AN EXAMPLE OF A RANK 2 POISSON STRUCTURE WHICH IS STABLE UNDER DEFORMATIONS

RENAN LIMA

ABSTRACT. Let Π be a rank 2 Poisson Structure in the Projective Space defined by the dimension 2 foliation F in the pull-back component. We prove that for a generic choice of F, the irreducible component of the Poisson structures containing Π and the irreducible component of Foliations containing F coincides.

1. INTRODUCTION

1.1. Poisson structures. Let X be a complex manifold. A Poisson structure on X is a bivector field Π ∈ \( H^0(X, \Lambda^2 TX) \) such that the Schouten bracket \([\Pi, \Pi] \) ∈ \( H^0(X, \Lambda^3 TX) \) vanishes identically. The vanishing of the Schouten bracket implies that the image of the morphism

\[ \Omega^1_X \stackrel{\eta}{\longrightarrow} TX \]

\[ \eta \longmapsto \langle \eta, \Pi \rangle \]

is an involutive subsheaf of \( TX \), and the induced foliation is called the symplectic foliation of \( \Pi \). The most basic invariant attached to \( \Pi \) is its rank \( \text{rk}(\Pi) \), which is the generic rank of this involutive subsheaf of \( X \). Thanks to the anti-symmetry of \( \Pi \), its rank is an even integer \( 2r \) where \( r \) is the largest integer such that \( \Pi^r \) does not vanish identically.

We say that \( \Pi_\epsilon \in H^0(X, \Lambda^2 TX) \) is a (small) deformation of \( \Pi \) if, locally, we have \( \Pi_\epsilon = \Pi + \epsilon \Pi_1 + \epsilon^2 \Pi_2 + \ldots \), where \( \Pi_i \) are bivector fields (not necessarily Poisson) and we have \( [\Pi_\epsilon, \Pi_\epsilon] = 0 \), i.e., \( \Pi_\epsilon \) is a Poisson structure in \( X \). We denote by \( F \) the foliation induced by \( \Pi \) and by \( F_\epsilon \) the foliation induced by \( \Pi_\epsilon \).

1.2. Space of Poisson Structures. The space \( \text{Pois}(\mathbb{P}^n) \subseteq \mathbb{P}H^0(\mathbb{P}^n, \Lambda^2 T\mathbb{P}^n) \) of Poisson structures on a given projective space \( \mathbb{P}^n \) consists of a system of homogeneous quadratic equations given by the integrability condition \([\Pi, \Pi] = 0 \). It is an algebraic variety and the description of the irreducible components is a challenging problem.

In dimension three, a Poisson structure \( \Pi \) on a smooth projective 3-fold, if not zero, have rank 2 and defines a codimension one foliation \( F \) on \( \mathbb{P}^3 \). Therefore the study of Poisson structures on \( \mathbb{P}^3 \) is equivalent to the study of codimension one foliations. The description of the irreducible components of \( \text{Pois}(\mathbb{P}^3) \) has been carried out in [2]. An analogue description is presented in [5].

In dimension bigger than 3, even in \( \mathbb{P}^4 \), very little is known about \( \text{Pois}(\mathbb{P}^n) \). One of the main difficulties is the increase in the complexity of the symplectic foliations. One may encounter, for the first time, the possibility that the foliation could have leaves of different dimensions (zero, two, four, etc.). Another difficulty is that if \( \Pi \)
is not Poisson structure of maximal rank in $\mathbb{P}^n$, it may exists small deformations $\Pi_\epsilon$ of $\Pi$ such that $\text{rk}(\Pi_\epsilon) > \text{rk}(\Pi)$ as the following example shows.

**Example 1.1.** Let $\Pi$ be a rank 2 Poisson structure in $\mathbb{P}^4$ defined, in the affine chart $x_i = \frac{X_i}{X_0}$, by $\Pi = x_1 x_2 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}$. Then, for any $\epsilon \neq 0$, consider the small deformation, defined in the same chart $x_i = \frac{X_i}{X_0}$, by $\Pi_\epsilon = x_1 x_2 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} + \epsilon x_3 x_4 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_4}$. We see that $\Pi_\epsilon$ is a rank 4 Poisson Structure.

Due to this fact, it is common to impose that $\Pi$ has maximal rank and work with the so called Log Symplectic structure. The description of some irreducible components can be found in [7] and in [9].

In this paper, we describe an irreducible component of $\text{Pois}(\mathbb{P}^n)$, for $n \geq 4$ which the generic element is a rank 2 Poisson structure.

1.3. **Rank 2 Poisson Structure in the Projective Space.** Let $(X_0 : \ldots : X_n)$ be homogeneous coordinates of $\mathbb{P}^n$, $\phi : \mathbb{P}^n \to \mathbb{P}^{n-1}$ be the projection from the last coordinate and $Y \in H^0(\mathbb{P}^{n-1}, T\mathbb{P}^{n-1}(1))$ be a homogeneous quadratic vector field in $\mathbb{P}^n$. The bivector field $\Pi \in H^0(\mathbb{P}^n, \wedge^2 T\mathbb{P}^n)$ defined by

$$\Pi = \frac{\partial}{\partial X_n} \wedge Y$$

is a rank 2 Poisson Structure in $\mathbb{P}^n$.

An alternative description of this Poisson structure is the following: Thinking $Y$ as a foliation of dimension 1, we pull it back by $\phi$, defining a foliation $\mathcal{F}$ of dimension 2. In this case, the tangent sheaf $T\mathcal{F} = \mathcal{O}_{\mathbb{P}^n}(-1) \oplus \mathcal{O}_{\mathbb{P}^n}(1)$, i.e., the tangent sheaf totally splits.

**Theorem 1.** For a very generic quadratic vector field $Y$ in $\mathbb{P}^{n-1}$ with $n \geq 4$, let $\Pi$ be the Poisson structure in $\mathbb{P}^n$ defined by $\Pi = \frac{\partial}{\partial x_n} \wedge Y$. Then, for any sufficiently small deformation $\Pi_\epsilon$ of $\Pi$ we have that $\text{rk}(\Pi_\epsilon) = 2$ and there exists homogeneous coordinates $(X'_0 : X'_1 : \ldots : X'_n)$ and a quadratic vector field $Y'_\epsilon$ in $\mathbb{P}^{n-1}$ such that $\Pi_\epsilon = \frac{\partial}{\partial X'_n} \wedge Y'_\epsilon$.

In particular, $\Pi$, as in the Theorem 1, is a regular point of $\text{Pois}(\mathbb{P}^n)$ and the generic element of this irreducible component is a rank 2 Poisson structure such that the associated foliation $\mathcal{F}$ has split tangent sheaf given by $T\mathcal{F} = \mathcal{O}_{\mathbb{P}^n}(-1) \oplus \mathcal{O}_{\mathbb{P}^n}(1)$.

The idea is to compute the tangent space $T_\Pi \text{Pois}(\mathbb{P}^n)$ and conclude that, for a $Y$ well chosen, any small deformations of $\Pi$ is rank 2 Poisson. So small deformations of $\Pi$ as Poisson structures coincides with small deformations of $\mathcal{F}$ as foliations. And then we apply stability theorem of foliations with split tangent sheaf proved in [3].

The author thanks Jorge Vitório Pereira for the stimulating discussions about this problem and to my wife Maria Brito de Lima for stimulating me to return my studies in holomorphic foliations.
2. The Tangent Space of the Poisson Structure

We refer [5, page 27] to see the sign convention of the Schouten Bracket we use.

If \( \alpha \) is a \( p \)-vector field, \( \beta \) is a \( q \)-vector field, \( \xi \) is an \( r \)-vector field and \( f \) is a function, then the Schouten bracket satisfies the following properties:

\[
\langle \alpha, \beta \rangle = \langle \alpha, \beta \rangle + (-1)^{p-1}q \langle \alpha, \xi \rangle \\
\langle \alpha \rangle = \alpha \langle \beta \rangle \langle \tau \rangle = (-1)^{(r-1)q} \langle \alpha \rangle \langle \beta \rangle
\]

\[
\left[ \frac{\partial}{\partial x_i}, f \right] = \frac{\partial f}{\partial x_i}
\]

If \((x_1, \ldots, x_n)\) is a local system of coordinates and \( \alpha = \sum a_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \), we write

\[
\frac{\partial \alpha}{\partial x_k} = \sum_{ij} \frac{\partial a_{ij}}{\partial x_k} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}
\]

and we have that \( \left[ \frac{\partial}{\partial x_i}, \alpha \right] = \frac{\partial \alpha}{\partial x_i} \).

Let \( \text{Fol}(\mathbb{F}^n) \) be the space of dimension 2 foliations. If \( \Pi \) is a rank 2 Poisson structure in \( \mathbb{F}^n \) and \( \mathcal{F} \) is the foliation induced by \( \Pi \), the tangent space of \( \text{Pois}(\mathbb{F}^n) \) at \( \Pi \) and the tangent space of \( \text{Fol}(\mathbb{F}^n) \) at \( \mathcal{F} \) are given by

\[
T_{\Pi}(\text{Pois}(\mathbb{F}^n)) = \{ \xi \in H^0(\mathbb{F}^n, \bigwedge^2 T\mathbb{F}^n); [\Pi, \xi] = 0 \}.
\]

\[
T_{\mathcal{F}}(\text{Fol}(\mathbb{F}^n)) = \{ \xi \in H^0(\mathbb{F}^n, \bigwedge^2 T\mathbb{F}^n); [\Pi, \xi] = 0 \text{ and } \Pi \wedge \xi = 0 \}.
\]

Let \((X_0 : \ldots : X_n)\) be homogeneous coordinates in \( \mathbb{F}^n \) and \( \phi : \mathbb{F}^n \to \mathbb{F}^{n-1} \) be the projection of the last coordinate. Let \( Y \in H^0(\mathbb{F}^{n-1}, T\mathbb{F}^{n-1}(1)) \) be any quadratic vector field and \( \Pi = \phi^*Y = \frac{\partial}{\partial X_n} \wedge Y \) be a Poisson structure. Before choosing \( Y \), we will make some simple computations of the tangent space and from now on, we will work in the affine chart \( x_i = \frac{X_i}{X_n} \).

**Lemma 2.1.** Let \( \xi \) be in \( T_{\Pi}(\text{Pois}(\mathbb{F}^n)) \) and write \( \xi = \alpha_0 + x_n \alpha_1 + x_n^2 \alpha_2 + x_n^3 \alpha_3 + \frac{\partial}{\partial x_n} \wedge \beta \), where \( \alpha_i \) do not depend on \( \frac{\partial}{\partial x_n} \) nor \( x_n \) and \( \beta \) does not depend on \( \frac{\partial}{\partial x_n} \).

Then \( \alpha_i \wedge Y = 0 \) for \( i = 1, 2, 3 \) and \( [\alpha_0, Y] \wedge B = 0 \). In particular, we have \( \xi = 0 \) if and only if \( \alpha_0 \wedge Y = 0 \).

**Proof.** Write \( \alpha = \alpha_0 + x_n \alpha_1 + x_n^2 \alpha_2 + x_n^3 \alpha_3 \), then

\[
0 = [\Pi, \xi] = \left[ \frac{\partial}{\partial x_i}, \wedge Y, \xi \right]
\]

\[
= \frac{\partial}{\partial x_n} \wedge [Y, \xi] - \frac{\partial \xi}{\partial x_n} \wedge Y
\]

\[
= \frac{\partial}{\partial x_n} \wedge [Y, \alpha] + \frac{\partial}{\partial x_n} \wedge \left[ Y, \frac{\partial}{\partial x_n} \wedge \beta \right] - \frac{\partial \alpha}{\partial x_n} \wedge Y - \frac{\partial}{\partial x_n} \wedge \frac{\partial \beta}{\partial x_n} \wedge Y
\]

\[
= \frac{\partial}{\partial x_n} \wedge [Y, \alpha] - \frac{\partial \alpha}{\partial x_n} \wedge Y - \frac{\partial}{\partial x_n} \wedge \frac{\partial \beta}{\partial x_n} \wedge Y.
\]

Since \( Y \) and \( \frac{\partial \alpha}{\partial x_n} \) do not depend on \( \frac{\partial}{\partial x_n} \), we have
(1) \[ \frac{\partial \alpha}{\partial x_n} \wedge Y = 0, \text{ and} \]

(2) \[ [Y, \alpha] - \frac{\partial \beta}{\partial x_n} \wedge Y = 0. \]

Since \( Y \) does not depend on \( x_n \) by equation (1) we have \( \alpha_i \wedge Y = 0 \) for \( i = 1, 2, 3 \).

Since \([Y, x_n] = 0\), making the wedge product with \( Y \) and comparing the terms with \( x_n \) in the equation (2), we have

\[ [Y, \alpha_0] \wedge Y = 0 \]

\[ \square \]

We can interpret \( \alpha_0 \) and \( Y \) as, respectively, bivector field and vector field on \( \mathbb{P}^{n-1} \). The idea is to impose some conditions in \( Y \) to prove \( \alpha_0 \wedge Y = 0 \).

3. THE POINCARÉ’S DOMAIN

In this section, we work in a Euclidean neighborhood of the Poincaré Singularity of \( Y \).

**Definition 3.1.** Let \( Y \) be a germ of holomorphic vector field in \((\mathbb{C}^{n-1}, 0)\) with isolated singularity at the origin. We say that \( Y \) is in the Poincaré’s domain if the eigenvalues \( \lambda_i \) of the linear part of \( X \) at 0 are non-resonant and 0 is not in the convex hull of \((\lambda_1, \ldots, \lambda_{n-1})\). We say that \( \lambda = (\lambda_1, \ldots, \lambda_{n-1}) \) is non-resonant if whenever \( \sum_i a_i \lambda_i = 0 \) with \( a_i \in \mathbb{Z} \) for every \( i \), then \( a_i = 0 \) for every \( i \).

We state the well-known Poincaré’s Linearization Theorem.

**Theorem 3.2** (Poincaré’s Linearization Theorem). Let \( Y \) be a holomorphic vector field in \((\mathbb{C}^{n-1}, 0)\) with isolated singularity at the origin. If \( Y \) is in the Poincaré’s domain, then there exists a holomorphic change of coordinates \((y_1, \ldots, y_{n-1})\) such that \( Y = \sum_i \lambda_i y_i \frac{\partial}{\partial y_i} \).

Let \( \mathfrak{X}^2 \) be the germ of bivector fields in \((\mathbb{C}^{n-1}, 0)\) and let \( Y = \sum_i \lambda_i y_i \frac{\partial}{\partial y_i} \) be a vector field in \((\mathbb{C}^{n-1}, 0)\) in Poincaré’s domain. We have the following proposition.

**Lemma 3.3.** The \( \mathbb{C} \)-linear map \( \Delta : \mathfrak{X}^2 \to \mathfrak{X}^2 \) which sends \( \alpha_0 \) to \([Y, \alpha_0]\) satisfies \( \ker \Delta \oplus \text{Im} \Delta = \mathfrak{X}^2 \) and \( \ker \Delta \) is generated as \( \mathbb{C} \)-vector space by the diagonal bivector fields \( y_i y_j \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j} \).

**Proof.** For every multi-index \( I = (i_1, \ldots, i_{n-1}) \) and \( \lambda = (\lambda_1, \ldots, \lambda_{n-1}) \), we denote by \( y^I := y_1^{i_1} y_2^{i_2} \cdots y_{n-1}^{i_{n-1}} \) and \( \langle \lambda, I \rangle = \lambda_1 i_1 + \ldots + \lambda_{n-1} i_{n-1} \). We have the following formula

\[ \left[ \lambda_j y_j \frac{\partial}{\partial y_j}, y^I \right] = \lambda_j i_j y^I \]

and

\[ \left[ Y, y^I \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j} \right] = (\langle \lambda, I \rangle - \lambda_i - \lambda_j) y^I \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j}. \]
Since $\mathfrak{X}^2$ is generated, as $\mathbb{C}$-vector space, by the bivector fields $y^i \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j}$ and $\lambda$ is non-resonant, we have $\langle \lambda, I \rangle - \lambda_i - \lambda_j = 0$ if and only if $y^i = y_i y_j$. This readily concludes the proof of the proposition.

**Proposition 3.4.** If $Y$ is in the Poincaré domain and $\alpha_0 \in \mathfrak{X}^2$ is a bivector field satisfying $[\alpha_0, Y] \wedge Y = 0$, then there exists a vector field $Z$ and $a_{ij} \in \mathbb{C}$ such that $\alpha_0 = Y \wedge Z + \sum_{i,j} a_{ij} y_i y_j \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j}$.

**Proof.** Since $[Y, \alpha_0] \wedge Y = 0$, by the de Rham Lemma (see [11]), there exists a vector field $V$ such that $[Y, \alpha_0] = Y \wedge V$.

Consider the application $\delta : \mathfrak{X}^1 \rightarrow \mathfrak{X}^1$ which sends $Z$ to $[Y, Z]$, where $\mathfrak{X}^1$ is the space of germs of vector fields. In the same way as lemma 3.3, we can prove that $\ker \delta$ is generated by the diagonal $y_i \frac{\partial}{\partial y_i}$ as a $\mathbb{C}$-vector space and $\text{Im} \delta \oplus \ker \delta = \mathfrak{X}^1$.

So, there exists $V_1 = \sum a_{ij} y_i \frac{\partial}{\partial y_i}$ and $Z \in \mathfrak{X}^1$ such that $V = V_1 + [Y, Z]$.

We have

$$\Delta(\alpha_0 - (Y \wedge Z)) = [Y, \alpha_0 - (Y \wedge V)] = (Y \wedge V) - (Y \wedge [Y, Z]) = (Y \wedge V) - (Y \wedge (V - V_1)) = Y \wedge V_1.$$ 

With direct computations, we have $Y \wedge V_1 \in \ker \Delta = 0$ and since $\ker \Delta \cap \text{Im} \Delta = 0$, we have $\alpha_0 - Y \wedge Z \in \ker \Delta$.

□

4. **Quadratic vector field and the Proof of the Theorem**

The theorem 3.5 of [8] implies readily the following:

**Theorem 4.1.** A very generic homogeneous quadratic vector field has only isolated singularities, one of the singularities satisfies the hypothesis of the Poincaré’s Linearization Theorem and, with the notation of the previous section, the integral curves of the vector fields $\frac{\partial}{\partial y_i}$ are Zariski dense in $\mathbb{P}^n$.

Note that, in a Euclidean neighborhood of the Poincaré singularity of $Y$, the integral curve $C$ defined by $\frac{\partial}{\partial y_i}$ is the zero locus of $(y_2, \ldots, y_{n-1})$. In particular, for every $p \in C$, we have

$$\sum_{i,j} a_{ij} y_i y_j \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j}(p) = 0.$$

**Lemma 4.2.** (Identity Principle) Let $C$ be a integral curve defined by the vector field $Y$. If there exists an open $V \in \mathbb{P}^n$ in the Euclidean Topology such that the bivector field $\sigma \in H^0(\mathbb{P}^n-1, \wedge^2 T_{\mathbb{P}^n-1})$ satisfies $\sigma|_{V \cap C} = 0$, then $\sigma|_C = 0$.

**Proof.** By the tubular flow theorem, we see that $C$ is a an immersed variety in $\mathbb{P}^n-1$. Fix $p \in C \cap V$ and let $q \in C$ be any point. Consider a real curve $\gamma : [0, 1] \rightarrow C$ with $\gamma(0) = p$ and $\gamma(1) = q$. For any $t \in [0, 1]$, there exists $V_t$ Euclidean neighborhood of $\gamma(t)$ in $\mathbb{P}^n$ and a local system of coordinates $(y^t_1, \ldots, y^t_{n-1})$ such that the connected
component of $C \cap V_t$ containing $\gamma(t)$ is given by the equations $y^t_2 = \ldots = y^t_{n-1} = 0$. So $W = \bigcup C \cap V_t$ is a analytic variety and, by the identity principle, we have $\sigma|_W = 0$. So $\sigma(q) = 0$.

Proof of the Theorem. Let $Y$ be as in theorem 4.1 and $\Pi = \frac{\partial}{\partial x_n} \wedge Y$. If $\xi$ is any bivector field in $T_{11}(\text{Pois}(\mathbb{P}^n))$, then, with the notation of lemma 2.1 we have $[\alpha_0, Y] \wedge Y = 0$. We can interpret $\alpha_0$ as a bivector field in $\mathbb{P}^{n-1}$. Working in the Euclidean neighborhood $V$ of Poincaré singularity with local system of coordinates $(y_1, \ldots, y_{n-1})$ and let $C$ be a integral curve defined $\frac{\partial}{\partial y_1}$, then, by proposition 3.4 we have that $(\alpha_0 \wedge Y)(p) = 0$ for every $p \in C \cap V$. By lemma 4.2, we have $(\alpha_0 \wedge Y)(p) = 0$ for every $p \in C$. Since $C$ is Zariski dense in $\mathbb{P}^n$, we have $\alpha_0 \wedge Y = 0$. By lemma 2.1 we have $\xi \wedge \Pi = 0$.

So, deforming a Poisson structure $\Pi$ is the same as deforming its associated foliation $F$. Since $TF = O_{p^n}(1) \oplus O_{p^n}(-1)$, by the main theorem of 3, any small deformation $F_\epsilon$ has $TF_\epsilon = O_{p^n}(1) \oplus O_{p^n}(-1)$. So the foliation $F_\epsilon$ is a direct sum of two foliations $F_1$ and $F_2$. The first one has degree 0 and the second one has degree 2. We can choose homogeneous coordinates $(X'_0 : \ldots : X'_n)$ such that $F_1$ is given by $\frac{\partial}{\partial X'_n}$. Let $Y_\epsilon$ be a quadratic vector field, such that the $F_\epsilon$ is defined by the bivector field $\Pi_\epsilon = \frac{\partial}{\partial X'_n} \wedge Y_\epsilon$. We can choose $Y_\epsilon$ in such a way that does not depend on $\frac{\partial}{\partial X'_n}$. The integrability condition of $F_\epsilon$ shows that $Y_\epsilon$ does not depend on $X'_n$. □

REFERENCES

1. M. Brunella, Birational Geometry of Foliations. Publicações Matemáticas do IMPA. Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 2004.
2. D. Cerveau, A. Lins Neto, Irreducible components of the space of holomorphic foliations of degree two in $\mathbb{C}P(n)$, $n \geq 3$. Ann. of Math. 143 (1996) p.577-612. 1,2
3. F. Cukierman and J. V. Pereira, Stability of holomorphic foliations with split tangent sheaf. Amer. J. Math. 130 (2008), no. 2, 413-439. 1,2,3
4. S. Druel, Structures de Poisson sur les variétés algébriques de dimension 3. Bull. Soc. Math. France 127 (1999), no. 2, p.229-253.
5. J.P. DuFour and N.T. Zung, Poisson structures and their Normal forms. Birkhäuser Basel, 2005. 1,2
6. F. Loray, J. V. Pereira and F. Touzet, Foliations with trivial canonical bundle on Fano 3-folds. Mathematische Nachrichten 286 (2013), issue 8-9, 921940. 1,2
7. R. Lima and J.V. Pereira A Characterization of Diagonal Poisson Structures. Bull. London Math. Soc. 46 (2014), no. 6, 1203-1217. 1,2
8. J. V. Pereira, The characteristic variety of a generic foliation. J. Eur. Math. Soc. (JEMS) 14 (2012), no. 1, 307-319. 1,2
9. B. Pym Elliptic singularities on log symplectic manifolds and FeiginOdesskii Poisson brackets arXiv:1507.05668v1 1,2
10. A. Polishchuk, Algebraic geometry of Poisson brackets. Algebraic geometry, 7. J. Math. Sci. 84 (1997), no. 5, p.1413-1444.
11. K. Saito, On a generalization of de-Rham lemma. Ann. Inst. Fourier (Grenoble) 26 (1976), no. 2, vii, p.165-170. 1,2