Tunneling decay of false vortices

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November 12, 2014
Based on the paper *Tunneling decay of false vortices* by Richard Mackenzie, Manu Paranjape and others (hep-th 1308.3501)

I worked on this during the summer of 2013

Some modifications to the paper

Review of the paper *Fate of the false vacuum: Semiclassical theory* by Sidney Coleman (Phys.Rev. D15 (1977) 2929-2936)
Overview

1. Main idea
2. Decay of false vacuum
   - Setting
   - Calculation
3. False vortices
   - Setting
   - Vortex solutions
4. Decay of false vortices
5. Decay of false vacuum 2
6. Conclusion
Theory in $2+1$ dimensions

- Universe is in the false vacuum (phase transition)
- Vacuum bubbles can form and decay
- Vortices are present (disk of true vacuum)
- Decay of vortices useful or not?
Main idea
Decay of false vacuum
False vortices
Decay of false vortices
Decay of false vacuum 2
Conclusion

Setting
Calculation

\[ \mathcal{L} = \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - U(\phi) \]

\[ \phi_+ = \text{false vacuum} \]
\[ \phi_- = \text{true vacuum} \]

- Universe in \( \phi_+ \) (phase transition)
- Locally (bubbles), \( \phi_+ \rightarrow \phi_- \) can happen because of QM, with probability:

\[ \frac{\Gamma}{V} = A e^{-\frac{B}{\hbar}} (1 + O(\hbar)) \]
Interesting because age of the universe $< \infty$

Consider $t$ such that $\Gamma/V$ times 4-volume of past light cone is of order 1

- If $t \ll 1$ year: inapplicable (too hot)
- If $t \sim 1$ year: secondary Big Bang
- If $t \sim 10^9$ years: we should worry!
We compute only $B$ ($A$ is tougher)

Start from QM, but I skip to field theory directly

$$B = S_E = \int d\tau d^3x \mathcal{L}_E$$

$E$ means Euclidean ($\tau = it$) and we solve with the conditions:

$$\lim_{\tau \to \pm \infty} \phi(\tau, \vec{x}) = \phi_+$$

$$\frac{\partial \phi}{\partial \tau}(0, \vec{x}) = 0$$

$$\lim_{|\vec{x}| \to \infty} \phi(\tau, \vec{x}) = \phi_+ \quad \text{(finite energy)}$$
The bounce:

- Not physical
- Not unique (lowest one counts)
We suppose that $\phi$ is invariant under $O(4)$

$$\phi(x) = \phi(\rho), \quad \rho = \sqrt{\tau^2 + |\mathbf{x}|^2}$$

The equation of motion is:

$$\frac{d^2 \phi}{d\rho^2} + \frac{3}{\rho} \frac{d\phi}{d\rho} = U'(\phi)$$

With the conditions:

$$\lim_{\rho \to \infty} \phi(\rho) = \phi_+$$

$$\left. \frac{d\phi}{d\rho} \right|_{\rho=0} = 0 \quad \text{(regular at the origin)}$$

$$\Longrightarrow B = 2\pi^2 \int_{0}^{\infty} \rho^3 d\rho \left( \frac{1}{2} \left( \frac{d\phi}{d\rho} \right)^2 + U(\phi) \right)$$
Equation of motion of a particle in the potential $-U$ with damping

If released far from $\phi_-$
$\implies$ undershoot

If released near $\phi_-$
$\implies$ overshoot

$\implies$ There is a solution (Supposedly with the lowest action of all)
We consider a complex scalar field $\phi$ with a gauge field $A_\mu$.

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \phi)^*(D^\mu \phi) - V(|\phi|)
\]

with $D_\mu = \partial_\mu - ieA_\mu$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

The potential that we use is, after rescaling:

\[
V(|\phi|) = (|\phi|^2 - \epsilon)(|\phi|^2 - 1)^2
\]

This theory is invariant under a local $U(1)$ transformation.
- True vacuum at $\phi = 0$
- Circle of symmetry breaking false vacua at $|\phi| = 1$
We want rotationally symmetric solutions of the form

\[ \phi(t, r, \theta) = f(t, r)e^{in\theta}, \quad A_i(t, r, \theta) = -\frac{n}{e} \epsilon^{ij} r_j r^2 a(t, r) \]

The energy is then

\[ E = 2\pi \int_0^\infty r dr \left( \frac{n^2(\dot{a}^2 + a'^2)}{2e^2 r^2} + \dot{f}^2 + f'^2 + \frac{n^2}{r^2}(1 - a)^2 f^2 + (f^2 - \epsilon)(f^2 - 1)^2 \right) \]

Must have \( f \to 1 \) and \( a \to 1 \) as \( r \to \infty \) for finite energy
Must have \( f \to 0 \) and \( a \to 0 \) as \( r \to 0 \) for continuity
This solution is a VORTEX

- Metastable for certain parameters
- Unstable for other parameters
- Topological defect of winding number $n$
- Section of cosmic string
- Quantized magnetic flux
We solve numerically the static EOM

\[ f'' + \frac{f'}{r} - \frac{n^2}{r^2} (1 - a)^2 f - (f^2 - 1)(3f^2 - (1 + 3\epsilon))f = 0 \]

\[ a'' - \frac{a'}{r} + 2e^2(1 - a)f^2 = 0 \]

With the vortex boundary conditions and a set of parameters \((n, e, \epsilon)\)
Main idea
Decay of false vacuum
False vortices
Decay of false vortices
Conclusion

Setting
Vortex solutions

Tunneling decay of false vortices

Vortex solutions

Vortex \( n=1, e=1.00, \varepsilon=0.10 \)

\begin{align*}
\rho_{\text{grad}} &\quad \rho_{\text{mag}} &\quad \rho_{\text{pot}} &\quad \rho_{\text{tot}} \\
0 &\quad 0.2 &\quad 0.4 &\quad 0.6 \\
0 &\quad 2 &\quad 4 &\quad 6 \\
\end{align*}

Vortex \( n=50, e=1.00, \varepsilon=0.005 \)

\begin{align*}
\rho_{\text{grad}} &\quad \rho_{\text{mag}} &\quad \rho_{\text{pot}} &\quad \rho_{\text{tot}} \\
0 &\quad 0.3 &\quad 0.2 &\quad 0.1 \\
0 &\quad 10 &\quad 20 &\quad 30 &\quad 40 \\
\end{align*}

Vortex \( n=1, e=1.00 \)

\begin{align*}
\varepsilon &\quad \text{no stable vortex} &\quad \text{thick-wall vortex} \\
0 &\quad 0.1 &\quad 0.2 &\quad 0.3 \\
0 &\quad 1 &\quad 2 &\quad 3 &\quad 4 \\
\end{align*}

Tunneling decay of false vortices
We generalize Coleman for the decay of the vortices

Must find the bounce, which respects:

- Vortex as $\tau \to -\infty$
- Turning point at $\tau = 0$
- Vortex as $\tau \to \infty$

Really tough so we restrict to a one parameter family of deformations

We parametrize by the radius $R$ of the vortex to get a bigger action
We must find the extremal solution to

\[ S_E = \int d\tau (T + E) \]

\[ T = 2\pi \int_0^\infty r \, dr \left( \dot{f}^2 + \frac{n^2 \dot{a}^2}{2e^2 r^2} \right) \]

\[ E = 2\pi \int_0^\infty r \, dr \left( \frac{n^2 a'^2}{2e^2 r^2} + f'^2 + \frac{n^2}{r^2} (1 - a)^2 f^2 + (f^2 - \epsilon)(f^2 - 1)^2 \right) \]

With the conditions of the bounce
Thin wall case ($n \gg 1$) most interesting: separate static energy

$$E(R) = E_{int} + E_{wall} + E_{ext}$$

Interior ($r < R - \delta/2$):
- $f = 0$
- $a = \left(\frac{r}{R}\right)^2$

Exterior ($r > R + \delta/2$):
- $f = a = 1$

$$\Rightarrow E_{int} = \frac{2\pi n^2}{e^2 R^2} - \epsilon \pi R^2$$

$$\Rightarrow E_{ext} = 0$$
Wall \((R - \delta/2 < r < R + \delta/2)\):

- \(r \sim R \gg 1\)
- \(f \approx 1\)
- \(1 - a = \frac{1}{R}\)

\[\Rightarrow E_{\text{wall}} = \pi R\]
No stable solution for $\epsilon > 0.24 \left( \frac{e}{2n} \right)^{2/3} \equiv \epsilon_c$ because $R_0$ disappears

At first order in $\epsilon$:

$$R_0 = \left( \frac{2n}{e} \right)^{2/3} , \quad E_0 = \frac{3\pi}{2} \left( \frac{2n}{e} \right)^{2/3} , \quad R_1 = \frac{1}{\epsilon}$$

Decay of vortex: $R_0 \rightarrow R_1$ quantum mechanically and $R_1 \rightarrow \infty$ classically
Recall the solution for $n = 50$, $e = 1$ and $\epsilon = 0.005$:

![Graph showing Vortex (n=50, e=1.00, $\epsilon=0.005$)]

|     | Approximation | Numerical |
|-----|---------------|-----------|
| $E_0$ | 101.5         | 92.5      |
| $R_0$ | 21.5          | 22        |
| $\epsilon_c$ | 0.0109     | 0.0105    |

⇒ Good approximation!!
To have the bounce we need $T$, with $R = R(\tau)$

$$T(R) = T_{int} + T_{wall} + T_{ext}$$

**Interior ($r < R - \delta/2$):**
- $f = 0$
- $a = \left(\frac{r}{R}\right)^2$

$$\Rightarrow T_{int} = \frac{\pi n^2}{e^2} \frac{\dot{R}^2}{R^2}$$

**Exterior ($r > R + \delta/2$):**
- $f = a = 1$

$$\Rightarrow T_{ext} = 0$$

**Wall ($R - \delta/2 < r < R + \delta/2$):**
- $f = f(r - R)$

$$\Rightarrow T_{wall} = \frac{\pi R}{2} \dot{R}^2$$
Must shift energy by $E_0$ to fit with Coleman

$$S_E^{\text{thin}} = \int d\tau \left( B(R)\dot{R}^2 + E(R) - E_0 \right)$$

with $R(-\infty) = R(\infty) = R_0$ and $\dot{R}(0) = 0$

We find the first integral

$$B(R)\dot{R}^2 - E(R) + E_0 = 0$$

$$\implies S_E^{\text{thin}} = 2 \int_{R_0}^{R_1} dR \sqrt{B(R)(E(R) - E_0)}$$
After some approximations we find (for small $\epsilon$)

$$S_{E}^{thin} \sim \frac{4\sqrt{2} \pi}{15\epsilon^2} \left(1 - \frac{45\epsilon}{8} \left(\frac{2n}{e}\right)^{2/3}\right)$$

and we get a lower bound

$$\Gamma^{thin} = A^{thin} \left(\frac{S_{E}^{thin}}{2\pi}\right)^{1/2} e^{-S_{E}^{thin}}$$
Decay rate of vacuum $\phi = 1$ and $A_\mu$ not excited

We already saw the starting point, with $\phi(\tau, \vec{x}) = \phi(\rho)$ and in (2+1) d:

$$S_{E}^{\text{vac}} = 4\pi \int_{0}^{\infty} d\rho \rho^2 (\phi'^2 + V(\phi))$$

$$\phi'' + \frac{2}{\rho} \phi' = V'(\phi)$$

with $\phi(\infty) = 1$ (so $\phi(0) \simeq 0$) and $\phi'(0) = 0$
If $\epsilon$ is small, the friction gives

$$\phi(\rho) = \begin{cases} 0 & \text{for } \rho \lesssim \rho_0 \\ \phi_k & \text{for } \rho \gtrsim \rho_0 \end{cases}$$

with $\phi_k$ respecting (neglecting $\epsilon$ in the potential)

$$\phi'_k = \phi_k(1 - \phi_k^2)$$

$$\Rightarrow S^\text{vac}_E(\rho_0) = \pi \left( \rho_0^2 - \frac{4}{3} \epsilon \rho_0^3 \right)$$
The minimal action is
\[ S_{E}^{\text{vac}} = \frac{\pi}{12\epsilon^2} \]
and the decay rate for a volume \( \Omega \) is about
\[ \Gamma^{\text{vac}} = \Omega A^{\text{vac}} \left( \frac{S_{E}^{\text{vac}}}{2\pi} \right)^{1/2} e^{-S_{E}^{\text{vac}}} \]
Two effects of the vortices:
- Replace existing false vacuum by true vacuum (core)
- Can decay themselves

We compare:
- Universe (volume $\Omega$) in false vacuum
- Universe with $N$ non-interacting vortices (same volume, neglect false vacuum between them)
\[ \frac{\Gamma_{\text{vac}}}{N \Gamma_{\text{thin}}} \sim \exp(S_{E \text{thin}}^{\text{thin}} - S_{E \text{vac}}^{\text{vac}}) = \exp \left( \frac{\pi}{\epsilon^2} \left( \frac{4\sqrt{2}}{15} - \frac{3\sqrt{2}\epsilon}{2} \left( \frac{2n}{e} \right)^{2/3} - \frac{1}{12} \right) \right) \]

For \( n = 50 \) and \( e = 1 \) we get (remember \( \epsilon_c = 0.01 \))

- For \( \epsilon < 0.006 \), \( \frac{\Gamma_{\text{vac}}}{N \Gamma_{\text{thin}}} > 1 \) \( \Rightarrow \) vortices work against the decay
- For \( \epsilon > 0.006 \), \( \frac{\Gamma_{\text{vac}}}{N \Gamma_{\text{thin}}} < 1 \) \( \Rightarrow \) vortices help the decay

Generically \( \epsilon \sim \epsilon_c \) helps the decay
Thank you!