BENDERS ADAPTIVE-CUTS METHOD FOR TWO-STAGE
STOCHASTIC PROGRAMS

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ABSTRACT. Benders decomposition is one of the most applied methods to solve
two-stage stochastic problems (TSSP) with a large number of scenarios. The
main idea behind the Benders decomposition is to solve a large problem by
replacing the values of the second-stage subproblems with individual variables,
and progressively forcing those variables to reach the optimal value of the sub-
problems, dynamically inserting additional valid constraints, known as Benders
cuts. Most traditional implementations add a cut for each scenario (multi-cut)
or a single-cut that includes all scenarios. In this paper we present a novel Ben-
ders adaptive-cuts method, where the Benders cuts are aggregated according
to a partition of the scenarios, which is dynamically refined using the LP-dual
information of the subproblems. This scenario aggregation/disaggregation is
based on the Generalized Adaptive Partitioning Method (GAPM), which has
been successfully applied to TSSPs. We formalize this hybridization of Ben-
ders decomposition and the GAPM, by providing sufficient conditions under
which an optimal solution of the deterministic equivalent can be obtained in
a finite number of iterations. Our new method can be interpreted as a com-
promise between the Benders single-cuts and multi-cuts methods, drawing on
the advantages of both sides, by rendering the initial iterations faster (as for
the single-cuts Benders) and ensuring the overall faster convergence (as for
the multi-cuts Benders). Computational experiments on three TSSPs (the
Stochastic Electricity Planning, Stochastic Multi-Commodity Flow and CVaR
Facility Location) validate these statements, showing that the new method
outperforms the other implementations of Benders method, as well as other
standard methods for solving TSSPs, in particular when the number of sce-
narios is very large. Moreover, our study demonstrates that the method is not
only effective for the risk-neutral decision makers, but also that it can be used
in combination with the risk-averse CVaR objective.

1. INTRODUCTION

In two-stage linear stochastic programming (TSSP), a class of problems studied
in this article, decisions are split between those made before and after the uncer-
tainty of some modeling parameters is revealed [Birge and Louveaux, 2011]. TSSPs
are used for modeling many optimization problems that naturally appear in trans-
portation, logistics, or telecommunications. Typical examples include stochastic
multi-commodity flow problems, where a set of given commodities has to be routed
between different pairs of origin and destination nodes at minimum cost [Gendron
et al., 1999; Barnhart et al., 2001]. In a more realistic setting, the demand of these

Date: November 22, 2022.
Key words and phrases. Two-stage Stochastic Programming, Benders Decomposition,
Adaptive-Partition Method, Electricity Planning, Stochastic Multi-Commodity Flow, Conditional
Value-at-Risk, Facility Location.
Supported by ANID through grants Fondecyt 1200809 and STIC-AmSud STIC19007.
commodities is uncertain, however decisions concerning the network structure including the capacity of its nodes/arcs have to be made here-and-now (i.e., in the first-stage, before the actual demand is realized), whereas in the second-stage, once the demand is revealed, the routing of each commodity can be calculated in a wait-and-see fashion (Sarayloo et al. 2021, Rahmaniani et al. 2018).

TSSPs are particularly challenging when the support of the uncertainty space is continuous and high-dimensional since there is usually no closed formula to express the second-stage optimization model. As an alternative, the sample average approximation (SAA) method (Mak et al. 1999, Kleywegt et al. 2002) has been widely used to generate a deterministic equivalent formulation by sampling scenarios to obtain an approximately optimal solution using a (potentially large) number of discrete scenarios. This large number of scenarios, which is needed for obtaining a reliable representation of data uncertainty, is a major challenge for solving practical applications of TSSPs using the SAA technique.

Decomposition methods have shown strong capabilities in the design and implementation of efficient solution algorithms for TSSPs with a large number of scenarios. A large portion of work on these decomposition methods relies on Benders decomposition (Benders 1962), which was further extended to stochastic programming by [Van Slyke and Wets 1969] and [Birge and Louveaux 1988] under the name of the L-Shaped method. This strategy consists of executing three steps, namely, projecting, relaxing and linearizing some components of the TSSP, so that the original problem can be solved via an iterative procedure that utilizes the dualization of the projected terms. Dual extreme rays and extreme points of the second-stage subproblems are then used to generate valid inequalities, commonly known as feasibility and optimality cuts, respectively. When included into the so-called master problem, these constraints result in an alternative valid formulation of the original optimization problem. We highlight the importance of an iterative method to identify only some of the extreme rays and extreme points since the enumeration of all of them is not practical in computational terms, and most of these extreme points and rays of the dual polyhedron are not active in an optimal solution.

A vanilla implementation of Benders decomposition is usually not sufficient to solve large TSSPs directly. However, the idea of having a master problem and a set of subproblems solved separately, plus a set of strategies to improve and stabilize the solution obtained throughout the iterations of the algorithm, have made Benders decomposition method (or Benders method, for short) a successful technique. A detailed review on enhancing techniques for Benders methods is presented in Rahmaniani et al. (2017), from which we highlight stabilization (Rubiales et al. 2013, Zaourar and Malick 2014, Fischetti et al. 2016, 2017), valid inequalities (Saharidis et al. 2011), cut removal and cut selection (Pacqueau et al. 2012, Yang and Lee 2012), parallelization (Rahmaniani et al. 2019), and normalization (Fischetti et al. 2010).

In recent years, in the context of TSSPs, improvements to Benders methods have focused on aggregating scenarios using different distance measures between scenarios. For example, Vandenbussche et al. (2019) use clustering techniques to aggregate scenarios, Trukhanov et al. (2010) take those cuts that marginally contribute at a given iteration and group scenarios associated with them, and Beltran-Royo (2019) proposes aggregation based on conditional expectation over subsets of scenarios. Furthermore, Crainic et al. (2021) propose to retain some important
scenarios in the master problem and to generate some artificial ones in order to derive strengthening valid inequalities. Lastly, Biel and Johansson (2020) generalize many of previous ideas and deliver a general framework to study different scenario aggregations for Benders methods based on these distance measures.

The idea of aggregating scenarios has been explored not only in the context of Benders decomposition, but also for solving the deterministic equivalent formulation of two-stage stochastic programs. In this context, dimension of the deterministic equivalent can be reduced by applying scenario aggregation using partitions of the scenario set. Such partitions are then iteratively refined to improve the obtained lower and upper bounds. One of the first methods based on this idea is the sequential approximation method (Birge and Wets 1986, Frauendorfer 1992). In this method a partition of the scenarios is proposed using rectangular regions of random space. Then, a lower bound of the expected value of the second-stage subproblem is obtained using Jensen’s approximation over the conditional expectation with respect to a partition of the scenarios, and an upper bound is obtained using generalizations of the Edmundson-Mandasky inequality (Huang et al. 1977). An iterative refinement of the partition is proposed on the cell with the largest difference between these bounds, converging to the optimal solution. A more computationally efficient method is proposed in Pierre-Louis et al. (2011) replacing the computation of the bound with a variance-reducing Monte Carlo estimator.

Several recent approaches for aggregating scenarios in order to produce bounds for TSSPs are based on measuring similarity between scenarios. Hewitt et al. (2022) introduce the notion of opportunity costs of predicting the wrong scenario, and use it to measure the similarity/distance between pairs of scenarios. The obtained distance measure is then embedded in a graph structure and graph clustering techniques are applied to partition the scenarios into smaller groups and to derive different upper and lower bounds for the TSSPs. Keutchayan et al. (2021) seek to find a clustering of the scenario set and a representative of each cluster, so that these representatives can be used to solve a smaller approximative TSSP. They introduce a discrepancy measure which assesses how well a representative scenario within the cluster matches the average cost. The goal is to find a partition of the set of scenarios into $K$ clusters (and a representative of each cluster) so that the discrepancy is minimized. Finally, Bertsimas and Mundru (2022) introduce the concept of problem-dependant divergence as a means to evaluate the difference between scenarios and propose an algorithm to partition the scenario set and solve the scenario reduction problem. They also provide conditions under which their approach for scenario reduction reproduces the SAA results.

Another aggregation method similar to the sequential approximation method but with a different refinement of the partition of scenarios is proposed by Espinoza and Moreno (2014) for a risk-minimization portfolio problem. The authors group scenarios and disaggregate them by considering only those that are critical for the risk measure, i.e., those that have different dual values for a given portfolio solution. Later, Song and Luedtke (2015) generalize this concept in the so-called adaptive partition method (APM), ensuring the existence of a sufficient partition for any two-stage stochastic program with fixed recourse and discrete uncertainty space. More recently, Ramirez-Pico and Moreno (2022) develop a generalization of the previous algorithm called the generalized adaptive partition method (GAPM),
proving the existence of a finite and sufficient partition and providing an implementation of the aggregation technique in a more general setting with a continuous uncertainty set. In the simplest case, the idea behind GAPM is to aggregate all scenarios to obtain a smaller and easier master problem, and then iteratively solve the subproblems to disaggregate the scenarios into subsets sharing, e.g., the same dual optimal solutions. Contrary to the similarity-based methods mentioned above, where heuristic partitions and valid bounds are provided, the GAPM guarantees to find an optimal solution of the TSSP using a potentially smaller number of aggregated scenarios.

Our Contribution. Following the theory of the GAPM, we formalize the idea of aggregating scenarios in the Benders decomposition context. The purpose of this paper is twofold: (1) to combine two methodologies, i.e., Benders decomposition and the GAPM, that have proven to be successful for solving two-stage stochastic problems and (2) to provide sufficient conditions under which a given set of scenarios can be aggregated without sacrificing the optimality. We develop the underlying theory that enables to apply the Benders method to a smaller set of aggregated scenarios, based on the dual optimal solutions of the subproblems. We name this new approach the Benders adaptive-cuts method. Due to aggregation, smaller TSSPs are obtained, thus, impacting the number of cuts that are needed to solve the problem. The aggregated scenario cuts resemble the single-cuts Benders method, the disaggregated ones resemble the multi-cuts Benders method, and using a guided scenario refinement procedure (stemming from the GAPM), we manage to draw on the advantage of both worlds. We also show how to deal with infeasible subproblems, thus requiring only a fixed recourse for the TSSPs.

In our computational study, we focus on three practically relevant examples of stochastic network flow problems, for which Benders decomposition has been successfully implemented in the past. These problems are: Stochastic Electricity Planning, Stochastic Multi-Commodity Flow, and Stochastic Facility Location with the CVaR objective. The obtained computational results support our theoretical findings, demonstrating that the Benders adaptive-cuts method outperforms the other standard implementations of Benders decomposition, as well as the GAPM or the deterministic equivalent formulation. The computational advantages of the new method are particularly pronounced for TSSPs with a large number of scenarios.

To the best of our knowledge, the only time the idea of using APM for Benders decomposition has been mentioned is in Pay and Song (2020). In this paper, the authors propose to generate coarse optimality cuts to improve the computational performance of their Benders decomposition implementation. These cuts are generated from an adaptive partition of the scenarios, but are not used as a stand-alone approach, possibly due to the lack of theoretical arguments for its convergence. Our paper closes this gap and provides formal theoretical arguments for hybridizing (G)APM with Benders decomposition, including a derivation of feasibility and optimality adaptive cuts from a partition of the scenarios and a proof of sufficient conditions needed to obtain the optimal solution. Integration of the APM with other decomposition methods is proposed by van Ackooij et al. (2018) for level decomposition, and Siddig and Song (2022) for the SDDP algorithm for multistage stochastic problems.

The remainder of this paper is organized as follows. The problem statement is presented in Section 2, along with the overview of two standard implementations of
the Benders method, based on single-cuts and multi-cuts, respectively. Our theoretical framework for integrating the GAPM into the Benders method and a generic implementation procedure are given in Section 3. Section 4 presents three classical stochastic network flow optimization problems for which we applied the new methodology, and Section 5 shows the results of our computational experiments. Finally, we draw some final conclusions in Section 6.

2. Problem Statement and Mathematical Framework

In this section, we provide a formal definition of two-stage linear stochastic programs that are subject of this study. We then provide an overview of two classical implementations of the Benders method based on a separation of multi-cuts and single-cuts, respectively, before we present the theoretical framework of the novel Benders adaptive-cuts approach.

2.1. Problem Formulation. We study the following two-stage linear stochastic program with fixed recourse:

$$\min_{x \in \mathcal{X}} c^T x + \sum_{s \in \mathcal{S}} p^s Q(x, \xi^s)$$

where $x$ is a first-stage variable to which we associate a cost vector $c \in \mathbb{R}^n$, and $\mathcal{X} \subseteq \mathbb{R}^n$ is a non-empty closed (polyhedral) set describing feasible first-stage solutions. The uncertainty is modeled using a discrete sample space which is composed of a set of scenarios $\mathcal{S}$, each with associated probability $p^s > 0$ for $s \in \mathcal{S}$. In practice, the set of scenarios is composed by equally probable scenarios sampled from a more complex distribution, as presented in the sample average approximation method (Mak et al. 1999, Kleywegt et al. 2002). For a given realization $\xi^s := (T^s, h^s)$ of the random variable $\xi$ from $\mathcal{S}$, $Q(x, \xi^s)$ is the associated second-stage subproblem, defined as the following linear program (LP):

$$\min_{y \geq 0} q^Ty$$
$$Wy = h^s - T^sx$$

In this LP, $y \in \mathbb{R}^m$ is a second-stage variable, $W \in \mathbb{R}^{p \times m}$ is a fixed recourse matrix, $q \in \mathbb{R}^m$ is a deterministic cost vector. A random technology matrix $T^s \in \mathbb{R}^{p \times n}$ and a random right-hand side vector $h^s \in \mathbb{R}^p$ come from the discrete sample space $\mathcal{S}$. We assume that there exists some $\bar{x}$ such that $Q(\bar{x}, \xi^s)$ is feasible and bounded for all $s \in \mathcal{S}$. Most commonly, first-stage decision space corresponds to a polytope $\mathcal{X} = \{x \in \mathbb{R}^n : Ax = b\}$, however it can comprise more complex structures, including constraints that restrict the values of (some of) $x$ variables to integer numbers. A deterministic equivalent of the problem (1) is obtained after introducing a copy of the second-stage variable $y^s \in \mathbb{R}^m$, for each scenario $s \in \mathcal{S}:

$$\min_{x \in \mathcal{X}, y^s \geq 0} \{c^T x + \sum_{s \in \mathcal{S}} p^s q^T y^s \mid Wy^s = h^s - T^sx, s \in \mathcal{S}\}$$

2.2. Classical Way(s) of Deriving Benders Cuts. In this section we briefly review the classical Benders decomposition approach created to address large TSSPs. This method is also known as the L-Shaped method. Its main idea is to solve a large TSSP by replacing the values of second-stage subproblems with single variables. The algorithm progressively forces these variables to reach the optimal value
of the subproblems, by inserting in a dynamic fashion additional valid constraints, known as Benders cuts.

To present the method, we rewrite problem (\(\Pi\)) using the value-function reformulation:

\[
(4a) \quad \min_{x \in \mathcal{X}, \theta \in \mathbb{R}^{|S|}} c^\top x + \sum_{s \in S} p^s \theta^s
\]

\[
(4b) \quad Q(x, \xi^s) \leq \theta^s \quad s \in S.
\]

The value of \(Q(x, \xi^s)\) can be replaced by the optimal solution of its dual problem (assuming that the problem is well-defined and the optimal solution exists):

\[
(5a) \quad Q(x, \xi^s) = \max_{\lambda \in \mathbb{R}^p} (h^s - T^s x)^\top \lambda
\]

\[
(5b) \quad W^\top \lambda \leq q.
\]

In general, let \(\Lambda = \{\lambda \in \mathbb{R}^p : W^\top \lambda \leq q\}\) be the feasible region of the dual problem (notice that \(\Lambda\) does not depend on \(x\)). Assuming that the problem has a relatively complete recourse, \(\Lambda\) is not empty, so let \(\text{XP}(\Lambda)\) and \(\text{XR}(\Lambda)\) be the sets of its extreme points and extreme rays, respectively. Hence, we can reformulate problem (\(\Pi\)) as

\[
(6a) \quad \min_{x \in \mathcal{X}, \theta \in \mathbb{R}^{|S|}} c^\top x + \sum_{s \in S} p^s \theta^s
\]

\[
(6b) \quad (h^s - T^s x)^\top \hat{\lambda}^s \leq \theta^s \quad \hat{\lambda}^s \in \text{XP}(\Lambda), \ s \in S
\]

\[
(6c) \quad (h^s - T^s x)^\top \tilde{\lambda} \leq 0 \quad \tilde{\lambda} \in \text{XR}(\Lambda), \ s \in S
\]

In this model, constraints (6b) are known as Benders optimality cuts – they ensure that \(\theta^s\) is a lower bound for \(Q(x, \xi^s)\) for any extreme point of \(\Lambda\), so, in particular, it is a bound for its maximum value. In the case that \(Q(\hat{x}, \xi^s)\) is infeasible for a given \(\hat{x}\), its dual is unbounded, so constraints (6c) (also known as Benders feasibility cuts) ensure that \(\hat{x}\) is no longer considered a feasible solution. This problem reformulation contains an exponential number of constraints, but it can be solved using a cutting plane approach by iteratively selecting a candidate solution \(\hat{x}\) of the restricted master problem (which is the problem (6) with a subset of constraints (6b)-(6c)) and solving the subproblems \(Q(\hat{x}, \xi^s)\) to add possibly violated feasibility or optimality cuts on the fly. This method is known as the multi-cuts Benders method (or multi-cuts L-shaped method) in stochastic programming, and it converges in a finite number of iterations [Birge and Louveaux 1988].

Alternatively, it is possible to represent the total second-stage cost using a single variable \(\Theta\), by aggregating optimality cuts (6b) into a single Benders optimality cut of type (7b). That way, we obtain the following equivalent problem reformulation:

\[
(7a) \quad \min_{x \in \mathcal{X}, \theta \in \mathbb{R}^{|S|}} c^\top x + \Theta
\]

\[
(7b) \quad \sum_{s \in S} p^s (h^s - T^s x)^\top \hat{\lambda}^s \leq \Theta \quad (\hat{\lambda}^1, \ldots, \hat{\lambda}^{|S|}) \in \text{XP}(\Lambda)^{|S|}
\]

\[
(7c) \quad (h^s - T^s x)^\top \tilde{\lambda} \leq 0 \quad \tilde{\lambda} \in \text{XR}(\Lambda), \ s \in S
\]
in which $(\hat{\lambda}^1, \ldots, \hat{\lambda}^{S(d)})$ corresponds to a Cartesian product of $|\mathcal{S}|$ extreme points of $\mathcal{X}P(\Lambda)$. Problem (7) was considered in the original formulation of the L-shaped method (Van Slyke and Wets 1969), also known as the single-cuts Benders method. A notable difference between the single-cuts and the multi-cuts variant is in the separation of optimality cuts: while multi-cuts can be separated for each feasible scenario independently, to separate constraint (7b), the first-stage solution $x$ must yield feasible subproblems $Q(x, \xi)$ for all scenarios $s \in \mathcal{S}$.

The advantages of the single-cuts reformulation include a smaller number of variables and potentially a smaller number of cuts separated during the cutting plane algorithm. Thus, solving the master problem at each cutting plane iteration is faster than solving the problem via the multi-cuts implementation. An important disadvantage is that optimality cuts (7b) can be added only when the whole set of subproblems is feasible. Moreover, at any given iteration, the cuts obtained from this implementation are weaker than those generated via the multi-cuts implementation, which explains why the multi-cuts implementation may converge in a fewer number of iterations. In addition, for the latter one, it is possible to add optimality cuts for only a subset of scenarios, saving time due to the smaller number of subproblems solved. In general, however, the multi-cuts approach entails additional cost of adding many more cuts to the master problem, thus making the master problem larger and slower to solve.

3. GAPM and Benders Adaptive-Cuts

We start this section by briefly summarizing the major ideas behind the generalized adaptive partition method, from which we then derive Benders adaptive-cuts.  

3.1. The Generalized Adaptive Partition Method. In Section 1, we briefly mentioned some of the techniques that aim to reduce the size of stochastic two-stage programs. The Generalized Adaptive Partition Method has been developed by (Espinoza and Moreno 2014, Song and Luedtke 2015, Ramirez-Pico and Moreno 2022), where the authors provide formal conditions under which any TSSP with fixed recourse can be solved exactly using a smaller deterministic equivalent reformulation of problem (1). We briefly summarize the major ideas behind the GAPM and introduce the necessary notation.

Given a two-stage stochastic program with fixed recourse as defined in problem (1), let us assume that the subproblems are feasible for any $x \in \mathcal{X}$ and any realization of uncertain parameters (i.e., we have a relatively complete recourse). Let $\mathcal{P}$ be a partition of the uncertainty space $\Omega$. Note that $\Omega$ can be a discrete set (like $\mathcal{S}$ in our previous problem) or continuous. For each element $P \in \mathcal{P}$ of this partition, let $h^P = \mathbb{E}[h|\xi|P]$ and $T^P = \mathbb{E}[T^|\xi|P]$ be the conditional expectations, given $P$, of the right-hand side vector and the technology matrix, respectively. Consequently, we can define the aggregated subproblem associated to element $P \in \mathcal{P}$ as follows:

$$Q(x, \mathbb{E}[\xi|P]) = \min \left\{ q^Ty \mid Wy = h^P - T^Px, y \geq 0 \right\}.$$  

Ramirez-Pico and Moreno (2022) provide conditions on the dual of this aggregated subproblem to ensure its equivalence with the conditional expectation of $Q(x, \xi)$ given $P$, denoted as $\mathbb{E}[Q(x, \xi)|P]$.

Proposition 1 (Ramirez-Pico and Moreno 2022). Let $\bar{x} \in \mathcal{X}$ and $P \subseteq \Omega$ be such that $Q(\bar{x}, \xi)$ is feasible for all $\xi \in P$, and let $\bar{x}$ be optimal solutions of the LP-dual
of the subproblem \( Q(\bar{x}, \xi) \). If \( \hat{\lambda}^\xi \) for \( \xi \in P \) satisfies
\[
(8a) \quad \left( E[h^\xi | P]\right)^\top \left( E[\hat{\lambda}^\xi | P]\right) = E\left[ h^\xi \hat{\lambda}^\xi | P \right]
\]
\[
(8b) \quad \bar{x}^\top \left( E[T^\xi | P]\right)^\top E[\hat{\lambda}^\xi | P] = \bar{x}^\top E\left[ T^\xi \hat{\lambda}^\xi | P \right]
\]
then
\[
Q(\bar{x}, E[\xi | P]) = E[Q(\bar{x}, \xi) | P].
\]

From this result, given a partition \( \mathcal{P} \) of \( \Omega \) such that its elements satisfy the conditions (8), by the law of total expectation we can rewrite
\[
E[Q(\bar{x}, \xi)] = E[Q(\bar{x}, \xi) | P] \cdot P(\mathcal{P}) = \sum_{\mathcal{P} \in \mathcal{P}} Q(\bar{x}, E[\xi | \mathcal{P}]) \cdot P(\mathcal{P}).
\]

Therefore, if \( \bar{x} \in \arg \min_{x \in X} \{ c^T x + \sum_{\mathcal{P} \in \mathcal{P}} Q(x, E[\xi | \mathcal{P}]) \cdot P(\mathcal{P}) \} \) for a partition \( \mathcal{P} \) whose all elements \( \mathcal{P} \in \mathcal{P} \) satisfy the condition (8), then \( \bar{x} \) is also optimal for (1). Moreover, there always exists a finite partition \( \mathcal{P} \) satisfying the required conditions (8). Indeed, for a discrete uncertainty space, this is true at least for \( \mathcal{P} = S \). For continuous uncertainty space, see Ramirez-Pico and Moreno (2022) for more details. The deterministic equivalent reformulation with respect to \( \mathcal{P} \) is given as:

\[
\min_{x \in X} \{ c^T x + \sum_{\mathcal{P} \in \mathcal{P}} p^p q^p y^p \mid W y^p = h^p - T^p x, \ y^p \geq 0, \ \mathcal{P} \in \mathcal{P} \}
\]

In what follows, we propose a Benders adaptive-cuts method which aims to solve the problem (9) in a cutting plane fashion in which the scenario disaggregation is guided according to the GAPM scheme. While Proposition 1 assumes that for a given \( \bar{x} \) all subproblems are feasible, we also show how to deal with a more general setting, in which certain realizations of \( x \) may render the subproblems infeasible.

3.2. Benders Adaptive-Cuts. The GAPM leads to the idea of reformulating the TSSPs using Benders decomposition with a potentially smaller set of scenarios. This results into what we call a Benders adaptive-cuts method, in which we generate feasibility and optimality cuts at each iteration while avoiding an increase in the size of the master problem, as often occurs in the multi-cuts implementation. Our method aims to benefit from the advantages of both, the Benders multi-cuts and single-cuts methods. The reformulation of the original stochastic program (1) is similar to (6), but injecting fewer cuts than in the multi-cuts case. We add more optimality cuts compared to the single-cuts implementation, with the advantage of addressing both optimality and feasibility during the same iteration, which is not the case when we refer to the single-cuts implementation.

We first show how to aggregate a set of optimality cuts for a subset \( \mathcal{P} \subseteq \mathcal{S} \) of scenarios and demonstrate the validity of this new cut for reformulation (6).

**Proposition 2.** Let \( \mathcal{P} \subseteq \mathcal{S} \), and let \( p^p = \sum_{s \in \mathcal{P}} p^s \), \( h^p = \sum_{s \in \mathcal{P}} p^s h^s \) and \( T^p = \sum_{s \in \mathcal{P}} p^s T^s \). Then, the following inequalities
\[
(10) \quad p^p \cdot (h^p - T^p x) \hat{\lambda}^p \leq \sum_{s \in \mathcal{P}} p^s \theta^s \quad \hat{\lambda}^p \in \mathbb{X}^p(\Lambda)
\]
are valid for problem (6).
Proof. For a given $x \in X$ and $s \in S$, let $\hat{\lambda}^s$ be an optimal solution of the subproblem $Q(x, \xi^s)$ (assuming the associated subproblem is feasible). Then, $(\hat{\lambda}^s)_{s \in P}$ is also an optimal solution for the problem

$$
\sum_{s \in P} p^s Q(x, \xi^s) = \max_{\lambda^s \in \mathbb{R}^{|P|}} \sum_{s \in P} p^s \cdot (h^s - T^s x)^\top \lambda^s
$$

with $W^\top \lambda^s \leq q \quad s \in P$.

On the other hand, for any $\hat{\lambda}^P \in XP(\Lambda)$, since $\hat{\lambda}^P$ is a feasible solution for the dual of $Q(x, \xi^s)$ for all $s \in S$, by optimality of $(\hat{\lambda}^s)_{s \in P}$, we have:

$$
\sum_{s \in P} p^s \cdot (h^s - T^s x)^\top \hat{\lambda}^P \geq \sum_{s \in P} p^s \cdot (h^s - T^s x)^\top \hat{\lambda}^P = \left( \left( \sum_{s \in P} p^s h^s \right) - \left( \sum_{s \in P} p^s T^s \right) x \right)^\top \hat{\lambda}^P = p^P \cdot (h^P - T^P x)^\top \hat{\lambda}^P
$$

Since the optimality cuts for the multi-cuts case enforce that

$$(h^s - T^s x)^\top \hat{\lambda}^s \leq \theta^s \quad s \in S,$$

then by aggregating these cuts for all $s \in P$, we obtain

$$p^P \cdot (h^P - T^P x)^\top \hat{\lambda}^P \leq \sum_{s \in P} p^s (h^s - T^s x)^\top \hat{\lambda}^s \leq \sum_{s \in P} p^s \theta^s$$

so (10) is a valid cut for problem (6).

Note that $h^P$ and $T^P$ can be interpreted as weighted averages of the random components $h^s$ and $T^s$ among all scenarios $s \in P$, and $p^P$ is the aggregated probability of these scenarios. Hence, (10) can be seen as an aggregation of the optimality cuts (6b) from the original multi-cuts Benders method. This type of aggregated Benders cuts has been also studied as a way to a priori create artificial scenarios to tighten the master problem (Crainic et al. 2021), or to generate Benders cuts (Rahmaniani et al. 2021), for a predefined clustering of scenarios.

Next, we present sufficient conditions under which Benders optimality cuts imposed with respect to a partition $P$ of $S$ provide a proper problem reformulation, and thus, an optimal solution is guaranteed to be found. These conditions correspond to the conditions required for the GAPM (cf. Proposition 1).

**Proposition 3.** Let $x^*$ be an optimal solution of the problem

(11a) $\min_{x \in X, \theta \in \mathbb{R}^{|S|}} c^\top x + \sum_{s \in S} p^s \theta^s$

(11b) $p^P (h^P - T^P x)^\top \tilde{\lambda}^P \leq \sum_{s \in P} p^s \theta^s \quad \tilde{\lambda} \in XP(\Lambda), P \in P$

(11c) $(h^s - T^s x)^\top \tilde{\lambda} \leq 0 \quad \tilde{\lambda} \in XR(\Lambda), s \in S$,
where $\mathcal{P}$ is a partition of $\mathcal{S}$ such that there exist optimal dual solutions $\hat{\lambda}^s$ of subproblems $Q(x^s; \xi^s)$ that satisfy the following conditions:

\begin{equation}
(12a) \quad \left(\sum_{s \in P} p^s\right) \cdot \sum_{s \in P} p^s \left(h^s \hat{T} \hat{\lambda}^s\right) = \left(\sum_{s \in P} p^s h^s\right) \left(\sum_{s \in P} p^s \hat{\lambda}^s\right) \quad \text{for all } P \in \mathcal{P}
\end{equation}

\begin{equation}
(12b) \quad \left(\sum_{s \in P} p^s\right) \cdot \sum_{s \in P} p^s \left(T^s x^s \hat{\lambda}^s\right) = \left(\sum_{s \in P} p^s T^s x^s\right) \left(\sum_{s \in P} p^s \hat{\lambda}^s\right) \quad \text{for all } P \in \mathcal{P}.
\end{equation}

Then, $x^*$ is also an optimal solution of $\Pi$.

**Proof.** For each $P \in \mathcal{P}$, since $\hat{x}^s \in \text{XP}(\Lambda)$ for all $s \in P$, then constraint $(11b)$ implies that

\[ p^P \cdot (h^P - T^P x)^\top \hat{x}^{s'} \leq \sum_{s \in P} p^s \theta^s \quad s' \in P \]

and hence, by multiplying the above inequalities by $p^{s'}/p^P$ and summing them up over all $s' \in P$, we obtain:

\[ p^P \cdot (h^P - T^P x)^\top \left(\sum_{s \in P} p^s \hat{\lambda}^s \right) \leq \sum_{s \in P} p^s \theta^s. \]

The left-hand side term of this expression can be rewritten as

\[ \left(\sum_{s \in P} p^s\right) \left(\sum_{s \in P} p^s h^s - \sum_{s \in P} p^s T^s x^s\right) \left(\sum_{s \in P} p^s \hat{\lambda}^s\right) \]

and, if $\hat{x}^s$ satisfies conditions $(12a)$ and $(12b)$, then this term is equal to

\[ \left(\sum_{s \in P} p^s\right) \left(\frac{1}{\sum_{s \in P} p^s} \sum_{s \in P} p^s (h^s - T^s x)^\top \hat{\lambda}^s\right) \]

Thus, constraint $(11b)$ implies that

\[ \sum_{s \in P} p^s (h^s - T^s x)^\top \hat{\lambda}^s \leq \sum_{s \in P} p^s \theta^s \quad P \in \mathcal{P}. \]

By aggregating these cuts for all $P \in \mathcal{P}$, this also implies that

\[ \sum_{P \in \mathcal{P}} \sum_{s \in P} p^s (h^s - T^s x)^\top \hat{\lambda}^s \leq \sum_{P \in \mathcal{P}} \sum_{s \in P} p^s \theta^s, \]

which is equivalent to

\[ \sum_{s \in S} p^s (h^s - T^s x)^\top \hat{\lambda}^s \leq \sum_{s \in S} p^s \theta^s := \Theta, \]

which are Benders cuts for the single-cuts reformulation. Hence, $x^*$ is also an optimal solution of the single-cuts Benders reformulation. Since the problem $(11)$ is a relaxation of the original multi-cuts Benders reformulation $(\Pi)$, and we just showed that an optimal solution $x^*$ of $(11)$ is also optimal for the problem $(\Pi)$, it follows that $(11)$ is a proper reformulation of problem $(\Pi)$. \hfill \Box

For two partitions $\mathcal{P}^1, \mathcal{P}^2$ of $\mathcal{S}$, we call $\mathcal{P}^2$ a refinement with respect to $\mathcal{P}^1$, denoted by $\mathcal{P}^1 \trianglelefteq \mathcal{P}^2$, if and only if for any two subsets $P_r, P_q \subset \mathcal{S}$, $P_r, P_q \in \mathcal{P}^2$, there exists a set $P \in \mathcal{P}^1$ such that $P_r \cup P_q \subseteq P$. The following result follows from Proposition 2.
Corollary 1. Let $v(P)$ denote the optimal solution value of the problem (11). For a given family of partitions $\{P^1, \ldots, P^t \}$ such that $P^1 \preceq \cdots \preceq P^t \preceq S$, we have

$$v(P^1) \leq \cdots \leq v(P^t) \leq v(S).$$

Hence, by starting with a union of all scenarios and by iteratively refining this partition, Benders adaptive-cuts method provides a valid and non-decreasing lower bound, which eventually ensures the convergence of the method. The convergence follows from two aspects: first, from the convergence of the GAPM, and second from the convergence of the Benders method applied within each GAPM iteration. In Section 3.4, we discuss how to implement the Benders adaptive-cuts method based on an iterative refinement of partitions of $S$ guided by the GAPM.

### 3.3. Benders Adaptive-Single-Cuts.

We notice that formulation (11) includes one optimality cut for each element $P \in P$, and thus, it can be interpreted as Benders multi-cuts method applied to the given partition $P$. Similarly, one can consider its single-cuts counterpart, which is given as

\begin{align}
(13a) \quad & \min_{x \in X, \theta, \nu \in \mathbb{R}} c^T x + \theta \\
(13b) \quad & \sum_{P \in P} \nu^P (h^P - T^P x)^T \lambda^P \leq \theta \quad (\lambda^1, \ldots, \lambda^{|P|}) \in XP(A)^{|P|} \\
(13c) \quad & (h^s - T^s x)^T \lambda \leq 0 \quad \lambda \in XR(A), s \in S.
\end{align}

In this model, we refer to constraints (13b) as Benders adaptive-single-cuts. Using similar arguments as above, it is not difficult to see that conditions given in Proposition 3 guarantee that the optimal solution $x^*$ of (13) is also optimal for the original problem (1).

It is worth mentioning that constraints (13b) have been originally used by Pay and Song (2020) under the name coarse cuts. The authors generated these cuts in the early stage of their Benders decomposition scheme, when the initial solutions of the master problem are far from the optimal solution. The cuts have been used within a generation of additional valid inequalities (and not as a stand-alone problem formulation), with the aim of enhancing the convergence and improving the computational performance. Thus, with our results derived in Proposition 3, we show that also coarse cuts, when applied to a partitioning scheme guided by the GAPM, can lead to an alternative exact solution approach, to which we refer as Benders adaptive-single-cuts method.

### 3.4. Implementation of Benders Adaptive-Cuts.

The result provided in Proposition 3 indicates that it is sufficient to find a partition of the scenarios, say $P^*$, satisfying the conditions (12a) and (12b) in order to correctly reformulate the problem (1) and find its optimal solution. Since $P^*$ depends on the optimal solution $x^*$, it is not possible to construct it a priori. However, we can start with a single aggregated scenario $P = \{S\}$ and iteratively refine the partitioning of $S$ until the convergence criteria are met. That way, we are generating new partition-based cuts, in a similar way to other Benders approaches, using a cutting plane procedure. At the same time, we will be potentially improving the incumbent solution. An algorithmic implementation of this methodology is presented in Algorithm 1.

In each iteration $t$, we denote by $P^{(t)}$ the current partition of the scenario set $S$. At the beginning, all the scenarios are aggregated, so we have $P^{(0)} = \{S\}$.
Algorithm 1 Implementation of the Benders Adaptive-cuts Method

**Input:** Set of scenarios $\mathcal{S}$, with probabilities $p^s$ for $s \in \mathcal{S}$

**Output:** Optimal solution $\hat{x}$ and optimal value $z^*$

1: Set $t := 0$, $z_L^{(0)} := -\infty$, $z_U := \infty$, $z^* := \infty$ and $\mathcal{P}^{(0)} = \{\mathcal{S}\}$.
2: Let MP be the problem
   $$\min_{x \in X, \theta \geq LB} c^T x + \sum_{s \in \mathcal{S}} p^s \theta^s$$
3: Solve MP and let $(x^{(t)}, \theta^{(t)})$ and $z_L^{(t)}$ be its optimal solution and its optimal value.
4: if $z_U = z_L^{(t)}$ then return optimal solution $\hat{x} := x^{(t)}$ and optimal value $z^* := z_L^{(t)}$
5: for all $P \in \mathcal{P}^{(t)}$ do
   6: Set $p^P := \sum_{s \in P} p^s$, $h^P := \sum_{s \in P} p^s h^s$, $T^P := \sum_{s \in P} p^s T^s$ and $\xi^P := (T^P, h^P)$
   7: if subproblem $Q(x^{(t)}, \xi^P)$ is infeasible then
      8: Get a dual extreme ray $\hat{\lambda}^P$ and add the following cut to MP $\triangleright$ feasibility cut
      $$\langle h^P - T^P x \rangle^T \hat{\lambda}^P \leq 0$$
   else
      9: Let $\hat{\lambda}^P$ be an optimal dual solution of $Q(x^{(t)}, \xi^P)$
     10: if $p^P(h^P - T^P x)^T \hat{\lambda}^P > \sum_{s \in P} p^s \theta^s(t)$ then
        11: Add to MP the following cut $\triangleright$ optimality cut
        $$p^P(h^P - T^P x)^T \hat{\lambda}^P \leq \sum_{s \in P} p^s \theta^s$$
     else
        12: for all $s \in \mathcal{S}$ do
           13: Solve and store either extreme ray $\hat{\lambda}^s$ or optimal solution $\hat{\lambda}^s$ of $Q(x^{(t)}, \xi^s)$
        14: end for
     end if
   end if
   15: Set $z_U := \min \{z_U, c^T x^{(t)} + \sum_{s \in \mathcal{S}} p^s Q(x^{(t)}, \xi^s)\}$
   16: if $\exists P \in \mathcal{P}^{(t)}, s \in P$ s.t. $Q(x^{(t)}, \xi^s)$ is infeasible, or s.t. $(\hat{\lambda}^s)_{s \in P}$ do not satisfy \[12\] then
      17: Run refinement procedure which refines $\mathcal{P}^{(t)}$ to obtain a new partition $\mathcal{P}^{(t+1)}$
      18: Set $t := t + 1$ and go to Step \[2\]
   else
      19: return $z^* := z_L^{(t)}, \hat{x} := x^{(t)}$
      20: end if

\{\mathcal{S}\}. When solving the initial LP given in Step \[2\] we ensure that the problem is bounded by restricting $\theta^s$ variables from below ($\theta \geq LB$), where e.g., $LB = \sum_{i=1} \min \{q_i LB_1, q_i UB_1\}$, and $LB_1$ and $UB_1$ correspond to a global lower and upper bound of $y_i$, respectively. In each iteration $t$ we are solving the associated problem relaxation given by \[11\] with respect to $\mathcal{P}^{(t)}$. We then check if the conditions of Proposition \[3\] are met, and if yes, we terminate with a guarantee that an optimal solution is found. Otherwise, we refine the partition $\mathcal{P}^{(t)}$ and repeat the above procedure. We emphasize that we do not start solving \[11\] with respect to
\( P(t) \) from scratch, but we keep all previously added cuts instead. Moreover, for any solution \( x \in X \) encountered along this process, we calculate the upper bound obtained by evaluating the expected value of the recourse, and potentially update the global upper bound. Thus, the algorithm either terminates because the condition of Proposition 3 is met, or because the lower bound at iteration \( t \), corresponding to \( v(P(t)) \) (see Corollary 1) matches the best found upper bound.

At a more detailed level, in the first set of instructions (Steps 3-13), we apply the usual Benders approach with the aggregated cuts, adding optimality and feasibility cuts at each iteration. Note that this step includes fewer optimality cuts (Step 12) than does the Benders multi-cuts method because we consider only each element of the partition, not each scenario. Similarly, we also add fewer feasibility cuts (Step 8) because the subproblem has to be feasible for the aggregated subproblems, not for each individual scenario. Once the aggregated Benders problem is optimized, we revise each element of the partition \( P \) to verify that conditions (12a) and (12b) are satisfied for this subset of scenarios (Steps 15-20) for the current solution \( x(t) \). Note that this is the only part of the algorithm where all second-stage subproblems are required to be optimized. If there exists a set \( P \in P(t) \) such that at least one of its subproblems \( s \in P \) is infeasible, or if conditions of Proposition 3 are not met, then the set \( P \) has to be refined to meet these conditions, and a new partition has to be created. The underlying refinement procedure (Step 19) is problem specific, and we discuss a few examples in the next section. When the conditions stated in Step 18 are satisfied, then \( x(t) \) is the optimal solution for problem (1), according to Proposition 3. Note that if \( P = S \), then the algorithm is equivalent to the Benders multi-cuts method.

A key part of the algorithm consists in refining of the current partition to satisfy conditions stated in Step 18. Let \( P \) be an element from \( P(t) \), we distinguish the following two cases:

- When all scenarios \( s \in P \) are feasible, one way to perform the refinement of \( P \) is to group all scenarios with the same dual solution \( \hat{\lambda} \), as presented in Song and Luedtke (2015). However, this condition might be too strong, as shown in Ramirez-Pico and Moreno (2022), generating a partition with too many subsets. On the other hand, to compute the minimal partition \( P^{(t+1)} \) satisfying (12) for \( x^{(t)} \) can be computationally difficult. Less restricting conditions can be obtained when the subproblem associated to the uncertainty parameters have particular substructures. One such example can be found in the stochastic flow problems, where many of the components of \( h^s \) are equal to 0 for all scenarios. In this case, we only need to consider dual variables associated to the remaining components (see Section 4.2 for more details).

- If the current solution \( x^{(t)} \) produces infeasible subproblems for some \( s \in P \), we can group all these infeasible scenarios sharing the same dual extreme ray \( \hat{\lambda} \). In fact, since \( (h^s - T^sx)\hat{\lambda} > 0 \) for these scenarios, then the aggregated feasibility cut \( p^P(h^P - T^P x)\hat{\lambda} \leq 0 \) will eliminate \( x^{(t)} \) from the set of feasible solutions. Note that we only need to add one feasible cut for each subset of scenarios sharing the same dual extreme ray. Indeed, this ensures that at the end of the algorithm all subproblems \( s \in P \) are feasible, satisfying constraints (11c) from Proposition 3.
4. Benders Adaptive-Cuts Applied to Stochastic Network Flow Problems

Since the 1950’s, network flow optimization problems have been comprehensively studied (Costa 2005, Zargham et al. 2013, Bertsimas and Sim 2003, Ahuja et al. 1988). The problems are highly relevant to a variety of applications, including transportation (Zapfel and Wasner 2002), telecommunications (Amaldi et al. 2008, Leitner et al. 2020, Ljubić et al. 2021), energy (Jin et al. 2013, Geidl and Andersson 2005), and others fields. Even though efficient combinatorial algorithms exist for specific problems such as max-flow or min-cost-flow (Ahuja et al. 1988), for a plethora of related (sometimes NP-hard) problems, efficient MIP-based exact solution methods need to be developed. This is especially true when network flow problems appear in more complex optimization settings, such as mixed-integer linear programming, nonlinear programming, large-scale optimization and, in particular, stochastic programming.

Stochastic network flows appear as a natural choice for many applications arising in transportation and logistics, where networks have to be designed subject to uncertain transportation demand. A similar situation occurs when a subset of facilities has to be open in order to serve customers’ demands, but the actual values of this demand are only discovered in the future. In our study, we focus on three stochastic network flow problems: a) the capacity planning problem (CPP), b) the stochastic multicommodity flow problem (SMCF), and c) the stochastic facility location problem with risk aversion (FL-CVaR). We choose the first two problems because they also frequently appear in the related literature when it comes to efficient implementations of the Benders decomposition approach (Rahmaniani et al. 2018, Crainic et al. 2021). The third problem complements the former two, as it involves binary first-stage variables and models a risk-averse objective function. Moreover, Benders single-cuts and multi-cuts methods for these three problems behave differently, as we will see in the computational results.

All three problems are defined on a directed graph $G = (V, E)$, where in the first-stage a decision is made concerning the structure of the network and the capacity of some of its nodes or arcs. In the second-stage, a routing decision is made once the uncertainty (e.g., random demand) is revealed. In both cases, the cost of the first-stage decisions plus the expected routing cost of the second-stage, are minimized. In the remainder of this section, for each of the three problems, we provide a mathematical formulation and a specific recipe for the refinement procedure. Results of our implementation and a comparison with alternative methods are given in Section 5.

4.1. The Capacity Planning Problem (CPP). Our first example is a stochastic network flow problem, originally proposed by Louveaux (1988), in which we are given a bipartite graph $G = (V, E)$, representing an electricity planning network with the set of nodes $V = V_L \cup V_R$ and the set of arcs $E = V_L \times V_R$. The nodes from $V_L$ represent power terminals, and the nodes from $V_R$ are the customer nodes with uncertain demand. There are $|K|$ different resources, and a unit of capacity allocated to terminal $i \in V_L$ uses $a_{ik}$ units of resource $k \in K$. Total availability of each resource $k \in K$ limited to $r_k > 0$. In the first stage one has to decide about the capacities of each power terminal $i \in V_L$, while respecting the resource limitations. In the second stage, once the demands at the nodes $j \in V_R$ are revealed, the
electricity can be transported from the terminals to demand nodes. Transporting
one unit of demand between nodes \(i \in V_L\) and \(j \in V_R\) invokes the cost/profit of \(e_{ij}\), which can include the transportation cost and revenue from sales, so \(e_{ij}\) can be positive (cost) or negative (profit). Note that the demand does not need to be
satisfied, so installing zero capacity at the terminals is a feasible solution for any
scenario. The objective function consists of minimizing the capacity installation
cost at the first stage plus the expected transportation cost/profit at the second
stage.

Mathematical formulation: Following our notation, the CPP can be stated as fol-

\[
\begin{align*}
\text{min} & \quad x \in \mathbb{R}^{\vert V_L \vert} + \sum_{i \in V_L} c_i x_i + \sum_{s \in S} p^s Q(x, \xi^s) \\
\text{subject to} & \quad \sum_{i \in V_L} a_{ik} x_i \leq r_k, \quad k \in K \\
& \quad \sum_{j \in V_R} y_{ij}^s \leq d_{ij}^s, \quad j \in V_R \\
& \quad \sum_{j \in V_R} y_{ij}^s \leq \hat{x}_i, \quad i \in V_L \\
\end{align*}
\]

In this model, the first-stage variable \(x \in \mathbb{R}^{\vert V_L \vert}\) represents the capacity installed
at each supply node \(i \in V_L\) with unit cost \(c_i\). The budget constraints (14b) are
imposed for each resource \(k \in K\). Given \(\hat{x}\), the second-stage subproblem
\(Q(\hat{x}, \xi^s)\) is a min-cost flow problem

\[
\begin{align*}
\text{min} & \quad \sum_{i \in V_L} e_{ij} y_{ij}^s \\
\text{subject to} & \quad \sum_{j \in V_R} y_{ij}^s \leq d_{ij}^s, \quad j \in V_R \\
& \quad \sum_{j \in V_R} y_{ij}^s \leq \hat{x}_i, \quad i \in V_L \\
\end{align*}
\]

where the second-stage variable \(y^s \in \mathbb{R}^{\vert E \vert}\) represents the flow from supply nodes
to demand nodes. This flow must satisfy the capacity constraint (15c) provided
by first-stage variables \(\hat{x}\) at each supply node and the maximum (random) demand
\(d^s\) at each demand node, cf. constraints (15b). Note that \(e_{ij}\) can be negative,
representing a profit for assigning flow on arc \(ij \in E\). Given the first-stage decision
\(\hat{x}\), the dual of the subproblem associated to scenario \(s \in S\) is as follows:

\[
\begin{align*}
\text{min} & \quad \mu_k^s + \nu_i^s \geq -e_{ij}, \quad ij \in E \\
\mu_k^s + \nu_i^s \geq -e_{ij}, \quad ij \in E \\
\end{align*}
\]

Benders cuts: An important remark about the capacity planning problem is that
the subproblem solution \(y^s = 0\) is feasible for any feasible first-stage decision \(\hat{x}\),
so there always exists an optimal solution for dual of the subproblem \(Q(\hat{x}, \xi^s)\).
Hence, there are only Benders optimality cuts to be dealt with when implementing
a Benders decomposition procedure.

Given a first-stage solution \(\hat{x}\) and the corresponding optimal solutions \((\hat{\mu}^s, \hat{\nu}^s)\)
of the dual of \(Q(\hat{x}, \xi^s)\) for each \(s \in S\), we present the inequalities that are added
to reformulations (6) and (7) for the cases of the multi-cuts and single-cuts implement-
mentation, respectively:
(CPP multi-cuts) \[ - \sum_{j \in V_R} d_j^p \hat{\mu}_j^p - \sum_{i \in V_L} \hat{\nu}_i^p x_i \leq \theta^s \quad s \in S \]

(CPP single-cuts) \[ - \sum_{s \in S} p^s \left[ \sum_{j \in V_R} d_j^p \hat{\mu}_j^p + \sum_{i \in V_L} \hat{\nu}_i^p x_i \right] \leq \Theta. \]

Given a partition \( P \) of the set \( S \), for any \( P \in \mathcal{P} \), let \( d_j^P := \sum_{s \in P} p^d d_j^s / p^P \) be the expected demand of the node \( j \in V_R \) with respect to the aggregated scenarios from \( P \). Let \((\hat{\mu}^P, \hat{\nu}^P)\) be an optimal solution of the subproblem \( (16) \) using demands \( d^P \). It follows from Proposition 2 that the following cuts are valid for the CPP:

(CPP adaptive-cuts) \[ -p^P \left[ \sum_{j \in V_R} d_j^P \hat{\mu}_j^P + \sum_{i \in V_L} \hat{\nu}_i^P x_i \right] \leq \sum_{s \in P} p^s \theta^s \quad P \in \mathcal{P}. \]

Refinement procedure: In order to explain the refinement procedure for the CPP which we have implemented as step 20 of Algorithm 1, we start with the following result derived from Proposition 3:

Corollary 2. Given a partition \( P \) of the set \( S \), the following condition ensures that adding the (CPP adaptive-cuts) for each \( P \in \mathcal{P} \) provides an optimal solution of the CPP:

\[ \sum_{j \in V_R} \left[ \left( \sum_{s \in P} p^s d_j^s \right) \cdot \left( \sum_{s \in P} p^s \hat{\mu}_j^s \right) \right] = \sum_{j \in V_R} \frac{1}{p^P} \sum_{s \in P} p^s d_j^s \hat{\mu}_j^s, \quad \text{for all } P \in \mathcal{P} \]

Sufficient conditions for a given partition \( P \) to satisfy the above equation are, for example, when either \( d^s = d^{s'} \) or \( \hat{\mu}^s = \hat{\mu}^{s'} \) for all pairs of scenarios \( s, s' \in P \). Indeed, for the vector of dual variables \( (\hat{\mu}) \) and the vector of the associated right-hand-side \( (d^s) \), equation (17) requires that the weighted average (for a given \( P \)) of their scalar product is equal to the scalar product of their weighted averages. This condition is naturally satisfied when one of these vectors is a constant \((d, \hat{\mu}, \text{or both})\) across all scenarios \( s \in P \). Hence, for each \( P \in \mathcal{P} \), we construct \( \mathcal{P}' \) by solving the subproblem \((16)\) for each \( s \in P \) and group them into subsets where \( \hat{\mu}^s \) has the same value.

4.2. The Stochastic Multicommodity Flow Problem (SMCF). Our second problem is a network design optimization problem under uncertainty. We are given a set of commodities \( K \), such that for each \( k \in K \) its origin \( O(k) \in V \) and its destination \( D(k) \in V \), are known, but its demand \( d_k \) is subject to uncertainty. We are also given the cost of routing a single unit of demand of \( k \) along an arc \( ij \in E \), denoted by \( c_{ij} > 0 \). In the first stage, we need to decide on the fraction of the given capacity \( u_{ij} > 0 \) of each arc \( ij \in E \) that need to be installed in order to simultaneously route all the commodities. Once the uncertainties are revealed, the second-stage decision consists of finding the minimum-cost routing of the commodities over the network installed in the first stage. The problem has been widely studied both in its integer (Gendron et al. 1999) and linear (Barnhart et al. 2001) versions, that is, where the capacity of each arc is either 0 or a nominal capacity \( u_{ij} \) for the former, or a fraction of this nominal capacity, for the latter. Various Benders approaches...
have been developed to solve the problem; the most recent and state-of-the-art im-
plementation of Benders decomposition by Rahmaniani et al. (2018) can efficiently
address instances of the SMCF with a small number of scenarios (between 16 to
64). However, our aim is to focus on more challenging SMCF instances for which
the size of the uncertainty set can be very large (and could go up to 50,000 scenar-
ios, as we shall see in our numerical study). Our goal is to understand the impact
of the proposed Benders adaptive-cuts and to obtain possible improvements of the
state-of-the-art for solving these challenging instances.

Mathematical formulation: The SMCF is formulated as follows
\begin{equation}
\min_{x \in [0,1]} \sum_{ij \in E} f_{ij} x_{ij} + \sum_{s \in S} p^s Q(x, \xi^s)
\end{equation}
where the first-stage variable $0 \leq x_{ij} \leq 1$ indicates the fraction of the nominal
capacity $u_{ij}$ that should be installed on the arc $ij \in E$, and $f_{ij} > 0$ is the installation
cost per unit of capacity. We are given a discrete set of scenarios $S$ modeling the
uncertain realizations of the demand of each commodity $k \in K$. For a given first-
stage decision $\hat{x}$, the second-stage problem $Q(\hat{x}, \xi^s)$ associated to scenario $s \in S$,
is a multicommodity flow problem on the network $G$ with arc capacities defined as
$\hat{x}_{ij} u_{ij}$ and with demands $d^s_k \geq 0$, for each $k \in K$:
\begin{equation}
Q(\hat{x}, \xi^s) := \min \sum_{ij \in E} \sum_{k \in K} c_{ijk} y^s_{ijk}
\end{equation}
\begin{align}
\sum_{j : ij \in E} y^s_{ijk} - \sum_{j : ji \in E} y^s_{jik} &= \begin{cases} 
\quad d^s_k & \text{if } i = O(k) \\
\quad -d^s_k & \text{if } i = D(k) \\
\quad 0 & \text{otherwise}
\end{cases} 
\quad i \in V, k \in K
\end{align}
\begin{align}
\sum_{k \in K} y^s_{ijk} &\leq u_{ij} \hat{x}_{ij} 
\quad ij \in E
\end{align}
\begin{align}
y^s_{ijk} &\geq 0 
\quad ij \in E, k \in K
\end{align}

Notably, for this problem, $\hat{x}$ does not always yield a feasible subproblem (19), so
its dual (20) could be unbounded in which case we will need to derive feasibility
cuts from its extreme rays.

Benders cuts: From (20), we define the feasibility cuts, which are the same no
matter if the single-cuts or the multi-cuts version is utilized. For a given $\hat{x}$, where
$Q^D(\hat{x}, \xi^s)$ is unbounded for some scenario $s \in S$, let $(\hat{\lambda}^s, \hat{\mu}^s)$ be an extreme ray of
the corresponding subproblem. Hence, the following feasibility cuts are added to
master problem (6) or (7):
(Feasibility cuts) \[ \sum_{k \in K} d_k^s \left( \hat{\lambda}^{s}_{O(k), k} - \hat{\lambda}^{s}_{D(k), k} \right) - \sum_{i,j \in E} u_{ij} \hat{\mu}_{ij}^{s} x_{ij} \leq 0 \]

Similarly, in the case that \( Q^D(\hat{x}, \xi^s) \) is feasible, let \((\hat{\lambda}^s, \hat{\mu}^s)\) be the optimal solution of \( Q^D(\hat{x}, \xi^s) \) for each \( s \in S \); then, the following optimality cuts are added to the master problem, if violated:

(Optimality multi-cuts) \[ \sum_{k \in K} d_k^s \left( \hat{\lambda}^s_{O(k), k} - \hat{\lambda}^s_{D(k), k} \right) - \sum_{i,j \in E} u_{ij} \hat{\mu}_{ij}^{s} x_{ij} \leq \Theta^s \quad s \in S \]

(Optimality single-cuts) \[ \sum_{s \in S} \left[ \sum_{k \in K} d_k^s \left( \hat{\lambda}^s_{O(k), k} - \hat{\lambda}^s_{D(k), k} \right) - \sum_{i,j \in E} u_{ij} \hat{\mu}_{ij}^{s} x_{ij} \right] \leq \Theta \]

Whilst the Benders multi-cuts method may add both (Feasibility cuts) and (Optimality multi-cuts) in a single iteration, the single-cuts method can generate a new optimality cut only when all subproblems have an optimal solution.

On the other hand, following Proposition 2, for a given partition \( P \) of \( S \), the feasibility and optimality aggregated cuts that can be added to the master problem, if violated, are

(Feasibility adaptive-cuts) \[ p^P \left[ \sum_{k \in K} d_k^P \left( \hat{\lambda}^P_{O(k), k} - \hat{\lambda}^P_{D(k), k} \right) - \sum_{i,j \in E} u_{ij} \hat{\mu}_{ij}^{P} x_{ij} \right] \leq 0 \quad P \in \mathcal{P} \]

(Optimality adaptive-cuts) \[ p^P \left[ \sum_{k \in K} d_k^P \left( \hat{\lambda}^P_{O(k), k} - \hat{\lambda}^P_{D(k), k} \right) - \sum_{i,j \in E} u_{ij} \hat{\mu}_{ij}^{P} x_{ij} \right] \leq \sum_{s \in P} p^s \Theta^s \quad P \in \mathcal{P} \]

where \((\hat{\lambda}^P, \hat{\mu}^P)\) and \((\hat{\lambda}^P, \hat{\mu}^P)\) are either extreme rays or optimal extreme points of the subproblem \( (20) \) considering the aggregated demand \( d_k^P := \sum_{s \in P} p^s d_k^s / p^P \) for each \( P \in \mathcal{P} \).

Note that feasibility cuts are required to obtain a feasible solution for the weighted aggregated demand \( d^P \) for each \( P \in \mathcal{P} \), not for each scenario \( s \in S \), which potentially reduces the number of feasibility cuts required to obtain a feasible solution for the problem.

Refinement procedure: The following corollary follows directly from Proposition 3:

**Corollary 3.** Given a partition \( \mathcal{P} \) of the set \( S \), the following condition guarantees that adding Benders adaptive-cuts for each \( P \in \mathcal{P} \) is sufficient to obtain an optimal solution of the SMCF:

\[ \sum_{k \in K} \left[ \left( \sum_{s \in P} \frac{p^s}{p^P} d_k^s \right) \cdot \left( \sum_{s \in P} \frac{p^s}{p^P} \left( \lambda^s_{O(k), k} - \lambda^s_{D(k), k} \right) \right) \right] = \sum_{k \in K} \left[ \left( \sum_{s \in P} \frac{p^s}{p^P} \cdot d_k^s \cdot \left( \lambda^s_{O(k), k} - \lambda^s_{D(k), k} \right) \right) \right] \]
Indeed, this condition follows from (12a) because only the demand is uncertain. Note that a given element \( P \in \mathcal{P} \) satisfies condition (21), for example, when either \( d_s^k = d_s'^k \) or \( \lambda_{O(k),k} - \lambda_{O(k'),k} = \lambda_{D(k),k} - \lambda_{D(k'),k} \) \( k \in K \) for all pairs of scenarios \( s, s' \in P \). Hence, we apply the following rule to create a refinement of a given partition \( P \): For each \( P \in \mathcal{P} \), we solve the dual subproblem (20) for each \( s \in P \). If there are unbounded subproblems, we group the scenarios sharing the same extreme ray. For the feasible subproblems, we group them into subsets where the vector \( (\lambda_{O(k),k} - \lambda_{D(k),k})_{k \in K} \) has the same value.

4.3. The Stochastic Facility Location Problem under risk aversion (FL-CVaR). Our third problem is a facility location problem with uncertain demand and a risk-averse decision maker. We are given a set of potential locations \( I \) to install a facility, each of them with a fixed installation cost \( f_i \geq 0 \) and a capacity \( K_i \geq 0 \) for \( i \in I \). For each client \( j \in J \), and each facility \( i \in I \), a transportation cost \( c_{ij} \geq 0 \) is incurred for each unit of demand of client \( j \) if \( j \) assigned to facility \( i \). The demand \( d_j \geq 0 \) of each client \( j \in J \) is uncertain. In the first stage, the decision maker decides which facilities to open. In the second stage, upon the realization of the uncertain demand (say \( \tilde{d}_j \)), each client is allocated to one of the open facilities, so that their capacities are respected. The problem is to minimize the total cost, involving the facility installation cost (first stage) and the expected value of the transportation cost of clients to facilities (second stage). We study a risk-adverse version of this problem, replacing the expected value of the second stage with a risk-measure known as the Conditional Value-at-Risk (CVaR). Given a level of risk-aversion \( \sigma \), we search for an optimal subset of facilities to install so that their opening cost plus the expected value of the worst \( \sigma \)-quantile of the random distribution of the transportation cost is minimized.

Mathematical formulation: The FL-CVaR can be formulated as follows:

\[
\begin{align*}
\min_{x \in \{0,1\}^I} & \sum_{i \in I} f_i x_i + \text{CVaR}_\sigma \left[ Q(x, \xi) \right] \\
\sum_{i \in I} K_i x_i & \geq D
\end{align*}
\]  

where binary first-stage variables \( x_i \) indicate if the facility \( i \in I \) is installed or not. The only constraint to the problem ensures that there must be enough capacity installed to satisfy all the demand, where \( D := \max_{\xi \in \Omega} \{ \sum_{i \in J} d_j^\xi \} \). Given a set of installed facilities \( \hat{x} \), the second-stage problem minimizes the transportation cost between the clients and the facilities:

\[
\begin{align*}
Q(\hat{x}, \xi) := & \min_{y \in \mathbb{R}_+^{I \times J}} \sum_{i \in I} \sum_{j \in J} c_{ij} y_{ij} \\
\sum_{i \in I} y_{ij} & \geq d_j^\xi \quad j \in J \\
\sum_{j \in J} y_{ij} & \leq K_i \cdot \hat{x}_i \quad i \in I.
\end{align*}
\]

Here, the second-stage variables \( y_{ij} \) represent the units of demand of client \( j \in J \) assigned to facility \( i \in I \). Constraint \( \text{24b} \) indicates that all the demand of client \( j \) must be satisfied, and constraint \( \text{24c} \) enforces that the demand assigned to a given facility \( i \in I \) does not exceed its capacity \( K_i \) (if the facility \( i \) is installed).
or zero (if the facility is not installed). In case of a discrete set of scenarios \( S \), the risk-aversion measure CVaR with a given risk-level \( \sigma \) can be formulated as a linear program, see [Rockafellar et al. (2000)](#). That is, we can replace the objective function of the problem (23a) by

\[
(25) \min_{x \in \{0,1\}^I, \tau \in \mathbb{R}} \sum_{i \in I} f_i x_i + \tau + \frac{1}{1-\sigma} \sum_{s \in S} p^s Q(x, \tau, \xi^s)
\]

where the modified second-stage problem for a given \( s \in S \) reads as follows:

\[
(26a) \quad Q'(\hat{x}, \hat{\tau}, \xi^s) := \min_{y \in \mathbb{R}^{|I| \times |J|}, z \geq 0} \sum_{i \in I} \sum_{j \in J} c_{ij} y_{ij} \leq z + \hat{\tau}
\]

\[
(26b) \quad \sum_{i \in I} y_{ij} \geq d^s_j \quad j \in J
\]

\[
(26c) \quad \sum_{j \in J} y_{ij} \leq K_i \cdot \hat{x}_i \quad i \in I.
\]

The dual of this subproblem is as follows:

\[
(27a) \quad Q'(\hat{x}, \hat{\tau}, \xi^s) := \max_{\beta \in \mathbb{R}^{|J|}, \gamma \in \mathbb{R}^{|I|}} -\alpha \hat{\tau} + \sum_{j \in J} \beta_j d^s_j - \sum_{i \in I} \gamma_i K_i \hat{x}_i
\]

\[
(27b) \quad -\alpha c_{ij} + \beta_j - \gamma_i \leq 0 \quad i \in I, j \in J
\]

\[
(27c) \quad 0 \leq \alpha \leq 1.
\]

Benders cuts: Note that, given (23b), the subproblem (26) is always feasible, for any \( s \in S \) and any feasible first-stage decision \( \hat{x} \), so only Benders optimality cuts need to be constructed.

Given a first-stage solution \( (\hat{x}, \hat{\tau}) \) and a corresponding dual solution \( (\alpha^s, \beta^s, \gamma^s) \) of \( Q'(\hat{x}, \hat{\tau}, \xi^s) \) for each scenario \( s \in S \), the optimality cuts to be added to master problem (6) or (7) are:

(FL-CVaR multicut)

\[
-\alpha^s \hat{\tau} + \sum_{j \in J} \beta^s_j d^s_j - \sum_{i \in I} \gamma^s_i K_i \hat{x}_i \leq \theta^s \quad s \in S
\]

(FL-CVaR single-cut)

\[
-\left( \sum_{s \in S} p^s \alpha^s \right) \cdot \hat{\tau} + \sum_{j \in J} \sum_{s \in S} p^s \beta^s_j d^s_j - \sum_{i \in I} \left( \sum_{s \in S} p^s \gamma^s_i \right) K_i \hat{x}_i \leq \Theta.
\]

Given a partition \( P \) of \( S \), let \( d^P_j := \sum_{s \in P} p^s d^s_j/p^P \) be the expected aggregated demand of customer \( j \in J \) and let \( (\alpha^P, \beta^P, \gamma^P) \) be an optimal dual solution of the subproblem with demand \( d^P \). Hence, by Proposition 2 the following cuts are valid for the FL-CVaR problem:

(FL-CVaR adaptive-cuts)

\[
-p^P \left[ \alpha^P \hat{\tau} + \sum_{j \in J} \beta^P_j d^P_j - \sum_{i \in I} \gamma^P_i K_i \hat{x}_i \right] \leq \sum_{s \in P} p^s \theta^s \quad P \in \mathcal{P}.
\]
Refinement procedure: As before, we provide a corollary of Proposition 3 for this problem, which guide us on how to apply a proper refinement of $P$ for the FL-CVaR problem:

**Corollary 4.** Given a partition $P$ of the set $S$, the following condition guarantees that adding Benders adaptive-cuts for each $P \in P$ is sufficient to obtain an optimal solution of the FL-CVaR problem:

$$
\sum_{j \in J} \left[ \left( \sum_{s \in P} p^s d_j^s \right) \cdot \left( \sum_{s \in P} p^s \beta_j^s \right) \right] = 
\sum_{j \in J} \sum_{s \in P} p^s d_j^s \beta_j^s, \quad \text{for all } p \in P.
$$

Therefore, sufficient conditions for a given partition $P$ to satisfy this condition are, for example, when either $d^s = d^{s'}$ or $\beta^s = \beta^{s'}$ for all pair of scenarios $s, s' \in P$. So, for each $P \in P$ we refine the partition by solving the subproblem (27) for each $s \in P$ and group them into subsets where $\beta^s$ have the same value.

5. **Computational Experiments**

In this section we study computational performance of the proposed Benders adaptive-cuts method, for which we consider two strategies:

- **ADAPTIVE**: the Benders adaptive-cuts method described in Algorithm 1
- **ADAPTIVE-SINGLE**: the Benders adaptive-cuts method described in Section 3.3 in which we adapt Algorithm 1 by replacing the optimality adaptive-cuts (Step 12) by

$$
\sum_{P \in P(t)} p^P (h^P - T^P x)^\top \lambda^P \leq \sum_{s \in S} p^s \theta^s
$$

and by including this cut only when all aggregated subproblems (for all $P \in P$) are feasible. Additionally, we include the standard single cut $\sum_{s \in S} p^s (h^s - T^s x)^\top \lambda^s \leq \Theta$ into the master problem each time that we refine the partition $P(t)$ (Step 20).

Our computational study is conducted on three stochastic network flow problems presented in Section 4. We compare the Benders adaptive-cuts method against:

- **SINGLE**: the single-cuts Benders implementation described in Section 2.2
- **MULTI**: the multi-cuts Benders implementation described in Section 2.2
- **DE**: the deterministic equivalent problem reformulation given by (3), solved directly using an off-the-shelf solver, and
- **GAPM**: the GAPM proposed by Ramirez-Pico and Moreno (2022), where each iteration is solved using an off-the-shelf solver of the compact problem reformulation given by (9).

Algorithmic Considerations: None of the aforementioned algorithms is enhanced using valid inequalities, warm starts, stabilization or additional performance improvement strategies for decomposition algorithms. Besides, in all study cases the set of first-stage variables $X$ is a polytope. That way, the results are useful to identify the advantages and disadvantages of the three ADAPTIVE strategies, and measure the pure impact of the new methodology. Furthermore, the refinement of a partition $P$ (Line 20 from Algorithm 1) is adapted for each problem using ideas discussed in Section 4.
Finally, the maximum running time allowed for each of the above algorithms is set to 86,400 seconds (24 hours). All the methods were implemented using the Python programming language and using Gurobi v9.0.2 as off-the-shelf optimization solver with its default parameters. The codes and instances are available at https://github.com/borelian/AdaptiveBenders. All runs use four threads and 32 GB of RAM. The computers employed for experiments use CentOS Linux v7.6.1810 on x86_64 architecture, with four eight-core Intel Xeon E5-2670 processors and 128 GB of RAM.

5.1. Results for the CPP.

Dataset. Our benchmark set for the CPP consists of the electricity planning instances available at https://people.orie.cornell.edu/huseyin/research/sp_datasets/sp_datasets.html. They have $|V_L| = 20$ source nodes and $|V_R| = 50$ demand nodes. Furthermore, there are 10 resource constraints that limit the first-stage decisions. The stochastic demand is a discrete probability distribution that is pairwise independent for each demand node, with a total of more than $10^{42}$ possible scenarios. For this experiment, we were able to test instances with the following numbers of sampled scenarios: 1,000, 10,000, 100,000, and 1,000,000.

Results. To analyze the obtained results, we plot the percentage gap over time of the global lower bound $z_L$ obtained through the execution of our algorithm, i.e., $\text{gap} = (\text{OPT} - z_L)/|\text{OPT}|$, where $\text{OPT}$ refers to the optimal objective value for each instance. Figure 1 shows the gap (in log-scale) over time for the different implementations (in different line-styles and colors). In the case of the Adaptive and Adaptive-Single, we also plot a dot when a refinement of the partition is done. This figure shows that Adaptive not only outperforms the other methods, but also obtains better initial solutions in the early iterations. This is particularly evident for the largest instances, i.e., those with 100,000 and 1,000,000 samples, where the optimal solution is obtained much faster by Adaptive. Comparing the standard Benders’ strategies, Multi is faster than Single when $S = 1,000$, but this performance is reverted when the number of scenarios increases (which is due to the master problem being overloaded with a large number of cuts separated by Multi).
Nevertheless, neither Multi nor Single is able to deal with 1,000,000 scenarios. Not even the initial iteration can be solved by these two methods, due to the large time required to solve the 1,000,000 subproblems (recall that these two methods require all \(|S|\) subproblems to be solved in order to derive Benders cuts). On the contrary, Adaptive and Adaptive-Single are able to find the optimal solution because they require to solve only \(|P|\) subproblems in each separation phase. Given that Adaptive can be interpreted as a multi-cuts implementation with respect to the given partition \(P\), it is not surprising that its performance is superior to that of Adaptive-Single. Indeed, this can be explained by two factors: (a) in each iteration stronger lower bounds are obtained due to the generation of multiple cuts, and (b) the size of \(P\) remains small in the final iteration, so that the master problem is not overloaded with a large number of cuts (which is sometimes a disadvantage of the multi-cuts implementation, when applied to the whole set \(S\), see above).

As a confirmation of these statements, we have also summarized the outcomes concerning the number of added cuts and the growth of the partition size in Figure 2. In Figure 2a, we present the number of inserted optimality cuts over time, in log-scale. The superior performance of both Adaptive and Adaptive-Single on very large instances is tied to three factors: (1) their ability to generate violated optimality cuts faster than other methods (due to the aggregation of scenarios, which allows smaller number of subproblems to be solved, particularly in the initial iterations), (2) the fewer number of cuts required to reach the optimal solution, and (3) less subproblems to be solved even in the last iterations, where usually \(|P|\) remains much smaller than \(|S|\). By contrast, Multi requires more cuts, particularly for the largest instances, and Single adds fewer cuts but has a much slower convergence rate, requiring a longer time than Adaptive to reach optimality (but shorter than Multi for the largest instances). In Figure 2b the y-axis shows, as a percentage, the relative size of the partition \((|P|/|S|)\) over time. This plot shows that with 1,000 scenarios, the final partition has a size of 70% of \(|S|\), while for the remaining instances the relative size of the final partition keeps decreasing. For the largest instances, the final partition contains \(\approx 12,000\) scenarios, which is nearly 1.2% of the total number of scenarios. Hence, these instances demonstrate the great ability of Adaptive to keep small partitions over time, being able to find optimal
Table 1. Solving Time Required by the Different Methods for the CPP

| $|S|$ | Adaptive | Adaptive-Single | Multi | Single | GAPM | DE  |
|-----|---------|----------------|-------|--------|------|-----|
| 10  | 0.29    | 0.54           | 0.30  | 1.04   | 0.57 | 0.26|
| 100 | 2.42    | 7.00           | 3.22  | 4.75   | 6.73 | 2.41|
| 1,000| 10.4    | 43.1           | 34.9  | 47.1   | 20.1 | 30.0|
| 10,000| 89.7    | 397.9          | 435.2 | 426.0  | 172.6| 233.7|
| 100,000| 832.5  | 4253.9         | 28238.7 | 11416.4 | 685.5 | 2180.1|
| 1,000,000| 9475.4 | 44126.6        | –     | –      | 4710.3| –   |

Table 2. Solutions Found for the CPP

| $|S|$ | Optimal solution | Reported objective | True objective (95% confidence) |
|-----|-----------------|-------------------|---------------------------------|
| 10  | $[4,7,5,5,6,10,2]$ | -1527.96          | -1396.72 ± 0.80                |
| 100 | $[3,6,4,4,5,11,3]$ | -1446.82          | -1434.80 ± 0.70                |
| 1,000| $[4,6,4,5,11,3]$ | -1447.35          | -1438.23 ± 0.72                |
| 10,000| $[4,6,3,5,11,3]$ | -1438.45          | -1438.53 ± 0.71                |
| 100,000| $[4,6,3,5,11,3]$ | -1438.12          | -1438.53 ± 0.71                |
| 1,000,000| $[4,6,3,5,11,3]$ | -1438.72          | -1438.53 ± 0.71                |

solutions for instances with a huge number of scenarios that remain out of reach for standard implementations of Benders method.

Comparison with other methods. Finally, we compare the performance of Benders adaptive-cuts with that of the deterministic equivalent (DE) and the standard Generalized Adaptive-Partition Method (GAPM).

Table 1 presents the solving times (in seconds) required by each algorithm. All algorithms perform similarly for smaller instances. We notice that Adaptive outperforms the other Benders strategies for all instances. In addition, Adaptive has similar solving times with the deterministic equivalent formulation when the number of scenarios is smaller than 100. However, for larger instances, solving the DE becomes difficult, so that the largest instance cannot be solved within 24h. We also found that for the CPP the GAPM performs very well on large instances, because GAPM exploits the ability to keep small partitions over time, being faster than Adaptive on the largest ones.

To show the relevance of considering a large number of scenarios for this particular problem, Table 2 presents the optimal solution found for each problem instance obtained with a different number of scenarios. We also show the obtained objective values, and a 95% confidence interval of the true objective value for the found solutions. It can be seen that considering less than 10,000 scenarios leads to a different optimal solution in each case, which in fact are suboptimal compared to the solution found with more than 10,000 scenarios. Also, notice that the reported objective values of the problem differ considerably from the true objective value of the solution, and that at least 10,000 scenarios are needed in order to have a reported objective value inside the 95% confidence interval.

5.2. Results for the SMCF.

Dataset. We tested the algorithm on the linear version of Canad instances R from Crainic et al. (2001) available at http://groups.di.unipi.it/optimize/Data/MMCF.html#Canad, but with random demands generated as in Rahmanian.
et al. (2018). The instances originally proposed by Rahmaniani et al. (2018) contain up to 100 scenarios. Since our goal was to test the performance of the new method on much larger sets of scenarios, we also generated additional instances, following the same instance generation procedure. We focus on instances with $|V| = 10$ and $|E| = 60$, and with the number of commodities $|K| \in \{10, 25, 50\}$, referred to as R04 (small), R05 (medium) and R06 (large), respectively. Concerning the fixed costs and capacities, we focus on 5 different configurations from this dataset, denoted as l1, l3, l5, l7, l9, wherein the the values of $f$ and $u$ are uniform over all arcs and are given as $(f_{ij}, u_{ij}) \in \{(1, 1), (10, 1), (5, 2), (1, 8), (10, 8)\}$, respectively. Starting from a deterministic instance from this dataset, we generated 30 stochastic instances with random demands: five different levels of linear correlation ($\{0\%, 20\%, 40\%, 60\%, 80\%\}$) between the commodities are considered, and for each of these levels, seven instances are generated with $|S| \in \{16; 100; 1,000; 5,000; 10,000; 20,000; 50,000\}$. Overall, we obtained a dataset with a total of 525 instances to be solved.

Results. Following the results obtained for the CPP, and after some preliminary experiments, we decided to exclude Adaptive-Single from the computational comparison on the SMCF. Our preliminary experiments have shown that the performance of Adaptive-Single is inferior to alternative implementations of Benders method studied in this paper.

Due to the large number of instances, we first present the cumulative performance charts of solving all these instances with the different Benders strategies aggregated by the number of scenarios. In Figure 3, each line shows the percentage of the 75 instances (per group, for a given size of $|S|$, for more than 16 scenarios) solved to optimality over time for different number of scenarios. Each line corresponds to a different Benders method, namely Single in dot-dashed blue, Multi in dashed green and Adaptive in continuous red. It can be seen that Single is consistently
Table 3. Average Solving Times, Iterations and Number of Cuts of Each Method for Different Instances and Numbers of Scenarios

| Method | Scenarios | Average Solving Time (Time) | Iterations (Iter) | Number of Feasibility Cuts (FC) | Number of Optimality Cuts (OC) |
|--------|-----------|-----------------------------|------------------|-------------------------------|--------------------------|
| Adaptive | 5         | 25 1850.8 1579 1804 | 25 12.5 32 1004 | 25 10.7 32 997 | 25 14.0 32 1004 |
| Multi   | 5         | 25 27714.5 212647 379000 | 25 73.8 212647 379000 | 25 73.8 212647 379000 | 25 73.8 212647 379000 |
| Single  | 5         | 25 2319.0 7859 1067 | 25 12.5 32 1004 | 25 10.7 32 997 | 25 14.0 32 1004 |

The Table summarizes some important aspects, including the number of instances solved to optimality (#In), the average solving time required (Time) in minutes, the average number of iterations (Iter), the average number of feasibility (FC) and optimality (OC) cuts added. We present these values for the three considered methods and for different instance sizes and numbers of scenarios, with a total of 25 instances per row. We also report the relative slow-down of Adaptive Multi vs the Single and Multi Single, respectively, compared to Adaptive Multi for the same instance, and report the geometric mean over 25 instances. This factor is shown between parenthesis in the column Time. Finally, for Adaptive Multi, the table also shows the number of refinements of the partitions (#Ref).

We start by comparing the Single vs the Multi implementation. The Single approach requires more iterations, with a similar number of feasibility cuts and (as expected) a considerably smaller number of optimality cuts added. However, this reduction in the number of cuts does not help to solve more problems faster. In fact, Single can only solve all 25 instances per group when considering a small number of scenarios. On the contrary, Multi adds a large number of cuts per iteration, but solves all instances per group with up to 10,000 scenarios, and is overall faster than Single.

When we analyze these values for Adaptive Multi, we can see that Multi solves a similar number of instances, with shorter solving times for 16 and 100 scenarios, but it is between 1.2 to 2.4 times slower than Adaptive for larger number of scenarios. Note that Adaptive requires the most iterations, however each iteration of the master is solved faster, which is explained by the number of cuts added to the master problem. In fact, Adaptive requires considerably fewer feasibility cuts than do the other methods because feasibility cuts are added for the aggregated scenarios, not for each single scenario (see Section 3.4 for more details). Also the number of outperformed by the other methods (independently on the number of scenarios), and it is able to solve all instances only for 1,000 scenarios or less. Also, Adaptive clearly outperformed by the other methods (independently on the number of scenarios), and the computational advantage of Adaptive is more pronounced with the increasing number of scenarios. For 20,000 and 50,000 scenarios, Adaptive also solves more instances than the other two Bender’s methods.
Numerical results are presented in Figure 4 for different configurations of the Benders adaptive-cuts method. The figure shows the evolution of the gap over time (in log-scale) for the three methods: Adaptive, Multi, and Single. For simplicity, we present only those cases with no correlation and using 100 scenarios and 10,000 scenarios (the results are similar to the remaining configurations). For a small number of scenarios (Figure 4), we see that Adaptive and Multi yield similar results, being faster than Single, with a few cases where Adaptive reaches a smaller gap faster than either Multi or Single, but this is not significant given that most of the instances are solved in no more than 100 seconds. Conversely, for a large number of scenarios (Figure 5), we can see how Adaptive has a positive impact in the early stages, obtaining a lower gap much faster than the other two Benders methods.

Comparison with Other Methods. In Table 4, we compare the performance of Adaptive, DE, and GAPM. The table shows the number of instances solved to optimality for each configuration within a time-limit of 24 hours (#In), and the average time required to solve these instances (Av. Time) in seconds. We can see that Adaptive clearly outperforms the other two methods when the number of scenarios is large. In fact, while the other methods require a shorter time for the configurations with 16 and 100 scenarios, Adaptive is between two to four times faster when solving
Figure 5. Gap Over Time for Different Configurations (10,000 scenarios)

Table 4. Time Required by Adaptive-cut Benders Versus GAPM and the Deterministic Equivalent Formulation for the SMCF Problem

| Inst | Scen | Adaptive | GAPM | DE |
|------|------|----------|------|----|
|      |      | Av. Time | #In  | Av. Time | #In  | Av. Time | #In  |
| 16   | 17.1 | 25       |       | 2.9 (0.17) | 25   | 1.39 (0.08) | 25   |
| 100  | 28.3 | 25       |       | 15.7 (0.54) | 25   | 7.9 (0.28) | 25   |
| 1000 | 153.3| 25       |       | 215.2 (1.37) | 25   | 103.7 (0.68) | 25   |
| R04  | 5000 | 748.6    | 25    | 1730.5 (2.18) | 25   | 794.9 (1.02) | 25   |
| 10000| 1797.2| 25      |      | 5395.8 (2.68) | 25   | 2482.6 (1.26) | 25   |
| 20000| 4538.1| 25      |      | 19979.8 (3.70) | 25   | 9932.7 (1.71) | 23   |
| 50000| 18593.0| 25   |      | 57845.6 (2.92) | 8    | 15562.7 (0.71) | 1    |
| 16   | 66.6 | 25       |       | 6.7 (0.12) | 25   | 4.4 (0.08) | 25   |
| 100  | 102.4| 25       |       | 40.5 (0.51) | 25   | 19.5 (0.25) | 25   |
| 1000 | 407.7| 25       |       | 535.7 (1.43) | 25   | 264.8 (0.71) | 25   |
| R05  | 5000 | 2319.0   | 25    | 4672.2 (2.05) | 25   | 2790.0 (1.17) | 25   |
| 10000| 5583.0| 25     |      | 23609.6 (3.53) | 24   | 10656.2 (1.75) | 25   |
| 20000| 16563.1| 25    |      | 28052.9 (2.09) | 2    | 10138.9 (2.41) | 3    |
| 50000| 12351.0| 10    |      | – –               |       | – –               |       |
| 16   | 288.3| 25       |       | 14.5 (0.08) | 25   | 10.1 (0.05) | 25   |
| 100  | 357.3| 25       |       | 90.4 (0.38) | 25   | 42.7 (0.18) | 25   |
| 1000 | 1385.4| 25      |      | 1015.8 (0.96) | 25   | 517.9 (0.49) | 25   |
| R06  | 5000 | 7923.0   | 25    | 10942.7 (1.18) | 25   | 5252.9 (0.87) | 25   |
| 10000| 19306.2| 25     |      | 12765.6 (2.99) | 8    | 11793.2 (1.52) | 8    |
| 20000| 40475.0| 19     |      | 9082.5 (3.44) | 1    | – –               | 0    |
| 50000| 6357.0| 5        | –     | – –               |       | – –               | 0    |

the problems with more than 1,000 scenarios. Moreover, Adaptive can solve all instances from R04 and R05, and almost all instances of the R06 group with up to 20,000 scenarios. On the contrary, even if GAPM can solve more instances than DE, none of them is able to solve a single instance of R05 and R06 with 50,000
scenarios. This can be explained by the size of the problems that need to be solved for the alternative methods. In fact, even though the GAPM exploits the idea of scenario aggregation, in the final iterations the problems became too big if the final partition contains a large number of scenarios. Indeed, the final partition required by the GAPM in most of the cases is very large (97.6% in median), which prevents it from finding the optimal solution. On the contrary, ADAPTIVE manages to deal with such large partitions, as it draws the advantage of decomposition, and does not solve the final partition as a compact model, but uses a dynamic cut generation instead. This shows that the advantage of ADAPTIVE does not rely on finishing with a small partition (as we saw for the CPP experiments), making this method a promising alternative for a broad class of stochastic problems.

5.3. Results for the FL-CVaR problem.

Dataset. We use the classic capacitated warehouse location instances from OR-Lib (Beasley 1990) available at http://people.brunel.ac.uk/~mastjjb/jeb/orlib/capinfo.html. In particular, we tested the algorithm on the instances cap41-44, cap61-64 and cap71-74 which consider $|I| = 16$ facility locations and $|J| = 50$ clients. The only differences between these instances are the capacities and the installation costs of the facilities. For the random demands, we construct scenarios by sampling $d_j$ uniformly between 0 and the original demand $d_j$ of each instance. The number of sampled scenarios is $|S| \in \{100; 500; 1,000; 2,500\}$, and for each case we generate ten instances with different sampled demands. We set the risk-aversion level at $\sigma = 90\%$.

Computational implementation. Since we are dealing with a MIP problem, to avoid the excessive inclusion of cuts during the branch-and-bound process, we only add the corresponding Benders cuts when a new feasible incumbent solution has been found by the solver, i.e., using a lazy-cut callback. Also, we keep a global partition $\mathcal{P}$ over the whole branch-and-bound tree, which could be refined by the new incumbent solutions found on different branches.

Results. We compare the different Benders strategies (SINGLE, MULTI and ADAPTIVE) with the deterministic equivalent (DE) formulation of the problem. As before, we exclude ADAPTIVE-SINGLE due to its poor performance. Figure 6 shows the cumulative performance charts of the different strategies for the different number of scenarios. Each line shows the percentage of the 120 instances solved to optimality over time. More details are also presented in Table 5.

When $|S| = 100$, all strategies perform similarly solving all instances in less than 10 minutes. In this case, DE is faster than all Benders strategies. As expected, when
Table 5. Average Solving Times and Number of Cuts of Each Method for Different Instances and Numbers of Scenarios for FL-CVaR problems

| Inst | | ADAPTED | | MULTI | | SINGLE | | DE |
|------|------|-------|------|-------|------|-------|------|
|      | | Time | #Ref | OC  | Time | OC  | Time | OC  | Time |
| cap4x| 100 | 81.0 | 8.6 | 14726 | 201.8 | (2.73) | 24056 | 421.5 | (5.57) | 2856 | 13.2 | (0.20) |
|      | 500 | 361.5 | 15.1 | 53183 | 1223.9 | (3.30) | 129223 | 2285.7 | (6.68) | 3371 | 105.3 | (0.33) |
|      | 1000 | 833.1 | 18.2 | 106140 | 2363.9 | (3.04) | 229860 | 5037.7 | (6.82) | 3643 | 311.0 | (0.46) |
|      | 2500 | 2020.4 | 23.7 | 208466 | 7203.3 | (3.47) | 229660 | 13133.6 | (6.50) | 3818 | 1776.8 | (0.94) |
| cap6x| 100 | 63.9 | 9.0 | 10602 | 181.5 | (2.77) | 23881 | 495.6 | (8.10) | 3223 | 22.6 | (0.38) |
|      | 500 | 412.5 | 16.0 | 49268 | 1292.4 | (2.87) | 125685 | 2668.0 | (7.09) | 3745 | 338.2 | (0.88) |
|      | 1000 | 937.4 | 19.1 | 94975 | 2641.1 | (2.79) | 235090 | 5508.7 | (6.13) | 3785 | 1404.3 | (1.54) |
|      | 2500 | 2449.7 | 24.4 | 190433 | 9020.4 | (3.42) | 587653 | 13861.3 | (5.87) | 3612 | 11667.9 | (4.92) |
| cap7x| 100 | 28.8 | 9.2 | 5345 | 90.6 | (3.25) | 13696 | 337.9 | (12.2) | 2417 | 27.2 | (0.98) |
|      | 500 | 168.1 | 16.0 | 22455 | 580.6 | (3.57) | 60124 | 1875.8 | (11.5) | 2635 | 554.2 | (2.74) |
|      | 1000 | 354.2 | 20.0 | 42240 | 1319.3 | (3.88) | 138972 | 3452.2 | (10.1) | 2627 | 2453.0 | (5.56) |
|      | 2500 | 1098.1 | 24.6 | 95468 | 4074.4 | (4.04) | 324071 | 10227.2 | (9.59) | 2956 | 19975.8 | (14.4) |

the number of scenarios increases, DE becomes slower requiring considerably more time than Benders strategies to solve all instances, in particular for $|S| = 2, 500$. Interestingly, this increment on the solving times depends on the instance family. While DE is very efficient for $\text{cap41-cap44}$, even with a large number of scenarios, its performance is the worst among all methods for $\text{cap71-cap74}$ when $|S| = 2, 500$.

Among the different Benders strategies, SINGLE is the slowest and ADAPTED is the fastest for all instances and all the number of scenarios studied. It can be seen that the number of optimality cuts added by SINGLE is smaller, which can explain the slow convergence to the optimal solution. On the other hand, ADAPTED requires approximately half of the number of optimality cuts required by MULTI, so the master problems solved at each iteration are smaller, which can explain the better performance of ADAPTED. Note that compared with the other two problem studied above, ADAPTED requires more refinements of the partition to reach the optimal solution. This can be explained by the binary nature of the first-stage decision, because different incumbent solutions found during the branch-and-bound process of the solver refine the partition independently. Nevertheless, the size of the final partition does not grow too much. In fact, the final partition needed by ADAPTED to prove the optimality of the solution ranges between ≈ 22% of $|S|$ for 100 scenarios to ≈ 13% for 2,500 scenarios. This is somehow expected, because given the $\sigma = 90\%$ risk-level of the CVaR objective, there are roughly speaking ≈ 10% of scenarios that are the most relevant ones for finding the optimal solution.

To show the relevance of considering a large number of scenarios for this problem, we analyze the solutions found by the model. Recall that we sample ten different problems for each instance. When $|S| = 100$, almost all instances obtain different optimal solutions (up to 4 different optimal solutions for $\text{cap41-cap44}$). On the contrary, when $|S| = 2, 500$, almost all instances obtain the same optimal solution for the ten samples of the customers’ demand, showing that a large number of scenarios is required to correctly model the problem. A similar behavior is observed with respect to the objective value of the optimal solution found. Figure 7 shows the dispersion of the objective value among the ten sampled demands for each instance and for the different number of scenarios $|S|$. We conclude that adding more scenarios is not only required to consistently obtain the same optimal solution,
but it is also needed to obtain a better estimation of the true objective value of these solutions.

6. Concluding Remarks

We proposed a new Benders adaptive-cuts method based on the GAPM for two-stage stochastic problems. We conducted an extensive computational analysis to highlight both the strengths and weaknesses of the proposed approach, by focusing on stochastic network flow problems. The results show that the performance of Benders adaptive-cuts is considerably better than that of its two counterparts based on a separation of multi-cuts and single-cuts, respectively. This superior performance is particularly pronounced in the early iterations of the cutting plane algorithms. For the stochastic multicommodity flow problem, the Benders adaptive-cuts method tends to perform similarly to Benders multi-cuts in the long term because proving the optimality frequently requires almost complete disaggregation of the set of scenarios. This does not occur for the capacity planning problem, where the Benders adaptive-cuts method significantly outperforms the other two variants. Moreover, for very large instances of the CPP with a million of scenarios, the final partition does not exceed 2% of the total number of scenarios. For the FL-CVaR problem, Benders adaptive-cuts method can be an order of magnitude faster than the deterministic equivalent and the single-cuts approach, and it significantly reduces the number of cuts and the solving times of the multi-cuts approach as well, even for a small to moderate number of scenarios.

Overall, the Benders adaptive-cuts method is shown to outperform its multi-cuts and single-cuts counterparts, due to the following major factors: (1) its ability to generate violated optimality and feasibility cuts faster than the other two methods (due to the aggregation of scenarios, particularly in the initial iterations), and (2) the fewer number of cuts required to reach the optimal solution. The latter effect is amplified for problems whose size of the final partition remains relatively small compared to the total number of scenarios.

We recall that in our study we did not apply any of the standard Benders decomposition algorithmic enhancements, leaving open to the reader the possibility to apply our methodology to particular problems where acceleration techniques can increase the global efficiency of Benders decomposition. Several improvements can be applied to keep a partition of small size. For example, using dual stabilization or a similar technique to obtain similar duals in the case of degenerated problems, or
considering a small tolerance between duals to group them, or re-constructing the partition based on the last first-stage solutions obtained by the algorithm. When it comes to extending theory and methodology of our results, it would be interesting to study convex two-stage stochastic optimization problems (notably, with a non-linear but convex objective function in the recourse).

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