Influence of Heat Source on Thermal Behaviour of Solid Cylinder.

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Abstract. The present manuscript deals with determination of influence of internal heat source on thermal behaviour of a long solid cylinder. A direct, steady state, nonhomogeneous boundary value problem with internal heat generation is formulated to analyse heat transfer and thermal stresses. The cylindrical surface heat source situated in annular region of long cylinder results in internal heat generation. Finite Hankel transform, Goodier’s thermoelastic displacement potential and Michell function are adopted to obtain the analytical solution. The outcomes are represented in the form of infinite series comprising Bessel’s functions. By considering a mathematical model for steel plate, outcomes are verified numerically. Some numerical results for the temperature distribution, displacement components, and the associated stresses are shown in figures drawn using Matlab software.

Keywords –Thermal analysis, steady state, internal heat source, Finite Hankel transform.

1. Introduction

During last few decades, a remarkable development of thermoelasticity is motivated by various fields of engineering science and technology. Thermoelastic materials have many applications in various fields of science and technology, viz., atomic physics, industrial, nuclear, chemical, marine and automobile engineering, thermal power plants, aerospace and metallurgy. The effect of temperature on the behaviour of basic structures of engineering such as annular discs, solid and hollow cylinders, spheres, spherical shells and circular plates is an important engineering concern. Due to their widespread usage in our day today life and industry, it is very significant to study the deformation behavior of these structures under mechanical and thermal loads. The principles of thermoelasticity have affected the way engineers design a number of different structures. The importance of thermal stresses in causing structural damages and changes in functioning of the structure is well recognized whenever thermal stress environments are involved. Therefore, the ability to predict thermal stresses induced by sudden thermal loading in composite engineering structures is necessary for the proper and safe design and the knowledge of its response during the service in these severe thermal environments.

Many researchers have endeavored to analyse the thermal behaviour of different thermoelastic solids. A great deal of research attention has been focused on the thermoelastic problems of solid cylinders. Marchi E, Zgrablich G [1] highlighted the heat conduction in hollow cylinder with radiation. The heat conduction in sector of hollow cylinder with radiation is studied by Marchi E. and Fasulo A. [2]. The direct problem of elastic deformation of finite length hollow cylinder has assessed by Sirakowski and Sun [3] to obtain exact solution. Taking an anisotropic finite hollow cylinder, axi-symmetric thermal stresses are analysed by Chen P.Y.P [4]. An axisymmetric inverse steady state problem of thermoelastic deformation of a finite length hollow cylinder is discussed by Deshmukh and Wankhede [5] Quasi static thermoelastic problem of an infinitely long circular cylinder is outlined by Gaikwad K. R., Ghadle K.P [6]. Gahane T.T [7] focused on transient thermoelastic problem of a cylinder with heat sources. Walde and Khobragade [8] determined the effect of internal heat source on thermal stresses. A functionally graded thick hollow cylinder with temperature-dependent material properties is used by Manthena V.R., Kedar G.D. [9] to carry out transient thermal stress analysis. Recently, Ahire Y.M. et. al. [10] investigated the thermoelastic behavior in thin hollow cylinder using internal moving heat source.

The research presented in this paper focuses on steady state, nonhomogeneous heat conduction problem of a long solid cylinder subjected to internal heat source. On the cylindrical curved surface \( r = b \) of the solid cylinder, homogeneous boundary condition of third kind is prescribed. The upper
surface $z = h$ is subjected to arbitrary temperature $f(r)$ whereas the lower surface $z = -h$ is thermally insulated. The finite integral transform techniques are applied to determine the heat transfer, displacement and associated thermal stresses.

2. Mathematical formulation:

Consider a long solid cylinder enclosed by the surface $r = b$ and the planes $z = -h$ and $z = h$. A cylindrical surface heat source of radius $r = r_1$, $0 < r_1 < b$ and of strength $g_{cyl}$ of linear length of cylinder is situated concentrically inside the cylinder along $z$-axis. Heat is dissipated by convection from the boundary surface $r = b$ into surrounding at zero temperature. Temperature $f(r)$ is prescribed along the upper surface $z = h$ and insulated boundary condition is applied on the lower surface $z = -h$.

The boundary value problem of heat conduction is

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} + \frac{g(r)}{k} = 0$$

(2.1)

Subject to boundary conditions,

$$\left[ \frac{\partial T(r,z)}{\partial r} + T(r,z) \right]_{r=b} = 0 \quad -h \leq z \leq h$$

(2.2)

$$[T(r,z)]_{z=h} = f(r) \quad 0 \leq r \leq b$$

(2.3)

$$\left[ \frac{\partial T(r,z)}{\partial z} \right]_{z=-h} = 0 \quad 0 \leq r \leq b$$

(2.4)

Where, $g(r) = \frac{g_{cyl} \delta(r-r_1)}{2\pi r}$

(2.5)

$\delta(r-r_1)$ is the Dirac delta function with $0 < r_1 < b$.

The radial displacement $u_r$ and axial displacement $u_z$ are expressed in terms of the Goodier’s thermoelastic displacement potential $\phi$ and Michell’s function $M$ as in Noda[11] are

$$u_r = \frac{\partial \phi}{\partial r} - \frac{\partial^2 M}{\partial \alpha \partial z}$$

(2.6)

$$u_z = \frac{\partial \phi}{\partial z} + 2(1-\nu)\nabla^2 M - \frac{\partial^2 M}{\partial z^2}$$

(2.7)

$\phi$ should satisfy the governing equation

$$\nabla^2 \phi = K\tau$$

(2.8)

t.e. $\nabla^2 \phi = KT \\{ \tau = T - T_i = T_0 = 0 \}$

That is, $\nabla^2 \phi = \left( \frac{1-\nu}{1-\nu} \right) \alpha T$

where, the Laplacian operator $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$ and $K = \left( \frac{1}{1-\nu} \right) \alpha$

where $K, \nu$ and $\alpha$ are restraint coefficient, Poisson’s ratio and linear coefficient of thermal expansion of the material of the cylinder respectively.

Also, $M$ must satisfy the equation

$$\nabla^2 \nabla^2 M = 0$$

(2.9)

The component of the thermal stresses are represented by $\phi$ and $M$ as in Noda [11] are

$$\sigma_{rr} = 2G \left\{ \frac{\partial^2 \phi}{\partial r^2} - K\theta \right\} + \frac{\partial}{\partial r} \left( v\nabla^2 M - \frac{\partial^2 M}{\partial r^2} \right)$$

(2.10)

$$\sigma_{\theta \theta} = 2G \left\{ \frac{1}{r} \frac{\partial \phi}{\partial r} - K\theta \right\} + \frac{\partial}{\partial r} \left( v\nabla^2 M - \frac{\partial^2 M}{r \partial r} \right)$$

(2.11)

$$\sigma_{zz} = 2G \left\{ \frac{\partial^2 \phi}{\partial z^2} - K\theta \right\} + \frac{\partial}{\partial z} \left( (2-\nu)\nabla^2 M - \frac{\partial^2 M}{\partial z^2} \right)$$

(2.12)

$$\sigma_{rz} = 2G \left\{ \frac{\partial^2 \phi}{\partial r \partial z} + \frac{\partial}{\partial r} \left( (1-\nu)\nabla^2 M - \frac{\partial^2 M}{\partial r \partial z} \right) \right\}$$

(2.13)
where $G$ is shear modulus.

For traction free surface, the stress components

$$\sigma_{rr} = \sigma_{rz} = 0 \text{ at } r = 0 ; \quad \sigma_{xz} = \sigma_{rz} = 0 \text{ at } z = h$$  \hspace{1cm} (2.14)

The system of equations (2.1) to (2.14) constitutes the mathematical model of the thermoelastic problem for displacement and associated thermal stresses developed within the long solid cylinder due to surface heat source.

3. The integral (Finite Hankel) transform required to find solution:

To solve the equation (2.1) under the third kind boundary condition in equation (2.2), the finite Hankel transform over the variable $r$ and its inverse transform is introduced as in Özişik [12].

Let $T^*$ denotes the Hankel transform of $T$ and let $\xi_m$ be the eigenvalues. Then

- **Integral transform**: $T^*(\xi_m, z) = \int_0^b r' K_0(\xi_m, r') T(r', z) \, dr'$
- **Inversion formula**: $T(r, z) = \sum_{m=1}^{\infty} K_0(\xi_m, r) T^*(\xi_m, z)$

where, transform kernel

$$K_0(\xi_m, r) = \frac{\sqrt{\pi}}{b} \frac{\xi_m J_0(\xi_m b)}{(1 + \xi_m^2)^{1/2}}$$

and the eigen values $\xi_m$ are the positive roots of the transcendental equation

$$\xi J_0'(\xi b) + J_0(\xi b) = 0$$ \hspace{1cm} (3.4)

where $J_0(x)$ is Bessel function of the first kind of order zero.

The finite Hankel transform $H$ defined in equation (3.1) satisfies the relation

$$H \left[ \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right] = -\xi_m^2 T^*(\xi_m, z)$$ \hspace{1cm} (3.5)

4. The analytical solution in the integral transform domain:

4.1) Solution of temperature Distribution:

On Application of transformation defined in equation (3.1) to the equations (2.1) to (2.5), gives,

$$\left( -\xi_m^2 T^*(m, z) + \frac{\partial^2 T^*(m, z)}{\partial z^2} \right) = -\frac{\rho' T^*(m, z)}{k}$$ \hspace{1cm} (4.1.1)

$$\left[ \frac{\partial T^*(m, z)}{\partial r} + T^*(m, z) \right]_{r=b} = 0$$ \hspace{1cm} (4.1.2)

$$T^*(m, z)\big|_{z=h} = f^*(\xi_m)$$ \hspace{1cm} (4.1.3)

$$\left[ \frac{\partial T^*(m, z)}{\partial z} \right]_{z=-h} = 0$$ \hspace{1cm} (4.1.4)
\[ g^*(\xi_m) = \frac{\sqrt{T} \sinh b}{b} \left( \frac{\xi_m}{1 + \xi_m^2} \right)^2 J_0(\xi_m) \]  \hspace{1cm} (4.1.5) 

Solving equation (4.1.1) and using conditions in equations (4.1.3) and (4.1.4),

\[ T^* = \left[ f^*(\xi_m) - \frac{g^*(\xi_m)}{\xi_m^2 k} \right] \cosh(\xi_m h) + \frac{g^*(\xi_m)}{\xi_m^2 k} \]  \hspace{1cm} (4.1.6) 

Applying inverse transform defined in equation (3.2) to the equation (4.1.6), we have temperature \( T \) as

\[ T(r, z) = \sum_m \frac{\sqrt{T}}{b} \left( \frac{\xi_m}{1 + \xi_m^2} \right)^2 J_0(\xi_m) \left[ f^*(\xi_m) - \frac{g^*(\xi_m)}{\xi_m^2 k} \right] \left( \frac{\cosh(\xi_m h)}{\cosh(2\xi_m h)} \right) + \frac{g^*(\xi_m)}{\xi_m^2 k} \]  \hspace{1cm} (4.1.7) 

### 4.2) Determination of displacement components:

Assume the displacement function \( \phi(r, z) \) as

\[ \phi(r, z) = \sum_m E_m \left\{ \frac{1}{1 + \xi_m^2} \left[ \frac{1}{2} J_0(\xi_m h) \right] \left[ f^*(\xi_m) - \frac{g^*(\xi_m)}{\xi_m^2 k} \right] \left( \frac{\cosh(\xi_m h)}{\cosh(2\xi_m h)} \right) \right\} \]  \hspace{1cm} (4.2.1) 

Using equation (4.2.1) in equation (2.8), we get,

\[ E_m = \frac{K}{\sqrt{T} b} \]  \hspace{1cm} (4.2.2) 

Substituting equation (4.2.2) in equation (4.2.1),

\[ \phi(r, z) = \sum_m \frac{K}{\sqrt{T} b} \left( \frac{1}{1 + \xi_m^2} \right)^2 J_0(\xi_m h) \left[ f^*(\xi_m) - \frac{g^*(\xi_m)}{\xi_m^2 k} \right] \left( \frac{\cosh(\xi_m h)}{\cosh(2\xi_m h)} \right) \]  \hspace{1cm} (4.2.3) 

Now, the suitable form of Michell’s function \( M \), which satisfies condition in equation (2.9) is given as

\[ M(r, z) = \sum_m \frac{K}{\sqrt{T} b} \left( \frac{1}{1 + \xi_m^2} \right)^2 J_0(\xi_m h) \left[ f^*(\xi_m) - \frac{g^*(\xi_m)}{\xi_m^2 k} \right] \left( \frac{\cosh(\xi_m h)}{\cosh(2\xi_m h)} \right) \times \left\{ B_m\sinh(\xi_m(z + h)) + C_m\xi_m(z + h)\cosh(\xi_m(z + h)) \right\} \]  \hspace{1cm} (4.2.4) 

Where \( B_m \) and \( C_m \) are arbitrary functions and can be determined with the help of boundary conditions on the traction free surface, given by equation (2.14).

Now, in order to obtain \( u_r \) and \( u_z \), substitute equations (4.2.3) and (4.2.4) in equations (2.6) and (2.7), we have

\[ u_r = -\sum_m \frac{K}{\sqrt{T} b} \left( \frac{1}{1 + \xi_m^2} \right)^2 J_0(\xi_m h) \left[ f^*(\xi_m) - \frac{g^*(\xi_m)}{\xi_m^2 k} \right] \times \left\{ \left( \frac{\cosh(\xi_m h)}{\cosh(2\xi_m h)} \right) - 2\xi_m^2 \left[ B_m\sinh(\xi_m(z + h)) + C_m\xi_m(z + h)\sinh(\xi_m(z + h)) \right] + \cosh(\xi_m(z + h)) \right\} \]  \hspace{1cm} (4.2.5) 

\[ u_z = -\sum_m \frac{K}{\sqrt{T} b} \left( \frac{1}{1 + \xi_m^2} \right)^2 f^*(\xi_m) - \frac{g^*(\xi_m)}{\xi_m^2 k} \times \left\{ \left( \frac{\cosh(\xi_m h)}{\cosh(2\xi_m h)} \right) + 2(1 - 2\nu)\xi_m^2 \left[ B_m\sinh(\xi_m(z + h)) + C_m\xi_m(z + h)\cosh(\xi_m(z + h)) + 2\sinh(\xi_m(z + h)) \right] \right\} \times \left\{ \left( \frac{\cosh(\xi_m h)}{\cosh(2\xi_m h)} \right) + \frac{2\nu m}{\xi_m^2} \frac{1}{J_0(\xi_m h)} \right\} \]  \hspace{1cm} (4.2.6)
4.3) Determination of stress components:

Substituting equations (4.2.3) and (4.2.4) in equations (2.10), (2.11), (2.12) and (2.13), we have

\[
\sigma_{rr} = \sqrt{2} G K \sum_{m} \frac{1}{(1 + \xi_m^2)^{2}} \left( f^{*}(\xi_m) - \frac{g^{*}(\xi_m)}{\xi_m^2 k} \right) \times \left\{ \begin{array}{l}
\xi_m J_1(\xi_m r) \left[ (z + h) \sinh[\xi_m(z + h)] \right] \vspace{0.2cm}
- \left( \xi_m J_1(\xi_m r) / J_0(\xi_m b) \right) \left[ (z + h) \sinh[\xi_m(z + h)] + 2 \cosh[\xi_m(z + h)] \right] \end{array} \right\}
\]

\[
\times - \xi_m^4 (\nu - 1) \left\{ B_m \cosh[\xi_m(z + h)] + C_m[\xi_m(z + h) \sinh[\xi_m(z + h)] + \cosh[\xi_m(z + h)]] \right\} \frac{1}{r} \frac{J_1(\xi_m r)}{J_0(\xi_m b)}
\]

\[
+ 2 \xi_m^5 (\nu - 1) \left\{ B_m \cosh[\xi_m(z + h)] + C_m[\xi_m(z + h) \sinh[\xi_m(z + h)] + \cosh[\xi_m(z + h)]] \right\} \frac{1}{r} \frac{J_1(\xi_m r)}{J_0(\xi_m b)}
\]

(4.3.1)

\[
\sigma_{\theta \theta} = \sqrt{2} G K \sum_{m} \frac{1}{(1 + \xi_m^2)^{2}} \left( f^{*}(\xi_m) - \frac{g^{*}(\xi_m)}{\xi_m^2 k} \right) \times \left\{ \begin{array}{l}
\xi_m J_1(\xi_m r) \left[ (z + h) \sinh[\xi_m(z + h)] \right] \vspace{0.2cm}
- \left( \xi_m J_1(\xi_m r) / J_0(\xi_m b) \right) \left[ (z + h) \sinh[\xi_m(z + h)] + 2 \cosh[\xi_m(z + h)] \right] \end{array} \right\}
\]

\[
\times - 2(\nu - 1) \left\{ B_m \cosh[\xi_m(z + h)] + C_m[\xi_m(z + h) \sinh[\xi_m(z + h)] + \cosh[\xi_m(z + h)]] \right\} \frac{1}{r} \frac{J_1(\xi_m r)}{J_0(\xi_m b)}
\]

\[
+ \xi_m^4 (\nu) \left\{ B_m \cosh[\xi_m(z + h)] + C_m[\xi_m(z + h) \sinh[\xi_m(z + h)] + \cosh[\xi_m(z + h)]] \right\} \frac{1}{r} \frac{J_1(\xi_m r)}{J_0(\xi_m b)}
\]

(4.3.2)

\[
\sigma_{z z} = \sqrt{2} G K \sum_{m} \frac{1}{(1 + \xi_m^2)^{2}} \left( f^{*}(\xi_m) - \frac{g^{*}(\xi_m)}{\xi_m^2 k} \right) \times \left\{ \begin{array}{l}
\xi_m J_1(\xi_m r) \left[ (z + h) \sinh[\xi_m(z + h)] \right] \vspace{0.2cm}
- \left( \xi_m J_1(\xi_m r) / J_0(\xi_m b) \right) \left[ (z + h) \sinh[\xi_m(z + h)] + 2 \cosh[\xi_m(z + h)] \right] \end{array} \right\}
\]

\[
\times 4 \xi_m (2 - \nu) \left\{ B_m \sinh[\xi_m(z + h)] + C_m[\xi_m(z + h) \cosh[\xi_m(z + h)] + 2 \sinh[\xi_m(z + h)]] \right\} \frac{1}{r} \frac{J_1(\xi_m r)}{J_0(\xi_m b)}
\]

\[
+ 2 \xi_m^5 (1 - \nu) \left\{ B_m \sinh[\xi_m(z + h)] + C_m[\xi_m(z + h) \cosh[\xi_m(z + h)] + 4 \sinh[\xi_m(z + h)]] \right\} \frac{1}{r} \frac{J_1(\xi_m r)}{J_0(\xi_m b)}
\]

(4.3.3)
\[ \sigma_{re} = \sqrt{2} \frac{K}{b} \sum_{m} \frac{1}{(1 + \xi_m^2)^2} \left[ f^* (\xi_m) - \frac{g^* (\xi_m)}{\xi_m^2 k} \right] \times \left\{ \frac{\xi_m J_1 (\xi_m r) [\theta_m (z + h) \cosh [\theta_m (z + h)] + \sinh [\theta_m (z + h)]]}{J_0 (\xi_m b)} \right\} \]

\[ \times \left\{ \frac{\xi_m J_1 (\xi_m r) [\theta_m (z + h) \cosh [\theta_m (z + h)] + \sinh [\theta_m (z + h)]]}{J_0 (\xi_m b)} \right\} \times \left( 1 - v \right) \left\{ \frac{\xi_m J_1 (\xi_m r) [\theta_m (z + h) \cosh [\theta_m (z + h)] + \sinh [\theta_m (z + h)]]}{J_0 (\xi_m b)} \right\} \times \left\{ \frac{1}{r} J_0 (\xi_m b) \right\} \]

Now, in order to satisfy equation (2.14), Solving equations (4.3.1) and (4.3.4) for arbitrary functions \( B_m \) and \( C_m \), we get,

\[ \sigma_{re} = \frac{1}{2v \xi_m^2 \cosh (2 \xi_m h)} \]

Substituting these values of \( B_m \) and \( C_m \) in equations (4.2.5), (4.2.6), (4.3.1), (4.3.2), (4.3.3), (4.3.4) resp., we get,

\[ u_r = -\sum_m \frac{K}{\sqrt{2} b} \frac{1}{(1 + \xi_m^2)^2} \left[ f^* (\xi_m) - \frac{g^* (\xi_m)}{\xi_m^2 k} \right] \times \left\{ \frac{(v-1)(z+h) \sinh [\theta_m (z+h)]}{v \cosh (2 \xi_m h)} \right\} \]

\[ u_z = \sum_m \frac{K}{\sqrt{2} b} \frac{1}{(1 + \xi_m^2)^2} \left[ f^* (\xi_m) - \frac{g^* (\xi_m)}{\xi_m^2 k} \right] \times \left\{ \frac{\xi_m J_1 (\xi_m r) [\theta_m (z + h) \cosh [\theta_m (z + h)] + \sinh [\theta_m (z + h)]]}{J_0 (\xi_m b)} \right\} \times \left\{ \frac{1}{r} J_0 (\xi_m b) \right\} \]

\[ \sigma_{rr} = \sum_m \frac{\sqrt{2} \xi_m K}{b} \frac{1}{(1 + \xi_m^2)^2} \left[ f^* (\xi_m) - \frac{g^* (\xi_m)}{\xi_m^2 k} \right] \times \left\{ \frac{(1-v)\xi_m^2 (z+h) \cosh [\theta_m (z+h)] + \sinh [\theta_m (z+h)]}{2v \cosh (2 \xi_m h)} \right\} \frac{J_1 (\xi_m r)}{J_0 (\xi_m b)} \]

\[ \sigma_{gb} = \sum_m \frac{\sqrt{2} \xi_m K}{b} \frac{1}{(1 + \xi_m^2)^2} \left[ f^* (\xi_m) - \frac{g^* (\xi_m)}{\xi_m^2 k} \right] \times \left\{ \frac{1}{r} \xi_m (z+h) \cosh [\theta_m (z+h)] + \sinh [\theta_m (z+h)] \right\} \frac{J_1 (\xi_m r)}{J_0 (\xi_m b)} \]

\[ \sigma_{xe} = \sum_m \frac{\sqrt{2} \xi_m K}{b} \frac{1}{(1 + \xi_m^2)^2} \left[ f^* (\xi_m) - \frac{g^* (\xi_m)}{\xi_m^2 k} \right] \times \left\{ \frac{\xi_m \xi_m^2 (z+h) \cosh [\theta_m (z+h)] + \sinh [\theta_m (z+h)]}{2v \cosh (2 \xi_m h)} \right\} \frac{J_1 (\xi_m r)}{J_0 (\xi_m b)} \]

\[ + \xi_m^2 (1-v) \left\{ \frac{\xi_m (z+h) \cosh [\theta_m (z+h)] + 3 \sinh [\theta_m (z+h)]}{v \cosh (2 \xi_m h)} \right\} \frac{J_1 (\xi_m r)}{J_0 (\xi_m b)} \]
\[
\sigma_{rz} = \sum_{m} \frac{\sqrt{2} G K}{b} \frac{1}{(1 + \xi_m^2)^{1/4}} \left[ f'({\xi_m}) - \frac{g'({\xi_m})}{\xi_m^4 k} \right] \times \left\{ \frac{\{\xi_m(z + h) \cosh[\xi_m(z + h)] - \sinh[\xi_m(z + h)]\}}{\nu \cosh(2\xi_m h)} \right\} \\
\times \left\{ \frac{1}{\xi_m} \left[ -J_1(\xi_m r) + \frac{1}{r} J_0(\xi_m b) \right] + \frac{1}{\xi_m} \frac{1}{J_0(\xi_m b)} \right\} - \frac{(1-\nu)\xi_m}{2} \left\{ \frac{J_1'(\xi_m r)}{J_0(\xi_m b)} \right\}
\]

(4.3.10)

5. Numerical results:

5.1: Special Case:
Set \( f(r) = (r^2 - b^2)^2 \)

Applying transform as defined in equation (3.1) to the equation (5.1.1),

\[
f'({\xi_m}) = \int_0^b \frac{\sqrt{2}}{b} \frac{\xi_m}{(1 + \xi_m^2)^{1/2}} \frac{J_0(\xi_m r)}{J_0(\xi_m b)} [r(r^2 - b^2)^2] dr \\
= \frac{8\sqrt{2}}{\xi_m} \frac{\xi_m}{\xi_m (1 + \xi_m^2)^{1/2}} \frac{\{8-b^2\xi_m^2\} J_1(\xi_m b) - 4b\xi_m J_0(\xi_m b)}{J_0(\xi_m b)}
\]

(5.1.2)

As a special case, mathematical model is constructed for steel (0.5% carbon) with parameters:
Thermal conductivity \( k = 53.6 \text{ W/mK} \), Young’s modulus \( E = 130\text{GPa} \), Poisson ratio \( \nu = 0.35 \), Coefficient of linear thermal expansion \( \alpha = 16.5 \times 10^{-6} /\text{K} \)
Also, Set., Radius of cylinder = \( b = 3\text{m} \),
Thickness (height) = \( h = 2\text{m} \),
\( r_1 = 1.5 \),
Strength of internal heat source = \( g_{cyl} = 100^\circ\text{C} \).
Also, \( \xi_1 = 1.2558, \xi_2 = 4.0795, \xi_3 = 7.1558, \xi_4 = 10.2710, \xi_5 = 13.3984, \xi_6 = 16.5312, \xi_7 = 19.6667 \) are the positive roots of the transcendental equation \( J_0(\xi) + J_0'(\xi) = 0 \) as in [12].

5.2. Graphical analysis:
Graphs are plotted with the help of matlab for different values of \( z \).

\[\text{Fig. 2 : Distribution of temperature } T \text{ versus radius } r\]

From fig. 2, due to presence of cylindrical surface heat source in the annular part, heat is transferred from centre of cylinder to the outer circular boundary of the cylinder in radial direction. It attains peak value at the centre of cylinder (for \( r = 0 \)), become zero \( r = 2 \) and then gradually decreases towards the outer circular boundary.
From fig. 3, the radial displacement shows sinusoidal nature in radial direction. The radial displacement mostly occurs in the annular region, where the heat source is situated, as a result of internal heat generation. At centre and outer circular boundary, the displacement is nearly zero. It is maximum at upper surface and go on decreasing towards lower surface.

From fig. 4, the Axial displacement is maximum at center, becomes zero near middle part and decreases towards outer circular curved boundary. It is maximum at upper surface and decreases towards lower surface. But, as compared to radial displacement, it does not show more variation near upper and lower surface.
From fig. 5, the radial stress function develops compressive stresses near centre in the region \(0 \leq r \leq 1\) and for \(1 \leq r \leq 3\), it develops tensile stresses. At center and towards the outer traction free surface, it nearly attains equilibrium condition. It is maximum near upper plane and decreases as we move towards lower plane.

![Fig. 6: Distribution of angular stress \(\sigma_{\theta\theta}\) versus radius \(r\)](image1)

From fig. 6, the angular stresses take peak value near centre and upper surface, steep decrease in the value is observed for \(z = 0.5\) near outer curved boundary, where it nearly vanishes. It develops tensile stresses in annular region of cylinder.

![Fig. 7: Distribution of axial stress \(\sigma_{zz}\) versus radius \(r\)](image2)

From fig. 7, the axial stresses \(\sigma_{zz}\) develops tensile stresses at the centre, it attains peak value at centre, steep decrease takes place for \(z = 0.5\) and gradually stresses decreases towards outer circular boundary. In the region, \(1 \leq r \leq 3\), it develops compressive stress.

![Fig. 8: Distribution of resultant stress \(\sigma_{rz}\) versus radius \(r\)](image3)
From fig. 8, The resultant stress function $\sigma_{rz}$ is sinusoidal in nature, develops tensile stresses in annular region of cylinder. At center, it attains equilibrium condition. Also, towards the outer traction free surface, it nearly vanishes near lower surface. Peak is observed at $r = r_1 = 1.5$, near upper surface where heat source is situated.

6. Conclusion

The present paper elaborates the analysis of heat transfer, displacement components and associated thermal stresses in a long solid cylinder under steady state conditions. The effect of internal heat source which is taken as cylindrical surface heat source kept inside the cylinder along z axis is discussed. The analytical solution is outlined by adopting finite integral transform technique. A special case is considered, a mathematical model has been constructed for steel (0.5% carbon). Numerical calculations are done and interpreted graphically for different values of z. It is observed that, internal heat generated due to the surface heat source situated near $r = r_1 = 1.5$ influenced the temperature, displacement function and thermal stresses near centre mostly. Heat flow is observed from centre to outer circular boundary of the solid cylinder. The radial stress and stress function $\sigma_{rz}$ attains equilibrium condition at center.

Any particular Scenarios of thermoelastic behaviour of solid cylinders can be accomplished by allocating specific values to the parameters and functions in the obtained expressions. The results presented here will be useful and applicable to many practical engineering applications where thermal environment plays a primary and decisive role.

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