Path homology of directed hypergraphs

Yuri Muranov  \hspace{5em} Anna Szczepkowska
University of Warmia and Mazury  \hspace{5em} University of Warmia and Mazury
Olsztyn, Poland  \hspace{5em} Olsztyn, Poland

Vladimir Vershinin  
Université de Montpellier  
Montpellier, France

September 22, 2021

Abstract

We describe various path homology theories constructed for a directed hypergraph. We introduce the category of directed hypergraphs and the notion of a homotopy in this category. Also, we investigate the functoriality and the homotopy invariance of the introduced path homology groups. We provide examples of computation of these homology groups.

Keywords: path complex, directed hypergraph, path homology, digraph, homotopy of directed hypergraphs.

AMS Mathematics Subject Classification 2020: 18G90, 55N35, 05C20, 05C22, 05C25, 05C65, 55U05.

1 Introduction

Directed hypergraphs are the generalization of digraphs and have been widely used in discrete mathematics and computer science, see e.g. [1], [2], [5], and [7]. In particular, the directed hypergraphs give effective tools for the investigation of databases and structures on complicated discrete objects.

Recently, the topological properties of digraphs, hypergraphs, multigraphs, and quivers have been studied using various (co)homologies theories, consult e.g. [3], [4], [6], [10], [13], [14], [15], [16].
In this paper, we construct several functorial and homotopy invariant homology theories on the category of directed hypergraphs using the path homology theory introduced in [8], [10], [12], [13], and [14].

A rich structure of a directed hypergraph gives a number of opportunities to define functorially a path complex for the category of hypergraphs which we construct in the paper. We describe these constructions in Section 3. We introduce also a notion of a homotopy in the category of directed hypergraphs and describe functorial relations between homotopy categories of directed hypergraphs, digraphs, and path complexes.

The essential difference from the situation of the category of digraphs is the existence of the notion of the density of the path complex that we introduce for the two of the introduced path complexes of directed hypergraphs. This notion gives an opportunity to define a filtration on the corresponding path complex and hence a filtration on its path homology groups. We consider all homology groups with coefficients in a unitary commutative ring $R$.

In Section 2, we define a category of directed hypergraphs and introduce the notion of homotopy in this category.

In Section 3, we construct several path homology theories on the category of directed hypergraphs.

### 2 Path complexes and homotopy of directed hypergraphs

Let $\Pi = (V, P)$ be a path complex with $V = \{0, \ldots, n\}$ and $J = \{0, 1\}$ be a set, see [10], [13]. Define a path complex $\Pi' = (V', P')$ where $V' = \{0', \ldots, n'\}$ and $p' = (i_0' \ldots i_n') \in P'$ iff $p = (i_0 \ldots i_n) \in P$. We identify $V \times J = V \times \{0\} \amalg V \times \{1\}$ with $V' \amalg V'$. Define a path complex $\Pi^\uparrow = (V \times J, P^\uparrow)$ where

$$P^\uparrow = P \cup P' \cup P^\#,$$

$$P^\# = \{q^\# = (i_0 \ldots i_k i_{k+1}' \ldots i_n') | q = (i_0 \ldots i_k i_{k+1} \ldots i_n) \in P\}.$$

We have morphisms $i_\bullet : \Pi \to \Pi^\uparrow$ and $j_\bullet : \Pi \to \Pi^\uparrow$ that are induced by the natural inclusion $V$ onto $V \times \{0\}$ and onto $V \times \{1\}$, respectively.

**Definition 2.1** (i) A **hypergraph** is a pair $G = (V, E)$ consisting of a non-empty set $V$ and a set $E = \{e_1, \ldots, e_n\}$ of distinct and non-ordered subsets of $V$ such that $\bigcup_{i=1}^{n} e_i = V$ and every $e_i$ contains strictly more than one
element. The elements of $V$ are called vertices and the elements of $E$ are called edges.

(ii) A directed hypergraph $G$ is a pair $(V, E)$ consisting of a set $V$ and a set $E = \{e_1, \ldots, e_n\}$ where $e_i \in E$ is an ordered pair $(A_i, B_i)$ of disjoint non-empty subsets of the set $V$ such that $V = \bigcup_{e_i \in E}(A_i \cup B_i)$. The elements of $V$ are called vertices and the elements of $E$ are called arrows. The set $A = \text{orig}(A \to B)$ is called the origin of the arrow and the set $B = \text{end}(A \to B)$ is called the end of the arrow. The elements of $A$ are called the initial vertices of $A \to B$ and the elements of $B$ are called its terminal vertices.

For a finite set $X$ let $\mathcal{P}(X)$ denote as usual the power set. We define a set $\mathcal{P}(X) = \{\mathcal{P}(X) \setminus \emptyset\} \times \{\mathcal{P}(X) \setminus \emptyset\}$ consisting of ordered pairs of non-empty subsets of $X$. Every map of finite sets $f: V \to W$ induces a map $\mathcal{P}(f): \mathcal{P}(V) \to \mathcal{P}(W)$. For a directed hypergraph $G = (V, E)$, by Definition 2.1 we have the natural map $\varphi_G: E \to \mathcal{P}(V)$ defined by $\varphi_G(A \to B) = (A, B)$.

**Definition 2.2** Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two directed hypergraphs. A morphism $f: G \to H$ is given by a pair of maps $f_V: V_G \to V_H$ and $f_E: E_G \to E_H$ such that the following diagram

$$
\begin{array}{ccc}
E_G & \xrightarrow{\varphi_G} & \mathcal{P}(V_G) \\
\downarrow f_E & & \downarrow \mathcal{P}(f_V) \\
E_H & \xrightarrow{\varphi_H} & \mathcal{P}(V_H)
\end{array}
$$

is commutative.

Let us denote by $\mathcal{DH}$ the category whose objects are directed hypergraphs and whose morphisms are morphisms of directed hypergraphs.

For a directed hypergraph $G = (V_G, E_G)$, we can consider subsets $P_0(G) \subset P(V_G) \setminus \emptyset$, $P_1(G) \subset P(V_G) \setminus \emptyset$ and $P_{01}(G) = P_0(G) \cup P_1(G)$ by setting

$$
P_0(G) = \{A \in P(V_G) \setminus \emptyset \exists B \in P(V_G) \setminus \emptyset: A \to B \in E_G\},$$

$$
P_1(G) = \{B \in P(V_G) \setminus \emptyset \exists A \in P(V_G) \setminus \emptyset: A \to B \in E_G\}.$$

**Definition 2.3** Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be directed hypergraphs. We define the box product $G \boxtimes H$ as a directed hypergraph with the set of vertices $V_{G \boxtimes H} = V_G \times V_H$ and the set of arrows $E_{G \boxtimes H}$ consisting of the union of arrows $\{A \times C \to B \times C\}$ with $(A \to B) \in E_G$, $C \in P_{01}(H)$ and $\{A \times C \to A \times D\}$ with $(C \to D) \in E_H$, $A \in P_{01}(G)$.
Every connected digraph $H = (V_H, E_H)$ can be considered as a directed hypergraph with the same set of vertices and of a set of arrows of the form $\{v\} \to \{w\}$ with $v \to w \in E_H$. Hence, Definition 2.3 gives naturally a box product $G \square H$ of a directed hypergraph $G$ and a connected digraph $H$. Note that a line digraph $I_n$ defined for example in [11, Sec. 3.1] is connected and that we have two digraphs $I_1$, namely $0 \to 1$ and $1 \to 0$.

**Definition 2.4**

i) Two morphisms $f_0, f_1 : G \to H$ of directed hypergraphs are called **one-step homotopic** if there exists a line digraph $I_1$ and a morphism $F : G \square I_1 \to H$, such that

$$ F\big|_{G \square \{0\}} = f_0 : G \square \{0\} \to H, \quad F\big|_{G \square \{1\}} = f_1 : G \square \{1\} \to H. $$

If the appropriate morphism $F$ called a one-step homotopy exists, we write $f_0 \simeq_1 f_1$.

ii) Two morphisms $f, g : G \to H$ of directed hypergraphs are called **homotopic**, which we denote $f \simeq g$ if there exists a sequence of morphisms $f_i : G \to H$ for $i = 0, \ldots, n$ such that $f = f_0 \simeq_1 f_1 \simeq_1 \cdots \simeq_1 f_n = g$.

iii) Two directed hypergraphs $G$ and $H$ are **homotopy equivalent** if there exist morphisms $f : G \to H$ and $g : H \to G$ such that $fg \simeq \text{Id}_H$ and $gf \simeq \text{Id}_G$. In such a case, we write $G \simeq H$ and call the morphisms $f, g$ **homotopy inverses** of each other.

**Proposition 2.5**

Two morphisms $f, g : G \to H$ of directed hypergraphs are homotopic if and only if there is a line digraph $I_n$ with $n \geq 0$ and a morphism $F : G \square I_n \to H$ such that $F\big|_{G \square \{0\}} = f_0 : G \square \{0\} \to H, \quad F\big|_{G \square \{n\}} = g : G \square \{n\} \to H$. ■

The relation "to be homotopic" is an equivalence relation on the set of morphisms between two directed hypergraphs, and homotopy equivalence is an equivalence relation on the set of directed hypergraphs. Thus, we can consider a category $h\mathcal{DH}$ whose objects are directed hypergraphs and morphisms are the classes of homotopic morphisms. We shall call the category $h\mathcal{DH}$ by **homotopy category of directed hypergraphs**.

### 3 Path homology of directed hypergraphs

#### 3.1 k-connective path homology

For a directed hypergraph $G = (V, E)$ and $c = 1, 2, 3, \ldots$ define a path complex [13 S3.1] $\mathcal{C}^c(G) = (V^c, P^c_G)$ where $V^c = V$ and a path $(i_0 \ldots i_n) \in P^c_V$
lies in $P^c_G$ iff for any pair of consequent vertices $(i_k, i_{k+1})$ of the path, we have $i_k = i_{k+1}$ or there are at least $c$ different edges $e_1 = (A_1 \rightarrow B_1), \ldots, e_c = (A_c \rightarrow B_c)$ such that the vertex $i_k$ is the initial vertex and the vertex $i_{k+1}$ is the terminal vertex of every edge $e_i$. The number $c$ is called the density of the path complex $C^c(G)$. It is clear that we have a filtration

$$C(G) = C^1(G) \supset C^2(G) \supset C^3(G) \supset \ldots \quad (3.1)$$

**Proposition 3.1** For every morphism of directed hypergraphs $f: G \rightarrow H$ define a morphism

$$C(f) = (f^1_V, f^1_p): C(G) \rightarrow C(H)$$

of path complexes putting $f^1_V = f_V$ and $f^1_p = f_p|_{P^c_G}: P^1_G \rightarrow P^1_H$ where $f_p$ is defined by $f_p(i_0\ldots i_n) = (f(i_0)\ldots f(i_n))$. Then we have the functor $C$ from the category $\mathcal{DH}$ of directed hypergraphs to the category $\mathcal{P}$ of path complexes.

The functor $C$ provides the functorial path homology theory on the category $\mathcal{DH}$ of directed hypergraphs. For any directed hypergraph $G$ and $k \in \mathbb{N}$, we set $H^c_k(G) = H_n(C^k(G))$ as regular path homology groups of path complex $C^k(G)$, see [10, S2]. We denote $H^c(G) = H_{1}^c(G)$.

We call these homology groups the connective path homology groups and for $k \geq 2$ the $k$-connective path homology groups of the directed hypergraph $G$, respectively. The connective path homology theory is functorial by Proposition 3.1. However the $k$-connective homology theory $H^c_k(G)$ is not functorial for $k \geq 2$ as it follows from Example 3.2 below. For any directed hypergraph $G$ the filtration in (3.1) induces homomorphisms

$$H^c_n(G) = H^{c(1)}_n(G) \leftarrow H^{c(2)}_n(G) \leftarrow H^{c(3)}_n(G) \leftarrow \ldots .$$

Let $\mathcal{D}$ be a category of digraphs without loops [11, S2]. A category $\mathcal{G}$ of graphs is defined similarly [11, S6].

Let $G = (V, E)$ be a directed hypergraph. Define a digraph $\mathfrak{G}(G) = (V^d_G, E^d_G)$ where $V^d_G = V$ and an arrow $v \rightarrow w$ lies in $E^d_G$ iff there is a hyperedge $(A \rightarrow B) \in E$ such that $v \in A, w \in B$.

**Example 3.2** i) Let $G = (V, E)$ be a directed hypergraph such that $V$ is the union $A \cup B$ of two non-empty sets with empty intersection and the set $E$ consists of one element $e = (A \rightarrow B)$. Then $\mathfrak{G}(G)$ is a complete bipartite digraph with arrows from vertices lying in $A$ to vertices lying in $B$.  

5
The path complexes
For every directed hypergraph
Theorem 3.3
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D
category
pergraphs to the category
D
was constructed in [9], [15]. It is based on the natural funct or
of directed hypergraphs. However the map
f
defined. Thus we have a functor
G
f
we have (C
from
G
f
fine a map
G
V
with
V
notations of the connective path homology groups
H
Example 3.4
The following example illustrates the technique of compu-
tations of the connective path homology groups
H
∗
G
R
G
G
generated by all directed edges of the digraph
homology of the chain complex Ω
∗
∂
G
We have
R
G
of coefficients. Consider a hypergraph
G
V
with
V
\{1, 2, 3, 4\}, \{1\} \rightarrow \{2\}, \{2\} \rightarrow \{3\}, \{1\} \rightarrow \{2\}, \{2\} \rightarrow \{3\}, \{2\} \rightarrow \{4\}
We compute homology of the path complex \( C(G) \) as in [10]. We have \( R_{ij}^{reg} = \{1, 2, 3, 4\} = \Omega_0, \ R_1^{reg} = \{e_{12}, e_{13}, e_{23}, e_{24}, e_{41}\}. \) We get \( \partial(e_{ij}) \in R_0^{reg} \) for all basic elements \( e_{ij} \in R_1^{reg} \), so \( \Omega_1 = R_1^{reg}. \) Thus, \( \Omega_1 \) is generated by all directed edges of the digraph \( \mathcal{G}(G) \) presented below

\[
\begin{array}{c}
\cdot & \rightarrow & \bullet \\
\cdot & \leftarrow & \bullet \\
\cdot & \leftarrow & \bullet \\
\cdot & \rightarrow & \bullet
\end{array}
\]
From the definition of \( \mathcal{G}(G) \), it follows that \( \Omega_i = 0 \) for \( i \geq 2 \) and the homology of the chain complex \( \Omega_n \) coincides with the regular path homology \( \mathcal{G}(G) \). Hence \( H_0^{\mathcal{G}(1)}(G) = H_1^{\mathcal{G}(1)}(G) = \mathbb{R} \) and \( H_i^{\mathcal{G}(1)}(G) = 0 \) for \( i \geq 2 \).
For $H^c_i(G)$, we have $\Omega_0 = \{1, 2, 3, 4\}$ and by definition $\Omega_1 = \{e_{12}, e_{23}, e_{24}\}$. Moreover, $\Omega_i = 0$ for $i \geq 2$. Thus, homology groups $H^c_i(G)$ coincide with the homology groups of the digraph which has the set of vertices $V_\emptyset(G)$ and the set of arrows obtained from $E_\emptyset(G)$ by deleting arrows $(1 \rightarrow 3)$ and $(4 \rightarrow 1)$. Hence, $H^c_0(G) = \mathbb{R}$ and $H^c_1(G) = 0$ for $i \geq 1$. For $k \geq 3$ we have $H^c_0 = \mathbb{R}^4$ and $H^c_1 = 0$ for $i \geq 1$.

**Lemma 3.5** Let $G = (V, E)$ be a directed hypergraph and $I_1 = (0 \rightarrow 1)$. We have a natural isomorphism $\mathcal{C}(G \square I_1) \cong [\mathcal{C}(G)]^\uparrow$ of path complexes.

**Proof.** By Definition 2.3, a directed hypergraph $G \square I_1 = (V_{G \square I_1}, E_{G \square I_1})$ has the set of vertices $V_{G \square I_1} = V \times I = V \times \{0, 1\}$ which we identify with $V \cup V'$, where $V = \{0, \ldots, n\}$, $V' = \{0', \ldots, n'\}$ and the set of edges $E_{G \square I_1}$ is the union $E^0 \cup E^1 \cup E^0_{\text{of}}$ of sets $E^0 = \{A \times \{i\} \rightarrow B \times \{i\}\}$ with $(A \rightarrow B) \in E_G$ for $i = 0, 1$ and $E^0_{\text{of}} = \{C \times \{0\} \rightarrow C \times \{1\}\}$ with $C \in S_0(G)$. Let $q = (i_0 \ldots i_n)$ be a path lying in $\mathcal{C}(G \square I_1)$. It follows from definition, that there are only three possibilities, namely

1. all the vertices $i_j \in V \times \{0\}$ and, hence, $q$ determines the unique path in in $\mathcal{C}(G)$,
2. all the vertices $i_j \in V \times \{1\}$ and, hence, $q$ determines the unique path in in $[\mathcal{C}(G)]'$,
3. there exists exactly one pair $(i_k, i_{k+1})$ of consequent vertices in $q$ such that $i_k \in C \times \{0\}, i_{k+1} \in C \times \{1\}$ for $C \in S_0(G)$.

Thus, the union of paths from (1)-(3) on the set of vertices $V \times J$ defines the path complex $[\mathcal{C}(G)]^\uparrow$ and vice versa. □

**Theorem 3.6** For a directed hypergraph $G$, the connective path homology groups $H^c(G)$ are homotopy invariant.

**Proof.** By Definition 2.4, it is sufficient to prove homotopy invariance for a one-step homotopy. Then the result follows from Lemma 3.5 and [10] Th. 3.4. □

### 3.2 Bold path homology

Let $p = (i_0 \ldots i_n)$ and $q = (j_0 \ldots j_m)$ be two paths of a path complex $\Pi$ with $i_n = j_0$. The concatenation $p \vee q$ of these paths is a path given by $p \vee q = (i_0 \ldots i_n j_1 \ldots j_m)$. The concatenation is well defined only if $i_n = j_0$.

For a directed hypergraph $G = (V, E)$, define a path complex $\mathcal{B}(G) = (V^b_G, P^b_G)$ where $V^b_G = V$ and a path $q = (i_0 \ldots i_n) \in P_V$ lies in $P^b_G$ iff there
is a sequence of hyperedges \((A_0 \to B_0), \ldots, (A_r \to B_r)\) in \(E\) such that \(B_i \cap A_{i+1} \neq \emptyset\) for \(0 \leq i \leq r - 1\) and the path \(q\) has the presentation

\[
(p_0 \lor v_0 w_0 \lor p_1 \lor v_1 w_1 \lor p_2 \lor \cdots \lor p_r \lor v_r w_r \lor p_{r+1})
\]

(3.2)

where \(p_0 \in P_{A_0}, p_{r+1} \in P_{B_r}, v_i \in A_i, w_i \in B_i, p_i \in P_{B_{i-1}} \cap P_{A_i}\) for \(1 \leq i \leq r\) and all concatenations in (3.2) are well defined. Note, that in the case of an empty sequence of edges \(A_i \to B_i\) every path \(q \in P_A\) and every path \(q \in P_B\) for an edge \(A \to B\) lies in \(P_{G}^{b}\).

**Proposition 3.7** Let \(f : G = (V_G, E_G) \to H = (V_H, E_H)\) be a morphism of directed hypergraphs. Define a morphism of path complexes

\[
\mathcal{B}(f) = (f_b^b, f_p^b) : (V_G^b, P_G^b) \to (V_H^b, P_H^b)
\]

by \(f_V^b = f : V_G^b = V_G \to V_H = V_H^b\) and \(f_p^b = f_p|_{P_G^b}\), where \(f_p\) is defined as in Proposition 3.6. Thus, we obtain a functor \(\mathcal{B}\) from the category \(\mathcal{DH}\) of directed hypergraphs to the category \(\mathcal{P}\) of path complexes.

Let us define the **bold path homology groups** of directed hypergraph \(G\) by \(H_s^b(G) = H_s(\mathcal{B}(G))\). By Proposition 3.7 we obtain a functorial path homology theory on the category \(\mathcal{DH}\) of directed hypergraphs.

**Example 3.8** Let \(G = (V, E)\) be a directed hypergraph such that for every edge \(e = (A \to B) \in E\) the sets \(A\) and \(B\) are one-vertex sets, \(A = \{v\}, B = \{w\}\), \(v, w \in V\). We can consider the hypergraph \(G\) as a digraph and \(H_s^c(G) \cong H_*^c(G)\). On the category of connected digraphs that can be considered as the subcategory of directed hypergraphs, the bold path homology groups are naturally isomorphic to the connective path homology groups and to the regular path homology groups \(H_s(G)\) defined in [9].

**Example 3.9** Now we compute the bold path homology groups \(H_s^b(G)\) of the directed hypergraph \(G\) from Example 3.4 in dimensions 0, 1, 2 for \(R = \mathbb{R}\). First, we describe the modules \(R_n^\text{reg}(\mathcal{B}(G))\) for \(0 \leq n \leq 4\). We have

\[
R_0^\text{reg} = \langle e_1, e_2, e_3, e_4 \rangle, \quad R_1^\text{reg} = \langle e_{12}, e_{13}, e_{23}, e_{24}, e_{32}, e_{34}, e_{42}, e_{43} \rangle,
\]

\[
R_2^\text{reg} = \langle e_{123}, e_{124}, e_{132}, e_{234}, e_{241}, e_{243}, e_{323}, e_{343}, e_{412}, e_{413} \rangle,
\]

\[
R_3^\text{reg} = \langle e_{1232}, e_{1234}, e_{1241}, e_{1243}, e_{1323}, e_{1323}, e_{2343}, e_{2412}, e_{2413}, e_{2434}, e_{3232}, e_{3434}, e_{4123}, e_{4124}, e_{4132} \rangle,
\]

\[
R_4^\text{reg} = \langle e_{12323}, e_{12343}, e_{12412}, e_{12413}, e_{12434}, e_{13232}, e_{23232}, e_{23434}, e_{24123}, e_{24132}, e_{24343}, e_{32323}, e_{34343}, e_{41234}, e_{41243}, e_{41323} \rangle.
\]
\[ \Omega_n = \mathcal{R}_n^{\text{reg}} \text{ for } n = 0, 1. \]

Thus \( \Omega_0 \) is generated by all the vertices and \( \Omega_1 \) is generated by all directed edges of the digraph \( H \) on Fig. 1.

![Figure 1: The digraph \( H \).](image)

As it follows from the path homology theory of digraphs, the rank of the image \( \partial : \Omega_1 \to \Omega_0 \) is equal to 3, the rank of the kernel \( \partial \) is equal to 5, and hence \( H^b_0(G) = \mathbb{R} \).

By the direct computation \( \Omega_2 \) is the vector space with the following basis: \( \{e_{123}, e_{132}, e_{234}, e_{243}, e_{323}, e_{343}, e_{434}, e_{413}\} \). In this basis the matrix of homomorphism \( \partial : \Omega_2 \to \Omega_1 \) has the form:

\[
\begin{pmatrix}
e_{12} & e_{13} & e_{23} & e_{24} & e_{32} & e_{34} & e_{43} & e_{41} \\
e_{123} & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\
e_{132} & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\
e_{232} & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
e_{234} & 0 & 0 & 1 & -1 & 0 & 1 & 0 \\
e_{243} & 0 & 0 & -1 & 1 & 0 & 0 & 1 \\
e_{323} & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
e_{343} & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
e_{434} & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
e_{413} & 0 & 1 & 0 & 0 & 0 & -1 & 1
\end{pmatrix}
\]

Its rank is equal to 5. Hence the rank of the image of \( \partial \) is equal to 5, the rank of the kernel \( \partial \) is equal to 4, and hence \( H^b_1(G) = 0 \).

We have \( \Omega_3 = \langle e_{1232}, e_{1323}, e_{2323}, e_{2343}, e_{2434}, e_{3232}, e_{3434}, e_{4343} \rangle \). Similar to the previous calculation, the rank of the image \( \partial : \Omega_3 \to \Omega_2 \) is equal to 4, the rank of the kernel \( \partial \) is equal to 4, and hence \( H^b_2(G) = 0 \).

We have \( \Omega_4 = \langle e_{12323}, e_{13232}, e_{23232}, e_{23434}, e_{24343}, e_{32323}, e_{34343}, e_{43434} \rangle \) and, similar to the previous calculation, the rank of the image \( \partial : \Omega_4 \to \Omega_3 \) is equal to 4, the rank of the kernel \( \partial \) is equal to 4. Hence \( H^b_3(G) = 0 \).
Lemma 3.10 Let $G = (V, E)$ be a directed hypergraph and $I_1 = (0 \to 1)$ the digraph. There is an inclusion $\lambda: [\mathcal{B}(G)]^\uparrow \to \mathcal{B}(G \boxdot I_1)$ of path complexes. The restrictions of $\lambda$ to the images of the morphisms $i_\bullet$ and $j_\bullet$, defined in Section 2, are the natural identifications.

Proof. By definition in Section 2 we have $[\mathcal{B}(G)]^\uparrow = (V \times J, [P^b_G]^\uparrow)$, where $[P^b_G]^\uparrow = P^b_G \cup [P^b_G]^\uparrow \cup [P^b_G]^\uparrow \mathcal{V}$. We have $V_{G \boxdot I_1} = V \times J = V \times \{0, 1\} = V \cup V'$ with $V = \{0, \ldots, n\}$, $V' = \{0', \ldots, n'\}$ and $E_{G \boxdot I_1}$ is the union of sets $E_0 \cup E^1 \cup E^{01}$, where $E^1 = \{A \times \{i\} \to B \times \{i\} \mid (A \to B) \in E_G\}$ for $i = 0, 1$ and $E^{01} = \{C \times \{0\} \to C \times \{1\} \mid C \in P_01(G)\}$. Now it follows that $\mathcal{B}(G) = \mathcal{B}(G \boxdot \{0\})$, $\mathcal{B}(G)' = \mathcal{B}(G \boxdot \{1\})$, where $\mathcal{B}(G \boxdot \{0\}), \mathcal{B}(G \boxdot \{1\}) \subset \mathcal{B}(G \boxdot I_1)$. Let $q = (i_0 \ldots i_n)$ be $n$-path in $\mathcal{B}(G) = \mathcal{B}(G \boxdot \{0\})$. Consider its presentation in the form (3.2) and let $A_i, B_i$ be the corresponding sets of vertices. For $0 \leq k \leq n$, consider a path $q_k^\# = (i_0 \ldots i_k i_k' i_{k+1} \ldots i_n) \in [P^b_G]^\#$. We will prove now that this path in $P^b_{G \boxdot I_1}$. There are following possibilities for the path $q$.

(1) Vertices $i_k, i_{k+1} \in p_s$ for $1 \leq s \leq r + 1$ in presentation (3.2). Then we write path $q_k^\#$ in the form

$$q_k^\# = \left( p_0^\# \lor v_0^\# w_0^\# \lor p_1^\# \lor \ldots \lor p_{r+1}^\# \lor v_{r+1}^\# w_{r+1}^\# \lor p_{r+2}^\# \right) \quad (3.3)$$

putting

$$A_i^\# = \begin{cases} A_i \times \{0\} & \text{for } i \leq s - 1 \\ B_{s-1} \times \{0\} & \text{for } i = s \\ A_{s-1} \times \{1\} & \text{for } i \geq s + 1, \end{cases} \quad B_i^\# = \begin{cases} B_i \times \{0\} & \text{for } i \leq s - 1 \\ B_{s-1} \times \{1\} & \text{for } i \geq s. \end{cases}$$

We have the following arrows in $E_{G \boxdot I_1}$:

$$A_i^\# \to B_i^\# = \left( A_i \times \{0\} \to B_i \times \{0\} \right) \quad \text{for } 0 \leq i \leq s - 1,$$

$$A_i^\# \to B_i^\# = \left( B_{s-1} \times \{0\} \to B_{s-1} \times \{1\} \right),$$

$$A_i^\# \to B_i^\# = \left( A_{s-1} \times \{1\} \to B_{s-1} \times \{1\} \right) \quad \text{for } s + 1 \leq i \leq r + 2.$$

Using identifications $\mathcal{B}(G) = \mathcal{B}(G \boxdot \{0\}), \mathcal{B}(G)' = \mathcal{B}(G \boxdot \{1\})$, we obtain

$$p_i^\# = \begin{cases} p_i & \text{for } i \leq s - 1 \\ p_i' & \text{for } s + 2 \leq i \leq r + 2 \\ w_{s-1} \ldots i_s & \text{for } i = s \\ i_s' i_{s+1}' \ldots v_s' & \text{for } i = s + 1 \end{cases} \quad (3.4)$$
Similarly to case (1), we have 

\[ E \]

we write path \( q \) in the considered case.

\[ \text{presentation of } i \] where \((3.3)\) putting \( G \) in the form \((3.3)\) putting \( \) in the form \((3.2)\) in the hypergraph \( G \square I_1 \) and \( q_k^\# \in P_{G \square I_1}^b \) in the considered case.

(2) Vertices \( i_k, i_{k+1} \in p_0 \) in presentation \((3.2)\). Then we write path \( p_k^\# \) in the form \((3.3)\) putting

\[
A_i^\# = \begin{cases} 
A_0 \times \{0\} & \text{for } i = 0 \\
A_{i-1} \times \{1\} & \text{for } 1 \leq i \leq r + 2,
\end{cases}
\]

\[
B_i^\# = \begin{cases} 
A_0 \times \{1\} & \text{for } i = 1 \\
B_{i-1} \times \{1\} & \text{for } 2 \leq i \leq r + 2
\end{cases}
\]

and

\[
p_i^\# = \begin{cases} 
(i_0 \ldots i_k) & \text{for } i = 0 \\
(v_i^0 \ldots v_{i-1}^0) & \text{for } i = 1 \\
p_{i-1}^\# & \text{for } 2 \leq i \leq r + 2
\end{cases}
\]

where \( (i_0 \ldots i_k) \in P_{A_i^\#}^b, (v_i^0 \ldots v_{i-1}^0) \in P_{B_i^\#}^b \square A_i^\# = P_{A_0 \times \{1\}}^b \). Hence, \((3.3)\) gives a presentation of \( q_k^\# \) in the form \((3.2)\) in the hypergraph \( G \square I_1 \) and \( q_k^\# \in P_{G \square I_1}^b \) in the considered case.

(3) Let \( i_k = v_s, i_{k+1} = w_s \) for \( 0 \leq s \leq r \) in the presentation \((3.2)\). Then we write path \( q_k^\# \) in the form \((3.3)\) putting

\[
A_i^\# = \begin{cases} 
A_i \times \{0\} & \text{for } i \leq s \\
A_{i+1} \times \{1\} & \text{for } i = s + 1 \\
A_{i-1} \times \{1\} & \text{for } i \geq s + 2,
\end{cases}
\]

\[
B_i^\# = \begin{cases} 
B_i \times \{0\} & \text{for } i \leq s - 1 \\
A_s \times \{1\} & \text{for } i = s \\
B_{s-1} \times \{1\} & \text{for } i \geq s + 1.
\end{cases}
\]

We have the following arrows in \( E_{G \square I_1} \):

\[
(A_i^\# \to B_i^\#) = (A_i \times \{0\} \to B_i \times \{0\}) \text{ for } 0 \leq i \leq s - 1,
\]

\[
(A_i^\# \to B_i^\#) = (A_s \times \{0\} \to A_s \times \{1\}),
\]

\[
(A_i^\# \to B_i^\#) = (A_{i-1} \times \{1\} \to B_{i-1} \times \{1\}) \text{ for } s + 1 \leq i \leq r + 2.
\]

Similarly to case (1), we have

\[
p_i^\# = \begin{cases} 
p_i & \text{for } i \leq s \\
v_i^0 & \text{for } i = s + 1 \\
p_{i-1}^\# & \text{for } s + 2 \leq i \leq r + 2,
\end{cases}
\]
where \( w_s^# = v_s, v_s^# = v'_s, w_{s+1}^# = v'_s \). Hence, (3.3) gives a presentation of \( q_k^# \) in the form (3.2) in the hypergraph \( G \Box I_1 \) and \( q_k^# \in \mathcal{P}_G(\Box I_1) \) in the considered case. □

**Theorem 3.11** Let \( G \) be a directed hypergraph. The bold path homology groups \( \mathcal{H}^b_*(G) \) are homotopy invariant.

**Proof.** By Definition 2.3 it is sufficient to prove homotopy invariance for the one-step homotopy. Let \( f_0, f_1: G \to H \) be one-step homotopic morphisms of directed hypergraphs with homotopy \( F: G \Box I_1 \to H \), where \( I_1 = (0 \to 1) \). Since \( \mathfrak{B} \) is a functor, we obtain morphisms of path complexes \( \mathfrak{B}(f_0), \mathfrak{B}(f_1): \mathfrak{B}(G) \to \mathfrak{B}(H) \) and \( \mathfrak{B}(F): \mathfrak{B}(G \Box I_1) \to \mathfrak{B}(H) \). Consider the composition \( \{\mathfrak{B}(G)\}^+ \xrightarrow{\lambda} \mathfrak{B}(G \Box I_1) \xrightarrow{\mathfrak{B}(F)} \mathfrak{B}(H) \) which gives a homotopy between morphisms \( \mathfrak{B}(f_0) \) and \( \mathfrak{B}(f_1) \) of path complexes by using identifications of the top and the bottom of \( \{\mathfrak{B}(G)\}^+ \) described in Lemma 3.10. Now the result follows from [10] Th. 3.4. □

### 3.3 Non-directed path homology

In this subsection, we describe several path homology theories on the category of directed hypergraphs \( \mathcal{DH} \) that are based on functorial relations between hypergraphs and directed hypergraphs.

For a hypergraph \( G = (V_G, E_G) \), we have a natural map \( \phi_G: E_G \to P(V_G) \setminus \emptyset \). A morphism of hypergraphs \( f: G \to H = (V_H, E_H) \) is given by the pair of maps \( f_V: V_G \to V_H \) and \( f_E: E_G \to E_H \) such that \( P(f_V) \circ \phi_G = \phi_H \circ f_E \), where \( P(f_V): P(V_G) \setminus \emptyset \to P(V_H) \setminus \emptyset \) is the map induced by \( f_V \). So we may turn to the category of hypergraphs \( \mathcal{H} \) in [10].

First, define a functor from category \( \mathcal{DH} \) to category \( \mathcal{H} \). For a finite set \( X \), define a map \( \sigma_X: \mathbb{P}(X) \to \mathbb{P}(X) \) by setting \( \sigma_X(A, B) = A \cup B \). Let \( G = (V, E) \) be a directed hypergraph. Define a hypergraph \( \mathfrak{E}(G) = (V^e, E^e) \) where \( V^e = V \) and

\[
E^e = \{ C \in P(V) \setminus \emptyset | C = A \cup B, (A \to B) \in E \}.
\]

(3.5)

Recall that in Section 2 for a directed hypergraph \( G = (V, E) \) we defined a map \( \varphi_G: E \to \mathbb{P}(V) \) by \( \varphi_G(A \to B) = (A, B) \).

**Proposition 3.12** Let \( f = (f_V, f_E): G = (V_G, E_G) \to H = (V_H, E_H) \) be a morphism of directed hypergraphs. Define a map \( f^e_E: E^e_G \to P(V_H) \) putting \( f_G^e(C) = [P(f_V)](C) \) for every \( C = A \cup B \in E^e_G \). Then the map \( f^e_E \) is a well defined map \( E^e_G \to E^e_H \) and the pair \( (f^e_V, f^e_E) \) with \( f^e_V = f_V \) defines a
Proof. The map \( f_E^* \) is well defined. Now we prove that its image lies in \( E_H^* \). Let \( C \in E_G^* \), \( C = \sigma_{V_G} \circ \varphi_G(A \rightarrow B) = A \cup B \) and \( f_{E_G} (A \rightarrow B) = (A' \rightarrow B') \in E_H^* \). Then, by Definition 2.2, \( [P(f_{V_G})](A, B) = (A', B') \in P(V(G)) \) and, hence, \( A' = [P(f_{V})](A), B' = [P(f_{V})](B) \). We have

\[
[P(f_{V})](C) = [P(f_{V})](A \cup B) = ([P(f_{V})](A)) \cup ([P(f_{V})](B)) = A' \cup B'.
\]

However, \( A' \cup B' = \sigma_{V_H} \circ \varphi_H(A' \to B') \in E_H^* \) and the claim that morphism \( f_E^* \) is well defined is proved. The functoriality is evident. \( \blacksquare \)

For a hypergraph \( G = (V, E) \), define a path complex \( \tilde{\mathcal{P}}^q(G) = (V^q, P^q_G) \) of density \( q \geq 1 \) where \( V^q = V \) and a path \( (i_0, \ldots, i_n) \in P^q \) lies in \( P^q_G \) iff every \( q \) consequent vertices of this path lie in a hyperedge \( e \), see [10]. Thus, we obtain a collection of functors \( \tilde{\tilde{\mathcal{P}}}^q \) from the category \( \mathcal{H} \) to the category \( \mathcal{P} \). Composition \( \tilde{\tilde{\mathcal{P}}}^q \circ \mathcal{E} \) gives collection of functors from category \( \mathcal{D} \mathcal{H} \) to category \( \mathcal{P} \). For a directed hypergraph \( G \) define

\[
H_*^{q(q)}(G) = H_*^{q} [ \tilde{\tilde{\mathcal{P}}}^q \circ \mathcal{E}(G) ] \text{ for } q = 1, 2, \ldots .
\]

We call these groups by the non-directed path homology groups of density \( q \) of a directed hypergraph \( G \). We denote \( H_*^q(G) = H_*^{q(1)}(G) \).

Proposition 3.13 Let \( G = (V, E) \) be a directed hypergraph and \( \Pi_V \) be a path complex of all paths on the set \( V \). Then \( H_*^q(G) = H_*^q(\Pi_V) \).

Proof. By Definition 2.1, \( V = \cup_{e \in E} (A_i \cup B_i) \) and every vertex \( v \in V^e = V \) lies in an edge \( e \in E^e \). So path complexes \( \Pi_V \) and \( \tilde{\tilde{\mathcal{P}}}^1 \circ \mathcal{E}(G) \) coincide. \( \blacksquare \)

Example 3.14 Now we compute path homology groups \( H_*^{q(q)}(G) \) of density \( q = 1, 2, 3 \) with coefficients in \( \mathbb{R} \) of the directed hypergraph \( G \) with \( V_G = \{1, 2, 3, 4, 5, 6\} \), \( E_G = \{e_1, e_2, e_3, e_4, e_5\} \), where \( e_1 = (\{1\} \to \{2\}) \), \( e_2 = (\{1\} \to \{3\}) \), \( e_3 = (\{2\} \to \{4, 6\}) \), \( e_4 = (\{3\} \to \{5\}) \), \( e_5 = (\{4\} \to \{5, 6\}) \). Then the hypergraph \( \mathcal{E}(G) \) has the set of vertices \( V^e_G = \{1, 2, 3, 4, 5, 6\} \) and the set of hyperedges

\[
E^e_G = \{e_1' = \{1, 2\}, e_2' = \{1, 3\}, e_3' = \{2, 4, 6\}, e_4' = \{3, 5\}, e_5' = \{4, 5, 6\}\} .
\]

In the case of \( q = 1 \), the homology groups \( H_*^q(G) \) coincide with the path homology group of the complete digraph \( D = (V_D, E_D) \) which has six vertices
and for every two vertices \( v, w \in V_D \) there are two arrows \((v \to w), (w \to v) \in E_D\). This digraph is contractible, and hence, see [11] S3.3, \( H^0_0(G) = \mathbb{R} \) and groups \( H^i_0(G) \) are trivial for \( i \geq 1 \).

If \( q = 2 \), homology groups \( H^q_0(G) \) coincide with path homology group of the digraph \( D_2 \) on Fig. 2, where two-sided arrow \( a \leftrightarrow b \) means that there are arrows \( a \to b \) and \( b \to a \). The digraph \( D \) is homotopy equivalent to the induced sub-digraph \( D' \subset D_2 \) with the set of vertices \( \{1, 2, 3, 4, 5\} \).

We compute directly the path homology of \( D' \) and we obtain \( H^q_0(G) = \mathbb{R} \) and trivial groups \( H^i_0(G) \) for \( i \geq 2 \).

Now we consider the case of \( e(3) \). We have \( \Omega^e_0(G) = \Omega^e_0(G) \) for \( n = 0, 1 \) and this equality is also true for all \( n \geq 0 \). We have

\[
\mathcal{R}^e_2(3) = A \oplus A_{246} \oplus A_{456},
\]

where \( A = \langle e_{121}, e_{212}, e_{131}, e_{313}, e_{353}, e_{535} \rangle \) and \( A_{abc} \) is the module generated by all regular paths with three vertices in the full digraph with vertices \( a, b, c \). Hence \( \Omega^e_2 = \mathcal{R}^e_2(3) \). Considering the digraph \( D_2 \), we obtain that \( \Omega^e_2 = \Omega^e_2 \). The cases with \( n \geq 4 \) are similar and \( \Omega^e_n = \Omega^e_n \) for \( n \geq 4 \).

**Proposition 3.15** Let \( G = (V, E) \) be a directed hypergraph, \( I_1 = (0 \to 1) \), and \( I = (V_I, E_I) \) be the hypergraph with the set of vertices \( V_I = \{0, 1\} \) and the set of edges \( E_I = \{e'_0 = \{0\}, e'_1 = \{1\}, \epsilon'_2 = \{0, 1\} \} \). There is a natural inclusion of path complexes

\[
\mathcal{S}^q[\mathcal{E}(G \square I_1)] \subset \mathcal{S}^q[\mathcal{E}(G) \times I]
\]

for \( q \geq 2 \). Moreover, in general case complexes in (3.6) are not equal.

**Proof.** Recall that the product "\( \times \)" of hypergraphs is defined in [5], [10]. The directed hypergraph \( G \square I_1 = (V_{G \square I_1}, E_{G \square I_1}) \) has the set of vertices
The pair \((V, E)\) and the set of edges that can be presented as the union \(E_0 \cup E_1 \cup E_{01}\) of three pairwise disjoint sets

\[
E_0 = \{(C \times \{0\}) \to D \times \{0\} \mid (C \to D) \in E\}, \\
E_1 = \{(C \times \{1\}) \to D \times \{1\} \mid (C \to D) \in E\}, \\
E_{01} = \{A \times \{0\} \to A \times \{1\} \mid A \subset \mathbf{P}_{01}(G)\}.
\]

Hence, the hypergraph \(\mathcal{E}(G \Box I_1) = (V_{\mathcal{E}(G \Box I_1)}, E_{\mathcal{E}(G \Box I_1)})\) has the set of vertices \(V_{\mathcal{E}(G \Box I_1)} = V \times \{0, 1\}\) and the set of edges that can be presented as a union of three pairwise disjoint sets \(E'_0 \cup E'_1 \cup E'_{01}\) where

\[
\begin{align*}
E'_0 &= \{(C \times \{0\}) \cup (D \times \{0\}) \mid (C \to D) \in E\}, \\
E'_1 &= \{(C \times \{1\}) \cup (D \times \{1\}) \mid (C \to D) \in E\}, \\
E'_{01} &= \{(A \times \{0\}) \cup (A \times \{1\}) \mid A \subset \mathbf{P}_{01}(G)\}.
\end{align*}
\]

By definition of a hypergraph \(\mathcal{E}(G) = (V^e, E^e)\), we obtain that the hypergraph \(\mathcal{E}(G) \times I = (V_{\mathcal{E}(G) \times I}, E_{\mathcal{E}(G) \times I})\) has the set of vertices \(V_{\mathcal{E}(G) \times I} = V \times \{0, 1\}\) and the set of edges that can be presented as a union of three pairwise disjoint sets \(E''_0 \cup E''_1 \cup E''_{01}\) where

\[
\begin{align*}
E''_0 &= \{(C \cup D) \times \{0\}, C \cup D, \{0\} \mid (C \to D) \in E\}, \\
E''_1 &= \{(C \cup D) \times \{1\}, C \cup D, \{0\} \mid (C \to D) \in E\}, \\
E''_{01} &= \{(A, C \cup D, \{0, 1\}) \mid (C \to D) \in E, A \subset (C \cup D) \times \{0, 1\}\}.
\end{align*}
\]

Let \(p_1: V \times \{0, 1\} \to V, p_2: V \times \{0, 1\} \to \{0, 1\}\) be natural projections. Then \(p_1(A) = C \cup D\) and \(p_2(A) = \{0, 1\}\) by definition of the product of hypergraphs. Thus, path complexes \(\mathcal{S}^q[\mathcal{E}(\Box I_1)]\) and \(\mathcal{S}^q[\mathcal{E}(G) \times I]\) have the same vertex set and, by (3.7) and (3.8), \(E'_0 = E''_0, E'_1 = E''_1, E'_{01} \subset E''_{01}\) and follows.

Now we prove that in general case of (3.6) there is no equality. Let \((C \to D) \in E\) be a directed edge and \(v \in C, w \in D\) be such vertices that the pair \((v, w)\) does not lie in a set \(A \in \mathbf{P}_{01}(G)\). Then the two vertex path \((v \times \{0\}), (w \times \{1\}) \in \mathcal{S}^q[\mathcal{E}(G) \times I]\) for \(q = 2\) lies in \(E''_{01}\) and does not lie in \(E'_0 \cup E'_1 \cup E'_{01}\).

**Lemma 3.16** Let \(G = (V, E)\) be a directed hypergraph, \(I_1 = (0 \to 1)\). There is the inclusion \(\mu: \mathcal{S}^2 \circ \mathcal{E}(G)^\dagger \to \mathcal{S}^2 \circ \mathcal{E}(G \Box I_1)\) of path complexes.

**Proof.** By definition of the hypergraph \(\mathcal{E}(G) = (V^e, E^e)\) and the functor \(\mathcal{S}^2\), we obtain that the set \(P^2_\mathcal{E}(G)\) of \(\mathcal{S}^2 \circ \mathcal{E}(G)\) consists of paths \(p = (i_0 \ldots i_n)\) on the set \(V\) such that every two consequent vertices \(i_s, i_{s+1} \in p\) lie in a set
\{C \cup D \mid (C \to D) \in E\}$. By definition, the set of paths of the path complex $[\mathcal{H}^2 \circ \mathcal{E}(G)]^\uparrow$ is a union of paths

$$P_{\mathcal{E}(G)}^2 \cup [P_{\mathcal{E}(G)}^2]^1 \cup [P_{\mathcal{E}(G)}^2]^\#$$

(3.9)
on the set $V \times \{0, 1\} = V \cup V'$. A path $p = (i_0 \ldots i_n)$ on the set $V \times \{0, 1\}$ lies in $P_{\mathcal{E}(G \sqcup I_1)}^2$ if any two consequent vertices lie in the exactly one of the sets $E'_0, E'_1, E'_{01}$ defined in (3.7). From definition of the functor $\mathcal{E}$, we conclude that in (3.9) any pair of consequent vertices of a path from $P_{\mathcal{E}(G)}^2$ lies in an edge from $E'_0$, any pair of consequent vertices of a path from $[P_{\mathcal{E}(G)}^2]^1$ lies in an edge from $E'_1$, and any pair of consequent vertices of a path from $[P_{\mathcal{E}(G)}^2]^\#$ lies in an edge from $E'_0 \cup E'_1 \cup E'_{01}$. ■

**Theorem 3.17** For a directed hypergraph $G$, the non-directed path homology groups $H_*(\mathcal{E}^2(G))$ of density two are homotopy invariant.

**Proof.** It is sufficient to prove homotopy invariance for the one-step homotopy. Let $f_0, f_1: G \to H$ be one-step homotopic morphisms of directed hypergraphs with a homotopy $F: G \sqcup I_1 \to H$. Since $\mathcal{H}^2 \circ \mathcal{E}$ is a functor, we obtain morphisms of path complexes $\mathcal{H}^2 \circ \mathcal{E}(f_0), \mathcal{H}^2 \circ \mathcal{E}(f_1): \mathcal{H}^2 \circ \mathcal{E}(G) \to \mathcal{H}^2 \circ \mathcal{E}(H)$, and $\mathcal{H}^2 \circ \mathcal{E}(F): \mathcal{H}^2 \circ \mathcal{E}(G \sqcup I_1) \to \mathcal{H}^2 \circ \mathcal{E}(H)$. Using Lemma 3.16 we can consider the composition

$$[\mathcal{H}^2 \circ \mathcal{E}(G)]^\uparrow \xrightarrow{\mu} [\mathcal{H}^2 \circ \mathcal{E}(G \sqcup I_1)] \xrightarrow{\mathcal{H}^2 \circ \mathcal{E}(F)} [\mathcal{H}^2 \circ \mathcal{E}(H)]$$

which gives a homotopy between morphisms $\mathcal{H}^2 \circ \mathcal{E}(f_0)$ and $\mathcal{H}^2 \circ \mathcal{E}(f_1)$. Now the result follows from [10] Th. 3.4. ■

### 3.4 Natural path homology

Let $G = (V, E)$ be a directed hypergraph. Define a digraph $\mathcal{H}(G) = (V^n_G, E^n_G)$ where $V^n_G = \{C \in \mathcal{P}(V) \setminus \emptyset | C \subseteq \mathcal{P}_{01}(G)\}$ and $E^n_G = \{A \to B | (A \to B) \in E\}$. Thus a set $X \in \mathcal{P}(V) \setminus \emptyset$ is a vertex of the digraph $\mathcal{H}(G)$ iff $X$ is an origin or an end of an arrow $e \in E$. Any arrow $e = (A \to B) \in E$ gives an arrow $(A \to B) \in E^n$. 

**Proposition 3.18** Every morphism of directed hypergraphs $f: G \to H$ defines a morphism of digraphs $\mathcal{H}(G) = (f^n_V, f^n_E): (V^n_G, E^n_G) \to (V^n_H, E^n_H)$ by $f^n_V(C): = [\mathcal{P}(f)] \circ \phi_G(C)$ and $f^n_E(A \to B) = (f_V(A) \to f_V(B)) \in E^n_H$. Moreover, $\mathcal{H}$ is a functor from the category $\mathcal{DH}$ to the category $\mathcal{D}$ of digraphs. ■
The composition $\mathfrak{D} \circ \mathfrak{R}$ gives a functor from $\mathcal{DH}$ to the category of path complexes. For a directed hypergraph $G$, we set $H^n_p(G) = H_*(\mathfrak{D} \circ \mathfrak{R}(G))$. These homology groups will be called the natural path homology groups of $G$.

**Example 3.19** Now we compute $H^n_p(G)$ with coefficients in $\mathbb{R}$ of directed hypergraph $G = (V_G, E_G)$:

$$V_G = \{1, 2, 3, 4, 5, 6, 7, 8\}, \quad E_G = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9\},$$

$$e_1 = (\{1\} \to \{3, 4\}), e_2 = (\{1\} \to \{5, 6\}), e_3 = (\{1\} \to \{7, 8\}),$$

$$e_4 = (\{2\} \to \{3, 4\}), e_5 = (\{2\} \to \{5, 6\}), e_6 = (\{2\} \to \{7, 8\}),$$

$$e_7 = (\{3, 4\} \to \{5, 6\}), e_8 = (\{5, 6\} \to \{7, 8\}), e_9 = (\{7, 8\} \to \{3, 4\}).$$

The groups $H^n_p(G)$ coincide with the regular path homology groups of the digraph given on Fig. 3.

![Figure 3: The digraph of Example 3.19.](image)

Computation gives: $H^0_p(G) = H^2_p(G) = \mathbb{R}$ and $H^i_p(G) = 0$ for other $i$.

**Lemma 3.20** Let $G = (V, E)$ be a directed hypergraph and $I_1 = (0 \to 1)$. There is an equality $\mathfrak{R}(G) \Box I_1 = \mathfrak{R}(G \Box I_1)$ of digraphs.

**Proof.** The digraph $\mathfrak{R}(G) \Box I_1$ has $V_{\mathfrak{R}(G) \Box I_1} = \{C \times \{0, 1\} | C \in \mathbf{P}_{01}(G)\}$ and $E_{\mathfrak{R}(G) \Box I_1} = \cup \cup E_0, E_{0 \to 1}$ where $E_{0,1} = \{A \times \{i\} \to B \times \{i\} | A \to B \in E; i = 0, 1\}$, $E_{0 \to 1} = \{C \times \{0\} \to C \times \{1\} | C \in \mathbf{P}_{01}(G)\}$. The digraph
\( \mathcal{H}(G \Box I_1) \) has the set of vertices \( V_{\mathcal{H}(G \Box I_1)} = \{ C \times \{0, 1\} | C \in P_{01}(G) \} \) which coincides with \( V_{\mathcal{H}(G) \Box I_1} \) and the set of edges \( E_{\mathcal{H}(G \Box I_1)} = E_{0,1} \cup E_{0 \rightarrow 1} \) which coincides with \( E_{\mathcal{H}(G) \Box I_1} \).

**Theorem 3.21** For a directed hypergraph \( G \), the natural path homology groups \( H^*_\mathcal{H}(G) \) are homotopy invariant.

**Proof.** The path homology groups defined on the category of digraphs are homotopy invariant [10],[11] and thus, the result follows from Lemma 3.20.

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Yu.V. Muranov: *Faculty of Mathematics and Computer Science, University of Warmia and Mazury, Sloneczna 54 Street, 10-710 Olsztyn, Poland* e-mail: muranov@matman.uwm.edu.pl

Anna Szczepkowska: *Faculty of Mathematics and Computer Science, University of Warmia and Mazury, 10-710 Olsztyn, Poland* e-mail: anna.szczepkowska@matman.uwm.edu.pl

Vladimir Vershinin: *Département des Sciences Mathématiques, Université de Montpellier, Place Eugène Bataillon, 34095 Montpellier cedex 5, France* e-mail: vladimir.verchinine@univ-montp2.fr