Consider the Euler-Lagrange equation corresponding to the Poincaré-Sobolev inequality in the hyperbolic space $\mathbb{B}^n$:

$$-\Delta_{\mathbb{B}^n} u - \lambda u = |u|^{p-1} u, \quad u \in H^1(\mathbb{B}^n),$$

where $n \geq 3$, $1 < p \leq \frac{n+2}{n-2}$, and $\lambda \leq \frac{(n-1)^2}{4}$. It is well-known from the results of Mancini-Sandeep [41] that under appropriate assumptions on $n, p$ and $\lambda$ the positive solutions of the above equations are unique up to hyperbolic isometries. In the spirit of Struwe, Bhakta-Sandeep [6] proved that, if $u \geq 0$, and $\|\Delta_{\mathbb{B}^n} u + \lambda u + u^p\|_{H^{-1}} \to 0$, then $\delta(u) := \text{dist}(u, M_{\lambda}) \to 0$, where $\text{dist}(u, M_{\lambda})$ denotes the $H^1$-distance of $u$ from the manifold of sums of superpositions of hyperbolic bubbles and (localized) Aubin-Talenti bubbles.

In the subcritical case, $M_{\lambda}$ consists of only hyperbolic bubbles and surprisingly in the critical case, i.e. $p = \frac{n+2}{n-2}$, we establish that up to a countable set of $\lambda \in \left[\frac{(n-2)^2}{4}, \frac{(n-1)^2}{4}\right]$ there are only hyperbolic bubbles present in the manifold $M_{\lambda}$ at the energy level of an integer multiple of the energy of a hyperbolic bubble.

Adhering to the improved profile decomposition, we then proceed to study the quantitative stability of Struwe decomposition in the hyperbolic space. We prove under certain bounds on $\|\nabla u\|_{L^2(\mathbb{B}^n)}$ the inequality

$$\delta(u) \lesssim \|\Delta_{\mathbb{B}^n} u + \lambda u + u^p\|_{H^{-1}},$$

holds whenever $3 \leq n \leq 5$. This generalises a recent result of Figalli and Glaudo [28] on the Euclidean case to the context of the hyperbolic space. Our technique is an amalgamation of Figalli and Glaudo’s method and builds upon a series of new and novel estimates on the interaction of hyperbolic bubbles and their derivatives in the hyperbolic space. For the proof, we introduce a number of new ingredients, including novel interaction integral estimates involving hyperbolic bubbles where we make the most use of the geometry of the hyperbolic space in order to obtain optimal interaction. In particular, we notice a remarkable contrast in the arguments and techniques, since hyperbolic space does not admit any substitute of the conformal group of dilations as in the case of Euclidean geometry.

**Contents**

1. Introduction 2
1.1. Poincaré-Sobolev inequality on the hyperbolic space. 2
1.2. The Euclidean Sobolev inequality. 4
1.3. The quantitative stability problem 4
1.4. Known results in the hyperbolic space 7
1.5. Main results of the article 8
1.6. Main hurdles, Novelty and Strategy of the proof 9
1.7. Notations 12

*Key words and phrases.* Stability, critical points, Poincaré-Sobolev, hyperbolic space, $\delta$- interacting hyperbolic bubbles.
1. Introduction

In this article, we continue our study on the stability of Poincaré-Sobolev inequalities in the hyperbolic space. The focus of this article is on the multi-bubble case. Let us first introduce to the reader the Sobolev embedding, known generally by the Poincaré-Sobolev inequality on the hyperbolic space. Throughout this article, $\mathbb{B}^n$ denotes the Poincaré ball model of the hyperbolic space equipped with the metric $g := \left( \frac{2}{1-|x|^2} \right)^2 dx^2$ and $\Delta_{\mathbb{B}^n}, \nabla_{\mathbb{B}^n}$ denote the Laplace-Beltrami operator and the gradient operator respectively, and $dv_{\mathbb{B}^n}$ denotes the volume element.

1.1. Poincaré-Sobolev inequality on the hyperbolic space. Let $n \geq 3$ and $\lambda \leq \frac{(n-1)^2}{4}$ and $1 < p \leq 2^* - 1$. Then there exists a best constant $S_{\lambda,p} := S_{\lambda,p}(\mathbb{B}^n) > 0$ such that

$$
S_{\lambda,p} \left( \int_{\mathbb{B}^n} |u|^{p+1} \, dv_{\mathbb{B}^n} \right)^{\frac{2}{p+1}} \leq \int_{\mathbb{B}^n} \left( |\nabla_{\mathbb{B}^n} u|^2 - \lambda u^2 \right) \, dv_{\mathbb{B}^n}
$$

(1.1)

holds for all $u \in C^\infty_\text{c}(\mathbb{B}^n)$, where $2^* = \frac{2n}{n-2}$.

By density (1.1) continues to hold for every $u$ belonging to the the closure of $C^\infty_\text{c}(\mathbb{B}^n)$ with respect to the norm

$$
\|u\|_\lambda = \left( \int_{\mathbb{B}^n} \left( |\nabla_{\mathbb{B}^n} u|^2 - \lambda u^2 \right) \, dv_{\mathbb{B}^n} \right)^{\frac{1}{2}}.
$$

(1.2)

The quantity $\lambda_1(\mathbb{B}^n) := \frac{(n-1)^2}{4}$ is the $L^2$-bottom of the spectrum of $-\Delta_{\mathbb{B}^n}$ defined by

$$
\lambda_1(\mathbb{B}^n) = \inf_{\|u\|_{L^2(\mathbb{B}^n)} = 1} \|\nabla_{\mathbb{B}^n} u\|_{L^2(\mathbb{B}^n)}^2.
$$
As a result, if $\lambda < \left( \frac{(n-1)^2}{4} \right)$, then the closure is the classical Sobolev space $H^1(\mathbb{B}^n)$. Otherwise, the closure is a larger space that we will denote by $\mathcal{H}^1(\mathbb{B}^n)$ and which may contain elements which are not square-integrable functions. We refer to Section 2 for precise definition of $H^1(\mathbb{B}^n)$.

The inequality (1.1) was first obtained by Mancini and Sandeep in [41] and in the same article, they also proved the existence of optimizers under appropriate assumptions on $n, \lambda$ and $p$. In particular, they showed that under the assumption that

$$
(H1) \begin{cases}
\lambda < \left( \frac{(n-1)^2}{4} \right), & \text{when } 1 < p < \frac{n+2}{n-2}, \text{ and } n \geq 3, \\
\frac{n(n-2)}{4} < \lambda < \left( \frac{(n-1)^2}{4} \right), & \text{when } p = \frac{n+2}{n-2}, \text{ and } n \geq 4,
\end{cases}
$$

there always exists a strictly positive, radially symmetric and decreasing extremizer $U := U_{n,\lambda,p}$ in $H^1(\mathbb{B}^n)$. For simplicity of notations we will always denote the radially symmetric solution by $U$ and the dependence of $n, \lambda$ and $p$ will be implicitly assumed. It is straightforward to verify that subject to the normalization

- $\|U\|_{\lambda}^2 = S_{\lambda,p}^{\frac{p+1}{p-1}}$, in addition to $U \in H^1(\mathbb{B}^n)$, $U > 0$ radially symmetric and decreasing, (1.3)

the obtained extremizer is a positive solution to the Euler-Lagrange equation

$$
-\Delta_{\mathbb{B}^n} u - \lambda u = |u|^{p-1}u \quad u \in H^1(\mathbb{B}^n).
$$

The equation (1.4) as well as the inequality (1.1) is invariant under the conformal group of the ball model, which in this case coincides with the isometry group of the ball model and is generated by the hyperbolic translations $\tau_b, b \in \mathbb{B}^n$. In analogy with the Euclidean case $\tau_0$ can be thought of as the translation with takes 0 to $b$. In [41] Mancini and Sandeep also classified the positive solutions of (1.4) and which in turn provides the classification of the extremizers of (1.1). Their results are as follows: Under the assumptions $(H1)$ the $n$-dimensional manifold defined by

$$
\mathcal{Z}_0 := \{ U[b] := U \circ \tau_b : b \in \mathbb{B}^n \}
$$

consists of all the positive solutions to (1.4) and $c\mathcal{Z}_0$, $c \in \mathbb{R}\{0\}$ consists of all the nontrivial extremizers of (1.1). Henceforth we shall call the elements of $\mathcal{Z}_0$ a hyperbolic bubble.

Thanks to (1.1) the solutions of (1.4) are the critical points of the energy functional

$$
I_{\lambda}(u) = \frac{1}{2} \int_{\mathbb{B}^n} (|\nabla_{\mathbb{B}^n} u|^2 - \lambda u^2) \ dv_{\mathbb{B}^n} - \frac{1}{p+1} \int_{\mathbb{B}^n} |u|^{p+1} \ dv_{\mathbb{B}^n}.
$$

Using the conformal invariance of the norms stated in Section 2, in particular Lemma 2.1, it is easy to see that all the hyperbolic bubbles have the same energy

$$
I_{\lambda}(U \circ \tau_b) = \frac{p-1}{2(p+1)} S_{\lambda,p}^{\frac{p+1}{p-1}}.
$$
1.2. The Euclidean Sobolev inequality. The classical Sobolev inequality in \( \mathbb{R}^n \), \( n \geq 3 \), asserts that there exists a best constant \( S := S(\mathbb{R}^n) \) such that

\[
\frac{S}{(n-2)} \left( \int_{\mathbb{R}^n} |u|^2 \, dx \right)^{\frac{2}{n-2}} \leq \int_{\mathbb{R}^n} |\nabla u|^2 \, dx
\]

holds for all \( u \in C_0^\infty(\mathbb{R}^n) \), where \( 2^* = \frac{2n}{n-2} \). By density argument, the inequality (1.6) continues to hold for all \( u \) satisfying \( \|\nabla u\|_{L^2(\mathbb{R}^n)} < \infty \), and \( \mathcal{L}^n(\{ |u| > t \}) \) for every \( t > 0 \), where \( \| \cdot \|_{L^2(\mathbb{R}^n)} \) denotes the \( L^2 \)-norm and \( \mathcal{L}^n \) denotes the Lebesgue measure on \( \mathbb{R}^n \). Henceforth we shall denote the closure by \( H^1(\mathbb{R}^n) \). The value of \( S \) is known and the equality cases in (1.6) have been well studied. The inequality (1.6) is invariant under the action of the conformal group of \( \mathbb{R}^n \) composed of the translations, dilations and inversions: for \( z \in \mathbb{R}^n, \mu > 0, u \in C_c^\infty(\mathbb{R}^n) \) define \( T_{z,\mu}(u) = \mu^\frac{n-2}{2} u(\mu(\cdot - z)) \), then both the norms in (1.6) are preserved. It is well known that (1.6) is achieved if and only if \( u \) is a constant multiple of the Aubin-Talenti bubbles [1, 52]

\[
U[z, \mu](x) = (n(n-2))^{\frac{n-2}{2}} \mu^{\frac{n-2}{2}} \left( \frac{1}{(1 + \mu^2 |x - z|^2)^{\frac{n+2}{2}}} \right), \quad z \in \mathbb{R}^n, \mu > 0.
\]

Therefore the set of equality cases in (1.6) forms a \( (n+2) \) dimensional manifold. The choice of the dimensional constant in the definition of \( U[z, \mu] = T_{z,\mu}(U[0,1]) \) ensures that \( U \) is a positive solution of the corresponding Euler-Lagrange equation

\[
-\Delta u = |u|^{2^* - 2} u \quad \text{in} \quad \mathbb{R}^n.
\]

Thanks to the Sobolev inequality, all solutions to (1.8) are the critical points of

\[
J(v) = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla v|^2 \, dx - \frac{1}{2^*} \int_{\mathbb{R}^n} |v|^{2^*} \, dx, \quad v \in H^1(\mathbb{R}^n).
\]

and moreover all the Aubin-Talenti bubbles have the same energy \( J(U[z, \lambda]) = \frac{1}{n} S(\mathbb{R}^n)^{\frac{2}{n}} \).

1.3. The quantitative stability problem. In connection with Aubin and Talenti’s results on the equality cases of the Euclidean Sobolev inequality, Brezis and Lieb [12] asked the following: if \( u \) almost optimizes (1.6) i.e. if the deficit

\[
\delta_{\text{Eud}}(u) = \|\nabla u\|_{L^2(\mathbb{R}^n)}^2 - S \|u\|_{L^{2^*}(\mathbb{R}^n)}^2
\]

is small then is it true that \( u \) is close, in some appropriate sense, to the family of Aubin-Talenti bubbles? This question is positively answered by Bianchi and Egnell [7] nearly thirty years ago in the following quantitative form

\[
\inf_{c \in \mathbb{R}, z \in \mathbb{R}^n, \mu \in \mathbb{R}} \left( \frac{\|\nabla u - cU[z, \mu]\|_{L^2(\mathbb{R}^n)}}{\|\nabla u\|_{L^2(\mathbb{R}^n)}} \right) \leq C(n) \delta_{\text{Eud}}(u)^{\frac{1}{2}},
\]

strengthening the original results of Brezis and Lieb in bounded domains. Here it is important to remark that the norm \( \|\nabla u\|_{L^2(\mathbb{R}^n)} \) is the strongest possible norm and the exponent \( \frac{1}{2} \) is sharp, in the sense that if we replace \( \frac{1}{2} \) by any other exponent strictly bigger than \( \frac{1}{2} \) then the inequality fails as \( \delta_{\text{Eud}}(u) \to 0 \).

The stability problem has been a centre of attraction for several authors for quite some time. Many possible generalizations and improvements in several directions have appeared in the literature. A notable generalization in this direction is the stability for \( p \)-Sobolev inequality for \( p \neq 2 \). In [25] the authors obtained results similar to Brezis and Lieb but the Bianchi-Egnell
type result was open for a long time. After some remarkable results of Cianchi [14], Cianchi-Fusco-Maggi-Pratelli [15], Figalli-Neumayer [29] finally Figalli and Zhang [31] obtained sharp quantitative stability results for $p \neq 2$. Our goal for this article is somewhat different from what we are going to describe now.

Another remarkable direction is the stability problem from the Euler-Lagrange point of view initiated by Cirallo-Figalli and Maggi [17]. The stability question can also be posed as follows: if $u$ almost solves the corresponding Euler-Lagrange equation then is it true that $u$ is close to the family of Aubin-Talenti bubbles in a quantitative way? The problem is far more challenging and generally false unless some more hypotheses are imposed upon it. To see this note that $u = U[-R e_1, 1] + U[R e_1, 1]$ almost solves (1.8) i.e. $\| \Delta u + |u|^{p-1} u \|_{H^{-1}}$ is small but clearly $u$ is not close to the family of Aubin-Talenti bubbles. Here it is important to remark that the interaction between the bubbles

$$
\int_{\mathbb{R}^n} \nabla U[-R e_1, 1] \cdot \nabla U[R e_1, 1] \, dx = \int_{\mathbb{R}^n} U[-R e_1, 1]^p U[R e_1, 1] \, dx \approx R^{2-n} = o(1) \quad \text{as} \quad R \to \infty,
$$

which makes $\| \Delta u + |u|^{p-1} u \|_{H^{-1}}$ small.

This shortcoming can easily be addressed by rephrasing the question to ‘does $u$ close to a sum of Aubin-Talenti bubbles?’ Yet this is not true as there are examples of non-trivial sign-changing solutions which are not the sum of Aubin-Talenti bubbles [20]. However, the classification results of Gidas, Ni and Nirenberg ensure all the positive solutions to (1.8) are the Aubin-Talenti bubbles [37].

A seminal work of Struwe [51] (also see [50]) which dates back to the results of Bianchi and Egnell asserts that imposing a non-negativity assumption on $u$ one can get a non-quantitative stability result.

**Theorem A** (Struwe, 1984). Let $n \geq 3$ and $N \geq 1$ be positive integers. Let $\{u_m\} \subset H^1(\mathbb{R}^n)$ be a sequence of nonnegative functions such that $(N - \frac{1}{2}) S^n \leq \| \nabla u_m \|_{L^2(\mathbb{R}^n)} \leq (N - \frac{1}{2}) S^n$, and assume that

$$
\| \Delta u_m + u_m^{2^*-1} \|_{H^{-1}} \to 0 \quad \text{as} \quad m \to \infty.
$$

Then there exists a sequence $(z_m^i, \ldots, z_m^N)_{i \leq N}$ of $N$-tuple of points in $\mathbb{R}^n$ and sequence $(\lambda_1^m, \ldots, \lambda_N^m)$ of $N$-tuple of positive real numbers such that

$$
\left\| \nabla \left( u_m - \sum_{i=1}^{N} U[z_i^m, \lambda_i^m] \right) \right\|_{L^2} \to 0 \quad \text{as} \quad m \to \infty.
$$

Moreover the Aubin-Talenti bubbles $U[z_i^m, \lambda_i^m]$, $U[z_j^m, \lambda_j^m]$, for $i \neq j$ do not interact with each other at the $H^1$-level

$$
\int_{\mathbb{R}^n} \nabla U[z_i^m, \lambda_i^m] \cdot \nabla U[z_j^m, \lambda_j^m] \, dx = \int_{\mathbb{R}^n} U[z_i^m, \lambda_i^m]^{2^*-1} U[z_j^m, \lambda_j^m] \, dx \approx \min \left( \frac{\lambda_i^m}{\lambda_j^m}, \frac{\lambda_j^m}{\lambda_i^m}, \frac{1}{\lambda_i^m \lambda_j^m |z_i^m - z_j^m|^2} \right) \to 0 \quad \text{as} \quad m \to \infty.
$$

A sharp quantitative form of Struwe decomposition has recently been obtained by Figalli and Glauco [28]. A clever way of getting around the problem of sign-changing $u$ is by phrasing the problem locally around the sum of Aubin-Talenti bubbles as nicely done by Cirallo-Figalli-Maggi [17] for single bubble case and Figalli-Glauco [28] for the multi-bubble case in dim $3 \leq n \leq 5$, and more recently by Deng-Sun and Wei [19] in dimension $n \geq 6$. It is remarkable that for multi-bubble cases the linear dependence of $\| \Delta u + |u|^{p-1} u \|$ heavily relies on the underlying dimension.
In other words, unlike the single bubble case, the form of quantitative stability results for the multi-bubble case, the underlying dimension plays a significant role and it seems this difference results from the behaviour of the interacting bubbles [19, 28]. In order to state their results we recall the definition of weak interaction among the bubbles.

A family of Aubin-Talenti bubbles \( \{ U[z_i, \lambda_i] \}_{1 \leq i \leq N} \) is said to be \( \delta \)-interacting if the interaction between any two bubbles is less than \( \delta \) i.e.

\[
\max_{1 \leq i \neq j \leq N} \min \left( \frac{\lambda_i}{\lambda_j}, \frac{\lambda_j}{\lambda_i} \right) \frac{1}{\lambda_i \lambda_j |z_i - z_j|^2} \leq \delta.
\]

Note that the definition makes sense only if there are more than one bubble. The results obtained so far in the Euclidean case have been summarized in the following theorem.

**Theorem B.** Let \( n \geq 3 \) and \( N \geq 1 \) be positive integers. There exist a small constant \( \delta = \delta(n, N) \) and a large constant \( C = C(n, N) > 0 \) such that the following statement holds: let \( u \in H^1(\mathbb{R}^n) \) be a function such that

\[
\left\| \nabla u - \sum_{i=1}^N \nabla \bar{U}_i \right\|_{L^2} \leq \delta
\]

where \( \{ \bar{U}_i \}_{1 \leq i \leq N} \) is a \( \delta \)-interaction family\(^1\) of Aubin-Talenti bubbles. Then there exists \( N \) Aubin-Talenti bubbles \( U_1, \ldots, U_N \) such that

(a) Cirallo-Figalli-Maggi [17]. If \( N = 1 \) then

\[
\left\| \nabla u - \nabla U_1 \right\|_{L^2} \leq C \| \Delta u + |u|^{2^* - 2} u \|_{H^{-1}}.
\]

(b) Figalli-Glaudo [28]. If \( N > 1 \) and \( 3 \leq n \leq 5 \) then

\[
\left\| \nabla u - \sum_{i=1}^N \nabla U_i \right\|_{L^2} \leq C \| \Delta u + |u|^{2^* - 2} u \|_{H^{-1}}.
\]

Furthermore, for any \( i \neq k \), the interaction between the bubbles can be estimated as

\[
\int_{\mathbb{R}^n} U_i^{2^* - 1} U_k \, dx \leq C \| \Delta u + |u|^{2^* - 2} u \|_{H^{-1}}.
\]

Moreover, for any \( i \neq k \), the interaction between the bubbles can be estimated as

\[
\int_{\mathbb{R}^n} U_i^{2^* - 1} U_k \, dx \leq C \| \Delta u + |u|^{2^* - 2} u \|_{H^{-1}}.
\]

Furthermore, the dimensional restriction is optimal in the sense that for \( n \geq 6 \) and \( N = 2 \) there exists a sequence \( \{ u_R \} \subset H^1(\mathbb{R}^n) \) such that (1.9) fails to hold for any \( C = C(n) \) as \( R \to \infty \).

(c) Ding-Sun-Wei [19]. For \( n \geq 6 \) the following optimal\(^2\) inequality holds

\[
\left\| \nabla u - \sum_{i=1}^N \nabla U_i \right\|_{L^2} \leq C \left\{ \begin{array}{ll}
\| \Delta u + |u|^{2^* - 2} u \|_{H^{-1}} \ln \| \Delta u + |u|^{2^* - 2} u \|_{H^{-1}} \frac{1}{2}, & \text{if } n = 6, \\
\| \Delta u + |u|^{2^* - 2} u \|_{H^{-1}} \frac{n+2}{2n}, & \text{if } n \geq 7.
\end{array} \right.
\]

Before proceeding to the next section let us mention to the interested readers some existing stability results in the literature. We refer to the beautiful treatises [2, 10, 13–16, 21–23, 27, 29–31, 34, 35, 40, 42, 43, 46, 48, 54] which covers, for example, the stability of isoperimetric inequality, 1-Sobolev inequality, Caffarelli-Kohn-Nirenberg inequality, log-Sobolev inequality, Gagliardo-Nirenberg-Sobolev inequality, Hardy-Littlewood-Sobolev, log-growth Hardy-Littlewood-Sobolev

---

\(^1\)when \( N = 1 \) the underlined statement should be omitted.

\(^2\)optimality in the same sense as in (b).
inequality on the Euclidean space. On the values of optimal constant $C$ in Bianchi-Egnell result we refer to [24] (see also [38]). On the stability of isoperimetric inequality on the sphere see [8] and on the hyperbolic space we refer to [9]. See also the recent articles [32] for stability results on $S^1(1/\sqrt{n-2}) \times S^1(1)$ and [26, 44] for related stability on a general Riemannian manifold.

For the critical point of view, other than the results [17, 19, 28] mentioned above, one can consult the beautiful monographs [18, 39] for the Euclidean isoperimetric inequality. The beautiful survey article by Fusco [33] gives a broad description of the stability results concerning several other related geometric and functional inequalities.

1.4. **Known results in the hyperbolic space.** In a recent article [5], we studied the quantitative stability of the Poincaré-Sobolev inequality in the spirit of Bianchi and Egnell [7]. The bottom of the spectrum of $-\Delta_{\mathbb{H}^n}$ being positive, the hyperbolic space admits sub-critical Sobolev inequality as well (i.e. $p < 2^\ast - 1$) and we established the stability for both critical and subcritical Sobolev inequalities.

The non-quantitative form of the Poincaré Sobolev inequality on the hyperbolic space have been obtained nearly a decade ago by the first author jointly with Sandeep [6]. They establish the following Struwe-type profile decomposition both for the subcritical and the critical case. Needless to say in the subcritical case there are no Aubin-Talenti bubbles and hence there are only hyperbolic bubbles present in the profile decomposition. However, in the critical one can not exclude the possibility of the presence of suitably localized Aubin-Talenti bubbles as indicated below.

For the critical case $p = 2^\ast - 1$, there are two types of bubbles. The hyperbolic bubbles which are of the form $u_k = \mathcal{U} \circ b_k$ for some sequence $b_k \in \mathbb{H}^n$ and localized Aubin-Talenti bubbles which are of the form

$$v_k = \left(1 - \frac{|x|^2}{2}\right)^{\frac{n-2}{2}} \Phi(x)U[x_0, \epsilon_k]$$

where $\epsilon_k \to 0^+$ and $\Phi$ is a smooth cut-off function such that $0 \leq \Phi \leq 1$, and $\Phi \equiv 1$ in a neighbourhood of $x_0$, and $U[x_0, \epsilon_k]$ are the standard Aubin-Talenti bubbles defined above. Using the compact $L^p_{loc}$-embedding, it is easy to verify that $I_\lambda(v_k) = J(U[x_0, \epsilon_k]) + o(1)$ as $\epsilon_k \to 0$.

As a result the hyperbolic bubbles have energy $I_\lambda(u_k) = \frac{1}{n} S_{\lambda,p}^2 + o(1)$ for all $j$ while the Aubin-Talenti bubble has energy $I_\lambda(v_k) = \frac{1}{n} S_{\lambda,p}^2 + o(1)$ as $\epsilon_k \to 0^+$ for all $j$.

**Theorem C** (Bhakta and Sandeep [6]). Let $n \geq 3, \lambda, p$ satisfies the hypothesis (H1). Let $\{u_m\} \subset H^1(\mathbb{B}^n)$ be a sequence such that $I_\lambda(u_m) \to d$ and

$$\|\Delta_{\mathbb{B}^n} u_m + \lambda u_m + |u_m|^{p-1} u_m\|_{H^{-1}} \to 0, \quad as \quad m \to \infty.$$  

Then there exists $N_1, N_2 \in \mathbb{N}$ and functions $u^j_m \in H^1(\mathbb{B}^n), 1 \leq j \leq N_1, v^j_m, 1 \leq j \leq N_2$ and $u \in H^1(\mathbb{B}^n)$ such that up to a subsequence

$$u_m = u + \sum_{j=1}^{N_1} u^j_m + \sum_{j=1}^{N_2} v^j_m + o(1) \quad in \quad H^1(\mathbb{B}^n),$$

where $I'_\lambda(u) = 0$ in $H^{-1}(\mathbb{B}^n)$ and $u^j_m, v^j_m$ are defined as above. Furthermore, we have

$$d = \begin{cases} I_\lambda(u) + N_1 \frac{p-1}{2(p+1)} S_{\lambda,p}^{p+1}, & if \ p < 2^\ast - 1, \\
I_\lambda(u) + \frac{N_1}{n} S_{\lambda,p}^2 + \frac{N_2}{n} S_{\lambda,p}^2, & if \ p = 2^\ast - 1. \end{cases}$$
Remark 1.1. If one or more kinds of bubbles are absent we will write \( N_i = 0, \ i = 1, 2 \). For example if the Aubin-Talenti bubbles are absent from the profile decomposition we simply write \( N_2 = 0 \). Note that this is indeed the case if \( p < 2^* - 1 \) or \( d < \frac{1}{n} S_{T}^{\frac{n}{p}} \) for \( p = 2^* - 1 \).

Remark 1.2. It is worth noting that in the above theorem we have imposed no sign assumption on the sequence \( u_m \), and hence the \( u \), which is obtained by just taking the weak limit of \( u_m \), may be different than the standard hyperbolic bubble. However, if we impose \( u_m \) are non-negative, then from the uniqueness of the positive solutions to (1.4) we obtain either \( u \equiv 0 \) or \( u = U \circ \tau_b \) for some \( b \in \mathbb{B}^n \). In either case we have the energy quantization

\[
d = \frac{\bar{N}_1}{n} S_{\lambda,p}^{\frac{n}{p}} + \frac{N_2}{n} S_{T}^{\frac{n}{p}}, \quad \text{if} \quad p = 2^* - 1, \quad \text{or} \quad d = \frac{\bar{N}_1}{2(p+1)} S_{\lambda,p}^{\frac{n}{p+1}}, \quad \text{if} \quad p < 2^* - 1,
\]

where \( \bar{N}_1 \) equal to either \( N_1 \) or \( N_1 + 1 \).

Remark 1.3. In Section 3 we show that if \( p = 2^* - 1 \) and \( d = \frac{N}{n} S_{T}^{\frac{n}{p}} \) for some \( N \in \mathbb{N} \) then there exists an at most countable set \( \Lambda \) such that if \( \lambda \notin \Lambda \) and \( u_m \geq 0 \) satisfies the assumptions of Theorem C then \( N_2 = 0 \), i.e. the Aubin-Talenti bubbles are absent in the profile decomposition. This is the starting point of our result and quite remarkably the proof relies on a very simple observation of the strict decreasing property of the function \( \lambda \mapsto S_{\lambda,2^*-1} \).

1.5. Main results of the article. In Section 4 we will show that

\[
\int_{\mathbb{B}^n} \left( (\nabla z_i - \lambda)U[z_i], (\nabla z_k - \lambda)U[z_k] \right)_{\mathbb{B}^n} \, dv_{\mathbb{B}^n} = \int_{\mathbb{B}^n} U[z_i]^p U[z_k] \, dv_{\mathbb{B}^n} \approx e^{-c(n,\lambda)d(z_i,z_k)},
\]

where

\[
c(n, \lambda) = \frac{n - 1 + \sqrt{(n - 1)^2 - 4\lambda}}{2}, \quad (1.10)
\]

is the rate of decay of the hyperbolic bubbles: \( U(x) \approx e^{-c(n,\lambda)d(x,0)} \). This estimate led us to the following definition.

Definition 1.1. Let \( U_1 = U[z_1], \ldots, U_N = U[z_N] \) be a family of hyperbolic bubbles. We say that the family is \( \delta \)-interacting for some \( \delta > 0 \) if

\[
\max_{i \neq k} e^{-c(n,\lambda)d(z_i,z_k)} \leq \delta, \quad \text{for all} \quad 1 \leq i, k \leq N. \quad (1.11)
\]

If we consider for some \( \alpha_1, \ldots, \alpha_N \in \mathbb{R} \) the family \( \alpha_i U_i, \ 1 \leq i \leq N \), then the family is said to be \( \delta \)-interacting if in addition to (1.11)

\[
\max_i |\alpha_i - 1| \leq \delta
\]

holds.

Therefore \( U_i, U_k \) belongs to a \( \delta \)-interacting family if their \( H^1 \)-scalar product is bounded by a constant multiple of \( \delta \). Throughout the article, we shall use the following notations:

\[
Q_{ik} = e^{-c(n,\lambda)d(z_i,z_k)}, \quad Q_k = \max_{i \neq k} Q_{ik} \quad \text{and} \quad Q = \max_k Q_k
\]

where \( 1 \leq i, k \leq N \).

The main result of this article is the following
Theorem 1.1. Let $3 \leq n \leq 5$ and $p > 2$ such that $p$ and $\lambda$ satisfy (H1). For any $N \in \mathbb{N}$ there exists a small constant $\delta = \delta(n, \lambda, p, N) > 0$ and a large constant $C = C(n, p, \lambda, N) > 0$ such that the following statement holds: let $u \in H^1(\mathbb{B}^n)$ be a function such that

$$\|u - \sum_{i=1}^{N} \bar{U}_i\|_\lambda \leq \delta,$$

where $(\bar{U}_i)_{1 \leq i \leq N}$ is a $\delta$-interacting family of hyperbolic bubbles. Then there exists $N$ hyperbolic bubbles $(U_i)_{1 \leq i \leq N}$ such that

$$\|u - \sum_{i=1}^{N} U_i\|_\lambda \leq C\|\Delta_{\mathbb{B}^n} u + \lambda u + \|u\|^{p-1} u\|_{H^{-1}}.$$

Moreover, for any $i \neq k$, the interaction between the bubbles can be estimated as

$$\int_{\mathbb{B}^n} U_i^p U_k \, dv_{\mathbb{B}^n} \leq C\|\Delta_{\mathbb{B}^n} u + \lambda u + \|u\|^{p-1} u\|_{H^{-1}}.$$

As an application of Theorem 1.1 and the profile decomposition Theorem 3.1 proved in Section 3 we have the following corollary.

Corollary 1.1. Let $3 \leq n \leq 5$ and $p > 2$ such that $p$ and $\lambda$ satisfy (H1) and $N \in \mathbb{N}$. Then there exists a small constant $\delta = \delta(n, \lambda, p, N) > 0$ and a large constant $C = C(n, p, \lambda, N) > 0$ and a countable subset $\Lambda$ of $\mathbb{R}$ if $p = 2^* - 1$ such that for $\lambda \notin \Lambda$ the following statement holds: For any non-negative function $u \in H^1(\mathbb{B}^n)$ satisfying

$$(N - \delta) S_{\lambda, p}^{\frac{p+1}{p}} \leq \|u\|^2 \leq (N - \delta) S_{\lambda, p}^{\frac{p+1}{p}},$$

there exists $N$ hyperbolic bubbles $(U_i)_{1 \leq i \leq N}$ such that

$$\|u - \sum_{i=1}^{N} U_i\|_\lambda \leq C\|\Delta_{\mathbb{B}^n} u + \lambda u + u^p\|_{H^{-1}}.$$

Moreover, for any $i \neq k$, the interaction between the bubbles can be estimated as

$$\int_{\mathbb{B}^n} U_i^p U_k \, dv_{\mathbb{B}^n} \leq C\|\Delta_{\mathbb{B}^n} u + \lambda u + |u|^p\|_{H^{-1}}.$$

Remark 1.4. We remark that the case $N = 1$ has already been studied in [5] and the result holds in any dimension $n \geq 3$ and $p \in (1, 2^* - 1]$. For $N \geq 2$, note that in the statement of Corollary 1.1 for the critical case $p = 2^* - 1$ we claim to prove the absence of Aubin-Talenti bubbles which significantly improves the profile decomposition result of [6] stated in Theorem C. However, we have to pay the price by excluding a countable set which in our humble opinion is a nice result. Our study reveals that the stability of the hyperbolic space in low dimension $3 \leq n \leq 5$ bear resembles that of the Euclidean space despite the fact that the kernel of the linearized operator in the hyperbolic space is $n$-dimensional. In a forthcoming work we shall construct sharp counterexamples in dimension $n \geq 6$ and derive sharp quantitative stability results for any dimension $[4]$.

1.6. Main hurdles, Novelty and Strategy of the proof. We shall now discuss the main difficulties that we encounter while proving Theorem 1.1 and we briefly describe the new tools and the strategy of the proof.
• As discussed above, while considering the multi-bubble scenario in the critical case we cannot ignore the existence of Aubin-Talenti bubbles. Hence we must take into account the interaction between the hyperbolic bubbles and the Aubin-Talenti bubbles, which is a very challenging task. In this article as a first step, we look for the situation where the Aubin-Talenti bubbles are absent. To our surprise, and even more surprising the simplicity of the proof, we found that at energy level integer multiple of \( \frac{1}{n} S_{\lambda,p} \), \( p = 2^* - 1 \) and outside a countable set \( \Lambda \) of \( \lambda \)'s there exist only hyperbolic bubbles. The set is characterized by

\[
\Lambda = \left\{ \lambda \in \left[ \frac{n(n-2)}{4}, \frac{(n-1)^2}{4} \right] : \left( \frac{S_{\lambda,p}}{S} \right)^\frac{1}{2} \in [0,1] \cap \mathbb{Q} \right\}.
\]  

We prove in Section 3 that \( \Lambda \) is at most countable. As a result, we could make the situation slightly simpler by considering \( \lambda \) in the complement of \( \Lambda \). Then we have to only consider the interaction between the hyperbolic bubbles. However, working with hyperbolic bubbles poses many non-trivial and new difficulties which we try to explain briefly.

• To our knowledge, the interaction integrals of hyperbolic bubbles are not known. As a first step, we obtained all the necessary and sharp interaction estimates among the bubbles. We obtain a series of new and novel estimates on the interactions of bubbles and their derivatives. We want to stress that these estimates are quite geometric in nature, and very explicit, i.e., it depends largely on the geometry of the hyperbolic space. Another novelty of this article is that we used explicit formulas of the hyperbolic distance and we were able to reduce the computations to a manageable integral evaluation. We found that all the interaction estimates are exponentially decaying in the hyperbolic distance between the points where the bubbles are most concentrated. More precisely, we prove that if \( U[z_1], U[z_2] \) are two bubbles then

\[
\int_{\mathbb{H}^n} U[z_1]^\alpha U[z_2]^\beta \, dv_{\mathbb{H}^n} \approx \begin{cases} e^{-c(n,\lambda) \min\{\alpha,\beta\} d(z_1,z_2)} & \text{if } \alpha \neq \beta, \\ d(z_1,z_2) e^{-c(n,\lambda) \min\{\alpha,\beta\} d(z_1,z_2)} & \text{if } \alpha = \beta, \end{cases}
\]

whenever, \( \alpha + \beta \geq 2 \), and where \( d(z_1,z_2) \) is the hyperbolic distance between \( z_1 \) and \( z_2 \) and the constant in \( \approx \) depends on \( |\alpha - \beta| \) and other natural parameters.

• A significant challenge comes while estimating the interaction between a bubble \( U[z_1] \) and the spacial derivative \( V_j(U[z_2]) \) of another bubble \( U[z_2] \) (see Appendix for the definition of \( V_j(U[z_2]) \)) where \( j \)-denoted the direction (say) \( e_j \). Without loss of generality we may assume \( z_1 = z \) and \( z_2 = 0 \). It turns out that the interaction

\[
\int_{\mathbb{H}^n} U[z]^p V_j(U[0]) \, dv_{\mathbb{H}^n}
\]

is very sensitive to the position of the point \( z \). Some simple computations lead to the fact that if \( z \) lies on the hyperplane \( \{ x \mid x \cdot e_j = 0 \} \) then the interaction vanishes. More generally we can prove: if \( z \) lies in the negative half \( \{ x \mid x \cdot e_j < 0 \} \) then the interaction is \( \approx e^{-c(n,\lambda) d(z,0)} \) and if \( z \) lies in the positive half \( \{ x \mid x \cdot e_j > 0 \} \) then the interaction is \( \approx -e^{-c(n,\lambda) d(z,0)} \). This is not a good situation to be in because constant \( C \) in Theorem 1.1 should depend only on the weak interaction strength \( \delta \), not on the location of the bubbles on which we do not have any information whatsoever. However, we manage to prove that if \( z \) lies \( \kappa > 0 \) distance away from the hyperplane i.e. \( z \in \{ x \mid |x \cdot e_j| \geq \kappa \} \) then the...
interaction is $\approx_{\kappa} e^{-c(n,\lambda) d(z,0)}$, where the dependence of the constant $\approx_{\kappa}$ on $n, p, \lambda$ is implicitly assumed. As a result, the interaction of two bubbles and the interaction of a bubble and the derivative of another bubble has the same decay estimates up to a sign.

- The hyperbolic space does not admit any substitute of the conformal group of dilations as in the case of Euclidean geometry. This has both advantages and disadvantages. The advantage is that we have only bubble clusters and not bubble towers in the language of [19]. As a result, the computation of the interaction in the hyperbolic space turns out to be simpler than that of the Euclidean case. There is also a caveat while proving Theorem 1.1 as indicated in the next few bullet points.

- The proof of the Theorem 1.1 follows the same strategy as in [28]. Given $u$ we find a $\delta$-interacting family of hyperbolic bubbles $(\alpha_i, U_i)_1 \leq i \leq N$ by the minimization process $\inf_{\tilde{\alpha}_i \in \mathbb{R}, \tilde{z}_i \in \mathbb{R}^n} \| u - \sum_{i=1}^N \tilde{\alpha}_i U[\tilde{z}_i] \|_\lambda$ and we put $\rho = u - \sum_{i=1}^N \alpha_i U_i = u - \sigma$. We test $\Delta_{\mathbb{H}^n} u + \lambda u + |u|^{p-1} u$ against $\rho$ and performing some similar computations as in [28, Theorem 3.3] the proof boils down to the justification of the following two inequalities:
  
  (i) **Improved spectral inequality.** Provided the weakly interaction $\delta$ is sufficiently small, it holds that
  $$
  \int_{\mathbb{R}^n} \sigma^{p-1} \rho^2 \ d\nu_{\mathbb{H}^n} \leq \frac{\tilde{c}}{p} \| \rho \|_\lambda^2, \quad \text{where} \quad \tilde{c} = \tilde{c}(n, N, \lambda, p) < 1.
  $$

  (ii) **Interaction integral estimates.** Given $\varepsilon > 0$, if $\delta$ is sufficiently small then
  $$
  \max_{i \neq k} \int_{\mathbb{R}^n} U_i^p U_k \ d\nu_{\mathbb{H}^n} \lesssim \|(\Delta_{\mathbb{H}^n} + \lambda) u + |u|^{p-1} u\|_{H^{-1}} + \varepsilon \| \rho \|_\lambda + \| \rho \|_\lambda^2.
  $$

- The proof of (i) uses a localization argument as in [28]. The main difference is that the $L^n$-norm of the gradient is conformal and hence one can not expect the $L^n$-norm of the gradient of the localized function to be small. The substitute in the hyperbolic space is the $L^n$-norm of the gradient of the localized function. Thanks to the Poincaré inequality this poses no threat in our case.

- The proof of (ii) in the Euclidean case relies heavily upon the interaction among the bubbles and the $\mu$-derivatives of another bubble. Since $u = \sigma + \rho$ one can express $\Delta_{\mathbb{H}^n} u + \lambda u + |u|^{p-1} u$ in the following manner
  $$
  \Delta_{\mathbb{H}^n} \rho + \lambda \rho + p\sigma^{p-1} \rho + I_1 + I_2 + I_3 = \Delta_{\mathbb{H}^n} u + \lambda u + |u|^{p-1} u,
  $$
  where $I_1 = \sigma^p - \sum_{i=1}^N \alpha_i^p U_i^p$ contains all the interactions between the bubbles. In the Euclidean case one test the above equation against $\partial_{\mu} U_k$ and derive that $| \int_{\mathbb{R}^n} I_i \partial_{\mu} U_k \ dx | \approx \max_{i \neq k} \int_{\mathbb{R}^n} U_i^p U_k \ dx$. The main reason for testing by $\partial_{\mu} U_k$ over testing by a spacial derivative (i.e. $\partial_{\mu} U_k$) is that spatial derivative leads to much weaker interaction estimates $< < \max_{i \neq k} \int_{\mathbb{R}^n} U_i^p U_k$ in case the heights of the Aubin-Talenti bubbles are comparable i.e. if $U_i = U[z_i, \mu_i], 1 \leq i \leq N$, then $\mu_i \approx \mu_j$ (see A. Bahri [3, Estimate (F11)]). Note that the latter situation is precisely the case of the hyperbolic bubbles and the only possibility is to test by $V_j(U_k)$. Thankfully our spacial derivative interaction almost behaves like the interaction of bubbles, but there is one caveat regarding the sign. There might be cancellation and $\int_{\mathbb{R}^n} I V_j(U_k)$ might bring very weak interaction.
• We overcome this issue with the help of a geometric lemma 8.2. The lemma suggests that if the family \((U_i)_{1 \leq i \leq N}\) sufficiently small and weakly interacting, then there exists a direction \(e_j\) such that up to hyperbolic translations, orthogonal transformations and rearrangement of indices \(z_N = 0\) and all other \(z_i\) lie in the negative half of the plane \(\{x \mid x \cdot e_j < 0\}\). Moreover, all \(z_i, i \neq N\) lie at a positive distance away from the plane \(\{x \mid x \cdot e_j = 0\}\). The proof of the lemma is purely geometric and makes the most use of the properties of the reflection in plane and inversion in spheres.

The organization of the article is as follows:

Section 1: Introduction section, contains a brief background on the stability problem in the Euclidean space, known results in the hyperbolic space, the main result of this article and a short subsection about major challenges and how we addressed it.

Section 2: Contains the basics of the hyperbolic space.

Section 3: This section is devoted to the improvement of the Struwe-type profile decomposition on the hyperbolic space. In particular, we obtained instances when Aubin-Talenti bubbles are absent in the profile decomposition.

Section 4: This is one of the core sections of this article. In this section, we proved all the interaction estimates namely (a) interaction between two bubbles, (b) interaction among three bubbles and (c) interaction between a bubble and the spacial derivation of another bubble. This section is slightly technical but all the results obtained, to the best of the knowledge of the authors, are all new.

Section 5: Contains the proof of the main theorem, namely, Theorem 1.1 and Corollary 1.1 assuming (i) improved spectral inequality which is proved in Section 7 and (ii) Interaction integral estimates, which is proved in Section 8.

Section 6: Another slightly technical section containing the core of the localization argument.

Appendix: Recalled some known results needed for the article.

1.7. Notations.

• A point \(x \in \mathbb{R}^n\) will be denoted by \(x = (x_1, \ldots, x_n)\). For a point \(x \in \mathbb{R}^n\), \(x_j\) will denote the \(j\)-th component of \(x\).

• \(n\) = dimension, \(N\) = number of bubbles, \(\delta\) = interaction strength, \(p + 1\) = exponent in Sobolev inequality and \(\lambda\) = spectral parameter.

• The notation \(T \lesssim S\) (respectively \(T \gtrsim S\)) means \(T \leq CS\) (respectively \(T \geq CS\)) for some constant \(C\) that depends only on the given data \(n, N, p, \lambda\). We denote \(T \approx S\) if both \(T \lesssim S\) and \(T \gtrsim S\) hold.

• The notation \(T \lesssim_{p,q,r} S\) means \(T \leq CS\) for some constant \(C\) that depends on the parameters \(p, q, r\) as well as on \(n, N, \lambda, p\).

• \(x \cdot y = \sum_{j=1}^{n} x_j y_j\) denotes the standard dot product in \(\mathbb{R}^n\) and \(|\cdot|\) the associated norm.

• \(\{e_j\}_{1 \leq j \leq N}\) denotes the standard basis of \(\mathbb{R}^n\), i.e \(l\)-th component of \(e_j\) is \(\delta_{jl}\).

• We denote by \(B_E(a, r)\) the Euclidean ball \(\{x \in \mathbb{R}^n \mid |x-a| < r\}\) and \(S(a, r)\) the Euclidean sphere \(\{x \in \mathbb{R}^n \mid |x-a| = r\}\). The hyperbolic ball will be denoted by \(B(a, r)\) which is defined in the next section.

• The notation \(\|\cdot\|_{L^r}\) will denote the \(L^r\)-norm with respect to the measure \(dv_{B^n}\).

• Given dimension \(n \geq 3\), \(S\) denotes the best Sobolev constant in \(\mathbb{R}^n\) and given \(\lambda, p\) as in (1.1), \(S_{\lambda, p}\) denotes the best Poincaré-Sobolev inequality in \(\mathbb{B}^n\). In the later case if \(p = 2^* - 1\) then we denote \(S_{\lambda, p}\) by \(S_\lambda\).
2. Basics on the hyperbolic space

The Euclidean unit ball $\mathbb{B}^n := \{ x \in \mathbb{R}^n : |x|^2 < 1 \}$ equipped with the Riemannian metric

$$ds^2 = \left( \frac{2}{1-|x|^2} \right)^2 dx^2$$

constitute the ball model for the hyperbolic $n$-space, where $dx$ is the standard Euclidean metric and $|x|^2 = \sum_{i=1}^n x_i^2$ is the standard Euclidean length. By definition, the hyperbolic $n$-space is a $n$-dimensional complete, non-compact Riemannian manifold having constant sectional curvature equal to $-1$ and any two manifolds sharing the above properties are isometric \cite{45}. In this article, all our computations will involve only the ball model and will be denoted by $\mathbb{B}^n$. We denote the inner product on the tangent space of $\mathbb{B}^n$ by $\langle \cdot, \cdot \rangle_{\mathbb{B}^n}$ and the volume element is given by $d\nu_{\mathbb{B}^n} = \left( \frac{2}{1-|x|^2} \right)^n dx$, where $dx$ denotes the Lebesgue measure on $\mathbb{R}^n$.

Let $\nabla_{\mathbb{B}^n}$ and $\Delta_{\mathbb{B}^n}$ denote gradient vector field and Laplace-Beltrami operator respectively. Therefore, in terms of local (global) coordinates $\nabla_{\mathbb{B}^n}$ and $\Delta_{\mathbb{B}^n}$ takes the form:

$$\nabla_{\mathbb{B}^n} = \left( \frac{1-|x|^2}{2} \right)^2 \nabla, \quad \Delta_{\mathbb{B}^n} = \left( \frac{1-|x|^2}{2} \right)^2 \Delta + (n-2) \left( \frac{1-|x|^2}{2} \right) x \cdot \nabla,$$

where $\nabla, \Delta$ are the standard Euclidean gradient vector field and Laplace operator respectively, and $\cdot$ denotes the standard inner product in $\mathbb{R}^n$.

**Hyperbolic distance on $\mathbb{B}^n$.** The hyperbolic distance between two points $x$ and $y$ in $\mathbb{B}^n$ will be denoted by $d(x, y)$. The hyperbolic distance between $x$ and the origin can be computed explicitly

$$\rho := d(x, 0) = \int_0^{|x|} \frac{2}{1-s^2} \, ds = \log \frac{1+|x|}{1-|x|},$$

and therefore $|x| = \tanh \frac{\rho}{2}$. Moreover, the hyperbolic distance between $x, y \in \mathbb{B}^n$ is given by

$$d(x, y) = \cosh^{-1} \left( 1 + \frac{2|x-y|^2}{(1-|x|^2)(1-|y|^2)} \right).$$

In this article, we will use several simplifications of the above formula which can be derived easily. For example, the following two formulas will be used:

$$\cosh \left( \frac{d(x, y)}{2} \right) = \frac{\sqrt{1-2x \cdot y + |x|^2|y|^2}}{\sqrt{(1-|x|^2)(1-|y|^2)}} = \frac{|y||x-y^*|}{\sqrt{(1-|x|^2)(1-|y|^2)}},$$

where $y^* = y/|y|^2$. Alternatively, using the formula $\cosh^{-1}(s) = \ln(s + \sqrt{s^2-1})$, $s > 1$ one can deduce

$$\frac{d(x, y)}{2} = \ln \left( \frac{|y||x-y^*| + |x-y|}{\sqrt{(1-|x|^2)(1-|y|^2)}} \right).$$

Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be an orthogonal transformation i.e., $Ax \cdot Ay = x \cdot y$ for all $x, y \in \mathbb{B}^n$. Then it follows from the formula of hyperbolic distance that $d(Ax, Ay) = d(x, y)$ for all $x, y \in \mathbb{B}^n$. In other words, the hyperbolic distance is preserved under the action of the orthogonal group of $\mathbb{R}^n$.

Geodesic balls in $\mathbb{B}^n$ of radius $r$ centred at $x \in \mathbb{B}^n$ will be denoted by

$$B(x, r) := \{ y \in \mathbb{B}^n : d(x, y) < r \}.$$ 

Next, we introduce the concept of hyperbolic translation.
2.1. Hyperbolic translation. We first recall some basic notions of reflections and inversions.

2.1.1. Reflections about a plane. Let a be a non-zero vector in $\mathbb{R}^n$ and $t$ be a real number. Consider the hyperplane in $\mathbb{R}^n$

$$P(a, t) = \{x \in \mathbb{R}^n \mid \frac{a}{|a|} \cdot x = t\}.$$ 

The reflection $\rho_{a,t}$ of $\mathbb{R}^n$ about the plane $P(a,t)$ is defined by

$$\rho_{a,t}(x) = x + 2\left(t - \frac{a}{|a|} \cdot x\right)\frac{a}{|a|}.$$ 

When $t = 0$ we will simply denote the reflection by $\rho_a$. It is easy to verify that $\rho_a$ sends the positive half plane $\{x \mid a \cdot x > 0\}$ to the negative half $\{x \mid a \cdot x < 0\}$ and vice versa, and leaves the plane $P_{a,0}$ invariant.

2.1.2. Inversion about a sphere. Let $a$ be a point in $\mathbb{R}^n$ and let $r$ be a positive real number. The sphere of $\mathbb{R}^n$ of radius $r$ centered at $a$ is defined by

$$S(a, r) = \{x \in \mathbb{R}^n \mid |x - a| = r\}.$$ 

The inversion $\sigma_{a,r}$ of $\mathbb{R}^n$ in the sphere $S(a, r)$ is defined by the formula

$$\sigma_{a,r}(x) = a + \left(\frac{r}{|x - a|}\right)^2 (x - a).$$ 

It is easy to verify that $|\sigma_{a,r}(x) - a||x - a| = r^2$, for all $x \in \mathbb{R}^n$.

Hence $\sigma_{a,r}$ sends sphere $S(a, r_1)$ to the sphere $S(a, \frac{r_1}{r_1})$ for every $r_1 > 0$. When $r$ is determined by $a$ (for example $r^2 = |a|^2 - 1$) we will simply use the notation $\sigma_a$ instead of $\sigma_{a,r}$.

2.1.3. The hyperbolic translation. Given a point $a \in \mathbb{R}^n$ such that $|a| > 1$ and $r > 0$, let $S(a, r) := \{x \in \mathbb{R}^n \mid |x - a| = r\}$ be the sphere in $\mathbb{R}^n$ with center $a$ and radius $r$ that intersects $S(0, 1)$ orthogonally. It is known that it is the case if and only if $r^2 = |a|^2 - 1$, and hence $r$ is determined by $a$. Let $\rho_a$ denotes the reflection with respect to the plane $P_a := \{x \in \mathbb{R}^n \mid x \cdot a = 0\}$ and $\sigma_a$ denotes the inversion with respect to the sphere $S(a, r)$. Then $\sigma_a \rho_a$ leaves $\mathbb{B}^n$ invariant (see [45]).

For $b \in \mathbb{B}^n$, the hyperbolic translation $\tau_b : \mathbb{B}^n \to \mathbb{B}^n$ that takes 0 to $b$ is defined by $\tau_b = \sigma_{b^*} \rho_{b^*}$ and can be expressed by the following formula

$$\tau_b(x) := \frac{(1 - |b|^2)x + (|x|^2 + 2x \cdot b + 1)b}{|b|^2|x|^2 + 2x \cdot b + 1} \quad (2.1)$$

where $b^* = \frac{b}{|b|^2}$. It turns out that $\tau_b$ is an isometry and forms the Möbius group of $\mathbb{B}^n$ (see [45], Theorem 4.4.6) for details and further discussions on isometries. Note that $\tau_{-b} = \sigma_{-b^*} \rho_{-b^*}$ is the hyperbolic translation that takes $b$ to the origin. In other words, the hyperbolic translation that takes $b$ to the origin is the composition of the reflection $\rho_{-b^*}$ and the inversion $\sigma_{-b^*}$.

For the convenience of the reader, we list several well-known properties of the hyperbolic translation in the next lemma. The proof follows by now standard properties of the Möbius transformations on the ball model which we skip for brevity and refer to the book [49].

Lemma 2.1. For $b \in \mathbb{B}^n$, let $\tau_b$ be the hyperbolic translation of $\mathbb{B}^n$ by $b$. Then for every $u \in C_C^\infty(\mathbb{B}^n)$, there holds,

(i) $\Delta_{\mathbb{B}^n}(u \circ \tau_b) = (\Delta_{\mathbb{B}^n}u) \circ \tau_b$,  
(ii) $\nabla_{\mathbb{B}^n}(u \circ \tau_b), \nabla_{\mathbb{B}^n}(u \circ \tau_b))_{\mathbb{B}^n} = \langle (\nabla_{\mathbb{B}^n}u) \circ \tau_b, (\nabla_{\mathbb{B}^n}u) \circ \tau_b\rangle_{\mathbb{B}^n}$.
(ii) For every open subset $U$ of $\mathbb{B}^n$

$$\int_U |u \circ \tau_b|^p \, dv_{\mathbb{B}^n} = \int_{\tau_b(U)} |u|^p \, dv_{\mathbb{B}^n}, \text{ for all } 1 \leq p < \infty.$$ 

(iii) For every $\phi, \psi \in C^\infty_c(\mathbb{B}^n)$,

$$\int_{\mathbb{B}^n} \phi(x)(u \circ \tau_b)(x) \, dv_{\mathbb{B}^n} = \int_{\mathbb{B}^n} (\phi \circ \tau_{-b})(x) \psi(x) \, dv_{\mathbb{B}^n}.$$ 

By density, the above formulas hold as long as the integrals involved are finite.

2.2. The Sobolev space $H^1(\mathbb{B}^n)$: Thanks to the Poincaré inequality, we define the Sobolev space on $\mathbb{B}^n$, denoted by $H^1(\mathbb{B}^n)$, as the completion of $C^\infty_c(\mathbb{B}^n)$ with respect to the norm

$$\|u\|_{H^1(\mathbb{B}^n)} := \left( \int_{\mathbb{B}^n} |\nabla_{\mathbb{B}^n} u|^2 \, dv_{\mathbb{B}^n} \right)^{\frac{1}{2}},$$

where $|\nabla_{\mathbb{B}^n} u|$ is given by

$$|\nabla_{\mathbb{B}^n} u| := \langle \nabla_{\mathbb{B}^n} u, \nabla_{\mathbb{B}^n} u \rangle_{\mathbb{B}^n}^{\frac{1}{2}}.$$ 

Using Poincaré inequality once again, $\|u\|_\lambda$ defined by

$$\|u\|_{\lambda}^2 := \int_{\mathbb{B}^n} \left( |\nabla_{\mathbb{B}^n} u|^2 - \lambda u^2 \right) \, dv_{\mathbb{B}^n} \text{ for all } u \in H(\mathbb{B}^n),$$

is an equivalent $H^1(\mathbb{B}^n)$ norm for $\lambda < \frac{(n-1)^2}{4}$. However, for $\lambda = \frac{(n-1)^2}{4}$, $\|u\|_\lambda$ is not an equivalent norm because, in general, $u$ may fail to be square integrable. Moreover, thanks to Lemma 2.1, the norms $\|u\|_{H^1(\mathbb{B}^n)}$, $\|u\|_\lambda$ are invariant under the action of isometries of the ball model. To distinguish between the norms we will call $\| \cdot \|_\lambda$ by $H^1_\lambda$-norm. Similarly, $H^1_\lambda$-inner product will refer to the inner product generating the norm $\| \cdot \|_\lambda$.

We also define the subset of $H^1(\mathbb{B}^n)$ consisting of radial functions

$$H^1_{rad}(\mathbb{B}^n) := \{ u \text{ radial } | \int_0^\infty \left[ u^2(\rho) + (u'(\rho))^2 \right] (\sinh \rho)^{n-1} \, d\rho < \infty \},$$

where $\rho$ denotes the differentiation with respect $\rho$.

We will also be needing the space $H^1_0(\Omega)$ for $\Omega \subset \mathbb{B}^n$ which is defined by the closure of $C^\infty_c(\Omega)$ in $H^1(\mathbb{B}^n)$.

For a positive weight $w$ we define the weighted $L^2$-space by

$$L^2_w(\mathbb{B}^n) := \left\{ u \mid \int_{\mathbb{B}^n} |u|^2 w \, dv_{\mathbb{B}^n} < \infty \right\}.$$ 

3. On the absence of Aubin-Talenti bubbles

We investigate, in the critical case $p = 2^* - 1$, under what conditions only the hyperbolic bubbles are present in Struwe-type profile decomposition. Because of the non-existence results of Sandeep and Mancini [41] in low dimensions and in low spectrum regions, in this section, we assume $n$ and $\lambda$ satisfy

$$\text{(H2)} \quad n \geq 4 \quad \text{and} \quad \frac{n(n-2)}{4} < \lambda < \frac{(n-1)^2}{4}.$$
Since \( p = 2^* - 1 \) is fixed in this section, for the simplicity of notations we denote \( S_{p, \lambda} \) by \( S_\lambda \), and the unique positive radial solution of (1.4) will be denoted by \( U \). To begin with we recall the uniqueness result of [41] in its simplest form.

**Theorem A** (Theorem 1.3 of [41]). Assume \((H2)\), then all the positive solutions of (1.4) are of the form \( U \circ \tau_b \) for some \( b \in \mathbb{B}^n \).

Also recall that all the positive solutions to (1.4) have the same energy \( I_\lambda(U \circ \tau_b) = \frac{1}{n} S_\lambda^{\frac{2}{n}} \).

Below are a few applications of the above existence and uniqueness of solutions to (1.4).

Define \( h : [\frac{n(n-2)}{4}, \frac{(n-1)^2}{4}] \to (0, \infty) \) by \( h(\lambda) = S_\lambda \). Clearly \( h \) is monotonically decreasing. In fact, we have

**Lemma 3.1.** The function \( h \) is strictly decreasing.

**Proof.** It is well known that \( S_{\frac{n(n-2)}{4}} = S \) where recall \( S \) is the best Euclidean Sobolev constant and \( S_\lambda < S \) for \( \lambda > \frac{n(n-2)}{4} \) (see [53]). So, with out loss of generality we assume \( \frac{n(n-2)}{4} < \lambda_1 < \lambda_2 \leq \frac{(n-1)^2}{4} \) and we claim to show that \( S_{\lambda_1} > S_{\lambda_2} \). On the contrary we assume \( S_{\lambda_1} = S_{\lambda_2} \). Then \( S_{\lambda_2}(\mathbb{B}^n) \) is also attained by \( U_{\lambda_2} \) and hence by uniqueness Theorem A, \( U_{\lambda_1} = C U_{\lambda_2} \) for some positive constant \( C \). Because of the assumed normalization (1.3) and \( S_{\lambda_1} = S_{\lambda_2} \), we conclude \( C = 1 \). But \( U_{\lambda_1} \) satisfies the Euler-Lagrange (1.4) with \( \lambda = \lambda_i, i = 1, 2 \). Subtracting the two equations satisfied by \( U_{\lambda_i} \) we see that \( (\lambda_1 - \lambda_2) U_{\lambda_1} = 0 \) which implies \( U_{\lambda_1} \equiv 0 \), a contradiction. Hence the proof follows.

A direct application of Lemma 3.1 is the following corollary. Set

\[
\Lambda = \left\{ \lambda \in \left[ \frac{n(n-2)}{4}, \frac{(n-1)^2}{4} \right] : \left( \frac{S_{\lambda}}{S} \right)^{\frac{2}{n}} \in [0, 1] \cap \mathbb{Q} \right\}. 
\]  

(3.1)

**Corollary 3.1.** The set \( \Lambda \) is at most countable.

**Proof.** By Lemma 3.1, the function \( g(\lambda) = \left( \frac{S_{\lambda}}{S} \right)^{\frac{2}{n}} = \left( \frac{h(\lambda)}{S} \right)^{\frac{2}{n}} \) is strictly decreasing and hence there exists a one-to-one correspondence between \( \left[ \frac{n(n-2)}{4}, \frac{(n-1)^2}{4} \right] \) and the image of \( g \) denoted by \( Im(g) \). As a result, we can write

\[
\Lambda = g^{-1}(\mathbb{Q} \cap Im(g))
\]

and since \( \mathbb{Q} \cap Im(g) \) is countable and \( g \) is strictly decreasing \( \Lambda \) must be countable.

**On the absence of Aubin-Talenti bubbles in the profile decomposition:** Recall that under the hypothesis \((H2)\) the profile decomposition may contain both Aubin-Talenti bubbles and the hyperbolic bubbles. The following simple consequence of Lemma 3.1 and Theorem C isolates the situation preferable for us.

**Theorem 3.1.** Let \( n \geq 4, p = 2^* - 1 \) and \( \frac{n(n-2)}{4} < \lambda < \frac{(n-1)^2}{4} \) be such that \( \lambda \notin \Lambda \) and \( N \in \mathbb{N} \) be given. Let \( \{u_m\} \subset H^1(\mathbb{B}^n) \) be a sequence such that \( u_m \geq 0, I_\lambda(u_m) \to d = \frac{N}{n} S_\lambda^{\frac{2}{n}} \) and

\[
\|\Delta_{\mathbb{B}^n} u_m + \lambda u_m + u_m^p\|_{H^{-1}} \to 0, \quad \text{as} \quad m \to \infty.
\]
Then there exists $N$ sequences \( \{b_j^m\} \subset \mathbb{B}^n, 1 \leq j \leq N \) such that
\[
u_m = \sum_{j=1}^{N} U \circ \tau_{b_j^m} + o(1) \quad \text{in } H^1(\mathbb{B}^n).
\]
In other words, no Aubin-Talenti bubble exists in the profile decomposition of \( u_m \).

Proof. By Theorem C and the subsequent Remark 1.2, we have
\[
\frac{n}{N} \frac{S_\lambda^p}{S} = \frac{N_1}{n} \frac{S_\lambda^p}{S} + \frac{N_2}{n} \frac{S_\lambda^p}{S}.
\]
If \( N_2 \neq 0 \), (which corresponds to the existence of Aubin-Talenti bubbles) then \( N \neq \tilde{N}_1 \) and hence
\[
\left( \frac{S_\lambda}{S} \right)^{\frac{p}{2}} = \frac{N_2}{N - \tilde{N}_1} \in \mathbb{Q}.
\]
This contradicts the assumption that \( \lambda \notin \Lambda \) and completes the proof of the theorem.

At this point, we are tempted to conjecture that Theorem 3.1 is the best possible scenario we can get for \( n(\frac{n-2}{4}) < \lambda < (\frac{n-1}{4})^2 \). In other words, if \( \mathbf{h} \) is continuous then \( \text{Im}(g) \) must contain both rational and irrational numbers and thence we can not expect to do better than Theorem 3.1. However, proving rigorously the existence of both Aubin-Talenti bubbles and hyperbolic bubbles at the prescribed energy level mentioned in Theorem 3.1 is a formidable challenge. We don’t have any positive or negative answer to it yet.

4. Interaction of bubbles

4.1. Interaction of two bubbles. In this subsection, we shall compute the interaction of two hyperbolic bubbles. We use the following sharp bounds on the hyperbolic bubble: there exists constants \( \chi_i = \chi_i(n,p,\lambda), i = 1,2 \) such that
\[
\chi_1 e^{-c(n,\lambda)d(x,z)} \leq U[z](x) \leq \chi_2 e^{-c(n,\lambda)d(x,z)}, \quad \text{for all } x \in \mathbb{B}^n,
\]
and for every \( z \in \mathbb{B}^n \), where \( c(n,\lambda) \) is as defined in (1.10). The case \( \lambda = 0 \) and \( p < 2^* - 1 \) the decay estimate (4.1) have been obtained by Bonforte-Gazzola-Grillo and Vázquez [11] some time ago which is a slight improvement of the seminal result of Mancini-Sandeep [41] concerning the asymptotic behaviour of radial solutions. We came to know about the validity of (4.1) for all admissible \( \lambda \) from personal communication with Sandeep K and Ramya Dutta [47]. The result is going to appear in their forthcoming article.

Lemma 4.1. Given \( n \geq 3 \), let \( \alpha, \beta \geq 0 \) be such that \( \alpha + \beta \geq 2 \). Let \( U_i = U[z_i], i = 1,2 \) be two hyperbolic bubbles such that \( d(z_i, z_j) \geq \ln 3 \). Then
\[
\int_{\mathbb{B}^n} U_1^\alpha U_2^\beta \, dv_{\mathbb{B}^n} \approx \begin{cases} 
|\alpha - \beta| & e^{-c(n,\lambda)} \min(\alpha,\beta) d(z_1,z_2), \\
\approx & d(z_1,z_2) e^{-c(n,\lambda)\beta} d(z_1,z_2), \\
& \text{if } \alpha \neq \beta, \\
& \text{if } \alpha = \beta.
\end{cases}
\]
Here we follow the convention that the constant in \( \approx \) depends also on \( n,p \) and \( \lambda \).

Proof. Before beginning the proof let us recall that for any two points \( x, z \in \mathbb{B}^n \) we have
\[
\cosh \frac{d(x, z)}{2} = \frac{\sqrt{1 - 2x.z + |x|^2|z|^2}}{\sqrt{(1 - |x|^2)(1 - |z|^2)}}.
\]
So in particular for $z = 0$ we have
\[ e^{d(x,0)} \approx \frac{1}{(1 - |x|^2)}. \]

By Lemma 2.1(iii) it is enough to prove the result for $z_1 = 0, z_2 = z$.

Let us compute the following integral for $\alpha, \beta \in \mathbb{R}^+ \cup \{0\}$ such that $\alpha + \beta \geq 2$:
\[
I = \int_{\mathbb{B}^n} U[0]^\alpha U[z]^{\beta} \, dv_{\mathbb{B}^n}
\approx \int_{\mathbb{B}^n} e^{-c(n,\lambda)d(x,0)} e^{-c(n,\lambda)\beta d(x,z)} \, dv_{\mathbb{B}^n}
\approx \int_{\mathbb{B}^n} (1 - |x|^2)^{c(n,\lambda)\alpha} \left( \frac{1 - |z|^2}{1 - 2|x.z| + |x|^2 |z|^2} \right)^\beta \left( \frac{2}{1 - |x|^2} \right)^n \, dx
\approx (1 - |z|^2)^{c(n,\lambda)\beta} \int_{\mathbb{B}^n} \left( \frac{1 - |x|^2}{1 - 2|x.z| + |x|^2 |z|^2} \right)^{c(n,\lambda)\beta} \, dx
= \frac{(1 - |z|^2)^{c(n,\lambda)\beta}}{|z|^{2c(n,\lambda)\beta}} \int_{\mathbb{B}^n} \left( \frac{1 - |x|^2}{|x - z|^2} \right)^{c(n,\lambda)(\alpha + \beta) - n} \, dx. \tag{4.2}
\]

Note that the the values of $c(n, \lambda) > \frac{(n-1)}{2}$ for all $\lambda < \frac{(n-1)^2}{4}$ and hence the integral (4.2) makes sense only if $\alpha + \beta \geq 2$. Now without loss of generality, we may assume that $\alpha \geq \beta$. We further subdivide our proof into two cases.

**Case-1:** $\alpha > \beta$. We notice that $|z^*| \geq 1$ and $|x| < 1$ and therefore
\[
|x - z^*| \geq |z^*| - |x| \geq 1 - |x| \approx (1 - |x|^2).
\]
Since by our hypothesis $d(0, z) \geq \ln 3$, we have $|z| \geq \frac{1}{2}$ and hence $\mathbb{B}^n \subset B_E(z^*, 3)$. Exploiting the fact that $c(n, \lambda)(\alpha - \beta) > 0$ we get
\[
\int_{\mathbb{B}^n} \frac{(1 - |x|^2)^{c(n,\lambda)(\alpha + \beta) - n}}{|x - z^*|^{2c(n,\lambda)\beta}} \, dx \lesssim \int_{\mathbb{B}^n} |x - z^*|^{c(n,\lambda)(\alpha - \beta) - n} \, dx
\lesssim \int_{B_E(z^*, 3)} |x - z^*|^{c(n,\lambda)(\alpha - \beta) - n} \, dx \lesssim_{|\alpha - \beta|} 1. \tag{4.3}
\]
This proves the upper bound.

For the lower bound, first note that for fixed $z_0$ with $\frac{1}{2} \leq |z_0| \leq 1$ the same argument leads to
\[
0 < \int_{\mathbb{B}^n} \frac{(1 - |x|^2)^{c(n,\lambda)(\alpha + \beta) - n}}{|x - z_0^*|^{2c(n,\lambda)\beta}} \, dx \lesssim_{|\alpha - \beta|} 1.
\]

Taking a minimizing sequence and using Fatou’s lemma one can easily conclude that
\[
\inf_{\frac{1}{2} \leq |z| \leq 1} \int_{\mathbb{B}^n} \frac{(1 - |x|^2)^{c(n,\lambda)(\alpha + \beta) - n}}{|x - z|^2c(n,\lambda)\beta} \, dx \approx_{|\alpha - \beta|} 1.
\]
This completes the proof lower bound. Hence combining the upper bound and lower bound estimates we conclude that
\[
\int_{\mathbb{B}^n} U[0]^\alpha U[z]^{\beta} \, dv_{\mathbb{B}^n} \approx_{|\alpha - \beta|} (1 - |z|^2)^{c(n,\lambda)\beta} \approx_{|\alpha - \beta|} e^{-c(n,\lambda)\min(\alpha, \beta) e^{-d(0, z)}}.
\]
4.4

The Jacobian of the above transformation is \( \rho \). Substituting \((n, \lambda, (n+\beta) - n)\) in \((\lambda)\) we have absorbed the term 2n \( \cos \theta \) in the first line we have made the change of variable \( z^* = z^*|\epsilon_n = t \epsilon_n \), where \( t = \frac{1}{|z^*|} \). Now we transformation the integral in polar co-ordinates:

\[
\begin{align*}
\theta_1 & = \rho \sin \theta_1 \ldots \sin \theta_{n-3} \sin \theta_{n-2} \\
\theta_2 & = \rho \sin \theta_1 \ldots \sin \theta_{n-3} \cos \theta_{n-2} \\
\theta_3 & = \rho \sin \theta_1 \ldots \cos \theta_{n-3} \\
& \\
& \\
& \\
\theta_n & = \rho \cos \theta.
\end{align*}
\]

The Jacobian of the above transformation is \( \rho^{n-1} \sin^{n-2} \theta \sin^n \theta_1 \ldots \sin \theta_{n-3} \), where \( \rho \in [0, 1] \), \( \theta, \theta_1, \ldots, \theta_{n-3} \in [0, \pi] \) and \( \theta_{n-2} \in [0, 2\pi] \). With the above transformation, our integral now transforms to

\[
\int_{B^n} \frac{1}{|x - z^*|} dx \approx \int_0^1 \rho^{n-1} \int_0^\pi \frac{\sin^{n-2} \theta}{(t^2 - 2t \rho \cos \theta + \rho^2)^{n/2}} d\theta d\rho \\
= \int_0^1 \rho^{n-1} \int_0^\pi \frac{\sin^{n-2} \theta}{(1 - 2\rho \cos \theta + \rho^2)^{n/2}} d\theta d\rho \\
= \int_0^1 \rho^{n-1} \int_0^\pi \frac{\sin^{n-2} \theta}{(1/\rho^2 - 2/\rho \cos \theta + 1)^{n/2}} d\theta d\rho \\
= \int_0^1 \rho^{n-1} \int_0^\pi \frac{\sin^{n-2} \theta}{(1/\rho^2 - 1)^{n/2}} d\theta d\rho \\
= C \int_0^1 \rho^{n-1} d\rho,
\]

where in the first line we have absorbed the term 2n \( \prod_{l=1}^{n-3} \int_0^\pi \sin^l \theta \ d\theta \) into \( \approx \) sign and in the second line we have made the change of variable \( \rho \rightarrow t\rho \). In the last line \( C := \int_0^\pi \sin^{n-2} \theta \ d\theta \) and in the fourth line we have used the well-known formula

\[
\int_0^\pi \frac{\sin^{n-2} \theta}{(r^2 - 2r \cos \theta + 1)^{n/2}} d\theta = \frac{1}{r^{n-2}(r^2 - 1)} \int_0^\pi \sin^{n-2} \theta \ d\theta, \quad \text{for } r > 1.
\]

Further, we estimate using \( t = \frac{1}{|z^*|} \approx 1 \) as follows

\[
\frac{C}{t} \int_0^1 |z|^{n-2} d\rho \lesssim |z|^{n-2} \int_0^1 \frac{\rho}{1 - \rho^2} d\rho \lesssim \log \left( \frac{1}{1 - |z|^2} \right) \approx d(0, z). \tag{4.4}
\]

Substituting (4.4) in (4.2) we obtain the desired upper bound

\[
\int_{B^n} U[0]^{\alpha} |U|^\beta \ d\mu \lesssim d(0, z) e^{-c(n, \lambda)d(0, z)} = d(0, z) e^{-c(n, \lambda)\min(\alpha, \beta)d(0, z)}.
\]

The lower estimate. Now we shall compute the lower bound. We fix \( \theta_0 \) small to be determined later (actually, \( \theta_0 = \pi/4 \) will work). We consider the following cone defined by the collection of
all \( x \in B_E(z, |z|) \) such that the angle between \( (x - z) \) and \(-z\) is less than or equal to \( \theta_0 \).

\[ C := \{ x \in B_E(z, |z|) : - (x - z) \cdot z \geq |x - z||z| \cos \theta_0 \} , \]

where recall \( \cdot \) denotes the standard Euclidean inner product. We emphasise that we shall choose \( \theta_0 \) independent of the point \( z \).

**Figure 1.** The above picture demonstrating the bounded cone \( C \) where the terms \((1 - |x|^2)\) and \( |x - z^*| \) are comparable for all \( x \in C \).

First we claim that if \( |z| \sim 1 \), then

\[ |x - z^*| \lesssim (1 - |x|^2) \quad \text{for all } x \in C. \]

**Proof of the claim:** By triangle inequality

\[ |x - z^*| \leq |x - z| + |z - z^*| \leq |x - z| + \frac{1 - |z|^2}{|z|} \lesssim |x - z| + (1 - |z|^2). \]  

(4.5)

On the other hand \(|x|^2 = |x - z|^2 + 2(x - z) \cdot z + |z|^2\), and hence

\[ 1 - |x|^2 = 1 - |z|^2 - |x - z|^2 - 2(x - z) \cdot z \geq (1 - |z|^2) - |x - z|^2 + 2|x - z||z| \cos \theta_0 \]

\[ = (1 - |z|^2) + |x - z| (2|z| \cos \theta_0 - |x - z|). \]  

(4.6)

In the \( C \), we have \(|x - z| \leq |z|\), therefore

\[ 2|z| \cos \theta_0 - |x - z| \geq |z|(2 \cos \theta_0 - 1) \geq (\sqrt{2} - 1)|z| \]

if \( \theta_0 = \pi/4 \). We get from (4.6)

\[ 1 - |x|^2 \geq (\sqrt{2} - 1) \left[(1 - |z|^2) + |x - z|\right]. \]  

(4.7)

Hence from (4.5) and (4.7) the claim follows. Now in view of the above, we simplify the required integral as follows

\[ \int_C \frac{(1 - |x|^2)^{c(n,\lambda)(\alpha + \beta) - n}}{|x - z^*|^{2c(n, \lambda)\beta}} \, dx \gtrsim \int_C \frac{1}{|x - z^*|^n} \, dx. \]

Now we again make the following change of variable in polar co-ordinate

\[ x_1 = -\rho \sin \theta \sin \theta_1 \ldots \sin \theta_{n-3} \sin \theta_{n-2} \]
\[ x_2 = -\rho \sin \theta \sin \theta_1 \ldots \sin \theta_{n-3} \cos \theta_{n-2} \]
\[ x_3 = -\rho \sin \theta \sin \theta_1 \ldots \cos \theta_{n-3} \]
\[ x_n = |z| \cos \theta, \quad z = |z|e_n, \quad t = \frac{1}{|z|}. \]

With the above change of variable, we can write
\[
|x - z^*|^2 = |x|^2 + |z^*|^2 - 2x_n z_n^* = \varrho^2 + |z|^2 - 2|z|\varrho \cos \theta + \frac{1}{|z|^2} - 2 (|z| - \varrho \cos \theta) \frac{1}{|z|}.
\]

This yields for \(|z| \approx 1\)
\[
\int_{\mathbb{C}} \frac{1}{|x - z^*|^n} \, dx \approx \int_0^{\frac{|z|}{|z|^2}} \varrho^{n-1} \, d\varrho \int_0^{\theta_0} (\sin \theta)^{n-2} \, d\theta \\
= \int_0^{\frac{|z|}{|z|^2}} \left( \frac{1-|z|^2}{|z|^2} \right)^{n-1} \varrho^{n-1} \, d\varrho \int_0^{\theta_0} (\sin \theta)^{n-2} \, d\theta \\
= \int_0^{\frac{|z|}{|z|^2}} \varrho^{n-1} \, d\varrho \int_0^{\theta_0} (\sin \theta)^{n-2} \, d\theta \\
\geq \int_0^{\frac{|z|}{|z|^2}} \varrho^{n-1} \, d\varrho \int_0^{\theta_0} \sin^{-2} \theta \, d\theta \, d\varrho \\
\geq C \theta_0 \int_0^{\frac{|z|}{|z|^2}} \, d\varrho \geq \log \left( \frac{|z|^2}{1 - |z|^2} \right) \geq \log \left( \frac{1}{1 - |z|^2} \right)
\]
\[
\approx d(0, z).
\]

where in the second line we used the change of variable \(\varrho \rightarrow \left( \frac{1-|z|^2}{|z|^2} \right) \varrho\). Therefore from the above computation, we conclude
\[
\int_{\mathbb{B}} u[0]^{\alpha} u[z]^{\beta} \, dv_{\mathbb{B}} \approx d(0, z) e^{-(\alpha, \lambda)\beta d(0, z)}.
\]

Remark 4.1. The sharpness of the exponent in Lemma 4.1 in case \(\alpha \neq \beta\) can also be seen from the following heuristic explanation. Note that
\[
ed^{d(x,z)-d(0,z)} \approx \frac{1 - 2x_z + |x|^2 z}{1 - |x|^2}.
\]

Then as \(|z| \to 1\), i.e. \(z \to e\) with \(|e| = 1\), there holds
\[
ed^{d(x,z)-d(0,z)} \to \frac{1 - 2x \cdot e + |x|^2}{1 - |x|^2} \approx |x - e|^2 e^{d(0,x)}
\]

Therefore for \(\alpha > \beta\), denoting \(c = c(n, \lambda)\)
\[
e^{\beta d(0,z)} \int_{\mathbb{B}} e^{-c d(0,x)} e^{-c \beta d(x,z)} \, dv_{\mathbb{B}}(x) \to \int_{\mathbb{B}} e^{-c d(0,x)} |x - e|^{-2c \beta} e^{-c \beta d(0,x)} \, dv_{\mathbb{B}}(x) \\
\approx \int_{\mathbb{B}} \left( \frac{1 - |x|^2}{|x - e|^{2\beta}} \right)^{(\alpha + \beta) - n} \, dv_{\mathbb{B}}(x) \approx 1.
\]
By replacing $\beta$ in $e^{c\beta d(0,z)}$ by $\nu$, we observe that as $|z| \to 1$,

$$e^{c\beta d(0,z)} \int_{\mathbb{B}^n} e^{-c\beta d(0,x,z)} e^{-c\beta d(x,z)} \, dv_{\mathbb{B}^n}(x) \to \begin{cases} 0 & \text{if } \nu < \beta \\ +\infty & \text{if } \nu > \beta. \end{cases}$$

This proves that $\nu = \beta$ is the precise asymptotic estimate for the interaction terms.

4.2. Interaction of three bubbles. In this section, we will derive the interaction of three hyperbolic bubbles. We denote $U_i = U[z_i]$, $i = 1, 2, 3$. The main focus of this section is to derive upper bounds on the integral of interactions. We recall

$$Q_{ij} := e^{-c(n,\lambda d(z_i,z_j)} \quad \text{and} \quad Q := \max_{i \neq j} Q_{ij}.$$

We prove the following lemma.

**Lemma 4.2.** Let $U_i = U[z_i], i = 1, 2, 3$ be three hyperbolic bubbles and $p > 1$ and assume $Q << 1$. Then

$$\int_{\mathbb{B}^n} u_i^{x-1} u_2 u_3 \, dv_{\mathbb{B}^n} \begin{cases} \lesssim Q^{\frac{p}{2}} (\ln \frac{1}{Q})^{\frac{1}{p}}, & \text{if } p > 1, \\ \lesssim Q^{\frac{p}{2}} \ln \frac{1}{Q}, & \text{if } p = 1, \\ \lesssim Q^{\nu}, & \text{for any } \nu < \frac{p+1}{2}, \quad \text{if } p < 1. \end{cases}$$

**Proof.**

Case (i). $p - 1 > 1$. We apply Hölder inequality as follows

$$\int_{\mathbb{B}^n} u_i^{x-1} u_2 u_3 \, dv_{\mathbb{B}^n} = \int_{\mathbb{B}^n} (u_i^{x-1} u_2^2) (u_i^{x-1} u_3^2) (u_i^{x-1} u_2 u_3) \, dv_{\mathbb{B}^n}$$

$$\leq \left( \int_{\mathbb{B}^n} u_i^{x-1} u_2^3 \, dv_{\mathbb{B}^n} \right)^{\frac{1}{3}} \left( \int_{\mathbb{B}^n} u_i^{x-1} u_3^3 \, dv_{\mathbb{B}^n} \right)^{\frac{1}{3}} \left( \int_{\mathbb{B}^n} u_i^{x-1} u_2 u_3 \, dv_{\mathbb{B}^n} \right)^{\frac{1}{3}}$$

$$\lesssim Q_{12}^{\frac{1}{3}} \left( Q_{23}^{\frac{2}{3}} \ln \frac{1}{Q_{23}} \right)^{\frac{1}{3}} Q_{13}^{\frac{1}{3}}$$

$$\lesssim Q^{\frac{1}{2}} \left( \ln \frac{1}{Q} \right)^{\frac{1}{3}},$$

where we have used $t \to t^{1/\ln(1/t)}$ is increasing near $t = 0$.

Case (ii). $p - 1 = 1$. The idea is the same as in case (i). We apply Hölder inequality as follows

$$\int_{\mathbb{B}^n} u_i u_2 u_3 \, dv_{\mathbb{B}^n} = \int_{\mathbb{B}^n} (u_i u_2^2) (u_i u_3^2) (u_i u_2 u_3) \, dv_{\mathbb{B}^n}$$

$$\leq \left( \int_{\mathbb{B}^n} u_i u_2^3 \, dv_{\mathbb{B}^n} \right)^{\frac{1}{3}} \left( \int_{\mathbb{B}^n} u_i u_3^3 \, dv_{\mathbb{B}^n} \right)^{\frac{1}{3}} \left( \int_{\mathbb{B}^n} u_i u_2 u_3 \, dv_{\mathbb{B}^n} \right)^{\frac{1}{3}}$$

$$\lesssim Q_{12}^{\frac{1}{3}} \left( Q_{23}^{\frac{2}{3}} \ln \frac{1}{Q_{23}} \right)^{\frac{1}{3}} \left( Q_{13}^{\frac{2}{3}} \ln \frac{1}{Q_{13}} \right)^{\frac{1}{3}}$$

$$\lesssim Q^{\frac{1}{2}} \ln \frac{1}{Q}.$$

Case (iii). We fix $\beta > \alpha > 1$ and $s_1, s_2 \in (0, 1)$ that to be decided later and apply Hölder

$$\int_{\mathbb{B}^n} u_i^{x-1} u_2 u_3 \, dv_{\mathbb{B}^n} \leq \left( \int_{\mathbb{B}^n} u_i^s u_2^s \, dv_{\mathbb{B}^n} \right)^{s_1} \left( \int_{\mathbb{B}^n} u_i^s u_3^s \, dv_{\mathbb{B}^n} \right)^{s_1} \left( \int_{\mathbb{B}^n} u_2^{x+1} u_3^{x+1} \, dv_{\mathbb{B}^n} \right)^{s_2}. $$
A quick sanity check gives
\[ 2\alpha s_1 = p - 1, \quad \beta s_1 + \frac{p + 1}{2} s_2 = 1. \] (4.9)
Since we assumed \( \alpha < \beta \), the interaction estimates of the previous section gives
\[
\int_{\mathbb{R}^n} u_1^{p-1} u_2 u_3 \, d\nu_{\mathbb{R}^n} \leq Q_{12}^{\alpha s_1} Q_{13}^{\alpha s_1} \left( \frac{Q_{23}^{\alpha s_1}}{Q_{23}} \ln \frac{1}{Q_{23}} \right)^{s_2} \\
\leq Q^{2\alpha s_1 + \frac{p+1}{2} s_2} \left( \ln \frac{1}{Q} \right)^{s_2} \\
= Q^{p-\beta s_1} \left( \ln \frac{1}{Q} \right)^{s_2}.
\]
If we want the interaction to be \( o(Q) \) then we need \( p - \beta s_1 > 1 \). Now let us check that all the above requirements are feasible. Choose \( \epsilon > 0 \) small and set \( \alpha = \frac{(p+1)(1-\epsilon)}{2(1-\epsilon)}, \beta = \frac{(p+1)(1+\epsilon)}{2} \). Then (4.9) gives for sufficiently small \( \epsilon \)
\[
s_1 = \frac{p - 1}{(p+1)(1-\epsilon)} < 1, \quad \beta s_1 = \frac{(1+\epsilon)(p-1)}{2(1-\epsilon)} < p - 1, \\
p + \frac{1}{2} s_1 = 1 - \beta s_1 = \frac{3 - \epsilon - (1+\epsilon)p}{2(1-\epsilon)} < \frac{p + 1}{2}.
\]
Next we show that for any \( \nu < \frac{p+1}{2} \) we can choose \( \epsilon > 0 \) such that \( \nu < p - \beta s_1 \). Indeed, direct computation gives
\[
p - \beta s_1 = p - \frac{(1+\epsilon)(p-1)}{2(1-\epsilon)} \to \frac{p + 1}{2}, \quad \text{as} \ \epsilon \to 0
\]
and hence we can choose such \( \epsilon \). As a result
\[
Q^{p-\beta s_1} \left( \ln \frac{1}{Q} \right)^{s_2} = Q' Q^{p-\beta s_1-\nu} \left( \ln \frac{1}{Q} \right)^{s_2} \lesssim_{\nu} Q'
\]
if \( Q << 1 \). This completes the proof.

\[ \square \]

**Remark 4.2.** The constant in the third case i.e \( p - 1 < 1 \) depends on \( \nu \). In our first case if \( p - 1 > \frac{3}{2} \) then we can refine the upper bound to \( Q^{\min(p-1,2)} \). Since our interest is in \( p \leq 2^* - 1 \) such assumptions are void in dimension \( n \geq 5 \). The proof follows in the same manner by clubbing
\[
u \in \left( 0, 1 \right) 
\]
and using Hölder inequality.

4.3. **Interaction of bubbles and the derivative of bubbles.** In this section we find the estimates on the interaction of a bubbles and the space derivatives of other bubbles. Before proceeding let us first simplify the formula of the derivative. In coordinates \( \frac{d}{dt}|_{t=0}(U \circ \tau_{t\epsilon}) = V_j(U) \), where \( V_j = (1 + |x|^2) \frac{\partial}{\partial x_j} - 2x_j \sum_{l=1}^n x_l \frac{\partial}{\partial x_l} \) and the derivatives involved are in the sense of Euclidean (see Appendix A).

Recall the Euclidean norm \( r = |x| \) and the hyperbolic distance \( \rho = d(x,0) \) is related by \( r = \tanh \frac{\rho}{\rho} \). Since a hyperbolic ball with center 0 is also a Euclidean ball with center 0 but possibly different radius, we deduce \( U \) is a function of \( r \). By abuse of notation we denote \( U(x) = U(r) = U_1(r) \). Then direct computation gives
\[
V_j(U) = \frac{x_j}{r}(1 - r^2) \frac{d}{dr}(U_1(r)) = 2 \frac{x_j}{|x|} U'(\rho)
\] (4.10)
where \( t \) denotes the derivative with respect to \( \rho \) and recall \( \rho = d(x, 0) \). It follows directly from the above expression and the bound \(|U'(\rho)| \lesssim U(\rho)|\) that \(|V_j(U)| \lesssim U|\) on \( B^n \). Moreover, note that as \( U' < 0 \) for all \( \rho > 0 \), \( V_j(U) \) changes sign in \( B^n \).

We prove the following

**Lemma 4.3.** Let \( z = (z_1, \ldots, z_n) \in B^n \) does not lie on the hyperplane \( P_j := \{ x \in \mathbb{R}^n \mid x_j = 0 \} \). Then

\[
\int_{B^n} U[z]^p V_j(U[0]) \, dv_{B^n} \approx_z \begin{cases} e^{-c(n,\lambda)d(z,0)}, & \text{if } z_j < 0, \\ -e^{-c(n,\lambda)d(z,0)}, & \text{if } z_j > 0, \end{cases}
\]

where \( \approx_z \) indicates the constant depends on (the position of) \( z \).

**Remark 4.3.** Before proving the lemma let us first make a remark about why it should be true. Also, this is a good time to remark that this is in sharp contrast to the Euclidean case. In the previous section, we observed that the interaction is an exponentially decaying function of the distance between the points where the bubbles are most concentrated. The derivative of an exponential function is an exponential function while the derivative of the distance function is of the absolute value of approximately 1 and that precisely plays the decisive role here. Formally,

\[
\int_{B^n} U[z]^p V_j(U[0]) \, dv_{B^n} = \int_{B^n} U[z]^p \frac{d}{dt} \bigg|_{t=0} U[te_j] \, dv_{B^n} \\
= \frac{d}{dt} \bigg|_{t=0} \int_{B^n} U[z]^p U[te_j] \, dv_{B^n} \\
\approx \frac{d}{dt} \bigg|_{t=0} e^{-c(n,\lambda)d(z,te_j)} \\
\approx e^{-c(n,\lambda)d(z,0)} \frac{d}{dt} \bigg|_{t=0} d(z,te_j) \\
\approx_z \pm e^{-c(n,\lambda)d(z,0)}.
\]

On the other hand, in the Euclidean case, the interaction of Aubin-Talenti bubbles has polynomial decay:

\[
\int_{\mathbb{R}^n} U[z,1]^p U[z_1,1] \, dx \approx |z - z_1|^{-(n-2)}
\]

and hence

\[
\left| \int_{\mathbb{R}^n} U[z,1]^p \partial_{x_j} U[0,1] \, dx \right| = \left| \frac{d}{dt} \bigg|_{t=0} \int_{\mathbb{R}^n} U[z,1]^p U[te_j,1] \, dx \right| \\
\approx \left| \frac{d}{dt} \bigg|_{t=0} |z - te_j|^{-(n-2)} \right| \\
\approx |z|^{-(n-1)} << |z|^{-(n-2)} \text{ if } |z| >> 1.
\]

See the treatise by A. Bahri [3, Estimate (F11)]. However, the interaction of bubbles and the \( \mu \)-derivative of another bubble has the same decay as the interaction of bubbles if the height of the bubbles is comparable [3, Estimate (F16)].

**Proof of Lemma 4.3.**

**Proof.** First, we simplify the expression as follows

\[
\int_{B^n} U[z]^p V_j(U[0]) \, dv_{B^n} = \int_{B^n} (\Delta_{B^n} U[z]) V_j(U[0]) \, dv_{B^n}
\]
\[= \int_{\mathbb{B}^n} u[z](-\Delta_{\mathbb{B}^n} V_j(u[0])) \, dv_{\mathbb{B}^n} \]
\[= p \int_{\mathbb{B}^n} u[z]u[0]^{p-1}V_j(u[0]) \, dv_{\mathbb{B}^n} \]
\[= 2p \int_{\mathbb{B}^n} u[z]u[0]^{p-1} \frac{x_j}{|x|} u[0]' \, dv_{\mathbb{B}^n}. \quad (4.11)\]

Recall that \(u[0]' \leq 0\) on \(\mathbb{B}^n\) and \(\rho\) and hence the term \(\frac{x_j}{|x|}\) will play the central role in our estimate. Also, note that the upper bound follows directly from the bound \(|u'(\rho)| \lesssim u(\rho)\) and the estimates on the interaction of bubbles obtained in the last subsection.

For \(x \in \mathbb{B}^n\), we denote by \(\hat{x}\) the reflection of \(x\) with respect to the plane \(P_j\). We denote by \(\mathbb{B}^n_+\) the negative half of the ball \(\{x \in \mathbb{B}^n \mid x_j < 0\}\). \(\mathbb{B}^n_+\) is similarly defined with \(< 0\) replaced by \(> 0\). Without loss of generality, we assume that \(z \in \mathbb{B}^n_+\). First, we claim that

**Claim:** \(d(x, z) \leq d(\hat{x}, z)\) for all \(x \in \mathbb{B}^n_+\).

The proof follows from the formula of the hyperbolic distance.

\[\cosh^2 \left( \frac{d(\hat{x}, z)}{2} \right) - \cosh^2 \left( \frac{d(x, z)}{2} \right) = \frac{2(x - \hat{x}) \cdot z}{(1 - |x|^2)(1 - |z|^2)} > 0\]

as both \(x, z \in \mathbb{B}^n_+\). Since \(\cosh\) is strictly increasing the claim follows. Since \(u[z]\) is strictly decreasing in \(d(x, z)\) we deduce \(u[z](x) > u[z](\hat{x})\) for all \(x \in \mathbb{B}^n_+\). We further simplify the integral \((4.11)\)

\[\int_{\mathbb{B}^n} u[z]pV_j(u[0]) \, dv_{\mathbb{B}^n} = 2p \int_{\mathbb{B}^n} u[z]u[0]^{p-1} \frac{x_j}{|x|} u[0]' \, dv_{\mathbb{B}^n} \]
\[= 2p \int_{\mathbb{B}^n} u[z]u[0]^{p-1} \frac{x_j}{|x|} u[0]' \, dv_{\mathbb{B}^n} + 2p \int_{\mathbb{B}^n_+} u[z]u[0]^{p-1} \frac{x_j}{|x|} u[0]' \, dv_{\mathbb{B}^n} \]
\[= 2p \int_{\mathbb{B}^n} (u[z](x) - u[z](\hat{x}))u[0]^{p-1}(x) \frac{x_j}{|x|} u[0]'(x) \, dv_{\mathbb{B}^n}(x). \quad (4.12)\]

Notice that the integrand in \((4.12)\) is non-negative. The proof can be finished if we show that \(u[z](x) - u[z](\hat{x}) \gtrsim u[z](x)\) on a compact subdomain \(B\) of \(\mathbb{B}^n_+\). We achieve this by comparison principle. Let us denote \(w(x) = u[z](x) - u[z](\hat{x})\) and fix a compact domain \(B \subset \mathbb{B}^n_+\). Then \(w\) satisfies

\[(-\Delta_{\mathbb{B}^n} - \lambda)w = u[z]^p(x) - u[z]^p(\hat{x}) \geq pu[z]^{p-1}(x)w(x) \geq u[z]^{p-1}(x)w(x) \geq 0 \text{ in } B.\]

Let \(c_1 = \inf_{x \in \partial B} w(x) > 0\) and let \(c_2\) be such that \(c_2 \sup_{x \in \partial B} u[z](x) \leq c_1\). Note that the constants \(c_1, c_2\) depends on \(B\) and hence on the location of \(z\). Since \(c_2u[z]\) satisfies \((-\Delta_{\mathbb{B}^n} - \lambda)(c_2u[z]) = u[z]^{p-1}(c_2u[z]),\) weak comparison principle gives \(w \geq c_2u[z] \text{ in } B.\)

Indeed, denoting \(v(x) = w(x) - c_2u[z](x),\) we see that \(v\) satisfies

\[\begin{cases} -\Delta_{\mathbb{B}^n} v - \lambda v \geq u[z]^{p-1}v, & \text{in } B, \\ v \geq 0 & \text{on } \partial B. \end{cases} \quad (4.13)\]

Multiplying \((4.13)\) by \(v^{-} \in H^1_0(B)\) and integrating by parts gives

\[\int_B (|\nabla_{\mathbb{B}^n} v^{-}|^2 - \lambda |v^{-}|^2) \, dv_{\mathbb{B}^n} \leq \int_B u[z]^{p-1}|v^{-}|^2 \, dv_{\mathbb{B}^n}. \quad (4.14)\]
But (4.14) is possible only when \( v = 0 \) because the first Dirichlet eigenvalue of the operator 
\(-\Delta_{B^n} - \lambda)/U[z]^{p-1} \) on \( B^n \) is \( p > 1 \) (see Lemma A.1) and so on \( B \) it must be bigger than 1. Hence \( v \geq 0 \) on \( B \). Moreover, if the compact set stays away from the hyperplane \( \{x_j = 0\} \) then we have the lower bound 
\[ \frac{\partial}{\partial x^j} U[0]'(x) > 0 \] on \( B \). Therefore
\[
\int_{B^n} U[z]^p V_j(U[0]) = 2p \int_{B^n} (U[z](x) - U[z](\hat{x})) U[0]^{p-1}(x) \frac{x_j}{|x|} U[0]'(x) \, dv_{B^n}(x)
\geq 2pc_2 \int_{B} U[z] U[0]^{p-1}(x) \frac{x_j}{|x|} U[0]'(x) \, dv_{B^n}(x)
\geq z \int_{B} U[z] U[0]^p \, dv_{B^n}
\geq z e^{-c(n,\lambda)d(z,0)}.
\]
This completes the proof. \( \square \)

**Remark 4.4.** Note that we can not do better than \( \approx z \), i.e., the constant must depend on \( z \). To view this note that if \( z_j = 0 \) then it follows from the formula of the hyperbolic distance that 
\[ d(x,z) = d(\hat{x},z) \] where recall \( \hat{x} \) is the reflection of \( x \) with respect to the plane \( \{x_j = 0\} \). As a result, the interaction vanishes and hence the constant must depend on the position of \( z \). Indeed, in the next lemma we show that if \( z_j \) stays away from \( 0 \), then we can make the bound uniform in the sense that if \( |z_j| \geq \kappa > 0 \) then the constant depends only on \( \kappa \).

**Lemma 4.4.** Let \( z = (z_1, \ldots, z_n) \in B^n \) satisfy \( |z_j| \geq \kappa \) for some constant \( \kappa \in (0,1) \). Then there exists a \( \delta_0 > 0 \) such that for all \( z \) satisfying \( 1 - |z| < \delta_0 \)
\[
\int_{B^n} U[z]^p V_j(U[0]) \, dv_{B^n} \approx_k \delta_0 \begin{cases} 
-\epsilon d(z,0), & \text{if } z_j < -\kappa, \\
-\epsilon d(z,0), & \text{if } z_j > \kappa,
\end{cases}
\]
where \( \approx_k \delta_0 \) indicates a constant that depends on \( \kappa, \delta_0 \) and the parameters \( n, \lambda, p \).

**Proof.** As in Lemma 4.3, we may assume \( z_j \leq -\kappa \) and we only need to obtain the lower bound. Given \( z \) as in the statement of the lemma, we set \( e = \frac{z_j}{\kappa} \). We fix \( R_0 \) large whose value will be decided later. We fix a compact set \( B (= B_e, \text{see Figure 2}) \) satisfying the following two conditions:

(i) \( |x - e| < \frac{1}{R_0} \) for all \( x \in B \),

(ii) the Euclidean diameter of \( B \leq \frac{\kappa}{2} \) and \( x_j \leq -\frac{\kappa}{2} \), for all \( x \in B \).

![Figure 2](image-url). The above picture demonstrating the choice of the compact set \( B_e \) which depends only on the direction \( e \) of \( z \).
Note that $B$ depends only on the direction of $z$. As before we denote by $\hat{x}$ the reflection of $x$ with respect to the plane $\{x_j = 0\}$. We claim that if $R_0$ is sufficiently large and $\delta_0$ is sufficiently small then $d(\hat{x}, z) - d(x, z)$ is large. We compute

$$d(\hat{x}, z) - d(x, z) = 2 \left( \ln \left( \frac{|z| |\hat{x} - z^*| + |\hat{x} - z|}{\sqrt{(1 - |x|^2)(1 - |z|^2)}} \right) - \ln \left( \frac{|z| |x - z^*| + |x - z|}{\sqrt{(1 - |x|^2)(1 - |z|^2)}} \right) \right)$$

$$= 2 \ln \left( \frac{|z| |\hat{x} - z^*| + |\hat{x} - z|}{|z| |x - z^*| + |x - z|} \right)$$

Recalling $e = \frac{\hat{x}}{|\hat{x}|}$, $z_k \leq -\kappa$ and $1 - |z| < \delta_0$ we estimate

$$|z||x - z^*| + |\hat{x} - z^*| \leq |z||x - e| + |z||e - z^*| + |x - e| + |e - z| \leq (1 + |z|)|x - e| + (1 + |z|^{-1})(1 - |z|) \leq (1 + \kappa^{-1})(R_0^{-1} + \delta_0).$$

On the other hand as $x_k \leq -\frac{\kappa}{2}$ we see that $|z||\hat{x} - z^*| + |\hat{x} - z| \geq \kappa$ and hence

$$d(\hat{x}, z) - d(x, z) \geq 2 \ln \left( \frac{\kappa}{(1 + \kappa^{-1})(R_0^{-1} + \delta_0)} \right) =: R_1.$$

Clearly if $R_0$ is sufficiently large and if $\delta_0$ is sufficiently small then $R_1$ can be made large. Now recall that there exists universal positive constants $\chi_1, \chi_2$ such that $\chi_1 e^{-c(n, \lambda) d(x, z)} \leq U[z](x) \leq \chi_2 e^{-c(n, \lambda) d(x, z)}$ for all $x \in B^n$. On the boundary of the fixed ball $B, x \in \partial B$ we have

$$U[z](x) - U[z](\hat{x}) \geq \chi_1 e^{-c(n, \lambda) d(x, z)} - \chi_2 e^{-c(n, \lambda) d(\hat{x}, z)} \geq \chi_1 e^{-c(n, \lambda) d(x, z)} - \chi_2 e^{-c(n, \lambda) d(x, z) + R_1} \geq (\chi_1 - \chi_2 e^{-c(n, \lambda) R_1}) e^{-c(n, \lambda) d(x, z)} \geq \left( \frac{\chi_1 - \chi_2 e^{-c(n, \lambda) R_1}}{\chi_2} \right) U[z](x).$$

Hence the choice of $R_1$ such that $\chi_1 - \chi_2 e^{-c(n, \lambda) R_1} \geq \chi_1/2$ would suffice to ensure uniform lower bound $U[z](x) - U[z](\hat{x}) \gtrsim_{\kappa, R_0} U[z](x)$ on $B$. Moreover, as $x_k \leq -\frac{\kappa}{2}$ on $B$ we have $\frac{\hat{x}}{|\hat{x}|} U[0]' \gtrsim_{\kappa} U[0]$ on $B$. Hence

$$\int_{B^n} U[z]^p V_j(U[0]) \, dv_{B^n} = 2p \int_{B^n} (U[z](x) - U[z](\hat{x})) U[0]^{p-1}(x) \frac{x_j}{|x|} U[0]'(x) \, dv_{B^n}(x) \gtrsim_{\kappa, \delta_0} \int_B U[z] U[0]^p \, dv_{B^n} \gtrsim_{\kappa, \delta_0} e^{-c(n, \lambda) d(z, 0)} \int_B \frac{(1 - |x|^2)^{c(n, \lambda)(p+1) - n}}{|x - z^*|^{2c(n, \lambda)}} \, dx. \quad (4.15)$$

Note that the same $B$ can be chosen for any points on the line segment $[z, e]$. Finally, to conclude the proof we note that given any two points $z_1$ and $z_2$ satisfying the hypothesis of the lemma and $|z_1| = |z_2|$, let $A$ be the orthogonal transformation that takes $z_1$ to $z_2$. If $B_1$ is the ball that is chosen for the point $z_1$ according to the properties (i) and (ii) then $B_2 = A(B_1)$ would satisfy the same properties for the point $z_2$. Hence for each such $z$ we can choose $B$ so that the integral involved in (4.15) can be bound from below uniformly with respect to $z$. This completes the proof of the lemma. \qed
5. Proof of the main stability theorem

In this section we prove the main theorem of this paper, namely Theorem 1.1.

Proof of Theorem 1.1: Let $U_i := U(z_i), i = 1, \ldots, N$ and $\sigma = \sum_{i=1}^{N} \alpha_i U_i$ be the linear combination of hyperbolic bubbles that is closest to $u$ in $\| \cdot \|_{\lambda}$, i.e.,

$$
\| u - \sigma \|_{\lambda} = \min_{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_N \in \mathbb{R}, \tilde{z}_1, \ldots, \tilde{z}_N \in B^n} \left( \int_{B^n} |\nabla_{B^n}(u - \tilde{\sigma})|^2 \, dv_{B^n} - \lambda \int_{B^n} |u - \tilde{\sigma}|^2 \, dv_{B^n} \right)^{\frac{1}{2}},
$$

and $\tilde{\sigma} = \sum_{i=1}^{N} \tilde{\alpha}_i U(\tilde{z}_i)$. Let $\rho := u - \sigma$. From the hypothesis, it directly follows that $\| \rho \|_{\lambda} \leq \delta$.

Furthermore, the family $(\alpha_i, U_i)_{1 \leq i \leq N}$ is $\delta'$-interacting for some $\delta'$ that goes to 0 as $\delta$ goes to 0.

Summing up, we can say that qualitatively $\sigma$ is a sum of weakly-interacting hyperbolic bubbles and $\| \rho \|_{\lambda}$ is small.

Since $\sigma$ minimizes $H^1_{\lambda}$ distance from $u$, $\rho$ is $H^1_{\lambda}$ orthogonal to the manifold composed of linear combination of $N$ hyperbolic bubbles (see [5, (5.1)]), namely the following orthogonality conditions hold:

$$
\int_{B^n} (\nabla_{B^n} \rho, \nabla_{B^n} U_i)_{B^n} - \lambda \rho U_i \, dv_{B^n} = 0 \quad (5.1)
$$

$$
\int_{B^n} (\nabla_{B^n} \rho, \nabla_{B^n} V_j(U_i))_{B^n} - \lambda \rho V_j(U_i) \, dv_{B^n} = 0 \quad (5.2)
$$

for all $i = 1, \ldots, N, j = 1, \ldots, n$, and where $V_j(U_i) = \frac{d}{dt} U_i \circ \tau_{t \delta_j}|_{t=0}$.

Since $U_i, V_j(U_i)$ are eigenfunctions for $-\frac{\Delta_{B^n} - \lambda}{\rho} \rho$ (see [5, Proposition 3.1] or Appendix) the above orthogonality conditions are equivalent to

$$
\int_{B^n} U_i^p \rho \, dv_{B^n} = 0, \quad (5.3)
$$

$$
\int_{B^n} V_j(U_i) U_i^{p-1} \rho \, dv_{B^n} = 0. \quad (5.4)
$$

for all $i = 1, \ldots, N, j = 1, \ldots, n$. Our goal is to show that $\| \rho \|_{\lambda}$ is controlled by $\| \Delta_{B^n} u + \lambda u + |u|^{p-1} u \|_{H^{-1}(B^n)}$. To achieve that let us start testing $\Delta_{B^n} u + \lambda u + |u|^{p-1} u$ by $\rho$, exploiting the orthogonality condition (5.1) yields

$$
\| \rho \|_{\lambda}^2 = \int_{B^n} [\nabla_{B^n} \rho, \nabla_{B^n} (u - \sigma)]_{B^n} - \lambda \rho (u - \sigma) \, dv_{B^n}
$$

$$
= \int_{B^n} [\nabla_{B^n} \rho, \nabla_{B^n} u]_{B^n} - \lambda \rho u \, dv_{B^n}
$$

$$
= \int_{B^n} (\Delta_{B^n} u - \lambda u - u|^{p-1}) \rho \, dv_{B^n} + \int_{B^n} u|^{p-1} \rho \, dv_{B^n}
$$

$$
\leq \| \Delta_{B^n} u - \lambda u - u|^{p-1} \|_{H^{-1}(B^n)} \| \rho \|_{H^1(B^n)} + \int_{B^n} u|^{p-1} \rho \, dv_{B^n}
$$

$$
\lesssim \| \Delta_{B^n} u + \lambda u + u|^{p-1} \|_{H^{-1}(B^n)} \| \rho \|_{\lambda} + \int_{B^n} u|^{p-1} \rho \, dv_{B^n}. \quad (5.5)
$$

To control the last term in (5.5), we use the following elementary estimates

$$
|a + b||a + b|^{p-1} - a|a|^{p-1} \leq p|a|^{p-1}|b| + C_p \left(|a|^{p-2}|b|^2 + |b|^p \right), \quad (5.6)
$$
\[
\left| \left( \sum_{i=1}^{N} a_i \right) \left| \sum_{i=1}^{N} a_i |a_i|^{p-1} \right| - \sum_{i=1}^{N} a_i |a_i|^{p-1} \right| \lesssim \sum_{1 \leq i \neq k \leq N} |a_i|^{p-1} |a_k|, \quad (5.7)
\]

that holds for any \( a, b \in \mathbb{R} \) and for any \( a_1, \ldots, a_N \in \mathbb{R} \). Applying (5.6) with \( a = \sigma \) and \( b = \rho \), and (5.7) with \( a_i = \alpha_i \mathcal{U}_i \) yields

\[
\left| u|u|^{p-1} - |\sigma|^{p-1}\sigma \right| \leq p|\sigma|^{p-1}|\rho| + C(|\sigma|^{p-2}|\rho|^2 + |\rho|^p)
\]

where \( C = C_{n, \lambda, N, p} \) and

\[
\left| \sigma|\sigma|^{p-1} - \sum_{i=1}^{N} \alpha_i \mathcal{U}_i|\alpha_i \mathcal{U}_i|^{p-1} \right| \lesssim \sum_{1 \leq i \neq k \leq N} |\alpha_i \mathcal{U}_i|^{p-1} |\alpha_k \mathcal{U}_k|.
\]

Combining the above two estimates we deduce

\[
\left| u|u|^{p-1} - \sum_{i=1}^{N} \alpha_i |\alpha_i|^{p-1} \mathcal{U}_i^p \right| \leq p|\sigma|^{p-1}|\rho| + C\left( |\sigma|^{p-2}|\rho|^2 + |\rho|^p + \sum_{1 \leq i \neq k \leq N} \mathcal{U}_i^{p-1} \mathcal{U}_k \right).
\]

Therefore, using (5.3), we get

\[
\int_{\mathbb{B}^n} u|u|^{p-1} \rho \, dv_{\mathbb{B}^n} \leq p \int_{\mathbb{B}^n} \sigma|\sigma|^{p-1} \rho^2 \, dv_{\mathbb{B}^n} + C \left( \int_{\mathbb{B}^n} |\sigma|^{p-2} \rho^3 \, dv_{\mathbb{B}^n} \right. \\
+ \int_{\mathbb{B}^n} |\rho|^{p+1} \, dv_{\mathbb{B}^n} + \sum_{1 \leq i \neq k \leq N} \int_{\mathbb{B}^n} |\rho| \mathcal{U}_i^{p-1} \mathcal{U}_k \, dv_{\mathbb{B}^n} \right). \quad (5.8)
\]

Using Proposition 7.1 from the forthcoming section, we have

\[
p \int_{\mathbb{B}^n} \sigma|\sigma|^{p-1} \rho^2 \, dv_{\mathbb{B}^n} \leq \tilde{c} \|\rho\|_{L^3}^3, \quad \text{with} \quad \tilde{c} = \tilde{c}(n, \lambda, N, p) < 1.
\]

Using Hölder and Poincaré-Sobolev inequality we estimate the other terms on the RHS of (5.8) as follows

\[
\int_{\mathbb{B}^n} |\sigma|^{p-2} \rho^3 \, dv_{\mathbb{B}^n} \leq \|\rho\|_{L^{p+1}}^3 \|\sigma\|_{L^{p+1}}^{p-2} \lesssim \|\rho\|_{L^3}^3,
\]

\[
\int_{\mathbb{B}^n} |\rho|^{p+1} \, dv_{\mathbb{B}^n} \lesssim \|\rho\|_{L^\lambda}^{p+1},
\]

\[
\int_{\mathbb{B}^n} |\rho| \mathcal{U}_i^{p-1} \mathcal{U}_k \, dv_{\mathbb{B}^n} \leq \|\rho\|_{L^{p+1}} \|\mathcal{U}_i^{p-1} \mathcal{U}_k\|_{L^{(p+1)\prime}} \lesssim \|\rho\|_{L^\lambda} \|\mathcal{U}_i^{p-1} \mathcal{U}_k\|_{L^{(p+1)\prime}}.
\]

Substituting these estimates into (5.8) yields

\[
\left| \int_{\mathbb{B}^n} u|u|^{p-1} \rho \, dv_{\mathbb{B}^n} \right| \leq \tilde{c}(n, \lambda, N) \|\rho\|_{L^3}^3 \\
+ C\left( \|\rho\|_{L^3}^3 + \|\rho\|_{L^\lambda}^{p+1} + \sum_{1 \leq i \neq k \leq N} \|\rho\|_{L^\lambda} \|\mathcal{U}_i^{p-1} \mathcal{U}_k\|_{L^{(p+1)\prime}} \right). \quad (5.9)
\]

Next, we control \( \|\mathcal{U}_i^{p-1} \mathcal{U}_k\|_{L^{(p+1)\prime}} \) for \( i \neq k \). Since \( p > 2 \) implies \( \min\{(p-1)(p+1), (p+1)\prime\} = (p+1)\prime \). Therefore, by Lemma 4.1, we find

\[
\|\mathcal{U}_i^{p-1} \mathcal{U}_k\|_{L^{(p+1)\prime}} = \left( \int_{\mathbb{B}^n} \mathcal{U}_i^{p-1}(p+1)\prime \mathcal{U}_k^{(p+1)\prime} \, dv_{\mathbb{B}^n} \right)^{\frac{1}{(p+1)\prime}} \approx \left(e^{-c(n, \lambda)(p+1)\prime d(z_i, z_k)}\right)^{\frac{1}{(p+1)\prime}} \\
= e^{-c(n, \lambda)d(z_i, z_k)}
\]
Moreover, using Lemma 8.1 we get
\[
\max_{i \neq k} \int_{\mathbb{B}_n} u_i^p u_k d\nu_{\mathbb{B}_n} \lesssim \|(\Delta_{\mathbb{B}_n} + \lambda)u + |u|^{p-1}u\|_{H^{-1}} + \epsilon \|\rho\|_\lambda + \|\rho\|_\lambda^2 .
\] (5.11)
Hence, combining (5.9), (5.10) and (5.11), we find
\[
\int_{\mathbb{B}_n} u|u|^{p-1}\rho d\nu_{\mathbb{B}_n} \leq \left(\tilde{c}(n, \lambda, N, p) + \varepsilon N^2 C_{n,\lambda,N,p} \tilde{C}\right) \|\rho\|^2_\lambda + C'_{n,\lambda,N,p}\left(\|\rho\|^3_\lambda + \|\rho\|^{p+1}\right)
+ \|\Delta_{\mathbb{B}_n} u + \lambda u + u|u|^{p-1}\|_{H^{-1}(\mathbb{B}_n)} \|\rho\|_\lambda ,
\] (5.12)
where \(\tilde{C}\) is the constant hidden in (5.11). We choose \(\varepsilon > 0\) small such that
\[
\tilde{c}(n, \lambda, N, p) + \varepsilon N^2 C_{n,\lambda,N,p} \tilde{C} < 1.
\]
Hence, substituting (5.12) in (5.5) yields
\[
\left(1 - \tilde{c}(n, \lambda, N, p) - \varepsilon N^2 C_{n,\lambda,N,p} \tilde{C}\right) \|\rho\|^2_\lambda \lesssim \|\rho\|^3_\lambda + \|\rho\|^{p+1}_\lambda + \|\Delta_{\mathbb{B}_n} u + \lambda u + u|u|^{p-1}\|_{H^{-1}(\mathbb{B}_n)} \|\rho\|_\lambda .
\]
Since we can assume \(\|\rho\|_\lambda \ll 1\), the last inequality implies
\[
\|\rho\|_\lambda \lesssim \|\Delta_{\mathbb{B}_n} u + \lambda u + u|u|^{p-1}\|_{H^{-1}(\mathbb{B}_n)} .
\] (5.13)
So, now we are just left to show that the value of all the \(\alpha_i\) can be replaced by 1; note that thanks to (5.13), this fact follows from Lemma 8.1. More precisely, \(|\alpha_i - 1| \lesssim \|\Delta_{\mathbb{B}_n} u + \lambda u + u|u|^{p-1}\|_{H^{-1}(\mathbb{B}_n)}\); so it suffices to consider \(\sigma' = \sum_{i=1}^N u_i\) to get that \(\sigma'\) satisfies all the desired conditions.

**Proof of Corollary 1.1**

**Proof.** First assume \(p < 2^* - 1\). Then the result immediately follows from the profile decomposition Theorem C, Remark 1.2 and Theorem 1.1.

Now assume \(p = 2^* - 1\), and let \(\Lambda \not\subset \Lambda\). We know that that \(S_\Lambda\) is not a rational multiple of \(S\) and hence there exists \(\delta = \delta(n, N, \lambda) > 0\) such that the interval \((\lambda - \delta)S_\Lambda, (\lambda + \delta)S_\Lambda\) does not contain an integer multiple of \(S\). Should the statement of the corollary be false there must exist a sequence \(\{u_m\}\) in \(H^1(\mathbb{B}_n)\) such that \(u_m \geq 0, I_\Lambda(u_m) \to \frac{N}{n} S^2_\Lambda\) and \(\|\Delta_{\mathbb{B}_n} u_m + \lambda u_m + u_m|u_m|^{p-1}\|_{H^{-1}(\mathbb{B}_n)} \lesssim \frac{1}{m} \to 0\) as \(m \to \infty\). Then by Theorem 3.1 and Theorem 1.1 the desired bound follows. □

6. LOCALIZATION OF FAMILY OF BUBBLES

In this section, we collect some prerequisites to prove the improved spectral inequality stated in the next section.

**Lemma 6.1.** Let \(n \geq 1\). Given a point \(x_0 \in \mathbb{B}_n\) and radii \(0 < r < R\), there exists a Lipschitz bump function \(\varphi = \varphi_{r,R} : \mathbb{B}_n \to [0,1]\) such that \(\varphi = 1\) on the geodesic ball \(B(x_0, r)\), \(\varphi = 0\) in \(B(x_0, R)\) and
\[
\sup_{x \in \mathbb{B}_n} |\nabla_{\mathbb{B}_n} \varphi(x)|_{\mathbb{B}_n} \lesssim \frac{1}{R - r}.
\]

**Proof.** Without loss of generality, we can assume that \(x_0 = 0\). As before let \(\rho(x) = d(x,0)\) denote the geodesic distance of \(x\) to \(0\). Define
\[
\approx \int_{\mathbb{B}_n} u_i^p u_k d\nu_{\mathbb{B}_n} .
\] (5.10)
that the followings hold:

Let we can also assume $U$ be a large parameter (we will fix it later depending on $\varepsilon$).

Proof. Without loss of generality, we may assume

where $\phi_0$ is radial, by abuse of notation we can write $\phi(x) = \phi(\rho)$. Therefore, it only remains to show that

$$\sup_{x \in \mathbb{B}^n} |\nabla_{\mathbb{B}^n} \phi(x)|_{\mathbb{B}^n} = \sup_{\rho} |\phi'(\rho)| \lesssim \frac{1}{R - r},$$

where $'$ denotes the derivative w.r.t. $\rho$. Clearly, for $r < \rho(x) < R$

$$|\phi'(\rho)| = \frac{\pi}{2(R - r)} \cos \left[ \frac{\pi}{2} \left( \frac{\rho(x) + R - 2r}{R - r} \right) \right] \lesssim \frac{1}{R - r},$$

and $\phi'(\rho) = 0$ elsewhere. This completes the proof.

As a corollary of Lemma 6.1, choosing $R$ large enough and $r = R/2$, it follows

$$|\nabla_{\mathbb{B}^n} \phi(x)|_{\mathbb{B}^n} = o(1) \quad \text{as} \quad R \to \infty.$$

Lemma 6.2. Assume $3 \leq n \leq 5$, $p \in (2, 2^* - 1)$ and $\lambda$ satisfies (H1). For $N \in \mathbb{N}$ and $\varepsilon > 0$, there exists $\delta = \delta(n, N, \lambda, p, \varepsilon) > 0$ such that if $U_i = U_i[\varepsilon_i]$, $i = 1, \ldots, N$ are $\delta$–interacting hyperbolic bubbles, then for any $1 \leq i \leq N$, there exists a Lipschitz bump functions $\phi_i : \mathbb{B}^n \to [0, 1]$ such that the followings hold:

(i) Almost all mass of $U_i^{p+1}$ is in the region $\{\phi_i = 1\}$, i.e.,

$$\int_{\{\phi_i = 1\}} U_i^{p+1} \, dv_{\mathbb{B}^n} \geq (1 - \varepsilon)S_{\lambda, p}^{\frac{p+1}{2}}.$$

(ii) In the region $\{\phi_i > 0\}$, there holds

$$\varepsilon U_i > U_k \quad \text{for all} \quad k \neq i.$$

(iii) The $L^\infty$ norm of $|\nabla_{\mathbb{B}^n} \phi_i|_{\mathbb{B}^n}$ is small, i.e.,

$$\sup_{x \in \mathbb{B}^n} |\nabla_{\mathbb{B}^n} \phi_i(x)|_{\mathbb{B}^n} \leq \varepsilon.$$

Proof. Without loss of generality, we may assume $i = N$ and applying hyperbolic translation we can also assume $U_N = U[0]$. Recall that,

$$U_N(x) \approx e^{-c(n, \lambda)d(x, 0)}, \quad U_k(x) \approx e^{-c(n, \lambda)d(x, z_k)}, \quad k \neq N.$$

Let $R > 1$ be a large parameter (we will fix it later depending on $\varepsilon$).

$$\int_{B(0,R/2)} U_N^{p+1} \, dv_{\mathbb{B}^n} \lesssim \int_{R/2}^\infty e^{-(p+1)c(n, \lambda)\rho} e^{(n-1)\rho} \, d\rho \leq C \frac{e^{-\left((p+1)c(n, \lambda)-(n-1)\right)R/2}}{(p+1)c(n, \lambda) - (n-1)},$$

as $(p + 1)c(n, \lambda) > n - 1$ (follows from the fact that $c(n, \lambda) > \frac{n+1}{2}$ and $p > 1$). We choose $R$ such that

$$C \frac{e^{-\left((p+1)c(n, \lambda)-(n-1)\right)R/2}}{(p+1)c(n, \lambda) - (n-1)} \leq \varepsilon S_{\lambda, p}^{\frac{p+1}{2}}.$$
R \geq \frac{2}{(p+1)c(n,\lambda) - (n-1)} \ln \left( \frac{C}{(p+1)c(n,\lambda) - (n-1)} \right).

Let \varphi be the Lipchitz bump function constructed in Lemma 6.1 with r = R/2, i.e., \varphi = 1 in B(0, R/2), \varphi = 0 in B(0, R)^c and |\nabla B^n \varphi|B^n \lesssim 2/R. We further enlarge R (in addition to the previous choice), if required, to ensure that \(2/R < \varepsilon\). Then,

\[
\int_{\{\varphi = 1\}} U_N^{p+1} \, dv_{B^n} = \int_{B^n} U_N^{p+1} \, dv_{B^n} - \int_{B(0,R/2)^c} U_N^{p+1} \, dv_{B^n} \geq S \frac{p+1}{\lambda p} R^p - \varepsilon S \frac{p+1}{\lambda p} = (1-\varepsilon) S \frac{p+1}{\lambda p}.
\]

Moreover, \(\sup_{B^n} |\nabla B^n \varphi(x)|B^n \lesssim 2/R \lesssim \varepsilon\). Hence (i) and (iii) hold.

Now we will determine \(\delta\). Note that \(\{\varphi > 0\} = B(0, R)\). If \(x \in B(0, R)\), using the decay estimate of \(U_N\) and \(U_k\), \(k \neq N\), it follows

\[
U_N(x) \gtrsim e^{-c(n,\lambda)d(x,0)} \gtrsim e^{-c(n,\lambda)R} U_k(x) \lesssim e^{-c(n,\lambda)d(x,z_k)} \lesssim e^{-c(n,\lambda)d(0,z_k)},
\]

as \(d(x, z_k) \geq d(0, z_k) - d(x, 0)\). To hold \(\varepsilon U_N > U_k, \ k \neq N\) in \(B(0, R)\), we need to show that

\[
\varepsilon e^{-c(n,\lambda)} \geq C_1 e^{-c(n,\lambda)} R \varepsilon e^{-c(n,\lambda)},
\]

for some constant \(C_1 > 0\). In other words, we need \(d(0, z_k) \geq \frac{1}{c(n,\lambda)} \ln \left( \frac{C_1 e^{2c(n,\lambda)}}{\varepsilon} \right)\). Since \(z_i\)’s are \(\delta\)-interacting bubbles, from the definition we already have

\[
\max_{i \neq k} e^{-c(n,\lambda) \delta(z_i, z_k)} \leq \delta,
\]

i.e., \(d(z_i, z_k) \geq \frac{\ln \delta^{-1}}{c(n,\lambda)}\) for all \(k \neq i\). So we choose \(\delta > 0\) such that

\[
\ln \delta^{-1} > \ln \left( \frac{C_1 e^{2c(n,\lambda)}}{\varepsilon} \right),
\]

then clearly \(U_i\)’s are \(\delta\)-interacting bubbles imply,

\[
\varepsilon U_N > U_k, \ \ k \neq N,
\]

on the set \(\{\varphi > 0\}\). Note that \(\delta \lesssim \varepsilon\). Taking \(\varphi_N = \varphi\) completes the proof. \(\Box\)

7. Improved Spectral Inequality

**Proposition 7.1.** Let \(3 \leq n \leq 5\), \(p \in (2,2^*-1]\) and \(\lambda\) satisfies (H1) and \(N \in \mathbb{N}\). There exists a positive constant \(\delta = \delta(n,N,\lambda,p)\) such that if \(\sigma = \sum_{i=1}^{N} \alpha_i U[z_i] \) is a linear combination of \(\delta\)-interacting hyperbolic bubbles and \(\rho \in H^1(\mathbb{B}^n)\) satisfy the orthogonality conditions

\[
\int_{\mathbb{B}^n} \rho \, U_i^p \, dv_{B^n} = 0 \quad \text{and} \quad \int_{\mathbb{B}^n} \rho \, U_i^{p-1} V_j(U_i) \, dv_{B^n} = 0 \quad \text{for all} \ 1 \leq i \leq N, \ 1 \leq j \leq n,
\]

where \(U_i = U[z_i]\). Then

\[
\int_{\mathbb{B}^n} \sigma^{p-1} \rho^2 \, dv_{B^n} \leq \bar{c} \|\rho\|_\lambda^2,
\]

where \(\bar{c} = \bar{c}(n,N,\lambda,p)\) is a positive constant strictly less than 1.
Proof. Fix a parameter $\varepsilon > 0$ and in this proof, we will denote with $o(1)$ any quantity that goes to 0 as $\varepsilon \to 0$. By Lemma 6.2, there exists $\delta$, $\phi$, satisfying (i), (ii) and (iii) of Lemma 6.2. Note that (i) implies the sets $\{\phi_i = 1\}$. It is almost disjoint as almost all the masses of $U_i$ is concentrated on $\{\phi_i = 1\}$. Moreover, (ii) implies $\phi_i$'s have disjoint support. Therefore,

$$\\mathbb{B}^n \subset \cup_{i=1}^N \{\phi_i = 1\} \cup \left\{ \sum_{i=1}^N \phi_i < 1 \right\}.$$  

By Lemma 6.2(ii) and $\delta$-interaction of $U_i$, on the set $\{\phi_i > 0\}$,

$$\sigma_i^{p-1} = (\alpha_i U_i + \sum_{k \neq i} \alpha_k U_k)^{p-1} \leq (1 + CN\varepsilon)^{p-1} U_i^{p-1} = (1 + o(1)) U_i^{p-1}.$$  

Therefore,

$$\int_{\mathbb{B}^n} \sigma_i^{p-1} \rho^2 \, d\nu_{\mathbb{B}^n} \leq \sum_{i=1}^N \int_{\{\phi_i = 1\}} \sigma_i^{p-1} \rho^2 \, d\nu_{\mathbb{B}^n} + \int_{\{\sum_{i=1}^N \phi_i < 1\}} \sigma_i^{p-1} \rho^2 \, d\nu_{\mathbb{B}^n}$$

$$\leq (1 + o(1)) \sum_{i=1}^N \int_{\{\phi_i = 1\}} U_i^{p-1} \rho^2 \, d\nu_{\mathbb{B}^n} + \left( \int_{\{\sum_{i=1}^N \phi_i < 1\}} \sigma_i^{p+1} \, d\nu_{\mathbb{B}^n} \right)^{\frac{p-1}{p+1}} \|\rho\|_{L^{p+1}}^2$$

$$\leq (1 + o(1)) \sum_{i=1}^N \int_{\mathbb{B}^n} (\rho \phi_i)^2 U_i^{p-1} \, d\nu_{\mathbb{B}^n} + \left( \sum_{i=1}^N \int_{\{\phi_i < 1\}} U_i^{p+1} \, d\nu_{\mathbb{B}^n} \right)^{\frac{p-1}{p+1}} \|\rho\|_\lambda^2$$

$$\leq (1 + o(1)) \sum_{i=1}^N \int_{\mathbb{B}^n} (\rho \phi_i)^2 U_i^{p-1} \, d\nu_{\mathbb{B}^n} + \varepsilon \frac{p-1}{p+1} \|\rho\|_\lambda^2 \quad (7.2)$$

Claim: $\int_{\mathbb{B}^n} (\rho \phi_i)^2 U_i^{p-1} \, d\nu_{\mathbb{B}^n} \leq \frac{1}{\Lambda} \|\rho \phi_i\|_\Lambda^2 + o(1) \|\rho\|_\lambda^2$ for $i = 1, \ldots, N$,

where $\Lambda$ is the smallest eigenvalue of $-\Delta_{\mathbb{B}^n}/U_i$, which is bigger than $p$.

To prove the claim, let $\psi$ be either $U_i$ or $V_j(U_i)$, $1 \leq j \leq n$ normalized so that

$$\int_{\mathbb{B}^n} \psi^2 U_i^{p-1} \, d\nu_{\mathbb{B}^n} = 1.$$  

Hence, thanks to the orthogonality condition (7.1), it holds that

$$\left| \int_{\mathbb{B}^n} \rho \phi_i \psi U_i^{p-1} \, d\nu_{\mathbb{B}^n} \right| = \left| \int_{\mathbb{B}^n} \rho \psi U_i^{p-1} (1 - \phi_i) \, d\nu_{\mathbb{B}^n} \right| \leq \left| \int_{\{\phi_i < 1\}} \rho \psi U_i^{p-1} \, d\nu_{\mathbb{B}^n} \right|$$

$$\leq \left( \int_{\mathbb{B}^n} |\rho|^{p+1} \, d\nu_{\mathbb{B}^n} \right)^{\frac{p-1}{p+1}} \left( \int_{\mathbb{B}^n} \rho^2 U_i^{p-1} \, d\nu_{\mathbb{B}^n} \right)^{\frac{1}{2}} \left( \int_{\{\phi_i < 1\}} U_i^{p+1} \, d\nu_{\mathbb{B}^n} \right)^{\frac{p-1}{2(p+1)}}$$

$$\leq o(1) \|\rho\|_\lambda,$$

where in the last inequality we have applied Lemma 6.2(i).

This proves that $\rho \phi_i$ is almost orthogonal to $\psi$. Hence, using [5, Proposition 3.1], it follows that if $\Lambda$ is the smallest eigenvalue of $-\Delta_{\mathbb{B}^n} - \lambda)/U_i^{p-1}$ bigger than $p$ then

$$\int_{\mathbb{B}^n} (\rho \phi_i)^2 U_i^{p-1} \, d\nu_{\mathbb{B}^n} \leq \frac{1}{\Lambda} \|\rho \phi_i\|_\Lambda^2 + o(1) \|\rho\|_\lambda^2.$$  

$$\quad (7.3)$$

This proves the claim.

We now estimate RHS of (7.3).

$$\int_{\mathbb{B}^n} \rho_\mathbb{B}^n (\rho \phi_i)^2 \, d\nu_{\mathbb{B}^n} = \int_{\mathbb{B}^n} \rho_\mathbb{B}^n (\rho \phi_i)^2 \, d\nu_{\mathbb{B}^n} + \int_{\mathbb{B}^n} \rho_\mathbb{B}^n \phi_i^2 \, d\nu_{\mathbb{B}^n} + 2 \int_{\mathbb{B}^n} \rho_\mathbb{B}^n \phi_i \, d\nu_{\mathbb{B}^n}.$$  

$$\quad (7.4)$$
Using Lemma 6.2(iii) and Poincaré inequality,
\[
\int_{\mathbb{R}^n} \rho^2 |\nabla_{\mathbb{B}^n} \varphi_i|^2 \, dv_{\mathbb{B}^n} \leq \varepsilon^2 \int_{\mathbb{R}^n} \rho^2 \, dv_{\mathbb{B}^n} = o(1)\|\rho\|_{\lambda}^2.
\]
Next,
\[
\int_{\mathbb{R}^n} \rho \varphi_i \nabla_{\mathbb{B}^n} \rho \cdot \nabla_{\mathbb{B}^n} \varphi_i \, dv_{\mathbb{B}^n} \leq \left( \int_{\mathbb{R}^n} \rho^2 |\nabla_{\mathbb{B}^n} \varphi_i|^2 \, dv_{\mathbb{B}^n} \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^n} |\nabla_{\mathbb{B}^n} \rho|^2 \, dv_{\mathbb{B}^n} \right)^{\frac{1}{2}} \\
\leq \varepsilon \|\rho\|_{\lambda} \|\nabla_{\mathbb{B}^n} \rho\|_{L^2} \leq o(1)\|\rho\|_{\lambda}^2.
\]
Inserting the above two estimates into (7.4), we find
\[
\|\rho \varphi_i\|_{\lambda}^2 = \int_{\mathbb{R}^n} \left( |\nabla_{\mathbb{B}^n}(\rho \varphi_i)|^2 - \lambda (\rho \varphi_i)^2 \right) \, dv_{\mathbb{B}^n} \leq \int_{\mathbb{R}^n} \left( |\nabla_{\mathbb{B}^n} \rho|^2 - \lambda \rho^2 \right) \varphi_i^2 \, dv_{\mathbb{B}^n} + o(1)\|\rho\|_{\lambda}^2.
\]
Taking a sum of \( i = 1 \) to \( N \) and using the fact that \( \varphi_i \)'s have disjoint support, it follows
\[
\sum_{i=1}^{N} \|\rho \varphi_i\|_{\lambda}^2 \, dv_{\mathbb{B}^n} \leq (1 + o(1))\|\rho\|_{\lambda}^2. \tag{7.5}
\]
Thus combining (7.2), (7.3) and (7.5), we deduce
\[
\int_{\mathbb{R}^n} \sigma^{p-1} \rho^2 \, dv_{\mathbb{B}^n} \leq \frac{1 + o(1)}{\Lambda} \|\rho\|_{\lambda}^2 \leq \frac{\tilde{c}}{p} \|\rho\|_{\lambda}^2,
\]
with \( \tilde{c} < 1 \) (since \( \Lambda > p \)). \( \square \)

8. Interaction integral estimates

In this section we prove the following interaction integral estimate:

**Lemma 8.1.** Let \( 3 \leq n \leq 5 \), \( N \in \mathbb{N} \), \( p \in (2, 2^* - 1] \) and \( \lambda \) be as in (H1). For every \( \varepsilon > 0 \), there exists a \( \delta > 0 \) depending on \( n, p, N, \lambda, \varepsilon \) such that if \( \{\alpha_i, \mathcal{U}[z_i]\}_{1 \leq i \leq N} \) is a family of \( \delta \)-interacting hyperbolic bubbles satisfying the orthogonality conditions (5.3) and (5.4) then setting \( u = \sum_{i=1}^{N} \alpha_i \mathcal{U}[z_i] + \rho \) the following estimates hold:
\[
\max_{i \neq k} \int_{\mathbb{R}^n} U_i^\dagger U_k \, dv_{\mathbb{B}^n} \lesssim \| (\Delta_{\mathbb{B}^n} + \lambda) u \|_{H^{1-1}} + \| u \|^2_{H^{-1}} + \varepsilon \|\rho\|_{\lambda} + \|\rho\|_{\lambda}^2,
\]
and
\[
\max_i |\alpha_i - 1| \lesssim \| (\Delta_{\mathbb{B}^n} + \lambda) u \|_{H^{1-1}} + \| u \|^2_{H^{-1}} + \varepsilon \|\rho\|_{\lambda} + \|\rho\|_{\lambda}^2.
\]

8.1. A geometric lemma. To prove the above lemma we take help of the following geometrically intuitive lemma.

**Lemma 8.2.** Let \( \mathcal{U}_i = \mathcal{U}[z_i], 1 \leq i \leq N \) be a family of \( \delta \)-interacting hyperbolic bubbles. Then there exists \( \delta_0 > 0 \) such that for every \( \delta < \delta_0 \), there exists an indices \( k \in \{1, \ldots, N\} \) and a direction \( e_j \) such that after applications of hyperbolic translations and rotations the followings hold:
\[
\begin{align*}
(i) & \quad z_k = 0, \\
(ii) & \quad \text{all } z_i, i \neq k \text{ lie in the negative half } \mathbb{B}^n := \{ x \in \mathbb{R}^n \mid x_j < 0 \}, \\
(iii) & \quad \text{There exists a constant } \kappa = \kappa(\delta) > 0 \text{ such that } (z_i)_j \text{ (the } j \text{-th component if } z_i \text{) satisfies } (z_i)_j \leq -\kappa \text{ for all } i \neq k.
\end{align*}
\]
Proof. Since there are only $N$-many indices we can find two indices $k_1, k_2 \in \{1, \ldots, N\}$ such that
\[ d(z_{k_1}, z_{k_2}) = \max_{i \neq k} d(z_i, z_k). \]
There may be other choices of such indices, but we only need a pair of them. By rearranging the indices and by applying a hyperbolic translation we may assume $z_1 = 0$ and $d(z_1, z_N) = \max_{i \neq k} d(z_i, z_k)$. Further applying an orthogonal transformation we may assume $z_N = |z_N|e_j$. Recall that the hyperbolic distances are preserved under the orthogonal group of transformations.

We will show that if $u[z_i]$ are sufficiently small $\delta$-interacting then $\tau_{-z_N}(z_i), 1 \leq i \leq N$ satisfies all the properties of the stated lemma.

For the simplicity of notations we denote $z_N = z$. Recall that $\tau_{-z} = \sigma_{-z^*}\rho_{-z^*}$ is the composition of reflection $\rho_{-z^*}$ with respect to the plane $\{x \in \mathbb{R}^n \mid x \cdot z^* = 0\}$ and the inversion with respect to the sphere $S(-z^*, r)$ where $r$ is determined by $r^2 = |z^*|^2 - 1 = \frac{|z|^2}{|z|^2} - 1$. We divide $\mathbb{B}^n$ into the regions $I_1, I_2$ and $I_3$ defined below (see Figure 3):

\[ I_1 = \mathbb{B}^n \cap \overline{B_E(z^*, r)} \quad I_3 = \mathbb{B}^n \cap \overline{B_E(-z^*, r)} \quad \text{and} \quad I_2 = \mathbb{B}^n \setminus (I_1 \cup I_3). \]

The following observations are easy to verify:

(a) $\sigma_{-z^*}$ leaves $S(-z^*, r)$ invariant.
(b) $z \in B_E(z^*, r)$ as direct computation shows $|z - z^*|^2 = r^2(1 - |z|^2) < r^2$.
(c) $\rho_{-z^*}(I_1) = I_3, \rho_{-z^*}(I_3) = I_1$ and $\rho_{-z^*}$ leaves $I_2$ invariant.
(d) $\sigma_{-z^*}(I_1 \cup I_2) \subset I_3$ and $\sigma_{-z^*}(I_3 \setminus S(-z^*, r)) \subset I_1 \cup I_2$.
(e) $\sigma_{-z^*}$ sends spheres $S(-z^*, r_1)$ to spheres $S(-z^*, \frac{r_1}{n^2})$.

According to our choice all $z_i$'s lie with in $\overline{B_E(0, |z|)}$. The above observation indicates all points $z_i$ such that $z_i \notin \overline{B_E(0, |z|)} \cap B_E(z^*, r)$ would translate under the hyperbolic translation $\tau_{-z}$ to the region $I_3$. Clearly $\tau_{-z}(z) = 0$. Now let $z_i \in \overline{B_E(0, |z|)} \cap B_E(z^*, r)$. Then $\rho_{-z^*}(z_i) \in B_E(-z^*, r) \cap B_E(0, |z|)$. Let us now look at the action of $\sigma_{-z^*}$ on the set $B_E(-z^*, r) \cap B_E(0, |z|)$.

According to (e), $\sigma_{-z^*}$ maps $B_E(-z^*, r) \cap B_E(0, |z|)$ to the region $B_E(-z^*, \frac{|z|}{n^2})$. To see that we let $r_0 = |z - z^*| = |z|r^2$. Then by (e), $\sigma_{-z^*}$ maps $S(-z^*, r_0)$ to $S(-z^*, \frac{r_0}{n^2})$ and leaves $S(-z^*, r)$
invariant. Since all the \( \rho_{-z}(z_i) \)s are trapped with in the spheres \( S(-z^*, r) \) and \( S(-z^*, r_0) \) we conclude \( \tau_{-z}(z_i), i \neq N \) would lie with in \( B(-z^*, \frac{1}{|z|}) \setminus \{0\} \) (see Figure 4). This proves (i) and (ii).

![Figure 4](image.png)

**Figure 4.** Under the action of hyperbolic translation \( \tau_{-z} \), \( z \) will map to 0 and all other \( z_i \)s which lie in the region \( R_0 \) will map to the largest ball centered at \( -z^* \) and radius \( \frac{1}{|z|} \). After the translation if \( x \) denotes the point \( \tau_{-z}(z_i) \) then \( x \) is at least \( \kappa \) distance apart from the plane \( \{x_j = 0\} \).

Now we claim that if \( U[z_i], 1 \leq i \leq N \) are sufficiently small \( \delta \)-interacting then (iii) holds (cf. Figure 4). Fix \( i \neq N \), for simplicity we denote \( x = \tau_{-z}(z_i) \) and we write \( x = (x', x_j) \) to indicate the \( j \)-th component. The above discussions yields \( x \in B(-z^*, \frac{1}{|z|}) \setminus \{0\} \) i.e.,

\[
| x_j + \frac{z_j}{|z|^2} |^2 + | x' |^2 \leq \frac{1}{|z|^2}. \tag{8.1}
\]

If \( \delta \) is small then \( d(z_i, z_N) = d(x, 0) = \ln \left( \frac{1 + |x|}{1 - |x|} \right) \) is sufficiently large. That is there exists small \( \eta = \eta(\delta) > 0 \) such that \( |x| > 1 - \eta \). Hence from (8.1) denoting \( t = \frac{1}{|z|^2} \) we deduce

\[
(x_j + t z_j)^2 + |x'|^2 = |x|^2 + 2tx_j z_j + t^2 z_j^2 \leq t
\]

i.e. \( x_j \leq \frac{t - t^2 z_j^2 - |x|^2}{2t z_j} \leq - \frac{1 - \eta)^3}{2} =: -\kappa, \)

where we have used \( z_j = |z| > 1 - \eta \) as \( d(0, z) = d(z_1, z_N) \gg 1 \). This completes the proof. \( \square \)

**Remark 8.1.** The point (ii) is redundant given point (iii) of the above lemma. We wanted to emphasise that (i) and (ii) are the properties of the hyperbolic translation while (iii) is the property of the \( \delta \)-interaction.

We are now in a position to prove the interaction integral estimates Lemma 8.1.

8.2. **The proof of interaction integral estimates.** Given \( u \) and \( (\alpha_i, U[z_i])_{i=1}^N \) a family of \( \delta \)-interacting hyperbolic bubbles as in Lemma 8.1. We put

\[
u = \sum_{i=1}^N \alpha_i U_i + \rho := \sigma + \rho, \tag{8.2}
\]
where $\mathcal{U}_i = \mathcal{U}[z_i]$ and $\sigma = \sum_{i=1}^{N} \alpha_i \mathcal{U}_i$. Recall that the following orthogonality conditions hold

$$\int_{\mathbb{R}^n} (\Delta_{\mathbb{R}^n} + \lambda) \mathcal{U}_i \rho \, dv_{\mathbb{R}^n} = \int_{\mathbb{R}^n} \mathcal{U}_i^p \rho \, dv_{\mathbb{R}^n} = 0 \quad (8.3)$$

$$\int_{\mathbb{R}^n} (\Delta_{\mathbb{R}^n} + \lambda) V_j(\mathcal{U}_i) \rho \, dv_{\mathbb{R}^n} = p \int_{\mathbb{R}^n} \mathcal{U}_i^{p-1} V_j(\mathcal{U}_i) \rho \, dv_{\mathbb{R}^n} = 0, \quad \text{for } 1 \leq j \leq n, \quad (8.4)$$

for all $i = 1, \ldots, N$. We compute

$$\Delta_{\mathbb{R}^n} u + \lambda u + |u|^p u = \Delta_{\mathbb{R}^n} \rho + \lambda \rho - \sum_{i=1}^{N} \alpha_i \mathcal{U}_i^p + |\sigma + \rho|^{p-1}(\sigma + \rho). \quad (8.5)$$

Putting $f = -\Delta_{\mathbb{R}^n} u - \lambda u - |u|^p u$ we can rewrite the equation (8.5) as

$$\Delta_{\mathbb{R}^n} \rho + \lambda \rho + p\sigma^{p-1} \rho + I_1 + I_2 + I_3 + f = 0, \quad (8.6)$$

where

$$I_1 = \sigma^p - \sum_{i=1}^{N} \alpha_i \mathcal{U}_i^p, \quad I_2 = |\sigma + \rho|^{p-1}(\sigma + \rho) - \sigma^p - p\sigma^{p-1} \rho, \quad I_3 = \sum_{i=1}^{N} (\alpha_i^p - \alpha_i) \mathcal{U}_i^p \quad (8.7)$$

We follow the approach of [19] and divide the proof of Lemma 8.1 into several small lemmas. We point out that the next four lemmas hold true in any dimension $n \geq 3$.

**Lemma 8.3.** We have

$$\int_{\mathbb{R}^n} I_j V_j(\mathcal{U}_k) \, dv_{\mathbb{R}^n} = \sum_{i \neq k} \alpha_i \mathcal{U}_i \int_{\mathbb{R}^n} \mathcal{U}_i^p V_j(\mathcal{U}_k) \, dv_{\mathbb{R}^n} + o(Q). \quad (8.8)$$

for all $1 \leq k \leq N$ and $j = 1, \ldots, n$.

**Proof.** We denote $\sigma_k = \sum_{i \neq k} \alpha_i \mathcal{U}_i$ so that $\sigma = \sigma_k + \alpha_k \mathcal{U}_k$ holds. We decompose the integral $\int_{\mathbb{R}^n} I_j V_j(\mathcal{U}_k) \, dv_{\mathbb{R}^n}$ into the following four integrals:

$$J_1 := \int_{\{\mathcal{U}_k \geq \sigma_k\}} \left( p\alpha_k^{p-1} \mathcal{U}_k^{p-1} \sigma_k - \sum_{i \neq k} \alpha_i \mathcal{U}_i^p \right) V_j(\mathcal{U}_k) \, dv_{\mathbb{R}^n}$$

$$J_2 = \int_{\{\mathcal{U}_k \geq \sigma_k\}} \left( \sigma^p - \alpha_k \mathcal{U}_k^p - p\alpha_k^{p-1} \mathcal{U}_k^{p-1} \sigma_k \right) V_j(\mathcal{U}_k) \, dv_{\mathbb{R}^n}$$

$$J_3 = \int_{\{\mathcal{U}_k \geq \sigma_k\}} \left( p\alpha_k \mathcal{U}_k \sigma_k^{p-1} + \sigma^p - \sum_{i=1}^{N} \alpha_i \mathcal{U}_i^p \right) V_j(\mathcal{U}_k) \, dv_{\mathbb{R}^n}$$

$$J_4 = \int_{\{\mathcal{U}_k \geq \sigma_k\}} \left( \sigma^p - p\alpha_k \mathcal{U}_k \sigma_k^{p-1} \right) V_j(\mathcal{U}_k) \, dv_{\mathbb{R}^n}.$$

We now estimate each integral one by one. Using $|V_j(\mathcal{U}_k)| \lesssim \mathcal{U}_k$ and using the inequality $|(a + b)^p - a^p - p a^{p-1} b| \lesssim a^{p-2} b^2$ whenever $0 \leq b \lesssim a$ we estimate

$$|J_2| \lesssim \int_{\{\mathcal{U}_k \geq \sigma_k\}} \mathcal{U}_k^{p-2} \sigma_k^2 \mathcal{U}_k \, dv_{\mathbb{R}^n}$$

$$\lesssim \int_{\{\mathcal{U}_k \geq \sigma_k\}} \mathcal{U}_k^{p-\epsilon} \sigma_k^{1+\epsilon} \, dv_{\mathbb{R}^n}$$

$$\lesssim Q_k^{1+\epsilon}. \quad (8.9)$$
Similarly, \[ |J_4| \lesssim \int_{\{\sigma_k > N\alpha_k u_k\}} \sigma_k^{p-2} u_k^2 u_{k} \, dv_B \]
\[ \lesssim \int_{\{\sigma_k > N\alpha_k u_k\}} \sigma_k^{p-\epsilon} u_k^{1+\epsilon} \, dv_B \]
\[ \lesssim Q_k^{1+\epsilon}. \tag{8.10} \]

For \( J_3 \) we approach as follows.
\[ |J_3| \lesssim \int_{\{\sigma_k > N\alpha_k u_k\}} \sigma_k^{p-1} u_k^2 \, dv_B + \int_{\{\sigma_k > N\alpha_k u_k\}} \left| \sigma_k^p - \sum_{i \neq k} \sigma_i^p u_i \right| u_k \, dv_B \]
\[ + \int_{\{\sigma_k > N\alpha_k u_k\}} u_k^{p+1} \, dv_B. \tag{8.11} \]

The first term in (8.11) can be estimated as in (J2), (J4) to obtain \( O(Q_k^{1+\epsilon}) \). For the second term we use the inequality \( |(\sum_i a_i)^p - \sum_i a_i^p| \lesssim \sum_i |a_i|^{p-1} \min\{|a_i|, |a_j|\} \) to get
\[ \int_{\{\sigma_k > N\alpha_k u_k\}} \left| \sigma_k^p - \sum_{i \neq k} \sigma_i^p u_i \right| u_k \, dv_B \lesssim \sum_{i<j, i \neq j \neq k} \int_{\{\sigma_k > N\alpha_k u_k\}} u_k^{p-1} u_j u_k \]
\[ \lesssim Q_k^{1+\epsilon}, \]
where in the last line we used Lemma 4.2. For the last term in (8.11) we see that \( \{\sigma_k > N\alpha_k u_k\} \subset \cup_{i \neq k} \{\alpha_i u_i > \alpha_k u_k\} \). Hence
\[ \int_{\{\sigma_k > N\alpha_k u_k\}} u_k^{p+1} \, dv_B \lesssim \sum_{i \neq k} \int_{\{\alpha_i u_i > \alpha_k u_k\}} u_k^{p+1} \, dv_B \]
\[ \lesssim \int_{\{\alpha_i u_i > \alpha_k u_k\}} u_k^{p-\epsilon} u_i^{1+\epsilon} \, dv_B \lesssim Q_k^{1+\epsilon}. \]

Now for the estimation of \( J_1 \) we see that
\[ J_1 - \int_{B^n} po_k^{p-1} u_k^{p-1} \sigma_k V_j(u_k) \, dv_B = \]
\[ - \int_{\{N\alpha_k u_k < \sigma_k\}} po_k^{p-1} u_k^{p-1} \sigma_k V_j(u_k) \, dv_B - \sum_{i \neq k} \int_{\{N\alpha_k u_k \geq \sigma_k\}} \sigma_i^p u_i^p V_j(u_k) \, dv_B = J_1^1 + J_1^2. \tag{8.12} \]

\( J_1^1, J_1^2 \) can be estimated in a similar way
\[ |J_1^1| \lesssim \int_{\{N\alpha_k u_k < \sigma_k\}} u_k^p \sigma_k \, dv_B \lesssim \int_{\{N\alpha_k u_k < \sigma_k\}} u_k^{p-\epsilon} \sigma_k^{1+\epsilon} \, dv_B \lesssim Q_k^{1+\epsilon}, \]
and since on \( \{\sigma_k \leq N\alpha_k u_k\}, \ U_i \lesssim U_k \) holds for all \( i \neq k \)
\[ |J_1^2| \lesssim \sum_{i \neq k} \int_{\{\sigma_k \leq N\alpha_k u_k\}} u_k^p u_i \, dv_B \lesssim \sum_{i \neq k} \int_{\{\sigma_k \leq N\alpha_k u_k\}} u_i^{p-\epsilon} u_k^{1+\epsilon} \, dv_B \lesssim Q_k^{1+\epsilon}. \]

Combining all the estimates and integrating by parts gives the result as indicated below
\[ \int_{B^n} I_1 V_j(u_k) \, dv_B = \sum_{l=1}^4 J_l = \int_{B^n} po_k^{p-1} u_k^{p-1} \sigma_k V_j(u_k) \, dv_B + o(Q) \]
Lemma 8.4. Denoting $\xi = \text{either } U_k \text{ or } V_j(U_k)$ for $1 \leq k \leq N$, the following estimate holds.

\[ \left| \int_{\mathbb{R}^n} I_2 \xi \, dv_{\mathbb{R}^n} \right| \lesssim \| \rho \|^\text{min}(p, 2). \tag{8.13} \]

Proof. We again use the elementary inequality $|(a + b)|a + b|^{p-1} - a^p - p\sigma^{p-1}b| \leq |b|^p + |a|^{p-2}|b|^2$ and $|\xi| \lesssim U_k$

\[ \left| \int_{\mathbb{R}^n} I_2 \xi \, dv_{\mathbb{R}^n} \right| \lesssim \int_{\mathbb{R}^n} |\rho|^p U_k \, dv_{\mathbb{R}^n} + \int_{\mathbb{R}^n} \sigma^{p-2} |\rho|^2 U_k \, dv_{\mathbb{R}^n} \]

\[ \lesssim \| \rho \|_{L^{p+1}} U_k \|_{L^{p+1}} + \| \sigma \|_{L^{p+1}} \| \rho \|_{L^{p+1}} U_k \|_{L^{p+1}} \]

\[ \lesssim \| \nabla \rho \|_{L^2}^2 + \| \nabla \rho \|_{L^2}^2 \]

\[ \lesssim \| \rho \|^\text{min}(p, 2) \quad \text{(as } \| \rho \|_\lambda \text{ is small).} \]

This completes the proof. \[ \square \]

Lemma 8.5. We have

\[ \int_{\mathbb{R}^n} I_3 V_j(U_k) \, dv_{\mathbb{R}^n} = \sum_{i \neq k} (\alpha_i^p - \alpha_i) \int_{\mathbb{R}^n} U_i^p V_j(U_k) \, dv_{\mathbb{R}^n} \tag{8.14} \]

for all $1 \leq k \leq N$ and $j = 1, \ldots, n$.

Proof. We note that

\[ (p + 1) \int_{\mathbb{R}^n} U_k^p V_j(U_k) \, dv_{\mathbb{R}^n} = (p + 1) \int_{\mathbb{R}^n} U_k^p \frac{d}{dt} \bigg|_{t=0} (U_k \circ \tau_{t_1}) \bigg) \, dv_{\mathbb{R}^n} \]

\[ = \frac{d}{dt} \bigg|_{t=0} \int_{\mathbb{R}^n} (U_k \circ \tau_{t_1})^{p+1} \, dv_{\mathbb{R}^n} = 0. \]

Hence only the terms $i \neq k$ will survive

\[ \int_{\mathbb{R}^n} I_3 V_j(U_k) \, dv_{\mathbb{R}^n} = \sum_{i=1}^N (\alpha_i^p - \alpha_i) \int_{\mathbb{R}^n} U_i^p V_j(U_k) \, dv_{\mathbb{R}^n} = \sum_{i \neq k} (\alpha_i^p - \alpha_i) \int_{\mathbb{R}^n} U_i^p V_j(U_k) \, dv_{\mathbb{R}^n}. \]

\[ \square \]
Lemma 8.6. Given $\epsilon > 0$ there exists a $\delta > 0$ depending on $n, N, p, \lambda, \epsilon$ such that if $(\alpha_i, U_i)_{i=1}^N$ is a family of $\delta$-interacting hyperbolic bubbles satisfying the orthogonality conditions (8.3), (8.4) then

$$|\int_{\mathbb{B}^n} p\sigma^{p-1}\rho \xi \, dv_\mathbb{B}| \lesssim (Q_k^{\text{min}(p-1,1)} + \epsilon)\|\rho\|_\lambda, \quad (8.15)$$

where $\xi = \text{either } U_k$ of $V_j(U_k)$, for every $1 \leq k \leq N, j = 1, \ldots, n$, and $Q_k$ is defined by $Q_k = \max_{i \neq k} Q_{ik}$.

Proof. Given $\epsilon > 0$, we use the following elementary inequality $|(a+b)^{p-1} - a^{p-1}| \leq C(\epsilon)b^{p-1} + \epsilon a^{p-1}$ for every $a, b > 0, p > 1$. Using the orthogonality conditions and applying the above elementary inequality with $a = \alpha_k U_k$ and $b = \sum_{i \neq k} \alpha_i U_i$ so that $a + b = \sigma$ we get

$$|\int_{\mathbb{B}^n} p\sigma^{p-1}\rho \xi \, dv_\mathbb{B}| = |\int_{\mathbb{B}^n} p(\sigma^{p-1} - \alpha_k^{p-1} U_k^{p-1})\rho \xi \, dv_\mathbb{B}|$$

$$\lesssim \sum_{i \neq k} \int_{\mathbb{B}^n} U_i^{p-1} U_k |\rho| \, dv_\mathbb{B} + \epsilon \int_{\mathbb{B}^n} U_k^n |\rho| \, dv_\mathbb{B}$$

$$\lesssim \sum_{i \neq k} \|U_i^{p-1} U_k\|_{L^{p+1}}\|\rho\|_{L^{p+1}} + \epsilon \|U_k^n\|_{L^{p+1}}\|\rho\|_{L^{p+1}}$$

$$\lesssim (Q_k^{\text{min}(p-1,1)} + \epsilon)\|\rho\|_\lambda.$$ 

This completes the proof. \qed

Proof of Lemma 8.1.

Proof. We proceed inductively. First note that in dimension $3 \leq n \leq 5$, $2^* - 2 > 1$, and recall our assumption is $2 < p \leq 2^* - 1$, so that in either case $p - 1 > 1$. Let $k$ be the indices and $e_j$ be the direction as obtained in Lemma 8.2. We test (8.6) against $V_j(U_k)$ and integrate and use the orthogonality conditions to get

$$\int_{\mathbb{B}^n} I_1 V_j(U_k) \, dv_\mathbb{B} + \int_{\mathbb{B}^n} I_3 V_j(U_k) \, dv_\mathbb{B}$$

$$= - \int_{\mathbb{B}^n} p\sigma^{p-1}\rho V_j(U_k) \, dv_\mathbb{B} - \int_{\mathbb{B}^n} I_2 V_j(U_k) \, dv_\mathbb{B} - \int_{\mathbb{B}^n} f V_j(U_k) \, dv_\mathbb{B}.$$ 

By Lemma 8.3-8.6 and using $\text{min}\{p-1,1\} = 1$ we get

$$\sum_{i \neq k} (\alpha_k^{p-1} \alpha_i + \alpha_i^{p-1} - \alpha_i) \int_{\mathbb{B}^n} U_i^n V_j(U_k) \, dv_\mathbb{B}$$

$$\lesssim \|f\|_{H^{-1}} + (Q_k + \epsilon)\|\rho\|_\lambda + \|\rho\|^2_\lambda + o(Q). \quad (8.16)$$

By Lemma 4.4 we see that $\int_{\mathbb{B}^n} U_i^n V_j(U_k) \, dv_\mathbb{B} \approx Q_{ik}$ for all $i \neq k$ and thanks to Lemma 8.2 the constant in $\approx$ depends only on $n, N, \lambda, p$ and $\epsilon$. Moreover, since the family $(\alpha_i, U_i)$ are $\delta$-interacting with $\delta$ sufficiently small we conclude $(\alpha_k^{p-1} \alpha_i + \alpha_i^{p-1} - \alpha_i) \approx 1$ for all $1 \leq i \leq N$. Hence (8.16) gives

$$Q_k = \max_{i \neq k} Q_{ik} \lesssim \|f\|_{H^{-1}} + (Q_k + \epsilon)\|\rho\|_\lambda + \|\rho\|^2_\lambda + o(Q)$$

$$\lesssim \|f\|_{H^{-1}} + \epsilon\|\rho\|_\lambda + \|\rho\|^2_\lambda + o(Q), \quad (8.17)$$
where we have used $\delta$ is sufficiently small to absorb the term $Q_k \| \rho \|_\lambda$. Now testing (8.6) against $U_k$ and using the orthogonality conditions we get

$$
\sum_i (\alpha_i^p - \alpha_i) \int_{\mathbb{R}^n} U_i^p U_k \, dv_{\mathbb{R}^n} = - \int_{\mathbb{R}^n} (I_1 + I_2 + p\sigma^{p-1}\rho + f) U_k \, dv_{\mathbb{R}^n}.
$$

(8.18)

It follows from Lemma 8.4 and Lemma 8.6 that if $\delta$ is small then

$$
\left| \int_{\mathbb{R}^n} (I_2 + p\sigma^{p-1}\rho + f) U_k \, dv_{\mathbb{R}^n} \right| \lesssim \| f \|_{H^{-1}} + \epsilon \| \rho \|_\lambda + \| \rho \|^2_\lambda.
$$

(8.19)

On the other hand, the inequality $|\sum_i a_i^p - \sum_i a_i^p| \lesssim \sum_i (a_i^p - a_i)$ and using the estimate (8.17) we get

$$
\left| \int_{\mathbb{R}^n} I_1 U_k \, dv_{\mathbb{R}^n} \right| \lesssim \sum_{i \neq j} \int_{\mathbb{R}^n} \alpha_i U_i U_j U_k \, dv_{\mathbb{R}^n} \lesssim Q_k + o(Q)
$$

$$
\lesssim \| f \|_{H^{-1}} + \epsilon \| \rho \|_\lambda + \| \rho \|^2_\lambda + o(Q).
$$

(8.20)

Combining (8.18), (8.19), (8.20) and using (8.17) we get

$$
|\alpha_k^p - \alpha_k| \int_{\mathbb{R}^n} U_k^{p+1} \, dv_{\mathbb{R}^n} \lesssim \sum_{i \neq k} |\alpha_i^p - \alpha_i| \int_{\mathbb{R}^n} U_i^p U_k + \int_{\mathbb{R}^n} (I_1 + I_2 + p\sigma^{p-1}\rho + f) U_k \, dv_{\mathbb{R}^n}
$$

$$
\lesssim \| f \|_{H^{-1}} + \epsilon \| \rho \|_\lambda + \| \rho \|^2_\lambda + o(Q).
$$

(8.21)

This gives the desired estimate on $\alpha_k$ i.e.

$$
|\alpha_k - 1| \lesssim \| f \|_{H^{-1}} + \epsilon \| \rho \|_\lambda + \| \rho \|^2_\lambda + o(Q).
$$

(8.22)

We proceed inductively as follows. Discard the indices $k$ from $\{1, \ldots, N\}$ and apply Lemma 8.2 to find indices $l$ ($\neq k$) and a direction which we again call $\epsilon_j$. We proceed as before and get

$$
\sum_{i \neq l} (\alpha_i^p - \alpha_i) \int_{\mathbb{R}^n} U_i V_j(U_l) \, dv_{\mathbb{R}^n} \lesssim \| f \|_{H^{-1}} + \epsilon \| \rho \|_\lambda + \| \rho \|^2_\lambda + o(Q).
$$

(8.23)

The terms corresponding to $i = k$ in (8.23) can be estimated by $Q_k$ and hence

$$
\max_{i \neq l, i \neq k} Q_i \lesssim \sum_{i \neq l, i \neq k} (\alpha_i^p - \alpha_i) \int_{\mathbb{R}^n} U_i V_j(U_k) \, dv_{\mathbb{R}^n}
$$

$$
\lesssim \left| (\alpha_i^p - \alpha_k) \int_{\mathbb{R}^n} U_i V_j(U_l) \, dv_{\mathbb{R}^n} \right| + \| f \|_{H^{-1}} + \epsilon \| \rho \|_\lambda + \| \rho \|^2_\lambda + o(Q)
$$

$$
\lesssim Q_k + \| f \|_{H^{-1}} + \epsilon \| \rho \|_\lambda + \| \rho \|^2_\lambda + o(Q)
$$

$$
\lesssim \| f \|_{H^{-1}} + \epsilon \| \rho \|_\lambda + \| \rho \|^2_\lambda + o(Q).
$$

Similarly we obtain the estimate $|\alpha_l - 1| \lesssim \| f \|_{H^{-1}} + \epsilon \| \rho \|_\lambda + \| \rho \|^2_\lambda + o(Q)$. Proceeding in this way and by deleting the indices one by one we get

$$
Q = \max_i Q_i \lesssim \| f \|_{H^{-1}} + \epsilon \| \rho \|_\lambda + \| \rho \|^2_\lambda + o(Q).
$$

Since $\delta$ is sufficiently small we get the desired estimate on $Q$ which in turn gives the desired estimate on $\max_i |\alpha_i - 1|$ as derived in (8.20)-(8.22), completing the proof of the lemma. □
Appendix A. Non-degeneracy and spectral gap of the linearized operator

In this subsection, we collect a few key lemmas needed for the proof related to the eigenvalues and eigenfunctions of the linearized operator

\[ L := (-\Delta_B^n - \lambda)/U[z]^{p-1} \]

We know that if \( \tau_b \) is a hyperbolic translation then \( U[z] \circ \tau_b \) also solves (1.4) and hence the kernel of the linearized operator contains non-trivial elements. It was shown in [36] that the degeneracy happens only along an \( n \)-dimensional subspace characterized by the vector fields

\[ V_j(x) := (1 + |x|^2) \frac{\partial}{\partial x_j} - 2x_j \sum_{l=1}^n x_l \frac{\partial}{\partial x_l}. \]

for \( j = 1, \ldots, n \). More precisely, we define

\[ \Phi_j(x) := \left. \frac{d}{dt} \right|_{t=0} U[z] \circ \tau_{t e}, \quad 1 \leq i \leq n, \]

then \( \Phi_j(x) = V_j(U[z]) \), \( \Phi_j \) solves the eigenvalue problem

\[ -\Delta_B^n \Phi_j - \lambda \Phi_j = p U[z]^{p-1} \Phi_j, \quad (A.1) \]

and the degeneracy in the solution space to (A.1) can occur only along the directions \( \Phi_j, 1 \leq j \leq n \).

**Theorem A ([36]).** Let \( V_j \) be the vector fields in \( \mathbb{B}^n \) defined above and \( \Phi_j = V_j(U[z]) \). Then \( \{\Phi_i\}_{j=1}^n \) forms a basis for the kernel of \( (-\Delta_B^n - \lambda - pU[z]^{p-1}) \).

As a result, we obtain complete information on the first and the second eigenvalues and corresponding eigenspaces of the operator \( L \). We recall the following result from our earlier work [5]

**Proposition A.1.** The first and the second eigenvalues of the operator

\[ L := (-\Delta_B^n - \lambda)/U[z]^{p-1} \]

are respectively 1 and \( p \). Moreover, the first eigenspace is one dimensional and spanned by \( U[z] \) and the second eigenspace is \( n \)-dimensional and spanned by \( \{\Phi_i\}_{1 \leq i \leq n} \).

In this sequel, we also recall another relevant result used in this article whose proof can be found in [36].

**Lemma A.1.** The first Dirichlet eigenvalue of the operator

\[ L := (-\Delta_B^n - \lambda)/U[z]^{p-1} \]

in the negative half \( \mathbb{B}^n_- := \{x = (x_1, \ldots, x_n) \in \mathbb{B}^n : x_j < 0\} \) is \( p \) for every \( j = 1, \ldots, n \) and the corresponding eigenspace is one dimensional spanned by \( V_j(U[z]) \).

**Acknowledgments.** The research of M. Bhakta is partially supported by the SERB WEA grant (WEA/2020/000005) and DST Swarnajayanti fellowship (SB/SIF/2021-22/09). D. Ganguly is partially supported by the INSPIRE faculty fellowship (IFA17-MA98). D. Karmakar acknowledges the support of the Department of Atomic Energy, Government of India, under project no. 12-R&D-TFR-5.01-0520. S. Mazumdar is partially supported by IIT Bombay SEED Grant RD/0519-IRCCSH0-025.
References

[1] T. Aubin, Problèmes isopérimétriques et espaces de Sobolev. J. Differ. Geom 11(4), (1976) 573–598.

[2] S. Aryan, Stability of Hardy-Littlewood-Sobolev inequality under bubbling. Preprint arXiv:2109.12610.

[3] A. Bahri, Critical points at infinity in some variational problems. Pitman Research Notes in Mathematics Series, 182. Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1989. vi+115+307 pp. ISBN: 0-582-02164-2.

[4] M. Bhakta, D. Ganguly, D. Karmakar, S. Mazumdar, Quantitative stability of Poincaré-Sobolev equation in the hyperbolic space in $n \geq 6$. Work in progress.

[5] M. Bhakta, D. Ganguly, D. Karmakar, S. Mazumdar, Sharp quantitative stability of Poincaré-Sobolev inequality in the hyperbolic space and applications to fast diffusion flows, Preprint, arXiv:2207.11024.

[6] M. Bhakta, K. Sandeep, Poincaré-Sobolev equations in the hyperbolic space. Calc. Var. Partial Differential Equations 44 (2012), no. 1-2, 247–269.

[7] G. Bianchi, H. Egnell, A note on the Sobolev inequality. J. Funct. Anal. 100 (1991), no. 1, 18–24.

[8] V. Bögelein, F. Duzaar, and N. Fusco, A quantitative isoperimetric inequality on the sphere, Adv. Calc. Var. 10 (2017), no. 3, 223–265.

[9] V. Bögelein, F. Duzaar, and C. Scheven, A sharp quantitative isoperimetric inequality in hyperbolic $n$-space, Calc. Var. Partial Differential Equations 54 (2015), no. 4, 3967–4017.

[10] M. Bonforte, J. Dolbeault, B. Nazaret and N. Simonov, Stability in Gagliardo-Nirenberg-Sobolev inequalities: flows, regularity and the entropy method Preprint hal-02887010 and arXiv: 2007.03674.

[11] M. Bonforte, F. Gazzola, G. Grillo, J. L. Vázquez, Classification of radial solutions to the Emden-Fowler equation on the hyperbolic space. Calc. Var. Partial Differential Equations 46 (2013), no. 1-2, 375–401.

[12] H. Brezis, E. H. Lieb, Sobolev inequalities with remainder terms, J. Funct. Anal, 62 (1985), no. 1, 73–86.

[13] E. A. Carlen, A. Figalli, Stability for a GNS inequality and the log-HLS inequality, with application to the critical mass Keller-Segel equation, Duke Math. J. 162 (2013), no. 3, 579–625.

[14] A. Cianchi, A quantitative Sobolev inequality in BV, J. Funct. Anal, 237 (2006), no. 2, 466–481.

[15] A. Cianchi, N. Fusco, F. Maggi, A. Pratelli, The sharp Sobolev inequality in quantitative form. J. Eur. Math. Soc.(JEMS) 11 (2009), no. 5, 1105–1139.

[16] M. Cicalese, G. P. Leonardi, A selection principle for the sharp quantitative isoperimetric inequality. Arch. Ration. Mech. Anal. 206(2), 617–643, 2012

[17] G. Ciraolo, A. Figalli, F. Maggi, A quantitative analysis of metrics on $\mathbb{R}^N$ with almost constant positive scalar curvature, with applications to fast diffusion flows, Int. Math. Res. Not. IMRN 2018, no. 21, 6780–6797.

[18] G. Ciraolo and F. Maggi, On the shape of compact hypersurfaces with almost-constant mean curvature, Comm. Pure Appl. Math. 70 (2017), no. 4, 665–716.

[19] B. Deng, L. Sun, J. C. Wei, Sharp quantitative estimates of Struwe’s Decomposition, arXiv:2103.15360.

[20] W. Y. Ding, On a conformally invariant elliptic equation on $\mathbb{R}^n$. Commun. Math. Phys. 107(2), 331–335, 1986

[21] J. Dolbeault and G. Toscani, Stability results for logarithmic Sobolev and Gagliardo-Nirenberg inequalities. Int. Math. Res. Not. IMRN (2016), no. 2, 473–498.

[22] J. Dolbeault, M. J. Esteban, Improved interpolation inequalities and stability. Advanced Nonlinear Studies 20 (2020), no. 2, 277–291.

[23] J. Dolbeault, M. J. Esteban, Hardy-Littlewood-Sobolev and related inequalities: stability, arXiv:2202.02972.
[24] J. Dolbeault, M. J. Esteban, A. Figalli, R. L. Frank, M. Loss, Stability for the Sobolev inequality with explicit constants. Preprint, arXiv:2209.08651.
[25] H. Egnell, F. Pacella, M. Tricarico, Some remarks on Sobolev inequalities. Nonlinear Anal. 13 (1989), no. 6, 671–681.
[26] M. Engelstein, R. Neumayer, L. Spolaor, Luca Quantitative stability for minimizing Yamabe metrics. Trans. Amer. Math. Soc. Ser. B 9 (2022), 395–414.
[27] A. Figalli, F. Maggi, A. Pratelli, A mass transportation approach to quantitative isoperimetric inequalities. Invent. Math. 182(1), 167–211, 2010
[28] A. Figalli, F. Glaudo, On the sharp stability of critical points of the Sobolev inequality, Arch. Ration. Mech. Anal. 237 (2020), no. 1, 201–258.
[29] A. Figalli, R. Neumayer, Gradient stability for the Sobolev inequality: the case $p \geq 2$, J. Eur. Math. Soc. (JEMS) 21 (2019), no. 2, 319–354.
[30] A. Figalli, F. Maggi, A. Pratelli, Sharp stability theorems for the anisotropic Sobolev and log-Sobolev inequalities on functions of bounded variation, Adv. Math. 242 (2013), 80–101.
[31] A. Figalli, Y. Ru-Ya Zhang Sharp gradient stability for the Sobolev inequality, Duke Math. J., to appear, arXiv:2003.04037
[32] R. L. Frank, Degenerate stability of some Sobolev inequalities. Ann. Inst. H. Poincaré Anal. Non Linéaire, to appear, Preprint arXiv: 2003.04037, 2021
[33] N. Fusco, The quantitative isoperimetric inequality and related topics, Bull. Math. Sci. 5 (2015), no. 3, 517–607.
[34] N. Fusco, F. Maggi, A, Pratelli, The sharp quantitative isoperimetric inequality. Annals of mathematics, pp. 941–980 (2008)
[35] N. Fusco, F. Maggi, A. Pratelli, The sharp quantitative Sobolev inequality for functions of bounded variation, J. Funct. Anal. 244 (2007), no. 1, 315–341.
[36] D. Ganguly, K. Sandeep, Nondegeneracy of positive solutions of semilinear elliptic problems in the hyperbolic space. Commun. Contemp. Math. 17 (2015), no. 1, 1450019, 13 pp.
[37] B. Gidas, W. M. Ni, and L. Nirenberg, Symmetry and related properties via the maximum principle. Commun. Math. Phys. 68(3), 209–243, 1979.
[38] T. König, On the sharp constant in the Bianchi–Egnell stability inequality. Preprint arXiv:2210.08482
[39] B. Krummel and F. Maggi, Isoperimetry with upper mean curvature bounds and sharp stability estimates, Calc. Var. Partial Differential Equations 56 (2017), no. 2, Paper No. 53, 43.
[40] F. Maggi, Some methods for studying stability in isoperimetric type problems. Bull. Am. Math. Soc. 45(3), 367–408, 2008.
[41] G. Mancini and K. Sandeep, On a semilinear elliptic equation in $B^N$. Ann. Sc. Norm. Super. Pisa Cl. Sci. 7 (2008), no. 4, 635–671.
[42] R. Neumayer, A note on strong-form stability for the Sobolev inequality, Calc. Var. Partial Differential Equations 59 (2020), no. 1, Paper No. 25, 8.
[43] V. H. Nguyen, The sharp Gagliardo-Nirenberg Sobolev inequality in quantitative form. J. Funct. Anal. 277, no.7 (2019), 2179–2208.
[44] F. Nobili, I. Y. Violo, Stability of Sobolev inequalities on Riemannian manifolds with Ricci curvature lower bounds, arXiv:2210.00636.
[45] J. G Ratcliffe, Foundations of Hyperbolic Manifolds, Graduate Texts in Mathematics, vol. 149. (1994) Springer, New York.
[46] B. Ruffini, Stability theorems for Gagliardo-Nirenberg-Sobolev inequalities: a reduction principle to the radial case. Rev. Mat. Complut. 27 (2014), 509–539.
[47] K. Sandeep, R. Dutta, Personal communication.
[48] F. Seuffert, An extension of the Bianchi-Egnell stability estimate to Bakry, Gentil, and Ledoux’s generalization of the Sobolev inequality to continuous dimensions, *J. Funct. Anal.*, 273(10):3094–3149, 2017.

[49] M. Stoll, Harmonic and subharmonic function theory on the hyperbolic ball. London Mathematical Society Lecture Note Series, 431. *Cambridge University Press, Cambridge*, 2016. xv+225 pp. ISBN: 978-1-107-54148-1.

[50] M. Struwe, Variational methods. Applications to nonlinear partial differential equations and Hamiltonian systems, (1990) Springer-Verlag, Berlin.

[51] M. Struwe, A global compactness result for elliptic boundary value problems involving limiting nonlinearities. *Math. Z.* 187 (1984), no. 4, 511–517.

[52] G. Talenti, Best constant in Sobolev inequality. *Ann. Mat. pura Appl.* 110(1), (1976) 353–372.

[53] A. Tertikas, K. Tintarev On the existence of minimizers for the Hardy-Sobolev-Ma’zya inequality. *Ann. Mat. Pura Appl.* (4) 186 (2007), no. 4, 645–662.

[54] J. C. Wei, Y. Wu, On the stability of the Caffarelli-Kohn-Nirenberg inequality. *Math. Ann., to appear*, arXiv:2106.09253.

Mousomi Bhakta, Department of Mathematics, Indian Institute of Science Education and Research Pune (IISER-Pune), Dr Homi Bhabha Road, Pune-411008, India

Email address: mousomi@iiserpune.ac.in

Debdip Ganguly, Department of Mathematics, Indian Institute of Technology Delhi, IIT Campus, Hauz Khas, New Delhi, Delhi 110016, India

Email address: debdipmath@gmail.com, debdip@maths.iitd.ac.in

Debabrata Karmakar, Tata Institute of Fundamental Research, Centre For Applicable Mathematics, Post Bag No 6503, GKV K Post Office, Sharada Nagar, Chikkabommsandra, Bangalore 560065, India

Email address: debabrata@tifrbng.res.in

Saikat Mazumdar, Department of Mathematics, Indian Institute of Technology Bombay, Mumbai 400076, India

Email address: saikat.mazumdar@iitb.ac.in