The theory of deformed plates in mechanical equilibrium is formulated and properties of circular plates lifted at the center are discussed.

I. Introduction

The interest in the mechanics of plates dates back to 1809 when the French Academy of Sciences donated an award for the best work on oscillating plates (Timoshenko 1952). The donation was initiated by Napoleon Bonaparte who was impressed by the demonstration of Chladni patterns at one of the Academy meetings. Three years later the only contestant, Sophie Germain, submitted a treatise where she proposed a variation principle corresponding to a distinctly nonphysical value of Poisson’s ratio \( \nu = 1 \). The resulting differential equation, when rederived by Lagrange, was correct, since the correct equation does not contain Poisson’s ratio. Nevertheless the physical basis of the underlying variation principle remained unclear, which was hardly surprising since the theory of elasticity did not exist yet. First step towards its formulation was Poisson’s memoir on the theory of plates published in 1814, shortly after the second, revised version of Germain’s work appeared. Although two years later Sophie Germain was finally, at third attempt, given the award, the theory of elastic plates in its present form was formulated by Kirchhoff not earlier than in 1850. The extension to plastic plates appeared still later, after the concept of plasticity emerged in the experiments by Tresca.

The theory of elastic plates is nowadays a standard part of textbooks on the mechanics of deformable bodies; see, for example, Landau and Lifshitz (1965). The theory of plasticity is explained in classical books by Hill (1950) and Kachanov (1956), and mathematics of the deformation theory of plasticity is described in detail in the monograph by Temam (1983). Properties of plastic plates are discussed in Kachanov’s book in the chapter entitled Plane Stress. Plates are considered in the framework of the theory of plastic flow, but most results can be carried over into the deformation theory simply by replacing velocities by deflections. Other results on plastic plates can be found in papers

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published in engineering journals, and still other results can be obtained by applying dimensional reduction to what is known from the theory of extended plastic bodies.

In this work all pieces are put together and a concise and self-contained deformation theory of plastic plates is developed. Throughout the work, the theory of elastic plates is used as a model. In section II the concepts of elasticity and plasticity are introduced, in section III constitutive equations are constructed, in section IV differential equations for plates in mechanical equilibrium are formulated and boundary conditions for them are discussed, in section V deformation energy is computed, in section VI variational principle for deformed plates is investigated, in section VII two mixed kind of deformations are discussed and in section VIII some simple examples are presented.

II. Elasticity and plasticity

Elastic deformation is a reversible process: the body returns to its initial state after the deforming forces have been lifted. Moreover, the deformation is proportional to the applied load. This is just a demonstration of the general law stating that the response is linear in its cause in case the cause is small. Linear relation between the stress inside the body to which an external load is applied and the resulting deformation of the body is called generalized Hooke’s law. Elastic properties of isotropic materials are characterized by two parameters only, Young’s modulus of elasticity $E$ (with the physical dimension of pressure) and Poisson’s ratio $\nu$ (dimensionless). Consider an homogenous and isotropic beam fastened at one end and subject to the action of an aligned load at the other end, and denote relative dilation of the beam in the direction of its axis by $\epsilon$, relative dilation of the beam in the lateral direction by $\epsilon'$, and pressure load (force per unit area) by $\tau$. In the elastic regime it holds

$$\epsilon = \frac{1}{E} \tau,$$

(1)

and

$$\epsilon' = -\frac{\nu}{E} \tau.$$

(2)

The first equation is the original Hooke’s law, the famous *ut tensio, sic vis* (as the extension, so the force) stated by Robert Hooke in 1660. The generalized Hooke’s law follows from equations (1) and (2), if one applies them to an infinitesimal volume of the body.

The values of Poisson’s ratio are restricted by the requirement that the deformation energy per unit volume is positively definite. This requirement implies that Poisson’s ratio must be from the interval $\langle -1, 1/2 \rangle$. If it is 0, the lateral section of the beam does not change when the beam is
stretched; if it is 1/2, the volume of the beam remains constant; and if it is from the interval (0, 1/2), the beam contracts in the lateral direction and its volume increases. For known materials, Poisson’s ratio assumes values typically from 0.3 to 0.4. It equals, for instance, 0.30 for steel, 0.34 for copper and 0.42 for gold.

After the pressure load applied to the beam reaches a certain critical value, called yield limit, the beam becomes plastic. In this regime, the deformation is irreversible; after the external force is removed, some residual dilation of the beam remains. The linear character of the deformation is lost, too. For a large class of materials, stretching of the beam proceeds in two clearly distinguished stages. First the beam dilates while the force remains more or less unchanged; then the beam continues to dilate only if the force starts to grow again. This two stages are called yielding and hardening. (The latter term refers to the fact that if we lift the force and then apply it again, plasticity takes over only after the force reaches the value from which it previously jumped to zero. Consequently, a larger force is needed to start the plastic deformation than for the first time.) In the course of yielding, Hooke’s law in the form (1) is replaced by an even simpler law

\[ \tau = \mathcal{E}, \]

where \( \mathcal{E} \) is the yield limit. The law is true for \( \epsilon_1 < \epsilon < \epsilon_2 \), where \( \epsilon_1 \) and \( \epsilon_2 \) are the relative dilations at which yielding and hardening take over respectively; particularly, \( \epsilon_1 = \mathcal{E}/E \). One introduces two limit kinds of plasticity: rigid plasticity, in which the beam does not deform at all until the stress reaches the yield limit, and perfect plasticity, in which no hardening takes place so that the beam stretches to infinity at the yield limit. Rigid plasticity means \( E = \infty, \epsilon_1 = 0 \), while perfect plasticity means \( \epsilon_2 = \infty \). In fig. 1, the behavior of a deformable beam is represented schematically by the solid line, with ‘E’ standing for elastic deformation, ‘Y’ standing for yielding, and ‘H’ standing for hardening. The rigid-plastic and perfect plastic limits are depicted by the dotted-line segments denoted ‘RP’ and ‘PP’ respectively.

A realistic description of plasticity is provided by the incremental theory, or the theory of plastic flow, in which the stress inside the body is related to the increment of the plastic deformation rather that to the plastic deformation itself. However, if we are interested in the initial stage of plastic deformation only, we can obtain satisfactory results also from the deformation theory in which the stress is related to the plastic part of the deformation in a similar way as to the elastic part. In other words, plasticity is regarded as a kind of a nonlinear elasticity. Particularly, one assigns a deformation
energy to a body, which means that one resigns on the description of the relaxation of the body to the new equilibrium state after the load has been lifted. The incremental and the deformation theory coincide if the body is *simply loaded*, which means if it undergoes, for instance, a dilation or a shear, but not a dilation followed by a shear. A close connection between the two theories exists even if the body is arbitrarily loaded, provided it is rigid-plastic. The deformation in the former theory is then obtained by integrating the deformation in the latter theory over time.

The simplest theory of plasticity is the deformation theory of perfectly plastic rigid-plastic bodies. The theory can be viewed as the rigid limit of the *Hencky theory* (the deformation theory of perfect elastic plasticity), or as the deformation variant of the *Saint-Venant–Levi–von Mises theory* (the incremental theory of perfect rigid plasticity). Both theories are discussed in detail in Kachanov (1956). The deformation theory of perfect rigid plasticity is, despite all its simplifications, extensively used in engineering. It is the basis of *limit analysis*, evaluation of the loads at which the plasticity takes over and the corresponding deformations of the body. In the limit analysis, the assumptions about perfect plasticity and validity of the deformation theory do not play any role, and the only remaining simplification, the assumption about rigid plasticity, provides us with results that can be interpreted as limit results for real bodies.

**III. Constitutive equation**

The full system of equations determining the form of a deformable body in equilibrium consists of *constitutive equation*, *equation of balance of forces*, and boundary conditions for the latter equation. If the body is perfectly plastic, constitutive equation must be supplemented by *constitutive inequality*. 
Neither constitutive equation nor equation of balance of forces are formulated for the plate directly. One postulates them for the extended body and then rewrites them into the form valid for the plate. This procedure can be called *dimensional reduction* since one effectively passes from three to two dimensions. When doing so one assumes that the plate is thin (its thickness is much smaller than the characteristic scale of deformation), and that the bending of the plate is small (the deflection is much smaller than the thickness of the plate; see Landau and Lifshitz 1965). In the literature sometimes a different approach is used, based on the original work by Kirchhoff (see Timoshenko and Woinowsky-Krieger 1959). One starts from *Kirchhoff hypotheses* stating, first, that the sections that were orthogonal to the mid-plane of the plate at the beginning remain orthogonal to it after the deforming forces have been applied, and second, that the mid-plane is neutral, that means it is bent but neither stretched nor compressed. Both hypotheses are, however, immediate consequences of the assumptions cited above.

Consider a plate with a constant thickness $h$ whose mid-plane is placed in the plane $(x, y)$ before the deformation, and denote the deflection of the mid-plane of the deformed plate from the plane $(x, y)$ by $w$. The stress inside the plate and the resulting deformation are described by two symmetric $2 \times 2$ matrices, *tensor of moments* $M$ and *Hessian matrix* $W$,

$$
M = \begin{pmatrix}
M_{xx} & M_{xy} \\
M_{yx} & M_{yy}
\end{pmatrix},
W = \begin{pmatrix}
W_{xx} & W_{xy} \\
W_{yx} & W_{yy}
\end{pmatrix},
$$

The components of the Hessian matrix are the second partial derivatives of the function $w(x, y)$,

$$
W = \begin{pmatrix}
\frac{\partial^2 w}{\partial x^2} & \frac{\partial^2 w}{\partial x \partial y} \\
\frac{\partial^2 w}{\partial x \partial y} & \frac{\partial^2 w}{\partial y^2}
\end{pmatrix}.
$$

The matrix is symmetric and its trace equals Laplacian acting on $w$,

$$
\text{tr} W = \Delta w.
$$

The components of the tensor of moments are moments per unit width induced by the stresses acting parallel to the plate. The moments $M_{xx}$ and $M_{yy}$ bend the plate in the direction of the axes $x$ and $y$, respectively, and the moments $M_{xy}$ and $M_{yx}$ twist the plate in the planes $(x, z)$ and $(y, z)$ respectively. The twisting moments are identical, so that the tensor of moments is symmetric just like the Hessian matrix.
Constitutive equation determines the stress arising inside the body in terms of deformation. For an elastic body, the equation reduces to the generalized Hooke’s law. Constitutive equation of a plate is a matrix equation relating the tensor of moments to the Hessian matrix. It is obtained from the constitutive equation of an extended body via dimensional reduction, by which 3 × 3 matrices are replaced by 2 × 2 ones. If the deformation is elastic and the material of the plate is isotropic, the constitutive equation reads

\[ M = -D[(1 - \nu)W + \nu \text{tr}W I], \]

where \(D = Eh^3/[12(1 - \nu^2)]\). The constant \(D\) is called flexural rigidity. We can see that the constitutive equation of the elastic isotropic plate is the most general relation between two symmetric matrices that is both linear and isotropic. Since it is obtained by dimensional reduction of the generalized Hooke’s law, it can be called two-dimensional Hooke’s law.

The starting point for the formulation of the constitutive equation of the plastic body is the yield criterion. It is a constraint on the stress inside the body defining the state of yielding or, if the body is perfectly plastic, the plastic state itself. The first yield criterion was proposed in 1868 by Tresca, a French engineer who pioneered the research of plasticity. Tresca’s criterion was later modified by von Mises. The purpose was to simplify the analysis; however, the new criterion happened to do even better than the original one when confronted with experimental data. The yield criterion can be represented by a surface in the space of main stresses called yield diagram. For von Mises’ criterion, the yield diagram is an infinite cylinder with the radius of the base \(\sqrt{2/3E}\), whose axis passes through the origin and is deflected from all three coordinate axes by the angle \(\pi/4\). Von Mises’ criterion can be slightly generalized so that an improved description of some materials, as marble and sandstone, is achieved. The generalization was proposed in Yang (1980a); more specifically, it was the first of the two generalizations proposed there, describing the effect of hydrostatic stress on yielding. In the generalized von Mises’ criterion materials are characterized, in addition to the yield limit \(E\), by the dimensionless parameter \(\kappa\) assuming values from the interval \((0, 1/2)\). (In the notation of Yang, \(\kappa = \alpha(2 - \alpha)/(1 - \alpha)^2\).) For \(\kappa = 0\) the yield diagram is sphere, for \(\kappa\) increasing from 0 to 1/2 it is a gradually stretching rotational ellipsoid whose axis is deflected by the angle \(\pi/4\) from all coordinate axes, and for \(\kappa = 1/2\) it is von Mises’ cylinder. Referring to these shapes we can call the generalized von Mises’ criterion with an arbitrary value of \(\kappa\) ellipsoid criterion, and the criteria with \(\kappa = 0\) and \(\kappa = 1/2\) spherical criterion and cylindric criterion respectively.

If we pass from an extended body to a plate, the yield criterion undergoes dimensional reduction
just like Hooke’s law. As a result, a constraint on the tensor of moments arises. Particularly, the
generalized von Mises’ criterion reduces to

\[(1 + \kappa) \text{tr} M^2 - \kappa (\text{tr} M)^2 = M_0^2.\]  

(4)

where \(M_0 = \mathcal{E} h^2 / 4\). Expressed in terms of main moments (eigenvalues of the matrix \(M\)), this equation reads

\[M_1^2 + M_2^2 - 2\kappa M_1 M_2 = M_0^2.\]  

(5)

The yield diagram is now a planar curve that can be obtained as an intersection of the three-
dimensional yield diagram with the horizontal plane. For \(\kappa = 0\) the yield diagram is circle, while
for \(\kappa\) increasing from 0 to 1/2 inclusive it is a gradually stretching ellipse. In fig. 2, three diagrams
of this class are shown, drawn by the solid line. The form of the diagrams suggests that we call the

![Fig. 2: Yield diagrams](image)

yield criterion in question *elliptical criterion*; or, if \(\kappa = 0\), *circular criterion*. One can formulate other
theories of plasticity starting from other yield criteria. By doing so one does not need to care about
the three-dimensional theory; instead, one can postulate yield criteria that are two-dimensional from
the beginning. Let us mention an alternative to the elliptical criterion obtained in this way, considered
appropriate for concrete plates. It is the *square criterion* proposed by Johanson (see Mansfield 1957),

\[\max (|M_1|, |M_2|) = M_0.\]  

(6)

In fig. 2, this criterion is depicted by the dotted line.
If one inserts stresses at the given point of the body into the yield criterion, one can decide what kind of deformation occurs there. For a perfectly plastic body in equilibrium, only two regimes of deformation are possible. If the stresses fit inside the yield diagram, the deformation is either elastic or there is no deformation at all depending on whether the body is elastoplastic or rigid-plastic; if, however, the stresses are placed on the surface of the yield diagram, the deformation possibly contains a plastic part. With the stresses outside the yield diagram there exists no equilibrium. We are talking of an extended body, but all we have said can be applied to a plate, too, if the word 'stresses' is replaced by the word 'moments'. Define the norm of the tensor of moments \( \|M\| \) as the left hand side of the two-dimensional yield criterion, if written with \( M_0 \) on the right hand side. Then the perfectly plastic plate can be in equilibrium only if the tensor of moments satisfies the constraint called \textit{constitutive inequality},

\[
\|M\| \leq M_0. \tag{7}\]

The plate is deformed elastically or remains flat if \( \|M\| < M_0 \), and possibly undergoes plastic deformation if \( \|M\| = M_0 \).

Let us proceed to the constitutive equation for the plastic plate. Once again, the equation is obtained by dimensional reduction. We will restrict ourselves to the perfectly plastic rigid-plastic plate obeying the deformation theory. For such plate, the Hessian matrix in the regime of plastic deformation is proportional to minus gradient of the norm \( \|M\| \),

\[
W = -C \frac{\partial \|M\|}{\partial M}, \quad C > 0. \tag{8}\]

In components, the equation reads

\[
W_{xx} = -C \frac{\partial \|M\|}{\partial M_{xx}}, \quad W_{xy} = -C \frac{\partial \|M\|}{\partial M_{xy}}, \quad W_{yy} = -C \frac{\partial \|M\|}{\partial M_{yy}}.
\]

For the elliptical criterion we obtain

\[
W = -C_1 [(1 + \kappa)M - \kappa \text{tr} \, M I], \tag{9}\]

where \( C_1 = C/\|M\| = C/M_0 \). Next we solve for \( M \) and fix the constant of proportionality by the yield criterion. As a result we find

\[
M = -D \frac{(1 - \kappa)W + \kappa \text{tr} \, W I}{\sqrt{(1 - \kappa) \text{tr} \, W^2 + \kappa (\text{tr} \, W)^2}}, \tag{10}\]

8
where $D = M_0/\sqrt{1 - \kappa^2}$. This is the constitutive equation we have sought. After comparing it with the two-dimensional Hooke’s law we conclude that $D$ can be regarded as ‘plastic flexural rigidity’ and $\kappa$ as ‘plastic Poisson’s ratio’.

To complete the theory we must specify the behavior of the plate if the stress is not large enough to produce plastic deformation. A deformed perfectly plastic rigid-plastic plate consists, in general, of two parts that are to be treated separately: the plastic domain, where $W \neq 0$ and $M$ is given by the constitutive equation, which means that $\|M\| = M_0$, and the rigid domain, where $W = 0$ and $\|M\| \leq M_0$. Obviously, in the rigid domain the plate is flat; or at least, as we will see, piecewise flat.

IV. Equation of balance of forces

In equilibrium, the forces acting on a column of matter reaching from one face of the plate to the other must compensate each other. Obviously, this requirement is the same for elastic and plastic plates. Suppose the plate is bent by the lateral pressure load $q$. Equation of balance of forces then reads

$$\nabla \nabla \cdot M = -q,$$

where the expression on the left hand side is the sum of the second derivatives of $M$,

$$\nabla \nabla \cdot M = \frac{\partial^2 M_{xx}}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yy}}{\partial y^2}.$$

The double gradient is a symmetric $2 \times 2$ matrix just like the tensor of moments. In general, we can define the scalar product of matrices as the trace of their matrix product. Thus, for symmetric matrices we have

$$A \cdot B = \text{tr}(AB) = A_{xx}B_{xx} + 2A_{xy}B_{xy} + A_{yy}B_{yy},$$

which immediately yields the expression for $\nabla \nabla \cdot M$ above.

By combining the equation of balance of forces with the constitutive equation, we arrive at the differential equation for deflection only. Consider first an elastic plate. If the material of the plate is not only isotropic but also homogeneous, $D$ and $\nu = \text{const}$, we find

$$\Delta \text{tr} W \equiv \Delta \Delta w = \frac{q}{D},$$

In such a way, the deflection of a homogenous and isotropic elastic plate in equilibrium obeys an inhomogeneous biharmonic equation, called Lagrange equation, with the source proportional to the
pressure load. Note that, as mentioned in the introduction, the equation does not contain Poisson’s ratio.

Since the deformation of the plate is governed by a differential equation of fourth order, we have to impose two boundary conditions on it. Let us introduce three basic sets of boundary conditions, corresponding to clamped, simply supported and free plate. Denote the edge of the mid-plane of the relaxed plate (a closed curve in the \((x, y)\) plane) by \(C\). The clamped plate is fixed steadily at the edge, hence it satisfies the conditions

\[
w = 0 \text{ on } C,
\]

and

\[
\frac{\partial w}{\partial n} = 0 \text{ on } C,
\]

where \(\partial/\partial n\) is the derivative in the direction normal to the curve. The remaining two sets of boundary conditions contain also components of the matrices \(M\) and \(W\). Denote the unit vectors normal and tangential to the curve by \(\mathbf{n}\) and \(\mathbf{t}\) respectively, and define normal, mixed and tangential components of a symmetric matrix as

\[
A_{nn} = A \cdot \mathbf{n} \mathbf{n} = A_{xx}n_x^2 + 2A_{xy}n_xn_y + A_{yy}n_y^2,
\]

\[
A_{nt} = A \cdot \mathbf{n} \mathbf{t} = A_{xx}n_x t_x + A_{xy}(n_x t_y + n_y t_x) + A_{yy}n_y t_y\]

and

\[
A_{tt} = A \cdot \mathbf{t} \mathbf{t} = A_{xx} t_x^2 + 2A_{xy} t_x t_y + A_{yy} t_y^2.
\]

The simply supported plate is fixed by a bar that allows it to rotate freely, and satisfies the condition (3) together with the condition

\[
M_{nn} = 0 \text{ on } C.
\]

The constraint on the matrix \(M\) follows from the very concept of mechanical equilibrium: the moment of the bar acting on the plate must be zero, otherwise the bar would make the plate rotate around the groove. Rewritten in terms of the matrix \(W\), the constraint reads

\[
W_{nn} + \nu W_{tt} = 0 \text{ on } C.
\]

The free plate is not fixed at all, so that the bending moment \(M_{nn}\) as well as some effective shearing force \(Q_n\) at its edge must be zero. As a result, the free plate shares the condition (5) with the simply supported plate, and in addition it obeys the condition

\[
Q_n \equiv \mathbf{n} \nabla \cdot M - \frac{\partial M_{nt}}{\partial t} = 0 \text{ on } C,
\]

where the first term on the left hand side is the linear combination of the first derivatives of \(M\), with the coefficients equal to the components of the vector \(\mathbf{n}\),

\[
\mathbf{n} \nabla \cdot M = n_x \frac{\partial M_{xx}}{\partial x} + n_x \frac{\partial M_{xy}}{\partial y} + n_y \frac{\partial M_{xy}}{\partial x} + n_y \frac{\partial M_{yy}}{\partial y},
\]
and $\partial/\partial t$ is the gradient in the direction tangential to the curve $C$. (The scalar product $n \nabla \cdot M$ is a sum of four terms instead of three, because the matrix $n \nabla$ is not symmetric.) Passing to the Hessian matrix, we find

$$\frac{\partial \text{tr} W}{\partial n} - (1 - \nu) \frac{\partial W_{nt}}{\partial t} = 0 \text{ on } C.$$  \hspace{1cm} (8)

In conditions (6) and (8) Poisson’s ratio, after all, enters the theory.

Let us summarize. An elastic plate in equilibrium satisfies the differential equation (2) and two boundary conditions: conditions (3) and (4), if it is clamped, conditions (3) and (6), if it is simply supported, and conditions (6) and (8), if it is free. From the theory of partial differential equations it follows that the function solving this problem exists and is unique.

Now we wish to formulate conditions of equilibrium of the plastic plate. Consider a perfectly plastic rigid-plastic plate whose behavior is described by the deformation theory, and suppose it obeys the elliptical criterion. Suppose furthermore that the part of the plate we are interested in lays in the plastic domain. By inserting from the constitutive equation into the equation of balance of forces we obtain

$$\nabla \nabla \cdot W = \frac{q}{D},$$ \hspace{1cm} (9)

where

$$W = \frac{(1 - \kappa)W + \kappa \text{tr} WI}{\sqrt{(1 - \kappa)\text{tr} W^2 + \kappa (\text{tr} W)^2}}.$$  \hspace{1cm} (10)

In such a way, the deflection in the plastic domain is given by an equation of fourth order again, but unlike in the elastic case this equation is nonlinear. Boundary conditions remain the same as in the elastic case, provided we express the 'moment' and the 'force' conditions in terms of the tensor of moments. After we pass from the tensor of moments to the Hessian matrix, the 'moment' condition reduces to (6) with $\nu$ replaced by $\kappa$, but the 'force' condition is more complicated than (8) due to the presence of the square root in the constitutive equation.

In the limit analysis, the deformation of a rigid-plastic plate is called collapse. If we adopt this terminology, we can say that the differential equation we have just obtained is valid everywhere only if total collapse takes place. In case of partial collapse we must formulate the conditions of equilibrium in the rigid domain, too. In this domain, the equation for deflection is just $W = 0$, however the deflection is constrained indirectly by the conditions imposed on the tensor of moments. These include the equation of balance of forces, the constitutive inequality, and the 'moment' and/or the 'force' condition at the edge of the plate, provided the edge is rigid, or partly rigid, and the plate
is simply supported or free. The tensor of moments in the rigid domain is independent on the Hessian matrix locally, but it is correlated with it globally, due to the requirement that it matches smoothly enough the tensor of moments in the plastic domain. As a result, the conditions on the tensor of moments listed above determine the size and shape of the rigid domain, as well as the shape of the plate in this domain.

The fact that the matrix $\mathcal{W}$ is a homogenous function of the Hessian matrix of degree zero, and not of degree one as in the elastic case, forces us to extend the class of possible deformations. Consider a deformation by which the plate has a corner along some line $\hat{C}$, so that the first derivative of $w$ in the direction normal to $\hat{C}$ takes a finite jump at $\hat{C}$, and the second derivative has a $\delta$-function type singularity. In the elastic case, such deformations are of course forbidden because the corner would produce a term proportional to the double gradient of the $\delta$-function in the equation governing the deflection. Now, however, we have an equation that is not singular at the corner. Formally, we can see this if we take the square root of the expression proportional to the $[\delta$-function]$^2$ in the denominator of the matrix $F$, and cancel the $\delta$-functions in the numerator and the denominator. As a result, functions with jumps in the first derivatives must be regarded as potential solutions to the equation in question. The functional space containing such functions is called space of functions with bounded Hessian. In the rigid domain the plate can have corners like anywhere else, so that the connected rigid domain is either flat or composed of several flat pieces sewed together. Particularly, if the relaxed plate is a polygon, the deformed plate can be, in the sense it is understood here, entirely rigid. The borderline between a rigid and a plastic domain can be smooth (the first derivatives of $w$ can be continuous there), but can be corner-like as well.

If the plate contains corners, the tensor of moments must satisfy two additional boundary conditions. It must hold

$$M = \pm \mathcal{D}[(1 - \kappa)nn + \kappa I] \text{ on } \hat{C}, \quad (11)$$

where the plus and the minus sign correspond to a 'ridge' and a 'canyon' respectively, and

$$\left( \frac{\partial M_{nn}}{\partial n} \right)_1 = \left( \frac{\partial M_{nn}}{\partial n} \right)_2 \text{ on } \hat{C}, \quad (12)$$

where the indices '1' and '2' refer to the limits from the two sides of the corner. The conditions follow from the requirement that moments and shear forces are balanced along the corner. Note that if the plate is clamped, the corner can appear at its edge, or a part of its edge, and if this is the case, condition (4) must be replaced by condition (11). In the plastic domain, we must insert for $M$
from the constitutive equation; condition (11) then forbids jumps in the second derivatives of \( w \), while condition (12) restricts jumps in the third derivatives of \( w \). The limit matrix in the former condition is the matrix we arrive at by the formal procedure described above. Note that we actually obtain a condition that is apparently much weaker than (11), namely

\[
M_{nn} = \pm \mathcal{D} \text{ on } \hat{C}.
\]  

However, if we take into account that \( M \) obeys the yield criterion in the plastic domain and the constitutive inequality in the rigid domain, we can prove that this condition is equivalent to (11).

A closer look at the conditions of equilibrium we have established reveals an ambiguity that has no analogue in the elastic case. If some function is the solution to all conditions, the same function multiplied by an arbitrary positive constant is the solution, too. This ambiguity is, however, harmless from the physical point of view. We obtain a unique solution if we take into account the hardening. (Formally, there will be still infinitely many solutions, but only the one corresponding to the maximal deformation in the yielding regime will become reality.) Since all solutions differing by a multiplicative factor are physically equivalent, we can impose an arbitrary normalization condition on the deflection.

From the physical considerations it follows that the theory has another unusual property that is in a sense dual to the scaling property discussed above. While for some loads there exist infinitely many solutions, for other loads there exists no solution at all. Consider a load of the form \( q = \lambda Q \), where \( Q \) is a function of coordinates characterizing the distribution of the load, and \( \lambda \) is a nonnegative dimensionless parameter characterizing the size of the load. For a given \( Q \), if \( \lambda \) is too small, the stresses are not able to deform the plate, while if \( \lambda \) is too large, the stresses cannot saturate the constitutive inequality. In such a way, a nontrivial state of equilibrium (such that at least some part of the plate becomes deformed) occurs only for some limit size, or sizes, of the load \( \lambda_{\text{lim}} \). Later we will show that for any distribution of the load there exists just one \( \lambda_{\text{lim}} \).

V. Deformation energy

If we wish to formulate the mechanics of a deformable body via the variation principle, we need an expression for the deformation energy (the work done by the stress in the course of deformation). For the plate, the deformation energy per unit area is

\[
\chi = -\int_0^W M(\bar{W}) \cdot d\bar{W}.
\]  

(1)
We can rewrite this in terms of the Hessian matrix, if we use the constitutive equation. Consider first an elastic plate. By inserting (III.4) into (1) we obtain
\[ \chi = \frac{D}{2} [(1 - \nu) \text{tr} W^2 + \nu (\text{tr} W)^2]. \] (2)

Rewritten in terms of the eigenvalues of the Hessian matrix, the equation reads
\[ \chi = \frac{D}{2} (W_1^2 + W_2^2 + 2\nu W_1 W_2). \] (3)

For a rigid-plastic plate obeying the elliptical criterion, we insert (III.10) into (1) and find
\[ \chi = D \sqrt{(1 - \kappa) \text{tr} W^2 + \kappa (\text{tr} W)^2}, \] (4)

or
\[ \chi = D \sqrt{W_1^2 + W_2^2 + 2\kappa W_1 W_2}. \] (5)

For a rigid-plastic plate obeying the square criterion, the procedure is a bit more delicate. According to (III.8), the eigenvalues of the Hessian matrix \( W_1 \) and \( W_2 \) are proportional to the derivatives of \( \|M\| \) with respect to the main moments \( M_1 \) and \( M_2 \). At the angles of the square, the derivatives must be understood in the generalized sense, as consisting of two limit vectors in the horizontal and vertical direction and the set of vectors pointing between them. As a result, we find for nonzero \( W_1 \) and \( W_2 \) that \( M_1 \) and \( M_2 \) are equal to \( \pm M_0 \) with the sign opposite to that of the corresponding element of the pair \( (W_1, W_2) \), and the deformation energy per unit area is
\[ \chi = M_0 (|W_1| + |W_2|). \] (6)

Besides the norm \( \|\cdot\| \) in the yield criterion we can introduce the norm \( \|\cdot\|_\star \) in the deformation energy. For the elliptical criterion these norms are
\[ \|M\| = \sqrt{(1 + \kappa) \text{tr} M^2 - \kappa (\text{tr} M)^2} \] (7)

and
\[ \|W\|_\star = \sqrt{(1 - \kappa) \text{tr} W^2 + \kappa (\text{tr} W)^2}. \] (8)

If \( \kappa = 0 \), both norms reduce to the standard (Frobenius) matrix norm, while for other values of \( \kappa \) they differ from the standard norm as well as from each other. However, there still exists a simple relation between them. The norms are dual in the sense that the square of the latter is, up to a multiplicative factor, the dual conjugate to the square of the former and vice versa. (To see that, note
that for smooth functions of $n \times n$ matrices the dual conjugation is just the $n$-dimensional Legendre transformation.) The norm in the square criterion and that in the resulting expression for deformation energy are dual, too. This is in agreement with the general concept of duality in the mechanics of deformable bodies, discussed in detail in Temam (1983).

VI. Variational principle

With the expression for deformation energy at hand we can proceed to the variational principle. Denote the region occupied by the mid-plane of the plate before the deformation by $S$. The full energy of the plate at rest is

$$U = U_d + U_p,$$  

(1)

where $U_d$ is the deformation energy of the plate,

$$U_d = \int_S \chi \, dS,$$  

(2)

and $U_p$ is the potential energy of the plate in the field of external forces,

$$U_p = -\int_S qw \, dS.$$  

(3)

In general, the energy includes also a ‘force’ and a ‘moment’ boundary term; they are, however, both absent for plates that are fixed in one of the three ways discussed in section IV. Equilibrium of the plate is determined by the variational principle

$$U \text{ is minimal for admissible } w\text{'s},$$  

(4)

where the class of admissible $w$’s depends on how the plate is fixed. Particularly, $w$ must satisfy the conditions (III.3) and (III.4), if the plate is clamped, the former condition only, if the plate is simply supported, and no condition at all, if the plate is free. The plate can be fixed not only at the edge but also at separate points in the bulk, or along the edge that is otherwise free. Consider a plate that is fixed at a given set of points $X_a$ at the heights $h_a$. Clearly, the forces supporting the plate can be regarded as Lagrange multipliers in the constrained variation principle

$$U_d \text{ is minimal for admissible } w\text{'s satisfying } w(X_a) = h_a.$$  

(5)

Euler’s equation resulting from the variational principle (4) or (5), when written in terms of the tensor of moments, is the same for the elastic and plastic plate. This is immediately seen if we notice
that, according to the definition of the deformation energy, the tensor of moments can be written as

\[ M = -\frac{\partial \chi}{\partial W}. \]  

(6)

By using this relation it is straightforward to show that Euler’s equation for both kinds of plates is just the equation of balance of forces (IV.1). Of course, if we express the tensor of moments from the constitutive equation, the resulting equation will depend on whether the plate is elastic or plastic, and in the latter case also on the choice of yield criterion. Moreover, for plastic plates Euler’s equation, in general, does not hold everywhere. It surely cannot hold in the rigid domain, since if we attempt to write it down there, we end up with an undefined expression of the type 0/0. For certain yield criterions the ‘non-Eulerian’ domain can be even more extended. For example, for the square criterion Euler’s equation can be written only in the part of the plastic domain where the main moments are placed at the angles of the yield diagram, and in the rest of the plate we must get along with the constraint on the Hessian matrix and the equation of balance of forces. In what follows, we will restrict ourselves to the elliptical criterion, for which the ‘non-Eulerian’ and rigid domain coincide.

To write down Euler’s equation does not mean to solve the variation principle completely. When we compute \( \delta U \), from integration by parts we obtain, in addition to the surface integral yielding Euler’s equation, the boundary term at the edge of the plate. If there are corners somewhere throughout the plate, we get also boundary terms at them. Finally, if the plate is partly rigid, we arrive at a surface integral over the rigid domain that cannot be treated in a standard way. We will discuss these items separately, considering plates of various kinds in the order of growing complexity: first an elastic plate, then a rigid-plastic plate that is plastic as a whole, and then a rigid-plastic plate with a finite rigid domain.

For an elastic plate, the only question to discuss is the form of the boundary term at the edge of the plate. The term can be written as

\[ \delta U(C) = \oint_C \left( M_{mn} \frac{\partial \delta w}{\partial n} - Q_n \delta w \right) dl. \]  

(7)

If the plate is clamped, both terms in the brackets are identically zero and we obtain no boundary condition in addition to those determining admissible \( w \)'s. If the plate is simply supported or free, the requirement that \( \delta U(C) \) is zero leads to an additional constraint, or constraints, on \( w \): we obtain condition (III.5) in the former case and conditions (III.5) and (III.7) in the latter case. In this way we recover the full system of conditions of equilibrium of an elastic plate established in section IV.
Denote the solution to all conditions by \( w_0 \). The functional \( U[w] \) has an extreme at \( w_0 \), \( \delta U[w_0] = 0 \); and since the functional is convex, the extreme is absolute minimum.

If we use the equation of balance of forces to express the pressure load in the definition of the potential energy, integrate two times by parts and use the boundary conditions, we find

\[
U_d = -\frac{1}{2} U_p, \tag{8}
\]

hence

\[
U = -U_d = \frac{1}{2} U_p. \tag{9}
\]

We can see that the two parts of the full energy, when evaluated at the solution to the variation principle, obey a simple identity that can be named *virial theorem* after the well-known theorem from the mechanics of material points. Applying this identity to a plate that is fixed at separate points, we express the deformation energy in the form

\[
U_d = \frac{1}{2} \sum_a P_a h_a, \tag{10}
\]

where \( P_a \) is the force in the \( a \)th point.

Consider now a fully plastic plate obeying the elliptical criterion. If the plate contains corners, the integral defining the full deformation energy has to be understood as

\[
\int_S \|W\| \, dS = \int_{S\setminus \hat{C}} \|W\| \, dS + \int_{\hat{C}} \left[ \frac{\partial w}{\partial n} \right] \, ds, \tag{11}
\]

where \( \hat{C} \) is the union of all corners and the square brackets denote the jump in the quantity inside of them. After calculating the contributions of the corners to \( \delta U \) and putting them equal to zero, we arrive at conditions (IV.11) and (IV.12). As a result, we obtain the full system of conditions of equilibrium of a plastic plate formulated in section IV. The variation of the minimized functional, if evaluated at the solution, is non-negative, \( \delta U[w_0] \geq 0 \). (It is positive if the plate has corners.) Since for the plastic plate the minimized functional is, just as for the elastic plate, convex, the solution is again absolute minimum. The minimized functional is convex but not strictly convex, hence the scaling property of solutions discussed at the end of section IV. The virial theorem for the plastic plate states

\[
U_d = -U_p, \tag{12}
\]

so that

\[
U = 0, \tag{13}
\]
and for the plate fixed at a given set of points it holds

\[ U_d = \sum_a P_a h_a. \]  

(14)

We can use the scaling property of solutions for a fast derivation of the virial theorem. Since any solution to the conditions of equilibrium is absolute minimum of \( U \), \( U \) must be the same for all solutions; and since \( U \) scales in the same way as \( w \) if the scaling constant is positive, the value of \( U \) for all solutions must be zero.

Let us now pass to a partly rigid plate assuming again that the elliptical criterion is valid. Our starting point will be an inequality that must be true for any physically acceptable yield criterion. If \( M \) is an arbitrary tensor of moments obeying the constitutive inequality, and \( \hat{M}(W) \) is the tensor of moments in the plastic domain corresponding to the given Hessian matrix \( W \), it must hold

\[ M \cdot W \geq \hat{M}(W) \cdot W. \]  

(15)

This inequality, called Drucker’s condition, guarantees that, when an elastoplastic plate relaxes after the deforming forces have been removed, the dissipated energy is non-negative. As can be seen from the expression of the Hessian matrix in the form (III.8), the inequality (15) is equivalent to the convexity of the yield diagram in the three-dimensional space \((M_{11}, M_{22}, M_{12})\); and according to the two-dimensional version of the theorem proven in Yang (1980b), the sufficient condition for such convexity is the convexity of the yield diagram in the two-dimensional space \((M_1, M_2)\) and the symmetry of \(\|M\|\) with respect to the exchanges of \(M_1\) and \(M_2\). Consequently, the proof of (15) for the elliptical criterion consists in the observation that the criterion obviously has both properties.

To describe the rigid domain in the framework of the variation principle, let us extend the tensor of moments into it. When doing so, we must realize that the tensor of moments has different meaning in the two parts of the plate. In the plastic domain, it is a \(2 \times 2\) matrix formed from the Hessian matrix, \( M = \hat{M}(W) \); in the rigid domain, it is an auxiliary \(2 \times 2\) matrix that does not depend on deflection. Suppose the tensor of moments satisfies the equation of balance of forces throughout the rigid domain. Then we can rewrite the contribution of this domain to \( \delta U \) into the form

\[ \delta U(S_r) = \int_{S_r} (\mathcal{D}\|\delta W\| + M \cdot \delta W) \, dS + \text{boundary terms}, \]

where \( S_r \) is the rigid part of \( S \). (We obtain this relation in the same way as we have obtained the virial theorem.) Suppose, in addition, that the tensor of moments obeys the constitutive inequality,

\[ (1 + \kappa) \text{tr} M^2 - \kappa (\text{tr} M)^2 \leq M_0^2, \]  

(16)
so that (15) is valid. For the elliptical criterion, the matrix \( \hat{M}(W) \) is given by the expression on the right hand side of (III.10). Using this expression, we can rewrite (15) into the form

\[
M \cdot W \geq -\mathcal{D} \|W\|_{\ast},
\]

hence

\[
M \cdot \delta W \geq -\mathcal{D} \|\delta W\|_{\ast},
\]

and the first term in \( \delta U(S_r) \) is non-negative. We can easily verify that if (16) does not hold, the first term in \( \delta U(S_r) \) can be negative, so that the validity of (16) is necessary and sufficient condition of the non-negativeness of this term. The boundary term coming from the border between the plastic and the rigid domain is zero if the matrices \( M \) of the two domains match smoothly on the border. The boundary terms coming from the rigid part of the edge of the plate, as well as from the corners that lay inside the rigid domain or at the boundary between the rigid and plastic domain, are zero if the same boundary conditions hold as for a fully plastic plate. In such a way, we have obtained the complete theory of plastic plates of section IV starting from the variation principle. In the rigid domain we can compute the total deformation energy in a similar way as in the plastic domain, and we find that the same virial theorem holds for a partly rigid plate as for a fully plastic one.

We conclude the discussion of the variation principle for the plastic plate by the proof of uniqueness of the limit load, mentioned at the end of section IV. Normalize the deflection of the plate by the condition

\[
\int_S wQ \ dS = A, \tag{17}
\]

where \( A \) is a constant with the physical dimension of energy. If \( w_0 \) is a solution to the variation principle for a given \( \lambda_{lim} \), from the virial theorem it follows

\[
\lambda_{lim} = \frac{1}{A} U_d[w_0]. \tag{18}
\]

Consider a constrained variation principle

\[
U_d \text{ is minimal for admissible normalized } w's. \tag{19}
\]

This principle is equivalent to the unconstrained variation principle (4) if we identify the parameter \( \lambda \) with the Lagrange multiplier. We are accustomed that the Lagrange multiplier is determined by the corresponding constraint; now, however, this is not the case. The value assumed by the Lagrange multiplier is given by an expression analagous to (18),

\[
\lambda^* = \frac{1}{A} U_d[w^*], \tag{20}
\]
where \( w^* \) is the solution to the constrained variation principle. Obviously, \( \lambda^* \) is the least of \( \lambda_{\text{lim}} \)'s. Define the admissible size of the load \( \lambda_{\text{adm}} \) as the value of \( \lambda \) for which there exists a tensor of moments obeying the equation of balance of forces (IV.1), the constitutive inequality (16), and the boundary condition, or conditions, for the given problem. According to this definition, all \( \lambda_{\text{lim}} \)'s are admissible.

Furthermore, since \( \lambda_{\text{adm}} Q = -\nabla \nabla \cdot M \) and \( w^* \) is normalized, it holds

\[
\lambda_{\text{adm}} = -\frac{1}{A} \int_S \nabla \nabla \cdot M \; w^* \, dS = -\frac{1}{A} \int_{S^*_p} M \cdot W^* \, dS \leq -\frac{1}{A} \int_{S^*_p} M(W^*) \cdot W^* \, dS = \frac{1}{A} U_d[w^*] = \lambda^*,
\]

where \( S^*_p \) is the plastic part of \( S \) under the deformation described by \( w^* \). (When rewriting the expression for \( \lambda_{\text{adm}} \) we have assumed that there are no corners in the rigid domain, but the argument can be easily generalized to include them.) In such a way, \( \lambda_{\text{adm}} \) is not greater than \( \lambda^* \), hence \( \lambda_{\text{lim}} \) is not greater than \( \lambda^* \), and hence there exists only one \( \lambda_{\text{lim}} \), equal to \( \lambda^* \) as well as to the maximal value of \( \lambda_{\text{adm}} \).

**VII. Two mixed kinds of deformation**

A straightforward generalization of the theory of elastic plates is obtained if one considers a plate that is stretched. If such plate is bent, an additional lateral force arises due to the action of stretching forces. Suppose the stresses induced by the stretching of the plate are homogenous and isotropic. The additional force per unit area is \(-N \text{tr} W\), where the constant \( N \), called tension, is proportional to the stretching forces. Consequently, the equation of balance of forces reads

\[
\nabla \nabla \cdot M + N \text{tr} W = -q. \tag{1}
\]

By inserting for \( M \) from the two-dimensional Hooke’s law we obtain

\[
D \Delta \text{tr} W - N \text{tr} W = q. \tag{2}
\]

To write down the corresponding expression for the deformation energy, we must determine the work done by the stretching forces in the course of the bending of the plate. The work equals

\[
\frac{N}{2}(\nabla w)^2,
\]

where the gradient squared is the sum of the first derivatives squared,

\[
(\nabla w)^2 = \left( \frac{\partial w}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2.
\]
As a result, the total deformation energy per unit surface is

\[ \chi = \frac{D}{2} \left[(1 - \nu)\text{tr}W^2 + \nu(\text{tr}W)^2\right] + \frac{N}{2}(\nabla w)^2. \]  

(3)

For \( N = 0 \) we return to equations (IV.2) and (V.2) for a plate without tension, while for \( D = 0 \) we obtain the theory of an perfectly flexible plate, or a membrane. Note that the deformation energy of the membrane is approximately proportional to the increment of the surface of the membrane due to the bending. Consequently, the theory of the membrane coincides with the theory of the bubble, if we consider 'bubbles' in the form of slightly bent planar layers.

In a similar way as we have mixed the deformation of the plate without tension and the deformation of the membrane, we can mix elastic and perfectly plastic deformations of the plate, too. The physical object we arrive at is Hencky’s plate, or the perfectly plastic elastoplastic plate described by the deformation theory. In perfectly plastic elastoplastic bodies, a pure elastic deformation takes place if the stresses lay inside the yield diagram, and a combined elastic and plastic deformation possibly takes place if the stresses are placed at the surface of the yield diagram. Furthermore, the plastic part of deformation is the same as in rigid-plastic bodies. When passing from three to two dimensions, we find that the theory is more involved than in the rigid-plastic case since the plasticity does not take over in the bulk of the plate at once, but it extends throughout the plate gradually, penetrating from the faces to the mid-plane. We can, however, interpolate between the strictly elastic behavior of the weakly deformed plate and the approximately rigid-plastic behavior of the strongly deformed plate by adopting the two-dimensional yield criterion, as well as the expression for the Hessian matrix of the pure plastic deformation, from the theory of rigid plasticity. The idea is, just as in the formulation of the square criterion, to use the pattern of the three-dimensional theory rather than take this theory as a starting point and derive all the formulas from it. However, while in the formulation of the square criterion this shift in perspective was just a shortcut to the exact theory (we would get the same results from the three-dimensional theory if we postulated an appropriate three-dimensional criterion), here it is an approximation. For the elliptical criterion, we arrive in this way at the previous formulas describing the plastic domain in the case \( \kappa = \nu \), and formulas with fourth roots otherwise. If \( \kappa = \nu \), the equation for deflection reads

\[ D_2W = q, \quad D_2W \equiv \begin{cases} D\Delta\text{tr}W, & \text{if } \|W\| < D/D \\ D
abla\nabla \cdot W, & \text{if } \|W\| \geq D/D \end{cases}, \]

(4)
and the deformation energy per unit area is

\[ \chi = \begin{cases} D \| W \|^2 / 2, & \text{if } \| W \|_* < \mathcal{D}/D \\ \mathcal{D} \| W \|_* - \mathcal{D}^2/(2\mathcal{D}), & \text{if } \| W \|_* \geq \mathcal{D}/D \end{cases}. \] (5)

For \( \mathcal{D} \to \infty \) and \( D \to \infty \) we arrive at the theory of the elastic and rigid-plastic plate respectively. For finite \( D \) as well as \( \mathcal{D} \), the whole plate is 'Eulerian' and the sufficient condition for the existence of the solution to the problem with the pressure load \( \lambda Q \) is the \textit{safe load condition} \( \lambda < \lambda_{\text{lim}} \) (Temam 1983).

**VIII. Examples**

The simplest problem in the mechanics of plates is to determine the shape of the circular plate lifted at the center. Solution to this problem for an elastic simply supported plate without tension was found as early as in 1829 by Poisson. In what follows we will consider plates fixed at the radius 1 with the center lifted to the height 1, and we will put \( D = 1 \) for the elastic plate and \( \mathcal{D} = 1 \) for the plastic plate. Note that the theory is valid only if the deflection is much smaller than the typical scale on which the plate is deformed, therefore if we want to obtain physically sensible solutions, we must rescale \( w \) by a constant much less than 1. From the symmetry of the problem it follows \( w = w(r) \), where \( r \) is the radial coordinate, \( r = \sqrt{x^2 + y^2} \). Deformation of the elastic plate without tension is given by the equation

\[ \Delta_r \Delta_r w = P\delta(r), \] (1)

where \( \Delta_r \) is the radial part of the Laplace operator,

\[ \Delta_r = \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr}, \]

\( r \) is radius vector and \( \delta(r) \) is 2-dimensional \( \delta \)-function, \( \delta(r) = \delta(x)\delta(y) \). The equation must be supplemented by boundary conditions at the center and the edge of the plate. By assumption, at the center it holds

\[ w(0) = 1. \] (2)

Suppose the plate is simply supported at \( r = 1 \). Then the conditions at the edge are

\[ w(1) = w''(1) + \nu w'(1) = 0. \] (3)

The general solution to (1) is

\[ w = Ar^2 \log r + Br^2 + C \log r + D. \]
The condition (2) implies $C = 0$, $D = 1$, the first condition (3) implies $B = -1$ and the second condition (3) implies $A = 2(1 + \nu)/(3 + \nu)$. As a result we obtain

$$w = 1 + r^2 \left[ \frac{2(1 + \nu)}{3 + \nu} \log r - 1 \right].$$

(4)

Since $\Delta_r w = 4A \log r$ and $\Delta_r \log r = 2\pi \delta(r)$, the force is $P = 8\pi A$ and the deformation energy is $U_d = P/2 = 4\pi A$. By inserting the expression for $A$ into $U_d$ we find

$$U_d = \frac{8\pi(1 + \nu)}{3 + \nu}.$$  

(5)

Suppose now that the plate is clamped at $r = 1$. Then we have, instead of (3),

$$w(1) = w'(1) = 0.$$  

(6)

This coincides with (3) in the limit $\nu \to \infty$, therefore the expressions for $w$ and $U_d$ for a clamped plate can be obtained by performing the limit $\nu \to \infty$ in the corresponding expressions for a simply supported plate. In particular, we find that the energy of the plate is $8\pi$. In fig. 3, two simply supported plates are depicted, with the values of Poisson’s ratio given next to them. (The nonphysical

Sophie Germain’s value $\nu = 1$ is used in order to obtain solutions that are not too close to each other.) In addition, the curve representing the clamped plate (denoted by 'C') is included into the figure.

The one-dimensional version of the problem we are interested in is the bending of the beam. A well-known result of the theory of beams is that an elastic beam fixed at the given set of points assumes the form of cubic spline, a piecewise smooth curve used in interpolation problems. The most
favored version of cubic spline is the natural spline which corresponds to the beam that is either simply supported at the endpoints or infinite. The two-dimensional analogue of the natural spline is the solution describing an infinite elastic plate fixed at the given set of points, found by Harder and Desmarais (1972) and known as the thin-plate spline. One can demonstrate some features of the thin-plate spline on an infinite plate lifted at the center and simply supported at its original height by a ringlike bar that is freely applied to it from above. The deflection must satisfy, in addition to the condition (2) and the first condition (3), the condition \( w''(\infty) = 0 \) (in order that the deformation energy is finite) and the condition that \( w' \) and \( w'' \) are continuous at \( r = 1 \) (in order that the bar does not rotate the plate or produce a corner on it). By applying these conditions to the general solution written with different coefficients \( A, B, C, D \) in the regions \( r \leq 1 \) and \( r > 1 \), we obtain

\[
\begin{align*}
w &= \begin{cases} 
1 + r^2(\log r - 1) & \text{if } r \leq 1 \\
-\log r & \text{if } r > 1 
\end{cases}.
\end{align*}
\]

(7)

The plate is depicted in fig. 4 by the solid line. For comparison, some finite free plates with \( \nu = 0 \) are presented in the figure, too, drawn by the dotted lines. Their deflection is

\[
w = \begin{cases} 
1 + r^2(p \log r - 1) & \text{if } r \leq 1 \\
-p \log r - q(r^2 - 1) & \text{if } 1 < r \leq R 
\end{cases}
\]

(8)

where \( p = 1/(1 + \Delta), q = 1 - p, \Delta = 1/(2R^2) \) and \( R \) is the radius of the plate. The energy of the infinite plate is \( 4\pi \) and that of the finite plate is \( 4\pi p \). Note that inside the circle at which the plate is fixed, the solution for the infinite plate coincides with that for the simply supported plate with the Sophie Germain’s value of Poisson’s ratio 1.

Fig. 4: An infinite elastic plate

![Diagram of an infinite elastic plate](image-url)

are presented in the figure, too, drawn by the dotted lines. Their deflection is

\[
w = \begin{cases} 
1 + r^2(p \log r - 1) & \text{if } r \leq 1 \\
-p \log r - q(r^2 - 1) & \text{if } 1 < r \leq R 
\end{cases}
\]

(8)

where \( p = 1/(1 + \Delta), q = 1 - p, \Delta = 1/(2R^2) \) and \( R \) is the radius of the plate. The energy of the infinite plate is \( 4\pi \) and that of the finite plate is \( 4\pi p \). Note that inside the circle at which the plate is fixed, the solution for the infinite plate coincides with that for the simply supported plate with the Sophie Germain’s value of Poisson’s ratio 1.
If we include the term describing tension into equation (1), the solution can be expressed in terms of special functions. The modified equation reads

$$\Delta_r \Delta_r w - \alpha^2 \Delta_r w = P \delta(r),$$

where $$\alpha = \sqrt{N}$$. To get rid of $$\alpha$$, replace $$r$$ by $$u = \alpha r$$ and $$P$$ by $$Q = \alpha^{-2}P$$. In this way we obtain

$$\Delta_u \Delta_u w - \Delta_u w = Q \delta(u),$$

which is equivalent to

$$\Delta_u \text{tr} W - \text{tr} W = Q \delta(u), \quad \Delta_u w = W.$$  

Equation for $$\text{tr} W$$ is the modified Bessel equation of zeroth order with the $$\delta$$-function source. Outside $$u = 0$$, the solution is

$$\text{tr} W = A I_0(u) + B K_0(u),$$

where $$I_0$$ and $$K_0$$ are modified Bessel functions of the first and second kind, both of zeroth order. The asymptotics of $$I_0$$ and $$K_0$$ at $$u \sim 0$$ are

$$I_0 = 1 + O(u^2), \quad K_0 = -\log \frac{u}{2} - \gamma + O(u^2 \log u),$$

where $$\gamma = 0.5772...$$ is the Euler-Mascheroni constant. By using $$\Delta_u \log u = 2\pi \delta(u)$$ we find that the equation for $$\text{tr} W$$ is satisfied at $$u = 0$$, too, provided $$Q = -2\pi B$$. Equation for $$w$$ yields

$$w = A I(u) + B K(u) + C \log u + D,$$

where $$I$$ and $$K$$ are arbitrarily chosen solutions to the equations

$$\Delta_u I = I_0, \quad \Delta_u K = K_0.$$  

The equations have obvious solutions

$$I = I_0, \quad K = K_0.$$  

If we use these solutions and asymptotics of $$I_0$$ and $$K_0$$ given above, we find from the condition (2) that $$C = B$$ and $$D = 1 - A + B[\log(1/2) + \gamma]$$, so that

$$w = 1 + AF(u) + BG(u), \quad \begin{cases} F = I_0(u) - 1 \hfill \\
G = K_0(u) + \log \frac{u}{2} + \gamma \end{cases}.$$  

(10)
The constants $A$ and $B$ must be determined from the conditions (3). For the plate with $\nu = 0$ we find
\[
w = 1 - \frac{qF(\alpha r) - pG(\alpha r)}{qF(\alpha) - pG(\alpha)}, \quad \begin{cases} p = F''(\alpha) = I_0(\alpha) - \frac{1}{\alpha}I_1(\alpha) \\ q = G''(\alpha) = K_0(\alpha) + \frac{1}{\alpha}K_1(\alpha) - \frac{1}{\alpha^2} \end{cases}.
\]

(11)

The force is given by the coefficient $B$ appearing in front of the function $G$ in the expression for $w$, $P = Q\alpha^2 = -2\pi B\alpha^2$. The deformation energy can be computed again as $U_d = -U_p/2 = P/2$, hence its value for the plate with $\nu = 0$ is
\[
U_d = \frac{\pi p\alpha^2}{qF(\alpha) - pG(\alpha)}.
\]

(12)

In fig. 5, some plates with $\nu = 0$ are depicted by the solid line with the values of $\alpha$ given next to them. An extended version of thin-plate splines that includes tension has been proposed by Franke (1985). The base functions entering these splines are drawn in the figure by the dotted lines. They are obtained by replacing the condition $w''(1) = 0$ (the second condition (3)) by the nonphysical condition $w''(\infty) = 0$, and are given by the formula
\[
w = 1 - \frac{G(\alpha r)}{G(\alpha)};
\]
see Mitáš and Mitášová (1988).

Let us now pass to the plastic plate. If the elliptical criterion is valid, the deflection of the plate is given by the equation
\[
\nabla \cdot \nabla W = P\delta(r).
\]

(13)
If, moreover, the plate is simply supported, \( w \) obeys the same boundary conditions as before, with \( \nu \) replaced by \( \kappa \). Consider \( w \) of the form
\[
w = 1 - r^p. \tag{14}\]

From the definition of \( \mathcal{W} \) we obtain
\[
\mathcal{W} = \text{const} + Ann,
\]
where \( \mathbf{n} = \mathbf{r}/r \) and \( A = (1 - \kappa)(1 + q)/\tilde{q}, \) \( q = 1 - p, \) \( \tilde{q} = \sqrt{q^2 - 2\kappa q + 1}. \) This yields
\[
\nabla \nabla \cdot \mathcal{W} = A \left[ \frac{\partial}{\partial x} \left( \frac{n_x}{r} \right) + \frac{\partial}{\partial y} \left( \frac{n_y}{r} \right) \right] = 2\pi A \delta(r),
\]
so that the ansatz (14) solves equation (13) if \( P = 2\pi A \). From the second condition (3) with \( \nu \) replaced by \( \kappa \) we obtain
\[
p = 1 - \kappa, \tag{15}\]
and if we insert this into the definition of \( A \) and use \( U_d = P = 2\pi A \), we find
\[
U_d = 2\pi \sqrt{1 - \kappa^2}. \tag{16}\]

For \( \kappa = 0 \) the plate has the shape of conus, and with increasing \( \kappa \) it bends inside. Note that the conus is obtained also in the theory with the square criterion, see Kachanov (1956).

Solution (15) applies also to obliquely clamped plates which satisfy, in addition to the condition \( w(1) = 0 \), the condition \( w'(1) = p \) with an arbitrary positive \( p \). The shape of such plates is given by equation (15) irrespective of their value of \( \kappa \). This rises a question as to what is the shape of the ordinary clamped plate that has \( w'(1) = 0 \). To answer that, introduce the function
\[
\xi = \lim_{p \to 0^+} (1 - r^p) \approx \begin{cases} 1 & \text{if } r = 0 \\ 0 & \text{if } 0 < r \leq 1 \end{cases}.
\tag{17}\]

The explicit expression for \( \xi \) is symbolic only, since the limit is understood in the weak sense, as an operation to be performed after the rest of the computation has been completed. In particular, if we define the integral norm of the function \( w(r) \) as the integral of the Frobenius norm of its Hessian matrix,
\[
\|w\| = \int \sqrt{\text{tr} W^2} dS = 2\pi \int_0^1 \sqrt{r^2 w''^2 + w'^2} dr,
\]
we find that the norm of \( \xi \) is finite and equals \( 2\sqrt{2}\pi \). In fact, we can define \( \xi \) by completing the space of \( C^2 \) functions with respect to the norm \( \|\cdot\| \). With the function \( \xi \) at hand, we immediately solve the
problem with clamped plate we have started with. The deflection of the plate is \( w = \xi \); that is, the plate remains flat, only the point \( r = 0 \) is pulled out of it. The deformation energy is obtained most easily from \( U_d = 2\pi A \) by inserting \( p = 0 \) into the formula for \( A \). In this way we find

\[ U_d = 2\pi \sqrt{2(1 - \kappa)}. \tag{18} \]

The search for the solution describing an infinite plate leads to the conclusion that an infinite plate fixed on the height 1 at \( r = 0 \) and on the height 0 at \( r = 1 \) has the deflection \( w = \xi_\infty \equiv \xi \) extended to all \( r \). In fig. 6, the infinite rigid-plastic plate lifted at the center is depicted by the solid line and by the black bullet on the vertical axis. To demonstrate the transition to such plate, several finite rigid-plastic plates with \( \kappa = 0 \) are shown in the figure, too, drawn by the dotted lines. The curves were obtained by matching the solution (15) for \( r \leq 1 \) with a quite intricate analytic solution for \( r > 1 \). For the radius of the plate this procedure yields

\[ R = \frac{\sqrt{1 + q}}{(1 + q^2)^{1/4}} \exp \left( \frac{1}{2} \arctan q \right). \]

If \( p \) decreases from 1 to 0, \( q \) increases from 0 to 1 and \( R \) increases from 1 to

\[ R_{\text{crit}} = 2^{1/4} e^{\pi/8} \approx 1.76. \]

The radius \( R_{\text{crit}} \) is depicted in the figure by the bullet on the horizontal axis. For \( R \geq R_{\text{crit}} \) the deflection of the plate is \( w = \xi_R \equiv \xi \) extended to the radius \( R \), so that the plate is represented by the solid line cut at \( r = R \) and the bullet on the vertical axis. Finally, in fig. 7 several infinite

![Fig. 6: An infinite rigid-plastic plate](image-url)
elastoplastic plates with $\kappa = 0$ are shown, with the values of the ratio $D/D$ given next to them. The solution can be expressed in terms of a set of parameters that are fixed by a system of algebraic equations. Deformation energy is $2\pi(\tilde{q} - \beta R_{plast}^2/2)$, where $\beta = D/D$ and $R_{plast}$ is the radius of the plastic domain in the central part of the plate.

**IX. Conclusion**

The mechanics of an elastic plate and the mechanics of a rigid-plastic plate with the elliptical criterion look at first glance similar: the latter differs from the former just by the square root in the expression for the deformation energy. However, this difference has far reaching consequences. Analyzing the differential equation for deflection in the neighborhood of the $\delta$-function source, one finds that the elastic plate interpolates smoothly between the points at which it is fixed, while the rigid-plastic plate has sharp vertices at these points. Moreover, the solution for a circular rigid-plastic plate suggests that if the size of the plate exceeds some limit value, the forces fixing the plate at the given set of points fail to deform the plate in the ordinary sense. The plate remains flat, and the forces just pull the points out of it. Using the limit analysis it can be shown that if the plate is infinite, it is not able to reach equilibrium but in this peculiar way.

The behavior of the plastic plate can be also compared to that of the plastic beam. Deformation energy of the plastic beam is the total variation of deflection, which implies that the beam fixed at the given set of points relaxes to a broken line. We can see that if we pass from one dimension to two, the plastic behavior becomes more singular.
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