Dissipative Structures of the Kuramoto–Sivashinsky Equation

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Received December 20, 2014

Abstract—The features of dissipative structure formation, which is described by the periodic boundary value problem for the Kuramoto–Sivashinsky equation, are investigated. A numerical algorithm based on the pseudospectral method is presented. The efficiency and accuracy of the proposed numerical method are proved using the exact solution of the equation under study. Using the proposed method, the process of dissipative structure formation, which is described by the Kuramoto–Sivashinsky equation, is studied. The quantitative and qualitative characteristics of this process are described. It is shown that there is a value of the control parameter for which the dissipative structure formation processes occur. Via cyclic convolution, the average value of the control parameter is found. In addition, the dependence of the amplitude of the formed structures on the value of the control parameter is analyzed.

Keywords: Kuramoto–Sivashinsky equation, self-organization, patterns, pseudospectral method, numerical simulation

DOI: 10.3103/S0146411615070147

INTRODUCTION

In nonlinear mathematical physics, processes of stable structures formation, which occur due to the evolution of instability in nonequilibrium systems, is of interest for several decades. Currently, these structures are called dissipative.

The processes of dissipative structure formation are known to occur only in systems described by nonlinear mathematical models. As one of the mathematical models describing these processes, we consider a model based on the Kuramoto–Sivashinsky nonlinear equation in the form

$$u_t + uu_x + u_{xx} + \delta u_{xxx} + u_{xxxx} = 0,$$

(1)

which was derived in works [1, 2]. This equation occurs in many applications, in particular, when describing the self-organization of nanostructure clusters on the surface of semiconductor substrates during ion bombardment [3], when a fluid flows on an inclined plane [4], etc. Equation (1) was studied from different point of views. In [5], exact solutions of this equation were found in the form of solitary waves and kinks; in [6], solutions to this equation were expressed via elliptic functions. The wave process described by Eq. (1) and its generalizations were simulated in [7–10]. In these papers, the dispersive term $\delta u_{xxx}$ in the Kuramoto–Sivashinsky equation was shown to result in spatially homogeneous wave structures when $\delta \leq 0.3$. However, to determine the control parameter $\delta^*$, qualitative estimates rather than quantitative criteria were used, which resulted in an inaccurate estimate of $\delta^*$. In this paper, we find a quantitative criterion for refining parameter $\delta^*$ and develop an effective numerical algorithm that allows one to solve the periodic boundary value problem for Eq. (1).

1. NUMERICAL ALGORITHM

For Eq. (1), let us consider the problem

$$u_t + uu_x + u_{xx} + \delta u_{xxx} + u_{xxxx} = 0, \quad u(x, 0) = u_0(x), \quad u_{ix}(0, t) = u_{ix}(H, t), \quad i = 0...3,$$

(2)

where $u_{ix} = \frac{\partial u}{\partial x^i}$ and $H$ is the period. To solve this problem, the pseudospectral method is used [11]. This method consists in reducing the problem under study, which is formulated for the partial differential equation, to the Cauchy problem for the ordinary differential equation by means of Fourier transform with respect to the spatial variable.

Thus, problem (2) is written as
where $\mathbf{L}$ and $\mathbf{N}$ are the linear and nonlinear operators in Eq. (1), respectively. By applying Fourier transform to (3), we obtain

$$v_t = \tilde{\mathbf{L}}v + \tilde{\mathbf{N}}[\tilde{v}], \quad v(x, 0) = v_0,$$

(4)

where $v$, $v_0$, $\tilde{\mathbf{L}}$, and $\tilde{\mathbf{N}}$ are Fourier transforms of $u$, $u_0$, $\mathbf{L}$, and $\mathbf{N}$, respectively. The Fourier transform of the linear operator is found by the formula

$$\tilde{\mathbf{L}} = k^2 - k^4 + i\delta k^3,$$

(5)

where $k$ is the wave vector. To construct the Fourier transform of the nonlinear term, the following sequence of equalities is used:

$$\tilde{\mathbf{N}} = \text{FT}[u u_x] = \text{FT}\left[\frac{u^2}{2}\right] = \frac{ik}{2} \text{FT}[u^2],$$

(6)

where FT is Fourier transform. To avoid computational errors and other negative effects associated with applying Fourier transform to nonlinear terms, we use exponential smoothing of wavenumbers [12].

There are a large number of methods for solving problem (4), which differ in approximation order, stability, and efficiency. In this paper, we consider three widely known methods: the integrating factor method (IFRK4) [13], the exponential time differencing method (ETDRK4) [14], and the Adams–Bashforth method (AB4BD4) [15]. All of them are fourth-order approximation methods. These methods are based on different approaches to numerically solve the Cauchy problem, so, upon testing the methods on the exact solution of problem (2), we select the most effective. As the exact solution, we use the solution of Eq. (1) in the form of a solitary wave, which is obtained for $\delta = 4$ and is written as

$$u_0(x) = C_0 + 9 - 15\tanh\left(\frac{x-x_0}{2}\right) - 15\tanh^2\left(\frac{x-x_0}{2}\right) + 15\tanh^3\left(\frac{x-x_0}{2}\right),$$

(7)

where $C_0$ and $x_0$ are arbitrary constants.

The result of comparison of the numerical and exact solution is shown in Fig. 1. The accuracy of the proposed algorithm was measured using the relative integrated error [13], which is defined as
The ETDRK4 has the smallest error for short and long computation times (see Fig. 1), thus making it the most accurate method. Moreover, it enables the most efficient computation, which is due to lower requirements for the mesh parameters to achieve the required accuracy (see Fig. 2).

2. NUMERICAL SIMULATION RESULTS

In [7–9], it is shown that, for certain values of control parameter $\delta$, the processes of dissipative structure formation occur in the systems described by Eq. (2) with the initial condition in the form of uniform distribution (see Fig. 3b). If the value $\delta$ is below a certain threshold, stable spatial structures are not formed either for short nor or long times (see Fig. 3a).

However, as mentioned earlier, to estimate $\delta$, qualitative criteria were used. Thus, the problem of refining the value of this parameter on the basis of quantitative criteria arises. To solve this problem, we define a structure as a formation that does not change its profile accurate to the shift along the spatial axis taking into account its periodicity.

Let $u(x, t^n) = u^n$ be the numerical solution to the problem at instant $t^n$. Below, we formulate the algorithm to solve the problem on the formation of dissipative structures.

1. At the selected time instants $t_k (k = 1, \ldots, M)$ such that $t_k - t_{k-1} = \Delta = \text{const}$, find the discrete cyclic convolution

   \[ C(u^k, u^{k-1}) = \text{FT}^{-1}(\text{FT}(u^k) \cdot \text{FT}(u^{k-1})) , \]

   where FT is fast Fourier transform and FT$^{-1}$ is inverse fast Fourier transform.

2. Find the coordinate of the maximum of the convolution:

   \[ i_{\text{max}}: C_{\text{max}} = \max_{i = 1 \ldots N_x} C_i . \]

Fig. 2. Dependence of $\varepsilon$ on time step $dt$ at instants (a) $t = 10$ and (b) $t = 100$ for different numbers of spatial mesh points $N$: (1$^{1,2}$) IFRK4, (2$^{1,2}$) AB4BD4, and (3$^{1,2}$) ETDRK4 for $N = 256$ and 1024, respectively.
3. Find $u^{k \rightarrow k-1}$, which is the cyclic shift of $u^k$ by $i_{\text{max}}$.

4. Find the value of the relative integral error $\varepsilon(u^{k \rightarrow k-1}, u^{k-1}) = \varepsilon_{k, k-1}$, which quantitatively characterizes the minimum difference between $u^{k \rightarrow k-1}$ and $u^{k-1}$.

5. Based on the comparison between $\varepsilon_{k, k-1}$ and a certain threshold $\varepsilon_0$, make sure that the dissipative structure is formed by the time $t_k$. Note that the value of threshold $\varepsilon_0$ is arbitrary.

Using the proposed algorithm, we find the value of the control parameter $\delta^*$ for which the structure formation processes occur. For each fixed initial condition given in the form of a uniformly distributed random quantity, series of numerical experiments are carried out for various values of $\delta$, from which the value of $\delta^*$ is determined. Then, the initial condition is changed and the procedure is repeated. As a result, we find the estimate of the average value $\delta_{\text{avg}}^*$ and its variance:

$$
\delta_{\text{avg}}^* = \frac{1}{M} \sum_{i=1}^{M} \delta_i^* = 0.43, \quad \sqrt{\sigma} = \sqrt{\frac{1}{M} \sum_{i=1}^{M} (\delta_i^* - \delta_{\text{avg}}^*)^2} = 0.03, \quad (11)
$$

where $i$ and $M$ are the index and the number of numerical experiments. In this work, we set $M = 10$ and $\varepsilon_0 = 0.1$.

Figure 4 shows the result of estimating the quantitative characteristics of the process of dissipative structure formation. It can be seen (Figs. 4a–4c) that, for $\delta = 4 > \delta^*$, the number of peaks, amplitude, and relative metric cease to change with time, which is not the case for $\delta = 0.15 < \delta^*$. In the latter case, the number of peaks $N$ changes constantly (becoming alternately larger and smaller), and the amplitude, having reached a certain average value, oscillates about it. Note also that the number of peaks of the formed structure depends on the length of the computational interval: the number of peaks increases with increasing length $H$.
Numerical experiments show that the amplitude of the dissipative structure increases linearly with a change in the value of parameter $\delta$; the amplitude is a well approximated by the linear function $A \approx 4.15\delta$.

This dependence is shown in Fig. 5.
Remark: due to the fact that the peaks are uniformly distributed over the computational domain, a two-fold increase in $H$ leads to a twofold increase in the number of peaks.

**CONCLUSIONS**

In this paper, we investigated the formation of dissipative structures described by the Kuramoto–Sivashinsky equation. Using the pseudospectral method, we constructed the numerical algorithm for simulating the processes of dissipative structure formation. A quantitative criterion was proposed for estimating the average value of the control parameter $\delta^*$ for which these processes occur: $\delta^* = 0.43$. The quantitative characteristics of the dissipative structure formation processes were analyzed. The dependence of the amplitude on parameter $\delta$ ($A = 4.15\delta$) was obtained.

**ACKNOWLEDGMENTS**

This work was supported by the Russian Foundation for Basic Research (project nos. 14-01-00493-a and 14-01-31078), a Grant for Support of Leading Scientific Schools of the Russian Federation (no. 2296.2014.1), and a Grant for Support of Young Scientists (no. 3694.2014.1).

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*Translated by Yu. Kornienko*