Abstract. In 1980, White conjectured that the toric ideal associated to a matroid is generated by quadrics. Blum defined the base-sortable matroid and showed that the class of base-sortable matroids is closed under duality, contraction, deletion, series and parallel extension, and direct sums. In this paper, we prove that the class of matroids for which the toric ideal is generated by quadrics and that has quadratic Gröbner bases, is closed under series and parallel extensions, series and parallel connections, and 2-sums.

1. Introduction

A matroid has multiple equivalent definitions. We define a matroid as a collection of subsets that satisfies the exchange axiom: A matroid $M$ is a pair $(E, B)$, where $E = [d] = \{1, \ldots, d\}$ and $B$ is a collection of subsets of $E$, that satisfies

- for every $B$ and $B'$ in $B$, for any $x \in B$, there exists $y \in B'$ such that $(B \cup \{y\}) \setminus \{x\}$ is a member of $B$.

For a detailed introduction to matroid theory, see [10]. We call a member of $B$ a basis of $M$. All the members of $B$ have the same cardinality. This cardinality is said to be the rank of $M$ and is denoted by $r(M)$. Let $B(M) = \{B_1, \ldots, B_n\}$ be the collection of bases of a matroid $M$ on $E$. Let $K$ be a field, and let $K[X] = K[x_1, \ldots, x_n]$ be the polynomial ring over $K$. We consider the ring homomorphism

$$
\pi_M : K[X] \to K[S] = K[s_1, \ldots, s_d] \quad x_j \mapsto \prod_{l \in B_j} s_l.
$$

The toric ideal $J_M$ is the kernel of $\pi_M$. The semigroup ring $R_M = K[X]/J_M$ is called the bases monomial ring of $M$, and it was introduced by N. White. White proved that the bases monomial ring $R_M$ is normal and, in particular, it is a Cohen-Macaulay ring for any matroid $M$ (see [14]).

Let $\mathcal{M}_{\text{QG}}$ be the class of matroids for which the toric ideal $J_M$ has a Gröbner basis consisting of quadratic binomials, let $\mathcal{M}_{\text{Q}}$ be the class of matroids for which $J_M$ is generated by quadrics, and let $\mathcal{M}$ be the class of all matroids. Clearly, the inclusions $\mathcal{M}_{\text{QG}} \subset \mathcal{M}_{\text{Q}} \subset \mathcal{M}$ hold. In the toric ideal of a matroid, there is the following conjecture:

**Conjecture 1.1.** The equalities $\mathcal{M}_{\text{QG}} = \mathcal{M}_{\text{Q}} = \mathcal{M}$ hold.

The equality $\mathcal{M}_{\text{Q}} = \mathcal{M}$ was conjectured by White [15, Conjecture 12]. Classes of uniform matroids and matroids with rank $\leq 2$ belong to $\mathcal{M}_{\text{QG}}$ [2] [9] [11]. Blum proved that the toric ideal of graphic matroids without $M(K_4)$-minor has a quadratic
Gröbner basis. In the case of $\mathcal{M}_Q$, classes of graphic matroids, matroids with rank $\leq 3$, and transversal polymatroids are included in $\mathcal{M}_Q [1, 4, 6]$.

Let $M$ be a matroid on $E$, and let $\mathcal{B}(M)$ be the collection of bases of $M$. An element $i \in E$ is called a **loop** of $M$ if it does not belong to any basis of $M$. Dually, an element $i \in E$ is said to be a **coloop** of $M$ if it is contained in all the bases of $M$. Let

$$B^*(M) = \{ E \setminus B \mid B \in \mathcal{B}(M) \}.$$  

Then a pair $(E, B^*(M))$ is a matroid, and it is called the **dual** of $M$ and is denoted as $M^*$.

Let $M$ and $\mathcal{B}(M)$ be as above, and let $c \in E$. We consider the following collection of subsets of $E \setminus \{c\}$:

$$\mathcal{B}(M) \setminus c = \begin{cases} 
\{B \setminus \{c\} \mid B \in \mathcal{B}(M)\} & \text{if } c \text{ is a coloop of } M \\
\{B \mid c \notin B \in \mathcal{B}(M)\} & \text{otherwise.}
\end{cases}$$

A pair $(E \setminus \{c\}, \mathcal{B}(M) \setminus c)$ is a matroid, and it is called the **deletion** of $c$ from $M$ and is denoted as $M \setminus c$. Dually, let $M/c$, the **contraction** of $c$ from $M$, be given by $M/c = (M^* \setminus c)^*$. We call a matroid $M'$ a **minor** of a matroid $M$ if $M'$ can be obtained from $M$ by a finite sequence of contractions and deletions.

Let $M_1$ and $M_2$ be matroids with $E_1 \cap E_2 = \emptyset$. Let $\mathcal{B}(M_1)$ and $\mathcal{B}(M_2)$ be collections of bases of $M_1$ and $M_2$, and let

$$\mathcal{B}(M_1) \oplus \mathcal{B}(M_2) = \{ B \cup D \mid B \in \mathcal{B}(M_1), D \in \mathcal{B}(M_2) \}.$$  

Then a pair $(E, \mathcal{B}(M_1) \oplus \mathcal{B}(M_2))$, where $E = E_1 \cup E_2$, is a matroid. This matroid is called the **direct sum** of $M_1$ and $M_2$, and it is denoted as $M_1 \oplus M_2$.

**Proposition 1.2.** [2, 15] Classes $\mathcal{M}_{QG}$ and $\mathcal{M}_Q$ are closed under duality, contraction, deletion, and direct sums.

Note that the associated semigroup ring $R_{M^*}$ is isomorphic to $R_M$ as $K$-algebras, and $R_{M/P}$ and $R_{M/P'}$ are combinatorial pure subrings of $R_M$, as defined in [8]. Furthermore, $R_{M_1 \oplus M_2}$ is the Segre product $R_{M_1} \ast R_{M_2}$ of $R_{M_1}$ and $R_{M_2}$. Blum defined base-sortable matroids and proved that the class of base-sortable matroids is closed under series and parallel extensions. In particular, the class of base-sortable matroids belongs to $\mathcal{M}_{QG}$ (See [2]).

The outline of this paper is as follows. In Section 2, we describe how to compute generating sets and Gröbner bases for the toric ideal of the series and parallel extensions. In Section 3, we use the results from Section 2 to form generating sets and Gröbner bases for the toric ideal of the series and parallel connections and the 2-sum.

We study the classes $\mathcal{M}_{QG}$ and $\mathcal{M}_Q$ with respect to series and parallel extensions, series and parallel connections, and 2-sums of matroids. From the theories of toric fiber products and combinatorial pure subrings, we have

**Theorem.** Classes $\mathcal{M}_{QG}$ and $\mathcal{M}_Q$ are closed under series and parallel extensions, series and parallel connections, and 2-sums.
This theorem follows as a corollary of Theorem 2.2, Corollary 2.3, and Theorem 3.1.

2. A series and parallel extension of a matroid

Let $M$ be a matroid on $E = [d]$, and let $\mathcal{B}(M)$ be the collection of bases of $M$. Then a series extension of $M$ at $c \in E$ by $d + 1$ is a matroid on $E \cup \{d + 1\}$ that has

$$\{B \cup \{d + 1\} \mid B \in \mathcal{B}(M)\} \cup \{B \cup \{c\} \mid c \notin B \in \mathcal{B}(M)\}$$

as the collection of bases and is denoted as $M \oplus_c (d + 1)$. Dually, we call a matroid $(M^* \oplus_c (d + 1))^*$ a parallel extension of $M$ at $c$ by $d + 1$. A series-parallel extension of $M$ is any matroid derived from $M$ by a finite sequence of series and parallel extensions. We suppose that $M$ does not have $c \in E$ as a coloop. Let $\mathcal{B}(M) = \{B_1, \ldots, B_\gamma, \ldots, B_n\}$ be the collection of bases of $M$, where $c \notin B_j$ for $j \in [\gamma]$ and $c \in B_j$ for $j \in [n] \setminus [\gamma]$. We renumber the bases of $M$, if necessary. Let $\mathcal{D}_M = \{b^1_j \mid j \in [n]\}$ denote a vector configuration satisfying $b^1_j = \sum_{i \in B_j} e_i$, where $e_i$ is the $l$-th standard vector. As necessary, we consider $\mathcal{D}_M$ as a collection of vectors or as a matrix. Now we consider a new vector configuration $\tilde{\mathcal{D}}_M = \{(b^1_j, a^i) \mid i \in [2], j \in [\alpha_i]\}$ that satisfies $b^1_j = b^1_l$ for $j \in [\gamma]$, where $(\begin{smallmatrix} \alpha_1 \\ \alpha_2 \end{smallmatrix}) = (\begin{smallmatrix} \gamma \\ 1 \end{smallmatrix})$, and $\mathbf{a}^2 = (\begin{smallmatrix} 1 \\ 1 \end{smallmatrix})$. We define the ring homomorphism $\tilde{\pi}_M$ as follows:

$$\tilde{\pi}_M : K[X] = K[x^i_j \mid i \in [2], j \in [\alpha_i]] \to K[S, W] = K[s, w_l \mid k \in [d], l \in [2]]$$

$$x^i_j \mapsto S^{b^1_j} W^{a^i}.$$

Then $J_{\tilde{\mathcal{D}}_M} = \ker(\tilde{\pi}_M)$.

Let $\omega \in \mathbb{Z}_{\geq 0}^n$, and let $\prec$ be an arbitrary monomial order. We define a new monomial order $\prec_\omega$ as follows:

$$X^a \prec_\omega X^b \iff \left\{ \begin{array}{l}
\omega \cdot a < \omega \cdot b ; \\
\omega \cdot a = \omega \cdot b \text{ and } X^a \prec X^b,
\end{array} \right. $$

for $a, b \in \mathbb{Z}_{\geq 0}^n$. We call a monomial order $\prec_\omega$ a weight order on $K[x_1, \ldots, x_n]$. We use the following useful result:

**Proposition 2.1.** [11, Proposition 1.11] For any monomial order $\prec$ and any ideal $I \subset K[x_1, \ldots, x_n]$, there exists a vector $w \in \mathbb{Z}_{\geq 0}^n$ such that $\text{in}_\omega(I) = \text{in}_\prec(I)$.

Let $\mathbf{F}$ be a homogeneous generating set for $J_{\mathcal{D}_M}$, and let

$$f = \prod_{l=1}^{u_f} x^{1_{j_l}}_l \prod_{l=1}^{v_f} x^{1_{k_l}}_l - \prod_{l=1}^{u'_f} x^{1_{j'_l}}_l \prod_{l=1}^{v'_f} x^{1_{k'_l}}_l \in \mathbf{F},$$

where $j_l, j'_l \in [\gamma], k_l, k'_l \in [n] \setminus [\gamma]$. However, if $u_f \neq u'_f$, then $\pi_M(f) \neq 0$ since the $c$-th entry of $\sum_{l=1}^{u_f} b^1_{j_l}$ does not coincide with the $c$-th entry of $\sum_{l=1}^{u'_f} b^1_{j'_l}$, and the $c$-th
entries of $\sum_{l=1}^{u_f} b_{kl}^l$ and $\sum_{l=1}^{u_f'} b_{kl}^l$ are zero. Therefore $u_f = u_f'$ and $v_f = v_f'$. Now let $I = (i_1, \ldots, i_{u_f}) \in \{1, 2\}^{u_f'}$ and consider the binomial $f^I \in K[X]$ defined by

$$f^I = \prod_{l=1}^{u_f} x_{j_l}^{i_l} \prod_{l=1}^{v_f} x_{k_l}^{1} - \prod_{l=1}^{u_f} x_{j_l}^{i_l'} \prod_{l=1}^{v_f} x_{k_l}^{1}.$$ 

Since $f \in J_{\tilde{D}_M}$, the new homogeneous binomial $f^I \in J_{\tilde{D}_M}$. We set

$$\tilde{F} = \{ f^I \mid f \in F, I \in \{1, 2\}^{u_f'} \} \cup \{ x_{j_1}^{1}x_{j_2}^{2} - x_{j_1'}^{1}x_{j_2'}^{2} \mid 1 \leq j_1 < j_2 \leq \gamma \}.$$

**Theorem 2.2.** Let $M$ be a matroid on $E$, and let $F$ be a Gröbner basis for $J_{\tilde{D}_M}$. Then $\tilde{F}$ is a Gröbner basis for $J_{\tilde{D}_M}$.

**Proof.** First, it is easy to see that $\tilde{F} \subset J_{\tilde{D}_M}$. Let $\omega = (\omega_1, \ldots, \omega_{\gamma})$ be a weight vector. We denote the underlined monomial of $f$ as the initial monomial of $f$ with respect to a weight order $\omega$. Let $\tilde{\omega} = (\omega_1', \ldots, \omega_{\gamma}')$ denote a weight vector satisfying $\omega_j = \omega_j'$ for $j \in [\gamma]$. Then the underlined monomial of $f^I$ is the initial monomial of $f^I$ with respect to a weight order $\prec_{\omega}$. We choose a tie-breaking monomial order on $K[X]$ that makes the monomial $x_{j_2}^{1}x_{j_1}^{2}$ for $1 \leq j_1 < j_2 \leq \gamma$ the initial monomial. Let $\text{in}(F) = \{ (\text{in}_{\omega}(f) \mid f \in F) \}$ and $\text{in}(F) = \{ (\text{in}_{\omega\prec_{\omega}}(f) \mid f \in \tilde{F}) \}$. Let $u$ and $v$ be monomials that are not in $\text{in}(F)$:

$$u = \prod_{l=1}^{m_1} (x_{i_l}^{1})^{p_l} \prod_{l=1}^{m_2} (x_{j_l}^{2})^{q_l} \prod_{l=1}^{m_3} (x_{k_l}^{1})^{r_l},$$

$$v = \prod_{l=1}^{m_1'} (x_{i_l}^{1})^{p_l'} \prod_{l=1}^{m_2'} (x_{j_l}^{2})^{q_l'} \prod_{l=1}^{m_3'} (x_{k_l}^{1})^{r_l'},$$

where $p_l, q_l, r_l, p_l', q_l', r_l' \in \mathbb{Z}_{\geq 0}$ for any $l$, and $I = \{i_1, \ldots, i_{m_1}\}$, $I' = \{i_1', \ldots, i_{m_1}'\}$, $J = \{j_1, \ldots, j_{m_2}\}$, and $J' = \{j_1', \ldots, j_{m_2}'\}$ are subsets of $[\gamma]$ with cardinalities $m_1$, $m_1'$, $m_2$, and $m_2'$, respectively; and $K = \{k_1, \ldots, k_{m_3}\}$ and $K' = \{k_1', \ldots, k_{m_3}'\}$ are subsets of $[n] \setminus [\gamma]$ with cardinalities $m_3$ and $m_3'$, respectively. Since neither $u$ nor $v$ is divided by $x_{j_2}^{1}x_{j_1}^{2}$ for $1 \leq j_1 < j_2 \leq \gamma$, it follows that $i_l \leq j_{l'}$ for $l \in [m_1]$ and $l' \in [m_2]$, and $i_{l'} \leq j_{l'}'$ for $l \in [m_1']$ and $l' \in [m_2']$. We suppose that $\pi_M(u) = \pi_M(v)$:

$$\pi_M(u) = w_1^q w_2^{p+q} r_1 \prod_{l=1}^{m_1} S^{p_l b_{kl}} \prod_{l=1}^{m_2} S^{q_l b_{kl}} \prod_{l=1}^{m_3} S^{r_l b_{kl}},$$

$$\pi_M(v) = w_1^{q'} w_2^{p'+q'} r_1' \prod_{l=1}^{m_1'} S^{p_l' b_{kl}} \prod_{l=1}^{m_2'} S^{q_l' b_{kl}} \prod_{l=1}^{m_3'} S^{r_l' b_{kl}}.$$
Here we set \( p = \sum_{i=1}^{m_1} p_i, \ q = \sum_{i=1}^{m_2} q_i, \ r = \sum_{i=1}^{m_3} r_i, \ p' = \sum_{i=1}^{m_1'} p_i, \ q' = \sum_{i=1}^{m_2'} q_i, \) and \( r' = \sum_{i=1}^{m_3'} r_i. \) Since \( b_j^1 = b_j^2 \) for \( j \in [\gamma], \) it follows that \( \pi_M(u') = \pi_M(v'), \) where

\[
\begin{align*}
  u' &= \prod_{i=1}^{m_1} (x_{i_1})^{p_i} \prod_{i=1}^{m_2} (x_{i_2})^{q_i} \prod_{i=1}^{m_3} (x_{i_3})^{r_i} \\
  v' &= \prod_{i=1}^{m_1'} (x_{i_1}')^{p_i} \prod_{i=1}^{m_2'} (x_{i_2}')^{q_i} \prod_{i=1}^{m_3'} (x_{i_3}')^{r_i}.
\end{align*}
\]

Hence \( u' - v' \) belongs to \( J_{\mathcal{D}}. \) If \( u' \) and \( v' \) belong to \( \text{in}(F), \) then \( u' \) and \( v' \) are in \( \text{in}(\tilde{F}). \) This is a contradiction. Therefore neither \( u' \) nor \( v' \) belongs to \( \text{in}(F). \) Since \( F \) is a Gröbner basis for \( J_{\mathcal{D}}, \) it follows that \( u' = v'. \)

In particular, \( L = L', \ J = J', \ K = K', \ p_i = p_i', q_i = q_i', \) and \( r_i = r_i' \) for any \( l. \) Thus \( u = v. \) Therefore \( \tilde{F} \) is a Gröbner basis for \( J_{\mathcal{D}}. \) \hfill \( \Box \)

**Corollary 2.3.** Let \( M \) be a matroid on \( E. \) If \( F \) is a homogeneous generating set for \( J_{\mathcal{D}}, \) then \( \tilde{F} \) is a generating set for \( J_{\mathcal{D}}. \)

**Proof.** We assume that \( F \) and \( F' \) are generating sets for \( \mathcal{D}. \) Then \( \tilde{F} \) and \( \tilde{F}' \) generate the same ideal. In particular, this holds if \( F' \) is a Gröbner basis for \( \mathcal{D}. \) Thus \( \langle \tilde{F} \rangle = \langle \tilde{F}' \rangle. \) By Theorem 22, if \( F' \) is a Gröbner basis for \( J_{\mathcal{D}}, \) then \( \tilde{F}' \) is a generating set for \( J_{\mathcal{D}}, \) since \( \tilde{F}' \) is a Gröbner basis for \( J_{\mathcal{D}}. \) \hfill \( \Box \)

**Corollary 2.4.** Let \( M \) be a matroid on \( E, \) and let \( M +c \ (d+1) \) denote a series extension of \( M \) at \( c \) by \( d+1. \) Then, by replacing variables, \( \tilde{F} \) becomes a generating set (resp. a Gröbner basis) for \( J_{M+c(d+1)}. \)

**Proof.** By elementary row operations on \( \tilde{D}_M, \) we obtain the vector configuration arising from \( M +c \ (d+1). \) \hfill \( \Box \)

**Remark 2.5.** If \( c \) is a coloop of \( M, \) then \( J_{M+c(d+1)} = J_M. \)

**Corollary 2.6.** Classes \( \mathcal{M}_Q \) and \( \mathcal{M}_Q \) are closed under series and parallel extensions.

3. A SERIES AND PARALLEL CONNECTION OF MATROIDS

Let \( M_1 \) and \( M_2 \) be matroids with \( E_1 \cap E_2 = \{ c \} \) and \( E = E_1 \cup E_2. \) Suppose that for both \( M_1 \) and \( M_2, \ c \) is neither a loop nor a coloop. Let

\[
\begin{align*}
  \mathcal{B}_S &= \{ B \cup D \mid B \in \mathcal{B}(M_1), D \in \mathcal{B}(M_2), B \cap D = \emptyset \} \\
  \mathcal{B}_P &= \{ B \cup D \mid B \in \mathcal{B}(M_1), D \in \mathcal{B}(M_2), c \in B \cap D \} \\
  &= \cup \{(B \cup D) \setminus \{ c \} \mid B \in \mathcal{B}(M_1), D \in \mathcal{B}(M_2), \text{c is in exactly one of } B \text{ and } D \}.
\end{align*}
\]

Then pairs \((E, \mathcal{B}_S)\) and \((E, \mathcal{B}_P)\) are matroids. These matroids are said to be the *series* and *parallel* connections of \( M_1 \) and \( M_2 \) with respect to the basepoint \( c. \) We denote them as \( S(M_1; c), (M_2; c) \) and \( P((M_1; c), (M_2; c)), \) or briefly, \( S(M_1, M_2) \) and \( P(M_1, M_2) \) [10, Proposition 7.1.13].
On the other hand, when $c$ is a loop of $M_1$, then we define

$$P(M_1, M_2) = M_1 \oplus (M_2/c) \quad \text{and} \quad S(M_1, M_2) = (M_1/c) \oplus M_2.$$  

When $c$ is a coloop of $M_1$, then we define

$$P(M_1, M_2) = (M_1 \setminus c) \oplus M_2 \quad \text{and} \quad S(M_1, M_2) = M_1 \oplus (M_2 \setminus c)$$

(see [10] 7.1.5 - 7.1.8]. Moreover, the 2-sum $M_1 \oplus_2 M_2$ of $M_1$ and $M_2$ is $S(M_1, M_2)/c$, or equivalently, $P(M_1, M_2) \setminus c$, where $c$ is neither a loop nor a coloop of either $M_1$ or $M_2$.

Let $M_1$ and $M_2$ be matroids on $E_1 = [d_1]$ and $E_2 = [d_2]$. We identify the set $[d_2]$ with the set $\{d_1 + 1, \ldots, d_1 + d_2\}$. Assume that $c_i \in E_i$ is not a coloop of $M_i$ for $i \in [2]$. Let

$$
B(M_1) = \{ B_1, \ldots, B_{\gamma_1}, \ldots, B_{n_1} \} \quad \text{and} \quad B(M_2) = \{ D_1, \ldots, D_{\gamma_2}, \ldots, D_{n_2} \}
$$

be collections of bases of $M_1$ and $M_2$, where $c_1 \notin B_j$ for $j \in [\gamma_1]$ and $c_2 \notin D_k$ for $k \in [\gamma_2]$. Let $\mathcal{D}_{M_1} = \{ b_j^1 \mid j \in [n_1] \}$ and $\mathcal{D}_{M_2} = \{ d_k^2 \mid k \in [n_2] \}$ be two vector configurations satisfying $b_j^1 = \sum_{i \in B_j} e_i$ and $d_k^2 = \sum_{i \in D_k} e_i$. We define the ring homomorphisms $\pi_{M_1}$ and $\pi_{M_2}$ by setting

$$
\pi_{M_1} : K[x_j^1 \mid j \in [n_1]] \to K[S] \quad x_j^1 \mapsto S^{b_j^1},
$$

$$
\pi_{M_2} : K[y_k^2 \mid k \in [n_2]] \to K[T] \quad y_k^2 \mapsto T^{d_k^2}.
$$

Similar to what we did in Section 2, we consider two new vector configurations

$$
\tilde{\mathcal{D}}_{M_1} = \{ (b_j^1, a^i) \mid i \in [2], j \in [\alpha_i] \} \quad \text{and} \quad \tilde{\mathcal{D}}_{M_2} = \{ (d_k^2, a^i) \mid i \in [2], k \in [\beta_i] \},
$$

such that $b_j^1 = b_j^2$ for $j \in [\gamma_1]$ and $d_k^1 = d_k^2$ for $k \in [\gamma_2]$, where $\binom{n_1}{\alpha_i} = \binom{\gamma_1}{\gamma_1}, \binom{n_2}{\beta_i} = \binom{\gamma_2}{\gamma_2}, a^1 = (0),$ and $a^2 = (1)$. We define the ring homomorphisms $\tilde{\pi}_{M_1}$ and $\tilde{\pi}_{M_2}$ as follows:

$$
\tilde{\pi}_{M_1} : K[X] = K[x_j^1 \mid j \in [2], j \in [\alpha_i]] \to K[S, W] \quad x_j^i \mapsto S^{b_j^i} W^{a^i},
$$

$$
\tilde{\pi}_{M_2} : K[Y] = K[y_k^2 \mid i \in [2], k \in [\beta_i]] \to K[T, W] \quad y_k^2 \mapsto T^{d_k^2} W^{a^i}.
$$

Then $J_{\tilde{\mathcal{D}}_{M_i}} = \ker(\tilde{\pi}_{M_i})$ for $i \in [2]$. Moreover, consider the vector configuration

$$
\tilde{\mathcal{D}} = \{ (b_j^1, d_k^2, a^i) \mid i \in [2], j \in [\alpha_i], k \in [\beta_i] \}.
$$

Let $K[Z] = K[z_{jk}^j \mid i \in [2], j \in [\alpha_i], k \in [\beta_i]]$ be the polynomial ring over $K$. The ring homomorphism $\pi$ is defined by

$$
\pi : K[Z] \to K[S, T, W] \quad z_{j,k}^i \mapsto S^{b_j^i} T^{d_k^i} W^{a^i}.
$$

Then $J_{\tilde{\mathcal{D}}} = \ker(\pi)$.

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be homogeneous generating sets for $J_{\mathcal{D}_{M_1}}$ and $J_{\mathcal{D}_{M_2}}$, respectively. Then we define $\tilde{\mathcal{F}}_1$ and $\tilde{\mathcal{F}}_2$ in a way analogous to what we did in Section 2. Let

$$f = \prod_{l=1}^{u_f} x_{j_l^1}^{i_l^1} - \prod_{l=1}^{u_f} x_{j_l^2}^{i_l^2} \in \tilde{\mathcal{F}}_1,$$
and let $k = (k_1, \ldots, k_{uf})$, with $k_l \in [\beta_l]$ for $1 \leq l \leq uf$. We consider the binomial $f_k \in K[Z]$ defined by

$$f_k = \prod_{i=1}^{uf} z_j^i k_l - \prod_{i=1}^{uf} z_j^i k_l.$$

Since $f \in J_{\tilde{D}_{M_1}}$, the new homogeneous binomial $f_k \in J_{\tilde{D}}$. If $\tilde{F}_1$ is any set of binomials in $J_{\tilde{D}_{M_1}}$, then

$$\text{Lift}(\tilde{F}_1) = \left\{ f_k \mid f \in \tilde{F}_1, k = \prod_{i=1}^{uf} [\beta_i] \right\}.$$

We define Lift$(\tilde{F}_2)$ in an analogous way. Furthermore, the quadratic binomial set Quad$(\tilde{D}_{M_1}, \tilde{D}_{M_2})$ is defined by

$$\text{Quad}(\tilde{D}_{M_1}, \tilde{D}_{M_2}) = \left\{ z_j^i k_l z_j^i k_l - z_j^i k_l - z_j^i k_l \mid i \in [2], 1 \leq j_1 < j_2 \leq \alpha_i, 1 \leq k_1 < k_2 \leq \beta_i \right\}.$$

We set $\tilde{N} = \text{Lift}(\tilde{F}_1) \cup \text{Lift}(\tilde{F}_2) \cup \text{Quad}(\tilde{D}_{M_1}, \tilde{D}_{M_2})$.

**Theorem 3.1.** Let $M_1$ and $M_2$ be matroids on $E_1 = [d_1]$ and $E_2 = [d_2]$, respectively; and assume that $c_i \in E_i$ is not a coloop of $M_i$ for $i \in [2]$. Let $S(M_1, M_2)$ be a series connection of $M_1$ and $M_2$ with respect to the basepoint $c = c_1 = c_2$. Then, by replacing variables,

$$N = \tilde{N} \cap K[\tilde{Z}]$$

is a generating set for $J_{S(M_1, M_2)}$. Here we set $K[\tilde{Z}] = K[z_j^i k_l \mid i \in [2], j \in [\alpha_i], k \in V_i]$, where $V_1 = [\gamma_2]$ and $V_2 = [\gamma_2] \setminus [\gamma_2]$. Moreover, if $F_1$ and $F_2$ are Gröbner bases for $J_{\tilde{D}_{M_1}}$ and $J_{\tilde{D}_{M_2}}$, then there exists a monomial order such that $N$ is a Gröbner basis for $J_{S(M_1, M_2)}$.

For the proof of Theorem 3.1, we use the results in [5, 13]. Let $r > 0$ be a positive integer, and let $\alpha, \beta \in \mathbb{Z}_{>0}$ be two vectors of positive integers. Let

$$K[X] = K[x_j^i \mid i \in [r], j \in [\alpha_i]] \quad \text{and} \quad K[Y] = K[y_k^i \mid i \in [r], k \in [\beta_i]]$$

be two multigraded polynomial rings with the multigrading $\deg(x_j^i) = \deg(y_k^i) = a_i^j \in \mathbb{Z}^d$. We write $\mathcal{A} = \{a^1, \ldots, a^u\}$ and assume that $\mathcal{A}$ is linearly independent. If $I$ and $J$ are homogeneous ideals of $K[X]$ and $K[Y]$, then the quotient rings $R_1 = K[X]/I$ and $R_2 = K[Y]/J$ are also multigraded by $\mathcal{A}$. Consider the polynomial ring

$$K[Z] = K[z_j^i k_l \mid i \in [r], j \in [\alpha_i], k \in [\beta_i]]$$

and consider the ring homomorphism

$$\phi_{I,J} : K[Z] \to R_1 \otimes_K R_2 \quad z_j^i k_l \mapsto x_j^i \otimes y_k^i.$$

The kernel of $\phi_{I,J}$ is called the toric fiber product of $I$ and $J$. It is denoted as $I \times_{\mathcal{A}} J = \ker(\phi_{I,J})$. The following result is in [13, Theorem 12 and Corollary 14].
Theorem 3.2. Suppose that the set \( A \) of degree vectors is linearly independent. Let \( F_1 \) and \( F_2 \) be homogeneous generating sets for \( I \) and \( J \), respectively. Then

\[
N = \text{Lift}(F_1) \cup \text{Lift}(F_2) \cup \text{Quad}_A
\]

is a homogeneous generating set for \( I \times_A J \). Moreover, if \( F_1 \) and \( F_2 \) are Gröbner bases of \( I \) and \( J \), then there exists a monomial order such that \( N \) is a Gröbner basis for \( I \times_A J \). The sets \( \text{Lift}(F_1) \), \( \text{Lift}(F_2) \), and \( \text{Quad}_A \) are defined in [13].

On the other hand, if \( I \) and \( J \) are toric ideals, then \( I \times_A J \) is also a toric ideal. If \( K[S] \) and \( K[T] \) are polynomial rings, and

\[
\phi : K[X] \rightarrow K[S], \quad x_i^j \mapsto f^j_i(S)
\]

\[
\psi : K[Y] \rightarrow K[T], \quad y_k^i \mapsto g^i_k(T)
\]

are ring homomorphisms, then we can form the toric fiber product homomorphism

\[
\phi \times_A \psi : K[Z] \rightarrow K[S, T], \quad z_{jk}^i \mapsto f^j_i(S)g^i_k(T).
\]

If \( I = \ker(\phi) \) and \( J = \ker(\psi) \) and both ideals are homogeneous with respect to the grading by \( A \), then \( I \times_A J = \ker(\phi \times_A \psi) \) (see [5]).

Proof of Theorem 3.1. Let \( F_1 \) and \( F_2 \) be generating sets (resp. Gröbner bases) for \( J_{D_{M_1}} \) and \( J_{D_{M_2}} \). From Theorem 2.2, Corollary 2.3, and Theorem 3.2, \( N \) is a generating set (resp. a Gröbner basis) for \( J_{\hat{D}} \). Now we consider two vector configurations

\[
\hat{D}' = \{(b^i_j, d^i_k, c^i) \mid i \in [2], j \in [\alpha], k \in [\beta], \}
\]

\[
D = \{(b^i_j, d^i_k, a^i) \mid i \in [2], j \in [\alpha], k \in V_i, \}
\]

where \( c^1 = a^1 \) and

\[
c^2 = \begin{cases} 
a^2 & \text{if } k \in [\gamma_2] \\
a^1 & \text{otherwise.}
\end{cases}
\]

Then \( J_{\hat{D}'} = J_{\hat{D}} \) because \( \hat{D}' \) can be obtained by an elementary row operation on \( \hat{D} \). Let \( \delta = (0, \ldots, 0, -1, 0) \in \mathbb{Z}^{d_1 + d_2 + 2} \). Since the usual inner product \( \delta \cdot (b^i_j, d^i_k, c^i) \) equals

\[
\begin{cases} 
-1 & \text{if } i = 2 \text{ and } k \in [\gamma_2] \\
0 & \text{otherwise,}
\end{cases}
\]

it follows that a subring \( K[\hat{Z}] / J_D \) of \( K[Z] / J_{\hat{D}'} \) is a combinatorial pure subring of \( K[Z] / J_{\hat{D}'} \) (see [7]). Thus \( J_D = J_{\hat{D}'} \cap K[\hat{Z}] \). In particular, \( N \) is a generating set (resp. a Gröbner basis) for \( J_D \). Furthermore, by elementary row operations on \( D \), we can obtain the vector configuration arising from \( S(M_1, M_2) \) with respect to the basepoint \( c \). Therefore, by replacing variables, \( N \) is a generating set (resp. a Gröbner basis) for \( J_{S(M_1, M_2)} \).

Corollary 3.3. Classes \( \mathcal{M}_{\text{QG}} \) and \( \mathcal{M}_{\text{Q}} \) are closed under series and parallel connections and 2-sums.
Proof. Let $M_1$ and $M_2$ be matroids with $E_1 \cap E_2 = \{c\}$. Let $S(M_1, M_2)$ (resp. $P(M_1, M_2)$) denote a series (resp. parallel) connection of $M_1$ and $M_2$ with respect to the basepoint $c$.

In the case of series and parallel connections, if $c$ is a loop or a coloop of $M_1$, then $\mathcal{M}_{\mathcal{QG}}$ and $\mathcal{M}_\mathcal{Q}$ are closed under series and parallel connections. Suppose that neither $M_1$ nor $M_2$ has $c$ as a loop or a coloop. Then by Theorem 2.2 and Theorem 3.1, $\mathcal{M}_{\mathcal{QG}}$ and $\mathcal{M}_\mathcal{Q}$ are closed under series connections. Also, $\mathcal{M}_{\mathcal{QG}}$ and $\mathcal{M}_\mathcal{Q}$ are closed under parallel connections from Proposition 1.2, and $P(M_1, M_2) = [S(M_1^*, M_2^*)]^*$ for any matroids $M_1$ and $M_2$ [10, Proposition 7.1.14].

In the case of the 2-sum, since $M_1 \oplus_2 M_2 = S(M_1, M_2)/c$, $\mathcal{M}_{\mathcal{QG}}$ and $\mathcal{M}_\mathcal{Q}$ are closed under 2-sums. □

Using the above results, we have

**Theorem 3.4.** Let $M$ be a matroid. Then if $M$ has no minor isomorphic to any of $M(K_4)$, $W^3$, $P_6$, or $Q_6$, then the toric ideal $J_M$ has a Gröbner basis consisting of quadratic binomials.

Theorem 3.4 immediately holds from the following result:

**Theorem 3.5.** [3, Corollary 3.1] A matroid $M$ is a minor of direct sums and 2-sums of uniform matroids if and only if $M$ has no minor isomorphic to any of $M(K_4)$, $W^3$, $P_6$, or $Q_6$.

Let $M$ be a matroid on $E$, and let

$$\text{rk} : 2^E \to \mathbb{Z}_{\geq 0} \quad X \mapsto |B_X|,$$

where $B_X$ is a basis for $M \setminus (E - X)$. A function $\text{rk}$ is said to be the rank function of $M$. Let $\lambda_M(X) = \text{rk}(X) + \text{rk}(E - X) - r(M)$ for $X \subset E$. We call $\lambda_M(X)$ the connectivity function of $M$. For $X \subset E$, if $\lambda_M(X) < k$, where $k$ is a positive integer, then both $X$ and $(X, E - X)$ are called $k$-separating. A $k$-separating pair $(X, E - X)$ for which $\min\{|X|, |E - X|\} \geq k$ is called a $k$-separation of $M$ with sides $X$ and $E - X$. For all $n \geq 2$, we say that $M$ is $n$-connected if, for any $k < n$, it has no $k$-separation.

Any matroid that is not 3-connected can be constructed from 3-connected proper minors of itself by a sequence of the operations of direct sum and 2-sum. Therefore, in order to prove Conjecture 1.1, it is enough to prove the following conjecture:

**Conjecture 3.6.** The class of all 3-connected matroids belongs to $\mathcal{M}_\mathcal{Q}$ and $\mathcal{M}_{\mathcal{QG}}$.

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