GENERAL HYPERPLANE SECTIONS OF NONSINGULAR FLOPS IN DIMENSION 3

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Let \( X \) be a 3-dimensional complex manifold, and \( f : X \to Y \) a proper bimeromorphic morphism to a normal complex space which contracts an irreducible curve \( C \subset X \) to a singular point \( Q \in Y \) while inducing an isomorphism \( X \setminus C \cong Y \setminus \{Q\} \). We assume that the intersection number with the canonical divisor \( (K_X \cdot C) \) is zero. In this case, it is known that the singularity of \( Y \) is Gorenstein terminal, and there exists a flop \( f^\# : X^\# \to Y([R]) \), which we call a nonsingular flop because \( X \) is nonsingular.

In order to investigate \( f \) analytically, we replace \( Y \) by its germ at \( Q \), and consider a general hyperplane section \( H \) of \( Y \) through \( Q \). Then \( H \) has only a rational double point, its pull-back \( L \subset X \) by \( f \) is normal, and the induced morphism \( f_H : L \to H \) factors the minimal resolution \( g : M \to H ([R]) \). The dual graph \( \Gamma \) of the exceptional curves of \( g \) is a Dynkin diagram of type \( A_n, D_n \), or \( E_n \). Let \( F = \sum_{k=1}^n m_k C_k \) be the fundamental cycle for \( g \) on \( M \). The natural morphism \( h : M \to L \) is obtained by contracting all the exceptional curves of \( g \) except the strict transform \( C_k^0 \) of \( C_k \).

Kollár defined an invariant of \( f \) called the length as the length of the scheme theoretic fiber \( f^{-1}(Q) \) at the generic point of \( C \). It coincides with the multiplicity \( m_{k_0} \) of the fundamental cycle at \( C_{k_0} \).

Katz and Morrison proved the following theorem ([KM, Main Theorem]). The purpose of this paper is to give its simple geometric proof.

**Theorem.** Let \( f : X \to Y \) be as above. Then the singularity of the general hyperplane section \( H \) and the partial resolution \( f_H : L \to H \) are determined by the length \( \ell \) of \( f \). More precisely, \( H \) has a rational double point of type \( A_1, D_4, E_6, E_7, E_8 \), or \( E_8 \), if \( \ell = 1, 2, 3, 4, 5 \) or \( 6 \), respectively.

We note that there is only one irreducible component of \( g^{-1}(Q) \) whose multiplicity in \( F \) coincides with \( \ell \) in the above cases.

**Proof.** Let \( H' \) be another general hyperplane section of \( Y \) through \( Q \), and \( f_{H'} : L' \to H' \) the induced morphism. \( H \) and \( H' \) have the same type of singularities, and so do \( L \) and \( L' \). Let \( P \) and \( P' \) be the singular points of \( L \) of \( L' \), respectively.

Let \( D \) be the effective Cartier divisor on \( L \) given by \( L' \cap L \). Then \( D \) is a general member of the linear system of effective Cartier divisors on \( L \) which contain \( C \) and such that \( (D \cdot C) = 0 \). In fact, if \( s_0 \) is the global section of \( \mathcal{O}_L(-C) \subset \mathcal{O}_L \) corresponding to \( D \), then from an exact sequence

\[
0 \to \mathcal{O}_X(-L) \to \mathcal{O}_X \to \mathcal{O}_L \to 0
\]
there exists a lifting $s \in H^0(X, \mathcal{O}_X)$ of $s_0$ which defines $L'$, because $H^1(X, \mathcal{O}_X(-L)) = 0$.

Let $\tilde{D}$ be the total transform of $D$ on $M$. Then we can write $\tilde{D} = F + D'$ for some $D'$ which is reduced, nonsingular and has no common irreducible components with $F$. If $\Gamma$ is of type $A_n$, then $D'$ has 2 irreducible components each of which intersects transversally one of the end components of $F$. Otherwise, $D'$ is irreducible, and intersects transversally a component $C_{k_1}$ of $F$ such that $m_{k_1} = 2$ ($k_1$ may be equal to $k_0$).

Let $t$ be the global section of $\mathcal{O}_X$ corresponding to $L$. Then $s + ct$ is also a lifting of $s_0$ for any $c \in \mathbb{C}$. Let $L'(c)$ be the corresponding divisor on $X$.

Let $P$ be a point on $C$ which is different from the $P_i$. For local analytic coordinates $\{x, y, t\}$, we can write $s + ct = F(x, y) + t(G(x, y, t) + c)$. For a general choice of $c$, $(G(x, y, t) + c)|_C$ does not vanish at the singular points $P'_i$ of $L' = L'(0)$ other than the $P_i$, and has only simple zeroes at some points $P''_j$. Then $L'' = L'(c)$ has singularities only at the $P''_j$, besides possibly at the $P_i$, with equations of the type $x^\ell + ty = 0$.

If we replace $L'$ by $L''$, we conclude that $L'$ has only singularities of type $A_{\ell-1}$ outside the $P_i$. We shall investigate the singularities of $L'$ at the $P_i$ case by case.

Let $\Gamma_i$ be the dual graph of the exceptional curves of $h$ over $P_i$, and $F_i$ the corresponding fundamental cycle. From the description of $\tilde{D}$ above, we can calculate the multiplicity $d_i$ of $D$ at the point $P_i$ by $d_i = ((m_{k_0}C_{k_0} + D') \cdot F_i)$.

If $\ell = 1$ or 2, then we can check that $d_i \leq 3$, hence $L'$ is nonsingular at the $P_i$. Then it follows that $\Gamma = A_1$ or $D_4$, respectively.

But if $\ell \geq 3$, then $d_i$ can be bigger, and we should look at the singularity of $L'$ more closely.

We assume first that $\ell = 3$. If $\Gamma = E_6$, then there is nothing to prove. We have to prove that $\Gamma \neq E_7, E_8$. If $\Gamma = E_7$, then $L$ has two singular points $P_1$ and $P_2$, where $F_1$ meets $D'$. We have 2 cases; $\Gamma_1 = A_1$ and $\Gamma_2 = A_5$, or $\Gamma_1 = A_4$ and $\Gamma_2 = A_2$. In the former case, $L'$ has at most $A_1$ singularity at $P_1$ because of the symmetry of $L$ and $L'$, while being nonsingular at $P_2$, since $d_2 = 3$. Therefore, $L'$ has simpler singularities than $L$, a contradiction. In the latter case, it has at most $A_2$ at $P_2$. We shall prove that $L'$ has $A_1$ at $P_1$.

Let $\mu : X^{(1)} \to X$ be the blowing-up at $P_1$, $E \simeq \mathbb{P}^2$ the exceptional divisor, and $L^{(1)}$ (resp. $L^{(1)'}$) the strict transform of $L$ (resp. $L'$). $B = L^{(1)} \cap E$ consists of 2 lines $B_1$ and $B_2$, which correspond to the 2 end components of $F_1$. Since their multiplicities in $F$ are 2 and $L^{(1)} \cdot E = B$, we deduce that $\mu^* L' = L^{(1)'} + 2E$, and neither of the $B_1$ are contained in $B' = L^{(1)'} \cap E$. Thus the intersection of 2 conics $B$ and $B'$ is equal to $(L^{(1)} \cap L^{(1)'}) \cap E$. We see from the description of $\tilde{D}$ that it consists of 2 points, one at $B_1 \cap B_2$ and the other on one component $B_1$. Then $B'$ must be a nonsingular conic, and $L'$ has $A_1$ singularity at $P_1$.

If $\Gamma = E_8$, then we have again 2 cases; $\Gamma_1 = A_1$ and $\Gamma_2 = E_6$, or $\Gamma_1 = A_7$. In the former case, $L'$ has at most $A_1$ singularity at $P_1$, while being nonsingular at $P_2$, since $d_2 = 3$. In the latter case, it has $A_1$ at $P_1$ as in the case of $E_7$.

Next we assume that $\ell = 4$. If $\Gamma = E_7$, then there is nothing to prove. If $\Gamma = E_8$, then we have 2 cases; $\Gamma_1 = D_5$ and $\Gamma_2 = A_2$, or $\Gamma_1 = A_6$ and $\Gamma_2 = A_1$.

In the former case, $L'$ has at most $A_1$ singularity at $P_1$. By the symmetry of $L$.
and $L', L'$ has $D_5$ at $P_1$. Let $\mu : X^{(1)} \to X, E, L^{(1)}$ and $L^{(1)'}$ as before. $B = L^{(1)} \cap E$ is a line, and $L^{(1)'} \cdot E = 2B$. Since the corresponding curve has multiplicity 4 in $F$, we have $\mu^* L' = L^{(1)'} + 2E$, and $B$ is not contained in $B' = L^{(1)'} \cap E$. $L^{(1)}$ has 2 singular points $P_1^{(1)}$ and $P_2^{(1)}$ on $B$ which are of types $A_3$ and $A_1$, respectively. We have $B \cap B' = P_1^{(1)}$ by the description of $\tilde{D}$.

Let $\nu : X^{(2)} \to X^{(1)}$ be the blowing-up at $P_1^{(1)}$, $E^{(1)} \simeq \mathbb{P}^2$ the exceptional divisor, and $L^{(2)}$ (resp. $L^{(2)'}$) the strict transform of $L^{(1)}$ (resp. $L^{(1)'}$). $B^{(1)} = L^{(2)} \cap E^{(1)}$ consists of 2 lines, and one of the corresponding curves on $M$ has multiplicity 3 in $F$, hence $\nu^* L^{(1)'} = L^{(2)'} + E^{(1)}$, and $L^{(1)'}$ is nonsingular at $P_1^{(1)}$. But this contradicts the symmetry of $L$ and $L'$.

In the latter case, $L'$ has at most $A_1$ singularity at $P_2$. Let $\mu : X^{(1)} \to X$, etc., as before. $B = L^{(1)} \cap E$ consists of 2 lines $B_1$ and $B_2$, which correspond to the 2 end components of $F_1$. Since their multiplicities in $F$ are 3 and 2, $B_1$ is contained in $B' = L^{(1)'} \cap E$, while $B_2$ is not. Thus we have $B' = B_1 + B_2'$ with $B_2 \neq B_2'$. The strict transform of $C_{k_0}$ passes through the point $B_1 \cap B_2$, a contradiction to the symmetry.

Finally, if $\ell \geq 5$, the assertion of the theorem is clear. Q.E.D.

References

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