Abstract

We obtain a formula for the number of horizontal equilibria of a planar convex body $K$ with respect to a center of mass $O$ in terms of the winding number of the evolute of $\partial K$ with respect to $O$. The formula extends to the case where $O$ lies on the evolute of $\partial K$ and a suitably modified version holds true for non-horizontal equilibria.

1 Introduction

We study the number of static equilibria of a planar convex body $K$ supported by a horizontal line subject to a uniform vertical gravity field. It is well-known that the number of static equilibria with respect to the centroid of a homogeneous body $K$ is $\geq 4$ (see [1] and Proposition 3.4 below). It was pointed out in [10], that this result is equivalent to the Four-vertex Theorem. For an arbitrary center of mass, one can find planar convex bodies with only one stable and one unstable equilibrium – the 3-dimensional counterparts of such objects are known as roly-poly toys. In [10] it is shown that there exists a homogeneous convex roly-poly toy with exactly one stable and one unstable equilibrium, the so-called gömböc – thus answering a long-standing conjecture by Arnol’d in the affirmative.

In this article, we provide a geometric characterisation of the number $n$ of static equilibria of a planar convex body $K$ in terms of the winding number of the evolute of $\partial K$ with respect to a given center of mass $O$ of $K$: If $\partial K$ is parametrized by a positively oriented curve $\gamma$ and $O$ is not a point of the evolute of $\partial K$, then the winding number of the evolute of $\partial K$ is an integer $m \leq 0$ and the formula

$$n = 2 - 2m$$

(1.1)
holds true. We will show that this formula remains valid, if \( O \) is a point of the evolute of \( \partial K \), possibly even a cusp, but in this case, \( m \) might be half-integer valued. Our main theorem is the following:

**Theorem 1.1**

Let \( K \) be a strongly convex compact set with \( C^3 \)-boundary \( \partial K \) such that the curvature of \( \partial K \) has only finitely many stationary points, and let \( O \) be a point in the plane. Then the number \( n \) of horizontal equilibria of \( K \) with respect to \( O \) is given by

\[
   n = 2 - 2m,
\]

where \( 0 \geq m \in \frac{1}{2} \mathbb{Z} \) is the winding number of the evolute of \( \partial K \) with respect to \( O \).

The strategy of the proof is to identify the horizontal equilibria as zeros of the first derivative of a support function that parametrizes \( \partial K \) and using the zero-counting integral developed in [3] in order to count its zeros. The resulting integral can then be related to the generalized winding number (see [4]) of the evolute of \( \partial K \).

In Section 4 we replace the horizontal supporting line of the body \( K \) by an inclined line with inclination angle \( \alpha \in (-\frac{\pi}{2}, \frac{\pi}{2}) \). It is interesting that for any angle \( \alpha \neq 0 \), there exist homogeneous bodies \( K \) such that the inequality \( n \geq 4 \) fails. In fact, for every \( \alpha \neq 0 \), there are such bodies with exactly one metastable equilibrium and also bodies with exactly one stable and one unstable equilibrium with respect to the centroid (see Proposition 4.1 below). Furthermore, a formula like (1.1) holds true for \( \alpha \neq 0 \), where \( m \) is the winding number of the evolute of a suitable modification of \( \partial K \).

## 2 Support Functions

For \( x, y \in \mathbb{R}^2 \), let \( (x, y) = \{tx + (1-t)y, t \in (0,1)\} \) denote the line segment between the points \( x \) and \( y \). A set \( K \subset \mathbb{R}^2 \) is called convex if for any \( x, y \in K \) it holds that \( (x, y) \cap K = (x, y) \). The set \( K \) is called strictly convex if \( (x, y) \cap \bar{K} = (x, y) \) for any \( x, y \in K \). A bounded convex set \( K \subset \mathbb{R}^2 \) with \( C^n \)-boundary, \( n \geq 2 \) is called strongly convex if \( \partial K \) can be parametrized by a curve \( \gamma : S^1 \to \partial K \) such that \( \|\dot{\gamma}\| = 1 \) and \( \ddot{\gamma} \) does not vanish. We will use the identification \( S^1 \cong \mathbb{R}/2\pi \mathbb{Z} \) and the notation \( u(\varphi) = (\cos(\varphi), \sin(\varphi))^\top \) throughout this article.

The boundary \( \partial K \) of a strictly convex compact set \( K \) admits a parametrization by support functions \( p \) and \( q \), i.e., there exists a parametrization \( z : S^1 \to \partial K \) such that
In fact, for fixed \( \varphi \), the orthogonal projection of \( K \) to the line \( g = \{ \lambda u(\varphi) \mid \lambda \in \mathbb{R} \} \) is a compact interval (see Figure 2), and for its endpoint \( P \) we have
\[
P = pu(\varphi) \quad \text{with} \quad p = \max \{ \langle X, u(\varphi) \rangle \mid X \in K \}.
\]
Since \( K \) is strictly convex, \( p = \langle Z, u(\varphi) \rangle \) for a unique \( Z \in K \). Hence, by choosing \( p(\varphi) = p \), and \( q(\varphi) \) as the oriented distance of \( Z \) and \( P \) we have indeed
\[
Z = z(\varphi) = p(\varphi)u(\varphi) + q(\varphi)u'(\varphi).
\]
The connection between the regularity of the boundary curve \( \partial K \) and the support functions is described in the following Lemma. Note that here we need that \( K \) is strongly convex.

**Lemma 2.1**

Let \( K \) be a strongly convex compact set with \( C^n \) boundary \( \partial K \), \( n \geq 2 \). Then \( \partial K \) can be parametrized by \( \varphi \mapsto z(\varphi) = p(\varphi)u(\varphi) + q(\varphi)u'(\varphi) \), where \( p \in C^n(S^1, \mathbb{R}^2) \).

This result is remarkable in that \( p \) as a function of arc length \( s \) along \( \partial K \) instead of \( \varphi \) is only in \( C^{n-1} \) in general.

**Proof.** Let \( \gamma \) be a \( C^n \) arc-length parametrization of \( \partial K \) and let \( J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \). Observe that \( \{-J\dot{\gamma}(s), \dot{\gamma}(s)\} \) forms an orthonormal basis of \( \mathbb{R}^2 \) for every \( s \), where the dot indicates
the derivative with respect to arc length. Hence we may write
\[
\gamma(s) = -p(s)\dot{\gamma}(s) + q(s)\ddot{\gamma}(s), \tag{2.1}
\]
where
\[
p(s) = -\langle \gamma(s), J\dot{\gamma}(s) \rangle \in C^{n-1},
\]
\[
q(s) = \langle \gamma(s), \dot{\gamma}(s) \rangle \in C^{n-1}.
\]
See Figure 3. It holds that \(\varphi(s) = \arg \dot{\gamma}(s) - \frac{\pi}{2} = -\arctan \left( \frac{\dot{\gamma}_1(s)}{\dot{\gamma}_2(s)} \right)\) is of class \(C^{n-1}\).

Hence \(\varphi \mapsto q(s(\varphi))\) is of class \(C^{n-1}\) and
\[
\frac{d\varphi}{ds} = -\frac{1}{1 + \left( \frac{\dot{\gamma}_1}{\dot{\gamma}_2} \right)^2} \frac{\ddot{\gamma}_1\dot{\gamma}_2 - \ddot{\gamma}_2\dot{\gamma}_1}{\dot{\gamma}_2^2} = -\frac{\ddot{\gamma}_1\dot{\gamma}_2 - \ddot{\gamma}_2\dot{\gamma}_1}{\dot{\gamma}_1^2 + \dot{\gamma}_2^2} = -\langle \dot{\gamma}, J\dddot{\gamma} \rangle. \tag{2.2}
\]
We now show that the derivative \((p \circ s)'(\varphi) = (q \circ s)(\varphi)\) which implies that \(\varphi \mapsto p(s(\varphi))\) is of class \(C^n\): Indeed we have
\[
(p \circ s)'(\varphi) = -\frac{d}{d\varphi} \left( \gamma(s(\varphi)), J\dddot{\gamma}(s(\varphi)) \right)
= -\left( \gamma(s(\varphi)), J\dddot{\gamma}(s(\varphi)) \right) \frac{ds}{d\varphi} - \left( \gamma(s(\varphi)), J\dddot{\gamma}(s(\varphi)) \right) \frac{ds}{d\varphi}
= \left( \gamma(s(\varphi)), J\dddot{\gamma}(s(\varphi)) \right) \frac{ds}{d\varphi} = (q \circ s)(\varphi),
\]
where we have used \(J\dddot{\gamma} \parallel \dddot{\gamma}\) in the last line. \(\square\)
Corollary 2.2

Let $K$ be a strongly convex compact set with $C^n$ boundary $\partial K$, $n \geq 2$. If $\partial K$ is parametrized by $z(\varphi) = p(\varphi)u(\varphi) + p'(\varphi)u'(\varphi)$, then $z'(\varphi) = u'(\varphi)\rho(\varphi)$, where $\rho(\varphi) = p(\varphi) + p''(\varphi)$ is the radius of curvature of $\partial K$ in $z(\varphi)$.

Proof. It follows from Lemma 2.1 that $p$ is of class $C^n$. First, by direct calculation, we see that $z' = (p+p'')u'$, because $u'' = -u$. If $n \geq 3$, we compute $z'' = (p+p'')u'' + (p' + p'''u)'$. Since the radius of curvature $\rho$ is the projection of $z''$ onto $u''$ we obtain the desired result. If $n = 2$ we consider again a parametrization $\gamma$ of $\partial K$ by arc length and use $\rho\gamma = J\gamma$ and $J^2 = -id$ to compute by (2.1)

$$\dot{\gamma} = -pJ\gamma - \ddot{p}J\dot{\gamma} + q\dot{\gamma} + \dot{q}\gamma$$

$$= \frac{p}{\rho}\dot{\gamma} - \ddot{p}J\dot{\gamma} + \frac{q}{\rho}J\dot{\gamma} + \dot{q}\gamma$$

$$= \left(\frac{p}{\rho} + \dot{q}\right)\dot{\gamma} + \left(\frac{q}{\rho} - \ddot{p}\right)J\dot{\gamma}.$$

Hence we have $\frac{p}{\rho} + \dot{q} \equiv 1$ and $\frac{q}{\rho} - \ddot{p} \equiv 0$. Using $\dot{q} = 1 - \frac{p}{\rho}$ we find

$$p + p'' = p + q' = p + \dot{q} \cdot \frac{ds}{d\varphi} = p + \frac{1 - \frac{p}{\rho}}{-\langle \dot{\gamma}, J\dot{\gamma} \rangle} = p + \frac{1 - \frac{p}{\rho}}{\frac{1}{\rho} \langle \dot{\gamma}, \dot{\gamma} \rangle} = \rho.$$

We will now collect a few expressions for relevant geometric quantities in terms of the parametrization for $\partial K$ from Lemma 2.1: First of all the arc length $s(\varphi)$ of $z|_{[0,\varphi]}$ is...
given by
\[ s(\varphi) = \int_0^{\varphi} |z'| \, d\varphi = \int_0^{\varphi} (p + p'') \, d\varphi = \int_0^{\varphi} p \, d\varphi + p'(\varphi) - p'(0) \] (2.3)
and hence the perimeter of \( K \) is
\[ s(2\pi) = \int_0^{2\pi} p \, d\varphi =: L. \]

The center of mass of the curve \( \partial K \) is given by
\[ \frac{1}{L} \int_0^{2\pi} z|z'| \, d\varphi = \frac{1}{L} \int_0^{2\pi} (pu' + p''u'')p + p'' \, d\varphi = \frac{1}{L} \int_0^{2\pi} \left( p^2 - \frac{p'^2}{2} \right) u \, d\varphi, \]
where we have integrated by parts. Similarly, the area \( A \) of \( K \) is given by
\[ A = \frac{1}{2} \int_0^{2\pi} (p^2 - p'^2) \, d\varphi, \]
and the centroid \( O \) of \( K \) by
\[ O = \frac{1}{3A} \int_0^{2\pi} (pu' + p''u')p + p'' \, d\varphi. \]

3 The evolute of \( \partial K \)

Let \( K \) be a strongly convex set of class \( C^2 \). Then, the evolute of \( \partial K \) is given by
\[ e(\varphi) = z(\varphi) - \rho(\varphi)u(\varphi) \]
\[ = z(\varphi) + (p(\varphi) + p''(\varphi))u''(\varphi) \]
\[ = p(\varphi)u(\varphi) + p'(\varphi)u'(\varphi) - (p(\varphi) + p''(\varphi))u(\varphi) \]
\[ = p'(\varphi)u'(\varphi) - p''(\varphi)u(\varphi). \] (3.1)
Thus, the evolute is obtained from the original curve \( \partial K \) by replacing its support function \( p \) by \( p' \) and a rotation about 90°. Formula (3.1) shows that all parallel curves of \( \partial K \), which have support function \( p + \) constant, have the same evolute as \( K \).

3.1 Curves of Constant Width

Suppose that \( K \) is a strongly convex set with \( C^2 \)-boundary \( \partial K \) and assume in addition that \( \partial K \) is a curve of constant width \( d > 0 \). Then \( \partial K \) can be parametrized by a
support function \( p \) that satisfies \( p(\varphi) + p(\varphi + \pi) \equiv d \). This equation implies that 
\[
p^{(k)}(\varphi) = -p^{(k)}(\varphi + \pi) \quad \text{for } k = 1, 2
\]
and it follows for the evolute \( e \) of \( \partial K \)
\[
e(\varphi + \pi) = p'(\varphi + \pi)u'(\varphi + \pi) - p''(\varphi + \pi)u(\varphi + \pi)
\]
\[
= p'(\varphi)u'(\varphi) - p''(\varphi)u(\varphi)
\]
\[
= e(\varphi).
\]
Hence \( e : S^1 \to \mathbb{R}^2 \) is \( \pi \)-periodic. This means that the evolute of a curve of constant width is traversed twice.

### 3.2 Cusps of the evolute

Even if \( \partial K \) is a smooth regular curve, its evolute has necessarily at least four singular points (cusps). The situation is described in the following lemma:

**Lemma 3.1**

Let \( K \) be strongly convex and compact with \( \partial K \) of class \( C^3 \) parametrized by \( \varphi \mapsto z(\varphi) = p(\varphi)u(\varphi) + p'(\varphi)u'(\varphi) \). We assume, that the curvature of \( \partial K \) has only finitely many stationary points. Then, the evolute of \( \partial K \), given by \( \varphi \mapsto e(\varphi) = p'(\varphi)u'(\varphi) - p''(\varphi)u(\varphi) \), is regular and of class \( C^2 \) except for points where the radius of curvature \( \rho \) of \( \partial K \) is stationary. More precisely:

- If \( \rho \) has a local minimum in \( \varphi_0 \), then \( e \) has a cusp in \( \varphi_0 \) pointing towards the point \( z(\varphi_0) \) (see Figure 4).

- If \( \rho \) has a local maximum in \( \varphi_0 \), then \( e \) has a cusp in \( \varphi_0 \) pointing away from the point \( z(\varphi_0) \) (see Figure 4).

- If \( \rho \) has a saddle point in \( \varphi_0 \), then \( e \) is \( C^1 \) in \( \varphi_0 \).

**Remarks.**

- By the Four-vertex Theorem (see [7], [5] or [8]), it follows that the evolute of \( \partial K \) has at least four cusps. Since maxima and minima alternate, the number of cusps is always even.

- Note that the \( C^2 \)-regularity of \( e \) is not evident, since the parametrization of \( e \) with respect to \( \varphi \) is obviously only \( C^1 \) in general.
The connection between the cusps of the evolute and strict local extrema of the base curve has first been observed by G. H. Light [6].

Proof. First of all, note that
\[ z' = \rho u' \quad \text{and} \quad e' = -\rho' u \]
which shows that \( \langle z', e' \rangle = 0 \). Suppose now, that \( \rho'(\varphi_0) = 0 \).

1. case: \( \rho \) has a local minimum in \( \varphi_0 \). Then
\[
\lim_{\varphi \to \varphi_0} \frac{e'(\varphi)}{\|e'(\varphi)\|} = u(\varphi_0) \quad \text{and} \quad \lim_{\varphi \to \varphi_0} \frac{e'(\varphi)}{\|e'(\varphi)\|} = -u(\varphi_0).
\]

2. case: \( \rho \) has a local maximum in \( \varphi_0 \). Then
\[
\lim_{\varphi \to \varphi_0} \frac{e'(\varphi)}{\|e'(\varphi)\|} = -u(\varphi_0) \quad \text{and} \quad \lim_{\varphi \to \varphi_0} \frac{e'(\varphi)}{\|e'(\varphi)\|} = u(\varphi_0).
\]

3. case: \( \rho \) has a saddle point in \( \varphi_0 \), i.e., \( \rho' \) does not change sign in \( \varphi_0 \). Then \( \lim_{\varphi \to \varphi_0} \frac{e'(\varphi)}{\|e'(\varphi)\|} \) exists, and \( e \) is \( C^1 \) in \( \varphi_0 \).

To check the regularity of the evolute, we interpret the curve locally as a graph of a function \( x_2(x_1) \) or \( x_1(x_2) \). Then, by the chain rule, we have for \( x_1(x_2) \)
\[
\frac{dx_2}{dx_1}(\varphi) = \frac{dz_2(\varphi)}{dz_1(\varphi)} = \tan(\varphi)
\]
and
\[
\frac{d^2x_2}{dx_1^2}(\varphi) = \frac{d^2\tan(\varphi)}{d\varphi^2} = -\frac{1}{\rho'(\varphi) \cos^3(\varphi)}.
\]

The case \( x_1(x_2) \) is similar. Since \( \rho' \) is \( C^0 \), we conclude that locally, in points \( \varphi \) where \( \rho'(\varphi) \neq 0 \), the curve \( e \) is \( C^2 \).

Corollary 3.2
If we count the arc length of the evolute \( e \) between two cusps alternating positive and negative, the resulting sum vanishes (see Figure 4).
Proof. The factor $\rho'$ in $e' = -\rho'u$ changes its sign in every cusp. The length of $e$ is
\[
\int_0^{2\pi} \|e'\| \, d\varphi = \int_0^{2\pi} |\rho'| \, d\varphi
\]
and hence the alternating sum of the lengths between cusps equals
\[
\int_0^{2\pi} \rho' \, d\varphi = 0.
\]

Figure 4: The blue points are maxima of the curvature of $z$, the magenta points are minima. The sum of the lengths of the red arcs of the evolute $e$ equals the sum of lengths of the green arcs.

3.3 Equilibria

We now choose a measure $\mu$ with support in the compact convex set $K \subset \mathbb{R}^2$ which models the density of a distribution of mass. The center of mass of $\mu$ is a point $O \in K$. Vice versa, given a point $O \in K$, there is a measure supported in $K$ with center of mass $O$ (e.g. a Dirac mass in $O$). In a physical model, this scenario can be realized by fixing a heavy lead ball in the point $O$ on a thin, lightweight plate which has shape $K$. If the
density in $K$ is constant the center of mass is usually called the centroid. If we allow signed measures, the center of mass can be any point $O$ in $\mathbb{R}^2$, and vice versa, given an arbitrary point $O \in \mathbb{R}^2$, there is a signed measure supported in $K$ with center of mass in $O$. A physical model can be manufactured be glueing a long, thin batten to $K$ joining $K$ to a point $O \notin K$ and to fix a heavy lead ball at its far end in $O$.

We are interested in the following question: Suppose $K$ is equipped with a center of mass $O$, as discussed above, and is rolling along a horizontal straight line $\ell$. Horizontal means, that $\ell$ is perpendicular to the direction of the gravitational force $g$. What can we say about the number of equilibria with respect to $O$ in terms of the geometry of $\partial K$? In particular, how many equilibrium positions are there?

Physically, an equilibrium position is characterized by the fact, that the vector $v$ from the center of mass $O$ of $K$ to the contact point of $\partial K$ with the supporting straight line $\ell$ is parallel to the gravitational force. This follows from Varignon’s Theorem of the resulting torque and the principle of angular momentum. In case of a horizontal supporting line $\ell$, this means that $v$ is orthogonal to $\ell$. The equilibrium is stable, if the potential energy of $K$ (i.e. of its center of mass) has a strict local minimum with respect to the direction $-g$, and it is unstable, if the potential energy has a strict local maximum. This translates into the following definition:

**Definition 3.3**

Let $K$ be strongly convex and compact with $\partial K$ of class $C^3$ parametrized by $\varphi \mapsto z(\varphi) = p(\varphi)u(\varphi) + p'(\varphi)u'(\varphi)$, where the origin is chosen in the center of mass $O$ of $K$. Then, a horizontal equilibrium position with respect to $O$ is a point $z(\varphi_0) \in \partial K$ such that $p'(\varphi_0) = 0$. The equilibrium is stable if $p$ has a strict local minimum in $\varphi_0$, and unstable if $p$ has a strict local maximum in $\varphi_0$.

A horizontal equilibrium $z \in \partial K$ is therefore a point where the tangent at $z$ and the line joining $z$ and the center of mass are perpendicular. Figure 5 shows a shape $K$ which has one stable and one unstable horizontal equilibrium with respect to the center of mass $O$. We start by investigating the number of equilibria for the special case of the centroid $O$ of a homogeneous body.

**Proposition 3.4**

Let $K$ be a convex and compact set with $C^1$ boundary. Then $K$ has at least four horizontal equilibria with respect to its centroid.

**Proof.** Suppose the boundary $\partial K$ is given in polar coordinates as $r : S^1 \to (0, \infty), \varphi \mapsto$
Figure 5: $K$ rolling along the horizontal line $\ell$. The dashed line is the trace of the red center of mass $O$. The solid red line emanating from $O$ corresponds to the angle $\varphi = 0$. Stable equilibrium on the left ($p$ has a strict local minimum), non-equilibrium in the middle ($p'(\varphi) \neq 0$), unstable equilibrium on the right ($p$ has a strict local maximum).

$r(\varphi)$, such that the origin is the centroid of $K$. The tangent in a point $z(\varphi) = r(\varphi)(\cos(\varphi), \sin(\varphi))^T \in \partial K$ is perpendicular to the line joining $z(\varphi)$ with the origin if and only if $r'(\varphi) = 0$. So we have to show that $r'$ has at least four zeros on $[0, 2\pi)$.

The condition that the centroid is at the origin leads upon integrating by parts to

$$\int_0^{2\pi} r^2(\varphi)r'(\varphi) \left( \frac{\sin(\varphi)}{-\cos(\varphi)} \right) d\varphi = 0. \quad (3.2)$$

This implies that (3.2) remains valid if $g(\varphi) = r^2(\varphi)r'(\varphi)$ is replaced by any translation $\varphi \mapsto g(\varphi - c)$, where $c \in \mathbb{R}$. We will now assume that there is no interval on which $r$ is constant, otherwise there is nothing to show. If $r'$ has only two zeros, then $r' > 0$ on an interval of length $l \in (0, \frac{\pi}{2}]$, or $r' < 0$ on an interval of length $l \in (0, \frac{\pi}{2}]$. We only discuss the first case (the second case is analogue). By a suitable translation we may assume that $r' > 0$ on $(a, \pi - a)$, where $0 \leq a < \pi/2$. By periodicity of $r^3$, we find

$$\int_0^{2\pi} g(\varphi) \, d\varphi = 0,$$

and therefore

$$\int_0^\pi g(\varphi) \, d\varphi > 0.$$
It follows that
\[ \int_0^\pi g(\varphi) \sin(\varphi) \, d\varphi > \sin(a) \int_0^\pi g(\varphi) \, d\varphi \geq 0. \] (3.3)

On the other hand
\[ \int_0^{2\pi} g(\varphi) \sin(\varphi) \, d\varphi > 0, \] (3.4)

and (3.3) and (3.4) contradict (3.2). Observe that the argument goes through if \( r' \) has a third zero either in \((a, \pi - a)\) or in \([0, 2\pi) \setminus (a, \pi - a)\) and hence we conclude that \( r' \) must have at least 4 zeros as claimed. \( \square \)

The previous proposition already appears in [1] and could also be obtained using the Sturm-Hurwitz Theorem (Theorem 5.16 in [9]).

The next theorem reveals a connection between the number of equilibrium points of \( K \) with respect to an arbitrary point \( O \) which is not a point of the evolute of \( \partial K \) and the winding number of the evolute of \( \partial K \) around \( O \).

**Theorem 3.5**

Let \( K \) be a strongly convex compact set with \( C^3 \)-boundary \( \partial K \), and \( O \) a point in the plane. Suppose that \( O \) is not a point of the evolute of \( \partial K \). Then the number \( n \) of horizontal equilibria of \( K \) with respect to \( O \) is given by

\[ n = 2 - 2m, \]

where \( 0 \geq m \in \mathbb{Z} \) is the winding number of the evolute of \( \partial K \) with respect to \( O \).

**Proof.** We consider the parametrisation \( z(\varphi) = p(\varphi)u(\varphi) + p'(\varphi)u'(\varphi) \) of \( \partial K \) with origin \( O \). The function \( p \) is of class \( C^3 \) by Lemma 2.1 and hence \( p' \) is of class \( C^2 \). The evolute \( e \) of \( \partial K \) is then given by \( e(\varphi) = p'(\varphi)u'(\varphi) - p''(\varphi)u(\varphi) \). In particular, since \( O \) is not a point on \( e \), \( p' \) can only have simple zeros and by periodicity of \( p \), the number \( n \) of zeros of \( p' \) is at least 2. Then according to Lemma 1.1 in [3], \( n \) and hence the number of horizontal equilibria of \( K \) is given by

\[ n = \frac{1}{\pi} \int_0^{2\pi} \frac{p''(\varphi)^2 - p'(\varphi)p'''(\varphi)}{p'(\varphi)^2 + p''(\varphi)^2} \, d\varphi. \] (3.5)

Hence \( n \) equals twice the winding number of the curve \( \varphi \mapsto (p''(\varphi), p'(\varphi)) \) with respect to \( O \). The evolute can be rewritten as follows:

\[ e(\varphi) = \begin{pmatrix} \cos(\varphi) - \sin(\varphi) \\ \sin(\varphi) \cos(\varphi) \end{pmatrix} \begin{pmatrix} -p''(\varphi) \\ p'(\varphi) \end{pmatrix} =: R(\varphi). \]
Since $R(\varphi)$ causes one counterclockwise rotation around the origin and since the winding number of $\varphi \mapsto (-p''(\varphi), p'(\varphi))$ equals $-\frac{n}{2}$, the winding number $m$ of the evolute is given by $m = 1 - \frac{n}{2}$. The claim follows immediately. \hfill \square

Remark. According to Section 3.1 the number of equilibria of a curve of constant width with respect to a point not on the evolute is 2 modulo 4.

The foregoing proof can be obtained by a direct computation which remains valid in a more general setting: Since $e$ is a piecewise $C^2$ immersion under the assumptions of Lemma 3.1, the winding number of $e$ with respect to $O$ is given (see Proposition 2.3 in [4]) by
\[
m = \frac{1}{2\pi} \int_0^{2\pi} \langle Je, e' \rangle \frac{1}{\|e\|^2} \, d\varphi = \frac{1}{2\pi} \int_0^{2\pi} \frac{p'(p' + p''')}{p'^2 + p''^2} \, d\varphi \tag{3.6}
\]
and the corresponding integrand is bounded. In the case of simple zeros of $p'$ as discussed in Theorem 3.5, the integrand is even continuous. Then it holds that
\[
-2m + 2 \overset{(3.6)}{=} \frac{1}{\pi} \int_0^{2\pi} \frac{-p'(p' + p''')}{p'^2 + p''^2} \, d\varphi + \frac{1}{\pi} \int_0^{2\pi} \frac{p'^2 + p''^2}{p'^2 + p''^2} \, d\varphi \tag{3.7}
\]
\[
= \frac{1}{\pi} \int_0^{2\pi} \frac{p'' - p'p'''}{p'^2 + p''^2} \, d\varphi = n. \tag{3.5}
\]
The last equality of this computation also holds true by Theorem 2.4 in [3] in a more general setting: In particular, the computation remains valid if $p'$ has zeros of order at most 2 and the relevant integrands are continuous by the following lemma:

Lemma 3.6
If $p \in C^k$, $k \geq 3$ and $p'$ only has zeros of order at most $k - 1$, then the integrands in (3.7) are continuous.

Proof. It suffices to show the continuity of the integrands in 0 provided $\varphi = 0$ is a zero
of \( p' \) of multiplicity \( k - 1 \). Using Proposition 2.5 in [3] we find by Taylor expansion

\[
\begin{align*}
p'(\varphi) &= \left( \frac{p^{(k)}(0)}{(k-1)!} + r_0(\varphi) \right) \varphi^{k-1}, \\
p''(\varphi) &= \left( \frac{p^{(k)}(0)}{(k-2)!} + r_1(\varphi) \right) \varphi^{k-2}, \\
p'''(\varphi) &= \left( \frac{p^{(k)}(0)}{(k-3)!} + r_2(\varphi) \right) \varphi^{k-3},
\end{align*}
\]

where \( r_i \) are continuous functions with \( \lim_{\varphi \to 0} r_i(\varphi) = 0 \). Then

\[
\lim_{\varphi \to 0} \frac{p''(\varphi)^2 - p'(\varphi)p'''(\varphi)}{p'(\varphi)^2 + p''(\varphi)^2} = \frac{1}{k-1}
\]

and

\[
\lim_{\varphi \to 0} \frac{p'(\varphi)(p'(\varphi) + p''(\varphi))}{p'(\varphi)^2 + p''(\varphi)^2} = \frac{k-2}{k-1}.
\]

In order to prove Theorem 1.1 it remains to discuss the cases where the center of mass \( O \) of \( K \) is possibly a point of the evolute. We continue to assume, as in Lemma 3.1, that the radius of curvature of \( \partial K \) has only finitely many stationary points and \( \partial K \) is of class \( C^3 \). We will distinguish two cases:

1. If \( O \) is a regular point of the evolute of \( \partial K \), then whenever \( e(\varphi_0) = 0 \), it holds that \( e'(\varphi_0) \neq 0 \). This corresponds to the two black points in Figure 6 which are labeled by 3 and 4. Since \( e' = -\rho' \) this means that \( \varphi_0 \) is not a stationary point of \( \rho \) and hence \( p''(\varphi) \neq 0 \). Therefore the set \( e^{-1}(0) \) consists of zeros of \( p' \) of multiplicity 2 and we conclude that \( p' \) has zeros of order at most 2.

In this case, computation (3.7) remains valid by Proposition 2.3 in [4] and Theorem 2.4 in [3] and the integrands are continuous according to Lemma 3.6. Proposition 2.2 in [4] tells us (since the angles in \( O \) are equal to \( \pi \)) that \( 2m \in \mathbb{Z} \) and we conclude that \( n = 2 - 2m \), but \( m \) might be half-integer valued.

2. If \( O \) is a singular point of the evolute of \( \partial K \), there exist values \( \varphi_0 \) such that \( e(\varphi_0) = e'(\varphi_0) = 0 \). See, e.g., the black point in Figure 6 which is labeled by 2. In this case, \( \varphi_0 \) is a stationary point of \( \rho \) which is either a saddle point or a cusp of \( e \).
according to Lemma 3.1. Since \( e = p' u' - p'' u \) and \( e' = -(p' + p'') u \) we conclude that such points are zeros of \( p' \) of order at least 3.

The computation (3.7) remains valid in this case if we can show that \( p' \) is an admissible function in the sense of Definition 2.4 in [3]. More precisely, the first equality is then justified by Proposition 2.3 in [4] and the last one by Theorem 2.4 in [3]. According to Proposition 2.2 in [4] (since the angles in \( O \) are 0, \( \pi \) or 2\( \pi \)) we find again \( 2m \in \mathbb{Z} \).

Since \( p' \in C^2 \), it suffices to show that the zeros of \( p' \) are admissible in the sense of Definition 2.1 in [3], i.e. we have to show that whenever \( p'(\varphi_0) = 0 \), then

\[
\lim_{\varphi \nearrow \varphi_0} \frac{p''(\varphi)}{p'(\varphi)} = -\infty \quad \text{and} \quad \lim_{\varphi \searrow \varphi_0} \frac{p''(\varphi)}{p'(\varphi)} = +\infty.
\]

Let \( \varphi_0 = 0 \) be a zero of \( p' \). If this zero is of multiplicity one or two, then the admissibility follows immediately from the 5th point in the Remark after Definition 2.1 in [3]. In the present case we assume that \( p'(0) = p''(0) = p'''(0) = 0 \). In this case, \( \rho(0) = \rho(0) \) and \( \rho'(0) = 0 \) and we can solve the ODE \( \rho = p + p'' \) with initial value \( p'(0) = 0 \) in order to obtain

\[
p(\varphi) = \int_0^\varphi \sin(\varphi - t) \rho(t) \, dt + p(0) \cos(\varphi) = \int_0^\varphi \sin(\varphi - t)(\rho(t) - \rho(0)) \, dt + \rho(0).
\]

Upon integrating by parts (since \( \rho \) is of class \( C^1 \)) we get the formulas

\[
p(\varphi) = \rho(\varphi) - \int_0^\varphi \cos(\varphi - t) \rho'(t) \, dt,
\]

\[
p'(\varphi) = \int_0^\varphi \sin(\varphi - t) \rho'(t) \, dt,
\]

\[
p''(\varphi) = \int_0^\varphi \cos(\varphi - t) \rho'(t) \, dt.
\]

Since the number of zeros of \( \rho' \) is finite, we can consider the case where e.g. \( \rho' > 0 \) on \((0, \varphi)\) provided \( \varphi > 0 \) is small enough. Then

\[
p''(\varphi) \geq \int_0^\varphi (1 - (\varphi - t)^2) \rho'(t) \, dt \geq (1 - \varphi^2) \int_0^\varphi \rho'(t) \, dt
\]

and

\[
p'(\varphi) \leq \varphi \int_0^\varphi \rho'(t) \, dt.
\]

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We conclude that

\[ \frac{p''(\varphi)}{p'(\varphi)} \geq \frac{1 - \varphi^2}{\varphi} \xrightarrow{\varphi \to 0} +\infty. \]

The remaining cases are similar and we find

\[ \lim_{\varphi \not\to 0} \frac{p''(\varphi)}{p'(\varphi)} = -\infty \quad \text{and} \quad \lim_{\varphi \to 0} \frac{p''(\varphi)}{p'(\varphi)} = +\infty \]

and therefore, the zeros of \( p' \) are admissible.

This concludes the proof of Theorem 1.1.

**Remarks.**

1. According to Section 3.1 the number of equilibria of a curve of constant width with respect to a point on the evolute is even.

2. It follows from the conclusion of Theorem 1.1, that the number of zeros of \( p' \) is finite. This also follows a priori from the fact that the number of extrema of \( \rho \) is finite. Indeed, if \( p'(\varphi_0) = 0 \), then the tangent of \( z \) in \( \varphi_0 \) is perpendicular to \( z(\varphi_0) \) and \( z(\varphi_0) \) is parallel to \( e'(\varphi_0) \). Since every arc of the evolute \( e \) is convex, there are at most 2 tangents to such an arc through \( z(\varphi_0) \). Since the curvature of \( \partial K \) has only finitely many stationary points, \( e \) is made of only finitely many arcs and there are only twice as many zeros of \( p' \) as there are extrema of \( \rho \).

3. For points on the evolute, one can formulate the result alternatively as follows: If the center of mass \( O \) lies on the evolute, then the number of equilibrium positions with respect to \( O \) is the average of the number of equilibrium positions in the neighbouring areas defined by the evolute, where each neighbouring area is weighted by its angle in \( O \). For example, the number of equilibrium positions in the black points in Figure 6 can be obtained in this way: The 3 is the average of 2 and 4, the 4 is the average of 4 and 4 (with equal weight) and 2 and 6 (with equal weight), and the 2 is the average of 2 (with full weight) and 4 (with weight zero).

4 **Oblique equilibria**

Here we investigate the equilibrium positions of \( K \) with respect to a center of mass \( O \) on an oblique line \( \ell \) with angle of inclination \( \alpha \neq 0 \). The situation is shown in Figure 7.
We can immediately read off the condition for an equilibrium position in terms of the support function $p$: An equilibrium point is characterised by the condition

$$p'(\varphi) = \tan(\alpha)p(\varphi),$$

(4.1)
or, if $p(\varphi) \neq 0$, equivalently by

$$\tan(\alpha) = \frac{p'(\varphi)}{p(\varphi)} = (\ln |p(\varphi)|)'.$$

In particular, the number $n_\alpha$ of solutions of (4.1) on $[0, 2\pi)$ corresponds to the number of equilibrium points. This number varies with $\alpha$: See Figure 8.

If we denote by $v = \begin{pmatrix} \cos(\alpha) \\ -\sin(\alpha) \end{pmatrix}$ the vector in the downhill direction of $\ell$ and by $s(\varphi)$ the arclength on $\partial K$ corresponding to the parameter interval $[0, \varphi]$ we can express the position of $O$ in coordinates with respect to fixed horizontal and vertical axis as

$$O(\varphi) = \begin{pmatrix} O_1(\varphi) \\ O_2(\varphi) \end{pmatrix} = (s(\varphi) - p'(\varphi))v + p(\varphi)v^\perp,$$
where \( v^\perp = \begin{pmatrix} \sin(\alpha) \\ \cos(\alpha) \end{pmatrix} \). An equilibrium corresponds to a point with stationary potential energy, i.e., \( O_2'(\varphi) = 0 \). A sufficient condition for an equilibrium to be stable is \( O_2''(\varphi) > 0 \), corresponding to a strict local minimum of the potential energy. Similarly, \( O_2''(\varphi) < 0 \) implies that an equilibrium is unstable. According to (2.3) we have

\[
O_2'(\varphi) = p'(\varphi) \cos(\alpha) - p(\varphi) \sin(\alpha), \quad \text{and} \quad O_2''(\varphi) = p''(\varphi) \cos(\alpha) - p'(\varphi) \sin(\alpha).
\]

Thus, for an equilibrium \( O_2'(\varphi) = 0 \), we obtain

- if \( p''(\varphi) > p(\varphi) \tan^2(\alpha) \), then \( \varphi \) is a stable equilibrium,
- if \( p''(\varphi) < p(\varphi) \tan^2(\alpha) \), \( \varphi \) is an unstable equilibrium.

In particular a center of mass \( O \) on \( \partial K \) is always a stable equilibrium.

Interesting observations are

**Proposition 4.1** 1. There are shapes \( K \) which have oblique equilibrium points with respect to the centroid for angle of inclination \( \alpha \), but no equilibrium for angle \(-\alpha\).
2. For all $\alpha \in (-\pi/2, \pi/2)$ there exist shapes $K$ which have stable equilibrium positions with respect to $\alpha$ for the centroid.

3. For all small $\alpha > 0$ there exist shapes $K$ which have only one metastable equilibrium, and no other equilibrium, with respect to $\alpha$ for the centroid.

4. For all small $\alpha > 0$ there exist shapes $K$ which have only one stable and one unstable equilibrium with respect to $\alpha$ for the centroid.

**Remark.** The last two properties are in sharp contrast to Proposition 3.4 for $\alpha = 0$.

**Proof.** Consider the support function

$$p(\varphi) = 3 - \frac{279}{8570} \sin(\varphi) + \frac{36}{857} \cos(\varphi) + \frac{4}{10} (\sin(2\varphi) + \cos(2\varphi)) + \frac{1}{5} \cos(3\varphi).$$

One can check, that $p + p'' > 0$ and that $\max(\ln p)' + \min(\ln p)' > 0$. Moreover, the centroid is at the origin. So, for $\alpha$ such that $-\min(\ln p)' < \tan(\alpha) < \max(\ln p)'$ the shape with this support function $p$ has the property mentioned in the first part of the proposition.

For the second part, observe that the ellipse with half axis $a > 1$ and $b = 1$ has two stable and two unstable equilibria with respect to its center for every angle $\alpha < \arctan(\frac{a^2-1}{2a})$.

For the rest, let

$$p_c(\varphi) = 3 + 3c(\cos(2\varphi) + \sin(2\varphi)) + 2c \cos(3\varphi) + \frac{36c^2}{9-43c^2} \cos(\varphi) - \frac{9(4-9c)c^2}{9-43c^2} \sin(\varphi).$$
Let $c > 0$ be sufficiently small, so that $z = p_c u + p'_c u'$ parametrizes the boundary of a convex body $K$. By construction, the centroid of $K$ lies at the origin. One can check that the function $p'_c(\varphi)/p_c(\varphi)$ has a unique maximum for each such $c$. Choose $\alpha_c$ in such a way that $\tan(\alpha_c) = \max p'_c/p_c$. Then $K$ has exactly one equilibrium for $\alpha_c$ and for a slightly smaller angle one stable and one unstable equilibrium. Since $p'_c/p_c$ converges uniformly to 0 for $c \searrow 0$, the claim follows.

In view of Theorem 1.1 it is natural to ask, if the number $n_\alpha$ of oblique equilibria with respect to angle $\alpha > 0$ can be obtained as $n_\alpha = 2 - 2m_\alpha$, where $m_\alpha$ is the winding number of the evolute of a suitable modification of $\partial K$. Consider therefore again a strongly convex and compact set $K$ with $C^3$ boundary and such that the radius of curvature of $\partial K$ has only finitely many stationary points. Let $z = pu + p'u'$ be the usual $C^2$ parametrization of $\partial K$ and let $e = p'u' - p''u$ be the evolute of $\partial K$. Define $e_\alpha = e - \tan(\alpha) Jz$ and $p'_\alpha = p' - \tan(\alpha) p$. Let $p_\alpha$ be a primitive of $p'_\alpha$ with constant of integration large enough such that $p_\alpha + p''_\alpha =: p_\alpha > 0$. In this case, $p_\alpha$ is again the support function of a curve $C_\alpha$ and the evolute of $C_\alpha$ is precisely $e_\alpha$.

**Proposition 4.2**

If the curvature of $C_\alpha$ admits only finitely many stationary points, then the number $n_\alpha$ of oblique equilibria of $\partial K$ with respect to $O \in \mathbb{R}^2$ and angle of inclination $\alpha$ is given by $n_\alpha = 2 - 2m_\alpha$, where $m_\alpha \in \frac{1}{2}\mathbb{Z}$ is the winding number of the evolute of $C_\alpha$ with respect to $O$.

**Proof.** Observe that $e_\alpha$ is a piecewise $C^2$ immersion, since $e$ is piecewise $C^2$, $z$ is of class $C^2$ and the number of zeros of $e'_\alpha$ is finite. In this case, the winding number $m_\alpha$ of $e_\alpha$ is given by

$$m_\alpha = \frac{1}{2\pi} \int_0^{2\pi} \frac{\langle Je_\alpha, e'_\alpha \rangle}{\|e_\alpha\|^2} \, d\varphi$$

and using $p'_\alpha = p' - \tan(\alpha) p$ we obtain

$$-2m_\alpha + 2 = \frac{1}{\pi} \int_0^{2\pi} \frac{p''_\alpha^2 - p'_\alpha p''_\alpha}{p'_\alpha^2 + p''_\alpha^2} \, d\varphi = n_\alpha,$$

in analogous manner to the case where $\alpha = 0$.

**Remark.** It is clear by definition that $e_\alpha$ diverges as $\alpha \to \pm \frac{\pi}{2}$, however, the renormalized perturbed evolute $e_\alpha/\tan(\alpha)$ converges to $-Jz$ as $\alpha \to \pm \frac{\pi}{2}$. Moreover $m_\alpha \to 1$ as $\alpha \to \pm \frac{\pi}{2}$ so that $n_{\pm \frac{\pi}{2}} = 0$, as expected.
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