Differential Type Operators and Gröbner-Shirshov Bases

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Abstract

A long standing problem of Gian-Carlo Rota for associative algebras is the classification of all linear operators that can be defined on them. In the 1970s, there were only a few known operators, for example, the derivative operator, the difference operator, the average operator, and the Rota-Baxter operator. A few more appeared after Rota posed his problem. However, little progress was made to solve this problem in general. In part, this is because the precise meaning of the problem is not so well understood. In this paper, we propose a formulation of the problem using the framework of operated algebras and viewing an associative algebra with a linear operator as one that satisfies a certain operated polynomial identity. This framework also allows us to apply theories of rewriting systems and Gröbner-Shirshov bases. To narrow our focus more on the operators that Rota was interested in, we further consider two particular classes of operators, namely, those that generalize differential or Rota-Baxter operators. As it turns out, these two classes of operators correspond to those that possess Gröbner-Shirshov bases under two different monomial orderings. Working in this framework, and with the aid of computer algebra, we are able to come up with a list of these two classes of operators, and provide some evidence that these lists may be complete. Our search has revealed quite a few new operators of these types whose properties are expected to be similar to the differential operator and Rota-Baxter operator respectively.

Recently, a more unified approach has emerged in related areas, such as difference algebra and differential algebra, and Rota-Baxter algebra and Nijenhuis algebra. The similarities in these theories can be more efficiently explored by advances on Rota’s problem.

Key words: Rota’s Problem; rewriting systems, Gröbner-Shirshov bases; operators; classification; differential type operators, Rota-Baxter type operators.
1. Introduction

Throughout the history of mathematics, objects are often understood by studying operators defined on them. Well-known examples are found in Galois theory, where a field is studied by its automorphisms, and in analysis and geometry, where functions and manifolds are studied through derivatives and vector fields. These operators abstract to the following linear operators on associative algebras.

automorphism \[ P(xy) = P(x)P(y), \] (1)
derivation \[ \delta(xy) = \delta(x)y + x\delta(y). \] (2)

By the 1970s, several more special operators, denoted by \( P \) below with corresponding name and defining property, had been studied in analysis, probability and combinatorics, including, for a fixed constant \( \lambda \),

average \[ P(x)P(y) = P(xP(y)), \] (3)

inverse average \[ P(x)P(y) = P(P(x)y), \] (4)

(Rota—)Baxter (weight \( \lambda \)) \[ P(x)P(y) = P(xP(y) + P(x)y + \lambda xy), \] (5)

Reynolds \[ P(x)P(y) = P(x P(y) + P(x)y - P(x)P(y)). \] (6)

Rota (1995) posed the question of finding all the identities that could be satisfied by a linear operator defined on associative algebras. He also suggested that there should not be many such operators other than these previously known ones. Even though there was some work on relating these different operators (Freeman, 1972), little progress was made on finding all such operators. In the meantime, new identities for operators have emerged from physics, algebra and combinatorial studies, such as

Nijenhuis \[ P(x)P(y) = P(xP(y) + P(x)y - P(xy)), \] (7)

Leroux’s TD \[ P(x)P(y) = P(xP(y) + P(x)y - xP(1)y), \] (8)

derivation (weight \( \lambda \)) \[ \delta(xy) = \delta(x)y + x\delta(y) + \lambda \delta(x)\delta(y). \] (9)

The previously known operators continue to find remarkable applications in pure and applied mathematics. For differential operators, we have the development of differential algebra (Kolchin, 1985), difference algebra (Cohn, 1965; Levin, 2008), and quantum differential operators (Lunts and Rosenberg, 1997, 1999). For Rota-Baxter algebras, we note their relationship with the classical Yang-Baxter equation, operads, combinatorics, and most prominently, the renormalization

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1 The following is quoted from Rota’s paper. “In a series of papers, I have tried to show that other linear operators satisfying algebraic identities may be of equal importance in studying certain algebraic phenomena, and I have posed the problem of finding all possible algebraic identities that can be satisfied by a linear operator on an algebra. Simple computations show that the possibility are very few, and the problem of classifying all such identities is very probably completely solvable. A notable step forward has been made in the unpublished (and unsubmitted) Harvard thesis of Alexander Doohovskoy.” He also remarked that a partial (but fairly complete) list of such identities are Eq. (1)-(6).
of quantum field theory through the Hopf algebra framework of Connes and Kreimer (Connes and Kreimer, 2000; Guo and Keigher, 2006, 2008; Andrews, Guo, Keigher, and Ono, 2003; Ebrahimi-Fard, Guo, and Kreimer, 2004; Ebrahimi-Fard, Guo, and Manchon, 2006; Guo and Sit, 2006; Bai, 2007; Ebrahimi-Fard and Guo, 2008; Guo and Zhang, 2008).  

1.1. Our approach  

These interesting developments motivate us to return to Rota’s question and try to understand the problem better. In doing so, we found that two key points in Rota’s question deserve further thoughts. First, we need a suitable framework to formulate precisely what is an “operator identity,” and second, we need to determine key properties that characterize the classes of operator identities that are of interest to other areas of mathematics, such as those listed above.  

For the first point, we note that a simplified but analogous framework has already been formulated in the 1960s and subsequently explored with great success. This is the study of PI-rings and PI-algebras, whose elements satisfy a set of polynomial identities, or PIs for short (Procesi, 1973; Rowen, 1980; Drensky and Fromanek, 2004).  

Let $k$ be a commutative unitary ring. In this paper, all algebras are unitary, associative $k$-algebras that are generally non-commutative, and all algebra homomorphisms will be over $k$, unless the contrary is noted or obvious.  

Recall that an algebra $R$ satisfies a polynomial identity if there is a non-zero (non-commutative) polynomial $\phi(X)$ in a finite set $X$ of indeterminates over $k$ (that is, $\phi(X) \in k(X)$, the free algebra on $X$) such that $\phi(X)$ is sent to zero under any algebra homomorphism $f : k(X) \to R$. To generalize this framework to the operator case, we shall introduce formally in Section 2 the notion of operated algebras and the construction of the free operated algebra $k[X]$ on $X$, which shall henceforth be called the operated polynomial algebra on $X$. An operator identity will correspond to a particular element $\phi(X)$ in $k[X]$. Analogous to PI-algebras, an OPI-algebra $R$ is an algebra with a $k$-linear operator $P$, a finite set $X$, and an operated polynomial $\phi(X) \in k[X]$ such that $\phi(X)$ is sent to zero under any morphism (of operated algebras) $f : k[X] \to R$. The operated polynomial $\phi$, or the equation $\phi(X) = 0$, is called an operated polynomial identity (OPI) on $R$ and we say $P$ (as well as $R$) satisfies the OPI $\phi$ (or $\phi(X) = 0$).  

As a first example, a differential algebra is an OPI-algebra $R$ with operator $\delta$, where the OPI is defined using $X = \{x, y\}$ and $\phi(x, y) := [xy] - [x]y - x[y]$, where $[\ ]$ denotes the operator in $k[X] = k[x, y]$. As a second example, a difference algebra $S$ is an OPI-algebra where the $k$-linear operator $P$ is an endomorphism, that is, $(S, P)$ satisfies $P(r)P(s) = P(rs)$ for all $r, s \in S$. A common difference algebra (taken from (Levin 2008, pp. 104–5)) is the following: Let $z_0 \in C$, where $C$ is the field of complex numbers, and let $S$ be the field of all functions $f(z)$ of one complex variable $z$ meromorphic in the region $U = \{z \in C \mid (\text{Re } z)(\text{Re } z_0) \geq 0\}$ (so that $z + z_0 \in U$ for all $z \in U$), then the shift (or translation) operator $P$ taking $f(z) \in S$ to $f(z + z_0) \in S$ is an automorphism of $S$, making $(S, P)$ an (inversive) difference algebra.  

With all operator identities understood to be OPIs in $k[X]$, the second point mentioned above may at first be interpreted as follows: among all OPIs, which ones are particularly consistent with

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2 Disclaimer: We are still exploring the best way to formulate Rota’s problem and nothing in this paper is meant to provide a definitive formulation.  

3 We illustrate only with an ordinary differential algebra, where the common notation for the derivation is $\delta$. In this paper, we have three symbols for the operator: $[\ ]$, $P$, and $\delta$, to be used respectively for a general (or bracketed word) setting, the Rota-Baxter setting, and the differential/difference setting; often, they are interchangeable. We use $[\ ]$ for $k[X]$ to emphasize that $k[X]$ is not the differential polynomial ring. Any dependence of the operator on parameters is suppressed, unless clarity requires otherwise.
the associative algebra structure so that they are singled out for study? This is a subtle question since one might argue (correctly, see Proposition 2.10) that any OPI defines a class of (perhaps trivial) operated algebras, just like any PI defines a class of algebras. We approach this by making use of two related theories: rewriting systems and Gröbner-Shirshov bases.

First, we shall regard an OPI as a rule that defines a rewriting system and study certain properties of this rewriting system, such as termination and confluence, that will characterize OPIs of interest. Termination and confluence are essential and desirable properties since we discovered our lists of OPIs by symbolic computation. As a rewriting rule, an OPI $\phi$ can be applied recursively and if not carefully done, such applications may lead to infinite recursion, in which case, it is no longer computationally feasible to derive meaningful consequences on the associative algebra from the OPI $\phi$. An example is the Reynolds operator identity in Eq. (6), where, if taken as a rewriting rule by replacing the equal sign with $\rightarrow$, the right hand side contains the expression $P(x)P(y)$, which equals the left-hand-side, leading to more and more complicated expressions as the rewriting rule is applied repeatedly $\textit{ad infinitum}$.

By putting aside for now OPIs like the Reynolds identity, we in effect restrict the class of OPIs under investigation and this allows us to apply symbolic computation to search for a list of identities for two broad families that include all the (other) previously mentioned OPIs. One family of operators consists of the OPIs of differential type, which include derivations, endomorphisms, differential operators of weight $\lambda$, and more generally operators $\delta$ satisfying an OPI of the form $\phi := [xy] = N(x, y)$, where $N(x, y)$ is a formal expression in $k\langle x, y \rangle$ in differentially reduced form, that is, it does not contain any subexpression of the form $[uv]$ for any $u, v \in k\langle x, y \rangle$.

The other family consists of the OPIs of Rota-Baxter type, which include those defining the average, Rota-Baxter, Nijenhuis, Leroux’s TD operators, and more generally OPIs of the form $\phi := [x[y] - M(x, y)]$ where $M(x, y)$ is an expression in $k\langle x, y \rangle$ in Rota-Baxter reduced form, that is, it does not involve any subexpression of the form $[uv]$ for any $u, v \in k\langle x, y \rangle$.

These two families share a common feature: each OPI involves a product: $xy$ for differential type, and $[x][y]$ for Rota-Baxter type. These families of OPIs thus provide properties arising from the associativity of multiplication, which we can explore in our computational experiments. More generally, for an OPI that gives rise to a terminating rewriting system, the associative law imposes various confluence constraints that may be satisfied by some operated algebras, but not by others. Thus, another advantage of the rewriting system approach is that we may use such constraints as criteria to screen OPI-algebras for further research.

In Section 2 of this paper, we begin the construction of the free operated algebras $k\langle X \rangle$ using a basis of bracketed words in $X$. This will be the universal space for OPIs by which we formulate Rota’s problem precisely in a general setting of a free operated algebra satisfying an OPI $\phi$. In Section 3, we develop Gröbner-Shirshov bases for free operated algebras and prove the Composition-Diamond Lemma (Theorem 3.13). In Section 4, we define operators and operated algebras of differential type and propose a conjectural answer to Rota’s Problem in this case with a list of differential type OPIs. As evidence of our conjecture, we verify in Section 4.2 that the operators in our list all satisfy the properties prescribed for a differential type operator, and in Section 5, we prove several equivalent criteria for an OPI $\phi$ in $k\langle x, y \rangle$ in differentially reduced form to be of differential type (Theorem 5.7), a result that connects together the rewriting

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4. We remind the reader that a term rewriting rule is a one-way replacement rule that depends on a term-order, unlike an equality or a congruence.

5. This, by definition, excludes the Reynolds operator as it stands. However, if we rewrite the Reynolds identity as $P(P(x)P(y)) = P(xP(y)) + P(xP(y)) = P(x)P(y)$, then it would be computationally feasible to explore its interaction with associativity, and would suggest that the Reynolds operator belongs to a “higher order” class.
system induced by $\phi$, the Gröbner-Shirshov bases of the operated ideal induced by $\phi$, and the free operated algebras satisfying $\phi$. In Section 6, we define similarly operators and operated algebras of Rota-Baxter type and give a conjecture for the complete list of Rota-Baxter type OPIs. In Section 7, we give a description of an empirical Mathematica program by which we obtained the lists. In the final Section 8, we explain our approach in the context of varieties of algebras, providing research directions towards a further understanding of Rota’s Problem, leading possibly to new tools and theoretical proofs of our conjectures.

2. Operator identities

In this section we give a precise definition of an OPI in the framework of operated algebras. We review the concept of operated (associative) monoids, operated algebras, and bracketed words, followed by a construction for the free operated monoids and algebras using bracketed words. Bracketed words are related to Motzkin words and decorated rooted trees (Guo 2009).

2.1. Operated monoids and algebras

**Definition 2.1.** An operated monoid is a monoid $U$ together with a map $P : U \to U$. A morphism from an operated monoid $U$ to an operated monoid $V$ is a monoid homomorphism $f : U \to V$ such that $f \circ P = P \circ f$, that is, the diagram below is commutative:

```
  U  P  \downarrow f
    \downarrow f
  U   V  P  \downarrow f
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Let $k$ be a commutative unitary ring. In Definition 2.1, we may replace “monoid” by “semigroup,” “$k$-algebra,” or “nonunitary $k$-algebra” to define operaded semigroup, operated $k$-algebra and operated nonunitary $k$-algebra, respectively. For example, the semigroup $\mathcal{F}$ of rooted forests, with the concatenation product and the grafting map $\lfloor \rfloor$, turns $\mathcal{F}$ into an operated semigroup (Guo 2009). The $k$-module $k\mathcal{F}$ generated by $\mathcal{F}$ is an operated nonunitary $k$-algebra. The unitarization of this algebra has appeared in the work of Connes and Kreimer (1998) on renormalization of quantum field theory.

The adjoint functor of the forgetful functor from the category of operated monoids to the category of sets gives the free operated monoids in the usual way. More precisely, a free operated monoid on a set $X$ is an operated monoid $U$ together with a map $j : X \to U$ with the property that, for any operated monoid $V$ together with a map $f : X \to V$, there is a unique morphism $\overline{f} : U \to V$ of operated monoids such that $f = \overline{f} \circ j$. Any two free operated monoid on the same set $X$ are isomorphic via a unique isomorphism.

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6 The concepts, construction of free objects and results in this section are covered in more generality in texts on universal algebra (Burris and Sankappanavar 1981; Cohn 1991; Baader and Nipkow 1998). Our review makes this paper more accessible and allows us to establish our own notations.

7 As remarked in Footnote 3, we use the same symbol $P$ for all distinguished maps and hence we shall simply use $U$ for an operated monoid. In this paper, all semigroups and monoids are associative but generally non-commutative.

8 To adapt Definition 2.1 for operated $k$-algebra categories, $P$ is assumed to be a $k$-linear map and $f$ is a morphism of the underlying $k$-algebras.
We similarly define the notion of a free operated (nonunitary) $k$-algebra on a set $X$. As shown in [Guo (2009)], the operated non-unitary $k$-algebra of rooted forests mentioned above is the free operated non-unitary $k$-algebra on one generator.

An **operated ideal** in an operated $k$-algebra $R$ is an ideal closed under the operator. The operated ideal generated by a set $\Phi \subseteq R$ is the smallest operated ideal in $R$ containing $\Phi$.

2.2. **Free operated monoids**

For any set $Y$, let $M(Y)$ be the free monoid generated by $Y$ and let $[Y]$ be the set $\{y \mid y \in Y\}$, which is just another copy of $Y$ whose elements are denoted by $[y]$ for distinction.

We now construct the free operated monoid over a given set $X$, which is just another copy of $X$ and the embedding $\iota : X \hookrightarrow M(X)$.

We similarly define the notion of a free operated (nonunitary) $k$-algebra on a set $X$. The notation $[X]$, suggested by a reviewer, is simpler and more natural, but $\mathcal{M}(X)$ is consistent with prior literature and occasionally, typographically more pleasing, as in $\{[y] \mid y \in X\}$.

**Theorem 2.2.** [Guo (2009), Corollaries 3.6 and 3.7]

1. The monoid $\llbracket X \rrbracket$, with operator $P := [\cdot]$ and natural embedding $j : X \to \llbracket X \rrbracket$, is the free operated monoid on $X$.
2. The unitary (associative) $k$-algebra $k[\llbracket X \rrbracket]$, with the $k$-linear operator $P$ induced by $[\cdot]$ and the natural embedding $j : X \to k[\llbracket X \rrbracket]$, is the free operated unitary $k$-algebra on $X$.

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9 We adopt two notations for the free operated monoid on $X$. The notation $[X]$, suggested by a reviewer, is simpler and more natural, but $\mathcal{M}(X)$ is consistent with prior literature and occasionally, typographically more pleasing, as in $\{[y] \mid y \in X\}$, when compared to $\llbracket [y] \rrbracket$.
Definition 2.3. An element $w \in \mathcal{M}(X)$ is called a \textbf{bracketed word on the generator set} $X$. If $X = \{x_1, \ldots, x_k\}$, we also write $k[X]$ simply as $k[x_1, \ldots, x_k]$. An element $\phi \in k[X]$ but not in $k$ is called a \textbf{bracketed polynomial in} $X$.

A nonunit element $w$ of $\mathcal{M}(X)$ can be uniquely expressed in the form

$$w = w_1 \cdots w_k \quad \text{for some } k \text{ and some } w_i \in X \cup \{0\}, 1 \leq i \leq k. \quad (13)$$

Definition 2.4. For a nonunit element $w \in \mathcal{M}(X)$, the decomposition in Eq. (13) is called the \textbf{standard decomposition} of $w$ and elements in $X \cup \{0\}$ are called \textbf{indecomposable}. The integer $|w| := k$ is called the \textbf{breadth} of $w$. The integer $d(w) := \min \{n \mid w \in \mathcal{M}_n\}$ is called the \textbf{depth} of $w$. We also consider $\mathbf{1}$ (the empty product in $\mathcal{M}(X)$ and Eq. (13)) to be indecomposable and define $|(\mathbf{1})| = d(\mathbf{1}) = 0$.

Remark 2.5. Alternatively (Guo, 2009), $\mathcal{M}(X)$ can be viewed as the set of \textbf{bracketed words} $w$ of the free monoid $M(X \cup \{[, ]\})$ generated by $X \cup \{[, ]\}$, in which the brackets $[, ]$ form balanced pairs, or more explicitly,

1. the total number of $[$ in the word $w$ equals to the total number of $]$ in $w$; and
2. counting from the left to the right of $w$, the number of $[$ is always greater than or equal to the number of $]$.

For example, for the set $X = \{x\}$, the element $w := [x|x|x]_X$ is a bracketed word in $M([x[,]])$, with $|w| = 3$ and $d(w) = 2$, while neither $[x|]_X$ (failing the first condition) nor $[|x]_X$ (failing the second condition) is.

2.3. \textit{Operated polynomial identity algebras}

We recall the concept of a polynomial identity algebra. Let $k(X)$ be the free non-commutative $k$-algebra on a finite set $X = \{x_1, \ldots, x_k\}$. A given $\phi \in k(X)$, $\phi \neq 0$, defines a category $\text{Alg}_\phi$ of algebras, whose objects are $k$-algebras $R$ satisfying $\phi(r_1, \ldots, r_k) = 0$ for all $r_1, \ldots, r_k \in R$.

The non-commutative polynomial $\phi$ (formally, the equation $\phi(x_1, \ldots, x_k) = 0$, or its equivalent $\phi_1(x_1, \ldots, x_k) = \phi_2(x_1, \ldots, x_k)$ if $\phi := \phi_1 - \phi_2$) is classically called a \textbf{polynomial identity} (PI) and we say $R$ is a $\textbf{PI-algebra}$ if $R$ satisfies $\phi$ for some $\phi$. For any set $Z$, we may define the free PI-algebra on $Z$ in $\text{Alg}_\phi$ by the obvious universal property.

We extend this notion to operated algebras. Let $\phi \in k[x_1, \ldots, x_k]$, let $R$ be an operated algebra, and let $r = (r_1, \ldots, r_k) \in R^k$. The substitution map $f_r : \{x_1, \ldots, x_k\} \rightarrow R$ that maps $x_i$ to $r_i$ induces a unique morphism $\overline{f_r} : k[x_1, \ldots, x_k] \rightarrow R$ of operated algebras that extends $f_r$. Let $\phi_R : R^k \rightarrow R$ be defined by

$$\phi_R(r_1, \ldots, r_k) := \overline{f_r}(\phi(x_1, \ldots, x_k)). \quad (14)$$

Definition 2.6. Let $\phi \in k[x_1, \ldots, x_k]$ and $R$ be an operated algebra. If

$$\phi_R(r_1, \ldots, r_k) = 0, \quad \forall r_1, \ldots, r_k \in R,$$

then $R$ is called a $\phi$-\textbf{algebra}, the operator $P$ defining $R$ is called a $\phi$-\textbf{operator}, and $\phi$ (or $\phi = 0$) is called an \textbf{operated polynomial identity} (OPI). An \textbf{operated polynomial identity algebra} or an $\textbf{OPI-algebra}$ is a $\phi$-algebra for some $\phi \in k[x_1, \ldots, x_k]$ for some positive integer $k$. 

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Example 2.7. When \( \phi := [xy] - x[y] - [x]y \), then a \( \phi \)-operator on a \( k \)-algebra \( R \) is a derivation on \( R \), usually denoted by \( \delta \), and \( R \) is an ordinary, possibly non-commutative, differential algebra in which \( \delta(a) = 0 \) for all \( a \in k \).

Example 2.8. When \( \phi := [x][y] - [x]y - [x]y \), then a \( \phi \)-operator (resp. \( \phi \)-algebra) is a Rota-Baxter operator (resp. Rota-Baxter algebra) of weight \( \lambda \). We denote such operators by \( P \).

Example 2.9. When \( \phi \) is from the noncommutative polynomial algebra \( k(X) \), then a \( \phi \)-algebra is an algebra with polynomial identity, which we may view as an operated algebra where the operator is the identity map.

The next proposition is a consequence of the universal property of free operated algebras and can be regarded as a special case of a very general result on \( \Omega \)-algebras, where \( \Omega \) is a set called the signature and \( \Omega \) represents a family of operations on the algebra (see e.g. Cohn [1991], Chapter I, Proposition 3.6). We caution the reader that there are two sets involved: the set \( X \) in terms of which an OPI is expressed, and the set \( Z \) on which the free \( \phi \)-algebra is constructed.

Proposition 2.10. (Baader and Nipkow [1998], Theorem 3.5.6) Let \( Z \) be a set, let \( R = k[Z] \), and let \( j_Z : Z \to R \) be the natural embedding. Let \( X = \{x_1, \ldots, x_k \} \) and \( \phi \in k[X] \). Let \( \phi_R : R^k \to R \) be as defined in Eq. (14), let \( I_\phi(Z) \) be the operated ideal of \( R \) generated by the set
\[
\{ \phi_R(r_1, \ldots, r_k) \mid r_1, \ldots, r_k \in R \},
\]
and let \( \pi_\phi : R \to R/I_\phi(Z) \) be the quotient morphism. Let
\[
i_Z := \pi_\phi \circ j_Z : Z \to R/I_\phi(Z).
\]
Then the quotient operated algebra \( R/I_\phi(Z) \), together with \( i_Z \) and the operator \( P \) induced by \([ ]\), is the free \( \phi \)-algebra on \( Z \).

For a specific proof of Proposition 2.10, see [Jou, Sit and Zhang (2011)]. Proposition 2.10 shows that for any non-zero \( \phi \in k[X] \), there is always a (free, associative, but perhaps trivial) \( \phi \)-algebra. Thus the “formulation” below of Rota’s Problem would not be helpful.

Find all non-zero \( \phi \in k[X] \) such that the OPI \( \phi = 0 \) can be satisfied by some linear operator on some associative algebra.

While the construction in Proposition 2.10 is general, we note that the free \( \phi \)-algebra may have hidden consequences.

Example 2.11. Let \( \phi(x, y) := [xy] - y[x] \). Let \( Z \) be a set and let \( Q = k[Z]/I_\phi(Z) \) be the free \( \phi \)-algebra with the operator \( P \) induced by \([ ]\) on \( R = k[Z] \). Let \( a, b, c \in Q \) be arbitrary. We must have \( P((ab)c) = P(a(bc)) \). Applying the identity \( \phi = 0 \) on \( Q \) to both sides, we find that \( (bc - cb)P(a) = 0 \). We do not know if \( I_\phi(Z) \) is completely prime or not, but if it is, then we would have two possibilities: \( Q \) is commutative, or \( Q \) is not commutative but \( P(a) = 0 \) for all \( a \in Q \). We also note that any commutative algebra with the identity as operator is a \( \phi \)-algebra.

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\[9] Recall that there are two notions of primeness for an ideal \( I \) of a not-necessarily commutative ring \( R \): \( I \) (\( I \neq R \)) is completely prime if \( uv \in I \) for \( u, v \in R \) implies that either \( u \in I \) or \( v \in I \); and \( I \) is prime if for any ideals \( U \) and \( V \), \( UV \subseteq I \) implies either \( U \subseteq I \) or \( V \subseteq I \). When \( R \) is commutative, the two definitions are equivalent.
3. Gröbner-Shirshov bases for free operated algebras

We now introduce the framework of Gröbner-Shirshov bases for the free operated algebra \( k\langle X \rangle \) on \( X \). Shirshov basis was first studied by Zhukov (1950) and then by Shirshov (1962). For a historic review, we refer the reader to the Introduction and Bibliography sections of Bokut, Chen, and Qiu (2010). We also provided a sketchy summary in Guo and Sit (2010). Recently, these bases have been obtained by Bokut, Chen, and Qiu (2010), who gave a good survey of methods to construct linear bases, and in particular, Gröbner-Shirshov bases, in algebras under various combinations of commutativity and associativity. Dotsenko and Khosorkin (2010) has further details on the relationship of Gröbner-Shirshov bases with the well-known work of Buchberger (1965) and Bergman (1978). We will consider the case of free unitary operated algebras.

With the notation in Bokut, Chen, and Qiu (2010), let \( k(X; \Omega) \) denote the free nonunitary associative algebra on \( X \) with a set \( \Omega \) of linear operators. When \( \Omega \) consists only of one unary operator \([\ ]\), \( k(X; \Omega) \) is the non-unitary version of \( k\langle X \rangle \) and may be constructed as \( k\langle \mathcal{S} \rangle \), where \( \mathcal{S} = \lim_{n \to \infty} \mathcal{S}_n \)

with \( \mathcal{S}_n \) defined recursively by

\[
\mathcal{S}_n := S(X \cup \{ \mathcal{S}_{n-1} \}), \quad \mathcal{S}_0 := S(X),
\]

and where, for any set \( Y \), \( S(Y) \) is the semigroup generated by \( Y \).

As is well-known, the difference between an associative algebra \( A \) and its unitarization \( \widehat{A} \) is very simple: \( \widehat{A} = \overline{A} \oplus k 1 \). For an operated algebra, the difference is much more significant, as we can already see from their constructions. Since we are studying operators on unitary algebras, we need to be careful adapting results from Bokut, Chen, and Qiu (2010). For this reason and for introducing notation, we establish here some results that will lead to the Composition-Diamond Lemma (Theorem 3.13) and construction of Gröbner-Shirshov bases for unitary operated algebras.

**Definition 3.1.** Let \( * \) be a symbol not in \( X \) and let \( X^* = X \cup \{ * \} \). By a \( * \)-bracketed word on \( X \), we mean any expression in \( \|X^*\| \) with exactly one occurrence of \( * \). The set of all \( * \)-bracketed words on \( X \) is denoted by \( \|X^*\| \). For \( q \in \|X^*\| \) and \( u \in \|X\| \), we define

\[
q|_u := q|_{* \to u},
\]

(15)

to be the bracketed word obtained by replacing the letter \( * \) in \( q \) by \( u \), and call \( q|_u \) a \( u \)-bracketed word on \( X \). Further, for \( s = \sum c_i u_i \in k\langle X \rangle \), where \( c_i \in k \) and \( u_i \in \|X\| \), and \( q \in \|X^*\| \), we define

\[
q|_s := \sum_i c_i q|_{u_i},
\]

(16)

and extend by linearity to define the symbol \( q|_s \) for any \( q \in k\langle X \rangle^* \). Note that \( q|_s \) is in general not a bracketed word but a bracketed polynomial.

This process is the same as the process of replacing subterms in Baader and Nipkow (1998, Definition 3.1.3). We note the following simple relationship between operator replacement and ideal generation.

**Lemma 3.2.** Let \( S \) be a subset of \( k\langle X \rangle \). Let \( \text{Id}(S) \) be the operated ideal of \( k\langle X \rangle \) generated by \( S \). Then

\[
\text{Id}(S) = \left\{ \sum_{i=1}^k c_i q_{i|_{s_i}} \mid c_i \in k, q_i \in \|X^*\|, s_i \in S, 1 \leq i \leq k, k \geq 1 \right\}.
\]
Proof. It is clear that the right hand side is contained in the left hand side. On the other hand, the right hand side is already an operated ideal of $k[X]$ that contains $S$. \hfill \square

Definition 3.3. For distinct symbols $\star_1, \star_2$ not in $X$, let $X^{\star_1 \star_2} = X \cup \{\star_1, \star_2\}$. We define a $\star_1 \star_2$-bracketed word on $X$ to be a bracketed word in $[\|X^{\star_1 \star_2}\|]$ with exactly one occurrence of $\star_1$ and exactly one occurrence of $\star_2$. The set of $\star_1 \star_2$-bracketed words on $X$ is denoted by $\|X\|^{\star_1 \star_2}$. For $q \in \|X\|^{\star_1 \star_2}$ and $u_1, u_2 \in k[X]$, we define

$$q|_{u_1, u_2} = q|_{\star_1 = u_1, \star_2 = u_2},$$

(17)

to be the bracketed word obtained by replacing the letter $\star_1$ (resp. $\star_2$) in $q$ by $u_1$ (resp. $u_2$) and call it a $(u_1, u_2)$-bracketed word on $X$.

A $(u_1, u_2)$-bracketed word on $X$ can also be recursively defined by

$$q|_{u_1, u_2} := (q^*|_{u_1})|_{u_2},$$

(18)

where $q^*$ is a $\star_1$-bracketed word on the set $X^{\star_1}$. Then $q^*|_{u_1}$ is in $\|X\|^{\star_2}$ and we can apply Eq. (15). Similarly, treating $q$ first as a $\star_2$-bracketed word $q^*$ on the set $X^{\star_1}$, we have

$$q|_{u_1, u_2} := (q^*|_{u_2})|_{u_1},$$

(19)

Definition 3.4. A monomial ordering on $\|X\|$ is a well-ordering $\preceq$ on $\|X\|$ satisfying the two conditions:

1. $1 \preceq u; \ u < v \Rightarrow q|_u < q|_v$, for all $u, v \in \|X\|$ and all $q \in \|X\|^*$.

(20)

Here, as usual, we denote $u < v$ if $u \preceq v$ but $u \neq v$. Given a monomial ordering $\preceq$ and a bracketed polynomial $s \in k[\|X\|]$, we let $\overline{s}$ denote the leading bracketed word (monomial) of $s$. If the coefficient of $\overline{s}$ in $s$ is 1, we call $s$ monic with respect to the monomial order $\preceq$.

Examples of such orderings will be considered later in this paper. For now, we fix a monomial ordering $\preceq$ on $\|X\|$. Let $s, s' \in k[\|X\|], let t \in \|X\|$ and suppose $\overline{s} < t$. Then

(1) For any $q \in \|X\|^*$, we have $q|_{\overline{s}} = q|_{\overline{s}} < q|_t$.

(2) For $q \in \|X\|^{\star_1 \star_2}$, we have

$$q|_{\overline{s}, \overline{s'}} = q|_{\overline{s}, \overline{s'}} = q|_{\overline{s}, \overline{s'}} < q|_{\overline{t}, \overline{t}}$$

and

$$q|_{\overline{s'}, \overline{s}} < q|_{\overline{t'}, \overline{t}}.$$

Proof. (1) Let $s = \sum_{i=1}^k a_is_i$ where $0 \neq a_i \in k$ and $s_i \in \|X\|$ with $s_1 > \cdots > s_k$. Thus $\overline{s} = s_1$. By definition, $q|_{\overline{s}} = \sum_{i=1}^k a(q|_{s_i})$, and by Eq. (20) for a monomial order, $q|_{s_1} > \cdots > q|_{s_k}$. Thus $q|_{\overline{s}} = q|_{s_1} = q|_{\overline{t}}$. The inequality follows by the property in Eq. (20) of a monomial order.

(2) Let $s = \sum_{i=1}^k a_is_i \in k[\|X\|]$ be as in Part (1). Thus $\overline{s} = s_1$. By Eq. (18) and Part (1), we have

$$q|_{\overline{s}, \overline{s'}} = (q^*|_{\overline{s}})|_{\overline{t'}} = (q^*|_{\overline{s}})|_{\overline{t'}} = q|_{\overline{s}, \overline{s'}},$$

(1 \leq i \leq k).

(21)

By Eq. (19) and the property in Eq. (20) of a monomial order, we have

$$q|_{s, \overline{s}} = (q^*|_{\overline{s}})|_{s_1} > \cdots > (q^*|_{\overline{s}})|_{s_k} = q|_{s, \overline{s}}.$$
It follows from Eqs. (21) and (22) that \( q_{i_1, \cdots, i_k} > \cdots > q_{i_k} \) and since by linearity, \( q_{i, \nu} = \sum_{v=1}^k a_q q_{i, \nu} \), we have
\[
q_{i, \nu} = \max\{ q_{i, \nu} \mid 1 \leq i \leq k \} = q_{i_1, \nu} = q_{i, \nu},
\]
(23)
The first equality in Part (2) follows by replacing \( s \) with \( \tau \) in Eq. (23), and the second by replacing \( s' \) with \( \tau \). By the equalities just proved and Eq. (19), we have
\[
q_{i, \nu} = q_{i, \nu} = (q^*_{i, \nu})_\tau < (q^*_{i, \nu})_\tau = q_{i, \nu}.
\]
The other inequality follows similarly. \( \square \)

The following concepts of intersection and including compositions are adapted from Bokut, Chen, and Qiu (2010). For operated algebras, they are analogous to the concepts of overlap and inclusion \( S \)-polynomials for associative algebras, as in Bergman (1978). Here we pay careful attention to ensure these concepts are well-defined.

**Definition 3.6.** Let \( f, g \in k[X] \) be two bracketed polynomials monic with respect to \( \leq \).

1. If there exist \( \mu, \nu, w \in [X] \) such that \( w = f_\mu = v g \) with \( |w| < |f| + |g| \), then we define
   \[(f, g)_w := (f, g)^{\mu, \nu}_w := f \mu - vg \]
   and call it the **intersection composition** of \( f \) and \( g \) with respect to \( (\mu, \nu) \).

2. If there exist a \( q \in [X]^* \) and \( w \in [X] \) such that \( w = f = q_{\tau} \), then we define
   \[(f, g)^q_w := f - q_{\tau} \]
   and call it an **including composition of \( f \) and \( g \) with respect to \( q \).**

**Remark 3.7.** We note that the superscripts \( \mu, \nu \) for the intersection composition \( (f, g)^{\mu, \nu}_w \) is not necessary, since \( \mu \) and \( \nu \) are uniquely defined by \( w \), indeed, by \(|w|\), because of the uniqueness of the standard decompositions of \( \tau, \mu, \nu, g \). However, the superscript \( q \) in the including composition \( (f, g)^q_w \) is needed to ensure that the notation is well-defined. For example, if \( g \) occurs in \( \tau \) more than once, we might have two different \( q \)'s that give the same \( q_{\tau} \) but different including compositions. To illustrate, take \( f = x y x, g = x - 1 \in k[X, y] \) and \( q_1 = * y x, q_2 = x y * \in [X]^* \).

Then we have
\[
xyx = w = q_1_{\tau} = q_2_{\tau}.
\]
But \((f, g)^q_{\tau} = f - q_1_{\tau} = xy \) and \((f, g)^q_{\tau} = f - q_2_{\tau} = xy \) are not the same.

**Remark 3.8.** If Definition 3.6(1) holds with \( \mu = 1 \), then the intersection composition is also an including composition. For if \( \tau = v g \), then \( \tau = q_{\tau} \) where \( q = v * \). Hence \((f, g)^q_{\tau} = (f, g)^v_{\tau} \). However, if \( v = 1 \) but \( \mu \neq 1 \), then since \( \tau = \mu g \), there is no \( q \in [X]^* \) satisfying \( \tau = q_{\tau} \). As should have been clear, Definition 3.6(2) is not symmetric with respect to \( f \) and \( g \).

**Definition 3.9.** Let \( S \) be a set of monic bracketed polynomials and let \( w \in [X] \).

1. For \( u, v \in k[X] \), we call \( u \) and \( v \) congruent modulo \((S, w)\) and denote this by
   \[ u \equiv v \mod (S, w) \]
   if \( u - v = \sum c_i q_{i, \nu} \), with \( c_i \in k, q_i \in [X]^*, s_i \in S \) and \( q_{i, \nu} < w \).

2. For \( f, g \in k[X] \) and suitable \( w, \mu, \nu \) or \( q \) that give an intersection composition \((f, g)^{\mu, \nu}_w \) or an including composition \((f, g)^q_w \), the composition is called **trivial modulo** \((S, w)\) if
   \[
   (f, g)^{\mu, \nu}_w \text{ or } (f, g)^q_w \equiv 0 \mod (S, w).
   \]
(3) The set $S \subseteq k[X]$ is a Gröbner-Shirshov basis if, for all $f, g \in S$, all intersection compositions $(f, g)_{S_{w}}^{w}$ and all including compositions $(f, g)_{S_{w}}^{\emptyset}$ are trivial modulo $(S, w)$.

**Definition 3.10.**

(1) Let $u, w$ be two bracketed words in $k[X]$. We call $u$ a subword of $w$ if $w$ is in the operated ideal of $[X]$ generated by $u$. In terms of $\star$-words, $u$ is a subword of $w$ if there is a $q \in |X|^{*}$ such that $w = q_{u}$. A subword $u$ of $w$ is a subword when viewed as a string in the free monoid $M(X \cup \{, \})$ as in Remark 2.5: namely the string of letters forming $u$ is a substring of the string of letters forming $w$.

(2) Let $u_{1}$ and $u_{2}$ be two subwords of $w$.

(a) $u_{1}$ and $u_{2}$ are called separated if there is a $q \in |X|^{*}$ such that $w = q_{u_{1}, u_{2}}$. In terms of strings in $M(X \cup \{, \})$, this means that the substrings $u_{1}$ and $u_{2}$ of $w$ have no overlap.

(b) $u_{1}$ and $u_{2}$ are called overlapping if there are subwords $a, b, c$ of $w$ such that $au_{1} = c = u_{2}b$ or $au_{1} = c = u_{1}b$ with $|c| < |u_{1}| + |u_{2}|$. In terms of strings in $M(X \cup \{, \})$, this means that the strings of $u_{1}$ and $u_{2}$ have an overlap.

We note there is a third relative location of $u_{1}$ and $u_{2}$ in $w$, namely either $u_{1}$ or $u_{2}$ is nested in (i.e., a subword of) the other.

**Proposition 3.11.** Let $S \subseteq k[X]$. Let $s_{1}, s_{2} \in S$ and suppose there exist $q_{1}, q_{2} \in |X|^{*}$ and $w \in k[X]$ such that $w = q_{1}w_{q_{1}} = q_{2}w_{q_{2}}$, by which we may view $w_{q_{1}}, w_{q_{2}}$ as subwords of $w$ and suppose as such, $w_{q_{1}}, w_{q_{2}}$ are separated in $w$. Then $q_{1}|_{s_{1}} \equiv q_{2}|_{s_{2}} \mod (S, w)$.

**Proof.** Let $q \in |X|^{*}$ be the $(\star_{1}, \star_{2})$-bracketed word obtained by replacing this occurrence of $w_{q_{1}}$ in $w$ by $\star_{1}$ and this occurrence of $w_{q_{2}}$ in $w$ by $\star_{2}$. Then we have

$$q^{\star_{1}}w_{q_{1}} = q_{2}, \quad q^{\star_{2}}w_{q_{2}} = q_{1}, \quad \text{and} \quad q^{\star_{1}}w_{q_{2}} = q_{1}w_{q_{2}} = q_{2}w_{q_{2}} = w,$$

where in the first two equalities, we have identified $|X|^{*}$ and $|X|^{*}$ with $|X|^{*}$ with $|X|^{*}$. Let

$$s_{1} = w_{q_{1}} + \sum_{i} c_{i}u_{i}, \quad s_{2} = w_{q_{2}} + \sum_{j} d_{j}v_{j}$$

where $c_{i}, d_{j} \in k, u_{i}, v_{j} \in |X|$, and $w_{q_{1}} = u_{i} < w_{q_{2}}$ and $v_{j} < w_{q_{2}}$. Then by the linearity of $s_{1}, s_{2}$ in $q_{1}, s_{2}$, we have

$$q_{2}|s_{2} - q_{1}|s_{1} = (q^{\star_{1}}w_{q_{1}})|s_{2} - (q^{\star_{0}}w_{q_{1}})|s_{1}$$

$$= q_{1}|s_{1} - q_{1}|s_{1},$$

$$= (q^{\star_{1}}w_{q_{1}})|s_{2} + q_{1}|s_{1} - q_{1}|s_{1},$$

$$= -q_{1}|s_{1} - q^{\star_{1}}w_{q_{1}}|s_{2} + q_{1}|s_{1} - q^{\star_{1}}w_{q_{1}}|s_{1},$$

$$= -q_{1}|s_{1} - q^{\star_{1}}w_{q_{1}}|s_{2} + q^{\star_{1}}w_{q_{1}}|s_{2}$$

$$= -q_{1}|s_{1} - q^{\star_{1}}w_{q_{1}}|s_{2} + q^{\star_{1}}w_{q_{1}}|s_{2}$$

$$= -q_{1}|s_{1} - q^{\star_{1}}w_{q_{1}}|s_{2} + q^{\star_{1}}w_{q_{1}}|s_{2}$$

$$= -\sum_{i} c_{i}(q^{\star_{1}}|u_{i})|s_{2} + \sum_{j} d_{j}(q^{\star_{1}}|v_{j})|s_{1},$$

Since $u_{i} = w_{q_{1}}$ and $v_{j} = w_{q_{2}}$, by Eqs. (18) and (19), we have

$$(q^{\star_{1}}|u_{i})|w_{q_{1}} = q_{1}w_{q_{1}} < q_{2}w_{q_{2}} = w \quad \text{and} \quad (q^{\star_{1}}|v_{j})|w_{q_{2}} = q_{1}w_{q_{2}} < q_{1}w_{q_{2}} = w.$$

This means

$$q_{1}|s_{1} \equiv q_{2}|s_{2} \mod (S, w),$$

completing the proof. □
Lemma 3.12. Let \( \preceq \) be a monomial ordering of \( k[\![X]\!] \) and let \( S \) be a set of monic bracketed polynomials in \( k[\![X]\!] \). Then the following conditions on \( S \) are equivalent:

(1) \( S \) is a Gröbner-Shirshov basis.
(2) For every \( s_1, s_2 \in S \) and \( w \in [\![X]\!] \) for which there exist \( q_1, q_2 \in [\![X]\!]^* \) such that \( w = q_1|_{\overline{\pi}} = q_2|_{\overline{\pi}} \), we have \( q_1|_{s_1} \equiv q_2|_{s_2} \mod (S, w) \).

Proof. (2) \( \Rightarrow \) (1): This is clear since the congruences include those from intersection composition and inclusion composition.

(1) \( \Rightarrow \) (2): Let \( s_1, s_2 \in S \) and \( w \in [\![X]\!] \), and suppose there exist \( q_1, q_2 \in [\![X]\!]^* \) such that \( w = q_1|_{\overline{\pi}} = q_2|_{\overline{\pi}} \). We fix one such occurrence of \( \overline{\pi} \) and one such occurrence of \( \overline{\pi} \). We distinguish three cases according to the relative location of these particular occurrences of \( \overline{\pi} \) and \( \overline{\pi} \) in \( w \).

Case I. Suppose the bracketed words \( \overline{\pi} \) and \( \overline{\pi} \) are separated in \( w \). This case is Proposition 3.11.

Case II. Suppose the bracketed words \( \overline{\pi} \) and \( \overline{\pi} \) overlap in \( w \). Then by switching \( s_1 \) and \( s_2 \) if necessary, we might assume that there exist some bracketed subwords \( \overline{\pi}_1, \overline{\pi}_2 \in [\![X]\!] \) of \( w \) such that \( \overline{\pi}_1 = \overline{\pi}_2 = \overline{\pi} \) with \( |\overline{\pi}_1| < |\overline{\pi}_2| + |\overline{\pi}| \). Thus there is \( p \in [\![X]\!]^* \) such that \( p|_{\overline{\pi}_1} = p|_{\overline{\pi}_2} = w \) and then \( q_1 = p|_{\overline{\pi}_1} \). Let \( q := q_1|_{\overline{\pi}_1} \) be obtained from \( q_1 \) by replacing \( \ast \) by \( \ast_1 \) and \( \ast_2 \) by \( \ast \). Then we have

\[
q^*|_{\overline{\pi}_1} = q_1, \quad q^*|_{\overline{\pi}_2} = q_2, \quad \text{and} \quad p|_{\overline{\pi}_1, \overline{\pi}_2} = q_1|_{\overline{\pi}_1} = w.
\]

where in the first two equalities, we have identified \( [\![X]\!]^* \) and \( [\![X]\!]^* \) with \( [\![X]\!]^* \). Thus, we have

\[
q_1|_{s_1} - q_2|_{s_2} = (q^*|_{\overline{\pi}_1})|_{s_1} - (q^*|_{\overline{\pi}_2})|_{s_2} = p|_{s_1, s_2}.
\]

Since \( S \) is a Gröbner-Shirshov basis, we have

\[
s_1 \mu - v s_2 = \sum c_j p|_{\pi_j},
\]

where each \( c_j \in k \), \( p_j \in [\![X]\!]^* \), \( \pi_j \in S \) and \( p_j|_{\overline{\pi}} < w_1 \). By linearity,

\[
q_1|_{s_1} - q_2|_{s_2} = p|_{s_1, s_2} = \sum c_j p|_{p|_{\pi_j}}.
\]

By Lemma 3.5.1, \( p|_{\pi_j} = p_j|_{\overline{\pi}} < w_1 \). Thus

\[
p|_{p|_{\pi_j}} = p|_{p|_{\overline{\pi}} < w_1} = p|_{\overline{\pi}} = q_1|_{\overline{\pi}} = w.
\]

Therefore

\[
q_1|_{s_1} \equiv q_2|_{s_2} \mod (S, w).
\]

Case III. Suppose one of the bracketed words \( \overline{\pi}_1 \), \( \overline{\pi}_2 \) is a subword of the other. Without loss of generality, suppose \( \overline{\pi}_1 = q_1|_{\overline{\pi}} \) for some \( \ast \)-bracketed word \( q \). Then we have an inclusion composition \( (s_1, s_2, q) \).

\[\text{\textsuperscript{11}}\text{We note that there might be multiple occurrences of } \overline{\pi}_1 \text{ and/or } \overline{\pi}_2 \text{ in } w, \text{ with different relative locations. If so, then we need to consider each of them separately. For example, take } s_1 = ab, s_2 = bc \text{ and } w = abca. \text{ Then } \overline{\pi}_1 = ab \text{ and } \overline{\pi}_2 = ba \text{ both appear twice in } w, \text{ as shown below.}\]

\[
w = \frac{1}{1} ab - \frac{2}{1} ab = \frac{1}{1} bc \frac{2}{1} a \frac{1}{2} bc.
\]

Then we need to consider the four (pairs of) occurrences of \( \overline{\pi}_1 \) and \( \overline{\pi}_2 \) in \( w \), two of which are separated and two of which overlap.
Then the following statements are equivalent:

\[(s_1, s_2)_{\overline{1}} = s_1 - q_{l_12} = \sum_j c_j p_j b_j,\]

with \(c_j \in k, \ p_j \in \|X\|^*, \ t_j \in S\) and \(p_j b_j \overline{1} < \overline{1}\). Then

\[w = q_{l_21} = q_{l_12} = q_{l_1q_{l_22}} = q_{l_1q_{l_22}} = p_{l_1r_1}. \tag{24}\]

where \(p \in \|X\|^*\) is obtained from \(q_1\) by replacing \(*\) with \(q\).

Now \(S\) is a Gröbner-Shirshov basis. Hence we may write, by Case II that has been proved and in which we take \(q_1 = p\) and \(s_1 = s_2\),

\[p_{l_12} - q_{l_21} = \sum_i d_i r_{i} b_i,\]

where \(d_i \in k, \ r_i \in \|X\|^*\) and \(v_i \in S\) and \(r_i b_i < w\). Hence

\[q_{l_21} - q_{l_11} = p_{l_12} - \sum_i d_i r_{i} b_i - q_{l_11},\]

\[= q_{l_1q_{l_22}} - q_{l_11} - \sum_i d_i r_{i} b_i\]

\[= -q_{l_11} - q_{l_1q_{l_22}} - \sum_i d_i r_{i} b_i\]

\[= -\sum_j c_j q_{l_11} p_j b_j - \sum_i d_i r_{i} b_i\]

\[= -\sum_j c_j (q_{l_11} p_j b_j - \sum_i d_i r_{i} b_i).\]

Now \(t_j\) is in \(S\) and

\[(q_{l_11} p_j b_j)_{\overline{1}} = q_{l_11} p_j b_j \overline{1} < q_{l_11} \overline{1} = w.\]

Thus, we obtain \(q_{l_21} - q_{l_11} \equiv 0 \mod (S, w). \square\)

The following version of Composition-Diamond lemma can also be proved by the same argument as its nonunitary analogue ([Bokut, Chen, and Qiu 2010, Theorem 3.2]).

**Theorem 3.13.** (Composition-Diamond lemma) Let \(S\) be a set of monic bracketed polynomials in \(k\|X\|\), > a monomial ordering on \(\|X\|\) and \(\text{Id}(S)\) the operated ideal of \(k\|X\|\) generated by \(S\). Then the following statements are equivalent:

1. \(S\) is a Gröbner-Shirshov basis in \(k\|X\|\).
2. If \(f \neq 0\) is in \(\text{Id}(S)\), then \(f = q_{l_11} \overline{1}\) for some \(q \in \|X\|^*\) and \(s \in S\).
3. If \(f \neq 0\) is in \(\text{Id}(S)\), then

\[f = c_1 q_{l_11} + c_2 q_{l_22} + \cdots + c_n q_{l_n1}, \tag{25}\]

where \(c_i \in k, \ s_i \in S, \ q_i \in \|X\|^*, \ q_{l_11} \overline{1} > q_{l_21} \overline{1} > \cdots > q_{l_n1} \overline{1}\).
4. \(k\|Z\| = k\text{Irr}(S) \oplus \text{Id}(S)\) where

\[\text{Irr}(S) := \|X\| \setminus \{q_{l_11} \overline{1} q \in \|X\|^*, s \in S\}\]

and \(\text{Irr}(S)\) is a \(k\)-basis of \(k\|X\|/\text{Id}(S)\).
Before providing its proof, we give the following immediate corollary of the theorem.

**Corollary 3.14.** Let $I$ be an operated ideal of $k\llbracket X \rrbracket$. If $I$ has a generating set $S$ that is a Gröbner-Shirshov basis, then $\text{Irr}(S)$ is a $k$-basis of $k\llbracket X \rrbracket/I$.

**Proof.** (1) $\implies$ (2) Let $0 \neq f \in \text{Id}(S)$. Then by Lemma 3.2, $f$ is of the form

$$f = \sum_{i=1}^{k} c_i q_i|s_i, \quad 0 \neq c_i \in k, q_i \in M^*(X), s_i \in S, 1 \leq i \leq k. \quad (26)$$

Let $w_i = q_i|s_i$. We rearrange them in non-increasing order by

$$w_1 = w_2 = \cdots = w_m > w_{m+1} > \cdots > w_k.$$ 

If for each $0 \neq f \in \text{Id}(S)$ there is a choice of the above sum such that $m = 1$, then $\overline{f} = q_1|_{s_1}$ and we are done. So suppose the implication $(1) \implies (2)$ does not hold. Then there is $0 \neq f \in \text{Id}(S)$ such that for any expression in Eq. (26), we have $m \geq 2$. Fix such an $f$ and choose an expression in Eq. (26) such that $w_1 = q_1|_{s_1}$ is minimal and such that $m$ is minimal for this choice of $w_1$, that is, with the fewest $q_i|s_i$ such that $q_i|_{s_i} = q_1|_{s_1}$. Since $m \geq 2$, we have $q_1|_{s_1} = w_1 = w_2 = q_2|_{s_2}$.

Since $S$ is a Gröbner-Shirshov basis in $k\llbracket X \rrbracket$, by Lemma 3.12, we have

$$q_{2|s_2} - q_1|_{s_1} = \sum_{j} d_j p_j|_{r_j},$$

where $d_j \in k$, $r_j \in S$, $p_j \in \llbracket X \rrbracket^*$ and $p_j|_{r_j} < w_1$. Thus

$$f = \sum_{i=1}^{k} c_i q_i|s_i$$

$$= (c_1 + c_2) q_1|_{s_1} + c_3 q_3|s_3 + \cdots + c_m q_m|_{s_m} + \sum_{i=m+1}^{k} c_i q_i|s_i + \sum_{j} d_j p_j|_{r_j}.$$ 

By the minimality of $m$, we must have $c_1 + c_2 = c_3 = \cdots = c_m = 0$. We then obtain an expression of $f$ in the form of Eq. (26) for which $q_1|_{s_1}$ is even smaller. This is a contradiction.

(2) $\implies$ (3). Suppose the implication does not hold. Let $F$ be the set of counterexamples, namely those $0 \neq f \in \text{Id}(S)$ that cannot be written in the form of Eq. (25). Then the set $\{ \overline{f} | f \in F \}$ of leading terms is not empty. Then there is an $f$ such that $\overline{f}$ is minimal in this set. By Item (2), there are $q \in \llbracket X \rrbracket^*$ and $s \in S$ such that $\overline{f} = q|_{s}$. Since $f$ is in $F$ and $q|s$ is not in $F$, $f - q|s$ is not zero. But $\overline{f - q|s} = \overline{f} - q|s = 0$ means that $\overline{f - q|s}$ is less than $\overline{f}$. By the minimality of $\overline{f}$ in $F$, $\overline{f - q|s} \neq 0$ is not in $F$ and hence can be written in the form of Eq. (25). But this means that $f$ can also be written in the form of Eq. (25). This is a contradiction.

(3) $\implies$ (4). Obviously $0 \in k\text{Irr}(S) + \text{Id}(S) \subseteq k\llbracket X \rrbracket$. Suppose the inclusion is proper. Then $k\llbracket X \rrbracket/(k\text{Irr}(S) + \text{Id}(S))$ contains only nonzero elements. Let $f \in k\llbracket Z \rrbracket/(k\text{Irr}(S) + \text{Id}(S))$ be such that

$$\overline{f} = \min \{ \overline{g} | g \in k\llbracket X \rrbracket/(k\text{Irr}(S) + \text{Id}(S)) \}. \quad (27)$$
Suppose $\overline{f}$ is in Irr$(S)$, then $f \neq \overline{f}$ since $f \notin$ Irr$(S)$. So $0 \neq f - \overline{f}$ is in $k[\mathbb{Z}] \setminus (k\text{Irr}(S) + \text{Id}(S))$ with $f - \overline{f} < 1$. This is a contradiction. But suppose $\overline{f}$ is not in Irr$(S)$. Then $\overline{f} = q|_{\mathbb{P}}$ for some $q \in [\mathbb{Z}]^*$ and $s \in S$. Then $\overline{f} - q|_{\mathbb{P}} < 1$. If $f = q|_{\mathbb{P}}$, then $f$ is in Id$(S)$, a contradiction. Thus $f \neq q|_{\mathbb{P}}$. Then $0 \neq f - q|_{\mathbb{P}}$ with $f - q|_{\mathbb{P}} < 1$. By the minimality of $\overline{f}$ in Eq. (27), we see that $f - q|_{\mathbb{P}} \in k$ Irr$(S) + \text{Id}(S)$ and hence also $f \in k$ Irr$(S) + \text{Id}(S)$, again a contradiction. Therefore, $k[\mathbb{Z}] = k$ Irr$(S) + \text{Id}(S)$.

Suppose $k$ Irr$(S) \cap \text{Id}(S) \neq 0$ and let $0 \neq f \in k$ Irr$(S) \cap \text{Id}(S)$. Then $f = c_1v_1 + \cdots + c_kv_k$ with $v_1 > \cdots > v_k \in$ Irr$(S)$. Then by $f \in$ Id$(S)$ and Part (3), $\overline{f} = v_1$ is of the form $q|_{\mathbb{P}}$ for some $q \in [\mathbb{Z}]^*$ and $s \in S$. This is a contradiction to the construction of Irr$(S)$.

Therefore, $k[\mathbb{Z}] = k$ Irr$(S) \oplus \text{Id}(S)$ and hence Irr$(S)$ is a basis of $k[\mathbb{Z}] / \text{Id}(S)$.

(4) $\implies$ (1). We first prove a lemma.

**Lemma 3.15.** Suppose Item (4) holds. Let $0 \neq h \in \text{Id}(S)$ and let $w \in [\mathbb{Z}]$ such that $w > \overline{h}$. Then $h = \sum_i d_iq_i|_{\mathbb{P}}$ with $q_i|_{\mathbb{P}} < w$.

**Proof.** Denote $\text{Lead}(S) := \{ q|_{\mathbb{P}} \mid q \in [\mathbb{Z}]^*, s \in S \}$. Then by Item (4), we have the disjoint union $[\mathbb{Z}] = \text{Lead}(S) \sqcup \text{Irr}(S)$. Then for $0 \neq h \in \text{Id}(S)$, we can write

$$h = c_1u_1 + \cdots + c_ku_k$$

in which $u_1 > \cdots > u_k \in [\mathbb{Z}]$ and there is $1 \leq i_0 \leq k$ such that $u_{i_0} \in \text{Lead}(S)$ and all the previous terms, if there are any, are in Irr$(S)$. We call $u_{i_0}$ the first monomial of $h$ in Lead$(S)$. Suppose the conclusion of the lemma does not hold. Then we can choose our counter example $h$ such that the first monomial $u_{i_0}$ of $h$ is minimal with respect to the order $<$. Then we have $u_{i_0} = q|_{\mathbb{P}}$ for some $s \in S$. Consider

$$h' := h - q|_{\mathbb{P}} = c_1u_1 + \cdots + c_{i_0-1}u_{i_0-1} + c_{i_0}q_{i_0-1} + c_{i_0+1}u_{i_0+1} + \cdots + c_ku_k.$$  

Then we still have $\overline{h'} < w$. Since $h$ is a counter example, $h' \neq 0$. Since $q|_{\mathbb{P}}$ is in Id$(S)$, $h'$ is still in Id$(S)$. Since

$$q|_{\mathbb{P}} = q|_{\mathbb{P}} < q|_{\mathbb{P}} = u_{i_0},$$

the first monomial of $h'$ in Lead$(S)$ is smaller than $u_{i_0}$. By the minimality of $h$, we have $h' = \sum_i d_iq_i|_{\mathbb{P}}$ with $q_i|_{\mathbb{P}} < w$. Then $h = h' + q|_{\mathbb{P}}$ also has this property. This is a contradiction. □

Now suppose $f, g \in S$ give a composition. Let $F = f\mu$ and $G = vg$ in the case of intersection composition and let $F = f$ and $G = q|_{\mathbb{P}}$ in the case of including composition. Then we have $w := \overline{F} = \overline{G}$. If $(f, g)_w = F - G = 0$, then there is nothing to prove. If $(f, g)_w \neq 0$, then by Lemma 3.15, there are $q_j \in [\mathbb{Z}]^*$ and $s_j \in S$ such that $(f, g)_w = \sum_j d_jq_j|_{\mathbb{P}}$ with $q_j|_{\mathbb{P}} < w$. Hence $(f, g)_w$ is trivial modulo $(S, w)$.

4. **Differential type operators**

As remarked in the Introduction, we restrict our attention to those OPIs that are computationally feasible, in particular, to two families that are broad enough to include all the operators in Rota’s list, except the Reynolds operator. These families are identified by how they behave with respect to multiplication for which associativity is assumed. As differentiation is easier than integration, we progress more on differential type OPIs than on Rota-Baxter type ones.
4.1. Concepts and conjecture

Our model for differential type operators is the free differential algebra and its weighted generalization as considered in [Guo and Keigher 2008]. We refer the reader there for further details on construction of free (noncommutative) differential algebras of weight $\lambda$.

4.1.1. The concepts

The known OPIs that define an endomorphism operator, a differential operator, or a differential operator of weight $\lambda$ share a common pattern, based on which we will define OPIs of differential type. For this family of operators, we shall use the prefix notation $\delta(r)$ (or $\delta r$) for the image of $r$ in such an algebra, which is more traditional, but we shall continue to use the infix notation $\{r\}^\lambda$ in $k[[X]]$ to emphasize the string nature of bracketed expressions.

**Definition 4.1.** We say an expression $E(X) \in k[[X]]$ is in **differentially reduced form** (DRF) if it does not contain any subexpression of the form $[uv]$ for any non-units $u, v \in k[[X]]$. Let $\Sigma$ be a rewriting system [Baader and Nipkow 1998] in $k[[X]]$. We say $E(X)$ is $\Sigma$-**reducible** if $E(X)$ can be reduced to zero under $\Sigma$.

Let a set $X$ be given. Define $x^{(n)} \in [[X]], n \geq 0$, recursively by

$$x^{(0)} = x, x^{(k+1)} = [x^{(k)}], k \geq 0.$$  

Then

$$\Delta(X) := \{x^{(n)} | x \in X, n \geq 0\},$$

(28)

generates a monoid $M(\Delta(X))$ in $[[X]]$ and hence $k\langle \Delta(X) \rangle := kM(\Delta(X))$ (the noncommutative differential polynomial ring) is a subalgebra of $k[[X]]$. Then $E(X) \in k[[X]]$ is in DRF if and only if it is in $k\langle \Delta(X) \rangle$.

**Definition 4.2.** Let $\phi(x, y) := [xy] - N(x, y) \in k[[x, y]]$.

(1) Define an associated rewriting system

$$\Sigma_\phi := \{[ab] \mapsto N(a, b) | a, b \in \mathcal{M}(Z) \setminus \{1\}\},$$

(29)

where $Z$ is a set. More precisely, for $g, g' \in k[[Z]]$, denote $g \rightarrow_{\Sigma_\phi} g'$ if there are $q \in \mathcal{M}^*(Z)$ and $a, b \in \mathcal{M}(Z)$ such that

(a) $q|_{[ab]}$ is a monomial of $g$ with coefficient $c \neq 0$,

(b) $g' = g - cq|_{[ab]-N(a, b)}$.

In other words, $g'$ is obtained from $g$ by replacing a subword $[ab]$ in a monomial of $g$ by $N(a, b)$.

(2) An expression $E(a, b) \in k[[Z]]$ is **differentially $\phi$-reducible** if it is $\Sigma_\phi$-reducible.

The non-unit requirement in Eq. (29) is to avoid infinite rewriting of the form such as $[u] = [u \cdot 1] \mapsto N(u, 1)$, when $N(u, 1)$ may involve $[u]$. See Section 5 for this rewriting system in terms of reduction relations.

**Definition 4.3.** We say an OPI $\phi \in k[[x, y]]$, or the expression $\phi = 0$, is of **differential type** (OPIDT) if $\phi$ has the form $[xy] - N(x, y)$, where $N(x, y)$ satisfies the three conditions:
Remark 4.4. Condition 1 is imposed since we are only interested in linear operators. Condition 1 for all type, then so is Example 4.8. Note that Condition 3 is not equivalent to Example 4.5. which is always true. Here I and and Conjecture 4.7. (OPIs of Di 4.1.2. The OPIDT conjecture of differential type by finding all expressions Conjecture 4.7. (OPIs of Di 4.1.2. The OPIDT conjecture of differential type by finding all expressions Conjecture 4.7. (OPIs of Di 4.1.2. The OPIDT conjecture of differential type by finding all expressions

\[ \phi_{k[Z]}(uv, w) - \phi_{k[Z]}(u, vw) \in I_\phi([Z]) \forall u, v, w \in k[[Z]], \]

which is always true. Here \( I_\phi([Z]) \) is the operated ideal of \( k[[Z]] \) generated by the set

\[ \{ \phi_{k[Z]}(a, b) \mid a, b \in k[[Z]] \}. \]

Example 4.5. For any \( \lambda \in k \), the expressions \( \lambda xy \), \( \lambda [x] [y] \) (operators that are semi-endomorphisms), and \( \lambda [x] x \) (operators that are semi-antimorphisms) are of differential type. A differential operator of weight \( \lambda \) satisfies an OPI of differential type (Eq. (9)). This can be easily verified.

4.1.2. The OPIDT conjecture

We can now state the classification problem of differential type OPIs and operators

Problem 4.6. (Rota’s Problem: the Differential Case) Find all operated polynomial identities of differential type by finding all expressions \( N(x, y) \in k[[x, y]] \) of differential type.

We propose the following answer to this problem.

Conjecture 4.7. (OPIs of Differential Type) Let \( k \) be a field of characteristic zero. Every expression \( N(x, y) \in k[[x, y]] \) of differential type takes one (or more) of the forms below for some \( a, b, c, e \in k \):

1. \( b(x[y] + [x]y) + c[x][y] + e xy \) where \( b^2 = b + ce \),
2. \( ce^2 yx + e xy + c[y][x] - ce(y[x] + [y]x) \),
3. \( \sum_{i,j \geq 0} a_{ij} [1]^j [x][y]^j \) with the convention that \( [1]^0 = 1 \).
4. \( x[y] + [x]y + ax[1] + by \),
5. \( [x]y + a(x[1] y - xy[1]) \),
6. \( x[y] + a(x[1] y - [1]xy) \).

Note that the list is not symmetric in \( x \) and \( y \). One might think that if \( N(x, y) \) is of differential type, then so is \( N(y, x) \). But this is not true.

Example 4.8. \( N_1(x, y) := x[y] \) is of differential type since

\[
N_1(uv, w) - N_1(u, vw) = uv[w] - u[vw] \\
\rightarrow uv[w] - u[vw] = 0
\]

for all \( u, v, w \in k[Z] \). However, \( N_2(x, y) := y[x] \) is not, since
Theorem 4.9. will be given in Section 5.

4.2. Evidence for the conjecture

Proof. Case 1 by Eq. (30). Again, by Eq. (30), for a non-unit

\[ \phi = b \]

Then the rewriting rule \[ u \rightarrow (uv)[u] = (uv - vu)[u] \],

which is in DRF (no further reduction using \( \Sigma_\phi \) is possible, where \( \phi := [xy] - N_2(x, y) \)) but non-zero. See also Example 2.11.

4.2.1. Verification of the operators

Theorem 4.9. The OPI \( \phi := [xy] - N(x, y) \), where \( N(x, y) \) is any expression listed in Conjecture 4.7 is of differential type.

Proof. Clearly, all six expressions are in DRF. We check \( \phi \)-reducibility for the first two cases.

**Case 1.** Here \( N(x, y) := b(x[y] + [x]y) + c[x][y] + exy \), where \( b^2 = b + ce \). We have

\[
N(x, y) = cN(x, y) + bxy = (c[x] + bx)(c[y] + by).
\]

Let \( \alpha \) be the operator defined by \( \alpha(u) := c[u] + bu \) for \( u \in k[x, y] \). Then for any non-units \( u, v \in k[x, y] \), the rewriting rule \( [uv] \rightarrow N(u, v) \) gives the rewriting rules

\[
\alpha(uv) = c[uv] + buv \rightarrow cN(u, v) + buv = \alpha(u)\alpha(v)
\]

by Eq. (30). Again, by Eq. (30), for a non-unit \( w \), we have

\[
N(uv, w) + b(\alpha(w)uv) = \alpha(u)(\alpha(w)v) \rightarrow \alpha(u)(\alpha(w)v)
\]

Then \( N(uv, w) - N(u, vw) \) is differentially \( \phi \)-reducible by associativity. If \( c \neq 0 \), then \( N(x, y) \) is of differential type. Suppose \( c = 0 \). The constraint \( b^2 = b + ce \) becomes \( b^2 = b \) and either \( b = 0 \) or \( b = 1 \). When \( b = 0 \), \( \phi = [xy] - exy \) (semi-endomorphism case), and when \( b = 1 \), \( \phi = [xy] - (x[y] + [x]y + exy) \). These are easily verified directly to be OPIs of differential type.

**Case 2.** Here \( N(x, y) := ce^2xy + exy + c[y][x] - ce(y[x] + [y]x) \) and we have \( N(x, y) - exy = c([y] - ey)([x] - ex) \). Let \( \alpha(u) = [u] - eu \) and the rest of the proof is similar to Case 1.

For the remaining cases, it is routine to check that \( N(uv, w) - N(u, vw) \) is differentially \( \phi \)-reducible for \( \phi := [xy] - N(x, y) \). For example, for Case 5, we have, using associativity,

\[
N(uv, w) = [uv]w + a(uv[1]w - uvw[1])
\]

\[ \rightarrow (u[1]v + a(u[1]v - uv[1]))w + a(uv[1]w - uvw[1]) = [u]vw + a(u[1]vw - uvw[1]) = N(u, vw).
\]
4.2.2. Computational evidence

Definition 4.10. The operator degree of a monomial in $k\langle X \rangle$ is the total number that the operator $\{ \}$ appears in the monomial. The operator degree of a polynomial $\phi$ in $k\langle X \rangle$ is the maximum of the operator degrees of the monomials appearing in $\phi$.

Theorem 4.11. Let $k$ be a field. The only expressions $N(x, y)$ of differential type for which the total operator degrees $\leq 2$ are the ones listed in Conjecture 4.7. More precisely, the only expressions of differential type in the form

$$\begin{align*}
N(x, y) &:= a_{0,0}xy + a_{0,1}x[y] + a_{0,2}x[y][y] + a_{1,0}[x]y + a_{1,1}[x][y] \\
&\quad + a_{1,2}[x][y][y] + a_{2,0}[x][y] + a_{2,1}[x][y] + a_{2,2}[x][y][y] \\
&\quad + b_{0,0}yx + b_{0,1}y[x] + b_{0,2}y[x][x] + b_{1,0}[y]x + b_{1,1}[y][x] \\
&\quad + b_{1,2}[y][x][x] + b_{2,0}[y][y][x] + b_{2,1}[y][y][x] + b_{2,2}[y][y][x][x]
\end{align*}$$

where $a_{i,j}, b_{i,j} \in k$ ($0 \leq i, j \leq 2$), are the ones listed.

Proof. This is obtained and verified by computations in Mathematica [Wolfram (2008)]. See Section 7 for a brief description and Sit (2010) for details and results.

5. Relationship of differential type operators with convergent rewriting and Gröbner-Shirshov bases

We now characterize OPIDT in terms of convergent rewriting systems and Gröbner-Shirshov bases as we have discussed in Section 3. We quote the following basic result of well order for reference.

Lemma 5.1. (1) Let $A$ and $B$ be two sets with well-orderings. Then we obtain an extended well order on the disjoint union $A \sqcup B$ by defining $a < b$ for all $a \in A$ and $b \in B$.

(2) Let $A$ be a set with a well order. Then the lexicographic order on $M(A)$ is a well order.

Let $>$ be a well-ordering on a set $Z$. We extend $>$ to a well-ordering on $\mathfrak{M}(Z) = \lim \mathfrak{M}_n(Z)$ by recursively defining a well-ordering $>_n$, on $\mathfrak{M}_n := \mathfrak{M}_n(Z)$ for each $n \geq 0$. Denote by $\deg_2(u)$ the number of $x \in Z$ in $u$ with repetition. When $n = 0$, we have $\mathfrak{M}_0 = M(Z)$. In this case, we obtain a well-ordering by taking the lexicographic order $>_\text{lex}$ on $M(Z)$ induced by $>$ with the convention that $u >_{\text{lex}} 1$ for all $u \in M(Z) \setminus \{1\}$. Suppose $>_n$ has been defined on $\mathfrak{M}_n := M(Z \sqcup \{\mathfrak{M}_{n-1}\})$ for an $n \geq 0$. Then $>_n$ induces

(1) a well-ordering $>_n'$ on $[\mathfrak{M}_n]$ by

$$[u] >_n' [v] \iff u >_n v;$$

(2) then a well-ordering $>_n''$ on $Z \sqcup [\mathfrak{M}_n]$ by Lemma 5.1.(1);
(3) then a well-ordering $>_n^{''''}$ on $Z \cup \langle M_n \rangle$ by

$$u>_n^{''''}v \iff \begin{cases} 
\text{either } \deg_z(u) > \deg_z(v) \\
\text{or } \deg_z(u) = \deg_z(v) \text{ and } u>_n^{'''}v. 
\end{cases}$$

(32)

(4) then the lexicographic well-ordering $>_n$ on $M_n = M(Z \cup \langle M_n \rangle)$ induced by $>_n^{''''}$. The orders $>_n$ are compatible with the direct system $(M_n)_{n \geq 0}$ and hence induces a well-ordering, still denoted by $>$, on $M(Z) = \lim_n M_n$.

**Example 5.2.** Under this order, $[xy]$ is greater than $1$, $x$, $y$ and their iterated operations under $\lfloor \rfloor$. Thus $[xy]$ is the leading term for $\phi(x, y) = [xy] - N(x, y)$ when $N(x, y)$ is in DRF, in particular, for those $N(x, y)$ listed in Conjecture 4.7.

**Lemma 5.3.** The order $>$ on $M(Z)$ is a monomial order.

*Proof.* We prove by induction on $n \geq 0$ the claim that for any $q \in M^*(Z) \cap M_n(Z \cup \bullet)$, $u > v$ in $M(Z)$ implies $q|_u > q|_v$.

When $n = 0$, we have $q \in M(Z \cup \bullet)$ in which $\bullet$ only appears once. Thus $q = a \star b$ with $a, b \in M(Z)$. Thus $u > v$ in $M(Z)$ implies that $aub > avb$ by the definition of lexicographic order.

Suppose the claim has been proved for all $q \in M^*(Z) \cap M_n(Z \cup \bullet)$ for an $n \geq 0$. Consider $q \in M^*(Z) \cap M_{n+1}(Z \cup \bullet)$. Then $q = apb$ with $p \in M^*(Z) \cap M_{n+1}(Z \cup \bullet)$ being indecomposable and $a, b \in M_n(Z)$. Thus $p \in Z$ is impossible. So we have $p \in M_n(Z \cup \bullet)$. Then $p = [p']$ and $p$ is in $M^*(Z) \cap M_n(Z \cup \bullet)$. Thus by the induction hypothesis, if $u > v$, then $p'|_u > p'|_v$. Then by Eq. (31), we also have $p|_u > p|_v$ and hence $q|_u > q|_v$ by the lexicographic order. This completes the induction. \[\square\]

We next extend the concept of reduction relation from polynomial algebras $k[Z]$ (Baader and Nipkow, 1998, Section 8.2) to operated polynomial algebras $k\langle Z \rangle$.

**Definition 5.4.** Let $Z$ be a set and let $<$ be a monomial well-ordering on $M(Z)$. Let $f \in k\langle Z \rangle$ be monic. We use $f$ to define the following reduction relation $\rightarrow_f$: For $g, g' \in k\langle Z \rangle$, define $g \rightarrow_f g'$ if there is $q \in M^*(Z)$ such that

1. $q|_f$ is a monomial of $g$ with coefficient $c$,
2. $g' = g - cq|_f$.

In other words, $g'$ is obtained by replacing a subword $\overline{f}$ in a monomial of $g$ by $\overline{f} - f$. If $F$ is a set of monic bracketed polynomials, we define $\rightarrow_F := \cup_{f \in F} \rightarrow_f$.

We refer the reader to Baader and Nipkow (1998) for concepts in rewriting systems, such as joinable and convergence.

**Proposition 5.5.** Let $Z$ be a set and let $M(Z)$ be equipped with a monomial well-ordering $<$. Let $F$ be a set of monic bracketed polynomials. Then the reduction relation $\rightarrow_F$ is a terminating relation.

See Baader and Nipkow (1998, Prop. 8.2.9) for the case of polynomials.
Proof. For each $f \in \mathbb{k}[\mathbb{Z}]$, let $M(f)$ denote the set of monomials in $f$. Let $>_\text{mul}$ denote the multiset order on the set $\mathcal{M}(\mathbb{N}(\mathbb{Z}))$ of finite multisets over $\mathbb{N}(\mathbb{Z})$ induced by $>$ on $\mathbb{N}(\mathbb{Z})$. Then by (Baader and Nipkow, 1998, Theorem 2.5.5), the order $>_\text{mul}$ is terminating. Thus we just need to show that if $g \rightarrow_F g'$, then $M(g) >_{\text{mul}} M(g')$. If $g \rightarrow_F g'$, then there are $f \in F$, $q \in \mathbb{N}^*(\mathbb{Z})$ such that $q_f$ is a monomial of $g$ with coefficient $c \neq 0$ and such that $g' = g - cq_f$. Since $<$ is a monomial well order, all terms in $q_f \mathbb{Z}$ are smaller than $q_f \mathbb{Z}$. Thus $M(g')$ is obtained from $M(g)$ by replacing the monomial $q_f \mathbb{Z}$ by smaller monomials. This implies $M(g) >_{\text{mul}} M(g')$. \hfill $\square$

We also prove the following variation of (Baader and Nipkow, 1998, Lemma 8.3.3).

Lemma 5.6. Let $f, g \in \mathbb{k}[\mathbb{Z}]$. If $f - g$ is reduced to zero, then $f$ and $g$ are joinable.

Proof. We use induction on the number $n$ of iterations of applying $\rightarrow_F$ to $f - g$ to get zero. If $n = 0$, then $f - g = 0$ and there is nothing to prove. Suppose the conclusion of the lemma holds with $n \geq 0$ iterations and consider the case of $n + 1$. Suppose the first reduction relation is $\rightarrow_f$ for an $f_i \in F$ by applying $f_i$ to a monomial $m$ and $m$ appears in $f$ (resp. $g$) with coefficient $a$ (resp. $b$). So $m = q_f$ for some $q \in \mathbb{N}^*(\mathbb{Z})$. Then we obtain $f - g \rightarrow_F h$ where

$$h = (f - g) - (a - b)q_f = (f - aq_f) - (g - bq_f).$$

Since $h$, that is the right hand side, is reduced to zero with $n$ iterations of reductions, by the induction hypothesis, $f - aq_f$ and $g - bq_f$ are joinable. Then it follows that $f$ and $g$ are joinable. \hfill $\square$

Theorem 5.7. Let $\phi(x, y) := \delta(xy) - N(x, y) \in \mathbb{k}[x, y]$ with $N(x, y)$ in DRF and totally linear in $x, y$. The following statements are equivalent:
(1) $\phi(x, y)$ is of differential type;
(2) The rewriting system $\Sigma_\phi$ is convergent;
(3) Let $Z$ be a set with a well-ordering. With the order $>$ in Eq. (32), the set

$$S := S_\phi := \{\phi(u, v) = \delta(uv) - N(u, v) | u, v \in \mathcal{M}(\mathbb{Z}) \setminus \{1\}\}$$

is a Gröbner-Shirshov basis in $\mathbb{k}[\mathbb{Z}]$.
(4) The free $\phi$-algebra on a set $Z$ is the noncommutative polynomial $\mathbb{k}$-algebra $\mathbb{k}(\Delta(Z))$ where $
\Delta(Z)$ is defined in Eq. (28), together with the operator $\partial := d_Z \in \mathbb{k}(\Delta(Z))$ defined by the following recursion:

Let $u = u_1u_2 \cdots u_k \in \Delta(Z)$, where $u_i \in \Delta(Z)$, $1 \leq i \leq k$.
(a) If $k = 1$, i.e., $u = \partial(x)$ for some $x \in \mathbb{Z}$, then define $d(u) = \partial(x)$.
(b) If $k \geq 1$, then recursively define $d(u) = N(u_1, u_2 \cdots u_k)$.

By Theorem 4.9, we have

Corollary 5.8. Let $N(x, y)$ be from the list in Conjecture 4.7. Then all the statements in Theorem 5.7 hold.

When $N(x, y) = x\delta(y) + \delta(x)y + \lambda \delta(x)\delta(y)$, we obtain (Bokut, Chen, and Qiu, 2010, Theorem 5.1).

Proof. (1) $\implies$ (2) We first note that the rewriting system $\Sigma_\phi$ in Definition 4.2 is the same as the reduction relation $\rightarrow_{S_\phi}$ with

$$S_\phi := \{\phi(u, v) | u, v \in \mathbb{k}[\mathbb{Z}]\},$$

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Then we have\(q_{1|\Sigma} = f = q_{2|\Sigma},\)

\[q_{1} = q_{2|\Sigma}, \quad q_{1} = q_{2|\Sigma} = q_{2|\Sigma}.
\]

Since \(s_{1}, s_{2}\) are in \(S_{\phi}(Z)\), we can write

\[s_{1} = \phi(u, v) = \delta(uv) - N(u, v) = \delta(uv) - \sum_{i} c_{i}\phi_{i}(u, v),\]

\[s_{2} = \phi(r, s) = \delta(rs) - N(r, s) = \delta(rs) - \sum_{i} c_{i}\phi_{i}(r, s),\] \hspace{1cm} (33)

for some \(u, v, r, s \in \mathcal{M}(Z)\backslash\{1\}\). Here we have used the notation

\[N(x, y) = \sum_{i=1}^{k} c_{i}\phi_{i}(x, y), \phi(x, y) \in \mathcal{M}(x, y), 1 \leq i \leq k.
\]

As in the proof of Lemma 3.12, there are three cases to consider.

**Case I.** Suppose the bracketed words \(\Sigma_{1}\) and \(\Sigma_{2}\) are disjoint in \(f\). Let \(q \in \mathcal{M}^{*} \ast (X)\) be the \((\ast_{1}, \ast_{2})\)-bracketed word obtained by replacing this occurrence of \(\Sigma_{1}\) (resp. \(\Sigma_{2}\)) in \(f\) by \(\ast_{1}\) (resp. \(\ast_{2}\)). Then we have

\[f = q_{1|\Sigma_{1}} = q_{2|\Sigma_{2}} = q_{2|\Sigma_{1}}.
\]

Then we have

\[q_{1} = q_{1|\Sigma_{1}} - q_{2|\Sigma_{2}}, \quad q_{2} = q_{2|\Sigma_{1}} - q_{1|\Sigma_{2}}.
\]

Hence

\[g_{1} = g_{1} = q_{1|\Sigma_{1}} - q_{2|\Sigma_{2}} - q_{1|\Sigma_{2}} - q_{2|\Sigma_{1}}.
\]

Similarly, \(g_{2} \rightarrow_{\Sigma_{1}} q_{1|\Sigma_{1}} - q_{2|\Sigma_{2}} - q_{2|\Sigma_{1}} - q_{1|\Sigma_{2}}\). This proves the local confluence.

**Case II.** Suppose the bracketed words \(\Sigma_{1}\) and \(\Sigma_{2}\) have nonempty intersection in \(f\) but are not a proper subword of each other. Since \(\Sigma_{1} = \delta(uv)\) and \(\Sigma_{2} = \delta(rs)\) are indecomposable in \(\mathcal{M}(Z)\), this is possible only when \(\delta(uv) = \delta(rs)\). Thus \(uv = rs\). Factoring each of \(u, v, r, s\) into standard decompositions, we see that there are \(a, b, c \in \mathcal{M}(Z)\) such that \(u = ab, v = c\) and \(r = a, s = bc\). Then we have \(\Sigma_{1} = \delta(abc) = \Sigma_{2}\) and

\[g_{1} = g_{2} = N(ab, c) - N(a, bc).
\]

Since \(u, v, r, s \neq 1\), we have \(a, c \neq 1\). If \(b = 1\), then \(g_{1} = g_{2}\) is already zero. If \(b \neq 1\), then since \(\phi\) is of differential type, \(g_{1} - g_{2}\) is reduced to zero. Then by Lemma 5.6, \(g_{1}\) and \(g_{2}\) are joinable.

**Case III.** Suppose one of the bracketed words \(\Sigma_{1}\) and \(\Sigma_{2}\) is contained in the other. Without loss of generality, suppose \(\Sigma_{1} = q_{l|\Sigma_{2}}\) for some \(\ast\)-bracketed word \(q \in \mathcal{M}^{*}(Z)\). This means that \(\delta(uv) = \Sigma_{1} = q_{l|\Sigma_{2}} = q_{l|\Sigma_{3}}\). Then \(q = \delta(q')\) for some \(\ast\)-bracketed word \(q'\) and hence \(\delta(uv) = q_{l|\Sigma_{3}} = \delta(q'_{l|\Sigma_{3}})\). This gives \(uv = q_{l|\Sigma_{3}}\). Since \(u, v \in \mathcal{M}(Z)\backslash\{1\}\), we have either \(q' = pv\) with \(p_{l|\Sigma_{3}} = u\) or \(q' = up\) with \(p_{l|\Sigma_{3}} = v\), where \(p \in \mathcal{M}^{*}(Z)\). Without loss of generality, suppose \(q' = pv\) with \(p_{l|\Sigma_{3}} = u\). Then we have

\[N(p_{l|\Sigma_{3}}, v) = N(u, v) \Sigma_{3} = \delta(uv) = \delta(p_{l|\Sigma_{3}}v) \rightarrow_{\Sigma_{3}} \delta(p_{l|\Sigma_{3}}v).
\]
Using the notations in Eq. (33), we obtain

\[ N(p|_{\delta(r)}, v) - \delta(p|_{\delta(r)}, v) = \sum_{i=1}^{k} c_i \phi_i(p|_{\delta(r)}, v) - \sum_{i=1}^{k} c_i \delta(p|_{\delta(r)}, v) \]

\[ \mapsto \sum_{i=1}^{k} c_i \phi_i(p|_{\delta(r)}, v) - \sum_{i=1}^{k} c_i N(p|_{\delta(r)}, v) \]

\[ = \sum_{i=1}^{k} c_i \sum_{j=1}^{k} c_j \phi_i(p|_{\delta(r)}, v) - \sum_{i=1}^{k} c_i \sum_{j=1}^{k} c_j \phi_j(p|_{\delta(r)}, v) \]

\[ = 0 \]

since the two double sums become the same after exchanging \( i \) and \( j \).

Note that \( N(p|_{\delta(r)}, v) = s_1 - s_1 \) and \( \delta(p|_{\delta(r)}, v) = q|_{\delta(r)} \). We then see that \( s_1 - s_1 = q \). Then \( N(p|_{\delta(r)}, v) \) and \( \delta(p|_{\delta(r)}, v) \) are joinable by (Baader and Nipkow, 1998, Lemma 8.3.3). Then

\[ H_{\mu, \nu} = X \]

resulting composition is joinable by (Baader and Nipkow, 1998, Lemma 8.3.3). Then

\[ N \in \{ \mu, \nu \} \]

Since \( H_{\mu, \nu} = X \), we have

\[ \delta \]

The case of intersection compositions. By the definition of \( N(x, y) \) being in DRF, we have

\[ \phi(x, y) = \delta(xy). \]

Let two elements of \( S \) be given. They are of the form

\[ f := \phi(u, v), \quad g := \phi(r, s), \quad u, v, r, s \in \mathcal{M}(Z) \setminus \{1\}. \]

Hence \( \overline{f} = \delta(uv) \) and \( \overline{g} = \delta(rs) \). Suppose \( w = \overline{f} \mu = \overline{g} \) gives an intersection composition, where \( \mu, \nu \in \mathbb{X} \). Since \( \overline{(f)} = \overline{(g)} = 1 \), we must have \( |w| < |\overline{f}| + |\overline{g}| = 2 \). Thus \( |w| = 1 \). This means that \( |\mu| = |\nu| = 0 \). Since \( f, g \) are monic, we have \( \mu = \nu = 1 \). Thus \( w = \overline{f} = \overline{g} \). That is, \( \delta(uv) = \delta(rs) \).

Thus \( uv = rs \). Factoring each of \( u, v, r, s \) into standard decompositions, we see that there are \( a, b, c \in \mathbb{X} \) such that \( u = ab, v = c \) and \( r = a, s = bc \). Therefore, \( f = \phi(ab, c) \) and \( g = \phi(a, bc) \) is the only pair that gives intersection composition. Then we have \( w = \overline{f} = \overline{g} = \delta(abc) = \overline{\gamma} \) and the resulting composition is

\[ (f, g)_w := f - g = -N(ab, c) + N(a, bc) \]  

(34)

Since \( N(ab, c) \xrightarrow{\Sigma} \delta(abc) \xrightarrow{\Sigma} N(a, bc) \) and \( \Sigma_{\phi} \) is confluent, we find that \( N(ab, c) \) and \( N(a, bc) \) are joinable. Hence \( N(ab, c) - N(a, bc) \) is reduced to zero. In particular, \( N(ab, c) - N(a, bc) \) is in \( \text{Id}(S) \). Since \( \phi(ab, c) = \delta(abc) = \phi(a, bc) \), we have \( N(ab, c) < \delta(abc) \) and \( N(a, bc) < \delta(abc) \).

Thus \( N(ab, c) - N(a, bc) \) is trivial modulo \( (S, \delta(abc)) \).

The cases of including compositions. On the other hand, \( f \) and \( g \) could only have the following including compositions:

1. If \( u = p|_{\delta(r)} \) for some \( p \in \mathcal{M}(Z) \), then

\[ w := \overline{f} = q|_{\delta(r)} = \delta(p|_{\delta(r)}, v) \]

with \( q := \delta(pv) \).
(2) If \( v = p|_{\phi(r,s)} \) for some \( p \in \mathfrak{M}^*(Z) \), then
\[
w := f - q|_g = \delta(u \, p|_{\phi(r,s)})
\]
with \( q := \delta(up) \).

So we just need to check that in both cases these compositions are trivial modulo \((S, w)\). Consider the first case. Using the notation in Eq. (33), this composition is

\[
(f, g)_w := f - q|_g = \delta(uv) - \sum_{i=1}^{k} c_i \phi(u, v) - \delta(p|_g v)
\]

\[
= \delta(p|_{\phi(r,s)} v) - \sum_{i=1}^{k} c_i \phi(p|_{\phi(r,s)}), v) - \left( \delta(p|_{\phi(r,s)} v) - \sum_{i=1}^{k} c_i \delta(p|_{\phi(r,s)} v) \right)
\]

\[
= - \sum_{i=1}^{k} c_i \phi(p|_{\phi(r,s)} v) + \sum_{i=1}^{k} c_i \delta(p|_{\phi(r,s)} v)
\]

\[
= - \sum_{i=1}^{k} c_i \phi(p|_{\phi(r,s)} v) - \sum_{i=1}^{k} c_i \phi(p|_{\phi(r,s)} v) + \sum_{i=1}^{k} c_i \phi(p|_{\phi(r,s)} v) + \sum_{i=1}^{k} c_i N(p|_{\phi(r,s)} v)
\]

\[
= - \sum_{i=1}^{k} c_i \phi(p|_{\phi(r,s)} v) + \sum_{i=1}^{k} c_i \phi(p|_{\phi(r,s)} v)
\]

\[
= - \sum_{i=1}^{k} c_i \phi(p|_{\phi(r,s)} v) + \sum_{i=1}^{k} c_i \phi(p|_{\phi(r,s)} v)
\]

since the double sums become the same after exchanging \( i \) and \( j \). Since \( \phi(r, s) = \delta(rs) \) we have \( \phi|_{\phi(r,s)} = \phi|_{\phi(r,s)} v < w \). Thus the first sum is trivial modulo \((S, w)\). Further every term \( u_i := \phi(p|_{\phi(r,s)} v) \) in the second sum is already in \( S \). So it is just \( \star|_{uc} \) for the \( \star \)-bracketed word \( \star \). We have

\[
\overline{u_i} = \overline{\phi(p|_{\phi(r,s)} v)} = \delta(p|_{\phi(r,s)} v) < w.
\]

Thus the second sum is also trivial modulo \((S, w)\). This proves \( (f, g)_w \equiv 0 \mod (S, w) \).

The proof of the second case is the same.

(3) \( \Rightarrow \) (1) Suppose that a \( \phi(x, y) := \delta(xy) - N(x, y) \in k\|x, y\| \) with \( N(x, y) \) in DRF is such that

\[
S := \{ \phi(u, v) \mid u, v \in k\|Z\| \}
\]

is a Gröbner-Shirshov basis in \( k\|Z\| \) for any \( Z \) with the order \( > \) in Eq. (32). Let \( a, b, c \in \mathfrak{M}(Z) \{1\} \).

For \( f = \phi(ab, c), g = \phi(a, bc) \), we have \( w := f = \phi = \delta(ab) = v|_\phi \) with \( \mu = \nu = 1 \). Thus we have an intersection composition

\[
(f, g)_w^{1,1} := f - g = -N(ab, c) + N(a, bc).
\]
If $N(ab, c) = N(a, bc)$, then there is nothing to prove. If $N(ab, c) - N(a, bc) \neq 0$, then since $-N(ab, c) + N(a, bc)$ is in $\text{Id}(S)$ and $S$ is a Gröbner-Shirshov basis, by Theorem 3.13, we have

$$-N(ab, c) + N(a, bc) = \sum_{i=1}^{n} a_i q_i s_i,$$

where $a_i \in k$, $q_i \in \mathfrak{M}(Z)$ and $s_i \in S$, $1 \leq i \leq n$. This means that $-N(ab, c) + N(a, bc)$ is reduced to zero by the rewriting system $\Sigma_\phi$ defined in Eq. (29). Hence $\phi$ is of differential type.

$(4) \implies (3)$ Suppose Item 4 holds. Then in particular $M(\Delta(Z))$ is a linear basis of $k[Z]/I_\phi(Z)$. Then the conclusion follows from $(4) \implies (1)$ in Theorem 3.13.

$(3) \implies (4)$ By Theorem 3.13 and Corollary 3.14, $M(\Delta(Z))$ is a basis of the free $\phi$-algebra $k[Z]/I_\phi(Z)$ in Proposition 2.10. Therefore, the restriction map

$$k(\Delta(Z)) = kM(\Delta(Z)) \rightarrow k[Z] \rightarrow k[Z]/I_\phi(Z)$$

is a linear isomorphism. Since $kM(\Delta(Z))$ is closed under the multiplication on $k[Z]$, we see that this linear isomorphism is an algebra isomorphism. The recursive definition of the operator $d$ follows from the fact that it is the operator $\delta$ on $k[Z]$ modulo $I_\phi(Z)$ and hence satisfies

$$\delta(uv) = N(u, v), \forall u, v \in M(\Delta(Z)).$$

$\square$

6. Rota-Baxter type operators

We just give a brief discussion of Rota-Baxter type operators. Their study is more involved than differential type operators and will be left to a future work.

**Definition 6.1.** We say an expression $E(X) \in k[[X]]$ is in Rota-Baxter reduced form (RBRF) if it does not contain any subexpression of the form $[u][v]$ for any $u, v \in k[[X]]$.

**Definition 6.2.** An OPI $\phi \in k[[x, y]]$ is of Rota-Baxter type if it has the form $[x][y] - [M(x, y)]$ for some $M(x, y) \in k[[x, y]]$ that satisfies the two conditions:

1. $M(x, y)$ is totally linear in $x, y$ in the sense that $x$ (resp. $y$) appears exactly once in each monomial of $M(x, y)$;
2. $M(x, y)$ is in RBRF;
3. $M(M(u, v), w) - M(u, M(v, w))$ is $\Pi_\phi$-reducible for all $u, v, w \in k[[x, y]]$, where $\Pi_\phi$ is the rewriting system

$$\Pi_\phi := \{|[a][b] \mapsto [M(a, b)] \mid a, b \in k[[x, y]]\}.$$  

If $\phi := [x][y] - [M(x, y)]$ is of Rota-Baxter type, we also say the expression $M(x, y)$, and the defining operator $P$ of a $\phi$-algebra $S$ are of Rota-Baxter type.

**Example 6.3.** The expression $M(x, y) := x[y]$ that defines the average operator is of Rota-Baxter type since

$$M(M(u, v), w) - M(u, M(v, w)) = M(u, v)[w] - u[M(v, w)] = u[v][w] - u[v][w] \mapsto u[v][w] - u[v][w] = 0.$$
Other examples are OPIs corresponding to a Rota-Baxter operator or a Nijenhuis operator.

**Problem 6.4. (Rota’s Classification Problem: Rota-Baxter Case)** Find all Rota-Baxter type operators. In other words, find all Rota-Baxter type expressions $M(x, y) \in k[[x, y]]$.

We propose the following answer to this problem.

**Conjecture 6.5. (OPIs of Rota-Baxter Type)** For any $d, \lambda \in k$, the expressions $M(x, y)$ in the list below are of Rota-Baxter type (new types are underlined). Moreover, any OPI of Rota-Baxter type is necessarily of the form

$$\phi := [x][y] - [M(x, y)],$$

for some $M(x, y)$ in the list.

1. $x[y]$ (average operator),
2. $[x]y$ (inverse average operator),
3. $x[y] + y[x],$
4. $[x]y + y[x],$
5. $x[y] + [x]y - [xy]$ (Nijenhuis operator),
6. $x[y] + [x]y + \lambda xy$ (Rota-Baxter operator of weight $\lambda$),
7. $x[y] - x[1][y] + \lambda xy,$
8. $[x]y - x[1][y] + \lambda xy,$
9. $[x]y + [x]y - x[1][y] + \lambda xy$ (generalized Leroux TD operator with weight $\lambda$),
10. $x[y] + [x]y - x[1][y] - x[1][y] + \lambda xy,$
11. $x[y] + [x]y - x[1][y] - [xy] + \lambda xy,$
12. $x[y] + [x]y - x[1][y] - [1][xy] + \lambda xy,$
13. $d[x][1][y] + \lambda xy$ (generalized endomorphisms),
14. $d[y][1][x] + \lambda xy$ (generalized antimorphisms).

**Remark 6.6.** Let $Z$ be any set. Recall that the bracketed words in $Z$ that are in RBRF when viewed as elements of $k[[Z]]$ are called Rota-Baxter words and they form a $k$-basis of the free Rota-Baxter $k$-algebra on $Z$. See [Guo and Sit (2006)]. Every expression in RBRF is a $k$-linear combination of Rota-Baxter words in $k[[x, y]]$.

More generally, if $\phi(x, y)$ is of Rota-Baxter type, then the free $\phi$-algebras on a set $Z$ in the corresponding categories of Rota-Baxter type $\phi$-algebras have special bases that can be constructed uniformly. Indeed, if $k[Z]'$ denotes the set of Rota-Baxter words in $k[[Z]]$, then the map

$$k[Z] \to k[[Z]] \to k[[Z]]/I_\phi(Z)$$

is bijective. Thus a suitable multiplication on $k[Z]'$ makes it the free $\phi$-algebra on $Z$. This is in fact how the free Rota-Baxter $k$-algebra on $Z$ is constructed when $\phi(x, y)$ is the OPI corresponding to the Rota-Baxter operator, the Nijenhuis operator [Lei and Guo (2012)] and the TD operator [Zhou (2011)].

7. Computational experiments

In this section, we give a brief description of the computational experiments in Mathematica that result in Conjectures 4.7 and 6.5. The programs consist of several Notebooks, available at [Sit (2010)] in a zipped file.

Basically, the non-commutative arithmetic for an operated algebra was implemented ad hoc, using bracketed words and relying as much as possible on the built-in facilities in Mathematica.
for non-commutative multiplication, list operations, rewriting, and equation simplification. Care was taken to avoid infinite recursions during rewriting of expressions. An elaborate ansatz with indeterminate coefficients (like the expression \( N(x, y) \) in Theorem 4.11) is given as input, and to obtain differential type OPIs, the difference \( N(uv, w) - N(u, vw) \) is differentially \( \phi \)-reduced using the rewrite rule system \( \Sigma \). The Rota-Baxter type OPIs are obtained similarly using an ansatz \( M(x, y) \) and reducing the difference \( M(M(u, v), w) - M(u, M(v, w)) \) with the rewrite rule system \( \Pi \). The resulting reduced form is equated to zero, yielding a system of equations in the indeterminate coefficients. This system is simplified using the method of Gröbner bases (a heuristic application of Divide and Conquer has been automated). Once the ansatz is entered, the “algebras” can either be obtained in one command \texttt{getAlgebras}, or the computation can be stepped through.

The programs provided 10 classes of differential type based on an ansatz of 14 terms, which is then manually merged into the 6 classes in Conjecture 4.7. We obtain no new ones after expanding the ansatz to 20 terms, including terms such as \( \lfloor \lfloor x \rfloor \rfloor \lfloor \lfloor y \rfloor \rfloor \). The list for Rota-Baxter type OPIs are obtained from an ansatz with 14 terms, some involving \( P(1) \) (or \( \lfloor 1 \rfloor \), in bracket notation) in a triple product.

We are quite confident that our list of differential type operators is complete. For Rota-Baxter type operators, our list may not be complete, since in our computations, we have restricted our rewriting system \( \Pi \) to disallow units in order to get around the possibly non-terminating reduction sequences modulo the identities. This is especially the case when the OPIs involve \( \lfloor 1 \rfloor \). Typically, for Rota-Baxter type OPIs, we do not know how to handle the appearance of \( \lfloor 1 \rfloor \lfloor 1 \rfloor \) computationally (they may cancel, or not, if our rewriting system \( \Pi \) is expanded to include units as in Definition 6.2). While expressions involving \( \lfloor 1 \rfloor \) alone may be reduced to zero using an expanded rewriting system, monomials involving a mix of bracketed words and \( \lfloor 1 \rfloor \) are often linearly independent over \( \mathbf{k} \).

The \texttt{Mathematica} Notebook \texttt{DTOrderTwoExamples.nb} shows the computations for differential type operators and the Notebook \texttt{VariationRotaBaxterOperators.nb} does the same for Rota-Baxter type ones. Non-commutative multiplication is printed using the symbol \( \otimes \) instead of \( \ast \). It is known that the output routines fail to be compatible with \texttt{Mathematica}, Version 8, and we will try to fix this incompatibility and post updated versions on-line.

8. Summary and outlook

We have studied Rota’s classification problem by considering algebras with a unary operator that satisfies operated polynomial identities. For this, we have reviewed the construction of the operated polynomial algebra.

A far more general theory called variety of algebras exists, of which the theories of PI-rings, PI-algebras, and OPI-algebras are special cases \cite{Drensky:2004}. An “algebra” is any set with a set of functions (operations), together with some identities perhaps. A Galois connection between identities and “variety of algebras” is set up similar to the correspondence between polynomial ideals and algebraic varieties. Thus, differential algebra is one variety of algebra, Rota-Baxter algebra is another, and so on.

In mathematics, specifically universal algebra \cite{Burris:1981, Cohn:1991}, a variety of algebras [or a finitary algebraic category] is the class of all algebraic structures of a given signature satisfying a given set of identities. Equivalently, a variety is a class of algebraic structures of the same signature which satisfies the HSP properties: closed under the taking of homomorphic images, subalgebras and (direct) products.
This equivalence, known as the HSP Theorem, is a result of G. Birkhoff, which is of fundamental importance in universal algebra. We refer interested readers to (Cohn, 1991, Chap. I, Theorem 3.7), (Burris and Sankappanavar, 1981, Theorem 9.5) and, for computer scientists with a model theory background, (Baader and Nipkow, 1998, Theorem 3.5.14). It is simple to see that the class of algebras satisfying some set of equations will be closed under the HSP operations. Proving the converse—classes of algebras closed under the HSP operations must be equational—is much harder.

By restricting ourselves to those special cases of Rota’s Problem that Rota was interested in, and by exploiting the structures of operated algebra and compatibility of associativity on one hand, and symbolic computation (Mathematica) on the other, we are able to give two conjectured lists of OPI-algebras.

The project arose from our belief that the construction of free objects in each class of the varieties should be uniformly done. Currently, similar results for the known classes are proved individually.

We also believe that there is a Poincaré-Birkhoff-Witt type theorem, similar to the enveloping algebra of a Lie algebra, where a canonical basis of the enveloping algebra is constructed from a basis of the Lie algebra. Here, the free algebra of the variety is constructed from the generating set Z with Rota-Baxter words or terms (see Remark 6.6).

The theory of OPI-rings needs to be studied further and there are many open problems. We end this discussion by providing just one. A variety is Schreier if every subalgebra of a free algebra in the variety is free. For example, the variety of all groups (resp. abelian groups) is Schreier. A central problem in the theory of varieties is whether a particular variety of algebras is Schreier. Which of the varieties of differential type algebras or Rota-Baxter type algebras are Schreier?

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