The Dabkowski-Sahi invariant and 4-moves for links

Haruko A. Miyazawa¹ · Kodai Wada² · Akira Yasuhara³

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Abstract
Dabkowski and Sahi defined an invariant of a link in the 3-sphere, which is preserved under 4-moves. This invariant is a quotient of the fundamental group of the complement of the link. It is generally difficult to distinguish between the Dabkowski-Sahi invariants of given links. In this paper, we give a necessary condition for the existence of an isomorphism between the Dabkowski-Sahi invariant of a link and that of the corresponding trivial link. Using this condition, we provide a practical obstruction to a link to be trivial up to 4-moves.

Keywords 4-move · Link · Welded link · Link-homotopy · Magnus expansion

Mathematics Subject Classification Primary 57K10; Secondary 57M05 · 57K12

1 Introduction

A 4-move is a local move as shown in Fig. 1. Two links are 4-equivalent if they are related by a finite sequence of 4-moves and isotopies. In the late 1970s, Nakanishi first systematically studied the 4-move, see [18] for a good survey of this topic. In 1979, he [9, Problem 1.59(3)(a)] conjectured that every knot is 4-equivalent to the trivial one. This conjecture remains open, although it has been verified for several classes of knots, e.g. 2-bridge knots [16], knots with 12 or less crossings [2], and closed 3-braid knots [18].

Since the linking number modulo 2 is a 4-equivalence invariant, not every link of two or more components is 4-equivalent to a trivial one. Moreover, Nakanishi [17, Proposition 4] proved that the Borromean rings is not 4-equivalent to the trivial 3-component link. In
1985, Kawauchi [9, Problem 1.59(3)(b)] proposed the following question: if two links are link-homotopic, then are they 4-equivalent? Here, two links are link-homotopic if they are related by a finite sequence of self-crossing changes and isotopies [12].

In [4, Theorem 3], Dabkowski and Przytycki showed that the 3-component link \( W \) in Fig. 4, which is link-homotopically trivial, is not 4-equivalent to the trivial one by using the 4th Burnside group of links introduced in [3]. This gives a negative answer to the above question of Kawauchi in the case of links of three or more components. However, the question in the case of 2-component links still remains open. Since two 2-component links are link-homotopic if and only if they have the same linking number, this open question can be rephrased as follows.

**Problem 1.1** (Dabkowski and Przytycki [4]) *Is it true that two 2-component links with the same linking number are 4-equivalent?*

This problem has been shown to be true for several classes of 2-component links, e.g. links with 11 or less crossings [2], 2-algebraic links [18], closed 3-braid links [18], and alternating links with 12 crossings [18]. On the other hand, there is a growing belief that the problem has a negative answer. In fact, it is very likely that there is a 2-component link with linking number 0 that is not 4-equivalent to the trivial one. While the 4th Burnside group of links is a powerful 4-equivalence invariant, it cannot distinguish any 2-component link from the trivial one [4, Theorem 4]. Therefore, in [5], Dabkowski and Sahi defined a new 4-equivalence invariant \( R_4(L) \) of a link \( L \) in the 3-sphere \( S^3 \), which is a quotient of \( \pi_1(S^3 \setminus L) \). For a given link \( L \), it is generally difficult to distinguish \( R_4(L) \) from \( R_4(O) \), where \( O \) is the trivial link of the same components as \( L \). In [5], by comparing the associated Lie rings of the groups, Dabkowski and Sahi proved that \( R_4(L) \) and \( R_4(O) \) are not isomorphic when \( L \) is either the link \( W \) or the Brorromean rings. (In [1], Brittenham, Hermiller, and Todd developed techniques for computing the invariant \( R_4 \) in the case of knots by considering a quotient of a Coxeter group.)

In this paper, we give a necessary condition for the existence of an isomorphism between \( R_4(L) \) and \( R_4(O) \) by using the nilpotent quotient of \( R_4 \) (Lemma 2.3). This condition deeply relies on the fact that, for an \( m \)-component link \( L \), the nilpotent quotient of \( R_4(L) \) has a “good” group presentation with \( m \) generators (Lemma 2.2). We use Lemma 2.3 to obtain a practical obstruction to \( L \) to be 4-equivalent to \( O \) (Theorem 3.1). Moreover, when \( L \) is a 3-component link, we provide a stronger obstruction than Theorem 3.1 (Theorem 3.4). The latter obstruction allows us to give a general construction of 3-component links that are link-homotopically trivial but not 4-equivalent to \( O \) (Proposition 4.4). This result widely generalizes [4, Theorem 3] (Remark 4.5).

The same result as Theorem 3.1 also holds for welded links. As its application, we show that there is a 2-component welded link with linking number 0 that is not 4-equivalent to the trivial one (Proposition 5.1). This gives a negative answer to Problem 1.1 for 2-component welded links. We remark that for any classical 2-component link with linking number 0, Theorem 3.1 cannot be used to show that the 2-component link is not 4-equivalent to the trivial one (Remark 3.5).
2 The Dabkowski-Sahi invariant of links

Throughout this paper, for a set $X$, we denote by $F(X)$ the free group on the alphabet $X^\pm = \{ x, x^{-1} \mid x \in X \}$. Let $q \geq 1$ be an integer. For a group $G$, we denote by $\Gamma_q G$ the $q$th term of the lower central series of $G$ and by $N_q G$ the $q$th nilpotent quotient $G / \Gamma_q G$. For two normal subgroups $H$ and $I$ of $G$, we denote by $H \cdot I$ the subgroup of $G$ generated by all $ab$ with $a \in H$ and $b \in I$.

Let $m \geq 1$ be an integer and $L$ an $m$-component link in $S^3$. All links in this paper are assumed to be ordered and oriented. We denote by $G$ the fundamental group of the complement $S^3 \setminus L$. Let $\langle X \mid R \rangle$ be a Wirtinger presentation of $G(L)$. Then we define a set of words in $F(X)$ as

$$S(X) = \{(awbw^{-1})^2(wbw^{-1}a)^{-2} \mid a, b \in X^\pm, w \in F(X)\} \subset F(X).$$

The Dabkowski-Sahi invariant of $L$, defined in [5], is the group given by the presentation $\langle X \mid R, S(X) \rangle$. We denote it by $\mathcal{R}_4(L)$. It is shown in [5, Proposition 2.3] that $\mathcal{R}_4(L)$ is a 4-equivalence invariant of $L$.

In this paper, we introduce a quotient of $\mathcal{R}_4(L)$. For an integer $n \geq 1$, we define a set of words in $F(X)$ as

$$W_n(X) = \{ w^n \mid w \in F(X) \} \subset F(X).$$

The $n$th reduced Dabkowski-Sahi invariant of $L$ is the group given by the presentation $\langle X \mid R, S(X), W_n(X) \rangle$. We denote it by $N^q_4(L)$. The Dabkowski-Sahi invariant $\mathcal{R}_4(L)$ and the reduced one $\mathcal{R}_4^q(L)$ are not always finite, but the nilpotent quotient $N^q_4 \mathcal{R}_4^q(L)$ is always a finite group for any $q \geq 1$, see [19, Chapter 2]. Since $\mathcal{R}_4(L)$ is invariant under 4-moves, the following result is obtained immediately.

Proposition 2.1 The group $N^q_4 \mathcal{R}_4^q(L)$ and its cardinality $|N^q_4 \mathcal{R}_4^q(L)|$ are invariant under 4-moves.

Now consider a diagram of $L$. For each $1 \leq i \leq m$, we choose one arc of the $i$th component of the diagram and denote it by $x_{i1}$. As shown in Fig. 2, let $x_{i2}, x_{i3}, \ldots, x_{ir_i}$ be the other arcs of the $i$th component in turn with respect to the orientation, where $r_i$ denotes the number of arcs of the $i$th component. In this figure, $u_{ij} \in \{ x_{kl} \}$ denotes the arc that separates $x_{ij}$ and $x_{ij+1}$. Let $\varepsilon_{ij}$ be the sign of the crossing among $x_{ij}, x_{ij+1}$, and $u_{ij}$. For $1 \leq i \leq m$ and $1 \leq j \leq r_i$, we put

$$v_{ij} = u_{i1}^{\varepsilon_{i1}} u_{i2}^{\varepsilon_{i2}} \cdots u_{ij}^{\varepsilon_{ij}} \in F(X),$$

where $X = \{ x_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq r_i \}$. In particular, we denote by $\lambda_i$ the word $v_{i r_i}$. By the geometric construction of the Wirtinger presentation, the word $\lambda_i$ corresponds to an $i$th longitude of $L$. In this sense, we call $\lambda_i$ an $i$th longitude word of $L$. Let $A = \{ \alpha_1, \alpha_2, \ldots, \alpha_m \}$. Following [13], we define a sequence of homomorphisms $\eta_q : F(X) \to F(A)$ inductively by

$$\eta_1(x_{i1}) = \alpha_i, \quad \eta_{q+1}(x_{i1}) = \alpha_i, \quad \text{and} \quad \eta_{q+1}(x_{ij+1}) = \eta_q(v_{ij}^{-1}) \alpha_i \eta_q(v_{ij}).$$

In [13, Theorem 4], Milnor used the homomorphism $\eta_q$ to give an explicit presentation of the nilpotent quotient $N^q G(L)$ as follows:

$$N^q G(L) \cong \langle \alpha_1, \ldots, \alpha_m \mid [\alpha_1, \eta_q(\lambda_1)], \ldots, [\alpha_m, \eta_q(\lambda_m)], \Gamma_q F(A) \rangle,$$
where \([\alpha_i, \eta_q(\lambda_i)] = \alpha_i^{-1} \eta_q(\lambda_i)^{-1} \alpha_i \eta_q(\lambda_i)\). For the groups \(N_q \mathcal{R}_4(L)\) and \(N_q \mathcal{R}_4^n(L)\), we obtain the following presentations.

**Lemma 2.2** The group \(N_q \mathcal{R}_4(L)\) has the presentation

\[
\langle \alpha_1, \ldots, \alpha_m \mid [\alpha_1, \eta_q(\lambda_1)], \ldots, [\alpha_m, \eta_q(\lambda_m)], S(A), \Gamma_q F(A) \rangle
\]

and the group \(N_q \mathcal{R}_4^n(L)\) has the presentation

\[
\langle \alpha_1, \ldots, \alpha_m \mid [\alpha_1, \eta_q(\lambda_1)], \ldots, [\alpha_m, \eta_q(\lambda_m)], S(A), W_n(A), \Gamma_q F(A) \rangle.
\]

The proof of this lemma is a minor modification of that of [13, Theorem 4], and therefore we omit it.

**Lemma 2.3** Let \(L\) be an \(m\)-component link and \(\lambda_i\) its \(i\)th longitude word \((1 \leq i \leq m)\). Let \(O\) be the trivial \(m\)-component link. If \(\mathcal{R}_4(L)\) and \(\mathcal{R}_4(O)\) are isomorphic, then the set \([\alpha_i, \eta_q(\lambda_i)] \mid 1 \leq i \leq m\) is a subset of the normal subgroup \(\langle (S(A)) \cdot \langle (W_n(A)) \cdot \Gamma_q F(A) \rangle\) of \(F(A)\), where \(\langle (\cdot) \rangle\) denotes the normal closure.

**Proof** The proof is parallel to that of [14, Lemma 2.1]. By Lemma 2.2, we have

\[
N_q \mathcal{R}_4^n(L) \cong \langle \alpha_1, \ldots, \alpha_m \mid [\alpha_1, \eta_q(\lambda_1)], \ldots, [\alpha_m, \eta_q(\lambda_m)], S(A), W_n(A), \Gamma_q F(A) \rangle
\]

and

\[
N_q \mathcal{R}_4^n(O) \cong \langle \alpha_1, \ldots, \alpha_m \mid S(A), W_n(A), \Gamma_q F(A) \rangle.
\]

Consider the following sequence of two natural projections \(\psi\) and \(\phi\):

\[
F(A) \xrightarrow{\psi} \langle \alpha_1, \ldots, \alpha_m \mid S(A), W_n(A), \Gamma_q F(A) \rangle
\]

\[
\xrightarrow{\phi} \langle \alpha_1, \ldots, \alpha_m \mid [\alpha_1, \eta_q(\lambda_1)], \ldots, [\alpha_m, \eta_q(\lambda_m)], S(A), W_n(A), \Gamma_q F(A) \rangle.
\]

Then it follows that \(\psi([\alpha_i, \eta_q(\lambda_i)]) \in \ker \phi\) for any \(1 \leq i \leq m\). On the other hand, we have

\[
|N_q \mathcal{R}_4^n(O)| = |\langle \alpha_1, \ldots, \alpha_m \mid S(A), W_n(A), \Gamma_q F(A) \rangle| = |N_q \mathcal{R}_4^n(L)| \times |\ker \phi|.
\]

Since \(\mathcal{R}_4(L)\) and \(\mathcal{R}_4(O)\) are isomorphic, \(N_q \mathcal{R}_4^n(L)\) and \(N_q \mathcal{R}_4^n(O)\) are also isomorphic, and moreover, they are finite groups. This implies that \(|\ker \phi|\) must be equal to 1. Therefore \([\alpha_i, \eta_q(\lambda_i)] \in \langle (S(A)) \cdot \langle (W_n(A)) \cdot \Gamma_q F(A) \rangle\). □

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3 Obstruction to a link to be trivial up to 4-moves

Denote by \( \mathbb{Z}_2 \) the cyclic group \( \mathbb{Z}/2\mathbb{Z} \) of order 2. The Magnus \( \mathbb{Z}_2 \)-expansion \( E^{(2)} \) is a homomorphism from \( F(A) \) to the ring of the formal power series in noncommutative variables \( X_1, \ldots, X_m \) with coefficients in \( \mathbb{Z}_2 \) defined by

\[
E^{(2)}(\alpha_i) = 1 + X_i \quad \text{and} \quad E^{(2)}(\alpha_i^{-1}) = 1 + X_i^2 + X_i^3 + \cdots
\]

for \( 1 \leq i \leq m \).

**Theorem 3.1** Let \( n \geq 8 \) be a power of 2 and \( q \geq 5 \) an integer. Let \( L \) be an \( m \)-component link and \( \lambda_i \) its \( i \)-th longitude word \( (1 \leq i \leq m) \). Suppose that

\[
E^{(2)}([\alpha_i, \eta_q(\lambda_i)]) = 1 + \sum_{j \geq 1} \sum_{k_1, \ldots, k_j \in \{1, \ldots, m\}} c(k_1, \ldots, k_j)X_{k_1} \cdots X_{k_j}
\]

for certain coefficients \( c(k_1, \ldots, k_j) \in \mathbb{Z}_2 \). If \( L \) is \( 4 \)-equivalent to the trivial \( m \)-component link, then the following hold.

(i) \( c(k) = c(k, k) = c(k, k, k) = c(k, k, k, k) = 0 \) for any \( k \).

(ii) \( c(k, l, l) = c(l, k, l) = c(k, l, l) = c(l, k, l) = c(k, l, k) \) for any \( k \neq l \).

(iii) \( c(k, l, l) = c(l, l, k) \) and \( c(k, k, k, l) = c(l, k, k) = c(k, l, k) \) for any \( k \neq l \).

(iv) If \( c(k_1, k_2, k_3, k_4) \neq 0 \) and \( |\{k_1, k_2, k_3, k_4\}| = 3 \), then \( k_1 = k_2, k_2 = k_3, \) or \( k_3 = k_4 \) for any \( k_1, k_2, k_3, k_4 \).

(v) \( c(k_1, \ldots, k_j) = 0 \) for any pairwise distinct indices \( k_1, \ldots, k_j \) with \( j < \min\{n, q\} \).

To show this theorem, we prepare the following two lemmas.

**Lemma 3.2** Let \( n \) be a positive power of 2. If \( t \in \langle \langle W_n(A) \rangle \rangle \), then \( E^{(2)}(t) = 1 + D(n) \), where \( D(n) \) denotes the terms of degree \( \geq n \).

**Proof** Since \( \langle \langle W_n(A) \rangle \rangle \) is a verbal subgroup, the element \( t \) can be written in the form \( \prod_i w_i^n \) for some \( w_i \in F(A) \). For each \( i \), the Magnus \( \mathbb{Z}_2 \)-expansion \( E^{(2)}(w_i) \) has the form \( 1 + D(1) \). Since \( n \) is a power of 2, all the binomial coefficients \( \binom{n}{r} \) are even for \( 0 < r < n \). Therefore

\[
E^{(2)}(w_i^n) = (1 + D(1))^n = 1 + (D(1))^n = 1 + D(n)
\]

for each \( i \). This shows that \( E^{(2)}(t) = 1 + D(n) \).

**Lemma 3.3** If \( u \in \Gamma_q F(A) \), then \( E^{(2)}(u) = 1 + D(q) \).

**Proof** This follows from [10, Corollary 5.7] directly.

**Proof of Theorem 3.1** Since \( L \) is \( 4 \)-equivalent to the trivial \( m \)-component link \( O, \mathcal{R}_q(L) \) and \( \mathcal{R}_q(O) \) are isomorphic. By Lemma 2.3, we have

\[
\{[\alpha_I, \eta_q(\lambda_I)] \mid 1 \leq I \leq m \} \subset \langle \langle S(A) \rangle \rangle \cdot \langle \langle W_n(A) \rangle \rangle \cdot \Gamma_q F(A).
\]

This implies that for each \( i \), the commutator \([\alpha_I, \eta_q(\lambda_I)]\) can be written in the form \( stu \) for some \( s \in \langle \langle S(A) \rangle \rangle, t \in \langle \langle W_n(A) \rangle \rangle \), and \( u \in \Gamma_q F(A) \). By Lemmas 3.2 and 3.3, we have

\[
E^{(2)}(t) = 1 + D(n) \quad \text{and} \quad E^{(2)}(u) = 1 + D(q).
\]

Since \( n \) and \( q \) can be chosen arbitrarily large, the condition \( j < \min\{n, q\} \) is no real restriction.
Since \( \min\{n, q\} \geq 5 \), it suffices to show that \( E^{(2)}(s) \) satisfies (i)--(v).

The element \( s \in \langle (S(A)) \rangle \) can be written in the form \( \prod_i g_i^{-1} s_i^{e_i} g_i \) for some \( g_i \in F(A) \), \( s_i \in S(A) \), and \( e_i \in \{1, -1\} \). Put \( E^{(2)}(s_i^{e_i}) = 1 + S \). Since the Magnus \( \mathbb{Z}_2 \)-expansions \( E^{(2)}(g_i) \) and \( E^{(2)}(g_i^{-1}) \) have the same terms of degree 1, we denote the terms by \( G_1 \). By a straightforward computation, it follows that

\[
E^{(2)}(g_i^{-1} s_i^{e_i} g_i) = 1 + S + S G_1 + G_1 S + \mathcal{D}((\min \deg S) + 2). \quad (1)
\]

Now we consider \( E^{(2)}(s_i^{e_i}) \). The element \( s_i^{e_i} \) can be written in the form

\[
(\alpha_k^\varepsilon w \alpha_i^\delta w^{-1})^2 (w \alpha_i^\varepsilon w^{-1} \alpha_k^\delta)^{-2}
\]

for some \( w \in F(A), k, l \in \{1, \ldots, m\} \), and \( \varepsilon, \delta \in \{1, -1\} \). We define a formal power series \( f_r^\varepsilon \) as

\[
f_r^\varepsilon = \begin{cases} 0 & (\varepsilon = 1), \\ X_r^2 + X_r^3 + X_r^4 + \cdots & (\varepsilon = -1). \end{cases}
\]

Putting \( E^{(2)}(w) = 1 + F \) and \( E^{(2)}(w^{-1}) = 1 + \overline{F} \), we have

\[
E^{(2)}(\alpha_k^\varepsilon w \alpha_i^\delta w^{-1}) = (1 + X_k + f_k^\varepsilon) (1 + X_l + f_l^\delta) (1 + \overline{F})
\]

\[
= (1 + X_k + f_k^\varepsilon) (1 + X_l + X_l \overline{F} + f_l^\delta + f_l^\delta \overline{F} + F X_l + F X_l \overline{F} + F f_l^\delta + F f_l^\delta \overline{F})
\]

\[
= 1 + X_k + X_l + P_1,
\]

where \( P_1 = X_k X_l + f_k^\varepsilon + f_l^\delta + F X_l + X_l \overline{F} + X_k F X_l + X_k X_l \overline{F} + F X_l \overline{F} + X_k f_l^\delta + f_k^\varepsilon X_l + F f_l^\delta + f_l^\delta \overline{F} + \mathcal{D}(4). \) Similarly we have

\[
E^{(2)}((w \alpha_i^\varepsilon w^{-1} \alpha_k^\delta)^{-1}) = E^{(2)}(\alpha_k^{-\varepsilon} w \alpha_i^{-\delta} w^{-1}) = 1 + X_k + X_l + P_2,
\]

where \( P_2 \) is obtained from \( P_1 \) by replacing \( \varepsilon \) and \( \delta \) with \( -\varepsilon \) and \( -\delta \), respectively. Therefore it follows that

\[
E^{(2)}(s_i^{e_i}) = (1 + X_k + X_l + P_1)^2 (1 + X_k + X_l + P_2)^2
\]

\[
= (1 + (X_k + X_l)^2 + (X_k + X_l) P_1 + P_1 (X_k + X_l) + P_1^2) \times (1 + (X_k + X_l)^2 + (X_k + X_l) P_2 + P_2 (X_k + X_l) + P_2^2).
\]

For \( r = 1, 2 \) we put \( Q_r = (X_k + X_l) P_r + P_r (X_k + X_l) + P_r^2 \ (\in \mathcal{D}(3)) \), and then have

\[
E^{(2)}(s_i^{e_i}) = 1 + (X_k + X_l)^4 + Q_1 + Q_2 + \mathcal{D}(5). \quad (2)
\]

We here observe

\[
Q_1 + Q_2 = (X_k + X_l)(P_1 + P_2) + (P_1 + P_2)(X_k + X_l) + P_1^2 + P_2^2.
\]

Firstly we focus on \( (X_k + X_l)(P_1 + P_2) + (P_1 + P_2)(X_k + X_l) \). Since \( f_k^\varepsilon + f_k^{-\varepsilon} = X_k^2 + X_k^3 + \cdots \)

and \( f_l^\delta + f_l^{-\delta} = X_l^2 + X_l^3 + \cdots \), we have

\[
P_1 + P_2 = X_k^2 + X_l^2 + X_k^3 + X_k^3 + X_k X_l^2 + X_k^2 X_l + F_1 X_l^2 + X_l^2 F_1 + \mathcal{D}(4),
\]
where $F_1$ denotes the terms of degree 1 in $F$. Note that $F_1$ also coincides with the terms of degree 1 in $\overline{F}$. By the equation above, we have

\[
\begin{align*}
(X_k + X_l)(P_1 + P_2) + (P_1 + P_2)(X_k + X_l) \\
= X_kX_l^2 + X_lX_k^2 + X_k^2X_l + X_l^2X_k \\
+ X_kX_lX_kX_l + X_kX_kX_kX_l + X_lX_kX_kX_l + X_kX_lX_kX_l \\
+ X_lF_1X_l^2 + X_kX_l^2F_1 + X_lF_1X_l^2 + X_l^2F_1 \\
+ F_1X_l^2X_k + F_1X_l^2 + X_l^2F_1X_k + X_l^2F_1X_l + \mathcal{D}(S).
\end{align*}
\]

Secondly we focus on $P_1^2 + P_2^2$. It follows that

\[
P_1^2 = (X_kX_l + X_lX_k + X_kF_1 + f_k^\varepsilon + f_l^\delta + \mathcal{D}(3))^2
\]

\[
= (X_kX_l + X_lX_k + X_kF_1 + f_k^\varepsilon + f_l^\delta)(X_kX_l + X_lX_k + X_kF_1 + f_k^\varepsilon + f_l^\delta)
\]

\[+(f_k^\varepsilon + f_l^\delta)(X_kX_l + X_lX_k + X_kF_1 + f_k^\varepsilon + f_l^\delta)^2 + \mathcal{D}(S)
\]

and

\[
P_2^2 = (X_kX_l + X_lX_k + X_kF_1)^2 + (X_kX_l + X_lX_k + X_kF_1)(f_k^{\varepsilon -} + f_l^{\delta -})
\]

\[+(f_k^{\varepsilon -} + f_l^{\delta -})(X_kX_l + X_lX_k + X_kF_1 + f_k^{\varepsilon -} + f_l^{\delta -})^2 + \mathcal{D}(S).
\]

This gives that

\[
P_1^2 + P_2^2
\]

\[= (X_kX_l + X_lX_k + X_kF_1)(X_k^2 + X_l^2) + (X_k^2 + X_l^2)(X_kX_l + X_lX_k + X_kF_1)
\]

\[+ \mathcal{D}(S) + X_k^4 + X_l^4 + \begin{cases}
X_k^2X_l^2 + X_l^2X_k^2 & (\varepsilon\delta = 1), \\
0 & (\varepsilon\delta = -1).
\end{cases}
\]

Therefore we have

\[
Q_1 + Q_2 = X_kX_l^2 + X_lX_k^2 + X_k^2X_l + X_l^2X_k
\]

\[+ X_kX_lX_kX_l + X_lX_kX_kX_l + I_{kl}(\varepsilon, \delta) + (X_k + X_l)^4
\]

\[+ X_kF_1X_l^2 + X_kX_l^2F_1 + F_1X_l^2X_k + X_l^2F_1X_k
\]

\[+ F_1X_lX_k^2 + X_lF_1X_k^2 + X_k^2F_1X_l + X_l^2F_1X_k + \mathcal{D}(S),
\]

\[\text{(3)}\]

where

\[
I_{kl}(\varepsilon, \delta) = \begin{cases}
0 & (\varepsilon\delta = 1), \\
X_k^2X_l^2 + X_l^2X_k^2 & (\varepsilon\delta = -1).
\end{cases}
\]

By Eqs. (2) and (3), it follows that

\[
E^{(2)}(s^\varepsilon_l^\delta) = 1 + X_kX_l^2 + X_lX_k^2 + X_k^2X_l + X_l^2X_k
\]

\[+ X_kX_lX_kX_l + X_lX_kX_kX_l + I_{kl}(\varepsilon, \delta)
\]

\[+ X_kF_1X_l^2 + X_kX_l^2F_1 + F_1X_l^2X_k + X_l^2F_1X_k
\]

\[+ F_1X_lX_k^2 + X_lF_1X_k^2 + X_k^2F_1X_l + X_l^2F_1X_k + \mathcal{D}(S).
\]

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Applying this to Eq. (1), we have
\[
E^{(2)} \left( g_i^{-1} s_i^e g_i \right) = 1 + X_k X_l^2 + X_l X_k^2 + X_k^2 X_l + X_l^2 X_k \\
+ X_k X_l X_i X_l + X_l X_k X_i X_k + I_{kl}(\varepsilon, \delta) \\
+ X_k F_1 X_l^2 + X_l X_k^2 F_1 + F_1 X_l X_k^2 + X_l^2 F_1 X_k \\
+ F_1 X_l X_k^2 + X_l F_1 X_k^2 + X_l^2 F_1 X_l + X_l^2 X_l F_1 \\
+ X_k X_l^2 G_1 + X_l X_k^2 G_1 + X_k^2 X_l G_1 + X_l^2 X_k G_1 \\
+ G_1 X_k X_l^2 + G_1 X_l X_k^2 + G_1 X_k^2 X_l + G_1 X_l^2 X_k + D(5). 
\]

This shows that \( E^{(2)} \left( g_i^{-1} s_i^e g_i \right) \) satisfies (i)–(iv) for each \( j \), and hence \( E^{(2)}(s) = E^{(2)} \left( \prod_i g_i^{-1} s_i^e g_i \right) \) also satisfies them.

Let \( \mathcal{O}(2) \) be the ideal generated by monomials containing \( X_r \) at least twice for some \( r \in \{1, \ldots, m\} \). Then we have
\[
E^{(2)} \left( a_k^e w a_k^\delta w^{-1} \right) \\
≡ (1 + X_k)(1 + F)(1 + X_l) \left( 1 + F \right) \equiv E^{(2)} \left( (w a_k^\delta w^{-1} a_k^e)^{-1} \right) \quad (\text{mod } \mathcal{O}(2)).
\]

This gives that
\[
E^{(2)}(s_i^e) \equiv ((1 + X_k)(1 + F)(1 + X_l) \left( 1 + F \right))^4 \equiv 1 \quad (\text{mod } \mathcal{O}(2)),
\]
and therefore \( E^{(2)}(s) - 1 \in \mathcal{O}(2) \). Consequently \( E^{(2)}(s) \) satisfies (v).

Let \( M \) be the free \( \mathbb{Z}_2 \)-module generated by the set
\[
\left\{ X_j^2 X_k X_l, X_j X_k^2 X_l, X_j X_k X_l^2 \mid \{j, k, l\} = \{1, 2, 3\} \right\}.
\]

In the case \( \mathcal{A} = \{a_1, a_2, a_3\} \), we denote by \( E^{(2)}|_M \) the corestriction of \( E^{(2)} \) on \( M \), i.e.
\[
E^{(2)}|_M : F(a_1, a_2, a_3) \longrightarrow E^{(2)}(F(a_1, a_2, a_3)) \cap M.
\]

When \( L \) is a 3-component link, by further analyzing Eq. (\( \star \)) in the proof of Theorem 3.1, we obtain the following theorem.

**Theorem 3.4** Let \( q \geq 5 \) be an integer, \( L \) a 3-component link, and \( \lambda_i \) its \( i \)th longitude word \( (i = 1, 2, 3) \). If \( L \) is 4-equivalent to the trivial 3-component link, then
\[
E^{(2)}|_M \left( [a_i, \eta_q(\lambda_i)] \right)
\]
\[
= \delta_1(2 X_2 X_3^2 X_1 + X_1 X_3^2 X_2) + \delta_2(2 X_1 X_3 X_2^2 + X_3^2 X_2 X_1) \\
+ \delta_3(2 X_3 X_2 X_1 + X_1 X_2 X_3) + \delta_4(2 X_3 X_2 X_1 + X_1 X_2 X_3) \\
+ (\delta_1 + \delta_2)(2 X_2 X_3^2 X_1 + X_1 X_3^2 X_2) + (\delta_3 + \delta_4)(2 X_3 X_2 X_1^2 + X_1^2 X_2 X_3) \\
+ (\delta_1 + \delta_3)(2 X_2 X_1^2 + X_1^2 X_2 X_1) + (\delta_2 + \delta_4)(2 X_3 X_2^2 + X_2^2 X_1 X_3) \\
+ (\delta_1 + \delta_2 + \delta_3 + \delta_4)(2 X_2 X_1 X_2 + X_2^2 X_1 X_3)
\]
for some \( \delta_1, \delta_2, \delta_3, \delta_4 \in \mathbb{Z}_2 \). In particular, the number of terms in \( E^{(2)}|_M \left( [a_i, \eta_q(\lambda_i)] \right) \) is 0, 8, or 12 (see Table 1).
Table 1 The number \( N \) of terms in \( E^{(2)}|_{M\left(\left[\alpha_i, \eta q(\lambda_i)\right]\right)} \)

| \( \delta_1 \) | \( \delta_2 \) | \( \delta_3 \) | \( \delta_4 \) | \( \delta_1 + \delta_2 \) | \( \delta_3 + \delta_4 \) | \( \delta_1 + \delta_3 \) | \( \delta_2 + \delta_4 \) | \( \delta_1 + \delta_2 + \delta_3 + \delta_4 \) | \( N \) |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 8 |
| 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 8 |
| 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 8 |
| 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 1 | 8 |
| 1 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 8 |
| 0 | 1 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 8 |
| 1 | 1 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 12 |
| 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 8 |
| 1 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 12 |
| 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 8 |
| 1 | 1 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | 12 |
| 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 8 |
| 1 | 0 | 1 | 1 | 1 | 0 | 1 | 0 | 1 | 12 |
| 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 8 |

**Proof** We will use the same notation as in the proof of Theorem 3.1. As shown in the proof, it suffices to prove that for \( s = \prod g_i^{-1} s_i^e i g_i \in \langle \langle S(\alpha_1, \alpha_2, \alpha_3) \rangle \rangle \), the corestriction \( E^{(2)}|_{M\left(s\right)} \) has the required form in the statement. As seen in Eq. (\( \star \)), each term in the Magnus \( \mathbb{Z}_2 \)-expansion \( E^{(2)}(g_i^{-1} s_i^e i g_i) \) is of degree \( \geq 3 \). Therefore we have

\[
E^{(2)}(s) - 1 = E^{(2)}\left(\prod g_i^{-1} s_i^e i g_i\right) - 1 = \sum_i \left(E^{(2)}\left(g_i^{-1} s_i^e i g_i\right) - 1\right) + \mathcal{D}(6).
\]

This shows that

\[
E^{(2)}|_{M\left(s\right)} = \sum_i E^{(2)}|_{M\left(g_i^{-1} s_i^e i g_i\right)}.
\]

By Eq. (\( \star \)), we have

\[
E^{(2)}|_{M\left(g_i^{-1} s_i^e i g_i\right)} = p(X_k X_j X_l^2 + X_k X^2_j X_j + X_j X^2_k X_k + X^2_j X_j X_k
+ X_j X_j X_k^2 + X_j X_j X_k^2 + X^2_j X_j X_l + X^2_j X_j X_l) + q(X_k X_j X_j + X_j X_k^2 X_j + X^2_j X_j X_j + X^2_j X_k X_j
+ X_j X_j X_k^2 + X_j X_j X_k^2 + X_j X_j X_k^2)
\]

for some \( p, q \in \mathbb{Z}_2 \). Here we put

\[
F_1(X_j, X_k, X_l) = X_k X_j X_l^2 + X_k X^2_j X_j + X_j X^2_k X_k + X^2_j X_j X_k
+ X_j X_j X_k^2 + X_j X_j X_k^2 + X_j X_j X_k^2 + X_j X_j X_k^2.
\]

and

\[
F_2(X_j, X_k, X_l) = X_k X_j X_j + X_k X^2_j X_j + X^2_k X_j X_j + X^2_j X_k X_j
+ X_j X_j X_k^2 + X_j X_j X_k^2 + X_j X_j X_k^2 + X_j X_j X_k^2.
\]
By using $F_1(X_j, X_k, X_l)$ and $F_2(X_j, X_k, X_l)$, it follows that

$$\sum_i E^{(2)}_{i|M} \left( g_{i}^{-1} s_{i}^{j} g_{i} \right)$$

$$= p_1 F_1(X_1, X_2, X_3) + q_1 F_2(X_1, X_2, X_3) + p_2 F_1(X_1, X_3, X_2) + q_2 F_2(X_1, X_3, X_2)$$

$$+ p_3 F_1(X_2, X_1, X_3) + q_3 F_2(X_2, X_1, X_3) + p_4 F_1(X_2, X_3, X_1) + q_4 F_2(X_2, X_3, X_1)$$

$$+ p_5 F_1(X_3, X_1, X_2) + q_5 F_2(X_3, X_1, X_2) + p_6 F_1(X_3, X_2, X_1) + q_6 F_2(X_3, X_2, X_1)$$

for some $p_i, q_i \in \mathbb{Z}_2$.

Now we first observe the top four terms in the right-hand side of the equation above:

$$p_1 F_1(X_1, X_2, X_3) + q_1 F_2(X_1, X_2, X_3) + p_2 F_1(X_1, X_3, X_2) + q_2 F_2(X_1, X_3, X_2)$$

$$= (p_1 + p_2)(X_2 X_1 X_3^2 + X_2^2 X_1 X_2 + X_3 X_1 X_2^2 + X_1 X_2 X_3)$$

$$+ (p_1 + q_1 + p_2)(X_2 X_3^2 X_1 + X_1 X_3^2 X_2 + X_1 X_3 X_2^2 + X_2^2 X_3 X_1)$$

$$+ (q_1 + p_2)(X_3 X_2^2 X_1 + X_3^2 X_2 X_1 + X_1 X_2 X_3^2 + X_1 X_2^2 X_3)$$

$$= (r_1 + r_2)(X_2 X_1 X_3^2 + X_2^2 X_1 X_2 + X_3 X_1 X_2^2 + X_1 X_2 X_3)$$

$$+ r_1 X_2 X_3^2 X_1 + X_1 X_3^2 X_2 + X_1 X_3 X_2^2 + X_2^2 X_3 X_1)$$

$$+ r_2 X_3 X_2^2 X_1 + X_3^2 X_2 X_1 + X_1 X_2 X_3^2 + X_1 X_2^2 X_3),$$

where $r_1 = p_1 + q_1 + p_2, r_2 = q_1 + p_2 + q_2 \in \mathbb{Z}_2$. Next we observe the middle four terms:

$$p_3 F_1(X_2, X_1, X_3) + q_3 F_2(X_2, X_1, X_3) + p_4 F_1(X_2, X_3, X_1) + q_4 F_2(X_2, X_3, X_1)$$

$$= (p_3 + p_4)(X_1 X_2 X_3^2 + X_2^2 X_2 X_1 + X_3 X_2 X_1^2 + X_1^2 X_2 X_3)$$

$$+ (p_3 + q_3 + p_4)(X_1 X_3^2 X_2 + X_2 X_3^2 X_1 + X_2 X_3 X_1^2 + X_1^2 X_3 X_2)$$

$$+ (q_3 + p_4 + q_4)(X_3 X_2^2 X_1 + X_3^2 X_2 X_1 + X_2 X_1 X_3^2 + X_2 X_1 X_3^2)$$

$$= t_2(X_1 X_2 X_3^2 + X_2^2 X_2 X_1 + X_3 X_2 X_1^2 + X_1^2 X_2 X_3)$$

$$+ t_1(X_1 X_3^2 X_2 + X_2 X_3^2 X_1 + X_2 X_3 X_1^2 + X_1^2 X_3 X_2)$$

$$+ (t_1 + t_2)(X_3 X_2^2 X_1 + X_3^2 X_2 X_1 + X_2 X_1 X_3^2 + X_2 X_1 X_3^2),$$

where $t_1 = p_3 + q_3 + p_4, t_2 = p_3 + p_4 \in \mathbb{Z}_2$. Finally we observe the bottom four terms:

$$p_5 F_1(X_3, X_1, X_2) + q_5 F_2(X_3, X_1, X_2) + p_6 F_1(X_3, X_2, X_1) + q_6 F_2(X_3, X_2, X_1)$$

$$= (p_5 + p_6)(X_1 X_3 X_2^2 + X_2^2 X_3 X_1 + X_2 X_3 X_1^2 + X_1^2 X_3 X_2)$$

$$+ (p_5 + q_5 + p_6)(X_1 X_2^2 X_3 + X_3 X_2^2 X_1 + X_3 X_2 X_1^2 + X_1^2 X_2 X_3)$$

$$+ (q_5 + p_6 + q_6)(X_2 X_1^2 X_3 + X_2^2 X_1 X_3 + X_3 X_1 X_2^2 + X_3 X_1 X_2^2)$$

$$= u_1(X_1 X_3 X_2^2 + X_2^2 X_3 X_1 + X_2 X_3 X_1^2 + X_1^2 X_3 X_2)$$

$$+ u_2(X_1 X_2^2 X_3 + X_3 X_2^2 X_1 + X_3 X_2 X_1^2 + X_1^2 X_2 X_3)$$

$$+ (u_1 + u_2)(X_2 X_1^2 X_3 + X_2^2 X_1 X_3 + X_3 X_1 X_2^2 + X_3 X_1 X_2^2),$$
where \( u_1 = p_5 + p_6, u_2 = p_5 + q_5 + q_6 \in \mathbb{Z}_2 \). As a result, it follows that

\[
E^{(2)}|_{M}(s) = \sum E^{(2)}|_{M} \left( g_i^{-1}s_i g_i \right)
\]

\[
= (r_1 + t_1)(X_2X_3^2X_1 + X_1X_2^2X_2) + (r_1 + u_1)(X_1X_3X_2^2 + X_2X_3X_1) + (r_2 + t_2)(X_3^2X_2X_1 + X_1X_2X_3^2) + (r_2 + u_2)(X_3X_2X_1^2 + X_1^2X_2X_3) + (t_1 + u_1)(X_2X_3X_2^2 + X_1^2X_3X_2) + (t_2 + u_2)(X_3X_2X_1^2 + X_1^2X_2X_3)
\]

Putting

\[
\delta_1 = r_1 + t_1, \quad \delta_2 = r_1 + u_1, \quad \delta_3 = r_2 + t_2, \quad \text{and} \quad \delta_4 = r_2 + u_2,
\]

we obtain the conclusion.

\[\square\]

**Remark 3.5**

(i) For an \( m \)-component link \( L \) and a sequence \( I \) of elements in \( \{1, \ldots, m\} \), Milnor [12, 13] defined an integer \( \mu_L(I) \) and proved that the residue class \( \overline{\mu}_L(I) \) modulo an integer \( \Delta_L(I) \) determined by the \( \mu \), is an invariant of the link \( L \). Let \( \lambda_i \) be an \( i \)th longitude word obtained from a diagram \( D \) of \( L \). If \( \Delta_L(k_1 \ldots k_j i) = 0 \), then by definition \( \mu_L(k_1 \ldots k_j i) \) modulo 2 coincides with the coefficient of \( X_{k_1} \cdots X_{k_j} \) in \( E^{(2)} \left( a_i^{w_i(D)} \eta_q(\lambda_i) \right) \), where \( w_i(D) \) denotes the sum of signs of self-crossings of the \( i \)th component of the diagram \( D \). Therefore a straightforward computation gives that the coefficient of \( X_{i}X_{k_1} \cdots X_{k_j} \) in \( E^{(2)}([\alpha_i, \eta_q(\lambda_i)]) = E^{(2)} \left( [a_i, \alpha_i^{-w_i(D)} \eta_q(\lambda_i)] \right) \) coincides with \( \mu_L(k_1 \ldots k_j i) \) modulo 2.

(ii) Let \( L \) be a 2-component link with linking number 0. From [13, Section 4], it follows that \( \overline{\mu}_L(I) = 0 \in \mathbb{Z} \) for any sequence \( I \) of length \( \leq 4 \) excluding 1122, 1221, 2211, 2112, 1212, and 2121. Moreover, the cyclic symmetry of \( \overline{\mu} \) gives that

\[
\overline{\mu}_L(1122) = \overline{\mu}_L(1221) = \overline{\mu}_L(2211) = \overline{\mu}_L(2112)
\]

and

\[
\overline{\mu}_L(1212) = \overline{\mu}_L(2121) \equiv 0 \pmod{2}.
\]

Therefore by (i) the link \( L \) satisfies the conditions given in Theorem 3.1.

**4 Applications**

**Proposition 4.1** Let \( m \geq 3 \) be an integer. Then the \( m \)-component link \( L_m \) in Fig. 3 is not \( 4 \)-equivalent to the trivial \( m \)-component link.

**Proof** Let \( \sigma \) be any permutation of the set \( \{1, 2, \ldots, m - 2\} \). It follows from [12, Section 5] that \( \Delta_{L_m}(I) = 0 \) for any sequence \( I \) with length \( \leq m \) and

\[
\overline{\mu}_{L_m}(\sigma(1) \ldots \sigma(m - 2) m - 1 m) = \mu_{L_m}(\sigma(1) \ldots \sigma(m - 2) m - 1 m) = \begin{cases} 1 & (\sigma = \text{id}), \\ 0 & \text{(otherwise)}. \end{cases}
\]
Fig. 3 The Milnor link of \( m \) components

Let \( \lambda_m \) be an \( m \)th longitude word of \( L_m \). By Remark 3.5(i), the coefficient of \( X_m X_1 X_2 \cdots X_{m-1} \) in \( E^{(2)}([\alpha_m, \eta_q(\lambda_m)]) \) is equal to 1 for \( \min\{n, q\} \geq m \). Therefore Theorem 3.1(v) completes the proof.

\[ \square \]

Remark 4.2
(i) As mentioned in Sect. 1, Nakanishi [17] showed Proposition 4.1 only for the case \( m = 3 \). (Note that the link \( L_3 \) is the Borromean rings.) In [5, Corollary 2.13], Dabkowski and Sahi obtained the same result by using the invariant \( \mathcal{R}_4 \).

(ii) Proposition 4.1 also follows from [15, Proposition 6.2] directly. Moreover, it follows that \( L_m \) and the trivial \( m \)-component link cannot be deformed into each other by a finite sequence of 4-moves and link-homotopies.

For a link \( L \), we denote by \( W(L; i) \) (resp. \( P(L; i) \)) a link obtained by replacing the \( i \)th component of \( L \) with a Whitehead double (resp. two zero-framed parallel copies) of it. The following lemma directly follows from [11, page 377] and [13, page 297].

Lemma 4.3 Let \( L \) be an \( m \)-component link and \( \lambda_m \) an \( m \)th longitude word obtained from a diagram \( D \) of \( L \). Suppose that

\[ E^{(2)}(\alpha_{\omega_m}(D)\eta_q(\lambda_m)) = 1 + F(X_1, \ldots, X_m). \]

Then the following hold.

(i) For an \( m \)th longitude word \( \omega_m \) obtained from a diagram \( D' \) of \( W(L; m) \),

\[ E^{(2)}(\alpha_{\omega_m}(D')\eta_q(\omega_m)) = 1 + (F(X_1, \ldots, X_{m-1}, 0))^2 X_m + X_m(F(X_1, \ldots, X_{m-1}, 0))^2 + D(2k + 2), \]

where \( k \leq \deg F(X_1, \ldots, X_{m-1}, 0) \).

(ii) For an \( (m+1) \)st longitude word \( \rho_{m+1} \) obtained from a diagram \( D'' \) of \( P(L; i) \) (1 \( \leq i \leq m \)),

\[ E^{(2)}(\alpha_{\rho_{m+1}}(D'')\eta_q(\rho_{m+1})) = 1 + F(X_1, \ldots, X_{i-1}, X_i + X_{i+1}, X_{i+2}, \ldots, X_m) + D(l + 1), \]

where \( l \leq \deg F(X_1, \ldots, X_m) \).

Proposition 4.4 Let \( L \) be a 2-component link with odd linking number. For any \( i, j \in \{1, 2\} \), the 3-component link \( P(W(L; i); j) \) is link-homotopically trivial, but not 4-equivalent to the trivial 3-component link.
Proof} We may assume that \( i = 2 \). Since the linking number of \( W(L; 2) \) is 0, all Milnor invariants of \( W(L; 2) \) with length \( \leq 3 \) vanish (see Remark 3.5(ii)). Combining this and [13, Theorem 7], we have that all Milnor invariants of \( P(W(L; 2); j) \) with length \( \leq 3 \) vanish, and therefore \( P(W(L; 2); j) \) is link-homotopically trivial [12].

Let \( \lambda_2 \) be a 2nd longitude word obtained from a diagram \( D \) of \( L \). Since \( L \) has an odd linking number, we have

\[
E^{(2)} \left( \alpha_2^{-w_2(D)} \eta_q(\lambda_2) \right) = 1 + X_1 + D(2).
\]

This together with Lemma 4.3(i) gives that for a 2nd longitude word \( \omega_2 \) obtained from a diagram \( D' \) of \( W(L; 2) \),

\[
E^{(2)} \left( \alpha_2^{-w_2(D')} \eta_q(\omega_2) \right) = 1 + X_1^2X_2 + X_2X_1^2 + D(4).
\]

For \( j = 1, 2 \), let \( \rho_3^{(j)} \) be a 3rd longitude word obtained from a diagram \( D_j \) of \( P(W(L; 2); j) \). Then it follows from Lemma 4.3(ii) that

\[
E^{(2)} \left( \alpha_3^{-w_3(D_j)} \eta_q(\rho_3^{(1)}) \right) = 1 + (X_1 + X_2)^2X_3 + X_3(X_1 + X_2)^2 + D(4)
\]

and

\[
E^{(2)} \left( \alpha_3^{-w_3(D_2)} \eta_q(\rho_3^{(2)}) \right) = 1 + X_1^2(X_2 + X_3) + (X_2 + X_3)X_1^2 + D(4).
\]

Therefore, by Remark 3.5(i), we have

\[
E^{(2)} \left|_{M} \right( \left[ \alpha_3, \eta_q(\rho_3^{(1)}) \right] \right) = X_3^2X_1X_2 + X_3^2X_2X_1 + X_1X_2X_3^2 + X_2X_1X_3^2
\]

and

\[
E^{(2)} \left|_{M} \right( \left[ \alpha_3, \eta_q(\rho_3^{(2)}) \right] \right) = X_3X_1^2X_2 + X_3X_2X_1^2 + X_1^2X_2X_3 + X_2X_1^2X_3.
\]

Since the number of terms in \( E^{(2)} \left|_{M} \right( \left[ \alpha_3, \eta_q(\rho_3^{(j)}) \right] \right) \) is 4 for each \( j \), Theorem 3.4 completes the proof. \( \square \)

Remark 4.5 The Hopf link \( L \) yields the 3-component link \( W \) in Fig. 4 as \( P(W(L; 2); 1) \). By Proposition 4.4, the link \( W \) is link-homotopically trivial, but not 4-equivalent to the trivial 3-component link. This result was first given by Dabkowski and Przytycki [4] using the 4th Burnside group as mentioned in Sect. 1, and reproved by Dabkowski and Sahi [5, Corollary 2.19] using the invariant \( \mathcal{R}_4 \).

5 Welded links

An \( m \)-component virtual link diagram is the image of an immersion of \( m \) circles into the plane, whose singularities are only transverse double points. Such double points are divided into classical crossings and virtual crossings. Welded Reidemeister moves consist of eight types of local moves. An \( m \)-component welded link is an equivalence class of \( m \)-component virtual link diagrams under welded Reidemeister moves [6].

An arc of a virtual link diagram \( D \) proceeds from an undercrossing to the next one, where overcrossings and virtual crossings are ignored. The group of \( D \) is defined by the Wirtinger presentation, i.e. each arc of \( D \) yields a generator and each classical crossing gives a relation.
Fig. 4 The 3-component link $W$ obtained from the Hopf link $L$ as $P(W(L; 2); 1)$

Note that no new generators or relation are added at a virtual crossing. It is easily shown that the group of $D$ is an invariant of welded links. The group of a welded link is given by the group for any virtual link diagram of the welded link [8, Section 4]. For a welded link $L$, we similarly define the Dabkowski-Sahi invariant $R_4(L)$ and the reduced one $R_4^r(L)$, via the group of $L$. Clearly the same result as Theorem 3.1 also holds for welded links. Even when applying it, we will refer to Theorem 3.1.

The $(i, j)$-linking number $\text{lk}_{ij}$ of a welded link is the sum of signs, in a diagram for the welded link, of classical crossings where the $i$th component passes over the $j$th one [7, Section 1.7]. For classical links in $S^3$, the half of $\text{lk}_{ij} + \text{lk}_{ji}$ coincides with the usual linking number between the $i$th and $j$th components.

**Proposition 5.1** There is a 2-component welded link with $\text{lk}_{1/2} = \text{lk}_{2/1} = 0$ such that it is not 4-equivalent to the trivial 2-component link.

**Proof** Let $L$ be the 2-component welded link represented by a virtual link diagram as shown in Fig. 5. By definition, this link $L$ satisfies that $\text{lk}_{1/2} = \text{lk}_{2/1} = 0$. Now consider the arcs $x_{ij}$ of the diagram in Fig. 5. Since $\lambda_1 = x_{21}^{-1}x_{21}$ and $\lambda_2 = x_{11}x_{12}^{-1}$, we have

$$[\alpha_2, \eta_5(\lambda_2)] = \alpha_2^{-1}\alpha_1\alpha_2^{-1}\alpha_1^{-1}\alpha_2\alpha_1\alpha_2^{-1}.$$  

After computing $E^{(2)}([\alpha_2, \eta_5(\lambda_2)])$, we see that the coefficient of $X_1X_2X_2$ is 1 and that of $X_2X_1X_1$ is 0. By Theorem 3.1(ii), $L$ is not 4-equivalent to the trivial 2-component link. □

**Remark 5.2** The 4th Burnside group of classical links, defined by Dabkowski and Przytycki in [3], naturally extends to welded links. Although we do not give here the precise definition, one can easily verify that the 4th Burnside group of the welded link given in Proposition 5.1 is isomorphic to that of the trivial 2-component link.
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