Paradoxical characterization of Lebesgue nonmeasurable sets

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ABSTRACT

We show that a set $A \subseteq \mathbb{R}^n$ is nonmeasurable in the sense of Lebesgue if and only if it has a common density point with its complement $A'$. Moreover, if there exists a density point of both $A$ and $A'$, then the set of such points has a positive Lebesgue measure.

1. Introduction

Notions of the Lebesgue measure, Lebesgue measurability and integral are of key importance for contemporary mathematics and constitute an indispensable part of virtually any graduate course in real analysis (see, e.g., [1, 2, 3]). One of fundamental results in this area is the celebrated Lebesgue Differentiation Theorem (see, e.g., [3], Theorem (7.2)), implying that almost every point of a Lebesgue measurable set in the $n$-dimensional Euclidean space $\mathbb{R}^n$ is a point of density of this set (see, e.g., [3], Theorem (7.13)). The latter result found numerous applications and inspired a considerable body of increasingly sophisticated research in various areas, like measure theory, analysis, topology, logic, theory of algorithms and computability. The interested reader is referred to [4, 5, 6, 7, 8, 9, 10, 11] and the references given there for just a few samples of classic and more recent activity in this area.

The aim of this note is to provide a characterization of Lebesgue nonmeasurability of sets in terms of their density points. Our main result, Theorem 3.2, states that a set $A \subseteq \mathbb{R}^n$ is nonmeasurable in the sense of Lebesgue if and only if it has a common density point with its complement $A'$. Moreover, it turns out that if there exists a density point of both $A$ and $A'$, then the set of such points actually has a positive Lebesgue measure. In fact, the latter quantity indicates how big the “fully nonmeasurable” part of $A$ is, because it is equal to the Lebesgue measure of the difference between the measurable superset of $A$ with the smallest measure and its measurable subset with the largest measure.

In our opinion, these findings are rather counterintuitive. Indeed, it is not easy to visualize a set “almost filling” a small ball and such that the “holes” in this set also “almost fill” it! They also show that measurability in the sense of Lebesgue is essentially a local property. Finally, they give insight into the extent to which nonmeasurable sets are “irregular”, or even somewhat “pathological”. Note that known constructions of Lebesgue nonmeasurable sets are based on the axiom of choice, so our results may be regarded as additions to a list of paradoxical consequences of this axiom.

This paper is organized as follows. In Section 2, we provide an introductory discussion on minimal measurable supersets and maximal measurable subsets of a general set in $\mathbb{R}^n$. Section 3 contains our main results. In the concluding Section 4, we briefly discuss some consequences of our findings.

1.1. Notation

The following notation will be used throughout. Let $\mathbb{N}$ denote the set of positive integers and let $\mathbb{R}$ denote the set of real numbers. For $n \in \mathbb{N}$, let $\mathbb{R}^n$ denote the $n$-dimensional Euclidean space. For a set $A \subseteq \mathbb{R}^n$, let $A'$ denote the complement of $C$, i.e., $A' = \mathbb{R}^n \setminus A$. For $A, B \subseteq \mathbb{R}^n$, $A \triangle B$ denotes the symmetric difference of $A$ and $B$, i.e., $A \triangle B = (A \setminus B) \cup (B \setminus A)$. For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x, r)$ denote the ball of radius $r$ centered at $x$. The $\sigma$-field of Lebesgue measurable subsets of $\mathbb{R}^n$ will be denoted by $\mathcal{M}$. For $A \in \mathcal{M}$, let $\mu(A)$ denote the $(n$-dimensional) Lebesgue measure of $A$. Also, for any $A \subseteq \mathbb{R}^n$, let $\mu'(A)$ denote the $(n$-dimensional) Lebesgue outer measure of $A$.

2. Measurable hulls and kernels

Recall that a set $H \subseteq \mathbb{R}^n$ is of type $G_\delta$, if it is an intersection of a countable collection of open sets in $\mathbb{R}^n$. $G_\delta$ sets are clearly Borel, and hence Lebesgue measurable. The following result will be our starting point.

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Theorem 2.1 ([3], Theorem (3.32)). Let $A \subseteq \mathbb{R}^n$. There exists a set $H$ of type $G_δ$ such that $A \subseteq H$ and
\[ \mu^*(A \cap H) = \mu(H \cap M), \quad M \in \mathcal{M}. \] (1)

In what follows, for a given $A \subseteq \mathbb{R}^n$, a superset $H \in \mathcal{M}$ of $A$ satisfying (1) will be called a measurable hull of $A$. In particular, Theorem 2.1 assures the existence of a $(G_δ)$ measurable hull of any set.

It is easy to see that if $M_1$, $M_2$ are measurable hulls of $A$, then $\mu(M_1 \Delta M_2) = 0$. Indeed putting $M = M_1 \cap M_2$, respectively, substituted for $H$, we get $0 = \mu(\emptyset) = \mu(M_2 \cap M) = \mu^*(A \cap M) = \mu(M_1 \cap M) = \mu(M)$, and, similarly, $\mu(M_2 \setminus M) = 0$. Slightly abusing the notation, we will denote any measurable hull of $A$ by $A^\ast$. Note that null sets (i.e., sets with zero Lebesgue outer measure) are Lebesgue measurable, and hence $\mu(A^\ast \setminus A) = 0$ if and only if $A \in \mathcal{M}$. 

Proposition 2.2. For any $A \subseteq \mathbb{R}^n$, we have $A \in \mathcal{M}$ if and only if 
\[ \mu(A^\ast \cap (A)^m) = 0. \] (2)

Proof. If $A \in \mathcal{M}$, we can take $A = A^\ast$, $(A)^m = A^\ast$, and (2) follows.

Conversely, assume (2). It is well known that $A \in \mathcal{M}$ if and only if $A$ satisfies the Caratheodory condition
\[ \mu^*(E \cap A) + \mu^*(E \setminus A) = \mu^*(E), \quad E \subseteq \mathbb{R}^n. \] (3)

see, e.g., [3], Theorem (3.30). Let $E \subseteq \mathbb{R}^n$. By the definition of the measurable hull, we have 
\[ \mu^*(E \cap A) + \mu^*(E \setminus A) \leq \mu^*(E \cap A^\ast) + \mu^*(E \cap (A)^m) \]
\[ = \mu(E \cap A^\ast) + \mu(E \cap (A)^m) \]
\[ = \mu(E \cap A^\ast) + \mu((E^\ast \cap A^m) \cup (E \cap (A)^m)) \]
\[ = \mu(E^\ast) \cap (E \cap (A)^m)) \]
\[ \leq \mu(E^\ast) + \mu(A^\ast \cap (A)^m) \]
\[ = \mu(E^\ast) = \mu^*(E). \]

where (2) was used in the last line. We have shown the inequality 
\[ \mu^*(E \cap A) + \mu^*(E \setminus A) \leq \mu^*(E). \]

This, together with subadditivity of $\mu^*$, implies (3), and hence $A \in \mathcal{M}$. □

The following notion of a measurable kernel of a set is, in some sense, dual to the notion of its measurable hull. Loosely speaking, the measurable hull of a given set is its smallest measurable superset, while its measurable kernel is its greatest measurable subset.

Definition 2.3. Let $A \subseteq \mathbb{R}^n$ and let $\mathcal{M}_A$ be the family of Lebesgue measurable subsets of $A$. A set $K \in \mathcal{M}_A$ such that
\[ \mu(K \cap M) = \sup_{E \in \mathcal{M}_A} \mu(E \cap M), \quad M \in \mathcal{M}, \] (4)

is called a measurable kernel of $A$.

Note that $\emptyset \in \mathcal{M}_A$ for each $A$, so $\mathcal{M}_A \neq \emptyset$ and hence the maximization problem (4) makes sense. It is easy to see that in the verification of (4), we may restrict ourselves to sets $M \in \mathcal{M}$ satisfying $\mu(M) < \infty$. Indeed, if (4) holds with $M$ replaced by $M \cap B(0,r)$ for each $r > 0$, then it holds also for $M$, since $\mu(E \cap M \cap B(0,r)) \uparrow \mu(E \cap M)$ as $r \to \infty$ for each $E \in \mathcal{M}_A$.

Proceeding similarly as in the case of the measurable hull above, it is easy to check that a measurable kernel of $A$ (if it exists) is unique up to a null set, i.e., if both $K_1$ and $K_2$ are measurable kernels of $A$, then $\mu(K_1 \setminus K_2) = 0$. Accordingly, slightly abusing the notation, we will denote any measurable kernel of $A$ by $A_m$. Clearly, $A = A_m$ if and only if $A \in \mathcal{M}$. The following result assures the existence of $A_m$ for a general $A$ and characterizes $A_m$ in terms of measurable hulls.

Lemma 2.4. For any $A \subseteq \mathbb{R}^n$, we have 
\[ A_m = ((A)^m)^\ast. \] (5)

Proof. If $M \in \mathcal{M}$ and $\mu(M) < \infty$, then maximizing $\mu(E \cap M)$ over $E \in \mathcal{M}_A$ is equivalent to minimizing $\mu(F \cap M)$ over $F \in \mathcal{M}_A$ such that $A^\ast \subseteq F$ (take $F = E'$). However, the latter minimum equals $\mu^*(A^\ast \cap M)$, so, by the definition of a measurable hull, it is attained at $(A^\ast)^m$. Consequently, the maximizer of (4) is (5). □

By Theorem 2.1 and Lemma 2.4, for any $A \subseteq \mathbb{R}^n$, we can choose $A_m$ which is a set of type $F_\gamma$ (i.e., it is a countable union of closed sets).

Also, Proposition 2.2 and Lemma 2.4 immediately yield the following simple corollary (which also follows easily from completeness of the Lebesgue measure).

Corollary 2.5. For any $A \subseteq \mathbb{R}^n$, $A \in \mathcal{M}$ if and only if $\mu(A^\ast \setminus A_m) = 0$.

Theorem 2.1, together with Corollary 2.5 and the discussion preceding it, immediately yields a well known characterization of any Lebesgue measurable set as a difference (resp., union) of a $G_δ$ (resp., $F_\gamma$) set and a null set, see, e.g., [3], Theorem (3.28).

3. Main results

Definition 3.1 (see [12], p. 251). We say that $x \in \mathbb{R}^n$ is a point of (exterior) density of a set $A \subseteq \mathbb{R}^n$, if
\[ \lim_{r \to 0} \frac{\mu^*(A \cap B(x,r))}{\mu(B(x,r))} = 1. \]

In the above definition, balls may be replaced by $n$-dimensional cubes centered at $x$ or, more generally, by any family of measurable sets shrinking regularly to $x$ in the sense defined in Section 7.2 of [3].

The set of all points of density of a set $A$ will be denoted by $D(A)$. By definition,
\[ D(A) = D(A^\ast), \quad A \subseteq \mathbb{R}^n \] (6)

(compare the discussion in Subsection 3.1.1 on p. 251 of [12]). The celebrated Lebesgue Differentiation Theorem implies that for $A \in \mathcal{M}$, $\mu(A \Delta D(A)) = 0$ (see, e.g., Theorems (7.2) and (7.13) of [3] or Proposition 1 on p. 12 of [12]). This result, together with (6), implies that for every (not necessarily measurable) $A \subseteq \mathbb{R}^n$, we have 
\[ \mu((A^\ast)^m \Delta D(A)) = 0. \] (7)

and hence, by completeness of the Lebesgue measure, $D(A) \in \mathcal{M}$.

Using (7) with $A^\ast$ instead of $A$, we get 
\[ \mu((A^\ast)^m \Delta D(A^\ast)) = \mu((A^\ast)^m \Delta D(A^\ast)) = 0. \]

Thus, by Lemma 2.4, we get the following counterpart of (7):
\[ \mu(A_m \Delta (D(A^\ast))^\ast) = 0. \] (8)

The following theorem is the main result of this paper.

Theorem 3.2. For $A \subseteq \mathbb{R}^n$, we have $A \in \mathcal{M}$ if and only if 
\[ D(A) \cap D(A^\ast) = \emptyset. \] (9)
Proof. If \( A \in \mathcal{M} \) and \( x \in D(A) \), then for any \( r > 0 \), we have
\[
\mu(A \cap B(x, r)) + \mu(A' \cap B(x, r)) = \mu(B(x, r)),
\]
and hence \( \lim_{r \to 0} \mu(A) = \mu(B(x, r)) = 0 \), so \( x \notin D(A') \) and (9) follows.

Conversely, assume that for a given \( A \subseteq \mathbb{R}^n \), (9) holds. Thus, by (6), we have \( D(A') \cap D(A') = \emptyset \) and hence, by the Lebesgue differentiation theorem, (2) holds. Consequently, \( A \in \mathcal{M} \) by Proposition 2.2. \( \square \)

The above argument can be easily refined to yield the following, apparently stronger, result.

**Corollary 3.3.** Let \( A \subseteq \mathbb{R}^n \). The following conditions are equivalent.

(i) \( \lim_{r \to 0} \frac{\mu(A \cap B(x, r))}{\mu(B(x, r))} = 1 \) for some \( x \in \mathbb{R}^n \),

(ii) \( A \notin \mathcal{M} \),

(iii) \( \mu(A) \cap \mu(A') = 0 \),

(iv) \( \mu(D(A) \cap D(A')) > 0 \).

**Proof.** If \( A \in \mathcal{M} \), then (10) holds for any \( x \in \mathbb{R}^n \), \( r > 0 \) and thus
\[
1 = \lim_{r \to 0} \frac{\mu(A \cap B(x, r))}{\mu(B(x, r))} + \lim_{r \to 0} \frac{\mu(A' \cap B(x, r))}{\mu(B(x, r))}.
\]

Clearly, (i) implies (ii).

The equivalence of (ii) and (iii) is a reformulation of Theorem 3.2. Moreover, by Proposition 2.2, (ii) implies that \( \mu(A') \cap (A')^c = 0 \), from which (iv) follows by (7). Finally, if (iv) holds, then clearly for any \( x \in D(A) \cap D(A') \), we have (i). \( \square \)

The following proposition further elaborates on the condition (iv) of Corollary 3.3 as an equivalent condition for nonmeasurability of a set.

**Proposition 3.4.** Let \( A \subseteq \mathbb{R}^n \) be such that \( \mu^*(A) < \infty \). Then
\[
\mu^*(A) - \sup_{E \in \mathcal{M}_A} \mu(E) = \mu(A') - \mu(A) = \mu(D(A) \cap D(A')).
\]

**Proof.** The first equality in (11) follows from the definitions of \( A^m \) and \( A_n \), (put \( M = \mathbb{R}^n \) into (1) and (4), respectively). For the proof of the second one, let us observe that by Lemma 2.4 and (7), we get
\[
\mu(A') - \mu(A_n) = \mu(A \setminus A_n) = \mu(A \cap ((A')^c)^c) = \mu(A \cap (A'^c)^c) = \mu(D(A) \cap D(A')). \square
\]

Corollary 2.5 and Proposition 3.4 provide an alternative proof of equivalence of conditions (ii) and (iv) in Corollary 3.3. Furthermore, these results suggest that for a set \( A \subseteq \mathbb{R}^n \), the quantity \( \mu(D(A) \cap D(A')) \) may be regarded as a “measure of Lebesgue nonmeasurability of \( A \), somewhat analogous to commonly used measures of noncompactness (see [13]).

4. Discussion and conclusion

One somewhat striking consequence of Theorem 3.2 is that Lebesgue nonmeasurability of a set is, in some sense, a local property, because it may be characterized by the “behaviour” or “shapes” of truncations of this set to arbitrarily small balls centered around a single point. On the other hand, Corollary 3.3 assures that if there is a single “peculiar” point, at which both the set under consideration and its complement are dense, then the family of such points has a positive Lebesgue measure, and hence it is uncountable.

In the light of the Carathéodory condition (3) for measurability, one may wonder how “bad” nonmeasurable sets are in this regard. This question may be formalized as a search for the smallest possible constant \( c \) such that
\[
\mu^*(E \cap A) + \mu^*(E \setminus A) \leq c \mu^*(E), \quad A, E \subseteq \mathbb{R}^n.
\]

Clearly, \( c = 2 \) satisfies (12), due to monotonicity of the outer measure. A somewhat surprising consequence of Theorem 3.2 is that it is actually the smallest constant satisfying this condition. Indeed, if \( A \notin \mathcal{M} \) and \( x \in D(A) \cap D(A') \), then for every \( c > 0 \) and \( r > 0 \) small enough, we have
\[
\mu^*(B(x, r) \cap A) + \mu^*(B(x, r) \setminus A) \geq (2 - c) \mu(B(x, r)).
\]