The spectrum, radiation conditions and the Fredholm property for the Dirichlet Laplacian in a perforated plane with semi-infinite inclusions

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Abstract

We consider the spectral Dirichlet problem for the Laplace operator in the plane $\Omega^\circ$ with double-periodic perforation but also in the domain $\Omega^\bullet$ with a semi-infinite foreign inclusion so that the Floquet-Bloch technique and the Gelfand transform do not apply directly. We describe waves which are localized near the inclusion and propagate along it. We give a formulation of the problem with radiation conditions that provides a Fredholm operator of index zero. The main conclusion concerns the spectra $\sigma^\circ$ and $\sigma^\bullet$ of the problems in $\Omega^\circ$ and $\Omega^\bullet$, namely we present a concrete geometry which supports the relation $\sigma^\circ \subsetneq \sigma^\bullet$ due to a new non-empty spectral band caused by the semi-infinite inclusion called an open waveguide in the double-periodic medium.

Keywords: periodic perforated plane, Dirichlet problem, semi-infinite open waveguide, radiation conditions, Fredholm operator of index zero.

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1 Introduction

1.1 Formulation of problems.

Let $\omega \subset \mathbb{R}^2$ be a non-empty open set with smooth boundary $\partial \omega$ such that the closure $\overline{\omega} = \omega \cup \partial \omega$ belongs to the rectangle

$$Q = \{x = (x_1, x_2) : |x_j| < l_j, \ j = 1, 2\}, \ l_j > 0.$$  \hfill (1.1)

An infinite domain $\Omega^\circ$, fig. \text{1} a, is the perforated plane

$$\Omega^\circ = \mathbb{R}^2 \setminus \bigcup_{\alpha \in \mathbb{Z}^2} \overline{\omega(\alpha)}$$ \hfill (1.2)

where $\alpha = (\alpha_1, \alpha_2), \ Z = \{0, \pm 1, \pm 2, \ldots\}$ and

$$\omega(\alpha) = \{x : (x_1 - 2\alpha_1 l_1, x_2 - 2\alpha_2 l_2) \in \omega\}.$$  \hfill (1.3)
Figure 1: The double-periodic perforated plane (a) and the semi-infinite inclusion in it (b). The periodicity cell is shaded.

Another domain $\Omega^\bullet$, fig. 1 b, is obtained from $\Omega^\circ$ by filling one ($j = 1$) or several semi-infinite rows of holes (1.3) with $\alpha_1 \in \mathbb{N} = \{1, 2, 3, \ldots\}$ and $\alpha_2 = 1, \ldots, J$, that is,

$$\Omega^\bullet = \Omega^\circ \cup \Xi^+ := \Omega^\circ \cup \{x_1 : x_1 > l_1, x_2 \in (l_2, 2jl_2 + l_2)\}.$$  \hfill (1.4)

The spectral Dirichlet problem

$$-\Delta u(x) = \lambda u(x), \ x \in \Omega^\bullet,$$  \hfill (1.5)

$$u(x) = 0, \ x \in \partial \Omega^\bullet,$$  \hfill (1.6)

and its weak formulation

$$(\nabla u, \nabla v)_{\Omega^\bullet} = \lambda (u, v)_{\Omega^\bullet}, \ \forall v \in H^1_0(\Omega^\bullet)$$  \hfill (1.7)

are associated with an unbounded positive definite self-adjoint operator $A^\bullet$ in $L^2(\Omega^\bullet)$ because the bilinear form on the left-hand side of the integral identity (1.7) is a positive definite form, closed in $H^1_0(\Omega^\bullet)$, see, e.g., [1, Ch 10]. Since the boundary $\partial \Omega^\bullet$ is smooth, the domain of this operator becomes

$$\mathcal{D}(A^\bullet) = H^2(\Omega^\bullet) \cap H^1_0(\Omega^\bullet).$$  \hfill (1.8)

In (1.5) and (1.7), $\nabla = \text{grad}, \ \Delta = \nabla \cdot \nabla$ is the Laplace operator, $\lambda$ a spectral parameter, $(\cdot, \cdot)_{\Omega^\bullet}$ the natural scalar product in the Lebesgue space $L^2(\Omega^\bullet)$, $H^2(\Omega^\bullet)$ the Sobolev space, and $H^1_0(\Omega^\bullet)$ the subspace of functions $u \in H^1(\Omega^\bullet)$ satisfying the Dirichlet condition (1.6).

The Dirichlet problem in the double-periodic domain (1.2) is also supplied with the operator $A^\circ$ in $L^2(\Omega^\circ)$ possessing the same general properties as $A^\bullet$. It is known, cf. [2, 3, 4] and others, that the spectrum $\sigma^\circ$ of $A^\circ$ has the band-gap structure

$$\sigma^\circ = \bigcup_{n \in \mathbb{N}} B^\circ_n$$  \hfill (1.9)

where the bands $B^\circ_n$, finite closed connected segments (2.6), will be described in Section 2.1.

According to [5], the spectrum $\sigma^\bullet$ of the operator $A^\bullet$ (and problems (1.5), (1.6) or (1.7)) gets much more complicated structure. One of goals in our paper is to find geometrical shapes in (1.4) such that the spectrum $\sigma^\bullet$ of $A^\circ$ has at least one additional band

$$B^\bullet_0 \subset (0, \lambda^\circ_1]$$  \hfill (1.10)

below the cutoff point $\lambda^\circ_1 = \sigma^\circ := \min \{\lambda : \lambda \in \sigma^\circ\}$ of the spectrum (1.9). However, the main purpose is to describe oscillatory waves which are localized near the semi-infinite intact strip, cf. (1.4),

$$\Xi^+ = (l_1, +\infty) \times (l_2, (2j + 1)l_2)$$  \hfill (1.11)
and travel along it. These waves decay exponentially as $x_2 \to \pm \infty$ and require for a radiation principle to detect direction of their propagation, see Section 5. Moreover, radiation conditions provide the problem (1.5), (1.6) with a Fredholm operator of index zero.

Using a primitive trick we also indicate geometries, fig. 2, a and b, which support trapped modes, i.e. eigenfunctions with the exponential decay in all directions. Our approach can be readily adapted to other shapes of open waveguides, see Section 5.4.

1.2 Preliminary discussion

Artificial experiments and natural phenomena demonstrate that waves may propagate along rows of foreign inclusions in homogeneous and periodic composite media. Classical mathematical tools to describe such wave processes used to consider cases when corresponding boundary-value problems keep periodicity at least in one direction, cf. the review papers [6, 7, 8] and others. In our case the Dirichlet problem

$$-\Delta v (x) - \lambda v (x) = f (x), \quad x \in \Omega^\sharp, \quad v (x) = 0, \quad x \in \partial \Omega^\sharp,$$

must be posed in the domain, fig. 2 a, $\Omega^\sharp = \Omega^\circ \cup \Xi$ (1.14)

with the infinite strip $\Xi = \mathbb{R} \times (l_2, (2j + 1)l_2)$, cf. (1.11). Reducing size $l_1$ to 1/2 by rescaling, we express the remaining periodicity along the $x_1$-axis as follows:

$$\Omega^\sharp = \{ x : (x_1 \pm 1, x_2) \in \Omega^\sharp \}.$$

(1.15)
Taking (1.15) into account, we apply the partial Gelfand transform [9], see also [10] and [11] §3.4,

\[ v(x) \mapsto V(x; \zeta) = \frac{1}{\sqrt{2\pi}} \sum_{\alpha_1 \in \mathbb{Z}} e^{-i\zeta \alpha_1} v(x_1 + \alpha_1, x_2) \]  

(1.16)

where \( \zeta \in [-\pi, \pi] \) is the dual variable or the Floquet parameter, and reduce (1.12), (1.13) to the

parameter-dependent problem

\[ -\Delta V(x; \zeta) - M(\zeta)V(x; \zeta) = F(x; \zeta), \quad x \in \Pi^\sharp, \]  

\[ V(x; \zeta) = 0, \quad x \in \Gamma^\sharp, \]  

(1.17)

(1.18)

with the quasi-periodicity conditions

\[ V\left(\frac{1}{2}, x_2; \zeta\right) = e^{i\zeta} V\left(-\frac{1}{2}, x_2; \zeta\right), \quad \frac{\partial V}{\partial x_1}\left(\frac{1}{2}, x_2; \zeta\right) = e^{i\zeta} \frac{\partial V}{\partial x_1}\left(-\frac{1}{2}, x_2; \zeta\right), \quad x_2 \in \mathbb{R}, \]  

(1.19)

in the perforated strip which is shaded in fig. 3 a, and redrawn in fig. 3 b,

\[ \Pi^\sharp = \left\{ x \in \Omega^\sharp : |x_1| < 1/2 \right\}. \]  

(1.20)

Here, \( M(\zeta) \) is a new notation for the spectral parameter, the conditions (1.19) are imposed on the lateral

sides of the strip and \( \Gamma^\sharp \) stands for the interior part of the boundary \( \partial \Pi^\sharp ) , \)

\[ \Gamma^\sharp = \left\{ x \in \partial \Pi^\sharp : |x_1| < 1/2 \right\} = \bigcup_{\alpha_2 \in \mathbb{Z}\{1, \ldots, J\}} \partial \omega(0, \alpha_2). \]  

(1.21)

Several fruitful approaches have been developed to indicate trapped modes, namely solutions \( V \in H^2(\Pi^\sharp) \) of the homogeneous \( (F = 0) \) problem (1.17)–(1.19), see again the review papers [6, 7, 8] and

many other publications. If \( M(\zeta) \) is an eigenvalue of this problem in \( \Pi^\sharp \) and \( V(\cdot; \zeta) \) is the corresponding

eigenfunction, then the Floquet wave

\[ v(x; \zeta) = e^{i\zeta \alpha_1} V(x; \zeta), \quad (x_1 - \alpha_1, x_2) \in \Pi^\sharp, \quad \alpha_1 \in \mathbb{Z}, \]  

(1.22)

becomes smooth in the 1-periodic domain (1.15) due to the quasi-periodicity conditions (1.19) and satisfies the homogeneous \( (f = 0) \) problem (1.12), (1.13). Moreover, it gains the exponential decay as \( x_2 \to \pm \infty \) but oscillates in the \( x_1 \)-direction. In other words, the wave is localized near the horizontal strip \( \Xi \) and propagates along it.

However, this direct and inherent way to detect localized propagative waves does not work in a case

when the periodicity in the \( x_1 \)-coordinate is disturbed even inside a finite volume, cf. fig. 2 a, because

the partial Gelfand transform (1.16) no longer applies. On this issue, there exists quite few results,

e.g., [12, 13], in particular about the absence of trapped waves decaying in all directions. The paper [5]

provides a description of the essential spectrum of semi-infinite and broken open waveguides but gives

neither concrete examples of localized waves, nor necessary radiation conditions but our paper partly

eliminates these omissions.

In the sequel we display localized waves under certain restrictions, some of which, especially shape and

homogeneous structure of the open waveguide (1.11), can be easily avoided and have been introduced in

order to simplify demonstration. The principal requirement concerns the position of the special spectral

band \( B^\natural_0 \) which gives rise to localized waves, namely it is situated below the spectrum \( \sigma^\sharp \) in the double-

periodic domain (1.2) while general results in [3] permit for nucleation of new bands inside each of the

spectral bands in \( \sigma^\sharp \). To provide the existence of the band (1.10) we need the assumption (2.17) below

which means that the foreign semi-infinite inclusion filling some holes, see fig. 1 b, is sufficiently wide. This assumption imposes an upper bound for the spectral parameter

\[ \lambda \in (0, \lambda^\sharp) \]  

(1.23)
so that the method developed in this paper does not allow us to examine the whole spectrum $\sigma^\bullet$ but only its bottom part.

The key point in our analysis of problem (1.5), (1.6) in domain $\Omega^\bullet$ in fig. 1 b, is a localization weighted estimate derived in Section 5.1 that proves the exponential decay of a solution in $\Omega^\bullet$ in each direction to infinity, except along the semi-infinite inclusion $\Xi^+$. A trick we use below to derive this estimate, is based on integration by parts and simple algebraic operations but works only in the case (1.23) where $\lambda^j \in (0, \lambda^\circ)$ is a certain bound, see (2.17), due to a technical reason. In this way, it remains an open question to construct an elemental example of specific propagative waves with the spectral parameter $\lambda^\circ$ inside a non-empty gap between the bands $B^\circ_n$ and $B^\circ_{n+1}$ with $n \geq 1$ in the spectrum (1.9). In [14] examples of arbitrarily many non-empty spectral bands for the Dirichlet Laplacian in a double-periodic perforated plane are given. Investigation of the spectral bands for other geometries of double-periodic two-dimensional structures are performed in [15] and [16].

1.3 Structure of the paper

In Sections 2 and 3 we present some mainly known information of the spectra of the Dirichlet problem in the domains $\Omega^\circ$, $\Omega^\bullet$ and Floquet waves localized near the infinite inclusion $\Xi$. In Section 4 we derive asymptotics at infinity of solutions to the inhomogeneous problem (1.12), (1.13). Although we follow the scheme in [10], [11, § 3.4], we have to repeat all arguments because the model problem (1.17)-(1.19) is posed in the infinite periodicity cell $\Pi^\circ$, (1.20). We also verify in Section 2.3 that this problem with the spectral parameter (1.23) supports just one trapped mode, i.e., an eigenfunction in $H^2(\Pi^\circ)$ with the exponential decay at infinity.

We start the last but central section with proving the localization estimate which demonstrates that a solution of problem (1.12), (1.13) with the permitted exponential growth in all directions in the plane actually decays in all direction except along the semi-infinite inclusion $\Xi^+$. Together with asymptotic formulas from Section 4.4 which helps to detach the above-mentioned localized Floquet waves, that estimate allows us to formulate in Section 5.2 radiation conditions which supply the operator of problem in $\Omega^\bullet$ with index zero. It should be emphasized that the classification ”outgoing/ incoming” for waves propagating along the open waveguide $\Xi^+$ is based on calculation of the Poynting vector and application of the Mandelstam energy principle. We also construct in Section 5.3 the (right) parametrix for the operator of problem (1.12), (1.13) in order to confirm its Fredholm property and in Section 5.4 we demonstrate the existence of trapped modes, that is, eigenfunctions enjoying the exponential decay in all directions. We finish the paper with mentioning available generalizations.

2 Spectra

2.1 The periodicity cell

In the framework of the Floquet-Block theory the Gelfand transform [9], see also [2, 3, 4] and others, applied to the Dirichlet problem in $\Omega^\circ$, see (1.2), generates the following spectral problem in the perforated rectangle (1.1), the periodicity cell $\omega = \mathbb{Q} \setminus \varpi$ shaded in fig. 1

\[-\Delta U(x; \eta) = \Lambda(\eta) U(x; \eta), \quad x \in \omega, \quad (2.1)\]
\[U(x; \eta) = 0, \quad x \in \partial \omega, \quad (2.2)\]

with the quasi-periodicity conditions

\[\left. \frac{\partial^p U(x; \eta)}{\partial x_j^p}(x; \eta) \right|_{x_j = l_j} = e^{i\eta_j l_j} \left. \frac{\partial^p U(x; \eta)}{\partial x_j^p}(x; \eta) \right|_{x_j = -l_j}, \quad p = 0, 1, \quad j = 1, 2. \quad (2.3)\]

Here,

\[\eta \in (\eta_1, \eta_2) \in Y = \left[ -\frac{\pi}{2l_1}, \frac{\pi}{2l_1} \right] \times \left[ -\frac{\pi}{2l_2}, \frac{\pi}{2l_2} \right] \quad (2.4)\]
is the dual variable of the Gelfand transform, the Floquet parameter. Note that we have set \( \ell_1 = 1/2 \) in Section 1.2. Problem (2.1)–(2.3) has the discrete spectrum composing the monotone unbounded sequence

\[
0 < \Lambda_1 (\eta) \leq \Lambda_2 (\eta) \leq \ldots \leq \Lambda_n (\eta) \leq \ldots \to +\infty
\]

where eigenvalues are listed according to their multiplicity. The functions \( \mathcal{Y} \ni \eta \mapsto \Lambda_n (\eta) \) are continuous and \( \pi \ell_j^{-1} \)-periodic in \( \eta_j, j = 1, 2 \), while the bands in (1.9) are the connected, closed and finite segments

\[
B^\circ_n = \{ \Lambda_n (\eta) : \eta \in \mathcal{Y} \}.
\]

It is known, see, e.g., [2, 3, 4], that the union \( \sigma^\circ \) of the bands (2.6) represents the whole spectrum of the Dirichlet problem [13, 14] in the double-periodic domain \( \Omega^\circ \).

### 2.2 The lower bound \( \lambda^\circ_1 \) of the spectrum \( \sigma^\circ \)

The next assertion is a piece of the mathematical folklore and the authors do not know the very origin of this result which is supplied with a condensed proof for reader’s convenience since it will be of further use in Lemmas 3 and 4. We, for example, refer to [17] where a similar trick was used in homogenization.

**Lemma 1** There holds the relationship

\[
\Lambda_1 (0) < \Lambda_1 (\eta) \quad \forall \eta \in \mathcal{Y}, \quad \eta \neq 0 = (0, 0).
\]

**Proof.** Since [23] with \( \eta = 0 \) turns into the periodicity conditions, by virtue of the strict maximum principle, the principal eigenfunction \( U_1 (x; 0) \) can be fixed real positive for \( x \in \mathcal{W} \) as well as \( \partial_\nu U_1 (x; 0) < 0 \) for \( x \in \partial \omega \) where \( \partial_\nu \) is the outward normal derivative on \( \partial \mathcal{W} \). Hence, \( Z (x; \eta) = U_1 (x; 0)^{-1} U_1 (x; \eta) \) is continuously differentiable in \( \mathcal{W} \) up to the smooth boundary \( \partial \omega \), in particular belongs to \( H^1 (\mathcal{W}) \). Then the integral identity serving for problem (2.1)–(2.3), assures that

\[
\Lambda_1 (\eta) \left\| U_1^\eta; L^2 (\mathcal{W}) \right\|^2 = \left( \nabla (ZU_1^\eta), \nabla (ZU_1^\eta) \right)_\mathcal{W}
\]

\[
= (Z \nabla U_1^\eta, Z \nabla U_1^\eta)_{\mathcal{W}} + \left( U_1^\eta \nabla Z, Z \nabla U_1^\eta \right)_{\mathcal{W}} + \left( U_1^\eta \nabla Z, U_1^\eta \nabla Z \right)_{\mathcal{W}}
\]

\[
= - (Z \triangle U_1^\eta, U_1^\eta)_{\mathcal{W}} - 2 (Z \nabla U_1^\eta, U_1^\eta \nabla Z)_{\mathcal{W}}
\]

\[
+ (Z \nabla U_1^\eta, U_1^\eta \nabla Z)_{\mathcal{W}} + \left( U_1^\eta \nabla Z, Z \nabla U_1^\eta \right)_{\mathcal{W}} + \left\| U_1^\eta \nabla Z; L^2 (\mathcal{W}) \right\|^2
\]

\[
= \Lambda_1 (0) \left\| U_1^\eta; L^2 (\mathcal{W}) \right\|^2 + \left\| U_1^\eta \nabla Z; L^2 (\mathcal{W}) \right\|^2 + \left( U_1^\eta \nabla Z, Z \nabla U_1^\eta \right)_{\mathcal{W}} - (Z \nabla U_1^\eta, U_1^\eta \nabla Z)_{\mathcal{W}}
\]

where \( U_1^\eta (x) = U_1 (x; \eta) \). Being pure imaginary, the latter difference vanishes because all other terms on the left and right in (2.8) are real. The function \( Z = U_1^\eta / |U_1^\eta| \) cannot be constant in \( \mathcal{W} \) for \( \eta \neq 0 \) due to the periodicity of \( U_1^\eta \) and the authentic quasi-periodicity of \( U_1^\eta \). This concludes with (2.7).

We further need the principal eigenvalue \( \Lambda^* > 0 \) of the mixed boundary-value problem

\[
- \triangle U (x) = \Lambda^* U (x), \quad x \in \mathcal{W}, \quad \partial_\nu U (x) = 0, \quad x \in \partial \Omega = \partial \mathcal{W} \setminus \partial \omega,
\]

\[
U (x) = 0, \quad x \in \partial \omega,
\]

together with the Friedrichs inequality

\[
\left\| \nabla U; L^2 (\mathcal{W}) \right\|^2 \geq \Lambda^* \left\| U; L^2 (\mathcal{W}) \right\|^2 \quad \forall U \in H^1_0 (\mathcal{W}, \partial \omega)
\]

where \( H^1_0 (\mathcal{W}, \partial \omega) \) consists of functions in \( H^1 (\mathcal{W}) \) which verify (2.10). By the min principle, cf. [11] Thm. 10.2.1, we have

\[
\lambda^\circ_1 = \Lambda_1 (0) \geq \Lambda^*.
\]

Notice that \( \Lambda_1 (0) = \Lambda^* \) when the cell \( \mathcal{W} \) is symmetric with respect to both axes \( x_1 \) and \( x_2 \).
2.3 Trapped modes

We fix some $\zeta \in [-\pi, \pi]$ and consider the Helmholtz equation

$$- \Delta V(x; \zeta) = M(\zeta) V(x; \zeta), \quad x \in \Pi^2,$$

(2.13)

with the Dirichlet (1.18) and quasi-periodicity (1.19) conditions. This problem is associated, see [1, §10.1], with a positive definite self-adjoint operator $A^\sharp(\zeta)$ in $L^2(\Pi^2)$ with the domain

$$\mathcal{D}(A^\sharp(\zeta)) = \{ V \in H^2(\omega) : \text{ (1.18) and the first equation in (1.19) are met} \}.$$  

(2.14)

According to [10], see also [11, §3.4], the essential spectrum $\sigma^\sharp_e(\zeta)$ of $A^\sharp(\zeta)$ takes the form

$$\sigma^\sharp_e(\zeta) = \bigcup_{n \in \mathbb{N}} B^\sharp_n(\zeta), \quad B^\sharp_n(\zeta) = \{ \Lambda_n(\zeta, \eta_2) : \eta_2 \in [-\frac{\pi}{l_2}, \frac{\pi}{l_2}] \}. $$

(2.15)

Hence, we recall that $l_1 = 1/2$ and, in view of (1.9) and (2.6), write

$$\sigma^\circ_e = \sigma = \bigcup_{\zeta \in (-\pi, \pi]} \sigma^\sharp_e(\zeta), \quad B^\circ_n(\zeta) = \bigcup_{\zeta \in [-\pi, \pi]} B^\sharp_n(\zeta).$$

(2.16)

Using a standard argument, see [18] and, e.g., [19], we examine the discrete spectrum $\sigma^\circ_d(\zeta)$ of $A_d(\zeta)$ at $\zeta = 0$ inside interval (1.29) and then discuss the case $\zeta \neq 0$. In what follows we fix width $2l_2J$ of the strip $[1,11]$ sufficiently large.

Lemma 2 Under the condition

$$\lambda^\sharp := M^\sharp := \pi^2 l_2^{-2} (2J)^{-2} < \min \{ \pi^2, \Lambda^* \} $$

(2.17)

the interval $(0, M^\sharp)$ contains a unique eigenvalue $M_1(0)$, see (1.11) and (2.12), of the operator $A^\sharp(0)$.

Proof. We employ the max-min principle, cf. [1, Thm 10.2.2],

$$M_p(\zeta) = \max_{E^\sharp_{p}(\zeta) \setminus \{0\}} \inf_{V \in E^\sharp_{p}(\zeta) \setminus \{0\}} \frac{\| \nabla V : L^2(\Pi^2) \|^2}{\| V : L^2(\Pi^2) \|^2},$$

(2.18)

where $E^\sharp_{p}(\zeta)$ is any subspace of codimension $p - 1$ in the space

$$E^\sharp(\zeta) = \{ V \in H^1_0(\Pi^2; \Pi^2) : V \left(\frac{1}{2}, x_2\right) = e^{i\zeta} V \left(-\frac{1}{2}, x_2\right), \quad x_2 \in \mathbb{R} \}. $$

(2.19)

Namely, if the right-hand side of (2.18) with a $p \in \mathbb{N}$ is strictly smaller than the lower bound $\sigma^\sharp_d(\zeta)$ of the essential spectrum in (2.15), then the discrete spectrum $\sigma^\circ_d(\zeta)$ contains eigenvalues $M_1(\zeta), \ldots, M_p(\zeta)$ computed by (2.18). It should be emphasized that the space (2.19) differs from (2.14) and, according to [1, Ch.10], coincides with the domain of the bi-linear form $\langle \nabla U, \nabla V \rangle_{\Pi^2}$ in the weak formulation of problem (1.17)-(1.19). Taking $\zeta = 0$ and $p = 1$, we insert into the Rayleigh quotient on the right-hand side of (2.18) the function

$$V_0(x) = \sin \left(\pi (2J)^{-1} (l_2^{-1} x_2 - 1) \right)$$

(2.20)

extended as null from $Q^\sharp = \Pi^2 \cap \Xi$ onto $\Pi^2$. This function lives in $E^\sharp(0)$ and makes the quotient equal to $M^\sharp$ from (2.16). As a result, there exists an eigenvalue, $M_1(0) < M^\sharp < \Lambda^* \leq \sigma^\sharp_d(0)$, the lower bound of the essential spectrum, cf. (2.7), (2.12) and (2.15), while the first inequality is strict because in contrast to our test function, an eigenfunction cannot vanish at a set of positive area. Hence $M_1(0) \in \sigma^\sharp_d(0)$.

Let $p = 2$ in (2.18). The second, that is, first positive eigenvalue $N$ of the Neumann problem in the rectangle $Q^\sharp = (-1/2, 1/2) \times (l_2, (2J + 1) l_2)$, satisfies $N = \pi^2 \min \{1, (2J l_2)^{-2} \} = \pi^2 (2J l_2)^{-2}$, see (2.16), while the orthogonality condition

$$\int_{Q^\sharp} V(x) \, dx = 0$$

(2.21)
assures the Poincaré inequality
\[ \int_{Q^1} |\nabla V(x)|^2 \, dx \geq N \int_{Q^1} |V(x)|^2 \, dx. \] (2.22)

Adding to (2.22) the Friedrichs inequalities (2.11) in the cells \( \varpi(0, \alpha_2) \subset \Pi^2 \) with \( \alpha_2 \in \mathbb{Z} \setminus \{1, \ldots, J\} \), we conclude that any function \( V \) in the subspace
\[ E^\perp(0) = \{ V \in C^2(0) : (2.21) \text{ is fulfilled} \} \] (2.23)
of codimension 1 due to one orthogonality condition imposed, verifies the estimate
\[ \|\nabla V; L^2(\Pi^2)\|^2 \geq \min \{N, \Lambda^*\} \|V; L^2(\Pi^2)\|^2. \] (2.24)

Thus, relations (2.17), (2.24) and (2.18), \( p = 2 \), show that the second eigenvalue \( M_2(0) \), if exists, lays outside the interval \( (0, M^2) \).

In the case \( \zeta \in (0, \pi) \) we replace the test function (2.20) by \( V_\zeta(x) = e^{i\zeta x}; V_0(x) \) which clearly satisfies the first quasi-periodicity condition in (1.19), compare with (2.19). We have \( \|V_\zeta; L^2(\Pi^2)\| = \|V_0; L^2(\Pi^2)\| \) and
\[ \|\nabla V_\zeta; L^2(\Pi^2)\|^2 = \|\nabla V_0; L^2(\Pi^2)\|^2 + \zeta^2 \|V_0; L^2(\Pi^2)\|^2. \]

The max-min principle (2.18), \( p = 1 \), ensures the existence of an eigenvalue \( M_1(\zeta) \in \sigma^1_\delta(\zeta) \) together with the estimate
\[ M_1(\zeta) < M^2 + \zeta^2 \]
but this inference is surely true only under the restriction \( M^2 + \zeta^2 < \sigma^2_\delta(\zeta) \). In view of (2.15) and Lemma 1 the latter is valid for \( |\zeta| < \zeta^2 \) with some \( \zeta^2 > 0 \). Since inequalities (2.22) and (3.1) do not take into account boundary conditions at the lateral sides of the perforated strip \( \Pi^2 \), formula (2.24) does not involve the parameter \( \zeta \) and, therefore, the interval \( (0, M^2) \) may include at most one eigenvalue.

**Lemma 3** There exists \( \zeta^2 > 0 \) such that, for \( |\zeta| < \zeta^2 \), the interval \( (0, M^2) \) contains the only eigenvalue \( M_1(\zeta) \) in the discrete spectrum \( \sigma^1_\delta(\zeta) \) of the operator \( A^\delta(\zeta) \). Moreover,
\[ M_1(\zeta) > M_1(0) \text{ for } \zeta \in (-\zeta^2, 0) \cup (0, \zeta^2). \] (2.25)

**Proof.** It remains to verify (2.26) and we apply the same argument as in Lemma 1. Let \( V_1^\delta(x) = V_1(x, \zeta) \) be the eigenfunction of problem (2.13), (1.13), (1.19) corresponding to \( M_1(\zeta) \). Again, by the strong maximum principle, \( V^\delta \) can be fixed positive in \( \Pi^2 \) with the negative outward normal derivative \( \partial_\nu V_1^0 \) on \( \Gamma^\delta \). The fraction \( Z^\delta(x) = V_1^0(x)^{-1} V_1^\delta(x) \) is continuously differentiable in \( \Pi^2 \) but does not belong to \( H^1(\Pi^2) \). However, all integrals in the modified calculation (2.3),
\[ M_1(\zeta)||V_1^\delta; L^2(\Pi^2)||^2 = M_1(0)||V_1^\delta; L^2(\Pi^2)||^2 + ||V_1^0\nabla Z^\delta; L^2(\Pi^2)||^2 \]
\[ + \left( (V_1^0\nabla Z^\delta, Z^\delta\nabla V_1^0)_{\Pi^2} - (Z^\delta\nabla V_1^0, V_1^0\nabla Z^\delta)_{\Pi^2} \right) \]
\[ = M_1(0)||V_1^\delta; L^2(\Pi^2)||^2 + ||V_1^0\nabla Z^\delta; L^2(\Pi^2)||^2, \] (2.26)
converge owing to the exponential decay of \( V_1^0(x) \) and \( V_1^\delta(x) \) as \( x_2 \to \pm \infty \), cf. Section 7.2. The last norm in (2.20) is positive because \( Z^\delta \) cannot be constant in view of the periodicity of \( V_1^0 \) and the quasi-periodicity of \( V_1^\delta \).

Two typical dispositions of the eigenvalue \( M_1(\zeta) \) below the essential spectrum are depicted in fig. 1 a and b, while \( \zeta^2 = \pi \) in the first case but \( \zeta^2 < \pi \) in the second one.
Figure 4: Several possible positions of the curve $M = M_1(\zeta)$.

When $M_1(\zeta)$ stays in the discrete spectrum, the function $\zeta \mapsto M_1(\zeta)$ is continuous and even (the latter is verified by complex conjugation in problem (2.13), (1.18), (1.19) so that the graphs in fig. 4 are symmetric with respect to the ordinate axis). Hence, the set

$$B^1 = \{ M_1(\zeta) : \zeta \in (-\zeta_0, \zeta_0) \} = [M_1(0), M^2]$$

(2.27)

with $M^2 = \pi^2 (2l_2 J)^{-1}$, is a semi-open segment. According to [5], this set is just a part of the lowest additional segment (1.10) in the essential spectrum $\sigma_e$ of problem (1.5), (1.6).

We are not able to reject the graph $M = M_1(\zeta)$ with several local extrema, cf. fig. 4, c, but the upper bound $M^2$ in (2.27) is fixed such that, for any $M \in (0, M^2) \subset B^1$ (the thick line in fig. 4 a-c), there exist exactly two points $\pm \zeta(M)$ with $M(\pm \zeta(M)) = M$. This restriction is introduced to simplify further notation in Section 5.

3 Localized Floquet waves

3.1 Propagative, standing and resonance waves

First of all, we make the change

$$V(x; \zeta) \mapsto W(x; \zeta) = e^{-i\xi x} V(x; \zeta)$$ (3.1)

and rewrite problem (2.13), (1.18), (1.19) as follows:

$$-\left( \frac{\partial}{\partial x_1} + i\zeta \right)^2 W(x; \zeta) - \frac{\partial^2}{\partial x_2^2} W(x; \zeta) = M(\zeta) W(x; \zeta), \quad x \in \Pi^2,$$

$$W(x; \zeta) = 0, \quad x \in \Gamma^2,$$

$$W\left(\frac{1}{2}, x_2; \zeta\right) = W\left(-\frac{1}{2}, x_2; \zeta\right), \quad \frac{\partial W}{\partial x_1}\left(\frac{1}{2}, x_2; \zeta\right) = \frac{\partial W}{\partial x_1}\left(-\frac{1}{2}, x_2; \zeta\right), \quad x \in \mathbb{R}. \quad (3.4)$$

Now the boundary-value problem (3.2)-(3.4) can be interpreted as a polynomial, actually quadratic, pencil in the complex variable $\zeta \in \mathbb{C}$, that is,

$$\mathbb{C} \ni \zeta \mapsto (\mathfrak{A}^2(\zeta; M) : H^1_{\text{per}}(\Pi^2) \cap H^1_0(\Pi^2; \Gamma^2) \rightarrow L^2(\Pi^2)),$$

(3.5)

see [20] and, e.g., [11, Ch.1] for summary of results. Notice that change (3.1) purposes to make the domain of the pencil independent of $\zeta$.

If, for some $\zeta \in [-\pi, \pi]$, $M$ is an eigenvalue of problem (2.13), (1.18), (1.19) and, therefore, of problem (3.2)-(3.4), then $\zeta$ is an eigenvalue of the pencil $\mathfrak{A}^2(\zeta; M)$ with the same eigenfunction, namely an eigenvector of $\mathfrak{A}(\zeta; M)$. However, the pencil may get associated vectors $W^1, ..., W^{n-1}$ in addition to
an eigenvector \( W^0 \) which all together form a Jordan chain and have to be found out from the abstract equations
\[
\mathfrak{A}^k (\zeta; M) W^k = - \sum_{l=1}^{k} \frac{\partial \mathfrak{A}^l (\zeta; M) W^{k-l}}{\partial \zeta} \quad \text{for} \quad k = 0, \ldots, \kappa - 1. \tag{3.6}
\]
Taking \( \zeta = 0 \) and \( M = M_1 (0) \), we write the corresponding boundary-value problem at \( k = 1 \) as the differential equation
\[
- \Delta W^1 (x) - M_1 (0) W^1 (x) = F^1 (x) := 2i \frac{\partial W^0}{\partial x_1} (x), \quad x \in \Pi^\sharp, \tag{3.7}
\]
with the Dirichlet \([3.3]\) and the periodicity \([3.4]\) conditions. Since \( M_1 (0) \) is a simple eigenvalue due to Lemma \([3]\) and the problem is formally self-adjoint, the Fredholm alternative brings the only compatibility condition in problem \([3.7], [3.3], [3.4]\)
\[
\int_{\Pi^\sharp} W^0 (x) F^1 (x) \, dx = 0 \tag{3.8}
\]
which is easily verified by integration by parts because \( W^0 \) is real. Thus, the problem admits a solution which is determined up to an addendum \( c W^0 \) and can be made pure imaginary. The constructed Jordan chain \( \{ W^0, W^1 \} \) gives rise to two Floquet waves, standing and resonance, namely bounded and with the linear growth,
\[
\begin{align*}
w^0 (x) &= V_1 (x; 0) = W^0 (x), \tag{3.9} \\
w^1 (x) &= ix_1 W^0 (x) + W^1 (x) \tag{3.10}
\end{align*}
\]
which satisfy the homogeneous \((f = 0)\) problem \([1.12], [1.13]\) in the periodic domain \([1.14]\), fig. 3 a.

The associated vector \( W^2 \) of rank 2 must fulfil the differential equation
\[
- \Delta W^2 (x) - M_1 (0) W^2 (x) = F^2 (x) := 2i \frac{\partial W^1}{\partial x_1} (x) - W^0 (x), \quad x \in \Pi^\sharp, \tag{3.11}
\]
with the usual conditions \([3.2] \) and \([3.3]\). This is nothing but the differential form of the abstract equation \([3.7]\) with \( k = 2 \). Note that the right-hand sides of \([3.7]\) and \([3.11]\) involve the first and second-order derivatives of \( \mathfrak{A} (\zeta; M_1 (0)) \) in the variable \( \zeta \). In the next lemma we will prove that this problem has no solution and, therefore, the Jordan chain \( \{ W^0, W^1 \} \) cannot be extended and there is no Floquet wave at \( M = M_1 (0) \) with the quadratic growth as \( x_1 \to \pm \infty \).

**Lemma 4** There holds the formula
\[
b := - (F^2, W^0)_{\Pi^\sharp} > 0. \tag{3.12}
\]

**Proof.** We recall that \( W^0 (x) = V_1 (x, 0) > 0, \quad x \in \Pi^\sharp \), and, hence, the function \( Z = (W^0)^{-1} W^1 \) is pure imaginary and continuously differentiable while all integrals below converge due to the exponential decay of \( W^0 \) and \( W^1 \) as \( x_2 \to \pm \infty \), see Remark \( 7 \). Equation \([3.7]\) turns into
\[
- \nabla \cdot (W^0 \nabla Z) - \nabla W^0 \cdot \nabla Z = 2i \frac{\partial W^0}{\partial x_1} \quad \text{in} \quad \Pi^\sharp.
\]
Multiplying it with \( W^0 Z \) and integrating by parts with the help of the boundary conditions yield
\[
(W^0 \nabla Z, W^0 \nabla Z)_{\Pi^\sharp} = 2i (\partial_1 W^0, W^0 Z)_{\Pi^\sharp} = (\partial_1 W^0, W^0 Z)_{\Pi^\sharp} = -i (W^0, \partial_1 (W^0 Z))_{\Pi^\sharp}. \tag{3.13}
\]
where $\partial_j = \partial/\partial x_j$. Furthermore,

$$b = (W^0, W^0)_{1^2} - 2i \{ Z \partial_t W^0, W^0 \}_{1^2} - 2i \{ W^0 \partial_t Z, W^0 \}_{1^2}$$

$$= (W^0, W^0)_{1^2} - i(Z, \partial_t (W^0)^2)_{1^2} - 2i \{ W^0 \partial_t Z, W^0 \}_{1^2}$$

$$= (W^0, W^0)_{1^2} + i(R, \partial_t Z, W^0)_{1^2} - 2i \{ W^0 \partial_t Z, W^0 \}_{1^2}.$$  \hfill (3.14)

Combining (3.13) and (3.14), we have

$$b = (W^0, W^0)_{1^2} - i \{ W^0 \partial_t Z, W^0 \}_{1^2} + i \{ W^0, W^0 \partial_t Z \}_{1^2} + (W^0 \partial_t Z, W^0 \partial_t Z \}_{1^2}$$

$$= \| W^0 (\partial_t Z + i) + L^2 (1^2) \|^2 + \| W^0 \partial_t Z; L^2 (1^2) \|^2 > 0.$$  \hfill (3.15)

The strict inequality is valid because the $1$–periodic in $x_1$ function $Z$ cannot be equal to $-ix_1$.  \hspace{1cm} \exists

If $M = M_1 (\zeta) \in B^2$ and $0 < |\zeta| < \zeta^2$, then problem (3.12), (3.13) has the Floquet waves

$$w^\pm (x, \zeta) = e^{\pm i\zeta x_1} W (x),$$

(3.16) cf. (3.2) and (3.4), where $W^+$ is an eigenfunction of problem (3.2)–(3.4) with $M = M_1 (\zeta)$, $\zeta \in (0, \pi)$ and $W^- (x) = W^+ (x)$. In contrast to the staying wave (3.9), which is just $1$–periodic in $x_1$, waves (3.16) are oscillatory due to the factors $e^{\pm i\zeta x_1}$ with the period $2\pi/\zeta \neq 1$. The absence of other Floquet waves occurs by virtue of our assumption in Section 2.3 and, in particular, consider the semi-open segment (2.27). A different argument will confirm this fact in Section 5.3.

**Remark 5** The graph in fig. 4, c, which is probably impossible, furnishes four Floquet waves (3.10) for $M \in (M (\pm \pi), M (\pm \pi))$ where

$$M (\zeta) = \max M (\zeta), \quad \zeta \in (0, \pi).$$

Moreover, at $M = M (\pi)$ there appear two linear Floquet waves

$$e^{i\zeta x_1} \left( \pm iW_1^\pm (x) + W_1^\pm (x) \right)$$

(3.18) in addition to waves (3.16) with $\zeta = \zeta$ and $W^\pm = W_1^\pm$. To avoid the incipient inconsistency in the notation, we had introduced a supplementary restriction on $M$ in Section 2.3 and, in particular, consider the semi-open segment $B^2$ even in the case $\zeta = \pi$ because the linear Floquet waves (3.18) are attributed to the point $\zeta = \pm \pi$ in fig. 4, a, too. At the same time, a clear modification of our notation is only needed to cover all the discarded situations.  \hspace{1cm} \exists

### 3.2 Outgoing and incoming waves

In [20] [27], [11] [5.4] it was shown that the symplectic, that is, sesquilinear and anti-Hermitian form

$$q_R (u, v) = \int_{\{x \in \Omega^2; x_1 = R\}} \left( \frac{v (R, x_2)}{\partial x_1} (R, x_2) - u (R, x_2) \right) \frac{\partial u}{\partial x_1} (R, x_2) \, dx_2$$

(3.19) is proportional to the mean-value of the projection on the $x_1$-axis of the Poynting vector [21] which indicates the direction of energy transfer by a propagative wave. We use this observation to classify Floquet waves according to the Mandelstam energy radiation principle [22], see also [23] Ch.1, [24], [25] and others.

Since the form $q_R (u, v)$ appears as a line integral in the Green formula for the Helmholtz operator and, hence, does not depend on the parameter $R > 0$ if $u$ and $v$ satisfy the homogeneous $(f = 0)$ problem (1.12), (1.13). Moreover, after integration in $R \in (n, n + 1)$ we obtain

$$q (u, v) = \int_{\Omega(n)} \left( \frac{v (x)}{\partial x_1} (x) - u (x) \frac{\partial v}{\partial x_1} (x) \right) \, dx,$$

(3.20)
where $\Pi^2 (n) = \{ x : (x_1 - n, x_2) \in \Pi^2 \}$ is a shifted perforated strip $\Pi^2$ and $n \in \mathbb{N}$.

According to the Mandelstam energy principle \[22\] in the interpretation \[26, 27, 11 \S 5.3\], we call a wave $w$ outgoing to infinity in the case $\text{Im} \, q (w, w) > 0$ and incoming from infinity in the case $\text{Im} \, q (w, w) < 0$.

For the Floquet waves (3.10), we have

$$ q (w^\pm, w^\pm) = \int_{\Pi^2} \left( \overline{W^\pm (x)} \frac{\partial W^\pm}{\partial x_1} (x) \pm i \zeta W^\pm (x) - W^\pm (x) \left( \frac{\partial W^\pm}{\partial x_1} (x) \pm i \zeta W^\pm (x) \right) \right) dx \quad (3.21) $$

$$ = 2i \text{Im} \int_{\Pi^2} \overline{W^\pm (x)} \left( \frac{\partial W^\pm}{\partial x_1} (x) \pm i \zeta W^\pm (x) \right) dx = ia^\pm. $$

Recalling the formula $w^+ = \overline{w^+}$, we see that $a^\pm = \pm a$ and $q (w^\pm, w^\pm) = 0$. It should be underlined that the equality $a = 0$ means that the differential equation

$$ - \left( \frac{\partial}{\partial x_1} \pm i \zeta \right)^2 + \frac{\partial^2}{\partial x_2^2} + M (\zeta) W^\pm (x) = 2i \left( \frac{\partial}{\partial x_1} \pm i \zeta \right) W^\pm (x), \quad x \in \Pi^2, \quad (3.22) $$

with conditions (3.3), (3.4) is solvable and, hence, the eigenvalue $\pm \zeta$ of the pencil $\mathfrak{A} (\cdot; M (\zeta))$ has the associated vector $W^\pm (x)$ in addition to the eigenvector $W^\pm$. For $\lambda \in (M_1 (0), M^2)$, the latter contradicts Lemma 2 and our definition of the upper bound $M^2$. In the next section we will show that $a^+ > 0$ if and only if the function $\zeta \mapsto M (\zeta)$ is strictly growing at the point $+\zeta$ so that in accord with fig. 4 a and b, the waves $w^+$ and $w^-$ are outgoing and incoming respectively.

The waves (3.9) and (3.10) at $\lambda = M_1 (0)$ satisfy $q (w^0, w^0) = q (w^1, w^1) = 0$ because $w^0$ and $iw^1$ are fixed real. At the same time, we derive from (3.12), (3.11) that

$$ q (w^1, w^0) = \int_{\Pi^2} \left( \overline{W^0 (x)} \left( ix_1 \frac{\partial W^0}{\partial x_1} (x) + \frac{\partial W^1}{\partial x_1} (x) + iW^0 (x) \right) \right. $$

$$ - \left. \left( ix_1 W^0 (x) + W^1 (x) \right) \frac{\partial W^0}{\partial x_1} (x) \right) dx $$

$$ = \int_{\Pi^2} \left( \overline{W^0 (x)} \left( \frac{\partial W^1}{\partial x_1} (x) + iW^0 (x) \right) - W^1 (x) \frac{\partial W^0}{\partial x_1} (x) \right) dx = ib. $$

Following \[26, 27\] and \[11 \S 5.3\], we introduce the linear wave packets

$$ w^\pm (x; 0) = w^\pm (x) \pm w^0 (x) \quad (3.24) $$

and recognize $w^+$ and $w^-$ are outgoing and incoming, respectively, because $b > 0$ in (3.23) due to Lemma 4 and

$$ q (w^\pm, w^\pm) = \pm 2ib, \quad q (w^\pm, w^\mp) = 0. \quad (3.25) $$

We have classified the Floquet waves (3.24) and (3.16) for the parameter $\lambda \in (M_1 (0), M^2)$ as in fig. 4 a and b.

### 3.3 Some asymptotic formulas

Let

$$ M^\varepsilon = M^0 + \varepsilon \quad (3.26) $$

where $M^0 \in B^2$ and $\varepsilon > 0$. We first set $M^0 = M_1 (0)$ and accept the standard asymptotic ansatzes \[28\] Ch. 9] for eigenvalues and eigenvectors of operator pencils \[4\]

$$ \zeta_{\varepsilon} = \pm \varepsilon^{1/2} \zeta' + \varepsilon \zeta'' + ..., \quad (3.27) $$

$$ W_{\varepsilon} (x) = W^0 (x) \pm \varepsilon^{1/2} \zeta' W^1 (x) + \varepsilon W'' (x) + ... \quad (3.28) $$

\[1\] The book \[28\] deals with linear but non self-adjoint pencils and reduction of our quadratic pencil to that one is obvious.
where denominator in the exponent \(1/2\) of the parameter \(\varepsilon\) is nothing that length 2 of the Jordan chain. We insert them into the equation
\[
\mathcal{A}^2(\zeta^{\pm}; M^0) W^{\pm} = 0
\]
and collect coefficients of \(1, \pm \zeta', \varepsilon^{1/2}\) and \(\varepsilon\). We obtain
\[
\mathcal{A}^2(0; M^0) W^0 = 0, \quad \mathcal{A}^2(0; M^0) W^1 = -\frac{d\mathcal{A}^2}{d\zeta} (0; M^0) W^0,
\]
\[
\mathcal{A}^2(0; M^0) W'' = -\frac{d\mathcal{A}^2}{d\zeta} (0; M^0) \left( \zeta'' W^0 + \zeta' W^1 - \frac{1}{2} (\zeta')^2 \right) - \frac{d^2\mathcal{A}^2}{d\zeta^2} (0; M^0) W^0 + W^0
\]
while the last term is due to the perturbation \(\varepsilon\) in \([3.20]\).

Equations \([3.30]\) are nothing but \([3.6]\) with \(k = 0, 1\) and, by definition, elements of the Jordan chain fulfill them. In view of \([3.26], [3.7]\) and \([3.12], [3.11]\) the compatibility condition in equation \([3.31]\) reads
\[
- (\zeta')^2 b + \|W^0; L^2 (\pi)\|^2 = 0.
\]
The positive root of this quadratic equation
\[
\zeta' = b^{-1/2} \|W^0; L^2 (\pi)\|
\]
specifies the main terms of ansätze \([3.27]\) and \([3.28]\) while general results in \([28\text{ Ch.9}]\) provide estimates of the asymptotic remainders. The obtained formula
\[
\left| \zeta^{\pm} \mp \varepsilon^{1/2} \zeta' \right| \leq c \varepsilon
\]

Together with \([3.20]\) imply that in the vicinity of the point \((0, M_1 (0))\) the graph of the function \(M_1 (\zeta)\) is approximated by the parabola \(M_1 (0) + b \|W^0; L^2 (\pi)\|^2 \zeta^2\) as it is depicted in fig. 4.

Let now \(M^0 \in (M_1 (0), M^1)\). Then the pencil \(\mathcal{A}^2 (\cdot; M^0)\) has two simple eigenvalues \(\pm \zeta^0\) with \(\zeta^0 \in (0, \pi)\) and the corresponding eigenfunctions are denoted by \(W^{\pm}\) while \(W^+ = W^-\). The asymptotic ansätze from the book \([28\text{ Ch.9}]\)
\[
\zeta^{\pm} = \pm \zeta^0 \pm \varepsilon \zeta' + ... \quad W^{\pm} (x) = W^{\pm} (x) + \varepsilon W^{\pm} (x) + ...
\]
inserted into \([3.20]\) leads to the abstract equation
\[
\mathcal{A}^2 (\pm \zeta^0; M^0) W^{\pm} = -\frac{d\mathcal{A}^2}{d\zeta} (\pm \zeta^0; M^0) W^\pm + W^\pm
\]
which turns into the differential equation
\[
- \left( \left( \frac{\partial}{\partial x_1} \pm i \zeta^0 \right)^2 + \frac{\partial^2}{\partial x_2^2} + M^0 \right) W^{\pm} (x) = \pm 2i \zeta' \left( \frac{\partial}{\partial x_1} \pm i \zeta^0 \right) W^{\pm} (x) + W^{\pm} (x), \quad x \in \Pi^2,
\]
with the boundary \([3.33]\) and periodicity \([3.4]\) condition.

In view of \([3.22], [3.21]\) the compatibility condition in this problem becomes
\[
\mp a^\pm \zeta' + \|W^\pm; L^2 (\pi)\|^2 = 0,
\]
where \(a^\pm = \pm a\) are taken from \([3.21]\). This formula furnishes the asymptotic ansätze \([3.32]\) while estimates of remainder are given by \([28\text{ Ch.9}]\).

According to \([3.20]\) and \([3.33]\) we have
\[
\frac{dM_1}{d\zeta} (\pm \zeta) = \pm \frac{1}{\zeta'} = \pm a \|W^\pm; L^2 (\pi)\|^{-2}
\]
and observe that the outgoing Floquet wave in (3.10) corresponds to the point $(+\zeta, M(\zeta))$ at an ascending arc of the graph of the function $M_1$ while the incoming wave to the point $(-\zeta, M(\zeta))$ on a descending arc. This observation in elasticity and acoustics is well-known, cf. [23] Ch.1, [24], [25] and [29], and has two important inferences. First, the Sommerfeld principle which indicates the direction of propagation on waves (3.16) by their wavenumbers $\pm \zeta$, may become wrong, see, e.g., the right descending arc in fig. 4. Second, the limiting absorption principle provides the same classification of waves as the Mandelstam energy principle but may fall through at points of extrema and inflexion. The latter is the real reason why we have chosen the universal energy principle. Notice that restriction $M < M^2$ has been introduced in order to unify our notation and to deal with only a couple of Floquet waves.

4 Detaching asymptotics

4.1 Weighted spaces

Following [10] and [11] §3.4, we study problems (1.17)-(1.19) and (3.2)-(3.4) in the Kondratiev spaces $W^l_{\beta}(\Pi^2)$ obtained by the completion of $C_\infty(\Pi^2)$ (infinitely differentiable functions with compact supports) in the weighted norm

$$||V; W^l_{\beta}(\Pi^2)|| = \left( \sum_{k=0}^{l} ||e^{\beta |x_2|} \nabla^k V; L^2(\Pi^2)||^2 \right)^{1/2},$$

(4.1)

where $l \in \{0, 1, 2, \ldots\}$ and $\beta \in \mathbb{R}$ are the smoothness and weight indexes while $\nabla^k V$ stands for a family of all order $k$ derivatives of $V$. This space consists of all functions in $H^l_{loc}(\Pi^2)$ with the finite norm (4.1) and coincides with $H^l(\Pi^2)$ in the case $\beta = 0$. However, for $\beta > 0 (\beta < 0)$ functions in $W^l_{\beta}(\Pi^2)$ decay exponentially as $x_2 \to \pm \infty$ (some growth at infinity is permitted) while the decay/growth rate is governed by $\beta$. The subspaces $W^l_{\beta,\text{per}}(\Pi^2)$ and $W^l_{\beta,0}(\Pi^2; \Gamma^2)$ are composed from functions satisfying (3.3) and (3.4), respectively.

Problem (1.12), (1.13) in the domain $\Omega^2$ which is infinite in two direction, requires for the weighted space $W^{l,\pm}_{\beta,\gamma}(\Omega^2)$ obtained by the completion of $C_\infty(\Omega^2)$ in the norm

$$||v; W^{l,\pm}_{\beta,\gamma}(\Omega^2)|| = \left( \sum_{k=0}^{l} ||e^{\beta |x_2| + \gamma x^+_1 \cdot \nabla^k v; L^2(\Omega^2)||^2 \right)^{1/2},$$

(4.2)

depending on two weight indexes and using the variables $x^+_1 = |x_1|$ and $x^-_1 = x_1$. The subspace $W^{l,\pm}_{\beta,\gamma,0}(\Omega^2)$ takes into account the Dirichlet condition (1.13).

To derive the key estimates, we also will use the space $W^{l,\pm}_{\beta,\gamma}(\Omega^*)$ in the domain (1.4) in fig. 1 b. We underline a crucial difference between norms (4.2) caused by the superscripts $\pm$. If both $\beta$ and $\gamma$ are positive, the weight with plus in (4.2) grows exponentially in all directions but the weight with minus gets the exponential decay when $|x_2| < \text{const}$ and $x_1 \to +\infty$ but still grows in other radial directions. These properties will allow us to describe asymptotics of solutions near the open waveguide $\Xi^\pm$.

4.2 The problem in the perforated strip

The inhomogeneous problem (3.2)-(3.4) is associated with the mapping

$$W^{2}_{\beta,\text{per}}(\Pi^2) \cap W^{1}_{\beta,0}(\Pi^2; \Gamma^2) =: W^0_{\beta} \ni W \mapsto A_{\beta} (\zeta, M) W = -((\partial_1 + i \zeta)^2 + \partial_2^2 + M W) \in W^0(\Pi^2)$$

(4.3)

which evidently is continuous for any $\beta \in \mathbb{R}$ but, according to [10] and [11] Thm.3.4.6, 5.1.4, is Fredholm if and only if the segments

$$\mathcal{T}^*_{\beta} = \left\{ \xi \in \mathbb{C} : \text{Re} \xi \in \left[ -\frac{\pi}{l_2}, \frac{\pi}{l_2} \right], \text{Im} \xi = \beta \right\}$$

(4.4)
in the complex plane is free of the $\xi$-spectrum of the quadratic pencil \[\text{[20 Ch. 1]}\]

$$\mathbb{C} \ni \xi \mapsto (\mathfrak{A}(\xi; \zeta, M) = - (\partial_1 + i\xi)^2 - (\partial_2 + i\xi)^2 - M : H^2_{\text{per}}(\mathcal{W}) \cap H^1_0(\mathcal{W}; \partial \omega) \to L^2(\mathcal{W})) \quad (4.5)$$

where $\zeta \in \mathbb{C}$, $M \in \mathbb{R}$ are fixed and $H^2_{\text{per}}(\mathcal{W})$ is the Sobolev space of functions which are $2l_j$-periodic in $x_j$, $j = 1, 2$ (recall that $l_1 = 1/2$ and compare \[\text{[1.3]}\] with \[\text{[3.3]}\]).

**Remark 6** Results in \[\text{[10]}\] and \[\text{[11, Ch. 3 and 5]}\] are obtained for general boundary-value problems for elliptic systems in smooth $n$-dimensional domains with periodic outlets to infinity. The presence of the periodicity conditions \[\text{[3.4]}\] does not impair the applicability of those results since the lateral sides $\{\pm 1/2\} \times \mathbb{R}$ of the strip $\Pi^r$ can be identified so that the problem can be posed on a perforated cylindrical surface in $\mathbb{R}^3$. In this way, a literal repetition of arguments in \[\text{[10]}\] proves all assertions in use below. $\boxempty$.

In the case $M < \lambda_0^2$ the segment $\Upsilon^1_0$ is free of the spectrum of the pencil \[\text{[4.3]}\] due to definition of the cutoff value $\lambda_0^2$ and formulas \[\text{[2.15]}, \text{[2.16]}\]. Notice that the Fredholm property of $A_0(\zeta; M)$ with any $\zeta \in [-\pi, \pi]$ implies the formula $M \notin \sigma^0_\lambda$ and the inclusion $M \in \sigma^1_\lambda$ means that the subspace $\ker A_0(\zeta; M)$ is not trivial,

$$\ker A_\beta (\zeta; M) = \{W \in W^2_{\beta,\text{per}}(\Pi^r) \cap W^1_{\beta,0}(\Pi^r) : W \text{ satisfy } (3.2)\}. \quad \text{(4.6)}$$

By inequality \[\text{[2.11]}\], problem \[\text{[2.1]}, \text{[2.3]}\] with $\Lambda(\eta) = M \in (0, \Lambda_1)$ and $\eta \in \mathcal{Y}$ has only trivial solution. Hence, the analytic Fredholm alternative, see, e.g., \[\text{[20, Thm 1.5.1]}\] shows that, for $M \in (0, M^2)$, the $\xi$-spectrum of the pencil $\mathfrak{A}(\cdot; \zeta, M)$, \[\text{[4.3]}\], is a countable set of normal eigenvalues without finite accumulation points. This spectrum is invariant with respect to shifts $\pm \pi/l_2$ along the real axis because the eigenpairs $\{\xi, U^0(x)\}$ and $\{\xi \pm 2\pi/l_2, e^{\mp i\pi x/l_2}U^0(x)\}$ occur simultaneously. Thus, there exists a positive $\beta^2(M)$ such that, for any $\zeta \in [-\pi, \pi]$, the rectangle

$$\Xi^1_{\beta^2(M)} = \{\xi \in \mathbb{C} : |\text{Re}\, \xi| \leq \pi/2l_2, |\text{Im}\, \xi| < \beta^2(M)\} \supset \Upsilon^1_0$$

is free of the $\xi$-spectrum, too. Besides, the theorem on asymptotics, see \[\text{[10, Thm 4]}, \text{[11, Thm 3.4.7 and 5.1.4]}\], ensures that the kernel of the operator $A_\beta (\zeta; M)$ is independent of $\beta \in (-\beta^2(M), \beta^2(M))$,

$$\ker A_\beta (\zeta; M) = \ker A_0 (\zeta; M) \quad \forall \zeta \in [-\pi, \pi]. \quad \text{(4.7)}$$

**Remark 7** In view of \[\text{[3.7]}\] a trapped mode $W^0 \in \ker A_0 (\zeta; M) \subset H^2_{\text{per}}(\Pi^r) \cap H^1_0(\Pi^r; \Gamma^r)$ falls into $W^2_{\beta,\text{per}}(\Pi^r) \cap W^1_{\beta,0}(\Pi^r; \Gamma^r)$ and therefore decays exponentially at infinity. Then the right-hand side $F^1$ of equation \[\text{[3.7]}\] belongs to $W^0_{\beta}(\Pi^r)$. The formally self-adjoint problem \[\text{[3.3]}\] admits a solution $W^1 \in W^2_{\beta,\text{per}}(\Pi^r) \cap W^1_{\beta,0}(\Pi^r; \Gamma^r)$ if and only if

$$\int_{\Pi^r} F^1(x)W(x)dx = 0 \quad \forall W \in \ker A_{-\beta} (0; M). \quad \text{(4.8)}$$

At the same time, $\ker A_{-\beta} (0; M)$ is spanned over the eigenfunction $W^0$ because of \[\text{[4.7]}\] and, thus, \[\text{[4.8]}\] converts into \[\text{[3.8]}\]. $\boxempty$

### 4.3 The problem in the periodic perforated plane.

The composition of the Gelfand transform \[\text{[1.10]}\] and the change \[\text{[3.4]}\] takes the form

$$v(x) \mapsto W(x; \zeta) = (G v)(x; \zeta) = \frac{1}{\sqrt{2\pi}} \sum_{\alpha_1 \in \mathbb{Z}} e^{-i\zeta(x_1 + \alpha_1)}v(x_1 + \alpha_1, x_2). \quad \text{(4.9)}$$

Note that $x \in \Omega^2$ on the left but $x \in \Pi^r$ on the right in \[\text{[4.9]}\]. By a direct calculation, cf. \[\text{[11, §3.4]}\], transform \[\text{[4.9]}\], establishes the isometric isomorphism

$$W^0_{\beta,0} (\Omega^2) \cong L^2(\Omega^2; W^0_{\beta}(\Pi^r)), \quad \text{(4.10)}$$
where
\[ \Upsilon_\gamma = \{ \zeta \in \mathbb{C} : |\text{Re} \, \zeta| \leq \pi, \text{Im} \, \zeta = \gamma \} \]  
(4.11)
and \( L^2 (\Upsilon_0; \mathcal{B}) \) stands for the Lebesgue space of abstract functions with values in a Banach space \( \mathcal{B} \) and the norm
\[ \| W; L^2 (\Upsilon_0; \mathcal{B}) \| = \left( \int_{\Upsilon_0} \| W (\zeta) ; \mathcal{B} \|^2 \, ds \right)^{1/2}. \]

The change
\[ v(x) \mapsto w(x) = e^{-\gamma x_1} v(x) \]
which provides the equivalency of the norms \( \| w; W^0_{\beta, 0}(\Omega^2) \| \) and \( \| v; W^0_{\beta, 0}(\Omega^2) \| \), cf. definition (4.2), passes property (4.10) to the Gelfand transform (4.9) with \( \zeta \in \Upsilon_\gamma \), that is, with a complex-valued dual variable. As a result, we come across the isomorphism, not necessarily isometric,
\[ W^0_{\beta, \gamma}(\Omega^2) \approx L^2 (\Upsilon_\gamma; W^0_{\beta}(\Pi^2)). \]

Let \( \lambda \in (0, M_1 (0)) \). According to Lemma 3, problem (3.2)-(3.4) has no trivial solution in \( H^2 (\Pi^2) \). Owing to the above-mentioned properties of the \( \zeta \)-spectrum of \( \mathfrak{M} (\cdot, \zeta, \lambda) \) the unique solvability of the differential equation
\[ -(\partial_1 + i\zeta)^2 W(x, \zeta) - \partial_2^2 W(x, \zeta) - \lambda W(x, \zeta) = F(x, \zeta), \quad x \in \Pi^2, \]  
(4.12)
with the usual conditions (3.3), (3.4) is also kept in \( W^2_{\beta, \gamma}(\Pi^2) \cap W^1_{\beta, \gamma}(\Pi^2; \Gamma^2) \) for \( F \in W^0_{\beta}(\Pi^2) \) and \( \beta \in (\beta^\gamma (\lambda), \beta^\gamma (\lambda)) \) where \( \beta^\gamma (\lambda) > 0 \) depends on \( \lambda \) and vanishes when \( \lambda \to M_1 (0) - 0 \).

Taking \( f \in W^2_{\beta, \gamma}(\Omega^2) \) and applying the Gelfand transform (4.9), we solve problem (4.12), (3.3), (3.4) with the right-hand side \( F = Gf \) and obtain a unique solution \( W(\cdot, \zeta) = -A_{\beta} (\zeta; M)^{-1} F(\cdot, \zeta) \) together with the estimate
\[ \| W(\cdot, \zeta) ; L^2 (\Upsilon_0; W^2_{\beta}(\Pi^2)) \|^2 \leq c \| F(\cdot, \zeta) ; L^2 (\Upsilon_0; W^0_{\beta}(\Pi^2)) \|^2 \leq C \| f; W^0_{\beta}(\Omega^2) \|^2. \]

The inverse Gelfand transform acts as follows:
\[ W(x, \zeta) \mapsto v(x) = \frac{1}{\sqrt{2\pi}} \int_{\Upsilon_\gamma} e^{i\zeta x_1} W(x_1 - [x_1], x_2, \zeta) \, d\zeta, \]  
(4.13)
see, e.g., [10] and [11] §3.4; here, \( [t] = \max \{ m \in \mathbb{Z} : m \leq t \} \) while \( t = \max \{ \Omega_1, \Omega_2 \} \). It gives us a solution \( v = G^{-1} W \in W^2_{\beta, 0}(\Omega^2) \cap W^1_{\beta, 0}(\Omega^2) \) of problem (1.12), (1.13) which meets the estimate
\[ \| v; W^2_{\beta, 0}(\Omega^2) \|^2 \leq C \int_{-\pi}^{\pi} \| F(\cdot, \zeta) ; W^0_{\beta}(\Pi^2) \|^2 \, d\zeta \leq C \| f; W^0_{\beta}(\Omega^2) \|^2 \]
and is unique because of the above-mentioned uniqueness of \( W(\cdot, \zeta) \).

This standard scheme to solve boundary-value problems in periodic domains breaks in the case
\[ \lambda \in [M_1 (0), M^f) \]  
(4.14)
since problem (3.2)-(3.4) gets a trapped mode for some \( \zeta \in (-\pi, \pi) \).

### 4.4 Asymptotics at infinity

As was deduced in Sections (2.3) and (3.4) problem (2.19), (3.18), (4.19) gains a trapped mode which gives rise to the Floquet waves (3.24) and (3.16) in the homogeneous problem (1.5), (1.6).

Let \( M = \lambda \) in (1.12) be fixed and let \( \beta^\gamma (M) > 0 \) be chosen such that rectangle (4.6) in the complex plane includes just two real points \( \zeta = \pm \zeta (M) \in \Upsilon^f_\gamma \), recall the notation in Section 3.1 and an argument in Section 4.2. The problem (3.2)-(3.4) with \( \zeta = \pm \zeta (M) \) gets the eigenfunction \( W^\pm \in H^2_{\text{per}}(\Pi^2) \cap \)
Moreover, there is no other eigenvalue of \( (4.15) \) in the segment \( \Upsilon_0 \), \((4.11)\), and we can fix \( \gamma > 0 \) such that \( \Upsilon_{\pm \gamma} \) is free of the \( \zeta \)-spectrum of the pencil. As a result, repeating an argumentation in the end of Section 4.3 with the replacement \( \Upsilon_0 \mapsto \Upsilon_{\pm \gamma} \) and using the Gelfand transform with complex dual variable deliver two solutions

\[
v^\pm (x) = \frac{1}{\sqrt{2\pi}} \int_{\pm \gamma} e^{i\zeta x_1} A_\beta (\zeta; M)^{-1} F (x_1 - [x_1], x_2; \zeta) d\zeta
\]

of problem \((4.12), (4.13)\) with the right-hand side

\[
f \in W^0_{\beta,\gamma}(\Omega^2) \cap W^0_{\beta,-\gamma}(\Omega^2) = W^0_{\beta,\gamma}(\Omega^2).
\]

Due to the exponential decay of the function \( f \) as \( x_1 \to \pm \infty \), compare \((4.17)\) and \((4.2)\), the Gelfand transform \( F = \mathcal{G} f \) is an analytic abstract function in the variable \( \zeta \in \Xi_\gamma \) with values in \( W^0_{\beta}(\Omega^2) \); moreover, \( F \) is 2\(\pi\)-periodic in \( \Re \zeta \) and continuous up to the boundary of the open rectangle

\[
\Xi_\gamma = \{ \zeta : |\Re \zeta| < \pi, |\Im \zeta| < \gamma \}.
\]

Notice that each of solutions \((4.16)\) is unique in its own class \( W^2_{\beta,-\gamma}(\Omega^2) \cap W^1_{\beta,\gamma}(\Omega^2) \).

We are in position to apply the Cauchy residue theorem to the contour integral

\[
\frac{1}{\sqrt{2\pi}} \int_{\partial \Xi_\gamma} e^{iz x_1} A_\beta (\zeta; M)^{-1} F (x - [x_1], x_2; \zeta) d\zeta.
\]

We observe that in view of 2\(\pi\)-periodicity along the real axis, the integrals along the vertical sides of the rectangle cancel each other while the sum of the integrals along the horizontal sides equals the difference \( v^- (x) - v^+ (x) \) (in both cases direction of integration was taken into account). At the same time, the contour integral turns in the sum of residuals which are to be computed according to the formula

\[
A_\beta (\zeta; M)^{-1} = (\zeta - \zeta_{\pm}^\pm)^{-1} W^\pm R^\pm + R^\pm (\zeta; M), \quad \zeta \in \mathbb{B}_\rho (\zeta^\pm),
\]

where \( \mathbb{B}_\rho (\zeta^\pm) = \{ \zeta \in \mathbb{C} : |\zeta - \zeta_{\pm}^\pm| < \rho \} \) is a disk of a small radius \( \rho > 0 \), \( \zeta_{\pm}^\pm = \pm \zeta \) and \( W^\pm \) respectively are simple eigenvalues and the corresponding eigenvectors of pencil \((3.5)\), see Section 3.3 \( R^\pm \) is a continuous functional in \( W^0_{\beta}(\Omega^2) \) and \( R (\cdot; M) \) is analytic in \( \mathbb{B}_\rho (\zeta^\pm) \). As a result, we conclude the representation

\[
v^- (x) = v^+ (x) + a_+ w^+ (x) + a_- w^- (x)
\]

where \( w^\pm \) are the Floquet waves \((3.16)\) and \( a_\pm \) are some coefficients satisfying the estimate

\[
|a_+| + |a_-| \leq C_\gamma \| f ; W^0_{\beta,\gamma}(\Omega^2) \|.
\]

Recalling the notation \((3.5), (3.10)\) and \((3.24)\), we derive the same formulas \((4.20)\) and \((4.21)\) also in the case \( \lambda = M_1 (0) \) when the eigenvalue \( \zeta = 0 \) of the pencil \( \mathfrak{A} (\cdot; \lambda) \) is of algebraic multiplicity 2 and generates the Jordan chain \( \{ W, W^1 \} \). We only mention that the new resolvent \((4.19)\) gains a pole of degree 2 at the point \( \zeta = 0 \).

**Theorem 8** For \( \lambda \in [M_1 (0), M^2) \), problem \((4.12), (4.13)\) in \( \Omega^2 \), see \((4.14)\) and fig. 3, a, with the right-hand side \((4.17)\) has two solutions \( v^\pm \in W^2_{\beta,\gamma}(\Omega^2) \cup W^1_{\beta,\gamma}(\Omega^2) \) which are given in \((4.16)\) and are related by the asymptotic formula \((4.20)\), where \( w^\pm \) are the Floquet waves \((3.24)\) or \((3.16)\) and the coefficients \( a_\pm \) enjoy estimate \((4.21)\).

In the next section we will interpret \((4.20)\) as an asymptotic decomposition of the growing solution \( v^- (x) \) with the decaying remainder \( v^+ (x) \) when \( x_1 \to +\infty \).
5 Solvability of the problem with radiation conditions

5.1 The localization estimates

The integral identity
\[
(\nabla u, \nabla v)_{\Omega^*} - \lambda (u, v)_{\Omega^*} = f (v) \quad \forall v \in W^{1+}_{\beta, \gamma, 0} (\Omega^*)
\]
(5.1)
serves for the inhomogeneous problem (1.3), (1.6) in the weighted space $W^{1+}_{\beta, \gamma, 0} (\Omega^*) \ni u$. According
to definition (4.2) all terms in (5.1) are defined properly if $f \in W^{1+}_{\beta, \gamma, 0} (\Omega^*)^*$ is an (anti)linear functional
in $W^{1+}_{\beta, \gamma, 0} (\Omega^*)$ and $( \cdot, \cdot )_{\Omega^*}$ is understood as an extension of the scalar product in $L^2 (\Omega^*)$ up to the duality
between $L_{-\beta, -\gamma} (\Omega^*)$ and $L_{\beta, \gamma} (\Omega^*)$. Here, $L_{\beta, \gamma} (\Omega^*)$ is a weighted Lebesgue space with the norm
\[
\| f ; L_{\beta, \gamma} (\Omega^*) \| = \| e^{\beta |x_1| + \gamma |x_2|} f ; L^2 (\Omega^*) \|.
\]  
(5.2)
The weak formulation (5.1) of the problem in $\Omega^*$ generates the continuous mapping
\[
W^{1+}_{-\beta, -\gamma, 0} (\Omega^*) \ni u \mapsto A_{-\beta, -\gamma} (M) u = f \in W^{1+}_{\beta, \gamma, 0} (\Omega^*)^*
\]  
(5.3)
while $A_{\beta, \gamma} (M)$ is adjoint for $A_{-\beta, -\gamma} (M)$.

The following assertion provides the key localization estimate which demonstrates that a growing
solution of the problem with a decaying right-hand side gets the decay property outside a sectorial
neighborhood of the inclusion (1.11).

Lemma 9 Let
\[
\lambda < \Lambda^*, \quad \beta, \gamma > 0 \quad \text{and} \quad \lambda + (\beta + \gamma)^2 < \Lambda^*.
\]  
(5.4)
Then a solution $u \in W^{1+}_{-\beta, -\gamma, 0} (\Omega^*)$ of problem (5.1) with the right-hand side
\[
f (v) = (f, v)_{\Omega^*}, \quad f \in L^2_{\beta, \gamma} (\Omega^*)
\]  
(5.5)
belongs to the space $W^{1+}_{-\beta, -\gamma, 0} (\Omega^*)$ and obeys the estimate
\[
\| u; W^{1+}_{-\beta, -\gamma, 0} (\Omega^*) \| \leq c \| f ; L^2_{\beta, \gamma} (\Omega^*) \| + \| u; W^{1+}_{\beta, \gamma, 0} (\Omega^*) \|
\]  
(5.6)
where the factor $c$ depends on $\lambda$ and $\beta, \gamma$ but is independent of $f$ and $u$.

Proof. Borrowing a trick from [30], we introduce the continuous function $R_R (x) = R_{R_1} (x_1) \times
R_{R_2} (x_2)$ where $R > 0$ is a big parameter and
\[
R_{R_1} (x) = \begin{cases} 
 e^{-\gamma x_1}, & x_1 \geq -R, \\
 e^{2 \beta R - \gamma |x_1|}, & x_1 \leq -R,
\end{cases} \quad R_{R_2} (x) = \begin{cases} 
 e^{\beta |x_2|}, & |x_2| \leq R, \\
 e^{2 \beta R - \beta |x_2|}, & |x_2| \geq R.
\end{cases}
\]  
(5.7)
We set $u_R = R_R u$, $v_R = R_R u_R$ and observe that $u_R \in H^1_0 (\Omega^*)$, $v_R \in W^{1+}_{-\beta, -\gamma, 0} (\Omega^*)$ because
\[
R_R (x) \leq c R e^{-\beta |x_2| - \gamma |x_1|}, \quad |\nabla R_R (x)| \leq (\beta + \gamma) R_R (x).
\]  
(5.8)
Inserting $v_R$ as a test function into (5.1) and performing simple algebraic transformations yield
\[
(R_R f, u_R)_{\Omega^*} = \| \nabla u_R ; L^2 (\Omega^*) \|^2 - \lambda \| u_R ; L^2 (\Omega^*) \|^2
\]
\[
- \| u_R R^{-1}_R \nabla R_R ; L^2 (\Omega^*) \|^2 + \left( (\nabla u_R, u_R R^{-1}_R \nabla R_R)_{\Omega^*} - (u_R R^{-1}_R \nabla R_R, \nabla u_R)_{\Omega^*} \right).
\]
The last difference in brackets is pure imaginary. Hence,
\[
\| \nabla u_R ; L^2 (\Omega^* \setminus \Xi^+) \|^2 - \lambda \| u_R ; L^2 (\Omega^* \setminus \Xi^+) \|^2
\]
\[
- \| u_R R^{-1}_R \nabla R_R ; L^2 (\Omega^*) \|^2 = \text{Re} \left( (R_R f, u_R)_{\Omega^*} - \| \nabla u_R ; L^2 (\Xi^+) \|^2 + \lambda \| u_R ; L^2 (\Xi^+) \|^2 + \| u_R R^{-1}_R \nabla R_R ; L^2 (\Xi^+) \|^2 \right)
\]  
(5.9)
By (5.7), we have \( R_R(x) = e^{-\gamma x_1} \) in \( \Omega^* \) for a big \( R \) and \( R_R(x) \leq e^{\gamma |x_1|+\beta |x_2|} \) in \( \Omega^* \). Hence, the right-hand side of (5.9) does not exceed the expression

\[
c \left( \| f; L_{\beta, \gamma}^2 (\Omega^*) \|, \| u_R; L^2 (\Omega^*) \| + \| u; W_{-\beta, -\gamma}^1 (\Xi^+) \| \right)^2 .
\]

(5.10)

According to formula (5.4), the set \( \Omega^* \setminus \Xi^+ \) consists of the cells \( \varpi (\alpha) \), with \( \alpha \in \mathbb{Z} = \mathbb{Z}^2 \), while the Friedrichs inequality (2.11) leads to the relation

\[
\| \nabla u_R; L^2 (\Omega^* \setminus \Xi^+) \|^2 = \sum_{\alpha \in \mathbb{Z}} \| \nabla u_R; L^2 (\varpi (\alpha)) \|^2 \geq \Lambda \sum_{\alpha \in \mathbb{Z}} \| u_R; L^2 (\varpi (\alpha)) \|^2 = \Lambda \| u_R; L^2 (\Omega^* \setminus \Xi^+) \|^2 .
\]

Taking the last formulas in (5.11) and (5.8) into account, we find some \( \delta > 0 \) such that the left-hand side of (5.13) is bigger than

\[
\delta \| \nabla u_R; L^2 (\Omega^* \setminus \Xi^+) \|^2 + \delta \| u_R; L^2 (\Omega^* \setminus \Xi^+) \|^2 = \delta \| R_R \nabla u + u \nabla R_R; L^2 (\Omega^* \setminus \Xi^+) \|^2 + \delta \| R_R u; L^2 (\Omega^* \setminus \Xi^+) \|^2 \geq \delta \tau \| R_R \nabla u; L^2 (\Omega^* \setminus \Xi^+) \|^2 + \delta \left( 1 - \frac{(\beta + \gamma)^2}{1 - \tau} \right) \| R_R u; L^2 (\Omega^* \setminus \Xi^+) \|^2 .
\]

Here, we applied the second relation in (5.8) and the simple formula \( (\alpha + \beta)^2 \geq \tau a^2 - \tau (1 - \tau)^{-1} b^2 \) with \( \tau \in (0, 1) \) and \( (1 - \tau)^{-1} (\beta + \gamma)^2 \geq 1/2 \). We add the expression

\[
\| R_R \nabla u; L^2 (\Xi^+) \|^2 + \| R_R u; L^2 (\Xi^+) \|^2 = \| e^{-\gamma x_1} \nabla u; L^2 (\Xi^+) \|^2 + \| e^{-\gamma x_1} u; L^2 (\Xi^+) \|^2
\]

to both sides of equality (5.9) and estimate its fragments by means of (5.10) and (5.11). As a result, we obtain the inequality

\[
\| R_R \nabla u; L^2 (\Omega^*) \|^2 + \| R_R u; L^2 (\Omega^*) \|^2 \leq C \left( \| f; L_{\beta, \gamma}^2 (\Omega^*) \| + \| u; W_{-\beta, -\gamma}^1 (\Xi^+) \| \right)^2
\]

(5.12)

where \( C \) is independent of \( R, f, u \) and \( R \). Comparing (5.7) and (4.2), we see that the left-hand side of (5.12) exceeds \( \| u; W_{-\beta, -\gamma}^1 (\Omega_R^*) \|^2 \) where \( \Omega_R^* = \{ x \in \Omega^* : |x_j| < R, j = 1, 2 \} \). Thus, the limit passage \( R \to \infty \) in (5.12) provides estimate (5.6) and, therefore, the inclusion \( u \in W_{-\beta, -\gamma}^1 (\Omega^*) \) is valid, too.

To apply in the next section Theorem \# an asymptotics, we prove the following lemma which lifts the smoothness of the weak solution.

**Lemma 10** Under the condition of Lemma \# the solution \( u \) of problem (5.7) falls into \( W_{-\beta, -\gamma}^2 (\Omega^*) \) and fulfills the estimate

\[
\| u; W_{-\beta, -\gamma}^2 (\Omega^*) \| \leq c \| f; L_{\beta, \gamma}^2 (\Omega^*) \| + \| u; W_{\beta, -\gamma}^0 (\Omega^*) \| .
\]

(5.13)

**Proof.** For \( \alpha \in \mathbb{Z}^2 \) and \( p = 0, 1 \), we determine the subdomains \( \Omega^p_\alpha \) \( \{ x \in \Omega^* : |x_j| < 1 + p/2 \} \) and apply local estimates (5.11) of solutions to the Dirichlet problem for the inhomogeneous Helmholtz equation, namely

\[
\| \nabla^2 u; L^2 (\Omega^p_\alpha) \| \leq c \left( \| f; L^2 (\Omega^p_\alpha) \| + \| u; L^2 (\Omega^p_\alpha) \| \right). \]

(5.14)

Owing to the periodic structure of \( \Omega^* \), we detect only finite number of homothetically different couples \( \Omega_{\alpha}^p \in \Omega^1_\alpha \) and therefore can fix the factor \( c \) in (5.14) independent of \( \alpha \) and, of course, of \( f \) and \( u \). Moreover,

\[
0 < c_{\beta, \gamma} \leq \left( \sup_{x \in \Omega_{\alpha}^p} e^{\beta |x_2| - \gamma x_1} \right)^{-1} = \inf_{x \in \Omega_{\alpha}^p} e^{\beta |x_2| - \gamma x_1} \leq C_{\beta, \gamma}.
\]

Inserting the exponential weights inside norms in (5.14) and summing in \( \alpha \) we yield

\[
\| e^{\beta |x_2| - \gamma x_1} \nabla^2 u; L^2 (\Omega^p_\alpha) \| \leq C \left( \| e^{\beta |x_2| - \gamma x_1} f; L^2 (\Omega^p_\alpha) \| + \| e^{\beta |x_2| - \gamma x_1} u; L^2 (\Omega^p_\alpha) \| \right).
\]

(5.15)

Enlarging the weight of \( f \) and taking (5.10) into account lead us to (5.13).
5.2 The problem with radiation conditions.

Let

\[ u \in W^{2+}_{-\beta,-\gamma} (\Omega^*) \cap W^1_{-\beta,-\gamma,0} (\Omega^*) \]  \hspace{1cm} (5.15)

be a solution of the problem

\[ -\Delta u (x) - \lambda u (x) = f (x), \quad x \in \Omega^*, \quad u(x) = 0, \quad x \in \partial \Omega^*, \]  \hspace{1cm} (5.16)

cf. (1.5), (1.6), with the right-hand side

\[ f \in L^2_{\beta,\gamma} (\Omega^*) \]  \hspace{1cm} (5.17)

while the spectral parameter and the weight indexes satisfy (4.14), i.e., \( \beta > 0 \) and \( \gamma > 0 \) are sufficiently small. We multiply the solution (5.15) which, by Lemmas 9 and 10, belongs to \( W^2_{-\beta,-\gamma} (\Omega^*) \) with the cut-off function

\[ \chi \in C^\infty (\mathbb{R}), \quad \chi (x_1) = 1 \quad \text{for} \quad x_1 \geq 2, \quad \chi (x_2) = 0 \quad \text{for} \quad x_1 \leq 1 \]  \hspace{1cm} (5.18)

and arrive at problem (1.12), (1.13) in \( \Omega^\sharp \) for \( u^\chi = \chi u \) with the new right-hand side

\[ f^\chi = \chi f + [\Delta, \chi] u \in L^2_{\beta,\gamma} (\Omega^\sharp). \]

The inclusion holds true because the commutator \([\Delta, \chi] u = 2 \nabla u \cdot \nabla \chi + u \Delta \chi \) has a support in the closed perforated strip \( \overline{\Omega^\star} (1) = \{ x \in \Omega^\star : 1 \leq x_1 \leq 2 \} \subset \overline{\Omega^\sharp} \) where the multipliers \( e^{\gamma x_1} \) do not affect weights in norms (4.2). Hence, we can apply Theorem 8 and conclude the representation (4.20) for \( u^\chi \), namely

\[ u^\chi (x) = a_+ w^+ (x) + a_- w^- (x) + u^\chi (x) \]  \hspace{1cm} (5.19)

with the remainder \( \tilde{u}^\chi \in W^{2-}_{\beta,\gamma} (\Omega^\sharp) \cap W^{1-}_{\beta,\gamma,0} (\Omega^\sharp) \). Observing that \( e^{\gamma x_1} = e^{\gamma x_1} \) for \( x_1 > 1 \) and \( e^{-\gamma x_1} \geq e^{-2\gamma x_1} \) for \( x_1 < 1 \), we have

\[ \chi \tilde{u}^\chi \in W^{2+}_{\beta,\gamma} (\Omega^*), \quad (1 - \chi^2) u \in W^{2+}_{\beta,\gamma} (\Omega^*). \]

Then we multiply (5.19) by \( \chi \), add \((1 - \chi^2) u \) to the result and derive the following representation of solution (5.15):

\[ u (x) = \chi (x_1) \left( a^+ w^+ (x) + a^- w^- (x) \right) + \tilde{u} (x) \]  \hspace{1cm} (5.20)

together with the estimate

\[ |a^+| + |a^-| + ||\tilde{u}; W^{2+}_{\beta,\gamma} (\Omega^*)|| \leq c \left( ||f; L^2_{\beta,\gamma} (\Omega^*)|| + ||u; W^{2+}_{-\beta,-\gamma} (\Omega^*)|| \right). \]  \hspace{1cm} (5.21)

To conclude with (5.21), it should be mentioned that all inclusions written above are accompanied with estimates of the corresponding norms by the same majorants as in (5.21).

**Theorem 11** Let \( \lambda \) and \( \beta, \gamma \) satisfy (5.4). A solution (5.16) of problem (5.16) with the right-hand side (5.17) takes the form (5.20) and estimate (5.21) is valid.

In the case \( a^- = 0 \) we say that solution (5.20) satisfies the Mandelstam radiation conditions. Indeed, it loses the incoming wave \( w^- \) and differs from the outgoing wave \( a^+ \chi w^+ \) localized near the semi-infinite inclusion \( \Xi^+ \), by a function with the exponential decay in all directions.

5.3 Solvability of the problem with the radiation condition

We proceed with the following assertion.
Theorem 12 Let $\lambda$ and $\beta, \gamma$ meet the conditions

$$\lambda \in [M_1(0), M^2] \text{ and } \beta, \gamma > 0, \quad \lambda + \beta^2 + \gamma^2 < \Lambda^*,$$

(5.22)
cf. (5.4). The operators of problem (5.19)

$$\mathcal{A}_{\pm, \pm, \pm}^2 : W_{\pm, \pm, \pm}^2 (\Omega^*) \cap W_{\pm, \pm, \pm, 0}^1 (\Omega^*) \rightarrow L_{\pm, \pm, \pm}^2 (\Omega^*)$$

are Fredholm and their indexes are as follows:

$$\text{Ind} \mathcal{A}_{\pm, \pm, \pm}^2 = \text{dim ker} \mathcal{A}_{\pm, \pm, \pm}^2 - \text{coker} \mathcal{A}_{\pm, \pm, \pm}^2 = \mp 1.$$  
(5.24)

**Proof.** To verify the Fredholm property, we follow a scheme proposed in [5] and construct a (right) parametrix $\mathcal{R}_{\pm, \pm, \pm}^2$ for operator (5.23), that is, a continuous mapping $L_{\pm, \pm, \pm}^2 (\Omega^*) \rightarrow W_{\pm, \pm, \pm}^2 (\Omega^*) \cap W_{\pm, \pm, \pm, 0}^1 (\Omega^*)$ such that $\mathcal{A}_{\pm, \pm, \pm}^2 \mathcal{R}_{\pm, \pm, \pm}^2 - \text{Id}$ is a compact operator in $L_{\pm, \pm, \pm}^2 (\Omega^*)$. Let us outline this scheme with minor modifications. First of all, thanks to the "lifting procedure" in our proof of Lemma (10) we may consider the weak formulation of problem (5.16) in the space $W_{\pm, \pm, \pm, 0}^1 (\Omega^*)$, namely

$$(\nabla u^*, \nabla v^*)_{\Omega^*} - \lambda (u^*, v^*)_{\Omega^*} = f^* (v^*) \quad \forall v^* \in W_{\mp, \mp, \mp}^1 (\Omega^*)$$

(5.25)
and the corresponding operator

$$\mathcal{A}_{\pm, \pm, \pm}^{1*} : W_{\pm, \pm, \pm, 0}^1 (\Omega^*) \rightarrow W_{\mp, \mp, \mp, 0}^1 (\Omega^*)^*.$$  
(5.26)

We also will need formula (5.25) and (5.26) with the change $\bullet \mapsto \circ$ of the superscript. Taking $f^* \in L_{\pm, \pm, \pm}^2 (\Omega^*)$, we annul this function on the foreign inclusion by setting $f^* = X f^* \in L_{\pm, \pm, \pm}^2 (\Omega^*)$ where $X \in C^\infty (\mathbb{R}^2)$ is a cut-off function such that

$$X (x) = 0 \quad \text{for } x \in \mathbb{R}^+, \text{ see (1.11), and}$$

$$X (x) = 1 \quad \text{for } x \in \mathbb{R}^2 \setminus \mathbb{R}^+, \quad \mathcal{X}^o = \{x : x_1 > 1, 0 < x_2 < 2 (J + 1) l_2\}.$$  

Then we perform the substitutions

$$u^o (x) = e^{\mp \beta |x_2| + \gamma |x_1|} u^* (x), \quad v^o (x) = e^{\pm \beta |x_2| + \gamma |x_1|} v^* (x),$$

(5.27)

and obtain from the integral identity (5.25) in $\mathcal{X}^o$ the new one posed in the Sobolev space $H^1_0 (\mathcal{X}^o)$

$$a^o (u^o, v^o) = (\nabla u^o, \nabla v^o)_{\mathcal{X}^o} \mp (\theta u^o, \theta v^o)_{\mathcal{X}^o} \pm (\nabla u^o, \theta v^o)_{\mathcal{X}^o} - (\theta u^o, \theta v^o)_{\mathcal{X}^o} - \lambda (u^o, v^o)_{\mathcal{X}^o}$$

(5.28)
where $\theta (x) = (\gamma \text{sign} x_1, \beta \text{sign} x_2)$. Since $\lambda \notin \sigma^o$, the operator $\mathcal{A}_{0, 0}^o = \mathcal{A}^o$ is an isomorphism and the problem (5.28) at $\theta = 0$ is uniquely solvable in $H^1_0 (\mathcal{X}^o)$. Furthermore, in view of formulas (5.11) and (5.22) we have

$$\text{Re } a^o (u^o, v^o) > (\Lambda^* - (\beta^2 + \gamma^2) - \lambda) \left| u^o ; L^2 (\mathcal{X}^o) \right|^2$$

so that the Lax-Milgram Lemma ensures the unique solvability of problem (5.28), too. The inverse changes (5.27) give us a solution $u^1 \in W_{\pm, \pm, \pm, 0}^1 (\mathcal{X}^o)$ which falls into $W_{\pm, \pm, \pm}^2 (\Omega^o)$ due to an argument in the proof of Lemma (10) with a slight modification. We now multiply $u^1$ with the cut-off function $X$ and observe that the difference $u^* - \mathcal{X} u^1$ must be find from the problem (5.16) with the new right-hand side

$$f^1 = (1 - \mathcal{X}^2) f^* + [\Delta, \mathcal{X}] u^1 \in L_{\pm, \pm, \pm}^2 (\Omega^o)$$

(5.29)
which has a support in the strip $\mathcal{X}^o \supset \mathcal{X}^o$. The latter allows us to fix a sufficiently small $\delta > 0$ such that $\mathcal{X} f^2 \in L_{\pm, \pm, \pm}^2 (\Omega^o)$ and the scheme (10) still works with the weight indexes $\pm \beta + \delta$ and gives a
solution \( u^2 \in W^{2-}_{\pm, \pm \delta, \pm \gamma}(\Omega^2) \cap W^{1-}_{\pm, \pm \delta, \pm \gamma, \delta}(\Omega^2) \). Multiplying this solution with the cut-off function \( \delta \), we observe that, first, \( \chi u^2 \in W^{2+}_{\pm, \pm \delta, \pm \gamma}(\Omega^*) \subset W^{2+}_{\pm, \pm \delta, \pm \gamma}(\Omega^*) \) and, second, it remains to determine the difference

\[
u^3 = u \cdot - \chi u^2 \in W^{2+}_{\pm, \pm \delta, \pm \gamma}(\Omega^*)
\]

from problem \( \text{[5.10]} \) with the right-hand side

\[
f^2 = (1 - \chi^2) f^1 + \lfloor \Delta, \chi \rfloor u^2.
\]

The last commutator belongs to \( W^{1+}_{\pm, \pm \delta, \pm \gamma}(\Omega^*) \) and has a support in the strip \( \Pi^2(1) \) while the embedding \( W^{1+}_{\pm, \pm \delta, \pm \gamma}(\Pi^2(1)) \subset L^2_{\pm, \pm \delta, \pm \gamma}(\Pi^2(1)) \) is compact due to negative increments of the smoothness and weight exponents. The first term on the right-hand side of \( \text{[5.30]} \) has a compact support and a classical construction of a parametrix in a finite smooth domain gives a compactly supported function \( u^3 \in H^2_{\Omega^*}(\Omega^*) \) such that the operator \( f^* \to R^2_{\pm, \pm \delta, \pm \gamma} f^* = \chi u^1 + \chi u^2 + u^3 \) gains the necessary properties. We repeat that a detailed explanation of the above procedure is given in \( [5] \). The operator \( A_{\Omega^*, -\gamma} \) in \( \text{[5.20]} \) is adjoint for \( A_{\Omega^*, \gamma} \) because the form on the left-hand side of \( \text{[5.23]} \) is symmetric. Hence,

\[
\text{Ind} A_{\Omega^*, -\gamma} = -\text{Ind} A_{\Omega^*, \gamma} \Rightarrow \text{Ind} A_{\Omega^*, -\gamma}^2 = -\text{Ind} A_{\Omega^*, \gamma}^2.
\]

The implication is supported by the lifting smoothing procedure in Lemma \( \text{[10]} \). Moreover, the theorem on the index increment, cf. \( [11] \text{ §3.3 and §5.1} \) ensures the equality

\[
\text{Ind} A_{\Omega^*, -\gamma}^2 = -\text{Ind} A_{\Omega^*, \gamma}^2 + 2
\]

where 2 is nothing but the total multiplicity of the spectrum of the pencil \( A(\cdot; \lambda) \) in the rectangle \( \{ \zeta \in \mathbb{C}: \Re \zeta \in (-\pi, \pi], |\Im \zeta| < \beta \} \), see formula \( \text{[3.35]} \) and recall our choice of the upper bound \( M^2 \) in Section \( \text{[2]} \). Combining \( \text{[5.31]} \) and \( \text{[5.32]} \) leads to \( \text{[5.24]} \). ☒

Let us prove the main result of our paper which, owing to the obtained results, can be obtained in a standard way, see, e.g., \( [11] \text{ Ch.5} \).

**Theorem 13** Let \( \text{[5.32]} \) and \( \text{[7.17]} \) be met. Problem \( \text{[5.10]} \) with the Mandelstam radiation condition has a solution

\[
u(x) = \chi(x_1) a^+ w^+(x) + \bar{u}(x)
\]

with \( a^+ \in \mathbb{C} \) and \( \bar{u} \in W^{2+}_{\beta_\gamma, \Omega^*} \cap W^{1+}_{\beta_\gamma, 0, \Omega^*} \) if and only if the right-hand side \( f \) satisfies the compatibility conditions

\[
(f, v)_{\Omega^*} = 0 \quad \forall v \in \ker A_{\Omega^*, \gamma}^2.
\]

This solution is defined up to a trapped mode in \( \ker A_{\Omega^*, \gamma}^2 \) with the exponential decay in all directions. The solution satisfying the orthogonality conditions

\[
(u, v)_{\Omega^*} = 0 \quad \forall v \in \ker A_{\Omega^*, \gamma}^2
\]

becomes unique and enjoys the estimate

\[
|a^+| + ||\bar{u}; W^{2+}_{\beta_\gamma, \Omega^*}|| \leq c_{\beta_\gamma}(\lambda) \| f; L^2_{\beta_\gamma, \Omega^*} \|.
\]

**Proof.** Owing to Theorems \( \text{[12]} \) and \( \text{[8]} \) formulas \( \text{[5.31]} \) and \( \text{[5.17]} \) provide a solution \( u \in W^{2+}_{\beta_\gamma, -\gamma}(\Omega^*) \cap W^{1+}_{\beta_\gamma, 0, \Omega^*} \) together with the representation \( \text{[5.20]} \) where we need to eliminate the coefficient \( a^- \), cf. Section \( \text{[5.2]} \). To this end, we observe that \( \dim(\ker A_{\Omega^*, -\gamma}^2 \ominus \ker A_{\Omega^*, \gamma}^2) = 1 \) and there exists a solution \( z \) of the homogeneous problem \( \text{[1.5]}, \text{[1.6]} \) in the form

\[
z(x) = \chi(x_1)(w^-(x) + sw^+(x)) + \bar{z}(x)
\]

(5.37)
where \( \tilde{z} \in W^{2+}_\beta(\Omega^*) \) and \( s \in \mathbb{C} \) is the reflection coefficient, \( |s| = 1 \). Finally, the difference \( u - a^-\zeta \) satisfies the radiation condition and takes the form (5.33). A solution \( z \) in \( \ker A^{2-}_\beta, \gamma \setminus \ker A^2_\beta, \gamma \neq \emptyset \) has at least one non-trivial coefficient \( a^\pm \) in its representation (5.20). Let us assume that \( a^+ = 0 \) and, therefore, representation (5.37) is not possible. We truncate the domain \( \Omega^* \) like \( \Omega^*_R = \{ x \in \Omega^* : |x| < R, j = 1, 2 \} \). We also denote \( T^*_R = \{ x \in \Omega^* : x_1 = R \} \) and insert \( z \) into the Green formula on \( \Omega^*_R \). Taking the Dirichlet condition (1.4) into account, we have

\[
0 = \int_{\partial \Omega^*_R \setminus \partial \Omega^*} \left( z(x) \partial_n z(x) - z(x) \overline{\partial_n z(x)} \right) dx
\]

(5.38)

where \( \partial_n \) is the outward normal derivative. The exponential decay of \( \tilde{z}(x) \) as \( |x| \to +\infty \) allows us to get rid in (5.38) of the remainder \( \tilde{z} \) and the whole integral over \( (\partial \Omega^*_R \setminus \partial \Omega^*) \setminus T^*_R \). Moreover, we add the integrals along \( \{ x \in \Omega^* : x_1 = R, \pm x_2 > R \} \) and write

\[
|a^+|^2 \int_{T^*_R} \left( w^+(x) \frac{\partial w^+}{\partial x_1}(x) - w^+(x) \frac{\partial w^+}{\partial x_2}(x) \right) dx_2 = O(e^{-\min(\beta, \gamma)R}).
\]

Finally, we integrate in \( R \in (N, N + 1) \) and send \( N \in \mathbb{N} \) to infinity to obtain \( |a^+|^2 q(w^+, w^+) = 0 \) according to (3.20). Recalling (3.21) and (3.25), (3.13), we conclude that \( a^+ = 0 \), \( z \in \ker A^2_\beta, \gamma \), and come across a contradiction to our assumption. The equality \( |s| = 1 \) for the reflection coefficient is verified by a similar calculation based on bi-orthogonality conditions of type (3.25).

5.4 Available generalizations

Many unnecessary restrictions were introduced in our paper to simplify demonstration only. In particular, the Laplace operator \( \Delta \) can be replaced by a formally self-adjoint second-order differential operator in the divergence form \( \nabla \cdot A(x) \nabla \) with a positive definite symmetric \( 2 \times 2 \)-matrix \( A \) with periodic measurable bounded coefficients. The boundary \( \partial \sigma \) of the periodicity cell can be, e.g., Lipschitz while the semi-infinite foreign inclusion can be formed by varying the boundary and the coefficients.

As in [5], the open waveguide may have several outlets to infinity, cf. fig. 5.

Each of open waveguides may enjoy a local perturbation, cf. fig. 2 a and b. By means of the classical approach [13], one can readily detect a point of the discrete spectrum. Let us show the existence of at least one eigenvalue \( \lambda^\bullet \in (0, M_1(0)) \) in the spectrum of the operator \( A^\bullet \) of the Dirichlet problem (1.4), (1.0) in the domain \( \Omega^\bullet = \Omega^* \cup \Xi^+ \cup \Xi_- \) where \( \Xi^\bullet \) is the rectangle \((-J_1l_1, J_1l_1) \times (-J_2l_2, J_2l_2)\) where \( J_1, J_2 \in \mathbb{N} \), see fig. 2 a. The lower bound of the spectrum \( \sigma^\bullet \) of \( A^\bullet \) is still equal to \( M_1(0) \). We choose \( J_1 \) and \( J_2 \) such that the Dirichlet problem in \( \Xi^\bullet \) has the principal eigenpair

\[
\lambda^\bullet = \frac{\pi^2}{4} \left( \frac{1}{J_1^2} + \frac{1}{J_2^2} \right) < M_1(0), \quad u^\bullet(x) = \cos \left( \frac{\pi}{J_1} x_1 \right) \cos \left( \frac{\pi}{J_2} x_2 \right)
\]

\( 23 \)
Extending $u^\bullet$ as null from $\Xi^\bullet$ onto $\Omega^\bullet$, we apply the min principle, cf. [1, Thm 10.2.1], and obtain

$$\sigma^\bullet = \inf_{u \in H^1_0(\Omega^\bullet) \setminus \{0\}} \frac{\|\nabla u; L^2(\Omega^\bullet)\|^2}{\|u; L^2(\Omega^\bullet)\|^2} \leq \frac{\|\nabla u^\bullet; L^2(\Xi^\bullet)\|^2}{\|u^\bullet; L^2(\Xi^\bullet)\|^2} = \lambda^\bullet < M_1(0).$$

Hence, the point $\sigma^\bullet$ of the spectrum of the operator $A^\bullet$ belongs to its discrete spectrum.

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