Pioneer’s Anomaly and the Solar Quadrupole Moment

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Abstract

The trajectories of test particles moving in the gravitational field of a non-spherically symmetric mass distribution become affected by the presence of multipole moments. In the case of hyperbolic trajectories, the quadrupole moment of an oblate mass induces a displacement of the trajectory towards the mass source, an effect that can be interpreted as an additional acceleration directed towards the source. Although this additional acceleration is not constant, we perform a general relativistic analysis in order to evaluate the possibility of explaining Pioneer’s anomalous acceleration by means of the observed Solar quadrupole moment, within the range of accuracy of the observed anomalous acceleration. We conclude that the Solar quadrupole moment generates an acceleration which is of the same order of magnitude of Pioneer’s constant acceleration only at distances of a few astronomical units.

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I. INTRODUCTION

Pioneer 10/11 are spacecraft which were launched more than thirty years ago to explore the outer region of the Solar system. They were also the first missions to enter the edge of interstellar space. Due to their especial structure, they also represent an ideal system to carry out precision celestial mechanics experiments which permit acceleration estimations to the level of $10^{-8}\text{cm/s}^2$. After encountering the outer planets Jupiter and Saturn, the two spacecraft followed hyperbolic orbits near the plane of the ecliptic to opposite sides of the Solar system. A careful analysis of orbital data from Pioneer 10/11 has been reported in [1, 2] which indicates the existence of a very weak, long-range acceleration, $a_P = (8.74 \pm 1.33) \times 10^{-8}\text{cm/s}^2$, directed towards the Sun. The most intriguing result about this acceleration is that it is constant, with a single measurement accuracy averaged over 5 days. Moreover, it has been measured at distances contained within the approximate range of 20 to 60 astronomical units (AU) from the Sun.

The most conservative hypothesis about the origin of this anomalous acceleration is that this kind of effects on Pioneer’s tracking data must be due to some technological reasons related to the configuration and equipment of the spacecraft. However, the analysis reported in [1, 2] seems to have ruled out all possible non-gravitational forces generated inside and outside the spacecraft. Moreover, the authors have taken into account the accepted values of the errors in the determination of the planetary ephemeris, Earth’s orientation, precession, and nutation. More adventurous explanations of the anomalous acceleration suggest the existence of new physical phenomena, especially inspired by the fact that $a_P$ is of the order of $cH$, where $c$ is the velocity of light in empty space and $H$ is the Hubble constant. For instance, in [3, 4, 5] it is suggested that the expansion of the Universe induces local tidal forces which generate the anomalous acceleration. In [6] a Kaluza-Klein model is proposed in which the fifth dimension becomes dynamical with an expansion rate that generates $a_P$. A scale dependent cosmological term has been used in [7] which affects the motion of test particles. A different explanation is suggested in [8] by means of a time-dependent Newtonian gravitational constant that produces a long-range acceleration. More sophisticated explanations have been proposed in the literature (see, for instance [2], for a recent review) which include dark matter and stringy inspired scalar fields [9, 10].

In this work, we investigate the possibility of explaining the anomalous acceleration by
means of the Solar quadrupole moment. Clearly, the additional acceleration due to the quadrupole is not constant, but decreases with distance. In this sense the, quadrupole moment could immediately be ruled out as an explanation for \( a_P \). However, our approach is different. Since \( a_P \) has been observed only at distances less than 60 AU, with an accuracy of around 15%, our goal is to determine whether the acceleration generated by the quadrupole is of the same order of \( a_P \) and can be fitted within its range of accuracy. This idea is based upon the fact that the quadrupole moment of the Sun has been determined in an exact manner only recently [11], and has been given the value of \( J_2 = -2.28 \times 10^{-7} \pm 15\% \), where \( J_2 \) is a dimensionless parameter to be introduced below. Before this value was known, different methods generated completely different values of \( J_2 \), sometimes with differences of several orders of magnitude. (In fact, such a great ambiguity in the value of \( J_2 \) could easily be used to explain \( a_P \) in the complete range of observation!) With this new value, we will show that the quadrupole moment of the Sun produces an acceleration which is of the same order of \( a_P \) at distances of a few AU’s, but is several orders of magnitude less than \( a_P \) at distances between 20 and 60 AU’s.

We will analyze the relativistic hyperbolic motion of test particles in the gravitational field of a mass with quadrupole moment. For the sake of simplicity, we will assume that the quadrupole is due only to the asphericities of the mass distribution, preserving the axial symmetry, and will neglect the rotation of the source. This allows us to use a static, axisymmetric solution of Einstein’s vacuum field equations as the metric which describes the spacetime. In particular, we will use the Erez–Rosen metric for our analysis. Moreover, we will assume that the motion is confined to the equatorial plane of the source. This is in good agreement with the actual path of Pioneer 10/11 which move near the plane of the ecliptic.

II. GRAVITATIONAL FIELD OF A MASS WITH QUADRUPOLE MOMENT

In this section we present a static axisymmetric solution of Einstein’s vacuum field equations which was first discovered by Erez and Rosen [12]. This solution generalizes Schwarzschild solution to include an arbitrary mass quadrupole moment. Originally, this solution was obtained in prolate spheroidal coordinates, but for the purposes of this work spherical coordinates \((t, r, \theta, \varphi)\) are more appropriate. In these coordinates, it can be shown
that it is symmetric with respect to the equatorial plane, $\theta = \pi/2$, so that it becomes a geodesic plane. For the sake of simplicity we will limit ourselves to the investigation of free motion on the equatorial plane. Accordingly, the Erez-Rosen solution can be written as

$$ds^2 = \left(1 - \frac{2m}{r}\right)e^{2q\psi}dt^2 - \left(1 - \frac{2m}{r}\right)^{-1}e^{2q(\gamma-\psi)}dr^2 - r^2e^{-2q\psi}d\varphi^2,$$

(1)

where the constant $m$ represents the Schwarzschild mass, $q$ is a constant parameter that determines the quadrupole moment, and $\psi$ and $\gamma$ are, in general, functions of $r$ and $\theta$. On the equatorial plane, these functions take the form

$$\psi = -\frac{1}{4} \left[\frac{1}{2} \left(\frac{3r^2}{m^2} - 6\frac{r}{m} + 2\right) \ln \left(1 - \frac{2m}{r}\right) + 3 \left(\frac{r}{m} - 1\right)\right],$$

(2)

$$\gamma \approx \frac{1}{10} \frac{m^4}{r^4}.$$  

(3)

The general form of the function $\gamma$ is rather cumbersome. Here we quote only the leading term in the limit $r \to \infty$ which is sufficient for the analysis we will perform in the next section.

In general, the Erez-Rosen solution is asymptotically flat and this allows us to determine its multipole moments in a covariant manner. For instance, using the Geroch-Hansen definition of multipole moments one can show that the monopole is $m$ and the quadrupole moment is $(2/15)qm^3$. Higher multipoles are given in terms of the monopole and quadrupole moments. So the parameters $m$ and $q$ acquire a clear physical meaning. As expected, in the limiting case $q = 0$, the Erez-Rosen metric coincides with the Schwarzschild solution on the equatorial plane. The hypersurface $r = 2m$ turns out to be singular, in accordance with the black hole uniqueness theorems. An additional singularity is situated at $r \to 0$. Outside the hypersurface $r = 2m$, the spacetime is completely regular.

At large distances from the source, the expansion of the metric component $g_{tt}$ leads to

$$g_{tt} \approx 1 - \frac{2m}{r} \left[1 - \frac{2}{15}q \left(\frac{m}{r}\right)^2\right].$$

(4)

Later on we will use this expression to determine the acceleration generated by the quadrupole moment.

III. MOTION OF TEST PARTICLES

Let us consider the motion of test particles on the equatorial plane of the Erez-Rosen metric. The geodesic equation can be reduced to a set of first order differential equations
due to the existence of two constants of motion $p_t$ and $p_\varphi$ associated with the time-translation and the axial symmetry, respectively. For the case of a massive test particle, the geodesic equations can be written as:

\[
\left(1 - \frac{2m}{r}\right) e^{2\psi} \dot{t} = p_t ,
\]

\[
r^2 e^{-2\psi} \dot{\varphi} = p_\varphi ,
\]

\[
r^2 = e^{-2\psi} \left[ p_t^2 - \left(1 - \frac{2m}{r}\right) e^{2\psi} \left(1 - \frac{2m}{r}\right) \frac{p_\varphi^2}{r^2} e^{4\psi} \right] ,
\]

where a dot represents the derivative with respect to the affine parameter along the geodesic. It is possible to perform a detailed analysis of these set of geodesic equations and find out all the effects of the quadrupole moment on the motion of test particles. Here, however, we are interested only on hyperbolic motion. Hence, let us consider the special case $r = r(\varphi)$ so that \(\dot{r} = (dr/d\varphi) \dot{\varphi}\), and introduce the following notations:

\[
u = \frac{2m}{u} , \quad \alpha = \frac{2m}{p_\varphi} .
\]

Then, the geodesic equation reduces to

\[
(u')^2 = F(u) := e^{-2\psi} \left[u^3 - u^2 + \alpha^2 u e^{-2\psi} + \alpha^2 e^{-2\psi} \left(p_t^2 e^{-2\psi} - 1\right)\right] ,
\]

where a prime denotes the derivative with respect to the angle coordinate $\varphi$. Since the expression $(u')^2$ is positive for a timelike geodesic, we must demand that the condition $F(u) \geq 0$ be satisfied. Accordingly, from Eq.(9) we obtain

\[
\alpha^2 p_t^2 \geq (1 - u) e^{2\psi} \left(\alpha^2 + u^2 e^{2\psi}\right) .
\]

Because the left-hand side of this inequality is a positive definite constant, this implies that the roots of the equation $F(u) = 0$ must be contained in the interval $(1 - u) > 0$ or, according to Eq.(8), in the interval $r \in (2m, \infty)$. As mentioned in the last section, the hypersurface $r = 2m$ contains in general a curvature singularity. Thus, hyperbolic motion is confined to the region where the Erez-Rosen metric is regular.

On the other hand, simple algebraic manipulations show that the equation $F(u) = 0$ can be expressed as a quadratic algebraic equation for the function $\exp(2\psi)$, i.e.,

\[
e^{4\psi} + \frac{\alpha^2}{u^2} e^{2\psi} - \frac{\alpha^2 p_t^2}{(1 - u)u^2} = 0 .
\]
Since \( \exp(2q\psi) > 0 \), we need to consider only the positive root of this equation which can be expressed as

\[
q = \frac{1}{2\psi} \ln \left[ \frac{\alpha^2}{2u^2} \left( \sqrt{1 + \frac{p_t^2 u^2}{\alpha^2 (1 - u)}} - 1 \right) \right]. \tag{12}
\]

Now, the function \( \psi \) is positive in the interval \( r \in (2m, \infty) \). Then, from Eq. \((12)\) we obtain that

\[
\frac{\alpha^2}{2u^2} \left( \sqrt{1 + \frac{p_t^2 u^2}{\alpha^2 (1 - u)}} - 1 \right) > 1 \quad \text{for} \quad q > 0 , \tag{13}
\]

\[
\frac{\alpha^2}{2u^2} \left( \sqrt{1 + \frac{p_t^2 u^2}{\alpha^2 (1 - u)}} - 1 \right) < 1 \quad \text{for} \quad q < 0 . \tag{14}
\]

These conditions can be represented in an equivalent manner as

\[
f(u) := u^2(u - 1) + \alpha^2(u - 1) + \alpha^2 p_t^2 > 0 \quad \text{or} \quad < 0 \quad \text{for} \quad q > 0 \quad \text{or} \quad q < 0 , \tag{15}\]

respectively. On the other hand, as can be seen from Eq. \((12)\), the function \( f(u) \) coincides with the function \( F(u) \) for vanishing quadrupole moment \( (q = 0) \), i.e., \( f(u) \) determines the motion of test particles in the Schwarzschild spacetime. This is a nontrivial result. In fact, suppose that the equation \( F(u) = 0 \) has a root at \( u = u_0 \). Then at \( u_0 \) the value of \( q \) is related to the value of \( u_0 \) by means of \((12)\) with \( u \) replaced by \( u_0 \). If, for instance, \( q < 0 \), then according to Eq. \((15)\) the Schwarzschild function \( f(u_0) \) is negative. In the case of hyperbolic motion, the value \( u = u_0 \) corresponds to a vanishing “velocity”, i.e., \( u'(u = u_0) = 0 \) or, equivalently, \( \dot{r} = 0 \), a condition which can be satisfied only at the perihelion of the hyperbolic trajectory.

At this point, the motion of test particles in the Schwarzschild spacetime is not allowed, because there \( f(u_0) < 0 \). From the form of the functions \( F(u) \) and \( f(u) \) one can show that in general

\[
f(u) < F(u) \quad \text{for} \quad q < 0 \quad \text{and} \quad f(u) > F(u) \quad \text{for} \quad q > 0 . \tag{16}\]

Let us consider the case of an oblate configuration \( (q < 0) \). Hyperbolic motion corresponds to the case where \( F(u = 0) > 0 \) and possesses (at least) one positive root, say \( u_2 \). The motion is confined to the region \( 0 < u < u_2 \) with \( r_2 = 2m/u_2 \) being the perihelion distance. At \( u = u_2 \), according to \((16)\), we have that \( f(u_2) < 0 \) and so no motion is allowed in the Schwarzschild spacetime. If we “move” from \( u_2 \) towards \( u = 0 \), we will find a point, say \( u_1 \), where \( f(u_1) = 0 \). Starting from this point and up to the point \( u = 0 \), where \( f(u = 0) > 0 \), the motion is allowed in the Schwarzschild as well as in the Erez–Rosen
spacetime. For the trajectory in the Schwarzschild spacetime the perihelion distance is given by
\[ r_1 = \frac{2m}{u_1} \] and at this point, according to (10), \( F(u = u_1) > 0 \), that is, the trajectory in the Erez–Rosen spacetime would correspond to a non-zero radial velocity. So we see that the perihelion distance of a hyperbolic path becomes affected by the presence of an oblate quadrupole moment in such a way that
\[ r_2 < r_1, \quad (17) \]
a result that trivially follows from the fact that \( u_1 < u_2 \). Thus an oblate quadrupole moment reduces the perihelion distance.

Let us denote by \( r_{ER}(\varphi) \) the hyperbolic path of a test particle in the Erez–Rosen spacetime with mass \( m \) and a negative quadrupole \( q \), and by \( r_S(\varphi) \) the corresponding path in the Schwarzschild spacetime which is obtained from \( r_{ER}(\varphi) \) by setting \( q = 0 \). Notice that the mass \( m \) must be the same in order to be able to compare both trajectories. Clearly, \( r_S(\varphi) \) must be a solution of the equation \( (u')^2 = f(u) \), whereas \( r_{ER}(\varphi) \) is a solution of \( (u')^2 = F(u) \) with the same mass. As we have shown above, at the perihelion we have that \( r_{ER}(\varphi_{per}) < r_S(\varphi_{per}) \). Moreover, since in this case \( f(u) < F(u) \) for all values of \( u \), we obtain that in general
\[ r_{ER}(\varphi) < r_S(\varphi), \quad (18) \]
at all points of the trajectory. The difference between these two trajectories will become smaller as \( r \) increases. Moreover, \( r_{ER} \) will approach \( r_S \) asymptotically at \( r \to \infty \), where the contribution of the mass quadrupole moment becomes negligible. If we were analyzing the motion of a test particle in the gravitational field of a mass with negative quadrupole moment, ignoring the contribution of the quadrupole, we would expect a path \( r_S(\varphi) \), whereas the measured path would be \( r_{ER}(\varphi) \), which is always less than \( r_S(\varphi) \). Then, one could interpret this result as due to an additional acceleration directed towards the source of the gravitational field. In fact, the presence of a quadrupole moment generates an “effective” acceleration which leads to deformations of the trajectories of test particles. In the next section we will estimate this effective acceleration.

To finish this section we mention that in the case of a prolate quadrupole moment \( (q > 0) \) the effect is opposite, that is, the perihelion distance increases and the effective acceleration would be directed outwards the gravitational source. This case, however, is not expected to
occur in astrophysical bodies because any deviations from spherical symmetry are usually generated by rotation which leads to oblate configurations.

IV. THE EFFECTIVE ACCELERATION

To find the exact hyperbolic orbit of a test particle in the gravitational field described by the Erez–Rosen metric one must solve the differential equation (9) in the range where this kind of motion is allowed. This can be done by applying standard numerical methods. Another possibility is to consider the case of motion at large distances from the source, where we can use a Taylor expansion of the functions ψ and γ that enter the function $F(u)$ in Eq. (9), and preserve the leading terms only. The resulting function $F(u)$ turns out to be a cubic polynomial of $u$. In this case, the differential equation (9) can be solved analytically as $\varphi = \varphi(u)$ (see [15] for an analysis of the resulting solutions), an expression from which, in principle, we could derive the acceleration due to the quadrupole moment $q$. Nevertheless, the resulting expression is rather cumbersome and an analysis seems to be possible only by applying numerical procedures.

To evaluate the acceleration $a_q$ induced by the quadrupole moment, we will use the analogy with the method applied to measure the Solar quadrupole. Indeed, for an axially symmetric distribution of matter, for instance the Sun, the contribution of the quadrupole moment to the exterior gravitational field can be expressed in the metric component $g_{tt}$ as

$$g_{tt} = 1 - 2\frac{m}{r} \left[ 1 + J_2 \left( \frac{R_0}{r} \right)^2 P_2(\cos \theta) \right],$$

(19)

where $J_2$ is a dimensionless constant, $P_2(\cos \theta)$ is the Legendre polynomial, and $R_0$ is the radius of the Sun. Evaluating this expression on the equatorial plane ($\theta = \pi/2$) and comparing it with Eq. (19), we can see that both expressions coincide if

$$q = 60 J_2 \left( \frac{R_0}{r} \right)^2.$$

(20)

Then, from Eq. (19) we can read the expression for the gravitational potential whose derivative with respect to the radial coordinate produces the leading term of the acceleration. Thus, we obtain

$$a_q = -\frac{3J_2 R_0 m}{2r^4}.$$

(21)
For the evaluation of this quantity we take, according to \[11\], the value of $J_2 = -2.28 \times 10^{-7}$ for the Solar quadrupole moment and $R_0 \approx 6.96 \times 10^{12} cm$ for the photospheric radius of the Sun. Moreover, we express the distance $r$ in astronomical units, i.e., $r = n \times 1.496 \times 10^{13} cm$, where $n$ is a positive real number. Thus, we obtain

$$a_q = \frac{1}{n^4} \times 4.33 \times 10^{-8} cm/s^2. \quad (22)$$

So we see that at a distance of one astronomical unit, the order of magnitude of $a_q$ coincides with $a_P$. Considering the uncertainty in the values of $J_2$, $R_0$ and $m$, one can show that $a_q$ and $a_P$ are still of the same order of magnitude at distances smaller than $n \approx 2.1$. In the interval between 20 and 60 AU’s, where the anomalous acceleration has been measured, $a_q$ is several orders of magnitude less than $a_P$.

V. CONCLUSIONS

In this work we have investigated the possibility of explaining Pioneer’s anomalous acceleration by means of the Solar quadrupole moment. In the analysis of the hyperbolic motion of a test particle in the gravitational field of a mass with a negative quadrupole moment, we have shown that the complete trajectory becomes closer to the gravitational source, an effect that can be interpreted as due to the presence of an additional acceleration, $a_q$, generated by the quadrupole moment.

Although $a_q$ is not a constant quantity, its value at distances very closed to one AU is of the same order of magnitude of $a_P$. The discrepancy between $a_q$ and $a_P$ increases drastically with distance, indicating that it is not possible to explain the anomalous acceleration at distances where it has been measured for Pioneer 10/11 spacecraft.

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