Results of a perturbation theory generating a one-parameter semigroup

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Abstract: This paper consists of the results about $\omega$-order preserving partial contraction mapping using perturbation theory to generate a one-parameter semigroup. We show that adding a bounded linear operator $B$ to an infinitesimal generator $A$ of a semigroup of the linear operator does not destroy $A$’s property. Furthermore, $A$ is the generator of a one-parameter semigroup, and $B$ is a small perturbation so that $A + B$ is also the generator of a one-parameter semigroup.

Keywords: $\omega$ – OCP\textsubscript{$n$}; Analytic semigroup; $C_0$-semigroup; Perturbation.

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1. Introduction

Perturbation theory comprises methods for finding an approximate solution to a problem; in perturbation theory, the solution is expressed as a power series in a small parameter $\varepsilon$. The first term is the known solution to the solvable problem. Successive terms in the series at higher powers of $\varepsilon$ usually become smaller. Assume $X$ is a Banach space, $X_n \subseteq X$ is a finite set, $T(t)$ the $C_0$-semigroup, $\omega$ – OCP\textsubscript{$n$} the $\omega$-order preserving partial contraction mapping, $M_m$ be a matrix, $L(X)$ be a bounded linear operator on $X$, $P_n$ a partial transformation semigroup, $\rho(A)$ a resolvent set, $\sigma(A)$ a spectrum of $A$ and $A \in \omega$ – OCP\textsubscript{$n$} is a generator of $C_0$-semigroup. This paper consists of results of $\omega$-order preserving partial contraction mapping generating a one-parameter semigroup.

Akinyele et al., [1] introduced perturbation of the infinitesimal generator in the semigroup of the linear operator. Batty [2] established some spectral conditions for stability of one-parameter semigroup and also in [3] Batty et al., revealed some asymptotic behavior of semigroup of the operator. Balakrishnan [4] obtained an operator calculus for infinitesimal generators of the semigroup. Banach [5] established and introduced the concept of Banach spaces. Chill and Tomilov [6] deduced some resolvent approaches to stability operator semigroup. Davies [7] obtained linear operators and their spectra. Engel and Nagel [8] introduced a one-parameter semigroup for linear evolution equations. Ribiger and Wolf [9] deduced some spectral and asymptotic properties of the dominated operator. Rauf and Akinyele [10] introduced $\omega$-order preserving partial contraction mapping and established its properties, also in [11], Rauf et al., deduced some results of stability and spectra properties on semigroup of a linear operator. Vrabie [12] proved some results of $C_0$-semigroup and its applications. Yosida [13] established and proved some results on differentiability and representation of one-parameter semigroup of linear operators.

In this paper, we show that adding a bounded linear operator $B$ to an infinitesimal generator $A$ of a semigroup of the linear operator does not destroy $A$’s property. Furthermore, $A$ is the generator of a one-parameter semigroup, and $B$ is a small perturbation so that $A + B$ is also the generator of a one-parameter semigroup.

2. Preliminaries

Definition 1. ($C_0$-Semigroup) [8] A $C_0$-Semigroup is a strongly continuous one parameter semigroup of bounded linear operator on Banach space.
Definition 2. (ω-OCPn)[11] A transformation $a \in P_n$ is called $\omega$-order preserving partial contraction mapping if $\forall x, y \in \text{Dom } a : x \leq y \implies ax \leq ay$ and at least one of its transformation must satisfy $ay = y$ such that $T(t+s) = T(t)T(s)$ whenever $t, s > 0$ and otherwise for $T(0) = I$.

Definition 3. (Perturbation) [1] Let $A : D(A) \subseteq X \to X$ be the generator of a strongly continuous semigroup $(T(t))_{t \geq 0}$ and consider a second operator $B : D(B) \subseteq X \to X$ such that the sum $A + B$ generates a strongly continuous semigroup $(S(t))_{t \geq 0}$. We say that $A$ is perturbed by operator $B$ or that $B$ is a perturbation of $A$.

Definition 4. (Analytic Semigroup) [12] We say that a $C_0$-semigroup $\{T(t); t \geq 0\}$ is analytic if there exists $0 < \theta \leq \pi$, and a mapping $S : C_\theta \to L(X)$ such that:

1. $T(t) = S(t)$ for each $t \geq 0$;
2. $S(z_1 + z_2) = S(z_1)S(z_2)$ for $z_1, z_2 \in C_\theta$;
3. $\lim_{z_1 \in C_\theta, z_1 \to 0} S(z_1)x = x$ for $x \in X$; and
4. the mapping $z_1 \to S(z_1)$ is analytic from $C_\theta$ to $L(X)$. In addition, for each $0 < \delta < \theta$, the mapping $z_1 \to S(z_1)$ is bounded from $C_\delta$ to $L(X)$, then the $C_0$-Semigroup $\{T(t); t \geq 0\}$ is called analytic and uniformly bounded.

Definition 5. (Perturbation class) [7] We say that operator $B$ is a class $P$ perturbation of the generator $A$ of the one-parameter semigroup $T(t)$ if:

$$
\begin{align*}
A & \text{ is a closed operator; } \\
\text{Dom}(A) & \supseteq \cup_{t \geq 0} T(t)(X); \\
\int_0^1 \|BT(t)\|dt & < \infty.
\end{align*}
$$

Note that $BT(t)$ is bounded for all $t > 0$ under conditions (1) and (1) by the closed graph theorem.

Example 1 ($2 \times 2$ matrix $M_m(\mathbb{N} \cup \{0\})$). Suppose

$$
A = \begin{pmatrix}
2 & 0 \\
1 & 2
\end{pmatrix}
$$

and let $T(t) = e^{tA}$, then

$$
e^{tA} = \begin{pmatrix}
e^{2t} & e^t \\
e^t & e^{2t}
\end{pmatrix}.
$$

Example 2 ($3 \times 3$ matrix $M_m(\mathbb{N} \cup \{0\})$). Suppose

$$
A = \begin{pmatrix}
2 & 2 & 3 \\
2 & 2 & 2 \\
1 & 2 & 2
\end{pmatrix}
$$

and let $T(t) = e^{tA}$, then

$$
e^{tA} = \begin{pmatrix}
e^{2t} & e^{2t} & e^{3t} \\
e^{2t} & e^{2t} & e^{3t} \\
e^t & e^t & e^{2t}
\end{pmatrix}.
$$

Example 3 ($3 \times 3$ matrix $M_m(\mathbb{C})$). Since we have for each $\lambda > 0$ such that $\lambda \in \rho(A)$ where $\rho(A)$ is a resolvent set on $X$. Suppose we have

$$
A = \begin{pmatrix}
2 & 2 & 3 \\
2 & 2 & 2 \\
1 & 2 & 2
\end{pmatrix}
$$

and let $T(t) = e^{tA}$, then

$$
e^{tA} = \begin{pmatrix}
e^{2t\lambda} & e^{2t\lambda} & e^{3t\lambda} \\
e^{2t\lambda} & e^{2t\lambda} & e^{2t\lambda} \\
e^{t\lambda} & e^{t\lambda} & e^{2t\lambda}
\end{pmatrix}.
$$
3. Main results

This section presents results of one-parameter semigroup generated by $\omega$-OCP$_n$ using perturbation theory.

**Theorem 1.** Let $A \in \omega - \text{OCP}_n$ be the generator of a one-parameter semigroup $T(t)_{t \geq 0}$ on the Banach space $X$ and suppose that

$$
\|T(t)\| \leq Me^{at}
$$

for all $t \geq 0$. If $B$ is a bounded operator on $X$, then $(A + B)$ is the generator of a one-parameter semigroup $S(t)_{t \geq 0}$ on $X$ such that

$$
\|S(t)\| \leq Me^{(a + M\|B\|)t}
$$

for all $t \geq 0$ and $B \in \omega - \text{OCP}_n$.

**Proof.** We define the operators $S(t)$ by

$$
S(t)f := T(t)f + \int_{s=0}^{t} T(t-s)BT(s)ds + \int_{s=0}^{t} \int_{u=0}^{s} T(t-s)BT(s-u)BT(u)f dv du
$$

$$
+ \int_{s=0}^{t} \int_{u=0}^{s} \int_{v=0}^{u} T(t-s)BT(s-u)BT(u-v)BT(v)f dv du dv + \cdots. \quad (2)
$$

The $n$th term is an $n$-fold integral whose integrand is a norm continuous function of the variables. It is easy to verify that the series is norm convergent and that

$$
\|S(t)f\| \leq Me^{at}\|f\| \sum_{n=0}^{\infty} (tM\|B\|)^n / n! = Me^{(a + M\|B\|)t}, \quad (3)
$$

for all $f \in X$, $t \geq 0$ and $B \in \omega - \text{OCP}_n$.

Since $S(s)S(t) = S(s + t)$ and if $f \in X$, then

$$
\lim_{t \to 0} \|s(t)f - f\| = \lim_{t \to 0} \left\{ \|T(t)f - f\| + \sum_{n=1}^{\infty} Me^{at}\|f\|(tM\|B\|)^n / n! \right\} \geq 0,
$$

so that $s(t)$ is a one-parameter semigroup. If $f \in X$ and $B \in \omega - \text{OCP}_n$, then

$$
\lim_{t \to 0} \|t^{-1}(s(t)f - f) - t^{-1}(T(t)f - f) - Bf\|
$$

$$
\leq \lim_{t \to 0} t^{-1} \int_{0}^{t} T(t-s)BT(s)f ds - Bf\| + \lim_{t \to 0} t^{-1}Me^{at}\|f\| \sum_{n=2}^{\infty} (tM\|B\|)^n / n! \geq 0. \quad (4)
$$

It follows that $f$ lies in the domain of the generator $Y$ of $S(t)$ if and only if it lies in the domain of $A$, and that

$$
Yf := Af + Bf, \quad (5)
$$

for such $f$.

As well as being illuminating in its own right, (2) easily leads to the identities

$$
S(t)f = T(t)f + \int_{s=0}^{t} S(t-s)BT(s)f ds
$$

$$
= T(t)f + \int_{s=0}^{t} S(t-s)BT(s)f ds
$$

$$
= T(t)f + \int_{s=0}^{t} T(t-s)BS(s)f ds. \quad (6)
$$

Hence the proof is complete. □

**Theorem 2.** Suppose $B$ is a class $P$ perturbation of the generator $A$, then

$$
\text{Dom}(B) \supseteq \text{Dom}(A).
$$
If \( \epsilon > 0 \) and \( A, B \in \omega - \text{OCP}_n \), then
\[
\|BR(\lambda, A)\| \leq \epsilon,
\]
for all large enough \( \lambda > 0 \). Hence \( B \) has relative bound 0 with respect to \( A \).

**Proof.** Combining (1) with the bound
\[
\|BT(t)\| \leq \|BT(t)\| Me^{\alpha(t-1)},
\]
valid for all \( t \geq 1 \), we then see that
\[
\int_0^\infty \|BT(t)\| e^{-\alpha t} dt < \infty,
\]
for all \( \lambda > a \). Suppose \( \epsilon > 0 \) and \( A, B \in \omega - \text{OCP}_n \), then for all large enough \( \lambda \) we have
\[
\int_0^\infty \|BT(t)\| e^{-\lambda t} dt \leq \epsilon.
\]
Now,
\[
\int_0^\infty T(t)e^{-\lambda t} f dt = R(\lambda, A)f,
\]
for all \( f \in X \), so by the closedness of \( B \), we see that \( R(\lambda, A)f \in \text{Dom}(B) \) and
\[
\|BR(\lambda, A)f\| \leq \epsilon \|f\|,
\]
as required to prove (7).

If \( g \in \text{Dom}(A) \) and we put \( f := (\lambda I - A)g \), then we deduce from (7) that
\[
\|Bg\| \leq \epsilon \|(\lambda I - A)g\| \leq \epsilon \|Ag\| + \epsilon \lambda \|g\|,
\]
for all large enough \( \lambda > 0 \). This implies the last statement of the theorem and hence the proof is complete. \( \square \)

**Theorem 3.** Assume \( B \) is a class \( P \) perturbation of the generator \( A \) of the one-parameter semigroup \( T(t) \) on \( X \), then \( B + A \) is the generator of a one-parameter semigroup \( S(t) \) on \( X \) and \( A, B \in \omega - \text{OCP}_n \).

**Proof.** Let \( a \) be small enough that
\[
c := \int_0^{2a} \|BT(t)\| dt < 1.
\]
We may define \( S(t) \) by the convergent series (2) for \( 0 \leq t \leq 2a \), and verify as in the proof of Theorem 1 that \( S(s)S(t) = S(s + t) \) for all \( s, t \geq 0 \) such that \( s + t \leq 2a \). We now extend the definition of \( S(t) \) inductively for \( t \geq 2a \) by putting
\[
S(t) := (S(a))^n S(t - na),
\]
if \( n \in \mathbb{N} \) and \( na < t \leq (n + 1)a \). It is straightforward to verify that \( S(t) \) is a semigroup. Now suppose that
\[
\|T(t)\| \leq N \text{ for } 0 \leq t \leq a.
\]
Assume \( f \in X \) and \( B \in \omega - \text{OCP}_n \), then
\[
\|S(t)f - f\| \leq \|T(t)f - f\| + \sum_{n=1}^\infty N \left( \int_0^1 \|BS(t)\| ds \right)^n \|f\|,
\]
so that
\[
\lim_{t \to 0} \|S(t)f - f\| = 0,
\]
and \( S(t) \) is a one-parameter semigroup on \( X \). It is an immediate consequence of the definition that
\[
S(t)f = T(t)f + \int_0^1 S(t - s)BS(s)f ds,
\]
for all \( f \in X, B \in \omega - \text{OCP}_n \) and all \( 0 \leq t \leq a \). Suppose that this holds for all \( t \) such that \( 0 \leq t \leq na \). If \( na \leq u \leq (n+1)a \), then

\[
S(u)f = S(a)S(u-a)f
= S(a) \left\{ T(u-a)f + \int_0^{u-a} S(u-a-s)BT(s)f \, ds \right\}
= T(a)T(u-a)f + \int_0^a S(a-s)BT(s)(T(u-a)f) \, ds + \int_0^{u-a} S(u-s)BT(s)f \, ds
= T(u)f + \int_0^u S(u-s)BT(s)f \, ds.
\] 

(13)

By induction, (12) holds for all \( t \geq 0 \).

We finally have to identify the generator \( Y \) of \( S(t) \). The subspace

\[
D := \bigcup_{t \geq 0} T(t) \{ \text{Dom}(A) \},
\]

is contained in \( \text{Dom}(A) \) and is invariant under \( T(t) \) and so is a core for \( A \). If \( f \in D \), then there exists \( g \in \text{Dom}(A) \) where \( A \in \omega - \text{OCP}_n \) and \( \varepsilon > 0 \) such that \( f = T(\varepsilon)g \). Hence,

\[
\lim_{t \to 0} t^{-1}(S(t)f - f) = \lim_{t \to 0} (T(t)f - f) + \lim_{t \to 0} t^{-1} \int_0^t T(t-s)(BT(\varepsilon))T(\varepsilon)g \, ds
= Af + (BT(\varepsilon))g
= (A + B)f.
\] 

(14)

Therefore, \( \text{Dom}(Y) \) contains \( D \) and \( Yf(B + A) \) for all \( f \in D \) and \( A, B \in \omega - \text{OCP}_n \). If \( f \in \text{Dom}(A) \), then there exists a sequence \( f_n \in D \) such that \( \|f_n - f\| \to 0 \) and \( \|Af_n - Af\| \to 0 \) as \( n \to \infty \). It follows by Theorem 2 that \( \|Bf_n - Bf\| \to 0 \) and hence that \( Yf_n \) converges. Since \( Y \) is a generator that is closed, then we deduce that

\[
Yf = (B + A)f,
\]

for all \( f \in \text{Dom}(A) \) and \( A, B \in \omega - \text{OCP}_n \). Multiplying (12) by \( e^{-\lambda t} \) and integrating over \((0, \infty)\), we see as in the proof of Theorem 2 that if \( \lambda > 0 \) is large enough, then

\[
R(\lambda, Y)f = R(\lambda, A)f + R(\lambda, Y)BR(\lambda, A)f,
\]

for all \( f \in Y \) and \( A, B \in \omega - \text{OCP}_n \).

If \( \lambda \) is also large enough that

\[
\|BR(\lambda, A)\| < 1,
\]

we deduce that

\[
R(\lambda, Y) = R(\lambda, A)(I - BR(\lambda, A))^{-1}.
\]

Hence,

\[
\text{Dom}(Y) = \text{Ran}(R(\lambda, Y)) = \text{Ran}(R(\lambda, A)) = \text{Dom}(A),
\]

and \( Y = A + B \), and this achieve the proof. \( \square \)

**Theorem 4.** Let \( A := -H \) where \( H = (-\Delta)^n \geq 0 \) acts in \( L^2(\mathbb{R}^N) \). Also let \( B \) be a lower order perturbation of the form

\[
(Bf)(x) := \sum_{|\alpha| < 2n} a_\alpha(x)(D^\alpha f)(x).
\]

If \( a_\alpha \in L^p_\sigma(\mathbb{R}^N) + L^\infty(\mathbb{R}^N) \) for each \( \alpha \), where \( p_\sigma \geq 2 \) and \( p_\sigma > N/(2n - |\alpha|) \), the \( A + B \) is the generator of a one-parameter semigroup and \( B \) has relative bound \( 0 \) with respect to \( A \) where \( A, B \in \omega - \text{OCP}_n \).

**Proof.** Suppose \( A \in \omega - \text{OCP}_n \) is the generator of holomorphic semigroup \( T(t) \) such that

\[
\|T(t)\| \leq c_1, \quad \|AT(t)\| \leq c_2/t,
\]

for all \( t > 0 \). Then

\[
\frac{d}{dt} \|T(t)\| \leq c_1 \|T(t)f\| + c_2 \|T(t)f\|/t,
\]

for all \( f \). Integrating (12) by \( e^{-\lambda t} \) and integrating over \((0, \infty)\), we see as in the proof of Theorem 2 that if \( \lambda > 0 \) is large enough, then

\[
R(\lambda, Y)f = R(\lambda, A)f + R(\lambda, Y)BR(\lambda, A)f,
\]

for all \( f \in Y \) and \( A, B \in \omega - \text{OCP}_n \).

If \( \lambda \) is also large enough that

\[
\|BR(\lambda, A)\| < 1,
\]

we deduce that

\[
R(\lambda, Y) = R(\lambda, A)(I - BR(\lambda, A))^{-1}.
\]

Hence,

\[
\text{Dom}(Y) = \text{Ran}(R(\lambda, Y)) = \text{Ran}(R(\lambda, A)) = \text{Dom}(A),
\]

and \( Y = A + B \), and this achieve the proof. \( \square \)
for all $t \in (0, 1)$. And also the operator $B \in \omega - OCP_n$ has domain containing $Dom(A)$ and there exists $\alpha \in (0, 1)$, such that
\[ \|Bf\| \leq \varepsilon \|Af\| + c_3 \varepsilon^{-\alpha/(1-\alpha)} \|f\|, \] (15)
for all $f \in Dom(a)$ and $0 < \varepsilon \leq 1$. Then
\[ \|BT(t)\| \leq (c_2 + c_1 c_3) t^{-\alpha}, \] (16)
for all $t \in (0, 1)$ so that $B$ is a class $P$ perturbation of $A$ and by Theorem 3 under the stated conditions on $t$ and $\varepsilon$, we have
\[ \|BT(t)f\| \leq \varepsilon \|AT(t)f\| + c_3 \varepsilon^{-\alpha/(1-\alpha)} \|T(t)f\| \leq (\varepsilon c_2 t^{-1} + c_1 c_3 \varepsilon^{-\alpha/(1-\alpha)}) \|f\|. \]
By putting $\varepsilon = t^{1-\alpha}$, then we obtain (16).

Assume $\alpha \in (0, 1)$, $H$ is a non-negative self-adjoint operator on $P$ and $B$ is a linear operator with $Dom(B) \supseteq (H)$, we have
\[ \|Bf\| \leq \varepsilon \|Af\| + c_3 \varepsilon^{-\alpha/(1-\alpha)} \|f\|, \]
for all $\varepsilon > 0$ if and only if there is a constant $c_4$ such that
\[ \|Bf\| \leq c_4 \|Af\| \|f\|^{1-\alpha}, \]
for all $f \in Dom(A)$ and $A, B \in \omega - OCP_n$.

By Theorem 3, it is sufficient to prove that for each $\alpha$ there exists $\beta < 1$ for which
\[ X_\alpha := a_\alpha(\cdot)D^\alpha (H + 1)^{-\beta} \]
is bounded.

Let $X_\alpha = a_\alpha(Q)b_\alpha(P)$, where
\[ b_\alpha(\varepsilon) = \frac{\varepsilon^{\mid \alpha \mid} e^{\alpha}}{(\varepsilon^{2\alpha} + 1)^{\beta}}. \]
If $a_\alpha \in L^\infty(\mathbb{R}^N)$, then $\|X\| \leq \|a_\alpha\|_{\infty} \|b_\alpha\|_{\infty} < \infty$ provided $|\alpha|/2n < \beta < 1$. On the other hand, if $a_\alpha \in L^p(\mathbb{R}^N)$ where $P \geq 2$ and $P > N/(2n - |\alpha|)$, then there exists $\beta$ such that
\[ \frac{N + |\alpha| P}{2np} < \beta < 1. \]
This implies that $(|\alpha| - 2n \beta)p + N < 0$ and hence $b_\alpha \in L^p(\mathbb{R}^N)$. \qed

4. Conclusion

In this paper, it has been established that $\omega$-order preserving partial contraction mapping generates a one-parameter semigroup using a perturbation theory on Banach space by showing that the semigroup of a linear operator is bounded, that $B$ has a relative bound 0 with respect to $A$, and also that $B + A$ is a generator of the one-parameter semigroup.

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