SPDES LAW EQUVALENCE AND THE COMPACT SUPPORT
PROPERTY: APPLICATIONS TO THE ALLEN-CAHN SPDE

HASSAN ALLOUBA

Abstract. Using our uniqueness in law transfer result for SPDEs, described
in a recent note, we prove the equivalence of laws of SPDEs differing by a
drift, under vastly applicable conditions. This gives us the equivalence in the
compact support property among a large class of SPDEs. As an important
application, we prove the equivalence in law of the Allen-Cahn and the associ-
ated heat SPDEs; and we give a criterion for the compact support property to
hold for the Allen-Cahn SPDE with diffusion function $a(t, x, u) = C u^\gamma$, with
$C \neq 0$ and $1/2 \leq \gamma < 1$.

1. Statements and discussions of results.

We start by considering the pair of parabolic SPDEs

\begin{equation}
\begin{aligned}
\frac{\partial U}{\partial t} &= \Delta_x U + b(t, x, U) + a(t, x, U) \frac{\partial^2 W}{\partial t \partial x}; \\
U_x(t, -\infty) &= U_x(t, \infty) = 0; \\
U(0, x) &= h(x);
\end{aligned}
\end{equation}

(1.1)

and

\begin{equation}
\begin{aligned}
\frac{\partial V}{\partial t} &= \Delta_x V + (b + d)(t, x, V) + a(t, x, V) \frac{\partial^2 W}{\partial t \partial x}; \\
V_x(t, -\infty) &= V_x(t, \infty) = 0; \\
V(0, x) &= h(x);
\end{aligned}
\end{equation}

(1.2)

on $\mathcal{R}_T := [0, T] \times \mathbb{R}$, where $W(t, x)$ is the Brownian sheet corresponding to the driving
space-time white noise, written formally as $\partial^2 W/\partial t \partial x$. As in Walsh [15], white noise
is regarded as a continuous orthogonal martingale measure, which we denote by $W$.
The diffusion $a(t, x, u)$ and the drifts $b(t, x, u)$ and $d(t, x, u)$ are Borel-measurable
$\mathbb{R}$-valued functions on $\mathcal{R}_T \times \mathbb{R}$; and $h : \mathbb{R} \to \mathbb{R}$ is a bounded continuous function.
Henceforth, we will denote (1.1) and (1.2) by $w_{\text{Neu}}(a, b, h)$ and $w_{\text{heat}}(a, b + d, h)$,
respectively. When $b \equiv 0$, we denote (1.1) by $w_{\text{heat}}(a, 0, h)$. In the interest of
getting quickly to our main results, we refer the reader to [5] for the rigorous
interpretation of all SPDEs considered in this paper, with the obvious modifications
to accommodate the change of space from $\mathcal{R}_{T,L} = [0, T] \times [0, L]$ to $\mathcal{R}_T$. Also, the law of
a random variable $X$ under the probability measure $\mathbb{P}$ is denoted by $\mathbb{L}_X$. Proceeding
toward a precise statement of our results, let $R_a(t, x) \triangleq d(t, x, u)/a(t, x, u)$, for any
$(t, x, u) \in \mathcal{R}_T \times \mathbb{R}$, whenever the ratio is well defined. Let $\lambda$ denote Lebesgue
measure. Our law equivalence result for the pair $w_{\text{heat}}(a, b, h)$ and $w_{\text{heat}}(a, b + d, h)$
can now be stated as

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Theorem 1.1. Let \((V, W^{(1)})\) be a solution (weak or strong) to \(e^{\text{heat}}_{a,b+d,h}(a,b,h)\) on some probability space \((\Omega^{(1)}, \mathcal{H}, \{\mathcal{F}_t\}, \mathbb{Q})\). Assume that \(R_U\) and \(R_V\) are in \(L^2(\mathbb{R}_T, \lambda)\), almost surely, whenever the random fields \(U\) and \(V\) solve (weakly or strongly) \(e^{\text{heat}}_{a,b,h}(a,b,h)\) and \(e^{\text{heat}}_{a,b+d,h}\), respectively. Assume further that there is a unique-in-law solution \((U, W^{(2)})\) to the heat SPDE \(e_{\text{heat}}(a,b,h)\) on \((\Omega^{(2)}, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})\). Then \(L^2_{\mathbb{Q}}\) and \(L^2_{\mathbb{P}}\) are mutually absolutely continuous (on \((C(\mathbb{R}_T; \mathbb{R}))\)).

Remark 1.1. By Theorem 1.1 in [5], which can trivially be extended from \(\mathcal{R}_{T,L}\) to \(\mathcal{R}_T\) (replacing \([0,L]\) with \(\mathbb{R}\) in the \(L^2\) condition and in the proof), uniqueness in law for the SPDE \(e_{\text{heat}}(a,b,h)\) is equivalent to uniqueness in law for \(e_{\text{heat}}(a,b,d,h)\) under the \(L^2(\mathbb{R}_T, \lambda)\) condition on \(R_U\) and \(R_V\). So, we can replace the uniqueness assumption on \(e_{\text{heat}}(a,b,h)\) in Theorem 1.1 above by that on \(e_{\text{heat}}(a,b,d,h)\). Also, our Neumann conditions may be replaced by Dirichlet conditions without affecting the conclusion of Theorem 1.1. Finally, Theorem 1.1 and its proof are valid when \(\mathcal{R}_T\) is replaced by \(\mathcal{R}_{T,L}\) (replacing \(\mathbb{R}\) with \([0,L]\)).

As an immediate consequence of Theorem 1.1 and Remark 1.1, we get the following law equivalence between the Allen-Cahn SPDE

\[
\begin{align*}
\frac{\partial V}{\partial t} &= \Delta_x V + 2V(1-V^2) + CV \frac{\partial^2 W}{\partial t \partial x}; \quad (t,x) \in \mathcal{R}_T, \ C > 0, \\
V_x(t, -\infty) &= V_x(t, \infty) = 0; \quad 0 < t \leq T, \\
V(0,x) &= h(x); \quad x \in \mathbb{R},
\end{align*}
\]

and its associated heat SPDE (the one obtained from the Allen-Cahn SPDE) by removing the Allen-Cahn nonlinearity. We note here that the proof of the uniqueness for the Allen-Cahn SPDE in Theorem 1.2 in [5] works just as well for the case \(\gamma = 1/2\), in addition to \(\frac{1}{2} < \gamma < 1\), because the SPDE in (1.3) with \(b = 0\) and \(a(t,x,u) = Cu^{1/2}\) admits uniqueness in law as discussed in [12] p. 326 and in [14].

Corollary 1.1. Suppose that \(V\) and \(U\) solve (weakly or strongly) the Allen-Cahn SPDE and its associated heat SPDE, respectively, on \(\mathcal{R}_{T,L}\) (see (0.3) in [5]) and with \(1/2 \leq \gamma \leq 1\). Then the laws of \(U\) and \(V\) are equivalent (on \((C(\mathbb{R}_{T,L}; \mathbb{R}))\)). If \(\mathcal{R}_{T,L}\) is replaced with \(\mathcal{R}_T\), if \(1/2 \leq \gamma < 1\), if \(h(x)\) has compact support, and if \(R_V\) is in \(L^2(\mathbb{R}_T, \lambda)\) a.s.:

\[
\int_{\mathbb{R}_T} R^2_U(t,x)dtdx = \frac{4}{C^2} \int_{\mathbb{R}_T} V^2(1-\gamma)(V^4-2V^2+1)dtdx < \infty; \text{ almost surely},
\]

then the laws of \(U\) and \(V\) are equivalent (on \((C(\mathbb{R}_T; \mathbb{R}))\)).

Remark 1.2. In the first part of Corollary 1.1, the continuity of \(U\) and \(V\) insures that \(R_U\) and \(R_V\) are in \(L^2(\mathbb{R}_{T,L}, \lambda)\), for \(0 \leq \gamma \leq 1\) (see the proof of Theorem 1.2 in [5]). When \(\mathcal{R}_{T,L}\) is replaced by \(\mathcal{R}_T\), we do not require that \(R_U\) be in \(L^2(\mathbb{R}_T, \lambda)\) (1.4) with \(U\) instead of \(V\). This is because \(R_U\) is already in \(L^2(\mathbb{R}_T, \lambda)\), since \(U(t, \cdot)\) has compact support for each \(t\) in the range \(1/2 \leq \gamma < 1\) by [12] [10]. Also, when we replace \(\mathcal{R}_{T,L}\) with \(\mathcal{R}_T\), \(\gamma \leq 1\) is replaced with \(\gamma < 1\); since, in this case, when \(\gamma = 1\), the integrability assumption in (1.1) has obvious problems for both \(U\) and \(V\).

Let \(\Omega = C(\mathbb{R}_T; \mathbb{R})\) and denote elements of \(\Omega\) by \(\omega\). Let \(X\) be the coordinate mapping process on \(\Omega\): \(X_\omega(t,x) = \omega(t,x)\). Denote by \(\mathcal{G}_i^X\), \(\mathcal{G}_i^\omega\), and \(\mathcal{G}_i^\omega\) the sigma
fields of subsets of $\Omega$ generated by $X$ when $(t, x)$ is fixed, when $t$ is fixed but $x$ is not, and when both $t$ and $x$ are not fixed, respectively. I.e.,

$$G_{t,x}^X = \sigma\left\{ \omega \in \Omega; X_\omega(t, x) = \omega(t, x) \in A \right\}; \quad (t, x) \in \mathcal{R}_T, \quad A \in \mathcal{B}(\mathbb{R}), \quad x \in \mathbb{R};$$

$$G_{t_i}^X = \sigma\left\{ \omega \in \Omega; (X_\omega(t_1, x_1) = \omega(t_1, x_1), \ldots, X_\omega(t_n, x_n) = \omega(t_n, x_n)) \in A \right\}; \quad n \geq 1, \quad A \in \mathcal{B}(\mathbb{R}^n), \quad x_i \in \mathbb{R}, \quad i = 1, \ldots, n; \quad t \in [0, T],$$

$$G_{t_i}^X = \sigma\left\{ \omega \in \Omega; (X_\omega(t_1, x_1) = \omega(t_1, x_1), \ldots, X_\omega(t_n, x_n) = \omega(t_n, x_n)) \in A \right\}; \quad n \geq 1, \quad A \in \mathcal{B}(\mathbb{R}^n), \quad (t_i, x_i) \in \mathcal{R}_T, \quad i = 1, \ldots, n \right).$$

Then, clearly, $G_{t,x}^X \subseteq G_{t_i}^X \subseteq \mathcal{B}(C(\mathcal{R}_T; \mathbb{R})) = G_{t_i}^X$. In this case, the last equality is trivial, and so absolute continuity on $B(C(\mathcal{R}_T; \mathbb{R}))$ implies absolute continuity on $G_{t,x}^X$ and $G_{t_i}^X$. This observation along with Theorem 1.1 easily give us

**Corollary 1.2.** Under the conditions of Theorem 1.1 $L^V_{\mathcal{Q}}^{(t,x)}$ is equivalent to $L^V_{\mathcal{P}}^{(t,x)}$ on $\mathbb{R}$, for every $(t, x) \in \mathcal{R}_T$ (in particular, if one is absolutely continuous with respect to Lebesgue measure then so is the other); and $L^V_{\mathcal{Q}}^{(t,\cdot)}$ is equivalent to $L^U_{\mathcal{P}}^{(t,\cdot)}$ on $C(\mathbb{R}; \mathbb{R})$, for every $t \in [0, T]$.

By proving law equivalence between $c^{N_{\text{heat}}(a, b, h)}_{\text{heat}}$ and $c^{N_{\text{heat}}(a, b + d, h)}_{\text{heat}}$ under considerably weaker conditions, Theorem 1.1 and Corollary 1.2 extend and make more applicable the notion of relative absolute continuity in our earlier work (Theorem 3.3.3 in [2] or Theorem 4.3 in [3]). Like Theorem 4.3 in [3], Theorem 1.1 and Corollary 1.2 (and thus the first assertion of Theorem 1.2 below) are equally valid for wave SPDEs, space-time SDEs, and SDEs (cf. Theorem 3.7, Theorem 4.3, and their proofs under the stronger conditions of [3]). An interesting application of Theorem 1.1 and Corollary 1.2 is to allow us to prove the following theorem about the compact support property of solutions to a large class of SPDEs containing the Allen-Cahn SPDE:

**Theorem 1.2.** Assume that the conditions of Theorem 1.1 hold. Then, $U(\cdot, \cdot)$ (or $V(\cdot, \cdot)$) $V(t, \cdot)$ and $U(t, \cdot)$ have compact support iff $V(\cdot, \cdot)$ and $U(t, \cdot)$ do. In particular, if $V(t, x)$ is a solution to the Allen-Cahn SPDE (1.3), $h(x)$ has compact support, and $1 \leq \gamma < 1$; then, for each $t \in [0, T]$, $V(t, \cdot)$ has compact support as a function of $x$ iff (1.4) holds.

It is noteworthy that all the Allen-Cahn SPDE results and their proofs here and in [5] are valid for the KPP SPDE, obtained by replacing the Allen-Cahn term $2V(1 - V^2)$ by the KPP term $V(1 - V)$. In [2, 3], we gave a proof of the existence of solutions to heat SPDEs with continuous diffusion coefficient $a$ and measurable drift $b$—with $a$ satisfying a linear growth condition and $b/a$ satisfying Novikov’s condition—using a system of stochastic differential-difference equations (SDDEs). In [5], we use our SDDE approach and the results of this note and [5] to further investigate the existence and some properties of SPDEs considered here.
2. Proofs of results

Proof of Theorem 1.1 It follows from the uniqueness in law assumption for $e_{\text{heat}}^n(a, b, h)$, the almost sure $L^2(\mathcal{R}_T, \lambda)$ condition on $R_V$, and a trivial extension of Theorem 1.1 in [5] to the space $\mathcal{R}_T$ that we have uniqueness in law for $e_{\text{heat}}^n(a, b + d, h)$ (see Remark 1.1).

Now, take $\{\tau^U_n\}$ and $\{\tau^V_n\}$ to be the sequences of stopping times

$$
\tau^U_n = T \land \inf \left\{ 0 \leq t \leq T; \int_{[0,t] \times \mathbb{R}} R^2_U(s, x)dsdx = n \right\}; \quad n \in \mathbb{N},
$$

and $\{\tau^V_n\}$ is gotten from (2.1) by replacing $U$ with $V$. Let $\tilde{\mathcal{W}} = \{\tilde{\mathcal{W}}_t(B), \mathcal{F}_t; 0 \leq t \leq T, B \in \mathcal{B}(\mathbb{R})\}$ be given by

$$
\tilde{\mathcal{W}}_t(B) \triangleq \mathcal{W}_t(2)(B) - \int_{[0,t] \times B} R_U(s, x)dsdx.
$$

Novikov’s condition and Girsanov’s theorem for white noise (see Corollary 3.1.3 in [4]) imply that, for $n \in \mathbb{N}$, $\tilde{\mathcal{W}}_n = \{\tilde{\mathcal{W}}_{\tau^U_n}(B), \mathcal{F}_t; 0 \leq t \leq T, B \in \mathcal{B}(\mathbb{R})\}$ is a white noise stopped at time $\tau^U_n$, under the probability measure $\tilde{\mathbb{P}}_n$ defined on $\mathcal{F}_T$ by the recipe

$$
d\tilde{\mathbb{P}}_n = \mathbb{E}_{T^2}^{R_U, \mathcal{W}}(\mathbb{R})
$$

(2.2)

$$
\triangleq \exp \left[ \int_{[0,T \land \tau^U_n] \times \mathbb{R}} R_U(s, x) \mathcal{W}^2(ds, dx) - \frac{1}{2} \int_{[0,T \land \tau^U_n] \times \mathbb{R}} R^2_U(s, x)dsdx \right].
$$

It then follows that $(U, \tilde{\mathcal{W}}_n)$, $(\Omega(2), \mathcal{F}_T, \{\mathcal{F}_t\}, \tilde{\mathbb{P}}_n)$ is a solution to the $e_{\text{heat}}^n(a, b + d, h)$ on $\mathbb{R}^{T \land \tau^U_n}$ for each $n \in \mathbb{N}$. Consequently for an arbitrary set $\Lambda \in \mathcal{B}(\mathbb{R})$ we get

$$
\mathbb{Q}[V(\cdot, \cdot) \in \Lambda, \tau^V_n = T] = \tilde{\mathbb{P}}_n[U(\cdot, \cdot) \in \Lambda, \tau^U_n = T]
$$

(2.3)

$$
= \mathbb{E}_\mathbb{P} \left[ 1_{\{U(\cdot, \cdot) \in \Lambda, \tau^U_n = T\}} \mathbb{E}_{T^2}^{R_U, \mathcal{W}}(\mathbb{R}) \right]; \quad n \in \mathbb{N}.
$$

To see (2.3) observe that, on the event $\Omega_n^U \triangleq \{\omega \in \Omega(2); \tau^U_n(\omega) = T\}$, $(U, \tilde{\mathcal{W}}_n)$ is a solution to $e_{\text{heat}}^n(a, b + d, h)$ on $\mathbb{R}_T$, under $\tilde{\mathbb{P}}_n$, and so the uniqueness in law for $e_{\text{heat}}^n(a, b + d, h)$ and the definitions of $\tau^U_n$ and $\tau^V_n$ give the first equality in (2.3). By the $L^2$ assumption on $R_V$ and the definition of $\tau^V_n$, we have $\lim_{n \to \infty} \mathbb{Q}[\tau^V_n = T] = 1$ so that taking limits in (2.3) we get

$$
\mathbb{Q}[V(\cdot, \cdot) \in \Lambda] = \lim_{n \to \infty} \tilde{\mathbb{P}}_n[U(\cdot, \cdot) \in \Lambda, \tau^U_n = T] = \lim_{n \to \infty} \mathbb{E}_\mathbb{P} \left[ 1_{\{U(\cdot, \cdot) \in \Lambda, \tau^U_n = T\}} \mathbb{E}_{T^2}^{R_U, \mathcal{W}}(\mathbb{R}) \right].
$$

(2.4)

Clearly, if $\mathbb{P}[U(\cdot, \cdot) \in \Lambda] = 0$ then $\mathbb{E}_\mathbb{P} \left[ 1_{\{U(\cdot, \cdot) \in \Lambda, \tau^U_n = T\}} \mathbb{E}_{T^2}^{R_U, \mathcal{W}}(\mathbb{R}) \right] = 0$ for each $n$, and so

$$
\mathbb{Q}[V(\cdot, \cdot) \in \Lambda] = \lim_{n \to \infty} \mathbb{E}_\mathbb{P} \left[ 1_{\{U(\cdot, \cdot) \in \Lambda, \tau^U_n = T\}} \mathbb{E}_{T^2}^{R_U, \mathcal{W}}(\mathbb{R}) \right] = 0.
$$

I.e., $L^Y_Q$ is absolutely continuous with respect to $L^U_Q$ (on $\mathcal{B}(\mathbb{R}_T; \mathbb{R})$). A similar argument yields the absolute continuity of $L^U_Q$ with respect to $L^Y_Q$, and we will omit
Our compact support result for the Allen-Cahn SPDE (1.3) can now be proved.

Proof of Theorem 1.2. To see the compact support transfer among (1.1) and (1.2), observe that if \( \mathbb{P}[U(\cdot, \cdot) \in C_c(\mathbb{R}; \mathbb{R})] = 1 \) then by Theorem 1.1 and Corollary 1.2 we have \( \mathbb{Q}[V(\cdot, \cdot) \in C_c(\mathbb{R}; \mathbb{R})] = 1 \), respectively, and vice versa.

If \( \frac{1}{2} \leq \gamma < 1 \) and the integrability condition (1.4) is satisfied by solutions of the Allen-Cahn SPDE (1.3), then by Corollary 1.1 the law of (1.3) is equivalent to that of the associated heat SPDE (without the Allen-Cahn nonlinearity). Now, observe that if \( U \) is a solution to the heat SPDE associated with (1.3); then by \([10, 12]\) we have that, for each \( t \in [0, T] \), \( U(t, \cdot) \) has compact support (in the space variable) almost surely whenever \( h(x) \) has compact support and if \( 0 < \gamma < 1 \). It then follows, as in the proof of the first part of Theorem 1.2 (the compact support transfer among (1.1) and (1.2)), that if \( V \) is a solution to the Allen-Cahn SPDE (1.3); then, for each \( t \in [0, T], V(t, \cdot) \) has compact support (in space) almost surely whenever \( h(x) \) is compactly supported and \( \frac{1}{2} \leq \gamma < 1 \). In the opposite direction, the compact supportedness of \( V(t, \cdot) \) for each \( t \in [0, T] \) trivially implies the integrability in (1.4) for \( \frac{1}{2} \leq \gamma < 1 \).

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Department of Mathematics and Statistics, University of Massachusetts, Amherst, MA 01003-4515
E-mail address: allouba@math.umass.edu