Bias Correction in Deterministic Policy Gradient Using Robust MPC

Arash Bahari Kordabad, Hossein Nejatbakhsh Esfahani, Sebastien Gros

Abstract—In this paper, we discuss the deterministic policy gradient using the Actor-Critic methods based on the linear compatible advantage function approximator, where the input spaces are continuous. When the policy is restricted by hard constraints, the exploration may not be Centred or Isotropic (non-CI). As a result, the policy gradient estimation can be biased. We focus on constrained policies based on Model Predictive Control (MPC) schemes and to address the bias issue, we propose an approximate Robust MPC approach accounting for the exploration. The RMPC-based policy ensures that a Centered and Isotropic (CI) exploration is approximately feasible. A posterior projection is used to ensure its exact feasibility, we formally prove that this approach does not bias the gradient estimation.

I. INTRODUCTION

Reinforcement learning (RL) provides powerful tools for tackling Markov Decision Processes (MDPs) without depending on the probability distribution underlying the state transition [1], [2]. RL methods attempt to enhance the closed-loop performance of a control policy deployed on the MDP, using observed realisation of the state transitions and of the corresponding stage cost. RL methods are usually either direct, based on an approximation of the optimal policy (e.g., deterministic and stochastic policy gradient methods [3]) or indirect, based on an approximation of the action-value function (e.g., Q-learning). Unstructured function approximation techniques (e.g., Deep Neural Networks) are often used to carry these approximations. Unfortunately, the closed-loop behavior of such approximators can be challenging to analyze formally. In contrast, structured function approximations such as Model Predictive Control (MPC) schemes provide a formal framework to analyse the stability and feasibility of the closed-loop system [4]. Recent research have focused on MPC-based policy approximation for RL [5], [6], [7], [8], [9], [10].

For computational reasons, simple models are usually preferred in the MPC scheme. Hence, the MPC model often does not have the structure required to correctly capture the real system dynamics and stochasticity. As a result, while MPC can deliver a reasonable approximation of the optimal policy, it is usually suboptimal [11]. Choosing the MPC model parameters that maximise the closed-loop performance of the MPC scheme is a difficult problem, and the parameters that best fit the MPC model to the real system are not guaranteed to yield the best MPC policy [6]. In [9], [6], it is shown that adjusting not only the MPC model, but also the cost and constraints can be beneficial to achieve the best closed-loop performances, and RL is proposed as a possible approach to perform that adjustment in practice. In the presence of uncertainties and stochasticity, if constraints satisfaction is critical, Robust Model Predictive Control (RMPC) provides tools to ensure that the constraints are satisfied, and can be used in the RL context [12].

Actor-Critic (AC) techniques combine the strong points of actor-only (policy search methods) and critic-only (e.g., Q-learning) methods [13]. AC approaches are based on genuine optimality conditions of the closed-loop policy and typically deliver less noisy policy gradients than direct policy search. The deterministic policy gradient is built based on an approximation of the advantage function associated with the policy. To this end, a linear compatible advantage function approximator is a convenient choice, because it provides a correct policy gradient estimation with a given structure and a low number of parameters [3]. For deterministic policies, exploration is required in order to estimate the corresponding policy gradient. In the presence of hard constraints, this exploration can be restricted. As a result, the exploration may become non-CI. In [14] it is shown that a linear compatible advantage function approximator can deliver an incorrect policy gradient estimation for a non-CI exploration.

In this paper, we propose to use a RMPC scheme that is robust with respect to a bounded disturbance of its first control input to enable the feasibility of a CI exploration. Because RMPC is computationally expensive, we use an inexpensive approximate RMPC instead, feasible to a first-order approximation. To ensure the feasibility of the exploration, a posterior projection technique is used. As a main result of this paper, we formally prove that the exploration resulting from RMPC scheme delivers an unbiased policy gradient estimation.

The paper is structured as follows. Section II provides background material on RL and details the bias problem. Section III presents the RMPC-based approach that tackles the problem. For the sake of simplicity, we will consider a formulation robust with respect to the exploration only, while in practice the formulation can also be robust against model uncertainties and the stochasticity of the real system, as in [12]. Section IV presents the projection approach for nonlinear problems. Section V describes the main theorem in the gradient bias correction using RMPC-based policy and proves that the resulting approach asymptotically yields a correct policy gradient. Section VI provides numerical examples of the method. Section VII delivers a conclusion.
II. BACKGROUND

For a given MDP with continuous state-input space, a deterministic policy parameterized by \( \theta \) delivers an input \( a \in \mathbb{R}^m \) as a function of state \( s \in \mathbb{R}^n \) as, \( \pi_\theta(s) : \mathbb{R}^n \rightarrow \mathbb{R}^m \). If delivered by an MPC scheme, this policy is obtained as:

\[
\pi_\theta(s) = u_0^*(s, \theta),
\]

where \( u_0^* \) is the first element of the solution \( u^* \) given by:

\[
\min_{u, x} V_\theta(x_N) + \sum_{k=0}^{N-1} \gamma^k \ell_\theta(x_k, u_k),
\]

subject to:

\[
x_{k+1} = f_\theta(x_k, u_k), \quad x_0 = s,
\]

\[
ah_\theta(x_k, u_k) \leq 0, \quad h_\theta(x_N) \leq 0,
\]

where \( V_\theta \) and \( \ell_\theta \) are the terminal costs and stage costs, respectively. Function \( f_\theta \) is the model dynamics and \( h_\theta \) are the stage and terminal inequality constraints, respectively. Vector \( x = \{x_0, \ldots, x_N\} \) is the predicted state trajectory and \( u = \{u_0, \ldots, u_{N-1}\} \) is the input profile. State \( s \) is the current state of the system, \( N \) is the horizon length and \( \gamma \in [0, 1] \) is the discount factor. For the following theoretical developments, it will be useful to consider a single-shooting formulation of MPC (2) resulting in a parametric Nonlinear Program (NLP):

\[
\min_u \Phi_\theta(s, u),
\]

subject to:

\[
H_\theta(s, u) \leq 0,
\]

delivering the input profile of (2) for all \( \theta, s \) for some cost \( \Phi_\theta \) and inequality constraints \( H_\theta \). We seek the policy parameters \( \theta \) that minimize the overall closed-loop cost \( J \) of the policy \( \pi_\theta \) defined as follows:

\[
J(\pi_\theta) = \mathbb{E}_{\pi_\theta} \left[ \sum_{k=0}^{\infty} \gamma^k L(s_k, a_k) \mid a_k = \pi_\theta(s_k) \right],
\]

where \( L(s, a) \in \mathbb{R} \) is the baseline stage cost evaluating the policy performance. It is shown in [6] that using an MPC stage cost \( \ell_\theta \) different from the baseline stage cost \( L \) can be beneficial when the MPC model is not exact. The expectation \( \mathbb{E}_{\pi_\theta} \) is taken over the distribution of the Markov chain in closed-loop with the policy \( \pi_\theta \). The policy gradient for the deterministic policy \( \pi_\theta \) is obtained as follows [3]:

\[
\nabla_\theta J(\pi_\theta) = \mathbb{E}_{\pi_\theta} \left[ \nabla_\theta \pi_\theta(s) \nabla_a A_{\pi_\theta}(s, \pi_\theta(s)) \right],
\]

where \( A_{\pi_\theta}(s, a) = Q_{\pi_\theta}(s, a) - V_{\pi_\theta}(s) \) is the advantage function associated to \( \pi_\theta \), and where \( Q_{\pi_\theta} \) and \( V_{\pi_\theta} \) are the action-value and value functions for the policy \( \pi_\theta \), respectively. In a non-episodic context, the expectation \( \mathbb{E}_{\pi_\theta} \) is taken over the steady-state distribution of the Markov chain. In an RL context, the advantage function \( A_{\pi_\theta} \) must be approximated and evaluated from data. In the following, we label the advantage function approximation as \( A_{\pi_\theta}^w \) with parameter vector \( w \). The corresponding estimation of the policy gradient in (5) reads as:

\[
\widehat{\nabla_\theta J(\pi_\theta)} = \mathbb{E}_{\pi_\theta} \left[ \nabla_\theta \pi_\theta(s) \nabla_a A_{\pi_\theta}^w(s, \pi_\theta(s)) \right].
\]

The following theorem provides the condition allowing one to replace the exact advantage \( A_{\pi_\theta} \) in (5) by an approximation \( A_{\pi_\theta}^w \), without affecting the policy gradient.

**Theorem 1.** [3] If \( A_{\pi_\theta}^w \) satisfies

i. \( \nabla_a A_{\pi_\theta}^w = \nabla_\theta \pi_\theta^w \)

ii. \( w \) minimizes the following mean-squared error:

\[
w = \arg \min_w \frac{1}{2} \mathbb{E}_s \left[ \| \nabla_a A_{\pi_\theta} - \nabla_a A_{\pi_\theta}^w \|^2 \right],
\]

where the gradients are evaluated at \( a = \pi_\theta \), then we have:

\[
\nabla_\theta J(\pi_\theta) = \nabla_\theta J(\pi_\theta^w).
\]

**Proof.** See [3].

An advantage function approximator that achieves (3) is labelled compatible. A linear compatible advantage function approximator \( A_{\pi_\theta}^w \), parametrized by \( w \) can read as [3]:

\[
A_{\pi_\theta}^w(s, a) = w^T \nabla_\theta \pi_\theta(a - \pi_\theta).
\]

It is well known that estimating \( \nabla_a A_{\pi_\theta} \) directly is very difficult [3]. As a surrogate to (7), the least-squares problem:

\[
w = \arg \min_w \frac{1}{2} \mathbb{E}_s \left[ (Q_{\pi_\theta} - V_{\pi_\theta} - A_{\pi_\theta}^w)^2 \right],
\]

is used, where the value function estimation \( \hat{V}_{\pi_\theta} \approx V_{\pi_\theta} \) is a baseline supporting the evaluation of \( w \). In order to obtain \( w \) from (10), the input \( a \) applied to the real system must be different from the actual policy \( \pi_\theta \), i.e. the input \( a \) applied to the real system should include some exploration in order to depart from the given policy \( \pi_\theta \). One common choice of exploration is to add a random disturbance \( e \) to the policy as follows:

\[
a = \pi_\theta(s) + e.
\]

For the sake of clarity, we define hereafter a CI exploration.

**Definition 1.** An exploration \( e \) is Centred and Isotropic (CI) if \( \mathbb{E}_e[ee^T] = 0 \), and there exists a scalar \( p \) such that \( \mathbb{E}_e[ee^T] = pI \). Otherwise it is non-CI.

Since the policy \( \pi_\theta \) is subject to the hard constraints (3b), an arbitrary input \( a \) resulting from a random exploration \( e \) may not be feasible. Hence the exploration ought to be restricted such that it respects the constraints. A possible solution for this problem is, e.g., to use a projection of \( a \) on the feasible set of NLP (3). In the following we provide a definition for the projection operator.

**Definition 2.** For an arbitrary input \( a \), the projection operator \( P(s, a) \) is defined as follows:

\[
P(s, a) = u_0^+, \quad (12a)
\]

\[
u^+ = \arg \min_u \frac{1}{2} \| u_0 - a \|^2, \quad (12b)
\]

subject to:

\[
H_\theta(s, u) \leq 0, \quad (12c)
\]

where \( u_0^+ \) is the first element of the input profile \( u_0^+, \ldots, N-1 \) solution of (12b)-(12c).
In particular, at a given state \( s \), the input \( a_\perp \) resulting from projecting the exploration is given by:

\[
a_{\perp} = P(s, \pi_\theta(s) + e).
\] (13)

Then the projected exploration \( e_{\perp} \) is given by:

\[
e_{\perp} = a_{\perp} - \pi_\theta.
\] (14)

Unfortunately, even if the selected exploration \( e \) is CI, the projected exploration \( e_{\perp} \) may not be [14]. It is shown in [14] that the linear compatible function approximator \( \tilde{\theta} \) using the fitting problem \( (10) \) delivers a correct estimated policy gradient \( \tilde{\theta} \) only for a CI exploration.

In this paper, we modify (3) to find a policy \( \hat{\pi}_\theta(s) \) for which a CI exploration is feasible. This policy \( \hat{\pi}_\theta(s) \) is based on creating a small distance from the boundaries of the constraints so that a small CI exploration is feasible. To perform this modification, in the next section we will introduce an approximate RMPC scheme having a computational complexity similar to a standard MPC scheme. This RMPC scheme delivers a policy that can be disturbed with an additive perturbation in a given ball while keeping feasibility to a first-order approximation.

III. RMPC-BASED DETERMINISTIC POLICY

In this section, we propose a modified policy \( \hat{\pi}_\theta \) based on an RMPC-scheme such that any input \( \hat{a} \) resulting from:

\[
\hat{a} = \hat{\pi}_\theta(s) + \hat{e}, \quad \forall \hat{e} \in B(0, \eta),
\] (15)

is feasible for the MPC (2), where \( B(0, \eta) \) is a ball of radius \( \eta \). For the sake of brevity, we consider in the following that the exploration \( \hat{e} \) is uniformly distributed in the ball \( B(0, \eta) \). In that specific case, the exploration \( \hat{e} \) is CI with \( \eta = \frac{1}{2} \hat{e}^2 \).

To generate \( \hat{\pi}_\theta \) we tighten the inequality constraint \( (1b) \) of NLP (3) as follows:

\[
\min_{u^i} \Phi_\theta(s, u),
\] (16a)

s.t. \( H_\theta(s, u) + \Delta_\theta(s, u) \leq 0 \),

\( (16b) \)

where \( \Delta_\theta(s, u) \geq 0 \) is a back-off term added to ensure that the NLP (3) is feasible for any additive perturbation \( \hat{e} \in B(0, \eta) \) of the input \( u_0 \) obtained from (16). In general, evaluating \( \Delta_\theta(s, u) \) is difficult. To address this issue, we propose to compute \( \Delta_\theta(s, u) \) using a first-order approximation of the constraint \( (1b) \). More specifically, we will impose the approximated constraint:

\[
H_\theta(s, \hat{u}) \approx H_\theta(s, u) + \frac{\partial H_\theta}{\partial u_0} \bigg|_u \hat{e} \leq 0,
\] (17)

where \( \hat{u} \) is the input profile resulting from perturbing \( u \) with the exploration \( \hat{e} \in B(0, \eta) \) in the first input \( u_0 \). The following Lemma provides an explicit form for (17).

**Lemma 1.** Inequality (17) holds tightly for all \( \hat{e} \in B(0, \eta) \) if

\[
H_\theta(s, u) + \left\| \frac{\partial H_\theta^i}{\partial u_0} \right\|_u \eta \leq 0
\] (18)

holds, where \( H^i_\theta \) is the \( i \)th element of the vector \( H_\theta \).

Proof. The following inequality

\[
\frac{\partial H_\theta}{\partial u_0} \bigg|_u \hat{e} \leq \left\| \frac{\partial H_\theta}{\partial u_0} \right\|_u \eta, \quad \forall \hat{e} \in B(0, \eta),
\] (19)

holds and is tight, where \( \| \cdot \| \) indicates an Euclidean norm.

The principles detailed above readily apply to MPC scheme (2). More specifically, an input disturbance \( \hat{e} \) yields:

\[
h_{\theta} \approx h_{\theta}(x_k, u_k) + \left( \frac{\partial h_{\theta}}{\partial x_k} + \frac{\partial h_{\theta}}{\partial u_0} \right) \hat{e}, \quad \forall \hat{e} \in B(0, \eta),
\] (20)

where the left hand side is evaluated of the perturbed trajectory and \( \frac{\partial h_{\theta}}{\partial x_k} \) is obtained from the following linear dynamics:

\[
\frac{\partial x_k}{\partial u_0} = \left( \frac{\partial f_{\theta}}{\partial x_k} + \frac{\partial f_{\theta}}{\partial u_0} \right) \bigg|_{x_{k-1}, u_{k-1}},
\] (21)

with the initial condition \( \frac{\partial x_0}{\partial u_0} = 0 \).

Imposing an arbitrary exploration radius \( \eta \) may be infeasible for some state \( s \). To avoid this issue, we consider the radius as a decision variable \( \nu \in [0, \bar{\eta}] \) whose optimal solution is \( \eta \). We label \( \bar{\eta} \) the maximum desired radius for the exploration. The RMPC-based policy \( \hat{\pi}_\theta \) is then obtained as the first element of the input sequence given by:

\[
\min_{u, x, \nu} - w \nu + V_\theta(x_N) + \sum_{k=0}^{N-1} \gamma^k e_{\theta}(x_k, u_k),
\] (22a)

s.t. \( x_{k+1} = f_{\theta}(x_k, u_k), \quad x_0 = s, \) (22b)

\[
\nu_k \leq 0, \quad \forall k \in [0, N-1]
\] (22c)

\[
\hat{h}_{\theta}(x_N) + \left( \frac{\partial \hat{h}_{\theta}}{\partial x_N} \right) \bigg|_{x_N} \nu \leq 0
\] (22d)

\[
\frac{\partial x_k}{\partial u_0} = \frac{\partial f_{\theta}}{\partial u_0}, \quad k = 2, \ldots, N-1
\] (22e)

\[
\nu \leq \nu_k \leq \bar{\eta}
\] (22f)

where \( w \) is a positive constant weight, chosen large enough such that \( \eta = \bar{\eta} \) when feasible. Index \( i \) indicates the \( i \)th element of the vectors.

One can observe that this RMPC scheme is feasible if the original MPC scheme (2) is feasible. Indeed, the choice \( \nu = 0 \) makes the RMPC and MPC schemes equivalent. It follows that the RMPC scheme (22) inherits the recursive feasibility of (2). We ought to stress again here that the recursive feasibility of (2) may require the robust formulation to be extended to take the stochastic disturbances and model errors into account, as e.g. in [12]. We have omitted this aspect here for the sake of brevity and simplicity. The theory presented hereafter is applicable to that extension. Additionally, one ought to note that RMPC (22) is accounting for a disturbance on the initial input only. However, exploration is meant to be applied on all times. This could be reflected in
the RMPC by accounting for a disturbance of the entire input profile, with minor modifications of the formulation. These modifications would, however, unnecessarily reduce the feasible domain of (22). The proposed formulation arguably avoids that issue, and ensures feasibility via introducing the exploration radius as a decision variable in the NLP. Finally, a stabilizing feedback ought to be considered when forming the sensitivities (21), especially when the dynamics (22c) are unstable. This additional feedback is a classic tool to reduce the conservatism of the RMPC schemes. It is not presented here for the sake of brevity.

Since a first-order approximation of the constraints is used when forming (22), its solution may not ensure the feasibility of all exploration $\hat{e} \in B(0, \eta)$. In the next section, we will address this problem with a posterior projection technique. We will show that this projection does not bias the policy gradient estimation.

IV. ENSURING FEASIBILITY

Because we considered a first-order approximation of the constraints when forming the RMPC (22), a posterior projection ought to be used to ensure the feasibility of the exploration. Using (12), we apply the projection of $\hat{a}$ on the feasible set as:

$$\hat{a}_\perp = P(s, \hat{\pi}_\theta + \hat{e}) .$$

(23)

Using (15), let us define the projection correction $\epsilon$ as:

$$\epsilon := \hat{a}_\perp - \hat{a},$$

(24)

and using (14), the feasible projected exploration $\hat{e}_\perp$ can be written as follows:

$$\hat{e}_\perp := \hat{a}_\perp - \hat{\pi}_\theta = \hat{e} + \epsilon .$$

(25)

In the following we will show that the norm of $\epsilon$ is in the order of $\eta^2$ for small enough $\bar{\eta}$. To this end, we make the following mild assumption for the constraints.

Assumption 1. $H_{\theta}$ is a second order differentiable function and we have:

$$\forall i, \left\| \frac{\partial H_{\theta}^i}{\partial u_0} \right\|_{\pi_\sigma} \neq 0 .$$

(26)

Note that if the constraints satisfy Linear Independence Constraint Qualification (LICQ), then (26) is satisfied.

Lemma 2. For the projection error $\epsilon$ defined in (24) and small enough $\bar{\eta}$, there exists a positive $\alpha$ such that:

$$\|\epsilon\| \leq \alpha \eta^2 .$$

(27)

Proof. Let us define $H_{\theta}$ as,

$$H_{\theta}(s, u_0) := H(s, \hat{u}),$$

(28)

where $\hat{u} := \{u_0, u_1, \ldots, u_{N-1}\}$. We define $H_{\theta}^i$ as the $i^{th}$ element of vector $H_{\theta}$. Consider the exploration described by its unitary direction $v$, i.e. $\|v\| = 1$, and magnitude $\zeta \leq \eta$, i.e. $\epsilon = \zeta v$. We observe that:

$$H_{\theta}(s, \hat{\pi}_\theta + \hat{e}) \leq H_{\theta}(s, \hat{\pi}_\theta) + (H_{\theta}')\hat{e} + R(\hat{e}) ,$$

(29)

where $(H_{\theta}') := \frac{\partial H_{\theta}}{\partial u_0} |_{\pi_\sigma}$. The inequality (29) holds for all $\hat{e} \in B(0, \eta)$ for some continuous function $R(\hat{e})$, and there is a constant $c$ such that:

$$|R(\hat{e})| \leq c \|\hat{e}\|^2 .$$

(30)

Additionally:

$$H_{\theta}(s, \hat{\pi}_\theta) + (H_{\theta}')\hat{e} + (1 - \alpha \eta k)\epsilon \leq -\eta \|H_{\theta}''\|_{\pi_\sigma} (1 - \alpha \eta k)\epsilon \leq -\eta \|H_{\theta}''\|_{\pi_\sigma} (1 - \alpha \eta k)\epsilon \leq -\eta \|H_{\theta}''\|_{\pi_\sigma} (1 - \alpha \eta k)\epsilon \leq -\eta \|H_{\theta}''\|_{\pi_\sigma} (1 - \alpha \eta k)\epsilon \leq 0 ,$$

(32)

where the first inequality uses (30) and (31). The second inequality holds for all $\eta \leq \bar{\eta}$.

Additionally:

$$(\hat{H}_{\theta}'(s, \hat{\pi}_\theta) + (\hat{H}_{\theta}'\hat{e}) + R(\hat{e})) \leq -\eta \|H_{\theta}''\|_{\pi_\sigma} (1 - \alpha \eta k)\epsilon \leq -\eta \|H_{\theta}''\|_{\pi_\sigma} (1 - \alpha \eta k)\epsilon \leq -\eta \|H_{\theta}''\|_{\pi_\sigma} (1 - \alpha \eta k)\epsilon \leq 0 .$$

(33)

The following theorem provides some useful properties on the statistics of $\hat{e}_\perp$.

Theorem 2. The projected exploration $\hat{e}_\perp$ defined in (23)-(25), for the policy resulting from RMPC (22), has the following properties:

$$\lim_{\eta \to 0} \mathbb{E}[\hat{e}_\perp] = 0 ,$$

(33a)

$$\lim_{\eta \to 0} \mathbb{E}[\hat{e}_\perp] = \frac{1}{\eta^2} \hat{e}_\perp \hat{e}_\perp^\top = \frac{1}{3} I ,$$

(33b)

$$\lim_{\eta \to 0} \mathbb{E}[\hat{e}_\perp] = \frac{1}{\eta^2} \hat{e}_\perp \hat{e}_\perp^\top = 0 ,$$

(33c)

where $\eta$ is the solution of $\nu$ in the RMPC (22) and $\xi$ is any scalar function satisfying $|\xi(\cdot)| \leq \eta^2$ for some positive $\eta$.

Proof. We have $\lim_{\eta \to 0} \eta = 0$, because $\eta \in [0, \bar{\eta}]$. Using Lemma 2, we have:

$$\lim_{\eta \to 0} \mathbb{E}[\epsilon] = 0 .$$

(34)

Taking the expectation from (25) and using that the exploration $\hat{e}$ is CI, we have:

$$\lim_{\eta \to 0} \mathbb{E}[\hat{e}_\perp] = \lim_{\eta \to 0} (\mathbb{E}[\hat{e}] + \mathbb{E}[\epsilon]) = 0 .$$

(35)
Using \( \text{Assumption 2} \), the second moment can be written as follows:

\[
\lim_{\eta \to 0} E \left[ \frac{1}{\eta^2} \hat{e}_\perp \hat{e}_\perp^\top \right] = \lim_{\eta \to 0} \left( E \left[ \frac{1}{\eta^2} \hat{e} \hat{e}^\top \right] + E \left[ \frac{1}{\eta^2} \hat{e} \hat{e}^\top \right] + E \left[ \frac{1}{\eta^2} \hat{e} \hat{e}^\top \right] \right). \tag{36}
\]

For the first term we use that the exploration \( \hat{e} \) is CI with \( p = \frac{1}{3} \eta^2 I \), i.e.

\[
E[\hat{e} \hat{e}^\top] = \frac{1}{3} \eta^2 I = \frac{1}{3} E[I]. \tag{37}
\]

Using \( \text{Assumption 2} \), for the second term we have:

\[
\lim_{\eta \to 0} \left\| E \left[ \frac{1}{\eta^2} \hat{e} \hat{e}^\top \right] \right\| \leq \lim_{\eta \to 0} E \left[ \frac{1}{\eta^2} \| \hat{e} \| \| \hat{e} \| \right] \leq \lim_{\eta \to 0} \alpha \eta \| \hat{e} \| = 0 \Rightarrow \lim_{\eta \to 0} E \left[ \frac{1}{\eta^2} \hat{e} \hat{e}^\top \right] = 0. \tag{38}
\]

The third term will vanish in the similar way and for the forth term we can write:

\[
\lim_{\eta \to 0} \left\| E \left[ \frac{1}{\eta^2} \hat{e} \hat{e}^\top \right] \right\| \leq \lim_{\eta \to 0} E \left[ \frac{1}{\eta^2} \| \hat{e} \| \| \hat{e} \| \right] \leq \lim_{\eta \to 0} \alpha \eta \| \hat{e} \| \| \hat{e} \| \leq \alpha \eta^2 = 0 \Rightarrow \lim_{\eta \to 0} E \left[ \frac{1}{\eta^2} \hat{e} \hat{e}^\top \right] = 0. \tag{39}
\]

Then, they deliver \( \text{Assumption 2b} \). Finally for \( \text{Assumption 2c} \), we have:

\[
\| \hat{e}_\perp \| = \| \hat{e} + \epsilon \| \leq \| \hat{e} \| + \| \epsilon \| \leq \eta + \alpha \eta^2. \tag{40}
\]

Then:

\[
\lim_{\eta \to 0} \left\| E \left[ \frac{1}{\eta^2} \hat{e}_\perp \xi(\hat{e}_\perp) \right] \right\| \leq \lim_{\eta \to 0} E \left[ \frac{1}{\eta^2} \| \hat{e}_\perp \| \| \xi(\hat{e}_\perp) \| \right] \leq \lim_{\eta \to 0} \eta(1 + \alpha \eta)^3 = 0, \tag{41}
\]

which delivers \( \text{Assumption 2c} \).

V. CORRECTED POLICY GRADIENT

In this section, we will show that the robust policy \( \hat{\pi}_\theta \) delivers the true gradient as \( \eta \to 0 \). Indeed, the deterministic policy gradient method uses “small” exploration and all results are valid in the sense of \( \eta \to 0 \). We propose the compatible advantage function:

\[
A^w_{\hat{\pi}_\theta}(s, \hat{a}_\perp) = \frac{\eta^2}{\eta^2} w^\top \nabla_\theta \hat{\pi}_\theta(\hat{a}_\perp - \hat{\pi}_\theta), \tag{42}
\]

where the factor \( \frac{\eta^2}{\eta^2} \) is required to account for the varying exploration radius \( \eta \) and \( w \) is obtained as follows:

\[
w = \arg \min \sum_{s} \frac{1}{2} E_{\hat{\pi}_\theta, \hat{e}_\perp} \left[ \frac{1}{\eta^2} \left( Q_{\hat{\pi}_\theta} - \hat{V}_{\hat{\pi}_\theta} - A^w_{\hat{\pi}_\theta} \right)^2 \right]. \tag{43}
\]

where \( E_{\hat{\pi}_\theta, \hat{e}_\perp} = E_{\hat{\pi}_\theta}[|E_{\hat{e}_\perp}|, |s|] \) and \( \eta^{-2} \) is introduced such that \( 43 \) remains well posed for \( \eta \to 0 \).

Assumption 2. \( Q_{\hat{\pi}_\theta} \) is analytic and at least twice differentiable for almost every feasible \( s \) and \( \nabla_\theta^2 Q_{\hat{\pi}_\theta} \) is bounded.

Assumption 2 is usually satisfied in practice, as \( Q_{\hat{\pi}_\theta} \) tends to be at least piecewise smooth for the many problems based on continuous state-input spaces. This assumption can be relaxed, but it requires more technical developments.

Theorem 3. The RMPC-based policy gradient estimation using the compatible advantage function in \(42\) with \( w \) given by \(43\) asymptotically converges to exact gradient, i.e.:

\[
\lim_{\eta \to 0} \nabla_\theta J(\hat{\pi}_\theta) = \nabla_\theta J(\hat{\pi}_\theta). \tag{44}
\]

Proof. The solution of \(45\) is given by:

\[
E_{\hat{\pi}_\theta, \hat{e}_\perp} \left[ \frac{1}{\eta^2} \nabla_\theta \hat{\pi}_\theta \hat{e}_\perp \left( Q_{\hat{\pi}_\theta} - \hat{V}_{\hat{\pi}_\theta} - A^w_{\hat{\pi}_\theta} \right) \right] = 0. \tag{45}
\]

Using \( \text{Assumption 2} \), the Taylor expansions of \( Q_{\hat{\pi}_\theta} \) and \( A^w_{\hat{\pi}_\theta} \) are valid almost everywhere. They read as:

\[
Q_{\hat{\pi}_\theta}(s, \hat{a}_\perp) = V_{\hat{\pi}_\theta}(s) + \nabla_\theta A_{\hat{\pi}_\theta}(s, \hat{\pi}_\theta(s)) \hat{a}_\perp + \xi, \tag{46}
\]

where \( \xi \) is the second-order remainder of the Taylor expansion of \( Q_{\hat{\pi}_\theta} \) at \( \hat{a}_\perp = 0 \) and the identity \( \nabla_\theta Q_{\hat{\pi}_\theta} = \nabla_\theta A_{\hat{\pi}_\theta} \) was used. Using \( \text{Assumption 2} \), \( \xi \) is of order \( \| \hat{e}_\perp \| \) for almost every feasible \( s \). By substitution of \(45 \) in \(44 \), we have:

\[
E_{\hat{\pi}_\theta, \hat{e}_\perp} \left[ \frac{1}{\eta^2} \nabla_\theta \hat{\pi}_\theta \hat{e}_\perp \left( \nabla_\theta A_{\hat{\pi}_\theta} - \nabla_\theta A^w_{\hat{\pi}_\theta} \right) \right] + E_{\hat{\pi}_\theta} \left[ \frac{1}{\eta^2} \nabla_\theta \hat{\pi}_\theta \hat{a}_\perp \right] + E_{\hat{\pi}_\theta, \hat{e}_\perp} \left[ \frac{1}{\eta^2} \nabla_\theta \hat{\pi}_\theta \hat{e}_\perp \right] \cdot \left( V_{\hat{\pi}_\theta} - \hat{V}_{\hat{\pi}_\theta} \right) = 0. \tag{47}
\]

Using Theorem 3, the second and third terms will vanish in the sense of \( \eta \to 0 \) and the first term will be:

\[
\lim_{\eta \to 0} E_{\hat{\pi}_\theta} \left[ \nabla_\theta \hat{\pi}_\theta \left( \nabla_\theta A_{\hat{\pi}_\theta} - \nabla_\theta A^w_{\hat{\pi}_\theta} \right) \right] = 0, \tag{48}
\]

which delivers \(44 \).

In addition, under mild conditions, the RMPC-based policy resulting from \(22 \) converges to the main MPC-based policy resulting from \(2 \) as \( \eta \to 0 \). For the sake of brevity, we do not formalise this statement here.

VI. NUMERICAL SIMULATION

In this section, we propose two numerical examples in order to illustrate the theoretical developments. The first example directly compares the MPC-based policy and the RMPC-based policy and the optimal policy with a nonlinear constraint. We consider the deterministic scalar MDP \( s^* = s + a \) with stage cost \( L(s, a) = s^2 + a^2 \), constraint \( s^2 + 5a^2 \leq 1 \) and discount factor \( \gamma = 0.9 \). Then we use the following MPC scheme to extract the approximated policy:

\[
\min_{x, u} x_{k+1}^2 + \sum_{k=0}^{N-1} \gamma^k \left( \theta x_k^2 + u_k^2 \right), \tag{49a}
\]

s.t. \( x_{k+1} = x_k + u_k, x_0^2 + 5u_k^2 \leq 1, x_0 = s \), \( \tag{49b} \)

then \( \pi_\theta(s) = u_0^*(s) \) obtained from the first element of the input solution. We can build RMPC-scheme according \(22 \) and extract the modified policy \( \hat{\pi}_\theta(s) \). Fig. 1(top) illustrates the posterior projected error \( \| \epsilon \| \) and the approximated feasible
radius $\eta$ for $\hat{y} = 0.05$. As it can be seen, e.g., a fixed radius exploration with $\eta = 0.05$ may be infeasible at $s = \pm 1$. Fig. 1 (bottom) compares these policies with the MPD optimal policy. This simple example shows that the RMPC-based policy makes a distance with the feasible bound to guarantee the feasibility of the exploration in both directions. While classic-MPC is on the feasible set bound, a feasible exploration should only be in one direction.

The second example compares the gradient of the RMPC-based and MPC-based policies with the true policy gradient. We consider linear scalar dynamics $s^+ = 0.97s + 0.1a + d$ where $d \sim \mathcal{U}(-10^{-5}, 10^3)$ is a scalar uniform noise. RL stage cost is $L(s,a) = 20(s-0.5)^2 + (a-2)^2$ with $\gamma = 0.9$. The policy is extracted from the following MPC scheme:

$$\min_u \sum_{k=0}^{50} \gamma^k \left( 10(x_k - 1/3)^2 + (u_k - u_{ref}(\theta))^2 \right),$$

subject to

$$x_{k+1} = 0.97x_k + 0.1u_k, \quad x_0 = s,$$

$$u_k \leq \theta,$$

where $u_{ref}(\theta) = 0.2 - \theta$. The initial RL parameter $\theta = 0.1$ is selected. The MPC policy can be adjusted by increasing the input bound in (50c) and raising the input reference $u_{ref}$. However, raising the input bound in (50c) by increasing $\theta$ results in decreasing the input reference $u_{ref}$, such that these terms are in contradiction to find the optimal policy. Fig. 1 shows the policy gradient over the RL iterations. The red (dashed) curve is the outcome of learning from the classic MPC, while the blue (solid) curve is the one from the RMPC. As it can be seen, the RMPC gradient $\nabla_\theta \hat{J}(\pi_\theta)$ delivers a very close gradient to the true gradient $\nabla_\theta J(\pi_\theta)$. However, the MPC policy gradient $\nabla_\theta J(\pi_\theta)$ has an obvious bias in both cases. Note that the closed-loop performance loss from this bias issue is not necessarily large for this example.

A more complex example demonstrating the theory on a nonlinear example would be useful. For the sake of brevity, such an example will be considered in the future.

VII. CONCLUSION

This paper presented the AC approach using a linear compatible advantage function approximation for the MPC-based deterministic policies. When the policy is restricted by hard constraints, the exploration may be non-CL and delivers a bias in the policy gradient. We proposed RMPC using constraint tightening to provide an approximated feasible and CI exploration. A posterior projection is used to ensure feasibility and formally we showed that the RMPC-based policy gradient converges to the true policy gradient for a small enough radius of exploration.

REFERENCES

[1] D. P. Bertsekas, Reinforcement learning and optimal control. Athena Scientific Belmont, MA, 2019.
[2] R. S. Sutton and A. G. Barto, Reinforcement learning: An introduction. MIT press, 2018.
[3] D. Silver, G. Lever, N. Heess, T. Degris, D. Wierstra, and M. Riedmiller, “Deterministic policy gradient algorithms,” in Proceedings of the 31st International Conference on International Conference on Machine Learning - Volume 32, ser. ICML’14. JMLR.org, 2014, p. 1–3.
[4] K. P. Wabersich and M. N. Zeilinger, “Safe exploration of nonlinear dynamical systems: A predictive safety filter for reinforcement learning,” arXiv preprint arXiv:1812.05506, 2018.
[5] T. Koller, F. Berkenkamp, M. Turchetta, and A. Krause, “Learning-based model predictive control for safe exploration,” in 2018 IEEE Conference on Decision and Control (CDC), 2018, pp. 6059–6066.
[6] S. Gros and M. Zanon, “Data-driven economic mmpc using reinforcement learning,” IEEE Transactions on Automatic Control, vol. 65, no. 2, pp. 636–648, 2019.
[7] M. Zanon, V. Kungurtsev, and S. Gros, “Reinforcement learning based on real-time iteration NMPC,” arXiv preprint arXiv:2005.05225, 2020.
[8] A. Bahari Kordabad, H. Nejatbakhsh Esfahani, A. M. Lekkas, and S. Gros, “Reinforcement learning based on scenario-tree MPC for ASVs,” arXiv e-prints, pp. arXiv–2103, 2021.
[9] S. Gros and M. Zanon, “Reinforcement learning for mixed-integer problems based on MPC,” arXiv preprint arXiv:2004.01430, 2020.
[10] H. Nejatbakhsh Esfahani, A. Bahari Kordabad, and S. Gros, “Reinforcement learning based on MPC/MHE for unmodeled and partially observable dynamics,” arXiv e-prints, pp. arXiv–2103, 2021.
[11] J. B. Rawlings, D. Q. Mayne, and M. Diehl, Model predictive control: theory, computation, and design. Nob Hill Publishing Madison, WI, 2017, vol. 2.
[12] M. Zanon and S. Gros, “Safe reinforcement learning using robust MPC,” IEEE Transactions on Automatic Control, 2020.
[13] V. R. Konda and J. N. Tsitsiklis, “Actor-critic algorithms,” in Advances in neural information processing systems, 2000, pp. 1008–1014.
[14] S. Gros and M. Zanon, “Bias correction in reinforcement learning via the deterministic policy gradient method for MPC-based policies,” in 2021 American Control Conference (ACC) [Submitted], 2021.