Intense quasiperiodic beam dynamics investigation in accelerating system

D A Ovsyannikov¹, I D Rubtsova¹ and N V Lomonosova¹

¹St. Petersburg State University, 7/9 Universitetskaya nab., St. Petersburg, 199034, Russia

E-mail: d.a.ovsyannikov@spbu.ru, i.ribtsova@spbu.ru, st032429@student.spbu.ru

Abstract. The paper is devoted to intense quasiperiodic beam dynamics investigation on the basis of integro-differential mathematical model. Space charge density representation by trigonometric polynomial makes it possible to obtain Coulomb field analytical expression in the same form. Space charge field is presented by the integral over particle phase states domain. Beam quality criteria are suggested in the form of integral functionals. Beam dynamics optimization problem is formalized as trajectory ensemble control problem. Analytical expression of objective functional variation is obtained; it makes possible directed optimization methods using. Numerical simulation and optimization results are presented for linear waveguide accelerator.

1. Introduction

Main topics discussed in this paper are as follows: intense beam dynamics mathematical modeling with quasiperiodicity account; formalization and investigation of beam dynamics optimization problem; numerical results. We introduce further development of the research presented in [1-4].

Our study is connected with self-consistent particle distributions modeling [1-9]. Aiming at mathematical model formulating, we use Coulomb field analytical expression; such an approach is applied in [1-5,8,9]. Trigonometric polynomials provide the convenient instrument for particle interaction description and investigation [1-3,9]. Space charge field integral representation makes it possible to introduce integro-differential beam dynamics model [1-5,9-11].

In view of Coulomb field account, independent variable in beam dynamics equations is the analogue of time; so beam dynamics quality criteria formalization presents the special problem [8,11].

Using the approach given in [5,10,12,13] we formulate the problem of beam dynamics optimization as trajectory ensemble control problem and obtain analytical expression for quality functional variation. It makes possible to apply directed optimization methods. Such an approach is successfully applied for solving different beam dynamics optimization problems [2,4,11,14-17].

Numerical results are presented for travelling-wave linear accelerator.

2. Beam dynamics equations

Consider intense beam longitudinal dynamics model in some accelerating system:

\[
\frac{dz_i}{d\tau} = p_i(1 + p_i^2)^{-1/2}, \quad \frac{dp_i}{d\tau} = \zeta c^{-2} E^{(RF)}(\tau, z_i, u_i) + |\zeta| c^{-2} E^{(int)}(\tau, z_i, p_i, z_c, p_c), \quad i = \overline{1,N},
\]

(1)
where \( z_c = N^{-1} \sum_{i=1}^{N} z_i, \) \( p_c = N^{-1} \sum_{i=1}^{N} p_i, \) with initial conditions \( z_i(0) = z_{0i}, \) \( p_i(0) = p_{0i}, \) \( i = 1, N. \) \( z \) 

Here \( = ct, c \) is the speed of light, \( t \) is the time; \( z \) is longitudinal coordinate; \( p \) is reduced impulse; \( (z_i, p_i) \) is \( i \)-th particle phase vector; \( (z_c, p_c) \) presents bunch center phase state; \( \zeta \) is particle specific charge; \( E^{(RF)}, E^{(int)} \) characterize respectively the effect of RF field and space charge field on model particle; \( \mathbf{u}(\tau) \) is program control.

3. Integro-differential beam dynamics model

Let us generalize beam dynamics model (1)-(2). For Coulomb field modeling we approximate particle density by trigonometric polynomial and derive the intensity \( E^{(int)}(z_i, z_c, p_c) \) expression in the same form [3-9]. After the transformation it is represented in the form: \( E^{(int)}(z_i, z_c, p_c) = N^{-1} \sum_{i=1}^{N} V(z_i, z_c, p_c, z_n), \) where the function \( V(z_i, z_c, p_c, z_n) \) is determined by particle interaction account method. Applying the law of large numbers we introduce mathematical model of Coulomb field intensity:

\[
\hat{E}^{(int)}(z_i, z_c, p_c) = \int_{M_{t,u}} V(z_i, z_c, p_c, z_n) \rho(\tau, z_n, p_n) d\tau d p_n, \tag{3}
\]

where \( M_{t,u} \) is particle phase states domain, \( \rho(\tau, z, p) \) is phase density. The analogic integral formulae is used for average phase coordinates. Beam dynamics model is generalized with due account of (3).

Consider dynamic controlled process described by the equations [1,3]:

\[
\frac{dx}{d\tau} = f(\tau, x, x_c, u) = f_1(\tau, x, u) + \int_{M_{t,u}} f_2(\tau, x, x_c, y_\tau) \rho(\tau, y_\tau) dy_\tau, \tag{4}
\]

\[
\frac{d\rho(\tau,x)}{d\tau} + \frac{\partial \rho(\tau,x)}{\partial x} f(\tau, x, x_c, u) + \rho(\tau, x) div_x f(\tau, x, x_c, u) = 0, \tag{5}
\]

where \( x_\tau(\tau) = \int_{M_{t,u}} y_\tau \, \rho(\tau, y_\tau) dy_\tau, \) with initial conditions

\[
x(0) = x_0 \in M_0, \, \rho(0, x) = \rho_0(x). \tag{6}
\]

Here \( \tau \in [0, T] \) is independent variable; \( T \) is fixed; \( x \) is phase vector; \( \mathbf{u}(\tau) \) is a control; vector function \( f_1 \) is determined by external field representation; vector function \( f_2 \) is determined by the method of particle interaction account; \( \rho(\tau, x) \) is phase density defined on system (4) trajectories; \( M_0 \) is initial phase domain; \( \rho_0(x) \) is initial phase density; \( M_{t,u} = \{ x_\tau = x(\tau, x_0, u): x_0 \in M_0 \}. \) All the functions in model (4)-(6) are supposed to be rather smooth to use mathematical optimization methods [5]. Control function \( \mathbf{u}(\tau) \) is supposed to be piecewise-continuous and taking values in a compact.

4. Trajectory ensemble control problem

Let us give the examples of beam dynamics quality criteria and suggest control problem formulation on the basis of their generalization.

The criterion characterizing beam phase spread at device cross-section \( z = \xi \) is as follows:

\[
K_1(\xi, \mathbf{u}) = \bar{N}^{-1} \sum_{i=1}^{\bar{N}} (\varphi(\tau_i(\xi), \xi, \mathbf{u}) - \bar{\varphi}(\xi, \mathbf{u}))^2. \tag{7}
\]

Here \( \bar{N} \) is the number of particles crossing the section \( z = \xi; \varphi(\tau, z, \mathbf{u}) \) is particle phase; \( \tau_i(\xi) = \xi; \) \( \bar{\varphi}(\xi, \mathbf{u}) = \bar{N}^{-1} \sum_{i=1}^{\bar{N}} \varphi(\tau_i(\xi), \xi, \mathbf{u}) \) is averaged phase at cross-section \( z = \xi \).

Let us introduce integral functional corresponding the criterion (7):

\[
J_1(\xi, \mathbf{u}) = \int_0^T \int_{M_{t,u}} (\varphi(\tau, z, \mathbf{u}) - \bar{\varphi}(\xi, \mathbf{u}))^2 \delta_e(z, \xi) \rho(\tau, z, p_n) d\tau d z d p_n d \tau ,
\]

\[
\bar{\varphi}(\xi, \mathbf{u}) = \int_0^T \int_{M_{t,u}} \varphi(\tau, z, \mathbf{u}) \delta_e(z, \xi) \rho(\tau, z, p_n) d\tau d z d p_n d \tau.
\]

The value of functional \( \bar{\varphi}(\xi, \mathbf{u}) \) presents average phase at \( z = \xi \) cross-section; \( \delta_e(z) \) is smooth approximation of Dirac delta function \( \delta(z). \)
To estimate phase scatter on the segment \([a,b]\) the following functional may be used:

\[
I_\tau(u, a, b) = \int_0^T \int_{M_{r,u}} \left( \varphi(\tau, z, u) - \bar{\varphi}(z, u) \right)^2 \Pi_\varepsilon(z, a, b) \rho(\tau, z, p_\tau) dz dt d\tau
\]

Here \( \Pi_\varepsilon(z, a, b) \) is smooth approximation of the function \( U(z - a)U(b - z) \), where \( U(z) \) is Heaviside function. The similar expressions may be introduced to estimate beam energy spread.

Generalizing the examples given we introduce quality criterion of dynamic control process (4)-(6):

\[
I(u) = \int_0^T \int_{M_{r,u}} \Phi(\tau, x, u, \Lambda(x, u)) \rho(\tau, x, t) dx dt, \quad (8)
\]

\[
\Lambda(x, u) = \int_0^T \int_{M_{r,u}} \lambda(t, v, u, x) \rho(t, v) dv dt.
\]

Let us consider the problem of objective functional (8) minimization with respect to control \( u \). Assume \( \Phi(\tau, x, u, \Lambda) \) and \( \lambda(t, v, u, x) \) to be smooth functions. It makes possible mathematical methods [5,10,12] applying and obtaining analytical expression for quality functional variation.

5. Quality functional variation

Objective functional (8) variation is derived on the basis of the approach [5]:

\[
\delta I(u, \Delta u) = \int_0^T \int_{M_{r,u}} \left( \Delta_u \Phi(\tau, x, u, \Lambda(x, u)) - \Psi^*(\tau, x, t) \frac{\partial \Lambda}{\partial \Lambda} \right) \rho(\tau, x, t) dx dt + \frac{\partial \Phi(\tau, x, u, \Lambda)}{\partial \lambda} \int_0^T \int_{M_{r,u}} \Delta_u \lambda(t, v, u, x) \rho(t, v) dv dt \right) \rho(\tau, x, t) dx dt. \quad (9)
\]

Here \( \Delta u \) is control \( u \) variation; \( \Delta_u \) designates the increment of any function with respect to argument \( u \) only; vector functions \( \Psi(\tau, x) \) satisfies on the trajectories of dynamic process (4)-(6) the auxiliary system of integro-differential equations

\[
\frac{d\Psi(\tau, x)}{d\tau} = \left( \frac{\partial \Phi(\tau, x, u, \Lambda(x, u))}{\partial x} \right)^* + \frac{\partial \Phi(\tau, x, u, \Lambda(x, u))}{\partial \lambda} \frac{\partial \Lambda(x, u)}{\partial x}
\]

\[
+ \int_0^T \int_{M_{r,u}} \frac{\partial \Phi(\tau, x, u, \Lambda(x, u))}{\partial \lambda} \frac{\partial \Lambda(x, u)}{\partial x} \rho(t, x, t, y) dy dt - \frac{\partial \Phi(\tau, x, u, \Lambda(x, u))}{\partial \lambda} \frac{\partial \Lambda(x, u)}{\partial x} \Psi(\tau, x, t)
\]

\[
- \int_{M_{r,u}} \left( \frac{\partial \Phi(\tau, x, u, \Lambda(x, u))}{\partial x} \right)^* \rho(t, x, t, y) dy dt + \left( \frac{\partial \Phi(\tau, x, u, \Lambda(x, u))}{\partial x} \right)^* \Psi(\tau, x, t) \rho(t, x, t, y) dy dt\]

with the following conditions at \( \tau = T \):

\[
\Psi(T, x(T)) = 0.
\]

Note that * denotes transposition.

The analytical representation (9) of quality criterion variation provides the possibility of directed optimization methods applying in beam dynamics optimization problems. It may be beneficial to combine gradient optimization with some Monte Carlo method of global extremum search [18-20].

6. Numerical results

The approach presented is applied for beam dynamics investigation in linear waveguide accelerator with injection energy 40 keV, accelerating wavelength 0.1 m, accelerator length 0.78 m. Numerical experiments were performed for beam current 5A, model particles number 64 [3].

Figure 1 presents the adequate approximation of bunch charge density by trigonometric polynomial. One can detect bunching process development along the structure.

At figure 2 one can see Coulomb field intensity distribution along quasiperiod for the middle and the end of the accelerating tract. The intensity reduce is caused by particle velocity growth.

The objectives of beam dynamics optimization are as follows: obtaining average reduced energy \( \bar{\gamma} = 10 \) at accelerator exit; phase spread minimization during beam movement; capture coefficient maximization. Some beam characteristics for initial (\( u_0 \)) and optimized (\( \bar{u} \)) controls are presented in the table 1.
Figure 1. Piecewise-constant charge density $\tilde{S}$ and trigonometric polynomial $S$ in bunching part (a) and accelerating part (b) of the structure

Figure 2. Space charge field intensity distribution along bunch length in the middle (a) and near the output section (b) of the accelerator

Table 1. Main beam characteristics obtained for initial ($u_0$) and optimized ($\tilde{u}$) controls

|                         | $u_0$  | $\tilde{u}$ |
|-------------------------|--------|-------------|
| Phase spread $\Delta \varphi_{\text{exit}}$ at device exit (rad) | 0.400  | 0.389       |
| Average reduced energy $\gamma_{\text{exit}}$ at device exit | 9.462  | 10.311      |
| Energy spread $\Delta \gamma_{\text{exit}}$ at device exit | 4.135  | 3.525       |
| Capture coefficient $K_c$ | 0.922  | 0.938       |

References
[1] Rubtsova I D 2016 On modeling and optimization of intense quasiperiodic beam dynamics Proc. XXV Russian Particle Accelerator Conf. RuPAC-2016 (21-25 November 2016 St. Petersburg)(Geneva: JACoW) http://www.JACoW.org pp 363-66
[2] Rubtsova I D 2016 Analytical Approach to Quasiperiodic Beam Coulomb Field Modeling, II Conference on Plasma&Laser Research and Technologies (2016), Journal of Physics: Conference Series 747 No 1 012074 http://iopscience.iop.org/1742-6596/747/1/012074
[3] Rubtsova I D, Lomonosova N V and Chuprynina T A 2017 Investigation of quasiperiodic intense beam longitudinal dynamics in linear waveguide accelerator Vestnik St. Petersburg State University of Technology and Design, Ser. 1: Natural and technical sciences 3, pp 15-23
[4] Rubtsova I D and Ovsyannikov D A 2018 Intense quasiperiodic beam dynamics in accelerating system: mathematical model and optimization method III International Conference on Laser and Plasma Researches and Technologies (2017), Journal of Physics: Conference Series
941 No 1 012092 http://iopscience.iop.org/article/10.1088/1742-6596/941/1/012092

[5] Ovsyannikov D A 1990 Modeling and Optimization of Charged Particle Beam Dynamics (Leningrad: Leningrad State University Press) p 312

[6] Drivotin O I and Ovsyannikov D A 2009 Self-consistent distributions for charged particle beam in magnetic field Int. J. Mod. Phys. A 24 (5) pp 816-42

[7] Bondarenko T V Polozov S M and Sumbaev A P 2016 Numerical simulation of the beam loading effect at the LUE-200 accelerator Physics of Particles and Nuclei Letters 13 (7), pp 919-22

[8] Rubtsova I D and Suddenko E N 2012 Investigation of program and perturbed motions of particles in linear accelerator Proc. XXIII Russian Particle Accelerator Conf. RuPAC-2012 (24-28 September 2012 St. Petersburg)(Geneva: JACoW http://www.JACoW.org) pp 367-69

[9] Rubtsova I D 2014 Integral-differential model of quasi-periodic beam longitudinal dynamics Proc. 20th Int. Workshop: Beam Dynamics & Optimization (BDO) (30 June-4 July 2014 St. Petersburg) (St. Petersburg: IEEE) p 144

[10] Ovsyannikov D A 2012 Mathematical modeling and optimization of beam dynamics in accelerators Proc. XXIII Russian Particle Accelerator Conf. RuPAC-2012 (24-28 September 2012 St. Petersburg)(Geneva: JACoW http://www.JACoW.org) pp 68-72

[11] Rubtsova I D 2014 Mathematical optimization model of longitudinal beam dynamics in klystron-type buncher Proc. XXIV Russian Particle Accelerator Conf. RuPAC-2014 (6-10 October 2014 Obninsk)(Geneva: JACoW http://www.JACoW.org) pp 66-68

[12] Ovsyannikov D A 1997 Modeling and Optimization Problems of Charged Particle Beam Dynamics Proc. European Control Conf. ECC’97 (1-4 July 1997 Brussels Belgium) 4 pp 1463-67

[13] Ovsyannikov D A, Ovsyannikov A D, Vorogushin M F, Svishtunov Yu A and Durkin A P 2006 Beam dynamics optimization: models, methods and applications Nuclear Instruments and Methods in Physics Research A 558 (1) 11-19

[14] Rubtsova I D 2015 Optimization of iterative beam dynamic process Proc. III Int. Conf. “Stability and Control Processes” in Memory of V.I.Zubov (SCP) (5-9 October 2015 St. Petersburg) ed L A Petrosyan and A P Zhabko (St. Petersburg: IEEE) pp 198-200 DOI: 10.1109/SCP.2015.7342092

[15] Zavadsky S V, Ovsyannikov D A and Chung S-L 2009 Parametric Optimization Methods for the Tokamak Plasma Control Problem Int. J. Mod. Phys. A 24 No 5 pp.1040–47

[16] Ovsyannikov A D, Ovsyannikov D A, Altsybeyev V V, Durkin A P and Papkovich V G 2014 Application of Optimization Techniques for RFQ Design Problems of Atomic Science and Technology 91 No 3 pp 116-19

[17] Ovsyannikov D A and Altsybeyev V V 2013 Mathematical Optimization Model for Alternating-Phase Focusing (APF) Linac Problems of Atomic Science and Technology No 4 p 93

[18] Vladimirrova L V 2014 Global Extremum Search on the Basis of Density and Its Mode Estimation Proc. 20th Int. Workshop: Beam Dynamics & Optimization (BDO) (30 June-4 July 2014 St. Petersburg) (St. Petersburg: IEEE) p 186

[19] Vladimirrova L V 2016 Multicriterial Approach to Beam Dynamics Optimization Problem II Conference on Plasma&Laser Research and Technologies (2016), Journal of Physics: Conference Series 747 No 1 012070 http://iopscience.iop.org/10.1088/1742-6596/747/1/012070

[20] Vladimirrova L V and Fatyanova I A 2015 Construction of optimal regression experiment plan based on random search with "memory" using parallel calculations Proc. III Int. Conf. “Stability and Control Processes” in Memory of V.I.Zubov (SCP) (5-9 October 2015 St. Petersburg) ed L A Petrosyan and A P Zhabko (St. Petersburg: IEEE) pp 303-304.