Analytical and Numerical Methods and Test Calculations of One-dimensional Force-free Magnetodynamics on Arbitrary Magnetic Surfaces across Horizons of Spinning Black Holes

Shinji Koide¹ and Tomoki Imamura²

¹ Department of Physics, Faculty of Science, Kumamoto University, 2-39-1, Kurokami, Kumamoto, 860-8555, Japan; koidesin@kumamoto-u.ac.jp
² Algorithm Laboratory, 2-6-14, Ebisu-minami, Shibuya-ku, Tokyo, 150-0022, Japan

Received 2019 June 4; revised 2019 June 12; accepted 2019 June 27; published 2019 August 16

Abstract

Numerical simulations of the force-free magnetodynamics (FFMD) of the electromagnetic field around a spinning black hole are useful to investigate the dynamic electromagnetic processes around a spinning black hole, such as the emergence of the Blandford–Znajek mechanism. To reveal the basic physics of magnetic fields around a black hole through the dynamic process, we use one-dimensional (1D) FFMD along the axisymmetric magnetic surface, which provides a relatively simple, sufficiently precise, and powerful tool to analyze the dynamic process around a spinning black hole. We review the analytic and numerical aspects of 1D FFMD for an arbitrary magnetic surface around a black hole. In addition, we also show some numerical simulation test results for three types of magnetic surfaces at the equatorial plane of the black hole.

Key words: black hole physics – Galaxy: nucleus – magnetic fields – methods: analytical – methods: numerical – plasmas

1. Introduction

Recently, images of the supermassive black hole shadow in the center of the giant elliptical galaxy M87, whose central core emits a powerful relativistic jet almost toward us (Biretta et al. 1999), have been reconstructed by the team of the Event Horizon Telescope (EHT) collaboration (Event Horizon Telescope Collaboration 2019a). To extract information on the dynamics of the plasma, the black hole’s gravitational field, and the black hole itself, it is necessary to develop fully general relativistic models of the accretion flow, associated winds and relativistic jets, and emission properties of the plasmas. The general relativistic models are also required to understand the formation of relativistic jets from stellar-mass black holes, such as microquasars (black hole binaries; Mirabel & Rodríguez 1994; Tingay et al. 1995) and gamma-ray bursts (Kulkarni 1999; LIGO Scientific Collaboration et al. 2017) and gravitational waves emitted from merging stellar-mass black holes (Abbott et al. 2016). The most common approach to dynamical relativistic source modeling uses the ideal general relativistic magnetohydrodynamic (GRMHD) approximation. Over the last decades, a number of GRMHD codes have been developed and applied to a large variety of astrophysical scenarios (Koide et al. 1998, 1999, 2000, 2002, 2006; Gammie et al. 2003; Koide 2003; McKinney 2006; Del Zanna et al. 2007; McKinney & Blandford 2009; McKinney et al. 2013; Radice & Rezzolla 2013). The EHT team also found that images produced from GRMHD simulations with general relativistic ray-tracing calculations (Event Horizon Telescope Collaboration 2019b; Porth et al. 2019) are consistent with images of the asymmetric ring seen in the EHT data. From a comparison between GRMHD simulations and EHT images, the EHT collaboration team concluded that the asymmetry in brightness in the ring can be explained in terms of relativistic beaming of the emission from a plasma rotating close to the speed of light around a black hole spinning clockwise. At the same time, in those models that produce the relativistic jet of M87, the jet is powered by extraction of black hole spin energy through the Blandford–Znajek mechanism (Blandford & Znajek 1977; McKinney 2006). The relativistic jet is anchored to the “funnel” region near the polar axis where low angular momentum material will be swallowed up by the black hole. The strong magnetic fields that permeate the black hole make the Blandford–Znajek mechanism work. The ensuing near-vacuum and magnetic dominance are difficult for GRMHD simulations to handle. In such a region near the polar axis, the force-free condition becomes a good approximation.

To investigate the dynamic processes of extremely strong magnetic fields around a spinning black hole, two-dimensional calculations of the force-free magnetodynamics (FFMD) have been performed (Komissarov 2001, 2002, 2004). Such simulations showed the emergence of energy radiation from a spinning black hole via an axisymmetric magnetic field. However, sufficient analysis of the emergence of energy radiation has not yet been shown because we have no analytic solution for a two-dimensional electromagnetic field around a spinning black hole to compare with the numerical results. When we assume that an axisymmetric magnetic surface is fixed around a black hole, we can reduce FFMD to one-dimensional FFMD (1D FFMD), where we can obtain the analytic solutions of the steady-state force-free fields to compare the numerical results. To investigate the fundamental physics of the magnetic fields dynamically, we can use numerical simulations of 1D FFMD for the axisymmetric magnetic surfaces across the black hole effectively. In this paper, we review a method of 1D FFMD for an arbitrary fixed axisymmetric magnetic surface around a black hole. The 1D FFMD has the following advantages.

1. Developing the numerical calculation code is easy, and small computer resources are sufficient to run the code.
2. Analyses of the numerical calculations are relatively easy.
3. An analytic solution of the steady state can be given for an arbitrary magnetic surface around a rapidly rotating black hole (Section 3).
However, 1D FFMD has some disadvantages.

1. An axisymmetric stationary magnetic surface should be assumed. The results strongly depend on the configuration of the magnetic surfaces.
2. We cannot treat the interaction between the magnetic surfaces and should neglect energy transport and wave propagation across the magnetic surfaces.

It is noted that the energy transport and wave propagation across the magnetic surface are negligible near the horizon. Therefore, we expect that the essential physics of force-free fields around a black hole could be seized via 1D FFMD.

We have performed 1D FFMD numerical simulations of a magnetic field around a spinning black hole at its equatorial plane to investigate the dynamic process of the energy extraction from the black hole via the magnetic field (Koide & Imamura 2018; Imamura & Koide 2019). In the case of the radial magnetic surface around an equatorial plane, the Poynting flux emerges from the ergosphere, and the region of the finite Poynting flux expands toward infinity like a tsunami (Koide & Imamura 2018). On the other hand, in cases other than the radial magnetic surface, such as incurvature- or excurvature-flared magnetic surfaces, the Poynting flux emerges around the ergosphere transiently at a very early stage of the simulation; however, the finite Poynting flux region spreads outward very slowly or is reduced and eventually vanishes (Imamura & Koide 2019). We explained the drastic difference between these cases based on the analytic solution of the steady-state force-free fields.

In this paper, we generalize the 1D FFMD equations to perform numerical simulations of the force-free field along an arbitrary magnetic surface. In addition, we show the analytic solutions of the steady-state field along an arbitrary magnetic surface. Test calculations are shown for cases of magnetic surfaces at the equatorial plane with three types of shapes.

We review generalized 1D FFMD equations for arbitrary magnetic surfaces in Section 2. In Section 3, we provide analytical solutions of the 1D steady-state field along an arbitrary axisymmetric magnetic surface around a spinning black hole. In Section 4, we present the numerical test results of 1D FFMD for cases of three types of magnetic surfaces along the equatorial plane. A summary and discussions are shown in Section 5.

2. Review of FFMD

2.1. Covariant Form of FFMD Equations

The FFMD equations are based on the Maxwell equations with force-free conditions (Komissarov 2001, 2004). Here we assume that the magnetic field is so strong that we can neglect the plasma inertia. We also assume that the electric resistivity and self gravity of the electromagnetic field vanish. The line element in the spacetime $x^μ = (t, x^1, x^2, x^3)$ is written by $ds^2 = g_{μν}dx^μdx^ν$. Here Greek subscripts like $μ$ and $ν$ run from 0 to 3, while Roman subscripts like $i$ and $j$ run from 1 to 3. We employ the natural unit system, where the speed of light $c$, gravitational constant $G$, and magnetic permeability and electric permittivity in the vacuum $μ_0$, $ε_0$ are unity. We also often set the black hole mass unity, $M = 1$. The covariant forms of the Maxwell equations are

\[ \nabla_μ F^μν = \frac{1}{\sqrt{-g}} \frac{∂}{∂x^μ} (\sqrt{-g} F^μν) = -J^ν, \]  

\[ \nabla_μ *F^μν = \frac{1}{\sqrt{-g}} \left( \frac{∂}{∂x^μ} (\sqrt{-g} *F^μν) \right) = 0, \]

where $\nabla_μ$ is the covariant derivative, $F^μν$ is the electromagnetic field tensor, $*F^μν = ε^{μνρσ}F_ρσ/2$ is the dual tensor of $F_μν$, and $J^ν = (ρ_e, J^i, J^j, J^k)$ is the four-current density ($ρ_e$ is the electric charge density). Here $ε^{μνρσ} = η^{μνρσ}/\sqrt{-g}$ is the Levi–Civita tensor, $g = det(g_{μν})$ is the determinant of $(g_{μν})$, and $η^{μνρσ}$ is the totally asymmetric symbol. We use the electric field $E_μ$ and magnetic field $B^μ$ given by

\[ E_μ = F_μ0, \quad B^μ = *F^0μ = \frac{1}{2} ε^{0μνσ}F_νσ. \]

The Maxwell equations read the conservation law of the electromagnetic energy and momentum,

\[ \nabla_μ T^μν = -f^L_μ, \]

where $T^μν = F^0μF^μν - \frac{1}{2} g^μ_ν F_ρσF^ρσ$ is the four-electromagnetic energy-momentum tensor and $f^L_μ = J^μ - *F^μν$ is the four-Lorentz force density. In the case of the force-free condition $f^L_μ = 0$, Equation (4) yields

\[ \nabla_μ T^μν = 0. \]

To perform FFMD numerical simulations, we use Equation (4) instead of Equation (1)

2.2. Degeneracy of Force-free Electromagnetic Field

The degeneracy of the force-free electromagnetic field is imposed on the FFMD supplementarily. This degeneracy is due to the condition of the magnetospheric plasma, where the charged particles are plentiful enough to support a strong electric current and screen the electric field (Goldreich & Julian 1969). To derive the degeneracy, we consider the low-inertia limit of relativistic magnetohydrodynamics (RMHD). The relativistic Ohm’s law is

\[ u^ν F_{μν} = η(J_μ - ρ'_e u_μ), \]

where $u^ν$ is the four-velocity of the plasma, $η$ is the resistivity, and $ρ'_e$ is the proper electric density. Equation (6) yields

\[ E_i + ε_{ijk}v^jB^k = \frac{η}{γ_L} (J_i - ρ'_e u_i) = ηJ_i, \]

where $γ_L = u^0$, $v^i = u^i/γ_L$, and $J_i = \frac{1}{η}(J_i - ρ'_e u_i)$ is similar to the current density observed by the plasma rest frame but not exactly. We have

\[ *F^μν F_μν = *F^μν F_μν = -4E_iB^i, \]

\[ = -4\frac{η}{γ_L} (J_i - ρ'_e u_i)B^i, \]

where $\gamma^{μνρσ} = 1$ if the order $[μνρσ]$ is an even permutation of $[0123]$, $\gamma^{μνρσ} = -1$ if the order $[μνρσ]$ is an odd permutation of $[0123]$, and $η^{μνρσ} = 0$ if $μ, ν, ρ, σ$ are not all different.
When \( \eta \) vanishes, Equations (8) and (9) yield the degeneracy

\[
F^{\rho\sigma}F_{\rho\sigma} = F^{\rho\sigma}F_{\rho\sigma} = 0,
\]

(10)

\[
F^{\rho\sigma}F_{\rho\sigma} > 0.
\]

(11)

In the case of the resistive plasma, the degeneracy is not always guaranteed. Because, for example, when \( \eta \) and \( J_i J^i \) are finite and \( u_J \) vanishes, \( *F^{\rho\sigma}F_{\rho\sigma} \neq 0 \) from Equation (8). Furthermore, when \( B^i \) vanishes, Equation (9) yields \( F^{\rho\sigma}F_{\rho\sigma} = -2\eta^2 j^2 < 0 \) where the degeneracy is broken.

2.3. 3+1 Formalism of FFMD

We review the 3+1 formalism of the FFMD equations derived from the covariant Equations (1), (2), and (4). In order to introduce the 3+1 formalism, we use the local coordinate frame called the “normal observer frame” \( x^\mu \), in which the line element is written by

\[
ds^2 = -d\tilde{t}^2 + \gamma_{ij}dx^idx^j.
\]

(12)

Here we treat \( \gamma_{ij} = g_{ij} \) as the elements of a 3 \times 3 matrix \( (\gamma_{ij}) \) and \( \gamma = \det(g_{ij}) \). Then \( \gamma^i \) indicates the elements of the inverse of the matrix \( (g_{ij}) \), that is, \( \gamma^i \delta_{kj} = \delta^i_j \). When we define the lapse function \( \alpha \) and shift vector \( \beta^i \) as \( \alpha = \sqrt{g_{tt}} = g_{tt} \) or \( \beta^i = \eta^i = 0 \), \( \alpha^2 = -g_{00} + g_{ij}\gamma^i\gamma^j \), we write the line element as

\[
ds^2 = -\alpha^2 + g_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt).
\]

(13)

Comparing Equations (12) and (13), we obtain \( d\tilde{t} = \alpha dt \), \( dx^i = dx^i + \beta^i dt \), and we have \( \partial \alpha = \alpha^{-1}(\partial \alpha - \beta^i \partial \beta_i) \), \( \partial \beta_i = \partial \beta_i \).

This local coordinate system is not always orthonormal in the space. Then the components of the vectors and tensors in the local reference frame are not intuitive in general. In the Boyer–Lindquist (BL) coordinates, the local frame of space is already orthogonal, \( g_{ij} = h_i^2 \delta_{ij} \), and then we can orthonormalize easily as

\[
ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu = -d\tilde{t}^2 + \sum_{i=1}^{3}(dx^i)^2,
\]

\[
= -d\tilde{t}^2 + \sum_{i=1}^{3}h_i h_j \delta_{ij}dx^i dx^j,
\]

(14)

where we set \( d\tilde{t} = \tilde{d}t \), \( dx^i = h_i dx^i \). We have \( \partial \tilde{t} = \partial t \), \( \partial_i = h_i^{-1} \partial \tilde{t} \). Here \( \eta_{\mu\nu} \) is the metric of Minkowski spacetime. This orthonormal local reference frame \( x^\mu \) is called the zero angular momentum observer (ZAMO) frame.

The 3+1 formalism of the Maxwell equations is

\[
\frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial \tilde{t}} (\gamma B^\tilde{t}) = 0,
\]

(15)

\[
\frac{\partial}{\partial \tilde{t}} B^\tilde{t} = -\epsilon^{ijk} \frac{\partial}{\partial x^j} (\alpha (E_i - \epsilon_{kpi} N^k B^i)),
\]

(16)

\[
\frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial \tilde{t}} (\sqrt{\gamma} E^\tilde{t}) = \tilde{p}_e.
\]

(17)

Here we derive the equation of energy conservation from Equation (4) with the Killing vector of temporal boost transformation \( \xi^\mu = (1, 0, 0, 0) \). Using the Killing equation \( \nabla_{\mu} \xi_{\nu} + \nabla_{\nu} \xi_{\mu} = 0 \), we obtain

\[
\nabla_{\mu} (T^{\mu\nu} \xi_{\nu}) = \frac{1}{\sqrt{-g}} \partial_{\mu} (\sqrt{-g} T^{\mu\nu} \xi_{\nu}) = 0.
\]

(22)

When we use the energy flux density \( S^{\mu} = T^{\mu\nu} \xi_{\nu} \) and the energy-at-infinity density \( \epsilon^{\infty} = S^{0} \), we have the equation of energy conservation,

\[
\frac{\partial}{\partial \tilde{t}} \epsilon^{\infty} + \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^i} (\alpha \sqrt{\gamma} S^i) = 0.
\]

(23)

2.5. Conditions from Axisymmetry on Flux Coordinates

Here we introduce flux coordinates for a given axisymmetric magnetic surface along an arbitrary axisymmetric surface imaged by Figure 1. Using the vector potential \( A^\mu \) for the given axisymmetric electromagnetic field, the magnetic surface is described by \( \Psi = A_\phi = \text{constant} \). The flux coordinates are given by \( x^\mu = (r, t, \Psi, \phi) \). Here \( r, t, \phi, \) and \( \Psi \) are the time, radial, and azimuthal coordinates, respectively. Using the flux coordinates, we have \( B^\phi = 0 \). For a radial magnetic surface as shown in Figure 2, we use the colatitude coordinate \( \theta \) as the coordinate \( \Psi = \Psi_0 + \theta_0 \) where the radial magnetic surface is given by \( \theta = \theta_0 \). Note that, in this case, we use the unit system so that \( \epsilon_{ijk} = 1 \). In general, the flux coordinates are not orthogonal, while the coordinates are orthogonal at the equatorial plane.

When we use the flux coordinates \( (r, \Psi, \phi) (B^\psi = 0) \), we have the following three conditions with respect to \( E_0, E_r, \) and \( J^\psi \) in general. It is noted that the BL and Kerr–Schild (KS) coordinates are used for the flux coordinates of radial magnetic surfaces around the equatorial plane. In the case of a radial magnetic surface, \( \Psi \) is directly given by \( \theta \) without the addition of \( \Psi_0 - \theta_0 \).
2.6. 1D FFMD Equations

We derive a general form of 1D FFMD equations for an arbitrary axisymmetric magnetic surface of the force-free electromagnetic field. Equations (15)–(18), (21), and (23) yield the following “1D” equations:

\[
\frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial r} (\sqrt{\gamma} B^r) = 0, \quad \frac{\partial}{\partial t} B^r = 0, \tag{27}
\]

\[
\frac{\partial}{\partial t} E^\psi = -\frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial r} [\alpha (B_\phi + \sqrt{\gamma} N^\psi E^\psi - \sqrt{\gamma} N^\phi E^\phi)], \tag{28}
\]

\[
\frac{\partial}{\partial t} B^\phi = -\frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial r} [\alpha \sqrt{\gamma} (E_\phi - \sqrt{\gamma} N^\phi B^r + \sqrt{\gamma} N^r B^\phi)], \tag{29}
\]

\[
\frac{\partial}{\partial r} (\alpha N^r) S_\phi = \frac{\partial}{\partial r} (\alpha N^\phi) S_\phi - \frac{\partial}{\partial \Psi} (\alpha N^\phi) S_\Psi
\]

\[
+ \frac{1}{2} \frac{\partial}{\partial r} \gamma_r T_\phi^\phi, \tag{30}
\]

\[
\frac{\partial}{\partial r} S_\phi = -\frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial r} [\alpha \sqrt{\gamma} (T_\phi^\phi + N^r S_\phi)], \tag{31}
\]

\[
\frac{\partial}{\partial r} e^\infty = -\frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial r} (\alpha \sqrt{\gamma} S_\phi). \tag{32}
\]

Note that other equations from Equations (17), (18), (20), (21), and (23) yield non-1D equations that include the \(B\)-derivative as shown in Appendix B. Equation (27) shows \(\sqrt{\gamma} B^r\) is uniform and constant. In fact, we have \(B^r = E^0 = \frac{1}{\sqrt{\gamma}} (\partial_\Psi A_\phi - \partial_\phi A_\Psi) = \frac{1}{\sqrt{\gamma}}\). The poloidal component of the magnetic field \(B_\phi\) is defined by \(B_\phi = B^\phi = \sqrt{\gamma} r B^r = \sqrt{\gamma} \frac{E^0}{\sqrt{\gamma}}\).

The primitive variable \(E^\psi\) is given by conservation quantities \(B^r, B^\phi, S_\phi,\) and \(S_\phi\) as

\[
E^\psi = \frac{1}{\sqrt{\gamma}} S_\phi B^\phi - S_\phi B^r \tag{33}
\]

where \(B_\phi = \gamma_\phi B^\phi + \gamma_\phi B^\phi, \quad B_\phi = \gamma_\phi B^\phi + \gamma_\phi B^\phi, \quad (\tilde{B})^2 = B^r B^r + B^\phi B^\phi.\) We have \(E^\Psi = \frac{1}{\sqrt{\gamma}} (E^\psi - \sqrt{\gamma} N^\psi E^\phi)\) using \(E_\psi = \gamma_\phi E_\psi\) with Equations (24) and (26) and \(\gamma_\phi = 0\). We can write

\[
T_\phi^\phi = -\tilde{E}^\phi E_\phi - B^\phi B_\phi + \tilde{u} \delta^\phi_\phi, \tag{34}
\]

\[
\tilde{u} = \frac{1}{2} (B^r B^r + B^\phi B_\phi + \tilde{E}^2), \tag{35}
\]

where \((\tilde{E})^2 = \tilde{E}^\phi E_\phi\).

Using Equations (29)–(31) and (33)–(35) with \(B^r = B_0 / \sqrt{\gamma}\) (\(B_0\) is constant), we perform 1D FFMD numerical simulations for an arbitrary magnetic surface. When we use the flux coordinates, we have \(B_0 = 1\).

2.7. Constants of Steady State in 1D FFMD

Here we show the constants for the steady state in 1D FFMD. We assume that the force-free field is stationary and axisymmetric. First, the equation of energy conservation
(Equation (32)) reads the energy flux constant,

\[ P = \alpha \sqrt{\gamma} S'. \tag{36} \]

Here \( e^\infty = -\chi^r T^r = \alpha (\dot{a} + NS') \), \( S' = -\chi^r T^r = \frac{1}{\alpha} e^\infty B_c \), where \( \chi' = (1, 0, 0, 0) \) is the Killing vector with respect to the time boost symmetry and \( E_\phi = \alpha (E_\phi + \epsilon_{\phi \beta} N^B B^\beta) = \alpha (E_\phi + \sqrt{\gamma} N^B B^\beta), B_\phi = \alpha (B_\phi - \epsilon_{\phi \beta} N^B B^\beta) = \alpha (B_\phi + \sqrt{\gamma} N^E E^\phi). \) We have

\[ P = E_\phi B_\phi = \alpha^2 (E_\phi + \sqrt{\gamma} N^B B^\beta) (B_\phi + \sqrt{\gamma} N^E E^\phi). \tag{37} \]

Next, the vanishing right-hand side of Equation (29) provides the constant,

\[ \Omega_F = -\alpha (E_\phi - \sqrt{\gamma} N^B B^\beta + \sqrt{\gamma} N^E E^\phi). \tag{38} \]

Using the constant \( \sqrt{\gamma} B^\beta = 1 \) in the flux coordinates, we also have

\[ \Omega_F = -\alpha (E_\phi - \sqrt{\gamma} N^B B^\beta + \sqrt{\gamma} N^E E^\phi). \tag{39} \]

Here \( \Omega_F \) represents the angular velocity of the magnetic field line. Last, Equation (28) in the situation with \( \beta = -\sqrt{\gamma} N^E \) yields the constant

\[ I = \alpha (B_\phi + \sqrt{\gamma} N^E E^\phi - \sqrt{\gamma} N^E E^\phi). \tag{40} \]

Here \( I \) is the current inside of the magnetic surface of \( \Psi = \Psi_0 \); \( \Psi \leq \Psi_0 \).

Using Equations (39) and (40), we obtain the analytic solution of the steady-state force-free field with \( I \) and \( \Omega_F \),

\[ B^\beta = \frac{1}{\alpha \sqrt{\gamma} D} [ \sqrt{\gamma} I - \gamma \Omega_F - \alpha (\gamma_\mu + \gamma N^\gamma)], \tag{41} \]

\[ E_\phi = \frac{1}{\alpha D} [ \gamma_\phi \Omega_F - \sqrt{\gamma} N^\gamma I + \alpha (\gamma_\phi N^\gamma + \gamma N^\gamma)], \tag{42} \]

where \( D = \gamma N_\gamma ^\gamma \) and \( D = \gamma_\phi N_\phi + \gamma N^\gamma \). The \( I \) and \( \Omega_F \) of the steady state are given in Section 3. In the BL and KS coordinates, we have \( \gamma = 0 \) and \( \gamma_\phi = \gamma_\phi \) respectively. Equation (37) yields the relation

\[ P = -I \Omega_F. \tag{43} \]

In the BL coordinates, these constants become

\[ \Omega_F = \omega - \frac{\alpha E_\phi}{RB'}, I = \alpha RB_\phi. \tag{44} \]

In the KS coordinates, they are

\[ \Omega_F = -\frac{\alpha}{\sqrt{\gamma} B'} (E_\phi + \sqrt{\gamma} N^B B^\beta), I = \alpha (B_\phi + \sqrt{\gamma} N^E E^\phi). \tag{45} \]

2.8. Spacetime with Flux Coordinates for an Arbitrarily Given Magnetic Surface

To describe the spacetime around the spinning black hole with mass \( M \) and angular momentum \( J \), we use the two coordinate systems, BL and KS, defined below. Here we use \( a = (J/J_{max}) M \), where \( J_{max} = M^2 \) is the maximum angular momentum of a spinning black hole with mass \( M \). The metric of the BL coordinates \( x'^\mu = (t, r, \theta, \phi) \) is given by

\[ ds^2 = g_{\mu \nu} dx'^\mu dx'^\nu = -\left( 1 - \frac{2Mr}{\rho^2} \right) dt^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \sin^2 \theta d\phi^2 dt, \tag{46} \]

where \( \rho^2 = r^2 + a^2 \cos^2 \theta, \Delta = r^2 - 2Mr + a^2, \Sigma^2 = (\rho^2 + a^2)^2 - \Delta \sin^2 \theta, R^2 = \frac{\Sigma^2}{\rho^2} \sin^2 \theta \). The horizon is given by \( \Delta = 0 \), which yields \( r = M \pm \sqrt{M^2 - a^2} \equiv \eta_i \). In this coordinate system, \( \alpha = \frac{\Delta^2}{\Sigma^2}, \beta = \frac{2M}{\rho^2} \sin^2 \theta, g = -\rho^2 \sin^2 \theta \).

The metric of the KS coordinates \( x'^\mu = (t, r, \theta, \phi) \) is given by

\[ ds^2 = g_{\mu \nu} dx'^\mu dx'^\nu = -\left( 1 - \frac{2Mr}{\rho^2} \right) dt^2 + \left( 1 + \frac{2M}{\rho^2} \right) dr^2 + \rho^2 d\theta^2 + \rho^2 d\phi^2 \]

\[ + \frac{2Mr}{\rho^2} dr + \frac{2M}{\rho^2 \sin^2 \theta} d\phi^2 \tag{47} \]

At the horizon \( r = \eta_i \), all of the metrics of the KS coordinates are finite, while some metric of the BL coordinates, for example, \( g_{rr} \), is infinite. In the KS coordinates, we have \( \beta = \frac{2Mr}{\rho^2}, \beta = 0, \beta = 0, \gamma = \rho^2 (\rho^2 + 2Mr) \sin^2 \theta, g = -\rho^4 \sin^2 \theta \). The BL and KS coordinates are used for the base of the flux coordinates. In the cases of radial magnetic surfaces as shown in Figure 2, we directly use these coordinates as the flux coordinates.

It is noted that the BL \( x'^\mu_{\text{BL}} = (t_{\text{BL}}, r_{\text{BL}}, \theta_{\text{BL}}, \phi_{\text{BL}}) \) and KS \( x'^\mu_{\text{KS}} = (t_{\text{KS}}, r_{\text{KS}}, \theta_{\text{KS}}, \phi_{\text{KS}}) \) coordinates are related as

\[ d\psi_{\text{KS}} = d\psi_{\text{BL}} + \frac{2Mr}{\Delta} dr_{\text{BL}}, d\phi_{\text{KS}} = d\phi_{\text{BL}} + \frac{a}{\Delta} dr_{\text{BL}}, \]

\[ r_{\text{KS}} = r_{\text{BL}}, \theta_{\text{KS}} = \theta_{\text{BL}}. \tag{48} \]

Then we note that at the horizon and infinitely far from the black hole, the times on the KS and BL coordinates are infinitely different. That is, the finite time on the BL coordinates corresponds to the infinite past on the KS coordinates. Conversely, at an infinitely far point from the black hole, the finite time on the BL coordinates corresponds to the infinite future on the KS coordinates.

We introduce the flux coordinates \( x'^\mu = (t, r, \Psi, \phi) \) around an arbitrary axisymmetric magnetic surface \( \Psi = \Psi_0 \), where \( \Psi_0 \) is a constant, which is described by \( \theta = \theta_0(r) \), where \( \theta_0(r) \) is function of \( r \). When the magnetic surface \( \Psi = \Psi_0 \) is described by \( \theta = \theta(r, \Psi) \), the Taylor expansion of \( \theta(r, \Psi) \) with an infinitesimally small variable \( \Psi - \Psi_0 \) yields \( \theta(r, \Psi) = \theta(r, \Psi_0) + \left( \frac{\partial \theta}{\partial \Psi} \right)_{\Psi=\Psi_0} (\Psi - \Psi_0) + \cdots \). Then we have

\[ \Psi = \Psi_0 + b(r)(\theta - \theta_0(r)), \tag{49} \]
where \( b(r) = \left[ \left( \frac{\partial \theta(r, \psi)}{\partial r} \right)_{\theta=\psi=0} \right]^{-1} \) and \( \theta_0(r) = \theta(r, \Psi_0) \). The function \( b(r) \) determines the flared shape of the magnetic surfaces. The metric of the flux coordinates is given by the metric of KS coordinates \( x_{KS}^i \),

\[
\gamma_{rr} = \gamma_{rr}^{KS} + K^2 \gamma_{r0}^{KS}, \quad \psi \psi = \frac{1}{b^4} \gamma_{00}^{KS}, \quad \psi \phi = \gamma_{0\phi}^{KS}, \quad \gamma_{rr} = \gamma_{rr}^{KS} \gamma_{\phi\phi} = \gamma_{\phi\phi}^{KS},
\]

\[
\gamma_{\phi\phi} = K \gamma_{r0}^{KS}, \quad \gamma_{\phi\phi} = 0, \quad \gamma_{00} = \gamma_{r0}^{KS}, \quad \alpha^2 = \gamma_{r0}^{KS}, \quad \beta^r = \gamma_{r0}^{KS}, \quad \beta = b K \gamma_{r0}^{KS}, \quad \beta = 0,
\]

where \( K = \theta_0(r) - \frac{b(r)}{b(r)} \). We also have

\[
\gamma_{rr}^{KS}, \quad \gamma_{\phi\phi}^{KS}, \quad \gamma_{\phi\phi}^{KS} = b^2 (\gamma_{r0}^{KS} + K^2 \gamma_{rr}^{KS}), \quad \psi \phi = \gamma_{0\phi}^{KS}, \quad \gamma_{r0}^{KS} = b K \gamma_{r0}^{KS}, \quad \gamma_{r0}^{KS} = 0.
\]

3. Analytic Solution of a Steady-state Force-freeField along an Arbitrary Magnetic Surface

Here we derive the analytic solution of the steady-state force-free field along an arbitrary magnetic surface \( \Psi = \Psi_0 \) using the flux coordinates \( (r, \theta, \Psi, \phi) \). We derive the constants \( I \) and \( \Omega_F \) of the steady state along the magnetic surface \( \Psi = \Psi_0 \) which is indicated by \( \theta = \theta_0(r) \). Because the constants of the steady-state force-free field do not depend on the coordinates, we use the KS coordinates for the base coordinates without loss of generality. First, we derive the condition with respect to \( I \) and \( \Omega_F \) at the horizon, which is called the “Znajek condition” (Znajek 1977). Using Equation (41), we have

\[
B_\theta = \frac{I + (\gamma_{r0}^{KS} + \gamma_{r0}^{KS}) \Omega_F - \alpha \gamma_{r0}^{KS} B_\theta^2}{\alpha (\gamma_{00}^{KS} - (\gamma_{r0}^{KS} + \gamma_{r0}^{KS})^2 \gamma_{r0}^{KS})}.
\]

Because the magnetic surface is radial at the horizon, \( b \) in Equation (49) is a constant \( b_H \) around the horizon. We have \( \gamma_{\phi\phi} = b^2 \gamma_{r0}^{KS} \) and \( N_{\phi} = N_{r}^{KS} = 0 \) around the horizon. We obtain

\[
I = \frac{I + (\gamma_{r0}^{KS} + \gamma_{r0}^{KS}) \Omega_F - \alpha \gamma_{r0}^{KS} N_{\phi} B_\theta^2}{\alpha (\gamma_{00}^{KS} - (\gamma_{r0}^{KS} + \gamma_{r0}^{KS})^2 \gamma_{r0}^{KS})}.
\]

Because the denominator of Equation (53), \( \gamma_{00}^{KS} - (\gamma_{r0}^{KS} + \gamma_{r0}^{KS})^2 \gamma_{r0}^{KS} = \Delta \sin^2 \theta (1 + 2Mr/\rho^2) \), vanishes at the horizon, the continuity of \( B_\theta \) at the horizon yields the Znajek condition,

\[
I = \frac{1}{\rho_H^2} (2Mr \Omega_F - a) b_H \sin \theta_H,
\]

where \( \rho_H^2 = \frac{a^2}{\gamma_0^2} \). Here \( b_H \) and \( \theta_H \) are the values of \( b \) and \( \theta \) at the horizon, respectively.

Next, we derive the condition with respect to \( I \) and \( \Omega_F \) at infinity. The condition is given by the condition of outward propagation of the force-free electromagnetic wave,

\[
E_\phi = B_\phi^2,
\]

where \( \wedge \) indicates the orthonormal (proper) frame of the normal observer, in which the \( r \)-coordinate is parallel to the magnetic surface \( \Psi = \Psi_0 \). At infinity, the proper frame \( x^\phi \) is given by \( \hat{d}_r = \hat{d}_r, \quad \hat{d} \Psi = h_1(d\hat{r} + k d\hat{\Psi}), \quad \hat{d} \Psi = h_2(d\hat{r} + l d\hat{\Psi}), \) and \( d\hat{\phi} = h_3(d\hat{r} + m d\hat{\Psi}) \), where \( h_1 = \gamma_{rr} = 1 + K^2 \gamma_{r0}^{KS} - h_2^2 l^2, \quad h_2^2 k = \frac{\gamma_{rr}^2}{\gamma_{r0}^{KS}} - \gamma_{r0}^{KS}, \quad h_2^2 = \gamma_{r0}^{KS} = r^2 \sin^2 \theta, \quad h_2^2 = -a \sin^2 \theta, \) and \( h_3^2 = h_2^2 - h_2^2 k^2 \) at infinity. Using the proper frame at infinity, we have \( ds^2 = -d\hat{\tau}^2 + d\hat{r}^2 + d\hat{\Psi}^2 + d\hat{\phi}^2 \). At infinity, we have \( h_2^2 = 1 - K^2 \gamma_{r0}^{KS} + k \gamma_{r0}^{KS} \), and \( l = 0 \). The relationships between the variables of the proper frame and the normal observer frame,

\[
E_\phi = \frac{1}{h_2^2} (E_\phi - k E_r), \quad B_\phi = h_3 B^\phi,
\]

and Equation (55) yield

\[
I = -b_H \sin \theta_H \Omega_F
\]

at infinity, where \( b_H \sin \theta_H = \lim b \sin \theta \). Using Equations (54) and (57), we obtain the constant

\[
\Omega_F = \frac{a}{2M r_H + \rho_H^2 b_H \sin \theta_H / b_H \sin \theta_H}.
\]

The constant \( I \) is given by Equations (57) and (58) as

\[
I = - \frac{a b_H \sin \theta_H}{2M r_H + \rho_H^2 b_H \sin \theta_H / b_H \sin \theta_H}.
\]

Equations (58) and (59) are the generalization of the solution given by Menon & Dermer (2005). The electric field \( E_\phi \) and magnetic field \( B^\phi \) of the steady-state solution are given by Equations (41) and (42) with \( \Omega_F \) and \( I \) given by Equations (58) and (59), respectively, for the flux coordinates on the BL and KS coordinates.

In general, when the magnetic surface is radial \( (b_\infty = b_H, 0 < \theta_\infty = \theta_H < \pi) \), we have

\[
\Omega_F = \frac{a}{(r_H + 2M) r_H} = \frac{a}{4M r_H - a^2}.
\]

In the case of \( a < M \), Equation (60) yields the well-known relation \( \Omega_F = \frac{\omega_H}{2} \), where \( \omega_H = -\frac{\omega_0}{\omega_0} = \frac{a}{2Mr_0} \) is the angular velocity of the normal observer at the horizon.

3.1. Derivation of Two Blandford–Znajek Solutions with the Perturbation Method

To show the validity of the 1D FFMD analytic solution, we derive the analytic solutions of the steady-state force-free field around a extremely slowly spinning black hole \( (a < M) \) given by Blandford & Znajek (1977), Blandford & Znajek (1977) resorted to a perturbation method in which they expanded on the powers of \( a/M \). Such a technique can only be of use when the change in the poloidal field caused by spinning up a nonrotating field configuration (supported by currents in an equatorial disk) can be regarded as small. They wrote an exact axisymmetric vacuum solution for the magnetic field in a Schwarzschild metric by \( \psi(r, \theta) = X(r, \theta) \) used as the unperturbed function. According to the perturbation method, the perturbed variables, the electromagnetic angular frequency and the current \( I \) are written by \( \Omega_F = \frac{a}{M} W(r, \theta) \) and

\[
I = \frac{b}{\mathfrak{g}} Y(r, \theta),
\]

respectively. They showed two examples of the perturbation technique: (a) the radial magnetic field (of
opposite polarity in the two hemispheres) and (b) a force-free magnetosphere in which the magnetic field lines lie on paraboloidal surfaces (cutting an equatorial disk). We derive the expression of \( W \) and \( Y \) using the magnetic surfaces for both cases as follows.

### 3.1.1. The Case of the Radial Magnetic Surface

The vector potential
\[
\Psi(r, \theta) = X(r, \theta) = -C \cos \theta \ (0 \leq \theta \leq \pi/2)
\]
with a constant \( C \) is an exact solution of the vacuum Maxwell equations in a Schwarzschild metric, which describes the unperturbed radial magnetic field (Equation (6.1) in Blandford & Znajek 1977). Blandford & Znajek (1977) used this solution as the unperturbed vector potential \( \Psi \) of the split monopole field around an extremely slowly spinning black hole. It is noted that in the case of the split monopole, the solution is used except on an equatorial disk containing a toroidal surface current density \( I_{sm} = 2C/r^2 \). Equation (61) yields \( \frac{d}{d\Omega} \frac{d\Omega}{\Psi} = \frac{\Psi}{2\pi \sin \theta d\theta} \), \( b = bH \sin \thetaH = \frac{bH}{\sin \thetaH} \) because \( \Psi = -C \cos \thetaH = -C \cos \theta \) reads \( \thetaH = \theta \). Equation (58) yields \( \OmegaF = \frac{4\piCH}{a^2 \cos \thetaH} \). Using \( a \ll M \), we have \( \OmegaF = \frac{a}{8M^2} = \frac{d}{Y} \). Equation (59) yields \( I = -\frac{aC}{8M^2} \sin^2 \theta = \frac{a}{8M^2} \). Then we obtain
\[
W = \frac{1}{8}, \quad Y = -\frac{C}{8} \sin^2 \theta.
\]
These \( W \) and \( Y \) are the same expressions given by Equation (6.5) in Blandford & Znajek (1977) in the radial magnetic surface case.

The power per solid angle from the horizon is given by
\[
P_{BZ} = \frac{dL}{d\Omega} = \frac{2\pi d\Psi}{d\Omega} \frac{d\Omega}{2\pi d\Psi} = \frac{2\pi d\Psi}{2\pi \sin \theta d\theta} P = \frac{bH}{\sin \thetaH} P,
\]
where \( L \) is the power radiated by the black hole in the region between the pole and the magnetic surface \( \Psi \) and \( \Omega \) is the solid angle. We have
\[
P_{BZ} = \left( \frac{a^2 C}{8M^2} \right)^2 \sin^2 \theta.
\]

### 3.1.2. The Case of the Paraboloidal Magnetic Surface

The vector potential of the paraboloidal magnetic surface is given by Equation (7.1) in Blandford & Znajek (1977) as
\[
\Psi(r, \theta) = X(r, \theta) = \frac{1}{2}[r(1 - \cos \theta) + 2M(1 + \cos \theta)(1 - \sin(1 + \cos \theta))].
\]
Equation (65) yields \( b = \frac{1}{2}[\partial r(\Psi/\partial r) - 2C \sin \theta/r + 2M \ln(1 + \cos \theta)] \). With \( a \ll M \), we have
\[
\begin{align*}
\PsiH &= \frac{C}{2}(2M(1 - \cos \thetaH) + 2M(1 + \cos \thetaH)(1 - \ln(1 + \cos \thetaH))), \\
\Psi &= \frac{C}{2}(\frac{1}{2} \ln \theta \sin^2 \theta + 4M(1 - \ln 2)).
\end{align*}
\]
Using \( \PsiH = \Psi \), we get \( \frac{1}{2}r_\infty \theta \sin 1 \ln 1 \cos 1 \sin \theta \approx C \frac{1}{2} \ln \theta \sin^2 \theta. \)

Using Equations (58) and (59), we obtain
\[
\OmegaF = \frac{a}{M^2} = I = -\frac{a}{M^2} h \sin \theta,\quad N = \frac{a}{4} \sin^2 \thetaH[1 + \ln(1 + \cos \thetaH)], \quad D = 4 \ln 2 + \sin^2 \thetaH + \{\sin^2 \thetaH - 2(1 + \cos \thetaH) \} \ln(1 + \cos \thetaH).
\]
Then we have
\[
W = \frac{N}{D},
\]
\[
Y = 2MC[2 \ln 2 + (1 + \cos \thetaH) \ln(1 + \cos \thetaH)] \left(\frac{N}{D}\right).
\]

The expression \( W \) is identified with the perturbed solution given by Equation (7.5) in Blandford & Znajek (1977).

Using Equation (63), we obtain the power per unit solid angle,
\[
P_{BZ} = 2 \left( \frac{a}{M} \right)^2 C^2 \left[1 + \ln(1 + \cos \thetaH)\right] \left(\frac{N}{D}\right). \quad (67)
\]

### 4. Test Numerical Calculations of 1D FFMD on the Magnetic Surface around the Equatorial Plane

#### 4.1. 1D FFMD Equations and Flux Coordinates for the Magnetic Surface along the Equatorial Plane for Numerical Simulations

We show the 1D FFMD equations for different types of magnetic surfaces along the equatorial plane for numerical calculations. In the KS coordinates, using \( x^\mu = (t, r, \theta, \phi) \), we have the Maxwell equation, and the conservation law of momentum at the equatorial plane as plane:}
\[
\frac{\partial}{\partial t} B^\phi = -\frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial r} [\alpha \sqrt{\gamma}(E_\theta + \sqrt{\gamma} N^\gamma B^\phi)],
\]
\[
\frac{\partial}{\partial t} S_\phi = -\frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial r} [\alpha \sqrt{\gamma}(T^*_T + N^\gamma S_\phi)] - \frac{\partial}{\partial r} (\gamma u),
\]
\[
\frac{\partial}{\partial t} S_\phi = -\sqrt{\gamma} \frac{\partial}{\partial r} [\alpha \sqrt{\gamma}(T^*_T + N^\gamma S_\phi)].
\]

Here we use \( N^0 = 0 \) and \( N^\phi = 0 \) at the equatorial plane in the KS coordinates.

In the BL coordinates, we have \( N^r = 0, N^\theta = 0, \alpha N^\phi = \omega, \) and \( g_{ij} = h_i h_j \). We use the ZAMO frame \( x^i \) easily and obtain the 3+1 formalism equation with the ZAMO frame along the equatorial plane as follows:
\[
\frac{\partial}{\partial t} B^\phi = -\frac{1}{h_i h_j} \frac{\partial}{\partial r} [h_\theta(\alpha E_\theta - h_\phi \omega B^\phi)].
\]
follows:

\[ \frac{\partial}{\partial t} S_\phi = -\frac{1}{2\alpha^2 \sqrt{T^j}} \frac{\partial}{\partial r} \left[ \frac{\alpha^2 \sqrt{T^j}}{T^r} \right] - \frac{1}{h_r} \frac{\partial}{\partial r} \left( \alpha - T^r \right) \]

\[ \frac{\partial}{\partial t} S_\phi = -\frac{1}{2\alpha^2 \sqrt{T^j}} \frac{\partial}{\partial r} \left[ \frac{\alpha^2 \sqrt{T^j}}{T^r} \right], \]  

(72)

(73)

where \( \sigma = (h_\phi/h_r)\partial (\alpha N^r) / \partial r \). For the numerical calculation of 1D FFMD, we use Equations (68)–(70) and (71)–(73) for the numerical calculations in the KS and BL coordinates, respectively.

The flux coordinates for the three types of magnetic surfaces along the equatorial plane as shown in Figure 3 are given as follows:

\[ \theta_0(r) = \frac{\pi}{2}, \]

\[ b(r) = \begin{cases} [1 + 0.25(r - r_H)^3]m & (r > r_H) \\ 1 & (r \leq r_H). \end{cases} \]  

(74)

The incurvature-flared, radial, and excurrence-flared magnetic surfaces are given by Equation (74) with \( m > 0, m = 0, \) and \( m < 0, \) respectively.

### 4.2. Numerical Method

We use the Lax–Wendroff scheme and the simplified total variation diminishing scheme (Davis 1984) for the 1D FFMD numerical calculations. The mesh number of the standard case is 1000 (and 10,000, if necessary). In the KS coordinates, we use the even interval mesh with the range \( r_{\text{min}} \leq r \leq r_{\text{max}} \) \( (r_{\text{min}} = 0.9r_H, r_{\text{max}} = (40 \sim 50)r_H). \)

In the BL coordinates, we use the even interval mesh on the (pseudo-)tortoise coordinate for the radial coordinate \( r_a \) such as \( dr_a \propto \frac{1}{r - r_{\text{min}}} dr \) \( (r_0 < r_{\text{min}}) \) is a constant, and we typically set \( r_0 = 1.9999 \) for the \( a = 0.01M \) case. When \( r_0 = r_H, \) the
coordinate \( r_* \) becomes the exact tortoise coordinate. We use the equations of the Faraday law and the conservation equation of energy flux \( S_i (i = r, \phi) \).

4.3. Test Numerical Calculations of 1D FFMD along the Equatorial Plane

Here we show some test calculations of 1D FFMD simulations for variously shaped magnetic surfaces around the equatorial plane, as shown in Figure 3. In the case of the radial magnetic surface, we can use the BL or KS coordinates as the flux coordinates. The 1D FFMD calculations along the equatorial plane are listed in Table 1.

4.3.1. Test Calculations of the Blandford–Znajek Solution

First we calculate the Blandford–Znajek solution on the monopole radial magnetic field as a test calculation of our numerical code in the BL and KS coordinates. We use the Blandford–Znajek solutions for the initial condition of time-development numerical simulations to check the 1D FFMD code due to the solutions being stationary. Note that we assume that the spin parameter of the black hole \( a_* \equiv a/M \) is much smaller than unity \((a_* = 0.01)\). In the BL coordinates, the solution is given by

\[
B^r = \frac{B_0}{h_0 R} [1 - a_*^2 f(r)] \approx \frac{B_0}{h_0 R},
\]

\[
B^\theta = 0,
\]

\[
B^\phi = - \frac{a B_0}{8 \alpha R M^2}, \quad \Omega_r = \frac{R}{\alpha (\Omega_F - \omega) B^\phi}, \quad E_r = E^\phi = 0, \tag{75}
\]

where \( \Omega_F = a/(8M^2) \) and \( f(r) \) is a complex function, which is given in Section 6 of Blandford & Znajek (1977). Here we use the flux coordinates \((t, r, \Psi, \phi) = (t, r, \theta, \phi)\); thus, we must set \( B_0 = 1 \). In the Blandford–Znajek solution, the slowly spinning black hole limit condition \((a_* \ll 1)\) is used, and we neglect the term of \( f(r) \).

In the KS coordinates, the Blandford–Znajek solution is given by

\[
\tilde{B}^r = \frac{B_0}{\sqrt{\gamma}}, \quad B^\theta = 0,
\]

\[
B^\phi = - \frac{a \alpha B_0}{2 M r} \left( 1 + \frac{r}{4M} \right), \tag{77}
\]

\[
E_\delta = - \frac{a B_0}{8 M^2 \alpha} - \sqrt{\gamma N} B^\phi, \quad E_r = E^\phi = 0. \tag{78}
\]

In both the BL and KS coordinates, the numerical results demonstrate that the solutions are stationary, as shown in Figures 4 and 5, respectively.

4.3.2. Test Calculation of the Electromagnetic Field in a Vacuum

In the force-free condition, we have the solution of the electromagnetic field in a vacuum. It is obtained from the Maxwell equations with \( \rho = 0 \) and \( J = 0 \). In the BL coordinates, we have

\[
B^r = \frac{B_0}{h_0 R}, \quad B^\theta = 0, \quad B^\phi = 0, \tag{79}
\]

\[
E_\delta = h_0 N^\delta B^r = \frac{R \omega}{\alpha} B^\phi, \quad E_r = E^\phi = 0. \tag{80}
\]

In the KS coordinates, the solution of the electromagnetic field in a vacuum is given by

\[
B^r = \frac{B_0}{\sqrt{\gamma}}, \quad B^\theta = 0, \quad B^\phi = - \frac{a B_0}{\Delta(1 - 2M/r)r^2}. \tag{81}
\]
Note that \( B^\phi \) diverges at the horizon in the KS coordinates, whose metrics are all finite. This indicates that divergence of the solution is physical, and we cannot use this solution.

We perform a test calculation with the vacuum solution in the BL coordinates, as shown in Figure 6. The result shows that \( I, P, \) and \( \Omega_F \) are negligibly small, and, as expected, the solution is nearly stationary. The numerical calculations in Sections 4.3.1 and 4.3.2 indicate the reliability of the 1D FFMD code.

### 4.3.3. Test Calculations of the Dynamic Process toward the Blandford–Znajek Solution with Small \( a_* \)

Here we perform numerical test calculations for the emergence of the Blandford–Znajek mechanism around a slowly spinning black hole of \( a_* = 0.01 \) with both BL and KS
coordinates. Note that the initial condition $I = 0$ ($P = 0$) is used in the cases of both the BL and KS coordinates. First, we show the result of the BL coordinates. When $I = 0$ ($P = 0$) in the BL coordinates of $a_\ast = 0.01$, we have $B^\phi = 0$. Figure 7 shows that no quantity ever changes at the horizon due to the time-freezing at the horizon. A wave is caused around the ergosphere, and the front propagates outward like a “tsunami” and inward to the horizon. The constants of the stationary state in the wave are realized as the Blandford–Znajek solution with $\Omega_\nu = \omega_H/2 = a_\ast/8M = 0.00125$. The energy flux is provided from the front of the inward wave near the horizon. In the front of the inward wave, the energy at infinity becomes negative. To observe this effect, we check the above statement near the black hole, as shown in Figure 8. We find that the profile of $P$
at $t = 4M$ has its maximum value at $r = 2.2M$, and its value at the horizon is zero. Comparing the profile of $P$ at $t = 4M$ and $8M$ indicates that the profile spreads both outward and inward. The outward spread is shown in Figure 7. Note that the inward spread continues to approach to the horizon but never actually reaches it. Then the energy flux emerges from the ergosphere. As shown in Figure 7, the energy at infinity $e^\infty$ becomes negative at $t = 40M$.

In the KS coordinates, we have the initial condition $I = 0$ ($P = 0$) and $E_\hat{3} = 0$. The results are shown in Figure 9. The quantities at the horizon rapidly converge to the values of the Blandford–Znajek solution, and the region similar to the Blandford–Znajek solution spreads outward. It appears that the horizon behaves like a boundary, while “causality” prohibits outward propagation of the information through the horizon. This result also demonstrates that the outward energy flux at the horizon increases spontaneously to reach the value of the Blandford–Znajek solution even in the case of initially no current and energy flux.

### 4.3.4. Test Calculations of the Blandford–Znajek Mechanism with Finite $a_*$

We perform 1D FFMD numerical simulations with the steady-state solution for finite $a_*$ at the equatorial plane in the flux coordinates based on the BL and KS coordinates. In the force-free steady state, we have constants $I$, $P$, and $\Omega_F$, and the azimuthal magnetic field component and colatitude electric...
field component are given by Equations (41) and (42), respectively.

When the magnetic surface locates at the equatorial plane ($\theta_0 (r) = \pi/2$ and $K = 0$), using Equations (58) and (59) with $\theta_H = \theta_\infty = \pi/2$, we obtain the following constants:

$$\Omega_H = \frac{a}{(b_\infty/b_H) r_H + 2M r_H},$$  \hspace{1cm} (83)

Figure 11. Simulation with steady-state solution with Equations (83) and (84) for $a*=0.95$. Dashed lines show quantities at $t = 0$, dotted lines show results at $t = 5M$, and solid lines show results at $t = 10M$. Dashed, dotted, and solid lines overlap in this case.

Figure 12. Simulation of the propagation of the pulse of the magnetic field with $A = 10^{-3}$ of Equation (85) at $r = 10$ outside the ergosphere. The background field is given by the steady-state solution with Equations (83) and (84) ($b_\infty/b_H = 1$ (radial magnetic surface), $a_\infty = 0.95$). Dashed lines show quantities at $t = 0$, dotted lines show results at $t = 5M$, and solid lines show results at $t = 10M$. 

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Figures 10 and 11 show the simulation results with the steady-state solution (Equations (83) and (84)) for the case of a radial magnetic surface ($b_\infty = b_H$), where the flux coordinates are the BL and KS coordinates, respectively, for $a_*=0.95$. Here, the horizon locates at $r=1.31M$. This clearly confirms that the solution gives the steady state of the electromagnetic field with the outward power predicted by $P = \Omega F = 0.048$, where we use the unit system such that $b_H = 1$. Hereafter, we use the unit system with $b_H = 1$.

\begin{equation}
I = -\frac{ab_\infty}{((b_\infty/b_H)\Omega_H + 2M)\Omega_H}.
\end{equation}

Figure 13. Simulation of the propagation of the pulse of the magnetic field with $A = 3 \times 10^{-2}$ of Equation (85) at $r = 1.8$ inside the ergosphere. The background field is given by the steady-state solution with Equations (83) and (84) ($b_\infty/b_H = 1$ (radial magnetic surface), $a_* = 0.95$). Dashed lines show quantities at $t = 0$, dotted lines show results at $t = 1M$, and solid lines show results at $t = 2M$.

Figure 14. Simulation of the propagation of the pulse of the magnetic field with $A = 10^{-2}$ of Equation (85) at $r = 0.9$ inside the horizon. The background field is given by the steady-state solution with Equations (83) and (84) ($b_\infty/b_H = 1$ (radial magnetic surface), $a_* = 0.95$). Dashed lines show quantities at $t = 0$, dotted lines show results at $t = 0.1M$, and solid lines show results at $t = 0.2M$. 

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4.3.5. Test Calculations of Pulse Propagation on the Blandford–Znajek Solution with Finite $a_*$

To investigate causality, we follow the propagation of the pulse initially added to the steady-state field around a rapidly spinning black hole ($a_*=0.95$). Here we initially add the perturbation of the pulse to the azimuthal component of the magnetic field. The additional azimuthal component of the magnetic field of the pulse to the stationary fields at $r=r_0$ with width $\delta$ and amplitude $A$ is given by

$$B^\phi_{\text{pul}} = \begin{cases} \frac{A}{2} \left( 1 + \cos \pi \frac{r-r_0}{\delta/2} \right) & \left( r_0 - \frac{\delta}{2} \leq r \leq r_0 + \frac{\delta}{2} \right), \\ 0 & \left( r < r_0 - \frac{\delta}{2}, r > r_0 + \frac{\delta}{2} \right) \end{cases}$$

(85)

Figure 15. Simulation of FFMD with zero power and current (initial condition: $P=0, I=0$) as initial conditions in terms of the BL coordinates ($a_*=0.95$). Dashed lines show quantities at $t=0$, dotted lines show results at $t=20M$, and solid lines show results at $t=40M$.

Figure 16. Enlarged plot near the horizon at an early stage of the simulation (initial conditions: $P=0, I=0$) of Figure 7 in terms of the BL coordinates ($a_*=0.95$). Dashed lines show quantities at $t=0$, dotted lines show results at $t=4M$, and solid lines show results at $t=8M$.

4.3.5. Test Calculations of Pulse Propagation on the Blandford–Znajek Solution with Finite $a_*$

Figure 12 shows the propagation of the pulse initially at $r=10M$ on the background steady-state electromagnetic field as obtained by Equations (83) and (84) for $a_*=0.95$. Two pulses propagate outward and inward with velocity 0.7 and 1.2, respectively. This demonstrates that the information can propagate both inward and outward at nearly the speed of light outside of the ergosphere.

Figure 13 shows the propagation of the pulse initially located at $r=1.8M$ inside the ergosphere, where the background is the same as that in Figure 11. As shown, two pulses also propagate outward and inward; however, the inward pulse runs fast, and the outward pulse propagates very slowly.

Figure 14 shows the propagation of the pulse initially located at $r=0.95M$ inside the horizon, where the background is the same as that in Figure 11. Here we find two pulses at $t=0.1M$ (dotted line). One pulse at $r=0.82M$ propagates inward rapidly. At $t=0.2M$, this pulse passes through the horizon. The pulse at $r=0.945$ propagates inward very slowly and...
Figure 17. Simulation of the case with initial conditions of $B^0 = 0$, $E_\theta = 0$ along the radial magnetic surface at the equatorial plane around a rapidly spinning black hole ($a_*=0.95$). Dashed lines show quantities at $t = 0$, dotted lines show results at $t = 30M$, and solid lines show results at $t = 60M$.

Figure 18. Simulation of the case with initial conditions of a steady-state solution of the force-free electromagnetic field along the incurvature-flared magnetic surface at the equatorial plane ($m = 0.25$) around a rapidly spinning black hole ($a_*=0.95$). Dashed lines show quantities at $t = 0$, dotted lines show results at $t = 5M$, and solid lines show results at $t = 10M$. Dashed, dotted, and solid lines overlap in this case.
appears to be nearly at rest. This confirms that the energy can be transported outward even inside of the horizon, but information is never transported outward. This result is consistent with the notion of causality around a black hole.

4.3.6. Test Calculations of the Emerging Blandford–Znajek Process with Finite $a_*$

To investigate the emergence process of the Blandford–Znajek mechanism, we run several simulations with the KS coordinates and observe the time-dependent process of the force-free field inside, at, and outside the horizon. To find the details of the process between the static limit and the horizon, we put the spin parameter of the black hole as $a_* = 0.95$. We also perform simulations with the BL coordinates to investigate the dependence on the coordinate systems. Here we use the initial condition with $B^\phi = 0$ and $E_b = 0$ around the spinning black hole.

Figure 15 shows the time development of the power $P$, current $I$, and angular velocity of the magnetic field lines $\Omega_F$ on the BL coordinates. The power and angular velocity of the magnetic field lines are zero at the initial condition. The results obtained at $t = 20M$ (dotted line) and $t = 40M$ (solid line) show that finite regions of $P$ and $\Omega_F$ spread outward at the speed of light. Note that $P$ and $\Omega_F$ converge to the values expected by the steady state, i.e., $P = \Omega_F^0 = 0.048$ with Equations (83) and (84). Then it seems that the steady-state outward energy flux is supplied in the vicinity of the black hole. Figure 16 shows an enlargement of Figure 15 near the horizon. Here we observe inward propagation of the tsunami of $P$ and $\Omega_F$ in the BL coordinates, which is caused by a time lapse of the coordinates near the horizon.

Figure 17 shows the time development of $P$, $I$, and $\Omega_F$ on the KS coordinates. The power and angular velocity of the magnetic field lines are zero in the initial condition. The results at $t = 30M$ (dotted line) and $t = 60M$ (solid line) show that finite regions of $P$ and $\Omega_F$ spread outward at the speed of light. Note that $P$ and $\Omega_F$ converge to the values expected by the steady state, i.e., $P = \Omega_F^0 = 0.048$ with Equations (84) and (83). Then it seems that the steady-state outward energy flux comes from the vicinity of the black hole.

4.3.7. Test Calculations of the Emergence of the Blandford–Znajek Mechanism along the Incurvature-flared Magnetic Surface at the Equatorial Plane

Here we show a numerical calculation with an incurvature-flared magnetic surface given by Equation (74) with $m = 0.25$ imaged in Figure 3(c).

Figure 18 shows the steady-state solution of $P$, $I$, and $\Omega_F$ along an incurvature-flared magnetic surface at the equatorial plane around a spinning black hole with $a_* = 0.95$.

Figure 19 shows the time development of the $P$, $I$, and $\Omega_F$ of the force-free electromagnetic field along an incurvature-flared magnetic surface at the equatorial plane around a spinning black hole with $a_* = 0.95$. Here the horizontal solid red lines
show the steady-state solution. In the incurvature-flared magnetic surface case, the electromagnetic field tends to converge very gradually to the steady state (details are provided by Imamura & Koide 2019).

4.3.8. Test Calculations of the Emergence of the Blandford–Znajek Mechanism along the Excurvature-flared Magnetic Surface at the Equatorial Plane

Here we show a numerical calculation with an excurvature-flared magnetic surface given by Equation (74) with \( m = -0.25 \) imaged in Figure 3(b). Figure 20 shows the time development of the \( P, I, \) and \( \Omega_F \) of the force-free electromagnetic field along an excurvature-flared magnetic surface at the equatorial plane around a spinning black hole with \( \alpha_* = 0.95 \) and \( a^* = 0.95 \). The horizontal solid red lines show the steady-state solution. In the excurvature-flared magnetic surface case, the electromagnetic field does not appear to converge to the steady state (details are given by Imamura & Koide 2019).

5. Summary

We have reviewed the basic method of 1D FFMD and showed the analytic solution of the steady-state force-free field for an arbitrary magnetic surface with Equations (41), (42), (57), and (58). To confirm the validity of the analytic solution, we derived two analytic solutions given by Blandford & Znajek (1977) using Equations (58) and (59). We have demonstrated test numerical calculations of 1D FFMD for three types of magnetic surfaces along the equatorial plane (in Table 1). The numerical simulations of 1D FFMD are used to investigate the mechanism of energy extraction from a spinning black hole via the magnetic field and transport of the extracted energy toward infinity (Koide & Imamura 2018; Imamura & Koide 2019). First, in the case of a slowly spinning black hole with \( a^* \ll 1 \), we have confirmed the analytic solution for the radial magnetic surface of a magnetic monopole derived by Blandford & Znajek (1977) with both the BL and KS coordinates. We have also performed simulations of nonstationary fields on the magnetic surface along the equatorial plane around rapidly spinning black holes. The electromagnetic field approaches those of the Blandford–Znajek solution spontaneously, except for the following excurvature-flared magnetic surface cases.

With the radial magnetic surface in the KS coordinates, under any initial conditions, the quantities at the horizon converge rapidly to the value given by the Blandford–Znajek solution, and the region of the Blandford–Znajek solution spreads toward infinity. Conversely, with the radial magnetic surface in the BL coordinates, except for the case of the initial condition of the Znajek condition at the horizon, no energy radiates at the horizon due to time-freezing at the horizon. The tsunami-like wave spreads outward to infinity and inward to the horizon. In the tsunami-like wave region, the Poynting flux is directed outward. At the front of the wave near the horizon, the energy at infinity becomes significantly negative to provide energy to

Figure 20. Simulation of the case with initial conditions of \( \tilde{B} = 0, \tilde{E}_r = 0 \) along the excurvature-flared magnetic surface at the equatorial plane (\( m = -0.25 \)) around a rapidly spinning black hole (\( \alpha_* = 0.95 \)). Dashed lines show quantities at \( t = 0 \). Horizontal solid red lines show static solutions. The top panels show results at early stages (dotted lines: \( t = 100M \); solid lines: \( t = 200M \)). The bottom panels show results at late stages (dotted lines: \( t = 1000M \); solid lines: \( t = 2000M \)).
the tsunami-like outward wave. In other words, the energy source of the outward Poynting flow is the negative energy region at the front of the inward wave near the horizon. This clearly demonstrates the difference between the results obtained in the BL and KS coordinates. The numerical results of the BL coordinates become the same as those of the KS coordinates distant from but not near the horizon. This difference is reasonable because the time is different between the BL and KS coordinates at the horizon and infinitely distant from the horizon (Equation (48)). In other words, the initial condition of the BL coordinate frames at the horizon corresponds to the condition at infinite past in the KS coordinates. In other cases with incurvature- and excurvature-flared magnetic surfaces, the regions of finite power density \((P > 0)\), where \(P\) is initially zero, spread gradually outward. In the incurvature-flared magnetic surface case, the electromagnetic field state \((\Omega_e, I, P)\) converges to the expected solution of the steady-state solution after a very long period \((t = 1000M)\). In the excurvature-flared magnetic surface case, the electromagnetic field state \((\Omega_e, I, P)\) appears not to converge to the steady state. Here the steady state is never achieved. Detailed analyses of incurvature- and excurvature-flared magnetic surfaces have been performed previously by Imanura & Koide (2019).

The results shown in Figures 11–14 provide important insight into the causality relative to the Blandford–Znajek mechanism. These results suggest that there are two kinds of energy flux with the information. One kind of energy flux can bring information in the direction of the outward energy flux, similar to a pulse outside the horizon. The other kind of energy flux never brings information in the direction of the outward energy flux, such as a pulse inside the horizon. This appears to explain causality in the Blandford–Znajek mechanism.

In conclusion, we have reviewed the basic method of 1D FFMD, given the analytic solutions of the steady-state force-free field for an arbitrary magnetic surface, and demonstrated test calculations of 1D FFMD with three types of magnetic surfaces at the equatorial plane. Using the analytic solutions of 1D FFMD, we may be able to analytically solve the Grad–Shafranov equation relative to \(\Psi(r, \theta)\) around a spinning black hole. This may become a remarkable challenge. We expect that 1D FFMD will be employed frequently as a standard tool in various fields.

S.K. thanks Mika Koide for her useful comments on this paper. We also thank Masaki Takahashi and Fumio Takahara for their plentiful discussion and suggestions on this study.

**Appendix A**

### 3+1 Formalism of Energy and Momentum Conservation Laws in 1D FFMD

We show the derivations of the 3+1 formalism of conservation laws of energy and momentum in FFMD (Equations (20) and (21)). The world line of the normal observer frame is perpendicular to the hypersurface of constant time in spacetime. The four-velocity of the normal observer frame is \(N^\mu = (1/\alpha, -\beta/\alpha, 0)\). The projection operator to a space-like hypersurface is written by \(P^\mu_\nu = S^\mu_\nu + N^\mu N^\nu\). The temporal component of an arbitrary vector \(F^\mu\) observed by the normal observer frame is

\[
\tilde{F}^t = F^\mu N_\mu.
\]

The projection of the vector \(F^\mu\) to the space-like hypersurface

\[
\tilde{F}^\mu = P^\mu_\nu F^\nu,
\]

and \(\tilde{F}^t\) produces \(F^\mu\) as

\[
F^\mu = \tilde{F}^\mu + \tilde{F}^t N^\mu.
\]

Components of the vector measured by the normal observer frame are written by \(F^\mu = (F^t, \tilde{F}^\mu)\). With respect to an arbitrary tensor \(T^\mu_\nu\), using

\[
\tilde{T}^a_\mu = T^a_\nu N_\nu, \quad \tilde{T}^t_\mu = T^t_\nu N_\nu P^\nu_\mu,
\]

we have the expansion form

\[
T^\mu_\nu = \tilde{T}^a_\mu N^a_\nu + \tilde{T}^t_\mu N^t_\nu + \tilde{T}^t_\nu N^t_\mu + \tilde{T}^t_\nu P^\nu_\mu.
\]

When \(T^\mu_\nu\) is a symmetric tensor, we have \(\tilde{T}^a_\mu = T^a_\mu\). When \(T^\mu_\nu\) is an antisymmetric tensor, we have \(\tilde{T}^a_\mu = 0\) and \(\tilde{T}^t_\mu = -\tilde{T}^t_\mu\). Components of the tensor measured by the normal observer frame are written by \(\tilde{F}^\mu = \tilde{T}^t_\mu + \tilde{T}^1_\mu, \quad \tilde{T}^t_\mu = \tilde{T}^t_\mu, \quad \tilde{T}^t_\mu = \tilde{T}^1_\mu, \quad \tilde{T}^t_\mu = \tilde{T}^t_\mu\).

Applying the above relations to the electromagnetic field tensor \(F^\mu_\nu\), its dual tensor \(*F^\mu_\nu\), and current density \(J^\mu\), we have

\[
F^{\mu t} = \frac{1}{\alpha} F^{\mu \tilde{t}} = \frac{1}{\alpha} E^\mu,
\]

\[
F^{t \mu} = F^{t \mu} + N^{t \nu} F^{\nu 0} - N^{\nu 0} F^{\nu t} = \epsilon^{\mu \nu \rho} (B_\nu + \epsilon_{\kappa \mu \nu} N^\mu E^\rho),
\]

\[
* F^{\mu t} = \frac{1}{\alpha} * F^{\mu \tilde{t}} = \frac{1}{\alpha} B^\mu,
\]

\[* F^{t \mu} = - \epsilon^{\mu \nu \rho} (E_\nu - \epsilon_{\kappa \rho \nu} N^\nu B^\rho),
\]

\[
J^t = \frac{1}{\alpha} J^t = \frac{1}{\alpha} \tilde{J}^\mu,
\]

\[
J^\mu = J^\mu + \epsilon^{\alpha \mu \nu} p_\nu.
\]

Here we use \(\epsilon^{\mu \nu \rho} = -N_\rho \epsilon^{\nu \rho \mu} = \frac{\epsilon^{\mu \nu \rho}}{\sqrt{\gamma}}\).

With respect to a symmetric energy-momentum tensor \(T^\mu_\nu\), the 3+1 formalism of the energy-momentum Equations (4) and (5), \(\nabla_\mu T^\mu_\nu = F^{\nu t}\) is much more complicated compared with that of an antisymmetric tensor of electromagnetic field \(F^{\mu t}\). When we use \(\tilde{a} = \tilde{T}^t_\mu, \quad \tilde{S}^t_\mu = \tilde{T}^t_\mu + \tilde{T}^1_\mu, \quad \tilde{T}^t_i = \tilde{T}^i, \quad \tilde{T}^1_i = \tilde{T}^i, \quad \tilde{T}^t_\mu = \tilde{T}^{\mu t}\), we obtain

\[
\alpha F^t = \frac{\partial \tilde{a}}{\partial t} + \frac{1}{\sqrt{\gamma}} \partial_t [\sqrt{\gamma} \alpha (S^t - N^t \tilde{a})] + (\partial_\alpha) S^t - \alpha K_{ij} T^t_j,
\]

\[
\alpha F_i = \frac{\partial S_i}{\partial t} + \frac{1}{\sqrt{\gamma}} \partial_t [\sqrt{\gamma} \alpha (T^t_j - N^t S_j)] + \tilde{a} \partial_t \alpha - (\partial_\alpha N^i) S_j - \frac{1}{2} \alpha (\partial_\gamma S_j) T^t j,
\]

where \(K_{ij} = -T^{\mu t} P_i^\mu P_j^t - T^{\mu t} N^t_{(i} N^t_{j)} = -(1/2\alpha)(\partial_\gamma S_i) + \gamma_{ij} (\alpha N^k)_{(i} + \alpha N^k (\partial_\gamma S_j)_{j)}\) is the external curvature of the spacetime. In the case of the force-free field, setting \(F^{\mu t} = 0\), we have

\[
\frac{\partial \tilde{a}}{\partial t} - \frac{1}{\sqrt{\gamma}} \partial_t [\sqrt{\gamma} \alpha (S^t - N^t \tilde{a})] - (\partial_\alpha) S^t + \alpha K_{ij} T^t j,
\]
\[
\frac{\partial S_i}{\partial t} = -\frac{1}{\sqrt{\gamma}}\partial_j[\sqrt{\gamma} \alpha(T^j_i - N^i S_j)] - \tilde{u}\partial_i \alpha
+ (\partial\alpha N^j) S_j + \frac{1}{2} \alpha (\partial\gamma_{jk}) T^{j k}.
\]

Here we used \( N_{
u} N^\nu = \partial_j (\ln \alpha) \) and \( N_{\alpha} N^\nu = -\alpha N^i \partial_i (\ln \alpha) \), where the semicolon indicates the covariant derivative.

**Appendix B**

**Non-1D Equations of FFMD**

Here we list non-1D equations of FFMD. Non-1D equations include the \( \Psi \)-derivative of the electromagnetic field variables, and information on the electromagnetic field of the neighbor magnetic surface is required to use the non-1D equations.

The non-1D equation from Equation (17) is
\[
\bar{p}_c = \frac{1}{\sqrt{\gamma}} \left[ \frac{\partial}{\partial r} (\sqrt{\gamma} E^i) + \frac{\partial}{\partial \Psi} (\sqrt{\gamma} E^\psi) \right].
\]

The non-1D equations from Equation (18) are
\[
\frac{\partial}{\partial t} E^i + \alpha (J^i + \bar{p}_c N^i) = \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial r} \left[ \alpha (B^i + \sqrt{\gamma} (N^i E^\psi - N^\psi E^i)) \right],
\]
\[
\frac{\partial}{\partial t} \bar{E}^i + \alpha (J^i + \bar{p}_c N^i) = \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial r} \left[ \alpha (B^i + \sqrt{\gamma} (N^i E^\psi - N^\psi E^i)) \right]
- \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial \Psi} \left[ \alpha (B^\psi + \sqrt{\gamma} (N^\psi E^i - N^i E^\psi)) \right].
\]

The non-1D equation of Equation (20) is
\[
\frac{\partial u}{\partial t} = -\frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial r} \left[ \alpha \sqrt{\gamma} (S^i + N^i \bar{u}) \right]
- \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial \Psi} \left[ \alpha \sqrt{\gamma} (S^\psi + N^\psi \bar{u}) \right]
- \frac{\partial \alpha}{\partial r} S^i - \frac{\partial \alpha}{\partial \Psi} S^\psi \alpha K_{ij} T^{j i}.
\]

Last, the non-1D equation from Equation (21) is
\[
\frac{\partial}{\partial t} S_\psi = -\frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial r} [\alpha \sqrt{\gamma} (T^i_\psi + N^i S_\psi)]
- \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial \Psi} [\alpha \sqrt{\gamma} (T^\psi_\psi + N^\psi S_\psi)]
- \bar{u} \frac{\partial \alpha}{\partial \Psi} \alpha N^j S_j + \frac{1}{2} \alpha \partial_{\gamma_{jk}} T^{j k}.
\]