Q-polynomial expansion for Brézin-Gross-Witten tau-function

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Abstract

In this paper, we prove a conjecture of Alexandrov that the generalized Brézin-Gross-Witten tau-functions are hypergeometric tau functions of BKP hierarchy after re-scaling. In particular, this shows that the original BGW tau-function, which has enumerative geometric interpretations, can be represented as a linear combination of Schur Q-polynomials with simple coefficients.

1 Introduction

The BGW tau-function, denoted by $\tau_{BGW}$, was introduced by Brézin, Gross, and Witten for studying lattice gauge theory in 1980 (c.f. [BG] and [GW]). Mironov, Morozov, and Semenoff showed that the BGW model can be considered as a particular case of the generalized Kontsevich model and the partition function $\tau_{BGW}$ is a tau function of the KdV hierarchy (c.f. [MMS]). In [N], Norbury gave a conjectural enumerative geometric interpretations for $\tau_{BGW}$, i.e. it is the generating function of intersection numbers of certain classes on the moduli spaces of stable curves (see Section 2.1 for precise definition). Hence $\tau_{BGW}$ shares many similar properties with Kontsevich-Witten tau function. In [MM], Mironov and Morozov conjectured a simple expansion formula for Kontsevich-Witten tau function in terms of Schur’s Q-polynomials. A proof of this conjecture using Virasoro constraints was recently given in [LY]. Inspired by Mironov-Morozov’s conjecture, Alexandrov proposed similar conjectures for the BGW tau function and its generalizations in [A20]. The main purpose of this paper is to prove Alexandrov’s conjectures.

Let $Q_\lambda$ be the Schur Q-polynomial associated to a partition $\lambda$ (see Section 2.2 for precise definition). We will consider $Q_\lambda$ as a polynomial of variables $t := (t_1, t_3, \cdots)$. These polynomials are tau functions of BKP hierarchy (c.f. [Y] and [KL]). In this paper we will prove the following formula for BGW tau function which was conjectured by Alexandrov (i.e. Conjecture 1 in [A20]):

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Theorem 1.1 The Brézin-Gross-Witten tau-function has the following expansion
\[ \tau_{BGW}(t) = \sum_{\lambda \in DP} 2^{-l(\lambda)} \left( \frac{\hbar}{16} \right)^{|\lambda|} \frac{Q_\lambda(\delta_{k,1})^3}{Q_{2\lambda}(\delta_{k,1})^2} \cdot Q_\lambda(t), \]
where \( DP \) is the set of all strict partitions, \( l(\lambda) \) and \(|\lambda|\) are the length and size of \( \lambda \) respectively, and \( \hbar \) is a formal parameter.

In the above formula, \( Q_\lambda(\delta_{k,1}) \) is the value of \( Q_\lambda \) at the point \( t = (1,0,0,\cdots) \). It is given by a simple formula \([18]\) which is related to the hook length formula.

The generalized BGW model was introduced by Mironov, Morozov, and Semenoff in \([MMS]\). This is a family of matrix models indexed by a complex number \( N \). When \( N = 0 \), it is the original BGW model. The partition functions of these models, denoted by \( \tau_{BGW}^{(N)} \), are also tau functions of the KdV hierarchy and satisfy the Virasoro constraints. A description of \( \tau_{BGW}^{(N)} \) using cut-and-join operators was given by Alexandrov in \([A18]\). These tau functions can be considered as a deformation of \( \tau_{BGW} \) analogous to the Kontsevich-Penner deformation of the Kontsevich-Witten tau function. In this paper, we will prove the following formula which was also conjectured by Alexandrov (i.e. Conjecture 2 in \([A20]\)):

Theorem 1.2 The generalized Brézin-Gross-Witten tau-function has the following expansion
\[ \tau_{BGW}^{(N)}(t) = \sum_{\lambda \in DP} \left( \frac{\hbar}{16} \right)^{|\lambda|} 2^{-l(\lambda)} \theta_\lambda Q_\lambda(\delta_{k,1})Q_\lambda(t), \tag{1} \]
where
\[ \theta_\lambda := \prod_{j=1}^{l} \prod_{k=1}^{\lambda_j} \theta(k) \tag{2} \]
for \( \lambda = (\lambda_1, \cdots, \lambda_l) \) and
\[ \theta(z) := (2z - 1)^2 - 4N^2. \tag{3} \]

In particular, this formula implies that \( \tau_{BGW}^{(N)}(t/2) \) are hypergeometric tau functions of BKP hierarchy as defined by Orlov in \([O1]\). When \( N = 0 \), this theorem is equivalent to Theorem 1.1 due to equation (18).

This paper is organized as follows. In section 2 we review definitions and basic properties of generalized BGW tau-functions and Schur Q-polynomials. In section 3 we prove that the right hand side of equation (1) satisfies the Virasoro constraints of \( \tau_{BGW}^{(N)}(t) \). Since the Virasoro constraints uniquely fix the tau function up to a constant (c.f. \([A18]\)), this gives a proof of Theorems 1.1 and 1.2.

The result of this paper was forecasted in \([LY]\) which focused on the Kontsevich-Witten tau function. Three days before the current paper was posted on the arXiv (c.f. arXiv:2104.01357), a new preprint \([A21]\) appeared which also gives a proof of Theorems 1.1 and 1.2 via a completely different approach using Boson-Fermion correspondence.

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2 Preliminaries

2.1 Brézin-Gross-Witten tau-function and its generalizations

The BGW model was originally proposed in [BG] and [GW] as a unitary matrix model. It was conjectured in [N] that the partition function of this model, i.e. $\tau_{BGW}$, has the following geometric interpretation.

Let $\mathcal{M}_{g,n}$ be the moduli space of stable genus $g$ curves $C$ with $n$ distinct smooth marked points $x_1, \ldots, x_n \in C$. For each $i \in \{1, \ldots, n\}$, there is a tautological class $\psi_i \in H^2(\mathcal{M}_{g,n})$, which is the first Chern class of the line bundle on $\mathcal{M}_{g,n}$ whose fiber at a point $(C; x_1, \ldots, x_n) \in \mathcal{M}_{g,n}$ is the cotangent space of $C$ at the marked point $x_i$. The Kontsevich-Witten tau function is the generating function of intersection numbers of these $\psi_i$-classes (c.f. [W] and [K]). To recover the BGW tau function, Norbury constructed a new family of classes $\Theta_{g,n} \in H^{4g-4+2n}(\mathcal{M}_{g,n})$, which are well behaved with respect to pullbacks of gluing and forgetful maps among moduli spaces of stable curves. Following notations in [A20], define intersection numbers

$$\langle \tau_{k_1} \tau_{k_2} \cdots \tau_{k_n} \rangle_{\Theta} := \int_{\mathcal{M}_{g,n}} \Theta_{g,n} \psi_1^{k_1} \psi_2^{k_2} \cdots \psi_n^{k_n}$$

for non-negative integers $k_1, k_2, \ldots, k_n$. Since the complex dimension of $\mathcal{M}_{g,n}$ is $3g-3+n$, these intersection numbers are non-zero only if $\sum_{i=1}^{n} k_i = g - 1$.

Let

$$F_{g,n}^{\Theta}(t) := \frac{1}{n!} \sum_{k_1, \ldots, k_n \geq 0} \langle \tau_{k_1} \cdots \tau_{k_n} \rangle_{\Theta} \prod_{i=1}^{n} (2k_i + 1)!! t^{2k_i+1}.$$ 

Assign degree of $t^k$ to be $k$ for all $k$. Then $F_{g,n}^{\Theta}(t)$ is a homogeneous polynomial of degree $2g-2+n$ by the above dimension constraint. Norbury’s conjecture (stated as a theorem in the first three versions of [N]) can be stated as

$$\tau_{BGW}(t) = \exp \left( \sum_{g=0}^{\infty} \sum_{n=0}^{\infty} \hbar^{2g-2+n} F_{g,n}^{\Theta}(t) \right).$$

This conjecture has been verified up to genus 7. The Virasoro constraints for $\tau_{BGW}$ were obtained in [GN].

Like BGW model, the generalized BGW model proposed in [MMS] is a family of matrix models indexed by a complex number $N$. The partition function $\tau_{BGW}^{(N)}(t)$ of this model is a tau function of KdV hierarchy for all $N$. When $N = 0$,

$$\tau_{BGW}^{(0)} = \tau_{BGW}$$

is the original BGW tau function explained above. However, for $N \neq 0$, enumerative geometric interpretation of $\tau_{BGW}^{(N)}$ is not known at the moment, although existence of such an interpretation is expected (see, for example, [A20]). It was shown in [AC] that a similar one-parameter deformation of BGW model can be transformed to the Kontsevich model.
after a shift of times except for the original BGW model. On the other hand, Alexandrov
proved in [A1] that for any \( N \), the generalized BGW tau function \( \tau_{BGW}^{(N)}(t) \) is uniquely
determined by the normalization condition
\[
\tau_{BGW}^{(N)}(0) = 1
\] (4)
and the Virasoro constraints
\[
L_m^{(N)} \tau_{BGW}^{(N)} = 0 \quad \text{for} \quad m \geq 0,
\] (5)
where
\[
L_m^{(N)} := \frac{1}{4} \sum_{a+b=2m} \frac{\partial^2}{\partial t_a \partial t_b} + \frac{1}{2} \sum_{k \geq 1, k \text{ odd}} kt_k \frac{\partial}{\partial t_{k+2m}} - \frac{1}{2\hbar} \frac{\partial}{\partial t_{2m+1}} + \frac{1-4N^2}{16} \delta_{m,0}.
\]
In this paper, we will take this property as the definition of \( \tau_{BGW}^{(N)} \).

The operators \( L_m^{(N)} \) satisfy the bracket relation
\[
[L_k^{(N)}, L_l^{(N)}] = (k-l)L_{k+l}^{(N)}
\]
for all \( k, l \geq 0 \). So they form a half branch of the Virasoro algebra. In particular, the first three operators \( L_0^{(N)}, L_1^{(N)} \) and \( L_2^{(N)} \) generate all other operators \( L_k^{(N)} \) for \( k > 2 \). Thus, to prove a function satisfying the Virasoro constraints (5), we just need to prove that it satisfies the \( L_k^{(N)} \)-constraint for \( k = 0, 1, 2 \). Since \( L_k^{(N)} \) does not depend on \( N \) for \( k > 0 \), we will simply write it as \( L_k \). More explicitly, the first three Virasoro operators are given by
\[
\begin{align*}
L_0^{(N)} &= \frac{1}{2} \sum_{k \geq 1, k \text{ odd}} \frac{kt_k}{2h} \frac{\partial}{\partial t_k} - \frac{1}{2\hbar} \frac{\partial}{\partial t_1} + \frac{1-4N^2}{16}, \\
L_1 &= \frac{1}{2} \sum_{k \geq 1, k \text{ odd}} \frac{kt_k}{2h} \frac{\partial}{\partial t_{k+2}} + \frac{1}{4} \frac{\partial^2}{\partial t_1 \partial t_1} - \frac{1}{2\hbar} \frac{\partial}{\partial t_3}, \\
L_2 &= \frac{1}{2} \sum_{k \geq 1, k \text{ odd}} \frac{kt_k}{2h} \frac{\partial}{\partial t_{k+4}} + \frac{1}{2} \frac{\partial^2}{\partial t_1 \partial t_3} - \frac{1}{2\hbar} \frac{\partial}{\partial t_5}.
\end{align*}
\] (6)

### 2.2 Schur Q-polynomial

In this paper, we will follow basic conventions for partitions and Q-polynomials in [LY]. A partition of length \( l(\lambda) = l \) is a sequence of non-negative integers \( \lambda = (\lambda_1, \cdots, \lambda_l) \). The size of \( \lambda \) is defined to be \( |\lambda| := \sum_{i=1}^l \lambda_i \). A partition \( \lambda \) is positive if all its parts \( \lambda_i \) are positive. It is weakly positive if it has at most one part equal to 0. It is strict if \( \lambda_1 > \lambda_2 > \cdots > \lambda_l > 0 \). In particular, a strict partition is always positive. The set of all strict partitions is denoted by \( DP \). For convenience, the empty partition \( \emptyset \), i.e. the
partition which has no parts, is considered to be a strict partition. We will also use the following notations: For any integers \(1 \leq i_1 < \cdots < i_n \leq l\),
\[
\lambda^{(i_1, \cdots, i_n)} := (\lambda_1, \cdots, \tilde{\lambda}_{i_1}, \cdots, \tilde{\lambda}_{i_n}, \cdots, \lambda_l),
\]
where \(\tilde{\lambda}_i\) means that the \(i\)-th part should be deleted from the partition. If \(\lambda\) is a partition and \(a_1, \cdots, a_k\) are non-negative integers, then
\[
(\lambda, a_1, \cdots, a_k) := (\lambda_1, \cdots, \lambda_l, a_1, \cdots, a_k).
\]

Q-polynomials were introduced by Schur in the study of projective representations of the symmetric group. As explained in Macdonald’s book [Mac], there are several equivalent definitions for such polynomials. For most of time, Schur’s original definition using Pfaffian would be sufficient for us. It starts with a sequence of polynomials \(q_k(t)\) defined by
\[
\sum_{k=0}^{\infty} q_k(t) z^k = \exp \left( 2 \sum_{k=0}^{\infty} t_{2k+1} z^{2k+1} \right),
\]
where \(z\) is a formal parameter. For a pair of non-negative integers \((r, s) \neq (0, 0)\), define
\[
A_{r,s} := q_r(t)q_s(t) + 2 \sum_{i=1}^{s} (-1)^i q_{r+i}(t)q_{s-i}(t),
\]
and set \(A_{0,0} := 0\). It turns out that \(A_{r,s}\) is skew symmetric with respect to \(r\) and \(s\). If \(\lambda = (\lambda_1, \cdots, \lambda_{2m})\) is a weakly positive partition of even length, we can define the associated Schur Q-polynomial by the Pfaffian:
\[
Q_\lambda(t) := \text{Pf}(A_{\lambda_i, \lambda_j})_{1 \leq i, j \leq 2m}.
\]
By properties of Pfaffian, \(Q_\lambda\) is skew symmetric with respect to permutations of parts of \(\lambda\). In particular \(Q_\lambda = 0\) if \(\lambda\) has two equal positive parts.

In [Mac] pages 262-263, a larger system of functions \(Q_\lambda\) were defined for all \(\lambda = (\lambda_1, \cdots, \lambda_l) \in \mathbb{Z}^l\). In this paper, we only need a portion of such functions, i.e. \(Q_\lambda\) for \(\lambda\) with at most one negative part. These functions are uniquely determined by the following rules:

- If \(\lambda\) is weakly positive of even length, it is given by equation (10).
- \(Q(\lambda,0) = Q_\lambda\) for all \(\lambda\).
- If for some \(i < l(\lambda)\), a partition \(\tilde{\lambda}\) is obtained from \(\lambda\) by switching \(\lambda_i\) and \(\lambda_{i+1}\) which are not both equal to 0, then \(Q_{\tilde{\lambda}} = -Q_\lambda\).
- Assume \(\lambda\) has exactly one part \(\lambda_i < 0\) and \(\lambda_j \geq 0\) for all \(j \neq i\). If there exists \(j > i\) such that \(\lambda_j = -\lambda_i\) and \(\lambda_k \neq -\lambda_i\) for all \(k > i\) and \(k \neq j\), then define
  \[
  Q_\lambda := (-1)^{j-i-1+\lambda_j} 2 Q_{\lambda^{(i,j)}}.
  \]
Otherwise define \(Q_\lambda := 0\).
Note that the notion of Q-polynomial associated with $\lambda$ used in [MM] and [A20] is equal to $2^{-l(\lambda)/2}Q_\lambda$. So formulas in this paper differ from corresponding formulas in [MM] and [A20] by a suitable factor.

Assign the degree of $t_k$ to be $k$ for all $k$. Then $Q_\lambda(t)$ is a homogeneous polynomial of degree $|\lambda|$. Moreover $\{Q_\lambda(t) \mid \lambda \in DP\}$ form a basis of $\mathbb{Q}[t_1, t_3, ...]$, which has a standard inner product such that

$$\langle Q_\lambda, Q_\mu \rangle = 2^{l(\lambda)}\delta_{\lambda,\mu} \quad \text{if } \lambda, \mu \in DP. \quad (12)$$

For any operator $f$ on $\mathbb{Q}[t_1, t_3, ...]$, the adjoint operator $f^\perp$ is defined by

$$\langle f^\perp p_1(t), p_2(t) \rangle = \langle p_1(t), fp_2(t) \rangle$$

for all $p_1, p_2 \in \mathbb{Q}[t_1, t_3, ...]$. Each polynomial $p \in \mathbb{Q}[t_1, t_3, ...]$ can be considered as an operator which acts on $\mathbb{Q}[t_1, t_3, ...]$ by multiplication.

Let $r$ be a positive odd integer. We then have

$$t_r^\perp = \frac{1}{2r} \frac{\partial}{\partial t_r}, \quad (13)$$

$$\frac{1}{2} \frac{\partial}{\partial t_r} Q_\lambda = \sum_{i=1}^{l(\lambda)} Q_{\lambda - r\epsilon_i}, \quad (14)$$

and

$$rt_r Q_\lambda = \sum_{i=1}^{l(\lambda)} Q_{\lambda + r\epsilon_i} + \frac{1}{2} Q_{\lambda,r} + \sum_{k=1}^{r-1} \frac{(-1)^{r-k}}{4} Q_{\lambda,k,r-k} \quad (15)$$

for any partition $\lambda$, where

$$\lambda \pm r\epsilon_i := (\lambda_1, ..., \lambda_i \pm r, ..., \lambda_l)$$

(see, for example, Lemma 2.2 and Corollary 2.3 in [LY], and [Mac] page 266).

The following formulas were proved by Aokage, Shinkawa, and Yamada in [ASY]: For all strict partitions $\lambda$,

$$L'_1 Q_\lambda(t) = \sum_{i=1}^{l(\lambda)} (\lambda_i - 1)Q_{\lambda - 2\epsilon_i}(t) \quad \text{and} \quad L'_2 Q_\lambda(t) = \sum_{i=1}^{l(\lambda)} (\lambda_i - 2)Q_{\lambda - 4\epsilon_i}(t), \quad (16)$$

where

$$L'_1 := \sum_{k \geq 1 \text{ odd}} kt_k \frac{\partial}{\partial t_{k+2}} + \frac{1}{4} \frac{\partial^2}{\partial t_1 \partial t_1} \quad \text{and} \quad L'_2 := \sum_{k \geq 1 \text{ odd}} \frac{1}{2} \frac{\partial^2}{\partial t_{k+4} \partial t_3}. \quad (17)$$

Note that iterated brackets of $L'_1$ and $L'_2$ also generate a half branch of Virasoro algebra. These Virasoro operators are different from the Virasoro operators for generalized BGW tau functions given by equation (6). Since the definition of Q-polynomials in [ASY] is slightly different from the definition in this paper, we have modified the coefficients of $L'_1$ and $L'_2$ to accommodate such difference (see [LY] Section 2.2 for more explanations).
2.3 Properties of $Q_\lambda(\delta_{k,1})$

For any partition $\lambda$, define

$$B_\lambda := Q_\lambda(\delta_{k,1}).$$

It is well known that

$$B_\lambda = \frac{2^{l(\lambda)}}{\lambda!} \prod_{i<j} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}$$

(18)

for weakly positive partitions $\lambda$, where $\lambda! := \prod_{i=1}^{l(\lambda)} \lambda_i!$ (see, for example, equation (3.3) in [A20]). This formula is related to the hook length formula. For most cases, we only need this formula for $l(\lambda) = 1$ or $l(\lambda) = 2$. They are given by

$$B_{(k)} = \frac{2^k}{k!}$$

(19)

and

$$B_{(k,m)} = \frac{2^{k+m}}{k!m!} \frac{k - m}{k + m}$$

(20)

for all non-negative integers $k$ and $m$ which are not both equal to 0. Equation (18) implies

$$\frac{B_\lambda}{B_{2\lambda}} = \prod_{j=1}^{l(\lambda)} (2\lambda_j - 1)!!$$

(21)

for all weakly positive partitions $\lambda$.

Since for any weakly positive partition $\lambda$ with even length, $B_\lambda$ is given by the Pfaffian of a skew symmetric matrix, it satisfies the standard recursion relation for Pfaffian

$$B_\lambda = \sum_{i=2}^{l(\lambda)} (-1)^i B_{(\lambda_1, \lambda_i)} B_\lambda^{(1,i)}$$

(22)

(c.f. Equation (2.4) in [O] and Theorem 9.14 in [HH]). By skew symmetry, we can expand $B_\lambda$ with respect to any part $\lambda_j$ to obtain

$$B_\lambda = (-1)^{j-1} \sum_{i=1}^{l(\lambda)} (-1)^{\tilde{i}(j)} B_{(\lambda_j, \lambda_i)} B_{\lambda^{(i,j)}},$$

(23)

where

$$\tilde{i}(j) := i - \delta_{i<j}$$

(24)

with $\delta_{i<j}$ equal to 1 if $i < j$ and equal to 0 otherwise.
If $\lambda$ is a positive partition with odd length, then we can still use equations (22) and (23) after replacing $\lambda$ by $(\lambda, 0)$. In particular, applying equation (23) with $j = l(\lambda) + 1$ for $(\lambda, 0)$, we obtain
\[
B_\lambda = \sum_{i=1}^{l(\lambda)} (-1)^{i+1} B_{(\lambda_i)} B_{\lambda^{(i)}}
\]  
for any positive partition $\lambda$ with odd length.

The same proof for Lemma 3.6 in [LY] shows the following

**Lemma 2.1** If $\lambda$ is a positive partition with even length $l$,
\[
\sum_{i=1}^{l(\lambda)} (-1)^i B_{(\lambda_i)} B_{\lambda^{(i)}} = 0.
\]

The same proof for Lemma 3.7 in [LY] shows the following

**Lemma 2.2** Let $\mu = (\mu_1, \cdots, \mu_l)$ be a weakly positive partition of odd length. Then equation (23) holds for $\lambda = (\mu, 0)$ with $j = 1, \cdots, l$.

### 3 Proof of main theorems

Let $\tau_N$ be the function given by the right hand side of equation (1), i.e.
\[
\tau_N(t) := \sum_{\lambda \in DP} \left( \frac{\hbar}{16} \right)^{|\lambda|} 2^{-l(\lambda)} \theta_{\lambda} B_{\lambda} Q_{\lambda}(t)
\]  
where $\theta_{\lambda}$ is defined by equations (2) and (3), and $B_{\lambda} = Q_{\lambda}(\delta_{k,1})$. In particular, $\theta_{\lambda}$ depends on $N$. Alexandrov's conjecture for generalized BGW tau functions can be restated as $\tau_{BGW}^{(N)} = \tau_N$ for all complex numbers $N$.

In this section, we will show that $\tau_N$ satisfies the Virasoro constraints
\[
L_m^{(N)} \tau_N = 0
\]  
for $m \geq 0$. Since $\tau_{BGW}^{(N)}$ satisfies the same Virasoro constraints and the same normalization condition, this will imply Theorem [1.2].

Since $\{Q_{\lambda} \mid \lambda \in DP\}$ form an orthogonal basis of $Q[t_1, t_3, \cdots]$, equation (27) is equivalent to
\[
\left\langle L_m^{(N)} \tau_N, Q_{\lambda} \right\rangle = \left\langle \tau_N, (L_m^{(N)})^\perp Q_{\lambda} \right\rangle = 0
\]
for all $\lambda \in DP$. Therefore instead of computing the action of $L_m^{(N)}$ on $\tau_N$, we will compute action of the adjoint operator $(L_m^{(N)})^\perp$ on each $Q_{\lambda}$ with $\lambda \in DP$.

Due to the Virasoro bracket relation, it suffices to prove equation (27) for $m = 0, 1, 2$. 

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3.1 Action of Virasoro operators

Recall operators $L_0^{(N)}$, $L_1$, $L_2$ are defined by equation (6), and operators $L'_1$, $L'_2$ are defined by equation (17). By equation (13), we have

**Lemma 3.1**

\[
(L_0^{(N)})^\perp = \frac{1}{2} \sum_{k \geq 1 \text{ odd}} k t_k \frac{\partial}{\partial t_k} + \frac{1 - 4N^2}{16} - \frac{t_1}{h},
\]

\[
(L_1)^\perp = \frac{1}{2} (L'_1)^\perp + \frac{t_1 t_1}{2} - \frac{3t_3}{h},
\]

\[
(L_2)^\perp = \frac{1}{2} (L'_2)^\perp + 3t_1 t_3 - \frac{5t_5}{h}.
\]

Note that the action of $(L_0^{(N)})^\perp$ is easy to compute since

\[
\sum_{k \geq 1 \text{ odd}} k t_k \frac{\partial}{\partial t_k} Q_\lambda = |\lambda| Q_\lambda,
\]

which follows from the fact that $Q_\lambda$ is homogeneous of degree $|\lambda|$. Multiplications by $t_1$, $t_3$, $t_5$ are given by equation (15). We also need to compute actions of other operators in the right hand sides of equations in Lemma 3.1. We start with computing actions of $(L'_1)^\perp$ and $(L'_2)^\perp$ first.

**Lemma 3.2** For any strict partition $\lambda = (\lambda_1, \ldots, \lambda_l)$,

\[
(L'_1)^\perp \cdot Q_\lambda = \sum_{i=1}^{l(\lambda)} (\lambda_i + 1) Q_{\lambda+2\epsilon_i} + \frac{1}{2} Q_{(\lambda,2)},
\]

\[
(L'_2)^\perp \cdot Q_\lambda = \sum_{i=1}^{l(\lambda)} (\lambda_i + 2) Q_{\lambda+4\epsilon_i} + Q_{(\lambda,4)} - \frac{1}{2} Q_{(\lambda,3,1)}.
\]

**Proof:** The formula for the action of $(L'_2)^\perp$ has been proved in [LY] Lemma 4.3. We only need to prove the first formula here.

Since $\{Q_\mu \mid \mu \in DP\}$ form an orthogonal basis of $Q[t_1, t_3, \cdots]$, we have

\[
(L'_1)^\perp \cdot Q_\lambda = \sum_{\mu \in DP} 2^{-l(\mu)} \langle (L'_1)^\perp \cdot Q_\lambda, Q_\mu \rangle Q_\mu.
\]

By skew symmetry of $Q$-polynomials, each summand on the right hand side of this equation is symmetric with respect to permutations of $\mu$. Hence we can replace each $\mu$ in the above equation by any permutation of $\mu$. After permutation, $\mu$ is still a partition with positive distinct parts.
By equation (16), for $\mu = (\mu_1, \cdots, \mu_{l(\mu)})$,

$$\langle (L'_1)^\perp \cdot Q_{\lambda}, Q_{\mu} \rangle = \langle Q_{\lambda}, L'_1 \cdot Q_{\mu} \rangle = \sum_{j=1}^{l(\mu)} (\mu_j - 1) \langle Q_{\lambda}, Q_{\mu - 2\epsilon_j} \rangle. \quad (29)$$

For this inner product to be non-zero, $\mu$ must have one of the following two forms.

Case (1), $\mu$ is a permutation of $\lambda + 2\epsilon_i$ for some $i$ between 1 and $l(\lambda)$. We may assume $\mu = \lambda + 2\epsilon_i$. By equation (29),

$$\langle (L'_1)^\perp \cdot Q_{\lambda}, Q_{\mu} \rangle = \sum_{j=1}^{l(\lambda)} (\lambda_j + 2\delta_{j,i} - 1) \langle Q_{\lambda}, Q_{\lambda + 2\epsilon_i - 2\epsilon_j} \rangle. \quad (30)$$

For $\langle Q_{\lambda}, Q_{\lambda + 2\epsilon_i - 2\epsilon_j} \rangle \neq 0$, $\lambda + 2\epsilon_i - 2\epsilon_j$ must be a permutation of $\lambda$. If $j \neq i$, this implies $\lambda_i + 2 = \lambda_j$ which is not possible since $\mu = \lambda + 2\epsilon_i$ must have distinct parts. Therefore we must have $j = i$ and

$$\langle (L'_1)^\perp \cdot Q_{\lambda}, Q_{\mu} \rangle = (\lambda_i + 1) \langle Q_{\lambda}, Q_{\lambda} \rangle = 2^{l(\lambda)} (\lambda_i + 1) = 2^{l(\mu)} (\lambda_i + 1) \quad (30)$$

for $\mu = \lambda + 2\epsilon_i$.

Case (2), $\mu$ is a permutation of $(\lambda, 2)$. We may assume $\mu = (\lambda, 2)$. Since $\mu$ must have distinct parts, $\lambda$ can not have parts equal to 2. By equation (29),

$$\langle (L'_1)^\perp \cdot Q_{\lambda}, Q_{\mu} \rangle = \sum_{j=1}^{l(\lambda)} (\lambda_j - 1) \langle Q_{\lambda}, Q_{(\lambda - 2\epsilon_j, 2)} \rangle + (2 - 1) \langle Q_{\lambda}, Q_{(\lambda, 0)} \rangle. \quad (31)$$

Note that $l(\lambda - 2\epsilon_j, 2) = l(\lambda) + 1$. Since $\lambda$ does not have parts equal to 2, $\langle Q_{\lambda}, Q_{(\lambda - 2\epsilon_j, 2)} \rangle = 0$ for $1 \leq j \leq l(\lambda)$. Moreover $Q_{(\lambda, 0)} = Q_{\lambda}$. So we have

$$\langle (L'_1)^\perp \cdot Q_{\lambda}, Q_{\mu} \rangle = \langle Q_{\lambda}, Q_{\lambda} \rangle = 2^{l(\lambda)} = 2^{l(\mu) - 1} \quad (31)$$

for $\mu = (\lambda, 2)$.

Combining equations (28), (30), (31), we obtain the desired formula. The lemma is thus proved. □

By repeatedly applying equation (15), we have

Lemma 3.3

$$t_1 t_2 Q_{\lambda}(t) = \sum_{i,j=1}^{l(\lambda)} Q_{\lambda + \epsilon_i + \epsilon_j}(t) + \sum_{i=1}^{l(\lambda)} Q_{\lambda + \epsilon_i, 1} + \frac{1}{2} Q_{(\lambda, 2)},$$

$$3t_1 t_3 Q_{\lambda}(t) = \sum_{i,j=1}^{l(\lambda)} Q_{\lambda + \epsilon_i + 3 \epsilon_j} + \frac{1}{2} \sum_{i=1}^{l(\lambda)} Q_{\lambda + 3 \epsilon_i, 1} + \frac{1}{2} Q_{(\lambda, 4)}$$

$$+ \frac{1}{2} \sum_{i=1}^{l(\lambda)} Q_{(\lambda + \epsilon_i, 3)} - \frac{1}{2} \sum_{i=1}^{l(\lambda)} Q_{(\lambda + \epsilon_i, 2, 1)} - \frac{1}{4} Q_{(\lambda, 3, 1)}. $$

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Combining above results from Lemmas 3.1, 3.2, 3.3 and applying equation (15), we obtain

**Proposition 3.4** For any strict partition \( \lambda = (\lambda_1, ..., \lambda_t) \), we have

\[
(L_0^{(N)})^\perp \cdot Q_{\lambda} = \left( \frac{|\lambda|}{2} + \frac{1 - 4N^2}{16} \right) Q_{\lambda} - \frac{1}{\hbar} \left( \sum_{i=1}^{l(\lambda)} Q_{\lambda+\epsilon_i} + \frac{1}{2} Q_{(\lambda,1)} \right),
\]

\[
(L_1)^\perp \cdot Q_{\lambda} = \frac{1}{2} \sum_{i,j=1 \atop i \neq j}^{l(\lambda)} Q_{\lambda+\epsilon_i+\epsilon_j} + \sum_{i=1}^{l(\lambda)} \frac{\lambda_i + 2}{2} Q_{\lambda+2\epsilon_i} + \frac{1}{2} \sum_{i=1}^{l(\lambda)} Q_{(\lambda+\epsilon_i,1)} + \frac{1}{2} Q_{(\lambda,2)} - \frac{1}{\hbar} \left( \sum_{i=1}^{l(\lambda)} Q_{\lambda+3\epsilon_i} + \frac{1}{2} Q_{(\lambda,3)} - \frac{1}{2} Q_{(\lambda,2,1)} \right),
\]

\[
(L_2)^\perp \cdot Q_{\lambda} = \sum_{i,j=1 \atop i \neq j}^{l(\lambda)} Q_{\lambda+3\epsilon_i+\epsilon_j} + \sum_{i=1}^{l(\lambda)} \frac{\lambda_i + 4}{2} Q_{\lambda+4\epsilon_i} + \frac{1}{2} \sum_{i=1}^{l(\lambda)} Q_{(\lambda+3\epsilon_i,1)} + \frac{1}{2} \sum_{i=1}^{l(\lambda)} Q_{(\lambda+\epsilon_i,3)} - \frac{1}{2} \sum_{i=1}^{l(\lambda)} Q_{(\lambda+\epsilon_i,2,1)} + Q_{(\lambda,4)} - \frac{1}{2} Q_{(\lambda,3,1)} - \frac{1}{\hbar} \left( \sum_{i=1}^{l(\lambda)} Q_{\lambda+5\epsilon_i} + \frac{1}{2} Q_{(\lambda,5)} - \frac{1}{2} Q_{(\lambda,4,1)} + \frac{1}{2} Q_{(\lambda,3,2)} \right).
\]

Now, we are ready to compute \( \langle L_k^{(N)} \tau_N, Q_{\mu} \rangle \) for \( k = 0, 1, 2 \). Essentially they are given by the following functions of \( \mu \):

\[
\Phi(\mu) := (8|\mu| + \theta(1)) B_{\mu} - \sum_{i=1}^{l(\mu)} \theta(\mu_i + 1) B_{\mu+\epsilon_i} - \frac{1}{2} \theta(1) B_{(\mu,1)},
\]

\[
\Psi(\mu) := \sum_{i,j=1 \atop i \neq j}^{l(\mu)} \theta(\mu_i + 1) \theta(\mu_j + 1) B_{\mu+\epsilon_i+\epsilon_j} + \theta(2) B_{(\mu,2)} + \sum_{i=1}^{l(\mu)} (\mu_i + 2) \theta^2(\mu_i) B_{\mu+2\epsilon_i} + \sum_{i=1}^{l(\mu)} \theta(1) \theta(\mu_i + 1) B_{(\mu+\epsilon_i,1)} + \frac{1}{16} \left( 2 \sum_{i=1}^{l(\mu)} \theta^3(\mu_i) B_{\mu+3\epsilon_i} + \theta(3) B_{(\mu,3)} - \theta(2,1) B_{(\mu,2,1)} \right),
\]

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and

\[ \Gamma(\mu) := 2 \sum_{i,j=1}^{l(\mu)} \theta(\mu_j + 1)\theta[3](\mu_i) B_{\mu + 3\epsilon_i + \epsilon_j} + \sum_{i=1}^{l(\mu)} \theta[4](\mu_i) B_{\mu + 4\epsilon_i} \]
\[ + \sum_{i=1}^{l(\mu)} \theta(1)\theta[3](\mu_i) B_{(\mu + 3\epsilon_i, 1)} + \sum_{i=1}^{l(\mu)} \theta(3)\theta(\mu_i + 1) B_{(\mu + \epsilon_i, 3)} \]
\[ - \sum_{i=1}^{l(\mu)} \theta(2,1)\theta(\mu_i + 1) B_{(\mu + \epsilon_i, 2, 1)} + 2\theta(4) B_{(\mu, 4)} - \theta(3,1) B_{(\mu, 3, 1)} \]
\[ - \frac{1}{16} \left( 2 \sum_{i=1}^{l(\mu)} \theta[5](\mu_i) B_{\mu + 5\epsilon_i} + \sum_{r=0}^{2} (-1)^r \theta(5-r, r) B_{(\mu, 5-r, r)} \right), \tag{37} \]

where

\[ \theta[k](r) := \prod_{j=1}^{k} \theta(r + j) \]

for all integers \( k \geq 1 \) and \( r \geq 0 \). Note that \( (k) \) is considered as a partition with only one part. So by definition, \( \theta(0) = 1 \) and \( \theta(\lambda, 0) = \theta(0, \lambda) = \theta_\lambda \) for all positive partitions \( \lambda \).

**Theorem 3.5** For all strict partitions \( \mu \),

\[ \langle L_0^{(N)} \cdot \tau_N, Q_\mu \rangle = \left( \frac{\hbar}{16} \right)^{|\mu|} \frac{\theta_\mu}{16} \cdot \Phi(\mu), \]
\[ \langle L_1 \cdot \tau_N, Q_\mu \rangle = \left( \frac{\hbar}{16} \right)^{|\mu|+2} \frac{\theta_\mu}{2} \cdot \Psi(\mu), \]
\[ \langle L_2 \cdot \tau_N, Q_\mu \rangle = \left( \frac{\hbar}{16} \right)^{|\mu|+4} \frac{\theta_\mu}{2} \cdot \Gamma(\mu). \]

**Proof:** The following formula was proved in [LY] Lemma 4.1: For any partition \( \mu \),

\[ \left\langle \sum_{\lambda \in DP} 2^{-l(\lambda)} f(\lambda) Q_\lambda, Q_\mu \right\rangle = f(\mu), \tag{38} \]

where \( f(\lambda) \) is any function of \( \lambda \) such that \( f(\lambda) \) is skew symmetric with respect to permutations of two parts of \( \lambda \) which are not both 0 and \( f((\lambda, 0)) = f(\lambda) \) for all partitions \( \lambda \). Set

\[ f(\lambda) := \left( \frac{\hbar}{16} \right)^{|\lambda|} \theta_\lambda B_\lambda. \]

Then \( f(\lambda) \) satisfies the above condition, and

\[ \tau_N = \sum_{\lambda \in DP} 2^{-l(\lambda)} f(\lambda) Q_\lambda. \]
Hence by equation (38), we have
\[ \langle \tau_N, Q_\mu \rangle = f(\mu) \]
for any partition \( \mu \). In particular, we can use this formula to compute
\[ \langle L_k^{(N)} \tau_N, Q_\mu \rangle = \langle \tau_N, (L_k^{(N)})^\perp Q_\mu \rangle \]
for \( k = 0, 1, 2 \) and \( \mu \in DP \). By Proposition 3.4, we have
\[ \langle L_0^{(N)} \cdot \tau_N, Q_\mu \rangle = \left( \frac{|\mu|}{2} + \frac{1 - 4N^2}{16} \right) f(\mu) - \frac{1}{\hbar} \left( \sum_{i=1}^{l(\mu)} f(\mu + \epsilon_i) + \frac{1}{2} f((\mu, 1)) \right), \quad (39) \]
\[ \langle L_1 \cdot \tau_N, Q_\mu \rangle = \frac{1}{2} \left( \sum_{\substack{i,j=1 \atop i \neq j}}^{l(\mu)} f(\mu + \epsilon_i + \epsilon_j) + \sum_{i=1}^{l(\mu)} (\mu_i + 2) f(\mu + 2\epsilon_i) \right) \]
\[ + \sum_{i=1}^{l(\mu)} f((\mu + \epsilon_i, 1)) + f((\mu, 2)) \]
\[ - \frac{1}{\hbar} \left( \sum_{i=1}^{l(\mu)} f(\mu + 3\epsilon_i) + \frac{1}{2} f((\mu, 3)) - \frac{1}{2} f((\mu, 2, 1)) \right), \quad (40) \]
and
\[ \langle L_2 \cdot \tau_N, Q_\mu \rangle = \sum_{\substack{i,j=1 \atop i \neq j}}^{l(\mu)} f(\mu + 3\epsilon_i + \epsilon_j) + \sum_{i=1}^{l(\mu)} \frac{\mu_i + 4}{2} f(\mu + 4\epsilon_i) + \frac{1}{2} \sum_{i=1}^{l(\mu)} f((\mu + 3\epsilon_i, 1)) \]
\[ + \frac{1}{2} \sum_{i=1}^{l(\mu)} f((\mu + \epsilon_i, 3)) - \frac{1}{2} \sum_{i=1}^{l(\mu)} f((\mu + \epsilon_i, 2, 1)) + f((\mu, 4)) - \frac{1}{2} f((\mu, 3, 1)) \]
\[ - \frac{1}{\hbar} \left( \sum_{i=1}^{l(\mu)} f(\mu + 5\epsilon_i) + \frac{1}{2} f((\mu, 5)) - \frac{1}{2} f((\mu, 4, 1)) + \frac{1}{2} f((\mu, 3, 2)) \right). \quad (41) \]

Note that \( f(\mu) \) contains a factor \( \theta_\mu \), and
\[ \theta_{(\mu,a_1,\ldots,a_k)} = \theta_\mu \theta(a_1,\ldots,a_k), \quad \theta_{\mu+k\epsilon_i} = \theta_\mu \theta^{[k]}(\mu_i) \]
for any integers \( k \geq 1 \) and \( a_1, \ldots, a_k \geq 0 \). After factoring out suitable factors from the right hand sides of equations (39), (40), and (41), we obtain the desired formulas. □

By Theorem 3.5, to prove Virasoro constraints for \( \tau_N \), we only need to show \( \Phi(\mu) = 0 \), \( \Psi(\mu) = 0 \), and \( \Gamma(\mu) = 0 \) for all strict partitions \( \mu \). These equations will be proved in Theorem 3.6, Theorem 3.9 and Theorem 3.11 respectively.
3.2 $L_0^{(N)}$ constraint

In this subsection, we will prove the following theorem, which implies the $L_0^{(N)}$ constraint for $\tau_N$.

**Theorem 3.6** Let $\Phi(\mu)$ be the function defined by equation (35). We have

$$\Phi(\mu) = 0$$

for all positive partitions $\mu$.

**Proof:** We first simplify $\Phi(\mu)$ using the following formula:

$$B_{(\lambda,1)} = 2B_{\lambda} - 2\sum_{i=1}^{l(\lambda)} B_{\lambda+\epsilon_i}$$

(42)

for any partition $\lambda$. This formula is obtained by setting $t = (1, 0, 0, \cdots)$ in equation (15) with $r = 1$. Using this formula, we can remove $B_{(\mu,1)}$ in equation (35) and obtain

$$\Phi(\mu) = 8|\mu|B_{\mu} - 4\sum_{i=1}^{l(\mu)} \mu_i(\mu_i + 1)B_{\mu+\epsilon_i}.$$  

(43)

Note that the right hand side of this equation does not depend on $N$ any more and all partitions involved have equal length.

Assume $\mu = (\mu_1, \ldots, \mu_l)$. We prove this theorem by induction on $l$.

If $l = 0$, then $\mu = \emptyset$ and $\Phi(\mu) = 0$ holds trivially since $|\mu| = l(\mu) = 0$.

If $l = 1$, $\Phi(\mu)$ is equal to

$$\Phi((\mu_1)) = 8\mu_1B_{\mu_1} - 4\mu_1(\mu_1 + 1)B_{(\mu_1+1)} = 0,$$

(44)

where the last equality follows from equation (19).

If $l = 2$, $\Phi(\mu)$ is equal to

$$\Phi((\mu_1, \mu_2)) = 8(\mu_1 + \mu_2)B_{(\mu_1,\mu_2)} - 4\mu_1(\mu_1 + 1)B_{(\mu_1+1,\mu_2)} - 4\mu_2(\mu_2 + 1)B_{(\mu_1,\mu_2+1)} = 0,$$

(45)

where the last equality follows from straightforward calculations using formula (20).

For any even integer $l > 2$, we apply recursion formula (22) to each term in $\Phi(\mu)$ to obtain

$$\Phi(\mu) = \sum_{j=2}^{l} (-1)^j B_{(\mu_1,\mu_j)}\Phi(\mu^{(1,j)}) + \sum_{i=2}^{l} (-1)^i B(\mu^{(1,i)}) \cdot \Phi((\mu_1, \mu_i)).$$

(46)

For any odd integer $l > 1$, we first replace each partition $\nu$ appeared in the right hand side of equation (43) by $(\nu, 0)$, then apply equation (22) to expand each term in $\Phi(\mu)$. The calculations are similar to the $l$ even case except that an extra term

$$B_{(\mu_1)}\Phi(\mu^{(1)}) + B_{\mu^{(1)}}\Phi((\mu_1))$$
should be added to the right hand side of equation (46).

Since the lengths of $\mu^{(1)}$ and $\mu^{(1,i)}$ are less than $l$, the theorem is reduced to the cases of $l = 1$ and $l = 2$, which have been considered in equations (44) and (45). The theorem is thus proved. □

3.3 $L_1$ constraint

In this subsection, we will prove $\Psi(\mu) = 0$, which is equivalent to the $L_1$ constraint of $\tau_N$. We will need the following two lemmas.

Lemma 3.7 For any partition $\mu = (\mu_1, \mu_2, ..., \mu_l)$ with $l \geq 3$, define

$$M_1(\mu) := \sum_{i,j=2 \atop i \neq j}^l (-1)^j B_{(\mu+\epsilon_j)^{(1,i)}} \cdot \omega(\mu_1, \mu_i, \mu_j),$$  \hspace{1cm} (47)

where

$$\omega(\mu_1, \mu_i, \mu_j) := a_1(\mu_1, \mu_j) B_{(\mu_1+1, \mu_i)} + a_1(\mu_i, \mu_j) B_{(\mu_1, \mu_1+1)},$$  \hspace{1cm} (48)

and

$$a_1(k, m) := \theta(k + 1) \theta(m + 1) - \theta(1) \{ \theta(k + 1) + \theta(m + 1) \}$$  \hspace{1cm} (49)

for all non-negative integers $k$ and $m$. Then

$$M_1(\mu) = 0$$

for all weakly positive partitions $\mu$ with even length.

Proof: For $j \neq 1, i$, we use recursion formula (23) to expand each term $B_{(\mu+\epsilon_j)^{(1,i)}}$ in the definition of $M_1(\mu)$ to obtain

$$B_{(\mu+\epsilon_j)^{(1,i)}} = \sum_{m=2 \atop m \neq i,j}^l (-1)^{\tilde{j}(i) + \tilde{m}(i,j)} B_{(\mu_j+1, \mu_m)} B_{\mu_1^{(1,i,j,m)}},$$  \hspace{1cm} (50)

where

$$\tilde{j}(i) := j - \delta_{j<i}, \text{ and } \tilde{m}(i,j) := m - \delta_{m<i} - \delta_{m>j}.$$  \hspace{1cm} (51)

After the expansion, we can compute all factors of type $B_\nu$ with $l(\nu) = 2$ using equation (20) and obtain

$$M_1(\mu) = \sum_{(a,b,c) \subseteq \{2, ..., l\}} \frac{2^{\mu_1+\mu_a+\mu_b+\mu_c+2} B_{\mu_1^{(1,a,b,c)}}}{(\mu_1+1)! (\mu_a+1)! (\mu_b+1)! (\mu_c+1)!} \sum_{(i,j,m) \in P(a,b,c)} (-1)^{\tilde{j}(i) + \tilde{m}(i,j)} \cdot \left\{ 16 \mu_1^2 \cdot \rho_2(\mu_i, \mu_j, \mu_m) + \mu_1 \cdot \rho_1(\mu_i, \mu_j, \mu_m) + \rho_0(\mu_i, \mu_j, \mu_m) \right\},$$  \hspace{1cm} (52)
where \( P(a, b, c) \) denotes the set of all permutations of \((a, b, c)\), and
\[
\begin{align*}
\rho_0(\mu_i, \mu_j, \mu_m) &= C_{j, m} \mu_i \left\{ -16 \mu_j (\mu_j + 1) (\mu_i + 1) + \theta(1)^2 \right\}, \\
\rho_1(\mu_i, \mu_j, \mu_m) &= C_{j, m} \left\{ 16 \mu_j (\mu_j + 1) (1 - \mu_i^2) - \theta(1)^2 \right\}, \\
\rho_2(\mu_i, \mu_j, \mu_m) &= C_{j, m} (\mu_i + 1) \mu_j (\mu_j + 1),
\end{align*}
\]
(53)
with
\[
C_{j, m} := \frac{(\mu_m + 1)(\mu_j + 1 - \mu_m)}{\mu_j + \mu_m + 1}.
\]

In Lemma A.1 in the appendix, we will prove three elementary identities
\[
\sum_{(i, j, m) \in P(a, b, c)} (-1)^{i + j + m} \rho_k(\mu_i, \mu_j, \mu_m) = 0
\]
for \( k = 0, 1, 2 \). This lemma follows from these identities. \(\square\)

Lemma 3.8 For any partition \( \mu = (\mu_1, \mu_2, ..., \mu_l) \) with \( l \geq 3 \), define
\[
M_2(\mu) := M_1(\mu) + \sum_{i=2}^{l} B_{(\mu + \epsilon_i)(1)} \cdot \tilde{\omega}(\mu_1, \mu_i),
\]
(54)
where
\[
\tilde{\omega}(\mu_1, \mu_i) := a_1(\mu_1, \mu_i) B_{(\mu_1 + 1)} ,
\]
(55)
with \( a_1(\mu_1, \mu_i) \) defined by equation (49), and \( M_1(\mu) \) is defined by equation (47). Then
\[
M_2(\mu) = 0
\]
for all positive partitions \( \mu \) with odd length.

Proof: As in the proof of Lemma 3.7. We first use recursion formula (23) to expand \( M_1(\mu) \) with respect to the part \((\mu_j + 1)\). Since all partitions \( \nu \) involved have odd length, we need replace \( \nu \) by \((\nu, 0)\) before doing expansion. This will produce extra terms for the expansion of \( M_1(\mu) \). More precisely, we need add an extra term
\[
(-1)^{\tilde{\mu}_j + 1} B_{(\mu_j + 1)} B_{\mu(1, s, i)}
\]
to the right hand side of equation (50), and an extra term
\[
\sum_{i, j=2}^{l} \sum_{i \neq j} (-1)^{i + j + m} B_{(\mu_j + 1)} B_{\mu(1, s, i)} \omega(\mu_1, \mu_i, \mu_j)
\]
should be added to the right hand side of equation (52).
On the other hand, we also use recursion formula (23) to expand the second part in the definition of $M_2(\mu)$ with respect to the part $(\mu_i + 1)$ and obtain

$$\sum_{i=2}^{l} B_{(\mu+\epsilon_i)(1)} \cdot \tilde{\omega}(\mu_1, \mu_i) = \sum_{i,j=2}^{l} (-1)^{i+j} B_{(\mu_i+1, \mu_j)} B_{(\mu_i+1, \mu_j)} \cdot \tilde{\omega}(\mu_1, \mu_i).$$

As in the proof of Lemma 3.7, we can still use Lemma A.1 to show that the contribution from the right hand side of equation (52) is 0. The remaining terms in $M_2(\mu)$ are

$$M_2(\mu) = \sum_{i,j=2}^{l} (-1)^{i+j} B_{(\mu_i+1, \mu_j)} \left( B_{(\mu_j+1)} \omega(\mu_1, \mu_i, \mu_j) + B_{(\mu_j+1, \mu_j)} \tilde{\omega}(\mu_1, \mu_i) \right).$$

Note that

$$a_1(k, m) = 16km(k+1)(m+1) - \theta(1)^2,$$

which is included in the definition of $\omega$ and $\tilde{\omega}$. After separating terms containing the factor $\theta(1)^2$ from terms not containing this factor, a straightforward calculation using above equation and equations (19) and (20) shows

$$B_{(\mu_j+1)} \omega(\mu_1, \mu_i, \mu_j) + B_{(\mu_j+1, \mu_j)} \tilde{\omega}(\mu_1, \mu_i) = \frac{2^{\mu_1+\mu_i+\mu_j+2}}{\mu_1! \mu_i! \mu_j!} \left( \frac{\mu_1(\mu_i^2 + \mu_i + \mu_j^2 + \mu_j)}{\mu_i + \mu_j + 1} - \mu_i \mu_j \right)$$

$$- \frac{\theta(1)^2 2^{\mu_1+\mu_i+\mu_j+2}}{(\mu_1+1)! (\mu_i+1)! (\mu_j+1)!} \left( \mu_1 + \frac{1 - \mu_i^2 - \mu_j^2}{1 + \mu_i + \mu_j} \right).$$

Note that this expression is symmetric with respect to $i$ and $j$. Since $(-1)^{i+j} i$ is skew symmetric with respect to $i$ and $j$, equation (56) implies that $M_2(\mu) = 0$. □

We are now ready to prove

**Theorem 3.9** Let $\Psi(\mu)$ be the function of $\mu$ defined by equation (36). We have

$$\Psi(\mu) = 0$$

for all strict partitions $\mu$. In particular, $\tau_N$ satisfies the $L_1$ constraint.

**Proof:** We first remove terms of form $B_{(\lambda,1)}$ in $\Psi(\mu)$ using equation (12) and remove terms of form $B_{(\lambda,2,1)}$ using following formula

$$B_{(\lambda,2,1)} = 2 \sum_{i=1}^{l(\lambda)} B_{\lambda+3\epsilon_i} + B_{(\lambda,3)},$$

(58)
which is obtained by evaluating both sides of equation (15) with \( r = 3 \) at the point \( t = (1, 0, 0, \cdots) \). After simplification, \( \Psi(\mu) \) can be written as

\[
\Psi(\mu) := \sum_{i,j=1 \atop i \neq j}^{l(\mu)} a_1(\mu_i, \mu_j) B_{\mu+i+\epsilon_j} + \theta(2) B_{(\mu, 2)} + 2 \sum_{i=1}^{l(\mu)} \theta(1) \theta(\mu_i + 1) B_{\mu+\epsilon_i} + \sum_{i=1}^{l(\mu)} a_2(\mu_i) B_{\mu+2\epsilon_i} - \frac{1}{8} \sum_{i=1}^{l(\mu)} a_3(\mu_i) B_{\mu+3\epsilon_i} - \frac{3}{2} \theta(2) B_{(\mu, 3)},
\]

where \( a_1(k, m) \) is defined by equation (49) and

\[
a_2(k) := (k + 2) \theta^{[2]}(k) - 2 \theta(1) \theta(k + 1),
\]

\[
a_3(k) := \theta^{[3]}(k) - \theta_{[2, 1]}
\]

for any non-negative integer \( k \).

Assume \( \mu = (\mu_1, \ldots, \mu_l) \). We prove this theorem by induction on \( l \).

**Step 1:** Prove \( \Psi(\mu) = 0 \) if \( l = 0, 1, 2 \).

If \( l = 0 \), we have

\[
\Psi(\emptyset) = \theta(2) \left( B_{(2)} - \frac{3}{2} B_{(3)} \right) = 0,
\]

since \( B_{(2)} = 2 \) and \( B_{(3)} = \frac{4}{3} \).

If \( l = 1 \), we have

\[
\Psi((\mu_1)) = \theta(2) B_{(\mu_1, 2)} + 2 \theta(1) \theta(\mu_1 + 1) B_{(\mu_1 + 1)} + a_2(\mu_1) B_{(\mu_1 + 2)} - \frac{1}{8} a_3(\mu_1) B_{(\mu_1 + 3)} - \frac{3}{2} \theta(2) B_{(\mu_1, 3)} = 0,
\]

where the last equality follows from straightforward calculations using equations (19) and (20).

If \( l = 2 \), we first remove \( B_{(\mu_1, \mu_2, 2)} \) and \( B_{(\mu_1, \mu_2, 3)} \) in \( \Psi((\mu_1, \mu_2)) \) using formula

\[
B_{(\mu_1, \mu_2, k)} = B_{(\mu_1)} B_{(\mu_2, k)} - B_{(\mu_2)} B_{(\mu_1, k)} + B(k) B_{(\mu_1, \mu_2)}
\]

for any positive integer \( k \), which is obtained using equation (25). We then have

\[
\Psi((\mu_1, \mu_2)) = g_1(\mu_1, \mu_2) + g_2(\mu_1, \mu_2) + \Psi(\emptyset) B_{(\mu_1, \mu_2)},
\]

where

\[
g_1(\mu_1, \mu_2) := 2a_1(\mu_1, \mu_2) B_{(\mu_1 + 1, \mu_2 + 1)} + \theta(2) \left\{ B_{(\mu_1)} B_{(\mu_2, 2)} - B_{(\mu_2)} B_{(\mu_1, 2)} \right\} + 2 \theta(1) \theta(\mu_1 + 1) B_{(\mu_1 + 1, \mu_2)} + 2 \theta(1) \theta(\mu_2 + 1) B_{(\mu_1, \mu_2 + 1)} + a_2(\mu_1) B_{(\mu_1 + 2, \mu_2)} + a_2(\mu_2) B_{(\mu_1, \mu_2 + 2)},
\]

and

\[
g_2(\mu_1, \mu_2) := \theta_{[2]}(\mu_1) B_{(\mu_1 + 1)} + \theta_{[2]}(\mu_2) B_{(\mu_2, 1)} + \theta_{[2]}(\mu_1) \theta_{[2]}(\mu_2) B_{(\mu_1 + 1, \mu_2, 1)}.
\]
and
\[ g_2(\mu_1, \mu_2) := -\frac{3}{2} \theta(2) \{ B(\mu_1)B(\mu_2,3) - B(\mu_2)B(\mu_1,3) \} - \frac{1}{8} \{ a_3(\mu_1)B(\mu_1+3,\mu_2) + a_3(\mu_2)B(\mu_1,\mu_2+3) \}. \]

By straightforward calculations using equations (19) and (20), we have
\[ g_1(\mu_1, \mu_2) = -g_2(\mu_1, \mu_2) = \frac{2^{\mu_1+\mu_2+4}(\mu_1 - \mu_2)}{\mu_1!\mu_2!} \cdot \{-12N^2 + 4(\mu_1^2 + \mu_2^2) + 12(\mu_1 + \mu_2) - 4\mu_1\mu_2 + 11\}. \]

Hence
\[ \Psi((\mu_1, \mu_2)) = 0. \] (64)

**Step 2:** Prove \( \Psi(\mu) = 0 \) if \( l \) is an even integer bigger than 2.

We use recursion formula (22) to expand \( \Psi(\mu) \) in equation (59). For those partitions \( \nu \) with odd length, we need to replace them by \( (\nu, 0) \) before applying the recursion formula (22). If after the first expansion, we obtain some terms containing a factor \( B(\mu, s) \}, \) for \( s \in \{0, 2, 3\} \),

we will use formula (23) with respect to the \( l \)-th part to expand them again. After such expansions, we obtain
\[ \Psi(\mu) = \sum_{i=2}^{l} (-1)^i B_{\mu_1, \mu_i} \Psi(\mu_1^{(1,i)}) + M_1(\mu) \]
\[ + \sum_{i=2}^{l} (-1)^i B_{\mu_1, \mu_i} \cdot \{ \Psi((\mu_1, \mu_i)) - B_{\mu_1, \mu_i} \cdot \Psi(\emptyset) \}, \] (66)

where \( M_1(\mu) \) is defined by equation (47). Since \( M_1(\mu) = 0 \) by Lemma 3.7, this reduces the proof of \( \Psi(\mu) = 0 \) to the \( l = 0 \) and \( l = 2 \) cases, which have been considered in equations (61) and (64) respectively.

**Step 3:** Prove \( \Psi(\mu) = 0 \) if \( l \) is an odd integer bigger than 1.

As in step 2, we expand \( \Psi(\mu) \) by recursion formula (22). For those partitions \( \nu \) appeared in the right hand side of equation (59) which have odd length, we need to replace them by \( (\nu, 0) \) before applying recursion formula (22). This will produce some extra terms containing the factor \( B(\mu_1, 0) \). Since \( (\mu, 2) \) and \( (\mu, 3) \) have even length, expansion of corresponding terms do not produce such factors. After factoring out \( B_{\mu_1, 0} \) from these terms, we obtain an expression which coincides with most terms of \( \Psi(\mu_1^{(1)}) \) but with terms \( \theta(2)B_{(\mu_1, 2)} - \frac{3}{2}\theta(2)B_{(\mu_1, 3)} \) missing. We can express summation of such terms as
\[ B_{(\mu_1, 0)} \left\{ \Psi(\mu_1^{(1)}) - \theta(2)B_{(\mu_1, 2)} + \frac{3}{2}\theta(2)B_{(\mu_1, 3)} \right\}. \]
We then expand $B_{(\mu^{(1)},k)} = B_{(\mu^{(1)},k,0)}$ using recursion formula (23) with $j = l$ for $k = 2, 3$. After regrouping terms, we obtain

$$\Psi(\mu) = \sum_{i=2}^{l} (-1)^i B_{(\mu_1,\mu_i)} \Psi(\mu^{1,i}) + B_{(\mu_1,0)} \Psi(\mu^{1}) + B_{\mu^{(1)}} \cdot \left\{ \Psi((\mu_1)) - B_{(\mu_1)} \Psi(\emptyset) \right\}$$

$$+ M_2(\mu) + \sum_{i=2}^{l} (-1)^i B_{\mu^{(1,i)}} \cdot \left\{ \Psi((\mu_1,\mu_i)) - B_{(\mu_1,\mu_i)} \Psi(\emptyset) \right\}$$

$$+ \theta(2) \left( B_{(\mu_1,2)} - \frac{3}{2} B_{(\mu_1,3)} \right) \sum_{i=2}^{l} (-1)^i B_{\mu^{(1,i)}} B_{(\mu_i)},$$

(67)

where $M_2(\mu)$ is defined by equation (54), which is equal to 0 by Lemma 3.8. The last term on the right hand side of the above equation is also equal to 0 by Lemma 2.1 applied to $\lambda = \mu^{(1)}$. By induction, the theorem is reduced to the cases discussed in step 1. This theorem is thus proved. □

### 3.4 $L_2$ constraint

In this subsection, we will prove $\Gamma(\mu) = 0$, which is equivalent to the $L_2$ constraint for $\tau_N$. We will need the following

**Lemma 3.10** For any partition $\mu = (\mu_1, ..., \mu_l)$ with $l \geq 3$, define

$$R(\mu) := \sum_{i=2}^{l} (-1)^i (\mu_1 + \mu_i) \left( \sum_{j=2 \atop j \neq i}^{l} a_3(\mu_j) B_{(\mu_1+3\epsilon_j)(1,1)} + 12\theta(2) B_{(\mu_3)(1,1)} \right) B_{(\mu_1,\mu_i)}$$

$$+ \sum_{i=2}^{l} (-1)^i B_{\mu^{(1,i)}} \left\{ (|\mu| - \mu_1 - \mu_i) \xi(\mu_1,\mu_i) - 12(\mu_1 + \mu_i) \theta(2) B_{(\mu_3)} B_{(\mu_1,\mu_i)} \right\},$$

(68)

where $a_3(k)$ is defined by equation (60), and

$$\xi(\mu_1,\mu_i) := a_3(\mu_1) B_{(\mu_1+3,\mu_i)} + a_3(\mu_i) B_{(\mu_1,\mu_i+3)} + 12\theta(2) \left\{ B_{(\mu_1)} B_{(\mu_3)} - B_{(\mu_1,3)} B_{(\mu_i)} \right\}.$$

Then

$$R(\mu) = 0$$

for all weakly positive partition $\mu$ with even length $l \geq 4$ such that $\mu_i > 0$ for all $2 \leq i \leq l$.

**Proof:** We first write the factor $|\mu| - \mu_1 - \mu_i$ in the definition of $R(\mu)$ as

$$|\mu| - \mu_1 - \mu_i = \sum_{j=2 \atop j \neq i}^{l} \mu_j.$$
We then use recursion formula (23) to expand following terms in $R(\mu)$ with respect to parts which have different forms from other parts

$$B_{(\mu+3\varepsilon)}^{(1,i)} = \sum_{m=2}^{l} ( -1)^{\tilde{j}(i)+\tilde{m}(i,j)} B_{(\mu_j+3,\mu_m)} B_{\mu}^{(1,i,j,m)},$$

$$\sum_{j=2}^{l} \mu_j B_{\mu}^{(1,i)} = \sum_{j,m=2}^{l} ( -1)^{\tilde{j}(i)+\tilde{m}(i,j)} \mu_j B_{(\mu_j,\mu_m)} B_{\mu}^{(1,i,j,m)},$$

$$B_{(\mu,3,0)}^{(1,i)} = \sum_{j,m=2}^{l} ( -1)^{\tilde{j}(i)+\tilde{m}(i,j)+1} B_{(\mu_j,3)} B_{(\mu_m,0)} B_{\mu}^{(1,i,j,m)} + B_{(3,0)} B_{\mu}^{(1,i)}.$$

Note that partition $\{\mu, 3, 0\}^{(1,i)}$ is weakly positive since the only possible zero part of $\mu$ is $\mu_1$ which is excluded in this partition. When expanding $B_{(\mu,3,0)}^{(1,i)}$, we first expand them with respect to the part 3, then expand again with respect to the zero part. The terms $B_{(3,0)} B_{\mu}^{(1,i)}$ in the last equation are cancelled with corresponding terms in the second line of the definition of $R(\mu)$.

After above expansions, we obtain

$$R(\mu) = \sum_{i,j,m=2}^{l} ( -1)^{i+j(i)+\tilde{m}(i,j)} B_{\mu}^{(1,i,j,m)} \cdot \left\{ \eta_1(\mu_1, \mu_i, \mu_j, \mu_m) + \eta_2(\mu_1, \mu_i, \mu_j, \mu_m) \right\},$$

where

$$\eta_1(\mu_1, \mu_i, \mu_j, \mu_m) = (\mu_1 + \mu_i) B_{(\mu_1,\mu_i)} \left( a_3(\mu_j) B_{(\mu_j+3,\mu_m)} - 12 a_2 B_{(\mu_j,3)} B_{(\mu_m)} \right),$$

$$\eta_2(\mu_1, \mu_i, \mu_j, \mu_m) = \mu_j B_{(\mu_j,\mu_m)} \xi(\mu_1, \mu_i).$$

By straightforward calculations using equations (19) and (20), we have

$$\eta_1(\mu_1, \mu_i, \mu_j, \mu_m) - \eta_1(\mu_1, \mu_i, \mu_m, \mu_j) = \frac{2^{\mu_1+\mu_i+\mu_j+\mu_m+7}(\mu_1-\mu_i)(\mu_j-\mu_m)}{\mu_1!\mu_i!\mu_j!\mu_m!} \cdot (-12N^2 + 4\mu_j^2 + 4\mu_m^2 + 12\mu_j + 12\mu_m - 4\mu_j\mu_m + 11),$$

$$\eta_2(\mu_1, \mu_i, \mu_j, \mu_m) - \eta_2(\mu_1, \mu_i, \mu_m, \mu_j) = \frac{2^{\mu_1+\mu_i+\mu_j+\mu_m+7}(\mu_1-\mu_i)(\mu_j-\mu_m)}{\mu_1!\mu_i!\mu_j!\mu_m!} \cdot (-12N^2 + 4\mu_1^2 + 4\mu_i^2 + 12\mu_1 + 12\mu_i - 4\mu_1\mu_i + 11).$$

Since $( -1)^{i+j(i)+\tilde{m}(i,j)} B_{\mu}^{(1,i,j,m)}$ is skew symmetric with respect to $j$ and $m$, equation (69) then implies

$$R(\mu) = \sum_{\mu_a, \mu_b, \mu_c \in \{1, \ldots, l\}} \frac{2^{\mu_1+\mu_a+\mu_b+\mu_c+7}(-1)^{a+b+c+1} B_{\mu}^{(1,a,b,c)}}{\mu_1!\mu_a!\mu_b!\mu_c!} \cdot \sum_{(i,j,m) \in \sigma(a,b,c)} ((\mu_1 - \mu_i)(\mu_j - \mu_m) \{ h_{a,b,c}(\mu) - 4(\mu_1\mu_i + \mu_j\mu_m) \}).$$
where $\sigma(a, b, c) := \{(a, b, c), (b, c, a), (c, a, b)\}$, and
\[
h_{a,b,c}(\mu) = -24N^2 + 22 + 4(\mu_1^2 + \mu_a^2 + \mu_b^2 + \mu_c^2) + 12(\mu_1 + \mu_a + \mu_b + \mu_c).
\]
The lemma then follows from elementary identities in Lemma A.2 in the appendix. □

We are now ready to prove the following

**Theorem 3.11** Let $\Gamma(\mu)$ be the function of $\mu$ defined by equation (37). We have
\[
\Gamma(\mu) = 0
\]
for all strict partitions $\mu$. In particular, $\tau_N$ satisfies the $L_2$ constraint.

**Proof:** We first use Theorem 3.6 to simplify $\Gamma(\mu)$ defined in equation (37).

By Theorem 3.6, $\Phi((\mu, 3)) = 0$ for any positive partition $\mu$. Hence we have
\[
\sum_{i=1}^{l} \theta(\mu_i + 1)B_{(\mu + \epsilon_i, 3)} = (8|\mu| + 24 + \theta(1))B_{(\mu, 3)} - \theta(4)B_{(\mu, 4)} - \frac{1}{2}\theta(1)B_{(\mu, 3, 1)}.
\]

Since $\Phi((\mu, 2, 1)) = 0$, we have
\[
\sum_{i=1}^{l} \theta(\mu_i + 1)B_{(\mu + \epsilon_i, 2, 1)} = (8|\mu| + 24 + \theta(1))B_{(\mu, 2, 1)} - \theta(3)B_{(\mu, 3, 1)}.
\]

Since $\Phi(\mu + 3\epsilon_i) = 0$ for all $1 \leq i \leq l(\mu)$, we have
\[
\sum_{j \neq i} \theta(\mu_j + 1)B_{(\mu + 3\epsilon_i + \epsilon_j)} = (8|\mu| + 24 + \theta(1))B_{(\mu + 3\epsilon_i)} - \theta(\mu_i + 4)B_{(\mu + 4\epsilon_i)} - \frac{1}{2}\theta(1)B_{(\mu + 3\epsilon_i, 1)}.
\]

We first use the above formulas to reduce corresponding summations in $\Gamma(\mu)$, then remove $B_{(\mu, 2, 1)}$ using equation (58). We obtain the following simplification of $\Gamma(\mu)$:
\[
\Gamma(\mu) = (8|\mu| + 24 + \theta(1)) \left(2 \sum_{i=1}^{l(\mu)} a_3(\mu_i)B_{(\mu + 3\epsilon_i)} + 24\theta(2)B_{(\mu, 3)}\right)
\]
\[
+ \sum_{i=1}^{l(\mu)} (\mu_i + 2)\theta[4](\mu_i)B_{\mu + 4\epsilon_i} + \theta(4)B_{(\mu, 4)} - \frac{1}{2}\theta(3, 1)B_{(\mu, 3, 1)}
\]
\[
- \frac{1}{16} \cdot \left(2 \sum_{i=1}^{l(\mu)} \theta[5](\mu_i)B_{(\mu + 5\epsilon_i)} + \sum_{r=0}^{2} (-1)^r \theta(5-r, r)B_{(\mu, 5-r, r)}\right),
\] (70)
where $a_3(\mu_i)$ is defined by equation (60).

Using the simple fact that
\[
\theta[3](0) - \theta(2, 1) = 24\theta(2) \quad \text{and} \quad \theta[4](0) = \theta(4)
\]
for all $k > 0$, it is straightforward to show that the right hand side of equation (70) does not change value if we replace $\mu$ by $(\mu, 0)$. Hence we have $\Gamma(\mu) = \Gamma((\mu, 0))$ for any partition $\mu$. Since the right hand side of equation (70) is skew symmetric with respect to permutations of weakly positive partitions, we have $\Gamma(\mu) = \pm \Gamma((0, \mu))$ for any positive partition $\mu$. Hence $\Gamma(\mu) = 0$ if and only if $\Gamma((0, \mu)) = 0$. In particular, if $\mu$ is a strict partition with odd length, instead of considering $\Gamma((\mu, 0))$, we will consider $\Gamma((0, \mu))$. In a summary, to prove $\Gamma(\mu) = 0$ for all strict partitions $\mu$, it suffices to show that $\Gamma((0, \mu)) = 0$ for all weakly positive partitions $\mu = (\mu_1, \mu_2, \ldots, \mu_l)$ with $l$ even and $\mu_i > 0$ for $2 \leq i \leq l$. We will prove this fact by induction on $l$.

If $l = 0$, we have

$$\Gamma(\emptyset) = c_1 + c_2, \quad (71)$$

where

$$c_1 := 24\{24 + \theta(1)\} \theta(2) B(3) + \theta(4) B(4) - \frac{1}{2} \theta(3,1) B(3,1),$$

$$c_2 := - \frac{1}{16} \sum_{r=0}^{2} (-1)^r \theta(5-r,r) B_{(5-r,r)}.$$

A straightforward calculation using equations (19) and (20) shows that $c_1 = -c_2 = 64\theta(3)$. Hence

$$\Gamma(\emptyset) = 0. \quad (72)$$

If $l = 2$, we use equations (25) and (22) to expand $B_\nu$ occurred in $\Gamma((\mu_1, \mu_2))$ with $l(\nu) = 3$ or 4 and obtain

$$\Gamma((\mu_1, \mu_2)) = f_1(\mu_1, \mu_2) + f_2(\mu_1, \mu_2) + B_{(\mu_1, \mu_2)} \Gamma(\emptyset), \quad (73)$$

where

$$f_1(\mu_1, \mu_2) := \{8(\mu_1 + \mu_2) + 24 + \theta(1)\} \left(2\alpha_3(\mu_1)B_{(\mu_1+3, \mu_2)} + 2\alpha_3(\mu_2)B_{(\mu_1+2, \mu_2+3)}
+ 24\theta(2) \left(B_{(\mu_1)}B_{(\mu_2,3)} - B_{(\mu_2)}B_{(\mu_1,3)}\right)\right) + 192(\mu_1 + \mu_2)\theta(2) B(3) B_{(\mu_1, \mu_2)}
+ (\mu_1 + 2)\theta[4](\mu_1)B_{(\mu_1+4, \mu_2)} + (\mu_2 + 2)\theta[4](\mu_2)B_{(\mu_1, \mu_2+4)}
+ \theta(4) \left(B_{(\mu_1)}B_{(\mu_2,4)} - B_{(\mu_2)}B_{(\mu_1,4)}\right) - \frac{1}{2} \theta(3,1) \left(B_{(\mu_1,1)}B_{(\mu_2,3)} - B_{(\mu_2,1)}B_{(\mu_1,3)}\right),$$

$$f_2(\mu_1, \mu_2) := - \frac{1}{8} \left(\theta[5](\mu_1)B_{(\mu_1+5, \mu_2)} + \theta[5](\mu_2)B_{(\mu_1, \mu_2+5)}\right)
- \frac{1}{16} \sum_{r=0}^{2} (-1)^r \theta(5-r,r) \left(B_{(\mu_1, r)}B_{(\mu_2, 5-r)} - B_{(\mu_2, r)}B_{(\mu_1, 5-r)}\right).$$
A straightforward calculation using equations (19) and (20) shows that

\[ f_1(\mu_1, \mu_2) = - f_2(\mu_1, \mu_2) \]

\[ = \frac{2^{\mu_1 + \mu_2 + 9}(\mu_1 - \mu_2)}{\mu_1! \mu_2!} \sum_{k=1}^{2} \left\{ 40 N^4 - 20(2\mu_k^2 + 10\mu_k - \mu_1\mu_2 + 11)N^2 \\
+ 2(4\mu_k^4 + 40\mu_k^3 + 145\mu_k^2 + 225\mu_k) - \mu_1\mu_2(8\mu_k^2 - 4\mu_1\mu_2 - 55) + \frac{297}{2} \right\}. \]

Hence

\[ \Gamma((\mu_1, \mu_2)) = 0. \]

(74)

For any even integer \( l > 2 \), we use recursion formula (22) to expand \( \Gamma(\mu) \) in equation (70). For those partitions \( \nu \) occurred in \( \Gamma(\mu) \) with odd length, we need to replace it by \((\nu, 0)\) before applying recursion formula (22). Since \( \nu \) is weakly positive, Lemma 2.2 guarantees equation (22) can still be used for \((\nu, 0)\) in this case. After the first expansion, we obtain some terms containing a factor \( B_{(\mu,s)}^{(1)} \) with \( 0 \leq s \leq 5 \). We can use formula (23) to expand such terms again with respect to the \( l \)-th part. After such expansions, we obtain

\[ \Gamma(\mu) = \sum_{i=2}^{l} (-1)^i B_{(\mu_1,\mu_i)}^{(1)} \Gamma^{(1,i)}(\mu^{(1,i)}) + \sum_{i=2}^{l} (-1)^i \{ B_{(\mu_1,\mu_i)}^{(1)} \Gamma^{(1,i)}((\mu_1,\mu_i)) - B_{(\mu_1,\mu_i)}^{(1)} \Gamma(\emptyset) \} + 16R(\mu), \]

where \( R(\mu) \) is defined by equation (68) and it is equal to 0 by Lemma 3.10. By induction, the theorem is reduced to the \( l = 0 \) and \( l = 2 \) cases, which have been considered in equations (72) and (74) respectively. The theorem is thus proved. \( \square \)

Proof of Theorems 1.1 and 1.2: Since \( L_{0}^{(N)} \), \( L_1 \), \( L_2 \) generate all Virasoro operators \( L_m^{(N)} \) for \( m \geq 0 \), Theorems 3.6, 3.9, 3.11 imply that \( \tau_N \) satisfies the Virasoro constraints (5) for all \( N \). Moreover, since \( Q_\lambda \) is a homogeneous polynomial of degree \( |\lambda| \), at \( t = 0, \)

\[ \tau_N(0) = Q_\emptyset = 1. \]

Hence \( \tau_N \) and \( \tau_{BGW}^{(N)} \) satisfy the same Virasoro constraints with the same normalization condition. By Alexandrov’s theorem in [A18], we have

\[ \tau_{BGW}^{(N)} = \tau_N \]

for all \( N \). This completes the proof of Theorem 1.2.

By equation (21), Theorem 1.1 follows from Theorem 1.2 with \( N = 0 \). Note that the dimension constraint for the geometric interpretation of \( \tau_{BGW} \) corresponds to the fact that the coefficient of \( h^m \) in \( \tau_N \) is a homogeneous polynomial of \( (t_1, t_3, \cdots) \) of degree \( m \) for all \( m \). \( \square \)

Appendix
A Some elementary identities

Lemma A.1 For any three positive integers $a < b < c$, and three non negative integers $\mu_a, \mu_b, \mu_c$ labeled by $a, b, c$, we have

$$\sum_{(i,j,m) \subseteq P(a,b,c)} (-1)^{i+j(i)+\tilde{m}(j,i)} \rho_k(\mu_i, \mu_j, \mu_m) = 0$$

for $k = 0, 1, 2$, where $P(a,b,c)$ is the set of all permutations of $(a,b,c)$, $\tilde{j}(i)$ and $\tilde{m}(i, j)$ are defined by equation (51), $\rho_k$ are defined by equation (53).

Proof: Set

$$K_{abc} := (\mu_a + 1)(\mu_b + 1)(\mu_c + 1).$$

Note that

$$\rho_0(\mu_i, \mu_j, \mu_m) = -16K_{abc} + \theta(1)^2 \cdot \mu_i(\mu_j - \mu_m),$$

$$\rho_1(\mu_i, \mu_j, \mu_m) = 16K_{abc}(1 - \mu_i) - \theta(1)^2 \cdot (\mu_j - \mu_m),$$

$$\rho_2(\mu_i, \mu_j, \mu_m) = K_{abc} \cdot (\mu_j - \mu_m).$$

The lemma then follows from the following identities

$$\mu_i(\mu_j - \mu_m) + \mu_j(\mu_m - \mu_i) + \mu_m(\mu_i - \mu_j) = 0,$$

$$(\mu_j - \mu_m) + (\mu_m - \mu_i) + (\mu_i - \mu_j) = 0.$$ 

□

Lemma A.2 For any three positive integers $a < b < c$, and three non negative integers $\mu_a, \mu_b, \mu_c$ labeled by $a, b, c$, we have

$$\sum_{(i,j,k) \in \sigma(a,b,c)} (\mu_j - \mu_m) = \sum_{(i,j,k) \in \sigma(a,b,c)} \mu_i(\mu_j - \mu_m) = \sum_{(i,j,k) \in \sigma(a,b,c)} (\mu_j\mu_m - \mu_i^2)(\mu_j - \mu_m) = 0,$$

where $\sigma(a,b,c) = \{(a, b, c), (b, c, a), (c, a, b)\}$.

The proof of this lemma is straightforward.

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