EXISTENCE AND SOAP FILM REGULARITY OF SOLUTIONS TO PLATEAU’S PROBLEM

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Abstract. Plateau’s soap film problem is to find a surface of least area spanning a given boundary. We begin with a compact orientable \((n-2)\)-dimensional submanifold \(M\) of \(\mathbb{R}^n\). If \(M\) is connected, we say a compact set \(X\) “spans” \(M\) if \(X\) intersects every Jordan curve whose linking number with \(M\) is 1. Picture a soap film that spans a loop of wire. Using \((n-1)\)-dimensional Hausdorff spherical measure as the measure of the size of a compact set \(X\) in \(\mathbb{R}^n\), we prove there exists a smallest compact set \(X_0\) that spans \(M\). We also show that \(X_0\) is almost everywhere a real analytic \((n-1)\)-dimensional minimal submanifold and if \(n=3\), then \(X_0\) has the structure of a soap film as predicted by Plateau. We provide more details about the minimizer \(X_0\). Primarily, \(X_0\) is the support of a current \(S_0\) and \(M\) is the support of the algebraic boundary of \(S_0\). We also discuss the more general case where \(M\) has codimension \(>2\).

Introduction

One of the classical problems in the Calculus of Variations is to prove the existence of a surface of least area spanning a given Jordan curve in Euclidian space. The problem was first posed by Lagrange [Lag61] in 1760, and Lebesgue [Leb02] called it “Plateau’s Problem” after Joseph Plateau [Pla73] who experimented with films of oil and wire frames.

The problem was formulated in many interesting ways:\footnote{See [Rad33] for an extensive report of work before 1930.} In 1930, Douglas [Dou31] solved the problem for surfaces which arise as the image of a disk, via minimization of an energy functional. His method was extended by Douglas and Courant [Cou50] to surfaces of higher topological type bounding disjoint systems of Jordan curves. In 1960, Federer and Fleming [FF60] used integral currents with a given algebraic boundary to model films, and minimized current mass (see §1.2) instead of energy or area of a surface. In dimension 7 and lower, their solutions turned out to be oriented submanifolds.

In the same year, Reifenberg [Rei60] approached the problem in an entirely different way. For certain types of boundaries \(M\), he chose a collection \(S\) of sets “spanning” \(M\) and concluded there was a set \(X \in S\) that had smallest Hausdorff spherical measure. For each boundary \(M\) he chose a subgroup \(L\) of the Čech homology of \(M\) with coefficients in some compact abelian group (sometimes \(\mathbb{Z}/2\mathbb{Z}\), \(\mathbb{Z}/3\mathbb{Z}\), or \(\mathbb{R}/\mathbb{Z}\).) A set \(X \supset M\) was said to span \(M\) if this subgroup was in the kernel of the
homomorphism induced by the inclusion \( M \hookrightarrow X \). His approach yielded a case-by-case analysis, and one case has considerable historical interest. Using \( \mathbb{Z}/2\mathbb{Z} \) coefficients, he found a minimizer in a collection containing all orientable and non-orientable surfaces in \( \mathbb{R}^3 \) with a given boundary, excluding from consideration non-manifold surfaces with triple junctions or other such singularities. This theorem improved that of Douglas and Courant, because it considered surfaces of arbitrary genus simultaneously, and solved the non-orientable problem.

Reifenberg proved a second special case, using \( \mathbb{R}/\mathbb{Z} \) coefficients, that dealt with more complicated spanning sets. A set \( X \supset M \) spanned \( M \) if there was no retraction \( X \rightarrow M \). This collection contained surfaces with triple junctions and other singularities, but his theorem was limited in that the boundary was required to be a single circle (or in higher dimensions, a sphere.) For more general boundaries, his approach was deficient. For example, consider the disjoint union of a disk and a circle in \( \mathbb{R}^3 \). There is no retraction to the pair of circles, yet we would not want to consider this as an admissible spanning set. As another example, consider the surfaces \( X_i, i = 1, 2, 3 \), in Figure 1. Any one could be a minimal surface, depending on the distance between the circles, but a simple computation shows there is no non-trivial collection of Reifenberg which contains all three simultaneously. One would like to have a theorem just like Reifenberg’s arbitrary genus solution, that simultaneously dealt with all reasonable spanning surfaces, including those with singularities.

![Figure 1](image-url)  

**Figure 1.** \( M \) is the disjoint union of three circles. \( X_1 \) is homeomorphic to a cylinder, and \( X_2 \) and \( X_3 \) are both homeomorphic to a cylinder union a disc. Each is a Reifenberg surface for some collection, but no non-trivial collection of Reifenberg surfaces spanning \( M \) contains \( X_1, X_2, \) and \( X_3 \) simultaneously.

After Reifenberg’s paper, attention shifted to regularity theory (e.g., [Alm76], [Gio61], [Sim57], [HS79], [Hil69],) higher genus problems (e.g., [TT88],) and mass minimization problems using different types of de Rham currents (e.g., [Fle66].) With currents, one can multiply films by density functions, test against differential forms, and apply functional analytic operators such as boundary. Thus, the use of currents ties Plateau’s problem more closely to algebraic topology than the purely set-theoretical approach of Reifenberg, and gives solutions additional structure. Moreover, mass is much easier to work with than “size,” which is defined as some reasonable measure of the support of a current, usually a variant of Hausdorff measure. One reason is that mass is lower semicontinuous in the weak topology, while size is lower semicontinuous in only very special circumstances. The trade-off is that the minimization of mass gives a less physically realistic model than the minimization of size. Mass takes into account multiplicity of a surface, whereas size does not. So, in the mass
minimization problem there is no benefit to combining surfaces together to form triple junctions, since the joined surfaces would retain a higher multiplicity.

Ideally, one would solve the size-minimization problem using currents, combining the best of both worlds. However, doing so has proved elusive. Morgan has promoted this problem in the context of rectifiable currents, and proved in [Mor89] an existence theorem for a prescribed \((n - 2)\)-dimensional compact oriented submanifold of the unit sphere in \(\mathbb{R}^n\). Size minimization problems bring with them the possibility of a sequence of long thin tentacles whose size tends to zero, but which converge in the Hausdorff metric to a set of full Lebesgue measure, and thus infinite size (see Figure 2.) Federer and Fleming faced a similar problem for mass-minimizing integral currents and developed a cut-and-paste procedure to remove such tentacles. De Pauw [dP09] has made progress on Morgan’s problem, but pointed out that it appears difficult to find both a compactness theorem and a cut-and-paste procedure for size-minimizing rectifiable currents. A different collection of currents was needed and he suggested a range of possibilities.

![Figure 2. Long, thin tentacles with small Hausdorff 2-measure can converge in the Hausdorff metric to a set of infinite Hausdorff 2-measure.](image)

The present paper tackles Plateau’s problem in two new ways. First, we replace Reifenberg’s collections with a single definition of span that is better suited to more general boundaries and surfaces. In particular, our collection of surfaces contains all the surfaces as in Figure 1. Using our new definition, it is also more readily apparent whether a given surface spans a given boundary.

Second, we replace rectifiable currents in the size minimization problem of Morgan with a collection of currents \(\mathcal{T}\) which roughly correspond to “dipole surfaces,” (see Figures 3 and 4.) Dipole surfaces nicely model the hydrophobic/hydrophylic nature of soap films, and bypass completely the problem of triple junctions contributing to the boundary. Our collection \(\mathcal{T}\) has good compactness properties (Theorem 6.0.9) and comes with a cut-and-paste procedure (Theorem 7.2.7.) We also define a continuous area functional corresponding to size, and a continuous boundary operator. The collection \(\mathcal{T}\) is a subset of \(\hat{\mathcal{B}}\), a subspace de Rham currents called “differential chains.” A topology on \(\hat{\mathcal{B}}\) was established and analyzed by the authors in [Har93], [Har], [Har12a], [Pug09] and [HP12], and there is now a growing theory of differential chains with a number of other applications in progress.

Let \(M\) be an \((m - 1)\)-dimensional compact orientable submanifold of \(\mathbb{R}^n\), where \(1 \leq m \leq n\) (for the classical case, set \(m = 2\) and \(n = 3\).) We say that an embedded sphere \(S\) of dimension \(n - m\)
disjoint from $M$ is a **simple link of** $M$ if the absolute value of the linking number\(^2\) $L(S, M_i)$ of $S$ with one of the connected components $M_i$ of $M$ is equal to one, and $L(S, M_j) = 0$ for the other connected components $M_j$ of $M$, $j \neq i$. We say that a subset $X \subset \mathbb{R}^n$ **spans** $M$ if every simple link of $M$ intersects $X$. One can drop the condition that $M$ be a manifold by specifying a particular $(m-1)$-cycle for which $S$ to link. Another possibility would be to drop the requirement that $S$ be a sphere. However, doing so would shrink the collection of spanning sets, and we do not wish to do this. Via Alexander duality, this is a cohomological spanning condition, rather than the homological spanning condition used by Reifenberg.

Note that in Figure 1, all three sets $X_i$ span $M$. If $M$ is a Jordan curve in $\mathbb{R}^3$, $X$ is a CW complex of finite Hausdorff 2-measure, $M$ embeds in $X$ as a subcomplex and $X$ does *not* span $M$, then there exists a retraction of $X$ onto $M$ (see Proposition 5.0.2.) Thus, apart from some pathological surfaces, possibly, our collection of surfaces contains Reifenberg’s $\mathbb{R}/\mathbb{Z}$ collection.

\(^2\)Given some choice of orientation of $M$ and $S$, the definition being independent of this choice.
In our Main Theorem, we assume \( m = n - 1, n \geq 3 \), and “size” means Hausdorff spherical measure of the support. Let \( \mathcal{X}(M) \) be the collection of all compact subsets \( X \) of \( \mathbb{R}^n \) which span \( M \). In particular, \( \mathcal{X}(M) \) includes orientable and non-orientable manifolds of all topological types, and manifolds with multiple junctions.

We will define a subcollection \( \mathcal{F}(M) \subset \mathcal{T} \) whose elements \( T \) satisfy \( \text{supp}(T) \in \mathcal{X}(M) \) and \( \text{supp}(\partial T) = M \). We show that if \( X \in \mathcal{X}(M) \), then its core\(^3\) \( X^* \) supports a current \( T \in \mathcal{F}(M) \). Thus minimizing size in \( \mathcal{F}(M) \) solves the problem of minimizing \( m \)-dimensional Hausdorff spherical measure in \( \mathcal{X}(M) \).

Here is our solution to Plateau’s Problem:

**Main Theorem.** If \( M \) is a \((n - 2)\)-dimensional compact orientable submanifold of \( \mathbb{R}^n \), then there exists a size-minimizing element \( S_0 \in \mathcal{F}(M) \). If \( S_0 \) is any such minimizer, then its support \( X_0 \) is almost everywhere a real analytic \( m \)-dimensional minimal submanifold. For \( n = 3 \), \( X_0 \) has the local structure of a soap film\(^4\).

The current \( S_0 \) satisfies additional properties, namely its support is contained in the convex hull of \( M \) and is reduced\(^5\). The current \( S_0 \) is also a differential chain (see \S1.3.) Furthermore, \( \text{supp}(\kappa S_0) = X_0 \), where \( \kappa \) is a continuous cone\(^6\) operator. The property \( \text{supp}(\kappa S_0) = X_0 \) may be surprising at first, for \( \kappa \) increases the dimension of \( S_0 \) by one, yet its support is unchanged. One can think of this operator as “filling in” the infinitesimal fluid between oppositely oriented surfaces of the dipole model of the film (see Figure 3.) This construction is important for it leads to a continuous area functional by way of the volume form \( dV \) acting on top dimensional currents. Finally, the measure \( S^m[X_0] \) corresponds to \( \kappa S_0 \) in the sense that the volume of \( \kappa S_0 \) restricted\(^7\) to a Borel set \( B \) is the same as the Hausdorff measure of \( X_0 \cap B \).

There are several obstacles in the way of achieving our result for higher codimension. First, we do not yet have a proof of Lemma 5.0.5 for higher codimension. Second, in several lemmas we make use of the fact that when \( m = n - 1 \), the normal bundle to a simple link is trivial. This is not necessarily the case in general. Finally, we adapted several of Reifenberg’s lemmas to our definition of “span,” and his computations are somewhat exhausting in higher codimension. Our methods proving existence build upon, develop, and go beyond those found in [Har04] and [Har12b], where linking numbers were first used to define spanning currents and a preliminary compactness theorem was established.

**Notation**

- \( \partial X \) is the frontier of \( X \);
- \( \bar{X} \) is the closure of \( X \);
- \( \, \text{int} \, X \) is the interior of \( X \);

\(^3\)The **core** of \( X \) is defined by \( X^* := \{ p \in X : \mathcal{H}^m(X \cap \Omega(p, r)) > 0 \text{ for all } r > 0 \} \), where \( \Omega(p, r) \) is the open ball of radius \( r \) about \( p \) and \( \mathcal{H}^m \) is \( m \)-dimensional Hausdorff measure.

\(^4\)In fact, \( X_0 \) is “restricted” in the sense of Almgren and this implies \( X_0 \) is almost everywhere a real analytic \( m \)-dimensional minimal submanifold by [Alm68] (1.7.) Soap film regularity for \( n = 3 \) follows from [Toy76] (see also [Dav11].)

\(^5\)We say \( X \) is **reduced** if \( X = X^* \).

\(^6\)See \S1.11 and \S2

\(^7\)See Definition 3.2.4
• $X^c$ is the complement of $X$ in $\mathbb{R}^n$;
• $h(X)$ is the convex hull of $X$;
• $diam(X)$ is the diameter of $X$;
• $\Omega(X, \epsilon)$ is the open epsilon neighborhood of $X$;
• $\overline{\Omega}(X, \epsilon)$ is the closed epsilon neighborhood of $X$;
• $C_q X$ is the inward cone over $X \subset \mathbb{R}^n$ with vertex $q \in \mathbb{R}^n$.

1. Differential chains

In this preliminary section we provide a quick survey of the space of differential chains, its topology, and the continuous operators used throughout the paper.

1.1. Dirac chains.

**Definition 1.1.1.** For $U$ open in $\mathbb{R}^n$, let $A_k(U)$ denote the vector space of finitely supported functions $U \rightarrow \Lambda^k(\mathbb{R}^n)$. We call $A_k(U)$ the space of Dirac $k$-chains in $U$.

We write an element $A$ of $A_k(U)$ using formal sum notation $A = \sum_{i=1}^{N} (p_i; \alpha_i)$ where $p_i \in U$ and $\alpha_i \in \Lambda^k(\mathbb{R}^n)$. We call $(p; \alpha)$ a $k$-element in $U$ if $\alpha \in \Lambda^k(\mathbb{R}^n)$ and $p \in U$.

1.2. Mass norm.

**Definition 1.2.1.** An inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$ determines the mass norm on $\Lambda^k(\mathbb{R}^n)$ as follows: Let $\langle u_1 \wedge \cdots \wedge u_k, v_1 \wedge \cdots \wedge v_k \rangle := det((u_i, v_j))$. The **mass** of a simple $k$-vector $\alpha = v_1 \wedge \cdots \wedge v_k$ is defined by $\|\alpha\| := \sqrt{\langle \alpha, \alpha \rangle}$. The mass of a $k$-vector $\alpha$ is $\|\alpha\| := \inf \left\{ \sum_{j=1}^{N} \| (\alpha_i) \| : \alpha_i \text{ are simple, } \alpha = \sum_{j=1}^{N} \alpha_i \right\}$. Define the mass of a $k$-element $(p; \alpha)$ by $\|(p; \alpha)\|_{B^0} := \|\alpha\|$.

The **mass** of a Dirac $k$-chain $A = \sum_{i=1}^{N} (p_i; \alpha_i) \in A_k(U)$ is given by

$$\|A\|_{B^0} := \sum_{i=1}^{N} \|(p_i; \alpha_i)\|_{B^0}.$$
1.3. Difference chains and the $B^r$ norm.

**Definition 1.3.1.** Let $h(X)$ denote the convex hull of $X$. Given $u \in \mathbb{R}^n$ and a $k$-element $(p; \alpha) \in \mathcal{A}_k(\mathbb{R}^n)$, let $T_u(p; \alpha) := (p + u; \alpha)$ be translation through $u$, and $\Delta_u(p; \alpha) := (T_u - I)(p; \alpha)$. Extend both operators linearly to $\mathcal{A}_k(\mathbb{R}^n)$. If $u_1, \ldots, u_j \in \mathbb{R}^n$, define the $j$-difference $k$-chain $\Delta_{\{u_1, \ldots, u_j\}}(p; \alpha) := \Delta_{u_1} \circ \cdots \circ \Delta_{u_j}(p; \alpha)$. The operators $\Delta_{u_i}$ and $\Delta_{u_k}$ commute, so $\Delta_{\{u_1, \ldots, u_j\}}$ depends only on the set $\{u_1, \ldots, u_j\}$ and not on the ordering. Say $\Delta_{\{u_1, \ldots, u_j\}}(p; \alpha)$ is inside $U$ if $h(\text{supp}(\Delta_{\{u_1, \ldots, u_j\}}(p; \alpha))) \subset U$.

To simplify our next definition, let $\Delta := \{\Theta := \text{image of } \Theta \} := \{\text{image of } \Theta \}$, $B := \text{translation}$, $\|\cdot\|$ be an identity map ($\|\cdot\|$). For simplicity, we often write $\|\cdot\|$. For $A \in \mathcal{A}_k(U)$ and $r \geq 0$, define the norm

$$\|A\|_{B^r} := \inf \left\{ \sum_{i=0}^{w} |\Delta_{j_i}(p_i; \alpha_i)|_{B^{j_i}} : A = \sum_{i=0}^{w} \Delta_{j_i}(p_i; \alpha_i), 0 \leq j_i \leq r \text{ and } \Delta_{j_i}(p_i; \alpha_i) \text{ is inside } U \right\}.$$

That is, the infimum is taken over all ways of writing $A$ as a finite sum of $j$-difference $k$-chains, for $j$ between 0 and $r$. It is shown in (Har12a, Theorem 3.2.1) that $\|\cdot\|_{B^r}$ is a norm on $\mathcal{A}_k(U)$. For simplicity, we often write $\|\cdot\|_{B^r} := \|\cdot\|_{B^r}$ if $U$ is understood from the context.

**Definition 1.3.2.** Let $\tilde{\mathcal{B}}_k^r = \tilde{\mathcal{B}}_k^r(U)$ be the Banach space obtained by completing the normed space $(\mathcal{A}_k(U), \|\cdot\|_{B^r})$. Elements of $\tilde{\mathcal{B}}_k^r(U)$, $0 \leq r < \infty$, are called differential $k$-chains of class $B^r$ in $U$. The maps $\tilde{\mathcal{B}}_k^r(U_1) \to \tilde{\mathcal{B}}_k^r(U_2)$ induced by inclusions $U_1 \hookrightarrow U_2$ are continuous. Moreover, if $r \leq s$, the identity map $(\mathcal{A}_k(U), \|\cdot\|_r) \to (\mathcal{A}_k(U), \|\cdot\|_s)$ is continuous and thus extends to a continuous map $u_k^{r,s} : \tilde{\mathcal{B}}_k^r(U) \to \tilde{\mathcal{B}}_k^s(U)$ called the linking map.

**Proposition 1.3.4** (Har12a, Lemma 5.0.2, Corollary 5.0.5). The linking maps $u_k^{r,s} : \tilde{\mathcal{B}}_k^r(U) \to \tilde{\mathcal{B}}_k^s(U)$ satisfy

(a) $u_k^{r,r} = \text{Id}$;
(b) $u_k^{r,t} \circ u_k^{r,s} = u_k^{r,t}$ for all $r \leq s \leq t$;
(c) The image $u_k^{r,s}(\tilde{\mathcal{B}}_k^r(U))$ is dense in $\tilde{\mathcal{B}}_k^s(U)$;
(d) $u_k^{r,s}$ is injective for each $r \leq s$.

In [Har12a] we study the inductive limit space $\hat{\mathcal{B}}_k^{\infty}(U) := \lim_{\rightarrow} \tilde{\mathcal{B}}_k^r(U)$.

**Theorem 1.3.5.** If $U \subset \mathbb{R}^n$ is bounded, the linking map $u_0^{0,1} : \hat{\mathcal{B}}_0^0(U) \to \hat{\mathcal{B}}_0^1(U)$ is compact.

**Proof.** Since $\tilde{\mathcal{B}}_0^1(U)$ is a Banach space, and since $\mathcal{A}_n(U)$ is dense in $\tilde{\mathcal{B}}_0^0(U)$, it suffices to show that the image of $\Theta := \{J \in \mathcal{A}_n(U) : \|J\|_{B^0} \leq 1\}$ is totally bounded.
For $k \in \mathbb{N}$ let $\Xi(k)$ be a covering of $U$ by finitely many balls $\Omega(x_j, 2^{-k})$ centered at points $x_j \in U$, $1 \leq j \leq N_k$. Let $\Omega(k) = \{y2^{-k}N_k^{-1} : y \in \mathbb{Z}, 0 \leq y \leq 2^kN_k\}$ and let

$$\mathcal{Z}(k) = \left\{ \sum_{j=1}^{N_k} (x_j; \beta_j) \in u_n^0(\Theta) : \|\beta_j\| \in \Omega(k) \text{ for all } 1 \leq j \leq N_k \right\}.$$

Let $A = \sum_{s=1}^r (p_s; \alpha_s) \in u_n^0(\Theta)$. We approximate $A$ with an element of $\mathcal{Z}(k)$ as follows:

Fix $1 \leq s \leq r$. Since $\Xi(k)$ covers $U$, we have $p_s \in \Omega(x_{j_s}, 2^{-k})$ for some $1 \leq j_s \leq N_k$. Set $p_s' = x_{j_s}$. Put $A' = \sum_{s=1}^N (p_s'; \alpha_s)$. By summing the $n$-vectors $\alpha_s$ at the same point $x_j$ and inserting zeros as necessary, we can write $A' = \sum_{j=1}^{N_k} (x_j; \mu_j)$, where $\sum_{j=1}^{N_k} \|\mu_j\| \leq 1$.

Now let $\mu_j'$ be the $n$-vector whose mass $\|\mu_j'\| \in \Omega(k)$ satisfies $0 \leq \|\mu_j\| - \|\mu_j'\| < 2^{-k}N_k^{-1}$. Set $A'' = \sum_{j=1}^{N_k} (x_j; \mu_j') \in \mathcal{Z}(k)$. It follows that

$$\|A - A''\|_{B^1} \leq \sum_{s=1}^r \|(p_s; \alpha_s) - (p_s'; \alpha_s)\|_{B^1} + \sum_{j=1}^{N_k} \|((x_j; \mu_j) - (x_j; \mu_j'))\|_{B^1}$$

$$\leq \sum_{s=1}^r \|p_s - p_s'\| \cdot \|\alpha_s\| + \sum_{j=1}^{N_k} \|\mu_j - \mu_j'\|$$

$$\leq 2^{1-k}.$$

**Definition 1.3.6.** Let $\mathcal{B}_r^s(U)$ be the Banach space of bounded $k$-forms on $U$ equipped with the comass norm $\|\omega\|_{B^0} = \sup_{\|\alpha\|=1} \omega(\alpha)$. For each $r \geq 1$, let $\mathcal{B}_r^s(U)$ be the Banach space of $(r-1)$-times differentiable $k$-forms with comass bounds on the $s$-th order directional derivatives for $0 \leq s \leq r - 1$ with the $(r-1)$-st derivatives satisfying a bounded Lipschitz condition,

8 with norm given by $\|\omega\|_{B^r} = \sup_{|t| \leq r-1} \{|D^t\omega|_{B^0}, |D^{r-1}\omega|_{Lip}\}$. Elements of $\mathcal{B}_r^s(U)$ are called **differential $k$-forms of class $B^r$ in $U$**.

We always denote differential forms by lower case Greek letters such as $\omega, \eta$ and differential chains by upper case Roman letters such as $J, K$, so there is no confusion when we write $\|\omega\|_{B^r}$ or $\|J\|_{B^r}$.

**Proposition 1.3.7** (Isomorphism Theorem, [Har12a] Theorem 3.5.2). For $r \geq 0$, the continuous dual space $\hat{\mathcal{B}}_r^s(U)'$ equipped with the dual norm is isometric to $\mathcal{B}_r^s(U)$ via the restriction of covectors in $\hat{\mathcal{B}}_r^s(U)'$ to $k$-elements.

Thus there is a jointly continuous bilinear pairing $\mathcal{B}_r^s \times \hat{\mathcal{B}}_r^s \to \mathbb{R}$ given by $(J, \omega) \mapsto \int_J \omega := \omega(J)$ satisfying $|\omega(J)| \leq \|\omega\|_{B^r} \|J\|_{B^r}$. By a slight abuse of notation, we write $\omega(J)$ to mean the evaluation on $J$ of the covector corresponding to $\omega$ given by the isomorphism of Proposition 1.3.7.

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8Here, $U$ is equipped with the intrinsic metric induced by the standard metric on $\mathbb{R}^n$. That is, $d_U(p,q) = \inf L(\gamma)$, where the infimum is taken over all paths $\gamma$ in $U$ from $p$ to $q$. We say that a form $\omega$ is Lipschitz if $|\omega|_{Lip} := \sup_{\|\alpha\|=1} \frac{|\omega(\alpha) - \omega(q,\alpha)|}{\|\alpha\|d_U(p,q)} < \infty$. C.f. Whitney’s Lipschitz comass constant, [Whi57], V, §10.
1.4. Support of a chain.

Definition 1.4.1. If $J \in \mathcal{B}_k^r(U)$, define the support of $J$ to be

$$\text{supp}(J) := \left\{ p \in \mathbb{R}^n : \forall \epsilon > 0, \exists \eta \in \mathcal{B}_k^r(U) \text{ with } \eta \text{ supported in } \Omega(p, \epsilon) \text{ s.t. } \int_J \eta \neq 0 \right\}.$$ 

The support of a nonzero differential chain is a nonempty closed subset of $\overline{U}$ ( [Har12a], Theorems 6.3.4). This definition also agrees with the usual notion of support on the subspace $A_k(U)$ of $\mathcal{B}_k^r(U)$.

Lemma 1.4.2. If $J_i \rightarrow J$ in $\mathcal{B}_k^r(U)$ and $p \in \text{supp}(J)$, then there exists $p_i \in \text{supp}(J_i)$ such that $p = \lim_{i \to \infty} p_i$.

Proof. Each $\Omega(p, \epsilon)$ must intersect all $\text{supp}(J_i)$ for sufficiently large $i$, for if not, there exists a differential form $\eta$ supported in $\Omega(p, \epsilon)$ with $\int_J \eta \neq 0$, and a subsequence $i_j \rightarrow \infty$ such that $\int_{J_{i_j}} \eta = 0$ for each $i_j$, contradicting continuity of the integral $\int$.

1.5. Pushforward.

Definition 1.5.1. Suppose $U_1 \subseteq \mathbb{R}^n$ and $U_2 \subseteq \mathbb{R}^w$ are open and $F : U_1 \rightarrow U_2$ is a differentiable map. For $p \in U_1$ define $F_\ast(p, \alpha) := (F(p), F_{\ast} \omega)$ for all $k$-elements $(p; \alpha)$ and extend to a linear map $F_\ast : A_k(U_1) \rightarrow A_k(U_2)$ called pushforward.

Definition 1.5.2. For $r \geq 1$ let $\mathcal{M}^r(U, \mathbb{R}^w)$ be the vector space of differentiable maps $F : U \rightarrow \mathbb{R}^w$ whose coordinate functions $F_i$ satisfy $\partial F_i / \partial e_j \in B^{r-1}(U)$ for all $j$. Define the seminorm

$$\rho_r(F) := \max_{i,j} \{ \| \partial F_i / \partial e_j \|_{B^{r-1}} \}.$$ 

Let $\mathcal{M}^r(U_1, U_2) := \{ F \in \mathcal{M}^r(U_1, \mathbb{R}^w) : F(U_1) \subset U_2 \subset \mathbb{R}^w \}$.

A map $F \in \mathcal{M}^1(U, \mathbb{R}^w)$ may not be bounded, but its directional derivatives must be. An important example is the identity map $x \mapsto x$ which is an element of $\mathcal{M}^1(\mathbb{R}^n, \mathbb{R}^n)$.

Proposition 1.5.3 ( [Har12a], Theorem 7.0.6, Corollary 7.0.18). If $F \in \mathcal{M}^r(U_1, U_2)$, then $F_\ast$ satisfies

$$\| F_\ast(A) \|_{B^{r-1}} \leq n^r \max \{ 1, \rho_r(F) \} \| A \|_{B^{r-1}}$$

for all $A \in A_0(U_1)$. It follows that $F_\ast$ extends to a well defined continuous linear map $F_\ast : \mathcal{B}_k^r(U_1) \rightarrow \mathcal{B}_k^r(U_2)$, whose dual map $F^\ast : \mathcal{B}_k^r(U_2) \rightarrow \mathcal{B}_k^r(U_1)$ is the usual pullback of forms. Thus $\int_{F_\ast(J)} \omega = \int_J F^\ast \omega$ for all $J \in \mathcal{B}_k^r(U_1)$ and $\omega \in \mathcal{B}_k^r(U_2)$.

Definition 1.5.4. A $k$-cell $\sigma$ in $U \subseteq \mathbb{R}^k$ is a bounded finite intersection of closed half-spaces, such that $\sigma \subseteq U$. An oriented affine $k$-cell in $U \subseteq \mathbb{R}^n$ is a $k$-cell in $U$ which is also contained in some affine $k$-subspace $K$ of $\mathbb{R}^n$, and which is equipped with a choice of orientation of $K$.

Definition 1.5.5. We say that $J \in \mathcal{B}_k^r(U)$ represents an oriented $k$-dimensional submanifold (possibly with boundary) $M$ of $U$ if $\int_M \omega = \int_J \omega$ for all $\omega \in \mathcal{B}_k^r(U)$.

Proposition 1.5.6 (Representatives of $k$-cells, [Har12a], Theorem 4.2.2). If $\sigma$ is an oriented affine $k$-cell in $U$, then there exists a unique differential $k$-chain $\tilde{\sigma} \in \mathcal{B}_k^r(U)$ which represents $\sigma$. 

Definition 1.5.7. Let
\[ P_k(U) := \left\{ J \in \hat{B}_k^1(U) : J = \sum_{i=1}^{N} a_i \sigma_i, \quad \sigma_i \text{ oriented affine } k \text{-cell}, a_i \in \mathbb{R} \right\}. \]

Elements of \( P_k(U) \) are called **polyhedral \( k \)-chains** in \( U \).

The subspace \( P_k(U) \subset \hat{B}_k^1(U) \) is dense in \( \hat{B}_k^1(U) \), for \( r \geq 1 \) ([Har12a], Theorem 4.2.5).

**Remark 1.5.8.** Polyhedral chains and Dirac chains are both dense subspaces of Whitney’s sharp chains and \( B_k^r \), \( r \geq 1 \) ([Whi57], VII, §8, Theorem 8A). The sharp norm of Whitney is comparable to the \( B^1 \) norm for \( k \)-chains, and they are identical for \( k = 0 \).

**Definition 1.5.9.** If \( \sigma \) is an oriented affine \( k \)-cell in \( U \) and \( F \in \mathcal{M}^1(U,W) \), then \( F_\sigma \in \hat{B}_k^1(W) \), and is called an **algebraic \( k \)-cell**.

We remark that an algebraic \( k \)-cell \( F_\sigma \) is not the same as a singular \( k \)-chain \( F\sigma \). For example, if \( F(x) = x^2 \) and \( \sigma = (-1,1) \), then the algebraic 1-cell \( F_\sigma = 0 \), but the singular 1-cell \( F\sigma \neq 0 \). In particular, singular chains have no relations, whereas algebraic \( k \)-cells inherit relations from the topology on \( \hat{B}^r_k \).

**Lemma 1.5.10** ([Har12a] Theorem 4.2.2, Corollary 7.0.18, Proposition 7.0.19). If \( \sigma \) is an oriented \( k \)-cell in \( U \subseteq \mathbb{R}^k \) and \( F \in \mathcal{M}^1(U,W) \) is a smooth embedding, then \( \int_{F_\sigma} \omega = \int_{F(\sigma)} \omega \) for all \( k \)-forms \( \omega \in \mathcal{B}_k^1(W) \). Moreover, \( \text{supp}(F_\sigma) = F(\sigma) \).

The next result is a consequence of Lemma 1.5.10.

**Proposition 1.5.11.** If \( M \) is a compact oriented \( k \)-submanifold with boundary, smoothly embedded in \( U \subseteq \mathbb{R}^n \), then there exists \( \tilde{M} \in \hat{B}_k^1(U) \) with
\[ \int_{\tilde{M}} \omega = \int_M \omega \]
for all \( \omega \in \mathcal{B}_k^1(U) \). Furthermore \( \text{supp}(\tilde{M}) = M \). If \( U \subseteq \mathbb{R}^n \) is a bounded open set equipped with an orientation, then there exists a unique differential \( n \)-chain \( \tilde{U} \in \hat{B}_n^1(U) \) which represents \( U \).

1.6. **Vector fields.** Let \( \mathcal{V}^r(U) \) be the Banach space of vector fields \( X \) on \( U \) such that \( X^b \in \mathcal{B}_1^r(U) \), equipped with the norm \( \|X\|_{B^r} := \|X^b\|_{B^r} \).

1.7. **Extrusion.** The interior product \( i_X \) of differential forms \( \mathcal{B}_k^r(U) \) with respect to a vector field \( X \in \mathcal{V}^r(U) \) is dual to an operator \( E_X \) on differential chains \( \hat{B}_k^r(U) \).

**Definition 1.7.1.** Let \( X \in \mathcal{V}^r(U) \). Define the graded operator \( \text{extrusion} \ E_X : A_k(U) \to A_{k+1}(U) \) by \( E_X(p;\alpha) := (p; X(p) \wedge \alpha) \) for all \( p \in U \) and \( \alpha \in \Lambda^k(\mathbb{R}^n) \).

**Theorem 1.7.2** ([Har12a], Theorems 8.2.2, 8.2.3). If \( X \in \mathcal{V}^r(U) \) and \( A \in A_k(U) \), then
\[ \|E_X(A)\|_{B^r} \leq n^2 2^r \|X\|_{B^r} \|A\|_{B^r}. \]
Furthermore, \( E_X : \hat{\mathcal{B}}_k^r(U) \to \hat{\mathcal{B}}_{k+1}^r(U) \) and \( i_X : \mathcal{B}_{k+1}^r(U) \to \mathcal{B}_k^r(U) \) are continuous graded operators satisfying

\[
\int_{E_X \cdot J} \omega = \int_J i_X \wedge \omega
\]

for all \( J \in \hat{\mathcal{B}}_k^r(U) \) and \( \omega \in \mathcal{B}_{k+1}^r(U) \).

Therefore, \( E_X \) extends to a continuous linear map \( E_X : \hat{\mathcal{B}}_k^r(U) \to \hat{\mathcal{B}}_{k+1}^r(U) \), and the dual operator \( i_X : \mathcal{B}_{k+1}^r(U) \to \mathcal{B}_k^r(U) \) is also continuous.

1.8. Retraction.

**Definition 1.8.1.** For \( \alpha = v_1 \wedge \cdots \wedge v_k \in \Lambda^k(\mathbb{R}^n) \), let \( \hat{\alpha}_i := v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_1 \in \Lambda^{k-1}(\mathbb{R}^n) \).

For \( X \in \mathcal{V}^r(U) \) define the graded operator **retraction** \( E_X^r : \mathcal{A}_k(U) \to \mathcal{A}_{k-1}(U) \) by \( (p; \alpha) \mapsto \sum_{i=1}^k (-1)^{i+1}(X(p), v_i)(p; \hat{\alpha}_i) \), for \( p \in U \) and \( \alpha \) simple, extending linearly to all of \( \mathcal{A}_k(U) \).

A straightforward calculation in [Har12a] shows this to be well-defined. The dual operator on forms is wedge product with the 1-form \( X\). That is to say, it is the operator \( X^\flat \wedge \cdot : \mathcal{A}_{k-1}(U)^* \to \mathcal{A}_k(U)^* \).

**Theorem 1.8.2** ([Har12a], Theorem 8.3.3, 8.3.4). If \( r \geq 1 \), \( X \in \mathcal{V}^r(U) \) and \( J \in \hat{\mathcal{B}}_k^r(U) \), then

\[
\|E_X^r(J)\|_{\mathcal{B}^r} \leq k \binom{n}{k} \|X\|_{\mathcal{B}^r} \|J\|_{\mathcal{B}^r}.
\]

Furthermore, \( E_X^r : \hat{\mathcal{B}}_k^r(U) \to \hat{\mathcal{B}}_{k-1}^r(U) \) and \( X^\flat \wedge \cdot : \mathcal{B}_{k-1}^r(U) \to \mathcal{B}_k^r(U) \) are continuous graded operators satisfying

\[
\int_{E_X^r \cdot J} \omega = \int_J X^\flat \wedge \omega
\]

for all \( J \in \hat{\mathcal{B}}_k^r(U) \) and \( \omega \in \mathcal{B}_{k-1}^r(U) \).

The commutation relation

\[
E_V E_W^r + E_W^r E_V = \langle V, W \rangle \text{Id}
\]

can be found in [Har12a] Proposition 8.3.5(d).

1.9. Boundary. There are several equivalent ways to define the boundary operator \( \partial : \hat{\mathcal{B}}_k^r(U) \to \hat{\mathcal{B}}_{k-1}^{r+1}(U) \) for \( r \geq 1 \). We have found it very useful to define boundary on Dirac chains directly. For \( v \in \mathbb{F}^n \), and a \( k \)-element \( (p; \alpha) \) with \( p \in U \), let \( P_v(p; \alpha) := \lim_{t \to 0} (p + tv; \alpha/t) - (p; \alpha/t) \). It is shown in ([Har12a], Lemma 8.4.1) that this limit exists as a well-defined element of \( \hat{\mathcal{B}}_k^r(U) \). We may then linearly extend this to a map \( P_v : \mathcal{A}_k(U) \to \hat{\mathcal{B}}_k^r(U) \). Moreover, the inequality \( \|P_v(A)\|_{\mathcal{B}^{r+1}} \leq \|v\| A \|_{\mathcal{B}^r} \) holds for all \( A \in \mathcal{A}_k(U) \) ([Har12a], Lemma 8.4.3(a)). For an orthonormal basis \( \{e_i\} \) of \( \mathbb{F}^n \), set

\[
\partial := \sum P_{e_i} E_{e_i}^r
\]
Since $P_e$ and $E_{e_i}^r$ are continuous, **boundary** $\partial$ is a well-defined continuous operator $\partial : \hat{B}_k^r(U) \to \hat{B}_{k-1}^{r+1}(U)$ that restricts to the classical boundary operator on polyhedral $k$-chains $\mathcal{T}_k(U)$, and is independent of choice of $\{e_i\}$. Furthermore, $\partial \circ F_* = F_* \circ \partial$ ([Har12a], Theorems 8.5.1 and Proposition 8.5.6).

**Theorem 1.9.1** (Stokes’ Theorem, [Har12a] Theorems 8.5.1, 8.5.2, 8.5.4). The bigraded operator boundary $\partial : \hat{B}_k^r(U) \to \hat{B}_{k-1}^{r+1}(U)$ is continuous with $\partial \circ \partial = 0$, and $\|\partial J\|_{B^{r+1}} \leq k\|J\|_{B^r}$ for all $J \in \hat{B}_k^r$ and $r \geq 1$. Furthermore, if $\omega \in \mathcal{B}_{k-1}^{r+1}(U)$ and $J \in \hat{B}_k^r(U)$, then

$$\int_{\partial J} \omega = \int_J d\omega.$$  

**Lemma 1.9.2.** If $J \in \hat{B}_k^r(U)$, then $\text{supp}(\partial J) \subset \text{supp}(J)$.

**Proof.** Suppose $p \in \text{supp}(\partial J)$ and $p \notin \text{supp}(J)$. For sufficiently small $r > 0$, $\Omega(p, r) \cap \text{supp}(J) = \emptyset$. Let $\eta$ be a smooth $k$-form supported in $\Omega(p, r)$ and with $\int_{\partial J} \eta \neq 0$. Then $\int_J \eta \neq 0$. But $d\omega$ is also supported in $\Omega(p, r)$, yielding a contradiction. $\square$

**Lemma 1.9.3.** Suppose $J \in \hat{B}_n^r(U)$, $p \in \text{supp}(J) \cap U$, and $q \notin \text{supp}(J)$. If $\alpha : [0,1] \to \mathbb{R}^n$ is any smoothly embedded path connecting $p$ and $q$, then $\alpha([0,1]) \cap \text{supp}(\partial J) \neq \emptyset$.

**Proof.** Suppose there exists such a path $\alpha$, but $\alpha([0,1]) \cap \text{supp}(\partial J) = \emptyset$. Let $T$ be a tubular neighborhood of $\alpha([0,1])$ disjoint from $\text{supp}(\partial J)$. Let $\epsilon > 0$ such that $\Omega(p, \epsilon) \subset T \cap U$, $\Omega(q, \epsilon) \subset T$ and $\Omega(q, \epsilon) \cap \text{supp}(J) = \emptyset$. Since $p \in \text{supp}(J)$, there exists $\eta \in \mathcal{B}_n^r(U)$ supported in $\Omega(p, \epsilon)$ with $\int_T \eta \neq 0$. Since $\Omega(p, \epsilon) \subset U$, we may extend $\eta$ by $0$ to all of $\mathbb{R}^n$. By Theorem 5.0.3 in [Har12a], we can assume that $\eta$ is smooth. Let $\alpha$ be a smooth $n$-form supported in $\Omega(q, \epsilon)$ such that $\int_T \alpha = \int_T \eta$.

Then $\int_T \eta - \alpha = 0$, and so by compactly supported de Rham theory, $\eta - \alpha = d\omega$ for some smooth $(n-1)$-form $\omega$ supported in $T$. But $\int_T \alpha = 0$, and so $\int_T d\omega \neq 0$, yielding a contradiction with the assumption that $T$ is disjoint from $\text{supp}(\partial J)$.

**Corollary 1.9.4.** Suppose $J \in \hat{B}_n^r(U)$ satisfies $\text{supp}(J) \subset U$ and $\text{supp}(J)$ has empty interior. Then $\text{supp}(\partial J) = \text{supp}(J)$.

**Proof.** We know that $\text{supp}(\partial J) \subset \text{supp}(J)$ by Lemma 1.9.2. Let $p \in \text{supp}(J)$. By hypothesis, every $\Omega(p, \epsilon) \subset U$ contains a point $q \notin \text{supp}(J)$, and hence a point $q' \in \text{supp}(\partial J)$ by 1.9.3.

**Corollary 1.9.5.** If $\tilde{U} \in \hat{B}_n^1(U)$ is as in Proposition 1.5.11 (or more generally, if we consider $\tilde{U}$ as the representative of $U$ in $\mathcal{B}_n^1(W)$ for any $U \subseteq W$) then $\text{supp}(\partial \tilde{U}) = \text{fr} U$.

1.10. **Prederivative.** The topological dual to Lie derivative $\mathcal{L}_X$ of differential forms $\mathcal{B}_n^r(U)$ restricts to a continuous operator $P_X$ on differential chains $\hat{B}_k^r(U)$.

**Definition 1.10.1.** Suppose $X \in \mathcal{V}(U)$. Define the continuous graded linear operator **prederivative** $P_X : \hat{B}_k^r(U) \to \hat{B}_{k+1}^{r+1}(U)$ by

$$P_X := E_X \partial + \partial E_X.$$  

---

$^9$This can be made more general. Namely, it is not necessary for some open sets $U$ to assume that $p \in U$.  

This agrees with the previous definition of $P_v$ for $v \in \mathbb{R}^n$ in the first paragraph of §1.9 since

$$E_v \partial + \partial E_v = \sum_i P_{e_i}(E_v E_{e_i}^\dagger - E_{e_i}^\dagger E_v) = \sum_i P_{e_i}(v, e_i)I = P_v.$$ 

Its dual operator is Lie derivative $\mathcal{L}_X$ by Cartan’s formula, and $\mathcal{L}_X : \mathcal{B}_k^r(U) \to \mathcal{B}_k^{r-1}(U)$ is continuous. Furthermore, $P_X \partial = \partial P_X$.

**Proposition 1.10.2** ([Har12a], Theorem 8.6.2). If $1 \leq r < \infty$ and $X \in \mathcal{V}^r(U)$, then

$$\|P_X(J)\|_{B^{r+1}} \leq 2kn^32^r\|X\|_{B^r}\|J\|_{B^r}$$

for all $J \in \mathcal{B}_k^r(U)$.

**Proposition 1.10.3** ([Har12a], Theorem 8.6.5). If $r \geq 1$, $X \in \mathcal{V}^{r+1}(U)$ and $J \in \mathcal{B}_k^r(U)$, $r \geq 0$ has compact support, then

$$P_XJ = \lim_{t \to 0}(\phi_t J/t - J/t)$$

where $\phi_t$ is the time-$t$ map of the flow of $X$.

The next lemma is proved similarly to Lemma 1.9.2:

**Lemma 1.10.4.** The supports of $E_Y J$ and $P_Y J$ satisfy $\text{supp}(E_Y J) \subset \text{supp}(J)$ and $\text{supp}(P_Y J) \subset \text{supp}(J)$.

1.11. **Cone operator.** Let $U \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^m$ be open sets, and for $q \in W$ let $\iota^q : U \to U \times W$ the map $p \mapsto (p, q)$. Similarly define $\iota^p : W \to U \times W$ for $p \in U$. If $P = \sum_i (p_i; \alpha_i) \in A_k(U)$ and $Q = \sum_j (q_j; \beta_j) \in A_l(W)$ define the Cartesian wedge product $\times : A_k(U) \times A_l(W) \to A_{k+l}(U \times W)$ by $P \times Q := \sum_{i,j} ((p_i, q_j); \iota^q_{i*} \alpha_i \wedge \iota^p_{j*} \beta_j)$. A calculation shows $\|P \times Q\|_{B^{r+s}} \leq \|P\|_{B^r} \|Q\|_{B^s}$, and so $\times$ extends to a jointly continuous bilinear map $\hat{\times} : \mathcal{B}_k^r(U) \times \mathcal{B}_l^s(W) \to \mathcal{B}_{k+l}^{r+s}(U \times W)$. We can improve on the order $r+s$ in some cases. In particular, if $(a, b) \in \mathcal{B}_k^1((a, b))$ represents the interval $(a, b) \subset \mathbb{R}$, and $J \in \mathcal{B}_k^r(U)$, then $(a, b) \hat{\times} J \in \mathcal{B}_k^{r+s}((a, b) \times U)$, and as Proposition 11.1.9 in [Har12a] shows,

$$
\text{(5)} \quad \|((a, b) \hat{\times} J)\|_{B^r} \leq (b - a)\|J\|_{B^r}.
$$

**Definition 1.11.1.** Suppose $F : [0, 1] \times U \to U$ and $q \in U$ satisfy $F(0, p) = p$ and $F(1, p) = q$ for all $p \in U$. Suppose further that $F|_{[0,1] \times U} \in \mathcal{M}^r((0, 1) \times U, U)$. Then define the cone operator $\kappa_F : \mathcal{B}_k^r(U) \to \mathcal{B}_k^{r+s}(U)$ by $\kappa_F(J) := F_*(0, 1) J$.

According to Proposition 1.5.3 and (5), $\kappa_F$ is continuous, with

$$\|\kappa_F(J)\|_{B^r} \leq (n + 1)2^r \max\{1, \rho_r(F)\}\|J\|_{B^r}.$$ 

Moreover, by Theorem 5.1.1 of [Har12b] we have

$$\kappa_F \partial + \partial \kappa_F = \text{Id}.$$
2. Constructions with embedded submanifolds

Let \( M \subset \mathbb{R}^n \) be a compact oriented, embedded \((n-2)\)-dimensional submanifold, and let \( U \supset M \) be convex, bounded and open. Let \( Y \) be a vector field on \( U \) supported in a neighborhood of \( M \) such that \( Y(p) \) is unit and normal to \( M \) for all \( p \in M \). Such a vector field exists since \( M \) necessarily has trivial normal bundle (Sec., e.g., [Mas59].) Let \( F: [0,1] \times U \to U \) be a smooth null-homotopy to \( q \in U \) satisfying \( F'_p(0) = Y(p) \) for all \( p \in M \). Let \( \kappa_Y := \kappa_F \) (See Definition 1.11.1.)

**Lemma 2.0.1.** The supports of \( P_Y \tilde{M} \in \dot{B}^2_{n-2}(U) \) and \( E_Y \tilde{M} \in \dot{B}^1_{n-1}(U) \) satisfy \( \text{supp}(P_Y \tilde{M}) = \text{supp}(E_Y \tilde{M}) = \text{supp}(\tilde{M}) = M \).

*Proof.* Since \( \tilde{M} \) represents \( M \), it follows from Proposition 1.5.11 that \( M = \text{supp}(\tilde{M}) \). Furthermore, \( \text{supp}(P_Y \tilde{M}) \subseteq \text{supp}(\tilde{M}) \) by Lemma 1.10.4. Conversely, let \( p \in M \), and let \( r > 0 \). Since \( p \in \text{supp}(\tilde{M}) \), there exists \( \omega \in \dot{B}^2_{n-2}(U) \) with \( \text{supp}(\omega) \subset \Omega(p,r) \) and \( \int_{\tilde{M}} \omega \neq 0 \). Since the vector field \( Y \) is normal to \( M \), a computation in coordinates \((x_1, \ldots, x_n)\) on \( \Omega(p,r) \) such that \( M \cap \Omega(p,r) \) can be written as the set \( x_{n-1} = x_n = 0 \) (if necessary, shrink \( r \)) and such that \( Y(x_1, \ldots, x_{n-2}, 0, 0) = \frac{\partial}{\partial x_{n-1}} \) shows that there exists \( \eta \in \dot{B}^2_{n-2}(U) \) supported in \( \Omega(p,r) \) such that \( \iota_Y \eta \gamma(p', v) = \omega(p', v) \) for any \( p' \in M \). In particular, this shows by Stokes’ Theorem that \( 0 = \int_{\tilde{M}} \omega = \int_{\tilde{M}} \iota_Y \eta = \int_{P_Y \tilde{M}} \eta \), and so \( p \in \text{supp}(P_Y \tilde{M}) \). A similar computation shows \( \text{supp}(E_Y \tilde{M}) = \text{supp}(\tilde{M}) \). \( \square \)

**Lemma 2.0.2.** If \( \partial S = P_Y \tilde{M} \), then \( S = \partial \kappa_Y S + E_Y \tilde{M} \).

*Proof.* It follows from the continuity of \( \kappa_Y \) and \( E_Y \) and since \( E^2_Y = 0 \) that \( \kappa_Y E_Y \tilde{M} = 0 \) (write \( \tilde{M} \) as a limit of Dirac chains \( A_i \) supported on \( M \) and compute directly \( \kappa_Y E_Y A_i = 0 \).) The result then follows from the relations \( P_Y = E_Y \partial + \partial E_Y \) (Definition (1.10.1)), and \( \kappa_Y \partial + \partial \kappa_Y = \text{Id} \) (Equation (6)). \( \square \)

**Corollary 2.0.3.** If \( \partial S = P_Y \tilde{M} \), then \( M \subseteq \text{supp}(S) \) and \( \text{supp}(\partial \kappa_Y S) \subseteq \text{supp}(S) \).

*Proof.* The first inclusion follows from Lemmas 1.9.2 and 2.0.1. The second inclusion follows from Lemma 2.0.2 and the inclusion \( \text{supp}(A + B) \subseteq \text{supp}(A) \cup \text{supp}(B) \). \( \square \)

**Corollary 2.0.4.** Suppose \( S \) satisfies \( \partial S = P_Y \tilde{M} \), \( \text{supp}(\kappa_Y S) \subseteq U \) has empty interior, and \( M \subseteq \text{supp}(\kappa_Y S) \setminus \tilde{M} \). Then \( \text{supp}(S) = \text{supp}(\kappa_Y S) \).

*Proof.* We know \( \text{supp}(\partial \kappa_Y S) = \text{supp}(\kappa_Y S) \) by Corollary 1.9.4 and \( \text{supp}(\partial \kappa_Y S) \subseteq \text{supp}(S) \) by Corollary 2.0.3.

Suppose \( p \in \text{supp}(S) \setminus M \) and \( p \notin \text{supp}(S - E_Y \tilde{M}) \). Then for each \( \epsilon > 0 \) such that \( \Omega(p, \epsilon) \cap \text{supp}(S - E_Y \tilde{M}) = \emptyset \), there exists a \((n-1)\)-form \( \eta \) supported in \( \Omega(p, \epsilon) \) such that \( \int_{S} \eta \neq 0 \). But \( \int_{S - E_Y \tilde{M}} \eta = 0 \) implies \( \int_{E_Y \tilde{M}} \eta \neq 0 \). This shows that \( p \in \text{supp}(E_Y \tilde{M}) = M \) by Lemma 2.0.1, a contradiction.
Now suppose \( p \in M \). By assumption, there exist \( p_i \in \text{supp}(S) \setminus M \) with \( p_i \to p \). By the preceding argument, it follows that \( p_i \in \text{supp}(S - E_Y M) \). Since this set is closed, we have \( p \in \text{supp}(S - E_Y M) \). \( \square \)

3. Borel measures and positive chains

In this section we describe a particularly well-behaved subclass differential chains called “positive chains.” We will use these later in Definition 6.0.2.

3.1. The convex cone of positive chains.

Definition 3.1.1. Define the \textit{mass} of \( J \in \hat{B}_k^1(U) \) to be the (possibly infinite) quantity

\[
M(J) := \inf \{ \liminf \| A_i \|_{B^0} : A_i \to J \text{ in } B^1, A_i \in A_k(U) \}.
\]

Proposition 3.1.2. If \( J \in \hat{B}_k^1(U) \), then \( M(J) = \sup \{ \int_J \omega : \omega \in B_k^1(U), \| \omega \|_{B^0} \leq 1 \} \).

Proof. See [Whi57], V, §16, Theorem 16A. \( \square \)

Corollary 3.1.3. If \( J \in \hat{B}_n^0(U) \), then \( M(J) = \| J \|_{B^0} \).

Definition 3.1.4. A Dirac \( n \)-chain \( A = \sum (p_i; \alpha_i) \) is \textit{positive} if each \( n \)-vector \( \alpha_i \) is positively oriented with respect to the standard orientation of \( \mathbb{R}^n \). We say \( J \in \hat{B}_n^r(U) \) is \textit{positive} if \( J \) is a limit of positive Dirac \( n \)-chains \( A_i \to J \).

Lemma 3.1.5. If \( r \geq 0 \) and \( J \in \hat{B}_n^r(U) \) is positive, then \( \| J \|_{B^r} = \int_J dV \). If \( r = 1 \), then \( M(J) = \int_J dV \).

Proof. Let \( A \) be a positive Dirac \( n \)-chain. By Proposition 1.3.7 and Corollary 3.1.3,

\[
\| A \|_{B^r} = \sup_{\| \omega \|_{B^r} = 1} \int_A \omega \leq \int_A dV \leq \| A \|_{B^r}.
\]

Since \( J \) is positive there exists a sequence positive Dirac \( n \)-chains \( \{ A_i \} \) with \( J = \lim_{i \to \infty} A_i \). Thus,

(7) \[
\int_J dV = \lim_{i \to \infty} \int_{A_i} dV = \lim_{i \to \infty} \| A_i \|_{B^r} = \| J \|_{B^r}.
\]

For the second part let \( r = 1 \). By Definition 3.1.1, Proposition 3.1.2, and Corollary 3.1.3 we have

\[
M(J) \leq \liminf \| A_i \|_{B^0} = \liminf \int_{A_i} dV = \int_J dV \leq M(J). \]

(8) \[ M(J) < \infty. \]

In particular, if \( J \in \hat{B}_n^1(U) \) is positive, then \( M(J) < \infty \).

Corollary 3.1.6. If \( U \) is bounded and \( J \in \hat{B}_n^r(U) \) is positive, then \( J = u_1, r(J') \) where \( J' \in \hat{B}_n^1(U) \) is positive.
Proposition 3.2.2. Let $A_i \to J$ be a sequence of positive Dirac chains converging to $J$ in the $B^r$ norm. By Lemma 3.1.5, $\|A_i\|_{B^r} = \|A_i\|_{p=0} = \int J_i dV \to \|J\|_{B^r}$. By Theorem 1.3.5, there exists a subsequence $A_{i_j} \to J'$ in $\widehat{B}_1^1(U)$. But $u_{1,r}^1(A) = A$ for any Dirac chain $A$, hence $u_{1,r}^1(J') = J$ by continuity of $u_{1,r}^1$. □

This lemma shows that all the positive chains lie in $\widehat{B}_1^1(U)$. A similar argument shows:

Corollary 3.1.7. If $U$ is bounded and if $J_i \to J$ in $\widehat{B}_1^1(U)$ where $J_i$ are positive, then $J$ is positive, and writing $J_i = u_{1,r}^1 J_i'$, $J = u_{1,r}^1 J'$ as in Corollary 3.1.6, there is a subsequence $J_{i_j} \to J'$ in $\widehat{B}_1^1(U)$.

3.2. An isomorphism of convex cones. The next result is a special case of Theorem 11A of ([Whi57], XI) which produces an isomorphism between sharp $k$-chains with finite mass and $k$-vector valued countably additive set functions (see [Whi57], XI, §2). We specialize to positive chains and Borel measures in the case $k = n$.

Let $\mathcal{C}(U)$ be the convex cone consisting of positive elements $J$ of $\widehat{B}_1^1(U)$ such that $\text{supp}(J) \subset U$. Let $\mathcal{M}(U)$ be the convex cone consisting of all finite Borel measures $\mu$ on $U$ such that $\text{supp}(\mu) \subset U \subset \mathbb{R}^n$ is closed as a subset of $\mathbb{R}^n$. Note that such measures $\mu$ are automatically Radon measures (Theorem 7.8 in [Fol99].)

Proposition 3.2.1. There exists a bijection $\Psi : \mathcal{C}(U) \to \mathcal{M}(U)$ with $J \mapsto \mu_J$ satisfying

(a) $\mu_J(U) = \|J\|_{B^1} = \int J dV = M(J)$ for all $J \in \mathcal{C}(U)$;
(b) $\mu_{aJ+bK} = a\mu_J + b\mu_K$ for all non-negative $a, b \in \mathbb{R}$ and $J, K \in \mathcal{C}(U)$;
(c) $\mu_A(X) = M(A\lfloor_X)$ for all $A \in \mathcal{A}_n(U)$ and Borel sets $X \subset U$;
(d) $\int_U f d\mu_J = \int J f dV$ for all $f \in B_0^1(U)$ and $J \in \mathcal{C}(U)$;
(e) $\text{supp}(J) = \text{supp}(\mu_J)$ for all $J \in \mathcal{C}(U)$.

Proposition 3.2.2. Let $J_i \to J \in \mathcal{C}(U)$. Then

(a) $\mu_{J_i}$ converges weakly to $\mu_J$;
(b) If $F$ is closed and $G$ is open, then $\mu_{J_i}(F) \geq \limsup \mu_{J_i}(F)$ and $\mu_{J_i}(G) \leq \liminf \mu_{J_i}(G)$;
(c) If $E$ is a Borel set with $\mu_J(\text{fr } E) = 0$, then $\mu_J(E) = \lim \mu_{J_i}(E)$.

Proof. It follows from Proposition 3.2.1 (d) that $\int_U f d\mu_{J_i} \to \int_U f d\mu_J$ for all $f \in B_0^1(U)$. Thus, the Portmanteau theorem\(^{10}\) gives (a)-(c). □

Definition 3.2.3. Let $J \in \mathcal{C}(U)$. A Borel set $B \subset U$ is $J$-compatible if $B$ is a continuity set for $\mu_J$, i.e. if $\mu_J(\text{fr } B) = 0$.

Since $\mu_J$ is finite, “most” cubes and balls are $J$-compatible.

If $f$ is a bounded positive Borel function and $J \in \mathcal{C}(U)$ let $f : J := \Psi^{-1}(f \mu_J) \in \mathcal{C}(U)$. Then $\|f : J\|_{B^1} = \int_U f d\mu_J$ by Proposition 3.2.1 (a). If $f \in B_0^1(U)$, then $f : J = m_f J$ where $m_f$ is

\(^{10}\)See, for example, [Mag12], Proposition 4.26.
the continuous operator “multiplication by a function” ([Whi57], VII, §1 (4), XI §12 (3), [Har12a] Theorem 6.1.3.)

**Definition 3.2.4.** Let \( J \in \mathcal{C}(U) \) and \( B \subset U \) a Borel set. The **part of \( J \) in \( B \)** is defined by 
\[
J|_B := \chi_B \cdot J.
\]

4. Basic results of Hausdorff and Hausdorff spherical measures

Most of the results in the section are due to Besicovitch [Bes48, Bes49] and appear the first sequence of lemmas in [Rei60]. Let \( \text{diam}(X) \) denote the **diameter** of \( X \subset \mathbb{R}^n \) and \( \alpha_m \) denote the Lebesgue measure of the unit \( m \)-ball in \( \mathbb{R}^m \). For any \( X \subset \mathbb{R}^n \), \( 0 \leq m \leq n \) and \( \delta > 0 \), define
\[
H^m_\delta(X) := \inf \left\{ \sum_{i=1}^{\infty} \alpha_m \left( \frac{\text{diam}(Y_i)}{2} \right)^m : X \subset \bigcup_{i=1}^{\infty} Y_i, Y_i \subset \mathbb{R}^n, \text{diam}(Y_i) < \delta \right\}.
\]

Then \( H^m(X) := \lim_{\delta \to 0} H^m_\delta(X) \) is the (normalized) Hausdorff \( m \)-**measure** of \( X \). Closely related to Hausdorff measure is (normalized) Hausdorff spherical measure \( S^m(X) \). This is defined in the same way, except that the sets \( Y_i \) are required to be \( n \)-balls. In particular, \( H^m(X) \leq S^m(X) \leq 2^m H^m(X) \). Both are metric outer measures, and so the Borel sets are measurable.

For \( m \)-rectifiable sets (see [Mat99] Definition 15.3), \( H^m \) and \( S^m \) are the same (Theorem 3.2.26 of [Fed69]). Most results which hold for one of these measures also holds for the other. There are two notable exceptions, giving Hausdorff spherical measure the edge for the calculus of variations in situations where it might be undesirable to assume that sets being considered are rectifiable a priori (See Lemmas 4.0.3 and 4.0.4, as well as Lemma 4.0.5.)

**Lemma 4.0.1** (Vitali Covering Theorem for Hausdorff Measure). Suppose \( H^m(X) < \infty \) and \( \mathcal{B} \) is a collection of closed \( n \)-balls such that for each \( x \in X \) and \( \epsilon > 0 \) there exists a ball \( \bar{\Omega}(x, \delta) \in \mathcal{B} \) with \( \delta < \epsilon \). Then there are disjoint balls \( B_i \in \mathcal{B} \) such that \( H^m(X \setminus \bigcup_i B_i) = 0 \).

**Proof.** Apply Theorem 2.8 of [Mat99] to the measure \( \mu = H^m|_X \), which is a finite Borel measure, hence a Radon measure by Theorem 7.8 of [Fol99]. \( \square \)

Since \( H^m \) and \( S^m \) are comparable, one can replace \( H^m \) with \( S^m \) in Lemma 4.0.1.

**Lemma 4.0.2.** Suppose \( X \subset \Omega(X, \epsilon) \subset Y \subset \mathbb{R}^m \) for some \( \epsilon > 0 \), and \( f : Y \to Y \) is Lipschitz with Lipschitz constant \( \lambda \). Then \( \lambda^m \leq \frac{\text{diam}(f(Y))}{\text{diam}(Y)} \leq \lambda^m \text{diam}(X) \) and \( \lambda^m \leq \frac{\text{diam}(f(X))}{\text{diam}(X)} \leq \lambda^m \text{diam}(X) \).

**Lemma 4.0.3.** Suppose \( S^m(X) < \infty \). For \( S^m \) almost every \( p \in X \),
\[
\lim_{r \downarrow 0} \sup_{a_m r^m} \frac{S^m(X \cap \bar{\Omega}(p, r))}{a_m r^m} = 1.
\]

See [Mat99] Theorem 6.6.

Contrast this with the corresponding result for Hausdorff measure:
Lemma 4.0.4. Suppose $\mathcal{H}^m(X) < \infty$. For $\mathcal{H}^m$ almost every $p \in X$,

$$2^{-m} \leq \limsup_{r \downarrow 0} \frac{\mathcal{H}^m(X \cap \bar{\Omega}(p,r))}{\alpha_m r^m} \leq 1.$$ 

See [Mat99] Theorem 6.2.

Lemma 4.0.5. Let $X_h$ be the set of points of $X$ at a distance $h$ from a fixed $M$-plane in $\mathbb{R}^n$, $0 \leq M < n$. Then

$$\int_0^\infty S^{m-1}(X_h) dh \leq S^m(X).$$

This is Lemma 4 in [Rei60]. The inequality does not hold if we replace Hausdorff spherical measure with Hausdorff measure. A counterexample and discussion can be found in [War64]. Yet the sharp inequality, as stated, is important in our proof of regularity. Our proof of regularity uses Hausdorff spherical measure because of this.

Lemma 4.0.6. If $X \subset \bar{\Omega}(p,r)$, then the cone $C_p(X)$ satisfies

$$S^m(C_p(X)) \leq r^{2m} \frac{\alpha_m}{\alpha_{m-1}} S^{m-1}(X).$$

If $X$ is an $(m-1)$-dimensional polyhedron, then

$$\mathcal{H}^m(C_p(X)) \leq \frac{r}{m} \mathcal{H}^{m-1}(X).$$

See Lemmas 5 and 6 of [Rei60]. More generally:

Lemma 4.0.7. Suppose $\Pi$ is a $k$-dimensional plane, $0 \leq k < n$, and $X \subset \mathbb{R}^n$ a compact set. Let $C = \cup_{x \in X} I_x$ where $I_x$ is the line connecting $x$ to its projection on $\Pi$. If $X \subset \Omega(\Pi,r)$, then

$$S^m(C) \leq 2^m \frac{\alpha_m}{\alpha_{m-1}} r S^{m-1}(X).$$

See Lemma 6 of [Rei60].

5. Spanning Sets

Please refer to the introduction for our definition of “span.” Here we give some examples and prove some results needed to work with this definition. In this section, fix $M$ a compact oriented, embedded $m-1$-dimensional submanifold. Unless otherwise noted, we assume $1 \leq m \leq n$.

Proposition 5.0.1. The cone $C_q(M)$ spans $M$ for all $q \in \mathbb{R}^n$.

Proof. If not, there is a simple link $N$ of $M$ such that $N \cap C_q(M) = \emptyset$. Let $M_i$ be the connected component of $M$ with $L(M_i, N) \neq 0$, and recall $L(M_i, N)$ is the degree of the map

$$f : M_i \times N \to S^{n-1},$$

$$(s, r) \mapsto \frac{s - r}{\|s - r\|}.$$
Since $C_q(M)$ is disjoint from $N$, the map $f$ factors as

$$f = g \circ h : M_i \times N \xrightarrow{h} C_q(M) \times N \xrightarrow{g} S^{n-1}$$

where $h(s, r) = Id$ and $g(p, r) = (p - r)/\|p - r\|$. So, it suffices to show that $h_*$ is trivial on $H_{m-1}(M_i \times N; \mathbb{Z}) \simeq H_{m-1}M_i \otimes H_{n-m}N$. Write $h = \iota \times Id$, where $\iota : M_i \to C_q(M)$ is given by $\iota(s) = s$. Since $\iota_* : H_{m-1}(M_i; \mathbb{Z}) \to H_{m-1}C_q(M); \mathbb{Z})$ is itself trivial, it follows from naturality of the Künneth formula that $h_* = 0$. \hfill \Box

In the above lemma, the only property of $C_q(M)$ needed is that the map on homology $i_* : H_{m-1}(M; \mathbb{Z}) \to H_{m-1}(C_q(M); \mathbb{Z})$ induced by the inclusion $i : M \hookrightarrow C_q(M)$ is trivial. So, any other compact set $X$ containing $M$ with that condition on homology will also span $M$. Note that $M \subset X$ if $X$ spans $M$. We can say slightly more, that $M \subset X \setminus M$.

Now suppose $M$ is a Jordan curve in $\mathbb{R}^3$. A compact set $X$ spans $M$ in the sense of Reifenberg if $X$ contains $M$ and there is no retraction of $X$ onto $M$. When $X$ is sufficiently “nice,” spanning in the sense of Reifenberg implies spanning in our sense. The following proposition is due to M.W. Hirsch, D. Sullivan and one of us (H.P.):

**Proposition 5.0.2.** Suppose $X \subset \mathbb{R}^3$ is a compact CW complex, $\mathcal{H}^2(X) < \infty$ and $M$ is the image of a homeomorphism $\gamma$ of $S^1$ onto a subcomplex of $X$. If $M$ is not a retract of $X$, then $X$ spans $M$.

**Proof.** Suppose $g : X \to S^1$ is a continuous map such that $g|_M$ has degree 1. Then the composition $\gamma \circ g : X \to M$ restricts to a degree-one map $M \to M$, and this map is homotopic to the identity.
Since \((X,M)\) has the homotopy extension property, it follows that \(\gamma \circ g\) is homotopic to a map \(f : X \to M\) whose restriction to \(M\) is the identity, so \(f\) is a retraction.

Thus, if \(g : X \to S^1\) is continuous, then \(g|_M\) does not have degree 1.

Since \(X\) is a CW complex, the universal cover of \(S^1\) is contractible and \(\pi_1S^1\) is abelian, it follows from obstruction theory that every homomorphism \(G : H_1(X,Z) \to H_1(S^1,Z)\) is \(g_*\) for some continuous map \(g : X \to S^1\). Therefore, identifying \(H_1(S^1,Z)\) with \(Z\) by the natural orientation of \(S^1\), we have the following:

If \(G : H_1(X,Z) \to Z\) is a homomorphism, then \(G([M]) \neq 1\).

That \(X\) spans \(M\) follows from the Universal Coefficient Theorem and Alexander duality. \(\square\)

The next lemma is not used to prove our main theorem, but we include it here anyway.

**Lemma 5.0.3.** Let \(m = n - 1\). If \(X = Z \cup L\) is compact and spans \(M\) where \(Z\) is closed and \(L\) is purely \(m\)-unrectifiable, then \(Z\) also spans \(M\).

**Proof.** Suppose \(N\) is a simple link of \(M\) and \(N \cap Z = \emptyset\). Choose a tubular neighborhood \(T\) of \(N\) with \(Z \cap T = \emptyset\). Parameterize \(N\) by \(\Theta : [0, 2\pi] \to S^1\), where \(\Theta(x)\) gives the \(\theta\)-coordinate of \(x \in N \simeq S^1 \subset \mathbb{R}^2\) in polar coordinates, where \(S^1\) is the unit circle in \(\mathbb{R}^2\). The pullback bundle of the tubular neighborhood via \(\Theta\) is isomorphic to a cylinder \(C\). Let \(\hat{\Theta} : C \to N\) denote the resulting map, which is a diffeomorphism except at the endpoints, where the map is 2-1. Let \(H = \hat{\theta}^{-1}(L \cap T)\). Since \(\theta\) is Lipschitz, it follows that \(H\) is also purely \(m\)-unrectifiable, and so by the Besicovitch-Federer projection theorem ( [Mat99] 18.1,) almost every orthogonal projection \(\pi : H \to K\) where \(K\) is an \(m\)-plane sends \(H\) to a set with Lebesgue \(m\)-measure zero. Thus, there exists an arc \(\rho \subset T\) which is disjoint from \(L \cap T\), whose endpoints \(p,q\) lie in the same fiber of \(T\) and which is given as a smooth section of the normal bundle to \(N\), except at the endpoints, where the section is discontinuous.

Let \(\sigma \subset D\) be the line segment joining \(p\) and \(q\). Since \(X\) is closed and \(\rho \cap X = \emptyset\), there exists \(\epsilon > 0\) such that \(\Omega(p,\epsilon) \cup \Omega(q,\epsilon) \subset T\) is disjoint from \(L\). Let \(T' \subset T\) be the \(\epsilon\)-neighborhood of \(\sigma\), and apply the projection theorem again to find a line segment \(\sigma'\) disjoint from \(L \cap T\) with one endpoint in \(\Omega(p,\epsilon)\) and the other in \(\Omega(q,\epsilon)\). Finally connect the appropriate endpoints of \(\rho\) and \(\sigma'\) inside \(\Omega(p,\epsilon)\) and \(\Omega(q,\epsilon)\) to create a simple link of \(M\) which is disjoint from \(X\). \(\square\)

**Definition 5.0.4.** Suppose \(X\) is compact and spans \(M\). A **competitor of \(X\) with respect to \(M\)** is a set \(\phi(X)\) where \(\phi : \mathbb{R}^n \to \mathbb{R}^n\) is a Lipschitz map that is the identity on \(B^c\) and \(\phi(B) \subset B\), where \(B \subset \mathbb{R}^n\) is some closed ball disjoint from \(M\). The map \(\phi\) is called a **deformation of \(X\)**.

**Lemma 5.0.5.** If \(m = n - 1\) and \(X \subset \mathbb{R}^n\) is compact and spans \(M\), and \(\phi(X)\) is a competitor of \(X\), then \(\phi(X)\) is compact and spans \(M\).

**Proof.** If not, then there is a simple link \(\eta\) of \(M\) such that \(\eta(S^1) \cap \phi(X) = \emptyset\). Let \(B = B_r\) be a closed ball of radius \(r\) satisfying \(M \cap B = \emptyset\), \(\phi \equiv \text{Id}\) on \(B^c\) and \(\phi(B) \subset B\). Expanding \(B\) slightly, we may also assume \(\phi \equiv \text{Id}\) on an \(\epsilon\)-neighborhood of \(frB\) and that \(\phi(B_{r-\epsilon}) \subseteq B_{r-\epsilon}\). We can assume without loss of generality that \(\phi\) is smooth: Approximate \(\phi\) uniformly to within \(\min\{\epsilon/2, d(\eta(S^1), \phi(X))/2\}\)
by a smooth function $\tilde{\phi}$. Let $\{f,g\}$ be a partition of unity subordinate to $\{\hat{B}_{r-\epsilon/2}, B_{r-\epsilon}\}$, and define $\hat{\phi} = f \cdot \tilde{\phi} + g \cdot \text{Id}$. Then $\hat{\phi}$ is smooth, $\hat{\phi}(X) \cap \eta(S^1) = \emptyset$, $\hat{\phi} \equiv \text{Id}$ on $B^c$ and $\hat{\phi}(B) \subseteq B$. Finally, expand $B$ slightly again and we may assume $\hat{\phi} \equiv \text{Id}$ on an $\epsilon'$-neighborhood of $frB$ and that $\hat{\phi}(B_{r-\epsilon'}) \subseteq B_{r-\epsilon'}$.

By compactness, there exists a tubular neighborhood $T$ of $\eta(S^1)$ so that $T \cap \hat{\phi}(X) = \emptyset$. Since we may perturb $\eta$ within $T$, let us assume without loss of generality that the intersection $\eta(S^1) \cap frB$ is transverse. By considering a possibly smaller $\epsilon'$ we may assume slightly more, that within the $\epsilon'$-neighborhood of $frB$, the curve $\eta(S^1)$ consists of radial line segments. So, the set $\eta(S^1) \cap B$ consists of a finite collection of arcs $\{\eta_1, \ldots, \eta_N\}$, each of which intersects $frB$ radially. Note that $T \cap \hat{B}$ consists of pairwise disjoint neighborhoods $T_i$ of the arcs $\eta_i$ (minus their endpoints.) For each $i$, let $\pi_i : T_i \to D$ be the projection onto the normal $m$-disk $D$ determined by the tubular neighborhood $T$ and some trivialization of the normal bundle of $\eta(S^1)$. Consider the smooth maps $\psi_i := \pi_i \circ \hat{\phi} : W_i \to D$ where $W_i := \hat{\phi}^{-1}(T_i)$. For each $i$ fix a regular value $x_i$ of $\psi_i$ and consider the compact sets $Y_i := \pi_i^{-1}(x_i) \cap B_{r-\epsilon'/2}$. Since $\hat{\phi}$ is proper, each preimage $Z_i := \hat{\phi}^{-1}(Y_i)$ is also compact as a subset of $\mathbb{R}^n$.

For each $i$, there exists by compactness of $Z_i$ a smooth compact $n$-manifold with boundary $P_i \subset W_i$, such that $Z_i \subset P_i$, and such that both line segments constituting $\pi_i^{-1}(x_i) \setminus B_{r-\epsilon'}$ meet $\partial P_i$ transversally and at one point each. Now $x_i$ is still a regular value of the restricted map $\psi_i|_{P_i} : P_i \to D$. But also $x_i$ is a regular value of $\psi_i|_{\partial P_i}$, since $\psi_i|_{\partial P_i}(x_i) = (\pi_i^{-1}(x_i) \setminus B_{r-\epsilon'}) \cap \partial P_i$, and this intersection is transverse and contained in a region on which $\hat{\phi} \equiv \text{Id}$. So, by the inverse function theorem for manifolds with boundary ( [Hir76], Theorem 4.1) the set $\psi_i|_{\partial P_i}^{-1}(x_i)$ is a collection of circles and arcs that meet the boundary $\partial P_i$ “neatly.” Since by construction the set $\psi_i|_{\partial P_i}(x_i)$ consists of exactly two points, there is exactly one arc $\beta_i$ in $\psi_i^{-1}(x_i)$, and it joins the two points. Moreover, the arcs $\beta_i$ are pairwise disjoint, since any point in an intersection of two must get mapped by $\hat{\phi}$ into disjoint tubular neighborhoods.

Furthermore, the arcs $\beta_i$ terminate inside the tubular neighborhood $T$ of $\eta$. So, we may smoothly extend each $\beta_i$ within $T$, linking the arcs together to form an embedding $\beta : S^1 \to M^c$. The curve $\beta$ is also a simple link of $M$, since $\beta$ is by construction regularly homotopic to $\eta$. Thus, $\beta(S^1) \cap X \neq \emptyset$, and so $\hat{\phi}(\beta(S^1)) \cap \hat{\phi}(X) \neq \emptyset$. But $\hat{\phi}(\beta(S^1)) \subset T$, yielding a contradiction. \Box

The core of $X \subset \mathbb{R}^n$ is defined by

$$X^* := \{ p \in X | \forall \epsilon(\epsilon(X \cap \Omega(p,r)) > 0 \text{ for all } r > 0\}.$$ 

The core $X^*$ of a closed set $X$ is closed, and $\mathcal{K}^m(X \setminus X^*) = 0$. The definition of core and its properties are unaltered if $\mathcal{O}^{m-1}$ is replaced by $\mathcal{O}^{n-1}$. We say $X$ is reduced if $X = X^*$.

Lemma 5.0.6. Let $m = n - 1$. If $X \subset \mathbb{R}^n$ is compact and spans $M$, then the core $X^*$ is compact and spans $M$.

Proof. If not, there is a simple link $N$ of $M$ and a tubular $\epsilon$-neighborhood $T$ of $N$ whose closure is disjoint from $X^*$. So, $\mathcal{K}^m(X \setminus T) = 0$. Let $\pi : T \to D$ be the projection onto the normal $m$-disk $D$ determined by the tubular neighborhood $T$ and some trivialization of the normal bundle of $N$. 

Then $\mathcal{H}^n(\pi(X \cap T)) = 0$, and so there exists a point $x \in D \setminus (\pi(X \cap T))$. Then $\pi^{-1}(x)$ is a simple link of $M$ missing $X$, giving a contradiction. $\square$

The next lemma is an easy consequence of the definition of the Hausdorff metric:

**Lemma 5.0.7.** If $X \subset \mathbb{R}^n$ is compact and $X_i \to X$ in the Hausdorff metric, where $X_i \subset \mathbb{R}^n$ is compact and spans $M$ for all $i$, then $X$ spans $M$.

**Lemma 5.0.8.** Let $m = n - 1$. Suppose $X \subset \mathbb{R}^n$ is compact and spans $M$. Let $p \in \mathbb{R}^n$, $r > 0$ and suppose $\Omega(p, r) \cap M = \emptyset$. If $p' \in \Omega(p, r)$, then $X' := (X \cap \Omega(p, r)^c) \cup C_{p'}(X \cap \text{fr} \Omega(p, r))$ is compact and spans $M$.

**Proof.** It is clear that $X'$ is compact. If $X'$ does not span $M$, then let $\eta$ be a simple link of $M$ such that $\eta(S^1) \cap X' = \emptyset$. Then $\eta(S^1) \cap X \subset \Omega(p, r)$. First, we observe that $X \cap \text{fr} \Omega(p, r) \neq \emptyset$. If not, there is by compactness an $\epsilon$-neighborhood $T$ of $\text{fr} \Omega(p, r)$ disjoint from $X$, so we may construct a diffeomorphism of $\Omega(p, r)$ which fixes $\text{fr} \Omega(p, r)$ and sends $\eta$ to a simple link $\eta'$ of $M$ such that $\eta'(S^1) \cap \Omega(p, r) \subset T$. Thus, $\eta'(S^1) \cap X = \emptyset$, yielding a contradiction.

This implies that $C_{p'}(X \cap \text{fr} \Omega(p, r)) \neq \emptyset$, and in particular, $p' \in X'$. Thus, $p' \notin \eta(S^1)$. Let $\delta > 0$ such that $\Omega(p', 2\delta) \cap \eta(S^1) = \Omega(p', 2\delta) \cap M = \emptyset$. Let $\rho : \Omega(p', \delta)^c \to \mathbb{R}^n$ be the identity on $\Omega(p, r)^c$ and elsewhere the radial projection away from $p'$ and onto the frontier of $\Omega(p, r)$. By definition of $C_{p'}$, it is enough to show $\rho(\eta(S^1))$ intersects $X$.

Suppose not. Since $\rho(\eta(S^1))$ is compact, there is an $\epsilon'$-neighborhood $T'$ of $\rho(\eta(S^1))$ such that $T' \cap X = \emptyset$. Let $V$ be a smooth radial vector field on $\mathbb{R}^n$, with center point $p'$, supported in $\Omega(p, r)$, and normalized so that if $\phi$ is the time-1 flow of $V$, then $\phi(\eta(S^1)) \subset T'$. Then $\phi \circ \eta$ is a simple link of $M$ disjoint from $X$, a contradiction. $\square$

### 6. Film chains

Throughout this section, fix $U \subset \mathbb{R}^n$, $M \subset U$, $q \in U$ and $Y$ as in §2.

**Definition 6.0.1.** Let

$$\mathcal{X}(M, U) := \{X \subset \mathbb{R}^n : X \text{ is reduced, compact and spans } M, S^{n-1}(X) < \infty, \text{ and } X \subset U\},$$

and set

$$m := \inf \{S^{n-1}(X) : X \in \mathcal{X}(U)\}.$$

Note that by Lemma 4.0.2, $m$ is independent of our choice of $U$.

**Definition 6.0.2.** A **film chain** is an element $S \in \mathcal{B}_{n-1}^2(U)$ satisfying

(a) $\partial S = P_Y \widetilde{M}$;
(b) $\kappa_Y S \in \mathcal{C}(U)$;
(c) $\mu_{\kappa Y} S = S^{n-1}|_X$ for some $X \in \mathcal{X}(U)$.
Let \( F(M,Y,U) \) denote the collection of all film chains.

**Theorem 6.0.3.** Suppose \( X \in \mathfrak{X}(M,U) \). Then there exists a unique film chain \( S_X \in F(M,Y,U) \) with \( \text{supp}(S_X) = X \). Moreover, \( S_X \) satisfies \( \text{supp}(\kappa_Y S_X) = \text{supp}(S_X) \) and \( \text{supp}(\partial S_X) = M \).

**Proof.** Let \( J \in \mathcal{C}(U) \) correspond to the measure \( S^{n-1}|_X \), and set \( S_X = \partial J + E_Y \tilde{M} \). Then \( \partial S_X = P_Y \tilde{M} \) since \( P_Y = \partial E_Y \). As in the proof of Lemma 2.0.2, we have \( \kappa_Y S_X = \kappa_Y (\partial J + E_Y \tilde{M}) = \kappa_Y \partial J = J \), since \( \kappa_Y \partial + \partial \kappa_Y = Id \), and \( J \) is top dimensional, so \( \kappa_Y J = 0 \). This establishes existence. For uniqueness, suppose \( S'_X \) also satisfies \( \partial S'_X = P_Y \tilde{M} \) and \( \mu_{\kappa_Y S'_X} = S^{n-1}|_X \). Then by Proposition 3.2.1, \( \kappa_Y S'_X = \kappa_Y S_X \). Thus, \( S_X = \partial \kappa_Y S_X + \kappa_Y \partial S_X = \partial \kappa_Y S'_X + \kappa_Y P_Y \tilde{M} = S'_X \).

To see that \( \text{supp}(S_X) = \text{supp}(\kappa_Y S_X) = X \), we apply Corollary 2.0.4 and Proposition 3.2.1 (e), noting that \( \text{supp}(\kappa_Y S_X) = \text{supp}(S^{n-1}|_X) = X \) since \( X \) is closed and reduced. The last equality is Lemma 2.0.1. \( \square \)

By Theorem 6.0.3, there is a 1-1 correspondence between \( F(M,Y,U) \) and \( \mathfrak{X}(M,U) \). Also by Theorem 6.0.3, film chains \( S \in F(M,Y,U) \) automatically satisfy \( \text{supp}(S) \in \mathfrak{X}(M,U) \), \( \text{supp}(S) = \text{supp}(\kappa_Y S) \) and \( \text{supp}(\partial S) = M \).

It is with this construction in mind that we define a continuous area functional on \( \hat{B}^2_{n-1}(U) \):

**Definition 6.0.4.** For \( S \in \hat{B}^2_{n-1}(U) \), define

\[
A(S) = A_Y(S) := \int_{\kappa_Y S} dV.
\]

We call \( A \) an area functional since

\[
S^{n-1}(\text{supp}(S)) = A(S)
\]

for all \( S \in F(M,Y,U) \). This is a consequence of Corollary 2.0.4 and Proposition 3.2.1 (a).

In particular,

\[
m = \inf\{S^{n-1}(X) : X \in \mathfrak{X}(M,U)\} = \inf\{A(S) : S \in F(M,Y,U)\}.
\]

**Proposition 6.0.5.** There exists a constant \( a_0 > 0 \) such that \( A(S) > a_0 \) for all \( S \in F(M,Y,U) \).

**Proof.** Let \( N = \gamma(S^1) \) be a simple link of \( M \) and let \( T \) be an \( \epsilon \)-tubular neighborhood of \( N \) whose closure is disjoint from \( M \). Let \( D \) be the \((n-1)\)-disk of radius \( \epsilon \) and \( \rho : T \to D \) the canonical projection given by \( T \) and some trivialization of the normal bundle of \( N \). It follows from Lemma 4.0.2 that there exists a constant \( C > 0 \) such that if \( B \subset T \), then \( S^{n-1}(\rho(B)) \leq CS^{n-1}(B) \).

Suppose \( X \) is compact and spans \( M \). Then \( \rho(X \cap T) = D \), otherwise the preimage \( \rho^{-1}(x) \) of a point \( x \in D \setminus \rho(X \cap T) \) would be a simple link of \( M \), and missing \( X \). Putting this together, we get
\( S^{n-1}(X) \geq S^{n-1}(X \cap T) \geq \frac{1}{\kappa} S^{n-1}(\rho(X \cap T)) = \frac{1}{\kappa} \alpha_{n-1} \epsilon^{n-1} \). The result follows from (8), setting \( a_0 = \frac{1}{\kappa} \alpha_{n-1} \epsilon^{n-1} \). \( \square \)

Let \( c_0 = \text{diam}(U) \text{vol}(M) \), where \( \text{vol}(M) \) is the volume of \( M \) and \( \text{diam}(U) \) is the diameter of \( U \).

**Proposition 6.0.6.** There exists \( S \in \mathcal{F}(M,Y,U) \) such that \( \mathcal{A}(S) \leq c_0 \).

**Proof.** We know \( C_q(M) \) spans \( M \) by Proposition 5.0.1. Since \( S^{n-1}C_q(M) \leq R \cdot \text{vol}(M) < \infty \), we may apply Theorem 6.0.3 to the set \( C_q(M)^* = C_q(M) \) to obtain a film chain \( S = S_{C_q(M)} \). Furthermore, \( \mathcal{A}(S) = S^{n-1}(C_q(M)) \leq c_0 \). \( \square \)

**Definition 6.0.7.** Define
\[
\mathcal{F}(M,Y,U) := \{ S \in \hat{\mathcal{B}}^2_n(U) : S = \lim_{i \to \infty} S_i, S_i \in \mathcal{F}(M,Y,U), \mathcal{A}(S_i) \leq c_0 \text{ for all } i \}
\]
and let \( \mathcal{C}(U,c_0) := \{ T \in \hat{\mathcal{B}}^1_n(U) : T \text{ is positive, } M(T) \leq c_0 \} \).

**Proposition 6.0.8.** If \( S \in \mathcal{F}(M,Y,U) \), then
\[
(a) \, \mathcal{A}(S) \leq c_0;
(b) \, \partial S = P_Y \hat{M};
(c) \, \kappa_Y S \in \mathcal{C}(U,c_0);
(d) \, \text{supp}(\kappa_Y S) \text{ spans } M.
\]

**Proof.** Choose \( S_i \to S \) in \( \mathcal{F}(M,Y,U) \) with \( S_i \in \mathcal{F}(M,Y,U) \) and \( \mathcal{A}(S_i) \leq c_0 \). Parts (a) and (b) follow from continuity of \( \kappa, \partial \) and \( \hat{f} \). A diagonal sequence of positive Dirac chains \( A_{i,j} \) approximating \( \kappa_Y S_i \) shows \( \kappa_Y(S) \) is positive. Since \( U \) is bounded, Corollary 3.1.6 shows \( \kappa_Y S \in \mathcal{B}^1_n(U) \) and hence by Lemma 3.1.5 that \( M(\kappa_Y S) = \int_{\kappa_Y S} \text{d}V = \lim_{i \to \infty} M(\kappa_Y S_i) \leq c_0 \).

Proof of (d): If not, there is a simple link \( N = \eta(S^1) \) of \( M \) and an \( \epsilon \) tubular neighborhood \( T = \Omega(N, \epsilon) \) of \( N \) such that \( \text{supp}(\kappa_Y S) \cap T = \emptyset \). Let \( f : U \to \mathbb{R} \) be a smooth function satisfying \( f(x) = 1 \) for \( x \in \Omega N, \epsilon/2, \supp(f) \subset T \) and \( f \geq 0 \). Then by Proposition 3.2.1 (d),
\[
\int_{\kappa_Y S} f \text{d}V = \lim_{i \to \infty} \int_{\kappa_Y S_i} f \text{d}V \geq \limsup_{i \to \infty} \mu_{\kappa_Y S_i}(\Omega N, \epsilon/2)
\]
\[
= \lim_{i \to \infty} S^{n-1}(\text{supp}(S_i) \cap \Omega N, \epsilon/2)
\]
\[
\geq \alpha_{n-1}(\epsilon/2)^{n-1},
\]
where the last inequality follows since \( \text{supp}(S_i) \) spans \( M \). This yields a contradiction, since \( f \text{d}V \) is supported away from \( \text{supp}(\kappa_Y S) \). \( \square \)

**Theorem 6.0.9.** \( \mathcal{C}(U,c_0) \subset \hat{\mathcal{B}}^1_n(U) \) is compact.

**Proof.** The set \( \mathcal{C}(U,c_0) \) is closed since the limit \( J \) in the \( B^1 \) norm of a sequence \( J_i \) of positive chains is again positive, and \( M(J) = \lim_{i \to \infty} M(J_i) \leq c_0 \) by Lemma 3.1.5. By Theorem 1.3.5 the \( u_{0,1}^{0,1} \)

\[\text{Here we think of } \hat{\mathcal{B}}^1_n(U) \text{ as a subspace of } \hat{\mathcal{B}}^2_n \text{ via the canonical injection map } u_{0,1}^{1,2} \].
image \( I \) of the ball of radius \( c_0 \) in \( \mathbb{B}_0^0 \) is totally bounded. Since \( \mathcal{C}(U, c_0) \) is a subset of the closure of \( I \), it is therefore compact.

**Corollary 6.0.10.** \( \kappa_Y(\mathcal{T}(M,Y,U)) \subset \mathbb{B}_0^1(U) \) is compact.

**Proof.** By Proposition 6.0.8 (c) and Theorem 6.0.9 it suffices to show \( \kappa_Y(\mathcal{T}(M,Y,U)) \) is closed. Suppose \( \{T_i\} \subset \kappa_Y(\mathcal{T}(M,Y,U)) \) and \( T_i \to T \). Write \( T_i = \kappa_Y(S_i) \) where \( S_i \in \mathcal{T}(M,Y,U) \). Then \( \partial T_i = S_i - E_Y M \) by Proposition 6.0.8 (b) and Lemma 2.0.2, so \( S_i \to \partial T + E_Y M =: S \in \mathcal{T}(M,Y,U) \) since \( \mathcal{T}(M,Y,U) \) is closed. It follows that \( T = \kappa_Y S \in \kappa_Y(\mathcal{T}(M,Y,U)) \).

**Corollary 6.0.11.** There exists \( S_0 \in \mathcal{T}(M,Y,U) \) with \( A(S_0) = m \). Any such \( S_0 \) is called an area minimizer.

**Theorem 6.0.12.** If \( S_0 \in \mathcal{T}(M,Y,U) \) is an area minimizer, then\(^{12}\) \( \text{supp}(\kappa_Y S_0) \subset h(M) \).

**Proof.** Let \( S_i \to S_0 \), where \( \{S_i\} \subset \mathcal{T}(M,Y,U) \) and set \( X_i := \text{supp}(S_i) \). If \( x \in \text{supp}(\kappa_Y S_0) \) and \( x \notin h(M) \), let \( \epsilon > 0 \) be small enough so that \( \Omega(x, 3\epsilon) \cap h(M) = \emptyset \). Let \( f \in \mathcal{B}_0^1(U) \) be supported in \( \Omega(x, \epsilon) \) such that \( f \) is non-increasing along flow-lines.

Let \( A = h(M), B = \Omega(h(M), \epsilon) \setminus A \) and \( C = U \setminus (A \cup B) \). Fix \( p \in A \) and let \( G : \mathbb{R}^n \to \mathbb{R}^n \) be a diffeomorphism given by the \((t = 1)\)-flow of a vector field \( V \) which is everywhere pointing towards \( p \) (except where it is zero), and such that

- \( V|_A \equiv 0; \)
- There is some \( 0 < s < 1 \) such that if \( y \in C \), then \( G(y) = sy + (1 - s)p; \)
- The magnitude of \( V \) is non-increasing along flow-lines.

Then the Lipschitz constant of \( G \) is equal to 1, since the divergence of \( V \) is everywhere non-positive, and \( G(U) \subset U \) since \( U \) is convex. Since \( G \) is a diffeomorphism, it follows from Lemma 5.0.6 that \( \bar{X}_i := (G(X_i))^* \subset \mathfrak{X}(M,U) \). Let \( \{S_i\} \subset \mathcal{T}(M,Y,U) \) be the corresponding sequence of film chains given by Theorem 6.0.3. By Lemma 4.0.2, we have:

\[
A(S_i) = \mathcal{S}_{n-1}(G(X_i)) = \mathcal{S}_{n-1}(G(X_i \cap A)) + \mathcal{S}_{n-1}(G(X_i \cap B)) + \mathcal{S}_{n-1}(G(X_i \cap C)) \\
\leq \mathcal{S}_{n-1}(X_i \cap A) + \mathcal{S}_{n-1}(X_i \cap B) + s^n \mathcal{S}_{n-1}(X_i \cap C) \\
= \mathcal{S}_{n-1}(X_i) - (1 - s^n) \mathcal{S}_{n-1}(X_i \cap C) \\
\leq A(S_i) - (1 - s^n) \mathcal{S}_{n-1}(X_i \cap \Omega(x, \epsilon)) \leq A(S_i) - (1 - s^n) \delta,
\]

thus contradicting the minimality of the sequence \( \{S_i\} \).

\(^{12}\)Recall \( h(M) \) is the convex hull of \( M \).

\(^{13}\)Recall \( \mathcal{B}_0^1 \) consists of bounded Lipschitz functions.
So, in particular, $\kappa_Y S_0 \in \mathcal{C}(U)$, so we may define its corresponding measure $\mu_{\kappa_Y S_0}$. Unfortunately, this is not the end of the story, since $S_0$ is not a priori an element of $\mathcal{F}(M,Y,U)$. There are two problems to take care of:

(a) We don’t know that $supp(S_0)$ spans $M$. This will be resolved by Corollary 2.0.4 upon fixing the next problem:
(b) The Hausdorff spherical $(n-1)$-measure of $supp(\kappa_Y S_0)$ may not be equal to $m$ (indeed it might not be finite at all,) and the measure $\mu_{\kappa_Y S_0}$ may not be $S^{n-1}|_X$ for some $X \in \mathcal{X}(M,U)$.

We prove in the next section that this cannot happen.

7. Minimizing sequences

Our main goal of this section if to prove:

**Theorem 7.0.1.** If $S_0 \in \mathcal{F}(M,Y,U)$ is an area minimizer, then $S^{n-1}(supp(S_0)) = m$. Moreover, $S_0 \in \mathcal{F}(M,Y,U)$.

**Definition 7.0.2.** If $X \subset \mathbb{R}^n$, we say a set $B \subset \mathbb{R}^n$ is $X$-compatible if $\mathcal{H}^{n-2}|_X(frB) < \infty$. Let $\Gamma(X)$ be the set of closed balls which are disjoint from $M$, are contained in $U$, and whose centers lie in $X$. Let $\Gamma(X) \subset \Gamma(X)$ consist of those balls which are $X$-compatible. Note that Lemma 4.0.5 implies that if $\mathcal{H}^{n-1}(X) < \infty$, then almost all cubes and almost all balls are $X$-compatible.

**Definition 7.0.3.** If $\{X_k\} \subset \mathcal{X}(U)$ converges to $X_0$ in the Hausdorff metric, let

$$\beta = \beta(\{X_k\}) := \inf \left\{ \liminf_{k \to \infty} \frac{S^{n-1}(X_k(p,r))}{\alpha_{n-1} r^{n-1}} : \Omega(p,r) \in \Gamma(X_0) \right\}.$$

**Definition 7.0.4.** A sequence $\{X_k\} \subset \mathcal{X}(U)$ is lower regular minimizing\textsuperscript{14} if $S^{n-1}(X_k) \to m$ and there exists $a > 0$ such that

$$S^{n-1}(X_k(p,r)) \geq a r^{n-1}$$

for all $k$, $\Omega(p,r) \in \Gamma(X_k)$ and $r > 2^{-k}$.

7.1. Overview of the proof of Theorem 7.0.1. The proof has five main steps:

(a) Start with any sequence $S_k \to S_0$ in $\mathcal{F}(M,Y,U)$ where $\{S_k\} \subset \mathcal{F}(M,Y,U)$. Let $X_k := supp(\kappa_Y S_k)$, $k \geq 0$. In §7.2 we “cut hairs” and create a modified sequence $\{\hat{X}_k\} \subset \mathcal{X}(M,U)$ which converges to $X_0$ in the Hausdorff metric, and such that $S^{n-1}(\hat{X}_k) \to m$. We then apply Theorem 6.0.3 to create a corresponding sequence $\{\hat{S}_k\} \subset \mathcal{F}(M,Y,U)$ of film chains.

(b) Next, we show that then there exists a subsequence $\{T_j\}$ of $\{\hat{S}_k\}$ whose supports form a lower regular minimizing sequence. This is Corollary 7.3.3.

(c) We then prove Lemma 7.4.9 which states that for this sequence, $\beta \geq 1$.

(d) Lower semicontinuity of Hausdorff spherical measure for lower regular minimizing sequences is then established in Theorem 7.4.1. It is easy to deduce from this that $S^{n-1}(X_0) = m$. That $S_0 \in \mathcal{F}(M,Y,U)$ follows shortly thereafter.

\textsuperscript{14}The $\{X_k\}$ we use will turn out to be uniformly locally Ahlfors regular, that is, there exists $a > 0$ with $a r^{n-1} \leq S^{n-1}(X_k(p,r)) \leq r^{n-1}/a$, but we do not use the right hand inequality. We also require the sequence be minimizing.
A number of lemmas in this section are adaptations of results of [Rei60].

7.2. Haircuts. Lemmas 7.2.1-7.2.4 are used to modify a spanning set \( X \in \mathcal{X}(M,U) \) inside a closed ball \( \Omega(p,r) \) disjoint from \( M \). These are followed by Lemma 7.2.5 which modifies \( X \) inside a closed cube. These lemmas will assist us in our proofs of Theorem 7.2.7 and Lemma 7.3.1. Lemma 7.2.1 is an adaptation of Lemma 8 in [Rei60].

Lemma 7.2.1. There exists a constant \( 1 \leq K_n < \infty \) such that if \( X \in \mathcal{X}(M,U) \) and \( \bar{\Omega}(p,r) \) is \( X \)-compatible and disjoint from \( M \), then there exists a compact set \( Z \subset \mathbb{R}^n \) spanning \( M \) satisfying

\[
\begin{align*}
(a) \quad Z \cap \Omega(p,r)^c &= X \cap \Omega(p,r)^c; \\
(b) \quad Z(p,r) &\subset h(x(p,r)); \\
(c) \quad Z(p,r) &\subset \Omega(x(p,r), K_n s^{n-2}(x(p,r))^{1/(n-2)}); \text{ and} \\
(d) \quad s^{n-1}(Z(p,r)) &\leq K_n s^{n-2}(x(p,r))^{(n-1)/(n-2)}.
\end{align*}
\]

Proof. Let \( K_n \) be the constant \( K_{n-1} \) in Lemma 8 of [Rei60], and let \( Z \) be the set defined as follows: let \( Z \setminus \Omega(p,r) := X \setminus \Omega(p,r) \) and let \( Z(p,r) \) be the “surface” defined in Lemma 8 of [Rei60], setting \( A = x(p,r) \). Since \( Z(p,r) \) is contained in the convex hull of \( x(p,r) \), it follows that \( z(p,r) = x(p,r) \), and so \( Z \) satisfies (a)–(d)). It remains to show that \( Z \) spans \( M \). In dimension \( n = 3 \) this is easiest to see: in this case, there is a grid of cubes and the set \( x(p,r) \) is a disjoint union of subsets \( x_i \), each contained in the interior of one such cube. The “surface” obtained by Reifenberg is then the disjoint union of the cones \( c_{p_i}(x_i) \), for some \( p_i \in x_i \). To see that \( Z \) spans in this case, suppose it does not. Then there exists a simple link \( N = \eta(S^1) \) of \( M \) disjoint from \( Z \). We modify the proof of Lemma 5.0.8 by projecting \( \eta \) radially to the frontier of each cube, then radially onto edges, and finally orthogonally onto the frontier of \( \Omega(p,r) \). The new curve is still a simple link of \( M \), but is now disjoint from \( X \) as well, giving a contradiction.

For \( n > 3 \), the construction in Lemma 8 of [Rei60] is considerably more difficult, but the basic idea to showing \( Z \) spans \( M \) is the same: Using an inductive argument, Reifenberg “chops” \( \bar{\Omega}(p,r) \) into a number of \( n \)-cubes \( \Omega = \{Q_i\}_{i=1}^n \), forming a cover of \( \Omega(p,r) \) by \( n \)-cubes with mutually disjoint interiors. Pairs of opposite faces of each \( n \)-cube \( Q \in \Omega \) are ordered by the coordinate vector to which they are perpendicular, and a face in the \( k \)-th pair lies in the same hyperplane as the corresponding face of any other \( n \)-cube neighboring \( Q \) in any of the remaining \( n - k \) directions.

For a given face \( F \) in the \( k \)-th pair of faces of a \( n \)-cube \( Q \), let \( \mathcal{N}(F) \) denote the maximal union, containing \( F \), of faces neighboring in the above manner\(^{15}\). Let

\[
\mathcal{N}_k := \{ \mathcal{N}(F) : F \text{ is in the } k \text{-th pair of faces for some cube } Q \},
\]

Likewise, let \( N_k := \cup F \), where the union is taken over all \( k \)-th faces \( F \) of the cubes \( Q \in \Omega \).

Now, let \( \eta \) be as above. We first project \( \eta \) radially to the frontier of each cube, so now \( \eta(S^1) \cap \Omega(p,r) \subset \bigcup_{s=1}^n N_s \). Now suppose \( \eta(S^1) \cap \Omega(p,r) \) is contained in \( \bigcup_{s=1}^k N_s \). Then by projecting \( \eta(S^1) \cap \mathcal{N}(F) \cap \Omega(p,r) \) onto the frontier\(^{16}\) of \( \mathcal{N}(F) \cap \Omega(p,r) \), for each \( \mathcal{N}(F) \in \mathcal{N}_k \) (which we may do

\(^{15}\)i.e., \( F \in \mathcal{N}(F) \), and if \( F' \in \mathcal{N}(F) \) is in the \( k \)-th pair of faces of some \( n \)-cube \( Q' \), then \( \mathcal{N}(F') \) also contains the corresponding face of any cube \( Q'' \) neighboring \( Q' \) in any of the remaining \( n - k \) directions. So, \( \mathcal{N}(F) = F \) when \( F \) is a face in the \( n \)-th pair of faces of \( Q \), and \( \mathcal{N}(F) \) is the entire hyperplane containing \( F \) when \( F \) is a face in the \( 1 \)-st pair.

\(^{16}\)considered as a subset of the hyperplane containing \( \mathcal{N}(F) \)
by induction on \( n \), starting at \( n = 4 \), by the \( n = 3 \) case of the proof,) it will follow that \( \eta(S^1) \cap \Omega(p, r) \) is contained in \( \bigcup_{s=1}^{k-1} N_s \). Thus, by downward induction on \( k \), this will result in a new simple link of \( M \) whose intersection with \( \Omega(p, r) \) lies entirely within \( fr \Omega(p, r) \), and is thus disjoint from \( X \) by 7.2.1 ((a)), giving a contradiction.

The following lemma is an adaptation of lemma 9 in [Rei60]:

**Lemma 7.2.2.** Suppose \( X \in \mathcal{X}(M, U) \) and \( \Omega(p, r) \) is \( X \)-compatible and disjoint from \( M \), with 
\[
S^{n-2}(x(p, r)^{1/(n-2)} < r/(2K_n).
\]

There exists a compact set \( \tilde{X} \subset \mathbb{R}^n \) spanning \( M \) such that

\[
\begin{align*}
(a) & \quad \tilde{X} \cap \bar{\Omega}(p, r)^c = X \cap \bar{\Omega}(p, r)^c; \\
(b) & \quad \tilde{X}(p, r) \subset fr\Omega(p, r); \text{ and} \\
(c) & \quad S^{n-1}(\tilde{X}(p, r)) \leq 2^{n-1}K_nS^{n-2}(x(p, r))^{(n-1)/(n-2)}.
\end{align*}
\]

**Proof.** Let \( Z \) be as in Lemma 7.2.1. Lemma 7.2.1 ((c)) implies \( Z(p, r) \subset \bar{\Omega}(p, r) \cap \Omega(p, r/2)^c \). So, we can radially project \( Z(p, r) \) onto \( fr\Omega(p, r) \), and there exists a Lipschitz extension \( \pi : \mathbb{R}^n \to \mathbb{R}^n \) of this projection with Lipschitz constant 2, \( \pi |_{\Omega(p, r)} = Id \), and \( \pi(\Omega(p, r)) \subset \bar{\Omega}(p, r) \). Then \( \tilde{X} := \pi Z \) is compact and spans \( M \) by Lemma 5.0.5. Lemmas 4.0.2 and 7.2.1(d) yield
\[
S^{n-1}(\tilde{X}(p, r)) = S^{n-1}(\pi(Z(p, r))) \leq 2^{n-1}S^{n-1}(Z(p, r)) \leq 2^{n-1}K_nS^{n-2}(x(p, r))^{(n-1)/(n-2)}.
\]

**Lemma 7.2.3.** Suppose \( X \in \mathcal{X}(M, U) \) and \( \bar{\Omega}(p, r) \) is disjoint from \( M \) with
\[
S^{n-1}(X(p, r)) \leq \frac{r^{n-1}}{(2(n-1))^{n-1}(2^n K_n)^{n-2}}.
\]

Suppose \( W \subset (r/2, r) \) has full Lebesgue measure. There exists \( r' \in W \) such that
\[
S^{n-2}(x(p, r'))^{(n-1)/(n-2)} \leq \frac{1}{2^n K_n} \int_0^{r'} S^{n-2}(x(p, t))dt.
\]

**Proof.** Suppose there is no such \( r' \). Let \( F(p, s) := \int_0^s S^{n-2}(x(p, t))dt \). By the Lebesgue Differentiation Theorem,
\[
\frac{d}{ds}F(p, s) > \frac{1}{(2^n K_n)^{(n-2)/(n-1)}}
\]
for almost every \( r/2 < s < r \). Integrating, this implies
\[
\int_{r/2}^r \frac{d}{ds} \left( F(p, s)^{1/(n-1)} \right) ds > \frac{r}{2(n-1)(2^n K_n)^{(n-2)/(n-1)}}.
\]

Since \( F(p, s) \) is increasing and absolutely continuous, the function \( F(p, s)^{1/(n-1)} \) is also absolutely continuous, and so the left hand side of (10) is equal to \( F(p, r)^{1/(n-1)} - F(p, r/2)^{1/(n-1)} \) by the Fundamental Theorem for Lebesgue Integrals. Lemma 4.0.5 gives
\[
S^{n-1}(X(p, r)) \geq F(p, r) > \frac{r^{n-1}}{(2(n-1))^{n-1}(2^n K_n)^{n-2}},
\]
Proof. By Lemmas 7.2.3 and 4.0.5, there exists $$\hat{X}\subset \mathbb{R}^n$$ spanning $$M$$ such that

$$(11) \quad \mathcal{S}^{n-1}(\hat{X}(p, r')) \leq \frac{r^{n-1}}{2^n K_n} \mathcal{S}^{n-2}(x(p, t))dt \leq \frac{1}{2^n K_n} \mathcal{S}^{n-1}(X(p, r')).$$

Thus, $$\mathcal{S}^{n-2}(x(p, r'))^{(n-1)/(n-2)} \leq \frac{r^{n-1}}{2^n (n-1) K_n} \leq r'/(2 K_n)$$, and so we may apply Lemma 7.2.2 to get the required set $$\hat{X}$$, since (11) and Lemma 7.2.2 ((c)) give

$$\mathcal{S}^{n-1}(\hat{X}(p, r')) \leq 2^{-1} K_n \mathcal{S}^{n-2}(x(p, r'))^{(n-1)/(n-2)} \leq \frac{1}{2} \mathcal{S}^{n-1}(X(p, r')).$$

\[\square\]

Lemma 7.2.4. Suppose $$X\in \mathcal{X}(M, U)$$ and $$\hat{\Omega}(p, r)$$ is disjoint from $$M$$, with

$$\mathcal{S}^{n-1}(X(p, r)) \leq \frac{r^{n-1}}{(2(n-1))^{n-1} (2^n K_n)^{n-2}}.$$ 

If $$W \subset (r/2, r)$$ has full Lebesgue measure, then there exist $$r/2 < r' < r$$ such that $$r' \in W$$, and a compact set $$\tilde{X} \subset \mathbb{R}^n$$ spanning $$M$$ such that

(a) $$\tilde{X} \cap \hat{\Omega}(p, r')^c = X \cap \hat{\Omega}(p, r')^c$$;

(b) $$\tilde{X}(p, r') \subset \text{fr} \hat{\Omega}(p, r')$$; and

(c) $$\mathcal{S}^{n-1}(\tilde{X}(p, r')) \leq \mathcal{S}^{n-1}(X(p, r'))/2.$$

Proof. By Lemmas 7.2.3 and 4.0.5, there exists $$r' \in W$$ such that

$$\mathcal{S}^{n-2}(x(p, r'))^{(n-1)/(n-2)} \leq \frac{1}{2^n K_n} \mathcal{S}^{n-1}(X(p, r')).$$

Thus, $$\mathcal{S}^{n-2}(x(p, r'))^{(n-1)/(n-2)} \leq \frac{r^{n-1}}{2^n (n-1) K_n} \leq r'/(2 K_n)$$, and so we may apply Lemma 7.2.2 to get the required set $$\tilde{X}$$, since (11) and Lemma 7.2.2 ((c)) give

$$\mathcal{S}^{n-1}(\tilde{X}(p, r')) \leq 2^{-1} K_n \mathcal{S}^{n-2}(x(p, r'))^{(n-1)/(n-2)} \leq \frac{1}{2} \mathcal{S}^{n-1}(X(p, r')).$$

\[\square\]

Lemma 7.2.5. Suppose $$X\in \mathcal{X}(U)$$ and $$Q$$ is a closed $$n$$-cube of side length $$\ell$$ disjoint from $$M$$, with

$$\mathcal{S}^{n-1}(X \cap Q) \leq \frac{\ell^{n-1}}{(4(n-1))^{n-1} (2^n K_n)^{n-2}}.$$ 

Then there exists a compact set $$X'$$ spanning $$M$$ such that

(a) $$X' \cap Q = X \cap Q^c$$;

(b) $$X' \cap Q \subset \text{fr} Q$$;

(c) $$X' \cap Q = (X \cap \text{fr} Q) \cup Y$$,

where $$Y \subset \text{fr} Q$$ satisfies

$$\mathcal{S}^{n-1}(Y) \leq (2 \sqrt{n})^{n-1} \mathcal{S}^{n-1}(X \cap Q).$$

Proof. Let $$p$$ be the center point of $$Q$$. Then $$\hat{\Omega}(p, \ell/2) \subset Q$$ satisfies the conditions of Lemma 7.2.4, and so there exists $$\ell/4 < r < \ell/2$$ and a compact set $$\tilde{X}$$ spanning $$M$$ such that

- $$\tilde{X} \cap \hat{\Omega}(p, r)^c = X \cap \hat{\Omega}(p, r)^c$$;
- $$\tilde{X}(p, r) \subset \text{fr} \hat{\Omega}(p, r)$$; and
- $$\mathcal{S}^{n-1}(\tilde{X}(p, r)) \leq \mathcal{S}^{n-1}(X(p, r))/2.$$

The remaining part of $$\tilde{X} \cap \hat{Q}$$ is contained in the region $$\hat{Q} \cap \hat{\Omega}(p, r)^c$$, so we may radially project $$\tilde{X} \cap \hat{Q}$$ from $$p$$ to a set $$Q \subset \text{fr} Q$$, and the image $$X'$$ of $$\tilde{X}$$ under this map spans $$M$$ by Lemma 5.0.5.
Moreover, since the Lipschitz constant of the projection is bounded above by $2\sqrt{n}$, by Lemma 4.0.2 and Lemma 7.2.4 ((c)), we have
\[
S^{n-1}(Y) \leq (2\sqrt{n})^{n-1}S^{n-1}(\tilde{X} \cap \tilde{Q}) = (2\sqrt{n})^{n-1} \left[ S^{n-1}(\tilde{X} \cap (\tilde{Q} \cap \Omega(p,r)^c)) + S^{n-1}(\tilde{X}(p,r)) \right] \\
\leq (2\sqrt{n})^{n-1}S^{n-1}(X \cap \tilde{Q}).
\]

\[\square\]

**Definition 7.2.6.** A collection $\mathcal{S}$ of closed $n$-cubes is a dyadic subdivision of $\mathbb{R}^n$ if $\mathcal{S} = \bigcup_{k \in \mathbb{Z}} \mathcal{S}_k$, where each $\mathcal{S}_k$ is a cover of $\mathbb{R}^n$ by $n$-cubes of side length $2^{-k}$ that intersect only on faces, and such that $\mathcal{S}_{k+1}$ is a refinement of $\mathcal{S}_k$.

**Theorem 7.2.7.** If $S_0 \in \mathcal{T}(M,Y,U)$ is an area minimizer, then there exists a sequence $\{T_k\} \subset \mathcal{T}(M,Y,U)$ such that $T_k \to S_0$ in $\mathcal{B}^2_{n-1}(U)$, $\kappa_Y T_k \to \kappa_Y S_0$ in $\mathcal{C}(U)$, and $\text{supp}(\kappa_Y T_k) \to \text{supp}(\kappa_Y S_0)$ in the Hausdorff metric.

**Proof.** Let $S_j \to S_0$ in $\mathcal{B}^2_{n-1}(U)$ with $\{S_j\} \subset \mathcal{T}(M,Y,U)$. Since $\{\kappa_Y S_j\} \subset \mathcal{C}(U)$ (Definition 6.0.2 ((c))) and $\kappa_Y S_j \to \kappa_Y S_0$ in $\mathcal{B}^2_{n}(U)$ (Definition 1.111.1) it follows from Corollary 3.1.7 that there exists a subsequence $\kappa_Y S_{j_k} \to \kappa_Y S_0$ in $\mathcal{B}^1_{n}(U)$. So, let us assume without loss of generality that this is the case, that $S_j \to S_0$ in $\mathcal{B}^2_{n-1}(U)$ with $\{S_j\} \subset \mathcal{T}(M,Y,U)$ and $\kappa_Y S_j \to \kappa_Y S_0$ in $\mathcal{C}(U)$. Let $X_j = \text{supp}(\kappa_Y S_j)$ for $j \geq 0$.

Let $\mathcal{S} = \bigcup_{k \in \mathbb{Z}} \mathcal{S}_k$ be a dyadic subdivision of $\mathbb{R}^n$ and let $\mathcal{D}_k = \{Q \in \mathcal{S}_k : Q \cap X_0 = \emptyset, Q \cap U \neq \emptyset\}$, let $d_k < \infty$ be the cardinality of $\mathcal{D}_k$ and let $\mathcal{N}_k = \{Q \in \mathcal{S}_k : Q \cap X_0 \neq \emptyset\}$. Note that $\mathcal{D}_k \cup \mathcal{N}_k$ covers $\tilde{U}$. By Theorem 6.0.12, there exists $N \in \mathbb{N}$ such that if $k \geq N$ and $Q \in \mathcal{N}_k$, then $Q \subset U$. So, $U_k := \cup_{Q \in \mathcal{N}_k} Q$ is a neighborhood of $X_0$ and $U_k \subset \Omega(X_0, \sqrt{n}2^{-k+1})$. If $k \geq N$, then $U_k \subset U$. For the rest of this proof, let us assume $k \geq N$.

Since $\text{supp}(\mu_{\kappa_Y S_0}) = X_0$ by Proposition 3.2.1 ((e)), it follows that $\mu_{\kappa_Y S_0}(Q) = 0$ for all $Q \in \mathcal{D}_k$. In particular, $\mu_{\kappa_Y S_0}(\text{fr}Q) = 0$, so we may apply Proposition 3.2.2 to deduce $\lim_{j \to \infty} \mu_{\kappa_Y S_j}(Q) = \mu_{\kappa Y S_0}(Q) = 0$. Since $S_j \in \mathcal{T}(M,Y,U)$, we have $\mu_{\kappa_Y S_j}(Q) = S^{n-1}|_{X_j}(Q)$, hence $\lim_{j \to \infty} S^{n-1}|_{X_j}(Q) = 0$.

Let $\rho_k := \min\{2^{-(n-1)k}/((2^{n}K_n)^{n-2}(4(n-1))^{-1}), 2^{-k}/d_k\}$. Since $0 < d_k < \infty$, there exists $N_k \in \mathbb{N}$ such that if $j \geq N_k$ then $S^{n-1}(X_j \cap Q) < \rho_k$ for all $Q \in \mathcal{D}_k$. In particular,
\[
\sum_{Q \in \mathcal{D}_k} S^{n-1}(X_{N_k} \cap Q) < 2^{-k},
\]
for each $k$. By Lemma 7.2.5 there exists a compact set $Y_k \subset \mathbb{R}^n$ spanning $M$, equal to $X_{N_k}$ outside $\cup_{Q \in \mathcal{D}_k} Q$, such that $Y_k \cap Q = \emptyset$ for each $Q \in \mathcal{D}_k$ and
\[
S^{n-1}(Y_k \cap F) \leq (2(2\sqrt{n})^{n-1} + 1)\rho_k
\]
for each face $F$ of $Q \in \mathcal{D}_k$. (This is because each face $F$ is shared by at most two adjacent cubes in $\mathcal{D}_k$.)

Let $Q \in \mathcal{D}_k$ and suppose $F$ is a face of $Q$ not contained in $U_k$. Since $S^{n-1}(Y_k \cap F) < 2^{-k(n-1) - 1} = 1/2S^{n-1}(F)$ and $Y_k \cap F$ is closed, there exists a finite sequence of deformations (see Definition 5.0.4)
\{\phi_i\}$ of $Y_k$ which sends $Y_k \cap F$ to the $(n - 2)$-skeleton of $F$. Repeating this process for each face $F$ of each $Q \in \mathcal{D}_k$ such that $F$ is not contained in $U_k$, we get a compact set $Z_k$ spanning $M$ by Lemma \ref{compactness of supp}.

Furthermore, the core $Z_k^\ast$ is contained in $U_k \subset U$, since $Z_k \cap U_k^\ast$ is a subset of the $(n - 2)$-skeletons of cubes in $\mathcal{D}_k$. By \eqref{inequality 12}, Lemma \ref{inequality 7.2.5} (12) and (14), we have
\[
S^{n-1}(Z_k^\ast) \leq S^{n-1}(Z_k) \leq S^{n-1}(Y_k) \leq S^{n-1}(X_{N_k}) + (2\sqrt{n})^{n-1} \sum_Q S^{n-1}(X_{N_k} \cap Q) \\
\leq S^{n-1}(X_{N_k}) + (2\sqrt{n})^{n-1}2^{-k} < \infty.
\]
In particular, $Z_k^\ast \in \mathcal{X}(M, U)$ by Lemma \ref{finnies measure} and $S^{n-1}(Z_k^\ast) \to m$.

Now, for each $k \geq 1$ let $S_k^r$ be the film chain corresponding to $Z_k^\ast$ in Theorem \ref{maximal regular minimizing sequences}. Since \{\{S_k^r\} \subset \mathcal{J}(M, Y, U)$, it follows from Corollary \ref{maximal regular minimizing sequences} that there exists a subsequence \{\{S_k^r\} converging to some $S_0^r$ in $\mathcal{J}(M, Y, U)$. As in the beginning of the proof, by taking a further subsequence, we can also ensure that $\kappa_Y S_k^r \to \kappa_Y S_0^r$ in $\mathcal{C}(U)$.

We show $\kappa_Y S_0^r = \kappa_Y S_0$, and hence $S_0^r = S_0$ by Proposition \ref{maximal regular minimizing sequences} ((b)). Let $X_0^r = \text{supp}(\kappa_Y S_0^r)$. Since $Z_k^\ast \subset U_k$ and $U_k \subset \Omega(X_0, \sqrt{n}2^{-k+1})$, it follows from Lemma \ref{maximal regular minimizing sequences} that $X_0^r \subset X_0 \subset U$. Therefore, $\kappa_Y S_0^r \in \mathcal{C}(U)$, and we may apply Proposition \ref{maximal regular minimizing sequences} to get a corresponding measure $\mu_{\kappa_Y S_0}$. It suffices to show $\mu_{\kappa_Y S_0} = \mu_{\kappa_Y S_0}$.

Let $\mathcal{S}'$ be a new dyadic subdivision of $\mathbb{R}^n$ such that each cube $Q \in \mathcal{S}'$ is $(\kappa_Y S_0^r)$-compatible\footnote{See Definition \ref{compatibility}} and $(\kappa_Y S_0^r)$-compatible. Fix $Q \in \mathcal{S}'$. Then by Proposition \ref{compatibility},
\[
\mu_{\kappa_Y S_0^r}(Q) = \lim_{i \to \infty} \mu_{\kappa_Y S_k^i}(Q) = \lim_{i \to \infty} S^{n-1}(Z_k^i \cap Q) \leq \lim_{i \to \infty} S^{n-1}(X_k \cap Q) + (2\sqrt{n})^{n-1}2^{-k_i} \\
= \lim_{i \to \infty} \mu_{\kappa_Y S_k^i}(Q) = \mu_{\kappa_Y S_0}(Q).
\]
Likewise, by \eqref{inequality 12},
\[
\mu_{\kappa_Y S_0}(Q) = \lim_{i \to \infty} S^{n-1}(X_k \cap Q) \leq \lim_{i \to \infty} S^{n-1}(Z_k^i \cap Q) + 2^{-k_i} = \mu_{\kappa_Y S_k^i}(Q).
\]
Now if $W \subset \mathbb{R}^n$ is open, by taking a Whitney decomposition of $W$ using cubes from $\mathcal{S}'$ we conclude that $\mu_{\kappa_Y S_0^r}(W) = \mu_{\kappa_Y S_0}(W)$. Since both measures are finite Borel measures on $\mathbb{R}^n$ and hence Radon measures, outer regularity proves the two measures are equal.

Finally, since $\text{supp}(\kappa_Y S_0^r) = Z_k^\ast \subset U_k \subset \Omega(\text{supp}(\kappa_Y S_0), \sqrt{n}2^{-k+1})$, it follows from Lemma \ref{maximal regular minimizing sequences} and compactness of $\text{supp}(\kappa_Y S_0)$ that $\text{supp}(\kappa_Y S_k)$ converges to $\text{supp}(\kappa_Y S_0)$ in the Hausdorff metric.

\begin{flushright}$\square$
\end{flushright}

\subsection{Lower regular minimizing sequences.}

\textbf{Lemma 7.3.1.} Suppose $S_0 \in \mathcal{J}(M, Y, U)$ is an area minimizer and $S_k \to S_0$ in $\hat{\mathcal{B}}_{n-1}^2(U)$ with \{\{S_k\} \subset \mathcal{J}(M, Y, U) and $\kappa_Y S_k \to \kappa_Y S_0$ in $\mathcal{C}(U)$. Suppose further that $X_k \to X_0$ in the Hausdorff
metric, where \( X_k := \text{supp}(\kappa_Y S_k) \) for \( k \geq 0 \). Then there exists a subsequence \( k_i \to \infty \) such that

\[
S^{n-1}(X_k(p, r)) > \frac{2^{(-i-1)(n-1)}}{(2(n-1))^{n-1} (2^n K_n)^{n-2}} \quad \text{for all } \bar{\Omega}(p, r) \in \Gamma(X_k) \text{ and } r > 2^{-i-1}.
\]

Proof. If not, there exist \( N_1 > 0 \) such that for all \( k \geq N_1 \), there exists \( \bar{\Omega}(p_k, r_k) \in \Gamma(X_k) \) with \( r_k > 2^{-N_2-1} \) such that

\[
S^{n-1}(X_k(p_k, r_k)) \leq \frac{2^{(-N_2-1)(n-1)}}{(2(n-1))^{n-1} (2^n K_n)^{n-2}}.
\]

Since \( U \) is bounded, there exists a subsequence \( p_{k_j} \to p \in X_0 \). By Lemma 7.2.4, there exist \( 2^{-N_2-2} < r'_{k_j} < r_{k_j} \) and a compact spanning set \( X_{k_j} \) satisfying the conclusions of Lemma 7.2.4. In particular,

\[
S^{n-1}(X_{k_j}(p_{k_j}, r'_{k_j})) \leq \frac{1}{2} S^{n-1}(X_{k_j}(p_{k_j}, r_{k_j})).
\]

Let \( \bar{\Omega}(p, r) \in \Gamma(X_0) \) be \((\kappa_Y S_0)\)-compatible, with \( r < 2^{-N_2-2} \). Then for \( j \) large enough, \( \bar{\Omega}(p, r) \subset \Omega(p_{k_j}, r'_{k_j}) \). Since \( p \in X_0 \), Propositions 3.2.2 and 3.2.1 ((e)) imply

\[
0 < \mu_{\kappa_Y S_0}(\bar{\Omega}(p, r)) = \lim_{j \to \infty} S^{n-1}(X_{k_j}(p, r)).
\]

Thus, for \( j \) large enough,

\[
0 < \frac{1}{2} \mu_{\kappa_Y S_0}(\bar{\Omega}(p, r)) < S^{n-1}(X_{k_j}(p, r)) \leq S^{n-1}(X_{k_j}(p_{k_j}, r_{k_j})).
\]

But since \( X_{k_j} \cap \bar{\Omega}(p_{k_j}, r'_{k_j})^c = \bar{X}_{k_j} \cap \bar{\Omega}(p_{k_j}, r'_{k_j})^c \), we may use (16) and (17) to deduce

\[
S^{n-1}(\bar{X}_{k_j}) = S^{n-1}(\bar{X}_{k_j}(p_{k_j}, r'_{k_j})) + S^{n-1}(X_{k_j} \cap \bar{\Omega}(p_{k_j}, r'_{k_j})^c)
\leq \frac{1}{2} S^{n-1}(X_{k_j}(p_{k_j}, r'_{k_j})) + S^{n-1}(X_{k_j} \cap \bar{\Omega}(p_{k_j}, r'_{k_j})^c)
\leq S^{n-1}(X_{k_j}) - \frac{1}{2} S^{n-1}(X_{k_j}(p_{k_j}, r'_{k_j}))
< S^{n-1}(X_{k_j}) - \frac{1}{4} \mu_{\kappa_Y S_0}(\bar{\Omega}(p, r)).
\]

Since \( S^{n-1}(X_{k_j}) \to m \), we have \( S^{n-1}(\bar{X}_{k_j}) < m \) for \( j \) large enough, a contradiction.

\[\]

Corollary 7.3.2. Suppose \( S_0 \in \mathcal{F}(M, Y, U) \) is an area minimizer and \( S_k \to S_0 \in \hat{\mathcal{B}}_{2-2}^2(U) \) with \( \{S_k\} \subset \mathcal{F}(M, Y, U) \) and \( \kappa_Y S_k \to \kappa_Y S_0 \in \hat{\mathcal{C}}(U) \). Suppose further that \( X_k \to X_0 \) in the Hausdorff metric, where \( X_k := \text{supp}(\kappa_Y S_k) \) for \( k \geq 0 \). Then there exists a subsequence \( X_{k_i} \to X_0 \) and a constant \( a > 0 \) such that

\[
F_{k_i}(p, r) \geq ar^{n-1}
\]

for all \( i > 0 \), \( \bar{\Omega}(p, r) \in \Gamma(X_{k_i}) \) and \( r > 2^{-i} \).

Proof. Since \( S^{n-1}(X_k) \to m \), let us assume without loss of generality that \( S^{n-1}(X_k) \leq m + \frac{2^{(-i+1)(n-1)-1}}{(2(n-1))^{n-1} (2^n K_n)^{n-2}} \). Let \( X_{k_i} \) be the subsequence determined by Lemma 7.3.1. Fix \( i \), and suppose \( \bar{\Omega}(p, r) \in \Gamma(X_{k_i}) \) and \( r > 2^{-i} \). Let \( \bar{\Omega}(p, s) \in \Gamma_c(X_{k_i}) \) where \( r > s > 2^{-i-1} \). Let \( Z_{k_i} \) be the
set determined by Lemma 7.2.1 using $X_k$, and $\bar{\Omega}(p,s)$. Since $S^{n-1}(Z_k) \geq m$ and $Z_k \cap \bar{\Omega}(p,s)^c = X_k \cap \bar{\Omega}(p,s)^c$, we have

$$S^{n-1}(X_k(p,s)) - \frac{2(-k,-1)(n-1)}{(2(n-1))^{n-1}(2^nK_n)^{n-2}} \leq S^{n-1}(Z_k(p,s)).$$

Thus, it follows from Lemmas 4.0.5, 7.3.1 and 7.2.1 that

$$F_k(p,s) \leq S^{n-1}(X_k(p,s)) \leq 2S^{n-1}(X_k(p,s)) - \frac{2(-k,-1)(n-1)}{(2(n-1))^{n-1}(2^nK_n)^{n-2}} \leq 2K_nS^{n-2}(x_k(p,s))^{(n-1)/(n-2)}.$$ 

In other words, for almost every $x \in (2^{-i-1}, r)$,

$$\frac{\partial}{\partial x} F_k(p,x) \in (2^{-i-1}, (n-2)/(n-1)) \geq (2K_n)^{-i-1}.$$ 

Integrating, this implies $F_k(p,r)^{1/(n-1)} - F_k(p,2^{-i-1})^{1/(n-1)} \geq \frac{1}{n-1} (2K_n)^{-i-1} (r - 2^{-i-1})$, and thus

$$F_k(p,r) \geq \frac{r^{n-1}}{(2(n-1))^{n-1}(2^nK_n)^{n-2}}.$$ 

Setting $a = (2(n-1))^{-(n-1)} (2K_n)^{-(n-2)}$ completes the proof. \qed

Theorem 7.2.7, Corollary 7.3.2 and Lemma 4.0.5 imply:

**Corollary 7.3.3.** Suppose $S_0 \in \mathcal{T}(M,Y,U)$ is an area minimizer. Then there exists a sequence $S_k \to S_0$ in $\mathcal{B}_{n-1}^2(U)$ where $\{S_k\} \subset \mathcal{T}(M,Y,U)$ such that

(a) $\kappa Y S_k \to \kappa Y S_0$ in $\mathcal{C}(U)$;
(b) $X_k \to X_0$ in the Hausdorff metric, where $X_i := \text{supp}(\kappa Y S_i)$;
(c) $\{X_k\}$ is lower regular minimizing.

Recall $\beta = \beta(\{X_k\}) := \inf \left\{ \liminf_{k \to \infty} \frac{S^{n-1}(X_k(p,r))}{\alpha_{n-1}^{n-1}r^{n-1}} : \bar{\Omega}(p,r) \in \Gamma(X_0) \right\}$.

**Lemma 7.3.4.** If $\{X_k\} \subset \mathcal{X}(M,U)$ is lower regular minimizing and $X_k \to X_0$ in the Hausdorff metric, then $\beta \geq a/\alpha_{n-1} > 0$.

**Proof.** Let $\bar{\Omega}(p,r) \in \Gamma(X_0)$ and $0 < \delta < 1$. For sufficiently large $k$, there exists $\bar{\Omega}(p_k,r_k) \in \Gamma(X_k)$ such that

$$2^{-k} < \delta \cdot r < r_k < r$$

and $\bar{\Omega}(p_k,r_k) \subset \bar{\Omega}(p,r)$. So by Corollary 7.3.2 and Lemma 4.0.5, $S^{n-1}(X_k(p,r)) \geq S^{n-1}(X_k(p_k,r_k)) \geq a\delta^{n-1}r_k^{n-1}$ and thus $\beta \geq a\delta^{n-1}/\alpha_{n-1}$. Now let $\delta \to 1$. \qed

**Theorem 7.3.5.** If $\{X_k\} \subset \mathcal{X}(M,U)$ is lower regular minimizing and $X_k \to X_0 \subset U$ in the Hausdorff metric, then $S^{n-1}(X_0) < \infty$. 

Proof. Suppose \( \{\tilde{\Omega}(p_i, r_i)\}_{i \in I} \subset \Gamma(X_0) \) is a collection of disjoint balls. If \( J \subset I \) is finite, then
\[
\beta \sum_{j \in J} \alpha_{n-1} r_j^{n-1} \leq \liminf_{k \to \infty} \sum_{j \in J} S^{n-1}(X_k(p_j, r_j)) \leq \liminf_{k \to \infty} \sum_{j \in J} S^{n-1}(X_k(p_j, r_j)) \\
\leq \liminf_{k \to \infty} S^{n-1}(X_k) = m.
\]
Since \( I \) is necessarily countable, it follows from Lemma 7.3.4 that
\[(19) \quad \alpha_{n-1} \sum_{i \in I} r_i^{n-1} \leq \frac{m}{\beta}.
\]
Now fix \( \delta \) and \( \delta' \) such that \( \delta' > \delta > 0 \) and \( \Omega(X_0, \delta) \subset U \). Then the subcollection of \( \Gamma(X_0) \) consisting of balls of radius \( r < \delta \) covers \( X_0 \cap \Omega(M, \delta') \). So, by the Vitali Covering Lemma and (19), it follows that
\[
S^{\alpha}_{100}(X_0 \cap \Omega(M, \delta')) \leq S^{n-1}\frac{m}{\beta}.
\]
Letting \( \delta \to 0 \) and then \( \delta' \to 0 \), we deduce that
\[
S^{n-1}(X_0 \setminus M) < \infty.
\]
Since \( M \) has finite \( S^{n-2} \)-measure, we conclude that \( S^{n-1}(X_0) < \infty \). \( \square \)

7.4. Lower semicontinuity of Hausdorff spherical measure. In this section we establish lower semicontinuity of Hausdorff spherical measure for lower regular minimizing sequences of compact spanning sets:

**Theorem 7.4.1.** Suppose \( \{X_k\} \subset X(M, U) \) is lower regular minimizing and \( X_k \to X_0 \subset U \) in the Hausdorff metric. If \( W \subset \mathbb{R}^n \) is open, then
\[
S^{n-1}(X_0 \cap W) \leq \liminf S^{n-1}(X_k \cap W).
\]

For this we make use of versions of several technical lemmas appearing in [Rei60]. Theorem 7.4.1 will follow once we have proved that \( \beta \geq 1 \) (Lemma 7.4.9.) The key inequality needed appears in Lemma 7.4.3 below: If \( X \in X(M, U) \) with \( S^{n-1}(X) \leq m + \delta \) for some \( \delta > 0 \), and \( \tilde{\Omega}(p, r) \subset U \setminus M \) is \( X \)-compatible for all \( k \geq 1 \), then
\[
S^{n-1}(X_k(p, r)) \leq \frac{rS^{n-2}(x_k(p, r))}{n-1} + \epsilon_k.
\]
Lemma 7.4.3 relies on the next result which is modeled after Lemma 7 in [Rei60] and makes use of a modified cone construction:

**Lemma 7.4.2.** Let \( Z \in X(M, U) \) and suppose \( \tilde{\Omega}(p, r) \) is disjoint from \( M \) and \( Z \)-compatible. If \( P_0 \in \tilde{\Omega}(p, r) \) and \( z(p, r) = \bigcup_{j=1}^N \tilde{z}_j \), where \( \tilde{z}_j \subset \tilde{\Omega}(P_0, r_j) \) for some \( r_j, j = 1, \ldots, N \), then for each \( \epsilon > 0 \), there exists a compact set \( \tilde{Z} \) spanning \( M \) such that
\[
(a) \quad \tilde{Z} \setminus \Omega(p, r) = Z \setminus \Omega(p, r);
(b) \quad S^{n-1}(\tilde{Z}(p, r)) \leq (1 + \epsilon) \sum_{j=1}^N \frac{r_j}{n-1} S^{n-2}(\tilde{z}_j);
(c) \quad \tilde{Z}(p, r) \subset h(z(p, r) \cup P_0);
(d) \quad \text{There exists an } (n-1)\text{-dimensional polyhedron } P \subset \tilde{Z}(p, r) \text{ such that } S^{n-1}(\tilde{Z}(p, r) \setminus P) < \epsilon.
\]
Proof. Let \( \tilde{Z}(p,r) \) be the surface \( X \) determined by \( A = x(p,r) \) in Lemma 7 of [Rei60]. Then \( \tilde{Z} := \tilde{Z}(p,r) \cup (Z \setminus \Omega(p,r)) \) satisfies (1)-(4). To see that \( \tilde{Z} \) spans \( M \), we use a modification of the argument in Lemma 5.0.8. Suppose \( N = \eta(S^1) \) is a simple link of \( M \) and is disjoint from \( \tilde{Z} \).

First project the portion of \( \eta \) inside

\[
E_0 := \Omega(p,r) \setminus \cup_i C_{P_i} \left( Fr \Omega(p,r) \cap \tilde{\Omega}(P_i, q_i) \right)
\]

radially away from \( P_0 \in E_0 \) onto \( Fr E_0 \), where \( P_i \) and \( q_i \) are defined in Reifenberg’s proof. Repeating this initial step as the inductive step for each \( X_i \subset \tilde{\Omega}(P_i, q_i) \) as defined in Reifenberg’s proof, we obtain by induction a new simple link \( \eta' \) of \( M \) with \( \eta'(S^1) \subset \Omega(p,r)^c \) disjoint from \( \tilde{Z} \), hence also disjoint from \( Z \), yielding a contradiction. \( \square \)

The following is an adaptation of Lemma 1* of [Rei60].

**Lemma 7.4.3.** Suppose \( X \in \mathcal{X}(M, U) \), with \( S^{n-1}(X) \leq m + \delta \) for some \( \delta > 0 \). If \( \tilde{\Omega}(p,r) \subset U \setminus M \), then

\[
S^{n-1}(X(p,r)) \leq \frac{r}{n-1} S^{n-2}(x(p,r)) + \delta.
\]

Proof. If \( \tilde{\Omega}(p,r) \) is not \( X \)-compatible, we are done. Otherwise, let \( \epsilon > 0 \) and let \( \tilde{X}_k \in \mathcal{X}(M, U) \) be defined as in Lemma 7.4.2. Then

\[
S^{n-1}(X(p,r)) \leq S^{n-1}(\tilde{X}(p,r)) + \delta \leq (1 + \epsilon) \frac{r}{n-1} S^{n-2}(x(p,r)) + \delta.
\]

\( \square \)

The following is an adaptation of Lemma 3* of [Rei60].

**Lemma 7.4.4.** Suppose \( \{X_k\} \subset \mathcal{X}(M, U) \) is lower regular minimizing, and \( X_k \to X_0 \) in the Hausdorff metric. Then

\[
\beta = \inf \left\{ \liminf_{k \to \infty} \frac{F_k(p,r)}{\alpha_{n-1} r^{n-1}} : \tilde{\Omega}(p,r) \in \Gamma(X_0) \right\}.
\]

Proof. Fix \( \tilde{\Omega}(p,r) \in \Gamma(X_0) \). By Lemma 4.0.5, for almost every \( t \in (0,r) \), the ball \( \tilde{\Omega}(p,t) \) is \( X_k \)-compatible for all \( k > 0 \). Let \( \epsilon_k \to 0 \) such that \( S^{n-1}(X_k) \leq m + \epsilon_k \). Using the definition of \( \beta \) and Lemma 7.4.3 we have

\[
\alpha_{n-1} r^{n-1} = \int_0^r \frac{n-1}{t} \alpha_{n-1} t^{n-1} dt \leq \int_0^r \frac{n-1}{t \beta} \liminf_{k \to \infty} S^{n-1}(X_k(p,t)) dt
\]

\[
= \int_0^r \frac{n-1}{t \beta} \liminf_{k \to \infty} (S^{n-1}(X_k(p,t)) - \epsilon_k) dt
\]

\[
\leq \frac{1}{\beta} \int_0^r \liminf_{k \to \infty} S^{n-2}(x_k(p,t)) dt
\]

\[
\leq \frac{1}{\beta} \liminf_{k \to \infty} \int_0^r S^{n-2}(x_k(p,t)) dt
\]

\[
= \frac{1}{\beta} \liminf_{k \to \infty} F_k(p,r).
\]
Thus, by Lemma 4.0.5,

\[
\beta \leq \inf_{\Omega(p,r) \in \Gamma(X_0)} \left\{ \liminf_{k \to \infty} \frac{F_k(p,r)}{\alpha_{n-1} r^{n-1}} \right\} \leq \inf_{\Omega(p,r) \in \Gamma(X_0)} \left\{ \liminf_{k \to \infty} \frac{S^{n-1}(X_k(p,r))}{\alpha_{n-1} r^{n-1}} \right\} = \beta.
\]

The following lemma is an adaptation of Lemma 4* of [Rei60].

**Lemma 7.4.5.** Suppose \( \{X_k\} \subset \mathcal{X}(M,U) \) is lower regular minimizing and \( X_k \to X_0 \) in the Hausdorff metric. If \( 0 < r_1 < r_2 \) and \( \Omega(p,r_2) \in \Gamma(X_0) \), then

\[
\liminf_{k_i \to \infty} \frac{F_{k_i}(p,r_1)}{\alpha_{n-1} r_1^{n-1}} \leq \liminf_{k_i \to \infty} \frac{F_{k_i}(p,r_2)}{\alpha_{n-1} r_2^{n-1}}
\]

and

\[
\limsup_{k_i \to \infty} \frac{F_{k_i}(p,r_1)}{\alpha_{n-1} r_1^{n-1}} \leq \limsup_{k_i \to \infty} \frac{F_{k_i}(p,r_2)}{\alpha_{n-1} r_2^{n-1}}
\]

for every subsequence of integers \( k_i \to \infty \).

**Proof.** By Lemma 7.4.4, for sufficiently large \( k \), \( \beta \alpha_{n-1} r_1^{n-1}/(n-1) \leq F_k(p,r) \) for all \( r \geq r_1 \). Let \( \epsilon_k \to 0 \) such that \( S^{n-1}(X_k) \leq m + \epsilon_k \). By Lemmas 7.4.3 and 4.0.5, for almost every \( r \in (r_1, r_2) \),

\[
\frac{n-1}{r} \left( 1 - \frac{\epsilon_k}{1 - \beta \alpha_{n-1} r_1^{n-1}} \right) \leq \frac{F_k(p,r)}{F_k(p,r_1)},
\]

so integrating from \( r_1 \) to \( r_2 \), we get

\[
\exp \left( \frac{-\epsilon_k}{1 - \beta \alpha_{n-1} r_1^{n-1}} \log \frac{r_2^{n-1}}{r_1^{n-1}} \right) \frac{F_k(p,r_1)}{\alpha_{n-1} r_1^{n-1}} \leq \frac{F_k(p,r_2)}{\alpha_{n-1} r_2^{n-1}}.
\]

In other words,

\[
\exp \left( \frac{-\epsilon_k}{1 - \beta \alpha_{n-1} r_1^{n-1}} \log \frac{r_2^{n-1}}{r_1^{n-1}} \right) \frac{F_k(p,r_1)}{\alpha_{n-1} r_1^{n-1}} \leq \frac{F_k(p,r_2)}{\alpha_{n-1} r_2^{n-1}}.
\]

The result follows since \( \epsilon_k \to 0 \) as \( k \to \infty \). \( \square \)

**Definition 7.4.6.** Suppose \( \{X_k\} \subset \mathcal{X}(M,U) \) is lower regular minimizing and \( X_k \to X_0 \) in the Hausdorff metric. Let \( \epsilon > 0 \). We say that \( \tilde{\Omega}(p',r') \in \Gamma(X_0) \) is \((\{X_k\}, \epsilon)\)-uniform if

\[
\beta \leq \liminf_{k \to \infty} \frac{F_k(p,r)}{\alpha_{n-1} r^{n-1}} \leq \limsup_{k \to \infty} \frac{F_k(p,r)}{\alpha_{n-1} r^{n-1}} \leq \beta + \epsilon
\]

and

\[
\beta \leq \liminf_{k \to \infty} \frac{S^{n-1}(X_k(p,r))}{\alpha_{n-1} r^{n-1}} \leq \limsup_{k \to \infty} \frac{S^{n-1}(X_k(p,r))}{\alpha_{n-1} r^{n-1}} \leq \beta + \epsilon
\]

for every \( \tilde{\Omega}(p,r) \in \Gamma(X_0) \) with \( \tilde{\Omega}(p,r) \subset \tilde{\Omega}(p',r') \).

The following lemma is an adaptation Lemma 5* of [Rei60], setting \( p = n \).

**Lemma 7.4.7.** Suppose \( \{X_k\} \subset \mathcal{X}(M,U) \) is lower regular minimizing and \( X_k \to X_0 \) in the Hausdorff metric. If \( \eta > 0 \), there exist \( \beta > \epsilon > 0 \) and \( 1 > \lambda > 0 \) such that if \( \tilde{\Omega}(p',r') \in \Gamma(X_0) \) is \((\{X_k\}, \epsilon)\)-uniform, then there exist \( P \in X_0(p',r') \) and a hyperplane \( \Pi_P \) through \( P \) such that \( X_0(P, \lambda r') \subset \Omega(\Pi_P, \eta \lambda r') \), and \( \tilde{\Omega}(P, \lambda r') \subset \tilde{\Omega}(p',r') \).
Proof. The proof is identical to that of Lemma 5* of [Rei60], with the exception that Equation (6) on p.30 of [Rei60] follows from Lemma 7.4.2. The reader interested in details can find them in [Rei60]. There are several ambiguities in his proof, however, so we provide the reader with some helpful notes: First, the sets \( \ell_{0}^{n} \) and \( C_{\theta} \) consist of those points whose joins to \( P \) make an an angle not greater than \( \theta \) with the line passing through the points \( P \) and \( Q \), not just the line segment \( PQ \). Second, the point \( P_{0} \) should be on the opposite side of \( P \) from \( Q \), not between the two as stated. Third, Equation (6) is valid only for \( n \) large enough. Fourth, the square roots in Equation (6) arise from the law of cosines applied three different times, and the inequality follows from applying Lemma 7.4.2 using center point \( P_{0} \).

The next result is based on Lemma 12, p. 22 of [Rei60].

**Lemma 7.4.8.** Let \( X \in \mathcal{X}(M,U) \) and \( \bar{\Omega}(q,r) \in \Gamma(X) \). If \( \Pi \) is a hyperplane containing \( q \) and \( x(q,r) \subset \Omega(\Pi,er) \) for some \( \epsilon < 1/2 \), then either

\[
S^{n-1}(X(q,r)) \geq \alpha_{n-1}r^{n-1} - 2^{2(n-1)}\epsilon \frac{\alpha_{n-1}}{\alpha_{n-2}} S^{n-2}(x(q,r))
\]

or there exists a compact set \( X' \) spanning \( M \) equal to \( X \) outside \( \bar{\Omega}(q,r) \) such that

\[
S^{n-1}(X'(q,r)) \leq 2^{2(n-1)}\epsilon \frac{\alpha_{n-1}}{\alpha_{n-2}} S^{n-2}(x(q,r)).
\]

**Proof.** Let \( C \) be the radial projection from \( q \) onto \( fr\bar{\Omega}(q,r) \) of \( \bigcup_{x \in x(q,r)} I_{x} \), where \( I_{x} \) denotes the line segment joining \( x \) to its orthogonal projection on \( \Pi \). By Lemma 4.0.2 and Lemma 4.0.7,

\[
S^{n-1}(C) \leq 2^{2(n-1)}\epsilon \frac{\alpha_{n-1}}{\alpha_{n-2}} S^{n-2}(x(q,r)).
\]

There are two possibilities: Either \( Y := (X \setminus X(q,r)) \cup C \) spans \( M \), or it does not (see Figure 7.) Suppose \( Y \) does not span \( M \). We show the orthogonal projection of \( C \cup X(q,r) \) onto \( \Pi \) contains \( \Pi \cap \Omega(q,r) \), and thus by Lemma 4.0.1 the first conclusion of the lemma will be satisfied. Let \( N = \eta(S^{1}) \) be a simple link of \( M \) disjoint from \( Y \). Let us assume without loss of generality that the intersection \( N \cap fr\bar{\Omega}(q,r) \) is transverse. The intersection \( N \cap \bar{\Omega}(q,r) \) consists of a finite collection of arcs \( \{\eta_{i}\} \) whose endpoints \( \{p_{1}^{i}, p_{2}^{i}\} \) lie on \( fr\bar{\Omega}(q,r) \). Since \( C \subset fr\bar{\Omega}(q,r) \) we may replace each arc \( \eta_{i} \) with a pair of line segments, \( p_{1}^{i}q \) followed by \( qp_{2}^{i} \) (so now, the curve \( \eta \) will only be piecewise smooth.) However, the intersection \( N \cap \bar{\Omega}(q,r) \) is now disjoint from \( \bigcup_{x \in x(q,r)} I_{x} \) as well.

The hyperplane \( \Pi \) divides \( \bar{\Omega}(q,r) \) into two hemispheres; north with pole \( N \), and south with pole \( S \). Pick an endpoint \( p_{1}^{j} \), \( j = 1,2 \), lying in some hemisphere with pole \( P \). Observe that for any \( x \) in the geodesic arc between \( p_{1}^{j} \) and \( P \), the line segment between \( x \) and \( q \) will be disjoint from \( \bigcup_{x \in x(q,r)} I_{x} \). This gives a regular homotopy of \( N \), and so we may assume without loss of generality that each \( p_{1}^{j} \in \{N,S\} \).

Now, suppose the orthogonal projection of \( C \cup X(q,r) \) onto \( \Pi \) does not contain \( \Pi \cap \Omega(q,r) \). Let \( a \in \Pi \cap \Omega(q,r) \) be such a point and let \( a_{N}, a_{S} \) be the points in the northern and southern hemispheres of \( fr\Omega(p,r) \), respectively, whose orthogonal projection onto \( \Pi \) is \( a \). By sliding the \( p_{1}^{j} \) along the geodesic arcs from \( N \) to \( a_{N} \) and \( S \) to \( a_{S} \), we can, as above, assume each \( p_{1}^{j} \in \{a_{N}, a_{S}\} \). Now replace the pair of segments \( p_{1}^{j}q, qp_{2}^{j} \) with the single segment \( p_{1}^{j}p_{2}^{j} \) (it may be degenerate.) Smoothing the resulting curve, this gives a simple link \( N \) of \( M \) disjoint from \( X \), a contradiction.
Finally, if $Y$ does span $M$, then the set $X' := Y^*$ satisfies the second conclusion of the lemma. \qed

The proof of the following Lemma (with slightly different notation and without a formal statement) can be found in [Rei60] Chapter 3, pp. 34-36. The proof relies on Lemmas 7.4.3, 7.4.4, 7.4.5, 7.4.7, and 7.4.8.

**Lemma 7.4.9.** Suppose $\{X_k\} \subset \mathcal{X}(M,U)$ is lower regular minimizing. Then $\beta \geq 1$.

**Proof of Theorem 7.4.1.** Let $\delta > 0$. Since $S^{n-1}(M) = 0$ it follows from Theorem 7.3.5 and Lemma 4.0.1 that we may cover $S^{n-1}$ almost all of $X_0 \cap W$ by a collection $\{\bar{\Omega}(p_i, r_i)\} \subset \Gamma(X_0)$ of disjoint balls of diameter $2r_i < \delta$ and contained in $W$. Thus by Lemma 7.4.9,

$$S^{n-1}_\delta(X_0 \cap W) = S^{n-1}_\delta(X_0 \cap W \cap \bigcup_i \bar{\Omega}(p_i, r_i))$$

\begin{align*}
&\leq \sum_i \alpha_{n-1} r_i^{n-1} \\
&\leq \sum_i \liminf_{k \to \infty} S^{n-1}(X_k(p_i, r_i)) \\
&\leq \liminf_{k \to \infty} \sum_i S^{n-1}(X_k(p_i, r_i)) \\
&\leq \liminf_{k \to \infty} S^{n-1}(X_k \cap W).
\end{align*}

\qed
Proof of Theorem 7.0.1. Let $S_0 \in \mathcal{F}(M, U)$ be an area minimizer. By Corollary 7.3.3 and Theorem 6.0.12, we may apply Theorem 7.4.1. We set $W = \mathbb{R}^n$, and this yields, by Definition 9,

$$S^{n-1}(\text{supp}(\kappa_Y S_0)) \leq m.$$ 

In particular, $\text{supp}(\kappa_Y S_0)$ will have empty interior, and hence by Corollary 2.0.4, $\text{supp}(\kappa_Y S_0) = \text{supp}(S_0)$. On the other hand, by Theorem 6.0.12 and Proposition 6.0.8(d), $\text{supp}(S_0)^* \in \mathcal{X}(M, U)$ and hence $S^{n-1}(\text{supp}(S_0)) = m$. We next prove that $S_0 \in \mathcal{F}(M, Y, U)$.

Pick a sequence $\{S_k\}$ as in Corollary 7.3.3 and a dyadic subdivision $\mathcal{G}$ of $\mathbb{R}^n$ such that each $Q \in \mathcal{G}$ is $\kappa_Y S_k$-compatible and $X_k := \text{supp}(S_k)$-compatible for all $k \geq 0$. Theorem 7.4.1 implies

$$S^{n-1}(X_0 \cap Q) \leq \lim inf S^{n-1}(X_k \cap Q)$$

for all $Q \in \mathcal{G}$. This is in fact an equality, for if $S^{n-1}(X_0 \cap Q) < \lim inf S^{n-1}(X_k \cap Q)$ for some $Q \in \mathcal{G}$, then by Theorem 7.4.1 applied to $W = X_0 \cap Q^c$,

$$m = S^{n-1}(X_0) = S^{n-1}(X_0 \cap Q) + S^{n-1}(X_0 \cap Q^c)$$

$$\leq S^{n-1}(X_0 \cap Q) + \lim inf S^{n-1}(X_k \cap Q^c)$$

$$< \lim inf S^{n-1}(X_k \cap Q) + \lim inf S^{n-1}(X_k \cap Q^c)$$

$$\leq \lim inf S^{n-1}(X_k) = m.$$ 

So, by Proposition 3.2.2, $S^{n-1}|_{X_0}(Q) = \lim inf \mu_{\kappa_Y S_k}(Q) = \mu_{\kappa_Y S_0}(Q)$. Using a Whitney decomposition, this equality extends to all open sets $W$, and hence by outer regularity of the finite Borel measure $\mu_{\kappa_Y S_0}$,

$$(22) S^{n-1}|_{X_0} = \mu_{\kappa_Y S_0}.$$ 

It follows from Corollary 2.0.4 and Proposition 3.2.1 (e) that $X_0$ is reduced. Thus, $X_0 \in \mathcal{X}(M, U)$, and so by Proposition 6.0.8 (b)-(c) and Theorem 6.0.12, $S_0 \in \mathcal{F}(M, Y, U)$. 

8. Regularity

Let $\mathcal{X}(M) := \cup_U \mathcal{X}(M, U)$ where the union ranges over all convex open sets $U$ containing $M$. Likewise, let $\mathcal{F}(M) := \cup_U \mathcal{F}(M, Y, U)$ where the union ranges over all convex open sets $U$ containing $M$ and all vector fields $Y$ as in §2. By Corollary 6.0.11 and Theorem 7.0.1, there exists an element $S_0 \in \mathcal{F}(M)$ such that $A(S_0) = S^{n-1}(\text{supp}(S_0)) = m$. Thus, we have proved the bulk of our main theorem: there exists a size-minimizing element of $\mathcal{F}(M)$. We now prove some regularity results for such size-minimizing elements.

Lemma 8.0.1. If $S_0 \in \mathcal{F}(M)$ is a size minimizer, and if $\bar{\Omega}(p, r)$ is $X_0 := \text{supp}(S_0)$-compatible and disjoint from $M$, then $S^{n-1}(X_0(p, r))/r^{n-1} \geq K_n^{-(n-2)}(n-1)^{(-(n-1))}$.

Proof. By Lemmas 7.2.1 and 4.0.5 we have

$$F_0(p, r) = \int_0^r S^{n-2}(x_0(p, t))dt \leq S^{n-1}(X_0(p, r)) \leq K_n S^{n-2}(x_0(p, r))^{(n-1)/(n-2)}.$$ 

18See Definition 3.2.3. This is possible by Proposition 6.0.8.
In other words,
\[
\frac{d}{dr} F_0(p,r) \geq \frac{K_n^{-n/(n-1)}}{F_0(p,r)^{(n-2)/(n-1)}} \geq K_n^{-n/(n-1)}
\]
Integrating both sides from 0 to \( r \) yields
\[
F_0(p,r)^{(1/(n-1)} \geq K_n^{-n/(n-1)} r/(n-1).
\]
Applying Lemma 4.0.5 again we have
\[
S^{n-1}(X_0(p,r)) \geq F_0(p,r) \geq K_n^{-n/(n-1)} r^{n/(n-1)}(n-1)^{1/(n-1)}.
\]
**Theorem 8.0.2.** The support \( X_0 \) of any size minimizer \( S_0 \in \mathcal{S}(M) \) is \((n-1)\)-rectifiable.

**Proof.** Say \( S_0 \in \mathcal{S}(M, Y, U) \) for some \( Y \) and \( U \). By Preiss’s theorem, (Corollary 17.9 of [Mat99]) it suffices to show the density \( \Theta^{n-1}(\mu_{\mathcal{Y}, S_0}, p) \) exists and is positive and finite for \( S^{n-1} \) almost every \( p \in X_0 \). The density \( \Theta^{n-1}(\mu_{\mathcal{Y}, S_0}, p) \) exists and is finite by Proposition 5.16 of [Dav99], which is valid for both \( \mathcal{H}^{n-1} \) and \( S^{n-1} \). In the notation of [Dav99], the set \( U \) is an open ball about \( p \) whose closure is disjoint from \( M \), and \( X_0 \) is minimal (with respect to Hausdorff spherical measure) in \( U \) by Lemma 5.0.5. That \( \Theta^{n-1}(\mu_{\mathcal{Y}, S_0}, p) \) is non-zero follows from Lemma 8.0.1.

**Corollary 8.0.3.** If \( S_0 \in \mathcal{S}(M) \) is a size-minimizer and \( \overline{O}(p, r) \) is disjoint from \( M \), then \( \text{supp}(S_0) \cap \Omega(p, r) \) is \((1, \delta)\)-restricted\(^{19}\) with respect to \( \Omega(p, r)^c \) for all \( \delta > 0 \).

**Proof.** Let \( X_0 = \text{supp}(S_0) \). By Lemma 8.0.2 and Theorem 3.2.26 of [Fed69], it follows that \( \mathcal{H}^{n-1}(X_0) = S^{n-1}(X_0) = \mu \). Suppose \( \phi(X_0) \) is a competitor\(^{20}\) of \( X_0 \) with respect to \( M \). Since \( \phi \) is Lipschitz, the set \( \phi(X_0) \) is also \((n-1)\)-, and so \( \mathcal{H}^{n-1}(\phi(X_0)) = S^{n-1}(X_0) \). By Lemmas 5.0.5 and 5.0.6, \( \phi(X_0)^c \in \mathcal{S}(M) \), and so putting this together we have
\[
\mathcal{H}^{n-1}(X_0) \leq \mathcal{H}^{n-1}(\phi(X_0)).
\]
This gives the result, since every permissible deformation in the definition of restricted sets is of this type. \( \square \)

It follows from Corollary 8.0.3 and [Alm68] (1.7) that the support of any size minimizer \( S_0 \in \mathcal{S}(M) \) is almost everywhere a real analytic \((n-1)\)-dimensional minimal submanifold of \( \mathbb{R}^n \). Soap film regularity for \( n = 3 \) follows from [Tay76], and this completes the proof of our main theorem.

\begin{thebibliography}{99}

[Alm68] Frederick Almgren, *Existence and regularity almost everywhere of solutions to elliptic variational problems among surfaces of varying topological type and singularity structure*, Annals of Mathematics 87 (1968), no. 2, 321–391.

[Alm75] , *Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints*, Bulletin of the American Mathematical Society 81 (1975), no. 1, 151–155.

[Alm76] , *Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints*, vol. 4, Mem. Amer. Math. Soc., 1976.

\(^{19}\)See [Alm76], also [Alm75] and [Dav99].

\(^{20}\)See Definition 5.0.4.

\end{thebibliography}
[Bes48] Abram Samoilovitch Besicovitch, *Parametric surfaces III: Surfaces of minimum area*, Journal London Math. Soc. **23** (1948), 241–246.

[Bes49] Abram Samoilovitch Besicovitch, *Parametric surfaces I: Compactness*, Proc. Cambridge Phil. Soc. **45** (1949), 1–13.

[Cou50] Richard Courant, *Dirichlet’s principle, conformal mapping, and minimal surfaces*, Interscience, 1950.

[Dav09] Guy David, *Hölder regularity of two-dimensional almost-minimal sets in $\mathbb{R}^n$*, Annales de la faculte des sciences de Toulouse mathematiques **XVIII** (2009), no. 1, 65–246.

[Dav11] Guy David, *Regularity of minimal and almost minimal sets and cones; J. TAYLOR’S theorem for beginners*, 2011.

[Dou31] Jesse Douglas, *Solutions of the problem of Plateau*, Transactions of the American Mathematical Society **33** (1931), 263–321.

[dP09] Thierry de Pauw, *Size minimizing surfaces with integral coefficients*, Annales scientifiques de l’Ecole normale superieure **42** (2009), no. 1, 37–101.

[Fed69] Herbert Federer, *Geometric measure theory*, Springer, Berlin, 1969.

[FF60] Herbert Federer and Wendell H. Fleming, *Normal and integral currents*, The Annals of Mathematics **72** (1960), no. 3, 458–520.

[Fle66] Wendell H. Fleming, *Flat chains over a finite coefficient group*, Transactions of the American Mathematical Society **121** (1966), no. 1, 160–186.

[Fol99] Gerald Folland, *Real analysis*, John Wiley and Sons, 1999.

[Gio61] De Giorgi, *Frontiere orientate di misura minima*, Sem. Mat. Scuola Norm. Sup. Pisa (1960-1961).

[Har] Jenny Harrison, *Ravello lecture notes*, 2005.

[Har93] _______., *Stokes’ theorem on nonsmooth chains*, Bulletin of the American Mathematical Society **29** (1993), 235–242.

[Har04] _______., *On Plateau’s problem for soap films with a bound on energy*, Journal of Geometric Analysis **14** (2004), no. 2, 319–329.

[Har12a] _______., *Operator calculus of differential chains and differential forms*, Journal of Geometric Analysis (to appear) (2012), no. arXiv:1106.5839.

[Har12b] _______., *Soap film solutions of Plateau’s problem*, Journal of Geometric Analysis (2012).

[Hil69] Stefan Hildebrandt, *Boundary behavior of minimal surfaces*, Arch. Rational Mech. Ana **35** (1969), 47–81.

[Hir76] Morris W. Hirsch, *Differential topology*, Springer-Verlag, Berlin, 1976.

[HP12] Jenny Harrison and Harrison Pugh, *Topological aspects of differential chains*, Journal of Geometric Analysis **22** (2012), no. 3, 685–690.

[HS79] Robert Hardt and Leon Simon, *Boundary regularity and embedded solutions for the oriented Plateau problem*, Bulletin of the American Mathematical Society **1** (1979), no. 1, 263–265.

[Lag61] Joseph-Louis Lagrange, *Essai d’une nouvelle méthode pour déterminer les maxima et les minima des formules intégrales indéfinies*, Miscellanea Taurinensia (1760-1761), 335–362.

[Leb02] Henri Lebesgue, *Intégrale, longueur, aire*, Annali di Matematica (1902).

[Mag12] Francesco Maggi, *Sets of finite perimeter and geometric variational problems*, Cambridge University Press, 2012.

[Mas59] William Massey, *On the normal bundle of a sphere imbedded in euclidean space*, Proc. Amer. Math. Soc. **10** (1959), no. 6, 959–964.

[Mat99] Pertti Mattila, *Geometry of sets and measures in Euclidean spaces: Fractals and rectifiability*, Cambridge University Press, 1999.

[Mor89] Frank Morgan, *Size minimizing rectifiable currents*, Inventiones mathematicae **96** (1989), 333–348.

[Pla73] Joseph Plateau, *Experimental and theoretical statics of liquids subject to molecular forces only*, Gauthier-Villars, 1873.

[Pug09] Harrison Pugh, *Applications of differential chains to complex analysis and dynamics*, Harvard senior thesis (2009).

[Rad33] Tibor Radó, *On the problem of Plateau*, Ergebnisse d. Math Vol **2** (1933).

[Rei60] Ernst Robert Reifenberg, *Solution of the Plateau problem for m-dimensional surfaces of varying topological type*, Acta Mathematica **80** (1960), no. 2, 1–14.

[Sim57] James Simons, *Minimal cones, PLATEAU’S problem, and the BERNSTEIN conjecture*, Proc Natl Acad Sci **2** (1957), no. 410-411.

[Tay76] Jean Taylor, *The structure of singularities in soap-bubble-like and soap-film-like minimal surfaces*, Annals of Mathematics **103** (1976), no. 2, 489–539.

[TT88] Friedrich Tomi and Anthony Tromba, *Existence theorems for minimal surfaces of non-zero genus spanning a contour*, Memoirs of the AMS **302** (1988).
[War64] D.J. Ward, *A counterexample in area theory*, Mathematical Proceedings of the Cambridge Philosophical Society 60 (1964), no. 4, 821–845.

[Whi57] Hassler Whitney, *Geometric Integration Theory*, Princeton University Press, Princeton, NJ, 1957.