Method of estimation of cusp catastrophe occurrence possibility based on limited statistical information

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Abstract. This paper deals with the improvement of probabilistic methods of design by using the catastrophe theory and the possibility theory application. A brief description of the catastrophe theory is presented, control parameters are viewed as fuzzy variables with unknown distribution laws in the first part of the paper. The definition of the possibility measure of the cusp catastrophe occurrence is given in the second part of the paper. The condition of the cusp catastrophe occurrence is formulated according to Zadeh’s extension principle. The proposed method can be applied to solve the problems of the technical systems reliability and to give recommendations for engineering of forest machine and equipment elements in the case of limited statistical information.

1. Introduction
Probabilistic methods of design are sufficiently developed and, for now, are widely used in engineering practice. It is supposed that input statistical information is full, i.e. the distribution laws of loads, dimensions, mechanical properties and other characteristics are known and can be applied to perform design calculations with standard accuracy. But in real practice actually machine operating conditions vary greatly and depend on a great number of natural, social and individual factors. Therefore, it often is quite difficult to obtain full statistical information about the level and conditions of loading. The load-bearing characteristics of materials depend on special features of manufacturing process and working conditions (for example, temperature, impact conditions etc.). Thus, these characteristics are also not clearly determined and, in many cases, their distribution laws are unknown. Consequently, the application of traditional probability methods with the use of limited statistical data can lead to the wrong result. In this case, the possibility theory methods [1-3] can be applied to solve this problem. It should be noted that Dempster-Shafer theory [4], Bayesian analysis [5], interval average methods [6] can also provide acceptable solution with limited information. Current theories and their application features were particularly analyzed and used for estimation of reliability measures by L V Utkin in his fundamental work [3]. In contrast to common practice, catastrophe theory methods are taken as a principle of limit-state models formulation by Pitukhin A.V. [7, 8]. The cusp catastrophe is used as the basis, control parameters are viewed as Gaussian stochastic quantities (functions) with known distribution laws.

The aim of this study is to present the method of estimation of the cusp catastrophe occurrence possibility with limited input statistical information on control parameters for solving the problems of the technical systems reliability.
2. Methods and Materials

The catastrophe theory was first recognized in the middle of the XX\textsuperscript{th} century as a special branch of the dynamical systems theory [9-11]. Catastrophe theory methods provide an opportunity to research sudden quantitative changes of a system after smooth variation of external conditions. According to Thom’s classification [11], there are seven types of elementary catastrophes: fold, cusp, swallowtail, butterfly, hyperbolic umbilic, elliptic umbilic, parabolic umbilic. The cusp catastrophe potential function is the function of one state variable and two control variables, therefore it is most often used to model a system’s state.

The canonical form of the cusp catastrophe potential function

\[ V_{ab}(x) = \frac{1}{4} x^4 + \frac{1}{2} ax^2 + bx, \]  

(1)

where, \( x \) is a state variable; \( a, b \) – control variables.

Cusp catastrophe manifold \( M \) can be expressed through the relationship

\[ \frac{d}{dx} V_{ab}(x) = x^3 + ax + b. \]  

(2)

Cubic equation (2) may have from one to three real roots, depending on discriminant

\[ D = 4a^3 + 27b^2, \]

\[ D < 0 \text{ – three different real roots, } D > 0 \text{ – one real root, } \]

\[ D = 0 - \begin{cases} a \neq 0, b \neq 0 \text{ – three real roots, two are equal;} \\ a = b = 0 \text{ – three equal real roots.} \end{cases} \]

Manifold \( M \) points located in on the internal cusp surface (unstable space, \( I \) domain) correspond to an unstable equilibrium system state. The trajectory that defines the system state can leave \( I \) domain due to changing variance \( a, b \). It causes a sudden system change or catastrophe. According to Thom’s perfect delay principle [11], the catastrophe happens only when the trajectory leaves \( I \) domain (figure 1).

![Figure 1. Manifold M and cusp catastrophe representation.](image-url)
3. Results and Discussion

In the general case, \(a\) and \(b\) are control variables changing during the time \(t\) and system state, which will be defined by the random process. The problem of the random process (function) \(D(a,b,t)\) overshoot was solved in [8].

Catastrophe happens when \((a, b)\) points trajectory is leaving \(I\) domain, and \(D(a,b)\) sign is changing from negative to positive.

If \(a\) and \(b\) are stochastic quantities with known mean values and unknown distribution laws, the condition of cusp catastrophe occurrence is

\[ D = 4\tilde{a}^3 + 27\tilde{b}^2 \geq 0, \]  

the condition of cusp catastrophe nonoccurrence is

\[ D = 4\tilde{a}^3 + 27\tilde{b}^2 < 0, \]

where \(\tilde{a}, \tilde{b}\) are fuzzy variables, i.e. variables with limited information (unknown distribution laws).

The solution of this problem includes the determination of the possibility measure of cusp catastrophe occurrence \(Q\), the possibility measure of cusp catastrophe nonoccurrence \(R\) and the necessity measure of cusp catastrophe nonoccurrence \(N\).

Fuzzy function \(D(d)\) of variables \(\tilde{a}, \tilde{b}\), determining discriminant \(D\) from the point of possibility theory, is

\[ D(d) = 4\tilde{a}^3 + 27\tilde{b}^2. \]  

Control variables possibility distribution laws are supposed to be known

\[ \pi_a(a) = \exp \left[ - \left( \frac{a - m_a}{h_a} \right)^2 \right]; \]  
\[ \pi_b(b) = \exp \left[ - \left( \frac{b - m_b}{h_b} \right)^2 \right], \]

where, \(m_a, m_b\) are mean values of control parameters; \(h_a, h_b\) are measures of dispersion of control parameters.

Fuzzy variable \(D\) possibility distribution law is

\[ \pi_D(d) = \exp \left[ - \left( \frac{d - m_d}{h_d} \right)^2 \right], \]

where \(m_d\) and \(h_d\) are parameters under evaluation.

According to [3], the inverse function is

\[ \pi_D^{-1}(d) = d = m_d \pm h_d (- \ln \alpha)^{1/2} = m_d \pm h_d \beta; \]  
\[ \alpha = \pi_D(d) = \exp(-\beta^2); \]  
\[ \beta = (- \ln \alpha)^{1/2}. \]

The graph of fuzzy variable \(D\) possibility distribution law is presented in figure 2.
According to Zadeh’s extension principle [2]

\[ d = 4(m_a - h_a \beta)^3 + 27(m_b - h_b \beta)^2. \]  

(11)

Negative sign before \( h_a \) and \( h_b \) means, that the function \( D(d) \) of variables \( a, b \) is an increasing function.

The condition of cusp catastrophe occurrence according to (3) and (11) is:

\[ d = 4(m_a - h_a \beta)^3 + 27(m_b - h_b \beta)^2 \geq 0. \]  

(12)

This expression contains unknown arguments \( \beta \) and \( d \). It is necessary to determine \( m_d \), then \( \alpha \) and \( h_d \) to find the full solution of the problem.

By setting \( \beta \) to zero (\( \beta = 0 \)), according to equations (8) and (11) we can obtain:

\[ m_d = 4m_a^3 + 27m_b^2. \]  

(13)

The value \( d = 0 \) corresponds to the condition of an unstable equilibrium system state. The value of parameter \( \beta \) is defined by the solution of cubic equation (11) or (12) by setting \( d \) to zero (\( d = 0 \)).

Using equation (9), we can obtain \( \alpha \), and, according to equation (8) we can obtain \( h_d \):

\[ h_d = \frac{m_d}{\beta}. \]

The possibility measure of cusp catastrophe occurrence can be expressed as:

\[ Q = \exp \left[ -\left( \frac{m_d}{h_d} \right)^2 \right]. \]  

(14)

The necessity measure of cusp catastrophe nonoccurrence in this case is:

\[ N = 1 - Q. \]  

(15)

The possibility measure of cusp catastrophe nonoccurrence \( R \) is determined by the method of estimation of possibility of mechanical systems failure-free operation [3].
4. Conclusion
Catastrophe theory methods are used to solve stability problems of shells, beams, plates etc. The proposed method is based on mathematical catastrophe theory and possibility theory. Therefore, it can be used to estimate the system failure possibility. The proposed method differs from statistical catastrophe theory methods [7, 8], based on probability approach (probability distribution functions of control variables must be known). In the case of limited statistical information we assume control variables possibility distribution law, in this case the order (algorithm) of calculations is almost identical.

This approach to designing enables us to solve the problems of the systems reliability and to give recommendations for engineering of forest machine and equipment elements in terms of the possibility theory.

Acknowledgments
This work was supported by the Strategic Development Program of Petrozavodsk State University (2017-2022).

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