Strong Asymptotic Properties of Kernel Smooth Density and Hazard Function Estimation for Right Censoring NA Random Variable

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Abstract

Most studies for NA random variable is under complete sampling setting, which is actually an relatively ideal condition in application. The paper relaxes this condition to the censoring incomplete sampling data and considers the topic for kernel estimation of the density function together with the hazard function based on the Kaplan-Meier estimator. The strong asymptotic properties for the two estimators are firstly established.

Keywords: Kaplan-Meier estimator; right-censoring data; kernel density distribution; NA random variable.

1 Introduction

Definition (Joag-Dev and Proschan, 1983) Random sequences $\{T_i; 1 < i \leq n\}$ are said to be negatively associated (NA) if for every pair of disjoint subsets $B_1$ and $B_2$ from $\{1, 2, \ldots, n\}$,

$$\text{cov} (f_1(T_i; i \in B_1), f_2(T_j; j \in B_2)) \leq 0,$$

where there exists the covariance for $f_1(\cdot)$ and $f_2(\cdot)$ with increasing for every variable (or decreasing for every variable). A sequence of random variables $\{T_i; i \geq 1\}$ is said to be NA if every finite subfamily is NA.
Obviously, NA random variables includes independent random variable as a special case, and also describes many other random variable, for example the random sampling without replacement in a finite population. Many researchers have studied the property of NA random variables and published a large number of articles. For example, Su et al. (1997) established a probability inequality and some moment inequalities for the partial sum of a NA sequence, which contributed to prove some properties for strictly stationary NA sequences such as weak invariance principle. The results in Shao (2000) showed that most of the well-known inequalities, such as the Kolmogorov exponential inequality and the Rosenthal maximal inequality, are still hold for NA random variables. Wu and Chen (2013) presented two strong representation results of the Kaplan-Meier estimator for NA data with censoring, which will be the key results in the paper. Zhou and Lin (2015) considered a nonparametric regression model with repeated negative associated (NA) error structures, they proposed the wavelet procedures to estimate the regression function. Thuan and Quang (2016) gave some properties for the constructed notions of negative association and obtained inequalities which formed maximal inequality and Hájek- Rényi’s type inequality. Tang et al. (2018) studied the asymptotic normality of the wavelet estimator of an unknown Borel measurable function in the nonparametric regression model, where the random errors are asymptotically negatively associated random variables. Meng (2018) established two general strong laws of large numbers in which the coefficient of sum and the weight are both general functions for NA random variables, et al.

Most studies for NA random variable is under complete sampling setting, however, which is actually an relatively ideal condition in application. When study survival data, censoring incomplete situation is often encountered in data sampling. Let \((T_i, Y_i), i = 1, \cdots, n,\) denote a sequence of nonnegative random variable vector where \(T_i\) is the true survival time of interest which is right censored by the censoring random variable \(Y_i\). It is assumed that \(T_i\) is independent of \(Y_i\), but there are not assumed to be mutually independent for \(T_i\)’s and \(Y_i\)’s, which are all NA in our paper. And then the observed data is \((X_i, \delta_i)\) in the censorship model, where

\[ X_i = \min(T_i, Y_i) = T_i \land Y_i \] \[ \text{and} \quad \delta_i = I(T_i \leq Y_i), \quad i = 1, \cdots, n, \]

and \(I(A)\) is the indicator of the random event \(A\). For its simplicity, assumed that \(T_i\) have a common unknown continuous marginal distribution function \(F(x) = P(T_i \leq x)\) and denote its survival distribution \(S_T(t) = 1 - F(t)\). The random censoring times \(Y_i, i = 1, \cdots, n,\) being independent of the random variables \(T_i\)’s, are assumed to have a common distribution.
function $G(y) = P(Y_i \leq y)$ with its survival distribution $S_Y(t) = 1-G(t)$. Meanwhile, denote $L(\cdot)$ as the distribution of the observed variable $X_i$’s, and write its survival distribution as $S_X(t) = 1-L(t)$. For any distribution function $H(\cdot)$, we define the left and right endpoints of its support as $a_H$ and $\tau_H$ by $a_H = \inf\{x : H(x) > 0\}, \tau_H = \sup\{x : H(x) < 1\}$ in our paper.

The distribution function $L(\cdot)$ can be consistently estimated by the empirical distribution function $L_n(t)$, which is defined as follows:

$$L_n(t) = \frac{1}{n} \sum_{k=1}^{n} I(X_k < t) = 1 - \frac{Y_n(t)}{n} \triangleq \bar{Y}_n(t)$$

where $Y_n(t) = \sum_{k=1}^{n} I(X_k < t)$, while

$$Y_n(t) = \sum_{k=1}^{n} I(X_k \geq t),$$

the number of uncensored or censored observations no less than time $t$.

For drawing nonparametric inference about $F(\cdot)$ based on the censored observations $(X_i, \delta_i), i = 1, \ldots, n$, introduce a stochastic process on $[0, \infty)$ as follows:

$$N_n(t) = \sum_{k=1}^{n} I(T_k \leq t \land Y_k) = \sum_{k=1}^{n} I(X_k \leq t, \delta_k = 1),$$

the number of uncensored observations no larger than time $t$. The well-known nonparametric maximum likelihood estimation $\hat{F}_n(\cdot)$ of $F(\cdot)$ was the Kaplan-Meier (K-M) estimator (Kaplan and Meier, 1958), which is usually used to estimate $F(\cdot)$ for the incomplete data $(X_i, \delta_i)$:

$$1 - \hat{F}_n(x) = \prod_{s \leq x} (1 - \frac{dN_n(s)}{Y_n(s)}),$$

where the jump $dN_n(s) = N_n(s) - N_n(s-)$. 

Define the sub-distribution function $F_*(t) = P(T_1 \leq t \land Y_1) = P(X_1 \leq t, \delta_1 = 1)$. Since $F(0) = 0$, hence we have by integration by parts that

$$F_*(t) = \int_{0}^{\infty} \left[ \int_{s \leq t \land z} dF(s) \right] dG(z)$$

$$= \int_{0}^{t} F(z) dG(z) + \int_{t}^{\infty} F(t) dG(z)$$

$$= -\int_{0}^{t} F(z) dS_Y(z) + F(t)S_Y(t) = \int_{0}^{t} S_Y(z) dF(z),$$

and then

$$dF_*(t) = S_Y(t) dF(t).$$
The estimation for the hazard function $h(\cdot)$ is also an interesting issue in survival analysis, which is defined as follows when there is further assumed that $F(\cdot)$ has a density $f(\cdot)$:

$$h(x) = \frac{d}{dx}(- \log S_T(x)) = \frac{f(x)}{S_T(x)} = \frac{f(x)}{1 - F(x)} \text{ for } F(x) < 1.$$  

Its correspondingly cumulative hazard function is defined as

$$H(x) = \int_0^x h(s)ds = \int_0^x \frac{dF_*(s)}{S_X(s)}.$$  

(1.1)

The above representation of $H(\cdot)$ in term of $F_*(\cdot)$ and $S_X(\cdot)$ suggests the empirical estimation for $H(\cdot)$ by

$$\hat{H}_n(x) = \int_0^x \frac{dN_n(s)}{Y_n(s)} = \int_0^x \frac{dF_n(s)}{L_n(s)},$$  

(1.2)

where $\bar{L}_n(s) = 1 - \hat{L}_n(s)$, and

$$F_{sn}(t) = \frac{1}{n} \sum_{k=1}^{n} I(X_k \leq t, \delta_k = 1) = \frac{N_n(t)}{n},$$

the empirical distribution functions of $F_*(\cdot)$.

Note that $dN_n(X_{(k)}) = \sum_{j=1}^{n} [\delta_j I(X_j = X_{(k)})] = \delta_{(k)}, k = 1, 2, \cdots, n$, we may verify that the estimators $\hat{F}_n(\cdot)$ and $\hat{H}_n(\cdot)$ can be respectively represented as

$$1 - \hat{F}_n(x) = \prod_{X_{(k)} \leq x} (1 - \frac{dN_n(X_{(k)})}{n - k + 1}) = \prod_{X_{(k)} \leq x} (1 - \frac{\delta_{(k)}}{n - k + 1}),$$  

(1.3)

and

$$\hat{H}_n(x) = \sum_{X_{(k)} \leq x} \frac{\delta_{(k)}}{n - k + 1},$$  

(1.4)

where $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ are the order statistics of $X_1, X_2, \ldots, X_n$ and $\delta_{(k)}$ is the concomitant of $X_{(k)}$. The K-M estimator $\hat{F}_n(x)$ and the estimator $\hat{H}_n(x)$ have been generally accepted as a substitute for the usual empirical estimator of distribution function $F(\cdot)$ and the cumulative hazard function $H(\cdot)$ in the case of right censoring, respectively, which help to study other estimators such as the kernel density estimator and the kernel hazard estimator in the following.

A kernel estimator for $f(\cdot)$ based on $\hat{F}_n(\cdot)$ can be constructed as

$$f_n(t) = b_n^{-1} \int_{a_F}^{+\infty} k\left(\frac{t-x}{b_n}\right)d\hat{F}_n(x),$$

where

$$\int_{a_F}^{+\infty} k\left(\frac{t-x}{b_n}\right)dt = \int_{a_F}^{+\infty} k\left(\frac{t-x}{b_n}\right)1_{t \geq a_F} dt,$$

is a constant, and

$$b_n = \sqrt{\frac{\log n}{n}},$$

and

$$\hat{F}_n(x) = \frac{1}{n} \sum_{i=1}^{n} I(X_i \leq x).$$

(4.17)
where \( k(\cdot) \) is a smooth probability kernel function and \( \{b_n, n \geq 1\} \) is a sequence of bandwidth tending to zero at appropriate rates.

Similarly, we can also construct an kernel estimator for the hazard function \( h(\cdot) \) under the NA sampling data, which is defined by

\[
h_n(t) = b_n^{-1} \int_{a_F}^{+\infty} k(\frac{t - x}{b_n})d\hat{H}_n(x).
\]

The smoothed estimators \( f_n(\cdot) \) and \( h_n(\cdot) \) have attracted the attention of many investigators. See, for example, Mielniczuk (1986) investigated kernel estimator of a density function using the K-M estimator for censored data. When the data was sampled from \( \alpha \)-mixing and censoring, Cai (1998) explored the uniform consistency (with rates) and the asymptotic normality of the kernel estimators for density and hazard function. Zhou (1999) successfully established several asymptotic uniformly strong and weak representations for kernel estimators of the density function and the hazard function under left truncation. Antoniadis, et al. (1999) proposed a wavelet method for estimating density and hazard rate functions from randomly right-censored data. Some other results, one may refer to Diehl and Stute (1988), Gijbels and Wang (1993), Arcones and Giné (1995), Zhou and Yip (1999), Lemdani and Ould-Saïd (2007), Shen and He (2008) among others.

For present our main results, define

\[
\bar{f}_n(t) = b_n^{-1} \int_{a_F}^{+\infty} k(\frac{t - x}{b_n})dF_*(x), \quad \bar{h}_n(t) = b_n^{-1} \int_{a_F}^{+\infty} k(\frac{t - x}{b_n})dH(x),
\]

\[
f_n^*(t) = b_n^{-1} \int_{a_F}^{+\infty} k(\frac{t - x}{b_n})dF_{sn}(x), \quad F_{sn}(x) = \frac{1}{n} \sum_{i=1}^{n} I(T_i \leq x).
\]

The main purpose of this paper is to study the asymptotic properties of kernel smoothing density estimator \( f_n(\cdot) \) and hazard estimator \( h_n(\cdot) \) based on censoring NA data. Under certain regularity conditions, we establish the strong asymptotic properties for the two estimators with the convergent rates being \( O(b_n^{-1}(n^{-1} \ln n)^{1/2}) \) a.s., where \( \{b_n, n \geq 1\} \) will be defined in the main results.

2 Main results and their proofs

**Theorem 1** Under the conditions of Lemma 1, and assume that \( k(\cdot) \) is the bounded variation probability kernel density on the finite interval \((r,s)\), where \( r < 0 < s \). Suppose that density
Lemma 1 Let \( f(\cdot) = F'(\cdot) \) and \( g(\cdot) = G'(\cdot) \) are bounded on the closed interval \([0, \tau]\) for some \( a_F < \tau < \tau_L \). Then there is

\[
\sup_{0 < t \leq \tau} \left| f_n(t) - \bar{f}_n(t) - \frac{f_n(t) - Ef_n(t)}{1 - L(t)} \right| = O\left(b_n^{-1} (n^{-1} \ln n)^{1/2}\right) \text{ a.s.,} \tag{2.1}
\]

where the sequence \( \{b_n; n \geq 1\} \) satisfies \( b_n^{-1} = o((n \ln^{-1} n)^{1/2}) \).

Theorem 2 Under the conditions of Lemma 1, and assume that \( k(\cdot) \) is the bounded variation probability kernel density on the finite interval \((r, s)\), where \( r < 0 < s \). The density functions \( f(\cdot) = F'(\cdot) \) and \( g(\cdot) = G'(\cdot) \) are bounded on \([0, \tau]\) with \( a_F < \tau < \tau_L \), then there is

\[
\sup_{0 < t \leq \tau} \left| h_n(t) - \bar{h}_n(t) - \frac{f_n(t) - Ef_n(t)}{1 - L(t)} \right| = O\left(b_n^{-1} (n^{-1} \ln n)^{1/2}\right) \text{ a.s.,} \tag{2.2}
\]

where the sequence \( \{b_n; n \geq 1\} \) satisfies \( b_n^{-1} = o((n \ln^{-1} n)^{1/2}) \).

Remark 1 Theorem 1 and Theorem 2 are fundamental results in studying censoring NA data, which can be useful in deriving some asymptotic properties for the kernel density estimator \( f_n(\cdot) \) and the hazard function estimator \( h_n(\cdot) \), respectively. Take for example, if one can establish the similar main results in Hall (1981) for NA data, then using Theorem 1, the following proposition may hold, which is the next issue we will consider.

**Proposition** Suppose that the sequence \( \{b_n; n \geq 1\} \) satisfies \( b_n \to 0 \) and

(a) \( (\ln n)^2/(nb_n \ln \ln n) \to 0 \),

(b) \( nb_n \to \infty \) in such a way that

\[
\lim_{n \to \infty} \sup_{m: \vert m-n \vert \leq n} \left| \frac{b_m}{b_n} - 1 \right| \to 0, \text{ for } \epsilon \to 0,
\]

then there will be

\[
\limsup_{n \to \infty} \pm \left( \frac{nb_n}{2 \ln \ln n} \right)^{1/2} (f_n(t) - \bar{f}_n(t)) = [\varphi(f, G) \int k^2(s)ds]^{1/2}, \text{ a.s.}
\]

where \( \varphi(f, G) \) is some functional for \( f(\cdot) \) and \( G(\cdot) \).

We firstly present two lemmas (Wu and Chen, 2013) that help to prove our theorem.

**Lemma 1** Let \( \{T_n; n \geq 1\} \) and \( \{Y_n; n \geq 1\} \) be two sequences of NA random variables. Suppose that the sequences \( \{T_n; n \geq 1\} \) and \( \{Y_n; n \geq 1\} \) are independent. Then, for any \( 0 < \tau < \tau_L = \tau_F \wedge \tau_G \),

\[
\sup_{0 < t < \tau} \left| \hat{F}_n(t) - F(t) \right| = O((n^{-1} \ln n)^{1/2}) \text{ a.s.} \tag{2.3}
\]
and
\[
\sup_{0 < t < \tau} \left| \hat{H}_n(t) - H(t) \right| = O((n^{-1} \ln n)^{1/2}) \text{ a.s.} \tag{2.4}
\]

For positive reals \( x \) and \( t \), and \( \delta \) taking value 0 or 1, write
\[
\eta(x, t, \delta) = \int_0^{x/t} \frac{dF_*(s)}{L^2(s)} - \frac{I(x \leq t, \delta = 1)}{L(x)}.
\]

**Lemma 2** Let \( \{T_n; n \geq 1\} \) and \( \{Y_n; n \geq 1\} \) be two sequences of NA random variables. Suppose that the sequences \( \{T_n; n \geq 1\} \) and \( \{Y_n; n \geq 1\} \) are independent. Then, for any \( 0 < \tau < \tau_L \),
\[
\hat{F}_n(t) - F(t) = -\frac{S_T(t)}{n} \sum_{i=1}^n \eta(X_i, t, \delta_i) + r_{1n}(t), \tag{2.5}
\]
and
\[
\hat{H}_n(t) - H(t) = -\frac{1}{n} \sum_{i=1}^n \eta(X_i, t, \delta_i) + r_{2n}(t), \tag{2.6}
\]
where \( \sup_{0 < t < \tau < \tau_L} |r_{in}(t)| = O((n^{-1} \ln n)^{1/2}) \) a.s., \( i = 1, 2 \).

**Remark 2** Note that by the definition of \( \eta(X, t, \delta) \),
\[
-\frac{1}{n} \sum_{i=1}^n \eta(X_i, t, \delta_i) = \frac{1}{n} \sum_{i: X_i \leq t} \frac{N_n(X_i) - N_n(X_i-)}{L_n(X_i)} - \frac{1}{n} \int_0^t \frac{\sum_{i=1}^n I(X_i \geq s)}{L^2(s)} dF_*(s)
\]
\[
= \int_0^t \frac{1}{L(s)} dF_{en}(s) - \int_0^t \frac{\hat{L}_n(s)}{L^2(s)} dF_*(s).
\]
Therefore, we can obtain by Lemma 2 that
\[
\hat{F}_n(t) - F(t) = S_T(t) \left[ \int_0^t \frac{1}{L(s)} dF_{en}(s) - \int_0^t \frac{\hat{L}_n(s)}{L^2(s)} dF_*(s) \right] + r_{1n}(t)
\]
\[
= (1 - F(t)) \left[ \int_0^t \frac{1}{L(s)} dF_{en}(s) - \int_0^t \frac{\hat{L}_n(s)}{L^2(s)} dF_*(s) \right] + r_{1n}(t)
\]
\[
= (1 - F(t)) \left[ \int_0^t \frac{1}{L(s)} d[F_{en}(s) - F_*(s)] - \int_0^t \frac{\hat{L}_n(s) - \bar{L}(s)}{L^2(s)} dF_*(s) \right] + r_{1n}(t).
\]

Meanwhile, one can establish the following result by the fact that \( \{X_n; n \geq 1\} \) and \( \{(X_n, \delta_n); n \geq 1\} \) are all NA random variable sequences according to Joag-Dev and Proschan (1983).
Lemma 3 Under the conditions of Lemma 1, for any $0 < \tau < \tau_L$,

$$\sup_{0 < t < \tau} |F_n(t) - F(t)| = O((n^{-1} \ln n)^{1/2}) \ a.s. \quad (2.7)$$

and

$$\sup_{0 < t < \tau} |L_n(t) - L(t)| = O((n^{-1} \ln n)^{1/2}) \ a.s. \quad (2.8)$$

For simplicity and without loss of generality, it can be assumed that $a_F = 0$ in the following proof procedure.

Proof of Theorem 1 According to Remark 2, $f_n(x) - \bar{f}_n(x)$ can be expressed as

$$f_n(x) - \bar{f}_n(x) = b_n^{-1} \int_0^{+\infty} k\left(\frac{x-t}{b_n}\right) d[\tilde{F}_n(t) - F(t)]$$

$$= b_n^{-1} \int_0^{+\infty} k\left(\frac{x-t}{b_n}\right) d\left\{(1 - F(t)) \int_0^t \frac{1}{L(s)} d[F_n(s) - F_s(s)]ight\}$$

$$+ \left\{-b_n^{-1} \int_0^{+\infty} \left\{(1 - F(t)) \int_0^t \frac{1}{L(s)} d[F_n(s) - F_s(s)]\right\} d[L(t)]ight\} + \left\{b_n^{-1} \int_0^{+\infty} \left\{(1 - F(t)) \int_0^t \frac{1}{L^2(s)} d[F_n(s) - F_s(s)]\right\} d[L(t)]\right\}$$

$$= -b_n^{-1} \int_0^{+\infty} \left\{(1 - F(t)) \int_0^t \frac{1}{L(s)} d[F_n(s) - F_s(s)]\right\} d\left\{(x-t) / b_n\right\}$$

$$+ b_n^{-1} \int_0^{+\infty} \left\{(1 - F(t)) \int_0^t \frac{1}{L_n(s) - L(s)} d[F_n(s) - F_s(s)]\right\} d\left\{(x-t) / b_n\right\}$$

$$- b_n^{-1} \int_0^{+\infty} \left\{r_n(t) d\left\{(x-t) / b_n\right\}\right\}$$

$$\triangleq - I_1 + I_2 - I_3 \quad (2.9)$$

Consider $I_1$ firstly, there is

$$\int_0^t \frac{1}{L(s)} d[F_n(s) - F_s(s)] = \frac{F_n(t) - F_s(t)}{L(t)} + \int_0^t \frac{F_n(s) - F_s(s)}{L^2(s)} d[L(s)].$$

Thus, we have the following formula

$$I_1 = \left\{b_n^{-1} \int_0^{+\infty} (1 - F(t)) \frac{F_n(t) - F_s(t)}{L(t)} d\left\{(x-t) / b_n\right\}$$

$$+ b_n^{-1} \int_0^{+\infty} (1 - F(t)) \int_0^t \frac{F_n(s) - F_s(s)}{L^2(s)} d[L(t)] d\left\{(x-t) / b_n\right\}\right\}$$

$$\triangleq I_{11} + I_{12} \quad (2.10)$$
Using the partial integration for $I_{11}$, we have
\[
I_{11} = b_n^{-1} \int_0^{+\infty} \frac{F_{sn}(t) - F_s(t)}{1 - G(t)} dk(x - t) b_n
\]
\[
= \frac{1}{b_n(1 - G(x))} \int_0^{+\infty} [F_{sn}(t) - F_s(t)] dk(x - t)
\]
\[
+ \frac{1}{b_n(1 - G(x))} \int_0^{+\infty} \frac{F_{sn}(t) - F_s(t)}{1 - G(t)} (G(t) - G(x)) dk(x - t)
\]
\[
\triangleq - \frac{1}{1 - G(x)} [f_n^*(x) - Ef_n^*(x)] + I_{11}^*. \tag{2.11}
\]

Then, combine the property of empirical process and (2.6), when $n$ is large enough,
\[
|I_{11}^*| \leq \frac{1}{b_n(1 - G(\tau))^2} \sup_{0 < x \leq \tau} |F_{sn}(t) - F_s(t)| \int_0^x |G(x) - G(x - ub_n)| |dk(u)|
\]
\[
= O((n^{-1} \ln n)^{1/2}) \ a.s. \tag{2.12}
\]

and
\[
|I_{12}| = \left| b_n^{-1} \int_r^x (1 - F(x - b_n u)) \int_0^x \frac{F_{sn}(s) - F_s(s)}{L^2(s)} d\tilde{L}(s) dk(u) \right|
\]
\[
\leq \left| b_n^{-1} \int_r^x (1 - F(x - b_n u)) \int_0^x \frac{F_{sn}(s) - F_s(s)}{L^2(s)} d\tilde{L}(s) dk(u) \right|
\]
\[
+ \left| b_n^{-1} \int_r^x (1 - F(x - b_n u)) \int_x^s \frac{F_{sn}(s) - F_s(s)}{L^2(s)} d\tilde{L}(s) dk(u) \right|
\]
\[
= b_n^{-1} \left| \int_0^x \frac{F_{sn}(s) - F_s(s)}{L^2(s)} d\tilde{L}(s) \right| \left| \int_r^x (1 - F(x - b_n u)) dk(u) \right|
\]
\[
+ b_n^{-1} \left| \int_r^x (1 - F(x - b_n u)) \int_x^s \frac{F_{sn}(s) - F_s(s)}{L^2(s)} d\tilde{L}(s) dk(u) \right|.
\]

Again, note that
\[
\int_r^x (1 - F(x - b_n u)) dk(u) = -b_n \int_r^x k(u) f(x - b_n u) du.
\]

Integrating by parts for $I_{12}$, we have for $0 < \tau < \tau_L$
\[
|I_{12}| \leq b_n^{-1} \sup_{0 < x \leq \tau} |F_{sn}(x) - F_s(x)| \left| \int_0^x \frac{1}{L^2(s)} d\tilde{L}(s) \right| b_n \left| \int_r^x k(u) f(x - b_n u) du \right|
\]
\[
+ b_n^{-1} \sup_{0 < x \leq \tau} |F_{sn}(x) - F_s(x)| \cdot \left| \int_r^x (1 - F(x - b_n u)) \int_x^s \frac{d[(1 - F(s))(1 - G(s))]}{(1 - F(s))^2(1 - G(s))^2} dk(u) \right|
\]
\[
\leq \sup_{0 < x \leq \tau} |F_{sn}(x) - F_s(x)| \left| - \frac{1}{L(s)} \right| \left| \sup_{0 < x \leq \tau} f(x) \int_r^x |k(u)| du \right|
\]
\[
+ b_n^{-1} \sup_{0 < x \leq \tau} |F_{sn}(x) - F_s(x)| \cdot \left| \int_r^x (1 - F(x - b_n u)) \int_x^s \frac{f(s)(1 - G(s)) + g(s)(1 - F(s))}{(1 - F(s))^2(1 - G(s))^2} ds dk(u) \right|.
\]
Since density function \( f(\cdot) = F'(\cdot) \) and \( g(\cdot) = G'(\cdot) \) are bounded in the closed interval \([0, \tau]\), where means that \( \bar{L}'(s) = f(s)(1 - G(s)) + g(s)(1 - F(s)) \) is also bounded in the interval \([0, \tau]\), therefore by empirical process

\[
|I_{12}| \leq M \cdot \sup_{0 < s \leq \tau} |F_{*n}(x) - F_*(x)| \left| \frac{1}{L(T)} - \frac{1}{L(0)} \right| \sup_{0 < x \leq \tau} f(x) \\
+ b_n^{-1} \sup_{0 < x \leq \tau} |F_{*n}(x) - F_*(x)| \int_0^s \left| \frac{\sup_{0 < x \leq \tau} \bar{L}'(x)}{(1 - F(\tau))^2(1 - G(\tau))^2} \int_x^{x-b_n u} ds \right| dk(u) \\
= M \cdot \sup_{0 < x \leq \tau} |F_{*n}(x) - F_*(x)| \left| \frac{1}{L(T)} - \frac{1}{L(0)} \right| \sup_{0 < x \leq \tau} f(x) \\
+ \sup_{0 < x \leq \tau} |F_{*n}(x) - F_*(x)| \left| \frac{\sup_{0 < x \leq \tau} \bar{L}'(x)}{(1 - F(\tau))^2(1 - G(\tau))^2} \int_\tau^s |u| \, dk(u) \right| \\
= O((n^{-1} \ln n)^{1/2}) \text{ a.s.,} \quad (2.13)
\]

where \( M \) is some positive constant number.

Thus, combining equations (2.10), (2.11), (2.12) and (2.13), we have

\[
\sup_{0 < x \leq \tau} \left| I_1 + \frac{f_{*n}(x) - E f_{*n}(x)}{1 - G(x)} \right| = O((n^{-1} \ln n)^{1/2}) \text{ a.s.} \quad (2.14)
\]

On the other hand, similar to the discussion of \( I_{12} \),

\[
|I_2| = \left| b_n^{-1} \int_0^{\infty} \left\{ (1 - F(t)) \int_0^x \frac{\bar{L}_n(s) - \bar{L}(s)}{L^2(s)} dF_*(s) \right. \right. \\
+ \left. \left. \int_x^t \frac{\bar{L}_n(s) - \bar{L}(s)}{L^2(s)} dF_*(s) \right\} dk(\frac{x-t}{b_n}) \right| \\
\leq b_n^{-1} \left| \int_0^x \frac{\bar{L}_n(s) - \bar{L}(s)}{L^2(s)} dF_*(s) \right| \left| \int_\tau^s (1 - F(x-b_n u)) dk(u) \right| \\
+ b_n^{-1} \left| \int_\tau^s \left\{ (1 - F(x-b_n u)) \int_x^{x-b_n u} \frac{\bar{L}_n(s) - \bar{L}(s)}{L^2(s)} dF_*(s) \right\} dk(u) \right| \\
\triangleq I_{21} + I_{22},
\]

where

\[
I_{21} = \sup_{0 \leq x \leq \tau} \left| \bar{L}_n(x) - \bar{L}(x) \right| \left| \int_0^x \frac{f(s) ds}{(1 - F(s))^2(1 - G(s))} \right| \left| -b_n \int_\tau^s k(u) f(x-b_n u) du \right| \\
\leq \sup_{0 \leq x \leq \tau} \left| \bar{L}_n(x) - \bar{L}(x) \right| \cdot \sup_{0 \leq x \leq \tau} f(x) \cdot \frac{1}{(1 - F(\tau))^2(1 - G(\tau))},
\]
and

\[ I_{22} \leq b_n^{-1} \sup_{0 \leq x \leq \tau} |\bar{L}_n(x) - \bar{L}(x)| \]
\[ \cdot \left| \int_r^s \{(1 - F(x - b_nu)) \left[ \int_x^{x-b_nu} \frac{(1 - G(s))dF(s)}{(1 - F(s))^2(1 - G(s))^2} \right] dk(u) \right| \]
\[ \leq b_n^{-1} \sup_{0 \leq x \leq \tau} |\bar{L}_n(x) - \bar{L}(x)| \left( \int_r^s \left| \int_x^{x-b_nu} \frac{f(s)ds}{(1 - F(s))^2(1 - G(s))} \right| dk(u) \right) \]
\[ \leq \sup_{0 \leq x \leq \tau} |\bar{L}_n(x) - \bar{L}(x)| \int_r^s |u|dk(u) \sup_{0 \leq x \leq \tau} f(x) \left( \frac{1}{(1 - F(\tau))^2(1 - G(\tau))} \right). \]

It can be obtained by (2.7) that

\[ \sup_{0 \leq x \leq \tau} |I_2| = O((n^{-1} \ln n)^{1/2}) \ a.s. \quad (2.15) \]

As for term \( I_3 \), it follows from Lemma 2 that

\[ \sup_{0 < x \leq \tau} |I_3| = b_n^{-1} \sup_{0 < x \leq \tau} r_{1n}(t) \int_0^{+\infty} |dk(x-t)| = O(b_n^{-1}(n^{-1} \ln n)^{1/2}) \ a.s. \quad (2.16) \]

This completes the proof by combining (2.9), (2.14), (2.15) and (2.16).

**Proof of Theorem 2** Note the strong asymptotic expression from (2.6) that

\[ \hat{H}_n(t) - H(t) = \int_0^t \frac{1}{L(s)} dF_n(s) - \int_0^t \frac{\bar{L}_n(s)}{L^2(s)} dF_n(s) + r_2n(t), \]

and the counterpart to the term \( I_{11} \),

\[ I'_{11} = b_n^{-1} \int_0^{+\infty} \frac{F_n(t) - F(t)}{L(t)} dF_n(t) \]
\[ = \frac{1}{b_nL(x)} \int_0^{+\infty} [F_n(t) - F(t)] dk(x-t) \]
\[ + \frac{1}{b_nL(x)} \int_0^{+\infty} \frac{F_n(t) - F(t)}{L(t)} [\bar{L}(t) - \bar{L}(x)] dk(x-t). \]

Then following the proof procedure of Theorem 1, we can also establish Theorem 2, and this ends the proof.
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