ON WARPED PRODUCT GRADIENT YAMABE SOLITONS

TOKURA, W. 1, ADRIANO, L. 2, PINA, R. 3, AND BARBOZA, M. 4

Abstract. The purpose of this article is to study gradient Yamabe soliton on warped product manifolds. First, we prove triviality results in the case of noncompact base with limited warping function, and for compact base. In order to provide nontrivial examples, we consider the base conformal to a semi-Euclidean space, which is invariant under the action of a translation group, and then we characterize steady solitons. We use this method to give infinitely many explicit examples of complete steady gradient Yamabe solitons.

1. Introduction and main results

A Yamabe soliton is a semi-Riemannian manifold \((M, g)\) admitting a smooth vector field \(X \in \mathfrak{X}(M)\) such that
\[
(S_g - \rho)g = \frac{1}{2} \mathcal{L}_X g,
\]
where \(S_g\) denotes the scalar curvature of \(M\), \(\rho\) is a real number and \(\mathcal{L}_X g\) denotes the Lie derivative of \(g\) with respect to \(X\). We say that \((M, g)\) is shrinking, steady or expanding, if \(\rho > 0\), \(\rho = 0\), \(\rho < 0\), respectively. When \(X = \nabla h\) for some smooth function \(h \in C^\infty(M)\), we say that \((M, g, \nabla h)\) is a gradient Yamabe soliton with potential function \(h\). In this case the equation (1) turns out
\[
(S_g - \rho)g = \text{Hess}_g(h),
\]
where \(\text{Hess}_g(h)\) denote the Hessian of \(h\). When \(h\) is constant, we call it a trivial Yamabe soliton.

Adding the condition of constant \(\rho\) in definition (1) to be a differentiable function on \(M\) we obtain the extension of Yamabe solitons called Almost Yamabe soliton. In particular, for gradient vector field, we call it a Almost gradient Yamabe soliton [25].

After their introduction in the Riemannian sense, the study of semi-Riemannian Yamabe solitons attracted a growing number of authors, showing many differences with respect to the Riemannian case. Calviño et. al. in [4] study Yamabe solitons and left-invariant Yamabe soliton on three-dimensional homogeneous space and showed that the class of Yamabe solitons is strictly greater than the class of left-invariant Yamabe solitons. On the other hand, Neto et. al. in [14] showed that the class of semi-Riemannian Yamabe solitons is strictly greater than the Riemannian Yamabe soliton. Moreover, infinitely many proper semi-Riemannian gradient Yamabe solitons are exhibit.

In recent works on solitons and equations, the notion of warped product introduced by Bishop and O’neil in [5] has attracted major research activities [3, 10, 11, 12, 13].

Definition 1.1. (25) Let \((B^n, g_B)\) and \((F^d, g_F)\) be two semi-Riemannian manifolds, as well as a positive smooth function \(f\) on \(B\). On the product manifold \(B \times F\), we define the...
metric
\[ g = \pi^* g_B + (f \circ \pi)^2 \sigma^* g_F, \]
where \( \pi : B \times F \to B \), \( \sigma : B \times F \to F \) are the projections on the first and second factor, respectively. The product space \( B \times F \) furnished with metric tensor \( g \) is called warped product. We denote it by \( B \times_f F \). The function \( f \) is called warping function, \( B \) is called the base and \( F \) the fiber.

Brozos-Vázquez et al. in [3] provide a special warped product structure for gradient Yamabe solitons, its results establish that a gradient Yamabe soliton \((M, g)\) with potential function \( h \) and such that \(|\nabla h| \neq 0\), is locally isometric to a warped product of unidimensional base and constant scalar curvature fiber. In the Riemannian context a global structure result was given in [26].

Sousa and Pina investigated gradient Ricci soliton on warped product and proved that the potential function only depends on base or the warping function is constant (see [10]). The same technique can be used to prove this result for gradient Yamabe solitons.

These results make it interesting for further investigation of gradient Yamabe solitons with warped product structure \( B \times_f F \) where the potential function only depends on \( B \), and \( F \) with constant scalar curvature.

Notation 1.2. Throughout this paper, we will consider the following:
\[ \mathcal{M}^{n+d} = (B^n \times F^d, \bar{g}), \quad S_F = \lambda_F = \text{constant}, \quad \bar{h} = h \circ \pi, \quad h \in C^\infty(B). \]
where \( S_F \) is the scalar curvature of \( F \), \( \bar{h} \) is the potential function of \( \mathcal{M}^{n+d} \) and \( \bar{g} \) is given by (3).

We start by focusing our attention on compact base gradient Yamabe soliton \( \mathcal{M}^{n+d} \). It has been known that every compact Riemannian gradient Yamabe soliton is of constant scalar curvature, hence, trivial since \( h \) is harmonic, see [27], [28]. In this direction, we have the following theorem.

Theorem 1.3. Let \( \mathcal{M}^{n+d} \) be a gradient Yamabe soliton with compact Riemannian base. Then \( \mathcal{M}^{n+d} \) is trivial.

By the above theorem, in order to obtain examples of nontrivial gradient Yamabe solitons, we must relax the hypothesis of compactness of the base. Since the continuous images of compact spaces are compact, that is, limited functions, it then follows that a condition weaker than compactness is considered to be limited. Then we consider the following question.

Question: Does there exist a gradient Yamabe soliton \( \mathcal{M}^{n+d} \) with nonconstant limited warping function?

In the sequel, we give a negative partial answer as follows:

Theorem 1.4. Let \( \mathcal{M}^{n+d} \) be a gradient Yamabe soliton with soliton constant \( \rho \) and Riemannian base with scalar curvature \( S_B \geq \rho - \frac{\lambda^2}{2} \). If \( f \) reaches the maximum, then \( \mathcal{M}^{n+d} \) must be a standard semi-Riemannian product.

Next, it is interesting to know under which conditions an gradient Yamabe soliton \( \mathcal{M}^{n+d} \) has non compact base. In this case, we obtain the following

Proposition 1.5. Let \( \mathcal{M}^{n+d} \) be a gradient Yamabe soliton with potential function \( \bar{h} \) and complete Riemannian base \((B^n, g_B)\). If \( \langle \nabla \log f, \nabla h \rangle = \text{constant} \neq 0 \), then \((B^n, g_B)\) is isometric to the standard Euclidean space \((\mathbb{R}^n, g_0)\).
Recently, the conformal semi-Euclidean space has become an interesting space to display examples of steady gradient Yamabe solitons and steady gradient Ricci solitons. Neto and Tenenblat in [14] treated the conformally flat semi-Riemannian space \((\mathbb{R}^n, \frac{1}{f} g_0)\), where \(g_0\) is the canonical semi-Riemannian metric, and obtain a necessary and sufficient condition to this manifold be a gradient Yamabe soliton. In order to exhibit solutions they consider the invariant action of an \((n-1)\)-dimensional translation group, and as a result geodesically complete (see definition 2.2) example was obtained. The same technique is used to obtain all invariant solutions of steady gradient Ricci soliton [15].

The warped product has proved its efficiency in constructing new examples of manifolds with certain geometric characteristics [21] [20] [19]. Considering invariant solutions, Neto in [11] provide an explicit example of a complete static vacuum Einstein space-time. On the other hand, in the same invariant solution context Sousa in [10] provide examples of non conformally flat Ricci solitons.

In the remainder of this article, we focus our attention on the warped product \(\overline{M}^{n+d}\) where the base is conformal to an \(n\)-dimensional semi-Euclidean space, invariant under the action on an \((n-1)\)-dimensional translation group. As application, we will construct 5 examples of steady gradient yamabe solitons. Besides, we provide a way to construct infinitely many explicit examples of geodesically complete steady gradient Yamabe solitons, with base conformal to the Lorentzian space (see example 1.18).

More precisely, consider the semi-Riemannian metric

\[
\delta = \sum_{i=1}^{n} \varepsilon_i dx_i \otimes dx_i
\]

in coordinates \(x = (x_1, \ldots, x_n)\) of \(\mathbb{R}^n\), where \(n \geq 3\), \(\varepsilon_i = \pm 1\). For an arbitrary choice of non zero vector \(\alpha = (\alpha_1, \ldots, \alpha_n)\) we define the function \(\xi : \mathbb{R}^n \to \mathbb{R}\) by

\[
\xi(x_1, \ldots, x_n) = \alpha_1 x_1 + \cdots + \alpha_n x_n.
\]

Next, we consider that \(\mathbb{R}^n\) admits a group of symmetries consisting of translations [12] and we then look for positive smooth functions \(\varphi, f, h : (a, b) \subset \mathbb{R} \to (0, \infty)\) such that \(f = f \circ \xi, \varphi = \varphi \circ \xi, h = h \circ \xi : M = \xi^{-1}(a, b) \to \mathbb{R}\), satisfies [2] with \(M = \mathbb{R}^n \times F^d\) and metric tensor

\[
g = \frac{\delta}{\varphi(x)^2} + f(x)^2 g_F.
\]

**Theorem 1.6.** With \((\mathbb{R}^n, \delta)\) and \(f = f \circ \xi, \varphi = \varphi \circ \xi, h = h \circ \xi\) as above, the manifold \(M = \mathbb{R}^n \times F^d\), furnished with the metric tensor

\[
g = \frac{\delta}{\varphi(x)^2} + f(x)^2 g_F
\]

is a gradient Yamabe soliton if, and only if,

\[
h'' + 2 \frac{\varphi' h'}{\varphi} = 0,
\]

\[
||\alpha||^2 (n-1)(2\varphi'' - n(\varphi')^2) - 2 \frac{d}{f} (\varphi f'' - (n-2)\varphi' f') - \frac{d(d-1)}{f^2} \varphi^2 (f')^2 + \varphi' h' \varphi = \rho - \frac{\lambda_F}{f^2},
\]

\[
||\alpha||^2 (n-1)(2\varphi'' - n(\varphi')^2) - 2 \frac{d}{f} (\varphi f'' - (n-2)\varphi' f') - \frac{d(d-1)}{f^2} \varphi^2 (f')^2 + \varphi^2 \frac{f' h'}{f} = \rho - \frac{\lambda_F}{f^2},
\]
when $||\alpha||^2 \neq 0$, that is, $\alpha$ is a timelike or spacelike vector.

And

\[
(8) \quad h'' + 2 \frac{\varphi'h'}{\varphi} = 0,
\]

\[
(9) \quad \rho - \frac{\lambda F}{f^2} = 0,
\]

when $||\alpha||^2 = 0$, that is, $\alpha$ is a lightlike vector.

**Corollary 1.7.** If $||\alpha||^2 = 0$ and $\lambda_F \neq 0$, then the warped product $\mathbb{R}^n \times f F^d$ become a standard semi-Riemannian product.

In this case, we have the following obstruction on the constant soliton $\rho$

**Corollary 1.8.** If $||\alpha||^2 = 0$ and $\lambda_F > 0$, then there is no expanding or steady gradient Yamabe soliton with product metric [4]. Similarly, if we assume that $||\alpha||^2 = 0$ and $\lambda_F < 0$, then there is no shrinking or steady gradient Yamabe soliton with product metric [4].

Now, by equations (5) and (8) in Theorem 1.6 we easily see that a necessary condition for the above manifold be a gradient Yamabe soliton, is that $h$ is a monotone function. That is,

\[
h' (\xi) = \frac{k_1}{\varphi^2 (\xi)},
\]

for some $k_1 \in \mathbb{R}$.

We provide steady solutions for ODE in Theorem 1.6 in the cases: $h' = 0$ and $h' \neq 0$ with $n + d = 6$, or $||\alpha||^2 = 0$.

**Theorem 1.9.** If $||\alpha||^2 \neq 0$ and $\lambda_F \neq 0$, then the warped product metric $g$ given by [4] with $n + d = 6$ is a steady gradient Yamabe soliton with potential function $h$ with $h' \neq 0$ if, and only if, the functions $f$, $h$ and $\varphi$, satisfies

\[
(10) \quad f (\xi) = \frac{k_2}{\varphi (\xi)},
\]

\[
(11) \quad h (\xi) = k_1 \int \frac{1}{\varphi^2 (\xi)} d\xi,
\]

\[
(12) \quad \int \frac{pd\varphi}{q\varphi^3 W (k_3 e^{\frac{p}{q}\varphi^2} - 1) + q\varphi^3} = \xi + k_4,
\]

where $k_1 \neq 0$, $k_2 \neq 0$, $k_3 \neq 0$, $k_4$ are constants, $p = \frac{k_1}{10}$, $q = \frac{\lambda F}{10k_2^2 ||\alpha||^2}$ and $W$ is the product log function.

**Theorem 1.10.** If $||\alpha||^2 \neq 0$ and $\lambda_F = 0$, then the warped product metric $g$ given by [4] with $n + d = 6$ is a steady gradient Yamabe soliton with potential function $h$ with $h' \neq 0$ if, and only if, the functions $f$, $h$ and $\varphi$, satisfies

\[
(13) \quad f (\xi) = \frac{k_2}{\varphi (\xi)},
\]

\[
(14) \quad h (\xi) = k_1 \int \frac{1}{\varphi^2 (\xi)} d\xi,
\]
(15) \[ 40 \int \frac{\varphi d\varphi}{k_1 - 20k_3\varphi^4} = \xi + k_4, \]

where \( k_1 \neq 0, k_2 \neq 0, k_3 \) and \( k_4 \) are constants.

**Theorem 1.11.** If \( ||\alpha||^2 \neq 0 \) and \( \lambda_F = 0 \), then given a smooth function \( \varphi > 0 \) the warped product metric \( g \) given by (14) is a steady gradient Yamabe soliton with potential function \( h \) with \( h' = 0 \) if, and only if, the functions \( h \) and \( f \), satisfies

\[ h(\xi) = \text{constant}, \]

(16) \[ f(\xi) = \varphi^{\frac{2}{n+2}}(\xi)e^{\Phi(\xi)} \left( \int e^{-(d+1)\Phi(\xi)} d\xi + \frac{2}{d+1} C \right)^{\frac{n}{n+2}}, \]

where \( \Phi(\xi) = \int z_p(\xi)d\xi \) and \( z_p \) is a particular solution of

(17) \[ z^2 + \frac{2}{d+1} z' + \left( \frac{n+d-1}{d(d+1)^2} \left( n \left( \frac{\varphi'}{\varphi} \right)^2 - 2\frac{\varphi''}{\varphi} \right) \right) = 0. \]

In the null case \( ||\alpha||^2 = 0 \) we obtain

**Theorem 1.12.** If \( ||\alpha||^2 = 0 \) and \( \lambda_F = 0 \), then given two smooth functions \( \varphi(\xi) \) and \( f(\xi) \), the warped product metric \( g \) given by (14) is a steady gradient Yamabe soliton with potential function \( h \) if, and only if

\[ h(\xi) = k_1 \int \frac{1}{\varphi^2(\xi)} d\xi. \]

**Remark 1.13.** As we can see in the proof of Theorem 1.6, if \( \rho \) is a function defined only on the base, then we can easily extend Theorem 1.6 into context of almost gradient Yamabe solitons. In the particular case of lightlike vectors there are infinitely many solutions, that is, given \( \varphi \) and \( f \)

\[ \rho(\xi) = \frac{\lambda_F}{f(\xi)^2}, \]

\[ h(\xi) = k_1 \int \frac{1}{\varphi^2(\xi)} d\xi, \]

provide a family of almost gradient Yamabe soliton with warped product structure.

Before proving our main results, we present some examples illustrating the above theorems.

**Example 1.14.** In Theorem 1.9, consider \( \mathbb{M}^6 = \mathbb{R}^3 \times \mathbb{H}^3 \), \( k_1 = 1, k_2 = 1, k_3 = 0, k_4 = 0 \), then we get

\[ f(\xi) = \frac{1}{\varphi(\xi)}, \quad h(\xi) = \int \frac{1}{\varphi^2(\xi)} d\xi, \quad \int \frac{d\varphi}{\varphi^3 W(e^{-\frac{1}{10\varphi^2}} - 1)} + \varphi^3 = -10\xi. \]

The family of solutions of \( \varphi \) is described in the follow phase portrait

![Figure 1. Sampling \( \varphi(0) \) and \( \varphi'(0) \)](image-url)
Example 1.15. In Theorem 1.10 consider \( k_1 = 1, k_2 = 1, k_3 = 0, k_4 = 0 \), then the functions
\[
f(\xi) = \sqrt{\frac{20}{\xi}}, \quad h(\xi) = 20 \ln \xi, \quad \varphi(\xi) = \sqrt{\frac{\xi}{20}},
\]
provide a steady gradient Yamabe soliton defined in the semi-space \( \xi > 0 \) of Euclidean space \( \mathbb{R}^{n-d}, d = 1, 2, 3 \).

Example 1.16. In Theorem 1.10 consider \( k_1 = 1, k_2 = 1, k_3 = -\frac{1}{2}, k_4 = 0 \), then the functions
\[
f(\xi) = \frac{1}{\sqrt{\tan(\frac{\xi}{20})}}, \quad h(\xi) = 20 \ln \left[ \sin \left( \frac{\xi}{20} \right) \right], \quad \varphi(\xi) = \sqrt{\tan(\frac{\xi}{20})},
\]
provide a steady gradient Yamabe soliton defined in the slice \( 0 < \xi < 10\pi \) of Euclidean space \( \mathbb{R}^{n-d}, d = 1, 2, 3 \).

Example 1.17. In Theorem 1.11 consider the Lorentzian space \( (\mathbb{R}^4, g) \) with coordinates \( (x_1, x_2, x_3, x_4) \) and signature \( \varepsilon_1 = -1, \varepsilon_k = 1 \) for \( k = 2, 3, 4 \), and \( F^3 \) scalar flat fiber. Let \( \xi = x_2 + x_3 + x_4 \) and choose \( \varphi(\xi) = |\sec(\xi)| \) where \( \xi \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z} \), then \( \varepsilon_p(\xi) = -\frac{1}{2} \) is a particular solution of (18), and by Theorem 1.11
\[
f(\xi) = (2|\sec(\xi)|\varepsilon^\frac{1}{2}), \quad h(\xi) = \text{constant}, \quad \varphi(\xi) = |\sec(\xi)|,
\]
provide a steady gradient Yamabe soliton in warped metric defined on \( x_1 + x_2 + x_3 \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z} \).

Example 1.18. In Theorem 1.12 consider the Lorentzian space \( (\mathbb{R}^n, g) \) with coordinates \( (x_1, \ldots, x_n) \) and signature \( \varepsilon_1 = -1, \varepsilon_i = 1 \) for all \( i \geq 2 \), and fiber \( (\mathbb{R}^d, g_0) \) where \( g_0 \) is the Euclidean metric. Let \( \xi = x_1 + x_2 \) and choose \( k \in \mathbb{R} \setminus \{0\} \). Then
\[
f(\xi) = e^{k\xi}, \quad h(\xi) = -\frac{k_1 e^{-2k\xi}}{2k}, \quad k_1 \neq 0, \quad \varphi(\xi) = e^{k\xi},
\]
defines a family of complete steady gradient Yamabe soliton on \( (\mathbb{R}^n, \varphi^{-2}g) \times_f (\mathbb{R}^d, g_0) \) with potential function \( h \) and warping function \( f \) (see section 3).

2. Preliminaries

In this section we shall present some preliminaries which will be used in the paper. We shall follow the notation and terminology of Bishop and O’Neill [2].

Lemma 2.1. \( M^{n+d} \) is a gradient Yamabe soliton with potential function \( \tilde{h} \) and soliton constant \( \rho \) if, and only if, \( (B^n, g_B) \) is an almost gradient Yamabe soliton with potential function \( h \), soliton function
\[
\lambda = -\frac{\lambda_f}{f^2} + \frac{2d}{f} \Delta f + d(d-1) |\nabla f|^2 f^2 + \rho,
\]
and scalar curvature
\[
S_B = \frac{\langle \nabla f, \nabla h \rangle}{f} + \lambda.
\]
Proof of Lemma 2.1: Using that $S_F = \lambda_F$, we have by the well known formula of scalar curvature on warped product that

$$S_{\bar{g}} = \pi^* \left[ S_B + \frac{\lambda_F}{f^2} - 2d \frac{\Delta f}{f} - d(d-1) \frac{|\nabla f|^2}{f^2} \right]$$

where $\Delta$ denote the Laplacian on $B$, and we use $g_B = \langle \cdot, \cdot \rangle = |\cdot|^2$ for simplicity.

Then we have that $M^{n+d}$ is a gradient Yamabe soliton with potential function $\bar{h}$ and soliton constant $\rho$ if, and only if,

$$\pi^* \left[ S_B + \frac{\lambda_F}{f^2} - 2d \frac{\Delta f}{f} - d(d-1) \frac{|\nabla f|^2}{f^2} \right] - \rho \bar{g} = Hess(\bar{h}).$$

Let $\mathcal{L}(B), \mathcal{L}(F)$ the spaces of lifts of vector fields on $B$ and $F$ to $B \times F$, respectively. Consider $X \in \mathcal{L}(B)$ and $V \in \mathcal{L}(F)$, then

$$\bar{g}(X,V) = 0 = Hess(\bar{h})(X,V).$$

Hence, we just need to look at equation (20) for pair of fields in $\mathcal{L}(B)$, and $\mathcal{L}(F)$.

Taking $X,Y \in \mathcal{L}(B)$ in (20) and using $Hess(\bar{h}) = \pi^*(Hess(h))$ we obtain the following equivalent condition

$$\pi^* \left( S_B + \frac{\lambda_F}{f^2} - 2d \frac{\Delta f}{f} - d(d-1) \frac{|\nabla f|^2}{f^2} \right) \pi^* g_B = \pi^*(Hess(h)),$$

which says that $(B^n, g_B)$ is an almost gradient Yamabe soliton with soliton function

$$\lambda = -\frac{\lambda_F}{f^2} + \frac{2d}{f} \Delta f + d(d-1) \frac{|\nabla f|^2}{f^2} + \rho,$$

and potential function $h$.

Now, consider $V,W \in \mathcal{L}(F)$, then we obtain the Hessian expression

$$Hess(\bar{h})(V,W) = V(W(\bar{h})) - (\nabla_V W)^M \bar{h} = V(W(\bar{h})) + \frac{\pi(V,W)}{f} \nabla(\bar{f})(\bar{h}) - \nabla^F_V W(\bar{h}) = \bar{f} \sigma^* g_F(V,W) (\nabla \bar{f}) \bar{h} = \bar{f} \sigma^* g_F(V,W) [d\pi(\nabla \bar{f})(h) \circ \pi]$$

(21)

Substituting $V,W \in \mathcal{L}(F)$ into (20) and considering equation (21) we obtain

$$\pi^* \left( S_B + \frac{\lambda_F}{f^2} - 2d \frac{\Delta f}{f} - d(d-1) \frac{|\nabla f|^2}{f^2} \right) \bar{f}^2 \sigma^* g_F = \bar{f} \sigma^* g_F [d\pi(\nabla \bar{f})(h) \circ \pi]$$

And using $\nabla f = \pi_*(\nabla \bar{f})$ we obtain the following equivalent condition

$$S_B + \frac{\lambda_F}{f^2} - \frac{2d}{f} \Delta f - d(d-1) \frac{|\nabla f|^2}{f^2} - \rho = \frac{(\nabla f, \nabla h)}{f}$$

which is equation (19). This concludes the proof. □

Definition 2.2. A semi-Riemannian manifold for which every geodesic is defined on the entire real line is said to be geodesically complete, or just complete.

Given a curve $\gamma$ in $M \times F$, we can write $\gamma(s) = (\gamma_B(s), \gamma_F(s))$, where $\gamma_B = \pi \circ \gamma$ and $\gamma_F = \sigma \circ \gamma$. The following proposition guarantees a condition for curve $\gamma$ to be geodesic.

Proposition 2.3. ([3]) A curve $\gamma = (\gamma_B, \gamma_F)$ in $B \times F$ is a geodesic if, and only if,
γ''_B = g_F(γ'_F, γ'_F) f ◦ γ_B ▽ f in B,

(2) γ''_F = \frac{-2}{f ◦ γ_B} \frac{d(f ◦ γ_B)}{ds} γ'_F in F.

3. PROOF OF THE MAIN RESULT

Proof of Theorem 1.3: By Lemma 2.1, we have that

(22) (S_B − λ)g_B = Hess(h),  \quad S_B − λ = \frac{⟨∇f, ∇h⟩}{f}.

Combining this equations we obtain

(23) Δ h − ⟨∇w, ∇ h⟩ = 0,

where w = ln f.

Denoting Δ_w := Δ − ⟨∇w, ∇⟩, it follows from integration by parts and (23) that

∫_M |∇h|^2 e^{-w} dv = − ∫_M h(Δ_w h) e^{-w} dv = 0.

Hence, |∇h| = 0, which show that h is constant.

□

Proof of Theorem 1.4: Using lemma 2.1, we have that

(24) S_B + \frac{λ_F}{f^2} − 2f \Delta f − d(d − 1) \frac{|∇f|^2}{f^2} − \rho = \frac{⟨∇f, ∇h⟩}{f}.

Since S_B ≥ ρ − \frac{λ_F}{f^2}, we obtain by equation (24), that

(25) Δ f + ⟨∇w, ∇ f⟩ = \frac{(S_B − ρ)f^2 + λ_F}{2fd} ≥ 0,

where w = \frac{h}{2d} + ln f^{\frac{d−1}{2}}.

Now, consider x_0 the point where f attains its maximum f_0, and define

Ω_0 := \{x ∈ B : f(x) = f_0\}

Ω_0 is closed and nonempty since x_0 ∈ Ω_0. Let now y ∈ Ω_0, then applying the maximum principle (see [31] p. 35) to (25) we obtain that, f(x) = f_0 in a neighborhood of y so that Ω_0 is open. Connectedness of B yields Ω_0 = B. Thus f is constant.

□

Proof of Proposition 1.5: Consider θ = ⟨∇ log f, ∇ h⟩, then applying Lemma 2.1 we obtain θg_B = Hess(h). The result follow by lemma:

Lemma 3.1. ([30], Theorem 1) Let (N^n, ĝ) be a complete manifold. Suppose that there exist a smooth function h : N → ℝ satisfying Hess(h) = \θ ĝ for some constant θ ≠ 0. Then N^n is isometric to ℝ^n.

□

Proof of Theorem 1.6: The equivalence given by Lemma 2.1 states that a necessary and sufficient condition to M_{i+^d} be a gradient Yamabe soliton with potential function h is

(26) S_B + \frac{λ_F}{f^2} − 2f \Delta f − d(d − 1) \frac{|∇f|^2}{f^2} − \rho = \frac{⟨∇f, ∇h⟩}{f},

and

(27) \left( S_B + \frac{λ_F}{f^2} − 2f \Delta f − d(d − 1) \frac{|∇f|^2}{f^2} − \rho \right) g_B = Hess(h).

We will use the above equations in combination with the invariant solutions technique to obtain equations (5), (6), (7), (8) and (9).
First, for an arbitrary choice of a non zero vector $\alpha = (\alpha_1, \ldots, \alpha_n)$, consider $\xi : \mathbb{R}^n \to \mathbb{R}$ given by $\xi(x_1, \ldots, x_n) = \alpha_1 x_1 + \cdots + \alpha_n x_n$. Since we are assuming that $\varphi(\xi)$, $h(\xi)$ and $f(\xi)$ are functions of $\xi$, then we have

$$
\varphi_{,x_i} = \varphi' \alpha_i, \quad f_{,x_i} = f' \alpha_i, \quad h_{,x_i} = h' \alpha_i,
$$

(28)

$$
\varphi_{,x_i x_j} = \varphi'' \alpha_i \alpha_j, \quad f_{,x_i x_j} = f'' \alpha_i \alpha_j, \quad h_{,x_i x_j} = h'' \alpha_i \alpha_j.
$$

It is well known that for the conformal metric $g_B = \varphi^{-2} \delta$, the Ricci curvature is given by (21):

$$
Ric_{g_B} = \frac{1}{\varphi^2} \left\{ (n - 2) \varphi Hess(\varphi) + |\varphi \Delta_{\delta} \varphi - (n - 1)|\nabla_{\delta} \varphi|)^2 g \right\}.
$$

(29)

And then combining (28) with (29) we easily see that the scalar curvature on conformal metric is given by

$$
S_B = \sum_{k=1}^{n} \varphi^2 \varepsilon_k (Ric_{g_B})_{kk} = (n - 1)(2 \varphi \sum_{k=1}^{n} \varepsilon_k \varphi_{,x_k x_k} - \sum_{k=1}^{n} \varepsilon_k \varphi^2_{,x_k x_k}) = ||\alpha||^2 (n - 1)(2 \varphi \varphi'' - n (\varphi')^2).
$$

(30)

Now, in order to compute the Hess($h$) of $h$ relatively to $g_B$ we evoke the expression

$$
(Hess(h))_{ij} = h_{,x_i x_j} - \sum_{k=1}^{n} \Gamma^k_{ij} h_{,x_k},
$$

where the Christoffel symbol $\Gamma^k_{ij}$ for distinct $i, j, k$ are given by

$$
\tilde{\Gamma}^k_{ij} = 0, \quad \tilde{\Gamma}_{ij} = \varepsilon_i \varepsilon_k \frac{\varphi_{,x_k}}{\varphi}, \quad \tilde{\Gamma}^i_{ij} = - \frac{\varphi_{,x_i}}{\varphi}.
$$

Therefore,

$$
(Hess(h))_{ij} = h_{,x_i x_j} + \varphi^{-1}(\varphi_{,x_i} h_{,x_j} + \varphi_{,x_j} h_{,x_i}) - \delta_{ij} \varepsilon_i \varepsilon_k \varphi_{,x_k} h_{,x_k} = \alpha_i \alpha_j h'' + (2 \alpha_i \alpha_j - \delta_{ij} ||\alpha||^2) \varphi^{-1} \varphi' h'.
$$

(32)

And the Laplacian $\Delta f = \sum_k \varphi^2 \varepsilon_k (Hess(f))_{kk}$ of $f$ with respect to $g_B$ is

$$
\Delta f = ||\alpha||^2 \varphi^2 (f'' - (n - 2) \varphi^{-1} \varphi' f').
$$

(33)

On the other hand, the expression of $\langle \nabla f, \nabla h \rangle$ and $|\nabla f|^2$ on conformal metric $g_B$ are given by

$$
\langle \nabla f, \nabla h \rangle = \varphi^2 \sum_k \varepsilon_k f_{,x_k} h_{,x_k}, \quad |\nabla f|^2 = \varphi^2 \sum_k \varepsilon_k f_{,x_k}^2,
$$

(34)

Then substituting (30) and (34) into (26) we obtain (7).

Now, for $i \neq j$ we obtain by (27) and (32) that

$$
\alpha_i \alpha_j \left( h'' + \frac{2 \varphi' h'}{\varphi} \right) = 0.
$$

If there exist $i, j$, $i \neq j$ such that $\alpha_i \alpha_j \neq 0$, then we get

$$
h'' + \frac{2 \varphi' h'}{\varphi} = 0
$$

which is equation (5). And for $i = j$, substituting (30), (32) and (34) into (27) we obtain (6).
Now, we need to consider the case $\alpha_{k_0} = 1$, $\alpha_k = 0$ for $k \neq k_0$. In this case, substituting (30), (32) and (34) into (27) we obtain

$$
\left[ \varepsilon_{k_0} (n-1) (2 \varphi'' - n(\varphi')^2) + \frac{\lambda_F}{f^2} - \frac{2d}{f} \varepsilon_{k_0} (\varphi^2 f' - (n-2)\varphi \varphi') + \varepsilon_{k_0} \frac{d(d-1)}{f^2} \frac{\varphi^2 (f')^2 - \rho}{\varphi^2} \right] = 0
$$

when $i \neq k_0$, that is, $\alpha_i = 0$ and

$$
\left[ \varepsilon_{k_0} (n-1) (2 \varphi'' - n(\varphi')^2) + \frac{\lambda_F}{f^2} - \frac{2d}{f} \varepsilon_{k_0} (\varphi^2 f'' - (n-2)\varphi \varphi') + \varepsilon_{k_0} \frac{d(d-1)}{f^2} \frac{\varphi^2 (f')^2 - \rho}{\varphi^2} \right] = 0
$$

for $i = k_0$, that is, $\alpha_{k_0} = 1$.

However, this equations are equivalent to equations (5) and (6). The Lightlike case follow by taking $||\alpha||^2 = 0$ into (6) and (7). This completes the demonstration.

Proof of Theorem 1.9 and 1.10: Since $\rho = 0$ and $h' \neq 0$ we have by equation (6) and (7) of Theorem 1.6 that

$$
\frac{\varphi'}{\varphi} = -\frac{f'}{f}.
$$

Integrating this equation we have

$$
f(\xi) = \frac{k_2}{\varphi(\xi)},
$$

for some $k_2 \in \mathbb{R} \setminus \{0\}$, which is equation (10) and (13).

Integrating the equation (5), we have that

$$
h'(\xi) = \frac{k_1}{\varphi^2(\xi)},
$$

for some $k_1 \neq 0$, and

$$
h(\xi) = k_1 \int \frac{1}{\varphi^2(\xi)} d\xi,
$$

which is equation (11) and (14).

Substituting equation (36) into (6) and considering (35), we obtain the follow differential equation

$$
\varphi^2 \varphi'' - \frac{(n+d)}{2} \varphi (\varphi')^2 + \frac{k_1}{2(n+d-1)} \varphi' = -\varphi^3 \lambda_F = \frac{k_2}{k_2^2 ||\alpha||^2}.
$$

Then since, $n + d = 6$, we obtain

$$
\varphi^2 \varphi'' - 3 \varphi (\varphi')^2 + \frac{k_1}{10} \varphi' = -\varphi^3 \lambda_F = \frac{k_2}{k_2^2 ||\alpha||^2}.
$$

If $\lambda_F = 0$, considering $\varphi(\xi)^{-2} = v(\xi)$, we obtain by (37) the following equivalent condition

$$
v'' + \frac{k_1}{10} v v' = 0.
$$

Integrating equation (38) we get

$$
v' + \frac{k_1}{20} v^2 = k_2, \quad k_2 = \text{constant}.
$$

This implies that

$$
- \int \frac{1}{k_2 v^2 - k_2} dv = \xi + k_3, \quad k_3 = \text{constant}.$$
Therefore, it follows from \( \varphi(\xi)^{-2} = v(\xi) \) that
\[
40 \int \frac{\varphi d\varphi}{k_1 - 20k_2 \varphi^4} = \xi + k_3,
\]
which is equation (15).

For \( \lambda_F \neq 0 \), consider the change \( u(\varphi) = \varphi^{-3} \varphi' \), then (37) is equivalent to the following separable variables differential equation
\[
\frac{du}{d\varphi} = -\frac{q - pu}{\varphi^3 u},
\]
where \( p = \frac{k_1}{10} \), \( q = \frac{\lambda_F}{k_2^{\frac{n}{2}} \varphi^2} \), and which solution is given by
\[
u(\varphi) = \frac{q}{p} \left( W \left( k_3 e^{\frac{\varphi^2}{4\varphi^2 - 1}} \right) + 1 \right), \quad k_3 = \text{constant} \neq 0,
\]
where \( W \) is the product log function.

Substituting back for \( u(\varphi) = \varphi^{-3} \varphi' \), we obtain
\[
\frac{pd\varphi}{q\varphi^3 W \left( k_3 e^{\frac{\varphi^2}{4\varphi^2 - 1}} \right) + q\varphi^3} = dt,
\]
which provide equation (12). The converse is a straightforward computation. This concludes the proof of Theorems 1.9 and 1.10.

**Proof of Theorem 1.11**: Since \( h' = 0 \) and \( \lambda_F = \rho = 0 \) we have by equation (6) and (7) of Theorem 1.6 that
\[
(n - 1)(2\varphi \varphi'' - n(\varphi')^2) - 2 \frac{d}{f} (\varphi^2 f'' - (n - 2) \varphi \varphi' f') - \frac{d(d - 1)}{f^2} \varphi^2 (f')^2 = 0,
\]
which is equivalent to
\[
\left( f' \right) - \frac{(n - 2)}{(d + 1)} \varphi' \right)^2 + \frac{2}{d + 1} \left( f' \right) \frac{(n - 2)}{(d + 1)} \varphi' \right)^2 + \frac{(n + d - 1)}{d(d + 1)^2} \left( \left( \frac{\varphi'}{\varphi} \right)^2 - 2 \frac{\varphi''}{\varphi} \right) = 0.
\]
Consider \( z = f' \frac{f'}{(d + 1) \varphi} \), then
\[
z^2 + \frac{2}{d + 1} z' + \frac{(n + d - 1)}{d(d + 1)^2} \left( \left( \frac{\varphi'}{\varphi} \right)^2 - 2 \frac{\varphi''}{\varphi} \right) = 0.
\]

Recall that the Ricatti differential equation is a differential equation of the form
\[
(z(\xi)') = f_2(\xi)z(\xi)^2 + f_1(\xi)z(\xi) + f_0(\xi),
\]
where \( f_0, f_1 \) and \( f_2 \) are smooth functions on \( \mathbb{R} \), and by Picard theorem, given a particular solution \( z_0 \) of (40), we have that the general solution of Riccati equation is given by
\[
z(\xi) = z_0(\xi) + \Phi(\xi) \left[ C - \int \Psi(\xi) f_2(\xi) d\xi \right]^{-1},
\]
where
\[
\Psi(\xi) = \exp \left\{ \int [2f_2(\xi)z_0(\xi) + f_1(\xi)] d\xi \right\}, \quad C = \text{constant}.
\]

Observe that (39) is a Ricatti differential equation with
\[
f_1(\xi) = 0, \quad f_2(\xi) = -\frac{d + 1}{2} \quad \text{and} \quad f_0(\xi) = -\frac{(n + d - 1)}{2d(d + 1)} \left( \left( \frac{\varphi'}{\varphi} \right)^2 - 2 \frac{\varphi''}{\varphi} \right).
\]
Then we obtain
\[
\frac{f'(\xi)}{f(\xi)} = \frac{(n-2)}{(d+1)} \varphi'(\xi) + \frac{e^{-}\frac{dz(\frac{\xi})}{\xi}}{2} + \frac{e^{-(d+1)}f z(\xi)_{d\xi}}{2}\frac{2}{d+1}
\]
And thus
\[
f(\xi) = \varphi \frac{e}{2} f z(\xi)_{d\xi}\left(\int e^{-(d+1)}f z(\xi)_{d\xi} + \frac{2}{d+1}C\right)^{\frac{d+1}{2}},
\]
where \(z(\xi)_{d\xi}\) is a particular solution of \(\ref{39}\). This expression is equation \(\ref{17}\).

Now, since \(h' = 0\), we have that \(h(\xi) = constant\), which is equation \(\ref{16}\). Then we prove the necessary condition. A direct calculation shows us the converse implication. This concludes the proof of Theorem.

\[\Box\]

**Proof of Theorem 1.12**: It follows immediately from \(\ref{8}\) and \(\ref{9}\).

\[\Box\]

**Proof of completeness of example 1.18**: Let \((R^n, g)\) be the standard semi-Euclidean space where \(g = -dx_1^2 + \sum_{i=2}^n dx_i^2\). Take \(k \in R \setminus \{0\}\), and consider the functions
\[
\varphi(\xi) = e^{2\xi}, \quad f(\xi) = e^{2\xi}, \quad h(\xi) = -\frac{k e^{-2\xi}}{2}, \quad k_1 \neq 0.
\]

Call \(\hat{g} := \varphi^{-2} g = e^{-2\xi} g\), then the gradient \(\nabla_{\hat{g}} f\) is given by
\[
\nabla_{\hat{g}} f = \sum_{r,s=1}^n \hat{g}^{rs} f_{,r} \partial_r = \sum_{r,s=1}^n \varphi^2 \varepsilon_r \delta_{rs} f' \alpha_s \partial_s = \sum_{s=1}^n k \varepsilon_s \alpha_s e^{3\xi} \partial_s.
\]
Since \(\alpha_1 = \alpha_2 = 1, \alpha_i = 0, \text{ for } i \geq 3, \text{ and } \varepsilon_1 = -1, \varepsilon_i = 1, \text{ for } i \geq 2, \text{ we obtain}
\[
\nabla_{\hat{g}} f = (-k e^{3\xi}, k e^{3\xi}, 0, \ldots, 0).
\]
Then, considering \(\gamma_B(s) = (y_1(s), \ldots, y_n(s))\) and \(\gamma_F(s) = (y_{n+1}, \ldots, y_{n+p}(s))\) in Proposition 2.3 we have
\[
\begin{align*}
y''_1(s) &= -k \left[y''_{n+1}(s)^2 + \cdots + y''_{n+p}(s)^2\right] e^{4k(y_1(s) + y_2(s))}, \quad (I) \\
y''_2(s) &= k \left[y''_{n+1}(s)^2 + \cdots + y''_{n+p}(s)^2\right] e^{4k(y_1(s) + y_2(s))}, \quad (II) \\
y''_r(s) &= 0, \quad \text{for } r \in \{3, \ldots, n\}, \quad (III) \\
y''_{n+l}(s) &= -2k[y''_1(s) + y''_2(s)] y''_{n+l}(s), \quad \text{for } l \in \{1, \ldots, d\}. \quad (IV)
\end{align*}
\]
The sum of differential equation (I) and (II) gives \(y''_1(s) + y''_2(s) = 0\), then by integration (41)
\[
y'_1(s) + y'_2(s) = c_1, \quad y_1(s) + y_1(s) = c_1 s + c_2, \quad c_1, c_2 \in R.
\]
Substituting (41) into (IV), we obtain the second order linear ordinary differential equation
\[
y''_{n+l}(s) + 2kc_1y'_{n+l}(s) = 0 \quad \text{for each } l \in \{1, \ldots, d\},
\]
whose general solutions is
\[
y_{n+l}(s) = \begin{cases} c_{3,l} + c_{4,l} s & \text{if } c_1 = 0 \\ c_{3,l} + c_{4,l} e^{-2c_1 s} & \text{if } c_1 \neq 0 \end{cases}
\]
where \(c_{3,l}, c_{4,l} \in R\). This shows that for each \(l \in \{1, \ldots, d\}\), the functions \(y_{n+l}(s)\) are defined on the entire real line \(R\). Since the solutions of (III) are given by \(y_r(s) = c_{5,r} + c_{6,r} s\), for \(c_{5,r}, c_{6,r} \in R\), whose domain is \(R\), it is only necessary to prove that the solutions of (I) and (II) are also defined in \(R\).
Integrating (42) and replacing its result into (I), we have

\[ y''_1(s) = -k\left[c^2_{7,1} + c^2_{7,2} + \cdots + c^2_{7,r}\right]e^{-4kc_1s}e^{4k(c_1s + c_2)} \]

where \(c_{7,1}, c_{7,2}, \ldots, c_{7,r} \in \mathbb{R}\).

Now, by (41) we obtain that

\[ y''_1(s) = -k[c^2_{7,1} + c^2_{7,2} + \cdots + c^2_{7,p}]e^{-4kc_1s}e^{4k(c_1s + c_2)} = -k[c^2_{7,1} + c^2_{7,2} + \cdots + c^2_{7,p}]e^{4k c_2} \]

(43)

\[ = c_{8,1} \in \mathbb{R}. \]

Then \( y_1(s) = \frac{c_{8,1}}{4}s^2 + c_9s + c_{10}, \) whose domain is \(\mathbb{R}\). Except for the signal, the same occurs for \(y_2(s)\). Thus, all the geodesics \(\gamma = (\gamma_B, \gamma_F)\) are defined for the entire real line, which means that \((\mathbb{R}^n, \varphi^{-2}g) \times_f (\mathbb{R}^d, g_0)\) is geodesically complete. \(\square\)

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1 Universidade Federal de Goiás, IME, 131, 74001-970, Goiânia, GO, Brazil. E-mail address: williamisotokura@hotmail.com

2 Universidade Federal de Goiás, IME, 131, 74001-970, Goiânia, GO, Brazil. E-mail address: levi@ufg.br

3 Universidade Federal de Goiás, IME, 131, 74001-970, Goiânia, GO, Brazil. E-mail address: romildo@ufg.br

4 Instituto Federal Goiano, 75790-000, Rodovia Geraldo Silva Nascimento Km 2.5, Urutáí, GO, Brazil. E-mail address: marcelo.barboza@ifgoiano.edu.br