Spinning and Spinning Deviation Equations of Bi-metric Type Theories

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Abstract

Spinning equations of bi-metric types theories of gravity, the counterpart of the Papapetrou spinning equations of motion have been derived as well as their corresponding spinning deviation equations. Due to introducing different types of bi-metric theories, the influence of different curvatures based upon different affine connections, have been examined. A specific Lagrangian function for each type theory has been proposed, in order to derive the set of spinning motions and their corresponding spinning deviation equations.

1 Introduction

Bi-metric theories of gravity are considered promising theories of gravity in strong fields. These equations were existed in different stages during the last century and the current one. In 1940 Rosen introduced such a challenging gravitational theory, wider than the orthodox general theory of relativity. The Theory is called the bi-metric theory of gravity of the core of the theory, based on considering any point in the manifold is identified by two reference frames, the first is a gravitational one described in a curved space; while the second is an inertial one, defined in a flat space [1] and [2]. Subsequently, Rosen developed its corresponding geodesic equations [3]. Also, Israelit (1975) [4] solved these equations of motion for test particle, while Falikand Rosen (1981) extended this study to examine the motion of charged particles [5].

Yet, the concept of imposing two metrics, has inspired Moffat [6] to present another version of bi-metric theory of gravity, based on regarding one combined metric produced of these two previous ones. His theory has is based on considering variable speed of light (VSL), as an alternative to dismiss dark energy as mentioned in different theories of gravity [7].

Consequently, the concept of bi-metric theory has been extended to define a bi-metric paradigm of Modified Newtonian gravity MOND by Milgrom to find an alternative remedy

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to apparent constant speed of rotation curves in spiral galaxies instead of proposing dark matter particles [8-10]. Such a treatment is called Bimetric Modified dynamics BIMOND [11].

Nevertheless, another wave of developing bi-metric theories of gravity was established in 2012 by Hassan and Rosen [12] to consider bigravity theory, in this case, there are two types of matter produced from two parallel field equations one is the normal matter, and the other is called twin matter. Due to introducing this concept of bi-gravity, the theory has to regarded as a ghost free one [13]. Arkami et al (2014) have performed their corresponding path equations [14].

Moreover, an alternative version of bi-metric theory of gravity has been performed by Verozub based geodesic mappings on to obtain a gravitational theory, invariant under gauge transformations [15-16], to examine the behavior very strong fields, based on combining the two bi-gravity metrics into one able to examine the stability of super-massive objects in strong gravitational fields such as the active galactic nuclei [17].

The aim of our present work is to extend our study which was presented to derive the corresponding path and path deviation equations from the above mentioned set of bi-metric theories of gravity Kahil(2017) [18]. Accordingly, it has been essential to find the analogous parts of Papapetrou equations [19] in bi-metric theory as if it was performed in Einstein-Cartan theories [20] and a specific class of Non-Riemannian geometry called absolute parallelism (AP)-geometry [21].

From this perspective, we are going to derive the set of spinning objects with precession in the presence of strong gravitational field, for different versions of the above mentioned type theories. The reason for studying spinning equation is due to its actual existence rather than focusing hypothetical existence of a test particle of discarded its intrinsic properties. This makes our work more feasible to examine the behavior of particles in different versions of bi-metric theory of gravity. This study may be considered in order to set a scheme of examining the stability of these spinning objects in strong gravitational fields in our future work.

The paper is organized as follows, Section 1 Showing the transformation from geodesic into a spinning path for short, Section 2 deriving of the spinning and spinning deviations for Rosen’s bi-metric theory, Section 3 obtaining the corresponding spinning and spinning deviation equations for Moffat’s of variable speed of light, Section 4 presenting the proposed spinning and spinning deviation equations that maybe existed on dealing with BIMOND paradigm, Section 5 obtaining the associated versions of spinning and spinning deviation equations for separate and combined metrics in Bi-gravity models. Section 6 displaying the spinning and spinning deviation equations for the Verozub bi-metric theory of geodesic-invariant. Section 7, implementing a schematic overview on the impact of bi-metric effects on various sets of spinning and spinning deviation equations comparing with their counterparts in general relativity.
2 Transformation From Test Particles to Spinning Objects

Equations of motion for a spinning objects may be derived in twofold, one of them from geodesic equations as being a deviated path from geodesic satisfying the following relation [23]:

\[ V^\alpha = U^\alpha + \beta \frac{D\Psi^\alpha}{DS} \]  

where \( V^\alpha = \frac{dx^\alpha}{d\tau} \) is the unit tangent vector associated to path \( \tau \) and \( U^\alpha \) is a unit tangent vector of a geodesic defined with the parameter \( S \). \( \beta \) is an arbitrary parameter and \( \Psi^\alpha \) is its corresponding deviation vector and \( \frac{D}{DS} \) is the covariant derivative with respect to parameter \( S \).

By taking the covariant derivative on both sides, and proposing the following relation

\[ S^{\alpha\beta} = S(U^\alpha\Psi^\beta - U^\beta\Psi^\alpha) \]  

where \( S \) is the magnitude of a spin, and \( S^{\alpha\beta} \) is the spin tensor.

Taking into account, \( \beta = \frac{S}{m} \), \( m \) is mass of the spinning object, such that the geodesic equation,

\[ \frac{DU^\alpha}{DS} = 0, \]  

and its corresponding geodesic deviation equation,

\[ \frac{D^2\Psi^\alpha}{DS^2} = R_{\rho\sigma}^\alpha U^\rho U^\sigma \Psi^\delta, \]  

where, \( R_{\rho\sigma}^\alpha \) is the Riemann curvature tensor, and

\[ \frac{dS}{d\tau} = 1. \]

Thus, equation (1) becomes, after simple calculations, Papapertrou equation for short [22],

\[ \frac{DV^\alpha}{D\tau} = \frac{1}{2m} R_{\rho\sigma}^\alpha S^{\alpha\delta} V^\rho \]  

from this perspective, it can be found that this type of formulation is feasible for deriving spinning objects with no precession with their intrinsic properties. However, the process of deriving a generalized method able to obtain the translational and rotational equations need to propose a rival method based applying the action principle on a specific Lagrangian as shown in the following subsection.

The Papapertrou Equation in General Relativity: Lagrangian Formalism

It is well known that equation of spinning objects in the presence of gravitational field have been studied extensively. This led us to suggest its corresponding Lagrangian formalism,
using a modified Bazanski Lagrangian [24], for a spinning and precessing object and their corresponding deviation equation in Riemannian geometry in the following way [22]

\[ L = g_{\alpha\beta} P^\alpha \frac{D\Psi^\beta}{DS} + S_{\alpha\beta} \frac{D\Psi^\alpha}{DS} + \frac{1}{2} R_{\alpha\nu\rho\delta} S^{\rho\delta} U^\nu \Psi^\alpha + 2 P_{[\alpha} U_{\beta]} \Psi^{\alpha\beta} \]  

(6)

where \( P^\mu \) is the momentum vector, and \( \Psi^{\mu\nu} \) is the spin deviation tensor, in which,

\[ P^\alpha = m U^\alpha + U_\beta \frac{DS^{\alpha\beta}}{DS}. \]

Thus, applying the Euler-Lagrange equations

\[ \left( \frac{d}{dS} \frac{\partial L}{\partial \dot{\Psi}^\mu} - \frac{\partial L}{\partial \Psi^\mu} \right) = 0, \]  

(7)

and

\[ \left( \frac{d}{dS} \frac{\partial L}{\partial \dot{\Psi}^{\mu\nu}} - \frac{\partial L}{\partial \Psi^{\mu\nu}} \right) = 0, \]  

(8)

to obtain the set of spinning for the spinning object,

\[ \frac{DP^\mu}{DS} = \frac{1}{2} R^{\mu\nu\rho\delta} S_{\rho\delta} U^\nu, \]  

(9)

and,

\[ \frac{DS^{\mu\nu}}{DS} = 2 P^{[\mu} U^{\nu]}. \]  

(10)

Also, applying the following identity on both equations (9) and (10), to obtain the set of equations may be derived their corresponding deviation equations, using the following identity [25].

\[ A^\mu_{\nu\rho} - A^\mu_{\rho\nu} = R^\mu_{\beta\nu\rho} A^\beta, \]  

(11)

and

\[ A^{\mu\nu}_{\nu\rho} - A^{\mu\nu}_{\rho\nu} = A^{[\mu} R^{|\nu]}_{\beta\nu\rho}, \]  

(12)

where \( A^\mu \) and \( A^{\mu\nu} \) are both arbitrary vector and tensor respectively.

Multiplying both sides with arbitrary vectors, \( U^\rho \Psi^\nu \) as well as using the following condition .

\[ U^\alpha_{\nu} \Psi^\rho = \Psi^\alpha_{\nu} U^\rho, \]  

(13)

and \( \Psi^\alpha \) is its deviation vector associated to the unit vector tangent \( U^\alpha \). Also in a similar way:

\[ S^{\alpha\beta}_{\nu\rho} \Psi^\rho = \Psi^{\alpha\beta}_{\nu\rho} U^\rho, \]  

(14)

one obtains the corresponding deviation equations [26]

\[ \frac{D^2 \Psi^\mu}{DS^2} = R^{\mu\nu\rho\sigma}_{\nu\rho\sigma} P^{\nu\rho} U^\sigma \Psi^\sigma + \frac{1}{2} (R^{\mu\nu\rho\delta} S^{\rho\delta} U^\nu)_{\nu\rho} \Psi^\rho, \]  

(15)

and

\[ \frac{D^2 \Psi^{\mu\nu}}{DS^2} = S^{[\rho\mu}_{\nu\rho\sigma} U^\sigma \Psi^\nu + 2 (P^{[\mu} U^{\nu]})_{\nu\rho} \Psi^\rho. \]  

(16)
3 Spin and Spin deviation equations of Bimetric Theories

In this approach we are going to obtain the set of spinning objects as an extension to their counterparts of general relativity and alternative theories of gravity in bi-metric type theories for a test particle.

3.1 Spinning Equation and Spin Deviation of Rosen’s Approach

(i) Case \( P = mU^{\alpha} \)

In this approach, we are corresponding spinning equations for short, that are obtained from the following Lagrangian:

\[
L = (g_{\alpha\beta} - \gamma_{\alpha\beta})U^{\alpha} \nabla_{\alpha} \Psi^{\beta} + S_{\mu\nu} \nabla_{\mu} \Psi^{\nu} + \frac{1}{2m}(R_{\alpha\beta\gamma\delta}U^{\alpha} \Psi^{\beta} S^{\gamma\delta})
\]  

(17)

where \( g_{\mu\nu} \) is the metric tensor of the curved space and \( \gamma_{\mu\nu} \) the corresponding metric tensor of the flat space.

where, \( \nabla_{\alpha} \Psi^{\beta} \) a specific covariant derivative defined as follows [27]:

\[
\frac{\nabla A^\mu}{\nabla S} = \frac{dA^\mu}{dS'} + \Delta^\mu_{\nu\rho} A^\nu U^\rho
\]

such that

\[
\Delta^\mu_{\nu\rho} = \Gamma^\mu_{\nu\rho} - \hat{\Gamma}^\mu_{\nu\rho}
\]

where \( \Gamma^\mu_{\nu\rho} \) the affine connection of curved space and \( \hat{\Gamma}^\mu_{\nu\rho} \) is the affine connection of the flat space. Using the Bazanski approach [23] to obtain its path equation by taking the variation with respect to \( \Psi^{\alpha} \) and \( \Psi^{\alpha\beta} \) respectively.

\[
\nabla U^\mu_{\nabla S} = \frac{1}{2m} R_{\mu\nu\rho\sigma}^{\alpha} S^\nu U^\mu
\]

(18)

and

\[
\nabla S^{\alpha\beta}_{\nabla S} = 0
\]

(19)

Applying the law of commutation relations (11), (12), (13) and (14) we find their corresponding set of deviation equation to become

\[
\frac{\nabla \Psi^{\alpha}}{\nabla S} = R_{\mu\nu\rho}^{\alpha} U^{\mu} U^{\nu} \Psi^{\rho} + \frac{1}{2m} (R_{\mu\nu\rho}^{\alpha} S^{\mu\nu} U^{\rho})_{\mid \Psi^{\rho}}
\]

(20)

where \( ; \) is the covariant for curved space.

and

\[
\frac{\nabla \Psi^{\alpha\beta}}{\nabla S} = S^{\alpha\beta}_{\mid \Psi^{\rho}} R_{\mu\rho}^{\beta} U^{\mu} \Psi^{\rho}
\]

(21)

The above set of deviation equation behaves identically as their counterparts in general relativity. where \( ; \) is the covariant and absolute derivatives for curved space.

(ii) Case \( P \neq mU^{\alpha} \)
In this approach, we are corresponding spinning equations whose momentum is \( P^\mu \), that are obtained from the following Lagrangian:

\[
L = (g_{\alpha\beta} - \gamma_{\alpha\beta}) \frac{\nabla \Psi^\beta}{\nabla S} + S_{\mu\nu} \frac{\nabla \Psi^{\mu\nu}}{\nabla S} + \frac{1}{2m} (R_{\alpha\beta\gamma\sigma} U^\alpha \Psi^\beta S^{\gamma\sigma} + 2P_{[\mu \nu]} \Psi^{\mu\nu}) \tag{22}
\]

where \( g_{\mu\nu} \) is the metric tensor of the curved space and \( \gamma_{\mu\nu} \) the corresponding metric tensor of the flat space

\[
P^\mu = (mU^\mu + U_\nu \frac{\nabla S^{\mu\nu}}{\nabla S})
\]

The analogous momentum described as by Papapetrou [16].

Thus, we can apply the Bazanski approach to obtain its path equation by taking the variation with respect to \( \Psi^\alpha \) and \( \Psi^{\alpha\beta} \) respectively [24]

\[
\nabla_P^\mu \frac{\nabla S}{\nabla} = \frac{1}{2} R_{\mu
u\rho\sigma} S^{\rho\sigma} U^\mu U^\nu \tag{23}
\]

and

\[
\nabla S^{\alpha\beta} \frac{\nabla S}{\nabla} = 2P^{[\alpha U^{\beta}]} \tag{24}
\]

where \( | \) is the associated covariant derivative flat space.

From the previous equation, we find a new effect of covariant derivative for flat spaces appears even if its associated curvature is zero. This gives the spinning deviation equations are quite different than their counterpart of general relativity [25].

Applying the law of commutation relation as shown in equations (11), (12), (13) and (14), we find their corresponding set of deviation equation to become

\[
\nabla \Psi^\alpha \frac{\nabla S}{\nabla} = R_{\mu
u\rho\sigma}^\alpha P^\mu U^\nu \Psi^\rho + \frac{1}{2m} [ (R_{\mu
u\rho\sigma}^\alpha S^{\rho\sigma} U^\mu U^\nu)_{,\rho} + (R_{\mu
u\rho\sigma}^\alpha S^{\rho\sigma} U^\mu U^\nu)_{,\rho}] \Psi^\rho \tag{25}
\]

and

\[
\nabla \Psi^{\alpha\beta} \frac{\nabla S}{\nabla} = S^{[\alpha \delta} R_{\delta\mu\rho\sigma}^\beta U^\mu \Psi^\rho + 2(P^{[\alpha U^{\beta}]} + P^{[\alpha U^{\beta}]} |_{\rho} \Psi^\rho). \tag{26}
\]

Comparing (20) and (21) with and (25) and (26), we find out that the effect of different covariant derivatives appear effective, if the object is regarded its intrinsic properties on the spinning deviation equations.

### 3.2 Spin and Spin Deviation Equations of Moffat’s Approach

Moffat [6] presented the framework of variable speed of light VSL satisfying bimetric theory and its causality to reveal the problem of dark energy due to VSL by introducing such a metric in the following way [28].

\[
\hat{g}_{\mu\nu} = g_{\mu\nu} + B \partial_\mu \phi \partial_\nu \phi \tag{27}
\]

While the inverse metrics \( g^{\mu\nu} \) satisfies that

\[
\hat{g}^{\mu\nu} = g^{\mu\nu} + \frac{B}{K} \delta^{\mu}_{\nu} \phi^{\nu} + KB \sqrt{T_{\mu\nu}}, \tag{28}
\]
where $\hat{g}_{\mu\nu}$ defines a specific matter metric tensor of a given matter field, $B$, $\phi$ is a bi-scalar field, $K$ is an arbitrary constant and $T_{\mu\nu}$ is a given energy-momentum tensor [29].

i Case $P^\mu = m U^\mu$

The appropriate Lagrangian which can work for obtaining path and path deviations of the set of equations of motion are suggested as follows:

Accordingly, we suggest its corresponding Lagrangian to obtain the sets of equations for spinning and spinning deviation.

$$\hat{L} = \hat{g}_{\mu\nu} U^\mu \frac{DU^\nu}{DS} + S_{\mu\nu} \frac{D\Psi_{\mu\nu}}{DS} + \frac{1}{2m} \hat{R}_{\alpha\beta\gamma\delta} S^{\gamma\delta} U^\beta \Psi^\alpha$$  \hfill (29)

where

$$\hat{R}_{\beta\gamma\delta} = \hat{\Gamma}_{\beta\delta,\gamma} - \hat{\Gamma}_{\beta\gamma,\delta} + \hat{\Gamma}_{\beta\gamma} \hat{\Gamma}_{\delta,\phi} - \hat{\Gamma}_{\beta\delta} \hat{\Gamma}_{\gamma,\phi}$$  \hfill (30)

For deriving the spinning equation is obtained by taking the variation with respect to $\Psi^\alpha$ and $\Psi^{\alpha\beta}$ on (29) to become

$$\frac{DU^\alpha}{DS} = \frac{1}{2m} \hat{R}_{\beta\gamma\delta} S^{\gamma\delta} U^\beta$$  \hfill (31)

and

$$\frac{DS^{\alpha\beta}}{DS} = 0.$$  \hfill (32)

Similarly we can obtain their corresponding deviation equation, using the commutation relations as shown in the (11), (12),(13) and (14) to get

$$\frac{D^2\Psi^\alpha}{DS^2} = \hat{R}_{\beta\gamma\delta} U^\gamma U^\beta \Psi^\delta + \frac{1}{2m} (\hat{R}_{\beta\gamma\delta} S^{\gamma\delta} U^\beta)_{\gamma\delta} \Psi^\rho$$,  \hfill (33)

and

$$\frac{D^2\Psi^{\mu\nu}}{DS^2} = S^{(\alpha\delta} \hat{R}_{\delta\gamma\rho)} U^\gamma \Psi^\rho.$$  \hfill (34)

ii Case $P \neq m U$

The appropriate Lagrangian which can work for obtaining path and path deviations of the set of equations of motion are suggested as follows:

$$\hat{L} = \hat{g}_{\alpha\beta} P^\alpha \frac{D\Psi^\beta}{DS} + S_{\mu\nu} \frac{D\Psi^{\mu\nu}}{DS} + \frac{1}{2} \hat{R}_{\alpha\beta\gamma\delta} U^\alpha \Psi^\beta S^{\gamma\sigma} + 2 P_{\mu\nu} U^\mu U^\nu$$  \hfill (35)

Thus, taking the variation with respect to $\Psi^\delta$ and $\Psi^{\delta\sigma}$ respectively to get

$$\frac{DP^\alpha}{DS} = \frac{1}{2} \hat{R}_{\alpha\mu\nu\rho} S^{\mu\nu} U^\mu U^\nu$$  \hfill (36)

and

$$\frac{DS^{\alpha\beta}}{DS} = 2 P^{[\alpha U^\beta]}.$$  \hfill (37)
In this case, it can be found that the problem of obtaining the corresponding deviation equation using the commutation relations as shown in the (11), (12), (13) and (14)1:

\[ \frac{\hat{D}^2 \Psi^\alpha}{DS^2} = \hat{R}_{\mu\nu\rho} P^\mu U^\nu \Psi^\rho + \frac{1}{2} (\hat{R}_{\mu\nu\rho} S^\nu U^\mu)_{,\sigma} \Psi^\sigma \]  

(38)

and

\[ \frac{\hat{D}^2 \Psi^{\alpha\beta}}{DS^2} = (S^{[\alpha\delta} \hat{R}_{\mu\nu\delta]} \Psi^\mu U^\nu + 2 (P^{[\alpha} U^{\beta]})_{,\rho} \Psi^\rho \]  

(39)

The above equations are similar to their counterparts of general relativity due to combining the two metric tensor into one metric.

### 3.3 Spin and Spin deviation Equations of BIMOND Type Theories

In this section, we present the equations for spinning objects in the presence of bi-metric modified Newtonian dynamics paradigm [9]. This type of theories is the bimetric version of Milgram’s Modified Newtonian Dynamics, which gives a the relationship between the two affine connections defined by \( g_{\mu\nu} \) and \( \gamma_{\mu\nu} \) respectively to become:

\[ C^\alpha_{\beta\rho} = \Gamma^\alpha_{\beta\rho} - \bar{\Gamma}^\alpha_{\beta\rho}, \]  

(40)

such that

\[ g_{\mu\nu,\rho} = g_{\delta\nu} C^\delta_{\mu\rho} + g_{\delta\mu} C^\delta_{\nu\rho}, \]

and

\[ \gamma_{\mu\nu,\rho} = -\gamma_{\delta\nu} C^\delta_{\mu\rho} - \gamma_{\delta\mu} C^\delta_{\nu\rho}. \]

Thus, we suggest the following Lagrangian spinning path and spinning deviation equations for the following cases:

i. \( P^\alpha = m U^\alpha \)

\[ L = \hat{g}_{\mu\nu} U^\mu \frac{\hat{D}^2 \Psi^\nu}{DS^2} + S_{\mu\nu} \frac{\hat{D}^2 \Psi^{\mu\nu}}{DS^2} + \frac{1}{2m} (R_{\alpha,\mu\nu\rho} - \bar{R}_{\alpha,\mu\nu\rho}) S^{\nu\rho} U^\mu \Psi^\alpha \]  

(41)

Taking the variation with respect \( \Psi^\alpha \) and \( \Psi^{\alpha\beta} \) we obtain

\[ \frac{\hat{D} U^\alpha}{DS} = \frac{1}{2m} (R^\alpha_{\mu\nu\rho} - \bar{R}^\alpha_{\mu\nu\rho}) S^{\rho\nu} U^\mu \]  

(42)

and

\[ \frac{\hat{D} S^{\alpha\beta}}{DS} = 0 \]  

(43)

While their corresponding set of deviation equations may be derived using the commutation relations (11), (12), (13) and (14), to become
\[
\bar{D}^2 \Psi^\alpha = (R^\alpha_{\mu\nu\rho} - \bar{R}^\alpha_{\mu\nu\rho})U^\mu U^\nu \Psi^\rho + \frac{1}{2m} ([R^\alpha_{\mu\nu\rho} S^\rho U^\mu]_\delta - [\bar{R}^\alpha_{\mu\nu\rho} S^\rho U^\mu]_\delta) \Psi^\delta \tag{44}
\]

and

\[
\bar{D}^2 \Psi^{\alpha\beta} = (S^{[\alpha\delta} R^{\beta]_{\delta\rho\sigma}} - S^{[\alpha\delta} \bar{R}^{\beta]_{\delta\rho\sigma}}) U^\alpha \Psi^\rho \tag{45}
\]

\[\text{ii. } P^\alpha \neq m U^\alpha\]

\[
L = \tilde{g}_{\mu\nu} P^\mu \bar{D}^2 \Psi^\nu + S_{\mu\nu} \bar{D}^2 \Psi^{\mu\nu} + \frac{1}{2} (R_{\alpha,\nu\rho} - \bar{R}_{\alpha,\nu\rho}) S^\rho U^\alpha + 2 P_{[\mu} U_{\nu]} \Psi^{\mu\nu} \tag{46}
\]

Taking the variation with respect \(\Psi^\alpha\) and \(\Psi^{\alpha\beta}\) we obtain

\[
\frac{\bar{D} U^\alpha}{DS} = \frac{1}{2} (R^\alpha_{\mu\nu\rho} - \bar{R}^\alpha_{\mu\nu\rho}) S^\rho U^\mu \tag{47}
\]

and

\[
\frac{\bar{D} S^{\alpha\beta}}{DS} = 2 P^{[\alpha U_{\beta]} \tag{48}
\]

While the set of deviation equations may be derived using the commutation relations as expressed in \((11), (12),(13)\) and \((14)\) to become

\[
\frac{\bar{D}^2 \Psi^\alpha}{DS^2} = (R^\alpha_{\mu\nu\rho} - \bar{R}^\alpha_{\mu\nu\rho}) P^\mu U^\nu \Psi^\rho + \frac{1}{2} ([R^\alpha_{\mu\nu\rho} S^\rho U^\mu]_\delta - [\bar{R}^\alpha_{\mu\nu\rho} S^\rho U^\mu]_\delta) \Psi^\delta \tag{49}
\]

and

\[
\frac{\bar{D}^2 \Psi^{\alpha\beta}}{DS^2} = (S^{[\alpha\delta} R^{\beta]_{\delta\rho\sigma}} - S^{[\alpha\delta} \bar{R}^{\beta]_{\delta\rho\sigma}}) U^\alpha \Psi^\rho + 2 (P^{[\alpha U_{\beta]}})_{\rho} \Psi^\rho + 2 (P^{[\alpha U_{\beta]}})_{\rho} \Psi^\rho \tag{50}
\]

Equations (49) and (50) have shown that the effect of two different covariant derivatives is regarded for an object regarding its intrinsic properties. Such a relationship makes, the bi-metric theory different the conventional general relativity.

### 3.4 Generalized Spin and Spin Deviation Equations of Bi-metric Theories

Hossenfelder [30] has introduced an alternative version of bi-metric theory, having two different metrics \(g\) and \(h\) of Lorentzian signature on a manifold \(M\) one is defined in tangential space \(TM\) and the other is in its co-tangential space \(T^*M\) respectively. These can be regarded as two sorts of matter and twin matter, existing individually, each of them has its own field equations as defined within Riemannian geometry.

\[
dS^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{51}
\]

and

\[
d\tau^2 = h_{\mu\nu} dx^\mu dx^\nu \tag{52}
\]
In this part we are going to present a generalized form which can be present different types of path and path deviation which can be explained for any bimetric theory which has two different metrics and curvatures as defined by Riemannian geometry.

Such an approach may work to derive the spinning and spinning equations as well as their corresponding deviation equations for Hassan-Rosen bigravity theory. From this point of view, it is worth mentioning to suggest corresponding Lagrangian in the following way:

Accordingly, we suggest a Lagrangian able to describe two independent sets of a generalized spinning and spinning deviation equations, after applying a specific action principle, with taking into considerations new additive terms: twin matter \( \bar{m} \), the twin momentum \( \bar{P}_\alpha \), the twin unit tangent vector \( V^\alpha \), the twin deviation a vector \( \Phi^\alpha \), the twin spinning tensor \( \bar{S}^{\alpha\beta} \) and the spinning deviation tensor \( \Phi^{\alpha\beta} \), provided that \( \frac{d\tau}{dS} = 0 \)

i. Case \( P^\mu = mU^\mu \) and \( \bar{P}^\mu = \bar{m}V^\mu \)

\[
L = g_{\mu\nu}U^\mu\Psi_{\alpha\beta}U^\nu + \tilde{h}_{\mu\nu}V^\mu\Phi_{\alpha\beta}V^\nu + S_{\mu\rho}\Psi^{\mu\rho}U^\rho + \tilde{S}_{\mu\rho}\Phi^{\mu\rho}V^\rho + \frac{1}{2m}R_{\alpha\beta\gamma\delta}U^\beta S^{\gamma\delta}\Psi^\alpha + \frac{1}{2\bar{m}}S_{\alpha\beta\gamma\delta}V^\beta \tilde{S}^{\gamma\delta}\Phi^\alpha, \tag{53}
\]

Taking the variation with respect to \( \Psi^\alpha \) and \( \Phi^\alpha \) to get

\[
\frac{DU^\alpha}{DS} = \frac{1}{2m}R_{\beta\gamma\delta}^\alpha U^\beta S^{\gamma\delta}, \tag{54}
\]

and

\[
\frac{DV^\alpha}{D\tau} = \frac{1}{2\bar{m}}S_{\beta\gamma\delta}^\alpha V^\beta \tilde{S}^{\gamma\delta}, \tag{55}
\]

Also, taking the variation with respect to \( \Psi^{\alpha\beta} \) and \( \Phi^{\alpha\beta} \) to get

\[
\frac{DS^{\alpha\beta}}{DS} = 0, \tag{56}
\]

and

\[
\frac{D\tilde{S}^\alpha}{D\tau} = 0 \tag{57}
\]

and their corresponding Spin deviation equations are obtained using the commutation relations (11),(12, (13) and (14) to become:

\[
\frac{D^2\Psi^\alpha}{DS^2} = R_{\beta\gamma\delta}^\alpha U^\gamma U^\delta \Psi^\beta + \frac{1}{2m}(R_{\beta\gamma\delta}^\alpha U^\beta S^{\gamma\delta})_{;\delta} \Psi^\beta \tag{58}
\]

and

\[
\frac{D^2\Phi^\alpha}{D\tau^2} = S_{\beta\gamma\delta}^\alpha V^\gamma V^\delta \Phi^\beta + \frac{1}{2\bar{m}}(S_{\beta\gamma\delta}^\alpha V^\beta \tilde{S}^{\gamma\delta})_{;\delta} \Phi^\beta \tag{59}
\]

as well as

\[
\frac{D^2\Psi^{\alpha\beta}}{DS^2} = S_{\sigma\gamma\delta}^{\alpha\beta} U^\gamma \Psi^\delta, \tag{60}
\]

and

\[
\frac{D^2\Phi^{\alpha\beta}}{D\tau^2} = S_{\sigma\gamma\delta}^{\alpha\beta} V^\gamma \Phi^\delta \tag{61}
\]
ii. Case $P^\mu \neq mU^\mu$ and $P^\mu \neq \bar{m}V^\mu$

$$L = g_{\mu\nu}P^\mu \Psi_{\nu}U^\nu + h_{\mu\nu}\bar{P}^\mu \Phi_{\nu}V^\nu + S_{\mu\nu}\Psi_{\mu}U^\nu + \bar{S}_{\mu\nu}\Phi_{\mu}V^\nu + f_\alpha \Psi^\alpha + \bar{f}_\alpha \Phi^\alpha + 2P_{[\mu}U^{\nu]} \Psi^{\mu\nu} + 2\bar{P}_{[\mu}V^{\nu]} \Phi^{\mu\nu},$$

(62)

where $f_\mu = \frac{1}{2}R_\mu\gamma\delta U^\gamma S^\delta$ and $\bar{f}_\mu = \frac{1}{2}S_\mu\gamma\delta V^\gamma \bar{S}^\delta$

Taking the variation with respect to $\Psi^\alpha$ and $\Phi^\alpha$ to get

$$\frac{DU^\alpha}{DS} = \frac{1}{2m}R_{\beta\gamma\delta}^{\alpha} U^\beta S^\gamma \delta,$$

(63)

and

$$\frac{DV^\alpha}{DT} = \frac{1}{2m}S_{\beta\gamma\delta}^{\alpha} V^\beta \bar{S}^\gamma \delta$$

(64)

Also, taking the variation with respect to $\Psi^{\alpha\beta}$ and $\Phi^{\alpha\beta}$ to get

$$\frac{DS^{\alpha\beta}}{DS} = 0,$$

(65)

and

$$\frac{D\bar{S}^\alpha}{DT} = 0$$

(66)

and their corresponding Spin deviation equations are obtained using the commutation relations to become:

$$\frac{D^2\Psi^\alpha}{DS^2} = R_{\beta\gamma\delta}^{\alpha} U^\gamma U^\beta \Psi^\delta + \frac{1}{2m}(R_{\beta\gamma\delta}^{\alpha} U^\beta S^\gamma \delta)_{;\delta} \Psi^\delta,$$

(67)

and

$$\frac{\bar{D}^2\Phi^\alpha}{D\tau^2} = S_{\beta\gamma\delta}^{\alpha} V^\gamma V^\beta \Phi^\delta + \frac{1}{2m}(S_{\beta\gamma\delta}^{\alpha} V^\beta \bar{S}^\gamma \delta)_{;\delta} \Phi^\delta,$$

(68)

as well as

$$\frac{D^2\Psi^{\alpha\beta}}{DS^2} = S^{[\alpha\rho} R_{\rho\gamma\delta]}^{\beta} U^\gamma \Psi^\delta + 2(P^{[\alpha} U^{\beta])_{;\delta} \Psi^\delta,$$

(69)

and

$$\frac{\bar{D}^2\Phi^{\alpha\beta}}{D\tau^2} = \bar{S}^{[\alpha\rho} R_{\rho\gamma\delta]}^{\beta} U^\gamma \Phi^\delta + 2(P^{[\alpha} U^{\beta})_{;\delta} \Phi^\delta,$$

(70)

3.5 Spin and Spin Deviation Equations of Bi-gravity Type Theories

Recently, Arkami et al [14] have suggested the two metrics $g_{\mu\nu}$ and $h_{\mu\nu}$ are connected with each other by a specific quasi-metric free from ghost in the following manner,

If one considers the two metrics can be related to each other, they can be combined in one metric as quasimetric one [30] such that:

$$\tilde{\gamma}_{\mu\nu} = \alpha^A g_{\mu\nu} + \alpha^B h_{\mu\nu} + \alpha^C \alpha^D \frac{d\tau}{DS} (g_{\mu\nu} - U_{\mu} U_{\nu}) + \frac{dS}{d\tau} (h_{\mu\nu} - V_{\mu} V_{\nu}) + 2U_{(\mu} V_{\nu)},$$

(71)
where $\alpha_g$ and $\alpha_h$ are the coupling strengths, and their corresponding line element becomes

\[ dS^2 = (\alpha_g^2 g_{\mu\nu} + \alpha_h^2 h_{\mu\nu}) dx^\mu dx^\nu + 2\alpha_g\alpha_f \sqrt{(g_{\mu\rho} h_{\rho\sigma} dx^\mu dx^\nu dx^\sigma)} \] (72)

However, applying the action principle on the Lagrangian function (53) to obtain the corresponding spinning equations of bi-gravity theory is expressed for the case $P^m u = mU^\mu$ and $P^m u = \bar{m}V^\mu$ in the following way:

\[
\left( \frac{d}{dS} \frac{\partial L}{\partial \dot{\Psi}^\alpha} - \frac{\partial L}{\partial \Psi^\alpha} \right) + \left( \frac{d}{d\tau} \frac{d}{dS} \frac{\partial L}{\partial \dot{\Phi}^\alpha} - \frac{\partial L}{\partial \Phi^\alpha} \right) = 0,
\] (73)

to give the spinning analog whose geodesic-like has mentioned by Arkani et al (2014)

\[
g_{\mu\alpha}(\frac{DU^\mu}{DS} + \frac{1}{2m} R^\mu_{\nu\rho\sigma} S^{\rho\sigma} U^\nu) + h_{\mu\alpha}(\frac{DV^\mu}{D\tau} + \frac{1}{2\bar{m}} S^\mu_{\nu\rho\sigma} \bar{S}^{\rho\sigma} V^\nu) \frac{d\tau}{ds} = 0 \] (74)

Yet, Extending the same technique of the Bazanski approach, we obtain its deviation equations to obtain:

\[
g_{\mu\alpha}[\frac{D^2 \Psi^\alpha}{DS^2} + R^\alpha_{\beta\gamma\delta} U^\beta U^\gamma \Psi^\delta] + \frac{d\tau}{ds} \gamma_{\mu\alpha}[\frac{D^2 \Phi^\alpha}{D\tau^2} + R^\alpha_{\beta\gamma\delta} V^\gamma V^\beta \Phi^\delta], = 0 \] (75)

Thus, applying the action principle on the Lagrangian function (62) to obtain the corresponding spinning equations of bi-gravity theory is expressed in the following way:

\[
\left( \frac{d}{dS} \frac{\partial L}{\partial \dot{\Psi}^\alpha_{\beta}} - \frac{\partial L}{\partial \Psi^\alpha_{\beta}} \right) + \left( \frac{d}{d\tau} \frac{d}{dS} \frac{\partial L}{\partial \dot{\Phi}^\alpha_{\beta}} - \frac{\partial L}{\partial \Phi^\alpha_{\beta}} \right) = 0,
\] (76)

to give the same an extended results to what was mentioned by Arkani et al (2014) for spinning objects for short

\[
\alpha_g g_{\mu\nu}(\frac{DS^\mu}{DS} + S^{[\mu\nu} R_{\rho\sigma]} U^\nu) + \alpha_h h_{\mu\nu}(\frac{DV^\mu}{D\tau} + \frac{1}{2\bar{m}} S^\mu_{\nu\rho\sigma} \bar{S}^{\rho\sigma} V^\nu) \frac{d\tau}{ds} = 0 \] (77)

Applying the same technique of the Bazanski approach, we obtain its deviation equations to obtain:

\[
g_{\mu\alpha}g_{\nu\beta}[\frac{D^2 \Psi^\alpha}{DS^2} + F^\alpha_{\beta\gamma} U^\gamma] + h_{\mu\alpha} h_{\nu\beta}[\frac{D^2 \Psi^\alpha}{D\tau^2} + \bar{F}^\alpha_{\beta\gamma} V^\gamma] = 0 \] (78)

where

\[ F^\alpha_{\beta\gamma} = 2S^{[\alpha\sigma} R^\beta_{\gamma\delta]} U^\gamma \Psi^\delta \]

and

\[ \bar{F}^\alpha_{\beta\gamma} = 2\bar{S}^{[\alpha\sigma} S^\beta_{\gamma\delta]} V^\gamma \Phi^\delta \]

In a similar process, we can find the following equations for the case $P^\mu \neq mU^\mu$ and $P^\mu \neq mV^\mu$

Thus, we get the corresponding spinning equations for precessing objects as mentioned by Arkani et al (2014)

\[
\alpha_g g_{\mu\nu}(\frac{DP^\mu}{DS} + R^\mu_{\nu\rho\sigma} S^{\rho\sigma} U^\nu) + \alpha_h h_{\mu\nu}(\frac{D\bar{P}^\mu}{D\tau} + \frac{1}{2} S^\mu_{\nu\rho\sigma} \bar{S}^{\rho\sigma} V^\nu) \frac{d\tau}{ds} = 0. \] (79)
Applying the same technique of the Bazanski approach, we obtain its deviation equations to obtain:

Thus, applying the action principle to obtain the corresponding spinning equations of bi-gravity theory is expressed in the following way:

\[
\left( \frac{d}{dS} \frac{\partial L}{\partial \dot{\Psi}^{\alpha\beta}} - \frac{\partial L}{\partial \Psi^{\alpha\beta}} \right) + \left( \frac{d\tau}{dS} \right) \left( \frac{d}{d\tau} \frac{\partial L}{\partial \dot{\Phi}^{\alpha\beta}} - \frac{\partial L}{\partial \Phi^{\alpha\beta}} \right) = 0, \tag{80}
\]

Applying the same technique of the Bazanski approach, we obtain its deviation equations to obtain:

\[
g_{\mu\alpha}g_{\nu\beta}\left[ D^2 \Psi^{\alpha\beta} + F^{\alpha\beta}_\gamma P^\gamma + 2\left( P^{[\alpha}_{\mu\nu}\Psi^{\beta]} + h_{\mu\alpha}h_{\nu\beta}\frac{\tilde{D}^2\Phi^{\alpha}}{D\tau^2} + \bar{F}^{\alpha\beta}\bar{P}^\gamma + 2(\bar{P}^{[\alpha}_{\mu\nu}\Psi^{\beta]} + 2S^{[\alpha\beta]} - \tilde{g}_{\alpha\beta}) \right) \right]
\]

\[
g_{\mu\alpha}g_{\nu\beta}\left[ \tilde{D}^2 \Phi^{\alpha\beta} - \tilde{D}^2 \Phi^{\alpha\beta} \right] = 0, \tag{81}
\]

From the above set of equations (77) and (81), we find that these equations are different than their counterparts in Riemannian equation, but if we used the metric as defined in (71), we obtain the following equations for spinning and spinning deviations for bi-gravity theory as similar to the Moffat version of bi-metric theory.

Thus, the corresponding Lagrangian for \( P^\alpha = mU^\alpha \) may be expressed as

\[
\tilde{L} = \tilde{g}_{\mu\nu}U^\mu \left( \frac{d\Psi^\nu}{dS} + \tilde{\Gamma}^\nu_{\rho\delta}U^\rho U^\delta + \frac{1}{2m} \tilde{R}^{\alpha\beta\nu\sigma}S^{\rho\sigma}U^\nu \Psi^\mu \right), \tag{82}
\]

such that, the affine connection may be expressed in terms of \( \tilde{\Gamma}^\alpha_{\beta\sigma} \) as defined as

\[
\tilde{\Gamma}^\alpha_{\beta\sigma} = \frac{1}{2} \tilde{g}^{\alpha\delta}(\tilde{g}_{\delta\beta,\sigma} + \tilde{g}_{\delta\sigma,\beta} - \tilde{g}_{\beta\sigma,\delta})
\]

provided that its corresponding curvature,

\[
\tilde{R}^{\alpha}_{\beta\gamma\sigma} = \tilde{\Gamma}^\alpha_{\beta\gamma,\sigma} - \tilde{\Gamma}^\alpha_{\beta\gamma}\tilde{\Gamma}^\gamma_{\alpha,\sigma} + \tilde{\Gamma}^\beta_{\gamma,\sigma}\tilde{\Gamma}^\gamma_{\beta\alpha} - \tilde{\Gamma}^\rho_{\gamma,\sigma}\tilde{\Gamma}^\alpha_{\beta\rho}.
\]

Taking the variation respect to \( \psi^\mu \) to obtain the spinning equation for short in the following way

\[
\tilde{D}U^\alpha = \frac{1}{2m} \tilde{R}^{\alpha}_{\mu\rho\sigma}S^\rho\sigma U^\nu, \tag{83}
\]

Consequently, applying the commutation laws (9) and (10) on equation (83) to obtain its corresponding path deviation equation,

\[
\tilde{D}^2 \Psi^\alpha = \tilde{R}^{\alpha}_{\mu\rho\sigma}U^\mu U^\nu \Psi^\rho + \frac{1}{2m} (\tilde{R}^{\alpha}_{\mu\rho\sigma}S^\rho\sigma U^\nu)_{\sigma} \Psi^\sigma. \tag{84}
\]

Moreover, the spinning and spinning deviation equations with precession become,

\[
\tilde{L} = \tilde{g}_{\alpha\beta}U^\alpha \frac{\tilde{D}\Psi^\beta}{DS} + S_{\alpha\beta} \frac{\tilde{D}\Psi^\alpha}{DS} + f_\alpha \Psi^\alpha + f_{\alpha\beta} \Psi^\alpha \Psi^\beta \tag{85}
\]

to give

\[
\frac{\tilde{D}U^\mu}{DS} = f^\mu, \tag{86}
\]
and
\[ \frac{\bar{D}S_{\mu\nu}}{DS} = f_{\mu\nu}. \] (87)

Accordingly, its corresponding deviation equation becomes:
\[ \frac{\bar{D}^2 \Psi^\alpha}{\bar{D}S^2} = \bar{R}_{\mu\nu\rho}^\alpha U^\mu U^\nu \Psi^\rho + f_{\rho}^\alpha \Psi^\rho \] (88)
and
\[ \frac{\bar{D}^2 \Psi_{\mu\nu}}{\bar{D}S^2} = S_{\mu\delta} \bar{R}_{\delta\sigma\rho}^\sigma U^\sigma U^\rho + f_{\mu\rho} \Psi^\rho \] (89)

The above equations of spinning objects in different types of bi-gravity metrics as two independent metrics and one combined one having a free-ghost particles, may give rise to visualize the effect of different curvatures affecting the proper mass and twin mass as well as representing their accompanied spin and twin spin tensors. The combined metric tensors of bigravity as expressed in Riemannian geometry may be expressed geometrically using Finsler geometry. Such a type of work will be assigned in our future work.

4 Spinning and Spinning Deviation Equations of Bi-metric Invariant-Gravitation Theory

It is well known Einstein’s field equations are not invariant under the transformation of field variables; yet its equation of motion become invariant. Yet the question of gauge invariance of field variables was primarily addressed by Weyl who obtained the following transformation for the Christoffel symbol [31]
\[ \bar{\Gamma}^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} + \delta^\alpha_\beta \psi^\gamma(x) + \delta^\alpha_\gamma \psi^\beta(x) \] (90)

where \( \psi(x) \) are arbitrary differentiable function, describing the same gravitational field, due to its geodesic equations,
\[ \frac{dx^2}{ds^2} + (\Gamma^\alpha_{\beta\gamma} - c^{-1} \Gamma^0_{\beta\gamma} \frac{dx^\alpha}{ds}) \frac{dx^\beta}{ds} \frac{dx^\gamma}{ds} = 0 \] (91)

which is invariant under transformation equation (..). Following this approach Verozub (1991) suggested the following tensor to represent invariance.
\[ B^\alpha_{\beta\gamma} = \Pi^\alpha_{\beta\gamma} - \Pi^\alpha_{\beta\gamma} \] (92)
such that
\[ \Pi^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} - \frac{1}{n+1} (\delta^\alpha_\beta \Gamma^\gamma(x) + \delta^\alpha_\gamma \Gamma^\beta(x)) \]
and
\[ ^o\Pi^\alpha_{\beta\gamma} = ^o\Gamma^\alpha_{\beta\gamma} - \frac{1}{n+1} (\delta^\alpha_\beta ^o\Gamma^\gamma(x) + \delta^\alpha_\gamma ^o\Gamma^\beta(x)) \]

where \( \Pi^\alpha \) and \( ^o\Pi^\alpha \) are the Thomas symbol for Riemannian Space-time and Minkowski Space time respectively , And
\[ \Gamma_{\gamma}(x) = \Gamma_{\gamma}^\alpha \quad \text{and} \quad \circ \Gamma_{\gamma}^\alpha = \circ \Gamma_{\gamma}^\alpha \]

\[ G_{\alpha\beta}(\psi) = g_{\alpha\beta} - \frac{1}{n+1}(\Gamma_{\alpha} - \circ \Gamma_{\alpha})(\Gamma_{\beta} - \circ \Gamma_{\beta}) \]

such that

\[ \phi_{\alpha} = \frac{1}{n+1}(\Gamma_{\alpha} - \circ \Gamma_{\alpha}) = \frac{1}{2(n+1)} \frac{\partial}{\partial x^\alpha} \ln |\bar{g}| \]

If one defines its associate Christoffel symbol

\[ \bar{\Gamma}_{\gamma}^{\alpha\beta} = \frac{1}{2} G^{\sigma\alpha}(G_{\sigma\beta,\alpha} + G_{\alpha\sigma,\beta} - G_{\alpha\beta,\sigma}) \]

to define its curvature tensor to become

\[ R_{\alpha\beta\delta}^{\gamma} = \bar{\Gamma}_{\gamma}^{\alpha\delta,\beta} - \bar{\Gamma}_{\gamma}^{\alpha\beta,\delta} + \bar{\Gamma}_{\rho}^{\beta} \bar{\Gamma}_{\rho,\delta}^{\alpha} - \bar{\Gamma}_{\alpha\delta}^{\rho} \bar{\Gamma}_{\rho,\beta}^{\gamma} \quad (93) \]

i. Case \( \bar{P}^{\alpha} = m \bar{U}^{\alpha} \)

The spinning equations can be expressed by the following Lagrangian

\[ \bar{L} = G_{\alpha\beta}(\psi) \bar{U}^{\alpha} \frac{D\bar{\Psi}^\beta}{DS} + \bar{S}_{\mu\nu} \frac{D\bar{\Psi}^{\alpha\beta}}{DS} + \frac{1}{2m} \bar{R}_{\mu\nu}\bar{S}^{\rho\delta}\bar{U}^{\nu}\bar{\Psi}^{\rho} \]

where

Taking the variation with respect to \( \bar{\Psi}^{\alpha} \) and \( \bar{\Psi}^{\alpha\beta} \) to obtain

\[ \frac{D\bar{U}^{\alpha}}{DS} = \frac{1}{2m} \bar{R}_{\beta\sigma\rho}^{\alpha} \bar{S}^{\sigma\rho}\bar{U}^{\beta} \]

and

\[ \frac{D\bar{S}^{\alpha\beta}}{DS} = 0 \quad (96) \]

In a similar way, using the commutation relations as shown in (11),(12), (13) and (14) ,
we obtain their corresponding deviation equations

\[ \frac{D^2\bar{\Psi}^{\alpha}}{S^2} = \bar{R}^{\alpha}_{\beta\sigma\rho} \bar{U}^{\beta} \bar{\Psi}^{\rho} + \frac{1}{2m} (\bar{R}^{\alpha}_{\beta\sigma\rho} \bar{S}^{\sigma\rho}\bar{U}^{\beta})_{\beta} \bar{\Psi}^{\rho} \]

and

\[ \frac{\bar{D}^2\bar{\Psi}^{\alpha\beta}}{S^2} = \bar{S}^{[\alpha\rho} \bar{R}^{\beta]}_{\rho\sigma\delta} \bar{U}^{\sigma} \bar{\Psi}^{\delta} \]

ii. Case \( \bar{P}^{\alpha} \neq m \bar{U}^{\alpha} \)

If we take the corresponding the Bazanski-like Lagrangian for a spinning object to become ,

\[ \bar{L} = G_{\mu\nu}(\psi) \bar{P}^{\mu} \frac{D\bar{\Psi}^{\nu}}{DS} + \bar{S}_{\mu\nu} \frac{D\bar{\Psi}^{\mu\nu}}{DS} + \bar{f}_{\mu} \bar{\Psi}^{\mu} + \bar{f}_{\mu\nu} \bar{\Psi}^{\mu\nu} \]

(99)
where,
\[
\bar{f}^\alpha = \frac{1}{2} \bar{R}^\alpha_{\beta\sigma\rho} \bar{S}^{\sigma\rho} \bar{U}^\beta
\]
\[
\bar{f}^{\alpha\beta} = \bar{S}^{[\alpha\rho} \bar{R}^\beta_{\rho\sigma]} \bar{U}^\sigma \bar{\Psi}^\delta + 2(\bar{P}^{[\alpha} \bar{U}^{\beta]},_{\delta}) \bar{\Psi}^\delta
\]
Thus, taking the variation with respect to \(\Psi^\alpha\) and \(\Psi^{\alpha\beta}\) to obtain
\[
\frac{D\bar{P}^{\alpha}}{DS} = \frac{1}{2} \bar{R}^\alpha_{\beta\sigma\rho} \bar{S}^{\sigma\rho} \bar{U}^\beta,
\]  
(100)
and
\[
\frac{D\bar{S}^{\alpha\beta}}{DS} = 2 \bar{P}^{[\mu} \bar{U}^{\nu]},
\]  
(101)
Also, following the same technique as mentioned in (..) and (..) to obtain
\[
\frac{D^2\Psi^\alpha}{SS^2} = \bar{R}^\alpha_{\beta\sigma\rho} \bar{P}^{\beta} \bar{U}^\sigma \bar{\Psi}^\rho + \bar{f}^\alpha_{\rho} \bar{\Psi}^\rho
\]  
(102)
and
\[
\frac{D^2\Psi^{\alpha\beta}}{SS^2} = \bar{S}^{[\alpha\rho} \bar{R}^\beta_{\rho\sigma]} \bar{U}^\sigma \bar{\Psi}^\delta
\]  
(103)
From the above equations, we can figure out the in this part, the spinning and spinning deviation equations as similar look as similar as their counterpart of general relativity, this is due to embedding all properties of bi-metric effects in terms of a modified metric tensor \(G_{\mu\nu} = g_{\mu\nu}(\psi)\), as a function of a field \(\psi\) rather than coordinates of space time.

5 Concluding remarks

Equations of motion of spinning objects in different types of bi-metric theories of gravity have revealed the effect of different curvatures associated with the moving particles have been discussed. These curvatures are not the mere Riemanian curvature, due to the involvement of different factors affecting its appearance.

Accordingly, It has been shown that, in Rosen’s theory that the Papapetrou equation, for short, is expressed with one single curvature, while the other curvature is neglected due to its flatness of the associated space time as shown in (18) which appears its r.h.s. similar to (9); even with a combined affine derivatives. Yet, the system of equations are quite different than their counterparts in general relativity. This difference is clarified on dealing with \(P^\mu \neq m U^\mu\) as shown by comparing equations (9) and (10) with their counterparts in (23), (24), (36),(37),(47),(48),(63),(64),(65),(66), (79), (81),(100) and (101). We also, have found the effect of the other type of absolute covariant derivative, on the spinning deviating object, despite the vanishing of its curvatures as shown in equations (24), (25) and (26).

On displaying the set of spinning and spinning deviation equations of Moffat’s version, as regarded to be a bi-metric theory of having instead of two separate metrics, one combined metric, with an amended affine connection, having its own curvature (31), (32), (36) and (37) leading to similar appearance of the Papapetrou equations and their deviation
ones as in equations (33), (34), (36) and (37). In Moffat’s model, has led us to propose the system of equations of rotating objects of the same appearance like the Papapetrou equations for objects in general relativity.

Since, the problem of bi-metric theory is assigned to define strong fields of gravity, therefore, it is worth mentioning to examine the bi-metric analog of MOND due to its role to find solutions to problems that have no general relativity explanation i.e. the rotation curves of spiral galaxies. This may give rise to consider, the derivation of spinning equations of BIMOND, has given rise, to investigate to what extend different metrics, may affect on the behaviorism of spinning and spinning deviations equations as a result of finding a tensor connecting the two affine connections (40) has an impact on the two different types of curvature. From this perspective, it was necessary to obtain their possible set of of equation to spinning equations (42), (43), (47), and (48) and their deviation ones as in equations (44), (45), (49) and (50).

Nevertheless, the tendency of studying bi-metric theory has been developed by proposing two different sources of gravity, have emerged the need to study the behavior of particles subject to these two effects, based on regarding by-gravity as a massive theory of ghost-free [13]. On dealing with spinning equations, we have proposed, such a hypothetical definition of twin spin tensor, associated with twin matter. Yet one should take into consideration the proposed the has kept link between geodesic deviation vector and spin tensor unchanged as in (2). Owing to this illustration, we have figured out that there are two independent sets of spinning equations and their deviations ones stemmed from one Lagrangian. Such an approach may give rise to search for an appropriate geometry able to express these two types of matter together.

However, there is an alternative way on dealing with bi-gravity theory, having a combined metric, as similar to Moffat’s treatment of VSL, to be geomtrized using Finsler geometry. This type of treatment has obtained the system of spinning equations to behave as similar as the conventional general relativity, but different from their composition, related to the combined metric. We can compare the similarity between the equations (9),(10),(15) and (16), and their counterparts (86),(87),(88) and (89).

Finally, we have derived the set of spinning and spinning deviation equations for Verozub’s version a bi-metric theory of gravity. In this type theories expressed the metric tensor is no longer a function of coordinates of space time $g_{\mu\nu}(x)$; but a function from a proposed by a field variable $g_{\mu\nu}(\psi)$, leading to its own affine connection $\bar{\Gamma}_\alpha^\gamma$ and associated curvature $\bar{R}_\alpha^\beta$. This can be found plainly on deriving their associate spinning equations and spinning deviation equations, different equations of spinning (95), (96), (100) and (101), as well as there a spinning deviation equations (97), (98), (102) and (103).

Thus, from the results obtained, it is essential to search for an a wider geometry than the Riemanian treatment to express all quantities of bigravity theory. Such a request can be found by applying Finsler geometry, which will determined in our future work.

6 References

[1]N. Rosen, Ann. Physics, 455 (1974).
[2] A. Papapetrou, Proceedings of Royal Irish Academy Section A Vol 52, 11 (1948).
[3] N. Rosen, Gen. Relativ. and Gravit., 4, 435 (1973).
[4] M. Israelit, Gen. Relativ. and Gravit., 7, 623 (1976).
[5] R. Falik and N. Rosen, Gen. Relativ. and Gravit., 13, 599 (1981).
[6] J.W. Moffat, arXiv: hep-th/0208122 (2002)
[7] J.W. Moffat arXiv: quant-ph/0204151 (2002)
[8] M. Milgrom, Astrophys. J. 270, 365 (1983)
[9] M. Milgrom, Phys.Rev.D80:123536,2009 arXiv: 0912.0790 (2009)
[10] M. Milgrom, arXiv: 1404.7661 (2014)
[11] M. Milgrom, Phys. Rev. D 89, 024027 (2014); arXiv:1308.5388 (2014)
[12] S.F. Hassan and Rachel.A Rosen, arXiv 1109.3515 (2012)
[13] E. Babichev and R. Rito, arXiv 1503.07529 (2015)
[14] Y. Akrami, T. Kovisto, and A.R. Solomon, arXiv.1404.0006 (2014)
[15] L.V. Verozub, arXiv 0911.5512 (2009)
[16] L.V. Verozub, Space-time Relativity and Gravitation, Lambert Academic Publishing. (2015)
[17] L.V. Verozub, 322,3, 1.(2001).
[18] Magd E. Kahil, Gravit. Cosmol., Vol 23, 70 (2017).
[19] A. Papapetrou, Proceedings of Royal Society London A 209, 248(1951).
[20] Magd E. Kahil, Gravit. Cosmol., Vol 24, 83 (2018).
[21] Magd E. Kahil, ADAP, 3, 136 (2018).
[22] Magd E. Kahil, Odessa Astronomical Publications, vol 28/2, 126 (2015).
[23] S.L. Bazanski, J. Math. Phys., 30, 1018 (1989).
[24] M.E.Kahil, J. Math. Physics 47,052501 (2006).
[25] M. Heydrai-Fard, M. Mohseni, and H.R. Sepanigi, H.R. (2005) Phys. Lett. B, 626, 230.
[26] M. Roshan, Phys.Rev. D87,044005 (2013).
[27] J. Foukzoun, S.A. Podosenov, A.A. Potpov, and E. Menkova, arXiv: 1007.3290 (2010).
[28] J.W. Moffat, arXiv: 1110.1330 (2011)
[29] J.W. Moffat, arXiv: 1306.5470 (2013)
[30] S. Hossenfelder, arXiv: 0807.2838 (2008).
[31] J.D. Bekenstein, arXiv: gr-qc/9211017 (1992).
[32] C. Romero, J. B. Fonseca-Neto, and M. L. Pucheu, arXiv: 1201.1469 (2012).