NON-LOCALIZABILITY AND ASYMPTOTIC COMMUTATIVITY

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ABSTRACT

The mathematical formalism commonly used in treating nonlocal highly singular interactions is revised. The notion of support cone is introduced which replaces that of support for nonlocalizable distributions. Such support cones are proven to exist for distributions defined on the Gelfand-Shilov spaces $S^{\beta}$, where $0 < \beta < 1$. This result leads to a refinement of previous generalizations of the local commutativity condition to nonlocal quantum fields. For string propagators, a new derivation of a representation similar to that of Källen-Lehmann is proposed. It is applicable to any initial and final string configurations and manifests exponential growth of spectral densities intrinsic in nonlocalizable theories.
1 Introduction

There are several reasons that make it desirable to improve mathematical tools used in the nonlocal theory of highly singular quantum fields with an exponential or faster high-energy behaviour. First of all string theory gives us new indications [1-3] that the concept of space-time manifold is only approximate and valid at length scales coarser than the Planck scale. Certainly, one or another kind of nonlocality arises at any attempt to unify quantum gravity with other fundamental interactions, however it is of great interest that just the exponential growth is characteristic of the spectral densities which occur in the Källen-Lehmann representations for string propagators with pointlike boundary conditions [4,5].

A second reason concerns the problem of formulating causality which is crucial for any nonlocal theory. Recall that the exponential bound on the off-mass-shell amplitudes has been found by Meiman [6] just from the microcausality considerations. For faster momentum-space growth, no definite criterion of macrocausality in terms of observables has been obtained as yet. However a mathematical analog of local commutativity has been found which ensures a number of important physical consequences for arbitrary high-energy behaviour. These include the existence of the unitary scattering matrix [7] and the polynomial boundedness of the scattering amplitudes within the physical region of variables [8,9]. It is surprising enough that the connection between spin and statistics and the $TCP$-invariance derived previously [10-12] for quasilocal fields can also be established [13-15] in the essentially nonlocalizable case when the holomorphy domain of vacuum expectation values becomes empty and radically new proofs are needed. Thirdly, the use of highly singular nonlocal form-factors [16] turns out to be effective for a phenomenological description of strong interactions [17-18].

The main purpose of the present paper is a more precise formulation of the above-mentioned generalization of the local commutativity condition. This is accomplished by adopting two ideas of the Sato-Martineau theory of hyperfunctions which can briefly be named complexification and compactification of space-time. This enables one to simplify significantly the mathematical techniques used in the papers devoted to nonlocal quantum fields and develop a convenient functional calculus effective beyond the theory of hyperfunctions [19].

The work is organized as follows. In Sec.2, we redefine in terms of complex variables the Gelfand-Shilov test function spaces that used in the nonlocal quantum field theory. In Secs.3 and 4 the key notions of a carrier-cone and support cone are introduced for nonlocalizable distributions. The proof of the existence of the support cones leads naturally to an asymptotic commutativity condition formulated in Sec.5 as a fall-off property of the field commutator in the spacelike directions. As an application, generalization of the Paley-Wiener-Schwartz theorem to the nonlocalizable case is presented in Sec.6. In Sec.7, a new derivation of the Källen-Lehmann representation for propagators of open and closed strings is proposed. It is essential that this derivation is applicable to arbitrary initial and final string configurations and shows the same character of causal and local properties as
that established in [4,5] under pointlike boundary conditions. These considerations are somewhat heuristic and serve as a motivation and background for the previous rigorous analysis. Sec.8 is devoted to concluding remarks.

2 Redefinition of the presheaf of Gelfand-Shilov spaces

In the framework of Wightman axiomatic approach [20], quantum fields are treated as operator-valued distributions over the Schwartz space $S$. Wider classes of quantum fields are obtained if $S$ is replaced by a dense subset with a finer topology. For instance, the Jaffe strictly local fields [21], whose vacuum expectation values are ultradistributions, are defined on the Gelfand-Shilov spaces $S^\beta$, where $\beta > 1$, with the Schwartz space being roughly $S^\infty$. The Meiman fields [6] are defined on $S^1$ and correspond to hyperfunctions.

For still wider classes of distributions the notion of support loses its sense and nonlocality comes into play. The space $S^{1,b}$ is adequate to the exponential high-energy behaviour characteristic of the quasilocal field theory investigated in [8,10-12]. Although exactly this behaviour shows itself in string theories, we wish to develop here a mathematical formalism covering so-called essentially nonlocalizable fields defined on $S^\beta$ with $\beta < 1$, which may be of advantage in deriving the TCP-invariance and the connection between spin and statistics by Lücke's method [13,14]. We recall [22] that $S^\beta$ consists of those complex valued infinitely differentiable functions that satisfy the inequalities

$$|x^k \partial^q \varphi(x)| \leq C_k b^{|q|} q^\beta q,$$

where $x \in \mathbb{R}^n$, the constants $C_k$ and $b$ depend on the function $\varphi$; $k$ and $q$ are multi-indices, and the standard notation of the theory of functions of several variables are used. Accordingly, $S^\beta$ can be made into a topological space by means of the projective limit $|k| \to \infty$ and the inductive limit $b \to \infty$, proceeding from the norms

$$||\varphi||_{N,b} = \sup_{|k| \leq N} \frac{|x^k \partial^q \varphi(x)|}{b^{|q|} q^\beta q}$$

It perhaps should be noted that $\sup_{|k| \leq N} |x^k|$ can be replaced here by $(1 + |x|)^N$. Besides $S^\beta$, it is advisable to use the spaces $S^\beta(O)$, with $O$ an open set in $\mathbb{R}^n$, which are defined in the same manner but with $x \in O$. Such a collection of spaces is customary referred to as a presheaf. For $\beta < 1$, every function belonging to $S^\beta(O)$ can be analytically continued into the whole of $C^n$ as an entire function whose order of growth less than or equal to $1/(1 - \beta)$, see [22]. We shall rewrite analogously the definition of $S^\beta(O)$ for conelike domains $O$ which will play a leading role below. For this purpose, we introduce one more functional space.

**Theorem 1.** Let $U$ be an open connected cone in $\mathbb{R}^n$ and $d(x,U)$ be the distance from the point $x$ to $U$. For $\beta < 1$, the space $S^\beta(U)$ is isomorphic to the space $E^\beta(U)$ of entire functions satisfying the inequalities

$$|x^k \varphi(x + iy)| \leq C_k \exp\{ |by|^{\beta'} + d(bx, U)^{\beta'} \}$$

(3)
where the designation $\beta' = 1/(1 - \beta)$ is used for brevity. (Note that $\beta' \geq 1$.)

Remark. The choice of the norm in $\mathbb{R}^n$ is unessential here because all these are equivalent and the inductive limit $b \to \infty$ is implicit. Furthermore $d(bx, U) = bd(x, U)$ since by a cone is meant one with its vertex at the origin.

Proof. Let us denote by $S^{\beta,b,N}(U)$ the Banach space with the norm $\| \cdot \|_{U,N,b}$. Owing to the assumption $\beta < 1$ the Taylor series expansion of $\varphi \in S^{\beta,b,N}(U)$ is convergent for all $z \in \mathbb{C}^n$ and since its center $\xi \in U$ can be taken arbitrarily, the behaviour of the analytic continuation is estimated by

$$\inf_{\xi \in U} (1 + |\xi|)^{-N} \sum_q b^q d(z - \xi)^q |q^q / q!$$

We replace the sum by the supremum with a slightly greater $b$. An easy evaluation shows that

$$\text{const} \cdot \exp\{|h'z|^\beta\} \leq \sup_q |z^q |q^q / q! \leq \exp\{|hz|^\beta\}$$

where $| \cdot |$ is the Lebesque $t^\beta$-norm, $h = e(e\beta')^{-1/\beta'}$ and $h'$ is arbitrarily near to $h$. The inequality $|z|^\beta \leq |2x|^\beta + |2y|^\beta$ enables the exponential be factorized. Further, let $\xi_0$ be a point of $U$ such that $d(x, U) = |x - \xi_0|$. Then

$$\inf_{\xi \in U} (1 + |\xi|)^{-N} \exp\{|x - \xi|^\beta\} \leq (1 + |\xi_0|)^{-N} \exp\{d(x, U)^\beta\}$$

If $x \in U$, then $\xi_0 = x$ and $d(x, U) = 0$, whereas for the points $x$ whose distance to $U$ does not exceed some $\theta > 0$, we have $|\xi_0| \geq (1 - \theta)|x|$. For the remainder points, the inequality

$$(1 + |\xi_0|)^{-N} \leq C_x(1 + |x|)^{-N} \exp\{d(\varepsilon x, U)^\beta\}$$

holds with arbitrarily small $\varepsilon > 0$. Combining our estimates, we see that $S^{\beta,b,N}(U)$ is continuously embedded into the Banach space $E^{\beta,b',N}(U)$ of entire functions, with the norm

$$\|\varphi\|_{U,N,b'} = \sup_z (1 + |x|)^N |\varphi(z)| \exp\{-|b'y|^\beta - d(b'x, U)^\beta\}$$

where $b' > 2hb$.

To prove the converse, let $\varphi \in E^{\beta,b,N}(U)$. Due to the inequality

$$1 + |\xi| \leq (1 + |x|)(1 + |x - \xi|)$$

we have

$$(1 + |x|)^{-N} \exp\{d(bx, U)^\beta\} \leq C_{N,b'} \inf_{\xi \in U} (1 + |x|)^{-N} \exp\{|b'(x - \xi)|^\beta\},$$

where $b'$ is arbitrarily near to $b$. Furthermore

$$\exp\{|x|^\beta + |y|^\beta\} \leq \exp\{2 \sum_j |z_j|^\beta\}$$

where $\beta = 1/(1 - \beta)$ is used for brevity. (Note that $\beta' \geq 1$.)
Let us fix \( \xi \in U \) and apply the Cauchy formula for the polydisk

\[
D = \{ z \in \mathbb{C}^n : |z_j - \xi_j| \leq r_j \}.
\]

Making use of (7), (8), we obtain

\[
(1 + |\xi|)^N |\partial^q \varphi(\xi)| = \frac{q^!}{(2\pi)^n} \left| \int_{\partial D} \frac{(1 + |\xi|)^N \varphi(z) \, dz}{(z - \xi)^{q+1}} \right| \\
\leq C_{N,N'} \| \varphi \|_{U,N,b} q! r^{-q} \exp \{ 2 \sum_j (b_j r_j)^{\beta'} \}
\]

Now one can exploit the freedom in the choice of \( r \) and take the lower bound, which gives

\[
(1 + |\xi|)^N |\partial^q \varphi(\xi)| \leq C_{N,N'} \| \varphi \|_{U,N,b} q! (2 b' e/h)^{q!} q^{-q/\beta'}.
\]

Finally we use the Stirling formula and conclude that \( E^{\beta,b,N}(U) \) is continuously embedded into \( \hat{S}^{\beta,2b'/h,N}(U) \). Taking the projective and inductive limits completes the proof.

### 3 Carrier cones

Now we introduce test function spaces associated with closed cones in \( \mathbb{R}^n \). Recall that by the projection of a cone is meant its intersection with the unit sphere and that a cone \( V \) is called a compact subcone of another cone \( U \) if \( \text{pr} V \subset \text{pr} U \). Then the notation \( V \in U \) is commonly used. Let \( B \) be a ball in \( \mathbb{R}^n \) centered about the origin. Given a closed cone \( K \), we set

\[
S^\beta(\mathbb{R}^n) = \lim_{\rightarrow U} S^\beta(U \cup B)
\]

where \( U \) runs over open cones such that \( K \in U \).

**Definition 1.** Let \( f \in (S^\beta)' \), \( \beta < 1 \). A closed cone \( K \subset \mathbb{R}^n \) will be said to be a carrier cone of \( f \) if this linear form is continuous under the topology induced on \( S^\beta \) by that of \( S^\beta(\mathbb{R}^n) \).

Let us dwell on the motivation of this definition. For \( \beta < 1 \), the space \( S^\beta(U \cup B) \) does not change with variations of the radius of the ball \( B \) whose role is only to provide connectedness. From the proof of Theorem 1, it is obvious that this space coincides with \( E^\beta(U) \). It is worthwhile to point out the particular case of the degenerate cone \( K = \{ 0 \} \). Then \( S^\beta(K) \) is identical to the space of entire functions satisfying the bound (3) with \( d(bx,U) \) replaced by \( |bx| \). The inductive limit (9) is actually taken over special neighbourhoods of \( K \) which can be regarded as traces of the neighbourhoods of its closure in the radially compactified space \( \hat{\mathbb{R}}^n \). Using such a compactification, we follow to the paper [23] where the local properties of the Fourier hyperfunctions was investigated. The point is that for them there exist smallest carriers among compact sets in \( \hat{\mathbb{R}}^n \), while there are no minimal carriers in \( \mathbb{R}^n \). We shall show that this important fact holds true for distributions on \( S^\beta, \beta < 1 \), providing we restrict ourselves by those compact sets that are
closures of cones. It should be noted that there is a natural one-to-one correspondence between the cones in $\mathbb{R}^n$ and the subsets of the compactifying sphere and that the family of spaces $E^\beta(U)$ may be thought of as a presheaf over this sphere which is really a sheaf in contrast to the initial $S^\beta(O)$. Keeping this motivation at the back of our mind, we prefer to say below about cones in $\mathbb{R}^n$ rather than about compact sets in $\mathbb{R}^n$. To this end, we shall smooth $f$ by convolution with test functions decreasing rapidly enough and we shall use one more class of Gelfand-Shilov space, defining them in terms of complex variables from the outset. Namely, the space $S^\beta(U)$, where $\alpha > 0, \beta < 1$, and $U$ is an open cone in $\mathbb{R}^n$, consists of all entire functions with the property that

$$|\varphi(z)| \leq C \exp\{|by|^{\beta'} + d(bx, U)^{\beta'} - |x/a|^{\beta'}\} \tag{10}$$

Here the constants $a, b,$ and $C$ depend on $\varphi$. Accordingly, $S^\beta_a(U)$ is equipped with the inductive limit topology by $a, b \to \infty$. We recall [22], that this space is only nontrivial provided $\alpha + \beta \geq 1$.

**Theorem 2.** Let $f \in (S^\beta)'$ and let $K$ be a carrier cone of $f$. Suppose that $\varphi \in S^\beta_a$. Then the convolution $(f * \varphi)(x) = (f(\xi), \varphi(x - \xi))$ belongs to every space $S^\beta_a(C)$ with $C$ an open compact subcone of $\hat{C}K$. Moreover the mapping $S^\beta_a \rightarrow S^\beta_a(C) : \varphi \rightarrow f * \varphi$ is continuous.

This theorem proven in [24] is valid for any $\beta \geq 0$, but here we would like to present for $\beta \leq 1$ an alternative proof using the isomorphism $S^\beta(K) = E^\beta(K)$. Let $\varphi \in S^\beta_{a,b}$. We proceed from the formula

$$|(f * \varphi)(z)| \leq \|f\|_{U,a',b'} \|\varphi(z - \xi)\|_{U,a',b'} \tag{11}$$

which holds for any $a', b'$ and each cone $U$ such that $K \subseteq U$. By definition of the norms involved in (11), we have

$$|\varphi(z - \xi)|_{U,a',b'} \leq \|\varphi\|_{a,b} \sup_{\zeta = \xi + in} \exp\{|b(y - \eta)|^{\beta'} - |(x - \xi)/a|^{\beta'}\}$$

$$- |b'y|^{\beta'} - d(b'\xi, U)^{\beta'} + |\xi/a|^{1/\alpha}.$$

Let us show that if $a', b'$ are sufficiently large and the cone $U$ is separated from $C$ by a nonzero angle distance, the right-hand side of this inequality can be majorized by

$$C \|\varphi\|_{a,b} \exp\{|2by|^{\beta'} + d(2b, C)^{\beta'} - |\theta x/a|^{1/\alpha}\} \tag{12}$$

In fact, in estimating the supremum over $\eta$ one can exploit convexity of the function $|\cdot|^{\beta'}$ and take $b' \geq 2b$. In considering the supremum over $\xi$, we assume at first that $x$ lies in the cone $C_\theta = \{x : d(x, C) \leq \theta |x|\}$. If $\theta$ is small enough, then

$$|x - \xi| \geq 2^\alpha \max(\theta |x|, \theta |\xi|) \quad \text{for all } x \in C_\theta, \xi \in U_\theta$$
and one can replace $|x-\xi|^{1/\alpha}$ by $|\theta x|^{1/\alpha} + |\theta\xi|^{1/\alpha}$. For $\xi \in U_\theta$, we have $|x/2|^{1/\alpha} \geq |x-\xi|^{1/\alpha} + |\xi|^{1/\alpha}$. Taking $a' > a/\theta$, $b' > 2^{\alpha}/(a\theta)$, we reveal the decrease $\exp\{-|\theta x/a|^{1/\alpha}\}$ within the cone $C_\theta$. If $x \in C_\theta$, then $|\xi|^{1/\alpha} \leq |x-\xi|^{1/\alpha} + |\xi/(1-\varepsilon)|^{1/\alpha}$ as follows from the elementary inequality $(\varepsilon u + (1-\varepsilon)v)^{1/\alpha} \leq u^{1/\alpha} + v^{1/\alpha}$ valid for positive $u$, $v$ due to the monotone behaviour of this function in $\varepsilon \in [0,1]$. Choosing $\varepsilon$ and $a'$ so that $\varepsilon/(1-\varepsilon) < ab\theta$, $\theta' > a/\varepsilon$, we obtain a growth no faster than $\exp\{|b\theta x|^{1/\alpha}\}$ outside $C_\theta$. It remains to notice that under the conditions $b > 1/a$, $|x| > 1/(b\theta)$ the function equal to $-|\theta x/a|^{1/\alpha}$ on $C_\theta$ and $|b\theta x|^{1/\alpha}$ elsewhere is bounded by the sum of the last two terms in the exponent (12).

In particular, setting $\alpha = 1 - \beta$, we obtain

$$|(f \ast \varphi)(x)| \leq \mathcal{C}_{a,b}(C) \|\varphi\|_{a,b} \exp\{-|\theta x/a|^{\beta'}\} \quad (x \in C \in \mathcal{C}K, \; \varphi \in S^{\beta,b}_{a,a})$$

This formula provides the basis for all the following study and makes evident the meaning of the generalization of local commutativity presented in Sec.5. In the next section, we shall show that this fall-off property is completely equivalent to the fact that $f$ is carried by the cone $K$.

4 Support cones

**Theorem 3.** Let $f \in (S^\beta)'$, $0 < \beta < 1$, and let $K$ be a closed cone in $\mathbb{R}^n$. Assume that for each cone $C \in \mathcal{C}K$ there exists a constant $\theta > 0$ such that the estimate (13) holds with any $a$, $b$. Then $K$ is a carrier cone of $f$.

**Proof.** We have to extend the linear form $f$ to every space. $S^{\beta,b'}(U) = \bigcap_N S^{\beta,b',N}(U)$ with $U$ an open cone such that $K \subset U$. The extension can be defined by the formula

$$(\hat{f}, \varphi) = \int (f(\zeta), \varphi_0(x-\zeta)\varphi(\zeta)) \, dx,$$

where $\varphi_0 \in S^{\beta,b}_1$, $\int \varphi_0(x)dx = 1$ and $a_0$, $b_0$ will be chosen below. Let us denote the integrand by $J(x)$, fix a cone $V$ so that $K \subset V \subset U$, and consider at first the behaviour of $J(x)$ inside $V$. For each $b$, $N'$ and some $N$ depending on $b$,

$$|J(x)| \leq \|f\|_{N,b} \|\varphi_0(x-\zeta)\varphi(\zeta)\|_{N,b} \leq$$

$$\leq \|f\|_{N,b} \|\varphi_0\|_{a_0,b_0} \|\varphi\|_{U,N',b'} \sup_{\xi,\eta}(1+|\xi|)^{N-N'} \times$$

$$\times \exp\{|b_0\eta|^{\beta'} - |(x-\xi)/a_0|^{\beta'} + |b_0\eta|^{\beta'} + d(b_\delta U,\xi)\beta' - |b_\delta|^{\beta'}\}$$

If $\xi \in U$, we have $|x-\xi|^{\beta'} \geq |\theta x|^{\beta'} + |\theta' \xi|^{\beta'}$ since the cone $\mathcal{C}U$ is separated from $V$ by a nonzero angle distance. For $\xi \in U$, we use the inequality $1 + |x| \leq (1 + |x - \xi|)(1 + |\xi|)$. Choosing $a_0 < \theta'/b'$ and $b > 2(b_0 + b')$, we see that $J(x)$ decreases within $V$ faster than any inverse power of $|x|$. In estimating this function outside $V$ we regard it as $f \ast \chi_x$, where $\chi_x(\zeta) = \varphi_0(\zeta)\varphi(x-\zeta)$, denote $\mathcal{C}V$ by $C$ and apply (13). This time we deal with the $S^{\beta,b}_{1-\beta,a}$-norm of $\chi_x$ which involves

$$\sup_{\xi}(1 + |x-\xi|)^{-N'} \exp\{|\xi/a_0|^{\beta'} + d(b'(x-\xi),U)^{\beta'} + |\xi/a|^{\beta'}\}. (16)$$
(We do not write out the supremum over $\eta$, assuming $b$ subject to the above condition.)

Using the inequality $d(\cdot, U) \leq |\cdot|$ and choosing $a_0 < 1/(2b')$, $a > 2a_0$, we find that $\|\chi_x\|_{a,b}$ increases no faster than $\exp\{|by|^\beta\}$. Hence, if $a < \theta/b$, then $J(x)$ decreases exponentially everywhere outside the cone $V$. Thus, the integral (14) is convergent under the condition $a_0 < \min(\theta', \theta/2)/\beta'$ and determines a linear form on $S^{\beta,b'}(U)$ whose continuity is ensured by the presence of the factor $\|\varphi\|_{U,N',b'}$ in our estimates. This distribution coincides with the initial one, when restricted to $S^\beta$. In fact, let $\varphi \in S^\beta$. Then the term $d(b'\xi, U)$ is absent from (15) and this estimate shows that not only $J(x)$ but also the function $(g(\zeta), \varphi_0(x - \zeta)\varphi(\zeta))$ is integrable for each $g \in (S^\beta)'$. Therefore the sequence of Riemann sums corresponding to the integral $\int \varphi_0(x - \zeta)\varphi(\zeta)dx$ is weakly fundamental in $S^\beta$ and, this space being Montel, converges in its topology to some element which is of necessity $\varphi$. Now we recall [22] that the space $S^{\beta,b}_{1-\beta,a}$ is only nontrivial provided $ab \geq \gamma$ where $\gamma$ depends on $\beta$. Hence different $\varphi_0$ occur in the formula (14) for different $b'$ and we have to make sure of consistency of all the extensions.

**Lemma.** Let $U$ be an open cone in $\mathbb{R}^n$ and $0 < \beta < 1$ as before. The space $S^\beta_{1-\beta}$ is dense in $S^\beta(U)$.

It perhaps should be recalled that we always add a ball $B$ to the cone to ensure connectedness. Let $\varphi \in S^{\beta,b'}(U)$ and let $\varphi_0 \in S^{\beta,b_0}_{1-\beta,a_0}$, where $a_0 = 1/(2b')$ and $b_0 = \gamma/a_0$. The estimate (16) with $\xi$ replaced by $x - \xi$ shows that, for each $x$, the function $\varphi_0(x - \xi)\varphi(\xi)$ belongs to $S^{\beta,b}_{1-\beta,a}$ with $a > 1/b'$ and $b > 2(b_0 + b')$. The above-mentioned sequence of Riemann sums lies in this space too. Moreover it is weakly fundamental in $S^{\beta,b}(U)$ as can be seen from a norm estimate similar to (15). Hence it converges to $\varphi$ in the topology of every space $S^{\beta,b}(U)$ with $b_1 > b$ since their intersection is a Montel space. This completes the proof of Theorem 3.

**Corollary.** For each distribution $f$ defined on $S^\beta$, $0 < \beta < 1$, there exist a smallest carrier cone which can be called the support cone of $f$.

**Proof.** Let us denote by $K$ the intersection of all the carrier cones and consider an open cone $C \subseteq \mathbb{R}^n$. The complements of the carrier cones form an open covering of $C$ from which one can choose a finite subcovering. Slightly shrinking the cones involved in the latter, we obtain covering cones where (13) is satisfied. Therefore this estimate holds everywhere in $C$ with some nonzero $\theta$ and so $K$ is a carrier too.

## 5 Asymptotic commutativity

Now we are in a position to formulate the following.

**Definition 2.** Let $\{A^{(\kappa)}\}$ be a finite or infinite set of nonlocal quantum fields defined on the test function space $S^{\beta}(\mathbb{R}^n)$, $0 < \beta < 1$, and transforming according to finite-dimensional representations of the proper orthochronous Lorentz group. We say Lorentz components $A^{(\kappa)}_j$ and $A^{(\kappa')}_j$ are asymptotically (anti)commute if for each vectors $\Phi$, $\Psi$ from their common domain in the Hilbert space, the distribution

$$f = \langle \Phi, [A^{(\kappa)}_j(x), A^{(\kappa')}_j(x')] \rangle \Psi$$

is carried by the cone $\nabla = \{(x, x') : (x - x')^2 \geq 0\}$ or, equivalently, fulfills the condition
(13) outside this cone. The principle of asymptotic commutativity means that every two field components asymptotically commute or anticommute. We assume that the type of this relation depends only on the type of the fields, \(i.e.,\) on \(\kappa, \kappa'\).

It should be mentioned that the term "asymptotic commutativity" has been introduced in Ref.[25] but with some other meaning. Namely, Lücke called nonlocal tempered fields \(A, A'\) \(\varphi\)-asymptotically commuting, where \(\varphi \in S(\mathbb{R}^n)\) and decreases rapidly enough, if the function

\[
F(x) = \left\| \int d\xi d\xi' [A(\xi), A'(\xi')]\varphi(x - \xi)\varphi(\xi)\Phi \right\|
\]

has finite the norm \(\|F\|_{\lambda, \varepsilon} = \sup\{|F(x)|\exp(\varepsilon|x|) : |x| > \lambda x^0\}\) for all \(\lambda, \varepsilon > 1\). Generalizations of the principle of local commutativity to fields breaking the Jaffe strict localizability condition [21] were considered by several authors [6-18, 26-28]. A formulation exploiting the presheaves of Gelfand-Shilov spaces and the requirement of continuity under an induced topology has been proposed in [26], but there the main attention was paid to the conditions under which this generalization keeps sense of microcausality. In [9], this approach was applied to nonlocal fields and the triviality of \(S^\beta_0\) for \(\alpha + \beta < 1\) was interpreted in such a way that the functions belonging to \(S^\beta\) enable one to test the causal properties to an accuracy of \(\exp\{-|x - x'|^{1/(1-\beta)}\}\). Theorems 2 and 3 confirm this view. Closely related definitions have been proposed independently in [7,27]. Bümmerstede and Lücke [7] suggested to replace the local commutativity by an axiom of "essential locality" which implies continuity of the field commutator under the topology of \(\lim \rightarrow S^\beta \cap S^{\beta, b}(\mathbb{V})(b \rightarrow \infty)\). Constantinescu and Taylor [27] introduced a classification of extensions of the field commutator outside the light cone. Namely, for a field \(A\) on \(S^{\beta_0}\), they defined the order of the extension to be the greatest \(\beta' = 1/(1-\beta)\) such that all the distributions \(\langle \Phi, [A(x), A(x')]\Psi \rangle\) can be extended to \(S^\beta(\mathbb{V})\). A comparison of these two formulations can be found in [28]. The papers [7,9,26-28] proceeded from the definition (1), (2) while Efimov [17] took advantage of the redefinition of \(S^\beta\) in terms of complex variables. The spaces (9) and the idea of compactification were used neither in our or other above-mentioned papers. From the technical standpoint the compactification procedure means taking an additional inductive limit over conelike neighborhoods of \(\nabla\). It is worthwhile to point out that the continuity under this topology is a weaker requirement than that of continuity under the topology of \(S^\beta(\mathbb{V})\). It is the compactification that enables one to derive Theorem 3, to establish the existence of support cones and to prove the equivalence of the two versions of asymptotic commutativity formulated as a support property and as a fall-off property of the field commutator.

6 Generalization of the Paley-Wiener-Schwartz theorem

It will be recalled that this theorem [29] relates the support properties of tempered distributions to the growth properties of their Laplace transforms. In [30], it has been extended to the ultradistributions defined on \(S^\beta_0, \beta > 1\). A generalization to the nonlocalizable case
\(\beta < 1\) was also formulated there, but without a proof. The results of Sec.4 enables us to present such a proof here. It is worthwhile to note that this generalization, interesting by itself, is needed to derive the connection between spin and statistics for nonlocalizable fields, see [13,15].

**Theorem 4.** Let \(K\) be a closed convex cone in \(R^n\) which does not contain a straight line, and let \(V\) be the interior of its dual cone \(K^* = \{q : qx \geq 0, x \in K\}\). Suppose that \(K\) is a carrier of \(K\) and let \(V\) be the interior of its dual cone \(V^*\). Let \(f \in (S^\beta)'\), \(0 < \beta < 1\). Then for the distribution \(f\) there exists a Laplace transform \(g(s)(s = p + iq)\) holomorphic in the tubular cone \(TV = R^n + iV\) and satisfying the inequality

\[
|g(s)| \leq C_\varepsilon(V')|q|^{-N}\exp\{|\varepsilon s|^{1/\beta}\} \quad (q \in V')
\]

for any \(\varepsilon > 0, V' \subseteq V\) and some \(N\) depending on \(\varepsilon, V'\). Conversely, if \(g\) is a function holomorphic in a tube \(T'V\), with \(V\) an open connected cone, and satisfying the growth condition (17), then it serves as the Laplace transform of a distribution \(f \in (S^\beta)'\) carried by the cone \(V^*\).

**Proof.** We shall show that \(e^{isz} \in S^\beta(K)\) for any \(q \in V = \text{int}K^*\) and then define the Laplace transformation by the formula

\[
g(s) = (f(z), e^{isz})
\]

After that the estimate (17) can be derived from the inequality

\[
|g(s)| \leq \|f\|_{U,N,b} \|e^{isz}\|_{U,N,b}
\]

where \(K \subseteq U, b\) is arbitrarily large and \(N\) depends on \(U, b\). We have

\[
\|e^{isz}\|_{U,N,b} = \sup_{z}(1 + |x|^N)\exp\{-qx - py - |by|^\gamma - d(bx, U)^\gamma\}
\]

The inclusion \(V' \subseteq V\) implies \(K \subseteq \text{int}V^*\). We choose the cone \(U\) and another open auxiliary cone \(U'\) so that \(K \subseteq U \subseteq U' \subseteq \text{int}V^*\). In evaluating the supremum over \(y\) we use the inequality \(\gamma y \leq |p||y|\) and obtain the factor \(\exp\{|\varepsilon p|^{1/\beta}\}\) where \(\varepsilon\) is arbitrarily small. Next we estimate the supremum over \(x\) inside and outside the cone \(U'\). There exists \(\theta > 0\) such that \(-qx \leq -\theta|q||x|\) for all \(q \in V', x \in U'\). Indeed, the function \(i_{V'}(x) = \sup\{-qx : q \in \text{pr}V'\}\) called the indicatrix of \(V'\) is continuous negative in \(\text{int}V^*\) and takes the maximal value \(-\theta\) on the compact set \(\text{pr}U\). This gives the factor \(\|q\|^{-N}\), while for \(x \in U'\) we have \(d(x, U) > \delta|x|, \delta > 0\) and so obtain the factor \(\exp\{|\varepsilon q|^{1/\beta}\}\). Further, it can be easily verified that the difference quotients corresponding to the partial derivatives \(\partial e^{isz}/\partial s_j\) converge to them in the topology of \(S^\beta(U)\) and so all the derivatives \(\partial g/\partial s_j\) exist. By the Hartogs theorem, this means that \(g\) is holomorphic in \(TV\).

Conversely, given a function \(g\) analytic in \(TV\) and satisfying (17), we define \(f \in (S^\beta)'\) by the formula

\[
(f, \varphi) = \int g(p + iq) \psi(p + iq) dp \quad (q \in V)
\]

\(^2\)The case of Fourier hyperfunctions \(\alpha = \beta = 1\) had been considered by Kawai [23].
where $\psi(p) = (2\pi)^{-n} \int \varphi(x)e^{-ipx}dx$. Taking into account that the Fourier operator is an isomorphism from $S^{\beta}_{1-\beta}$ onto $S^{1-\beta}_{\beta}$ (and maps $S^{\beta,b}_{1-\beta,a}$ into $S^{1-\beta,a'}_{\beta,b'}$ with $a',b'$ arbitrarily near to $a,b$) and using (10), (17), we see that the right-hand side of (20) is meaningful and independent of $q$. Then we consider the convolution

$$(f \ast \varphi)(x) = \int g(s) \psi(-s) e^{-isx} ds \quad (\text{Im} s = q \in V)$$

and exploit the latitude in the choice of $q$, which gives

$$|(f \ast \varphi)(x)| \leq C_{\varepsilon,b}(V') \|\psi\|_{a,b} \inf_{q \in V'} |q|^{-N} \exp\{\varepsilon q^{1/\beta} + |aq|^{1/\beta} + qx\}.$$ 

Evaluating the infimum under the additional condition $|q| \leq 1$, we obtain the following polynomial bound

$$|(f \ast \varphi)(x)| \leq C_{a,b} \|\psi\|_{a,b} (1 + |x|)^N$$

(21)

Here $N$ depends on $b$ since $\varepsilon$ must be sufficiently small compared to the number $1/b$ which characterizes the rate of decrease of $\psi$. The continuity of the Fourier operator allows $\|\psi\|$ to be replaced by $\|\varphi\|$ in the inequality (21). Applying the trick (14) again, we make sure that this restriction on the growth enables the distribution $f$ to be extended to the space $S^\beta$. Next we fix a cone $C \subseteq \overline{CV}$ and estimate the infimum over $q \in V', |q| \geq 1$. The cone $V'$ can be taken so that $C \subseteq \overline{CV'}$. Then there exists $\theta' > 0$ such that $qx \leq -\theta'|q||x|$ for all $q \in V', x \in C$. Namely, $\theta'$ is the minimal value of the indicatrix $i_{V'}(x)$ on the compact set $pr\overline{C}$. Finding this extremum, we arrive at the estimate (13) with $\theta$ arbitrarily near to $\theta'\beta(1-\beta)^{1-\beta}$. According to Theorem 3 and Lemma, $f$ has a unique extension to a distribution $\hat{f}$ on the space $S^\beta(V^*)$. The exponential $e^{isz}$ belongs to this space for any $\text{Im} s \in \text{ch}V$, where ch signifies the convex hull, since $V^{**} = \overline{\text{ch}V}$. Approximating the exponential by $\varphi_\nu \in S^\beta_{1-\beta}$ and passing to the limit $\nu \to \infty$ in (20), we see that the initial function $g$ and $\hat{f}$ are connected by the formula (18). This completes the proof.

**Corollary.** If $g$ is holomorphic in $TV$ and fulfills the condition (17), then it can be analytically continued into the tubular cone $T^{\text{ch}V}$ where an analogous estimate holds.

Note that if $\text{ch}V = \mathbb{R}^n$, then $V^* = \{0\}$ and the function $g$ is entire analytic. In this case the term $d(bx, U)$ in (19) should be replaced by $|bx|$ and an evaluation of extremum shows that (17) holds in the whole of $\mathbb{C}^n$ without the factor $|q|^{-N}$ of course.

### 7 Causality and Källen-Lehmann-like representations for propagators of open and closed strings

As shown in [12], the Källen-Lehmann representation for particle propagators in Minkowski space-time holds true up to the exponential growth of the spectral density: $\rho(M^2) \leq Ce^lM$. By the Meiman criterion, the theories with such a growth are nonlocalizable, with $S^{1,1/l}$ being the adequate test function space and $l$ playing the role of an elementary length. Analogous representations for string propagators with the pointlike boundary conditions were considered in [4,5]. As would be expected [32], the growth of their associated spectral
density turns out to be exponential with the fundamental length \( l_{\text{Pl}} \approx \sqrt{\alpha'} \). In investigating the string propagators, one usually employs the operator formalism or the Polyakov method \([32]\). In this section, we present another approach closely related to the secondary quantization. It is applicable to arbitrary initial and final string configurations and shows that the character of causal and singular properties of the string propagators is independent of the boundary conditions. We shall set forth the main steps of the proof for the case of open bosonic string and outline final results for closed ones. We shall follow the notation of \([34]\) and use in intermediate laying out the system of units for which \( \sqrt{2\alpha'} = 1 \).

Let us proceed from the following BRST-invariant expression for the Hamiltonian of the open bosonic string in \( D \)-dimensional space-time

\[
H = L_0 - 1 = \frac{1}{2}(p^2 + M^2), \quad M^2 = 2 \sum_{n=1}^{\infty} n (a_{-n} a_{n\mu} + c_{-n} b_n + b_{-n} c_n) - 2 \tag{22}
\]

(In fact we take \( D \) to be equal to 26.) Here \( p_\mu = -i\partial/\partial x_\mu \) is the momentum operator canonically conjugate to the string center mass position and playing a significant part in our approach. The operators \( a_{\pm n}^\mu \) are Fourier modes of the string coordinates \( X_\mu(\tau, \sigma) \) and \( c_{\pm n}, b_{\pm n} \) are ghost and antighst modes satisfying the usual anticommutation relations. Keeping in mind the zeta-function regularization, one might replace the normal ordering constant \(-2\) by \((D-2)\sum_{n=1}^{\infty} n\), restrict the summation in (22) by a large \( N \), and then treat \( H \) as a limit of Hamiltonians describing dynamic systems with finite number of degrees of freedom. This would give a more accuracy to the following study and put off the appearance of the tachyon but we will not go into such details for brevity. We use the Berezin holomorphic representation which is best suited to the Euclidean continual integration and normalize the corresponding measure to unity, \( \int d\omega = \prod_n \int \frac{da_{-n} da_n}{2\pi i} e^{-a_{-n} a_n} \int dc_{-n} db_n dc_n e^{-c_{-n} b_n - b_{-n} c_n} = 1 \).

The index \( \mu \) will henceforth be omitted and the notation \( A \) will be used for the collection of variables \( \{a_1, b_1, c_1, \ldots, a_n, b_n, c_n, \ldots\} \). Let us denote by \( \delta(x, A^*; x', A') \) the kernel of unity operator defined by

\[
\delta(x, A^*; x', A') = \delta(x - x') \sum_k \frac{(A^k)^* A^k}{k!} \tag{23}
\]

where \( k \) runs through the multi-indices which have only finite number of nonzero components \( k_n \), each being \((D+2)\)-tuple of nonnegative integers. Of course, the indices corresponding to the Grassmann variables \( b_n \) and \( c_n \) make a contribution to the sum only being equal to 0 or 1. The operation \( * \) consisting in the conjugation \( a_n \rightarrow a_{-n}, b_n \rightarrow c_{-n}, c_n \rightarrow b_{-n} \) implies the inversion of the order in the products. The Green function for the operator \( H \) is defined by

\[
HG(x, A^*; x', A') = \delta(x, A^*; x', A')
\]

where \( H \) acts on the former pair of variables, while the latter is regarded as parameters. One can readily verify that
\[ G(x, A^*; x', A') = \int \frac{dp}{(2\pi)^D} 2\pi D^2 \sum_k \frac{e^{ip(x-x')}}{p^2 + 2[k] - 2} \frac{(A^k)^* A'^k}{k!} \] (24)

where \([k]\) stands for \(\sum_n n|k_n|\). Using the summation over the spectrum of the mass operator, one may reduce this representation to the Källen-Lehmann form

\[ G(x, A^*; x', A') = \sum_{M^2} \int \frac{dp}{(2\pi)^D} 2\pi D^2 \rho(M^2; A^*, A') \]

where the spectral density matrix is determined by

\[ \rho(M^2; A^*, A') = \int \frac{dp}{(2\pi)^D} 2\pi D^2 \frac{\rho(M^2; A^*, A')}{p^2 + M^2} \]

Note that the \(\varphi\)-integration yields the Kronecker symbol since eigenvalues of \(M^2\) are even integers. From (26), one can immediately obtain an explicit expression for the spectral density

\[ \rho(M^2) = \text{Sp}(M^2; A^*, A) = \int d\varphi \rho(M^2; A^*, A) \]

in the case of pointlike boundary conditions. Namely,

\[ \rho(M^2) = \frac{1}{\varphi} \rho(M^2; A^*, A) = \frac{1}{\varphi} \left( \int d\varphi e^{\pi i \varphi (M^2 + 2)} \prod_n \frac{(A^k_n)^* A'^k_n}{k_n!} \right) \]

(25)

(26)

It perhaps is worthwhile to note that the integration over \(a_n, a_n^*\) gives the power \(-26\) which is reduced to \(-24\) by virtue of the integration over the Grassmann variables. Let us denote by \(f(e^{-2\pi i \varphi})\) the product that occurs in (27). Substituting the power series expansion \(f(w) = \sum m d_m w^m\) and performing the \(\varphi\)-integration, we obtain the equality \(\rho(M^2) = d_{M^2/2 + 1}\). The asymptotic behaviour of the coefficients \(d_m\) is well known [34, Sec.2.3.5] and we arrive at the formula

\[ \rho(M^2) \approx M^{-25/2} \exp(M/M_0) \] (28)

where \(M_0 = 1/(4\pi \sqrt{\alpha'})\). It should be pointed out that the representation (25) enables one to write the function \(\rho\) in another form

\[ \rho(M^2) = \sum_k \delta_{M^2+2}^{[k]} \]

(29)

expressing it as the degree of degeneracy of the mass operator eigenvalues. From this viewpoint the problem of deriving the formula (28) looks pure combinatorial. Thus in the
case of pointlike initial and final string configurations, the growth of the spectral density is indeed exponential. For more general boundary conditions, one can make use of the following simple facts. Let $R$ and $\rho$ be operators in a Hilbert space. Assume that $R$ is bounded, i.e., $\|R\| = \sup_{\varphi \neq 0} \|R\varphi\|/\|\varphi\| < \infty$, and $\rho$ is compact. Then the positive self-adjoint operator $(\rho^* \rho)^{1/2}$ is also compact. Let $\lambda_m$ be its eigenvalues. The nuclear norm is defined by $\|\rho\|_1 = \sum_m \lambda_m$ and a compact operator is called nuclear if this norm is finite. The inequalities $|Sp\rho| \leq \|\rho\|_1$ and $\|R\rho\|_1 \leq \|R\| \|\rho\|_1$ are valid. It is easy to see that in our case the nuclear norm of $\rho(M^2; A^*, A')$ is precisely the sum (29). It follows that the spectral density $Sp R \rho$ behaves exponentially as well for any bounded $R$ in the Källen-Lehmann-like representation.

The next step in studying the string propagator causal properties is passing to the Minkowski space-time by means of analytic continuation in $p_0$. Strictly speaking, this should be performed before the limit $N \to \infty$ which breaks down microcausality due to the appearance of the tachyon eigenvalue $-1/\alpha'$ in the spectrum of $M^2$. On doing so, one can see that the Källen-Lehmann representation for string propagators make no sense inside the hyperboloid $(x-x')^2 \approx \alpha'$ where singularities present themselves independently of the boundary conditions. It is natural to assume that this conclusion is unaffected by taking into account the string interaction. For the closed bosonic string, analogous formulae can be derived. This time, however, we deal with right and left modes, i.e., double of the variables and the $\delta$-symbol must satisfy the additional condition $(L_0 - \tilde{L}_0) \delta = 0$ because of which formula (24) is replaced by

$$G(x, A^*, \tilde{A}^*; x', A', \tilde{A}') = \int \frac{dp}{(2\pi)^D} 2 \sum_{k, \tilde{k}} \delta^{[k]}(p^2 + 4|k| - 4) \frac{(A^k \tilde{A}^\tilde{k})^* A^k \tilde{A}^\tilde{k}}{k! \tilde{k}!}$$

and the spectral density matrix acquires the form

$$\rho_{cl}(A^*, \tilde{A}^*; A', \tilde{A}') = \int_{-1/2}^{1/2} d\varphi e^{i\varphi(M^2+4)} \sum_{k, \tilde{k}} \delta^{[k]}(4\pi i\varphi[k]) \frac{(A^k \tilde{A}^\tilde{k})^* A^k \tilde{A}^\tilde{k}}{k! \tilde{k}!}$$

In this case, too, the exponential asymptotic behaviour takes place independently of the initial and final string configurations and so the causal and singular properties of the open and closed string propagators are practically identical. We would like to point out that in the one loop approximation the representation (29), making allowance for the constraint $L_0 = \tilde{L}_0$, leads immediately to the formally modular invariant expression [35] for the cosmological constant. The approach presented here can be extended to the open and closed fermionic Neveu-Schwarz-Ramond strings and the final results are qualitatively the same but their derivation will be discussed in some detail elsewhere.

8 Concluding remarks

In this paper, we have no attempt to prove the existence of support cones for those distributions which are defined on the space $S^0$ whose Fourier transform is the Schwartz
space $D$. Such a proof would be extremely desirable because this space consisting of functions with compact momentum-space support is universal for nonlocal quantum fields and one imposes no restrictions on the high-energy behaviour when incorporates it into the Wightman scheme. The problem is unfortunately complicated by triviality of the space $S_0^1$ which makes inapplicable the foregoing elementary methods aimed at clarity in presenting the concept of asymptotic commutativity. However the modern theory of functions of several complex variables enables one to solve it as we hope to show before long. The most interesting applications of the mathematical techniques developed here are improvements in Lücke’s derivations [13, 14] of the connection between spin and statistics and the TCP-invariance for essentially nonlocal quantum fields. This topic is covered in considerable detail in Ref. [36]. Of course, there are many unanswered questions concerning the existence of a minimum spatial resolution in the string theory whereas string interactions look local and this fact is commonly believed to be enough to preserve causality. One of the principal features of the superstring theory is the remarkable role of the modular invariance in the banishment of the ultraviolet divergences. For this reason the main omission of the above consideration seems to be lack of any examination how this invariance could affect the causal and singular properties of closed string propagators. However this is probably somewhat premature because so far there has been no consistent formulation of the closed string field theory developed in such a manner that the integration over the modular parameters is naturally restricted to the fundamental region of the modular group. Another question of great importance is the relation between the superstring theory and the effectively nonlocal quantum theory of fields with infinitely many components and arbitrarily high spins. These two approaches might turn out to be complementary standpoints on the evolution of Universe, describing its different phases, and this was one of the main inspirations of our investigation.

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