Abstract. In the note we prove that all composition factors of a finite group possessing a Carter subgroup of odd order either are abelian, or are isomorphic to $L_2(3^{2n+1})$.

**Keywords**: group of induced automorphisms, $(rc)$-series.

**Introduction**

A known result by Glauberman and Thompson states, that a finite simple group can not includes a self-normalizing Sylow $p$-subgroup for $p \geq 5$ (see [1, Theorem X.8.13], for example). Later, in [2, Corollary 1.2] Guralnick, Malle, and Navarro obtain a generalization of this result, proving that in any simple group $G$ for a Sylow subgroup $P$ of odd order the equality

$$N_G(P) = PC_G(P)$$

can not be fulfilled. This result is obtained by the authors as a corollary to the following theorem.

**Theorem 1.** [2, Theorem 1.1] Let $p$ be an odd prime and $P$ a Sylow $p$-subgroup of the finite group $G$. If $p = 3$, assume that $G$ has no composition factors of type $L_2(3^f)$, $f = 3^a$ with $a \geq 1$.

1. If $P = N_G(P)$, then $G$ is solvable.
2. If $N_G(P) = PC_G(P)$, then $G/O_p'(G)$ is solvable.

In the paper we prove a generalization of the first statement of the theorem.

**Theorem 2.** (Main Theorem) Assume that $G$ possesses a Carter subgroup of odd order, then each composition factor of $G$ either is abelian, or is isomorphic to $L_2(3^{2n+1})$, $n \geq 1$. Moreover, if 3 does not divide the order of a Carter subgroup, then $G$ is solvable.

Clearly, item (1) of Theorem follows from Lemmas [3] and [5] (see the proof in the end of the paper).

**1 Notations**

In the paper only finite groups are considered, so the term “group” is always used in the meaning “finite group”.

The notation in the paper agrees with that of [3]. Recall that a nilpotent selfnormalizing subgroup is called a **Carter subgroup**. A non-refinable normal series of a group is called a **chief series**. A composition series is called an **$(rc)$-series** if it is a refinement of a chief series.

Let $A, B, H$ be subgroups of $G$ such that $B$ is normal in $A$. Define $N_H(A/B) := N_H(A) \cap N_H(B)$ to be the **normalizer** of $A/B$ in $H$. If $x \in N_H(A/B)$, then $x$ induces

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1 this term is introduced by V.A.Vedernikov in [4]
an automorphism on $A/B$ acting by $Ba \mapsto Bx^{-1}ax$. Thus there exists a homomorphism $N_H(A/B) \to \text{Aut}(A/B)$. The image of $N_H(A/B)$ under the homomorphism is denoted by $\text{Aut}_H(A/B)$ and is called the \textit{group of $H$-induced automorphisms} of $A/B$, while the kernel of the homomorphism is denoted by $C_H(A/B)$ and is called the \textit{centralizer} of $A/B$ in $H$. If $B = 1$, then we use the notation $\text{Aut}_H(A)$. Notice that $\text{Aut}_G(A)$ sometimes is called the automizer of $A$ in $G$. Groups of induced automorphisms are introduced by F.Gross in [5], where the author says that this notion is taken from unpublished Wielandt’s lectures.

Evidently, $C_H(A/B) = C_G(A/B) \cap H$, so $\text{Aut}_H(A/B) = \mathcal{N}_H(A/B)/C_H(A/B) \cong \mathcal{N}_H(A/B)C_G(A/B)/C_G(A/B) \leq \text{Aut}_G(A/B)$, i.e. $\text{Aut}_H(A/B)$ can be naturally considered as a subgroup of $\text{Aut}_G(A/B)$, and we think of $\text{Aut}_H(A/B)$ as a subgroup of $\text{Aut}_G(A/B)$ without additional clarifications.

We need the following result.

**Lemma 3.** [6, Theorem 1] (Generalized Jordan-Hölder theorem) Let

$$G = G_0 \supset G_1 \supset \ldots \supset G_n = 1$$

be an $(rc)$-series of $G$, denote $G_{i-1}/G_i$ by $S_i$. Assume that

$$G = H_0 \supset H_1 \supset \ldots \supset H_n = 1$$

is a composition series of $G$ and $T_i = H_{i-1}/H_i$. Then there exists a permutation $\sigma \in \text{Sym}_n$ such that for every section $T_i$ the inclusion $\text{Aut}_G(T_i) \leq \text{Aut}_G(S_{i\sigma})$ holds. Moreover, if the second series is also an $(rc)$-series, then $\sigma$ can be chosen so that the isomorphisms $\text{Aut}_G(T_i) \cong \text{Aut}_G(S_{i\sigma})$ holds.

## 2 Proof of the main theorem

We divide the proof of the main theorem into several lemmas.

**Lemma 4.** Let $K$ be a Carter subgroup of $G$ and

$$G = G_0 \supset G_1 \supset \ldots \supset G_n = 1$$

be an $(rc)$-series of $G$. Then for every nonabelian composition factor $S$ of $G$ there exists $i$ such that $G_{i-1}/G_i \cong S$ and $\text{Aut}_K(G_{i-1}/G_i)$ is a Carter subgroup of $\text{Aut}_G(G_{i-1}/G_i)$.

**Proof.** The claim follows by induction on the length of the chief series, whose refinement is the $(rc)$-series, and [7, Lemma 3].

**Lemma 5.** (mod CFSG) Let $G$ be a finite almost simple group, possessing a Carter subgroup $K$ of odd order. Then $G \cong L_2(3^{2n+1}) \wr \langle \varphi \rangle$, where $n \geq 1$ and $\varphi$ is a field automorphism of $G$ of order $2n+1$.

In particular, if a Sylow 3-subgroup of $G$ is a Carter subgroup, then $G \cong L_2(3^{3n}) \wr \langle \varphi \rangle$, where $n \geq 1$ and $\varphi$ is a field automorphism of $G$ of order $3^n$.

**Proof.** The claim follows from the classification of Carter subgroups given in [8, Tables 7–10]. Notice that only this lemma in the paper uses the classification of finite simple groups.

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Now we are ready to proof the main result of the paper (Theorem 2). Assume that a finite group $G$ possesses a Carter subgroup $K$ of odd order. Assume that there exists a nonabelian composition factor $S$ of $G$. Then by Lemma 3 there exist subgroups $A, B$ of $G$ such that $A/B \simeq S$ and $\text{Aut}_K(A/B)$ is a Carter subgroup of $\text{Aut}_G(A/B)$. By Lemma 5 we obtain $S \simeq L_2(3^{2n+1})$. Notice that by [8, Table 10] it follows that in this case $|\text{Aut}_K(A/B)|$ is divisible by 3, i.e. $|K|$ is divisible by 3 as well. Therefore, if $|K|$ is not divisible by 3, then $G$ is solvable.

Notice that statement (1) in Theorem 1 can be obtained by exactly the same arguments.

References

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