ON THE HAMILTONIAN APPROACH AND PATH INTEGRATION FOR A POINT PARTICLE MINIMALLY COUPLED TO ELECTROMAGNETISM

Kostas Skenderis* and Peter van Nieuwenhuizen†

Institute for Theoretical Physics, State University of New York at Stony Brook, Stony Brook, NY 11794-3840

May 15, 2019

Abstract

We derive the exact configuration space path integral, together with the way how to evaluate it, from the Hamiltonian approach for any quantum mechanical system in flat spacetime whose Hamiltonian has at most two momentum operators. Starting from a given, covariant or non-covariant, Hamiltonian, we go from the time-discretized path integral to the continuum path integral by introducing Fourier modes. We prove that the limit $N \to \infty$ for the terms in the perturbation expansion (“Feynman graphs”) exists, by demonstrating that the series involved are uniformly convergent. All terms in the expansion of the exponent in $\langle x|\exp(-\Delta t \hat{H}/\hbar)|y\rangle$ contribute to the propagator (even at order $\Delta t$!). However, in the time-discretized path integral the only effect of the terms with $\hat{H}^2$ and higher is to cancel terms which naively seem to vanish for $N \to \infty$, but, in fact, are nonvanishing. The final result is that the naive correspondence between the Hamiltonian and the Lagrangian approach is correct, after all. We explicitly work through the example of a point particle coupled to electromagnetism. We compute the propagator to order $(\Delta t)^2$ both with the Hamiltonian and the path integral approach and find agreement.

* e-mail: kostas@max.physics.sunysb.edu
† e-mail: vannieu@max.physics.sunysb.edu
1 Introduction.

Path integrals are often first written down in a symbolic way as an integral over paths of the exponent of an action, and then defined by some time-discretization. Of course, there are many ways in which to implement time-discretization. In some instances, rules have been discovered which lead to desirable answers for the path integral. A well-known example is the midpoint rule for the interaction $\int dt A_j(x) \dot{x}^j$ of a point particle coupled to an electromagnetic potential. This rule leads to gauge invariance of the terms proportional to $\Delta t$ in the propagator but it breaks gauge invariance in the terms of order $(\Delta t)^2$. The terms of higher order in $\Delta t$ are needed for the evaluation of anomalies (see below). In general, starting from a continuum path integral, there is no preferred way to discretize it. One might take the point of view that different discretizations simply correspond to different theories.

In this article we take a different point of view. We take the Hamiltonian formalism as starting point, and shall deduce both the action $S_{\text{config}}$ to be used in the configuration space path integral, and the way this path integral should be evaluated (“the measure”). We mean by the expressions “to be used” and “should be” that in this way the path integral formalism exactly reproduces the propagator of the Hamiltonian formalism. Of course, in the Hamiltonian $\hat{H}(\hat{x}^i, \hat{p}_j)$ there is a priori a corresponding ambiguity in the ordering of the operators $\hat{x}^i$ and $\hat{p}_j$. However, in several examples, the Hamiltonian of a quantum mechanical model is, in fact, the regulator of the Jacobians for symmetry transformations of a corresponding field theory, and these regulators are uniquely fixed by requiring certain symmetries of the field theory to be maintained at the quantum level\cite{2, 3, 5, 6}. For example, in regulators are constructed which maintain Weyl (local scale) invariance but as a consequence break Einstein (general coordinate) invariance. Thus, the field theory and the choice of which symmetries are free of anomalies fixes the regulator, the regulator is the Hamiltonian of a corresponding quantum mechanical model, and the operator ordering of this Hamiltonian is thus fixed. For these reason we consider Hamiltonians of the form $\hat{H} = \hat{p}^2 + a'(\hat{x}) \hat{p}_i + b(\hat{x})$ whose operator ordering is fixed in this way but whose coefficients $a'(\hat{x})$ and $b(\hat{x})$ are not restricted except that we assume that they are regular functions; they may correspond to covariant or non-covariant Hamiltonians. The results of this paper prove which path integral (including, of course, the way how to evaluate it) corresponds to which regulator (Hamiltonian). For chiral anomalies\cite{2} this precise correspondence
was not needed due to their topological nature, but for trace anomalies\cite{5, 6} and other anomalies of non-topological nature, the precise correspondence is crucial.

Having obtained the 1-1 correspondence, it is then also possible to start with a particular action in the path integral (the latter to be evaluated as derived below) and to find the corresponding Hamiltonian operator. This will usually be the case when one is dealing with quantum field theories. For example, when one is dealing with renormalizable field theories or when the theory has certain symmetries the action may be known, and this will fix the operator ordering and the terms in the Hamiltonian.

In the Hamiltonian approach the propagator is defined by

$$< x, t_2 | y, t_1 > = < x | \exp \left( -\frac{\Delta t}{\hbar} \hat{H} \right) | y >, \quad \Delta t = t_2 - t_1, \quad (1)$$

and evaluated, following Feynman, by inserting a complete set of momentum eigenstates

$$< x, t_2 | y, t_1 > = \int dp < x | \exp \left( -\frac{\Delta t}{\hbar} \hat{H} \right) | p > < p | y > . \quad (2)$$

Expanding the exponent, and moving in each term $(-\Delta t \hat{H}/\hbar)^n/n!$ the $\hat{x}$ operators to the left and the $\hat{p}$ operators to the right, one obtains an unambiguous answer for the propagator. No regularization is needed. However, one must keep track of the commutators between $\hat{x}^i$ and $\hat{p}_j$. It is often assumed that it is sufficient to expand the exponent only to first order in $\Delta t$, and to reexpontiate the result

$$< x | \exp \left( -\frac{\Delta t}{\hbar} \hat{H} \right) | p > = \exp \left[ -\frac{\Delta t}{\hbar} h(x, p) \right] < x | p > \quad (3)$$

$$< x | \hat{H} | p > \equiv h(x, p) < x | p > . \quad (4)$$

This is incorrect for Hamiltonians with derivative coupling: for nonlinear sigma models where the $\hat{p}^2$ term is multiplied by a function of $\hat{x}$ (“the metric”)\cite{7, 8, 9} or for Hamiltonians with a term $A(\hat{x}) \cdot \hat{p}$. We shall consider the Hamiltonian

$$\hat{H} = \frac{1}{2} \left( \hat{p}_i - \left( \frac{\epsilon}{c} \right) A_i(\hat{x}) \right) \left( \hat{p}^i - \left( \frac{\epsilon}{c} \right) A^i(\hat{x}) \right) + V(\hat{x}), \quad (5)$$

for arbitrary but nonsingular $A_i(\hat{x})$ and $V(\hat{x})$ which is obviously the most general Hamiltonian of the form $\hat{H} = \hat{p}^2 + a^i(\hat{x}) \hat{p}_i + b(\hat{x})$, and show that
there are terms proportional to $\Delta t$ in the propagator which are due to commutators between $\hat{p}_i$ and $A_j(\hat{x})$. In fact, all terms in the expansion of the exponential give such contributions! Because the commutators $[\hat{p}_i, \hat{x}_j] = -i\hbar\delta^i_j$ are proportional to $\hbar$, the propagator becomes a series in $\hbar$, $\Delta t$ and $(x - y)^i$ with coefficients which are functions of $x$. When we use the term “of order $(\Delta t)^k$” we mean all terms which differ from the leading term by a factor $(\Delta t)^k$, counting $(x - y)^i$ as $(\Delta t)^{1/2}$. The terms of order $\hbar$ w.r.t. the classical result correspond to one-loop corrections in the path integral approach, and can be written in terms of the classical action as the Van Vleck determinant. Terms of higher order in $\hbar$ in the propagator can be computed straightforwardly (though tediously) in the Hamiltonian approach, again without need to specify a regularization. This indicates that the details of the path integral should follow straightforwardly from the Hamiltonian starting point. In particular it should not be necessary to fix a free constant in the overall normalization of the path integral by hand, for example by dividing by the path integral for a free particle.

One begins by defining the path integral as

$$< x, 0 | y, -T > = \lim_{N \to \infty} \int \left[ \prod_{\alpha=1}^{N-1} dx_\alpha \right] \left[ \prod_{\alpha=1}^{N} < x_{\alpha-1}, t_{\alpha-1} | x_{\alpha}, t_{\alpha} > \right],$$  \hspace{1cm} (6)

where $x_0 = x$ and $x_N = y$. This particular time-discretization follows from the Hamiltonian approach; it is due to the operator identity

$$\exp(-\frac{T}{\hbar} \hat{H}) = \left( \exp(-\frac{T/N}{\hbar} \hat{H}) \right)^N.$$  \hspace{1cm} (7)

The main result of this paper is a proof that the $N \to \infty$ limit exists, and defines a continuum action $S_{\text{config}}$ and an unambiguous and simple way to evaluate the path integral perturbatively.

We begin by splitting $x_\alpha$ into a background part $z_\alpha$ and a quantum part $\xi_\alpha$. We shall also decompose the time-discretized action $S$ into a part $S_0$ which yields the propagator on the world line, and the rest which yields the interaction terms $S_{\text{int}}$. The $z_\alpha$ satisfy the equation of motion of $S_0$ and the boundary conditions $z_\alpha = y$ at $\alpha = N$ and $z_\alpha = x$ at $\alpha = 0$, so that $\xi_\alpha = 0$ both at $\alpha = 0$ and at $\alpha = N$. Since $S_0$ is not equal to $S$, there are terms linear in $\xi_\alpha$ in the expansion of $S(z + \xi, z + \dot{\xi})$. Notice that the time-discretized action $S$ has not been obtained by some ad-hoc rule, but rather it is determined from the Hamiltonian approach.
The final result for the path integral should not depend on the choice of \( S_0 \). We choose \( S_0 \) as the action of a free particle because that leads to simple perturbation theory, but other choices of \( S_0 \) should lead to the same final result although the Feynman rules for the perturbative expansion of the path integral will be different.

One now expands \( \xi_\alpha \) into eigenfunctions of \( S_0 \), i.e., in terms of trigonometric functions

\[
\xi_\alpha = \sum_{k=1}^{N-1} y^k \sin \alpha k \pi / N, \quad (\alpha = 1, \ldots, N - 1).
\]

(8)

Changing integration variables from \( \xi_\alpha \) to \( y^k \), the Jacobian is essentially unity, while \( S_0 \) is quadratic and diagonal in these \( y^k \). Rescaling these \( y^k \) such that the kinetic term in terms of the rescaled variables \( v^k \) becomes the one of the continuum theory, the Jacobian of this rescaling leads to a non-trivial factor in the measure. At this point, the path integral has the generic form

\[
\int d\mu \exp\left[ -\frac{1}{\hbar}(S_0 + S_{\text{int}}(N)) \right],
\]

(9)

where the measure \( \mu \) and the kinetic term \( S_0 \) are already in the form of the continuum theory, but the interaction \( S_{\text{int}} \) still depend on \( N \).

By the term “continuum theory” we mean the path integral with

(i) the classical Lagrangian \( L = \frac{1}{2} \dot{z}^2 - i(\dot{z}) \dot{z}^i A_i + V \),

(ii) the expansion \( x(t) = z(t) + \sum_{k=1}^{\infty} v^k \sin k \pi t / T \), where \( z(t) \) is a solution of the equation of motion \( \ddot{z} = 0 \) with the boundary conditions \( z(0) = x \) and \( z(T) = y \), and

(iii) the measure which normalizes the Gaussian integration with \( S_0 \) over the modes \( v^k \) to \((2\pi \hbar T)^{-1/2}\) (not to unity because there is always one more intermediate set \( |p><p| \) in Feynman’s approach than intermediate sets \( |x><x| \). The remaining factor \((2\pi \hbar T)^{-1/2}\) combines with classical part \( \exp(-x - y)^2 / 2\hbar T \) to yield a representation of \( \delta(x - y) \) for small \( T \).

One must then show that the limit \( N \to \infty \) in \( S_{\text{int}} \) yields the interaction of the continuum theory. This is a well-known complicated problem, but we shall present here a totally elementary proof which uses only trigonometric relations such as \( 2 \sin \alpha \sin \beta = \cos(\alpha - \beta) - \cos(\alpha + \beta) \) and the fact that the infinite series we encounter are uniformly convergent as functions of \( N \). This property allows us to take the limit \( N \to \infty \) before the summations are performed. For the Hamiltonians of the form \( T(p) + V(x) \) such a proof is quite simple, but for non-vanishing vector potential \( A(x) \), we need rather laborious algebra.
The result is surprisingly simple. All terms in the propagator which in the Hamiltonian approach are due to commutators, are only needed to make sure that in the limit $N \to \infty$ one obtains the classical action. To be more explicit, consider the discretized action in (40). The last three lines vanish in the naive limit $N \to \infty$ (the limit $N \to \infty$ for fixed mode index $k$) since they have extra factors $1/N$ w.r.t. the two lines above. These latter two lines naively yield the term $\int dt A_j(x) \dot{x}^j$ in the classical action because $1/N$ becomes $dt$ and $(\xi_{\alpha-1} - \xi_{\alpha})$ becomes $\dot{\xi} dt$. The claim is that if one does not take the naive limit but carefully evaluates the sums, then the non-naive terms in the first two lines cancel all of the last three lines in (40). To discuss in more detail what we mean by the naive limit $N \to \infty$ consider the interaction terms

$$S_{\text{int}} = \frac{1}{N} \sum_{k,l=1}^{N-1} v^k v^l \lambda(k) \lambda(l) \sum_{\alpha=1}^{N-1} \sin \alpha k \pi/N \sin \alpha l \pi/N,$$

where $\lambda(k) = k \pi [2N^2(1 - \cos k \pi/N)]^{-1/2}$. For fixed $k$ and $N$ tending to infinity one finds, $\lambda(k) = 1$ while for $k \sim N$ tending to infinity one has $\lambda(k) = \pi/2$. One may take the limit $N \to \infty$ in $S_{\text{naive}}(N)$ for given fixed $k$ and $l$, because the error thus committed cancels against the terms in $S(N) - S_{\text{naive}}(N)$. Here $S_{\text{naive}}(N)$ is the time-discretized action we would have obtained, if we had ignored the terms coming from commutators in the Hamiltonian approach.

The non-trivial measure factorizes into a factor for each mode $v^k$. One can then easily compute propagators $\langle v^k v^l \rangle$ and Feynman graphs in terms of modes. One can also use the quantum “fields” $\xi(\tau)$ and find that $\langle \xi(\tau_1) \xi(\tau_2) \rangle$ is the expected world line propagator (the inverse of $\partial^2/\partial \tau^2$ with the correct boundary conditions). However, the mode representation is to be preferred because mode cut-off is the natural regularization scheme\[5, 6\]. Actually, all one-loop diagrams we evaluate are already finite by themselves since the divergences of the tadpole graphs are put to zero by mode regularization. Although we do not consider here curved space, we mention for completeness that in curved spacetime there are extra “ghosts” obtained by exponentiating a factor $(\det g_{ij})^{1/2}$ in the measure and that with these ghosts all loop calculations become finite if one uses mode regularization\[5, 6, 9\].

From our point of view, the ambiguities often encountered in the definition of path integrals are due to starting “halfway”. Starting from the beginning, which means for us starting with the Hamiltonian approach, no ambiguities result and one derivesthe action to be used in the path integral.
The result is the 1-1 correspondence

\[ \hat{H} = \frac{1}{2} \left( \hat{p}_i - \left( \frac{e}{c} \right) \hat{A}_i \right) \delta^{ij} \left( \hat{p}_j - \left( \frac{e}{c} \right) \hat{A}_j \right) + \hat{V} \]

and the path integral is perturbatively evaluated by computing Feynman graphs with given propagators and vertices.

In section 2 we discuss the Hamiltonian approach for a point particle coupled to electromagnetism. Although we only need the propagator to order \( \Delta t \) in order to construct the corresponding path integral, we evaluate it to order \( (\Delta t)^2 \) in order to compare later with a similar result obtained from the path integral. A useful check is that it factorizes into a classical part and a Van Vleck determinant. In section 3, the path integral is cast into a form where the only \( N \)-dependence resides in the interaction terms \( S_{\text{int}} \). In section 4, we discuss the limit \( N \to \infty \) in \( S_{\text{int}} \). We organize the discussion by giving six examples which cover all possible cases one encounters in a perturbative evaluation of the path integral. In section 5 we evaluate as a check the path integral to order \( (\Delta t)^2 \) at the one-loop level. Here we discuss how to evaluate the continuum path integrals in general in perturbation theory. The result agrees with the one obtained in section 2 from the Hamiltonian approach. In section 6 we note that our work straightforwardly extends to field theories with derivative coupling like Yang-Mills theory. We discuss how our work might be extended to curved spacetime, and also to phase space path integrals.

### 2 Hamiltonian operator approach.

We wish to evaluate the propagator in Euclidean space

\[ < x | \exp(-\Delta t \hat{H}/\hbar)|y >, \]  

where

\[ \hat{H} = \frac{1}{2} \left( \hat{p}_i - \left( \frac{e}{c} \right) A_i(\hat{x}) \right) \left( \hat{p}^i - \left( \frac{e}{c} \right) A^i(\hat{x}) \right), \ i = 1, \ldots, n. \]  

We do not add a term \( V(x) \) since the analysis for this term is the same as for the term \( (A_i(x))^2 \). Indices are raised and lowered by \( \delta_{ij} \), so for notational
simplicity we write all indices down. We shall only use the commutation relation $[\hat{p}_i, \hat{x}_j] = \hbar \delta_{ij}$ and $\hat{p}_i|p> = p_i|p>$. We rewrite the Hamiltonian as

$$\hat{H} = \hat{\alpha} - (\frac{e}{c})\hat{\beta} + (\frac{e}{c})^2\hat{\gamma},$$

(14)

where

$$\hat{\alpha} = \frac{1}{2}\hat{p}^2, \quad \hat{\beta} = \hat{A} \cdot \hat{\mathbf{p}}, \quad \hat{\gamma} = \frac{1}{2} \left( i\frac{\hbar c}{e} \partial \cdot \hat{A} + \hat{A}^2 \right).$$

Following Feynman we insert a complete set of $|p>$ states

$$<x|\exp(-\Delta t\hat{H}/\hbar)|y> = \int d^n p <x|\exp(-\Delta t\hat{H}/\hbar)|p><p|y>.$$ (15)

We expand the exponential and define

$$<x|\hat{H}^k|p> = \sum_{l=0}^{2k} B^k_l(x)p^l <x|p>,$$ (16)

where $B^k_l(x)p^l$ is a polynomial of degree $l$ in $p$'s, and

$$<x|p> = (2\pi\hbar)^{-n/2} \exp(i\frac{\hbar}{x\cdot p}).$$ (17)

After rescaling the momenta as $p = \sqrt{\hbar/\Delta t} q$ we have

$$<x|\exp(-\Delta t\hat{H}/\hbar)|y> = (4\pi^2\hbar\Delta t)^{-n/2} \int d^n q \exp(i\frac{q \cdot (x-y)}{\sqrt{\hbar\Delta t}}) \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \sum_{l=0}^{2k} B^k_l(x)q^l (\frac{\Delta t}{\hbar})^{k-l/2}.$$ (18)

The leading term comes from summing all the terms with $l = 2k$ and has the simple form $\exp(-\frac{1}{2}q^2)$. For this reason we introduced the $q$ variable. It follows that only a finite number of $B$'s need to be calculated in order to obtain the result up to desired order in $\Delta t$. In particular, the result up to and including $(\Delta t)^2$ needs the first five $B$'s ($l = 2k$ through $l = 2k - 4$). A detailed discussion of the combinatorics is given in [9]. Here we merely give our result.

$$B^k_{2k}(x)q^{2k} = \alpha^k,$$ (19)

$$B^k_{2k-1}(x)q^{2k-1} = -(\frac{e}{c})k\alpha^{k-1}\beta,$$ (20)
\[
B_{2k-2}(x)q^{2k-2} = (\frac{e}{c})^2 k \alpha^{k-1} \gamma + \\
(\frac{e}{c})^2 \left( \frac{k}{2} \right) \alpha^{k-2} \left[ \left( \frac{i\hbar c}{e} \right) q_i (\partial_i \beta) + \beta^2 \right], \quad (21)
\]

\[
B_{2k-3}(x)q^{2k-3} = - (\frac{e}{c})^3 \left( \frac{k}{2} \right) \alpha^{k-2} \left[ \frac{1}{2} \left( \frac{i\hbar c}{e} \right)^2 \partial^2 \beta + \left( \frac{i\hbar c}{e} \right) q_i \partial_i \gamma \\
+ 2 \beta \gamma + \left( \frac{i\hbar c}{e} \right) A_i (\partial_i A_j) q_j \right] \\
- (\frac{e}{c})^3 \left( \frac{k}{3} \right) \alpha^{k-3} \left[ \left( \frac{i\hbar c}{e} \right)^2 q_i q_j \partial_i \partial_j \beta \\
+ 3 \left( \frac{i\hbar c}{e} \right) \beta q_i \partial_i \beta + \beta^3 \right]. \quad (22)
\]

\[
B_{2k-4}(x)q^{2k-4} = (\frac{e}{c})^4 \left( \frac{k}{2} \right) \alpha^{k-2} \left[ \frac{1}{2} \left( \frac{i\hbar c}{e} \right)^2 \partial^2 \gamma + \left( \frac{i\hbar c}{e} \right) A_i \partial_i \gamma + \gamma^2 \right] \\
+ (\frac{e}{c})^4 \left( \frac{k}{3} \right) \alpha^{k-3} \left[ \left( \frac{i\hbar c}{e} \right)^3 q_i (\partial_i \partial^2 \beta) \\
+ \left( \frac{i\hbar c}{e} \right)^2 \left[ q_i q_j (\partial_i \partial_j \gamma) + (\partial_i \beta)(\partial_i \beta) \right] + 3 \beta \partial^2 \beta + q_i (\partial_i A_j) (\partial_j \beta) + 2 q_i A_j (\partial_i \partial_j \beta) \right] \\
+ \left( \frac{i\hbar c}{e} \right) \left[ 3 \beta q_i (\partial_i \gamma) + 3 \gamma q_i (\partial_i \beta) \right] \\
+ 3 A_i (\partial_i \beta) + 3 \beta^2 \gamma \right] \\
+ (\frac{e}{c})^4 \left( \frac{k}{4} \right) \alpha^{k-4} \left[ \left( \frac{i\hbar c}{e} \right)^3 q_i q_j q_k (\partial_i \partial_j \partial_k \beta) \\
+ \left( \frac{i\hbar c}{e} \right)^2 \left[ 3 (\partial_i \beta)(\partial_j \beta) q_i q_j + 4 (\partial_i \partial_j \beta) \beta q_i q_j \right] \\
+ 6 \left( \frac{i\hbar c}{e} \right) \beta^2 (\partial_i \beta) q_i + \beta^4 \right]. \quad (23)
\]
The combinatorial factors
\[ \binom{k}{s} \]
indicate that only \( s \) out of \( k \) factors of \( \hat{H} \) are involved in yielding commutators. For example, the last term in (21) is due to picking two factors \( \beta \) and \( k - 2 \) factors of \( \alpha \) out of \( k \) factors \( \hat{H} \). Clearly this can be done in
\[ \binom{k}{2} \]
ways and there are two powers of \( q \) less than in the leading term. Similarly, the one but last term in (21) is due to one commutator of \( \hat{\alpha} \) and \( \hat{\beta} \).

Next we perform the summations over \( k \) which is easy and the Gaussian integrals which are straightforward but tedious. The result reads

\[
<x | \exp(-\Delta t \hat{H}/\hbar)|y> = (2\pi\hbar\Delta t)^{-n/2} \exp \left( -\frac{1}{2\Delta t\hbar}(x - y)^2 \right)
\]

\[
\begin{align*}
&\left\{ 1 - \left( \frac{-ie}{\hbar c} \right) A_i(x - y)_i \\
&\quad + \frac{1}{2} \left[ \left( \frac{-ie}{\hbar c} \right) A_{i,j} + \left( \frac{-ie}{\hbar c} \right)^2 A_i A_j \right] (x - y)_i(x - y)_j \\
&\quad + \frac{1}{3!} \left[ \left( \frac{-ie}{\hbar c} \right) A_{i,j,k} + 3 \left( \frac{-ie}{\hbar c} \right)^2 A_{i,j} A_k \right] (x - y)_i(x - y)_j(x - y)_k \\
&\quad + \frac{1}{4!} \left[ \left( \frac{-ie}{\hbar c} \right) A_{i,j,k,l} + 3 \left( \frac{-ie}{\hbar c} \right)^2 A_{i,j} A_{k,l} + 4 \left( \frac{-ie}{\hbar c} \right)^2 A_{i,j,k} A_l \\
&\quad \quad + 6 \left( \frac{-ie}{\hbar c} \right)^4 A_{i,j} A_{k,l} A_l \right] (x - y)_i(x - y)_j(x - y)_k(x - y)_l \\
&\quad + \frac{1}{4!} \frac{\Delta t}{\hbar} \left( \frac{e}{c} \right)^2 F_{ik} F_{kj}(x - y)_i(x - y)_j \\
&\quad + \frac{i\Delta t}{12} \frac{e}{c} \left[ F_{ki,k}(x - y)_i - \frac{1}{2} F_{ki,kj}(x - y)_i(x - y)_j \right] \\
&\quad - \frac{1}{12} \frac{\Delta t}{\hbar} \left( \frac{e}{c} \right)^2 A_i F_{kj,k}(x - y)_i(x - y)_j - \frac{(\Delta t)^2}{48} \left( \frac{e}{c} \right)^2 F^2 \right\}. 
\end{align*}
\] (24)
Before going on, we briefly compare this result with the incorrect result which one would have obtained by the linear approximation mentioned in the introduction and widely used. In the latter case we find instead of the terms in the curly brackets the following expression,

\[
< x | \exp(-\Delta t \hat{H}/\hbar) | y > = (2\pi \hbar \Delta t)^{-n/2} \exp \left( -\frac{1}{2\hbar \Delta t} (x - y)^2 \right) \{ 1 - \left( -\frac{ie}{\hbar c} \right) A_i(x - y)_i + \frac{1}{2} \left( -\frac{ie}{\hbar c} \right)^2 A_i A_j(x - y)_i(x - y)_j \\
- \frac{1}{2} i \Delta t \left( \frac{e}{c} \right) \partial \cdot A \} \text{(false).} \tag{25}
\]

This result is obtained by replacing \( \int dp < x | \exp(-\Delta t \hat{H}/\hbar) | p > < p | y > \) by \( \int dp \exp(-\Delta t \hbar(x, p)/\hbar) < x | p > < p | y > \), where \( h(x, p) \) is defined in (4).

The \( p \)-dependence in the exponent is coming from the \( p^2 \) term in \( \alpha \), a single \( p \) in \( \beta \) and the inner product \( p \cdot (x - y) \) from the plane waves. Integration over \( p \) yields (25) to order \( \Delta t \). To order \( \Delta t \) we thus find the same number of terms in both cases, but the term \( \partial \cdot A \) is present only in the linear approximation, whereas in the correct approach it is cancelled by a commutator \([p^2, \beta]\). This commutator yields a term \( p_i p_j \partial_i A_j \) whose integration over \( p \) gives a \( \delta_{ij} \) term which cancel the \( \partial \cdot A \) term, and a term with \( (x - y)_i(x - y)_j \) which is the term with \( A_{i,j} \) in (24). The corrections due to commutators already show up at order \( \Delta t \). Clearly, the linear approximation gives already incorrect results for the propagator at order \( \Delta t \).

We expect the result in (24) to contain a factor of \( \exp\left[-\frac{1}{\pi} S_{cl}\right] \), where \( S_{cl} \) is the classical action evaluated along a classical trajectory. We claim that to order \( (\Delta t)^2 \) it reads

\[
S_{cl} = \frac{1}{2\Delta t} (x - y)_i(x - y)_i - i \left( \frac{e}{c} \right) \{ A_i(x - y)_i - \frac{1}{2} A_{i,j}(x - y)_i(x - y)_j \\
+ \frac{1}{3!} A_{i,j,k}(x - y)_i(x - y)_j(x - y)_k \\
- \frac{1}{4!} A_{i,j,k,l}(x - y)_i(x - y)_j(x - y)_k(x - y)_l \}
- \frac{\Delta t}{24} \left( \frac{e}{c} \right)^2 F_{ik} F_{kj}(x - y)_i(x - y)_j + 0((\Delta t)^{5/2}). \tag{26}
\]

To obtain this result, we used that the classical Lagrangian corresponding to (13) is given by

\[
L = \frac{1}{2} \dot{x}_i \dot{x}^i - i \left( \frac{e}{c} \right) \dot{x}^i A_i, \tag{27}
\]
where the factor ‘i’ is due to our working in Euclidean space. (The simplest way to see this is to note that one has the same Hamiltonian in both the Minkowski and the Euclidean case, but one uses \(\exp(-i\Delta t\hat{H}/\hbar)\) in the former and \(\exp(-\Delta t\hat{H}/\hbar)\) in the latter case. In both cases \(\langle x|p >\) and \(< p|y >\) are plane waves.) The dynamical equations of motion are

\[
\ddot{x}_i = -i(e/c)F_{ij}\dot{x}_j.
\]  

(28)

We then evaluated \(S_{cl}\) by expanding all fields around the endpoint \(x\),

\[
S_{cl} = \int_{-\Delta t}^{0} L dt = \Delta t L(x) - \frac{1}{2}(\Delta t)^2 \frac{dL(x)}{dt} + \frac{1}{3!}(\Delta t)^3 \frac{d^2L(x)}{dt^2} - \frac{1}{4!}(\Delta t)^4 \frac{d^3L(x)}{dt^3} + \cdots.
\]  

(29)

We only need \(x(t)\) and \(\dot{x}(t)\) at \(t = 0\) since higher time derivatives of \(x(t)\) can be obtained by using the equations of motion (28). To obtain \(\dot{x}(t = 0)\) in terms of \(x\) and \(y\), we expand \(x(-\Delta t)\) around \(t = 0\), and use (28). This yields a series in power of \(\dot{x}(0)\), \(x_i\) and \(y_i\) which is inverted to yield \(\dot{x}_i(0)\) in terms of \(x_i\) and \(y_i\). For our purposes it is sufficient to determine \(\dot{x}_i(0)\) to order \((\Delta t)^{3/2}\). We find

\[
\dot{x}_i(t = 0) = \frac{1}{\Delta t}(x-y)_i - i\left(\frac{e}{c}\right)\left\{\frac{1}{2}F_{ij}(x-y)_j - \frac{1}{6}F_{ij,l}(x-y)_j(x-y)_l + \frac{1}{24}F_{ij,kl}(x-y)_j(x-y)_k(x-y)_l \right\}.
\]  

(30)

This result combined with (27) and (29) leads to (26).

Factoring out \(\exp[-\frac{1}{\hbar}S_{cl}]\) from (24) we left with

\[
\langle x|\exp(-\Delta t\hat{H}/\hbar)|y > = (2\pi\hbar\Delta t)^{-n/2}\exp[-\frac{1}{\hbar}S_{cl}]
\exp\left(i\frac{\Delta t}{12}\left(\frac{e}{c}\right)[F_{ki,k}(x-y)_i - \frac{1}{2}F_{ki,kj}(x-y)_i(x-y)_j]\right) - \frac{\Delta t^2}{48}\left(\frac{e}{c}\right)^2F^2.
\]  

(31)

We expect also a factor of \((\det D_{ij})^{1/2}\) to be present in the propagator, where \(D_{ij}\) is the Van Vleck matrix

\[
D_{ij}(x,y;\Delta t) = -\frac{\partial}{\partial x_i}\frac{\partial}{\partial y_j}S_{cl}(x,y;\Delta t).
\]  

(32)
In fact, the remaining terms are just the Van Vleck determinant, and no other terms are present to order \((\Delta t)^2\). Note that the \(F^2\) term which yields the trace anomaly in two dimensions\(^3\), is part of the Van Vleck determinant, whereas a corresponding term \(\bar{h}R\) in curved spacetime is not contained in the corresponding Van Vleck determinant. This is not surprising since the \(F^2\) is a one-loop effect whereas the \(\bar{h}R\) is a two loop effect.

Our final result, thus, reads

\[
<x|\exp(-\Delta t \hat{H}/\hbar)|y> = (2\pi\hbar)^{-n/2}(\det D_{ij})^{1/2}\exp[-\frac{1}{\hbar}S_{cl}][1 + O((\Delta t)^{5/2})].
\]  

(33)

3 Derivation of the path integral.

The time-discretized path integral with \(N - 1\) intermediate steps is given by

\[
<x|\exp(-T \hat{H}/\hbar)|y> = \lim_{N \to \infty} \left(\frac{N}{2\pi\hbar T}\right)^{n/2} \int \prod_{i=1}^{N} \prod_{\alpha=1}^{N-1} \left(dx_{\alpha i}\left(\frac{N}{2\pi\hbar T}\right)^{1/2}\right)\exp\left\{-\frac{1}{2\epsilon\hbar}\sum_{\alpha=1}^{N}(x_{\alpha-1} - x_{\alpha})^2 + \frac{ie}{\hbar c} \sum_{\alpha=1}^{N} A_i(x_{\alpha-1})(x_{\alpha-1} - x_{\alpha})_i - \frac{ie}{2\hbar c} \sum_{i=1}^{N} A_{i,j}(x_{\alpha-1})(x_{\alpha-1} - x_{\alpha})_i(x_{\alpha-1} - x_{\alpha})_j\right\},
\]  

(34)

where \(\alpha\) is the discretization index and letters from the middle of the latin alphabet like i, j, k etc. are spacetime indices. \(\epsilon \equiv T/N\) and \(x_0 \equiv x, x_N \equiv y\).

To obtain this result we inserted \(N - 1\) complete sets of states \(|x_{\alpha}>\) and used the result \((24)\) for the matrix element \(<x_{\alpha-1}|\exp(-\epsilon\hat{H}/\hbar)|x_{\alpha}>\) obtained from the Hamiltonian approach. We kept only the terms up to order \(\epsilon\) (the first three lines in \((24)\)), because only these terms will contribute in the limit \(N \to \infty\). We decompose \(x_{\alpha i}\) as

\[
x_{\alpha i} = z_{\alpha i} + \xi_{\alpha i}.
\]  

(35)

The \(z_{\alpha i}\) yield the classical trajectory of a free particle and satisfy the equation

\[
z_{(\alpha+1)i} - 2z_{\alpha i} + z_{(\alpha-1)i} = 0,
\]  

(36)

with boundary conditions

\[
z_{0i} = x_i, z_{Ni} = y_i.
\]  

(37)
In the limit $N \to \infty$ (36) becomes the field equation of the action for a free particle. (36) with the boundary conditions (37) can be solved to yield

$$z_{\alpha i} = x_i + \frac{\alpha}{N}(y - x)_i. \quad (38)$$

The $\xi$'s are the quantum fluctuations with boundary conditions $\xi_0i = \xi_Ni = 0$. We go over to the mode variables using the transformation

$$\xi_{\alpha i} = \sum_{k=1}^{N-1} y_i^k \sin \frac{\alpha k \pi}{N}. \quad (39)$$

The path integral becomes

$$\exp\left(-\frac{(x - y)^2}{2\hbar T}\right) \lim_{N \to \infty} \left(\frac{N}{2\pi \hbar T}\right)^{n/2} \int \prod_{i=1}^{n} \prod_{k=1}^{N-1} \left[dy_i^k \left(\frac{N^2}{4\pi \hbar T}\right)^{1/2}\right]$$

$$\exp\left(-\frac{N^2}{2\hbar T} \sum_{k=1}^{N-1} (y_i^k)^2 (1 - \cos k \pi / N)\right)$$

$$\exp\left(\frac{ie}{\hbar c} \sum_{\alpha=1}^{N} \left\{ A_i(z_{\alpha - 1}) + A_{i,j}(z_{\alpha - 1})\xi_{(\alpha - 1)j} + \frac{1}{2} A_{i,jk}(z_{\alpha - 1})\xi_{(\alpha - 1)j}\xi_{(\alpha - 1)k} + \cdots \right\} \left[ \frac{1}{N} (x - y)_i + (\xi_{(\alpha - 1)i} - \xi_{\alpha i}) \right]$$

$$- \frac{1}{2} \left[ A_{i,j}(z_{\alpha - 1}) + A_{i,jk}(z_{\alpha - 1})\xi_{(\alpha - 1)k} \right]$$

$$+ \frac{1}{2} A_{i,jkl}(z_{\alpha - 1})\xi_{(\alpha - 1)k}\xi_{(\alpha - 1)l} + \cdots \right\} x \left[ \frac{1}{N} (x - y)_i + (\xi_{(\alpha - 1)i} - \xi_{\alpha i}) \right] \left[ \frac{1}{N} (x - y)_j + (\xi_{(\alpha - 1)j} - \xi_{\alpha j}) \right] \right\}, \quad (40)$$

where the $\xi$'s are functions of the modes $y_i^k$ as in (38). A summation over $i = 1, \ldots, n$ is understood in all terms in the exponent. We have used that the matrix $M_{\alpha k} = \sqrt{2/N} \sin \alpha k \pi / N$ is an orthogonal matrix. This produces the extra factor of $N/2$ in the measure for $y_i^k$. We rescale the modes

$$v_i^k = \left(\frac{2N^2(1 - \cos \frac{k \pi}{N})}{k^2 \pi^2}\right)^{1/2} y_i^k$$

$$\equiv \lambda(k)^{-1} y_i^k. \quad (41)$$

The kinetic term becomes

$$\exp \left( - \sum_{k=1}^{N-1} \frac{(k \pi)^2}{4\hbar T} (v_i^k)^2 \right), \quad (42)$$
while the measure becomes
\[
\left( \frac{N}{2\pi \hbar T} \right)^{n/2} \prod_{i=1}^{n} \prod_{k=1}^{N-1} \left( \frac{N^2}{4\pi \hbar T} \right)^{1/2} \left( \frac{k^2 \pi^2}{2N^2 (1 - \cos \frac{k\pi}{N})} \right)^{1/2} dv_i^k. \tag{43}
\]
This expression can be simplified by using the product formula
\[
\prod_{k=1}^{N-1} 2(1 - \cos k\pi/N) = N, \tag{44}
\]
which is a special case \((x \to 1)\) of the formula
\[
\left[ \prod_{k=1}^{N-1} (x - \cos k\pi/N) \right]^2 = \frac{2^{1-2N}}{x^2 - 1} \text{Re} \left[ -1 + (x + i\sqrt{1 - x^2})^{2N} \right]. \tag{45}
\]
To derive this formula, one uses that \(x^2 - 1\) times the left hand side is proportional to \(\prod_{k=1}^{2N-1} (x - \cos k\pi/N)\). The measure now becomes
\[
(2\pi \hbar T)^{-n/2} \prod_{i=1}^{n} \prod_{k=1}^{N-1} \left( \frac{\pi k^2}{4\hbar T} \right)^{1/2} dv_i^k. \tag{46}
\]
Thus, the \(N\) dependence of the kinetic term and the measure have disappeared after the rescaling (the \(N\) appears only in the upper limit of the sum) and the \(N \to \infty\) limit can be easily taken. One finds
\[
(2\pi \hbar T)^{-n/2} \prod_{i=1}^{n} \prod_{k=1}^{\infty} \left( \frac{\pi k^2}{4\hbar T} \right)^{1/2} dv_i^k, \tag{47}
\]
for the measure and
\[
\exp \left( -\frac{1}{2\hbar T} \int_{-1}^{0} d\tau \dot{\xi}^2 \right) = \exp \left( -\sum_{k=1}^{\infty} \frac{(k\pi)^2}{4\hbar T} (v_i^k)^2 \right), \tag{48}
\]
for the kinetic term, where \(\xi_i\) is the continuum limit of (39)
\[
\xi_i(\tau) = \sum_{k=1}^{\infty} v_i^k \sin k\pi \tau. \tag{49}
\]
The propagator for the modes obtained from the kinetic term reads
\[
\langle v_i^m v_j^n \rangle = \frac{2\hbar T}{\pi^2 \hbar^2} \delta_{ij} \delta^{mn}. \tag{50}
\]
At this point we have obtained the measure of the continuum theory, given in (4), and the kinetic term and the propagators for the modes are also that of the continuum theory. It remains to take the limit in the interaction terms.
4 The limit $N \to \infty$ in the interaction terms.

The interaction terms in (40) can be recast as follows

$$\exp \frac{ie}{\hbar c} \sum_{\alpha=1}^{N} \left\{ A_i(z_{\alpha-1}) + A_{i,j}(z_{\alpha-1})\xi(\alpha-1)j + \cdots \right\},$$

where according to (39) and (41) we have now

$$\xi_{\alpha i} = \sum_{k=1}^{N-1} v_k^i \lambda(k) \sin \alpha k \pi/N.$$ (52)

The first line in (51) is coming solely from the $A_i(x-y)_i$ term in (40), whereas the rest is a combination of both the $A_i(x-y)_i$ and the $A_{i,j}(x-y)_i(x-y)_j$ terms. Actually only one part of the latter contributes, namely the one which is proportional to $(\xi(\alpha-1)_i - \xi_{\alpha i})(\xi(\alpha-1)_j - \xi_{\alpha j})$. The rest tends to zero when $N \to \infty$, as will be clear at the end of this section. These terms are not shown in (51).

We now proceed to show that the first line in (51) limits to

$$\exp \frac{ie}{\hbar c} \int_{-1}^{0} d\tau A_i(x(\tau))(x-y)_i,$$ (53)

whereas the rest of (51) limits to

$$\exp \frac{ie}{\hbar c} \int_{-1}^{0} d\tau A_i(x(\tau))\dot{\xi}_i.$$ (54)

The $x(\tau)$ is the continuum limit of (35)

$$x_i(\tau) = z_i(\tau) + \xi_i(\tau),$$ (55)

where $z_i(\tau)$ is the continuum limit of (38)

$$z_i(\tau) = x_i - \tau(y-x)_i,$$ (56)
and $\xi_i(\tau)$ is given in (49).

There are eight different kinds of terms which we encounter trying to take the limit $N \to \infty$ in (51). We can have terms with or without $\dot{\xi}$. Each of them can contain an even or odd number of quantum fields. (In the latter case only interference terms can be studied since the expectation value of odd number of quantum fields is trivially zero). Finally, in each case we can also have an additional factor of $(\alpha/N)^p$ coming from the expansion of $A_i(z_\alpha)$ around $x_i$ (see (38)). We will illustrate with six examples how the limit $N \to \infty$ can be rigorously taken in all cases. The basic idea is that expanding (51) leads to uniformly convergent series for $N$ in the whole interval $1 \leq \hat{N} < \infty$, and therefore the limit $N \to \infty$ can be taken before the summation over the modes will be performed.

4.1 Examples with only $\xi$’s.

The case of only $\xi$’s is relatively easier than the case where $\dot{\xi}$’s are involved. This case covers all the terms in the first line in (51) and also all the extra terms we would have if we had started with an additional scalar potential $V(x)$. We give two examples where two $\xi$’s are involved. In the first one the two $\xi$’s are coming from the same $S_{\text{int}}$ whereas in the second case we deal with an interference term. We use the latter case to illustrate how one deals with factors like $(\alpha/N)^p$.

4.1.1 Example 1.

We shall show that

$$\lim_{N \to \infty} \left( \frac{1}{N} \sum_{\alpha=1}^{N-1} \xi_{\alpha i} \xi_{\alpha j} \right) = \left\langle \int_{-1}^{0} d\tau \xi_i(\tau) \xi_j(\tau) \right\rangle,$$  \hspace{1cm} (57)

where $\langle \rangle$ means path integral average. We start with the left hand side

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k,l=1}^{N-1} \langle v^k v^l \rangle \lambda(k) \lambda(l) \sum_{\alpha=1}^{N-1} \sin \alpha k \pi / N \sin \alpha l \pi / N.$$  \hspace{1cm} (58)

The propagator is given in (50) and the $\lambda(k)$ in (41). Combining the product of the two sines into a sum of two cosine functions, the summation over $\alpha$ yields $N/2 \delta_{kl}$. Hence, the left-hand side of (57) yields

$$\frac{1}{2} (2\hbar T) \delta_{ij} \lim_{N \to \infty} \sum_{k=1}^{N-1} \frac{1}{4N^2 \sin^2 k \pi / 2N}.$$  \hspace{1cm} (59)
We remove the $N$-dependence in the summation symbol by extending the sum to infinity, rewriting the sum as
\[ \sum_{k=1}^{\infty} f_k(N), \] (60)
where
\[ f_k(N) = \begin{cases} 0 & \text{if } k > N - 1 \\ 1/(4N^2 \sin^2 k\pi/2N) & \text{if } k \leq N - 1. \end{cases} \]

We view $f_k(N)$ as a function of $N$. Since $k \leq N - 1$, clearly $k\pi/2N < \pi/2$. Using the inequality $2\theta/\pi \leq \sin \theta \leq \theta$ for $0 \leq \theta \leq \pi/2$ we get an upper bound for the summands
\[ |f_k(N)| \leq \frac{1}{4N^2(k^2/N^2)} = \frac{1}{4k^2}. \] (61)

Since the series $\sum_{k=1}^{\infty} (2k)^{-2}$ is convergent, we conclude that (60) is uniformly convergent in $N$ for the whole interval $1 \leq N < \infty$. Thus, we can interchange the limit of $N$ tending to infinity with the summation over $k$.

Using (50), we obtain
\[ \frac{1}{2}(2\bar{h}T)\delta_{ij} \sum_{k=1}^{\infty} \frac{1}{k^2\pi^2} = \sum_{k,l=1}^{\infty} \langle v_k^i v_j^l \rangle \left( \frac{1}{2} \delta^k \right) = \langle \int_{-1}^{0} d\tau \xi_i(\tau)\xi_j(\tau) \rangle. \] (62)

This proves (57).

**4.1.2 Example 2.**

In our second example we will prove that
\[ \lim_{N \to \infty} \frac{1}{N^2} \sum_{\alpha,\beta=1}^{N} \left( \frac{\alpha - 1}{N} \right) \left( \frac{\beta - 1}{N} \right) \xi_{(\alpha - 1)j} \xi_{(\beta - 1)j} = \langle \int_{-1}^{0} d\tau d\tau' \xi_i(\tau)\xi_j(\tau') \rangle. \] (63)

This term is encountered when we expand the term with $A_{i,j}(z_{\alpha-1})\xi_{(\alpha-1)j}$ around $x_i$ in the first line in (61), and then use two $S_{int}$. We start again with the left hand side
\[ \lim_{N \to \infty} \sum_{k,l=1}^{N-1} \langle v_i^k v_j^l \rangle \lambda(k)\lambda(l) \frac{1}{N^2} \sum_{\alpha,\beta=1}^{N-1} \left( \frac{\alpha}{N} \right) \left( \frac{\beta}{N} \right) \sin \alpha k \pi/N \sin \beta l \pi/N. \] (64)
The summation over $\alpha$ and $\beta$ can be easily performed by observing that all the cases with $(\alpha/N)^p$ factors can be obtained from the ones with no such factors by just introducing temporarily an extra parameter $r$ in the argument of one of the sines and then differentiating appropriate number of times. So, in our case we write

$$\frac{1}{N} \sum_{\alpha=1}^{N-1} \left( \frac{\alpha}{N} \right) \sin \frac{\alpha k \pi}{N} = -\frac{1}{k \pi N} \frac{d}{dr} \left. \sum_{\alpha=1}^{N-1} \cos r \frac{\alpha k \pi}{N} \right|_{r=1}. \tag{65}$$

The summation over $\alpha$ can be easily performed by writing the cosine as the real part of an exponential. The result is

$$\frac{1}{N} \sum_{\alpha=1}^{N-1} \left( \frac{\alpha}{N} \right) \sin \frac{\alpha k \pi}{N} = -\frac{(-1)^k}{2N \tan k \pi/2N}. \tag{66}$$

Using (66), (50) and (41), (64) becomes

$$\lim_{N \to \infty} \delta_{ij} \frac{h(T)}{N} \sum_{k=1}^{N-1} \frac{1}{4N^2 \sin^2 k \pi/2N^2} \frac{\cos^2 k \pi/2N}{4N^2 \sin^2 k \pi/2N}. \tag{67}$$

Using the same arguments as in the first example we conclude that the series over $k$ is uniformly convergent. Therefore the limit $N \to \infty$ can be taken keeping $k$ fixed. The result is

$$\sum_{k,l=1}^{\infty} \frac{2h(T)}{k^2 \pi^2} \delta_{ij} \delta_{kl} \left[ \frac{(-1)^k}{k \pi} \right] \left[ \frac{(-1)^l}{l \pi} \right] = \int_{-1}^{0} d\tau d\tau' \tau' \xi_i(\tau) \xi_j(\tau'), \tag{68}$$

which proves (63).

The generalization of these two examples to many $\xi$’s is straightforward. In every case we first reduce the summation of product of sines to the summations of a single sine or cosine by using the trigonometric formulas

$$\sin a \sin b = \frac{1}{2} \left[ \cos(a - b) - \cos(a + b) \right], \tag{69}$$
$$\sin a \cos b = \frac{1}{2} \left[ \sin(a + b) + \sin(a - b) \right]. \tag{70}$$

Then we use the results of our previous examples.
4.2 Examples with $\xi$’s and $\dot{\xi}$’s.

The case where a $\dot{\xi}$ is involved is more complicated. One naively expects that in the limit $N \to \infty$ the sum $\sum_{\alpha=0}^{N-1} (\xi_{\alpha j_1} - \xi_{(\alpha+1)j_1}) \xi_{\alpha j_2} \cdots \xi_{\alpha j_q}$ becomes $(-1)^{q+1} \int d\tau \dot{\xi}_{j_1}(\tau) \xi_{j_2}(\tau) \cdots \xi_{j_q}(\tau)$. (The factor $(-1)^{q+1}$ is due to the fact that $\xi_{(\alpha+1)j}$ corresponds to a $\tau$-value which is smaller than that of $\xi_{\alpha j}$.) Actually this would have been true if we were allowed to take the limit $N \to \infty$ inside the summation (for $k$ fixed). To see this we insert the mode expansion for $\xi$ into the sum

$$\sum_{\alpha=0}^{N-1} (\xi_{\alpha j_1} - \xi_{(\alpha+1)j_1}) \xi_{\alpha j_2} \cdots \xi_{\alpha j_q} = \sum_{k_1, \ldots, k_q=1}^{N-1} v_{j_1}^{k_1} \cdots v_{j_q}^{k_q} \lambda(k_1) \cdots \lambda(k_q)$$

$$\left[(1 - \cos k_1 \pi/N) \sum_{\alpha=0}^{N-1} \sin \alpha k_1 \pi/N \cdots \sin \alpha k_q \pi/N \right.$$  
$$- \sin k_1 \pi/N \sum_{\alpha=0}^{N-1} \cos \alpha k_1 \pi/N \sin \alpha k_2 \pi/N \cdots \sin \alpha k_q \pi/N \right] \tag{71}$$

The sums over $\alpha$ is of order $N$. For fixed $k_1$ the factor $(1 - \cos k_1 \pi/N)$ tends to $1/N^2$, whereas the $\sin k_1 \pi/N$ goes as $1/N$. Hence, the first term inside the square brackets in (71) naively tends to zero for $N$ going to infinity and the second one gives the correct continuum limit. However, a more careful analysis shows that this naive limit is not correct. Consider, for example, the expectation value for the case $q = 2$. In the term which is naively zero the sum over $\alpha$ of $\sin \alpha k_1 \pi/N \sin \alpha k_2 \pi/N$ gives $(N/2)\delta_{k_1k_2}$, while the propagator combines with the $\lambda$’s and cancels the factor $(1 - \cos k_1 \pi/N)$. The final result is that this term has a limit $(1/4)(2\overline{T}h)\delta_{j_1j_2}$. The same is true for any $q$, namely both terms have non-vanishing finite limit. Similar results hold for the terms which were produced by commutators in the Hamiltonian approach (the last three lines in (40)). Naively all these terms tend to zero for $N$ going to infinity, but careful analysis reveals a finite result. In fact the terms coming from commutators just cancel the contribution from the first term in the square brackets in (71), so that at the end the naive limit gives the correct result!
4.2.1 Example 3.

Consider the terms in the third line in (51)
\[
\sum_{\alpha=0}^{N-1} \left[ (\xi_{\alpha i} - \xi_{(\alpha+1)i})\xi_{\alpha j} - \frac{1}{2} (\xi_{\alpha i} - \xi_{(\alpha+1)i})(\xi_{\alpha j} - \xi_{(\alpha+1)j}) \right] = \\
\sum_{\alpha=0}^{N-1} (\xi_{\alpha i} - \xi_{(\alpha+1)i}) \frac{(\xi_{\alpha j} + \xi_{(\alpha+1)j})}{2}. 
\]
(72)

We will show that it limits to \( \int_{-1}^{0} d\tau \dot{\xi}_i(\tau) \xi_j(\tau) \). We insert the mode expansion for the \( \xi \)'s and we use the trigonometric formula for the decomposition of \( \sin(a+1)k\pi/N \). The terms proportional to \( (1 - \cos k\pi/N) \) indeed cancel each other. One is left with
\[
\frac{1}{2} \sum_{k,l=1}^{N-1} v_i^k v_j^l \lambda(k)\lambda(l)\left[ \sin l\pi/N \sum_{\alpha=0}^{N-1} \cos \alpha l\pi/N \sin \alpha k\pi/N \\
- \sin k\pi/N \sum_{\alpha=0}^{N-1} \cos \alpha k\pi/N \sin \alpha l\pi/N \right]. 
\]
(73)

The expectation value of (73) vanishes since the expression within the square brackets is antisymmetric in \( k, l \) whereas the propagator for the modes provides a \( \delta^{kl} \). Thus, it is trivially equal to \( \langle \int_{-1}^{0} d\tau \dot{\xi}_i(\tau) \xi_j(\tau) \rangle \) which is also equal to zero.

4.2.2 Example 4.

The case with one \( \dot{\xi} \) and three \( \xi \)'s is more delicate. It corresponds to the case \( q = 3 \) in (51). We will prove that
\[
\lim_{N \to \infty} \frac{1}{3!} A_{i,jkl}(x) \left( \sum_{\alpha=0}^{N-1} \left[ (\xi_{\alpha i} - \xi_{(\alpha+1)i})\xi_{\alpha j} - \frac{3}{2} (\xi_{\alpha i} - \xi_{(\alpha+1)i})(\xi_{\alpha j} - \xi_{(\alpha+1)j}) \right] \xi_{\alpha k}\xi_{\alpha l} \right) \\
= \frac{1}{3!} A_{i,jkl}(x) \sum_{\alpha=0}^{N-1} \left[ \left( (\xi_{\alpha i} - \xi_{(\alpha+1)i})(\xi_{\alpha j} + \xi_{(\alpha+1)j}) \right) \xi_{\alpha k}\xi_{\alpha l} \right] \\
+ \text{cyclic in } j, k, l \\
= 0 = \frac{1}{3!} A_{i,jkl}(x) \langle \int_{-1}^{0} d\tau \dot{\xi}_i(\tau) \xi_j(\tau) \xi_k(\tau) \xi_l(\tau) \rangle 
\]
(76)
Since the $A_{i,jkl}$ is symmetric in $j,k,l$, we can symmetrize the second line in (74). This yields three terms, each with a factor $1/2$. Applying Wick's theorem we expect each of them to give three contractions. However, only one contraction is non-zero, namely $\langle (\xi_\alpha - \xi_{(\alpha+1)}) (\xi_\alpha - \xi_{(\alpha+1)}) \rangle \langle \xi_\alpha \xi_\alpha \rangle$. We now show that the other two possible contractions are zero. Consider the case

$$\langle (\xi_\alpha - \xi_{(\alpha+1)}) \xi_\alpha \rangle \langle (\xi_\alpha - \xi_{(\alpha+1)}) \xi_\alpha \rangle =$$

$$\sum_{k_1,k_2=1}^{N-1} \frac{2hT}{2N^2(1 - \cos k_1 \pi/N)} \frac{2hT}{2N^2(1 - \cos k_2 \pi/N)} \left[ (1 - \cos k_1 \pi/N)(1 - \cos k_2 \pi/N) \sum_{\alpha=0}^{N-1} \sin^2 \alpha k_1 \pi/N \sin^2 \alpha k_2 \pi/N \right] (77)$$

$$- (1 - \cos k_1 \pi/N) \sin k_2 \pi/N \frac{1}{2} \sum_{\alpha=0}^{N-1} \sin^2 \alpha k_1 \pi/N \sin 2\alpha k_2 \pi/N \quad (78)$$

$$- (1 - \cos k_2 \pi/N) \sin k_1 \pi/N \frac{1}{2} \sum_{\alpha=0}^{N-1} \sin^2 \alpha k_2 \pi/N \sin 2\alpha k_1 \pi/N \quad (79)$$

$$+ \sin k_1 \pi/N \sin k_2 \pi/N \frac{1}{4} \sum_{\alpha=0}^{N-1} \sin 2\alpha k_1 \pi/N \sin 2\alpha k_2 \pi/N \right], \quad (80)$$

where we have suppressed the spacetime indices. The terms (78) and (79) are clearly zero due to the summation over $\alpha$. Furthermore, (77) and (80) each vanish in the limit $N \to \infty$. Using this result, Wick’s theorem, and the symmetrization in $j,k,l$, (74) becomes

$$\frac{1}{3!} A_{i,jkl}(x) \sum_{\alpha=0}^{N-1} \left[ \langle (\xi_{\alpha i} - \xi_{(\alpha+1)i}) \xi_{\alpha j} \rangle \langle \xi_{\alpha k} \xi_{\alpha l} \rangle - \frac{1}{2} \langle (\xi_{\alpha i} - \xi_{(\alpha+1)i}) (\xi_{\alpha j} - \xi_{(\alpha+1)j}) \rangle \langle \xi_{\alpha k} \xi_{\alpha l} \rangle + \text{(cyclic in } j,k,l) \right]. \quad (81)$$

This indeed agrees with (75). We will show that (75) is equal to zero. We substitute the mode expansion for the $\xi$’s into (75). After some trigonometry we get

$$\langle (\xi_\alpha - \xi_{(\alpha+1)}) (\xi_\alpha + \xi_{(\alpha+1)}) \rangle \langle \xi_\alpha \xi_\alpha \rangle =$$

$$\sum_{k_1,k_2=1}^{N-1} \frac{2hT}{2N^2(1 - \cos k_1 \pi/N)} \frac{2hT}{2N^2(1 - \cos k_2 \pi/N)} \sin^2 \alpha k_2 \pi/N$$

22
\[
\left[- \sin^2 k_1 \pi/N \cos 2\alpha k_1 \pi/N - \frac{1}{2} \sin 2k_1 \pi/N \sin 2\alpha k_1 \pi/N \right],
\]
where we have again suppressed the spacetime indices. The second term in (82) vanishes due to the summation over \( \alpha \). The first one tends to zero in the limit \( N \to \infty \). Here we use the summation formula
\[
\sum_{\alpha=0}^{N-1} \cos \alpha k_1 \pi/N \sin^2 \alpha k_2 \pi/N = -\frac{N}{4} \delta_{k_1, k_2}.
\]
To prove this formula we first use trigonometric formulas to reduce the summation in (83) to summations over a single cosine function and then we perform these summations. Thus indeed (75) is equal to zero. In (76), combining the cosine with a sine, and combining the two remaining sine functions, leads to double-angle sine functions whose integral vanishes. This proves the continuum limit for the term \( q = 3 \) in (51).

It is straightforward to generalize to the case of one \( \dot{\xi} \) and arbitrary number of \( \xi \)’s. In every case we first use the symmetry of \( A_{i,j_1 \ldots j_q} \) to symmetrize the \( (\xi_\alpha - \xi_{\alpha+1}) \xi_\alpha \ldots \xi_\alpha \) term, so that \( q \) terms are obtained. Then Wick’s theorem gives \( q \) contractions for the \( (\xi_\alpha - \xi_{\alpha+1}) \xi_\alpha \ldots \xi_\alpha \) term, but just one contraction for each of the \( q \) terms since all but one contraction vanish. The \( q \) terms from \( (\xi_\alpha - \xi_{\alpha+1}) \xi_\alpha \ldots \xi_\alpha \) term combine with the \( q \) terms from the symmetrization of \( (\xi_\alpha - \xi_{\alpha+1})^2 \xi_\alpha \ldots \xi_\alpha \) to yield \( q \) terms of the form \( (\xi_\alpha - \xi_{\alpha+1}) \frac{(\xi_\alpha + \xi_{\alpha+1})}{2} \xi_\alpha \ldots \xi_\alpha \). Using similar arguments as in the case of (73) one can show that the generalization of (73) also vanishes and, therefore, is trivially equal to the continuum case.

4.2.3 Example 5.

We now consider examples of interference. The first example concerns with the interference of two terms, each with an even number of \( \xi \) fields. We take twice the third line in (51). We will show that
\[
\lim_{N \to \infty} \sum_{\alpha, \beta=0}^{N-1} \frac{1}{2} \langle (\xi_{\alpha i_1} - \xi_{(\alpha+1) i_1}) (\xi_{\alpha j_1} + \xi_{(\alpha+1) j_1}) \rangle
\]
\[
\quad + \frac{1}{2} \langle (\xi_{\beta i_2} - \xi_{(\beta+1) i_2}) (\xi_{\beta j_2} + \xi_{(\beta+1) j_2}) \rangle,
\]
\[
\langle \int_{-1}^{0} d\tau d\tau' \dot{\xi}_{i_1}(\tau) \xi_{j_1}(\tau) \dot{\xi}_{i_2}(\tau') \xi_{j_2}(\tau') \rangle.
\]

23
As we have shown in (72), the contraction of the first two (or last two) factors in (84) vanishes, so that only two contractions remain. The summations over \( \alpha \) and \( \beta \) can be performed (use (73) twice and that (73) vanishes for \( k = l \)) to yield

\[
\lim_{N \to \infty} \sum_{k_1 \neq l_1, k_2 \neq l_2} [\langle v_{i_1}^{k_1} v_{i_2}^{k_2} \rangle \langle v_{j_1}^{l_1} v_{j_2}^{l_2} \rangle + \langle v_{i_1}^{k_1} v_{j_2}^{l_2} \rangle \langle v_{j_1}^{l_1} v_{i_2}^{k_2} \rangle] \lambda(k_1) \lambda(l_1) \lambda(k_2) \lambda(l_2) \frac{1}{4} \left[ 1 - (-1)^{k_1 + l_1} \right] \left[ 1 - (-1)^{k_2 + l_2} \right] \\
\left[ \frac{\sin k_1 \pi / N \sin l_1 \pi / N}{\cos k_1 \pi / N - \cos l_1 \pi / N} \right] \left[ \frac{\sin k_2 \pi / N \sin l_2 \pi / N}{\cos k_2 \pi / N - \cos l_2 \pi / N} \right].
\]

(86)

Each propagator gives a \( \delta \)-function, so we left with a double sum. Combining each propagator with the corresponding two factors of \( \lambda \), we get

\[
(2\hbar T)^2 (\delta_{i_1 i_2} \delta_{j_1 j_2} - \delta_{i_1 j_2} \delta_{j_1 i_2}) \left[ \sum_{k,l=1}^{N-1} \frac{1}{4N^2 \sin^2 k \pi / 2N} \left( \frac{1}{4N^2 \sin^2 l \pi / 2N} \right) \frac{1}{2} \left[ 1 - (-1)^{k+l} \right] \right] \\
\frac{1}{4N^4 \sin^2 (l-k) \pi / 2N} \frac{1}{\sin^2 (k+l) \pi / 2N}.
\]

\[
\equiv (2\hbar T)^2 (\delta_{i_1 i_2} \delta_{j_1 j_2} - \delta_{i_1 j_2} \delta_{j_1 i_2}) \lim_{N \to \infty} I(N).
\]

(87)

Using the trigonometric formula for the sine of double angle and the one which expresses the difference of cosines as a product of sines we get

\[
I(N) = \sum_{k,l=1}^{N-1} \frac{1}{2} \left[ 1 - (-1)^{k+l} \right] \frac{1}{4N^4 \sin^2 (l-k) \pi / 2N} \frac{\cos^2 k \pi / 2N \cos^2 l \pi / 2N}{\sin^2 (k+l) \pi / 2N}.
\]

(88)

We split this sum into two sums according to whether \( k + l \) is smaller or larger than \( N \)

\[
I(N) = I_1(k + l \leq N) + I_2(k + l > N),
\]

where

\[
I_1(N) = \sum_{k,l=1}^{\infty} g_{kl}^{(1)}(N),
\]

(90)

and

\[
g_{kl}^{(1)} = \begin{cases} 0 & \text{if } k + l > N \\ \frac{1}{2} \left[ 1 - (-1)^{k+l} \right] \frac{\cos^2 k \pi / 2N \cos^2 l \pi / 2N}{4N^4 \sin^2 (l-k) \pi / 2N \sin^2 (k+l) \pi / 2N} & \text{if } k + l \leq N. \end{cases}
\]

24
In $I_2$ we make the transformation

\[ k' = N - k, \quad l' = N - l, \]

so $k' + l' < N$ and $I_2$ becomes

\[ I_2(N) = \sum_{k',l'=1}^{\infty} g^{(2)}_{k'l'}(N), \]

where

\[ g^{(2)}_{k'l'} = \begin{cases} 0 & \text{if } k' + l' \geq N \\ \frac{1}{2} \left[1 - (-1)^{k'+l'}\right] \frac{\sin^2 k'\pi/2N \sin^2 l'\pi/2N}{\sin^2(l'-k')\pi/2N \sin^2(k'+l')\pi/2N} & \text{if } k' + l' < N. \end{cases} \]

We shall now again prove that these series converge uniformly in $N$.

For $0 < k + l \leq N$, we have the upper bound

\[ \sin(k + l)\pi/2N \geq (k + l)/N, \]

using the inequality $\sin \theta \geq 2\theta/\pi$ valid for $0 \leq \theta \leq \pi/2$. From the same inequality we also get

\[ |\sin(l - k)\pi/2N| \geq |l - k|/N, \]

since $-N \leq (l - k) \leq N$. Hence, an upper limit for the summands in $I_1$ can be found which is independent of $N$

\[ |g^{(1)}_{kl}(N)| \leq \frac{1}{4(k^2 - l^2)^2} \left[1 - (-1)^{k+l}\right]. \]

The same upper limit holds for $g^{(2)}_{k'l'}(N)$. (From (73) it follows that $g^{(1)}_{kl}$ and $g^{(2)}_{kl}$ vanish at $k = l$). The double series $\sum_{k,l=1}^{\infty} [1 - (-1)^{k+l}]/[8(k^2 - l^2)^2]$ is convergent. Actually, apart for the factor $1/8$, this is exactly the series we analytically evaluate in section 5. Thus, the limit $N \to \infty$ can be taken keeping fixed $k$ and $l$. The result is that $I_2(N)$ tends to zero whereas $I_1(N)$ tends to

\[ \sum_{k,l=1}^{\infty} \left[1 - (-1)^{k+l}\right] \frac{2}{\pi^4(l^2 - k^2)^2}. \]

Going back to (77) we get

\[ (\delta_{i_1,i_2}\delta_{j_1,j_2} - \delta_{i_1,j_2}\delta_{j_1,i_2}) \sum_{k,l=1}^{\infty} \left(\frac{2hT}{k^2\pi^2}\right)\left(\frac{2hT}{l^2\pi^2}\right) \left(1 - [1 - (-1)^{k+l}]\frac{kl}{l^2 - k^2}\right)^2. \]
Using
\[ \int_{-1}^{0} d\tau (k\pi) \cos k\pi \sin l\pi \tau = \begin{cases} 0 & \text{if } k = l \\ -[1 - (-1)^{k+l}] \frac{kl}{(k+l)^2} & \text{if } k \neq l, \end{cases} \]

we find that (97) indeed reproduces (85).

4.2.4 Example 6.

We now give an example of interference with two terms, each with an odd number of quantum fields. The basic features are the same, the algebra though is much more laborious. We take the second term in the first line of (51) and the term with \( q = 2 \). We expand \( A_{l,n}(x_{\alpha-1}) \) around \( x_i \). We will prove that
\[ \lim_{N \to \infty} \frac{1}{2!} A_{i,jk}(x) A_{l,mn}(x)(x - y)_l(y - x)_m \left( \frac{1}{N} \sum_{\alpha,\beta = 0}^{N-1} \left[ (\xi_{\alpha i} - \xi_{(\alpha+1)i})\xi_{\alpha j} - \frac{2}{N} (\xi_{\alpha i} - \xi_{(\alpha+1)i}) (\xi_{\alpha j} - \xi_{(\alpha+1)j}) \right] \xi_{\alpha k}(\frac{\beta}{N})\xi_{\beta n} \right) \]

(98)

\[ = -\frac{1}{2!} A_{i,jk}(x) A_{l,mn}(x)(x - y)_l(y - x)_m \]

\[ \langle \int_{-1}^{0} d\tau d\tau' \xi_i(\tau)\xi_j(\tau)\xi_k(\tau)\xi_n(\tau') \rangle, \]

(99)

where the relative minus sign is due to the difference in sign between (38) and (56). Following the same procedure as in the case with one \( \xi \) and odd number of \( \xi \)'s we first symmetrize w.r.t \( j, k \). We will study only the contraction \( \langle i j \rangle \langle k n \rangle \) since the contraction \( \langle i k \rangle \langle j n \rangle \) is equal to this one and the last one, \( \langle i n \rangle \langle j k \rangle \), can be studied in a similar way, where we abbreviate the \( \xi \)'s by their spacetime indices. From now on the factor \( \frac{1}{N} \) in (98) becomes implied and the spacetime indices are supressed. We substitute the mode expansion for the \( \xi \)'s in (98). The summation over \( \beta \) is given in (56). After some trigonometry (98) becomes

\[ \lim_{N \to \infty} \frac{(2\pi T)^2}{2} \sum_{k,l=1}^{N-1} \frac{1}{2N^2(1 - \cos k\pi/N)} \frac{1}{2N^2(1 - \cos l\pi/N)} \frac{(-1)^l}{2N \tan l\pi/2N} \]

26
\[
\left\{ (1 - \cos k\pi/N) \frac{1}{2} \sum_{\alpha=0}^{N-1} (1 - \cos 2\alpha k\pi/N) \sin \alpha l\pi/N \right.
\]
\[
- \sin k\pi/N \left( \frac{N}{4} \delta_{2k,l} \right) \] (100)
\[
- \frac{1}{2} (1 - \cos k\pi/N)^2 \frac{1}{2} \sum_{\alpha=0}^{N-1} (1 - \cos 2\alpha k\pi/N) \sin \alpha l\pi/N \] (102)
\[
- \frac{1}{2} (1 - \cos k\pi/N)(1 - \cos l\pi/N)
\]
\[
\frac{1}{2} \sum_{\alpha=0}^{N-1} (1 - \cos 2\alpha k\pi/N) \sin \alpha l\pi/N \] (103)
\[
+(1 - \cos k\pi/N) \sin k\pi/N \left( \frac{N}{4} \delta_{2k,l} \right) \] (104)
\[
+ \frac{1}{2} (1 - \cos k\pi/N) \sin l\pi/N \left( - \frac{N}{4} \delta_{2k,l} \right) \] (105)
\[
+ \frac{1}{2} (1 - \cos l\pi/N) \sin k\pi/N \left( \frac{N}{4} \delta_{2k,l} \right) \] (106)
\[
- \frac{1}{2} \sin^2 k\pi/N \frac{1}{2} \sum_{\alpha=0}^{N-1} (1 + \cos 2\alpha k\pi/N) \sin \alpha l\pi/N \] (107)
\[
- \frac{1}{2} \sin k\pi/N \sin l\pi/N \frac{1}{2} \sum_{\alpha=0}^{N-1} \sin 2\alpha k\pi/N \cos \alpha l\pi/N \} \] (108)

The terms (100) and (101) are coming from the \((\xi_\alpha - \xi_{\alpha+1})\xi_\alpha\) term. The former is cancelled by the \((\xi_\alpha - \xi_{\alpha+1})^2\) term and the latter gives the continuum limit. Indeed, the term (100) is cancelled exactly by the terms (102), (103), (107) and (108). The terms (104), (105) and (106) vanish in the limit \(N \to \infty\). It remains to take the limit \(N \to \infty\) in (101). The term (101) can be rewritten as
\[
-(2\hbar T)^2 \sum_{k=1}^{N-1} \frac{1}{128N^4} \frac{\cos k\pi/N}{\sin^2 k\pi/2N \sin^2 k\pi/N}. \] (109)

We split the sum in two sums, the first running from 1 to \(N/2 - 1\) and the second from \(N/2\) to \(N-1\). In the first one an upper bound for the summands can be found by using the same inequalities as in the first example,
\[
\left| \frac{1}{128N^4} \frac{\cos k\pi/N}{\sin^2 k\pi/2N \sin^2 k\pi/N} \right| \leq \frac{1}{512k^4}. \] (110)
In the second sum we make the transformation \( k' = N - k \). The series becomes
\[
\sum_{k' = 1}^{N/2} \frac{1}{128N^4 \cos^2 k'/2N \sin^2 k'/N} = \sum_{k' = 1}^{N/2} \frac{1}{32N^4} \frac{\sin^2 k'/2N \cos k'/N}{\sin^2 k'/N}.
\]
(111)

An upper bound for the summands of this series is given by
\[
\left| \frac{1}{32N^4} \frac{\sin^2 k'/2N \cos k'/N}{\sin^2 k'/N} \right| \leq \frac{1}{512k'^4}.
\]
(112)

Therefore the series (109) is uniformly convergent. So the limit \( N \to \infty \) in (101) can be performed before the summation. The result reads
\[
-(2\hbar T)^2 \sum_{k,l=1}^{\infty} \frac{1}{k^2 \pi^2 l^2 \pi^2} \left[ -\frac{(-1)^l}{l\pi} \right] (k\pi)(\frac{1}{4} 0^{2k,l})
\]
\[
= - \int_{-1}^{0} \int_{-1}^{0} d\tau d\tau' \langle \dot{x}_i(\tau) \dot{x}_j(\tau) \rangle \dot{x}_i(\tau) \dot{x}_j(\tau') \rangle
\]
(113)

, which is what we wanted to prove.

One can easily check now that the terms of (10) which were omitted in (51) indeed tend to zero. These terms are those in the last three lines in (10) except for the terms proportional to \( (\xi_{(a-1)i} - \xi_{ai})(\xi_{(a-1)j} - \xi_{aj}) \). All of them are equal to \( 1/N \) times terms which were proven finite in the limit \( N \to \infty \).

Combining (48), (53) and (54) we get the continuum action \( S_{\text{config}} \)
\[
S_{\text{config}} = \int_{-T}^{0} d\tau \left[ \frac{1}{2} \dot{x}_i \dot{x}_i - i \left( \frac{e}{c} \right) T \dot{x}_i A_i \right] = \int_{-T}^{0} dt \left[ \frac{1}{2} \dot{x}_i \dot{x}_i - i \left( \frac{e}{c} \right) \dot{x}_i A_i \right].
\]
(114)

, where in the last step we have rescaled the the time \( \tau = t/T \).

5 Evaluation of the path integral.

In the continuum path integral with action (114) we set \( T = \Delta t \) and then we evaluate it to order \( (\Delta t)^2 \). The derivation of the path integral indicates how to evaluate it. First we decompose \( x_i(\tau) \) into a function \( z_i(\tau) \) and a quantum part \( \xi_i(\tau) \)
\[
x_i(\tau) = z_i(\tau) + \xi_i(\tau).
\]
(115)
The function $z_i(\tau)$ is not a solution of the classical field equations, but rather of the field equations corresponding to $L_0 = \dot{x}^2 / 2$. It satisfies the same boundary conditions as $x_i(\tau)$ and hence is given by

$$z_i = x_i - \tau(y - x)_i. \quad (116)$$

It follows that the quantum field vanishes at the boundary

$$\xi_i(\tau = 0) = \xi_i(\tau = -1) = 0. \quad (117)$$

Since the eigenfunctions of $S_0$ with these boundary conditions are the functions $\sin(n\pi \tau)$, we expand the quantum field on a trigonometric basis

$$\xi_i = \sum_{n=1}^{\infty} v^n_i \sin(n\pi \tau). \quad (118)$$

The propagator for the modes is obtained by using only the part quadratic in velocities and reads

$$\langle v^m_i v^n_j \rangle = \frac{2\Delta \hbar}{\pi^2 n^2} \delta_{ij} \delta^{mn}, \quad (119)$$

as follows from the measure in (117). If we would multiply this result with two sine functions and sum over $m$ and $n$, we would recover the result of (6, 9) for $\langle \xi_i(\tau_1) \xi_j(\tau_2) \rangle$. However, we shall work here entirely in terms of modes.

The $S_{\text{int}}$, up to the order we are interested in, is given by

$$S_{\text{int}} = \frac{i}{\hbar} \left( \frac{e}{c} \right) \int_{-1}^{0} d\tau \left[ (x - y)_i \dot{\xi}_i \right] \{ A_i(z(\tau)) + A_{i,j}(z(\tau)) \xi_j + \frac{1}{2} A_{i,jk}(z(\tau)) \xi_j \xi_k + \frac{1}{3!} A_{i,jkl}(z(\tau)) \xi_j \xi_k \xi_l + \cdots \}. \quad (120)$$

We factor out all the terms which do not depend on quantum fields

$$\exp \left[ - \frac{1}{2\hbar\Delta \tau} (x - y)^2 + \frac{i}{\hbar} \left( \frac{e}{c} \right) \left\{ A_i(x - y)_i - \frac{1}{2} A_{i,j}(x - y)_i(x - y)_j + \frac{1}{3!} A_{i,jk}(x - y)_i(x - y)_j(x - y)_k - \frac{1}{4!} A_{i,jkl}(x - y)_i(x - y)_j(x - y)_k(x - y)_l \right\} \right]. \quad (121)$$

Observe that (121) differs from $\exp(-S_{\text{cl}}/\hbar)$ by just one term (namely the $F^2$ term in (26)). We will recover this missing term from a tree graph
The reason for the absence of the $F^2$ term from (121) is that $z_i(\tau)$ does not satisfy the full field equations but rather the field equations of $L_0 = \frac{\dot{x}^2}{2}$.

Using only one factor of $S_{\text{int}}$ we get the following contribution

\[
\left(\frac{ie}{\hbar c}\right) \int_{-1}^{0} d\tau \left[ A_{i,j}(z(\tau))\langle \dot{\xi}_i(\tau)\xi_j(\tau) \rangle + \frac{1}{2}A_{i,jk}(z(\tau))\langle \xi_j(\tau)\xi_k(\tau) (x-y)i \rangle + \frac{1}{3!}A_{i,jkl}(z(\tau))\langle \dot{\xi}_i(\tau)\xi_j(\tau)\xi_k(\tau)\xi_l(\tau) \rangle \right].
\]

The first two terms are one loop contributions and are of order $\Delta t$ and higher (since $(x-y)i$ is of order $(\Delta t)^{1/2}$), while the last one is a 2-loop contribution of order $(\Delta t)^2$ and higher. However, performing the $\tau$-integration the 2-loop contribution of order $(\Delta t)^2$ vanishes because combining the sine and cosine functions one always ends up with a sine of a double angle whose integral vanishes. The first term is a superficially divergent tadpole, but using mode regularization (i.e., first evaluate the integrals for a finite number of modes, and then let the number of modes tend to infinity) one finds that it is, in fact, finite. This is thus a property of our regularization scheme, similar to the property of dimensional regularization which puts equal to zero all divergences which are not logarithmic divergences. The first two terms in (122) yield

\[
i\frac{\Delta t}{12}\left(\frac{e}{c}\right)[F_{ki,k}(x-y)i - \frac{1}{2}F_{ki,kj}(x-y)i(x-y)j].
\]

To get this result we used the known sum $\zeta(2) = \sum_{n=0}^{\infty} n^{-2} = \pi^2/6$.

Two factors of $S_{\text{int}}$ yield

\[-\frac{1}{2}\left(\frac{e}{\hbar c}\right)^2 \int_{-1}^{0} d\tau d\tau' \left\{ [A_{i,j}(z(\tau))A_{k,l}(z(\tau'))\langle \dot{\xi}_i(\tau)\xi_l(\tau') \rangle (x-y)i(x-y)k + A_{i}(z(\tau))A_{k,l}(z(\tau'))\langle \dot{\xi}_i(\tau)\xi_l(\tau') \rangle (x-y)k + A_{i,j}(z(\tau))A_{k}(z(\tau'))\langle \dot{\xi}_i(\tau)\xi_k(\tau') \rangle (x-y)i + A_{i}(z(\tau))A_{k,l}(z(\tau'))\langle \dot{\xi}_i(\tau)\xi_k(\tau') \rangle] + A_{i,j}(z(\tau))A_{k,l}(z(\tau'))\langle \dot{\xi}_i(\tau)\xi_j(\tau)\dot{\xi}_k(\tau') \rangle \right\} \]
tree graphs and combine to give

\[ \frac{4}{\pi^4} \left( \frac{\Delta t}{\hbar} \right) \left( \frac{e}{c} \right)^2 \sum_{k=0}^{\infty} \frac{1}{(2k+1)^4} F_{ik} F_{kj} (x - y)_i (x - y)_j. \quad (125) \]

The sum which appears in (125) is known and is equal to \( \lambda(4) = (1 - 2^{-4}) \zeta(4) = \pi^4/96 \). Using this result we identify (125) as the missing term of the classical action. The last term in (124) is a one-loop graph and gives

\[ -\frac{2}{\pi^4} \left( \frac{e}{c} \right)^2 F^2 (\Delta t)^2 \sum_{m,n=1;m\neq n}^{\infty} \frac{1 - (-1)^{m+n}}{(m^2 - n^2)^2}. \quad (126) \]

The double sum which appears in (126) seems not tabulated. Here we give an analytic evaluation of it. The idea is to extend the limits of summation to \( \pm \infty \) so that linear shifts of the summation variable are allowed. This can be done by observing that the summand is symmetric under \( n \rightarrow -n, m \rightarrow -m, m \leftrightarrow n \). Notice also that only \( m + n = \text{odd} \) contributes. The sum becomes

\[ \sum_{k,l=-\infty}^{\infty} \frac{1}{(2k+1)^2 - (2l)^2} = \sum_{k,l=-\infty}^{\infty} \frac{1}{(2k+1)^2 - (2l)^2} - \pi^4/48. \quad (127) \]

The last double sum can be rewritten by substituting \( 2k = 2l + p - 1 \) for \( p \) odd

\[ \sum_{l=-\infty}^{\infty} \sum_{p=1, \text{odd}}^{\infty} \frac{1}{p^2 (4l + p)^2} = 2 \left( \sum_{p=1, \text{odd}}^{\infty} p^{-2} \right)^2 = 2(\lambda(2))^2 = \pi^4/32. \quad (128) \]

Hence, the sum is equal to \( \pi^4/96 \).

(126) together with (124) give the Van Vleck determinant. The final result is that the path integral correctly reproduces the propagator found from the Hamiltonian operator approach. There are further one-loop diagrams which give contribution of higher order than \( (\Delta t)^2 \). For example, taking twice the term \( (x - y)_i A_{i,jk} \xi_j \xi_k \) we get a one-loop result proportional to \( (\Delta t)^3 \).
6 Conclusions.

We have proven the 1-1 correspondence between the Hamiltonian approach and path integration for Hamiltonians of the form \( \hat{H} = \hat{p}^2 + a'(\hat{x})\hat{p}_i + b(\hat{x}) \). The correspondence we found is this: casting the Hamiltonian into the form

\[
\hat{H} = \frac{1}{2}(\hat{p}_i - \frac{\epsilon}{c}A_i(\hat{x}))(\hat{p}^i - \frac{\epsilon}{c}A^i(\hat{x})) + V(\hat{x}),
\]

(129)

the action to be used in the path integral is

\[
S_{\text{config}} = \int_{-T}^{0} dt \left[ \frac{1}{2}\delta_{ij}\dot{x}^i\dot{x}^j - i(\frac{\epsilon}{c})\dot{x}^iA_i(x) + V(x) \right].
\]

(130)

This result holds for any Hamiltonian, whether it is covariant or not. The path integral is perturbatively evaluated by treating the term \( \dot{x}^2/2 \) as the free part \( S_0 \), decomposing \( x(\tau) \) into a background part \( z(\tau) \) and a quantum part \( \xi(\tau) \), and expanding \( \xi(\tau) \) in terms of eigenfunctions of \( S_0 \). The measure in the path integral as well as the action \( S_0 \) determines the world line propagator (see (50)). (This is, in fact, the only place where the measure plays a role for us). The other two terms in (130) yield the vertices, and one can now evaluate (as we did) the path integral in a perfectly straightforward and standard manner (“Feynman graphs”).

Of course the expansion of \( \xi_i(\tau) \) into modes is well-known

\[
\xi_i(\tau) = \sum_{k=1}^{\infty} v_i^k \sin k\pi \tau.
\]

(131)

What we have shown is that all arbitrariness (such as the overall normalization of the path integral) can be fixed by starting with the Hamiltonian approach. Moreover, we have given an elementary (though at times somewhat tedious) proof that the \( N \rightarrow \infty \) limit of the time-discretized path integral exists as far as perturbation theory is concerned, and indeed yields the continuum path integral with its measure.

The actual proof that the limit \( N \rightarrow \infty \) exists was given by carefully analyzing six examples which cover all cases one encounters in the perturbative evaluation of the path integral. In each example we found upper bounds for the infinite series which showed that these series are uniformly convergent as a function of \( N \). This allowed us to take the limit \( N \rightarrow \infty \) inside the summation symbols (i.e., at fixed mode index \( k \)).
Our results confirm the lore about path integrals that the naive $N \to \infty$ limit in the discretized path integral yields the correct continuum path integral. However, this came about by an interesting “conspiracy”: the higher order terms in the Hamiltonian evaluation of $\langle x \mid \exp(-\Delta t \hat{H}/\hbar)\mid y \rangle$ which are due to expanding the exponent and taking into account the commutators between $\hat{x}$ and $\hat{p}$ operators, cancel against the terms in the time-discretized action which seem to (but do not) vanish in the $N \to \infty$ limit. Due to this conspiracy it is, after all, correct to use (3) and (4), omitting all commutators, to obtain the action to be used in the path integral. Namely, this yields $h(x, p) = \frac{1}{2}p^2 - \frac{\xi}{\hbar} A^i(x)p_i + \frac{1}{2}(\frac{\xi}{\hbar})^2 A^2(x) + V(x)$, and after integrating out the momenta, the naive $N \to \infty$ limit yields the correct action $S_{\text{config}}$ in (130). However, (3) and (4) by themselves do not yield the correct propagator.

Our results immediately generalize to quantum field theories with derivative interactions, such as Yang-Mills theory with gauge fixing term and ghost action. The Hamiltonian for the gauge and ghost fields contain again terms linear in momenta. For example, in the Lorentz gauge, the Hamiltonian reads

$$H(\text{gauge}) = \frac{1}{2}p(A^a_0)^2 - \frac{1}{2}p(A^a_0)^2 + p(A^a_k)\partial_k A^a_0 + \frac{1}{2}(G_{kl})^2 - gp(A^a_k)f^a_{bc}A^b_0A^c_k,$$

and

$$H(\text{ghost}) = p(b_a)p(c^a) + p(c^a)gf^a_{bc}A^b_0A^c + (\partial_k b_a)(D_k c^a),$$

where $b_a(c^a)$ are the antighost (ghost) fields. Following the results of this paper one can find the 1-1 correspondence between operator Hamiltonians and path integral actions. In particular, one may determine the operator ordering of $H(\text{gauge})$ and $H(\text{ghost})$ which corresponds to the usual BRST invariant quantum action in the configuration space path integral. However, again, the linear approximation in the Hamiltonian approach yields incorrect results if one uses it to compute the propagator in the Hamiltonian approach.

Another extension of our results would be to consider phase space path integrals. In the discretized action we first integrated at some point over the momenta, and then studied the limit $N \to \infty$. One might leave the discretized momenta in the action, and consider the limit $N \to \infty$ with momenta present. The continuum action is expected to be $p\dot{x} - H(p, x)$, i.e., the Legendre transform of the classical Lagrangian in (130). Again one could introduce classical trajectories for $p$ and $x$ satisfying $x(0) = x,$
\( x(T) = y \) and satisfying the Hamilton equations of motion of a suitable Hamiltonian \( H_0 \) contained \( H \). (This will, of course, fix \( p(0) \) and \( p(T) \) as well). The quantum deviations \( \xi(\tau) \) and \( \pi(\tau) \) vanish then at the boundaries and can be expanded into a complete set, for example again \( \sin k\pi t/T \). The measure is expected to come out unity (except for the factor \( (2\pi\hbar T)^{-1/2} \) mentioned in the introduction) and propagators and vertices would then be defined. The problem would be to prove that the limit \( N \to \infty \) of the discretized theory indeed produce this continuum theory.

We are interested in extending our results to models in curved spacetime (nonlinear sigma models). This is a well-known problem to which partial answers have been given in [12] and [9]. In the propagator to order \( \Delta t \) one finds a term proportional to Ricci tensor \( R_{ij} \) contracted with \( (x^i - y^i)(x^j - y^j) \), which cannot be written as the action of a local functional. Thus, it is not immediately clear what the continuum action is, and which terms in the limit \( N \to \infty \) will cancel. However, one can still exponentiate this term and obtain the discretized action. The \( R_{ij} \) term should become an \( R \) term in the continuum theory since at the perturbative level \( (x^i - y^i)(x^j - y^j) \) should be equivalent to \( g^{ij} \Delta t \). However, this would only produce a factor \( R/6 \) into the action whereas one needs a factor \( R/8 \) [4, 5, 12]. Further cancellations of type studied in section 4 should then indeed reduce \( R/6 \) to \( R/8 \). Note that this analysis might be done without the need of using Einstein invariance to go to Riemann normal coordinates, and hence problems with time-ordering in arbitrary coordinates [12] would be avoided.

Acknowledgments: We thank Bas Peeters and Jan de Boer for numerous discussions. Eduard Brézin and Jan Pierre Zuber told us that they had also discovered that it is incorrect to use the linear approximation to obtain the propagator (see [3], [4]). We hope that our solution as based on the “conspiracy” described in the conclusions will satisfy them and others.

This work was supported by NSF grant 92-11367.

References

[1] See, for example, L. Schulman, ‘Techniques and Applications of Path Integration’, John Wiley and Sons, New York, 1981.

[2] L. Alvarez-Gaumé and E. Witten, Nucl. Phys. B234, (1984) 269.
[3] A. Diaz, W. Troost, P. van Nieuwenhuizen and A. van Proeyen, Int. J. Mod. Phys. A4 (1989) 3959; M. Hatsuda, P. van Nieuwenhuizen, W. Troost and A. van Proeyen, Nucl. Phys. B335, (1990) 166.

[4] H. Dekker, Physica 103A (1980) 586.

[5] F. Bastianelli, Nucl. Phys. B376, (1992) 113.

[6] F. Bastianelli and P. van Nieuwenhuizen, Nucl. Phys. B389, (1993) 53.

[7] R. Graham, Z. Phys. B26, (1977) 281.

[8] F. Langouche, D. Roekaerts, and E. Tirapegui, ‘Functional Integration and Semiclassical Expansions’, D. Reidel Publishing Company, Dordrecht, Holland, 1982.

[9] B. Peeters, P. van Nieuwenhuizen, ITP-SB-93-51.

[10] J. Van Vleck, Proc. Natl. Acad. Sci. 24, (1928) 178; C. Morette, Phys. Rev. 81 (1951) 848.

[11] B. DeWitt, Rev. Mod. Phys. 29, (1957) 377.

[12] B. DeWitt, ‘Supermanifolds’, 2nd ed., Cambridge University Press, 1992.