Generating Minimum-Snap Quadrotor Trajectories Really Fast

Declan Burke, Airlie Chapman, Iman Shames

Abstract—We propose an algorithm for generating minimum-snap trajectories for quadrotors with linear computational complexity with respect to the number of segments in the spline trajectory. Our algorithm is numerically stable for large numbers of segments and is able to generate trajectories of more than 500,000 segments. The computational speed and numerical stability of our algorithm makes it suitable for real-time generation of very large scale trajectories. We demonstrate the performance of our algorithm and compare it to existing methods, in which it is both faster and able to calculate larger trajectories than state-of-the-art. We also show the feasibility of the trajectories experimentally with a long quadrotor flight.

I. INTRODUCTION

Quadrotor trajectory optimization for has been extensively studied with a variety of approaches suggested [1]. Mellinger and Kumar [2] pioneered minimum-snap trajectory generation for quadrotors. They used smooth trajectories to control quadrotors and formulated a quadratic program (QP) to calculate continuous splines. A method of solving this QP was proposed by Richter et al. [3], who also included a procedure for making trajectories safer. Mellinger et al. [4] and Deits and Tedrake [5] further developed the use of minimum-snap splines and incorporated collision avoidance by formulating mixed-integer programs to generate trajectories. Another approach to trajectory generation is in the use of motion primitives generated by solving optimal control problems, often to minimize the snap of the primitive. Lui et al. [6] discretize the state space using primitives and then perform a graph search, while Muller et al. [7] evaluate many primitives at each controller update step and execute the feasible primitive with the lowest cost.

In this paper, we revisit the equality constrained QP formulation of [2], [3] used to generate a minimum snap trajectory. The aforementioned QP formulation leads to conceptually simple optimisation problems. However, solving the resultant optimization program can be challenging in practice. Richter et al. [3] note numerical instability in using constrained optimisation methods to solve for trajectories of more than four segments. De Almeida and Akella further investigate the numerical instability of the problem formulation and identify the ill-conditioned matrix that is the most significant source of error [8]. Beyond numerical stability, other properties of solutions to minimum-snap trajectory generation problems such as their computational complexity remain unexplored.

In this paper, we propose a method that solves the QP problem corresponding to a minimum snap trajectory generation with a very large number of segments for long, albeit finite, flight times. We achieve this by explicitly identifying and exploiting the algebraic structure of the problem. Particularly, first, we show that our solution is of linear computational complexity in the number of segments of the trajectory. Second, inspired by the properties of confluent Vandermonde matrices we introduce a new change of variable to arrive at an equivalent yet better conditioned problem. Our work differs from that of [8] as we address the numerical instability of the original QP formulation of [2], [3] rather than proposing a new model for minimising the snap of the trajectory.

In summary, our algorithm solves problems many times larger than state-of-the-art methods, demonstrating with trajectories of more than 500,000 segments (see Fig. 1).

Fig. 1. The computation time required to generate minimum-snap trajectories for splines of a range of polynomial segments using the proposed algorithm (Algorithm 1).

The paper’s outline is as follows. In the next section, we introduce notations along with the minimum snap problem. In Section III the method for solving the minimum snap problem is described. In Section IV we study the conditioning problems inherent to the vanilla formulation of the minimum snap problem and propose a change of variable that alleviate this issue. Experimental results are presented in Section V. Concluding remarks come in the end.

The first and second authors are with the Department of Mechanical Engineering and third with the Department of Electrical and Electronic Engineering, The University of Melbourne, VIC, 3010 Australia. Emails: {declanb@student., airlie.chapman@, iman.shames@junimelb.edu.au}. This work is partially supported by Defence Science and Technology Group, through agreement MyIP: ID9156 entitled “Verifiable Hierarchical Sensing, Planning and Control”, the Australian Government, via grant AUSMURI000001 associated with ONR MURI grant N00014-19-1-2571.

The authors note that an inequality constrained QP should instead be employed to introduce position, velocity and acceleration constraints useful for obstacle avoidance. See [8] for a problem formulation and further details.
II. THE MINIMUM SNAP PROBLEM

A. Notation

We use the following pieces of notation throughout the paper. We write the dot product as \( \cdot \), and the identity matrix as \( I_k \in \mathbb{R}^{k \times k} \). The condition number of a matrix \( V \) is defined as \( \kappa(V) = ||V|| \cdot ||V^{-1}|| \). The cardinality of a set \( \Xi \) is written as \( |\Xi| \). A function \( \pi \) of \( t \) that is parameterised by \( p \) is written \( \pi(t;p) \). We use the order notation for the sequences \( \{\eta_i\} \) and \( \{\nu_i\} \), to write \( \eta_i = O(\nu_i) \) if there exists a positive constant \( C \in \mathbb{R} \) such that \( |\eta_i| \leq C |\nu_i| \) for \( i \) sufficiently large. Given a sequence of column vectors \( \{x_i\} \) and matrices \( \{V_i\} \) for \( i = 1, \ldots, k \), let

\[
\mathrm{vec}(\{x_i\}) := \begin{bmatrix}
    x_1 \\
    \vdots \\
    x_k
\end{bmatrix}, \quad \mathrm{diag}(\{V_i\}) := \begin{bmatrix}
    V_1 & & \\
    & \ddots & \\
    & & V_k
\end{bmatrix}.
\]

B. Formulating an Optimization Program

Mellinger and Kumar established that a smooth trajectory in terms of position \( [x, y, z]^T \in \mathbb{R}^3 \) and yaw \( \psi \in SO(2) \) can be used to calculate motor commands for a quadrotor, i.e., the system’s control inputs [2]. Their method of generating this trajectory involves formulating and solving four QPs. The QPs are of the same form and they may be solved separately, so we consider the general problem of generating a spline \( \pi(t) \) that can represent any of \( x(t), y(t), z(t) \) or \( \psi(t) \) [2]

We now introduce the representation of the system used to formulate minimum-snap trajectory generation as an optimization program. We consider nondecreasing times as \( T = [t_0, \ldots, t_k]^T \in \mathbb{R}^{k+1} \) such that \( t_{i-1} \leq t_i \) for \( i = 1, \ldots, k \). The conventional choice for the trajectory is a continuous spline \( \pi(t;p) : [t_0, t_k] \rightarrow \mathbb{R} \) of \( k \geq 1 \), \( k \) integer, segments

\[
\pi(t;p) = \begin{cases} 
    \pi_1(t;p_1) & t_0 \leq t < t_1, \\
    \vdots \\
    \pi_k(t;p_k) & t_{k-1} \leq t < t_k,
\end{cases}
\]

where each segment \( \pi_i(t;p_i) : [t_{i-1}, t_i] \rightarrow \mathbb{R} \) is a polynomial of order \( n-1 \), where \( n \) is a positive integer. We represent the segments as \( \pi_i(t;p_i) = [1, t, \ldots, t^{n-1}]p_i \), where \( p_i = [p_{0,i}, \ldots, p_{n-1,i}]^T \) for all \( i = 1, \ldots, k \), and the entire spline as \( p = \mathrm{vec}(\{p_i\}) \in \mathbb{R}^{kn} \), \( i = 1, \ldots, k \).

Constraints on the trajectory \( \pi(t;p) \) are introduced by setting the value of the segments \( \pi_i(t;p_i) \) and their derivatives at \( t_{i-1} \) and \( t_i \), \( i = 1, \ldots, k \). The value of \( j \)th derivative of the segment \( \pi_i(t;p_i) \) at time \( t \) is the scalar \( \delta_{i,j}, \) and we consider up to the \( (s-1) \)th derivative, where \( s \) is a positive integer. We represent the first \( s-1 \) derivatives as the vector \( d_{i,t} = \mathrm{vec}(\{\delta_{i,j,t}\}) \in \mathbb{R}^s \), \( j = 0, \ldots, s-1 \), and for many derivative values \( d_{i,t} \) at \( t_{i-1} \) and \( t_i \), we write the vector \( d = \mathrm{vec}(\{d_{i,t_{i-1}}, d_{i,t_i}^T\}) \in \mathbb{R}^{2sk} \) for \( i = 1, \ldots, k \). We require continuity between segments, so we equate \( d_{i,t_i} = d_{i+1,t_i} \) for all \( i = 1, \ldots, k-1 \). Further

\[\text{Problem 1:} \text{ Generate a minimum snap spline } \pi(t;p^*) \text{ with } k \text{ polynomial segments of order } n-1 \text{ where } p^* \text{ is obtained by solving (1) using an algorithm with computational complexity of } O(kn^3).\]
Algorithm 1: Algorithm of Linear Computational Complexity for Solving Problem [1]

\[
\begin{align*}
\text{for } i = 1, \ldots, k & \text{ do} \\
& \text{construct } A_i, Q_i, \Sigma_i, b_{i,t-1}, b_{i,t} \\
& \text{partition } B_i, C_i, D_i, g_{i,t-1}, g_{i,t} \\
& \bar{B}_i \leftarrow \begin{cases} B_1, & i = 1, \\
B_1 + D_{i-1} - C_T^{-1} B_1^{-1} C_{i-1}, & i = 2, \ldots, k, \end{cases} \\
& \bar{g}_{i,t-1} \leftarrow \begin{cases} g_{i,t-1}, & i = 1, \\
g_{i,t-1} + g_{i-1,t-2}, & i = 2, \ldots, k, \end{cases} \\
\text{for } i = k, \ldots, 1 & \text{ do} \\
& f_{i,t-1} = \bar{f}_{i,t-1}(\bar{g}_{i,t-1} - C_i b_{i,t}) \\
& f_{i,t} = \bar{f}_{i,t} - D_{k-1} \bar{B}_k^{-1} \bar{g}_{k,t-1} \\
& \text{solve for } d_{i,t-1} \text{ and } d_{i,t} \\
& \text{solve for } p_i \\
\end{align*}
\]

In the next section, we propose a solution to Problem [1]. Later in Section IV, we further introduce a modification that considerably improves the numerical stability of the proposed solution.

III. A STRUCTURED SOLUTION

A. The Algorithm

We begin by presenting our algorithm for solving Problem [1]. We start by introducing the variables required by the algorithm’s calculations. Let

\[ d = b + \Sigma f, \quad (2) \]

where \( b \in \mathbb{R}^{2k} \), \( f \in \mathbb{R}^m \), and \( \Sigma \in \mathbb{R}^{2k \times m} \) is such that \( \Pi \Sigma = \beta \) and \( \Pi \Sigma = 0 \). We note that \( b \) is constant in the sense its entries are either the parameters \( \beta_k \), for \( k \in \Xi \), or zeros. We partition \( b \), \( f \), and \( \Sigma \) into vectors and matrices of size \( b_{i,t} \in \mathbb{R}^n \), \( f_{i,t} \in \mathbb{R}^{m_{i,t}} \), and \( \Sigma_{i,t} \in \mathbb{R}^{m_{i,t} \times m_{i,t}} \), for \( i = 1, \ldots, k \) and \( t = t_{i-1}, \ldots, t_i \), such that \( b = \text{vec}((b_{i,t-1}, f_{i,t})^T) \in \mathbb{R}^{2k} \) and \( f = \text{vec}((f_{i,t-1}, f_{i,t})^T) \in \mathbb{R}^m \). Further, let \( \Sigma_t = \text{diag}(\{\Sigma_{i,t-1, i,t}\}) \) such that \( \Sigma = \text{diag}(\{\Sigma_t\}) \in \mathbb{R}^{2k \times m} \) for \( i = 1, \ldots, k \).

For \( i = 1, \ldots, k \), let

\[
\begin{align*}
\Sigma_i^T A_i^{-T} Q_i A_i^{-1} \Sigma_i & = \begin{bmatrix} B_i & C_i \\
C_i^T & D_i \end{bmatrix}, \\
\Sigma_i^T A_i^{-T} Q_i A_i^{-1} \begin{bmatrix} b_{i,t-1} & b_{i,t} \\
0 & 0 \end{bmatrix} & = \begin{bmatrix} g_{i,t-1} & g_{i,t} \end{bmatrix},
\end{align*}
\]

where \( B_i \in \mathbb{R}^{m_{i,t-1} \times m_{i,t-1}}, \Sigma_i \in \mathbb{R}^{m_{i,t-1} \times m_{i,t-1}}, C_i \in \mathbb{R}^{m_{i,t-1} \times m_{i,t-1}}, D_i \in \mathbb{R}^{m_{i,t} \times m_{i,t}}, \) and \( g_{i,t} \in \mathbb{R}^{m_{i,t}} \) for \( t = t_{i-1}, \ldots, t_i \).

**Proposition 2:** Algorithm [1] solves Problem [1] in \( O(kn^3) \). 
**Proof:** See Appendix [2].

**Remark 3:** We note that in practice \( s \) is usually chosen to be 5, such that continuity is enforced up to the snap of position [2]. The number of segments \( k \) however depends on the size of the trajectory and can be very large. This is then an improvement over other methods for solving Problem [1] which typically have a cubic computational complexity in \( k \). The upper bound on the size of the equations ([1]) is \( kn + (k - 1)s \). Solving with Gaussian Elimination would then require \( O(\frac{1}{2}kn^3(n^2 + s^2)) \) operations. The best current method, proposed by Richter et al. [3], includes the inversion of a matrix of size \( m \), and thereby needs \( O(2kn^3) \) operations. Linear in \( k \), our approach dramatically reduces the computation time required to solve Problem [1], particularly for large trajectories.

IV. IMPROVING CONDITIONING

We now present a reformulation that results in a QP with well-conditioned matrices. To motivate this reformulation, we take a short detour to explore Vandermonde matrices. The \( A_i \) are transposed confluent Vandermonde matrices and are notoriously ill-conditioned [9]. These matrices are ultimately the culprits behind the numerical instability of [1].

A Vandermonde matrix is defined by the scalars \( z_0, z_1, \ldots, z_n \in \mathbb{C} \) as

\[
V(z_0, z_1, \ldots, z_n) = \begin{bmatrix} 1 & \mathbf{z} & \cdots & \mathbf{z}^{n-1} \end{bmatrix} \in \mathbb{C}^{(n+1) \times (n+1)}.
\]

One way of generalising the standard Vandermonde matrix is to allow confluency, that is, including columns with elements that are the differentiated elements of other columns. One strategy to improve the conditioning of \( V \) is to carefully select the points \( z_j \), for \( j = 0, \ldots, n \). Ideally, \( V \) is perfectly conditioned, i.e., \( \kappa(V) = 1 \), which occurs when the points are roots of unity \( z_j = \exp(\frac{2\pi i}{n}) \) (here \( i \) is the imaginary unit), for \( j = 0, \ldots, n - 1 \) [10]. Taking the points of a confluent Vandermonde matrix to be roots of unity has also been shown to improve its condition number [11].

To formulate a program with better conditioned matrices than [1], we thus scale the problem such that all the \( A_i \) and \( Q_i \) have the nondimensional points \( t_i - 1 \) and \( t_i \) (with associated matrices \( A_{\pm} \) and \( Q_{\pm} \)). The nondimensional program is

\[
\min_{\bar{p},d} \quad \bar{p}^T \bar{Q} \bar{p},
\]

s.t. \( \bar{A} \bar{p} = d \),

\[
\Delta d = 0,
\]

\[ \Pi d = \beta, \]

where \( \bar{A} = \text{diag}(\Gamma_1 A_{\pm}, \ldots, \Gamma_k A_{\pm}) \in \mathbb{R}^{2kn \times nk} \) and \( \bar{Q} = \text{diag}(Q_{\pm}, \ldots, Q_{\pm}) \in \mathbb{R}^{nk \times nk} \) are constructed from elements \( \Gamma_i A_{\pm} \in \mathbb{R}^{nk \times nk} \) and \( Q_{\pm} \in \mathbb{R}^{nk \times nk} \) are constructed from elements \( \Gamma_i A_{\pm} \in \mathbb{R}^{nk \times nk} \) and \( Q_{\pm} \in \mathbb{R}^{nk \times nk} \). The matrix \( \Gamma_i = \text{diag}(\Delta_0^i, \Delta_{i-1}^i, \ldots, \Delta_{n-i}^i, \Delta_0^{n-i}) \) scales the rows of the \( A_{\pm} \) with elements \( \Delta_t = (t_i - t_{i-1})/2 \) for \( i = 1, \ldots, k \).

The following proposition is needed to show why [4] can be used to find a solution to Problem [1]. It considers the program for a minimum snap trajectory of a single segment in both a dimensional \( (t_i - 1 \leq t < t_i) \) and nondimensional form \((-1 \leq t < 1)\).
Proposition 4: Consider the two optimisation programs
\[
\begin{align*}
\min_{\pi(t; p_i)} & \int_{t_0}^{t_1} \left[ \frac{d^4\pi_i(t; p_i)}{dt^4} \right]^2 dt, \\
\text{s.t.} & \quad \frac{d^3\pi_i(t; p_i)}{dt^3} = \delta_{i,j,t}, \quad j = 0, \ldots, s - 1, \quad t = t_{i-1}, t_i,
\end{align*}
\]
and
\[
\begin{align*}
\min_{\pi(t; p_i)} & \int_{t_0}^{t_1} \left[ \frac{d^4\pi_i(t; p_i)}{dt^4} \right]^2 dt, \\
\text{s.t.} & \quad \frac{d^3\pi_i(t; p_i)}{dt^3} = \beta_{i,j,t}, \quad j = 0, \ldots, s - 1, \quad t = t_{i-1}, t_i,
\end{align*}
\]

The polynomial segment \( \pi_i(t; p_i) \) solves (6) and \( \pi_i(t; p_i) = \pi_i(\Delta_j t + \Delta_i t + t_{i-1}; p_i) \) solves (5), if
\[
\begin{align*}
\Delta_j^\tau \delta_{i,j,1} & = \beta_{i,j,t1}, \quad j = 0, \ldots, s - 1 \quad (7a) \\
\Delta_j^\tau \delta_{i,j,-1} & = \beta_{i,j,t0}, \quad j = 0, \ldots, s - 1. \quad (7b)
\end{align*}
\]

Proof: Under the change of variable \( \tau = \Delta_j^{-1}(t - \Delta_i t - t_{i-1}) \), let \( \pi_i(t; p_i) = \pi_i(\Delta_j^{-1}(\tau - \Delta_i t - t_{i-1}); p_i) \). With this change of variable, (5) becomes
\[
\begin{align*}
\min_{\pi_i(t; p_i)} & \quad \Delta_j^{-1} \int_{t_0}^{t_1} \left[ \frac{d^4\pi_i(t; p_i)}{dt^4} \right]^2 dt, \\
\text{s.t.} & \quad \frac{d^3\pi_i(t; p_i)}{dt^3} = \Delta_j^\tau \delta_{i,j,t}, \quad j = 0, \ldots, s - 1, \quad \tau = -1, 1.
\end{align*}
\]

The minimization of (8) is independent of the constant factor. Therefore, if the constraints (7) are satisfied then (8) is equivalent to (5). If \( \pi_i(t; p_i) \) solves (6) then \( \pi_i(\Delta_j t + \Delta_i t + t_{i-1}; p_i) \) is the solution to (5).

Now we state a relationship that allows (6) to be used to find a solution to (5).

Lemma 5: Let \( \overline{p} \) and \( d \) solve (3), \( p \) and \( d \) solve (1), and \( \pi_i(t; p_i) \) and \( \pi(t; p) \) be splines. Then
\[
\pi(t; p) = \begin{cases} 
\pi_i(\Delta_j t + \Delta_i t + t_0; p) & t_0 \leq t < t_1, \\
\vdots & \\
\pi_k(\Delta_j t + \Delta_k + t_{k-1}; p_k) & t_{k-1} \leq t < t_k.
\end{cases}
\]

Proof: Temporarily omitting the constraints (1c) and (1d), the program in the nondimensional form is
\[
\begin{align}
\min_{\overline{p}, d} & \quad \overline{p}^T Q \overline{p}, \\
\text{s.t.} & \quad \overline{A}' \overline{p} = \overline{d},
\end{align}
\]
where \( \overline{A}' = \text{diag} \{ A_{\pm 1}, \ldots, A_{\pm 1} \} \in \mathbb{R}^{kn \times kn} \) is constructed from elements \( A_{\pm 1} \in \mathbb{R}^{n \times n} \). We will next substitute (7) into (2) to satisfy Proposition 4. Stacking the derivatives \( d_{i,j,t} \) and \( d_{i,j,t} \) for \( j = 0, \ldots, s - 1 \), the conditions (7) can be expressed as the matrix equation \( \Gamma_1 \overline{d} = \overline{d} \). Substituting this expression into (9b) yields (10b). We introduce the constraints (4a) and (4d) to constrain the dimensional derivatives as desired. The conditions (7) of Proposition 4 are satisfied by construction, hence the solution to (4), \( \overline{p}^T(t) \), can be used to calculate the solution to (1) using Lemma 5.

Remark 6: Replacing the original QP (1) with the better conditioned (4) in Problem 1, Algorithm 1 can still be used for solution. With \( A_{\pm 1} = \Gamma_1 A_{\pm 1}, Q_{\pm 1} = Q_{\pm 1}, \text{ and } \Sigma_{\pm 1}, b_{i,t_{i-1}} \) and \( b_{i,t_{i}} \) as defined, Algorithm 1 calculates \( p \). Then the solution to Problem 1, \( \pi(t) \), can be calculated using (5).

The improvement of the condition number of the reformation of \( \Gamma_1 A_{\pm 1} \) compared to the original \( A_i \) is demonstrated by Table IV. The condition number \( \kappa(\Gamma_1 A_{\pm 1}) \) rapidly grows to more than \( 10^{10} \) for \( t_{i-1} = 10 \) and \( t_i = 12 \), where the matrices become practically unusable in calculations given numerical errors [12]. This helps in explaining the instability observed in generating large trajectories, as the \( t_i \) increase with the number of segments. We also note the \( \kappa(\Gamma_i A_{\pm 1}) \) remain constant in Table IV due to constant \( 1 \). The \( t_{i-1} \) and \( t_i \) only appear in \( \Gamma_1 A_{\pm 1} \) as part of the difference terms \( \Delta_i \). Hence, \( \kappa(\Gamma_i A_{\pm 1}) \) does not increase with \( t_{i-1} \) and is only prone to introducing numerical error for large \( \Delta_i \).

V. EXPERIMENTAL RESULTS

A. Large Scale Trajectory Generation

We demonstrated the performance of Algorithm 1 with the nondimensional program (4) by generating a minimum-snap trajectory of a large number of segments. Implementing Algorithm 1 with (4) in C++, we were able to generate a spline of more than 50000 segments in 156.01 s. The algorithm was implemented and run on a laptop with an Intel Core i7-8650U CPU running at 1.9 GHz, with 16 GB of RAM. The limitation of algorithm was memory allocation on the computer, and the implementation was not optimized to increase performance.
This method was chosen for comparison given it is compared with solving the equations from the KKT conditions to demonstrate the performance of Algorithm 1 with (4) compared with solving (11). The computation time of Algorithm 1 ranges from 2.56 to 12.41 ms for calculating splines of 10 to 50 segments, whereas it takes 5.93 s to calculate a 50 segment spline by solving (11).

VI. CONCLUSIONS AND FUTURE WORKS

In this paper we have presented a new algorithm to generate minimum-snap spline trajectories for quadrotors in linear computational complexity in the number of segments. We also proposed a reformulation of the associated optimization program with better conditioned matrices than the original program. With these two developments, we are able to generate large trajectories not limited by computational time or number of segments but instead by computer memory. The performance of the proposed algorithm furthers the applicability of minimum-snap trajectory planning for real-time applications, enabling the faster calculation of larger trajectories onboard quadrotors.

In a sequel, we will explore further applications enabled by the algorithm’s computational complexity. For example, the algorithm can be used as a fast oracle for zeroth-order optimization algorithms. Such approaches have been suggested to minimize the snap of a trajectory by optimizing the time parameterization $T$.

APPENDIX

A. Constructing Hessian Matrices

There are two possible constructions of the Hessian matrices $Q_i(T) \in \mathbb{R}^{nk \times nk}$ in (1), corresponding to $\pi_i(t; p_i) = x_i(t)$, $y_i(t)$ or $z_i(t)$, and $\pi_i(t; p_i) = \psi_i(t)$. Both are constructed by calculating the quadratic form that results from the following integral in representing $\pi_i(t; p_i)$ as a vector of coefficients $p_i$

$$\int_{t_0}^{t_1} \frac{d^{s-1} \pi_i(r; p_i)^2}{d\tau^{s-1}} \, dr = p^T_i Q_i(T) p_i,$$

where $s = 5$ for $\pi_i(t; p_i) = x_i(t)$, $y_i(t)$ or $z_i(t)$, such that the integrand is the square of the snap of position, or $s = 3$ for $\pi_i(t; p_i) = \psi_i(t)$, such that the integrand is the square of the acceleration of position.

B. Proof of Proposition 3

Before stating the proof, we first formulate (1) so that the desirable structure of the matrices in the program is revealed. From this reformulation, we obtain an efficient method in a similar fashion as Cantoni et al. [16], from which we derive the steps of Algorithm 1.

Similar to Richter et al. [3], we reformulate (1) by replacing (1b) with $p = A^{-1} d$. We further replace (1d) with (2). By construction $\Pi d = \Pi (b + \Sigma f) = \beta$, so we substitute (2) into (1) and are left with

$$\min \int f^T \Sigma^T A^{-T} Q A^{-1} b + \frac{1}{2} f^T \Sigma^T A^{-T} Q A^{-1} \Sigma f, \quad \text{(10a)}$$

s.t. $\Lambda \Sigma f = 0. \quad \text{(10b)}$
The KKT conditions for (10) yield
\[
\begin{bmatrix}
\Sigma^T A^{-T} QA^{-1} \Sigma & \Sigma^T A^{-T} \\
A \Sigma & 0
\end{bmatrix}
\begin{bmatrix}
\lambda \\
0
\end{bmatrix}
= \begin{bmatrix}
\Sigma^T A^{-T} QA^{-1} b \\
0
\end{bmatrix},
\]  \quad (11)
where the Lagrangian multipliers are \( \lambda = [\lambda^T_1, \ldots, \lambda^T_{k-1}]^T \).

Choosing the partition of variables as (3), we may permute the variables and columns of (11) to reveal the block-tridiagonal structure
\[
\begin{bmatrix}
R_1 & -\Psi^T & 0 & 0 \\
-\Psi & R_2 & -\Psi^T & 0 \\
0 & -\Psi & R_3 & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & \cdots & \cdots
\end{bmatrix}
\begin{bmatrix}
y_1 \\
y_2 \\
y_3 \\
\vdots \\
y_N
\end{bmatrix}
= \begin{bmatrix}
c_1 \\
c_2 \\
c_3 \\
\vdots \\
c_N
\end{bmatrix},
\]  \quad (12)
where
\[
R_i = \begin{bmatrix}
B_i & C_i & 0 \\
C_i^T & D_i & I \\
0 & I & 0
\end{bmatrix}, \quad \Psi = \begin{bmatrix}
0 & 0 & I \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix},
\]
and
\[
y_i = [f_{i,i-1}^T, f_{i,t_i}, \lambda_i^T]^T \quad \text{and} \quad c_i = [g_{i,i-1}^T, g_{i,t_i}^T, 0]^T.
\]

Block-tridiagonal matrices such as (12) are commonly solved through Block LU factorisation [17]. Derived from this factorisation, the following set of recursions solve (11).

In satisfying the optimality conditions for (1), the following is our efficient method for solving the optimization program in Problem 1.

**Lemma 7:** For \( i = 1, \ldots, k \), let \( f_{i,i-1}, f_{i,t_i} \), and \( \lambda_i \) satisfy
\[
\begin{align*}
&f_{i,i-1} = B_i^{-1} (g_{i,i-1} - C_i f_{i,t_i}), \quad (13a) \\
&f_{i,t_i} = \begin{cases} 
(D_i - C_i^T B_i^{-1} C_i)^{-1} f_{i+1,t_i} & i = k, \\
(D_i - C_i^T B_i^{-1} C_i)^{-1} g_{i,t_i} & i = k-1, \ldots, 1, 
\end{cases} \quad (13b) \\
g_{i,t_i} = C_i^T f_{i,i-1} + D_i f_{i,t_i} + \lambda_i, \quad (13c)
\end{align*}
\]
where
\[
\begin{align*}
B_i &= \begin{cases} 
B_1 & i = 1, \\
B_i + D_{i-1} - C_i^T B_{i-1}^{-1} C_{i-1} & i = 2, \ldots, k,
\end{cases} \\
\bar{g}_{i,t_{i-1}} &= \begin{cases} 
g_{i,t_{i-1}} & i = 1, \\
g_{i,t_{i-1}} + g_{i-1,t_{i-1}} - C_i^T B_{i-1}^{-1} \bar{g}_{i-1,t_{i-1}} & i = 2, \ldots, k.
\end{cases}
\end{align*}
\]  \quad (14a)
\[
\begin{align*}
\bar{g}_{i,t_{i-1}} &= \begin{cases} 
g_{i,t_{i-1}} & i = 1, \\
g_{i,t_{i-1}} + g_{i-1,t_{i-1}} - C_i^T B_{i-1}^{-1} \bar{g}_{i-1,t_{i-1}} & i = 2, \ldots, k.
\end{cases}
\end{align*}
\]  \quad (14b)

Then (13) solves (11).

**Proof:** Let the LU factorisation of the block tridiagonal matrix \( R \) in (12) be \( R = LU \) as
\[
\begin{bmatrix}
\Phi_1 & I \\
\Phi_2 & \Phi_3 \\
\vdots & \vdots \\
\Phi_{k-1} & I
\end{bmatrix}
\begin{bmatrix}
R_1 & -\Psi^T & 0 & 0 \\
-\Psi & R_2 & -\Psi^T & 0 \\
0 & -\Psi & R_3 & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & \cdots & \cdots
\end{bmatrix}
\begin{bmatrix}
y_1 \\
y_2 \\
y_3 \\
\vdots \\
y_N
\end{bmatrix}
= \begin{bmatrix}
c_1 \\
c_2 \\
c_3 \\
\vdots \\
c_N
\end{bmatrix},
\]
with the block elements for \( i = 1, \ldots, k - 1 \)
\[
\Phi_i = \begin{bmatrix}
C_i^T B_i^{-1} & -I & D_i - C_i^T B_i^{-1} C_i \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix},
\]
and for \( i = 2, \ldots, k \)
\[
\Phi_i = \begin{bmatrix}
B_i + D_{i-1} - C_i^T B_{i-1}^{-1} C_{i-1} & C_i & 0 \\
C_i^T & D_i & I \\
0 & I & 0
\end{bmatrix}.
\]

Solving \( Lx = c \) yields the iteration (14b). Calculating each \( B_i + D_{i-1} - C_i^T B_{i-1}^{-1} C_{i-1} \) gives rise to (14a). These matrices are used to solve \( Uy = x \), providing expressions for \( f_{i,i-1} \) and \( f_{i,t_i} \) as (13a) and (13b). The solution to \( Uy = x \) also governs the values of \( \lambda_i \) with (13c).

Lemma 7 can be used to solve Problem 1 with linear computational complexity in \( k \). There are \( 2k \) matrix calculations required in computing (14a) and (14b), while \( 2k \) systems of equations need to be solved in (13a) and (13b). All the matrices involved are square and at largest \( s \), and the computational complexity is then \( O(\frac{3}{2}ks^3) \).

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