We study lattice SU(2) Yang-Mills theory with dimension $d \geq 4$. The model can be expressed as a $(d - 1)$-dimensional O(4) non-linear $\sigma$-model in a $d$-dimensional heat bath. As is well known, the non-linear $\sigma$-model alone shows a phase transition. If the quark confinement is a consequence of absence of a phase transition for the Yang-Mills theory, then the fluctuations of the heat bath must destroy the long-range order of the non-linear $\sigma$-model. In order to clarify whether this is true, we replace the fluctuations of the heat bath with Gaussian random variables, and obtain a Langevin equation which yields the effective action of the non-linear $\sigma$-model through analyzing the Fokker-Planck equation. It turns out that the fluctuations indeed destroy the long-range order of the non-linear $\sigma$-model within a mean field approximation estimating a critical point, whereas for the corresponding U(1) gauge theory, the phase transition to the massless phase remains against the fluctuations.
1 Introduction

We study Euclidean SU(2) Yang-Mills theory on the hypercubic lattice \( \mathbb{Z}^d \) with dimension \( d \geq 4 \). It is widely believed that\(^1\) the gauge theory shows a quark confinement phase with a mass gap for all the values of the coupling in dimensions \( d = 4 \). On the other hand, the corresponding U(1) gauge theory in dimensions \( d = 4 \) is proven to show the existence of a deconfining transition to a massless phase \(^2\)\(^3\). Thus it is expected that there exists a crucial difference between SU(2) and U(1) gauge theories.

In this paper, we explore the origin of this difference. For this purpose, we go back to the paper by Durhuus and Fröhlich \(^4\). They showed that the \( d \)-dimensional Yang-Mills system can be interpreted as many \( (d-1) \)-dimensional non-linear \( \sigma \)-models which are stacked up in the \( d \)-th direction and coupled through \( (d-1) \)-dimensional external Yang-Mills fields.\(^5\) When we give our eye to one of the \( (d-1) \)-dimensional non-linear \( \sigma \)-models, the system can be interpreted as a \( (d-1) \)-dimensional non-linear \( \sigma \)-model in a \( d \)-dimensional heat bath. When we turn off the interaction between the non-linear \( \sigma \)-model and the heat bath, the non-linear \( \sigma \)-model becomes the standard O(4) non-linear \( \sigma \)-model because the gauge group SU(2) is homeomorphic to 3-sphere \( \mathbb{S}^3 \). As is well known, the O(4) non-linear \( \sigma \)-model is proven to show a phase transition \(^7\) in dimensions greater than or equal to three. This implies that, if the quark confinement is a consequence of absence of a phase transition for the Yang-Mills theory, then the fluctuations of the external Yang-Mills fields must destroy the long-range order of the O(4) non-linear \( \sigma \)-model.

The effective action of the \( (d-1) \)-dimensional non-linear \( \sigma \)-model can be derived by integrating out the degrees of freedom of the heat bath. However, carrying out the integration is very difficult. Instead of doing so, we replace the fluctuations of the external Yang-Mills fields with Gaussian random variables. Within this approximation, the spins of the non-linear \( \sigma \)-model can be interpreted as “particles” which move on \( \mathbb{S}^3 \), acted by the two-body interaction and the random forces. Namely the dynamics of the “particles” obeys a Langevin equation \(^8\). As is well known, a Langevin dynamics yields a Fokker-Planck equation which describes the time evolution of the distribution of the “particles”. In the present system, the effective action of the non-linear \( \sigma \)-model can be derived from the steady state solution to the corresponding Fokker-Planck equation. In the effective action so obtained, the attractive potential between the two “particles” is modified by the fluctuations of the external Yang-Mills fields.

We show that the height and the width of the barrier of the attractive potential depend on the coupling constant of the Yang-Mills theory. Roughly speaking, the critical value of the coupling constant for the phase transition to a massless phase can be estimated by the height and the width of the barrier of the attractive potential. Therefore the critical value becomes a function of the coupling constant. In consequence, we obtain that within a certain mean field approximation, the critical value is always strictly less than the value of the coupling constant itself for weak couplings. This implies that the critical value must be equal to zero, i.e., there is no phase transition to a massless phase for non-zero coupling constants.

On the other hand, the corresponding U(1) gauge theory shows that the attractive

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\(^1\)See, for example, the book \(^1\).

\(^2\)See also related articles \(^5\).
potential does not depend on the coupling constant for weak coupling constants within the same approximation. Namely the fluctuations of the external Yang-Mills fields does not affect the critical behavior of the O(2) non-linear σ-model.

This paper is organized as follows. In the next section, we express SU(2) Yang-Mills theory in the form of the O(4) non-linear σ-model with a large heat bath, following Durhuus and Fröhlich [4]. In Section 3, we obtain the Langevin equation for the “particles” moving on S^3, by replacing the fluctuations of the heat bath with Gaussian random variables. In the standard procedure, the Langevin equation yields the Fokker-Planck equation for the distribution of the “particles”. In Section 4, a steady state solution to the Fokker-Planck equation is obtained. The result immediately yields the effective action of the non-linear σ-model. Further, we show that the phase transition of the O(4) non-linear σ-model disappears, owing to the fluctuations, within a mean field approximation for the effective action so obtained. In Section 5, we apply the same method to the corresponding U(1) gauge theory, and show that the phase transition to the massless phase remains against the fluctuations.

2 Yang-Mills theory as a σ-model in a heat bath

Let Λ be a sublattice of Z^d. The SU(2) gauge field on Λ is a map from the oriented links or nearest neighbour pairs ⟨q, q’⟩ of sites, q, q’, of the lattice Λ into the Lie group G = SU(2),

⟨q, q’⟩ → U_{qq’} ∈ G,  \hspace{1cm} \text{(2.1)}

obeying

U_{q’q} = (U_{qq’})^{-1}. \hspace{1cm} \text{(2.2)}

Let γ be an oriented path which is written γ = ⟨q_1, q_2⟩⟨q_2, q_3⟩⋯⟨q_{n-1}, q_n⟩ with the oriented links, ⟨qi, qi+1⟩ of the neighboring sites, q_i, q_{i+1}, for i = 1, 2, …, n − 1. When q_1 = q_n, the path γ is a loop. For an oriented path γ, we write

U_γ = U_{q_1q_2}U_{q_2q_3}⋯U_{q_{n-1}q_n}. \hspace{1cm} \text{(2.3)}

The Euclidean action of pure Yang-Mills theory on the lattice Λ ⊂ Z^d is given by

A_d^{YM}(Λ) := -\frac{1}{2} \sum_{p⊂Λ} \text{Re} \text{Tr} U_{∂p}, \hspace{1cm} \text{(2.4)}

where p denotes an oriented plaquette (unit square) of Λ, and ∂p is the oriented loop formed by the four sides of p. The orientation of the loop ∂p obeys the orientation of the plaquette p. The expectation value is given by

⟨⋯⟩_Λ := Z_Λ^{-1} \int \prod_{b⊂Λ} dU_b⟨⋯⟩ \exp \left[ -\beta A_d^{YM}(Λ) \right] \hspace{1cm} \text{(2.5)}

with the inverse temperature β and the normalization Z_Λ, where b is a link in Λ and dU_b is the Haar measure of the gauge group G = SU(2).
Following Durhuus and Fröhlich [4], we use the relation between the $d$-dimensional Yang-Mills action and a $(d-1)$-dimensional non-linear \( \sigma \)-model. The coordinates of a lattice site \( q \) are denoted \( (x(1), x(2), \ldots, x(d-1), x(d)) = (i, x(d)) \) with \( i = (x(1), \ldots, x(d-1)) \in \mathbb{Z}^{d-1} \). Write \( \Lambda_{\tau} = \Lambda \cap \{q : x(d) = \tau\} \) for the \((d-1)\)-dimensional hyperplane, and \( \Lambda^0 = \Lambda \cap \mathbb{Z}^{d-1} \times \{0\} \) for the projection onto \( \mathbb{Z}^{d-1} \) lattice. Let \( U_{ij}^h(\tau) \) denote the gauge field \( U_{qq'} \) assigned to the link \( \langle q, q' \rangle \) in \( \Lambda_\tau \) with \( q = (i, \tau) \) and \( q' = (j, \tau) \), and \( U_{ij}^v(\tau) \) the gauge field \( U_{qq'} \) with \( q = (i, \tau) \) and \( q' = (i, \tau + 1) \). The former are called horizontal gauge fields localized at \( x(d) = \tau \), and the latter are called vertical gauge fields localized in the slice \([\tau, \tau + 1]\). Now the Yang-Mills action can be rewritten as

\[
\mathcal{A}_d^{YM}(\Lambda) = -\frac{1}{2} \sum_{\tau} \sum_{p \subset \Lambda_{\tau}} \text{Re Tr} \ U_{ij}^h(\tau) - \frac{1}{2} \sum_{\tau} \sum_{(ij) \subset \Lambda^0} \text{Re Tr} \ U_{ij}^v(\tau) - U_{ij}^v(\tau) U_{ij}^h(\tau + 1). \tag{2.6}
\]

The first term in the right-hand side is a sum of Yang-Mills actions which depend on the horizontal gauge fields in \((d-1)\)-dimensional hyperplane at \( x(d) = \tau \). As to the second term, the vertical gauge fields in different slices are not coupled to each other. Therefore the summand about \( \tau \) in the second term is written in an action of a \((d-1)\)-dimensional non-linear \( \sigma \)-model for the vertical gauge fields as

\[
\mathcal{A}_{d-1}(\Lambda^0; U^h(\tau), U^h(\tau + 1)) = -\frac{1}{2} \sum_{(ij) \subset \Lambda^0} \text{Re Tr} \ U_{ij}^v(\tau) - U_{ij}^v(\tau) U_{ij}^h(\tau + 1) \tag{2.7}
\]

in the external gauge fields, \( U^h(\tau) = \{U_{ij}^h(\tau)\} \) and \( U^h(\tau + 1) = \{U_{ij}^h(\tau + 1)\} \).

Let \( S^3 \) denote the 3-sphere. In order to express the gauge fields in terms of spins \( S \in S^3 \), we use the homeomorphism \( \varphi : S^3 \to \text{SU}(2) \) which is defined by [4]

\[
\varphi(S) = \varphi \left( \begin{array}{c} S(0) \\ S(1) \\ S(2) \\ S(3) \end{array} \right) = \begin{pmatrix} S(0) + iS(3) & -S(1) + iS(2) \\ S(1) + iS(2) & S(0) - iS(3) \end{pmatrix} \tag{2.8}
\]

with the radius \( (S(0))^2 + (S(1))^2 + (S(2))^2 + (S(3))^2 = 1 \). Then the interaction potential \( V_{12} \) between two spins \( S_1 \) and \( S_2 \) in the non-linear \( \sigma \)-model \((2.7)\) can be written

\[
V_{12} = -\frac{1}{2} \text{Re Tr} \ \varphi^{-1} \varphi (S_1) \varphi (S_2) \varphi (\sigma_1) \varphi (\sigma_2)^{-1}, \tag{2.9}
\]

where we have written \( \sigma_1 \) and \( \sigma_2 \) for the external horizontal gauge fields. When the external gauge fields, \( \sigma_\ell \), take the vacuum configurations, \( \sigma_1 = \sigma_2 = (1, 0, 0, 0) \), the interaction becomes that of the \( O(4) \) non-linear \( \sigma \)-model in \((d-1)\) dimensions as

\[
V_{12} = -\frac{1}{2} \text{Re Tr} \ \varphi^{-1} \varphi (S_1) \varphi (S_2) = -S_1 \cdot S_2 = -\sum_{k=0}^{3} S_1^{(k)} S_2^{(k)}. \tag{2.10}
\]

As is well known, the \( O(4) \) non-linear \( \sigma \)-model shows a long-range order of spins at low temperatures in three or higher dimensions [7]. The long-range order leads to the perimeter law of the decay of the Wilson loop [4]. The perimeter law implies deconfinement of quarks. If the confinement of quarks indeed occurs in the \( SU(2) \) gauge theory, the fluctuations of
the external gauge fields around the vacuum must destroy the long-range order of the O(4) non-linear $\sigma$-model.

In order to take account of the fluctuations around the vacuum configuration of the external gauge fields, we approximate $\sigma_\ell$ as

$$\sigma_\ell = \left( \sqrt{1 - |\hat{\sigma}_\ell|^2}, \hat{\sigma}_\ell \right) \approx (1, \hat{\sigma}_\ell)$$  \hfill (2.11)

with small fluctuations,

$$\hat{\sigma}_\ell = \left( \sigma^{(1)}_\ell, \sigma^{(2)}_\ell, \sigma^{(3)}_\ell \right), \quad \text{for } \ell = 1, 2.$$  \hfill (2.12)

We write $\delta \sigma_\ell = (0, \hat{\sigma}_\ell)$. Then the two-body potential is written

$$V_{12} \approx -S_1 \cdot S_2 - \frac{1}{2} \text{Re Tr} \varphi(S_1)^{-1} \varphi'(\delta \sigma_1) \varphi(S_2) - \frac{1}{2} \text{Re Tr} \varphi(S_1)^{-1} \varphi(S_2) \varphi'(-\delta \sigma_2),$$  \hfill (2.13)

dropping the second order in the fluctuations $\delta \sigma_\ell$. Here we have written

$$\varphi'(\delta \sigma) = \begin{pmatrix} i\sigma^{(3)} & -\sigma^{(1)} + i\sigma^{(2)} \\ \sigma^{(1)} + i\sigma^{(2)} & -i\sigma^{(3)} \end{pmatrix}.$$  \hfill (2.14)

The right-hand side of (2.13) can be written

$$V_{12} \approx V_0 + V_R$$  \hfill (2.15)

with

$$V_0 = -S_1 \cdot S_2$$  \hfill (2.16)

and

$$V_R = -\sqrt{2} \hat{\sigma}_+ \cdot \left( \hat{S}_1 \times S_2 \right) - \sqrt{2} \hat{\sigma}_- \cdot \left( S^{(0)}_1 \hat{S}_2 - S^{(0)}_2 \hat{S}_1 \right),$$  \hfill (2.17)

where

$$\hat{\sigma}_\pm = \frac{1}{\sqrt{2}} (\sigma_2 \pm \sigma_1),$$  \hfill (2.18)

and

$$\hat{S}_\ell = \left( S^{(1)}_\ell, S^{(2)}_\ell, S^{(3)}_\ell \right), \quad \ell = 1, 2.$$  \hfill (2.19)

Thus the present system can be expressed as the O(4) non-linear $\sigma$-model in the heat bath. The interaction between the non-linear $\sigma$-model and the heat bath is given by $V_R$.

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3The contributions of the second order of the fluctuations $\delta \sigma_\ell$ give order of temperature $T = \beta^{-1}$ in the potential $V_{12}$. Therefore one can expect that the contributions of the second order slightly modifies the coupling constants of the interaction potentials at low temperatures.
3 Langevin dynamics for two particles on $\mathbb{S}^3$.

If we can integrate out the degrees of freedom of the heat bath, then we can obtain the effective action of the non-linear $\sigma$-model. However, it is very difficult problem. Instead of this way, we replace the fluctuations of the external gauge field $s$ with Gaussian random variables. Then, the spins of the $\sigma$-model can be interpreted as the “particles” which move on $\mathbb{S}^3$, acted by the two-body interaction and the random forces.

In order to derive the effective two-body interaction between two spins of the $\sigma$-model within this approximation, we first introduce the Langevin equation for the two “particles”. We write $\hat{x}_\ell = (x^{(1)}_\ell, x^{(2)}_\ell, x^{(3)}_\ell)$, $\ell = 1, 2$, for the local coordinates of the two 3-spheres $\mathbb{S}^3$. Then the Langevin equation is given by

$$\frac{d}{dt} x^{(i)}_\ell = F^{(i)}_{0,\ell} + F^{(i)}_{R,\ell}, \quad \ell = 1, 2; \quad i = 1, 2, 3. \quad (3.1)$$

with the forces, $F^{(i)}_{0,\ell}, F^{(i)}_{R,\ell}$, which are given by the gradient of the potentials as

$$F^{(i)}_{0,\ell} = -g^{ij}_\ell \partial_j V_0$$

and

$$F^{(i)}_{R,\ell} = -g^{ij}_\ell \partial_j V_R, \quad (3.2)$$

where $g^{ij}_\ell$ is the matrix inverse of the metric tensor $g_{ij,\ell}$ for the “particle” $\ell$, and we have used the Einstein summation convention and written

$$\partial_{i,\ell} = \frac{\partial}{\partial x^{(i)}_\ell}. \quad (3.4)$$

Let $\rho_t(\hat{x}_1, \hat{x}_2)$ be the distribution of the two “particles” on $\mathbb{S}^3 \times \mathbb{S}^3$. The expectation value of the function $f(\hat{x}_1, \hat{x}_2)$ on $\mathbb{S}^3 \times \mathbb{S}^3$ at time $t$ is given by

$$\langle f \rangle_t := \int_{\mathbb{S}^3 \times \mathbb{S}^3} f(\hat{x}_1, \hat{x}_2) \rho_t(\hat{x}_1, \hat{x}_2) d\mu_1 d\mu_2, \quad (3.5)$$

where we have written

$$d\mu_\ell = \sqrt{\det g_\ell} \, dx^{(1)}_\ell dx^{(2)}_\ell dx^{(3)}_\ell \quad \text{for} \quad \ell = 1, 2. \quad (3.6)$$

For a small $\Delta t > 0$, the following relation must hold:

$$\langle f \rangle_{t+\Delta t} = \mathbb{E} \int_{\mathbb{S}^3 \times \mathbb{S}^3} f(\hat{x}_1(\Delta t), \hat{x}_2(\Delta t)) \rho_t(\hat{x}_1, \hat{x}_2) d\mu_1 d\mu_2 + O((\Delta t)^2), \quad (3.7)$$

where $\mathbb{E}$ stands for the average over the fluctuations $\hat{\sigma}_\ell$, $\ell = 1, 2$, and $\hat{x}_\ell(t + \Delta t)$ is the solution of the Langevin equation (3.1) with the initial conditions $\hat{x}_\ell(t) = \hat{x}_\ell$ at time $t$. As

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4See, for example, the book [9].
usual, we assume that, for the short interval \([t, t + \Delta t]\), the fluctuations \(\dot{\sigma}^{(i)}_\ell\) are constant, and satisfy

\[
\mathbb{E} \left[ \sigma^{(i)}_\ell \right] = 0, \quad \mathbb{E} \left[ \sigma^{(i)}_\ell \sigma^{(j)}_\ell \right] = \frac{\alpha}{\Delta \ell} \delta^{ij} \quad \text{and} \quad \mathbb{E} \left[ \sigma^{(i)}_1 \sigma^{(j)}_2 \right] = \frac{\alpha'}{\Delta \ell} \delta^{ij}, \tag{3.8}
\]

where \(\alpha\) and \(\alpha'\) are a nonnegative constant, and \(\delta^{ij}\) is the Kronecker delta. Physically, a natural assumption is that \(\alpha\) and \(\alpha'\) satisfy the condition \(\alpha > \alpha' > 0\). From the relation between the fluctuations and the temperature of the heat bath, both of \(\alpha\) and \(\alpha'\) are proportional to the temperature \(\beta^{-1}\) of the heat bath.

From the Langevin equation (3.1), we have

\[
x^{(i)}_\ell(s) - x^{(i)}_\ell(t) = \int_t^s dt' \frac{dx^{(i)}_\ell(t')}{dt} = \int_t^s dt' F^{(i)}_\ell(\tilde{x}(t')),
\]

where we have written \(F^{(i)}_\ell = F^{(i)}_{0,\ell} + F^{(i)}_{R,\ell}\) and \(\tilde{x}(t) = (\tilde{x}_1(t), \tilde{x}_2(t))\). Using this relation, we obtain

\[
F^{(i)}_\ell(\tilde{x}(t')) = F^{(i)}_\ell(\tilde{x}(t)) + \sum_{m,k} \frac{\partial F^{(i)}_\ell(\tilde{x}(t))}{\partial x_m^{(k)}} \int_t^{t'} dt'' F^{(k)}_m(\tilde{x}(t'')) + \cdots. \tag{3.10}
\]

Combining these, the expansion with respect to \(\Delta t\) is derived as

\[
x^{(i)}_\ell(t + \Delta t) = x^{(i)}_\ell(t) + F^{(i)}_\ell(\tilde{x}(t)) \Delta t + \frac{1}{2} \sum_{m,k} \frac{\partial F^{(i)}_\ell(\tilde{x}(t))}{\partial x_m^{(k)}} F^{(k)}_m(\tilde{x}(t)) (\Delta t)^2 + \cdots. \tag{3.11}
\]

Substituting this into (3.7) and using (3.8), the order of \(\Delta t\) yields

\[
\int_M d\mu f(\bar{x}) \frac{\partial \rho_t(\bar{x})}{\partial t} = \int_M d\mu \sum_{\ell,i} \frac{\partial f(\bar{x})}{\partial x^{(i)}_\ell} F^{(i)}_{0,\ell}(\bar{x}) \rho_t(\bar{x})
+ \frac{\Delta t}{2} \int_M d\mu \sum_{\ell,i,m,j} \frac{\partial^2 f(\bar{x})}{\partial x^{(i)}_\ell \partial x^{(j)}_m} \mathbb{E} \left[ F^{(i)}_{R,\ell}(\bar{x}) F^{(j)}_{R,m}(\bar{x}) \right] \rho_t(\bar{x})
+ \frac{\Delta t}{2} \int_M d\mu \sum_{\ell,i,m,k} \frac{\partial f(\bar{x})}{\partial x^{(i)}_\ell} \mathbb{E} \left[ \frac{\partial F^{(i)}_{R,\ell}(\bar{x})}{\partial x^{(k)}_n} F^{(k)}_{R,n}(\bar{x}) \right] \rho_t(\bar{x}), \tag{3.12}
\]

where we have written \(M = S^3 \times S^3\) and \(d\mu = d\mu_1 d\mu_2\). Since this equation holds for any function \(f\), we can derive the equation of the time evolution for the distribution \(\rho_t\), i.e., the Fokker-Planck equation.

To this end, consider first the first term in the right-hand side of (3.12). Note that

\[
\sum_i \frac{\partial f(\bar{x})}{\partial x^{(i)}_\ell} F^{(i)}_{0,\ell}(\bar{x}) \rho_t(\bar{x}) = \sum_i \frac{1}{\sqrt{\det g_\ell}} \frac{\partial}{\partial x^{(i)}_\ell} \sqrt{\det g_\ell} F^{(i)}_{0,\ell}(\bar{x}) f(\bar{x}) \rho_t(\bar{x})
- \sum_i f(\bar{x}) \frac{1}{\sqrt{\det g_\ell}} \frac{\partial}{\partial x^{(i)}_\ell} \sqrt{\det g_\ell} F^{(i)}_{0,\ell}(\bar{x}) \rho_t(\bar{x})
= \mathrm{div}_\ell \left[ F_{0,\ell}(\bar{x}) f(\bar{x}) \rho_t(\bar{x}) \right] - f(\bar{x}) \mathrm{div}_\ell \left[ F_{0,\ell}(\bar{x}) \rho_t(\bar{x}) \right], \tag{3.13}
\]
where \( \text{div}_\ell \) stands for the divergence for the “particle” \( \ell \). Combining this with the divergence theorem\(^5\)

\[
\int_{S^3} d\mu_\ell \text{div}_\ell v_\ell = 0, \tag{3.14}
\]

for a vector field \( v_\ell \) on \( S^3 \), the first term in the right-hand side of (3.12) is written as

\[
\sum_{\ell,i} \int_M d\mu (\partial_i,\ell f) F_{0,\ell,\rho_t}^{(i)} = -\sum_{\ell} \int_M d\mu \text{div}_\ell (F_{0,\ell,\rho_t}). \tag{3.15}
\]

As to the second and third terms in the right-hand side of (3.12), we must compute the second moments of the random forces. But one can treat these terms in the same way as in the above. The detail is given in Appendix A\(^5\). As a result, the Fokker-Planck equation is given by

\[
\frac{\partial \rho_t}{\partial t} = -\sum_{\ell} \text{div}_\ell (F_{0,\ell,\rho_t}) + (\alpha + \alpha') \sum_{\ell} \{ \Delta_\ell \rho_t - \text{div}_\ell [\xi_\ell \text{div}_\ell (\xi_\ell \rho_t)] \}
- (\alpha + \alpha') \{ \text{div}_1 [\eta_1 W \cdot \text{div}_2 (\eta_2 \rho_t)] + \text{div}_2 [\eta_2 W \cdot \text{div}_1 (\eta_1 \rho_t)] \}
- 2\alpha' \sum_{m,n} \text{div}_m \left[ \hat{\zeta}_m \cdot \text{div}_n (\hat{\zeta}_n \rho_t) \right], \tag{3.16}
\]

where \( \Delta_\ell \) is the Laplacian for the “particle” \( \ell \), and we have written \( W = S_1 \cdot S_2 \); the vector fields, \( \xi_\ell \), \( \eta_\ell \) and \( \hat{\zeta}_\ell \), are given by

\[
\xi_\ell^i := g_{ij}^\ell \partial_j W, \tag{3.17}
\]

\[
\eta_\ell^i := g_{ij}^\ell \partial_j S_\ell \tag{3.18}
\]

and

\[
\hat{\zeta}_\ell^i := g_{ij}^\ell \partial_j,\ell \left( S_1^{(0)} S_2 - S_2^{(0)} S_1 \right) \tag{3.19}
\]

for \( i = 1, 2, 3 \) and \( \ell = 1, 2 \). Here the vectors \( \eta_\ell^i \) have four components like \( S_\ell \), and \( \hat{\zeta}_\ell^i \) have three components like \( \hat{S}_\ell \). This Fokker-Planck equation can be written

\[
\frac{\partial \rho_t}{\partial t} = -\text{div} \ J \quad \text{with} \quad \text{div} \ J = \text{div}_1 J_1 + \text{div}_2 J_2 \tag{3.20}
\]

in terms of the current \( J = (J_1, J_2) \) which is given by

\[
J_\ell^i = g_{ij}^\ell J_j,\ell \tag{3.21}
\]

with

\[
J_{j,1} = - (\partial_{j,1} V_0) \rho_t - (\alpha + \alpha') \{ \partial_{j,1} \rho_t - [(\partial_{j,1} W) \text{div}_1 (\xi_1 \rho_t) + W (\partial_{j,1} S_1) \cdot \text{div}_2 (\eta_2 \rho_t)] \}
+ 2\alpha' \hat{\zeta}_{j,1} \cdot \left[ \text{div}_1 (\hat{\zeta}_1 \rho_t) + \text{div}_2 (\hat{\zeta}_2 \rho_t) \right] \tag{3.22}
\]

and with \( J_{j,2} \) given by exchanging the subscripts 1 and 2 in \( J_{j,1} \). Here we have written

\[
\hat{\zeta}_{i,\ell} := \partial_{i,\ell} \left( S_1^{(0)} \hat{S}_2 - S_2^{(0)} \hat{S}_1 \right). \tag{3.23}
\]

\(^5\)See, for example, Theorem 5.11 in Chap. II of the book [3].
4 A steady state for the Fokker-Planck dynamics

The effective potential $V_{\text{eff}}$ between the two “particles” is derived from a steady distribution $\rho_t = \rho$ for the Fokker-Planck equation (3.20), as in (4.7) below. For a steady distribution $\rho_t = \rho$, the Fokker-Planck equation (3.20) becomes $\text{div } J = 0$. In order to obtain the solution near the north pole, $S_{\ell} = (1, 0, 0, 0)$, for $\ell = 1, 2$, we introduce the local coordinates, $(x_{\ell}, y_{\ell}, z_{\ell})$ for $\ell = 1, 2$, as

\[
S_{\ell} = \left( \sqrt{1 - x_{\ell}^2 - y_{\ell}^2 - z_{\ell}^2}, x_{\ell}, y_{\ell}, z_{\ell} \right).
\]  

(4.1)

We write

\[
r = (x, y, z) = (x_1 - x_2, y_1 - y_2, z_1 - z_2)
\]  

and

\[
R = (X, Y, Z) = (x_1 + x_2, y_1 + y_2, z_1 + z_2).
\]  

(4.3)

We also write $r = |r|$ and $R = |R|$. In order to solve the partial differential equation $\text{div } J = 0$, we employ the Cauchy-Kowalevski type expansion with respect to small $x_{\ell}, y_{\ell}, z_{\ell}$.

Let us compute the $x$-component $J_{x,1}$ of the current $J_1$ for the particle 1. Note that

\[
V_0 = -S_1 \cdot S_2 = -1 + \frac{1}{2} r^2 + \frac{1}{8}(r \cdot R)^2 + \cdots.
\]  

(4.4)

Immediately,

\[
\frac{\partial V_0}{\partial x_1} = x + \frac{1}{4}(r \cdot R)x + \frac{1}{4}(r \cdot R)X + \cdots.
\]  

(4.5)

Therefore, the first term of $J_{x,1}$ of (3.22) becomes

\[
- (\partial_{x_1} V_0) \rho = \left[ -x - \frac{1}{4}(r \cdot R)x - \frac{1}{4}(r \cdot R)X + \cdots \right] \rho.
\]  

(4.6)

In order to treat the rest of the terms of $J_{x,1}$, we assume that the steady state solution $\rho_t = \rho$ of $\text{div } J = 0$ has the form,

\[
\rho = \exp[-\beta V_{\text{eff}}],
\]  

(4.7)

where $V_{\text{eff}}$ is the effective potential to be determined, and $\beta$ is the inverse temperature of the heat bath. Both of $\alpha$ and $\alpha'$ are proportional to the temperature $\beta^{-1}$ as mentioned in the preceding section. The effective potential $V_{\text{eff}}$ must be vanishing for $r = 0$ because the two-body potential (2.13) becomes constant irrespective of the external fluctuations for $S_1 = S_2$. From this and taking account of the spherical and exchange symmetries, we assume that the effective potential $V_{\text{eff}}$ can be expended as

\[
V_{\text{eff}} = C_{20} r^2 + C_{40} r^4 + C_{22} r^2 R^2 + C_{22}' (r \cdot R)^2 + \cdots,
\]  

(4.8)

\[^6\text{See, for example, Sec. D of Chap. 1 in the book [11].}\]
where \( C_{20}, C_{40}, C_{22} \) and \( C'_{22} \) are the coefficients to be determined. In the following, we take \( \alpha \) and \( \alpha' \) to be small, and ignore the order of \( \alpha \) and \( \alpha' \).

For small \( x_t, y_t, z_t \), the current \( J_{x,1} \) is written

\[
J_{x,1} = \left[-x - \frac{1}{4}(r \cdot R)x - \frac{1}{4}(r \cdot R)X\right] \rho - (\alpha - \alpha') \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right) \rho
+ (\alpha + \alpha') \left[x \left(\frac{x}{\partial x_1} + \frac{y}{\partial y_1} + \frac{z}{\partial z_1}\right) + x \left(\frac{x}{\partial x_2} + \frac{y}{\partial y_2} + \frac{z}{\partial z_2}\right) - \frac{r^2}{2} \frac{\partial \rho}{\partial x_2}\right]
+ 2 \alpha' \left[-x_1 \left(\frac{x}{\partial x_1} + \frac{y}{\partial y_1} + \frac{z}{\partial z_1}\right) + x_2 \left(\frac{x}{\partial x_1} + \frac{y}{\partial y_1} + \frac{z}{\partial z_1}\right)\right]
- \left(\frac{3}{2} \frac{x}{x} + \frac{1}{2} X\right) \left(\frac{x_2}{\partial x_2} + \frac{y_2}{\partial y_2} + \frac{z_2}{\partial z_2}\right) - \frac{r^2}{2} \frac{\partial \rho}{\partial x_1} + \frac{1}{2} (r_1^2 + r_2^2) \frac{\partial \rho}{\partial x_2}\right] + \cdots.
\]

The derivation is given in Appendix B. Let us substitute \( \rho \) of (4.7) with the effective potential (4.8) into this right-hand side. First of all, since the leading order which is proportional to \( x \exp[-\beta V_{\text{eff}}] \) must be vanishing, we have

\[
4\beta (\alpha - \alpha') C_{20} = 1. \tag{4.10}
\]

Since we can choose

\[
\beta = \frac{1}{\alpha - \alpha'}, \tag{4.11}
\]

without loss of generality, we have

\[
C_{20} = \frac{1}{4}. \tag{4.12}
\]

Using these, we get

\[
- (\alpha - \alpha') \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right) \exp[-\beta V_{\text{eff}}] = \left(\frac{\partial V_{\text{eff}}}{\partial x_1} - \frac{\partial V_{\text{eff}}}{\partial x_2}\right) \exp[-\beta V_{\text{eff}}] \tag{4.13}
\]

with

\[
\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right) V_{\text{eff}} = x + 8C_{40} r^2 x + 4C_{22} R^2 x + 4C'_{22} (r \cdot R) X + \cdots. \tag{4.14}
\]

Moreover we have

\[
\left(x \frac{\partial}{\partial x_1} + y \frac{\partial}{\partial y_1} + z \frac{\partial}{\partial z_1}\right) \rho = \left(-\frac{1}{2} \beta r^2 + \cdots\right) \exp[-\beta V_{\text{eff}}], \tag{4.15}
\]

\[
\left(x_1 \frac{\partial}{\partial x_1} + y_1 \frac{\partial}{\partial y_1} + z_1 \frac{\partial}{\partial z_1}\right) \rho = \left(-\frac{1}{4} \beta r^2 - \frac{1}{4} \beta (r \cdot R) + \cdots\right) \exp[-\beta V_{\text{eff}}], \tag{4.16}
\]

\[
\left(x_2 \frac{\partial}{\partial x_2} + y_2 \frac{\partial}{\partial y_2} + z_2 \frac{\partial}{\partial z_2}\right) \rho = \left(-\frac{1}{4} \beta r^2 + \frac{1}{4} \beta (r \cdot R) + \cdots\right) \exp[-\beta V_{\text{eff}}], \tag{4.17}
\]

\[
- \frac{r^2}{2} \frac{\partial \rho}{\partial x_2} = \left[-\frac{\beta}{4} x r^2 + \cdots\right] \exp[-\beta V_{\text{eff}}] \tag{4.18}
\]
and

\[- r_2^2 \frac{\partial \rho}{\partial x_1} + \frac{1}{2} (r_1^2 + r_2^2) \frac{\partial \rho}{\partial x_2} = \frac{\beta}{4} x [r^2 + R^2 - (r \cdot R)] \exp[-\beta V_{\text{eff}}] + \cdots. \] (4.19)

Substituting these into (4.9), we obtain

\[J_{x,1} \exp[\beta V_{\text{eff}}] = [8C_{40} - 1] r^2 x + \frac{\alpha' \beta}{2} [r^2 X - (r \cdot R)x] + \left[ 4C_{22} + \frac{\alpha' \beta}{5} \right] R^2 x + \left[ 4C'_{22} - \frac{(\alpha + \alpha') \beta}{4} \right] (r \cdot R)x + \cdots. \] (4.20)

From \( \text{div} J = 0 \), the coefficients must satisfy the relations,

\[5(8C_{40} - 1) + \alpha' \beta = 0 \] (4.21)

and

\[3 \left[ 4C_{22} + \frac{\alpha' \beta}{2} \right] + \left[ 4C'_{22} - \frac{(\alpha + \alpha') \beta}{4} \right] = 0. \] (4.22)

Using these relations, the current \( J_{x,1} \) can be written

\[J_{x,1} = \left\{ - \frac{\alpha' \beta}{5} r^2 x + \frac{\alpha' \beta}{2} [r^2 X - (r \cdot R)x] + A \left[ R^2 x - 3 (r \cdot R)x \right] \right\} \exp[-\beta V_{\text{eff}}] + \cdots \] (4.23)

with the constant,

\[A = 4C_{22} + \frac{\alpha' \beta}{2}, \] (4.24)

which we cannot determine in the present method. Clearly one notices that in \( \text{div} J \), there appear the other terms,

\[\frac{1}{5} \alpha' \beta^2 r^4 \quad \text{and} \quad - A \beta [r^2 R^2 - 3 (r \cdot R)^2]. \] (4.25)

These are higher order in powers of the local coordinates but order of \( \beta \). Since the equation \( \text{div} J = 0 \) must hold, this implies that there must exist some terms of order of \( \beta \) in the effective potential \( V_{\text{eff}} \) so as to cancel the above terms of (4.25).

When both of the coefficients \( C_{22} \) and \( C'_{22} \) depend on \( \beta \), the corresponding terms may appear in the expansion. In this case, from (1.22), we have

\[C_{22} \sim C \beta \quad \text{and} \quad C'_{22} \sim -3C' \beta \] (4.26)

with some constant \( C \) for a large \( \beta \). Substituting these into \( V_{\text{eff}} \), we have

\[V_{\text{eff}} \sim \frac{1}{4} r^2 + C_{40} r^4 + C' \beta [r^2 R^2 - 3 (r \cdot R)^2]. \] (4.27)

This leads to instability of binding of the two particles because the value of \( R^2 \) is expected to become larger than order of \( \beta^{-1} \) in the thermal equilibrium. Thus we require that both of \( C_{22} \) and \( C'_{22} \) are order of 1.
In consequence, we need the following terms in the effective potential $V_{\text{eff}}$:

$$C_{60}r^6, \quad C_{42}r^4R^2, \quad C'_{42}r^2(r \cdot R)^2. \quad (4.28)$$

Here all the coefficients, $C_{60}, C_{42}, C'_{42}$, are proportional to $\beta$ for a large $\beta$. In the same way as in the above, we can determine these coefficients as

$$C_{60} = -\frac{3!}{7!}\alpha'\beta^2, \quad C_{42} = \frac{1}{56}A\beta \quad \text{and} \quad C'_{42} = -\frac{3}{56}A\beta \quad (4.29)$$

so as to cancel the above terms (4.25) which appear in $\text{div} \ J$. As a result, the dominant contributions in the effective potential $V_{\text{eff}}$ are given by

$$V_{\text{eff}} \sim \frac{1}{4}r^2 - \frac{3!}{7!}\alpha'\beta^2 r^6 + \frac{1}{56}A\beta r^2[r^2R^2 - 3(r \cdot R)^2] \quad (4.30)$$

for a large $\beta$ because the second, third and fourth terms in the right-hand side of (4.8) do not affect the critical behavior.

Now we discuss the critical behavior of the $(d-1)$-dimensional $\sigma$ model with the above two-body interaction $V_{\text{eff}}$. Consider first the case of $A = 0$. Namely the effective potential is given by

$$V_{\text{eff}} \sim \frac{1}{4}r^2 - \frac{3!}{7!}\alpha'\beta^2 r^6 \quad (4.31)$$

for small $r$ and large $\beta$. The second term lowers the potential barrier. Within a mean-field approximation [12], the critical temperature $T_C$ can be estimated by the volume and the height of the potential well. More precisely, $T_C \sim \text{(volume)} \times \text{(height)}$. In the present case, the width $w$ and the height $h$ of the effective potential $V_{\text{eff}}$ are estimated as

$$w \sim (\lambda\beta)^{-1/4}, \quad h \sim (\lambda\beta)^{-1/2}, \quad (4.32)$$

where we have written

$$\lambda = 12 \cdot \frac{3!}{7!}\alpha'\beta. \quad (4.33)$$

Therefore the critical temperature $T_C$ is estimated as

$$T_C \sim w^3 \times h \sim (\lambda\beta)^{-5/4}. \quad (4.34)$$

This is lower than $\beta^{-1}$ for small temperature $T = \beta^{-1}$. This implies that the true critical temperature must be equal to zero.

In the case of $A \neq 0$, the third term in the right-hand side of (4.30) may heighten the potential barrier if $R^2$ does not take a small value. But it is impossible that the term heightens the potential barrier in all the directions of $r$. Thus we reach the same conclusion, $T_C = 0$.

Let us make the following two remarks:

1. Our argument can be applied to the systems in arbitrary dimensions. Therefore a reader might think that our method suggests no phase transition for non-Abelian lattice gauge theory also in five or higher dimensions. On this point, we should remark the following: We used the two-body approximation, considering only a
single plaquette. When dealing with two plaquettes within our method, three- and four-body interactions would appear in the effective potential for the non-linear $\sigma$-model. The resulting interactions may change the conclusion of this section. Namely a high-dimensional system may exhibit a phase transition. Actually, in five or higher dimensions, the effect of the three- or four-body interactions may not be ignored because the number of the neighboring plaquettes for a fixed plaquette becomes large, compared to low-dimensional systems. However, taking account of such interactions is not so easy.

2. Consider the O(4) non-linear $\sigma$-model on the three-dimensional lattice with the effective two-body interaction which we obtained. Then the correlation length of the model leads to an estimate of the string tension \[4, 5\]. Does the scaling limit so obtained give the standard continuum? This problem must be very important. But it is very difficult to compute the low temperature asymptotics of the correlation length for such a weakly attractive potential.

5 Difference between U(1) and SU(2) gauge theories

Let us see difference between U(1) and SU(2) gauge theories.

For this purpose, we apply the present method to the abelian case $G = U(1)$. In the case, the gauge field $U_b$ on a link $b$ is written

$$U_b = \exp[i\theta_b]$$

in terms of the angle variable $\theta_b \in [0, 2\pi)$. Therefore the two-body interaction $V_{12}$ between $\theta_1$ and $\theta_2$ is written

$$V_{12} = -\cos(\theta_1 - \theta_2 + \sigma_1 - \sigma_2),$$

where $\sigma_1$ and $\sigma_2$ are the angle variables of the external fields. We write $\theta = \theta_1 - \theta_2$ and $\delta\sigma = \sigma_1 - \sigma_2$, and assume that $\delta\sigma$ is a small fluctuation. Under this assumption, the potential can be approximated as

$$V_{12} \approx -\cos \theta + \delta\sigma \sin \theta.$$  \(5.3\)

Then the Langevin equation is given by

$$\frac{d\theta}{dt} = -\sin \theta - \delta\sigma \cos \theta.$$  \(5.4\)

As usual, we assume

$$\mathbb{E}[(\delta\sigma)^2] = \frac{\alpha}{\Delta t}.$$  \(5.5\)

for a small $\Delta t$. In the same way as in the SU(2) case, we obtain the Fokker-Planck equation,

$$\frac{\partial \rho_t}{\partial t} = \left[ \frac{\partial}{\partial \theta} \sin \theta + \frac{\alpha}{2} \frac{\partial}{\partial \theta} \sin \theta \cos \theta + \frac{\alpha}{2} \frac{\partial^2}{\partial \theta^2} \cos^2 \theta \right] \rho_t.$$  \(5.6\)
For a steady state \( \rho_t = \rho \), we have
\[
\left[ \sin \theta + \frac{\alpha}{2} \sin \theta \cos \theta + \frac{\alpha}{2} \frac{\partial}{\partial \theta} \cos^2 \theta \right] \rho = 0. \tag{5.7}
\]
One can easily find the solution,
\[
\rho = \begin{cases} 
(\cos \theta)^{-1} \exp \left[ -2\alpha^{-1} \cos \theta \right], & \text{for } -\pi/2 < \theta < \pi/2; \\
0, & \text{otherwise}. \end{cases} \tag{5.8}
\]
Since the diffusion disappears at \( \theta = \pm \pi/2 \) in the right-hand side of (5.4), the “particle” cannot move beyond the points. Clearly, we have
\[
\rho \sim \text{const.} \exp \left[ -\alpha^{-1} \theta^2 \right] \tag{5.9}
\]
for a small \( \theta \). Thus there is no term which is proportional to \( \alpha^{-1} \) or higher powers of \( \alpha^{-1} \) in the effective potential, and the critical behavior can be expected to be the same as the standard O(2) nonlinear-\( \sigma \) model. This is consistent with the rigorous result of [2, 3].

### A Derivation of the Fokker-Planck equation

Consider first the case with \( \alpha' = 0 \) in (3.8). We introduce \( \sigma^{ij} \) satisfying \( \sigma^{ji} = -\sigma^{ij} \) with
\[
(\sigma^{01}, \sigma^{02}, \sigma^{03}) = (\sigma_+^{(1)}, \sigma_+^{(2)}, \sigma_+^{(3)}), \quad \text{and} \quad (\sigma^{23}, \sigma^{31}, \sigma^{12}) = (\sigma_-^{(1)}, \sigma_-^{(2)}, \sigma_-^{(3)}). \tag{A.1}
\]
Then the random potential \( V_R \) of (2.17) can be written
\[
V_R = -\frac{1}{\sqrt{2}} \varepsilon_{ijkl} \sigma^{ij} S_1^{(k)} S_2^{(l)}, \tag{A.2}
\]
where \( \varepsilon_{ijkl} \) is completely antisymmetric and satisfies \( \varepsilon_{0123} = +1 \), and we have used the Einstein summation convention. From \( \alpha' = 0 \), we have
\[
\mathbb{E} \left[ \sigma^{\alpha \beta} \sigma^{mn} \right] = \frac{\alpha}{\Delta t} (\delta^{\alpha m} \delta^{\beta n} - \delta^{\alpha n} \delta^{\beta m}). \tag{A.3}
\]
Using (A.2) and (A.3), we obtain
\[
\begin{align*}
\mathbb{E} \left[ (\partial_{t,1} V_R) (\partial_{k,1} V_R) \right] &= \frac{1}{2} \mathbb{E} \left[ \varepsilon_{\alpha \beta \gamma \delta} \sigma^{\alpha \beta} (\partial_{t,1} S_1^{(\gamma)}) S_2^{(\delta)} \varepsilon_{mnst} \sigma^{mn} (\partial_{k,1} S_1^{(s)}) S_2^{(t)} \right] \\
&= \frac{\alpha}{2\Delta t} \varepsilon_{\alpha \beta \gamma \delta} \varepsilon_{mnst} (\delta^{\alpha m} \delta^{\beta n} - \delta^{\alpha n} \delta^{\beta m}) (\partial_{t,1} S_1^{(\gamma)}) S_2^{(\delta)} (\partial_{k,1} S_1^{(s)}) S_2^{(t)} \\
&= \frac{2\alpha}{\Delta t} \sum_{\gamma, \delta} \left[ (\partial_{t,1} S_1^{(\gamma)}) (\partial_{k,1} S_1^{(\gamma)}) S_2^{(\delta)} S_2^{(\delta)} - (\partial_{t,1} S_1^{(\gamma)}) S_2^{(\gamma)} (\partial_{k,1} S_1^{(\delta)}) S_2^{(\delta)} \right]. \tag{A.4}
\end{align*}
\]
Using the metric
\[
g_{ij,t} = \frac{\partial S_t}{\partial x_t^{(i)}} \frac{\partial S_t}{\partial x_t^{(j)}} \tag{A.5}
\]
of $S^3$ for the “particle” $\ell$, the above result is written

$$
\mathbb{E} \left[ (\partial_{\ell,1} V_R) (\partial_{\ell,1} V_R) \right] = \frac{2\alpha}{\Delta t} \left[ g_{\ell k,1} - (\partial_{\ell,1} W)(\partial_{\ell,1} W) \right]
$$

(A.6)

and

$$
\mathbb{E} \left[ (\partial_{\ell,2} V_R) (\partial_{\ell,2} V_R) \right] = \frac{2\alpha}{\Delta t} \left[ g_{\ell k,2} - (\partial_{\ell,2} W)(\partial_{\ell,2} W) \right],
$$

(A.7)

where we have written $W = S_1 \cdot S_2$. Similarly, we have

$$
\mathbb{E} \left[ (\partial_{\ell,1} \partial_{\ell,1} V_R) (\partial_{\ell,1} V_R) \right] = \frac{2\alpha}{\Delta t} \sum_{\gamma, \delta} \left[ \frac{\partial^2 S_1^{(\gamma)}}{\partial x_1^{(k)} \partial x_1^{(j)}} \frac{\partial S_1^{(\gamma)}}{\partial x_1^{(t)}} S_2^{(\delta)} - \frac{\partial^2 S_1^{(\gamma)}}{\partial x_1^{(k)} \partial x_1^{(j)}} S_2^{(\gamma)} \frac{\partial S_1^{(\delta)}}{\partial x_1^{(t)}} S_2^{(\delta)} \right].
$$

(A.8)

Combining this with

$$
\sum_{\gamma} \frac{\partial^2 S_1^{(\gamma)}}{\partial x_1^{(k)} \partial x_1^{(j)}} \frac{\partial S_1^{(\gamma)}}{\partial x_1^{(t)}} = \Gamma_{k \ell,1}^{m} g_{m \ell,1},
$$

(A.9)

we obtain

$$
\mathbb{E} \left[ (\partial_{\ell,1} \partial_{\ell,1} V_R) (\partial_{\ell,1} V_R) \right] = \frac{2\alpha}{\Delta t} \left[ \Gamma_{k \ell,1}^{m} g_{m \ell,1} - (\partial_{\ell,1} \partial_{\ell,1} W)(\partial_{\ell,1} W) \right],
$$

(A.10)

where $\Gamma_{k \ell,1}^{m}$ are the Christoffel symbols $[9]$. In the same way, we get

$$
\mathbb{E} \left[ (\partial_{\ell,1} V_R) (\partial_{\ell,2} V_R) \right] = -\frac{2\alpha}{\Delta t} W (\partial_{\ell,1} \partial_{\ell,2} W)
$$

(A.11)

and

$$
\mathbb{E} \left[ (\partial_{\ell,2} \partial_{\ell,1} V_R) (\partial_{\ell,2} V_R) \right] = -\frac{2\alpha}{\Delta t} (\partial_{\ell,2} W)(\partial_{\ell,1} \partial_{\ell,2} W).
$$

(A.12)

Using (A.6), we have

$$
\mathbb{E} \left[ F_{R,1}^{(i)} F_{R,1}^{(j)} \right] = \mathbb{E} \left[ g_{1}^{ij} (\partial_{\ell,1} V_R) g_{1}^{jk} (\partial_{\ell,1} V_R) \right] = \frac{2\alpha}{\Delta t} g_{1}^{ij} g_{1}^{jk} [g_{tk,1} - (\partial_{\ell,1} W)(\partial_{\ell,1} W)] = \frac{2\alpha}{\Delta t} (g_{1}^{ij} - \xi_{1}^{i} \xi_{1}^{j}),
$$

(A.13)

where $\xi_{1}^{i}$ is the vector field which is given by (3.17). From (A.6) and (A.10), we obtain

$$
\sum_{k} \mathbb{E} \left[ \frac{\partial F_{R,1}^{(i)}}{\partial x_1^{k}} F_{R,1}^{(k)} \right] = \mathbb{E} \left[ (\partial_{k,1} g_{1}^{ij} \partial_{\ell,1} V_R) (g_{1}^{k} \partial_{\ell,1} V_R) \right] = (\partial_{k,1} g_{1}^{ij}) g_{1}^{k} \mathbb{E} [(\partial_{\ell,1} V_R)(\partial_{\ell,1} V_R)] + g_{1}^{ij} g_{1}^{k} \mathbb{E} [(\partial_{k,1} \partial_{\ell,1} V_R)(\partial_{\ell,1} V_R)]
$$

$$
= \frac{2\alpha}{\Delta t} (\partial_{k,1} g_{1}^{ij}) g_{1}^{k} [g_{j,1}^{ij} + g_{1}^{ij} \Gamma_{k \ell,1}^{m} g_{m \ell,1} - (\partial_{k,1} \xi_{1}^{i}) \xi_{1}^{j}]
$$

$$
= \frac{2\alpha}{\Delta t} \left[ \frac{1}{\sqrt{\det g_{1}}} \partial_{j,1} g_{1}^{ij} \sqrt{\det g_{1}} - (\partial_{k,1} \xi_{1}^{i}) \xi_{1}^{k} \right]
$$

(A.14)
where we have used \[ \Gamma_{kj,1}^k = \frac{1}{\sqrt{\det g_1}} \partial_{j,1} \sqrt{\det g_1}. \] (A.15)

In the same way, the relations (A.11) and (A.12) yield

\[
\mathbb{E} \left[ F_{R,1}^{(i)} F_{R,2}^{(j)} \right] = g_i^j g_2^k \mathbb{E} \left[ (\partial_{\ell,1} V_R) (\partial_{k,2} V_R) \right] \\
= -\frac{2\alpha}{\Delta t} g_i^j g_2^k W (\partial_{\ell,1} \partial_{k,2} W) \tag{A.16}
\]

and

\[
\sum_k \mathbb{E} \left[ \frac{\partial F_{R,1}^{(i)}}{\partial x_2^k} F_{R,2}^{(k)} \right] = \mathbb{E} \left[ (\partial_{k,2} g_1^i \partial_{j,1} V_R) (g_2^k \partial_{\ell,2} V_R) \right] \\
= g_i^j g_2^k \mathbb{E} \left[ (\partial_{k,2} \partial_{j,1} V_R) (\partial_{\ell,2} V_R) \right] \\
= -\frac{2\alpha}{\Delta t} g_i^j g_2^k (\partial_{j,1} \partial_{k,2} W), \tag{A.17}
\]

respectively. The contribution from the two random forces \( F_{R,\ell} \) with the same indexes \( \ell = 1 \) in the right-hand side of (3.12) becomes

\[
I_{11} := \frac{\Delta t}{2} \left\{ \sum_{i,j} \int_M d\mu \frac{\partial^2 f}{\partial x_1^{(i)} \partial x_1^{(j)}} \mathbb{E} \left[ F_{R,1}^{(i)} F_{R,1}^{(j)} \right] + \sum_{i,k} \int_M d\mu \frac{\partial f}{\partial x_1^{(i)}} \mathbb{E} \left[ \frac{\partial F_{R,1}^{(i)}}{\partial x_1^{(k)}} F_{R,1}^{(k)} \right] \right\} \rho_t \\
= \alpha \sum_{i,j} \int_M d\mu \frac{\partial^2 f}{\partial x_1^{(i)} \partial x_1^{(j)}} (g_i^j - \xi_1^j \xi_1^i) \rho_t \\
+ \alpha \sum_i \int_M d\mu \frac{\partial f}{\partial x_1^{(i)}} \left[ \frac{1}{\sqrt{\det g_1}} \partial_{j,1} g_i^j \sqrt{\det g_1} - (\partial_{k,1} \xi_1^i) \xi_1^k \right] \rho_t, \tag{A.18}
\]

where we have used (A.13) and (A.14). Note that

\[
\int_M d\mu \left[ \sum_{i,j} g_i^j \frac{\partial^2 f}{\partial x_1^{(i)} \partial x_1^{(j)}} + \sum_i \left( \frac{1}{\sqrt{\det g_1}} \partial_{j,1} g_i^j \sqrt{\det g_1} \right) \frac{\partial f}{\partial x_1^{(i)}} \right] \rho_t \\
= \int_M d\mu \left( \frac{1}{\sqrt{\det g_1}} \partial_{j,1} g_i^j \sqrt{\det g_1} \partial_{j,1} f \right) \rho_t \\
= \int_M d\mu (\Delta_1 f) \rho_t = \int_M d\mu f (\Delta_1 \rho_t), \tag{A.19}
\]

where the second equality follows from the property of the Laplacian \( \Delta_\ell \). The rest of the contributions in the right-hand side of (A.18) are computed as

\[
\int_M d\mu \left[ \sum_{i,j} \frac{\partial^2 f}{\partial x_1^{(i)} \partial x_1^{(j)}} \xi_1^i \xi_1^j + \sum_i \frac{\partial f}{\partial x_1^{(i)}} (\partial_{k,1} \xi_1^i) \xi_1^k \right] \rho_t
\]

\footnote{See, for example, Sec.7 of Chap. I of the book \[10\].}

\footnote{See, for example, Corollary 5.13 in Chap. II of the book \[9\].}
where we have used the divergence theorem (3.14). Substituting this and (A.19) into (A.18), we obtain
\[ I_{11} = \alpha \int_M d\mu \{ \Delta_1 \rho_t - \text{div}_1 [\xi_1 \text{div}_1 (\xi_1 \rho_t)] \}. \tag{A.21} \]

Next consider the contribution from the two random forces \( F_{R,\ell} \) with different indexes, \( \ell = 1 \) and \( \ell = 2 \), in the right-hand side of (3.12). Using (A.16) and (A.17), we obtain
\[
I_{12} := \frac{\Delta t}{2} \left\{ \sum_{i,j} \int_M d\mu \left[ \frac{\partial^2 f}{\partial x_1^{(i)} \partial x_2^{(j)}} E \left[ F_{R,1}^{(i)} F_{R,2}^{(j)} \right] + \sum_{i,k} \int_M d\mu \frac{\partial f}{\partial x_1^{(i)}} E \left[ \frac{\partial F_{R,1}^{(i)}}{\partial x_2^{(k)}} F_{R,2}^{(k)} \right] \right] \rho_t \right\} \\
= -\alpha \int_M d\mu \sum_{i,j} \frac{\partial^2 f}{\partial x_1^{(i)} \partial x_2^{(j)}} g^{ij}_W (\partial \ell_1, \partial k_2) W \rho_t \\
-\alpha \int_M d\mu \sum_i \frac{\partial f}{\partial x_1^{(i)}} g^{ij}_W (\partial \ell_1, \partial k_2) W \rho_t \\
= -\alpha \int_M d\mu \frac{1}{\sqrt{\det g_2}} \partial_j \sqrt{\det g_2} g^{jk}_2 (\partial \ell_1, \partial k_2) W \rho_t \\
+\alpha \int_M d\mu (\partial \ell_1 f) \frac{1}{\sqrt{\det g_2}} \partial_j \sqrt{\det g_2} g^{jk}_2 (\partial \ell_1, \partial k_2) W \rho_t \\
-\alpha \int_M d\mu (\partial \ell_1 f) g^{ij}_W (\partial \ell_1, \partial k_2) W \rho_t \\
= \alpha \int_M d\mu (\partial \ell_1 f) g^{ij}_W \frac{1}{\sqrt{\det g_2}} \partial_j \sqrt{\det g_2} g^{jk}_2 (\partial \ell_1, \partial k_2) W \rho_t, \tag{A.22} \]

where we have used the divergence theorem (3.14). Recalling \( W = S_1 \cdot S_2 \), we have
\[ \partial \ell_1 \partial k_2 W = (\partial \ell_1 S_1) \cdot (\partial k_2 S_2). \tag{A.23} \]

Substituting this into the above result, we get
\[
I_{12} = \alpha \int_M d\mu (\partial \ell_1 f) g^{ij}_W (\partial \ell_1 S_1) W \frac{1}{\sqrt{\det g_2}} \partial_j \sqrt{\det g_2} g^{jk}_2 (\partial k_2 S_2) \rho_t. \]
Similarly, further we have Fluctuations in SU(2) Yang-Mills Theory. It is sufficient to calculate the corrections from the second term in this right-hand side, where \( \eta_i \) is given by (3.18). From (3.12), (3.15), (A.18), (A.21), (A.22) and (A.24), we obtain the Fokker-Planck equation,

\[
\frac{\partial \rho_t}{\partial t} = -\sum \text{div}_\ell (F_{0,\ell} \rho_t) + \alpha \sum \{ \Delta_t \rho_t - \text{div}_\ell [\xi_\ell \text{div}_\ell (\xi_\ell \rho_t)] \}
- \alpha \{ \text{div}_1 [\eta_1 W \cdot \text{div}_2 (\eta_2 \rho_t)] + \text{div}_2 [\eta_2 W \cdot \text{div}_1 (\eta_1 \rho_t)] \},
\]

for \( \alpha' = 0 \).

Next consider the case with \( \alpha' \neq 0 \). To begin with, we note that

\[
\mathbb{E} \left[ \sigma_+^{(i)} \sigma_+^{(j)} \right] = \frac{1}{2} \mathbb{E} \left[ \left( \sigma_2^{(i)} + \sigma_1^{(i)} \right) \left( \sigma_2^{(j)} + \sigma_1^{(j)} \right) \right]
= \frac{1}{2} \left\{ \mathbb{E}[\sigma_2^{(i)} \sigma_2^{(j)}] + \mathbb{E}[\sigma_1^{(i)} \sigma_1^{(j)}] + \mathbb{E}[\sigma_2^{(i)} \sigma_1^{(j)}] + \mathbb{E}[\sigma_1^{(i)} \sigma_2^{(j)}] \right\}
= \frac{\alpha + \alpha'}{\Delta t} \delta^{ij}.
\]

Similarly,

\[
\mathbb{E} \left[ \sigma_-^{(i)} \sigma_-^{(j)} \right] = \frac{\alpha - \alpha'}{\Delta t} \delta^{ij}.
\]

Further, we have

\[
\mathbb{E} \left[ \sigma_+^{(i)} \sigma_-^{(j)} \right] = \frac{1}{2} \mathbb{E} \left[ \left( \sigma_2^{(i)} + \sigma_1^{(i)} \right) \left( \sigma_2^{(j)} - \sigma_1^{(j)} \right) \right]
= \frac{1}{2} \left\{ \mathbb{E}[\sigma_2^{(i)} \sigma_2^{(j)}] - \mathbb{E}[\sigma_1^{(i)} \sigma_1^{(j)}] + \mathbb{E}[\sigma_2^{(i)} \sigma_1^{(j)}] - \mathbb{E}[\sigma_1^{(i)} \sigma_2^{(j)}] \right\}
= 0.
\]

Since we can write

\[
\mathbb{E} \left[ \sigma_-^{(i)} \sigma_-^{(j)} \right] = \frac{\alpha + \alpha'}{\Delta t} \delta^{ij} - \frac{2\alpha'}{\Delta t} \delta^{ij},
\]

it is sufficient to calculate the corrections from the second term in this right-hand side, with replacing \( \alpha \) with \( \alpha + \alpha' \) in the above result (A.25).

In (A.13), the correction to \( \mathbb{E} \left[ g^\ell_1 (\partial_{\ell,1} V_R) g^{jk}_1 (\partial_{k,1} V_R) \right] \) is given by

\[
-\frac{4\alpha'}{\Delta t} \zeta^i_1 \cdot \zeta^j_1.
\]

(A.30)
where $\hat{\zeta}_i^\ell$ is given by (3.19). Similarly, the correction to $\mathbb{E} \left[ (\partial_{k,1}g^{ij}\partial_{j,1}V_R)(g^{k\ell}\partial_{\ell,1}V_R) \right]$ in (A.14) is given by

$$-\frac{4\alpha'}{\Delta t} \left( \partial_{k,1} \hat{\zeta}_1^i \right) \cdot \hat{\zeta}_1^k.$$  \hspace{1cm} (A.31)

Therefore the same calculations as those from (A.18) to (A.21) yield the correction,

$$-2\alpha' \text{div}_1 \left[ \hat{\zeta}_1 \cdot \text{div}_1 (\hat{\zeta}_1 \rho_t) \right],$$  \hspace{1cm} (A.32)

in the right-hand side of the Fokker-Planck equation (A.25).

In (A.16), the correction to $\mathbb{E} \left[ g^{ij} \left( \partial_{\ell,1} V_R \right) \right.$

$$\left. \left( g^{k\ell} \partial_{\ell,2} V_R \right) \right]$ in (A.17) is given by

$$-\frac{4\alpha'}{\Delta t} \hat{\zeta}_1^i \cdot \hat{\zeta}_2^j.$$  \hspace{1cm} (A.33)

Further, the correction to $\mathbb{E} \left[ (\partial_{k,2}g^{ij}\partial_{j,2}V_R)(g^{k\ell}\partial_{\ell,2}V_R) \right]$ in (A.17) is given by

$$-\frac{4\alpha'}{\Delta t} \left( \partial_{k,2} \hat{\zeta}_1^i \right) \cdot \hat{\zeta}_2^k.$$  \hspace{1cm} (A.34)

Therefore similar calculations to those from (A.22) to (A.24) yield the correction,

$$-2\alpha' \text{div}_1 \left[ \hat{\zeta}_1 \cdot \text{div}_2 (\hat{\zeta}_2 \rho_t) \right],$$  \hspace{1cm} (A.35)

in the right-hand side of the Fokker-Planck equation (A.25). In consequence, the Fokker-Planck equation is given by

$$\frac{\partial \rho_t}{\partial t} = - \sum_{\ell} \text{div}_\ell \left( F_{0,\ell} \rho_t \right) + (\alpha + \alpha') \sum_{\ell} \left\{ \Delta_\ell \rho_t - \text{div}_\ell \left[ \xi_\ell \text{div}_\ell (\xi_\ell \rho_t) \right] \right\}$$

$$- (\alpha + \alpha') \left\{ \text{div}_1 \left[ \eta_1 W \cdot \text{div}_2 (\eta_2 \rho_t) \right] + \text{div}_2 \left[ \eta_2 W \cdot \text{div}_1 (\eta_1 \rho_t) \right] \right\}$$

$$-2\alpha' \sum_{m,n} \text{div}_m \left[ \hat{\zeta}_m \cdot \text{div}_n (\hat{\zeta}_n \rho_t) \right].$$  \hspace{1cm} (A.36)

### B Derivation of the expansion (4.9)

The metric $g_{ij,\ell}$ of $\mathbb{S}^3$ is computed as

$$g_{ij,\ell} = \begin{pmatrix}
1 + \gamma_\ell x_\ell^2 & \gamma_\ell x_\ell y_\ell & \gamma_\ell x_\ell z_\ell \\
\gamma_\ell y_\ell x_\ell & 1 + \gamma_\ell y_\ell^2 & \gamma_\ell y_\ell z_\ell \\
\gamma_\ell z_\ell x_\ell & \gamma_\ell z_\ell y_\ell & 1 + \gamma_\ell z_\ell^2
\end{pmatrix} = \begin{pmatrix}
1 + x_\ell^2 & x_\ell y_\ell & x_\ell z_\ell \\
y_\ell x_\ell & 1 + y_\ell^2 & y_\ell z_\ell \\
z_\ell x_\ell & z_\ell y_\ell & 1 + z_\ell^2
\end{pmatrix} + \cdots,$$  \hspace{1cm} (B.1)

where we have written

$$\gamma_\ell = \frac{1}{\sqrt{1 - r_\ell^2}} \quad \text{with} \quad r_\ell = \sqrt{x_\ell^2 + y_\ell^2 + z_\ell^2}.$$  \hspace{1cm} (B.2)
Therefore, the inverse $g^{ij}_\ell$ is given by

$$g^{ij}_\ell = \begin{pmatrix} 1 - x^2_\ell & -x_\ell y_\ell & -x_\ell z_\ell \\ -y_\ell x_\ell & 1 - y^2_\ell & -y_\ell z_\ell \\ -z_\ell x_\ell & -z_\ell y_\ell & 1 - z^2_\ell \end{pmatrix} + \cdots. \quad (B.3)$$

Using this, we have

$$(\partial_{x,1} W) \text{div}_1(\xi_1 \rho) = \frac{\partial S_1 \cdot S_2}{\partial x_1} \frac{1}{\sqrt{\det g_{1}}} \partial_{i,1} \sqrt{\det g} g^{ij}_1 (\partial_{j,1} S_1 \cdot S_2) \rho$$

$$= -x g^{ij}_1 (\partial_{j,1} S_1 \cdot S_2) \partial_{i,1} \rho + \cdots$$

$$= x \left( x^\rho \frac{\partial \rho}{\partial x_1} + y^\rho \frac{\partial \rho}{\partial y_1} + z^\rho \frac{\partial \rho}{\partial z_1} \right) + \cdots. \quad (B.4)$$

Similarly,

$$W(\partial_{x,1} S_1) \cdot \text{div}_2(\eta_2 \rho) = W(\partial_{x,1} S_1) \cdot g^{ij}_2 (\partial_{j,2} S_2) \partial_{i,2} \rho + \cdots$$

$$= W(\partial_{x,1} S_2) \cdot g^{ij}_2 (\partial_{j,2} S_2) \partial_{i,2} \rho + \cdots$$

$$= W \left\{ \partial_{j,2} \left[ -x - \frac{1}{2} (\mathbf{r} \cdot \mathbf{R}) x_1 + \cdots \right] \right\} g^{ij}_2 \partial_{i,2} \rho + \cdots$$

$$= W g^{ij}_2 \partial_{i,2} \rho + W \left[ x_1 x_2 \frac{\partial \rho}{\partial x_2} + x_1 y_2 \frac{\partial \rho}{\partial y_2} + x_1 z_2 \frac{\partial \rho}{\partial z_2} \right] + \cdots$$

$$= \left( 1 - \frac{1}{2} r^2 \right) \left[ g^{11}_2 \frac{\partial \rho}{\partial x_2} + g^{21}_2 \frac{\partial \rho}{\partial y_2} + g^{31}_2 \frac{\partial \rho}{\partial z_2} \right]$$

$$+ \left[ x_1 x_2 \frac{\partial \rho}{\partial x_2} + x_1 y_2 \frac{\partial \rho}{\partial y_2} + x_1 z_2 \frac{\partial \rho}{\partial z_2} \right] + \cdots$$

$$= \frac{\partial \rho}{\partial x_2} - \frac{1}{2} r^2 \frac{\partial \rho}{\partial x_2} - \left[ x_1 x_2 \frac{\partial \rho}{\partial x_2} + x_1 y_2 \frac{\partial \rho}{\partial y_2} + x_1 z_2 \frac{\partial \rho}{\partial z_2} \right] + \cdots$$

$$= \frac{\partial \rho}{\partial x_2} - \frac{1}{2} r^2 \frac{\partial \rho}{\partial x_2} + x \left[ x_2 \frac{\partial \rho}{\partial x_2} + y_2 \frac{\partial \rho}{\partial y_2} + z_2 \frac{\partial \rho}{\partial z_2} \right] + \cdots. \quad (B.5)$$

We write

$$\hat{\zeta}_{i,\ell} = \left( \zeta^{(1)}_{i,\ell}, \zeta^{(2)}_{i,\ell}, \zeta^{(3)}_{i,\ell} \right). \quad (B.6)$$

Note that

$$\zeta^{(a)}_{x,1} = \frac{\partial}{\partial x_1} \left( S^{(a)}_1 S^{(a)}_2 - S^{(0)}_2 S^{(a)}_1 \right)$$

$$= \frac{-x_1}{\sqrt{1 - r^2}} g^{(0)}_{2} - \sqrt{1 - r^2} \frac{\partial S^{(a)}_1}{\partial x_1}. \quad (B.7)$$

Therefore, we have

$$\hat{\zeta}_{x,1} = \left( \frac{-x_1 x_2}{\sqrt{1 - r^2}} - \sqrt{1 - r^2}, \frac{-x_1 y_2}{\sqrt{1 - r^2}}, \frac{-x_1 z_2}{\sqrt{1 - r^2}} \right).$$
\[
\left(-x_1 x_2 - \sqrt{1 - r_2^2}, -x_1 y_2, -x_1 z_2\right) + \cdots.
\]  
(B.8)

In the same way,

\[
\hat{\zeta}_{y,1} = \left(-y_1 x_2, -y_1 y_2 - \sqrt{1 - r_2^2}, -y_1 z_2\right) + \cdots
\]  
(B.9)

and

\[
\hat{\zeta}_{z,1} = \left(-z_1 x_2, -z_1 y_2, -z_1 z_2 - \sqrt{1 - r_2^2}\right) + \cdots.
\]  
(B.10)

From these results, we obtain

\[
\hat{\zeta}_{x,1} \cdot \hat{\zeta}_{x,1} = 1 - r_2^2 + 2x_1 x_2 + \cdots.
\]  
(B.11)

\[
\hat{\zeta}_{x,1} \cdot \hat{\zeta}_{y,1} = y_1 x_2 + x_1 y_2 + \cdots
\]  
(B.12)

and

\[
\hat{\zeta}_{x,1} \cdot \hat{\zeta}_{z,1} = z_1 x_2 + x_1 z_2 + \cdots.
\]  
(B.13)

Using these, we have

\[
\hat{\zeta}_{x,1} \cdot \text{div}_1(\hat{\zeta}_1 \rho) = \hat{\zeta}_{x,1} \cdot g^{ij}_1 \hat{\zeta}_{j,1} \partial_{i,1} \rho + \cdots
\]
\[
= (1 - r_2^2 + 2x_1 x_2)g^{ij}_1 \partial_{i,1} \rho + (y_1 x_2 + y_1 y_2)g^{ij}_1 \partial_{i,1} \rho + (z_1 x_2 + z_1 z_2)g^{ij}_1 \partial_{i,1} \rho + \cdots
\]
\[
= (1 - r_2^2 + 2x_1 x_2) \left[ (1 - x_1^2) \frac{\partial \rho}{\partial x_1} - x_1 y_1 \frac{\partial \rho}{\partial y_1} - x_1 z_1 \frac{\partial \rho}{\partial z_1} \right]
\]
\[
+ (y_1 x_2 + y_1 y_2) \frac{\partial \rho}{\partial y_1} + (z_1 x_2 + z_1 z_2) \frac{\partial \rho}{\partial z_1} + \cdots
\]
\[
= \frac{\partial \rho}{\partial x_1} - r_2^2 \frac{\partial \rho}{\partial x_1} - x_1 \left( x \frac{\partial \rho}{\partial x_1} + y \frac{\partial \rho}{\partial y_1} + z \frac{\partial \rho}{\partial z_1} \right)
\]
\[
+ x_2 \left( x_1 \frac{\partial \rho}{\partial x_1} + y_1 \frac{\partial \rho}{\partial y_1} + z_1 \frac{\partial \rho}{\partial z_1} \right) + \cdots.
\]  
(B.14)

In the same way,

\[
\hat{\zeta}_{x,2} = \left(x_1 x_2 + \sqrt{1 - r_1^2}, x_2 y_1, x_2 z_1\right) + \cdots,
\]  
(B.15)

\[
\hat{\zeta}_{y,2} = \left(y_2 x_1, y_1 y_2 + \sqrt{1 - r_1^2}, y_2 z_1\right) + \cdots
\]  
(B.16)

and

\[
\hat{\zeta}_{z,2} = \left(z_2 x_1, z_2 y_1, z_2 z_1 + \sqrt{1 - r_1^2}\right) + \cdots.
\]  
(B.17)

Combining these, (B.8), (B.9) and (B.10), we obtain

\[
\hat{\zeta}_{x,1} \cdot \hat{\zeta}_{x,2} = - \left(1 - \frac{1}{2} r_1^2 - \frac{1}{2} r_2^2 + 2x_1 x_2\right) + \cdots,
\]  
(B.18)
\[ \hat{\zeta}_{x,1} \cdot \hat{\zeta}_{y,2} = -2x_1y_2 + \cdots \] (B.19)

and

\[ \hat{\zeta}_{x,1} \cdot \hat{\zeta}_{z,2} = -2x_1z_2 + \cdots . \] (B.20)

Using these, we have

\[ \hat{\zeta}_{x,1} \cdot \text{div}_2(\hat{\zeta}_2\rho) = \hat{\zeta}_{x,1} \cdot g^{ij}_{2} \hat{\zeta}_{j,2} \partial_i \rho + \cdots \]

\[ = -\left( 1 - \frac{1}{2}r_1^2 - \frac{1}{2}r_2^2 + 2x_1x_2 \right) g^{i2}_{1} \partial_i \rho 
- 2x_1y_2g^{22}_{2} \partial_i \rho - 2x_1z_2g^{32}_{2} \partial_i \rho + \cdots \]

\[ = -g^{i2}_{2} \partial_i \rho + \frac{1}{2}(r_1^2 + r_2^2) \frac{\partial \rho}{\partial x_2} \]

\[ - 2x_1 \left( x_2 \frac{\partial \rho}{\partial x_2} + y_2 \frac{\partial \rho}{\partial y_2} + z_2 \frac{\partial \rho}{\partial z_2} \right) + \cdots \]

\[ = - \frac{\partial \rho}{\partial x_2} + \frac{1}{2}(r_1^2 + r_2^2) \frac{\partial \rho}{\partial x_2} \]

\[ - \left( \frac{3}{2} + \frac{1}{2}X \right) \left( x_2 \frac{\partial \rho}{\partial x_2} + y_2 \frac{\partial \rho}{\partial y_2} + z_2 \frac{\partial \rho}{\partial z_2} \right) + \cdots . \] (B.21)

Substituting (4.6), (B.4), (B.5), (B.14) and (B.21) into (3.22), we obtain the expansion (4.9).

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