Critical curves of plane Poiseuille flow with slip boundary conditions

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Abstract
We investigate the linear stability of plane Poiseuille flow in 2D under slip boundary conditions. The slip is defined by the tangential velocity at the wall in units of the maximal flow velocity. As it turns out, the critical Reynolds number depends smoothly on but increases quite rapidly.

1 Introduction
No-slip boundary conditions are a convenient idealization of the behavior of viscous fluids near walls. In real systems there is always a certain amount of slip which, however, is hard to detect experimentally because of the required space resolution. In high precision measurements Elrick and Emrich detected slip of the order 0.1% in laminar pipe flow with Reynolds numbers of 16 to 4300. The measuring error in was nearly as low as the fluctuations due to Brownian motion. Very recently, Archer et al. observed the existence of slip in plane laminar Couette flow with added polymers.

We examine how the linear instability of the steady plane Poiseuille flow depends on the slip defined by

\[ s := \frac{u_{\text{wall}}}{u_{\text{max}}} \]  

where \( u_{\text{wall}} \) is the tangential velocity at the wall and \( u_{\text{max}} \) is the midstream velocity. As boundary conditions we adopt

\[ \frac{\partial u}{\partial z} \pm bu = 0, \quad w = 0, \quad \text{at } z = \pm 1 \]  

where \( z \) is measured in units of the channel half-width. The slip \( s \) is implicitly determined by the parameter \( b > 0 \) with \( s \to 0 \) in the limit \( b \to \infty \).

2 Orr-Sommerfeld equation with slip boundary conditions
The continuity equation in two dimensions is most conveniently satisfied by introducing a stream function \( \Psi(x, z, t) \) where \( x \) denotes the streamwise direction and \( z \) the direction normal to the boundaries (see Fig. 1). The velocity \((u, w)\) is connected to \( \Psi \) through

\[ u = \frac{\partial \Psi}{\partial z}, \quad w = -\frac{\partial \Psi}{\partial x}. \]  

In terms of \( \Psi \) the Navier-Stokes equations for plane Poiseuille flow in two dimensions read in dimensionless form

\[ \frac{\partial}{\partial t} \Delta \Psi + \frac{\partial \Psi}{\partial z} \frac{\partial \Delta \Psi}{\partial x} - \frac{\partial \Psi}{\partial x} \frac{\partial \Delta \Psi}{\partial z} = \frac{1}{R} \Delta^2 \Psi. \]  

As usually, \( \Psi \) is decomposed in the stream function \( \Psi_b \) of the steady profile and a Fourier ansatz in \( x \)-direction for the disturbance field with the wave number \( \alpha \):

\[ \Psi(x, z, t) = \Psi_b(z) + \sum_{q=-\infty}^{\infty} e^{iq\alpha x} \Psi_q(z, t). \]  

However, with slip the basic flow is now given by

\[ \Psi_b = z - \frac{bs}{6} z^3; \quad s = \frac{2}{2+b}. \]
The linearized part of (4) leads to the Orr-Sommerfeld equation

$$L \Psi_q = R \frac{\partial}{\partial t} (D^2 - q^2 \alpha^2) \Psi_q$$

(7)

where

$$L = (D^2 - q^2 \alpha^2)^2 - i \alpha q R \left[ U(z) (D^2 - q^2 \alpha^2) - U''(z) \right]$$

(8)

with $$U(z) = \partial \Psi_b/\partial z$$ and $$D := \frac{\partial}{\partial z}$$.

Figure 1: Geometry of the basic flow with slip boundary conditions.

3 Numerical method

We determine the critical (neutral) curves in the parameter space of the Reynolds number $$R$$ and the wave number $$\alpha$$ of the disturbance. The Reynolds number is based on the channel half-width and on the midstream velocity of the steady flow.

The solution of the differential equation (7) leads to a generalized eigenvalue problem that we solve numerically as in [3] using up to 70 Chebyshev polynomials as basis functions. The critical curve is the set of points $$(R, \alpha)$$ for which the most critical eigenvalue has zero real part with all other modes decaying exponentially.

4 Results

In Fig. 2 we present the critical curves for different slips $$s$$. The critical Reynolds number $$R_c$$ is the lowest Reynolds number on the critical curve. We define also the slip $$s_c$$ by the corresponding normalized tangential velocity of the critical mode at the wall. The results are listed in Tab. 1.

![Critical Curves](image)

Figure 2: Critical curves of plane Poiseuille flow for different slips $$s = 0\%, 1\%, 2\%, 3\%, 4\%, 5\%$$ and 6%.

Table 1: Slip $$s_c$$ of the critical mode and critical Reynolds number $$R_c$$ at different slips $$s$$ of the steady flow.

| $$s$$  | 0% | 0.1% | 0.2% | 0.5% | 1% |
|--------|----|------|------|------|----|
| $$s_c$$ | 0% | 0.9% | 1.8% | 4.5% | 8% |
| $$R_c$$ | 5772 | 5773 | 5781 | 5847 | 6070 |
| $$s$$  | 2% | 3% | 4% | 5% | 6% |
| $$s_c$$ | 21% | 31% | 39% | 47% | 55% |
| $$R_c$$ | 6960 | 8600 | 11060 | 15310 | 23230 |

5 Literature

[1] Elrick R.M., Emrich R.J., Phys. Fluids 9 (1966), 28

[2] Archer L.A., Larson R.G., Chen Y.-L., J.Fluid Mech. 301 (1995), 133

[3] Rauh A., Zachrau T., Zoller J., Physica D 86 (1995), 603