ON THE ANALYTICITY OF PERIODIC GRAVITY WATER WAVES WITH INTEGRABLE VORTICITY FUNCTION

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Abstract. We prove that the streamlines and the profile of periodic gravity water waves traveling over a flat bed with wavespeed which exceeds the horizontal velocity of all fluid particles are real-analytic graphs if the vorticity function is merely integrable.

1. Introduction and the main result

This paper is dedicated to the study of the regularity properties of two-dimensional periodic gravity water waves traveling over a flat bed when gravity is the sole driving mechanism. The waves we consider are rotational and the vorticity function has a general form. While there are no known non-trivial solutions that describe periodic gravity water waves traveling over a flat bed, in the case when the fluid has infinite depth there is an explicit solution which is due to Gerstner [2, 14, 16, 28]. This solution has a non-trivial vorticity function, the streamlines being real-analytic trochoids. It is interesting that Gerstner’s example is the only possible non-trivial solution for gravity water waves without stagnation points and with the pressure constant along all streamlines, cf. [31]. Small amplitude gravity water waves possessing a general vorticity were constructed first in [11] by using power series expansions, the existence of waves of large amplitude and without stagnation points being established more recently [7] by using local and global bifurcation techniques. While in [7] the vorticity function is assumed to have a Hölder continuous derivative, the same authors construct in [9] gravity water waves with a vorticity function which is merely bounded. This new result is obtained by taking advantage of the weak formulation for the water wave problem. In the irrotational setting, when the vorticity is zero and the waves travel over still water or uniformly sheared currents, the existence theory is related to Nekrasov’s equation. Based on this formulation of the problem, irrotational water waves with small and large amplitude were constructed in [1, 22, 37] by using global bifurcation theory, some of the waves possessing sharp crests with a stagnation point at the top.

When proving the real-analyticity of gravity water waves, the assumption that the flow does not possess stagnation points, that is fluid particles that travel horizontally with the same speed as the wave, is crucial. Indeed, as a consequence of elliptic maximum principles, irrotational waves cannot possess stagnation points beneath the wave profile (see e.g. [8]), but the Stokes wave of extreme form has a stagnation point at each crest and the wave profile is only Lipschitz continuous in a neighbourhood of that point as it forms an angle of $2\pi/3$. On the other hand, it is known that waves with vorticity may possess stagnation points inside the fluid layer, see [12, 38] for the case when the vorticity function is constant or linear, and that the streamlines containing the stagnation points are not real-analytic. For irrotational

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waves, it is shown in [24] by using the Schwartz reflection principle that in the absence of stagnation points the profile of the waves is real-analytic. In the context of rotational waves without stagnation points, it was proved first for waves with Hölder continuous vorticity [6] and later for waves with merely bounded vorticity [33] that all the streamlines beneath the wave surface are real-analytic. The authors of [6] establish the real-analyticity of the wave profile only under the additional assumption that the vorticity function is real-analytic. The methods rely strongly on invariance of the problem with respect to horizontal translations and they have been generalized to prove similar results for capillary and capillary-gravity waves [17, 18, 19], deep-water waves [34], solitary waves [21, 32], and stratified waves [20] (see also the survey [13]). While in these references the real-analyticity of the wave profile is established only for real-analytic vorticity functions, it is shown [25, 35, 39] that any of the waves mentioned before has a real-analytic profile if the vorticity function is merely Hölder continuous. In the later papers the authors introduce additionally an iteration procedure and estimate all partial derivatives of an associated height function with respect to the horizontal variable in order to obtain the desired regularity result. We enhance that capillary and capillary-gravity waves with a real-analytic and decreasing vorticity function have a real-analytic profile, cf. [29], the result being true for waves with or without stagnation points. The smoothness of the free surface for gravity waves without stagnation points on the surface and for waves with capillary effects has been established in [10, 26, 27] in the regime when the vorticity is constant but the profile is not necessarily a graph (waves with overhanging profiles).

We establish herein the real-analyticity of the free surface and of the streamlines for solutions of the weak formulation of the water wave problem derived in [9], when assuming only integrability of the vorticity function. Such solutions describe water waves traveling over currents which present sudden changes with respect to the depth in form of a discontinuous vorticity or waves generated by wind that possess a thin layer of high vorticity at the surface (see [23, 36]). We enhance that the regularity results obtained in [6, 25, 33] appear as a particular case of our analysis. The relevance of our result can be viewed in the light of [30, Theorem 3.1 and Remark 3.2] as we can now state: within the set of all periodic gravity waves without stagnation points the symmetric waves with one crest and trough per period are characterized by the property that all the streamlines have a global minimum on the same vertical line. Another characterization for symmetric gravity water waves with one crest and trough per period was obtained in [4, 5] where it is shown that merely the fact that the wave profile has a unique crest per period ensures the symmetry of the wave. The proof of our main result Theorem 1.1 combines the invariance of the problem with respect to horizontal translations with Schauder estimates for weak solutions of elliptic boundary value problems. Particularly, we show that even under this weak integrability condition on the vorticity function, all derivatives of the height function with respect to the horizontal variable have Hölder continuous derivatives and are weak solutions of certain elliptic problems. Estimating their Hölder norm, we obtain the desired regularity result.

To complete this section we present the governing equations and our main result Theorem 1.1. We consider herein the water wave problem in the formulation for the height function $h$

\[
\begin{align*}
(1 + h_q^2)h_{pp} - 2h_p h_q h_{pq} + h_p^2 h_{qq} - \gamma(p)h_p^3 &= 0 \quad \text{in} \quad \Omega, \\
1 + h_q^2 + (2gh - Q)h_p^2 &= 0 \quad \text{on} \quad p = 0, \\
h &= 0 \quad \text{on} \quad p = p_0,
\end{align*}
\]

(1.1)
where $\Omega := \mathbb{S} \times (p_0, 0)$ and the function $h$ is assumed to satisfy additionally
\begin{equation}
\inf_{\Omega} h_p > 0. \tag{1.2}
\end{equation}
We denoted by $\gamma$ the vorticity function, $g$ is the gravity constant, $p_0 < 0$ is the relative mass flux, and $Q$ is the total head. We set $\mathbb{S} := \mathbb{R}/(2\pi \mathbb{Z})$ to denote the unit circle. The equivalence of the height function formulation (1.1)-(1.2) to the Euler equations is discussed in detail in [3, 7] in the context of smooth solutions and in [9, 13] for $L_r$-solutions\footnote{Given $1 \leq r \leq \infty$, we denote the usual Lebesgue spaces by $L_r$.}, see also the remarks subsequent to Theorem 1.1 below.

Solutions of (1.1)-(1.2) describe two-dimensional $2\pi$-periodic gravity water waves traveling over the flat bed $y = -d$, the wave profile being the graph $y = h(q, 0) - d$. Hereby, $d$ is an arbitrary real constant. The value $h(q, p)$ represents the exact height of the water particle determined by the coordinate $(q, p)$ above the horizontal bed. Because of (1.2), these solutions correspond to waves without stagnation points and critical layers, each streamline being a graph $y = h(q, p) - d$, for some unique $p \in [p_0, 0]$. Indeed, following the wave from a frame moving with the wavespeed $c$, which does not appear in (1.1) as a solution of (1.1) solves the water wave problem for any value of $c$, cf. [7], the velocity field $(u, v)$ of the fluid is given by
\begin{equation}
(u - c, v) = \left( -\frac{1}{h_p}, -\frac{h_q}{h_p} \right),
\end{equation}
and it follows readily from (1.2) that the horizontal speed of each individual particle is less than the wave speed $c$. Moreover, since the streamlines are curves in the fluid which are tangent to the velocity field, it is easy to see that these are the graphs $y = h(q, p) - d, p \in [p_0, 0]$, as the tangent to each graph is always parallel to the velocity field and, by letting $p$ vary between $p_0$ and 0, the graphs foliate the fluid domain.

Assuming only boundedness and a sign condition on the vorticity function $\gamma$, the authors establish in [9] the existence of weak solutions of (1.1)-(1.2) which form a $C^1$-bifurcation curve. The local branch can be continued if one assumes that the vorticity is Hölder continuous close to the free surface and the bed, a characterization of the global branch being also included. These weak solutions belong to the space $C^{1+\alpha}(\Omega)$ for some $\alpha \in (0, 1)$ and are solutions of (1.1)-(1.2) in the sense that they satisfy the last two equations of (1.1) and the condition (1.2) pointwise, while the first equation is satisfied in a weak sense. More precisely, introducing the anti-derivative
\begin{equation}
\Gamma(p) := \int_0^p \gamma(s) \, ds \quad \text{for } p \in (p_0, 0),
\end{equation}
the first equation of (1.1) may be written in the weak form
\begin{equation}
\left( \frac{h_q}{h_p} \right)_q - \left( \Gamma(p) + \frac{1 + h_q^2}{2h_p^2} \right)_p = 0 \quad \text{in } \Omega, \tag{1.3}
\end{equation}
and the weak solutions found in [9] satisfy (1.3) when testing with functions from $C^1(\Omega)$ that have compact support. In fact, having a bounded vorticity function, the authors of [9] prove that their weak solutions belong to $W^{2,r}_r(\Omega)$, with $r := 2/(1 - \alpha)$. We come now the the main result of the paper.
Theorem 1.1. Assume that $\gamma \in L_1((p_0,0))$. Given a weak solution $h \in C^{1+\alpha}(\Omega)$ of (1.1), we have that $\partial_q^m h \in C^{1+\alpha}(\Omega)$ for all $m \in \mathbb{N}$. Moreover, there exists a constant $L > 1$ with the property that
\[
\|\partial_q^m h\|_{1+\alpha} \leq L^{m-2}(m-3)!
\]
for all integers $m \geq 3$.

We enhance that if $\gamma \in L_r((p_0,0))$ for some $r > 2$ then any solution $h \in W^2_r(\Omega)$ of (1.1)-(1.2) is a weak solution\(^2\) of the same problem in the sense of (1.3). Particularly, our result applies for all $L_r$-solutions of (1.1)-(1.2) found in [9] and studied also in [13], improving the previous regularity results [6, 25, 33]. Furthermore, consider the classical water wave problem for a $2\pi$-periodic traveling gravity wave with profile $\eta$, i.e.
\[
\begin{aligned}
(u-c)u_x + vu_y &= -P_x \quad \text{in } \Omega_\eta, \\
(u-c)v_x + vv_y &= -P_y - g \quad \text{in } \Omega_\eta, \\
ux + vy &= 0 \quad \text{in } \Omega_\eta, \\
P &= 0 \quad \text{on } y = \eta, \\
v &= (u-c)\eta_x \quad \text{on } y = \eta, \\
v &= 0 \quad \text{on } y = -d,
\end{aligned}
\]  
\tag{1.5}

where we denote by $\Omega_\eta := \{(x,y); x \in \mathbb{S} \text{ and } -d < y < \eta(x)\}$ the $(2\pi$-periodic) fluid body beneath the wave $\eta$, by $P$ the pressure in $\Omega_\eta$, and by $g$ the gravitational constant. Assume that
\[
(u,v,P,\eta) \in W^1_r(\Omega_\eta) \times W^1_r(\Omega_\eta) \times W^1_r(\Omega_\eta) \times W^{2-1/r}_r(\mathbb{S})
\]
satisfies (1.5) in $L_r(\Omega_\eta)$. If we assume additionally that $u < c \in \overline{\Omega_\eta}$, then it is shown in [13] that the height function $h$ is well-defined, belongs to $W^2_r(\Omega)$, satisfies (1.2), and is a $L_r$-solution of (1.1). Besides, the vorticity function $\gamma$ is also well-defined and belongs to $L_r((p_0,0))$. Thus, Theorem 1.1 is applicable for such solutions of (1.5).

As a direct consequence of Theorem 1.1 we obtain that the streamlines of the wave corresponding to a weak solution $h$ of (1.1)-(1.2) are graphs of real-analytic functions. Indeed, for any fixed $p \in [p_0,0]$, the function $h(\cdot,p)$ is a smooth function on $\mathbb{S}$. Moreover, in virtue of (1.4), we can use the Lagrange formula for the remainder to obtain that
\[
|h(q,p) - \sum_{m=0}^{N} \frac{\partial_q^m h(q_0,p)}{m!}(q - q_0)^m| \leq \frac{|\partial_q^{N+1} h|_0}{(N+1)!}|q - q_0|^{N+1} \leq L^{-2}(L|q - q_0|)^{N+1} \rightarrow_{N \rightarrow \infty} 0
\]
if $L|q - q_0| < 1$. Thus, the Taylor series of $h(\cdot,p)$ at $q_0$ converges on a small interval containing $q_0$, the length of the interval being independent of $q_0$ and $p$. Summarizing, we have:

Corollary 1.2. Under the assumptions of Theorem 1.1, given $p \in [p_0,0]$, each streamline $\Psi_p := \{(q,h(q,p)) - d) \in \mathbb{R}^2; q \in \mathbb{S}\}$ is a real-analytic curve. Particularly, the water wave’s free surface $\Psi_0$ is a real-analytic periodic curve.

Remark 1.3. In the proof of Theorem 1.1 we did not make use of the periodicity of $h$ in the variable $q$. Therefore, the argument preceding Corollary 1.2 yields also the real-analyticity of the streamlines and of the wave profile of gravity solitary waves of finite depth.

\(^2\)Recall that Sobolev’s embedding theorem ensures that $W^2_r(\Omega) \subset C^{1+\beta}(\Omega)$ for any $\beta \in (0, (r - 2)/r)$. 
Proof. For every $\varepsilon \in (0,1)$, let $\tau_\varepsilon f$ denote the translation in the $q$-direction of a given function $f : \overline{\Omega} \to \mathbb{R}$, i.e. $\tau_\varepsilon f(q,p) := f(q+\varepsilon, p)$ for $(q,p) \in \overline{\Omega}$. Further we define the difference quotient $u^{\varepsilon} := (\tau_\varepsilon h - h)/\varepsilon \in C^{1+\alpha}(\overline{\Omega})$ and, because $\tau_\varepsilon h$ is also a weak solution of (1.1), we obtain that $u^{\varepsilon}$ solves the following generalized elliptic problem

\begin{align}
\begin{cases}
(a_{11}^\varepsilon \partial_q u^{\varepsilon})_q + (a_{12}^\varepsilon \partial_p u^{\varepsilon})_q + (a_{21}^\varepsilon \partial_q u^{\varepsilon})_p + (a_{22}^\varepsilon \partial_p u^{\varepsilon})_p = 0 & \text{in } \Omega, \\
(h_q + (\tau_\varepsilon h)_q) u^{\varepsilon}_q + (2g(\tau_\varepsilon h)_p - Q)(h_p + (\tau_\varepsilon h)_p) u^{\varepsilon}_p = -2gh^2 \varepsilon & \text{on } p = 0, \\
u^{\varepsilon} = 0 & \text{on } p = p_0,
\end{cases}
\end{align}

whereby $f_m, g_m, \varphi_m \in C^\alpha(\overline{\Omega})$ are given by

\begin{align}
f_m := \sum_{n=1}^{m-1} \binom{m-1}{n} \left[ -\partial_q \left( \frac{1}{h_p} \right) \partial_q (\partial_q^{m-n} h) + \partial_q \left( \frac{h_q}{h_p^2} \right) \partial_p (\partial_q^{m-n} h) \right],
\end{align}

\begin{align}
g_m := \sum_{n=1}^{m-1} \binom{m-1}{n} \left[ \partial_q \left( \frac{h_q}{h_p^2} \right) \partial_q (\partial_q^{m-n} h) - \partial_q \left( \frac{1 + h_q^2}{h_p^3} \right) \partial_p (\partial_q^{m-n} h) \right],
\end{align}

and

\begin{align}
\varphi_m := -2^{-1}(2g - Q) \sum_{n=1}^{m-1} \binom{m}{n} (\partial_q^n h_p)(\partial_q^{m-n} h_p) - g \sum_{n=1}^{m} \binom{m}{n} (\partial_q^n h)(\partial_q^{m-n} h^2).
\end{align}
whereby

$$
(a_{ij}^\varepsilon) = \begin{pmatrix}
\frac{1}{(\tau h_p)} & \frac{h_q}{h_p(\tau h_p)} \\
\frac{h_q + (\tau h_q)}{2(\tau h_p)^2} & \frac{(h_p + (\tau h_p))(1 + h_q^2)}{2h_p^2(\tau h_p)^2}
\end{pmatrix}.
$$

If $\varepsilon \in (0, \varepsilon_0)$ and $\varepsilon_0$ is sufficiently small, then the matrix $(a_{ij}^\varepsilon)$ has positive eigenvalues bounded away from zero uniformly in $\Omega$ and $\varepsilon \in (0, \varepsilon_0)$, while the boundary condition of (2.2) on $p = 0$ is uniformly oblique as

$$
\inf_{\varepsilon \in (0,1)} \inf_{\Omega} |(2g(\tau h_p) - Q)(h_p + (\tau h_p))| > 0.
$$

Moreover, all the coefficients appearing in (2.2), as well the right-hand side of the second equation of (2.2), are uniformly bounded, with respect to $\varepsilon \in (0, \varepsilon_0)$, in $C^{\alpha}(\Omega)$. Schauder estimates for elliptic problems, cf. [9, Theorem 3], guarantee now the existence of a constant $K > 0$ such that

$$
\|u^\varepsilon\|_{1+\alpha} \leq K\|u^\varepsilon\|_\alpha \leq K\|h^\varepsilon\|_{1+\alpha}
$$

for all $\varepsilon \in (0, \varepsilon_0)$. Consequently, $(u^\varepsilon)_\varepsilon$ is bounded in $C^{1+\alpha}(\overline{\Omega})$ and a subsequence of it converges in $C^1(\overline{\Omega})$ towards $h_q$. In view of the estimate (2.4), it follows that $h_q \in C^{1+\alpha}(\overline{\Omega})$. Letting $\varepsilon \to 0$ in (2.2), we find due to the convergence in $C^1(\overline{\Omega})$ that $\partial^m_p h$ solves (2.1) when $m = 1$.

For the general case, let us assume that $\partial^m_p h \in C^{1+\alpha}(\overline{\Omega})$ for all $1 \leq n \leq m - 1$. We are left to prove that $\partial^m_p h \in C^{1+\alpha}(\overline{\Omega})$ is the solution of (2.1). Similarly as before, we define the quotient $w^\varepsilon := [\tau (\partial^m_q - \partial^m_p h)] / \varepsilon \in C^{1+\alpha}(\overline{\Omega})$, and we observe that it solves the following problem

$$
\begin{cases}
\left(\frac{1}{h_p} \partial_q w^\varepsilon\right)_q - \left(\frac{h_q}{h_p^2} \partial_p w^\varepsilon\right)_q + \left(\frac{1 + h_q^2}{h_p^2} \partial_p w^\varepsilon\right)_p = (f^\varepsilon_m)_q + (g^\varepsilon)_p & \text{in } \Omega, \\
(h_q + (\tau h_q)w^\varepsilon + (2g(\tau h_p) - Q)(h_p + (\tau h_p))w^\varepsilon = \varphi^\varepsilon_m & \text{on } p = 0, \\
\varepsilon = 0 & \text{on } p = p_0,
\end{cases}
$$

with

$$
\begin{align*}
f^\varepsilon_m := & \partial^m_q (\tau h)\left(\tau h_p - h_p\right) - \partial_p \partial^m_q (\tau h)\left(\frac{h_q}{h_p} - \frac{(\tau h_q)}{(\tau h_p)^2}\right) + \tau \varepsilon f_{m-1} - f_{m-1}, \\
g^\varepsilon := & \partial_p \partial^m_q (\tau h)\left(\frac{1 + h_q^2}{h_p^2} - \frac{1 + (\tau h_q)^2}{\tau h_p^2}\right) - \partial^m_q (\tau h)\left(\frac{h_q}{h_p} - \frac{(\tau h_q)}{(\tau h_p)^2}\right) + \tau \varepsilon g_{m-1} - g_{m-1}, \\
\varphi^\varepsilon_m := & \partial_p \partial^m_q (\tau h)\left(\tau h - (2gh - Q)h_p - (2g\tau h - Q)(\tau h)_p\right) - \partial^m_q (\tau h)\left(\frac{(\tau h_q)}{(\tau h_p)^2}\right) \\
& + \tau \varepsilon \psi_{m-1} - \varphi_{m-1}.
\end{align*}
$$

We note that the induction assumption ensures that all coefficients and the terms on the right-hand side of the equations of (2.5) are bounded in $C^{\alpha}(\overline{\Omega})$, uniformly in $\varepsilon \in (0,1)$. In virtue of (2.3) and noticing that the matrix $(a_{ij}^\varepsilon)$ has positive eigenvalues bounded away from zero.
uniformly in $\Omega$, we may apply the Schauder estimate [9, Theorem 3] to problem (2.5) to find a positive constant $M$ such that

$$\|w^\varepsilon\|_{C^{1+\alpha}(\bar{\Omega})} \leq M$$

for all $\varepsilon \in (0, 1)$. Consequently, a subsequence of $(w^\varepsilon)_\varepsilon$ converges in $C^1(\bar{\Omega})$ towards $\partial_q^m h$. The same arguments as above yield that $\partial_q^m h \in C^{1+\alpha}(\bar{\Omega})$ and, letting $\varepsilon \to 0$ in the equations of (2.5), we find that $\partial_q^m h$ is the weak solution of (2.1). \hfill \Box

Before proving the main theorem, we need the following auxiliary results.

**Lemma 2.2.** Let $N \geq 3$ and assume that $\partial_q^m u_i \in C^{\alpha}(\bar{\Omega})$ for all $0 \leq n \leq N$ and $1 \leq i \leq 5$.

(i) If $L \geq 1$ and $\|\partial_q^nu_i\|_{\alpha} \leq L^{n-3/2}(n-2)!$ for all $2 \leq n \leq N$, then there exists a constant $C_0 > 1$ with the property that

$$\|\partial_q^n(u_1u_2u_3u_4u_5)\|_{\alpha} \leq C_0 \left( 1 + \sum_{i=1}^{5} \sum_{l=0}^{1} \|\partial_q^lu_i\|_{\alpha} \right)^{16} L^{n-3/2}(n-2)! \quad \text{for all } 2 \leq n \leq N. \quad (2.6)$$

(ii) If $L \geq 1$ and $\|\partial_q^nu_i\|_{\alpha} \leq L^{n-1}(n-2)!$ for all $2 \leq n \leq N$, then there exists a constant $C_1 > 1$ such that

$$\|\partial_q^n(u_1u_2u_3)\|_{\alpha} \leq C_1 \left( 1 + \sum_{i=1}^{3} \sum_{l=0}^{1} \|\partial_q^lu_i\|_{\alpha} \right)^{6} L^{n-1}(n-2)! \quad \text{for all } 2 \leq n \leq N. \quad (2.7)$$

(iii) If $L \geq 1$ and $\|\partial_q^nu_i\|_{\alpha} \leq L^{n-2}(n-3)!$ for all $3 \leq n \leq N$, then there exists a constant $C_2 > 1$ with the property that

$$\|\partial_q^n(u_1u_2)\|_{\alpha} \leq C_2 \left( 1 + \sum_{i=1}^{2} \sum_{l=0}^{2} \|\partial_q^lu_i\|_{\alpha} \right)^{2} L^{n-2}(n-3)! \quad \text{for all } 3 \leq n \leq N. \quad (2.8)$$

The constants $C_0$, $C_1$, and $C_2$ do not depend on $L$.

**Proof.** We prove only (i), the proof of (ii) and (iii) being similar. Given $2 \leq n \leq N$, we have

$$\partial_q^n(u_1u_2) = \left( \sum_{k=0}^{n} \sum_{n-k=1}^{n} \frac{n!}{k!(n-k)!} (\partial_q^ku_1)(\partial_q^{n-k}u_2) \right), \quad (2.9)$$

and, since $\|u_1u_2\|_{\alpha} \leq \|u_1\|_{\alpha} \|u_2\|_{\alpha}$, we find

$$\left\| \left( \sum_{k=0}^{n} \sum_{k=n-1}^{n} \frac{n!}{k!(n-k)!} (\partial_q^ku_1)(\partial_q^{n-k}u_2) \right) \right\|_{\alpha} \leq 2 \left( 1 + \sum_{i=1}^{2} \sum_{l=0}^{1} \|\partial_q^lu_i\|_{\alpha} \right)^{2} L^{n-3/2}(n-2)!.$$

On the other hand, if $n \geq 4$ the middle sum in (2.9) does not vanish, and we use the convergence of the series $\sum_n n^{-2}$ to find that

$$\left\| \sum_{k=2}^{n-2} \frac{n!}{k!(n-k)!} (\partial_q^ku_1)(\partial_q^{n-k}u_2) \right\|_{\alpha} \leq \sum_{k=2}^{n-2} \frac{n!}{k!(n-k)!} L^{k-3/2}(k-2)!L^{n-k-3/2}(n-k-2)! \leq L^{n-3}(n-2)! \sum_{k=2}^{n-2} \frac{n^2}{(k-1)^2(n-k-1)^2} = CL^{n-3}(n-2)!.$$
Summarizing, we have shown that there exists a constant $C > 1$ with

$$
\| \partial_n^q (u_1 u_2) \|_\alpha \leq C \left( 1 + \sum_{i=1}^{2} \sum_{l=0}^{1} \| \partial_l^i u_i \|_\alpha \right)^2 L^{n-3/2} (n - 2)! \quad \text{for all } 2 \leq n \leq N. \quad (2.10)
$$

Applying the estimate (2.10) to the functions

$$
v_1 := \frac{u_1 u_2}{C \left( 1 + \sum_{i=1}^{2} \sum_{l=0}^{1} \| \partial_l^i u_i \|_\alpha \right)^2} \quad \text{and} \quad v_2 := \frac{u_3 u_4}{C \left( 1 + \sum_{i=1}^{2} \sum_{l=0}^{1} \| \partial_l^i u_i \|_\alpha \right)^2},
$$

which verify $\| \partial_n^q v_i \|_\alpha \leq L^{n-3/2} (n - 2)!$ for all $2 \leq n \leq N$, we get that

$$
\| \partial_q^n (u_1 u_2 u_3 u_4) \|_\alpha \leq C \left( 1 + \sum_{i=1}^{2} \sum_{l=0}^{1} \| \partial_l^i u_i \|_\alpha \right)^8 L^{n-3/2} (n - 2)!
$$

for all $3 \leq n \leq N$. Finally, we use the estimate (2.10) for the functions

$$
u_5 \quad \text{and} \quad u_6 := \frac{u_1 u_2 u_3 u_4}{C \left( 1 + \sum_{i=1}^{2} \sum_{l=0}^{1} \| \partial_l^i u_i \|_\alpha \right)^8}
$$

and obtain the desired estimate (2.6). \hfill \Box

We use now Lemma 2.2 to prove the following estimate.

**Lemma 2.3.** Assume that $\partial_n^q u \in C^\alpha(\Omega)$ for all $0 \leq n \leq N$, with $N \geq 3$, and let $C_1$ be the constant determined in Lemma 2.2 (ii). If there exists a constant

$$
L \geq \| \partial_q^2 (1/u) \|_\alpha^2 + C_1^2 \left( 1 + \sum_{l=0}^{1} (2 \| \partial^l_0 (1/u) \|_\alpha + \| \partial_0^{l+1} u \|_\alpha) \right)^{12}, \quad (2.11)
$$

such that $\| \partial_n^q u \|_\alpha \leq L^{-2} (n - 3)!$ for all $3 \leq n \leq N$ and $\inf_{\Omega} u > 0$, then we have

$$
\| \partial_q^n (1/u) \|_\alpha \leq L^{-3/2} (n - 2)! \quad \text{for all } 2 \leq n \leq N. \quad (2.12)
$$

**Proof.** The proof follows by induction. By the choice of $L$, it is clear that the relation (2.12) is satisfied when $n = 2$. So, let us assume that (2.12) is satisfied for all $2 \leq n \leq m - 1$, with $3 \leq m \leq N$. In order to prove the assertion for $n = m$, we write

$$
\partial_q^m (1/u) = \partial_q^{m-1} (u_1 u_2 u_3)
$$

whereby $u_1 := -\partial_q u$ and $u_2 = u_3 := 1/u$. Our hypothesis and the induction assumption ensure that for all $2 \leq n \leq m - 1$ we have

$$
\| \partial_q^n u_1 \|_\alpha = \| \partial_q^{n+1} u \|_\alpha \leq L^{-1} (n - 2)!, \quad \| \partial_q^n u_2 \|_\alpha = \| \partial_q^n (1/u) \|_\alpha \leq L^{-3/2} (n - 2)! \leq L^{-1} (n - 2)!,
$$

and therefore,

$$
\| \partial_q^n v_1 \|_\alpha \leq L^{-3/2} (n - 2)!, \quad \| \partial_q^n v_2 \|_\alpha \leq L^{-3/2} (n - 2)!.
$$

Using these estimates and the induction assumption (2.12), we obtain

$$
\| \partial_q^n (1/u) \|_\alpha \leq L^{-3/2} (n - 2)!.
$$

This completes the proof. \hfill \Box
the last inequality being a consequence of the fact that $L > 1$. Whence, Lemma 2.2 (ii) and the relation (2.11) combine to

$$\|\partial_q^m (1/u)\|_\alpha \leq C_1 \left( 1 + \sum_{l=0}^1 (2\|\partial_q^l (1/u)\|_\alpha + \|\partial_q^{l+1} u\|_\alpha) \right)^6 L^{m-2}(m - 3)!$$

$$\leq L^{m-3/2}(m - 2)!,$$

which completes the proof. \hfill \Box

We come now to the proof of our main result.

**Proof of Theorem 1.1.** Let $h$ be a weak solution of (1.1)-(1.2) and let $L$ be a positive constant such that

$$L \geq \|\partial_q^2 (1/h_p)\|_\alpha^2 + \|\partial_q^2 h_q\|_\alpha^2 + C_1^2 \left( 1 + \sum_{l=0}^1 (2\|\partial_q^l (1/h_p)\|_\alpha + \|\partial_q^{l+1} h_p\|_\alpha) \right)^{12} + \sum_{l=0}^4 \|\partial_q^l h\|_{1+\alpha}. \tag{2.13}$$

Then, it is clear that $L \geq 1$. Moreover, the inequality (2.13) guarantees that (1.4) is satisfied for $m = 3$ and $m = 4$. So, let us presuppose that (1.4) is true for all $3 \leq n \leq m - 1$, whereby $m \geq 5$. We need to show only that (1.4) is satisfied for $m$. To this end, let us observe that

$$\max\{\|\partial_q^n h_q\|_\alpha, \|\partial_q^n h_p\|_\alpha\} \leq \|\partial_q^n h\|_{1+\alpha} \leq L^{n-2}(n - 3)! \quad \text{for } 3 \leq n \leq m - 1. \tag{2.14}$$

This estimate together with the Lemma 2.3, which we may apply to the function $u = h_p$, cf. (1.2) and (2.13), yield

$$\|\partial_q^n (1/h_p)\|_\alpha \leq L^{n-3/2}(n - 2)! \quad \text{for all } 2 \leq n \leq m - 1. \tag{2.15}$$

With our choice of $L$ and in view of the induction assumption, we also have that

$$\|\partial_q^n h_q\|_\alpha \leq L^{n-3/2}(n - 2)! \quad \text{for } 2 \leq n \leq m - 1. \tag{2.16}$$

Recall that $\partial_q^n h$ is the solution of the elliptic problem (2.1). The arguments used in the proof of Proposition 2.1 and the Schauder estimate derived in [9, Theorem 3] ensure the existence of a positive constant $C_3$ such that

$$\|\partial_q^m h\|_{1+\alpha} \leq C_3 (\|\partial_q^m h\|_0 + \|f_m\|_\alpha + \|g_m\|_\alpha + \|\varphi_m\|_\alpha), \tag{2.17}$$

meaning that we are left to prove that the right-hand side of relation (2.17) may be estimated by $L^{m-2}(m - 3)!$.

The supremum norm $\|\partial_q^m h\|_0$ can be bounded by using the induction assumption only

$$\|\partial_q^m h\|_0 \leq \|\partial_q^{m-1} h\|_\alpha \leq L^{m-3}(m - 4)!. \tag{2.18}$$

The terms appearing in the definition of $f_m$ and $g_m$ can be estimated by using the same scheme. Indeed, let us notice that the estimate (2.6) of Lemma 2.2 together with the estimates (2.15) and (2.16) ensure the existence of a constant $K_0 > 1$ with the property that

$$\max \left\{ \left\| \partial_q^n \left( \frac{1}{h_p} \right) \right\|_\alpha, \left\| \partial_q^n \left( \frac{h_q}{h_p^2} \right) \right\|_\alpha, \left\| \partial_q^n \left( \frac{1 + h_q^2}{h_p^2} \right) \right\|_\alpha \right\} \leq K_0 L^{n-3/2}(n - 2)! \tag{2.19}$$
for all $2 \leq n \leq m - 1$. With this preparation, we write the first sum appearing in the definition of $f_m$ as follows

$$
\sum_{n=1}^{m-1} \binom{m-1}{n} \partial^m_q \left( \frac{1}{h_p} \right) \partial_q (\partial_q^{m-n} h) = \left( \sum_{n=1}^{m-3} + \sum_{n=2}^{m-1} \sum_{n=m-2}^{m-1} \right) \binom{m-1}{n} \partial^m_q \left( \frac{1}{h_p} \right) \partial_q (\partial_q^{m-n} h),$

and observe that the convergence of the series $\sum_n n^{-2}$ implies

$$
\left\| \sum_{n=2}^{m-3} \binom{m-1}{n} \partial^m_q \left( \frac{1}{h_p} \right) \partial_q (\partial_q^{m-n} h) \right\|_\alpha \leq \sum_{n=2}^{m-3} \binom{m-1}{n} \left\| \partial^m_q \left( \frac{1}{h_p} \right) \right\|_\alpha \left\| \partial_q^{m-n} h \right\|_{1+\alpha}
$$

$$
\leq K_0 L^{m-7/2} \sum_{n=2}^{m-3} \binom{m-1}{n} (n-2)! (m-n-3)!
$$

$$
\leq K_0 L^{m-7/2} (m-3)! \sum_{n=2}^{m-3} \frac{(m-1)^2}{(n-1)^2 (m-n-2)^2}
$$

$$
\leq K_0 L^{m-7/2} (m-3)!.
$$

On the other hand, it follows readily from (2.15) and the induction assumption that

$$
\left\| \left( \sum_{n=1}^{m-3} + \sum_{n=m-2}^{m-1} \right) \binom{m-1}{n} \partial^m_q \left( \frac{1}{h_p} \right) \partial_q (\partial_q^{m-n} h) \right\|_\alpha \leq K_1 L^{m-5/2} (m-3)!
$$

Since the remaining sums that appear in the definition of $f_m$ and $g_m$ can be estimated in a similar way, we conclude that

$$
\left\| f_m \right\|_\alpha + \left\| g_m \right\|_\alpha \leq K_2 L^{m-5/2} (m-3)!
$$

(2.20)

It remains to estimate the norm $\left\| \varphi_m \right\|_\alpha$. Because $C^\alpha(\overline{\Omega})$ is an algebra, we need to estimate only the terms

$$
T_1 := \sum_{n=1}^{m-1} \binom{m}{n} \partial^n_q h_p \partial_q^{m-n} h_p
$$

and

$$
T_2 := \sum_{n=1}^{m-1} \binom{m}{n} \partial^n_q h_p \partial_q^{m-n} h_p^2.
$$

In order to deal with $T_1$, we write

$$
T_1 = \left( \sum_{n=1}^{2} + \sum_{n=3}^{m-3} + \sum_{n=m-2}^{m-1} \right) \binom{m}{n} \partial^n_q h_p \partial_q^{m-n} h_p,
$$

and obtain from the induction assumption that

$$
\left\| \left( \sum_{n=1}^{2} + \sum_{n=m-2}^{m-1} \right) \binom{m}{n} \partial^n_q h_p \partial_q^{m-n} h_p \right\|_\alpha \leq K_3 L^{m-3} (m-3)!.\]
On the other hand, if \( m \geq 6 \), the induction assumption together with the convergence of the series \( \sum_n n^{-3} \) imply

\[
\left\| \sum_{n=3}^{m-3} \binom{m}{n} (\partial_q^n h_p) \partial_q^{m-n} h_p \right\|_\alpha \leq \sum_{n=3}^{m-3} \binom{m}{n} \|\partial_q^n h\|_{1+\alpha} \|\partial_q^{m-n} h\|_{1+\alpha} \\
\leq \sum_{n=3}^{m-3} \binom{m}{n} L^{n-2}(n-3)!L^{m-n-2}(m-n-3)! \\
\leq L^{m-4}(m-3)! \sum_{n=3}^{m-3} \frac{m^3}{(n-2)^3(m-n-2)^3} \\
\leq C_4 L^{m-4}(m-3)!,
\]

and we conclude that

\[
\|T_1\|_\alpha \leq K_4 L^{m-3}(m-3)!. \tag{2.21}
\]

Finally, in order to estimate \( T_2 \), we write

\[
T_2 = \left( \sum_{n=1}^{2} \sum_{n=3}^{m-3} \sum_{n=m-2}^{m} \binom{m}{n} (\partial_q^n h) \partial_q^{m-n} h_p^2 \right)
\]

and infer from the relation (2.14) and Lemma 2.2 (iii) that there exists a constant \( K_5 > 1 \) such that \( \|\partial_q^n h_p^2\|_\alpha \leq K_5 L^{n-2}(n-3)! \) for all \( 3 \leq n \leq m - 1 \). This estimate combined with the induction assumption guarantee the existence of a constant \( K_6 > 1 \) with the property that

\[
\left\| \left( \sum_{n=1}^{2} \sum_{n=m-2}^{m} \binom{m}{n} (\partial_q^n h) \partial_q^{m-n} h_p^2 \right) \right\|_\alpha \leq K_6 L^{m-3}(m-3)!
\]

and, when \( m \geq 6 \), we use again the convergence of the series \( \sum_n n^{-3} \) to find

\[
\left\| \sum_{n=3}^{m-3} \binom{m}{n} (\partial_q^n h) \partial_q^{m-n} h_p^2 \right\|_\alpha \leq C_5 \sum_{n=3}^{m-3} \binom{m}{n} \|\partial_q^n h\|_{1+\alpha} \|\partial_q^{m-n} h_p^2\|_\alpha \\
\leq K_5 \sum_{n=3}^{m-3} \binom{m}{n} L^{n-2}(n-3)!L^{m-n-2}(m-n-3)! \\
\leq K_5 L^{m-4}(m-3)!
\]

Thus, we have found a constant \( K_7 > 1 \) such that

\[
\|T_2\|_\alpha \leq K_7 L^{m-3}(m-3)!. \tag{2.22}
\]

Combining (2.17), (2.18), (2.20), (2.21), and (2.22) we conclude that there exists a constant \( K_8 > 1 \) such that \( \|\partial_q^m h\|_{1+\alpha} \leq K_8 L^{m-5/2}(m-3)! \). Since \( K_8 \) is independent of \( m \) and \( L \), we may require, additionally to (2.11), that \( L \geq K_8^2 \). But then \( \|\partial_q^m h\|_{1+\alpha} \leq L^{m-2}(m-3)! \), and the induction principle guarantees that (1.4) is satisfied for all \( m \in \mathbb{N} \) with \( m \geq 3 \). This finishes the proof. \( \square \)
References

[1] C. J. Amick, L. E. Fraenkel, and J. F. Toland: On the Stokes conjecture for the wave of extreme form, *Acta Math.*, 148 (1982), 193–214.

[2] A. Constantin: On the deep water wave motion, *J. Phys. A*, 34 (2001), 1405–1417.

[3] A. Constantin: *Nonlinear Water Waves with Applications to Wave-Current Interactions and Tsunamis*, CBMS-NSF Conference Series in Applied Mathematics, Vol. 81, SIAM, Philadelphia, 2011.

[4] A. Constantin, M. Ehrnström, and E. Wahlén: Symmetry of steady periodic gravity water waves with vorticity, *Duke Math. J.*, 140 (2007), 591–603.

[5] A. Constantin and J. Escher: Symmetry of steady periodic surface water waves with vorticity, *J. Fluid Mech.*, 498 (2004), 171–181.

[6] A. Constantin and J. Escher: Analyticity of periodic travelling free surface water waves with vorticity, *Ann. Math.*, 173 (2011), 559–568.

[7] A. Constantin and W. Strauss: Exact steady periodic water waves with vorticity, *Comm. Pure Appl. Math.*, 57 (2004), 481–527.

[8] A. Constantin and W. Strauss: Pressure beneath a Stokes wave, *Comm. Pure Appl. Math.*, 63(4) (2010), 533–557.

[9] A. Constantin and W. Strauss: Periodic traveling gravity water waves with discontinuous vorticity, *Arch. Ration. Mech. Anal.*, 202(1) (2011), 133–175.

[10] A. Constantin and E. Varvaruca: Steady periodic water waves with constant vorticity: regularity and local bifurcation, *Arch. Ration. Mech. Anal.*, 199(1) (2011), 33–67.

[11] M.-L. Dubreil-Jacotin: Sur la détermination rigoureuse des ondes permanentes périodiques d’amplitude finie, *J. Math. Pures. Appl.*, 13 (1934), 217–291.

[12] M. Ehrnström, J. Escher, and E. Wahlén: Steady water waves with multiple critical layers, *SIAM J. Math. Anal.*, 43(3) (2011), 1436–1456.

[13] J. Escher: Regularity of rotational travelling water waves, *Philos. Trans. R. Soc. Lond. A*, 370 (2011), 1602–1615.

[14] F. Gerstner: Theorie der Wellen samt einer daraus abgeleiteten Theorie der Deichprofile, *Ann. Phys.*, 2 (1809), 412–445.

[15] D. Gilbarg and N. S. Trudinger: *Elliptic partial differential equations of second order*, Springer Verlag, Berlin, 2001.

[16] D. Henry: On Gerstner’s water wave, *J. Nonlinear Math. Phys.*, 15 (2008), 87–95.

[17] D. Henry: Analyticity of the streamlines for periodic travelling free surface capillary-gravity water waves with vorticity, *SIAM J. Math. Anal.*, 42 (2010), 3103–3111.

[18] D. Henry: Analyticity of the free surface for periodic travelling capillary-gravity water waves with vorticity, *J. Math. Fluid Mech.*, 14(2) (2012), 249–254.

[19] D. Henry: Regularity for steady periodic capillary water waves with vorticity, *Philos. Trans. R. Soc. Lond. A*, 370 (2012), 1616–1628.

[20] D. Henry and B.-V. Matioc: On the regularity of steady periodic stratified water waves, *Commun. Pure Appl. Anal.*, 11(4) (2012), 1453–1464.

[21] V. M. Hur: Analyticity of Rotational Flows Beneath Solitary Water Waves *Int. Math. Res. Not.*, doi:10.1093/imrn/rnr1232 (2011).

[22] G. Keady and J. Norbury: On the existence theory for irrotational water waves, *Math. Proc. Camb. Phil. Soc.*, 83 (1978), 137–157.

[23] J. Ko and W. Strauss: Effect of vorticity on steady water waves, *J. Fluid Mech.*, 608 (2008), 197–215.

[24] H. Lewy: A note on harmonic functions and a hydrodynamical application, *Proc. Amer. Math. Soc.*, 3 (1952), 111–113.

[25] H. Chen, W.-X. Li and L.-J. Wang: Regularity of traveling free surface water waves with vorticity, *J. Nonlinear Sci.*, 2013, DOI 10.1007/s00332-013-9181-6.

[26] C. I. Martin: Regularity of steady periodic capillary water waves with constant vorticity, *J. Nonlinear Math. Phys.*, 19(supp01) (2012), 1240006, 7 p.

[27] C. I. Martin: Local bifurcation and regularity for steady periodic capillary-gravity water waves with constant vorticity, *Nonlinear Anal. Real World Appl.*, 14(1) (2013), 131–149.
[28] A.-V. Matioc: An explicit solution for deep water waves with Coriolis effects, *J. Nonlinear Math. Phys.*, 19(supp.01) (2012), 1240005, 8p.
[29] A.-V. Matioc: Steady internal water waves with a critical layer bounded by the wave surface, *J. Nonlinear Math. Phys.*, 19(supp.01) (2012), 1250008, 21p.
[30] A.-V. Matioc and B.-V. Matioc: On the symmetry of periodic gravity water waves with vorticity, *Differential Integral Equations*, 26(1-2) (2013), 129–140.
[31] A.-V. Matioc and B.-V. Matioc: On periodic water waves with Coriolis effects and isobaric streamlines, *J. Nonlinear Math. Phys.*, 19(supp01) (2012), 1240009, 15 p.
[32] A.-V. Matioc and B.-V. Matioc: Regularity and symmetry properties of rotational solitary water waves, *J. Evol. Equ.*, 12 (2012), 481–494.
[33] B.-V. Matioc: Analyticity of the streamlines for periodic traveling water waves with bounded vorticity, *Int. Math. Res. Not.*, 17 (2011), 3858–3871.
[34] B.-V. Matioc: On the regularity of deep-water waves with general vorticity distributions, *Quart. Appl. Math.*, LXX(2) (2012), 393–405.
[35] B.-V. Matioc: Regularity results for deep-water waves with Hölder continuous vorticity, *Appl. Anal.*, 92(10) (2013), 2144–2151.
[36] O. M. Philipps and M. L. Banner: Wave breaking in the presence of wind drift and swell, *J. Fluid Mech.*, 66 (1974), 625–640.
[37] J. F. Toland: Stokes waves, *Topol. Methods Nonlinear Anal.*, 1(7) (1996), 1–48.
[38] E. Wahlén: Steady water waves with a critical layer, *J. Differential Equations*, 246(6) (2009), 1468–2483.
[39] L.-J. Wang: Regularity of traveling periodic stratified water waves with vorticity, *Nonlinear Anal.*, 81 (2013), 247–263.

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