Resummation of transverse momentum distributions in distribution space.

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Motivation.
Motivation

The LHC physics program.

- Discovery of the Higgs boson in 2012 was a spectacular success
  - Electroweak symmetry breaking generates masses ✓
  - Completed particle content of the Standard Model ✓
- But many puzzles remain:
  - Hierarchy / naturalness problem
  - Is there only one Higgs boson?
  - What is dark matter?
  - What causes neutrino masses?
  - Where is the missing CP-violation needed for baryogenesis?

Hope:

LHC should find hints for physics beyond the Standard Model
How to search for new particles at the LHC.

Direct searches:
- Sufficiently light particles can be created at the LHC → Search for resonances
- Example: Higgs discovery
- Often used in SUSY-searches
- So far: No hints for new particles

Indirect searches:
- Search for deviations from SM predictions
- Most prominent: Higgs couplings
- So far: No significant deviations from SM
- Way out: More differential observables → Shapes provide more information than inclusive measurements

[ CMS '14 ]

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How to search for new particles at the LHC.

Direct searches:
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Higgs $p_T$ spectrum.

- Important observable of the LHC Higgs program
- Tests SM predictions
- Theory and experiment are $\sim$compatible
- Large uncertainties currently limited by statistics
- $\sim$10 times more data by 2018
  $\rightarrow$ Theory uncertainties important soon

Need precision predictions for the Higgs $p_T$-spectrum.
Motivation
Z-boson $p_T$ spectrum.

- Standard candle of SM
  - Important test of QCD
- Measured to $\lesssim 1\%$ accuracy
- Theory prediction (NNLO+NNLL):
  - $\sim 10\%$ at peak
  - $\sim 4\%$ at $p_T^Z = 10$ GeV
- $N^3LL$ soon available

Theory prediction (NNLO+NNLL):
- $\sim 1\%$ theory precision around the corner

How reliable is the theory uncertainty at this level of precision?
Z-boson $p_T$ spectrum.

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- Theory prediction (NNLO+NNLL): [Catani, de Florian, Ferrera, Grazzini '15]
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  [Li, Zhu '16] [Vladimirov '16]
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How reliable is the theory uncertainty at this level of precision?
Theoretical challenges.

- Bulk of cross section $\sigma(q_T)$ located at $q_T \ll Q$ \hspace{1cm} ($Q = m_H, m_Z$)
- Large Sudakov logarithms $\alpha_s \ln^2 (Q/q_T)$ spoil expansion in $\alpha_s$
  $\rightarrow$ Resummation to all orders is necessary

Illustration for Higgs production: [Bozzi, Catani, de Florian, Grazzini ’03]
**Theoretical challenges.**

- Bulk of cross section $\sigma(q_T)$ located at $q_T \ll Q$ ($Q = m_H, m_Z$)
- Large Sudakov logarithms $\alpha_s \ln^2(Q/q_T)$ spoil expansion in $\alpha_s$
  $\rightarrow$ Resummation to all orders is necessary
- Resummation is well understood since [Collins, Soper, Sterman '85]
- Resummation carried out (partially) in Fourier space $\vec{b}_T$
  DYRes [Catani, de Florian, Ferrera, Grazzini ’15 ...], HRes [de Florian, Ferrera, Grazzini, Tommasini '12 ...], ResBos [Wang, Li$^3$, Yuan '12 ...], CuTe [Becher, Neubert, Wilhelm ’12], [D’Alesio, Echevarria, Melis, Scimemi ’14], [Echevarria, Kasemets, Mulders, Pisano ’15], [Neill, Rothstein, Vaidya ’15], . . .
  - Resums $\ln(Qb_T)$ rather than $\ln(Q/q_T)$
  - Theory uncertainties are estimated in Fourier space:
    Scale variations probe $\ln(Qb_T)$ rather than $\ln(Q/q_T)$
  - How reliable is this (when we go to 1% precision)?
- Is it possible to carry out resummation directly in momentum space?
  [Frixione, Nason, Ridolf ’97] [Ellis, Veseli ’98] [Kulesza, Stirling ’00] [Monni, Re, Torrielli ’16]

**Goal:** Resummation in momentum space as a complementary approach.
Factorization of transverse momentum distributions.
Factorization theorem.

\[ \sigma(q_T) = \sigma_0 H(Q, \mu) \int d^2k_1 \, d^2k_2 \, d^2k_s \, \delta(q_T - k_1 - k_2 - k_s) \times B(Qe^Y, k_1, \mu, \nu)B(Qe^{-Y}, k_2, \mu, \nu)S(k_s, \mu, \nu) \]

- **Hard function**: Describes hard process, e.g. \( gg \rightarrow H \rightarrow VV \)
- **Beam functions**: Describe collinear radiation along beam axes
  Often referred to as transverse-momentum dependent PDFs (TMDPDFs)
- **Soft function**: Describes isotropic, soft radiation
- **Corrections**: Are power suppressed by \( q_T/Q \)
Resummation of large logarithms.

- **Hard**, **beam** and **soft** functions contain UV and rapidity divergences.
- Renormalization induces unphysical scales $\mu$ and $\nu$ ($\zeta$ in CSS).
  \[ \sigma(\vec{q}_T) \sim \sigma_0 H(Q, \mu) [B(\mu, \nu) \otimes B(\mu, \nu) \otimes S(\mu, \nu)](\vec{q}_T) \]
- Large Sudakov logarithms split into
  \[ \ln^2 \frac{Q}{q_T} = \ln^2 \frac{Q}{\mu} + 2 \ln \frac{q_T}{\mu} \ln \frac{\nu}{Q} + \ln \frac{q_T}{\mu} \ln \frac{\mu q_T}{\nu^2} \]
- All-order logarithmic structure encoded in renormalization group equations (RGE).
- RGEs allow to resum large logarithms individually in **hard**, **beam** and **soft** functions.
  - Logarithms in spectrum can be resummed to all orders in $\alpha_s$.
- Resummation accuracy only specified by boundary terms / anomalous dimensions.

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Resummation through RG-evolution.

Example: Hard function

- **Hard** function contains UV-divergences
- Renormalization induces unphysical scale $\mu$:
  \[ H^{(\text{bare})}(Q) = Z_H(Q, \mu) \cdot H(Q, \mu) \]

  - counterterm
  - Renormalized hard function

- $\mu$-Independence of *bare* function induces RGE:
  \[ \mu \frac{dH(Q, \mu)}{d\mu} = -\frac{d \ln Z_H}{d \ln \mu} H(Q, \mu) \equiv \gamma_H \cdot H(Q, \mu) \]

- RGE describes all-order logarithmic structure
- Solution ($\mu_H \sim Q$):
  \[ H(Q, \mu) = H(Q, \mu_H) \exp \left[ \int_{\mu_H}^{\mu} \frac{d \mu'}{\mu'} \gamma_H(Q, \mu') \right] \]

  - Free of large logs
  - Exponentiated (resummed) logs

- Only requires fixed-order result of $H(Q, \mu_H)$ and $\gamma_H$
RG structure of the cross section.

RGEs capture all-order logarithmic structure:

- **$\mu$-RGE:**
  \[
  \mu \frac{dH(Q, \mu)}{d\mu} = \gamma_H H(Q, \mu)
  \]
  \[
  \mu \frac{dB(\omega, \vec{q}_T, \mu, \nu)}{d\mu} = \gamma_B B(\omega, \vec{q}_T, \mu, \nu)
  \]
  \[
  \mu \frac{dS(\vec{q}_T, \mu, \nu)}{d\mu} = \gamma_S S(\vec{q}_T, \mu, \nu)
  \]

- **$\nu$-RGE (RRGE):**
  \[
  \nu \frac{dB(\omega, \vec{q}_T, \mu, \nu)}{d\nu} = -\frac{1}{2} \int d^2\vec{k}_T \gamma_{\nu}(\vec{q}_T - \vec{k}_T, \mu) B(\omega, \vec{k}_T, \mu, \nu)
  \]
  \[
  \nu \frac{dS(\vec{q}_T, \mu, \nu)}{d\nu} = \int d^2\vec{k}_T \gamma_{\nu}(\vec{q}_T - \vec{k}_T, \mu) S(\vec{k}_T, \mu, \nu)
  \]

- Commutativity of $\mu$ and $\nu$ RGE:
  \[
  \mu \frac{d\gamma_{\nu}(\vec{q}_T, \mu)}{d\mu} = \nu \frac{d\gamma_{S}(\vec{q}_T)}{d\nu} \delta(\vec{q}_T) = -4\Gamma_C[\alpha_s(\mu)] \delta(\vec{q}_T)
  \]
Difficulties with resummation in momentum space.
Difficulties with resummation in momentum space

- Convolutions become products in Fourier space
- RGEs are easily solved:
  (Simplification: $\mu_T = \mu_B = \mu_S$)

\[
\sigma(\vec{q}_T) = \sigma_0 \int d^2\vec{b}_T \ e^{i\vec{b}_T \cdot \vec{q}_T}
\]

Fixed order boundary

\[
\times \ H(Q, \mu_H) B(\omega, \vec{b}_T, \mu_T, \nu_B)^2 S(\vec{b}_T, \mu_T, \nu_S)
\]

RG evolution

\[
\times \ \exp \left[ \int_{\mu_H}^{\mu_T} \frac{d\mu'}{\mu'} \gamma_H(Q, \mu') \right] \ \exp \left[ \ln \frac{\nu_S}{\nu_B} \gamma_{\nu}(\vec{b}_T, \mu_T) \right]
\]

- Resummation in Fourier space
  \[
  \mu_H = Q, \quad \mu_T = 1/b_T, \quad \nu_B = Q, \quad \nu_S = 1/b_T
  \]

- “Natural” scales in momentum space (?)
  \[
  \mu_H = Q, \quad \mu_T = q_T, \quad \nu_B = Q, \quad \nu_S = q_T
  \]
First attempt at momentum space resummation.

- “Natural” choice in momentum space:
  \[
  \mu_H = Q, \quad \mu_T = q_T \\
  \nu_B = Q, \quad \nu_S = q_T
  \]

- At LL, this gives
  \[
  \sigma(q^2_T) = \frac{\sigma_0}{q^2_T} \exp \left[ - \int_{q_T}^{Q} \frac{d\mu'}{\mu'} \gamma_H(Q, \mu') \right] e^{-2\gamma E \omega} \frac{\Gamma(1 - \omega)}{\Gamma(\omega)}
  \]
  \[
  \omega = 2\Gamma_C[\alpha_s(q_T)] \ln \frac{Q}{q_T}
  \]

  The spectrum contains a divergence when \( \Gamma_C[\alpha_s(q_T)] = \ln^{-1} \frac{Q^2}{q^2_T} \)
  - For \( Q = m_H \): \( q_T \approx 8 \text{ GeV} \) (\( \Gamma_C \sim C_A \))
  - For \( Q = m_Z \): \( q_T \approx 2 \text{ GeV} \) (\( \Gamma_C \sim C_F \))

- Simple momentum space resummation ill-defined!
  Already noted in [Frixione, Nason, Ridolfi '97; Chiu, Jain, Neill, Rothstein '12]
Problems in the momentum space resummation.

- Divergence arises from $\nu$-evolution of soft function:

$$S(\vec{p}_T, \mu, \nu_B) = \int d^2\vec{b}_T e^{i\vec{b}_T \cdot \vec{p}_T} \tilde{S}(\vec{b}_T, \mu, \nu_S) \exp \left[ \ln \frac{\nu_B}{\nu_S} \tilde{\gamma}_\nu(\vec{b}_T, \mu) \right]$$

$$= S(\vec{p}_T, \mu, \nu_S) + \sum_{n=1}^{\infty} \frac{1}{n!} \ln^n \frac{\nu_B}{\nu_S} [(\gamma_\nu \otimes^n) \otimes S](\vec{p}_T, \mu, \nu_S)$$

- Obviously fulfills RGE

$$\nu \frac{d}{d\nu} S(\vec{p}_T, \mu, \nu) = \int d^2\vec{k}_T \gamma_\nu(\vec{p}_T - \vec{k}_T, \mu) S(\vec{k}_T, \mu, \nu)$$

- A priori, this only shifts $\ln(p_T/\nu_S)$ into $\ln(p_T/\nu_B)$

- Resummation assumes that $S(\vec{p}_T, \mu, \nu_S)$ has no large logs, i.e.

$$S(\vec{p}_T, \mu, \nu_S) = \delta(\vec{p}_T) + \cdots$$

$\Rightarrow$ All logs are predicted through RG evolution

- Does $\nu_S = q_T$ eliminate all logarithms in the boundary $S(\vec{p}_T, \mu, \nu_S)$?
Problems in the momentum space resummation.

Origin of divergence in Fourier space:

\[ S(\vec{p}_T, \mu, \nu_B) = \int d^2\vec{b}_T \, e^{i\vec{b}_T \cdot \vec{p}_T} \, \tilde{S}(\vec{b}_T, \mu, \nu_S) \exp \left[ \ln \frac{\nu_B}{\nu_S} \tilde{\gamma}_{\nu}(\vec{b}_T, \mu) \right] \]

- \( \nu_S = p_T \) assumes \( \ln(b_T \nu_S) = \ln(b_T p_T) \sim 0 \) in Fourier transformation:

\[ \tilde{S}(\vec{b}_T, \mu, \nu_S = p_T) \sim 1 + \alpha_s \ln(b_T \nu_S) \ln(b_T \mu) + \cdots \]

\[ = 1 + \alpha_s \ln(b_T p_T) \ln(b_T \mu) + \cdots \]

\[ \approx 1 + \cdots \]

- In fact: region \( b_T \ll p_T^{-1} \) induces the explicit divergence:

\[ S(\vec{p}_T, \mu, \nu_B) \approx \int d^2\vec{b}_T \, \exp \left[ -2\Gamma_C \ln \frac{\nu_B}{\nu_S} \ln(b_T^2 \mu^2) \right] \]

\[ \sim \frac{1}{1 - \omega} , \quad \omega = 2\Gamma_C [\alpha_s(\mu)] \ln \frac{\nu_B}{\nu_S} \]

- Divergence results from energetic emissions \( b_T^{-1} \gg p_T \)!
Problems in the momentum space resummation.

Illustration in momentum space:

- Investigate the first convolution in more detail:

\[
\ln \frac{\nu_B}{\nu_S} \int d^2 \vec{k}_1 \, d^2 \vec{k}_2 \, \gamma_\nu(\vec{k}_1, \mu) \, S(\vec{k}_2, \mu, \nu_S) \, \delta(\vec{p}_T - \vec{k}_1 - \vec{k}_2)
\]

- Which momenta $|\vec{k}_1|, |\vec{k}_2|$ contribute to the convolution?

- Kinematically forbidden
Difficulties with resummation in momentum space

Problems in the momentum space resummation

Illustration in momentum space:

- Investigate the first convolution in more detail:
  \[
  \ln \frac{\nu_B}{\nu_S} \int d^2 \vec{k}_1 \ d^2 \vec{k}_2 \ \gamma_\nu(\vec{k}_1, \mu) \ S(\vec{k}_2, \mu, \nu_S) \ \delta(\vec{p}_T - \vec{k}_1 - \vec{k}_2)
  \]

- Which momenta $|\vec{k}_1|, |\vec{k}_2|$ contribute to the convolution?

- Kinematically forbidden

- Soft contributions $k_1 \sim k_2 \sim p_T$
  - Correctly described by $\nu_S \sim p_T$
  - Induces large $\ln(\nu_B/\nu_S)$

- Hard contributions $k_1, k_2 \gg p_T$
  - Contributes due to kinematic cancellation
  - Should not give a large log...
  - But receives spurious $\ln(\nu_B/\nu_S)$
Illustration in momentum space:

- Investigate the first convolution in more detail:
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  \ln \frac{\nu_B}{\nu_S} \int d^2\vec{k}_1 \, d^2\vec{k}_2 \, \gamma_{\nu}(\vec{k}_1, \mu) \, S(\vec{k}_2, \mu, \nu_S) \, \delta(\vec{p}_T - \vec{k}_1 - \vec{k}_2)
  \]
- Which momenta $|\vec{k}_1|, |\vec{k}_2|$ contribute to the convolution?

  - Kinematically forbidden
  - Soft contributions $k_1 \sim k_2 \sim p_T$
    - Correctly described by $\nu_S \sim p_T$
    - Induces large $\ln (\nu_B/\nu_S)$
  - Hard contributions $k_1, k_2 \gg p_T$
    - Contributes due to kinematic cancellation $\vec{k}_1 + \vec{k}_2 = \vec{p}_T$
    - Should not give a large log...
    - but receives spurious $\ln \frac{\nu_B}{\nu_S}$
Problems in the momentum space resummation.

- The first convolution
  \[ \ln \frac{\nu_B}{\nu_S} \int d^2 k_1 d^2 k_2 \gamma_\nu(k_1, \mu) S(k_2, \mu, \nu_S) \delta(p_T - k_1 - k_2) \]

- should actually behave as
  \[ \int d^2 k_1 d^2 k_2 \left( \ln \frac{\nu_B}{\nu_S} \sim k_1 \right) \gamma_\nu(k_1, \mu) S(k_2, \mu, \nu_S \sim k_2) \delta(p_T - k_1 - k_2) \]

  Rapidity logarithm of emission \( k_1 \)

  Minimize logs \( \ln(k_2/\nu_s) \) inside \( S(k_2, \mu, \nu_S) \)

- Can not be achieved with simple exponential
  \[ S(p_T, \mu, \nu_B) = S(p_T, \mu, \nu_S) + \sum_{n=1}^{\infty} \frac{1}{n!} \ln^n \frac{\nu_B}{\nu_S} [(\gamma_\nu \otimes^n) \otimes S](p_T, \mu, \nu_S) \]

  \[ \text{Requires a “generalized” exponential solution} \]

- Soft function is a distribution \( \rightarrow \) How to set \( \nu_S = k_2 \) ?

  \[ \text{Requires “distributional scale setting”} \]
Distributional scale setting.

(For simplicity: for one-dimensional distributions)
Toy example.

- Toy function $F(k, \mu)$ containing logarithms $\ln(k/\mu)$
- Logarithmic structure governed by toy RGE:
  \[
  \mu \frac{dF(k, \mu)}{d\mu} = -\alpha_s F(k, \mu)
  \]
- Formal solution: (neglecting $\alpha_s$ running)
  \[
  F(k, \mu) = F(k, \mu_0) \exp\left(\alpha_s \ln \frac{\mu_0}{\mu}\right)
  \]
  - shifts logarithms $\ln(k/\mu_0)$ into $\ln(k/\mu)$
  - sufficient if $F(k, \mu_0)$ is known “exactly” at reference scale $\mu_0$
    Example: PDF evolution
- Purpose of resummation:
  - predict all logarithms of $F(k, \mu)$
  - such that $F(k, \mu_0)$ is free of large logs $\ln(k/\mu_0)$
    $\rightarrow F(k, \mu_0)$ can be calculated perturbatively
Simple toy function:

- Toy function (neglecting $\alpha_s$ running)
  \[ F(k, \mu) = F(k, \mu_0) \exp(\alpha_s \ln \frac{\mu_0}{\mu}) \]

- $\mu_0 = k$ eliminates all logarithms $\ln(k/\mu_0)$ in boundary term
  \[ F(k, \mu_0 = k) = 1 + \cdots \]

- All logarithms are now fully resummed:
  \[ F(k, \mu) = (1 + \cdots) \exp(\alpha_s \ln \frac{k}{\mu}) = 1 + \alpha_s \ln \frac{k}{\mu} + \frac{1}{2} \alpha_s^2 \ln^2 \frac{k}{\mu^2} + \cdots \]

- All-order structure of $F(k, \mu)$ easily derived from evolution equation
Toy example in distribution space.

Plus distributions:
- For many observables: $F(k, \mu)$ contains plus distributions
  (Encodes the cancellation of IR divergences $k \to 0$)

\[
\left[ \theta(k)g(k, \mu) \right]^\mu_+ = \theta(k)g(k, \mu) \quad \text{for } k \neq 0
\]

\[
\int^\mu_+ d\mu \left[ \theta(k)g(k, \mu) \right]^\mu_+ = 0
\]

- Important type:

\[
\mathcal{L}_n(k, \mu) = \left[ \theta(k) \ln^n \frac{k}{\mu} \right]^\mu_+
\]

Distributional toy example:

\[
F(k, \mu) = \delta(k) + \alpha_s \mathcal{L}_0(k, \mu) + \alpha_s^2 \mathcal{L}_1(k, \mu) + \cdots
\]

- fulfills the RGE

\[
\mu \frac{dF(k, \mu)}{d\mu} = -\alpha_s F(k, \mu)
\]
Toy example in distribution space.

- How to derive the toy distribution

\[ F(k, \mu) = \delta(k) + \alpha_s \mathcal{L}_0(k, \mu) + \alpha_s^2 \mathcal{L}_1(k, \mu) + \cdots \]

- from its RGE?

\[ \mu \frac{dF(k, \mu)}{d\mu} = -\alpha_s F(k, \mu) \]

- Formal solution:

\[ F(k, \mu) = F(k, \mu_0) \exp \left( \alpha_s \ln \frac{\mu_0}{\mu} \right) \]

- Need to mimic normal scale setting:

  ▶ Minimize boundary term

\[ F(k, \mu_0 = k|_+ ) = \delta(k) + \cdots \]

  ▶ Predict higher logarithmic distributions

\[ \delta(k) \exp \left( \alpha_s \ln \frac{\mu_0}{\mu} \right) \bigg|_{\mu_0 = k|_+} = \delta(k) + \alpha_s \mathcal{L}_0(k, \mu) + \cdots \]
Distributional scale setting.

Definition:

\[
D(k, \mu = k|_+) \equiv \frac{d}{dk} \left[ \int^{k} dk' D(k', \mu = k) \right]
\]

- Setting \( \mu = k \) inside integral well-defined
- No effect for arbitrary \( \mu = \mu_0 \): \( D(k, \mu = \mu_0|_+) = D(k, \mu = \mu_0) \)

Illustration:

\[
\mathcal{L}_n(k, \mu = k|_+) = \frac{d}{dk} \int^{k} dk' \left[ \frac{\theta(k')}{k'} \ln^n \frac{k'}{\mu} \right]^{\mu}_{\mu=k}
\]

\[
= \frac{d}{dk} \left[ \frac{\theta(k)}{n+1} \ln^{n+1} \frac{k}{\mu} \right]_{\mu=k}
= \frac{d}{dk} 0 = 0
\]

- Minimizes distributions \( \mathcal{L}_n(k, \mu) \) like ordinary logarithms \( \ln^n (k/\mu) \)

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Distributional scale setting.

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\]

- Setting \( \mu = k \) inside integral well-defined
- No effect for arbitrary \( \mu = \mu_0 \): \( D(k, \mu = \mu_0|_+) = D(k, \mu = \mu_0) \)

Properties: \((n, m \geq 0)\)

\[
\delta(k) \ln^{n+1} \frac{\mu_0}{\mu} \bigg|_{\mu_0=k|_+} = (n + 1) \left[ \frac{\theta(k)}{k} \ln^n \frac{k}{\mu} \right]^{\mu}_{+}
\]

\[
(m + 1) \mathcal{L}_m(k, \mu) \ln^n \frac{\mu_0}{\mu} \bigg|_{\mu_0=k|_+} = (n + m + 1) \mathcal{L}_{m+n}(k, \mu)
\]

- Distributions are essentially treated like ordinary logarithms
Illustration: The distributional exponential.

- Formal solution: (neglecting $\alpha_s$ running)

\[
F(k, \mu) = F(k, \mu_0) \exp \left( \alpha_s \ln \frac{\mu_0}{\mu} \right)
\]

- Correct minimization of distributional boundary term: $\mu_0 = k|_+$:

\[
F(k, \mu) = \frac{d}{dk} \int_k^k dk' \ F(k', \mu_0) \exp \left( \alpha_s \ln \frac{\mu_0}{\mu} \right) \bigg|_{\mu_0=k}
\]

\[
= \frac{d}{dk} \int_k^k dk' \ F(k', k) \exp \left( \alpha_s \ln \frac{k}{\mu} \right)
\]

\[
= \frac{d}{dk} \theta(k)(1 + \cdots) \exp \left( \alpha_s \ln \frac{k}{\mu} \right)
\]

\[
= \delta(k) + \alpha_s \left[ \frac{\theta(k)}{k} \exp \left( \alpha_s \ln \frac{k}{\mu} \right) \right] \mu
\]

\[
= \delta(k) + \alpha_s L_0(k, \mu) + \alpha_s^2 L_1(k, \mu) + \cdots
\]

- $\mu_0 = k|_+$ correctly produces the distributional exponential £
Summary: distributional scale setting.

- Minimizing distributional logarithms requires distributional scale setting

\[ D(k, \mu = k_+) \equiv \frac{d}{dk} \left[ \int_{k_0}^{k} dk' D(k', \mu = k) \right] \]

- Two-dimensional generalization:

\[ D(\vec{p}_T, \mu = p_T^+ ) \equiv \frac{1}{2\pi p_T} \frac{d}{dp_T} \left[ \int_{|\vec{k}_T| \leq p_T} d^2 \vec{k}_T D(\vec{k}_T, \mu = p_T) \right] \]

- Treats distributional logarithms essentially like ordinary logarithms

- Allows to solve RGEs (i.e. resummation) directly in distribution space
$q_T$-Resummation in distribution space.
Strategy.

- Focus on the soft function \( S(p_T, \mu, \nu) \)
- Beam and hard functions are analogous (and much simpler)
- Soft rapidity anomalous dimension \( \gamma_{\nu} \) is governed by cusp anom. dim
  \[
  \mu \frac{d\gamma_{\nu}(p_T, \mu)}{d\mu} = -4\Gamma_C[\alpha_s(\mu)]\delta(p_T) \quad (1)
  \]
  - Crucial input to guarantee path independence of \((\mu, \nu)\)-running
- Soft function evolution is governed by
  \[
  \mu \frac{dS(p_T, \mu, \nu)}{d\mu} = \gamma_S(\mu, \nu) S(p_T, \mu, \nu) \quad (2)
  \]
  \[
  \nu \frac{dS(p_T, \mu, \nu)}{d\nu} = (\gamma_{\nu} \otimes S)(p_T, \mu, \nu) \quad (3)
  \]
- Equations increase in complexity → solve step by step
Formal solution of rapidity anomalous dimension $\gamma_\nu$:

$$\gamma_\nu(\vec{p}_T, \mu) = \gamma_\nu(\vec{p}_T, \mu_0) - \delta(\vec{p}_T) \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} 4\Gamma C[\alpha_s(\mu')]$$

Boundary is distributionally minimized with $\mu_0 = p_T|_+:$

$$\gamma_\nu(\vec{p}_T, \mu) = \left[ \frac{4\Gamma C[\alpha_s(p_T)]}{2\pi p_T^2} \right]^{\mu} + \left[ \frac{1}{2\pi p_T^2} \frac{d\gamma_\nu[\alpha_s(p_T)]}{d \ln p_T} \right]^{\xi} + \delta(\vec{p}_T) \gamma_\nu[\alpha_s(\xi)]$$

- **Cusp piece:**
  - Predicted by RGE
  - Fulfills required $\mu$-dependence

- **Noncusp-piece:**
  - Fixed-order boundary terms
  - $\xi$-dependence cancels exactly
  - Distribution required to regulate $1/p_T^2$ divergence

- Virtual corrections to real emission $\vec{p}_T$ resummed in $\alpha_s(p_T)$
Soft $\mu$-evolution.

- $\mu$-evolution:

$$\mu \frac{dS(\vec{p}_T, \mu, \nu)}{d\mu} = \gamma_S(\mu, \nu) S(\vec{p}_T, \mu, \nu)$$

- Formal solution:

$$S(\vec{p}_T, \mu, \nu_0) = S(\vec{p}_T, \mu_0, \nu_0) \exp \left[ \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \gamma_S(\mu', \nu_0) \right]$$

- To minimize boundary term: $\mu_0 = \nu_0 = p_T|$+

$$S(\vec{p}_T, \mu, p_T|+) = \delta(\vec{p}_T) S[\alpha_s(\mu)]$$

$$+ \left[ \frac{1}{2\pi p_T} \frac{d}{dp_T} S[\alpha_s(p_T)] \exp \left\{ \int_{p_T}^{\mu} \frac{d\mu'}{\mu'} \gamma_S(\mu', p_T) \right\} \right]_{p_T}^{\mu}$$

- Boundary term $S[\alpha_s(\mu)]$ arises as coefficient of $\delta(\vec{p}_T)$
Soft $\nu$-evolution.

- Rapidity-RGE:
  \[
  \nu \frac{dS(\vec{p}_T, \mu, \nu)}{d\nu} = (\gamma_\nu \otimes S)(\vec{p}_T, \mu, \nu)
  \]

  Solution is a “generalized” exponential
  \[
  S(\vec{p}_T, \mu, \nu) = S(\vec{p}_T, \mu, p_T|_+ )
  + \int_{p_T|_+}^{\nu} \frac{d\nu_1}{\nu_1} \int d^2 \vec{k}_1 \, \gamma_\nu(\vec{p}_T - \vec{k}_1, \mu) S(\vec{k}_1, \mu, k_1|_+)
  
  + \int_{p_T|_+}^{\nu} \frac{d\nu_1}{\nu_1} \int d^2 \vec{k}_1 \, \gamma_\nu(\vec{p}_T - \vec{k}_1, \mu)
  \times \int_{k_1|_+}^{\nu_1} \frac{d\nu_2}{\nu_2} \int d^2 \vec{k}_2 \, \gamma_\nu(\vec{k}_1 - \vec{k}_2, \mu) S(\vec{k}_2, \mu, k_2|_+)
  + \cdots
  \]

- Fulfills the rapidity RGE $\checkmark$
- $\nu = p_T|_+$ reproduces correct boundary $S(\vec{p}_T, \mu, p_T|_+)$
Resummation in distribution space.

\[
\sigma(\vec{q}_T) = \sigma_0 \frac{1}{2\pi q_T} \frac{d}{dq_T} \int d^2 \vec{p}_T \left| \vec{p}_T \right| \leq q_T H(Q, \mu_H) \frac{d}{d\mu'} \gamma_H(Q, \mu') \int d^2 \vec{k}_a d^2 \vec{k}_b d^2 \vec{k}_s \delta(\vec{p}_T - \vec{k}_a - \vec{k}_b - \vec{k}_s)
\]

\[
\times \exp \left[ \int_{\mu_H}^{\mu_T} \frac{d\mu'}{\mu'} \gamma_H(Q, \mu') \int d^2 \vec{k}_s' \delta(\vec{k}_s - \vec{k}_s') \right]
\]

\[
+ \sum_{n=1}^{\infty} \prod_{i=1}^{n} \int_{k_{i-1}}^{\nu_i-1} \frac{d\nu_i}{\nu_i} \int d^2 \vec{k}_i \gamma_{\nu}(\vec{k}_{i-1} - \vec{k}_i, \mu_T) \delta(\vec{k}_s - \vec{k}_s' - \sum_i \vec{k}_i)
\]

\[
\times B_a(\omega_a, \vec{k}_a, \mu_T, \nu_a) B_b(\omega_b, \vec{k}_b, \mu_T, \nu_b) S(k_s', \mu_T, k_s'|+)
\]

- Complicated iterative distributional structure
- Intrinsically nonperturbative due to \( \gamma_{\nu}(\vec{k}_T, \mu) \sim \alpha_s(k_T) \)
  - Nonperturbative effects suppressed by \( \mathcal{O}(\Lambda_{\text{QCD}}^2/q_T^2) \), but require nonperturbative model
- No numerical solution available yet
Verification: LL cross section without $\alpha_s$-running.

Solution in distribution space:

\[
\sigma(q_T) = \sigma_0 \frac{1}{2\pi q_T} \frac{d}{dq_T} \theta(q_T) f_a(\omega_a, q_T) f_b(\omega_b, q_T) \exp \left[ -\frac{\Gamma_C}{2} \ln^2 \frac{Q^2}{q_T^2} \right] \\
\times \left[ 1 - 2\Gamma_C^2 \zeta_3 \ln \frac{Q^2}{q_T^2} + \Gamma_C^3 \left( \frac{2\zeta_3}{3} \ln^3 \frac{Q^2}{q_T^2} + 6\zeta_5 \ln \frac{Q^2}{q_T^2} \right) \\
+ \Gamma_C^4 \left( -4\zeta_5 \ln^3 \frac{Q^2}{q_T^2} + 10\zeta_3^2 \ln^2 \frac{Q^2}{q_T^2} - 30\zeta_7 \ln \frac{Q^2}{q_T^2} \right) \\
+ \mathcal{O}(\Gamma_C^5) \right]
\]

- Exponential resums $\ln(Q/q_T)$ at LL
- Many apparent-subleading terms arise from rapidity evolution
- These have no simple exponential structure
Solution in Fourier space:

\[
\sigma(\vec{q}_T) = \sigma_0 \frac{1}{2\pi q_T} \frac{d}{dq_T} \theta(q_T) f_a(\omega_a, q_T) f_b(\omega_b, q_T) \exp\left[-\frac{\Gamma_C}{2} \ln^2 \frac{Q^2}{q_T^2}\right] \\
\times \left[1 - 2\Gamma_C^2 \zeta_3 \ln \frac{Q^2}{q_T^2} + \Gamma_C^3 \left(\frac{2\zeta_3}{3} \ln^3 \frac{Q^2}{q_T^2} + 6\zeta_5 \ln \frac{Q^2}{q_T^2} - \frac{10}{3} \zeta_3^2\right)\right. \\
\left. + \Gamma_C^4 \left(-4\zeta_5 \ln^3 \frac{Q^2}{q_T^2} + 10\zeta_3^2 \ln^2 \frac{Q^2}{q_T^2} - 30\zeta_7 \ln \frac{Q^2}{q_T^2} + 28\zeta_3 \zeta_5\right)\right] \\
+ \mathcal{O}(\Gamma_C^5)
\]

\[
= \sigma_0 f_a(\omega_a, \mu) f_b(\omega_b, \mu) \int \frac{d^2 \vec{b}_T}{(2\pi)^2} e^{i\vec{b}_T \cdot \vec{q}_T} \exp\left[-\frac{\Gamma_C}{2} \ln^2 \frac{Q^2 b_T^2}{4e^{-2\gamma_E}}\right]
\]

- Simple Sudakov exponential in Fourier space → solution well-defined
- Induces same apparent-subleading terms as distributional solution
- Differ only by constant terms
  → Intrinsically different boundary condition than distribution space
Verification: LL cross section without $\alpha_s$-running.

Naive solution in momentum space:

$$\sigma(q_T) = \sigma_0 \frac{1}{2\pi q_T} \frac{d}{dq_T} \theta(q_T) f_a(\omega_a, q_T) f_b(\omega_b, q_T) \exp \left[ -\frac{\Gamma_C}{2} \ln^2 \frac{Q^2}{q^2_T} \right]$$

$$\times \left[ 1 + \frac{2}{3} \Gamma_C^3 \ln^3 \frac{Q^2}{q^2_T} \zeta_3 + \mathcal{O}(\Gamma_C^5) \right]$$

- Naive solution misses many apparently subleading terms
- These are crucial to cancel the observed divergence:
  
  *Hence, we find the peculiar feature that apparent-NNLL and higher terms cancel the divergence caused by the apparent-NLL terms in the strict LL spectrum.*

- Previous attempts tried to obtain an exponential form by neglecting apparently subleading terms $\times$  
  [Frixione, Nason, Ridolfi '97] [Ellis, Veseli '98]
- Resummation accuracy can not be specified by counting $\ln(Q/q_T)$!
Comparison to DDT-formula.

- Structure of the LL resummed cross section by [Dokshitzer, Diakonov, Troian '80]

\[
\sigma_{DDT}(\vec{q}_T) = \sigma_0 \frac{d}{dq_T^2} f_a(\omega_a, q_T) f_b(\omega_b, q_T) e^{S(Q,q_T)}
\]

- Comparison to LL in strict RGE counting (not counting \(\ln(Q/q_T)\)):

\[
\sigma_{LL}(\vec{q}_T) = \sigma_0 \frac{1}{2\pi q_T} \frac{d}{dq_T} f_a(\omega_a, \mu_T) f_b(\omega_b, \mu_T) \int d^2\vec{p}_T \\
|\vec{p}_T|\leq q_T \times \exp\left[\int_{\mu_H}^{\mu_T} \frac{d\mu'}{\mu'} \gamma_H(Q, \mu')\right] \left[\int d^2\vec{k}_s \delta(\vec{k}_s - \vec{k}'_s) \right]
\]

\[
+ \sum_{n=1}^{\infty} \prod_{i=1}^n \int_{k_{i-1}+}^{\nu_{i-1}} \frac{d\nu_i}{\nu_i} \int d^2\vec{k}_i \gamma_\nu(\vec{k}_{i-1} - \vec{k}_i, \mu_T) \delta(\vec{k}_s - \vec{p}_T - \sum_i \vec{k}_i) \right]
\]

\[
\times \left(\delta(\vec{k}_s) + \left[\frac{1}{2\pi k_s} \frac{d}{dk_s} \exp\left\{\int_{k_s}^{\mu_T} \frac{d\mu'}{\mu'} \frac{4\Gamma_C[\alpha_s(\mu')]}{\mu'} \ln\frac{\mu'}{k_s}\right\}\right]_{\mu_T}\right)
\]

- Simple structure of DDT-formula can not hold due to rapidity logarithms ✗
Comparison to coherent branching formalism.

- [Monni, Re, Torrielli '16] recently carried out resummation in momentum space using the coherent branching formalism [Banfi, Salam, Zanderighi '04]:

\[
\sigma(\vec{q}_T) = \sigma_0 \int d^2 \vec{k}_1 \frac{R'(k_1)}{2\pi k_1^2} e^{-R(\epsilon k_1)} \times \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=2}^{n+1} \int d^2 \vec{k}_i \frac{R'(k_i)}{2\pi k_i^2} \delta(\vec{q}_T - \sum_j \vec{k}_j) \]

- Comparison to our result:
  - \(\mu\)-evolution: \(R(\epsilon k_T) = \int_{\epsilon k_T}^{Q} \frac{d\mu'}{\mu'} \gamma_H(Q, \mu')\)
  - \(\nu\)-evolution: \(\frac{R'(k_i)}{2\pi k_i^2} = \int_{k_i}^{Q} \frac{d\nu'}{\nu'} \gamma_{\nu}(\vec{k}_i, \mu)\)
  - \(\epsilon\) acts as IR-regulator (similar to plus distributions)

Both approaches are closely related.
**Comparison to coherent branching formalism (NLL).**

- [Monni, Re, Torrielli '16] recently carried out resummation in momentum space using the coherent branching formalism [Banfi, Salam, Zanderighi '04]:

\[
\sigma(\vec{q}_T) = \sigma_0 \int d^2\vec{k}_1 \frac{R'(k_1)}{2\pi k_1^2} e^{-R(k_1) + \ln(\epsilon)R'(k_1)} \times \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=2}^{n+1} \int d^2\vec{k}_i \frac{R'(k_1)}{2\pi k_i^2} \delta(\vec{q}_T - \sum_j \vec{k}_j)
\]

- In practice: expand all $\vec{k}_i$ around hardest emission $\vec{k}_1$

- Resums $\ln(Q/k_1)$ rather than $\ln(Q/q_T)$
  - Formally equivalent?

- Great simplification: Avoids most nonperturbative effects

- Formally subleading difference to our method
  - Differences should be checked numerically
Conclusion.
Conclusion.

Resummation in momentum space:

- “Naive” resummation in distribution space yields spurious divergences from energetic emissions
- Distributional solution requires distributional scale setting to minimize boundary terms and is no simple exponential anymore
- Accuracy is entirely specified through anomalous dimensions: Accuracy can not be specified through counting of $\ln(Q/q_T)$
- Solved in theory, but numerical implementation is challenging

Phenomenological impact:

- Boundary terms intrinsically different from Fourier space resummation $\Rightarrow$ expect insight into (non)perturbative uncertainties
- $\bar{q}_T$-spectrum is intrinsically nonperturbative (no flaw of $\vec{b}_T$-space), but nonperturbative contributions are suppressed as $\mathcal{O}(\Lambda_{QCD}^2/q_T^2)$
- Numerical study will be interesting
Conclusion.

Resummation in momentum space:

- “Naive” resummation in distribution space yields spurious divergences from energetic emissions
- Distributional solution requires distributional scale setting to minimize boundary terms and is no simple exponential anymore
- Accuracy is entirely specified through anomalous dimensions: Accuracy can not be specified through counting of $\ln(Q/q_T)$
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Phenomenological impact:

- Boundary terms intrinsically different from Fourier space resummation $\Rightarrow$ expect insight into (non)perturbative uncertainties
- $\vec{q}_T$-spectrum is intrinsically nonperturbative (no flaw of $\vec{b}_T$-space), but nonperturbative contributions are suppressed as $O(\Lambda_{QCD}^2/q_T^2)$
- Numerical study will be interesting

Thank you for your attention!
Backup slides.
Alternative solution of toy function.

Toy RGE

\[ \mu \frac{dF(k, \mu)}{d\mu} = -\alpha_s F(k, \mu) \]

Ansatz

\[ F(k, \mu) = f_0(\mu)\delta(k) + [\theta(k)f_1(k, \mu)]^\mu \]

Apply derivative:

\[ \mu \frac{dF(k, \mu)}{d\mu} = \delta(k)\left( \mu \frac{df_0(\mu)}{d\mu} - \mu f_1(\mu, \mu) \right) + \left[ \theta(k)\mu \frac{df_1(k, \mu)}{d\mu} \right]^\mu \]

\[ \overset{!}{=} -\alpha_s f_0(\mu)\delta(k) - \alpha_s [\theta(k)f_1(k, \mu)]^\mu \]

Coupled system of RGEs:

\[ \mu \frac{df_1(k, \mu)}{d\mu} = -\alpha_s f_1(k, \mu), \]

\[ \mu \frac{df_0(\mu)}{d\mu} = -\alpha_s f_0(\mu) + \mu f_1(\mu, \mu). \]

\( f_0(\mu) \) is pure boundary, determines \( f_1(k, \mu) \):

\[ f_1(k, \mu) = \frac{d}{dk} f_0(k) \exp\left( \alpha_s \ln \frac{k}{\mu} \right). \]
Backup slides

Alternative solution of toy function.

- Toy RGE

\[ \mu \frac{dF(k, \mu)}{d\mu} = -\alpha_s F(k, \mu) \]

- Ansatz

\[ F(k, \mu) = f_0(\mu)\delta(k) + [\theta(k)f_1(k, \mu)]^\mu \]

- Simplest boundary: \( f_0(\mu) = 1 \)

\[
F(k, \mu) = \delta(k) + \left[ \theta(k) \frac{d}{dk} \exp\left(\alpha_s \ln \frac{k}{\mu}\right) \right]^\mu
\]

\[
= \delta(k) - \alpha_s \left[ \theta(k) \exp\left(\alpha_s \ln \frac{k}{\mu}\right) \right]^\mu
\]

- Exactly matches the solution using \( \mu_0 = k |^+ \)

- Illustrates equivalence of both techniques
Comparison: Resummation in Fourier space.

- Toy function is governed by \( \mu \frac{dF(k, \mu)}{d\mu} = -\alpha_s F(k, \mu) \)
- Formal solution in Fourier space: (neglecting \( \alpha_s \) running)
  \[
  \tilde{F}(y, \mu) = \tilde{F}(y, \mu_0) \exp \left( \alpha_s \ln \frac{\mu_0}{\mu} \right)
  \]
- \( \tilde{F}(y, \mu_0) \) depends on \( \ln(iy\mu_0e^{\gamma E}) \) \( \Rightarrow \) Choose \( \mu_0 = -ie^{-\gamma E}/y \)
- Transform back to distribution space:
  \[
  F(k, \mu) = \frac{e^{-\gamma E\alpha_s}}{\Gamma(1 + \alpha_s)} \left\{ \delta(k) + \alpha_s \left[ \frac{\theta(k)}{k} \exp \left( \alpha_s \ln \frac{k}{\mu} \right) \right]^\mu \right\}
  \]
- Subleading correction
- Distributional solution
- Fourier spaces induces different boundary term:
  \[
  \frac{e^{-\gamma E\alpha_s}}{\Gamma(1 + \alpha_s)} = 1 - \frac{\pi^2}{12} \alpha_s^2 + \cdots
  \]

Resummation in distribution space / Fourier space intrinsically probe different boundary conditions!
Illustration of soft $\nu$-evolution.

- Solve $\nu$-RGE
  \[ \nu \frac{dS(\vec{p}_T, \mu, \nu)}{d\nu} = (\gamma_\nu \otimes S)(\vec{p}_T, \mu, \nu) \]
  starting with boundary term $S(\vec{p}_T, \mu, \nu = p_T|_+)$

- To illustrate solution, expand
  \[ S(\vec{p}_T, \mu, \nu) = \sum_{n=0}^{\infty} S^{[n]}(\vec{p}_T, \mu, \nu) \]
  - $S^{[n]} = $ soft function after $n$ real emissions
  - $\gamma_\nu$ counts as one real emission

- $\nu$-RGE now becomes
  \[ \nu \frac{d}{d\nu} S^{[n]}(\vec{p}_T, \mu, \nu) = (\gamma_\nu \otimes S^{[n-1]})(\vec{p}_T, \mu, \nu) \]

- Recursive solution:
  \[ S^{[n]}(\vec{p}_T, \mu, \nu) = S^{[n]}(\vec{p}_T, \mu, \nu_0) + \int_{\nu_0}^{\nu} \frac{d\nu'}{\mu'} (\gamma_\nu \otimes S^{[n-1]})(\vec{p}_T, \mu, \nu') \]

- This requires $\nu_0 = p_T|_+$ at each order $n$!
Perturbativity of $\nu$-kernel.

- The $\nu$-kernel involves convolutions $\gamma_\nu \otimes^n$
- Problematic due to Landau pole in: $\gamma_\nu(\vec{k}_T, \mu) \sim \left[ \frac{4 \Gamma_C[\alpha_s(k_T)]}{2\pi k_T^2} \right]^\mu$ +
  - Expect $(\gamma_\nu \otimes \gamma_\nu)(\vec{p}_T, \mu) = \int d^2\vec{k}_T \gamma_\nu(\vec{p}_T - \vec{k}_T, \mu)\gamma_\nu(\vec{k}_T, \mu)$ to be non-perturbative
- $p_T, \mu \gg \Lambda_{QCD}$: Landau pole effectively regulated by plus prescription
- Illustration:

$$\int d^2\vec{k}_T \gamma_\nu(\vec{p}_T - \vec{k}_T, \mu)\gamma_\nu(\vec{k}_T, \mu)$$

$$\sim -2 \left[ \frac{\Gamma_C[\vec{p}_T]}{p_T^2} + \mathcal{O}\left(\frac{\Lambda}{p_T}\right) \right]^\mu \int_\Lambda^\mu \frac{dk_T}{k_T}\Gamma_C[\alpha_s(k_T)]$$

$$+ (\gamma_\nu \otimes \gamma_\nu)(\vec{p}_T, \mu) \mid_{\mathbb{R}^2\setminus (B_\Lambda(\vec{0}) \cup B_\Lambda(\vec{p}_T))}$$
In practice: Vary scales to test perturbative uncertainties

Also want to transition between

- Canonical scale setting: \( \mu_0 = p_T|_+ \), \( p_T \ll Q \)
- Fixed order regime: \( \mu_0 = \mu_{FO} \), \( p_T \sim Q \)

Implementation through a profile \( \mu_0(p_T) \) with distributional scale setting

Example: Rapidity anomalous dimension

\[
\gamma_\nu(\vec{p}_T, \mu) = \left\{ \gamma_\nu^{FO}(\vec{p}_T, \mu_0) - \delta(\vec{p}_T) \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} 4\Gamma_C[\alpha_s(\mu')] \right\}_{\mu_0 = \mu_0(p_T)|_+}
\]

Profile function:

- Canonical scale setting: \( \mu_0(p_T) \propto p_T \), \( p_T \ll Q \)
- Fixed order regime: \( \mu_0(p_T) \propto \mu_{FO} \), \( p_T \sim Q \)
Example: Implementation of profiles for $\gamma_\nu$.

- Resum $\gamma_\nu$ at lowest order (i.e. keep only $\beta_0$, $\Gamma_0$)
- together with the full $\mathcal{O}(\alpha_s)$ boundary term

$$2\Gamma_0 \mathcal{L}_0(\vec{p}_T, \mu) \frac{\alpha_s(\mu)}{4\pi}$$

Result:

$$\gamma^{(N)LL(0)}_\nu(\vec{p}_T, \mu) = 2\Gamma_0 \mathcal{L}_0(\vec{p}_T, \mu)$$

$$\gamma^{(N)LL(1)}_\nu(\vec{p}_T, \mu) = -2\beta_0 \Gamma_0 \frac{1}{\pi \mu^2} \left[ \frac{\mu^2}{\vec{p}_T^2} \ln \frac{\mu_0(p_T)^2}{\mu^2} \right]^\mu + 2\beta_0 \Gamma_0 \frac{1}{\pi \mu^2} \left[ \frac{\mu^2}{\vec{p}_T^2} \ln \frac{\mu_0(p_T)^2}{p_T^2} \right]^\mu + 4\Gamma_0 \beta_0 \ln^2 \frac{\mu}{\mu_0(\mu)} \delta(\vec{p}_T)$$

- Full result at $\mathcal{O}(\alpha_s^2)$:

$$\gamma^{(1)}_\nu(\vec{p}_T, \mu) = -2\beta_0 \Gamma_0 \frac{1}{\pi \mu^2} \left[ \frac{\mu^2}{\vec{p}_T^2} \ln \frac{p_T^2}{\mu^2} \right]^\mu + 2\Gamma_1 \frac{1}{\pi \mu^2} \left[ \frac{\mu^2}{\vec{p}_T^2} \right]^\mu + \gamma_{\nu 1} \delta(\vec{p}_T)$$

- Vanishes for canonical scale $\mu_0(x) = x$ → probes subleading terms
- Reproduces exact result to this order for canonical scale $\mu_0(x) = x$
Nonperturbative modelling of $\gamma_\nu$. 

- Consider for simplicity only $\gamma_\nu(\vec{p}_T, \mu) = \left[ \frac{4 \Gamma_C [\alpha_s(p_T)]}{2 \pi p_T^2} \right]^{\mu} + \cdots$

- Split using a profile function $\mu_0(p_T)$:
  
  
  
  $\gamma_\nu(\vec{p}_T, \mu) = \left[ \frac{1}{2 \pi p_T^2} 4 \Gamma_C [\alpha_s(\mu_0(p_T))] \right]^{\mu} + \left[ \frac{1}{2 \pi p_T^2} (4 \Gamma_C [\alpha_s(p_T)] - 4 \Gamma_C [\alpha_s(\mu_0(p_T))]) \right]^{\mu}$

- For suitable profiles $\mu_0$, we can expand in moments:
  
  
  
  $\gamma_\nu(\vec{p}_T, \mu) = \left[ \frac{1}{2 \pi p_T^2} 4 \Gamma_C [\alpha_s(\mu_0(p_T))] \right]^{\mu} + \sum_{n=1}^{\infty} \Omega_n \Delta^n \delta(\vec{p}_T)$

- Moments become more intuitive in $\vec{b}_T$-space:
  
  $\int d^2 \vec{p}_T e^{i \vec{p}_T \cdot \vec{b}_T} \sum_{n=1}^{\infty} \Omega_n \Delta^n \delta(\vec{p}_T) = \sum_{n=1}^{\infty} \Omega_n (-b_T^2)^n$

- Leading nonperturbative effect on spectrum is a Gaussian:
  
  $\sigma(\vec{q}_T) \sim \int d^2 \vec{b}_T e^{i \vec{q}_T \cdot \vec{b}_T} H(Q) B(b_T)^2 S(b_T) e^{\ln \frac{\nu_S}{\nu_B} \tilde{\gamma}_{\nu_{(pert)}}(\vec{b}_T, \mu)} e^{-\ln \frac{\nu_S}{\nu_B} \Omega_1 b_T^2 + \cdots}$
Backup slides

RG-evolved cross section (1).

\[ \sigma(\vec{q}_T) = \sigma_0 H(Q, \mu_H) \frac{1}{2\pi q_T} \frac{d}{dq_T} \int d^2 \vec{p}_T \]

\[ \times \exp \left[ \int_{\mu_H}^{\mu_T} \frac{d\mu'}{\mu'} \gamma_H(Q, \mu') \right] \int d^2 \vec{k}_a d^2 \vec{k}_b d^2 \vec{k}_s \delta(\vec{p}_T - \vec{k}_a - \vec{k}_b - \vec{k}_s) \]

\[ \times \int d^2 \vec{k}_s' \left[ \delta(\vec{k}_s - \vec{k}_s') \right] \]

\[ + \sum_{n=1}^{\infty} \prod_{i=1}^{n} \int_{k_{i-1}^+}^{\nu_i-1} \frac{d\nu_i}{\nu_i} \int d^2 \vec{k}_i \gamma_{\nu}(\vec{k}_{i-1} - \vec{k}_i, \mu_T) \delta(\vec{k}_s - \vec{k}_s' - \sum_{i} \vec{k}_i) \]

\[ \times B_a(\omega_a, \vec{k}_a, \mu_T, \nu_a) B_b(\omega_b, \vec{k}_b, \mu_T, \nu_b) S(\vec{k}_s', \mu_T, k_s'|+) \]

\( \nu \)-evolution

\( \nu \)-logs minimized
R.G.-evolved cross section (2).

\[ B_a(\omega_a, \vec{k}_a, \mu_T, \nu_a) \, B_b(\omega_b, \vec{k}_b, \mu_T, \nu_b) \, S(\bar{k}'_s, \mu_T, \bar{k}'_s|+) \]

\[ \supset \left[ \frac{1}{2\pi k_a} \frac{d}{dk_a} \frac{1}{2\pi k_b} \frac{d}{dk_b} \frac{1}{2\pi k_s} \frac{d}{dk_s} \right] B(\omega_a, k_a) \, B(\omega_b, k_b) \, S[\alpha_s(k_s)] \]

\[ \times \exp \left\{ \int_{k_a}^{\mu_T} \frac{d\mu'}{\mu'} \, \gamma_B(\omega_a, \mu', \omega_a) + \int_{k_b}^{\mu_T} \frac{d\mu'}{\mu'} \, \gamma_B(\omega_b, \mu', \omega_b) \right. \]

\[ + \left. \int_{k_s}^{\mu_T} \frac{d\mu'}{\mu'} \, \gamma_S(\mu', k_s) \right\}^{\mu_T} \]

All logs minimized (pure boundary)
Comparison to Fourier resummation.

- Fourier resummation has analogous form of [Collins, Soper, Sterman '85]:
  \[
  \sigma(\bar{q}_T) = \sigma_0 \int \frac{d^2 \bar{b}_T}{(2\pi)^2} e^{i\bar{b}_T \cdot \bar{q}_T} 
  \times \exp \left[ -\int_{(1/b_T)^2}^{Q^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2} \left( \ln \frac{Q^2}{\bar{\mu}^2} A[\alpha_s(\bar{\mu})] + 2B[\alpha_s(\bar{\mu})] \right) \right] 
  \times H(Q, Q) \tilde{B}^2(\bar{b}_T, 1/b_T, Q) \tilde{S}(\bar{b}_T, 1/b_T, 1/b_T) 
  \]

  All scales minimized in Fourier space

- Relation to our notation:
  - \[ A(\alpha_s) = \Gamma_C(\alpha_s) + \beta(\alpha_s) \frac{d\tilde{\gamma}_\nu(\alpha_s)}{\alpha_s} \]
    \[ \rightarrow \text{Rapidity evolution contributes to cusp-term [Becher, Neubert '10]} \]
  - \[ B(\alpha_s) = \gamma_H(\alpha_s) - \tilde{\gamma}_\nu(\alpha_s) \]

- Remark: Beam and soft functions are often combined into TMDPDFs

- In practice: \( \ln(Q^2 b^2) \rightarrow \ln(1 + Q^2 b^2) \) differs from canonical resummation, but suppresses small \( \bar{b}_T \rightarrow 0 \), e.g. in DYRes [Catani, de Florian, Ferrera, Grazzini '15 ...], HRes [de Florian, Ferrera, Grazzini, Tommasini '12 ...]
Comparison to partial Fourier resummation.

- Many groups employ hybrid scale choice $\mu_T$ [Becher, Neubert, Wilhelm '12 ’13], [D'Alesio, Echevarria, Melis, Scimemi '14], [Echevarria, Kasemets, Mulders, Pisano ’15]
  
  $\mu_H = Q, \mu_B = \mu_T, \mu_S = \mu_T$

  $\nu_B = Q, \nu_S \sim 1/b_T$

- where $\mu_T \sim Q_0 + q_T$ is chosen in momentum space and $Q_0 \approx 2 \text{ GeV}$ [D'Alesio, Echevarria, Melis, Scimemi '14] / $Q_0 \approx 8 \text{ GeV}$ [Becher, Neubert, Wilhelm ’12]

- RG-evolved cross section:
  
  $\sigma(q_T) = \sigma_0 \int \frac{d^2\vec{b}_T}{(2\pi)^2} e^{i\vec{b}_T \cdot \vec{q}_T}$

  $\times \exp\left[-\int_{\mu_T}^{Q} \frac{d\mu'}{\mu'} \gamma_H(Q, \mu')\right] \exp\left[-\ln(b_T Q) \tilde{\gamma}_\nu(\vec{b}_T, \mu_T)\right]

  \times H(Q, Q) \tilde{B}^2(\vec{b}_T, \mu_T, Q) \tilde{S}(\vec{b}_T, \mu_T, 1/b_T)$

  $\triangleright$ $\nu$-scale chosen in Fourier space $\rightarrow$ no spurious divergence

  $\triangleright$ $\mu$-scale chosen in physical space $\rightarrow$ no nonperturbative effects

- Differs from canonical resummation for small $\vec{q}_T$