The V-filtration for tame unit $F$-crystals

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Abstract  Let $X$ be a smooth variety over an algebraically closed field of characteristic $p > 0$, $Z$ a smooth divisor, and $j : U = X \setminus Z \to X$ the natural inclusion. We introduce in an axiomatic way the notion of a $V$-filtration on unit $F$-crystals and prove such axioms determine a unique filtration. It is shown that if $\mathcal{M}$ is a tame unit $F$-crystal on $U$, then such a $V$-filtration along $Z$ exists on $j_* \mathcal{M}$. The degree zero component of the associated graded module is proven to be the (unipotent) nearby cycles functor of Grothendieck and Deligne under the Emerton–Kisin Riemann–Hilbert correspondence. A few applications to $\mathbb{A}^1$ and gluing are then discussed.

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1 Introduction

An important construction for $\mathcal{D}$-modules in characteristic zero is the $V$-filtration. While this filtration has many interesting applications, this paper mostly focuses on only a single aspect. The $V$-filtration gives a purely algebraic method for discussing the nearby and vanishing cycles functors for $\mathcal{D}$-modules without first passing through the Riemann–Hilbert correspondence. This paper will prove the existence of a unique $V$-filtration in the case when $\mathcal{M}$ is a tame unit $F$-crystal, a critical first step to a more intricate theory. As in the characteristic zero setting, this $V$-filtration can be used to define a (unipotent) nearby cycles functor on the category of tame $F$-crystals. This functor is compatible under the Emerton–Kisin Riemann–Hilbert correspondence to

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Grothendieck and Deligne’s (unipotent) tame nearby cycles functor. As an application, the unipotent nearby cycles functor will be combined with a naive vanishing cycles functor to give a gluing theorem for tame unit $F$-modules on $\mathbb{A}^1$.

Let $X$ be a smooth variety over $\mathbb{k}$ a field of positive characteristic. Let $Z$ be a smooth divisor and $U = X \setminus Z$. By a theorem of N. Katz, the category of $\mathcal{O}_X$-coherent unit $F$-modules on $U$ (a certain type of $\mathcal{D}$-module) is equivalent to the category of étale locally constant sheaves of finite dimensional $\mathbb{F}_p$ vector spaces. The latter category is well known to be equivalent to the category of representations of $\pi_{\text{et}}^1(U)$.

Under a natural tameness condition, the $V$-filtration developed in this manuscript successfully captures the behavior of the unit $F$-modules as one étale localizes near any point $z \in Z$ and encodes it as a representation of $\pi_{\text{tame}}^1(\mathcal{O}_X, z)$. Unlike in the case of characteristic zero, it does not detect and was not intended to detect, étale information on all of $X$. For example, if $W$ is a representation of $\pi_{\text{et}}^1(X)$, it does not distinguish between the nontrivial $\mathcal{O}_X$-coherent unit $F$-module on $X$ associated to $W$ and the trivial module of rank $\dim(W)$. This is because they are isomorphic as $\mathcal{O}_X$-modules after passing to an étale cover (forgetting descent data).

We show that this $V$-filtration gives data equivalent to that of the unipotent vanishing cycles functor for étale sheaves. The information provided by the $V$-filtration permits us to understand the category of unit $F$-modules on curves in a more simplistic manner, as is done in Sect. 5. There are some limitations to the theory as developed in this paper, for example, it is not known if a $V$-filtration exists in the wildly ramified case, and it does not distinguish between many interesting extensions constructed in Sect. 5.5. These are likely not irremovable limitations but ones which likely require a more sophisticated and refined development of the $V$-filtration treated in this paper. This paper is intended to serve as a stepping stone into understanding the local analysis of unit $F$-modules. Let us briefly review some of the classical development of the $V$-filtration.

The $V$-filtration in characteristic zero was first studied by Kashiwara and Malgrange. Let $X$ be a smooth complex variety over $\mathbb{C}$ and fix a simple normal crossings divisor $D$ on $X$. Assume furthermore that when working locally, $D$ is cut out by the equation $f = 0$. In [13] and [16], it was shown that there is a unique $\mathbb{C}$-indexed filtration on holonomic $\mathcal{D}$-modules satisfying formal properties and these properties may be checked locally. The four most important local formal properties are related to coherence, multiplication by $f$ being an operator of degree one, $\partial f$ being an operator of degree minus one, and the automorphism $f \partial f - r$ acting on the $r$th component of the associated graded being nilpotent. A consequence of the formal properties is that the degree zero component of the associated graded corresponds under the Riemann–Hilbert correspondence to the unipotent nearby cycles functor. Taking different components of the associated graded, one can recover the nearby and vanishing cycles functors. Moreover, on the graded objects, the morphism “multiplication by $f$” will correspond to the variation map, $\partial f$ the canonical map, and $f \partial f$ the monodromy action.

Unfortunately, in positive characteristic, the techniques from the classical setting do not translate well. Most notably, there is not a single Euler operator $f \partial f$ but infinitely many Euler operators $f^{p^r} \partial_f^{p^r}$ which all have integral eigenvalues. Additionally, the
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The ring $\mathbb{D}$ is not Noetherian, which impedes proving any statements which depend on finite
generation. The most general definition for nearby and vanishing cycles is given in [3],
where it is defined on the derived category of constructible sheaves with coefficients
in a finite abelian group $\Gamma$. Even though the definition does not require the residue
characteristic of the base field be invertible in $\Gamma$, for many theorems this hypothesis
is critical. This was an impediment to defining the notion of a perverse sheaf for $p$
not invertible in $\Gamma$. This obstruction was removed independently by Gabber and Emerton–
Kisin [6,7]. Gabber showed that the category of constructible sheaves admits an exotic $t$-structure
whose heart is known as the category of perverse sheaves. Emerton and
Kisin created a Riemann–Hilbert correspondence between the derived category of unit
$F$-modules and the category of constructible $\mathbb{F}_p$-sheaves, which translates the trivial
$t$-structure to an exotic $t$-structure on the derived category of constructible sheaves.
It turns out that the abelian categories obtained by these two methods are the same.
Therefore, it is natural to ask whether there is a purely algebraic notion of $V$-filtration
for these unit $F$-modules, which are a very special type of left $\mathbb{D}$-module. The main
results of this paper are

**Theorems 3.15 and 4.5**

1. If $\mathcal{M}$ is a tame $F$-crystal on $X \setminus Z$ where $Z$ is a smooth divisor, then $j_* \mathcal{M}$ admits
a unique $V$-filtration along $Z$. This filtration has the property that $\text{Gr}_V^0 \mathcal{M}$ is a unit
$F$-crystal on $Z$.

2. Let $\text{Sol}(\cdot)$ denote the Riemann–Hilbert solution functor taking unit $F$-modules
to perverse sheaves on the étale site, $f$ a smooth divisor, and $Z$ its support. Let
$\Psi_{\text{un, tame}}^f (\cdot)$ be the tame unipotent nearby cycles functor of Grothendieck–Deligne
(see Definition 4.1) and $\Psi_{\text{un}}^Z (\mathcal{M}) = \text{Gr}_V^0 \mathcal{M}$. There is a natural isomorphism

$$
\Psi_{\text{un, tame}}^f (\text{Sol}_X (\cdot)) \simeq \text{Sol}_Z (\Psi_{\text{un}}^Z (\cdot))[1]
$$

of functors from the category of tame unit $F$-crystals on $X$ to the category of
sheaves on $Z_{\text{ét}}$.

The nearby and vanishing cycles functors were used by Deligne in both the complex and $l \neq p$
cases to understand perverse sheaves stratified along divisors. This
approach was modified by Verdier [20] and used to glue perverse sheaves along a
divisor. An alternative approach with similar results was provided by Beilinson [2],
whose approach reduced some of the redundancy of the gluing data by using the unipotent
nearby cycles functor. The most general approach to gluing was completed by
MacPherson–Vilonen [19]. The easiest case of gluing in characteristic zero is using the
nearby cycles functor to recover the equivalence of categories between local systems
on $X \setminus Z$ and representations of $\pi_1 (X \setminus Z)$. The $V$-filtration developed in this paper
yields a positive characteristic version of this very important theorem. It is proven in
Sect. 5 using the $V$-filtration developed in Sect. 3.

Even though a notion of Bernstein–Sato polynomial has been discovered by Mustaţă
in [18], the approach taken in this paper is an approach by flat descent. In Sect. 3, formal
properties of a $V$-filtration are explored, and a descent argument is used to reduce
existence to an étale local question. After the formal properties are developed, the
definition of tame $F$-crystals is given. The work of Grothendieck–Murre is employed with the hypothesis of tame ramification to further reduce the question of existence to the case of a Kummer covering, which is explored in detail by Theorem 3.2. After discussing existence in Sect. 3, it is proven in Sect. 4 that the filtration discovered recovers the (unipotent) nearby cycles functor. As a final application in Sect. 5, there is a discussion of gluing split unit $F$-crystals on $\mathbb{A}^1$ and placing the $V$-filtration on extensions.

**Generalizations** This paper concentrates only on unit $F$-modules tamely ramified along smooth divisors. There are obvious generalizations to the proofs and definitions for unit $F^n$-modules. One can also generalize Sect. 3 from smooth divisors to divisors with normal crossings. It is the author’s hope that via continued development of the $V$-filtration in the case of singular divisors will follow from Kashiwara’s theorem and the smooth divisor case as it does in characteristic zero.

**Notation** Throughout this paper, $k$ will denote an algebraically closed field of positive characteristic. As $k$ is perfect, the Frobenius endomorphism $a \mapsto a^p$ is an isomorphism. Let $Fr : \text{Spec}(k) \to \text{Spec}(k)$ be the associated morphism of ringed spaces. We denote by $\text{Spec}(k)_{(1)}$ the $\text{Spec}(k)$-scheme whose structure morphism is $Fr^{-1}$. Since $\text{Spec}(k)_{(1)}$ is $\text{Spec}(k)$ as a ringed space, any $\text{Spec}(k)$-scheme $X$ can be naturally considered as a $\text{Spec}(k)_{(1)}$-scheme. We denote $X^{(1)} = X \times_{\text{Spec}(k)_{(1)}} \text{Spec}(k)$ where $\text{Spec}(k) \to \text{Spec}(k)_{(1)}$ is given by $Fr$. Notice that, $X^{(1)}$ and $X$ are isomorphic as $\text{Spec}(k)$-schemes but not as $\text{Spec}(k)$-schemes. The Frobenius morphism $F_X : X \to X^{(1)}$ is the pair $(id, Fr)$ where $id$ is the identity map on spaces and $Fr : \mathcal{O}_{X^{(1)}} \to \mathcal{O}_X$ is the map $f \mapsto f^p$. $F_X$ is a morphism of $\text{Spec}(k)$-schemes. We will also use the common abuse of notation, namely for an $\mathcal{O}_X$-module $\mathcal{M}$, $F_X^* \mathcal{M} = \mathcal{O}_X \otimes_{\mathcal{O}_{X^{(1)}}} \mathcal{M}$ will denote the tensor product where $\mathcal{M}$ is considered as a module on $X^{(1)}$ via realizing $X^{(1)}$ and $X$ are the same as ringed spaces. Lastly, if $Z \subset X$ is a closed subscheme, then $\mathcal{I}_Z$ will denote the reduced sheaf of ideals defining $Z$.

**2 Preliminary information**

In the first subsection, two fundamental theorems from algebraic geometry are recounted, strengthened, and worded specifically for use in the latter sections. In the definition of the objects, important facts from Grothendieck’s theory of tame ramification will be used. A subsection is provided outlining the main theorems used from the text by Grothendieck and Murre [8]. The last subsection features a restricted case of the Emerton–Kisin Riemann–Hilbert correspondence due to Katz [14]. This version of the theorem will be used to show compatibility of the definitions from Sect. 4 under this correspondence.

**2.1 Theorems from algebraic geometry**

**Theorem 2.1** [10, 8.12] (Zariski’s Main Theorem) Let $X$ be a normal locally Noetherian integral scheme and $Z \subset X$ a closed subscheme of codimension one. If $U = X \setminus Z$ and $\pi_Y : Y \to U$ with $K(Y) \supset K(U)$ a finite separable extension, then
there exists a normal integral scheme $Y$ and maps $\pi : Y \to X$ and $\iota : V \to Y$ such that:

1. The following diagram is a commutative.

\[
\begin{array}{ccc}
V & \xrightarrow{\iota} & Y \\
\downarrow{\pi_V} & & \downarrow{\pi} \\
U & \xrightarrow{=} & X
\end{array}
\]

2. $\pi$ is finite and $\iota$ is an open immersion.
3. If $y \in \pi^{-1}(Z)$ and of codimension one, then $\pi(y)$ is also of codimension one.
4. If $X$ is finite type over $\text{Spec}(k)$, then so is $Y$.

**Proof** Most of this statement is contained in the reference, and therefore, only a sketch is provided.

1. $Y$ is constructed as the normalization of $X$ in $K(V)$ and the diagram follows from this construction.
2. These statements are local so assume $X$ is affine. Since $K(X) = K(U) \subset K(V)$ is finite separable and $X$ is Noetherian and normal, $O_Y = \overline{O_X}^{K(V)}$ is a finitely generated module over $O_X$ by Eisenbud [5, 13.14].
3. If $y \in \pi^{-1}(Z)$, then since $X$ is normal and any integral extension of a normal domain satisfies the going down axiom, there is some $y'$ with $y \in \{y'\}$ such that $\pi(y')$ is codimension one. However, $y$ has codimension one that implies $y' = y$ since $Y$ is integral over $X$.
4. Clear from 2.

\[\square\]

**Theorem 2.2** [1] (Zariski–Nagata Purity Theorem) Let $X$ be a smooth $k$-scheme and $\pi : Y \to X$ be a quasi-finite morphism, which is generically étale. If $\pi$ is étale at all codimension one points of $Y$, then $\pi$ is étale.

2.2 Tame ramification along smooth divisors

In this section, $X$ is a locally Noetherian normal scheme.

**Definition 2.3** Recall that an extension of discrete valuations rings $(A, \pi) \to (B, \Pi)$ is called tamely ramified if

1. $\pi B = \Pi e$ and $e$ is invertible in $A/(\pi)$.
2. The extension $A/(\pi) \subset B/(\Pi)$ is separable.

**Definition 2.4** [8, 2.2.2] Let $Z \subset X$ be a closed subscheme of codimension one. A covering $\pi : Y \to X$ is said to be tamely ramified with respect to $Z$ if

1. $Y$ is normal.
2. Every irreducible component of $Y$ dominates an irreducible component of $X$. 
3. $\pi$ is finite.
4. $\pi$ is étale over $U = X \setminus Z$.
5. For all generic points $z \in Z$, the extension $O_{X,z} \subset O_{Y,y}$ is tamely ramified as discrete valuation rings for all $y \in \pi^{-1}(z)$.

**Lemma 2.5** [8, 2.2.8] If $Z \subset X$ is a closed subscheme of codimension one and $\pi : Y \to X$ is a finite map which is étale over $X \setminus Z$, then the following are equivalent.

1. $\pi$ is tamely ramified with respect to $Z$.
2. $\pi_z : Y \times_X \text{Spec}(O_{X,z}) \to \text{Spec}(O_{X,z})$ is tamely ramified with respect to $Z \times_X \text{Spec}(O_{X,z})$ for all $z \in Z$.
3. $\pi_\mathbb{Z} : Y \times_X \text{Spec}(O_{X,Z}^{\text{ét}}) \to \text{Spec}(O_{X,Z}^{\text{ét}})$ is tamely ramified with respect to $Z \times_X \text{Spec}(O_{X,Z}^{\text{ét}})$ for all geometric points $\mathbb{Z} \in Z$.

**Remark 2.6** [8, 2.4.2–2.4.4] Let $X$ be connected, $Z \subset X$ a normal crossings divisor and $U = X \setminus Z$. Similar to defining the usual étale fundamental group of $U$, by considering only the covers of $U$ which are tamely ramified with respect to $Z$, one can construct a profinite group $\pi_1^t(X, Z)$. It is called the tame fundamental group with respect to $X$. The continuous actions of $\pi_1^t(X, Z)$ on finite sets correspond to finite tamely ramified covers.

**Theorem 2.7** [9, XIII: Appendix 1] (Consequence of Abhyankar’s lemma) If $S$ is a scheme which is the spectrum of a strictly Henselian regular local ring and $D \subset S$ is a divisor with normal crossings, then $\pi_1^t(S, D)$ is abelian.

**Definition 2.8** Let $X$ be a scheme and $Z \subset X$ a normal crossings divisor. Suppose further $\{a_i\}_{i \in I}$ is a finite regular sequence of sections determining $Z$ and $\{n_i\}_{i \in I}$ a set of integers coprime to the residue characteristic of $X$. Define a Kummer covering (of $X$ with respect to $Z$) to be any covering isomorphic (as $X$-schemes) to the covering $O_X[\{T_i\}]/(\{T_i^{n_i} - a_i\})$.

If $X$ is normal, then any Kummer covering of $X$ is a normal covering [8, 1.7.2]. The next corollary uses Lemma 2.5 to conclude that if $Z$ is the disjoint union of smooth divisors, then étale locally one may replace a tamely ramified covering with a Kummer covering. The latter objects are easier to understand.

**Corollary 2.9** [8, 2.3.4] $\pi : Y \to X$ is tamely ramified with respect to a simple normal crossings divisor $Z$ if and only if for every point $x \in X$ there is an étale neighborhood $U \to X$ of $x$ such that $Y_U = Y \times_X U \to X_U = X \times_X U$ is a finite disjoint union of Kummer coverings.

2.3 Unit $F$-modules, $F$-crystals, and the Riemann–Hilbert correspondence

In the characteristic zero setting, there is a well-known special case of the Riemann–Hilbert correspondence: algebraic vector bundles with connection correspond to local systems for the analytic topology. This correspondence is given by the functor of flat
sections. The analog of this correspondence in characteristic \( p \) is explained in [14]. There is an equivalence of categories between unit \( F \)-crystals and étale local systems by taking the étale kernel of \( 1 - F \).

In this section, \( X \) is a smooth variety over \( \text{Spec}(k) \), which is now assumed to be of characteristic \( p > 0 \).

**Definition 2.10** A unit \( F \)-module is a quasi-coherent sheaf \( M \) equipped with an isomorphism \( \phi : F^* M \cong M \) where \( F^* \) denotes pullback along the absolute Frobenius. There is an important natural \( p \)-linear map

\[
F\phi : M \rightarrow F^* M \xrightarrow{\phi} M.
\]

Since \( \phi \) is typically not explicitly mentioned, the symbol \( F \) is implicitly understood to be \( F\phi \).

**Remark 2.11** Let \( D \) be Grothendieck’s ring of differential operators. A unit \( F \)-module is a special type of \( D \)-module. To equip a unit \( F \)-module with the structure of a left \( D \)-module, one first expresses \( D \) in terms of the divided power filtration \( D = \cup D^{(m)} \). Through Morita theory, \( D^{(m)} \) acts on \( (F^m)^* M \), which is isomorphic to \( M \) by \( \phi \). The actions obtained this way are all compatible and give \( M \) the structure of a \( D \)-module. See [12] or [15] for more details.

**Definition 2.12** A unit \( F \)-module whose underlying quasi-coherent sheaf \( M \) is \( O_X \)-coherent is called a unit \( F \)-crystal.

**Theorem 2.13** [14, 4.1.1] There is an equivalence of categories between \( \mathbb{P}_p \)-local systems on \( X_{\text{ét}} \) (\( \mathbb{P}_p \)-representations of \( \pi_{\text{ét}}^{\text{\acute{e}t}}(X) \)) and unit \( F \)-crystals on \( X \).

The correspondence is given in the following manner: To a representation \( W \) of \( \pi_{\text{ét}}^{\text{\acute{e}t}}(X) \), factor the representation through \( G = \text{Aut}_X(\tilde{X}) \) for \( \tilde{X} \) some finite Galois covering of \( X \). Equip \( W \otimes_{\mathbb{P}_p} O_{\tilde{X}} \) with the \( p \)-linear map \( \tilde{F}(w \otimes f) = w \otimes f^p \) and give this sheaf the diagonal tensor product representation structure over \( G \). Taking the \( G \)-invariants (which is just Galois descent), one obtains an \( O_X \)-coherent unit \( F \)-module on \( X \).

The inverse functor is given by the kernel sheaf of \( 1 - F : M \rightarrow M \) on the étale site. Call this functor \( K \). The category of \( \mathbb{P}_p \)-local systems has duality (denoted by \( \vee \)) so we could get an equivalence between the categories of unit \( F \)-crystals on \( X \) and the opposite category of \( \mathbb{P}_p \)-local systems on \( X_{\text{ét}} \) by considering the functor \( \mathcal{K}^\vee(\mathcal{M}) = \mathcal{K}(\mathcal{M})^\vee \). Emerton and Kisin generalized this functor to the category of unit \( F \)-modules, which is denoted \( \text{Sol}(\cdot) \), see [6, 9.2]. The convention of this paper will be to adopt the notation \( \text{Sol}(\cdot) \) even though we only intended to use this simple case when \( \text{Sol} \) is restricted to the full subcategory of unit \( F \)-crystals. On this subcategory, the functors \( \text{Sol}(\cdot) \) and \( \mathcal{K}^\vee \) are naturally isomorphic.

**Definition 2.14** A unit \( F \)-crystal is called trivial if it is isomorphic (as a unit \( F \)-module) to a free \( O_X \)-module with the usual Frobenius.

The last technical notion of this section is the pullback functor for unit \( F \)-modules.
Definition 2.15 [6, 2.3.1] Let $\pi : Y \to X$ be a map of schemes and $\mathcal{M}$ a unit $F$-module on $X$. Ignoring a standard shift, define the unit $F$-module pullback $\pi^! \mathcal{M}$ to be the sheaf of $\mathcal{O}_Y$ modules $\pi^! \mathcal{M}$ equipped with the map $F_{\pi^! \mathcal{M}}(f \otimes m) = f \otimes F_{\mathcal{M}}(m)$. The notation $\pi^!$ is selected to emphasize the extra $F$-module structure.

3 The $V$-filtration for tame unit $F$-crystals on $X$

Throughout this section, $X$ will denote a smooth variety over $\text{Spec}(\mathbb{k})$, $Z$ a smooth divisor, and $U = X \setminus Z$.

3.1 Motivating example: Kummer coverings ramified along a smooth divisor

Consider the characteristic zero situation where $X = \mathbb{A}^1_{\mathbb{C}}$, $Z$ is the origin, and $U = X \setminus Z$. Let $\pi_n : Y \to X$ be the degree $n$ Kummer covering of the origin given by $Y = X$ and $\pi_n$ the map $x \mapsto x^n$ on points. Let $\mathcal{M}$ an $\mathcal{O}_U$-coherent $\mathbb{D}_U$-module on $U$ such that $\text{Sol}(\mathcal{M})$ is a trivial local system on $V = Y \setminus \pi^{-1}(Z)$. $\mathcal{M}|_V \cong \mathcal{O}_V^n$ is not just a $\mathbb{D}_V$-module but also has a compatible action of $\text{Aut}_U(V)$. $\mathcal{M}$ is determined by a complex valued representation of $\text{Aut}_U(V)$ and is obtained by Galois descent. To create a $V$-filtration on every algebraic vector bundle with rational connection on $X$, it is enough to create a $V$-filtration on $j_* \mathcal{O}_Y^n |_Y$ for every character $\chi$ of $\text{Aut}_U(V)$. In order to do this, first consider the valuation $\nu$ on $K(X)$ determined by $Z$ and consider this as a valuation on $K(Y)$ (which we can do because $X = Y$). Define a filtration on $K(Y)$ by $V^i K(Y) = \{ s \in K(Y) \mid \nu(s) \geq i \}$ and define a filtration on $j_* \mathcal{O}_Y^n |_Y \subset K(Y)$ by intersection with $V^i K(Y)$. This filtration will have the properties which uniquely determine a $V$-filtration, and hence, this must be the $V$-filtration on $j_* \mathcal{O}_Y^n |_Y \subset K(Y)$.

Attempting to mimic this situation in positive characteristic, this section will consider degree $n$ cyclic coverings of $U$, which are totally ramified along $Z$. For the technical purpose of forcing the representation theory to be semisimple, the limitation to tamely ramified coverings will be imposed. This should not be viewed as an arbitrary restriction, as wildly ramified coverings are likely to correspond to the notion of irregular $\mathbb{D}$-modules in characteristic zero. The tamely ramified coverings that are of interest are the Kummer coverings. It will now be shown that the characteristic zero process described above survives for Kummer coverings. It will be generalized to the tamely ramified case in Theorem 3.15.

Definition 3.1 The standard filtration on $\mathcal{O}_Y$. Assume that $Z$ is smooth and irreducible with generic point $\eta$. Let $Y \to X$ be a Kummer covering of $X$ with respect to $Z$. Let $\bar{\eta}$ be the unique point lying over $\eta$ and $\nu = \nu_{\bar{\eta}} : K(Y) \to \mathbb{Z}$ its associated valuation. Define $V^i K(Y) = \{ c \mid \nu(c) \geq i \}$, where $d = \nu(\eta)$ and let $j_* \mathcal{O}_Y \subset K(Y)$ have the induced filtration where $j : U_Y = \pi^{-1}(U) \to Y$ is the natural inclusion.

Theorem 3.2 Assume that $Z$ is smooth and irreducible with generic point $\eta$, $Y \to X$ a connected Kummer covering, and $\bar{\eta}$ the unique point lying over $\eta$. Let $V \to U$ be the restriction of this covering; it is an étale Galois covering since $\mathbb{k}$ is algebraically closed. Let $W$ be an $\mathbb{F}_p$-representation of $G = \text{Aut}_U(V)$ and consider the unit
$F$-module $\tilde{M}_W = W \otimes_{F_p} O_V$ equipped with the standard Frobenius morphism $\tilde{F}$. Endow $j_*\tilde{M}_W$ with the filtration induced by the standard filtration on the second factor. If $\mathcal{M}_W = M^G_W$ and $F = \tilde{F}^G$ (see Theorem 2.13) and $(\mathcal{M}_W, F)$ is equipped with the descended filtration, then $F$ induces an isomorphism

$$O_Z \otimes O_{Z(1)} \operatorname{Gr}^{\frac{j}{\mathcal{M}_W}} \mathcal{M}_W \cong \operatorname{Gr}^{\frac{d}{\mathcal{M}_W}} \mathcal{M}_W.$$  

**Proof** $G$ is isomorphic to group of $d$-th roots of unity in $\overline{F}_p$ since $k$ is algebraically closed. In particular, it is abelian and its representations over $\overline{F}_p$ are semisimple. For any such representation $A$, denote by $A^X = \{a \in A | \sigma(a) = \chi(\sigma)a\}$ the weight space for the character $\chi \in \operatorname{Hom}_{\text{group}}(G, \overline{F}_p^\times)$.

Let $\overline{W} = W \otimes_{F_p} \overline{k}$ and decompose $\overline{W} = \oplus_{X \in G^*} \overline{W}^X$ into weight spaces. Then,

$$M_W = \tilde{M}^G_W = \oplus_{X \in G^*} \left( \overline{W}^X \otimes_{\overline{k}} O_V \right)^G = \oplus_{X \in G^*} \left( \overline{W}^X \otimes_{\overline{k}} O_V^{X^{-1}} \right)$$

$$\Rightarrow \operatorname{Gr}^{\frac{j}{\mathcal{M}_W}} \mathcal{M}_W \cong \oplus_{X \in G^*} \overline{W}^X \otimes_{\overline{F}_p} \operatorname{Gr}^{\frac{j}{\mathcal{M}_W}} \left( O^{X^{-1}}_V \right)$$

Thus, the claim will follow if it is shown that the restriction of $F$

$$O_Z \otimes O_{Z(1)} \overline{W}^X \otimes_{\overline{k}} \operatorname{Gr}^{\frac{j}{\mathcal{M}_W}} \left( O^{X^{-1}}_V \right) \rightarrow \overline{W}^X \otimes_{\overline{F}_p} \operatorname{Gr}^{\frac{j}{\mathcal{M}_W}} \left( O^{X^{-1}}_V \right)$$

is an isomorphism. Recall that $\overline{W}$ is not just a representation of $G$ over $\overline{k}$, but one which came from a $\overline{F}_p$-vector space and hence has a compatible Frobenius action $F : \overline{W} \rightarrow \overline{W}$. In particular, there is a canonical isomorphism $\overline{k} \otimes_{\overline{k}(1)} \overline{W}^X \rightarrow \overline{W}^{X_p}$ given by $b \otimes w \mapsto bF(w)$. From this, it is necessary and sufficient to show that for every $\chi \in G^*$, $F$ induces an isomorphism

$$O_Z \otimes O_{Z(1)} \overline{W}^X \otimes_{\overline{k}} \operatorname{Gr}^{\frac{j}{\mathcal{M}_W}} \left( O^{X_1}_V \right) \cong \operatorname{Gr}^{\frac{j}{\mathcal{M}_W}} \left( O^{X_1}_{V}\right).$$

Let $\pi$ be a uniformizing parameter of $O_{Y, \overline{Y}}$. Let $\sigma$ be a generator of $G$ and $\xi$ the $d$th root of unity such that $\sigma(\pi) = \xi \pi$. Let $\epsilon$ be the smallest natural number such that $\xi^\epsilon = \chi(\sigma)$. First, it will be proven that for every $k \geq 0$

$$V^{\frac{k}{\mathcal{M}_W}} \left( O^{X_1}_V \right) = \left( V^{\frac{k(j-\epsilon)}{d}} O_V \right)^{\chi(\sigma)} \pi^{p^k \epsilon}.$$  

Multiplication by $\pi^{p^k \epsilon}$ is clearly an isomorphism on $O_V$, and it is an isomorphism between $V^{\frac{j-k}{d}} K(Y)$ and $V^{\frac{j}{d}} K(Y).$ Thus, $V^{\frac{k}{\mathcal{M}_W}} O_V = (V^{\frac{k(j-\epsilon)}{d}} O_V) \pi^{p^k \epsilon}$.

The statement then follows because for $c = c' \pi^{p^k \epsilon} \in V^{\frac{j}{d}} O_V$, $\sigma(c) = \chi(\sigma)c \Leftrightarrow \sigma(c') \pi^{p^k \epsilon} = \chi(\sigma)p^k c' \pi^{p^k \epsilon} \Leftrightarrow \sigma(c') \pi^{p^k \epsilon} \pi^{p^k \epsilon} = \sigma(c') \chi(\sigma) p^k \pi^{p^k \epsilon} \Leftrightarrow \chi(\sigma)p^k c' \pi^{p^k \epsilon} \Leftrightarrow \sigma(c') = c'$
Considering the cases \( k = 0 \) and \( k = 1 \) and recalling that \( V^* O_V^G = (V^* O_V)^G \) by definition, we obtain \( \text{Gr}_j^G O_V^X = \text{Gr}_{j \cdot \alpha}^G O_V^{\pi^e} \), \( \text{Gr}_{ \frac{p}{j}}^G O_V^\pi^p = \text{Gr}_{\frac{p(j-\epsilon)}{j}}^G O_V^{\pi^p} \). This respects the map induced by \( F \) in that the diagram

\[
\begin{array}{c}
\text{Gr}_{i - \epsilon}^G O_V^{\pi^e} \xrightarrow{F} \text{Gr}_{\frac{p(j-\epsilon)}{j}}^G O_V^{\pi^p} \\
\downarrow \cong \downarrow \\
\text{Gr}_{i}^G O_V^X \xrightarrow{F} \text{Gr}_{\frac{p}{j}}^G O_V^\pi^p 
\end{array}
\]

commutes. This gives the following commutative diagram,

\[
\begin{array}{c}
\text{Gr}_{\frac{j-\epsilon}{\alpha}}^G O_V^{\pi^e} \xrightarrow{F} \text{Gr}_{\frac{p(j-\epsilon)}{j}}^G O_V^{\pi^p} \\
\downarrow \cong \downarrow \\
\text{Gr}_{\frac{i}{\alpha}}^G (O_V^X) \xrightarrow{F} \text{Gr}_{\frac{p}{j}}^G (O_V^\pi^p) 
\end{array}
\]

Thus, it is enough to show that \( F \) induces this isomorphism precisely when \( \chi = 1 \).

Working locally, assume that \( I_Z = (t) \). \( O_V^G = O_U \) and \( O_U \) is given the standard Frobenius and the filtration \( V \cdot O_U = I_Z = O_X t^i \). \( Gr^j O_U \cong O_Z \) naturally respecting \( F \). It is then clear that \( F \) induces \( O_Z \otimes O_Z^{(1)} \cong O_Z \).

\[\square\]

Remark 3.3 It is not true that the pullback of the descended filtration is the trivial filtration.

3.2 Definition and uniqueness

Definition 3.4 Let \( \mathcal{M} \) be a unit \( F \)-module on \( X \). A filtration \( V \cdot \mathcal{M} \) is called specializing (along \( Z \) of depth \( k \), see 2) if it is an exhaustive, separated, discrete, and left continuous \( \mathbb{Q} \)-indexed descending filtration such that for every \( i \in \mathbb{Q} \):

1. \( V^i \mathcal{M} \) is a quasi-coherent \( O_X \)-module.
2. \( I_Z^i V^i \mathcal{M} \subset V^{i+1} \mathcal{M} \).
3. \( F(V^i \mathcal{M}) \subset V^{ip} \mathcal{M} \).
4. \( F \) induces an isomorphism \( F_Z^i Gr^i \mathcal{M} \cong Gr^{ip} \mathcal{M} \) whenever \( Gr^i \mathcal{M} \neq 0 \).

An important property of the \( V \)-filtration in characteristic zero is that it is inconsequential whether the theory is developed for \( \mathbb{C} \)-indexed or \( \mathbb{Z} \)-indexed filtrations. The next lemma gives an analog of this phenomena in positive characteristic.

Lemma 3.5 Let \( \mathcal{M} \) be a unit \( F \)-module with \( V \cdot \mathcal{M} \) and \( W \cdot \mathcal{M} \) filtrations specializing \( \mathcal{M} \) (along \( Z \)).

1. If \( V^n \mathcal{M} \subset W^n \mathcal{M} \) for all integers \( n \), then \( V \cdot \mathcal{M} \subset W \cdot \mathcal{M} \).
2. If \( V^n \mathcal{M} = W^n \mathcal{M} \) for all integers \( n \), then \( V \cdot \mathcal{M} = W \cdot \mathcal{M} \).
Proof Let \( r \in \mathbb{Q}, m \in V^r \mathcal{M} \) and let \( s \) be the unique rational number so that \( m \in W^s \mathcal{M} \) and \( 0 \neq m \in Gr^s_W \mathcal{M} \). By condition 4 of the definition, \( F : Gr^s_W \mathcal{M} \rightarrow Gr^{sp^s}_W \mathcal{M} \) is injective so \( F^n(m) \notin W^{sp^n} \mathcal{M} = \cup_{j > sp^n} W^j \mathcal{M} \) for all \( n \). Yet \( F^n(m) \in V^{rp^n} \mathcal{M} \) for all \( n \). Thus for every \( n \), there can exist no integer \( t \) with the property that \( rp^n \geq t > sp^n \), as this would imply \( F^n(m) \notin W^{sp^n} \mathcal{M} \supset W^t \mathcal{M} \supset V^t \mathcal{M} \supset V^{rp^n} \mathcal{M} \equiv F^n(m) \). The property that for every \( n \), no such integer exists which is equivalent to the condition that \( s \geq r \). Hence, \( m \in W^s \mathcal{M} \subset V^r \mathcal{M} \). The choice of \( m \) and \( r \) was arbitrary, so \( V^r \mathcal{M} \subset V^t \mathcal{M} \) for all \( r \in \mathbb{Q} \). The latter statement clearly follows from the former. \( \Box \)

Definition 3.6 A specializing filtration is called super-specializing if it also satisfies the following properties:

1. (SS1) \( V^0 \mathcal{M} \) is \( \mathcal{O}_X \)-coherent.
2. (SS2) \( \mathcal{I}_{Z} \mathcal{V}^i \mathcal{M} = \mathcal{V}^{i+1} \mathcal{M} \) for all \( i \in \mathbb{Q}, i \neq -1 \).
3. (SS3) If locally \( \mathcal{I}_{Z} = (f) \), then multiplication by \( f, m_f : \text{Gr}^i_{V} \mathcal{M} \rightarrow \text{Gr}^{i+1}_{V} \mathcal{M} \), is an isomorphism for all \( i \neq -1 \).

Remark 3.7 The definition of specializing could use further axiomatization. The use of depth is to preserve the property of having a specializing filtration after pulling back along flat covers. The definition of super-specializing is too strong for the category of all unit \( F \)-modules when \( X \) has dimension greater than one but is a correct definition when either \( X \) is a curve or we consider only unit \( F \)-modules which are \( \mathcal{O}_U \)-coherent on \( U \); namely axiom (SS1) seems too strong of a condition to impose on arbitrary unit \( F \)-module in higher dimensions. It should also be more properly called super-specializing of depth one. In the more general situation, one should look for filtrations which are super-specializing when pulled-back along covers.

Proposition 3.8 If \( \mathcal{M} \) is a unit \( F \)-module with super-specializing filtration \( V^r \mathcal{M} \) and specializing filtration \( W^r \mathcal{M} \), then \( V^r \mathcal{M} \subset W^r \mathcal{M} \). In particular, when super-specializing filtrations exist, they are unique.

Proof Let \( m \in V^0 \mathcal{M} \) and let \( l \) be the greatest integer so that \( W^l \mathcal{M} \supset V^0 \mathcal{M} \). Such an \( l \) exists since \( W^\mathcal{M} \) is exhaustive and \( V^0 \mathcal{M} \) is \( \mathcal{O}_X \)-coherent. Let \( s \in \mathbb{Q} \) be such that \( m \in W^s \mathcal{M} \) and \( 0 \neq m \in Gr^s_W \mathcal{M} \). Then, \( F^n(m) \subset V^0 \mathcal{M} \subset W^l \mathcal{M} \) and \( F^n : Gr^s_W \mathcal{M} \rightarrow Gr^{sp^n}_W \mathcal{M} \) is injective. Hence, \( F^n(m) \notin W^{sp^n} \mathcal{M} \) for all \( n \). This can only happen if \( l \leq sp^n \) for all \( n \). Hence, \( s \geq 0 \) and \( m \in W^0 \mathcal{M} \). As \( m \) was chosen arbitrarily, \( V^0 \mathcal{M} \subset W^0 \mathcal{M} \). By property (SS2) of \( V^r \mathcal{M}, V^n \mathcal{M} \subset W^n \mathcal{M} \) for all integers \( n \geq 0 \).

Let \( m \in V^{-j} \mathcal{M} \) and \( s \in \mathbb{Q} \) so that \( m \in W^s \mathcal{M} \) and \( 0 \neq m \in Gr^s_W \mathcal{M} \). In local coordinates, \( t^{jm} \neq 0 \in Gr^{s+j}_W \mathcal{M} \) unless \( -j = s \). If \( t^{jm} \neq 0 \), then \( F^n(t^{jm}) \notin W^{(s+j)p^n} \mathcal{M} \) by injectivity of the map \( F^n : Gr^{s+j}_W \mathcal{M} \rightarrow Gr^{(s+j)p^n}_W \mathcal{M} \). Yet \( t^{jm} \in V^0 \mathcal{M} \subset W^0 \mathcal{M} \) and so it must be that \( (s+j) \geq 0 \). In either situation, it follows that \( s \geq -j \) and hence that \( m \in W^{-j} \mathcal{M} \). As \( m \) was arbitrary, \( V^n \mathcal{M} \subset W^n \mathcal{M} \) for all \( n \in \mathbb{Z} \). Now, use the previous Lemma 3.5.
Lemma 3.9 Let \( \pi : Y \to X \) be a flat morphism with \( Y \) Noetherian and \( E = \pi^{-1}(Z) \). If \( \mathcal{M} \) is a unit \( F \)-module on \( X \) with specializing filtration \( V \cdot \mathcal{M} \), then \( \pi^*V \cdot \mathcal{M} \) is a specializing filtration for \( \pi^1 \mathcal{M} \) (see Definition 2.15). If \( V \cdot \mathcal{M} \) is super-specializing and \( \pi \) is an étale covering, then \( \pi^*V \cdot \mathcal{M} \) is super-specializing.

Proof First note because the map \( \pi \) is flat, \( \pi^*V \cdot \mathcal{M} \) is a subsheaf of \( \pi^1 \mathcal{M} \). It is obviously exhaustive, discrete, left continuous, \( \mathbb{Q} \)-indexed, and descending. As \( \pi \) is a flat covering, \( \pi^*(\cap_i V^i \mathcal{M}) = \cap_i \pi^*V^i \mathcal{M} \) so it is also separated. For every \( i \in \mathbb{Q}, \pi^*V^i \mathcal{M} \) is quasi-coherent. Since \( Y \) is Noetherian, there exists an integer \( k_E \) such that \( \mathcal{I}_E^{k_E} \subset \pi^*\mathcal{I}_Z \) and thus \( \mathcal{I}_E^{k_E} \pi^*V^i \mathcal{M} \subset \pi^*\mathcal{I}_Z V^i \mathcal{M} \subset \pi^*V^i+1 \mathcal{M} \) and \( F(\pi^*V^i \mathcal{M}) \subset \pi^*F(V^i \mathcal{M}) \subset \pi^*V^{i+1} \mathcal{M} \). By flatness, there is a natural isomorphism \( \pi^1 \mathcal{M} \cong \mathcal{I}_E^{k_E}\mathcal{G}_V \cong \mathcal{G}_V \pi^1 \mathcal{M} \), which is compatible with the action of \( F \). The natural isomorphism \( F^\infty \mathcal{O}_{E(1)} \cong \mathcal{O}_E \) confirms condition 4 of the definition is still true.

When \( \pi \) is étale, \( E \) is the disjoint union of smooth divisors since \( Z \) is smooth. As (SS2) and (SS3) are local conditions, we may replace \( Y \) by a neighborhood of one of the components of \( E \) and \( X \) by the image of the neighborhood. In this case, \( \mathcal{I}_E = \pi^*\mathcal{I}_Z \) and hence \( \mathcal{I}_E \pi^*V^i \mathcal{M} = \pi^*\mathcal{I}_Z V^i \mathcal{M} \). Thus, when \( i \neq -1 \), we obtain

\[
\mathcal{I}_E \pi^*V^i \mathcal{M} = \pi^*\mathcal{I}_Z V^i \mathcal{M} = \pi^*V^{i+1} \mathcal{M}
\]

which is condition (SS2). Notice that if locally \( \mathcal{I}_Z = (f) \), then \( \mathcal{I}_E = \pi^*(f) \). Thus, we have the commuting diagram,

\[
\begin{array}{ccc}
\text{Gr}^i_{\pi^*V^i \mathcal{M}} \pi^1 \mathcal{M} & \xrightarrow{m_f} & \text{Gr}^{i+1}_{\pi^*V^i \mathcal{M}} \pi^1 \mathcal{M} \\
\downarrow \cong & & \downarrow \cong \\
\pi^*|E \text{Gr}^i_V \mathcal{M} & \xrightarrow{m_f} & \pi^*|E \text{Gr}^{i+1}_V \mathcal{M}
\end{array}
\]

from which condition (SS3) follows. Condition (SS1) is obvious. \( \square \)

Compatible descent data Recall Grothendieck’s theorem of descent: Let \( \pi : Y \to X \) be faithfully flat and quasi-compact. Define the category of descent data to be the category whose objects are pairs \((\mathcal{N}, \phi)\) where \( \mathcal{N} \) is a quasi-coherent \( \mathcal{O}_Y \)-module, \( \phi \), the descent data, is an isomorphism between \( p_1^*\mathcal{N} \) and \( p_2^*\mathcal{N} \) where \( p_i : Y \times_X Y \to Y \) the projection maps. The morphisms are maps of \( \mathcal{O}_Y \)-modules respect the descent data under pullback to \( Y \times_X Y \). The pullback functor \( \pi^*(-) \) gives an equivalence of categories between the category of quasi-coherent \( \mathcal{O}_X \)-modules and the category of descent data on \( Y \). See [17, Ch2] for a detailed treatment of descent.

Let \((\mathcal{N}, \phi)\) be an object of the category of descent data. For an \( \mathcal{O}_Y \)-submodule \( A \subset \mathcal{N} \), we will say that \( A \) is compatible with respect to the descent data \( \phi \) if \( \phi(p_1^*A) \subset p_2^*A \). In this case, because \( \phi \) is an isomorphism, it is clear that \((A, \phi)\) is again an object of the category of descent data and the inclusion map \( A \to \mathcal{N} \) a morphism in this category. If \( \mathcal{M} \) is the module associated with \((\mathcal{N}, \phi)\) under the descent theorem, then \((A, \phi)\) determines a submodule of \( \mathcal{M} \). A morphism will be called compatible with descent data if it determines a morphism in the category of descent data.
Lemma 3.10 Let $\pi : Y \to X$ be a finite faithfully flat quasi-compact morphism and $E = \pi^{-1}(Z)$. Suppose that $\mathcal{N}$ is a unit $F$-module on $Y$ with descent data over $X$ and $\mathcal{M}$ the descent of $\mathcal{N}$ to $X$.

1. If $\mathcal{N}$ has a specializing filtration (along $E$) $V \cdot \mathcal{N}$ with compatible descent data, then $\mathcal{M}$ has a specializing filtration (along $Z$) given by descent.

2. If $\pi$ is étale and $\mathcal{N}$ has a super-specializing filtration (along $E$) $V \cdot \mathcal{N}$, then it is automatically stable under the descent data and the descended filtration $V \cdot \mathcal{M}$ is also super-specializing.

Proof 1. Let $V \cdot \mathcal{M}$ be the descended filtration. It clearly satisfies axiom 1 of Definition 3.4 because the codomain of the descent functor is quasi-coherent $\mathcal{O}_X$-modules. Axiom 2 of Definition 3.4 follows because $\pi^*I_Z = I_E$. Thus, $I_E$ must descend to $I_Z$. By assumption, the descent data are compatible with $F$ so axiom 3 is satisfied. Axiom 4 follows from the exactness of descent.

2. Let $p_1$ and $p_2$ be the usual projection maps $Y \times_X Y \to Y$. By Lemma 3.9, there are unique super-specializing filtration on $p_1^*\mathcal{N}$ and $p_2^*\mathcal{N}$ obtained by pullback. The covering isomorphism $\phi : p_1^*\mathcal{N} \to p_2^*\mathcal{N}$ must then interchange these two filtrations since all the axioms of super-specializing filtrations are preserved under isomorphism. Hence, there are a descent data of the filtration which is compatible with the descent data for $\mathcal{N}$.

By the previous part of this proof, it is only left to check that the descended filtration satisfies the axioms (SS1), (SS2), and (SS3). The first is a classical result in commutative algebra [10, 2.5.2], and the latter two follow directly from $\pi$ being unramified.

3.3 Existence for tame unit $F$-crystals on $X$

Let $j : U \to X$ be the inclusion. Given a unit $F$-module $(\mathcal{M}, F)$ on $U$, we can define the pushforward of $\mathcal{M}$ to $X$ to be the $\mathcal{O}_X$-module $j_*\mathcal{M}$ equipped with the Frobenius $j_*F$.

Definition 3.11 A tame unit $F$-crystal $\mathcal{M}$ on $X$ is a unit $F$-module on $X$ satisfying the following two conditions.

1. $\mathcal{M}$ is isomorphic to the pushforward of a unit $F$-crystal on $U$. (Recall that, a unit $F$-crystal on $U$ is a unit $F$-module on $U$, which is $\mathcal{O}_U$-coherent.)

2. For every closed point $\overline{z} \in Z$, there is a Zariski open neighborhood $W$ of $\overline{z}$ and a tamely ramified covering $Y \to W$ which trivializes $\mathcal{M}|_{U \cap W}$ as a unit $F$-crystal.

3.3.1 Reduction to Kummer coverings

Lemma 3.12 A unit $F$-module on $X$ is a tame unit $F$-crystal if and only if is étale locally trivialized by a Kummer covering.

Proof ($\Rightarrow$) Let $Y$ be a tamely ramified covering of a Zariski neighborhood $W$ trivializing $\mathcal{M}|_{U \cap W}$. By Corollary 2.9, for each closed point $\overline{w} \in W$, there is an étale
neighborhood $U'$ of $\overline{w}$ for which $Y_{U'} \to U'$ is a disjoint union of Kummer covering. ($\Leftarrow$) Obvious.

**Theorem 3.13** If $\mathcal{M}$ is a tame unit $F$-crystal, then $\mathcal{M}$ has a super-specializing $V$-filtration (along $Z$) if and only if for every closed point $\overline{z} \in Z$ there exists a pair $(Y, U')$ of an étale neighborhood $U'$ of $\overline{z}$ such that $\mathcal{M}|_{U'}$ has a super-specializing filtration and $Y \to U'$ a Kummer covering trivializing $\mathcal{M}|_{U'}$.

**Proof** Necessity follows from Lemma 3.9 applied to $U' \to X$.

To prove sufficiency, the critical observation is that condition 2 of Definition 3.4 requires $\text{Gr}^V_1 \mathcal{M}$ that is supported on $Z$. Thus, $V\mathcal{M}|_U$ (if it exists) has to be the trivial filtration $V\mathcal{M} = \mathcal{M}$. By hypothesis, $Z$ may be covered by a finite number of étale neighborhoods $U'_i$ where a super-specializing $V$-filtration exists on $\mathcal{M}|_{U'_i}$. By Lemma 3.10, each $U'_i$ determines a $V$-filtration on a Zariski open neighborhood $U_i$. By the unicity of super-specializing filtrations, the filtrations must agree on double and triple intersections of the $\{U_i\}$. As the collection $\{U_i\}$ covers $Z$, the sheaf condition ensures that there is a Zariski open set $W' \supset Z$ where a super-specializing $V$-filtration exists on $\mathcal{M}|_{W'}$. Hence, by the earlier observation, it is known how to extend the $V$-filtration from a Zariski neighborhood of $W$ of $Z$ to $W \cup U$. $\square$

### 3.3.2 Existence

**Lemma 3.14** In the situation of Theorem 3.2, the filtration created is super-specializing for the module $\mathcal{M}_W$.

**Theorem 3.15** (Main Theorem) If $\mathcal{M}$ is a tame $F$-crystal and $Z$ is smooth, then $\mathcal{M}$ admits a unique super-specializing $V$-filtration.

**Proof** First use Theorem 3.13 to reduce this question to a question about Kummer coverings. Now, use the previous lemma. $\square$

**Remark 3.16** One can replace smooth by normal crossings divisors by reworking Theorem 3.2.

### 4 The tame nearby cycles functors

We now embark on proving an analog of arguably the most important aspect of the $V$-filtration in characteristic zero, its compatibility with the nearby cycles functor under the Riemann–Hilbert correspondence.

In this section, assume that $X$ is affine for simplicity. It will be necessary to work on the étale site in both the context of local systems (étale sheaves of finite dimensional $\mathbb{F}_p$-vector spaces) and unit $F$-modules. To differentiate the functors $p^*$ and $p_*$ on the étale site from the Zariski versions, the notation $p^*_\text{ét}$ and $p^*_{\text{ét}}$ will be used.
4.1 Definition

4.1.1 The tame nearby cycles functor of Grothendieck and Deligne

**Definition 4.1** [11, Éxpose I 2.7] Let $f : X \to \mathbb{A}^1_k$ be a morphism of schemes and $S$ the spectrum of the étale local ring of $\mathbb{A}^1_k$ at the origin, i.e., the spectrum of the strict Henselization of the local ring at the origin. If $X_S = X \times_{\mathbb{A}^1_k} S$, then $f$ defines a function $\tilde{f} : X_S \to S$. Let $s \in S$ be the closed point, $\eta \in S$ the generic point, $\eta'$ a point defining the tame closure of $s$, and $S'$ the normalization of $S$ in $\eta'$. Let $i$ be the inclusion of the special fiber into $X'_S := X_S \times S S'$, $j : U'_S := U_S \times S S' \to X'_S$, and $g : X'_S \to X$ the natural map. For an étale local system on $\mathcal{L}$ on $X$, define the (underived) tame nearby cycles functor by

$$\Psi_{\text{tame}}^f(L) = i_* \overset{\text{et}}{\cdot} j_* \overset{\text{et}}{\cdot} j^* \overset{\text{et}}{\cdot} g^* \mathcal{L}.$$  

**Remark 4.2** It is important to remember that $\Psi_{\text{tame}}^f(L)$ is not just a sheaf on the special fiber, but that it is also equipped with a continuous action of the group $\text{Gal}(\eta'/\eta)$.

**Definition 4.3** Using this action, the hypothesis of Definition 4.1, and following Beilinson’s presentation [2], define the tame unipotent nearby cycles functor $\Psi_{\text{un}}^f(-)$ to be the composition of $\Psi_{\text{tame}}^f$ with the functor of invariants $(-)^{\text{Gal}(\eta'/\eta)}$.

4.1.2 Unipotent nearby cycles functor for tame $F$-crystals

**Definition 4.4** (The unipotent nearby cycles functor for tame $F$-crystals) Let $\mathcal{M}$ be a tame $F$-crystal on $U$ and equip $\mathcal{M}$ with its unique super-specializing $V$-filtration $V\mathcal{M}$. Define the nearby cycles of $\mathcal{M}$ as

$$\Psi_{\text{un}}^f(Z)(\mathcal{M}) = \text{Gr}^0_{V}(\mathcal{M}) \overset{F}{\longrightarrow} \text{Gr}^0_{V}(\mathcal{M}).$$

4.2 Compatibility of definitions under the Riemann–Hilbert correspondence

**Theorem 4.5** Suppose that $X$ is smooth affine and $Z = f^{-1}(0)$ is smooth where $f : X \to \mathbb{A}^1$. For any smooth Spec($k$) variety $S$, let $\text{Sol}_S(-)$ be the functor Riemann–Hilbert solution functor of Emerton–Kisin [6, 9.2]. There is a natural isomorphism

$$\Psi_{\text{tame,un}}^f(\text{Sol}_X(-)) \simeq \text{Sol}_Z(\Psi_{\text{un}}^f(Z)(-))[1]$$

of functors from tame unit $F$-crystals on $X$ to $\mathbb{F}_p$ sheaves on $Z_{\text{ét}}$.

**Strategy of proof** After constructing a natural transformation between these two functors, we will show it is an isomorphism by showing it is an isomorphism at the generic point $\hat{z}$ of $Z$. This allows us to simplify to the case when $\mathcal{M}|_{Y_U} \cong W \otimes_{\mathbb{F}_p} O_{Y_U}$ for some Kummer covering $\pi : Y \to X$ of the form $Y = Y' \times_{\mathbb{A}^1_k} X$ for $Y' \to \mathbb{A}^1_k$ a
Kummer covering of the origin. The map \( g_{\text{ét}} \) from the definition of \( \Psi_f^{\text{lame,un}} \) will then factor through \( \pi \) and imply that \( \Psi_f^{\text{lame,un}}(\text{Sol}_X(\mathcal{M}))^\vee = W^{\text{Gal}(\eta'/\eta)} \) in cohomological degree \(-\dim X\). We will also use \( Y \) to compute the generic stalk the other functor, namely we will compute that \( \text{Gr}^0_{\pi^*} \pi^* \mathcal{M} = W^{\text{Aut}_X(Y)} \otimes_{\mathbb{F}_p} \mathcal{O}_{\pi^{-1}(Z)} \) and hence \( \text{Sol}_Z(\Psi^\text{un}_Z(\mathcal{M}))^\vee = W^{\text{Aut}_X(Y)} \) in cohomological degree \(-\dim Z = -\dim X + 1\). These two identifications will imply our natural transformation is an isomorphism.

**Proof** To ease notations, we will ignore cohomological shifts/indices and replace \( \Psi_f^{\text{lame,un}} \) and \( \Psi^\text{un}_Z \) by \( \Psi_f \) and \( \Psi_Z \), respectively. Notice \( \Psi_f(\text{Sol}_X(\mathcal{M})) = \Psi_f(\text{Sol}_U(j^*\mathcal{M})) \) so it will be sufficient to prove there is a natural isomorphism

\[
\Psi_f(\text{Sol}_U(j^*(-))) \simeq \text{Sol}_Z(\Psi_Z(-)).
\]

The functor \( i^*_\text{ét} j^* i^*_\text{ét} g_{\text{ét}}^*(\pi^* \text{Sol}_Z(\mathcal{M}))^\vee \) is left exact, \( \text{Sol}_U(j^*\mathcal{M}) = \text{Ker}(j^*\mathcal{M} \rightarrow j^*\mathcal{M})^\vee \), and \( \text{Sol}_Z(\Psi_Z(\mathcal{M})) = \text{Ker}(\text{Gr}_V^0(\mathcal{M}) \rightarrow \text{Gr}_V^0(\mathcal{M}))^\vee \) so there is a natural isomorphism

\[
\Psi_f(\text{Sol}_X(\mathcal{M}))^\vee = \Psi_f(\text{Sol}_U(j^*\mathcal{M}))^\vee \simeq \text{Ker}(i^*_\text{ét} j^* i^*_\text{ét} g_{\text{ét}}^* \mathcal{M} \rightarrow i^*_\text{ét} j^* i^*_\text{ét} g_{\text{ét}}^* \mathcal{M})^{\text{Gal}(\eta'/\eta)}.
\]

The conclusion of the theorem will follow by proving the two following statements.

1. The map \( \text{Sol}_U(j^*\mathcal{M}) \rightarrow \mathcal{M} \) has image in the subsheaf \( V^0\mathcal{M} \). This will prove that \( \Psi_f(\text{Sol}_U(j^*\mathcal{M})) \rightarrow i^*_\text{ét} j^* i^*_\text{ét} g_{\text{ét}}^* \mathcal{M} \) has image in the subsheaf \( i^*_\text{ét} j^* i^*_\text{ét} g_{\text{ét}}^* V^0\mathcal{M} \). Hence, there is a natural transformation

\[
\Psi_f(\text{Sol}_U(j^*(-))) \simeq \text{Sol}_Z(\Psi_Z(-)).
\]

2. The induced map

\[
\Psi_f(\text{Sol}_U(j^*\mathcal{M}))^\vee \rightarrow \text{Ker}
\]

\[
\left( Gr^0_{\text{ét}} j^* \text{Sol}_X(\mathcal{M})^\vee \rightarrow i^*_\text{ét} j^* i^* g_{\text{ét}}^* \mathcal{M} \rightarrow Gr^0_{\text{ét}} j^* g_{\text{ét}}^* \mathcal{M}\right)
\]

is an isomorphism of sheaves, and therefore, the natural transformation in 1 is a natural isomorphism.

1. It is enough to prove the assertion étale locally on \( X \). Since \( X \) is arbitrary, it will be enough to show \( \text{Ker}(1 - F)(X) \subset V^0 \mathcal{M}(X) \). Moreover, we may use Lemma 3.12 to assume that \( \mathcal{M} \) is such that \( \mathcal{M}|_{\mathcal{Y}_U} = W \otimes_{\mathbb{F}_p} \mathcal{O}_{\mathcal{Y}_U} \) for \( W \) a finite dimensional \( \mathbb{F}_p \)-representation of \( G = \text{Aut}_X(Y) \) with \( Y \rightarrow X \) a Kummer covering. \( \text{Ker}(1 - F)(X) \) is generated by \( G \)-invariant sums \( \sum_i w_i \otimes f_i \) (with \( w_i \in W, \ f_i \in \mathcal{O}_V \)), which are annihilated by \( 1 - F \). We may assume that the summands are linearly independent over \( \mathbb{F}_p \). By virtue of being annihilated by \( 1 - F \), we have

\[
\sum_i w_i \otimes f_i = \sum_i w_i \otimes f_i^p.
\]
If $j$ is such that $f_j = \min_i \{ \nu(f_i) \}$ where $\nu$ is the valuation determined by $Z$, then $p \nu(f_j) \geq \nu(f_j)$, or equivalently, $\nu(f_j) \geq 0$. As $j$ was chosen such that $f_j$ had minimal valuation and the filtration $V, M$ constructed in Theorem 3.2 was determined by the valuation $\nu$, it follows that $\text{Ker}(1 - F)(X) \subset V^0 M(X)$.

2. We want to check that a map of locally constant sheaves on $Z$ is an isomorphism so it is enough to check it is an isomorphism at the stalk of generic point $\hat{z} \in Z$. Thus, when checking stalks, we may use Lemma 3.12 to assume that $M$ is trivialized by a Kummer covering $Y \rightarrow X$. If $V = Y_U$, then $M|_V \cong W \otimes \mathbb{F}_p \mathcal{O}_V$ for some finite dimensional $\mathbb{F}_p$-representation $W$ of $G = \text{Aut}_X(Y)$. From the construction of the proof 3.2, it is easy to check that in this case $\text{Gr}^0_{V,M} |_V \cong W^{\text{Aut}_X(Y)} \otimes \mathbb{F}_p \mathcal{O}|_Z V$ and the $F$ structure is given by the Frobenius in the second factor. Utilizing that we are working étale locally near $\hat{z}$, we may further assume that $Y = Y' \times \mathbb{A}^1_k X$ for $Y' \rightarrow \mathbb{A}^1_k$ a Kummer covering of the origin. Factoring $g_\text{ét}$ through $\pi$, it follows that the generic stalk

$$\text{Sol}_Z(\Psi_Z(M)) \cong \text{Ker} \left( \text{Gr}^0_{z} i_{\text{ét}}^* j_{\text{ét}}^* g_\text{ét}^* M \rightarrow \text{Gr}^0_{z} i_{\text{ét}}^* j_{\text{ét}}^* g_\text{ét}^* M \right) \cong W^{\text{Aut}_X(Y)}.$$ 

On the other hand, the stalk of $i_{\text{ét}}^* j_{\text{ét}}^* g_\text{ét}^* M$ at the generic point of $Z$ is $W \otimes \mathcal{O}^{\text{tame}}_Z$ where $\mathcal{O}^{\text{tame}}_Z$ is the ring of integers of the maximal tame extension of the strict Henselization of $\mathcal{O}_X$ at $\hat{z}$. Under this identification, the action of $\text{Gal}(\eta^f/\eta)$ is the diagonal action determined by $\text{Gal}(\eta^f/\eta)$ acting naturally on the second factor and acting on $W$ via the quotient $\text{Gal}(\eta^f/\eta) \rightarrow \text{Aut}_X(Y)$. Therefore, the generic stalk

$$\Psi_f(\text{Sol}_U(j^* M)) \cong \text{Ker} \left( W \otimes \mathcal{O}^{\text{tame}}_Z \rightarrow W \otimes \mathcal{O}_Z \right)^{\text{Gal}(\eta^f/\eta)} \cong W^{\text{Gal}(\eta^f/\eta)} = W^{\text{Aut}_X(Y)}.$$ 

These two identifications naturally pass through the map from 1, which proves it is an isomorphism on the generic point and hence an isomorphism of locally constant sheaves.

\[\square\]

5 Applications to $\mathbb{A}^1_k$

Now, some applications to the case when $X = \mathbb{A}^1_k$ and $Z$ is the origin are explored to indicate that the $V$-filtration created in Sect. 3 has some of typical properties of the characteristic zero situation; namely it will be shown how to use the $V$-filtration to recover information without passing through the Riemann–Hilbert correspondence. A classical application in characteristic zero is to use the $V$-filtration to recover representations of the fundamental group. This task is too difficult in our setting, as the $V$-filtration only keeps track information étale locally near $Z$. While every tame unit
$F$-module is étale locally trivialized by a Kummer extension, it is not globally trivialized by a Kummer extension. However, in Theorem 5.9, it is shown that one can use the information provided by the nearby cycles functor to recover representations, which were trivialized by a Kummer covering. This leads naturally into considering a gluing construction. We can recover information very near $Z$ by the nearby cycles functor and glue it to sheaf on the complement.

There is a technical challenge to this process. First, taking the associated graded module of $(M, V^\bullet M)$ gives a map of $\mathcal{O}_Z$-modules $F : \mathcal{N} \to \mathcal{N}'$ with the property $F^* \mathcal{N} \to \mathcal{N}'$ is an isomorphism. If $\mathcal{N}'$ was equal to $\mathcal{N}$, this would be a unit $F$-module but it is not so we must take a small detour. In the first section, some technical categorical equivalences are explored to make the data extraction of data in section two rigorous. For a gluing theorem, one wants to put a $V$-filtration on the entire category on unit $F$-modules which is easiest to do for extensions which are étale locally split. This is the approach taken in Sect. 3 in Theorem 5.16. In the last section, nontrivial extensions possessing $V$-filtrations are explored.

5.1 Preliminary categorical equivalences

For a topological group $G$, denote the set of continuous multiplicative $k$-valued characters (where $k$ is equipped with the discrete topology) by $G^\vee$. A map of vector spaces graded by an abelian group $A$ is said to be $p$-graded if it maps the $a$th component to the $pa$th component.

Suppose that $G$ is an abelian profinite group and every finite quotient of $G$ by open subgroups has order coprime to $p$.

**Definition 5.1** Let $\mathcal{C}(G)$ be the category with objects $(V \overset{\tau}{\to} W)$ such that:

1. $V$ is a $G^\vee$-graded finite dimensional $k$-vector space.
2. $W$ is $G^\vee$-graded finite dimensional $k$-vector space.
3. $\tau$ is a $p$-graded $p$-linear map.
4. The natural map $\tilde{\tau} : k \otimes_k (1) V \to W$ is an isomorphism of vector spaces.

A morphism between $\tau : V \to W$ and $\tau' : V' \to W'$ in this category is given by a pair $(f, g)$ of graded maps such that $g \circ \tau = \tau' \circ f$.

**Definition 5.2** Define a functor $F : \text{Rep}_{\text{cont}}(G, \mathbb{F}_p) \to \mathcal{C}(G)$ by the rule:

For every object $V$ of $\text{Rep}_{\text{cont}}(G, \mathbb{F}_p)$ define $F(V) = \{ V \otimes_{\mathbb{F}_p} k \overset{\tau}{\to} V \otimes_{\mathbb{F}_p} k \}$ where $\tau_V(v \otimes c) = v \otimes c^p$ and $V \otimes_{\mathbb{F}_p} k$ is given a $G^\vee$-grading by defining $(V \otimes_{\mathbb{F}_p} k)^x = \{ x \in V \otimes_{\mathbb{F}_p} k | gv = \chi(g)v \}$. For every morphism $f : V \to V'$ define $F(f) = (f \otimes id, f \otimes id)$.

**Proposition 5.3** The functor $F$ is well-defined.

**Proof** First, it will be checked that the functor is well defined on objects. Let $V$ be a continuous $G$ representation over $\mathbb{F}_p$.

By the hypothesis on $G$ that every finite quotient has order coprime to $p$, the $G$ representation $V \otimes_{\mathbb{F}_p} k$ will be semisimple. Moreover, as $G$ is abelian, $V \otimes_{\mathbb{F}_p} k$ will
be a finite direct sum of character spaces. In particular, it has a grading by elements of the abelian group $G^\vee$. Thus, $F(V)$ satisfies conditions 1 and 2 of Definition 5.1.

Axiom 4 of Definition 5.1 is clearly satisfied, and it remains to show axiom 3.

$\tau_V$ is clearly a $p$-linear map. It is also a map of $G$ representations. Suppose that $x \in V \otimes_{\mathbb{F}_p} k$ is in the weight space of weight $\chi \in G^\vee$. For any $g \in G$

$$g \tau_V(x) = \tau_V(gx) = \tau_V(\chi(g)x) = \chi(g)^p \tau_V(x),$$

showing that $\tau_V$ is $p$-graded.

\[\square\]

**Theorem 5.4** $F$ gives an equivalence of categories

$$\text{Rep}_{\text{cont}}(G, \mathbb{F}_p) \cong \mathcal{C}(G)$$

where $\text{Rep}_{\text{cont}}(G, \mathbb{F}_p)$ is the category of continuous finite dimensional representations over $\mathbb{F}_p$.

**Definition 5.5** Define a functor $G : \mathcal{C}(G) \to \text{Rep}_{\text{cont}}(G, \mathbb{F}_p)$ by the following rule:

For every object $V \to W$, $G(\tau) = \text{Ker}(id_V - n \circ \tilde{\tau}^{-1} \circ \tau)$ with reference to the diagram,

$$V \xrightarrow{\tau} W \xrightarrow{\tilde{\tau}^{-1}} k \otimes_{k(1)} V \xrightarrow{m} V$$

where $m$ is the $k$-vector space isomorphism $f \otimes v \mapsto f v$.

$V$ is given a $G$ action by specifying that $G$ acts on the $\chi \in G^\vee$ graded piece via the character $\chi$.

For every morphism $(f, g)$ set $G((f, g)) = f|_{G(\tau)}$.

**Proposition 5.6** The functor $G$ is well defined.

**Proof** The only thing to show is that $m \circ \tilde{\tau}^{-1} \circ \tau$ respects the action of $G$. $\tau$ is $p$-graded so it respects the $G$-action. The multiplication map also respects the $G$ action. It is enough to show that $\tilde{\tau}$ respects the $G$ action. It will respect the $G$ action since $\tau$ did.

\[\square\]

**Proof** (of Theorem 5.4) First, it is shown that $id \simeq G \circ F$.

Fix $V$ a $G$ representation over $\mathbb{F}_p$. There is a map $n_V : V \to V \otimes_{\mathbb{F}_p} k$ given by $v \mapsto v \otimes 1$.

One needs to check that image of $n_V$ is inside $G(F(V))$. This will be shown by showing that $m \circ \tilde{\tau}_V^{-1} \circ \tau_V = \tau_V$.

$$(m \circ \tilde{\tau}_V^{-1} \circ \tau_V)(v) = m(\tilde{\tau}_V^{-1}(1 \cdot \tau(v))) = m(1 \otimes \tau(v)) = \tau(v)$$

Thus, $n_V : V \to (G \circ F)(V)$ is a well-defined map of sets. It is clearly a well-defined map of $G$ representations over $\mathbb{F}_p$.  


It is easy to see that $n_V$ is an isomorphism.

It remains to show that $n_V$ satisfies the naturality condition to be a natural isomorphism. The following diagram commutes for any map of representations $f : V \rightarrow V'$.

\[
\begin{array}{ccc}
V & \xrightarrow{n_V} & (G \circ F)(V) \\
\downarrow f & & \downarrow (G \circ F)(f) \\
V' & \xrightarrow{n_{V'}} & (G \circ F)(V')
\end{array} \xrightarrow{=} \Ker(id - \tau_V)
\]

This completes the proof of $G \circ F \simeq 1$.

The case of $F \circ G \simeq Id$ is now considered.

For an object $V \rightarrow W$, define $n_\tau = (m, g_\tau)$ where $m$ and $g_\tau$ are defined by the following commutative diagram:

\[
\begin{array}{ccc}
(F \circ G)(\tau) \otimes_{F_p} k & \xrightarrow{m} & V \\
\downarrow \tau_{F(\tau)} & & \downarrow \tau \\
(F \circ G)(\tau) \otimes_{F_p} k & \xrightarrow{g_\tau} & W \\
\downarrow m & \cong & \downarrow (m \circ \tau^{-1} \circ \tau)^{-1} \\
V & & V
\end{array}
\]

That is, define $n_\tau = (m, g_\tau)$.

It is clear that the maps $m$ and $g_\tau$ respect the $G^V$ grading.

One can observe that by Katz’s theorem applied to the point $\text{Spec}(k)$, the map $m$ is an isomorphism. By the diagram, $g_\tau$ is an isomorphism.

The naturality condition is the last item to check. For this, consider the following diagram which is easily confirmed as commutative for any morphism $(f, g)$ of $\mathcal{C}(G)$.

\[
\begin{array}{ccc}
(F \circ G)(\tau) \otimes_{F_p} k & \xrightarrow{m} & V \\
\downarrow \tau_{F(\tau)} & & \downarrow \tau \\
(F \circ G)((f, g)) & \xrightarrow{g_\tau} & W \\
\downarrow f & & \downarrow g \\
(F \circ G)((f, g)) & \xrightarrow{m} & V' \\
\downarrow \tau' & & \downarrow \tau' \\
(F \circ G)((f, g)) & \xrightarrow{g_{\tau'}} & W' \\
\downarrow f & & \downarrow g \\
(F \circ G)((f, g)) & \xrightarrow{m} & V' \\
\end{array}
\]

$\square$
5.2 Equivalence of categories $\text{Rep}_{\mathbb{F}_p}(\pi^K_1(X, Z))$ and Kummer $\mathbb{F}$-crystals

This section will prove the analog of the following well-known characteristic 0 theorem. Let $\text{Loc}(U)$ be the category of $\mathcal{D}_X$-modules of the form $j_* \mathcal{M}$ for $\mathcal{M}$ an $\mathcal{O}_U$-coherent $\mathcal{D}_U$-module. From the Riemann–Hilbert correspondence, we know that $\text{Loc}(U)$ is equivalent to the category of representations of $\pi_1(U) \cong \mathbb{Z}$. Moreover, we know that we can obtain this theorem from the theory of the $V$-filtration in characteristic 0 by considering $\text{Gr}^{[0,1]} M$ where we use the grading to determine the action of the generator of $\mathbb{Z}$. That is, $1 \in \mathbb{Z}$ acts on $\text{Gr}^F M$ by $e^{2\pi i r}$. Unfortunately, it is not possible to reproduce such a global theorem in positive characteristic. This is because the $V$-filtration is étale invariant and $\mathbb{A}^1$ has many nontrivial étale covers. We present a theorem by restricting the type of covers we permit. Alternatively, one could instead opt to work on projective space to obtain the same global result.

We remain in the setting $X = \mathbb{A}^1$ and $Z$ is the origin.

**Definition 5.7** If $\mathcal{M}$ is a tame $\mathbb{F}$-crystal, then define the nearby cycles functor as

$$\Psi_Z(\mathcal{M}) = \text{Gr}^{[0,1]} M \xrightarrow{F} \text{Gr}^{p[0,1]} M \equiv \bigoplus_{i \in [0,1]} \text{Gr}^{ip} M.$$ 

**Definition 5.8** Let $\pi^K_1(X, Z)$ denote the profinite quotient subgroup of $\pi^\text{tame}_1(X, Z)$ obtained by taking the inverse limit of Kummer extensions of $X$ ramified only at $Z$. There is an obvious natural isomorphism

$$\pi^K_1(X, Z) \cong \prod_{p \nmid l} \mathbb{Z}_l.$$ 

A unit $\mathbb{F}$-crystal will be called Kummer tame if it is trivialized by a Kummer covering of $X$.

**Theorem 5.9** The composition of functors $\mathbb{G} \circ \Psi_Z$ (where $\mathbb{G}$ is as in Definition 5.5) gives an equivalence of categories

$$\{\text{Kummer } \mathbb{F}\text{-crystals}\} \cong \text{Rep}_{\text{cont}}(\pi^K_1(X, Z), \mathbb{F}_p).$$

**Proof** The functor is clearly fully faithful and essentially surjective. □

5.3 The $V$-filtrations on some special modules on $\mathbb{A}^1$

Thus far, we have seen that if $\mathcal{M}$ is a unit $\mathbb{F}$-module on $X$ of the form $\mathcal{M} = j_* j^! \mathcal{M}$ when $j^! \mathcal{M}$ a unit $\mathbb{F}$-crystal on $U$ trivialized by a tame cover of $X$, then $\mathcal{M}$ has a $V$-filtration. We will now show in the case of $X = \mathbb{A}^1$ and $Z$ the origin that other modules also possess a $V$-filtration. In this subsection, $\Delta_Z = j_* \mathcal{O}_U / \mathcal{O}_X (= \mathbb{k}[t, t^{-1}]/\mathbb{k}[t]$ in local coordinates) denotes the unit $\mathbb{F}$-module of delta functions on $Z$.

**Proposition 5.10** If $\mathcal{N}$ is a locally finitely generated unit $\mathbb{F}$-module with support contained in $Z$, then $\mathcal{N}$ has a $V$-filtration.
Proof First, it is necessary to observe that \( N \cong W \otimes_{\mathbb{F}_p} \Delta \) for \( W \) a finite dimensional \( \mathbb{F}_p \)-vector space. This follows by one of two arguments utilizing Kashiwara’s theorem [6, 5.10.1] and the Riemann–Hilbert correspondence. The main point of either argument is that the category of locally finitely generated unit \( F \)-modules with supported contained in \( Z \) is equivalent to the category of unit \( F \)-modules on a point by Kashiwara’s theorem. Applying Riemann–Hilbert, we obtain that the category of locally finitely generated unit \( F \)-modules with support in \( Z \) is equivalent to the category of finite dimensional \( \mathbb{F}_p \)-vector spaces. The first avenue of proof is to compute \( f^i(W \otimes_{\mathbb{F}_p} k \otimes \Delta) \cong W \otimes_{\mathbb{F}_p} k \) with the given Frobenius corresponds to \( W \) under Riemann–Hilbert. The second more abstract proof is to consider \( \Delta \) is a simple object. Thus, the equivalence must follow because there is only one simple object in the category of \( \mathbb{F}_p \) vector spaces, and these categories are equivalent.

Give \( W \otimes_k \Delta \) the filtration \( V^{-i}(W \otimes_k \Delta) = \{x|T^i_{\Delta}x = 0\} \) for all \( i > 1 \) and \( V^i(W \otimes_k V) = 0 \) otherwise. It is not difficult to see that \( V^*M \) is a super-specializing \( V \)-filtration. □

**Proposition 5.11** If \( M \) is a unit \( F \)-module on \( X \) such that the natural adjunction morphism \( M \rightarrow j_*j^!M \) is injective and \( j_*j^!M \) is a tame \( F \)-crystal on \( U \), then \( M \) has a super-specializing \( V \)-filtration.

**Proof** If we can show étale locally on \( Z \) that \( \mathcal{M} \cong \mathcal{O}_X^{\oplus k} \oplus \mathcal{O}_U^{\oplus l} \oplus (W \otimes_{\mathbb{F}_p} \mathcal{O}_{\pi^{-1}(U)})^G \) where \( G = \text{Aut}_X(Y) \) is automorphism group of a Kummer covering of \( \pi : Y \rightarrow X \) and \( W \) is a representation of \( G \) over \( \mathbb{F}_p \) with \( W^{\text{triv}} = 0 \) (the eigenspace of the trivial character), then the result will follow from Theorem 3.2 and Lemma 3.10.

Replacing \( X \) by an étale neighborhood, we know that there exists a Kummer covering \( \pi : Y \rightarrow X \) such that

\[
j_*j^!M \cong \mathcal{O}_U^{\oplus k} \oplus (W_1 \otimes_{\mathbb{F}_p} \mathcal{O}_{\pi^{-1}(U)})^G
\]

for \( W_1 \) a \( G = \text{Aut}_X(Y) \) representation over \( \mathbb{F}_p \) with \( W_1^{\text{triv}} = 0 \). Let \( Q = \text{Coker}(\mathcal{M} \rightarrow j_*j^!M) \). \( Q \) is again a locally finitely generated unit \( F \)-module and is supported on the point \( Z \). By the work in the previous proposition, \( Q \cong W_2 \otimes_{\mathbb{F}_p} \Delta Z \).

The claim will follow if we show for any surjective homomorphism \( j_*j^!M \rightarrow Q \) that the kernel is of the form proposed in the first line of the proof. The result will follow by several computations.

**Computation 1:** \( \text{Hom}_{\mathcal{O}_X(F)}(\mathcal{O}_U^{\oplus k}, Q) \cong W_2^{\oplus k} \)

Let \( e_i \) denote the standard \( \mathcal{O}_U \)-basis for \( \mathcal{O}_U^{\oplus k} \). Given a map \( f : \mathcal{O}_U^{\oplus k} \rightarrow Q \), we must have \( f(e_i) = f(Fe_i) = f(e_i) \). Hence, \( f(e_i) = 0 \) for all \( i \) since no nonzero elements of \( W_2 \otimes_{\mathbb{F}_p} \Delta Z \) have this property. Moreover, we must have \( tf(e_i t^{-1}) = 0 \) so that \( f(e_i t^{-1}) \in W_2 \otimes_{\mathbb{F}_p} \mathcal{O}_{\pi^{-1}(U)} \). Finally, \( f \) must also satisfy \( (1 - t^{p^{-1}}F)(f(e_i t^{-1})) = f(0) = 0 \). Therefore, \( f(e_i t^{-1}) \in W_2 \otimes_{\mathbb{F}_p} \mathcal{O}_{\pi^{-1}(U)} \). The result follows.

**Computation 2:** \( \text{Hom}_{\mathcal{O}_X(F)}((W_1 \otimes_{\mathbb{F}_p} \mathcal{O}_{\pi^{-1}(U)})^G, Q) = 0 \)

By pulling back along the cover \( Y \), any morphism in \( \text{Hom}_{\mathcal{O}_X(F)}((W_1 \otimes_{\mathbb{F}_p} \mathcal{O}_{\pi^{-1}(U)})^G, Q) = 0 \) would become a morphism which respected the \( G \) actions on \( W_1 \otimes_{\mathbb{F}_p} \mathcal{O}_{\pi^{-1}(U)} \).
and $Q \otimes_{O_X} O_Y \cong W_2 \otimes_{\mathbb{F}_p} \Delta_{\pi^{-1}(Z)}$. $O_Y$ is faithfully flat so it is enough to show that there are no $O_Y[F]$-morphisms between these two objects, which also respect the $G$-action.

As in Theorem 3.2, define $\overline{W}_1 = \overline{W}_i \otimes_{\mathbb{F}_p} k$ and decompose $\overline{W}_1 = \bigoplus \chi W^\chi$ as a direct sum of nonzero character spaces. For each $\chi$, choose a basis $\{e_{\chi,1}, \ldots, e_{\chi,n_\chi}\}$ of $W^\chi$. By computation 1, we have

$$\text{Hom}_{O_Y[F]}(\overline{W}_1 \otimes_k O|_{\pi^{-1}(U)}, \Delta_{\pi^{-1}(Z)}) \cong \bigoplus \chi(W_2^{\otimes n_\chi}).$$

This isomorphism is given by associating in the coordinate $\chi, i$ of $\bigoplus \chi(W_2^{\otimes n_\chi})$ the unique vector $a_{\chi,i} \in W_2$ such that $\phi(e_{\chi, i} \otimes t^{-1}) = a_{\chi,i} \otimes t^{-1}$.

This morphism can only be $g$ stable if for each $\chi$ and $i$ we have $\chi(g)e_{\chi,i} \otimes (g t^{-1}) = \chi(g)a_{\chi,i} \otimes gt^{-1}$. As we assumed none of the $\chi$ are trivial, this can only happen if each $a_{\chi,i} = 0$. In other words, the entire morphism is 0.

**Computation 3:** $\text{Aut}_{O_Y[F]}(Q) \cong GL(W_2)$ and $\text{Aut}_{O_Y[F]}(O_{U}^{\oplus k}) \cong GL_k(\mathbb{F}_p)$, and these isomorphisms are compatible with the obvious actions on each side of the isomorphism from computation 1.

The first statement is a consequence of Kashiwara’s embedding theorem and Riemann–Hilbert: $\text{Aut}_{O_Y[F]}(Q) \cong \text{Aut}_{\mathbb{F}_p}(W_2^\chi) \cong GL(W_2)$. The second computation follows from Riemann–Hilbert on $U$,

$$\text{Aut}_{O_Y[F]}(O_{U}^{\oplus k}) \cong \text{Aut}_{\pi_1^\text{et}(U)}((\mathbb{F}_p^{\oplus k})^\vee) \cong GL_k(\mathbb{F}_p),$$

where $\pi_1^\text{et}(U)$ has trivial action $(\mathbb{F}_p^{\oplus k})^\vee$.

The result about the decomposition $\mathcal{M}$ follows from these computations because $\text{Aut}_{O_Y[F]}(O_{U}^{\oplus k})$ and $\text{Aut}_{O_Y[F]}(Q)$ act on $\text{Hom}_{O_X[F]}(j_* j^1 \mathcal{M}, Q) \cong W_2^{\otimes k}$. These actions force kernels of morphisms in the same orbit to be isomorphic. It is easy to see that if $g \in \text{Hom}_{O_Y[F]}(j_* j^1 \mathcal{M}, Q)$ corresponds to $(w_1, \ldots, w_n) \in W_2^{\oplus k}$ under $\text{Hom}_{O_X[F]}(j_* j^1 \mathcal{M}, Q) \cong W_2^{\otimes k}$, then $g$ is surjective if and only if $\{w_1, \ldots, w_n\}$ spans $W$. If we let $n = \dim_{\mathbb{F}_p} W_2$, then by a basis for $W_2$ we see that giving a surjective morphism is the same as choosing a $n \times k$ matrix of column rank $n$. We can put this matrix in normal form by using the actions of $GL_n(\mathbb{F}_p)$ and $GL_k(\mathbb{F}_p)$. Thus, the kernel of every surjective map $j_* j^1 \mathcal{M} \to Q$ is isomorphic to $O_X^{\oplus n} \oplus O_U^{\oplus (k-n)} \oplus (W_1 \otimes_k O_{\pi^{-1}(U)})^G$.

\[\square\]

5.4 Quiver/gluing description of locally split finitely generated tame unit $F$-modules on $\mathbb{A}^1_k$

The category of tame local systems on $\mathbb{A}^1_k$ (locally constant étale sheaves of finite dimensional $\mathbb{F}_p$ vector spaces on $U = \mathbb{A}^1 \setminus \{0\}$ trivialized by a cover of $\mathbb{A}^1$ tamely ramified at the origin) is too complicated to be expressed as a finite dimensional $k$-vector space with the action of an operator as it is in characteristic 0. This is because
\( \pi_1'(X, Z) \) has quotient groups, which are not cyclically generated. Likewise, the category of tame finitely generated \( F \)-modules is too complex to be described by a quiver of finite dimensional vector spaces as it is in the characteristic 0 case. See [20] for more information about gluing and [4] for information about representations of quivers. However, a theorem is now provided which allows one to recover information from the nearby and vanishing cycles quiver if some of the global data are remembered as in the classical gluing constructions.

**Definition 5.12** With the notation of Definition 4.1, a unit \( F \)-module on \( X \) (possibly not tame), \( M \), is said to be locally split near \( Z \) if for every point \( z \in Z \), there exists an \( \text{étale} \) neighborhood \( U \to X \) of \( z \) such that the extension class \( [M|_U] = 0 \in \text{Ext}^1_{\mathcal{O}_U[F]}(\text{Im}(\alpha)|_U, \text{Ker}(\alpha)|_U) \) where \( \alpha : M \to j_*j^!M \) is the natural adjunction map. A unit \( F \)-module \( M \) on \( X \) will be called a tame locally split unit \( F \)-module if it is locally split near \( Z \), and for every \( z \in Z \), there exists a Zariski neighborhood \( U' \) of \( z \) such that \( j_{U'}^*j_*^{1!}M \) is tame \( F \)-crystal on \( U' \).

**Proposition 5.13** Every object in the category of tame locally split unit \( F \)-modules on \( \mathbb{A}_k^1 \) admits a unique \( V \)-filtration.

**Proof** By previous considerations, one only needs to show it exists in a Zariski neighborhood of every point \( z \in Z \). Take this Zariski neighborhood \( V \) of \( z \) to be as in the definition of tame locally split. Using the splitting and tameness conditions, in an \( \text{étale} \) neighborhood of \( Z \), \( M \cong \text{Im}(\alpha) \oplus \text{Ker}(\alpha) \). \( \text{Im}(\alpha) \) has a super-specializing \( V \)-filtration by Proposition 5.11, and \( \text{Ker}(\alpha) \) has a super-specializing filtration by Proposition 5.10. The direct sum filtration is super-specializing so by Lemma 3.10 it determines a \( V \)-filtration in a Zariski neighborhood of \( Z \).

**Definition 5.14** Define the (unipotent) vanishing cycles sheaf to be \( \text{Gr}^{-1}(\mathcal{M}) \to \text{Gr}^{-p}(\mathcal{M}) \). There is a natural map \( (t, t^p) \) from this object to \( \Psi_{Z}^{\text{un}}(\mathcal{M}) \) given by the following diagram.

\[
\begin{array}{ccc}
\text{Gr}^{-1} \mathcal{M} & \xrightarrow{f} & \text{Gr}^{0} \mathcal{M} \\
\downarrow F & & \downarrow F \\
\text{Gr}^{-p} \mathcal{M} & \xrightarrow{t^p} & \text{Gr}^{0} \mathcal{M}
\end{array}
\]

**Definition 5.15** Let \( \mathcal{G} \) be the category of whose objects are triples \( (N, V \to W, a) \) where \( N \) is a locally finitely generated unit \( F \)-module on \( U \), \( V \to W \) is a \( p \)-linear map of finite dimensional \( k \)-vector spaces inducing \( k \otimes_{k(t)} V \cong W \) and \( a = (a_1, a_2) : (V \to W) \to \Psi_{Z}^{\text{un}}(j_*N) \) a morphism of quivers. \( \mathcal{G} \) is the category of gluing data.

**Theorem 5.16** The functor \( \mathcal{M} \mapsto (\mathcal{M}|_U, \Psi_{Z}^{\text{un}}(\mathcal{M}), (t, t^p)) \) induces an equivalence of categories

\[ \{\text{Locally finitely generated tame locally split } F \text{-modules}\} \simeq \mathcal{G}. \]
Proof. We will prove this is a well-defined functor that is fully faithful and essentially surjective.

First, we will prove that if \( M \) and \( N \) are tame locally split \( F \)-crystals on \( U \) where \( M \) and \( N \) are equipped with their \( V \)-filtrations, every map of unit \( F \)-modules \( M \to N \) is a map of filtered modules. Let \( f \) be such a map.

It is enough to check this locally and even \( \text{étale} \) locally. Thus, we may assume \( M \cong \text{Im}(\alpha_M) \oplus \text{Ker}(\alpha_M) \) and \( N \cong \text{Im}(\alpha_N) \oplus \text{Ker}(\alpha_N) \). We have that

\[
\text{Hom}_{\mathcal{O}_X[F]}(M,N) \cong \text{Hom}_{\mathcal{O}_F}(\text{Im}(\alpha_N), \text{Im}(\alpha_N)) \oplus \text{Hom}_{\mathcal{O}_F}(\text{Ker}(\alpha_M), \text{Ker}(\alpha_N))
\]

Therefore, it suffices to check two extreme cases, when both \( \alpha_M \) and \( \alpha_N \) are injective and when they are zero.

Case 1: \( \alpha_M \) and \( \alpha_N \) are injective.

In this case, \( f \) is completely determined by \( j_*j^! f \), and by the construction from Proposition 5.11, the filtration on \( M \) is just the intersection of the filtration on \( j_*j^! \mathcal{M} \) with \( \text{Im}(\alpha_M) \). Therefore, we only need to prove \( j^! f \) is a filtered morphism. However, this is obvious because after passing to an \( \text{étale} \) neighborhood and a pulling back along a Kummer covering \( \pi : Y \to X \), \( j^! f \) is a map of \( \mathcal{O}_Y[F] \)-modules that respects the \( \text{Aut}_X(Y) \)-action. By the Riemann–Hilbert correspondence, we know \( j^! f \) is completely determined by a map of representations \( W_1 \to W_2 \) with \( \mathcal{M}|_Y \cong W_1 \otimes_{\mathcal{O}_F} \mathcal{O}|_{\pi^{-1}(U)} \) and \( \mathcal{N}|_Y \cong W_2 \otimes_{\mathcal{O}_F} \mathcal{O}|_{\pi^{-1}(U)} \). It is clear that at this level, \( j^! f|_Y \) is a filtered morphism from Theorem 3.2. As \( j^! f \) is obtained by descent and the filtrations are obtained by descent, we have that \( f \) is a filtered map.

Case 2: \( \alpha_M \) and \( \alpha_N \) are zero.

In this case, we utilize the construction from Proposition 5.10. For any \( s \in V^{-i} \mathcal{M} \) and \( x \in T^i_U \), we have \( xf(s) = f(xs) = 0 \). Thus, \( f(s) \in V^i \mathcal{N} \) which proves that \( f \) is a map of filtered modules.

Thus, the functor is well defined.

It is faithful because if \( f : \mathcal{M} \to \mathcal{N} \) is a map of tame split \( F \)-modules, then it is filtered. If \( f|_U \) vanishes, then \( f \) is zero as a map \( \text{Im}(\alpha_M) \to \text{Im}(\alpha_N) \). Thus, after passing to an \( \text{étale} \) cover, we are only left to consider the case when \( \mathcal{M} = \text{Ker}(\alpha_M) \) and \( \mathcal{N} = \text{Ker}(\alpha_N) \). If \( \text{Gr}(f) = 0 \), then we must have \( f(V^{-1} \mathcal{M}) = 0 \). Yet, we have already seen in Proposition 5.10 that any module supported on \( Z \), such as \( \mathcal{M} \), is generated by \( V^{-1} \mathcal{M} \) as an \( \mathcal{O}_X[F] \)-module. Hence, \( f = 0 \).

To see it is essentially surjective, let \( (\mathcal{N}, V \to W, a) \) be an object of the target category. Write \( a = (a_1, a_2) \). In the notation of the proof of Proposition 5.11 that \( \mathcal{N} \cong \mathcal{O}_U^{\oplus k} \oplus (W_1 \otimes_{\mathcal{O}_F} \mathcal{O}|_{\pi^{-1}(U)})^G \). Now, let \( k = \dim_k \text{Gr}^0 j_* \mathcal{N}, W_2 = \text{Gr}^0 \mathcal{N}/\text{Im}(a_1) \) and consider \( Q = W_2 \otimes_k \Delta_Z \). As \( k \geq \dim_k W_2 \), there is a surjective map onto \( \mathcal{O}|_U^{\oplus k} \to Q \). Let \( K \) be the kernel of this map and define \( \mathcal{M} = \oplus(W_1 \otimes_{\mathcal{O}_F} \mathcal{O}|_{\pi^{-1}(U)})^G \oplus K \oplus (\text{Ker}(a_1) \otimes_k \Delta_Z) \) where we give \( \text{Ker}(a_1) \) the Frobenius map \( F : \text{Ker}(a_1) \to \text{Ker}(a_2) \cong F^* \text{Ker}(a_1) \). Under this functor, \( \mathcal{M} \) goes to a triple isomorphic to the triple \((\mathcal{N}, V \to W, a)\).

Given the above construction, it is clear how to prove that the functor is full. \( \square \)
Remark 5.17 The above gluing construction is clearly valid in the case when \( X \) is any curve and \( Z \) is a finite number of points as all computations are étale local.

5.5 The \( V \)-filtration for some nonsplit extensions

We now turn our investigation to modules which are not locally split. We will show that in a very interesting case, the \( V \)-filtration still exists. However, we will quickly notice that the gluing information is no longer faithful on objects. This indicates that a further refined definition of \( V \)-filtration will need to be created to understand these instances.

An important takeaway from this section is to notice the striking difference between characteristic \( p \) and 0. In characteristic 0, the fact that \( \text{Ext}^1 \mathcal{O}_U, \Delta_Z \cong \mathbb{C} \) plays an important role in the gluing constructions of [2]. We will observe in the positive characteristic that \( \text{Ext}^1 \mathcal{O}_U, \Delta_Z \) is an infinite dimensional \( \mathbb{F}_p \) vector space with the same cardinality as the set \( \mathbb{k} \).

Recall that \( X = \mathbb{A}^1 \), \( Z \) is a point, and \( \Delta_Z = j_* \mathcal{O}_U / \mathcal{O}_X = (\mathbb{k}[t, t^{-1}] / \mathbb{k}[t] \) in local coordinates\) denotes the unit \( F \)-module of delta functions on \( Z \).

Theorem 5.18 1. Let \( \{e_\lambda\}_{\lambda \in \Lambda} \) be an \( \mathbb{F}_p \)-basis for \( \mathbb{k} \). The \( \mathbb{F}_p \)-vector space \( \text{Ext}^1_{\text{unit}} j_* \mathcal{O}_U, \Delta_Z \) has an \( \mathbb{F}_p \)-basis given by the classes \( \left[ \frac{e_\lambda}{t^n + 1} \right] \) for \( n \in \mathbb{N} \), \( p \nmid n \), and \( \lambda \in \Lambda \).

2. For each representative \( c(t) \) of a class in \( \text{Ext}^1_{\text{unit}} j_* \mathcal{O}_U, \Delta_Z \), define \( \mathcal{M}_c \) as the \( \mathcal{O}_X \)-module \( j_* \mathcal{O}_U \oplus \Delta_Z \) equipped with the Frobenius \( F(f, g) = (f^p, tf^p c(t) + g^p) \). Equipped with the obvious inclusion and projection maps, \( \mathcal{M}_c \) is an extension of unit \( F \)-modules and its class in \( \text{Ext}^1_{\text{unit}} j_* \mathcal{O}_U, \Delta_Z \) is \( [c(t)] \). Note \( \mathcal{M}_c \) is automatically unit by [6, 5.2].

3. Each extension \( \mathcal{M}_c \) determined by a class \( c(t) \in \text{Ext}^1_{\text{unit}} j_* \mathcal{O}_U, \Delta_Z \) admits a super-specializing \( V \)-filtration when \( p \nmid v_t(c) + 1 \) and \( v_t(c) < 0 \). This filtration induces a filtration on the exact sequence

\[
0 \to \Delta_Z \to \mathcal{M}_c \to j_* \mathcal{O}_U \to 0.
\]

The induced filtration on \( \Delta_Z \) coincides with the \( V \)-filtration on \( \Delta_Z \), and the filtration on \( j_* \mathcal{O}_U \) is the \( V \)-filtration on \( j_* \mathcal{O}_U \) shifted by \( -\frac{n}{p} \).

Proof 1. Follows directly from the free \( \mathcal{O}_X[F] \) resolutions of \( j_* \mathcal{O}_U \) given by the root morphism in [15] and the associated complex from [6, 5.3.3]. It is the free resolution given by \( \mathcal{O}_X[F] \) acting on \( t^{-1} \). The kernel of this action is the left \( \mathcal{O}_X[F] \)-module generated by \( 1 - t^{p-1} F \). Thus, there is a quasi-isomorphism

\[
j_* \mathcal{O}_U \cong (\mathcal{O}_X[F] \xrightarrow{x^{p-1} t^{-1} - x} \mathcal{O}_X[F])
\]

Since \( j_* \mathcal{O}_U \) and \( \Delta_Z \) are unit \( F \)-modules, any extension of them is a unit \( F \)-module by Emerton and Kisin [6, 5.2]. Also, homomorphisms between unit \( F \)-modules are precisely \( \mathcal{O}_X[F] \)-module homomorphisms. Therefore, we obtain

\[
\text{Ext}^1_{\text{unit}} j_* \mathcal{O}_U, \Delta_Z \cong \text{Ext}^1_{\mathcal{O}_X[F]} j_* \mathcal{O}_U, \Delta_Z.
\]
The resolution above implies

\[ \operatorname{Ext}^1_{\text{unit} \mathcal{F}}(j_* \mathcal{O}_U, \Delta Z) = \text{Cokern} \left( \Delta Z \xrightarrow{x \mapsto t^{p-1} Fx - x} \Delta Z \right). \]

It is obvious that the set of classes \([\frac{e_i}{\sqrt[p]{t^l}}]_n\) such that \(n \in \mathbb{N}\) and \(\lambda \in \Lambda\) generates the cokernel as a \(\mathbb{F}_p\)-vector space because they generated \(\Delta Z\) as a \(\mathbb{F}_p\)-vector space. Notice that if \(n = p^k l\) (with \(k \geq 1\)) and \(p \nmid l\) and \(a \in \mathbb{k}\), then since \(\mathbb{k}\) is perfect, \(\sqrt[p]{a} \in \mathbb{k}\) and \(t^{p-1} F(\sqrt[p]{a} t^{-1} p^{-k-1} - \sqrt[p]{a} t^{-l} p^{-k-1}) = at^{-n-1} - \sqrt[p]{a} t^{-l-1}\).

Therefore, we see that \([\frac{a}{\sqrt[p]{t^l}}]_n = [\frac{\sqrt[p]{a}}{t^{l+1}}]_n\). It is easy to see by applying this method \(k\) times,

\[ \left[ \frac{a}{t^{n+1}} \right] = \left[ \frac{\sqrt[p]{a}}{t^{l+1}} \right]. \]

Now, we are left to show that the proposed set is linearly independent. Suppose \(\sum_{i=1}^{m} [\frac{a_i e_{\lambda_i}}{\sqrt[p]{t^{n_i+1}}}] = [0]\) where \(n_{i+1} = n_i \geq 1\), \(p \nmid n_i\), \(0 \neq a_i \in \mathbb{F}_p\) and \(\{e_{\lambda_i}\}_{n_i=n}\) is linearly independent for every \(n\). This is the same as assuming there exists \(g(t) = g_{\alpha} t^{-\alpha} + \cdots + g_{\beta} t^{-\beta}\) where the coefficients are in \(\mathbb{k}\) and

\[ \sum_{i=1}^{m} a_i e_{\lambda_i} = t^{p-1} F(g(t)) - g(t). \]

From this equation, it is clear that \(\alpha \geq 2\) which leaves that \(p\) must divide \(n_{m}\), which is a contradiction.

2. The equivalence between \(\operatorname{Ext}^1(\Delta Z, j_* \mathcal{O}_U)\) computed as a derived functor and the group of equivalence classes of extensions under Baer product is given by \([\mathcal{M}_c] \mapsto \delta_c(id_{j_* \mathcal{O}_U})\) where \(\delta_c\) is the connecting morphism induced by \(\mathcal{M}_c\) in the snake lemma. We will now recall the construction of the connecting morphism. Let \(\mathcal{O}_X[F] \to \mathcal{O}_X[F]\) be the resolution of \(j_* \mathcal{O}_U\) from part 1. We get the following diagram with exact rows.

\[
\begin{array}{ccccccccc}
0 & \xrightarrow{} & \text{Hom}_{\mathcal{O}_X[F]}(\mathcal{O}_X[F], \Delta Z) & \xrightarrow{} & \text{Hom}_{\mathcal{O}_X[F]}(\mathcal{O}_X[F], \mathcal{M}_c) & \xrightarrow{} & \text{Hom}_{\mathcal{O}_X[F]}(\mathcal{O}_X[F], j_* \mathcal{O}_U) & \xrightarrow{} & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \xrightarrow{} & \text{Hom}_{\mathcal{O}_X[F]}(\mathcal{O}_X[F], \Delta Z) & \xrightarrow{} & \text{Hom}_{\mathcal{O}_X[F]}(\mathcal{O}_X[F], \mathcal{M}_c) & \xrightarrow{} & \text{Hom}_{\mathcal{O}_X[F]}(\mathcal{O}_X[F], j_* \mathcal{O}_U) & \xrightarrow{} & 0
\end{array}
\]

The identity map on \(j_* \mathcal{O}_U\) is in the lower right hand corner of this diagram and is given by \(\phi(1) = t^{-1}\). Notice that \(\tilde{\phi}(1) = (t, 0)\) is a lift of \(\phi\) to the middle term of the bottom row. The snake lemma constructs \(\delta_c(id_{j_* \mathcal{O}_U}) = [x \mapsto \tilde{\phi}((t^{p-1} F - 1)x)]\).

Under the identification of \(\operatorname{Ext}^1_{\mathcal{O}_X[F]}(j_* \mathcal{O}_U, \Delta Z)\) with \(\text{Cokern}(\Delta Z \to \Delta Z)\) (which is given by evaluation at 1), we have

\[ [\mathcal{M}_c] \mapsto [(t^{p-1} F - 1)\tilde{\phi}(1)] = [c(t)]. \]
3. Set \( n = -v_t(c(t)) - 1 \) and using the description of the previous proposition, define a filtration by \( (f, \overline{g}) \in V^j \mathcal{M}_c \) if only if \( \min \{v_t(f) - n/p, v_t(\overline{g})\} \geq j \) for all \( j \in \mathbb{Q} \) and \( j \leq 0 \). For \( j \geq 0 \), one requires \( g = 0 \) and \( v_t(f) - n/p \geq j \). Axioms 1 and 2 of a specializing filtration are clear. Axiom 3 follows through from a case-by-case analysis. Moreover, the nonzero components of the associated graded module are \( \text{Gr}^i \mathcal{M}_c \) with \( r = i - n/p \) or \( r = i \) for any integer \( i \). In the noninteger \( r = i - n/p \) case, it is a single vector space generated by \( (t^i, 0) \), and in the integer case \( r = -i \leq 1 \), it is generated by \( (0, 1/p) \). It is easy to confirm axiom 4 and the axioms of super-specialization.

\[\square\]

**Remark 5.19** For each class \( c \) with \( p \nmid v_t(c) + 1 \), the extension \( \mathcal{M}_c \) is a type of maximal extension in the category of super-specializable modules. The shift in the filtration appears to account for the Tate twist for perverse sheaves in the \( l \neq p \) situation. See [2].

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