Quark solitons as constituents of hadrons

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ABSTRACT

We exhibit static solutions of multi-flavour QCD in two dimensions that have the quantum numbers of baryons and mesons, constructed out of quark and anti-quark solitons. In isolation the latter solitons have infinite energy, corresponding to the presence of a string carrying the non-singlet colour flux off to spatial infinity. When $N_c$ solitons of this type are combined, a static, finite-energy, colour singlet solution is formed, corresponding to a baryon. Similarly, static meson solutions are formed out of a soliton and an anti-soliton of different flavours. The stability of the mesons against annihilation is ensured by flavour conservation. The static solutions exist only when the fundamental fields of the bosonized Lagrangian belong to $U(N_c \times N_f)$ rather than to $SU(N_c) \times U(N_f)$. Discussion of flavour symmetry breaking requires a careful treatment of the normal ordering ambiguity. Our results can be viewed as a derivation of the constituent quark model in QCD$_2$, allowing a detailed study of constituent mass generation and of the heavy quark symmetry.

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1. Introduction

One of the key outstanding problems in strong interaction physics is the derivation of hadron spectroscopy from QCD, the underlying theory. Quarks were first postulated as constituents of hadrons to describe qualitatively the spectroscopy of mesons and baryons containing the three lightest \( u, d \) and \( s \) quark flavours.\(^1\) Subsequently, it was realized that the short-distance properties of strongly-interacting matter could be described exactly in terms of current quarks and the asymptotic freedom of QCD. The phenomenological successes of current algebra\(^2\) and chiral symmetry\(^3\) implied that the current light \((u, d, s)\) quarks must be much lighter than the original constituent quarks, and the relation between current and constituent light quarks awaits clarification.\(^4–8\) The distinction between current and constituent heavy quarks \((c, b, t)\) is also not clear, but is not so crucial. QCD-inspired potential models work well for mesons made out of heavy quarks, and lattice techniques provide fair understanding of the effective heavy quark–anti-quark potential.

The mystery of the relationship between light current and apparently the heavier constituent quarks is only deepened by the successes of calculations of baryon properties made using the Skyrme model\(^9,10\), a soliton in the low-energy chiral approximation\(^11\) to QCD in terms of bosonic matrix variables. Constituent quarks do not appear in the Skyrme model, their rôles being usurped by coherent states of current quarks.

In \(1 + 1\) dimensions, the spectrum and interactions of mesons in QCD\(_2\) were first discussed in the framework of the large-\(N_c\) expansion.\(^12–15\) For baryons, non-Abelian bosonization methods\(^16\) applied to QCD\(_2\)\(^17\) have made it possible to obtain the low-lying spectrum\(^18–21\) in the case of an unbroken light flavour symmetry, again without any reference to the idea of constituent quarks.* More recently, explicit asymptotic static soliton solutions of the bosonized heavy quark theory have

* The one-flavour case in the discretized light-cone formalism is discussed in ref. [22].
been exhibited. These have the quantum numbers of quarks and an infinite energy associated with a colour flux tube of infinite length.†

In this paper we extend this approach by exhibiting static soliton solutions of QCD$_2$ that have the quantum numbers of baryons and mesons. These new solutions are colour singlets and have finite energy. The solutions with baryon number zero are bound states of the quark and anti-quark solitons, while those with non-zero baryon number are bound states of $N_c$ quark solitons, corresponding to mesons and baryons, respectively. They provide a theoretical laboratory in which the concept of a constituent quark can be dissected. They also provide insight into the QCD description of heavy-light $Qar{q}$ mesons such as the $D$ and $B$, and baryons with one or two heavy quarks, to which the previous heavy-quark effective potential and light-quark chiral approaches have not been applicable. We show that the $D$ and $B$ mesons are likely to contain OZI-evading densities of quark–anti-quark pairs that are absent in the naïve constituent quark description, and could play observable rôles in their dynamics and decays.

The paper is organized as follows. In section 2 we show how mesons emerge as electron-positron solitons in QED$_2$, developing intuition for the QCD$_2$ case discussed in section 3. Section 4 discusses explicit meson and baryon solutions in QCD$_2$, in terms of quark and anti-quark solitons. Such solutions turn out to exist only when the fundamental fields of the bosonized Lagrangian belong to $U(N_c \times N_f)$ rather than to $SU(N_c) \times U(N_f)$. Section 5 discusses their semi-classical quantization. Section 6 contains comments on $D$- and $B$-meson physics, and section 7 is a summary and outlook. Formal aspects of mass splitting and normal ordering ambiguities are discussed in an Appendix.

† There are also qualitative and group-theoretical indications that such a mechanism could be responsible for appearance of constituent quarks in QCD$_4$, but the relevant dynamics is as yet unknown.[$^5$]
2. Mesons from solitons in QED$_2$

Some of the interesting nonperturbative phenomena in QCD have close analogues in QED$_2$ and are easy to derive, once the bosonized form of the Lagrangian is known. In this section we present a rederivation of the relevant results obtained long ago, via Abelian bosonization, by Coleman$^{[23]}$, and add some new results of our own, namely explicit solutions in the case of broken flavour symmetry. We believe the reader will find this section useful for developing physical intuition for the discussion of QCD$_2$ to follow in the next section.

The Lagrangian of multi-flavour massive QED in two dimensions is

\[\mathcal{L} = \sum_k \bar{\psi}_k (i \not{D} - m_k) \psi_k - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}\]

\[i D_\mu = i \partial_\mu - e A_\mu\]

where $k$ is the flavour index, $A_\mu$ is the gauge potential, and

\[F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \equiv \epsilon_{\mu\nu} F\]

The bosonized version of (1) reads$^{[23]}$

\[\mathcal{L} = \frac{1}{2} \sum_k (\partial_\mu \chi_k)^2 + \frac{1}{2} F^2 + \frac{e}{\sqrt{\pi}} F \sum_k \chi_k + \sum_k \frac{C}{\pi} m_k \mu_k N_{\mu_k} \cos \sqrt{4\pi} \chi_k\]

\[\bar{\psi}_k \gamma_\mu \psi_k = -\frac{1}{\sqrt{\pi}} \epsilon_{\mu\nu} \partial^\nu \chi_k\]

\[\bar{\psi}_k \psi_k = -\frac{C}{\pi} \mu_k N_{\mu_k} \cos \sqrt{4\pi} \chi_k\]

\[Q = \int_{-\infty}^{\infty} dx \sum_k \bar{\psi}_k^\dagger \psi_k = \frac{1}{\sqrt{\pi}} \sum_k [\chi_k(\infty) - \chi_k(-\infty)]\]

where $Q$ is the total electric charge, $C = 1/2 \ e^\gamma \simeq 0.891$ ($\gamma$ is Euler’s constant),
$N_{\mu_k}$ stands for normal ordering with respect to a scale $\mu_k$, and we take the $\theta$ vacuum angle \cite{23} to be zero, for simplicity.

We integrate out $F$ and take $\mu_k = m_k$, and then “absorb” a factor of $C/\pi$ in a redefinition of $m_k^2$. Then the Lagrangian (3) becomes

$$\mathcal{L} = \frac{1}{2} \sum_k (\partial_\mu \chi_k)^2 - \frac{e^2}{2\pi} (\sum_k \chi_k)^2 + \sum_k m_k^2 \cos \sqrt{4\pi} \chi_k$$

The equations of motion in the static case read

$$\chi_k'' - 4\alpha (\sum_l \chi_l) - \sqrt{4\pi} m_k^2 \sin \sqrt{4\pi} \chi_k = 0$$

where $\alpha \equiv \frac{e^2}{4\pi}$. The static potential is given by

$$V = 2\alpha (\sum_l \chi_l)^2 + \sum_k m_k^2 (1 - \cos \sqrt{4\pi} \chi_k)$$

Multiplying (5) by $\chi_k'$, summing over $k$ and integrating, we get

$$\frac{1}{2} \sum_k \chi_k'^2 - V(x) = \text{(const.)}$$

We are looking for finite energy static solutions, therefore the gradient terms must vanish asymptotically,

$$\chi_k'(\pm \infty) = 0$$

so that

$$V(\infty) = V(-\infty)$$

\* For a detailed discussion of normal ordering in the presence of different masses, see the Appendix.
Taking $\chi_k(-\infty) = 0$, we see that $V(-\infty) = 0$, and hence $V(\infty) = 0$. Thus

$$\sum_l \chi_l(\infty) = 0$$

$$\cos \sqrt{4\pi \chi_k(\infty)} = 1$$

(10)

We thus see that only states with zero total charge $Q$ are allowed, which is what one expects, since QED$_2$ is confining. From (10) it follows

$$\chi_k(\infty) = \sqrt{\pi} n_k, \quad n_k = 0, \pm 1, \pm 2, \ldots$$

(11)

For two flavours, from (10) and (11), taking the lowest non-trivial $n_k$’s, namely $n_1 = 1, n_2 = -1$, we obtain

$$\chi_1(\infty) = \sqrt{\pi}$$

$$\chi_2(\infty) = -\sqrt{\pi}$$

(12)

The boundary conditions (12) correspond to a meson built out of a soliton and an anti-soliton. Equations (5) with boundary conditions (12) can be solved explicitly. When $m_1 = m_2$ it is easy to see that $\chi_1(x) = -\chi_2(x)$ and the “string tension” term proportional to $\alpha$ in (5) vanishes, leading to two “mirror” decoupled sine-Gordon equations for $\chi_1, \chi_2$. When $m_1 \neq m_2$, a solution can be found numerically. It is particularly interesting to examine the solutions for widely unequal masses. Some representative solutions are shown in Fig. 1. It is worthwhile pointing out that for $\alpha \ll m_1^2, m_2^2$, the widths of $\chi_1$ and $\chi_2$ are governed by $1/m_1$ and $1/m_2$, respectively. The lighter “quark”, say $\chi_2$, will be more influenced by the string tension, getting heavier with increasing $\alpha/m_2^2$. This is just the intuitive picture we usually associate with a light current quark getting a constituent mass of the order of the gauge scale.

Let us now consider the case $m_1 \to \infty$. Then, in (5) with $k = 1$, we can neglect the $\alpha$ term and get a free soliton of mass $m_1$, which in the limiting case tends to
a theta function

\[ \chi_1(x) \xrightarrow{m_1 \to \infty} \sqrt{\pi} \theta(x) \]  

(13)
as follows from the explicit form of the sine-Gordon solution,

\[ \chi_1 = \frac{2}{\sqrt{\pi}} \tan^{-1} \left( \sqrt{4\pi m_1 x} \right) \]  

(14)

Then, for the light flavour, \( k = 2 \),

\[ \chi''_2 - 4\alpha \chi_2 - \sqrt{4\pi m_2^2} \sin \sqrt{4\pi \chi_1} = 4\alpha \sqrt{\pi} \theta(x) . \]  

(15)

We see that the light “anti-quark” field \( \chi_2 \) feels a point-like “source” term due to the heavy “quark” \( \chi_1 \). In the following sections we shall demonstrate that a very similar phenomenon occurs in QCD\(_2\).

3. Hadronic Solitons

Two non-Abelian bosonizations of QCD\(_2\) have been developed, one in terms of \( SU(N_c) \times U(N_f) \) bosonic variables\(^{[18]} \) where \( N_c \) is the number of colours and \( N_f \) is the number of flavours, and the other\(^{[19]} \) in terms of \( U(N_c \times N_f) \) bosonic variables.*

We will now show that in the \( U(N_c \times N_f) \) scheme, but not in the \( SU(N_c) \times U(N_f) \) scheme, there are static solutions that have the quantum numbers of baryons and mesons, constructed out of quark and anti-quark solitons. In isolation the latter solitons have infinite energy, corresponding to the presence of a string carrying the non-singlet colour flux off to spatial infinity. When \( N_c \) solitons of this type are combined, a static, finite-energy, colour singlet solution is formed, corresponding to a baryon. Similarly, static meson solutions are formed out of a soliton and an anti-soliton of different flavours. The stability of the mesons against annihilation is ensured by flavour conservation.

* The specific case of \( SU(N_c) \), \( N_f = 2 \) has also been considered in a mixed Abelian – non-Abelian formalism in Ref. 24.
The effective exact $U(N_c \times N_f)$ flavour symmetric bosonic action with gauge fields integrated out reads

$$S_{\text{eff}}[u] = S_0[u] + \frac{e_c^2 N_f}{8 \pi^2} \int d^2 x \text{Tr} \left[ \partial^{-1}_- \left( u \partial_- u^\dagger \right) \right]_c^2 + m'^2 \tilde{N}_m \int d^2 x \text{Tr} \left( u + u^\dagger \right)$$

(16)

where

$$S_0[u] = \frac{1}{8\pi} \int d^2 x \text{Tr} \left( \partial_\mu u \right) \left( \partial^\mu u \right) + \frac{1}{12\pi B} \int d^3 x \text{Tr} \epsilon^{ijk} \left( u^\dagger \partial_i u \right) \left( u^\dagger \partial_j u \right) \left( u^\dagger \partial_k u \right)$$

(17)

e_c is the strong coupling constant (having dimensions of mass in 1+1 space time), $u$ is an element in $U(N_c \times N_f)$, the subscript $c$ denotes projection onto the colour part, i.e. averaging over flavour and subtracting the $U(1)$ part, and

$$m'^2 = m_q \tilde{m} \frac{C}{2\pi}.$$  

(18)

where $\tilde{m}$ is a normal ordering scale, and $N_m$ stands for normal ordering with respect to $\tilde{m}$ (see the discussion in the Appendix). We shall look for solutions with $u$ in diagonal form (see the discussion in ref. [18]),

$$u_{\alpha\alpha'jj'} = \delta_{\alpha\alpha'} \delta_{jj'} e^{-i\sqrt{4\pi} \chi_{\alpha j}}$$

(19)

The Wess-Zumino term vanishes for either static or diagonal solutions. In general, for non-equal masses, in the static case,

$$S_{\text{eff}}[u] = -\frac{1}{2} \sum_{\alpha j} \left[ \chi'_{\alpha j}(x) \right]^2 - 2\alpha_c \sum_\alpha \int \left[ \sum_l \chi_{\alpha l} - \frac{1}{N_c} \sum_{\beta l} \chi_{\beta l} \right]^2$$

$$+ \sum_{\alpha j} \int m_j^2 \cos \sqrt{4\pi} \chi_{\alpha j}$$

(20)

where $\alpha_c = e_c^2/4\pi$ and $m_j$ stands for the $j$-th mass. The relation of $m_j$'s to the
“current” quark masses and to \( \alpha_c \), as obtained through the normal ordering, is discussed in the Appendix.

The Hamiltonian density is

\[
H = \frac{1}{2} \sum_{\alpha j} \left( \chi'_{\alpha j} \right)^2 + V,
\]

\[
V = 2\alpha_c \sum_\alpha \left[ \sum_l \chi_{\alpha l} \left( 1 - \frac{1}{N_c} \sum_\beta \chi_{\beta l} \right) \right]^2 + \sum_{\alpha j} m_j^2 \left[ 1 - \cos \sqrt{4\pi} \chi_{\alpha j} \right]
\]

where we have added a constant to \( V \), to make \( V \) vanish for \( \chi_{\alpha j} = 0 \). The equations of motion are

\[
\chi''_{\alpha j} - 4\alpha_c \left[ \sum_l \chi_{\alpha l} \left( 1 - \frac{1}{N_c} \sum_\beta \chi_{\beta l} \right) \right] - \sqrt{4\pi} m_j^2 \sin \sqrt{4\pi} \chi_{\alpha j} = 0
\]

To obtain an integral of motion, we multiply by \( \chi'_{\alpha j} \) and sum over \( \alpha, j \), to obtain in the static case

\[
\sum_{\alpha j} -\frac{1}{2} \left( \chi'_{\alpha j} \right)^2 + 2\alpha_c \sum_\alpha \left[ \sum_l \chi_{\alpha l} \left( 1 - \frac{1}{N_c} \sum_\beta \chi_{\beta l} \right) \right]^2 + \sum_{\alpha j} m_j^2 \left[ 1 - \cos \sqrt{4\pi} \chi_{\alpha j} \right] = (\text{const.})
\]

Namely, the left-hand side of (23) is independent of \( x \). For static solutions, the existence of such an \( x \)-independent integral of motion is the analogue of energy conservation for solutions evolving in time.

If we choose the boundary condition \( \chi_{\alpha j}(-\infty) = 0 \), then the constant on the right-hand side of (23) vanishes, and hence also for \( \chi_{\alpha j}(+\infty) \) (denoted hereafter simply by \( \chi_{\alpha j} \)), we must have

\[
2\alpha_c \sum_\alpha \left[ \sum_l \chi_{\alpha l} \left( 1 - \frac{1}{N_c} \sum_\beta \chi_{\beta l} \right) \right]^2 + \sum_{\alpha j} m_j^2 \left[ 1 - \cos \sqrt{4\pi} \chi_{\alpha j} \right] \bigg|_{+\infty} = 0
\]

Note that this condition is also obtained by requiring finite energy, as follows from
eq. (21). We infer from (24) that

$$\frac{1}{\sqrt{\pi}} \chi_{\alpha j} = n_{\alpha j} \quad \text{integers}$$

$$\sum_l \chi_{\alpha l} = \sqrt{\pi} \sum_l n_{\alpha l} = \sqrt{\pi} n \quad \text{independent of } \alpha$$

(25)

The baryon number of any given flavour $l$ is given by

$$B_l = \sum_\alpha n_{\alpha l}$$

(26)

Combining eqs. (25) and (26) we get the total baryon number

$$B = \sum_l B_l = nN_c$$

(27)

which clearly is an integer multiple of $N_c$.

4. Explicit solutions and bosonization schemes

For the mesonic solutions, we need $B = 0$ and hence $n = 0$. Let us consider the case of a meson containing a quark of flavour $l = 1$ and an anti-quark of flavour $l = 2$, and no other constituents. Thus, we need

$$\sum_l n_{\alpha l} = 0$$

$$\sum_\alpha n_{\alpha l} = \begin{cases} 1 & l = 1 \\ -1 & l = 2 \\ 0 & l \geq 3 \end{cases}$$

(28)

Note that in the product scheme, i.e. with $h$ in $SU(N_c)$ and $g$ in $U(N_f)$, we would
have
\[ n_{a\ell} = p_\alpha + q_l \]
\[ \sum_\alpha p_\alpha = 0 \] (29)

Then
\[ N_f p_\alpha + \sum_l q_l = 0 \] (30)

which implies that \( p_\alpha \) is independent of \( \alpha \) and hence must be zero. Thus
\[ n_{a\ell} = q_l \quad \text{integers} \] (31)
\[ \sum_l q_l = 0 \]
\[ N_c q_l = \begin{cases} 1 & l = 1 \\ -1 & l = 2 \\ 0 & l \geq 3 \end{cases} \] (32)

which is impossible. Thus there are no solutions with mesons as quark–anti-quark states in the “product scheme” of \( SU(N_c) \times U(N_f) \).

Let us now give an example of a solution in the \( U(N_c \times N_f) \) scheme. In view of the discussion above, such a solution will obviously have components in what we called \( l \) in ref. [19], namely a non-factorizable part in the colour-flavour space.

As obtained above, the asymptotic boundary conditions are:
\[ \chi_{\alpha j}(-\infty) = 0 \]
\[ \chi_{\alpha j}(+\infty) = \sqrt{\pi} n_{\alpha j} \] (33)

where the set \( \{ n_{\alpha j} \} \) must satisfy the constraints (28). A possible solution is
\[ n_{11} = 1 \]
\[ n_{12} = -1 \] (34)

with all other \( n_{\alpha j} \) being zero. Having specified the asymptotic boundary conditions
at $x \to \pm \infty$, we must now see whether a solution exists for all $x$. In general such solutions can only be found numerically. The case of an exact $SU(N_f)$ symmetry, i.e. equal quark masses, is an exception where an explicit analytical solution can be found:

$$
\chi_{11}(x) = -\chi_{12}(x) = \frac{2}{\sqrt{\pi}} \tan^{-1} \left[ \exp \left( \sqrt{4\pi m} x \right) \right],
$$

with all others vanishing identically. In order to show that this is a solution, we note that with (35) the string tension term proportional to $\alpha_c$ in (22) vanishes identically, and the equations for the various $\chi_{\alpha j}$ decouple, yielding a set of independent sine-Gordon equations with equal masses. It is then obvious that with the boundary conditions (33), (34), $\chi_{11}(x)$ is the usual sine-Gordon soliton.

Recently there has been much discussion in the literature of the so-called heavy quark symmetry\textsuperscript{[26–31]}. Here this symmetry manifests itself in a rather clear way. Consider a $Q\bar{q}$ meson made out of a heavy quark $Q$ and a light anti-quark $\bar{q}$, such as the $D$- or $B$-mesons. When $m_Q$ is much larger than the scale of the theory, $m_Q \gg \varepsilon_c$, its profile tends to a theta function,

$$
\chi_{1Q} \xrightarrow{m_Q/\varepsilon_c \to \infty} \sqrt{\pi} \theta(x)
$$

as in the QED$_2$ case, while the baryonic current of the heavy quark tends to a delta function,

$$
J^B_Q = \frac{1}{\sqrt{\pi}} \sum_\alpha \partial_x \chi_{\alpha Q} \xrightarrow{m_Q/\varepsilon_c \to \infty} \delta(x)
$$

Thus the heavy quark acts as a static colour source, while the profile $\chi_{aq}$ of the light quark becomes independent of $m_Q$, as shown in Fig. 2. The physics of $B$ and $D$ mesons in the context of the heavy quark symmetry will be discussed in more detail in section 6.

\textsuperscript{*} We have used the subroutine package COLSYS.\textsuperscript{[23]}
The presence of the static colour source (37) makes the total energy of the $Q\bar{q}$ system finite. From (21), (26), (33) and (34) we see that had there been no other colour source, the light flavour profile $\chi_{1\bar{q}}$ on its own would correspond to a configuration with a net baryon number $-1$, one “unit” of colour charge and a finite energy density per unit length, associated with a colour flux tube of infinite length,

$$\chi_{1\bar{q}} \xrightarrow{x \to \infty} -\sqrt{\pi}$$

resulting in the total energy being infinite. This is reminiscent of the single quark solution discussed in ref. [8]. It gives precise meaning to the intuitive concept of quark confinement: isolated quarks have infinite energy because flux conservation forces them to emit a flux tube which has no sink to absorb it.

Multi-quark baryonic solutions can be obtained in a similar way to the meson solutions. For example, taking $N_c = 3$ and $B = 3$ (a 3-quark state*) we find $n = 1$ (cf. eq. (27)). One possible solution is

$$n_{11} = n_{21} = n_{31} = 1 \quad (39)$$

Corresponding to a “$uuu$”-like baryon, i.e. a “$\Delta^{++}$”. When quark masses are equal, $m_1 = m_2 = \ldots = m_{N_f}$, this solution coincides with one found earlier in the “product scheme”[18], in which the colour part is “frozen”, i.e. the string tension vanishes identically. The vanishing of string tension in (39) is caused by a mechanism similar to the one occurring in the meson with equal quark masses described above, leading to

$$\chi_{11}(x) = \chi_{21}(x) = \chi_{31}(x) = \frac{2}{\sqrt{\pi}} \tan^{-1} \left[ \exp \left( \sqrt{4\pi m_x} \right) \right], \quad (40)$$

This baryon solution is plotted in Fig. 3(a).

* In our normalization a single quark carries one unit of baryon number.
There is an important difference, however, between the meson and the baryon cases. The vanishing of the string tension term in the meson is a manifestation of the fact that the chromoelectric fluxes of the quark and anti-quark cancel each other. This phenomenon has its counterpart in QED$_{2}$\textsuperscript{[23]}, as discussed in section 2. The cancellation of fluxes of $N_c$ quarks has no such counterpart and can only occur in a non-Abelian theory. Another difference between the baryonic and mesonic solitons is that the latter do not exist in the “product scheme”.

When the boundary conditions (39) are taken together with non-equal quark masses, we still obtain a “uuu”-like baryon, with the classical solution (40), as in the “product scheme”. The quantum fluctuations, however, must be treated differently, due to flavour symmetry breaking (see the Appendix for additional discussion).

An intrinsically new 3-quark solution occurs when more than one flavour appears in the nontrivial solution, for example,

$$n_{11} = n_{22} = n_{33} = 1$$

(41)

This corresponds to a baryon in which each quark has a different flavour. Such a solution is particularly interesting when quark masses are taken to be non-equal, corresponding to a “uds”-like baryon, or to a baryon in which one quark is much heavier than the QCD scale, such as the $\Sigma_c$ or $\Sigma_b$,\textsuperscript{†} again serving as a theoretical laboratory for the study of the heavy quark symmetry discussed earlier.\textsuperscript{‡}

One can also study baryons containing two heavy quarks, as shown in Fig. 3(b). For $N_c = 3$, the light quark distribution in such a baryon is the same as in a $\bar{Q}q$ meson. The physical reason for this is that the two heavy quarks are essentially at rest and act as a static colour source. They are colour triplets and combine to

\textsuperscript{†} In 1+1 dimensions there is no spin and therefore baryons must be in the completely symmetric representation of $SU(N_f)$, as the space part of the wave function is symmetric for lowest energy states. Thus with $N_c = 3$ and three light flavours there is only decuplet and no octet, and there is no analogue of $\Lambda_c$.

\textsuperscript{‡} For a different approach to heavy-quark baryons in the chiral soliton framework, see ref. 32.
a colour anti-triplet, $3 \otimes 3 \rightarrow 3^*$, i.e. the effective field seen by the light quark $q$ in a $QQq$ baryon is that of a static anti-quark. This gives precise meaning to the concept of constituent quark in QCD$_2$. The $QQq$ case is particularly clear, since this type of baryon contains only one light constituent quark, while in $Qqq$ or $qqq$ baryons there are two or three such objects, superimposed nonlinearly. A solution of the $QQq$ type with $m_1 \ll m_2 \ll m_3$ is shown in Fig. 3(c).

5. Semi-classical quantization

As has already been mentioned, one of the specific applications which motivated this study was that to mesons containing just one heavy quark $Q$. In the constituent quark language, these would be described as $Q\bar{q}$ mesons, where $\bar{q}$ represents some light constituent quark ($u,d$ or $s$). Examples include the $D,D_s$ and $B$ mesons. To describe such mesons within our QCD$_2$ approach, we need to consider a quark mass matrix with one heavy eigenvalue $M \gg e_c$, and $N_f - 1$ (typically three) light eigenvalues $m \ll e_c$. In such a case, the light quark degrees of freedom should be quantized semi-classically, as was already done for baryons made out of light current quarks.$^{[18]}$ The resulting lump is the best QCD model we can derive for the concept of a light constituent quark. However, clearly it is a coherent state containing in some sense an infinite number of current quarks, at least in the massless limit.

When discussing quantum fluctuations around the classical solution $u_c(x)$, we write

$$U = A u_c(x) A^{-1}$$

(42)

with $A$ unitary. Now, we take $A$ to be a product

$$A = A_c A_f$$

(43)

where $A_c$ acts only in the colour space and $A_f$ in flavour space. This factorization is implied by the fact that although the bosonic variables $u(x)$ belong to $U(N_c \times N_f)$,
the full theory is not $U(N_c \times N_f)$ invariant. Only the free-fermion theory has the full $U(N_c \times N_f)$ symmetry, and gauging of the colour part results in this symmetry being broken down to $SU(N_c) \times U(N_f) \times U(1)$. If we take $A_c$ to be a function of $x_-$ only, we see that it will not change the interaction term in $S_{\text{eff}}[u]$ of eq. (16), thus having no $\alpha_c$ effects. Thus we use “light-cone quantization” for the colour part. For $A_f$ we consider a function of $t$ only, thus reducing the treatment of quantum fluctuations to that which has already been treated in ref. [18], for the equal mass case. The unequal mass case is discussed in the Appendix.

Taking the colour part $A_c$ to be a function of $x_-$, while the flavour part $A_f$ is taken to be a function of $t$, might turn out to be problematic. A more natural choice would have been to take both $A_c$ and $A_f$ as functions of $t$, since we are performing the quantization around a static classical solution, $u_c = u_c(x)$.

In any case, we expect that for equal quark masses the fluctuations in the colour space will not influence ratios such as flavour content of hadrons (sec. 6), since the physical states are colour singlets. The situation might change, though, when masses are different. The flavour content will then in general depend on the mass ratios, and the different masses could in general be influenced differently by the colour fluctuations. This is because the flavours which are light compared with the typical scale of the theory are in the strong-coupling regime, and will be more affected than the flavours which are much heavier than the dynamical scale of the theory. In our case, we will be interested in the $u$, $d$ and $s$ contents in the approximation where they are all very light.
6. Comments on $D$ and $B$ physics

In the previous sections we have described QCD$_2$ solitons which could serve as models for $Qar{q}$ mesons such as the $D$ or $B$. In addition to giving some insight into the concept of a constituent quark from the point of QCD, this study may also give some new insights into the dynamics and weak decays of $D$ and $B$ mesons.\* In particular, we would like to comment on the existence and possible phenomenological rôle of non-valence quarks in the $D$ and $B$ meson wave functions.

There are various phenomenological indications that the proton wave function contains a significant density of non-valence $\bar{s}s$ quarks. Moreover, their abundance relative to $\bar{u}u$ and $\bar{d}d$ quarks is qualitatively reproduced by Skyrme model soliton calculations. The relative abundances of $\bar{s}s, \bar{u}u$ and $\bar{d}d$ have also been calculated in baryonic QCD$_2$ solitons,$^{[21]}$ and found to be qualitatively similar to the QCD$_4$ results.$^{[33–35]}$ In this chapter we invert the logic: QCD$_2$ mesonic solitons contain calculable non-valence quark densities, and we would expect QCD$_4$ mesonic solitons and hence physical $D$ and $B$ mesons to contain similar non-valence quark densities.

The calculation of the different light quark densities in a QCD$_2$ mesonic soliton parallels very closely that in a baryonic soliton:$^{[21]}

\begin{equation}
\frac{\langle M|\bar{q}iq_i|M \rangle}{\langle M|\bar{q}jq_j|M \rangle} = \frac{\ll z^*_iz_i \rr}{\ll z^*_jz_j \rr} \tag{44}\end{equation}

where $\ll \gg$ denotes an average over the collective coordinate representation of the mesonic soliton. In the physical case with one heavy and three light quarks we find for a $Q\bar{q}_1$ meson:

$$\langle M|\bar{q}_1q_1|M \rangle : \langle M|\bar{q}_2q_2, \bar{q}_3q_3|M \rangle = 2 : 1 \tag{45}$$

This parallels the previous result$^{[21]}$ for the sea- and valence-quark content of baryons in the semi-classical quantization of QCD$_2$ with $N_l$ light flavours and

\* $Q\bar{q}$ mesons in QCD$_2$ have also been recently studied in the large-$N_c$ limit.$^{[36]}$
For a baryon $B$ containing $k$ light valence quarks of flavour $v$

$$\langle \bar{v} v \rangle_B = \frac{k + 1}{N_l + N_c} \quad (46)$$

and

$$\langle (\bar{q} q)_{\text{sea}} \rangle_B = \frac{1}{N_l + N_c}, \quad (47)$$

where $(\bar{q} q)_{\text{sea}}$ refers to the non-valence quarks in the baryon $B$, and the total flavour content is normalized to 1. Equations (46) and (47) suggest an “equipartition” for valence and sea, each with a content of $1/(N_l + N_c)$. Since the semiclassical quantization of light flavours in $\bar{Q}q$ mesons parallels that in the baryons, the “equipartition” result applies to the light flavour content of $\bar{Q}q$ mesons as well. This implies, for example, that

$$\langle D|\bar{s}s|D\rangle/\langle D|\bar{u}u|D\rangle = \frac{1}{2} \quad (48)$$

and similarly for $B$ mesons. We might expect the ratio (48) to be qualitatively similar, though possibly smaller by about a factor 2, for the realistic case of QCD with light flavour $SU(3)$ breaking.$^{[33-35]}$

The presence of significant amounts of $\bar{s}s$ quarks in the $D$ and $B$ mesons implies that the naïve OZI rule forbidding disconnected quark diagrams can be evaded.$^{[37]}$ This might also have implications for $D$ and $B$ production dynamics, but here we only emphasize some possible implications for $D$ and $B$ decays.

1. Annihilation diagrams: $c\bar{s} \rightarrow u\bar{d}$ could be more important for $D^0$ and $D^+$ mesons than is normally supposed when only the $D_s$ wave function is assumed to contain strange quarks.

2. The final states from $D$ and $B$ decays could contain more strange particles than is normally supposed when $\bar{s}s$ pairs need to pop out of the QCD vacuum or be created at the weak vertex. This could help explain the surprisingly
large \cite{38,39,40} branching ratios for $D^0 \to \phi K^0$ and $D^+ \to \phi K^+$. This observation could also have implications for attempts to estimate the ratio $|V_{cs}/V_{cd}|$ of Kobayashi-Maskawa matrix elements on the basis of strange final states. It might also have implications for the ratios of $D \to K\bar{K}$ and $\pi\pi$ final states.

3. The $\bar{s}s$ pairs could provide an additional source for $B \to \phi + X$. At this time it is premature to compare this with the data.

Detailed investigations of these possibilities should await more realistic calculations in QCD$_4$, however.

### 7. Summary and outlook

We have shown in this paper that the spectrum of QCD$_2$ includes finite-energy mesonic and baryonic solitons which contain one heavy quark. These solitons can be regarded as bound states of the infinite-energy single-quark solitons that we found previously. They provide meaning for the previously fuzzy concept of a constituent quark, at least in QCD$_2$. A particularly interesting application is to the study of mesons and baryons containing both heavy and light quarks. A constituent light quark is seen to be a semi-classical coherent state containing an indefinite number of light $\bar{q}q$ pairs, among which non-valence flavours have as much as one half of the abundance of the valence flavour. This observation could have phenomenological implications for the dynamics and weak decays of $D$ and $B$ mesons.

The next step in the programme of developing accurate QCD descriptions of constituent quarks and $Q\bar{q}$ mesons is to extend the analysis of this paper to the realistic case of four dimensions. This may be possible for the lowest-lying $Q\bar{q}$ mesons if they are describable by spherically symmetric wave functions, which could be analyzed using an effective two-dimensional field theory in $(r, t)$ coordinates. We are now investigating this possibility.
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APPENDIX

We discuss here the more formal aspects of mass splitting. In the following we will focus mainly on the bosonization in the product scheme, since the technicalities are simpler there, while the conclusions are the same as in the \( U(N_c \times N_f) \) scheme. We recall the initial bosonic action in the product scheme \(^{18}\)

\[
\tilde{S} [g, h, A_{\mu}] = N_c S [g] + N_f S [h, A_{\mu}] - \frac{1}{2e^2} \int d^2 x \text{Tr} \, F_{\mu\nu} F^{\mu\nu} + C \tilde{m} N_{\tilde{m}} \int d^2 x \left[ \text{Tr} (M g) \right] \text{Tr} h + h.c. \]  
\tag{A.1}

where \( g \in U(N_f), h \in SU(N_c), A_{\mu} \) is colour gauge field, \( M = \text{diag}(m_1, \ldots, m_{N_f}) \) is the mass matrix, \( m_i \) are the quark masses, \( N_{\tilde{m}} \) stands for normal ordering with respect to \( \tilde{m} \), and \( F_{\mu\nu} \) is the usual non-Abelian field-strength tensor. Now define the “dimensionless mass matrix” \( D \) as

\[
D = \frac{1}{m_0} M \]  
\tag{A.2}

where \( m_0 \) is an arbitrary mass parameter. In the gauge \( A_- = 0 \), integrating out
and then taking the strong-coupling limit, $e_c \gg m_i$, we get\[^{18}\]

$$
\tilde{S}[g] = N_c S[g] + m^2 N_m \int d^2 x \text{Tr} \left[ D(g + g^\dagger) \right] \quad (A.3)
$$

$$
m = \left[ CN_c m_0 \left( \frac{e_c^2 N_f}{2\pi} \right)^{\frac{1}{2p+1}} \right] \quad (A.4)
$$

$$
p = \frac{N_c^2 - 1}{N_c(N_c + N_f)} \quad (A.5)
$$

In the $U(N_c \times N_f)$ bosonization scheme the mass term, denoted M.T., has the form

$$
\text{M.T.} = m_0 C \tilde{m} N_m \text{Tr} \left[ \hat{D}(u + u^\dagger) \right] \quad (A.6)
$$

where $u \in U(N_c \times N_f)$ and $\hat{D}$ is the dimensionless mass matrix (in flavour only). In the strong-coupling limit the mass term (A.6) exactly coincides with that of (A.3).\[^{19}\] The classical low-lying solutions to the action (A.3) are given by the time-independent form

$$
g = \text{diag} \left[ \exp \left( -i \sqrt{\frac{4\pi}{N_c}} \phi_1(x) \right), \ldots, \exp \left( -i \sqrt{\frac{4\pi}{N_c}} \phi_{N_f}(x) \right) \right] \quad (A.7)
$$

yielding an action density

$$
\tilde{S}_d[g] = - \int dx \sum_{i=1}^{N_f} \left[ \frac{1}{2} \left( \frac{d\phi_i}{dx} \right)^2 - 2 \tilde{m}_i^2 \left( \cos \sqrt{\frac{4\pi}{N_c}} \phi_i - 1 \right) \right] \quad (A.8)
$$

with mass parameters

$$
\tilde{m}_i^2 = \frac{m_i}{m_0} m^2 \quad (A.9)
$$

Each $\phi_i$ has a classical solution

$$
\phi_i(x) = \sqrt{\frac{4N_c}{\pi}} \arctan \left[ \exp \left( \sqrt{\frac{8\pi}{N_c}} \tilde{m}_i x \right) \right] \quad (A.10)
$$
with an energy

$$E_i = 4\tilde{m}_i \sqrt{\frac{2N_c}{\pi}}$$  \hspace{1cm} (A.11)

The minimum classical energy solution is given by a solution with exactly one non-trivial entry,

$$g_0(x) = \text{diag} \left[1, 1, \ldots, \exp \left(-i \sqrt{\frac{4\pi}{N_c}} \phi_{N_f}(x)\right)\right]$$  \hspace{1cm} (A.12)

with $m_{N_f}$ chosen to be the smallest mass.

In order to quantize the system we introduce the collective coordinates $A(t)$,

$$g(x, t) = A(t) \, g_0(x) \, A^\dagger(t), \quad A(t) \in U(N_f)$$  \hspace{1cm} (A.13)

Writing $A = (A_1, \ldots, A_{N_f-1}, z)$ where $A_1, \ldots, A_{N_f-1}, z$ are column vectors, we get

$$\tilde{S}[g] - \tilde{S}[g_0] = \frac{1}{2M_{N_f}} \int dt (Dz)_i^\dagger (Dz)_i + iN_c \int dt \dot{z}_i^\dagger \dot{z}_i$$

$$- \frac{2\pi}{M_{N_f} N_c} \int dt \sum_{i=1}^{N_f} (\tilde{m}_i^2 - \tilde{m}_{N_f}^2)|z_i|^2$$

$$Dz = \dot{z} - z(z^\dagger \dot{z})$$

$$\frac{1}{2M_i} = \frac{N_c}{2\pi} \int_{-\infty}^{\infty} \left(1 - \cos \sqrt{\frac{4\pi}{N_c}} \phi_i\right) dx = \frac{\sqrt{2}}{\tilde{m}_i} \left(\frac{N_c}{\pi}\right)^\frac{1}{2}, \quad i = 1 \ldots N_f.$$

The resulting Hamiltonian is

$$H = 4\tilde{m}_{N_f} \left(\frac{2N_c}{\pi}\right)^\frac{1}{2} \left\{1 + \left(\frac{\pi}{2N_f}\right)^2 \left[ C_2(R) - \frac{N_c^2}{2N_f} (N_f - 1)\right] + \sum_{i=1}^{N_f} \frac{m_i - m_{N_f}}{m_{N_f}} |z_i|^2 \right\}$$

(A.15)

The Hamiltonian depends on $m_0$ only through $\tilde{m}_{N_f}$, therefore the overall mass scale is undetermined, and only mass ratios are meaningful. The quantity $|z_i|^2$
is proportional to the $\bar{q}_i q_i$ operator. Therefore the last term in (A.15), due to $m_i \neq M_{N_f}$, is proportional to the weighted average over flavours of the $\bar{q}q$ operators, the weights being proportional to the mass differences. Since this term comes from quantum fluctuations around the classical solution, consistency with the semi-classical approximation requires that it be very small compared to one.

Generalizing eq. (A.2), by introducing an extra undetermined mass parameter $m_{i0}$ for each flavour, we define $\rho_i = m_i/m_{i0}$. The mass term has the form

$$M.T. = C \hat{m}_N \hat{m} \text{Tr} (M g) = C \sum_{i=1}^{N_f} m_{i0} \hat{m}_N \rho_i g_{ii}$$

(A.16)

By integrating out the gauge fields and taking the strong-coupling limit, we get

$$\tilde{S} [g] = N_c S [g] + \sum_{i=1}^{N_f} \hat{m}_i^2 N_{\hat{m}_i} \int d^2 x \frac{m_i}{m_{i0}} (g_{ii} + g_{ii}^*)$$

(A.17)

$$\hat{m}_i = \left[ C N_c m_{i0} \left( \frac{e_c N_f}{2\pi} \right)^{\frac{1}{2p}} \right]^{\frac{1}{p+1}}$$

(A.18)

The classical low mass solutions to the action (A.17) are given by (A.7) with an action density (A.8), but now with mass parameters

$$\hat{m}_i^2 = \rho_i \hat{m}_i^2 = \frac{m_i}{m_{i0}} \hat{m}_i^2$$

(A.19)

Each $\phi_i$ has a classical solution (A.10) with energy (A.11). The minimum classical energy solution is given by the ansatz (A.12), with $\hat{m}_{N_f}$ chosen such that it is the smallest mass. Define now $A(t)$ as in (A.13), resulting in an effective action like in
(A.14), and a Hamiltonian

\[
H = M \left\{ 1 + \left( \frac{\pi}{2N_c} \right)^2 \left[ C_2(R) - \frac{N_c^2}{2N_f} (N_f - 1) \right] + \sum_{i=1}^{N_f} \left[ \frac{m_i}{m_{N_f}} \left( \frac{m_{i0}}{m_{N0}} \right)^{\frac{1-p}{1+p}} - 1 \right] |z_i|^2 \right\}
\]

(A.20)

\[
M = 4\tilde{m}_{N_f} \left( \frac{2N_c}{\pi} \right)^{\frac{1}{2}}.
\]

There are several problems connected with these arbitrary mass parameters. Since the choice of the parameters \( m_{i0} \) is arbitrary, we can choose

\[
m_{i0} = m_{N0} \left( \frac{m_i}{m_{N_f}} \right)^{\frac{p+1}{p-1}},
\]

in which case \( H \) has no explicit contribution due to the mass differences, which is unacceptable.

The question now arises at what stage should one apply the semi-classical approximation.\cite{20} The classical solution is governed by \( \tilde{m}_i \) but the lightest \( \tilde{m}_i \) is not necessarily the lightest \( m_i \). In fact, the normal ordering ambiguity is present even in a non-interacting theory. To see this, we formally set \( N_c = 1 \), i.e. we have the field \( g \) with level 1 which means free quarks. There is no gauge group, so the dependence on \( e_c \) should vanish and indeed \( p = 0, \tilde{m}_i = Cm_{i0} \) and \( M = 4C \left( \frac{2m_{N_f}m_{N0}}{\pi} \right)^{\frac{1}{2}} \). The lowest multiplet corresponds to a Young tableaux of one box and \( C_2(R) = \frac{N_f^2 - 1}{2N_f} \), therefore

\[
H = M \left[ \left( \frac{\pi}{2} \right)^2 \frac{N_f}{2} - 1 \right] + \sum_{i=1}^{N_f} \frac{m_i m_{i0}}{m_{N_f} m_{N0}} |z_i|^2 \right]^{(A.21)}
\]

which in general does not look like a non-interacting theory, unless one takes \( m_{i0} = m_{N0} \) for all \( i \).
One could argue that the above example is not relevant for a generic case, since for small $N_c$ the semi-classical approximation is not justified. This can be seen from (A.15), as follows. Consistency with the semi-classical approximation requires the quantum corrections to the energy to be much smaller than the classical contribution. In the case of one box this means \( \left( \frac{\pi}{2} \right)^2 \frac{N_f - 1}{2} \ll 1 \), which for any $N_f \geq 2$ is not correct.

Nevertheless, since the choice of the $m_{i0}$’s is a priori arbitrary, we see that there is an ambiguity. In the non-interacting case it is obvious what is the right choice of the $m_{i0}$’s, but in general we do not know what is the choice leading to the optimal semi-classical approximation, as is best illustrated by the case of equal quark masses, which can be made to appear unequal by a suitable choice of the $m_{i0}$’s.

One can summarize the situation as follows. When all masses are equal, the treatment of ref. [18], in the strong coupling limit, yields an effective Lagrangian in terms of flavour degrees of freedom only, with a mass scale that involves the coupling $\alpha_c$, as in eq. (A.4). When the current mass $M$ of some quark is heavy, $M^2 \gg \alpha_c$, we expect the constituent mass of that quark to be heavy and close to the current mass. The problems arise in the intermediate cases, when we do not know what is the “best” starting point for the normal ordering scale, before going to the semi-classical approximation. Of course, if one were able to sum all corrections, one would obtain the full result, regardless of the starting point.
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FIGURE CAPTIONS

1) Mesons in QED\textsubscript{2} with two flavours, solutions of eqs. (5), (12).
   (a) $\alpha = 0.1, m_1 = 0.5, m_2 = 0.5$;
   (b) $\alpha = 0.1, m_1 = 2.0, m_2 = 0.5$;
   (c) $\alpha = 0.1, m_1 = 4.0, m_2 = 0.5$;
   (d) $\alpha = 2.0, m_1 = 4.0, m_2 = 0.5$.
In all four plots the upper line corresponds to $\chi_1$, and the lower line to $\chi_2$.
(a), (b) and (c) show how, with increasing $m_1$, the $\chi_1$ profile converges to a step function, while the light flavour profile, $\chi_2$, remains almost unaffected.
The $\alpha$ dependence can be inferred from the difference between (c) and (d): $\chi_2$ gets heavier with increasing $\alpha/m_2^2$. The effect is not a sharp one, because part of the $\alpha$ dependence is already included in the value of $m_2$, through the normal ordering (see Appendix).

2) Mesons in QCD\textsubscript{2} with $N_c = 2, N_f = 2$, solutions of eqs. (22), (34).
   (a) $\alpha_c = 1.0, m_1 = 0.1, m_2 = 0.1$;
   (b) $\alpha_c = 1.0, m_1 = 0.1, m_2 = 0.5$;
   (c) $\alpha_c = 1.0, m_1 = 0.1, m_2 = 2.0$;
   (d) $\alpha_c = 10.0, m_1 = 0.1, m_2 = 2.0$.
$\chi_{11}$ and $\chi_{12}$ are the upper and lower continuous curves, respectively. The “non-valence” components $\chi_{21}$ and $\chi_{22}$ are denoted by dot-dashed and dashed lines, respectively, except for (a), where they are exactly zero.

3) Baryons in QCD\textsubscript{2} with $N_c=3, N_f=3$, solutions of eq. (22).
   (a) a “uuu”-like baryon, eq. (39), with $\alpha_c=1.0, m_1=m_2=m_3=0.1$;
   (b) a “ucb”-like baryon, eq. (41), with $\alpha_c=1.0, m_1=0.1, m_2=0.5, m_3=0.8$;
   (c) a “ubt” -like baryon, eq. (41), with $\alpha_c=1.0, m_1=0.1, m_2=1.0, m_3=2$.
In (a) the “non-valence” components, $\chi_{21}, \chi_{31}$, etc. are exactly zero. In (b) and (c) the continuous lines denote the “valence” components, while the dashed and dot-dashed lines denote the “non-valence” components. The “non-valence” components of the heavy flavours are clearly much more suppressed than those of the light flavour.