LOCALIZED $L^p$-ESTIMATES OF EIGENFUNCTIONS: A NOTE ON AN ARTICLE OF HEZARI AND RIVIÈRE

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Abstract. We use a straightforward variation on a recent argument of Hezari and Rivièr [8] to obtain localized $L^p$-estimates for all exponents larger than or equal to the critical exponent $p_c = \frac{2(n+1)}{n-1}$. We are able to this directly by just using the $L^p$-bounds for spectral projection operators from our much earlier work [12]. The localized bounds we obtain here imply, for instance, that, for a density one sequence of eigenvalues on a manifold whose geodesic flow is ergodic, all of the $L^p$, $2 < p \leq \infty$, bounds of the corresponding eigenfunctions are relatively small compared to the general ones in [12], which are saturated on round spheres. The connection with quantum ergodicity was established for exponents $2 < p < p_c$ in the recent results of the author [13] and Blair and the author [3]; however, the article of Hezari and Rivièr [8] was the first one to make this connection (in the case of negatively curved manifolds) for the critical exponent, $p_c$. As is well known, and we indicate here, bounds for the critical exponent, $p_c$, imply ones for all of the other exponents $2 < p \leq \infty$. The localized estimates involve $L^2$-norms over small geodesic balls $B_r$ of radius $r$, and we shall go over what happens for these in certain model cases on the sphere and on manifolds of nonpositive curvature. We shall also state a problem as to when one can improve on the trivial $O(r^{\frac{1}{2}})$ estimates for these $L^2(B_r)$ bounds. If $r = \lambda^{-1}$, one can improve on the trivial estimates if one has improved $L^p(M)$ bounds just by using Hölder’s inequality; however, obtaining improved bounds for $r \gg \lambda^{-1}$ seems to be subtle.

1. Introduction.

Let $(M, g)$ be an $n$-dimensional compact manifold without boundary with $n \geq 2$. Then if $\Delta_g$ is the associated Laplace-Beltrami operator, we shall consider $L^2$-normalized eigenfunctions of $\sqrt{-\Delta_g}$, i.e., functions $e_\lambda$ satisfying

\begin{equation}
\Delta_g e_\lambda(x) = \lambda^2 e_\lambda(x), \quad \text{and} \quad \int_M |e_\lambda|^2 \, dV_g = 1.
\end{equation}

Here $dV_g$ denotes the volume element for $(M, g)$, and, in what follows, all of the $L^p$-norms are taken with respect to this measure.

Our main result says that one can control the critical $L^p$-norms of eigenfunctions in terms of local $L^2$-estimates over balls of possibly small size.

**Theorem 1.1.** For $r > 0$ smaller than the injectivity radius of $(M, g)$, let $B_r(x)$ denote the geodesic ball of radius $r$ centered at $x$. Then there is a uniform constant $C$, depending
only on \((M,g)\), so that for \(\lambda \geq 1\) and eigenfunctions as in (1.1) we have

\[
\|e_\lambda\|_{L^{2(n+1)/n-1} (M)} \leq C\lambda^{\frac{n-1}{n-1}} \left( r^{-\frac{n-1}{n-1}} \sup_{x \in M} \|e_\lambda\|_{L^2(B_r(x))} \right)^{\frac{2}{n-1}}, \quad \lambda^{-1} \leq r \leq \text{Inj} M,
\]

where \(\text{Inj} M\) denotes the injectivity radius of \((M,g)\).

The special case of (1.2) corresponding to \(r \approx 1\) is equivalent to the earlier estimates of the author [12],

\[
\|e_\lambda\|_{L^{2(n+1)/n-1} (M)} \leq C\lambda^{\frac{n-1}{n-1}}, \quad \lambda \geq 1,
\]

which are saturated on round spheres both by zonal spherical harmonics and highest weight spherical harmonics. Note that by a Bernstein inequality, (1.3) yields the sup-norm estimates

\[
\|e_\lambda\|_{L^\infty (M)} \leq C\lambda^{\frac{n-1}{n-1}}, \quad \lambda \geq 1,
\]

and so by interpolating between this estimate, the trivial \(L^2\) estimate and (1.3), we obtain the results of [12]:

\[
\|e_\lambda\|_{L^p (M)} \leq C\lambda^{\sigma(p)}, \quad \lambda \geq 1,
\]

where

\[
\sigma(p) = \begin{cases} 
\frac{n}{2} \left(1 - \frac{1}{p}\right) - \frac{1}{2}, & 2 \leq \frac{2(n+1)}{n-1} \leq p \leq \infty, \\
\frac{n-1}{2} \left(1 - \frac{1}{p}\right), & 2\leq p \leq \frac{2(n+1)}{n-1}.
\end{cases}
\]

As was shown in [11], these estimates are also saturated on the round sphere. To be more specific, for \(2 < p \leq \frac{2(n+1)}{n-1}\) they are saturated by the highest weight spherical harmonics, while for \(\frac{2(n+1)}{n-1} \leq p \leq \infty\), they are saturated by zonal spherical harmonics.

In [12], a stronger version of (1.4)–(1.5) was obtained for general \((M,g)\). Specifically, if \(E_j\) denotes the projection onto the eigenspace of \(\sqrt{-\Delta} g\) with eigenvalue \(\lambda_j\), and if \(\chi_\lambda\) denotes the spectral projection operator,

\[
\chi_\lambda f = \sum_{\lambda_j \in [\lambda, \lambda+1)} E_j f, \quad \lambda \geq 0,
\]

projecting onto unit bands of frequencies, then it was shown in [12] that

\[
\|\chi_\lambda f\|_{L^p(M)} \leq C(1+\lambda)^{\sigma(p)}\|f\|_{L^2(M)},
\]

if \(\sigma(p)\) is as in (1.5). Here

\[
0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots,
\]

denotes the spectrum of \(\sqrt{-\Delta} g\) labeled with respect to multiplicity, to which we can associate an orthonormal basis of eigenfunctions \(\{e_{\lambda_j}\}_{j=0}^\infty\).

By the argument that we just gave showing how (1.4) follows from (1.3), the preceding estimates just follow from the special case

\[
\|\chi_\lambda f\|_{L^{2(n+1)/n-1} (M)} \leq C(1+\lambda)^{\frac{n-1}{n-1}}\|f\|_{L^2(M)}.
\]

We shall prove the localized estimates (1.2) for eigenfunctions just by using (1.7), and, thus, unlike the arguments in [8], avoid the use of semi-classical analysis.
Before doing this, let us record a corollary of (1.2).

**Corollary 1.2.** Assume that the geodesic flow on \((M, g)\) is ergodic. Then there exists a density one subsequence of eigenvalues \(\lambda_{jk}\) so that for every \(2 < p \leq \infty\) we have
\[
\|e_{\lambda_{jk}}\|_{L^p(M)} = o(\lambda_{jk}^{\sigma(p)}),
\]
if \(\sigma(p)\) is as in (1.5).

To see this, we note that, by the argument that we just gave, (1.8) follows from the special case
\[
\|e_{\lambda_{jk}}\|_{L^2(M)} = o(\lambda_{jk}^{\sigma(2)}).
\]
To prove this, we use the quantum ergodicity theorem of Colin de Verdière–Shnirelman–Zelditch [5]–[9]–[19] to select a density one subsequence of eigenvalues so that the corresponding eigenfunctions satisfy (see e.g., [14, Corollary 6.2.4])
\[
\int_{\Omega} |e_{\lambda_{jk}}|^2 \, dV_g \to \frac{1}{|M|} |\Omega|, \quad k \to \infty,
\]
for Jordan measurable subsets \(\Omega\) of \(M\), where \(|\Omega|\) denotes its \(dV_g\)-measure. Since \(|B_r(x)| \approx r^n\) with bounds independent of \(x \in M\) if \(r \ll 1\), we get that
\[
\lim_{k \to \infty} r^{-\frac{n+1}{4}} \|e_{\lambda_{jk}}\|_{L^2(B_r(x))} \approx r^{-\frac{n+1}{4}}.
\]
This along with (1.2) yields (1.9) since for a given fixed \(r \ll 1\) we can find \(O(r^{-n})\) points \(x_\ell \in M\) so that the resulting balls \(B_r(x_\ell)\) cover \(M\) and have overlap of at most a constant \(N = N((M, g))\), which can be chosen independent of \(r\).

As we shall see in §4, for all \((M, g)\) there is the trivial uniform bounds
\[
\|e_{\lambda}\|_{L^2(B_r(x))} \leq Cr\frac{2}{\lambda}, \quad \lambda^{-1} \leq r \leq \text{Inj} M, \ x \in M.
\]
As we shall also show, this bound is saturated by the \(L^2\)-normalized zonal functions on \(S^n, Z_\lambda, \lambda = \sqrt{k(k+n-1)}\), \(k \in \mathbb{N}\), centered at a given \(x_0 \in S^n\), since
\[
\|Z_\lambda\|_{L^2(B_r(x_0))} \approx r\frac{2}{\lambda}, \quad \lambda^{-1} \leq r \leq \text{Inj} M.
\]
On the other hand, in the case of negative curvature, recent results of Han [6] and Hezari and Rivièrè [8] improve upon (1.11) considerably in the sense that, if all the sectional curvatures of \((M, g)\) are negative then there is always a density one sequence of eigenvalues \(\{\lambda_{jk}\}\) such that one has the small-scale quantum ergodic estimates
\[
\|e_{\lambda_{jk}}\|_{L^2(B_r(x))} \approx r^n, \quad \forall x \in M, \text{ if } r = (\log \lambda_{jk})^{-\kappa},
\]
for a range of powers \(\kappa > 0\) depending on the dimension. Also, Bérard’s [1] proof of improved error estimates for the Weyl formula for manifolds of nonpositive curvature imply that one can always improve (1.11) for the smallest allowable \(r\) if \((M, g)\) has nonpositive curvature, since then one has
\[
\|e_{\lambda}\|_{L^2(B_r(x))} \leq Cr\frac{2}{(\log \lambda)^{\frac{1}{2}}}, \quad \text{if } r = \lambda^{-1}.
\]
It would be interesting to find other general cases where (1.11) can be improved.
We note just by using H"older's inequality, if one has the improved estimates (1.9) for the critical exponent $p_c = \frac{2(n+1)}{n-1}$, then one automatically has an improvement over (1.11) at the smallest possible scale, i.e.,

$$||e_{\lambda_j}||_{L^2(B_{r_j}(x))} = o(\lambda_j^{-\frac{1}{2}}), \quad x \in M.$$  

The converse assertion need not hold, though. For the $L^2$-normalized highest weight spherical harmonics satisfy $||Q_{\lambda}||_{L^2(S^n)} \approx \lambda^{\frac{n-1}{2(n+1)}}$ as well as $||Q_{\lambda}||_{L^2(B_r(x))} = o(r^{\frac{1}{2}})$ for all $\lambda^{-1} \leq r \ll \lambda^{-\frac{1}{2}}$ (see \S 4).

2. Proof of the localized $L^p_c$-bounds.

Choose a nonnegative function $\rho \in S(\mathbb{R})$ satisfying

(2.1) \[ \rho(0) = 1 \quad \text{and} \quad \rho(t) = 0, \quad \text{if} \quad |t| \geq 1. \]

It then follows that if we let $P = \sqrt{-\Delta_g}$ then the operator defined by

(2.2) \[ T_{\lambda,r} = \frac{1}{\pi} \int_{-\infty}^{\infty} r^{-1} \rho(r^{-1}t) e^{it\lambda} \cos(tP) \, dt, \]

by Euler's formula equals $\rho(r(\lambda - P)) + \rho((r(\lambda + P))$. Therefore, by (2.1)

$$T_{\lambda,r} e_{\lambda} = [1 + \rho(2r\lambda)] e_{\lambda},$$

and since we are assuming that $\rho$ is nonnegative we have $|T_{\lambda,r} e_{\lambda}| \geq |e_{\lambda}|$, and so

(2.3) \[ ||e_{\lambda}||_{L^2(B_{r}(B_{r}(x)) \leq ||T_{\lambda,r} e_{\lambda}||_{L^2(B_{r}(B_{r}(x))}. \]

Next, we note that by Huygens' principle, the kernel $(\cos tP)(x,y)$ vanishes if the geodesic distance between $x$ and $y$ is greater than $t$. Therefore, we conclude from the second part of (2.1) that the kernel $K_{\lambda,r}(x,y)$ of $T_{\lambda,r}$ satisfies

(2.4) \[ K_{\lambda,r}(x,y) = 0, \quad \text{if} \quad d_g(x,y) > r, \]

where $d_g$ denotes geodesic distance in $(M,g)$. Consequently, if we could show that there is a uniform constant $C$ so that when $\lambda \geq 1$ and $\lambda^{-1} \leq r \leq \text{Inj} \ M$

(2.5) \[ ||T_{\lambda,r} f||_{L^2(B_{r}(B_{r}(x)))} \leq C r^{-\frac{1}{2}} \lambda^{\frac{n-1}{2(n+1)}} ||f||_{L^2(M)}. \]

Note that if we combine (2.3) and (2.5) we obtain the following uniform localized estimates.

Proposition 2.1. Let $(M,g)$ be as above. Then there is a uniform constant $C = C(M,g)$ so that

(2.6) \[ ||e_{\lambda}||_{L^2(B_{r}(B_{r}(x)))} \leq C r^{-\frac{1}{2}} \lambda^{\frac{n-1}{2(n+1)}} ||e_{\lambda}||_{L^2(B_{r}(x))}, \quad \lambda \geq 1, \quad \lambda^{-1} \leq r \leq \text{Inj} \ M. \]

Let us postpone the proof of (2.5) for the moment, which will be a simple consequence of the author’s earlier estimate (1.7), and see, now, how Proposition 2.1 implies our main estimate (1.2). Clearly, it suffices to prove the latter when $\lambda^{-1} \leq r \leq \delta$ where $\delta$ is a fixed positive number since the bounds for $\delta < r < \text{Inj} \ M$ follow from (1.3). To do this we use the fact that if $r$ is small enough we can cover $M$ by balls $\{B_r(x_\ell)\}_{\ell=1}^{N(r)}$ where $N(r) \approx r^{-n}$ and where the doubled balls $\{B_{2r}(x_\ell)\}_{\ell=1}^{N(r)}$ overlap at most $A$ times, with $A
being a constant that depends on \((M, g)\) but not on small \(r > 0\). We then conclude from (2.6) that, if \(C_0 = C \frac{2(n+1)}{n(n-1)}\), then
\[
\|e_\lambda\|_{L^{\frac{2(n+1)}{n-1}}(M)} \leq \sum_{\ell=1}^{N(r)} \|e_\lambda\|_{L^{\frac{2(n+1)}{n-1}}(B_r(x_\ell))}
\leq C_0 r^{-\frac{n+4}{n-1}} \sum_{\ell=1}^{N(r)} \|e_\lambda\|_{L^2(B_{2r}(x_\ell))}
\leq C_0 r^{-\frac{n+4}{n-1}} \left( \sup_{1 \leq \ell \leq N(r)} \|e_\lambda\|_{L^2(B_{2r}(x_\ell))} \right) \sum_{\ell=1}^{N(r)} \|e_\lambda\|^2_{L^2(B_{2r}(x_\ell))}
\leq AC_0 r^{-\frac{n+4}{n-1}} \left( \sup_{1 \leq \ell \leq N(r)} \|e_\lambda\|_{L^2(B_{2r}(x_\ell))} \right) \|e_\lambda\|^2_{L^2(M)},
\]
which of course yields (1.2) because of (1.1).

**End of Proof of Theorem 1.1.** To complete the proof of our main result we just need to prove (2.5).

We first recall that
\[
T_{\lambda, r} f = \sum_{j=0}^{\infty} \left( \rho(\lambda - \lambda_j) + \rho(\lambda + \lambda_j) \right) E_j f.
\]
Since \(\rho \in \mathcal{S}(\mathbb{R})\), for every \(N = 1, 2, 3, \ldots\), we have for \(\lambda \geq 1\)
\[
|\rho(\lambda - \lambda_j)| + |\rho(\lambda + \lambda_j)| \leq C_N(1 + r|\lambda - \lambda_j|)^{-N}.
\]
Therefore,
\[
\|\chi_k T_{\lambda, r} f\|_{L^2(M)} \leq C_N(1 + r|\lambda - k|)^{-N}\|\chi_k f\|_{L^2(M)}, \quad N = 1, 2, \ldots.
\]

To exploit this, let, for \(\ell \in \mathbb{Z}\),
\[
I_\ell = \left[ \lambda + r^{-1}(2\ell - 1), \lambda + r^{-1}(2\ell + 1) \right).
\]
Then \(\mathbb{R} = \bigcup I_\ell\). Also, since \(O(r^{-1})\) intervals \([k-1, k)\) intersect a given \(I_\ell\) as \(k\) ranges over \(\mathbb{N}\), we can use Minkowski’s inequality and the Cauchy-Schwarz inequality to see that
\[
\left\| \sum_{\{k: [k-1,k) \cap I_\ell \neq \emptyset}\}} \chi_k h \right\|_{L^{\frac{2(n+1)}{n-1}}(M)} \leq C r^{-\frac{1}{2}} \left( \sum_{\{k: [k-1,k) \cap I_\ell \neq \emptyset\}} \|\chi_k h\|^2_{L^{\frac{2(n+1)}{n-1}}(M)} \right)^{\frac{1}{2}}.
\]
If we use this along with (1.6) and (2.8), we deduce that
\[
\left\| \sum_{\{k: [k-1,k) \cap I_\ell \neq \emptyset\}} \chi_k T_{\lambda, r} h \right\|_{L^{\frac{2(n+1)}{n-1}}(M)} \leq r^{-\frac{1}{2}} \left( \sum_{\{k: [k-1,k) \cap I_\ell \neq \emptyset\}} \|k T_{\lambda, r} h\|^2_{L^2(M)} \right)^{\frac{1}{2}}
\leq C_N r^{-\frac{1}{2}} (1 + |\lambda + r^{-1}(2\ell + 1)|)^{-\frac{n-1}{n+1}} (1 + |\ell|)^{-N}\|h\|_{L^2(M)},
\]
for each \(N \in \mathbb{N}\), using (2.9) and the fact that if \(\lambda_j \in [k-1, k)\) and \([k-1, k) \cap I_\ell \neq \emptyset\) then
\(1 + r|\lambda_j - \lambda| \leq 1 + |\ell|\) in the last step.
From this we get
\[
\|T_{\lambda,r}f\|_{L^2(M)}^{2(n+1)} \leq \sum_{\ell \in \mathbb{Z}} \left\| T_{\lambda,r} \left( \sum_{\lambda_j \in I_\ell} E_j f \right) \right\|_{L^2(M)}^{2(n+1)} \\
= \sum_{\ell \in \mathbb{Z}} \left\| \sum_{k : |k-1,k| \neq 0} \chi_k T_{\lambda,r} \left( \sum_{\lambda_j \in I_\ell} E_j f \right) \right\|_{L^2(M)}^{2(n+1)} \\
\leq C N r^{-\frac{\lambda}{2}} \sum_{\ell \in \mathbb{Z}} (1 + |\ell|)^{-N} \left( 1 + |\lambda - 1(2\ell + 1)| \right) \frac{n-1}{2(n+1)} \|f\|_{L^2(M)} \\
\leq C r^{-\frac{\lambda}{2}} \|f\|_{L^2(M)},
\]
if \( N \geq 2 \), since, by our assumption that \( \lambda^{-1} r^{-1} \leq 1 \), we have \( |\lambda + r^{-1}(2\ell + 1)| = \lambda + \lambda^{-1} r^{-1}(2\ell + 1) | \lesssim \lambda(1 + |\ell|) \).
This concludes the proof of (2.5). \( \square \)

3. Localized eigenfunction estimates for other exponents.

Note that if we use (1.6), we could repeat the proof of (2.5) to get that for exponents \( p > 2 \)
\[
(3.1) \quad \|T_{\lambda,r}f\|_{L^p(M)} \leq C r^{-\frac{\lambda}{2}} \lambda^{\sigma(p)} \|f\|_{L^2(M)}, \quad \lambda \geq 1, \ \lambda^{-1} \leq r \leq \text{Inj} M,
\]
which by our earlier arguments, yields the following generalization of (2.6)
\[
\|e_\lambda\|_{L^p(B_o(x))} \leq C r^{-\frac{\lambda}{2}} \lambda^{\sigma(p)} \|e_\lambda\|_{L^2(B_o(x))}, \quad \lambda \geq 1, \ \lambda^{-1} \leq r \leq \text{Inj} M.
\]
We then could use these bounds to obtain
\[
(3.2) \quad \|e_\lambda\|_{L^p(M)} \leq \lambda^{\sigma(p)} \left[ \sup_{x \in M} r^{-\frac{\lambda}{2}} \|e_\lambda\|_{L^2(B_o(x))} \right]^{\frac{p-2}{p}}, \quad \lambda \geq 1, \ \lambda^{-1} \leq r \leq \text{Inj} M, \ 2 < p < \infty,
\]
as well as
\[
(3.3) \quad \|e_\lambda\|_{L^\infty(M)} \leq C \lambda^{\frac{n-1}{4}} \sup_{x \in M} r^{-\frac{\lambda}{4}} \|e_\lambda\|_{L^2(B_o(x))}, \quad \lambda \geq 1, \ \lambda^{-1} \leq r \leq \text{Inj} M.
\]
By the remarks we shall make about the relationship between these estimates and the \( L^p \)-norms of the highest weight spherical harmonics, (3.2) cannot be improved when \( p_c = \frac{2(n+1)}{n-1} \leq p < \infty \) at least for \( r = \lambda^{-1} \), while the observations we shall make about sup-norms of zonal functions and the right side of (3.3) show that this estimate cannot be improved for the full range of radii, \( \lambda^{-1} \leq r \leq \text{Inj} M \). In all likelihood (3.2) is also sharp for this full range of \( r \) if \( p \geq p_c \) since its counterpart (3.1) is best possible for this range of \( r \) and all exponents \( p \geq p_c \). The estimate (3.1) is not sharp for \( 2 \leq p < p_c \), though.

These estimates are only of potential use for the range of exponents \( p_c \leq p \leq \infty \). This is because, for the range of exponents \( p_c \leq p < \infty \) the eigenfunction estimates (1.5) are saturated by eigenfunctions concentrating at points, such as zonal functions on the sphere; however, for the complementary range of exponents \( 2 < p \leq p_c \) the bounds in (1.5) are saturated by eigenfunctions concentrating along periodic geodesics, such as the
highest weight spherical harmonics on the sphere. We shall have more to say about these two cases and the estimates (3.2)–(3.3) in the next section.

For the range of exponents \( p_c < p \leq \infty \) the author and Zelditch in [15] showed that one has

\[
\|e_\lambda\|_{L^p(M)} = o(\lambda^{\sigma(p)})
\]

for generic \((M, g)\), and in recent papers [17]–[18] gave necessary and sufficient conditions in the real analytic case for a stronger version of (3.4) involving quasimodes.

For the complementary range of exponents \( 2 < p < p_c \), because of reasons that we just alluded to, one would not expect bounds like (3.3) to be useful for proving (3.4). Instead, in a series of papers of the author [13] and Blair and the author [2]–[3], motivated by earlier related work of Bourgain [4], the strategy was to prove a variation of (3.2) which controls the \( L^p \)-norms of eigenfunctions in terms of their \( L^2 \)-mass on small tubes about geodesics. Specifically if \( \Pi \) denotes the space of unit length geodesics in \( M \) and if \( T_{\lambda^{-\frac{1}{2}}} (\gamma) \) denotes a \( \lambda^{-\frac{1}{2}} \)-tube about a given \( \gamma \in \Pi \), it was shown that

\[
\|e_\lambda\|_{L^p(M)} \leq C \lambda^{\sigma(p,n)} \left[ \sup_{\gamma \in \Pi} \int_{T_{\lambda^{-\frac{1}{2}}} (\gamma)} |e_\lambda|^2 \, dV_g \right]^{\theta(p,n)}, \quad \lambda \geq 1, \quad 2 < p < p_c,
\]

for some \( \theta(p,n) > 0 \).

When \( n = 2 \) the author and Zelditch [16] were able to show that if \((M, g)\) has nonpositive curvature one has

\[
\sup_{\gamma \in \Pi} \int_{T_{\lambda^{-\frac{1}{2}}} (\gamma)} |e_\lambda|^2 \, dV_g = o(1),
\]

and, therefore, by (3.5), one gets the improved eigenfunction bounds (3.4) when \( n = 2 \) if \( 2 < p < p_c \). For higher dimensions, Blair and the author [3] obtained (3.6) and hence (3.4) under this curvature assumption.

For the elusive endpoint case \( p = p_c \) Hezari and Révière [8] were able to obtain a stronger version of (3.4) involving logarithmic improvements for a density one subsequence of eigenfunctions on a given manifold of negative curvature. They did this by proving results like Theorem 1.1 when \( r \) is a power of \( 1/\log \lambda \) and then obtaining, for a density one sequence of eigenfunctions, very natural \( L^2(B_r) \)-norms for such \( r \). We shall say more about the latter results, which were also obtained independently by Han [6], in the next section.

4. Remarks on the size of \( L^2(B_r) \).

Let us conclude with a few remarks about the size of \( L^2(B_r(x)) \)-norms of eigenfunctions.

The first is that for any \((M, g)\) we have the trivial estimates

\[
\|e_\lambda\|_{L^2(B_r(x))} \leq C r^{\frac{1}{2}}, \quad x \in M, \quad \lambda \geq 1, \quad \lambda^{-1} \leq r \leq \text{Inj } M.
\]

An interesting question would be to determine when one can improve on this easy estimate for \( r = r(\lambda) \to 0 \) as \( \lambda \to \infty \) (either through all eigenvalues or subsequences) using
some dynamical or geometric assumption, such as \((M, g)\) having everywhere nonpositive curvature. We shall go over a few model cases after presenting the proof of (4.1).

To prove this inequality, we may of course assume that \(\text{Inj} \ M \geq 4\). Then, if \(\rho \in S(\mathbb{R})\) is as in (2.1), then it suffices to show that
\[
\|\rho(\lambda - P)f\|_{L^2(B_r(x))} \leq Cr^{\frac{n}{2}}\|f\|_{L^2(M)}, \quad x \in M, \ \lambda \geq 1, \ \lambda^{-1} \leq r \leq \text{Inj} \ M,
\]
since \(e_\lambda = \rho(\lambda - P)e_\lambda\). By a routine \(TT^*\) argument, if \(\eta = |\rho|^2\), this is equivalent to
\[
(4.2) \quad \|\eta(\lambda - P)h\|_{L^2(B_r(x))} \leq Cr\|h\|_{L^2(B_r(x))},
\]
if \(\text{supp} \ h \subset B_r(x)\), and \(\lambda \geq 1, \ \lambda^{-1} \leq r \leq \text{Inj} \ M\).

If we argue as in the proof of [10, Lemma 5.1.2] we find that, since \(\text{supp} \ \hat{\eta} \subset [-2, 2]\) and we are assuming that \(\text{Inj} \ M \geq 4\), the kernel of \(\eta(\lambda - P)\) can be written as
\[
\lambda^{\frac{n}{2}} \sum_{\pm} a_\pm(\lambda, d_g(x, y)) \left(d_g(x, y)\right)^{-\frac{n}{2}} e^{\pm i\lambda d_g(x, y)}, \quad \text{if} \ d_g(x, y) \geq \lambda^{-1},
\]
where
\[
|\partial^j_s a_\pm(\lambda, s)| \leq C_j s^{-j}, \quad s \geq \lambda^{-1}, \quad \text{and} \ j = 0, 1, 2, \ldots,
\]
and this kernel is \(O(\lambda^{n-1})\) when \(d_g(x, y) \leq \lambda^{-1}\). Using this, it is routine to obtain (4.2) using Hörmander’s \(L^2\)-oscillatory integral estimates (e.g. [10, Theorem 2.1.1]). The argument one uses is very similar to the proof of [10, Theorem 5.2.1].

As one would expect, the estimates (4.1) are saturated on the standard spheres. In that case the \(L^2\)-normalized zonal eigenfunctions \(Z_\lambda\) centered at a given point \(x_0 \in S^n\) with \(\lambda = \sqrt{k(n + n - 1)}\) are given by the formula
\[
Z_\lambda(x) = (d_k|S^n|)^{-\frac{1}{2}} \sum_{j=1}^{d_k} e_{k,j}(x)e_{k,j}(x_0)
\]
where \(d_k \approx \lambda^{n-1}\) denotes the dimension of the space of spherical harmonics of degree \(k\) and \(\{e_{k,j}\}_{j=1}^{d_k}\) is an orthonormal basis of this space (see e.g., [14, §3.4] for more details). One can use the classical Darboux formula (see e.g. [11, 4.7]) to see that
\[
Z_\lambda(x) \approx (d_g(x, x_0))^{-\frac{n+1}{2}} \left[\cos(N_kd_g(x, x_0) + \gamma) + O(1)(\lambda d_g(x, x_0))^{-1}\right],
\]
if \(N_k = (2k + n - 1)/2, \ \gamma = -(n-1)\pi/4, \ \text{and} \ d_g(x, x_0) \geq \lambda^{-1}\),
as well as
\[
Z_\lambda(x, x_0) = O(\lambda^{\frac{n-1}{2}}) \quad \text{if} \ d_g(x, x_0) \leq \lambda^{-1}.
\]
Using these facts, we find that
\[
\|Z_\lambda\|_{L^2(B_r(x_0))} \approx r^{\frac{n}{2}}, \quad \text{if} \ \lambda^{-1} \leq r \leq \pi,
\]
showing, as claimed, that (4.1) is saturated on \(S^n\).

For the zonal functions, we have just shown that we get no improvement for the size of \(r^{-\frac{n}{2}}\|e_\lambda\|_{L^2(B_r)}\) by taking \(r\) to be very small. For the other extreme spherical harmonics, the highest weight spherical harmonics, \(Q_\lambda\), as it turns out, one does achieve an improvement by taking \(r\) to be small. Recall that the \(Q_\lambda\) are the restriction of
the harmonic polynomials $c_k(x_1 + ix_2)^k$ to the unit sphere $|(x_1, x_2, \ldots, x_{n+1})| = 1$, and $c_k \approx k^{n-1}$ for $L^2$-normalization. Therefore,

$$|Q_\lambda| \approx k^{n-1} e^{-\frac{k}{2}(x_1^2 + \cdots + x_{n+1}^2)}$$

and $Q_\lambda$ is an eigenfunction with frequency $\sqrt{k(n-1)} = \lambda$, as above. Thus, $Q_\lambda$ behaves like a Gaussian beam concentrated along a $\lambda^{-\frac{1}{2}}$ neighborhood of the set on the unit sphere centered at the origin in $\mathbb{R}^{n+1}$, where $x_3 = \cdots = x_{n+1} = 0$, which, of course is a periodic geodesic. Using these size estimates we find that if $y$ is a point on this geodesic then

$$\|Q_\lambda\|_{L^2(B_r(y))} \approx r^\frac{1}{2}, \quad \lambda^{-\frac{1}{2}} \leq r \leq \pi$$

meaning that there is no improvement over (4.1) for this range of $r$, while on the other hand, we have the uniform bounds as $x$ ranges over $S^n$:

$$r^{-\frac{1}{2}}\|Q_\lambda\|_{L^2(B_r(x))} \leq Cr^{-\frac{1}{2}}, \quad \text{if } r = \lambda^{-1}.$$  

Curiously, if we use this fact along with (3.2)–(3.3) we get

$$\|Q_\lambda\|_{L^p(S^n)} \lesssim \lambda^{\frac{n-1}{2n}} (\frac{1}{r^\frac{1}{2}}), \quad \frac{2(n-1)}{n-1} \leq p \leq \infty,$$

which is sharp since, in view of the above $\|Q_\lambda\|_{L^p(S^n)}$ is comparable to $\lambda^{\frac{n-1}{2n}} (\frac{1}{r^\frac{1}{2}})$ for all $p \geq 2$. One also gets smaller improvements on these $L^2(B_r)$-norms for $Q_\lambda$ if $\lambda^{-1} < r \ll \lambda^{-\frac{1}{2}}$.

In the case of manifolds of nonpositive curvature, we can also beat the trivial estimate (4.1) if we use a result of Bérard [1] if $(M, g)$ is of nonpositive curvature. Recall that, in this case, he showed that the error term in the local Weyl law is $O(\lambda^{n-1}/\log \lambda)$, which implies that

$$|\lambda(x)| \leq C\lambda^{\frac{n-1}{2n}}/(\log \lambda)^{\frac{1}{2}}.$$  

Using this, we deduce that we have

$$\|e_\lambda\|_{L^2(B_r(x))} \leq C\frac{r^\frac{1}{2}}{(\log \lambda)^{\frac{1}{2}}, \quad \text{if } r = \lambda^{-1},}$$

under this curvature assumption. The argument is circular, but we can then use (3.3) to obtain (4.4). Unfortunately we cannot use (4.5) along with (3.2) to get any improvements over (1.4) if $p < \infty$ for manifolds of nonpositive curvature, although, it was already known by a recent result of Hassell and Tacy [7] that for $\frac{2(n+1)}{n-1} < p < \infty$, like in (4.4), one gets a $(\log \lambda)^{-\frac{1}{2}}$ improvement over (1.4) in this case.

If one works on the much larger scale $r = 1/(\log \lambda)^\kappa$, where the power $\kappa = \kappa_n$ depends on $n$, Han [6] and Hezari and Rivière [8] showed that there is a density one sequence of eigenvalues, $\{\lambda_{j_k}\}$ so that

$$\int_{B_r(x)} |e_{\lambda_{j_k}}|^2 dV_g = \frac{|B_r(x)|}{|M|} + o(r^n), \quad r = 1/(\log \lambda)^\kappa.$$

Since the radii shrink as $\lambda$ increases, this of course does not follow from the classical quantum ergodic identity (1.10). The latter holds whenever the geodesic flow is ergodic, while special dynamical properties of the geodesic flow on negatively curved manifolds was
used in the aforementioned results to obtain (4.6). Using (1.2) one can obtain certain log-power improvements of (1.3) for density one subsequences of eigenfunctions on negatively curved manifolds as was done in Hezari and Riviére [7].

Han asked in [6] whether one could break the logarithmic barrier and prove the variant of (4.6) where $r = \lambda^{-\kappa}$ for some $0 < \kappa < 1$. An affirmative answer to this seemingly difficult question would similarly lead to $\lambda^{-\frac{\kappa+1}{n+1}}$ improvements of (1.3). We remark, though, that one does not need the full strength of (4.6) to get improvements over (1.3). Indeed, if one could show that

\begin{equation}
\sup_{x \in M} \|e_{\lambda\alpha}\|_{L^2(B_r(\lambda\alpha)(x))} \leq C(r(\lambda))^{\frac{n}{2}}, \quad r(\lambda) \to 0,
\end{equation}

as $\lambda$ ranges over a subsequence $\lambda_j$ of eigenvalues, then Theorem 1.1 would yield

\begin{equation}
\|e_{\lambda}\|_{L^\frac{2(n+1)}{n+1+1}(M)} \leq C(r(\lambda)\lambda)^{\frac{n+1}{n+1+1}}
\end{equation}

for this subsequence. To obtain the missing $1/(\log \lambda)^{1/2}$ endpoint result with $p = p_c$ of Hassell and Tacy [7] that we mentioned before would require $r(\lambda) = 1/(\log \lambda)^{\frac{n+1}{n}}$, which involves a power, $\frac{n+1}{n-1} > 1$, which is larger than the ones occurring in [6] and [8].

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References

[1] P. Bérard: On the wave equation on a compact Riemannian manifold without conjugate points, Math. Z. 155 (1977) 249–276.
[2] M. Blair and C. D. Sogge: Refined and microlocal Kakeya-Nikodym bounds for eigenfunctions in two dimensions, Anal. PDE 8 (2015), 747–764.
[3] M. Blair and C. D. Sogge: Kakeya-Nikodym averages, $L^p$-norms and lower bounds for nodal sets of eigenfunctions in higher dimensions, J. Euro. Math. Soc. (JEMS), 17 (2015), 2513–2543.
[4] J. Bourgain: Geodesic restrictions and $L^p$-estimates for eigenfunctions of Riemannian surfaces, Linear and complex analysis, 27–35, Amer. Math. Soc. Tranl. Ser. 2, 226, Amer. Math. Soc., Providence, RI, 2009.
[5] Y. Colin de Verdière: Ergodicité et fonctions propres du laplacien, Comm. Math. Phys. 102 (1985), 497–502.
[6] X. Han: Small scale quantum ergodicity on negatively curved manifolds, arXiv:1410.3911 (2014).
[7] A. Hassell and M. Tacy: Improvement of eigenfunction estimates on manifolds of non-positive curvature, Forum Math. 27 (2015), 1435–1451.
[8] H. Hezari and G. Riviére: $L^p$ norms, nodal sets, and quantum ergodicity, arXiv:1411.4078 (2014), Adv. Math., to appear.
[9] A. I. Shnirelman: Ergodic properties of eigenfunctions, Uspeksi Math. Nauk 29 (1974), 181–182.
[10] C. D. Sogge: Fourier integrals in classical analysis, Cambridge Tracts in Mathematics, 105 Cambridge University Press, Cambridge, 1993.
[11] C. D. Sogge: Oscillatory integrals and spherical harmonics, Duke Math. J. 53 (1986), 43–65.
[12] C. D. Sogge: Concerning the $L^p$ norm of spectral clusters for second-order elliptic operators on compact manifolds, J. Funct. Anal. 77 (1988), 123–138.
[13] C. D. Sogge: Kakeya-Nikodym averages and $L^p$-norms of eigenfunctions, Tohoku Math. J. 63 (2011), 519–538.
[14] C. D. Sogge: Hangzhou lectures on eigenfunctions of the Laplacian, Annals of Mathematics Studies, 188 Princeton University Press, Princeton, NJ, 2014.
[15] C. D. Sogge and S. Zelditch: Riemannian manifolds with maximal eigenfunction growth, Duke Math. J. 114 (2002), 387–437.

[16] C. D. Sogge and S. Zelditch: On eigenfunction restriction estimates and $L^4$-bounds for compact surfaces with nonpositive curvature, Advances in Analysis: The Legacy of Elias M. Stein, Princeton Mathematical Series, Princeton Univ. Press, 2014, 447–461.

[17] C. D. Sogge and S. Zelditch: Focal points and sup-norms of eigenfunctions, Rev. Mat. Iberoam., to appear.

[18] C. D. Sogge and S. Zelditch: Focal points and sup-norms of eigenfunctions on analytic Riemannian manifolds II: the two-dimensional case, Rev. Mat. Iberoam., to appear.

[19] S. Zelditch: Uniform distribution of eigenfunctions on compact hyperbolic surfaces, Duke Math. J. 55 (1987), 919–941.

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