Research Article

The Commutators of Multilinear Maximal and Fractional-Type Operators on Central Morrey Spaces with Variable Exponent

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Received 16 February 2022; Accepted 18 March 2022; Published 5 April 2022

Academic Editor: Dumitru Motreanu

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We show that the maximal operator associated with multilinear Calderón-Zygmund singular integrals and its commutators are bounded on products of central Morrey spaces with variable exponent. Moreover, some bounded properties are obtained for the commutators of multilinear Calderón-Zygmund operators as well as for the corresponding fractional integra ls.

1. Introduction

Let $T$ be a multilinear operator initially defined on the $m$-fold product of Schwartz spaces and taking values into the space of tempered distributions,

$$T : \mathcal{S}^\prime(\mathbb{R}^n) \times \cdots \times \mathcal{S}^\prime(\mathbb{R}^n) \rightarrow \mathcal{S}^\prime(\mathbb{R}^n).$$

Following Grafakos and Torres [1], we say that $T$ is an $m$-linear Calderón-Zygmund operator if, for some $1 \leq p_1, \cdots, p_m < \infty$, it extends to a bounded multilinear operator from $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$, where $1/p = 1/p_1 + \cdots + 1/p_m$, and if there exists a function $K$, defined away from the diagonal $x = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$, satisfying

$$T(f_1, \cdots, f_m)(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \cdots, y_m)f_1(y_1) \cdots f_m(y_m)dy_1 \cdots dy_m,$$

for all $x \notin \bigcap_{i=1}^m \text{supp } f_j$ and $f_1, \cdots, f_m$ are $C^\infty$ functions with compact support,

$$|K(x, y_1, \cdots, y_m)| \leq \frac{A}{(\sum_{k=1}^m |x - y_k|)^{\frac{\alpha}{m}}}.$$

for some $A > 0$ and all $(x, y_1, \cdots, y_m) \in (\mathbb{R}^n)^{m+1}$ with $x \neq y_k$ for some $k$, and

$$\left|K(x, y_1, \cdots, y_m) - K(x', y_1, \cdots, y_m)\right| \leq \frac{A|x - x'|^\varepsilon}{(\sum_{k=1}^m |x - y_k|)^{\frac{\alpha}{m}}},$$

for some $\varepsilon > 0$, whenever $|x - x'| \leq (1/2)$ max $\{1, |x - y_1|, \cdots, |x - y_m|\}$, and also that

$$\left|K(x, y_1, \cdots, y_i, \cdots, y_m) - K(x, y_1, \cdots, y_i', \cdots, y_m)\right| \leq \frac{A|y_i - y_i'|^\varepsilon}{(\sum_{k=1}^m |x - y_k|)^{\frac{\alpha}{m}}}.$$

whenever $|y_i - y_i'| \leq (1/2)$ max $\{1, |x - y_1|, \cdots, |x - y_m|\}$ for all $1 \leq i \leq m$.

Multilinear Calderón-Zygmund operators were introduced by Coifman and Meyer [2, 3] in the 70s and were systematically studied by Grafakos and Torres [1]. They showed that multilinear Calderón-Zygmund operators map $L^{p_1} \times \cdots \times L^{p_m}$ into $L^p$ for some $1 < p_1, \cdots, p_m < \infty$ and some $0 < p < \infty$ with $1/p = 1/p_1 + \cdots + 1/p_m$. Moreover, multilinear Calderón-Zygmund operators satisfy weak endpoint bounds when at least one $p_i$ is equal to one. Lerner et al. [4]
established the multiple weights theory for multilinear Calderón-Zygmund operators and obtained some weighted estimates for these operators and their commutators. Grafakos and Kalton [5] studied the boundedness of these operators on products of Hardy spaces. Subsequently, under some additional conditions, Hu and Meng [6] proved the boundedness of these operators from product Hardy spaces into Hardy spaces.

For convenience, we will write \((y_1, \ldots, y_m) = \tilde{y}\) and \(dy_1, \ldots, dy_m = d\tilde{y}\). Grafakos and Torres [7] defined the maximal multilinear Calderón-Zygmund operator

\[
T^* \left( \vec{f} \right) (x) = \sup_{\delta > 0} |T_\delta (f_1, \ldots, f_m) (x)|,
\]

where \(\vec{f} = (f_1, \ldots, f_m)\) and \(T_\delta\) are the smooth truncations of \(T\) given by

\[
T_\delta (f_1, \ldots, f_m) (x) = \int_{|x-y| < \delta} K(x, y_1, \ldots, y_m) f_1(y_1) \cdots f_m(y_m) dy.
\]

Moreover, they established the boundedness of \(T^*\) on product of Lebesgue spaces and also obtained some weighted norm inequalities for this maximal operator. Later on, the topic was studied by several authors; see [8–16].

A function \(b \in L_{loc} (\mathbb{R}^n)\) is said to belong to space BMO (\(\mathbb{R}^n\)), if

\[
\|b\|_{\text{BMO} (\mathbb{R}^n)} = \sup_{x \in B} \frac{1}{|B|} \int_B |b(y) - b_B| dy < \infty,
\]

where \(B\) is a ball of \(\mathbb{R}^n\) with center at the origin and radius \(R\), \(|B|\) is the Lebesgue measure of \(B\), and \(b_B = (1/|B|) \int_B f (x) dx\).

**Definition 1.** Given a collection of locally integrable functions \(\vec{b} = (b_1, \ldots, b_n)\), the commutators of \(\vec{b}\) and the multilinear Calderón-Zygmund operator \(T\) are defined by

\[
T_{\vec{b}} (f_1, \ldots, f_m) = \sum_{j=1}^{m} T_{b_j}^j \left( \vec{f} \right),
\]

where each term is the commutator of \(b_j\) and \(T\) in the \(j\)th entry of \(T\), that is,

\[
T_{b_j}^j \left( \vec{f} \right) = b_j T (f_1, \ldots, f_j, \ldots, f_m) - T (f_1, \ldots, b_j f_j, \ldots, f_m).
\]

Also, the commutators of \(\vec{b}\) and the maximal multilinear Calderón-Zygmund operator \(T^*\) are defined by

\[
T_{\vec{b}}^* (f_1, \ldots, f_m) = \sum_{j=1}^{m} T_{b_j}^* \left( \vec{f} \right),
\]

where each term is the commutator of \(b_j \) and \(T^*\) in the \(j\)th entry of \(T^*\), that is,

\[
T_{b_j}^* \left( \vec{f} \right) = \frac{1}{|B|} \int_B \left| b_j T (f_1, \ldots, f_j, \ldots, f_m) - T (f_1, \ldots, b_j f_j, \ldots, f_m) \right| dx.
\]

This definition coincides with the commutator of Coifman et al. [17] when \(m = 1\). The multilinear commutators \(T_{\vec{b}}\) were considered by Lerner et al. in [4]; they proved that if \(\vec{b} \in (\text{BMO})^m, 1 < p_1, \ldots, p_m < \infty,\) and \(p\) defined by \(1/p = 1/p_1 + \cdots + 1/p_m\), then

\[
T_{\vec{b}} : L^{p_1} (\omega_1) \times \cdots \times L^{p_m} (\omega_m) \rightarrow L^p (\nu_w),
\]

where \(\nu_w = \prod_{j=1}^{m} \omega_j^{1/p_j}\) and \(\omega = (\omega_1, \ldots, \omega_m)\) satisfies the \(A_{p_1, \ldots, p_m}\) condition, that is,

\[
\sup_Q \left( \frac{1}{|Q|} \int_Q \nu_w^{1/p_j} \prod_{j=1}^{m} \left( \frac{1}{|Q|} \int_Q \omega_j^{1/p_j} \right)^{1/p_j} \right)^{1/p} < \infty.
\]

For further contributions in the study of the multilinear commutators \(T_{\vec{b}}\) and \(T_{\vec{b}}^*\), we refer to [18, 19] and the references therein.

On the other hand, as one of the most important operators, the multilinear fractional operator has also been intensively studied in the recent years. Kenig and Stein [20] considered the following multilinear fractional operator \(I_a\),

\[
I_a (\vec{f}) (x) = \int_{(\mathbb{R}^n)^m} f_1 (y_1) f_2 (y_2) \cdots f_m (y_m) dy_1 \cdots dy_m / |x-y_1, \ldots, y_m-x|^{m+a} dy_1 \cdots dy_m,
\]

where \(m, n\) denote the nonnegative integers with \(m \geq 1, n \geq 2\). They proved that \(I_a\) is of strong type \((L^{p_1} \times \cdots \times L^{p_m}, L^p)\) and weak type \((L^{p_1} \times \cdots \times L^{p_m}, L^p_{\text{weak}})\). Moen [21] developed a weighted theory that adapts to the multilinear fractional integral operators. He established the multiple weighted norm inequalities for the multilinear fractional integral operators and the corresponding multilinear fractional maximal operators. Following the work of [4], we define the commutators of \(\vec{b}\) and the multilinear fractional operator \(I_a\) as follows:

\[
I_{a,\vec{b}} (f_1, \ldots, f_m) = \sum_{j=1}^{m} I_{a, b_j} \left( \vec{f} \right),
\]

where each term is the commutator of \(b_j\) and \(I_a\) in the \(j\)th entry of \(I_a\), that is,

\[
I_{a, b_j} \left( \vec{f} \right) = b_j I_a (f_1, \ldots, f_j, \ldots, f_m) - I_a (f_1, \ldots, b_j f_j, \ldots, f_m).
\]
The commutators $I_{a,b}$ were first studied by Chen and Xue [22], in which some weighted strong bounds and $L(\log L)$ type endpoint estimates for $I_{a,b}$ are obtained.

Lu and Yang [23, 24] introduced the central bounded mean oscillation space $\text{C}^k_\text{MO}(\mathbb{R}^n)$, $1 < q < \infty$, which satisfies the following condition:

$$\|f\|_{\text{C}^k_\text{MO}(\mathbb{R}^n)} = \sup_{R > 0} \left( \frac{1}{|B(0,R)|} \int_{B(0,R)} |f(y) - f_{B(0,R)}|^q \, dy \right)^{1/q} < \infty.$$  

(18)

This space can be regarded as a local version of $\text{BMO}(\mathbb{R}^n)$ at the origin, but they have quite different properties. For example, the well-known John-Nirenberg inequality shows that the functions in $\text{BMO}(\mathbb{R}^n)$ can be described by means of the condition

$$\sup_{R \in \mathbb{R}} \left( \frac{1}{|B(0,R)|} \int_{B(0,R)} |f(y) - f_{B(0,R)}|^q \, dy \right)^{1/q} < \infty.$$  

(19)

However, the space $\text{C}^k_\text{MO}(\mathbb{R}^n)$ depends on $q$. More precisely, if $q_1 < q_2$, then $\text{C}^k_\text{MO}(\mathbb{R}^n) \subset \text{C}^{k_1}_\text{MO}(\mathbb{R}^n)$. Therefore, there is no analogy of the classical John-Nirenberg inequality of $\text{BMO}(\mathbb{R}^n)$ for the space $\text{C}^{k_1}_\text{MO}(\mathbb{R}^n)$. One can imagine that the behavior of $\text{C}^k_\text{MO}(\mathbb{R}^n)$ may be quite different from that of $\text{BMO}(\mathbb{R}^n)$.

**Definition 2.** Let $\lambda < 1/n$ and $1 < q < \infty$. The $\lambda$-central bounded mean oscillation space $\text{CMO}^{\lambda_\text{MO}}(\mathbb{R}^n)$ is defined by

$$\|f\|_{\text{CMO}^{\lambda_\text{MO}}(\mathbb{R}^n)} = \sup_{R > 0} \left( \frac{1}{|B(0,R)|} \int_{B(0,R)} |f(y) - f_{B(0,R)}|^q \, dy \right)^{1/q} < \infty.$$  

(20)

**Remark 3.** When $\lambda = 0$, the space $\text{CMO}^{\lambda_\text{MO}}(\mathbb{R}^n)$ is just the space $\text{CMO}^{k_\text{MO}}(\mathbb{R}^n)$ defined above. If two functions which differ by a constant are regarded as a function in the space $\text{CMO}^{k_\text{MO}}(\mathbb{R}^n)$, then it becomes a Banach space. Obviously, (20) is equivalent to the following condition

$$\sup_{R > 0} \inf_{c \in \mathbb{R}} \left( \frac{1}{|B(0,R)|} \int_{B(0,R)} |f(y) - c|^q \, dy \right)^{1/q} < \infty.$$  

(21)

**Definition 4.** Let $\lambda \in \mathbb{R}$ and $1 < q < \infty$. The central Morrey space $\text{BMO}^{\lambda}(\mathbb{R}^n)$ consists of all $f \in L^q_{\text{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{\text{BMO}^{\lambda}(\mathbb{R}^n)} = \sup_{R > 0} \left( \frac{1}{|B(0,R)|} \int_{B(0,R)} |f(y)|^q \, dy \right)^{1/q} < \infty.$$  

(22)

The above $\lambda$-central BMO spaces and central Morrey spaces were introduced in Alvarez et al. [25]; since then, various operators were studied in these spaces. For some recent development, we mention that Fu et al. [26] established $\lambda$-central BMO estimates for a class of multisublinear operators on the product of central Morrey spaces. As its special cases, the corresponding results of multilinear Calderón-Zygmund operators can be deduced. Si and Xue [18] established $\lambda$-central BMO estimates for commutators of maximal multilinear Calderón-Zygmund operators and multilinear fractional operators on central Morrey spaces.

In the last two decades, following the fundamental work of Kováčik and Rákosnk [27] on variable Lebesgue spaces $L^p(\cdot)(\mathbb{R}^n)$ and variable Sobolev spaces $W^{k,p,\cdot}(\mathbb{R}^n)$ (here and below, the exponent $p(\cdot)$ is a function and not a constant), function spaces with variable exponent, such as variable Morrey spaces, variable Herz spaces, and variable Hardy spaces, have been widely studied by a significant number of authors; see [28–37] and the references therein. These spaces are of interest in their own right and also have applications to image restoration [38], fluid dynamics [39], and PDEs with nonstandard growth conditions [40].

In 2019, Fu et al. [41] introduced the $\lambda$-central BMO spaces and the central Morrey spaces with variable exponent and proved the boundedness of the fractional singular integrals and its commutator on those spaces. Subsequently, Wang and Xu [42] further obtained the boundedness of multilinear fractional integral operators and their commutators on central Morrey spaces with variable exponent. The $\lambda$-central BMO estimates for $m$-linear Calderón-Zygmund operators and their commutators on the product of central Morrey spaces with variable exponent are independently obtained by Wang [43] and Wang et al. [44].

Motivated by [18, 41], the aim of this paper is to establish $\lambda$-central BMO estimates for the maximal multilinear Calderón-Zygmund operators $T^*$ and its commutators $T^*_b$ on central Morrey spaces with variable exponent. Moreover, the similar boundedness properties for the multilinear commutators $T^*_b$ and $I_{a,b}$ are obtained.

Throughout this paper, the symbol $\mathbb{N}$ stands for the set of all natural numbers. We denote $B(0,R) = \{ y \in \mathbb{R}^n : |y| < r \}$ simply by $B$, and $B(0,|R|)$ by $B$ for $|R| \in \mathbb{N}$. $f_B$ denotes the integral average of $f$ on $B$, i.e., $f_B = (1/|B|) \int_B f(x) \, dx$. If $E \subset \mathbb{R}^n$ is a measurable set, then $|E|$ means the Lebesgue measure of $E$ and $\chi_E$ denotes its characteristic function. $p'(\cdot)$ is the conjugate exponent function defined by $1/p(\cdot) + 1/p'(\cdot) = 1$. The letter $C$ stands for a positive constant whose value may change from appearance to appearance.

### 2. Preliminaries and Main Results

Let us first recall some basic properties of Lebesgue spaces with variable exponent; we refer to the surveys [45, 46] and the monographs [47, 48] for further details.

The variable Lebesgue spaces are a generalization of the classical $L^p$ spaces with the exponent $p$ replaced by a measurable function $p(\cdot) : \mathbb{R}^n \to (0,\infty)$. It consists of all measurable functions $f$ on $\mathbb{R}^n$ such that
Lemma 5. Let \( p(\cdot) \in \mathcal{B}(\mathbb{R}^n) \), then we have
\[
\frac{1}{|B|} \| \chi_B \|_{L^{p(\cdot)}(\mathbb{R}^n)} \| \chi_B \|_{L^{1/p'(\cdot)}(\mathbb{R}^n)} \leq C.
\] (31)

Lemma 6. Let \( p(\cdot) \in \mathcal{B}(\mathbb{R}^n) \), then we have for all measurable subsets \( E \subset B \),
\[
\frac{\| \chi_E \|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\| \chi_E \|_{L^{1/p'(\cdot)}(\mathbb{R}^n)}} \leq C. 
\] (32)

where \( \delta_1, \delta_2 \) are constants with \( 0 < \delta_1, \delta_2 < 1 \).

Lemma 7. Let \( p(\cdot), q(\cdot), r(\cdot) \in \mathcal{D}(\mathbb{R}^n) \) be such that
\[
\frac{1}{r(x)} = \frac{1}{p(x)} + \frac{1}{q(x)},
\] (33)

for almost every \( x \in \mathbb{R}^n \). Then, we have
\[
\| fg \|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq 2 \| f \|_{L^{p(\cdot)}(\mathbb{R}^n)} \| g \|_{L^{r(\cdot)}(\mathbb{R}^n)}
\] (34)

for all \( f \in L^{p(\cdot)}(\mathbb{R}^n) \) and all \( g \in L^{r(\cdot)}(\mathbb{R}^n) \).

Lemma 8. Suppose that \( p(\cdot) \in LH(\mathbb{R}^n) \) and \( 0 < p_\pm \leq p(x) \leq p_\mp < \infty \).
(i) For all cubes (or balls) \( |Q| \leq 2^n \) and any \( x \in Q \), we have
\[
\| \chi_Q \|_{L^{p(\cdot)}(\mathbb{R}^n)} \approx |Q|^{1/p(x)}.
\] (35)

(ii) For all cubes (or balls) \( |Q| \geq 1 \), we have
\[
\| \chi_Q \|_{L^{p(\cdot)}(\mathbb{R}^n)} \approx |Q|^{1/p_\infty},
\] (36)

where \( p_\infty = \lim_{x \to -\infty} p(x) \).

The proofs of Lemmas 7 and 8 can be found in [48]. Tan et al. [50] obtain the following result.

Lemma 9. Let \( m \in \mathbb{N}, 0 < \alpha < mn \). If \( q(\cdot), p_1(\cdot), \ldots, p_m(\cdot) \in \mathcal{D} \)
\( (\mathbb{R}^n) \cap LH(\mathbb{R}^n) \) such that \( 1/q(\cdot) = 1/p_1(\cdot) + \cdots + 1/p_m(\cdot) - \alpha/n \), then we have
\[
\| I_m (f) \|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq C \prod_{i=1}^{m} \| f_i \|_{L^{p_i(\cdot)}(\mathbb{R}^n)},
\] (37)

with the constant \( C > 0 \) independent of \( f_1, \ldots, f_m \).

Now, we recall that the \( \lambda \)-central bounded mean oscillation space and the central Morrey space in [41] are defined as follows.

Definition 10. Let \( \lambda < 1/n \) and \( q(\cdot) \in \mathcal{D}(\mathbb{R}^n) \). The \( \lambda \)-central bounded mean oscillation space with variable exponent \( C \mathcal{M}O^{q(\cdot), \lambda}(\mathbb{R}^n) \) is defined by
\[
C\mathcal{M}O^{q(\cdot), \lambda}(\mathbb{R}^n) = \left\{ f \in L^{q(\cdot)}_{\text{loc}}(\mathbb{R}^n) : \| f \|_{C\mathcal{M}O^{q(\cdot), \lambda}(\mathbb{R}^n)} < \infty \right\},
\] (38)
where
\[ \|f\|_{\text{CMO}^{(\ell)}(\mathbb{R}^n)} := \sup_{R>0} \frac{\|f - f_{B(0,R)}\|_{L^\infty(B(R))}}{|B(0,R)|^{\frac{1}{n}}} \] (39)

**Definition 11.** Let \( \lambda \in \mathbb{R} \) and \( q(\cdot) \in \mathscr{P}(\mathbb{R}^n) \). The central Morrey space with variable exponent \( B^{\theta(\cdot),\lambda}(\mathbb{R}^n) \) is defined by
\[ B^{\theta(\cdot),\lambda}(\mathbb{R}^n) = \left\{ f \in L^q(\mathbb{R}^n) : \|f\|_{B^{\theta(\cdot),\lambda}(\mathbb{R}^n)} < \infty \right\}, \] (40)

where
\[ \|f\|_{B^{\theta(\cdot),\lambda}(\mathbb{R}^n)} := \sup_{R>0} \frac{\|f_{B(0,R)}\|_{L^\theta(\mathbb{R}^n)}}{|B(0,R)|^{\frac{1}{n}}} \] (41)

**Remark 12.** When \( \lambda = 0 \), the space \( \text{CMO}^{(\ell)}(\mathbb{R}^n) \) is just the space \( \text{CBMO}^{(\ell)}(\mathbb{R}^n) \) defined in [51]. If \( q(\cdot) = q \) is constant, then we can immediately get Definitions 2 and 4, respectively.

**Remark 13.** Denote by \( \text{CMO}^{(\ell)}(\mathbb{R}^n) \) and \( B^{(\ell)}(\mathbb{R}^n) \) the inhomogeneous versions of the \( \lambda \)-central bounded mean oscillation space and the central Morrey space with variable exponent, which are defined, respectively, by taking the supremum over \( R \geq 1 \) in Definitions 10 and 11 instead of \( R > 0 \) there. Obviously, \( \text{CMO}^{(\ell)}(\mathbb{R}^n) \subset \text{CMO}^{(\ell),\lambda}(\mathbb{R}^n) \) for \( q(\cdot) \in \mathscr{P}(\mathbb{R}^n) \) and \( \lambda < 1/n \), and \( B^{(\ell),\lambda}(\mathbb{R}^n) \subset B^{(\ell)}(\mathbb{R}^n) \) for \( q(\cdot) \in \mathscr{P}(\mathbb{R}^n) \) and \( \lambda \in \mathbb{R} \).

Our main results can be stated as follows.

**Theorem 14.** Suppose that \( p(\cdot), p_1(\cdot) \in \mathscr{P}(\mathbb{R}^n) \cap L^H(\mathbb{R}^n), \lambda_i < 0, i = 1, \ldots, m, \lambda = \lambda_1 + \cdots + \lambda_m \) and \( 1/p(\cdot) = 1/p_1(\cdot) + \cdots + 1/p_m(\cdot) \), then the maximal multilinear Calderón-Zygmund operator \( T^*_b \) is bounded from \( B^{\theta(\cdot),\lambda}(\mathbb{R}^n) \times \cdots \times B^{\theta(\cdot),\lambda}(\mathbb{R}^n) \) to \( B^{\theta(\cdot),\lambda}(\mathbb{R}^n) \) and satisfies
\[ \left\| T^*_b \left( \vec{f} \right) \right\|_{B^{\theta(\cdot),\lambda}(\mathbb{R}^n)} \leq C \sup_j \|b_j\|_{\text{CMO}^{(\ell)}(\mathbb{R}^n)} \prod_{i=1}^m \|f_i\|_{B^{\theta(\cdot),\lambda}(\mathbb{R}^n)}. \] (42)

**Theorem 15.** Suppose that \( p(\cdot), p_1(\cdot), s_j(\cdot) \in \mathscr{P}(\mathbb{R}^n) \cap L^H(\mathbb{R}^n), \) \( 1 < p_j(\cdot) \leq p_j(\cdot) < \infty \), \( 0 < u_j < 1/m \), \( b_j \in \text{CMO}^{(\ell),u_j}(\mathbb{R}^n), \) \( j = 1, \ldots, m \), and \( u_j + v_j < 0 \), then the commutators of the multilinear Calderón-Zygmund operator \( T^*_b \) defined as in (11) are bounded from \( B^{\theta(\cdot),\lambda}(\mathbb{R}^n) \times \cdots \times B^{\theta(\cdot),\lambda}(\mathbb{R}^n) \) to \( B^{\theta(\cdot),\lambda}(\mathbb{R}^n) \) and satisfy
\[ \left\| T^*_b \left( \vec{f} \right) \right\|_{B^{\theta(\cdot),\lambda}(\mathbb{R}^n)} \leq C \sup_j \|b_j\|_{\text{CMO}^{(\ell)}(\mathbb{R}^n)} \prod_{i=1}^m \|f_i\|_{B^{\theta(\cdot),\lambda}(\mathbb{R}^n)}. \] (43)

**Theorem 16.** Let \( 0 < \alpha < mn \). Suppose that \( q(\cdot), p_1(\cdot), s_j(\cdot) \in \mathscr{P}(\mathbb{R}^n) \cap L^H(\mathbb{R}^n), 1 < p_j(\cdot) \leq p_j(\cdot) < \infty, 0 < u_j < 1/m, b_j \in \text{CMO}^{(\ell),u_j}(\mathbb{R}^n), j = 1, \ldots, m, \gamma_i < -\alpha/mn, \beta_j = \beta_m + \sum_{i=1}^m \gamma_i + (a/n) \) with \( \beta_j + \gamma_i < -\alpha/n \), \( 1/p(\cdot) = \sum_{i=1}^m (1/p_1(\cdot)) - (a/n) > 0 \) and \( 1/q(\cdot) = (1/s_j(\cdot)) + (1/p(\cdot)) \), then the commutators of the multilinear fractional operator \( I_{\alpha,b} \) defined as in (16) are bounded from \( B^{p_1(\cdot),q(\cdot)}(\mathbb{R}^n) \times \cdots \times B^{p_m(\cdot),q(\cdot)}(\mathbb{R}^n) \) to \( B^{\theta,\lambda}(\mathbb{R}^n) \) and satisfy
\[ \left\| I_{\alpha,b} \left( \vec{f} \right) \right\|_{B^{\theta,\lambda}(\mathbb{R}^n)} \leq C \sup_j \|b_j\|_{\text{CMO}^{(\ell)}(\mathbb{R}^n)} \prod_{i=1}^m \|f_i\|_{B^{p_i(\cdot),q(\cdot)}(\mathbb{R}^n)} \] (44)

**Remark 17.** In fact, under the same conditions of Theorem 15, the commutators of the multilinear Calderón-Zygmund operator \( T^*_b \) are bounded from \( B^{p_1(\cdot),q(\cdot)}(\mathbb{R}^n) \times \cdots \times B^{p_m(\cdot),q(\cdot)}(\mathbb{R}^n) \) to \( B^{\theta,\lambda}(\mathbb{R}^n) \). We omit its proof since it is essentially similar to the proof of Theorem 2.2.

**Remark 18.** We mention that, even when \( q(\cdot) \) and \( p_j(\cdot)(i = 1, \ldots, m) \) are constant, Theorem 14 is completely new. Theorems 15 and 16 can be seen as generalizations of the corresponding main results in [18] with different approaches based on the theory of variable exponent and on generalization of the \( \lambda \)-central BMO norms. Moreover, our results remain true for the inhomogeneous versions of \( \lambda \)-central BMO spaces and central Morrey spaces with variable exponent.

### 3. Proof of Theorem 2.1

Without loss of generality, we may assume that \( m = 2 \). Let \( f_1, f_2 \) be functions in \( B^{p_1(\cdot),\lambda}(\mathbb{R}^n) \) and \( B^{p_2(\cdot),\lambda}(\mathbb{R}^n) \), respectively. For simplicity, we denote by \( L^p(\mathbb{R}^n) = L^p(\mathbb{R}^n) \) and \( B^{p(\cdot),\lambda} = B^{p(\cdot),\lambda}(\mathbb{R}^n) \). We need to prove
\[ \|X_B T^*_b (f_1, f_2)\|_{L^p(\mathbb{R}^n)} \leq C \|X_B\|_{L^p(\mathbb{R}^n)} \prod_{i=1}^2 \|f_i\|_{B^{p(i),\lambda}(\mathbb{R}^n)}. \] (45)

It follows from the Minkowski inequality that
\[ \|X_B T^*_b (f_1, f_2)\|_{L^p(\mathbb{R}^n)} \leq \|X_B T^*_b (f_1 X_B, f_2 X_B)\|_{L^p(\mathbb{R}^n)} + \|X_B T^*_b (f_1, f_2 X_B)\|_{L^p(\mathbb{R}^n)} + \|X_B T^*_b (f_1 X_B, f_2)\|_{L^p(\mathbb{R}^n)} \] (46)
\[ = E_1 + E_2 + E_3 + E_4. \]
For $E_1$, noting that $T^*$ is of type $(L^p_1 \times \cdots \times L^p_n, L^p)$ (see [44, Theorem 1.3]), by Lemmas 6 and 8, we deduce that

\begin{equation}
E_1 \leq C \left| f_1 \chi_{2B} \right|_{L^p_1} \left| f_2 \chi_{2B} \right|_{L^p} \leq C \sum_{i=1}^{2} \left| B \right|^\alpha \left| \chi_{2B} \right|_{L^p_1} \left| f_i \right|_{L^p} \left( \frac{\left| \chi_{2B} \right|_{L^p}}{\left| f_i \right|_{L^p}} \right)^{\frac{1}{\alpha}}
\end{equation}

(47)

For $E_2$, since $\left| (x-y_1, x-y_2) \right|^{2n} \geq \left| x-y_2 \right|^{2n} \sim \left| 2B \right|^{2n}$ for $x \in B$ and $y_2 \in (2^2B)^c$, using Hölder’s inequality and Lemma 5, we have

$$
\left| T^* \left( f_1 \chi_{2B}, f_2 \chi_{2B} \right) (x) \right| \leq C \left| f_i \chi_{2B} \right|_{L^p} \left| f_2 \chi_{2B} \right|_{L^p} \left| \chi_{2B} \right|_{L^p} \left| f_1 \chi_{2B} \right|_{L^p} \left( \frac{\left| \chi_{2B} \right|_{L^p}}{\left| f_i \chi_{2B} \right|_{L^p}} \right)^{\frac{1}{\alpha}}
$$

(51)

Hence, we derive the estimate

$$
E_4 \leq C \left| B \right|^\alpha \left| \chi_B \right|_{L^p} \sum_{i=1}^{2} \left| f_i \right|_{L^p} \left( \frac{\left| \chi_B \right|_{L^p}}{\left| f_i \right|_{L^p}} \right)^{\frac{1}{\alpha}}.
$$

(52)

The proof of Theorem 2.1 is complete.

### 4. Proof of Theorem 2.2

By linearity, it is enough to consider the operator with only one symbol. Fix then $b_1 \in \text{CMO}^{\alpha, \lambda}(\mathbb{R}^n)$ and consider the operator

$$
T_{b_1}^* \left( \tilde{f} \right) (x) = \sup_{\delta \in \mathbb{R}} \left| b_1 (x) \right| \left| T_{b_1} (f_1, \cdots, f_m) - T_{b_1} (b_1 f_1, \cdots, b_1 f_m) \right|.
$$

(53)

Without loss of generality, we may assume that $m = 2$. Let $f_1, f_2$ be functions in $B^{p,1}_{\alpha,\lambda}(\mathbb{R}^n)$ and $B^{p,1}_{\alpha,\lambda}(\mathbb{R}^n)$, respectively. We need to prove

(54)

(55)

Applying the Minkowski inequality, we obtain

$$
\left\| b_1 \right\|_{\text{CMO}^{\alpha,\lambda} (\mathbb{R}^n)} \leq C \left\| b_1 \right\|_{\text{CMO}^{\alpha,\lambda} (\mathbb{R}^n)} \left\| b_1 \right\|_{\text{CMO}^{\alpha,\lambda} (\mathbb{R}^n)} \sum_{i=1}^{2} \left\| f_i \right\|_{L^p} \left( \frac{\left\| \chi_B \right\|_{L^p}}{\left\| f_i \right\|_{L^p}} \right)^{\frac{1}{\alpha}}
$$

(54)

$$
\left\| T_{b_1}^* \left( \tilde{f} \right) \right\|_{L^p} \leq \left\| T_{b_1} \left( f_1, f_2 \right) \right\|_{L^p} + \left\| T_{b_1} \left( b_1 - (b_1)_{B} \right) \right\|_{L^p} + \left\| T_{b_1} \left( (b_1 - (b_1)_{B}) f_1, f_2 \right) \right\|_{L^p}
$$

(55)

$$
= I + II.
$$
For $I$, we have

$$I \leq \left\| X_B (b_1 - (b_1)_B) T^* (f_1 X_{2B}^2 f_2 X_{2B}) \right\|_{L^p}$$
$$+ \left\| X_B (b_1 - (b_1)_B) T^* \left( f_1 X_{2B}^2 f_2 X_{2B} \right) \right\|_{L^p}$$
$$+ \left\| X_B (b_1 - (b_1)_B) T^* \left( f_1 X_{2B}^2 f_2 X_{2B}^2 \right) \right\|_{L^p}$$
$$= I_1 + I_2 + I_3 + I_4.$$  \hspace{1cm} (56)

For $I_1$, set $1/r(\cdot) = (1/p_1(\cdot)) + (1/p_2(\cdot))$, then $1/p(\cdot) = (1/s_1(\cdot)) + (1/r(\cdot))$. By Lemmas 7, 6, and 8, we get

$$I_1 \leq \left\| X_B (b_1 - (b_1)_B) \right\|_{L^{p_1}} \left\| X_B T^* (f_1 X_{2B}^2 f_2 X_{2B}) \right\|_{L^{p_2}}$$
$$\leq C \left\| b_1 \right\|_{CMO(p_1, p_1)} \left\| X_B \right\|_{L^{p_1}} \left\| f_1 X_{2B}^2 f_2 X_{2B} \right\|_{L^{p_2}}$$
$$\leq C \left\| b_1 \right\|_{CMO(p_1, p_1)} \left\| X_B \right\|_{L^{p_1}} \left( \left\| f_1 \right\|_{L^{p^{*}}} \left\| f_2 \right\|_{L^{p^{*}}} \right)$$
$$\leq C \left\| b_1 \right\|_{CMO(p_1, p_1)} \left\| X_B \right\|_{L^{p_1}} \left( \left\| f_1 \right\|_{L^{p^{*}}} \left\| X_B \right\|_{L^{p_2}} \right)$$
$$\leq C \left\| b_1 \right\|_{CMO(p_1, p_1)} \left\| X_B \right\|_{L^{p_1}} \left( \left\| f_1 \right\|_{L^{p^{*}}} \left\| X_B \right\|_{L^{p_2}} \right).$$  \hspace{1cm} (57)

For $I_2$, noting that $\left| (x - y_1, x - y_2) \right|^{2n} \geq \left| x - y_1 \right| \sim \left| x \right| B^2$ for $x \in B$ and $y_1 \in (2B)^2$, by Hölder’s inequality and Lemma 5, we get

$$\left| T^* (f_1 X_{2B}^2 f_2 X_{2B})_B (x) \right| \leq C \int_{\mathbb{R}^d} \left| f_1 (y_1) \right| dy_1 \int_{\mathbb{R}^d} \left| f_2 (y_2) \right| dy_2 \int_{\mathbb{R}^d} \left| f_1 (y_1) \right| dy_1 \int_{\mathbb{R}^d} \left| f_2 (y_2) \right| dy_2$$
$$\leq C \int_{\mathbb{R}^d} \left| f_1 (y_1) \right| dy_1 \int_{\mathbb{R}^d} \left| f_2 (y_2) \right| dy_2 \int_{\mathbb{R}^d} \left| f_1 (y_1) \right| dy_1 \int_{\mathbb{R}^d} \left| f_2 (y_2) \right| dy_2$$
$$\leq C \left\| f_1 X_{2B} \right\|_{L^{p_1}} \left\| X_B \right\|_{L^{p_1}} \left\| f_2 X_{2B} \right\|_{L^{p_1}}$$
$$\leq C \left\| b_1 \right\|_{CMO(p_1, p_1)} \left\| X_B \right\|_{L^{p_1}} \left\| f_1 X_{2B} \right\|_{L^{p_1}} \left\| f_2 X_{2B} \right\|_{L^{p_1}}$$
$$\leq C \left\| b_1 \right\|_{CMO(p_1, p_1)} \left\| X_B \right\|_{L^{p_1}} \left\| f_1 X_{2B} \right\|_{L^{p_1}} \left\| f_2 X_{2B} \right\|_{L^{p_1}}.$$  \hspace{1cm} (58)

Thus, from Lemmas 7 and 8, it follows that

$$I_2 \leq C \left\| B \right\|^{v_{1, p_{1}}} \prod_{i=1}^{2} \left\| f_i \right\|_{L^{p_{1}}},$$
$$I_3 \leq C \left\| b_1 \right\|_{CMO(p_1, p_1)} \left\| B \right\|^{v_{1, p_{1}}} \left\| X_B \right\|_{L^{p_1}} \prod_{i=1}^{2} \left\| f_i \right\|_{L^{p_{1}}},$$
$$I_4 \leq C \left\| b_1 \right\|_{CMO(p_1, p_1)} \left\| B \right\|^{v_{1, p_{1}}} \left\| X_B \right\|_{L^{p_1}} \prod_{i=1}^{2} \left\| f_i \right\|_{L^{p_{1}}}.$$  \hspace{1cm} (60)

For $II$, we have the following decomposition

$$II \leq \left\| X_B T^* (b_1 - (b_1)_B) \right\|_{L^{p_2}}$$
$$+ \left\| X_B T^* (b_1 - (b_1)_B) \right\|_{L^{p_2}}$$
$$\leq \left\| (b_1 - (b_1)_B) X_B \right\|_{L^{p_2}} \left\| f_1 X_{2B} \right\|_{L^{p_2}} \left\| f_2 X_{2B} \right\|_{L^{p_2}}$$
$$+ \left\| (b_1 - (b_1)_B) X_B \right\|_{L^{p_2}} \left\| f_1 X_{2B} \right\|_{L^{p_2}} \left\| f_2 X_{2B} \right\|_{L^{p_2}}.$$  \hspace{1cm} (61)

For $II_1$, set $1/\phi(\cdot) = (1/s_1(\cdot)) + (1/p_2(\cdot))$, then $1/p(\cdot) = (1/\phi(\cdot)) + (1/p_2(\cdot))$. By Hölder’s inequality, Lemmas 7, 5, and 8, we get

$$II_1 \leq C \left\| (b_1 - (b_1)_B) X_B \right\|_{L^{p_2}} \left\| f_1 X_{2B} \right\|_{L^{p_2}} \left\| f_2 X_{2B} \right\|_{L^{p_2}}$$
$$\leq C \left\| (b_1 - (b_1)_B) X_B \right\|_{L^{p_2}} \left\| f_1 X_{2B} \right\|_{L^{p_2}} \left\| f_2 X_{2B} \right\|_{L^{p_2}}$$
$$\leq C \left\| (b_1 - (b_1)_B) X_B \right\|_{L^{p_2}} \left\| f_1 X_{2B} \right\|_{L^{p_2}} \left\| f_2 X_{2B} \right\|_{L^{p_2}}$$
$$\leq C \left\| (b_1 - (b_1)_B) X_B \right\|_{L^{p_2}} \left\| f_1 X_{2B} \right\|_{L^{p_2}} \left\| f_2 X_{2B} \right\|_{L^{p_2}}.$$  \hspace{1cm} (62)
where in the penultimate inequality we use the estimates

\[
\left| (b_1)_{2B} - (b_1)_B \right| \leq \frac{1}{|B|} \int_B |b_1 - (b_1)_{2B}| \, dx \\
\leq \frac{1}{|B|} \int_{2B} |b_1 - (b_1)_{2B}| \, dx \\
\leq \frac{1}{|B|} \| (b_1 - (b_1)_{2B}) X_{2B} \|_{L^{2}(\Omega)} \| X_{2B} \|_{L^{2}(\Omega)}.
\]

(63)

For \( H_2 \), set \( 1/p(\cdot) = (1/p'_1(\cdot)) - (1/s_1(\cdot)) \). By Hölder’s inequality, Lemmas 5–8, we deduce that

\[
T^* \left( (b_1 - (b_1)_B) f_1 X_{2B} f_2 X_{(2B')} \right)(x) \\
\leq C \int_{2B} |f_1(y_1)| |b_1(y_1) - (b_1)_B| dy_1 \int_{(2B')} |f_2(y)| \frac{|f_2(y)|}{|x-y_2|^{2\alpha}} dy_2 \\
\leq C \int_{2B} |f_1(y_1)| |b_1(y_1) - (b_1)_B| dy_1 \\
\times \sum_{i=1}^{\infty} 2^{2\alpha} \int_{2i \cdot |y_2|}^{(2i+1) \cdot |y_2|} |f_2(y)| dy_2 \\
\leq C \| (b_1 - (b_1)_B) X_{2B} \|_{L^{2}(\Omega)} \| f_1 X_{2B} \|_{L^{2}(\Omega)} \| X_{2B} \|_{L^{2}(\Omega)} \\
\times \sum_{i=1}^{\infty} 2^{2\alpha} \| f_2 X_{2i \cdot |y_2|} \|_{L^{2}(\Omega)} \| X_{2i \cdot |y_2|} \|_{L^{2}(\Omega)}
\]

(64)

This implies that

\[
H_2 \leq C \| b_1 \|_{\mathcal{CMO}^{1/(p^{-1})}} \| B \| \| b_1 \|_{L^{p}} \sum_{i=1}^{\infty} 2^{i \cdot \alpha} \| f_2 \|_{L^{2}(\Omega)} \| X_{i \cdot |y_2|} \|_{L^{2}(\Omega)}.
\]

(65)

For \( H_3 \), noting the fact that (see [17], Page 516)

\[
\left| (b_1)_{2^{i+1}B} - (b_1)_B \right| \leq C \| b_1 \|_{\mathcal{CMO}^{1/(p^{-1})}} (l+1) 2^{i+1} \| B \|^{\alpha}.
\]

(66)

Using Hölder’s inequality, Lemmas 5–8, we obtain

\[
T^* \left( (b_1 - (b_1)_B) f_1 X_{2B} f_2 X_{(2B')} \right)(x) \\
\leq C \int_{2B} |f_2(y)| dy_2 \int_{(2B')} \frac{|f_2(y)|}{|x-y_2|^{2\alpha}} dy_2 \\
\leq C \int_{2B} |f_2(y)| dy_2 \sum_{i=1}^{\infty} 2^{2\alpha} \int_{2i \cdot |y_2|}^{(2i+1) \cdot |y_2|} |f_2(y)| dy_2 \\
\leq C \| f_2 X_{2B} \|_{L^{2}(\Omega)} \| X_{i \cdot |y_2|} \|_{L^{2}(\Omega)} \sum_{i=1}^{\infty} 2^{i \cdot \alpha} \| f_2 X_{2i \cdot |y_2|} \|_{L^{2}(\Omega)} \| X_{2i \cdot |y_2|} \|_{L^{2}(\Omega)}
\]

(67)

Thus, we get

\[
H_3 \leq C \| b_1 \|_{\mathcal{CMO}^{1/(p^{-1})}} \| B \| \| X_{2B} \|_{L^{p}} \sum_{i=1}^{\infty} 2^{i \cdot \alpha} \| f_2 \|_{L^{2}(\Omega)} \| X_{i \cdot |y_2|} \|_{L^{2}(\Omega)}.
\]

(68)

For \( H_4 \), in view of \( u_j + v_j < 0 \), as argued before, we arrive at the inequality

\[
T^* \left( (b_1 - (b_1)_B) f_1 X_{2B} f_2 X_{(2B')} \right)(x) \\
\leq C \int_{2B} |f_2(y)| dy_2 \int_{(2B')} \frac{|f_2(y)|}{|x-y_2|^{2\alpha}} dy_2 \\
\leq C \int_{2B} |f_2(y)| dy_2 \sum_{i=1}^{\infty} 2^{2\alpha} \int_{2i \cdot |y_2|}^{(2i+1) \cdot |y_2|} |f_2(y)| dy_2 \\
\leq C \| f_2 X_{2B} \|_{L^{2}(\Omega)} \| X_{i \cdot |y_2|} \|_{L^{2}(\Omega)} \sum_{i=1}^{\infty} 2^{i \cdot \alpha} \| f_2 X_{2i \cdot |y_2|} \|_{L^{2}(\Omega)} \| X_{2i \cdot |y_2|} \|_{L^{2}(\Omega)}
\]

(69)
Thus, we get

$$H_4 \leq C \|b_1\|_{cMO^{\lambda/(\lambda_n)}_1} \|B\|^1 \|X_B\|_{L^{\psi}} \prod_{i=1}^{2} \|f_i\|_{\rho^{\lambda/(\lambda_n)}}. \quad (70)$$

The proof of Theorem 2.2 is complete.

5. Proof of Theorem 2.3

Fix $b_1 \in \text{CMO}^{\lambda/(\lambda_n)}_1 (R^n)$; as in the proof of Theorem 2.2, we consider the operator

$$I^1_{a,b} \left( f \right)(x) = (b_1(x) - (b_1)_B) I_a (f_1, f_2)(x) - I_a \left( b_1 - (b_1)_B \right) f_1 f_2 (x). \quad (71)$$

Without loss of generality, we may assume that $m = 2$. Let $f_1, f_2$ be functions in $B^s_{p,1}(R^n)$ and $B^s_{p,2}(R^n)$, respectively. We need to prove

$$\left\| X_B I^1_{a,b} \left( f \right) \right\|_{L^{\psi}} \leq C \|b_1\|_{cMO^{\lambda/(\lambda_n)}_1} \|B\|^1 \|X_B\|_{L^{\psi}} \prod_{i=1}^{2} \|f_i\|_{\rho^{\lambda/(\lambda_n)}}. \quad (72)$$

The Minkowski inequality implies that

$$\left\| X_B I^1_{a,b} \left( f \right) \right\|_{L^{\psi}} \leq \left\| X_B \left( b_1 - (b_1)_B \right) I_a (f_1, f_2) \right\|_{L^{\psi}} + \left\| X_B I_a \left( b_1 - (b_1)_B \right) f_1 f_2 \right\|_{L^{\psi}} \quad (73)$$

$$= J + JJ.$$

For $J$, we decompose

$$J \leq \left\| X_B \left( b_1 - (b_1)_B \right) I_a (f_1 X_2 B, f_2 X_2 B) \right\|_{L^{\psi}} + \left\| X_B \left( b_1 - (b_1)_B \right) I_a (f_1 X_2 B, f_2 X_2 B) \right\|_{L^{\psi}} + \left\| X_B \left( b_1 - (b_1)_B \right) I_a (f_1 X_2 B, f_2 X_2 B) \right\|_{L^{\psi}} \quad (74)$$

For $J_1$, since $\lambda = \beta_1 + \gamma_2 + (a/n)$, $1/p(\cdot) = (1/p_1(\cdot)) + (1/p_2(\cdot)) - (a/n)$, and $1/q(\cdot) = (1/s_1(\cdot)) + (1/p(\cdot))$, we use Lemmas 6–9 and obtain

$$J_1 \leq \left\| X_B \left( b_1 - (b_1)_B \right) \right\|_{L^{\psi}} \left\| X_B \right\|_{L^{\psi}} \prod_{i=1}^{2} \|f_i\|_{\rho^{\lambda/(\lambda_n)}} \leq C \|b_1\|_{cMO^{\lambda/(\lambda_n)}_1} \|B\|^1 \|X_B\|_{L^{\psi}} \prod_{i=1}^{2} \|f_i\|_{\rho^{\lambda/(\lambda_n)}}.$$
Similarly, we have

\[ J_3 \leq C \left\| b_1 \right\|_{C^{\infty}(\mathbb{R}^d)} \left\| \mathcal{M}^{\infty}(\mathbb{R}^d) \right\| \left\| f \right\|_{L^p(\mathbb{R}^d)}^2 \prod_{k=1}^d \left\| f_k \right\|_{L^p(\mathbb{R}^d)}. \] (78)

For \( J_4 \), noting that \( (x-y_1, x-y_2) \sim (x-y_1) \), \( |x-y_2| \sim |x-y_2| \), \( |x-y_2| = 2^k |B|^{1-\alpha/2(n+1)} \), and \( y_2 \in (2k^2)^\alpha \), by Hölder’s inequality and Lemma 5, we get

\[
\left| f \left( (f_1, f_2, f_3, f_4) \right) \right| \leq C \sum_{c=1}^{2^d} \sum_{c=1}^{2^d} \sum_{c=1}^{2^d} \sum_{c=1}^{2^d} \left| f_1 \right|_{L^q(\mathbb{R}^d)}^2 \left| f_2 \right|_{L^q(\mathbb{R}^d)}^2 \left| f_3 \right|_{L^q(\mathbb{R}^d)}^2 \left| f_4 \right|_{L^q(\mathbb{R}^d)}^2.
\]

Therefore, as in the estimation of \( J_2 \), we have

\[ J_4 \leq C \left\| b_1 \right\|_{C^{\infty}(\mathbb{R}^d)} \left\| \mathcal{M}^{\infty}(\mathbb{R}^d) \right\| \left\| f \right\|_{L^p(\mathbb{R}^d)}^2 \prod_{k=1}^d \left\| f_k \right\|_{L^p(\mathbb{R}^d)}. \] (80)

For \( JJ \), we use the following decomposition

\[
JJ \leq \left\| X_{b_1} f_1 \left( (b_1, b_2) f_1 X_{2, b_2} f_2 X_{2, b_2} \right) \right\|_{L^p(\mathbb{R}^d)}^2
+ \left\| X_{b_1} f_1 \left( (b_1, b_2) f_1 X_{2, b_2} f_2 X_{2, b_2} \right) \right\|_{L^p(\mathbb{R}^d)}^2
+ \left\| X_{b_1} f_1 \left( (b_1, b_2) f_1 X_{2, b_2} f_2 X_{2, b_2} \right) \right\|_{L^p(\mathbb{R}^d)}^2
+ \left\| X_{b_1} f_1 \left( (b_1, b_2) f_1 X_{2, b_2} f_2 X_{2, b_2} \right) \right\|_{L^p(\mathbb{R}^d)}^2
= JJ_1 + JJ_2 + JJ_3 + JJ_4.
\] (81)

For \( JJ_1 \), set \( 1/q(\cdot) = (1/s_1(\cdot)) + (1/p_1(\cdot)) \), then \( 1/q(\cdot) = (1/q(\cdot)) + (1/p_2(\cdot)) - (1/n(\cdot)) \). By Lemmas 9, 7, and 5, we get

\[
JJ_1 \leq C \left\| (b_1 - (b_1)_B) f_1 X_{2, 2} f_2 X_{2, 2} \right\|_{L^p(\mathbb{R}^d)}^2 \left\| f_1 X_{2, 2} f_2 X_{2, 2} \right\|_{L^q(\mathbb{R}^d)}^2
\leq C \left\| (b_1 - (b_1)_B) f_1 X_{2, 2} f_2 X_{2, 2} \right\|_{L^p(\mathbb{R}^d)}^2 \left\| f_1 X_{2, 2} f_2 X_{2, 2} \right\|_{L^q(\mathbb{R}^d)}^2
\leq C \left\{ \left\| (b_1 - (b_1)_B) X_{2, 2} \right\|_{L^q(\mathbb{R}^d)}^2 + \left\| (b_1)_B X_{2, 2} \right\|_{L^q(\mathbb{R}^d)}^2 \right\} \prod_{k=1}^d \left\| f_1 X_{2, 2} f_2 X_{2, 2} \right\|_{L^q(\mathbb{R}^d)}^2.
\]

For \( JJ_2 \), set \( 1/q(\cdot) = (1/s_1(\cdot)) - (1/p_1(\cdot)) \). By Hölder’s inequality, Lemmas 5–8, we derive

\[
JJ_2 \leq C \left\| (b_1 - (b_1)_B) f_1 X_{2, 2} f_2 X_{2, 2} \right\|_{L^p(\mathbb{R}^d)}^2 \left\| f_1 X_{2, 2} f_2 X_{2, 2} \right\|_{L^q(\mathbb{R}^d)}^2
\leq C \left\| (b_1 - (b_1)_B) X_{2, 2} \right\|_{L^q(\mathbb{R}^d)}^2 \left\| f_1 X_{2, 2} f_2 X_{2, 2} \right\|_{L^q(\mathbb{R}^d)}^2
\leq C \left\{ \left\| (b_1 - (b_1)_B) X_{2, 2} \right\|_{L^q(\mathbb{R}^d)} + \left\| (b_1)_B X_{2, 2} \right\|_{L^q(\mathbb{R}^d)} \right\} \prod_{k=1}^d \left\| f_1 X_{2, 2} f_2 X_{2, 2} \right\|_{L^q(\mathbb{R}^d)}^2.
\]
Therefore, we have

$$J_2 \leq C \|b_1\|_{C^{\alpha}(\mathbb{R})} \|B\| \|X_B\|_{L^p} \prod_{i=1}^{2} \|f_i\|_{g^{\alpha}(\mathbb{R})}. \quad (84)$$

For $J_3$, we note that $\beta_1 + \gamma_1 < -\alpha/n$ and derive

$$|J_a\left((b_1 - (b_1)_a)f_1 X_{2(a,\mu)} f_2 X_{2(\nu)}\right)(x)|$$

$$\leq C \left\|f_2(x)\right\| dy \left\|f_1\right\|_{L^p} \left\|f_2\right\|_{g^{\alpha}(\mathbb{R})} \prod_{i=1}^{2} \|f_i\|_{g^{\alpha}(\mathbb{R})}. \quad (85)$$

Then, we conclude that

$$J_3 \leq C \|b_1\|_{C^{\alpha}(\mathbb{R})} \|B\| \|X_B\|_{L^p} \prod_{i=1}^{2} \|f_i\|_{g^{\alpha}(\mathbb{R})}. \quad (86)$$

For $J_4$, since $\beta_1 + \gamma_1 < -\alpha/n$, then $\beta_1 + \gamma_1 + (\alpha/2n) < 0$, using Hölder’s inequality, Lemma 5, and the fact that $\gamma_2 < -\alpha/(2n)$, we have

$$|J_a\left((b_1 - (b_1)_a)f_1 X_{2(a,\mu)} f_2 X_{2(\nu)}\right)(x)|$$

$$\leq C \left\|f_2(x)\right\| dy \left\|f_1\right\|_{L^p} \left\|f_2\right\|_{g^{\alpha}(\mathbb{R})} \prod_{i=1}^{2} \|f_i\|_{g^{\alpha}(\mathbb{R})}.$$
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