Cocycles of nilpotent quotients of free groups

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Abstract
We focus on the cohomology of the $k$-th nilpotent quotient of the free group, $F/F_k$. This paper describes all the group 2-, 3-cocycles in terms of Massey products, and gives expressions for some of the 3-cocycles. We also give simple proofs of some of the results on Milnor invariants and the Johnson-Morita homomorphisms.

Keywords: nilpotent group, higher Massey products, group cohomology, mapping class group, link

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1 Introduction

Let $F$ be the free group of rank $q$. We define $F_1$ to be $F$, and $F_k$ to be the commutator subgroup $[F_{k-1}, F]$ by induction. Accordingly, we have the central extension,

$$0 \longrightarrow F_k/F_{k+1} \longrightarrow F/F_{k+1}F \xrightarrow{p_k} F/F_k \longrightarrow 0 \quad \text{(central extension)}$$

The abelian kernel is known to be free and of finite rank. We denote the rank by $N_k \in \mathbb{N}$.

The nilpotent quotient $F/F_k$, i.e., the free Lie algebra and its applications have been studied; see, e.g., [Mas, BC] and references therein. The group homology of $F/F_k$ plays a fruitful role in the study of low dimensional topology, including the Milnor (link) invariant, the Johnson-Morita homomorphism, and tree parts of the quantum invariants; see [GL, Ki, IO, Heap, KN, Tu, Po]; Massey products of manifolds appear in nilpotent obstructions of manifolds, as described in [CGO, FS, Ki, GL, Heap]. Moreover, the homology groups of degree 2 and 3 are computed as

$$H_2(F/F_k; \mathbb{Z}) \cong \mathbb{Z}^{N_k}, \quad H_3(F/F_k; \mathbb{Z}) \cong \bigoplus_{i=k}^{2k-2} \mathbb{Z}^{qN_i-N_{i+1}}.$$
The former is a result by Hopf, while the latter was shown by Orr and Igusa [10]. Corollaries 5.5 and 6.5. Meanwhile, (2) was shown by using spectral sequences [10]; the third homology seems far from being quantitative and computable in the viewpoint of group complex.

In this paper, we describe bases of the cohomology $H^2(F/F_k; \mathbb{Z})$ and $H^3(F/F_k; \mathbb{Z})$ as Massey products (see Theorem 3.2) and explicitly express some of the cocycles. Thus, this description interprets Massey products as (freely) nilpotent obstructions. Furthermore, we consider the Massey products to be an algorithm to produce many expressions of cocycles. In fact, Section 5 gives 3-cocycles of some nilpotent groups, and concretely describes their expressions. In so doing, it is reasonable to hope that the expressions may be useful for computing various things appearing in topology; including the Morita homomorphism [Mor] and the Orr link invariant [O]. Incidentally, the geometric realization of (1) is the iterated torus bundle; thus, we can express the 2-cocycles as differential 2-forms in the deRham complex; see Theorem A.4.

As corollaries, Section 4 gives simple proofs of four known results of [FS, Po, Tu, Heap, Ki], which are related to Massey products. The original proofs were discussed at the cohomology level, and they constructed Massey products according to circumstances. In contrast, we know the bases of $H^2(F/F_k; \mathbb{Z})$; therefore, we can give shorter proofs in situations that the Massey products can be obtained from cocycles in $H^2(F/F_k; \mathbb{Z})$ as universal objects.

This paper is organized as follows. Section 2 reviews Magnus expansions and Massey products, and Section 3 states our theorem with a basis. Section 4 explains the known results and gives our proofs. Section 5 discusses an algorithm to produce cocycles.

Conventional notation. Throughout this paper, let $F$ be the free group of rank $q$, and let an integer $N_k \in \mathbb{Z}$ be the rank of $F_k/F_{k+1}$. Given a group $G$, we denote $G$ by $G_1$ and define $G_k$ to be the commutator subgroup $[G_{k-1}, G]$ by induction. For a group $G$, we write $BG$ for the Eilenberg-MacLane space, i.e., $K(G, 1)$-space. Furthermore, we assume the basic properties of group (co)-homology as in [Bro, Sections I, II, and VII].

2 Review: Magnus expansions and higher Massey products

Let us begin by reviewing unipotent Magnus expansions and higher Massey products.

First, we review the Magnus expansion modulo degree $k$. Let $\mathbb{Z}\langle X_1, \ldots, X_q \rangle$ be the polynomial ring with non-commutative indeterminacy $X_1, \ldots, X_q$, and $J_k$ be the two-sided ideal generated by polynomials of degree $\geq k$. Then, the Magnus expansion (of the free group $F$) is the homomorphism $\mathcal{M}: F \rightarrow \mathbb{Z}\langle X_1, \ldots, X_q \rangle/J_k$ defined by

$$\mathcal{M}(x_i) = 1 + X_i, \quad \mathcal{M}(x_i^{-1}) = 1 - X_i + X_i^2 + \cdots + (-1)^{k-1}X_i^{k-1}. \quad (3)$$

As is known, $\mathcal{M}(F_k) = 0$. By passage to this $F_k$, this $\mathcal{M}$ further induces an injection

$$\mathcal{M}: F/F_k \longrightarrow \mathbb{Z}\langle X_1, \ldots, X_k \rangle/J_k.$$

Next, we review another description of the Magnus embedding [GG], which is a faithful linear representation of $F/F_k$. Let $\Omega_k$ be the polynomial ring $\mathbb{Z}[\lambda_i^{(j)}]$ over commuting indeterminates $\lambda_i^{(j)}$ with $i \in \{1, 2, \ldots, k - 1\}, j \in \{1, \ldots, q\}$. We define

$$\Upsilon_k: F \rightarrow GL_k(\Omega_k),$$
as a homomorphism by setting

\[
\Upsilon_k(x_j) = \begin{pmatrix}
1 & \lambda_1^{(j)} & 0 & \cdots & 0 \\
0 & 1 & \lambda_2^{(j)} & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & \lambda_{k-1}^{(j)} \\
0 & 0 & \cdots & 0 & 1
\end{pmatrix}.
\]

As is known [GG], the image \(\Upsilon_k(F_k)\) consists of the identity matrix, and the quotient map \(F/F_k \to GL_k(\Omega_k)\) is injective. Thus, we have an isomorphism \(F/F_k \cong \text{Im}(\Upsilon_k)\). It is shown [KN] Appendix that the correspondence \(x_j \mapsto \Upsilon_k(x_j)\) gives an equivalence to the former Magnus expansion \(\mathcal{M}\). The longer the word of \(x\) is, the harder the computation of \(\mathcal{M}(x)\) is; however, that of \(\Upsilon_k(x)\) is still simpler. Indeed, the map \(\Upsilon_k\) is defined over the commutative ring \(\Omega_k\) and is therefore compatible with computer programming.

Next, we review the higher Massey products, which were first defined by Kraines [Kra]. Here, we describe the products in the non-homogenous complex of a group \(G\) with a trivial coefficient ring \(A\). That is, as is shown in [Br], Chap. III.1, we define the cochain \(C^*(G; A)\) as \(\text{Map}(G^n, A)\) and the coboundary map \(\partial\) by setting

\[
\partial_n^*(f)(g_1, \ldots, g_n) = f(g_2, \ldots, g_n) + (-1)^n f(g_1, \ldots, g_{n-1}) + \cdots - f(g_1, g_2, g_3, \ldots, g_n).
\]

Furthermore, the cup product on \(C^*(G; A)\) can be described as a canonical product. More precisely, for \(u \in C^p(G; A)\) and \(v \in C^q(G; A)\), the product \(u \cup v \in C^{p+q}(G; A)\) is defined by

\[
(u \cup v)(g_1, \ldots, g_{p+q}) := (-1)^{pq} u(g_1, \ldots, g_p) \cdot v(g_{p+1}, \ldots, g_{p+q}) \in A.
\]

For \(i \leq n\), fix a cocycle \(\gamma_i \in C^p(G; A)\). Then, a defining system associated with \((\gamma_1, \ldots, \gamma_n)\) is a set of elements \((a_{s,t})\) for \(1 \leq s \leq t \leq n\) with \((s, t) \neq (1, n)\), satisfying

(i) \(a_{s,t} \in C^{p_s+p_{s+1}+\cdots+p_t-t+s}(G; A)\).

(ii) When \(s = t\), the diagram map \(a_{s,s}\) is a representative cocycle of \(\gamma_s\) in \(C^{p_s}(G; A)\).

(iii) \(\partial^*(a_{s,t}) = \sum_{r=s}^{t-1} (-1)^{p_s+p_{s+1}+\cdots+p_t-t+s} a_{s,r} \cup a_{r+1,t}\).

Given such a defining system, we can define a cocycle of the form,

\[
\sum_{r: 1 \leq r \leq n-1} (-1)^{p_1+p_2+\cdots+p_{r-1}+1} a_{1,r} \cup a_{r+1,n} \in C^{p_1+p_2+\cdots+p_{n+1}+2}(G; A).
\]

Following [Kra], the \(n\)-fold Massey product, \(\langle \gamma_1, \gamma_2, \ldots, \gamma_n \rangle\), is defined to be the set of cohomology classes of cocycles associated with all possible defining systems. There are many interpretations of the higher Massey product; this paper uses the Massey product as a nilpotent algorithm to yield cocycles from other cocycles of lower degree.

**Remark 2.1.** As is known [FS], if \(p_1 = p_2 = \cdots = p_n = 1\) and every \(m\)-fold Massey product with \(m < n\) chosen from \(\{\gamma_1, \ldots, \gamma_n\}\) is null-cohomologous, the \(n\)-fold Massey product \(\langle \gamma_1, \gamma_2, \ldots, \gamma_n \rangle\) is a singleton in \(H^2(G; A)\).
3 Main theorem: generators of $H^*(F/F_k)$

We will describe the bases of some cohomology of $F/F_k$ in terms of Massey products (Theorem 3.2).

Before stating the theorem, we should concretely define some of the defining systems. Let $G = F/F_k$. For $1 \leq t \leq q$, let $\alpha_t : F/F_k \to \mathbb{Z}$ be the $t$-th summand of the abelianization $F/F_k \to \mathbb{Z}^q$. We regard $\alpha_t$ as a 1-cocycle of $F/F_k$. Then, given a $k$-tuple $I = (j_1, \ldots, j_k) \in \{1, 2, \ldots, q\}^k$, we have 1-cocycles $\alpha_{j_1}, \ldots, \alpha_{j_k}$. Furthermore, for $1 \leq s \leq t \leq k$, let us consider the evaluation of the coefficient of $X_s \cdots X_t$. That is, we set up the linear map,

$$
\beta_{i_{s+1}\cdots i_t} : \mathbb{Z}(X_1, \ldots, X_q) \to \mathbb{Z}; \quad \sum a_{j_1\cdots j_s}X_{j_1}\cdots X_{j_s} \mapsto a_{i_{s+1}\cdots i_t}.
$$

(5)

Let us denote the composite $\beta_{i_{s+1}\cdots i_t} \circ \mathcal{M}$ by $c_{i_{s+1}\cdots i_t}$, which we will use many times.

Lemma 3.1. Consider the case $p_1 = p_2 = \cdots = p_n = 1$ with $n = k$. Let $a_{s,t} : F \to \mathbb{Z}$ be the composite $c_{i_{s+1}\cdots i_t}$. Then, the set of $(a_{s,t})$ is a defining system associated with $(\alpha_{i_1}, \ldots, \alpha_{i_k})$. In particular, the associated 2-cocycle is represented by

$$
F/F_k \times F/F_k \to \mathbb{Z}; \quad (x, y) \mapsto \sum_{1 \leq \ell \leq k-1} c_{i_{s+1}\cdots i_t}(x)c_{i_{t+1}\cdots i_k}(y).
$$

(6)

Proof. From the unipotent Magnus expansion, the right hand side in (iii) is equivalent to the product of upper triangular matrices. Thus, it is not so hard to check (iii) by direct computation. Thus, the formula of the Massey product (4) readily means (6). \qed

Moreover, let us review standard sequences from [CFL]. Equip the set of all sequences $\bigcup_{s=1}^{\infty} \{1, 2, \ldots, q\}^s$ with the lexicographical order. Then, a sequence $I = i_1i_2\cdots i_k$ is said to be standard, if $I < i_{s+1}\cdots i_k$ for any $2 \leq s \leq k$. Let $\mathfrak{U}_k$ be the set of standard sequences of length $k$. As is known (see, e.g., [CFL Theorem 1.5]), the order of $\mathfrak{U}_k$ is equal to $N_k$.

Furthermore, the following is known:

$$
N_k := \text{rank}(F_k/F_{k+1}) = \text{rank}(H^2(F/F_k; \mathbb{Z})) = |\mathfrak{U}_k| = \frac{1}{m} \sum_{d \mid k} \mu(k/d)q^d \in \mathbb{N},
$$

(7)

where $\mu$ is the Möbius function; see, e.g., [CFL Theorem 1.5] and [IO Corollary 5.5].

Now we show that the second and third cohomology of $F/F_k$ are generated by Massey products:

Theorem 3.2. (I) Every $j$-fold Massey product with $j < k$ is zero. In particular, for any standard index $i_1\cdots i_k \in \mathfrak{U}_k$, the $k$-fold one $(\alpha_{i_1}, \ldots, \alpha_{i_k})$ is unique in $H^2(F/F_k; \mathbb{Z})$ and is represented by a 2-cocycle in (5).

(II) The second cohomology $H^2(F/F_k; \mathbb{Z}) \cong \mathbb{Z}^{N_k}$ is spanned by the $k$-fold Massey products $(\alpha_{i_1}, \ldots, \alpha_{i_k})$ running over standard sequences $(i_1 \cdots i_k) \in \mathfrak{U}_k$.

(III) For any $k \leq \ell \leq 2k - 2$, consider the projection $p_\ell : F/F_\ell \to F/F_k$. Then, there are homomorphisms $\mathfrak{s}_\ell : Z^3(F/F_\ell; \mathbb{Z}) \to Z^3(F/F_k; \mathbb{Z})$ such that

$$
(\alpha_s \sim (\alpha_{i_1}, \ldots, \alpha_{i_k}) = p_\ell^* \circ \mathfrak{s}_\ell(\alpha_s \sim (\alpha_{i_1}, \ldots, \alpha_{i_k})) \in Z^3(F/F_k; \mathbb{Z}),
$$

(8)

for any $(i_1 \cdots i_\ell) \in \mathfrak{U}_\ell, 1 \leq s \leq q$, and that the following set of 3-cocycles is a basis of the third cohomology $H^3(F/F_k; \mathbb{Z}) \cong \bigoplus_{\ell=k}^{2k-2} N_{\ell-1}$.

$$
\bigcup_{1 \leq s \leq q} \{ \mathfrak{s}_\ell(\alpha_s \sim (\alpha_{i_1}, \ldots, \alpha_{i_k})) \mid (i_1 \cdots i_\ell) \in \mathfrak{U}_\ell, 1 \leq s \leq q, (i_1 \cdots i_\ell s) \not\in \mathfrak{U}_{\ell+1} \}. \quad (9)
$$
The statement of (1) might be classically known; see [FS] [T].

Proof. The reader who is pressed for time may skip this proof.

(I) Recall the definitions of (ii) and of ([5]). Thus, every lower Massey product is nullcohomologous by \( a_{s,t} \) for some \((s,t)\), by induction on \(k\).

(II) Denote by \( S \) the sum of \( c_{i_1\ldots i_k} \) on \( F_k/F_{k+1} \), where \( i_1\ldots i_k \in \mathcal{U}_k \). Namely, \( S := \oplus_{i \in \mathcal{U}_k} c_I : F_k/F_{k+1} \to \mathbb{Z}^{N_k} \). As is known [CF], Theorems 3.5 and 3.9, the sum \( S \) is surjective; hence, it is bijective. Accordingly, the centrally extended group operation on \( F/F_k \times F_k/F_{k+1} \) from the 2-cocycles \( \oplus_{i \in \mathcal{U}_k} \langle \alpha_{i_1}, \ldots, \alpha_{i_k} \rangle \) is, by definition, formulated as

\[
(g, \alpha) \cdot (h, \beta) = (gh, S^{-1} \left( \bigoplus_{i_1\ldots i_k} c_{i_1\ldots i_k}(\alpha + \beta) + \sum_{1 \leq \ell < k} c_{i_1\ldots i_{\ell}i_{\ell+1}\ldots i_k}(gh) c_{i_{\ell+1}\ldots i_k}(h) \right))
\]

for \( g, h \in F/F_k \) and \( \alpha, \beta \in F_k/F_{k+1} \). Here, it is worth noticing that the subsequences \( i_1i_2\ldots i_\ell \), \( i_{\ell+1}\ldots i_k \) are also standard. Therefore, this group is isomorphic to the nature \( F/F_k \) as central extensions over \( F/F_k \) (cf. matrix multiplications). Hence, the second cohomology \( H^2(F/F_k) \cong \mathbb{Z}^{N_k} \) is generated by the sum \( \oplus_{i \in \mathcal{U}_k} \langle \alpha_{i_1}, \ldots, \alpha_{i_k} \rangle \), which provides a basis of \( H^3(F/F_k) \), as desired.

(III) First, we will mention some of the results from [IO]. Igusa and Orr [IO, Theorem 6.7] constructed a certain filtration on the homology, \( \mathcal{F}^aH_3(F/F_k; \mathbb{Z}) \), such that two isomorphisms,

\[
\frac{\mathcal{F}^aH_3(F/F_k; \mathbb{Z})}{\mathcal{F}^a_{a+1}H_3(F/F_k; \mathbb{Z})} \cong \mathbb{Z}^{q_{N_{\ell-1}-N_\ell}}.
\]

\[
\mathcal{F}^aH_3(F/F_k; \mathbb{Z}) \cong \text{Im}((p_{\ell-1})_*: H_3(F/F_{\ell-1}; \mathbb{Z}) \to H_3(F/F_k; \mathbb{Z}))
\]

hold for \( k < \ell < 2k \), and the left quotient is zero otherwise. In particular, \( \mathcal{F}^aH_3(F/F_k; \mathbb{Z}) \) is a direct summand of \( H_3(F/F_k; \mathbb{Z}) \), and \( \text{Ker}((p_{\ell-1})_*) \) is a direct summand of \( H_3(F/F_{\ell-1}) \), if \( k < \ell < 2k \). Thus, for \( \ell \geq k \), we readily have the composite,

\[
H_3(F/F_k; \mathbb{Z}) \to \mathcal{F}^a_{a+1}H_3(F/F_k) \cong H_3(F/F_k)/\text{Ker}((p_{\ell})_*) \to H_3(F/F_k; \mathbb{Z}).
\]

The application of this composite to \( \text{Hom}(\bullet, \mathbb{Z}) \) is regarded as a map \( \iota_\ell : H^3(F/F_k; \mathbb{Z}) \to H^3(F/F_k; \mathbb{Z}) \). Accordingly, we consider the composite of the cup product on \( F/F_k \) and \( \iota_\ell \):

\[
\Theta_\ell : H^1(F/F_k; \mathbb{Z}) \otimes H^2(F/F_k; \mathbb{Z}) \xrightarrow{\cup} H^3(F/F_k; \mathbb{Z}) \xrightarrow{\iota_\ell} H^3(F/F_k; \mathbb{Z}).
\]

We will show that the direct sum \( \oplus_{\ell=k}^{2k-2} \Theta_\ell \) surjects onto \( H^3(F/F_k; \mathbb{Z}) \). Consider the Lyndon-Hochschild spectral sequence of the central extension \( F_k/F_{k+1} \to F/F_{k+1} \to F/F_k \). Then, as is shown in the proof of [IO, Lemma 5.8], we find a sequence,

\[
H_3(F/F_{k+1}; \mathbb{Z}) \to H_3(F/F_k; \mathbb{Z}) = E^2_{3,0} \xrightarrow{d^3_{3,0}} E^2_{1,1} \xrightarrow{\delta_*} H_2(F/F_k; \mathbb{Z}) \to 0 \quad \text{(exact)},
\]

satisfying \( d^3_{3,0}(\mathcal{F}^{k+1}H_3(F/F_k; \mathbb{Z})) = \text{Ker}(\delta_*) \) and \( d^3_{3,0}(\mathcal{F}^kH_3(F/F_k; \mathbb{Z})) = 0 \). Keep in mind that each term is free. Furthermore, consider the spectral sequence on the cohomology level. Noting the identity,

\[
E^1_{2,1} = H^1(F/F_k; F_k/F_{k+1}) \cong H^1(F/F_k; \mathbb{Z}) \otimes H^2(F/F_k; \mathbb{Z}),
\]

we dually obtain the following sequence from (III):

\[
0 \to H^2(F/F_k) \to H^1(F/F_k) \otimes H^2(F/F_{k-1}) \xrightarrow{d^3_{2,0}} \mathcal{F}^{k+1}H^3(F/F_k) \quad \text{(exact)}.
\]
As is usual with the cup$_1$-product, the differential map $d_2^{3,0}$ is equal to the cup product. To summarize, this sequence and the isomorphisms \( \text{III} \) imply the surjectivity of $\oplus_{\ell=k}^{2k-2} \Theta_{\ell}$, as required. Moreover, the construction of $\Theta_{\ell}$ admits the desired section $\varphi_2$ satisfying \( \text{II} \).

The proof is completed by showing the linear independence of \( \text{(9)} \). Notice from the definition of \( (i_1 \cdots i_\ell) \in \mathfrak{U}_\ell \) that \( (i_1 \cdots i_\ell s) \in \mathfrak{U}_{\ell+1} \) if and only if $s > i_1$. Thus, if $s > i_1$, then \( i_1 i_2 \cdots i_{k-1} s \) is not standard because of $i_k \geq s$. Here, we mention \[ \text{GL} \] Proposition 4.5] showing the equality,

$$
\alpha_s \cup (\alpha_{i_1}, \ldots, \alpha_{i_k}) = \alpha_{i_k} \cup (\alpha_{i_1}, \ldots, \alpha_{i_{k-1}}, \alpha_s) \in H^3(F/F_k; \mathbb{Z}).
$$

Hence, $\alpha_s \cup (\alpha_{i_1}, \ldots, \alpha_{i_k})$ with \( (i_1 \cdots i_k) \in \mathfrak{U}_\ell \) and \( (i_1 \cdots i_k s) \in \mathfrak{U}_{\ell+1} \) is cohomologous to a cocycle in \( \text{(9)} \). By \[ \text{III} \] and functoriality, a similar conclusion can be made even in the case $k < \ell \leq 2k - 2$. Hence, from the surjectivity of $\oplus_{\ell=k}^{2k-2} \Theta_{\ell}$, comparing the dimensions of $H^3(F/F_k)$ with the order of \( \text{(9)} \) leads to linear independence, as required. \( \square \)

We immediately obtain a corollary from the preceding proof.

**Corollary 3.3.** For any $s \leq q$, the map $H^2(F/F_k; \mathbb{Z}) \to H^3(F/F_k; \mathbb{Z})$ which sends $\beta$ to $\alpha_s \cup \beta$ is injective.

## 4 Applications: simple proofs

There are some topological invariants using $H^*(F/F_k)$, and some results on the Milnor link-invariant and Mapping class group. In this section, we give simple proofs of the results of Fenn-Sjerve \[ \text{FS} \], Turaev \[ \text{Tu} \], Porter \[ \text{Po} \], Kitano \[ \text{Ki} \], and Heap \[ \text{Heap} \].

### 4.1 Theorem of Fenn-Sjerve on the Massey product

We state the theorem \[ \text{FS} \] and give an alternate proof using Theorem 3.2.

Assume $k \geq 3$. Let $W_1, \ldots, W_\ell$ be words in $F_k$, and $R \subset F$ be the normal closure of $W_1, \ldots, W_\ell$ and $F_{k+1}$. Let $G$ be the quotient group $F/R$. Since $H_1(F) \cong H_1(G)$, the 1-cocycle $\alpha_j : F \to \mathbb{Z}$ induces a 1-cocycle $\alpha_j : G \to \mathbb{Z}$ for $j \leq q$.

We will regard the words $W_j$ as 2-cycles, and state Theorem 4.1 below. Consider the projection $p : G \to F/F_k$ with center $Z$. Recall Hopf’s theorem which claims the isomorphisms,

$$
H_2(G) \cong (R \cap [F,F])/[F,R], \quad H_2(F/F_k) \cong (F_k \cap F_2)/[F,F_k] = F_k/F_{k+1}.
$$

Moreover, note from \[ \text{Br} \] Exercise 6 in \[ \text{II.5} \] that the 5-term exact sequence is given by

$$(R \cap [F,F])/[F,R] \xrightarrow{p^*} F_k/F_{k+1} \longrightarrow Z \longrightarrow H_1(G) \xrightarrow{\cong} H_1(F/F_{k+1}) \to 0 \quad \text{ (exact).}$$

Noticing $W_j \in R \cap [F,F]$, we denote $p_*(W_j)$ by $\mathcal{W}_j$, and regard it a 2-cycle.

**Theorem 4.1 \[ \text{FS} \].** Suppose that all of $W_1, \ldots, W_\ell$ lies in $F_k$. For any $\ell < k$, every $\ell$-fold Massey product $\langle \alpha_{j_1}, \ldots, \alpha_{j_\ell} \rangle$ vanishes. On the other hand, Massey products of length $k$ are defined and evaluated on $\{\mathcal{W}_j\}$ according to the formula,

$$
\sum_{j_1, \ldots, j_k} [\langle \alpha_{j_1}, \ldots, \alpha_{j_\ell} \rangle, \mathcal{W}_{j_1} \cdots \mathcal{W}_{j_\ell}] = \mathcal{M}(W_j) \in \mathcal{M}(F_k/F_{k+1}). \quad \text{(12)}
$$

Here, the outer $[,]$ is the pairing of $H^2(G; \mathbb{Z})$ and $H_2(G; \mathbb{Z})$. 


Proof. Given a standard index $I = i_1 \cdots i_k \in \mathbf{U}_k$, let $W_I \in F_k/F_{k+1}$ be a generator of the $i_1 \cdots i_k$-th summand of $\mathbb{Z}^{N_k} \cong F_k/F_{k+1}$. A previous paper [CFL] §2 explicitly describes the word $W_I$ and refers to it as the standard commutator.

First, we focus on the case $G = F/(F_{k+1}, W_I)$. A previous paper [CFL, Lemma 3.4] showed that, for any standard index $j_1 \cdots j_k$, the $X_{j_1} \cdots X_{j_k}$-coefficient of $\mathcal{M}(W_I)$ is $\delta_{i_1,j_1} \cdots \delta_{i_k,j_k} \in \{0,1\}$. In particular, the cocycle expression of $\langle \alpha_{j_1}, \ldots, \alpha_{j_k}, W_I \rangle$ in Theorem 3.3 (II) says that the pairing $[\langle \alpha_{j_1}, \ldots, \alpha_{j_k}, W_I \rangle]$ is equal to $\delta_{i_1,j_1} \cdots \delta_{i_k,j_k}$. Thus, equality (12) holds for $G = F/(F_{k+1}, W_I)$.

Now we can complete the proof. Since $F_k/F_{k+1}$ is abelian, we can expand $p_*(W_I)$ as $\prod_{I \subset \mathbf{U}_k} (W_I)^{a_I}$ for some $a_I \in \mathbb{Z}$. Then, from the preceding discussion, equality (12) holds in the $X_{j_1} \cdots X_{j_k}$-coefficients with respect to every standard index $j_1 \cdots j_k$. Since the other coefficient is a linear sum of the such $X_{j_1} \cdots X_{j_k}$-coefficients, we conclude that the equality (12) in $\mathcal{M}(F_k/F_{k+1})$ is satisfied.

\begin{remark}
Finally, we should remark on continued subsections. Let us choose a connected CW complex $X$ such that $\pi_1(X) \cong G$ and fix a classifying map $c : X \to BG$ is obtained by killing the higher homotopy groups of $X$. From $H_*(BG; \mathbb{Z}) \cong H_*(G)$, we suppose 2-cycles $X_{j} \in H_2(X)$ with $c_X(X_{j}) = [W_I]$ and $\pi_1(X) \cong G$, the 1-cocycle $\alpha_j$ may be that of $H^1(X; \mathbb{Z})$, and the pairing $\langle [\alpha_{j_1}, \ldots, \alpha_{j_k}, W_I] \rangle$ is equivalent to the right hand side of (12). To conclude, we can deal with Massey products in $H^2(X; \mathbb{Z})$, as in Theorem 4.1.
\end{remark}

### 4.2 Milnor invariant and Massey product

Porter [Po] and Turav [Tu] independently showed that the Milnor link invariant is equivalent to some Massey product of the link complement space. For simplicity, this paper focuses on only the $m$-th leading terms of the Milnor invariant and gives another proof of their result.

To state the theorems, we begin by reviewing the Milnor invariant, according to [Mil, Tu]. We suppose that the reader has elementary knowledge of knot theory, as provided in [Po, Tu]. Let $M$ be an integral homology 3-sphere, that is, a closed 3-manifold such that $H_2(M; \mathbb{Z}) \cong H_2(S^3; \mathbb{Z})$. Choose a link $L \subset M$ with $q$-components, and a meridian-longitude pair $(m_\ell, t_\ell)$ for $\ell \leq q$, where $I_\ell$ may be preferred. Then, it is known ([Mil]; see also [Tu, Lemma 1.2]) that the $m$-th quotient $\pi_1(M \setminus L)/\pi_1(M \setminus L)_m$ has the group presentation

$$\langle x_1, \ldots, x_q \mid [x_\ell, w_j^{(m)}] = 1 \text{ for } \ell \leq q, \pi_1(M \setminus L)_m \rangle,$$

where $x_i$ and $w_j^{(m)}$ are represented by the $j$-th meridian and longitude, respectively.

For simplicity, let us assume the existence of $k \in \mathbb{Z}$ such that $w_j^{(m)}$ is trivial in the $k$-th quotient for any $m < k$, i.e., $w_j^{(m)} \in (\pi_1(M \setminus L))_k$. We call it Assumption $A_k$. By considering $w_i^{(k)}$ to be a word in $F/F_k$, we will focus on the value $\mathcal{M}(w_i^{(k)}) \in \mathbb{Z}[X_1, \ldots, X_q]/\mathcal{J}_k$. The coefficient of $X_{i_1} \cdots X_{i_k}$ of $\mathcal{M}(w_i^{(k)})$ is called the $k$-th Milnor $\mu$-invariant of $L$, and it is denoted by $\mu(i_1 \cdots i_k; \ell)$. Let $\alpha_j : \pi_1(M \setminus L; \mathbb{Z}) \to \mathbb{Z}$ be the $j$-th summand of the abelianization $H_1(M \setminus L; \mathbb{Z}) \cong \mathbb{Z}^q$.

\begin{theorem} [The minimal non-vanishing case of [Tu, Po]] Suppose $A_k$. Let $[I_\ell] \in H_2(M \setminus L; \mathbb{Z})$ be a 2-cycle corresponding to the $\ell$-th longitude $I_\ell$. For any index $I = i_1i_2 \cdots i_k$, the $(k + 1)$-fold Massey product $\langle \alpha_{i_1}, \ldots, \alpha_{i_k-1}, \alpha_{i_k}, \alpha_\ell \rangle$ is uniquely defined, and the following equality holds:

$$\mu(i_1 \cdots i_k; \ell) = \langle \alpha_{i_1}, \ldots, \alpha_{i_k-1}, \alpha_{i_k}, \alpha_\ell \rangle, [I_\ell] \in \mathbb{Z}. \tag{14}$$
\end{theorem}
Proof. Denote $\pi_1(M \setminus L)$ by $G$, and take the classifying map $\mathcal{C} : M \setminus L \to BG$, as in Remark 1.2. We claim that the pushforward $\mathcal{C}_{\ast}([l])$ corresponds to the relation $[m_k, l]$. Consider the $\ell$-th torus boundary, $\partial_\ell(M \setminus L)$, as a $K(\mathbb{Z}^2, 1)$-space. This $\pi_1(\partial_\ell(M \setminus L)) \cong \mathbb{Z}^2$ is presented by $\langle m_k, l \mid [m_k, l] \rangle$. From Hopf’s theorem, the generator of $H_2(\partial_\ell(M \setminus L)) \cong \mathbb{Z}$ corresponds to $[m_\ell, l]$. Considering the pushforwards via the inclusion $\partial_\ell(M \setminus L) \to M \setminus L$ and $\mathcal{C}$ proves the claim.

We are now in a position to complete the proof. From the above claim, the projection $p : G \to G/G_{k+1}$ takes the pushforward $(p \circ \mathcal{C})_{\ast}([l])$ corresponding to the word $[x_\ell, w_\ell^{(k+1)}]$. Then, by assumption, that (13) implies $w_\ell^{(k+1)}$ is trivial in $G/G_{k+1} \cong \mathbb{Z}/F_{k-1}$. Thus, $w_\ell^{(k)}$ may lie in $F_{k-1}$, leading to $[x_\ell, w_\ell^{(k)}]\in F_k$. Therefore, $G/G_{k+1} \cong F/F_k$, which implies that $[x_\ell, w_\ell^{(k+1)}]\in F_{k+1}$. In particular, $\langle \alpha_{i_k} \cdots, \alpha_{i_k}, \alpha_{i_\ell} \rangle$ of length $k+1$ is uniquely defined; see Remark 2.1. Moreover, by the theorem (14) of Fenn-Sjerve, the right hand side of (14) means the $X_{i_k} \cdots X_{i_k} X_{\ell}$-coefficient of $\mathcal{M}([x_\ell, w_\ell^{(k+1)}])$. Then, we can verify, by directly computing $\mathcal{M}(x_\ell w_\ell^{(k+1)} x_\ell^{-1}(w_\ell^{(k+1)})^{-1})$, that it is equal to the $X_{i_k} \cdots X_{i_k}$-coefficient of $\mathcal{M}(w_\ell^{(k)})$, that is, $\mu(i_1 \cdots i_k; \ell)$, as required.

Incidentally, Milnor defined the invariant of higher degree, and Porter [Po] and Turaev [Tu] described the higher invariant in terms of Massey product; see also the paper [KN], which provides a refinement of the higher invariant. Although we will omit showing the details, the higher degree relation can be proven in the same manner as Theorem 4.3.

4.3 Johnson homomorphism and Massey product

Now let us focus on the Johnson homomorphism of the mapping class group; see, e.g., [Day, GL, Heap, Mor] for the significance of this homomorphism. This section paraphrases [Ki], which describes the relation between the Johnson homomorphism and Massey products.

First, we should establish the notation. Suppose $q = 2g$. Let $\Sigma_{g,r}$ be a compact oriented surface of genus $g$ with $r$ boundaries. Let $\Gamma_{g,1}$ be the group of isotopy classes of orientation-preserving homeomorphisms of $\Sigma_{g,1}$. Then, the action of $\Gamma_{g,1}$ on $F = \pi_1(\Sigma_{g,1})$ can be regarded as a homomorphism $\Gamma_{g,1} \to \operatorname{Aut}(F)$. Subject to $F_k$, we have $\rho_k : \Gamma_{g,1} \to \operatorname{Aut}(F/F_k)$. We commonly denote the kernel $\operatorname{Ker}(\rho_k)$ by $\mathcal{T}(k)$. We have a filtration $\Gamma_{g,1} \supset \mathcal{T}(2) \supset \mathcal{T}(3) \supset \cdots$.

Next, let us review the homomorphism (15) below. Fix any $f \in \mathcal{T}(k)$. Given $[\gamma] \in H_1(\Sigma_{g,1}) = F/F_2$, choose a representative $\gamma \in \pi_1(\Sigma_{g,1})$. Then, $f_\gamma(\gamma)^{-1}$ lies in $F_k$ since the action of $f_\gamma$ on $F/F_k$ is trivial by $f \in \mathcal{T}(k)$. Then, the Johnson homomorphism is defined as the map,

$$\tau_k : \mathcal{T}(k) \to \operatorname{Hom}(H_1(\Sigma_{g,1}), F_k/F_{k+1}),$$

which sends $[f]$ to the homomorphism $[\gamma] \to [f_\gamma(\gamma)^{-1}]$. As is known, this $\tau_k$ is well-defined and a homomorphism. The following is also well-known; see [Joh, Mor].

Theorem 4.4. Let $f \in \mathcal{T}(k)$. Then, $f \in \mathcal{T}(k+1)$ if and only if $\tau_m(f) = 0$ for any $m \leq k$.

Next, let us examine the mapping torus $T_{f,1}$ for a fixed $f \in \mathcal{T}(k)$ with $k \geq 2$. Here, $T_{f,1}$ is the quotient space of $\Sigma_{g,1} \times [0, 1]$ subject to the relation $(y, 0) \sim (f(y), 1)$ for any $y \in \Sigma_{g,1}$. Since $f \in \mathcal{T}(k)$ with $k \geq 2$, we have $H_1(T_{f,1}) \cong H_1(\Sigma_{g,1} \times S^1)$. Furthermore, fix a basis $\{x_1, \ldots, x_{2g}\}$ of the free group $F = \pi_1(\Sigma_{g,1})$. Then, following the von-Kampen argument can verify the presentation,

$$\pi_1(T_{f,1}) \cong \langle x_1, \ldots, x_{2g}, \gamma \mid [x_1, \gamma][f_\gamma(x_1)x_1^{-1}, \ldots, [x_{2g}, \gamma][f_\gamma(x_{2g})x_{2g}^{-1}] \rangle.$$  

(16)
Here, $\gamma$ represents a generator of $\pi_1(S^1)$. Since $T_{f,1}$ is a $\Sigma_{g,1}$-bundle over $S^1$ by definition, it is a $K(\pi,1)$-space. Hence, $H_*(\pi_1(T_{f,1})) \cong H_*(T_{f,1}) \cong H_*(\Sigma_{g,1} \times S^1)$. Moreover, Hopf’s theorem implies that the relations $[x_1, \gamma]f_*(x_1)x_1^{-1}, \ldots, [x_{2g}, \gamma]f_*(x_{2g})x_{2g}^{-1}$ represent a basis $\{X_1, \ldots, X_{2g}\}$ of $H_2(T_{f,1}) \cong \mathbb{Z}^{2g}$.

Now let us state and prove Proposition 4.5. Since the boundary $\partial T_{f,1}$ is the torus $\partial \Sigma_{g,1} \times S^1$, we can define $T_f^\gamma$ to be the resulting space obtained by filling in the torus $\partial T_{f,1} \cong \partial \Sigma_{g,1} \times S^1$ with the solid torus $\Sigma_{g,1}$ by the pushforwards $T_f^\gamma$ denoting the inclusion $\Sigma_{g,1} \times D^2 \to T_f^\gamma$. This $T_f^\gamma$ is called the Dehn filling along a curve on $\partial T_{f,1}$. By denoting the inclusion $T_{f,1} \to T_f^\gamma$ by $\iota$, the homology $H_2(T_f^\gamma; \mathbb{Z}) \cong H^1(T_f^\gamma; \mathbb{Z}) \cong \mathbb{Z}^{2g}$ is spanned by the pushforwards $\{u_*(X_1), \ldots, u_*(X_{2g})\}$. Moreover, we should notice the presentation of $\pi_1(T_f^\gamma)$,

$$\pi_1(T_f^\gamma) \cong \langle x_1, \ldots, x_{2g} \mid f_*(x_1)x_1^{-1}, \ldots, f_*(x_{2g})x_{2g}^{-1} \rangle.$$  \hspace{1cm} (17)

**Proposition 4.5** (cf. [K] and [CL] Corollary 4.1). Let $\{X_1, \ldots, X_{2g}\}$ be the basis of $H_2(T_{f,1})$, and $\iota : T_{f,1} \to T_f^\gamma$ be the inclusion as above. Let $x_j^* : \pi_1(T_f^\gamma) \to \mathbb{Z}$ be the 1-cocycle which sends $x_i$ to $\delta_{j,i}$. Define a map $\tau_k^\gamma : \mathcal{T}(k) \to \text{Hom}(H_1(\Sigma_{g,1}), \mathcal{M}(F_k/F_{k+1}))$ by letting $\tau_k^\gamma(f)$ be the homomorphism,

$$x_i \mapsto \sum_{j_1, \ldots, j_k} [\langle x_{j_1}^*, \ldots, x_{j_k}^* \rangle, u_*(X_{j_1}) \cdots X_{j_k}].$$ \hspace{1cm} (18)

Here, the outer $[,]$ is the pairing of $H^2(T_{f,1})$ and $H_2(T_{f,1})$. Then, $\tau_k^\gamma(f)(x_i)$ is equal to $\mathcal{M}(\phi_{j,k})^*(f)(x_i))$.

**Proof.** As mentioned above, $f_*(x_i)x_i^{-1} \in F_k$, since $f \in \mathcal{T}(k)$. The statement readily follows from Theorem 4.4 with $W_j = f_*(x_j)x_j^{-1}$ and $t = 2g$. \hfill $\Box$

### 4.4 Vanishing condition of the Morita homomorphism

As a lift of the Johnson homomorphism $\tau_k$, Morita [Mor] defined a map $\tilde{\tau}_k : \mathcal{T}(k) \to H_3(F/F_k)$, which is called the Morita homomorphism. Furthermore, Heap [Heap] showed the vanishing condition of $\tilde{\tau}_k$ using by (relative) bordism theory. The purpose of this subsection is to give a simpler proof of Heap (Theorem 4.6).

Let us review the map $\tilde{\tau}_k$. Fix $f \in \mathcal{T}(k)$. From Theorem 4.4, the relation $f_*(x_j)x_j^{-1}$ vanishes in the $k$-th nilpotent quotient of $\pi_1(T_f^\gamma)$. Thus, from (17), we have a canonical surjection,

$$\phi_{j,k}^\gamma : \pi_1(T_f^\gamma) \longrightarrow \pi_1(T_f^\gamma)/(\pi_1(T_f^\gamma))_k \cong F/F_k.$$

Let $[T_f^\gamma] \in H_3(T_f^\gamma) \cong \mathbb{Z}$ be the fundamental 3-class. Then, $\tilde{\tau}_k(f)$ is defined to be the push-forward $(\phi_{j,k}^\gamma)_*[T_f^\gamma] \in H_3(F/F_k)$. It is known [Mor, Heap] that this $\tilde{\tau}_k(f)$ depends only on $f \in \mathcal{T}(k)$ and that $\tilde{\tau}_k$ is a lift of the Johnson map $\tau_k$.

**Theorem 4.6** ([Heap, Theorem 5]). Let $f \in \mathcal{T}(k)$. Then, $\tilde{\tau}_k(f) = 0$ if and only if $f \in \mathcal{T}(2k-1)$.

**Proof.** First, let us make some observations. Notice from Theorems 4.4 and 4.5 that $f \in \mathcal{T}(k+1)$ if and only if the pairing $\langle x_{j_1}^*, \ldots, x_{j_k}^* \rangle, u_*(X_{j_1}) \rangle$ in (18) is zero for any standard indexes $j_1 \cdots j_k$. Furthermore, Theorem 3.2 (III) implies that $\tilde{\tau}_k(f)$ is zero if and only if the pairing

$$\langle s_{\ell}(x_{j_1}^*, \ldots, x_{j_\ell}^*) \rangle, (\phi_{j,k}^\gamma)_*[T_f^\gamma] \rangle$$ \hspace{1cm} (19)

is zero for any sequences $(s, j_1, \ldots, j_\ell)$ in (9), where we replace $\alpha_j$ by $x_j^*$.\hfill \text{End of Document}
Now we can complete the proof. Suppose \( f \in \mathcal{T}(2k - 1) \). Then, we can define the Johnson homomorphisms \( \tau_\ell \) for \( \ell \leq 2k - 2 \) and have \( \tau_\ell = 0 \). Hence, all of the pairing \([x^*_j, \ldots, x^*_k], \iota^*_s(\mathcal{X}_s)\) is zero. Note the cap product \([T^*_f] \cap \alpha_s = \iota^*_s(\mathcal{X}_s)\) because \( H^*(T^*_f; \mathbb{Z}) \cong H^*(S^1 \times \sum_{g \geq 0} \mathbb{Z}) \). Thus, the pairing in (18) is zero. Hence, \( \tilde{\tau}_k(f) = 0 \). Conversely, let us assume \( \tilde{\tau}_k(f) = 0 \). Then, all of the pairing (19) is zero. When \( \ell = k \), Corollary 3.3 implies that the pairing (18) is zero. From Theorems 4.4 and 4.5, we have studied the map. Especially, Massuyeau [Mas] showed an equivalence of the map \( \tilde{\tau} \), thus, we can define it.

First, let us focus on the group \( T_f \). The simplest case is when \( \ell = \); in the introduction, we regard the higher Massey products as an algorithm to produce cocycles. However, our situation is with integral coefficients; if we can determine the homeomorphism type of \( T_f \), we can hope to compute \( \tilde{\tau}_k(f) \) by using the 3-cocycles of Theorem 3.2 (III).

5 Expressions of some 3-cocycles

It is practically important to give explicit expressions of group cocycles. This section focuses on quotient groups of \( F/F_k \) and gives an algorithm to describe their 3-cocycles. As mentioned in the introduction, we regard the higher Massey products as an algorithm to produce cocycles.

5.1 3-cocycles of \( F/F_k \)

First, let us focus on the group \( F/F_k \) and give presentations of the 3-cocycles \( s_k(\alpha_s \sim \langle \alpha_{i_1}, \ldots, \alpha_{i_k} \rangle) \) in Theorem 3.2 when \( \ell = k \) and \( \ell = k + 1 \) with \( k \geq 3 \).

Using the notation in §3, given an index \( i_1 i_2 \cdots i_\ell \) and \( s \in \mathbb{N} \), let us define a map,

\[
\Gamma_{i_1 i_2 \cdots i_\ell} : (F/F_k)^3 \rightarrow \mathbb{Z}; \quad (x, y, z) \mapsto c_s(x) \left( \sum_{\ell \leq j \leq \ell - 1} c_{i_1 i_2 \cdots i_j} (y) c_{i_{j+1} \cdots i_\ell} (z) \right).
\]

The simplest case is when \( \ell = k \), in which the 3-cocycle \( s_k(\alpha_s \sim \langle \alpha_{i_1}, \ldots, \alpha_{i_k} \rangle) \) is exactly presented as \( \Gamma_{i_1 i_2 \cdots i_k} \), since \( s_k = \text{id} \). Next, to get presentations of \( s_k(\alpha_s \sim \langle \alpha_{i_1}, \ldots, \alpha_{i_\ell} \rangle) \) with \( k < \ell \leq 2k - 2 \), it is enough to explicitly give a function \( b : (F/F_k)^2 \rightarrow \mathbb{Z} \) such that the difference \( \Gamma_{i_1 i_2 \cdots i_\ell} - \partial_s^2 b \) is a restricted map \( (F/F_k)^2 \rightarrow \mathbb{Z} \).

For example, we can describe the case \( \ell = k + 1 \). We take the map \( b \) by setting

\[
b(x, y) = c_s(x) c_{i_1 \cdots i_{k+1}}(y) c_{i_{k+2}}(y) + c_{i_1}(x) c_{i_2 \cdots i_{k+2}}(y) + c_{i_{k+2}}(x) c_{i_1 \cdots i_{k+1}}(y).
\]

Then, as a result of the difference \( (\Gamma_{i_1 i_2 \cdots i_{k+1}} - \partial_s^2 b)(x, y, z) \), we obtain

**Proposition 5.1.** The cohomology 3-class \( \alpha_s \sim \langle \alpha_{i_1}, \ldots, \alpha_{i_k} \rangle \) is represented by \( \Gamma_{i_1 i_2 \cdots i_k} \).

When \( \ell = k + 1 \), the 3-class \( s_{k+1}(\alpha_s \sim \langle \alpha_{i_1}, \ldots, \alpha_{i_{k+1}} \rangle) \) is represented by the map:

\[
(x, y, z) \mapsto \sum_{\ell = 2}^k \left( c_s(x) (c_{i_1 \cdots i_\ell} (y) c_{i_{\ell+1} \cdots i_{k+1}} (z) - c_{i_1 \cdots i_{\ell-1}} (y) c_{i_\ell \cdots i_{k+1}} (z) (c_{i_{k+1}} (y) + c_{i_{k+1}} (z)))) - c_{i_1}(x) c_{i_2 \cdots i_\ell} (y) c_{i_{\ell+1} \cdots i_{k+1}} (z) - c_{i_{k+1}}(x) c_{i_2 \cdots i_{\ell-1}} (y) c_{i_{\ell} \cdots i_k} (z) \right).
\]

Since the length of every sequence in each term is less than \( k \), this map \( \Gamma_{i_1 i_2 \cdots i_{k+1}} - \partial_s^2 b \) can be regarded as a map from \( (F/F_k)^3 \).
Here, we should mention that while Igusa and Orr [IO, §10] express the 3-cocycles \( \alpha_s \sim \langle \alpha_i, \ldots, \alpha_k \rangle \) by using Igusa’s picture, our description using Massey products is simpler and compatible with the non-homogenous complex of \( G \).

Concerning the higher case \( \ell > k + 1 \), the author attempted describing the 3-cocycles, but made little progress.

### 5.2 Quotient groups by central elements

This subsection deals with the situation in [4.2] or [FS]. Namely, we fix central elements \( W_1, \ldots, W_{\ell} \in F_k/F_{k+1} \) and let \( G \) be the quotient group of \( F/F_{k+1} \) subject to \( W_1, \ldots, W_{\ell} \). For simplicity, let us assume \( k > 2 \) and that there are standard sequences \( I^{(j)} = (i_1^{(j)}, \ldots, i_k^{(j)}) \in \mathfrak{U}_k \) with \( j \leq s \), which are mutually distinct, and that \( W_j \) corresponds to the Massey product \( \langle \alpha_i^{(j)}, \ldots, \alpha_k^{(j)} \rangle \) of length \( k \). Such an assumption appears in discussions on higher Milnor invariants; see [Tu, KN].

Then, as in (5), we can easily check that the following map is well-defined and a 2-cocycle:

\[
\phi_{\Lambda_j} : G \times G \to \mathbb{Z}; \quad ([X], [Y]) \mapsto \sum_{\ell=1}^{k-1} c_{i_1^{(j)}, i_2^{(j)}, \ldots, i_{\ell}^{(j)}}(X) c_{i_{\ell+1}^{(j)}, \ldots, i_k^{(j)}}(Y),
\]

where we represent any element of \( G \) by a representative from \( F/F_k \). Here, we should mention the 5-term sequence from the central extension \( F/F_k \to G \):

\[
0 \to H^1(G; \mathbb{Z}) \xrightarrow{\sim} H^1(F/F_k; \mathbb{Z}) \to \mathbb{Z}^m \xrightarrow{\delta^*} H^2(G; \mathbb{Z}) \to H^2(F/F_k; \mathbb{Z}).
\]

Actually, we can verify that the 2-cocycle \( \phi_{\Lambda_j} \) corresponds to the image of \( \delta^* \).

Now let us give some 3-cocycles of \( G \).

**Proposition 5.2.** Let \( G \) be the group \( F/\langle F_k, W_1, \ldots, W_s \rangle \), as above. Fix \( r, s \in \{1, \ldots, q\} \) such that \( \langle ri_1^{(j)} \ldots i_k^{(j)} \rangle \) and \( \langle i_2^{(j)} \ldots i_k^{(j)} \rangle \) are different from other indexes \( i_1^{(j')}, \ldots, i_k^{(j')} \) for \( j' \leq s \). Then, the Massey product \( \langle \alpha_r, \phi_{\Lambda_j}, \alpha_s \rangle \) is defined and is represented by the map,

\[
(x, y, z) \mapsto c_r(x) \left( \sum_{\ell: 1 \leq \ell < k} c_{i_1^{(j)}, \ldots, i_{\ell}^{(j)}}(y) c_{i_{\ell+1}^{(j)}, \ldots, i_k^{(j)}}(z) \right) \quad \left( \sum_{\ell: 1 \leq \ell < k} c_{ri_1^{(j)}, \ldots, i_{\ell}^{(j)}}(x) c_{i_{\ell+1}^{(j)}, \ldots, i_k^{(j)}}(y) \right) c_s(z).
\]

**Proof.** Notice the equalities,

\[
\alpha_r \sim \phi_{\Lambda_j} = \partial^* \left( \sum_{\ell: 1 \leq \ell < k} c_{ri_1^{(j)}, \ldots, i_{\ell}^{(j)}}(x) c_{i_{\ell+1}^{(j)}, \ldots, i_k^{(j)}}(y) \right), \quad \phi_{\Lambda_j} \sim \alpha_s = \partial^* \left( \sum_{\ell: 1 \leq \ell < k} c_{i_1^{(j)}, \ldots, i_{\ell}^{(j)}}(x) c_{i_{\ell+1}^{(j)}, \ldots, i_k^{(j)}}(y) \right).
\]

Then, from the definition of the triple Massey product, we have a representative. \( \square \)

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### A Cocycles in deRham complexes of the iterated torus bundle

We will describe the cocycles in Theorem 3.2 as differential forms in deRham complexes.

First, when \( G = F/F_k \), we can verify, by induction on \( k \), that the Eilenberg-MacLane space \( B(F/F_k) \) is a \( C^\infty \)-manifold. Precisely, since \( B(F_k/F_{k+1}) \) is the product of \( N_k \)-copies of \( S^1 \), the geometric realization of (1) implies that \( B(F/F_{k+1}) \) is a universal torus bundle over \( B(F/F_k) \).
We will address the manifold structure of $B(F/F_k)$ in detail. The Heisenberg (triangular) group is a good toy model from the viewpoint of the unipotent Magnus expansion $\Upsilon_k$. Moreover, we should review shuffles and the image of the Magnus expansion $\mathcal{M}$. As in [CFL], a sequence $(c_1 c_2 \cdots c_k) \in \{1, \ldots, q\}^k$ is called the resulting shuffle of two sequences $I = a_1 a_2 \cdots a_{|I|}$ and $J = b_1 b_2 \cdots b_{|J|}$ if there are $|I|$ indices $\alpha(1), \alpha(2), \ldots, \alpha(|I|)$ and $|J|$ indices $\beta(1), \beta(2), \ldots, \beta(|J|)$ such that

\begin{enumerate}
\item $1 \leq \alpha(1) < \alpha(2) < \cdots < \alpha(|I|) \leq k$, and $1 \leq \beta(1) < \beta(2) < \cdots < \beta(|J|) \leq k$.
\item $c_{\alpha(i)} = a_i$ and $c_{\beta(j)} = b_j$ for some $i \in \{1, 2, \ldots, |I|\}$ $j \in \{1, 2, \ldots, |J|\}$.
\end{enumerate}

Let the symbol $\text{Sh}(I, J)$ denote the set of the resulting shuffles of $I$ and $J$. Thanks to [CFL] Theorem 3.9], the image of the Magnus expansion $\mathcal{M}$ completely characterizes “the shuffle relation”. In fact, the image in $\mathbb{Z} \langle X_1, \ldots, X_q \rangle / \mathcal{J}_k$ is realized by

$$\left\{ \sum_{I=(i_1 \cdots i_n)} a_I \cdot X_{i_1} \cdots X_{i_n} \mid \text{For any indexes } J \text{ and } K, \ a_J \cdot a_K = \sum_{L \in \text{Sh}(J,K)} a_L \right\}. \quad (20)$$

We further examine the $\mathbb{R}$-extension of the image and state Theorem A.1. Let us consider the real extension $\mathbb{R} \langle X_1, \ldots, X_q \rangle / \mathcal{J}_k = \mathbb{R} \otimes \mathbb{Z} \langle X_1, \ldots, X_q \rangle / \mathcal{J}_k$ and identify it with the Euclidean space of dimension $\sum_{\ell=0}^{k-1} q^\ell$. Since the shuffle relation is linear, the subset of $\mathbb{R} \langle X_1, \ldots, X_q \rangle / \mathcal{J}_k$ satisfying the shuffle relation (20) is a subspace, which is contractible. Denote the subspace by $\text{Im}(\mathcal{M}_\mathbb{R})$. Since $F/F_k$ freely acts on the subspace via the Magnus expansion, we have

**Theorem A.1.** The quotient space of $\text{Im}(\mathcal{M}_\mathbb{R})$ subject to the free action of $F/F_k$ is a closed connected $C^\infty$-manifold and is an Eilenberg-MacLane space of $F/F_k$.

In particular, every $s$-form of $BF/F_k$ is identified with an $F/F_k$-invariant $s$-form of $\text{Im}(\mathcal{M}_\mathbb{R})$. Thus, via the de Rham theorem, we will describe the basis of $H^*(F/F_k; \mathbb{R}) \cong H^*_{dR}(BF/F_k)$ as $F/F_k$-invariant $s$-forms of $\text{Im}(\mathcal{M}_\mathbb{R})$ as follows:

To state Lemmas A.2 and A.3 we need some terminology. Consider the cotangent bundle of $\mathbb{R} \langle X_1, \ldots, X_q \rangle / \mathcal{J}_k$, and denote by $dX_{j_1 \cdots j_t}$ the dual basis corresponding to the coordinate $X_{j_1} X_{j_2} \cdots X_{j_t}$. Following the pullback, we regard the basis as $1$-forms in $T^*\text{Im}(\mathcal{M}_\mathbb{R})$. Furthermore, for $s \in \{1, \ldots, q\}$, the $1$-form $dX_s$ on $\text{Im}(\mathcal{M}_\mathbb{R})$ is $F/F_k$-invariant, and the resulting $1$-cocycle in $\bigwedge^1 BF/F_k$ corresponds to the $s$-th summand of the abelianization $\alpha_s$. Extend the map $\beta_{i\alpha u} \cdots i \in (5)$ as $\mathbb{R} \langle X_1, \ldots, X_q \rangle / \mathcal{J}_k \to \mathbb{R}$. In addition, for $(t, k_0) \in \mathbb{N}^2$, we prepare a set of the form,

$$S_{t, k_0} := \{ (k_1, \ldots, k_u) \in \mathbb{N}^u \mid u \geq 1, \ k_0 + k_1 + \cdots + k_u = t \}.$$  

**Lemma A.2.** Fix an index $(j_1, \ldots, j_t) \in \{1, \ldots, q\}^t$. Define the $1$-form of the formula,

$$\sum_{k_0=1}^{t} \sum_{(k_1, \ldots, k_u) \in S_{t, k_0}} (-1)^u \prod_{1 \leq w \leq u} \beta_{j_1+k_0+k_1+\cdots+k_{w-1}, j_2+k_0+k_1+\cdots+k_{w-1}, \ldots, j_{k_0+k_1+\cdots+k_u}} dX_{j_1 \cdots j_{k_0}}.$$  

We denote this $1$-form by $\gamma_{j_1 \cdots j_t}$. This $1$-form is $F/F_k$-invariant.

In what follows, we denote the sum $k_0 + k_1 + \cdots + k_u$ by $p_u$ for brevity.
Proof. It is enough to show that, for any \( h \leq q \), the pullback \( \mathcal{M}(x_h)^*(\gamma_{j_1 \cdots j_t}) \) is \( \gamma_{j_1 \cdots j_t} \) itself. Notice, by definition, the following pullback formula:

\[
\mathcal{M}(x_h)^*(\beta_{j_1 \cdots j_k}) = \beta_{j_1 \cdots j_k} + \delta_{j_k,h} \beta_{j_1 \cdots j_{k-1}},
\]

\[
\mathcal{M}(x_h)^*(dX_{j_1 \cdots j_k}) = dX_{j_1 \cdots j_k} + \delta_{j_k,h} dX_{j_1 \cdots j_{k-1}}.
\]

Thus, the pullback \( \mathcal{M}(x_h)^*(\prod_{w=1}^{u} \beta_{j_1+p_{w-1} \cdots j_{p_w}} dX_{j_1 \cdots j_{k_0}}) \) is formed as

\[
( \prod_{w: 1 \leq w \leq u} \beta_{j_1+p_{w-1} \cdots j_{p_w}} + \delta_{h,j_w} \beta_{j_1+p_{w-1} \cdots j_{p_w}})(dX_{j_1 \cdots j_{k_0}} + \delta_{h,j_0} dX_{j_1 \cdots j_{k_0-1}}).
\]

Denote the coefficients of \( dX_{j_1 \cdots j_{k_0}} \) and of \( dX_{j_1 \cdots j_{k_0-1}} \) by \( A_{k_0} \) and \( B_{k_0} \), respectively. Then, by a careful observation, we can check that

\[
(-1)^u \sum_{(k_1, \ldots, k_u) \in \mathcal{S}_{t,k_0}} (A_{k_0} - B_{k_0+1}) = (-1)^u \sum_{(k_1, \ldots, k_u) \in \mathcal{S}_{t,k_0}} \beta_{j_1+p_0 \cdots j_{p_1}} \beta_{j_1+p_1 \cdots j_{p_2}} \cdots \beta_{j_1+p_{u-1} \cdots p_u}.
\]

Since the sum of the left hand side running over \( 1 \leq k_0 \leq t \) is equal to \( \mathcal{M}(x_h)^*(\gamma_{j_1 \cdots j_t}) \), we have the desired \( \mathcal{M}(x_h)^*(\gamma_{j_1 \cdots j_t}) = \gamma_{j_1 \cdots j_t} \).

Lemma A.3. Fix an index \((j_1, \ldots, j_k)\). Then, for any \( s < t \) with \((s, t) \neq (1, k)\), we have

\[
d\gamma_{j_s \cdots j_t} = \sum_{r: s \leq r \leq t-1} \gamma_{j_s \cdots j_r} \wedge \gamma_{j_{r+1} \cdots j_t} \in \wedge^2 \text{Im}(\mathcal{M}_R).
\]

Proof. First, by the Leibniz rule, the left hand side \( d\gamma_{j_s \cdots j_t} \) is expressed as

\[
\sum_{k_0=s}^{t} \left( \sum_{(k_1, \ldots, k_u) \in \mathcal{S}_{r,k_0}} (-1)^u \beta_{j_1+p_0 \cdots j_{p_1}} \cdots \beta_{j_1+p_{u-1} \cdots j_{p_u}} dX_{j_1+p_0 \cdots j_{p_u}} \wedge dX_{j_{u+1} \cdots j_{k_0}} \right).
\]

Here, the check \( \beta_{j_{p_0+1} \cdots j_{p_u+1}} \) means the elimination of the term \( \beta_{j_{p_0+1} \cdots j_{p_u+1}} \). On the other hand, the right hand side becomes

\[
\sum_{r: s \leq r < t} \left( \sum_{k_0': r+1 \leq k_0' \leq t} (-1)^u \beta_{j_1+k_0' \cdots j_{p_1}} \cdots \beta_{j_1+k_{u-1}' \cdots j_{p_u}} dX_{j_{u+1} \cdots j_{k_0}} \right).
\]

By replacing \( r \) by \( k_0, k_0' \) by \( k_{u+1} \) and \( k_u' \) by \( k_{u+1} \), a careful comparison deduces that this sum equal to the preceding expansion of the left hand side.

This situation is the same as the defining system, as mentioned in [3]. Note that the deRham theorem preserves the cup product. Thus, in parallel to the main theorem 3.2, we readily obtain the basis of the 2-cocycle of \( H^2_{\text{dR}}(BF/F_k) \) as follows.

Theorem A.4. The second cohomology \( H^2_{\text{dR}}(BF/F_k) \cong \mathbb{R}^N_k \) is spanned by the Massey products \( \langle \alpha_{i_1}, \ldots, \alpha_{i_k} \rangle \) running over standard sequences \( (i_1 \cdots i_k) \in \mathcal{U}_k \). Here, \( \langle \alpha_{i_1}, \ldots, \alpha_{i_k} \rangle \) is represented by the 2-form,

\[
\sum_{k_0=1}^{k} \left( \sum_{(k_1, \ldots, k_u) \in \mathcal{S}_{k,k_0}} (-1)^{k_0} \beta_{j_1+k_0 \cdots j_{p_1}} \cdots \beta_{j_1+p_0 \cdots j_{p_u} \cdots j_{p_u}} dX_{j_{1+p_0} \cdots j_{1+p_u}} \wedge dX_{j_{1 \cdots j_{k_0}}} \right).
\]
Proof. We will only compute the sum $\sum_{r=1}^t \gamma_{j_1\cdots j_r} \wedge \gamma_{j_{r+1}\cdots j_t}$ from the definition of the Massey product. Here, we should remark that the sum is $F/F_k$-invariant by Lemma A.2. \qed

As a result for 3-cocycles in $H^3_{dr}(B(F/F_k))$, Theorem 3.2 (III) and Proposition 5.1 enable us to similarly describe 3-cocycles $\alpha_\ell(dX_j \wedge (\alpha_{i_1}, \ldots, \alpha_{i_\ell}))$ as 3-forms, where $\ell = k, k+1$. Day [Day] consider an extension of the Morita homomorphism from differential 3-forms of $B(F/F_k)$; thus, it seems interesting to view the work of Dat from Theorem A.4.

Finally, we conclude this appendix by describing some examples with $t \leq 4$.

**Example A.5.** (i) The 1-form $\gamma_{ab}$ is $dX_{ab} - \beta_a dX_{b}$. Hence, the Massey product $\langle \alpha_a, \alpha_b, \alpha_c \rangle = \gamma_{ab} \wedge \gamma_c + \gamma_a \wedge \gamma_{bc}$ is expressed as $dX_a \wedge dX_{bc} + dX_{ab} \wedge dX_c - \beta_a dX_b dX_c - \beta_b dX_a dX_c$.

(ii) Next, when $t = 3$, the 1-form $\gamma_{abc}$ is $dX_{abc} - \beta_c dX_{ab} - \beta_b \beta_c dX_a + \beta_{bc} dX_a$. Hence, the Massey product $\langle \alpha_a, \alpha_b, \alpha_c, \alpha_d \rangle$ is formulated as
\[
(dX_{abc} - \beta_c dX_{ab} - \beta_b \beta_c dX_a + \beta_{bc} dX_a) \wedge dX_d + (dX_{ab} - \beta_a dX_b) \wedge (dX_{cd} - \beta_c dX_d) + \beta_a \wedge (dX_{bcd} - \beta_d dX_{bc} - \beta_c \beta_d dX_b + \beta_{cd} dX_b).
\]

(iii) Next, when $t = 4$ and $(j_1, j_2, j_3, j_4) = (a, b, c, d)$, the 1-form $\gamma_{abcd}$ is
\[
dX_{abcd} - \beta_d dX_{abc} + \beta_c \beta_d dX_a + \beta_{cd} dX_{ab} - \beta_{bc} \beta_d dX_a - \beta_b \beta_{cd} dX_a - \beta_b \beta_{cd} dX_a.
\]

Then, $\langle \alpha_a, \alpha_b, \alpha_c, \alpha_d, \alpha_e \rangle$ can be similarly computed as $\gamma_{abcd} \gamma_{c} + \gamma_{abc} \gamma_{de} + \gamma_{ab} \gamma_{cde} + \gamma_{a} \gamma_{bcde}$.

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