The $\tau$ lepton anomalous magnetic moment

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We review the Standard Model prediction of the $\tau$ lepton $g-2$ presenting updated QED and electroweak contributions, as well as recent determinations of the leading-order hadronic term, based on the low energy $e^+e^-$ data, and of the hadronic light-by-light one.

1. INTRODUCTION

Numerous precision tests of the Standard Model (SM) and searches for its possible violation have been performed in the last few decades, serving as an invaluable tool to test the theory at the quantum level. They have also provided stringent constraints on many “New Physics” (NP) scenarios. A typical example is given by the measurements of the anomalous magnetic moment of the electron and the muon, $a_e$ and $a_\mu$, where recent experiments reached the fabulous relative precision of 0.7 ppb [1] and 0.5 ppm [2], respectively (the dimensionless quantity $a$ is defined as $a = (g - 2)/2$, where $g$ is a gyromagnetic factor). The anomalous magnetic moment of the electron, $a_e$, is rather insensitive to strong and weak interactions, hence providing a stringent test of QED and leading to the most precise determination of the fine-structure constant $\alpha$ to date [3,4]. On the other hand, the $g-2$ of the muon, $a_\mu$, allows to test the entire SM, as each of its sectors contributes in a significant way to the total prediction. Compared with $a_e$, $a_\mu$ is also much better suited to unveil or constrain NP effects. Indeed, for a lepton $l$, their contribution to $a_l$ is generally expected to be proportional to $m_l^2/\Lambda^2$, where $m_l$ is the mass of the lepton and $\Lambda$ is the scale of NP, thus leading to an $(m_\mu/m_e)^2 \sim 4 \times 10^4$ relative enhancement of the sensitivity of the muon versus the electron anomalous magnetic moment. The anomalous magnetic moment of the $\tau$ lepton, $a_{\tau}$, would suit even better; however, its relatively short lifetime makes a direct measurement impossible, at least at present.

Recent high-precision experiments at low-energy $e^+e^-$ colliders [5,6,7] allowed a significant improvement of the uncertainty of the leading-order hadronic contribution to $a_\mu$ [8,9]. In parallel to these efforts, many other improvements of the SM prediction for $a_\mu$ were carried on in recent years (see Refs. [10] for reviews). All these experimental and theoretical developments allow to significantly improve the SM prediction for the anomalous magnetic moment of the $\tau$ lepton as well, which is usually split into three parts: QED, electroweak and hadronic (see [11,12] for a very recent review).

2. QED CONTRIBUTION TO $a_\tau$

The QED contribution to $a_\tau$ arises from the subset of SM diagrams containing only leptons ($e, \mu, \tau$) and photons. The leading (one-loop) contribution was first computed by Schwinger more than fifty years ago [13]. Also the two- and three-loop QED terms are known (see Refs. [4,12]). Adding up these contributions and us-
One-loop Contribution

The QED contribution of the same order of magnitude as the three-loop one,
m by the ratio ($\tau$ is the mass of the lepton), where $F$ is the Fermi coupling constant \(18\), $Z$ is the mass of the Higgs, or other hypothetical bosons) can be derived in Refs. \([3,4]\), for $\alpha$ (small corrections of order $m_\tau^2/M_Z^2$), where $M_Z$ is the mass scale much smaller than $M_W$ \([26]\). This remarkable calculation leads to a significant reduction of the one-loop prediction because of large factors of $\ln(M_{Z,W,H}/m_\ell)$, where $m_\ell$ is a fermion mass scale much smaller than $M_W$. 

The two-loop contribution to $a_\tau^\text{EW}$ can be divided into fermionic and bosonic parts; the fermionic, $a_\tau^\text{EW}(2 \text{ loop ferm})$, includes all two-loop EW corrections containing closed fermion loops, whereas all other contributions are grouped into the latter, $a_\tau^\text{EW}(2 \text{ loop bos})$. The bosonic part was computed in Ref. \([24]\): its value, for $M_H = 150 \text{ GeV}$, is $a_\tau^\text{EW}(2 \text{ loop bos}) = -3.06 \times 10^{-8}$ \([12]\).

The hadronic uncertainties mainly arise from the latter ones. In Refs. \([23,25]\) these nonperturbative effects were modeled introducing effective quark masses as a simple way to account for strong interactions, and the diagrams with quark triangle loops were computed with simple approximate expressions. However, contrary to the case of the muon $g-2$, where all fermion masses (with the exception of $m_e$) enter in the evaluation of these contributions, in the case of the $\tau$ lepton these approximate expressions do not depend on any

3.1. One-loop Contribution

The one-loop EW term is \([17]\): $a_\tau^\text{EW}(1 \text{ loop}) = \frac{5G_\mu m_\tau^2}{24\sqrt{2}\pi^2} \left[ 1 + \frac{1}{6} (1 - 4\sin^2\theta_W)^2 + O(m_\tau^2/M_Z^2, \alpha/\pi) \right]$, where $G_\mu = 1.16637(1) \times 10^{-5} \text{ GeV}^{-2}$ is the Fermi coupling constant \([18]\), $M_Z$, $M_W$, and $M_H$ are the masses of the $Z$, $W$ and Higgs bosons, and $\theta_W$ is the weak mixing angle. Closed analytic expressions for $a_\tau^\text{EW}(1 \text{ loop})$ taking exactly into account the $m_\tau^2/M_Z^2$ dependence ($B = Z, W, H$) or other hypothetical bosons) can be found in Ref. \([19]\). Employing the on-shell definition $\sin^2\theta_W = 1 - M_Z^2/M_W^2$ \([20]\), where $M_Z = 91.1875(21) \text{ GeV}$ and $M_W$ is the SM prediction of the $W$ mass (which can be derived, for example, from the simple formula of \([21]\) leading to $M_W = 80.383 \text{ GeV}$ for $M_H = 150 \text{ GeV}$), and including the tiny $O(m_\tau^2/M_Z^2, \alpha/\pi)$ corrections of Ref. \([19]\), for $M_H = 150 \text{ GeV}$ one obtains $a_\tau^\text{EW}(1 \text{ loop}) = 55.1(1) \times 10^{-8}$ \([12]\). The uncertainty encompasses the shifts induced by variations of $M_H$ from 114.4 GeV, the current lower bound at 95% confidence level \([22]\), up to a few hundred GeV, and the tiny uncertainty due to the error in $m_\tau$. The estimate of the total EW contribution of Ref. \([15]\), $a_\tau^\text{EW} = 55.60(2) \times 10^{-8}$, obtained from the one-loop formula (without the small corrections of order $m_\tau^2/M_Z^2$), is similar to the one-loop value reported above. However, its uncertainty ($2 \times 10^{-10}$) is too small, and it doesn’t contain the two-loop contribution which, as we’ll now discuss, is not negligible.

3.2. Two-loop Contribution

The two-loop EW contributions $a_\tau^\text{EW}(2 \text{ loop}) (l = e, \mu$ or $\tau)$ were computed in 1995 by Czarnecki, Krause and Marciano \([23,24,25]\).
fermion mass lighter than $m_{\tau}$; apart from $m_{\tau}$, they only depend on \( M_{\text{top}} \) and \( m_b \), the masses of the top and bottom quarks (assuming the charm mass \( m_c < m_{\tau} \)). Very recently, this analysis was slightly refined in Ref. \cite{12} by numerically integrating exact expressions provided in Ref. \cite{27} for arbitrary values of \( m_f \), obtaining, for \( M_{\mu} = 150 \text{ GeV} \), \( a^{\text{EW}}_\tau(2 \text{ loop}) = -4.68 \times 10^{-8} \). This evaluation also included the tiny \( O(10^{-9}) \) contribution of the \( \gamma Z \) mixing diagrams, suppressed by \((1 - 4 \sin^2 \theta_W) \sim 0.1 \) for quarks and \((1 - 4 \sin^2 \theta_W)^2 \) for leptons, via the explicit formulae of Ref. \cite{28}.

The sum of the fermionic and bosonic two-loop EW contributions described above gives \( a^{\text{EW}}_\tau(2 \text{ loop}) = -7.74 \times 10^{-8} \), a 14% reduction of the one-loop result. The leading-logarithm three-loop EW contributions to the muon \( g-2 \) were determined to be extremely small via renormalization-group analyses \cite{28,29}. In Ref. \cite{12} an additional uncertainty of \( O[a^{\text{EW}}_\tau(2 \text{ loop})](\alpha/\pi) \ln(M^2_{\mu}/m^2_{\mu}) \sim O(10^{-9}) \) was assigned to \( a^{\text{EW}}_\tau \) to account for these neglected three-loop effects. Adding \( a^{\text{EW}}_\tau(2 \text{ loop}) \) to the one-loop value presented above, one gets the total EW correction (for \( M_{\mu} = 150 \text{ GeV} \)) \cite{12}:

\[
a^{\tau}_{\text{EW}} = 47.4(5) \times 10^{-8}. \tag{2}
\]

The uncertainty allows \( M_{\mu} \) to range from 114 GeV up to \( \sim 300 \text{ GeV} \), and reflects the estimated errors induced by hadronic loop effects, neglected two-loop bosonic terms and the missing three-loop contribution. It also includes the tiny errors due to the uncertainties in \( M_{\text{top}} \) and \( m_{\tau} \). The value in Eq. (2) is in agreement with the prediction \( a^{\tau}_{\text{EW}} = 47(1) \times 10^{-8} \) \cite{25,10}, with a reduced uncertainty. As we mentioned in Sec. 3.1 the EW estimate of Ref. \cite{15}, \( a^{\text{EW}}_\tau = 55.60(2) \times 10^{-8} \), mainly differs from Eq. (2) in that it doesn’t include the two-loop corrections.

4. THE HADRONIC CONTRIBUTION

In this section we will analyze \( a^{\text{HAD}}_\tau \), the contribution to the \( \tau \) anomalous magnetic moment arising from QED diagrams involving hadrons. Hadronic effects in (two-loop) EW contributions are already included in \( a^{\text{EW}}_\tau \) (see Sec. 3).

4.1. Leading-order Hadronic Contribution

Similarly to the case of the muon \( g-2 \), the leading-order hadronic contribution to the \( \tau \) lepton anomalous magnetic moment is given by the dispersion integral \cite{30}:

\[
a^{\text{HLO}}_{\tau} = \frac{m^2_\tau}{12 \pi^2} \int_{4m^2_e}^{\infty} ds \frac{\sigma^{(0)}(e^+e^-\rightarrow \text{hadrons})K_\tau(s)}{s}, \tag{3}
\]

in which the role of the low energies is very important, although not as strongly as in \( a^{\mu}_{\text{HLO}} \).

The history of the \( a_\tau \) calculations \cite{15,16,31,32,33,34,12} based mainly on experimental \( e^+e^- \) data is shown in Table 1. Purely theoretical estimates somewhat undervalue the hadronic contribution and have rather large uncertainties \cite{35,36,37,38}.

| Author                      | \( a^{\text{HLO}}_\tau \times 10^8 \) |
|-----------------------------|---------------------------------------|
| Narison                     | 370 \pm 40                            |
| Barish & Stroynowski        | \sim 350                               |
| Samuel et al.               | 360 \pm 32                            |
| Eidelman & Jegerlehner      | 338.4 \pm 2.0 \pm 9.1                 |
| Narison & Passera           | 337.5 \pm 3.7                         |

We updated the calculation of the leading-order contribution using the whole bulk of experimental data below 12 GeV, which include old data compiled in Refs. \cite{33,39}, recent results from the CMD-2 and SND detectors in Novosibirsk \cite{5,6,8}, and from the radiative return studies at KLOE in Frascati \cite{7} and BaBar at SLAC \cite{10}. The improvement is particularly strong in the channel \( e^+e^- \rightarrow \pi^+\pi^- \). Our result is

\[
a^{\text{HLO}}_{\tau} = 337.5(3.7) \times 10^{-8}. \tag{4}
\]

The overall uncertainty is 2.5 times smaller than that of the previous data-based prediction \cite{33,34}.

4.2. Higher-order Hadronic Contributions

The hadronic higher-order (\( \alpha^3 \)) contribution \( a^{\text{HHO}}_{\tau} \) can be divided into two parts: \( a^{\text{HHO}}_\tau(vp) + a^{\text{HHO}}_\tau(lll) \). The first one is the \( O(\alpha^3) \) contribution of diagrams containing hadronic self-
energy insertions in the photon propagators. It was determined by Krause in 1996 [11]:

\[ a^\text{HHO}(\text{vp}) = 7.6(2) \times 10^{-8}. \]  

(5)

Note that naively rescaling the muon result by the factor \( m_e^2/m_\mu^2 \) (as it was done in Ref. [15]) leads to the totally incorrect estimate \( a^\text{HHO}(\text{vp}) = (-101 \times 10^{-11}) \times m_e^2/m_\mu^2 = -29 \times 10^{-8} \) (the \( a^\text{HHO}(\text{vp}) \) value is from Ref. [14]); even the sign is wrong!

The second term, also of \( O(\alpha^3) \), is the hadronic light-by-light contribution. Similarly to the case of the muon \( g-2 \), this term cannot be directly determined via a dispersion relation approach using data (unlike the leading-order hadronic contribution), and its evaluation therefore relies on specific models of low-energy hadronic interactions with electromagnetic currents. Until recently, very few estimates of \( a^\text{HHO}(\text{lbl}) \) existed in the literature [15,40,41], and all of them were obtained simply rescaling the muon results \( a^\text{HHO}(\text{lbl}) \) by a factor \( m_e^2/m_\mu^2 \). Following this very naive procedure, the \( a^\text{HHO}(\text{lbl}) \) estimate varies between \( a^\text{HHO}(\text{lbl}) = 23(11) \times 10^{-8} \), and \( a^\text{HHO}(\text{lbl}) = 38(7) \times 10^{-8} \), according to the values chosen for \( a^\text{HHO}(\text{lbl}) \) from Refs. [12] and [43], respectively.

These very naive estimates fall short of what is needed. Consider the function \( A_2^{(6)}(m_l/m_j, \text{lbl}) \), the three-loop QED contribution to the \( g-2 \) of a lepton of mass \( m_l \) due to light-by-light diagrams involving loops of a fermion of mass \( m_j \). This function was computed in Ref. [14] for arbitrary values of the mass ratio \( m_l/m_j \). In particular, if \( m_j \gg m_l \), then \( A_2^{(6)}(m_l/m_j, \text{lbl}) \sim (m_l/m_j)^2 \). This implies that, for example, the (negligible) part of \( a^\text{HHO}(\text{lbl}) \) due to diagrams with a top-quark loop can be reasonably estimated simply rescaling the corresponding part of \( a^\text{HHO}(\text{lbl}) \) by a factor \( m_e^2/m_\mu^2 \). On the other hand, to compute the dominant contributions to \( a^\text{HHO}(\text{lbl}) \), i.e., those induced by the light quarks, we need the opposite case: \( m_j \ll m_l = m_\tau \). In this limit, \( A_2^{(6)}(m_l/m_j, \text{lbl}) \) does not scale as \( (m_l/m_j)^2 \), and a naive rescaling of \( a^\text{HHO}(\text{lbl}) \) by \( m_e^2/m_\mu^2 \) to derive \( a^\text{HHO}(\text{lbl}) \) leads to an incorrect estimate.

For these reasons, a parton-level estimate of \( a^\text{HHO}(\text{lbl}) \) was recently performed in Ref. [12], based on the exact expression for \( A_2^{(6)}(m_l/m_j, \text{lbl}) \), using the quark masses recently proposed in Ref. [45] for the determination of \( a^\text{HHO}(\text{lbl}) \): \( m_u = m_d = 176 \) MeV, \( m_s = 305 \) MeV, \( m_c = 1.18 \) GeV and \( m_b = 4 \) GeV (note that with these values the authors of Ref. [45] obtain \( a^\text{HHO}(\text{lbl}) = 136 \times 10^{-11} \), in perfect agreement with the value in Ref. [43]—see also Ref. [46] for a similar earlier determination). The result of this analysis is [12]

\[ a^\text{HHO}(\text{lbl}) = 5(3) \times 10^{-8}. \]  

(6)

This value is much lower than those obtained by simple rescaling above. The dominant contribution comes from the \( u \) quark; the uncertainty \( \delta a^\text{HHO}(\text{lbl}) = 3 \times 10^{-8} \) allows \( m_u \) to range from 70 MeV up to 400 MeV. Further independent studies (following the approach of Ref. [43], for example) would provide an important check of this result.

The total hadronic contribution to the anomalous magnetic moment of the \( \tau \) lepton can be immediately derived adding the values in Eqs. [4], [5] and [6],

\[ a^\text{had} = a^\text{PL} + a^\text{HHO}(\text{vp}) + a^\text{HHO}(\text{lbl}) = 350.1(4.8) \times 10^{-8}. \]  

(7)

Errors were added in quadrature.

5. THE STANDARD MODEL PREDICTION FOR \( a_\tau \)

We can now add up all the contributions discussed in the previous sections to derive the SM prediction for \( a_\tau \) [12]:

\[ a^\text{SM} = a^\text{QED} + a^\text{EW} + a^\text{HLO} + a^\text{HHO} = 117.721(5) \times 10^{-8}. \]  

(8)

Errors were added in quadrature.

The most stringent limit on the anomalous magnetic moment of the \( \tau \) lepton was derived in 2004 by the DELPHI collaboration from \( e^+e^- \rightarrow e^+e^-\tau^+\tau^- \) total cross section measurements at \( \sqrt{s} \) between 183 and 208 GeV at LEP2 [47]:

\[ -0.052 < a_\tau < 0.013 \]  

(9)

at 95% confidence level. Comparing this result with Eq. [8], it is clear that the sensitivity of the
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The best existing measurements is still more than an order of magnitude worse than needed. For other limits on $a_\tau$ see Ref. [18,48].

6. CONCLUSIONS

The $g - 2$ of the τ lepton is much more sensitive than the muon one to EW and NP loop effects that give contributions $\sim m_l^2$, making its measurement an excellent opportunity to unveil (or just constrain) NP effects.

Unfortunately, the very short lifetime of the τ lepton makes it very difficult to determine its anomalous magnetic moment by measuring its spin precession in the magnetic field, like in the muon $g - 2$ experiment [2]. Instead, experiments focus on high-precision measurements of the τ lepton pair production in various high-energy processes, comparing the measured cross sections with the QED predictions [47,18], but their sensitivity is still more than an order of magnitude worse than that required to determine $a_\tau$.

Nonetheless, there are many interesting suggestions to measure $a_\tau$, e.g., from the radiation amplitude zero in radiative τ decays [49] or from other observables. By employing such methods at B factories, one can hope to benefit from the possibility to collect very high statistics. A similar method to study $a_\tau$ using radiative W decays and potentially very high data samples at LHC was suggested in Ref. [50]. Yet another method would use the channeling in a bent crystal similarly to the measurement of magnetic moments of short-living baryons [51]. In the case of the τ lepton, it was suggested to use the decay $B^+ \rightarrow \tau^+ \nu_\tau$, which would produce polarized τ leptons [15] and was recently observed [52]. We believe that a detailed feasibility study of such experiments, as well as further attempts to improve the accuracy of the theoretical prediction for $a_\tau$, are quite timely.

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