Unifying the Hyperbolic and Spherical 2-Body Problem with Biquaternions

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Abstract—The 2-body problem on the sphere and hyperbolic space are both real forms of holomorphic Hamiltonian systems defined on the complex sphere. This admits a natural description in terms of biquaternions and allows us to address questions concerning the hyperbolic system by complexifying it and treating it as the complexification of a spherical system. In this way, results for the 2-body problem on the sphere are readily translated to the hyperbolic case. For instance, we implement this idea to completely classify the relative equilibria for the 2-body problem on hyperbolic 3-space and discuss their stability for a strictly attractive potential.

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1. BACKGROUND AND OUTLINE

The case of the 2-body problem on the 3-sphere has recently been considered by the author in [1]. This treatment takes advantage of the fact that $S^3$ is a group and that the action of $SO(4)$ on $S^3$ is generated by the left and right multiplication of $S^3$ by itself. This allows for a reduction in stages, first reducing by the left multiplication, and then reducing an intermediate space by the residual right-action. An advantage of this reduction by stages is that it allows for a fairly straightforward derivation of the relative equilibria solutions: the relative equilibria may first be classified in the intermediate reduced space and then reconstructed on the original phase space.

For the 2-body problem on hyperbolic space the same idea does not apply. Hyperbolic 3-space $H^3$ cannot be endowed with an isometric group structure as the symmetry group $SO(1,3)$ does not arise as a direct product of two such groups. This prevents us from reducing in stages.

Nevertheless, despite these differences, the sphere and hyperbolic space are both two sides of the same coin. The sphere $S^3$ is the real affine variety

$$u^2 + v^2 + w^2 + z^2 = 1,$$  

in Euclidean space, whereas $H^3$ is a connected component of

$$t^2 - x^2 - y^2 - z^2 = 1$$

in Minkowski space. If we complexify the variables each of them defines the same complex 3-sphere $\mathbb{CS}^3$ in $\mathbb{C}^4$ endowed with the standard complex-valued inner product. We call $\mathbb{CS}^3$ the complexification of $S^3$ and $H^3$, and refer to $S^3$ and $H^3$ as real forms of $\mathbb{CS}^3$. This is analogous to the notion of real forms and complexifications for vector spaces and Lie groups. To extend this analogy further, one can complexify an analytic Hamiltonian system on a real form to obtain a

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holomorphic Hamiltonian system on the complexification. The original real system appears as an invariant sub-system in the complexified phase space.

We shall consider the 2-body problem on $H^3$ and complexify it to obtain a holomorphic Hamiltonian system for the complexified 2-body problem on $\mathbb{C}S^3$. The group of complex symmetries is the complexification $SO_{4}\mathbb{C}$ of $SO(1, 3)$. Pleasingly, this puts us back into the same regime that we had earlier with the $SO(4)$-symmetry. In particular, we may take advantage of the $SL_2\mathbb{C} \times SL_2\mathbb{C}$-double-cover over $SO_{4}\mathbb{C}$ and reduce in stages. We can then apply exactly the same analysis for the spherical problem as performed in [1], except that now all variables are taken to be complex and one must remember to restrict attention to the invariant hyperbolic real form. This trick allows us to classify the relative equilibria solutions for the hyperbolic problem using the same methods which are used for the spherical case.

One-parameter sub-groups of $SO(1, 3)$ come in four types: elliptic, hyperbolic, loxodromic, and parabolic. The classification of relative equilibria for the 2-body problem on the Lobachevsky plane $H^2$ was carried out in [3, 4, 6]. For each configuration of the two bodies there exists precisely one elliptic and hyperbolic solution up to time-reversal symmetry. We extend this result and show that for each configuration of two bodies in $H^3$ there exists, up to conjugacy, a circle of loxodromic relative equilibria, including the elliptic and hyperbolic solutions from the $H^2$ case. There are no parabolic solutions for a strictly attractive potential. Furthermore, we piggy-back on the stability analysis performed in [3] and apply a continuity argument to derive complete results for the reduced stability of the relative equilibria.

The paper is arranged as follows. We first provide a short review of the theory of real forms for holomorphic Hamiltonian systems. The material is distilled from the more complete treatment given in [2], but is provided to ensure that the paper remains for the most part self-contained. The complexified 2-body problem is then formulated and conveniently presented in terms of biquaternions. Next, we consider in closer detail the hyperbolic real form. We discuss the hyperbolic spherical problem as performed in [1], except that now all variables are taken to be complex and we equip the manifold with a real-symplectic structure $\Omega$. A real structure $\eta$ on a complex manifold $M$ is an involution whose derivative is conjugate-linear everywhere. If $(M, \Omega)$ is a holomorphic symplectic manifold, then a real structure $\eta$ is called a real-symplectic structure if it satisfies $R^*\Omega = \Omega$.

If the fixed-point set $M^R$ is non-empty, then the restriction of $\Omega$ to $M^R$ defines a real symplectic form $\hat{\omega}_R$. A real structure $r$ on a complex manifold $C$ can be lifted to a real-symplectic structure $R$ on the cotangent bundle by setting

$$\langle R(\eta), X \rangle = \langle \eta, r_\ast X \rangle \quad (2.1)$$

for $\eta \in T^*_yC$ and for all $X \in T_{r(x)}C$. If $C^r$ is non-empty, then $(T^*C)^R$ is canonically symplectomorphic to $T^*C^r$.

**Theorem 1 ([2])**. Let $f$ be a holomorphic function on a holomorphic symplectic manifold $(M, \Omega)$ equipped with a real-symplectic structure $R$ with a non-empty fixed-point set $M^R$. If $f$ is purely real on $M^R$, then the Hamiltonian flow generated by $f$ leaves $M^R$ invariant. Moreover, the flow on $M^R$ is identically the Hamiltonian flow on $(M^R, \hat{\omega}_R)$ generated by the restriction $u$ of $f$. 

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In the situation described by this theorem we may refer to \((M^R, \hat{\omega}_R, u)\) as a “real form” of the holomorphic Hamiltonian system \((M, \Omega, f)\).

2.2. Compatible Group Actions

Let \(M\) be a complex manifold and \(G\) a complex Lie group acting holomorphically on \(M\). If \(M\) is equipped with a real structure \(R\), then the action is called \(R\)-compatible if there exists a real Lie group structure \(\rho\) on \(G\) which satisfies

\[
R(g \cdot x) = \rho(g) \cdot R(x)
\]

for all \(x\) in \(M\) and \(g\) in \(G\). Recall that a real Lie group structure is a real structure on \(G\) which is also a group homomorphism. For such a compatible group action the real sub-group \(G^\rho = \text{Fix } \rho\) acts on \(M^R\). One can show (see, for instance, [10, Proposition 2.3]) that for \(x\) in \(M^R\)

\[
T_x(G \cdot x \cap M^R) = T_x(G^\rho \cdot x).
\]

This fact is significant for the study of relative equilibria solutions for a Hamiltonian system.

Consider a Hamiltonian system \((M, \Omega, f)\), either real or complex, and suppose it admits a symplectic group action by \(G\) which preserves the Hamiltonian. The tuple \((M, \Omega, f, G)\) is a Hamiltonian \(G\)-system. A solution is called a relative equilibrium (RE) if it is contained to a \(G\)-orbit. Equivalently, the solution is the orbit of a one-parameter sub-group of \(G\) [8]. If we combine this with (2.3) and Theorem 1, we have the following.

Proposition 1. Let \((M, \Omega, f, G)\) be a holomorphic Hamiltonian \(G\)-system and suppose the action of \(G\) is \(R\)-compatible with respect to a real-symplectic structure \(R\) with a non-empty fixed-point set \(M^R\). If \(f\) is real on \(M^R\), then there is a one-to-one correspondence between RE of \((M, \Omega, f, G)\) which are contained to \(M^R\) and RE of the real form \((M^R, \hat{\omega}_R, u, G^\rho)\).

2.3. Holomorphic Momentum Maps

A holomorphic group action of a complex group \(G\) on a holomorphic symplectic manifold \((M, \Omega)\) is Hamiltonian if it admits an equivariant holomorphic momentum map \(J: M \to \mathfrak{g}^*\) satisfying \(\langle J(x), \xi \rangle = H_\xi(x)\) for each \(x\) in \(M\) and for all \(\xi\) in the Lie algebra \(\mathfrak{g}\) of \(G\). Here \(H_\xi\) is a holomorphic function whose Hamiltonian vector field is that generated by \(\xi\). If \(\rho\) is a real Lie group structure on \(G\) and \(R\), a real-symplectic structure on \(M\), then the momentum map \(J\) will be called \(R\)-compatible with respect to \(\rho\) if

\[
J \circ R = \overline{\rho^*} \circ J.
\]

The map \(\overline{\rho^*}\) is the conjugate-adjoint to \(\rho_* = D\rho\) defined by

\[
\langle \overline{\rho^*} \eta, \xi \rangle = \langle \eta, \rho_* \xi \rangle
\]

for each \(\eta\) in \(\mathfrak{g}^*\) and for all \(\xi\) in \(\mathfrak{g}\).

If \(G\) is connected, the \(R\)-compatibility of \(J\) implies that the action of \(G\) on \(M\) is also \(R\)-compatible in the sense of (2.2). In this case, the symplectic action of \(G^\rho\) on \(M^R\) is also Hamiltonian. Indeed, observe that \(J(x)\) belongs to \(\text{Fix } \overline{\rho^*}\) for \(x\) in \(M^R\). This set consists of all complex-linear forms on \(\mathfrak{g}\) which are real on \(\mathfrak{g}^\rho = \text{Fix } \rho_*\). We therefore have an identification of \(\text{Fix } \overline{\rho^*}\) with \((\mathfrak{g}^\rho)^*\). In this way the restriction

\[
\tilde{J} : = J|_{M^R}: M^R \longrightarrow \text{Fix } \overline{\rho^*} \cong (\mathfrak{g}^\rho)^*
\]

is the momentum map for the action of \(G^\rho\) on \((M^R, \hat{\omega}_R)\). (See also [2, Proposition 4.5].)
Indeed, since the polarisation identity we shall denote the symmetric, complex bilinear form which gives rise to
\[ \| \cdot \|_2 \]
in most cases it should be clear from the context or not make any difference. In any case, using which we shall denote by
\[ \det \]
the well-known double cover of the matrix representation this quadratic form corresponds to the determinant. Observe that the determinant of \[ \det \] is
\[ u^2 + v^2 + w^2 + z^2. \]
We may therefore identify the biquaternions with matrices \[ Q \] in \( M_2(\mathbb{C}) \) whose quadratic form is \( \det Q \), or with vectors \( q = (u, v, w, z)^T \) in \( \mathbb{C}^4 \) whose quadratic form is the standard \( q^T q \). We will not always specify whether a biquaternion \( q \) is understood to mean an element of \( M_2(\mathbb{C}) \) or \( \mathbb{C}^4 \). In most cases it should be clear from the context or not make any difference. In any case, using the polarisation identity we shall denote the symmetric, complex bilinear form which gives rise to \( \| \cdot \|_2 \) by \( \langle \cdot, \cdot \rangle \).

The set of biquaternions \( q \) with \( \| q \|^2 = 1 \) forms a group under multiplication. With respect to the matrix representation this corresponds to the special linear group \( SL_2(\mathbb{C}) \) of matrices with \( \det Q = 1 \). On the other hand, it also corresponds to the affine variety in \( \mathbb{C}^4 \) of vectors satisfying \( q^T q = 1 \). This is the complex 3-sphere \( \mathbb{C}S^3 \) defined in (1.1). Multiplication on the left and right by \( SL_2(\mathbb{C}) \) on \( M_2(\mathbb{C}) \) preserves the determinant. This provides an action of \( SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \) on \( M_2(\mathbb{C}) \) which we shall denote by
\[ (g_1, g_2) \cdot Q = g_1 Q g_2^{-1}. \]
This action preserves the complex quadratic form on \( \mathbb{C}^4 \cong M_2(\mathbb{C}) \), and consequently, establishes the well-known double cover of \( SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \) over \( SO_4(\mathbb{C}) \). We note that this action preserves \( \mathbb{C}S^3 \). Indeed, since \( \mathbb{C}S^3 \) may be identified with the group \( SL_2(\mathbb{C}) \), this action is just left and right multiplication of the group by itself.

3. THE COMPLEXIFIED 2-BODY PROBLEM

3.1. Biquaternions

Let \( \{1, I, J, K\} \) denote the standard basis for the real algebra of quaternions. A biquaternion is a linear combination
\[ q = u1 + vI + wJ + zK, \] (3.1)
where \( u, v, w, z \) each belong to \( \mathbb{C} \). As a complex algebra this admits a representation by identifying \( q \) with the matrix
\[ Q = \begin{pmatrix} u + iv & w + iz \\ -w + iz & u - iv \end{pmatrix}. \] (3.2)
The biquaternions are a composition algebra over \( \mathbb{C} \), meaning there exists a complex quadratic form \( \| \cdot \|^2 \) which satisfies \( \| q_1 q_2 \|^2 = \| q_1 \|^2 \| q_2 \|^2 \) for all biquaternions \( q_1 \) and \( q_2 \). With respect to the matrix representation this quadratic form corresponds to the determinant. Observe that the determinant of \( Q \)
\[ u^2 + v^2 + w^2 + z^2. \]

We may therefore identify the biquaternions with matrices \( Q \) in \( M_2(\mathbb{C}) \) whose quadratic form is \( \det Q \), or with vectors \( q = (u, v, w, z)^T \) in \( \mathbb{C}^4 \) whose quadratic form is the standard \( q^T q \). We will not always specify whether a biquaternion \( q \) is understood to mean an element of \( M_2(\mathbb{C}) \) or \( \mathbb{C}^4 \). In most cases it should be clear from the context or not make any difference. In any case, using the polarisation identity we shall denote the symmetric, complex bilinear form which gives rise to \( \| \cdot \|^2 \) by \( \langle \cdot, \cdot \rangle \).

The set of biquaternions \( q \) with \( \| q \|^2 = 1 \) forms a group under multiplication. With respect to the matrix representation this corresponds to the special linear group \( SL_2(\mathbb{C}) \) of matrices with \( \det Q = 1 \). On the other hand, it also corresponds to the affine variety in \( \mathbb{C}^4 \) of vectors satisfying \( q^T q = 1 \). This is the complex 3-sphere \( \mathbb{C}S^3 \) defined in (1.1). Multiplication on the left and right by \( SL_2(\mathbb{C}) \) on \( M_2(\mathbb{C}) \) preserves the determinant. This provides an action of \( SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \) on \( M_2(\mathbb{C}) \) which we shall denote by
\[ (g_1, g_2) \cdot Q = g_1 Q g_2^{-1}. \] (3.3)
This action preserves the complex quadratic form on \( \mathbb{C}^4 \cong M_2(\mathbb{C}) \), and consequently, establishes the well-known double cover of \( SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \) over \( SO_4(\mathbb{C}) \). We note that this action preserves \( \mathbb{C}S^3 \). Indeed, since \( \mathbb{C}S^3 \) may be identified with the group \( SL_2(\mathbb{C}) \), this action is just left and right multiplication of the group by itself.

3.2. The Hyperbolic Real Structure

We shall be interested in the following real structure \( r \) defined on \( \mathbb{C}S^3 \):
\[ r: (u, v, w, z) \mapsto (\overline{u}, -\overline{v}, -\overline{w}, -\overline{z}). \] (3.4)
According to the matrix representation in (3.2), this real structure corresponds to taking the conjugate transpose of \( Q \). The fixed-point set is therefore the set of Hermitian \( 2 \times 2 \) matrices with unit determinant. Alternatively, the fixed-point set consists of the points \((t, ix, iy, iz)\) for \( t, x, y, z \) real numbers satisfying (1.2). The solution set has two connected components corresponding to \( t \geq 1 \) and \( t \leq -1 \). We shall only be interested in the \( t \geq 1 \) component and write this as \( H^2 \), noting that this is the hyperboloid model of hyperbolic-3 space.

The action of \( SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \) in (3.3) is \( r \)-compatible with respect to the real Lie group structure
\[ \rho(g_1, g_2) = (g_2^{-1}, g_1^{-1}). \] (3.5)
The fixed-point set of $\rho$ is a conjugate-diagonal copy of $SL_2\mathbb{C}$ which acts by
\[ g \cdot Q = gQg^1. \] (3.6)
The hyperboloid $H^3 \subset \mathbb{R}^4$ is invariant with respect to this action. Therefore, the action preserves the indefinite orthogonal form of signature $(1,3)$. Incidentally, this establishes the double cover of $SL_2\mathbb{C}$ over $SO(1,3)$.

### 3.3. The Problem Setting

Consider the holomorphic symplectic manifold $T^*CS^3$. Thanks to the complex bilinear form on $\mathbb{C}^4$ the complex tangent spaces of $CS^3$ may be identified with their duals. In this way the cotangent bundle $T^*CS^3$ may be identified with the set
\[ \{(q,p) \in \mathbb{C}^4 \times \mathbb{C}^4 \mid \langle q,p \rangle = 0, \|q\|^2 = 1\}. \] (3.7)
The phase space for the complexified 2-body problem is $T^*CS^3 \times T^*CS^3$. Strictly speaking, the diagonal collision set should be removed, however, we will not reflect this in our notation. We consider a holomorphic Hamiltonian
\[ H(q_1,p_1,q_2,p_2) = -\frac{\|p_1\|^2}{2m_1} - \frac{\|p_2\|^2}{2m_2} + V(\langle q_1,q_2 \rangle) \] (3.8)
for $V: \mathbb{C} \to \mathbb{C}$, some holomorphic function which we call the potential.

We now take the product of the real structure $r$ on $CS^3 \times CS^3$ and lift this to a real-symplectic structure $R$ on the cotangent bundle. The hyperbolic phase space $T^*H^3 \times T^*H^3$ may be identified with a connected component of Fix $R$. Restricting the Hamiltonian to this component gives
\[ H(q_1,p_1,q_2,p_2) = \frac{|p_1|^2}{2m_1} + \frac{|p_2|^2}{2m_2} + V(\cosh \psi). \] (3.9)
Here $|p|^2$ denotes the modulus of the covector $p \in T^*_gH^3$ with respect to the hyperbolic metric on $H^3$. Recall that this metric is inherited by the negative of the ambient Minkowski metric of signature $(1,3)$ on $\mathbb{R}^4$. This explains the rather unfortunate presence of what appears to be negative kinetic energy in the holomorphic Hamiltonian.

For $q_1$ and $q_2$ belonging to $H^3 \subset \mathbb{C}^4$ the inner product $\langle q_1,q_2 \rangle$ is equal to $\cosh \psi$ where $\psi$ is the hyperbolic distance between $q_1$ and $q_2$. For the case of gravitational attraction we choose
\[ V(z) = -m_1m_2 \frac{z}{\sqrt{z^2 - 1}}, \] (3.10)
which corresponds to the potential $-m_1m_2 \coth \psi$. In this case the Hamiltonian $H$ is purely real on the real-symplectic form Fix $R$ and so we may apply Theorem 1 to the holomorphic Hamiltonian system on $T^*CS^3 \times T^*CS^3$. It follows that the Hamiltonian system for the 2-body problem on hyperbolic 3-space occurs as a real form of the holomorphic Hamiltonian system described above.

### 3.4. Symmetry and Momentum

The momentum map for the cotangent lift of left multiplication on a group is right translation of a covector to the identity [9]. Since $CS^3$ may be identified with the group $SL_2\mathbb{C}$, it follows that the momentum map for the action of left multiplication on phase space is the total left momenta
\[ L_{\text{tot}} = L_1 + L_2 \] (3.11)
where $L_k = p_kq_k^{-1}$ denotes the left momentum of the $k^{th}$ particle. Likewise, for the case of right multiplication the momentum is left translation to the identity. The momentum map for right multiplication on phase space is the total right momenta
\[ R_{\text{tot}} = R_1 + R_2 \] (3.12)
where \( R_k = q_k^{-1} p_k \) denotes the right momentum of the \( k \)th particle. The momentum map for the full group action of \( SL_2 \mathbb{C} \times SL_2 \mathbb{C} \) is therefore
\[
J: T^* \mathbb{C}^3 \times T^* \mathbb{C}^3 \rightarrow \mathfrak{sl}_2 \mathbb{C}^* \times \mathfrak{sl}_2 \mathbb{C}^*; \quad (q_1, p_2, q_2, p_2) \mapsto (L_{tot}, -R_{tot}).
\]
(3.13)
Notice that in the second factor we have negated the total right momenta. This is because for the action in (3.3) we have the inverse of right multiplication.

By once again using the complex bilinear form on the biquaternions to identify \( \mathfrak{sl}_2 \mathbb{C} \subset M_2(\mathbb{C}) \) with its dual, we may express the conjugate-adjoint \( \rho^* \) of the real Lie group structure \( \rho \) as
\[
\rho^*(L, R) = (-R^\dagger, -L^\dagger).
\]
(3.14)

The cotangent lift of \( r \) to \( T^* \mathbb{C}^3 \) is given by \((q, p) \mapsto (q^\dagger, p^\dagger)\), where \( q \) and \( p \) are here representing matrices in \( M_2(\mathbb{C}) \). It follows that the momentum map \( J \) is \( R \)-compatible with respect to \( \rho \), and hence, the group action is \( R \)-compatible. Furthermore, according to (2.6), the momentum \( C_{tot} \) for the action of \( SL_2 \mathbb{C} \) on \( T^* \mathbb{H}^3 \times T^* \mathbb{H}^3 \) may be identified with \( L_{tot} = R_{tot}^\dagger \).

4. SYMMETRY AND REDUCTION

4.1. One-Parameter Sub-Groups

A non-zero element of the Lie algebra \( \mathfrak{sl}_2 \mathbb{C} \) is conjugate up to the adjoint action to either a semi-simple or nilpotent matrix of the form
\[
S = \begin{pmatrix} \eta & 0 \\ 0 & -\eta \end{pmatrix} \quad \text{or} \quad N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\]
(4.1)
for \( \eta \) a non-zero complex number. The one-parameter sub-group generated by an element conjugate to \( S \) is called elliptic for \( \eta \) imaginary, hyperbolic for \( \eta \) real, and loxodromic for \( \eta \) a general complex number. A one-parameter sub-group generated by an element conjugate to \( N \) is called parabolic. This terminology is inherited from the group \( \text{PSL}_2 \mathbb{C} \) of Möbius transformations.

Recall that \((t, ix, iy, iz)\) in \( \mathbb{H}^3 \) corresponds via (3.2) to the Hermitian matrix
\[
\begin{pmatrix}
t - x & iy - z \\
-iy - z & t + x
\end{pmatrix}.
\]
(4.2)
By taking the exponential of the elements \( S \) and \( N \) we may use (3.6) to compute the action of the one-parameter sub-groups on \( \mathbb{H}^3 \). To help visualise these orbits, we can use the Poincaré ball model of hyperbolic 3-space to identify \( \mathbb{H}^3 \) with the open ball in \( \mathbb{R}^3 \) via the map which sends \((t, ix, iy, iz)\) to
\[
(X, Y, Z) = \left( \frac{x}{1 + t}, \frac{y}{1 + t}, \frac{z}{1 + t} \right).
\]
In Fig. 1 we include an illustration of these orbits on the Poincaré ball. In these diagrams the \( X \)-axis is the vertical line through the north and south poles.

4.2. Geodesics and the Lobachevsky Plane

Let \( \mathbb{H}^2 \) denote the sub-manifold of \( \mathbb{H}^3 \) given by setting \( y = 0 \). Note from (4.2) that this sub-manifold equivalently corresponds to those purely real, symmetric matrices with unit determinant. It also corresponds to the open disk in the Poincaré ball by setting \( Y = 0 \). More generally, we shall refer to any 2-dimensional sub-manifold of \( \mathbb{H}^3 \) as a Lobachevsky plane if it can be transformed to \( \mathbb{H}^2 \) by the action of \( SL_2 \mathbb{C} \).
Fig. 1. Orbits of one-parameter sub-groups on the Poincaré ball. Shown are examples of elliptic, hyperbolic, loxodromic, and parabolic orbits.

If we set $y = 0$ and $x = 0$, we obtain the curve $H^1$. In terms of the general form of a biquaternion given in (3.1) such elements of $H^1$ may be written as

$$e^{\psi j} = \cosh \psi + j \sinh \psi,$$

where $j = -iK$. Since $j^2 = 1$ we see that $H^1$ is none other than the unit hyperbola in the algebra of split-complex numbers, the analogue of the unit circle in the complex numbers. Finally, we remark that $H^1$ is a geodesic in $H^3$ and that the action of $SL_2 \mathbb{C}$ can send any geodesic to $H^1$.

**Proposition 2.** Let $\hat{J}$ denote the momentum map for the action of $SL_2 \mathbb{C}$ on $T^*H^3 \times T^*H^3$. The critical points of $\hat{J}$ are those points $(q_1, p_1, q_2, p_2)$ where $p_1$ and $p_2$ are each tangent to the geodesic through $q_1$ and $q_2$. The corresponding momentum $C_{tot}$ in $\mathfrak{s}l_2 \mathbb{C}^*$ has $||C_{tot}||^2$, a non-negative real number. Conversely, $||C_{tot}||^2$ is real if and only if $p_1$ and $p_2$ are each tangent to a Lobachevsky plane containing $q_1$ and $q_2$.

**Proof.** A point is a critical point of the momentum map if and only if the action at this point is not locally free. We may suppose $q_1$ and $q_2$ belong to $H^1$. The isotropy sub-group fixing $H^1$ is the group of rotations $SO(2) \subset SO(1, 3)$ in the $xy$-plane. Therefore, the action fails to be locally free when $p_1$ and $p_2$ have no $x$ or $y$ component. A calculation in terms of the split-complex numbers reveals that $||L_{tot}||^2$ is real and non-negative.

We may use the $SL_2 \mathbb{C}$-action to place the first particle at the centre of the Poincaré ball where $q_1 = I$. We can then rotate the ball so that $q_2$ belongs to $H^1$ and $p_2$ to $H^2$. Since $p_1, q_2$ and $p_2$ are all Hermitian, the imaginary part of $||C_{tot}||^2$ is

$$\det(L_1 + L_2) - \det(L_1^\dagger + L_2^\dagger) = 2\langle p_1, [p_2, q_2^{-1}] \rangle.$$ 

Observe that $p_1$ is a traceless, Hermitian matrix since it is tangent to $H^3$ at $q_1 = I$, and that $[p_2, q_2^{-1}]$ is a real, skew-symmetric matrix as $p_2$ and $q_2$ are each real and symmetric. It follows that $||C_{tot}||^2$ is real if and only if either $p_2$ commutes with $q_2$ or $p_1$ is real. If $p_1$ is real, then everything belongs to $H^2$. If $q_2$ commutes with $p_2$, then $p_2$ must also be a split-complex number, and hence, $p_2$ is tangent to $H^1$ at $q_1$. We can therefore rotate the ball about the $H^1$-axis to send $p_1$ into $H^2$. 

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4.3. The Right-Reduced Space

Instead of reducing by the full $SL_2 \mathbb{C} \times SL_2 \mathbb{C}$-symmetry, we shall instead obtain an intermediate reduced space by reducing by the action of right multiplication by $SL_2 \mathbb{C}$. The orbit quotient is given by sending $(q_1, p_1, q_2, p_2)$ to

$$(L_1, L_2, q_R),$$

where we have introduced the right-invariant quantity $q_R = q_1 q_2^{-1}$. One can directly perform this orbit quotient by using Corollary 3.8.5 in [8] or by applying the “semdirect product reduction by stages theorem” as in [1]. The reduced Poisson structure on $\mathfrak{sl}_2 \mathbb{C}^* \times \mathfrak{sl}_2 \mathbb{C}^* \times \mathbb{C}^4$ is the Poisson structure on the dual of the complexified special-Euclidean Lie algebra $\mathfrak{se}_4 \mathbb{C} = \mathfrak{so}_4 \mathbb{C} \times \mathbb{C}^4$. The Hamiltonian $H$ descends to

$$H(L_1, L_2, q_R) = -\frac{||L_1||^2}{2m_1} - \frac{||L_2||^2}{2m_2} + V(z),$$

where $z = \langle q_1, q_2 \rangle = \langle q_R, 1 \rangle$. We can use the explicit expression for the Lie–Poisson equations for a semidirect product [7] to obtain the equations of motion

$$\begin{align*}
\dot{L}_1 &= +f(z) \text{Im}(q_R), \\
\dot{L}_2 &= -f(z) \text{Im}(q_R), \\
\dot{q}_R &= -\frac{L_1}{m_1} q_R + \frac{q_R L_2}{m_2}.
\end{align*}$$

In this set of equations $f(z) = dV/dz$ and $\text{Im}(q) = q - \langle q, 1 \rangle$ is the “imaginary part” of a biquaternion $q$.

4.4. Relative Equilibria

The action of left multiplication by $SL_2 \mathbb{C}$ descends to the right-reduced space in $\mathfrak{se}_4 \mathbb{C}^*$ as

$$g \cdot (L_1, L_2, q_R) = (gL_1 g^{-1}, gL_2 g^{-1}, ggqrg^{-1}).$$

A RE of the system on $T^* \mathbb{C}^3 \times T^* \mathbb{C}^3$ is a solution contained to an orbit of $SL_2 \mathbb{C} \times SL_2 \mathbb{C}$. It must therefore descend to a RE on the right-reduced space with respect to the group action above. Combined with Proposition 1, we see that RE of the hyperbolic 2-body problem can be found by classifying those RE in $\mathfrak{se}_4 \mathbb{C}^*$ which lift to solutions on $T^* H^3 \times T^* H^3$.

Without any loss of generality, we may suppose the RE are orbits of one-parameter sub-groups generated by $S$ or $N$ given in (4.1). We separate this sub-section into two parts to handle both cases separately.

4.4.1. Semi-simple RE

By differentiating the action of the one-parameter sub-group $g(t) = \exp(St)$ in (4.6) and setting this to equal the equations of motion in (4.5) we obtain

$$\begin{align*}
[S, L_1] &= +f \text{Im}(q_R), \\
[S, L_2] &= -f \text{Im}(q_R), \\
[S, q_R] &= -\frac{L_1}{m_1} q_R + \frac{q_R L_2}{m_2}.
\end{align*}$$

The first two equations imply $\text{Im}(q_R)$ is orthogonal to $S = -i \eta I$, where we are now using the biquaternion notation from (3.1). Therefore, we may suppose $q_R$ is a biquaternion of the form $u1 + zK$. For hyperbolic RE we must have $\langle q_R, 1 \rangle = \langle q_1, q_2 \rangle = \cosh \psi$, and so it follows that $q_R$ is the split-complex number $e^{\psi j}$ from (4.3). Equations (4.7) and (4.8) are now satisfied for

$$L_1 = x_1 I + yJ \quad \text{and} \quad L_2 = x_2 I - yJ,$$
and hence, no such parabolic RE exist. This minus sign is a consequence of the negative kinetic energy term in (3.8). We can then calculate the solutions are uniquely given by

\[ x_1 = iy \left( \coth 2\psi + \frac{m_1}{m_2} \operatorname{csch} 2\psi \right) + im_1\eta, \]
\[ x_2 = iy \left( \coth 2\psi + \frac{m_2}{m_1} \operatorname{csch} 2\psi \right) + im_2\eta. \]

We remark that the linear system in \( x_1 \) and \( x_2 \) is non-degenerate for \( \psi \neq 0 \) independent of the masses \( m_1 \) and \( m_2 \). This is in contrast to the spherical case where the linear system is degenerate for an angular separation of \( \pi/2 \) and admits singular solutions only when \( m_1 = m_2 \).

### 4.4.2. Non-existence of parabolic RE

For a parabolic RE we replace \( q_R \) in Eqs. (4.7)–(4.9). By considering the action of the isotropy sub-group of \( N \) in \( SL_2\mathbb{C} \) we may suppose that \( q_R \) is of the form \( \psi + i \sinh \psi I \). In this case, Eqs. (4.7) and (4.8) are satisfied for \( L_s = y_sJ + z_s\bar{K} \) where

\[ y_1 - iz_1 = -y_2 + iz_2 = f \sinh \psi. \]

Expanding Eq. (4.9) reveals that we require

\[ 0 = f \sinh \psi (m_1 e^{\psi} + m_2 e^{-\psi}). \]

For a strictly attractive potential \( f \) is never zero, and so the equation above cannot hold for \( \psi \neq 0 \), and hence, no such parabolic RE exist.

### 4.5. Reconstruction and Classification

According to the matrix representation of biquaternions in (3.2), \( q_R = e^{\psi I} \) is Hermitian. For a hyperbolic RE \( q_1 \) and \( q_2 \) are also Hermitian. Therefore, since \( q_R = q_1 q_2^{-1} \) we see that this implies \( q_1 \) and \( q_2 \) each commute with \( q_R \). This is only possible if they too belong to the split-complex numbers. Therefore, we must have \( q_1 = e^{\chi_1 j} \) and \( q_2 = e^{-\chi_2} \) for \( \chi_1 \) and \( \chi_2 \) real numbers satisfying \( \chi_1 + \chi_2 = \psi \).

For each particle \( q_s \) we may differentiate the \( SL_2\mathbb{C} \)-action in (3.6) for the orbit of \( g(t) = \exp(S t) \) to find the velocity vector \( \dot{q}_s = S q + q S^\dagger \). The momentum \( p_s \) is then given by \( -m_s \dot{q}_s \). Recall that this minus sign is a consequence of the negative kinetic energy term in (3.8). We can then calculate \( L_1 \) and \( L_2 \) explicitly and compare this with the expression in (4.10). From this calculation we obtain

\[ x_s = im_s(\eta + \bar{\eta} \cosh 2\chi_s) \quad \text{and} \quad y = m_s \bar{\eta} \sinh 2\chi_s. \]

By setting these expressions to equal those in Eqs. (4.11) and (4.12) we obtain the full classification of RE for the 2-body problem on \( H^3 \).

**Theorem 2.** Every semi-simple RE for the 2-body problem on \( H^3 \) is conjugate to a RE generated by the biquaternion \( S = -i\eta I \) in \( sl_2\mathbb{C} \subset M_2(\mathbb{C}) \) for a complex number \( \eta \). These solutions are classified up to conjugacy by the separation \( \psi > 0 \) between the particles and the phase \( \theta = \arg \eta \in [0, \pi) \). For each \((\theta, \psi)\) we may suppose \( q_1 = e^{\chi_1 j} \) and \( q_2 = e^{-\chi_2} \) for \( \chi_1 \) and \( \chi_2 \) positive real numbers uniquely determined by \( \chi_1 + \chi_2 = \psi \) and

\[ m_1 \sinh 2\chi_1 = m_2 \sinh 2\chi_2. \]
The modulus of $\eta$ satisfies
\[ |\eta|^2 = \frac{f \sinh \psi}{2\zeta}, \tag{4.14} \]
where $\zeta = m_1 \sinh 2\chi_1 = m_2 \sinh 2\chi_2$. Here $f = dV/dz$ where $z = \cosh \psi$ and $V$ is the potential. For a strictly attractive potential there are no parabolic RE.

**Corollary 1.** For every separation $\psi > 0$ between two bodies on $H^3$ there exists a circle of loxodromic RE with generator $\eta \in \mathbb{C}$ satisfying (4.14). In particular, up to time reversal there is precisely one elliptic and one hyperbolic RE corresponding to when $\eta$ is purely imaginary and purely real, respectively.

### 5. STABILITY

#### 5.1. The Full Reduced Space

We may further reduce the right-reduced space by the residual action of left multiplication in (4.6) to obtain the full reduced space. This was done in [1] for the case of the 3-sphere, and so by complexifying the variables in [1, Theorem 2] we obtain the (holomorphic) orbit quotient
\[ \pi: (L_1, L_2, q_R) \mapsto \langle \{ k_{ij} \}_{i \leq j}, \delta, z \rangle \in \mathbb{C}^8. \tag{5.1} \]
Here we have introduced the invariants $k_{ij} = \langle v_i, v_j \rangle$ and $\delta = \langle v_3, [v_1, v_2] \rangle$, and where we are writing $(v_1, v_2, v_3)$ to denote $(L_1, L_2, \text{Im}(q_R))$. The image of $\pi$ satisfies the relations $\delta^2 = \det k_{ij}$ and $||q_R||^2 = 1$, and thus defines a 6-dimensional affine variety in $\mathbb{C}^8$.

Let $M$ denote the sub-set of $T^*\mathbb{C}^3 \times T^*\mathbb{C}^3$ for which $\pi$ is an orbit-map with respect to the $G = \text{SL}_2\mathbb{C} \times \text{SL}_2\mathbb{C}$-action. It can be shown that the fixed-point set $M^R$ is
\[ (T^*H^3 \times T^*H^3) \setminus \text{Crit} \tilde{J}. \]

The group $G$ acts freely on $M^R$, and therefore, thanks to $R$-compatibility, the intersection of a $G$-orbit with $M^R$ is an orbit of $G^\rho$. This implies that $M^R \hookrightarrow M$ descends to an injection of orbit spaces $M^R/G^\rho \hookrightarrow M/G$. It follows that $\pi$ restricted to $M^R$ defines an orbit map for the action of $\text{SL}_2\mathbb{C}$ on $(T^*H^3 \times T^*H^3) \setminus \text{Crit} \tilde{J}$.

The image $\pi(M^R/G^\rho)$ is a real 6-dimensional affine variety whose generic symplectic leaves are 4-dimensional and given by the level sets of the (complex-valued) Casimir function $||C_{\text{tot}}||^2$.

#### 5.2. Degenerate RE

Thanks to the classification in Theorem 2 we may parameterise the RE for the hyperbolic 2-body problem in the full reduced space by $(\theta, \psi)$. This parameterises a surface $\text{Fix} \, H$ of fixed points in the full reduced space. The critical values of the energy-Casimir map
\[ \mu: M^R/G^\rho \to \mathbb{R} \times \mathbb{C}; \quad \langle \{ k_{ij} \}_{i \leq j}, \delta, z \rangle \mapsto (H, ||C_{\text{tot}}||^2) \]
are precisely the image of $\text{Fix} \, H$. We call this image the energy-Casimir diagram. For the gravitational potential in (3.10) this image is included in Fig. 2. In practice, to obtain this image it is first necessary to show that
\[ \zeta = m_1 m_2 Z^{-1} e^\psi \sinh 2\psi, \tag{5.2} \]
where
\[ Z = \sqrt{(m_1 + m_2 e^{2\psi})(m_2 + m_1 e^{2\psi})}. \tag{5.3} \]
One can then express $\chi_1, \chi_2$, and $|\eta|^2$ in terms of $(\theta, \psi)$ and use (4.13) to carry out the computation. Observe that this image is pinched along a singular cusp. We claim that this is indicative of degenerate RE. By degenerate RE, we mean a fixed-point in the full reduced space whose linearisation of $H$ at this point is singular. As the next proposition shows, degenerate RE in the full reduced space are critical points of the Casimir map restricted to $\text{Fix} \, H$. The proof is immediate.
Fig. 2. Energy-Casimir diagram for $m_1 = 1$ and $m_2 = 2$. The lines emanating from the focal point are curves of constant $\theta$ and those transversal to them are of constant $\psi$.

Proposition 3. Let $P$ be a Poisson manifold and suppose the symplectic leaf $M$ through $x$ is the regular level set of a Casimir function $C = (C_1, \ldots, C_k)$. Consider the Hamiltonian flow generated by $H$ and suppose $\text{Fix} H$ is an immersed sub-manifold containing $x$. The sub-space $\ker D_x(C|_{\text{Fix} H}) \subset T_x M$ is fixed by the linearised flow of $H$ at $x$.

As a corollary we see that the cusp in Fig. 2 must correspond to degenerate RE. Indeed, since this cusp consists of singular values of $\mu$ restricted to $\text{Fix} H$, they must also be singular values of the Casimir map restricted to $\text{Fix} H$.

Fig. 3. The curve $\Gamma(\theta, \psi) = 0$ of degenerate RE.

Proposition 4. A RE with separation $\psi$ and phase $\theta$ is degenerate if and only if

$$\Gamma(\theta, \psi) = (m_1 + m_2)Z \cos 2\theta + \left( \frac{1 + \tanh \psi}{2 \cosh \psi} \right) (m_1 + m_2 \cosh 2\psi)(m_2 + m_1 \cosh 2\psi)$$

is equal to zero. These RE correspond to the singular points along the cusp of the energy-Casimir diagram.
Proof. Thanks to our formulation of the problem as the complexification of the 2-body problem on the sphere, the fully reduced equations of motion in $\mathbb{C}^8$ coincide with those in Eq. (2.10) of [1]. These may be linearised at a fixed point in a symplectic leaf. A RE is degenerate if and only if the constant term of the characteristic polynomial of this linearisation is zero. This polynomial is derived in Eq. (3.16) of [1], however, for a hyperbolic RE in Fix $H$ we must remember to replace every instance of a trigonometric function with its hyperbolic counterpart in $\psi$. The constant term $c_0$ is then given by

$$
\left( \frac{k_{11}}{m_1^2} - \frac{k_{22}}{m_2^2} \right)^2 + 2 \coth \theta \operatorname{csch}^2 \theta \left[ \frac{k_{11}}{m_1} \left( 1 + \frac{m_2}{m_1} \right) + \frac{k_{22}}{m_2} \left( 1 + \frac{m_1}{m_2} \right) \right] + \left[ (m_1 + m_2) \coth \theta \operatorname{csch}^2 \theta \right]^2.
$$

The expressions in (4.13) allow us to write

$$
k_{11} = -2|\eta|^2 m_1^2 (\cos 2\theta + \cosh 2\chi_1),
$$

which can then be rewritten using (5.2) and (5.3) as

$$
k_{11} = -\frac{m_1^2}{2 \sinh^3 \psi \cosh \psi} \left[ Z e^{-\psi} \cos 2\theta + (m_1 \cosh 2\psi + m_2) \right].
$$

A similar expression holds for $k_{22}$. If we substitute this into the constant term above, we find that $c_0 = -2e^{-\psi} \Gamma(\theta, \psi) \operatorname{csch}^6 \psi$. In Fig. 3 we illustrate the solutions to $\Gamma(\theta, \psi) = 0$. As this defines a connected curve in $(\theta, \psi)$-space, it must correspond exactly to the degenerate RE along the singular cusp in Fig. 2.

5.3. The Energy-Casimir Method

If $C = ||C_{\text{tot}}||^2$ is real, then by Proposition 2 the orbit-reduced space may be identified with the space of $SL_2 \mathbb{C}$-orbits which intersect

$$
\left\{ (q_1, p_1, q_2, p_2) \in T^*H^2 \times T^*H^2 \mid C_{\text{tot}} \in sl_2 \mathbb{R}, \, ||C_{\text{tot}}||^2 = C \right\}.
$$

Observe that $\{q_1, p_1, q_2, p_2\}$ spans the space of real, symmetric matrices in $M_2(\mathbb{C})$. From (3.6) we see that the orbits contained within this space are equivalently the orbits of the sub-group $SL_2 \mathbb{R}$. Therefore, thanks to the isomorphism $PSL_2 \mathbb{R} \cong SO(1, 2)$ the reduced space for $C = ||C_{\text{tot}}||^2$ real is identical to the orbit-reduced space for the 2-body problem on $H^2$ with Casimir $C$.

The Hessian and linearisation of the reduced Hamiltonian for the problem on $H^2$ is found in [3]. We can therefore apply a continuity argument to extend their results to $H^3$, noting that the behaviour of the fixed points can only change when they cross a degenerate point whereupon an eigenvalue becomes zero.

**Theorem 3.** The stability of a RE for the gravitational 2-body problem on $H^3$ is determined by $\Gamma(\theta, \psi)$ as follows:

1) For $\Gamma < 0$ the Hessian of the Hamiltonian at the fixed point in reduced space is positive definite, and thus, the RE is Lyapunov stable.

2) For $\Gamma > 0$ the Hessian at the fixed point in reduced space has signature $(+++)$, and thus, the RE is linearly unstable.

In particular, a RE is unstable whenever $\cos 2\theta > 0$, or whenever $\psi > \psi_{\text{crit}}$.

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