A NOTE ON QUASI-CONVEX FUNCTIONS

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Abstract. We present an example of smooth quasi-convex functions in the positive octant of $\mathbb{R}^3$ which cannot be obtained as the images of convex smooth functions under a monotone smooth mappings of $\mathbb{R}$.

1. Introduction

Quasi-convex functions play an important role in problems related to continuous optimization and mathematical programming such as generalizations of the von Neumann minimax theorem, the Kuhn-Tucker saddle-point theorem, and other optimization problems related to consumer demand and indirect utility function (see, for instance, [3], [5] and the references therein).

According to [3] a real-valued function $u(x)$ defined in a convex subset $E$ of the Euclidean space $\mathbb{R}^d$ is called quasi-convex if

$$u(\lambda x + (1 - \lambda)y) \leq \max[u(x), u(y)]$$

as long as $\lambda \in [0, 1], x, y \in E$. Convex functions in [3] are those for which

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

Numerous properties of quasi-convex functions and their relation to convex functions are discussed, for instance, in [3], [6], and [1], in particular, that $F[f(x)]$ is quasi-convex if $f$ is convex and $F$ is nondecreasing. Somehow the following very natural question is left untouched: can any quasi-convex function $u$ be represented as $F[f(x)]$ with convex $f$ and nondecreasing $F$?

This issue is also avoided in many other publications on the subject of quasi-convexity. The reason for that is probably because the answer to this question is negative and the corresponding counterexamples are given in [2]. In these examples, however, $u$ is not smooth and for almost any point of $E$ there exists a neighborhood such that in that neighborhood the above representation still holds.

We want to present an example of a smooth function $u$, which is quasi-convex in a convex domain $E$ such that there are no smooth and strictly monotone functions $F$ such that $F[u(x)]$ is convex in a ball in $E$. Then, of course, $u$ itself is not even locally an increasing smooth image of a smooth convex function. This directly contradicts the claim made in n. 1 of [2] that for any smooth quasi-convex function $u$ one can find strictly increasing $F$.

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such that $F[u(x)]$ is convex. Our arguments have much in common with Remark 5.14 of [4].

Here is our result.

**Theorem 1.1.** In $\mathbb{R}^3$ consider the domain

$$E := \{(x, y, z) : x, y, z > 0\},$$

fix $\alpha > 0$, and introduce the function

$$u(x, y, z) = \frac{x^\alpha(y^\alpha + z^\alpha)}{x^\alpha y^\alpha}.$$

Then $E$ is convex, $u$ is quasi-convex in $E$, and, if $\beta \in (0, 1]$, then for any $(x_0, y_0, z_0) \in E$ and any twice continuously differentiable function $F(t)$ on $\mathbb{R}$ such that $F'[u(x_0, y_0, z_0)] \neq 0$, the matrix of the second order derivatives of $F[u(x, y, z)]$ at $(x_0, y_0, z_0)$ is neither nonnegative nor nonpositive. Hence, $F[u(x, y, z)]$ is neither convex nor concave in any neighborhood of any point in $E$ if $F' > 0$ or $F' < 0$ on $\mathbb{R}$.

Proof. First observe that the function

$$v(x, y) = \frac{x^\alpha + y^\alpha}{x^\alpha y^\alpha} = \frac{1}{x^\alpha} + \frac{1}{y^\alpha}$$

is convex in $(0, \infty)^2$. Then, in light of homogeneity of $u$ for $(x_i, y_i, z_i) \in (0, \infty)^3$, $\lambda_i \in [0, 1]$, $i = 1, 2$, such that $\lambda_1 + \lambda_2 = 1$, we have

$$u(\lambda_1 x_1 + \lambda_2 x_2, \lambda_1 y_1 + \lambda_2 y_2, \lambda_1 z_1 + \lambda_2 z_2) = v\left(\frac{\lambda_1 x_1 + \lambda_2 x_2}{\lambda_1 z_1 + \lambda_2 z_2}, \frac{\lambda_1 y_1 + \lambda_2 y_2}{\lambda_1 z_1 + \lambda_2 z_2}\right)$$

$$= v\left(\mu_1 \frac{x_1}{z_1} + \mu_2 \frac{x_2}{z_2}, \mu_1 \frac{y_1}{z_1} + \mu_2 \frac{y_2}{z_2}\right),$$

where $\mu_i = \lambda_i z_i/(\lambda_1 z_1 + \lambda_2 z_2)$. Since $\mu_i \geq 0$ and $\mu_1 + \mu_2 = 1$ and $v$ is convex, the last expression above is less than

$$\mu_1 v\left(\frac{x_1}{z_1}, \frac{y_1}{z_1}\right) + \mu_2 v\left(\frac{x_2}{z_2}, \frac{y_2}{z_2}\right) = \mu_1 u(x_1, y_1, z_1) + \mu_2 u(x_2, y_2, z_2)$$

$$\leq \max\left[u(x_1, y_1, z_1), u(x_2, y_2, z_2)\right].$$

This shows that $u$ is quasi-convex in $E$.

We now come to analyzing $F[u]$. Denote by $Du$ the column-vector gradient of $v$ and by $D^2 v$ its matrix of the second-order derivatives. By $a^*$ we mean the transpose of a matrix $a$ and by $(a, b)$ we mean the scalar product of $a, b \in \mathbb{R}^3$. We have

$$D\{F[u]\} = F'[u]Du, \quad D^2\{F[u]\} = F''[u]Du(Du)^* + F'[u]D^2 u.$$

We fix $(x_0, y_0, z_0) \in E$ and take a column-vector $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ (written in a common abuse of notation as a row vector) such that at $(x_0, y_0, z_0)$ we have $\langle Du, \xi \rangle = 0$, which means that

$$-\xi_1 \alpha \frac{x_0^\alpha}{x_0^{\alpha+1}} - \xi_2 \alpha \frac{y_0^\alpha}{y_0^{\alpha+1}} + \xi_3 \alpha \frac{z_0^{\alpha-1}(x_0^\alpha + y_0^\alpha)}{x_0^\alpha y_0^\alpha} = 0,$$
which is
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Therefore, to prove our claim it suffices to show that the quadratic form

\[ R(\xi_1, \xi_2) = a^2(\alpha + 1)^2 - \alpha(\alpha + 1)(a^2 + 1) = \alpha(\alpha + 1)(\alpha - 1), \]

which is < 0 if \( \alpha < 1 \), and the theorem is proved in this case.
In case $\alpha = 1$, take $\xi$ from above, introduce $\eta = \xi + \kappa Du(x_0, y_0, z_0)$, and note that
\[
(F'[u])^{-1}\langle \eta, D^2\{F[u]\}\eta \rangle = \langle \xi, D^2u\xi \rangle + 2\kappa \langle D^2u\xi, Du \rangle + \left[F''[u](F'[u])^{-1}|Du|^4 + \langle D^2uDu, Du \rangle\right]\kappa^2,
\]
where at $(x_0, y_0, z_0)$ as is easy to check
\[
\langle \xi, D^2u\xi \rangle = \frac{2z_0}{x_0 + y_0}\left(\frac{\xi_1}{x_0} - \frac{\xi_2}{y_0}\right)^2.
\]
This quantity vanishes if $\xi_1 = tx_0, \xi_2 = ty_0, t \in \mathbb{R}$, and for the right-hand side of (1.2) not to change sign, say for $\kappa = 1$, when $t$ runs through $\mathbb{R}$ it is necessary to have $\langle D^2uDu, \xi \rangle = 0$ for those $\xi_1, \xi_2$.

However, with this choice at $(x_0, y_0, z_0)$ we have $\xi_3 = tz_0$ and
\[
D^2u\xi = t\left(\frac{z_0}{x_0^2}, \frac{z_0}{y_0^2}, \frac{1}{x_0} - \frac{1}{y_0}\right),
\]
\[
\langle D^2u\xi, Du \rangle = t\left(-\frac{z_0^2}{x_0^4} - \frac{z_0^2}{y_0^4} - \left(\frac{1}{x_0} + \frac{1}{y_0}\right)^2\right).
\]
This shows that for $\alpha = 1$ as well, the right-hand side of (1.2) has different signs for any fixed $(x_0, y_0, z_0)$ if we vary $\xi$ and $\kappa$ and finishes the proof of the theorem.

**Remark 1.1.** One can show that, if $\alpha > 1$, then for any point $(x_0, y_0, z_0) \in E$ one can find large $\lambda > 0$ such that $\exp(\lambda u)$ is strictly convex in a neighborhood of $(x_0, y_0, z_0)$. Of course, $\lambda \to \infty$ as $\alpha \downarrow 1$.

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