Semi-analytic Faddeev solution to the $N$-boson problem with zero-range interactions

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We study two-body correlations for $N$ identical bosons by use of the hyperspherical adiabatic expansion method. We use the zero-range interaction and derive a transcendental equation determining the key ingredient of the hyperradial potential. The necessary renormalization is for both repulsive and attractive interactions achieved with an effective range expansion of the two-body phase-shifts. Our solutions including correlations provide the properties of Bose-Einstein condensates exemplified by stability conditions as established by mean-field Gross-Pitaevskii calculations. The many-body Efimov states are unavoidable for large scattering lengths.

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Introduction. Solutions to the two-body problem are described in many textbooks. The three-body problem is solvable in practice at least for short-range interactions [1, 2]. Analytic solutions are found for specific potentials, e.g. square-well potentials confined to only $s$-waves, where the large distance behavior is particularly simple [3]. This limit is effectively the result for a zero-range interaction which is accurate when the small distances comparable to the interaction range are uninteresting or only needed to provide overall average properties. Therefore much effort has been devoted to the three-body problem by direct use of zero-range interactions, where regularization is needed to prevent unphysical collapsed wave functions [4, 5].

Not surprising the $N$-body problem has not been solved in general. The most popular approximation is the mean-field assumption which minimizes the technical difficulties but ignores correlations. Even then often the zero-range interaction is employed to exploit the additional simplifications allowing elaborate systematic investigations. Two prominent examples could be the Skyrme-Hartree-Fock fermion calculations for $N$-nucleon systems, see e.g. [6], and the analogous Gross-Pitaevskii calculations for dilute atomic Bose gases, see e.g. [7, 8].

Improvements within the mean-field approximation require better interactions, for example of finite range. This changes significantly solutions of high density as for nuclei but would not be visible for dilute condensed atomic gases. Another improvement to include correlations and go beyond the mean-field is usually much more difficult. For nuclei a perturbative treatment starting with the zero-range interaction and the mean-field solution is at the moment not very useful since a collapse into cluster components only can be avoided by a large additional phenomenological renormalization of the interaction. For dilute systems like atomic gases an appropriately renormalized zero-range interaction is not a severe approximation [9] and it would therefore be directly useful to include effects of correlations.

Transparent or even analytic extension beyond the mean-field approximation is highly desirable for several reasons, e.g. (i) provide simplicity and insight, (ii) allow systematics and access of complicated systems, (iii) provide intermediate and large distance properties, which in quantum mechanics often are directly responsible for the qualitative features of the entire solution, (iv) in designs of more efficient numerical methods by use of the available large distance asymptotic. The purpose of the present letter is to lay out the foundation for a series of applications and extensions by providing a semi-analytic solution to the $N$-boson problem.

Theory. We consider a dilute system of $N$ identical bosons of mass $m$, coordinates $r_i$ and momenta $\hat{p}_i$. They affect each other via a two-body short-range interaction $V(r)$ which can be approximated by a zero-range potential. The Hamiltonian is then

$$\hat{H} = \sum_{i=1}^{N} \left( \frac{\hat{p}_i^2}{2m} + \frac{1}{2} m \omega^2 r_i^2 \right) + \sum_{i<j}^{N} V(r_{ij}) , \quad (1)$$

where $r_{ij} = |r_j - r_i|$ and an external harmonic field of angular frequency $\omega$ is added. We choose the hyperspherical adiabatic expansion method where the principal coordinate is the hyperradius $\rho$ defined by $\rho$:

$$\rho^2 = \frac{1}{N} \sum_{i<j}^{N} r_{ij}^2 = \sum_{i}^{N} (r_i - \bar{R})^2 = \sum_{i}^{N} r_i^2 - NR^2 , \quad (2)$$

where $\bar{R}$ is the center of mass. The remaining degrees of freedom are the dimensionless hyperangles, collectively denoted by $\Omega$. The lowest adiabatic relative wave function $\Psi$ is given by

$$\Psi(\rho, \Omega) = \frac{f(\rho)\Phi(\rho, \Omega)}{\rho(2N-4)/2} , \quad \Phi(\rho, \Omega) = \sum_{i<j}^{N} \phi(\rho, r_{ij}) , \quad (3)$$

where the angular part $\Phi$ is expressed as a sum of two-body correlation amplitudes, Faddeev components, $\phi$ each assumed to depend only on the overall size $\rho$ and the distance between the pairs of particles. For zero-range interactions this dependence is no further restriction as two particles then only interact via $s$-waves. For fixed hyperradius the free variable in $r_{ij}$ can conveniently be substituted by an angle $\alpha_{ij}$ defined by $r = \sqrt{2\rho}\sin \alpha$, where we omitted the indices $ij$. 
The Faddeev component $\phi(\alpha) \equiv \phi(\rho, \sqrt{2}\rho \sin \alpha)$ is determined from the angular Faddeev equation \cite{1,2}, i.e.

$$0 = \left[ \hat{H}^2 + v(\alpha) \hat{R} - \lambda(\rho) \right] \phi(\alpha) ,$$

where $\hbar^2 \lambda/(2m^2 \rho^2)$ is the energy eigenvalue, $v(\alpha)$ is related to the two-body potential $V(r)$ by $v(\alpha) = 2m \rho^2 V(\sqrt{2}\rho \sin \alpha)/\hbar^2$, and the kinetic energy, $\hat{H}^2$, and rotation, $\hat{R}$, operators are given by

$$\hat{H}^2 = -\frac{\partial^2}{\partial \alpha^2} + \frac{3N - 9 - (3N - 5) \cos 2\alpha}{\sin 2\alpha} \frac{\partial}{\partial \alpha} ,$$

$$\hat{R} = 1 + 2(N - 2)\hat{R}_{13} + \frac{1}{2}(N - 2)(N - 3)\hat{R}_{34} .$$

The operators, $\hat{R}_{13}$ and $\hat{R}_{34}$, in eq. (5) “rotate” two-body Faddeev components between particles 1 (or 2) and a third particle $(2(N - 2)$ terms), and ones between particles 3 and 4 both different from 1 and 2 $( (N - 2)(N - 3)/2$ terms). The resulting radial equation is then

$$f(\rho) = 0 ,$$

where $E$ is the energy and the effective radial potential is

$$\frac{2mU(\rho)}{\hbar^2} \equiv \lambda(\rho) = \frac{(3N - 4)(3N - 6)}{4\rho^2} + \frac{\rho^2}{b^2} \left( N - 5 \right) ,$$

with the oscillator length $b^2 \equiv \hbar/(m\omega)$. This potential is therefore determined by the external trap $(\rho^2)$, the generalized centrifugal barrier $(\rho^2)$ and the interaction part $(\lambda(\rho))$ from the angular equation. The properties of the eigenvalue $\lambda$ from eq. (8) are then decisive for the radial potential and the corresponding energies and wave functions.

**Zero-range interaction.** We approximate $v$ by a zero-range interaction which means that eq. (8) has only kinetic energy terms for all $\alpha \neq 0$. The solution is well known as the Jacobi functions $P_{\alpha}^{(a,b)}(x)$, i.e.

$$\phi_\nu(\alpha) = P_{\nu}^{(3N/2 - 4,1/2)}(-\cos 2\alpha) ,$$

$$\lambda = 2\nu(2\nu + 3N - 5) ,$$

where the boundary condition $\phi_\nu(\alpha = \pi/2) = 0$ is fulfilled by this Jacobi function \cite{3,4}. The boundary condition of $\phi_\nu(\alpha = 0) = 0$ is only obeyed for these functions for integer values of $\nu$ which therefore fully determines the free solutions. However, at the point $\alpha = 0$ we now have an infinitely large potential which can be replaced by an appropriate boundary condition, i.e. obtained by using the observation that the wave function $\hat{R}\phi_\nu(\alpha)$ at small two-particle separation $r$ approaches the two-body wave function $u(r)$. Then $\nu$ does not have to be integer. The coordinate and wave function connections are $r = \sqrt{2}\rho \sin \alpha \approx \sqrt{2}\rho \alpha$ and $u(r) \propto \alpha \hat{R}\phi_\nu(\alpha)$. The boundary condition replacing the zero-range interaction for $u$ is \cite{5}

$$\left. \frac{1}{u(r)} \frac{du(r)}{dr} \right|_{r=0} = -\frac{1}{a_s} + \frac{1}{2} k^2 R_{eff} + O(k^4) ,$$

where $k$ is the wave number, $a_s$ is the scattering length and $R_{eff}$ is the effective range. For $R_{eff} = 0$ we then get

$$\left. \frac{\partial[\alpha \hat{R}\phi_\nu(\alpha)]}{\partial \alpha} \right|_{\alpha=0} = -\frac{\sqrt{2}\lambda}{a_s} \alpha \hat{R}\phi_\nu(\alpha) \left|_{\alpha=0} .$$

For small $\alpha$ the solutions \cite{6} behave as \cite{1}

$$\phi_\nu(\alpha) \approx \frac{A}{\alpha} + B , \quad A \equiv -\frac{\sin(\pi\nu)}{\sqrt{\pi}} \frac{\Gamma(\nu + \frac{3N-6}{2})}{\Gamma(\nu + \frac{3N-5}{2})} ,$$

$$B \equiv \cos(\pi\nu) \frac{2}{\sqrt{\pi}} \frac{\Gamma(\nu + \frac{1}{2})}{\Gamma(\nu + 1)} .$$

Then at the edge of the zero-range potential we get

$$\alpha \hat{R}\phi_\nu(\alpha) \left|_{\alpha=0} = A ,$$

$$\left. \frac{\partial[\alpha \hat{R}\phi_\nu(\alpha)]}{\partial \alpha} \right|_{\alpha=0} = B + (\hat{R} - 1)\phi_\nu(0) .$$

Combining eqs. (12), (15) and (16) we obtain

$$\frac{\rho}{a_s} = -\frac{1}{\sqrt{2A}} \left[ B + (\hat{R} - 1)\phi_\nu(0) \right] ,$$

where $(\hat{R} - 1)\phi_\nu(0)$ are given by eq. (6) and the explicit expressions

$$\hat{R}_{13}\phi_\nu(0) = \frac{4}{\sqrt{\pi}} \frac{\Gamma(\frac{3N-6}{2})}{\Gamma(\frac{3N-5}{2})} \left( \frac{3N-6}{2} \right) ,$$

$$\int_{-1}^{1} dx \frac{(1 + x)^{1/2} (1 - x)^{3(3N-11)/2}}{\sqrt{\pi} \Gamma(3N/2 - 4,1/2)} P_{\nu}(3N/2 - 4,1/2)(x) ,$$

$$\hat{R}_{34}\phi_\nu(0) = \frac{2}{\sqrt{\pi}} \frac{\Gamma(\frac{3N-4}{2})}{\Gamma(\frac{3N-9}{2})} \left( \frac{3N-8}{2} \right) ,$$

$$\int_{-1}^{1} dx \frac{(1 + x)^{1/2} (1 - x)^{3(3N-11)/2}}{\sqrt{\pi} \Gamma(3N/2 - 4,1/2)} P_{\nu}(3N/2 - 4,1/2)(x) .$$

The structure of eq. (17) is such that the right hand side depends on the particle number $N$ and the index $\nu$ on the Jacobi functions which in turn determines the angular eigenvalue $\lambda$ from eq. (8). Thus the left hand side, $\rho/a_s$, is a unique function of $\nu$ and in turn of $\lambda$ through eq. (10). By inversion the effective radial potential is determined as a function of hyperradius divided by the scattering length.

Extension to finite values of $R_{eff}$ amounts to replacing $1/a_s$ in eq. (17) by $1/a_s - \frac{1}{2} k^2 R_{eff}$, where $mE_2/\hbar^2 = k^2 = (\lambda + (9N - 19)/2)/(2\rho^2)$ with the two-body energy $E_2$. This substitution is obtained by eq. (4) in the limit of small $\alpha$ from the connection between $u(r)$ and $\phi_\nu(\alpha)$. Then eq. (17) becomes a second order equation in $\rho$ where the physical solution easily is extracted. Now the effective radial potential depends on both scattering length and effective range of the two-body interaction.
Angular eigenvalues. The expression in eq. (17) is straightforward to compute as a function of $\lambda$ via the (possibly complex) values of $\nu$. In fig. 1 we show the computed function $\rho/a_s$ as a function of $\lambda$ obtained as a sum of three contributions, i.e. the terms related to the Faddeev components between particles $1-2$ ($B$-term), $1-3$ (eq. (18)) and $3-4$ (eq. (19)). This procedure is obviously easier than solving the transcendental equation to get $\lambda$ as a function of $\rho/a_s$. The immediate implication is that the interaction only enters via the ratio $\rho/a_s$.

![FIG. 1: The angular eigenvalues in units of $\lambda_{\infty}$ = −1.65N^{7/3}(1 - 2/N) as functions of $\rho/(Na_s)$ obtained from eq. (17). The computation is for $N = 100$. The three different terms from eqs. (17), i.e. B-term (dotted), 13-term in eq. (18) (long-dashed) 34-term in eq. (19) (short-dashed), are shown individually along with the sum (solid).](image)

We confine ourselves to the most interesting cases of negative $\lambda$ corresponding to attractive two-body interactions. The terms $1 - 3$ and $3 - 4$ are both negative and vary from small $\lambda$ and large $\rho/a_s$ to large $\lambda$ and small $\rho/a_s$. The first term is much smaller but crosses the $\rho = 0$ axis and is therefore responsible for the same behavior of the sum of the three terms. However, this very important zero point for $\lambda$ is much larger for the sum than for the first term. We used units of $\lambda_{\infty}$ from [13] and a scaling by $N$ on the $\rho$-axis. Then the figure is fairly independent of particle number.

When both the index and $\lambda$ are small, $\nu \ll 1$, the Jacobi functions are almost constant allowing the analytic result

$$\lambda(\rho) \simeq \sqrt{2} N(N - 1) \frac{\Gamma \left( \frac{N - 3}{2} \right)}{\Gamma \left( \frac{N - 2}{2} \right)} \frac{a_s}{\rho},$$

where the validity condition, $\nu \ll 1$, can be translated into $\rho \gg N^{5/2} |a_s|$. This is the asymptotic behavior seen to the far left in fig. 1. For $N \gg 1$ where our center of mass separation is less important this result is identical to that derived in [14] for a constant angular wave function and a zero-range interaction renormalized for use in mean-field computations.

At the other limit of large $\rho/a_s$ and large negative $\lambda$ we also obtain a closed analytic expression, i.e.

$$\lambda(\rho) \simeq -2 \frac{\rho^2}{a_s^2},$$

where now only the first term in eq. (17) contributes. In fact this behavior corresponds precisely to the energy of a two-body bound state as $E_b = \hbar^2 \lambda/(2m_p^2)$. We emphasize that the interaction only enters via the ratio $\rho/a_s$. The scales in fig. 1 then implies that the results only exhibit the behavior for distances where $\rho$ is (much) larger than $a_s$.

![FIG. 2: The angular eigenvalues in units of $\lambda_{\infty}$ = −1.65N^{7/3}(1 - 2/N) for $N = 100$ as functions of hyperradius for several values of the scattering length $a_s$ both in units of the effective range $R_{eff}$.](image)

For many realistic systems the interesting region is $|a_s| < 1$, i.e. the region on fig. 1 where $\rho \approx 0$. To exhibit the behavior in this region we need to include the effective range in eq. (11) and solve the resulting second order equation in $\rho/R_{eff}$. The resulting angular eigenvalues for different scattering lengths are shown in fig. 2 as a function of hyperradius. No real solutions exist at small values of $\rho$ where the detailed behavior of the two-body interaction in any case is important. Higher order terms in the effective range expansion would also allow the small $\rho$-values.

As $\rho$ increases the eigenvalue levels off when $a_s$ is sufficiently large. The height of this plateau is independent of $a_s$ and in fact precisely equal to the value of $\lambda$ obtained from fig. 1 for $\rho = 0$. This value is rather accurately given by the unit used in fig. 1 $\lambda_{\infty} = −1.65N^{7/3}(1 - 2/N)$, extracted by elaborate numerical solutions of a variational equation with finite-range interactions [14].

As $\rho$ increases further to values exceeding $a_s$ the eigenvalues either bend up and approach zero ($a_s < 0$) as expressed in eq. (20) or bend down diverging ($a_s > 0$) as expressed in eq. (21). These two characteristics then reflect the different signs of the scattering length and the related ability of the two-body potential to support a bound state or not.

Radial potential and solutions. The angular eigenvalues are now inserted into the effective radial potential in eq. (5). The behavior for different scattering lengths is seen in fig. 3. The confining trap provides the large positive potential for large distances. The minimum appearing for relatively weakly attractive potentials of $|a_s|/R_{eff} \lesssim 10.5$ gradually disappears as the attraction increases. The barrier towards small distance disappears roughly when $|a_s|/b_t > 0.67$ as derived previously in
The improvement is towards less stability decreasing with increasing $N$. The minimum supports quasistationary states with characteristic features similar to condensates. The radial wave function $j$ obtained from eq. 7 is shown in fig. 3 as a distribution around the minimum. Other interaction parameters leaving the minimum would produce almost indistinguishable radial wave functions.

The plateau region for large $a_s$ in fig. 2 now appears in fig. 3 as a $\rho^{-2}$-potential with a strength of approximately $2.25N^2 - 1.65N^7/3$, which is negative already when $N > 3$. As this strength is less than $-0.25$ the Efimov conditions are fulfilled and a number of states related by simple scaling properties are solutions to the radial equation in eq. 7, see [11, 12]. These states are located in the plateau region far outside the range of the two-body interaction but before the confining wall of the trap. If created they may have a sufficiently long lifetime to be seen or perhaps play a role in some processes. They have obviously enough energy to decay into bound cluster states of much smaller size.

**Conclusions.** Zero-range interactions have been used extensively for two and three-body systems. It is a substantial simplification and very accurate for large distances compared to the range of the potential. We extend the application to $N$-boson systems by use of the adiabatic hyperspherical expansion method. We expect that two-body correlations are most important and dominated by $s$-waves. As zero-range interactions only are active in $s$-waves we use accordingly wave functions consisting of only $s$-wave Faddeev two-body amplitudes. This may then be viewed as the largest contribution in an expansion in both partial waves and many-body correlation amplitudes. For dilute systems we thereby obtain an accurate effective radial potential from a transcendental algebraic equation.

We derive the crucial equation and renormalize the zero-range interaction by an effective range expansion in terms of two-body phase-shifts. This is analogous to the field theoretical renormalization studied intensively for three-body systems. We extract and discuss the pertinent general scaling properties for both attractive and repulsive interactions. Use of parameters corresponding to systems forming condensates reveal the established properties and stability conditions.

The method has many applications and extensions which are beyond the scope of this letter, e.g. more systematic computations of various properties like one- and two-body densities, studies of correlations or effects beyond the mean-field approximation, investigations of dynamics in general and in particular the recombination process into bound cluster states, applications on more complicated systems like two-component boson systems, extensions to fermionic and mixed systems.

![Graph](image-url)

**FIG. 3:** The effective radial potential in eq. (8) for $N = 100$ as a function of $\rho$ for several values of the scattering length $a_s$ both in units of the effective range $R_{\text{eff}}$. The oscillator length is $b_s = 1442R_{\text{eff}}$ as in [11]. The radial wave function (two-dashed line) located in the minimum is shown for $a_s = -2R_{\text{eff}}$. 

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