New algorithms for solving third- and fifth-order two point boundary value problems based on nonsymmetric generalized Jacobi Petrov–Galerkin method

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ABSTRACT

Two families of certain nonsymmetric generalized Jacobi polynomials with negative integer indexes are employed for solving third- and fifth-order two point boundary value problems governed by homogeneous and nonhomogeneous boundary conditions using a dual Petrov–Galerkin method. The idea behind our method is to use trial functions satisfying the underlying boundary conditions of the differential equations and the test functions satisfying the dual boundary conditions. The resulting linear systems from the application of our method are specially structured and they can be efficiently inverted. The use of generalized Jacobi polynomials simplify the theoretical and numerical analysis of the method and also leads to accurate and efficient numerical algorithms. The presented numerical results indicate that the proposed numerical algorithms are reliable and very efficient.

Introduction

Techniques for finding approximate solutions for differential equations, based on classical orthogonal polynomials are popularly known as spectral methods. The term “spectral” was probably originated from the fact that the trigonometric functions \( \cos(kx) \) are the eigenfunctions of the Laplace operator with the periodic boundary conditions. This fact and the availability of Fast Fourier Transform (FFT) are the main advantages of the Fourier spectral method. Thus, using Fourier series to solve partial differential equations, with principal differential operator being the Laplace operator (or its power) with periodic boundary conditions, results in very alternative numerical algorithms [1–6].

The spectral methods aim to approximate functions (solutions of differential equations) by means of truncated series of orthogonal polynomials. There are three well-known methods of spectral methods, namely, tau, collocation and Galerkin methods [2,4]. The choice of the suitable spectral method suggested for solving the given equation depends certainly on the type of the differential equation and the type of the boundary conditions governed by it. The spectral...
methods take various orthogonal polynomials as trial functions. The use of the Chebyshev, Legendre, ultraspherical and classical Jacobi polynomials is suitable for non-periodic problems, while the use of Laguerre and Hermite polynomials is suitable for handling the problems defined respectively on the half line, and on the whole line [4,6,7].

Standard spectral methods are capable of providing very accurate approximations to well-behaved smooth functions with significantly less degrees of freedom when compared with finite difference or finite element methods [1,2,4,8]. Classical orthogonal polynomials are used successfully and extensively for the numerical solution of differential equations in spectral and pseudospectral methods [2,8–13].

The classical Jacobi polynomials $P_{n}^{(x)}(x)$ have great importance in analysis from both theoretical and practical points of view [14]. The six special polynomials of the classical Jacobi polynomials namely, ultraspherical, Chebyshev polynomials of the four kinds, and Legendre polynomials have been extensively employed in numerical analysis and in particular in spectral methods [15]. It is known that the Jacobi polynomials are precisely the only polynomials arising as eigenfunctions of a singular Sturm–Liouville problem (2, Section 9.2).

The construction of the generalized Jacobi polynomials was first introduced by Guo et al. [16]. They extended the definition of the classical Jacobi polynomials $P_{n}^{(x)}(x)$ to allow their parameters $\alpha$, $\beta$ to take negative integers. In Guo et al. [16], it has been shown that the generalized Jacobi polynomials, with parameters corresponding to the number of boundary conditions in a given differential equation, are considered as the natural basis functions for the spectral approximation of this problem. Moreover, it has been shown that the use of generalized Jacobi polynomials simplifies the numerical analysis for the spectral approximations of boundary value problems (BVPs) and also leads to very efficient numerical algorithms.

Recently, Abd-Elhameed et al. [17] and Doha et al. [18] have analyzed in detail some numerical algorithms for solving the differentiated and integrated forms of third and fifth-order boundary value problems based on the application of the spectral method namely Petrov–Galerkin method. In these two articles, the authors have employed two new families of general parameters generalized Jacobi polynomials.

A large number of books and research articles dealing with the theory of ordinary differential equations, or their practical applications in various fields, contain mainly results from the theory of second-order linear differential equations, and some results from the theory of some special linear differential equations of higher even order. However, there are few studies for handling third- and fifth-order BVPs. This is due to that the application of the collocation method on such kinds of BVPs leads to high condition numbers. More precisely, it leads to condition numbers of order $N^{6}$ for third-order BVPs and $N^{10}$ for fifth-order BVPs, respectively, where $N$ is the number of retained modes. These high condition numbers will lead to instabilities caused by rounding errors [9,19,20]. In this paper, we introduce some efficient spectral algorithms for reducing these condition numbers to be of $O(N^{2})$ and $O(N^{4})$ for third- and fifth-order BVPs, respectively, based on certain nonsymmetric generalized Jacobi Petrov–Galerkin method.

The study of odd-order equations is of mathematical and physical interest. As an example, third-order equation contains a type of operator which appears in many physical applications such as the Kortweg–de Vries equation. The oscillation properties of third-order differential equations can be found in the monographs of Mckelvey [21]. For more applications of odd-order differential equations, see the monograph by Gregus [22], in which many physical and engineering applications of third-order differential equations are discussed [22].

In the sequence of papers of Abd-Elhameed [23], Doha and Abd-Elhameed [24,25], Doha and Bhrawy [26] and Doha et al. [27], the authors handled second-, fourth-, 2nth- and $(2n+1)$th-order two point boundary value problems. In these articles, they suggested some numerical algorithms based on constructing combinations of various orthogonal polynomials together with the application of the Galerkin method. Recently, Doha and Abd-Elhameed [28] have introduced and used a family of orthogonal polynomials called “symmetric generalized Jacobi polynomials” for handling multidimensional sixth-order two point boundary value problems by the Galerkin method. For other studies on third- and fifth-order BVPs, one can be referred for example to [29,30].

Since, the main differential operator in odd-order differential equations is not symmetric, it is convenient to use a Petrov–Galerkin method. The main difference between the two spectral methods namely, Galerkin and Petrov–Galerkin methods, is that in case of Galerkin method, the test functions coincide with the trial functions, while in Petrov–Galerkin method, the trial and test functions are chosen in a way such that they satisfy respectively, the boundary conditions and their dual conditions of the differential equation.

The main objective in this article is to introduce new algorithms for handling third- and fifth-order BVPs, based on applying the nonsymmetric generalized Jacobi Petrov–Galerkin method (GJPGM). The linear systems resulted from the application of GJPGM are band and hence they can be efficiently inverted.

The structure of the paper is as follows. In “Some properties of classical and generalized Jacobi polynomials” Section, some properties of classical and generalized Jacobi polynomials are given. In “Dual Petrov-Galerkin algorithms for third-order elliptic linear differential equations” and “Dual Petrov-Galerkin algorithms for fifth-order elliptic linear differential equations” Sections, GJPGM is applied for the sake of solving third- and fifth-order linear BVPs with constant coefficients governed by homogenous boundary conditions. In “Structure of the coefficients matrices in the linear systems” Section, the linear systems resulting from the application of GJPGM are investigated. In “Condition number of the resulting matrices” Section, we discuss the condition numbers of the obtained systems. In “Convergence analysis” Section, we state and prove two theorems for the convergence of the proposed algorithms. In “Numerical results” Section, some numerical results accompanied by some comparisons with the other available algorithms appeared in literature are given. Conclusions are given in “Concluding remarks” Section.

Some properties of classical and generalized Jacobi polynomials

Classical Jacobi polynomials

The classical Jacobi polynomials associated with the real parameters $(\alpha > -1, \beta > -1)$ [14,31,32], are a sequence of
polynomials $R^{(a,\beta)}_n(x)$, $x \in (-1,1)$, $(n = 0, 1, 2, \ldots)$, each respectively of degree $n$. Now, and for the sake of simplicity in the upcoming computations, it is useful to define the following normalized classical Jacobi polynomials by $R^{(a,\beta)}_n(x) = \frac{\mu^{(a,\beta)}(n)}{\Delta^{(a,\beta)}(n)} P^{(a,\beta)}_n(x)$. This means that, $R^{(a,\beta)}_n(x) = \frac{\mu^{(a,\beta)}(n+1)}{\Gamma(n+1)} R^{(a,\beta)}_n(x)$. In such case $R^{(a,\beta)}_0(x) = C^{(a)}_0(x)$, where $C^{(a)}(x)$ is the ultraspherical polynomial. Moreover, $R^{(a,\beta)}_n(x)$ may be generated with the aid of the following three term recurrence relation:

$$
(2n + \lambda)(n + a + 1)R^{(a,\beta)}_n(x) = (2n + \lambda - 1)xR^{(a,\beta)}_{n+1}(x) + (2x^2 - \beta^2)(2n + \lambda)R^{(a,\beta)}_n(x) - 2n(n + \beta)(2n + \lambda + 1)R^{(a,\beta)}_{n-1}(x), \quad n = 1, 2, \ldots,
$$

starting from $R^{(a,\beta)}_0(x) = 1$ and $R^{(a,\beta)}_1(x) = \frac{1}{2\lambda - 1 + 2\lambda}$, or obtained from Rodrigue’s formula

$$R^{(a,\beta)}_n(x) = \left(\frac{-1}{\pi}\right)^n \frac{\Gamma(n + \beta)}{\Gamma(n + a + 1)} (1 - x)^{-a}(1 + x)^{-\beta} D^n

\times (1 - x)^{2n}(1 + x)^{2n},
$$

where $\lambda = a + \beta + 1$, $(z)_k = \frac{\Gamma(z + k)}{\Gamma(z)}$, $D \equiv \frac{d}{dx}$.

The orthogonality relation of $R^{(a,\beta)}_n(x)$ is

$$
\int_{-1}^{1} (1 + x)^\gamma (1 - x)^\gamma R^{(a,\beta)}_m(x) R^{(a,\beta)}_n(x) dx = \left\{ \begin{array}{ll}
0, & m \neq n, \\
h^{(a,\beta)}_n, & m = n,
\end{array} \right.
$$

(1)

where $h^{(a,\beta)}_n = 2\gamma n! \Gamma(n + \beta + 1)/(\Gamma(a + 1))^2 (2n + \lambda)\Gamma(n + a + 1)$.

The polynomials $R^{(a,\beta)}_n(x)$ are eigenfunctions of the singular Sturm–Liouville equation:

$$(1 - x^2)\phi''(x) + [\beta - x - (\lambda + 1)x]\phi'(x) + n(n + \lambda)\phi(x) = 0.
$$

The following relations are useful in the sequel:

$$R^{(a,\beta)}_0(x) = \frac{1}{k + 1} \left[(k + a + 1)R^{(a,\beta-1)}_{k+1}(x) - 2R^{(a,\beta-1)}_k(x)\right],
$$

(2)

$$R^{(a,\beta)}_1(x) = \frac{1}{k + a + \beta} \left[(k + \beta)R^{(a,\beta-1)}_k(x) + 2R^{(a,\beta-1)}_k(x)\right],
$$

(3)

$$R^{(a,\beta)}_2(x) = \frac{2(2k + x + \beta)}{k + a + \beta + 2} \left[R^{(a,\beta)}_k(x) - R^{(a,\beta)}_{k+1}(x)\right],
$$

(4)

$$(1 - x^2)R^{(a,\beta)}_{k+1}(x) = \frac{4(2k + x + 1)}{(k + a + 1)^2} \left[(k + \beta)(2k + \lambda + 1)R^{(a,\beta)}_k(x) - R^{(a,\beta)}_{k+1}(x) + (k + a + 1)(k + \lambda - 1)R^{(a,\beta)}_{k+2}(x) + (k + \lambda - 1)^2R^{(a,\beta)}_k(x)\right],
$$

(5)

$$D^\alpha R^{(a,\beta)}_k(x) = \left(\frac{k - q + 1}{2}\right)^\alpha \frac{\Gamma(a + q + \beta)\Gamma(k + \beta)}{\Gamma(a + q + \beta + k)} R^{(a,\beta+\alpha)}_k(x).
$$

Note 1. It is worth noting that $R^{(a,\beta+\alpha-4)}_k(x)$ is identical to the ultraspherical polynomials, $C^{(a+\alpha)}_k(x)$, which is explicitly defined by

$$C^{(a+\alpha)}_n(x) = (-\frac{1}{2})^\alpha \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+n+\beta)} D^\alpha \left[ (1 - x^2)^{\alpha-\beta} D^\beta \left[ (1 - x^2)^{\alpha-\beta} \right] \right],
$$

$C^{(a+\alpha)}_n(1) = 1, \quad n = 0, 1, 2, \ldots,
$$

This definition has the desirable properties that $C^{(a)}_n(x)$ is identical with the Chebyshev polynomials of the first kind $T_n(x)$, $C^{(a)}_n(x)$ is the Legendre polynomials $P_n(x)$, and $C^{(a)}_n(x)$ is equal to $\frac{d}{dx} U_n(x)$ is the Chebyshev polynomials of the second kind (see, [33]).

Now, the following theorem is useful in what follows.

**Theorem 1.** The $q$th derivative of the normalized Jacobi polynomial $R^{(a,\beta)}_n(x)$ is given explicitly by

$$D^q R^{(a,\beta)}_n(x) = (n + \lambda) \sum_{i=0}^{n-q} C_{n,q}(\alpha + q + \beta + q, \alpha, \beta) R^{(a,\beta)}_i(x),
$$

where

$$C_{n,q}(\alpha + q + \beta + q, \alpha, \beta) = \frac{(n + \lambda)(i + q + \alpha + 1)}{(n - i - q)! \Gamma(2i + \lambda + 1)! (i + x + 1)^{n-i} \times 3F_2 \left(\begin{array}{ccc}
Theorem 1, Doha [34].
$$

Non-symmetric general Jacobi polynomials

Following [16], a family of generalized Jacobi polynomials/functions with indexes $\alpha, \beta \in \mathbb{R}$ can be defined. Let $w(x) = (1 - x)^\alpha (1 + x)^\beta$. We denote by $L^{(a,\beta)}_{\alpha,\beta}(-1,1)$ the weighted $L^2$ space with inner product:

$$(u, v)_{\alpha,\beta} = \int_{-1}^{1} u(x)v(x)w(x)dx,
$$

and the associated norm $\|u\|_{\alpha,\beta} = (u, u)_{\alpha,\beta}^{1/2}$. Now, we aim to define Jacobi polynomials with parameters $\alpha$ and/or $\beta \leq -1$, which will be called “non-symmetric generalized Jacobi polynomials (GJPs)”. These polynomials will satisfy some selected properties that are essentially relevant to spectral approximations. In this work, the values of the two parameters $\alpha$ and $\beta$ are restricted to take negative integers.

Now, and if we assume that $\ell, m$ are two integers, then we can define GJPs by

$$f^{(\alpha,\beta)}_{\ell,m}(x) = \left\{ \begin{array}{ll}
(1 - x)^{-\ell}(1 + x)^{-m} R^{(\alpha,\beta)}_{\ell-m}(x), & k_0 = -(\ell + m), \ell, m \leq -1, \\
(1 - x)^{-\ell} R^{(\alpha,\beta)}_{\ell-m}(x), & k_0 = -\ell, \ell \leq -1, m > -1, \\
(1 - x)^{-m} R^{(\alpha,\beta)}_{\ell-m}(x), & k_0 = -m, \ell > -1, m \leq -1, \\
R^{(\alpha,\beta)}_{\ell-m}(x), & k_0 = 0, \ell, m > -1.
\end{array} \right.
$$

It should be noted here that the GJPs have the characterization that for $\ell, m \in \mathbb{Z}$ and $\ell, m \geq 1$,

$$D^\ell f^{(\alpha,\beta)}_{\ell-m}(1) = 0, \quad i = 0, 1, \ldots, \ell - 1;
$$

$$D\ell f^{(\alpha,\beta)}_{\ell-m}(\ell) = 0, \quad j = 0, 1, \ldots, m - 1.
$$

It is not difficult to verify that

$$f^{(-1,\beta)}_{k}(x) = \frac{4}{(k - 1)(k - 3)} \left[ L_{k-3}(x) + \frac{2k - 3}{2k - 1} L_{k-2}(x) \right] - L_{k-1}(x) + \frac{2k - 3}{2k - 1} L_{k-2}(x), \quad k \geq 3,
$$

$$f^{(-1,\beta)}_{k}(1) = 1, \quad \ell = 0, 1, 2, \ldots,
$$

$$f^{(-1,\beta)}_{\ell}(\ell) = 0, \quad j = 0, 1, \ldots, m - 1.
$$

Generalized Jacobi Solutions for odd-order BVPs 675
\[ J_k^{-1-2}(x) = \frac{2}{2k - 3} \left[ L_k^{-2}(x) + \frac{2k - 3}{2k - 1} L_k^{-2}(x) \right] \]

\[ J_k^{-3}(x) = \frac{24}{(2k - 5)(2k - 7)(k - 2)} \left[ L_k^{-5}(x) - \frac{2(2k - 7)}{2k - 3} L_k^{-2}(x) \right] \]

\[ J_k^{-3-2}(x) = \frac{8}{2(2k - 5)} \left[ L_k^{-3}(x) + \frac{2(2k - 7)}{2k - 3} L_k^{-2}(x) \right] \]

where \( L_k(x) \) is the Legendre polynomial of the \( k \)th degree.

\( \{ J_k^{-l-m}(x) \} \) are natural candidates as basis functions for PDFs with the following boundary conditions:

\[ D' u(1) = a_i, \quad i = 0, 1, \ldots, \ell - 1; \]

\[ D' u(-1) = b_j, \quad j = 0, 1, \ldots, m - 1. \]

**Dual Petrov–Galerkin algorithm for third-order elliptic linear differential equations**

This section is concerned with using GJPBM for solving the following third-order elliptic linear differential equation

\[ u^{(3)}(x) - z_1 u^{(2)}(x) - \beta_i u^{(1)}(x) + \gamma_j u(x) = f(x), \]

\( x \in (-1, 1), \) governed by the homogeneous boundary conditions

\[ u(\pm 1) = u^{(1)}(1) = 0. \]

We define the space

\[ V = \{ u \in H^{(2)}(I) : u(\pm 1) = u^{(1)}(1) = 0 \}, \]

and its dual space

\[ V' = \{ u \in H^{(2)}(I) : u(\pm 1) = u^{(1)}(-1) = 0 \}, \]

where

\[ H^{(2)}(I) = \{ u : \| u \|_{2, x, \delta} < \infty \}, \]

\[ \| u \|_{2, x, \delta} = \left( \sum_{k=0}^{2} \| \delta_k^j u \|_{o, x, \delta}^2 \right)^{1/2}. \]

Let \( P_N \) be the space of all polynomials of degree less than or equal to \( N \). Setting \( V_N = V \cap P_N \) and \( V'_N = V' \cap P_N \). We observe that:

\[ V_N = \text{span} \{ J_k^{-3}(x), J_k^{-2-1}(x), \ldots, J_k^{-1-2}(x) \}, \]

\[ V'_N = \text{span} \{ J_k^{-2-1}(x), J_k^{-2-2}(x), \ldots, J_k^{-1-2}(x) \}. \]

The dual Petrov–Galerkin approximation of (7) and (8) is to find \( u_N \in V_N \) such that

\[ (D^3 u_N(x), v(x)) - z_1(D^2 u_N(x), v(x)) - \beta_i(Du_N(x), v(x)) + \gamma_j(u_N(x), v(x)) = (f(x), v(x)), \quad \forall v \in V'_N. \]

\[ (D^3 u_N(x), v(x)) = (D^2 u_N(x), v(x)) - \beta_i(Du_N(x), v(x)) + \gamma_j(u_N(x), v(x)) = (f(x), v(x)), \quad \forall v \in V'_N. \]

**The choice of basis functions**

We can choose suitable basis functions and their dual basis by setting

\[ \phi_k(x) = J_{k+3}^{-2-1}(x) = (1 - x^2)(1 - x) R_k^{(2,1)}(x), \; k = 0, 1, \ldots, N - 3, \]

\[ \psi_k(x) = J_{k+3}^{-2}(x) = (1 - x^2)(1 + x) R_k^{(2,1)}(x), \; k = 0, 1, \ldots, N - 3. \]

It is worthy noting here that the basis \{ \phi_k(x) \} are orthogonal on \([-1, 1]\) in the sense that

\[ \int_{-1}^{1} \phi_k(x) \phi_j(x) dx = \left\{ \begin{array}{ll} \delta_k^j, & k \neq j, \\ \| R_k \|_1, & k = j. \end{array} \right. \]

The idea behind this choice is to use trial and test functions to guarantee the satisfaction of the underlying boundary and dual boundary conditions of the third-order differential equations under investigation. In contrast to other bases [1,2,24], these choices lead to linear systems with specially structured matrices that are well-conditioned, i.e. have bounded condition numbers and therefore, can be efficiently inverted. These and other items will be discussed in the section entitled “Condition numbers of the resulting matrices”.

It is clear that the two sets of orthogonal polynomials \{ \phi_k(x) \} and \{ \psi_k(x) \} are linearly independent, and therefore we have

\[ V_N = \text{span} \{ \phi_k(x) : k = 0, 1, 2, \ldots, N - 3 \}, \]

and

\[ V'_N = \text{span} \{ \psi_k(x) : k = 0, 1, 2, \ldots, N - 3 \}. \]

Now the following two important lemmas will be stated and proved.

**Lemma 1**

\[ D^3 J_{k+3}^{-2-1}(x) = (k + 1)(k + 3) R_k^{(2,1)}(x). \]

**Proof.** By using Leibnitz’s rule, we have

\[ D^3 J_{k+3}^{-2-1}(x) = (1 - x^2)(1 - x) D^2 R_k^{(2,1)}(x) \]

\[ + 3(3x^2 - 2x - 1) D R_k^{(2,1)}(x) \]

\[ + 6(3x - 1) D R_k^{(2,1)}(x) + 6 R_k^{(2,1)}(x). \]

Making use of the relation

\[ (1 - x^2)(1 - x) D^3 R_k^{(2,1)}(x) = (1 + 6x - 7x^2) D^2 R_k^{(2,1)}(x) \]

\[ + (k - 1)(k + 5)(x - 1) D R_k^{(2,1)}(x), \]

we obtain

\[ D^3 J_{k+3}^{-2-1}(x) = 2(x^2 - 1) D^2 R_k^{(2,1)}(x) + [(k - 1)(k + 5)(x - 1) \]

\[ + 6(3x - 1)] D R_k^{(2,1)}(x) + 6 R_k^{(2,1)}(x), \]

which in turn with Eq. (2), and after some rather lengthy manipulation, yields
Making use of the two relations (4) and (6), we have

\[ D^2 J_{k+3}^{-2,-1}(x) = \frac{1}{6}(k+1)(k+3) [k(k+4)(x-1) R_{k+1}^{(2)}(x) + 12 R_0^{(2)}(x)]. \]

Finally, and in virtue of (2) and (3), and after some manipulation, we get

\[ D^2 J_{k+3}^{-2,-1}(x) = 2(k+1)(k+3) R_0^{(2)}(x). \]  

By recalling the definition of Pochhammer’s symbol, \((z)_n = \frac{\Gamma(z+n)}{\Gamma(z)}\), we have

**Lemma 2**

\[ D^2 J_{k+3}^{-2,-1}(x) = \frac{2(k+3)}{(2k+5)} R_1^{(2)}(x) - \frac{(k+1)(k+3)}{(k+\frac{3}{2})} R_0^{(2)}(x), \]

\[ D J_{k+3}^{-2,-1}(x) = \frac{(k+1)(k+3)}{2(k+2)(k+\frac{3}{2})} R_1^{(2)}(x) + \frac{(k+2)}{(k+\frac{5}{2})} R_0^{(2)}(x), \]

\[ J_{k+3}^{-2,-1}(x) = \frac{(k+3)}{4(k+2)(k+\frac{3}{2})} R_1^{(2)}(x) - \frac{3(k+1)}{2(k+\frac{5}{2})} R_0^{(2)}(x), \]

Proof. The proof of Lemma 2 is rather lengthy and it can be accomplished by following the same procedure used in the proof of Lemma 1. □

Now, based on the two Lemmas 1 and 2, the following theorem can be obtained.

**Theorem 2.** For arbitrary constants \(a_k\), one has

\[ D^2 \left[ \sum_{k=0}^{N-3} a_k J_{k+3}^{-2,-1}(x) \right] = \sum_{k=0}^{N-3} N - 3b_k R_1^{(2)}(x), \]

where

\[ b_k = 2(k+1)(k+3), \]

Moreover, if

\[ D^2 \left[ \sum_{k=0}^{N-3} a_k J_{k+3}^{-2,-1}(x) \right] = \sum_{k=0}^{N-3} \varepsilon_k J_{k+3}^{(-2)}(x), \]

then

\[ \varepsilon_k = a_{k-1}J_{k+1}^{(-2)} + a_k \beta_k^{(1)} + a_{k+1} \gamma_k^{(2)}, \]

where

\[ \beta_k^{(2)} = \frac{2(k+3)}{(2k+5)}, \quad \beta_k^{(1)} = -\frac{(k+1)(k+3)}{(k+\frac{5}{2})}, \]

\[ \gamma_k^{(2)} = -\frac{2(k+3)}{(2k+5)}. \]

Also, if

\[ D^2 \left[ \sum_{k=0}^{N-3} a_k J_{k+3}^{-2,-1}(x) \right] = \sum_{k=0}^{N-3} \varepsilon_k J_{k+3}^{(-2)}(x), \]

then

\[ \varepsilon_k = a_{k-2}J_{k+1}^{(-2)} + a_{k-1} \beta_k^{(1)} + a_k \gamma_k^{(1)} + a_{k+1} \delta_k^{(1)} + a_{k+2} \mu_k^{(1)}, \]

where

\[ \beta_k^{(1)} = \frac{(k+3)}{(2k+5)(k+\frac{5}{2})}, \quad \beta_k^{(2)} = -\frac{(k+1)(k+3)}{(k+\frac{5}{2})}, \]

\[ \gamma_k^{(1)} = -\frac{(k+1)(k+3)}{(k+\frac{5}{2})}, \quad \delta_k^{(1)} = -\frac{(k+1)(k+3)}{(k+\frac{5}{2})}, \]

\[ \mu_k^{(1)} = -\frac{(k-1)}{2(k+2)(k+\frac{5}{2})}. \]

Finally, if

\[ \sum_{k=0}^{N-3} a_k J_{k+3}^{(-2)}(x) = \sum_{k=0}^{N-3} \varepsilon_k \mu_k^{(1)}, \]

then

\[ \varepsilon_k = a_{k-3} \omega_k^{(0)} + a_{k-2} \beta_k^{(0)} + a_{k-1} \gamma_k^{(0)} + a_k \delta_k^{(0)} + a_{k+1} \mu_k^{(0)} + a_{k+2} \eta_k^{(1)} + a_{k+3} \eta_k^{(0)}, \]

where

\[ \omega_k^{(0)} = \frac{(k+4)}{4(k+2)(k+\frac{5}{2})}, \quad \beta_k^{(0)} = -\frac{3(k+3)}{4(k+2)(k+\frac{5}{2})}, \]

\[ \gamma_k^{(0)} = -\frac{3(k+1)(k+3)}{4(k+2)(k+\frac{5}{2})}, \quad \delta_k^{(0)} = -\frac{3(k-1)(k+3)}{4(k+2)(k+\frac{5}{2})}, \]

\[ \mu_k^{(0)} = \frac{3(k+3)}{4(k+2)(k+\frac{5}{2})}, \quad \eta_k^{(0)} = \frac{3(k-1)}{4(k+2)(k+\frac{5}{2})}, \]

\[ \varepsilon_k = -\frac{(k-2)}{4(k+2)(k+\frac{5}{2})}. \]

Now, the application of Petrov–Galerkin method on Eq. (7), yields

\[ (D^2 u_N(x) - \beta_1 D^2 u_N - \beta_1 D u_N + \gamma_1 u_N, \psi_k(x)) = (f(x), \psi_k(x)), \]

where

\[ u_N(x) = \sum_{k=0}^{N-3} a_k \phi_k(x), \quad \phi_k(x) = J_{k+3}^{(-2)}(x), \]

\[ \psi_k(x) = J_{k+3}^{(-2)}(x), \quad k = 0, 1, \ldots, N - 3. \]
Substitution of formulae (11), (13), (15) and (17) into (19) yields
\[
\left( \sum_{j=0}^{N} b_j R_j^{(1,2)}(x) - \sum_{j=0}^{N-1} \beta_j e_j R_j^{(1,2)}(x) \right) + \gamma_1 \sum_{j=0}^{N} e_j b_j R_j^{(1,2)}(x, J_{k+1}^{(-1,-2)}(x)) = \left( f, J_{k+1}^{(-1,-2)}(x) \right),
\]
where \( b_k \) and \( e_k, 0 \leq k \leq 2 \) are given by (12), (14), (16) and (18), respectively.

Eq. (20) is equivalent to
\[
\left( \sum_{j=0}^{N} b_j R_j^{(1,2)}(x) - \sum_{j=0}^{N-1} \beta_j e_j R_j^{(1,2)}(x) \right) + \gamma_1 \sum_{j=0}^{N} e_j b_j R_j^{(1,2)}(x, J_{k}^{(2)}(x)) = \left( f, R_k^{(2)}(x) \right),
\]
where \( w = (1 - x^2)(1 + x) \). Making use of the orthogonality relation (1), it is not difficult to show that Eq. (20) is equivalent to
\[
f_k = (b_k - \beta_k e_k + \gamma_1 e_k, 0) b_{k+2}^2; \quad k = 0, 1, \ldots, N - 3,
\]
where
\[
f_k = \left( f, R_k^{(2)}(x) \right).
\]
This linear system may be put in the form
\[
(b_k^* - \beta_k e_k + \gamma_1 e_k, 0) = f_k^*, \quad k = 0, 1, \ldots, N - 3,
\]
where
\[
f_k^* = f_k / b_k^2, \quad b_k^2 = \frac{8}{(k + 1)(k + 2)(k + 3)},
\]
which may be written simply in the matrix form
\[
(B_1 + \alpha_1 E_2 + \beta_1 E_3 + \gamma_1 E_4) a = \mathbf{f},
\]
where
\[
a = (a_0, a_1, \ldots, a_{N-3})^T, \quad \mathbf{f} = (f_0, f_1, \ldots, f_{N-3})^T,
\]
and the nonzero elements of the matrices \( B_1, E_2, E_3, E_4 \) are given explicitly in the following theorem.

**Theorem 3.** The nonzero elements of the matrices \( B_k = (b_k^*) \) and \( E_k = (e_k^*) \), \( 0 \leq i \leq 2 \), for \( 0 \leq k, j \leq N - 3 \), are given as follows:
\[
b_{k+1}^2 = 2(k + 1)(k + 3), \quad e_{k+1}^2 = \frac{2(k+1)(k+2)}{2(k+3)}, \quad e_{k, k+1} = \frac{1}{4(k+1)(k+2)}(k+2)(k+3)\mathbf{2},
\]
\[
e_{k, k+1} = \frac{4(k+1)(k+2)}{2(k+3)(2k+5)}, \quad e_{k, k+1}^2 = \frac{4(k+1)(k+2)}{2(k+3)(2k+5)}.
\]
\[
e_{k, k+1}^2 = \frac{4(k+1)(k+2)}{2(k+3)(2k+5)}, \quad e_{k, k+1} = \frac{4(k+1)(k+2)}{2(k+3)(2k+5)}.
\]
\[
e_{k, k+2} = \frac{4(k+1)(k+2)}{2(k+3)(2k+5)}, \quad e_{k, k+2} = \frac{4(k+1)(k+2)}{2(k+3)(2k+5)}.
\]
\[
e_{k, k+1} = \frac{4(k+1)(k+2)}{2(k+3)(2k+5)}, \quad e_{k, k+1} = \frac{4(k+1)(k+2)}{2(k+3)(2k+5)}.
\]
\[
e_{k, k+1} = \frac{4(k+1)(k+2)}{2(k+3)(2k+5)}, \quad e_{k, k+1} = \frac{4(k+1)(k+2)}{2(k+3)(2k+5)}.
\]
\[
e_{k, k+1} = \frac{4(k+1)(k+2)}{2(k+3)(2k+5)}, \quad e_{k, k+1} = \frac{4(k+1)(k+2)}{2(k+3)(2k+5)}.
\]
\[
e_{k, k+1} = \frac{4(k+1)(k+2)}{2(k+3)(2k+5)}, \quad e_{k, k+1} = \frac{4(k+1)(k+2)}{2(k+3)(2k+5)}.
\]

**Dual Petrov–Galerkin algorithm for fifth-order elliptic differential equations**

In this section we aim to apply the GJPGM for solving the following fifth-order elliptic linear equation
\[
-u^{(5)}(x) + 2u^{(4)}(x) + 2u^{(3)}(x) - 2u^{(2)}(x) - \delta_2 u^{(1)}(x) + \mu_2 u(x) = f(x), \quad x \in (-1, 1),
\]
governed by the homogeneous boundary conditions
\[
u(\pm 1) = u^{(1)}(\pm 1) = u^{(2)}(1) = 0.
\]
We define the following two spaces
\[
V = \{ u \in H^5(I) : u(\pm 1) = u^{(1)}(\pm 1) = u^{(2)}(1) = 0 \},
\]
and
\[
V' = \{ u \in H^5(I) : u(\pm 1) = u^{(1)}(\pm 1) = u^{(2)}(-1) = 0 \},
\]
where
\[
H^5(I) = \{ u : \| u \|_{L^2} < \infty, \quad \| u \|_{L^2} = \left( \sum_{k=0}^{\infty} \| \partial^k u \|_{L^2}^2 \right)^{1/2} \}.
\]
Now, setting \( V_N = V \cap P_N \) and \( V' = V' \cap P_N \) we observe that:
\[
V_N = \text{span}\{ J_5^{(-3,-2)}(x), J_6^{(-3,-2)}(x), \ldots, J_{N-3}^{(-3,-2)}(x) \},
\]
\[
V'_N = \text{span}\{ J_5^{(-3,-2)}(x), J_6^{(-2,-3)}(x), \ldots, J_{N-3}^{(-2,-3)}(x) \}.
\]
The dual Petrov–Galerkin approximation of (24) and (25) is to find \( u_N \in V_N \) such that
\[
-(D^5 u_N(x), v(x)) + 2(D^4 u_N(x), v(x)) + 2(D^3 u_N(x), v(x)) - 2(D^2 u_N(x), v(x)) + \mu_2 (u_N(x), v(x)) = (f(x), v(x)), \quad \forall v \in V'_N.
\]

**The choice of basis functions**

We can choose suitable basis functions and their dual basis – in the same way as in the previous case and for the same reasons – by setting
\[
\phi_k(x) = J_k^{(-3,-2)}(x) - (1 - x^2)^2(1 - x^2)R_k^{(3,2)}(x), \quad k = 0, 1, \ldots, N - 5,
\]
\[
\psi_k(x) = J_k^{(-2,-3)}(x) - (1 - x^2)^2(1 + x)R_k^{(3,2)}(x), \quad k = 0, 1, \ldots, N - 5.
\]
It is worthy noting here that the basis \( \{ \phi_k(x) \} \) are orthogonal on \([-1, 1]\) in the sense that
\[
\int_{-1}^{1} \phi_{k}(x) \phi_{j}(x) \, dx = \begin{cases} 0, & k \neq j, \\ b_{k}^{2}, & k = j. \end{cases}
\]
It is clear that the two sets of orthogonal polynomials \( \{ \phi_k(x) \} \) and \( \{ \psi_k(x) \} \) are linearly independent, and therefore we have
\[
V_N = \text{span}\{ \phi_k(x) : k = 0, 1, 2, \ldots, N - 5 \},
\]
and
\[
V'_N = \text{span}\{ \psi_k(x) : k = 0, 1, 2, \ldots, N - 5 \}.
\]
The following two lemmas are needed.

Lemma 3

\[ D^3J_{k,5}^{3,-2}(x) = -3(k+1)(k+2)(k+4)(k+5)R_{k}^{(2,3)}(x). \]

**Proof.** Setting \( \alpha = 2, \beta = 1 \) in relation (5), we get

\[
(1-x^2)R_{k}^{(3,3)}(x) = \frac{12}{(2k+5)^3} [ (k+2)(2k+7)R_{k}^{(3,1)}(x) + 2(k+3)R_{k+1}^{(3,1)}(x) - (k+4)(2k+5)R_{k+1}^{(3,1)}(x) ]. \quad \square
\]

Making use of this relation and with the aid of the two relations (6) for \( q = 2 \) and (10), we obtain

\[
D^3J_{k,5}^{3,-2}(x) = \frac{1}{(2k+5)^3} [ (2k+7)(k-1)R_{k}^{(3,2)}(x) + 2(k)R_{k+1}^{(4,1)}(x) - (2k+5)(k+1)R_{k}^{(3,4)}(x) ].
\]

Finally, from the two relations (2) and (3), and after some lengthy manipulation, we get

\[
D^3J_{k,5}^{3,-2}(x) = -3(k+1)(k+2)(k+4)(k+5)R_{k}^{(2,3)}(x).
\]

Lemma 4

\[
D^4J_{k,5}^{3,-2}(x) = \frac{3(k+2)(k+4)}{2(k+7)}R_{k+1}^{(2,3)}(x) + \frac{3(k+1)(k+4)}{2(k+7)^2}R_{k+1}^{(2,3)}(x) + \frac{3}{2(k+7)^3}R_{k+1}^{(2,3)}(x), \quad (27)
\]

\[
D^4J_{k,5}^{3,-2}(x) = \frac{3(k+4)}{2(k+7)^2}R_{k+1}^{(2,3)}(x) + \frac{3(k+2)(k+4)}{2(k+7)^3}R_{k+1}^{(2,3)}(x) + \frac{3(k+1)(k+4)}{2(k+7)^4}R_{k+1}^{(2,3)}(x) + \frac{3(k-1)}{4(k+7)^4}R_{k-1}^{(2,3)}(x), \quad (28)
\]

\[
D^4J_{k,5}^{3,-2}(x) = \frac{3(k+4)}{8(k+3)(k+9)}R_{k+1}^{(2,3)}(x) + \frac{9(k+4)}{8(k+3)^2(k+9)}R_{k+1}^{(2,3)}(x) + \frac{9(k+1)(k+4)}{8(k+3)^3(k+9)}R_{k+1}^{(2,3)}(x) + \frac{9(k+1)(k+4)}{4(k+3)^4}R_{k+1}^{(2,3)}(x) + \frac{9}{8(k+3)^5}R_{k+1}^{(2,3)}(x) + \frac{9}{8(k+3)^6}R_{k+1}^{(2,3)}(x) + \frac{3(k-2)}{8(k+3)(k+3)(k+7)}R_{k-1}^{(2,3)}(x), \quad (29)
\]

\[
D^4J_{k,5}^{3,-2}(x) = \frac{3(k+5)}{16(k+3)(k+9)}R_{k+1}^{(2,3)}(x) + \frac{3(k+4)}{4(k+3)(k+9)}R_{k+1}^{(2,3)}(x) + \frac{3(k+4)}{4(k+3)^2(k+9)}R_{k+1}^{(2,3)}(x) + \frac{9(k+2)(k+4)}{4(k+3)^3(k+9)}R_{k+1}^{(2,3)}(x) + \frac{9(k+2)(k+4)}{8(k+3)^4}R_{k+1}^{(2,3)}(x) + \frac{9(k+2)}{8(k+3)^5}R_{k+1}^{(2,3)}(x) + \frac{9(k+2)}{8(k+3)^6}R_{k+1}^{(2,3)}(x) - \frac{3(k-2)}{4(k+3)(k+3)(k+7)}R_{k-1}^{(2,3)}(x) - \frac{3(k-2)}{16(k+3)(k+3)}R_{k-1}^{(2,3)}(x), \quad (30)
\]

and

\[
J_{k,5}^{3,-2}(x) = -\frac{3(k+6)(k+8)}{32(k+3)(k+3)(k+7)}R_{k+1}^{(2,3)}(x) + \frac{15(k+5)(k+7)}{32(k+3)(k+3)(k+7)}R_{k+1}^{(2,3)}(x) - \frac{15(k+4)(k+6)}{32(k+3)(k+3)(k+7)}R_{k+1}^{(2,3)}(x) + \frac{15(k+2)(k+6)}{16(k+3)^2}R_{k+1}^{(2,3)}(x) + \frac{45(k+1)(k+5)}{16(k+3)^2}R_{k+1}^{(2,3)}(x) + \frac{15(k+1)(k+5)}{16(k+3)^2}R_{k+1}^{(2,3)}(x) - \frac{15(k-1)(k+6)}{8(k+3)^2}R_{k-1}^{(2,3)}(x) - \frac{15(k-2)(k+6)}{32(k+3)(k+3)(k+7)}R_{k-1}^{(2,3)}(x) + \frac{15(k-3)(k+6)}{32(k+3)(k+3)(k+7)}R_{k-1}^{(2,3)}(x).
\]

(31)

Applying Petrov–Galerkin method to (24) and (25) and if we make use of the two Lemmas 3 and 4, and after performing some lengthy manipulation, then the numerical solution of (24) and (25) can be obtained. This solution is given in the following theorem.

**Theorem 4.** If \( u_N(x) = \sum_{k=0}^{N-5} a_k J_{k,5}^{3,-2}(x) \) is the Petrov–Galerkin approximation to (24) and (25), then the expansion coefficients \( \{ a_k : k = 0, 1, \ldots, N-5 \} \) satisfy the matrix system

\[
(B_2 + x_2G_2 + \beta_2G_3 + \gamma_2G_2 + \delta_2G_1 + \mu_2G_0)u = \Gamma,
\]

where the nonzero elements of the matrices \( B_2 = \{ b_{i,j} \} \) and \( G_i = \{ g_{i,j} \}; (0 \leq i \leq 4) \), for \( 0 \leq k, j \leq N-5 \), are given as follows:

\[
b_{k,k} = r_k, \quad g_{k,k} = \frac{r_k}{2(k+\frac{1}{2})^2}, \quad g_{k,k+1} = \frac{3(k+1)(k+5)}{2(k+7)},
\]

\[
g_{k,k+1} = \frac{-3+2}{(k+3)(2k+7)}, \quad g_{k,k+1} = \frac{-3+1}{2(k+\frac{1}{2})^2}, \quad g_{k,k+1} = \frac{-3+1}{2(k+\frac{1}{2})^2},
\]

\[
g_{k,k+1} = \frac{3+1}{4(k+\frac{1}{2})^2}, \quad g_{k,k+1} = \frac{3+1}{2(k+\frac{1}{2})^2}, \quad g_{k,k+1} = \frac{-3+1}{4(k+\frac{1}{2})^2},
\]

\[
g_{k,k+1} = \frac{3+1}{4(k+\frac{1}{2})^2}, \quad g_{k,k+1} = \frac{3+1}{8(k+\frac{1}{2})^3}, \quad g_{k,k+1} = \frac{-3+1}{8(k+\frac{1}{2})^3},
\]

\[
g_{k,k+1} = \frac{3+1}{8(k+\frac{1}{2})^3}, \quad g_{k,k+1} = \frac{-3+1}{8(k+\frac{1}{2})^3}, \quad g_{k,k+1} = \frac{3+1}{8(k+\frac{1}{2})^3},
\]

\[
g_{k,k+1} = \frac{3+1}{8(k+\frac{1}{2})^3}, \quad g_{k,k+1} = \frac{-3+1}{8(k+\frac{1}{2})^3}, \quad g_{k,k+1} = \frac{3+1}{8(k+\frac{1}{2})^3},
\]

\[
g_{k,k+1} = \frac{3+1}{8(k+\frac{1}{2})^3}, \quad g_{k,k+1} = \frac{-3+1}{8(k+\frac{1}{2})^3}, \quad g_{k,k+1} = \frac{3+1}{8(k+\frac{1}{2})^3}.
\]

\[
g_{k,k+1} = \frac{3+1}{8(k+\frac{1}{2})^3}, \quad g_{k,k+1} = \frac{-3+1}{8(k+\frac{1}{2})^3}, \quad g_{k,k+1} = \frac{3+1}{8(k+\frac{1}{2})^3}.
\]
where \( f_\ell = \int_{-1}^{1} (1-x^2)(1+x) f(x) R^{(1,2)}_\ell (x) dx \).

**Corollary 2.** If \( u_0(x) = \sum_{k=0}^{N-3} a_k f^{(2-k)}(x) \) is the Petrov–Galerkin approximation to problem (24) and (25), for \( x_2 = \beta_2 = \gamma_2 = \delta_2 = \mu_2 = 0 \), then the expansion coefficients \( \{a_k : k = 0, 1, \ldots, N - 5\} \) are given explicitly by

\[
a_k = \frac{k + 3}{384} f_\ell, \quad k = 0, 1, \ldots, N - 5,
\]

where

\[
f_\ell = \int_{-1}^{1} (1-x^2)(1+x) f(x) R^{(2,3)}_\ell (x) dx.
\]

Now, each of the matrices \( E_{3-q} \) \((1 \leq q \leq 3)\) and \( G_{5-q} \) \((1 \leq q \leq 5)\) is a band matrix and the total number of non-zero diagonals upper or lower the main diagonal for each matrix is \( q \). Thus the coefficient matrices \( D_1 \) and \( D_2 \) are at most four-band and six-band matrices, respectively. These special structures of \( D_1 \) and \( D_2 \) simplify greatly the solution of the two linear systems (23) and (32). These two systems can be decomposed by LU-factorization. Moreover, the operations required for constructing these factorizations are of order \( 21(N-2) \) and \( 55(N-4) \), respectively. Also, the number of operations required for solving the two decomposable triangular systems are of order \( 13(N-2) \) and \( 21(N-4) \) respectively.

**Note 2.** The total number of operations mentioned in the previous discussion includes the number of all subtractions, additions, divisions and multiplications [35].

**Treatment of nonhomogeneous boundary conditions**

This section is devoted to describe the way of how third- and fifth-order BVPs governed by nonhomogeneous boundary conditions can be converted to BVPs governed by homogeneous boundary conditions.

Now, let us consider the one-dimensional third-order equation

\[
u^{(3)}(x) = \sigma_1 u^{(2)}(x) - \beta_1 u^{(1)}(x) + \gamma_1 u(x) = f(x), \quad x \in I = (-1, 1),
\]

governed by the nonhomogeneous boundary conditions:

\[
u^{(1)}(-1) = a_0, \quad u^{(1)}(1) = a^1. \tag{33}
\]

Now, and if we make use of the transformation

\[
V(x) = u(x) + a_0 + a_1 x + a_2 x^2,
\]

where

\[
a_0 = -a_- + 3a_+ + 2a^1, \quad a_1 = a_- - a_+, \quad a_2 = -a_- + a_+ - 2a^1,
\]

then, the transformation (34) turns the nonhomogeneous boundary conditions (33) into the homogeneous boundary conditions:

\[
V^{(1)}(\pm 1) = V^{(1)}(1) = 0. \tag{35}
\]
Hence, it is sufficient to solve the following modified one-dimensional third-order equation:

\[ V^{(3)}(x) - x_0 V^{(2)}(x) - \beta V^{(1)}(x) + \gamma V(x) = f'(x), \]
\[ x \in I = (-1,1), \]  
(36)
governed by the homogeneous boundary conditions (35), where \( V(x) \) is given by (34), and

\[ f'(x) = f(x) + (-2x_0a_2 - \beta_1a_1 + \gamma_1a_0) + (-2\beta_1a_2 + \gamma_1a_0)x + \gamma_1a_2x^2. \]

Now, the application of the GJPGM to the modified Eq. (36), leads to the following equivalent system of equations

\[(B_1 + x_1E_2 + \beta_1E_1 + \gamma_1E_0)a = \Gamma, \]

where \( B_1, E_2, E_1 \) and \( E_0 \) are the matrices defined in Theorem 3, and \( \Gamma = \{f_0, f_1', \ldots, f_{N-3}'\} \), where

\[ f'_k = \begin{cases} 
-2x_0a_2 - \beta_1a_1 + \gamma_1a_0, & k = 0, \\
\frac{1}{2}(-2\beta_1a_2 + \gamma_1a_0), & k = 1, \\
\frac{1}{2}a_1, & k = 2, \\
f_k, & k \geq 3,
\end{cases} \]

and \( f_k = \int_{-1}^{1} (1 + x^2)(1 + x)R_{2k}^{(3)}(x)f(x)dx \).

The same procedure can be applied to solve the following fifth-order BVP:

\[ -u^{(5)}(x) + x_2u^{(4)}(x) + \beta_2u^{(3)}(x) - \gamma_2u^{(2)}(x) - \delta_2u^{(1)}(x) + \mu_2u(x) = f(x), \quad x \in (-1,1), \]  
(37)
governed by the nonhomogeneous boundary conditions

\[ u(\pm1) = a_b, \quad u^{(1)}(\pm1) = a'_b, \quad u^{(2)}(1) = b. \]  
(38)

In such case, (37) and (38) can be turned into

\[ -V^{(5)}(x) + x_2V^{(4)}(x) + \beta_2V^{(3)}(x) - \gamma_2V^{(2)}(x) - \delta_2V^{(1)}(x) + \mu_2V(x) = f'(x), \quad x \in I = (-1,1), \]  
(39)
governed by the homogenous boundary conditions

\[ V(\pm1) = V^{(1)}(\pm1) = V^{(2)}(1) = 0, \]

where

\[ V(x) = u(x) + a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4, \]

with

\[ a_0 = \frac{1}{16}(-2a_4 + 8a_4 + 2b - 5a_1 - 11a_2), \]
\[ a_1 = \frac{1}{4}(a_1 - a_1 + 3a_1 - 3a_2), \]
\[ a_2 = \frac{1}{8}(-6a_1 + 2b - 3a_1 + 3a_2), \]
\[ a_3 = \frac{1}{4}(-a_1 - a_1 - a_1 - a_1), \]
\[ a_4 = \frac{1}{16}(4a_4 + 3a_4 - 3a_4), \]

and

\[ f'(x) = (\mu_2a_0 - \delta_2a_1 - 2\gamma_2a_2 + 6b_2a_3 + 24a_2a_4) + (\mu_2a_1 - 2\delta_2a_2 - 6\gamma_2a_3 + 24b_2a_4a_2x + (\mu_2a_2 - 3\delta_2a_3 - 12\gamma_2a_4)x^2 + (\mu_2a_3 - 4\delta_2a_4)x^3 + \mu_2a_4x^4 + f(x). \]

If the GJPGM is applied to Eq. (39), then the following equivalent system of equations is obtained

\[(B_2 + \gamma G_4 + \beta G_3 + \gamma G_1 + \mu G_0)a = \Gamma, \]

where \( B_2, G_i \) (\( 0 \leq i \leq 4 \)) are the matrices defined in Theorem 4, and

\[ \Gamma = \{f_0, f_1', \ldots, f_{N-3}'\}, \]

and \( f_k = \int_{-1}^{1} (1 + x^2)(1 + x)R_{2k}^{(3)}(x)f(x)dx \).

**Condition numbers of the resulting matrices**

In the direct collocation method, the condition numbers behave like \( O(N^6) \) and \( O(N^6) \) for third- and fifth-order BVPs, respectively, \( N \): maximal degree of polynomials. In this article, improved condition numbers with \( O(N^4) \) and \( O(N^6) \) are obtained, respectively, for third- and fifth-order BVPs. The advantage with respect to propagation of rounding errors is demonstrated.

For GJPGM, the resulting systems obtained for the two differential equations \( u^{(5)}(x) = f(x) \) and \( -u^{(5)}(x) = f(x) \) are \( B_1a = \Gamma \) and \( B_2a = \Gamma \), where \( B_1 \) and \( B_2 \) are two diagonal matrices their elements are given by \( b_{1k}^1 \) and \( b_{1k}^2 \), where

\[ b_{1k}^1 = 2(k + 1)(k + 3), \quad b_{1k}^2 = 3(k + 1)(k + 2)(k + 4)(k + 5). \]

Thus we note that the condition numbers of the matrices \( B_1 \) and \( B_2 \) behave like \( O(k^3) \) and \( O(k^5) \) for large values of \( k \), respectively. The evaluation of the condition numbers for the matrices \( B_1 \) and \( B_2 \) is easy because of their special structures, since \( B_1 \) and \( B_2 \) are diagonal matrices, so their eigenvalues are their diagonal elements. In such case, the condition number can be defined as:

\[ \text{Condition number of the matrix} = \frac{\text{Max (eigenvalue of the matrix)}}{\text{Min (eigenvalue of the matrix)}}. \]

In Table 1, we list the values of the conditions numbers of the matrices \( B_1 \) and \( B_2 \), respectively, for different values of \( N \).

**Remark 1.** If we add \( \sum_{q=1}^{3} E_{3-q}(1 \leq q \leq 3) \) and \( \sum_{q=1}^{5} G_{5-q} \) \( (1 \leq q \leq 5) \), where the matrices \( E_{3-q} \) and \( G_{5-q} \) are the matrices their nonzero elements are given explicitly in Theorems 3 and 4, to the matrices \( B_1 \) and \( B_2 \), respectively, then we find that the eigenvalues of the matrices \( D_1 = B_1 + \sum_{q=1}^{3} E_{3-q} \), \( D_2 = B_2 + \sum_{q=1}^{5} G_{5-q} \) are all real and positive. Moreover, the effect of these additions does not significantly change the values of the condition numbers for the systems. This means that matrices
Theorem 5. Bernstein type Inequality [36]. The well-known Legendre polynomials \( L_k(x), k = 0, 1, 2, \ldots \), satisfy the following inequality:

\[
\sqrt{\sin \theta} L_k(\cos \theta) < \sqrt{\frac{2}{nk}}, \quad 0 < \theta < \pi.
\]

Theorem 6. A function \( u(x) = (1 - x)^2(1 + x) f(x) \in H^2((-1, 1)), \) with \( |f(x)| \leq L, \) can be expanded as an infinite sum of nonsymmetric generalized Jacobi polynomials \( \left\{ J_{k+1}^{(2-1)}(x) : k = 0, 1, 2, \ldots \right\}, \) and the series converges uniformly to \( u(x) \). Explicitly, the expansion coefficients in \( u(x) = \sum_{k=0}^{\infty} a_k J_{k+1}^{(2-1)}(x) \), satisfy the following inequality:

\[
|a_k| < \frac{9L}{nk^2}, \quad \forall k > 0.
\]

Proof. Since \( \left\{ J_{k+1}^{(2-1)}(x) : k = 0, 1, 2, \ldots \right\} \) are orthogonal basis of \( H^2((-1, 1)), \) then

\[
a_k = \frac{1}{h_k^{2.1}} \int_{-1}^{1} J_{k+1}^{(2-1)}(x) u(x) \, dx,
\]

where \( h_k^{2.1} \) is as defined in (1). With the aid of the relation

\[
J_{k+1}^{(2-1)}(x) = \frac{4}{(k-1)(k-3)} \left[ L_{k-3}(x) - \frac{2k-3}{2k-1} L_{k-2}(x) - \frac{2k-3}{2k-1} L_{k-1}(x) \right],
\]

and after integration by parts two times, we get,

\[
a_k = \frac{(k+1)(k+3)}{2(2k+3)} \int_{-1}^{1} I_k(x) f^{(2)}(x) \, dx,
\]

where

\[
I_k(x) = \frac{L_{k-2}(x)}{4(k-\frac{1}{2})(2k+1)(2k+3)} - \frac{L_{k-1}(x)}{(2k+1)(2k+3)} - \frac{3L_k(x)}{(2k-1)(2k+3)} + \frac{3(2k+1)L_{k+1}(x)}{4(2k+1)(k+\frac{1}{2})} + \frac{3L_{k+2}(x)}{(2k+1)(2k+7)} - \frac{3L_{k+3}(x)}{2(2k+5)(2k+7)} + \frac{(2k+3)L_{k+4}(x)}{8(k+\frac{3}{2})^2}.
\]

Now, making use of the substitution \( x = \cos \theta, \) yields

\[
a_k = \frac{(k+1)(k+3)\sum_{m=0}^{2m} b_{m,k} \left( \int_{0}^{\pi} L_{k+1-m-2}(\cos \theta) f^{(2)}(\cos \theta) \sin \theta \, d\theta \right)}{2(2k+3)}.
\]

Therefore, we have

\[
|a_k| \leq \frac{(k+1)(k+3)L_{2m} \sum_{m=0}^{2m} b_{m,k} \left( \int_{0}^{\pi} \left| L_{k+1-m-2}(\cos \theta) \right| \sin \theta \, d\theta \right)}{2(2k+3)}.
\]
With the aid of Theorem 5, we have,

$$|a_k| < \frac{(k+1)(k+3) L}{\sqrt{2}(k+2)} \sum_{m=0}^{7} \frac{|b_m|}{\sqrt{k+m-2}} \left( \int_{0}^{\pi} \sqrt{\sin \theta} \, d\theta \right)$$

$$< \frac{4(k+1)(k+3) L}{\sqrt{k+2}} \sum_{m=0}^{7} \frac{|b_m|}{m}$$

$$< \frac{4(k+1)(k+3) L}{\sqrt{k+2}} \frac{(k+2)(4k^2 + 16k + 3)}{(k-1)^2(k+2)^3}.$$ 

Now it can be easily shown that

$$(k+1)^2(4k^2 + 16k + 3) < 4(k+2)^5$$

and $$(k - \frac{1}{2}) > (k+1)^b,$$

and accordingly,

$$|a_k| < \frac{8L(k+2)^b}{\pi \sqrt{k-2}(k+1)^b} < \frac{9L}{\pi \sqrt{k-2}(k-2)} = \frac{9L}{\pi(k-2)^{\frac{1}{3}}} \sim \frac{9L}{mk^2}.$$ 

This completes the proof. $\square$

**Theorem 7.** A function $u(x) = (1-x)^3(1+x)^2f(x) \in H^2_{\text{sym}} (-1, 1),$ with $|f(x)| < L,$ can be expanded as an infinite sum of nonsymmetric generalized Jacobi polynomials $\{J_{k+3-2}(x) : k = 0, 1, 2, \ldots\},$ and the series converges uniformly to $u(x).$ Explicitly, the expansion coefficients in $u(x) = \sum_{k=0}^{\infty} a_k J_{k+3-2}(x),$ satisfy the following inequality:

$$|a_k| \leq \frac{L \pi}{k^3}, \quad \forall k \geq 0.$$

**Proof.** The proof is similar to the proof of Theorem 6. $\square$

**Numerical results**

**Example 1.** Consider the one dimensional third-order differential equation

$$\begin{align*}
\mu \frac{d^3 u}{dx^3}(x) + \sigma \frac{d^2 u}{dx^2}(x) - \beta_1 \frac{d u}{dx}(x) + \gamma_1 u(x) &= f(x), \\
\mu (u(\pm 1)) &= \beta_1 (u(\pm 1)),
\end{align*}$$

where $f(x)$ is chosen such that the exact solution for (40) is $u(x) = (1-x)^3(1+x)^2 \sin(m \pi x), j, m \in N.$ We have $u_N(x) = \sum_{i=0}^{N-3} a_i (1-x^2)(1-x) R_i^{(1)}(x)$ and the vector of unknowns $a = (a_0, a_1, \ldots, a_{N-3})^T$ is the solution of the system $$(B_1 + \sigma E_2) a = \Gamma,$$ where the nonzero elements of the matrices $B_1$ and $E_2$ are given explicitly in Theorem 3. In Table 3, the maximum pointwise error $E$ for $u - u_N$ to Eq. (40) is listed, using GJPGM for various values of $j, m$ and the coefficients $\sigma, \beta_1, \gamma_1.$

**Example 2.** Consider the one dimensional fifth-order differential equation

$$\begin{align*}
\mu \frac{d^5 u}{dx^5}(x) + \sigma \frac{d^4 u}{dx^4}(x) + \beta_3 \frac{d^3 u}{dx^3}(x) - \gamma_1 \frac{d^2 u}{dx^2}(x) - \delta_2 \frac{d u}{dx}(x) + \gamma_2 u(x) &= f(x), \\
\mu (u(\pm 1)) &= \beta_1 (u(\pm 1)) = \beta_2 (u(\pm 1)),
\end{align*}$$

where $f(x)$ is chosen such that the exact solution for (41) is given by $u(x) = (1-x^2)^2(1-x) R_i^{(12)}(x),$ and the vector of unknowns $a = (a_0, a_1, \ldots, a_{N-5})^T$ is the solution of the system

$$(B_2 + \sigma_2 G_3 + \beta_3 G_2 + \gamma_1 G_1 + \delta_2 G_3 + \mu G_0) a = \Gamma,$$ 

where the nonzero elements of the matrices $B_2$ and $G_i$ $(0 \leq i \leq 4)$ are given explicitly in Theorem 4. Table 4 lists the maximum pointwise error $E$ for $u - u_N$ to (41), using GJPGM for various values of $m$ and the coefficients $\beta_3, \gamma_1, \gamma_2,$ $\beta_2$ and $\mu_2.$

**Example 3.** Consider the one dimensional third-order nonhomogeneous equation

$$\begin{align*}
\mu \frac{d^3 u}{dx^3}(x) + \beta_1 \frac{d^2 u}{dx^2}(x) + \gamma_1 u(x) &= f(x), \\
\mu (u(\pm 1)) &= \beta_1 (u(\pm 1)),
\end{align*}$$

where $f(x)$ is chosen such that the exact solution for (42) is $u(x) = \sin(m \pi x).$ Setting $V(x) = u(x) - \sin(m \pi x) + \frac{1}{4} \mu \cos(m \pi x) \sin(m \pi x)(1 - x),$ then the differential Eq. (42) is equivalent to the differential equation

$$\begin{align*}
\mu \frac{d^3 u}{dx^3}(x) - \beta_1 \frac{d^2 u}{dx^2}(x) + \gamma_1 u(x) &= f(x), \\
\mu (u(\pm 1)) &= \beta_1 (u(\pm 1)) = 1.
\end{align*}$$

Table 5 lists the maximum pointwise error $E$ for $u - u_N$ to (42), using GJPGM for various values of $m$ and the coefficients $\beta_1, \gamma_1, \beta_2$ and $\gamma_2.$

In the following, and for the sake of comparison, we give two other numerical examples to show the effectiveness of GJPGM.

- First, consider the following third-order boundary value problem.

**Example 4** ($f(x) = 37, 38 J$).

$$\begin{align*}
\mu \frac{d^3 u}{dx^3}(x) + \beta_1 \frac{d^2 u}{dx^2}(x) + \gamma_1 u(x) &= f(x), \\
\mu (u(\pm 1)) &= \beta_1 (u(\pm 1)) = 1.
\end{align*}$$

The analytic solution to (43) is $u(x) = (t^2 - 1) \sin t.$ The transformation $t = \frac{1}{\sqrt{\mu}}$ turns Eq. (43) into

$$\begin{align*}
\mu \frac{d^3 u}{dx^3}(x) + \beta_1 \frac{d^2 u}{dx^2}(x) + \gamma_1 u(x) &= f(x), \\
\mu (u(0)) &= \beta_1 (u(0)) = 1.
\end{align*}$$

The analytic solution to (44) is $u(x) = \frac{1}{2} (x^2 + 2x - 3) \sin(\frac{1}{\sqrt{\mu}} x)$.

We set $u_N(x) = \sum_{k=0}^{N-3} a_k (1-x)^2(1-x)^2 R_i^{(12)}(x) + \frac{1}{2} (1 + x)$

In Table 6, the maximum pointwise error $E$ for $u - u_N$ to Eq. (44) is listed, using GJPGM. The maximum absolute errors by our algorithm and by the second- and fourth-order quintic nonpolynomial spline [37], and by the quartic nonpolynomial spline method [38], for Example 4 are presented in Table 7.
Second, consider the fifth-order boundary value problem.

**Example 5 (43–46).**

\[
\begin{align*}
\begin{cases}
\dot{y}(t) - y(t) &= -(15 + 10t)e^t, \quad t \in [0, 1], \\
y(0) &= 0, \quad \dot{y}(0) = 1, \quad y'(0) = 0, \\
y(1) &= 0, \quad \dot{y}(1) = -e. 
\end{cases}
\end{align*}
\]

The analytic solution of (45) is \(y(t) = t(1-t)e^t\). The transformation \(t = \frac{11}{12} \) turns Eq. (45) into

**Table 3** Maximum pointwise error for \(u - u_N\), and \(N = 8, 12, 16, 20, 24\).

| \(N\) | \(j\) | \(m\) | \(x_1\) | \(\beta_1\) | \(\gamma_1\) | \(E\) |
|---|---|---|---|---|---|---|
| 8  | 1  | 1  | 0  | 0  | 0  | 2.558 \times 10^{-3} |
| 12 | 12 | 12 | 12 | 12 | 12 | 1.909 \times 10^{-6} |
| 16 | 16 | 16 | 16 | 16 | 16 | 4.368 \times 10^{-10} |
| 20 | 20 | 20 | 20 | 20 | 20 | 2.811 \times 10^{-14} |
| 24 | 24 | 24 | 24 | 24 | 24 | 3.885 \times 10^{-16} |

**Table 4** Maximum pointwise error for \(u - u_N\), and \(N = 8, 12, 16, 20, 24\).

| \(N\) | \(m\) | \(x_2\) | \(\beta_2\) | \(\gamma_2\) | \(\delta_2\) | \(E\) |
|---|---|---|---|---|---|---|
| 8  | 3  | 0  | 0  | 0  | 0  | 1.135 \times 10^{-1} |
| 12 | 12 | 12 | 12 | 12 | 12 | 2.464 \times 10^{-4} |
| 16 | 16 | 16 | 16 | 16 | 16 | 8.165 \times 10^{-8} |
| 20 | 20 | 20 | 20 | 20 | 20 | 1.098 \times 10^{-11} |
| 24 | 24 | 24 | 24 | 24 | 24 | 5.551 \times 10^{-16} |

**Table 5** Maximum pointwise error for \(u - u_N\), and \(N = 8, 12, 16\).

| \(N\) | \(m\) | \(x_1\) | \(\beta_1\) | \(\gamma_1\) | \(E\) |
|---|---|---|---|---|---|
| 8  | 1  | 0  | 0  | 0  | 2.804 \times 10^{-9} |
| 12 | 12 | 12 | 12 | 12 | 1.110 \times 10^{-16} |
| 16 | 16 | 16 | 16 | 16 | 2.819 \times 10^{-8} |

**Table 6** The maximum absolute error for Example 4.

| \(N\) | \(5\) | 8 | 11 | 14 | \(E\) |
|---|---|---|---|---|---|
| \(E\) & \(1.077 \times 10^{-3}\) & \(1.131 \times 10^{-7}\) & \(3.567 \times 10^{-12}\) & \(2.579 \times 10^{-16}\) |

**Table 7** The best errors for Example 4.

| Method | Error |
|---|---|
| Present method | \(2.58 \times 10^{-16}\) |
| Quintic nonpolynomial spline method Lang [37] | \(2.39 \times 10^{-12}\) |
| Quartic nonpolynomial spline method [38] | \(2.196 \times 10^{-9}\) |
In this paper, some new algorithms for obtaining numerical spectral solutions for third- and fifth-order BVPs based on employing certain nonsymmetric generalized Jacobi–Galerkin method are presented and implemented. The algorithms are very efficient and applicable. The main advantage of our algorithms is that the linear systems resulted from the application of them are band and this of course reduces drastically the computational cost and effort. Moreover, it is found that, for some particular third- and fifth-order BVPs, diagonal systems are obtained. An advantage of the presented algorithms is that high accurate approximate solutions are achieved using a few number of terms of expansion of the nonsymmetric generalized Jacobi polynomials. The obtained numerical results are comparing favorably with the analytical ones.

### Conclusions

In this paper, some new algorithms for obtaining numerical spectral solutions for third- and fifth-order BVPs based on employing certain nonsymmetric generalized Jacobi–Galerkin method are presented and implemented. The algorithms are very efficient and applicable. The main advantage of our algorithms is that the linear systems resulted from the application of them are band and this of course reduces drastically the computational cost and effort. Moreover, it is found that, for some particular third- and fifth-order BVPs, diagonal systems are obtained. An advantage of the presented algorithms is that high accurate approximate solutions are achieved using a few number of terms of expansion of the nonsymmetric generalized Jacobi polynomials. The obtained numerical results are comparing favorably with the analytical ones.

| Method               | Error    |
|----------------------|----------|
| Present method       | 6.35 × 10⁻¹⁷ |
| Cubic B-spline method| 1.14 × 10⁻² |
| Sixth-degree B-spline method| 2.08 × 10⁻⁵ |
| Sextic spline method | 2.01 × 10⁻⁵ |
| Finite difference method| 1.15 × 10⁻² |
| Sextic spline method | 4.84 × 10⁻⁷ |
| Nonpolynomial sextic spline method| 1.61 × 10⁻¹³ |
| Sextic spline method | 5.28 × 10⁻⁷ |
| Quartic spline method| 7.66 × 10⁻⁵ |

### Conflict of interest

The authors have declared no conflict of interest.

### Compliance with Ethics Requirements

This article does not contain any studies with human or animal subjects.

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