Virtual links are algorithmically recognisable

Vassily O. Manturov

March 29, 2022

Virtual links were proposed by Kauffman in 1996, see [Kau], as a generalisation of classical links from the point of view of planar diagrams.

In the present paper we prove the following

**Theorem 1.** There is an algorithm to decide whether two virtual links are equivalent or not.

First of all, we use the result by Moise [Moi] that all 3-manifolds admit a triangulation. In the sequel, we suppose all manifold to be triangulated.

The proof will use methods of three-dimensional topology coming from Haken’s theory of normal surfaces [Ha], and its development by Matveev [Mat]. We shall deal with 3-manifolds and 2-surfaces in them. A compact surface $F$ in a 3-manifold $M$ is called proper if $F \cap \partial M = \partial F$. A proper submanifold is called **essential** if it does not cobound a ball together with a part of the boundary of the manifold. We are interested in essential spheres and essential annuli.

A 3-manifold $M$ is called **irreducible** if every 2-sphere in $M$ bounds a 3-ball in $M$.

We shall use the following definition of virtual link.

A **virtual link** $L$ is an equivalence class for embeddings of split union of circles into $M \times I$, modulo isotopy, fibre-preserving homeomorphisms of $M \times I$ and stabilisations/destabilisations of $M$; we shall use the same word “link” also for the image of the split union of circles for a given realisation of the virtual link.

Here by $M$ we mean a compact two-dimensional surface; this surface is not necessarily connected, but for each component $M_i$, the manifold $M_i \times I$ should contain at least one component of the link $L$. By stabilisations/destabilisations we mean addition/removal of 1-handles to $M$ (respectively, thick cylinders to $M \times I$) such that the handle to be added/removed is disjoint from the link.

Having a virtual link, one can try to minimise its representative with respect to the genus of the surface $M$ by destabilising the diagram while possible. Clearly, this can be done in different ways. Here we should take care that the destabilisation should be performed in such a way that no “empty” component appear; each $M_i \times I$ should contain at least one link component. If there is a destabilisation which divides the manifold into two components, one of which contains no components of the link, then we should remove this component and paste the remaining one: we attach $D^2 \times I$ to $S^1 \times I$. 

1
It is worth mentioning that classical links which are essentially contained in a ball, admit a presentation in $S^2 \times I$ (or, in several copies of $S^2 \times I$, if we deal with a split classical link).

One of the key points that we shall use is the following theorem due to Kuperberg, \cite{Kup}.

**Theorem 2.** The result of destabilisation is well defined up to isotopy.

Thus, in order to compare virtual knots, it is sufficient to find their minimal representatives to and compare them.

We shall use the several theorems from Haken’s theory of normal surfaces, see \cite{Mat}.

By a *pattern* we mean a fixed 1-polyhedron (graph) on the boundary of a 3-manifold without isolated vertices (we may assume that this graph is a sub-polyhedron of the 1-frame of the triangulation for the boundary). In the sequel, all proper surfaces with boundary in a manifold $M$ with a boundary pattern $\Gamma \subset \partial M$ are thought to be in general position to pattern $\Gamma$, unless otherwise specified.

A manifold $(M, \Gamma)$ with a boundary pattern is *boundary irreducible* if for every proper disc $D$ the curve $\partial D$ is in general position to $\Gamma$ and bounds a disc in $\partial M$.

By a *compressing disc* for a proper surface $F$ in a 3-manifold $M$ we mean a disc $D \subset M$, such that $D \cap F = \partial D$. A surface $F \subset M$ is called *incompressible* if for any compressing disc $D$ the curve $\partial D$ is trivial in $D$.

A 3-manifold is *sufficiently large* if there is an incompressible closed surface $F \subset M$ which is two-sided and different from $S^2, \mathbb{RP}^2$.

A 3-manifold without boundary is called *Haken* if it is irreducible, boundary irreducible and sufficiently large. An irreducible boundary irreducible 3-manifold $(M, \Gamma)$ with boundary-pattern $\Gamma$ is *Haken* if it is either sufficiently large or $\Gamma$ is non-empty (hence, $\partial M$ is non-empty) and $M$ is a handlebody but not a ball.

Note, that a solid torus with a non-empty boundary pattern is thus Haken.

A well-known statement \cite{Mat} says the following:

**Proposition 1.** An irreducible boundary irreducible 3-manifold with nonempty boundary is either sufficiently large or a handlebody.

**Lemma 1.** There is an algorithm to decide whether a 3-manifold $M$ is reducible; if it is, the algorithm constructs a 3-sphere $S \subset M$ which does not bound any ball in $M$.

**Lemma 2.** Classical links are algorithmically recognisable.

**Remark 1.** The proof for an algorithm for recognising classical links, goes in the same vein as the one we are performing for virtual links, see \cite{Mat}.

**Lemma 3.** There is an algorithm to decide whether a Haken manifold $M$ with a pattern $\Gamma$ on the boundary $\partial M$ has an essential proper annulus.
Lemma 4. There is an algorithm to recognise whether two Haken manifolds 
\((M, \Gamma)\) and \((M', \Gamma')\) with patterns on the boundary are homeomorphic by a home-
omorphism taking \(\Gamma\) to \(\Gamma'\).

Also, we need one more lemma.

Let \((M, L)\) be a representative of a virtual link \(L\), i.e. \(M = \tilde{M} \times I\) for a closed 2-surface \(\tilde{M}\). Let \(N\) be a small open tubular neighbourhood of \(L\). Let us cut \(N\) from \(M\). We obtain a manifold to be denoted by \(M_L\). Its boundary consists of boundary components of \(M\) (two if \(M\) is connected) and several tori; the number of tori is equal to the number of components of \(L\). Let us endow each such torus with a pattern \(\Gamma_L\) that represents the meridian of the corresponding component. We obtain a manifold \((M_L, \Gamma_L)\).

Clearly, virtual link \(L\) can be restored from \((M_L, \Gamma_L)\): we know how to make the manifold \(M\) by pasting solid tori to the boundary components of \(M_L\) since we know the meridians of these tori. Thus, we can restore the pair \((M, L)\).

Suppose the link is not a split sum of a classical link with another link.

Lemma 5. The manifold \((M_L, \Gamma_L)\) is Haken.

Proof. In virtue of Proposition \[, it is sufficient to prove that our manifold is irreducible and boundary irreducible: it never happens for such a manifold to be a handlebody.

Now, for any compact oriented 2-surface \(S_g\), the manifold \(S_g \times I\) is irreducible, unless \(g = 0\). So, if our link \(L\) is not classical, then for its neighbourhood \(N(L)\), the set \((S_g \times I) \setminus N(L)\) may be irreducible if it contains a sphere \(S\) such that \(S\) bounds a ball in \(S_g \times \{0, 1\}\) containing some components of \(N\). This means that those component form a classical link which is split from the remaining components of \(L\), which leads to a contradiction.

Further, since the link \(L\) is not a split sum of an unknot with a virtual link, the manifold \(M_L\) is boundary irreducible.

Thus, our manifold is irreducible, boundary irreducible, and hence (by Proposition \[) Haken. \qed

Now, let us prove the main theorem.

Lemma 6. Let \(L\) be a link that cannot be represented as a split union \(K \cup L'\) of a classical link \(K\) with a virtual link \(L'\). Then \(M_L\) is irreducible.

Let \(L, L'\) be a virtual links.

Step 1. Construct some representatives of links \((M, L), (M', L')\). Take the corre-
sponding manifolds with patterns on the boundary by \((M_L, \Gamma), (M'_{L'}, \Gamma').\)

Step 2. Find whether one of \((M_L, \Gamma)\) or \((M'_{L'}, \Gamma')\) is reducible. If yes, then it is possible to find a reducing sphere, and, thus, extract all classical splittable sublinks from \(M_L\). Classical links are algorithmically recognisable. We can compare classical split components for \((M_L, \Gamma)\) and \((M'_{L'}, \Gamma')\). If they do not represent isotopic classical links, we stop: the initial links are not equivalent. Otherwise, we go on.
Then, we reduce our problem to the case when no classical split sublinks are possible. From now on, all 3-manifolds are Haken.

Step 3. Now, each connected component of \((M_L, \Gamma)\) and \((M'_L, \Gamma')\) is a Haken manifold with a pattern on the boundary. Thus, we can algorithmically decide whether there is a homeomorphism \(f : M_L \to M'_L\), taking \(\Gamma\) to \(\Gamma'\). If there is any, then our virtual links \(L, L'\) are equivalent; if not, they are not then \(L\) and \(L'\) are not equivalent virtual links.

Performing the steps described above, we obtain the claim of the theorem.

**Remark 2.** The proof described above works equivalently for oriented virtual links and for framed virtual links.

**References**

[Ha] Haken, W. (1961), Theorie der Normalflächen, *Acta Mathematicae* 105, pp. 245–375.

[Kau] Kauffman, L. H. (1999), Virtual knot theory, *European Journal of Combinatorics* 20(7), pp. 662–690.

[Kup] Kuperberg, G. (2002), What is a Virtual Link?, www.arXiv.org, math-GT/0208039, *Algebraic and Geometric Topology*, 2003, 3, 587-591

[Moï] Moïse, E.E. (1952), Affine structures in 3–manifolds. V. The triangulation theorem and Hauptvermutung, *Annals of Mathematics*, 57, pp. 547–560.

[Mat] Matveev, S.V. (2003), *Algorithmic topology and classification of 3-manifolds*, (Springer Verlag).