Approximation Properties of Simple Lie Groups Made Discrete

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Abstract. In this paper we consider the class of connected simple Lie groups equipped with the discrete topology. We show that within this class of groups the following approximation properties are equivalent: (1) the Haagerup property; (2) weak amenability; (3) the weak Haagerup property (Theorem 1.10). In order to obtain the above result we prove that the discrete group $GL(2, K)$ is weakly amenable with constant 1 for any field $K$ (Theorem 1.11).

In the final part of the paper we give a contractive Schur multiplier characterization of locally compact groups coarsely embeddable into Hilbert spaces (Theorem 1.12). Consequently, all locally compact groups whose weak Haagerup constant is 1 embed coarsely into Hilbert spaces and hence the Baum-Connes assembly map with coefficients is split-injective for such groups.

1. Introduction

Amenability for groups was first introduced by von Neumann in order to study the Banach-Tarski paradox. It is remarkable that this notion has numerous characterizations and one of them, in terms of an approximation property by positive definite functions, is the following: a locally compact (Hausdorff) group $G$ is amenable if there exists a net of continuous compactly supported, positive definite functions on $G$ tending to the constant function 1 uniformly on compact subsets of $G$. Later, three weak forms of amenability were introduced: the Haagerup property, weak amenability and the weak Haagerup property. In this paper we will study these approximation properties of groups within the framework of Lie theory and coarse geometry.

Definition 1.1 (Haagerup property [10]). A locally compact group $G$ has the Haagerup property if there exists a net of positive definite $C_0$-functions on $G$, converging uniformly to 1 on compact sets.

Definition 1.2 (Weak amenability [17]). A locally compact group $G$ is weakly amenable if there exists a net $(\varphi_i)_{i \in I}$ of continuous, compactly supported Herz-Schur multipliers on $G$, converging uniformly to 1 on compact sets, and such that $\sup_i \|\varphi_i\|_{B_2} < \infty$.

The weak amenability constant $\Lambda_{WA}(G)$ is defined as the best (lowest) possible constant $\Lambda$ such that $\sup_i \|\varphi_i\|_{B_2} \leq \Lambda$, where $(\varphi_i)_{i \in I}$ is as just described.

Definition 1.3 (The weak Haagerup property [38]). A locally compact group $G$ has the weak Haagerup property if there exists a net $(\varphi_i)_{i \in I}$ of $C_0$ Herz-Schur multipliers on $G$, converging uniformly to 1 on compact sets, and such that $\sup_i \|\varphi_i\|_{B_2} < \infty$.

The weak Haagerup constant $\Lambda_{WH}(G)$ is defined as the best (lowest) possible constant $\Lambda$ such that $\sup_i \|\varphi_i\|_{B_2} \leq \Lambda$, where $(\varphi_i)_{i \in I}$ is as just described.

Clearly, amenable groups have the Haagerup property. It is also easy to see that amenable groups are weakly amenable with $\Lambda_{WA}(G) = 1$ and that groups with the Haagerup property have the weak Haagerup property with $\Lambda_{WH}(G) = 1$. Also, $1 \leq \Lambda_{WH}(G) \leq \Lambda_{WA}(G)$ for any locally compact group $G$, so weakly amenable groups have the weak Haagerup property.
It is natural to ask about the relation between the Haagerup property and weak amenability. The two notions agree in many cases, like generalized Baumslag-Solitar groups (see [14, Theorem 1.6]) and connected simple Lie groups with the discrete topology (see Theorem 1.10). In general, weak amenability does not imply the Haagerup property and vice versa. In one direction, the group $\mathbb{Z}/2 \times \mathbb{F}_2$ has the Haagerup property [19], but is not weakly amenable [44]. In the other direction, the simple Lie groups $\text{Sp}(1, n), n \geq 2$, are weakly amenable [17], but since these non-compact groups also have Property (T) [5, Section 3.3], they cannot have the Haagerup property. However, since the weak amenability constant of $\text{Sp}(1, n)$ is $2n - 1$, it is still reasonable to ask whether $\Lambda_{WA}(G) = 1$ implies that $G$ has the Haagerup property. In order to study this, the weak Haagerup property was introduced in [37, 38], and the following questions were considered.

**Question 1.4.** For which locally compact groups $G$ do we have $\Lambda_{WA}(G) = \Lambda_{WH}(G)$?

**Question 1.5.** Is $\Lambda_{WH}(G) = 1$ if and only if $G$ has the Haagerup property?

It is clear that if the weak amenability constant of a group $G$ is 1, then so is the weak Haagerup constant, and Question 1.4 has a positive answer. In general, the constants differ by the example $\mathbb{Z}/2 \times \mathbb{F}_2$ mentioned before. There is an another class of groups for which the two constants are known to be the same.

**Theorem 1.6** ([31]). Let $G$ be a connected simple Lie group. Then $G$ is weakly amenable if and only if $G$ has the weak Haagerup property. Moreover, $\Lambda_{WA}(G) = \Lambda_{WH}(G)$.

By the work of many authors [16, 17, 18, 24, 30, 33], it is known that a connected simple Lie group $G$ is weakly amenable if and only if the real rank of $G$ is zero or one. Also, the weak amenability constants of these groups are known. Recently, a similar result was proved about the weak Haagerup property [31, Theorem B]. Combining the results on weak amenability and the weak Haagerup property with the classification of connected Lie groups with the Haagerup property [10, Theorem 4.0.1] one obtains the following theorem, which gives a partial answer to both Question 1.4 and Question 1.5.

**Theorem 1.7.** Let $G$ be a connected simple Lie group. The following are equivalent.

1. $G$ is compact or locally isomorphic to $\text{SO}(n, 1)$ or $\text{SU}(n, 1)$ for some $n \geq 2$.
2. $G$ has the Haagerup property.
3. $G$ is weakly amenable with constant 1.
4. $G$ has the weak Haagerup property with constant 1.

The purpose of this paper is to consider the same class of groups as in theorem above, but made discrete. When $G$ is a locally compact group, we let $G_d$ denote the same group equipped with the discrete topology. The idea of considering Lie groups without their topology (or with the discrete topology, depending on the point of view) is not a new one. For instance, a conjecture of Friedlander and Milnor is concerned with computing the (co)homology of the classifying space of $G_d$, when $G$ is a Lie group (see [40] and the survey [45]).

Other papers discussing the relation between $G$ and $G_d$ include [13], [4] and [6]. Since our focus is approximation properties, will we be concerned with the following question.

**Question 1.8.** Does the Haagerup property/weak amenability/the weak Haagerup property of $G_d$ imply the Haagerup property/weak amenability/the weak Haagerup property of $G$?

It is not reasonable to expect an implication in the other direction. For instance, many compact groups such as $\text{SO}(n)$, $n \geq 3$, are non-amenable as discrete groups. It follows from Theorem 1.10 below (see also Corollary 4.3) that when $n \geq 5$, then $\text{SO}(n)$ as a discrete group does not even have the weak Haagerup property. It is easy to see that Question 1.8 has a positive answer for second countable, locally compact groups $G$ that admit a lattice $\Gamma$. Indeed, $G$ has the Haagerup property if and only if $\Gamma$ has the Haagerup property. Moreover, $\Lambda_{WA}(\Gamma) = \Lambda_{WA}(G)$ and $\Lambda_{WH}(\Gamma) = \Lambda_{WH}(G)$. 
Remark 1.9. A similar question can of course be asked for amenability. This case is already settled: if $G_d$ is amenable, then $G$ is amenable [46, Proposition 4.21], and the converse is not true in general by the counterexamples mentioned above. A sufficient and necessary condition of the converse implication can be found in [4].

Recall that $\text{SL}(2, \mathbb{R})$ is locally isomorphic to $\text{SO}(2, 1)$ and that $\text{SL}(2, \mathbb{C})$ is locally isomorphic to $\text{SO}(3, 1)$. Thus, Theorem 1.7 and the main theorem below together show in particular that Question 1.8 has a positive answer for connected simple Lie groups. This could however also be deduced (more easily) from the fact that connected simple Lie groups admit lattices [49, Theorem 14.1].

**Theorem 1.10 (Main Theorem).** Let $G$ be a connected simple Lie group, and let $G_d$ denote the group $G$ equipped with the discrete topology. The following are equivalent.

1. $G$ is locally isomorphic to $\text{SO}(3)$, $\text{SL}(2, \mathbb{R})$ or $\text{SL}(2, \mathbb{C})$.
2. $G_d$ has the Haagerup property.
3. $G_d$ is weakly amenable with constant 1.
4. $G_d$ is weakly amenable.
5. $G_d$ has the weak Haagerup property with constant 1.
6. $G_d$ has the weak Haagerup property.

The equivalence of (1) and (2) in Theorem 1.10 was already done by Cornulier [13, Theorem 1.14] and in greater generality. His methods are the inspiration for our proof of Theorem 1.10. That (1) implies (2) basically follows from a theorem of Guentner, Higson and Weinberger [26, Theorem 5.4], namely that the discrete group $\text{GL}(2, K)$ has the Haagerup property for any field $K$. Here we prove a similar statement about weak amenability.

**Theorem 1.11.** Let $K$ be any field. The discrete group $\text{GL}(2, K)$ is weakly amenable with constant 1.

Theorem 1.11 is certainly known to experts. The result was already mentioned in [48, p. 7] and in [43] with a reference to [26], and indeed our proof of Theorem 1.11 is merely an adaption of the methods developed in [26]. However, since no published proof is available, we felt the need to include a proof.

To obtain Theorem 1.10 we use the classification of simple Lie groups and then combine Theorem 1.11 with the following results proved in Section 4: If $G$ is one of the four groups $\text{SO}(5)$, $\text{SO}_0(1, 4)$, $\text{SU}(3)$ or $\text{SU}(1, 2)$, then $G_d$ does not have the weak Haagerup property. Also, if $G$ is the universal covering group of $\text{SU}(1, n)$ where $n \geq 2$, then $G_d$ does not have the weak Haagerup property.

In the final part of the paper we study coarse embeddability of locally compact groups into Hilbert spaces. An important application of this concept in [55], [50] and [21] is that the Baum-Connes assembly map with coefficients is split-injective for all locally compact groups that embed coarsely into a Hilbert space (see [3] for more information about the Baum-Connes assembly map). Here, we give a contractive Schur multiplier characterization of locally compact groups coarsely embeddable into Hilbert spaces (see also [22, Theorem 5.3] for the discrete case), and this characterization can be regarded as an answer to the non-equivariant version of Question 1.5. As a result, any locally compact group with weak Haagerup constant 1 embeds coarsely into a Hilbert space and hence the Baum-Connes assembly map with coefficients is split-injective for all these groups.

**Theorem 1.12.** Let $G$ be a $\sigma$-compact, locally compact group. The following are equivalent.

1. $G$ embeds coarsely into a Hilbert space.
2. There exists a sequence of Schur multipliers $\varphi_n : G \times G \to \mathbb{C}$ such that
   - $\|\varphi_n\|_S \leq 1$ for every natural number $n$;
   - each $\varphi_n$ tends to zero off tubes (Definition 6.1);
   - $\varphi_n \to 1$ uniformly on tubes.
If any of these conditions holds, one can moreover arrange that the coarse embedding is continuous and that each $\varphi_n$ is continuous.

From Theorem 1.12 together with [21, Theorem 3.5] we immediately obtain the following.

**Corollary 1.13.** If $G$ is a $\sigma$-compact, locally compact group with $\Lambda_{WH}(G) = 1$, then $G$ embeds coarsely into a Hilbert space. In particular, the Baum-Connes assembly map with coefficients is split-injective for all second countable, locally compact groups $G$ with $\Lambda_{WH}(G) = 1$.

## 2. Preliminaries

Throughout, $G$ will denote a locally compact group. A kernel $\varphi: G \times G \to \mathbb{C}$ is a Schur multiplier if there exist bounded maps $\xi, \eta: G \to \mathcal{H}$ into a Hilbert space $\mathcal{H}$ such that $\varphi(g, h) = \langle \xi(g), \eta(h) \rangle$ for every $g, h \in G$. The Schur norm of $\varphi$ is defined as

$$
\|\varphi\|_S = \inf\{\|\xi\|_{\infty}\|\eta\|_{\infty}\}
$$

where the infimum is taken over all $\xi, \eta: G \to \mathcal{H}$ as above. See [47, Theorem 5.1] for different characterizations of Schur multipliers. Clearly, $\|\varphi \cdot \psi\|_S \leq \|\varphi\|_S \cdot \|\psi\|_S$ and $\|\varphi\|_S = \|\varphi\|_S$ when $\varphi$ and $\psi$ are Schur multipliers and $\varphi(x, y) = \varphi(y, x)$. Also, any positive definite kernel $\varphi$ on $G$ which is normalized, i.e., $\varphi(x, x) = 1$ for every $x \in G$, is a Schur multiplier of norm 1. The unit ball of Schur multipliers is closed under pointwise limits.

A continuous function $\varphi: G \to \mathbb{C}$ is a Herz-Schur multiplier if the associated kernel $\hat{\varphi}(g, h) = \varphi(g^{-1} h)$ is a Schur multiplier. The Herz-Schur norm of $\varphi$ is defined as $\|\varphi\|_{B_2} = \|\hat{\varphi}\|_S$. When $\varphi$ is a Herz-Schur multiplier, the two bounded maps $\xi, \eta: G \to \mathcal{H}$ can be chosen to be continuous. The set $B_2(G)$ of Herz-Schur multipliers on $G$ is a unital Banach algebra under pointwise multiplication and $\|\cdot\|_{\infty} \leq \|\cdot\|_{B_2}$. Any continuous, positive definite function $\varphi$ on $G$ is a Herz-Schur multiplier with $\|\varphi\|_{B_2} = \varphi(1)$.

Below we list a number of permanence results concerning weak amenability and the weak Haagerup property, which will be useful later on. General references containing almost all of the results are [1], [17], [30] and [38]. Additionally we refer to [15, Theorem III.9] and [9, Corollary 12.3.12].

Suppose $\Gamma_1$ is a co-amenable subgroup of a discrete group $\Gamma_2$, that is, there exists a left $\Gamma_2$-invariant mean on $l^\infty(\Gamma_2/\Gamma_1)$. Then

$$
\Lambda_{WA}(\Gamma_1) = \Lambda_{WA}(\Gamma_2).
$$

(2.1)

If $(G_i)_{i \in I}$ is a directed family of open subgroups in a locally compact group $G$ whose union is $G$, then

$$
\Lambda_{WA}(G) = \sup \Lambda_{WA}(G_i).
$$

(2.2)

For any two locally compact groups $G$ and $H$

$$
\Lambda_{WA}(G \times H) = \Lambda_{WA}(G)\Lambda_{WA}(H).
$$

(2.3)

When $H$ is a closed subgroup of $G$

$$
\Lambda_{WA}(H) \leq \Lambda_{WA}(G) \quad \text{and} \quad \Lambda_{WH}(H) \leq \Lambda_{WH}(G).
$$

(2.4)

When $K$ is a compact normal subgroup of $G$ then

$$
\Lambda_{WA}(G/K) = \Lambda_{WA}(G) \quad \text{and} \quad \Lambda_{WH}(G/K) = \Lambda_{WH}(G).
$$

(2.5)

When $Z$ is a central subgroup of a discrete group $G$ then

$$
\Lambda_{WA}(G) \leq \Lambda_{WA}(G/Z).
$$

(2.6)
Recall that a lattice in a locally compact group $G$ is a discrete subgroup $\Gamma$ such that the quotient $G/\Gamma$ admits a non-trivial finite $G$-invariant Radon measure. When $\Gamma$ is a lattice in a second countable, locally compact $G$ then

$$\Lambda_{WA}(\Gamma) = \Lambda_{WA}(G) \quad \text{and} \quad \Lambda_{WH}(\Gamma) = \Lambda_{WH}(G). \tag{2.7}$$

When $H$ is a finite index, closed subgroup in a group $G$ then

$$\Lambda_{WH}(H) = \Lambda_{WH}(G). \tag{2.8}$$

3. Weak amenability of $\text{GL}(2, K)$

This section is devoted to the proof of Theorem 1.11 (see Theorem 3.7 below). The general idea of our proof follows the idea of [26, Section 5], where it is shown that for any field $K$ the discrete group $\text{GL}(2, K)$ has the Haagerup property. Our proof of Theorem 1.11 also follows the same strategy as used in [28].

Recall that a pseudo-length function on a group $G$ is a function $\ell : G \to [0, \infty)$ such that

- $\ell(e) = 0$,
- $\ell(g) = \ell(g^{-1})$,
- $\ell(g_1g_2) \leq \ell(g_1) + \ell(g_2)$.

Moreover, $\ell$ is a length function on $G$ if, in addition, $\ell(g) = 0 \implies g = e$.

**Definition 3.1.** We say that the pseudo-length group $(G, \ell)$ is weakly amenable if there exist a sequence $(\varphi_n)$ of Herz-Schur multipliers on $G$ and a sequence $(R_n)$ of positive numbers such that

- $\sup_n \|\varphi_n\|_{B_2} < \infty$;
- $\supp \varphi_n \subseteq \{g \in G \mid \ell(g) \leq R_n\}$;
- $\varphi_n \rightharpoonup 1$ uniformly on $\{g \in G \mid \ell(g) \leq S\}$ for every $S > 0$.

The weak amenability constant $\Lambda_{WA}(G, \ell)$ is defined as the best possible constant $\Lambda$ such that $\sup_n \|\varphi_n\|_{B_2} \leq \Lambda$, where $(\varphi_n)$ is as just described.

Notice that if the group $G$ is discrete and the pseudo-length function $l$ on $G$ is proper (in particular, $G$ is countable), then the weak amenability of $(G, l)$ is equivalent to the weak amenability of $G$ with same weak amenability constant. On other hand, every countable discrete group admits a proper length function, which is unique up to coarse equivalence ([53, Lemma 2.1]). If the group is finitely generated discrete, one can simply take the word-length function associated to any finite set of generators.

The next proposition is a variant of a well-known theorem, which follows from two classical results:

- The graph distance $\text{dist}$ on a tree $T$ is a conditionally negative definite kernel [29].
- The Schur multiplier associated with the characteristic function $\chi_n$ of the subset $\{(x, y) \in T^2 \mid \text{dist}(x, y) = n\}$ has Schur norm at most $2n$ for every $n \in \mathbb{N}$ [8, Proposition 2.1].

The proof below is similar to the proof of [9, Corollary 12.3.5].

**Proposition 3.2.** Suppose a group $G$ acts isometrically on a tree $T$ and that $\ell$ is a pseudo-length function on $G$. Suppose moreover that $\text{dist}(g.v, v) \to \infty$ if and only if $\ell(g) \to \infty$ for some (and hence every) vertex $v \in T$. Then $\Lambda_{WA}(G, \ell) = 1$.

**Proof.** Fix a vertex $v \in T$ as in the assumptions. For every $n \in \mathbb{N}$ we consider the functions $\psi_n(g) = \exp\left(-\frac{1}{n}\text{dist}(g.v, v)\right)$ and $\hat{\chi}_n(g) = \chi_n(g.v, v)$ defined for $g \in G$. Then

$$\hat{\chi}_m(g)\psi_n(g) = \exp(-m/n)\hat{\chi}_m(g)$$
holds for all \( g \in G \) and every \( n, m \in \mathbb{N} \). As \( G \) acts isometrically on \( T \), each \( \psi_n \) is a unital positive definite function on \( G \) by Schoenberg’s theorem and \( \| \hat{\psi}_n \|_{B_2} \leq 2n \) for every \( n \in \mathbb{N} \). It follows that \( \| \psi_n \|_{B_2} = 1 \) and \( \| \hat{\psi}_m \psi_n \|_{B_2} \leq 2m \cdot \exp(-m/n) \) for every \( n, m \in \mathbb{N} \). Therefore, for any \( M \in \mathbb{N} \), we have

\[
\left\| \sum_{m=0}^{M} \hat{\psi}_n \right\|_{B_2} \leq \| \psi_n \|_{B_2} + \left\| \sum_{m>M} \hat{\psi}_n \right\|_{B_2} \leq 1 + \sum_{m>M} 2m \cdot \exp(-m/n).
\]

Hence, if we choose \( M_n \) suitably for all \( n \in \mathbb{N} \), then the functions \( \varphi_n = \sum_{m=0}^{M_n} \hat{\psi}_n \) satisfy that \( \| \varphi_n \|_{B_2} \leq 1 + \frac{1}{n} \) and \( \text{supp} \varphi_n \subseteq \{ g \in G \mid \text{dist}(g.v, v) \leq M_n \} \). The assumption

\[
\text{dist}(g.v, v) \to \infty \iff \ell(g) \to \infty
\]

then insures that \( \text{supp} \varphi_n \subseteq \{ g \in G \mid \ell(g) \leq R_n \} \) for some suitable \( R_n \) and that \( \varphi_n \to 1 \) uniformly on \( \{ g \in G \mid \ell(g) \leq S \} \) for every \( S > 0 \), as desired. \( \square \)

**Remark 3.3.** The two classical results listed above have a generalization:

- The combinatorial distance \( \text{dist} \) on the 1-skeleton of a CAT(0) cube complex \( X \) is a conditionally negative definite kernel on the vertex set of \( X \) [42].
- The Schur multiplier associated with the characteristic function of the subset \( \{ (x, y) \in X^2 \mid \text{dist}(x, y) = n \} \) has Schur norm at most \( p(n) \) for every \( n \in \mathbb{N} \), where \( p \) is a polynomial and \( X \) is (the vertex set of) a finite-dimensional CAT(0) cube complex [41, Theorem 2].

To see that these results are in fact generalizations, we only have to notice that a tree is exactly a one-dimensional CAT(0) cube complex, and in this case the combinatorial distance is just the graph distance. Because of these generalizations and the fact that the exponential function increases faster than any polynomial, it follows with the same proof as the proof of Proposition 3.2 that the following generalization is true (see also [41, Theorem 3]): suppose a group \( G \) acts cellularly (and hence isometrically) on a finite-dimensional CAT(0) cube complex \( X \) and that \( \ell \) is a pseudo-length function on \( G \). Suppose moreover \( \text{dist}(g.v, v) \to \infty \) if and only if \( \ell(g) \to \infty \) for some (and hence every) vertex \( v \in X \). Then \( \Lambda_{WA}(G, \ell) = 1 \).

In our context, a norm on a field \( K \) is a map \( d: K \to [0, \infty) \) satisfying, for all \( x, y \in K \)

\[
\begin{align*}
&\text{(i)} \ d(x) = 0 \text{ implies } x = 0, \\
&\text{(ii)} \ d(xy) = d(x)d(y), \\
&\text{(iii)} \ d(x + y) \leq d(x) + d(y).
\end{align*}
\]

A norm obtained as the restriction of the usual absolute value on \( \mathbb{C} \) via a field embedding \( K \hookrightarrow \mathbb{C} \) is archimedean. A norm is discrete if the triangle inequality (iii) can be replaced by the stronger ultrametric inequality

\[
\text{(iii')} \ d(x + y) \leq \max\{d(x), d(y)\}
\]

and the range of \( d \) on \( K^\times \) is a discrete subgroup of the multiplicative group \( (0, \infty) \).

**Theorem 3.4 ([26, Theorem 2.1]).** Every finitely generated field \( K \) is discretely embeddable: For every finitely generated subring \( A \) of \( K \) there exists a sequence of norms \( d_n \) on \( K \), each either archimedean or discrete, such that for every sequence \( R_n > 0 \), the subset

\[
\{ a \in A \mid d_n(a) \leq R_n \text{ for all } n \in \mathbb{N} \}
\]

is finite.

Let \( d \) be a norm on a field \( K \). Following Guentner, Higson and Weinberger [26] define a pseudo-length function \( \ell_d \) on \( \text{GL}(n, K) \) as follows: if \( d \) is discrete

\[
\ell_d(g) = \log \max_{i,j} \{ d(g_{ij}), d(g^{ij}) \},
\]
where \( g_{ij} \) and \( g^{ij} \) are the matrix coefficients of \( g \) and \( g^{-1} \), respectively; if \( d \) is archimedean, coming from an embedding of \( K \) into \( \mathbb{C} \) then
\[
\ell_d(g) = \log \max\{\|g\|, \|g^{-1}\|\},
\]
where \( \| \cdot \| \) is the operator norm of a matrix in \( \text{GL}(n, \mathbb{C}) \).

**Proposition 3.5.** Let \( d \) be an archimedean or a discrete norm on a field \( K \). Then the pseudo-length group \( (\text{SL}(2, K), \ell_d) \) is weakly amenable with constant 1.

**Proof.** The archimedean case: it is clear that the pseudo-length function on \( \text{SL}(2, K) \) is the restriction of that on \( \text{SL}(2, \mathbb{C}) \), so clearly we only have to show \( (\text{SL}(2, \mathbb{C}), \ell_d) \) is weakly amenable with constant 1. Since \( \ell_d \) is continuous and proper, this follows from the fact that \( \text{SL}(2, \mathbb{C}) \) is weakly amenable with constant 1 as a locally compact group ([18, Remark 3.8]).

The discrete case: this is a direct application of [26, Lemma 5.9] and Proposition 3.2. Indeed, [26, Lemma 5.9] states that there exist a tree \( T \) and a vertex \( v_0 \in T \) such that \( \text{SL}(2, K) \) acts isometrically on \( T \) and
\[
\text{dist}(g.v_0, v_0) = 2 \max_{i,j} \frac{\log d(g_{ij})}{\log d(\pi)}
\]
for all \( g = [g_{ij}] \in \text{SL}(2, K) \). Here dist is the graph distance on \( T \) and \( \pi \) is certain element of \( \{x \in K \mid d(x) < 1\} \). Since the action is isometric, \( \text{dist}(g.v_0, v_0) \to \infty \) if and only if \( \ell_d(g) \to \infty \). Hence, we are done by Proposition 3.2. \( \square \)

**Corollary 3.6.** Let \( K \) be a field and \( G \) a finitely generated subgroup of \( \text{SL}(2, K) \). Then there exists a sequence of pseudo-length functions \( \ell_n \) on \( G \) such that \( \Lambda_{WA}(G, \ell_n) = 1 \) for every \( n \), and such that for any sequence \( R_n > 0 \), the set \( \bigcap_n \{g \in G \mid \ell_n(g) \leq R_n\} \) is finite.

**Proof.** As \( G \) is finitely generated, we may assume that \( K \) is finitely generated as well. Now, let \( A \) be the finitely generated subring of \( K \) generated by the matrix coefficients of a finite generating set for \( G \). Clearly, \( G \subseteq \text{SL}(2, A) \subseteq \text{SL}(2, K) \). Since \( K \) is discretely embeddable, we may choose a sequence \( d_n \) of norms \( d_n \) on \( K \) according to Theorem 3.4. It follows from Proposition 3.5 that \( \Lambda_{WA}(G, \ell_{d_n}) = 1 \). We complete the proof by observing that for any sequence \( R_n > 0 \),
\[
\bigcap_n \{g \in G \mid \ell_{d_n}(g) \leq R_n\} \subseteq \text{SL}(2, F),
\]
where \( F \) is the finite set \( \{a \in A \mid d_n(a) \leq \exp(R_n) \text{ for all } n \in \mathbb{N}\} \). \( \square \)

**Theorem 3.7.** Let \( K \) be a field. Every subgroup \( \Gamma \) of \( \text{GL}(2, K) \) is weakly amenable with constant 1 (as a discrete group).

**Proof.** By the permanence results listed in Section 2 we can reduce our proof to the case where \( \Gamma \) is a finitely generated subgroup of \( \text{SL}(2, K) \). It then follows from the previous corollary that there exists a sequence \( \ell_n \) of pseudo-length functions on \( \Gamma \) such that \( \Lambda_{WA}(\Gamma, \ell_n) = 1 \) and for any sequence \( R_n > 0 \), the set \( \bigcap_n \{g \in \Gamma \mid \ell_n(g) \leq R_n\} \) is finite. For each fixed \( n \in \mathbb{N} \) there is a sequence \( (\varphi_{n,k})_k \) of Herz-Schur multipliers on \( \Gamma \) and a sequence of positive numbers \( (R_{n,k})_k \) such that
\begin{enumerate}
  \item \( \|\varphi_{n,k}\|_{L_2} \leq 1 \) for all \( k \in \mathbb{N} \);
  \item \( \supp \varphi_{n,k} \subseteq \{g \in \Gamma \mid \ell_n(g) \leq R_{n,k}\} \);
  \item \( \varphi_{n,k} \to 1 \) uniformly on \( \{g \in \Gamma \mid \ell_n(g) \leq S\} \) for every \( S > 0 \) as \( k \to \infty \).
\end{enumerate}

Upon replacing \( \varphi_{n,k} \) by \( |\varphi_{n,k}|^2 \) we may further assume that \( 0 \leq \varphi_{n,k} \leq 1 \) for all \( n, k \in \mathbb{N} \).

Given any \( \varepsilon > 0 \) and any finite subset \( F \subseteq \Gamma \), we choose a sequence \( 0 < \varepsilon_n < 1 \) such that \( \Pi_{n}(1 - \varepsilon_n) > 1 - \varepsilon \). It follows from (3) that for each \( n \in \mathbb{N} \) there exists \( \ell_n \in \mathbb{N} \) such that \( 1 - \varepsilon_n < \varphi_{n,k_n}(g) \) for all \( g \in F \). Consider the function \( \varphi = \prod_n \varphi_{n,k_n} \). It is not hard to see
that \( \varphi \) is well-defined, since \( 0 \leq \varphi_{n,k_n} \leq 1 \). Additionally, since \( \| \varphi_{n,k_n} \|_{B_2} \leq 1 \) for all \( n \in \mathbb{N} \) we also have \( \| \varphi \|_{B_2} \leq 1 \). Moreover, supp \( \varphi \subseteq \bigcap_n \{ g \in \Gamma \mid \ell_n(g) \leq R_{n,k_n} \} \) and

\[
\varphi(g) = \prod_n \varphi_{n,k_n}(g) > \prod_n (1 - \varepsilon_n) > 1 - \varepsilon
\]

for all \( g \in F \). This completes the proof. \( \square \)

The remaining part of this section follows Cornulier’s idea from [12]. In [12] he proved the same results for Haagerup property, and the same argument actually works for weak amenability with constant 1.

**Corollary 3.8.** Let \( R \) be a unital commutative ring without nilpotent elements. Then every subgroup \( \Gamma \) of \( \text{GL}(2, R) \) is weakly amenable with constant 1 (as a discrete group).

**Proof.** Again by the permanence results in Section 2, we may assume that \( \Gamma \) is a finitely generated subgroup of \( \text{SL}(2, R) \), and hence that \( R \) is also finitely generated. It is well-known that every finitely generated ring is Noetherian and in such a ring there are only finitely many minimal prime ideals. Let \( p_1, \ldots, p_n \) be the minimal prime ideals in \( R \). The intersection of all minimal prime ideals is the set of nilpotent elements in \( R \), which is trivial by our assumption. So \( R \) embeds into the finite product \( \prod_{i=1}^n R/p_i \). If \( K_i \) denotes the fraction field of the integral domain \( R/p_i \), then \( \Gamma \) embeds into \( \text{SL}(2, \prod R/p_i) = \prod \text{SL}(2, K_i) \). Now, the result is a direct consequence of Theorem 3.7, (2.3) and (2.4). \( \square \)

**Remark 3.9.** In the previous corollary and also in Theorem 3.7, the assumption about commutativity cannot be dropped. Indeed, the group \( \text{SL}(2, \mathbb{H}) \) with the discrete topology is not weakly amenable, where \( \mathbb{H} \) is the skew-field of quaternions. This can be seen from Theorem 1.10. Moreover, \( \text{SL}(2, \mathbb{H})_{\text{ad}} \) does not even have the weak Haagerup property by the same argument.

**Remark 3.10.** In the previous corollary, the assumption about the triviality of the nilradical cannot be dropped. Indeed, we show now that the group \( \text{SL}(2, \mathbb{Z}[x]/x^2) \) is not weakly amenable. The essential part of the argument is Dorofaeff’s result that the locally compact group \( \mathbb{R}^3 \rtimes \text{SL}(2, \mathbb{R}) \) is not weakly amenable [23]. Here the action \( \text{SL}(2, \mathbb{R}) \curvearrowright \mathbb{R}^3 \) is the unique irreducible 3-dimensional representation of \( \text{SL}(2, \mathbb{R}) \).

Consider the ring \( R = \mathbb{R}[x]/x^2 \). We write elements of \( R \) as polynomials \( ax + b \) where \( a, b \in \mathbb{R} \) and \( x^2 = 0 \). Consider the unital ring homomorphism \( \varphi \colon R \to \mathbb{R} \) given by setting \( x = 0 \), that is, \( \varphi(ax + b) = b \). Then \( \varphi \) induces a group homomorphism \( \tilde{\varphi} \colon \text{SL}(2, R) \to \text{SL}(2, R) \). Embedding \( \mathbb{R} \subseteq R \) as constant polynomials, we obtain an embedding \( \text{SL}(2, \mathbb{R}) \subseteq \text{SL}(2, R) \) showing that \( \tilde{\varphi} \) splits. The kernel of \( \tilde{\varphi} \) is easily identified as

\[
\ker \tilde{\varphi} = \left\{ \begin{pmatrix} a_{11}x + 1 & a_{12}x \\ a_{21}x & a_{22}x + 1 \end{pmatrix} \mid a_{ij} \in \mathbb{R}, \ a_{11} + a_{22} = 0 \right\} \simeq \mathfrak{s}(2, \mathbb{R})
\]

We deduce that \( \text{SL}(2, R) \) is the semidirect product \( \mathfrak{s}(2, \mathbb{R}) \rtimes \text{SL}(2, \mathbb{R}) \). A simple computation shows that the action \( \text{SL}(2, \mathbb{R}) \curvearrowright \mathfrak{s}(2, \mathbb{R}) \) is the adjoint action. Since \( \mathfrak{s}(2, \mathbb{R}) \) is a simple Lie algebra, the adjoint action is irreducible. By uniqueness of the 3-dimensional irreducible representation of \( \text{SL}(2, \mathbb{R}) \) (see [39, p. 107]) and from [23] we deduce that \( \mathfrak{s}(2, \mathbb{R}) \rtimes \text{SL}(2, \mathbb{R}) \simeq \mathbb{R}^3 \rtimes \text{SL}(2, \mathbb{R}) \) is not weakly amenable.

It is easy to see that \( \text{SL}(2, \mathbb{Z}[x]/x^2) \) is identified with \( \mathfrak{s}(2, \mathbb{Z}) \rtimes \text{SL}(2, \mathbb{Z}) \) under the isomorphism \( \text{SL}(2, R) \simeq \mathfrak{s}(2, \mathbb{R}) \rtimes \text{SL}(2, \mathbb{R}) \). Since \( \mathfrak{s}(2, \mathbb{Z}) \rtimes \text{SL}(2, \mathbb{Z}) \) is a lattice in \( \mathfrak{s}(2, \mathbb{R}) \rtimes \text{SL}(2, \mathbb{R}) \), we conclude from (2.7) that \( \mathfrak{s}(2, \mathbb{Z}) \rtimes \text{SL}(2, \mathbb{Z}) \) and hence \( \text{SL}(2, \mathbb{Z}[x]/x^2) \) is not weakly amenable.

**Remark 3.11.** We do not know if \( \text{SL}(2, \mathbb{Z}[x]/x^2) \) also fails to have the weak Haagerup property. As \( \text{SL}(2, \mathbb{Z}[x]/x^2) \) may be identified with a lattice in \( \mathbb{R}^3 \rtimes \text{SL}(2, \mathbb{R}) \), by (2.7) the question is equivalent to the question [31, Remark 5.3] raised by Haagerup and the first author concerning the weak Haagerup property of the group \( \mathbb{R}^3 \rtimes \text{SL}(2, \mathbb{R}) \).
Recall that a group $\Gamma$ is residually free if for every $g \neq 1$ in $\Gamma$, there is a homomorphism $f$ from $\Gamma$ to a free group $F$ such that $f(g) \neq 1$ in $F$. Equivalently, $\Gamma$ embeds into a product of free groups of rank two. A group $\Gamma$ is residually finite if for every $g \neq 1$ in $\Gamma$, there is a homomorphism $f$ from $\Gamma$ to a finite group $F$ such that $f(g) \neq 1$ in $F$. Equivalently, $\Gamma$ embeds into a product of finite groups. Since free groups are residually finite, it is clear that residually free groups are residually finite. On the other hand, residually finite groups need not be residually free as is easily seen by considering e.g. groups with torsion.

**Corollary 3.12.** Every residually free group is weakly amenable with constant 1.

**Proof.** Since the free group of rank two can be embedded in $\text{SL}(2, \mathbb{Z})$, a residually free group embeds in $\prod_{i \in I} \text{SL}(2, \mathbb{Z}) = \text{SL}(2, \prod_{i \in I} \mathbb{Z})$ for a suitably large set $I$. We complete the proof by the previous corollary. \hfill $\square$

### 4. Failure of the weak Haagerup property

In this section we will prove the following result, which is the combination of Corollaries 4.3, 4.5 and 4.6.

**Proposition 4.1.** If $G$ is one of the four groups $\text{SO}(5)$, $\text{SO}_0(1,4)$, $\text{SU}(3)$ or $\text{SU}(1,2)$, then $G_\Delta$ does not have the weak Haagerup property.

Also, if $G$ is the universal covering group of $\text{SU}(1,n)$ where $n \geq 2$, then $G_\Delta$ does not have the weak Haagerup property.

When $p, q \geq 0$ are integers, not both zero, and $n = p + q$, we let $I_{p,q}$ denote the diagonal $n \times n$ matrix with 1 in the first $p$ diagonal entries and $-1$ in the last $q$ diagonal entries. When $g$ is a complex matrix, $g^t$ denotes the transpose of $g$, and $g^*$ denotes the adjoint (conjugate transpose) of $g$. We recall that

\[
\begin{align*}
\text{SO}(p, q) &= \{ g \in \text{SL}(p + q, \mathbb{R}) \mid g^t I_{p,q} g = I_{p,q} \} \\
\text{SO}(p, q, \mathbb{C}) &= \{ g \in \text{SL}(p + q, \mathbb{C}) \mid g^t I_{p,q} g = I_{p,q} \} \\
\text{SU}(p, q) &= \{ g \in \text{SL}(p + q, \mathbb{C}) \mid g^* I_{p,q} g = I_{p,q} \}.
\end{align*}
\]

When $p, q > 0$, the group $\text{SO}(p, q)$ has two connected components, and $\text{SO}_0(p, q)$ denotes the identity component.

To prove Proposition 4.1, we follow a strategy that we have learned from Cornulier [13], where the same techniques are applied in connection with the Haagerup property. The idea of the proof is the following. We consider the groups as real algebraic groups $G(\mathbb{R})$. Let $K$ be a number field of degree three over $\mathbb{Q}$, not totally real, and let $\mathcal{O}$ be its ring of integers. Then $G(\mathcal{O})$ embeds diagonally as a lattice in $G(\mathbb{R}) \times G(\mathbb{C})$. The group $G(\mathbb{C})$ will have real rank two, and we deduce that the group $G(\mathcal{O})$ does not have the weak Haagerup property by combining [31, Theorem B] with (2.7). As $G(\mathcal{O})$ is a subgroup of $G(\mathbb{R})$, (2.4) implies that $G(\mathbb{R})$ also does not have the weak Haagerup property, and we are done. We will now make this argument more precise.

Let $K$ denote the field $\mathbb{Q}(\sqrt[3]{2})$ and $\mathcal{O}$ its ring of integers $\mathbb{Z}[\sqrt[3]{2}]$. Let $\omega = e^{2\pi i / 3}$ be a third root of unity and let $\sigma: K \to \mathbb{C}$ be the field monomorphism uniquely defined by $\sigma(\sqrt[3]{2}) = \omega \sqrt[3]{2}$. If we denote the image of $\sigma$ by $K^\sigma$, then $\sigma$ induces a ring isomorphism, also denoted $\sigma$, of matrix algebras

\[
\sigma: M_n(K) \to M_n(K^\sigma)
\]

by applying $\sigma$ entry-wise.
The field $K$ is an algebra over $\mathbb{Q}$ with basis $1, 2^{1/3}, 2^{2/3}$. With respect to this basis, multiplication is given by
\[
\begin{pmatrix}
    a_1 \\
    b_1 \\
    c_1
\end{pmatrix} \odot \begin{pmatrix}
    a_2 \\
    b_2 \\
    c_2
\end{pmatrix} = \begin{pmatrix}
    a_1a_2 + 2b_1c_2 + 2c_1b_2 \\
    a_1b_2 + b_1a_2 + 2c_1c_2 \\
    a_1c_2 + b_1b_2 + c_1a_2
\end{pmatrix} \quad (4.2)
\]
where $a_i, b_i, c_i \in \mathbb{Q}$ and $i = 1, 2$. Multiplication by an element $x = a + 2^{1/3}b + 2^{2/3}c \in K$ where $a, b, c \in \mathbb{Q}$ defines an endomorphism of $K$, and it is clear from (4.2) that the matrix representation of $x$ is
\[
\begin{pmatrix}
    a & 2c & 2b \\
    b & a & 2c \\
    c & b & a
\end{pmatrix}. \quad (4.3)
\]

If $\pi(x)$ denotes the matrix in (4.3) then $\pi: K \to M_3(\mathbb{Q})$ is an algebra homomorphism.

4.1. The real case. Let $A$ be the $\mathbb{R}$-algebra $\mathbb{R}^3$ with multiplication $\circ$ given by (4.2) where $a_i, b_i, c_i \in \mathbb{R}$ and $i = 1, 2$. The unit of $A$ is $(1,0,0)$. Let $\xi_1: A \to \mathbb{R}$ and $\xi_2: A \to \mathbb{C}$ be the algebra homomorphisms defined by
\[
\xi_1(a, b, c) = a + 2^{1/3}b + 2^{2/3}c, \quad \text{and} \quad \xi_2(a, b, c) = a + \omega 2^{1/3}b + \omega^2 2^{2/3}c, \quad (4.4)
\]
where $a, b, c \in \mathbb{R}$. It is easily verified that $\xi = (\xi_1, \xi_2)$ is an algebra isomorphism of $A$ onto $\mathbb{R} \oplus \mathbb{C}$.

More generally, we define $\xi^n_1: M_n(A) \to M_n(\mathbb{R})$ and $\xi^n_2: M_n(A) \to M_n(\mathbb{C})$ by
\[
\xi^n_i([x_{jk}]) = [\xi_i(x_{jk})] \quad \text{when} \quad [x_{jk}] \in M_n(A)
\]
for $i = 1, 2$, and we let $\xi^n = (\xi^n_1, \xi^n_2)$. It follows that $\xi^n$ is an $\mathbb{R}$-algebra isomorphism of $M_n(A)$ onto $M_n(\mathbb{R}) \oplus M_n(\mathbb{C})$. We also denote the multiplication in $M_n(A)$ by $\circ$. We note that $\xi^n$ preserves transposition and the determinant in the sense that for every $x \in M_n(A)$
\[
\xi^n(x^t) = \xi^n(x)^t \quad \text{and} \quad \det_{\mathbb{R} \oplus \mathbb{C}} \xi^n(x) = \xi(\det_A x).
\]

**Proposition 4.2.** Let $p, q \geq 0$ be integers with $p + q \geq 3$. If $\sigma$ is the homomorphism in (4.1), then the homomorphism $1 \times \sigma$ embeds the group $SO(p, q, \mathbb{Z}[\sqrt{2}])$ as a lattice in $SO(p, q) \times SO(p, q, \mathbb{C})$.

**Proof.** We use the notation introduced before Proposition 4.2. We will show that
\[
\Lambda = \{(l, \sigma(l)) \in SO(p, q) \times SO(p, q, \mathbb{C}) \mid l \in SO(p, q, \mathbb{C})\}
\]
is a lattice in $SO(p, q) \times SO(p, q, \mathbb{C})$. We put $n = p + q$. Let $H$ be the group consisting of matrices $(a, b, c) \in M_n(A)$ such that
\[
(a^t, b^t, c^t) \circ (a, b, c) = (I_{p,q}, 0,0) \quad \text{and} \quad \det_A[(a, b, c)] = (1,0,0). \quad (4.5)
\]
Observe that
\[
\xi^n(I_{p,q}, 0,0) = (I_{p,q}, I_{p,q}).
\]
Then $(a, b, c) \in H$ if and only if
\[
\xi^n(a, b, c)^t \xi^n(a, b, c) = (I_{p,q}, I_{p,q}) \quad \text{and} \quad \det_{\mathbb{R} \oplus \mathbb{C}} \xi^n(a, b, c) = (1,1),
\]
that is, if and only if $\xi^n(a, b, c)$ belongs to $SO(p, q) \times SO(p, q, \mathbb{C})$. Thus, $\xi^n$ is a group isomorphism of $H$ onto $SO(p, q) \times SO(p, q, \mathbb{C})$.

The next idea is to identify $H$ with an algebraic subgroup of $M_{3n}(\mathbb{R})$ by adopting the matrix representation (4.3) of $K$. Let $\pi: M_n(A) \to M_{3n}(\mathbb{R})$ be the map sending $(a, b, c) \in M_n(A)$ to
\[
\begin{pmatrix}
    a & 2c & 2b \\
    b & a & 2c \\
    c & b & a
\end{pmatrix} \quad (4.6)
\]
where $a, b, c \in M_n(\mathbb{R})$. It is not hard to see that $\pi$ is an injective ring homomorphism.
We let $G = \pi(H)$. Then $G$ is the subgroup of $\text{SL}(3n, \mathbb{R})$ consisting of matrices of the form (4.6), where $a, b, c \in M_n(\mathbb{R})$ satisfies the relations (4.5). The crucial point is that the definition (4.2) of the multiplication $\circ$ in $A$ is given by integral polynomials in the entries, and hence the relations (4.5) are polynomial equations in the entries of $a, b, c$ with integral coefficients. This shows that $G$ is an algebraic subgroup of $\text{SL}(3n, \mathbb{R})$ defined over $\mathbb{Q}$. Moreover, $\rho = \xi^n \circ \pi^{-1}$ is a group isomorphism of $G$ onto $\text{SO}(p, q) \times \text{SO}(p, q, \mathbb{C})$, which is also a diffeomorphism. Since $\text{SO}(p, q) \times \text{SO}(p, q, \mathbb{C})$ is semisimple (here we use $p + q \geq 3$), we deduce that $G$ is semisimple.

By the Borel Harish-Chandra Theorem [7, Theorem 7.8], the subgroup $G_{\mathbb{Z}} = \text{SL}(3n, \mathbb{Z}) \cap G$ is a lattice in $G$, and hence $\rho(G_{\mathbb{Z}})$ is a lattice in $\text{SO}(p, q) \times \text{SO}(p, q, \mathbb{C})$. It remains to show that $\rho(G_{\mathbb{Z}}) = \Lambda$.

Suppose first that $g \in G_{\mathbb{Z}}$ is of the form (4.6) and put $l = \xi_1^n \circ \pi^{-1}(g) = \xi_1^n(a, b, c)$. Then $l \in \text{SO}(p, q, \mathbb{O})$ and $\xi_2^n(a, b, c) = \sigma(l)$. This shows that $\rho(g) = (l, \sigma(l)) \in \Lambda$.

Conversely, given $(l, \sigma(l)) \in \Lambda$ where $l \in \text{SO}(p, q, \mathbb{O})$, we can in a unique way write $l = a + 2^{1/3}b + 2^{2/3}c = \xi_1^n(a, b, c)$ where $a, b, c \in M_n(\mathbb{Z})$. Then $\sigma(l) = \xi_2^n(a, b, c)$ and if we define $g$ by (4.6) then $g \in G_{\mathbb{Z}}$ and $\rho(g) = l$.

This proves that $\Lambda = \rho(G_{\mathbb{Z}})$, and the proof is complete. 

**Corollary 4.3.** If $G$ is $\text{SO}(5)$ or $\text{SO}_0(1, 4)$, then $G_{\mathbb{A}}$ does not have the weak Haagerup property.

**Proof.** The Lie group $\text{SO}(5, \mathbb{C})$ has real rank two (see Table IV of [34, Ch.X §6]). It is thus a consequence of [31, Theorem B] that $\text{SO}(5, \mathbb{C})$ does not have the weak Haagerup property.

Suppose $(p, q) = (5, 0)$ or $(p, q) = (1, 4)$ and let $\Gamma = \text{SO}(p, q, \mathbb{Z}[\sqrt{2}])$. As $\text{SO}(p, q, \mathbb{C}) \simeq \text{SO}(p + q, \mathbb{C})$, we see that $\text{SO}(p, q, \mathbb{C})$ does not have the weak Haagerup property. Since $\Gamma$ is embedded via $1 \times \sigma$ as a lattice in $\text{SO}(p, q) \times \text{SO}(p, q, \mathbb{C})$, it follows from (2.7) that $\Gamma$ does not have the weak Haagerup property. Since $\Gamma$ is a subgroup of $\text{SO}(p, q)$, we conclude that $\text{SO}(p, q)_{\mathbb{A}}$ does not have the weak Haagerup property. We have now shown that $\text{SO}(5)_{\mathbb{A}}$ and $\text{SO}(1, 4)_{\mathbb{A}}$ do not have the weak Haagerup property. To finish the proof, recall that the group $\text{SO}_0(1, 4)$ has index two in $\text{SO}(1, 4)$, so that by (2.8) we conclude that $\text{SO}_0(1, 4)_{\mathbb{A}}$ also does not have the weak Haagerup property.

**4.2. The complex case.** To prove that $\text{SU}(3)$ and $\text{SU}(2, 1)$ do not have the weak Haagerup property we use the same technique as before, but in a complex version. Let $K = \mathbb{Q}(\sqrt{2}, i)$ and $\mathcal{O} = \mathbb{Z}[\sqrt{2}, i]$, and let $\sigma: K \to \mathbb{C}$ be the field homomorphism defined by

$$\sigma(\sqrt{2}) = \omega \sqrt{2}, \quad \sigma(i) = i.$$  

We also use $\sigma$ to denote the ring homomorphism

$$\sigma: M_n(K) \to M_n(\mathbb{C})$$  

obtained by applying $\sigma$ entry-wise.

Let $A$ be the $\mathbb{C}$-algebra $\mathbb{C}^3$ with multiplication $\circ$ given by (4.2) where $a_i, b_i, c_i \in \mathbb{C}$ and $i = 1, 2$. Let $\xi_1, \xi_2, \xi_3: A \to \mathbb{C}$ be the algebra homomorphisms defined by

$$\xi_1(a, b, c) = a + 2^{1/3}b + 2^{2/3}c,$$

$$\xi_2(a, b, c) = a + \omega 2^{1/3}b + \omega^2 2^{2/3}c,\quad (4.8)$$

$$\xi_3(a, b, c) = a + \omega^2 2^{1/3}b + \omega 2^{2/3}c.$$  

Then it is easily verified that $\xi = (\xi_1, \xi_2, \xi_3)$ is an isomorphism of $A$ onto $\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$. More generally, for $i = 1, 2, 3$ we define $\xi_i^n: M_n(A) \to M_n(\mathbb{C})$ by

$$\xi_i^n([x_{jk}]) = [\xi_i(x_{jk})] \quad \text{when} \quad [x_{jk}] \in M_n(A)$$

and let $\xi^n = (\xi_1^n, \xi_2^n, \xi_3^n)$. It follows that $\xi^n$ is a $\mathbb{C}$-algebra isomorphism of $M_n(A)$ onto $M_n(\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C})$. Multiplication in $M_n(A)$ is also denoted by $\circ$. Elements of $M_n(A)$ are
thought of as triples \((a, b, c)\), where \(a, b, c \in M_n(\mathbb{C})\). We note that for every \((a, b, c)\) in \(M_n(A)\)

\[
(\xi^0_n, \xi^\mathfrak{a}_n, \xi^\mathfrak{b}_n)(a^*, b^*, c^*) = ((\xi^0_n, \xi^\mathfrak{a}_n, \xi^\mathfrak{b}_n)(a, b, c))^*
\]

and

\[
\det_{\mathbb{C}[\mathfrak{G}]} \xi^n(a, b, c) = \xi(\det_{\mathbb{C}}(a, b, c)).
\]

Warning: did you notice the index switch in (4.9)?

**Proposition 4.4.** Let \(p, q \geq 0\) be integers with \(p + q \geq 2\). If \(\sigma\) is as in (4.7), then the homomorphism \(1 \times \sigma\) embeds the group \(SU(p, q, \mathbb{Z}[\sqrt{2}, i])\) as a lattice in \(SU(p, q) \times SL(p+q, \mathbb{C})\).

**Proof.** Put \(n = p + q\). Let \(H\) be the group consisting of matrices \((a, b, c) \in M_n(A)\) such that

\[
(a^*, b^*, c^*) \circ (a, b, c) = (I_{p, q}, 0, 0) \quad \text{and} \quad \det_{\mathbb{C}}[(a, b, c)] = (1, 0, 0).
\]

Observe that

\[
\xi^n(I_{p, q}, 0, 0) = (I_{p, q}, I_{p, q}, I_{p, q}).
\]

Using (4.9)-(4.10) we see that \((a, b, c) \in H\) if and only if

\[
((\xi^0_n, \xi^\mathfrak{a}_n, \xi^\mathfrak{b}_n)(a, b, c)) = (I_{p, q}, I_{p, q}, I_{p, q})
\]

and

\[
\det_{\mathbb{C}[\mathfrak{G}]} \xi^n(a, b, c) = (1, 1, 1).
\]

Thus, \(\xi^n\) is a group isomorphism of \(H\) onto the group \(L\) consisting of matrices \((s, z, w) \in M_{3n}(\mathbb{C})^\mathfrak{G}\) such that

\[
s^*s = I_{p, q}, \quad z^*w = I_{p, q}, \quad w^*z = I_{p, q}, \quad \det \mathbb{C}s = \det \mathbb{C}z = \det \mathbb{C}w = 1.
\]

It is easily seen that \(L\) is in fact

\[
L = \{(s, z, (z^*)^{-1}I_{p, q}) \in M_{3n}(\mathbb{C})^\mathfrak{G} \mid s \in SU(p, q), \ z \in SL(n, \mathbb{C})\}.
\]

Let \(\eta\) be the isomorphism of \(L\) onto \(SU(p, q) \times SL(n, \mathbb{C})\) given by \(\eta(s, z, (z^*)^{-1}) = (s, z)\). Let \(\pi : M_n(A) \to M_{3n}(\mathbb{C})\) be the map sending \((a, b, c) \in M_n(A)\) to

\[
\begin{pmatrix}
a & 2c & 2b \\
b & a & 2c \\
c & b & a
\end{pmatrix}
\]

(4.12)

where \(a, b, c \in M_n(\mathbb{C})\). It is not hard to see that \(\pi\) is an injective ring homomorphism. We let \(G = \pi(H)\). Then \(G\) is the subgroup of \(SL(3n, \mathbb{C})\) consisting of matrices of the form (4.12), where \(a, b, c \in M_n(\mathbb{C})\) satisfies the relations (4.11). The crucial point is that the definition (4.2) of the multiplication \(\circ\) in \(A\) is given by integral polynomials in the entries, and hence the relations (4.11) are polynomial equations in the real and imaginary parts of the entries of \(a, b, c\) with integral coefficients. This shows that \(G\) is an algebraic subgroup of \(SL(3n, \mathbb{C})\) defined over \(\mathbb{Q}\). Moreover, \(\rho = \eta \circ \xi^n \circ \pi^{-1}\) is a group isomorphism of \(G\) onto \(SU(p, q) \times SL(n, \mathbb{C})\), which is also a diffeomorphism. Since \(SU(p, q) \times SL(n, \mathbb{C})\) is semisimple (here we use \(p + q \geq 2\)), we deduce that \(G\) is semisimple.

By the Borel Harish-Chandra Theorem, the subgroup \(G_{\mathbb{Z}+i\mathbb{Z}} = SL(3n, \mathbb{Z}[i]) \cap G\) is a lattice in \(G\), and hence \(\rho(G_{\mathbb{Z}+i\mathbb{Z}})\) is a lattice in \(SU(p, q) \times SL(n, \mathbb{C})\).

We will finish the proof by showing that

\[
\Lambda = \{(l, \sigma(l)) \in SU(p, q) \times SL(n, \mathbb{C}) \mid l \in SU(p, q, \mathbb{O})\}
\]

coincides with \(\rho(G_{\mathbb{Z}+i\mathbb{Z}})\).

Suppose first that \(g \in G_{\mathbb{Z}+i\mathbb{Z}}\) is of the form (4.12) and put \(l = \xi^n \circ \pi^{-1}(g) = \xi^n(a, b, c)\). Then \(l \in SU(p, q, \mathbb{O})\) and \(\xi^\mathfrak{a}_n(a, b, c) = \sigma(l)\). This shows that \(\rho(g) = (l, \sigma(l)) \in \Lambda\).

Conversely, given \((l, \sigma(l)) \in \Lambda\) where \(l \in SU(p, q, \mathbb{O})\) we can in a unique way write \(l = a + 2^{1/3}b + 2^{2/3}c = \xi^\mathfrak{a}_n(a, b, c)\) where \(a, b, c \in M_n(\mathbb{Z} + i\mathbb{Z})\). Then \(\sigma(l) = \xi^\mathfrak{a}_n(a, b, c) = \sigma(l)\) and if we define \(g\) by (4.12) then \(g \in G_{\mathbb{Z}+i\mathbb{Z}}\) and \(\rho(g) = l\).
This proves that $\Lambda = \rho(G_{2+\mathbb{Z}})$, and the proof is complete. \hfill \Box

**Corollary 4.5.** If $G$ is SU(3) or SU(1, 2), then $G_d$ does not have the weak Haagerup property.

*Proof.* The Lie group SL(3, $\mathbb{C}$) has real rank two (see Table IV of [34, Ch.X §6]). It is thus a consequence of [31, Theorem B] that SL(3, $\mathbb{C}$) does not have the weak Haagerup property. Suppose $(p, q) = (3, 0)$ or $(p, q) = (1, 2)$ and let $\Gamma = SU(p, q, \mathbb{Z}[\sqrt{2}])$. Since $\Gamma$ is embedded via $1 \times \sigma$ as a lattice in $SU(p, q) \times SL(3, \mathbb{C})$, it follows from (2.7) that $\Gamma$ does not have the weak Haagerup property. Since $\Gamma$ is a subgroup of $SU(p, q)$, we conclude that $SU(p, q)_d$ does not have the weak Haagerup property. This completes the proof. \hfill \Box

**Corollary 4.6.** Let $\tilde{G}$ be the universal covering group $SU(1, n)$ of SU(1, $n$) where $n \geq 2$. Then $\tilde{G}_d$ does not have the weak Haagerup property.

*Proof.* Let $G = SU(1, n)$, and let $q: \tilde{G} \to G$ be the covering homomorphism. If $\Gamma$ denotes the image of $SU(p, q, \mathbb{Z}[\sqrt{2}])$ under $1 \times \sigma$, then $\Gamma$ is a lattice in $G \times SL(n + 1, \mathbb{C})$. Let $\tilde{\Gamma}$ be the lift of $\Gamma$ to $\tilde{G} \times SL(n + 1, \mathbb{C})$, that is, $\tilde{\Gamma} = (q \times 1)^{-1}(\Gamma)$. Since $q \times 1$ is a covering homomorphism $\tilde{G} \times SL(n + 1, \mathbb{C}) \to G \times SL(n + 1, \mathbb{C})$, it is then easy to check that $\tilde{\Gamma}$ is a lattice in $\tilde{G} \times SL(n + 1, \mathbb{C})$. The rest of the proof is now similar to the previous proof.

The Lie group $SL(n + 1, \mathbb{C})$ has real rank $n$ (see Table IV of [34, Ch.X §6]). It is thus a consequence of [31, Theorem B] that SL($n + 1, \mathbb{C}$) does not have the weak Haagerup property. It follows from (2.7) that $\tilde{\Gamma}$ does not have the weak Haagerup property. The projection $\tilde{G} \times SL(n + 1, \mathbb{C}) \to \tilde{G}$ is injective on $\tilde{\Gamma}$, and hence $\tilde{\Gamma}$ embeds as a subgroup of $\tilde{G}$. We conclude that $\tilde{G}_d$ does not have the weak Haagerup property. \hfill \Box

### 5. Proof of the Main Theorem

In this section we prove Theorem 1.10. The theorem is basically a consequence of Theorem 1.11 and Proposition 4.1 together with the permanence results listed in Section 2 and general structure theory of simple Lie groups.

We recall that two Lie groups $G$ and $H$ are *locally isomorphic* if there exist open neighborhoods $U$ and $V$ around the identity elements of $G$ and $H$, respectively, and an analytic diffeomorphism $f: U \to V$ such that

- if $x, y, xy \in U$ then $f(xy) = f(x)f(y)$;
- if $x, y, xy \in V$ then $f^{-1}(xy) = f^{-1}(x)f^{-1}(y)$.

When two Lie groups $G$ and $H$ are locally isomorphic we write $G \approx H$. An important fact about Lie groups and local isomorphisms is the following [34, Theorem II.1.11]: Two Lie groups are locally isomorphic if and only if their Lie algebras are isomorphic.

The following is extracted from [11, Chapter II] and [36, Section I.11] to which we refer for details. If $G$ is a connected Lie group, there exists a connected, simply connected Lie group $\tilde{G}$ and a covering homomorphism $\tilde{G} \to G$. The kernel of the covering homomorphism is a discrete, central subgroup of $\tilde{G}$, and it is isomorphic to the fundamental group of $G$.

The group $\tilde{G}$ is called the *universal covering group* of $G$. Clearly, $\tilde{G}$ and $G$ are locally isomorphic. Conversely, any connected Lie group locally isomorphic to $G$ is the quotient of $\tilde{G}$ by a discrete, central subgroup. If $N$ is a discrete subgroup of the center $Z(\tilde{G})$ of $\tilde{G}$, then the center of $\tilde{G}/N$ is $Z(\tilde{G})/N$.

Let $G_1$ and $G_2$ be locally compact groups. We say that $G_1$ and $G_2$ are *strongly locally isomorphic*, if there exist a locally compact group $G$ and finite normal subgroups $N_1$ and $N_2$
of $G$ such that $G_1 \cong G/N_1$ and $G_2 \cong G/N_2$. In this case we write $G_1 \sim G_2$. It follows from (2.5) that if $G \sim H$, then $\Lambda_{WH}(G_d) = \Lambda_{WH}(H_d)$.

A theorem due to Weyl states that a connected, simple, compact Lie group has a compact universal cover with finite center [35, Theorem 12.1.17], [34, Theorem II.6.9]. Thus, for connected, simple, compact Lie groups $G$ and $H$, $G \approx H$ implies $G \sim H$.

**Proof of Theorem 1.10.** Let $G$ be a connected simple Lie group. As mentioned, the equivalence $(1) \iff (2)$ was already done by Cornulier [13, Theorem 1.14] in a much more general setting, so we leave out the proof of this part. We only prove the two implications $(1) \implies (3)$ and $(6) \implies (1)$, since the remaining implications then follow trivially.

Suppose (1) holds, that is, $G$ is locally isomorphic to $SO(3)$, $SL(2, \mathbb{R})$ or $SL(2, \mathbb{C})$. If $Z$ denotes the center of $G$, then by assumption $G/Z$ is isomorphic to $SO(3)$, $PSL(2, \mathbb{R})$ or $PSL(2, \mathbb{C})$. It follows from Theorem 1.11 and (2.5) that the groups $SO(3)$, $PSL(2, \mathbb{R})$ and $PSL(2, \mathbb{C})$ equipped with the discrete topology are weakly amenable with constant 1 (recall that $SO(3)$ is a subgroup of $PSL(2, \mathbb{C})$). From (2.6) we deduce that $G_d$ is weakly amenable with constant 1. This proves (3).

Suppose (1) does not hold. We prove that (6) fails, that is, $G_d$ does not have the weak Haagerup property. We divide the proof into several cases depending on the real rank of $G$. We recall that with the Iwasawa decomposition $G = KAN$, the real rank of $G$ is the dimension of the abelian group $A$.

If the real rank of $G$ is at least two, then $G$ does not have the weak Haagerup property [31, Theorem B]. By a theorem of Borel, $G$ contains a lattice (see [49, Theorem 14.1]), and by (2.7) the lattice also does not have the weak Haagerup property. We conclude that $G_d$ does not have the weak Haagerup property.

If the real rank of $G$ equals one, then the Lie algebra of $G$ is isomorphic to a Lie algebra in the list [36, (6.109)]. See also [34, Ch.X §6]. In other words, $G$ is locally isomorphic to one of the classical groups $SO_0(1, n)$, $SU(1, n)$, $Sp(1, n)$ for some $n \geq 2$ or locally isomorphic to the exceptional group $F_{4(-20)}$. Here $SO_0(1, n)$ denotes the identity component of the group $SO(1, n)$.

We claim that the universal covering groups of $SO_0(1, n)$, $Sp(1, n)$ and $F_{4(-20)}$ have finite center except for the group $SO_0(1, 2)$. Indeed, $Sp(1, n)$ and $F_{4(-20)}$ are already simply connected with finite center. The $K$-group from the Iwasawa decomposition of $SO_0(1, n)$ is $SO(n)$ which has fundamental group of order two, except when $n = 2$, and hence $SO_0(1, n)$ has fundamental group of order two as well. As the center of the universal cover is an extension of the center of $SO_0(1, n)$ by the fundamental group of $SO_0(1, n)$, the claim follows.

The universal covering group $SU(1, n)$ of $SU(1, n)$ has infinite center isomorphic to the group of integers.

We have assumed that $G$ is not locally isomorphic to $SL(2, \mathbb{R}) \sim SO_0(1, 2)$ or $SL(2, \mathbb{C}) \sim SO_0(1, 3)$. If $G$ has finite center, it follows that $G$ is strongly locally isomorphic to one of the groups

$$
SO_0(1, n), \quad n \geq 4,
$$
$$
SU(1, n), \quad n \geq 2,
$$
$$
Sp(1, n), \quad n \geq 2,
$$
$$
F_{4(-20)},
$$

and if $G$ has infinite center, then $G$ is isomorphic to $SU(1, n)$. Clearly, there are inclusions

$$
SO_0(1, 4) \subseteq SO_0(1, n), \quad n \geq 4,
$$
$$
SU(1, 2) \subseteq SU(1, n), \quad n \geq 2,
$$
$$
SU(1, 2) \subseteq Sp(1, n), \quad n \geq 2.
$$
The cases where $G$ is strongly locally isomorphic to $\text{SO}(1, n)$, $\text{SU}(1, n)$ or $\text{Sp}(1, n)$ are then covered by Proposition 4.1. Since $\text{SO}(5) \subseteq \text{SO}(9) \sim \text{Spin}(9) \subseteq F_{4(-20)}$ ([52, §4.Proposition 1]), the case where $G \sim F_{4(-20)}$ is also covered by Proposition 4.1. Finally, if $G \simeq \tilde{\text{SU}}(1, n)$, then Proposition 4.1 shows that $G_d$ does not have weak Haagerup property.

If the real rank of $G$ is zero, then it is a fairly easy consequence of [35, Theorem 12.1.17] that $G$ is compact. Moreover, the universal covering group of $G$ is compact and with finite center.

By the classification of compact simple Lie groups as in Table IV of [34, Ch.X §6] we know that $G$ is strongly locally isomorphic to one of the groups $\text{SU}(n + 1) \ (n \geq 1)$, $\text{SO}(2n + 1) \ (n \geq 2)$, $\text{Sp}(n) \ (n \geq 3)$, $\text{SO}(2n) \ (n \geq 4)$ or one of the five exceptional groups $\text{E}_6$, $\text{E}_7$, $\text{E}_8$, $\text{F}_4$, $\text{G}_2$.

By assumption $G$ is not strongly locally isomorphic to $\text{SU}(2) \sim \text{SO}(3)$. Using (2.5) it then suffices to show that if $G$ equals any other group in the list, then $G_d$ does not have the weak Haagerup property. Clearly, there are inclusions

$$\text{SO}(5) \subseteq \text{SO}(n), \ n \geq 5,$$

$$\text{SU}(3) \subseteq \text{SU}(n), \ n \geq 3,$$

$$\text{SU}(3) \subseteq \text{Sp}(n), \ n \geq 3.$$

Since we also have the following inclusions among Lie algebras (Table V of [34, Ch.X §6])

$$\mathfrak{so}(5) \subseteq \mathfrak{so}(9) \subseteq \mathfrak{su}_1 \subseteq \mathfrak{su}_6 \subseteq \mathfrak{su}_7 \subseteq \mathfrak{su}_8$$

and the inclusion ([54])

$$\text{SU}(3) \subseteq \text{G}_2,$$

it is enough to consider the cases where $G = \text{SO}(5)$ or $G = \text{SU}(3)$. These two cases are covered by Proposition 4.1. Hence we have argued that also in the real rank zero case $G_d$ does not have the weak Haagerup property. \hspace{1cm} \square

6. A Schur multiplier characterization of coarse embeddability

In this section we give a characterization of coarse embeddability into Hilbert spaces in terms of contractive Schur multipliers. It is well-known that the notion of coarse embeddability into Hilbert spaces can be characterized by positive definite kernels (see [27, Theorem 2.3] for the discrete case and [20, Theorem 1.5] for the locally compact case).

If $G$ is a locally compact group, a (left) tube in $G \times G$ is a subset of $G \times G$ contained in a set of the form

$$\text{Tube}(K) = \{(x, y) \in G \times G \mid x^{-1}y \in K\}$$

where $K$ is a compact subset of $G$.

**Definition 6.1.** A kernel $\varphi: G \times G \to \mathbb{C}$ tends to zero off tubes, if for any $\varepsilon > 0$ there is a tube $T \subseteq G \times G$ such that $|\varphi(x, y)| < \varepsilon$ whenever $(x, y) \notin T$.

Note that if $\varphi: G \to \mathbb{C}$ is a function, then $\varphi$ vanishes at infinity, if and only if the associated kernel $\hat{\varphi}: G \times G$ defined by $\hat{\varphi}(x, y) = \varphi(x^{-1}y)$ tends to zero off tubes.

**Definition 6.2 (2, Definition 3.6).** Let $G$ be a $\sigma$-compact, locally compact group. A map $u$ from $G$ into a Hilbert space $H$ is said to be a coarse embedding if $u$ satisfies the following two conditions:

- for every compact subset $K$ of $G$ there exists $R > 0$ such that $$(s, t) \in \text{Tube}(K) \implies \|u(s) - u(t)\| \leq R;$$
- for every $R > 0$ there exists a compact subset $K$ of $G$ such that $\|u(s) - u(t)\| \leq R \implies (s, t) \in \text{Tube}(K)$. 


We say that a group \( G \) embeds coarsely into a Hilbert space or admits a coarse embedding into a Hilbert space if there exist a Hilbert space \( H \) and a coarse embedding \( u : G \to H \).

Every second countable, locally compact group \( G \) admits a proper left-invariant metric \( d \), which is unique up to coarse equivalence (see [51] and [32]). So the preceding definition is equivalent to Gromov’s notion of coarse embeddability of the metric space \((G, d)\) into Hilbert spaces. We refer to [21, Section 3] for more on coarse embeddability into Hilbert spaces for locally compact groups).

It is not hard to see that the countability assumption in [37, Proposition 4.3] is superfluous. We thus record the following (slightly more general) version of [37, Proposition 4.3].

**Lemma 6.3.** Let \( G \) be a group with a symmetric kernel \( k : G \times G \to [0, \infty) \). The following are equivalent.

1. For every \( t > 0 \) one has \( \|e^{-tk}\|_S \leq 1 \).
2. There exist a real Hilbert space \( \mathcal{H} \) and maps \( R, S : G \to \mathcal{H} \) such that
   \[ k(x, y) = \|R(x) - R(y)\|^2 + \|S(x) + S(y)\|^2 \]
   for every \( x, y \in G \).

Recall that a kernel \( k : G \times G \to \mathbb{R} \) is conditionally negative definite if \( k \) is symmetric \((k(x, y) = k(y, x))\), vanishes on the diagonal \((k(x, x) = 0)\) and
\[
\sum_{i,j=1}^{n} c_i c_j k(x_i, x_j) \leq 0
\]
for any finite sequences \( x_1, \ldots, x_n \in G \) and \( c_1, \ldots, c_n \in \mathbb{R} \) such that \( \sum_{i=1}^{n} c_i = 0 \). It is well-known that \( k \) is conditionally negative definite if and only if there is a function \( u \) from \( G \) to a real Hilbert space such that \( k(x, y) = \|u(x) - u(y)\|^2 \).

If \( G \) is a locally compact group we say that a kernel \( k : G \times G \to \mathbb{C} \) is proper, if the set \( \{(x, y) \in G \times G \mid |k(x, y)| \leq R\} \) is a tube for every \( R > 0 \).

Theorem 1.12 is contained in the following theorem, which extends both [22, Theorem 5.3] and [20, Theorem 1.5] in different directions.

**Theorem 6.4.** Let \( G \) be a \( \sigma \)-compact, locally compact group. The following are equivalent.

1. The group \( G \) embeds coarsely into a Hilbert space.
2. There exists a sequence of (not necessarily continuous) Schur multipliers \( \varphi_n : G \times G \to \mathbb{C} \) such that
   - \( \|\varphi_n\|_S \leq 1 \) for every natural number \( n \);
   - each \( \varphi_n \) tends to zero off tubes;
   - \( \varphi_n \to 1 \) uniformly on tubes.
3. There exists a (not necessarily continuous) symmetric kernel \( k : G \times G \to [0, \infty) \) which is proper, bounded on tubes and satisfies \( \|e^{-tk}\|_S \leq 1 \) for all \( t > 0 \).
4. There exists a (not necessarily continuous) conditionally negative definite kernel \( h : G \times G \to \mathbb{R} \) which is proper and bounded on tubes.

Moreover, if any of these conditions holds, one can arrange that the coarse embedding in (1), each Schur multiplier \( \varphi_n \) in (2), the symmetric kernel \( k \) in (3) and the conditionally negative definite kernel \( h \) in (4) are continuous.

**Proof.** We show \( (1) \iff (4) \iff (3) \iff (2) \).

That (1) implies (4) with \( h \) continuous follows directly from [21, Theorem 3.4].

That (4) implies (3) follows from Schoenberg’s Theorem and the fact that normalized positive definite kernels are Schur multipliers of norm 1.

Suppose (3) holds. We show that (4) holds. From Lemma 6.3 we see that there are a real Hilbert space \( \mathcal{H} \) and maps \( R, S : G \to \mathcal{H} \) such that
\[
k(x, y) = \|R(x) - R(y)\|^2 + \|S(x) + S(y)\|^2 \quad \text{for every } x, y \in G.
\]
As $k$ is bounded on tubes, the map $S$ is bounded. If we let
\[ h(x, y) = \|R(x) - R(y)\|^2, \]
then it is easily checked that $h$ is proper and bounded on tubes, since $k$ has these properties and $S$ is bounded. It is also clear that $h$ is conditionally negative definite. Thus (4) holds.

Suppose now that (4) holds. By the GNS construction there are a real Hilbert space $S$ and a map $u: G \to \mathcal{H}$ such that
\[ h(x, y) = \|u(x) - u(y)\|^2. \]
It is easy to check that the assumptions on $h$ imply that $u$ is a coarse embedding. Thus (1) holds.

If (3) holds, we set $\varphi_n = e^{-k/n}$ when $n \in \mathbb{N}$. It is easy to check that the sequence $\varphi_n$ has the desired properties so that (2) holds.

Conversely, suppose (2) holds. We verify (3). Essentially, we use the same standard argument as in the proof of [38, Proposition 4.4] and [10, Theorem 2.1.1].

Since $G$ is locally compact and $\sigma$-compact, it is the union of an increasing sequence $(U_n)_{n=1}^{\infty}$ of open sets such that the closure $K_n$ of $U_n$ is compact and contained in $U_{n+1}$ (see [25, Proposition 4.39]). Fix an increasing, unbounded sequence $(\alpha_n)$ of positive real numbers and a decreasing sequence $(\varepsilon_n)$ tending to zero such that $\sum_n \alpha_n \varepsilon_n$ converges. By assumption, for every $n$ we can find a Schur multiplier $\varphi_n$ tending to zero off tubes and such that $\|\varphi_n\|_S \leq 1$ and
\[ \sup_{(x,y) \in \text{Tube}(K_n)} |\varphi_n(x,y) - 1| \leq \varepsilon_n/2. \]
Upon replacing $\varphi_n$ by $|\varphi_n|^2$ one can arrange that $0 \leq \varphi_n \leq 1$ and
\[ \sup_{(x,y) \in \text{Tube}(K_n)} |\varphi_n(x,y) - 1| \leq \varepsilon_n. \]
Define kernels $\psi_i : G \times G \to [0, \infty]$ and $\psi : G \times G \to [0, \infty]$ by
\[ \psi_i(x,y) = \sum_{n=1}^i \alpha_n (1 - \varphi_n(x,y)), \quad \psi(x,y) = \sum_{n=1}^{\infty} \alpha_n (1 - \varphi_n(x,y)). \]
It is easy to see that $\psi$ is well-defined, bounded on tubes and $\psi_i \to \psi$ pointwise (even uniformly on tubes, but we do not need that).

To see that $\psi$ is proper, let $R > 0$ be given. Choose $n$ large enough such that $\alpha_n \geq 2R$. As $\varphi_n$ tends to zero off tubes, there is a compact set $K \subseteq G$ such that $|\varphi_n(x,y)| < 1/2$ whenever $(x, y) \notin \text{Tube}(K)$. Now if $\psi(x,y) \leq R$, then $\psi(x,y) \leq \alpha_n/2$, and in particular $\alpha_n (1 - \varphi_n(x,y)) \leq \alpha_n/2$, which implies that $1 - \varphi_n(x,y) \leq 1/2$. We have thus shown that
\[ \{(x, y) \in G \times G \mid \psi(x,y) \leq R\} \subseteq \{(x, y) \in G \times G \mid 1 - \varphi_n(x,y) \leq 1/2\} \subseteq \text{Tube}(K), \]
and $\psi$ is proper.

We now show that $\|e^{-t\psi}\|_S \leq 1$ for every $t > 0$. Since $\psi_i$ converges pointwise to $\psi$, it will suffice to prove that $\|e^{-t\psi_i}\|_S \leq 1$, because the set of Schur multipliers of norm at most 1 is closed under pointwise limits. Since
\[ e^{-t\psi_i} = \prod_{n=1}^i e^{-t\alpha_n (1 - \varphi_n)}, \]
it is enough to show that $e^{-t\alpha_n (1 - \varphi_n)}$ has Schur norm at most 1 for each $n$. And this is clear:
\[ \|e^{-t\alpha_n (1 - \varphi_n)}\|_S = e^{-t\alpha_n} \|e^{t\alpha_n \varphi_n}\|_S \leq e^{-t\alpha_n} e^{t\alpha_n} \|\varphi_n\|_S \leq 1. \]
The only thing missing is that $\psi$ need not be symmetric. Put $k = \psi + \tilde{\psi}$ where $\tilde{\psi}(x,y) = \psi(y,x)$. Clearly, $k$ is symmetric, bounded on tubes and proper. Finally, for every $t > 0$
\[ \|e^{-tk}\|_S \leq \|e^{-t\psi}\|_S \|e^{-t\tilde{\psi}}\|_S \leq 1, \]
Finally, the statements about continuity follow from [21, Theorem 3.4] and the explicit constructions used in our proof of (1) $\implies$ (4) $\implies$ (3) $\implies$ (2).

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