Shaping tail dependencies by nesting box copulas

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Abstract
We introduce a family of copulas which are locally piecewise uniform in
the interior of the unit cube of any given dimension. Within that family,
the simultaneous control of tail dependencies of all projections to faces
of the cube is possible and we give an efficient sampling algorithm. The
combination of these two properties may be appealing to risk modellers.

1 Introduction
Copulas have become an accepted tool for modelling dependencies in the fi-
nancial industry; an overview from an applications point of view is given by
P. Embrechts in [2] together with a comprehensive list of references. One rea-
son why risk modellers have become more used to working with copulas is the
fact that copulas allow the creation of models with increasing dependencies in
the tails of the marginal distributions.

We deal with two shortcomings of some of the currently used copula mod-
els. Firstly, practical algorithms for generating independent random samples of
copulas efficiently, particularly in higher dimensions, are relatively scarce (refer
to A. McNeil et al. [6] and A. McNeil [5] for various simulation algorithms).
Secondly, the parameters of the most prominent copulas such as the t-copulas,
archimedean copulas (for instance Clayton and Gumbel) or nested archimedean
copulas are related to pairwise dependencies of the marginal distributions so
that a copula in the corresponding family is uniquely determined by the pro-
jections onto the 2-dimensional faces.

In this paper we give a general construction principle, which we call tail
nesting, for copulas in any dimension r. The characteristics of the tails can
be shaped by prescribing the tail dependencies in a very flexible manner. The
resulting copulas have efficient simulation algorithms.

A copula C corresponds to a Borel probability measure c on the r-cube
$[0,1]^r$, which, when projected to any 1-dimensional face, yields the uniform
probability measure. The correspondence between C and c is given via $C(u) =
c([0,u])$ for $u = (u_1, \ldots, u_n)$ and $[0,u] := \prod_i [0,u_i]$. In the context of this paper,
working directly with the measure c turns out to be more convenient and we
call $c$ as well a copula or a copula measure. We refer to [3] or R. Nelsen [9] for an introduction to copulas.

Our main result is summarised below in this section. Some readers may prefer to read first the motivating examples in Section 2 and then return to the paragraph below.

To begin with, we define the notion of tail dependency in higher dimensions which we work with. It is motivated by Example 2.3 and the definition of lower tail dependency in [6].

**Definition 1.1.** For $c$ as above, we define the tail degree of $c$,

$$\text{td}(c) = \inf \left\{ \tau \mid \liminf_{s \to 0} \frac{c(s \cdot [0,1]^r)}{s^\tau} = \infty \right\}. \quad (1)$$

Its tail coefficient in case $\text{td}(c) < \infty$ is

$$\text{tc}(c) = \liminf_{s \to 0} \frac{c(s \cdot [0,1]^r)}{s^{\text{td}(c)}} \quad (2)$$

where $s \in (0,1]$. We observe that $\text{td}(c) \geq 1$ for $r \geq 1$. Formally, we set $\text{tc}(c) = 0$ if $\text{td}(c) = \infty$ and define the tail characteristic of $c$ as the function

$$\text{tchar}_c = (\text{tcoef}_c, \text{tdeg}_c): \mathcal{F} \to [0,\infty] \times [0,\infty]$$

on the set of front faces $\mathcal{F}$ of $[0,1]^r$ by $\text{tcoef}_c(F) = \text{tc}(c_F)$, $\text{tdeg}_c(F) = \text{td}(c_F)$. Here $c_F$ denotes the push forward measure of $c$ to $F \in \mathcal{F}$ with respect to the canonical projection $[0,1]^r \to F$. Front faces of $[0,1]^r$ are those faces which contain the origin. Hence the tail characteristic is the collection of all tail coefficients and tail degrees of the projections of $c$ to the front faces of $[0,1]^r$.

We say that a copula $c$ on $[0,1]^r$ has tail dependence of degree $\text{td}(c)$ if its tail degree satisfies $\text{td}(c) < r$; otherwise it has no tail dependence.

**Example 1.2.** The Clayton copula given by $\text{cl}([0,u]) = (u_1^{-\theta} + \cdots + u_r^{-\theta} - r + 1)^{-1/\theta}$ with parameter $\theta \in (0,\infty)$ satisfies $\text{tchar}_{\text{cl}}(F) = (\dim F^{-1/\theta}, 1)$ for $F$ with $\dim F \geq 1$.

The Gumbel copula $\text{gu}([0,u]) = \exp(-((-\ln u_1)^\theta + \cdots + (-\ln u_r)^\theta)^{1/\theta})$, for parameter $\theta \in [1,\infty)$ satisfies $\text{tchar}_{\text{gu}}(F) = (1, (\dim F)^{1/\theta})$ for $\dim F \geq 1$. Hence the Gumbel copula has tail dependence of degree $r^{1/\theta}$ provided $\theta > 1$.

This must not be confused with the tail dependencies at the opposite vertex $(1,\ldots,1)$.

For convenience and without loss of generality we work exclusively with tail dependencies at the origin. For $r = 2$, lower tail dependence in [6] implies tail dependence of degree 1 and tail coefficient $> 0$ for the face $[0,1]^2$. We refer to A. Charpentier & J. Segers [11] who have investigated the tails of archimedean copulas in a very general setting.

We consider now a face $F'$ and a face $F$ of $F'$. As projecting the copula first to $F'$ and the result to $F$ is the same as projecting the copula directly to $F$ we see that $\text{tdeg}$ is a non-decreasing map in the following sense: If $F, F'$ are two faces of $[0,1]^r$ and $F \subset F'$, then $\text{tdeg}(F) \leq \text{tdeg}(F')$. We call a map $b: \mathcal{F} \to \mathbb{R}$ with $b(F) \leq b(F')$ for $F \subset F'$ *non-decreasing* and *increasing* if the strict inequality $b(F) < b(F')$ holds for any $F \subset F'$.  

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Main Results. Let $a, b: \mathcal{F} \to (1, \infty)$ denote maps on the faces of $[0, 1]^r$ such that $(a(F), b(F)) = (1, \dim F)$ for those $F \in \mathcal{F}$ with $\dim F \leq 1$. Let $b$ be non-decreasing.

(i) If $b$ is increasing we construct a copula measure $c$ with $t_{char} c = (a, b)$. The copula is locally piecewise uniform in $(0, 1)^r$.

(ii) If $b$ is not increasing, we give necessary conditions for $a, b$ such that $t_{deg} c = (a, b)$ for some copula $c$. We investigate special cases where $t_{deg}$ is not increasing.

(iii) The construction for the proof generalises naturally to copulas of order $k$, i.e., measures on the unit cube which project to any face of dimension $k$ to the uniform probability measure.

(iv) Tail characteristics for risks $X_1, \ldots, X_r$ could be defined via the transformations of the $X_i$ to uniform random variables. The construction works just as well with any other transformation.

(v) The construction comes along with an efficient simulation algorithm.

Finally we remark that the construction is elementary and contributes to the understanding of dependence patterns for random variables. We can imagine many applications for risk modelling. Some of them we are going to discuss elsewhere.

The paper is organised as follows. We illustrate and motivate the copula construction principle in order to prove the main results by means of two simple examples in the next section. In Section 3 we introduce some notation and in Section 4 we study the spaces of the most simple non-trivial copulas in any dimension. They are the building blocks in the construction of our main result. We introduce the construction technique of nesting in Section 5. We explore it in Section 6 for shaping the tail characteristics and eventually state Theorem 6.5 about tail nesting. We derive some corollaries in Section 7 and discuss the construction further in Section 8.

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2 Motivation

This section illustrates some of the ideas and observations in this paper in a very elementary fashion. The reader may find the descriptions in this section helpful when going through the construction in any dimension.

Example 2.1 (Tail nesting in dimension 2). We decompose the unit 2-dimensional square $[0, 1]^2$ into four boxes by splitting each edge in the middle, i.e., into $[s, s + 1/2] \times [t, t + 1/2]$, for $s = 0, 1/2$ and $t = 0, 1/2$. Each of the four vertices $(i, j) \in \{0, 1\}^2$ corresponds to one of these squares. Now we choose a probability measure on $[0, 1]^2$ which has constant density in each square.
We describe this measure by the map \( c': \{0, 1\}^2 \rightarrow [0, 1] \) which assigns to each vertex the measure of the corresponding box in the decomposition. This measure is a copula if it projects to the uniform measure on each of its edges. In choosing a copula \( c \) of that type we have only one degree of freedom. We can set the probability \( c'(0, 0) = c([0, 1/2]^2) \) equal to any \( p \in [0, 1/2] \). Then, due to the copula condition, \( c'(0, 1) = c'(1, 0) = -p + 1/2 =: q \) and thus \( c'(1, 1) = p \). This is the most simple case of a grid copula. The application of grid copulas in risk management was suggested by D. Straßburger & D. Pfeifer [11].

![Figure 1: Nesting a 2-dimensional box copula into itself. Grey levels according to the density of probability for some \( p \in (1/4, 1/2) \).](image)

The following observation is simple, but crucial for the remainder of the paper: We set \( c^1 = c \) as above. Then we nest \( c \) into the square \([0, 2^{-1}]^2 \) of \( c^1 \) as follows: Decompose that square again into four equally sized squares \([s, s+2^{-2}] \times [t, t+2^{-2}] \) for \( s = 0, 2^{-2} \) and \( t = 0, 2^{-2} \) and modify \( c^1 \) on \([0, 2^{-1}]^2 \) by ‘multiplying’ it with \( c \), in order to obtain \( c^2 \), as illustrated in Fig. 1.

In this construction we have refined the initial decomposition of \([0, 1]^2 \). We call such decompositions box decompositions. It can be verified immediately that \( c^2 \) is again a copula. We call copulas of that type box copulas. Now we can repeat this construction by nesting \( c \) into the square \([0, 2^{-2}]^2 \) of \( c^2 \), in order to obtain \( c^3 \) and so on, by recursively nesting \( c \) into the square \([0, 2^{-n}]^2 \) of \( c^n \). The limit \( c^\infty \) of this sequence of copulas exists and is again a copula. Suppose we start with \( p = (1/2)^b \) and \( b > 1 \). Then \( c^\infty([0, 2^{-n}]^2) = p^n = (2^{-nb}) \) and thus \( \lim \sup c^\infty([0, u]^2)/u = 0 \) as \( u \to 0 \).

**Remark 2.2.** Key observations when studying the simple example are:

(i) The copulas \( c^n \) are asymmetric and the probability density increases as \( u \to 0 \) for \( p > 1/4 \).

(ii) As \( (u/2)^b < c^\infty([0, u]^2) \leq u^b \), the copula \( c^\infty \) has zero lower tail dependence\(^1\) for \( b > 1 \).

(iii) Nevertheless, given \( p \in [0, 2^{-n}] \), we can choose \( b \) such that \( c^\infty([0, 2^{-n}]) = p \). Hence this copula family is still good enough for sensitivity testing in risk modelling. Furthermore, there is a simple recursive algorithm to generate samples of \( c^\infty \).

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\(^1c^\infty \) is said to have lower tail dependence if \( \lim c^\infty([0, u]^2)/u > 0 \) as \( u \to 0 \), refer e.g. to [6] for details.
(iv) We can further modify the tail behaviour by nesting in the $n$-th step a copula of the same type but with $c'(0,0) = (1 + \delta_n) \cdot 2^{-b}$ where $|\delta_n|$ is close to 0. By choosing appropriate sequences $(1 + \delta_n, b)$ we can not only control the tail dependencies in the limit but also how the limit is approached as $n \to \infty$.

**Example 2.3.** Analogously to the decomposition in Example 2.1, we decompose now the unit 3-dimensional cube $[0,1]^3$ into 8 cubes, each isometric to $[0,1/2]^3$. Each of these cubes contains exactly one of the vertices $\nu \in \{0,1\}^3$. We assign to the ‘even’ cubes (i.e., those with $\nu_1 + \nu_2 + \nu_3 \equiv 0 \bmod 2$) the uniform measure with total probability equal to $1/4$. The ‘odd’ cubes get probability zero. When projected to any 2-dimensional face, the resulting probability measure is the uniform probability measure. In particular, $c$ is a copula.

![Illustration of $c^9$ from Example 2.3](image)

The pictures were generated by plotting one point in the centre of each of the 49 ‘even’ cubes. The grey level of a point $(u_1,u_2,u_3)$ is given by $-u_1$. The view in the left picture is along a diagonal. From left to right it is stepwise rotated around the third coordinate axis. The points in the picture on the right are uniformly distributed in the $(u_2,u_3)$-plane. Their grey level is merely an indication of the $u_1$-level.

Now set $c^1 = c$ and nest $c$ into the even cubes of the decomposition underlying $c^1$. In this way the even cubes decompose again into 8 cubes, each isometric to $[0,1/4]^2$; four are again ‘even’ and the others are odd. We obtain a copula $c^2$ on $[0,1]^3$, its support consisting of 16 cubes, each with uniform measure and probability equal to $(1/4)^2$. Observe now that $c^2$ still projects to the uniform measure on each of the 2-dimensional faces of $[0,1]^3$. We can continue this nesting and obtain a limit measure $c^\infty$.

The projection of the limit measure to any 2-face of $[0,1]^3$ is again the uniform measure and thus $c^\infty$ is in particular a copula. It is not difficult to see that the limit measure is, up to scaling, the 2-dimensional Hausdorff measure of the support of $c^\infty$. The latter is the intersection of all the ‘even’ cubes obtained during the recursive definitions of the $c^n$.

**Remark 2.4.** We summarise the main observations from the above example. To this end assume that we have three risks $X_1, X_2, X_3$ whose dependence structure is given by $c^\infty$, i.e., $P(X_i \leq Q_i(u_i), i = 1, 2, 3) = c^\infty([0,u])$ where $Q_i$ is the quantile function.

(i) Even if risks $X_1, X_2, X_3$ are pairwise independent, they can be heavily dependent overall. For the univariate margins of the above copula the
third margin is a function of the other two.

(ii) The probability that all three risks are worse than their \(2^{-n}\)-Quantile is \((2^{-n})^3\). A measure for tail dependencies in higher dimensions should show that \(c^\infty\) has some tail dependence. This is one motivation for Definition 1.1.

It has become more and more common that risk modellers focus on tail dependencies when modelling a portfolio. This example demonstrates nicely that in calibrating the corresponding dependence models it is not sufficient to focus on the estimation of pairwise dependencies alone.

The construction for shaping the tails of copulas in Section 6 is as in Example 2.1, but generalised to any dimension \(r \geq 2\). Roughly speaking, we can shape the projections to any lower dimensional faces, which are also of this type, simultaneously so that we can achieve any tail characteristic which is consistent with the condition for probability measures.

Before describing this aspect, we need to define some notation related to cubes, their vertices and faces in Section 3. Then we study maps \(\{0,1\}^r \to [0,1]\), the equivalents of those maps for \(r = 2\) in Example 2.1 which define copulas in Section 4.

### 3 Notation and basic definitions

By \(u, v, w\) we denote usually points in \(\mathbb{R}^r\) where \(u = (u_1, \ldots, u_r)\). The unit \(r\)-cube is \([0,1]^r \subset \mathbb{R}^r\). The set of its vertices is

\[
V := \{0,1\}^r.
\]

We use the letters \(\nu, \mu\) exclusively for elements of \(V\). There is a one-to-one correspondence between front faces \(F\) of the \(r\)-cube and the vertices \(\nu \in \{0,1\}^r\) given by

\[
F(\nu) = \{u \in [0,1]^r \mid u_i = 0 \text{ if } \nu_i = 0\} \tag{3}
\]

We set \(\mathbf{0} := (0, \ldots, 0)\) and \(\mathbf{1} := (1, \ldots, 1)\). and denote by

\[
\hat{} : u \mapsto \hat{u} = \mathbf{1} - u
\]

the reflection with \(\mathbf{0} = \mathbf{1}\). For a front face \(F\) the corresponding back face is \(\hat{F}\) and the complementary front face \(F^c\). We observe that the front face complementary to \(F(\nu)\) is \(F(\hat{\nu})\). We can identify each face \(F\) naturally with \([0,1]^{\dim F}\). Other faces of \([0,1]^r\) are of the form \(F(\nu) + \mu\) for \(\mu \in F(\hat{\nu})\). Given a front face \(F\) of \([0,1]^r\) we denote by

\[
\pi^F : [0,1]^r \to F^c \tag{4}
\]

the canonical projection along \(F\) to its complement \(F^c\).

An interval \(I\) in \(\mathbb{R}^r\) is an \(r\)-fold product of intervals in \(\mathbb{R}\). We call a compact interval in \(\mathbb{R}^r\) with non-empty interior an \(r\)-box in \(\mathbb{R}^r\). Hence \(r\)-boxes are of the form \([u_1, v_1] \times \cdots \times [u_r, v_r] =: [u,v]\) with \(u_i < v_i\) for each \(i\). A box
decomposition $\mathcal{I}$ of an $r$-box $I$ is a collection of finite $r$-boxes $\{I_1, \ldots, I_n\}$ such that their union is $I$ and their non-trivial intersections are of lower dimensions, i.e., $I_1 \cup \cdots \cup I_n = I$ and interior($I_k$) $\cap$ interior($I_l$) $\neq \emptyset$ for all $I_k \neq I_l$.

Given an $r$- and $r'$-box $I$ and $I'$ with box decompositions $\mathcal{I}$ and $\mathcal{I}'$, respectively, we can form the product box $I \times I'$ in $\mathbb{R}^{r+r'}$ which inherits a natural box decomposition $\mathcal{I} \times \mathcal{I}'$, the product decomposition. If a box decomposition of a box $I$ is the product of box decompositions of its 1-dimensional faces we call it a grid decomposition or simply a grid.

Given an $r$-box $I = [u, v]$ we denote by $\iota$ the canonical affine transformation from the unit $r$-cube $[0, 1]^r$ to $I$,

$$\iota: [0, 1]^r \to I, \quad w \mapsto (u_1 + w_1(v_1 - u_1), \ldots, u_r + w_r(v_r - u_r))$$

Via this transformation we define the corresponding faces of and projections (Box measure). Definition 3.1. Let $\mathcal{I} = \{I_1, \ldots, I_n\}$ be a box decomposition of $I$ and $[0, 1]^r$. We view elements $b \in \mathbb{R}^\mathcal{I}$, i.e., maps $b: \mathcal{I} \to \mathbb{R}$, as signed measures on $I$ such that $b|_{I_i}$ is a uniform measure, i.e., proportional to the Lebesgue measure of $I_i$.

Let $\mathcal{J}$ be a box decomposition of $[0, 1]^r$. The canonical transformations $\iota: [0, 1]^r \to I$ induces a canonical vector space isomorphism $\iota: \mathbb{R}^\mathcal{J} \to \mathbb{R}^{\iota(\mathcal{J})}$ by pushing forward the measures.

In the same way we can push forward box measures using the canonical projections $\pi^F$ along faces. In case $\mathcal{J}$ is not a grid decomposition, the elements of $\pi^F(\mathcal{J})$ do not define a box decomposition of $F^c$ (see e.g., Fig. 1). In this case, we denote by $\pi^F(\mathcal{J})$ the projection of some refinement of $\mathcal{J}$ into a grid.

In this way, we can push forward box measures in $\mathbb{R}^\mathcal{J}$ to box measures on $F^c$ and obtain a linear map

$$\pi^F: \mathbb{R}^\mathcal{J} \to \mathbb{R}^{\pi^F(\mathcal{J})}.$$  

Our ultimate goal is to construct copulas in $\mathbb{R}^\mathcal{J}$ satisfying some given conditions, such as a certain behaviour near the the origin $o \in [0, 1]^r$. To this end the following notion turns out to be quite useful.

Definition 3.2. Let $\mathcal{J}$ be a box decomposition of $[0, 1]^r$. Then define the following linear subspaces of $\mathbb{R}^\mathcal{J}$ for $k = 0, 1, \ldots, r$,

$$G_k(\mathcal{J}) = \bigcap \left\{ \ker(\pi^F: \mathbb{R}^\mathcal{J} \to \mathbb{R}^{\pi^F(\mathcal{J})}) \mid F \in \mathcal{F}, \ \text{codim}(F) = k \right\}$$

and set $G_{-1}(\mathcal{J}) = \mathbb{R}^c$.

Observe that $G_k(\mathcal{J})$ is the set of those box measures which project onto each face of dimension $k$ to the zero measure. Thus it does not depend on the choice of $\pi^F(\mathcal{J})$ in $[0]$. Furthermore, we obtain a filtration of linear subspaces of $\mathbb{R}^\mathcal{J}$,

$$\{0\} = G_r(\mathcal{J}) \subset G_{r-1}(\mathcal{J}) \subset \cdots \subset G_0(\mathcal{J}) \subset G_{-1}(\mathcal{J}) = \mathbb{R}^\mathcal{J}$$

We let $cu_{\mathcal{J}} \in \mathbb{R}^\mathcal{J}$ denote the element which corresponds to the uniform probability measure on $[0, 1]^r$. 

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Definition 3.3. For \( k = 0, 1, 2, \ldots, r \) we define the following subsets of \( \mathbb{R}^J \):

\[
C_k(J) = (\text{cu}_J + G_k(J)) \cap [0,1]^J.
\] (9)

The set of probability measures in \( \mathbb{R}^J \) is \( C_0(J) \subset \mathbb{R}^J \) and the subset of copulas is \( C_1(J) \), the box copulas in \( \mathbb{R}^J \). We call elements of \( C_k(J) \) box copulas of order \( k \). We observe that

\[
\pi^F: C_k(J) \to C_k(\pi^F(J))
\]

for \( k \leq \text{codim } F \). The sets \( C_k(J) \) are convex in \( \mathbb{R}^J \). We call elements of \( G_k(J) \) copula generators of order \( k \) for \( J \).

The limits of sequences of box copulas we work with later on are not box copulas. We extend some notions from above to Borel measures on \( [0,1]^r \) such as the property of being a copula of order \( k \) in the obvious way. We call a probability measure \( c \) on \( [0,1]^r \) locally piecewise uniform on \( E \subset [0,1]^r \) if for each \( w \in E \) there exists a neighbourhood \( W \) of \( w \) in \( [0,1]^r \) such that \( c|_W \) is equal to some box measure restricted to \( W \).

Remark 3.4. Note that by construction each box copula corresponds to a probability measure on \( [0,1]^r \) such that the projection to any edge yields the uniform probability measure. The notion of a box copula differs from the notion of a grid-type copula \[11\] or simply grid copula only by the underlying decomposition. We already made use of the fact that any box decomposition can be refined into a grid decomposition when defining projections. One reason for introducing the notion of a box copula is that it is useful when designing efficient simulation algorithms for nested copulas (see Algorithm 6.4). For that purpose, we may want to describe the measure with the smallest possible number of boxes (see Fig. 1 for an illustration).

Example 3.5 (Box copulas). It is not difficult to construct box copulas:

(i) The copulas \( c^n, n < \infty \), from Example 2.3 are box copulas of order 2.

(ii) Take any copula \( c' \) on \( [0,1]^r \), select some \( J \) and let \( c(J) \) be the \( c' \)-measure of \( J, J \in \mathcal{J} \). Then \( c \in C_1(J) \). The finer \( J \), the better the approximation of \( c' \) by \( c \).

(iii) Given some copula generator \( g \in G_k(J) \), some \( c \in C_k(J) \cap (0,1)^J \), then \( c + tg \in C_k(J) \) for \( |t| \leq \varepsilon \) and \( \varepsilon > 0 \) sufficiently small. This follows from the convexity of \( C_k(J) \).

Remark 3.6 (Copula surgery). If \( I \subset [0,1]^r \) is an \( r \)-box, \( g \in G_k(J) \) and \( \iota: [0,1]^r \to I \) the canonical map, then \( \iota(g) \) can be viewed as a copula generator on \([0,1]^r \) with support contained in \( I \) and with respect to any box decomposition \( \mathcal{I} \) which extends \( \iota(J) \). Observe that we can now build the sum \( c + \iota(g) \) as signed measures for \( c \in C_k(\mathcal{I}) \). Provided \( g \) is appropriate, e.g., \(|g(\nu)| \) sufficiently small for each \( \nu \), \( c + \iota(g) \) is a copula of order \( k \). It is indeed a box copula for any common refinement of \( \mathcal{I}' \) and \( \mathcal{I} \). This construction can be iterated with
appropriate sequences of generators. The reader may wish to compare this with some of the methods for the construction of copulas in [9].

Nesting and tail nesting which we introduce later on are special cases of ‘copula surgery’. In order to explore the range of such constructions, a detailed study of the vector spaces $G_k(\mathcal{J})$ for certain $\mathcal{J}$ turns out to be helpful. This is the subject of the next section.

4 Vertex decompositions and their copulas

In this section we will describe the box copula spaces for the most simple box decompositions. A point $u \in (0, 1)^r$ induces a decomposition of the $i$-th edge, namely $\{[0, u_i], [u_i, 1]\}$. The product of these decompositions is a box decomposition $\mathcal{J}(u)$. Each box of this decomposition contains exactly one vertex of $[0, 1]^r$. In this way, we identify $\mathcal{J}(u) \simeq V = \{0, 1\}^r$ via $u$. We call these decompositions vertex decompositions and corresponding box measures vertex measures. Vertex measures are defined by $u$ together with a map $\{0, 1\}^r \to \mathbb{R}$, i.e., by an element of $\mathbb{R}^V$. We denote the canonical basis of $\mathbb{R}^V$ by $(e_\nu)_{\nu \in V}$. Observe that the linear spaces $G_k(V)$ do not depend on $u$; but the copula spaces $C_k(u) := C_k(\mathcal{J}(u))$ do. We also point out that vertex decompositions project to vertex decompositions along the faces $F$ via the corresponding map $\pi_F : \mathbb{R}^V \to \mathbb{R}^V \cap F^c$.

Given a subset $W \subset V$ we view $\mathbb{R}^W$ naturally as the linear subspace of $\mathbb{R}^V$. In this way a map $W \to \mathbb{R}$ is extended to a map on $V$ by mapping elements in $V \setminus W$ to zero. The canonical projection $\mathbb{R}^V \to \mathbb{R}^W$ corresponds to restricting maps $V \to \mathbb{R}$ onto $W$, denoted by $x \mapsto x|_W$. We decompose $V$ into disjoint sets

$$V = V_r \cup V_{r-1} \cup \cdots \cup V_0$$

with $V_k = \{\nu \in V \mid \nu_1 + \cdots + \nu_r = r - l\}$. We write

$$V_{\geq k} = V_r \cup V_{r-1} \cup \cdots \cup V_k,$$

and likewise $V_{> k}$ or $V_{\leq k}$ and so forth. Accordingly, we decompose the set of front faces $\mathcal{F}$ of $[0, 1]^r$ into

$$\mathcal{F} = \mathcal{F}_r \cup \mathcal{F}_{r-1} \cup \cdots \cup \mathcal{F}_0$$

where $\mathcal{F}_l$ is the set of front faces of codimension $l$. Observe that $V_r = \{\text{o}\}$ and $\mathcal{F}_r = \{\text{o}\}$. We denote by $\mathcal{F}^k := \mathcal{F}_{r-k}$ the set of front faces of dimension $k$. We also write $\mathcal{F}_{> k}, \mathcal{F}_{< k}, \ldots$ along the lines of (11).
Remark 4.1. We summarise some observations for later purposes:

(i) The map map $V \rightarrow \mathcal{F}$, given by $\nu \mapsto F(\nu)$ is compatible with the above decompositions. It defines bijections $V_k \rightarrow \mathcal{F}_k$, $k = 0, 1, \ldots, r$. For $\nu \in V_k$ we have that $F(\nu) \subset V_{\geq k}$ and $F \cap V_k = \{\nu\}$. In the same way, the map $\nu \mapsto F(\nu)$ defines bijections $V_k \rightarrow \mathcal{F}_k$.

(ii) Choose $x \in \mathbb{R}^V$ and $\nu \in V$. Then $\pi^{F(\nu)}(x) \in \mathbb{R}^{F(\nu) \cap V}$ with

$$\pi^{F(\nu)}(x)(\mu) = \sum_{\nu' \in F(\nu) + \mu} x(\nu').$$

(iii) $G_k(\cap V) \subset G_k(V)$ for each $F \in \mathcal{F}$ and any $k$ where $G_k(\cap V)$ is defined. Indeed, given $x \in G_k(\cap V)$ and $F' \in \mathcal{F}_k$ we observe that $F' \cap F$ has codimension $\leq k$ in $F$. Hence $x \in G_k(\cap V) \subset \mathbb{R}^V$ projects to zero along $F'$.

(iv) We have that $\pi^F(G_k(V)) \subset G_k(F^c \cap V)$. Indeed, if $F'$ is a front face of codimension $k$ in $F^c$, then $F \times F'$ is a face of codimension $k$ in $[0, 1]^r$. Hence $\pi^{F'}(\pi^F(x)) = \pi^{F \times F'}(x) = 0$ for $x \in G_k(V)$.

Next we are describing the spaces $G_k(V)$ and the corresponding copula spaces.

Lemma 4.2. The vector spaces $G_k(V) \subset \mathbb{R}^V$ have the following properties:

$$G_k(V) = \{x \in \mathbb{R}^V \mid \pi^F(x)(o) = 0 \text{ for each } F \in \mathcal{F}_{\leq k}\} \quad (13)$$

In other words, $x$ projects to the zero-measure along any face of codimension $k$ if and only if the projection of $x$ along any face of codimension $\leq k$ is zero at the origin $o$.

Proof of Lemma 4.2. It is evident that $G_k(V)$ is contained in the set on the right hand side of (13) which we denote by $G'_k(V)$. We argue by induction over the dimension $r$. To this end we note that the statement is trivial for $r = 0$. We assume that $G'_k(V') = G_k(V')$ for the vertices $V'$ of cubes up to dimension $r - 1$ and for all $k = 0, \ldots, r - 1$. Suppose now that $x \in G'_k(V)$ and $F \in \mathcal{F}_k$. We need to show that $\pi^F(x) = 0$. We distinguish two cases.

Firstly, assume $k < r$. Then we observe that $\pi^F(x) \in G'_k(F^c \cap V)$ and by the induction hypothesis in $G_k(F^c \cap V)$ as $\dim F^c = k < r$. Note that $G_k(F^c \cap V) = \{0\}$ as $\dim F^c = k$ and hence $\pi^F(x) = 0$.

Secondly, If $k = r$ we see from the definition of $G_r(V)$ that $x$ restricted to any $F \in \mathcal{F}_1$ is in $G'_{r-1}(F \cap V)$ and thus in $G_{r-1}(F \cap V)$. Hence $x(\nu) = 0$ for $\nu \in \bigcup \mathcal{F}_1 = V \setminus \{1\}$. It remains to show that $x(1) = 0$. This follows by projecting $x$ along the $r$-dimensional front face $[0, 1]^r$ to the zero-dimensional face $\{0\}$ which yields 0 as $x \in G'_r(V)$ by assumption. Hence $x(1) = 0$. □

The following proposition describes the degrees of freedom one has in finding box copulas with prescribed properties for the tail.

\footnote{Drawing pictures for $r = 3$ may provide helpful illustrations.}
Proposition 4.3. There is a linear map $S: \mathbb{R}^V \rightarrow \mathbb{R}^V$ with the following properties:

(i) $S$ is an isomorphism between the vector space filtrations, i.e., the following diagram is commutative for $S$:

\[
\begin{align*}
\{0\} & = \mathbb{R}^{V>0} \subset \mathbb{R}^{V_{>r-1}} \subset \cdots \subset \mathbb{R}^{V>0} \subset \mathbb{R}^{V_{>1}} = \mathbb{R}^V \\
\{0\} & = G_r(V) \subset G_{r-1}(V) \subset \cdots \subset G_0(V) \subset G_{-1}(V) = \mathbb{R}^V
\end{align*}
\]

(ii) For $x \in \mathbb{R}^V$ and any $F \in \mathcal{F}$ we have $\pi^{F(\nu)}(S(x))(\mathbf{o}) = x(\nu)$.

Proof of Proposition 4.3. If $S$ with properties (i) and (ii) existed, it would be unique as its inverse would be given by the previous remark. To prove existence, we just construct $S$. First we are going to define $S'(e_\mu) := \sum_\nu (-1)^{\mu-\nu} e_\nu$ and it is readily verified that $S'(e_\nu) \in G_{r-1}(V)$. From the previous lemma we know that $\dim G_{r-1}(V) = 1$. For any $\mu \in V_k$ we denote by

\[
\hat{F}(\mu) := F(\hat{\mu})
\]

the back face corresponding to $F(\hat{\mu})$. Observe that $\dim \hat{F}(\mu) = k$ and that $\mu$ corresponds naturally to the origin in $\hat{F}(\mu)$. Exactly as with $S'(e_\mu)$ before, we define now $S'(e_\mu)$ to be the unique element in $G_{k-1}(\hat{F}(\mu) \cap V) \subset G_{k-1}(V)$ having the required property, and that is

\[
S'(e_\mu) = \sum_\nu (-1)^{\mu-\nu} e_\nu
\]

Therefore we obtain a candidate for the isomorphism in question, namely

\[
S'(x) = \sum_\mu \sum_\nu \in \hat{F}(\mu) (-1)^{\mu-\nu} x(\mu)e_\nu
\]

Now observe that $\nu \in \hat{F}(\mu)$ if and only if $\mu \in F(\nu)$ and thus $S' = S$ where $S$ is as in (14). As $S$ is compatible with the filtration on the basis $(e_\mu)$, it is indeed compatible with the filtration. As it satisfies $\pi^{F(\nu)}(S(e_\mu))(\mathbf{o}) = 1$ for $\nu = \mu$ and $\pi^{F(\nu)}(S(e_\mu))(\mathbf{o}) = 0$ otherwise it satisfies $\pi^{F(\nu)}(S(x))(\mathbf{o}) = x(\nu)$ by linearity. By Lemma 4.2, $S$ restricts to isomorphisms $\mathbb{R}^{V_{>k}} \simeq G_k(V)$. 

Remark 4.4. Observe that the inverse of $S: \mathbb{R}^V \rightarrow \mathbb{R}^V$ is given by

\[
S^{-1}(z)(\nu) = \pi^{F(\nu)}(z)(\mathbf{o}) = \sum_{\mu \in F(\nu)} z(\mu), \quad \nu \in V.
\]
Remark 4.5. For each $k$ there exists a unique map $T_k : \mathbb{R}^{V_{>k}} \to G_k(V)$ such that $T_k(x)(\nu) = x(\nu)$ for $\nu \in V_{>k}$. The map $T_k$ is a vector space isomorphism. As we do not make use of this result, we leave the proof as an exercise.

Next we describe the vertex copulas for $J(u)$. We abbreviate $u^\nu := \prod_{\{i \mid \nu_i = 1\}} u_i$ for any $\nu \in V$ and $u \in (0, 1)$ where $u^o = 1$. The uniform copula in $C_k(u)$ corresponds to the element $c_u := \sum_{\nu \in V} u^{1-\nu}(1 - u)^\nu e_\nu \in \mathbb{R}^V$ which depends on $u$. Observe that $\pi^F(\nu)(c_u)(o) = u^{1-\nu}$. Therefore the proposition implies that

$$S(x_u) = c_u \text{ for } x_u = \sum_{\nu} u^{1-\nu} e_\nu.$$ 

Corollary 4.6. The vertex copulas of order $k$ with respect to $J(u)$ are given by

$$C_k(u) = \{ S(x) \mid x \in \mathbb{R}^V, x(\nu) = u^{1-\nu} \text{ for } \nu \in V_{\leq k}, x \text{ satisfies (18)} \}$$

with

$$\sum_{\mu \in F(\nu)} (-1)^{\mu-\nu} x(\mu) \in [0, 1] \text{ for each } \nu \in V. \quad (18)$$

Condition (18) is the condition for $x$ being a probability measure.

Proof. A given element in $c \in C_k(u)$ is of the form $c = g + cu$ with $g \in G_k(V)$. Using the proposition, choose $y \in \mathbb{R}^{V_{>k}}$ such that $S(y) = g$ and set $x = y + x_u$. Then $S(x) = c$ and the statement follows from the proposition.

Remark 4.7. The reader may compare Condition (18) with the rectangle inequality for a copula in [6], p. 185. We see from the corollary that in case of vertex measures there is no need to check that property for every $r$-box in $[0, 1]^r$.

5 Nesting box copulas

In this section we define formally the nesting constructions used in the Examples 2.1 and 2.3. Related techniques have been used by G. Fredricks et al. [3] in dimension 2.

Definition 5.1 (Nesting box copulas). Suppose $z_j \in \mathbb{R}^{I_j}$, $j = 1, 2$ are two box measures on $[0, 1]^r$ and $I_1 \in I_1$ is a box for $z_1$. Let $\nu : [0, 1]^r \to I_1$ be the
Algorithm 5.3 (for sampling nested box copulas) Then nest($z_2, z_1, I_1$) is the box measure $y \in \mathbb{R}^J$ with $J = (I_1 \setminus I_1) \cup \nu(I_2)$ and

$$
y|_{I_1 \setminus I_1} = z_1|_{I_1 \setminus I_1} \quad \text{and} \quad y|_{I_1} = z_1(I_1)\nu(z_2).$$ (19)

In other words, nest($z_2, z_1, I_1$) is equal to $z_1$ outside of $I_1$ and on $I_1$, it is a copy of $z_2$, scaled by $z_1(I_1)$. We call nest($z_2, z_1, I_1$) the box measure obtained from nesting $z_2$ into $z_1$ along $I_1$.

If $S_1 \subset I_1$, we define nest($z_2, z_1, S_1$) to be the box measure obtained by consecutively nesting $z_2$ into the elements of $S_1$. Note that this construction does not depend on the order of the different nesting operations. We call nest($z_2, z_1, S_1$) the box measure obtained from nesting $z_2$ into $z_1$ along $S_1$.

Lemma 5.2. Assume that $z_j \in C_k(I_j)$, $j = 1, 2$ are box copulas of order $k$. Then nest($z_2, z_1, I_1$) is in $C_k(J)$ where $J$ is as in the definition.

Proof. The measure $z_1|_{I_1}$ is uniform. The projections of $z_1(I_1)\nu(z_2)$ along faces of $I_1$ up to codimension $k$ is by assumption a constant times the euclidean volume of the face for the corresponding dimension. The scale factor is arranged such that projections of $z_1|_{I_1}$ to these faces are equal to the projection of $z_1(I_1)\nu(z_2)$. To this end note that $\nu(z_2)(I_1) = 1$.

Algorithm 5.3 (for sampling nested box copulas). Suppose that we have a simulation algorithm for drawing random samples from copulas $z_1$ and $z_2$ as in the lemma, i.e., essentially an algorithm for simulating multinomial distributed variates. Then the following is a simulation algorithm for nest($z_2, z_1, I_1$):

(i) Draw a random variate $U \in [0, 1]^r$ from $z_1$.

(ii) If $U \notin I_1$ then return $U$.

(iii) otherwise generate a random sample $V$ from $z_2$. Return $\nu(V)$.

We can nest copulas iteratively. To this end we start with a box copula $c^1$. Then construct $c^n$ recursively. Suppose we have already constructed $c^{n-1} \in \mathbb{R}^{J_{n-1}}$. Then we choose a box copula $z_n$, a subset $S_{n-1} \subset J_{n-1}$ and set $c^n = \text{nest}(z_{n-1}, c^{n-1}, S_{n-1})$.

Remark 5.4. Suppose that $c^n \in \mathbb{R}^{J_n}$, $n > 1$, is a obtained from consecutive nestings. Intuitively, the sequence of measures $(c^n)$ converges as for $J \in J_n$, we have that $c^n(J) = c^n(J)$ for each $m \geq n$. Formally, given $E \subset [0, 1]^r$ define

$$c^\infty(E) = \lim_{n \to \infty} \inf \left\{ \sum_i c^{m(i)}(W_i) \middle| (W_i) \text{ a countable open covering of } E, m(i) \geq n \right\}$$

Arguing as in [8], p. 8, Eq. (1), we conclude that $c^\infty$ is an outer measure where Borel sets are measurable. Form the construction we see that $c^\infty(J) = c^n(J)$ for each $J \in J^n$. It is evident that $c^\infty$ is a copula measure. Note that in the interesting case where we have infinitely many non-trivial nestings, the limit measure is not given by a box copula.
Example 5.5 (generalising Example 2.3). We choose the vertex decomposition of \([0,1]^r\) given by \(u = (1/2, \ldots, 1/2)\) and start with the vertex copula \(c^1 = 2^{-r} S_{(\mathcal{C}(u))} + cu \in C_r(\mathcal{F}(u))\) on \([0,1]^r\). \(S\) as in Proposition 4.3. This copula assigns to each vertex \(v \in \{0,1\}^r\) with \(\text{mod}(\sum v_i, 2) = 0\) the probability \(2^{-r+1}\) and to the other vertices the probability 0. Proceeding as above, we choose \(z = e^1\) and \(S_{n-1} = \mathcal{J}_{n-1}\). The limit copula \(c^\infty\) is, up to scaling, equal to the \((r-1)\)-dimensional Hausdorff measure on the limiting set \(\bigcap_n \text{support}(c^n)\).

We consider now the canonical projections \(U_i: \text{support}(c^\infty) \rightarrow [0,1]\) onto the \(i\)-th coordinate and view it as a random variable with probability measure given by \(c^\infty\). We write

\[
U_i = \sum_{n=1}^{\infty} B_{in} \cdot 2^{-n}
\]

with \(B_{in}\) having values in \(\{0,1\} \simeq \mathbb{Z}/2\mathbb{Z}\). They are the digits for the binary representation of \(U_i\). It can be seen from the construction of \(c^\infty\) that \(B_{kn} = \sum_{i \neq k} B_{in}\), where the summation is in \(\mathbb{Z}/2\mathbb{Z}\). Hence \(U_k\) is the bitwise addition of the \(U_i, i \neq k\). Any subset of \(\{U_1, \ldots, U_r\}\) with less than \(r\) elements is independent.

This example is well known, at least for finitely many digits (where the \(U_i\) are ‘cut off’ after the \(m\)-th digit), refer e.g., to J. Stoyanov [10]. The set of \(U_i\) can be enlarged by \(U_N = \sum_{i \in N} U_i\) for any non-empty subset \(N \subset \{1, \ldots, r-1\}\) where \(\sum^2\) stands for bitwise addition as described before. In this way we obtain a longer finite sequence of pairwise independent random variables \(U_N\).

A similar construction works also for representations with respect to any base \(d \geq 2\), i.e., with any \(\mathbb{Z}/d\mathbb{Z}\). Just start with the regular grid for \([0,1]^r\) which decomposes each edge into \(d\) intervals of length \(1/d\) and assign to them the digits \(0, 1, \ldots, d-1\). Define the starting box copula as above for the case \(d = 2\) by adding the digits assigned to the edges \(1, \ldots, r-1\) in \(\mathbb{Z}/2\mathbb{Z}\). Finite versions of such constructions have applications in computer science (see e.g., M. Luby & A. Wigderson [4]).

6 Tail nesting

In this section we define and investigate tail nesting which is a specific way of nesting vertex copulas in order to shape tail dependencies.

Let \(\mathcal{J}\) be a grid decomposition of \([0,1]^r\). The o-box of \(\mathcal{J}\) is the unique box in \(\mathcal{J}\) which contains the origin \(o\). For \(\nu \in V\) we define the set \(\mathcal{J}(\nu)\) of o-boxes with respect to \(\nu\) as the set of all \(J \in \mathcal{J}\) such that \(\pi^{F(\nu)}(J)\) is an o-box of \(\pi^{F(\nu)}(\mathcal{J})\) and \(\pi^{F(\nu)}(J)\) is not an o-box of \(\pi^{F(\mu)}(\mathcal{J})\) for any \(\mu \in F(\nu) \setminus \{\nu\}\). In this way we obtain a disjoint decomposition of \(\mathcal{J}\),

\[
\mathcal{J} = \bigcup_{\nu \in V} \mathcal{J}(\nu).
\]

Thus elements in \(\mathcal{J}(\nu)\) project to the o-box in the decomposition of \(F(\hat{\nu})\) and \(F(\hat{\nu})\) is the face of maximal dimension having that property.
Definition 6.1 (Tail nesting for grid copulas). Suppose that \( z \in C_k(u) \) is a vertex copula of dimension \( r \) and that \( c \) is a grid copula of order \( k \) on \([0,1]^r\) with grid \( \mathcal{J} \). Then \( \text{tnest}(z,c) \) is the grid copula obtained from \( c \) by nesting successively for each \( \nu \in V \) the product measure of \( \pi^{F(\nu)}(z) \) with the uniform copula \( cu_{F(\nu)} \) as an element of \( C_k(u) \), that is
\[
\pi^{F(\nu)}(z) \times cu_{F(\nu)} \in C_k(u),
\]
into \( \mathcal{J}(\nu) \). Thus \( \text{tnest}(z,c) \) is given by the consecutive nestings
\[
\text{nest}\left(\pi^{F(\nu)}(z) \times cu_{F(\nu)}, c, \mathcal{J}(\nu)\right), \quad \nu \in V.
\]
The grid decomposition for \( \text{tnest}(z,c) \) is obtained from \( \mathcal{J} \) by assigning to each box \( J \in \mathcal{J} \) the vertex decomposition induced by the nesting.

Figure 3: Illustration of tail nesting a 3-dimensional vertex copula \( z \) once into itself. The copula measure \( z \) is pictured left. A front and back view of the resulting box copula \( \text{tnest}(z,z) \) is illustrated middle and right, respectively. The tail vertex \( o \) is the upper front corner in the left and middle picture. On the right hand side, the opposite vertex \( 1 \) corresponds to the lower front corner. Grey levels are set according to the density of probability.

Lemma 6.2. In the setting of the above definition, \( \text{tnest}(z,c) \) is a grid copula of order \( k \). Furthermore, tail-nesting commutes with projections along faces \( F \) of \([0,1]^r\),
\[
\pi^F(\text{tnest}(z,c)) = \text{tnest}(\pi^F(z), \pi^F(c))
\]
for the corresponding grid copulas.

Proof. It is a direct consequence of Lemma 5.2 that \( \text{tnest}(z,c) \) is a grid copula of order \( k \) provided \( z,c \) are of order \( k \).

We prove now the second part of the statement. For any face \( F \) we denote here by \( cu_F \) the uniform copula on \( F \) with grid decomposition \( \mathcal{J}_F = \pi^{F^c}(\mathcal{J}) \). Now let \( F \) be a given face. Suppose that \( \nu \in V \cap F^c \) and \( I \in \mathcal{J}_{F^c(\nu)} \). Observe that \( F(\nu) \subset F^c \). The elements in \( \mathcal{J} \) which project along \( F \) to \( I \) are of the form \( I \times K, K \in \mathcal{J}_F \) and in particular \( K \subset F \). Observe that \( I \times K \in \mathcal{J}(\mu) \) where \( F(\mu) = F(\nu) \times F' \) with \( F' \subset F \). Therefore we obtain
\[
\pi^{F(\nu)}\left(\pi^F(z)\right) \times cu_{F(\nu)} = \pi^F\left(\pi^{F(\nu)}(z)\right) \times cu_{F(\nu)}
\]
\[
= \pi^F\left(\pi^{F(\mu)}(z) \times cu_{F(\mu)}\right).
\]
This implies that \( tnest(\pi^F(z), \pi^F(c))|_I = \alpha \pi^F(tnest(z, c)|_{I \times F}) \) for a scaling factor \( \alpha \geq 0 \). From the definition of nesting, \( \alpha = 1 \). \( \square \)

In analogy to Example 2.1, we can now investigate iterated tail nestings in dimension \( r \). We pick a sequence \( z_n, n = 1, 2, \ldots \) with \( z_n \in C_k(u^{(n)}) \), for \( n \geq 1 \). We define recursively grid copulas \( c^n \) such that \( c^1 = z_1 \) and
\[
c^n = tnest(z_n, c^{n-1})
\]
with grid decomposition \( J^n \). We denote the limit copula \( c^\infty \) also by \( tnest((z_n)) \) and observe that \( c^\infty \) is a copula of order \( k \). Lemma 6.2 implies that
\[
\pi^F tnest((z_n)) = tnest((\pi^F(z_n))) \quad \text{for any face } F.
\]
Note that the \( o \)-box of \( c^n \) is \([0, d^{(n)}]\) with
\[
d_i^{(n)} = \prod_{l=1}^{n} u_i^{(l)} \quad i = 1, \ldots, r.
\]
The \( o \)-box of \( \pi^F(\nu)(c^n) \) in \( F(\hat{\nu}) \) is denoted by \([0, d^{(n)}(\nu)]\) and given by
\[
d_i^{(n)}(\nu) = d_i^{(n)} \quad \text{if } \nu_i = 0 \quad \text{and} \quad d_i^{(n)}(\nu) = 0 \quad \text{if } \nu_i = 1.
\]

**Remark 6.3.** Building a simulation algorithm for \( c^n \) based on this definition and Algorithm 5.3 is not efficient. Observe that the nestings for \( \nu \in V_{<k} \) do not change the measure and merely refine the decomposition. Including these trivial nestings formally in the definition has advantages in stating and proving properties of iterated tail nestings.

**Algorithm 6.4** (for sampling tail nested copulas). Generate the samples based on Algorithm 5.3 and do the iterative nestings only for \( o \)-boxes with respect to \( \nu \in V_{>k} \). In the first iteration step one obtains a box copula which is illustrated in Fig. 3. Then one needs to extend the definition of \( o \)-boxes to \( o \)-boxes with respect to \( \nu \in V \) for the resulting box decomposition, which is easy. Proceeding in that way yields the required algorithm. In the first iteration, one ends up in the ‘otherwise’-routine of Algorithm 5.3 with a probability of \( z_1(\bigcup_{\nu > k} J(u^{(1)})(\nu)) \) and conditioned on that, with a probability of \( z_2(\bigcup_{\nu > k} J(u^{(2)})(\nu)) \) in the ‘nested otherwise’-routine and so on. Hence generating \( N \) samples of the limit copula \( c^\infty \) requires the sampling of approximately
\[
N' := N \left( 1 + \sum_{n=1}^{\infty} \prod_{l=1}^{n} z_l \left( \bigcup_{\nu > k} J(u^{(l)})(\nu) \right) \right)
\]
random variates from a multinomial distribution for the vertices.\(^3\) If we have an upper bound \( p_{\text{max}} = \max_l(z_l(\bigcup_{\nu > k} J(u^{(l)})(\nu))) < 1 \), then
\[
N' \leq N(1 - p_{\text{max}})^{-1}
\]
and the algorithm converges. In practise, working with \( p_{\text{max}} \) sufficiently smaller than 1 and with a finite sequence \( (z_n) \) for shaping a desired tail behaviour should be sufficient.

\(^3\)In each ‘otherwise-routine’, the number of vertices is \( 2^k, k \leq r \)
Remark 6.6. We can coarsen the box decomposition $\mathcal{J}^n$ by building consecutively unions of two boxes, starting with boxes in $\mathcal{J}^n$, if they share a common face of codimension 1 and the probability density of $c^n$ on these boxes is the same, irrespective of the choice of probabilities $z_l(\nu)$, $l = 1, \ldots, n$ and $\nu \in V$. In this way we obtain the relevant box decomposition for Algorithm 6.4 and denote it by $\widetilde{\mathcal{J}}^n$. We observe that each box in $\mathcal{J}^n$ is also contained in $\widetilde{\mathcal{J}}^m$ for $m \geq n$ and that $c^m$ restricted to these boxes is equal to $c^n$ restricted to these boxes.

The theorem below summarises the main properties of iterated tail nestings and demonstrates the flexibility one has when ‘shaping the tail’ of $c^n$.

The constraint is given by Corollary 4.6 and in particular Condition (18) for $z_c$. This follows immediately from the definition of nesting.

Now assume that $d^{(n)} \to 0$ as $n \to \infty$ then $c^\infty$ is locally piecewise uniform on $[0, 1]^r \setminus \cup \mathcal{F}_{\geq k}$.

Proof. By Proposition 4.3, in particular (ii), and since nest commutes with $\pi^F$, it is sufficient to show that (25) holds for $\nu = \mathbf{o}$, i.e., that $c^\infty([0, d^{(n)}(\nu)]) = \prod_{i=1}^n x_i(\mathbf{o})$ where $d^{(n)} = d^{(n)}(\mathbf{o})$. This follows immediately from the definition of nesting.

Remark 6.6. We observe that (25) also holds for the limit copula obtained by nesting iteratively $\tilde{c}_n = \text{nest}(S(x_n), c^{n-1})$ with $\tilde{c}_1 = S(x_1)$. Tail nesting avoids those nestings which are not relevant for shaping the tail characteristic. Formally, this is done in (20) by averaging over those dimensions which do not matter for shaping the tail.

Theorem 6.5. Let $V$ be the set of vertices of $[0, 1]^r$ and $S: \mathbb{R}^V \to \mathbb{R}^V$ denote the isomorphism given by (14). Assume that $(x_n)_{n \geq 1}$ is a sequence in $\mathbb{R}^V$ such that $S(x_n) \in C_k(u^{(n)})$ is a sequence of vertex copulas of order $k$. Then $c^\infty := \text{nest}((S(x_n)))$ is a copula of order $k$ and satisfies

$$\left(\pi^F(c^\infty)([0, d^{(n)}(\nu)])\right) = \prod_{i=1}^n x_i(\nu), \quad \nu \in V,$$

where $d^{(n)}_i(\nu) = (1 - \nu_i)\prod_{l=i}^n u^{(l)}_i$ for $i = 1, \ldots, r$. Furthermore, if $d^{(n)} \to 0$ as $n \to \infty$ then $c^\infty$ is locally piecewise uniform on $[0, 1]^r \setminus \cup \mathcal{F}_{\geq k}$.

Proof. By Proposition 4.3, in particular (ii), and since nest commutes with $\pi^F$, it is sufficient to show that (25) holds for $\nu = \mathbf{o}$, i.e., that $c^\infty([0, d^{(n)}]) = \prod_{i=1}^n x_i(\mathbf{o})$ where $d^{(n)} = d^{(n)}(\mathbf{o})$. This follows immediately from the definition of nesting.

Now assume that $d^{(n)} \to 0$ as $n \to \infty$. As above, let $\mathcal{J}^n$ be the grid decomposition for $c^n = \text{nest}(z_n, c^{n-1})$ obtained in the process of iterated nestings. Then, given $\varepsilon > 0$, all elements of $\cup_{\nu > k}\mathcal{J}^n(\nu)$ are contained in the $\varepsilon$-tube $W$ around $\cup \mathcal{F}_{\geq k}$ for all $n \geq n_0$, where $n_0$ is sufficiently large. Indeed, if the $\varepsilon$-tube is taken with respect to the maximum norm, we may choose $n_0$ such that $\max_i\{d^{(n)}_i\} < \varepsilon$ for each $n \geq n_0$. We obtain $c^\infty|[0, 1]^r \setminus W = c^n|[0, 1]^r \setminus W$ for $n \geq n_0$. Note that $c^n, n < \infty$, is piecewise uniform by (24). Therefore $c^\infty$ is locally piecewise uniform in $[0, 1]^r \setminus \cup \mathcal{F}_{\geq k}$.

Remark 6.6. We observe that (25) also holds for the limit copula obtained by nesting iteratively $\tilde{c}_n = \text{nest}(S(x_n), c^{n-1})$ with $\tilde{c}_1 = S(x_1)$. Tail nesting avoids those nestings which are not relevant for shaping the tail characteristic. Formally, this is done in (20) by averaging over those dimensions which do not matter for shaping the tail.

The $\varepsilon$-tube around a set $E$ is the set of all points with distance $< \varepsilon$ to some point in $E$. 

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In this context we notice the following. Suppose we start with a grid copula of order \(k\). We refine possibly the corresponding grid before ‘tail-nesting’ a sequence of vertex copulas of order \(k\) into the grid copula. In this way the original grid copula measure is merely modified in an arbitrarily small neighbourhood of \(\cup \mathcal{F}_{\geq k}\), provided the refinement of the original grid was fine enough.

### 7 Tail characteristics

Before we apply the theorem to construct copulas with certain tail characteristics, we investigate properties of any tail characteristic. To this end we assume that \(c\) is any copula measure of order \(k\), where \(k \geq 1\), with \(\text{tdeg}_c < \infty\) and \(\text{tcoef}_c < \infty\). We denote \(c_{\mathcal{F}} := \pi_{\mathcal{F}}(c)\). For \(s \in (0, 1]\) we write

\[
c_{\mathcal{F}}(s[0, 1]^{\dim F}) = \tilde{a}(F, s) s^{\text{tdeg}_c(F)}
\]

The condition for probability measures (18) implies that

\[
\sum_{F' \supset F} (-1)^{\dim F' - \dim F} \tilde{a}(F', s) s^{\text{tdeg}_c(F')} \in [0, 1]
\]

From this equation we obtain necessary condition for maps which are tail characteristics of copulas (see the remark below).

As an application of the theorem we are going to state sufficient conditions. We have already introduced the properties non-decreasing and increasing for maps \(b: \mathcal{F} \to [0, \infty]\) in the introduction. We say \(b: \mathcal{F} \to [0, \infty]\) is increasing at \(F \in \mathcal{F}\) and eventually constant at \(F\) if \(b(F') < b(F')\) and \(b(F') = b(F')\), respectively, for each \(F' \in \mathcal{F}\) with \(F \subset F'\) and \(F \neq F'\).

If \(b\) is increasing and \(a\) is an accumulation point of the corresponding maps \(\tilde{a}(\cdot, s)\) as \(s \to 0\), we can see that the dominating summand in (27) is \(a(F, s) s^{\text{tdeg}_c(F)}\), provided \(a(F) > 0\).

**Remark 7.1** (Necessary conditions). From the above we obtain the following necessary conditions for maps \(a, b: \mathcal{F} \to (0, \infty)\) such that \(b = \text{tdeg}_c\) for some copula \(c\) of order \(k\) and such that \(a\) is an accumulation point of the corresponding maps \(\tilde{a}(\cdot, s)\) as \(s \to 0\).

(i) \(b\) is non-decreasing, i.e., \(b(F') \geq b(F)\) for any \(F' \supset F\),

(ii) \((a(F), b(F)) = (1, \dim F)\) for each \(F \in \mathcal{F}^{\leq k}\).

(iii) If \(b\) is eventually constant at \(F\), then \(\sum_{F' \supset F} (-1)^{\dim F' - \dim F} a(F') \geq 0\).

If a pair \(a, b\) satisfies (i)–(iii) for \(k\) we say that \(a, b\) satisfies \(\text{NC}_k\).

The tail characteristics of a copula \(c\) restricted to the faces of dimension \(l\) provide a measure for *tail dependencies of order \(l\)*. Given a map \(\mathcal{F} \to \mathbb{R}\) as above we view it as well as a map \(V \to \mathbb{R}\) by composing it with the bijection \(\nu \mapsto F(\hat{\nu})\).
Corollary 7.2. Let $a_m, b : F \to (0, \infty)$, $m \geq 1$, satisfy NC$_k$ for some $k \geq 1$. Suppose the sequence $(a_m)_{m \geq 1}$ converges to some $a : F \to (0, \infty)$. Assume in addition that $b$ is increasing. Let $(s_m)_{m \geq 1}$ be a sequence in $(0, 1)$ with $s_m \to 0$ as $m \to \infty$. Then, after passing to common subsequences, again denoted by $s_n, a_n, n \geq 1$, there exists a sequence $(z_n)_{n \geq 1}$ of vertex copulas of order $k$, such that we have for $c^\infty := \text{unest}((z_n)_{n \geq 1})$
that
\begin{equation}
 c^\infty_F (s_n[0, 1]^{\dim F}) = a_n(F) s_n^{b(F)} .
\end{equation}

Remark 7.3. Let $c$ be any copula $c$ of order $k$ with $t\deg_c$ increasing. Choosing $s_m \to 0$, such that $a_m(F) := \tilde{a}(F, s_m)$ converges as $m \to \infty$ where $\tilde{a}(F, s)$ is as in (26). Then the corollary states that after passing to a subsequence, $c^\infty_F (s_n[0, 1]^{\dim F}) = c_F(s_n[0, 1]^{\dim F})$.

Proof of Corollary 7.2. For $a$ and $\delta > 0$ we define open intervals $W_{\delta,a} := (\min(a) - \delta, \max(a) + \delta)$ and $W_{\delta,1} = (1 - \delta, 1 + \delta)$. We claim that there exists some $t_0 \in (0, 1)$ and $\delta \in (0, 1) \cap (0, \min(a))$ such that for any $t < t_0$, any map
\begin{equation}
 h : V \to W_{\delta,1} \cup W_{\delta,a} , \quad \text{with} \quad h(\nu) = 1 \text{ for } \nu \in V_{\leq k}
\end{equation}

the following is true: For
\begin{equation}
 x := h \cdot t^b \in \mathbb{R}^V
\end{equation}
with $x(\nu) = h(\nu)^b(\nu)$ we have $z = S(x) \in C_k(t1)$.

The idea for the proof of this claim if already laid out in Remark 7.1. Indeed, we need to check the condition for vertex copulas of order $k$ in Corollary 4.6. Observe that $b$ and $h$ restricted to $V_{\leq k}$ are such that $z \in C_k(t1)$ provided Condition (18) for a probability measure is fulfilled. We estimate the respective expression
\begin{equation}
 S(x)(\nu) = \sum_{\mu \in F(\nu)} (-1)^{\mu - \nu} x(\mu)
\end{equation}
for $\nu \in V$ from below and from above by
\begin{equation}
 x(\nu) - \sum_{\mu \in F(\nu) \setminus \{\nu\}} x(\mu) \leq S(x)(\nu) \leq x(\nu) + \sum_{\mu \in F(\nu) \setminus \{\nu\}} x(\mu) ,
\end{equation}
which is equivalent to
\begin{equation}
 h(\nu)t^{b(\nu)} - \sum_{\mu \in F(\nu) \setminus \{\nu\}} h(\mu)t^{b(\mu)} \leq S(x)(\nu) \leq h(\nu)t^{b(\nu)} + \sum_{\mu \in F(\nu) \setminus \{\nu\}} h(\mu)t^{b(\mu)} .
\end{equation}
Since $h(\nu) > 0$ and $b(\mu) > b(\nu)$ for each $\mu \in F(\nu) \setminus \{\nu\}$ by assumption on $b$ this proves that Condition (18) holds for $\nu \neq 1$ if $t > 0$ is small enough. Next observe that
\begin{equation}
 S(x)(1) = 1 - rt + \sum_{\nu \in V_{\geq 2}} (-1)^{1-\nu} h(\nu)t^{b(\nu)} \in [0, 1]
\end{equation}
provided that is sufficiently small. Now we choose such that the above claim holds. Given the sequences and we choose the subsequences, again denoted by and such that

$$t_1 := s_1 < t_0 \quad \text{and} \quad t_{n+1} := s_{n+1}/s_n < t_0,$$

$$h_1(\nu) := a(\nu) \in W_{\delta,a} \quad \text{and} \quad h_{n+1}(\nu) := a_{n+1}(\nu)/a_n(\nu) \in W_{\delta,1}$$

for any . Then we set

$$x_n := h_n \cdot (t_n)^b \in C_k(t_n 1)$$

and observe that

$$\prod_{l=1}^n q_l = s_n \quad \text{and} \quad \prod_{l=1}^n x_n(\nu) = a_n(\nu) \prod_{l=1}^n (t_l)^b(\nu) = a_n(\nu)(s_n)^b(\nu).$$

Now the corollary follows from the theorem.

**Corollary 7.4.** Suppose and satisfy the condition NC for some . Assume further that is increasing. Then there exists some and a sequence such that is the nest of some in , such that satisfies .

**Proof.** Given and as in the Corollary, we choose for and such that

$$(1 + \delta') \cdot t^b \in C_k(t 1) \quad \text{for each } \delta' : V \to (-\delta, \delta)$$

We choose now a sequence with , such that and for each . Then we set for and . We calculate next the tail degree of . Given we determine such that . We obtain for that

$$\frac{(\pi F(\nu) c^\infty)(t^m[0, 1(\nu)])}{t^{(m-1)\tau}} \leq \frac{(\pi F(\nu) c^\infty)(s[0, 1(\nu)])}{s^\tau} < \frac{(\pi F(\nu) c^\infty)(t^{m-1}[0, 1(\nu)])}{t^{(m-1)\tau}}.$$

Applying the theorem yields

$$\frac{t^\tau a_m(\nu)t^{m\nu(\nu)}}{t^{m\tau}} \leq \frac{(\pi F(\nu) c^\infty)(s[0, 1(\nu)])}{s^\tau} < \frac{t^{-\tau} a_{m-1} a(\nu)t^{(m-1)\nu(\nu)}}{t^{(m-1)\tau}}$$

and therefore . A continuity argument shows that there exists an adequate sequence such that .

**Remark 7.5.** We make the following observation in the previous proof. When nesting with where , in order to construct , we can arrange that and depending on . This enables us to obtain an upper bound for

$$\limsup a(\cdot, s)/\liminf a(\cdot, s)$$
depending only on $b$. Here $\tilde{a}$ is determined from $c^\infty$ as in (26).

Furthermore, if we had $t_{\infty} = 1$, then $\lim sup a(\cdot, s) = \lim inf a(\cdot, s)$ as $s \to 0$. In such a situation, the condition for the probability measure may become more difficult to control. Further below we will see that it can be easily controlled if the tail degree is equal to 1.

In the next application we weaken the condition that $b$ is increasing.

**Corollary 7.6.** Suppose that $a, b: \mathcal{F} \to (0, \infty)$ satisfy condition NC$_k$ for some $k \geq 1$. Assume that $b$ is increasing or eventually constant at every $F \in \mathcal{F}$. Then, given any sequence in $[0, 1]$ converging to 0, there is a subsequence $(s_n)_{n \geq 1}$, a sequence $(z_n)_{n \geq 1}$ of vertex copulas of order $k$ such that

$$\lim_{n \to \infty} \frac{c^\infty (s_n[0, 1]^{\dim F})}{(s_n)^{\nu_F}} = a(F).$$

for $c^\infty = \text{tnest}((z_n))$ and $\text{tdeg}_{c^\infty} = b$.

**Proof.** We proceed exactly as in the proof of Corollary 7.2 with $(s_m)$ the originally given sequence and with $a_m = a$. To begin with we obtain the estimate (32) for those $\nu$ with $b$ increasing at $F(\nu)$ by arguing as above.

Setting $x_1 = h_1 \cdot (t_1)^b$ for $t_1 < t_0$, we choose $h_1 = a$ and it remains to show that $S(x_1)(\nu) \in [0, 1]$ for those $\nu$ with $b$ eventually constant at $F(\nu)$. Observe that for these $\nu$, $b(\mu) = b(\nu)$ for each $\mu \in F(\nu)$. As $a$ satisfies (iii) by assumption we see that $S(x_1)(\nu) \in [0, 1]$ provided $t_0$ is sufficiently small. Without loss of generality assume $s_1 = t_1$.

Next we set $x_n = (t_n)^b$, $t_n = s_n/s_n^{-1}$, $n \geq 1$. We claim that $x_n \in C_k(t_n \mathbf{1})$ after passing to an appropriate subsequence $(s_n)_{n \geq 1}$. Indeed, suppose $b$ is eventually constant at $F(\nu)$ and $l = \dim F(\nu) > 0$. Then we decompose the vertices of $F(\nu)$ along the lines of $[10]$ into $(V \cap F(\nu))_{j1} \cup \cdots \cup (V \cap F(\nu))_0$ and observe that the number of elements in $(V \cap F(\nu))_j$ is $\binom{l}{j}$. As

$$\sum_{j=0}^{l} \binom{l}{j} (-1)^j = 0$$

we see that $x_n(\nu) = 0$. Hence $x_n$ satisfies the condition for a probability measure. For $n = S(x_n)$, $n \geq 1$, we see that $(z_n)$ has the desired properties. In view of Remark 7.5 we can achieve $\text{tdeg}_{c^\infty} = b$. If the ratios $s_n/s_{n+1}$ were not uniformly bounded from above we can enlarge the sequence $s_n$ by appropriate intermediate points in order to apply the arguments as in Remark 7.5. \qed

Recall that the Clayton copula has tail dependence of degree 1 and likewise the nested Clayton copulas which are described in [5]. We conclude this section by investigating necessary and sufficient conditions for tail coefficients in case of tail degree 1.

**Remark 7.7.** Suppose $c$ is a copula of order $k$ and $\tilde{a}(\cdot, s)$ as in Remark 7.1. Then any accumulation point $a$ of $\tilde{a}(\cdot, s)$ satisfies condition (ii) and (iii) in that remark. If $c$ has tail dependence of degree $k$, then (iii) is equivalent to
(iii)' \( S(a)|_{F \cap V} \in \mathbb{R}^{F \cap V} \) is a probability measure for \( F \in \mathcal{F}_k \)

with \( a(\nu) = a(F(\hat{\nu})) \). Indeed, given \( F \in \mathcal{F}^k \) choose \( \nu \in V_k \) with \( F(\hat{\nu}) = F \). By assumption, \( \text{tdeg}_c(F') = k \) for any face \( F' \supset F \) and thus \( \text{tdeg}_c \) is eventually constant at any \( F' \supset F \). We recall that \( F(\hat{\mu}) \supset F(\hat{\nu}) \) if and only if \( \mu \in F(\nu) \). Given any \( \nu' \in F \cap V \) we see from (iii) that

\[
S(a)(\nu') = \sum_{\mu \in F(\nu')} (-1)^{\mu-\nu'} a(\mu) \geq 0.
\]

As \( \sum_{\mu \in F(\nu)} S(a)(\mu) = a(\nu) = a(F(\hat{\nu})) = 1 \) this shows that (iii)' holds. The other direction is now evident as well.

**Corollary 7.8.** Let \( a: \mathcal{F} \to [0,1] \) with \( a(F) = 1 \) for \( F \in \mathcal{F}^\leq 1 \). Assume that \( S(a)|_{F \cap V} \) is a probability measure for any \( F \in \mathcal{F}_1 \). Then there exists a sequence \((z_n)_{n \geq 1}\) of vertex copulas such that \( c^\infty := \text{tnest}((z_n)) \) has tail dependence of degree 1 and \( \text{tcoef}_{c^\infty} = a \). Moreover,

\[
\lim_{s \to 0} \frac{c_F(s[0,1]^{\dim F})}{s} = a(F)
\]

In other words, by means of tail nesting we can achieve any possible tail coefficients in case of tail degree 1. As the above limit exists, the copulas \( c_F \) with \( \dim F = 2 \) have lower tail dependence \( a(F) \).

**Proof.** Along the lines of the construction above, we set

\[
x_1 := a \cdot (t_1)^b
\]

with \( b(\nu) = 1 \) for \( \nu \neq 1 \) and \( b(1) = 0 \). By the assumptions on \( a \), \( S(x_1) \in C_1(t_11) \) provided \( t_1 > 0 \) is sufficiently small. Next we claim that \( x := t^b \in C_1(t1) \) for any \( t \in (0,1) \). As \( b \) is constant on \( V_{>0} \) we need to verify only that \( S(t^b)(1) \in [0,1] \). And indeed,

\[
S(t^b)(1) = (t^0 - t^1) + \sum_{\mu \in V} (-1)^{\mu-1} t^1 = (1 - t) \in [0,1]
\]

for any \( t \in (0,1) \). Now choose \( t_n \in (0,1), n > 2 \), such that \( t_n \to 1 \) and set

\[
x_n := (t_n)^b.
\]

Then \( c^\infty := \text{tnest}((S(x_n))_{n \geq 1}) \) has the desired properties.

**Remark 7.9.** Other interesting examples of copulas with tail dependence of degree 1 are \( \text{tnest}((S((1+\delta_n)(t_n)^b))_{n \geq 1}) \) for appropriate \( \delta_n \) with and

\[
\prod_{n=1}^{\infty} (1 + \delta_n(\nu)) = a(\nu).
\]

In this way, we can control how the limits are approached, starting at an arbitrary \( t_1 \in (0,1) \).
8 Change of coordinates

When studying tail characteristics for random variables $X_1, \ldots, X_r$, we could study for a given decreasing sequence $(0, 1) \ni s_n \to 0$ and each $\nu \in V$ the asymptotic behaviour of

$$p_n(\nu) = P(X_i \leq Q_i(s_n) \text{ for } \nu_i = 0) \quad \text{as } n \to \infty$$

(34)

where $Q_i$ is the quantile function of $X_i$, i.e., the inverse of the cumulative distribution function $F_i : x \mapsto P(X_i \leq x)$. We assume for simplicity that the $F_i$ are continuous and strictly increasing. As an application of Theorem 6.5 we can use tail nesting in order to construct probability spaces with random variables $X_1, \ldots, X_r$ where the asymptotic behaviour of all the functions in (34) can be prescribed.

The transformation to the uniform variables appearing in this context is just one of many possible transformations. One of the critical comments about the use of copulas is that “(...) there is no particular mathematical or practical reason (...)” (Th. Mikosch [7]) for selecting this transformation.

More generally one may wish to look at the asymptotic behaviour of

$$p(\xi^{(n)}, \nu) = P(X_i \leq \xi_i^{(n)}(n) \text{ for } \nu_i = 0) \quad \text{as } n \to \infty$$

(35)

for a given sequence $\xi^{(n)} \in \mathbb{R}^r$ such that $P(X_i \leq \xi_i^{(n)}) > 0$ and strictly decreasing to 0. As above, we can apply Theorem 6.5 in order construct probability spaces together with random variables $X_1, \ldots, X_r$ where

(i) the cumulative distribution function $F_i$ of $X_i$ is given, and

(ii) the asymptotic behaviour of $p(\xi^{(n)}, \nu$) can be controlled simultaneously for all $\nu \in V$.

We can arrange the vertex copulas $x_n \in C_k(u^{(n)})$ in Theorem 6.5 such that

$$F_i(\xi_i^{(n)}) = d_i^{(n)}.$$ 

where $d_i^{(n)}$ is as in [22]. The sequence $(u^{(n)})$ is determined by the sequence $(\xi^{(n)})$ and we can choose $x_n \in \mathbb{R}^V$ subject to the conditions in Corollary 4.6. The transformation from $\xi^{(n)}$ to $d^{(n)}$ gives merely a nice coordinate system to carry out the geometric construction of nesting.

**Example 8.1.** Consider a collection of Pareto-distributions $F_i : (-\infty, -1] \to [0, 1], s \mapsto (-s)^{-\alpha_i}$ with $\alpha \in (0, \infty)^r$. Say we are choosing $\xi_i^{(n)} = (-t)^n$ for some $t > 1$, i.e., we aim to control the asymptotic behaviour of the probabilities $P(X_i \leq (-t)^n \text{ for } \nu_i = 0) \quad \text{as } n \to \infty$.

As $d_i^{(n)} = (-t)^{-\alpha_i}$ we see that $u^{(n)} = ((-t)^{-\alpha_1}, \ldots, (-t)^{-\alpha_r}) =: u$ which does not depend on $n$. We define now $x_n$ by

$$x_n = (1 + \delta_n) \cdot (-t)^{-b}$$
where \( b(1) = 0, \delta_n(\nu) = 0 \) for \( \nu \in V_{\leq 1}, |\delta_n(\nu)| < \delta \), and \( \prod_n (1 + \delta_n(\nu)) =: a(\nu) \in (0, \infty) \). Furthermore, we require that

\[
b(\nu) < b(\mu) \quad \text{for } F(\hat{\nu}) \subset F(\hat{\mu}) \text{ and } \nu \neq \mu,
\]

and

\[
\max\{\alpha_i \mid \nu_i = 0\} \leq b(\nu) \leq \sum_{\nu_i = 0} \alpha_i, \quad \text{for } \nu \neq 1.
\]

We observe next that \( S(x_n) =: z_n \in C_1(u), \) provided \(|t|\) is sufficiently large. We impose the condition on the right hand side of (37) in order to ensure that the probabilities of o-boxes as approaching the origin o are not smaller than in the case where the corresponding \( X_i \) are independent. Note that for \( \nu \in V_1 \) the upper and lower bound in (37) are equal and their values consistent with the requirements in Corollary 4.6 for copulas (of order 1). For the probability measure \( c^n := \text{cnest}((z_n)) \) on \([0, 1]^r\) and \( X_i(u) = F_i^{-1}(u_i), u \in [0, 1]^r \) we obtain

\[
P(X_i \leq -(t)^n \text{ for } \nu_i = 0) \sim a(\nu)(-(-t)^n)^{b(\nu)} \quad \text{as } n \to \infty.
\]

As in the previous section, we can weaken condition (36) by dealing directly with constraints for \((\delta_n, b)\) imposed by Corollary 4.6.

**Conclusion**

The construction and examples described in this paper provide insights into a variety of asymptotic dependence structures of random variables. Tail nested copulas enable us to deal with tail dependencies of any order. The behaviour of these copulas can be controlled along a sequence inside the unit \( r \)-cube which converges to the origin.

We believe that tail nested copulas are suitable for applications in risk management. They allow the risk modeller not only to take those dependencies into account which really matter in the specific application, but as well to generate corresponding stochastic samples numerically in an efficient manner.

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