Symmetries, constants of the motion and reduction of mechanical systems with external forces

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Abstract

This paper is devoted to the study of mechanical systems subjected to external forces in the framework of symplectic geometry. We obtain a Noether’s theorem for Lagrangian systems with external forces, among other results regarding symmetries and conserved quantities. We particularize our results for the so-called Rayleigh dissipation, i.e., external forces that are derived from a dissipation function, and illustrate them with some examples. Moreover, we present a theory for the reduction of Lagrangian systems subjected to external forces which are invariant under the action of a Lie group.

1 Introduction

In this paper, we study the geometry and symmetries of Hamiltonian and Lagrangian systems with external forces, focusing on the so-called systems

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with Rayleigh dissipation. Mechanical systems with external forces are usual in Engineering [4–6], but also can arise in a more sophisticated manner, for instance, after a process of reduction of a nonholonomic system with symmetries [6, 7, 11]. As it is well-known (see Refs. [16, 22]), external forces can be regarded as semibasic 1-forms on the tangent or cotangent bundle. Our approach is based on the symplectic structure obtained from a regular Lagrangian in the Lagrangian formulation as well as the geometry of the tangent bundle. There are other ways to treat with symmetries, for instance a variational approach like in Ref. [2].

The main result when we are in presence of symmetries is the celebrated Noether theorem. See Ref. [32] for the original paper by E. Noether (see also Refs. [18, 31]). In our case, in spite of the existence of a non-conservative external force, we are able to extend Noether theorem and, furthermore, to obtain new conserved quantities. Our approach is just an appropriate modification of the well-known results for conservative mechanical systems (that means with no external forces) [8–10, 12, 14, 19–23, 25, 29, 34–38]. So, we first define point-base symmetries (that is, those provided by vector fields on the configuration manifold $Q$), and then symmetries on the tangent bundle.

There are other approaches that can be found in the previous literature and have some relation with ours. For instance, Cantrijn [4] considers Lagrangian systems that depend explicitly on time, and defines a 2-form on $\mathbb{R} \times TQ$ that depends on the Poincaré-Cartan 2-form of the Lagrangian and the semibasic 1-form representing the external force. Alternatively, van der Schaft [39, 40] considers a framework stemming from system theory, in which an “observation” manifold appears together with the usual state space, and obtains a Noether's theorem for Hamiltonian system in this frame. Other approaches using variational tools can be found in Ref. [2]. However, in our approach no additional structure or objects are introduced besides the proper external force.

The paper is organized as follows. In Sections 2 and 4 we review Hamiltonian and Lagrangian systems with external forces, respectively. In Section 3 we cover the relation between fibre bundle morphisms and semibasic 1-forms. In Section 5, we present some (as far as we know) original results concerning symmetries and constants of the motion for mechanical systems with external forces. In Section 6 we study the symmetries and constants of the motion in the Hamiltonian framework. We relate these symmetries with the ones obtained for Lagrangian systems in the previous section. In Section 7 we
particularize the results of the previous section for the Rayleigh dissipation. Classically [15, 17, 41], only external forces that are linear on the velocities are regarded as examples of Rayleigh dissipation. However, following Lurie [24] and Minguzzi [30], we consider a wider family of external forces as Rayleigh dissipation, namely forces that are derived from a dissipation function (which is not necessarily quadratic on the velocities). Finally, in Section 8 we present a scheme for reduction in Lagrangian systems subjected to external forces which are invariant under the action of a Lie group.

2 Hamiltonian systems subject to external forces

An external force is geometrically interpreted as a semibasic 1-form on $T^*Q$. Let us recall [1, 16, 22] that a 1-form $\gamma$ on $T^*Q$ is called semibasic if

$$\gamma(Z) = 0$$

for all vertical vector fields $Z$.

Remark 1. This definition can be extended to any fibre bundle $\pi : E \to M$. Indeed, a 1-form $\gamma$ on $E$ is called semibasic if

$$\gamma(Z) = 0$$

for all vertical vector fields $Z$ on $E$. If $(x^i, y^a)$ are fibred (bundle) coordinates, then the vertical vector fields are locally generated by $\{\partial/\partial y^a\}$. So $\gamma$ is a semibasic 1-form if it is locally written as

$$\gamma = \gamma_i(x, y)dx^i.$$

A Hamiltonian system with external forces is given by a Hamiltonian function $H : T^*Q \to \mathbb{R}$ and a semibasic 1-form $\gamma$ on $T^*Q$. Let $\omega_Q = -d\alpha_Q$ be the canonical symplectic form of $T^*Q$. Locally these objects can be written as

$$\alpha_Q = p_i dq^i,$$
$$\omega_Q = dq^i \wedge dp_i,$$
$$\gamma = \gamma_i(q, p) dq^i,$$
$$H = H(q, p),$$
where \((q^i, p_i)\) are bundle coordinates in \(T^*Q\).

The dynamics of the system is given by the vector field \(X_{H,\gamma}\), defined by

\[
\iota_{X_{H,\gamma}}\omega_Q = dH + \gamma.
\]

If \(X_H\) is the Hamiltonian vector field for \(H\), that is,

\[
\iota_{X_H}\omega_Q = dH, \tag{1}
\]

and \(Z_\gamma\) is the vector field defined by

\[
\iota_{Z_\gamma}\omega_Q = \gamma,
\]

then we have

\[X_{H,\gamma} = X_H + Z_\gamma.\]

Locally, the above equations can be written as

\[
X_H = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i},
\]

\[\gamma = \gamma_i dq^i,
\]

\[Z_\gamma = -\gamma_i \frac{\partial}{\partial p_i},
\]

\[X_{H,\gamma} = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \left(\frac{\partial H}{\partial q^i} + \gamma_i\right) \frac{\partial}{\partial p_i}. \tag{2}
\]

Then, a curve \((q^i(t), p_i(t))\) in \(T^*Q\) is an integral curve of \(X_{H,\gamma}\) if and only if it satisfies the forced motion equations

\[
\frac{d}{dt} q^i = \frac{\partial H}{\partial p_i},
\]

\[
\frac{d}{dt} p_i = -\left(\frac{\partial H}{\partial q^i} + \gamma_i\right).
\]

### 3 Semibasic forms and fibred morphisms

Given a semibasic 1-form \(\gamma\) on \(TQ\), one can define the following morphism of fibre bundles \([16, 22]\):

\[
D_\gamma : TQ \to T^*Q,
\]

\[
\langle D_\gamma(v_q), w_q \rangle = \gamma(v_q)(u_{w_q}),
\]

4
for every \(v_q, w_q \in T_q Q\), \(u_{v_q} \in T_{v_q}(TQ)\), with \(T\tau_Q(u_{v_q}) = w_q\). In local coordinates, if
\[
\gamma = \gamma_i(q, \dot{q})dq^i,
\]
then
\[
D\gamma(q^i, \dot{q}^i) = \left(q^i, \gamma_i(q^i, \dot{q}^i)\right).
\]
Here \((q^i, \dot{q}^i)\) are bundle coordinates in \(TQ\).

Conversely, given a morphism of fibre bundles
\[
D : TQ \longrightarrow T^*Q
\]
we define a semibasic 1-form \(\gamma\) on \(TQ\) by
\[
\gamma_D(v_q)(u_{v_q}) = \langle D(v_q), T\tau_Q(u_{v_q}) \rangle,
\]
where \(v_q \in T_q Q\), \(u_{v_q} \in T_{v_q}(TQ)\).

If locally \(D\) is given by
\[
D(q^i, \dot{q}^i) = (q^i, D_i(q, \dot{q}))
\]
then
\[
\gamma_D = D_i(q, \dot{q})dq^i.
\]
So there exists a one-to-one correspondence between semibasic 1-forms and fibred morphisms from \(TQ\) to \(T^*Q\).

4 Lagrangian systems with external forces

We shall now consider a Lagrangian system with Lagrangian function \(L\) subjected to external forces. An external force is given by a semibasic 1-form \(\beta\) on \(TQ\). In bundle coordinates, we have
\[
\beta = \beta_i(q, \dot{q})dq^i.
\]
If \(L : TQ \rightarrow \mathbb{R}\), then \(\omega_L = -d\alpha_L\) is the Poincaré-Cartan 2-form, where \(\alpha_L = S^*(dL)\). Here, \(S\) is the vertical endomorphism of \(TQ\), which in local coordinates is given by
\[
S = dq^i \otimes \frac{\partial}{\partial \dot{q}^i},
\]
hence,
\[ \omega_L = dq^i \wedge d \left( \frac{\partial L}{\partial \dot{q}^i} \right). \]

Then, the dynamics is given by the vector field \( \xi_{L,\beta} \) via the equation
\[ t_{\xi_{L,\beta}} \omega_L = dE_L + \beta, \tag{3} \]
where \( E_L = \Delta(L) - L \) is the energy of the system and \( \Delta \) is the Liouville vector field:
\[ \Delta = q^i \frac{\partial}{\partial \dot{q}^i}. \]

Here, we are assuming that \( L \) is regular, that is, the Hessian matrix
\[ (W_{ij}) = \left( \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \right). \tag{4} \]
is invertible. It can be easily proven that \( \omega_L \) is symplectic if and only if \( L \) is regular [22]. Let \( \xi_\beta \) be the vector field given by
\[ t_{\xi_\beta} \omega_L = \beta, \]
and \( \xi_L \) be the vector field given by
\[ t_{\xi_L} \omega_L = dE_L, \tag{5} \]
then
\[ \xi_{L,\beta} = \xi_L + \xi_\beta. \]
We have
\[ \xi_\beta = -\beta_i W^{ij} \frac{\partial}{\partial \dot{q}^j}, \]
where \((W^{ij})\) is the inverse matrix of \((W_{ij})\). Then \( \xi_{L,\beta} \) is a second order differential equation (SODE), meaning that,
\[ S(\xi_{L,\beta}) = S(\xi_L) = \Delta. \tag{6} \]

We know that
\[ \xi_L = q^i \frac{\partial}{\partial q^i} + \dot{q}^i \frac{\partial}{\partial \dot{q}^i}, \]
where
\[ \dot{q}^i \frac{\partial p_j}{\partial \dot{q}^i} + q^i \frac{\partial p_j}{\partial q^i} - \frac{\partial L}{\partial \dot{q}^j} = 0. \tag{7} \]
Then
\[ \xi_{L,\beta} = \dot{q}^i \frac{\partial}{\partial q^i} + \left( \xi^i - \beta_j W^{ji} \right) \frac{\partial}{\partial \dot{q}^i}. \]

Hence, a solution of \( \xi_{L,\beta}, (q^i(t)) \), satisfies
\[
\frac{dq^i}{dt} = \dot{q}^i, \\
\frac{d\dot{q}^i}{dt} = \xi^i - \beta_j W^{ji}.
\]

Therefore, from Eq. (7), we get
\[
\ddot{q}^i \frac{\partial p_j}{\partial q^i} + \dot{q}^i \frac{\partial p_j}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^j} + \beta_k W^{ki} \frac{\partial p_j}{\partial \dot{q}^i} = 0.
\]

Since \( p_j = \partial L/\partial \dot{q}^j \), the term \( \partial p_j/\partial \dot{q}^i \) is equal to \( W_{ji} \), and thus we finally obtain
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i} = -\beta_i.
\]

If we construct the Legendre transform \([22]\)
\[
\begin{align*}
TQ & \xrightarrow{\text{Leg}} T^*Q \\
\tau_q & \downarrow \hspace{1cm} \pi_Q
\end{align*}
\]

(and assume \( L \) be hyperregular, that is, Leg is a diffeomorphism), then we can define the external force \( \gamma \) on \( T^*Q \) by
\[
\text{Leg}^* \gamma = \beta.
\]

Thus \( \xi_{L,\beta} \) and \( X_{H,\gamma} \) are Leg-related, that is, Leg takes \( \xi_{L,\beta} \) onto \( X_{H,\gamma} \), where \( H \) is defined by
\[
H \circ \text{Leg} = E_L.
\]

**Definition 1.** In what follows, we will refer to the pair \( (L, \beta) \) for a forced Lagrangian system given by a Lagrangian \( L \) and a semibasic 1-form \( \beta \). The corresponding vector field \( \xi_{L,\beta} \), given by Eq. (3), will be called **forced Euler-Lagrange vector field**.
Remark 2. Take

\[ \alpha_L = S^*(dL) = p_i dq^i, \]

where \( p_i = \partial L / \partial \dot{q}^i \), then \( \alpha_L \) is a semibasic 1-form on \( TQ \); and the corresponding fibred map is just the Legendre transform

\[ \text{Leg} : TQ \to T^*Q. \]

5 Symmetries and constants of the motion in the Lagrangian description

Let \( f : TQ \to \mathbb{R} \) be an arbitrary function and \( \tau_Q : TQ \to Q \) the projection. Then the vertical lift \([42, 43]\) of \( f \) is a function \( f^v : TQ \to Q \) given by

\[ f^v = f \circ \tau_Q. \]

Any 1-form \( \omega \) in \( Q \) can be naturally regarded as a function on \( TQ \), which we shall denote by \( \iota \omega \). If \( X \) is a vector field on \( Q \), its vertical lift is the unique vector field \( X^v \) on \( TQ \) such that

\[ X^v(\iota \omega) = (\alpha(X))^v \]

for every 1-form \( \alpha \) on \( Q \). The complete lift of a function \( f \) on \( Q \) is the function \( f^c \) on \( TQ \) given by

\[ f^c = \iota(df). \]

The complete lift of a vector field \( X \) on \( Q \) is the vector field \( X^c \) on \( TQ \) such that

\[ X^c(f^c) = (X(f))^c \]

for every function \( f \) on \( Q \). If \( X \) generates locally a 1-parameter group of transformations on \( Q \), then \( X^c \) generates the induced transformations on \( TQ \) \([22]\). Locally, if \( X \) is given by

\[ X = X^i \frac{\partial}{\partial q^i}, \]

then its vertical lift is

\[ X^v = X^i \frac{\partial}{\partial \dot{q}^i}. \]
and its complete lift is
\[
X^c = X^i \frac{\partial}{\partial q^i} + \dot{q}^i \frac{\partial}{\partial \dot{q}^i} \frac{\partial}{\partial q^i}.
\]

Let \((L, \beta)\) be a Lagrangian system with Lagrangian function \(L\) and external force \(\beta\); denote by \(\xi_{L,\beta}\) the corresponding forced Euler-Lagrange vector field.

**Definition 2.** A function \(f\) on \(TQ\) is called a constant of the motion (or a conserved quantity) if \(\xi_{L,\beta}(f) = 0\).

Suppose that, for a certain coordinate \(q^i\), \(\partial L/\partial q^i = \beta_i\). Then
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right) = 0,
\]
and \(p_i = \partial L/\partial \dot{q}^i\) is a constant of the motion. This motivates the following theorem.

**Theorem 1** (Noether’s theorem for dissipative systems). Let \(X\) be a vector field on \(Q\). Then \(X^c(L) = \beta(X^c)\) if and only if \(X^v(L)\) is a constant of the motion.

**Proof.** By Eq. (3), we can write
\[
(dE_L + \beta)(X^c) = (i_{\xi_{L,\beta}} \omega_L)(X^c) = -d\alpha_L(\xi_{L,\beta}, X^c)
\]
\[
= -\xi_{L,\beta}(\alpha_L(X^c)) + X^c(\alpha_L(\xi_{L,\beta})) + \alpha_L([\xi_{L,\beta}, X^c]).
\]

Now, since \(\xi_{L,\beta}\) is a SODE, we have
\[
\alpha_L(\xi_{L,\beta}) = i_{\xi_{L,\beta}} (S^*dL) = (S\xi_{L,\beta})L = \Delta L.
\]
It is easy to see that \(S X^c = X^v\). Moreover, \([\xi_{L,\beta}, X^c]\) is a vertical vector field, and thus \(S[\xi_{L,\beta}, X^c] = 0\). Then
\[
(dE_L + \beta)(X^c) = -\xi_{L,\beta}(X^v L) + X^c(\Delta L).
\]

On the other hand, we can write
\[
dE_L(X^c) = X^c(E_L) = X^c(\Delta L) - X^c(L).
\]
Combining these last two equations one deduces
\[
\xi_{L,\beta}(X^v L) = X^c(L) - \beta(X^c).
\]
In particular, the right-hand side vanishes if and only if the left-hand side does.
Definition 3. Consider the forced Lagrangian system \((L, \beta)\). Then

i) A symmetry of the forced Lagrangian system is a vector field \(X\) on \(Q\) such that \(X^c(L) = \beta(X^c)\).

ii) A Lie symmetry is a vector field \(X\) on \(Q\) such that \([X^c, \xi_{L,\beta}] = 0\).

iii) A Noether symmetry is a vector field \(X\) on \(Q\) such that \(X^c(E_L) + \beta(X^c) = 0\) and \(\mathcal{L}_{X^c\alpha_L}\) is exact.

Proposition 2. If \(X\) is a vector field on \(Q\) such that

\[
d(\mathcal{L}_{X^c\alpha_L}) = 0,
\]

then \(X\) is a Lie symmetry if and only if

\[
\mathcal{L}_{X^c\beta} = -d(X^c(E_L)).
\]

Proof. Indeed,

\[
\iota_{[X^c, \xi_{L,\beta}]}\omega_L = \mathcal{L}_{X^c}(\iota_{\xi_{L,\beta}}\omega_L) - \iota_{\xi_{L,\beta}}(\mathcal{L}_{X^c}\omega_L)
= \mathcal{L}_{X^c}(dE_L + \beta) + \iota_{\xi_{L,\beta}}d(\mathcal{L}_{X^c}\alpha_L)
= d(X^c(E_L)) + \mathcal{L}_{X^c}\beta.
\]

Since \(\omega_L\) is non-degenerate, then \([X^c, \xi_{L,\beta}]\) vanishes if and only if \(\iota_{[X^c, \xi_{L,\beta}]}\omega_L\) does. \(\square\)

Proposition 3. A Noether symmetry is a Lie symmetry if and only if

\[
\iota_{X^c}d\beta = 0.
\]

Proof. Since \(\mathcal{L}_{X^c\alpha_L}\) is exact, it can be written as \(\mathcal{L}_{X^c\alpha_L} = df\) for some function \(f : TQ \to \mathbb{R}\). Obviously, \(d(\mathcal{L}_{X^c\alpha_L}) = d(df) = 0\). In addition,

\[
\mathcal{L}_{X^c\beta} = \iota_{X^c}(d\beta) + d(\iota_{X^c}\beta) = \iota_{X^c}(d\beta) + d(\beta(X^c))
= \iota_{X^c}(d\beta) - d(X^c(E_L)).
\]

By Proposition 2, the result holds. \(\square\)

Proposition 4. Let \(X\) be a vector field on \(TQ\) such that

\[
\mathcal{L}_{X^c\alpha_L} = df,
\]

then \(X\) is a Noether symmetry if and only if \(f - X^u(L)\) is a conserved quantity.
Proof. Indeed,

\[ df = \mathcal{L}_{X^c} \alpha_L = \iota_{X^c} (d\alpha_L) + d(\iota_{X^c} \alpha_L) = \iota_{X^c} (d\alpha_L) + d(\iota_{X^c} S^* dL) \]
\[ = \iota_{X^c} (d\alpha_L) + d(\iota_S X^c dL) = \iota_{X^c} (d\alpha_L) + d(X^v L), \]

so

\[ \iota_{\xi_{L,\beta}} \iota_{X^c} (d\alpha_L) = \iota_{\xi_{L,\beta}} (d(f - X^v L)) = \xi_{L,\beta} (f - X^v(L)), \]

but

\[ \iota_{\xi_{L,\beta}} \iota_{X^c} d\alpha_L = \iota_{X^c} \iota_{\xi_{L,\beta}} \omega_L = \iota_{X^c} (dE_L + \beta) = X^c(E_L) + \beta(X^c), \]

and the result holds. \(\square\)

Observe that this last proposition is a generalisation of Theorem 1. In other words, every symmetry of the forced Lagrangian system is a Noether symmetry. In fact, if \( f \) is a constant function, clearly \( X^v L \) is a conserved quantity. Moreover,

\[ \mathcal{L}_{X^c} \alpha_L = 0, \]

so

\[ 0 = (\mathcal{L}_{X^c} \alpha_L)(\xi_{L,\beta}) = X^c(\alpha_L(\xi_{L,\beta})) - \alpha_L([X^c, \xi_{L,\beta}]) \]
\[ = X^c(\Delta L) - S[X^c, \xi_{L,\beta}]L = X^c(\Delta L), \]

and thus,

\[ 0 = X^c(E_L) + \beta(X^c) = X^c(\Delta L) - X^c(L) + \beta(X^c) = -X^c(L) + \beta(X^c). \]

Remark 3. A Noether symmetry is a symmetry of the forced Lagrangian system if and only if \( \mathcal{L}_{X^c} \alpha_L = 0 \).

We have just discussed infinitesimal symmetries on \( Q \), the so-called point-like symmetries [19]. We shall now cover symmetries which are not necessarily point-like, that is, vector fields on \( TQ \).

Definition 4. A dynamical symmetry of \( \xi_{L,\beta} \) is a vector field \( \tilde{X} \) on \( TQ \) such that \( [\tilde{X}, \xi_{L,\beta}] = 0 \). A Cartan symmetry is a vector field \( \tilde{X} \) on \( TQ \) such that \( \tilde{X}(E_L) + \beta(\tilde{X}) = 0 \) and \( \mathcal{L}_{\tilde{X}} \alpha_L = df \).

Remark 4. Let \( X \) be a vector field on \( Q \). Then

i) \( X \) is a Lie symmetry if and only if \( X^c \) is a dynamical symmetry.

ii) \( X \) is a Noether symmetry if and only if \( X^c \) is a Cartan symmetry.
Proposition 5. If $\tilde{X}$ is a vector field on $TQ$ such that
\[ d(\mathcal{L}_{\tilde{X}}\alpha_L) = 0, \]
then $\tilde{X}$ is a dynamical symmetry if and only if
\[ d(\tilde{X}(E_L)) = -\mathcal{L}_{\tilde{X}}\beta. \]

Proposition 6. A Cartan symmetry is a dynamical symmetry if and only if
\[ \iota_{\tilde{X}}d\beta = 0. \]

Proposition 7. Let $\tilde{X}$ be a vector field on $TQ$ such that
\[ \mathcal{L}_{\tilde{X}}\alpha_L = df. \]
Then $\tilde{X}$ is a Cartan symmetry if and only if $f - (S\tilde{X})L$ is a constant of the motion.

The proofs are completely analogous to those for point-like symmetries. Notice that Theorem 1 cannot be generalised for symmetries on $TQ$, since $[\xi_{L_\beta}, \tilde{X}]$ is not a vertical vector field for a general $\tilde{X}$ on $TQ$.

6 Symmetries and constants of the motion in the Hamiltonian description

Let $\alpha$ and $\tilde{X}$ be a 1-form and a vector field on $T^*Q$, respectively. We say that $\alpha$ is a first integral of $\tilde{X}$ if $\alpha(\tilde{X}) = 0$. Similarly, a function $F$ on $T^*Q$ is called a first integral of $\tilde{X}$ if $dF(\tilde{X}) = \tilde{X}(F) = 0$.

Let $(H, \gamma)$ be a Hamiltonian system with Hamiltonian function $H$ and external force $\gamma$. Let $X_{H,\gamma}$ be the corresponding Hamiltonian vector field. A first integral of $X_{H,\gamma}$ is called a constant of the motion or a conserved quantity.

Let $F$ and $G$ be two functions on $T^*Q$. Let $X_F$ and $X_G$ be their corresponding Hamiltonian vector fields, namely $\iota_{X_F}\omega_Q = dF$ and $\iota_{X_G}\omega_Q = dG$. The Poisson bracket of $F$ and $G$ is given by
\[ \{F, G\} = \omega_Q(X_F, X_G). \]
Let $\alpha$ and $\beta$ be 1-forms on $T^*Q$, with $X_\alpha$ and $X_\beta$ their corresponding Hamiltonian vector fields. Then their Poisson bracket is defined as

$$\{\alpha, \beta\} = -\iota_{[X_\alpha,X_\beta]}\omega_Q.$$  

Clearly,

$$X_{H,\gamma}(F) = \iota_{X_{H,\gamma}}dF = \iota_{X_{H,\gamma}}(\iota_{X_F}\omega_Q) = -\iota_{X_F}(\iota_{X_{H,\gamma}}\omega_Q) = -\omega_Q(X_H,X_F) - \gamma(X_F) = \{F,H\} - \gamma(X_F),$$

and hence $F$ is a constant of the motion if and only if

$$\{F,H\} = \gamma(X_F).$$

**Proposition 8.** If $\dot{X}$ is a vector field on $T^*Q$ such that $\mathcal{L}_{\dot{X}}\alpha_Q$ is closed, then $\dot{X}$ commutes with $X_{H,\gamma}$ if and only if

$$d(\dot{X}(H)) = -\mathcal{L}_{\dot{X}}\gamma.$$  

**Proposition 9.** Let $\dot{X}$ be a vector field on $T^*Q$ such that

$$\mathcal{L}_{\dot{X}}\alpha_Q = df.$$  

Then $\dot{X}(H) + \gamma(\dot{X}) = 0$ if and only if $f - \alpha_Q(\dot{X})$ is a constant of the motion. Additionally, $\dot{X}$ commutes with $X_{H,\gamma}$ if and only if

$$\iota_{\dot{X}}d\gamma = 0.$$

Now suppose that $(L, \beta)$ is a Lagrangian system such that $H \circ \text{Leg} = E_L$ and $\text{Leg}^*\gamma = \beta$. Let $\tilde{X}$ be a vector field on $TQ$ and $\dot{X}$ the Leg-related vector field on $T^*Q$. Then:

i) $\tilde{X}$ commutes with $X_{H,\gamma}$ if and only if $\tilde{X}$ is a dynamical symmetry of $(L, \beta)$.

ii) $\mathcal{L}_{\tilde{X}}\alpha_Q = df$ if and only if $\mathcal{L}_{\dot{X}}\alpha_L = dg$, where $g = f \circ \text{Leg}$.

iii) Suppose that $\mathcal{L}_{\tilde{X}}\alpha_Q = df$. Then the following assertions are equivalent.

a) $\dot{X}(H) + \gamma(\dot{X}) = 0$.

b) $f - \alpha_Q(\dot{X})$ is a conserved quantity.

c) $\dot{X}(E_L) + \beta(\dot{X}) = 0$.

d) $f \circ \text{Leg} - \alpha_L(\dot{X})$ is a conserved quantity.
7 Rayleigh dissipation

7.1 Rayleigh dissipation function and 1-form

Rayleigh [41] considers the hypothesis that there is a non-conservative force linear on the velocities. This external force can be described as a semibasic 1-form on $TQ$ as follows:

$$ R = R_{ij}(q)\dot{q}^i dq^j, \quad (9) $$

where $R_{ij}$ is symmetric. Of course, $R$ can be described as a bilinear form on $TQ$:

$$ R : TQ \times TQ \to \mathbb{R}, $$

$$ R(q^1, \dot{q}_1^1, \dot{q}_2^2) = R_{ij}\dot{q}_1^i \dot{q}_2^j, $$

or, in other words, a symmetric $(0, 2)$-tensor $R$ on $Q$. Since $R$ is a $(0, 2)$-tensor on $Q$, it defines a linear mapping

$$ \tilde{R} : TQ \to T^*Q $$

by

$$ \tilde{R}(v_q) = \iota_{v_q} R, $$

that is,

$$ \tilde{R}(v_q)(w_q) = R(v_q, w_q). $$

Therefore

$$ \tilde{R}(q^i, \dot{q}^i) = (q^i, R_{ij}(q)\dot{q}^i \dot{q}^j), $$

so $\tilde{R}$ defines a semibasic 1-form $\tilde{R}$ on $TQ$ given by

$$ \tilde{R} = R_{ij}(q)\dot{q}^i dq^j. \quad (10) $$

In the literature [15, 17] the Rayleigh dissipation function is defined as

$$ \mathcal{R}(q, \dot{q}) = \frac{1}{2}R_{ij}(q)\dot{q}^i \dot{q}^j, \quad (11) $$

so we can write

$$ \tilde{R} = \frac{\partial \mathcal{R}}{\partial \dot{q}^i} dq^i = S^*(d\mathcal{R}). $$

Notice that external forces of the form $S^*(d\mathcal{F})$, for some function $\mathcal{F}$, are quite more general than the form (9) originally proposed by Rayleigh [41]. That is, we do not need to require $\mathcal{R}$ to be of the form (11). In fact, more
general dissipation functions are studied in Refs. [24, 30]. This function $R$ can be physically interpreted as a potential that depends on the velocities from which the external force is derived.

Remark 5. $R$ and $\tilde{R} = R + f$ define the same 1-form $\tilde{R}$, for $f : Q \to \mathbb{R}$ arbitrary.

We shall now consider a Lagrangian system with hyperregular Lagrangian function $L$, and which is subject to an external force linear on the velocities. Suppose that the force can be described through the Rayleigh dissipation function $R$. Then the equations of motion of the system are the integral curves of the vector field $\xi_{L,\tilde{R}}$, given by

$$\iota_{\xi_{L,\tilde{R}}} \omega_L = dE_L + \tilde{R}.$$  

Let $\xi_{\tilde{R}}$ be the vector field given by

$$\iota_{\xi_{\tilde{R}}} \omega_L = \tilde{R},$$

then

$$\xi_{L,\tilde{R}} = \xi_L + \xi_{\tilde{R}},$$

where $\xi_L$ is the vector field given by Eq. (5). We have

$$\xi_{\tilde{R}} = -R_{ik} q^k W^{ij} \frac{\partial}{\partial q^j},$$

where $W^{ij}$ is the inverse of the Hessian matrix of the Lagrangian (4). Then $\xi_{L,\tilde{R}}$ is a SODE in the sense of Eq. (6), and the equations of motion of the system are

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i} = -R_{ij}(q) \dot{q}^j = -\frac{\partial R}{\partial \dot{q}^i}.$$  

It is easy to see that

$$\xi_{L,\beta}(E_L) + \Delta(R) = 0,$$

where $\Delta$ is the Liouville vector field. In particular, if $R$ is of the form (11), then $\Delta(R) = 2R$.

We can also consider the Hamiltonian formalism for the Rayleigh dissipation. Indeed, since we have assumed $L$ to be hyperregular, we can always define the external force $\tilde{R}$ on $T^*Q$ by

$$\tilde{R} = \text{Leg}^* \tilde{R},$$

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and consider the Hamiltonian function given by Eq. (8). Locally $\hat{R}$ can be written as

$$\hat{R} = R_{ij}(q)p_idq^j$$

Then the equations of motion of the system are the integral curves of the vector field $X_{H,\hat{R}}$, given by

$$\iota_{X_{H,\hat{R}}}\omega_Q = dH + \hat{R}.$$ 

If $Z_{\hat{R}}$ is the vector field defined by

$$\iota_{Z_{\hat{R}}}\omega_Q = \hat{R},$$

then we have

$$X_{H,\hat{R}} = X_H + Z_{\hat{R}},$$

where $X_H$ is the Hamiltonian vector field given by Eq. (1). In canonical coordinates, $X_H$ and $\hat{R}$ are given by Eqs. (2) and (10), respectively, and we have

$$Z_{\hat{R}} = -R_{ij}(q)p_i\frac{\partial}{\partial p_j},$$

$$X_{H,\hat{R}} = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \left( \frac{\partial H}{\partial q^i} + R_{ij}(q)p_j \right) \frac{\partial}{\partial p_i},$$

where we have made use of the fact that $R_{ij}$ is symmetric. Thus, the equations of motion are

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i},$$

$$\frac{dp_i}{dt} = -\left( \frac{\partial H}{\partial q^i} + R_{ij}(q)p_j \right).$$

As we have shown in Section 3, given the semibasic 1-form $\bar{R}$, we can define the following morphism of fibred bundles:

$$D_{\bar{R}} : TQ \xrightarrow{\tau_Q} Q \xleftarrow{\pi_Q} T^*Q,$$

$$\langle D_{\bar{R}}(v_q), w_q \rangle = \bar{R}(v_q)(u_{w_q}),$$

for every $v_q, w_q \in T_qQ, u_{w_q} \in T_{w_q}(TQ)$, with $T\tau_Q(u_{w_q}) = w_q$. In local coordinates, we have

$$D_{\bar{R}}(q^i, \dot{q}^i) = (q^i, R_{ij}(q)\dot{q}^j).$$
\section{7.2 Constants of the motion for Rayleigh dissipation}

We shall now consider the case in which the external force is derived from a dissipation function $\mathcal{R}$ (not necessarily quadratic in the velocities).

**Lemma 10.** Consider a semibasic 1-form $\bar{\mathcal{R}}$ on $TQ$ given by

$$\bar{\mathcal{R}} = S^*(d\mathcal{R})$$

for some function $\mathcal{R} : TQ \rightarrow \mathbb{R}$, where $S^*$ is the adjoint of the vertical endomorphism. Then, for each vector field $X$ on $Q$

$$\bar{\mathcal{R}}(X^c) = X^v(\mathcal{R}),$$

$$\mathcal{L}_X \bar{\mathcal{R}} = S^*(d(X^c(\mathcal{R}))). \tag{12}$$

Similarly, for each vector field $\tilde{X}$ on $TQ$,

$$\bar{\mathcal{R}}(\tilde{X}) = (S\tilde{X})(\mathcal{R}).$$

**Proof.** Indeed,

$$\bar{\mathcal{R}}(\tilde{X}) = \iota_X \bar{\mathcal{R}} = \iota_X (S^* d\mathcal{R}) = \iota_{S \tilde{X}} d\mathcal{R} = (S\tilde{X})(\mathcal{R}).$$

In particular, $SX^c = X^v$. Eq. (12) can be shown by direct computation in bundle coordinates. \hfill $\square$

**Proposition 11.** Let $X$ be a vector field on $Q$. Then $X^c(L) = X^v(\mathcal{R})$ if and only if $X^v(L)$ is a constant of the motion.

**Example 1** (Fluid resistance). Consider a body of mass $m$ moving through a fluid that fully encloses it. For the sake of simplicity, suppose that the motion takes place along one dimension. Then the drag force \cite{3,13} is given by

$$\bar{\mathcal{R}} = \frac{1}{2} \rho C A q^2 dq,$$

where $C$ is a dimensionless constant depending on the body shape, $\rho$ is the mass density of the fluid and $A$ is the area of the projection of the object on a plane perpendicular to the direction of motion. For the sake of simplicity, suppose that the density is uniform, and then $k = CA\rho/2$ is constant. The dissipation function is thus

$$\mathcal{R} = \frac{k}{3} q^3.$$
If the body is not subject to forces besides the drag, its Lagrangian is 
\[ L = m \ddot{q}^2 / 2. \]  
Consider the vector field \( X = e^{k/mq} \partial / \partial q \). We can verify that \( X^c(L) = X^v(\mathcal{R}) \), so \( X^v(L) = me^{k/mq} \dot{q} \) is a constant of the motion. In particular, when \( k \to 0 \) we recover the conservation of momentum.

**Proposition 12.** If \( \mathcal{L}_{X^c} \alpha_L \) is closed, then \( X \) is a Lie symmetry of \( (L, \bar{R}) \) if and only if
\[ d(X^c(E_L)) = -S^*(d(X^c\mathcal{R})). \]

**Proposition 13.** If \( \mathcal{L}_{X^c} \alpha_L = df \) for some function \( f : TQ \to \mathbb{R} \), then the following statements are equivalent:

i) \( X \) is a Noether symmetry.

ii) \( X^c(E_L) + X^v(\mathcal{R}) = 0. \)

iii) \( f - X^v(L) \) is a constant of the motion.

Moreover, a Noether symmetry is a Lie symmetry if and only if \( \iota_X d \bar{R} = 0. \)

Let \( \tilde{X} \) be a vector field on \( TQ \). If \( \mathcal{L}_{\tilde{X}} \alpha_L \) is closed, then \( \tilde{X} \) is a dynamical symmetry if and only if
\[ d(\tilde{X}(E_L) + (S\tilde{X})(\mathcal{R})) = -\iota_{\tilde{X}} d \bar{R}. \]

**Proposition 14.** If \( \mathcal{L}_{\tilde{X}} \alpha_L = df \), then the following statements are equivalent:

i) \( \tilde{X} \) is a Cartan symmetry.

ii) \( \tilde{X}(E_L) + (S\tilde{X})(\mathcal{R}) = 0. \)

iii) \( f - (S\tilde{X})(L) \) is a conserved quantity.

We shall now cover some examples proposed in Ref. [30] and obtain their constants of motion.

**Example 2** (A rotating disk). Let us consider a disk of mass \( m \) and radius \( r \) placed on a horizontal surface. Let \( \varphi \) be the angle of rotation of the disk with respect to a reference axis. The Lagrangian of the disk is \( L = T = mr^2 \dot{\varphi}^2 / 4 \) and its Rayleigh dissipation function is \( \mathcal{R} = \mumgr \dot{\varphi}^2 / 2. \) The Poincaré-Cartan 1-form is \( \alpha_L = mr^2 \dot{\varphi} / 2 \, d\varphi \). The external force is \( \bar{R} = \mumgr / 2 \, d\varphi. \)
Consider the vector field \( \tilde{X} = r \dot{\varphi} \partial/\partial \varphi + \mu g \partial/\partial \dot{\varphi} \). Clearly, \( \tilde{X}(E_L) = \tilde{X}(L) = (S \tilde{X})(\mathcal{R}) \). We have that

\[
\mathcal{L}_{\tilde{X}} \alpha_L = \frac{\mu mg r^2}{2} d\dot{\varphi} + \frac{mr^3}{2} \ddot{\varphi} d\dot{\varphi} = df,
\]

where

\[
f = \frac{\mu mg r^2}{2} \varphi + \frac{mr^3}{4} \dot{\varphi}^2
\]

modulo a constant, and \( (S \tilde{X})(L) = mr^3 \dot{\varphi}^2 / 2 \), so

\[
f - (S \tilde{X})(L) = \frac{\mu mg r^2}{2} \varphi - \frac{mr^3}{4} \dot{\varphi}^2
\]

is a constant of the motion. Since \( \mathcal{R} \) is closed, \( \iota_{\tilde{X}} d \mathcal{R} = 0 \) is trivially satisfied, so \( \tilde{X} \) is a dynamical symmetry as well as a Cartan symmetry.

However, since \( \mathcal{R} \) is closed, it is not strictly an external force. In fact, the Lagrangian

\[
\tilde{L} = L + \frac{\mu mg r^2}{2} \varphi
\]

leads to the same equations of motion as \( L \) with the external force \( \mathcal{R} \).

**Example 3** (The rotating stone polisher). Consider a system formed by two concentric rings of the same mass \( m \) and radius \( r \), which are placed over a rough surface, and rotate in opposite directions. Let \( (x, y) \) be the position of the centre and \( \theta \) the orientation of the machine. Let \( \omega \) be the angular velocity of the rings. The Rayleigh dissipation function is given by

\[
\mathcal{R} = 2 \mu m g r \omega + \frac{\mu mg}{2r \omega} (\dot{x}^2 + \dot{y}^2),
\]

and the Lagrangian is \( L = T = m(\dot{x}^2 + \dot{y}^2 + r^2 \dot{\theta}^2 + r^2 \omega^2) \). The Poincaré-Cartan 1-form is \( \alpha_L = 2m(\dot{x} dx + \dot{y} dy + r^2 d\dot{\theta}) \), and the external force is

\[
\mathcal{R} = \frac{\mu mg}{r \omega} (\dot{x} dx + \dot{y} dy).
\]

Let \( \tilde{X}^{(1)} = 2r \omega \partial/\partial x + \mu g \partial/\partial \dot{x} \) and \( \tilde{X}^{(2)} = 2r \omega \partial/\partial y + \mu g \partial/\partial \dot{y} \). We can check that \( \tilde{X}^{(i)}(E_L) = \tilde{X}^{(i)}(L) = (S \tilde{X}^{(i)})(\mathcal{R}) \) (for \( i = 1, 2 \)). We have that

\[
\mathcal{L}_{\tilde{X}^{(i)}} \alpha_L = df_i \text{ for } f_1 = 2 \mu mg x \text{ and } f_2 = 2 \mu mg y,
\]

along with \( (S \tilde{X}^{(1)})(L) = 4mr \dot{x} \) and \( (S \tilde{X}^{(2)})(L) = 4mr \dot{y} \), so \( 2mr \omega \dot{x} - \mu mg x \) and \( 2mr \omega \dot{y} - \mu mg y \) are constants of the motion.
8 Momentum map and reduction

It is well-known that if a \(d\)-dimensional symmetry group is acting over a physical system, then the number of independent degrees of freedom is reduced by \(d\). In other words, \(Q\) is reduced by \(d\) dimensions, so \(TQ\) and \(T^*Q\) are reduced by \(2d\) dimensions. Therefore \(2d\) variables can be eliminated from the equations of motion. This fact can be exploited in a systematic way by means of the procedure known as reduction, which is due to Marsden and Weinstein [1, 28].

Let \(G\) be a Lie group acting on \(Q\) and consider the lifted action to \(TQ\) using tangent prolongation, that is, if \(\Phi_g: Q \to Q\) is the diffeomorphism given by \(\Phi_g(q) = gq\) for each \(g \in G\) and \(q \in Q\), then the lifted action is defined by

\[
T\Phi_g: TQ \to TQ.
\]

In what follows, we shall assume every group action considered to be free and proper. Let \(g\) the the Lie algebra of \(G\) and \(g^*\) its dual. Let \(L: TQ \to \mathbb{R}\) be a Lagrangian function subjected to an external force \(\beta\). Suppose that the \(G\)-action leaves \(L\) invariant, and hence \(\alpha_L\) and \(\omega_L\) are invariant. Then the natural momentum map [1],

\[
J: TQ \to g^*,
\]

\[
J(v_q)(\xi) = \alpha_L(v_q)(\xi_Q(v_q)),
\]

is equivariant and Hamiltonian. For each \(\xi \in g\) and \(v \in TQ\), \(J\xi: TQ \to \mathbb{R}\) is the function given by

\[
J\xi(v_q) = \langle J(v_q), \xi \rangle.
\]

Lemma 15. Let \(\xi \in g\). Then

i) \(J\xi\) is a conserved quantity for \(\xi_L, \beta\) if and only if

\[
\beta(\xi_Q^c) = 0
\]

(13)

ii) If the previous equation holds, then \(\xi\) leaves \(\beta\) invariant if and only if

\[
\iota_{\xi_Q^c}d\beta = 0.
\]

(14)

In addition, the vector subspace of \(g\) given by

\[
g_\beta = \left\{ \xi \in g \mid \beta(\xi_Q^c) = 0, \ \iota_{\xi_Q^c}d\beta = 0 \right\}
\]

is a Lie subalgebra of \(g\).
Proof.  i) We have that
\[
J\xi = \alpha_L(\xi_Q) = \iota_{\xi_Q} \alpha_L,
\]
so
\[
d(J\xi) = d(\iota_{\xi_Q} \alpha_L) = \mathcal{L}_{\xi_Q} \alpha_L - \iota_{\xi_Q} d\alpha_L = \iota_{\xi_Q} \omega_L.
\]
Contracting this equation with $\xi_L,\beta$, one gets
\[
\iota_{\xi_L,\beta} (d(J\xi)) = \xi_L (J\xi),
\]
on the left-hand side, and
\[
\iota_{\xi_L,\beta} \iota_{\xi_Q} \omega_L = -\iota_{\xi_Q} \iota_{\xi_L,\beta} \omega_L = -\iota_{\xi_Q} (dE_L + \beta) = -\xi_Q^c (E_L) - \beta(\xi_Q^c),
\]
on the right-hand side.
Thus $J\xi$ is a conserved quantity for $\xi_{L,\beta}$ if and only if
\[
\xi_Q^c (E_L) + \beta(\xi_Q^c) = 0.
\]
(15)

Now observe that
\[
\xi_Q^c (E_L) = \xi_Q^c (\Delta(L)) - \xi_Q^c (L) = \xi_Q^c (\Delta(L)) = \mathcal{L}_{\xi_Q} (\Delta(L)) = \mathcal{L}_{\xi_Q} (\iota_\Delta dL)
\]
\[
= \iota_{\xi_Q,\Delta} dL + \iota_\Delta (\mathcal{L}_{\xi_Q} dL) = \iota_{[\xi_Q,\Delta]} dL = \xi_Q^c (\Delta(L)),
\]
since $\xi_Q^c (L) = 0$ by the $G$-invariance of $L$, but $[\xi_Q,\Delta] = 0$, and thus
\[
\xi_Q^c (E_L) = 0
\]
for each $\xi \in \mathfrak{g}$, that is, $E_L$ is $G$-invariant. By Eq. (15), $J\xi$ is a conserved quantity for $\xi_{L,\beta}$ if and only if
\[
\beta(\xi_Q^c) = 0.
\]
(16)

ii) For each $\xi \in \mathfrak{g}_\beta$, we have that
\[
\mathcal{L}_{\xi_Q} \beta = d(\iota_{\xi_Q} \beta) + \iota_{\xi_Q} d\beta = d(\beta(\xi_Q^c)) + \iota_{\xi_Q} d\beta.
\]
If Eq. (16) holds, then $\beta$ is $\mathfrak{g}_\beta$-invariant (i.e., $\mathcal{L}_{\xi_Q} \beta = 0$) if and only if
\[
\iota_{\xi_Q} d\beta = 0
\]
For $g_\beta$ being a Lie subalgebra it is necessary and sufficient that $[\xi, \eta] \in g_\beta$ for each $\xi, \eta \in g_\beta$. Since $\xi \in g \mapsto \xi_Q \in \mathcal{X}(Q)$ is a Lie algebra antihomomorphism [33], this is equivalent to

$$\beta([\xi_Q, \eta_Q]) = 0,$$

$$t_{[\xi_Q, \eta_Q]}^\alpha t^\beta = 0,$$

but $[\xi_Q, \eta_Q]^c = [\xi_Q^c, \eta_Q^c]$, since the complete lift is a morphism between Lie algebras. Then

$$\beta([\xi_Q, \eta_Q]) = \beta([\xi_Q^c, \eta_Q^c]) = t_{[\xi_Q, \eta_Q]}^\alpha t^\beta = \mathcal{L}_{\xi_Q} t_{\eta_Q}^\beta - t_{\eta_Q} \mathcal{L}_{\xi_Q}^\beta$$

$$= \xi_Q^c(\beta(\eta_Q^c)) - \eta_Q^c(\beta(\xi_Q^c)) - t_{\eta_Q} (t_{\xi_Q}^\beta) = 0,$$

by Eqs. (14) and (13). Similarly,

$$t_{[\xi_Q, \eta_Q]}^\alpha t^\beta = t_{[\xi_Q, \eta_Q]}^\alpha t^\beta = \mathcal{L}_{\xi_Q} t_{\eta_Q}^\beta - t_{\eta_Q} \mathcal{L}_{\xi_Q}^\beta$$

$$= \mathcal{L}_{\xi_Q}^c t_{\eta_Q} \mathcal{L}_{\xi_Q} - t_{\eta_Q} \mathcal{L}_{\xi_Q}^\beta = 0.$$

It is worth mentioning that our Lemma 15 i) was previously obtained by Marsden and West [26, Theorem 3.1.1], albeit from a variational approach.

**Corollary 16.** For each $\xi \in g_\beta$, $\xi_Q^c$ is a Noether symmetry and it is a symmetry of the forced Lagrangian system.

**Proof.** Since $\alpha_L$ is invariant,

$$\mathcal{L}_{\xi_Q}^c \alpha_L = 0$$

for each $\xi \in g$. In combination with Eq. (15), this implies that $\xi_Q^c$ is a Noether symmetry. By Remark 3, it is also a symmetry of the forced Lagrangian system.

**Theorem 17.** Let $G_\beta \subset G$ be the Lie subgroup generated by $g_\beta$ and $J_\beta : TQ \to g_\beta^*$ the reduced momentum map. Let $\mu \in g_\beta^*$ be a regular value of $J_\beta$ and $(G_\beta)_\mu$ the isotropy group in $\mu$. Then

i) $J_\beta^{-1}(\mu)$ is a submanifold of $TQ$ and $\xi_L, \beta$ is tangent to it.
ii) The quotient space $M_\mu := J^{-1}_\beta(\mu)/(G_\beta)_\mu$ is endowed with an induced symplectic structure $\omega_\mu$, namely

$$\pi^*_\mu \omega = \iota^*_\mu \omega_L,$$

where $\pi_\mu : J^{-1}_\beta(\mu) \to M_\mu$ and $\iota_\mu : J^{-1}_\beta(\mu) \hookrightarrow TQ$ denote the projection and the inclusion, respectively.

iii) $L$ induces a function $L_\mu : M_\mu \to \mathbb{R}$ defined by

$$L_\mu \circ \pi_\mu = L \circ \iota_\mu.$$

Moreover, we can introduce a function $E_{L_\mu} : M_\mu \to \mathbb{R}$, given by

$$E_{L_\mu} \circ \pi_\mu = E_L \circ \iota_\mu. \quad (17)$$

iv) $\beta$ induces a reduced semibasic 1-form $\beta_\mu$ on $M_\mu$, given by

$$\pi^*_\mu \beta = \iota^*_\mu \beta.$$

**Proof.** For a proof of the first three assertions see Refs. [1, 27, 33]. Observe that

$$\Delta(L) \circ \iota_\mu = \iota^*_\mu \Delta(L) = \iota^*_\mu (\iota_\mu dL) = \iota^*_\mu (\Delta (\iota_\mu dL)) = (\iota^*_\mu \Delta) (\iota^*_\mu L) = (\iota^*_\mu \Delta) (L \circ \pi_\mu) = (\pi^*_\mu \Delta_\mu) (L \circ \pi_\mu) = \pi^*_\mu (\Delta_\mu(L_\mu)) = \Delta_\mu(L_\mu) \circ \pi_\mu,$$

where $\Delta$ is the Liouville vector field on $TQ$ and $\Delta_\mu$ is a $\pi_\mu$-related vector field on $M_\mu$, namely

$$\pi^*_\mu \Delta_\mu = \iota^*_\mu \Delta.$$

Then we can introduce a function $E_{L_\mu} : M_\mu \to \mathbb{R}$, given by

$$E_{L_\mu} = \Delta_\mu(L_\mu) - L_\mu,$$

which satisfies Eq. (17). Since $\beta$ is $(G_\beta)_\mu$-invariant, it induces a reduced semibasic 1-form $\beta_\mu$ on $M_\mu$.

**Corollary 18.** The vector field $\xi_{L_\mu,\beta_\mu}$, defined by

$$\iota_{\xi_{L_\mu,\beta_\mu}} \omega = dE_{L_\mu} + \beta_\mu,$$

determines the dynamics on $M_\mu$. It is $\pi_\mu$-related to $\xi_{L,\beta}$. 23
Remark 6. In the Rayleigh dissipation case $\beta = S^*(d\mathcal{R})$, according to Lemma 10, one can equivalently define $g_\beta \equiv g_\mathcal{R}$ as

$$g_\mathcal{R} = \{ \xi \in g \mid \xi^\nu(\mathcal{R}) = 0, S^*d\xi^\nu(\mathcal{R}) = 0 \},$$

where the condition $S^*d\xi^\nu(\mathcal{R}) = 0$ means that $\xi^\nu(\mathcal{R})$ is basic: it does not depend on $\dot{q}^i$. If additionally $\mathcal{R}$ is $g_\mathcal{R}$-invariant, i.e., $\xi^\nu(\mathcal{R}) = 0$ for each $\xi \in g_\mathcal{R}$, then it induces a dissipation function $\mathcal{R}_\mu : M_\mu \to \mathbb{R}$ given by

$$\mathcal{R}_\mu \circ \pi_\mu = \mathcal{R} \circ \iota_\mu.$$ 

Remark 7 (Reconstruction of the dynamics). Knowing the integral curves on $M_\mu$, we want to obtain the integral curves on $J^{-1}(\mu)$. Let $c(t)$ and $[c(t)]$ be the integral curves of $\xi_{L,\beta}$ and $\xi_{L_\mu,\beta_\mu}$, respectively, with $c(0) = p_0$. Let $d(t) \in J^{-1}(\mu)$ be a smooth curve such that $d(0) = p_0$ and $[d(t)] = [c(t)]$. We can write

$$c(t) = \Phi_{g(t)}(d(t)), \quad (18)$$

for $g(t) \in (G_\beta)_\mu$. Then we have to find $g(t)$ in order to express $c(t)$ in terms of $[c(t)]$. Now [1]

$$\xi_{L,\beta}(c(t)) = \xi^\prime(t) = T\Phi_{g(t)}(d(t))(d^\prime(t)) + T \Phi_{g(t)}(d(t))(TL_{g(t)}(g^\prime(t)))\xi_Q(d(t)),$$

and using the $\Phi_g$-invariance of $\xi_{L,\beta}$ one gets

$$\xi_{L,\beta}(d(t)) = d^\prime(t) + (TL_{g(t)}(g^\prime(t)))\xi_Q(d(t)).$$

In order to solve this equation, we first solve the algebraic problem

$$\xi_Q(d(t)) = \xi_{L,\beta}(d(t)) - d^\prime(t),$$

for $\xi(t) \in g_\beta$, and then solve

$$g^\prime(t) = TL_{g(t)}\xi(t)$$

for $g(t)$. The integral curve sought is given by Eq. (18).

Example 4 (Angular momentum). Consider $Q = \mathbb{R}^n \setminus \{0\}$, a Lagrangian function $L$ on $TQ$ that is spherically symmetric, say $L(q, \dot{q}) = L(\|q\|, \|\dot{q}\|)$. Consider the Lie group $G = SO(n) = \{ O \in \mathbb{R}^{n \times n} \mid O^t O = \text{Id}, \det(O) = 1 \}$
acting by rotations on $Q$. The action can be lifted to $TQ$ via the tangent lift. Explicitly, for $O \in SO(3)$ we let

$$g_O : Q \rightarrow Q,$$

$$(q) \mapsto (O \cdot q)$$

$$Tg_O : TQ \rightarrow TQ,$$

$$(q, \dot{q}) \mapsto (O \cdot q, O \cdot \dot{q}).$$

The group $SO(n)$ acts freely and properly. The Lie algebra of the group is given by $\mathfrak{g} = so(n) = \{ o \in \mathbb{R}^{n \times n} \mid o^t + o = 0 \}$. In the case $n = 3$ this algebra will be identified with the algebra of three dimensional vectors with the cross product by taking

$$\begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix} \mapsto \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}.$$ 

The infinitesimal generator of $\xi \in \mathfrak{g}$ is given by

$$\xi_Q(q) = (\xi \times q),$$

$$\xi_Q^c(q, \dot{q}) = (\xi \times q, \xi \times \dot{q}),$$

$$\xi_v^c(q, \dot{q}) = (0, \xi \times \dot{q}).$$

One can identify $\mathfrak{g}$ with $\mathfrak{g}^*$ by using the inner product on $\mathbb{R}^3$. The moment map is then given by [1, Example 4.2.15]

$$J(q, \dot{q}) = q \times \dot{q}.$$ 

Identifying $\mathfrak{g}^* \simeq \mathbb{R}^3$ one sees that the coadjoint actions of $G$ is the usual one (by rotations). Let $\mu \in \mathfrak{g}^*$, $\mu \neq 0$ then the isotropy group $G_\mu \simeq S^1$ of $\mu$ under the coadjoint action, which are the rotations around the axis $\mu$.

We look for Rayleigh potentials $\mathcal{R}$ such that $\mathfrak{g}_\mathcal{R} = \mathfrak{g}$. The condition that $\xi_Q^v(\mathcal{R}) = 0$ implies that $\mathcal{R}$ is spherically symmetric on the velocities. Then, the condition that $\xi_Q^c(\mathcal{R}, \dot{\mathcal{R}})$ is semi-basic means that the terms which are not spherically symmetric on the positions cannot involve the velocities. That is:

$$\mathcal{R} = A(q) + B(\|q\|, \|\dot{q}\|).$$

Without loss of generality, one can take $\mu = (0, 0, \mu_0)$. Hence, if $(q, v) \in J^{-1}(\mu)$, both $q$ and $\dot{q}$ lie on the $xy$-plane. Moreover, they must satisfy the equation $\dot{q}_1 p_2 - \dot{q}_1 q_2 = \mu_0$. We can finally apply Theorem 17, to our system and find out that $(M_\mu, L_\mu)$, which is a Hamiltonian system over a 2-dimensional manifold [1, Example 4.3.4].
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Data Availability

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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