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GALAXY FORMATION IN TRIAXIAL HALOES: BLACK HOLE-BULGE-DARK HALO CORRELATION

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ABSTRACT

The masses of supermassive black holes (SBHs) show correlations with bulge properties in disk and elliptical galaxies. We study the formation of galactic structure within flat-core triaxial haloes and show that these correlations can be understood within the framework of a baryonic component modifying the orbital structure in the underlying potential. In particular, we find that terminal properties of bulges and their central SBHs are constrained by the destruction of box orbits in the harmonic cores of dark haloes and the emergence of progressively less eccentric loop orbits there. SBH masses, \( M_\bullet \), should exhibit a tighter correlation with bulge velocity dispersions, \( \sigma_B \), than with bulge masses, \( M_B \), in accord with observations, if there is a significant scatter in the \( M_H - \sigma_H \) relation for the halo. In the context of this model the observed \( M_\bullet - \sigma_B \) relation implies that haloes should exhibit a Faber-Jackson type relationship between their masses and velocity dispersions. The most important prediction of our model is that halo properties determine the bulge and SBH parameters. The model also has important implications for galactic morphology and the process of disk formation.

Subject headings: galaxies: evolution – galaxies: ISM – galaxies: kinematics & dynamics – galaxies: structure – hydrodynamics

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1. Introduction

The possibility that supermassive black holes (SBHs) inhabit the centers of many if not most galaxies, and the observed correlation between
SBH masses and galactic bulge properties has potentially a fundamental significance for our understanding of galaxy formation and evolution. The relationships between black hole and bulge properties include a loose relationship between SBH and bulge masses, $M_* \sim 0.001 M_B$, and an apparently much tighter one between the SBH mass and the velocity dispersion in the corresponding bulge, $M_* \sim \sigma_B^4$ (e.g., Ferrarese & Merritt 2000; Gebhardt et al. 2000; Tremaine et al. 2002; cf. reviews by Kormendy & Gebhardt 2001; Merritt & Ferrarese 2001).

In this paper we attempt to provide a physical explanation for these relationships between SBHs and their host galaxies. Our model is based on the interaction between the dark haloes of galaxies and the baryonic components settling in their midst. As baryonic matter accumulates to form the bulge and SBH, the orbital structure of the underlying gravitational potential is modified. This, in turn, affects the subsequent accumulation of gas, which is highly dissipative and therefore is sensitive to orbital geometry. Consequently, halo properties will determine those of the bulge and the SBH, and this will give rise to correlations between them that are compatible with the observed ones.

Cuspy, triaxial haloes appear to be a natural outcome of dissipationless collapse in a cold dark matter (CDM) dominated universe (e.g., Cole & Lacey 1996). Interactions with the baryonic component during the initial stages of collapse may affect the triaxiality, making it milder but still non-negligible (Dubinsky 1994). Before it even becomes dominant in the central regions, a clumpy baryonic component can also level off the central cusps of dark haloes, producing harmonic cores (El-Zant, Shlosman & Hoffman 2001). Even though this may affect the equidensity contours, again making them rounder, it need not symmetrize the equipotentials; these can remain asymmetric if triaxiality is not affected beyond some radius (for example, a homogeneous bar has a non-axisymmetric force contribution inside its density figure). From an observational standpoint it appears that haloes of fully formed galaxies tend to have nearly constant density cores (de Blok & Bosma 2002) and that residual potential axial ratios of about 0.9 in CDM haloes are plausible, even in present day galaxies (e.g., Kuijken & Tremaine 1994; Rix & Zaritsky 1995; Rix 1995).

The orbital structure of the inner regions of slowly rotating, non-axisymmetric potentials with harmonic cores is dominated by box orbits (e.g., Binney & Tremaine 1987), which have no particular sense of circulation. They are self-intersecting and, therefore, cannot be populated with gas. Dissipation causes material to sink quickly toward the only long-lived attractor available — the center (e.g., Pfenniger & Norman 1990; El-Zant 1999). In the process, interaction with the triaxial harmonic core causes the baryonic material to lose most of its angular momentum. The combined system would, therefore, also be slowly rotating. Thus, unless star formation terminates the collapse, the final concentration of the first baryonic material could be extremely large.

The onset of star formation is expected to occur when the baryonic material becomes self-gravitating — roughly speaking, when its density becomes larger than that of the halo core. This also happens to be the criterion for the destruction of the harmonic core and the emergence of loop (or tube) orbits, which do have a definite sense of circulation. The role of the SBH is to contribute to the emergence of these orbits in the very central region. It is the collusion between the SBH and the more extended hot baryonic component in creating the loop orbits that leads to the correlations claimed in this paper.

Whereas, in three dimensions, a box orbit can be represented as a superposition of three (generally) incommensurable radial oscillations along mutually perpendicular axes, which are efficiently attenuated by dissipation, loop orbits are best described in terms of modest radial and vertical excursions superposed on rotational motion about the center. While the vertical and radial excursions, like the oscillations characterizing box orbits, are attenuated by dissipation, the rotational motion is not efficiently dissipated among gas clouds populating such orbits (in the same sense of circulation).

The minimization of the radial and vertical oscillations results in closed periodic orbits that are confined to a plane. We expect the amount of gas dissipation on loop orbits to depend on their ax-
sial ratios — for, again, motion on highly eccentric orbits would lead to shocks and the accompanying loss of angular momentum on a short dynamical timescale, as observed in numerical simulations of gas flows in barred galaxies (e.g., Heller & Shlosman 1994). Lacking a detailed model for the dissipation rate associated with gaseous motion, we will assume that there is a critical eccentricity below which the loop orbits can serve as long-lived attractors for dissipative motion. In other words, gas populating these orbits will evolve only secularly and not dynamically. Fortunately, this dependence on critical eccentricity will turn out to be weak.

In the following we explore, within the above framework, the formation of galactic bulges and central compact objects. We show that it is possible to deduce a well-defined linear correlation between the masses of the central SBHs and corresponding bulges in given dark matter haloes, with mass ratios comparable to those observed. We demonstrate that, if dark matter halo cores follow a Faber-Jackson type relation (Faber & Jackson 1976) between their masses and velocity dispersions, then a similar relation also applies to the bulges. An $M_\bullet - \sigma_B$ relation between the SBH mass and the bulge velocity dispersion, which under certain assumptions can have smaller scatter than the $M_\bullet - M_B$ relation, also follows naturally. Within this framework, it is possible to make a number of testable predictions concerning the related structures of bulges, SBHs and their host haloes. We discuss these in the final section. Formal aspects of the perturbation analysis applied to bulge-halo systems have been deferred to Appendices A and B. Preliminary results of this work have been reported by Shlosman (2002).

2. The model

Since, at this stage, we are interested in generic dynamical phenomena related to orbital structure, the exact form of the halo potential is immaterial — as long as it exhibits a harmonic core where no loop orbits can exist. The exact distribution of the baryonic material is also not crucial, except, again, for its central density distribution. If this diverges, say as $\rho \propto r^{-1}$, loop orbits will be created all the way to the center, since the bulge-halo system no longer possesses a harmonic core. In this case there is no need to form the SBH. If, on the other hand, the (proto)-bulge has a harmonic core of its own, there will still be a nearly constant density region near the center with only box orbits — unless a central point mass is present. We will assume that ‘cuspy’ bulges, when they exist, are products of later evolution (for example, a result of star formation and cold dissipationless collapse: e.g., Lokas & Hoffman 2000).

We use a logarithmic standard form (e.g., Binney & Tremaine 1987) to represent the halo potential:

$$\Phi_H = \frac{1}{2} V_H^2 \log(\frac{R_H^2 + x^2 + \beta^{-2} y^2 + \gamma^{-2} z^2}{R_H^2}),$$  \hspace{1cm} (1)$$

where $V_H$ is the asymptotic (in the limit $R \gg R_H$ and $\beta, \gamma \to 1$) circular velocity, $R_H$ is the core radius and $\beta, \gamma < 1$ are the potential axis ratios. We will consider the process of bulge formation to be terminated, or at least substantially slowed down, when the combined (baryonic plus halo) potential admits sufficiently round non-intersecting loop orbits, which permit the long-term circulation of gas without excessive dissipation. Such motions necessarily take place in a symmetry plane determined by the (orbit-averaged) angular momentum (e.g., Frank, King & Raine 2002). Therefore, for our purposes, it will suffice to consider only orbits in the plane $z = 0$ and to ignore the vertical dimension. We fix $\beta$ at 0.9 (though the effects of its variation are discussed where relevant) and adopt a threshold axial ratio $p_{\text{crit}}$ for orbits that can be populated with gas. The value of $p_{\text{crit}}$ that best describes when this happens remains to be investigated, but its mere existence is what is important here. As we show below, our results are rather insensitive to the exact value of $p_{\text{crit}}$.

The halo core mass will be taken to be $M_H = V_H^2 R_H / G$ and we define the halo density to be $\rho_H = M_H / R_H^3 = V_H^2 / R_H G$. This is conveniently close to the value of the density in the region where the potential is effectively harmonic — that is, within the region where, in the absence of the bulge component, no loop orbits exist for $\beta = 0.9$. We note, however, that a bulge with a larger core radius probes a region of the halo core with smaller average density than the region probed by a bulge with a relatively small core radius. This effect will be discussed in Section 4.

Since our aforementioned criterion depends on
the potential in a chosen symmetry plane, the exact three-dimensional mass distribution of the bulge is unimportant, as the same planar potential may arise from a variety of these. A particularly convenient form for the potential in this situation is that of Miyamoto & Nagai (1975),

$$\Phi = -\frac{GM_D}{\sqrt{x^2 + y^2 + (A + \sqrt{B^2 + z^2})^2}}. \tag{2}$$

This potential can approximate a disk-bulge system, with the parameters $A$ and $B$ determining the scale length and height, respectively. In the symmetry plane, therefore, the potential can arise from a range of density distributions — from highly flattened to spherical ones. In general, when we refer to the “bulge” we will have a spherical system in mind with $A = 0$. In particular, all densities quoted are calculated under this assumption. We will comment on the effects of assuming a flattened distribution where relevant. In that case $M_D$ in eq. (2) is replaced by $M_B$, the spherical bulge mass. The bulge core is defined by the radius $R_B = A + B$ and encloses the core mass $M_{BC} = 2^{-3/2}M_B$. The bulge core density is defined as $\rho_B = M_{BC}/R_B^3$.

The scaling relations that arise do not depend on the absolute masses and lengths, but instead, on the ratios of these quantities among the various contributions to the potential. An important property of the dynamics in our model is that any transformation that leaves the relative masses and lengths of the components constant will also not affect the orbital structure once scale transformations are taken into account. Such transformations only change the time units, hence the dynamics remains invariant on a new characteristic timescale. In addition, in regions where the baryonic and halo components have the same mass distributions, i.e., inside their cores, only the density ratios of these components determine their relative contributions to the potential, and hence the orbital structure.

For the remainder of this paper, we will describe all components of the potential in terms of units scaled to the dimensions of the halo potential, eq. (1). We scale all distances to $R_H$, with $\tilde{R}_B = R_B/R_H$, and all masses to $M_H$ so that $\tilde{M}_B = M_B/M_H$. The normalized bulge core density is then $\tilde{\rho}_B = 2^{-3/2}\tilde{M}_B/\tilde{R}_B^3$. The bulge potential in scaled units is

$$\Phi_B = -V^2_H \frac{2^{3/2}\tilde{\rho}_B \tilde{R}_B^3}{\sqrt{x^2 + y^2 + \tilde{R}_B^3}}. \tag{3}$$

The black hole potential, in the same units, is

$$\Phi_\bullet = -V^2_H \frac{2^{3/2}K\tilde{\rho}_B \tilde{R}_B^3}{\sqrt{x^2 + y^2}}, \tag{4}$$

where $K \equiv M_\bullet/M_B$. Note that all orbits in the total potential $\Phi = \Phi_H + \Phi_B + \Phi_\bullet$ depend only on three parameters, which may be taken to be $\tilde{R}_B$, $\tilde{\rho}_B$, and $K$.

3. Bulge formation in triaxial haloes and the $M_\bullet - M_B$ correlation

In this section, we demonstrate that the arguments outlined above imply a linear relation between the SBH and bulge masses, provided that the baryonic and CDM density distributions are not completely axisymmetric and exhibit, at some stage, nearly constant density cores.

3.1. Critical eccentricities and the emergence of SBH-bulge correlations

3.1.1. The inner minimum and the role of bulge density

Fig. 1 (the result of orbital integrations) exhibits the axis ratios of the closed loop orbits $p = x_{\text{max}}/y_{\text{max}}$ as a function of the longer axis length ($y_{\text{max}}$). In this figure, we have fixed the scaled bulge core density, $\tilde{\rho}_B = 5.52$, and the SBH-to-bulge mass ratio, $K = M_\bullet/M_B = 10^{-3}$, but vary the scaled bulge core radius, $\tilde{R}_B$. Consequently, the bulge masses $\tilde{M}_B$ range over a factor of 100. As expected, when only the halo contributes to the potential (dotted line), $p$ quickly drops to zero inside the halo core, since no loop orbits can exist there. (For halo potential axis ratios smaller than the adopted value of 0.9, i.e., greater triaxiality, $p$ reaches 0 at radii that are progressively closer to the halo core radius, $y_{\text{max}} \rightarrow 1$. As the baryonic component, both in the bulge and the SBH, becomes progressively more massive, correspondingly rounder loop orbits appear. Loop orbits in the very central region are produced by the presence of the SBH. The latter is represented numerically by a Plummer sphere with a softening
scale of 0.002, in scaled units. Near the center, where the SBH contribution is dominant, the axis ratios of the loop orbits tend to $p \to 1$.

![Figure 1](image_url)

Fig. 1—Axis ratios of closed loop orbits $p = x_{\text{max}}/y_{\text{max}}$ as a function of $y_{\text{max}}$. The dotted line corresponds to a system with only a halo contribution (i.e., no bulge and no SBH), the solid line refers to a system with a "bulge" in the form of a Plummer sphere with $\hat{M}_B = 1$ and $\hat{R}_B = 0.4$, the dashed line represents a bulge with $\hat{M}_B = 0.1$ and $\hat{R}_B = 0.19$, while the dashed-dotted line represents a bulge with $\hat{M}_B = 0.01$ and a radius of $\hat{R}_B = 0.086$. In all cases with the bulge present, $\hat{\rho}_B = 5.52$ and an SBH with a fraction $K = 10^{-3}$ of the bulge mass is present.

The axis ratio curves exhibit two distinct minima. It is apparent that the inner minima of the curves in Fig. 1 have (nearly) the same values of $p$, but occur at different radii $\propto R_B$. This is a consequence of assuming a fixed ratio of SBH-to-bulge mass, $K$, and a constant bulge core density $\hat{\rho}_B > 1$. Because the bulge and halo densities are nearly uniform in the inner regions, their relative contributions to the potential are determined by their density ratio.

In the absence of the SBH contribution, $p$ in Fig. 1 would tend to zero within the effective harmonic core of the bulge-halo system (see eq. [B8]). When the relative density is kept constant (and $> 1$) this is proportional to $\hat{R}_B \sim \hat{M}_B^{1/3}$ (the effect of density variation will be examined in Section 3.2). The reason for this is that inside the nearly constant-density bulge core, the potential, which is now a superposition of two nearly harmonic potentials, is nearly harmonic (even if the new core is less triaxial than the halo alone, e.g., if the bulge is assumed to be spherical). The inner minima of the curves in Fig. 1, therefore, correspond to a transition from the region where the SBH provides the dominant contribution toward the creation of loop orbits to that where the bulge provides this contribution. Therefore, the minima occur at radii where the gravitational acceleration due to the SBH is proportional to that due to the bulge+halo, $R_\bullet \sim (M_\bullet/M_B)^{1/3}R_B = K^{1/3}R_B$ (since the minimum is located well within the bulge core radius). Since $K$ is taken to be constant, we expect the minima to occur at radii $\propto R_B$.

In Section 2, we have defined the critical value $p_{\text{crit}}$ above which gas circulation can be sustained for secular timescales. Here we have shown that given a critical value, $p_{\text{crit}}$, for the inner minimum and the relative bulge-to-halo core density ratio, a value of $K$ associated with this minimum is determined. However, satisfying the condition $p > p_{\text{crit}}$ at the inner minimum does not imply that the same condition is satisfied at all radii within $R_H$. Until it is satisfied at the outer minimum, as well, further evolution can occur. This is briefly sketched below and outlined in more detail in Section 3.4.

3.1.2. Outer minimum, minimal bulge mass, and final black hole mass

Until the critical value of $p$ is reached at all radii inside the halo core, the bulge will continue to grow, since outside the bulge core $p$ declines again. It is clear from Fig. 1 that, in contrast to that of the inner minimum, the value of $p$ at the outer minimum depends more sensitively on the bulge mass than on its density. The growth of the bulge, therefore, determines a minimal bulge mass fixed by the condition that $p \geq p_{\text{crit}}$ at all radii within the halo core.

If the SBH did not continue to grow along with the bulge, this growth in the bulge mass would decrease the linear correlation coefficient $K$, but, as we show in Section 3.4, this leads only to near-independence of the final value of $K$ on $p_{\text{crit}}$ and still keeps the values of $K$ within the observed range.
3.2. Scaling relationships determined by inner minima

As shown in the previous section, the creation of loop orbits with axis ratio above a given value depends, in the central region, solely on the ratio of the bulge-to-halo core density and the mass ratio \( K = M^* / M_B \). The actual value of the bulge mass determines only the position of the minimum (as a fraction of the halo core radius), not the axis ratio at the minimum. But even the radius of the minimum is largely insensitive to the bulge mass, i.e., \( \propto M_B^{1/3} \), provided that the minimum actually exists and the bulge core density relative to the halo harmonic core density remains constant. We will now examine the bounds on \( M^* \) and \( K \) which ensure the existence of a minimum \( p \geq p_{\text{crit}} \) inside the bulge core and observe the effect of varying the bulge density.

We calculate and plot \( K = K(\hat{\rho}_B) \) for different choices of \( p_{\text{crit}} \) (Fig. 2), in the following way. First, without any loss of generality, we choose a bulge mass \( M_B \) that is large enough that the outer minimum satisfies \( p \geq p_{\text{crit}} \), for all \( p_{\text{crit}} \leq 0.9 \). This is done in order to focus on the effects of the inner minima only. (Recall, from the previous section, that the value of the outer minimum depends on the mass of the bulge.) Fig. 2, therefore, exhibits the effect of \( \hat{\rho}_B \) and \( K \) on the inner minima of the \( p \) curves at \( \hat{R} < \hat{R}_B \). In order to obtain \( K = K(\hat{\rho}_B) \), we vary \( M^* \), which is some fraction \( K \) of the chosen bulge mass. The density is varied by contracting the bulge (by decreasing its core radius) until the condition \( p \geq p_{\text{crit}} \) is satisfied everywhere inside the bulge core.

One observes in Fig. 2 that at smaller densities \( K \) tends toward an asymptotic (and maximal) value associated with a given \( p_{\text{crit}} \). We refer to this hereafter as the “asymptotic” regime. In the transition to this regime, along a \( p_{\text{crit}} = \text{const.} \) curve, the value of \( K \) is monotonically increasing. In other words, the bulge contribution toward the creation of loop orbits of a given eccentricity gradually diminishes, and is compensated by a greater contribution from the SBH component. In the process, the minimum in \( p \) moves outward. The asymptotic regime corresponds to a situation in which the existence of loop orbits with the required elongation depends on the value of \( K \), irrespective of the bulge density (that is, effectively, only on the SBH mass). In this limit the SBH contribution to the potential is sufficient to create loop orbits with \( p \geq p_{\text{crit}} \) at all radii within the halo core (i.e., at \( \hat{R} \leq 1 \)), without additional contributions from the bulge component (cf. Fig. 3). Essentially, it corresponds to a rapid collapse to the center and formation of the SBH by a large fraction of the baryonic material, bypassing the formation of a bulge. This regime appears to be of academic interest only, as it implies \( \hat{\rho}_B < 1 \) all the way to the center; it will not be discussed further.

A second regime characterizes the dynamical state of the SBH-bulge-dark halo system with \( \hat{\rho}_B \gtrsim 1 \). In this “scaling regime” the \( \log K - \log \hat{\rho}_B \) curves are parallel straight lines with av-
erage slopes of about \(-2.5\). This can be explained in the following manner. From eq. (B8) we know that the radius of the effective harmonic core of the bulge-halo system, \(y_{\text{max}}\), is proportional to \(\tilde{R}_B^3 \tilde{M}_B^{-1/2}\). In the scaling regime, the location of the minimum in \(p\) will be proportional to \(y_{\text{max}}\). Now, in order to maintain the minimum at a specified value of \(p_{\text{crit}}\), the gravitational acceleration due to the SBH must be proportional to the acceleration due to the halo (which contains the nonaxisymmetry) at \(y_{\text{max}}\). This implies \(\tilde{\rho}_B \tilde{R}_B^3 K / y_{\text{max}}^2 \propto y_{\text{max}}\). Substituting for \(y_{\text{max}}\) and noting that \(\tilde{\rho}_B \propto M_B / \tilde{R}_B^3\), we obtain \(K \propto \tilde{\rho}_B^{-5/2}\). This is approximately what is found from Figure 2 and it also holds if we change the mass of the bulge keeping the radius constant and again using eq. (B8). Note, furthermore, that for constant \(\tilde{\rho}_B\) this relation predicts the radius of the effective harmonic core (and the location of the minimum) to be proportional to \(\tilde{R}_B\), as expected from the heuristic considerations of the previous section.

While the asymptotic values of \(K\) depend on \(M_B\), the values of \(K\) in scaling regime do not. The transition between these two regimes can, therefore, be characterized by a sharp change in the behavior of \(K\), as seen from Fig. 2.

We are mainly interested in the scaling region, because the onset of star formation can be tied to the baryonic component becoming self-gravitating, which corresponds roughly to the bulge core density exceeding that of the background halo, i.e., \(\tilde{\rho}_B \gtrsim 1\). In this case, and for \(p_{\text{crit}} \lesssim 0.8\), the SBH contributes significantly to the potential only at radii \(\tilde{R} \ll 1\). The \(p\) curves, therefore, exhibit a definite inner minimum well inside the halo core.

The values of \(\tilde{\rho}_B\) and the initial \(K\), in principle, also depend on the critical eccentricity, \(p_{\text{crit}}\), of the inner minimum, which is expected to be independent of the bulge mass, but which does depend on complex gas dynamics. If, after the SBH forms, the bulge mass falls short of the value required for creating sufficiently round closed loop orbits at all radii inside the halo core, it will continue to grow until the outer minimum of the axis ratio curve also attains \(p \gtrsim p_{\text{crit}}\). In Section 3.4, we show that the SBH growth, if continued, becomes intermittent. In the next section, we also demonstrate that one can place constraints on the possible range of \(K\) values by considering this constraint on the outer minimum.

### 3.3. Minimal bulge mass and bulge growth

To obtain loop orbits rounder than a given \(p_{\text{crit}}\) at all radii \(\tilde{R} < 1\), and not only in the central region, one in fact needs to take into account the bulge-to-halo core mass ratio and not only the density ratio. This can already be seen in Fig. 1, where all curves with SBH and bulge contributions exhibit an inner minimum with \(p \gtrsim 0.9\), while \(p\) declines significantly as one moves further out. For low bulge masses, an additional outer minimum emerges, within \(\tilde{R}_B < \tilde{R} < 1\), before \(p\) rises again outside the halo harmonic core. Only one of the \(p\) curves, corresponding to the most massive bulge, has \(p > 0.9\) at all radii. Therefore, for a given halo, there is a minimal bulge mass, \(M_{B\text{min}}\), that is necessary to create sufficiently round loop orbits at all radii \(\tilde{R} > \tilde{R}_B\).

If we assume that the bulge core density varies at most by a factor of a few (\(\tilde{\rho}_B = \mathcal{O}(1)\)), \(\tilde{R}_B\), which is only weakly dependent on the bulge mass and density (\(\tilde{R}_B \sim \tilde{M}_B^{1/3} \tilde{\rho}_B^{-1/3}\)), varies little. The minimal bulge mass is also nearly constant (calculations show, for example, that it varies by \(\sim 30\%\) when \(\tilde{\rho}_B\) changes from 1 to 2.5). There is more sensitivity to the assumed \(p_{\text{crit}}\). In Fig. 4 (asterisks) we show the bulge mass required to create...
loop orbits with $p \geq p_{\text{crit}}$ at all radii outside $R_B$, for $\hat{\rho}_B = 1$. These are the masses necessary, at this density, to produce outer minima with the required values. For bulge densities $\hat{\rho}_B \sim 1$ these minima lie at radii $R \gtrsim 0.5$. Thus, unless the SBH can contribute significantly to the potential at radii comparable to $R_H$, which seems implausible, the dominant contributions to the potential at the radii examined here should be only those due to the bulge and halo components. Therefore, once the halo parameters are fixed, the creation of loop orbits with given $p \geq p_{\text{crit}}$ outside the bulge core will depend only on the bulge parameters $R_B$ and $M_B$.

For $\hat{\rho}_B = 1$, a well-defined inner minimum in the $p$ curve exists inside the bulge core only if $p_{\text{crit}} \leq 0.8$. For larger $p_{\text{crit}}$, the inner minimum moves outward and merges with the outer one. Systems with these properties lie outside the scaling region in Fig. 2. In this case, for the condition $p \geq p_{\text{crit}}$ to be satisfied, the SBH has to contribute significantly at all radii — no matter how massive the bulge is. (For $\hat{\rho}_B = 1$, the curve with $p_{\text{crit}} = 0.9$ in Fig. 2 lies in this regime.) Furthermore, the minimal bulge density required to create loop orbits with $p \geq p_{\text{crit}}$, without appeal to an overmassive SBH, increases rapidly when $p_{\text{crit}} \gtrsim 0.8$ (Fig. 5), suggesting that the bulk of the material may form stars before the condition $p \geq p_{\text{crit}}$ is reached everywhere within the halo core. If we demand that $\hat{\rho}_B \leq 1$, then $p_{\text{crit}}$ must be smaller than 0.8 in order for the model to be plausible, implying that $K$ must lie in the range $10^{-4} - 10^{-2}$.

On the other hand, star formation can be somewhat delayed to higher densities. A reasonable range lies within $\hat{\rho}_B \approx 1 - 10$. Still higher densities in the bulge can be excluded based on the observed rotation curves. In this range of $\hat{\rho}_B$, values of $p_{\text{crit}} > 0.8$ become feasible without invoking unrealistically massive SBHs, as seen from Fig. 5. One may also reasonably assume that, due to enhanced dissipation, higher densities require larger $p_{\text{crit}}$ to maintain long-lived gaseous motion (detailed modeling, however, will be necessary to determine exactly how $p_{\text{crit}}$ depends on the system parameters). In this situation, the range in $K$ that can be inferred from Fig. 2, is again about $10^{-4} - 10^{-2}$ and compatible with the observed value of $\sim 10^{-3}$, which exhibits a significant scatter.
The values of $K$ obtained so far, which should be regarded as initial values, are fixed by the inner minimum of the axis ratio curves. As argued in Section 3.1, further evolution of the bulge will occur unless sufficiently round loop orbits also exist at the outer minimum. In the next section we examine how this evolution can modify the range of $K$, if at all.

3.4. Black hole growth and SBH-to-bulge mass ratio

There are basic differences between the conditions for satisfying $p \geq p_{\text{crit}}$ at the inner and outer minima in Fig. 1. As discussed in Section 3.1, the initial growth of the SBH is terminated after the inner loop orbits reach the critical eccentricity. The growth of the bulge, on the other hand, continues until the required $p_{\text{crit}}$ is reached at all radii inside the halo core. Since the increase of bulge mass, at roughly constant density, moves the inner minimum outward as described in Section 3.1, the SBH would have to continue growing in pace with the bulge mass if it is to maintain the critical value of $K$ (which is determined only by the values of $\hat{\rho}_B$ and $p_{\text{crit}}$). Does this actually happen, and if so, how?

To see how the parallel growth of the SBH and bulge could come about, consider the solid curve in Fig. 6, which represents the state of the SBH/bulge system at the end of the initial stage of infall. This is equivalent to one of the curves (say, the dash-dotted line) in Fig. 1, except that we have chosen $p_{\text{crit}} \approx 0.6$ for the inner minimum, instead of 0.9. This line has two maxima: a left-hand one at the origin, determined by the SBH, and a right-hand one, determined by the bulge core. Note that the outer part of the solid curve drops below $p_{\text{crit}}$, implying that the bulge will continue to grow. Now suppose this growth occurs without corresponding growth of the SBH (in contrast to the case in Fig. 1, where $K$ is kept constant). If the bulge continues to grow while the SBH growth is stopped, the right-hand maximum moves further to the right, while the left-hand maximum stays the same. This opens a widening ‘gap’ at the position of the inner minimum which drops below $p_{\text{crit}}$, allowing gas to flow again toward the radius of influence of the SBH, $R_\bullet$. This is illustrated by the dashed curve in Fig. 6, which corresponds to a factor of 10 increase in $M_B$ accompanied by a similar drop in $K$. The gas is expected to accumulate in this vicinity. However, once a substantial amount of gas has collected at $R_\bullet$, it becomes self-gravitating and is prone to global self-gravitating instabilities. The fastest of these instabilities, $m = 2$ modes or bar instability (e.g., Bardeen 1975), have been discussed in a similar context by Shlosman, Frank & Begelman (1989). They induce rapid (dynamical) gas inflow. Hence we expect that the growth of the SBH at this stage will be intermittent, but because the time-averaged conditions for infall at the inner and outer maximum do not change, we expect the value of $K$ to stay roughly constant within the range given in Fig. 2, namely $\sim 10^{-3} - 10^{-2}$, depending on the value of $p_{\text{crit}}$.

Is there always enough gas to accumulate at $\sim R_\bullet$ in order to trigger a bar instability, due to the opening of the gap between the SBH and the bulge? One can imagine the opposite extreme to that discussed above, in which insufficient gas enters the widening gap from outside, perhaps because star formation is efficient during the early
stages of infall. Let us suppose, for the sake of argument, that the SBH does not grow beyond its initial mass, as determined by the initial value of $K$. How does this affect the range of final $K$ values?

Figure 4 shows minimal bulge masses, $M_{\text{Bmin}} \sim 10^{-1} - 10^{-3}$, for a range of $p_{\text{crit}}$ values. While we have no estimate for the initial bulge masses which define the initial $K$ in Fig. 2, it is possible to rule out very small masses, e.g., $< 10^{-3}$, because the total baryonic mass within the halo core is taken to be about 10%. This means that for lower $p_{\text{crit}} \approx 0.3 - 0.4$, the initial bulge mass is equal to or larger than $M_{\text{Bmin}}$ estimated in Section 3.3, and, therefore, will not grow beyond its initial value. It is easy to understand this result, because one needs small baryonic masses to create loop orbits with such large eccentricities. This means that in this regime the initial value of $K \sim 10^{-4}$ (in Figures 2 and 4) is also its final value.

Alternatively, in the regime of larger $p_{\text{crit}} \approx 0.7 - 0.8$, the bulge can grow at most by a factor of $\sim 30 - 100$, reducing the initial $K$ by this amount. Luckily, the initial values of $K$ for high $p_{\text{crit}}$ lie around their high end, $\sim 10^{-2}$. Reduction by up to two orders of magnitude brings them again to about $10^{-4}$. Hence, whether the SBH grows after the initial stage or not does not destroy the $M_{\bullet} - M_{\text{B}}$ correlation and does not move the values of $K$ outside the observed range. A corollary of this discussion is that the final value of $K$ appears insensitive to the value of $p_{\text{crit}}$, if the growth of the SBH is supressed, and depends strongly on $p_{\text{crit}}$, if the SBH grows in tandem with the bulge. Note also that the variation of $M_{\text{Bmin}}(p_{\text{crit}})$ (represented by the asterisks in Fig. 4), follows closely that of $K(p_{\text{crit}})$ (represented by filled circles). The ratio of these two quantities, $M_{\bullet}M_{\text{B}}/M_{\text{Bmin}}^2 \sim 0.03$ is, therefore, largely independent of $p_{\text{crit}}$.

In other words, the minimal bulge mass is predicted to be about 5 times the geometric mean between the SBH and the halo masses, provided that the SBH grows along with the bulge.

To summarize, the initial value of the black hole-to-bulge mass ratio $K$ is fixed by the density at which the gaseous material in the inner region becomes self-gravitating and forms stars, and by the value of $p_{\text{crit}}$. However, unless loop orbits of sufficient eccentricity are created in the whole harmonic halo core region, the mass of the bulge will continue to grow. At the same time, the dynamical infall onto the SBH can choke, if the remaining gas mass is insufficient to cause bar instability and channel it to the SBH. In the latter case $K$ will decline below its initial value as the bulge grows, but will tend to approach a final value roughly independent of $p_{\text{crit}}$. In the opposite limit, the SBH will grow in proportion to the bulge, and $K$ will stay constant (as shown by Figs. 2 and 4). In either limit, we assume that the process stops when the mass of the baryonic component reaches the minimal mass required to achieve loop orbits with a given $p \geq p_{\text{crit}}$, at all radii.

For less triaxial haloes, and for a given bulge core density, lower SBH masses are required to produce loop orbits below a critical eccentricity. The minimal mass of the bulge needed to create the required loop orbits in the outer region, however, also decreases — especially since, for a mildly triaxial halo, the radius at which no loop orbits exist inside the harmonic core decreases with decreasing triaxiality. Thus, in the regime of mild halo triaxiality, the ratio of $K$ to (normalized) minimal bulge mass should not depend sensitively on the value of $\beta$. We do ignore the fact that triaxiality can be a function of radius, and choose a fixed value of $\beta$ for simplicity.

### 3.5. Constraints on morphology and halo properties

Once orbits of sufficiently large $p$ are present, dissipation will reduce radial motion relative to these orbits, as well as vertical motion away from the symmetry plane defined by the angular momentum vector. Infalling gas will start to populate the newly-formed, round, non-intersecting periodic loops, leading to the formation of a disk component inside the halo core. At this stage the bulge formation stops. Outside the halo core, gas can accumulate at any stage of the formation process on closed loop orbits, which always exist in strongly inhomogeneous density distributions. Provided that the halo triaxiality is small, these orbits will be nearly circular (for potential axis ratio $\beta$ mildly deviating from unity $p \sim \beta$: see, e.g., Rix 1995).

If the halo core radius is exceedingly large, however, it is possible that no significant disk component will form at all. For if the core is large relative to the total halo (virial) radius, the contracting
gaseous component will end up in the core, instead of spinning up and forming an extended disk. Losing most of its angular momentum to the halo, it will eventually end up as part of the bulge component. This would lead to the formation of an elliptical rather than a disk-dominated galaxy. In this context, one expects that haloes with larger core radii are host to larger spheroidal components. Since it appears that, in general, more massive galaxies are usually of earlier type (e.g., Persic, Salucci & Stel 1996), one could deduce that more massive cores have larger core radii. This would be expected if these cores followed a Faber-Jackson type relation (for example, if $M_H \sim \sigma_H^4$, then $M_H \sim R_H^2$) as tentatively suggested by observations (Burkert 1995; Salucci & Burkert 2000; Dalcanton & Hogan 2001) and deduced if halo cores formed via the destruction of the inner ($\rho \sim r^{-1}$) regions of NFW haloes (Navarro, Frenk & White 1997). In this case, lower mass haloes form when the Universe is denser and are, therefore, more concentrated. As a consequence, the region where $\rho \sim r^{-1}$ is smaller relative to the virial radius for low mass haloes (El-Zant, Shlosman & Hoffman 2001). The existence of a Faber-Jackson type relation for halo cores is also required in order to reproduce the $M_\bullet - \sigma_B$ relation as discussed in the next section.

Within the above framework, the formation of the SBH is intimately tied to the bulge component, whereas the formation of the disk component takes place after the processes leading to bulge and SBH formation are essentially complete. The bulk of the disk is also expected to form at scales larger than the halo harmonic core (but see Section 5). Thus, the deduced correlation involving the bulge and SBH does not simply generalize to one involving the disk as well, in accordance with observations (e.g., Kormendy & Gebhardt 2001).

4. The $M_\bullet - \sigma_B$ relation

The fact that all the orbital properties discussed in this paper depend only on the relative magnitudes of the masses and spatial scales of the galactic components involved has important consequences. For the $M_\bullet - M_B$ relationship it has the obvious implication that the derived correlation will hold for all halo masses. A more powerful prediction transpires in relation to the $M_\bullet - \sigma_B$ relation. For this to hold, it is necessary that the masses and velocity dispersions of halo cores are related in a similar manner.

Suppose that for a given halo there exist unique values for the SBH and bulge masses, and for the bulge scalelength. If a Faber-Jackson type relation between the halo core mass and velocity dispersion exists, a corresponding relation will exist between the bulge parameters. It also follows that a similar relation will exist between the SBH mass and the bulge (and halo) velocity dispersion. This can be deduced by simple scaling transformations, because the orbital properties we are interested in are all invariant with respect to spatial scale and mass transformations. This means that if we multiply the masses of the SBH, bulge and halo by some constant $\alpha_M$, thus effectively changing the mass units, the curves in Fig. 1 will remain invariant. In the same manner, if we multiply all length-scales by some factor $\alpha_R$, so that $R_H \rightarrow \alpha_R R_H$ and $R_B \rightarrow \alpha_R R_B$, thus effectively changing the length unit, all curves in Fig. 1 remain the same. This implies that if, for example, $M_H \propto R_H^2$, then $M_\bullet \propto M_B \propto R_H^2 \propto R_H^2$ are equivalent systems in the sense described above (they have the same axis ratio curves for their loop orbits with the same $p$ values as a function of rescaled radius). These systems will all follow the $M_H - \sigma_H$ relation for the halo. In this particular case, $\sigma_H \propto M_H^{1/4}$ implies a similar relationship between $\sigma_B$ and $M_B$, as well as $M_\bullet$.

It is of course possible that, for a given halo, the masses of the SBH and bulge, as well as the bulge lengthscale, are not unique. In other words, the subset of haloes with a given $M_H$ and $R_H$ may contain bulges and SBHs with a distribution of properties. In Section 3.2 we had assumed that the bulge collapse is largely terminated when its density is of the order of the halo core density. This in turn fixes $K$, once $p_{\text{crit}}$ is determined. In this section we further assume that the value of $K$ is not changed as a result of any subsequent bulge growth. That is, the SBH grows in tandem with the bulge (cf. Section 3.4). This relaxes the assumption that the bulge masses are determined by the minimal mass required to create closed loops with $p \geq p_{\text{crit}}$ at all radii, allowing for more massive bulges.

One would like to infer to what extent variations in the bulge and SBH properties, within a
given halo, affect the homology relations discussed above. We now show that when one accounts for variations of the bulge and the SBH parameters, the departure from the aforementioned relation is not dramatic. This comes about basically because the mass and core radius of the bulge are correlated, under the assumptions of our model. First, we note that the average velocity dispersion of the core of the baryonic component can be written as \( \sigma_B^2 = \alpha GM_\text{B}/R_B \), and the average density \( \rho_B = 2^{-3/2}M_\text{B}/R_B^3 \). In this case \( \sigma_B \propto \rho_B^{1/6}M_\text{B}^{1/3} \).

Here \( \alpha \) depends on the functional form of the density distribution. Its exact value is unimportant if all bulges are assumed to have the same functional form for their density distributions. Henceforth we set \( \alpha = 1 \).

For a constant bulge-to-halo density ratio \( \rho_B \) a relationship between bulge velocity dispersion and mass \( M_\text{B} \propto \sigma_B^3 \) results. In a toy model where the halo density does not vary at all with radius a constant bulge density determines a unique \( K \), given \( p_{\text{crit}} \), and an \( M - \sigma \) relation between bulge properties within a given halo arises, with index equal to 3. Considering more realistic models for the density distribution raises the index somewhat. When the core radius of the baryonic component is not very small compared to that of the halo, the density of the halo will not be strictly constant in the region of interest. As a result, the slope of the \( M - \sigma \) relationship will further increase.\(^3\) This occurs because larger bulges “see” a smaller mean halo density, and therefore smaller bulge densities suffice to produce the same relative contribution to the potential. Since \( M_\text{B} \propto \sigma_B^3 \rho_B^{-1/2} \), an inverse correlation between density and \( \sigma \) steepens the \( M - \sigma \) relation. (A marginal effect is already seen in Fig. 1 where the inner maxima produced by bulges with progressively larger \( R_B \) also have progressively larger inner minima of \( \rho \).)

If the baryonic component giving rise to the gravitational potential is significantly flattened there will be an increase in the \( M_\text{c} - \sigma \) slope to 4. In this case, the surface, rather than the volume density, will determine the gravitational field, resulting in a situation where \( M_\text{B} \propto \sigma_B^4 \).

\(^3\)If \( \rho_B \propto R_B^{-\alpha} \), where \( \alpha \geq 0 \), the slope of \( M_\text{c} - \sigma \) is given by \( 2[(3-\alpha)/(2-\alpha)] \) which stays between 3–5 for \( \alpha = 0 \)–1.5, and then rises rapidly. The resulting slope of \( M_\text{c} - \sigma \) is some weighted average of \( \rho_B \), and greater than 3.

The fact that the index of the relation between \( M_\text{B} \) and \( \sigma_\text{B} \) at constant bulge density is close to that of the Faber-Jackson relationship implies that, even if there is considerable variation in the bulge mass for a given halo, the departure from a Faber-Jackson relation defined by the halo parameters (as described above) is not too large. This is illustrated in Fig. 7, where we plot the relationship between the average velocity dispersion (simply defined as \( \sigma_B^2 = GM_\text{B}/R_\text{B} \)) of the baryonic component and its mass for systems having parameters \( K \) and \( p_{\text{crit}} \) corresponding to those in Fig. 2. The plots are obtained by keeping \( K \) and \( p_{\text{crit}} \) constant and, for a given bulge mass, decreasing its characteristic radius until \( p \geq p_{\text{crit}} \) at all radii inside the halo core. To obtain physical parameters we set \( V_H = 200 \text{ km s}^{-1} \) and \( R_H = 5 \text{ kpc} \).

The solid line in Fig. 7 is actually a superpo-
sition of several lines, corresponding to different values of $p_{\text{crit}}$ (and $K$), as used in Fig. 2. Its average slope is around 3.2 and increases slowly toward higher dispersion velocities. These lines, however, have different end-points, determined by the minimal bulge masses associated with the different $p_{\text{crit}}$ (cf. Section 3.3) which are denoted by circles in Fig 7. The maximum values of the bulge masses in these curves correspond to 2.5 times the halo core mass. It seems unlikely that bulges would be more massive than this. Their characteristic densities or radii would have to be much larger than those of the halo, as would their contribution to the potential. In systems with disks such extreme conditions would violate the shapes of observed rotation curves.

In reality, for a given halo, there is a range of possible bulge densities and associated values of the $K$ parameter. This, in principle, affects the normalization of the $M_B - \sigma_B$ relation. Nevertheless, the normalization turns out to be weakly dependent on $\rho_B$ ($\sim \sigma_B^2 \rho_B^{-1/2}$). Thus, variations in bulge densities within haloes of given mass and size do not significantly affect the relationship between bulge mass and velocity dispersion. In our model, the central (bulge) density required for the production of loop orbits with $p \geq p_{\text{crit}}$ in the scaling regime is also only weakly dependent on $K$ (as $K^{2/5}$, cf. Fig. 2). The above leads to an important corollary, that large variation in $K$ will cause only small changes in $M_B - \sigma_B$, as illustrated in Fig. 8. This relation, therefore, appears to be robust and is not heavily affected by changes in bulge and SBH parameters. Note also that, for less dense bulges the slope of the lines in Fig. 8 tends to $\sim 4$, especially in the limit of large values of the velocity dispersion (which is also where most of the observations lie). This is due to the the effect described above — massive bulges with smaller densities are more extended, and, therefore probe larger regions of the halo core.

To obtain an $M_* - \sigma_B$ relation, one has to multiply the values of the bulge masses in Fig. 7 by the appropriate $K$ factors, arriving at an $M_* - \sigma_B$ relation within a given halo. This introduces “scatter” by shifting the $K = \text{const.}$ lines, as can be seen from Fig. 9. For any fixed value of $\sigma_B$, this figure reveals that the (vertical) scatter in $M_*$ is similar to that in $K$.

This leads us to an important point. In the present formulation, the $M_B - \sigma_B$ relation is tighter than the $M_* - \sigma_B$ one. This appears to contradict observations suggesting that $M_* - \sigma_B$ is much tighter than the Faber-Jackson relation. However, this result has been obtained under the assumption that there is no scatter in the halo Faber-Jackson relation. A significant scatter in $M_H - \sigma_B$ would result in a corresponding scatter in $M_B - \sigma_B$, which follows from the homology scaling discussed in the beginning of this section. Moreover, if $p_{\text{crit}}$ correlates with $\sigma_B$ in such a way that gaseous systems embedded in haloes with...
higher velocity dispersion require larger values of $p_{\text{crit}}$ to remain stable, the scatter in the $M_\bullet - \sigma_B$ relation due to variations in $K$ can be reduced. For example, in Fig. 9, we have assumed a single halo for all the lines with different $p_{\text{crit}}$. If, however, lines with higher values of $p_{\text{crit}}$ are associated with haloes with larger $\sigma_H$ (and thus $\sigma_B$), the constant $p_{\text{crit}}$ lines would be shifted in such a way that the vertical distances between them at a given value of $\sigma_B$ decrease. We illustrate this effect in Fig. 10, where a linear relationship is assumed between $\sigma_B$ and $p_{\text{crit}}$.

We suggest that the relative tightness of $M_\bullet - \sigma_B$ could result from a loose relation between $\sigma_B$ and $p_{\text{crit}}$, coupled with scatter in $M_H - \sigma_H$ comparable to that of the Faber-Jackson relation for bulges. Such $p_{\text{crit}} - \sigma_B$ relation can result from a general trend that dissipation is increasing in more concentrated systems. Testing this will require detailed modeling of the way in which the dissipation rate depends on $p_{\text{crit}}$, velocity and density, and is outside the scope of this paper. Based on the illustrative example of Fig. 10, we only claim here a plausibility of a loose relationship between $p_{\text{crit}}$ and the system velocity dispersion. If the postulated correlation persists for haloes of different masses, a steepening of the $M_\bullet - \sigma_B$ relation relative to the halo (and bulge) mass velocity dispersion relation is expected, implying that the former should have a larger index.

5. Discussion and conclusions

In the model presented here, gaseous baryonic material settles inside a mildly non-axisymmetric halo with a nearly constant density core. Initially, no orbits with a definite sense of rotation exist.\(^5\) The first infalling baryonic material, therefore, efficiently loses its angular momentum to the core. This initial collapse terminates only when the gas

\(^5\)Or only very eccentric ones if one allows for slow figure rotation in the halo.
becomes self-gravitating and forms stars. The increased central density concentration which is produced in this first phase, however, destroys the harmonic core, paving the way for the existence of non-intersecting closed loops with a definite sense of rotation.

If the loop orbits are too eccentric, the gas will shock and depopulate them. These orbits, therefore, cannot represent long-lived attractors of the dissipative motion. Thus, if the baryonic component does not possess a central cusp initially, the absence of sufficiently round supporting closed loops will lead to the formation of a central mass concentration including an SBH. At larger radii the potential responsible for creating sufficiently round loop orbits is that of the extended baryonic component, in the form of a bulge. This leads to a linear relationship between the bulge and SBH masses.

It is crucial that both the onset of self-gravity and the appearance of increasingly circular loop orbits are subject to the same condition, both depending on the density ratio of the collapsing gas to that of the background halo core. This limits the allowable range in densities. For bulge core densities of order those of the halo core, and for plausible values of the critical eccentricity, the value of $K$ lies in the range $10^{-4} - 10^{-2}$, which is compatible with present observations. For densities up to an order of magnitude larger, the range of values is the same, provided that the critical values of the closed loop axis ratios are larger. This can be expected if the dissipation rate along loop orbits is dependent on density — a plausible assumption.

The most important prediction of the model outlined above is that the bulge and SBH parameters are determined by the halo properties. In particular, relationships between SBH and bulge masses and the velocity dispersion of the bulge necessarily arise, and with the right exponents, if the haloes should also exhibit a Faber-Jackson type relation between their masses and velocity dispersions. Moreover, within a given halo, variations in the bulge and SBH properties are not expected to destroy these relations — because imposing the condition of critical eccentricity requires that the bulge and SBH masses are related to the bulge velocity dispersion via a power law, with index also close to that of the Faber-Jackson relationship. Finally, if the Faber-Jackson relationship for the halo exhibits significant scatter and if, as again seems plausible, the critical eccentricities of the loops anti-correlate with the density of the system, less scatter should be present in the relationships between SBH masses and bulge velocity dispersions than the corresponding relationships between bulges masses and their own velocity dispersions — the standard Faber-Jackson relationship.

There is already tentative evidence that halo cores may indeed follow Faber-Jackson type relationships (Burkert 1995; Dalcanton & Hogan 2001). In addition, a halo core produced by flattening out the inner region (where $\rho \propto r^{-1}$) of the NFW profile would produce such a relation (El-Zant, Shlosman & Hoffman 2001). The model thus makes testable predictions concerning the relationships among the very inner regions of galaxies, their extended baryonic components, the dark matter haloes they are thought to be embedded in and the cosmology that predicts their existence.

In this framework, larger and more massive halo cores produce, on average, larger and more massive bulges. Disks form outside the core, or later, when the central concentration produced by the baryons destroys the core. If the core is very large, most of the baryonic material is consumed in the first phase and no significant disk forms. This effect is expected to be prominent in larger mass cores, since if these follow the Faber-Jackson relation, more massive haloes should have proportionally larger cores. Other predictions include the requirement that the average density of the bulge in the central region should be close to that of the halo core. There is also a minimal bulge mass associated with a given core, although this varies significantly with the critical loop orbit eccentricity assumed. A number of additional consequences for galaxy formation and evolution will be discussed elsewhere.

To obtain relationships between SBH masses and bulge properties, we have assumed that during the gaseous infall phase the baryonic component did not have a central density cusp. Most observed bulges and ellipticals, however, do have such cusps. One then has to assume that once star formation starts, cold dissipational collapse is initiated. This would lead to a central density cusp, as it does in the case of cosmological haloes.
Memory of the initial state is retained via the total energy, which determines the final velocity dispersion.

Merritt & Ferrarese (2001) have suggested that a “self-regulating” mechanism, related to the threshold mass necessary for the loss of triaxiality in the system, may be behind the close correlation between SBH and bulge properties. As these authors point out, however, the SBH masses required for strong chaotic behavior leading to rapid loss of triaxiality are in fact probably too large — being of the order of a few percent of the mass of the system’s baryonic component. This is actually of the order of the SBH mass needed to create round loop orbits at all radii inside the halo core. In our model, however, this is not assumed. Instead, an additional baryonic component plays the role of creating these orbits in the outer region. The collision between this “bulge” component and the SBH in destroying the harmonic core and creating a situation whereby stable gaseous motion can exist gives rise to the correlations described in this paper.

The correlations obtained here are compatible with the observed ones, and with acceptable scatter, despite our lack of knowledge of the values of such parameters as $p_{\text{crit}}$ and its variation with system properties. Detailed modeling of the gas dynamics will be required to further constrain this model. It is also possible that our distinction between the dynamical role played by the SBH and that played by the bulge core is too restrictive. We have introduced this to be able to obtain quantitative results, within the model, solely on the basis of the orbital characteristics. In general, the roles of the two components, i.e., the SBH and the bulge, may not be too distinct — formation of the SBH can take place simultaneously with a cuspy bulge. For this, the even more ambitious task of a self-consistent treatment, including gas and stellar dynamics and star formation, is required. We believe, however, that our results are generic and arise from fundamental dynamical phenomena which will manifest themselves in any formulation of galaxy formation in mildly triaxial haloes with harmonic cores.

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A. Perturbation analysis

Let the potential $\Phi$ be a function of the Cartesian coordinates in the halo symmetry plane, $\Phi = \Phi(x^2, y^2)$ with $\Phi(0, 0) = 0$. We follow de Zeeuw & Merritt (1983, hereafter dZM) and expand the potential in even powers of the coordinates:

$$\Phi = \frac{1}{2} \kappa_1^2 x^2 + \frac{1}{2} \kappa_2^2 y^2 + \frac{1}{4} a_5 x^4 + \frac{1}{2} a_7 x^2 y^2 + \frac{1}{4} a_9 y^4 + \ldots \quad \text{(A1)}$$

If no bulge exists, inside the halo’s harmonic core only the terms quadratic in the coordinates are important. If, due to an additional component, or at the boundaries of the core, weak nonlinearity is present the series can be truncated, as above, to second order, as higher order terms are unimportant. The coefficients are given by: $\kappa_1^2 = 2 \partial x \Phi$, $\kappa_2^2 = 2 \partial y \Phi$, $a_5 = 2 \partial_x^2 \Phi$, $a_7 = 2 \partial_{x y}^2 \Phi$, $a_9 = 2 \partial_y^2 \Phi$, where the derivatives are taken at $x = y = 0$. In terms of these one defines the auxiliary variables $\mu_{11} = \frac{2 a_{10} \kappa_1}{\kappa_2}$, $\mu_{12} = \frac{a_5 \kappa_1 - \kappa_2 a_9}{2 \kappa_1 \kappa_2}$, $\mu_{22} = \frac{a_{10} \kappa_2}{\kappa_1}$.

The condition for stability of the loop orbits is (see dZM; Table 2A third row)

$$\mu_{12}(\mu_{11} - \mu_{12} + \mu_{22}) > 0. \quad \text{(A2)}$$

We will be interested in systems that are both mildly nonlinear and mildly nonaxisymmetric: thus $\kappa_1 \sim \kappa_2$ and $a_5 \sim a_9 \sim a_7 < 0$. The above condition reduces to

$$a_7 > \frac{3}{2} \left( \frac{\kappa_2}{\kappa_1} a_5 - \frac{\kappa_1}{\kappa_2} a_9 \right), \quad \text{(A3)}$$

which, under the above conditions, is always satisfied.

It is, therefore, the condition for the existence of loop orbits and not their stability that will be of interest to us. In the absence of a central mass (SBH) or density cusp these cannot be found arbitrarily close to the center, instead there is a bifurcation radius beyond which these exist. This determines the effective core radius of the system. To second order, the condition for the existence of loop orbits is

$$\mu_{22} - \frac{1}{2} \mu_{12} \frac{\kappa_2 \delta}{Q} \leq \frac{1}{2} \mu_{12} - \mu_{11}, \quad \text{(A4)}$$

with $\delta = \kappa_1 / \kappa_2 - 1$ and (to first order) $Q = H / \kappa_2$, where $H$ is the Hamiltonian. For $H > 0$ (first order potential terms dominate in eq. A1) and $\kappa_1 \lesssim \kappa_2$, the above requires

$$H \geq \frac{4 \kappa_1 \kappa_2^2 (\kappa_2 - \kappa_1)}{\kappa_2 a_7 - \delta \kappa_1 a_9}. \quad \text{(A5)}$$

In the unperturbed case, the action variables (e.g., Binney & Tremaine 1987) are time-independent and (exact) solutions can be written in terms of these in Cartesian coordinates as $x = \sqrt{2I_1 / \kappa_1} \cos \theta_1$, etc. In the mildly nonlinear case the solutions of the equations of motion, averaged over a dynamical time (denoted below by a “bar”), approximate the true solution to first order in the (relative amplitude of the) perturbation and for a number of dynamical times inversely proportional to this (dZM; see also Bogoliubov & Mitropolsky 1961; Arnold 1989). In this case analogous approximate solutions can be given in terms of the corresponding action variables. In particular, for loop orbits in mildly nonlinear potentials one finds

$$\bar{I}_1 = \frac{Q(\frac{1}{2} \mu_{12} - \mu_{22}) + \kappa_1 - \kappa_2}{-\mu_{11} + \mu_{12} - \mu_{22}} \quad \text{(A6)}$$

and

$$\bar{I}_2 = Q - \bar{I}_1. \quad \text{(A7)}$$

Eliminating $Q$ one gets

$$\bar{I}_1 = \frac{\bar{I}_2 + F_2 / F_3}{F_1 / F_3 - 1}, \quad \text{(A8)}$$
where $F_1 = -\mu_{11} + \mu_{12} - \mu_{22}, F_2 = \kappa_1 - \kappa_2$ and $F_3 = \mu_{12} - \mu_{22}$.

At bifurcation, to first order,
\[ \bar{I}_2 = Q = \frac{H}{\kappa_2}, \quad (A9) \]
where $H$ is given by eq. (A5). These orbits are infinitely thin and represent oscillations along the $y$-axis, with amplitude
\[ y_{\text{max}} = \sqrt{2\bar{I}_2/\kappa_2}, \quad (A10) \]
which is the effective core of the bulge-halo system, as mentioned above. As one increases $H$ these become thicker with axis ratio $x_{\text{max}}/y_{\text{max}} = \sqrt{\frac{1}{\kappa_2} \frac{\bar{I}_2}{I_1}}$. The view we have taken in this paper is that when this ratio becomes large enough, $\gtrsim p_{\text{crit}}$, such loop orbits can support gaseous motion, and that stars formed on these orbits can constitute populations of stellar disks, thus ending the bulge formation stage. We note here that, beyond the bifurcation point, the above relation predicts a rather rapid (as a function of radius) transition to round loop orbits. This is confirmed by orbital integration, even though if the bulge mass is smaller than a certain minimal mass, in the fully nonlinear treatment, the orbital axis ratio can decrease again. The second minimum in axis ratio curves (e.g., Fig. 1) is thus not reproduced by this perturbation analysis.

**B. Application to the potential used in this paper**

We now apply the perturbation analysis to the superposition of potentials given by eqs. (1) and (3) with the goal of calculating the bifurcation radii of the loops. This is the effective harmonic core radius of the bulge-halo system. Since we are only interested in orbit shapes, we set $V_H^2 = G = 1$ and use scaled variables, as defined in Section 2. For the halo, we use
\[ \Phi_H = \frac{1}{2} \log(1 + x^2 + qy^2), \quad (B1) \]
where $q$, corresponding to $\beta^{-2}$ in eq. (1), parametrizes the nonaxisymmetry. For the bulge, we have
\[ \Phi_B = -\frac{\hat{M}_B}{\sqrt{\hat{R}_B^2 + x^2 + y^2}}. \quad (B2) \]

With the above definitions for the potential we have
\[ \kappa_1^2 = 2\partial_{xx}\Phi = 1 + \frac{\hat{M}_B}{\hat{R}_B^3}, \quad (B3) \]
\[ \kappa_2^2 = 2\partial_{yy}\Phi = q + \frac{\hat{M}_B}{\hat{R}_B^3}, \quad (B4) \]
\[ a_7 = 2\partial_{x^2y^2}\Phi = -q - \frac{3\hat{M}_B}{2\hat{R}_B^5}, \quad (B5) \]
and
\[ a_9 = 2\partial_{y^3}\Phi = -q^2 - \frac{3\hat{M}_B}{2\hat{R}_B^5}. \quad (B6) \]

From equations (A5), (A9), and (A10) bifurcation happens when the value of the long axis of the loops satisfies
\[ y_{\text{max}}^2 = \frac{8\kappa_1 \kappa_2 (\kappa_2 - \kappa_1)}{\kappa_2 a_7 - 3\kappa_1 a_9}. \quad (B7) \]
Substituting from the expressions above and taking the limits $\dot{M}_B/\dot{R}_B^4(\approx \dot{\rho}_B) \gg 1$, $\dot{M}_B/\dot{R}_B^5 \gg 1$, and assuming $q \sim 1$, we obtain
\[
y_{\text{max}} \approx 2(q - 1)^{1/2} \frac{\dot{R}_B^{5/2}}{\sqrt{3} \dot{M}^{1/2}}.
\]
This is the effective harmonic core radius of the bulge-halo system.