SOLUTION OPERATOR OF INHOMOGENEOUS DIRICHLET PROBLEM IN THE UNIT BALL

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Abstract. In this paper we estimate norms of integral operator induced with Green function related to the Poisson equation in the unit ball with vanishing boundary data.

1. Introduction and Notation

We denote by $B^n$ and $S^{n-1}$ the unit ball and unit sphere in $\mathbb{R}^n$ respectively. Throughout the paper we will assume that $n > 2$ (the case $n = 2$ has been already treated in [14, 15]). By the vector norm $|\cdot|$ we consider $|x| = (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$, and by the norm of an operator $T: X \to Y$ which acts between two normed spaces $X$ and $Y$ we mean

$$\|T\| = \sup\{\|Tx\| : \|x\| = 1\}.$$ 

Let $P$ be the Poisson kernel, i.e. function

$$P(x, \eta) = \frac{1 - |x|^2}{|x - \eta|^n},$$

and let $G$ be the Green function of the unit ball w.r.t. Laplace operator, i.e., the function

$$G(x, y) = c_n \left( \frac{1}{|x - y|^{n-2}} - \frac{1}{|x, y|^{n-2}} \right),$$

where

$$c_n = \frac{1}{(n - 2)\omega_{n-1}},$$

(1.1)

where $\omega_{n-1}$ is the Hausdorff measure of $S^{n-1}$ and

$$[x, y] := |x|y| - y|x| = |y|x| - x|y|.$$ 

As it is known, both functions $P$ and $G$ are harmonic for $|x| < 1$ with $x \neq y$. Let $f: S^{n-1} \to \mathbb{R}^n$ be a $L^1$ integrable function on the unit sphere $S^{n-1}$, and let $g: B^n \to \mathbb{R}^n$ be $L^1$ integrable function in the unit ball. The solution
of the Poisson equation $\triangle u = g$ (in the sense of distributions), in the unit ball, satisfying the boundary condition $u|_{S^{n-1}} = f \in L^1(S^{n-1})$ is given by
\begin{equation}
\begin{split}
u(x) &= P[f](x) - G[g](x) := \int_{S^{n-1}} P(x, \eta) f(\eta) d\sigma(\eta) - \int_{B^n} G(x, y) g(y) dy,
\end{split}
\end{equation}
\begin{equation}
|x| < 1.
\end{equation}
Here $d\sigma$ is the normalized Lebesgue $n - 1$ dimensional measure of the Euclid sphere.

We consider the Poisson equation with inhomogenous Dirichlet boundary condition
\begin{equation}
\begin{split}
\left\{ \begin{array}{l}
\triangle u(x) = g, x \in B^n \\
u|_{\partial B^n} = 0
\end{array} \right.
\end{split}
\end{equation}
where $g \in L^p(B^n)$, $p \geq 1$. The weak solution is then given by
\begin{equation}
\begin{split}
u(x) &= -G[g](x) = -\int_{B^n} G(x, y) g(y) dy, |x| < 1.
\end{split}
\end{equation}

The main goal of our paper is related to estimating various norms of the integral operator $G$. We call it the solution operator of Dirichlet’s problem. The compressive study of this problem for $n = 2$ has been done by the first author in [14]. In [15] it is considered its counterpart for differential operator of Dirichlet’s problem. For some related results concerning the planar case we refer to the papers [2, 4, 5, 6, 7]. In [3], Anderson, Khavinson and Lomonosov considered the $L^2$ norm of the operator
\begin{equation}
\begin{split}
N[f](x) =: \frac{1}{(n - 2)\omega_{n-1}} \int_{B^n} \frac{1}{|x - y|^{n-2}} f(y) dy.
\end{split}
\end{equation}

The following two results extend and generalize the corresponding results obtained in [14] and [3].

**Theorem 1.1.** Let $G : L^p(B) \to L^\infty(B)$, where $p > n/2$. Then
\begin{equation}
\begin{split}
\|G\| = c_n \left( \frac{\pi^{n/2} \Gamma(1 + q) \Gamma\left(\frac{n - q(-2 + n)}{-2 + n}\right)}{\Gamma\left(1 + \frac{n}{2}\right) \Gamma\left(\frac{n}{2 - 2 + n}\right)} \right)^{\frac{1}{q}}, 1 \leq q < \frac{n}{n - 2}
\end{split}
\end{equation}
where $n \geq 3$ and $1/p + 1/q = 1$. In particular for $p = \infty$
\begin{equation}
\begin{split}
\|G\|_\infty = \frac{1}{2n} (n \geq 3).
\end{split}
\end{equation}

**Remark 1.2.** The particular case $p = \infty$ ($q = 1$) of Theorem 1.1 is simply and follows from the following observation. Since the function $u(x) = -\frac{1}{2n} (1 - |x|^2)$ represents unique solution of Poisson equation
\begin{equation}
\begin{split}
\left\{ \begin{array}{l}
\triangle u(x) = 1, x \in \Omega \\
u|_{\partial \Omega} = 0
\end{array} \right.
\end{split}
\end{equation}
it follows that for any integer $n, n \geq 3$ we have
\begin{equation}
\begin{split}
\|G\|_\infty = \sup_{x \in B^n} \left| \int_{B^n} G(x, y) dy \right| = \frac{1}{2n} \sup_{x \in B^n} (1 - |x|^2) = \frac{1}{2n}.
\end{split}
\end{equation}
Theorem 1.3. For $p \geq 1$, the operator $G$ is a bounded operator of the space $L^p$ onto itself with the norm $\|G\|_p$ satisfying the inequalities
\[
\|G\|_p \leq (2n)^{\frac{2}{p} - 2} \lambda_1^{2(1-p)/p} \quad 1 \leq p \leq 2
\]
and
\[
\|G\|_p \leq \lambda_1^{-\frac{2}{p}} (2n)^{\frac{2}{p} - 2} \quad 2 \leq p \leq \infty
\]
which reduces to an equality for $p = 1, 2, \infty$, where $\lambda_1 = \lambda_1(B^n)$ is the first eigenvalues of Dirichlet Laplacian of the unit ball defined in Subsection 2.3.

The proof of Theorem 1.1 is postponed in section 4 and is obtained via M"obius transformations of the unit ball. It depends in Lemma 3.1, which is somehow very involved and presents itself a subtle integral inequality. The proof of Theorem 1.3 uses the eigenvalues of Dirichlet Laplacian and follows from Ries-Thorin interpolation theorem.

2. Preliminaries

2.1. Gauss hypergeometric function. Through the paper we will often use the properties of the hypergeometric functions. First of all, the hypergeometric function $F(a, b, c, t) = 2F_1(a, b; c; t)$ is defined by the series expansion
\[
\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} t^n, \quad |t| < 1,
\]
and by the continuation elsewhere. Here $(a)_n$ denotes shifted factorial, i.e. $(a)_n = a(a + 1)...(a + n - 1)$ and $a$ is any real number. The following identities will be used in the proof of the main results of this paper:

Euler’s identity:
\[
F(a, b; c; t) = (1 - t^2)^{c-a-b} F(c - a, c - b; c; t), \quad \text{Re}(c) > \text{Re}(b) > 0,
\]
Pfaff’s identity:
\[
F(a, b; c; t) = (1 - t^2)^{-a} F(a, c - b; c; \frac{t}{t - 1}), \quad \text{Re}(c) > \text{Re}(b) > 0,
\]
Differentiation identity:
\[
\frac{\partial}{\partial t} F(a, b; c; t) = \frac{ab}{c} F(a + 1, b + 1; c + 1; t),
\]
and Kummer’s Quadratic Transformation
\[
F \left( a, b; 2b; \frac{4t}{(1 + t)^2} \right) = (1 + t)^{2a} F(a, a + \frac{1}{2} - b; b + \frac{1}{2}; t^2),
\]
where above identity is true for every $t$ for which both series converge.

By using the Chebychev’s inequality one can easily obtain the following inequality for Gamma function (see [8]).
Proposition 2.1. Let $m, p$ and $k$ be real numbers with $m, p > 0$ and $p > k > -m$: If
\begin{equation}
    k(p - m - k) \geq 0 \ (\leq 0)
\end{equation}
then we have
\begin{equation}
    \Gamma(p)\Gamma(m) \geq (\leq)\Gamma(p - k)\Gamma(m + k).
\end{equation}

2.2. Möbius transformations of the unit ball. The set of isometries
of the hyperbolic unit ball $B^n$ is a Kleinian subgroup of all Möbius trans-
formations of the extended space $\mathbb{R}^n$ onto itself denoted by $\text{Conf}(B^n) = \text{Isom}(B^n)$. We refer to the Ahlfors’ book [1] for detailed survey to thi
s class of important mappings. In general a Möbius transform
$T_x : B^n \to B^n$ has the form
\begin{equation}
    z = T_x y = (1 - |x|^2) \frac{(y - x) - |y - x|^2 x}{|x|^2}.
\end{equation}
Then we have
\begin{equation}
    |T_x y| = \frac{|x - y|}{|x, y|}.
\end{equation}
If $dy$ denotes the volume measure in the ball, because $y = T_{-x} z$ is a confor-
mal mapping, in view of (2.8) we have
\begin{equation}
    dy = \left(1 - \frac{|x|^2}{|z, -x|^2}\right)^n dz.
\end{equation}

2.3. Eigenvalues of Dirichlet Laplacian. First of all, it is known that
there exist an orthonormal basis of $L^2(B^n)$ consisting of eigenfunctions $(\varphi_n)_n$
of Dirichlet Laplacian
\begin{equation}
    \begin{cases}
    -\Delta u = \lambda u, & z \in B^n \\
    u|_{\partial B^n} = 0
    \end{cases}
\end{equation}
with corresponding eigenvalues $\lambda_1 < \lambda_2 \leq \ldots \leq \lambda_n$... The functions $\varphi_n$ are
real valued.

It is well known that $\lambda_1(B^n)$ is given by the square of the first positive
zero of the Bessel function $J_{\alpha(n-1)/2}(t)$ of the first kind of order $\alpha = (n-1)/2$:
\begin{equation}
    J_{\alpha}(t) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m + \alpha + 1)} \left(\frac{t}{2}\right)^{2m+\alpha}.
\end{equation}

3. The main lemma

Lemma 3.1. Let
\begin{equation}
    I(t) = (1 - t^2)^{n-q(n-2)} \int_0^1 \frac{(1 - r^{n-2})^q}{(1 - r^2 t^2)^{n-q(n-2)+1}} dr, \quad 0 \leq t < 1,
\end{equation}
where $n \geq 3$ is a natural number and $1 < q < \frac{n}{n-2}$. Then the maximal value of function $I(t)$ is attained for $t = 0$, i.e.,
\[
\max_{0 \leq t < 1} I(t) = I(0) = \int_0^1 (1 - r^{n-2})^q r^{n-q(n-2)-1} dr
\]
\[
= \frac{\Gamma(1 + q) \Gamma \left( \frac{n-q(n-2)}{n-2} \right)}{(n-2) \Gamma \left( 1 + q + \frac{n-q(n-2)}{n-2} \right)}.
\]

**Proof.** At the beginning we will observe the case $n > 3$. For $a = n - q(n-2)$ we have $0 < a < 2$ and the next expansion
\[
I(t) = (1 - t^2)^a \int_0^1 (1 - r^{n-2})^\frac{n-a}{n-2} r^{a-1} (1 - r^{2t^2})^a dr
\]
\[
= (1 - t^2)^a \sum_{k=0}^{\infty} \frac{\Gamma(k+a+1)}{\Gamma(a+1)k!} t^{2k} \int_0^1 (1 - r^{n-2})^\frac{n-a}{n-2} r^{2k+a-1} dr
\]
\[
= \frac{\Gamma(2 + \frac{2-a}{n-2})}{(n-2)\Gamma(a+1)} \sum_{k=0}^{\infty} \frac{\Gamma(k+a+1)\Gamma(\frac{a+2k}{n-2})}{\Gamma\left(\frac{2(k+n-1)}{n-2}\right)k!} t^{2k}.
\]

Assume that $n \geq 3$ and $k \geq 0$. Let
\[
K = \frac{2k}{n-2}, \quad M = 2 + \frac{2k+a}{n-2}, \quad P = 2 + \frac{2}{n-2}.
\]

From (2.6) we have
\[
\Gamma(M)\Gamma(P) \leq \Gamma(M-K)\Gamma(P+K).
\]

By using the formula $\Gamma(x+1) = x\Gamma(x)$ and (3.3), we have
\[
\frac{\Gamma\left(\frac{a+2k}{n-2}\right)}{\Gamma\left(\frac{2(k+n-1)}{n-2}\right)} \leq \frac{\Gamma\left(2 + \frac{a+2k}{n-2}\right) 1}{ \Gamma\left(2 + \frac{2}{n-2}\right) \left(\frac{a+2k}{n-2} + 1\right)}.
\]
We obtain for \( a \in (0, 2) \)

\[
\frac{I(t)}{\Gamma(2 + \frac{a}{n-2})} \leq \frac{(1 - t^2)^a}{\Gamma(a + 1)} \sum_{k=0}^{\infty} \frac{\Gamma(a + k + 1)}{\Gamma(1 + k)} \frac{t^{2k}}{\left(\frac{a+2k}{n-2}\right)\left(\frac{a+2k}{n-2} + 1\right)}
\]

\[
\frac{(n-2)(1 - t^2)^a}{a} F\left(2, 1 + a, \frac{2 + a}{2}, t^2\right)
\]

\[
- \frac{(n-2)(1 - t^2)^a}{(n + a - 2)} F\left(1 + a, \frac{1}{2}(n + a - 2), \frac{a+n}{2}, t^2\right)
\]

\[
= \frac{(n-2)}{a(n + a - 2)} F\left(1, \frac{1}{2}(n - a - 2), \frac{a+n}{2}, t^2\right)
\]

\[
:= J(t).
\]

The last expression for the function \( J(t) \) was obtained by using the identity (2.4). Further we have

\[
\frac{\partial J(t)}{\partial t} = -\frac{2t(n-2)}{a + 2} F\left(2, 1 + a, \frac{2 + a}{2}, t^2\right)
\]

\[
- \frac{2t(n-2)(n-a-2)}{(n + a - 2)(a + n)} F\left(2, 1 + \frac{1}{2}(n-a-2), 1 + \frac{a+n}{2}, t^2\right) < 0.
\]

We conclude that the maximal value of the function \( I(t) \) for \( t = 0 \) is attained.

In order to prove the special case \( n = 3, 1 < q < 3 \) of Lemma 3.1 we should notice that

\[
\max_{0 \leq t < 1} I(t) = \max_{0 \leq t < 1} (1 - t^2)^{3-q} \int_0^1 \frac{(1 - r)^q r^{2-q}}{(1 - rt^2)^{4-q}} dr
\]

\[
\leq \max_{0 \leq t < 1} (1 - t^2)^{3-q} \int_0^1 \frac{(1 - r)^q r^{2-q}}{(1 - rt^2)^{4-q}} dr.
\]

Put

\[
J(t) := (1 - t^2)^{3-q} \int_0^1 \frac{(1 - r)^q r^{2-q}}{(1 - rt^2)^{4-q}} dr, 0 \leq t < 1.
\]

By using the Taylor expansion we obtain

\[
J(t) = \frac{\Gamma(1 + q)\Gamma(3 - q)}{6} (1 - t^2)^{3-q} F\left(4 - q, 3 - q, 4; t^2\right), 0 \leq t < 1.
\]

By using (2.1) and (2.2) respectively on expression for \( J(t) \) we have

\[
J(t) = \frac{\Gamma(1 + q)\Gamma(3 - q)}{6} F\left(q, 3 - q, 4; \frac{t^2}{t^2 - 1}\right), 0 \leq t < 1.
\]
So,

\[
\max_{0 \leq t < 1} J(t) = \frac{\Gamma(1 + q)\Gamma(3 - q)}{6} \max_{0 \leq t < 1} F\left( q, 3 - q, 4; \frac{t^2}{t^2 - 1} \right)
\]

\[
= \frac{\Gamma(1 + q)\Gamma(3 - q)}{6} \max_{0 \leq t < 1} F\left( q, 3 - q, 4; 0 \right).
\]

The last equality is a consequence of the fact that \( \frac{t^2}{t^2 - 1} < 0 \) and that coefficients

\[
\frac{(q)k(3 - q)k}{(1)k(4)k}
\]

of the hypergeometric function

\[
F\left( q, 3 - q, 4; \frac{t^2}{t^2 - 1} \right)
\]

are decreasing with respect to \( k \geq 1 \).

So,

\[
\max_{0 \leq t < 1} I(t) = I(0) = \int_0^1 (1 - r)^q r^{2-q} dr = \frac{\pi q(1 - q)(2 - q)}{6 \sin \pi q}.
\]

\[\square\]

4. Proof of Theorem 1.1

We start this section with an easy lemma.

**Lemma 4.1.** Let \( \|G\| := \|G : L^p(B^n) \to L^\infty(B^n)\| \) for \( p > \frac{n}{2} \). Then

\[
\|G\| = \sup_{x \in B^n} \left( \int_{B^n} |G(x, y)|^q dy \right)^{\frac{1}{q}}, \quad \frac{1}{p} + \frac{1}{q} = 1.
\]

**Proof.** Let \( u(x) = G[g](x), \ g \in L^p(B) \). Hölder inequality implies

\[
\|u\|_\infty \leq \sup_{x \in B} \left( \int_B |G(x, y)|^q dy \right)^{\frac{1}{q}} \left( \int_B |g(y)|^p dy \right)^{\frac{1}{p}},
\]

i.e.,

\[
\|G\| \leq \sup_{x \in B^n} \left( \int_{B^n} |G(x, y)|^q dy \right)^{\frac{1}{q}}.
\]

On the other hand, there exist \( x_0 \in B^n \) so that

\[
\left( \int_{B^n} |G(x_0, y)|^q dy \right)^{\frac{1}{q}} > \sup_{x \in B^n} \left( \int_{B^n} |G(x, y)|^q dy \right)^{\frac{1}{q}} - \epsilon.
\]

We fix \( x_0 \in B^n \). Let us consider the function

\[
g(y) = \frac{(G(x_0, y))^{q-1}}{\|(G(x_0, y))^{q-1}\|_p}.
\]
Then
\[
\|G\| \geq |G(g)(x_0)|
\]
\[
= \left( \int_{B^n} |G(x_0, y)|^q \, dy \right)^{\frac{1}{q}} \int_{B^n} |G(x_0, y)|^q \, dy
\]
\[
= \left( \int_{B^n} |G(x_0, y)|^q \, dy \right)^{\frac{1}{q}}
\]
\[
> \sup_{x \in B^n} \left( \int_{B^n} |G(x, y)|^q \, dy \right)^{\frac{1}{q}} - \epsilon,
\]
i.e.,
\[
\|G\| = \sup_{x \in B^n} \left( \int_{B^n} |G(x, y)|^q \, dy \right)^{\frac{1}{q}}, q > 1.
\]

Proof of Theorem 1.1. We divide the proof into two cases.

(i) This case includes the following range for \((n, q)\): \(n > 3\), with \(1 < q < \frac{n}{n-2}\) and \(n = 3\) with \(q \in (2, 3)\). According to Lemma 4.1,
\[
\|G\| = \sup_{x \in B^n} \left( \int_{B^n} |G(x, y)|^q \, dy \right)^{\frac{1}{q}}, q > 1.
\]

Further we have
\[
\|G\|^q = c_n^q \sup_{x \in B^n} \frac{1}{|x - y|^{q(n-2)}} \left| 1 - \frac{|x - y|^{n-2}}{|x, y|} \right|^q \, dy,
\]
where \(c_n\) is defined in (1.1). We use the change of variable \(z = T_{x,y} \) i.e. \(T_{x,z} = y\), in the previous integral where \(T_{x,y}\) Möbius transform defined in (2.7). By (2.9), denoting \(t = |x|\), we obtain,
\[
\sup_{x \in B^n} \int_{B^n} |G(x, y)|^q \, dy
\]
\[
= \sup_{x \in B^n} c_n^q \int_{B^n} \frac{1}{|x - T_{x,z} y|^{q(n-2)}} \left| 1 - \frac{|x - y|^{n-2}}{|x, y|} \right|^q (1 - t^2)^n \, dz
\]
\[
= c_n^q \sup_{x \in B^n} (1 - t^2)^n \int_{B^n} \frac{(1 - |z|^{n-2})^q}{\left| |z|^q(n-2) \, 1 - |x|^2 \right|^{2n}} \, dz
\]
\[
= c_n^q \sup_{x \in B^n} (1 - t^2)^n \int_{B^n} \frac{(1 - |z|^{n-2})^q}{\left| |z|^q(n-2) \, 1 - |x|^2 \right|^{2n}} \, dz.
\]
\[
\|G\|_q^a = c_n^a \sup_{x \in B^n} (1 - t^2)^a \int_{B^n} \left( 1 - \frac{|z|^{n-2}}{|z|^{n-2}} \right)^q [z, -z]^{q(n-2) - 2n} dZ
\]

\[
= c_n^a \sup_{x \in B^n} (1 - t^2)^a \int_0^1 \frac{(1 - r^{n-2})^q}{r^{1-a}} dr \int_S \frac{d\xi}{r x + \xi} ^2 |2n-q(n-2)-1|^{n+q-2} \]

\[
= c_n^a \sup_{x \in B^n} (1 - t^2)^a \int_0^1 \frac{(1 - r^{n-2})^q}{r^{1-a}} dr \int_1 \frac{(1 - r^{n-2})^q}{r^{1-a}} dr \int_{-1}^{1} \frac{(1 - r^{n-2})^q}{r^{1-a}} dr \int_{-1}^{1} \frac{(1 - s^2)^{\frac{n-3}{2}}}{s^{\frac{n+2}{2}}} ds,
\]

where

\[
a = n - q(n-2), \quad C_n = \frac{\omega_{n-1} \Gamma(n - 1)}{2^{n-2} \Gamma(n-2)}
\]

and in last two equalities it was assumed without loss of generality that

\[x = te, \xi = (\xi_1, ..., \xi_n)\]. If we take change of variable

\[
\tau = \frac{1 - s}{2}
\]

in the previous integral we have

(4.3)

\[
\|G\|_q^a = C_n^a \sup_{x \in B^n} (1 - t^2)^a \int_0^1 \frac{(1 - r^{n-2})^q}{r^{1-a}} dr \int_1 \frac{(1 - s^2)^{\frac{n-3}{2}}}{s^{\frac{n+2}{2}}} ds
\]

\[
= 2^{n-2} C_n \sup_{x \in B^n} (1 - t^2)^a \int_0^1 \frac{(1 - r^{n-2})^q r^{a-1}}{(1 + rt)^{n+a}} dr \int_0^1 \frac{\tau^{\frac{n-3}{2}} (1 - \tau)^{\frac{n-3}{2}}}{(1 - \frac{4rt}{(1+rt)^2})^{\frac{n+2}{2}}} d\tau.
\]

On the other hand, for fixed \(r\) we have \(\frac{4rt}{(1+rt)^2} < 1\) and

\[
\int_0^1 \frac{\tau^{\frac{n-3}{2}} (1 - \tau)^{\frac{n-3}{2}}}{(1 - \frac{4rt}{(1+rt)^2})^{\frac{n+2}{2}}} d\tau
\]

\[
= \sum_{k=0}^{\infty} \frac{\Gamma(\lambda + k)}{k! \Gamma(\lambda)} \left( \frac{4rt}{(1+rt)^2} \right)^k \int_0^1 \tau^{k+\frac{n-3}{2}} (1 - \tau)^{\frac{n-3}{2}} d\tau
\]

(4.4)

\[
= \Gamma \left( \frac{n-1}{2} \right) \sum_{k=0}^{\infty} \frac{\Gamma(\lambda + k) \Gamma(k + \frac{n-3}{2} + 1)}{k! \Gamma(\lambda) \Gamma(n-1 + k)} \left( \frac{4rt}{(1+rt)^2} \right)^k
\]

\[
= \frac{\Gamma^2 \left( \frac{n-1}{2} \right)}{\Gamma(n-1)} \left( \frac{4rt}{(1+rt)^2} \right)^{\frac{n+1}{2}},
\]

where \(\lambda = \frac{n+1}{2}\).

By using Kummer quadratic transformation and Euler’s transformation for
hypergeometric functions, for \( t = |x| \), we obtain

\[
\sup_{x \in B^n} (1 - t^2)^a \int_0^1 \frac{(1 - r^{n-2}) q r^{a-1}}{(1 + rt)^{n+a}} F \left( \frac{n - 1}{2}; n - 1; \frac{4rt}{1 + rt} \right) dr
\]

\[
= \sup_{x \in B^n} (1 - t^2)^a \int_0^1 (1 - r^{n-2}) q r^{a-1} F \left( \frac{n + a + 2}{2}; \frac{n - 1}{2}; r^2 t^2 \right) dr
\]

\[
= \sup_{x \in B^n} (1 - t^2)^a \int_0^1 (1 - r^{n-2}) q r^{a-1} (1 - r^2 t^2)^{-a-1} F(rt) dr
\]

\[
\leq \sup_{x \in B^n} (1 - t^2)^a \int_0^1 (1 - r^{n-2}) q r^{a-1} (1 - r^2 t^2)^{-a-1} \max_{t \leq 1} F(rt) dr,
\]

where

\[
F(s) = F \left( -\frac{a}{2}, \frac{q(n - 2) - 2}{2}; \frac{n}{2}; s^2 \right).
\]

Then by using the identity for the derivative of hypergeometric function we obtain

\[
\frac{\partial}{\partial t} F \left( -\frac{a}{2}, \frac{q(n - 2) - 2}{2}; \frac{n}{2}; r^2 t^2 \right)
\]

\[
= -2 r^2 t \frac{a}{2} q(n - 2) - 2 - 2 n \frac{2}{2} \frac{q(n - 2) - n + 2}{2} F \left( \frac{q(n - 2) - n + 2}{2}; \frac{n + 2}{2}; r^2 t^2 \right) < 0,
\]

for any \( t \in [0, 1] \), which implies

\[
\max_{|x| \leq 1} F \left( -\frac{a}{2}, \frac{q(n - 2) - 2}{2}; \frac{n}{2}; r^2 |x|^2 \right) = F \left( -\frac{a}{2}, \frac{q(n - 2) - 2}{2}; \frac{n}{2}; 0 \right).
\]

Finally, according to Lemma 3.1, for \( n > 3 \) the maximal value of the function

\[
\mathcal{I}(x) = \int_{B^n} |G(x,y)|^q dy
\]

\[
= c_n^q (1 - |x|^2)^a \int_0^1 (1 - r^{n-2}) q r^{a-1} dr \int_S \frac{d\xi}{|r x + \xi|^{2n-q(n-2)}}
\]
is attained for $x = 0$. So,

$$(4.8)$$

$$
\|G\|^q : c_h^q = \sup_{x \in B^n} (1 - |x|^2)^a \int_0^1 (1 - r^{n-2})^{-q} r^{a-1} dr \int_S \frac{d\xi}{|rx + \xi|^{n+a}} = \omega_{n-1} \sup (1 - |x|^2)^a \int_0^1 (1 - r^{n-2})^{-q} r^{a-1} dr \mathcal{F}(r|x)) dr
$$

$$= \omega_{n-1} \int_0^1 (1 - r^{n-2})^{-q} r^{n-q(n-2)-1} dr F\left(\frac{n+a}{2}, \frac{q(n-2)-2}{2}; \frac{n}{2}; 0\right)$$

$$= \omega_{n-1} \int_0^1 (1 - r^{n-2})^{-q} r^{a-1} dr = \frac{\omega_{n-1} \Gamma(1 + q) \Gamma\left(\frac{n-q(n-2)}{n-2}\right)}{(n-2) \Gamma(1 + q + \frac{n-q(n-2)}{n-2})}.$$

(ii) The case $n = 3$ with $1 < q \leq 2$. It is clear that

$$I(x) = \int_{B^3} |G(x, y)|^q dy = \frac{1}{(2\pi)^q} \int_{B^3} \left(\frac{1}{|x - y|} - \frac{1}{|x, y|}\right)^q dy,$$

and that the same transforms for $I(x)$ as in the previous general case give

$$I(x) = c_3 (1 - x^2)^{3-q} \int_0^1 (1 - r)^q r^{2-q} F((6 - q)/2, (5 - q)/2, 3/2, r^2 x^2) dr,$$

where $c_3$ is appropriate constant as in general case. Put $t = |x|$. We can represent $I(x)$ as

$$I(x) = c_3 \int_0^1 (1 - r)^q r^{1-q} (1 - t^2)^{3-q} \left((1 - rt)^{-4+q} - (1 + rt)^{-4+q}\right) dr.$$

So,

$$I(x) = c_3 \frac{(1 - t^2)^{3-q}}{2(4 - q) t} \sum_{n=0}^{\infty} t^n \int_0^1 (1 - r)^q r^{1-q} (r^n - (-r)^n) \left(\frac{-4+q}{n}\right) dr,$$

and this implies

$$I(x) = c_3 \frac{(1 - t^2)^{3-q}}{2(4 - q) t} \sum_{n=0}^{\infty} \frac{(-1 + e^{in\pi}) (\frac{-4+q}{n}) \Gamma(2 + n - q) \Gamma(1 + q)}{\Gamma(3 + n)} t^n.$$

Thus

$$I(x) = c_3 \frac{\pi(-1 + q) (1 - t^2)^{3-q} \left(F(2 - q, 4 - q; 3; t) - F(2 - q, 4 - q; 3; -t)\right)}{4 \sin(\pi q) (4 - q) t}.$$

Let

$$c'(q) := c_3 \frac{2^{-q} \pi^{2-q} (-1 + q) q}{(4 - q) \sin(\pi q)}.$$

Then

$$I(x)/|c'| \leq I_1(x) = \frac{1 - t^2}{t} \left(F(2 - q, 4 - q; 3; t) - F(2 - q, 4 - q; 3; -t)\right)$$
for $1 < q < 2$ and

$$I_1(x) = a_0 + \sum_{n=1}^{\infty} a_n t^n$$

where $a_0 > 0$ and

$$a_n = \frac{2(1 + (-1)^n) \Gamma(3 + n - q)(-(n - q))\Gamma(4 + n) + \Gamma(n)\Gamma(5 + n - q))}{\Gamma(n)\Gamma(2 + n)\Gamma(4 + n)\Gamma(2 - q)\Gamma(4 - q)}.$$ 

Further $a_n \leq 0$ because

$$\frac{(1 + n - q)(2 + n - q)(3 + n - q)(4 + n - q)}{n(1 + n)(2 + n)(3 + n)} \leq 1,$$

which again implies that maximal value of the function $I(x)$ is attained for $x = 0$. This finishes the proof of Theorem 1.1. \qed

5. Proof of Theorem 1.3

Let $\Omega$ be a domain of $\mathbb{R}^n$ and let $|\Omega|$ be its volume. For $\mu \in (0,1]$ define the operator $V_\mu$ on the space $L^1(\Omega)$ by Riesz potential

$$(V_\mu f)(x) = \int_{\Omega} |x - y|^{n(\mu - 1)} f(y) dy.$$ 

The operator $V_\mu$ is defined for any $f \in L^1(\Omega)$ and $V_\mu$ is bounded on $L^1(\Omega)$, or more generally we have the next lemma.

**Lemma 5.1.** [11, p. 156-159]. Let $V_\mu$ be defined on the $L^p(\Omega)$, $p > 0$. Then $V_\mu$ is continuous as a mapping $V_\mu : L^p(\Omega) \to L^q(\Omega)$, where $1 \leq q \leq \infty$, and

$$0 \leq \delta = \delta(p,q) = \frac{1}{p} - \frac{1}{q} < \mu.$$ 

Moreover, for any $f \in L^p(\Omega)$

$$\|V_\mu f\|_q \leq \left(\frac{1 - \delta}{\mu - \delta}\right)^{1-\delta} (\omega_{n-1}/n)^{1-\mu} |\Omega|^{\mu-\delta} \|f\|_p.$$ 

**Remark 5.2.** If instead of Riesz potential we consider the solution operator for a domain $\Omega$ with finite volume, then the operator is in general non-bounded. However it is bounded, if the boundary of $\Omega$ is enough regular. See [12] for an essential approach to the solution of this problem.

**Theorem 5.3.** Let $\|G\|_1 := \|G : L^1(B) \to L^1(B)\|$, then

$$\|G\|_1 = \frac{1}{2n}.$$ 

**Proof.** According to Theorem 1.1 we have

$$\|G\|_{L^\infty \to L^\infty} = \frac{1}{2n}.$$ 

On the other hand, Lemma 5.1 states that $G : L^1 \to L^1$ is bounded. Then

$$\|G\|_{L^1 \to L^1} = \|G^*\|_{L^\infty \to L^\infty},$$
where $\mathcal{G}^*$ is appropriate adjoint operator. Since 
\[ \mathcal{G}^* f(x) = \int_{B^n} \overline{G(y,x)} f(y) dy = \int_{B^n} G(x,y) f(y) dy, f \in L^\infty(B), \]
we have 
\[ \|\mathcal{G}\|_{L^1 \to L^1} = \|\mathcal{G}\|_{L^\infty \to L^\infty}. \]

In the sequel we are going to observe Hilbert case $p = 2$, $\mathcal{G} : L^2(B) \to L^2(B)$. It is well-known that $\mathcal{G}^{-1} = -\triangle$ on the Sobolev space $H^1_0(\Omega)$, so the Hilbert norm $\mathcal{G}$ is precisely the reciprocal value of the norm of $-\triangle$ (c.f. [3]). So we have the following theorem, whose proof is included for the sake of completeness.

**Theorem 5.4.** Let $\|\mathcal{G}\|_2 := \|\mathcal{G} : L^2(B^n) \to L^2(B^n)\|$, then 
\[ \|\mathcal{G}\|_2 = \frac{1}{\lambda_1}. \]

Thus
\[ \|\mathcal{G}g\|_2 \leq \frac{1}{\lambda_1} \|g\|_2, \ g \in L^2(B^n). \] (5.1)

Equality is attained in (5.1) for $g(x) = c\varphi_1(x)$, a.e. $x \in B^n$ where $c$ is a real constant.

**Proof.** If $f \in L^2(B^n)$, then under the previous notation 
\[ f(x) = \sum_{k=1}^{\infty} \langle f, \varphi_k \rangle \varphi_k(x). \]
Since $\mathcal{G}$ is bounded, we have 
\[ \mathcal{G}[f] = \sum_{k=1}^{\infty} \langle f, \varphi_k \rangle \mathcal{G}[\varphi_k]. \]
Also,
\[ \mathcal{G}[\varphi_k] = \frac{1}{\lambda_k} \mathcal{G}[\triangle \varphi_k] = -\frac{1}{\lambda_k} \varphi_k. \]

The fact that $(\varphi_k)$ is orthonormal implies
\[ \|\mathcal{G}f\|_2^2 = \sum_{k=1}^{\infty} \frac{|\langle f, \varphi_k \rangle|^2}{\lambda_k^2}. \]

Since $\lambda_1$ is a simple eigenvalue and $0 < \lambda_1 < \lambda_2 \leq ...$, we have 
\[ \|\mathcal{G}f\|_2 \leq \frac{1}{\lambda_1} \|f\|_2. \]
Finally,
\[ \|\mathcal{G}\|_2 = \frac{1}{\lambda_1}. \] \[ \square \]
By using the Riesz-Thorin interpolation theorem \[17\], we obtain the following estimates of the norm of the operator \( \mathcal{G} : L^p \rightarrow L^p \).

Let us denote by \( \| \mathcal{G} \|_{L^1 \rightarrow L^1} = \| \mathcal{G} \|_{L^\infty \rightarrow L^\infty} = \| \mathcal{G} \|_1 \) and \( \| \mathcal{G} \|_{L^2 \rightarrow L^2} = \| \mathcal{G} \|_2 \). Then

\[
\| \mathcal{G} \|_p \leq \| \mathcal{G} \|_2^{\frac{2p}{p+1}} \| \mathcal{G} \|_2^{\frac{p(1-p)}{p}} = (2n)^{\frac{p-2}{p}} \lambda_1^{\frac{2(1-p)}{p}} ,
\]

where \( \| \mathcal{G} \|_p \) represents the norm of the operator \( \mathcal{G} : L^p(B^n) \rightarrow L^p(B^n) \), \( 1 < p < 2 \). Similarly,

\[
\| \mathcal{G} \|_p \leq \| \mathcal{G} \|_2^{\frac{2}{p}} \| \mathcal{G} \|_1^{\frac{p-2}{p}} = \lambda_1^{\frac{2}{p}} (2n)^{\frac{p-2}{p}} ,
\]

where \( \mathcal{G} : L^p(B^n) \rightarrow L^p(B^n) \), \( 2 < p < \infty \). This yields the proof of Theorem \[13\].

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