Scalaron decay into a pair of scalar particles but not only

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Abstract

The particle production through the scalaron decays are considered for several different channels. The central part of the work is dedicated to a study of the decay probability into two complex minimally coupled massless scalars. The calculations are performed by two different independent methods to avoid possible errors. In addition we calculated the decay probability into massless minimally coupled real scalars, conformally coupled massive scalars, massive fermions, and gauge bosons. The results are compared with the published papers which in some cases disagree with each other.
1 Introduction

The popular now mechanism of the Starobinsky inflation is based on the introduction of the additional term quadratic in scalar curvature, $R$, into the canonical Hilbert-Einstein action [1]:

$$S(R^2) = -\frac{m_{Pl}^2}{16\pi} \int d^4x \sqrt{-g} \left[ R - \frac{R^2}{6M_R^2} \right],$$  \hspace{1cm} (1)

where $m_{Pl} \approx 1.2 \cdot 10^{19}$ GeV is the Planck mass and $M_R$ is a constant parameter with dimension of mass. According to the estimate of ref. [2] the magnitude of temperature fluctuations of CMB demands $M_R \approx 3 \cdot 10^{13}$ GeV.

Due to the nonlinear term in the action curvature, $R$ becomes a dynamical variable and we can speak about new gravitational scalar degree of freedom, scalaron, with mass equal to $M_R$.

As is argued e.g. in the review [3], cosmological evolution in $R^2$-modified theory is naturally divided into the following four epochs:

1) inflation, when $R$ slowly decreases from some large value $R/M_R^2 \gtrsim 10^2$,

2) curvature oscillations,

$$R(t) = 4M_R \frac{\cos(M_R t + \theta)}{t},$$  \hspace{1cm} (2)

leading to efficient particle production through the scalaron decay and consequently to the universe heating,

3) transition of the scalaron domination regime to the dominance of the produced matter of mostly relativistic particles, and

4) transition to the conventional cosmology governed by the General Relativity.

In this paper we confine ourselves to the epoch of the universe heating and calculate the rate of the production of different types of particles with the aim to resolve some discrepancies in the existing literature. The main attention is paid to the case of decays into massless minimally coupled scalars, which has not been previously considered in the literature. We perform the calculations in two different independent ways: the usual calculations of the matrix element of the external field $R(t)$ between vacuum and a pair of the scalar particle state and calculating quantum corrections to the scalaron equation of motion. The latter method is analogous to those considered in refs. [4–6]. We have also studied the decays into massless minimally coupled real scalars, conformally coupled massive scalars, massive fermions, and massless gauge bosons. In the latter case the decay is induced by the conformal anomaly.

The form of the action written as (1) is called the Jordan frame. We prefer to use it because the equation of motion of the scalaron field has the form of the usual Klein-Gordon equation, see eq. (1). However in several papers as e.g. in [7] the so-called Einstein frame is used. Both frame are presumably equivalent.

In the course of this paper we assume that the metric is the spatially-flat Friedmann-Lemaître-Robertson-Walker (FLRW) one with the interval:

$$ds^2 = dt^2 - a^2(t)\delta_{ij}dx^i dx^j,$$  \hspace{1cm} (3)
where $a(t)$ is the cosmological scale factor and the Hubble parameter is expressed through $a(t)$ as $H = \dot{a}/a$.

According to eq. (3) the metric tensor $g_{\mu\nu}$ is taken with the signature convention $(+,-,-,-)$. The Riemann tensor describing the curvature of space-time is determined according to $R^\alpha_{\mu\beta\nu} = \partial_\beta R^\alpha_{\mu\nu} + \cdots$, $R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}$, and $R = g^{\mu\nu}R_{\mu\nu}$.

Equation of motion for the curvature scalar which follows from action (1) has the form:

$$D^2 R + M_R^2 R = -\frac{8\pi M_R^2}{m_{Pl}^2} T_{\mu}^\mu,$$  

(4)

where $D^2 = g^\mu_\nu D_\mu D_\nu$, $D_\mu$ is the covariant derivative in metric (3), and $T_{\mu}^\mu$ is the trace of the energy-momentum tensor of matter, which comes from the canonical matter action omitted in eq (1), but for some fields presented below (9, 10). For homogeneous $R = R(t)$

$$D^2 R = \left(\partial_t^2 + 3H\partial_t\right) R.$$  

(5)

The effective action of the scalaron field leading to equation of motion (4) can be written as

$$A_R = \frac{m_{Pl}^2}{48\pi M_R^4} \int d^4 x \sqrt{-g} \left[ \frac{(D R)^2}{2} - \frac{M_R^2 R^2}{2} - \frac{8\pi M_R^2}{m_{Pl}^2} T_{\mu}^\mu R \right].$$  

(6)

To determine the energy density of the scalaron field we have to redefine this field in such a way that the kinetic term of the new field enters the action with the coefficient unity. So the canonically normalized scalar field is [8]:

$$\Phi = \frac{m_{Pl}}{\sqrt{48\pi M_R^4}} R.$$  

(7)

Correspondingly, the energy density of the scalaron field is equal to:

$$\epsilon_R = \epsilon_{\Phi} = \frac{\dot{\Phi}^2 + M_R^2 \Phi^2}{2} = \frac{m_{Pl}^2 (\dot{R}^2 + M_R^2 R^2)}{96\pi M_R^4}.$$  

(8)

2 Scalar decay products of the scalaron

We assume that the actions of the non-interacting, except for coupling to gravity, complex and real scalar fields with mass $m$ has respectively the forms:

$$S_c[\phi_c] = \int d^4 x \sqrt{-g} \left( g^{\mu\nu} \partial_\mu \phi_c^* \partial_\nu \phi_c - m^2 |\phi_c|^2 + \xi R |\phi_c|^2 \right),$$  

(9)

$$S_r[\phi_r] = \frac{1}{2} \int d^4 x \sqrt{-g} \left( g^{\mu\nu} \partial_\mu \phi_r \partial_\nu \phi_r - m^2 \phi_r^2 + \xi R \phi_r^2 \right).$$  

(10)

If the constant $\xi$ is zero, fields $\phi$’s are called minimally coupled to gravity; for $\xi = 1/6$ they are called conformally coupled, because in this case the trace of the energy-momentum tensor of the fields $\phi$’s become zero.
The equation of motion both for real and complex fields $\phi$’s have the form

$$D^2 \phi + m^2 \phi - \xi R \phi = 0,$$

(11)

which in metric (3) transforms to

$$\ddot{\phi} - \frac{\Delta \phi}{a^2} + 3H \dot{\phi} + m^2 \phi - \xi R \phi = 0,$$

(12)

where $\Delta$ is the three-dimensional Laplace operator in flat 3D-space.

The energy-momentum tensor of $\phi$ is defined as the variation of the action over the metric tensor:

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}}.$$ 

(13)

Correspondingly for the complex field

$$T^{(c)}_{\mu\nu} = (\partial_\mu \phi^*_c)(\partial_\nu \phi_c) + (\partial_\nu \phi^*_c)(\partial_\mu \phi_c) - g_{\mu\nu}(g^{\alpha\beta}\partial_\alpha \phi^*_c \partial_\beta \phi_c - m^2|\phi_c|^2)$$

$$+ \xi (2R_{\mu\nu} - g_{\mu\nu} R)|\phi_c|^2 - 2\xi (D_\mu D_\nu - g_{\mu\nu} D^2)|\phi_c|^2,$$

(14)

where $D_\mu$ is the covariant derivative in metric (3). The trace of this tensor is:

$$T^{(c)}_{\mu\mu} = 2(6\xi - 1)\partial_\mu \phi^*_c \partial^\mu \phi_c + 2\xi(6\xi - 1)R|\phi_c|^2 + 4(1 - 3\xi)m^2|\phi_c|^2.$$ 

(15)

Note that for $\xi = 1/6$ and $m = 0$ the trace vanishes.

For the real field $\phi$, the energy-momentum tensor has the same form with twice smaller coefficients.

Field $\phi$’s enter the equation of motion for $R$ (4) via the trace of their energy-momentum tensor. Taking quantum average of $T^{(c)}_\mu$ over background "filled" by classical scalaron field $R$, but devoid of $\phi$-particles, we can obtain equation for $R$ with an account for particle production. As we see in what follows, in the particular case of harmonic oscillations of the scalaron, particle production can be described by the simple term $\Gamma \dot{R}/2$.

### 2.1 Decay into a pair of minimally coupled massless scalars

The scalaron decay width into two massless (or very low mass) scalar bosons was calculated in refs. [1,6,9]. Here we follow our paper [6], where another approach was used based on papers [4,5], which allows to derive closed equation for an arbitrary time evolution of the source field (in our case the scalaron, $R(t)$), while the traditional methods are valid only for the harmonic oscillations of the source.

According to Eq. (9) the action for the complex massless scalar field with minimal coupling to gravity has the form:

$$S^{(00)}_{c}[\phi_c] = \int d^4x \sqrt{-g} g^{\mu\nu}\partial_\mu \phi^*_c \partial_\nu \phi_c.$$ 

(16)

and leads to the following equation of motion:

$$\ddot{\phi}_c + 3H\dot{\phi}_c - \frac{1}{a^2}\Delta \phi_c = 0.$$ 

(17)
It is convenient to study particle production in terms of the conformally rescaled field, and the conformal time according to the equations:

$$\chi_c = a(t)\phi_c, \quad d\eta = dt/a(t). \quad (18)$$

The curvature scalar is expressed through the scale factor as

$$R = -6 \left( \dot{H} + 2H^2 \right) = -6a''/a^3, \quad (19)$$

here and below prime denotes derivative with respect to conformal time.

The equation of motion for the conformally rescaled field $\chi$ takes the form:

$$\chi''_c - \Delta \chi_c + \frac{1}{6} a^2 R \chi_c = 0, \quad (20)$$

while action (16) turns into:

$$S_c^{(00)}[\chi_c] = \int d\eta d^3 x \left( \chi''_c \chi'_c - \vec{\nabla} \chi'_c \vec{\nabla} \chi_c - \frac{a^2 R}{6} |\chi_c|^2 \right). \quad (21)$$

Equation (4), which describes the scalaron evolution, can now be written as:

$$R'' + 2\frac{a'}{a} R' + a^2 M^2 R = \frac{16\pi}{a^2} M^4 M_{Pl}^2 \left( \chi''_c \chi'_c - \vec{\nabla} \chi'_c \vec{\nabla} \chi_c + \frac{a^2}{a^2} |\chi_c|^2 - \frac{a'}{a} (\chi''_c + \chi''_c) \right). \quad (22)$$

Our aim is to derive a closed equation for $R$ taking the average value of the $\chi$-dependent quantum operators in the r.h.s. of eq. (22), in presence of classical curvature field $R(\eta)$. The consideration essentially repeat those of ref. [4], where the equation was derived in one-loop approximation.

Equation (20) can be transformed into the following integro-differential equation convenient for perturbative solution:

$$\chi_c(x) = \chi_c^{(0)}(x) - \frac{1}{6} \int d^4 y G(x,y) a^2(y) R(y) \chi_c(y) \equiv \chi_c^{(0)}(x) + \delta \chi_c(x), \quad (23)$$

where $\chi_c^{(0)}$ satisfies the free equation $\chi''_c - \Delta \chi_c = 0$ and the massless Green’s function is

$$G(x,y) = \frac{1}{4\pi|x-y|} \delta((x_0 - y_0) - |x - y|) \equiv \frac{1}{4\pi\Delta r} \delta(\Delta \eta - \Delta r). \quad (24)$$

Here $\Delta \eta = x_0 - y_0$ and $\Delta r = |x - y|$. Since $\Delta r = |x - y| \geq 0$, the condition $\Delta \eta \geq 0$ is also to be fulfilled.

The free field $\chi_c^{(0)}$ is quantised in the usual way:

$$\chi_c^{(0)}(x) = \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2E_k} \left[ \hat{a}_k e^{-ik \cdot x} + \hat{b}^\dagger_k e^{ik \cdot x} \right], \quad (25)$$

where $x^\mu = (\eta, \mathbf{x})$, $k^\mu = (E_k, \mathbf{k})$, $E_k^2 - \mathbf{k}^2 = 0$, and $\hat{a}_k$ is the annihilation operator for particles, while $\hat{b}^\dagger_k$ is the creation operator for antiparticles. The creation-annihilation operators satisfy the commutation relations:

$$[\hat{a}_k, \hat{a}^\dagger_{k'}] = 2E_k (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k'}), \quad (26)$$
and analogously for $\hat{b}_k$.

The particle production effects are assumed to weakly perturb the free solution, so Eq. (23) can be solved in the first order approximation as

$$\chi_c(x) \simeq \chi_c^{(0)}(x) - \frac{1}{6} \int d^4y G(x - y) a^2(y) R(y) \chi_c^{(0)}(y) \equiv \chi_c^{(0)}(x) + \chi_c^{(1)}(x).$$  (27)

Now we calculate the vacuum expectation values of the various terms in the r.h.s. of equation (22), keeping only the contribution from the terms linear in $\chi^{(1)}$. The terms of zero order which are bilinear in $\chi^{(0)}$ and its derivatives have nothing to do with particle production and can only change the parameters of the theory through the renormalization procedure.

We need to calculate the products of the quantum operators of the kind:

$$\langle \chi_c^{(0)}(x) \chi_c^{(1)*}(x) \rangle = -\frac{1}{6} \int d^4y dy_0 G(x - y) a^2(y_0) R(y_0) \langle \chi_c^{(0)}(x) \chi_c^{(0)*}(y) \rangle,$$  (28)

where $dy_0 = d\eta_y$ is the time component corresponding to the space coordinate $dy$.

The vacuum expectation values of the creation/annihilation operators are

$$\langle \hat{b}^\dagger_k \hat{b}_{k_2} \rangle = 0, \quad \langle \hat{a}_{k_1}^{\dagger} \hat{a}_{k_2} \rangle = 2E_k (2\pi)^3 \delta^{(3)}(k_1 - k_2),$$  (29)

where in the last equation we have used commutator (26).

Now using expansion (25) we find

$$\langle \chi_c^{(0)}(x) \chi_c^{(1)*}(x) \rangle = -\frac{1}{6} \int \frac{d^3k}{2E_k (2\pi)^3} d^3y dy_0 G(x - y) a^2(y_0) R(y_0) e^{-iE_k(x_0 - y_0) + ik(x - y)}.$$  (30)

Let us first integrate over angles in $d^3k = E_k^2 dE_k d(\cos \theta) d\phi$:

$$\langle \chi_c^{(0)}(x) \chi_c^{(1)*}(x) \rangle = -\frac{1}{48\pi^2} \int d^3y dy_0 dk \frac{e^{ik\Delta r} - e^{-ik\Delta r}}{i\Delta r} e^{-ik\Delta \eta} G(x - y) a^2(y_0) R(y_0).$$  (31)

For brevity we used the notation $k = E_k = |k|$.

Next we integrate over $d^3y = d^3\Delta r$ using the delta-function in the expression for the Green function (24) and obtain:

$$\langle \chi_c^{(0)}(x) \chi_c^{(1)*}(x) \rangle = \frac{i}{48\pi^2} \int dy_0 dk \left( 1 - e^{-2ik\Delta \eta} \right) a^2(y_0) R(y_0).$$  (32)

Therefore,

$$\langle |\chi_c(x)|^2 \rangle \simeq \langle 2 \text{Re}(\chi_c^{(0)} \chi_c^{(1)*}(x)) \rangle = -\frac{1}{24\pi^2} \int dy_0 dk \text{Re} \left( i e^{-2ik\Delta \eta} a^2(y_0) R(y_0) \right).$$  (33)

The integral over $dk$ can be taken according to the equation:

$$\int_0^\infty dk e^{iak} = \pi \delta(\alpha) + i \mathcal{P} \left( \frac{1}{\alpha} \right),$$  (34)
so we arrive finally at

$$
\langle |\chi_c(x)|^2 \rangle \simeq -\frac{1}{48\pi^2} \int dy_0 a^2(y_0) R(y_0) \mathcal{P} \left( \frac{1}{\Delta \eta} \right).
$$

The upper limit of integration is imposed by the condition $\Delta \eta \geq 0$, see Eq. (24).

The dominant contribution to the particle production comes from the first term in the r.h.s. of Eq. (22). So we have to calculate the expectation value:

$$
\langle (\chi_c^{(0)}(x))' (\chi_c^{(1)}(x))'' \rangle = -\frac{1}{6} \int d^3y_0 \partial G(x-y) a^2(y_0) R(y_0) \langle (\chi_c^{(0)}(x))' (\chi_c^{(0)}(y))'' \rangle. 
$$

Taking into account that $\partial G(x-y)/\partial x_0 = -\partial G(x-y)/\partial y_0$ and integrating by part over $dy_0$ we get

$$
\langle (\chi_c^{(0)}(x))' (\chi_c^{(1)}(x))'' \rangle = -\frac{1}{6} \int d^3y_0 G(x-y)a^2(y_0) \times
$$

$$
\times \left[ R'(y_0) \langle (\chi_c^{(0)}(x))' (\chi_c^{(0)}(y))' \rangle + R(y_0) \langle (\chi_c^{(0)}(x))' (\chi_c^{(0)}(y))'' \rangle \right],
$$

where the derivative of $a^2(y_0)$ is neglected because it is slowly varying function of time.

In complete analogy with the calculations made above we find

$$
\langle (\chi_c^{(0)}(x))' (\chi_c^{(1)}(x))'' \rangle = -\frac{1}{6} \int \frac{d^3k}{2E_k(2\pi)^3} \int d^3y \, dy_0 \, G(x-y)a^2(y_0) \times
$$

$$
e^{-iE_k(x_0-y_0)k} e^{iE_k R' + E_k^2 R}. 
$$

Integration over directions of $d^3k$ leads as above to

$$
\langle (\chi_c^{(0)}(x))' (\chi_c^{(1)}(x))'' \rangle = -\frac{1}{48\pi^2} \int d^3y_0 dk \, \frac{e^{ik\Delta r} - e^{-ik\Delta r}}{i\Delta r} e^{-i\Delta \eta} G(x-y)a^2 \left( -iE_k R' + E_k^2 R \right).
$$

After integration over $d^3y$ with $G(x-y)$ given by Eq. (24) we arrive at

$$
\langle (\chi_c^{(0)}(x))' (\chi_c^{(1)}(x))'' \rangle = \frac{i}{48\pi^2} \int dy_0 dk \, (1 - e^{-2ik\Delta \eta}) a^2(y_0) \left( -iE_k R' + k^2 R \right).
$$

Using equations $2ik \exp(-2ik\Delta \eta) = \partial_{y_0} \exp(-2ik\Delta \eta)$ and $4k^2 \exp(-2ik\Delta \eta) = -\partial^2_{y_0} \exp(-2ik\Delta \eta)$ and integrating by parts we obtain

$$
\langle |\chi'_c(x)|^2 \rangle \simeq -\frac{1}{192\pi^2} \int dy_0 a^2(y_0) R''(y_0) \mathcal{P} \left( \frac{1}{\Delta \eta} \right).
$$

In a similar way we get

$$
\langle |\nabla \chi_c(x)|^2 \rangle \simeq \frac{1}{192\pi^2} \int dy_0 a^2(y_0) R''(y_0) \mathcal{P} \left( \frac{1}{\Delta \eta} \right),
$$

$$
\langle (\chi_c^*(x)\chi'_c(x) + \chi_c'(x)\chi_c(x)) \rangle \simeq -\frac{1}{48\pi^2} \int dy_0 a^2(y_0) R'(y_0) \mathcal{P} \left( \frac{1}{\Delta \eta} \right).
$$
Inserting these expressions into (22), we obtain a closed integro-differential equation for \( R \), which we will transform into ordinary differential equation for harmonic oscillations of \( R \) neglecting its slow power law decrease at the scale of very fast oscillation time.

By the same reason the scale factor \( a \) varies very little during many oscillation times \( \omega^{-1} = M_R^{-1} \). Thus, we expect that \( dt/\eta \sim dt/t \) and that the dominant part in the integrals in (27) is given by derivatives of \( R \), since \( R' \sim \omega R + t^{-1} R \simeq \omega R \), because \( \omega t \gg 1 \). So the dominant contribution of particle production is given by expression

\[
\langle (\chi'_c(x))^2 - |
\nabla \chi_c(x)|^2 \rangle \simeq - \frac{1}{96\pi^2} \int_{\eta_0}^{\eta} \frac{a^2(\eta_1)R''(\eta_1)}{\eta - \eta_1},
\]

and is reduced to

\[
\ddot{R} + 3H \dot{R} + M_R^2 R \simeq - \frac{1}{6\pi} \frac{M_R^2}{M_{Pl}^2} \frac{1}{a^4} \int_{\eta_0}^{\eta} \frac{a^2(\eta_1)R''(\eta_1)}{\eta - \eta_1} \simeq - \frac{1}{6\pi} \frac{M_R^2}{M_{Pl}^2} \int_{t_0}^{t} dt_1 \frac{\dot{R}(t_1)}{t - t_1} \quad (44)
\]

The equation is naturally non-local in time since the effect of particle production depends upon all the history of the system evolution.

Rigorous determination of the decay width of the scalaron is described in ref. [6]. Here we present it in a simpler and intuitively clear way. We will look for the solution of Eq. (44) in the form:

\[
R = R_{amp} \cos(\omega t + \theta) \exp(-\Gamma t/2), \quad (45)
\]

where \( R_{amp} \) is the slowly varying amplitude of \( R \)-oscillations, \( \theta \) is a constant phase depending upon initial conditions, and \( \omega \) and \( \Gamma \) is to be determined from the equation. The term \( 3H \dot{R} \) is not essential in the calculations presented below and will be neglected. The exponent is taken equal to \( \Gamma t/2 \) so the scalaron energy density would decrease as \( \exp(-\Gamma t) \).

Assuming that \( \Gamma \) is small, so the terms of order of \( \Gamma^2 \) are neglected and treating the r.h.s. of Eq (44) as perturbation we obtain:

\[
\left[ (-\omega^2 + M_R^2) \cos(\omega t + \theta) + \Gamma \omega \sin(\omega t + \theta) \right] e^{-\Gamma t/2} =
\frac{1}{6\pi} \frac{\omega^2M_R^2}{M_{Pl}^2} e^{-\Gamma t/2} \int_{t_0}^{t} \frac{d\tau}{\tau} \cos(\omega t + \theta) \cos \omega \tau + \sin(\omega t + \theta) \sin \omega \tau \quad (46)
\]

The first, logarithmically divergent, term in the integrand leads to mass renormalization and can be included into physical \( M_R \), while the second term is finite and can be analytically calculated at large upper integration limit \( \omega t \) according to the well-known integral

\[
\int_{0}^{\infty} \frac{d\tau}{\tau} \sin \omega \tau = \frac{\pi}{2}, \quad (47)
\]

Comparing the l.h.s. and r.h.s. of Eq. (46) we can conclude that \( \omega = M_R \) and the width of the scalaron decay into a pair of "charged" massless minimally coupled scalars is

\[
\Gamma_c = \frac{M_R^2}{12M_{Pl}^2} \quad (48)
\]
The width of the decay into a pair of neutral identical particles should evidently be twice smaller:

\[ \Gamma_r = \frac{M_r^2}{24 M_{Pl}^2}. \]  \hfill (49)

The latter result agrees with those presented e.g. in ref. [7, 9]. However, is it twice larger than the width of the scalaron decay into two real massless scalars calculated in paper [2], eq. (76).

### 2.2 Decay into minimally coupled complex scalars, another method

Now we calculate the rate of the scalaron decay into the same channel, as is studied in the previous subsection, in a different way dealing with the energy loss of the scalaron into the produced particles. To this end we will use the equation of motion of the decay products (20) and calculate the energy density of particles \( \chi \) and anti-\( \chi \) created by the oscillating gravitational field of the scalaron per unit time \( \dot{\rho}_\chi \). Then we compare it to the energy density of the canonically normalized scalaron field (7).

In what follows we use for \( R \) the solution (2):

\[ R = \frac{4}{M_R} \cos(M_R t + \theta)/t. \]

For this \( R \) the energy density of \( \Phi \) is equal to (8):

\[ \rho_\Phi = \frac{\dot{\Phi}^2 + M_R^2 \Phi^2}{2} = \frac{M_{Pl}^2 (\dot{R}^2 + M_R^2 R^2)}{96 \pi M_R^4} \approx \frac{M_{Pl}^2}{6 \pi t^2} \quad \text{(for } M_R t \gg 1). \]  \hfill (50)

Note that this is formally equal to the critical energy density for matter dominated universe.

Due to the energy conservation \( \dot{\rho}_\chi + \dot{\rho}_{\bar{\chi}} = 2 \dot{\rho}_\chi = -\dot{\rho}_\Phi \). So for the rate of the energy dissipation of the scalaron we find:

\[ \Gamma = 2 \dot{\rho}_\chi / \rho_\Phi, \]  \hfill (51)

where \( \dot{\rho}_\chi \) is calculated along the standard lines of particle production theory in external time-dependent fields, see e.g. [10–12].

According to the Parker theorem [13, 14] massless particles are not created by conformally flat FLRW metric. This is fulfilled for massless spin 1/2-fermions, massless gauge boson (up to conformal anomaly [15]), but is not always true for scalar bosons, because the later are conformally invariant only for \( \xi = 1/6 \).

The particle production in conformally flat FLRW background is convenient to study in terms of conformal time as described above in Eqs. (18)-(21). In what follows we closely follow book [12]. The quantum field operators describing the created particles is assumed to satisfy the equation:

\[ \chi'' - \Delta \chi + f(\eta) \chi = 0. \]  \hfill (52)

We omit here subindex \( c \) at \( \chi_c \), because only complex field \( \chi \) is considered in this section. Field \( \bar{\chi} \) satisfies the Hermitian conjugate equation. The function \( f(\eta) \) is a classical external field producing quanta of \( \chi \) and anti-\( \chi \) particles.
The amplitude of production of a pair of $\chi$ and anti-$\chi$ bosons with momenta $k_1$ and $k_2$, respectively, is equal to the matrix element of the interaction term between vacuum and the corresponding particle-antiparticle state:

$$A(k_1, k_2) = \int d\eta d^3 x f(\eta) \langle k_1, k_2 | \chi^\dagger(\eta, x) \chi(\eta, x) |0\rangle.$$  

(53)

Recall that we work in conformal time $d\eta = dt/a(t)$ and conformally transformed field $\chi = a\phi$.

The quantum field operators are expanded in terms of the creation/annihilation operators as given by Eq. (25). The “bra” state of the produced pair of $\chi$ and $\bar{\chi}$ quanta is defined in terms of these operators as

$$\langle k_1, k_2 | = \langle 0 | \hat{a}(k_1) \hat{b}(k_2).$$  

(54)

Now keeping in mind that the annihilation operator acting on the vacuum state to the right annihilates the state, $\hat{a}(k)|0\rangle = 0$, and $\langle 0|\hat{a}^\dagger(k) = 0$ and applying commutation relation (26) we obtain:

$$A(k_1, k_2) = \int d\eta d^3 x f(\eta) d\tilde{k} \tilde{d}k' \langle 0 | \hat{a}(k_1) \hat{b}(k_2) \hat{a}^\dagger(k) \hat{b}^\dagger(k') |0\rangle \times \exp \left[ i\eta(E_{k_1} + E_{k_2}) - i\cdot k + k' \right] = (2\pi)^3 \delta(k_1 + k_2) \int d\eta f(\eta) \exp \left[ i\eta(E_{k_1} + E_{k_2}) \right],$$

(55)

where $d\tilde{k} = d^3k/(2E_k(2\pi)^3)$ and $E_k = |k|$. Hence

$$|A(k_1, k_2)|^2 = (2\pi)^3 V \delta(k_1 + k_2) \left| \int d\eta f(\eta) \exp[i\eta(E_{k_1} + E_{k_2})] \right|^2,$$

(56)

here the following identities are used:

$$\int d^3 x e^{-ikx} = (2\pi)^3 \delta(k), \quad \text{and} \quad \delta(k = 0) = V/(2\pi)^3,$$

(57)

where $V$ is the total space volume.

The Fourier transform of the source can be simply calculated for the case of harmonic oscillations

$$f(\eta) = f_0 \cos(\omega\eta) = [\exp(i\omega\eta) + \exp(-i\omega\eta)].$$

(58)

Taking into account that the energy of the created particles should be positive, $E_{k_1,2} > 0$, we obtain

$$\left| \int d\eta f(\eta) \exp[i\eta(E_{k_1} + E_{k_2})] \right|^2 = \frac{\pi}{2} f_0^2 \delta(\omega - E_{k_1} - E_{k_2}) \Delta\eta,$$

(59)

where we used that $\delta(0) = \Delta\eta/(2\pi)$ and $\Delta\eta$ is the time duration of the process, presumably $\omega \Delta\eta \gg 1$. 

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So the $\chi$-particle production rate per unit volume and unit conformal time is

$$\frac{dn^{(\text{tot})}_\chi}{d\eta} = 2\frac{dn_\chi}{d\eta} = \frac{2}{V\Delta \eta} \int d^3k_1 d^3k_2 |A(k_1, k_2)|^2 = \frac{(2\pi)^3 f_0^2}{16\pi} \int d^3\tilde{k}_1 d^3\tilde{k}_2 \delta(k_1 + k_2) \delta(\omega - E_{k_1} - E_{k_2}) = \frac{f_0^2}{16\pi}. \quad (60)$$

Now we have to express $f(\eta) = f_0 \cos \omega \eta$ through $R(t)$ (2). Comparing equations (20) with (52) and using Eq. (2) we conclude that

$$f_0 \cos \omega \eta = \frac{2a^2 M_R}{3t} \cos M_R t. \quad (61)$$

We took here the initial phase $\theta = 0$. Evidently it simply corresponds to a fixation of the initial time moment and does not have any impact on the production probability. Since the product $p_\mu dx^\mu$ is invariant and in our case $p_\mu$ has only one component, namely, $p_\mu \rightarrow p_0 = M_R$, we can conclude that $\omega d\eta = M_R dt$ and thus

$$f_0 = \frac{2a^2 M_R}{3t}. \quad (62)$$

The r.h.s. of eq. (60) is proportional to $a^4$. The same is true for its l.h.s. because $d/d\eta = ad/dt$ and $n_\chi \sim \chi \chi' \sim a^3$. So returning to $dn_\phi/dt$ we find that it does not depend upon $a$. Since the energy of a pair of $\phi$-particles in the physical frame is equal to $M_R$, the time derivative of the energy lost to creation of $\phi\bar{\phi}$ pair is obtained from $\dot{n}_\phi$ by multiplication by $M_R$ and we find

$$2\dot{\phi} = \frac{M_R^3}{72\pi t^2}. \quad (63)$$

The energy density of the scalaron field (8) is $M_R^2/(6\pi t^2)$ and hence the decay width is

$$\Gamma_c = \frac{\dot{\phi}_\chi^{(\text{tot})}}{\theta R} = \frac{2\dot{\phi}_\chi}{\dot{\theta}_R} = \frac{M_R^3}{12M^2_{Pl}}. \quad (64)$$

It agrees with result (48) of the previous section.

### 2.3 Decays into neutral scalars

The action of a real scalar field $\phi_r$ with non-minimal coupling to gravity, $\xi \phi^2_r R$, has the form:

$$S_l^{(m)}[\phi_r] = \frac{1}{2} \int d^4x \sqrt{-g} \left(g^\mu_\nu \partial_\mu \phi_r \partial_\nu \phi_r + \xi R \phi^2_r - m^2_{\phi_r} \phi^2_r \right), \quad (65)$$

leading to the equation of motion:

$$\ddot{\phi}_r + 3H \dot{\phi}_r - \frac{1}{a^2} \Delta \phi_r + (m^2_{\phi_r} - \xi R) \phi_r = 0. \quad (66)$$

We skip the subindex $r$ in what follows.
The particle production will be considered, as above, in terms of the conformally rescaled field, and the conformal time according to the definitions:

\[ \chi = a(t)\phi, \]
\[ d\eta = dt/a(t). \]

Correspondingly equation (67) transforms into

\[ \chi'' - \Delta \chi + \left( \frac{1}{6} - \xi \right) a^2 R \chi + m_\phi^2 a^2 \chi = 0. \] (69)

Here prime means differentiation over \( \eta \) and \( R = -6a''/a^3 \). The temporal evolution of \( R(t) \) is given by eq. (2).

The particle production by external time-dependent field \( V(t) \) was studied in many works dedicated to the universe heating, see e.g. [11]. In particular the production rate was calculated perturbatively (but not only) in the book [12] for the (iflaton) field with the harmonic dependence on time:

\[ V(\eta) = V_0 \cos(\Omega_c \eta + \theta), \] (70)

where \( V_0 \) as well as \( \Omega_c \) may slowly depend on time. The field \( \chi \) satisfies the equation:

\[ \chi'' - \Delta \chi + V(\eta) \chi = 0. \] (71)

Since \( dt = a d\eta \), and \( \Omega dt = \Omega_c d\eta \), the conformal frequency is \( \Omega_c = a\Omega = aM_R \).

It has been shown, see e.g. eq. (6.40) in the book [12], that the number density of particles created per unit of conformal time is

\[ N_\chi' = \frac{V_0^2}{32\pi}, \] (72)

where the number density is expressed in canonical way through the transformed field \( \chi = a\phi \), namely

\[ N_\chi \sim \chi \partial_\eta \chi \sim a^3 N_\phi. \] (73)

Hence

\[ \dot{N}_\phi = \frac{V_0^2}{32\pi a^4}. \] (74)

Now we apply this result for \( \chi \) satisfying eq. (69) with \( \xi = 0 \) and \( m_\phi = 0 \), i.e. for

\[ V(\eta) = \frac{4a^2 M_R \cos(M_R t + \theta)}{6t}. \] (75)

Notice that this expression is written in terms of physical time \( t \) but not the conformal time \( \eta \).
Keeping in mind that each particle \( \chi \) carries an energy \( M_R/2 \) we find for the energy production rate per unit of physical time:

\[
\dot{\varepsilon}_\phi = \frac{1}{2} M_R N_\phi = \frac{M_R^3}{144\pi t^2}.
\]

And finally

\[
\Gamma(\xi = 0, m_\phi = 0) = \frac{\dot{\varepsilon}_\phi}{\varepsilon_\phi} = \frac{M_R^3}{24M_P^2},
\]

which coincides with the result obtained in the previous subsection, eq. (49).

### 2.4 Decay into conformally coupled massive scalars

Let us consider now the case of conformally coupled decay products, i.e. equation (69) with \( \xi = 1/6 \) and \( m_\phi \neq 0 \), but still \( m_\phi \ll M_R \), so the phase space suppression is not essential. The interaction leading to the particle production in this case has the form:

\[
V = m_\phi^2 a^2.
\]

Using the solution \[8\]

\[
H = \frac{\dot{a}}{a} = \frac{2}{3t}(1 + \sin(M_Rt + \theta))
\]

we find

\[
a^2 m_\phi^2 \approx m_\phi^2 t^{4/3} \exp\left(1 - \frac{4\cos(M_Rt + \theta)}{3tM_R}\right) \to a^2 \frac{4m_\phi^2}{3tM_R} \cos(M_Rt + \theta).
\]

Comparing it with the expression for \( R(t) \) \[2\] and the width \( \Gamma(\xi = 1/6, m_\phi \neq 0) \) we can conclude that the decay width of the scalaron energy density, if the decay is induced by \((m_\phi a)^2\), is equal to

\[
\Gamma(\xi = 1/6, m_\phi \neq 0) = \frac{m_\phi^4}{6M_RM_P^2}.
\]

This result coincide with those of ref. [9].

The energy release from \( \phi \) decay into the primeval plasma in this case is

\[
\dot{\varepsilon}_\phi = \Gamma(\xi = 1/6; m_\phi \neq 0)\varepsilon_\phi = \frac{m_\phi^4}{36\pi t^2 M_R}.
\]

### 2.5 Decay into fermions

Here we calculate the probability of fermion-antifermion pair production by scalar field

\[
\Phi = \Phi_0 \cos M_R t = \Phi_0 \frac{e^{iM_R t} + e^{-iM_R t}}{2}
\]

(83)
in the following way. First calculate the decay width of the Φ-meson with mass $M_R$ into a pair of light fermions. The decay amplitude is equal to

$$A_\psi = g\Phi \bar{\psi}\psi.$$  \hspace{1cm} (84)

To proceed with the standard calculations we need to take matrix element of operators in the above expression between one particle Φ initial state with zero momentum and fermion-antifermion final state, using the plane wave decomposition of the wave-function operators. In particular $\Phi \sim a_\Phi \exp(-iM_Rt)$, where $a_\Phi$ is the annihilation operator of one-particle Φ state. Taking matrix element squared and integrating over the phase space we find for the decay width the known expression

$$\Gamma_{\text{decay}} = \frac{1}{2M_R} \int d\tau_2 |A_\psi|^2 = \frac{|g|^2 M_R}{8\pi}.$$  \hspace{1cm} (85)

Here $d\tau_2$ is the invariant phase space of the fermion-antifermion pair and the factor $1/(2M_R)$ is related to normalization of the plane waves to one particle in the whole space volume $V$, so each plane wave contains factor $1/\sqrt{2EV}$ ($E$ is the particle energy).

Now let us turn to the number density of fermions produced by the oscillating field (83). In the case of the decay the initial scalar particle is described by the plane wave $\exp(-iM_Rt)$ with positive energy. The same is true also for the production by external field. This gives the factor 1/2 in the amplitude and 1/4 in the probability. Instead of $1/(2M_R)$ we need to substitute the density of the scalar particles in the field (83), which is equal to $2M_R\Phi_0^2$. So the density of fermions produced per unit conformal time can be obtained from Eq. (85) by multiplication with $2M_R\Phi_0^2/4$:

$$N'_\psi = \frac{|g|^2 M_R^2 \Phi_0^2}{16\pi}.$$  \hspace{1cm} (86)

Now we have to go to conformal time and conformally transformed field, $\psi = \bar{\psi}/a^{3/2}$. The relation between the conformally transformed quantities and the physical ones remains the same as above:

$$N'_\bar{\psi} = a^4 N_\psi.$$  \hspace{1cm} (87)

In the considered case the particle production in induced by the oscillation of the scale factor:

$$g\Phi_0 \rightarrow m_\psi a(t) \rightarrow \frac{2m_\psi a}{3tM_R},$$  \hspace{1cm} (88)

compare to Eq. (80).

Repeating arguments similar to those presented in the previous subsection we find:

$$\frac{\dot{\bar{\psi}}}{\dot{\psi}} = \frac{m_\psi^2 M_R}{36\pi t^2} \cdot \frac{6\pi t^2}{M_{Pl}^2} = \frac{m_\psi^2 M_R}{6M_{Pl}^2}.$$  \hspace{1cm} (89)
So the width of the scalaron decay into a pair of massive fermions is equal to:

\[ \Gamma = \frac{m_\phi^2 M_R}{6M_{Pl}^2}. \]  

(90)

This result coincides with that obtained in ref. [7].

A few words about the meaning of \( \Gamma \) are in order. From the derivation above it is clear that \( \Gamma \) determines the damping of the energy density of the scalaron field which is proportional to \( R^2 \). So we expect \( |R|^2 \sim \exp(-\Gamma t) \) and hence \( |R| \sim \exp(-\Gamma t/2) \). Let us check how this \( \Gamma \) is related to the damping coefficient \( \gamma \) in the equation:

\[ \ddot{R} + \gamma \dot{R} + M_R^2 R = 0. \]  

(91)

The solution of this equation has the form

\[ R \sim \cos(\Omega t) \exp(-\Gamma t/2). \]  

(92)

Substituting this expression into Eq. (91) we find:

\[ \gamma = \Gamma, \text{ and } \Omega^2 = M_R^2 - \Gamma^2/4. \]  

(93)

### 2.6 Decays into gauge bosons

Under conformal transformation vector gauge bosons are not transformed, \( A_\mu \rightarrow A_\mu \), and their equations of motion in terms of conformal time is the same as those in flat Minkowski metric. So in this approximation gauge bosons cannot be created by conformally flat gravitational field. This is truth but not all the truth. Conformal anomaly destroys this conclusion and allows for gauge boson to be created \([15]\).

Equation of motion of massless gauge field with an account of the anomaly, as derived in ref. [15], has the form:

\[ A'' - \Delta A + \alpha \kappa \xi'' A' = 0, \]

(94)

where \( \alpha \) is the gauge coupling constant squared (for electromagnetic \( U(1) \)-gauge group \( \alpha = 1/137 \) at low energies), \( \xi = \ln a \),

\[ \kappa = \frac{11}{3} N - \frac{2}{3} N_F, \]

(95)

\( N \) is the rank of the proper gauge group, and \( N_F \) is the number of fermion families (\( \kappa \) is usually denoted as \( \beta \) but here we follow the original paper).

According to the calculations of ref. [15] the number density of the produced gauge bosons per unit of physical time is

\[ \dot{N}_g = \frac{\alpha^2 \kappa^2}{32 \pi} \left( \frac{\ddot{a}}{a} \right)^2. \]

(96)
Note that
\[ R = -6 \left( \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) \approx -6 \frac{\ddot{a}}{a}, \]  
(97)

The last approximate equality is valid for quickly oscillating \( R \) given by Eq. (2).

In ref. [15] this equation was applied to particle production near singularity in Friedmann cosmology. Here we shall use it for \( R^2 \)-cosmology. To this end one needs to substitute the average value of \( R(t)^2 \) taking \( \langle \cos^2(M_R t) \rangle = 1/2 \). So

\[ \dot{N}_g = \frac{\alpha^2 \kappa^2 M_R^2}{144 \pi^2 t^2}, \]  
(98)

and finally the width of the scalaron decay into two gauge bosons is equal to

\[ \Gamma_g = \frac{\dot{\varphi}_g}{\varphi} = \frac{\alpha^2 \kappa^2 M_R^3}{144 \pi^2 t^2} \frac{6 \pi t^2}{M_{Pl}^2} = \frac{\alpha^2 \kappa^2 M_R^3}{24 M_{Pl}^2}. \]  
(99)

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