Notes on Coherent Feedback Control for Linear Quantum Systems

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Abstract—This paper considers some formulations and possible approaches to the coherent LQG and $H^\infty$ quantum control problems. Some new results for these problems are presented in the case of annihilation operator only quantum systems showing that in this case, the optimal controllers are trivial controllers.

I. INTRODUCTION

In recent years, a number of papers have considered the feedback control of systems whose dynamics are governed by the laws of quantum mechanics instead of classical mechanics; see e.g., [1]–[4]. Also, an important class of quantum system models are linear quantum stochastic systems which describe quantum optical devices such as optical cavities, linear quantum amplifiers, and finite bandwidth squeezers; e.g. see [5]. For such linear quantum system models an important class of quantum control problems are referred to as coherent control problems; e.g., see [5]. These coherent quantum control problems include the coherent LQG control problem (e.g., see [2]) and the coherent $H^\infty$ control problem (e.g., see [1]). In coherent quantum control problems, the controller itself is required to be a quantum system. One motivation for considering such coherent quantum control problems is that coherent controllers have the potential to achieve improved performance since quantum measurements inherently involve the destruction of quantum information.

In this paper, we discuss the formulation and possible approaches to the coherent LQG and $H^\infty$ quantum control problems for a class of linear stochastic quantum system models. Also, we present some new results for these problems in the case of annihilation operator only quantum systems; e.g., see [3].

II. QUANTUM SYSTEMS AND PHYSICAL REALIZABILITY

In this section, we describe the general class of quantum systems under consideration; see also [1], [4], [8].

We consider a collection of $n$ independent quantum harmonic oscillators. Corresponding to this collection of harmonic oscillators is a vector of annihilation operators $a = [a_1 \ a_2 \ \ldots \ a_n]^T$. The adjoint of the operator $a_i$ is denoted by $a_i^*$ and is referred to as a creation operator. The operators $a_i$ and $a_i^*$ are such that the following commutation relations are satisfied:

$$\begin{bmatrix} a & a^* \\ a^* & a \\ \end{bmatrix} = \Theta = T J T^\dagger$$

where $\Theta$ is a Hermitian commutation matrix of the form $\Theta = T J T^\dagger$ with $J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$ and $T = \Delta(T_1, T_2)$. Here $\Delta(T_1, T_2)$ denotes the matrix $\begin{bmatrix} T_1 & T_2 \\ T_2^\# & T_1^\# \end{bmatrix}$. Also, $T^\dagger$ denotes the adjoint transpose of a vector of operators or the complex conjugate transpose of a complex matrix. In addition, $^\#$ denotes the adjoint of a vector of operators or the complex conjugate of a complex matrix.

The quantum harmonic oscillators are assumed to be coupled to $m$ external independent quantum fields modelled by bosonic annihilation field operators $A_1, A_2, \ldots, A_m$. For each annihilation field operator $A_k$, there is a corresponding creation field operator $A_k^*$, which is the operator adjoint of $A_k$. The field annihilation operators are also collected into a vector of operators defined as follows: $A = [A_1, A_2, \ldots, A_m]^T$.

In order to describe the dynamics of a quantum linear system, we first specify the Hamiltonian operator for the quantum system which is a Hermitian operator on $\mathcal{H}$ of the form

$$\mathbf{H} = \frac{1}{2} \begin{bmatrix} a^\dagger & a^T \end{bmatrix} M \begin{bmatrix} a & a^\# \end{bmatrix}$$

where $M$ is a Hermitian matrix of the form

$$M = \Delta(M_1, M_2).$$

Also, we specify the coupling operator vector for the quantum system to be a vector of operators of the form

$$L = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} a & a^\# \end{bmatrix}$$

where $N_1 \in \mathbb{C}^{m \times n}$ and $N_2 \in \mathbb{C}^{m \times n}$. We can write

$$\begin{bmatrix} L \\ L^\# \end{bmatrix} = N \begin{bmatrix} a & a^\# \end{bmatrix},$$

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where \( N = \Delta(N_1, N_2) \). These operators then lead to the following quantum stochastic differential equations (QSDEs) which describe the dynamics of the quantum system under consideration:

\[
\begin{bmatrix}
\frac{da}{dt} \\
\frac{dA^\text{out}}{dt}
\end{bmatrix} = F\begin{bmatrix} a \\ a^\# \end{bmatrix} dt + G\begin{bmatrix} dA \\ dA^\# \end{bmatrix};
\]

\[
\begin{bmatrix}
\frac{dA^\text{out}}{dt}
\end{bmatrix} = H\begin{bmatrix} a \\ a^\# \end{bmatrix} dt + K\begin{bmatrix} dA \\ dA^\# \end{bmatrix},
\]

(3)

where

\[
F = \Delta(\dot{F}_1, \dot{F}_2), \quad G = \Delta(\dot{G}_1, \dot{G}_2),
\]

\[
H = \Delta(\dot{H}_1, \dot{H}_2), \quad K = \Delta(\dot{K}_1, \dot{K}_2),
\]

(4)

and

\[
F = -i\Theta M - \frac{1}{2}\Theta N^1 JN; \quad G = -\Theta N^1 J;
\]

\[
H = N; \quad K = I.
\]

(5)

Definition 1: A linear quantum system of the form (3), (4) is physically realizable if there exist complex matrices \( \Theta = \Theta^\dagger \), \( M = M^\dagger \), \( N \), such that \( \Theta \) is of the form

\[
\Theta = TJT^\dagger
\]

where \( T = \Delta(T_1, T_2) \) is non-singular, \( M \) is of the form in (3), and \( N \) is satisfied.

Theorem 1: (See [12]) The linear quantum system (3), (4) is physically realizable if and only if there exists a complex matrix \( \Theta = \Theta^\dagger \) such that \( \Theta \) is of the form in (5), and

\[
F\Theta + \Theta F^\dagger + GJG^\dagger = 0; \quad G = -\Theta H^\dagger J; \quad K = I. \quad (7)
\]

The physical realizability of the linear quantum system (3), (4) is related to the \((J, J)\)-unitary properties of the corresponding transfer function matrix

\[
\Gamma(s) = \begin{bmatrix} F & G \\ H & K \end{bmatrix} = H(sI - F)^{-1} G + K. \quad (8)
\]

Definition 2: (See [12], [13]) The transfer function \( \Gamma(s) \) is said to be \((J, J)\)-unitary if \( \Gamma(s)J\Gamma(s) = J \), for every \( s \in \mathbb{C} \).

Theorem 2: (see [12]) Suppose the linear quantum system (3), (4) is minimal, and that \( \lambda_i(F) + \lambda_j^*(F) \neq 0 \) for all eigenvalues \( \lambda_i(F), \lambda_j(F) \) of the matrix \( \hat{F} \). Then this linear quantum system is physically realizable if and only if the following conditions hold:

(i) The system transfer function matrix \( \Gamma(s) \) in (5) is \((J, J)\)-unitary;

(ii) The matrix \( K \) is of the form \( K = I \).

A. Annihilation Operator Quantum Systems

An important special case of the above class of linear quantum systems occurs when the QSDEs (3) can be described purely in terms of the vector of annihilation operators \( a \); e.g., see [3], [6]. In this case, we consider Hamiltonian operators of the form \( H = a^\dagger M a \) and coupling operator vectors of the form \( L = Na \) where \( M \) is a Hermitian matrix and \( N \) is a complex matrix. In this case, we replace the commutation relations (1) by the commutation relations

\[
[a, a^\dagger] = \Theta \quad (9)
\]

where \( \Theta \) is a positive-definite commutation matrix. Then, the corresponding QSDEs are given by

\[
da = Fadt + GdA;
da^\text{out} = Hadt + KdA
\]

(10)

where

\[
F = \Theta \left(-iM + \frac{1}{2}N^1N\right); \quad G = -\Theta N^1;
\]

\[
H = N; \quad K = I.
\]

(11)

Definition 3: A linear quantum system of the form (10) is physically realizable if there exist complex matrices \( \Theta \geq 0 \), \( M = M^\dagger \), \( N \), such that (11) is satisfied.

Theorem 3: (See [6]) The annihilation operator linear quantum system (10) is physically realizable if and only if there exists a complex matrix \( \Theta > 0 \) such that

\[
F\Theta + \Theta F^\dagger + GG^\dagger = 0; \quad G = -\Theta H^\dagger; \quad K = I. \quad (12)
\]

The physical realizability of the annihilation operator linear quantum system (10) is related to the lossless bounded real property of the corresponding transfer function matrix

\[
\Gamma(s) = \begin{bmatrix} F & G \\ H & K \end{bmatrix} = H(sI - F)^{-1} G + K. \quad (13)
\]

Definition 4: The transfer function \( \Gamma(s) \) is said to be lossless bounded real if all of the poles of \( \Gamma(s) \) are in the open left half of the complex plane and \( \Gamma^\sim(s)\Gamma(s) = I \) for every \( s \in \mathbb{C} \).

Theorem 4: (See [6]) A minimal annihilation operator linear quantum system (10) is physically realizable if and only if the system transfer function matrix \( \Gamma(s) \) in (13) is lossless bounded real and the matrix \( K \) is of the form \( K = I \).

III. COHERENT FEEDBACK CONTROL

We now consider problems of coherent feedback control for linear quantum systems. In these problems, both the plant and the controller are linear quantum systems. Moreover, in the plant, the input fields to the system are divided into two types, control input fields and vacuum noise input fields. Hence, we re-write the system (3) in the following form.

**Plant:**

\[
\begin{bmatrix}
da \\
da^\# 
\end{bmatrix}
= F\begin{bmatrix} a \\ a^\# \end{bmatrix} dt + \begin{bmatrix} G_{w1} \\ G_{w2} \end{bmatrix} \begin{bmatrix} dW \\ dW^\# \end{bmatrix}
\]

\[
+ \begin{bmatrix} K_{11} \\ K_{12} \end{bmatrix} \begin{bmatrix} a \\ a^\# \end{bmatrix} dt
\]

(14)

where \( \mathcal{U} \) denotes the control input field, \( W \) denotes the noise input field, and \( \mathcal{V} \) denotes the output field. In coherent feedback control, it is assumed that the output field is an
input field for another quantum linear system, which is the controller and the control input field is an output field for the controller system. Note that it is assumed that there is no direct feedthrough from the control input field \( U \) to the output field \( Y \). The transfer function matrix corresponding to the plant \( (4) \) will be denoted \( \Gamma_P(s) = \begin{bmatrix} \Gamma_{P1}(s) & \Gamma_{P2}(s) \end{bmatrix} \).

It will be assumed that the plant is physically realizable. That is, it is assumed that the plant (14) can be augmented with an additional (unused) output to obtain a quantum linear system of the form \( (3) \) which is physically realizable. This augmented plant is defined as follows:

\[
\begin{bmatrix}
d a \\
d a^\# 
\end{bmatrix} = \begin{bmatrix} F & a_c & a_c^\# \\
& a & a^\#
\end{bmatrix} dt + \begin{bmatrix} G_{u1} & G_{u2} \\
G_{cy1} & G_{cy2}
\end{bmatrix} \begin{bmatrix} dW \\
dY 
\end{bmatrix};
\]

\[
\begin{bmatrix}
d\dot{Y} \\
dY^\# 
\end{bmatrix} = \begin{bmatrix} H_1 & H_2 \\
H_1 & H_2
\end{bmatrix} \begin{bmatrix} a \\
a^\#
\end{bmatrix} dt + \begin{bmatrix} K_{11} & K_{12} \\
K_{12} & K_{12}
\end{bmatrix} \begin{bmatrix} dW \\
d\dot{Y} 
\end{bmatrix} + \begin{bmatrix} K_{11} & K_{12} \\
K_{21} & K_{22}
\end{bmatrix} \begin{bmatrix} d\dot{Y}^\# \\
\bar{d}Y^\#
\end{bmatrix}.
\]

Since physical realizability requires that the direct feedthrough matrix for this system is the identity, we have \( K_{21} = 0, K_{21} = 0, K_{12} = 0, \) and \( K_{12} = 0 \).

The system describing the controller is also assumed to be physically realizable. That is, it is assumed that the controller (16) can be augmented with an additional (unused) output to obtain a quantum linear system of the form \( (3) \) which is physically realizable. This augmented controller is defined as follows:

\[
\begin{bmatrix}
da_c \\
da_c^\#
\end{bmatrix} = \begin{bmatrix} F_c & a_c & a_c^\# \\
& a_c & a_c^\#
\end{bmatrix} dt + \begin{bmatrix} G_{cw1} & G_{cw2} \\
G_{cy1} & G_{cy2}
\end{bmatrix} \begin{bmatrix} d\dot{Y} \\
\bar{d}Y 
\end{bmatrix} + \begin{bmatrix} K_{cw11} & K_{cw12} \\
K_{cy11} & K_{cy12}
\end{bmatrix} \begin{bmatrix} \bar{d}Y^\# \\
\bar{d}Y^\#
\end{bmatrix};
\]

\[
\begin{bmatrix}
d\dot{U} \\
\dot{d}U^\#
\end{bmatrix} = \begin{bmatrix} H_{c1} & H_{c2} \\
H_{c1} & H_{c2}
\end{bmatrix} \begin{bmatrix} a_c \\
a_c^\#
\end{bmatrix} dt + \begin{bmatrix} K_{cw11} & K_{cw12} \\
K_{cy11} & K_{cy12}
\end{bmatrix} \begin{bmatrix} \bar{d}Y \\
\bar{d}Y^\#
\end{bmatrix} + \begin{bmatrix} K_{cw21} & K_{cw22} \\
K_{cy21} & K_{cy22}
\end{bmatrix} \begin{bmatrix} \bar{d}Y^\# \\
\bar{d}Y^\#
\end{bmatrix}.
\]

Cost Output: The coherent LQG control problem and the coherent \( H^\infty \) control problem are both defined in terms of the following cost output for the plant \( (4) \):

\[
dZ = C \begin{bmatrix} a \\
a^\#
\end{bmatrix} dt + D \begin{bmatrix} d\dot{U} \\
\dot{d}U^#
\end{bmatrix}.
\]
Note that no physical realizability restrictions are placed on this output equation at this stage. The transfer function matrix from the two inputs of the plant \([15]\) to the performance output \(Z(t)\) will be denoted \(\Gamma_c(s) = \begin{bmatrix} \Gamma_{z1}(s) & \Gamma_{z2}(s) \end{bmatrix} \). The resulting closed-loop transfer function matrix from the noise inputs to the cost output is then calculated to be

\[
\Gamma_{cl}(s) = \left[ \Gamma_{z1}(s) + M(s)\Gamma_{cl2}(s)\Gamma_{p1}(s) M(s)\Gamma_{cl1}(s) \right]
\]

where \(M(s) = \Gamma_{z2}(s)(I - \Gamma_{cl2}(s)\Gamma_{p2}(s))^{-1}\).

**Coherent Quantum LQG Control:** Using Theorem 2, a coherent quantum LQG control problem can be formulated purely in terms of transfer function matrices as follows:

\[
\min_{\Gamma_c(s)} \|\Gamma_{cl}(s)\|_2
\]

subject to the constraints that the closed-loop system is internally stable, \(\Gamma_c(s)\) is \((J,J)\)-unitary, and \(p_1 + p_1^* \neq 0\) for all poles \(p_1, p_1^*\) of the transfer function matrix \(\Gamma_c(s)\). Here \(\|\Gamma_{cl}(s)\|_2\) denotes the \(H_2\) norm of the transfer function matrix \(\Gamma_{cl}(s)\):

\[
\|\Gamma_{cl}(s)\|_2^2 := \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Tr}[\Gamma_{cl}(i\omega)\Gamma_{cl}^*(i\omega)]d\omega;
\]

e.g., see Section 3.3.3 of \([14]\). Here, we assume that the transfer function matrices \(\Gamma_{z2}(s)\Gamma_{cl1}(s)\) and \(\Gamma_{z2}(s)\Gamma_{cl2}(s)\Gamma_{p1}(s)\) are strictly proper. Also, it follows from the form of \([19]\) that the transfer function matrix \(\Gamma_{z1}(s)\) is strictly proper. These conditions ensure that the quantity \(\|\Gamma_{cl}(s)\|_2^2\) is finite provided that the closed loop system is stable.

**Coherent Quantum \(H^\infty\) Control:** Similarly to the above, we can also formulate a coherent \(H^\infty\) control problem purely in terms of transfer function matrices as follows:

\[
\min_{\Gamma_c(s)} \|\Gamma_{cl}(s)\|_{\infty}
\]

subject to the constraints that the closed-loop system is internally stable, \(\Gamma_c(s)\) is \((J,J)\)-unitary, and \(p_1 + p_1^* \neq 0\) for all poles \(p_1, p_1^*\) of the transfer function matrix \(\Gamma_c(s)\). Note that this coherent \(H^\infty\) control problem is different to the coherent \(H^\infty\) control problem considered in previous papers such as [1], [3] in that we consider all quantum noise inputs as disturbances, including the controller quantum noise inputs.

These frequency domain formulations of the coherent quantum LQG control problem and the coherent \(H^\infty\) control problem motivate numerical methods to solve these problems using the frequency domain optimization tools developed in [15]. These approaches are beyond the scope of the current paper but may be pursued in future research.

**Time Domain Formulations of Coherent Quantum \(H^\infty\) and LQG Control:** To develop time domain formulations of the coherent quantum \(H^\infty\) and LQG control problems, we first consider a time domain version of the physical realizability condition that \(\Gamma_c(s)\) is \((J,J)\) unitary. Note that that we can write

\[
\begin{bmatrix}
\dot{U}(s) \\
\dot{\bar{u}}(s) \\
\bar{U}^*(s) \\
\bar{U}^*(s)
\end{bmatrix}
= \begin{bmatrix}
\bar{V}(s) \\
\bar{Y}(s) \\
\bar{Y}(s) \\
\bar{Y}(s)
\end{bmatrix}
\]

for the following classical LTI system corresponding to the quantum system \([17]\) defined as follows:

\[
\begin{bmatrix}
\dot{U}(s) \\
\dot{\bar{u}}(s) \\
\bar{U}^*(s) \\
\bar{U}^*(s)
\end{bmatrix}
= \begin{bmatrix}
F_c \\
\bar{a}_c \\
\bar{a}_c
\end{bmatrix} + \begin{bmatrix}
G_{cu1} \\
G_{cy1} \\
G_{cu2} \\
G_{cy2}
\end{bmatrix} \begin{bmatrix}
\bar{V} \\
\bar{Y} \\
\bar{Y} \\
\bar{Y}
\end{bmatrix}.
\]

In this classical system, all quantities are complex vectors or matrices. Then as in [13], we can write

\[
\begin{bmatrix}
U(s) \\
\bar{U}(s) \\
\bar{U}^*(s) \\
\bar{U}^*(s)
\end{bmatrix} = \text{CHAIN}^{-1}\left(\Gamma_c(s)\right) \begin{bmatrix}
U(s) \\
\bar{U}(s) \\
\bar{U}^*(s) \\
\bar{U}^*(s)
\end{bmatrix}
\]

where \(\text{CHAIN}^{-1}\left(\Gamma_c(s)\right)\) denotes the inverse of the chain scattering representation of the transfer function matrix \(\Gamma_c(s)\). Then the transfer function matrix \(\Gamma_c(s)\) is \((J,J)\) unitary if and only if the transfer function matrix \(\text{CHAIN}^{-1}\left(\Gamma_c(s)\right)\) is lossless bounded real; e.g., see Lemma 4.4 of [13] and the proof of Theorem 2 in [12]. Furthermore, it follows from Lemma 2.6.1 of [16] that the transfer function matrix \(\text{CHAIN}^{-1}\left(\Gamma_c(s)\right)\) is lossless bounded real if and only if

\[
\int_0^T \left( \|U\|^2 + \|\bar{U}\|^2 + \|\bar{U}^*\|^2 + \|\bar{Y}^*\|^2 \right) dt \geq 0
\]

for all \(T > 0\) and all solutions to (20) with zero initial condition, and

\[
\int_0^\infty \left( \|U\|^2 + \|\bar{U}\|^2 + \|\bar{U}^*\|^2 + \|\bar{Y}^*\|^2 \right) dt = 0
\]

for all solutions to (20) with zero initial condition satisfying

\[
\int_0^\infty \left( \|U\|^2 + \|\bar{U}\|^2 + \|\bar{Y}^*\|^2 + \|\bar{Y}^*\|^2 \right) dt < \infty.
\]

To consider the time domain version of the coherent quantum \(H^\infty\) and LQG control problems, we now consider
In this classical system, all quantities are complex vectors or matrices. Then we have

\begin{align*}
   Z(s) &= \Gamma_{cl}(s) \begin{bmatrix} \mathcal{W}(s) \\ \mathcal{W}^*(s) \\ \mathcal{W}(s) \\ \mathcal{W}^*(s) \end{bmatrix}.
\end{align*}

The Time Domain Coherent Quantum LQG Control: We can calculate the LQG control cost function \( \|\Gamma_{cl}(s)\|_2 \) in the time domain as

\begin{equation}
   \|\Gamma_{cl}(s)\|_2 = \sum_{k=1}^{n+n} \int_0^\infty \|Z_k(t)\|^2 dt
\end{equation}

where \( Z_k(t) \) denotes the response of the closed loop system with zero initial condition, such that the \( k \)th input noise is a unit impulse, and all other noise inputs are zero; e.g., see Section 6.1 of [17]. Thus, the time domain formulation of the coherent quantum LQG control problem involves minimizing the cost in (24) for the system (23) over the controllers (20), subject to the constraints defined by (21), (22).

The Time Domain Coherent Quantum \( H^\infty \) Control: In the time domain, \( H^\infty \) control problems are conveniently considered as sub-optimal problems via a game theory approach; e.g., see [14], [18]. That is, the condition \( \|\Gamma_{cl}(s)\|_\infty < \gamma \) is equivalent to the condition

\begin{equation}
   \sup_{\mathcal{W}(\cdot),\mathcal{W}^*(\cdot),\mathcal{W}^!(\cdot),\mathcal{W}^!(\cdot)} J_\gamma < \infty
\end{equation}

for all \( T > 0 \). Here

\begin{equation}
   J_\gamma = \int_0^T \left( \|Z\|^2 - \gamma^2 \left( \|\mathcal{W}\|^2 + \|\mathcal{W}\|^2 \right) \right) dt.
\end{equation}

Hence, we can consider the following game problem for the system (23):

\begin{equation}
   \inf_{U(\cdot),U^*(\cdot),U^!(\cdot),U^!(\cdot)} \sup_{\mathcal{W}(\cdot),\mathcal{W}^*(\cdot),\mathcal{W}^!(\cdot),\mathcal{W}^!(\cdot)} J_\gamma < \infty
\end{equation}

subject to the constraints defined by (21), (22).

Restriction to a Strictly Proper Controller: In the standard LQG and \( H^\infty \) control problems, the controller is usually restricted to be strictly proper. However, in the quantum case, a strictly proper controller may not be physically realizable. However, once we have decided on the dimension of the controller noise vector, the direct feedthrough matrix in the system (17) is restricted to be the identity matrix. This determines the direct feedthrough matrices \( K_{cu} \) and \( K_{cy} \) in the controller (15). Since these matrices are fixed, these terms in the controller (17) can be incorporated into the plant (14). In addition, we assume that the dimension of the controller noise \( \mathcal{W} \) is greater than or equal to the dimension of the plant input \( U \) and we write

\begin{equation}
   \mathcal{W} = \begin{bmatrix} \mathcal{W}_a \\ \mathcal{W}_b \end{bmatrix}
\end{equation}

where the dimension of \( \mathcal{W}_a \) is equal to the dimension of \( U \). This leads to the following modified plant and controller classical systems:

**Modified Plant:**

\begin{align*}
   \begin{bmatrix} \dot{a} \\ \dot{a}^# \end{bmatrix} &= (F + G_u K_{cy} H) \begin{bmatrix} a \\ a^# \end{bmatrix} \\
   &+ (G_w + G_u K_{cy} K) \begin{bmatrix} \mathcal{W} \\ \mathcal{W}^# \end{bmatrix} + G_u \begin{bmatrix} U \\ U^# \end{bmatrix} ;
\end{align*}

\begin{align*}
   \begin{bmatrix} \mathcal{Y} \\ \mathcal{Y}^# \end{bmatrix} &= H \begin{bmatrix} a \\ a^# \end{bmatrix} + K \begin{bmatrix} \mathcal{W} \\ \mathcal{W}^# \end{bmatrix} ;
\end{align*}

**Modified Controller:**

\begin{align*}
   \begin{bmatrix} \dot{a}_c \\ \dot{a}_c^# \end{bmatrix} &= F_c \begin{bmatrix} a_c \\ a_c^# \end{bmatrix} + G_{cu} \begin{bmatrix} \mathcal{W}_a \\ \mathcal{W}_b \\ \mathcal{W}_a^# \\ \mathcal{W}_b^# \end{bmatrix} \\
   &+ G_{cy} \begin{bmatrix} \mathcal{Y} \\ \mathcal{Y}^# \end{bmatrix} ;
\end{align*}

\begin{equation}
   \begin{bmatrix} U \\ U^# \end{bmatrix} = H_c \begin{bmatrix} a_c \\ a_c^# \end{bmatrix} .
\end{equation}

In order to develop a condition for the physical realizability of the modified controller (27) purely in terms of the dynamics matrices \( F_c, G_{cy}, \) and \( H_c \), we first apply Theorem 1 to the augmented controller system (17). From this it follows that (27) is physically realizable if and only if there exists a complex matrix \( \Theta = \Theta^\dagger \) such that

\begin{equation}
   F_c \Theta + \Theta F_c^\dagger + G_c J G_c^\dagger = 0; \quad G_c = -\Theta \tilde{H}_c^\dagger J .
\end{equation}

Here,

\begin{align*}
   G_c &= \begin{bmatrix} G_{cu1} & G_{cy1} & G_{cu2} & G_{cy2} \end{bmatrix} ; \\
   \tilde{H}_c &= \begin{bmatrix} H_{c1} \\ H_{c1}^\dagger \\ H_{c2} \\ H_{c2}^\dagger \end{bmatrix} ;
\end{align*}

\begin{align*}
   G_{cu1} &= \begin{bmatrix} G_{cu1a} & G_{cu1b} \end{bmatrix} ; \\
   G_{cu2} &= \begin{bmatrix} G_{cu2a} & G_{cu2b} \end{bmatrix} .
\end{align*}
The conditions (28) can be rewritten as
\[ F_c \Theta + \Theta F_{c}^\dagger + G_{cw1a}G_{cw1a}^\dagger + G_{cw1b}G_{cw1b}^\dagger + G_{cy}G_{cy}^\dagger \]
\[ -G_{cw2a}G_{cw2a}^\dagger - G_{cw2b}G_{cw2b}^\dagger - G_{cy2}G_{cy2}^\dagger = 0 \tag{29} \]
and
\[ G_{cw1a} = -\Theta H_{c1}^\dagger; \quad [ G_{cw1b} G_{cy} ] = -\Theta H_{c1}^\dagger; \]
\[ G_{cw2a} = \Theta H_{c2}^\dagger; \quad [ G_{cw2b} G_{cy} ] = \Theta H_{c2}^\dagger. \tag{30} \]
Substituting from (30) into (29), we obtain the following necessary and sufficient condition for the physical realizability of a controller (27) with dynamics matrices \( F_c, G_{cy}, \) and \( H_c; \)
\[ F_c \Theta + \Theta F_{c}^\dagger - \Theta \left( H_{c2}H_{c2}^\dagger - H_{c1}H_{c1}^\dagger \right) \Theta \]
\[ + G_{cy} G_{cy}^\dagger - G_{cy2} G_{cy2}^\dagger = 0. \tag{31} \]
From this, we can make the following observations. Given a triple \( F_c, G_{cy}, H_c; \) and a Hermitian commutation matrix \( \Theta, \) we can always find matrices \( G_{cw1a}, \) and \( G_{cw2b} \) such that
\[ G_{cw2b}G_{cw1a}^\dagger - G_{cw1b}G_{cw2b}^\dagger = M \]
where
\[ M = F_c \Theta + \Theta F_{c}^\dagger - \Theta \left( H_{c2}H_{c2}^\dagger - H_{c1}H_{c1}^\dagger \right) \Theta \]
\[ + G_{cy} G_{cy}^\dagger - G_{cy2} G_{cy2}^\dagger = 0 \]
has a Hermitian solution \( \Theta \), then the controller dynamics defined by the triple \( F_c, G_{cy}, H_c; \) can be made physically realizable by the addition of a number of controller noises. This is essentially the result of Lemma 5.6 of [1]. However, the addition of these noises will be detrimental to closed loop performance in both the LQG and \( H^\infty \) cases being considered. Also, if the triple \( F_c, G_{cy}, H_c; \) is such that the Riccati equation
\[ F_c \Theta + \Theta F_{c}^\dagger - \Theta \left( H_{c2}H_{c2}^\dagger - H_{c1}H_{c1}^\dagger \right) \Theta \]
\[ + G_{cy} G_{cy}^\dagger - G_{cy2} G_{cy2}^\dagger = 0 \]
has a Hermitian solution \( \Theta \), then the controller dynamics defined by the triple \( F_c, G_{cy}, H_c; \) can be made physically realizable with a number of controller noises and the controller noises \( \hat{W}_b \) are not required. This is essentially the main result of [19].

We now give a game theory interpretation of the physical realizability condition (31). Indeed, it follows from (20) that the Riccati equation (31) will have a Hermitian solution if and only if the following game for the system
\[ \inf_{\mathcal{Y}(\cdot)} \sup_{\tilde{W}(\cdot)} J \]

has a finite value for all \( T > 0 \). Here
\[ J = \int_0^T \left( ||\dot{\tilde{Y}}||^2 + ||\dot{\tilde{W}}||^2 + ||\dot{\tilde{W}}||^2 + ||\tilde{W}_b||^2 \right) dt. \]
This constraint can be used in place of the constraints (21) and (22) in the time domain coherent LQG and \( H^\infty \) control problems.

IV. COHERENT LQG AND \( H^\infty \) CONTROL FOR ANNihilation OPERATOR SYSTEMS

In problems of coherent LQG control and coherent \( H^\infty \) control for annihilation operator systems, the plant (14) and controller (16) are replaced with the following QSDEs:

**Plant:**
\[ da = F_{ad} dt + G_a d\hat{W} + G_w dt; \]
\[ dy = H_{ad} dt + K_{ad} d\hat{W}. \tag{32} \]

**Controller:**
\[ dc_c = F_{c ad} dt + G_c d\hat{W} + G_{cy} d\hat{Y}; \]
\[ dU_c = H_{c ad} dt + K_{c ad} d\hat{W} + K_{cy} d\hat{Y}. \tag{33} \]

Also, the augmented controller (17) is replaced with the following QSDEs:
\[ dc_c = F_{c ad} dt + [ G_c G_y ] \left[ \begin{array}{c} d\hat{W} \\ dy \end{array} \right]; \]
\[ dU_c = \left[ \begin{array}{c} H_{c ad} \\ H_{c ad} \end{array} \right] a_c dt + [ K_{c c} K_{cy} ] \left[ \begin{array}{c} d\hat{W} \\ dy \end{array} \right]; \tag{34} \]
and the cost output (19) is replaced by the equation
\[ dZ = C_{ad} dt + d\hat{U}. \tag{35} \]

The transfer function matrices \( \Gamma_p(s), \Gamma_C(s), \tilde{\Gamma}_C(s), \) and \( \Gamma_z(s) \) are defined as in the previous section for the systems (32), (33), (34), and (35). Then, the frequency domain and \( H^\infty \) control problems are defined as in the previous section except that Theorem 3 is used in place of Theorem 2. That is, the physical realizability constraint becomes a constraint that the transfer function matrix \( \tilde{\Gamma}_C(s) \) is lossless bounded real.

In case of annihilation operator systems, the time domain physical realizability conditions (21), (22) are replaced by the conditions
\[ \int_0^T \left( ||\dot{\tilde{Y}}||^2 + ||\dot{\tilde{W}}||^2 + ||\dot{\tilde{Y}}||^2 + ||\tilde{W}_b||^2 \right) dt \geq 0 \tag{36} \]
for all \( T > 0 \) and all solutions to (20) with zero initial condition, and
\[ \int_0^\infty \left( ||\dot{\tilde{Y}}||^2 + ||\dot{\tilde{W}}||^2 + ||\dot{\tilde{Y}}||^2 + ||\tilde{W}_b||^2 \right) dt = 0 \tag{37} \]
for all solutions to (20) with zero initial condition satisfying
\[ \int_0^\infty \left( ||\dot{\tilde{Y}}||^2 + ||\dot{\tilde{W}}||^2 \right) dt < \infty. \]

In the time domain coherent quantum LQG control problem for the case of annihilation operator systems, the cost in (34) remains the same and physical realizability constraints (21), (22) are replaced by the constraints (35). In the time domain coherent quantum \( H^\infty \) problem for the case of annihilation operator systems, the game problem in (25) is replaced by the game problem
\[ \inf_{\mathcal{U}(\cdot)} \sup_{\tilde{W}(\cdot)} J_\gamma < \infty \tag{38} \]
subject to the constraints defined by \((36), (37)\) where

\[
J_\gamma = \int_0^T \left( \|Z\|^2 - \gamma^2 \left( \|W\|^2 + \|\tilde{W}\|^2 \right) \right) dt.
\]

For the coherent quantum LQG and \(H^\infty\) control problems for the case of annihilation operator systems, when we consider the restriction to a strictly proper controller, the modified plant and controller \((26), (27)\) reduce to the following:

**Modified Plant:**

\[
\dot{a} = (F + G_uK_{cy}H)a + (G_w + G_uK_{cy}K)W + G_uK_{cw}\tilde{W} + G_uU;
\]

\[
Y = Ha + KW;
\]

**Modified Controller:**

\[
\dot{a}_c = F_c a_c + G_{cw} \begin{bmatrix} \tilde{W}_a \\ \tilde{W}_b \end{bmatrix} + G_{cy}Y;
\]

\[
U = H_c a_c,
\]

where \(G_{cw} = [G_{cwa} \quad G_{cwb}]\). Also, the Riccati equation condition for physical realizability \((31)\) reduces in this case to the Riccati equation

\[
F_c \Theta + \Theta F_c^\dagger + \Theta H_c H_c^\dagger \Theta + G_{cy}G_{cy}^\dagger + G_{cwb}G_{cwb}^\dagger = 0
\]

where in this case a solution \(\Theta > 0\) is sought. This is a bounded real Riccati equation. From this, it follows that a controller with dynamic defined by a triple \(F_c, G_{cy}, H_c\) can be made physically realizable by the addition of suitable controller noises if and only if \(F_c\) is Hurwitz and \(\|H_c(sI - F_c)^{-1}\|_\infty \leq 1\). In this case, only the controller noise \(\tilde{W}_a\) is needed and there is never any advantage in using the controller noise \(\tilde{W}_b\). This is essentially the result of Theorem 3.2. of [3].

**Coherent LQG Control for Annihilation Operator Systems:** In order to consider the coherent LQG problem for annihilation operator systems, we first assume that all of the noises acting on the plant \((32)\) and controller \((33)\) are purely canonical quantum noises. That is \(dWd\tilde{W}^\dagger = Idt\) and \(dWd\tilde{W} = Idt\); e.g., see [3]. Also, assuming the plant \((32)\) is physically realizable, it is straightforward to verify using Theorem 3 that the modified plant corresponding to \((40)\) is physically realizable if we ignore the control input \(U\) in the sense that the augmented system

\[
da = (F + G_uK_{cy}H)adt + \begin{bmatrix} G_w + G_uK_{cy} \\ G_uK_{cw} \end{bmatrix} \begin{bmatrix} dW \\ d\tilde{W} \end{bmatrix};
\]

\[
\begin{bmatrix} dY \\ d\tilde{Y} \end{bmatrix} = \begin{bmatrix} H \\ H \end{bmatrix} adt + \begin{bmatrix} K & 0 \\ K & K \end{bmatrix} \begin{bmatrix} dW \\ d\tilde{W} \end{bmatrix}
\]

is physically realizable. It follows from Theorem 3 that there exists a matrix \(\Theta > 0\) such that

\[
F_a \Theta + \Theta F_a^\dagger + G_aG_a^\dagger = 0; \quad G_a = -\Theta H_a^\dagger,
\]

where

\[
F_a = F + G_uK_{cy}H; \quad G_a = [G_w + G_uK_{cy} \quad G_uK_{cw}];
\]

\[
H_a = \begin{bmatrix} H \\ H \end{bmatrix}.
\]

Now to consider the coherent LQG problem for the system \((39)\), we first ignore the physical realizability constraint on the controller \((40)\). In this case, the LQG problem can be solved by using the separation principle; e.g., see [21]. In order to do this, we first consider the Kalman Filter for the system \((39)\). Noting that \(dY = L \begin{bmatrix} d\tilde{Y} \\ d\tilde{Y} \end{bmatrix}\) where \(L = [I \quad 0]\), we can equivalently consider the Kalman filter for the following system derived from the system \((42)\):

\[
da = F_adt + G_a \begin{bmatrix} dW \\ d\tilde{W} \end{bmatrix};
\]

\[
\begin{bmatrix} d\tilde{Y} \\ dY \end{bmatrix} = LH_adt + L \begin{bmatrix} dW \\ d\tilde{W} \end{bmatrix}.
\]

As in Section 4.3.3 of [21], this Kalman filter is constructed by first finding the solution \(Q \geq 0\) to the Riccati equation

\[
F_aQ + QF_a^\dagger + G_aG_a^\dagger - (G_a + QH_a^\dagger)LL^\dagger(G_a + QH_a^\dagger)^\dagger = 0
\]

However, it follows immediately from \((43)\) that the matrix \(\Theta > 0\) satisfies this equation. Furthermore, the corresponding Kalman gain matrix is given by the formula

\[
K_g = (G_a + QH_a^\dagger)LL^\dagger(G_a + QH_a^\dagger)^\dagger
\]

which is equal to zero when \(Q = \Theta\) using \((43)\). Thus, we can conclude that the Kalman gain is zero and the Kalman state estimate is independent of the output \(Y\). This in turn means that the LQG controller obtained using the separation principle will have a transfer function of zero. However, a zero transfer function automatically satisfies the physical realizability constraint. That means that this must be the coherent optimal LQG controller. That is, we have proved that the dynamic part of the coherent optimal LQG controller is zero and thus, the total coherent optimal LQG controller must consist purely of a static direct feedthrough term. This leads to the following theorem, which is one of the main results of this paper.

**Theorem 5:** For any physically realizable annihilation operator plant of the form \((32)\) and cost \((35)\) with purely quantum noise inputs, then the solution to the corresponding coherent LQG problem will be a purely static controller of the form

\[
dU = K_{cw}d\tilde{W} + K_{cy}dY.
\]

The above derivation of this theorem also leads to the following corollary which is also one of the main results of this paper.

**Corollary 1:** For any physically realizable annihilation operator plant of the form \((32)\) with purely quantum noise inputs, then the solution to the corresponding coherent LQG problem will be a purely static controller of the form

\[
dU = K_{cw}d\tilde{W} + K_{cy}dY.
\]
inputs, then the corresponding Kalman filter dynamics will be independent of the output $Y$.

Note that here the “Kalman filter” is defined to be the set of stochastic differential equations constructed via the standard Kalman filter formulas applied to the matrices in the plant model \([12]\), not the “quantum filter” such as considered in \([22]\).

This corollary indicates that for the case under consideration, the output $Y$ contains no information about the system variables $x$. This conclusion will also hold for any measurements of the quadratures of $Y$ since such measurements contain less information than $Y$ itself.

**Coherent $H^\infty$ Control for Annihilation Operator Systems:** To consider this problem, we assume the plant is physically realizable and that the cost output is of the form

$$dZ = L \begin{bmatrix} dY \\ d\tilde{Y} \end{bmatrix}$$  \hspace{1cm} (45)$$

where $\tilde{Y}$ is the additional output introduced in the augmented plant (see \([15]\)) and $L$ is a matrix whose columns are standard unit vectors. That is, the cost output consists of a collection of physical outputs of the plant. Also, note that $L^\dagger L = I$.

We first consider the application of the trivial controller $dU = d\tilde{W}$ to the plant. It is straightforward to verify that the resulting system will also be physically realizable. Then, it follows that the resulting transfer function $\Gamma_Z(s)$ of the augmented system from $\begin{bmatrix} Y \\ \tilde{Y} \end{bmatrix}$ to $Z$ will be lossless bounded real. Hence, the transfer function $\Gamma_Z(s) = L\tilde{\Gamma}(s)$ from $\begin{bmatrix} Y \\ \tilde{Y} \end{bmatrix}$ to $Z$ will satisfy:

$$\Gamma_Z(j\omega)^\dagger\Gamma_Z(j\omega) = \Gamma(j\omega)^\dagger L^\dagger L\Gamma(j\omega) = \Gamma(j\omega)^\dagger\Gamma(j\omega) = I$$

for all $\omega \geq 0$. Hence, $||\Gamma_Z(s)||_\infty = 1$.

Now, we consider the application of any physically realizable controller of the form \([13]\) to the plant \([14]\). It is straightforward to verify using Theorem \([3]\) that the resulting closed loop system is physically realizable. It follows that the resulting transfer function $\tilde{\Gamma}(s)$ of the augmented system from $\begin{bmatrix} Y \\ \tilde{Y} \end{bmatrix}$ to $Z$ will be lossless bounded real. Hence, the transfer function $\tilde{\Gamma}_Z(s) = L\tilde{\Gamma}(s)$ from $\begin{bmatrix} W \\ \tilde{W} \end{bmatrix}$ to $Z$ will satisfy:

$$\tilde{\Gamma}_Z(j\omega)^\dagger\tilde{\Gamma}_Z(j\omega) = \tilde{\Gamma}(j\omega)^\dagger L^\dagger L\tilde{\Gamma}(j\omega) = \tilde{\Gamma}(j\omega)^\dagger\tilde{\Gamma}(j\omega) = I$$

for all $\omega \geq 0$. Hence, $||\tilde{\Gamma}_Z(s)||_\infty = 1$. That is, the trivial controller $dU = d\tilde{W}$ achieves the same closed loop $H^\infty$ norm as any other physically realizable controller. From this, we obtain the following theorem, which is one of the main results of this paper.

**Theorem 6:** For any physically realizable annihilation operator plant of the form \([12]\) and physical cost \([45]\), then the trivial controller $dU = d\tilde{W}$ will always be an optimal solution to the corresponding coherent $H^\infty$ quantum optimal control problem.

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