Topological Strings on $X_{N,M}$: Conifold Singularities and Degeneration of Mirror Curves

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ABSTRACT: In this paper we study certain degenerations of the mirror curves, associated with Calabi-Yau threefolds $X_{N,M}$, and the effect of these degenerations on the topological string partition function of $X_{N,M}$. We show that when the mirror curve degenerates and become the union of the lower genus curves the corresponding partition function factorizes into pieces corresponding to the components of the degenerate mirror curve. Moreover we show that using degeneration of a generalised mirror curve it is possible to obtain the partition function corresponding to $X_{N,M-1}$ from $X_{N,M}$. 

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1 Introduction: Topological strings, M-strings and quiver gauge theories

The non-compact Calabi-Yau 3-fold (CY3-fold) $X_{N,M}$ with $N, M \in \mathbb{N}$ [1–6] has the structure of a double elliptic fibration with an underlying $SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z})$ symmetry. One elliptic fibration has the Kodaira singularity of type $I_{N-1}$ and the other elliptic fibration has $I_{M-1}$ singularity. The topological string partition function on $X_{N,M}$ was computed in [1] and shown to be related to the Little string theories (LSTs) with eight supercharges. In the decompactification limit the low energy description of circle compactified LSTs of types $(M,N)$ and $(N,M)$ are described by quiver gauge theories with gauge groups $U(M)^N$ and $U(N)^M$ respectively. In the geometric engineering argument the M-theory compactification on a non-compact Calabi-Yau 3-fold $Y$ is described at low energies by the 5d $\mathcal{N}=1$ SCFTs. These SCFTs are the UV completions of the gauge theories we are interested in. The low energy gauge theory is completely specified by the requirement of supersymmetry, once the gauge group $G$, hypermultiplet representation $R$ and the 5d Chern-Simons level $k$ is fixed. In taking the QFT limit the gravitational interactions are tuned off. This is achieved by sending the volume of $Y$ to infinity while keeping the volumes of compact four-cycles and two-cycles finite. This is equivalent to the non-comactness condition of the CY 3-fold. The coulomb branch of the SCFT is identical to the extended Kähler cone of the 3fold $Y$ [2, 7]. $Y$ can be understood as the singular limit of a smooth 3-fold $\tilde{Y}$ in which certain number of compact four-cycles have shrunk to a point. The existence of a gauge theory description of $\tilde{Y}$ implies that the abelian gauge algebra is isomorphic to the quotient $H^2(\tilde{Y}, \mathbb{R})/H^2(\tilde{Y}, \mathbb{Z})$. The later enhances to a non-abelian gauge algebra in the singular limit $\tilde{Y} \rightarrow Y$. The BPS states of the 5d theory correspond to M2-branes wrapping holomorphic two-cycles and M5-branes wrapping holomorphic four-cycles. The volume of the two-cycles and four-cycles correspond to the masses of the BPS states. At a generic point of the Coulomb branch the two-cycles and four-cycles have non-zero volumes and the BPS spectra is massive. At the origin of the Coulomb branch some of the cycles may shrink to a point and indicate a local singularity on the 3-fold.

The prepotential of the 5d gauge theory is related to the triple intersection of the divisors (four-cycles) in $\tilde{Y}$. Specifically for a given basis $D_i$ the Kähler forms $J$ admits a linear expansion $J = \phi_i D_i$ for $i = 1, 2, ..., h^{1,1}(\tilde{Y})$. The Kähler moduli $\phi_i$ for $i = 1, ..., r$ are associated to compact 4-cycles, denote them by $D_i = M_i$, and parametrize the Coulomb branch. The rest of the Kähler moduli $\phi_i$ for $i = r + 1, ..., h^{1,1}(\tilde{Y})$ are associated to the non-compact four-cycles $D_i = \tilde{M}_i$ and parametrize the mass parameters of the 5d gauge theory. The tensions of the elementary monopole strings are proportional to the volumes of compact four-cycles and related [7] to the 5d gauge theory prepotential $F$ as

$$\partial_i F = vol(M_i) = \frac{1}{2} \int_{\tilde{Y}} J^2 \wedge M_i$$

(1.1)
Similarly the volume of compact two-cycles and the triple intersection numbers are encoded in the prepotential $F$ as follows

$$\partial_i \partial_j F = \text{vol}(M_i \cap M_j) = \int_{\tilde{Y}} J \wedge M_i \wedge M_j$$

$$\partial_i \partial_j \partial_k F = \int_{\tilde{Y}} D_i \wedge D_j \wedge D_k$$

(1.2)

This shows that the dynamics of the Coulomb branch of $5d$ SCFTs can be studied in terms of the mathematics of smooth 3-folds $\tilde{Y}$. The Coulomb branch of the 5d gauge theory is partitioned into chambers by real codimension-one walls, along which some matter fields become massless. These chambers in gauge theory description are analogues of the relative Kähler cones of CY 3-fold, where the cones are related to each other through flop transitions. The quantized Chern-Simons levels, denoted by $c_{lmn}$, of the 5d theory are discontinuous across the codimension one walls as do the classical intersection numbers $D_l \cdot D_m \cdot D_n$ under flop transition. The refined topological type IIA string partition function $Z_{N,M}$ on $X_{N,M}$ can efficiently be computed using the refined topological vertex formalism. The partition function $Z_{N,M}$ takes the form of an infinite series expansion. The expansion parameters depend on the choice of a preferred direction common to all vertices of the toric web diagram. Different choices of the preferred direction give equivalent but seemingly different representations of $Z_{N,M}$ [2]. The web diagram of $X_{N,M}$ contain either horizontal, vertical or diagonal directions as the preferred ones. In this way the 5-brane web is seen as composed of vertical strips, horizontal strips and diagonal strips suitably glued together. This allows the $Z_{N,M}$ to be expressed as three series representations denoted by $Z^{(N,M)}_{\text{hor}}$, $Z^{(N,M)}_{\text{ver}}$ and $Z^{(N,M)}_{\text{diag}}$. The Kähler moduli space of $X_{N,M}$ contains three special regions each of which can be interpreted as the weak coupling region of a quiver gauge theory related to either $Z^{(N,M)}_{\text{hor}}$, $Z^{(N,M)}_{\text{ver}}$ or $Z^{(N,M)}_{\text{diag}}$. The respective weak coupling regions are in general disjoint. The basis of independent parameters in each region is a complicated linear combinations of the Kähler parameters of the 5-branes web. The Kähler parameters of $X_{N,M}$ are interpreted either as the gauge coupling constants, Coulomb branch parameters or hypermultiplet masses. Since the three decompositions of the 5-brane web are mutually related through $SL(2,\mathbb{Z})$ transformations, the authors suggested in [2] a triality symmetry between the three quiver gauge theories and further conjectured the equivalence of $X_{N,M}$ and $X_{N',M'}$ for $MN = N'M'$ and $\text{gcd}(M,N) = \text{gcd}(N',M')$ This triality symmetry is a generalization of T-duality of the underlying LSTs. The elliptic Calabi-Yau 3-fold $X_{N,M}$ is dual to the brane web of type IIB $M$ NS5-branes and $N$ D5-branes wrapped on two $S^1$s. Denote by $\{y^0, y^1, y^2, y^3, ..., y^9\}$ the coordinates of type IIB string theory vacuum $\mathbb{R}^{1,9}$. The common worldvolume of the 5-branes along $\{y^0, y^1, y^2, y^3, y^4\}$ gives rise to the gauge theory under consideration. The $(p,q)$ brane web is arranged in the $\{y^5, y^6\}$ plane which is compactified to a torus $T^2$. The $(p,q)$-charges and their conservation encode the details of the five-dimensional mass deformed supersymmetric gauge theory. $SL(2,\mathbb{Z})$ duality symmetry of type IIB is translated to a duality between one gauge theory corresponding to $M$ NS5-branes and $N$ D5-branes and the second gauge theory corresponding to $N$ NS5-branes and $N$ D5-
branes. This $SL(2, \mathbb{Z})$ is the manifestation of the T-duality of underlying circle compactified LSTs as alluded to before. On lifting the type IIB superstring theory to F-theory the $SL(2, \mathbb{Z})$ duality is encoded in the topological equivalence of $X_{M,N}$ and $X_{N,M}$.

The compactified 5-brane web gives rise to a five dimensional $\mathcal{N} = 2$ supersymmetric gauge theory on the common worldvolume. This 5-branes web can be deformed to include also (1, 1) 5-branes. In string theory this is interpreted as the splitting of the D5-branes on the NS5-brane world volume. In other words the string tension is turned on for the strings that are stretched between D5-branes. It gives rise to the mass deformation of the bifundamental hypermultiplets in the five dimensional gauge theory. The mass deformation results in the breaking of supersymmetry to $\mathcal{N} = 1$ in five dimensions. Because of the toric compactification of the 5-branes web one gets affine $\hat{A}_{N-1}$ quiver gauge theory with an $SU(N)$ gauge group at each node and one bifundamental matter stretched between adjacent nodes. There are $M$ coupling constants $\tau_i, i = 1, ..., M$ for each node such that

$$\sum_{i=1}^{M} \tau_i = \frac{1}{R_1} \quad (1.3)$$

where $R_1$ is the radius of the $S^1$ on which M5-brane theory is compactified. In geometrical terms each gauge coupling constant is related to the area of a distinct curve in CY 3-fold. If there are more than one, though equivalent, choices of these curves, this gives rise to dual gauge theory formulations of the same system. In other words for the web of $M$ NS5-branes and $N$ D5-branes the gauge theory on the D5-branes is given by

$$\text{gauge group} : U(1) \times SU(M) \times SU(M) \times \ldots \times SU(M)$$

$$\text{hypermultiplet representation} : \oplus_{i=1}^{N} \left( (M_a, \bar{M}_a+1) \oplus (\bar{M}_a, M_a+1) \right) \quad (1.4)$$

where $N_a$ is the $SU(N)$ fundamental representation of the $a$-th node and $\bar{N}_a$ the complex conjugate one. Under the $SL(2, \mathbb{Z})$ duality the corresponding web consists of $N$ NS5-branes and $M$ D5-branes, and the gauge theory is given by

$$\text{gauge group} : U(1) \times SU(N) \times SU(N) \times \ldots \times SU(N)$$

$$\text{hypermultiplet representation} : \oplus_{i=1}^{M} \left( (N_a, \bar{N}_a+1) \oplus (\bar{N}_a, N_a+1) \right) \quad (1.5)$$

where $M_a$ is the $SU(N)$ fundamental representation of the $a$-th node and $\bar{M}_a$ the complex conjugate one. In the case that each bifundamental mass is different there are $MN$ number of them. Web compactification on $T^2$ results in $M + N - 2$ constraints. As a result the five dimensional affine quiver gauge theory depends on $MN + 2$ number of parameters. As alluded to before there is a third quiver gauge theory description related to the (1.4) and (1.5) through a triality symmetry

$$\text{gauge group} : U(1) \times SU(\frac{MN}{k}) \times SU(\frac{MN}{k}) \times \ldots \times SU(\frac{MN}{k})$$

$$\text{hypermultiplet representation} : \oplus_{i=1}^{k} \left( (P_a, \bar{P}_{a+1}) \oplus (\bar{P}_a, P_{a+1}) \right) \quad (1.6)$$
The UV completions of these gauge theories are described by LSTs.
The equivalence of $X_{MN}$ and $X_{M',N'}$ for $MN = N'M'$ and $gcd(M, N) = gcd(N', M') = k$
gives rise to a web of dualities between the quiver gauge theories

$$\left\{ U(N)^{M} \sim U(M)^{N} \sim U\left(\frac{MN}{k}\right)^k \right\} \sim \left\{ U(N')^{M'} \sim U(M')^{N'} \sim U\left(\frac{M'N'}{k}\right)^k \right\} \quad (1.7)$$

for $MN = N'M'$ and $gcd(M, N) = gcd(N', M') = k$.

The partition function of the quiver gauge theories given in (1.4) and (1.5) can be computed
directly by using Nekrasov instanton calculus as described in [3]. In doing so one has to take
into account the non-trivial winding of strings on the compact direction transverse to the
5-branes. In [8] it was proposed to consider the intersections of M2-branes and M5-branes
as independent degrees of freedom and were called M-strings. The table given in figure 1
summarises the coordinate labels and specifies the world volume directions of BPS M5-M2-
M-string configuration.

The M5-branes are separated along the compactified $x^6 \sim x^6 + 2\pi R_6$ dimension with the
positions parametrised by scalars $\{a_1, ..., a_M\}$ where $M$ denotes the total number of M5-branes
and $a_i - a_{i+1}$ are the vev of the scalars of 6d tensor multiplets. The M2-branes are stretched
between these M5-branes. For the transverse space $\mathbb{R}^4$ we can have only one stack of M2-
branes between M5-branes. However it is possible to perform an orbifolding of the transverse
$\mathbb{R}^4$ such that the mass deformation and supersymmetry remain preserved. The orbifolding
allows the multiple stacks of M2-branes with each stack charged under the orbifold action. For
the M-string dual to $(N, M)$ web diagram there will be $N$ stacks of M2-branes, with $i$-th stack consisting of $k_i$
number of them. In gauge theory $k_i$ characterises the instanton number. It
was shown subsequently in [3] that the M-string partition function $Z(N, M)$ is the generating
function of the equivariant $(2, 0)$ elliptic genus of the M-string world sheet,

$$Z(N, M) = \sum_{\vec{k}} Q_1^{k_1} Q_2^{k_2} ... Q_M^{k_M} \chi_{ell}(M(N, \vec{k}), V) \quad (1.8)$$

Its target space is the product of moduli spaces of $U(N)$ instantons of charge $k_i$ on $\mathbb{C}^2$:
$M(N, \vec{k}) := M(N, k_1) \times M(N, k_2) \times ... \times M(N, k_N)$ along with a vector bundle $V(N, M)$
on it. The mass deformation is taken care of by an extra $U(1)_m$ action with equivariant
parameter $m$. The vector bundle is special in the sense that only right moving fermions couple
to it. $M(N, \vec{k})$ is nothing other than the moduli space of M-strings. For example the specific
Figure 2: Web diagram of $X_{N,M}$. $t_i \in \{t_1, ..., t_N\}$ denotes the distance between $i$-th and $i+1$-th red lines and $T_i \in \{T_1, ..., T_M\}$ denotes the distance between $i$-th and $i+1$-th blue lines. $m$ denotes the Kähler parameter of the diagonal $\mathbb{P}^1$s. The double and single bars $||$, $|$, = indicate periodic identifications.

values $M = 1, N = k$ correspond to a single M5-brane wrapped on parallel $S^1$ and $k$ stack of M2-branes wrapped on the transverse $S^1$ and ending on the M5-branes. The stack of M2-branes appear as coloured points in the $\mathbb{R}^4||$ that resides inside the M5-brane world volume and transverse to the M-string world sheet. Thus for the configuration that involves $n_l$ number of M2-branes in the $l$-th stack, where $l = 1, ..., k$, the moduli space is obviously the product of Hilbert scheme of points as follows

$$H := \text{Hilb}^{n_1} [\mathbb{C}^2] \times \text{Hilb}^{n_2} [\mathbb{C}^2] \times ... \times \text{Hilb}^{n_k} [\mathbb{C}^2]$$

The vector bundle $V$ over $H$ that is required for $(2,0)$ world sheet theory has been determined in [8] and turns out to be the following

$$V_I = \bigoplus_{t,s=1}^N \text{Ext}^1(I_r, I_s) \otimes L^{-\frac{1}{2}}$$

where $I = (I_1, I_2, ..., I_N) \in H$. Roughly speaking Ext groups count the massless open string states for strings that are stretched between D-branes wrapped on complex submanifolds of CY spaces. Note that each factor $\text{Ext}^1(I_r, I_s) \otimes L^{-\frac{1}{2}}$ in the fibre denotes the contribution of a pair of stack of M2-branes ending on a single M5-brane from opposite sides. In other words there is an isomorphism between the degrees of freedom on the $(N, M)$ 5-branes web and the moduli space of M-strings, $M(N, \vec{k})$. Using equivariant fixed point theorems one only needs to know the fibres of the bundle $V(N, M)$ over the fixed points.
The weights of $V(N, M)$ at the fixed points $\vec{I}(1), \vec{I}(2), ..., \vec{I}(M)$ are given by the following Chern character expansion \[3\]

$$
\sum_{\text{weights}} e^w = \sum_{p=1}^{M} \sum_{r,s=1}^{N} Q_m e^{i(a_r-a_s)} \left( \sum_{(i,j) \in \nu_r^p} t^{r,i-p+\frac{1}{2}} q^{r,i-j+\frac{1}{2}} + \sum_{(i,j) \in \nu_{r+1}^p} t^{-r,j+i-p+\frac{1}{2}} q^{-r,i-j+\frac{1}{2}} \right)
$$

(1.11)

where $\nu_1(1), \nu_2(1), ..., \nu_N(1), \nu_1^{(1)}, ..., \nu_N^{(1)}$ label the fixed points. The elliptic genus is then given as follows

$$
Z = \int_M \prod_i x_i \frac{\theta_1(\tau, x_i+z)}{\theta_1(\tau, x_i)}
$$

(1.12)

where $x_i$ and $\tilde{x}_i$ denote the Chern roots respectively of the tangent bundle and vector bundle $V(N, M)$ as can be read from (1.11) and the theta function of first kind $\theta_1(\tau, z)$ is defined by

$$
\theta_1(\tau; z) = -ie^{\frac{i\pi}{4}} (e^{i\pi z} - e^{-i\pi z}) \prod_{k=1}^{\infty} (1 - e^{2\pi ik\tau})(1 - e^{2\pi ik\tau}e^{2\pi ikz})(1 - e^{2\pi ik\tau}e^{-2\pi ikz}).
$$

(1.13)

More succinctly, the Nekrasov partition function of the gauge theory on the D5-branes of the web is identical to the appropriately normalised topological string partition function of CY 3-fold $X_{N,M}$ and it is the generating function of the $(2,0)$ elliptic genus of the product of instanton moduli spaces on which the bundle $V(N, M)$ coupled to the right moving fermions exists.

**Presentation of the paper**

We summarising the type IIA/type IIB mirror symmetry conjecture in section (2). In sections (3) we construct the quantum mirror curve of $X_{N,M}$ and study the limits in which it can be reduced to a lower genus curve. In section (5) we show that in the splitting degeneration limit the partition function $Z_{X_{N,M}}$ can be constructed from the partition function $Z_{X_{N,M-1}}$ and also elaborated on pictorially. In the last section (6) we briefly mention some physical consequences of the degenerations discussed in the previous sections. In the appendix we reproduce the proof of an identity used in the main text.

**2 Mirror non-compact CY 3-folds, mirror curves and the Kodaira Spencer theory: a summary**

Consider the A-model topological strings on a toric CY 3-fold $M = \mathbb{C}^{l+3}/U(1)^l$. Algebraically $M$ is defined by the following set of constraints

$$
\sum_{i=1}^{l+3} Q_i a_i |x_i|^2 = k^a, \quad a = 1, ..., l
$$

(2.1)
modulo the action of $U(1)^l$, where each $X_i$ parametrizes a complex plane $\mathbb{C}$ and can be visualised as $S^1$-fibrations over $\mathbb{R}_+$. In this way $M$, as defined by (2.1), is a $T^3$-fibration over a non-compact convex and linearly bounded subspace in $\mathbb{R}^3$, with $T^3$ parametrised by $\{\theta_i\}$ coordinates. $k^a \in \mathbb{R}_+$ are called the Kähler parameters. The CY condition
\[ c_1(TM) = 0 \] holds iff
\[ \sum_{i=1}^{l+3} Q^a_i = 0, \quad a = 1, \ldots, l \] (2.3)
Inspecting equation (2.1) makes it clear that since $Q^a_i \in \mathbb{Z}$, all toric CY 3-folds are constrained to be non-compact. The second constraint (2.3) furnishes a representation of $M$ as a $\mathbb{R}_+ \times T^2$ fibered over $\mathbb{R}^3$. In this way the the toric three fold $M$ allows its construction by gluing patches of $\mathbb{C}^3$.

The toric diagram $\Gamma_M$ corresponding to $M$ specifies the loci along which the $S^1$ fibers degenerate. The boundary of the region $B$ is defined by $X_i = 0$. For each value of $i$ this zero locus defines a 2-plane in $\mathbb{R}^3$ whose normal vector satisfies
\[ \sum_{i=1}^{3+l} Q^a_i \vec{n}_i = 0 \] (2.4)
Obviously, the $S^1$ parametrised by $\theta_i$ shrinks at $|X_i| = 0$ and at the intersection of two such loci $S_{ij} = \{ |X_i| = 0 \} \cap \{ |X_j| = 0 \}$, two circles $S^1$s shrink to zero size. For $S_{ij}$ a closed line in $\partial B$ the open $S^1$ bundle over it is a $\mathbb{P}^1$. For $S_{ij}$ a half open line it represents a non-compact direction $\mathbb{C}$. It is clear now that the relative position of $X_i, X_j$ is determined by the length of the line segment $S_{ij}$ which is nothing other than the Kähler parameter of the corresponding $\mathbb{P}^1$. The CY condition (2.3) and the $T^2$ fibration structure allows to project the $S_{ij}$s onto $\mathbb{R}^2$ in such a way that all the information about the geometry of $M$ is contained in it. Projecting all the $S_{ij}$s onto $\mathbb{R}^2$ in this way constitute the toric diagram $\Gamma_M$.

To construct the mirror $N$ of the three fold $M$, consider variable $v_1, v_2 \in \mathbb{C}$, and the homogeneous coordinates $x_i = e^{y_i} \in \mathbb{C}^*, i = 1, \ldots, l + 3$ related to $X_i$ by $|x_i| = e^{-|X_i|^2}$. $x_i$ are constrained by $x_i \sim \lambda x_i$ for $\lambda \in \mathbb{C}^*$. The mirror geometry $N$ is then given by the algebraic equation
\[ v_1 v_2 = \sum_{i=1}^{l+3} x_i, \] (2.5)
constrained by
\[ \prod_{i=1}^{l+3} x_i^{Q^a_i} = e^{-r^a - i\theta_a}, \quad a = 1, \ldots, l \] (2.6)
All of these equations can be combined into one equation
\[ v_1 v_2 = H(x, y; r^a, \theta_a) \]  
(2.7)
where \( x, y \in \mathbb{C}^* \). \( H(x, y; r^a, \theta_a) \) can be decomposed into pant diagrams described by
\[ e^x + e^y + 1 = 0. \]  
(2.8)
The last equation describes a conic bundle over \( \mathbb{C}^* \times \mathbb{C}^* \) in which the fibers degenerate over two lines over the family of Riemann surfaces \( \Sigma : H(x, y; r^a, \theta_a) = 0 \in \mathbb{C}^* \times \mathbb{C}^* \) If the toric diagram of \( M \) is thickened, what emerges is nothing else but \( \Sigma \); the genus of \( \Sigma \) equals the number of closed meshes and the number of punctures equals the number of semi infinite lines in the toric diagram.\(^1\)
In the topological A-model the topological vertex computation can be interpreted as the states of a chiral boson on a three-punctured sphere. This chiral boson on each patch of the sphere is identified with the Kodaira Spencer field on the Riemann surface embedded in the CY 3-fold of mirror topological B-model. The A-model closed topological strings on toric CY 3-fold, with or without D-branes, is computable by gluing cubic topological vertex expressions. On the mirror B-model the gluing rules are equivalent to the operator formation of the Kodaira Spencer theory on the Riemann surface. Kodaira Spencer theory describes the dynamics of complex deformation of the CY 3-fold.

The holomorphic 3-form on the mirror CY is given by
\[ \Omega = \frac{dv_2 dx dy}{v_2} \]  
(2.9)
In studying the variation of complex structure, if one is only interested in the perturbations of \( H(x, y; r^a, \theta_a) \) with no variations in \( (v_1, v_2) \), the problem gets reduced to one complex dimension. In this special case the CY 3-fold has the structure of a fibration over the \( (x, y) \)-plane, with fiber given by (2.5). This fibration develops a node on the locus
\[ H(x, y; r^a, \theta_a) = 0 \]  
(2.10)
the mirror curve. The period integral over \( \Omega \) over 3-cycles reduce, using Cauchy’s theorem, to
\[ \int_S dx dy \]  
(2.11)
where \( \partial S \subset \Sigma \), which in turn reduces to
\[ \int_{\beta} y dx \]  
(2.12)
using Stoke’s theorem, where \( \beta \) denotes a one-cycle on \( \Sigma \). This shows that the complex structure deformation of \( H(x, y; r^a, \theta_a) = 0 \) depends on the 1-form
\[ \lambda = y dx \]  
(2.13)
\(^1\)It is a standard in literature to call \( \Sigma \) the mirror curve.
and \( \lambda \) is defined patch by patch. Solving \( H = 0 \) yields \( y = f(x) \). Under complex structure deformation we have

\[
y = f(x) + \delta f
\]

and it correspondingly changes \( \lambda \) by

\[
\delta \lambda = \delta f dx
\]

To put it in the context of QFT we can identify \( \delta f = \partial \phi \). This variation \( \delta \lambda \) is identified with the Kodaira Spencer field in [9]. It also satisfies a consistency relation

\[
\bar{\partial}_x \delta f = 0
\]

The Kodaira Spencer action has the following kinetic term

\[
\int_{CY} \omega \bar{\partial}^{-1} \bar{\partial} \omega \quad (2.17)
\]

where \( \omega \) is a \((2,1)\)-form which represents the change in the complex structure of CY. In each patch \( \omega = \partial \xi \) with \( \xi \) is a \((1,1)\)-form.

\[
\int_{CY} \partial \xi \bar{\partial} \xi \quad (2.18)
\]

and locally \(^2\) in each patch we have

\[
\int_{\Sigma} \partial \phi \bar{\partial} \phi \quad (2.19)
\]

the action for a free scalar field. For a higher genus Riemann surface, it is first decomposed into pants, each pant having three boundaries. Near each of the boundary, a local coordinate \( x \) is chosen such that \( x \to \infty \) at the boundary. The complex structure variation is studied in the limit \( x \to \infty \) at each boundary. The action given by (2.19) is free, however the the interaction part is encoded in the gluing data of various patches.

### 3 Degenerations of the Mirror curves of the Conifold Singularities

The CY 3fold \( X_{N,M} \) is a double elliptic fibration of type \( A_{N-1} \times A_{M-1} \) over a non-compact base \( \mathbb{C} \). It is toric with a web diagram (5) which is drawn on a torus with radii of the two circles of the torus being dual to the Kähler class of the elliptic fibers of \( X_{N,M} \). These 3folds were studied by [1, 3–5, 10] as examples of toric varieties of infinite type.

The toric CY 3fold \( X_{N,M} \) can be obtained by \( \mathbb{Z}_N \times \mathbb{Z}_M \) orbifolds of \( X_{1,1} \). This set up is dualizable to \((p, q)\) 5-brane webs and realise various five- and six-dimensional gauge theories. The 5-brane web is identical to the toric web underlying \( X_{N,M} \).

\(^2\) The local patch dependence of this formulation is related to the framing ambiguity in topological A model
The mirror curves of toric CY 3folds are determined by the corresponding Newton polygon. The line in the web orthogonal to the line in the Newton polygon joining \((k_1, \ell_1)\) and \((k_2, \ell_2)\) is given by (passing through \((x_0, y_0)\)),

\[
(\Delta \ell) y + (\Delta k) x = (\Delta \ell) y_0 + (\Delta k) x_0
\]

where \(\Delta \ell = \ell_2 - \ell_1\) and \(\Delta k = k_2 - k_1\). Since \((x_0, y_0)\) is arbitrary therefore we get

\[
(\Delta \ell) y + (\Delta k) x = \alpha
\]

The equation of the Riemann surface in this patch is given by exponentiating and complexifying \((x, y)\) to \((u, v)\),

\[
X^{\Delta k} Y^{\Delta \ell} = -e^{\tilde{\alpha}},
\]

where \(X = e^u\) and \(Y = e^v\) with \(u, v \in \mathbb{C}\) and \(\text{Re}(\tilde{\alpha}) = \alpha\). Since the imaginary part \(\tilde{\alpha}\) is not determined, we have introduced a factor of \(-1\) (shifting the imaginary part by \(i\pi\)) for later convenience. With this choice, \(\tilde{\alpha}\) will be identified with the complexified Kähler parameters.

In the mirror curve, we will have

\[
A_{k_1 \ell_1} X^{k_1} Y^{\ell_1} + A_{k_2 \ell_2} X^{k_2} Y^{\ell_2} = 0
\]

which implies

\[
X^{\Delta k} Y^{\Delta \ell} = -\frac{A_{k_1 \ell_1}}{A_{k_2 \ell_2}} \implies A_{k_2 \ell_2} = A_{k_1 \ell_1} e^{-\tilde{\alpha}}
\]

3.1 Example: Resolved Conifold

In this case, the Newton polygon is shown in figure (3) and the corresponding mirror curve is given by,

\[
A_{00} + A_{10} X + A_{01} Y + A_{11} XY = 0
\]

Let us choose so that the horizontal line in the web corresponding to \((0, 0)\) and \((0, 1)\) points in the Newton polygon goes through the origin so that \(\alpha = 0\) for this line, which gives,

\[
A_{01} = A_{00}
\]

Similarly \(A_{10} = A_{00}\) and \(A_{10} = A_{01}\). The line in the web corresponding to \((0, 1), (1, 1)\) has the equation \(x = T\) where \(T\) is the horizontal distance between the two vertices in the web (the vertical distance is also \(T\)). Thus we get \(A_{11} = A_{01} e^{-t}\) where \(\text{Re}(t) = T\), thus the mirror curve is given by

\[
1 + X + Y + e^{2\pi i t^*} XY = 0
\]

where \(t^* = \frac{i}{2\pi} t = \frac{i}{2\pi} T - \frac{\text{Im}(t)}{2\pi}\) so that \(\text{Im}(t^*) > 0\). This is the notation that we will use in the rest of the section.
\[ A_{2,3} = A_{2,2} e^{2\pi i (2\tau + 2t_2)} \]

\[ \tau = t_1 + t_2 \]
\[ \rho = t_3 + t_1 \]

**Figure 3:** tessellation of Newton polygons and web diagram of \( X_{1,1} \)

### 3.2 Mirror curve dual to \( X_{1,1} \)

In this case the mirror curve is given by,

\[ \sum_{(k,\ell)\in\mathbb{Z}^2} A_{k,\ell} X^k Y^\ell = 0. \] (3.9)

Let's take the origin of the web to be the vertex of the web corresponding to the triangle \((0,0), (1,0), (0,1)\) as shown in the figure below.

With this choice the equation of the horizontal line in the web corresponding to \((k,\ell)\) and \((k,\ell + 1)\) is given by

\[ y = \ell(t_1 + t_3) + k t_1 \] (3.10)

where \(\tau\) is the periodicity of the web in the vertical direction and \(t_1\) is the horizontal distance between two consecutive vertices on the diagonal in the web (see figure above). This gives

\[ A_{k,\ell+1} = A_{k,\ell} e^{2\pi i (\ell \tau + k z)} \implies A_{k,\ell+1} = A_{k,0} e^{2\pi i (\ell \frac{\tau + k}{2} + (\ell+1) k z)} \] (3.11)

where \(\text{Im}(\tau) = \frac{t_1 + t_3}{2\pi}\) and \(\text{Im}(z) = \frac{t_3}{2\pi}\). The equation of the line in the web corresponding to \((k,\ell), (k+1,\ell)\) is given by \(x = k(t_1 + t_2) + \ell t_1\) where \(\rho\) is the periodicity of the web in the horizontal direction. This gives

\[ A_{k+1,\ell} = A_{k,\ell} e^{2\pi i (k \rho + \ell z)} \implies A_{k+1,\ell} = A_{0,\ell} e^{2\pi i (\frac{k(k+1)}{2} + (k+1) \ell z)} \] (3.12)

From Eq(3.11) and Eq(3.12) it follows that:

\[ A_{k,\ell} = A_{0,0} e^{2\pi i (\frac{\ell (\ell-1)}{2} \tau + \frac{k(k+1)}{2} \rho + \ell k z)} \] (3.13)
This gives the following mirror curve:

$$\sum_{k,\ell \in \mathbb{Z}} e^{2\pi i \left( \frac{\ell(\ell-1)}{2} \tau + \frac{k(k-1)}{2} \rho + \ell \tau \right)} X^k Y^\ell = 0 \quad (3.14)$$

Note that in the limit $z \to 0$ we give the factorisation of this curve

$$\left( \sum_{k, \in \mathbb{Z}} e^{2\pi i \left( \frac{k(k-1)}{2} \rho \right)} X^k \right) \left( \sum_{\ell \in \mathbb{Z}} e^{2\pi i \left( \frac{\ell(\ell-1)}{2} \tau \right)} Y^\ell \right) = 0 \quad (3.15)$$

### 3.3 Mirror curve dual to $X_{1,2}$

Consider the periodic Newton polygon with vertices $(0,0), (1,0), (2,0), (2,1), (1,1), (0,1)$ as shown in figure (4). The mirror curve is given by

$$\sum_{k,\ell \in \mathbb{Z}} B_{k\ell} X^k Y^\ell = 0 \quad (3.16)$$

where the coefficients $B_{k,\ell}$ can be determined in the same way as for the genus two case and are functions of the four Kähler parameters $(\tau, \rho, z, w)$ (see figure below). They are related to each other as follows:

$$B_{2k+2,\ell} = B_{2k+1,\ell} e^{2\pi i (k \rho + (\ell+1) z + w)}, \quad B_{2k+1} = B_{2k,\ell} e^{2\pi i (k \rho + \ell z)} \quad (3.17)$$

$$B_{k,\ell+1} = B_{k,\ell} e^{2\pi i (\ell \tau + k z)} \quad (3.18)$$

![Figure 4: tessellation of Newton polygons and web diagram of $X_{1,2}$](image-url)
These recursive relations have the following solution:

\[
B_{2k,\ell} = \exp\left[2\pi i \left(k(k - 1)\rho + \frac{\ell(\ell - 1)}{2} \tau + 2k\ell z + kz + kw\right)\right]
\]

\[
B_{2k+1,\ell} = \exp\left[2\pi i \left(k^2 \rho + \frac{\ell(\ell - 1)}{2} \tau + (2k + 1)\ell z + k(z + w)\right)\right]
\]

If we define

\[
\Theta\left(\Omega(\rho, z, \tau)|(u, v)\right) = \sum_{k,\ell} \exp\left(2\pi i Q(k, \ell) / 2\right) X^k Y^\ell
\]

where

\[
\Omega(\rho, z, \tau) = \begin{pmatrix} \rho & z \\ z & \tau \end{pmatrix}, \quad Q(k, \ell) = (k \ell) \Omega \begin{pmatrix} k \\ \ell \end{pmatrix}
\]

then the mirror curve is given by

\[
\Theta\left(\Omega(2\rho, 2z, \tau)|(2u - \rho + z + w, v - \tau)\right) + e^{2\pi i u} \Theta\left(\Omega(2\rho, 2z, \tau)|(2u + z + w, v - \tau + z)\right) = 0
\]

To see the factorisation we rewrite the curve as

\[
\sum_{k,\ell \in \mathbb{Z}} \exp\left[2\pi i \left(k(k - 1)\rho + \frac{\ell(\ell - 1)}{2} \tau + 2k\ell z + kz + kw\right)\right] X^{2k} Y^\ell
\]

\[
+ \exp\left[2\pi i \left(k^2 \rho + \frac{\ell(\ell - 1)}{2} \tau + (2k + 1)\ell z + k(z + w)\right)\right] X^{2k+1} Y^\ell
\] = 0

(3.22)

In the limit \(z \to 0\) we get

\[
\sum_{k,\ell \in \mathbb{Z}} \left(\exp\left[2\pi i \left(k(k - 1)\rho + \frac{\ell(\ell - 1)}{2} \tau + kw\right)\right] X^{2k} Y^\ell + \exp\left[2\pi i \left(k^2 \rho + \frac{\ell(\ell - 1)}{2} \tau + kw\right)\right] X^{2k+1} Y^\ell\right) = 0
\]

(3.23)

Taking out the common factor we get the factorised form

\[
\sum_{\ell \in \mathbb{Z}} \left(\exp\left[2\pi i \left(\frac{\ell(\ell - 1)}{2} \tau\right)\right] Y^\ell\right) \left(\sum_{k \in \mathbb{Z}} X^{2k}\left(\exp\left[2\pi i \left(k(k - 1)\rho + kw\right)\right] + \exp\left[2\pi i \left(k^2 \rho + kw\right)\right] X\right)\right) = 0
\]

(3.24)

In the mirror construction this Riemann surface \(\Sigma\) is a part of the mirror CY 3-fold. Obviously for 6D theories the corresponding toric webs have no semi-infinite lines and hence no punctures. The periodicity of the web is taken into account by including all of its images under the periodic shift.
3.4 Mirror curve dual to $X_{(N,M)}$

Consider the $(N, M)$ web shown in figure (5). The Kähler class $\omega$ of $X_{N,M}$ is parametrized by $(m_{\alpha,\beta}, \tau, \rho, T, t) = (m_{\alpha,\beta}, \tau, \rho, m, T_1, T_2, \cdots, T_M, t_1, t_2, \cdots, t_N)$ with $\tau = \sum_{i=1}^{M} T_i$ and $\rho = \sum_{j=1}^{N} t_j$. In the partition function $Z_{N,M}$ the Kähler parameter are quantum corrected whereas in the mirror curve the Kähler parameter have to be quantum corrected. The factorisation properties of the mirror curve will in general be affected by the quantum corrections. The mirror curve is given by a sum over the monomials associated with the Newton polygon. In this case the Newton polygon tiles the plane therefore,

$$H_{N,M}(X,Y) := \sum_{(i,j) \in \mathbb{Z}^2} A_{i,j} X^i Y^j.$$  \hspace{1cm} (3.25)

The coefficients $A_{i,j}$ depend on the length of the various line segments in the web which are the Kähler parameters of the corresponding Calabi-Yau threefolds. As discussed before neighboring pair of points in the Newton polygon connected by a line give a relation between the associated coefficients $A_{i,j}$,

$$\frac{A_{i,k+1}}{A_{i,k}} = e^{\sum_{j=1}^{k-1} T_j + \sum_{\alpha=0}^{i-1} m_{\alpha,k}} \hspace{1cm} (3.26)$$

$$\frac{A_{i+1,k}}{A_{i,k}} = e^{\sum_{j=1}^{i-1} t_j + \sum_{\alpha=0}^{k-1} m_{i,\alpha}}$$
\[ A_{i+1,k+1} = A_{i+1,k+1} e^{T_1(T_1 + T_2) + (T_1 + T_2 + T_3) + \cdots + (T_1 + \cdots + T_{k-1}) + \sum_{\beta=1}^k \sum_{\alpha=0}^i m_{\alpha,\beta}} \]

\[ = A_{i+1,k+1} e^{\sum_{\gamma=1}^{k-1} (k-\gamma) T_\gamma + \sum_{\beta=1}^k \sum_{\alpha=0}^i m_{\alpha,\beta}} \]

\[ = A_{0,1} e^{t_1 + (t_1 + t_2) + \cdots + (t_1 + t_2 + \cdots + t_{i-1}) + \sum_{\gamma=1}^{k-1} (k-\gamma) T_\gamma + \sum_{\beta=0}^k \sum_{\alpha=0}^i m_{\alpha,\beta}} \]

Since \( A_{0,1} = A_{0,0} = 1 \) we get,

\[ A_{i+1,k+1} = e^{\sum_{\gamma=1}^{i-1} (i-\gamma) t_\gamma + \sum_{\gamma=1}^{k-1} (k-\gamma) T_\gamma + \sum_{\beta=0}^k \sum_{\alpha=0}^i m_{\alpha,\beta}} \]

Thus the curve is given by

\[ H_{N,M}(X,Y) = \sum_{(i,k) \in \mathbb{Z}^2} A_{i+1,k+1} X^{i+1} Y^{k+1} \]

\[ = \sum_{i=0,k=0}^{N-1,M-1} W_{i,k}(X,Y) \]

\[ W_{i,k}(X,Y) = \sum_{(a,b) \in \mathbb{Z}^2} A_{Na+i+1,Mb+k+1} X^{Na+i+1} Y^{Mb+k+1} \]

\[ A_{Na+i+1,Mb+k+1} = e^{\sum_{\gamma=1}^{Na+i} (Na+i-\gamma) t_\gamma + \sum_{\gamma=1}^{Mb+k} (Mb+k-\gamma) T_\gamma + \sum_{\beta=0}^{Mb+k} \sum_{\alpha=0}^{Na+i} m_{\alpha,\beta}} \]

Using

\[ t_\gamma = t_{\gamma'} \quad \text{if} \quad \gamma \equiv \gamma' \pmod{N} \]

\[ T_\gamma = T_{\gamma'} \quad \text{if} \quad \gamma \equiv \gamma' \pmod{M} \]

\[ m_{\alpha_1,\beta_1} = m_{\alpha_2,\beta_2} \quad \text{if} \quad \alpha_1 \equiv \alpha_2 \pmod{N} \quad \text{and} \quad \beta_1 \equiv \beta_2 \pmod{M} \]
we get
\[
\sum_{\gamma=1}^{Na+i-1} (Na + i - \gamma)t_\gamma = \sum_{\gamma=1}^{N} (Na + i - \gamma)t_\gamma + \sum_{\gamma=N+1}^{2N} (Na + i - \gamma)t_\gamma + \cdots \tag{3.32}
\]
\[
+ \sum_{\gamma=N(a-1)+1}^{Na} (Na + i - \gamma)t_\gamma + \sum_{\gamma=Na+1}^{Na+i-1} (Na + i - \gamma)t_\gamma
\]
\[
= \sum_{\gamma=1}^{N} \left[ (Na + i - \gamma) + (Na - 1 + i - \gamma) + (Na - 2 + i - \gamma) + \cdots + (N + i - \gamma) \right] t_\gamma + \sum_{\gamma=1}^{i-1} (i - \gamma)t_\gamma
\]
\[
= \sum_{\gamma=1}^{N} \left[ Na + a(i - \gamma) \right] t_\gamma + \sum_{\gamma=1}^{i-1} (i - \gamma)t_\gamma
\]
\[
= \left[ Na + a(i - \gamma) \right] \tau - \sum_{\gamma=1}^{N} \gamma t_\gamma + \sum_{\gamma=1}^{i-1} (i - \gamma)t_\gamma
\]
Similarly
\[
\sum_{\gamma=1}^{Mb+k-1} (Mb + k - \gamma)t_\gamma = \left[ M\frac{b(b+1)}{2} + bk \right] \rho - \sum_{\gamma=1}^{M} \gamma T_\gamma + \sum_{\gamma=1}^{k-1} (k - \gamma)T_\gamma \tag{3.33}
\]
\[
\sum_{\beta=0}^{Mb+k} \sum_{\alpha=0}^{Na+i} m_{\alpha,\beta} = \sum_{\beta=0}^{Mb+k} \left[ \sum_{\alpha=0}^{N-1} m_{\alpha,\beta} + \sum_{\alpha=N}^{2N-1} m_{\alpha,\beta} + \cdots + \sum_{\alpha=N(a-1)}^{Na-1} m_{\alpha,\beta} + \sum_{\alpha=Na}^{Na+i} m_{\alpha,\beta} \right] \tag{3.34}
\]
\[
= \sum_{\beta=0}^{Mb+k} \left[ a \sum_{\alpha=0}^{N-1} m_{\alpha,\beta} + \sum_{\alpha=N}^{i} m_{\alpha,\beta} \right]
\]
\[
= a \sum_{\alpha=0}^{N-1} \left[ b \sum_{\beta=0}^{M-1} m_{\alpha,\beta} + \sum_{\beta=0}^{k} m_{\alpha,\beta} \right] + \sum_{\alpha=N}^{i} \left[ b \sum_{\beta=0}^{M-1} m_{\alpha,\beta} + \sum_{\beta=0}^{k} m_{\alpha,\beta} \right]
\]
\[
= ab \sum_{\alpha=0}^{N-1} \sum_{\beta=0}^{M-1} m_{\alpha,\beta} + a \sum_{\alpha=N}^{i} \sum_{\beta=0}^{k} m_{\alpha,\beta} + b \sum_{\alpha=0}^{N-1} \sum_{\beta=0}^{M-1} m_{\alpha,\beta} + i \sum_{\alpha=0}^{k} \sum_{\beta=0}^{k} m_{\alpha,\beta}
\]
Since \( \sum_{\alpha=0}^{N-1} m_{\alpha,\beta} \) is independent of \( \beta \) by Lemma 5.4 of [10]\(^3\) therefore
\[
\sum_{\beta=0}^{Mb+k} \sum_{\alpha=0}^{Na+i} m_{\alpha,\beta} = \left( ab + \frac{a(k+1)}{M} + \frac{b(i+1)}{N} \right) \sum_{\alpha=0}^{N-1} \sum_{\beta=0}^{M-1} m_{\alpha,\beta} + \sum_{\alpha=0}^{i} \sum_{\beta=0}^{k} m_{\alpha,\beta} \tag{3.35}
\]
\[
= \left( ab + \frac{a(k+1)}{M} + \frac{b(i+1)}{N} \right) m + m^{i,k}
\]
\(^3\)We reproduce the proof in appendix (B)
\[ \sum_{\gamma=1}^{N_a+i-1} (Na + i - \gamma) t_\gamma + \sum_{\gamma=1}^{Mb+k-1} (Mb + k - \gamma) T_\gamma + \sum_{\beta=0}^{Mb+k} \sum_{\alpha=0}^{N_a+i} m_{\alpha,\beta} \]  (3.36)

\[ z_1(Na + i + 1) + z_2(Mb + k + 1) = \]

\[
\left[ N^a(a+1) + a \right] \tau - \sum_{\gamma=1}^{N} \gamma t_\gamma + \sum_{\gamma=1}^{i-1} (i - \gamma) t_\gamma + \left[ M^b(b+1) + bk \right] \rho - \sum_{\gamma=1}^{M} \gamma T_\gamma + \sum_{\gamma=1}^{k-1} (k - \gamma) T_\gamma +
\]

\[ (ab + (k+1) \frac{M}{M^2} + \frac{1}{M^2}) \right. \]

\[ m + m^i + m^\beta + z_1(Na + i + 1) + z_2(Mb + k + 1) \]

\[ G_{N,M}^{i,k}(t, T, m) + \frac{1}{2} (a + \frac{i+1}{N}, b + \frac{k+1}{M}) \left( m^i m^\beta + \left( a + \frac{i+1}{N} \right) \left( b + \frac{k+1}{M} \right) \right) + a \tau (\frac{N}{2} - 1) + b \rho (\frac{M}{2} - 1) \]

\[ \frac{Nz_1(m + u) + \frac{1}{2} (a + u) \cdot (\bar{z} + v) + (m + u) \cdot (\bar{z} + v)}{(Nz_1, Mz_2) + (\frac{i+1}{N}, \frac{k+1}{M})} \]

We define the genus two theta function as:

\[ \Theta_{\bar{u}, \bar{v}}(\bar{z}, \Omega) = \sum_{\bar{u} \in \mathbb{Z}^2} e^{\frac{1}{2} (\bar{u} \cdot \bar{u}) \Omega + (\bar{u} \cdot \bar{u}) \cdot (\bar{z} + \bar{v})} \]  (3.38)

Then

\[ W_{i,k}^{i,k}(X, Y) = e^{G_{N,M}^{i,k}} \Theta_{\bar{u}, \bar{v}}(\bar{z}, \Omega) \]  (3.39)

The genus of the mirror curve

\[ \sum_{i=0,k=0}^{N-1,M-1} W_{i,k}(X, Y) = 0 \]  (3.40)

is \( MN + 1 \). The underlying abelian surface has polarisation \((N, M)\) with the period \( \Omega = \left( \begin{array}{c} N \tau m \\ M \rho \end{array} \right) \). The theta functions form a basis of this \((N, M)\)-polarization of the abelian surface. The mirror curve for general values of \( N \) and \( M \) cannot be factorized in the limit \( z \to 0 \).
3.5 Geometric interpretation of the mirror curve

An interesting way to visualise the mirror curve $\Sigma$ is to see it as $N$ copies of the base torus glued together by $N-1$ branch cuts. The one cycles, $A_i$ and $B_i$, of the base torus are lifted to a basis of 1-cycles $A_i, B_i, i = 1, ..., N$ on $\Sigma$. Riemann-Hurwitz theorem is used to compute the genus of $\Sigma$ and is equal to $N$. Riemann-Roch theorem is handy in the computation of the number of moduli of $\Sigma$, which is equal to $N$ in this case.

In the case under consideration, the genus $N$ Riemann surface is seen as defined by theta divisor. A polarised abelian variety $U$ admits a line bundle $L$ with $c_1(L) = \omega$ where $0 \leq y_i \leq 1$ define the abelian variety where

$$\omega = [Ndy_1 \wedge dy_3 + Mdy_2 \wedge dy_4]$$  \hspace{1cm} (3.41)

$\omega$ is a $(1,1)$-form if the period matrix $\Omega$ of $\Sigma$ is symmetric and $Im(\Omega) > 0$. The line bundle $L$ admits $MN$ holomorphic sections. In the case of an abelian surface these sections are given by genus 2 theta functions

$$\Theta \left[ \begin{array}{ccc} i & j & \frac{1}{M} \\ 0 & 0 & \frac{1}{N} \end{array} \right] (z|\Omega) \hspace{0.5cm} 0 \leq i < M, \hspace{0.5cm} 0 \leq j < N. \hspace{1cm} (3.42)$$

A theta divisor is the zero locus of a linear combination of the above set of theta functions

$$\sum_{i}^{M} \sum_{j}^{N} A_{ij} \Theta \left[ \begin{array}{ccc} i & j & \frac{1}{M} \\ 0 & 0 & \frac{1}{N} \end{array} \right] (z|\bar{\Omega}) = 0$$  \hspace{1cm} (3.43)

where $A_{ij}$ are the moduli of the curve. This zero locus defines the mirror curve of genus $MN+1$ and is the Riemann surface $\Sigma$. For the special case of $M = 1$ the mirror curve can be expressed in the following form

$$\sum_{n=0}^{N} \frac{1}{n!} \left( \frac{m}{2\pi i} \right)^n \partial_x^n \theta_1(z|\tau) \partial_x^n h(x) = 0 \hspace{1cm} (3.44)$$

where $\theta_1$ is the Jacobi theta function and $h(x) = \prod_{j=1}^{N} \theta_1(x - \xi_j|\rho)$ with $\xi_j$ is the moduli of $\Sigma$. This can be reorganised into the following form

$$\Theta_{\frac{1}{2}, \ldots, \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}}(z, N\beta \frac{2\pi}{2\pi}, \ldots, \frac{2\pi}{2\pi}) = 0 \hspace{1cm} (3.45)$$

where $\bar{\Omega}$ is the period matrix of the genus $MN+1$ curve $\hat{\Sigma}$ which is an unbranched cover of a genus 2 curve and in general is given by

$$\hat{\Omega} = \left[ \begin{array}{cccccccc} \tau & \beta_{m_1} & \beta_{m_2} & \beta_{m_3} & \cdots & \beta_{m_{MN}} \\ \beta_{m_1} & 2\pi i & 0 & 0 & \cdots & 0 \\ \beta_{m_2} & 0 & 2\pi i & 0 & \cdots & 0 \\ \beta_{m_3} & 0 & 0 & 2\pi i & \cdots & 0 \\ \beta_{m_4} & \cdots & \cdots & \cdots & \cdots & \cdots \\ \beta_{m_{MN}} & 0 & 0 & 0 & \cdots & \rho \end{array} \right]$$  \hspace{1cm} (3.46)
It is easy to see from the following representation of genus \( g = MN + 1 \) theta function

\[
\Theta \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right] (Z|\hat{\Omega}) = \sum_{m \in \mathbb{Z}^g} \exp \left( \pi i (m + \alpha).\hat{\Omega}.(m + \alpha) + 2\pi i (Z + \beta).(m + \alpha) \right) \tag{3.47}
\]

where \( Z, \alpha, \beta, m \) are \( g \)-vectors and \( \Omega \) is a \( g \times g \) matrix with \( \text{Im} \Omega > 0 \). In the limit \( m_N \to 0 \) the genus \( N + 1 \) theta function gets split into the product of genus \( N \) theta function and a jacobi theta function \((g = 1)\) with characteristics. For instance the mirror curve for \( X_{1,2} \) is given by

\[
\Theta \left( \Omega(2\rho, 2z, \tau)|(2u - \rho + z + w, v - \tau) \right) + e^{2\pi i u} \Theta \left( \Omega(2\rho, 2z, \tau)|(2u + z + w, v - \tau + z) \right) = 0 \tag{3.48}
\]

In the limit of \( \text{Im}(u) \to \infty \) this curve collapses to

\[
\Theta \left( \Omega(2\rho, 2z, \tau)|(2u - \rho + z + w, v - \tau) \right) = 0 \tag{3.49}
\]

which, after performing \( Sp(2, \mathbb{Z}) \) transformations, is the curve mirror to \( X_{1,1} \).

### 3.6 Splitting of higher genus theta functions and M5-branes partition functions

To study the decomposition of generalised theta function [11] defined on the Jacobian of a genus \( g = M \) curve, we start from the following Fourier representation

\[
\Theta(\Omega|\vec{z}) = \sum_{m \in \mathbb{Z}^M} e^{2\pi i \sum_{i=1}^{M} m_i z_i + i\pi \sum_{i,j=1}^{M} m_i \Omega_{ij} m_j} \tag{3.50}
\]

where \( \Omega \) is the period matrix and satisfies the following constraints

\[
\sum_{i=1}^{M} \Omega_{ij} = \tau, \quad \text{and} \quad \sum_{j=1}^{M} \Omega_{ij} = \tau \tag{3.51}
\]

This constraint encodes various periodicity properties. In other words we can decompose \( \Omega \) as

\[
\Omega = \frac{\tau}{M} + \Omega' \tag{3.52}
\]

where \( \Omega' \) is the traceless part. Now redefine \( z_i \) as follows

\[
z_i = \frac{z}{M} + z'_i \quad \text{such that} \quad \sum_{i=1}^{M} z'_i = 0 \tag{3.53}
\]
Putting back these redefined variable in (3.50) we get
\[
\Theta(\Omega|\vec{z}) = \sum_{m \in \mathbb{Z}^M} e^{2\pi i \frac{\vec{z} \cdot \sum_{i=1}^M m_i + \pi i (\sum_{i=1}^M m_i)^2} {M}} + 2\pi i \sum_{i=1}^M m_i \Omega_{ij} m_j \tag{3.54}
\]

Next we use a trick, essentially a redefinition of indices, to write the exponential in a suggestive form. To this end we decompose the set of indices \(m\) into two parts. First we impose the constraint that \(\sum_{i=1}^M m_i = l\) which effectively reduces the set \(\{m_1, ..., m_M\}\) to \(\{m_1, ..., m_{M-1}\}\). Secondly we perform a sum over \(l\).

\[
\Theta(\Omega|\vec{z}) = \sum_{l \in \mathbb{Z}} e^{2\pi i \frac{\vec{z} \cdot \sum_{i=1}^M m_i + \pi i (\sum_{i=1}^M m_i)^2} {M}} \sum_{m \in \mathbb{Z}^M} e^{i2\pi i \sum_{i=1}^M m_i z_i' + \pi i \sum_{i,j=1}^M m_i \Omega_{ij} m_j} \tag{3.55}
\]

To be able to write the first summation as a Jacobi theta function with characteristics, we make another redefinition \(l = Ms + i\) where \(s \in \mathbb{Z}\) and \(i \in \mathbb{Z}_N\), resulting in

\[
\Theta(\Omega|\vec{z}) = \sum_{i \in \mathbb{Z}_N} e^{2\pi i (s + \frac{i}{M}) \vec{z} \cdot \sum_{i=1}^M m_i + \pi i (s + \frac{i}{M})^2} \sum_{m \in \mathbb{Z}^M} e^{i2\pi \sum_{p=1}^M m_p z_p' + \pi i \sum_{p,q=1}^M m_p \Omega_{pq} m_q} \tag{3.56}
\]

where \(\Theta_i\) is the second summation factor in the first line of (3.56).

The splitting of theta functions has important consequences for M5-branes partition function. On general grounds [12] the partition functions of M5-branes on a six-manifold \(X\) are actually sections of a line bundle \(L\) over the intermediate Jacobian \(J_X = H^3(X, \mathbb{R})/H^3(W, \mathbb{Z})\). The intermediate Jacobian for our CY 3-fold is an abelian surface. The line bundle \(L\) is uniquely specified by its first Chern class \(c_1(L) = \omega\), where \(\omega \in H^2(J, \mathbb{Z})\) gives the principal polarisation. For the case at hand we have \(M\) M5 branes that are probing the transverse space \(S^1 \times \mathbb{C}^2/\mathbb{Z}_N\).

The partition function of this theory correspond to sections of a line bundle \(L\) of polarisation \((M, N)\) and readily given by eq.(3.38)

\[
\Theta_{\vec{u}, \vec{v}}(\vec{z}, \Omega) = \sum_{\vec{n} \in \mathbb{Z}^2} e^{\frac{1}{2}(\vec{n} + \vec{u})^2 \Omega(\vec{n} + \vec{u}) + (\vec{n} + \vec{u}) \cdot (\vec{z} + \vec{v})} \tag{3.57}
\]

and there are \(MN\) of them. The eq.(3.56) then shows that a genus \(g\) theta function splits into a product of genus \(g - 1\) theta function and an ordinary theta function. Therefore same must be true of the theta functions (3.57) that describe M5-branes partition functions.
4 Degenerations and their Effect on the Partition Function

The partition function for the \((N,M)\) configuration of the CY 3fold which we denoted by \(X_{N,M}\) is given by

\[
Z_{(N,M)}(\tau, \rho, \epsilon_{1,2}, m, t) = \sum_{\alpha} \prod_{i=1}^{N} Q_i^{\alpha(i)} \prod_{a=1}^{M} \prod_{i=1}^{N} \frac{\vartheta_{\alpha_a^{i+1} \alpha_a^{i}}(m)}{\vartheta_{\alpha_a^{i}}(\epsilon_+)}
\]

\[
\prod_{1 \leq a < b \leq M} \prod_{i=1}^{N} \frac{\vartheta_{\alpha_a^{i+1} \alpha_a^{i}}(t_{ab} - m) \vartheta_{\alpha_b^{i+1} \alpha_b^{i}}(t_{ab} + m)}{\vartheta_{\alpha_a^{i}}(t_{ab} - \epsilon_+) \vartheta_{\alpha_b^{i}}(t_{ab} + \epsilon_+)}
\]

where the sum is over the \(N\) partitions of \(\alpha^i = \{\alpha_a^{(1)}, \alpha_a^{(2)}, ..., \alpha_a^{(N)}\}\) and \(\alpha_a^{(1)} \equiv \alpha_a^{(N+1)}\).

\[Q_i = e^{t_i}\] where \(t_i\) is the distance between red lines (M5 brane).

In this paper we will restrict our study of degeneration to the special point in the Kähler moduli space where \(Q_i := Q := e^{2\pi i \tau}\) and to the unrefined limit of the \(\Omega\)-background parameters \(\epsilon_1 = -\epsilon_2 = \epsilon\).

We defined

\[|\alpha^i| = \sum_{i=1}^{N} |\alpha_a^i|\]

where \(|\alpha_a^i|\) is the size of the partition \(\alpha_a^i\) which is the sum of the parts of partition, \(m\) is the mass parameter and \(t_{ab}\) is the distance between blue lines (M2 branes), \(t_{ab} = m, \epsilon_+ = \frac{\epsilon_1 + \epsilon_2}{2}\).
For two integer partitions $\mu$ and $\nu$, theta function $\vartheta_{\mu\nu}$ in above partition function (4.1) is defined as

$$\vartheta_{\mu\nu}(x) = \prod_{(i,j) \in \mu} \vartheta(\rho, e^{-x} t^{-\nu_j+i-\frac{1}{2}} q^{-\mu_i+j-\frac{1}{2}}) \prod_{(i,j) \in \nu} \vartheta(\rho, e^{-x} t^{\nu_j-i+\frac{1}{2}} q^{\mu_i-j+\frac{1}{2}})$$  \hspace{1cm} (4.3)

Here $t = e^{-i\tau_2}, q = e^{i\tau_1}, \nu^t$ represents the transpose of the partition $\nu$ and product $\prod_{(i,j) \in \nu}$ means that the product is over all the boxes of the Young diagram corresponding to the partition $\nu$ having length $\ell(\nu)$

$$(i, j) \in \nu, \text{ implies that } 1 \leq i \leq \ell(\nu), \ 1 \leq j \leq \nu_i.$$  

The Jacobi theta function $\vartheta(\rho, y)$ for $y = e^{2\pi iz}$ is defined as

$$\vartheta(\rho, y) = (y^{\frac{1}{2}} - y^{-\frac{1}{2}}) \prod_{k=1}^{\infty} (1 - y e^{2\pi ik\rho})(1 - y^{-1} e^{-2\pi ik\rho})$$

For $x = 0$ and in unrefined case

$$\vartheta_{\mu\nu}(0) = - \prod_{(i,j) \in \mu} \vartheta(\rho, \nu_j^\mu - i + \mu_i - j + 1) \prod_{(i,j) \in \nu} \vartheta(\rho, \mu_i^\nu - i + \nu_i - j + 1)$$

$$= - \prod_{(i,j) \in \mu} \vartheta(\rho, h_\mu(i, j) + \nu_i^\mu - \mu_j^\mu) \prod_{(i,j) \in \nu} \vartheta(\rho, h_\nu(i, j) + \mu_i^\nu - \nu_j^\nu)$$  \hspace{1cm} (4.4)

where $h_\mu(i, j) = \mu_i + \mu_j^\mu - i - j + 1$ is the hook length of the partition $\mu$. Since, the Jacobi theta function $\vartheta(\rho, z)$ is an odd function w.r.t. $z$ i.e., $\vartheta(\rho, 0) = 0$, therefore $\vartheta_{\mu\nu}(0) = 0$ if $h_\mu(i, j) + \nu_j^\mu - \mu_j^\mu = 0$. Since, $h_\mu(i, j) \neq 0$, therefore $\nu_j^\mu \neq \mu_j^\mu$ if $\mu = \nu$ then

$$\vartheta_{\mu\nu}(0) = \prod_{(i,j) \in \mu} \vartheta(\rho, h_\mu(i, j))^2$$

$h_\mu(i, j)$ is non zero therefore $\vartheta_{\mu\nu}(0) \neq 0$. To prove that $\mu \neq \nu$ implies $\vartheta_{\mu\nu}(0) = 0$ i.e., Either $h_\mu(i, j) + \nu_j^\mu - \mu_j^\mu = 0$ or $h_\nu(i, j) + \mu_i^\nu - \nu_j^\nu = 0$. Because $h_\mu(i, j) \neq 0$ therefore $\nu_j^\mu \neq \mu_j^\mu$. An interesting property of $\vartheta_{\mu\nu}(x)$ which we will use extensively in later sections is the following,

$$\vartheta_{\mu\nu}(0) = \delta_{\mu\nu} \prod_{(i,j) \in \mu} \vartheta(q^{h_\mu(i,j)}) \vartheta(q^{-h_\nu(i,j)})$$  \hspace{1cm} (4.5)

where $\delta_{\mu\nu}$ is the kronecker delta function.

4.1 \hspace{0.5cm} (N, M) = (1, 2)

We begin by looking at the case of $X_{1,2}$. The unrefined partition function is given by,

$$Z_{(1,2)}(\tau, \rho, m, t, \epsilon) = \sum_{\alpha_{1,2}} Q^{||\alpha_1||+||\alpha_2||} \frac{\vartheta_{\alpha_1\alpha_1}(m)\vartheta_{\alpha_2\alpha_2}(m) \vartheta_{\alpha_1(0)} \vartheta_{\alpha_2(0)}}{\vartheta_{\alpha_1\alpha_1}(t_m)\vartheta_{\alpha_2(0)} \vartheta_{\alpha_1\alpha_2}(t_m^\epsilon) \vartheta_{\alpha_2(0)}(t)^2}$$  \hspace{1cm} (4.6)
Here, $t^{-}_m = t - m$ and $t^{+}_m = t + m$. The above defined partition function $Z_{(1,2)}$ in (4.6) in the limit $t \mapsto m$ changes to

$$Z_{(1,2)}(\tau, \rho, m, t = m, \epsilon) = \sum_{\alpha_1, \alpha_2} Q^{(1)_{\alpha_1} + (2)_{\alpha_2}} \frac{\partial_{a_1a_1}(m) \partial_{a_2a_2}(m) \partial_{\alpha_1\alpha_2}(0) \partial_{\alpha_2\alpha_2}(2m)}{\partial_{a_1a_1}(0) \partial_{a_2a_2}(0) \partial_{\alpha_1\alpha_2}(m)^2}$$

Using the property of $\partial_{\mu\nu}(x)$ defined in eq.(4.5) we get

$$Z_{(1,2)}(\tau, \rho, m, t = m, \epsilon) = \sum_{\alpha_1} Q^{2(\alpha_1)} \frac{\partial_{a_1a_1}(2m)}{\partial_{a_1a_1}(0)} = Z_{(1,1)}(2\tau, \rho, 2m, \epsilon)$$

$$(4.7)$$

$$(4.8)$$

$$(4.9)$$

$$(4.10)$$

$$(4.11)$$

$$(4.12)$$
4.3 General \((N,M)\)

\[
Z_{(N,M)}(\tau, \rho, m, t, \epsilon) = \sum_{\alpha} \prod_{i=1}^{N} Q_{i}^{i(\alpha)} \prod_{i=1}^{N} M^{i(\alpha)} \frac{\partial^{i+1} \alpha_{a}^{i} (m)}{\partial^{i} \alpha_{a}^{i} (0)} \prod_{1 \leq a < b \leq M}^{N} \frac{\partial^{i+1} \alpha_{ab}^{i+1} (t_{ab} - m)}{\partial^{i} \alpha_{ab}^{i} (t_{ab})} \frac{\partial^{i+1} \alpha_{ab}^{i+1} (t_{ab} + m)}{\partial^{i} \alpha_{ab}^{i} (t_{ab})} \tag{4.12}
\]

As in the previous section we see that

\[
Z_{(1,M)}(\tau, \rho, m, \epsilon) = \sum_{\alpha}^{N} Q_{i}^{i(\alpha)} \frac{\partial^{i+1} \alpha_{a}^{i} (M m)}{\partial^{i} \alpha_{a}^{i} (0)}
\]

By generalizing the case of \(Z_{(1,M)}\),

\[
Z_{(N,M)}(\tau, \rho, t = m, \epsilon) = \sum_{\alpha}^{N} Q_{i}^{i(\alpha)} \frac{\partial^{i+1} \alpha_{a}^{i} (M m)}{\partial^{i} \alpha_{a}^{i} (0)} \prod_{1 \leq a < b \leq M}^{N} \frac{\partial^{i+1} \alpha_{ab}^{i+1} (t_{ab})}{\partial^{i} \alpha_{ab}^{i} (t_{ab})} \frac{\partial^{i+1} \alpha_{ab}^{i+1} (t_{ab})}{\partial^{i} \alpha_{ab}^{i} (t_{ab})} \tag{4.13}
\]

So, in general

\[
Z_{(N,M)}(\tau, \rho, t_{a.a+1} = m, \epsilon) = Z_{(1,1)}(\tau, \rho, (M m, \epsilon))^{N} \tag{4.14}
\]

4.4 Factorisation: \(Z_{(M,N)} \rightarrow Z_{(1,1)}^{N}\)

Recall that, on a arbitrary point of the Kähler cone, the number of independent Kähler parameters entering the partition function are

\[
\#(T_{a}s) + \#(t_{i}s) + \#(intersections) - \#(horizontal \ constraints) - \#(vertical \ constraints) + 2
\]

\[
= (M - 1) + (N - 1) + MN - (M - 1) - (N - 1) + 2
\]

\[
= MN + 2 \tag{4.15}
\]

In general we can have three different series representations of \(Z_{(M,N)}\) according to whether the toric web diagram of \(X_{M,N}\) is sliced into horizontal strips, vertical strips and diagonal strips

\[
Z_{(M,N)}(t, T, m, \epsilon_{1}, \epsilon_{2}) = Z^{pert}_{(T, m)} \sum_{k} e^{-\bar{k} \cdot t} Z_{k}^{(T, m)}
\]

\[
Z_{(M,N)}(t, T, m, \epsilon_{1}, \epsilon_{2}) = Z^{pert}_{(t, m)} \sum_{k} e^{-\bar{k} \cdot T} Z_{k}^{(t, m)}
\]

\[
Z_{(M,N)}(t, T, m, \epsilon_{1}, \epsilon_{2}) = Z^{pert}_{(t, t)} \sum_{k} e^{-\bar{k} \cdot m} Z_{k}^{(t, t)} \tag{4.16}
\]
These expansion have been interpreted as instanton expansions of three gauge theories which are dual to each other. For these to be consistent expansions it is assumed that there exists a region of the moduli space of $X_{(M,N)}$ in which either either $T$ or $t$ or $m$ become infinite, with all the rest of parameters kept finite. This region of the moduli space corresponds to the weak coupling limit of gauge theories.

At the special point in the moduli space where $t_{a,a+1} = m$, we are left with three independent Kähler parameters, $\tau, \rho, m$. Moreover due to the weak coupling expansion $\{T \to \infty\}$, $N$
horizontal strips gets decoupled and we get $Z_{1,1}^N$.

**Remark 1:**

After normalisation by the gauge theory perturbative part, the partition function $Z_{(1,1)}(\tau, \rho, m)$ can be written as [13]

$$Z_{(1,1)}(\tau, \rho, m) = e^{-\frac{\pi i (\tau + \rho + m)}{12}} \prod_{(k,l,m) > 0} \left( 1 - e^{2\pi i (k\tau + l\rho + pm)} - e(4kl - p^2) \right)$$

$$= \frac{1}{\Phi_{10}(\tau, \rho, m)}$$  \hspace{1cm} (4.17)

where $e(4kl - p^2)$ is the Fourier coefficient of the elliptic genus of $K3$

$$\chi(K3, \tau, z) = \sum_{h \geq 0, m \in \mathbb{Z}} 24c(4h - m^2)e^{2\pi i (h\tau + mz)}$$  \hspace{1cm} (4.18)

and $\Phi_{10}(\tau, \rho, m)$ is the unique weight 10 automorphic form of $Sp(2, \mathbb{Z})$. We have implicit used the fact that the large radius limit (universal part) of the Taub-NUT elliptic genus matches with the elliptic genus of $\mathbb{C}^2$ [14]. This allows us to write $Z_{(N,M)}(\tau, \rho, t_{a,a+1} = m)$ in the following way

$$Z_{(N,M)}(\tau, \rho, t_{a,a+1} = m) = e^{-\frac{N\pi i (\tau + \rho + m)}{12}} \prod_{(k,l,m) > 0} \left( 1 - e^{2\pi i (Mk\tau + l\rho + pm)} - Ne(4kl - p^2) \right)$$

$$= \frac{1}{\Phi_{10}(M\tau, \rho, MMm)}$$  \hspace{1cm} (4.19)

**Remark 2:**

The CY 3-fold $X_{1,1}$ has a nice interpretation in terms of the so-called banana curves [15]. A banana configuration of curves in the CY 3-fold is a union of three curves $C_i \equiv \mathbb{P}^1$ with the normal bundle given by $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Moreover $C_1 \cap C_2 = C_2 \cap C_3 = C_3 \cap C_1 = \{x, y\}$ for distinct point $x, y \in CY$ 3-fold and there exist a preferred coordinate patch in which $C_i$ are along the coordinate axis.

In other words the refined topological string partition function $Z_{X_{N,M}}(\omega, \epsilon_1, \epsilon_2)$ is factored into a product of $N$ copies of $Z_{X_{1,1}}(\tau, \rho, m)$, where the later is the topological partition function on a CY 3-fold with a single banana configuration of curves.

### 5 Splitting Degeneration

In the case of splitting degeneration we consider the following partition function for $(N, M)$

$$Z_{(N,M)}(\tau, \rho, m_{a,b}, \epsilon_{1,2}, t_{ab}) = \sum_{\alpha^1_a} \prod_{i=1}^N Q_{a_i}^{(i)} \prod_{i=1}^M \frac{\partial_{\alpha_{a}^i + 1} (m_a)}{\partial_{\alpha_{b}^i} (\epsilon_+)} \prod_{1 \leq a < b \leq M} \frac{\partial_{\alpha_{a}^i \alpha_{b}^{i+1}} (t_{ab} - m_a) \partial_{\alpha_{a}^i \alpha_{b}^{i+1}} (t_{ab} + m_b)}{\partial_{\alpha_{a}^i \alpha_{b}^{i}} (t_{ab} - \epsilon_+) \partial_{\alpha_{a}^i \alpha_{b}^{i}} (t_{ab} + \epsilon_+)}$$ \hspace{1cm} (5.1)
Here, For \( N = 1 \) the above defined partition function reduces to
\[
Z_{(1,M)}(\tau, \rho, m_{a,b}, \epsilon_{1,2}, t_{ab}) = \sum_{\alpha_1, \ldots, \alpha_M} Q^{|\alpha_1| + \cdots + |\alpha_M|} \prod_{a=1}^{M} \frac{\vartheta_{a_\alpha} (m_a)}{\vartheta_{a_a} (\epsilon_+)} \prod_{1 \leq a < b \leq M} \frac{\vartheta_{a_\alpha_b} (t_{ab} - m_a) \vartheta_{a_a_b} (t_{ab} + m_b)}{\vartheta_{a_a_b} (t_{ab} - \epsilon_+) \vartheta_{a_a_b} (t_{ab} + \epsilon_+)}
\]

Note that \( \sum_{\text{all partitions } \mu} Q^{\mid \mu \mid} = \frac{\vartheta_1}{\vartheta(\tau)} \). This factor appears in the degeneration limit as discussed below.

From mathematical viewpoint similar degenerations of certain CY 3-folds have been discussed in [16, 17].

5.1 \((N, M) = (1, 2)\)

For \( M = 2 \) in the unrefined case (\( \epsilon_1 = -\epsilon_2 \)), we have
\[
Z_{(1,2)}(\tau, \rho, m_{1,2}, \epsilon, t_{12}) = \sum_{\alpha_{1,2}} Q^{\mid \alpha_{1,2} \mid} \frac{\vartheta_{\alpha_1 \alpha_1} (m_1) \vartheta_{\alpha_2 \alpha_2} (m_2)}{\vartheta_{\alpha_1 \alpha_1} (0) \vartheta_{\alpha_2 \alpha_2} (0)} \frac{\vartheta_{\alpha_1 \alpha_2} (t_{12} - m_1) \vartheta_{\alpha_1 \alpha_2} (t_{12} + m_2)}{\vartheta_{\alpha_1 \alpha_2} (t_{12}) \vartheta_{\alpha_1 \alpha_2} (t_{12})}
\]

Here, \( m_1 \neq m_2, t_{12} - m_1 = t_{\sim} \) and \( t_{12} - m_2 = t_{\sim} + m_1 + m_2, t_{12} \) is the distance between the blue lines and \( t_{\sim} \) is the red line. For better understanding of this partition function for different limits we write it as
\[
Z_{(1,2)}(\tau, \rho, m_{1,2}, t_{\sim}, \epsilon) = \sum_{\alpha_{1,2}} Q^{\mid \alpha_{1,2} \mid} \frac{\vartheta_{\alpha_1 \alpha_1} (m_1) \vartheta_{\alpha_2 \alpha_2} (m_2)}{\vartheta_{\alpha_1 \alpha_1} (0) \vartheta_{\alpha_2 \alpha_2} (0)} \frac{\vartheta_{\alpha_1 \alpha_2} (t_{\sim}) \vartheta_{\alpha_1 \alpha_2} (t_{\sim} + m_1 + m_2)}{\vartheta_{\alpha_1 \alpha_2} (t_{\sim} + m_1) \vartheta_{\alpha_1 \alpha_2} (t_{\sim} + m_2)}
\]

In the limits any \( m_1 \rightarrow 0, m_2 \rightarrow 0 \) or \( t_{\sim} \rightarrow 0 \), \( Z_{(1,2)} \) reduces to the \( Z_{(1,1)} \). More explicitly
\[
Z_{(1,2)}(\tau, \rho, m_1 = 0, m_2, t_{\sim}, \epsilon) = \sum_{\alpha_{1,2}} Q^{\mid \alpha_{1,2} \mid} \frac{\vartheta_{\alpha_2 \alpha_2} (m_2)}{\vartheta_{\alpha_2 \alpha_2} (0)} = \sum_{\alpha_1} Q^{\mid \alpha_1 \mid} Z_{(1,1)}(\tau, \rho, m_2, \epsilon) \quad (5.2)
\]
\[
Z_{(1,2)}(\tau, \rho, m_1, m_2 = 0, t_{\sim}, \epsilon) = \sum_{\alpha_{1,2}} Q^{\mid \alpha_{1,2} \mid} \frac{\vartheta_{\alpha_2 \alpha_2} (m_1)}{\vartheta_{\alpha_2 \alpha_2} (0)} = \sum_{\alpha_1} Q^{\mid \alpha_1 \mid} Z_{(1,1)}(\tau, \rho, m_1, \epsilon) \quad (5.3)
\]
and
\[
Z_{(1,2)}(\tau, \rho, m_1, m_2, t_{\sim} = 0, \epsilon) = \sum_{\alpha_{1,2}} Q^{\mid \alpha_{1,2} \mid} \frac{\vartheta_{\alpha_2 \alpha_2} (m_2)}{\vartheta_{\alpha_2 \alpha_2} (0)} = \sum_{\alpha_1} Q^{\mid \alpha_1 \mid} Z_{(1,1)}(\tau, \rho, m_2, \epsilon) \quad (5.4)
\]
5.2 \( (N, M) = (1, 3) \)

For \( M = 3 \),

\[
Z_{(1,3)}(\tau, \rho, m_{1,2,3}, t_{12,23,13}, \epsilon) = \sum_{\alpha_{1,2,3}} Q^{\alpha_1+|\alpha_2|+|\alpha_3|} \frac{\partial_{\alpha_1\alpha_3}(m_1)\partial_{\alpha_2\alpha_2}(m_2)\partial_{\alpha_2\alpha_3}(m_3)}{\partial_{\alpha_1\alpha_1}(0)\partial_{\alpha_2\alpha_2}(0)\partial_{\alpha_2\alpha_3}(0)} \frac{\partial_{\alpha_1\alpha_2}(t_{12} - m_1)\partial_{\alpha_1\alpha_2}(t_{12} + m_2)\partial_{\alpha_2\alpha_2}(t_{23} - m_2)\partial_{\alpha_2\alpha_3}(t_{23} + m_3)\partial_{\alpha_1\alpha_3}(t_{13} - m_1)\partial_{\alpha_1\alpha_3}(t_{13} + m_3)}{\partial_{\alpha_1\alpha_1}(t_{12})\partial_{\alpha_1\alpha_2}(t_{12})\partial_{\alpha_2\alpha_3}(t_{23})\partial_{\alpha_2\alpha_3}(t_{23})\partial_{\alpha_1\alpha_3}(t_{13})\partial_{\alpha_1\alpha_3}(t_{13})}
\]

(5.5)

Here, all \( m_i's \) \( i = 1, 2, 3 \) are different, \( t_{13} = t_{12} + t_{23} \) with all \( t_{ab}'s \) \( a = 1, 2 \) and \( b = 2, 3 \) different. To make the calculation simple we rewrite the partition function in the following way after making the redefinitions \( \tilde{t}_{12} = t_{12} - m_1, \tilde{t}_{23} = t_{23} - m_2 \) and \( \tilde{t}_{13} = t_{13} - m_1 \):
Figure 8: web diagram for $Z_{1,3}$

Figure 9: degeneration of the web diagram of $Z_{1,3}$ in the limit $m_3 \to 0$

Figure 10: degeneration of the web diagram of $Z_{1,3}$ in the limit $m_2 \to 0$

$$Z_{(1,3)}(\tau, \rho, m_{1,2,3}, \tilde{t}_{12,23}, \epsilon) = \sum_{\alpha_{1,2,3}} Q^{\alpha_1 + \alpha_2 + \alpha_3} \frac{\partial \alpha_1 \partial \alpha_2 \partial \alpha_3 (m_1) \partial \alpha_2 \partial \alpha_3 (m_2) \partial \alpha_3 (m_3)}{\partial \alpha_1 (0) \partial \alpha_2 (0) \partial \alpha_3 (0)}$$

$$= \frac{\partial \alpha_1 \partial \alpha_2 \partial \alpha_3 (m_1 + m_2 + m_3 + \tilde{t}_{12} + \tilde{t}_{23}) \partial \alpha_2 \partial \alpha_3 (m_2 + m_3 + \tilde{t}_{12} + \tilde{t}_{23}) \partial \alpha_3 (m_3 + \tilde{t}_{23})}{\partial \alpha_1 \partial \alpha_2 \partial \alpha_3 (m_1 + m_2 + m_3 + \tilde{t}_{12} + \tilde{t}_{23}) \partial \alpha_1 \partial \alpha_2 \partial \alpha_3 (m_2 + \tilde{t}_{12} + \tilde{t}_{23}) \partial \alpha_3 (m_1 + m_2 + \tilde{t}_{12} + \tilde{t}_{23})}$$

(5.6)
In the limits $m_1 \to 0$ or $m_2 \to 0$ or $m_3 \to 0$, $Z_{(1,3)} \to Z_{(1,2)}$. For example in case of $m_3 = 0$

$$
Z_{(1,3)}(\tau, \rho, m_{1,2}, \tilde{t}_{12}, \epsilon) = \sum_{\alpha_{1,2,3}} Q^{|\alpha_1|+|\alpha_2|+|\alpha_3|} \frac{\vartheta_{\alpha_1 \alpha_1}(m_1)\vartheta_{\alpha_2 \alpha_2}(m_2)\vartheta_{\alpha_1 \alpha_2}(\tilde{t}_{12})\vartheta_{\alpha_1 \alpha_2}(\tilde{t}_{12} + m_1 + m_2)}{\vartheta_{\alpha_1 \alpha_1}(0)\vartheta_{\alpha_2 \alpha_2}(0)\vartheta_{\alpha_1 \alpha_2}(m_1 + \tilde{t}_{12})\vartheta_{\alpha_1 \alpha_2}(m_2 + \tilde{t}_{12})}
$$

$$
= \sum_{\alpha_3} Q^{\alpha_3} Z_{(1,2)}(\tau, \rho, m_{1,2}, \tilde{t}_{12}, \epsilon)
$$

(5.7)
Same degeneration of $Z_{1,3}$ results if one takes the limit for $t_{ab} \to 0$ for any $a,b$.

\[ (N, M) = (2, 3) \]

\[ Z_{(2,3)}(\tau, \rho, m_{a,b}, \epsilon_{1,2}, t_{ab}, \epsilon) = \sum_{\alpha} \prod_{i=1}^{2} Q_{\alpha}^{a_i} \prod_{i=1}^{3} \prod_{\alpha} \frac{\partial^{\alpha+1}_{\alpha} (m_{a})}{\partial^{\alpha} (0)} \times \prod_{1 \leq a < b \leq 3} \prod_{i=1}^{2} \frac{\partial^{\alpha+1}_{\alpha} (t_{ab} - m_{a})}{\partial^{\alpha}_{\alpha} (t_{ab} + m_{b})} \]  

(5.8)

In the degenerate limit $m_1 = 0$ and using the identity (4.5) we get

\[ Z_{(2,3)}(\tau, \rho, m_{2,3}, t_{ab}, \epsilon) = \sum_{\alpha} Q^{\alpha_1 + \alpha_2} \left( \frac{\partial^{\alpha_1}_{\alpha_1} (0)}{\partial^{\alpha_2}_{\alpha_1} (0)} \right)^2 \]

\[ \times \left[ \sum_{\alpha_2, \alpha_3, \alpha_1} Q^{\alpha_1 + \alpha_2 + \alpha_3} \prod_{i=1}^{2} \prod_{a=2}^{3} \frac{\partial^{\alpha+1}_{\alpha} (m_{a})}{\partial^{\alpha}_{\alpha} (0)} \right] \]

\[ \times \left[ \prod_{2 \leq a < b \leq 3} \prod_{i=1}^{2} \frac{\partial^{\alpha+1}_{\alpha} (t_{ab} - m_{a})}{\partial^{\alpha}_{\alpha} (t_{ab} + m_{b})} \right] \]

\[ \times \left( \frac{\partial^{\alpha_1}_{\alpha_1} (t_{12} + m_2) \partial^{\alpha_1}_{\alpha_1} (t_{12} + m_2)}{\partial^{\alpha_1}_{\alpha_2} (t_{12}) \partial^{\alpha_1}_{\alpha_2} (t_{12})} \frac{\partial^{\alpha_1}_{\alpha_1} (t_{13} + m_3) \partial^{\alpha_1}_{\alpha_3} (t_{13} + m_3)}{\partial^{\alpha_1}_{\alpha_3} (t_{13}) \partial^{\alpha_1}_{\alpha_3} (t_{13})} \right) \]  

(5.9)

Recognizing the $Z_{(2,2)}(\tau, \rho, m_{2,3}, t_{ab}, \epsilon)$ part, the last expression can be written more succinctly as

\[ Z_{(2,3)}(\tau, \rho, m_{2,3}, t_{ab}, \epsilon) = \sum_{\alpha} Q^{\alpha_1 + \alpha_2} \left( \frac{\partial^{\alpha_1}_{\alpha_1} (0)}{\partial^{\alpha_2}_{\alpha_1} (0)} \right)^2 \]

\[ \times \left[ \sum_{\alpha_2, \alpha_3, \alpha_1} Z_{(2,2)}^{\alpha_1, \alpha_2, \alpha_3} (\tau, \rho, m_{2,3}, t_{23}, \epsilon) \right] \]

\[ \times \left( \frac{\partial^{\alpha_1}_{\alpha_1} (t_{12} + m_2) \partial^{\alpha_1}_{\alpha_1} (t_{12} + m_2)}{\partial^{\alpha_1}_{\alpha_2} (t_{12}) \partial^{\alpha_1}_{\alpha_2} (t_{12})} \frac{\partial^{\alpha_1}_{\alpha_1} (t_{13} + m_3) \partial^{\alpha_1}_{\alpha_3} (t_{13} + m_3)}{\partial^{\alpha_1}_{\alpha_3} (t_{13}) \partial^{\alpha_1}_{\alpha_3} (t_{13})} \right) \]  

(5.10)

where $Z_{(2,2)}^{\alpha_1, \alpha_2, \alpha_3} (\tau, \rho, m_{2,3}, t_{23}, \epsilon)$ is defined by

\[ Z_{(2,2)}(\tau, \rho, m_{2,3}, t_{23}, \epsilon) := \sum_{\alpha} Z_{(2,2)}^{\alpha_1, \alpha_2, \alpha_3} (\tau, \rho, m_{2,3}, t_{23}, \epsilon) \]  

(5.11)

Similar degeneration results from taking the limits $m_2 = 0$ or $m_3 = 0$. 

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5.4 General \((N, M)\)

For the unrefined case \(\epsilon_1 = -\epsilon_2\)

\[
Z_{(N, M)}(\tau, \rho, m_i, \epsilon_{1,2}, t_{ab}) = \sum_{a_b^i} \prod_{i=1}^{N} Q_i^{[a_i^{(i)}]} \prod_{i=1}^{N} \prod_{a=1}^{M} \frac{\vartheta_{a_i^{a}a_i^{a}}(m_a)}{\vartheta_{a_i^{a}a_i^{a}}(0)} \times \prod_{1 \leq a < b \leq M} \frac{\vartheta_{a_i^{a}a_i^{a}}(t_{ab} - m_a)\vartheta_{a_i^{a}a_i^{a}}(t_{ab} + m_b)}{\vartheta_{a_i^{a}a_i^{a}}(t_{ab})} \times \frac{\vartheta_{a_i^{a}a_i^{a}}(t_{ab} + m_a + m_{a+1} + \ldots + m_{b-1})}{\vartheta_{a_i^{a}a_i^{a}}(t_{ab} + m_a + \ldots + m_{b-1})} = \sum_{a_b^i} \prod_{i=1}^{N} Q_i^{[a_i^{(i)}]} \prod_{i=1}^{N} \prod_{a=1}^{M} \frac{\vartheta_{a_i^{a}a_i^{a}}(m_a)}{\vartheta_{a_i^{a}a_i^{a}}(0)} \times \prod_{a=2}^{M} \prod_{b=a+1}^{M} \frac{\vartheta_{a_i^{a}a_i^{a}}(t_{aa+1} + t_{a+1a+2} + \ldots + t_{b-1b} + m_a + m_{a+1} + \ldots + m_{b-1})}{\vartheta_{a_i^{a}a_i^{a}}(t_{aa+1} + t_{a+1a+2} + \ldots + t_{b-1b} + m_a + m_{a+1} + \ldots + m_{b-1})} \times \frac{\vartheta_{a_i^{a}a_i^{a}}(t_{aa+1} + t_{a+1a+2} + \ldots + t_{b-1b} + m_a + m_{a+1} + \ldots + m_{b-1} + m_b)}{\vartheta_{a_i^{a}a_i^{a}}(t_{aa+1} + t_{a+1a+2} + \ldots + t_{b-1b} + m_a + m_{a+1} + \ldots + m_{b-1} + m_b)}
\]

More explicitly

\[
Z_{(N, M)}(\tau, \rho, m_a, \epsilon, t_{ab}) = \sum_{a_b^i} \prod_{i=1}^{N} Q_i^{[a_i^{(i)}]} \prod_{i=1}^{N} \prod_{a=2}^{M} \frac{\vartheta_{a_i^{a}a_i^{a}}(m_1)}{\vartheta_{a_i^{a}a_i^{a}}(0)} \times \frac{\vartheta_{a_i^{a}a_i^{a}}(t_{12} + m_1 + m_2)}{\vartheta_{a_i^{a}a_i^{a}}(t_{12} + m_2)} \times \frac{\vartheta_{a_i^{a}a_i^{a}}(t_{12} + t_{23} + m_2)\vartheta_{a_i^{a}a_i^{a}}(t_{12} + t_{23} + m_1 + m_2 + m_3)}{\vartheta_{a_i^{a}a_i^{a}}(t_{12} + t_{23} + m_1 + m_2 + m_3)} \times \frac{\vartheta_{a_i^{a}a_i^{a}}(t_{12} + t_{23} + \ldots + t_{M-1M} + m_1 + m_2 + \ldots + m_{M-1} + m_M)}{\vartheta_{a_i^{a}a_i^{a}}(t_{12} + t_{23} + \ldots + t_{M-1M} + m_1 + m_2 + \ldots + m_{M-1} + m_M)}
\]

(5.13)
Specializing to $N = 1$, $Q_i = Q$ and in the limit $m_1 = 0$ the last expression reduces to

$$Z_{(1,M)}(\tau, \rho, m, t_{ab}, \epsilon) = \sum_{\alpha} Q^{[\alpha]} \sum_{\alpha a} Q^{[\alpha a]} \prod_{a=2}^{M} \frac{\theta_{\alpha+1\alpha} (m_a)}{\vartheta_{\alpha+1\alpha} (0)}$$

$$\times \prod_{a=2}^{M-1} \prod_{b=a+1}^{M} \left( \frac{\theta_{\alpha a \alpha b} (t_{aa+1} + t_{a+1a+2} + \ldots + t_{b-1b} + m_{a+1} + \ldots + m_{b-1})}{\theta_{\alpha a \alpha b} (t_{aa+1} + t_{a+1a+2} + \ldots + t_{b-1b} + m_a + m_{a+1} + \ldots + m_{b-1})} \right) \times \frac{\theta_{\alpha a \alpha b} (t_{aa+1} + t_{a+1a+2} + \ldots + t_{b-1b} + m_a + m_{a+1} + \ldots + m_{b-1} + m_b)}{\theta_{\alpha a \alpha b} (t_{aa+1} + t_{a+1a+2} + \ldots + t_{b-1b} + m_a + m_{a+1} + \ldots + m_{b-1})}$$

$$= \sum_{\alpha} Q^{[\alpha]} Z_{(1,M-1)}(\tau, \rho, m, t_{ab}, \epsilon)$$  \hspace{1cm} (5.14)$$

where obviously $t_{ab}$ and $m_i$ do not include the moduli which are tuned to zero. More generally and at the same point in the moduli space we expect the structure similar to (5.11) for $Z_{(N,M)}$ by induction

$$Z_{(N,M)}(\tau, \rho, m, t_{ab}, \epsilon) = \sum_{\alpha} Q^{[\alpha]} \prod_{i=1}^{N} Q^{[\alpha_i]} \prod_{i=1}^{N} \prod_{a=2}^{M} \frac{\theta_{\alpha+1\alpha} (m_a)}{\vartheta_{\alpha+1\alpha} (0)}$$

$$\times \prod_{a=2}^{M-1} \prod_{b=a+1}^{M} \left( \frac{\theta_{\alpha a \alpha b} (t_{aa+1} + t_{a+1a+2} + \ldots + t_{b-1b} + m_{a+1} + \ldots + m_{b-1})}{\theta_{\alpha a \alpha b} (t_{aa+1} + t_{a+1a+2} + \ldots + t_{b-1b} + m_a + m_{a+1} + \ldots + m_{b-1})} \right) \times \frac{\theta_{\alpha a \alpha b} (t_{aa+1} + t_{a+1a+2} + \ldots + t_{b-1b} + m_a + m_{a+1} + \ldots + m_{b-1} + m_b)}{\theta_{\alpha a \alpha b} (t_{aa+1} + t_{a+1a+2} + \ldots + t_{b-1b} + m_a + m_{a+1} + \ldots + m_{b-1})}$$

$$\times \prod_{i=1}^{N} \frac{\theta_{\alpha+1\alpha} (m_1)}{\vartheta_{\alpha+1\alpha} (0)} \left( \frac{\theta_{\alpha+1\alpha} (t_{12}) \theta_{\alpha+1\alpha} (t_{12} + m_1 + m_2)}{\theta_{\alpha+1\alpha} (t_{12} + m_1) \theta_{\alpha+1\alpha} (t_{12} + m_2)} \right) \times \left( \frac{\theta_{\alpha+1\alpha} (t_{12} + t_{23} + m_2) \theta_{\alpha+1\alpha} (t_{12} + t_{23} + m_1 + m_2 + m_3)}{\theta_{\alpha+1\alpha} (t_{12} + t_{23} + m_1 + m_2) \theta_{\alpha+1\alpha} (t_{12} + t_{23} + m_2 + m_3)} \right) \times \cdots \times \left( \frac{\theta_{\alpha+1\alpha} (t_{12} + t_{23} + \ldots + t_{M-1M} + m_1 + m_2 + \ldots + m_{M-1} + m_M)}{\theta_{\alpha+1\alpha} (t_{12} + t_{23} + \ldots + t_{M-1M} + m_2 + \ldots + m_{M-1} + m_M)} \right)$$

$$= \sum_{\alpha} F^{\alpha}(\tau, \rho, m_{c,d}, t_{cd}) Z_{(N,M-1)}^{\alpha_1, \alpha_2, \ldots, \alpha_{M-1}}(\tau, \rho, m_{p,q}, t_{pq}, \epsilon)$$  \hspace{1cm} (5.15)$$

where $F^{\alpha}(\tau, \rho, m_{c,d}, t_{cd})$ is a collection of rest of the factors. Pictorially it is shown below in figure (14).
6 Physical consequences of the degenerations

Recall the following degeneration (4.14)

\[ Z_{(N,M)}(\tau, \rho, t_{a,a+1} = m, \epsilon) = Z_{(1,1)}(M \tau, \rho, M m, \epsilon)^N \]  

(6.1)

This degeneration corresponds to a $U(M)^N$ quiver gauge theory degenerating to a $U(1)^N$ gauge theory. Moreover the gauge coupling constant $\tau$ and the hypermultiplet mass parameter $m$ are scaled to $M \tau$ and $M m$ under the degeneration. This rescaling corresponds to multiple wrapping number of the D-branes along the $\tau$ and $m$ directions.

Similarly the second degeneration of the $Z_{N,M}$ (5.15) that we discussed and is given by

\[ Z_{(N,M)}(\tau, \rho, m_i, t_{ab}, \epsilon) = \sum_{\alpha_a} F^{\alpha_a^i} (\tau, \rho, m_{c,d}, t_{cd}) Z_{a_{\alpha_a i \alpha_a i \cdots M_{\alpha M} \alpha M}^{-1}} (\tau, \rho, m_{p,q}, t_{pq}, \epsilon) \]  

(6.2)

has an interesting physical interpretation. The limit $m_i \to 0$ corresponds to supersymmetry enhancement to $N = 4$ and we get a decoupling factor of $\eta(\tau)$. This is true only for $N \in \mathbb{N}, M = 1$. For $N \in \mathbb{N}, M \in \mathbb{N}_{\geq 2}$ the factorisation is only partial.

7 Conclusions

This paper explored some interesting consequences of the mirror symmetry of the local CY 3-fold $X_{N,M}$. We investigated some important properties of the type A topological string partition function on $X_{N,M}$ in special regions of the Kähler moduli space. We have called these degenerate limits, because in these limits the partition functions on $X_{N,M}$ collapse to those on $X_{N,M-1}$ in various ways. In accordance with mirror symmetry the degeneration behaviour on the type A side is reproduced on the type B side in the degeneration of the quantum mirror curves into lower genus curves.

For future directions it would be interesting to study the analogous properties of $Z_{N,M}$ and quantum mirror curves for the general $\Omega$-background i.e. $\epsilon_1 \neq 0$ and/or $\epsilon_1 \neq 0$ and $\epsilon_1 \neq \epsilon_2$ and at an arbitrary point of the Kähler moduli space of $X_{N,M}$. It will also be interesting to study the modular properties of the free energy $\log(\hat{Z}_{(N,M)}(\tau, \rho, \epsilon, m, t))$ and the single particle free energy \[ P\log(\hat{Z}_{(N,M)}(\tau, \rho, \epsilon, m, t)) \] along the lines of [19]. We hope to report on these matters in future. Also see recent interesting work [20, 21].

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A Geometry of $X_{N,M}$: a quick review

The non-compact CY 3-fold $X_{1,1}$ is defined as the partial compactification \cite{3, 10} of the resolved conifold geometry. The later is given by $\mathbb{C}^\times \times \mathbb{C}^\times$ fibered over the $z$-plane. The partial compactification is achieved by compactifying each of the two $\mathbb{C}^\times$ fibers to a $\mathbb{T}^2$ fiber. Of the three Kähler parameters $\tau, \rho, m$ of the CY 3-fold $X_{1,1}$, $\rho$ and $\tau$ correspond to the elliptic fibers and $m$ corresponds to the curve class of the exceptional $\mathbb{P}^1$ of the resolved conifold. We will define the non-compact CY 3-fold $X_{N,M}$ for $N, M \in \mathbb{N}$ as the $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifold of $X_{1,1}$.

In toric geometry the equation of the conifold
\[ z_1 z_2 - z_3 z_4 = 0, \quad z_1, z_2, z_3, z_4 \in \mathbb{C} \] (A.1)
is translated to an equation on integer lattices parametrised by 3-vectors $v_1, v_2, v_3, v_4$
\[ v_1 + v_2 - v_3 - v_4 = 0. \] (A.2)

The CY condition constrains the geometry to a plane. The irreducible toric rational curves of the 2-dimensional cone are given by
\[
C^1_{(a,b)} := R_{\geq 0} Conv\{(a + 1, b, 1), (a, b + 1, 1)\}, \quad C^2_{(a,b)} := R_{\geq 0} Conv\{(a, b, 1), (a, b + 1, 1)\}, \\
C^3_{(a,b)} := R_{\geq 0} Conv\{(a, b, 1), (a + 1, b, 1)\}.
\] (A.3)

for all $a, b \in \mathbb{Z}$. These curve classes satisfy the following relations
\[
\begin{align*}
C^1_{(a-1,b)} + C^3_{(a-1,b)} &= C^1_{(a,b-1)} + C^3_{(a,b-1)}, \\
C^1_{(a-1,b)} + C^2_{(a,b)} &= C^1_{(a,b-1)} + C^2_{(a,b-1)}.
\end{align*}
\] (A.4)

For the local CY 3-fold $X_{N,M}$ a modular covariant basis of generators can be given by
\[
\begin{align*}
C_{m,(a,b)} &= C^1_{(a,b)}, & C_{\tau,(a,b)} &= C^1_{(a,b)} + C^2_{(a,b)}, \\
C_{\rho,(a,b)} &= C^1_{(a,b)} + C^3_{(a,b)}
\end{align*}
\] (A.5)

where $a, b \in \mathbb{Z}$. In the fundamental domain of the $(N,M)$-web there are $3MN$ toric rational curves where $a \in \mathbb{Z}_N, b \in \mathbb{Z}_M$. Due to the $2NM$ constraints in (A.5) and torus periodicity the effective rank is $MN + 2$.

B $\sum_{a=0}^{N-1} m_{a,b}$ is independent of $b$: proof

Note that in our notation the curve classes $C^1_{(a,b)}$ are represented by the Kähler parameters $m_{a,b}$. Using the first relation in eq.(A.5), we can write the following summation
\[
\sum_{a=0}^{p-1} (C^1_{(a-1,b)} + C^3_{(a-1,b)}) = \sum_{a=0}^{p-1} (C^1_{(a,b-1)} + C^3_{(a-1,b)}),
\] (B.1)
Due to the compactification of web diagram on a torus there is periodicity relation $C^1_{(-1,b)} = C^1_{(p-1,b)}$. After simplification the second term cancels on both sides and we get

$$
\sum_{a=0}^{p-1} (C^1_{(a-1,b)}) = \sum_{a=0}^{p-1} (C^1_{(a,b-1)}),
$$

(B.2)

Expanding the left side

$$
\sum_{a=0}^{p-1} (C^1_{(-1,b)} + C^1_{(0,b)} + C^1_{(1,b)} + \ldots + C^1_{(p-3,b)} + C^1_{(p-2,b)}) = \sum_{a=0}^{p-1} (C^1_{(a,b-1)}),
$$

(B.3)

Rearranging the terms after using $C^1_{(-1,b)} = C^1_{(p-1,b)}$, we obtain the desired relation

$$
\sum_{a=0}^{p-1} C^1_{(a,b)} = \sum_{a=0}^{p-1} C^1_{(a,b-1)}.
$$

(B.4)

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Figure 14: Pictorial representation of the degeneration equation (5.15)