SYMPLECTIC EMBEDDINGS AND SPECIAL KÄHLER GEOMETRY OF $CP(n-1,1)$

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ABSTRACT

The embedding of the isometry group of the coset spaces $\frac{SU(1,n)}{U(1) \times SU(n)}$ in $Sp(2n+2,\mathbb{R})$ is discussed. The knowledge of such embedding provides a tool for the determination of the holomorphic prepotential characterizing the special geometry of these manifolds and necessary in the superconformal tensor calculus of $N = 2$ supergravity. It is demonstrated that there exists certain embeddings for which the homogeneous prepotential does not exist. Whether a holomorphic function exists or not, the dependence of the gauge kinetic terms on the scalars characterizing these coset in $N = 2$ supergravity theory can be determined from the knowledge of the corresponding embedding, à la Gaillard and Zumino. Our results are used to study some of the duality symmetries of heterotic compactifications of orbifolds with Wilson lines.

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1. INTRODUCTION

In recent years, special geometry has emerged as an important structure in the study of extended supergravity, superstrings and topological field theories*. The moduli space of (2, 2) superconformal field theories with central charge $c = 9$ exhibits special geometry. Special geometry played an important role in the analysis of Calabi-Yau threefolds in relation to mirror and generalized duality symmetries [3,4]. More recently, special geometry provided a useful tool in the study of the quantum moduli space and obtaining exact solutions of low energy effective actions for $N = 2$ rigid [5] and local $N = 2$ Yang Mills theories [6,7].

The concept of special Kähler geometry first appeared in the physics literature in the analysis of $N = 2$ supergravity models [8,9,10]. There special Kähler manifolds are defined by the coupling of $n$ vector multiplets of the gauge sector of the theory to $N = 2$ supergravity. The lagrangian of the theory was derived using the superconformal tensor calculus [8]. In this method one starts with an action invariant under the $N = 2$ superconformal group in four dimensions. Then with gauge fixing conditions, the resulting action is only invariant under super-Poincaré group as required. If we ignore the hypermultiplets, the theory contains $(n + 1)$ vector multiplets with scalar components $X^I$, $(I = 0, \cdots, n)$, where the multiplet labelled by 0 corresponds to the graviphoton and contains a scalar and a fermion to be gauge fixed in order to break dilatations and the $U(1)$ symmetry of the superconformal group and the a $R$-symmetry in the fermionic sector of the theory. It was found that the couplings can be described in terms of a prepotential $F$ which is a holomorphic function of degree two in terms of the scalar fields, each of weight one. The physical scalar fields define an $n$-dimensional complex hypersurface defined by the gauge fixing condition

$$i(X^I \bar{F}_I - F_I \bar{X}^I) = 1. \quad (1.1)$$

Letting the $X^I$ be proportional to holomorphic sections $Z^I(z)$ of a projective $(n + \ldots)$. 

* for a review see [1,2] and references therein.
1)-dimensional space [14], where $z$ is a set of $n$ complex coordinates, then the $z$ coordinates parametrize a Kähler space with metric $g_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} K$, where $K$, the Kähler potential is expressed by

$$K = -\log i \left( Z^I \bar{F}_I(Z) - F_I(Z) \bar{Z}^I \right)$$

$$= -\log \left( i < \Omega|\bar{\Omega} > \right) = -\log i \left( Z^I \ F_I(Z) \right) \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \left( \begin{array}{c} \bar{Z}^I \\ \bar{F}_I(\bar{Z}) \end{array} \right),$$

$$X^I = e^{K/2} Z^I, \quad \bar{X}^I = e^{K/2} \bar{Z}^I.$$  

Special coordinates correspond to the choice

$$z^\alpha = \frac{X^\alpha}{X^0}; \quad Z^0(z) = 1, \quad Z^\alpha(z) = z^\alpha. \quad (1.3)$$

The homogeneous symmetric manifolds with the special geometry structure were classified in [11] with appropriate holomorphic functions.

The gauge kinetic terms corresponding to the $(n+1)$ vector multiplets in an $N = 2$ supergravity encoded in the matrix $\mathcal{N}$ is given by

$$\mathcal{N}_{ij} = \bar{F}_{ij} + 2i \frac{(ImF_{im})(ImF_{jn})X^mX^n}{(ImF_{ab})X^aX^b}. \quad (1.4)$$

The intrinsic definition of special Kähler geometry in terms of symplectic bundles was later given [13] in connection with the geometry of the moduli of Calabi-Yau spaces where special Kähler manifolds were associated with the moduli space of the Kähler or complex structure. In this approach the symplectic symmetry is inherited from the symmetry of the homology cycles. Also a special coordinate-independent description was given in [14], where special geometry was obtained purely from the constraints of the extended $N = 2$ supersymmetry in the non-linear sigma models associated with an arbitrary number $n$ of vector multiplets of a four dimensional supergravity. In this formalism, the underlying symplectic $Sp(2n+2, \mathbb{R})$ symmetry is related to the duality transformations of the gauge sector as discussed in [15].
In fact the relation between the above mentioned approaches can be understood from the fact that on the same Calabi-Yau manifold, both heterotic and type-II superstrings can be compactified, giving low-energy effective theories with \( N = 1 \) and \( N = 2 \) respectively. Therefore although the effective action of heterotic compactification does not have the \( N = 2 \) structure, its moduli space must be compatible with the \( N = 2 \) supersymmetry, giving it the special structure \([16,17,18]\). The special geometry structure can also be shown to be a consequence of the underlying \((2,2)\) world-sheet supersymmetry \([19]\).

The only special Kähler manifolds which are of a direct product form are given by \([12]\)

\[
SK(n + 1) = \frac{SU(1,1)}{U(1)} \times \frac{SO(2,n)}{SO(2) \times SO(n)}.
\]

These cosets are of fundamental importance in superstring theories, those with \( n = 2, 4 \), describe the moduli spaces of \( N = 1 \) heterotic string compactifications. They also arise in the study of heterotic \( N = 2 \) superstring compactification, where the \( \frac{SU(1,1)}{U(1)} \) represents the dilaton-axion vector multiplets and the other factor parametrize toroidal and possible Wilson-line moduli.

A method of constructing the holomorphic prepotential for the manifolds \( SK(n+1) \) has been given in \([20]\). This construction is inspired by the method of Gaillard and Zumino \([15]\) of constructing the gauge couplings in an abelian gauge theory with scalars parametrizerizing a coset space. In this method, the duality transformations on the gauge fields are parametrized by the embedding of the isometry group of the coset space parametrized by the scalar fields in \( Usp(n,n) \). Such an embedding is crucial in fixing the lagrangian of the theory which produces duality invariant equations of motion.

In the context of determining the holomorphic function encoding the special geometry of the cosets \( SK(n+1) \)\([20]\), one introduces the symplectic section \((X^A, F_A)\) and demands that it transforms as a vector under the symplectic transformations induced by the embedding of the isometry group of \( SK(n+1) \) into the symplectic group \( Sp(2n + 2, \mathbb{R}) \). These transformations are then used to fix the relations.
between $F_\Lambda$ and $X^\Lambda$. Clearly different embeddings lead to different relations, and where an $F$ function exists, to different $F$ functions. In [22] the results of [20] were extended to the other infinite series of special manifolds, $CP_{n-1,1} = \frac{SU(1,n)}{U(1) \times SU(n)}$.

In this paper we concentrate on the special Kähler manifolds $CP_{n-1,1}$, the construction of the corresponding kinetic terms for the scalars and gauge fields in the corresponding $N = 2$ supergravity. We also discuss the relevance of our formalism to the study of duality symmetries in orbifold compactifications with Wilson lines.

This work is organized as follows. In section two we review the construction of [22]. We give three embeddings of the isometry group $SU(1,n)$ into the symplectic group $Sp(2n+2,\mathbb{R})$. This construction is then applied to the simplest example $SU(1,1)$ $U(1)$ and we rederive the familiar duality symmetry of this model. Section three contains an explicit calculation for the gauge kinetic couplings of the $N = 2$ supergravity theory whose scalars parametrize the coset $CP(n-1,1)$. This gives an emphasis on the importance of the embedding in the calculation of the lagrangian in cases where a holomorphic function $F$ does not exist. Moreover, $CP_{n-1,1}$ appear as submanifolds in heterotic string compactifications on orbifolds with Wilson lines [27,28]. The duality symmetry of these models can be determined from the study of their mass spectrum. In section four we will study these duality transformations using the methods of special geometry. In this formalism, the duality symmetries are much easier to analyze, in particular for the cases with more than one Wilson line. Section five contains a summary of our results and conclusions.
2. SYMPLECTIC EMBEDDINGS

The isometry group of the cosets $CP_{n-1,1}$ is given by the group $SU(1,n)$. An element of $SU(1,n)$ is represented by an $(n + 1) \times (n + 1)$ complex matrix $M$ satisfying

$$M^\dagger \eta M = \eta, \quad \det M = 1,$$  \hspace{1cm} (2.1)

with $\eta$ is the constant diagonal metric with signature $(+, -, \cdots, -)$. Decomposing the matrix $M$ into its real and imaginary part,

$$M = U + iV,$$  \hspace{1cm} (2.2)

then the first relation in (2.1) implies for the real $(n + 1) \times (n + 1)$ matrices $U$ and $V$, the following relations

$$U^t \eta U + V^t \eta V = \eta,$$

$$U^t \eta V - V^t \eta U = 0.$$  \hspace{1cm} (2.3)

An element $\Omega$ of $Sp(2n + 2, \mathbb{R})$ is a $(2n + 2) \times (2n + 2)$ real matrix satisfying

$$\Omega^t L \Omega = L, \quad L = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$  \hspace{1cm} (2.4)

If we write

$$\Omega = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$  \hspace{1cm} (2.5)

where the matrices $A, B, C$ and $D$ are $(n + 1) \times (n + 1)$ matrices, then in terms of these block matrices, (2.4) implies the following conditions

$$A^t C - C^t A = 0, \quad A^t D - C^t B = 1, \quad B^t D - D^t B = 0.$$  \hspace{1cm} (2.6)

An embedding of $SU(1,n)$ into the symplectic group $Sp(2n + 2, \mathbb{R})$ is given by

$$A = U, \quad C = -\eta V, \quad B = V \eta, \quad D = \eta U \eta.$$  \hspace{1cm} (2.7)

Consider the embedding $\Omega_e$ with matrix components given in (2.7), and introduce the symplectic section $(X^\Lambda, F_\Lambda)$ which transforms as a vector under the
symplectic transformations induced by $\Omega_e$. These transformation rules can then be used to determine the relation between $F_\Lambda$ and the coordinates $X^\Lambda$. In components, these transformations are given by

$$
X \rightarrow UX + V\eta \partial F, \\
\partial F \rightarrow -\eta VX + \eta U\eta \partial F,
$$

(2.8)

where $X$ and $\partial F$ are $(n + 1)$-dimensional vectors with components $X^\Lambda$ and $F_\Lambda$ respectively. It is clear that the transformation relations (2.8) implies that $\partial F$ can be identified with $i\eta X$, in which case, a holomorphic prepotential $F$ exists and is given, in terms of the coordinates $X$, by

$$
F = \frac{i}{2}X^t\eta X.
$$

(2.9)

With the above relation, the complex vector $X$ transforms as

$$
X \rightarrow (U + iV)X = MX,
$$

(2.10)

which implies that $X$ should be identified with the complex coordinates which parametrize the $SU(1, n) / U(1) \times SU(n)$ coset. These complex coordinates satisfy the following relation

$$
\phi^\dagger \eta \phi = 1, \quad \text{where} \quad \phi = \begin{pmatrix} \phi^0 \\ \vdots \\ \phi^{n+1} \end{pmatrix},
$$

(2.11)

and are parametrized in terms of unconstrained coordinates $z^\alpha$ by [26]

$$
\phi^0 = \frac{1}{\sqrt{Y}}, \quad \phi^j = \frac{z^\alpha}{\sqrt{Y}}, \quad \alpha = 1, \cdots, n,
$$

(2.12)

where $Y = 1 - \sum_\alpha z^\alpha \overline{z}^\alpha$. Here we identify $X$ with the complex vector $\frac{1}{\sqrt{2}}\phi$. The special coordinates in this case are given by $z^\alpha$ and thus $Z^0 = 1$, $Z^\alpha = z^\alpha$, and
the Kähler potential is given by

\[ K = -\log(1 - \sum_{\alpha} z^\alpha \bar{z}^\alpha). \]  

(2.13)

A different embedding of \( SU(1, n) \) into \( Sp(2n+2, \mathbb{R}) \) leads to a different relation between \( F^\Lambda \) and \( X^\Lambda \). In fact once an embedding \( \Omega_e \) is specified, then for all elements \( S \in Sp(2n+2, \mathbb{R}) \), the matrix

\[ \Omega'_e = S \Omega_e S^{-1}, \]  

(2.14)

provides another embedding with a corresponding symplectic section. As an example, consider the element

\[ S_1 = \begin{pmatrix} \Sigma & 0 \\ 0 & \Sigma \end{pmatrix}, \quad \text{with} \]
\[ \Sigma = \begin{pmatrix} \frac{1}{\sqrt{2}} \sigma & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \]  

(2.15)

Using (2.14) and (2.15) another embedding of \( SU(1, n) \) into \( Sp(2n+2, \mathbb{R}) \) can be obtained and is given by

\[ \Omega'_e = \begin{pmatrix} \Sigma U \Sigma & \Sigma V \eta \Sigma \\ -\Sigma \eta V \Sigma & \Sigma \eta U \eta \Sigma \end{pmatrix}. \]  

(2.16)

For this embedding, we define a new section \( (X', \partial F') \) which transforms as a vector under the action of \( \Omega'_e \). Clearly, the two sections \( (X, \partial F) \) and \( (X', \partial F') \) are related by the following relations

\[ X' = \Sigma X, \quad (\partial F)' = \Sigma \partial F. \]  

(2.17)
These in components lead to the relations

\[ X'^0 = \frac{1}{\sqrt{2}} (X^0 + X^1), \]
\[ X'^1 = \frac{1}{\sqrt{2}} (X^0 - X^1), \]
\[ X'^j = X^j, \quad j = 2, \ldots, n \]
\[ F'_0 = \frac{1}{\sqrt{2}} (F_0 + F_1) = \frac{i}{\sqrt{2}} (X^0 - X^1) = iX'^1, \]
\[ F'_1 = \frac{1}{\sqrt{2}} (F_0 - F_1) = \frac{i}{\sqrt{2}} (X^0 + X^1) = iX'^0, \]
\[ F'_j = F_j = -iX^j = -iX'^j. \]

From these relations, it can be easily seen that there exists a holomorphic prepotential \( F' \) which can be expressed in terms of \( X' \) by

\[ F' = i \left( X'^0 X'^1 - \frac{1}{2} \sum_{j=2}^{n} (X'^j)^2 \right). \]  \( (2.19) \)

For this parametrization, we have

\[ Z'^0 = 1, \quad Z'^1 = \frac{1 - z^1}{1 + z^1}, \quad Z'^j = \frac{\sqrt{2} z^j}{1 + z^1}, \quad j = 2, \ldots, n \]

and the Kähler potential is given by

\[ K = - \log(Z'^1 + \bar{Z}'^1 - \sum_j Z'^j \bar{Z}'^j). \] \( (2.20) \)

In general given two sections related by a symplectic transformations as follows

\[
\begin{pmatrix}
X' \\
\partial F'
\end{pmatrix} =
\begin{pmatrix}
X & Y \\
Z & T
\end{pmatrix}
\begin{pmatrix}
X \\
\partial F
\end{pmatrix}.
\] \( (2.22) \)

Then it can be shown that [1] the relation between the corresponding holomorphic
functions is given by

\[ F' = \frac{1}{2} \left( X \partial F \right) \begin{pmatrix} Z^t X & Z^t Y \\ T^t X & T^t Y \end{pmatrix} \left( X \partial F \right). \]  

(2.23)

Moreover, it can be shown [8] that a holomorphic prepotential \( F' \) exists such that

\[ F'_\Lambda = \frac{\partial F'}{\partial X^\Lambda}, \]  

(2.24)

provided the mapping \( X^\Lambda \rightarrow X'^\Lambda \) is invertible.

Using (2.9), (2.23) and the symplectic transformation given in (2.15), the expression of the holomorphic function \( F' \) given in (2.19) can be verified.

As a demonstration of the above calculations, we work out the familiar example \( SU(1,1)/U(1) \) and derive the duality symmetry action on the modulus of this coset which is the special coordinate corresponding to the two embedding described above. This example appears in string theory in the description of the moduli spaces of heterotic string compactification as well as the moduli space parametrized by the complex dilaton-axion field.

First, let us define the special coordinates \( t = \frac{X^1}{X^0} \). Eq. (2.8) give the following transformation for the coordinates \((X^0, X^1)\)

\[ \begin{pmatrix} X^0 \\ X^1 \end{pmatrix} \rightarrow \begin{pmatrix} z_1 & \bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix} \begin{pmatrix} X^0 \\ X^1 \end{pmatrix}, \quad |z_1|^2 - |z_2|^2 = 1. \]  

(2.25)

This gives for the special coordinates the following transformation

\[ t \rightarrow \frac{z_2 + \bar{z}_1 t}{z_1 + \bar{z}_2 t}. \]  

(2.26)

The Kähler potential in terms of \( t \) is given by

\[ K = - \log(1 - t\bar{t}). \]  

(2.27)

In the coordinate system \((X'^0, X'^1)\) corresponding to the embedding (2.16) for
the $SU(1, 1)$ case, define a new special coordinate $T = \frac{X^{r1}}{X^{r0}}$. If we write

$$
(z_1 + z_2) = (d + ic),
(z_1 - z_2) = (a - ib).
$$

(2.28)

Then the condition $z_1 \bar{z}_1 - z_2 \bar{z}_2 = 1$, implies that $ad - bc = 1$. Using (2.16) we obtain the following embedding in $Sp(4, \mathbb{R})$,

$$
\Omega'_{SU(1, 1)} = \begin{pmatrix}
    d & 0 & c & 0 \\
    0 & a & 0 & -b \\
    b & 0 & a & 0 \\
    0 & -c & 0 & d
\end{pmatrix}.
$$

(2.29)

This gives using (2.19), the following transformations

$$
X^{r0} \rightarrow dX^{r0} + icX^{r1},
X^{r1} \rightarrow -ibX^{r0} + aX^{r1},
$$

(2.30)

From which we obtain the familiar $SL(2, \mathbb{R})$ transformation for the $T$ moduli

$$
T \rightarrow \frac{aT - ib}{icT + d}.
$$

(2.31)

It is clear that the above two formalisms are related by the holomorphic field redefinition

$$
T = \frac{1 - t}{1 + t}.
$$

(2.32)

In what follows we will discuss a certain embedding for which a holomorphic prepotential does not exist [6]. Remaining with the $SU(1, 1)/U(1)$ example, consider the
matrix
\[
\begin{pmatrix}
d & 0 & c & 0 \\
0 & d & 0 & c \\
b & 0 & a & 0 \\
0 & b & 0 & a \\
\end{pmatrix},
\] (2.33)

it is clear that (using \(ad - bc = 1\)) it provides an embedding of \(SU(1, 1)\) in \(Sp(4, \mathbb{R})\). The transformation of the corresponding symplectic vector, \((X''_{\Lambda}, F''_{\Lambda})\), gives
\[
\begin{align*}
X''_0 &\to dX''_0 + cF''_0, \\
X''_1 &\to dX''_1 + cF''_1, \\
F''_0 &\to bX''_0 + aF''_0, \\
F''_1 &\to dX''_1 + aF''_1,
\end{align*}
\] (2.34)

which implies that \(F''_0\) and \(X''_0\) are respectively proportional to \(F''_1\) and \(X''_1\), and therefore a holomorphic prepotential does not exist.

For the \(SU(1, n)\) case, the embedding for which an \(F\) function does not exist can be obtained from the embedding \(\Omega_e\) using (2.14) for \(S = S_2 \in Sp(2n + 2, \mathbb{R})\) given by
\[
S_2 = \begin{pmatrix} X & Y \\ Z & T \end{pmatrix}, \quad \text{with} \quad X = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}, \quad Y = \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix},
\]
\[
Z = \begin{pmatrix} z & 0 \\ 0 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix},
\] (2.35)
\[
x = t = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad z = -y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}.
\]

The new section \((X''_{\Lambda}, F''_{\Lambda})\) corresponding to the embedding
\[
\Omega''_e = S_2 \Omega_e S_2^{-1},
\] (2.36)
can be expressed in terms of \((X^\Lambda, F_\Lambda)\) by

\[
X'^0 = \frac{1}{\sqrt{2}}(X^0 + X^1),
\]
\[
X'^1 = \frac{1}{\sqrt{2}}(F^1 - F^0) = -\frac{i}{\sqrt{2}}(X^0 + X^1),
\]
\[
X'^j = X_j, \quad j = 2, \cdots, n
\]
\[
F'^0 = \frac{1}{\sqrt{2}}(F^0 + F^1) = \frac{i}{\sqrt{2}}(X^0 - X^1),
\]
\[
F'^1 = \frac{1}{\sqrt{2}}(X^0 - X^1),
\]
\[
F'^j = F_j = X^j, \quad j = 2, \cdots, n
\]

Using (2.9), (2.23) and (2.35), it can be shown that \(F'' = 0\). Notice that the mapping \(X^\Lambda \rightarrow X''^\Lambda\) is not invertible.
3. Gauge Kinetic Couplings in $N = 2$ supergravity

In this section we discuss the gauge kinetic couplings of $N = 2$ supergravity in the formalism of the theory corresponding to the various embeddings considered in the previous section using the method of [15]. We demonstrate that the same results can be obtained directly using the method of special geometry where the formalism admits a holomorphic prepotential.

The action for the gauge part of the theory which also has a set of scalars spanning an $m$-dimensional manifold can be written using the notation of [1] as

$$\mathcal{L} = \frac{1}{4} (ImN_{IJ}) F^{+I}_{\mu\nu} F^{+I}_{\mu\nu} - \frac{i}{8} (ReN_{IJ}) \epsilon^{\mu\rho\sigma} F^{I}_{\mu\nu} F^{J}_{\rho\sigma} = \frac{1}{2} Im \left( N_{IJ} F^{+I} F^{+J} \right),$$

(3.1)

where $I, J$ label the gauge fields, and $F^{I}_{\mu\nu}$ denotes the field strength. Here $ImN_{IJ}$ and $ReN_{IJ}$ are, respectively, field dependent generalization of the inverse squared coupling constant and the $\theta$-angle in standard gauge theories.

The Bianchi identities and equations of motion are given, respectively, by

$$\partial^{\mu} Im F^{+I}_{\mu\nu} = 0,$$
$$\partial_{\mu} Im G^{+I}_{\mu\nu} = 0,$$

(3.2)

where

$$G^{+I}_{\mu\nu} \equiv 2i \frac{\partial \mathcal{L}}{\partial F^{+I}_{\mu\nu}} = N_{IJ} F^{+I \mu\nu},$$
$$G^{-I}_{\mu\nu} \equiv -2i \frac{\partial \mathcal{L}}{\partial F^{-I}_{\mu\nu}} = \tilde{N}_{IJ} F^{-I \mu\nu},$$

(3.3)

The set of equations in (3.2) is invariant under $GL(2n, \mathbb{R})$ transformations:

$$\begin{pmatrix} \tilde{F}^+ \\ \tilde{G}^+ \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} F^+ \\ G^+ \end{pmatrix}.$$  

(3.4)

To preserve the relations (3.3) under the action of the above transformation implies
that the gauge kinetic coupling matrix must transform as
\[ \hat{\mathcal{N}} = (C + DN) (A + BN)^{-1}. \] (3.5)

Moreover, the fact that the matrix \( \mathcal{N} \) is symmetric restricts the group of general linear transformations to \( Sp(2n, \mathbb{R}) \).

The construction of the lagrangian (3.1) amounts to the determination of the dependence of \( \mathcal{N} \) on the scalar fields. Such dependence should produce the transformation law of \( \mathcal{N} \) defined in (3.5). Suppose that the scalar fields are valued in the coset space \( \frac{G}{H} \). The part of the lagrangian involving the scalars only is invariant under the isometry group of the coset. The isometry group action on the gauge part of the theory must correspond to a duality transformation as given in (3.4). Therefore one has to imbed the isometry group into the group \( Sp(2n, \mathbb{R}) \) or \( Usp(n, n) \) [15, 2]. An element of \( Usp(n, n) \), we call \( S \), is a complex matrix which satisfies the symplectic condition (2.4) together with the condition
\[ S^\dagger J S = J, \quad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \] (3.6)

An element of \( Usp(n, n) \) can be given in terms of \( \Omega \in Sp(2n, \mathbb{R}) \) defined in (2.4) by
\[ S = C \Omega C^{-1}, \quad C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i1 \\ 1 & -i1 \end{pmatrix}. \] (3.7)

This gives, for instance, for \( \Omega_c \in Sp(2n, \mathbb{R}) \) defined in (2.7) an element in \( Usp(n, n) \) given by
\[ S = \frac{1}{2} \begin{pmatrix} U - iV \eta - i\eta V + \eta U \eta, & U + iV \eta - i\eta V - \eta U \eta \\ U - iV \eta + i\eta V - \eta U \eta, & U + iV \eta + i\eta V + \eta U \eta \end{pmatrix}. \] (3.8)

If we associate to the coset representative of \( \frac{G}{H} \) an element in \( Usp(n, n) \) given by
\[ \begin{pmatrix} a & b^* \\ b & a^* \end{pmatrix}, \] (3.9)
It can then be shown that the matrix $\mathcal{N}$ is given by [2, 15],

$$\mathcal{N} = i(a^\dagger + b^\dagger)^{-1}(a^\dagger - b^\dagger).$$  \hfill (3.10)

Let us consider the construction of the gauge kinetic matrix for the simplest case where we have one physical scalar field parametrizing the coset $\frac{SU(1, 1)}{U(1)}$. An element of $SU(1, 1)$ is given by the following matrix

$$M_1 = \begin{pmatrix} z_1 & \bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix}, \quad |\bar{z}_1|^2 - |\bar{z}_2|^2 = 1. \hfill (3.11)$$

Using (2.7) an embedding of $M_1$ in $Sp(4, \mathbb{R})$ can be given by

$$\Omega_1 = \begin{pmatrix} u_1 & u_2 & v_1 & v_2 \\ u_2 & u_1 & v_2 & v_1 \\ -v_1 & v_2 & u_1 & -u_2 \\ v_2 & -v_1 & -u_2 & u_1 \end{pmatrix}. \hfill (3.12)$$

where we have decomposed $M_1$ into its real and imaginary part

$$z_1 = u_1 + iv_1, \quad z_2 = u_2 + iv_2. \hfill (3.13)$$

Eq. (3.7) or (3.8) gives for $M_1$, the following embedding in $Usp(2, 2)$

$$\mathcal{S}_1 = \begin{pmatrix} \bar{z}_1 & 0 & 0 & z_2 \\ 0 & \bar{z}_1 & z_2 & 0 \\ 0 & \bar{z}_2 & z_1 & 0 \\ \bar{z}_2 & 0 & 0 & \bar{z}_1 \end{pmatrix}. \hfill (3.14)$$

A coset representative of $\frac{SU(1, 1)}{U(1)}$ can be given by

$$W_1 = \begin{pmatrix} \phi^0 & \bar{\phi}^1 \\ \phi^1 & \phi^0 \end{pmatrix}. \hfill (3.15)$$

where $\phi_0$ and $\phi_1$ are as given in (2.12). The corresponding element for the coset
representative in $Usp(n, n)$ is given by
\[
W_1 = \begin{pmatrix}
\phi^0 & 0 & 0 & \phi^1 \\
0 & \phi^0 & \phi^1 & 0 \\
0 & \bar{\phi}^1 & \phi^0 & 0 \\
\bar{\phi}^1 & 0 & 0 & \phi^0
\end{pmatrix}.
\tag{3.16}
\]

Using the expression (3.10), with $a$ and $b$ given by
\[
a = \begin{pmatrix}
\phi^0 & 0 \\
0 & \phi^0
\end{pmatrix}, \quad b = \begin{pmatrix}
0 & \bar{\phi}^1 \\
\bar{\phi}^1 & 0
\end{pmatrix},
\tag{3.17}
\]
we obtain for the gauge kinetic matrix
\[
\mathcal{N} = i \left( \frac{\phi^0}{\phi^1} \right) \begin{pmatrix}
\phi^0 & \phi^1 \\
\phi^1 & \phi^0
\end{pmatrix}^{-1} \begin{pmatrix}
\phi^0 & -\phi^1 \\
-\phi^1 & \phi^0
\end{pmatrix}
= \frac{i}{(\phi^0)^2 - (\phi^1)^2} \begin{pmatrix}
(\phi^0)^2 + (\phi^1)^2 & -2\phi^0\phi^1 \\
-2\phi^0\phi^1 & (\phi^0)^2 + (\phi^1)^2
\end{pmatrix}
= \frac{i}{(X^0)^2 - (X^1)^2} \begin{pmatrix}
(X^0)^2 + (X^1)^2 & -2X^0X^1 \\
-2X^0X^1 & (X^0)^2 + (X^1)^2
\end{pmatrix}
= \frac{i}{(1 - t^2)} \begin{pmatrix}
1 + t^2 & -2t \\
-2t & 1 + t^2
\end{pmatrix}.
\tag{3.18}
\]

Another embedding of $M_1$ in $Sp(4, \mathbb{R})$ can be obtained using (2.16) for the $SU(1, 1)$ case, this is given by
\[
\Omega'_1 = \begin{pmatrix}
u_1 + u_2 & 0 & v_1 + v_2 & 0 \\
0 & u_1 - u_2 & 0 & v_1 - v_2 \\
v_2 - v_1 & 0 & u_1 - u_2 & 0 \\
0 & -(v_1 + v_2) & 0 & u_1 + u_2
\end{pmatrix}.
\tag{3.19}
\]

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and the corresponding embedding in $Usp(2, 2)$ is given by

$$S'_1 = \begin{pmatrix} \bar{z}_1 & 0 & z_2 & 0 \\ 0 & \bar{z}_1 & 0 & -z_2 \\ \bar{z}_2 & 0 & z_1 & 0 \\ 0 & -\bar{z}_2 & 0 & z_1 \end{pmatrix},$$  \tag{3.20}$$

The corresponding element for the coset representative in $Usp(2, 2)$ is given by

$$W'_1 = \begin{pmatrix} \phi^0 & 0 & \phi^1 & 0 \\ 0 & \phi^0 & 0 & -\phi^1 \\ \phi^1 & 0 & \phi^0 & 0 \\ 0 & -\phi^1 & 0 & \phi^0 \end{pmatrix}.$$  \tag{3.21}$$

This gives using (3.10) the following gauge kinetic matrix

$$\mathcal{N}' = i \left( \begin{array}{cc} \frac{X'^1}{X'^0} & 0 \\ 0 & \frac{X'^0}{X'^1} \end{array} \right) = i \left( \begin{array}{cc} T & 0 \\ 0 & 1/T \end{array} \right) = i \left( \begin{array}{cc} 1-t & 0 \\ 0 & 1+t \end{array} \right).$$  \tag{3.22}$$

The embedding given in (2.33), for which a holomorphic function does not exist, give for an element of $SU(1, 1)$ an embedding in $Usp(2, 2)$ given by

$$S''_1 = \begin{pmatrix} \bar{z}_1 & 0 & z_2 & 0 \\ 0 & \bar{z}_1 & 0 & z_2 \\ \bar{z}_2 & 0 & z_1 & 0 \\ 0 & -\bar{z}_2 & 0 & z_1 \end{pmatrix},$$  \tag{3.23}$$

In this case, the coset representative of $\frac{SU(1, 1)}{U(1)}$ has the following embedding in $Usp(2, 2)$

$$W''_1 = \begin{pmatrix} \phi^0 & 0 & \phi^1 & 0 \\ 0 & \phi^0 & 0 & \phi^1 \\ \phi^1 & 0 & \phi^0 & 0 \\ 0 & \phi^1 & 0 & \phi^0 \end{pmatrix},$$  \tag{3.24}$$
and the gauge kinetic terms for this embedding are given by

\[ N'' = i \begin{pmatrix} X'^1 & 0 & 0 & X'^0 \\ 0 & X'^1 & 0 & X'^0 \end{pmatrix} = i \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} = \frac{i}{(1+t)} \begin{pmatrix} 1-t & 0 \\ 0 & 1-t \end{pmatrix}. \] (3.25)

Notice that the matrix \( N \) for the case where the \( F \) function does not exist can be determined from the knowledge of the corresponding embedding.

As another example, consider the coset \( \frac{SU(1,2)}{U(1) \times SU(2)} \), parametrized by two scalar fields \( Z^1 \) and \( Z^2 \). A coset representative is given by

\[ W_2 = \begin{pmatrix} \phi^0 & \phi^1 & \phi^2 \\ \phi^1 & a & b \\ \phi^2 & c & d \end{pmatrix}. \] (3.26)

This gives for the embedding (2.7), a corresponding element in \( Usp(3,3) \) given by

\[ W_2 = \begin{pmatrix} \phi^0 & 0 & 0 & 0 & \phi^1 & \phi^2 \\ 0 & a & b & \phi^1 & 0 & 0 \\ 0 & c & d & \phi^2 & 0 & 0 \\ 0 & \bar{\phi}^1 & \bar{\phi}^2 & \phi^0 & 0 & 0 \\ \phi^1 & 0 & 0 & 0 & \bar{a} & \bar{b} \\ \phi^2 & 0 & 0 & 0 & \bar{c} & \bar{d} \end{pmatrix}, \] (3.27)

which produces the following gauge kinetic matrix

\[ N = i \begin{pmatrix} \phi^0 & \phi^1 & \phi^2 \\ \phi^1 & \bar{a} & \bar{c} \\ \phi^2 & \bar{b} & \bar{d} \end{pmatrix}^{-1} \begin{pmatrix} \phi^0 & -\phi^1 & -\phi^2 \\ -\phi^1 & \bar{a} & \bar{c} \\ -\phi^2 & \bar{b} & \bar{d} \end{pmatrix}. \] (3.28)
which in components gives

\[
\mathcal{N}_{00} = i \frac{1}{\delta} \left( (\bar{a}d - \bar{b}c)\phi^0 - (\bar{b} + \bar{c})\phi^1\phi^2 + \bar{a}(\phi^2)^2 + \bar{d}(\phi^1)^2 \right), \\
\mathcal{N}_{01} = \frac{-2i(\bar{a}d - \bar{b}c)\phi^1}{\delta}, \\
\mathcal{N}_{02} = \frac{-2i(\bar{a}d - \bar{b}c)\phi^2}{\delta}, \\
\mathcal{N}_{10} = \frac{i(2\bar{c}\phi^0\phi^2 - 2\bar{d}\phi^1\phi^0)}{\delta}, \\
\mathcal{N}_{11} = \frac{i}{\delta} \left( (\bar{a}d - \bar{b}c)\phi^0 + (\bar{b} - \bar{c})\phi^1\phi^2 - \bar{a}(\phi^2)^2 + \bar{d}(\phi^1)^2 \right), \\
\mathcal{N}_{12} = \frac{i}{\delta} \left( -2\bar{c}(\phi^2)^2 + 2\bar{d}\phi^1\phi^2 \right), \\
\mathcal{N}_{20} = \frac{i}{\delta} \left( 2\bar{b}\phi^0\phi^1 - 2\bar{a}\phi^0\phi^2 \right), \\
\mathcal{N}_{21} = \frac{i}{\delta} \left( 2\bar{a}\phi^1\phi^2 - 2\bar{b}(\phi^1)^2 \right), \\
\mathcal{N}_{22} = \frac{i}{\delta} \left( (\bar{a}d - \bar{b}c)\phi^0 - (\bar{b} + \bar{c})\phi^1\phi^2 + \bar{a}(\phi^2)^2 - \bar{d}(\phi^1)^2 \right) \tag{3.29},
\delta = (\bar{a}d - \bar{b}c)\phi^0 + (\bar{b} + \bar{c})\phi^1\phi^2 - \bar{a}(\phi^2)^2 - \bar{d}(\phi^1)^2.
\]

Using the following conditions

\[
ad - bc = \phi^0, \quad a\bar{\phi}^2 - b\bar{\phi}^1 = \bar{\phi}^2, \quad d\bar{\phi}^1 - c\bar{\phi}^2 = \bar{\phi}^1, \tag{3.30}
\]

which arise from the fact that \(W_2\) is an element of \(SU(1, 2)\), eq. (3.29) gives

\[
\mathcal{N}_{00} = \left. i \frac{1 + (Z^1)^2 + (Z^2)^2}{1 - (Z^1)^2 - (Z^2)^2} \right|_{Z^1}, \\
\mathcal{N}_{01} = \mathcal{N}_{10} = -2i \left. \frac{Z^1}{1 - (Z^1)^2 - (Z^2)^2} \right|_{Z^1}, \\
\mathcal{N}_{02} = \mathcal{N}_{02} = -2i \left. \frac{Z^2}{1 - (Z^1)^2 - (Z^2)^2} \right|_{Z^2}, \\
\mathcal{N}_{11} = \left. i \frac{1 + (Z^1)^2 - (Z^2)^2}{1 - (Z^1)^2 - (Z^2)^2} \right|_{Z^1Z^2}, \\
\mathcal{N}_{12} = \mathcal{N}_{21} = 2i \left. \frac{Z^1Z^2}{1 - (Z^1)^2 - (Z^2)^2} \right|_{Z^1Z^2}, \\
\mathcal{N}_{22} = \left. i \frac{1 - (Z^1)^2 + (Z^2)^2}{1 - (Z^1)^2 - (Z^2)^2} \right|_{Z^2}. \tag{3.31}
\]
Another embedding of the coset representative in $Usp(3, 3)$ can be given using (2.16) by

$$W'_2 = \begin{pmatrix}
\frac{\phi^0 + a}{2} & \frac{\phi^0 - a}{2} & \frac{b}{\sqrt{2}} & \phi^1 & 0 & \frac{\bar{\phi}^2}{\sqrt{2}} \\
\frac{\phi^0 - a}{2} & \frac{\phi^0 + a}{2} & -\frac{b}{\sqrt{2}} & 0 & -\phi^1 & \frac{\bar{\phi}^2}{\sqrt{2}} \\
c & -c & d & \frac{\phi^2}{\sqrt{2}} & \frac{\phi^2}{\sqrt{2}} & 0 \\
\phi^1 & 0 & \phi^2 & \frac{\phi^0 + a}{2} & \frac{\phi^0 - a}{2} & \frac{\bar{b}}{\sqrt{2}} \\
0 & -\phi^1 & \phi^2 & \frac{\phi^0 - a}{2} & \frac{\phi^0 + a}{2} & -\frac{\bar{b}}{\sqrt{2}} \\
\frac{\bar{\phi}^2}{\sqrt{2}} & \frac{\bar{\phi}^2}{\sqrt{2}} & 0 & \frac{\bar{c}}{\sqrt{2}} & -\frac{\bar{c}}{\sqrt{2}} & \bar{d}
\end{pmatrix}. \quad (3.32)$$

The gauge couplings in this case are given by

$$N'_{00} = i \frac{(1 - Z^1)^2}{1 - (Z^1)^2 - (Z^2)^2},$$

$$N'_{01} = N'_{10} = i \frac{(Z^2)^2}{1 - (Z^1)^2 - (Z^2)^2},$$

$$N'_{02} = N'_{02} = -i \sqrt{2} \frac{Z^2(1 - Z^1)}{1 - (Z^1)^2 - (Z^2)^2},$$

$$N'_{11} = i \frac{(1 + Z^1)^2}{1 - (Z^1)^2 - (Z^2)^2},$$

$$N'_{12} = N'_{21} = -i \sqrt{2} \frac{Z^2(1 + Z^1)}{1 - (Z^1)^2 - (Z^2)^2},$$

$$N'_{22} = i \frac{1 - (Z^1)^2 + (Z^2)^2}{1 - (Z^1)^2 - (Z^2)^2}. \quad (3.33)$$

Finally for the embedding (2.33) we get the following element for the coset representative in $Usp(3, 3)$
\[ W_2'' = \begin{pmatrix} \phi^0 + a & -i\phi^0 - a & b & \bar{\phi}^1 & 0 & \bar{\phi}^2 \\ i\phi^0 - a & \phi^0 + a & -i\frac{b}{\sqrt{2}} & 0 & \bar{\phi}^1 & i\frac{\bar{\phi}^2}{\sqrt{2}} \\ \frac{c}{\sqrt{2}} & \frac{i\bar{c}}{\sqrt{2}} & d & \frac{\bar{\phi}^2}{\sqrt{2}} & i\frac{\bar{\phi}^2}{\sqrt{2}} & 0 \\ \phi^1 & 0 & \frac{\phi^2}{\sqrt{2}} & \frac{\phi^0 + \bar{a}}{2} & i\frac{\phi^0 - \bar{a}}{2} & \frac{\bar{b}}{\sqrt{2}} \\ 0 & \phi^1 & -i\frac{\phi^2}{\sqrt{2}} & -i\frac{\phi^0 - \bar{a}}{2} & \phi^0 + \bar{a} & i\frac{\bar{b}}{\sqrt{2}} \\ \frac{\phi^2}{\sqrt{2}} & -i\frac{\phi^2}{\sqrt{2}} & 0 & \frac{\bar{c}}{\sqrt{2}} & -i\frac{\bar{c}}{\sqrt{2}} & \bar{d} \end{pmatrix} \]  

(3.34)

and the gauge couplings in this case are given by

\[ N_{00}'' = i\frac{1 - (Z_1)^2 + (Z_2)^2}{(1 + Z_1)^2}, \]
\[ N_{01}'' = N_{10}'' = -\frac{(Z_2)^2}{1 + (Z_1)^2}, \]
\[ N_{02}'' = N_{20}'' = -i\sqrt{2}\frac{Z^2}{1 + (Z_1)}, \]
\[ N_{11}'' = i\frac{1 - (Z_1)^2 - (Z_2)^2}{(1 + Z_1)^2}, \]
\[ N_{12}'' = N_{21}'' = \sqrt{2}\frac{Z^2}{1 + Z_1}, \]
\[ N_{22}'' = i. \]  

(3.35)

In general we represent the cosets \( SU(1, n)/SU(n) \times U(1) \) by the element

\[ W_n = \begin{pmatrix} \phi^0 \\ \phi^1 \\ \cdots \\ \phi^n \end{pmatrix}, \]  

(3.36)

where

\[ \mathcal{X} = (\bar{\phi}^1 \quad \bar{\phi}^2 \quad \cdots \quad \bar{\phi}^n) \]  

(3.37)
and $\mathcal{Y}$ is an $(n \times n)$ complex matrix satisfying the conditions

\begin{align*}
\mathcal{Y}^\dagger \mathcal{Y} &= 1 + \mathcal{X}^\dagger \mathcal{X}, \\
\phi^0 \mathcal{X} - \mathcal{X} \mathcal{Y} &= 0, \\
\det W_n &= 1.
\end{align*}

(3.38)

The embedding of $W_n$ in $Usp(n + 1, n + 1)$ is given by

\[
W_n = \begin{pmatrix}
\phi^0 & 0 & 0 & \vec{X} \\
0 & \mathcal{Y} & \mathcal{X}^\dagger & 0 \\
0 & \mathcal{X} & \phi^0 & 0 \\
\mathcal{X}^t & 0 & 0 & \vec{Y}
\end{pmatrix}.
\]

(3.39)

The other two embeddings $W'_n$ and $W''_n$ can be obtained from $W_n$ by the following relations

\[
W'_n = C_1 W_n C_1^{-1}, \quad W''_n = C_2 W_n C_2^{-1},
\]

\[
C_1 = CS_1 C_1^{-1}, \quad C_2 = CS_2 C_1^{-1}.
\]

(3.40)

where $C$, $S_1$ and $S_2$ are given, respectively, given in (3.7), (2.15) and (2.35).

Having determined the form of the gauge kinetic matrix in the various embeddings considered in this paper, we demonstrate how our results can be obtained using the formalism of special geometry. For $\frac{SU(1, n)}{SU(n) \times U(1)}$, the holomorphic function in the basis $X$ and $X'$ where shown to be given by (2.9) and (2.19). Using the expressions of $F$ and $F'$ for the $n = 1$ case, one can reproduce the expressions for $\mathcal{N}$ and $\mathcal{N}'$ in (3.18) and (3.22) respectively. The reader could verify for herself that substituting the expressions for the holomorphic functions for the various cosets produces the results obtained by using the method of [15].

The matrix $\mathcal{N}'$ and $\mathcal{N}''$ in the basis $X'$ and $X''$ can also be obtained from $\mathcal{N}$ by performing a symplectic transformation which connects the section $(X^A, F_A)$...
with \((X^\Lambda, F^\Lambda_\Lambda)\) and \((X''^\Lambda, F''^\Lambda_\Lambda)\). This gives

\[
N' = \Sigma \Lambda \Sigma,
\]

\[
N'' = (Z + T \Lambda)(X + Y \Lambda)^{-1}.
\]  

(3.41)
4. Duality symmetries

In this section our analysis of the cosets $\frac{SU(1,n)}{U(1) \times SU(n)}$ is used to study a subgroup of the duality symmetry in heterotic string theories compactified on orbifolds with Wilson lines [24,27]. In toroidal compactifications [25], one has a set of scalar fields, the moduli, which are encoded in the metric of the lattice defining the torus and a possible antisymmetric tensor and Wilson lines. The moduli space of toroidal compactification [25] is given (locally) by the coset space $\frac{SO(d+16,d)}{SO(d+16,d) \times SO(d)}$, where $d$ is the dimension of the torus upon which the theory is compactified and the factor 16 comes from the inclusion of Wilson lines. In orbifold models, the twist freezes some of the moduli and thus the moduli spaces of orbifolds have smaller dimensions than those of their corresponding toroidal compactifications. It can be demonstrated [27] that the moduli spaces of orbifolds can be determined from the knowledge of the eigenvalues of the twist and their multiplicities. The moduli space of the orbifold (without Wilson lines) are parametrized by the $T$ moduli corresponding to the Kähler deformations and the $U$ moduli which correspond to the deformations of the complex structure. Both moduli spaces are given by a special Kähler manifold. The $U$ moduli space is described by the coset $[SU(1,1)]$, and except for the $\mathbb{Z}_3$ orbifold, whose $T$ moduli space is given by $\frac{SU(3,3)}{SU(3) \times SU(3) \times U(1)}$, the $T$ moduli spaces for all symmetric orbifolds yielding $N = 1$ space-time supersymmetry are given by the special Kähler manifolds.

$$SK(n+1) = \frac{SU(1,1)}{U(1)} \times \frac{SO(n,2)}{SO(n) \times SO(2)}, \quad n = 2, 4.$$ 

Duality symmetries are those discrete automorphisms of the moduli space which leave the underlying conformal field theory invariant. One method of determining the duality symmetry is to study the mass spectrum of the theory which depends on the moduli fields and quantum numbers. Duality transformations are identified by those transformations on the moduli fields and quantum numbers which leaves the spectrum invariant.
In what follows we shall analyze the duality symmetries of the cosets \( CP(n-1,1) \) which describe a submanifold in the moduli space of factorizable orbifolds with Wilson lines [27, 28]. The examples for \( n = 1 \) and \( n = 2 \) were studied in details in [28], relying on the mass spectrum of these cosets. Here, we shall reproduce these results using the methods of special geometry, as well as the duality symmetry and its action on the moduli for any value of \( n \). Clearly the number \( n \), related to the number of Wilson lines, is not arbitrary and is constrained by modular invariance.

The mass formula for any special Kähler manifold was provided in [21]. Once a section \((X, \partial F)\) is specified, based on symmetry arguments, the mass formula can be given by

\[
|m|^2 = |PX + Q\partial F|^2. \tag{4.1}
\]

where \((P, Q)\) is a vector encoding the windings and momenta. Under the target space duality group \( \Gamma \), the vector \((X, \partial F)\) transforms by

\[
\begin{pmatrix} X \\ \partial F \end{pmatrix} \to \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \begin{pmatrix} X \\ \partial F \end{pmatrix}, \tag{4.2}
\]

where the matrix \(\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}\) is the embedding of the duality group in \(Sp(2n + 2, \mathbb{Z})\). For \(|m|^2\) to be invariant under the duality transformations, \((P, Q)\) should transform as follows

\[
\begin{pmatrix} P \\ Q \end{pmatrix} \to \begin{pmatrix} D' & -C' \\ -B' & A' \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix}. \tag{4.3}
\]

For \(\Gamma = SU(1, n)\) and for the embedding defined in (2.7) where the holomorphic prepotential given by (2.9) we have

\[
\begin{pmatrix} X \\ \partial F \end{pmatrix} \to \begin{pmatrix} U' & V'\eta \\ -\eta V' & \eta U'\eta \end{pmatrix} \begin{pmatrix} X \\ \partial F \end{pmatrix},
\begin{pmatrix} P \\ Q \end{pmatrix} \to \begin{pmatrix} \eta U'\eta & \eta V' \\ -V'\eta & U' \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix}. \tag{4.4}
\]
with $U'$ and $V'$ are integer valued matrices. Using the form of the holomorphic prepotential in (2.9), the mass formula can be written as

$$|m|^2 = |P''X|^2, \quad \text{with} \quad P' = \begin{pmatrix} P'_0 = P_0 + iQ_0 \\ P'_i = P_i - iQ_i \end{pmatrix}. \quad (4.5)$$

Under the $SU(1,n)$ transformation

$$X \rightarrow MX, \quad M \in SU(1,n), \quad (4.6)$$

it can be easily shown that the following transformations for the quantum numbers hold,

$$\begin{pmatrix} P'_0 \\ -P'_i \end{pmatrix} \rightarrow M^* \begin{pmatrix} P'_0 \\ -P'_i \end{pmatrix}. \quad (4.7)$$

We now compare these calculations with those of string compactifications. First consider the simplest example of $\Gamma = SU(1,1)$. The mass formula is given by [28]

$$|m|^2 = 2 \frac{|m_c - n_c t|^2}{(1 - tt)} = u_1^\dagger \Xi_1 u_1,$$

where

$$\Xi_1 = \frac{2}{1 - tt} \begin{pmatrix} 1 & -t \\ -\bar{t} & tt \end{pmatrix}, \quad u_1 = \begin{pmatrix} m_c \\ n_c \end{pmatrix},$$

$$m_c = \frac{1}{2\sqrt{2}u_1}(m_2 - im_1U_0 + 2in_1u_1 - 2n_2u_1U_0), \quad (4.8)$$

$$n_c = \frac{1}{2\sqrt{2}u_1}(-m_2 + im_1U_0 + 2in_1u_1 - 2n_2u_1U_0),$$

where we have set the moduli $U_0 = u_1 + iu_2$, and performed the change of variables $t = \frac{1 - T'}{1 + T'}$, with $T' = \frac{T}{2u_1}$ and $(n_1,n_2,m_1,m_2)$ are the winding and momentum quantum numbers. Notice that in this example, the coset $\frac{SU(1,1)}{U(1)}$ appears as a truncation of the coset $\frac{SO(2,2)}{SO(2) \times SO(2)}$ when the moduli $U$ is fixed to a constant value by twisting. That is why $m_c$ and $n_c$ are not integer valued. This breaks the duality group $SU(1,1, \mathbb{Z})$ down to a subgroup which depends on the value of $U_0$. For more details the reader is referred to [28].
If we define the action of the duality group on the quantum numbers by

\[ u_1 \rightarrow M_1^* u_1, \quad (4.9) \]

then for \(|m|^2\) in (4.8) to be target space duality invariant, we get

\[ \Xi_1 \rightarrow M_1^{-1} \Xi_1 M_1^{*-1}. \quad (4.10) \]

From (4.10) one can extract the duality transformation for the moduli \(t\). This can be easily done by writing

\[ \Xi_1 = \begin{pmatrix} X^0 \\ -X^1 \end{pmatrix} (X^0 - X^1). \quad (4.11) \]

where we have defined \( t = \frac{X^1}{X^0} \). Then (4.10) together with the relation

\[ M_1^{*-1} = L M_1 L, \quad L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (4.12) \]

give

\[ \begin{pmatrix} X^0 \\ X^1 \end{pmatrix} \rightarrow M_1 \begin{pmatrix} X^0 \\ X^1 \end{pmatrix}, \quad (4.13) \]

a result which agrees with (4.6).

Next consider a less trivial example given by the coset \( SU(1,2)/SU(2) \times U(1) \). The mass formula in this case is given by [28]

\[ |m|^2 = \frac{2|m_c - n_c t - Q_c A|^2}{1 - \bar{t} t - \bar{A} A} = u_2^* \Xi_2 u_2, \]

\[ \Xi_2 = \begin{pmatrix} 1 & -t & -A \\ -\bar{t} & t & \bar{t} \bar{A} \\ -A & t \bar{A} & \bar{A} \bar{A} \end{pmatrix}, \quad u_2 = \begin{pmatrix} n_c \\ m_c \\ Q_c \end{pmatrix} \quad (4.14) \]

here \( Q_c \) depends on the embedding of the twist in \( E_8 \times E_8 \) [27] and \( A \) represents the Wilson line moduli. Again under the duality transformation which is given by
a subgroup of $SU(1,2,\mathbb{Z})$ and parametrized by $M_2$, we have

$$u_2 \rightarrow M_2^* u_2,$$  \hspace{1cm} (4.15)

and

$$\Xi_2 \rightarrow M_2^{-1} \Xi_2 M_2^{-1}.$$  \hspace{1cm} (4.16)

If we define the moduli fields by

$$t = \frac{X^1}{X^0}, \quad A = \frac{X^2}{X^0},$$ \hspace{1cm} (4.17)

$\Xi_2$ can be rewritten as

$$\Xi_2 = \begin{pmatrix} X^0 \\ -X^1 \\ -X^2 \end{pmatrix} \begin{pmatrix} X^0 & -X^1 & -X^2 \end{pmatrix}.$$  \hspace{1cm} (4.18)

and (4.16) then gives

$$\begin{pmatrix} X^0 \\ X^1 \\ X^2 \end{pmatrix} \rightarrow \Omega_2 \begin{pmatrix} X^0 \\ X^1 \\ X^2 \end{pmatrix},$$ \hspace{1cm} (4.19)

which agrees with (4.6). From (4.19) one can easily read off the duality transformations of the moduli, these are given by

$$\begin{align*}
t &\rightarrow \frac{z_3 + z_4 t + z_5 A}{z_0 + z_1 t + z_2 A}, \\
A &\rightarrow \frac{z_6 + z_7 t + z_8 A}{z_0 + z_1 t + z_2 A}.
\end{align*}$$  \hspace{1cm} (4.20)

Therefore, for a model with $n-1$ Wilson line moduli, where the duality group is a subgroup of $SU(1,n,\mathbb{Z})$, eq. (4.6) together with the identification

$$t = \frac{X^1}{X^0}, \quad A^j = \frac{X^j}{X^0},$$ \hspace{1cm} (4.21)

gives the duality transformation for any number of moduli.
5. Conclusions

We have analyzed the special Kähler manifolds, the so-called minimal coupling, \( SU(1,n)/U(1) \times SU(n) \) with regard to the construction of their prepotentials. The prepotential is a holomorphic function of degree two, expressed in terms of the scalar fields parametrizing these cosets, and is essential in the superconformal tensor calculus of \( N = 2 \) supergravity. The method employed which relies on the embedding of the isometry group \( SU(1,n) \) into the symplectic group \( Sp(2n+2, \mathbb{R}) \) provides a powerful tool in calculating the lagrangian of the model. There are certain embeddings for which a holomorphic function does not exist. In these cases the knowledge of the embedding is enough to determine the lagrangian of the vectors and scalars irrespective of the fact whether an \( F \) function exists or not.

The analysis of these cosets apart from being relevant to \( N = 2 \) supergravity models, is also important for the study of a subgroup of the duality symmetries in heterotic string compactifications with Wilson lines. The target space symmetries are much easier to analyze in terms of the symplectic sections. In terms of the coordinates of the section, the duality symmetries are linear and form a subgroup of the symplectic transformations.

Recently, in the analysis of physically interesting problems, the formalism of the theory requires the non-existence of the holomorphic function. This was shown to be the case in the study of perturbative corrections to vector couplings in \( N = 2 \) heterotic string vacua [6,7]. The non-existence of the holomorphic function also provided a new mechanism in the study of supersymmetry breaking of \( N = 2 \) supersymmetry down to \( N = 1 \) [23]. In this sense, our analysis will be relevant in the study of quantum corrections of the coupling in the minimal coupling models as well as in the study of supersymmetry breaking.
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