ON NEARLY RADIAL PRODUCT FUNCTIONS

MICHAEL CHRIST

Abstract. If \( \|f\|_{L^2(\mathbb{R}^d)} = 1 \) and if the function \( f(x)f(y) \) is close in \( L^2 \) norm to a radially symmetric function of \( (x, y) \) then \( f \) is close in \( L^2 \) norm to a centered Gaussian function. A quantitative form of this assertion is established.

1. Statement of principal result

It is well known that if \( f : \mathbb{R}^d \rightarrow \mathbb{C} \) then the function \( (f \otimes f)(x, y) = f(x)f(y) \) with domain \( \mathbb{R}^d \times \mathbb{R}^d \) is radially symmetric if and only if \( f \) is a radial complex Gaussian function, by which mean a function \( G : \mathbb{R}^d \rightarrow \mathbb{C} \) of the form

\[
G(x) = ce^{-\gamma |x|^2}
\]

where \( c, \gamma \in \mathbb{C} \).

In this note we establish a quantitative version of this uniqueness statement.

Denote by \( G \subset L^2(\mathbb{R}^d) \) the set of all square integrable complex radial Gaussian functions. By a radially symmetric function Lebesgue measurable function we mean one of the form \( f(x) = h(|x|) \) almost everywhere. Denote by \( \mathcal{P} : L^2(\mathbb{R}^d \times \mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d \times \mathbb{R}^d) \) the orthogonal projection onto the subspace of all radially symmetric \( L^2 \) functions. For \( f, g \in L^2(\mathbb{R}^d) \), denote by \( f \otimes g \in L^2(\mathbb{R}^d \times \mathbb{R}^d) \) the function

\[
(f \otimes g)(x, y) = f(x)g(y).
\]

Then for nonzero functions \( f, g \), \( \|\mathcal{P}(f \otimes g)\|_2 \leq \|f\|_2\|g\|_2 \) for all \( f, g \in L^2(\mathbb{R}^d) \), with equality if and only if \( f \) is a complex radial Gaussian and \( g \) is a scalar multiple of \( f \), up to redefinition on sets of Lebesgue measure zero.

\( L^2 \times L^2 \) denotes the Hilbert space of all ordered pairs of functions \( (f, g) \) with both \( f, g \in L^2(\mathbb{R}^d) \), with norm squared

\[
\|(f, g)\|_2^2 = \|f\|_2^2 + \|g\|_2^2.
\]

Define \( \mathcal{G}^\times \subset \mathcal{G} \times \mathcal{G} \subset L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \) to be

\[
\mathcal{G}^\times = \{(F, cF) : F \in \mathcal{G} \text{ and } 0 \neq c \in \mathbb{C}\}.
\]

We regard \( L^2 \times L^2 \) as a Hilbert space with norm defined by \( \|(f, g)\|_2^2 = \|f\|_2^2 + \|g\|_2^2 \), of which \( \mathcal{G}^\times \) is a closed subspace. The distance squared in \( L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \) from \( (f, g) \) to \( \mathcal{G}^\times \) is defined by

\[
\text{dist} ((f, g), \mathcal{G}^\times)^2 = \inf_{(F, cF) \in \mathcal{G}^\times} (\|f - F\|_2^2 + \|g - cF\|_2^2).
\]

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Theorem 1.1. For each $d \geq 1$ there exists $c_d > 0$ such that for all $(f, g) \in L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ satisfying $\|f\|_2 = \|g\|_2 = 1$,

$$\|\mathbb{P}(f \otimes g)\|_2 \leq 1 - c_d \text{dist}((f, g), \mathcal{G}^\times)^2.$$  

There exists $C_d < \infty$ such that whenever $0 \neq (f, g) \in L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ satisfy $\|f\|_2 = \|g\|_2 = 1$,

$$\|\mathbb{P}(f \otimes g)\|_2 \leq 1 - \frac{d}{2(d+1)} \text{dist}((f, g), \mathcal{G}^\times)^2 + C_d \text{dist}((f, g), \mathcal{G}^\times)^3.$$  

Other recent papers in which quantitative stability theorems in this spirit are proved, for other inequalities, include [1], [2], [3], [4], [5].

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2. Some notation

The notation $\|f\|$ with no subscript indicates the $L^2$ norm, over either $\mathbb{R}^d$ or $\mathbb{R}^d \times \mathbb{R}^d$, and for functions taking values either in $\mathbb{C}$ or in $\mathbb{C} \times \mathbb{C}$, with respect to Lebesgue measure.

For $r \in \mathbb{R}^+$, denote by $\sigma_r$ the unique probability measure on $S_r = \{z \in \mathbb{R}^d \times \mathbb{R}^d : |z| = r\}$ that is invariant under rotations of $\mathbb{R}^{2d} = \mathbb{R}^d \times \mathbb{R}^d$. For $0 \neq z \in \mathbb{R}^d \times \mathbb{R}^d$,

$$\mathbb{P}(f \otimes g)(z) = \iint f(x)g(y) \, d\sigma_{|z|}(x, y).$$  

Let $\omega_d \in \mathbb{R}^+$ denote the measure of the unit sphere in $\mathbb{R}^{2d}$. For each dimension $d \geq 1$, for any Lebesgue measurable subsets $A, B \subset \mathbb{R}^d$ with finite measures,

$$|A| \cdot |B| = |A \times B| = \omega_d \int_0^\infty \sigma_r(A \times B) \, r^{2d-1} \, dr$$

and

$$\mathbb{P}(1_A \otimes 1_B) = \omega_d \int_0^\infty \sigma_r(A \times B) \, r^{2d-1} \, dr.$$  

For any $E \subset \mathbb{R}^+$, let $A_E = \{z \in \mathbb{R}^d : |z| \in E\}$. Then

$$\langle \mathbb{P}(1_A \otimes 1_B), 1_{A_E} \rangle = \omega_d \int_E \sigma_r(A \times B) \, r^{2d-1} \, dr.$$  

For $E \subset \mathbb{R}^+$ define

$$\mu(E) = |A_E| = \omega_d \int_E r^{2d-1} \, dr.$$  

3. Preliminary lemmas

The orthogonal projection $\mathbb{P}$ is a bounded linear operator, indeed a contraction, from $L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d \times \mathbb{R}^d)$. A stronger form of boundedness will be proved in this section. For $a = (a_1, a_2, a_3) \in (0, \infty)^3$ define

$$\Lambda(a_1, a_2, a_3) = \min_{i \neq j} \frac{a_i}{a_j}.$$
Lemma 3.1. There exists an exponent \( \gamma \in \mathbb{R}^+ \) with the following property. Let \( d \geq 1 \). There exists \( C < \infty \) such that for any Lebesgue measurable sets \( A, B \subset \mathbb{R}^d \) and \( A \subset \mathbb{R}^d \times \mathbb{R}^d \) with positive, finite measures, if \( A \) is radially symmetric then

\[
\mathbb{P}(1_A \otimes 1_B), 1_A) \leq C\Lambda(|A|, |B|, |A|^{1/2}) \gamma \cdot |A|^{1/2}|B|^{1/2}|A|^{1/2}.
\]

This will be a consequence of the next three lemmas. Since \( (f, g) \mapsto f \otimes g \) is an isometry from \( L^2 \times L^2 \) into \( L^2 \), and \( \mathbb{P} \) is a contraction on \( L^2 \), one has \( \mathbb{P}(1_A \otimes 1_B), 1_A) \leq |A|^{1/2}|B|^{1/2}|A|^{1/2} \) for all Lebesgue measurable sets \( A, B \subset \mathbb{R}^d \) and \( A \subset \mathbb{R}^{d+2} \). Lemma 3.1 improves on this trivial bound, unless \( |A|, |B| \) are comparable and \( |A| \) is comparable to \( |A| \cdot |B| \).

Lemma 3.2. \( \sigma_r(A \times B) \leq C \min(1, r^{-d}|A|, r^{-d}|B|)^{1/2} \).

Proof. \( \sigma_r(A \times B) = \sigma(r^{-1}A \times r^{-1}B) \) where \( tE = \{ tx : x \in E \} \). Since \( |r^{-1}E| = r^{-d}|E| \) for \( E \subset \mathbb{R}^d \), it suffices to treat the case \( r = 1 \). It also suffices to treat the case in which \( |A| \leq |B| \). Thus it suffices to show that \( \sigma(A \times \mathbb{R}^d) \leq C|A|^{1/2} \) for any Lebesgue measurable set \( A \subset \mathbb{R}^d \) satisfying \( |A| \leq 1 \).

One has

\[
\sigma(A \times \mathbb{R}^d) = c_d \int_A (1 - |x|^2)^{(d-2)/2} dx.
\]

This gives \( \sigma(A \times \mathbb{R}^d) \leq c|A|^{1/2} \) for \( d = 1 \), and \( \leq C_d|A|^{1} \) for \( d \geq 2 \). \( \square \)

Lemma 3.3. Let \( d \geq 1 \). There exists \( C_d < \infty \) such that for any Lebesgue measurable sets \( A, B \subset \mathbb{R}^d \) with positive, finite measures,

\[
\int_0^\infty \sigma_r(A \times B)^2 r^{2d-1} dr \leq C_d \min(|A|/|B|, |B|/|A|)^{1/5} \cdot |A| \cdot |B|.
\]

Proof. Assume without loss of generality that \( |A| \leq |B| \). Define \( \rho \) by

\[
\rho^d = |A|^{3/5}|B|^{2/5}.
\]

Then

\[
\int_0^\infty \sigma_r(A \times B)^2 r^{2d-1} dr \leq \int_0^{\rho} r^{2d-1} dr + \int_{\rho}^\infty (r^{-d}|A|)^{1/2} \sigma_r(A \times B) r^{2d-1} dr
\]

\[
\leq \rho^{2d} + \rho^{-d/2} |A|^{1/2} \int_0^\infty \sigma_r(A \times B) r^{2d-1} dr
\]

\[
\leq \rho^{2d} + \rho^{-d/2} |A|^{1/2} \cdot |A| \cdot |B|
\]

\[
= 2|A|^{6/5}|B|^{4/5}.
\]

\( \square \)

Lemma 3.4. For any dimension \( d \geq 1 \) there exists \( C_d < \infty \) such that for any Lebesgue measurable sets \( A, B \subset \mathbb{R}^d \) and any radially symmetric Lebesgue measurable set \( A \subset \mathbb{R}^{d+2} \),

\[
\mathbb{P}(1_A \otimes 1_B), 1_A) \leq C_d \min \left( \frac{|A| \cdot |B|}{|A|}, \frac{|A|}{|A| \cdot |B|} \right)^{1/6} |A|^{1/2}|B|^{1/2}|A|^{1/2}.
\]
Corollary 3.5. Let \( A = A_E \) where \( E \subset \mathbb{R}^+ \). Then \( |A| = \mu(E) \) where the measure \( \mu \) is as defined in (2.6). We already know that

\[
\int_E \sigma_r(A \times B) r^{2d-1} dr \leq \int_{\mathbb{R}^d} \sigma_r(A \times B) r^{2d-1} dr = \omega_d^{-1} |A| \cdot |B| \\
\leq C(\mu(E))^{-1/2} \cdot |A|^{1/2} |B|^{1/2} \mu(E)^{1/2}.
\]

This provides a stronger upper bound than stated when \( \mu(E) \geq |A| \cdot |B| \).

Assume without loss of generality that \( |A| \leq |B| \). Set \( E^- = \{ r \in E : r \leq |A|^{1/d} \} \) and \( E^+ = E \setminus E^- \).

\[
\int_E \sigma_r(A \times B) r^{2d-1} dr \leq C \int_E \min(1, r^{-d/2} |A|^{1/2}) r^{2d-1} dr \\
\leq C |A|^{1/2} \int_{E^+} r^{-d/2} r^{2d-1} dr + C \int_{E^-} r^{2d-1} dr \\
= C |A|^{1/2} \int_{|A|^{1/d}} \infty 1_E(r) r^{-d/2} r^{2d-1} dr + C' \mu(E^-)
\]

Apply Hölder’s inequality with exponents \( 3 \) and \( \frac{3}{2} \) to obtain

\[
\int_{|A|^{1/d}} \infty 1_E(r) r^{-d/2} r^{2d-1} dr \leq \left( \int_{|A|^{1/d}} \infty r^{-3d/2} r^{2d-1} dr \right)^{1/3} \left( \int_E r^{2d-1} dr \right)^{2/3} \\
= C |A|^{1/6} \mu(E)^{2/3}
\]

where \( C < \infty \) depends only on the dimension \( d \). If \( \mu(E) \leq |A| \cdot |B| \) we have shown that

\[
(3.5) \quad \int_E \sigma_r(A \times B) r^{2d-1} dr \leq C |A|^{2/3} \mu(E)^{2/3} + C \mu(E) \\
\leq C |A|^{1/3} |B|^{1/3} \mu(E)^{2/3} = (\mu(E) / |A| \cdot |B|)^{1/6} \cdot |A|^{1/2} |B|^{1/2} \mu(E)^{1/2}.
\]

Lemma 3.3 is a straightforward combination of Lemmas 3.3 and 3.4.

Denote by \( L^{p,q} \) the Lorentz spaces, as defined in [7]. The next result is a simple consequence of Lemma 3.4.

Corollary 3.5. For any dimension \( d \geq 1 \) there exists a constant \( C < \infty \) such that for all \( f, g \in L^2(\mathbb{R}^d) \),

\[
(3.6) \quad \| P(f \otimes g) \| \leq C \| f \|_{L^{2,4}} \| g \|_{L^{2,4}}.
\]

The space \( L^{2,4} \) is strictly larger than \( L^2 \), so this strengthens the \( L^2 \otimes L^2 \to L^2 \) boundedness of \( P \).

4. Compactness

In this section we establish a preliminary, nonquantitative formulation of Theorem 1.1. Although this formulation is entirely superseded by the final result, its proof is an essential part of the reasoning.

Proposition 4.1. Let \( d \geq 1 \). For every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for any \( 0 \neq f, g \in L^2(\mathbb{R}^d) \),

\[
(4.1) \quad \| P(f \otimes g) \| \geq (1 - \delta) \| f \| \| g \| \quad \implies \quad \text{dist} (f, \mathcal{G}) \leq \varepsilon \| f \|.
\]
The proof involves a compactness argument and consequently yields no control over the
dependence of \( \delta \) on \( \varepsilon \).

The hypotheses are unchanged under interchange of \( f \) with \( g \), so likewise dist \((g, \mathcal{G})\) \( \leq \varepsilon ||g|| \). A stronger conclusion holds, and will be proved below: There exist a common element \( G \in \mathcal{G} \) and scalars \( a, b \in \mathbb{C} \) such that both \( ||f - aG|| < \varepsilon ||f|| \) and \( ||g - bG|| < \varepsilon ||g|| \).

Proposition \([1.1]\) together with the second conclusion of Theorem \([1.1]\) implies the first conclusion \([1.5]\); the second conclusion \([1.6]\) will be proved in \([6]\) and \([6]\).

An important property of the inequality \( ||P(f \otimes g)|| \leq ||f||||g|| \) is its dilation-invariance. Thus if \( \rho \in \mathbb{R}^+ \) and \( f, g \in L^2(\mathbb{R}^d) \) then the dilated functions \( \tilde{f}(x) = f(\rho x) \) and \( \tilde{g}(x) = g(\rho x) \) satisfy

\[
(4.2) \quad \frac{||P(f \otimes g)||}{||f||||g||} = \frac{||P(\tilde{f} \otimes \tilde{g})||}{||\tilde{f}||||\tilde{g}||}.
\]

**Lemma 4.2.** Let \( d \geq 1 \). There exists a continuous function \( \Theta : \mathbb{R}^+ \to \mathbb{R}^+ \) satisfying

\[
\lim_{t \to 0} \Theta(t) = 0
\]

with the following property. For any \( \delta > 0 \), \( t \in (0, 1] \), and any \( f, g \in L^2(\mathbb{R}^d) \) that satisfy \( ||P(f \otimes g)|| \geq (1 - \delta)||f||||g|| \), there exists \( \rho \in \mathbb{R}^+ \) such that the modified function \( f^*(x) = \rho^{d/2} f(\rho x) \) satisfies

\[
(4.3) \quad \int_{|f^*(x)| \geq t^{-1} ||f||} |f^*(x)|^2 \, dx \leq \Theta(t + \delta) ||f^*||^2
\]

\[
(4.4) \quad \int_{|f^*(x)| \leq t ||f||} |f^*(x)|^2 \, dx \leq \Theta(t + \delta) ||f^*||^2
\]

\[
(4.5) \quad \int_{|x| \geq t^{-1}} |f^*(x)|^2 \, dx \leq \Theta(t + \delta) ||f^*||^2
\]

\[
(4.6) \quad \int_{|x| \leq t} |f^*(x)|^2 \, dx \leq \Theta(t + \delta) ||f^*||^2.
\]

Moreover, the same conclusions hold with \( f, f^* \) replaced by \( g, g^* \) respectively, where \( g^*(x) = \rho^{d/2} g(\rho x) \) with the same value of \( \rho \) as for \( f \).

**Proof.** We may assume throughout that \( \delta \leq \delta_0(d) \) where \( \delta_0(d) \) is positive but may be chosen as small as desired. By multiplying \( f, g \) independently by positive constants we may assume without loss of generality that \( ||f|| = ||g|| = 1 \). The existence of \( \rho \) for which the first two conclusions hold simultaneously for \( f \) and for \( g \), follows from Lemma \([3.1]\) via the reasoning in \([3]\).

By dilation invariance of the inequality, we may replace \( f \) by \( f^*(x) = \rho^{d/2} f(\rho x) \) and \( g \) by \( g^*(x) = \rho^{d/2} g(\rho x) \) without affecting the hypotheses. Therefore \( \rho \) may be taken to equal \( 1 \) henceforth. The fourth conclusion for \( f \) is now a simple consequence of the first.

To obtain the third conclusion for \( f \) let \( \lambda < \infty \) be a parameter to be chosen below and let

\[
A = \{ x : \lambda^{-1} \leq |f(x)| \leq \lambda \} \quad \text{and} \quad A_t = \{ x \in A : |x| \geq t^{-1} \}.
\]

Decompose \( f = f_0 + f_1 \) where \( f_1(x) = f(x) 1_{\mathbb{R}^d \setminus A} \). Then \( |f| \leq \lambda 1_A + |f_1| \). Further decompose \( f_0 = f_{00} + f_01 \) where \( f_{01} = f_0 1_{A_t} \).

Likewise define

\[
B = \{ x : \lambda^{-1} \leq |g(x)| \leq \lambda \} \quad \text{and} \quad B_t = \{ x \in B : |x| \geq t^{-1} \},
\]

and decompose \( g = g_0 + g_1 \) where \( g_1(x) = g(x) 1_{\mathbb{R}^d \setminus B} \). Then \( |g| \leq \lambda 1_B + |g_1| \). Likewise decompose \( g_0 = g_{00} + g_{01} \) where \( g_{01} = g_0 1_{B_t} \).
The first two conclusions together imply that \( \|f_1\| + \|g_1\| = o_{\delta+\lambda^{-1}}(1) \). Therefore \( \|\mathbb{P}(f_0 \otimes g_0)\| \geq (1 - \delta - o_{\delta+\lambda^{-1}}(1))\|f_0\|\|g_0\| \).
Moreover
\[
\|\mathbb{P}(f_0 \otimes g_0)\|^2 \leq \|\mathbb{P}(\lambda 1_{A_t} \otimes \lambda 1_B)\|^2 = \lambda^2 \omega_d \int_{0}^{\infty} \sigma_r(A_t \times B)^2 r^{2d-1} \, dr
\]

with the last line holding because \((x, y) \in A_t \times B \Rightarrow |x| \geq t^{-1} \Rightarrow |(x, y)| \geq t^{-1} \). Therefore
\[
\|\mathbb{P}(f_0 \otimes g_0)\|^2 \leq \lambda^2 \omega_d (t^d |A_t|)^{1/2} \int_{0}^{\infty} \sigma_r(A_t \times B)^2 r^{2d-1} \, dr
\]
\[
\leq \lambda^2 (t^d |A|)^{1/2} |A| \cdot |B|.
\]

By Chebyshev’s inequality, \( |A| \leq \lambda^2 \|f\|^2 = \lambda^2 \) and likewise \( |B| \leq \lambda^2 \). Therefore
\[
\|\mathbb{P}(f_0 \otimes g_0)\|^2 \leq C \lambda^7 t^{d/2}.
\]

Choose \( \lambda = t^{-d/28} \) to obtain \( \|\mathbb{P}(f_0 \otimes g_0)\| \leq C t^{d/8} \). Likewise \( \|\mathbb{P}(f_00 \otimes 00)\| \leq C t^{d/8} \).

Therefore
\[
\|\mathbb{P}(f \otimes g)\| \leq \|\mathbb{P}(f_00 \otimes 00)\| + o_{t+\delta}(1).
\]

The right-hand side in this last inequality is \( \leq \|f_00\|\|000\| + o_{t+\delta}(1) \). By hypothesis, the left-hand side is \( \geq 1 - \delta \). Therefore
\[
\|f_00\|\|000\| \geq 1 - o_{t+\delta}(1).
\]

From this together with the identity \( 1 = \|f\|^2 = \|f_00\|^2 + \|f_01\|^2 + \|f_1\|^2 \) and the inequality \( \|g_00\| \leq \|g\| = 1 \), it follows that \( \|f_01\| = o_{t+\delta}(1) \). Since \( f_00 \) is supported where \( |x| \leq t^{-1} \) and \( \|f_1\| = o_{t+\delta}(1) \), the third conclusion follows for \( f \). The same reasoning applies to \( g \).

Define the Fourier transform by
\[
(4.7) \quad \hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi ix \cdot \xi} f(x) \, dx.
\]

This is a bijective isometry on \( L^2(\mathbb{R}^d) \).

**Lemma 4.3.** For any \( f, g \in L^2(\mathbb{R}^d) \),
\[
(4.8) \quad \left( \mathbb{P}(f \otimes g) \right)^\wedge = \mathbb{P}(\hat{f} \otimes \hat{g}) \]
where the left-hand side is the \( \mathbb{R}^{d+d} \) Fourier transform of \( \mathbb{P}(f \otimes g) \). Consequently
\[
(4.9) \quad \|\mathbb{P}(\hat{f} \otimes \hat{g})\| = \|\mathbb{P}(f \otimes g)\|.
\]

**Proof.** \( \mathbb{P}(f \otimes g) \) is the unique function \( h \in L^2(\mathbb{R}^{d+d}) \) of norm 1 that maximizes \( \text{Re}(\langle f \otimes g, h \rangle) \).
This quantity is equal by Plancherel’s theorem to
\[
\text{Re}(\langle \hat{f} \otimes \hat{g}, \hat{h} \rangle) = \text{Re}(\langle \hat{f} \otimes \hat{g}, \hat{h} \rangle).
\]

Since \( \hat{h} \) is also radial and has norm 1,
\[
\text{Re}(\langle \hat{f} \otimes \hat{g}, \hat{h} \rangle) = \text{Re}(\langle \mathbb{P}(\hat{f} \otimes \hat{g}), \hat{h} \rangle) \leq \|\hat{f}\| \cdot \|\hat{g}\|
\]
Thus we have shown that $\|P(f \otimes g)\| \leq \|P(\hat{f} \otimes \hat{g})\|$. The same reasoning gives the converse inequality, so
\[
\|P(f \otimes g)\| = \|P(\hat{f} \otimes \hat{g})\|
\]
and $h$ is the closest radial function of norm 1 to $f \otimes g$ if and only if $\hat{h}$ is the closest radial function of norm 1 to $\hat{f} \otimes \hat{g}$. Thus $(P(f \otimes g))^\wedge = P(\hat{f} \otimes \hat{g})$.

**Corollary 4.4.** Let $d \geq 1$. There exists a continuous function $\Theta : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $\lim_{t \to 0} \Theta(t) = 0$ with the following property. For any $\delta > 0$ and any nonzero functions $f, g \in L^2(\mathbb{R}^d)$ that satisfy $\|P(f \otimes g)\| \geq (1 - \delta)\|f\|\|g\|$, there exists $\rho \in \mathbb{R}^+$ such that if $f^*(x) = \rho^{d/2} f(\rho x)$ and $g^*(x) = \rho^{d/2} g(\rho x)$ then $f^*$ and $g^*$ satisfy the conclusions of Lemma 4.2.

If $\|P(f \otimes g)\| \geq (1 - \delta)\|f\|\|g\|$ then $f, g$ satisfy the conclusions of Lemma 4.2 for some $\rho > 0$, while $\hat{f}, \hat{g}$ also satisfy these conclusions, with respect to some other $\rho' \in \mathbb{R}^+$. It is clear from the uncertainty principle, broadly construed, that the product $\rho \rho'$ is bounded below by a constant that depends only on $d$ and on the auxiliary function $\Theta$. The next step is to show that this product is necessarily bounded above. The following lemma will be used for this purpose.

**Lemma 4.5.** For any $d \geq 1$ and any continuous function $\Theta : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $\lim_{t \to 0^+} \Theta(t) = 0$ there exist $\delta_0 > 0$ and $C \in [1, \infty)$ with the following property. Let $f \in L^2(\mathbb{R}^d)$ be a nonnegative function with positive norm which satisfies the conclusions of Lemma 4.2 with $\rho = 1$, with $\delta = \delta_0$, and with this auxiliary function $\Theta$. Then
\[
\int_{|\xi| \leq C} |\hat{f}(\xi)|^2 d\xi \geq C^{-1} \|f\|^2 \tag{4.10}
\]
\[
\int_{|\xi| \leq C^{-1}} |\hat{f}(\xi)|^2 d\xi \leq \frac{1}{2} \|f\|^2. \tag{4.11}
\]

To clarify the statement: The conclusions of Lemma 4.2 are stated in terms of $f^*(x) = \rho^{d/2} f(\rho x)$. The hypothesis of Lemma 4.5 is that if $\rho$ is taken to equal 1 then $f^*$ satisfies the four inequalities stated as conclusions of that lemma.

The first conclusion (4.10) implies that the dilated function $\xi \mapsto s^{d/2} \hat{f}(s \xi)$ cannot satisfy the conclusions of Lemma 4.2 with parameter $s$ very large. The second conclusion (4.11) implies that $s$ cannot be very small. Thus if $\rho, \rho'$ are as discussed above and if we dilate so that $\rho = 1$, then $\rho'$ is bounded both above and below by finite positive constants which depend only on the dimension $d$ and on a choice of an auxiliary function $\Theta$ satisfying the conclusions of Lemma 4.2.

**Proof of Lemma 4.5.** To prove (4.10) consider the auxiliary function $G(x) = e^{-\pi|x|^2}$. Assume without loss of generality that $\|f\| = 1$. Provided that $\delta$ is sufficiently small, the nonnegativity of $f$, the lower bound $\|f\| \geq 1$, and the upper bounds provided by the conclusions of Lemma 4.2 together provide a lower bound for $\int f G$. But since $G = \hat{G}$, $\int f G = \int f \hat{G} = \int \hat{f} G$. Therefore $\int e^{-\pi|\xi|^2} \hat{f}(\xi) d\xi \geq \eta$ for some positive constant $\eta$ which depends only on the dimension $d$. This easily implies (4.10) since $\|\hat{f}\| \leq 1$.

To prove (4.11) let $\lambda \in \mathbb{R}^+$ be large and consider
\[
\int |\hat{f}(\xi)|^2 e^{-\lambda \pi|\xi|^2} d\xi = \lambda^{-d/2} \int \int f(x) f(y) e^{-\pi|x-y|^2/\lambda} dx dy. \tag{4.12}
\]
The right-hand side is majorized by a constant, uniformly for all functions that satisfy \( \|f\| \leq 1 \). If \( f \) is supported in any fixed bounded region then the right-hand side is \( O(\lambda^{-d/2}\|f\|_2^2) \) as \( \lambda \to \infty \). It follows readily that if \( f \) satisfies the conclusions of Lemma 4.2 with \( \rho = 1 \), and if \( \|f\| \leq 1 \), then the right-hand side of (4.12) is majorized by a function of \( \lambda \) that tends to zero as \( \lambda \to \infty \). Therefore the same goes for the left-hand side. Now

\[
\int |\hat{f}(\xi)|^2 e^{-\lambda|\xi|^2} \, d\xi \geq c \int_{|\xi| \geq \lambda^{-1/2}} |\hat{f}(\xi)|^2 \, d\xi
\]

with \( c > 0 \) independent of \( \lambda \), establishing (4.11). \( \square \).

This type of argument, exploiting nonnegativity, is made in greater detail in [6].

Let \( d \geq 1 \) and let \( \delta > 0 \). Let \( \Theta : \mathbb{R}^+ \to \mathbb{R}^+ \) be a continuous function satisfying \( \lim_{t \to 0} \Theta(t) = 0 \). We say that a function \( f \), localized at a (second) common scale. Therefore once

Let \( \delta \). Call Corollary 4.7.

\[ f, g \]

because Lemma 4.2 says that \( f^*(x) = \rho^{d/2} f(\rho x) \) and \( g^*(x) = \rho^{d/2} g(\rho x) \), and the Fourier transforms of \( f^*, g^* \), are \( (\delta, \Theta) \)–normalized.

**Proposition 4.6.** For each \( d \geq 1 \) there exist \( \delta_0 > 0 \) and a continuous function \( \Theta : \mathbb{R}^+ \to \mathbb{R}^+ \) satisfying \( \lim_{t \to 0} \Theta(t) = 0 \) with the following property. Let \( \delta \in (0, \delta_0) \). Let \( f, g \in L^2(\mathbb{R}^d) \) have positive norms, and assume that \( f \) is nonnegative. Suppose that \( \|\mathbb{P}(f \otimes g)\| \geq (1 - \delta)\|f\|\|g\| \). Then there exists \( \rho \in \mathbb{R}^+ \) such that the functions \( f^*(x) = \rho^{d/2} f(\rho x) \) and \( g^*(x) = \rho^{d/2} g(\rho x) \), and the Fourier transforms of \( f^*, g^* \), are \( (\delta, \Theta) \)–normalized.

**Proof.** Lemma 4.5 forces the parameter \( \rho \) in Corollary 4.4 to be comparable to 1 if \( f \) is \( \delta \)–normalized for sufficiently small \( \delta \). \( \square \)

The reasoning did not require an assumption that both functions \( f, g \) were nonnegative, because Lemma 4.2 says that \( f, g \) are localized at a common scale, and likewise \( \hat{f}, \hat{g} \) are localized at a (second) common scale. Therefore once \( f, \hat{f} \) are shown to be localized at a pair of scales \( \rho, \rho' \) satisfying \( \rho \rho' \asymp 1 \), the same follows for \( g \).

**Corollary 4.7.** Let \( d \geq 1 \). For every \( \varepsilon > 0 \) there exists \( \delta > 0 \) with the following property. If \( 0 \neq f \in L^2(\mathbb{R}^d) \) is nonnegative, if \( 0 \neq g \in L^2(\mathbb{R}^d) \), and if \( \|\mathbb{P}(f \otimes g)\| \geq (1 - \delta)\|f\|\|g\| \) then

\[
(4.13) \quad \text{dist}((f, g), \mathfrak{S}^x) < \varepsilon \|(f, g)\|.
\]

**Proof.** Suppose the contrary. Then there exists a sequence of pairs \( (f_n, g_n) \) of functions in \( L^2(\mathbb{R}^d) \) satisfying \( \|f_n\| \equiv \|g_n\| \equiv 1 \), \( \|\mathbb{P}(f_n \otimes g_n)\| \to 1 \), \( f_n \) is nonnegative, and the distance \( \text{dist}((f_n, g_n), \mathfrak{S}^x) \) from \( (f_n, g_n) \) to the set \( \mathfrak{S}^x \) of all \( (F, cF) \) with \( F \in \mathfrak{S} \) and \( 0 \neq c \in \mathbb{C} \) is bounded below by a positive quantity independent of \( n \).

By Proposition 4.6 there exist sequences of numbers \( \rho_n, \delta_n \in \mathbb{R}^+ \) and an auxiliary function \( \Theta \) satisfying \( \lim_{t \to 0} \Theta(t) = 0 \) such that \( \lim_{n \to \infty} \delta_n = 0 \) and the sequences of functions \( f_n^*(x) = \rho_n^{d/2} f_n(\rho_n x) \) and \( g_n^*(x) = \rho_n^{d/2} g_n(\rho_n x) \) are \( (\delta_n, \Theta) \)–normalized. Moreover, the Fourier transforms \( \hat{f}_n, \hat{g}_n \) are also \( (\delta_n, \Theta) \)–normalized. By Rellich’s Lemma, the sequences \( (f_n^* : n \in \mathbb{N}) \) and \( (g_n^* : n \in \mathbb{N}) \) are each precompact in \( L^2(\mathbb{R}^d) \). Therefore there exists an increasing sequence of natural numbers \( n_k \) such that the subsequences \( f_{n_k}^*, g_{n_k}^* \) converge in \( L^2 \) norm to limits \( f_\infty, g_\infty \in L^2(\mathbb{R}^d) \), respectively.

By Proposition 4.6 there exist sequences of numbers \( \rho_n, \delta_n \in \mathbb{R}^+ \) and an auxiliary function \( \Theta \) satisfying \( \lim_{t \to 0} \Theta(t) = 0 \) such that \( \lim_{n \to \infty} \delta_n = 0 \) and the sequences of functions \( f_n^*(x) = \rho_n^{d/2} f_n(\rho_n x) \) and \( g_n^*(x) = \rho_n^{d/2} g_n(\rho_n x) \) are \( (\delta_n, \Theta) \)–normalized. Moreover, the Fourier transforms \( \hat{f}_n, \hat{g}_n \) are also \( (\delta_n, \Theta) \)–normalized. By Rellich’s Lemma, the sequences \( (f_n^* : n \in \mathbb{N}) \) and \( (g_n^* : n \in \mathbb{N}) \) are each precompact in \( L^2(\mathbb{R}^d) \). Therefore there exists an increasing sequence of natural numbers \( n_k \) such that the subsequences \( f_{n_k}^*, g_{n_k}^* \) converge in \( L^2 \) norm to limits \( f_\infty, g_\infty \in L^2(\mathbb{R}^d) \), respectively.

Now \( \|f_\infty\| = \lim_{n \to \infty} \|f_n^*\| = \lim_{n \to \infty} \|f_n\| = 1 \). Moreover, since \( \mathbb{P} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \) is a bounded linear operator,

\[
\|\mathbb{P}(f_\infty \otimes g_\infty)\| = \lim_{n \to \infty} \|\mathbb{P}(f_n^* \otimes g_n^*)\| = \lim_{n \to \infty} \|\mathbb{P}(f_n \otimes g_n)\| = 1.
\]

Therefore \( (f_\infty, g_\infty) \in \mathfrak{S}^x \). In particular, \( f_\infty, g_\infty \) are radial complex Gaussians. This contradicts the assumption that the distance from \( f_n \) to \( \mathfrak{S} \) does not tend to zero. \( \square \)
Since the functions $f_n$ are nonnegative, $f_\infty$ is necessarily close in norm to a positive Gaussian function in this argument. Therefore the conclusion can be refined: There exists a positive Gaussian $F$ such that $\|f - F\| \leq \varepsilon \|f\|$. □

**Lemma 4.8.** For any functions $f, g \in L^2(\mathbb{R}^d)$,

\[(4.14)\quad \|P(|f| \otimes |g|)\| \geq \|P(f \otimes g)\|.

**Proof.** If $h \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$ is radial then so is $|h|$. \[
\|f \otimes g - |h|\| = \|f \otimes g - |h|\| \leq \|f \otimes g - h\|.
\]

The next result is identical to Corollary 4.7, except that the restriction to nonnegative functions is removed.

**Corollary 4.9.** Let $d \geq 1$. For every $\varepsilon > 0$ there exists $\delta > 0$ with the following property. If $0 \neq f, g \in L^2(\mathbb{R}^d)$ satisfy $\|P(f \otimes g)\| \geq (1 - \delta)\|f\|||g||$ then there exists a radial complex Gaussian $G$ such that $\|\text{d} - G\| \leq \varepsilon \|f\|$ and $\|\text{d} - cG\| \leq \varepsilon \|g\|$, where $c = \|g\|/\|f\|$.

**Proof.** Let the pair $(f, g)$ satisfy the hypotheses for some small $\delta > 0$, and assume without loss of generality that $\|f\| = \|g\| = 1$. By Lemma 4.8, the pair $(|f|, |g|)$ satisfies the hypotheses, with the same parameter $\delta$. Corollary 4.7 guarantees that there exists a positive Gaussian function $F$ such that $\|(|f|, |g|) - (F, F)\|$ is small. By exploiting dilations we may reduce to the case in which $F(x) = e^{-\pi|x|^2/2}$.

Express $f = e^{i\varphi}|f|$ and $g = e^{i\psi}|g|$ where $\varphi, \psi$ are Lebesgue measurable real-valued functions. Set $\tilde{f} = e^{i\varphi}F$ and $\tilde{g} = e^{i\psi}F$. Then $\|(|f|, |g|) - (\tilde{f}, \tilde{g})\|$ is small, so $\|P(\tilde{f} \otimes \tilde{g})\|$ is nearly equal to $\|f\|||g||$ and hence nearly equal to $\|f\|||\tilde{g}||$.

Let $\varepsilon > 0, \delta > 0$. Choose $R \geq 1$ sufficiently large that \[\int_{|x_1| > R/2} e^{-\pi(|x|^2 + |y|^2)^2} dx \, dy < \varepsilon.\] Suppose that $\|P(f \otimes g)\| \geq (1 - \delta)\|f\|||g||$ and $\|(|f|, |g|) - (F, F)\| < \delta$. If $\delta$ is sufficiently small then there exists a function $h$ such that

\[
\int_{|x_1| \leq 2R} \left| e^{i[\varphi(x) + \psi(y)]} e^{-\pi(|x|^2 + |y|^2)^2/2} - h(|(x, y)|) \right|^2 dx \, dy < e^{-2\pi R^2 - R^2 \varepsilon}.
\]

The same holds with any $R$–dependent constant factor; this factor is chosen for the sake of convenience below. The same bound follows with $h(t) = e^{i\xi(t)}|h(t)|$ replaced by $e^{i\xi(t)}e^{-\pi t^2/2}$, with $\varepsilon$ replaced by $2\varepsilon$, for some real-valued measurable function $\xi$.

By Chebyshev’s inequality,

\[(4.15)\quad \{|z = (x, y) : |z| \leq 2R \text{ and } |e^{i[\varphi(x) + \psi(y)]} - e^{i[\varphi(x) + \psi(y) - \xi(|x|)]}| \geq \varepsilon^{1/4}\} \leq e^{-R \varepsilon^{1/2}},\]

where $|\cdot|$ denotes Lebesgue measure. By choosing a typical value of $y$ one concludes that there exists a real-valued measurable function $\hat{\varphi}$ defined on $\mathbb{R}^+$ such that

\[(4.16)\quad \{|x \in \mathbb{R}^d : |x| \leq R \text{ and } |e^{i\varphi(x)} - e^{i\hat{\varphi}(|x|^2)}| \geq \varepsilon^{1/4}\} \leq C e^{-R \varepsilon^{1/2}}.\]

Indeed, this holds with $\hat{\varphi}(|x|^2) = \xi(|x|^2 + |y|^2)^{1/2} - \psi(y)$ for any typical value of $y$ since $|e^{i\varphi(x)} - e^{i\xi(|x|^2 + |y|^2)\psi(y)}|$ is small for nearly all $x$ for typical $y$. By the same reasoning, $e^{i\hat{\psi}(y)}$ is nearly equal in the same sense to $e^{i\hat{\psi}(|y|^2)}$ for some real-valued measurable function $\hat{\psi}$. 

Thus for any $\eta > 0$,

$$
\left| \{(s, t) \in \mathbb{R}^+ \times \mathbb{R}^+ : |(s, t)| \leq R^2 \text{ and } |e^{i[\tilde{\varphi}(s) + \tilde{\psi}(t) - \xi(\sqrt{s + t})]} - 1| \geq 2\varepsilon^{1/4} \right| 
\leq C\eta^{2d} + C\eta^{-(2d-1)}e^{-R\varepsilon^{1/2}}.
$$

Here $|\cdot|$ denotes Lebesgue measure on $\mathbb{R} \times \mathbb{R}$, restricted to the quadrant $\mathbb{R}^+ \times \mathbb{R}^+$. Choosing $\eta$ to be an appropriate power of $e^{-R\varepsilon}$ yields an upper bound $Ce^{-\varepsilon^R\varepsilon^c}$ for some $c, C \in \mathbb{R}^+$.

Proposition 8.2 of [3] is concerned with ordered triples of functions $(\tilde{\varphi}, \tilde{\psi}, \tilde{\xi})$ for which $\tilde{\varphi}(s) + \tilde{\psi}(t) - \tilde{\xi}(s + t)$ is nearly zero for nearly all ordered pairs $(s, t)$ in an interval. By applying this proposition with $\tilde{\xi}(t) = \xi(t^{1/2})$ we conclude that there exists an affine function $L$ such that

$$
\left| \{(s, t) \in [0, R^2/4] : |e^{i\tilde{\varphi}(s)} - e^{iL(s)}| \geq C\varepsilon^{1/4} \right| 
\leq Ce^{-cR\varepsilon^c}.
$$

Replacing $L$ by its real part does not worsen the approximation since $\tilde{\varphi}$ is real-valued and hence $e^{i\tilde{\varphi}}$ is unimodular, so we may assume that $L$ is real-valued. The favorable factor $e^{-cR\varepsilon^c}$ on the right-hand side makes it possible to overcome the power $r^{d-1}$ that appears in the polar coordinate expression for Lebesgue measure in $\mathbb{R}^d$ to conclude that

$$
\left| \{x \in \mathbb{R}^d : |x| \leq \frac{1}{2}R \text{ and } |e^{i\varphi(x)} - e^{iL(|x|^2)}| \geq C\varepsilon^{1/4} \right| 
\leq C\varepsilon^c.
$$

Thus $f$ is nearly equal to the Gaussian function $G(x) = e^{-\pi|x|^2/2}e^{iL(|x|^2)}$. The same reasoning applies to $g$, which is consequently nearly equal to a Gaussian function $\tilde{G}(x) = e^{-\pi|x|^2/2}e^{i\tilde{L}(|x|^2)}$, where $\tilde{L}$ is another real-valued affine function.

Now $\|\mathbb{P}(G \otimes \tilde{G})\|_2$ is nearly equal to $\|G\|\|	ilde{G}\|$ since $(G, \tilde{G})$ is nearly equal to $(f, g)$. Thus

$$
e^{i[L(|x|^2) + \tilde{L}(|y|^2)]} \approx e^{i\xi(|x|^2 + |y|^2)},
$$

where $\approx$ denotes approximate equality in weighted $L^2$ norm with weight $e^{-\pi(|x|^2 + |y|^2)}$. Express $L(|x|^2) = \alpha'|x|^2 + \beta'$, $\tilde{L}(|y|^2) = \alpha''|y|^2 + \beta''$, and $\xi(|z|^2) = \alpha|z|^2 + \beta$. By choosing a typical value of $y$ and regarding both sides as functions of $x$ we conclude that $\alpha'$ is approximately equal to $\alpha$. Reversing the roles of the variables proves that $\alpha''$ is also approximately equal to $\alpha$, whence $\alpha', \alpha''$ are approximately equal. \qed

This completes the proof of Proposition 4.1.

**Remark 4.1.** Young’s convolution inequality and the Hausdorff-Young inequality are strongly bound up with additive structure, and the analyses of near extremizers of each of these inequalities [3, 5] relied on information from additive combinatorics. Additive structure apparently plays a less central role in the present work, but is the basis for the proof of Corollary 4.9.

5. Spectral analysis

Define

$$F(x) = e^{-\pi|x|^2/2}
$$

for $x \in \mathbb{R}^d$. This function satisfies $\|F\| = 1$.

Define a bounded linear operator $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ by

$$
Tf(x) = \int_{\mathbb{R}^d} F(y) \mathbb{P}(f \otimes F)(x, y) dy.
$$

This operator is related to the projection $\mathbb{P}$ by the identity

$$
\langle \mathbb{P}(f \otimes F), \mathbb{P}(g \otimes F) \rangle = \langle Tf, g \rangle.
$$
Indeed,

\[ (\mathbb{P}(f \otimes F), \mathbb{P}(g \otimes F)) = (\mathbb{P}(f \otimes F), g \otimes F) = \iint \mathbb{P}(f \otimes F)(x,y)\overline{\varphi}(x)F(y) \, dx \, dy = \int Tf(x)\overline{\varphi}(x) \, dx = \langle Tf, g \rangle. \]

For any \( f, g \in L^2(\mathbb{R}^d) \), \( \mathbb{P}(f \otimes g) \equiv \mathbb{P}(g \otimes f) \). Since \( (\mathbb{P}(f \otimes F), \mathbb{P}(g \otimes F)) \) is the complex conjugate of \( (\mathbb{P}(g \otimes F), \mathbb{P}(f \otimes F)) \), it follows from (5.2) that \( T \) is self-adjoint.

Define \( \mathcal{R} \subset L^2(\mathbb{R}^d) \) to be the subspace consisting of all radial functions, which is the closure of the span of all functions \( |x|^{2m}e^{-\pi|x|^2/2} \), where \( m \in \{0,1,2,\ldots\} \). The range of \( T \) is contained in \( \mathcal{R} \). Indeed, if \( g \in L^2(\mathbb{R}^d) \) is orthogonal to all radial functions then \( g \otimes F \) is orthogonal to all radial functions in \( L^2(\mathbb{R}^{d+2}) \), so

\[ \langle Tf, g \rangle = (\mathbb{P}(f \otimes F), \mathbb{P}(g \otimes F)) = (\mathbb{P}(f \otimes F), 0) = 0 \]

for all \( f \in L^2(\mathbb{R}^d) \). Since \( T \) is self-adjoint, \( T \) vanishes identically on \( \mathcal{R} \), as well.

We require an understanding of the eigenvalues and eigenvectors of \( T \). The relevant information is contained in the next result, together with the fact that \( T \equiv 0 \) on \( \mathcal{R} \).

**Proposition 5.1.** There exists an orthonormal basis for \( \mathcal{R} \) consisting of eigenfunctions of \( T \) of the form

\[ \psi_m(x) = q_m(|x|^2)e^{-\pi|x|^2/2} : m = 0,1,2,\ldots \]

where \( q_m \) is a polynomial of degree exactly \( m \). The corresponding eigenvalues are

\[ \lambda_{d,m} = \frac{\Gamma(m + \frac{d}{2})}{\Gamma(m + d)} \cdot \frac{\Gamma(d)}{\Gamma(\frac{d}{2})} \quad \text{for} \quad m = 0,1,2,\ldots. \]

Throughout the analysis we represent elements of \( \mathbb{R}^d \times \mathbb{R}^d \) as \( z = (x,y) \) where \( x,y \in \mathbb{R}^d \). Let \( G(z) = e^{-\pi|z|^2/2} \), so that \( G = F \otimes F = \mathbb{P}(F \otimes F) \). Denote elements \( \alpha \in \{0,1,2,\ldots\}^d \) by \( \alpha = (\alpha_1, \ldots, \alpha_d) \) and write \( |\alpha| = \sum_{j=1}^d \alpha_j \). \( x^\alpha = \prod_{j=1}^d x_j^{\alpha_j} \), and \( x^\alpha F \) indicates the function \( x \mapsto x^\alpha F(x) \).

**Lemma 5.2.** \( T(x^\alpha F) = 0 \) for any \( \alpha \in \{0,1,2,\ldots\}^d \setminus \{0,2,4,\ldots\}^d \). For any \( \alpha \in \{0,2,4,\ldots\}^d \), there exists a polynomial \( Q : \mathbb{R}^d \to \mathbb{C} \) of degree exactly \( |\alpha|/2 \) such that

\[ T(x^\alpha F) = Q(|x|^2)F(x). \]

**Proof.** \( \mathbb{P}(x^\alpha F \otimes F) \) is the projection onto the radial subspace of \( x^\alpha F(x)F(y) = x^\alpha G(z) \). Clearly \( \mathbb{P}(x^\alpha G(x,y)) \) is a scalar multiple of \( |z|^{\alpha}G(z) \). Moreover, \( \mathbb{P}(x^\alpha F \otimes F) = 0 \) if at least one component \( \alpha_j \) is odd, because the integral over \( S^{2d-1} \) of any function that is odd with respect to one or more coordinate variables must vanish.

If \( \alpha \in \{0,2,4,\ldots\}^d \) then \( x^\alpha \) is a nonnegative function which does not vanish identically, so for any \( r \in \mathbb{R}^+ \), \( \int x^\alpha G(x,y) \, d\sigma_r(x,y) = G(x,y) \int x^\alpha \, d\sigma_r(x,y) \) is strictly positive. Therefore \( \mathbb{P}(x^\alpha G) = c_{d,\alpha}|z|^{\alpha}G(z) \).
Continuing to assume that \( \alpha \in \{0, 2, 4, \ldots \}^d \), set \( m = \frac{1}{2}\alpha \). Then
\[
\int_{\mathbb{R}^d} |z|^{\alpha} G(z) F(y) \, dy = \int_{\mathbb{R}^d} F(y) (x, y)^{2m} G(x, y) \, dy \\
= \int_{\mathbb{R}^d} e^{-\pi|y|^2/2}(x, y)^2 e^{-\pi|y|^2/2} \, dy \\
= q_m(|x|^2) e^{-\pi|x|^2/2} \\
= q_m(|x|^2) F(x)
\]
where \( q_m : \mathbb{R} \to \mathbb{R} \) is a polynomial of degree exactly \( m \).

Proof of Proposition 5.1. We have shown that
\[
e^{-\pi|x|^2/2} = \lambda_{d,m} |x|^{2m} + \text{a polynomial in } |x|^2 \text{ of lower degree},
\]
where \( \lambda_{d,m} \neq 0 \) for each nonnegative integer \( m \). The Gram-Schmidt procedure therefore constructs an orthonormal basis for \( \mathcal{R} \) consisting of eigenfunctions of \( T \) of the indicated form.

The corresponding eigenvalue \( \lambda_{d,m} \) equals the coefficient of the highest power of \( |x|^{2m} \) in the polynomial \( e^{-\pi|x|^2/2} T(|x|^{2m} e^{-\pi|x|^2/2}) \). To compute this coefficient write
\[
\mathbb{P}(|x|^{2m} F) (z) = \gamma_{m,d} |x|^{2m} e^{-\pi|x|^2/2}
\]
where
\[
\gamma_{m,d} = \int_{S^{2d-1}} |x|^{2m} \, d\sigma(x, y)
\]
where \( S^{2d-1} \subset \mathbb{R}^{d+1} \) is the unit sphere and \( \sigma \) is surface measure on \( S^{2d-1} \), normalized so that \( \sigma(S^{2d-1}) = 1 \). Consequently
\[
T(|x|^{2m} F) = \int_{\mathbb{R}^d} e^{-\pi|y|^2/2} \mathbb{P}(|x|^{2m} F)(x, y) \, dy \\
= \int_{\mathbb{R}^d} e^{-\pi|y|^2/2} \gamma_{m,d} (|x|^2 + |y|^2)^m e^{-\pi(x|^2+|y|^2)/2} \, dy \\
= (\gamma_{m,d} \int_{\mathbb{R}^d} e^{-\pi|y|^2/2} \, dy |x|^{2m} + O(|x|^{2m-2})) e^{-\pi|x|^2/2} \\
= (\gamma_{m,d} |x|^{2m} + O(|x|^{2m-2})) e^{-\pi|x|^2/2}
\]
where \( O(|x|^{2m-2}) \) denotes a polynomial in \( |x|^2 \) of degree at most \( 2m - 2 \) as a polynomial in \( x \). Thus \( \lambda_{m,d} = \gamma_{m,d} \).

Define \( \omega_n \) by the relation \( \int_{\mathbb{R}^n} g(|z|) \, dz = \omega_n \int_0^\infty g(r) r^{n-1} \, dr \). One can compute \( \gamma_{m,d} \) by writing
\[
\int_{\mathbb{R}^{d+1}} e^{-\pi(|x|^2+|y|^2)/2} |x|^{2m} \, dx \, dy = \omega_{2d} \int_0^\infty r^{2d+2m} e^{-\pi r^2/2} r^{-1} \, dr \\
= \frac{1}{2} \omega_{2d} \gamma_{m,d} (\pi/2)^{-d-m} \int_0^\infty s^{d+m} e^{-s} s^{-1} \, ds \\
= \frac{1}{2} \omega_{2d} \gamma_{m,d} (\pi/2)^{-d-m} \Gamma(m + d).
\]
The left-hand side can be alternatively be evaluated as
\[
\int_{\mathbb{R}^{d+1}} e^{-\pi(|x|^2 + |y|^2)/2} |x|^{2m} \, dx \, dy = \int_{\mathbb{R}^d} e^{-\pi|x|^2/2} |x|^{2m} \, dx \times \int_{\mathbb{R}^d} e^{-\pi|y|^2/2} \, dy
\]
\[
= \frac{1}{2} \omega_d (\pi/2)^{-m - \frac{d}{2}} \Gamma(m + \frac{d}{2}) \cdot \frac{1}{2} \omega_d (\pi/2)^{-d/2} \Gamma(\frac{1}{2}d).
\]

Since \(\gamma_{0,d} = 1\), the same calculation with \(m = 0\) gives \(1 = \frac{1}{2} \omega_d (\pi/2)^{-d/2} \Gamma(d/2)\). Therefore
\[
\gamma_{m,d} = \frac{2^{-2} \omega_d^2 \Gamma(m + \frac{1}{2}d) \Gamma(\frac{1}{2}d)}{2^{-1} \omega_d \Gamma(m + d)} = \frac{\Gamma(m + \frac{1}{2}d) \Gamma(\frac{1}{2}d)}{\Gamma(m + d)} \cdot \Gamma(d) = \frac{\Gamma(m + \frac{1}{2}d)}{\Gamma(m + d)} \cdot \Gamma(d).
\]

\[\square\]

**Lemma 5.3.** If \(f, g \in L^2(\mathbb{R}^d)\) satisfy \(\langle f, \psi_0 \rangle = \langle g, \psi_0 \rangle = 0\) and \(\langle f + g, \psi_1 \rangle = 0\) then
\[
\|Tf\|^2 + 2 \text{Re} \langle Tf, g \rangle + \|Tg\|^2 \leq \frac{d + 2}{2(d + 1)} \|(f, g)\|^2.
\]

**Proof.** For \(m \in \{0, 1, 2, \ldots\}\) let \(\psi_m\) be \(L^2\)-normalized eigenfunctions of \(T\) with corresponding eigenvalues \(\lambda_{m,d}\) discussed above, and \(\psi_0 = F\).

For fixed dimension \(d\), the eigenvalue \(\gamma_{m,d}\) is a decreasing function of \(m\). Indeed,
\[
\frac{\gamma_{m+1,d}}{\gamma_{m,d}} = \frac{m + \frac{1}{2}d}{m + d}
\]
according to (5.4) and the functional equation of the Gamma function. The leading eigenvalues are
\[
\lambda_{0,d} = 1, \quad \lambda_{1,d} = \frac{d/2}{d} \lambda_{0,d} = \frac{1}{2}, \quad \lambda_{2,d} = \frac{1 + \frac{1}{2}d}{1 + d} \lambda_{1,d} = \frac{d + 2}{4(d + 1)}.
\]

Decompose
\[
f = \sum_{m=0}^{\infty} \hat{f}(m) \psi_m + \hat{f} \quad \text{and} \quad g = \sum_{m=0}^{\infty} \hat{g}(m) \psi_m + \hat{g}
\]
where \(\hat{f}, \hat{g} \perp \mathcal{R}\). It is given that \(\hat{f}(0) = \hat{g}(0) = 0\) and that \(\hat{g}(1) = -\hat{f}(1)\). Then
\[
\langle Tg, f \rangle = \langle T(g - \hat{g}), (f - \hat{f}) \rangle = \sum_{m=0}^{\infty} \lambda_{m,d} \hat{g}(m) \hat{f}(m)
\]
and
\[
\|f\|^2 = \|\hat{f}\|^2 + \sum_m |\hat{f}(m)|^2 \quad \text{and} \quad \|g\|^2 = \|\hat{g}\|^2 + \sum_m |\hat{g}(m)|^2.
\]
Since \( \hat{f}(1) = \hat{g}(1) = 0 \) and \( \hat{f}(1) + \hat{g}(1) = 0 \),

\[
\| Tf \|^2 + 2 \Re \langle Tf, g \rangle + \| Tg \|^2 = \sum_{m=0}^{\infty} \left( \lambda_{m,d}(|\hat{f}(m)|^2 + |\hat{g}(m)|^2 + 2 \Re (\hat{f}(m)\overline{\hat{g}(m)}) \right)
\]

\[
= \frac{1}{2}|\hat{f}(1) + \hat{g}(1)|^2 + \sum_{m=2}^{\infty} \lambda_{m,d}|\hat{f}(m) + \hat{g}(m)|^2
\]

\[
\leq \frac{d+2}{4(d+1)} \sum_{m=2}^{\infty} |\hat{f}(m) + \hat{g}(m)|^2
\]

\[
\leq \frac{d+2}{2(d+1)} \sum_{m=2}^{\infty} (|\hat{f}(m)|^2 + |\hat{g}(m)|^2)
\]

\[
\leq \frac{d+2}{2(d+1)} \|(f, g)\|^2.
\]

\[\square\]

6. Perturbation Analysis

For nonzero \( f, g \in L^2(\mathbb{R}^d) \) define

\[
(6.1) \quad \Phi(f, g) = \frac{\|P(f \otimes g)\|^2}{\|f\|^2\|g\|^2}.
\]

Continue to let \( F(x) = e^{-|x|^2/2} \).

Suppose that the ratio of the distance of \((u, v)\) to \( \mathcal{G}^\times \) to the norm of \((u, v)\) is small, and that the closest element of \( \mathcal{G}^\times \) to \((u, v)\) is \((F, F)\). Then the first variation at \((r, s, t) = 0\) of \( \|u - e^{r|x|^2+s}F\|^2 + \|v - e^{t} F\|^2 \) with respect to \((r, s, t)\) must vanish. Therefore \((u, v)\) can be expressed in the form \((u, v) = (F + f, F + g)\) where \((f, g)\) is unique and satisfies

\[
\langle f, F \rangle = \langle g, F \rangle = \langle f + g, |x|^2 F \rangle = 0.
\]

Equivalently,

\[
(6.2) \quad \langle f, \psi_0 \rangle = \langle g, \psi_0 \rangle = \langle f + g, \psi_1 \rangle = 0.
\]

One has

\[
\|P(u \otimes v)\|^2 = \|F\|^4 + 2 \Re \langle P(f \otimes F), F \otimes F \rangle + 2 \Re \langle P(F \otimes g), F \otimes F \rangle + 2 \Re \langle P(f \otimes g), F \otimes F \rangle + 2 \Re \langle P(f \otimes F), F \otimes g \rangle + \langle P(f \otimes F), P(f \otimes F) \rangle + \langle P(F \otimes g), P(F \otimes g) \rangle + O(||(f, g)||^3)
\]

as \( ||(f, g)|| \to 0 \). Observe that

\[
\langle P(f \otimes F), F \otimes F \rangle = \langle f \otimes F, P(F \otimes F) \rangle = \langle f \otimes F, F \otimes F \rangle = \langle f, F \rangle \cdot \langle F, F \rangle = 0
\]

and likewise \( \langle P(g \otimes F), F \otimes F \rangle = 0 \). Invoking the identity \( \langle P(f \otimes F), F \otimes g \rangle = \langle Tf, f \rangle \) and using the relations \( \langle f, F \rangle = \langle g, F \rangle = 0 \) we obtain

\[
\|P(u \otimes v)\|^2 = 1 + 2 \Re \langle Tf, g \rangle + \langle Tf, f \rangle + \langle Tg, f \rangle + \langle T(t), g \rangle + O(||(f, g)||^3).
\]

On the other hand,

\[
\|u\|^2\|v\|^2 = 1 + \|f\|^2 + \|g\|^2 + O(||(f, g)||^4)
\]
since \( f, g \perp F \). Therefore

\[
\frac{\| \mathbb{P}(u \otimes v) \|}{\| u \| \| v \|} = 1 + 2 \Re \langle T f, g \rangle + \langle T f, f \rangle + \langle T(g), g \rangle - \| (f, g) \|^2 + O(\| (f, g) \|^3).
\]

The inequality under investigation here has a useful group of symmetries. If \( \delta_r(f)(x) = f(rx) \), then \( \mathbb{P}(\delta_r(f) \otimes \delta_r(g)) = \delta_r(\mathbb{P}(f \otimes g)) \) where \( \delta_r \) acts on functions with domain \( \mathbb{R}^d \) on the left-hand side of the equation, and on functions with domain \( \mathbb{R}^d \times \mathbb{R}^d \) on the right. Likewise if \( e_t(f)(x) = e^{it|x|^2} f(x) \) for \( t \in \mathbb{R} \) then \( \mathbb{P}(e_t(f) \otimes e_t(g)) = e_t(\mathbb{P}(f \otimes g)) \). Of course \( \mathbb{P}(c' f \otimes c'' g) = c' c'' (\mathbb{P}(f \otimes g)) \) for scalars \( c', c'' \in \mathbb{C} \).

**Proof of Theorem 6.1.** Let \( (u, v) \in L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \). Suppose that the closest element of the closed subspace \( \mathfrak{G}^\times \) of \( L^2 \times L^2 \) to \( (u, v) \) is \( (F, F) \), and that the distance from \( (u, v) \) to \( (F, F) \) is much less than \( \| (u, v) \| \). The orthogonality relations \( (6.2) \) are consequently satisfied by \( (f, g) = (u - F, v - F) \). Therefore

\[
\frac{\| \mathbb{P}(u \otimes v) \|}{\| u \| \| v \|} = 1 + 2 \Re \langle T f, g \rangle + \langle T f, f \rangle + \langle T(g), g \rangle - \| (f, g) \|^2 + O(\| (f, g) \|^3)
\]

\[
\leq 1 + \frac{d + 2}{2(d + 1)} \| (f, g) \|^2 - \| (f, g) \|^2 + O(\| (f, g) \|^3)
\]

\[
\leq 1 - \frac{d}{2(d + 1)} \| (f, g) \|^2 + O(\| (f, g) \|^3)
\]

\[
= 1 - \frac{d}{2(d + 1)} \text{dist} ((u, v), \mathfrak{G}^\times)^2 + O(\text{dist} ((u, v), \mathfrak{G}^\times)^3).
\]

Consider next a general ordered pair \( (u, v) \in L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \) satisfying \( \| u \| = \| v \| = 1 \), with the distance from \( (u, v) \) to \( \mathfrak{G}^\times \) sufficiently small. The closest point in \( \mathfrak{G}^\times \) to \( (u, v) \) may be expressed as \( (aT(F), bT(F)) \) where \( T \) is a norm-preserving element of the group of transformations of \( L^2(\mathbb{R}^d) \) generated by the \( c_t \) and \( r^{d/2} \delta_r \), and \( a, b \in \mathbb{C} \). Therefore the closest element of \( \mathfrak{G}^\times \) to \( (\tilde{u}, \tilde{v}) = (a^{-1}T^{-1}(u), b^{-1}T^{-1}(v)) \) is \( (F, F) \). We have shown that

\[
\frac{\| \mathbb{P}(u \otimes v) \|}{\| u \| \| v \|} = \frac{\| \mathbb{P}(\tilde{u} \otimes \tilde{v}) \|}{\| \tilde{u} \| \| \tilde{v} \|} \leq 1 - \frac{d}{2(d + 1)} \text{dist} ((\tilde{u}, \tilde{v}), \mathfrak{G}^\times)^2 + O(\text{dist} ((\tilde{u}, \tilde{v}), \mathfrak{G}^\times)^3)
\]

\[
= 1 - \frac{d}{2(d + 1)} \text{dist} ((a^{-1}u, b^{-1}v), \mathfrak{G}^\times)^2 + O(\text{dist} ((a^{-1}u, b^{-1}v), \mathfrak{G}^\times)^3).
\]

Now

\[
1 = \| u \|^2 = |a|^2 \| F \|^2 + \| u - aT(F) \|^2 = |a|^2 + \| u - aT(F) \|^2,
\]

so \( |a|^2 = 1 - \| u - aT(F) \|^2 \) and consequently \( |a^{-1}| = 1 + O(\text{dist} (u, v), \mathfrak{G}^\times)^2 \). Likewise \( |b^{-1}| = 1 + O(\text{dist} (u, v), \mathfrak{G}^\times)^2 \). Therefore

\[
\text{dist} ((a^{-1}u, b^{-1}v), \mathfrak{G}^\times) = \text{dist} ((u, v), \mathfrak{G}^\times) + O(\text{dist} ((u, v), \mathfrak{G}^\times)^2).
\]

Inserting these estimates into the above result gives

\[
(6.3) \quad \frac{\| \mathbb{P}(u \otimes v) \|}{\| u \| \| v \|} = 1 - \frac{d}{2(d + 1)} \text{dist} ((u, v), \mathfrak{G}^\times)^2 + O(\text{dist} (u, v), \mathfrak{G}^\times)^3)
\]

as was to be shown. \( \square \)
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MICHAEL CHRIST, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CA 94720-3840, USA

E-mail address: mchrist@berkeley.edu