EXISTENCE OF LIFE IN LENIA

CRAIG CALCATERA & AXEL BOLDT

ABSTRACT. Lenia is a continuous generalization of Conway’s Game of Life. Bert Wang-Chak Chan has discovered and published many seemingly organic dynamics in his Lenia simulations since 2019. These simulations follow the Euler curve algorithm starting from function space initial conditions. The Picard-Lindelöf Theorem for the existence of integral curves to Lipschitz vector fields on Banach spaces fails to guarantee solutions, because the vector field associated with the integro-differential equation defining Lenia is discontinuous. However, we demonstrate the dynamic Chan is using to generate simulations is actually an arc field and not the traditional Euler method for the vector field derived from the integro-differential equation. Using arc field theory we prove the Euler curves converge to a unique flow which solves the original integro-differential equation. Extensions are explored and the modeling of entropy is discussed.

Keywords: arc fields; discontinuous vector fields; integro-differential equations; entropy models

1. Introduction

Lenia is a continuous model proposed by Bert Wang-Chak Chan around 2018 [2] which generalizes the Game of Life. The Game of Life (GoL) is a discrete dynamical system, proposed by Conway in 1970. Lenia follows simpler generalizations of GoL, such as “Larger than Life” [4], [5], “RealLife” [9], and “SmoothLife” [11].

The original Game of Life is a cellular automata, a discrete dynamical system on \( \mathbb{Z}^2 \). Its rules can be written with convolution on the state space \( \mathbb{Z}^2 \) in the following way. In this form the rules are then straightforward to generalize to continuous dynamical systems. Denote the Chebyshev distance, or \( L^\infty \) metric, on \( \mathbb{Z}^2 \) by

\[ \| x \|_\infty := \max_{i \in \{1, 2\}} | x_i |. \]

Define the kernel \( K : \mathbb{Z}^2 \to \mathbb{R} \) by

\[ K(x) := \begin{cases} 1 & \text{if } \| x \|_\infty \leq 1 \\ 0 & \text{otherwise.} \end{cases} \]

For another function \( g : \mathbb{Z}^2 \to \mathbb{R} \) define the discrete convolution operator * by

\[ (K * g)(x) := \sum_{y \in \mathbb{Z}^2} K(y) g(x - y) \]

Then the discrete dynamical system \( f : \mathbb{Z}^2 \times \mathbb{N} \to \{0, 1\} \) for GoL is given by

\[ f_{t+1}(x) = \begin{cases} 1 & \text{if } f_t(x) = 0 \text{ and } (K * f_t)(x) = 3 \ (\text{birth}) \\ 1 & \text{if } f_t(x) = 1 \text{ and } (K * f_t)(x) \in \{3, 4\} \ (\text{survival}) \\ 0 & \text{otherwise (death)}. \end{cases} \]
for \( t \in \mathbb{N} \) and \( x \in \mathbb{Z}^2 \). The initial condition is given by an arbitrary function \( f_0 : \mathbb{Z}^2 \to \{0, 1\} \).

What has made GoL so famous is that these finite, simple rules are enough to create infinitely complex, sometimes even beautiful, dynamics. In his public introduction of the GoL instructions in 1970, Martin Gardner \([6]\) wrote, “Because of Life’s analogies with the rise, fall and alterations of a society of living organisms, it belongs to a growing class of what are called ‘simulation games’ (games that resemble real-life processes).” Thanks to the existence of “gliders” in GoL, Conway showed it is possible to carefully set the initial conditions so that iterating \( f \) generates structures with the behaviors of the three logical gates, AND, OR, NOT. This proves GoL is a universal Turing machine, and therefore capable of approximating any mathematical operation. Therefore the most complex mathematical simulations imaginable may be faithfully represented by GoL with the right choice of initial condition.

GoL is a recursive function on \( \mathbb{Z}^2 \), and simply being well-defined is enough to guarantee the existence of the sequence of steps starting with any initial condition. So it is obvious “life exists” in GoL starting from any initial condition. But it is not so obvious solutions exist for its continuous generalizations.

Chan’s instructions generalize Conway’s by using decreasingly-small, discrete state space locations and decreasingly-small time steps until, in the limit, the instructions hopefully converge to a continuous dynamical system. The resulting simulation is intended to mimic some of the aspects of actual material life. Like GoL, Lenia appears to give complex dynamics with relatively simple instructions.

However, the existence of the limiting processes for these continuous instructions are difficult to prove. In fact, we show in this paper that the instructions for Lenia are an integro-differential equation which gives a discontinuous vector field on an infinite dimensional metric space. The basic existence theorem for solutions to ODEs on Banach spaces, the Picard-Lindelöf Theorem, requires Lipschitz continuous vector fields, and is therefore not applicable.

Instead of using a vector field on a Banach space, we generalize to an arc field on a metric space \([1]\) and show solutions exist there. Then we prove that the solution to the arc field solves the original integro-differential equation, thus proving the existence of the generalized Game of Life dynamics in Lenia.

Two major advantages arise from this effort to extend the discrete model to the continuous. First, there is the possibility of greater descriptive power and more complicated dynamics in a continuous model. Second, the continuous limit will have computable laws available, such as hard limits on measurements of the total mass or higher order moments under particular parameter choices. There are more theoretical results for continuous dynamical systems than discrete systems, because of the control that the continuum \( \mathbb{R} \) gives compared with the gaps in discrete \( \mathbb{Z} \).

In the next section we will review Chan’s Lenia model, and compare it to GoL. In the third section we interpret Chan’s instructions as an integro-differential equation. This gives rise to a vector field which, however, is discontinuous and incompatible with the Euler curve approach Chan uses to simulate the Lenia dynamic. We instead define an arc field and show that Chan’s simulation is the Euler approximation of solutions to this arc field. The main result of the paper is to show that the Euler approximations will converge to a solution of the integro-differential equation.
To achieve this we first review the theory of arc fields in Section 5, using it to prove the existence and uniqueness of the limit of the Lenia algorithm in Section 7. Then we show the resulting solutions also solve Lenia’s integro-differential equation in Section 8. In the final section, we build on those solutions to create more complicated models. We feed the creatures food, and put them in competition with each other using generalizations of the Lotka–Volterra predator-prey ODEs.

2. Lenia: continuous version of GoL

Chan’s formulation \[2\] (17), (18), p. 266 of the Lenia dynamic \(f : \mathbb{R}^n \times \mathbb{R} \to [0, 1]\) where \((x, t) \mapsto f_t(x)\) for \(x \in \mathbb{R}^n\) and \(t \in [0, \infty)\) is given by

\[
\begin{align*}
  f_{t+dt}(x) &= f_t(x) + dt \left(G(K * f_t)(x)\right)^{(1-f_t(x))/dt} \\
  f_t(x) &= f_t(x) + dt \left(G(K * f_t)(x)\right)^{(1-f_t(x))/dt}
\end{align*}
\]

for positive infinitesimal \(dt\). We discuss this formula in the next two paragraphs.

Here \(K\) is a kernel function \(K : \mathbb{R}^n \to \mathbb{R}\) which is assumed to be \(L^1\) so the continuous convolution

\[
(K * f_t)(x) := \int_{\mathbb{R}^n} K(x-y)f_t(y)\,dy
\]

is well defined. \(G : \mathbb{R} \to \mathbb{R}\) is the activation function which we will assume to be bounded and Lipschitz continuous with constant \(C_G \geq 0\) such that \(|G(x) - G(y)| \leq C_G |x-y|\). Finally, the square brackets denote the clip function

\[
[x]_a^b := \begin{cases} 
  b & \text{if } x \geq b \\
  x & \text{if } a < x < b \\
  a & \text{if } x \leq a
\end{cases}
\]

Basic choices of kernel \(K\) and activation function \(G\) used to create complicated dynamics satisfy \(K : \mathbb{R}^2 \to \mathbb{R}^+\) such as \(K(x, y) := \exp\left(4 - \frac{1}{r^2-1}\right)1_{\{0<r<1\}}(x, y)\) where \(r = \sqrt{x^2 + y^2}\) and \(1_S\) is the indicator function on the set \(S\), and \(G : [0, 1] \to [-1, 1]\) such as \(G(x) := 2\exp\left(-\left(x - \frac{1}{2}\right)^2/2\right) - 1\). Such choices were initially explored by Chan in \[2\].

Line \(2.1\) is meant to describe the time evolution of the creatures of Lenia. The creature at time \(t\) is described by the function \(f_t : \mathbb{R}^n \to [0, 1]\) where usually \(n = 2\) or \(3\).

For a given a creature \(f\) the value \(f_t(x) \in [0, 1]\) represents the density of mass of the creature at the location in space \(x \in \mathbb{R}^n\) at time \(t\).

How do we understand the meaning of the kernel and activation functions? The metaphor is not completely robust, because we are using an extremely simple mathematical model to attempt the simulation of the complexity of organic life, but the intuitive idea may be something like the following. The initial condition of a creature \(f_0\) represents its physical state at birth. The kernel \(K\) dictates how a creature with initial condition responds to its environment. And the activation function \(G\) encodes how the environment supports creatures. So \(K\) synthesizes and simplifies a great deal of information if we think of these creatures as simulations of biological organisms. This includes simplifying the result of the creature’s DNA acting toward expressing its morphology in response to its sustenance and neighbors and other environmental factors. In sum, if we change the formula for \(K\) then we change the species of creature we are studying. If we change the initial condition \(f_0\) we change the individual within that species. Changes in \(G\) may be seen as reflecting
the ability of the environment to support certain shapes of individuals. Recently in \cite{7} complicated dynamics have been discovered for kernels of the form

$$K(x, y) := \sum_{i=1}^{k} b_i \exp \left( -\frac{(x - a_i)^2}{2w_i} \right).$$

There they used neural networks to explore parameter space \{\(a_i, b_i, w_i\}\} for fixed \(k\) and \(c\) to discover initial conditions which simulate adaptivity to obstacles in their environment.

Chan’s formulation can be thought of as a continuous generalization of the original GoL instructions in the following way (cf. \cite[2, p. 258]{2}). The original GoL dynamical system \(f : \mathbb{Z}^2 \times \mathbb{N} \to \{0, 1\}\) can be rewritten as

$$f_{t+1}(x) = \left[ f_t(x) + G(K \ast f_t)(x) \right]_{0}^{1}$$

where the square brackets denote the clip function \([x]_{0}^{1} := \min \{\max \{x, 0\}, 1\}\) and the kernel \(K : \mathbb{Z}^2 \to \{0, 1\}\) is given by

$$K(x) := \begin{cases} 
1 & \text{if } \|x\|_{\infty} = 1 \\
1/2 & \text{if } \|x\|_{\infty} = 0 \\
0 & \text{if } \|x\|_{\infty} > 1
\end{cases}$$

and the activation function \(G : \mathbb{R} \to \mathbb{R}\) is given by

$$G(x) = 2 \cdot 1_{[2.5, 3.5]}(x) - 1.$$  

\section{3. Integro-differential equation}

Formula (\ref{2.1}) is not a traditional differential because of the infinitesimals in the bounds of the clip function. By replacing the infinitesimals \(dt\) with \(\Delta t > 0\) and taking the limit, we interpret Chan’s formula as a forward integro-differential equation as follows:

$$\frac{d}{dt} f_t(x) = \lim_{\Delta t \to 0^+} \frac{f_{t+\Delta t}(x) - f_t(x)}{\Delta t} = \lim_{\Delta t \to 0^+} \frac{[G((K \ast f_t)(x))]_{(1-f_t(x))/\Delta t} - [G((K \ast f_t)(x))]_{-f_t(x)/\Delta t}}{\Delta t}$$

so the Lenia formula is

$$\frac{d}{dt} f_t(x) = V(f_t)(x) := \begin{cases} 
G(K \ast f_t(x)) & \text{if } 0 < f_t(x) < 1 \\
[G(K \ast f_t(x))]_{0}^{\infty} & \text{if } f_t(x) = 0 \\
[G(K \ast f_t(x))]_{-\infty}^{0} & \text{if } f_t(x) = 1.
\end{cases}$$

Line (\ref{3.1}) is an integro-differential equation because of the convolution \(K \ast f_t\).

It is tempting to interpret formula (\ref{3.1}) as a vector field \(V\) on a Banach space then attempt to use traditional ODE theory to show the existence of integral curves to \(V\). There are several problems with this effort. The natural Banach space to use would be \(L^{\infty}(\mathbb{R}^n, \mathbb{R})\). Then \(V\) is defined on the closed subspace \(L^{\infty}(\mathbb{R}^n, [0, 1])\) which might be extended easily enough. However, the derivative in (\ref{3.1}) is taken pointwise and not in the Banach space sense. In fact you can’t make a vector field on \(L^{\infty}\) specifying \(V(f)\) by referring to particular values \(f(x)\) at particular positions \(x \in \mathbb{R}^n\) because the members of \(L^{\infty}\) are not merely functions but equivalence classes of functions up to measure 0. To be careful, we immediately need to pick a less commonly used Banach space.
We take as our context the Banach space \( B(\mathbb{R}^n, \mathbb{R}) \) of bounded Lebesgue measurable functions \( f : \mathbb{R}^n \to \mathbb{R} \) with the supremum norm
\[
\|f\|_\infty := \sup_{x \in \mathbb{R}^n} |f(x)|.
\]
We emphasize that we are not using the essential supremum on equivalence classes of functions. \( B(\mathbb{R}^n, \mathbb{R}) \) is a collection of actual functions\(^1\), so we can refer to \( f(x) \) when defining \( V(f) \). We take the domain of the vector field \( V \) to be the closed subset consisting of measurable functions bounded by \([0, 1]\)
\[
M := B([0, 1]) := \{ f : \mathbb{R}^n \to [0, 1] \mid f \text{ is Lebesgue measurable} \}
\]
with the supremum metric \( d_\infty(f, g) := \|f - g\|_\infty \). Then it is straightforward to check \( M \subset B(\mathbb{R}^n, \mathbb{R}) \) is closed and therefore \((M, d_\infty)\) is a complete metric space.

However, the greater obstacle to this effort is that the vector field \( V \) is not continuous as the following example shows.

**Example 1.** Analyzing \( V \) near the constant 0 function \( f(x) = 0 \) with the input \( g(x) := \epsilon 1_{(0, 1]}(x) \) for some small \( \epsilon > 0 \) shows \( V : M \to M \) is not continuous with respect to \( d_\infty \). For a typical growth function \( G \) described in Chan’s paper, we have \( G(0) = -1 = G(1) \) and \( G(1/2) = 1 \). (These numbers can be modified to fit many other choices of \( G \) and the argument still follows.) Then for kernel \( K \in L^1 \) we have \( K \ast f = 0 \approx K \ast g \) so \( G(K \ast f) = -1 \approx G(K \ast g) \). But even though \( f \) and \( g \) are \( \epsilon \)-close, the formula for \( V \) clips for \( f \) but not for \( g \), and we have
\[
\|V(f) - V(g)\| \approx \|0 - (-1)\| = 1
\]
So the distance between \( V(f) \) and \( V(g) \) is not small, even though \( f \) and \( g \) can be made arbitrarily close. \( V \) is not continuous in the \( L^p \) metric for any \( 1 \leq p \leq \infty \).

Therefore the basic differential equation theory for Lipschitz vector fields on Banach spaces is not sufficient to guarantee the existence of solutions. Another approach is required. The natural setting for this model is not vector fields on linear spaces, but arc fields on metric spaces. In fact, Chan’s formula \(^2\) (9, p. 256)
\[
(3.2) \quad f_{t+\Delta t}(x) = [f_t(x) + \Delta tG(K \ast f_t)(x)]^1_0
\]
is not the traditional Euler approximation for the integral curves to the vector field \( V \) which would instead start as \( f_{t+\Delta t} \approx f_t + \Delta tV(f_t) \). What Chan is actually doing

\(^1\)So why is \( L^\infty \) more commonly used in analysis? One answer is that \( L^\infty \) is the dual of \( L^1 \) and they occupy the ends of the \( L^p \) spectrum. The elements of \( L^1 \) require equivalence classes of functions for a clean definition, and, unlike \( L^\infty \), doesn’t have an analog metric which can distinguish each function and remain complete.

We have the following inequalities:
1. For \( f \in L^1 \) and \( g \in L^\infty \) we have
   \( \|fg\|_1 \leq \|f\|_1 \text{esssup} |g| \) and
2. For \( f \in L^1 \) and \( g \in B(\mathbb{R}, \mathbb{R}) \) we have
   \( \|fg\|_1 \leq \|f\|_1 \text{sup} |g| \)

However, the first inequality is sharper than the second as we can see by changing \( g \) arbitrarily on a set of measure 0.

Perhaps more importantly, the existence of the Hilbert space \( L^2 \) in the middle of the \( L^p \) spectrum has a basis which makes it crucial in functional analysis and its applications, such as signal analysis.
is using an arc field $X$ instead of the vector field $V$ for the Euler curve algorithm to simulate life in Lenia, as explained in the next section.

4. LENIA ARC FIELD

Lenia lives on the metric space $M := B(\mathbb{R}^n, [0, 1])$ with the supremum metric $d_{\infty}(f, g) := \|f - g\|_{\infty}$. Chan’s instruction (3.2) for the motion of these creatures $f$ is the Lenia arc field $X$ which is a map $X : M \times [0, 1] \to M$ given by

$$X_t(f) = [f + tG(K \ast f)]_0$$

Here $t$ represents time, and for each $t \in [0, 1]$ the arc field displays the tendency to change any creature $f \in M$ toward a new creature $X_t(f) \in M$. Notice $X_0(f) = f$ which means the arc field starting at time $t = 0$ with $f$ gives a “direction” for change, using a curve $X_t(f)$ in the metric space $M$.

The arc field $X$ is used to generate a continuous flow $F : M \times [0, \infty) \to M$ using the familiar Euler curve technique

$$F_t(f) = \lim_{n \to \infty} X_{t/n}^{(n)}(f)$$

where the superscript parentheses denote composition $n$ times:

$$X_{t}^{(n)} = X_t \circ X_t \circ \cdots \circ X_t = \bigcirc_{k=1}^{n} X_t.$$ 

The main result of this paper is to prove that this Euler curve algorithm converges to a flow. We also prove the resulting flow is forward tangent to the directions given by the arc field $X$, that the flow is unique, and the flow exists for all forward time. These facts are proven in Section 2 after reviewing the relevant results from arc field theory. Then we show this flow is the desired solution to the original integro-differential equation.

5. ARC FIELD THEORY

In this subsection we review the results of arc field theory [1] used to analyze Lenia throughout the rest of the paper.

**Definition 1.** An **arc field** on a metric space $M$ is a continuous map $X : M \times [0, 1] \to M$ such that for all $f \in M$, $X_0(f) = f$.

The **speed** of an arc field at $f \in M$ is

$$\rho(f) := \sup_{s \neq t} \frac{d(X_s(f), X_t(f))}{|s - t|}$$

i.e., the curve in $M$ starting at $f$ given by $X_{(\cdot)}(f) : [0, 1] \to M$ is Lipschitz with constant $\rho(f)$. An arc field $X$ has **linear speed growth** if there is a point $f \in M$ and positive constants $c_1$ and $c_2$ such that for all $r > 0$

$$\rho(f, r) := \sup \{\rho(g) \mid g \in B(f, r)\}$$

satisfies

$$\rho(f, r) \leq c_1 r + c_2. \quad (5.1)$$

Notice $M$ is an arbitrary metric space and does not need to be a function space, though that is the structure Lenia uses in its model.
**Definition 2.** A (forward) **flow** on a metric space \( M \) is a continuous map \( F : M \times [0, \infty) \to M \) which satisfies for all \( f \in M \)

(i) \( F_0(f) = f \)

(ii) \( F_t(F_s(f)) = F_{s+t}(f) \) for all \( s, t \geq 0 \).

\( F \) is a **flow of an arc field** \( X \) if in addition we have \( F \) is (forward) tangent to \( X \), meaning

\[
\lim_{h \to 0^+} \frac{d(F_{t+h}(f), X_h(F_t(f)))}{h} = 0
\]

for all \( t \geq 0 \) and \( f \in M \).

Due to tangency, each curve from the flow \( F_t(f) : [0, \infty) \to M \) is the analog of an integral curve (think of a solution to an evolutionary PDE), just as an arc field \( X \) is the metric space analog of a vector field.

The linear bound on speed in the definition of an arc field \( X \) is used to prove the resulting flow \( F \) tangent to \( X \) exists for all time. The ODE \( x' = x^2 \) for \( x(t) \in M := \mathbb{R} \) is the standard example used to show the linear speed growth assumption is natural, since this ODE has quadratic speed growth and solutions diverge to infinity in finite time.

**Proposition 1.** The Lenia arc field \( \textbf{(4.1)} \)

\[
X_t(f)(x) = [f(x) + tG(K \ast f)(x)]^1_0
\]

on \( M := B(\mathbb{R}^n, [0, 1]) \) with supremum metric \( d_\infty \) has speed globally bounded by \( \max |G| \).

**Proof.**

\[
\rho(f) = \sup_{s \neq t} \frac{d(X(f, s), X(f, t))}{|s - t|}
\]

\[
= \sup_{s \neq t} \frac{\| [f + sG(K \ast f)]^1_0 - [f + tG(K \ast f)]^1_0 \|}{|s - t|}
\]

\[
\leq \sup_{s \neq t} \frac{\| f + sG(K \ast f) - (f + tG(K \ast f)) \|}{|s - t|}
\]

\[
= \sup_{s \neq t} \| G(K \ast f) \| \leq \max |G| .
\]

\( \square \)

The following conditions are used to guarantee arc fields on arbitrary complete metric spaces generate flows via Euler curves.

**Condition E1:** For each \( f_0 \in M \), there are constants \( r > 0, \delta > 0 \) and \( \Lambda \in \mathbb{R} \) such that for all \( f, g \in B(f_0, r) \) and \( 0 \leq t < \delta \)

\[
d(X_t(f), X_t(g)) \leq (1 + t\Lambda) d(f, g) .
\]

**Condition E2:** For each \( f_0 \in M \), there are constants \( r > 0, \delta > 0 \) and \( \Omega \in \mathbb{R} \) such that for all \( f \in B(f_0, r) \) and \( 0 \leq t \leq s < \delta \)

\[
d(X_{s+t}(f), X_t(X_s(f))) = st\Omega .
\]

Infinitesimally E1 limits the spread of \( X \) in a linear fashion. E2 restrains \( X \) to be almost flow-like, since \( X \) would be a flow if it satisfied E2 with constant \( \Omega = 0 \).
Theorem 1. Let $X$ be an arc field on a complete metric space $M$ which satisfies $E1$ and $E2$ and has linear speed growth. Then there exists a unique forward flow $F$ tangent to $X$.

The Euler curve algorithm converges to this flow, meaning

$$F_t(f) = \lim_{n \to \infty} X_{t/n}^{(n)}(f)$$

where

$$X_{t/n}^{(n)} = X_t \circ X_{t/2} \circ \cdots \circ X_{t/n} = \bigcirc_{k=1}^n X_t.$$  

The proof of this fundamental theorem is given in [1].

When $X$ merely has locally bounded speed instead of linearly bounded speed, we can only guarantee a local flow, meaning time $t$ may be limited in different ways before solutions blow up, depending on the initial condition $f$.

Theorem 1 generalizes the classical Fundamental Theorem of ODEs (also known as the Picard-Lindelöf Theorem or the Cauchy-Lipschitz Theorem) which states that Lipschitz vector fields on Banach spaces have unique integral curves. One of the results of this paper is to show that Theorem 1 also guarantees solutions to some interesting non-continuous vector fields on function spaces.

6. Clip function properties

In this section we collect some facts about the clip function that will be used in the existence proof, Theorem 3, below. The clip function

$$[x]_a^b := \begin{cases} 
  b & \text{if } x \geq b \\
  x & \text{if } a < x < b \\
  a & \text{if } x \leq a 
\end{cases} = \min \{ \max \{a, x\}, b\}$$

is well-defined whenever $a \leq b$. We will also use the notation

$$[x]_a := \max \{x, a\}$$

$$[x]^b := \min \{x, b\}$$

for the one-sided clips. Notice $[x]_0 = x \cdot 1_{[0, \infty)}(x)$ is a low pass filter where $1_S$ is the indicator function on the set $S$

$$1_s(x) := \begin{cases} 
  1 & x \in S \\
  0 & \text{otherwise.}
\end{cases}$$

More generally $[x]_a = (x - a) \cdot 1_{[0, \infty)}(x) + a$ and $[x]^b = (x - b) \cdot 1_{(-\infty, 0]}(x) + b$. So composing them shows $[x]_a^b = [x]_a^b$ is a band pass filter.

Let $a \leq b$ and $c \leq d$ and $x, y \in \mathbb{R}$. It is straightforward to verify the following identities and inequalities

- $[x]_a^b = r [x]_{a/r}^{b/r}$ for $r > 0$ and $[r x]_a^b = r [x]_{a/r}^{a/r}$ for $r < 0$

- $[x + y]_a^b = [x]_{a-y}^{b-y} + y$

- $[x + y]_a^b = [x]_{a-y}^{b-y} + y$

- $[x]_a^b = [x]_a^b - [x]^a + a$ which gives

$$[x]_a^b - [x]^d = [x]_a^b - [x]^d - (d - a)$$ assuming $b \geq d$ and $a \geq c$
Therefore we mechanically choose $X$ discontinuous the traditional theory (Picard-Lindelöf Theorem) does not apply.

**Example 2.** Let the following simpler example which is not used in the remainder of the paper.

Fields can handle discontinuities in directions, we recommend first working through warmup for using the clip function on the full arc field, and to get a feel for how arc the arc field by Theorem 1. However the calculations get a bit complicated. As a warmup for using the clip function on the full arc field, and to get a feel for how arc fields can handle discontinuities in directions, we recommend first working through the following simpler example which is not used in the remainder of the paper.

7. Existence and Uniqueness of Solutions to the Lenia Arc Field

We will show the Lenia arc field (4.1) satisfies the regularity conditions, E1 and E2, which guarantee the existence and uniqueness of a forward flow tangent to the arc field by Theorem 1. However the calculations get a bit complicated. As a warmup for using the clip function on the full arc field, and to get a feel for how arc fields can handle discontinuities in directions, we recommend first working through the following simpler example which is not used in the remainder of the paper.

**Example 2.** Let $V : \mathbb{R} \to \mathbb{R}$ be the discontinuous vector field given by

$$V(x) := \begin{cases} 
-1 & \text{if } x > 0 \\
0 & \text{if } x \leq 0.
\end{cases}$$

$V(x)$ gives the direction for $x \in M := \mathbb{R}$. Even though the directions from $V$ are clear, and the resulting dynamic has a unique formula for all time, because $V$ is discontinuous the traditional theory (Picard-Lindelöf Theorem) does not apply.

To make an analog arc field to $V$ we replace the vectors with curves in the same direction. Therefore we mechanically choose $X : M \times [0,1] \to M$ with $M := \mathbb{R}$ defined by

$$X_t(x) := [x - t]_0$$

where the square brackets again denote the clip function. This arc field has the dynamic that solutions with initial conditions $x > 0$ move down until hitting 0, then stop. This immediate stop displays a discontinuous direction change, but the arc field is still continuous in $t$ for each $x$.

Let’s check E1 using $|[x]_0 - [y]_0| \leq |x - y|$.

$$d(X_t(x), X_t(y)) \leq |[x - t]_0 - [y - t]_0| \leq |(x - t) - (y - t)| = d(x, y).$$

so $\Lambda = 0$ is sufficient.
Then $E_2$ is calculated using the property of the clip $[x+y]_0 = [x]_y + y$

$$d((X_{s+t}(x), X_t(X_s(x))) = [x - (s + t)]_0 - [x - s]_0 - t_0$$

$$= [- (s + t)]_x + x - ([s]_x + x - t)_0$$

$$= [- (s + t)]_x + x - ([s]_x + x - t)$$

$$= [- (s + t)]_x - ([s]_x + t) = 0$$

where the third to fourth line calculation used $[x]_a := \max \{a, x\}$ so that $[s]_x = 0$ for $t \geq 0$.

Since the speed of $X$ is globally bounded by 1 on the complete metric space $M$, Theorem 1 guarantees the Euler curves generate a unique forward flow tangent to $X$. Notice $F = X$ in this trivial scenario.

N.b.: Precisely how uniqueness of solutions backward in time fails is quite complicated for the Lenia dynamic. Considering the reverse dynamic of the previous example is illuminating for understanding the issue. In particular notice that two different initial conditions can coincide after a finite amount of time. For instance $F_t(x) = 0$ for all $x \in M$ for any $t \geq |x|$. The forward flow is still unique, but the backward flow is not well-defined from $x = 0$ since there are multiple inverses to choose from. This doesn’t happen with Lipschitz continuous vector fields, where forward and backward flows exist uniquely, so integral curves cannot intersect at any finite time $t$. Similarly in the full Lenia model, depending on the choice of $K$ and $G$, it is possible for many different initial conditions $f$ to go extinct, i.e., $F_t(f) = 0$ after finite time $t$.

We next prove the Lenia arc field $X$ has a flow using similar calculations from the above toy example, Example 2. Now, however, there are further complications, because the number of variables to consider is infinite instead of 1. Since we are taking the supremum over all possible $x \in \mathbb{R}^n$ we have to account for all possible cases. The clip function

$$[x]_a^b = \begin{cases} b & \text{if } x \geq b \\ x & \text{if } a < x < b \\ a & \text{if } x \leq a \end{cases}$$

naturally breaks into 3 cases each time it’s used. When we check $E_2$ we need to consider $X_t(X_s)$ which means we need to deal with the clip of the clip, so the cases multiply, and the proof is inevitably complicated.

The major new result of the paper is the following theorem.

**Theorem 2.** Consider the complete metric space $M := \mathcal{B}(\mathbb{R}^n, [0, 1])$ with the supremum metric $d_{\infty}$. Let $K \in L^1$ meaning $K : \mathbb{R}^n \to \mathbb{R}$ satisfies $\int_{\mathbb{R}^n} |K(x)| \, dx < \infty$. Let $G : \mathbb{R} \to \mathbb{R}$ be bounded and Lipschitz continuous. Then the Lenia arc field $X : M \times [0, 1] \to M$ given by

$$X_t(f)(x) := \left[ f(x) + tG(K * f)(x) \right]_0^1$$

satisfies Conditions $E_1$ and $E_2$.

Thus $X$ generates a unique forward flow $F : M \times [0, \infty) \to M$ forward tangent to $X$. 

$F$ is approximated by Euler curves

$$F_t(f) = \lim_{n \to \infty} X_t^{(n)}(f).$$

**Proof.** This is a consequence of the more general Theorem B proven next. Specifically, the space of measurable functions $B(\mathbb{R}^n, [0,1])$ is a closed subset of $\mathcal{B}(\mathbb{R}^n, \mathbb{R})$ under the sup norm. Then remember the general fact from Fourier Theory about convolutions that if $K \in L^1(\mathbb{R}^n)$ and $f \in L^p(\mathbb{R}^n)$ for any $1 \leq p \leq \infty$ then $K * f \in L^p(\mathbb{R}^n)$ and $\|K * f\|_p = \|K\|_1 \|f\|_p$ by Fubini’s Theorem. Since $G$ is bounded we see $X$ is continuous in $t$ for each $f \in M$. Further $X_t(f)$ is continuous in $f$ since $G$ is, and $X$ is a well-defined arc field on $M$. Finally, the Lenia arc field is $X_t(f) := [f + tV(f)]_0^b$ where $V(f) := G(K * f)$ and this $V$ is Lipschitz with constant $C_G \|K\|_1$ where $C_G$ is the Lipschitz constant of $G$ since

$$\|V(f) - V(g)\|_\infty = \|G(K * f) - G(K * g)\|_\infty$$

$$\leq C_G \|K \|_1 \|f - g\|_\infty.$$

$\square$

**Theorem 3.** Let $S$ be a measure space and let $a, b : S \to [-\infty, \infty]$ be functions on $S$ separated by $m := \inf_{x \in S} (b(x) - a(x)) > 0$. Let $\mathcal{B} := \mathcal{B}(S, \mathbb{R})$ be the Banach space of bounded measurable functions $f : S \to \mathbb{R}$ with norm $\|f\|_\infty := \sup_{x \in S} |f(x)|$. Define the metric space subspace of $\mathcal{B}$

$$M := \{ f \in \mathcal{B} \mid a(x) \leq f(x) \leq b(x) \ \forall x \in S \}$$

with metric given by $d_\infty(f, g) := \|f - g\|_\infty$. Finally, let $V : M \to \mathcal{B}(S, \mathbb{R})$ be a locally Lipschitz map.

Then the arc field $X : M \times [0,1] \to M$ given by

$$X_t(f)(x) := [f + tV(f)]_a^b(x) \quad \text{and} \quad X_t(f)(x) := [f(x) + tV(f)(x)]_a^b(x)$$

satisfies Conditions E1 and E2 and generates a unique local forward flow $F$ tangent to $X$.

$F$ may be approximated by Euler curves

$$F_t(f) = \lim_{n \to \infty} X_t^{(n)}(f).$$

**Proof.** Notice $M$ is not necessarily a Banach space since scalar multiples of elements in $M$ may leave $M$, but it is a complete metric space since it is a closed subspace $M \subset \mathcal{B}(S, \mathbb{R})$ of the Banach space. So $X$ is a well-defined arc field on $M$ and the speed $\rho$ of $X$ is bounded by the norm of the vector field

$$\rho(f) := \sup_{s \neq t} \frac{d_\infty(X_s(f), X_t(f))}{|s - t|}$$

$$= \sup_{s \neq t} \frac{\| [f + sV(f)]_a^b - [f + tV(f)]_a^b \|_\infty}{|s - t|}$$

$$\leq \sup_{s \neq t} \frac{\| f + sV(f) - (f + tV(f)) \|_\infty}{|s - t|} = \|V(f)\|_\infty$$
When \( V \) is globally Lipschitz with constant \( C_V \), then \( X \) has linear speed growth since for any \( f_0 \in M \) and \( r > 0 \)

\[
\rho(f) \leq \|V(f)\|_\infty \leq \|V(f) - V(f_0)\|_\infty + \|V(f_0)\|_\infty
\]

gives

\[
\rho(f_0, r) := \sup \{ \rho(f) | f \in B(f_0, r) \} \leq C_V r + \|V(f_0)\|_\infty
\]

Let \( f_0 \in M \) and let \( \epsilon > 0 \) be such that \( V \) has local Lipschitz constant \( C_V \) meaning \( \|V(f) - V(g)\|_\infty \leq C_V \|f - g\|_\infty \) for all \( f, g \in B(f_0, \epsilon) \).

E1 follows from

\[
d_\infty(X_t(f), X_t(g)) = \|[f + tV(f)]_a^b - [g + tV(g)]_a^b\|_\infty
\]

\[
\leq \|f + tV(f) - (g + tV(g))\|_\infty
\]

\[
\leq \|f - g\|_\infty + t \|V(f) - V(g)\|_\infty
\]

\[
\leq \|f - g\|_\infty + tC_V \|f - g\|_\infty
\]

\[
= d_\infty(f, g)(1 + tC_V)
\]

So the \( \Lambda \) for E1 is the local Lipschitz constant \( C_V \) of \( V \).

The last thing to check is E2, which is significantly more complicated. Now let \( s, t \) be such that \( 0 \leq t \leq s \) satisfy \( s \leq \min \left\{ \frac{\epsilon}{r \rho}, \frac{m}{\rho} \right\} \) where \( m := \inf_{x \in S} \{b(x) - a(x)\} > 0 \) so that \( f + sV(f) \in B(f_0, \epsilon) \) for any \( f \in B(f_0, \epsilon/3) \). It will be more obvious later that this is sufficient restriction to guarantee \( X_t X_s (f) \) stays in \( B(f_0, \epsilon) \) and the subsequent calculations can count on the spread of \( a < b \) to remain valid.

Using the formula \([x + y]_a^b = [x]_a^b + y \) we calculate

\[
d_\infty(X_{s+t}(f), X_t(X_s(f))) = \|X_{s+t}(f) - X_t(X_s(f))\|_\infty
\]

\[
= \|[f + (s + t) V(f)]_a^b - [g + tV(f)]_a^b\|_\infty
\]

\[
\leq \|[f + sV(f)]_a^b + t V \left( \left[ f + sV(f) \right]_a^b \right)_a^b\|_\infty
\]

\[
\leq \|[f + sV(f)]_a^b + t V \left( f + sV(f) \right)\|_\infty
\]

\[
\leq \|[f + sV(f)]_a^b + t V \left( f + sV(f) \right)\|_\infty
\]

\[
(7.1)
\]

where \( \rho := \rho(f_0, \epsilon) \) which is the finite speed of \( X \). This last line follows because the bound on \( s \) keeps \( X_s (f) \) inside \( B(f_0, 2\epsilon/3) \) for any \( f \in B(f_0, \epsilon/3) \) by the definition of \( \rho \). Thus the same local Lipschitz constant \( C_V \) and the bound on the speed \( \rho \) of
\[ \left\| V(f) - V\left([f + sV(f)]^b_a\right) \right\|_\infty \leq C_V \left\| f - [f + sV(f)]^b_a \right\|_\infty \]

We will use this calculation twice more in this proof without comment noticing \( X_t X_s (f) \) also stays inside the ball \( B(f_0, \epsilon) \) since \( t < s \).

So the second term \((st) C_V \rho = O(st) \) of line (7.1) works for the E2 condition.

Now concentrating on similarly bounding the first term of (7.1)

\[ \left\| [f + sV(f)]^b_{a-tV(f)} - [f + sV(f)]^b_{a-tV([f + sV(f)]^b_a)} \right\|_\infty \]

we consider the cases when \(-tV\left([f + sV(f)]^b_a\right)(x)\) is greater and smaller than 0.

Remembering \( a \leq f \leq b \) and using the general fact of the clip that \([x]_{\min(a,c)}^{d} = [x]_{\max(a,c)}^{\min(b,d)} \) when \( \max\{a, c\} \leq \min\{b, d\} \) we have four cases to check:

\[ \left\{ 
\begin{aligned}
[f + sV(f)]^b_{a-tV([f + sV(f)]^b_a)}(x) \\
[f + sV(f)]^b_{a-tV([f + sV(f)]^b_a)}(x) \\
[f + sV(f)]^b_{a-tV([f + sV(f)]^b_a)}(x) \\
[f + sV(f)]^b_{a-tV([f + sV(f)]^b_a)}(x)
\end{aligned}
\]

Notice we are using the previously mentioned bound \( t \leq \frac{\|f\|_\infty}{\rho} \) which ensures that \( \|X_t X_s (f) - X_s (f)\|_\infty \leq \rho \leq b - a \) so \( a < b - tV\left([f + sV(f)]^b_a\right) \) in case III and \( a - tV\left([f + sV(f)]^b_a\right) < b \) in case IV so the clips are well-defined.

The first two cases will be handled in a similar way and are relatively easy, because the dynamics of \( X_s \) and \( X_t \) go in the same direction at those values of \( x \). The last two cases give a new twist.
Case I is $V\left(\left[f + sV(f)\right]^b_a\right) (x) \geq 0$ and $V(f)(x) \geq 0$ which gives

$$\left|\frac{[f + sV(f)]^b_a - [f + sV(f)]^b_a}{a - tV(f)}\right| (x) = O(st)$$

where the last line used $|x^v - [x]^v| \leq |v - w|$ and then the same calculation as line (7.2).

For Case III, we assume $V\left(\left[f + sV(f)\right]^b_a\right) (x) \geq 0$ and $V(f)(x) \leq 0$. We use the general formulas $|x + y|^b_a = |x|^b_a + y \text{ again and } |x|^b_a - |x|^c_a = |x|^b_a - |x|^c_a - (d - a)$ assuming $b \geq d \geq c$ and $b \geq a \geq c$ which will be true when we use it in the third line below.

$$\left|\frac{[f + sV(f)]^b_a - [f + sV(f)]^b_a}{a - tV(f)}\right| (x) = O(st)$$

where we used $b - tV\left(\left[f + sV(f)\right]^b_a\right) \geq a$ in the second line by the choice of $t \leq s \leq \frac{a}{n}$ at the beginning of the proof, and $[f + sV(f) - (b - a)]^a_{a - tV(f)} = 0$ in the fifth to sixth line because $f \geq b + sV(f) \leq a$.

The last line uses the trick where $|x|^b_a - |y|^b_a \leq |c - a|$ for $a \leq b \leq c$. 
Case IV is an analogous calculation to Case III and gives the same result. Since all four cases satisfy the $O(st)$ bound, E2 is satisfied. Therefore the Euler curves converge to a local flow tangent to $X$.

Under the global Lipschitz condition, the bound on the speed of $X$ is linear so solutions exist for all time. □

Theorem 2 is easily generalized to functions $f : S \to \mathbb{R}^l$ for arbitrary indexing sets $I$ by thinking of these more complicated functions as simply $f \in \mathcal{B}(S, \mathbb{R}^l) \simeq \mathcal{B}(S \times I, \mathbb{R})$ in the following corollary, which is used in the next section to give extensions of the Lenia model on $\mathcal{B}(S, \mathbb{R}^3)$.

**Corollary 1.** Let $S$ and $I$ be measure spaces, and define the Banach space $\mathcal{B} := \mathcal{B}(S \times I, \mathbb{R})$ of bounded measurable functions $f : S \times I \to \mathbb{R}$ with infinite norm $\|f\|_\infty := \sup_{x \in S, i \in I} |f_i(x)|$ where we write $f_i(x) = f(x,i)$. Define the clip function $[f]_a^b : \mathcal{B} \to \mathcal{B}$ by clipping coefficients independently, i.e., $[f]_a^b$ is defined for each $x$ and $i$ as

$$\left(\left([f]_a^b\right)(x)\right)_i := [f_i(x)]_{a_i(x)}^b(x)$$

where $a_i$ and $b_i$ are functions from $S$ to the extended reals $[-\infty, \infty]$ such that $-\infty \leq a_i(x) \leq b_i(x) \leq \infty$ for all $i \in I$ with $\inf_{i \in I, x \in S} (b_i(x) - a_i(x)) > 0$.

Define the metric space $M$ to be the functions bounded by the clip

$$M := \{f \in \mathcal{B} \mid [f]_a^b = f\}$$

Notice $M$ is simply the image of the clip, and so has metric derived from the norm on $\mathcal{B}$ given by $d_\infty(f, g) := \|f - g\|_\infty$ making $M$ a complete metric space.

Let $V : M \to M$ be a locally Lipschitz map. Then the arc field $X : M \times [0, 1] \to M$ given by

$$X_t(f) := [f + tV(f)]_a^b$$

satisfies Conditions E1 and E2.

Thus $X$ generates a unique local forward flow $F$ tangent to $X$. When $V$ is globally Lipschitz the flow $F$ exists for all time $F : M \times [0, \infty) \to M$.

8. **Solution to the integro-differential equation**

In the previous section, we interpreted Chan’s formulation of continuous Lenia as specifying an arc field on the space $M := \mathcal{B}(\mathbb{R}^n, [0, 1])$, namely the arc field $X : M \times [0, 1] \to M$ is given by

$$X_t(f) := [f + tG \circ (K * f)]_0^1.$$

Assuming $G : \mathbb{R} \to \mathbb{R}$ is a Lipschitz-continuous bounded function and $K \in L^1(\mathbb{R}^n)$, Theorem 2 guarantees a unique forward flow $F : M \times [0, \infty) \to M$ that is forward-tangent to this arc field. In this section we show $F$ is the sought-after solution of continuous Lenia by proving it solves the original integro-differential equation. Then we investigate some elementary properties of that solution.

Focusing on an initial condition $f_0 \in M$ we define

$$f_t := F_t(f_0).$$
For \( x \in \mathbb{R}^n \) and \( t \geq 0 \),
\[
\frac{d}{dt^+} f_t(x) := \lim_{h \to 0^+} \frac{f_{t+h}(x) - f_t(x)}{h}
\]
denotes the point-wise forward derivative of \( f_t(x) \) with respect to \( t \), at the spot \( x \). Note that this is not a limit of functions in the sup norm, but rather a pointwise limit of real numbers. Then we obtain the following forward integro-differential equation:

**Proposition 2.**
\[
\frac{d}{dt^+} f_t(x) = \begin{cases} 
G((K * f_t)(x)) & \text{if } 0 < f_t(x) < 1 \\
|G((K * f_t)(x))|^{0} & \text{if } f_t(x) = 1 \\
|G((K * f_t)(x))|_{0} & \text{if } f_t(x) = 0
\end{cases}
\]
for \( x \in \mathbb{R}^n \) and \( t \geq 0 \).

**Proof.** The tangency condition \( (5.2) \) gives pointwise
\[
0 = \lim_{h \to 0^+} \frac{f_{t+h}(x) - [f_t(x) + hG((K * f_t)(x))]^{1}}{h}
\]
and
\[
\lim_{h \to 0^+} |G((K * f_t)(x))|^{1-f_t(x)/h} = \begin{cases} 
G((K * f_t)(x)) & \text{if } 0 < f_t(x) < 1 \\
|G((K * f_t)(x))|^{0} & \text{if } f_t(x) = 1 \\
|G((K * f_t)(x))|_{0} & \text{if } f_t(x) = 0
\end{cases}
\]
holds since the argument of the clip does not depend on \( h \) while the bounds of the clip approach 0, \(+\infty\), or \(-\infty\). \( \square \)

Note that we cannot expect an equation involving the two-sided derivative of \( f_t(x) \), since it is easy to create examples where \( f_t(x) \) rises from 0 to 1 with increasing speed and then abruptly comes to a stop at \( f_t(x) = 1 \).

### 8.1. Basic Properties

**Example 3.** We will show that for certain choices of \( G \) and \( K \), an initial condition \( f_0 \) with bounded support can have a solution whose support explodes instantly, i.e., \( f_t(x) > 0 \) for all \( x \in \mathbb{R}^n \) and all \( t > 0 \), i.e., \( \text{supp}(f_t) = \mathbb{R}^n \).

For this, pick a kernel \( K \in L^1(\mathbb{R}^n) \) that is continuous, non-negative, and satisfies \( K(0) > 0 \); take a growth function \( G \) with \( G(u) > 0 \) for all \( u > 0 \) and an initial condition \( f_0 \) for which \( \{ y | f_0(y) > 0 \} \) has non-empty interior.

Since \( G((K * f_t)(x)) \) is non-negative and \( f_t(x) \) is continuous in \( t \), we know by our proposition that \( f_t(x) \) is monotone increasing as a function of \( t \). Pick \( t > 0 \) and define \( S_t = \text{int}\{ y | f_t(y) > 0 \} \), the interior of the set of values where \( f_t \) is positive. Let \( x \) be an element of the boundary of \( S_t \). By our choice of \( K \) there exists \( \varepsilon > 0 \) such that \( K(z) > 0 \) whenever \( |z| < \varepsilon \). Then there exists \( y \in S_t \) with \( |x - y| < \varepsilon \), and there is a neighborhood \( U \) of \( y \) where \( f_t > 0 \). By our assumptions on \( K \), we have \( (K * f_t)(x) > 0 \) (integral of a positive function over an open set is positive).

This gives \( G((K * f_t)(x)) > 0 \). The expression \( G((K * f_t)(x)) \) is continuous in \( t \) and \( x \), because \( G \) is continuous, \( f_t \) depends continuously on \( t \), and convolution with
K is a continuous operator \( L^\infty(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n) \) whose image consists entirely of continuous functions. We can thus find a neighborhood \( U \) of \( x \) and an open interval \( I \) containing \( t \) such that \( G((K * f_t)(y)) > 0 \) for all \( y \in U \) and \( s \in I \). Again by our proposition, and by monotonicity, this implies \( f_t(y) > 0 \) for all \( y \in U \). This gives \( x \in S_t \). So \( S_t \) is both open and closed: \( S_t = \mathbb{R}^n \) or \( S_t = \emptyset \). The latter is excluded by our choice of \( f_0 \) and by monotonicity.

Admittedly, the previous example is contrived, since normally we would consider growth functions \( G(0) < 0 \) as Chan assumes in [2] and [3]. It does however exhibit a new effect that does not occur in discrete Lenia, where a growth function with \( G(0) = 0 \) clearly cannot cause such an instantaneous support explosion.

The following proposition describes a much more typical behavior: an upper bound to the speed with which information can travel away from the initial condition; i.e., the support of \( f_t \) grows at a bounded rate.

We use \( \|\cdot\|_2 \) to denote the Euclidean distance, or \( L^2 \) norm, and \( \lfloor \cdot \rfloor \) denotes the integer floor function.

**Proposition 3.** Assume there exists \( a > 0 \) such that \( G(u) \leq 0 \) for \( |u| \leq a \), and let \( g \) be a finite and positive upper bound for \( G \). Further assume that the support of the kernel \( K \) is contained in the open Euclidean ball \( B(0, R) \subseteq \mathbb{R}^n \) with radius \( R > 0 \). For \( x \in \mathbb{R}^n \) define

\[
d(x) := \inf \{ \|x - y\|_2 \mid y \in \text{supp}(f_0) \}
\]

the distance from \( x \) to the support of \( f_0 \). We then have

\[
f_t(x) = 0
\]

for every \( t \) with

\[
0 < t \leq \frac{a |d(x)/R|}{g \|K\|_1}.
\]

**Proof.** The claim is equivalent to the statement: if \( k \in \mathbb{Z}_{\geq 1} \) and \( d(x) \geq kR \) and \( t \leq ak/(g \|K\|_1) \), then \( f_t(x) = 0 \). This can be shown by induction on \( k \), and the case \( k = 1 \) already contains the idea. If \( d(x) \geq R \), then \( f_0 \) is identically zero on \( B(x, R) \), which means \( f_t \leq gt \) on that ball (since \( f_t \) is continuous in \( t \) and Proposition 2 bounds the forward derivative by \( g \)). This in turn implies \( |(K * f_t)(x)| \leq gt \|K\|_1 \). So if \( gt \|K\|_1 \leq a \), we’ll have \( G((K * f_t)(x)) \leq 0 \), meaning that \( f_t(x) \) cannot grow and must stay at 0. \( \square \)

**Proposition 4.** Assume \( G \) is Lipschitz and bounded and that \( K \in L^1 \).

If \( f_0 \) is continuous, then \( f_t \) is continuous for all \( t > 0 \).

**Proof.** \( C(\mathbb{R}^n, [0, 1]) \) is a closed subspace of \( B(\mathbb{R}^n, [0, 1]) \) with supremum metric \( d_\infty \) and our arc field restricts to this space. By completeness, the solutions \( f_t \) we construct are therefore all continuous functions on \( \mathbb{R}^n \) provided the initial condition \( f_0 \) is. \( \square \)

9. **Comparison with Asymptotic Lenia**

Asymptotic Lenia [8] is an alternative model that avoids the use of the clip function, but has also been shown to bear complicated dynamics in simulations. Instead of using formula (2.1), Asymptotic Lenia is defined by

\[
f_{t+dt} = f_t + dt (T (K * f_t) - f_t)
\]
which is interpreted as
\[
\frac{d}{dt} f = V(f)
\]
where
\[
V(f) := T(K * f) - f
\]
and \(T\) is now restricted to \(T : \mathbb{R} \to [0, 1]\) and is described as unimodal, meaning it has a single maximum in \((0, 1)\). Now \(T = \frac{G + 1}{2}\) and \(T(0) = 0 = T(1)\) is a typical assumption which guarantees solutions starting with \(0 \leq f_0(x) \leq 1\) to remain bounded. To see this, notice if \(f(x) = 0\) then \(T \geq 0\) guarantees \(V(f)(x) \geq 0\) so the dynamic cannot move a positive initial condition function negative. Similarly if \(f(x) = 1\) then since \(T \leq 1\) we have \(V(f)(x) \leq 0\) so the dynamic cannot make an initial function bounded by 1 grow past 1.

Assuming \(G\) is Lipschitz with constant \(C_G\) we then have
\[
\|V(f) - V(g)\|_\infty = \|G(K * f) - G(K * g) - (f - g)\|_\infty \\
\leq C_G \|K * (f - g)\|_\infty + \|f - g\|_\infty \\
\leq (C_G \|K\|_1 + 1) \|f - g\|_\infty
\]
so \(V\) is Lipschitz on the Banach space \(B(\mathbb{R}^n, \mathbb{R})\). Now the traditional Picard-Lindelöf Theorem guarantees solutions to Asymptotic Lenia which restrict to the closed subset \(B(\mathbb{R}^n, [0, 1])\). This is a much simpler existence proof. Moreover these solutions exist forward and backward in time and are unique in both directions. Simulations illustrate the intuition that such solutions would be smoother in some sense, and more amenable to traditional mathematical analysis.

On the other hand, the original Lenia model yields unique solutions only forward in time. This means there are more complicated and interesting dynamics possible in original Lenia. One example is that creatures in Asymptotic Lenia cannot “die”. If an initial condition \(f_0\) is not the constant zero function, then \(f_t\) can never become the constant 0 function in any finite time \(t < \infty\) in Asymptotic Lenia. The reason for this is that uniqueness of solutions prevents any solution from crossing the constant 0 integral curve, which is a solution assuming \(T(0) = 0\). But in original Lenia the creatures can die in finite time. Solutions do intersect continually in the forward direction. That is very rare using classical continuous mathematical models.

What we find even more interesting about the original Lenia model is that it models the “arrow of time” in a novel way. Most physics equations admit time symmetry, meaning any system moving forward in time admits the possibility of moving backward in time as another valid solution. But the 2nd Law of Thermodynamics (as well as all human experience) says that is not a realistic model of the world. The clip function makes original Lenia different than typical continuous dynamics because it throws away information, it displays entropy, unlike Asymptotic Lenia. This quality of losing information is stronger than classic models of dissipation, especially the heat equation \(f_t = f_{xx}\) which is partially reversible, though Lenia never is. So Lenia reflects that aspect of reality better, yet still yields unique, continuous, deterministic solutions forward in time. Ilya Prigogine’s concluded that the quality of irreversibility that so many physical systems exhibit leads us to doubt the use of deterministic models in physics, since they typically do not display such behavior:

“The more we know about our universe, the more difficult it becomes to believe in
On the contrary, we can model very strong irreversibility in a continuous and deterministic dynamical system as the Lenia model proves.

10. Extensions

Studying dynamics on metric spaces gives a natural environment for extending or generalizing models. For instance, we can feed Lenia’s creatures, making regions for instance it doesn’t grow any faster than its current strength, so for instance it doesn’t grow where \( f(x) = 0 \). But because there is no lower bound, it can starve with any specified speed at regions where \( \phi(x) \) is negative. Notice \( X^\phi_t (f) := \left[ f + t \vert \phi \vert \right]_0^1 \) has \( V^f_1 := \vert \phi \vert \) which is Lipschitz, since

\[
\|V^f_1(f) - V^g_1(g)\|_\infty := \|\vert \phi \vert - \vert g \vert\|_\infty \leq \|f - g\|_\infty.
\]

Therefore it has a unique local flow for all time by Theorem 3.

Next we can combine the two dynamics

\[
(X^2)_t (f) := \left[ f + t \left( \vert \phi \vert^f + G(K \ast f) \right) \right]_0^1.
\]

and we still have that \( V^2 \) is Lipschitz and so the flow is guaranteed to exist uniquely for all forward time.

Now it’s easy to extend the model to a richer environment where we can keep track of the food, so that it is depleted when it is eaten. For simplicity we limit the food in range \([a, b]\). Now \( M := B(\mathbb{R}^n \times \mathbb{R}^n, [0, 1] \times [a, b]) \simeq B(\mathbb{R}^n, [0, 1]) \times B(\mathbb{R}^n, [a, b]) \) and, for instance,

\[
X^\phi_t (f) := \left( \left( f + t \left( \vert \phi \vert^f + G(K \ast f) \right) \right)_{[a, b]} \right)_0^1.
\]

Now \( \phi_t \) shrinks if \( f \) eats it.

Continuing to complicate the model, we generalize the Lotka-Volterra ODE by introducing predators and prey. Again we use \( M := B(\mathbb{R}^n, [0, 1]) \times B(\mathbb{R}^n, [0, 1]) \) but now allow the food itself to move and grow like the predators. Define the predator colony \( f \) as eating the prey colony \( g \) with

\[
X^f_t (f) := \left( \left( f + t \left( \vert g \vert^f + G^1(K^1 \ast f) \right) \right)_{[a, b]} \right)_0^1
\]

where we allow the predators and prey to have different growth functions \( G^i \) and \( K^i \) assuming they are fundamentally different creatures with different rules for how they evolve in time.

We can further complicate the model by allowing the predators \( f \) to eat the prey \( g \) which eat the food \( \phi \) with the arc field on \( M := B(\mathbb{R}^n, [0, 1]) \times B(\mathbb{R}^n, [0, 1]) \times \).
$B(\mathbb{R}^n, [a, b])$ defined by

\[
X_t^5 \left( \begin{array}{c} f \\ g \\ \phi \end{array} \right) := \left( \begin{array}{c} \left[ f + t \left( [g]_t^f + G^1 (K^1 * f) \right) \right]_0^1 \\ \left[ g + t \left( -[g]_t^f + [\phi]_t^g + G^2 (K^2 * g) \right) \right]_0^1 \\ \left[ \phi + t (-[\phi]_t^g)_b^a \right] \end{array} \right)
\]

\[
= C \left[ \begin{array}{c} f \\ g \\ \phi \end{array} \right] + tV^5 \left( \begin{array}{c} f \\ g \\ \phi \end{array} \right)
\]

where the clip function $C$ is

\[
C \left( \begin{array}{c} f \\ g \\ \phi \end{array} \right) = \left( \begin{array}{c} [f]_{0}^{1} \\ [g]_{0}^{1} \\ [\phi]_{a}^{b} \end{array} \right)
\]

and

\[
V^5 \left( \begin{array}{c} f \\ g \\ \phi \end{array} \right) := \left( \begin{array}{c} [g]_t^f + G^1 (K^1 * f) \\ -[g]_t^f + [\phi]_t^g + G^2 (K^2 * g) \\ -[\phi]_t^g \end{array} \right)
\]
which we demonstrate is Lipschitz using \( |x|_a^b - |y|_c^d | \leq \max \{ |x - y|, |a - c|, |b - d| \}

since

\[
\left\| V \left( \begin{array}{c} f^1 \\ g^1 \\ \phi^1 \\
 \end{array} \right) - V \left( \begin{array}{c} f^2 \\ g^2 \\ \phi^2 \\
 \end{array} \right) \right\|_\infty
\]

\[
= \left\| \left( \begin{array}{c}
 [g^1]^f_1 + G^1 (K^1 * f^1) \\
 - [g^1]^f_1 + [\phi^1]^g_1 + G^2 (K^2 * g^1) \\
 \end{array} \right) - \left( \begin{array}{c}
 [g^2]^f_2 + G^1 (K^1 * f^2) \\
 - [g^2]^f_2 + [\phi^2]^g_2 + G^2 (K^2 * g^2) \\
 \end{array} \right) \right\|_\infty
\]

\[
= \max \left\{ \frac{\left\| [g^1]^f_1 - [g^2]^f_2 + G^1 (K^1 * f^1) - G^1 (K^1 * f^2) \right\|_\infty}{\left\| [\phi^2]^g_2 - [\phi^1]^g_1 \right\|_\infty}, \frac{\left\| [g^2]^f_2 - [g^1]^f_1 + [\phi^1]^g_1 - [\phi^2]^g_2 + G^2 (K^1 * g^1) - G^2 (K^2 * g^2) \right\|_\infty}{\left\| [\phi^2]^g_2 - [\phi^1]^g_1 \right\|_\infty} \right\}
\]

\[
\leq \max \left\{ \frac{\max \{ \left\| g^1 - g^2 \right\|_\infty, \left\| f^1 - f^2 \right\|_\infty \} + \left\| G^1 (K^1 * f^1) - G^1 (K^1 * f^2) \right\|_\infty}{2 \max \{ \left\| g^1 - g^2 \right\|_\infty, \left\| f^1 - f^2 \right\|_\infty, \left\| [\phi^2]^g_2 - [\phi^1]^g_1 \right\|_\infty \} + \left\| G^2 (K^2 * g^1) - G^2 (K^2 * g^2) \right\|_\infty \}
\right\}
\]

\[
\leq 2 \left\| \begin{array}{c}
 f^1 \\
 g^1 \\
 \phi^1 \\
 \end{array} \right\|_\infty + \max \left\{ \left\| G^1 (K^1 * f^1) - G^1 (K^1 * f^2) \right\|_\infty, \left\| G^2 (K^2 * g^1) - G^2 (K^2 * g^2) \right\|_\infty \}
\right\}
\]

\[
\leq 2 \left\| \begin{array}{c}
 f^1 \\
 g^1 \\
 \phi^1 \\
 \end{array} \right\|_\infty + \max \left\{ \left\| G^1 (K^1 * f^1) - G^1 (K^1 * f^2) \right\|_\infty, \left\| G^2 (K^2 * g^1) - G^2 (K^2 * g^2) \right\|_\infty \}
\right\}\max \left\{ \left\| f^1 - f^2 \right\|_\infty, \left\| g^1 - g^2 \right\|_\infty \}
\right\}
\]

\[
\leq 2 \left\| \begin{array}{c}
 f^1 \\
 g^1 \\
 \phi^1 \\
 \end{array} \right\|_\infty + \left( 2 \max \left\{ \left\| G^1 (K^1 * f^1) - G^1 (K^1 * f^2) \right\|_\infty, \left\| G^2 (K^2 * g^1) - G^2 (K^2 * g^2) \right\|_\infty \}
\right\}\max \left\{ \left\| f^1 - f^2 \right\|_\infty, \left\| g^1 - g^2 \right\|_\infty \}
\right\}
\]

so \( V^5 \) is Lipschitz with constant \( C^5 := 2 + \max \left\{ \left\| \frac{dG^i}{dx} \right\|_\infty, \left\| K^j \right\|_1 \right\} \). Therefore

Theorem 1 guarantees \( X^5 \) (and therefore all the previous simpler models \( X^4 \) to \( X^4 \)) has a unique flow for all time. Extensions with results similar to these have been explored with simulations in [7]. That article and the references contained within it provide an excellent collection of speculations on the value and future directions of these models that are beyond the scope of the present paper.

Further opportunities for extending the models include time-dependence and delay dynamics. So, for instance, combined with the previous dynamics the growth of the creatures may not happen immediately when they encounter a fertile land. However, by adding a delay, they might store nutrients as long as they are in the fertile place and grow later. Or from another point of view, they may not notice the food immediately when they encounter it. The technical requirements are slightly more complex, but the resulting model is still just an arc field on a metric space, so the theory naturally supports the extension.
If we add time dependence, then Lenia becomes a faithful generalization of neural network models from discrete to continuous. Neural cellular automata or convolutional neural networks proceed from 2 steps: convolution, the activation. The activation step is precisely what the \( G \) function does in Lenia. In fact, clipping is commonly used.

Fully connected neural network layers are also generalizable to continuous models with this formalism using kernels that depend on time \( s \) and space \( x \)

\[
X_{s,t}(f)(x) = [f(x) + tG(K_{s,x} * f)(x)]_0^1
\]

where \( K_{s,x}(y) \) depends on \( x \) and \( s \). Put more simply,

\[
X_t(f) = [f + tC(f)]_0^1
\]

where \( C : \mathcal{B}(\mathbb{R}^n, \mathbb{R}) \times \mathbb{R} \rightarrow \mathcal{B}(\mathbb{R}^n, \mathbb{R}) \) because the input image \( f \) is transformed to \( C(f) \) in any way desired in the fully connected layer. (In a convolutional layer \( K_{s,x} \) is constant in \( x \).) Then given any finite set of training data, i.e., a collection of pairs of functions \( \{(f_i, g_i) \mid i \in I\} \subset (\mathcal{B}(\mathbb{R}^n, [0,1]))^2 \) we ask whether we can find a time dependent kernel \( K_s : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R} \) such that the flow \( F_{s,t} \) generated by the time-dependent arc field

\[
X_{s,t}(f) := [f + tK_s * f]_0^1
\]

satisfies \( F_{0,1}(f_i) = g_i \) for all \( i \in I \). I.e., the flow \( F \) interpolates the training data. This \( F \) is then the convolutional neural network which we seek to predict the result of new inputs \( f \) which will give outputs \( F_{0,1}(f) = g \). In this perspective we see a continuous neural network as an infinite dimensional control problem. Given the ability to control \( K_s \) at any time \( s \) can we find the \( K_s \) that guarantees \( F \) moves from \((f_i)\) to \((g_i)\)?

References

[1] Calcaterra, C., Bleecker, D., (2000) “Generating Flows on a Metric Space”, Journal of Mathematical Analysis and Applications, 248, pp. 645–677.
[2] Chan, B. (2019) “Lenia: Biology of Artificial Life”, Complex Systems, 28(3), pp. 251–286.
[3] Chan, B. (2020) “Lenia and Expanded Universe”, Proc. of ALIFE 2020, pp. 221-229.
[4] Evans, K. M. (2001) “Larger than Life: Digital Creatures in a Family of Two-Dimensional Cellular Automata”, Discrete Mathematics and Theoretical Computer Science Proceedings, Vol. AA, pp. 177–192.
[5] Evans, K. M. (2010) “Larger than Life’s Extremes: Rigorous Results for Simplified Rules and Speculation on the Phase Boundaries”, Chapter 11 in Game of Life Cellular Automata, edited by Andrew Adamatzky, Springer, pp. 179-221.
[6] Gardner, M. (October 1970), “The Fantastic Combinations of John Conway’s New Solitaire Game ‘Life’”, Mathematical Games, Scientific American, Vol. 223, no. 4, pp. 120–123.
[7] Hamon, G., Etcheverry, M., Chan, B., Moulin-Frier, C., Oudeyer, P. (2022) “Learning Sensorimotor Agency in Cellular Automata”, (unpublished) available online at https://developmentalsystems.org/sensorimotor-lenia/ Retrieved 3/15/2022
[8] Kawaguchi, T., Suzuki, R., Arita, T., Chan, B., (2021) “Introducing Asymptotics to the State-updating Rule in Lenia”, Proceedings of the ALIFE 2021: The 2021 Conference on Artificial Life. ALIFE 2021: The 2021 Conference on Artificial Life, online, pp. 91, ASME. https://doi.org/10.1162/isal_a_00425
[9] Pivato, M. (2007) “RealLife: The Continuum Limit of Larger than Life Cellular Automata”, Theoretical Computer Science, 372(1), 46–68.
[10] Prigogine, I. (1997) The End of Certainty: Time, Chaos, and the New Laws of Nature, Free Press.
[11] Radler, S. (2011) “Generalization of Conway’s ‘Game of Life’ to a continuous domain - SmoothLife”, arXiv preprint, arXiv:1111.1507
