Gravity tests and the Pioneer anomaly

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Experimental tests of gravity performed in the solar system show a good agreement with general relativity. The latter is however challenged by the Pioneer anomaly which might be pointing at some modification of gravity law at ranges of the order of the size of the solar system. We introduce a metric extension of general relativity which, while preserving the equivalence principle, modifies the coupling between curvature and stress tensors and, therefore, the metric solution in the solar system. The “post-Einsteinian extension” replaces Newton gravitation constant by two running coupling constants, which depend on the scale and differ in the sectors of traceless and traced tensors, so that the metric solution is characterized by two gravitation potentials. The extended theory has the capability to preserve compatibility with gravity tests while accounting for the Pioneer anomaly. It can also be tested by new experiments or, maybe, by having a new look at data of already performed experiments.

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I. INTRODUCTION

Most gravitation tests performed in the solar system show a good agreement with general relativity (GR). In particular, the equivalence principle, lying at the basis of GR, is one of the most accurately verified properties of nature \textsuperscript{1}. This entails that the gravitational field has to be identified with the metric tensor $g_{\mu\nu}$ in a Riemannian space-time. Then, the parametrized post-Newtonian (PPN) formalism allows one to give a quantitative form to the agreement of observations with the metric tensor predicted by GR, through its confrontation with a family of more general metric solutions. Alternatively, GR can be tested by looking for hypothetical deviations of gravity force law from its standard form \textsuperscript{2}, as predicted by unification models although not observed up to now.

Besides these successes, GR is challenged by observations performed at galactic and cosmological scales. Anomalies have been known for some time to affect the rotation curves of galaxies. They are commonly accounted for by keeping GR as the theory of gravity at galactic scales but introducing unseen “dark matter” to reproduce the rotation curves \textsuperscript{3, 4}. Anomalies have been seen more recently in the relation between redshifts and luminosities for type II supernovae. They are usually interpreted as an unexpected acceleration of cosmic expansion due to the presence of some “dark energy” of completely unknown origin \textsuperscript{5}. As long as the “dark side” of the universe is not observed through other means, these galactic and cosmic anomalies may also be interpreted as deviations from GR occuring at large scales \textsuperscript{6, 7, 8}.

The Pioneer anomaly constitutes a new piece of information in this puzzling context, which may already reveal an anomalous behaviour of gravity at scales of the order of the size of the solar system \textsuperscript{9}. The anomaly was discovered when Doppler tracking data from the Pioneer 10/11 probes were analyzed during their travel to the outer parts of the solar system. After the probes had reached a quieter environment, after flying by Jupiter and Saturn, a precise comparison of tracking data with predictions of GR confirmed that the Doppler velocity was showing an anomaly varying linearly with elapsed time (see Fig. 8 of \textsuperscript{10}). The deviation may be represented as an anomalous acceleration directed towards the Sun with an approximately constant amplitude over a large range of heliocentric distances (AU $\equiv$ astronomical unit)

$$a_P = (0.87 \pm 0.13) \text{ nm s}^{-2} , \quad 20 \text{ AU} \lesssim r \lesssim 70 \text{ AU} \quad (1)$$

Though a number of mechanisms have been considered to this aim \textsuperscript{11, 12, 13, 14}, the anomaly has escaped up to now all attempts of explanation as a systematic effect generated by the spacecraft itself or its environment. In particular, present knowledge of the outer part of the solar system does apparently preclude interpretations in terms of gravity \textsuperscript{15} or drag effects \textsuperscript{16} of ordinary matter. The inability of explaining the anomaly with conventional physics has given rise to a growing number of new theoretical propositions. It has also motivated proposals for new missions designed to study the anomaly and try to understand its origin \textsuperscript{17}. The importance of the Pioneer anomaly

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for space navigation already justifies it to be submitted to further scrutiny while its potential impact on fundamental physics, especially on gravitation theory, can no more be neglected. The possibility that the Pioneer anomaly be the first hint of a modification of gravity law at large scales cannot be let aside investigations. In this context, the compatibility of the Pioneer anomaly with other gravity tests appears to be a key question.

In order to discuss this point, we first recall that, though the interpretation of gravitation as the metric of space-time constitutes an extremely well tested basis, the precise form of the coupling between space-time curvature and gravity sources can still be discussed. Like the other fundamental interactions, gravitation may also be treated within the framework of field theory. Radiative corrections due to its coupling to other fields then naturally lead to embed GR within the larger class of fourth order theories. Modifications are thus expected to appear and they may affect large length scales. This suggests to consider GR as an effective theory of gravity valid at the length scales for which it has been accurately tested but not necessarily at smaller or larger scales. Note that, in contrast to GR, fourth order theories show renormalizability as well as asymptotic freedom at high energies. Hence, they constitute a strong basis for extending the gravitation theory at scales not already constrained by experiments, for instance using renormalization group trajectories. Renormalizability of these theories however comes with a counterpart, that is the problem of ghosts. It has however been convincingly argued that this problem does not constitute a definitive deadend for an effective field theory valid in a limited scale domain. In particular, the departure from unitarity is expected to be negligible at ordinary scales tested in present day universe.

In this paper, we will review the main features of a phenomenological framework which has been recently developed. It relies upon a theory of gravitation lying in the vicinity of GR, the deviation representing for example the radiative corrections due to the coupling of gravity with other fields. It is presented below in its linearized form, where its significance is more easily given, and then in its full non linear version. It is also interpreted as an extension of the PPN Ansatz with the Eddington parameters and being functions of heliospheric distances rather than mere constants. The extended framework is shown to have the ability to account for Pioneer anomaly while remaining compatible with other gravity tests. It also leads to the prediction of other anomalies related to Pioneer anomaly, which can be tested by new experiments or, in some cases, by having a new look at data of already performed experiments.

II. GRAVITY TESTS IN THE SOLAR SYSTEM

GR provides us with an excellent theoretical description of gravitational phenomena in the solar system. In order to discuss the experimental evidences in favor of this statement, we first recall a few basic features of this description. In order to apply the principle of relativity to accelerated motions, Einstein introduced what is now called the equivalence principle. A weak form of this principle is expressed by the universality of free fall, a property reflecting the universal coupling of all bodies to gravitation. With Einstein, this property acquires a geometrical significance, gravitation fields being identified with the metric tensor while freely falling motions are geodesics of the associated space-time. Universality of free fall is then a consequence of the metric nature of gravitation theory.

The equivalence principle is one of the best ever tested properties of nature. Potential violations are usually parametrized by a relative difference in the accelerations undergone by two test bodies of different compositions in free fall at the same location. Modern experiments constrain the parameter to stay below the level. These experiments test the principle at distances ranging from the millimeter in laboratory experiments and references in) to the sizes of Earth-Moon or Sun-Mars orbit.

In order to obtain GR, it remains to write the equations determining the metric tensor from the distribution of energy and momentum in space-time. GR corresponds to a particular choice of the form of the coupling between curvature tensor and stress tensor: the Einstein curvature tensor is simply proportional to the stress tensor, the proportionality constant being related to the Newton gravitation constant inherited from classical physics.

Note that this relation accounts in a simple manner for the fact that both and have a null covariant divergence: the first property comes with Riemannian geometry (Bianchi identities) while the second one expresses conservation of energy and momentum and is a necessary and sufficient condition for motions of test masses to follow geodesics.

The metric tensor in the solar system is then deduced by solving the Einstein-Hilbert equation. Here we consider the simple case where the gravity source, i.e. the Sun, is described as a point-like motion-less mass so that the metric is simply written in terms of the Newton potential. The solution is conveniently written in terms of spherical
coordinates \((c \text{ denotes light velocity, } t \text{ and } r \text{ the time and radius, } \theta \text{ and } \varphi \text{ the colatitude and azimuth angles})\) with the gauge convention of isotropic spatial coordinates

\[
\begin{align*}
\text{ds}^2 &= g_0 c^2 \text{d}t^2 + g_{rr} \left( \text{d}r^2 + r^2 \left( \text{d}\theta^2 + \sin^2 \theta \text{d}\varphi^2 \right) \right) \\
g_{00} &= 1 + 2\phi + 2\phi^2 + \ldots \\
g_{rr} &= -1 + 2\phi + \ldots \\
\phi &= -\frac{\kappa}{r}, \quad \kappa \equiv \frac{G_N M}{c^2}, \quad |\phi| \ll 1
\end{align*}
\]

GR is usually tested through its confrontation with the family of parametrized post-Newtonian (PPN) metric tensors introduced by Eddington \cite{51} and then developed by several physicists \cite{52, 53, 54, 55}.

\[
g_{00} = 1 + 2\alpha \phi + 2\beta \phi^2 + \ldots \\
g_{rr} = -1 + 2\gamma \phi + \ldots
\tag{4}
\]

The three parameters \(\alpha, \beta\) and \(\gamma\) are constants, the first of which can be set to unity by redefining Newton constant \(G_N\). Within the PPN family, GR corresponds to \(\gamma = \beta = 1\). Anomalous values of \(\gamma\) or \(\beta\) differing from unity affect geodesic motions and can therefore be evaluated from a comparison of observations with predictions deduced from GR.

Experiments which have now been performed for more than four decades have led to more and more strict bounds on the anomalies \(\gamma - 1\) and \(\beta - 1\). For example, Doppler ranging on Viking probes in the vicinity of Mars \cite{46} and deflection measurements using VLBI astrometry \cite{56} or radar ranging on the Cassini probe \cite{57} give smaller and smaller values of \(\gamma - 1\), with presently a bound of a few \(10^{-5}\). Analysis of the precession of planet perihelions \cite{58} and of the polarization by the Sun of the Moon orbit around the Earth \cite{59} allow for the determination of linear superpositions of \(\beta\) and \(\gamma\), resulting now to \(\beta - 1\) smaller than a few \(10^{-4}\). These tests are compatible with GR with however a few exceptions, among which notably the anomalous observations recorded on Pioneer probes. We will see below that this contradiction between Pioneer observations and other gravity tests may be cured in an extended framework, thanks to the fact that the anomaly \(\gamma - 1\) is no longer a constant but a function in this more general framework.

An alternative manner to test GR has been to check the \(1/r\) dependence of the Newton potential \(\phi\), that is also of the component \(g_{00}\) in \(\text{(4)}\). Hypothetical modifications of its standard expression are usually parametrized in terms of an additional Yukawa potential depending on two parameters, the range \(\lambda\) and the amplitude \(\alpha\) measured with respect to Newton potential \(\phi\):

\[
\Phi_N(r) = \phi(r) \left( 1 + \alpha e^{-\frac{r}{\lambda}} \right)
\tag{5}
\]

The presence of such a Yukawa correction has been looked for at various distances ranging from the millimeter in laboratory experiments \cite{44} to the size of planetary orbits \cite{60}. The accuracy of short range tests has been recently improved, as gravity experiments were pushed to smaller distances \cite{61, 62, 63} and as Casimir forces, which become dominant at submillimeter range, were more satisfactorily taken into account \cite{64, 65, 66, 67}. On the other side of the distance range, long range tests of the Newton law are performed by monitoring the motions of planets or probes in the solar system. They also show an agreement with GR with a good accuracy for ranges of the order of the Earth-Moon \cite{45} or Sun-Mars distances \cite{46, 47, 68, 69}. When the whole set of results is reported on a global figure (see fig.1 in \cite{70} reproduced thanks to a courtesy of the authors of \cite{60}), it appears that windows however remain open for violations of the standard form of Newton force law at short ranges, below the millimeter, as well as long ones, of the order of or larger than the size of the solar system.

One merit of the latter tests is to shed light on a potential scale dependence of violations of GR. As a specific experiment is only sensitive to a given range of distances, this has to be accounted for, especially in the context, recalled in the Introduction, where doubts arise about the validity of GR at galactic or cosmic scales. In order to discuss this scale dependence, it is worth rewriting the Yukawa perturbation \(\Phi_N\) in terms of a running constant replacing Newton gravitation constant. To this aim, we introduce the expression of the potential \(\Phi_N[k]\) in Fourier space, with \(k\) the spatial wavevector, and relate it to a coupling constant \(G_N[k]\):

\[
-k^2 \Phi_N[k] \equiv 4\pi \frac{G_N[k] M}{c^2} , \quad \tilde{G}_N[k] = G_N \left( 1 + \alpha \frac{k^2}{k^2 + \lambda^2} \right)
\tag{6}
\]

The left-hand side extends the standard Poisson equation \((-k^2\text{ is the Laplacian operator})\), with the constant \(G_N\) replaced by a “running coupling constant” \(G_N[k]\) which depends on spatial wavevector \(k\).

Note that the Yukawa correction \(\Phi_N\) gives rise in the domain \(r \ll \lambda\) to a perturbation \(\delta \Phi_N\) which is linear in the distance \(\sim - (\alpha/2\lambda^2)(\kappa r)\). In equation \(\text{(6)}\) equivalently, the correction of the running constant scales as \(\delta G_N \sim (\alpha/\lambda^2)(G_N/k^2)\) in the domain \(|kl| \gg 1\). In fact, experimental constraints obtained at scales of the order of
the size of the solar system can be written as bounds on the combination $\alpha/\lambda^2$, so that they can be translated into bounds on the anomalous acceleration $\partial_r \delta \Phi_N$ or, equivalently, on $k^2 \delta \hat{G}_N$. It is worth emphasizing that these bounds result in allowed anomalous accelerations which remain 2000 times too small to account for the Pioneer anomaly [70]. In other words, the Pioneer anomaly cannot be due to a modification of Newton law, as such a modification would be much too large to remain unnoticed by planetary tests. Anew, this contradiction between Pioneer observations and other gravity tests may be cured by the extended framework studied below, now thanks to the fact that the existence of an anomalous space-dependent potential will not only be considered for the metric component $g_{00}$ but also for $g_{rr}$.

To summarize this section, tests performed on gravity in the solar system confirm its metric character and provide strong evidence in favor of gravitation theory being very close to GR. They however still leave room for alternative metric theories, which deviate from GR in a specific way. Anomalies in the metric components remain allowed, as long as they modify spatial dependencies without strongly affecting the time component $g_{00}$. It is shown in next sections that such extensions of GR may in fact arise naturally, in particular when effects of radiative corrections are taken into account.

### III. LINEARIZED GRAVITATION THEORY

We come now to the description of the “post-Einsteinian” extension of GR. We first repeat that tests performed at various length scales have showed that the equivalence principle (EP) was preserved at a higher accuracy level, $10^{-12}$, than the EP violation which would be needed to account for the Pioneer anomaly. As a matter of fact, the standard Newton acceleration at 70 UA is of the order of $1 \mu\text{m s}^{-2}$ while the Pioneer anomaly is of the order of $1 \text{nm s}^{-2}$, which would correspond to a violation order of $10^{-3}$. This does not mean that EP violations are excluded, and they are indeed predicted by unification models [71, 72]. However, any such violations are bound to occur at a lower level than needed to affect the Pioneer anomaly. Hence, EP violations will be ignored in the following and we shall restrict our discussion to a confrontation of GR with alternative metric theories.

Furthermore, neither PPN extensions of GR nor mere modifications of Newton force laws have the ability to account for the Pioneer anomaly (see the previous section). However, such extensions do not cover the totality of possible extensions of GR. In particular, there exist extended metric theories characterized by the existence of two gravitation potentials instead of a single one [38, 39, 40]. The first one merely represents a modified Newton potential while the second one can be understood in terms of a space-dependent PPN parameter $\gamma$. In this larger family of extensions, there is enough room available for accommodating the Pioneer anomaly while preserving compatibility with other gravity tests. Let us stress that this larger family is not introduced as an adhoc solution to the Pioneer anomaly. It emerges as the natural extension of GR induced by radiative corrections due to the coupling of gravity with other fields, and some phenomenological consequences were explored [41] before noticing that they included Pioneer-like anomalies [38, 39, 40]. In order to present these ideas in a simple manner, we will start with the linearized version of gravitation theory, which is approximately valid for describing Pioneer-like probes having escape motions in the outer solar system [38, 39]. We will then present some salient features of the non linear theory [40].

In the linearized treatment, the metric field may be represented as a small perturbation $h_{\mu\nu}$ of Minkowski metric $\eta_{\mu\nu}$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

where $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$, $|h_{\mu\nu}| \ll 1$.

The field $h_{\mu\nu}$ is a function of position $x$ in spacetime or, equivalently in Fourier space, of wavevector $k$

$$h_{\mu\nu}(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx}h_{\mu\nu}[k]$$

Gauge invariant observables of the metric theory, i.e. quantities which do not depend on a choice of coordinates, are given by curvature tensors. In the linearized theory, i.e. at first order in $h_{\mu\nu}$, Riemann, Ricci, scalar and Einstein curvatures are written in momentum representation as

$$R_{\lambda\mu\rho} = \frac{k_\lambda k_\rho h_{\mu\nu} - k_\lambda k_\mu h_{\nu\rho} - k_\mu k_\rho h_{\lambda\nu} + k_\mu k_\nu h_{\lambda\rho}}{2}$$

$$R_{\mu\nu} = R^\lambda_{\lambda\mu\nu}$$

$$E_{\mu\nu} = R_{\mu\nu} - \eta_{\mu\nu} R$$

We use the sign conventions of [73], indices being raised or lowered using Minkowski metric.
These curvature fields are similar to the gauge invariant electromagnetic fields of electrodynamics so that, while being supported by its geometrical interpretation, GR shows essential similarities with other field theories \cite{21,22}. This suggests that GR may be considered as the low energy effective limit of a more complete unified theory \cite{27,28} which should describe the coupling of gravity with other fields. In any case, this theory should contain radiative corrections to the graviton propagator, leading to a modification of gravitation equations \cite{2} and to a momentum dependence of the coupling between curvature and stress tensors. In the weak field approximation, it is easily seen that Einstein tensor, which is divergenceless, has a natural decomposition on the two sectors corresponding to different conformal weights \cite{41}, that is also on traceless (conformal weight 0) and traced components (conformal weight 1).

The general coupling between curvature and stress tensors can be written in terms of a linear response function constrained by the transversality condition

\[ E_{\mu\nu}[k] = \chi_{\mu\nu\lambda\rho}[k] T^{\lambda\rho}[k] \quad , \quad k^\mu \chi_{\mu\nu\lambda\rho}[k] = 0 \]  \hspace{1cm} (10)

We consider as above the isotropic and stationary situation with a point-like and motion-less Sun of mass \( M \). We then deduce that the general coupling \( \Phi_N \) is described by two running constants \( \tilde{G}^0 \) and \( \tilde{G}^1 \), depending on the spatial wavevector \( \mathbf{k} \) and living in the two sectors (0) and (1), so that gravitation equations (11) become \( \Phi_N \) become \[ \Phi_N \]

\[ E_{\mu\nu}[k] = 2\pi\delta(k_0)E_{\mu\nu}[k], \quad \pi_{\mu\nu} \equiv \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \]

\[ E_{\mu\nu}[k] = \pi^0_\mu \pi^0_\nu \tilde{G}^0[k] \frac{8\pi M}{c^2} + \pi_{\mu\nu} \pi^{00} \tilde{G}^1[k] - \tilde{G}^0(k) \frac{8\pi M}{3 c^2} \]  \hspace{1cm} (11)

The Newton gravitation constant \( G_N \) in (2) has been replaced in (11) by two running coupling constants \( \tilde{G}^0 \) and \( \tilde{G}^1 \) which are related through Poisson-like equations to two potentials \( \Phi^0 \) and \( \Phi^1 \) (compare with (6))

\[ -k^2 \Phi^a[k] = \tilde{G}^a[k] \frac{4\pi M}{c^2}, \quad a = 0, 1 \]  \hspace{1cm} (12)

These two potentials determine the metric, that is the solution of the modified equations (11), written here with spatial isotropic coordinates

\[ g_{00} = 1 + 2\Phi_N, \quad \Phi_N \equiv \frac{4\Phi^0 - \Phi^1}{3} \]

\[ g_{rr} = -(1 - 2\Phi_N + 2\Phi_P), \quad \Phi_P \equiv \frac{2(\Phi^0 - \Phi^1)}{3} \]  \hspace{1cm} (13)

\( \Phi_N \) is defined from the difference \( (g_{00} - 1) \) and identified as an extended Newton potential while \( \Phi_P \) is defined from \( (g_{00} - g_{rr} - 1) \) and interpreted as measuring the difference between the potentials \( \Phi^0 \) and \( \Phi^1 \) in the two sectors of traceless and traced curvatures. As \( \Phi^0 \) and \( \Phi^1 \), \( \Phi_N \) and \( \Phi_P \) obey Poisson equations with running constants \( \tilde{G}_N \) and \( \tilde{G}_P \) written as linear combinations of \( \tilde{G}^0 \) and \( \tilde{G}^1 \)

\[ -k^2 \Phi_a[k] = \tilde{G}_a[k] \frac{4\pi M}{c^2}, \quad a = N, P \]  \hspace{1cm} (14)

\[ \tilde{G}_N \equiv \frac{4\tilde{G}^0 - \tilde{G}^1}{3}, \quad \tilde{G}_P \equiv \frac{2(\tilde{G}^0 - \tilde{G}^1)}{3} \]

Standard Einstein equation is recovered when the running constants \( \tilde{G}^0 \) and \( \tilde{G}^1 \) are momentum independent and equal to each other, that is also when

\[ \left[ \tilde{G}_N \right]_{st} \equiv G_N, \quad \left[ \tilde{G}_P \right]_{st} = 0 \]

\[ \left[ \Phi_N(r) \right]_{st} \equiv \phi(r), \quad \left[ \Phi_P(r) \right]_{st} = 0 \]  \hspace{1cm} (15)

The two potentials \( \Phi_a \) will be written as sums of these standard expressions and anomalies which, according to the discussions of the previous section, will remain small

\[ \Phi_a(r) \equiv \left[ \Phi_a(r) \right]_{st} + \delta \Phi_a(r), \quad |\delta \Phi_a(r)| \ll 1 \]  \hspace{1cm} (16)
IV. NON LINEAR GRAVITATION THEORY

Before embarking in the discussion of phenomenological consequences of these anomalous potentials, let us recall briefly that the linearized theory presented in the preceding section can be transformed into a full non linear theory. The linearized theory will indeed be sufficient to discuss the anomalous acceleration of Pioneer probes as well as potential effects on light-like waves [38, 39] but the non linear theory will be needed to address the case of planetary tests [40], which was valid only at first order around Minkowski space-time. We may nevertheless simplify this relation by working at first order in deviations from standard Einstein theory [40].

To this aim, we write the metric, now in terms of Schwartzschild coordinates [74]

\[ ds^2 = \tilde{g}_{00}(\tilde{r})c^2dt^2 + \tilde{g}_{rr}(\tilde{r})d\tilde{r}^2 - \tilde{r}^2 \left( d\theta^2 + \sin^2\theta d\varphi^2 \right) \]

\[ \tilde{g}_{\mu\nu}(r) = [\tilde{g}_{\mu\nu}(r)]_{st} + \delta\tilde{g}_{\mu\nu}(r) \quad , \quad |\delta\tilde{g}_{\mu\nu}(r)| \ll 1 \]

(17)

The standard GR solution is then treated exactly

\[ [\tilde{g}_{00}]_{st} = 1 - 2\kappa \tilde{u} = -\frac{1}{[\tilde{g}_{rr}]_{st}} \quad , \quad \tilde{u} = \frac{1}{\tilde{r}} \]

(18)

while the anomalous metric components are taken into account at first order. Proceeding in this manner, it is possible to define in the non linear theory two potentials \( \delta\tilde{\Phi}_N \) and \( \delta\tilde{\Phi}_P \) which generalize [10]

\[ \delta\tilde{g}_{rr} = \frac{2\tilde{u}}{(1 - 2\kappa \tilde{u})^2}(\delta\tilde{\Phi}_N - \delta\tilde{\Phi}_P)' \quad , \quad f' = \partial_\tilde{u}f \]

\[ \delta\tilde{g}_{00} = 2(1 - 2\kappa\tilde{u}) \int \frac{\delta\tilde{\Phi}_N - 2\kappa\tilde{u}\delta\tilde{\Phi}_P'}{(1 - 2\kappa\tilde{u})^2} d\tilde{u} \]

(19)

In the linearized approximation, corrections in \( \kappa \tilde{u} \) are disregarded and the simple relations of the preceding section are recovered. In the general case, equations (19) fully describe non linear effects of the Newton potential \( \kappa \tilde{u} \). The precise form of the non linear version (19) has been chosen so that potentials are related in a simple way to the corresponding anomalous Einstein curvatures

\[ \delta\tilde{E}_0^0 \equiv 2\tilde{u}^4(\delta\tilde{\Phi}_N - \delta\tilde{\Phi}_P)'' \]

\[ \delta\tilde{E}_r^r \equiv 2\tilde{u}^3\delta\tilde{\Phi}_P' \]

(20)

At this stage, it is worth noticing that the PPN metric [1] may be recovered as a particular case of the more general extension [10]. This particular case corresponds to the following expressions of anomalous potentials and anomalous Einstein curvatures [10]

\[ \delta\tilde{\Phi}_N = (\beta - 1)\kappa^2\tilde{u}^2 + O(\kappa^3\tilde{u}^3) \quad , \quad [\text{PPN}] \]

\[ \delta\tilde{\Phi}_P = (\gamma - 1)\kappa\tilde{u} + O(\kappa^2\tilde{u}^2) \]

(21)

\[ \delta\tilde{E}_0^0 = \tilde{u}^2O(\kappa^2\tilde{u}^2) \quad , \quad [\text{PPN}] \]

\[ \delta\tilde{E}_r^r = \tilde{u}^2(2(\gamma - 1)\kappa\tilde{u} + O(\kappa^2\tilde{u}^2)) \]

(22)

Note that the PPN metric already shows an anomalous behaviour of Einstein curvatures which have non null values apart from the gravity source. This is the case for \( \delta\tilde{E}_r^r \) at first order in \( \kappa \), and for \( \delta\tilde{E}_0^0 \) at higher orders. Relations (20) thus extend this anomalous behaviour to more general dependences of the curvatures \( \delta\tilde{E}_0^0 \) and \( \delta\tilde{E}_r^r \). Similar statements apply as well for anomalous potentials \( \delta\tilde{\Phi}_N \) and \( \delta\tilde{\Phi}_P \), which generalize the specific dependence of PPN potentials [24] where \( \beta - 1 \) and \( \gamma - 1 \) are constants. In other words, the post-Einsteinian metric [10] can be thought of as an extension of PPN metric where \( \beta - 1 \) and \( \gamma - 1 \) are no longer constants but rather functions of space.

V. PHENOMENOLOGICAL CONSEQUENCES

As already discussed, the new phenomenological framework is characterized by two anomalous potentials: the first one \( \delta\tilde{\Phi}_N \) is a modification of Newton potential while the second one \( \delta\tilde{\Phi}_P \) represents the difference of gravitational couplings in the two sectors of traceless and traced curvatures. The first potential is not able by itself to explain the
Pioneer anomaly: its anomalous part is indeed bound by planetary tests to be much smaller than would be needed to account for the Pioneer anomaly \textsuperscript{70}. This is why we will focus the attention in the following on the second potential which can produce a Pioneer-like anomaly for probes on escape trajectories in the outer solar system \textsuperscript{38, 39}. This second potential can also be understood as promoting the PPN parameter $\gamma$ to the status of a space dependent function and it has therefore other consequences which have to be evaluated with great care: It is clear that the modification of GR needed to produce the Pioneer anomaly should not spoil its agreement with other gravity tests.

We first discuss the effect of the second potential on Doppler tracking of Pioneer-like probes. To this aim, we calculate the Doppler velocity taking into account the perturbations on probe motions as well as on light propagation between stations on Earth and probes. We then write the time derivative of this velocity as an acceleration $a$ and finally subtract the expression obtained in standard theory from that obtained in extended one. We thus obtain the prediction of the post-Einsteinian extension \textsuperscript{38, 39} for the Pioneer anomalous acceleration $\delta a \equiv a - [a]_{\text{st}}$. In a configuration similar to that of Pioneer 10/11 probes, which follow nearly radial trajectories with a kinetic energy much larger than their potential energy, this anomalous acceleration takes the simplified form

$$\delta a \simeq 2 \frac{d\delta \Phi_P}{dr} v_P^2$$

Thus, an anomaly in Doppler tracking of Pioneer-like probes is a direct consequence of the presence of the second potential $\Phi_P$. Note that the anomalous acceleration comes out as proportional to the kinetic energy, which is a remarkable prediction of the new framework. Data on probes with very different kinetic energies (unfortunately not available at the moment) could thus be used to confirm or infirm this prediction.

Using the known velocity of the Pioneer probes ($v_P \sim 12\text{km s}^{-1}$), and identifying the acceleration \textsuperscript{24} with the recorded Pioneer anomaly \textsuperscript{11}, we deduce the value of the derivative $d\delta \Phi_P/dr$ in the outer solar system. The constancy of recorded anomaly over a large range of distances agrees with a simple parametrization of the second potential \textsuperscript{38}

$$\delta \Phi_P(r) \equiv - \frac{G_P M}{r c^2} + \frac{\zeta_P M r}{c^2}$$

This value of the parameter $\zeta_P M$ is much larger than that allowed for the parameter $\zeta_N M$ which could be defined on the first potential \textsuperscript{38}. This shows in a clear manner how the second potential $\Phi_P$ opens the possibility to account for the Pioneer anomaly. It has to be kept in mind for the forthcoming discussions that the simple model \textsuperscript{24} does not need to be exact in the whole solar system. However, the expression of $\delta \Phi_P$ can generally be given the form \textsuperscript{24}, provided $\zeta_P$ denotes a function of the heliocentric distance.

We come now to the discussion of the effects of the second potential $\delta \Phi_P$ on the propagation of light rays, which can be done in the linearized theory. Considering in particular deflection experiments usually devoted to the determination of Eddington parameter $\gamma$, we obtain the following expression for the anomaly $\delta \psi$ (with respect to GR) of the deflection angle of rays grazing the surface of the Sun \textsuperscript{38} (the same decomposition as in \textsuperscript{24} is used for $\delta \Phi_P$)

$$\delta \psi \simeq - \kappa \frac{\partial}{\partial \rho} \left( \delta \gamma(\rho) \ln \frac{4 r_1 r_2}{\rho^2} \right)$$

$$\delta \gamma(\rho) = - \frac{G_P}{G_N} + \frac{\zeta_P(\rho) P^2}{2 G_N}$$

Terms which are not amplified near occultation have been neglected; $r_1$ and $r_2$ correspond to the heliocentric distances of the receiver and emitter, and $\rho$ is the distance of closest approach of the light ray to the Sun; $\delta \gamma(\rho)$ is a range dependent anomalous part in Eddington parameter $\gamma$. These expressions are reduced to PPN ones when the function $\zeta_P$ vanishes. Otherwise, they show that Eddington deflection tests could reveal the presence of $\delta \Phi_P$ through a space dependence of the parameter $\gamma$.

We conclude this survey of phenomenological consequences of the new framework by discussing planetary tests and, in particular, those involving the perihelion precession of planets. As the latter are known to depend on the two PPN parameters $\beta$ and $\gamma$, this discussion has to be presented in the context of the non linear theory. Still focusing the attention on the effects of the second potential, one thus obtains the following expression for the anomaly $\delta \Delta \varphi$ (with respect to GR) of the perihelion precession \textsuperscript{40}

$$\delta \Delta \varphi \simeq \bar{u} (\bar{u} \delta \Phi_P)'' + \frac{e^2 \bar{u}^2}{8} (\bar{u}^2 \delta \Phi_P' + \bar{u} \delta \Phi_P)'$$

This expression has been truncated after leading and sub-leading orders in the eccentricity $e$; the function $\delta \Phi_P$ and its derivatives have thus to be evaluated at the inverse radius $\bar{u}$ of the nearly circular ($e \ll 1$) planetary orbit. As the
leading order vanishes for a contribution $\zeta_{P} M r / c^2$ with $\zeta_{P}$ constant, the main result is thus proportional to $e^2$ in this case. This means that perihelion precession of planets could be used as a sensitive probe of the value and variation of $\zeta_{P}$ for distances corresponding to the radii of planetary orbits [40].

VI. DISCUSSION

Gravity tests which have been performed up to now in the solar system firmly support the validity of the equivalence principle, that is also the metric nature of gravitation. They also strongly indicate that the actual gravitation theory should be very close to GR. Nonetheless, these tests still leave room for alternative metric theories of gravitation, and the anomaly observed on the trajectories of the Pioneer 10/11 probes may well be a first indication of a modification of gravity law in the outer part of the solar system. This possibility would have such a large impact on fundamental physics, astrophysics and maybe cosmology that it certainly deserves further investigations. We have discussed in the present paper a new extension of GR which allows one to address these questions in a well defined theoretical framework [39, 40].

When its radiative corrections are taken into account, GR appears as imbedded in a family of metric theories characterized, at the linearized level, by two running coupling constants which replace the single Newton gravitation constant or, equivalently, by two potentials which replace the standard Newton potential. When applied to the solar system, this post-Einsteinian extension of GR leads to a phenomenological framework which has the ability to make the Pioneer anomaly compatible with other gravity tests. Precisely, the first potential $\Phi_N$ remains close to its standard Newtonian form in order to fit planetary tests but the second potential $\Phi_P$ opens a phenomenological freedom which can be understood as an Eddington parameter $\gamma$ differing from unity, as in PPN metric, with now a possible space dependence.

In order to confirm, or infirm, the pertinence of this framework with respect to gravity tests in the solar system, it is now necessary to re-analyze the motions of massive or massless probes in this new context. Contrarily to what has been done here, it is particularly important to take into account the effects of anomalous potentials $\delta \Phi_N$ and $\delta \Phi_P$ simultaneously. Let us scan in the last paragraphs of this paper some ideas which look particularly promising.

The main novelty induced by the second potential $\delta \Phi_P$ is to produce an anomaly on Doppler tracking of Pioneer-like probes having highly eccentric motions in the outer solar system. As already discussed, if the recorded anomaly is identified with this effect, one deduces the value of the derivative $d\delta \Phi_P / dr$ of the second potential at the large distances explored by Pioneer probes. A natural idea is therefore to confront the more detailed prediction deduced from the new theory [39] against the larger set of data which will soon be available [75, 76]. It is particularly clear that the eccentricity of the orbits plays a key role in the evaluation of the Pioneer anomaly: it takes large values for Pioneer-like motions which sense $\delta \Phi_P$ whereas it is zero for circular orbits which do not. This suggests to devote a dedicated analysis to the intermediate situation, not only for the two categories of bound and unbound orbits, but also for the flybys used to bring Pioneer-like probes from the former category to the latter one. While Pioneer probes may sense the second potential at the large heliocentric distances they are exploring, planets or planetary probes may feel its presence at distances of the order of the astronomical unit. Then, it would be worth studying planetary probes on elliptical orbits, for example on transfer orbits from Earth to Mars or Jupiter. Another natural target for such a study is LISA with its three crafts on elliptical orbits [77].

The second potential $\delta \Phi_P$ also affects the propagation of light waves, and it could thus be detected as a range dependence of the anomalous Eddington parameter $(\gamma - 1)$ to be seen for example in deflection experiments. This might already be attainable through a reanalysis of existing data, given by the Cassini experiment [57], VLBI measurements [50] or HIPPARCOS [78]. It may also be reached in the future by higher accuracy Eddington tests, as for example the LATOR project [79], or global mapping of deflection over the sky (GAIA project [80]). Reconstruction of the dependence of $\gamma$ versus the impact parameter $\rho$ would directly provide the space dependence of the second potential $\delta \Phi_P$. This would then either produce a clear signature of the new framework presented in this paper or put constraints on the presence of the second potential at small heliocentric distances.

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