ON THE GLOBAL EXISTENCE AND STABILITY OF A THREE-DIMENSIONAL SUPERSONIC CONIC SHOCK WAVE

Li, Jun\(^1\)^{*}; Witt, Ingo\(^2\)^{**}; Yin, Huicheng\(^1\)^{*}

1. Department of Mathematics and IMS, Nanjing University, Nanjing 210093, P.R. China.
2. Mathematical Institute, University of Göttingen, Bunsenstr. 3-5, D-37073 Göttingen, Germany.

Abstract

We establish the global existence and stability of a three-dimensional supersonic conic shock wave for a perturbed steady supersonic flow past an infinitely long circular cone with a sharp angle. The flow is described by a 3-D steady potential equation, which is multi-dimensional, quasilinear, and hyperbolic with respect to the supersonic direction. Making use of the geometric properties of the pointed shock surface together with the Rankine-Hugoniot conditions on the conic shock surface and the boundary condition on the surface of the cone, we obtain a global uniform weighted energy estimate for the nonlinear problem by finding an appropriate multiplier and establishing a new Hardy-type inequality on the shock surface. Based on this, we prove that a multi-dimensional conic shock attached to the vertex of the cone exists globally when the Mach number of the incoming supersonic flow is sufficiently large. Moreover, the asymptotic behavior of the 3-D supersonic conic shock solution, that is shown to approach the corresponding background shock solution in the downstream domain for the uniform supersonic constant flow past the sharp cone, is also explicitly given.

Keywords: Steady potential equation, supersonic flow, multi-dimensional conic shock, global existence, Hardy-type inequality, tangential vector fields.

2010 Mathematical Subject Classification: 35L70, 35L65, 35L67, 76N15.

§1. Introduction

In this paper, we are concerned with the multi-dimensional steady and supersonic conic shock wave problem for a perturbed incoming supersonic flow past an infinitely long circular cone. This problem is fundamental in gas dynamics, for instance, for the supersonic flight of projectiles and rockets. It is also one of the basic models for the discussion of the theory of weak solutions to quasilinear hyperbolic equations and systems in multi-dimensions (see [3], [22]–[23], [30], [37]). Under suitable assumptions on the incoming supersonic flow with a small spherically symmetric perturbation and a spherically symmetric pointed body or artificial boundary conditions on the conic surface, there is an extensive literature studying supersonic flow past a pointed body (see [5], [7]–[10], [18], [21], [31], [35]–[36], and the references therein). The first rigorous mathematical analysis was given in [7] by Courant and Friedrichs, who proved that, for a uniform supersonic flow \((0,0,q_0)\) with constant density \(\rho_0 > 0\) which approaches from minus infinity, when the flow hits the sharp circular cone \(\sqrt{x_1^2 + x_2^2} = b_0 x_3, \ b_0 > 0\), in direction of the \(x_3\)-axis (see Figure 1 below), then there appears a supersonic

\(^*\) Li Jun and Yin Huicheng were supported by the NSFC (No. 10931007, No. 11025105, No. 11001122), by the Doctoral Program Foundation of the Ministry of Education of China (No. 20090091110065), and by the DFG via the Sino-German project “Analysis of PDEs and application.” This research was carried out while Li Jun and Yin Huicheng were visiting the Mathematical Institute of the University of Göttingen.

\(^{**}\) Ingo Witt was partly supported by the DFG via the Sino-German project “Analysis of PDEs and application.”
conic shock $\sqrt{x_1^2 + x_2^2} = s_0 x_3$, $s_0 > b_0$, attached to the tip of the cone provided that $b_0$ is less than some critical value $b^* > 0$, which is determined by the parameters of the incoming flow. When the incoming supersonic flow is multi-dimensionally perturbed, the basic problem of both mathematical and physical relevance that naturally arises is whether such a conic shock is globally stable. Or else, do there appear new shocks or other complicated singularities in the downstream domain? Here, we will focus on this problem when the Mach number of the incoming supersonic flow is appropriately large. It will be shown that a global supersonic conic shock exists uniquely in the whole space and that there are no other singularities between the conic shock and the conic surface for a multi-dimensionally perturbed supersonic polytropic gas past the sharp cone $\sqrt{x_1^2 + x_2^2} = b_0 x_3$ (see Figure 2 below). This result agrees with physical experiment and numerical simulations.

Figure 1. A uniform supersonic flow past a sharp circular cone

Figure 2. A multi-dimensionally perturbed supersonic flow past a sharp circular cone
In this paper, we will use the potential equation to describe the motion of a supersonic polytropic gas (this model is also favored in [22],[23],[30]), where polytropic gas means that the pressure \( P \) and the density \( \rho \) of the gas are related by the equation of state \( P = A \rho^\gamma \), with \( A > 0 \) being a constant and the adiabatic constant \( \gamma \) satisfying \( 1 < \gamma < 3 \) (for air, \( \gamma \approx 1.4 \)). Let \( \Phi(x) \) be a potential of the velocity \( u = (u_1, u_2, u_3) \), i.e., \( u_i = \partial_i \Phi \). Then it follows from Bernoulli’s law that

\[
\frac{1}{2} |\nabla \Phi|^2 + h(\rho) = C_0. \tag{1.1}
\]

Here, \( h(\rho) = \frac{c^2(\rho)}{\gamma - 1} \) is the specific enthalpy, \( c(\rho) = \sqrt{P'/(\rho)} \) is sound speed, \( \nabla = (\partial_1, \partial_2, \partial_3) \), and \( C_0 = \frac{1}{2} q_0^2 + h(\rho_0) \) is Bernoulli’s constant which is determined by the incoming uniform supersonic flow at minus infinity with velocity \( (0,0,q_0) \) and density \( \rho_0 > 0 \).

By (1.1) and the implicit function theorem, in view of \( h'(\rho) = \frac{c^2(\rho)}{\rho} > 0 \) for \( \rho > 0 \), the density \( \rho(x) \) can be expressed as

\[
\rho = h^{-1} \left( C_0 - \frac{1}{2} |\nabla \Phi|^2 \right) \equiv H(\nabla \Phi). \tag{1.2}
\]

Substituting (1.2) into the equation \( \sum_{i=1}^{3} \partial_i (\rho u_i) = 0 \), which expresses the conservation of mass, yields

\[
\sum_{i=1}^{3} \partial_i (H(\nabla \Phi) \partial_i \Phi) = 0. \tag{1.3}
\]

More intuitively, for any \( C^2 \)-solution \( \Phi \), (1.3) can be rewritten as a second-order quasilinear equation,

\[
\sum_{i=1}^{3} ((\partial_i \Phi)^2 - c^2) \partial_i^2 \Phi + 2 \sum_{1 \leq i < j \leq 3} \partial_i \Phi \partial_j \Phi \partial_{ij}^2 \Phi = 0; \tag{1.4}
\]

here \( c = c(\rho) = c(H(\nabla \Phi)) \). Note that (1.4) is strictly hyperbolic with respect to the \( x_3 \)-direction in case \( u_3 > c(\rho) \) holds.

For the geometry of the conic surface, it is convenient to work in the cylindrical coordinates \((z, r, \theta)\), where

\[
x_1 = r \cos \theta, \quad x_2 = r \sin \theta, \quad x_3 = z, \tag{1.5}
\]

\( r = \sqrt{x_1^2 + x_2^2} \), and \( 0 \leq \theta \leq 2\pi \). Under the change of coordinates (1.5), Eq. (1.4) becomes

\[
((\partial_r \Phi)^2 - c^2) \partial_r^2 \Phi + ((\partial_\theta \Phi)^2 - c^2) \partial_\theta^2 \Phi + \frac{1}{r^2} \left( \frac{(\partial_\theta \Phi)^2}{r^2} - c^2 \right) \partial_\theta \Phi + 2 \partial_z \Phi \left( \partial_r \Phi \partial_z^2 \Phi + \frac{1}{r^2} \partial_\theta \Phi \partial_{z\theta}^2 \Phi \right)
\]

\[+ \frac{2}{r^2} \partial_r \Phi \partial_\theta \Phi \partial_{r\theta}^2 \Phi - \frac{1}{r} \partial_\Phi \left( \frac{(\partial_\theta \Phi)^2}{r^2} + c^2 \right) = 0. \tag{1.6}
\]

Let \( \Phi^-(z, r, \theta) \) and \( \Phi^+(z, r, \theta) \) denote the velocity potential for the flow ahead and past the resulting shock front \( r = \chi(z, \theta) \), respectively, where \( \chi(0, \theta) = 0 \). Then (1.6) splits into two equations. That is, \( \Phi^\pm(z, r, \theta) \) satisfy the following equations in their corresponding domains,

\[
((\partial_r \Phi^\pm)^2 - (c^\pm)^2) \partial_r^2 \Phi^\pm + ((\partial_\theta \Phi^\pm)^2 - (c^\pm)^2) \partial_\theta^2 \Phi^\pm + \frac{1}{r^2} \left( \frac{(\partial_\theta \Phi^\pm)^2}{r^2} - (c^\pm)^2 \right) \partial_\theta \Phi^\pm
\]

\[+ 2 \partial_z \Phi^\pm \left( \partial_r \Phi^\pm \partial_z^2 \Phi^\pm + \frac{1}{r^2} \partial_\theta \Phi^\pm \partial_{z\theta}^2 \Phi^\pm \right) + \frac{2}{r^2} \partial_r \Phi^\pm \partial_\theta \Phi^\pm \partial_{r\theta}^2 \Phi^\pm
\]

\[- \frac{1}{r} \partial_\Phi^\pm \left( \frac{(\partial_\theta \Phi^\pm)^2}{r^2} + (c^\pm)^2 \right) = 0 \quad \text{in } \Omega^\pm. \tag{1.7}
\]
and

$$(\partial_t \Phi^+)^2 - (c^+)^2 \partial^2_z \Phi^+ + ((\partial_t \Phi^+)^2 - (c^+)^2) \partial^2_z \Phi^+ + \frac{1}{r^2} \left(\frac{(\partial_\theta \Phi^+)^2}{r^2} - (c^+)^2\right) \partial^2_\theta \Phi^+$$

$$+ 2 \partial_t \Phi^+ \left(\partial_\theta \Phi^+ \partial^2_z \Phi^+ + \frac{1}{r^2} \partial_\theta \Phi^+ \partial^2_\theta \Phi^+\right) + \frac{2}{r^2} \partial_t \Phi^+ \partial_\theta \Phi^+ \partial^2_\theta \Phi^+$$

$$- \frac{1}{r^2} \partial_\theta \Phi^+ \left(\frac{(\partial_\theta \Phi^+)^2}{r^2} + (c^+)^2\right) = 0 \quad \text{in } \Omega_+;$$

where $c^+ = c(\sqrt{b_0 \partial_\theta})$, $\Omega_+ = \{(z, r, \theta) : r > \chi(z, \theta), 0 \leq \theta \leq 2\pi, z > 0\}$, and $\Omega_+ = \{(z, r, \theta) : b_0 z < r < \chi(z, \theta), 0 \leq \theta \leq 2\pi, z > 0\}$. On the conic surface $r = b_0 z$, $\Phi^+$ satisfies the boundary condition

$$\partial_z \Phi^+ - b_0 \partial_\theta \Phi^+ = 0 \quad \text{on } r = b_0 z,$$

while on the conic shock $\Gamma = \{r = \chi(z, \theta)\}$, by Eq. (1.3) and the change of coordinates (1.5), the Rankine-Hugoniot condition becomes

$$[H(\nabla \Phi) \partial_z \Phi] - [H(\nabla \Phi) \partial_z \Phi] \partial_z \chi = \frac{1}{r^2} [H(\nabla \Phi) \partial_\theta \Phi] \partial_\theta \chi \quad \text{on } \Gamma.$$  \tag{1.10}

Moreover, the potential $\Phi(z, r, \theta)$ is continuous across the shock, i.e.,

$$\Phi^+(z, \chi(z, \theta), \theta) = \Phi^-(z, \chi(z, \theta), \theta) \quad \text{on } \Gamma.$$  \tag{1.11}

Furthermore, we impose initial conditions on $\Phi^-(z, r, \theta)$,

$$\Phi^-(0, r, \theta) = \varepsilon \Phi_0^- (r, \theta), \quad \partial_t \Phi^-(0, r, \theta) = q_0 + \varepsilon \Phi_1^- (r, \theta),$$  \tag{1.12}

where $\varepsilon > 0$ is a small parameter, $q_0 > c(\rho_0)$, and $\Phi_0^- (r, \theta), \Phi_1^- (r, \theta) \in C^\infty_0 ((0, l) \times [0, 2\pi])$ for some fixed number $l > 0$.

The main result states:

**Theorem 1.1.** For small $b_0 > 0$ and a sufficiently large speed $q_0$, there exists a small constant $\varepsilon_0 > 0$ depending on $q_0$, $\rho_0$, $b_0$, and $\gamma$ such that problem (1.7)-(1.8) together with (1.9)-(1.12) possesses a global $C^\infty$ supersonic shock solution $(\Phi^+(z, r, \theta), \chi(z, \theta))$ for any $\varepsilon < \varepsilon_0$. Moreover, $(\nabla_z \Phi^+, \frac{\chi(z, \theta)}{z})$ approaches the corresponding quantities for the incoming uniform supersonic flow $(0, 0, q_0)$ with density $\rho_0$ past the sharp circular cone $r = b_0 z$ with rate $(1 + z)^{-m_0}$ for any positive number $m_0 < \frac{1}{2}$.

**Remark 1.1.** The various smallness assumptions in Theorem 1.1 can be expressed by saying that

$$0 < \varepsilon \ll \min \left\{ \frac{1}{b_0^2} (b_0 q_0)^{-2}, \frac{1}{b_0^2} (b_0 q_0)^{-\frac{\delta}{2}} \right\} \ll b_0^2 \quad \text{and} \quad b_0 \ll b^*,$$

where $b^*$ is the critical value given in Remark 2.1 below.

**Remark 1.2.** As in [5], [9-10], and [31], where suitable symmetry assumptions on the incoming supersonic flow with small perturbation or artificial boundary condition on the conic surface were imposed, we emphasize that also in the case treated in this paper there are no discontinuities for the weak solution $\Phi(x) = \Phi^+(x)$ in $\Omega_+$ and $\Phi^-(x)$ in $\Omega_-$, but the main multi-dimensional conic shock front. This means that the supersonic conic shock is structurally stable in the whole space for a polytropic gas and arbitrary perturbations. This agrees with observations from physical experiment and numerical simulations.
Remark 1.3. The nonlinear hyperbolic equation (1.4) is actually a second-order quasilinear wave equation in two space dimensions if one regards $x_3$ as time, as the flow is supersonic in $x_3$-direction. By a direct verification, one sees that (1.4) does not fulfill the “null-condition” put forward in [6] and [19]. Therefore, in terms of the results of [1], [12–13], [15–17], [25–26], and [28], if there was no shock for the solution to Eq. (1.4), then the classical solution to (1.4) would blow up in finite time. Thus, the result of Theorem 1.1 asserts that the multi-dimensional supersonic shock absorbs all possible compressions in the flow and prevents the flow from forming further shocks as well as other singularities.

Remark 1.4. As energy estimates fail to hold in BV-spaces for multi-dimensional hyperbolic equations and systems as shown in [27], the method used in [21] (which was the Glimm scheme for a spherically symmetrically perturbed conic surface) cannot be applied to the genuinely multi-dimensional problem treated here.

Remark 1.5. It was indicated in [7, pages 317–318 and 414] that if a supersonic steady flow approaches from minus infinity and hits a sharp cone in direction of its axis, then it follows from the Rankine-Hugoniot conditions and the physical entropy condition, by an application of the method of the apple curve (see Figure 3 below) that there possibly occur a weak shock and a strong shock attached to the tip of the cone. These shocks are supersonic and transonic, respectively. It was frequently stated that the strong shock is unstable and that, therefore, only the weak shock is present in real situations. In [32-34], the global instability of an attached strong conic shock in the whole space was systematically studied (in this case, the corresponding subsonic potential equation is nonlinear elliptic and the steady Euler system becomes elliptic-hyperbolic) which especially showed that a global strong conic shock is actually unstable as long as the perturbation of the sharp circular cone satisfies some suitable assumptions. On the other hand, from the result in this paper, one infers the global stability of a supersonic conic shock. Consequently, in regard to the global stability or instability of weak and strong conic shocks, these results give a partial answer to the problem of Courant and Friedrichs [7] stated above.

![Figure 3. Apple curve showing all possible end velocity of a conic shock.](image)

Figure 3. Apple curve showing all possible end velocity of a conic shock.

Remark 1.6. For large $q_0$, the incoming flow is called hypersonic. The famous independence principle for large Mach numbers (that there exists a stable limit state for a hypersonic flow as the Mach number goes to infinity) is likewise illustrated by Theorem 1.1 for a hypersonic gas past a sharp cone. For more physical properties of hypersonic flow, see [2], [8], and [29].

Remark 1.7. In case the conic surface is also perturbed, where the multi-dimensional perturbation is small and of compact support (possibly including a compact perturbation near the vertex of the cone, as local existence
of such a conic shock was established in [4]) or decays sufficiently fast when \( z \) goes to infinity, a result analogous to Theorem 1.1 remains in force, where the proofs of this paper still work.

Let us mention some work which is directly related to this paper. In [5] and [9], under the assumptions of an incoming uniform supersonic flow and that the angle of the spherically symmetrically curved conic body is sufficiently small (smaller than the critical angle which guarantees that the supersonic shock is attached), it was shown that a spherically symmetric supersonic conic shock exists globally past the conic body for a supersonic polytropic flow. Z. Xin and H. Yin established in [31] the global existence of a multi-dimensional supersonic conic shock for an incoming uniform supersonic flow past a generally curved sharp cone under an artificial boundary condition – the Dirichlet condition for the potential on the conic surface (physically, this kind of boundary condition means that the conic body is perforated or porous). It should be emphasized here that the Dirichlet boundary condition for the potential in [31] played a crucial role in deriving \textit{a priori} energy estimates and further obtaining the global existence. It means that the Poincaré inequalities are available on the shock surface and in the interior of the downstream domain. By applying the Glimm scheme, in the case of a spherically symmetrically curved cone, W.-C. Lien and T.-P. Liu in [21] obtained the global existence of a weak solution and the long-distance asymptotic behavior under suitable restrictions on the large Mach number, the sharp vertex angle, and the shock strength. The main interest here is to establish the global existence of a genuinely multi-dimensional supersonic conic shock for a perturbed supersonic polytropic gas past an infinitely long cone with a sharp angle when the speed of the incoming flow is large. Especially, we remove the key assumption of spherically symmetry on the perturbed supersonic flow which was assumed in [5], [9], and [21] and which was essential in the proofs there.

Let us also comment on the proof of Theorem 1.1. In order to prove Theorem 1.1, we intend to use continuous induction to establish \textit{a priori} estimates of the solution and its derivatives. To achieve this objective, as in [5], [11], and [15], we need to derive global weighted energy estimates for the linearized problem (1.7)–(1.8) with (1.9)–(1.12). Based on such estimates, one then obtains the global existence, stability, and the asymptotic behavior of the shock solution to the perturbed nonlinear problem. The key ingredients in the analysis to obtain weighted energy estimates are an appropriate multiplier and a new Hardy-type inequality on the shock surface. Finding a suitable multiplier is a hard task for the following reasons: First, to obtain the global existence requires to establish global estimates, independent of \( z \), of the potential function and its derivatives on the boundaries as well as in the interior of the downstream domain. This implies strict constraints on the multiplier and makes the computations delicate and involved. Secondly, as our background solution is self-similar in a downstream domain and strongly depends on the location of the boundary of the cone, the angle at the vertex of the cone, the Mach number of the incoming flow, and the equation of state of the gas under consideration, one needs to take some measures to simplify the coefficients of the nonlinear equation together with the corresponding nonlinear boundary conditions so that the procedure to find the multiplier becomes manageable. Thirdly, for the multi-dimensional case, the Neumann-type boundary condition (1.9) fulfilled by \( \Phi^+ \) introduces additional difficulties compared to [5] and [31], where [5] only treats the case of a spherically symmetric conic shock with Neumann-type boundary condition on the conic surface, while [31] treats the case of an artificial Dirichlet-type boundary condition for the potential on a multi-dimensionally perturbed conic surface. The latter plays a key role in the analysis of [31], as the corresponding Poincaré inequality is available on the shock surface and the interior of the downstream domain, respectively, while this is not the case in the problem treated here. Thanks to some delicate analysis accompanied by a new Hardy-type inequality derived by making full use of the special structure of the shock boundary conditions (i.e., the sizes as well as the signs of the coefficients in (3.6)–(3.7)), we finally overcome all these difficulties and obtain a uniform estimate of \( \| \nabla_x \Phi^+ \|_{L^2(\Omega_+)} \). From this, higher-order estimates of \( \nabla_x \Phi^+ \) can be established by using Klainerman vector field and commutator arguments together with a careful verification that suitably chosen higher-order derivatives of the solution satisfy the Neumann-type boundary condition on the conic surface. This eventually establishes Theorem 1.1.

The paper is organized as follows: In §2, we derive some basic estimates of the background self-similar solution in case of an incoming hypersonic flow, which are required to treat the linearization of the nonlinear problem and for the construction of the multiplier. In §3, we reformulate problem (1.7)–(1.12) by decomposing
its solution as a sum of the background solution and a small perturbation \( \phi \) so that its linearization can be studied in a convenient way. In §4, we first establish a uniform weighted energy estimate for the corresponding linear problem, where also an appropriate multiplier is constructed. Based on such an energy estimate, we obtain a uniform weighted energy estimate of \( \nabla_x \phi \) for the nonlinear problem through establishing a new Hardy-type inequality. In §5, by the estimates derived in §4, we continue to establish uniform higher-order weighted estimates of \( \nabla_x \phi \). In §6, the proof of Theorem 1.1 is eventually completed by utilizing Sobolev’s embedding theorem and continuous induction. Some lengthy computations are carried out in an appendix.

In what follows, we will use the following conventions:

- \( C \) stands for a generic positive constant which does not depend on any quantity except the adiabatic constant \( \gamma \) (1 < \( \gamma \) < 3).
- \( C(\cdot) \) represents a generic positive constant which depends on its argument (or arguments).
- \( O(\cdot) \) means that \( |O(\cdot)| \leq C|\cdot| \) holds true. In particular, \( O(\varepsilon) \) abbreviates \( |O(\varepsilon)| \leq C(b_0, q_0)\varepsilon \).
- \( dS \) stands for the surface measure in the corresponding surface integral.

§2. Analysis of the self-similar background solution

In this section, we will provide, with more details than in [5], properties of the background solution when the Mach number of the incoming supersonic flow is large and the supersonic shock is attached. These properties will be applied again and again in the later analysis of §3–§5.

Following the illustrations of [7, page 407], the supersonic conic shock phenomenon for an incoming supersonic flow past a sharp circular cone is described as follows: Suppose that there is a uniform supersonic flow \((0, 0, q_0)\) with constant density \( \rho_0 > 0 \) which approaches from minus infinity. Let the flow hit the circular cone \( \{(r, z); r \leq b_0z, z \geq 0\} \) in the direction of the \( z \)-axis. Then there exists a critical value \( b^* > 0 \), which is determined by the parameters of the incoming flow, such that there occurs a supersonic conic shock \( r = s_0z, \) attached to the tip of the cone whenever \( b_0 < b^* \) holds true. Moreover, the solution to (1.3) with (1.1) past the shock surface is self-similar, that is, in cylindrical coordinates \((z, r, \theta)\), the density and velocity between the shock front and the conic surface are of the form \( \rho = \rho(s), u_1 = u_r(s) \frac{z_1}{r}, u_2 = u_r(s) \frac{z_2}{r}, \) and \( u_3 = u_z(s) \), where \( s = \frac{r}{z} \). In this case, Eq. (1.3) with (1.1) can be reduced to the following nonlinear ordinary differential system:

\[
\begin{aligned}
\rho'(s) &= -\frac{\rho u_r(s) u_{rr} - u_r}{s((1 + s^2)c^2(\rho) - (su_z - u_r)^2)}, \\
u_r'(s) &= -\frac{c^2(\rho) u_r}{s((1 + s^2)c^2(\rho) - (su_z - u_r)^2)}, \quad \text{for } b_0 \leq s \leq s_0. \\
u_z'(s) &= -\frac{c^2(\rho) u_r}{(1 + s^2)c^2(\rho) - (su_z - u_r)^2},
\end{aligned}
\tag{2.1}
\]

As explained in [5] or [7], for the denominator it holds \((1 + s^2)c^2(\rho) - (su_z - u_r)^2 > 0\) for \( b_0 \leq s \leq s_0 \), which implies that system (2.1) makes sense.

On the shock front \( r = s_0z \), it follows from the Rankine-Hugoniot conditions and Lax’s geometric entropy conditions on the 2-shock that

\[
\begin{aligned}
\left[ \rho u_r \right] - s_0 [\rho u_r] &= 0, \\
\left[ u_z \right] + s_0 [u_r] &= 0
\end{aligned}
\tag{2.2}
\]

and

\[
\begin{aligned}
\lambda_1(s_0) < s_0 < \lambda_2(s_0), \\
\frac{c(\rho_0)}{\sqrt{q_0} - c^2(\rho_0)/7} < s_0
\end{aligned}
\tag{2.3}
\]

Following the illustrations of [7, page 407], the supersonic conic shock phenomenon for an incoming super-
where
\[
\lambda_{1,2}(s) = \frac{u_r(s)u_z(s) + c(\rho(s))\sqrt{u_z^2(s) + u_r^2(s) - c^2(\rho(s))}}{u_z^2(s) - c^2(\rho(s))}.
\] (2.4)

Additionally, the flow satisfies the condition
\[
u_r(b_0) = b_0 u_z(b_0)
\] (2.5)
on the fixed boundary \(s = b_0\).

As indicated in \cite[pages 411–414]{7} or \cite{18}, the boundary value problem (2.1)–(2.4) can be solved by the shooting method as well as by the method of the apple curve. More specifically, for any given \(b_0 > 0\), which is smaller than the critical value \(b^*\), one can determine the solution to (2.1)–(2.4) by finding the integral curve of \(\frac{du_r}{du_z} = -\frac{u_z}{u_r}\) from the intersection point of the apple curve with the ray \(u_r = b_0 u_z\) to some point at the shock polar (see Figure 3 above or \cite[Fig. 8 on page 414]{7}). In this paper, such a supersonic solution past the shock is called the background solution.

For large \(q_0\), some detailed properties of the background solution are as follows:

**Lemma 2.1.** For \(q_0\) large enough, \(1 < \gamma < 3\), and \(0 < b_0 < b_*\) \(= \sqrt{\frac{1}{2}\left(\frac{\gamma + 7}{\gamma - 1} - 1\right)}\), one has, for \(b_0 \leq s \leq s_0\),

(i) \(s_0 = b_0 \left(1 + O((b_0q_0)^{-\frac{\gamma - 1}{2}})\right)\).

(ii) \(0 \leq su_z(s) - u_r(s) \leq O((b_0q_0)^{\frac{\gamma + 1}{2}})\).

(iii) \(u_r(s) = \frac{b_0q_0}{1 + b_0^2} \left(1 + O((b_0q_0)^{-\frac{\gamma - 1}{2}})\right)\).

(iv) \(u_z(s) = \frac{q_0^2}{1 + b_0^2} \left(1 + O((b_0q_0)^{-\frac{\gamma - 1}{2}})\right)\).

(v) \(\rho(s) = \left(\frac{\gamma - 1}{2A\gamma(1 + b_0^2)}\right)^{\frac{1}{\gamma - 2}} \left(s_0^2 - \frac{\gamma - 1}{2}\right) \left(1 + O((b_0q_0)^{-2}) + O((b_0q_0)^{-\frac{\gamma - 1}{2}})\right)\).

(vi) \(q^2(s) - c^2(\rho(s)) = q_0^2 \left(1 - \frac{2 - \gamma}{4}\right) \left(1 + O((b_0q_0)^{-2}) + O((b_0q_0)^{-\frac{\gamma - 1}{2}})\right)\).

(vii) \(u_z^2(s) - c^2(\rho(s)) = \frac{1 - b_0^2(1 + b_0^2)}{1 + b_0^2} q_0^2 \left(1 + O((b_0q_0)^{-2}) + O((b_0q_0)^{-\frac{\gamma - 1}{2}})\right) > 0\).

(viii) \((1 + s^2)c^2(\rho(s)) - (su_z(s) - u_r(s))^2 = s_0^2(1 + O((b_0q_0)^{-2}) + O((b_0q_0)^{-\frac{\gamma - 1}{2}})) > 0\).

**Remark 2.1.** \(b_* = \sqrt{\frac{1}{2}\left(\frac{\gamma + 7}{\gamma - 1} - 1\right)}\) is a root of the quartic equation \(1 - \frac{2 - \gamma}{2}\frac{1}{b_0^2}(1 + b_0^2) = 0\). This guarantees that the flow across the shock is still supersonic in direction of \(z\) for \(b_0 < b_*\) and large \(q_0\). It can be derived from the expression in Lemma 2.1 (vii).

**Proof.** Set \(\rho^+ = \lim_{s \to s_0^-} \rho(s), u_r^+ = \lim_{s \to s_0^-} u_r(s), u_z^+ = \lim_{s \to s_0^-} u_z(s),\) and \(\alpha = \frac{\rho^+}{\rho_0}\). It follows the Rankine-Hugoniot conditions (2.2) that
\[
\begin{align*}
u_r^+ &= \frac{q_0}{1 + b_0^2} \left(1 + \frac{s_0^2}{\alpha}\right), \\
u_z^+ &= \frac{q_0^2}{1 + b_0^2} \left(1 - \frac{1}{\alpha}\right).
\end{align*}
\] (2.6)
Substituting (2.6) into Bernoulli’s law (1.1) yields

$$\frac{A\gamma}{\gamma - 1} (\rho^+ \gamma + 1 - \rho_0^{-1} (\rho^+)^2) + \frac{s_0^2 q_0^2}{2(1 + s_0^2)} (\rho_0^2 - (\rho^+)^2) = 0. \quad (2.7)$$

Set

$$F(x) = \frac{A\gamma}{\gamma - 1} x^{\gamma + 1} - \frac{A\gamma}{\gamma - 1} \rho_0^{-1} x^2 + \frac{s_0^2 q_0^2}{2(1 + s_0^2)} (\rho_0^2 - x^2).$$

Then (2.7) implies that $F(\rho_0) = F(\rho^+) = 0$. It follows from a direct computation that, for $x \in (0, \rho_0)$,

$$F'(x) = \frac{A\gamma(\gamma + 1)}{(\gamma - 1) \rho_0^{-1} x} x - \frac{2A\gamma}{(\gamma - 1) \rho_0^{-1} x} x^2 - \frac{s_0^2 q_0^2}{(1 + s_0^2)} x^2 = x \left( \frac{A\gamma(\gamma + 1)}{(\gamma - 1) \rho_0^{-1} x} x^{\gamma - 1} - \frac{2A\gamma}{(\gamma - 1) \rho_0^{-1} x} x^{\gamma - 1} - \frac{s_0^2 q_0^2}{(1 + s_0^2)} x^{\gamma - 1} \right).$$

Due to Lax’s geometric entropy conditions (2.3), $F'(x) < 0$ for $x \in (0, \rho_0)$. Together with $F(\rho_0) = F(\rho^+) = 0$, this yields $\rho^+ > \rho_0$.

In this case, (2.7) is equivalent to

$$\frac{\alpha}{\alpha^2 - 1} \frac{(\gamma - 1) s_0^2 q_0^2}{2A\gamma(1 + s_0^2)} = (\gamma - 1) s_0^2 q_0^2,$$

where $\alpha > 1$. Since the left hand side of (2.8) is bounded if $\alpha > 1$ is bounded, for $q_0$ is large, $\alpha$ is also large. From this fact, one has

$$\alpha = \frac{1}{\rho_0} \left( \frac{(\gamma - 1) s_0^2 q_0^2}{2A\gamma(1 + s_0^2)} \right)^{\frac{1}{\gamma - 1}} (b_0 q_0)^{\frac{1}{\gamma - 1}} \left( 1 + O((b_0 q_0)^{-2}) + O((b_0 q_0)^{-\frac{8}{\gamma - 1}}) \right).$$

Substituting this into (2.6) yields

$$\begin{cases}
  u_x^+ = \frac{q_0}{1 + s_0^2} \left( 1 + O((b_0 q_0)^{-\frac{2}{\gamma - 1}}) \right), \\
  u_r^+ = \frac{s_0 q_0}{1 + s_0^2} \left( 1 + O((b_0 q_0)^{-\frac{4}{\gamma - 1}}) \right).
\end{cases} \quad (2.9)$$

Moreover, from $u_x'(s) < 0$ and $u_r'(s) > 0$ for $s \in [b_0, s_0]$, one has

$$u_x^+ \leq u_r(s) \leq u_r(b_0) = b_0 u_x(b_0) \leq b_0 u_x(s) \leq b_0 u_x^+. \quad (2.10)$$

Combining (2.9) with (2.10) yields (i) and further (iii)–(iv).

(ii) comes from (2.9) and the fact that

$$0 = b_0 u_x(b_0) - u_r(b_0) \leq su_x(s) - u_r(s) \leq s_0 u_x^+ - u_r^+ \quad \text{for } b_0 \leq s \leq s_0;$$

here $(su_x(s) - u_r(s))' \geq 0$ for $b_0 \leq s \leq s_0$ has been applied.

Furthermore, Bernoulli’s law (1.1) and (i)–(iv) show by a direct computation that (v)–(viii) hold. Therefore, the proof of Lemma 2.1 is complete. □

**Lemma 2.2.** Under the assumptions of Lemma 2.1, one has, for $b_0 \leq s \leq s_0$,

(i) $
\lambda_1(s) - s = \frac{\sqrt{\gamma - 1} (1 + b_0^2) (\sqrt{\gamma - 1} b_0^2 - \sqrt{2 - (\gamma - 1) b_0^2}) + \sqrt{2 - (\gamma - 1) b_0^2}}{2} \left( 1 + O((b_0 q_0)^{-2}) + O((b_0 q_0)^{-\frac{8}{\gamma - 1}}) \right) < 0.
$
below we will assume that $T$ is independent of the shock. Thus, for large $q$ (or see [16]), one easily derives the validity of Lemma 3.1.

Due to (2.1)–(2.3), we establish estimates of the coefficients which appear in the reformulated problem when $\{1+(b_0 q_0)^{-2}\} \sim O((b_0 q_0)^{-\frac{2}{\gamma-1}})$. 

(v) $|\rho'(s)| \leq O(\frac{1}{s})$.

**Proof.** (i)–(v) can be directly derived from (2.1), (2.4), and Lemma 2.1. For the reader’s convenience, we provide the detailed computation for (i) as an example.

It follows from (2.4), (iii)–(v) in Lemma 2.1, and a direct computation that

$$\lambda_1(s) = \frac{u_r(s)u_z(s) - c(\rho(s))\sqrt{u_z^2(s) + u_r^2(s)} - c^2(\rho(s))}{u_r^2(s) - c^2(\rho(s))}$$

$$= \frac{b_0q_0}{1+b_0^2} \left( 1 + O((b_0 q_0)^{-2}) + O((b_0 q_0)^{-\frac{2}{\gamma-1}}) \right) \left( 1 + O((b_0 q_0)^{-2}) + O((b_0 q_0)^{-\frac{2}{\gamma-1}}) \right)$$

$$= \frac{2b_0 - (1+b_0^2)\sqrt{1-b_0}\sqrt{2-(\gamma-1)b_0}}{2-\gamma(1-b_0^2)} \left( 1 + O((b_0 q_0)^{-2}) + O((b_0 q_0)^{-\frac{2}{\gamma-1}}) \right).$$

Due to $s_0 = b_0 \left( 1 + O((b_0 q_0)^{-\frac{2}{\gamma-1}}) \right)$ by (i) in Lemma 2.1, $\lambda_1(s) - s$ satisfies (i). 

**Remark 2.2.** Since the denominator of system (2.1) is positive in the interval $[b_0, s_0]$ (see Lemma 2.1 (viii)), one can extend the background solution $(\rho(s), u_z(s), u_r(s))$ to (2.1)–(2.3) and (2.5) to the interval $[b_0, s_0 + \tau_0]$ for some small positive constant $\tau_0$ satisfying $0 < \tau_0 \leq q_0 \left( s_0 - b_0 \right)$. In the following sections, we will denote this extension of the background solution to $\{(z, r): z > 0, b_0z \leq r \leq (s_0 + \tau_0)z\}$ by $(\tilde{\rho}(s), \tilde{u}_z(s), \tilde{u}_r(s))$, where $s = \frac{r}{z}$. The corresponding extension of the potential will be denoted by $\tilde{\Phi}(s)$.

§3. Reformulation of the nonlinear problem

In this section, we reformulate problem (1.8)–(1.11) by decomposing its solution as a sum of the background solution and a small perturbation. Moreover, based on the analysis of the background solution in Lemmas 2.1 and 2.2, we establish estimates of the coefficients which appear in the reformulated problem when $q_0$ is large.

We now provide a global existence result for the solution to Eq. (1.7) with initial data (1.12) ahead of the shock.

**Lemma 3.1.** Eq. (1.7) with (1.12) possesses a $C^\infty$-solution $\Phi^-(z, r, \theta)$ in $\Omega_- = \{(z, r, \theta): r > \chi(z, \theta), 0 \leq \theta \leq 2\pi, z > 0\}$. Moreover, $\Phi^-(z, r, \theta) - q_0 z \in C^\infty_0(\Omega_-)$ and, for any $k \in \mathbb{N}$, there exists a positive constant $C_k$ independent of $\varepsilon$ such that

$$\|\Phi^-(z, r, \theta) - q_0 z\|_{C^k(\Omega_-)} \leq C_k \varepsilon.$$

**Proof.** The quasilinear equation (1.7) is strictly hyperbolic with respect to the direction of $z$, for the supersonic flow condition $u_z^- > c(\rho^-)$. The initial condition (1.12) means a small perturbation with compact support away from the origin. Thus, for large $q_0$, by finite propagation speed and standard Picard iteration (or see [16]), one easily derives the validity of Lemma 3.1. 

Note that there exists a constant $T_0 > 0$ such that $\Phi^-(z, r, \theta) = q_0 z$ for $z > T_0$. Without loss of generality, below we will assume that $T_0 = 1$.

Next, we reformulate the nonlinear problem (1.8)–(1.11). For notational convenience, we will neglect all the superscripts “+” in (1.8)–(1.11) from now on.
Let $\Phi$ be the solution to (1.8)–(1.11) and $\dot{\varphi}$ be the perturbation of the background solution, that is, $\dot{\varphi} = \Phi - \dot{\Phi}$; here $\Phi$ is given in Remark 2.2. Then, by a direct computation, (1.7) is reduced to:

$$\mathcal{L}\dot{\varphi} = f_1(R, \nabla x)\partial^2_x \dot{\varphi} + f_2(R, \nabla x)\partial^2_\gamma \dot{\varphi} + f_3(R, \nabla x)\partial^2_\theta \dot{\varphi} + \frac{1}{r} f_4(R, \nabla x)\partial^2_\theta \dot{\varphi}$$

$$+ \frac{1}{r} f_5(R, \nabla x)\partial^2_\gamma \dot{\varphi} + \frac{1}{r} f_6(R, \nabla x)\partial^2_\gamma \dot{\varphi} + \frac{1}{r} f_7(R, \nabla x)\dot{\varphi}$$

in $\Omega_+$, (3.1)

where

$${\cal L}\dot{\varphi} = \partial^2_x \dot{\varphi} + 2P_1(s)\partial^2_\gamma \dot{\varphi} + P_2(s)\partial^2_\theta \dot{\varphi} - \frac{1}{r^2} P_3(s)\partial^2_\theta \dot{\varphi} + \frac{2}{r} P_4(s)\partial_\gamma \dot{\varphi} + \frac{2}{r} P_5(s)\partial_\gamma \dot{\varphi},$$

(3.2)

and

$$f_1(s, \nabla x) = \frac{1}{u^2(s) - c^2(\rho(s))} \left\{ -2u(s)\partial_x \dot{\varphi} - (\partial_x \dot{\varphi})^2 - \gamma - \frac{1}{2} \left( 2u(s) + \partial_x \dot{\varphi} \partial_x \dot{\varphi} + \frac{(\partial_y \dot{\varphi})^2}{r^2} \right) \right\},$$

$$f_2(s, \nabla x) = \frac{1}{u^2(s) - c^2(\rho(s))} \left\{ -2u(s)\partial_\gamma \dot{\varphi} - 2u(s)\partial_\gamma \dot{\varphi} - 2\partial_\gamma \dot{\varphi} \partial_\gamma \dot{\varphi} \right\},$$

$$f_3(s, \nabla x) = \frac{1}{u^2(s) - c^2(\rho(s))} \left\{ -2u(s)\partial_\gamma \dot{\varphi} - (\partial_\gamma \dot{\varphi})^2 - \gamma - \frac{1}{2} \left( 2u(s) + \partial_\gamma \dot{\varphi} \partial_\gamma \dot{\varphi} + \frac{(\partial_y \dot{\varphi})^2}{r^2} \right) \right\},$$

$$f_4(s, \nabla x) = -\frac{1}{u^2(s) - c^2(\rho(s))} \left\{ (\partial_y \dot{\varphi})^2 \right\} + \frac{1}{2} \left( 2u(s) + \partial_\gamma \dot{\varphi} \partial_\gamma \dot{\varphi} + \frac{(\partial_y \dot{\varphi})^2}{r^2} \right),$$

$$f_5(s, \nabla x) = -\frac{2}{u^2(s) - c^2(\rho(s))} \left\{ \frac{1}{r} u(s)\partial_y \dot{\varphi} + \frac{1}{r} \partial_\gamma \dot{\varphi} \partial_\gamma \dot{\varphi} \right\},$$

$$f_6(s, \nabla x) = -\frac{2}{u^2(s) - c^2(\rho(s))} \left\{ \frac{1}{r} u(s)\partial_\gamma \dot{\varphi} + \frac{1}{r} \partial_\gamma \dot{\varphi} \partial_\gamma \dot{\varphi} \right\},$$

(3.3)
where $s = \frac{r}{z}$.

On $r = b_0z$, one has

$$\partial_r \varphi = b_0 \partial_z \varphi. \quad \text{(3.3)}$$

On the free boundary $r = \chi(z, \theta)$, by the continuity condition (1.11), (1.10) can be written as

$$H(\nabla \Phi)((\partial_r \Phi)^2 + (\partial_z \Phi)^2 - q_0 \partial_z \Phi) - \rho_0 q_0 \partial_z \Phi + \rho_0 q_0^2 = -\frac{1}{\chi^2} H(\nabla \Phi)(\partial_\theta \Phi)^2 \quad \text{on} \ r = \chi(z, \theta). \quad \text{(3.4)}$$

Introducing the notation

$$\xi(z, \theta) = \frac{\chi(z, \theta) - s_0 z}{z}.$$ 

Then (3.4) can be rewritten as

$$B_1 \partial_r \varphi + B_2 \partial_z \varphi + B_3 \xi = \kappa(\xi, \nabla_x \varphi) \quad \text{on} \ r = \chi(z, \theta), \quad \text{(3.5)}$$

where

$$B_1 = -\frac{\rho(s_0)}{c^2(\rho(s_0))} \left( u_r^2(s_0) + u_z(s_0)(u_z(s_0) - q_0) \right) u_r(s_0) + 2\rho(s_0) u_r(s_0),$$

$$B_2 = -\frac{\rho(s_0)}{c^2(\rho(s_0))} \left( u_r^2(s_0) + u_z(s_0)(u_z(s_0) - q_0) \right) u_z(s_0) + 2\rho(s_0) u_z(s_0) - q_0 + (\rho(s_0) - \rho_0) q_0,$$

$$B_3 = \rho(s_0) \left( 2u_r(s_0) u_r'(s_0) + 2(u_z(s_0) - q_0) u_z'(s_0) + q_0 u_z'(s_0) \right) + \rho'(s_0) \left( u_r^2(s_0) + u_z(s_0)(u_z(s_0) - q_0) \right)$$

and the generic function $\kappa(\xi, \nabla_x \varphi)$ is used to denote a quantity dominated by $C(b_0, q_0) ||(\xi, \nabla_x \varphi)||^2$.

By Lemma 3.3 below, one has $B_1 \neq 0$ in (3.5) for large $q_0$. Thus, Eq. (3.5) can be rewritten as

$$B_0 \varphi + \mu_2 \xi = \kappa(\xi, \nabla_x \varphi) \quad \text{on} \ r = \chi(z, \theta); \quad \text{(3.6)}$$

here $B_0 \varphi = \partial_r \varphi + \mu_1 \partial_z \varphi$ with $\mu_1 = \frac{B_2}{B_1}$ and $\mu_2 = \frac{B_3}{B_1}$.

Furthermore, (1.11) and the properties of $\Phi^-$ for $z \geq 1$ and $\theta \in [0, 2\pi]$ listed in Lemma 3.1 imply

$$\varphi(z, \chi(z, \theta), \theta) = \Phi(z, \chi(z, \theta), \theta) - \hat{\Phi}(z, s_0 z, \theta) - \left( \Phi(z, \chi(z, \theta), \theta) - \hat{\Phi}(z, s_0 z, \theta) \right)$$

$$= -\left( \int_0^1 \hat{u}_r(s_0 + t \xi(z, \theta)) dt \right) z \xi(z, \theta). \quad \text{(3.7)}$$
On the other hand, it follows from Lemma 3.1 that \( \Phi^- - q_0 z \in C^\infty_0(\Omega_-) \) in (1.7) and that, near the vertex of the cone \( r = b_0 z \), the solution \( \Phi^+(x) \) is actually the background solution. Thus, in order to prove Theorem 1.1, by the local existence and stability result for the multi-dimensional shock solution to the potential flow equation in [24], we only need to solve problem (3.1) in the domain \( \{ (z, r, \theta) : z \geq 1, b_0 z \leq r \leq \chi(z, \theta), 0 \leq \theta \leq 2\pi \} \) with the boundary conditions (3.3), (3.6)–(3.7), and small initial data \( \varphi(z, r, \theta)|_{z=1}, \partial_z \varphi(z, r, \theta)|_{z=1}, \) and \( \xi(z, \theta)|_{z=1} \).

Here, smallness of initial data means that
\[
\sum_{l \leq k_0} |\nabla_z^l \varphi(1, r, \theta)| + \sum_{l \leq k_0} |\nabla_{z, \theta}^l \xi(1, \theta)| \leq C\varepsilon, \tag{3.8}
\]
where \( k_0 \in \mathbb{N}, k_0 \geq 7 \).

For later use, we now list specific estimates of the coefficients in (3.1), (3.2), and (3.6). Since these estimates result from a direct, but tedious computation that makes use of Lemmas 2.1 and 2.2, we postpone the proof to the appendix.

With respect to the coefficients of \( L\hat{\varphi} \) in (3.2), one has:

**Lemma 3.2.** For \( q_0 \) large enough, \( 1 < \gamma < 3, 0 < b_0 < b_\ast \), and \( b_0 \leq s \leq s_0 \), one has
\[
P_1(s) = \frac{b_0}{1 - \frac{2 - \gamma}{2} b_0^3 (1 + b_0^3) \left(1 + O((b_0 q_0)^{-2}) + O((b_0 q_0)^{-\frac{2}{1 - \gamma}})\right)},
\]
\[
P_2(s) = \frac{b_0^3 (\frac{2 - \gamma}{2} - 2 - \gamma)}{1 - \frac{2 - \gamma}{2} b_0^3 (1 + b_0^3)} \left(1 + O((b_0 q_0)^{-2}) + O((b_0 q_0)^{-\frac{2}{1 - \gamma}})\right),
\]
\[
P_3(s) = \frac{2 - \gamma b_0^3 (1 + b_0^3)}{1 - \frac{2 - \gamma}{2} b_0^3 (1 + b_0^3)} \left(1 + O((b_0 q_0)^{-2}) + O((b_0 q_0)^{-\frac{2}{1 - \gamma}})\right),
\]
\[
P_4(s) = O((b_0 q_0)^{-2}) + O((b_0 q_0)^{-\frac{2}{1 - \gamma}}),
\]
\[
P_5(s) = -\frac{1 + 2 b_0^3}{1 - \frac{2 - \gamma}{2} b_0^3 (1 + b_0^3)} \left(1 + O((b_0 q_0)^{-2}) + O((b_0 q_0)^{-\frac{2}{1 - \gamma}})\right),
\]
\[
P_1'(s) = -1 + \frac{2 - \gamma b_0^3 - 2 - \gamma b_0^3}{(1 + b_0^3)(1 - \frac{2 - \gamma}{2} b_0^3 (1 + b_0^3))} \left(1 + O((b_0 q_0)^{-2}) + O((b_0 q_0)^{-\frac{2}{1 - \gamma}})\right),
\]
\[
P_2'(s) = \frac{2 b_0 + 2(\gamma - 2) b_0^3}{(1 + b_0^3)(1 - \frac{2 - \gamma}{2} b_0^3 (1 + b_0^3))} \left(1 + O((b_0 q_0)^{-2}) + O((b_0 q_0)^{-\frac{2}{1 - \gamma}})\right),
\]
\[
P_3'(s) = \frac{1 + b_0^3}{(1 - \frac{2 - \gamma}{2} b_0^3 (1 + b_0^3))} \left(1 + O((b_0 q_0)^{-2}) + \frac{1}{b_0} O((b_0 q_0)^{-\frac{2}{1 - \gamma}})\right).
\]

Moreover, \( B_i, i = 1, 2, 3 \), in (3.5) and \( \mu_j, j = 1, 2 \), in (3.6) admit the following estimates:

**Lemma 3.3.** For large \( q_0 \), one has
\[
B_1 = \frac{2}{1 + b_0^3} \left(\frac{\gamma - 1}{2 A \gamma (1 + b_0^3)}\right) \left(1 + O((b_0 q_0)^{-2}) + O((b_0 q_0)^{-\frac{2}{1 - \gamma}})\right) > 0,
\]
\[
B_2 = \frac{1 - b_0^3}{b_0 (1 + b_0^3)^2} \left(\frac{\gamma - 1}{2 A \gamma (1 + b_0^3)}\right) \left(1 + O((b_0 q_0)^{-2}) + O((b_0 q_0)^{-\frac{2}{1 - \gamma}})\right),
\]
\[
B_3 = \frac{1}{b_0 (1 + b_0^3)^2} \left(\frac{\gamma - 1}{2 A \gamma (1 + b_0^3)}\right) \left(1 + \frac{1}{b_0} O((b_0 q_0)^{-2}) + \frac{1}{b_0} O((b_0 q_0)^{-\frac{2}{1 - \gamma}})\right),
\]
\[
\mu_1 = \frac{1 - b_0^3}{2 b_0^3} \left(1 + O((b_0 q_0)^{-2}) + O((b_0 q_0)^{-\frac{2}{1 - \gamma}})\right) > 0,
\]
\[
\mu_2 = \frac{q_0}{2(1 + b_0^3)} \left(1 + \frac{1}{b_0} O((b_0 q_0)^{-2}) + \frac{1}{b_0} O((b_0 q_0)^{-\frac{2}{1 - \gamma}})\right) < 0.
\]
Remark 3.1. We emphasize that there is a large factor \( q_0 \) in the coefficient \( \mu \) of the boundary condition (3.6) which shows that one has to be very careful in treating the shock boundary condition later on.

§4. A first-order weighted energy estimate

In this section, we establish a weighted energy estimate of \( \nabla_x \hat{\varphi} \) for the linear part of (3.1), together with (3.3) and (3.6)–(3.8), which will play a fundamental role in our subsequent analysis.

Set \( D_T = \{(z, r; 1 \leq z \leq T, b_0 z \leq r \leq \chi(z, \theta), 0 \leq \theta < 2\pi\} \) for any \( T > 1 \). \( \Gamma_T = \{(z, r; r = \chi(z, \theta), 1 \leq z \leq T, 0 \leq \theta < 2\pi\} \) and \( B_T = \{(z, r; r = b_0 z, 1 \leq z \leq T, 0 \leq \theta < 2\pi\} \) are the lateral boundaries of \( D_T \).

Theorem 4.1. Let \( \hat{\varphi} \in C^\infty(D_T) \) satisfy the boundary condition (3.3). Let \(|\xi(z, \theta)| + |z \partial_z \xi(z, \theta)| + |\partial_\theta \xi(z, \theta)| + |\nabla_x \xi| \leq M \varepsilon \) hold true for small \( \varepsilon \), \( z \in [1, T] \), and some positive constant \( M \). Then there exists a multiplier \( M \hat{\varphi} = A(z, r) \partial_z \hat{\varphi} + B(z, r) \partial_r \hat{\varphi} \) with smooth coefficients such that, for any fixed constant \( \mu < -1 \),

\[
C_1 T^{\mu + 1} \iint_{b_0 T \leq r \leq \chi(T, \theta)} (\nabla_x \hat{\varphi})^2 (T, r, \theta) d\theta d\theta + C_2 \iiint_{D_T} \mu (\nabla_x \hat{\varphi})^2 dz dr d\theta
+ C_3 \iint_{\Gamma_T} \mu^{\mu + 1} (\partial_r \hat{\varphi})^2 dS + C_4 \iiint_{\Gamma_T} \mu^{\mu + 1} \frac{(\partial_\theta \hat{\varphi})^2}{r^2} dS
\leq \iint_{D_T} \mathcal{L} \cdot M \hat{\varphi} dz d\theta + C_5 \iiint_{b_0 T \leq r \leq \chi(1, \theta)} (\partial_z \hat{\varphi})^2 + (\partial_r \hat{\varphi})^2 + \frac{(\partial_\theta \hat{\varphi})^2}{r^2} (1, r, \theta) d\theta d\theta
+ C_6 \iint_{\Gamma_T} \mu^{\mu + 1} (B_0 \hat{\varphi})^2 dS,
\]

(4.1)

where \( C_i \), \( 1 \leq i \leq 6 \), are some positive constants depending on \( b_0 \) and \( q_0 \). In particular,

\[
\begin{align*}
C_3 &= \frac{(\gamma - 1) b_0^3 (1 + b_0^2)}{8 (1 - \frac{\gamma - 1}{2} b_0^2 (1 + b_0^2))} + \frac{1}{b_0^6} O((b_0 q_0)^{-2}) + \frac{1}{b_0^6} O((b_0 q_0)^{-\frac{2}{\gamma - 1}}), \\
C_6 &= \frac{(\gamma - 1) b_0^3 (1 + b_0^2)}{2 (1 - \frac{\gamma - 1}{2} b_0^2 (1 + b_0^2))} + O((b_0 q_0)^{-2}) + O((b_0 q_0)^{-\frac{2}{\gamma - 1}}).
\end{align*}
\]

(4.2)

Remark 4.1. The values of constants \( C_3 \) and \( C_6 \) will play an important role in the energy estimates for the nonlinear problem (3.3) with (3.6)–(3.8). The most troublesome term \( \iint_{\Gamma_T} \mu^{\mu + 1} (B_0 \hat{\varphi})^2 dS \) in the right-hand side of (4.1) will be shown to be absorbed by positive integrals in the left-hand side of (4.1). The reason for which the term \( \iint_{\Gamma_T} \mu^{\mu + 1} (B_0 \hat{\varphi})^2 dS \) is hard is the following one: In view of the Neumann boundary condition (3.3), which is different from the artificial Dirichlet boundary condition used in [31], the usual Poincaré inequality is not available for the solution \( \hat{\varphi} \) on the shock surface which means that the \( L^2(\Gamma_T) \)-estimates of \( \nabla_x \hat{\varphi} \) cannot be obtained directly. (Note that the boundary condition (3.6) contains the function \( \xi \), and that \( \xi \) is roughly equivalent to \( \frac{\nabla_x \hat{\varphi}}{\hat{\varphi}} \) in view of (3.7), so that an estimate of \( \hat{\varphi} \) on the shock surface has to be established.)

Proof. We will determine the coefficients \( A(z, r) = z^\mu r a(\frac{z}{r}) \) and \( B(z, r) = z^{\mu + 1} b(\frac{z}{r}) \) for \( z \geq 1 \) later on. Set \( M \hat{\varphi} = A(z, r) \partial_z \hat{\varphi} + B(z, r) \partial_r \hat{\varphi} \). Then it follows from an integration by parts that

\[
\iint_{D_T} \mathcal{L} \cdot M \hat{\varphi} dz d\theta = \iiint_{D_T} \mu z \left( K_1 (\partial_z \hat{\varphi})^2 + K_2 \partial_z \hat{\varphi} \partial_r \hat{\varphi} + K_3 (\partial_r \hat{\varphi})^2 + K_4 \frac{(\partial_\theta \hat{\varphi})^2}{r^2} \right) dz d\theta d\theta
+ T^{\mu + 1} \iint_{b_0 T \leq r \leq \chi(T, \theta)} N_1 (T, r, \theta) dS - \iint_{b_0 T \leq r \leq \chi(1, \theta)} N_1 (1, r, \theta) dS
+ \iint_{\Gamma_T} \mu^{\mu + 1} (N_2 - \partial_z \chi N_1 - \partial_\theta \chi N_3) ds + \iint_{B_T} \mu^{\mu + 1} \left( b_0 N_1 - N_2 \right) ds
\]

(4.3)
This then implies the following five steps.

Step 1. Handling of \( \int_{B_T} z^{\mu+1} \left( b_0 N_1 - N_2 \right) dS \). By the boundary condition (3.3) and \( u_r(0) = b_0 u_z(0) \), one obtains that

\[
 b_0 N_1 - N_2 = \left( \frac{1}{2} + b_0^2 + \frac{b_0^2 (1 + b_0^2) u_z^2(0)}{2 (u_z^2(0) - c^2(\rho(0)))} \right) (b(0) - b_0^2 a(0))(\partial_\varphi) + (b(0) - b_0 a(0)) (\partial_\varphi)^2
\]

on \( B_T \). Since the terms \( \frac{b_0^2 (1 + b_0^2) u_z^2(0)}{2 (u_z^2(0) - c^2(\rho(0)))} \) and \( P_3(0) \) are positive according to (iv) and (vii) in Lemma 2.1 and Lemma 3.2, in order to control the integral \( \int_{B_T} z^{\mu+1} \left( b_0 N_1 - N_2 \right) dS \), one should choose

\[
 b(0) = b_0^2 a(0).
\]

This then implies

\[
 \int_{B_T} z^{\mu+1} \left( b_0 N_1 - N_2 \right) dS = 0.
\]

Step 2. Positivity of \( \int_{B_T \leq r \leq \chi(T, \theta)} N_1 dS \). To ensure the positivity of the terms in \( N_1 \), which are quadratic in \( (\partial_\varphi, \partial_r \varphi, \partial_{\theta \varphi}, \partial_{\theta r}) \), \( a(s) \) and \( b(s) \) should fulfill

\[
 \begin{cases}
 a(s) > 0, \\
 b^2(s) - 2s P_1(s) a(s) b(s) + s^2 P_2(s) a^2(s) < 0
\end{cases}
\]
which is equivalent to
\[ a(s) > 0, \quad \lambda_1(s) < \frac{b(s)}{sa(s)} < \lambda_2(s). \] (4.8)

In this case, one arrives at
\[
\iint_{b^0 \leq r \leq \chi(T, \theta)} N_1 \, dS \geq \iint_{b^0 \leq r \leq \chi(T, \theta)} \left( \lambda_{\min}(s) \left( (\partial_s \varphi)^2 + (\partial_r \varphi)^2 \right) + \frac{1}{2r^2} saP_3(s)(\partial_\theta \varphi)^2 \right) \, dS, \] (4.9)

where
\[
\lambda_{\min}(s) = \frac{1}{2} \left( \frac{1}{2} \frac{sa(s) + b(s)P_1(s)}{sa(s)P_2(s) - \frac{1}{2} sa(s) - b(s)P_1(s) + \frac{1}{2} sa(s)P_2(s)} + b(s)^2 \right).
\]

Moreover, by Lemmas 2.1, 2.2, and 3.2, the assumption on \( \xi(z, \theta) \) in Theorem 4.1, and the choices of \( a(s), b(s) \) to be made later on, one actually obtains that
\[
\lambda_{\min}(s) \geq C(b_0, q_0) + O(\varepsilon). \] (4.10)

**Step 3. Positivity of the integral over \( D_T \).** Under the constraints (4.6) and (4.8), we will choose \( a(s) \) and \( b(s) \) in such a way that
\[
K_1(\partial_s \varphi)^2 + K_2(\partial_s \varphi \partial_r \varphi) + K_3(\partial_r \varphi)^2 + K_4 \frac{(\partial_\theta \varphi)^2}{r^2} \geq 0.
\]

This will be true if the coefficients \( K_i, 1 \leq i \leq 4 \), satisfy
\[
\begin{cases}
K_1 > 0, \\
K_2 - 4K_1K_3 < 0, \\
K_4 > 0.
\end{cases} \tag{4.11}
\]

According to (4.6), one can choose
\[
\begin{cases}
a(s) = 1, \\
b(s) = s^2 \left( 1 + \frac{b}{b_0}(s - b_0) \right),
\end{cases} \tag{4.12}
\]

where the constant \( b \) will be determined in a moment.

It follows from Lemmas 2.1, 2.2, and 3.2 and a direct computation that
\[
\begin{align*}
K_1 &= \left( \frac{1}{2} b_0 + O(b_0^2) \right) b - \left( \frac{\mu}{2} - 1 \right) b_0 + O((b_0 q_0)^{-2}) + O((b_0 q_0)^{-\frac{4}{n}}), \\
K_2 &= \left( b_0^2 + O(b_0^3) \right) b + \left( 2 - \mu \right) b_0^2 + O((b_0 q_0)^{-2}) + O((b_0 q_0)^{-\frac{4}{n}}), \\
K_3 &= \left( \frac{\gamma + 1}{4} b_0^3 + O(b_0^4) \right) b + \left( 1 - \frac{\mu(\gamma + 1)}{4} \right) b_0^3 + O((b_0 q_0)^{-2}) + O((b_0 q_0)^{-\frac{4}{n}}), \\
K_4 &= \left( \frac{\gamma - 1}{4} b_0^3 + O(b_0^4) \right) b - \frac{4}{4} (\gamma - 1) b_0^3 + O((b_0 q_0)^{-2}) + O((b_0 q_0)^{-\frac{4}{n}}), \\
K_2^2 - 4K_1K_3 &= -\frac{\gamma - 1}{2} \left( \mu - b \right) \left( \mu - 2 \right) b_0^4 + O(b_0^5) + O((b_0 q_0)^{-2}) + O((b_0 q_0)^{-\frac{4}{n}}).
\end{align*} \tag{4.13}
\]
Therefore, one obtains from (4.11) that, for \(q_0\) large enough,
\[
\dot{b} + 2 - \mu > 0, \quad (\mu - \dot{b})(\mu - 2 - \dot{b}) > 0, \quad \dot{b} + \mu < 0.
\]
If one sets
\[
\dot{b} = 1, \tag{4.14}
\]
then
\[
\mu < -1. \tag{4.15}
\]
In this case, one arrives at
\[
\iint_{D_T} z^\mu \left\{ K_1(\partial_z \dot{\varphi})^2 + K_2 \partial_z \dot{\varphi} \partial_r \dot{\varphi} + K_3(\partial_r \dot{\varphi})^2 + K_4(\partial_\theta \dot{\varphi})^2 \right\} \, dz \, dr \, d\theta 
\geq C(b_0) \iint_{D_T} z^\mu \left( (\partial_z \dot{\varphi})^2 + (\partial_r \dot{\varphi})^2 + \frac{(\partial_\theta \dot{\varphi})^2}{r^2} \right) \, dz \, dr \, d\theta. \tag{4.16}
\]
**Step 4. Estimate of** \(\iint_{\Gamma_T} z^{\mu+1}(N_2 - \partial_z \chi N_1 - \partial_\theta \chi N_3) \, dS\). By the assumptions on \(\xi(z, \theta)\) in Theorem 4.1, it follows from the expressions for \(N_1, N_2, N_3\) and a direct computation that
\[
N_2 - \partial_z \chi N_1 - \partial_\theta \chi N_3 = \beta_0(\partial_z \dot{\varphi})^2 + \beta_1 \partial_z \dot{\varphi} \partial_r \dot{\varphi} + \beta_2(\partial_r \dot{\varphi})^2 + \beta_3 \frac{1}{r^2}(\partial_\theta \dot{\varphi})^2, \tag{4.17}
\]
where
\[
\begin{align*}
\beta_0 &= s_0 P_1(s_0) a(s_0) - \frac{1}{2} s_0^2 a(s_0) - b(s_0) + O(\varepsilon), \\
\beta_1 &= s_0 P_2(s_0) a(s_0) - s_0 b(s_0) + O(\varepsilon), \\
\beta_2 &= \frac{1}{2} P_2(s_0) b(s_0) - s_0 P_1(s_0) b(s_0) + \frac{1}{2} s_0^2 P_2(s_0) a(s_0) + O(\varepsilon), \\
\beta_3 &= \frac{1}{2} P_3(s_0) (b(s_0) - s_0^2 a(s_0)) + O(\varepsilon).
\end{align*}
\]
Due to \(\partial_r \dot{\varphi} = \mathcal{B}_0 \dot{\varphi} - \mu_1 \partial_z \dot{\varphi}\), from (4.17) one obtains that
\[
N_2 - \partial_z \chi N_1 - \partial_\theta \chi N_3 = (\beta_0 - \mu_1 \beta_1 + \beta_2 \mu_1^2)(\partial_z \dot{\varphi})^2 + (\beta_1 - 2 \mu_1 \beta_2) \partial_z \dot{\varphi} \mathcal{B}_0 \dot{\varphi} + \beta_2(\mathcal{B}_0 \dot{\varphi})^2 + \beta_3 \frac{1}{r^2}(\partial_\theta \dot{\varphi})^2.
\]
By Lemmas 2.1 and 3.2 and the expressions for \(a(s), b(s)\) in (4.12) and (4.14), one arrives at
\[
\begin{align*}
\beta_0 - \mu_1 \beta_1 + \beta_2 \mu_1^2 &= \frac{1}{4(1 - \frac{1}{2} b_0^2(1 + b_0^2))} \left( \frac{\gamma - 1}{2} b_0^2(1 + b_0^2)^3 + O((b_0 q_0)^{-2}) + O((b_0 q_0)^{-\frac{1}{2}}) + O(\varepsilon) \right), \\
\beta_1 - 2 \mu_1 \beta_2 &= O((b_0 q_0)^{-2}) + O((b_0 q_0)^{-\frac{1}{2}}) + O(\varepsilon), \\
\beta_2 &= \frac{1}{1 - \frac{1}{2} b_0^2(1 + b_0^2)} \left( - \frac{\gamma - 1}{2} b_0^2(1 + b_0^2) + O((b_0 q_0)^{-2}) + O((b_0 q_0)^{-\frac{1}{2}}) + O(\varepsilon) \right), \\
\beta_3 &\geq C(b_0, q_0) + O(\varepsilon).
\end{align*}
\]
Consequently,
\[
\begin{align*}
\iint_{\Gamma_T} z^{\mu+1}(N_2 - \partial_z \chi N_1 - \partial_\theta \chi N_3) \, dS 
\geq (Q_1(b_0, q_0) + O(\varepsilon)) \iint_{\Gamma_T} z^{\mu+1}(\partial_z \dot{\varphi})^2 \, dS - (Q_2(b_0, q_0) + O(\varepsilon)) \iint_{\Gamma_T} z^{\mu+1}(\mathcal{B}_0 \dot{\varphi})^2 \, dS \\
+ (C(b_0, q_0) + O(\varepsilon)) \iint_{\Gamma_T} z^{\mu+1}\frac{(\partial_\theta \dot{\varphi})^2}{r^2} \, dS, \tag{4.18}
\end{align*}
\]
where
\[
Q_1(b_0, q_0) = \frac{(\gamma - 1)b_0^2(1 + b_0^2)^3}{8(1 - \frac{4}{5}b_0^2(1 + b_0^2))} + O((b_0q_0)^{-2}) + O((b_0q_0)^{-\frac{2}{3}}) > 0,
\]
\[
Q_2(b_0, q_0) = \frac{(\gamma - 1)b_0^2(1 + b_0^2)}{2(1 - \frac{4}{5}b_0^2(1 + b_0^2))} + O((b_0q_0)^{-2}) + O((b_0q_0)^{-\frac{2}{3}}) > 0.
\]

**Step 5. Estimate of** \(\iint_{b_0 \leq \rho \leq \chi(1, \rho)} N_1(1, r, \theta) dS.\) From the expression for \(N_1\) and the initial condition (3.8), one easily obtains
\[
\left| \iint_{b_0 \leq \rho \leq \chi(1, \rho)} K\rho(1, r, \theta) d\rho d\theta \right| \leq C(b_0, q_0)\varepsilon^2. \tag{4.19}
\]

Finally, substituting the estimates (4.7), (4.9)–(4.10), (4.16), and (4.18)–(4.19) into (4.3), (4.1) and (4.2) are obtained in terms of the smallness of \(\varepsilon\) given in Remark 1.1. Therefore, Theorem 4.1 is proved. □

Based on Theorem 4.1, we will derive a first-order uniform energy estimate of \(\nabla_{x, \theta} \phi\) for the linear part of (3.1) together with (3.3) and (3.6)–(3.8). To this end, we require a Hardy-type inequality on \(\iint_{\Gamma_T} \mu^{-1} |\phi|^2 dS\), which is motivated by [14, Theorem 330] and derived utilizing the special structures of (3.6)–(3.7).

**Lemma 4.2. (Hardy-type inequality)** Under the assumptions of Theorem 4.1, for \(\mu < -1\), one has
\[
\iint_{\Gamma_T} \mu^{-1} |\phi|^2 dS \leq C(b_0, q_0)\varepsilon^2 + \frac{(1 + b_0^2)^2}{\mu^2} (1 + O((b_0q_0)^{-2}) + O((b_0q_0)^{-\frac{2}{3}})) \iint_{\Gamma_T} \mu^{-1} (\partial_\rho \phi)^2 dS + C(b_0, q_0)\varepsilon \left( \iint_{\Gamma_T} \mu^{-1} (B_0 \phi)^2 dS + \iint_{\Gamma_T} \mu^{-1} (\frac{\partial \phi}{r^2})^2 dS \right). \tag{4.20}
\]

**Proof.** We first assert that
\[
\iint_{\Gamma_T} \mu^{-1} |\phi|^2 dS \leq \left( \frac{4}{\mu^2} + O(\varepsilon) \right) \iint_{\Gamma_T} \mu^{-1} (1 - \mu_1 \partial_\rho \chi)^2 (\partial_\rho \phi)^2 dS
\]
\[
+ C(b_0, q_0)\varepsilon \left( \iint_{\Gamma_T} \mu^{-1} (B_0 \phi)^2 dS + \iint_{\Gamma_T} \mu^{-1} (\frac{\partial \phi}{r^2})^2 dS \right) + C(b_0, q_0)\varepsilon^2. \tag{4.21}
\]

Indeed, note that
\[
\iint_{\Gamma_T} \mu^{-1} \phi^2 dS = \int_0^{2\pi} d\theta \int_1^T \mu^{-1} \phi^2(z, \chi(z, \theta), \theta)dz. \tag{4.22}
\]

Set \(m(\theta) \equiv \int_1^T \mu^{-1} \phi^2(z, \chi(z, \theta), \theta)dz\). Then, by an integration by parts,
\[
m(\theta) = \frac{1}{\mu} \phi^2(1, \chi(1, \theta), \theta) \bigg|_{z=1}^T - \frac{2}{\mu} \int_1^T \mu \phi(z, \chi(z, \theta), \theta) (\partial_{\rho, \rho} \phi + \partial_{\rho, \chi} \phi + \partial_{\rho, \rho} \phi + (1 - \mu_1 \partial_\rho \chi) \partial_\rho \phi) (z, \chi(z, \theta), \theta)dz
\]
\[
\leq \frac{1}{\mu} \phi^2(1, \chi(1, \theta), \theta) - \frac{2}{\mu} \int_1^T \mu \phi(z, \chi(z, \theta), \theta) (\partial_\rho \chi B_0 \phi + (1 - \mu_1 \partial_\rho \chi) \partial_\rho \phi) (z, \chi(z, \theta), \theta)dz. \tag{4.23}
\]

By (3.7), one has
\[
\hat{\varphi}(z, \chi(z, \theta), \theta) = a_1(z, \chi(z, \theta), \theta) \xi, \tag{4.24}
\]
where
\[
a_1(z, \chi(z, \theta), \theta) \equiv - \left( \int_0^1 \hat{a}_r(s_0 + t\xi(z, \theta))dt \right) < 0. \tag{4.25}
\]
Due to the assumptions in Theorem 4.1 and Lemma 2.1, \( \partial_z \chi = b_0(1 + O((b_0q_0)^{-\frac{2}{r^T}}) + O(\varepsilon)) \), and thus because of the smallness of \( \varepsilon \),
\[
1 - \mu_1 \partial_z \chi = \frac{1 + b_0^2}{2}(1 + O((b_0q_0)^{-\frac{2}{r^T}})) > 0.
\] (4.26)

It follows from (3.8), (3.6), \( \mu_2 < 0 \) by Lemma 3.3, (4.24)–(4.25), and a direct computation that
\[
m(\theta) \leq C(b_0, q_0)\varepsilon^2 - \frac{2}{\mu} \int_1^T z^\mu \dot{\varphi}(z, \chi(\varepsilon, \theta), \theta)(1 - \mu_1 \partial_z \chi) \partial_z \dot{\varphi}(z, \chi(\varepsilon, \theta), \theta) \, dz
\]
\[
\quad + \frac{2\mu_2}{\mu} \int_1^T z^\mu \partial_z \chi \dot{\varphi}(z, \chi(\varepsilon, \theta), \theta) \xi \, dz - \frac{2}{\mu} \int_1^T z^\mu \partial_z \chi \dot{\varphi}(z, \chi(\varepsilon, \theta), \theta) \kappa(\xi, \nabla \dot{\varphi}) \, dz
\]
\[
= C(b_0, q_0)\varepsilon^2 - \frac{2}{\mu} \int_1^T z^\mu \dot{\varphi}(z, \chi(\varepsilon, \theta), \theta)(1 - \mu_1 \partial_z \chi) \partial_z \dot{\varphi}(z, \chi(\varepsilon, \theta), \theta) \, dz
\]
\[
\quad + \frac{2\mu_2}{\mu} \int_1^T z^\mu \partial_z \chi \dot{\varphi}(z, \chi(\varepsilon, \theta), \theta) \xi \, dz - \frac{2}{\mu} \int_1^T z^\mu \partial_z \chi \dot{\varphi}(z, \chi(\varepsilon, \theta), \theta) \kappa(\xi, \nabla \dot{\varphi}) \, dz
\]
\[
\leq C(b_0, q_0)\varepsilon^2 + \frac{1}{2} \int_1^T z^{\mu-1} \dot{\varphi}^2(z, \chi(\varepsilon, \theta), \theta) \, dz + \frac{2}{\mu^2} \int_1^T z^{\mu+1}(1 - \mu_1 \partial_z \chi)^2 (\partial_z \dot{\varphi})^2(z, \chi(\varepsilon, \theta), \theta) \, dz
\]
\[
\quad + C(b_0, q_0)\varepsilon \left( \int_1^T z^{\mu-1} \dot{\varphi}^2(z, \chi(\varepsilon, \theta), \theta) \, dz + \int_1^T z^{\mu+1}\xi^2 \, dz + \int_1^T z^{\mu+1}(\nabla_z \dot{\varphi})^2 \, dz \right)
\]
\[
\quad + \frac{(\partial_0 \dot{\varphi})^2}{r^2}(z, \chi(z, \theta), \theta) \, dz).
\]
Together with (4.22) and (4.26), this yields
\[
\int_{\Gamma_T} z^{\mu-1} \dot{\varphi}^2 \, dS
\]
\[
\leq C(b_0, q_0)\varepsilon^2 + \frac{4}{\mu^2} + O(\varepsilon) \int_{\Gamma_T} z^{\mu+1}(1 - \mu_1 \partial_z \chi)^2 (\partial_z \dot{\varphi})^2 \, dS
\]
\[
\quad + C(b_0, q_0)\varepsilon \left( \int_{\Gamma_T} z^{\mu+1}(B_0 \dot{\varphi})^2 \, dS + \int_{\Gamma_T} z^{\mu+1}(\partial_0 \dot{\varphi})^2 \frac{r}{r^2} \, dS \right)
\]
\[
\leq C(b_0, q_0)\varepsilon^2 + \frac{(1 + b_0^2)^2}{\mu^2} \left( 1 + O((b_0q_0)^{-2}) + O((b_0q_0)^{-\frac{2}{r^T}}) \right) \int_{\Gamma_T} z^{\mu+1}(\partial_z \dot{\varphi})^2 \, dS
\]
\[
\quad + C(b_0, q_0)\varepsilon \left( \int_{\Gamma_T} z^{\mu+1}(B_0 \dot{\varphi})^2 \, dS + \int_{\Gamma_T} z^{\mu+1}(\partial_0 \dot{\varphi})^2 \frac{r}{r^2} \, dS \right).
\]
Hence, Lemma 4.2 is proved. \( \square \)

**Theorem 4.3.** Under the assumptions of Theorem 4.1, for \( \mu < -1 \), one has
\[
C_0T^{\mu+1} \int_{b_0T \leq r \leq \chi(T, \theta)} |\nabla_x \dot{\varphi}(T, r, \theta)|^2 \, dr \, d\theta + C_0 \int_{D_T} z^\mu |
abla_x \dot{\varphi}|^2 \, dz \, dr \, d\theta
\]
\[
\quad + C_0 \int_{\Gamma_T} z^{\mu+1} |\nabla_z \dot{\varphi}|^2 \, dS \leq \int_{D_T} L \dot{\varphi} \cdot M \dot{\varphi} \, dz \, dr \, d\theta + C(b_0, q_0)\varepsilon^2,
\] (4.27)
where \( C_0 = C_0(b_0, q_0) > 0 \) is a generic constant.

**Proof.** To obtain (4.27), we are required to give a delicate estimate of the term \( C_0 \int_{\Gamma_T} z^{\mu+1}(B_0 \dot{\varphi})^2 \, dS \) in the right-hand side of (4.1) so that it can be absorbed by the positive terms in the left-hand side of (4.1).
We now treat the term $\int_\Gamma z^{\mu+1}(B_0\phi)^2\,dS$.

From (3.6) and the definition of $\kappa(\xi, \nabla_x\phi)$, one has

$$
\int_\Gamma z^{\mu+1}(B_0\phi)^2\,dS = \int_\Gamma z^{\mu+1}(\kappa(\xi, \nabla_x\phi) - \mu_2\xi)^2\,dS
\leq \mu_2^2(1 + O((b_0q_0)^{-2})) \int_\Gamma z^{\mu-1}(\xi)^2\,dS + C(b_0, q_0) \int_\Gamma z^{\mu+1}(\kappa(\xi, \nabla_x\phi))\,dS.
$$

(4.28)

Note that

$$
\int_\Gamma z^{\mu+1}\kappa^2(\xi, \nabla_x\phi)\,dS \leq C(b_0, q_0)\varepsilon^2 \int_\Gamma z^{\mu+1}(\xi^2 + |\nabla_x\phi|^2)\,dS
\leq C(b_0, q_0)\varepsilon^2 \int_\Gamma z^{\mu+1}\left(\xi^2 + (\partial_\phi\phi)^2 + \frac{(\partial_\phi\phi)^2}{r^2} + (B_0\phi)^2\right)\,dS.
$$

(4.29)

Furthermore, as $\varepsilon$ is small and by the boundary condition (3.7) together with Lemma 2.1 (iii), one obtains from (4.28)–(4.29) that

$$
\int_\Gamma z^{\mu+1}(B_0\phi)^2\,dS \leq \frac{\mu_2^2(1 + b_0^2)}{(b_0q_0)^2} \left(1 + O((b_0q_0)^{-2}) + O((b_0q_0)^{-\frac{2}{3}})\right) \int_\Gamma z^{\mu-1}\phi^2\,dS
+ C(b_0, q_0)\varepsilon^2 \int_\Gamma z^{\mu+1}\left((\partial_\phi\phi)^2 + \frac{(\partial_\phi\phi)^2}{r^2}\right)\,dS
\leq \frac{1}{4b_0^2} \left(1 + \frac{1}{b_0}O((b_0q_0)^{-2}) + \frac{1}{b_0}O((b_0q_0)^{-\frac{2}{3}})\right) \int_\Gamma z^{\mu-1}\phi^2\,dS
+ C(b_0, q_0)\varepsilon \int_\Gamma z^{\mu+1}\frac{(\partial_\phi\phi)^2}{r^2}\,dS.
$$

(4.30)

Therefore, combining (4.30) with (4.20) in Lemma 4.2 yields

$$
\int_\Gamma z^{\mu+1}(B_0\phi)^2\,dS
\leq C(b_0, q_0)\varepsilon^2 + \frac{(1 + b_0^2)}{4b_0^2\varepsilon^2} \left(1 + \frac{1}{b_0}O((b_0q_0)^{-2}) + \frac{1}{b_0}O((b_0q_0)^{-\frac{2}{3}})\right) \int_\Gamma z^{\mu+1}(\partial_\phi\phi)^2\,dS
+ C(b_0, q_0)\varepsilon \int_\Gamma z^{\mu+1}\frac{(\partial_\phi\phi)^2}{r^2}\,dS.
$$

(4.31)

Substituting (4.31) into (4.1), one obtains

$$
C_0T^{\mu+1} \int_{b_0T \leq r \leq \chi(T, \theta)} |\nabla_x\phi|^2\,drd\theta + C_0 \int_{D_T} z^{\mu} |\nabla_x\phi|^2\,dzd\theta
+ \left(Q_0(b_0, q_0) + \frac{1}{b_0^2}O((b_0q_0)^{-2}) + \frac{1}{b_0^2}O((b_0q_0)^{-\frac{2}{3}})\right) \int_{D_T} z^{\mu+1}(\partial_\phi\phi)^2\,dS
+ C_0 \int_{\Gamma_T} z^{\mu+1}\frac{(\partial_\phi\phi)^2}{r^2}\,dS
\leq \int_{D_T} \mathcal{L}\phi \cdot \mathcal{M}\phi\,dx + C(b_0, q_0)\varepsilon^2,
$$

(4.32)
where \( Q_0(b_0, q_0) = \frac{(\gamma - 1)b_0^2(1 + b_0^2)^3}{8(1 - \frac{\mu^2}{2}b_0^2(1 + b_0^2))} \left(1 - \frac{1}{\mu^2}\right) \) and \( C_0 = C_0(b_0, q_0) > 0 \) is a generic constant. Moreover,

\[ Q_0(b_0, q_0) > 0 \]

because of \( \mu^2 > 1 \). Together with (4.32), this finishes the proof of Theorem 4.3. □

§5. Higher-order weighted energy estimates

In this section, we derive a higher-order energy estimate so that one can establish the decay properties of \( \nabla_x \phi \) and \( \xi \) for large \( z \).

As in [20], we denote by

\[ S = \{ S_1, S_2 \}, \quad \text{where} \quad S_1 = z\partial_z + r\partial_r, \quad S_2 = \partial_\theta, \]  

(5.1)
certain Klainerman vector fields and by

\[ S_r = \{ S_{1r}, S_{2r} \}, \quad \text{where} \quad S_{1r} = z\partial_z + z\partial_z \chi(z, \theta)\partial_r, \quad S_{2r} = \partial_\theta + \partial_\theta \chi \partial_r, \]  

(5.2)
vector fields which are tangential to \( \Gamma \). We will use these vector fields, which are tangential to the cone surface and are nearly tangential to the shock surface, respectively, to act upon Eq. (3.1) and the boundary conditions (3.3) and (3.6). This allows us to raise the order of the energy estimates by a commutator argument. Let us point out that there is a difference to the usual commutator argument inasmuch as the vector fields \( S \) are only nearly tangential to the shock front which causes certain error terms to occur in the estimates which in turn is due to the perturbation of the shock surface with \( r = s_0z \). Furthermore, we cannot adopt the analysis of [31] as we have to deal with a Neumann-type boundary condition on the fixed boundary, while [31] treats an artificial Dirichlet-type boundary condition so that there a Poincaré-type inequality (see also [11, Lemma 1]) as one of the key elements of the analysis of [31] is available. However, by making use of the delicate energy estimate established in §4, we will be able to drive the desired estimates.

To prove Theorem 1.1, we require the following elementary estimate:

**Lemma 5.1.** Let \( \phi \) be a \( C^{k_0}(D_T) \) solution to (3.1), where \( k_0 \in \mathbb{N} \), and

\[
\sum_{0 \leq t \leq k_0 - 1} |\nabla_x S_t \phi| \leq C(b_0, q_0) \sum_{0 \leq t \leq k_0 - 1} |\nabla_x S_t \phi| \quad \text{in} \quad D_T.
\]  

(5.3)

where \( M > 0 \) is some constant and \( \varepsilon \) is sufficiently small. Then

\[ C \sum_{0 \leq t \leq k_0 - 1} |\nabla_x S_t \phi| \leq C(b_0, q_0) \sum_{0 \leq t \leq k_0 - 1} |\nabla_x S_t \phi| \quad \text{in} \quad D_T. \]  

(5.3)

**Proof.** The first inequality is immediate from the definition of the Klainerman vector fields \( S \) in (5.1).

We show the second inequality in (5.3). The case \( k_0 = 1 \) can be verified directly. Cases when \( k_0 \geq 3 \) can then be obtained by an inductive argument. Thus, we only need to deal with the case \( k_0 = 2 \).

Because of

\[
\partial_r = \frac{1}{r} \left( S_1 - z\partial_z \right),
\]

\[
\partial_r^2 = \frac{1}{r^2} \partial_r S_1 + \frac{1}{r^2} z\partial_z S_1,
\]

\[
\partial_r^2 = \frac{1}{r^2} \partial_r + \frac{1}{r} \partial_r S_1 - \frac{1}{r^2} z\partial_z S_1 + \frac{z^2}{r^2} \partial_z^2,
\]

(5.4)
it follows from Eq. (3.1) and the assumptions in Lemma 5.1 that
\[
\left| (1 - f_1) - (2P_1(s) - f_2) \frac{1}{s} + (P_2(s) - f_3) \frac{1}{s^2} \right| \partial_s^2 \dot{\varphi} \right| \leq C(b_0, q_0) \left( \frac{1}{r} |\nabla_x S_2 \dot{\varphi}| + \frac{1}{r} |\nabla_x \dot{\varphi}| \right). \tag{5.5}
\]
Moreover, by the assumptions in the Lemma 5.1 and the expression of \( f_i (1 \leq i \leq 3) \), one has
\[
\sum_{i=1}^{3} |f_i| \leq C(b_0, q_0) \varepsilon.
\]
Combining this with \( s^2 - 2P_1(s)s + P_2(s) < 0 \) from Lemma 2.2 and (5.4)–(5.5) yields
\[
r(|\partial_s^2 \dot{\varphi}, \partial_r^2 \dot{\varphi}, \partial_\theta^2 \dot{\varphi}|) \leq C(b_0, q_0) (|\nabla_x S_2 \dot{\varphi}| + |\nabla_x \dot{\varphi}|).
\]
Together with \( S_2 = \partial_\theta \) and the change of coordinates (1.5), this concludes the proof of Lemma 5.1. □

Now we begin to establish the higher-order energy estimates.

Since the vector fields \( S \) are tangential to the boundary \( r = b_0z, \partial_r S^m \dot{\varphi} = b_0 \partial_r S^m \dot{\varphi} \) holds on \( r = b_0z \) in view of the boundary condition (3.3). Therefore, one can apply Theorem 4.1 directly to \( S^m \dot{\varphi} (0 \leq m \leq k_0 - 1) \) to obtain:

**Lemma 5.2.** Let the assumptions of Theorem 4.1 be fulfilled. If \( \dot{\varphi} \) is a \( C^{k_0}(\overline{D_T}) \)-solution to problem (3.1) with (3.3) and (3.6)–(3.8), where \( k_0 \geq 7 \), then, for \( 0 \leq m \leq k_0 - 1 \) and \( \mu < -1 \),
\[
C_1 T^{\mu+1} \iint_{b_0 T \leq r \leq \chi(T, \theta)} |\nabla_x S^m \dot{\varphi}|^2 (T, r, \theta) \, dzdr \theta + C_2 \iiint_{D_T} \int_{D_T} \int_{D_T} z^\mu |\nabla_x S^m \dot{\varphi}|^2 \, dzdr \theta \\
+ C_3 \iint_{D_T} \int_{D_T} z^\mu (\partial_s S^m \dot{\varphi})^2 \, dsdr \\
\leq \iint_{D_T} \iint_{D_T} L S^m \dot{\varphi} \cdot MS^m \dot{\varphi} \, dzdr \theta + C_5 \iint_{b_0 \leq r \leq \chi(1, \theta)} \left( \partial_s S^m \dot{\varphi} \right) \\
+ (\partial_r S^m \dot{\varphi})^2 + \frac{1}{r^2} (\partial_\theta S^m \dot{\varphi})^2 \right) (1, r, \theta) \, drd \theta + C_6 \iint_{D_T} \int_{D_T} \int_{D_T} z^\mu B_0 S^m \dot{\varphi}^2 \, dzdr \theta. \tag{5.6}
\]
Here, the constants \( C_i, 1 \leq i \leq 6 \), are given in Theorem 4.1. □

As in Theorem 4.3, one needs to control the term \( \iint_{D_T} z^\mu (B_0 S^m \dot{\varphi})^2 \, ds \) in (5.6) to obtain related higher-order weighted energy estimates. In addition, the term \( \iint_{D_T} S^m L \dot{\varphi} \cdot MS^m \dot{\varphi} \, dzdr \theta \) appearing in the right-hand side of (5.6) has also to be paid attention to. This will produce some boundary terms on the shock surface and conic surface, respectively, by integration by parts.

We now establish:

**Theorem 5.3.** Let \( \dot{\varphi} \in C^{k_0}(\overline{D_T}) \) and \( \xi(z, \theta) \in C^{k_0}([0, 2\pi] \times [1, T]) \) be solutions to (3.1) with (3.3) and (3.6)–(3.8), where \( k_0 \geq 7 \), and
\[
\sum_{0 \leq l_1 + l_2 \leq \left( \frac{k_0}{2} \right) + 1} z^{l_1} |\partial_z^{l_1} \partial_\theta^{l_2} \xi| + \sum_{0 \leq l \leq \left( \frac{k_0}{2} \right) + 1} z^l |\nabla_x^{l+1} \dot{\varphi}| \leq M \varepsilon, \tag{5.7}
\]
where \( M > 0 \) is a constant. Then, for sufficiently small \( \varepsilon > 0 \) and \( \mu < -1 \),
\[
\iint_{b_0 T \leq r \leq \chi(T, \theta)} \sum_{0 \leq l \leq k_0 - 1} T^{2l+\mu+1} |\nabla_x^{l+1} \dot{\varphi}|^2 (T, r, \theta) \, dsdr \theta + \iint_{D_T} \sum_{0 \leq l \leq k_0 - 1} z^{2l+\mu} |\nabla_x^{l+1} \dot{\varphi}|^2 \, dzdr \theta \\
+ \iint_{D_T} \sum_{0 \leq l \leq k_0 - 1} z^{2l+\mu+1} |\nabla_x^{l+1} \dot{\varphi}|^2 \, ds \leq C(b_0, q_0) \varepsilon^2. \tag{5.8}
\]
Proof. Note that on the shock surface $\Gamma$
\[
B_0 S^m \phi = S^m_\Gamma B_0 \phi + B_0 (S^m - S^m_\Gamma) \phi + [B_0, S^m_\Gamma]
\]
and
\[
S^m_\Gamma \phi = -\left(\int_0^1 \dot{u}_r(s_0 + t\xi)dt\right)zS^m_\Gamma \xi + \left[ S^m_\Gamma, -\left(\int_0^1 \dot{u}_r(s_0 + t\xi)dt\right)z\right] \xi;
\]
here and below $[\cdot, \cdot]$ stands for the commutator. Together with Lemma 5.2 and the initial condition (3.8), this yields
\[
T^{\mu+1} \int_{b_0 T \leq r \leq \chi(T, \theta)} |\nabla_x S^m \phi| (T, r, \theta) drd\theta + \int \int \int_{D_T} z^\mu |\nabla_x S^m \phi|^2 dzd\theta dS + \int \int \int_{\Gamma_T} z^{\mu+1} |\nabla_x S^m \phi|^2 dS
\leq C(b_0, q_0)\xi^2 + C(b_0, q_0) \sum_{i=1}^4 I_k, \tag{5.9}
\]
where
\[
I_1 = \int \int \int_{D_T} L S^m \phi \cdot M S^m \phi dzd\theta dS,
\]
\[
I_2 = \int \int_{\Gamma_T} z^{\mu-1} \left| [S^m_\Gamma, -\left(\int_0^1 \dot{u}_r(s_0 + t\xi)dt\right)z] \xi \right|^2 dS,
\]
\[
I_3 = \int \int_{\Gamma_T} z^{\mu+1} |B_0 (S^m - S^m_\Gamma) \phi|^2 dS,
\]
\[
I_4 = \int \int \int_{\Gamma_T} z^{\mu+1} |[B_0, S^m_\Gamma] \phi|^2 dS.
\]
To prove Theorem 5.3, we treat each $I_k$ in (5.9) separately. With this aim, we divide the rest of the proof into four steps.

**Step 1. Estimate of $I_1$.** First we derive an explicit representation of $L S^m \phi$. Due to $S_1(\frac{r}{z}) = S_2(\frac{r}{z}) = 0$ and $S_1(\frac{1}{r}) = -\frac{1}{r}, S_2(\frac{1}{r}) = 0$,
\[
L S_1 \phi = S_1 L \phi - 2L \phi, \quad L S_2 \phi = S_2 L \phi.
\]
By induction, one obtains
\[
L S^m \phi = S^m L \phi + \sum_{0 \leq l \leq m-1} C_{lm} S^l \phi, \tag{5.10}
\]
Hence, it follows from Eq. (3.1) and (5.10) that
\[
S^m L \phi = I_{11} + I_{12} + I_{13}, \tag{5.11}
\]
where
\[
I_{11} = f_1(s, \nabla_x \phi) \partial^2_x S^m \phi + f_2(s, \nabla_x \phi) \partial^2_x S^m \phi + f_3(s, \nabla_x \phi) \partial^2_x S^m \phi + \frac{1}{r^2} f_4(s, \nabla_x \phi) \partial^2_\theta S^m \phi
\]
\[
+ \frac{1}{r} f_5(s, \nabla_x \phi) \partial^2_\theta S^m \phi + \frac{1}{r} f_6(s, \nabla_x \phi) \partial^2_\theta S^m \phi,
\]
\[
I_{12} = f_1(s, \nabla_x \phi) [S^m, \partial^2_x] \phi + f_2(s, \nabla_x \phi) [S^m, \partial^2_x] \phi + f_3(s, \nabla_x \phi) [S^m, \partial^2_x] \phi + \frac{1}{r^2} f_4(s, \nabla_x \phi) [S^m, \partial^2_\theta] \phi
\]
\[
+ \frac{1}{r} f_5(s, \nabla_x \phi) [S^m, \partial^2_\theta] \phi + \frac{1}{r} f_6(s, \nabla_x \phi) [S^m, \partial^2_\theta] \phi,
\]
\[
I_{13} = f_1(s, \nabla_x \phi) [S^m, \partial^2_x] \phi + f_2(s, \nabla_x \phi) [S^m, \partial^2_x] \phi + f_3(s, \nabla_x \phi) [S^m, \partial^2_x] \phi + \frac{1}{r^2} f_4(s, \nabla_x \phi) [S^m, \partial^2_\theta] \phi
\]
\[
+ \frac{1}{r} f_5(s, \nabla_x \phi) [S^m, \partial^2_\theta] \phi + \frac{1}{r} f_6(s, \nabla_x \phi) [S^m, \partial^2_\theta] \phi,
\]
\[
\]
\[ I_{13} = \sum_{0 \leq i \leq m} C_{in} \left\{ \sum_{i_{1} + i_{2} + i_{3} \geq 1} C_{i_{1}i_{2}} \left( S^{i_{1}}(f_{1}(s, \nabla_{x} \phi)) \partial_{x}^{2} S^{i_{2}} \phi + S^{i_{1}}(f_{2}(s, \nabla_{x} \phi)) \partial_{x}^{2} S^{i_{2}} \phi \right) + S^{i_{1}}(f_{3}(s, \nabla_{x} \phi)) \partial_{x}^{2} S^{i_{2}} \phi + S^{i_{1}}(\frac{1}{r} f_{4}(s, \nabla_{x} \phi)) \partial_{x}^{2} S^{i_{2}} \phi + S^{i_{1}}(\frac{1}{r} f_{5}(s, \nabla_{x} \phi)) \partial_{x}^{2} S^{i_{2}} \phi \right) + \sum_{0 \leq i \leq m} C_{in} \left\{ \sum_{i_{1} + i_{2} + i_{3} \geq 1} \left( -\frac{1}{r} \right) S^{i_{1}}(f_{7}(s, \nabla_{x} \phi)) \right\} \right\}. \]

Let us stress that only \( I_{11} \) contains derivatives of \( \phi \) of order \( m + 2 \), while \( I_{12} \) and \( I_{13} \) contain derivatives of \( \phi \) of order at most \( m + 1 \) which are thus lower-order terms.

From the expressions of \( f_{i}, 1 \leq i \leq 7 \), in (3.2), the inductive hypothesis (5.7), and Lemma 5.1, one obtains, for \( m \leq k_{0} - 1 \),

\[ |I_{12}| + |I_{13}| \leq C(b_{0}, q_{0}) \varepsilon \sum_{0 \leq i \leq m} |\nabla_{x} S^{i} \phi|^{2}, \]

which implies

\[ \left| \int \int \int_{D_{r}} (I_{12} + I_{13}) \cdot M S^{m} \phi dz dr d\theta \right| \leq C(b_{0}, q_{0}) \varepsilon \int \int \int_{D_{r}} z^{2} \sum_{0 \leq i \leq m} |\nabla_{x} S^{i} \phi|^{2} dz dr d\theta. \quad (5.12) \]

Next, we treat the troublesome term \( \int \int \int_{D_{r}} I_{11} \cdot M S^{m} \phi dz dr d\theta \). In view of the formulas \( \partial_{y}^{2} v \partial_{y} v = \frac{1}{2} \partial_{y} (\partial_{y} v)^{2} \), \( \partial_{y} v \partial_{y} v = \partial_{y_{1}} (\partial_{y_{1}} v \partial_{y_{2}} v) - \frac{1}{2} \partial_{y_{2}} (\partial_{y_{1}} v)^{2} \) and \( \partial_{y_{1}} v \partial_{y_{2}} v = \frac{1}{2} \left( \partial_{y_{1}} (\partial_{y_{1}} v \partial_{y_{2}} v) + \partial_{y_{2}} (\partial_{y_{1}} v \partial_{y_{2}} v) - \partial_{y_{1}} (\partial_{y_{2}} v \partial_{y_{1}} v) \right) \), one has

\[ I_{11} \cdot M S^{m} \phi = \partial_{r} I_{11} + \partial_{y} I_{11} + \partial_{y} I_{11} + I_{11}, \]

where

\[ I_{11} = \left\{ \frac{1}{2} f_{1} z^{\mu} r a (\partial_{x} S^{m} \phi)^{2} - \frac{1}{2} f_{3} z^{\mu} r a (\partial_{r} S^{m} \phi)^{2} - \frac{1}{2 r^{2}} f_{4} z^{\mu} r a (\partial_{x} S^{m} \phi)^{2} \right\} \]

\[ - \frac{1}{2 r} f_{6} z^{\mu} r a \phi \partial_{r} S^{m} \phi \partial_{r} \phi + f_{1} z^{\mu+1} b \partial_{x} S^{m} \phi \partial_{r} S^{m} \phi + \frac{1}{2} f_{2} z^{\mu+1} b (\partial_{x} S^{m} \phi)^{2} \]

\[ + \frac{1}{2 r^{2}} f_{5} z^{\mu+1} b \partial_{r} S^{m} \phi \partial_{r} S^{m} \phi, \]

\[ I_{11} = \left\{ \frac{1}{2} f_{2} z^{\mu} r a (\partial_{x} S^{m} \phi)^{2} + f_{3} z^{\mu} r a \partial_{r} S^{m} \phi \partial_{r} S^{m} \phi + \frac{1}{2 r^{2}} f_{6} z^{\mu} r a \partial_{x} S^{m} \phi \partial_{r} S^{m} \phi \right\} \]

\[ - \frac{1}{2 r} f_{1} z^{\mu+1} b (\partial_{r} S^{m} \phi)^{2} + \frac{1}{2} f_{2} z^{\mu+1} (\partial_{r} S^{m} \phi)^{2} - \frac{1}{2 r^{2}} f_{4} z^{\mu+1} b (\partial_{r} S^{m} \phi)^{2} \]

\[ + \frac{1}{2 r^{2}} f_{5} z^{\mu+1} b \partial_{r} S^{m} \phi \partial_{r} S^{m} \phi, \]

\[ I_{11} = \left\{ \frac{1}{r} f_{4} z^{\mu} r a \partial_{r} S^{m} \phi \partial_{x} S^{m} \phi + \frac{1}{2} f_{5} z^{\mu} r a (\partial_{x} S^{m} \phi)^{2} + \frac{1}{2 r^{2}} f_{6} z^{\mu} r a \partial_{x} S^{m} \phi \partial_{r} S^{m} \phi \right\} \]

\[ + \frac{1}{r} f_{4} z^{\mu+1} b \partial_{r} S^{m} \phi \partial_{x} S^{m} \phi + \frac{1}{2} f_{5} z^{\mu+1} b \partial_{x} S^{m} \phi \partial_{r} S^{m} \phi + \frac{1}{2 r^{2}} f_{6} z^{\mu+1} b (\partial_{r} S^{m} \phi)^{2} \ right\}. \]
\[
+ \left\{ \frac{1}{2} \partial_z (f_2 z^\mu r a) - \frac{1}{2} \partial_z (f_2 z^{\mu+1} b) - \frac{1}{2} \partial_t (f_3 z^{\mu+1} b) - \frac{1}{2r} \partial_b (f_6 z^{\mu+1} b) \right\} (\partial_t S^m \dot{\varphi})^2 \\
+ \left\{ -\partial_b \left( \frac{1}{r^2} f_4 z^\mu r a \right) - \frac{1}{2} \partial_r \left( f_6 z^\mu r a \right) + \frac{1}{2} \partial_r \left( f_5 z^{\mu+1} b \right) \right\} \partial_t S^m \dot{\varphi} \partial_b S^m \varphi \\
+ \left\{ \frac{1}{2} \partial_z \left( \frac{1}{r^2} f_6 z^\mu r a \right) - \frac{1}{2} \partial_b \left( f_4 z^{\mu+1} b \right) - \frac{1}{2} \partial_t \left( f_5 z^{\mu+1} b \right) \right\} \partial_t S^m \dot{\varphi} \partial_b S^m \varphi \\
+ \left\{ \frac{1}{2} \partial_z \left( \frac{1}{r^2} f_4 z^{\mu+1} b \right) + \partial_t \left( \frac{1}{r^2} f_4 z^{\mu+1} b \right) \right\} (\partial_b S^m \dot{\varphi})^2.
\]

Due to assumption (5.7) and in view of the properties of \( f_i \), \( 1 \leq i \leq 7 \), one has
\[
\sum_{i=1}^{3} z^{-\mu-1} |I_{1i}| + z^{-\mu} |I_{11}| \leq C(b_0, q_0) \varepsilon |\nabla_x S^m \dot{\varphi}|^2,
\]
which yields
\[
\left| \iint_{D_T} I_{11} \cdot B_0 S^m \dot{\varphi} \, dz \, dr \, d\theta \right| \\
\leq C(b_0, q_0) \varepsilon \left\{ \iint_{D_T} z^\mu |\nabla_x S^m \dot{\varphi}|^2 \, dz \, dr \, d\theta + T^{\mu+1} \iint_{b_0 T \leq r \leq \chi(T, \theta)} |\nabla_x S^m \dot{\varphi}|^2 (T, r, \theta) \, dr \, d\theta \\
+ \iint_{b_0 \leq r \leq \chi(1, \theta)} |\nabla_x S^m \dot{\varphi}|^2 (1, r, \theta) \, dr \, d\theta + \iint_{T_T} z^{\mu+1} |\nabla_x S^m \dot{\varphi}|^2 \, dS \right\} + \iint_{B_T} \left( b_0 I_{11} - I_{11}^2 \right) \, dS.
\]

The boundary condition \( \partial_b S^m \dot{\varphi} = b_0 \partial_z S^m \dot{\varphi} \) on \( B_T \), together with condition (4.6) and \( f_6|_{s=b_0} = b_0 f_5|_{s=b_0} \), gives by a direct computation
\[
\iint_{B_T} \left( b_0 I_{11} - I_{11}^2 \right) \, dS = 0,
\]
and further
\[
\left| \iint_{D_T} I_{11} \cdot B_0 S^m \dot{\varphi} \, dz \, dr \, d\theta \right| \\
\leq C(b_0, q_0) \varepsilon \left\{ \iint_{D_T} z^\mu |\nabla_x S^m \dot{\varphi}|^2 \, dz \, dr \, d\theta + T^{\mu+1} \iint_{b_0 T \leq r \leq \chi(T, \theta)} |\nabla_x S^m \dot{\varphi}|^2 (T, r, \theta) \, dr \, d\theta \\
+ \iint_{b_0 \leq r \leq \chi(1, \theta)} |\nabla_x S^m \dot{\varphi}|^2 (1, r, \theta) \, dr \, d\theta + \iint_{T_T} z^{\mu+1} |\nabla_x S^m \dot{\varphi}|^2 \, dS \right\}.
\]
Combining this with (5.11), (5.12), and the initial condition (3.8) yields
\[
|I_1| \leq C(b_0, q_0) \varepsilon \sum_{0 \leq l \leq m} \left\{ \iint_{D_T} z^\mu |\nabla_x S^l \dot{\varphi}|^2 \, dz \, dr \, d\theta + T^{\mu+1} \iint_{b_0 T \leq r \leq \chi(T, \theta)} |\nabla_x S^l \dot{\varphi}|^2 (T, r, \theta) \, dr \, d\theta \\
+ \iint_{T_T} z^{\mu+1} |\nabla_x S^l \dot{\varphi}|^2 \, dS + \varepsilon^2 \right\}.
\]

In particular, for \( m = 0 \), one has
\[
\left| \iint_{D_T} \mathcal{L} \dot{\varphi} \cdot M \dot{\varphi} \, dz \, dr \, d\theta \right| \leq C(b_0, q_0) \varepsilon \left\{ \iint_{D_T} z^\mu |\nabla_x \dot{\varphi}|^2 \, dz \, dr \, d\theta + T^{\mu+1} \iint_{b_0 T \leq r \leq \chi(T, \theta)} |\nabla_x \dot{\varphi}|^2 (T, r, \theta) \, dr \, d\theta \\
+ \iint_{T_T} z^{\mu+1} |\nabla_x \dot{\varphi}|^2 \, dS + \varepsilon^2 \right\}.
\]
Substituting this into (4.27) yields
\[
C_0 T^{\mu+1} \int_{b_0 T \leq r \leq T} |\nabla_x \dot{\varphi}|^2 (T, r, \theta) \, dr \, d\theta + C_0 \iint_{\Gamma_T} z^{\mu+1} |\nabla_x \dot{\varphi}|^2 \, dr \, d\theta \, dz \\
+ C_0 \iint_{\Gamma_T} z^{\mu+1} |\nabla_x \dot{\varphi}|^2 \, dS \leq C(b_0, q_0) \varepsilon^2. \tag{5.14}
\]

**Step 2. Estimate of $I_2$.** The vector fields $S_T$ coincide with the vector fields $S$ on $\Gamma$ and a direct computation yields
\[
\left| \left[ S_T^m, \int_0^1 \dot{u}_r(s_0 + t \xi) \, dt \cdot z \right] \right| = \left| \sum_{m_1 + m_2 = m, m_1 \geq 1} C_{m_1 m_2} S^{m_1} \left( \int_0^1 \dot{u}_r(s_0 + t \xi) \, dt \cdot z \right) S^{m_2} \right| \\
\leq C(b_0, q_0) z \left( M \varepsilon \sum_{1 \leq l \leq m} |S^l \dot{\varphi}| + |\xi| \right).
\]
Combining this with (3.7) gives
\[
\left| \left[ S_T^m, \left( \int_0^1 \dot{u}_r(s_0 + t \xi) \, dt \right) z \right] \right| \leq C(b_0, q_0) z \left( M \varepsilon \sum_{1 \leq l \leq m} |S^l \dot{\varphi}| + |\xi| \right),
\]
and further
\[
|I_2| \leq C(b_0, q_0) \left( M \varepsilon \sum_{1 \leq l \leq m-1} \iint_{\Gamma_T} z^{\mu+1} |\nabla_x S^l \dot{\varphi}|^2 \, ds + \iint_{\Gamma_T} z^{\mu+1} |\xi|^2 \, dS \right). \tag{5.15}
\]

**Step 3. Estimate of $I_3$.** One has that
\[
B_0 (S^m - S_T^m) \dot{\varphi} = [B_0, S^m - S_T^m] \dot{\varphi} + (S^m - S_T^m) B_0 \dot{\varphi}. \tag{5.16}
\]
It follows from assumption (5.7) and a direct computation that
\[
\left| [B_0, S^m - S_T^m] \dot{\varphi} \right| \leq C(b_0, q_0) M \varepsilon \sum_{0 \leq l \leq m-1} z^l |\nabla_x^{l+1} \dot{\varphi}| \tag{5.17}
\]
and
\[
| (S^m - S_T^m) B_0 \dot{\varphi} | \leq C(b_0, q_0) M \varepsilon \sum_{0 \leq l \leq m} z^l |\nabla_x^{l+1} \dot{\varphi}|. \tag{5.18}
\]
Combining (5.16)–(5.18) yields
\[
|I_3| \leq C(b_0, q_0) M \varepsilon \sum_{0 \leq l \leq m-1} \iint_{\Gamma_T} z^{2l+\mu+1} |\nabla_x^{l+1} \dot{\varphi}|^2 \, dS. \tag{5.19}
\]

**Step 4. Estimate of $I_4$.** As in Step 2, one has
\[
[B_0, S_T^m] \dot{\varphi} = [B_0, S^m] \dot{\varphi} \quad \text{on } \Gamma.
\]
Due to
\[
[B_0, S^m] \dot{\varphi} = \sum_{0 \leq l \leq m-1} C_{ml} S^l B_0 \dot{\varphi}
\]

and

\[ S^l B_0 \dot{\varphi} + \mu_2 S^l \xi = S^l \kappa(\xi, \nabla_x \dot{\varphi}) \text{ on } \Gamma, \]

one then arrives at

\[ |S^l B_0 \dot{\varphi}| \leq C(b_0, q_0) \left( |S^l \xi| + M \varepsilon \sum_{0 \leq k \leq l} |S^k \xi| + M \varepsilon \sum_{0 \leq k \leq l} |\nabla_x S^k \dot{\varphi}| \right) \]

and

\[ |I_4| \leq C(b_0, q_0) \left( \sum_{0 \leq l \leq m-1} \int \int_{\Gamma_T} z^{\mu+1} |\nabla_x S^l \dot{\varphi}|^2 ds + \int \int_{\Gamma_T} z^{\mu+1} |\xi|^2 dS \right). \quad (5.20) \]

Substituting (5.13)–(5.15), (5.19)–(5.20) into (5.9) and applying Lemma 5.1 yields

\[
\begin{align*}
&\int \int_{T_0 \leq r \leq \chi(T, \theta)} \sum_{0 \leq l \leq k_0-1} T^{2l+\mu+1} |\nabla_x^{l+1} \dot{\varphi}(T, r, \theta)|^2 dS + \int \int_{T_0 \leq r \leq k_0-1} z^{2l+\mu} |\nabla_x^{l+1} \dot{\varphi}|^2 dS \\
&+ \int \int_{T_0 \leq r \leq k_0-1} z^{2l+\mu+1} |\nabla_x^{l+1} \dot{\varphi}|^2 dS \leq C(b_0, q_0) \left( \int \int_{T_0 \leq r \leq k_0-1} z^{\mu+1} |\xi(\xi)|^2 dS + \varepsilon^2 \right). \quad (5.21)
\end{align*}
\]

On the other hand, by the boundary condition (3.7), Lemma 4.2, and (5.14), one has

\[
\int \int_{T_0 \leq r \leq \chi(T, \theta)} \sum_{0 \leq l \leq k_0-1} T^{2l+\mu+1} |\nabla_x^{l+1} \dot{\varphi}(T, r, \theta)|^2 dS + \int \int_{T_0 \leq r \leq k_0-1} z^{2l+\mu} |\nabla_x^{l+1} \dot{\varphi}|^2 dS \leq C(b_0, q_0) \varepsilon^2.
\]

Together with (5.21), this yields

\[
\begin{align*}
&\int \int_{T_0 \leq r \leq \chi(T, \theta)} \sum_{0 \leq l \leq k_0-1} T^{2l+\mu+1} |\nabla_x^{l+1} \dot{\varphi}(T, r, \theta)|^2 dS + \int \int_{T_0 \leq r \leq k_0-1} z^{2l+\mu} |\nabla_x^{l+1} \dot{\varphi}|^2 dS \\
&+ \int \int_{T_0 \leq r \leq k_0-1} z^{2l+\mu+1} |\nabla_x^{l+1} \dot{\varphi}|^2 dS \leq C(b_0, q_0) \varepsilon^2,
\end{align*}
\]

which shows that the conclusion of Theorem 5.3 holds. □

§6. Proof of Theorem 1.1.

Based on the higher-order energy estimate established in Theorem 5.3, we now prove the global existence of a conic shock wave, as asserted in Theorem 1.1, by using a local existence result and continuous induction. For any given \(z_0 > 0\), the solution to (1.8) with the initial data given on \(z = z_0\) and boundary conditions (1.9)–(1.11) exists in an interval \([z_0, z_0 + \zeta]\) for some \(\zeta > 0\) by the local existence result of [24] or [11, Appendix] provided that the initial data is smooth and satisfies the compatibility conditions. Moreover, if the perturbation of the initial data given on \(z = z_0\) is \(O(\varepsilon)\), then the lifespan of the solution is at least \(C/\varepsilon\), with some \(C > 0\). Therefore, as long as one can establish that the \(L^\infty\)-norms of \(\dot{\varphi}, \xi\), and their derivatives decay with a rate in \(z\), the solution can be continuously extended to the whole domain. That is, by the local existence result and the decay properties of the solution one obtains the uniform boundedness of \(\dot{\varphi}, \xi\), and their derivatives, and then one extends the solution continuously from \(z_0 \leq z \leq z_0 + \zeta\) to \(z_0 + \zeta \leq z \leq z_0 + 2\zeta\), with \(\zeta > 0\) being independent of \(z_0\). Hence, the key to proving Theorem 1.1 is to establish the decay of the \(L^\infty\)-norm of \(\dot{\varphi}, \xi\), and their derivatives.

It follows from Sobolev’s embedding theorem (see also [11, Lemma 14]) and the assumptions of Theorem 5.4 that, for \(b_0 z \leq r \leq \chi(z, \theta)\) and \(1 \leq z \leq T\), one has

\[
\sum_{0 \leq l \leq k_0-3} \int \int_{T_0 \leq r \leq \chi(z, \theta)} |z^{l+\mu+1} |\nabla_x^{l+1} \dot{\varphi}|^2 dz dr d\theta.
\]
On the other hand, (5.8) shows that
\[ \int_0^{b_0} \int_{0 \leq r \leq \chi(z, \theta)} \sum_{0 \leq k_0 
-2} \sum_{0 \leq k_0 \leq 1} |z^4 \nabla^{i+1}_x \phi|^2 \, drd\theta \leq C(b_0, q_0)\varepsilon z^{-\mu - 1}. \]
Hence, \[ \sum_{0 \leq t \leq k_0 - 2} \sum_{0 \leq k_0 \leq 1} |z^4 \nabla^{i+1}_x \phi|^2 \leq C(b_0, q_0)\varepsilon z^{-\mu - 2} \] for \( b_0z \leq r \leq \chi(z, \theta) \) and \( 1 \leq z \leq T \). For \( k_0 \geq 7 \), one has \[ \sum_{t \leq b_0^2 + 1} \sum_{0 \leq k_0 \leq 1} z^4 |\partial^2_t \partial^2_{x} \xi| \leq C(b_0, q_0)\varepsilon z^{-\frac{\mu}{2} - 1}. \] In addition, due to \( k_0 - 3 \geq \frac{k_0}{2} + 1 \), Eqs. (3.6) and (3.7) yield
\[ \sum_{0 \leq t \leq b_0^2 + 1} \sum_{0 \leq k_0 \leq 1} \varepsilon z^4 |\partial^2_t \partial^2_{x} \xi| \leq C(b_0, q_0)\varepsilon z^{-\frac{\mu}{2} - 1}. \] If one now chooses \( \mu \in (-2, -1) \), then it follows by continuous induction that the proof of Theorem 1.1 is complete. \( \square \)

Appendix

**Proof of Lemma 3.2.** First we estimate \( P_3(s) \) and \( P'_1(s) \). By (3.2), Lemmas 2.1-2.2 and Remark 2.2, then
\[ P_3(s) = \frac{1}{u_2^2(s) - c^2(\rho(s))} \left( -\frac{\gamma - 1}{2} s^2 \tilde{u}_z(s) \tilde{u}_r(s) + \frac{\gamma + 1}{2} s \tilde{u}_r(s) \tilde{u}'_r(s) \right) \]
\[ + s \tilde{u}_z(s) \tilde{u}'_r(s) + \frac{\gamma - 1}{2} \tilde{u}_r^2(s) - \frac{1}{2} c^2(\rho(s)) \right) \]
\[ = 1 + O((b_0q_0)^{-2}) + O((b_0q_0)^{-\frac{\mu}{2} - 1}) \left( -\frac{\gamma - 1}{2} \frac{b_0q_0}{1 + b_0^2} - \frac{b_0q_0}{(1 + b_0^2)^2} \right) \]
\[ = \frac{1}{2} \frac{b_0}{1 + b_0^2} b_0q_0 + \frac{q_0}{(1 + b_0^2)^2} \left( 1 + O((b_0q_0)^{-2}) \right) \left( 1 + O((b_0q_0)^{-\frac{\mu}{2} - 1}) \right). \]

Next we derive an expression for \( P'_1(s) \). Recall from (3.2) that
\[ P'_1(s) = \frac{\tilde{u}_z(s) \tilde{u}_r(s)}{u_2^2(s) - c^2(\rho(s))}. \]

It follows from Lemmas 2.1-2.2, Remark 2.2 and a direct computation that
\[ P'_1(s) = \frac{b_0q_0}{(1 + b_0^2)^2} \frac{b_0q_0}{(1 + b_0^2)} \left( 1 + O((b_0q_0)^{-\frac{\mu}{2} - 1}) \right) \]
\[ - \frac{q_0}{(1 + b_0^2)^2} \left( 1 + O((b_0q_0)^{-\frac{\mu}{2} - 1}) \right) \left( 1 + O((b_0q_0)^{-\frac{\mu}{2} - 1}) \right) \]
\[ = \frac{1}{2} \frac{b_0}{1 + b_0^2} \left( 1 + O((b_0q_0)^{-2}) \right) \left( 1 + O((b_0q_0)^{-\frac{\mu}{2} - 1}) \right). \]
Other estimates in Lemma 3.2 can be carried out analogously in terms of the expressions for $P_i$, $1 \leq i \leq 5$, in (3.2) and Lemmas 2.1 and 2.2; we omit the details here. The proof of Lemma 3.2 is complete. □

Next, we provide the proof of Lemma 3.3.

**Proof of Lemma 3.3.** From the expressions for $B_i$, $i = 1, 2, 3$, in (3.5) and Lemmas 2.1 and 2.2, one has

\[
B_1 = -\frac{(\gamma-1)}{2A\gamma(1+b_0^2)} (b_0q_0) + \frac{b_0q_0}{1+b_0^2} \left( (b_0q_0)^2 - \frac{q_0}{1+b_0^2} \right) \left( 1 + O((b_0q_0)^{-2}) + O((b_0q_0)^{-2}) \right)
\]

\[+ 2 \left( \gamma - 1 \right) \left( b_0q_0 \right) \left( \frac{b_0q_0}{1+b_0^2} \right) \left( 1 + O((b_0q_0)^{-2}) \right) \]

\[= \frac{2}{1+b_0^2} \left( \frac{\gamma - 1}{2A\gamma(1+b_0^2)} \right) \left( b_0q_0 \right) \left( \frac{b_0q_0}{1+b_0^2} \right) \left( 1 + O((b_0q_0)^{-2}) \right) \left( 1 + O((b_0q_0)^{-2}) \right).
\]

Similarly,

\[
B_2 = \frac{\gamma - 1}{2A\gamma(1+b_0^2)} \left( b_0q_0 \right) \left( \frac{b_0q_0}{1+b_0^2} \right) \left( 1 + O((b_0q_0)^{-2}) \right) \left( 1 + O((b_0q_0)^{-2}) \right),
\]

\[
B_3 = -\frac{1}{b_0(1+b_0^2)} \left( \frac{\gamma - 1}{2A\gamma(1+b_0^2)} \right) \left( b_0q_0 \right) \left( \frac{b_0q_0}{1+b_0^2} \right) \left( 1 + O((b_0q_0)^{-2}) \right) \left( 1 + O((b_0q_0)^{-2}) \right).
\]

On the other hand, expressions for $\mu_i (i = 1, 2)$ can be obtained from the estimates of $B_i$ which completes the proof of Lemma 3.3. □

**Acknowledgments.** Yin Huicheng wishes to express his gratitude to Professor Xin Zhoupine, Chinese University of Hong Kong, and Professor Chen Shuxing, Fudan University, Shanghai, for their constant interest in this problem and many fruitful discussions in the past.

**References**

1. S. Alinhac, *Blowup of small data solutions for a quasilinear wave equation in two space dimensions*, Ann. of Math. (2) **149** (1999), 97–127; II, Acta Math. **182** (1999), 1–23.
2. J.J. Bertin, *Hyperbolic aerothermodynamics*, AIAA, Washington, 1994.
3. Chang, Tung; Hsiao, Ling, *The Riemann problem and interaction of waves in gas dynamics*, Pitman Monographs and Surveys in Pure and Applied Mathematics, 41, Longman, New York, 1989.
4. Chen, Shuxing, *Existence of stationary supersonic flows past a pointed body*, Arch. Ration. Mech. Anal. **156** (2001), 141–181.
5. Chen, Shuxing; Xin, Zhouping; Yin, Huicheng, *Global shock wave for the supersonic flow past a perturbed cone*, Comm. Math. Phys. **228** (2002), 47–84.
6. D. Christodoulou, *Global solutions of nonlinear hyperbolic equations for small initial data*, Comm. Pure Appl. Math. **39** (1986), 267–282.
7. R. Courant; K.O. Friedrichs, *Supersonic flow and shock waves*, Interscience, New York, 1948.
8. R.N. Cox; L.F. Crabtree, *Elements of hypersonic aerodynamics*, Academic Press, New York, 1965.
9. Cui, Dacheng; Yin, Huicheng, *Global conic shock wave for the steady supersonic flow past a cone: Polytropic case*, J. Differential Equations **246** (2009), 641–669.
10. ———, *Global conic shock wave for the steady supersonic flow past a cone: Isothermal case*, Pacific J. Math. **233** (2007), 257–289.
11. P. Godin, *Global shock waves in some domains for the isentropic irrotational potential flow equations*, Comm. Partial Differential Equations **22** (1997), 1929–1997.
12. ———, *The lifespan of a class of smooth spherically symmetric solutions of the compressible Euler equations with variable entropy in three space dimensions*, Arch. Ration. Mech. Anal. **177** (2005), 479–511.
13. ———, *The lifespan of solutions of exterior radial quasilinear Cauchy-Neumann problems*, J. Hyperbolic Differ. Equ. **5** (2008), 519–546.
14. G.H. Hardy; J.E. Littlewood; G. Pólya, *Inequalities*, Cambridge University Press, London, 1964.
15. L. Hörmander, *Lectures on nonlinear hyperbolic differential equations*, Springer, 1997.
16. F. John, *Nonlinear wave equations, formation of singularities*, Univ. Lecture Series, 2, Amer. Math. Soc., Providence, RI, 1990.
17. M. Keel; H. Smith; C.D. Sogge, *Almost global existence for quasilinear wave equations in three space dimensions*, J. Amer. Math. Soc. 17 (2004), 109–153.
18. B.L. Keyfitz; G.G. Warnecke, *The existence of viscous profiles and admissibility for transonic shocks*, Comm. Partial Differential Equations 16 (1991), 1197–1221.
19. S. Klainerman, *The null condition and global existence to nonlinear wave equations*, Nonlinear systems of partial differential equations in applied mathematics, Part 1, Lectures in Appl. Math., 23, Amer. Math. Soc., Providence, RI, 1986, pp. 293–326.
20. S. Klainerman; T.C. Sideris, *On almost global existence for nonrelativistic wave equations in 3D*, Comm. Pure Appl. Math. 49 (1996), 307–321.
21. Lien, Wen-Ching; Liu, Tai-Ping, *Nonlinear stability of a self-similar 3-dimensional gas flow*, Comm. Math. Phys. 204 (1999), 525–549.
22. A. Majda, *One perspective on open problems in multi-dimensional conservation laws*, Multi-dimensional hyperbolic problems and computation, IMA, 29, Springer, 1990, pp. 217–237.
23. A. Majda, *Compressible fluid flow and systems of conservation laws*, Appl. Math. Sci., 53, Springer, New York, 1984.
24. A. Majda; E. Thomann, *Multi-dimensional shock fronts for second order wave equations*, Comm. Partial Differential Equations 12 (1987), 777–828.
25. J. Metcalfe; C.D. Sogge, *Global existence of null-form wave equation in exterior domains*, Math. Z. 256 (2007), 521–549.
26. J. Metcalfe; C.D. Sogge, *Long-time existence of quasilinear wave equations exterior to star-shaped obstacles via energy methods*, SIAM J. Math. Anal. 38 (2006), 188–209.
27. J. Rauch, *BV estimates fail for most quasilinear hyperbolic systems in dimension greater than one*, Comm. Math. Phys. 106 (1986), 481–484.
28. T.C. Sideris, *Formation of singularities in three-dimensional compressible fluids*, Comm.Math.Phys. 101 (1985), 475-487.
29. H.S. Tsien, *Similarity laws of hypersonic flows*, J. Math. Phys. 25 (1946), 247-251.
30. Xin, Zhouping, *Some current topics in nonlinear conservation laws*, Some current topics on nonlinear conservation laws, AMS/IP Stud. Adv. Math., 15, Amer. Math. Soc., Providence, RI, 2000, pp. xiii–xxxi.
31. Xin, Zhouping; Yin, Huicheng, *Global multi-dimensional shock wave for the steady supersonic flow past a three-dimensional curved cone*, Anal. Appl. 4 (2006), 101–132.
32. Xu, Gang; Yin, Huicheng, *Global transonic conic shock wave for the symmetrically perturbed supersonic flow past a cone*, J. Differential Equations 245 (2008), 3389-3432.
33. Xin, Zhouping; Yin, Huicheng, *Global multidi mensional transonic conic shock wave for the perturbed supersonic flow past a cone*, SIAM J. Math. Anal. 41 (2009), 178–218.
34. Xin, Zhouping; Yin, Huicheng, *Instability of one global transonic shock wave for the steady supersonic Euler flow past a sharp cone*, Nagoya J. Math. 199 (2010), 151–181.
35. Yin, Huicheng, *Global existence of a shock for the supersonic flow past a curved wedge*, Acta Math. Sin. (Engl. Ser.) 22 (2006), 1425–1432.
36. Yin, Huicheng, *Long shock for supersonic flow past a curved cone*, Geometry and nonlinear partial equations, Stud. Adv. Math., 29, Amer. Math. Soc., Providence, RI, 2002, pp. 207–215.
37. Zheng, Yuxi, *Systems of conservation laws. Two-dimensional Riemann problems*, Progr. Nonlinear Differential Appl., 38, Birkhäuser Boston, Boston, MA, 2001.