Material laws and related uncommon phenomena in the electromagnetic response of type-II superconductors in longitudinal geometry

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Abstract
Relying on our theoretical approach for the superconducting critical state problem in 3D magnetic field configurations, we present an exhaustive analysis of the electrodynamical response for the so-called longitudinal transport problem in the slab geometry. A wide set of experimental conditions have been considered, including modulation of the applied magnetic field either perpendicular or parallel (longitudinal) to the transport current density. The main objective of our work was to characterize the role of the macroscopic material law that should properly account for the underlying mechanisms of flux cutting and depinning. The intriguing occurrence of negative current patterns and the enhancement of the transport current flow along the center of the superconducting sample are reproduced as a straightforward consequence of the magnetically induced internal anisotropy. Moreover, we show that, related to a maximal projection of the current density vector onto the local magnetic field, a maximal transport current density occurs somewhere within the sample. The elusive measurement of the flux cutting threshold (critical value of such parallel component $J_c^\parallel$) is suggested on the basis of local measurements of the transport current density. Finally, we show that a high correlation exists between the evolution of the transport current density and the appearance of paramagnetic peak structures in terms of the applied longitudinal magnetic field.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

The high interest arising concerning the problem of magnetic flux depinning in type-II superconductors is markedly associated with its relevance to technological and industrial applications achieving elevated transport currents with no discernible energy dissipation. Thus, the distribution of vortices in a real type-II superconductor is determined by the balance between the electromagnetic driving forces perpendicular to the flux lines, and forces pinning the vortices to material inhomogeneities (lattice defects, impurities, and finite-size effects) [1–3]. Per unit volume, this reads $\mathbf{J} \times \mathbf{B} = \mathbf{F}_p$

(or $J_\perp B = F_p$).

On the other hand, it is well known that various striking phenomena occur when a transport current is applied to a nonideal type-II superconductor under the presence of a longitudinal magnetic field [4–33]. First, a remarkable enhancement of the critical current density has been observed in a wide number of conventional and high temperature superconducting systems [3, 8, 23–32]. Second, an intriguing negative longitudinal electric field along the direction of the transport current has been also reported [9–22]. Such a
resistive structure in the longitudinal field geometry has been intuitively understood in terms of helical domains, closely connected to the force-free current parallel to flux lines (recall that the magnetic force per unit volume is given by \( \mathbf{J} \times \mathbf{B} \)) [12, 19, 22, 26].

Nevertheless, although several facts suggest that the vortex lattice is arranged in a helical configuration, perhaps close to the force-free arrangement \((\mathbf{J} \parallel \mathbf{B})\), the arising voltage cannot be straightforwardly described as a critical flux-flow voltage if one is to be consistent with the experimental reports [4, 14]. On the other hand, although several proposals have been made in terms of the crossing and recombination of adjacent nonparallel flux lines, the so-called flux line cutting phenomenon has been basically recognized as the physical mechanism by which the longitudinal voltage is produced [19–21, 31–36]. In particular, remarkable numerical and conceptual difficulties related to the implementation of the above picture of the local dynamics for the transport current in realistic longitudinal geometries have led to unfortunate omissions of the phenomenon in practical calculations related to the applications of type-II materials.

Recently, we have shown that our general critical state theory [5, 6] is able to describe on a quantitative basis the anomalous features involved in the local electrodynamics of a current carrying superconducting sample subjected to variations of both longitudinal and transverse magnetic fields [7]. In brief, our work is developed at a phenomenological level which allows us to deal with 3D magnetic field configurations and nontrivial constraint conditions defining anisotropy effects on the superconducting material law. We have introduced a geometrical description for the constraints on the critical current density in terms of a closed region \( \Delta r \), such that the physically admissible states are given by the condition \( \mathbf{J} \in \Delta r \). When the physical limitations concerning the microscopic phenomena of flux depinning and cutting are considered to be independent, the region \( \Delta r \) becomes a cylinder, with a longitudinal rectangular section of size \( 2L_{\perp} \times 2L_{\parallel} \). This ansatz [5, 6] can be identified with the so-called double critical state model in type-II superconductivity. Hereafter, a related anisotropy parameter, that we shall name after bandwidth \( \chi \equiv \frac{J_{\parallel}}{J_{\perp}} \), will be used.

Preliminary work on the intriguing effects associated with the longitudinal transport problem was presented in [7]. Here, supported by numerical simulations that cover an extensive set of experimental conditions, we put forward a much more complete physical scenario. Thus, we shall show that the striking existence of negative flow domains, local and global paramagnetic structures, and emergence of peak-like structures in both the critical current density and the longitudinal magnetic moment, as well as the compression of the transport current in type-II superconductors under parallel magnetic fields, are all predicted by our general critical state model. In addition, we shall introduce some ideas that could be applied for the determination of the flux cutting threshold from local measurements of the current density flowing along specific layers of the superconducting sample, as correlated to the behavior of the magnetic moment components.

The paper is organized as follows. In section 2, the physical background for the critical state concept in longitudinal geometries under applied 3D magnetic fields is introduced (section 2.1). Then, the underlying approximations and the variational statement for the longitudinal transport problem in superconducting slabs are both described in detail (section 2.2). This encloses the time-discretized description of the magnetic field penetration process, and the local dynamics of the transport current in an arbitrary magnetization process. The theory is then applied for a set of experimental configurations and a collection of material laws as regards the interplay of the flux depinning and cutting mechanisms (section 3). On the one hand, the infinite bandwidth model (or T-state model) with \( J_{\parallel} \gg J_{\perp} \) is assumed (section 3.1). As the necessity of including a model with a well defined value for the threshold \( J_{\parallel} \) will become apparent, in sections 3.2 and 3.3 a more general description is presented as function of the anisotropy effects on the material law. These effects will be investigated in terms of the range of parameter values: \( \chi = 1, 2, 3, 4 \). The procedure will reveal both local and global properties of the magnetic moment and the transport current flow, indicating a possible reconstruction scheme of the underlying material law. Section 4 is devoted to summarizing the main findings of our work.

2. Longitudinal transport problem in the double critical state regime

This section is devoted to introducing the theoretical background which justifies our variational statement as an appropriate tool which allows us to implement a wide class of material laws in the investigation of the electromagnetic response of superconducting samples subjected to 3D magnetic field configurations and transport currents. The characteristic mathematical equations for the critical state problem are introduced in terms of a variational statement that is applied within the infinite slab geometry. The relevance of the parallel and perpendicular critical current limitations will be demonstrated.

2.1. General statements

The dynamics of the local magnetic field and transport current profiles in type-II superconductors is commonly obtained within the critical state model framework. In the basic formulation (Bean’s model, [37–39]) one considers that the magnetic flux lines, when driven by a macroscopic current density, will either penetrate or exit across the superconducting surface. In more detail, when the driving forces (local value of \( \mathbf{J} \times \mathbf{B} \)) overcome the pinning force in a local section of the superconducting sample, the system of vortices rearranges itself into a new metastable state such that the vortex lattice is pinned again when the equilibrium at the boundary is reestablished. Notice that this physical mechanism relates to the component of \( \mathbf{J} \) perpendicular to \( \mathbf{B} \). Since the displacement of the flux lines takes place with high associated resistivity, the system quickly adjusts itself to successive equilibrium states so as to avoid the high resistive losses. On the other hand, reversible energy terms related to the equilibrium properties of the vortex lattice are usually neglected (justified by the range
of interest for the local magnetic fields \( H_\perp \ll H \ll H_\parallel \), and the linear relation \( \mathbf{B} = \mu_0 \mathbf{H} \) is assumed.

For the particular case where the transport current is applied along the direction of the vortex lattice, rotation of vortices is induced by the component of the current density parallel to the magnetic field. Subsequent cross-joining and cutting will occur if the angle between neighboring vortices exceeds some critical value. Thus, in general, for nontrivial conditions, and related to the threshold values for the current density that trigger the displacement of vortex lines in some sense, a critical state may be defined as a given configuration with respect to the direction of the magnetic induction \( \mathbf{B} \). This fact introduces strong difficulties when trying to find an analytical solution for the critical state equation \( \mathbf{J} \). Thus, one can possibility for making the resolution of this system affordable is to find an equivalent variational statement. Thus, one can assume that the functional to be minimized can take the following form in terms of the current density:

\[
F[J_{t+1} \in \Delta \tau] = \frac{8\pi}{\mu_0} \int \int_{\Omega} \Delta A_0 \cdot J_{t+1} \, d\mathbf{r} + \int \int_{\Omega \times \Omega} \frac{J_{t+1} \cdot |J_{t+1} - 2\mathbf{J}|}{|\mathbf{r} - \mathbf{r}'|} \, d\mathbf{r} \, d\mathbf{r}'.
\]

It must be emphasized that equation (6) can be applied for any shape of the superconducting volume \( \Omega \) as well as for any general restriction (material law) for the current density \( J_{t+1} \in \Delta \tau \). It is also to be noticed that solenoidality \( (\nabla \cdot \mathbf{J} = 0) \) has to be imposed so as to be consistent with charge conservation in the quasi-steady regime.

2.2. Variational statement in slab geometry

In this subsection, we derive a specific variational formulation for the longitudinal transport problem in superconducting slabs for eventual 3D field configurations (figure 1), i.e., both in plane and transverse local magnetic field components emerge as derived effects of the external sources (transport currents and/or external magnetic sources).

We shall consider the time evolution of magnetic profiles \( H_{t+1}(z) \) within an infinite superconducting slab of thickness \( 2a \), cooled under the assumption of an initial state defined by a uniform vortex lattice perpendicular to the external surfaces, i.e., a constant magnetic field \( H_0 \). Then, the constraint for the current density (material law \( \mathbf{J} \in \Delta \tau \)) corresponding to the limitations on the parallel and perpendicular directions mentioned above may be visualized by a finite cylinder with its axis parallel to the local magnetic field \( H_0 \) (figure 1(a)). When a transport current is injected along the superconducting slab in the direction of the \( y \)-axis, a magnetic field component \( H_x \) appears, inducing a rotation of the critical current region \( \Delta \tau \) (see figure 1(b)). Finally, if a third magnetic field component \( H_z \) is switched, a new rotation of the region occurs (figure 1(c)). Notice that, by symmetry, the current density is confined to the \( XY \) plane, i.e., \( \mathbf{J} \in (J_x(z), J_y(z), 0) \).

In particular, this means that in practice one should impose the restriction that \( \mathbf{J} \) belongs to the projection of the critical current region onto the plane \( \mathbf{J} \parallel \Delta \tau \) and that \( \nabla \cdot \mathbf{J} = 0 \).

At this point, we must mention that, for numerical convenience, the material law is not strictly used in the form
of a cylinder, but smoothly reshaped as a superellipsoid \([5, 6]\) by means of the relation
\[
\left(\frac{J_0}{L_{\perp}}\right)^{2n} + \left(\frac{J_\parallel}{L_{\parallel}}\right)^{2n} \leq 1.
\]
(7)
The critical current region \(\Delta \parallel \) will be characterized in terms of its bandwidth \(\chi = |J_{\parallel}/J_{\perp}| \) and the superelliptic index \(n\).

We call the readers’ attention to the fact that two analytical approaches for the slab geometry may be found in the literature for extreme situations. The first one was introduced by Brandt and Mikitik in \([46]\) for the regime of strong pinning with very weak longitudinal current conditions, i.e., \(H_z\) must be very high as compared to \(H_y(a)\) (then \(J_{\parallel} \ll J_{\perp}\)). On the other hand, the opposite limit (\(H_y \rightarrow 0\)) was recently considered in \([7]\). Here, and based on the numerical resolution of the variational statement, a complete tour along the whole set of values for the perpendicular field will be presented.

In order to simplify the mathematical statements we shall normalize the electrodynamic quantities by defining \(h = H/J_{\perp} a\), \(j = I/J_{\parallel}\), and \(z = z/a\). In turn, our problem will be described in terms of \(N_s\) discretized layers, each one characterized by a current density function \(j(z) = j_{\parallel}(z) + j_{\perp}(z)\) distributed along \(|z| \leq N_s a/N_s\). For further consideration, notice that the value of the transport current assumed in this work, i.e. penetration to half-thickness, imposes the boundary condition \(h_{x}(0.5) = 0.5\) (figure 2).

In our discretized description, and in terms of Ampere’s law, \(h_{x}(z_i)\) will be evaluated from
\[
h_{x}(z_i) = \delta \sum_{j=i}^{2} \frac{j_{x}(z_j) - j_{x}(z_i)}{2},
\]
(8)
Similarly, the local profiles for the longitudinal magnetic field component \(h_{y}(z_i)\) can be obtained from
\[
h_{y}(z_i) = \delta \sum_{j=i}^{2} \frac{j_{y}(z_j) - j_{y}(z_i)}{2},
\]
(9)
with \(\delta = a/N_s\) the thickness of each layer over the local plane \(z_i\).

Then, for the magnetic process sketched in figure 1(c), equation (6) takes the following form in terms of discrete variables:
\[
F(l = l + 1) = \frac{1}{2} \sum_{i,j} I_{x,i+1}^2 M_{x,y,j}^i - \sum_{i,j} I_{x,i}^2 M_{x,y,j}^i + \frac{1}{2} \sum_{i,j} I_{x,i+1}^2 M_{x,y,j}^i - \sum_{i,j} I_{x,i}^2 M_{x,y,j}^i + \sum_{i} I_{x,i+1}^2 (l - 1/2)(h_{x,i+1}'(a) - h_{x,i}'(a)).
\]
(10)
In this expression we have introduced the sheet currents \(I_{x,i+1} = \delta j_{x}(z_i, t = l + 1)\) and \(I_{x,i} = \delta j_{x}(z_i, t = l + 1)\), and their mutual inductance coefficients \(M_{x,y,i}^j\). \(F\) has to be minimized along the different time steps \((l = 1, 2, \ldots)\) in the magnetic process under consideration. On the other hand, one can show that the parallel and perpendicular projections of the sheet current components are given by
\[
I_{\perp}^2 = (1 - h_{x,i}^2) I_{x,i}^2 + (1 - h_{y,i}^2) I_{y,i}^2 - 2 h_{x,i} h_{y,i} I_{x,i} I_{y,i},
\]
\[
I_{\parallel}^2 = h_{x,i} I_{x,i}^2 + h_{y,i} I_{y,i}^2 + 2 h_{x,i} h_{y,i} I_{x,i} I_{y,i}.
\]
(11)
These components have to be constrained according to the material law in equation (7). Further, for the transport problem, one has to consider the external constraint:
\[
\sum_{i} I_{x,i} = I_{tr}.
\]
(12)

The theoretical framework becomes closed by the following expressions for the mutual inductance coupling elements \([5, 6]\):
\[
M_{x,y,i}^{j} = 1 + 2[\text{min}(i, j)], \quad M_{x,y,i}^{j} = 2 \left(\frac{i}{2} + i - 1\right),
\]
\[
M_{x,y,i}^{j} = 1 + 2[N_s - \text{max}(i, j)], \quad M_{x,y,i}^{j} = 2 \left(\frac{i}{2} + N_s - i\right).
\]
(13)

In summary, the magnetic response of the superconductor is characterized by a collection of discretized current elements for the planar sheets \((j_{x,i}, j_{y,i})\) at the time steps \((l = 1, 2, 3, \ldots)\). Then, the magnetic field profiles may be obtained by means of equations (8) and (9), and one may define the sample’s magnetic moment per unit area as
\[
M_{l+1} = \sum_{i} z \times j_{i},
\]
(14)
where a factor of 2 has been introduced related to the contribution of the U-turns at infinity for each circuit. It is apparent that, as a global quantity, the component $M_X$ will be zero for the symmetry of our problem. However, when calculated for the half-thickness ($i = 1, \ldots, N_z$) it may serve as a measure of the distribution of $j_y$ and, thus, of the anisotropy ratio $\chi$, as will be shown below.

### 3. Influence of the magnetic anisotropy in the critical current

In the previous sections we have visualized the double critical state model as a phenomenological approach which can be formulated by means of a variational problem with physical constraints. Here, based on the above mentioned theoretical statements for the longitudinal transport current problem, we shall show the theoretical predictions for the magnetization process outlined in figure 1(c) as $h_z(a) \equiv H_{z0}$ is increased when one starts from a partially penetrated superconducting sample ($I_a = J_{c\perp} a/2$). Thus, we shall be able to identify the influence of a number of physical parameters along the different stages of the magnetization process. In addition, several initial states $h_{z0}$ will be focused on. Moreover, we shall concentrate on the effect of the flux cutting boundary ($j_{y0}$) considering several conditions for the material law. Two extreme cases ($\chi = 1$ and $\chi \to \infty$) will be considered first (sections 3.1 and 3.2), and the range in between in a second step ($\chi = 2, 3, \text{and } 4$ in section 3.3). Remarkably, our procedure will reveal the fingerprints of the cutting and depinning mechanism, thus being a theoretical pathway for the reconstruction of the material law, represented by the proper region $\Delta_y$.

Henceforth, we shall use the simplified notation T or CT$\chi$ with regards to the infinite bandwidth model (T for transport) or double critical state model (CT for cutting and transport) with anisotropy $\chi = |j_{c\parallel}/j_{c\perp}|$.

#### 3.1. T-states for the longitudinal transport configurations

**3.1.1. Field and current density penetration profiles.** Figure 2 shows the magnetic field profiles and the induced currents for three different initial conditions, i.e., $h_{z0} = 10, 2,$ and 0.5, all of them under assumption of the T-state model. The initial state for the transport current condition ($I_a = J_{c\parallel} a/2$) establishes the initial transport profile $j_y(0 \leq z < a/2) = 0$ and $j_y(a/2 \leq z < a) = 1$. As the transport current is no longer modified, the condition $h_z(a) = 0.5$ can be applied in what follows. On the other hand, by symmetry, one has the condition $h_z(0) = 0$ at the center of the slab.

Then, when the external magnetic field $h_z(a)$ is linearly increased from $h_z(a) = 0$, a current density $j_y$ is induced from the superconducting surface as an effect of the Faraday law. Simultaneously, the local component of the magnetic field $h_z(z)$ increases monotonically following two continuous stages fulfilling the aforementioned boundary conditions. First, the superconducting sample is fully penetrated by the transport current when $h_z^*(a) = 0.845 \pm 0.003$, and eventually the

![Figure 2](image-url)
condition $j_\parallel(0) = 1$ is reached as soon as $h_y(a) \to 0.860$. We notice that the value of $h^*_y(a)$ for the full penetration profile is basically independent of $h_{z0}$, at least to the numerical precision of our numerical calculations. This agrees with the analytical solution of [7]. Second, a remarkable enhancement of the transport current density occurs around the center of the slab as $h_y(a)$ increases over $h^*_y(a)$. Furthermore, an eventual negative current density appears, shielding the positive transport current around the center of the slab. In more detail, notice that the appearance of negative current flow is enhanced when the magnetic component $h_{z0}$ is decreased (figure 3).

In particular, profiles of magnetic field reentry (paramagnetism in the component $h_y$ around the center of the slab) are obtained for $h_{z0} \lesssim 1$ under relatively low applied magnetic fields $h_y(a)$ (see figure 2).

Another remarkable property is that, for the range of values $h_z(0) < h^*_z(a)$, negative surface current appears even for the partial penetration regime; e.g., for $h_{z0} = 0.5$ one has $j_y(a) < 0$ for $h_z(a) > 0.722$. Recall that, in [7], we have analytically shown that both effects, local paramagnetism and negative current zones, are also predicted in the limiting case $h_{z0} = 0$. Along this line, as a general rule, we can conclude that the smaller the value of $h_{z0}$, the sooner the surface of negative transport current and even paramagnetic local effects appear.

### 3.1.2. Analysis of the current density behavior

Figure 3 displays the evolution of the current density vector as a function of the longitudinal magnetic field $h_y = h_y(a)$ along the central and external sheets of the slab. The results are shown for the T-state model ($J_{\parallel} \to \infty$ and $J_{\perp} = 1.0$). Top: the components $j_y$ and $j_x$ at $(z = 0)$ and $(z = a)$. Middle: details of the above behavior. Bottom: behavior of the parallel and perpendicular components of $\mathbf{j}$ in the same conditions as above. The different curves correspond to the values of the perpendicular magnetic field given by $h_{z0} = 1, 2, 5, 10, 20, 50, 100, 200$ (all plots having the same color scale).

![Figure 3](image)

**Figure 3.** Evolution of the local current density as a function of the applied longitudinal magnetic field $h_y = h_y(a)$ along the central and external sheets of the slab. The results are shown for the T-state model ($J_{\parallel} \to \infty$ and $J_{\perp} = 1.0$). Top: the components $j_y$ and $j_x$ at $(z = 0)$ and $(z = a)$. Middle: details of the above behavior. Bottom: behavior of the parallel and perpendicular components of $\mathbf{j}$ in the same conditions as above. The different curves correspond to the values of the perpendicular magnetic field given by $h_{z0} = 1, 2, 5, 10, 20, 50, 100, 200$ (all plots having the same color scale).
(iii) The component of $\mathbf{J}$ perpendicular to $\hat{\mathbf{H}}$ and contained in the plane defined by the vectors $\hat{\mathbf{z}}$ and $\hat{\mathbf{H}}$ or so-called polar current component $J_{\perp}$:

$$J_{\perp} = \frac{H_J (H_x J_x + H_y J_y)}{H (H_x^2 + H_y^2)^{1/2}}. \quad (18)$$

Within the T-state model, the behavior of $\mathbf{J}$ is straightforwardly explained: the unbounded parallel current density allows unconstrained rotations for the flux lines as the applied magnetic field increases. In particular, this leads to negative values of $j_y(a)$ (slope of $h_y(a)$), simultaneous with high $j_z(0)$ (slope of $h_z(0)$). We call the readers' attention to the fact that negative values of the transport current are favored by smaller and smaller values of the field component perpendicular to the surface of the sample $h_{z0}$.

A property to be noticed in figure 3 is that, at the center of the slab, the flux line dynamics is mainly governed by the longitudinal transport current density $j_z(0)$. The basic idea is that, for moderate values of $h_z$, when $h_z$ increases $j_z$ practically becomes $j_z$. As this component is unconstrained, it grows indefinitely in the center.

3.1.3. Magnetic moment of the sample. Let us now concentrate on the magnetostatic properties by means of the global sample's magnetization curve $\mathbf{M}(\mathbf{H})$. Thus, we have calculated $\mathbf{M}$ as a function of the longitudinal magnetic field $h_z(a)$. Figure 4 displays the magnetic moment components $M_x(h_{z0})$ and $M_y(h_{z0})$ in units of $J_{\perp} a^2$. Notice first that within the partial penetration regime ($h_y(a) \leq h_y^*(a)$) the magnetic moment components are almost independent of the transversal magnetic field $h_{z0}$ (at least for non-small values of this quantity). In contrast, when $h_{z0} < 1$ and the patterns of negative current even occur before the full penetrated state, magnetization slightly increases. This is accompanied by faint field reentry effects that are also shown in the figure. Furthermore, as the threshold cutting current $j_{z\parallel}$ is unbounded for the T-state model, the magnetic moment $M_x$ always increases as related to the diverging behavior of $j_z(0)$.

We must emphasize that the unbounded behavior for the parallel current density assumed above (that leads to the prediction of arbitrarily high values of the transport current density) must be physically reconsidered. Thus, the trend of the magnetic moment $M_x$ and also the unbounded longitudinal current density $j_z$ disagree with the experimental evidence recollected in [9–11, 19, 22, 26–30]. In addition, in figure 3 one can notice that, as soon as the flow of negative current along the superconducting surface is reached, it never disappears although the longitudinal magnetic field remains increasing. By contrast, disappearance of the patterns of superficial negative current was detected in [12, 15, 16, 19, 22]. These observations have led us to consider $J_{\parallel}$-bounded descriptions as satisfactory solutions of the peculiar phenomena involved in the longitudinal transport current problem [9, 10, 12, 16].

On the other hand, the method described in this work suits the necessity of dealing with a physically acceptable description of both local and global issues concerning the electromagnetic quantities involved in the longitudinal transport current problem [7]. More realistic models for the material law are presented below.

3.2. CT1 states as a general approach

In this section, we show the results obtained for the square condition given by $\chi = J_{\parallel}/J_{\perp} = 1$ (CT1 in what follows). This can be considered as a lower bound for such a quantity because the experimental values reported in the literature are typically above unity.

In order to obtain continuity with the T-state results obtained above, the electrodynamic quantities of interest have been obtained under the same arguments as developed in section 2.2 with regards to the magnetic process shown at figure 1(c). On the other hand, with the aim of getting a detailed physical interpretation of how the longitudinal and transverse magnetic fields affect the dynamics of the transport current problem, we show the magnetic penetration profiles for low and high perpendicular fields, i.e., $h_{z0} = 0.5$ (figure 5) and $h_{z0} = 10$ (figure 6). Eventually, for completeness, the set of initial conditions $h_{z0}$ is extended in figure 7. It will be shown that the fingerprint of the CT model is identified as a peak effect in the magnetization curves (figure 8) caused by the maximal enhancement of the critical current transport density along the central layer.

In order to ease the interpretation of the intricate behavior of the magnetic profiles under CT conditions, we have divided the experimental process into three successive stages as the longitudinal magnetic field component $h_y(a)$ is increased, i.e. the following:
Figure 5. Profiles of the magnetic field components $h_x[z, h_y(\alpha)]$ and $h_y[z, h_y(\alpha)]$ for a perpendicular field component $h_0 = 0.5$. Also included are the corresponding current density profiles $j_y[z, h_y(\alpha)]$ and $j_x[z, h_y(\alpha)]$ for the CT1 model. For clarity, the longitudinal magnetic field is applied in three stages: top, $h_y(\alpha) = 0.005, 0.050, 0.170, 0.340, 0.680, 0.845, 1.0, 1.1, 1.3$; middle, $h_y(\alpha) = 1.3, 1.6, 1.9, 2.2, 3.0, 4.0, 5.0, 6.0$, and bottom, $h_y(\alpha) = 10, 20, 40, 80, 100, 125, 150, 1000$. Insets in the top panel show a zoom of the current density profiles close to the external sheet of the superconducting slab. Insets in either middle or bottom panels show the shape of one of the profiles $h_y[z, h_y(\alpha)]$ for the corresponding stage. Dashed arrows are included to supply a trace method from the initial profile until the last one corresponding to each one of the aforementioned stages.

(i) The current density at the center $j_y(0)$ increases until a maximum value is obtained (top of figures 5 and 6). Notice that the partial penetration regime is included within this stage.

(ii) The minimum value for the longitudinal current density along the superconducting surface is reached (middle row of figures 5 and 6).

(iii) The longitudinal current density $j_y(\alpha)$ stabilizes around $j_y(\alpha) \approx 0.5$ (bottom of figures 5 and 6).

3.2.1. Field and current density penetration profiles. To start with, notice that the trend of the profiles for the partial penetration regime is fairly independent of the perpendicular magnetic field $h_0$ (figures 5 and 6). Moreover, the partial penetration regime in which the transport current zone progressively penetrates the sample is practically independent of the magnetic anisotropy of the critical state (compare to figure 2). On the other hand, the negative current patterns are also found under the low applied magnetic fields $h_0 < 0.5$. However, by contrast to the results within the previous section, recall that for the T-state model the condition $j_0 \to \infty$ allows unbounded values for the longitudinal current $j_y$ at the center of the sample. For the bounded case, the magnetic anisotropy of the material law $\Delta_r$ defines the maximal current density for the critical state regime. In other words, the maximal length of the vector $J$ within the region $\Delta_r$ defines the maximal transport current allowed in the superconductor.

Recalling [5, 6], the maximal longitudinal transport current density corresponds to the optimal orientation in which the biggest distance within the superelliptic region is obtained, i.e.

$$j_y^{\text{max}} = (1 + \chi^{2n/(n-1)}(n-1)/2n).$$

Note that his equation allows us to obtain the maximum value expected for $j_y$ in terms of the actual critical state model in use. Thus, for $\chi = 1, n = 4$ one gets $j_y^{\text{max}} = 1.3$, a value that can be checked in figures 5 and 6.

3.2.2. Analysis of the current density behavior. Complementary to the field penetration profiles, figure 7 shows the dynamics of the main components of the current density along the central and external layers of the superconducting slab in the CT1 condition. Once again, notice that the full penetration regime can be clearly distinguished from the partial penetration regime.

Also, an interesting property to be noticed is that the value $j_y^{\text{max}}(0)$ is independent of the applied magnetic field at least as regards the existence of the peak effect in the transport current density (figure 7). Thus, the enhancement of the transport current density can be either accelerated or decelerated with the tilt of the applied magnetic field, but in general terms,
its maximum directly relates to the limitation introduced by the cutting mechanism. Physically, this means that the role played by the magnetic anisotropy of the material law may be characterized by the influence of the threshold cutting value on the enhancement of the critical current density. This point will be made clearer when material laws with different values of $\chi$ are considered.

3.2.3. Magnetic moment of the sample. Finally, notice that the limitation introduced by the flux cutting mechanism imposes a maximal compression of the current density within the sample. Thus, the peak effects both for the transport current density $j_y$ (figure 7) and for the magnetic moment component $M_x$ (figure 8) are defined by the instant at which the maximal transport current density occurs. Additionally, upon further increasing the longitudinal applied magnetic field component $h_z(a)$, the profile $h_z(z)$ will be forced to decrease from the central sheet ($z = 0$) toward the external surface ($z = a$). This reversal generates a local distortion of the longitudinal current density $j_y$ in a bow tie shape (see the middle row of figures 5 and 6). Likewise, as soon as the profile $j_y(0) = j_y(1)$ is reached, the magnetic moment $M_x$ starts decreasing (figure 8) as one can see by comparison of figures 7 and 8.

3.3. CT$\chi$ states: emergence of the cutting threshold

The intrinsic interplay between the cutting and depinning mechanisms introduces difficulties when one tries to extract the threshold values for a typical experimental arrangement. Nevertheless, as was argued in the previous section, several fingerprints of the actual material law can be identified, i.e. the longitudinal current density along the central and surface superconducting layers, and the behavior of the magnetic moment curves. In order to complement this scenario, we have simulated three additional experiments, defined by the so-called CT2, CT3, and CT4 conditions, in which the threshold value of the cutting current density is varied relative to the depinning limit ($\chi = 2, 3, 4$ respectively).

3.3.1. Field and current density penetration profiles. The behavior of the local electrodynamic quantities is displayed in terms of the aforementioned three stages. In order to allow comparison, the main features displayed in figures 5 and 6 for the CT1 condition with $h_{z0} = 0.5$ and 10, are also shown in figures 9 and 10 under the conditions CT2, CT3 and CT4. Several peculiarities are to be noted. To start with, the emergence of negative states for the transport current density close to the external surface of the superconducting sample will be more evident either when $h_{z0}$ is reduced and/or the current density anisotropy factor $\chi$ is increased.

3.3.2. Analysis of the current density behavior. In order to confirm the above interpretation, we show the magnetic dynamics of the longitudinal current density $j_y$, and the cutting current component $j_z$ (figure 11) for the conditions CT2, CT3, and CT4. We have taken a wide set of values for the perpendicular field component ($h_{z0}$). On the one hand,
as regards the sample’s surface, figure 11 shows that the longitudinal current density $j_y(a)$ does not display significant differences when one has $\chi \geq 2$ (see also figure 3 for $\chi = \infty$). Thus, the disappearance of the negative current flow along the external superconducting surface does not occur although a very high applied magnetic field has been considered ($h_y(a) = 1000$). On the other hand, it is important to notice that the patterns of the parallel current density along the superconducting surface ($j_{\parallel}(a)$) are almost indistinguishable as soon as the condition $\chi \geq 2$ (CT2) is reached (upper half of figure 11). This implies that for an accurate picture of the parallel critical current, surface properties do not provide useful information.

However, in particular, figure 11 shows that the threshold value for the cutting current density can be estimated from the experimental measurement of the transport current density along the central sheet of the superconducting sample. Notice further that, regardless of the experimental conditions ($h_y(0)$, $h_y(a)$) and also for different bandwidths $\chi$ no significant change occurs in the parallel current density around the central sheet of the sample (lower half of figure 11).

### 3.3.3. Magnetic moment of the sample.

The peak effects observed both in the local transport current density $j_y(0)$ (lower half of figure 11), and on the global magnetic moment $M_x$ (figure 12), are predicted to appear subsequent to the maximal longitudinal transport current density $j_y^{\text{max}}$ (equation (19)).

Furthermore, one additional feature is to be noted: the interval between the instant at which the maximal transport current condition is reached ($j_y(0) = j_y^{\text{max}}$), and the instant at which the slope of the magnetic moment $M_x$ changes sign, could be assumed as a transient or stabilization period required for an accurate determination of the value $j_{\parallel}$ when measurements are performed in terms of the applied longitudinal field $h_y$. Apparently, this transient increases with the value of the perpendicular field $h_y$. From this point on, $j_y(0)$ may be basically identified with $j_{\parallel}$.

### 4. Conclusions

Despite extensive experimental and theoretical studies of the electrodynamic response of type-II superconductors in
Figure 8. Magnetic moment \((M_x, M_y)\) of the slab as a function of the applied magnetic field component \(h_y\) for the CT1 model. The curves are labeled according to the perpendicular magnetic field component \(h_z\) for each.

Figure 9. Profiles of \(h_x[z, h_y(a)]\) and \(j_y[z, h_y(a)]\) for the first (top) and second (middle) stages of the magnetic dynamics described in the cases CT2 (left panel), CT3 (central panel) and CT4 (right panel), all under a field \(h_z = 0.5\). The third stage is only defined for the CT2 case (left pane—bottom). \(h_x(a)\) is as follows. CT2: first stage (left pane—top) \(h_x(a) = 0.005, 0.050, 0.170, 0.340, 0.500, 0.680, 0.845, 1.0, 1.1, 1.3, 1.6, 1.9, 2.2\), second stage (left pane—middle) \(h_x(a) = 5, 6, 8, 10, 20, 40\), and third stage (left pane—bottom) \(h_x(a) = 80, 120, 160, 200, 300, 400, 600, 800, 1000\). CT3: first stage (central pane—top) \(h_x(a) = 0.005, 0.050, 0.170, 0.340, 0.500, 0.680, 0.845, 1.0, 1.5, 1.8, 2.2, 2.6, 3.0\), and second stage (central pane—middle) \(h_x(a) = 4, 5, 6, 8, 15, 20, 40, 70, 100, 150, 200, 300, 400, 600, 1000\). CT4: first stage (right pane—top) \(h_x(a) = 0.005, 0.050, 0.170, 0.340, 0.500, 0.680, 0.845, 1.0, 1.5, 1.8, 2.2, 2.6, 3.0, 4.0\), and second stage (right pane—middle) \(h_x(a) = 5, 6, 8, 15, 20, 40, 100, 200, 400, 600, 1000\). The lower panels of the cases CT3 and CT4 show the profiles for \(j_x[z, h_y(a)]\).
Figure 10. Similar to figure 9, but for a perpendicular magnetic field component $h_z = 10$. The curves are labeled as follows. CT2 (left panel): first stage (top) $h_y(\alpha) = 0.005, 0.050, 0.170, 0.340, 0.500, 0.680, 0.845, 1.0, 5.0, 10, 15, 20, 30, 40$, and second stage (middle) $h_y(\alpha) = 40, 60, 80, 120, 160, 200, 300, 400, 600, 800, 1000$. CT3 (middle panel): first stage (top) $h_y(\alpha) = 0.005, 0.050, 0.170, 0.340, 0.500, 0.680, 0.845, 1.0, 5.0, 10, 15, 20, 25, 30, 40, 60$, and second stage (middle) $h_y(\alpha) = 60, 80, 120, 160, 200, 300, 400, 600, 800, 1000$. CT4 (right panel): first stage (top) $h_y(\alpha) = 0.005, 0.050, 0.170, 0.340, 0.500, 0.680, 0.845, 1.0, 5.0, 10, 15, 20, 35, 30, 40, 60, 80$, and second stage (middle) $h_y(\alpha) = 80, 120, 160, 200, 300, 400, 600, 800, 1000$. At the bottom of panels CT2, CT3 and CT4, the corresponding profiles of $j_x(z, h_y(\alpha))$ are shown.

Figure 11. Dynamics of the current density vector for the models CT2 (left panel), CT3 (middle panel) and CT4 (right panel). The first row of the plots corresponds to the patterns of transport current along the surface layers ($j_y(z = a)$) of the slab and the second one to their corresponding values for the parallel component of the current density ($j_\parallel(a)$). The dynamics of the transport current $j_y$ and the component $j_\parallel$ at the central layer of the superconducting slab ($z = 0$) is shown in the third and fourth rows, respectively. The curves are labeled according to the perpendicular magnetic field component $h_z = 1 \times 10^{-7}, 0.5, 1, 2, 5, 10$ with all plots having the same color scale.
longitudinal geometries, many uncertainties remain about the interaction between flux depinning and cutting mechanisms, and their influence on such striking observations as the appearance of negative transport current flow, the enhancement of the critical transport current density, and the observation of peak effects on the magnetization curves. In this paper, and based on the application of our general critical state theory, we have reproduced theoretically the existence of negative flow domains, local and global paramagnetic structures, and emergence of peak-like structures in the longitudinal magnetic moment, as well as the compression of the transport current density for a wide number of experimental conditions.

The longitudinal transport problem in superconducting slab geometry has been studied as follows: we have considered a superconducting slab lying in the $xy$ plane and subjected to a transport current density along the $y$ direction. The slab is assumed to be penetrated by a uniform vortex array along the $z$ direction, so that the local current density along the thickness of the sample is entirely governed by the depinning component $J_\perp$, perpendicular to the local magnetic field. Subsequently, a magnetic field source parallel to the transport current direction is switched on. Then, the experimental conditions have been changed through the value of the external magnetic field $H_{y0}$. The dynamical behavior of the transport current density is shown to rely on the interaction between the cutting and depinning mechanisms. Moreover, the intensity of the inherent effects has been shown to depend on the perpendicular component $H_{y0}$, being more prominent as this quantity is reduced. By means of our general critical state theory that allows us to modulate the influence of the different physical effects, we have been able to show that the peak structures observed in the magnetization curves and the patterns of the transport current along the central section of a superconducting sample are both directly associated with the local structure of the vortex lattice. Such dependence may become more pronounced as the extrinsic pinning of the material is reduced, in favor of the flux cutting interactions. The same conclusion was pointed out from the experimental measurements of Blamire et al. 

Going into detail, when the cutting threshold is high ($J_{c\parallel} \gg J_{c\perp}$ or $\chi \gg 1$) the emergence of negative current patterns is ensured because unbounded parallel current...
density allows unconstrained rotations for the flux lines as the longitudinal magnetic field increases. Thus, under a range of conditions, the peak effects in the magnetic moment and a modulation of the negative surface currents are predicted.

Concentrating on the local properties within the sample, a clear independence of the field penetration profiles relative to the anisotropy level of the material law has been obtained for the partial penetration regime. On the other hand, as soon as the full penetration state is reached, noticeable effects of the magnetic anisotropy law are predicted within both the central and external layers of the superconducting sample. Thus, according to our results (see figure 11) the elusive parallel critical current density parameter $J_{c\parallel}$ can be obtained from the local measurement of the transport current density along the central layer of the superconducting sample.

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