LAX PAIRS FOR LINEAR HAMILTONIAN SYSTEMS

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Abstract: We construct Lax pairs for linear Hamiltonian systems of differential equations. In particular, the Gröbner bases are used for computations. It is proved that the mappings in the construction of Lax pairs are Poisson. Under study are the various properties of first integrals of the system which are obtained from Lax pairs.

DOI: 10.1134/S0037446619040050

Keywords: Lax pairs, linear Hamiltonian systems, first integrals, Gröbner bases

1. Introduction

The aim of this paper is to calculate Lax pairs for a linear system of differential equations

$$\Gamma \dot{x} = -Px,$$

(1)

where $x$ is a column vector in $\mathbb{R}^{2n}$, while $\Gamma$ and $P$ are matrices in $\text{M}_{2n}(\mathbb{R})$. Here $P$ is assumed symmetric and $\Gamma$, skew-symmetric; i.e., $P^T = P$ and $\Gamma^T = -\Gamma$. We will also assume that $\det \Gamma \neq 0$ (what explains the parity of the dimension of the vector space) and $\det P \neq 0$.

On the other hand, if we consider an arbitrary system of differential equations

$$\dot{x} = Ax,$$

(2)

admitting a quadratic first integral $\frac{1}{2} x^T P x$, where $x \in \mathbb{R}^r$, $A \in \text{M}_{2n}(\mathbb{R})$, $P \in \text{M}_{2n}(\mathbb{R})$, $P^T = P$ and $\det A \neq 0$, $\det P \neq 0$; then $\Gamma = -PA^{-1}$ is a nondegenerate skew-symmetric matrix (see [1, 2]), and system (2) is immediately reduced to the form (1); therefore $r = 2n$.

We note that system (1) is Hamiltonian; see Section 2 below.

By a Lax pair of dimension $k$ for (1) we mean a smooth mapping

$$\text{LP}: \mathbb{R}^{2n} \longrightarrow \text{M}_k(\mathbb{C}) \times \text{M}_k(\mathbb{C}) \simeq \mathbb{R}^{4k^2}$$

(3)

that translates the solutions of (1) into the solutions of the following matrix differential equation:

$$\dot{L} = [B, L],$$

(4)

where $B \times L \in \text{M}_k(\mathbb{C}) \times \text{M}_k(\mathbb{C})$. In other words, if $\tau$ is a vector field on $\mathbb{R}^{2n}$ generated by (1), then for each $x \in \mathbb{R}^{2n}$ we have that the $L$-component of the tangent vector $d\text{LP}_x(\tau(x))$ at $\text{LP}(x)$ is defined by (4).

As it is known, this implies that for the function

$$f_l = \text{Tr}(L^l): \text{M}_k(\mathbb{C}) \longrightarrow \mathbb{C}, \text{ where } l \in \mathbb{N},$$

we have $\dot{f}_l = 0$; i.e., $f_l$ is constant on the solutions of (4). Therefore, the preimage of every functional combination of the real and imaginary parts of the function $f_l$ with respect to the mapping $\text{pr}_2 \circ \text{LP}$ is

The first author was supported by the Russian Foundation for Basic Research (Grants 16–01–00378a and 16–51–55012 China–a). The second author was partially supported by the Laboratory of Mirror Symmetry NRU HSE (RF Government Grant, Ag. No. 14.641.31.0001).

Original article submitted October 11, 2018; revised October 11, 2018; accepted December 19, 2018.
the first integral of (1). The main problem in this approach is to verify the nontriviality of the integrals, as well as the completeness of the system of the first integrals obtained by different Lax pairs.

We note that the study of Lax pairs for linear Hamiltonian systems is the first step to the nonlinear case, since each nonlinear system of differential equations gives a linear system in the first approximation. For instance, if the system is Hamiltonian, then we can decompose the Hamiltonian of the system into a Taylor series and leave for further consideration only terms with order of degree at most 2.

For $n = 1$, we investigate all possible Lax pairs of dimension 2 for (1) in Section 3. For the final answer in this case, we also use the computer calculations that involve Gröbner bases. The use of Gröbner bases for finding Lax pairs, as far as we know, has never been used before this work. They were also used to get Lax pairs in the general case. For a brief overview of the technology associated with the Gröbner bases see, for instance, [3] or [4].

For an arbitrary $n \geq 1$, as well as for a pair: an eigenvalue and an eigenvector with this eigenvalue of the matrix $(P\Gamma - 1)^2$, we build a Lax pair of dimension 2 in Section 4. We show that for different eigenvalues the first integrals of (1), constructed from the function $\text{Tr}(L^2)$, are in involution on $\mathbb{R}^{2n}$ with respect to the Poisson bracket given by the skew-symmetric matrix $-\Gamma$. These first integrals are quadratic functions on $\mathbb{R}^{2n}$, and in the case of a simple spectrum of $P\Gamma - 1$ they form a complete set of $n$ functionally independent first integrals of (1) (for complex eigenvalues and eigenvectors, we must take the real and imaginary parts of the corresponding functions, as we noted above). Thus, we obtain a new simple proof of Williamson’s result (see [5]) stating that (1) admits $n$ first quadratic integrals which are pairwise in involution and functionally independent; see Remark 4.3. (The detailed discussion of this matter, with additional links, is given at the end of [2, §2].)

In Section 5, we show that the mapping arising in the construction of Lax pairs is a morphism of Poisson manifolds with respect to the natural Poisson brackets.

2. Preliminaries

Recall the following well-known facts about symplectic structures and Poisson brackets.

Let a nondegenerate skew-symmetric matrix $W \in \mathbb{M}_s(\mathbb{R})$, with $s$ even, define the symplectic structure on $\mathbb{R}^s$ with the help of the 2-form

$$\Omega(x, y) = x^T Wy, \quad x, y \in \mathbb{R}^s.$$  

The vector field $v$ on $\mathbb{R}^s$ is called Hamiltonian with Hamiltonian $H$, which is a smooth function on $\mathbb{R}^s$, if

$$\Omega(v, w) = -dH(w),$$

where $w$ is an arbitrary vector field on $\mathbb{R}^s$. We will denote such a vector field as $v = \text{sgrad} H$. Their Poisson bracket $\{f, g\}$ of two smooth functions $f$ and $g$ on $\mathbb{R}^s$ is determined by the equations:

$$\{f, g\} = -\Omega(\text{sgrad} f, \text{sgrad} g) = -dg(\text{sgrad} f) = df(\text{sgrad} g).$$

Then, in coordinates,

$$\text{sgrad} g = W^{-1} \cdot \text{grad} g \quad \text{and} \quad \{f, g\} = (\text{grad} f)^T \cdot W^{-1} \cdot \text{grad} g,$$

where the column of functions, the gradient, is $\text{grad} f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_s}\right)^T$, and $x_1, \ldots, x_s$ are standard coordinates in $\mathbb{R}^s$.

Returning to (1) (in this case $s = 2n$), we can see by simple calculations that if $W = -\Gamma$, then the vector field $-\Gamma^{-1} Px$ is Hamiltonian with Hamiltonian $H = \frac{1}{2} x^T Px$, $x \in \mathbb{R}^{2n}$; see also [2, §1]. Therefore, system (1) is Hamiltonian with respect to the symplectic structure described above on $\mathbb{R}^{2n}$, specified by the skew-symmetric matrix $-\Gamma$.

Further we will use the following statements. We assume that the matrices $\Gamma$ and $P$ satisfy the conditions that are stated in the Introduction.
**Proposition 2.1.** (1) If $\lambda \in \mathbb{C}$ is an eigenvalue of $P \Gamma^{-1}$; then $\bar{\lambda}$, $-\lambda$, and $-\bar{\lambda}$ are eigenvalues of $P \Gamma^{-1}$ too.

(2) Let $v_1$ and $v_2$ be (complex) eigenvalues of $(P \Gamma^{-1})^2$ with corresponding eigenvalues $\lambda_1$ and $\lambda_2$ from $\mathbb{C}$. If $\lambda_1 \neq \lambda_2$, then $v_1^T \Gamma^{-1} v_2 = 0$.

(3) Let the spectrum of $P \Gamma^{-1}$ be simple. If $v$ is an eigenvector of $(P \Gamma^{-1})^2$, which is not an eigenvector of $P \Gamma^{-1}$; then $v^T \Gamma^{-1} (P \Gamma^{-1} v) \neq 0$.

**Proof.** (1): This property follows from the fact that $P \Gamma^{-1}$ is real, and from the equalities

$$0 = \det(P \Gamma^{-1} - \lambda E) = \det(P - \lambda \Gamma) = \det(P - \lambda \Gamma)^T$$

where $E$ is the identity matrix.

(2): This property follows from the chain of equalities:

$$v_1^T \Gamma^{-1} v_2 = (\lambda_1^{-1}(P \Gamma^{-1})^2 v_1)^T \Gamma^{-1} v_2 = \lambda_1^{-1} v_1^T \Gamma^{-1} P \Gamma^{-1} P \Gamma^{-1} v_2$$

$$= \lambda_1^{-1} v_1^T \Gamma^{-1} (P \Gamma^{-1})^2 v_2 = \lambda_1^{-1} \lambda_2 v_1^T \Gamma^{-1} v_2.$$

(3): Since the spectrum of $P \Gamma^{-1}$ is simple, from item (1) of Proposition 2.1 it follows that $(P \Gamma^{-1})^2$ is diagonalized, with a two-dimensional subspace of eigenvectors corresponding to each eigenvalue. Let $D$ be a matrix composed of columns that are linearly independent (over $\mathbb{C}$) eigenvectors of $(P \Gamma^{-1})^2$, so that the columns corresponding to one eigenvalue are adjacent. Then $D \Gamma^{-1} D$ has the number $v_i^T \Gamma^{-1} v_j$ as the $(i, j)$th entry, where $v_k$ is the $k$th column of $D$. Therefore, from item (2) of Proposition 2.1, we obtain that $G = D \Gamma^{-1} D$ is a block-diagonal matrix with $2 \times 2$ blocks on the diagonal. Note that $P \Gamma^{-1} v$ is also an eigenvector of $(P \Gamma^{-1})^2$ with the same eigenvalue as $v$, whereas $P \Gamma^{-1} v$ is not proportional to $v$. Therefore, if $v^T \Gamma^{-1} (P \Gamma^{-1} v) = 0$, then one of the $2 \times 2$ blocks of $G$ would be equal to zero. It would follow that $\det G = 0$. Hence, det $\Gamma^{-1} = 0$; a contradiction. □

3. Lax Pairs for $n = 1$

3.1. Using the square root of the matrix. In this subsection, we present the Lax pair of dimension 2 for (1) in the case $n = 1$, so that the function $\text{Tr}(L^2)$ will be the first integral of this system coinciding with $4H$, where the Hamiltonian is $H = \frac{1}{2} x^T P x$. Later, in Section 4 we give the general formulas for the Lax pairs of dimension 2 for (1), but these formulas will differ from this in the case of $n = 1$ and the Hamiltonian $H$ (for arbitrary $n$) will be obtained only as a linear combination of the first integrals connected with Lax pairs.

Since $P^T = P$, there exists a symmetric matrix $T \in M_2(\mathbb{C})$ such that $T^2 = P$. (Such a matrix always exists, since by conjugation by orthogonal matrices from $O(2, \mathbb{R})$ the matrix $P$ can always be reduced to a diagonal matrix, from which we take the root in the form of a diagonal matrix and then apply the reverse pairing.) Note that $\det T \neq 0$.

We will use the identity

$$\Gamma \cdot T = \det T \cdot T^{-1} \cdot \Gamma,$$

which is immediately verified by direct calculations for the explicit matrices

$$\Gamma = \begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} a & c \\ c & b \end{pmatrix}. \quad (6)$$

Let $x = (x_1, x_2)^T$ be a standard coordinate column in $\mathbb{R}^2$. We introduce the matrices

$$Z = \begin{pmatrix} x_1 & 0 \\ x_2 & 0 \end{pmatrix}, \quad L = T \cdot Z - \Gamma^{-1} \cdot T \cdot Z \cdot \Gamma,$$  \quad (8)

$$B = -\frac{1}{2} T \cdot \Gamma^{-1} \cdot T = -\frac{\det T}{2} \cdot \Gamma^{-1}, \quad (9)$$

where we used (6) in the last equality.
Note that from the explicit form (7) for $T$ and equality (8) by easy calculations we get the explicit form for $L$:

$$L = \begin{pmatrix} ax_1 + cx_2 & cx_1 + bx_2 \\ cx_1 + bx_2 & -ax_1 - cx_2 \end{pmatrix}.$$ 

This shows that $L^T = L$ and $\text{Tr} \ L = 0$. Thus, $L$ is uniquely determined by its first column.

The matrices $B$ and $L$ form a Lax pair of dimension 2 for (1), as shown in

**Proposition 3.1.** System (1) is equivalent to the system

$$\dot{L} = [B, L].$$

**Proof.** Notice that $\text{Tr}[B, L] = 0$ and

$$[B, L]^T = (BL - LB)^T = L^T B^T - B^T L^T = BL - LB = [B, L];$$

therefore, it is sufficient to verify (10) only for the first column of the resulting matrix on the left- and right-hand sides of this formula. Let us calculate

$$BL - LB = -\frac{1}{2} T \cdot \Gamma^{-1} \cdot T \cdot T \cdot Z + \frac{1}{2} T \cdot \Gamma^{-1} \cdot T \cdot Z \cdot \Gamma + \frac{1}{2} T \cdot Z \cdot \Gamma \cdot \Gamma^{-1} \cdot T - \frac{1}{2} \Gamma^{-1} \cdot T \cdot Z \cdot \Gamma \cdot \Gamma^{-1} \cdot T.$$

Since the multiplication of $Z$ on the left by any matrix gives some matrix with the zero second column, and the multiplication of the last matrix by the matrix $\Gamma$ or $\Gamma^{-1}$ on the right gives a matrix with the zero first column, from the rightmost equality in formula (9) we find that only the first and fourth terms of the sum affect the first column of $[B, L]$. Again by the second equality in (9) the sum of these terms is

$$-\frac{1}{2} T \cdot \Gamma^{-1} \cdot T^2 \cdot Z - \frac{\det T}{2} T \cdot \Gamma^{-1} \cdot P \cdot Z - \frac{\det T}{2} T \cdot T^{-1} \cdot \Gamma^{-1} \cdot T \cdot Z$$

$$= -\frac{1}{2} T \cdot \Gamma^{-1} \cdot P \cdot Z - \frac{1}{2} T \cdot \Gamma^{-1} \cdot T \cdot T \cdot Z = -T \cdot \Gamma^{-1} \cdot P \cdot Z = T \cdot \dot{Z},$$

where we used (1) in the last equality. □

Let us calculate $\text{Tr}(L^k)$, where $k$ is a natural. Note that, given $L = \begin{pmatrix} g & h \\ h & -g \end{pmatrix}$, we have $L^2 = (g^2 + h^2)E$, where $E$ is the identity matrix. Hence, $\text{Tr}(L^k) = 0$, if $k$ is odd, and $\text{Tr}(L^{2l}) = 2(g^2 + h^2)^l$. In our case $(g, h)^T = Tx$, where $x = (x_1, x_2)^T$, $T^2 = P$, and $T^T = T$. Therefore,

$$g^2 + h^2 = (Tx)^T \cdot Tx = x^T T^T T x = x^T T^2 x = x^T P x.$$

So, $\text{Tr}(L^{2l}) = 2(x^T P x)^l = 2(2H)^l$, where $H$ is the Hamiltonian of (1).

**3.2. Computer calculations.** We now consider the general problem of finding the Lax pairs of dimension 2 for systems of the form

$$\dot{x} = \Gamma \cdot P \cdot x,$$

where $\Gamma$ is skew-symmetric, and $P$ is an arbitrary (not necessarily symmetric) real $2 \times 2$ matrix. Without loss of generality (by changing $P$), we assume that

$$\Gamma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} p_1 & p_2 \\ p_3 & p_4 \end{pmatrix}.$$ (12)

We will look for the matrices $L$ and $B$ such that $L$ depends linearly on the coordinates $x$ in $\mathbb{R}^2$ (or $\mathbb{C}^2$), and $B$ does not depend on the coordinates $x$, and system (11) implies the system

$$\dot{L} = [B, L].$$ (13)
Let the first and second columns of $L$ be respectively $\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \cdot x$ and $\begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} \cdot x$. Assume that

$$B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}.$$

Then (13) can be rewritten as a system of 8 equations in 12 variables $a_i$, $y_i$, and $b_i$ (while $a_i$ and $y_i$ are determined up to multiplication by a constant). This system is amenable to research using a computer. In particular, it is possible to calculate its Gröbner basis. For instance, the first element of this basis that depends only on $y_3$ and $y_4$ has the form

$$p_1^2p_2p_4y_3^2 - p_1^2p_3p_4y_1^2 - p_1p_2^2p_3y_4^2 - p_1p_2^2p_4y_3y_4 + p_1p_2p_3^2y_4^2 + p_1p_2p_3y_4^3 + p_1p_3p_4y_3y_4 - p_1p_3p_4^2y_4^2 + p_2^2p_3p_4y_3^2 - p_2^2p_3p_4y_4^2.$$

(The remaining elements of the Gröbner basis, found by a computer, have a longer record.)

If we additionally assume that the matrix $P$ is symmetric, i.e., $p_2 = p_3$, then the Gröbner basis is simplified and the system has the following general solution:

$$a_1 = -y_3, \quad a_2 = -y_4, \quad a_3 = \frac{p_1y_4(y_1y_4 - 2y_2y_3) + 2p_2y_2y_3^2 - p_4y_1y_3^2}{y_2(p_1y_2 - 2p_2y_1) + p_4y_1^2},$$

$$a_4 = \frac{y_1^2(2p_2y_1 - p_1y_2) + p_4y_3(y_2y_3 - 2y_1y_4)}{y_2(p_1y_2 - 2p_2y_1) + p_4y_1^2},$$

$$b_2 = \frac{-p_1y_2^2 - 2p_2y_1y_2 + p_4y_1^2}{2(y_2y_3 - y_1y_4)}, \quad b_3 = \frac{p_1y_4^2 - 2p_2y_3y_4 + p_4y_3^2}{2(y_2y_3 - y_1y_4)},$$

$$b_1 = \frac{-b_4y_1y_4 + b_4y_2y_3 + p_1y_2y_4 - p_2y_3y_4 - p_2y_2y_3 + p_4y_1y_3}{y_2y_3 - y_1y_4},$$

dependent on the free variables $b_4$, $y_1$, $y_2$, $y_3$, and $y_4$. The matrix $L$ has the form

$$L = \begin{pmatrix} -x_1y_3 - x_2y_4 & x_1y_1 + x_2y_2 \\ q & x_1y_3 + x_2y_4 \end{pmatrix},$$

where

$$q = \frac{x_1(-p_4y_1y_3^2 + 2p_2y_2y_3^2 + p_1y_4(y_1y_4 - 2y_2y_3))}{p_4y_1^2 + y_2(p_1y_2 - 2p_2y_1)} + \frac{x_2((2p_2y_1 - p_1y_2)y_4^2 + p_4y_3(y_2y_3 - 2y_1y_4))}{p_4y_1^2 + y_2(p_1y_2 - 2p_2y_1)}.$$
4. Lax Pairs in the General Case

In this section, we construct Lax pairs of dimension 2 for an arbitrary natural $n$ for (1).

Suppose that the conditions on the matrices $\Gamma$ and $P$ are satisfied, as at the beginning of the Introduction.

Let $\lambda \in \mathbb{C}$ be an eigenvalue of the matrix $P\Gamma^{-1}$. Define the subspace $V_\lambda \subset \mathbb{C}^{2n}$ as

$$V_\lambda = \{ x \in \mathbb{C}^{2n} : (P\Gamma^{-1})^2 x = \lambda^2 x \}.$$  

We have that $\lambda \neq 0$, and from item (1) of Proposition 2.1 we get that $-\lambda$ is also an eigenvalue of $P\Gamma^{-1}$. Therefore, $\dim_{\mathbb{C}} V_\lambda \geq 2$, since the eigenvectors of $P\Gamma^{-1}$ with eigenvalues $\lambda$ and $-\lambda$ are linearly independent over $\mathbb{C}$ and belong to the subspace $V_\lambda$. Note that if the spectrum of $P\Gamma^{-1}$ is simple, then $\dim_{\mathbb{C}} V_\lambda = 2$.

Furthermore, $P\Gamma^{-1}V_\lambda = V_\lambda$, since $[P\Gamma^{-1}, (P\Gamma^{-1})^2] = 0$ and $\det(P\Gamma^{-1}) \neq 0$.

Choose an arbitrary nonzero column vector $w = (w_1, \ldots, w_{2n})^T \in V_\lambda \subset \mathbb{C}^{2n}$ such that $w$ is not an eigenvector of $P\Gamma^{-1}$. We will call such a pair $(\lambda, w)$ admissible. Define the column vector $\hat{w} = \lambda^{-1} P\Gamma^{-1} w \in V_\lambda \subset \mathbb{C}^{2n}$, where as usual $i^2 = -1$.

Put

$$a = x^T w = \sum_{1 \leq j \leq 2n} x_j w_j, \quad d = x^T \hat{w} = i\lambda^{-1} x^T P\Gamma^{-1} w = -i\lambda^{-1} (\Gamma^{-1} P x)^T w. \quad (18)$$

We introduce the matrices $B$ and $L$ of size $2 \times 2$ as

$$B = -\frac{i\lambda}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} a & d \\ d & -a \end{pmatrix}. \quad (19)$$

**Definition 4.1.** We define the quadratic function from $\mathbb{C}^{2n}$ to $\mathbb{C}$:

$$I_{\lambda, w} = \text{Tr}(L^2) = 2((x^T w)^2 + (x^T \hat{w})^2),$$

where the last equality ensues from the explicit form (19) of $L$.

It is easy that (see the end of 3.1) $\text{Tr}(L^k) = 0$, if $k$ is odd, and $\text{Tr}(L^{2l}) = 2(I_{\lambda, w}/2)^l$.

Each so constructed pair $B, L$ defines a Lax pair of dimension 2 with properties, as shown in

**Theorem 4.2.** (1) $B, L$ is a Lax pair of dimension 2 for system (1).

(2) Let $(\lambda_1, w_1)$ and $(\lambda_2, w_2)$ be admissible pairs, and assume that $\lambda_1^2 \neq \lambda_2^2$. Then $(\text{grad} I_{\lambda_1, w_1})^T \cdot \Gamma^{-1} \cdot \text{grad} I_{\lambda_2, w_2} = 0$.

(3) Let $(\lambda_j, w_j)$, where $1 \leq j \leq l \leq n$, be admissible pairs and all numbers $\lambda_j^2$ are pairwise distinct. Then the complex 1-forms $dI_{\lambda_j, w_j}$, where $1 \leq j \leq l$, are linearly independent over $\mathbb{C}$ on the complement of the union of at most $l$ linear subspaces of codimension 2 in $\mathbb{C}^{2n}$.

**Remark 4.3.** Note that if $(\lambda, w)$ is an admissible pair, then the complex conjugate pair $(\bar{\lambda}, \bar{w})$ is also an admissible pair. Moreover, $\overline{I_{\lambda, w}} = I_{\bar{\lambda}, \bar{w}}$. Therefore, for real and imaginary parts we have

$$\text{Re} I_{\lambda, w} = \frac{1}{2}(I_{\lambda, w} + I_{\bar{\lambda}, \bar{w}}), \quad \text{Im} I_{\lambda, w} = \frac{1}{2i}(I_{\lambda, w} - I_{\bar{\lambda}, \bar{w}}).$$

Therefore, from Section 2 (in particular, from formulas (5)) and items (2) and (3) of Theorem 4.2, taking, if necessary, the real and imaginary parts of the quadratic functions $I_{\lambda, w}$, we immediately obtain quadratic, functionally independent first integrals for (1), which are pairwise in involution with respect to the Poisson bracket for (1). In particular, if the spectrum of the matrix $P\Gamma^{-1}$ is simple, then we get $n$ of such first integrals. This, in particular, gives a new proof of the result from [5] (as we already noted in the Introduction).
We note here the difference in approaches: To prove the functional independence of the set of first integrals in [5] a rather cumbersome theorem was used from [7] on the classification of a pair of bilinear real forms: symmetric and skew-symmetric (the formulation of this theorem is also given in [8, Appendix 6]). In our case, we construct a set of n first quadratic integrals in an involution and give a very simple proof (see below), without the classification theorem for a pair of forms, that they are functionally independent. It follows from the general theory of Lie algebras that all families of first quadratic integrals in an involution generate the same vector space, and, consequently, the first integrals from [5] are also functionally independent. We will explain the latter in more detail.

Let M and N be two real symmetric matrices defining quadratic functions on the space \( \mathbb{R}^{2n} \). The Poisson bracket constructed from the symplectic form on \( \mathbb{R}^{2n} \) defined by the skew-symmetric matrix \(-\Gamma\), being applied to two quadratic functions will again be a quadratic function. This bracket will be rewritten on the set of symmetric matrices as follows:

\[
\{M, N\} \mapsto -2(M\Gamma^{-1}N - N\Gamma^{-1}M).
\]

Note that the mapping \( M \mapsto -2M\Gamma^{-1} \) defines an isomorphism between the Lie algebra of symmetric matrices with respect to the bracket described above, and the matrix Lie subalgebra consisting of matrices \( B \) satisfying the condition \( B^{T}\Gamma^{-1} + \Gamma^{-1}B = 0 \). The last algebra is isomorphic to the symplectic Lie algebra \( sp(2n, \mathbb{R}) \) through the mapping \( B \mapsto C^{T}B(C^{T})^{-1}, \) where \(-C^{T}\Gamma C\) is a standard symplectic unity.

If \( P/2 \) is a symmetric matrix defining a quadratic Hamiltonian and \( \Gamma^{-1} \) has a simple spectrum, then the matrix \(-C^{T}P\Gamma^{-1}(C^{T})^{-1}\) also has a simple spectrum, and therefore defines a regular element in the simple Lie algebra \( sp(2n, \mathbb{R}) \). Then the Cartan subalgebra containing this element consists of all elements commuting with it and is an abelian subalgebra of dimension \( n \). Therefore, there exists a unique real vector space of dimension \( n \) consisting of quadratic functions that are in involution with respect to the Poisson bracket and containing the Hamiltonian.

The explicit formulas, connecting our first integrals and the first integrals from [2, 5], are given in [9].

**Proof of Theorem 4.2.** (1): We note that

\[
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} a & d \\ d & -a \end{pmatrix} = \begin{pmatrix} d & -a \\ -a & -d \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & d \\ d & -a \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -d & a \\ a & d \end{pmatrix}.
\]

Therefore, \( BL, LB, \) and \([B, L]\) will be again symmetric matrices with zero trace. Consequently,

\[
[B, L] = BL - LB = BL - (LB)^T = 2BL.
\]

Hence we see that (4) is equivalent to the following system of equations:

\[
\begin{cases}
\dot{a} = -i\lambda d, \\
\dot{d} = i\lambda a.
\end{cases}
\]

(20)

From system (1) we obtain that

\[
\dot{x} = x^T \cdot w = (-\Gamma^{-1}Px)^T \cdot w, \quad \dot{\tilde{w}} = (-\Gamma^{-1}P\Gamma^{-1}w).
\]

Hence,

\[
\begin{cases}
(-\Gamma^{-1}P\Gamma^{-1}w) = -i\lambda x^T \cdot (i\lambda^{-1}P\Gamma^{-1}w), \\
(-\Gamma^{-1}P\Gamma^{-1}w) = i\lambda x^T \cdot w.
\end{cases}
\]

Therefore,

\[
\begin{cases}
x^T \Gamma^{-1}w = -i\lambda x^T \cdot (i\lambda^{-1}P\Gamma^{-1}w), \\
x^T \Gamma^{-1}w = i\lambda x^T \cdot w.
\end{cases}
\]

A similar approach using the iteration of various cases is also contained in [6].
The first equation from the last system is always satisfied. In order for the second equation to be satisfied, it suffices to prove that \((PT^{-1})^2 w = \lambda^2 w\). But we already chose \(w \in V_\lambda\) satisfying a condition.

(2): From Definition 4.1 we find

\[
I_{\lambda,w} = 2((x^T w)^2 + (\bar{x}^T \bar{w})^2) = 2(x^T w + i\bar{x}^T \bar{w}) \cdot (x^T w - i\bar{x}^T \bar{w})
= 2(x^T (E - \lambda^{-1} P T^{-1}) w) \cdot (x^T (E + \lambda^{-1} P T^{-1}) w),
\]

where \(E\) is the identity matrix. We note that from this formula we immediately obtain that \(I_{\lambda,w} \neq 0\), since \(w\) is not an eigenvector of \(PT^{-1}\). Now we calculate \(\text{grad} I_{\lambda,w}\):

\[
\text{grad} I_{\lambda,w} = 2(\text{grad}(a^2) + \text{grad}(d^2)) = 2(2a \cdot \text{grad} a + 2d \cdot \text{grad} d)
= 4((x^T w) \cdot w + (\bar{x}^T \bar{w}) \cdot \bar{w}) = 4((x^T w) \cdot E - \lambda^{-2} (x^T P T^{-1} w) \cdot P T^{-1} w)
= (g_1 \cdot E - g_2 \cdot P T^{-1}) w,
\]

where \(E\) is the identity matrix, and the linear functions \(g_1\) and \(g_2\) on \(\mathbb{C}^{2n}\) are given as

\[
g_1 = 4x^T \cdot w, \quad g_2 = 4\lambda^{-2} x^T \cdot P T^{-1} \cdot w.
\]

Now, since for every \(x \in \mathbb{C}^{2n}\) the matrix \(g_1 \cdot E - g_2 \cdot P T^{-1}\) commutes with \((PT^{-1})^2\), we see that for \(x \in \mathbb{C}^{2n}\) the vector \(I_{\lambda,w}\) is an eigenvector of \((PT^{-1})^2\) with eigenvalue \(\lambda^2\). Now item (2) of Theorem 4.2 follows from item (2) of Proposition 2.1.

(3): Since the nonzero eigenvectors corresponding to different eigenvalues are linearly independent, the statement of item (3) of Theorem 4.2 follows from the explicit calculation of the gradient \(\text{grad} I_{\lambda,w}\) performed during the proof of the previous item. Moreover, since the vector \(w\) is not an eigenvector of \(PT^{-1}\), the gradient \(\text{grad} I_{\lambda,w}\) will be equal to zero only on the intersection of the two linear complex hyperplanes in \(\mathbb{C}^{2n}\): \(g_1 = 0\) and \(g_2 = 0\). 

**Remark 4.4.** We note if we, when constructing a Lax pair, take an eigenvector \(w\) of \((PT^{-1})^2\) such that \(w\) is an eigenvector of \(PT^{-1}\), then we will infer that \(\text{Tr}(L^2) = 0\) (this follows at once from (21)). Hence we obtain the importance of the property \(\dim_{\mathbb{C}} V_\lambda \geq 2\). It is the absence of such a property that prevents us to construct a Lax pair with nonzero function \(\text{Tr}(L^2)\) for an arbitrary linear system \(\dot{x} = Ch\).

**Remark 4.5.** If the spectrum \(PT^{-1}\) is simple, we can construct a Lax pair of dimension \(2n\) such that system (4) will be equivalent to (1). To this end, it suffices to take the block diagonal matrices \(B\) and \(L\) with blocks that are matrices of size \(2 \times 2\), as in formula (19), with eigenvalues \(\lambda_i\) such that \(\lambda_i^2 \neq \lambda_j^2\) when \(i \neq j\).

Indeed, as we see from the proof of item (1) of Theorem 4.2, system (4) is equivalent to (20) with different pairs \((\lambda, w)\). But then the equations of these systems can be rewritten as the systems

\[
(\dot{x} + \Gamma^{-1} P x)^T \cdot w = 0, \quad (\dot{\bar{x}} + \Gamma^{-1} P \bar{x})^T \cdot \bar{w} = 0
\]

for \(n\) different eigenvectors \(w\). Since all these vectors \(w\) and \(\bar{w}\) are linearly independent, we obtain system (1):

\[
\dot{x} + \Gamma^{-1} P x = 0.
\]

**Remark 4.6.** We can construct Lax pairs of dimension \(2n\) in other ways. Let, for instance, the matrices \(B\) and \(L\) be from blocks of size \(n \times n\). We fix an eigenvalue \(\lambda\) of \(PT^{-1}\). We consider the following \(B\) and \(L\):

\[
B = -\frac{i\lambda}{2} \begin{pmatrix} E & 0 \\ -E & 0 \end{pmatrix}, \quad L = \begin{pmatrix} A & D \\ D & -A \end{pmatrix},
\]

where \(E\) is the identity matrix of size \(n \times n\), and \(A\) and \(D\) are symmetric matrices of size \(n \times n\). Let \(A = (a_{kl})_{1 \leq k,l \leq n}\) and \(D = (d_{kl})_{1 \leq k,l \leq n}\). Then it suffices to specify the matrix entries \(a_{k,l}\) and \(d_{k,l}\).
when \( l \geq k \). Given \( k, l \), we define \( a_{k,l} \) and \( d_{k,l} \) likewise and (see (18)) when we constructed the Lax pair. Note that \( \lambda \) must be the same, whereas the eigenvectors \( w \in V_\lambda \) can change. The function \( \text{Tr}(L^2) \) in this Lax pair is the sum of the functions \( I_{\lambda,w} \) above.

Moreover, we can write the Lax pair of dimension \( 2n \) with real entries and such that a few eigenvalues of \( \Gamma^{-1} \) are taken into account. We consider the matrix \( B \) of the form \( B = -\frac{1}{2} \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix} \), where \( J \) is the diagonal matrix with entries \( i\lambda_1, \ldots, i\lambda_n \) on the diagonal, and for all \( 1 \leq j \leq n \) the number \( \lambda_j \) is the eigenvalue of \( \Gamma^{-1} \) such that if \( \lambda_j^2 \notin \mathbb{R} \), then \( \lambda_j = \lambda_k \) for some \( k \neq j \). Now we take the matrix \( L \) as above, but with the diagonal matrices \( A \) and \( D \) such that the \( j \)th entries on the diagonals of \( A \) and \( D \) are constructed as \( a \) and \( d \) in (18) for the eigenvalue \( \lambda_j \). Then the pair \( B, L \) gives a Lax pair of dimension \( 2n \) for (1), and \( \text{Tr}(L^2) \) is the sum of the functions \( I_{\lambda_i,w} \) introduced above. Conjugating \( B \) and \( L \) by means of permutation matrices (i.e., by rearranging the vectors in the basis by some permutation), we obtain the block diagonal matrix \( B \) with blocks \(-\frac{1}{2} \begin{pmatrix} 0 & i\lambda \\ -i\lambda & 0 \end{pmatrix} \) of size \( 2 \times 2 \). If \( \lambda^2 \in \mathbb{R} \), then either the block itself, or its Jordan normal form will be real matrices. If \( \lambda^2 \notin \mathbb{R} \), then two of these blocks with complex conjugate numbers \( \lambda = a \pm bi \) have all different eigenvalues, and these eigenvalues are the same as the eigenvalues of the real matrix of size \( 4 \times 4 \) which can be written as the block matrix

\[
\begin{pmatrix}
0 & N_1 \\
N_2 & 0
\end{pmatrix}, \quad \text{where } N_1 = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \text{ and } N_2 = \begin{pmatrix} a & -b \\ -b & -a \end{pmatrix}.
\]

Therefore we can make a suitable replacement of the basis, and the matrix \( B \) will be real. Then the resulting Lax pair can be divided into the two real ones: with the real and imaginary parts of the matrix \( L \). Since the trace does not change under the conjugation of a matrix, we obtain the same integrals as for the original Lax pair (the real and imaginary parts of the trace of \( L^2 \)).

**Example 4.7.** In the case \( n = 1 \), we can always “normalize” (i.e., multiple by a complex number) the vector \( w \) from an admissible pair \((\lambda, w)\) in such a way that \( I_{\lambda,w} = 4H \), where \( H \) is the Hamiltonian of (1). Namely, \( w \) must satisfy a condition:

\[
w^T \Gamma \Gamma^{-1} w = -\lambda^2 \cdot \det \Gamma = \det P = p_1 p_4 - p_2^2, \tag{22}
\]

where the entries of \( P \) are denoted as on the right-hand side of (12) with the additional condition \( p_4 = p_2 \). (Since \( \Gamma^{-1} w \) is not proportional to the vector \( w \), the left-hand side of formula (22) is not equal to zero.)

In this case, by simple direct calculations we obtain

\[
a^2 = (w_1 x_1 + w_2 x_2)^2, \quad d^2 = \frac{(w_2 (p_1 x_1 + p_2 x_2) + w_1 (-p_2 x_1 - p_4 x_2))^2}{(p_1 p_4 - p_2^2)},
\]

\[
I_{\lambda,w} = 2(a^2 + d^2) = 2(x_1^2(p_1 p_4 w_1^2 - 2p_1 p_2 w_1 w_2 + p_1^2 w_2^2) + 2x_1 x_2 (p_2 p_4 w_1^2 - 2p_2^2 w_1 w_2 + p_1 p_2 w_2^2) + x_2^2(p_4^2 w_2^2 - 2p_2 p_4 w_1 w_2 + p_1 p_4 w_1^2))/(p_1 p_4 - p_2^2) = 4H.
\]

We note also that the condition in (22) can be rewritten as \( w^T P^{-1} w = 1 \) by means of identity (6) (where \( T \) must be replaced by \( P \)).

Moreover, in the notation of Subsection 3.2, the corresponding Lax pair from Theorem 4.2 is obtained if we take \( b_4 = 0 \), \( (y_3, y_4)^T = -w \), \((y_1, y_2)^T = \hat{w} \).

**Example 4.8.** In the case \( n = 2 \) we consider, for instance, the following matrices:

\[
P = \begin{pmatrix}
0 & a & 0 & b \\
a & 0 & -b & 0 \\
0 & -b & 0 & a \\
b & 0 & b & 0
\end{pmatrix}, \quad \Gamma = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{pmatrix}.
\]
We define, as in the first part of Remark 4.6,

\[
B = \begin{pmatrix}
0 & 0 & h & 0 \\
0 & 0 & 0 & h \\
h & 0 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix}, \quad L = \begin{pmatrix}
x^Tw & 0 & x^T\hat{w} & 0 \\
0 & -x^T\hat{w} & 0 & x^Tw \\
x^T\hat{w} & 0 & -x^Tw & 0 \\
0 & x^Tw & 0 & x^Tw
\end{pmatrix},
\]

where \( h = -i\lambda/2 = -\frac{1}{2}(-b + ai) \), \( \lambda = a + bi \), and the two eigenvectors with eigenvalue \( \lambda^2 \) for the matrix \( (PT^{-1})^2 \) are chosen: the first is \( w = (1, 1, i, -i)^T \) and the second is \( \hat{w} = -(i, i, 1, 1)^T \).

We have \( \hat{w} = (i, -i, -1, 1)^T \) and \( \hat{\hat{w}} = w \). Then \( B \) and \( L \) form a Lax pair for system (1). We define \( H_1 = x_1x_2 + x_3x_4 \) and \( H_2 = x_1x_4 - x_2x_3 \). Then it is not difficult to calculate that

\[
16H_1 = I_{\lambda,w} + \hat{I}_{\lambda,w}, \quad 16H_2 = i(I_{\lambda,w} - \hat{I}_{\lambda,w}).
\]

Hence, \( \text{Tr}(L^2) = 4((x^Tw)^2 + (x^T\hat{w})^2) = 2I_{\lambda,w} = 16H_1 - 16H_2i \).

5. The Mappings in the Construction of Lax Pairs Are Poisson

In this section, we show that on the image of the mapping arising in Lax pairs from Section 4 there is a symplectic structure such that the matrix equation of the Lax pair is Hamiltonian, and the mapping itself that appears in the construction of the Lax pair is Poisson; i.e. it preserves the Poisson brackets, where on \( \mathbb{R}^{2n} \) there is the Poisson bracket induced by (1); see Section 2.

By an admissible pair \((\lambda, w)\) such that \( w \in \mathbb{R}^{2n} \) if \( \lambda^2 \in \mathbb{R} \), and by the Lax pair from Theorem 4.2 we define the mapping

\[
\Phi_{\lambda,w} : \mathbb{R}^{2n} \rightarrow \mathbb{C}e_1 \oplus \mathbb{C}e_3, \quad x \mapsto a(x) \cdot e_1 + d(x) \cdot e_3.
\]

Depending on the three cases: 1) \( \lambda \in i\mathbb{R} \); 2) \( \lambda \in \mathbb{R} \); and 3) \( \lambda \in \mathbb{C}, \lambda^2 \notin \mathbb{R} \), this mapping defines some mappings of real spaces for which we keep the same notation:

\[
\begin{align*}
\Phi_{\lambda,w} : \mathbb{R}^{2n} &\rightarrow \mathbb{R}e_1 \oplus \mathbb{R}e_3, \\
\Phi_{\lambda,w} : \mathbb{R}^{2n} &\rightarrow \mathbb{R}e_1 \oplus \mathbb{R}ie_3, \\
\Phi_{\lambda,w} : \mathbb{R}^{2n} &\rightarrow \mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \mathbb{R}e_3 \oplus \mathbb{R}e_4,
\end{align*}
\]

where \( w = w_1 + iw_2, \hat{w} = \hat{w}_1 + i\hat{w}_2, e_2 = ie_1, \) and \( e_4 = ie_3 \).

**Theorem 5.1.** Let the spectrum of \( PT^{-1} \) be simple. We choose an admissible pair \((\lambda, w)\) such that \( w \in \mathbb{R}^{2n} \) if \( \lambda^2 \in \mathbb{R} \). Then on the image of the mapping \( \Phi_{\lambda,w} \) there is a natural symplectic structure such that the system obtained from the Lax pair is Hamiltonian with respect to this structure, and \( \Phi_{\lambda,w} \) is Poisson.

**Proof.** We define

\[
K_{\lambda,w} = -w^T\Gamma^{-1}w = -i\lambda^{-1}w^T\Gamma^{-1}PT^{-1}w = -i\lambda w^TP^{-1}w.
\]

From the conditions of the theorem and (3) of Proposition 2.1 it follows that \( K_{\lambda,w} \neq 0 \).

**Case 1.** Let \( z_1 \) and \( z_2 \) be the real coordinates on the vectors \( e_1 \) and \( e_3 \) correspondingly. To define the Poisson bracket \( \{\cdot, \cdot\} \) on smooth functions on the space \( \mathbb{R}(e_1, e_3) \), so that \( \Phi_{\lambda,w} \) will be a morphism of Poisson manifolds, we put

\[
\{z_1, z_2\} = \{z_3, z_2\} = 0, \quad \{z_1, z_3\} = -\{z_3, z_1\} = \{\Phi^*_{\lambda,w}(z_1), \Phi^*_{\lambda,w}(z_3)\} = \{x^Tw, x^T\hat{w}\} = -w\Gamma^{-1}\hat{w} = K_{\lambda,w},
\]

where for functions on \( \mathbb{R}^{2n} \) the Poisson bracket is given by the skew-symmetric matrix \( -\Gamma \), as for (1).
Now the nondegenerate skew-symmetric matrix
\[ \Gamma_{\lambda,w} = \begin{pmatrix} 0 & -K_{\lambda,w}^{-1} \\ K_{\lambda,w}^{-1} & 0 \end{pmatrix} \]
defines the symplectic structure on \( \mathbb{R} \langle e_1, e_3 \rangle \) such that the corresponding Poisson brackets (see Section 2) will be calculated by formulas (24)–(25) implying that for smooth functions \( h_1 \) and \( h_2 \) on \( \mathbb{R} \langle e_1, e_3 \rangle \) we have
\[
\{ h_1, h_2 \} = K_{\lambda,w} \frac{\partial h_1}{\partial z_1} \frac{\partial h_2}{\partial z_3} - K_{\lambda,w} \frac{\partial h_1}{\partial z_3} \frac{\partial h_2}{\partial z_1}.
\]
In result,
\[
\{ h_1, h_2 \} = \{ \Phi_{\lambda,w}^*(h_1), \Phi_{\lambda,w}^*(h_2) \}.
\]
We recall that the matrix equation defined by system (4) is equivalent to (20); i.e., to the system
\[
\begin{cases}
\dot{z}_1 = -i\lambda z_3, \\
\dot{z}_3 = i\lambda z_1,
\end{cases}
\]
which is clearly equivalent to the system
\[
\Gamma_{\lambda,w}(\dot{z}_1, \dot{z}_3)^T = P_{\lambda,w}(z_1, z_3)^T,
\]
where \( P_{\lambda,w} \) is the scalar matrix with the reals \( i\lambda K_{\lambda,w}^{-1} \) on the diagonal. From Section 2 we obtain at once that this system is Hamiltonian with respect to the symplectic form given by the skew-symmetric matrix \( \Gamma_{\lambda,w} \).

**Case 2.** Let \( z_1 \) and \( z_4 \) be the real coordinates on the vectors \( e_1 \) and \( ie_3 \) correspondingly. We define
\[
\begin{cases}
\{ z_1, z_4 \} = \{ z_1, z_4 \} = 0, \\
\{ z_1, z_4 \} = \{ \Phi_{\lambda,w}^*(z_1), \Phi_{\lambda,w}^*(z_4) \} = \{ x^T w, i^{-1} x^T \hat{w} \} = iw^T \Gamma^{-1} \hat{w} = -iK_{\lambda,w}.
\end{cases}
\]
Now the nondegenerate skew-symmetric matrix
\[ \tilde{\Gamma}_{\lambda,w} = \begin{pmatrix} 0 & -iK_{\lambda,w}^{-1} \\ iK_{\lambda,w}^{-1} & 0 \end{pmatrix} \]
gives the symplectic structure on \( \mathbb{R} \langle e_1, ie_3 \rangle \), so that for every two smooth functions \( h_1 \) and \( h_2 \) on \( \mathbb{R} \langle e_1, ie_3 \rangle \) we have
\[
\{ h_1, h_2 \} = \{ \Phi_{\lambda,w}^*(h_1), \Phi_{\lambda,w}^*(h_2) \}.
\]
Now the matrix equation defined by the Lax pair is equivalent to (20), and in our case (20) can be rewritten as
\[
\begin{cases}
\dot{z}_1 = \lambda z_4, \\
\dot{z}_4 = \lambda z_1;
\end{cases}
\]
i.e., (20) is equivalent to the system
\[
\tilde{\Gamma}_{\lambda,w}(\dot{z}_1, \dot{z}_4)^T = \tilde{P}_{\lambda,w}(z_1, z_4)^T,
\]
where \( \tilde{P}_{\lambda,w} \) is the diagonal matrix with the reals \( -i\lambda K_{\lambda,w}^{-1} \) and \( i\lambda K_{\lambda,w}^{-1} \) on the diagonal. The last system is Hamiltonian with respect to the symplectic form given by the skew-symmetric matrix \( \tilde{\Gamma}_{\lambda,w} \).

**Case 3.** Let \( z_k \) be the real coordinate on the vector \( e_k \), where \( 1 \leq k \leq 4 \). Then
\[
\Phi_{\lambda,w}^*(z_1) = x^T w_1, \quad \Phi_{\lambda,w}^*(z_2) = x^T w_2, \quad \Phi_{\lambda,w}^*(z_3) = x^T \hat{w}_1, \quad \Phi_{\lambda,w}^*(z_4) = x^T \hat{w}_2.
\]
We define the Poisson brackets
\[ \{z_1, z_2\} = -\{z_2, z_1\} = w_1^T(-\Gamma^{-1})w_2 = \left(-\frac{w + \bar{w}}{2}\right)^T \Gamma^{-1} \cdot \frac{w - \bar{w}}{2i} = \frac{i}{4}(w^T \Gamma^{-1}w - \bar{w}^T \Gamma^{-1}\bar{w} - 2w^T \Gamma^{-1}\bar{w}) = 0, \]
where we used that \(w^T \Gamma^{-1}\bar{w} = 0\) by item (2) of Proposition 2.1, because \(\bar{w}\) is an eigenvector with the eigenvalue \(\lambda^2\) of \((\Gamma \Gamma^{-1})^2\), and \(X^2 \neq \lambda^2\).

By similar calculations we obtain \(\{z_3, z_4\} = -\{z_4, z_3\} = 0\), since \(\hat{w}\) and \(\bar{w}\) are eigenvectors of \((\Gamma \Gamma^{-1})^2\) with eigenvalues \(\lambda^2\) and \(\bar{\lambda}^2\) correspondingly. We calculate now
\[ \{z_1, z_3\} = -w_1 \Gamma^{-1}\hat{w}_1 = -\left(\frac{w + \bar{w}}{2}\right)^T \Gamma^{-1} \cdot \frac{\hat{w} + \bar{w}}{2} = -\frac{1}{4}(w^T \Gamma^{-1}\hat{w} + \bar{w}^T \Gamma^{-1}\bar{w}) = \frac{1}{4}(K_{\lambda, w} + \bar{K}_{\lambda, w}), \]
where we used that \(\bar{w}^T \Gamma^{-1}\hat{w} = 0 = w^T \Gamma^{-1}\bar{w}\) by item (2) of Proposition 2.1. Similarly, we calculate
\[ \{z_2, z_4\} = -w_2 \Gamma^{-1}\hat{w}_2 = -\left(\frac{w - \bar{w}}{2i}\right)^T \Gamma^{-1} \cdot \frac{\hat{w} - \bar{w}}{2i} = \frac{1}{4}(w^T \Gamma^{-1}\hat{w} + \bar{w}^T \Gamma^{-1}\bar{w}) = \frac{1}{4}(K_{\lambda, w} + \bar{K}_{\lambda, w}) = -\{z_1, z_3\}, \]
\[ \{z_1, z_4\} = -w_1^T \Gamma^{-1}\hat{w}_1 = -\left(\frac{w + \bar{w}}{2}\right)^T \Gamma^{-1} \cdot \frac{\hat{w} + \bar{w}}{2i} = \frac{i}{4}(w^T \Gamma^{-1}\hat{w} - \bar{w}^T \Gamma^{-1}\bar{w}) = \frac{i}{4}(-K_{\lambda, w} + \bar{K}_{\lambda, w}), \]
\[ \{z_2, z_3\} = -w_2^T \Gamma^{-1}\hat{w}_2 = -\left(\frac{w - \bar{w}}{2i}\right)^T \Gamma^{-1} \cdot \frac{\hat{w} - \bar{w}}{2i} = \frac{i}{4}(w^T \Gamma^{-1}\hat{w} - \bar{w}^T \Gamma^{-1}\bar{w}) = \frac{i}{4}(-K_{\lambda, w} + \bar{K}_{\lambda, w}). \]

Thus, the matrix of the pairwise Poisson brackets between the coordinate functions \(z_1, \ldots, z_4\) is the following \(4 \times 4\) real skew-symmetric matrix, which we write as a block matrix with blocks of size \(2 \times 2\):
\[ Y = \begin{pmatrix} 0 & R \\ -R & 0 \end{pmatrix}, \]
where \(R\) is a real symmetric matrix of size \(2 \times 2\) with zero trace. We have
\[ \det R = \frac{1}{16}(-K_{\lambda, w} + \bar{K}_{\lambda, w})^2 + (-K_{\lambda, w} + \bar{K}_{\lambda, w})^2 = \frac{1}{16}((-2K_{\lambda, w})(2\bar{K}_{\lambda, w})) = -\frac{1}{4}|K_{\lambda, w}|^2 \neq 0. \]

We note that \(R^{-1}\) is again a symmetric matrix with zero trace. Now, from the calculations performed, it follows that the nondegenerate skew-symmetric matrix
\[ Y^{-1} = \begin{pmatrix} 0 & -R^{-1} \\ R^{-1} & 0 \end{pmatrix} \]
defines the symplectic structure on the space \(\mathbb{R}^4\), so that the mapping \(\Phi_{\lambda, w}\) will be Poisson with respect to this symplectic structure and the symplectic structure for the system (1) (see Section 2), i.e., the inverse image \(\Phi_{\lambda, w}^*\) preserves the Poisson bracket on functions.
Let \( \lambda = \alpha_1 + i\alpha_2 \), where \( \alpha_1 \) and \( \alpha_2 \) are reals. Then the matrix equation given by (4) is equivalent to (20), i.e. to the system
\[
\begin{align*}
\dot{z}_1 + i\dot{z}_2 &= (\alpha_2 - i\alpha_1)(z_3 + iz_4), \\
\dot{z}_3 + i\dot{z}_4 &= (\alpha_2 + i\alpha_1)(z_1 + iz_2).
\end{align*}
\]

The last system is equivalent to the following:
\[
(\dot{z}_1, \dot{z}_2, \dot{z}_3, \dot{z}_4)^T = C(z_1, z_2, z_3, z_4)^T, 
\]
where the block matrix \( C \) of size \( 4 \times 4 \) is
\[
C = \begin{pmatrix} 0 & F \\ -F & 0 \end{pmatrix}, \quad F = \begin{pmatrix} \alpha_2 & \alpha_1 \\ -\alpha_1 & \alpha_2 \end{pmatrix}
\]
Since the product of a symmetric matrix of size \( 2 \times 2 \) with zero trace on \( F \) is again a symmetric matrix with zero trace, the matrix \( Q = Y^{-1}C \) is a symmetric block diagonal matrix of the form
\[
Q = \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix},
\]
where \( M \) is a real symmetric matrix of size \( 2 \times 2 \) with zero trace, and \( \det M \neq 0 \). Therefore, (26) is equivalent to system
\[
-Y^{-1}(\dot{z}_1, \dot{z}_2, \dot{z}_3, \dot{z}_4)^T = -Q(z_1, z_2, z_3, z_4)^T,
\]
which is Hamiltonian with the symplectic structure defined by the skew-symmetric matrix \( Y^{-1} \) by Section 2. \( \square \)

From Theorem 5.1 and item (2) of Proposition 2.1 we obtain

**Corollary 5.2.** Under conditions of Theorem 5.1, we consider several admissible pairs \((\lambda_j, w_j)\), where \( 1 \leq j \leq l \), such that \( \lambda_j^2 \neq \lambda_{j_2}^2 \) and \( \lambda_{j_1}^2 \neq \lambda_{j_2}^2 \) for all naturals \( 1 \leq j_1 < j_2 \leq l \). We define on the image of \( \prod_{j=1}^l \Phi_{\lambda_j, w_j} \) the block diagonal symplectic structure, where each block corresponds to the symplectic structure for the pair \((\lambda_j, w_j)\) from Theorem 5.1. Then \( \prod_{j=1}^l \Phi_{\lambda_j, w_j} \) is Poisson.

We are grateful to A. N. Parshin for posing the problem and numerous discussions. The starting point of our research was the report by V. V. Kozlov “The Symplectic Geometry of Linear Hamiltonian Systems and the Solution of Algebraic Equations” at the joint seminar of the Departments of Algebra and Algebraic Geometry at the Steklov Mathematical Institute in September 2017.

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