omega) : v|_T \in P_k(T), \ T \in T_h \}.
\end{align}
Algorithm 1. A conforming DG finite element method for the problem (1.1)-(1.2) seeks \( u_h \in V_h \) satisfying \( u_h = I_k g \) on \( \partial \Omega \) and

\[
(\nabla_w u_h, \nabla_w v)_T = (f, v) \quad \forall v \in V_h^0, \tag{2.5}
\]

where \( I_k \) is the \( k \)-th order Lagrange interpolation.

Next we will discuss how to compute weak gradient \( \nabla_w u_h \) and \( \nabla_w v \) in (2.5). The concept of weak gradient \( \nabla_w \) was first introduced in [14, 15] for weak functions in WG methods and was modified in [16, 8] for the functions in \( V_h \) in (2.3) as follows. For a given \( T \in T_h \) and a function \( v \in V_h \), the weak gradient \( \nabla_w v \in RT_k(T) \) on \( T \) is the unique solution of the following equation,

\[
(\nabla_w v, \tau)_T = -(v, \nabla \cdot \tau)_T + \langle \{v\}, \tau \cdot n \rangle_{\partial T}, \quad \forall \tau \in RT_k(T), \tag{2.6}
\]

where \( RT_k(T) = [P_k(T)]^2 + xP_k(T) \) and \( \{v\} \) is defined in (2.1) and (2.2). The weak gradient \( \nabla_w \) is a local operator computed at each element.

3. Well Posedness. We start this section by introducing two semi-norms \( |v| \) and \( \|v\|_{1,h} \) for any \( v \in V_h \) as follows:

\[
|v|^2 = \sum_{T \in T_h} (\nabla_w v, \nabla_w v)_T, \tag{3.1}
\]
\[
\|v\|^2_{1,h} = \sum_{T \in T_h} \|\nabla v\|_T^2 + \sum_{e \in E_h^0} h_e^{-1} \|v\|_e^2. \tag{3.2}
\]

The following norm equivalences is proved in Lemma 3.2 [9] with \( v_0 = v \) and \( v_b = \{v\} \) that there exist two constants \( C_1 \) and \( C_2 \) independent of \( h \) such that

\[
C_1 \|v\|_{1,h} \leq |v| \leq C_2 \|v\|_{1,h}, \quad \forall v \in V_h^0. \tag{3.3}
\]

Lemma 3.1. The semi-norm \( |v| \) defined in (3.1) is a norm in \( V_h^0 \).

Proof. We only need to prove \( v = 0 \) if \( |v| = 0 \) for all \( v \in V_h^0 \). Let \( v \in V_h^0 \) and \( |v| = 0 \). By (3.3), we have \( \|v\|_{1,h} = 0 \) which implies that \( \nabla v = 0 \) in each \( T \in T_h \) and \( v = 0 \) on \( \partial e \in E_h^0 \). \( \nabla v = 0 \) on \( T \) implies that \( v \) is a constant on each \( T \). \( |v| = 0 \) on \( e \)
means that \( v \) is continuous. Thus \( v \) is a global constant on the whole domain. With \( v = 0 \) on \( \partial \Omega \), we conclude \( v = 0 \). This completes the proof of the lemma. \( \square \)

The well posedness of the conforming DG method follows immediately from the above lemma.

4. Error Equation. In this section, we will derive an error equation which will be used in the convergence analysis. First we define \( H(\text{div}; \Omega) \) space as the set of vector-valued functions on \( \Omega \) which, together with their divergence, are square integrable; i.e.,

\[
H(\text{div}; \Omega) = \left\{ v : v \in [L^2(\Omega)]^2, \nabla \cdot v \in L^2(\Omega) \right\}.
\]

Define an interpolation operator \( Q_h \) for \( \tau \in H(\text{div}, \Omega) \) (see [4]) such that \( Q_h \tau \in H(\text{div}, \Omega), Q_h \tau \in RT_k(T) \) on each \( T \in T_h \), and satisfies:

\[
(\nabla \cdot \tau, v)_T = (\nabla \cdot Q_h \tau, v)_T \quad \forall v \in P_k(T).
\]

**Lemma 4.1.** For any \( \tau \in H(\text{div}, \Omega) \),

\[
-(\nabla \cdot \tau, v)_T = (Q_h \tau, \nabla w v)_T \quad \forall v \in V_h^0.
\]

**Proof.** Since \( \{ v \} = v = 0 \) on \( \partial \Omega \) and \( Q_h \tau \in H(\text{div}, \Omega) \), then

\[
\langle Q_h \tau \cdot n, \{ v \} \rangle_{\partial T_h} = 0.
\]

It follows from (4.1), (2.6) and (4.3) that

\[
-(\nabla \cdot \tau, v)_T = -(\nabla \cdot Q_h \tau, v)_T
= -(\nabla \cdot Q_h \tau, v)_T + \langle \{ v \}, Q_h \tau \cdot n \rangle_{\partial T_h}
= (Q_h \tau, \nabla w v)_T,
\]

which proves the lemma. \( \square \)

Define a continuous finite element space \( \tilde{V}_h \), a subspace of \( V_h \), by

\[
\tilde{V}_h = \{ v \in H^1(\Omega) : v|_T \in P_k(T), \forall T \in T_h \}.
\]

**Lemma 4.2.** For any \( v \in \tilde{V}_h \),

\[
\nabla w v = \nabla v.
\]
Proof. By the definition of the weak gradient (2.4) and integration by parts, we have for any \( \tau \in RT_k(T) \),
\[
(\nabla_w v, \tau)_T = -(v, \nabla \cdot \tau)_T + \langle \{v\}, \tau \cdot n \rangle_{\partial T_T} = -(v, \nabla \cdot \tau)_T + \langle v, \tau \cdot n \rangle_{\partial T_T} = (\nabla v, \tau)_T,
\]
which implies
\[
(\nabla_w v - \nabla v, \tau)_T = 0, \quad \forall \tau \in RT_k(T).
\]
Since \( \nabla_w v - \nabla v \in RT_k(T) \), letting \( \tau = \nabla_w v - \nabla v \) in the above equation gives
\[
\|\nabla_w v - \nabla v\|_T^2 = 0,
\]
which proves the lemma. \( \square \)

Let \( e_h = I_h u - u_h \). Obviously, \( e_h \in V_h^0 \). Recall that \( I_h u \) is the \( k \)th order Lagrange interpolation of \( u \) and then \( I_h u \in \hat{V}_h \). By Lemma 4.2, we have
\[
(4.5)
\]
\[
\nabla w I_h u = \nabla I_h u.
\]

**Lemma 4.3.** Let \( e_h = I_h u - u_h \) be the error of the finite element solution arising from (2.5). Then we have
\[
(4.6)
\]
\[
(\nabla_w e_h, \nabla_w v)_{\mathcal{T}_h} = l(u, v), \quad \forall v \in V_h^0,
\]
where
\[
(4.7)
\]
\[
l(u, v) = (\nabla I_h u - Q_h \nabla u, \nabla_w v)_{\mathcal{T}_h}.
\]

Proof. Testing the equation (1.1) by \( v \in V_h^0 \) gives
\[
(4.8)
\]
\[
- (\nabla \cdot \nabla u, v) = (f, v).
\]
It follows from (4.2) that
\[
(4.9)
\]
\[
(Q_h \nabla u, \nabla_w v)_{\mathcal{T}_h} = (f, v).
\]

Adding \( (\nabla_w I_h u, \nabla_w v)_{\mathcal{T}_h} \) to the both sides of the equation (4.9) and using (4.5) yield
\[
(4.10)
\]
\[
(\nabla_w I_h u, \nabla_w v)_{\mathcal{T}_h} = (f, v) + (\nabla I_h u - Q_h \nabla u, \nabla_w v)_{\mathcal{T}_h},
\]
The difference of (4.10) and (2.6) gives (4.6). We have proved the lemma. \( \square \)
5. **Error Estimates.** In this section, we shall establish optimal order error estimates for $u_h$ in a discrete $H^1$ norm and the $L^2$ norm.

5.1. **An Estimate in a Discrete $H^1$ Norm.** We start this subsection by bounding the term $l(u, v)$ defined in (4.7).

**Lemma 5.1.** Let $u \in H^{k+1}(\Omega)$ and $v \in V_h^0$. Then, the following estimate holds,

$$|l(u, v)| \leq Ch^k|u|_{k+1}\|v\|.$$  

**Proof.** Using the Cauchy-Schwarz inequality and the definition of $I_h$ and $Q_h$, we have

$$l(u, v) = (\nabla I_h u - Q_h(\nabla u), \nabla_w v)_{T_h}$$

$$\leq \sum_{T \in T_h} \|\nabla I_h u - Q_h(\nabla u)\|_T \|\nabla_w v\|_T$$

$$\leq \left( \sum_{T \in T_h} \|\nabla I_h u - Q_h(\nabla u)\|_T^2 \right)^{1/2} \left( \sum_{T \in T_h} \|\nabla_w v\|_T^2 \right)^{1/2}$$

$$\leq \left( \sum_{T \in T_h} \|\nabla I_h u - \nabla u\|_T^2 + \|\nabla u - Q_h(\nabla u)\|_T^2 \right)^{1/2} \|v\|$$

$$\leq Ch^k|u|_{k+1}\|v\|,$$

which proves the lemma. $\blacksquare$

**Theorem 5.2.** Let $u_h \in V_h$ be the finite element solution of (2.5). Assume the exact solution $u \in H^{k+1}(\Omega)$. Then, there exists a constant $C$ such that

$$\|u_h - I_h u\| \leq Ch^k|u|_{k+1}.$$  

**Proof.** Letting $v = e_h$ in (4.6) gives

$$\|e_h\|^2 = l(u, e_h).$$  

Using (5.1), we arrive

$$\|e_h\|^2 \leq Ch^k|u|_{k+1}\|e_h\|,$$

which completes the proof. $\blacksquare$
5.2. An Estimate in the $L^2$ Norm. In this subsection, we will derive the error estimate for $u_h$ in the $L^2$ norm. First we define $\tilde{V}_h^0$ a subspace of $\tilde{V}_h$ in (4.4) as

\begin{equation}
\tilde{V}_h^0 = \{ v \in \tilde{V}_h : v|_{\partial \Omega} = 0 \}.
\end{equation}

Let $\tilde{u}_h \in \tilde{V}_h$ be the conforming finite element solution such that $\tilde{u}_h = I_h g$ on $\partial \Omega$ and satisfies

\begin{equation}
(\nabla \tilde{u}_h, \nabla v) = (f, v) \quad \forall v \in \tilde{V}_h^0.
\end{equation}

Since $\tilde{V}_h^0 \subset V_h^0$, by Lemma 4.2 (2.5) and (5.5), we have

\begin{equation}
(\nabla_w u_h - \nabla \tilde{u}_h, \nabla v) = 0, \quad \forall v \in \tilde{V}_h^0.
\end{equation}

Consider the dual problem: seek $\Phi \in H^1_0(\Omega)$ satisfying

\begin{equation}
-\nabla \cdot (\nabla \Phi) = u_h - \tilde{u}_h \quad \text{in } \Omega.
\end{equation}

Assume that the following $H^2$-regularity holds

\begin{equation}
\| \Phi \|_2 \leq C \| u_h - \tilde{u}_h \|.
\end{equation}

Now we are ready to derive the $L^2$ error estimate.

**Theorem 5.3.** Let $u_h \in V_h$ be the finite element solution of (2.5). Assume that the exact solution $u \in H^{k+1}(\Omega)$ and that (5.8) holds true. Then, there exists a constant $C$ such that

\begin{equation}
\| u - u_h \| \leq C h^{k+1} |u|_{k+1}.
\end{equation}

**Proof.** By the triangle inequality, we have

\begin{equation}
\| u - u_h \| \leq \| u - \tilde{u}_h \| + \| u_h - \tilde{u}_h \|.
\end{equation}

The definition of $\tilde{u}_h$ implies

\begin{equation}
\| u - \tilde{u}_h \| \leq C h^{k+1} |u|_{k+1}.
\end{equation}

Next we will estimate $\| u_h - \tilde{u}_h \|$. Let $\Phi_h \in V_h^0$ be the conforming DG approximation to the problem (5.7) satisfying

\begin{equation}
(\nabla_w \Phi_h, \nabla_w v) = (u_h - \tilde{u}_h, v), \quad \forall v \in V_h^0.
\end{equation}
Letting \( v = u_h - \tilde{u}_h \in V_h^0 \) in (5.12) and using Lemma 4.2 and (5.6), we have,
\[
\| u_h - \tilde{u}_h \|^2 = (\nabla w \Phi_h, \nabla w (u_h - \tilde{u}_h))_{\mathcal{T}_h} = (\nabla_w \Phi_h, \nabla w u_h - \nabla \tilde{u}_h)_{\mathcal{T}_h} \\
= (\nabla_w (\Phi_h - I_h \Phi), \nabla w u_h - \nabla \tilde{u}_h)_{\mathcal{T}_h}.
\]

By the Cauchy-Schwartz inequality, (5.2) and (5.8), then
\[
\| u_h - \tilde{u}_h \|^2 \leq \|
\Phi_h - I_h \Phi\| \left( \| u_h - I_h u \| + \| \nabla (I_h u - \tilde{u}_h) \| \right) \\
\leq C h |\Phi|_2 h |u|_{k+1} \\
\leq C h^{k+1} |u|_{k+1} \| u_h - \tilde{u}_h \|,
\]
which implies
\[
(5.13) \quad \| u_h - \tilde{u}_h \| \leq C h^{k+1} |u|_{k+1}.
\]

Combining (5.11) and (5.13) with (5.10), we have proved the theorem.

6. Numerical Example. We solve the following Poisson equation on the unit square:
\[
(6.1) \quad -\Delta u = 2 \pi^2 \sin \pi x \sin \pi y, \quad (x, y) \in \Omega = (0, 1)^2,
\]
with the boundary condition \( u = 0 \) on \( \partial \Omega \).

In computation, the first grid consists of two unit right triangles cutting from the unit square by a forward slash. The high level grids are the half-size refinement of the previous grid. We apply \( P_k \) finite element methods \( V_h \) and list the error and the order of convergence in the following table. The numerical results confirm the convergence theory.

REFERENCES

[1] D. Arnold, F. Brezzi, B. Cockburn and D. Marini, Unified analysis of discontinuous Galerkin methods for elliptic problems, SIAM J. Numer. Anal., 39 (2002), 1749-1779.
[2] I. Babuška, The finite element method with penalty, Math. Comp., 27 (1973), 221-228.
[3] S. Brenner, L. Owens and L. Sung, A weakly over-penalized symmetric interior penalty method, Ele. Trans. Numer. Anal., 30 (2008), 107-127.
[4] F. Brezzi and M. Fortin, Mixed and Hybrid Finite Elements, Springer-Verlag, New York, 1991.
[5] B. Cockburn, J. Gopalakrishnan, and R. Lazarov, Unified hybridization of discontinuous Galerkin, mixed, and conforming Galerkin methods for second order elliptic problems, SIAM J. Numer. Anal., 47 (2009), 1319-136.
[6] B. Cockburn and C. Shu, The local discontinuous Galerkin finite element method for convection-diffusion systems, SIAM J. Numer. Anal., 35 (1998), 2440-2463.
Table 6.1
Error profiles and convergence rates for (6.1)

| level | $\|u_h - u\|$ rate | $\|u_h - I_h u\|$ rate |
|-------|-----------------|-----------------|
| by $P_1$ elements |
| 6     | 0.7280E-03 2.09 | 0.7199E-01 0.91 |
| 7     | 0.1751E-03 2.06 | 0.3718E-01 0.95 |
| 8     | 0.4287E-04 2.03 | 0.1890E-01 0.98 |
| by $P_2$ elements |
| 6     | 0.6446E-05 2.94 | 0.1744E-02 1.95 |
| 7     | 0.8197E-06 2.98 | 0.4424E-03 1.98 |
| 8     | 0.1033E-06 2.99 | 0.1113E-03 1.99 |
| by $P_3$ elements |
| 6     | 0.4457E-07 4.02 | 0.2293E-04 2.97 |
| 7     | 0.2772E-08 4.01 | 0.2902E-05 2.98 |
| 8     | 0.1730E-09 4.00 | 0.3650E-06 2.99 |
| by $P_4$ elements |
| 5     | 0.2057E-07 5.03 | 0.4748E-05 3.95 |
| 6     | 0.6344E-09 5.02 | 0.3009E-06 3.98 |
| 7     | 0.1984E-10 5.00 | 0.1893E-07 3.99 |
| by $P_5$ elements |
| 4     | 0.2481E-07 6.04 | 0.3223E-05 4.94 |
| 5     | 0.3811E-09 6.02 | 0.1024E-06 4.98 |
| 6     | 0.5938E-11 6.00 | 0.3225E-08 4.99 |

[7] J. Douglas Jr. and T. Dupont, Interior penalty procedures for elliptic and parabolic Galerkin methods, Computing Methods in Applied Sciences, (1976), 207-216.
[8] L. Mu, X. Wang and X. Ye, A modified weak Galerkin finite element method for the Stokes equations, J. Comput. Appl. Math., 275 (2015), 79-90.
[9] L. Mu, J. Wang, Y. Wang and X. Ye, A weak Galerkin mixed finite element method for biharmonic equations, Numerical Solution of Partial Differential Equations: Theory, Algorithms, and Their Applications, 45 (2013), 247-277.
[10] K. Lipnikov, G. Manzini, F. Brezzi and A. Buffa, The mimetic finite difference method for the 3D magnetostatic field problems on polyhedral meshes, J. Comput. Phys., 230 (2011), 305-328.
[11] D. Pietro and A. Ern, Hybrid high-order methods for variable-diffusion problems on general meshes, Comptes Rendus Mathematique, 353 (2015), 31-34.
[12] W. Reed and T. Hill. Triangular mesh methods for the neutron transport equation. Technical Report LA-UR-73-0479, Los Alamos Scientific Laboratory, Los Alamos, NM, 1973.
[13] L. Veiga, F. Brezzi, A. Cangiani, G. Manzini, L. Marini and A. Russo, Basic principles of virtual element methods, Math. Models Methods Appl. Sci., 23 (2013), 119-214.
[14] J. Wang and X. Ye, A weak Galerkin finite element method for second-order elliptic problems. J. Comput. Appl. Math. 241 (2013), 103-115.

[15] J. Wang and X. Ye, A Weak Galerkin mixed finite element method for second-order elliptic problems, Math. Comp., 83 (2014), 2101-2126.

[16] X. Wang, N. Malluwawadu, F Gao and T. McMillan, A modified weak Galerkin finite element method, J. Comput. Appl. Math., 217 (2014), 319-327.

[17] M. Wheeler, An elliptic collocation-finite element method with interior penalties. SIAM J. Numer. Anal., 15 (1978), 152-161.