COHOMOLOGY AND OBSTRUCTIONS III: A VARIATIONAL FORM OF THE GENERALIZED HODGE CONJECTURE ON $K$-TRIVIAL THREEFOLDS

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Abstract. This paper studies the Hilbert scheme of a curve on a complete-intersection $K$-trivial threefold, in the case in which the curve is unobstructed in the ambient variety in which the threefold lives. The basic result is that the obstruction theory of the curve is completely determined by the scheme-theoretic Abel-Jacobi mapping. Several applications of this fact are given.

1. Introduction

This paper gives a proof of a variational version of the generalized Hodge conjecture in a very special case, namely for certain families of curves on certain complex projective threefolds $X_0$ with trivial canonical bundle. The (smooth) threefold $X_0$ must be the zero-scheme of a regular section of a vector bundle on a complex manifold $P$ and a maximal abelian subvariety of the intermediate Jacobian $J(X_0)$ must deform over some deformation $X/X'$ of $X_0$ as a sub-manifold of $P$. Finally, over a polydisk $Y_0'$, we must be given a family of (smooth) curves

$$p : Y_0 \to Y_0'$$

in $X_0$ whose fibers are (strongly) unobstructed in $P$, that is,

$$R^1p_*N_{Y_0'/Y_0} \to P.$$

The conclusion is then that $Y_0/Y_0'$ is the specialization of some family of curves on the generic fiber of $X/X'$. This result should be interpreted as generalizing the fact that a rigid curve on $X_0$ is always the specialization of a curve on the general fiber of $X/X'$ (because the relative dimension of the Hilbert scheme is bounded below by the Euler characteristic of the normal bundle which, for $K$-trivial threefolds, is alway zero).

A previous version of this paper appeared with the title “Cohomology and Obstructions II: Curves on Calabi-Yau threefolds.” C. Voisin subsequently pointed out that a key assertion in the proof of the main theorem, namely that an Abel-Jacobi map over a scheme takes values in an abelian variety defined over that scheme, was unsubstantiated. In developing a proof that claim in the case of intermediate Jacobians of $K$-trivial threefolds, the author was led to develop an algebro-geometric version of the physicists’ understanding of Hilbert schemes of Calabi-Yau threefolds as gradient schemes. That study developed a life of its own, resulting in a separate paper [C2], some of the results of which are necessary to correct the above-mentioned gap. To keep things in logical order, this manuscript has been renamed, and carries the label “Cohomology and Obstructions III.”

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1.1. **The tool: Gauss-Manin connection as obstruction.** This paper represents a concrete application of Theorem 13.1 of [C1] and of the realization of the relative Hilbert scheme as a relative gradient variant, which is the main result of [C2]. We first recall Theorem 13.1 of [C1]. In what follows,

\[ X/X' \]

will always be a deformation of a complex projective manifold \( X_0 \) where \( X' \) is a complex polydisk with distinguished basepoint 0. (Later in this paper, \( X_0 \) will be a \( K \)-trivial threefold, like a Calabi-Yau threefold or an abelian variety.) For any base extension

\[ A \to X' \]

we let

\[ X_A := X \times_{X'} A. \]

**Theorem 1.1.** Let \( \tilde{\Delta} \) denote a small disk around 0 in \( \mathbb{C} \) with parameter \( \tilde{t} \). Let \( X_{\tilde{\Delta}}/\tilde{\Delta} \) be a deformation of the complex projective manifold \( X_0 \). Suppose

\[ p': W_0 \to W'_0 \]

is a proper family of submanifolds of \( X_0 \) of fiber dimension \( q \) over a smooth (not necessarily compact) base \( W'_0 \). Suppose further that the family \( W_0/W'_0 \) deforms with \( X_0 \) to a family \( W_n/W'_n \) over the Artinian scheme \( \tilde{\Delta}_n \subseteq \tilde{\Delta} \) associated to the ideal \( \{ \tilde{t}^{n+1} \} \) and that

\[ \omega \in H^{p+q+1,q-1}(X_{\tilde{\Delta}}/\tilde{\Delta}) \]

lies in the kernel of the composition

\[ H^{p+q+1,q-1}(X_{\tilde{\Delta}}/\tilde{\Delta}) \xrightarrow{\Sigma} H^{p+q,q}(X_{\tilde{\Delta}}/\tilde{\Delta}) \xrightarrow{\text{pull-back}} R^q p'_* \left( \Omega^{p+q}_{W_n/\tilde{\Delta}_n} \right) \to \Omega^p_{W'_n} \]

induced by the Gauss-Manin connection \( \nabla \) and integration over the fiber. Let

\[ \varsigma \in R^1 p'_* \left( N_{W_0/W'_0 \times X_0} \right) \]

be the obstruction class measuring extendability of \( W_n/W'_n \) to a family \( W_{n+1}/W'_{n+1} \). Then, for \( \omega_0 = \omega|_{X_0} \),

\[ \langle \varsigma | \omega_0 \rangle|_{W_0} \in R^q p'_* \left( \Omega^{p+q}_{W_0} \right) \otimes \left\{ \tilde{t}^{n+1} \right\} / \left\{ \tilde{t}^{n+2} \right\} \]

(2)

goes to zero in

\[ \Omega^p_{W'_0} \otimes \left\{ \tilde{t}^{n+1} \right\} / \left\{ \tilde{t}^{n+2} \right\} \]

under the map

\[ \int_{W_0/W'_0} : R^q p'_* \left( \Omega^{p+q}_{W_0} \right) \to \Omega^p_{W'_0} \]

(3)

induced by the Leray spectral sequence. (This last map is commonly called integration over the fiber.)
1.2. **Abel-Jacobi mapping.** Suppose again that

\[ X_{\hat{\Delta}}/\hat{\Delta} \]

is a deformation of \( X_0 \) over the \( \hat{t} \)-disk. Let \( W'_0 \) be a complex polydisk. Suppose we have a diagram

\[
\begin{array}{ccc}
W_0 & \hookrightarrow & X_0 \\
\downarrow p_0 & & \downarrow p_0 \\
W'_0 & \to & \hat{\Delta} \\
\end{array}
\]

where \( p_0 \) is proper with fibers

\[ \{ Z_w \}_{w \in W'_0} \]

which are smooth and irreducible of dimension \( q \) and suppose, for simplicity, that the total space \( W_0 \) of the family is embedded in \( X_0 \). Let

\[ \hat{\Delta}_n := \text{Spec} \frac{C[[\hat{t}]]}{\hat{t}^{n+1}} \]

Suppose \( p_0 \) extends to

\[ p_n : W_n \to W'_n = W'_0 \times \hat{\Delta}_n \]

a proper and smooth family with fiber dimension \( q \) embedded in \( X_n \), the part of \( X_{\hat{\Delta}} \) over \( \hat{\Delta}_n \) via the commutative diagram

\[
\begin{array}{ccc}
W_n & \hookrightarrow & X_n \\
\downarrow p_n & & \downarrow p_n \\
W'_n & \to & \hat{\Delta}_n \\
\end{array}
\]

Let

\[ J_q \left( X_{\hat{\Delta}}/\hat{\Delta} \right) = \frac{\left( H^0 \left( \Omega^{2q+1}_{X_{\hat{\Delta}}/\hat{\Delta}} \right) + \ldots + H^{q+1} \left( \Omega^q_{X_{\hat{\Delta}}/\hat{\Delta}} \right) \right)^{\vee}}{H^{2q+1}_{X_{\hat{\Delta}}/\hat{\Delta}; Z}} \]

be the relative \( q \)-th intermediate Jacobian of \( X/\Delta \). Then we have an Abel-Jacobi mapping

\[ \varphi : W'_0 \times \hat{\Delta}_n \to J_q \left( X_{\hat{\Delta}}/\hat{\Delta} \right) \]

\[ (w, \hat{t}) \mapsto J^w_{Z_0} \]

defined as a morphism of analytic schemes. Further suppose

\[ A/\hat{\Delta} \subseteq J_q \left( X_{\hat{\Delta}}/\hat{\Delta} \right) \]

is an abelian subvariety orthogonal to \( H^0 \left( \Omega^{2q+1}_{X_{\hat{\Delta}}/\hat{\Delta}} \right) + \ldots + H^{q+1} \left( \Omega^q_{X_{\hat{\Delta}}/\hat{\Delta}} \right) \) such that

\[ \varphi(W'_n) \subseteq A_n \]

the abelian fiber over \( \hat{t} = 0 \). Then (5) induces a splitting of

\[ J_q \left( X_{\hat{\Delta}}/\hat{\Delta} \right) = \frac{\left( H^0 \left( \Omega^{2q+1}_{X_{\hat{\Delta}}/\hat{\Delta}} \right) + \ldots + H^{q+1} \left( \Omega^q_{X_{\hat{\Delta}}/\hat{\Delta}} \right) \right)^{\vee}}{H^{2q+1}_{X_{\hat{\Delta}}/\hat{\Delta}; Z}} \]
as follows. By Hodge theory there exists a lattice
\[ L = L_1 \oplus L_2 \]
in \( H_{2q+1}(X_{\Delta}/\Delta; \mathbb{Q}) \) such that \( H_{2q+1}(X_{\Delta}/\Delta; \mathbb{Z}) \) is of finite index and a direct sum decomposition
\[ H^0 \left( \Omega^{2q+1}_{X_{\Delta}/\hat{\Delta}} \right) + \ldots + H^{q+1} \left( \Omega^q_{X_{\Delta}/\hat{\Delta}} \right) = \left( V_1/\hat{\Delta} \right) \oplus \left( V_2/\hat{\Delta} \right) \]
such that
\[ A/\hat{\Delta} = \frac{V_1}{L_1} \]
and the above decomposition induces an isogeny
\[ J_q \left( X_{\Delta}/\hat{\Delta} \right) \rightarrow (A \oplus A^\perp)/\hat{\Delta}, \]
where
\[ A^\perp = \frac{V_2}{L_2}. \]
Furthermore the action of the Gauss-Manin connection respects the direct sum decomposition. Thus, since
\[ H^{q+2,q-1}_n \left( X_{\Delta}/\hat{\Delta} \right) \subseteq V_2/\hat{\Delta}, \]
differentiating via the Gauss-Manin connection, we have
\[ \frac{\nabla H^{q+2,q-1}_n \left( X_{\Delta}/\hat{\Delta} \right)}{\partial t} \subseteq V_2/\hat{\Delta}. \]
So from Theorem 13.1 of [C1] we have:

**Corollary 1.2.** If, under the Abel-Jacobi mapping
\[ \varphi(W'_n) \subseteq A_n, \]
the mapping
\[ R^1 p'_n \left( \Omega^3_{W_n} \right) \rightarrow \Omega^1_{W'_n} \]
given by \[ \{\} \] is zero.

### 1.3. The setting: Curves on K-trivial threefolds.
Let \( \Delta \) denote the t-disk and \( \Delta_n \subseteq \Delta \) the subscheme associated to the ideal \( \{t^{n+1}\} \). Our special situation is one in which there is a fixed projective manifold \( P \), a vector bundle
\[ \mathcal{E} \rightarrow P, \]
and
\[ X_{\Delta} \subseteq P \times \Delta \]
is a family of smooth threefolds \( \{X_t\}_{t \in \Delta} \) given as the zero scheme of a regular section
\[ \sigma : P \times \Delta \rightarrow \pi^* \mathcal{E} \]
where
\[ \pi : P \times \Delta \rightarrow P \]
is the standard projection. (In what follows, we will be working in the context of branched covers $\tilde{\Delta} \to \Delta$.) Then

$$\pi^*\mathcal{E}|_X = N_{X\Delta \setminus P \times \Delta}.$$  

Our final critical assumption is that

$$\omega_{X\Delta / \Delta} = \mathcal{O}_{X\Delta}.$$  

(6)

Let $J'$ denote a connected Zariski-open subscheme of the Hilbert scheme of curves on $P$ with universal curve

$$J \subset J' \times P \xrightarrow{q} P$$  

(7)

We assume:

i) For all $\{Z\} \in J'$, $Z$ is a connected smooth subscheme of $P$ of dimension 1.

ii) The natural map

$$T_{J'} \to p_*N_{J' \setminus J \times P}$$  

(8)

is an isomorphism and

$$R^1p_* \left( N_{J' \setminus J \times P} \right) = 0.$$  

(9)

A corollary of ii) is that

$$R^1p_* \left( q^*\mathcal{E} \right).$$  

(10)

Abusing notation we write

$$J \times \Delta \subset J' \times P \times \Delta \xrightarrow{q} P \times \Delta$$  

(11)

$$\downarrow \quad \quad p$$  

$$J' \times \Delta$$  

for the maps induced by (7) and the product structure. Thus we have that $E = p_*q^*\mathcal{E}$

is a vector bundle

$$\varepsilon : E \to J' \times \Delta$$

with a distinguished section

$$s =: p_*q^*\sigma : J' \times \Delta \to E.$$  

**Lemma 1.3.** The zero-scheme

$$I' \subseteq J' \times \Delta$$

of $s$ is the intersection of the Hilbert scheme of $X_{\Delta}$ with $J' \times \Delta$.

**Proof.** The assertion is exactly Theorem 1.5 of [K] where (10) is the needed hypothesis. \qed
1.4. **The result.** Our goal is to prove that, in the above setting, the only obstructions to finding a continuous family of curves $I'_t$ in $X_t$ are cohomological. That is, suppose that

$$M_0 \subseteq J_1 (X_0) = \frac{H^0 \left( \Omega^3_{X_0} \right) + H^1 \left( \Omega^2_{X_0} \right)}{H^3 (X_0; \mathbb{Z})}$$

is the maximal abelian subvariety of $J_1 (X_0)$ in the annihilator of $H^0 (\Omega^3_{X_0})$ and the relative intermediate Jacobian

$$J_1 (X_{\Delta}/\Delta) = \frac{H^0 \left( \Omega^3_{X_{\Delta}/\Delta} \right) + H^1 \left( \Omega^2_{X_{\Delta}/\Delta} \right)}{H^3 (X_{\Delta}/\Delta; \mathbb{Z})}$$

admits an isogeny

(11) \quad $J_1 (X_{\Delta}/\Delta) \rightarrow M \oplus M^\perp$

extending the isogeny

$$J_1 (X_0) \rightarrow M_0 \oplus M_0^\perp.$$ 

Then the relative Hilbert scheme $I'/\Delta$ of $X/\Delta$ is such that the closure of

$$I'|_{\Delta-(0)}$$

contains

$$I'_0 = I' \cap (J' \times \{0\}).$$

A surprising corollary is that a continuous family of rational curves on a complete-intersection Calabi-Yau threefold $X_0$ whose intermediate Jacobian has no abelian subvarieties always deforms.

1.5. **Organization of the proof.** In the above setting, if we have the isogeny (11), we will show that, for any map

$$\tilde{\Delta} \rightarrow X'$$

and $n$-th order extension $W'_n$ of a small open analytic subset $W'_0$ in the smooth locus of $(I'_0)_{\text{red}}$ into the relative Hilbert scheme of $X_{\tilde{\Delta}_n}/\tilde{\Delta}_n$, all sections

$$\omega \in H^{3,0} \left( X_{\tilde{\Delta}}/\tilde{\Delta} \right)$$

go to zero under the composition

$$H^{3,0} \left( X_{\tilde{\Delta}}/\tilde{\Delta} \right) \xrightarrow{\nabla} H^{2,1} \left( X_{\tilde{\Delta}}/\tilde{\Delta} \right) \xrightarrow{(\text{pull-back})} R^1 p'_* \left( \Omega^2_{W'_n/\tilde{\Delta}_n} \right) \rightarrow \Omega^1_{W'_n}.$$ 

We will show this by using the characterization of $I'$ as a gradient scheme, which is proved in [C2] to prove that, for the associated Abel-Jacobi mapping $\varphi$,

$$\varphi (W'_n) \subseteq M_n$$

where $M_n$ is the abelian variety summand of

$$J_1 \left( X_{\tilde{\Delta}_n}/\tilde{\Delta}_n \right).$$
Thus, by Theorem 1.1, if the $\varsigma \in \mathbb{R}^1 p'\ast (N_{W_0\setminus W_0'} \times X_0) / W_0 / W_0'$ is the (curvilinear) obstruction to extending $W_n / W_n'$ to a family of curves in $X_{\Delta_n+1} / \Delta_{n+1}$, the section
\[ \int_{W_0 / W_0'} (\varsigma | \omega_0) |_{W_0} \in \Omega^1_{W_0'} \otimes \{ \tilde{t}^{n+1} \} \{ \tilde{t}^{n+2} \} \]
vansishes. On the other hand, we show that the map
\[ R^1 p' (N_{W_0\setminus W_0'} \times X_0) \to R^1 p' (\Omega^2_{W_0}) \to \Omega^1_{W_0'} \varsigma' \mapsto \int_{W_0 / W_0'} (\varsigma' | \omega_0) \]
is surjective for any non-zero $\omega_0$. So our (curvilinear) obstructions $\varsigma$ to extension all lie in a proper sub-bundle of $R^1 p' (N_{W_0\setminus W_0'} \times X_0)$ of corank equal to dim $W_0'$. This gives an upper bound on the minimum number of equations defining $I'$ in $J' \times \Delta$ that will imply the desired result.

2. Curvilinear Obstructions

2.1. **Kuranishi theory.** Suppose, as in \[4\], we have families of submanifolds
\[ W_n / W_n' \]
of
\[ X_n / \Delta_n. \]
As in the proof of Theorem 12.1 of \[4\], we construct a $C^\infty$ “transversely holomorphic” trivialization
\[ F : W_0' \times X_{\Delta} \to W_0' \times X_0 \times \Delta, \]
which is “adapted to \[4\],” that is,
\[ p_n^{-1} (\{ w \} \times \Delta_n) \subseteq F^{-1} (Z_w \times \{ w \} \times \Delta) \]
for each $w \in W_0'$. Let
\[ \xi = \sum_{j>0} \xi_j \tilde{t}^j \]
be the Kuranishi data associated to this trivialization of the deformation $X_{\Delta} / \Delta$.
We consider the associated family of complex tori
\[ J_q (X_{\Delta} / \Delta) = \frac{(H^0 (\Omega^{q+1}_{X_{\Delta} / \Delta}) + \ldots + H^{q+1} (\Omega^q_{X_{\Delta} / \Delta})) \vee}{H_{2q+1} (X_{\Delta} / \Delta; \mathbb{Z})}. \]
If, as in \[3\], we define
\[ K^{p,q} = H^q (A^0_{X_0}, \Omega^p_{X_0} - L^1_{\xi} \Omega^0_{X_{\Delta} / \Delta}), \]
then in \[3\] it is shown that
\[ H^q (\Omega^p_{X_{\Delta} / \Delta}) = e^{(\xi)} K^{p,q} \]
and so we can rewrite $J_q \left( X_{\tilde{\Delta}} \right)$ as

$$J_q \left( X_{\tilde{\Delta}} \right) = \left( e^{\langle \xi \rangle} \left( K^{2q+1,0} + \ldots + K^{q+1,q} \right) \right)^\vee.$$ 

In this last formulation the Abel-Jacobi map

$$\varphi : W'_0 \times \tilde{\Delta}_n \to J_q \left( X_n / \tilde{\Delta}_n \right)$$

is given by the mapping

$$(w, t) \mapsto \left( \int_{Z_0}^{Z_w} : e^{\langle \xi \rangle} \left( K^{2q+1,0} + \ldots + K^{q+1,q} \right) \to \mathbb{C} \right).$$

Recapping the proof of Theorem 13.1 of [C1],

$$\xi = \sum_{j > 0} \xi_j \tilde{t}^j$$

has by construction the property that, for $j \leq n$,

$$(13) \quad \xi_j |_{W_0} \in A^{0,1}_{W_0} (T_{p_0})$$

where $T_{p_0}$ is the sheaf of tangent vectors to the fibers of the fibration $p_0$. Again following [C2] the action of the Gauss-Manin connection is given by the operator

$$\left\langle \frac{\partial \xi}{\partial t} \right\rangle.$$ 

Thus by type (12) restricted to

$$(14) \quad \int_{Z_0}^{Z_w} : \left\langle \frac{\partial \xi}{\partial t} \right\rangle H^{q+2,q-1} (X_0) \to \mathbb{C}$$

is given over $\tilde{\Delta}_n$ by

$$\int_{Z_0}^{Z_w} : \left\langle \frac{\partial \xi}{\partial t} \right\rangle H^{q+2,q-1} (X_0) \to \mathbb{C}$$

since, using (13), all other summands are zero through order $n$. The mapping (14) vanishes on $W'_0 \times \tilde{\Delta}_{n-1}$ by construction and the proof concludes by showing that the element in

$$R^q p'_* \left( \Omega_{W'_0}^{p+q} \otimes \{ \tilde{t}^{n+1} \} \right.$$ 

given by (14) is $n$ times the element $\mathfrak{F}$ given by capping with the obstruction class $\varsigma$ to extending the family $W'_n / W'_n$ to a family of submanifolds over $\Delta_{n+1}$.

2.2. **Curvilinear obstructions suffice.** Since we will need to work with obstructions arising from only curvilinear deformations, we will need the following general lemma before proceeding. Let

$$X \to X'$$

be a family of complex manifolds parametrized by a smooth space $X'$ with distinguished point $\{ X_0 \}$. 

Lemma 2.1. Let $Z_0$ be a submanifold of $X_0$ and let

$$Z_0 \subseteq I \subseteq X$$

$$\{0\} \in I' \subseteq X'$$

be a maximal flat deformation of $Z_0$ in $X/X'$. Suppose that, for all $n > 0$ and for all flat deformation

$$Z_n = \hat{\Delta}_n \times_I I \to X_{\hat{\Delta}} \to X$$

$$\hat{\Delta}_n \to \Delta \to X'$$

of $Z_0$, the obstructions $\nu$ to extending the family to a diagram

$$Z_{n+1} \to X_{\hat{\Delta}} \to X$$

$$\hat{\Delta}_{n+1} \to \Delta \to X'$$

lie in some fixed subspace

$$U \subseteq H^1(N_{Z_0 \setminus X_0}).$$

Then

$$\dim_{(Z_0)} I' \geq \dim X' - \dim U.$$ 

Proof. Let $I'$ denote the ideal in the local ring of $X'$ at 0 which locally defines $I'$. The obstructions to further extension of

$$I \to X$$

$$I' \subseteq X'$$

are given by a monomorphism

$$v : \left( \frac{I'}{m_{0,X'} \cdot I} \right)^\vee \to H^1(N_{Z_0 \setminus X_0}).$$

(See Chapter 1 of [Ko]). Let $g_1, \ldots, g_s \in I'$ give a basis for

$$\frac{I'}{m_{0,X'} \cdot I}.$$

Let $r = \text{codim}_0 (I', X') \leq s$ and, using the “Curve Selection Lemma” (Lemma 5.4 of [FM]), choose maps

$$f_i : \Delta \to X', \quad i = 1, \ldots, r,$$

such that

$$g_j \circ f_i = 0, \quad \forall j < i$$

and

$$g_i \circ f_i$$

generates the non-zero ideal $\{i^m_i\} = f_i^*((I')).$ By construction (see, for example, the proof of Proposition 4.7 of [BF]), each element

$$v \left( (f_i)_* \left( (i^m_i)^\vee \right) \right)$$

(15)
is non-zero in the quotient vector space
\[ H^1 \left( N_{Z_0 \setminus X_0} \right) \]
\[ \langle v \left( (f_j)_* \left( (t_i) \right) \right) \rangle_{j < i} \]

So the elements (17) are linearly independent in
\[ U \subseteq H^1 \left( N_{Z_0 \setminus X_0} \right). \]

So \( r \leq \dim U. \)

3. Local analytic obstructions

3.1. Algebraic obstruction map. For the rest of this paper we assume that we are in the situation of 1.3. As in [CK] we consider \( E^\vee \) as linear functions on \( E \) which we pull back to \( J' \) via the section \( s = p_* q^* \sigma \). Thus by Lemma 1.3 (16)
\[ s^* : E^\vee \to \mathcal{O}_{J' \times \Delta} \]
has image equal to the ideal sheaf
\[ \mathcal{I}' \]
of \( I' \) in \( J' \times \Delta \). we have
\[
\begin{array}{c}
E^\vee \\
\downarrow \\
\Omega_{J' \times \Delta / \Delta}^1 \\
\end{array} 
\]
\[ \to 0 \]
(17)
and so the restriction induced diagram
\[
\begin{array}{c}
E^\vee_{\Delta \times J'} \\
\downarrow \\
\Omega_{\Delta \times J' / \Delta}^1 \\
\end{array} 
\]
\[ \to 0 \]
(18)

Let \( S'_0 \) denote a reduced smooth quasi-projective scheme in \( I'_0 \) such that
\[ p_* N_{S_0 \setminus S'_0 \times X_0} \]
is locally free. Using this diagram
\[
\begin{array}{c}
E^\vee_{(S'_0)} \\
\downarrow \\
\Omega_{J' \times \Delta |_{(S'_0)}} \to 0 \\
\end{array} 
\]
\[ \Omega_{I'_0 |_{(S'_0)}} \to 0 \]

with exact rows and vertical surjections as well as (17) and (18), we can choose
\[ g_1, \ldots, g_{r'} \in \mathcal{T}' |_{(S'_0)} \]
such that
\[ dg_1, \ldots, dg_{r'} \]
restrict to a basis for the kernel of the morphism
\[ \Omega_{J' |_{(S'_0)}} \to \Omega_{I'_0 |_{(S'_0)}}. \]

The \( g_i \) define an analytic space
\[ U'_{(S'_0)} \subseteq \Delta \times J' \]
(19)
which is smooth over $\Delta$ and such that $I' \subseteq W_\Delta'$. Now
\[
\dim U_\langle S_0' \rangle = \dim J' - r' + 1
\]
and the dimension of the fiber of $U_\langle S_0' \rangle / \Delta$ is the rank of $p_*N_{S_0\setminus S_0'}$. Let
\[
S_0'
\]
be the ideal sheaf of $S_0'$ (on the appropriate open dense subset of $J'$). Shrinking the open dense subset if necessary, we can assume that
\[
I'_{S_0'} \subseteq W_\langle S_0' \rangle \Delta
\]
Now
\[
\dim U_\langle S_0' \rangle / \Delta = \dim J' - r' + 1
\]
and the dimension of the fiber of $U_\langle S_0' \rangle / \Delta$ is the rank of $p_*N_{S_0\setminus S_0'}$.

Let
\[
S_0'
\]
be the ideal sheaf of $S_0'$ (on the appropriate open dense subset of $J'$). Shrinking the open dense subset if necessary, we can assume that
\[
I'_{S_0'} \subseteq W_\langle S_0' \rangle \Delta
\]
is also locally free. Define
\[
S_0 = S_0' \times I'_{U_0} \Delta
\]
Applying $R\pi_*$ to the exact sequence
\[
0 \to N_{S_0\setminus S_0'} \to q^*N_{X_0\setminus p} \to 0
\]
we obtain the exact sequence
\[
0 \to p_*N_{S_0\setminus S_0'} \to T_\langle S_0' \rangle \to E_\langle S_0' \rangle \to R^1p_*N_{S_0\setminus S_0'} \to 0.
\]
Now from (17) we have exact
\[
E_{\langle S_0' \rangle} \to \frac{T'}{S_0' \cdot T'} \to 0
\]
from which we achieve an injection
\[
E_{\langle S_0' \rangle} \to \frac{T'}{S_0' \cdot T'} \to 0
\]
and so, composing with the surjection
\[
E_{\langle S_0' \rangle} \to R^1p_*N_{S_0\setminus S_0'} \times X_0
\]
in (21) we construct a map
\[
\alpha : \left(\frac{T'}{S_0' \cdot T'}\right) \to R^1p_*N_{S_0\setminus S_0'} \times X_0.
\]

3.2. Analytic obstruction map. On the other hand, by general obstruction theory we have an injective map
\[
\beta : \left(\frac{T'}{S_0' \cdot T'}\right) \to R^1p_*N_{S_0\setminus S_0'} \times X_0
\]
constructed via Kuranishi theory. (See §13 of [C1]; also, for example, Chapter 1 of [Kn]). As in (21) we start with a local versal deformation $X'/X'$ of $X_0$ and let $U'$ be smooth over $X'$ of minimal dimension containing the local (relative) Hilbert scheme $Y'$. Then
\[
Y'_\Delta = Y' \times X', \Delta \subseteq I'.
\]
Furthermore we can assume that
\[
U'_\Delta = U' \times X', \Delta
\]
is an open neighborhood of \((\{Y_0\}, \{X_0\})\) in the smooth space \(U'_{(S'_0)}\) constructed in \([19]\).

In \([22]\) one constructs a \(C^\infty\) “transversely holomorphic” trivialization
\[
F : X \times X', U' \to X_0 \times U',
\]
which is “adapted to the universal curve \(Y/Y'\) at \(\{Y_0\}\),” that is,
\[
Y \subseteq F^{-1}(Y_0 \times U').
\]

For each analytic map
\[
\vartheta : \tilde{\Delta} \to U'_\Delta
\]
sending 0 to a point \(s'_0 \in Y_\Delta \cap S'_0\) and having the property that
\[
\vartheta^*(T') = \{\tilde{t}^{n+1}\},
\]
the Kuranishi data associated to the trivialization \(F\) produces the obstruction element
\[
\varsigma \in H^1\left(N_{S_0/S'_0} \setminus \chi_0\right)
\]
in \([8]\) as the image of
\[
\left(\tilde{t}^{n+1}/\tilde{t}^{n+2}\right) \subseteq \left(T'/S'_0 \cdot T\right)^\vee
\]
under \(\beta\).

We wish to show that under certain base extensions, \(\alpha\) and \(\beta\) give the same obstruction map. But first we need to make precise the pull-backs we will need for this assertion. Let
\[
V = Y'_\Delta \cap S'_0
\]
as above, let \(D\) an auxiliary polydisk, and suppose we have an analytic isomorphism
\[
\psi : V \times D \to U'_\Delta
\]
such that
\[
\psi|_{V \times \{0\}} = \text{identity}_V.
\]
By flat stratification we can, after shrinking \(V\) if necessary, arrange that the family of ideals
\[
\psi^* (T')|_{\{v\} \times D}
\]
is flat over \(V\). We then use the “Curve Selection Lemma” (Lemma 5.4 of \([FM]\)) as in the proof of Lemma \([2,1]\) on each \(\{v\} \times D\), but this time with analytic parameter \(v\). Letting \(\tilde{t}\) denote the analytic parameter of a complex disk \(\tilde{\Delta}\), we can, after possibly again shrinking \(V\), choose analytic maps
\[
f_i : V \times \tilde{\Delta} \to V \times D, \quad i = 1, \ldots, r,
\]
such that
\[
g_j \circ f_i = 0, \quad \forall j < i
\]
and
\[
g_i \circ f_i
\]
generates the non-zero ideal \( \{ \tilde{t}_n \} = f^* (I') \). By construction (see again, for example, the proof of Proposition 4.7 of [BH]), each element
\[
\beta_* \left( \psi_* \left( (f_i)_* \left( (\tilde{t}_n^i)\right) \right) \right)
\]
gives a never-zero section of the quotient vector bundle
\[
R^1 p_* \left( N_Y \right) \langle \beta_* \left( \psi_* \left( (f_j)_* \left( (\tilde{t}_n^j)\right) \right) \right) \rangle_{j<i},
\]
3.3. Equality of the two obstruction maps. We claim that, for each \( i \),
\[
\alpha_* \left( \psi_* \left( (f_i)_* \left( (\tilde{t}_n^i)\right) \right) \right) = \beta_* \left( \psi_* \left( (f_i)_* \left( (\tilde{t}_n^i)\right) \right) \right)
\]
(24)
To prove the claim, it suffices to prove the following lemma.

**Lemma 3.1.** Let \( \{ Z_0 \} \in S'_0 \) and let
\[
\gamma : \tilde{\Delta} \to J' \times \Delta
\]
\[
\tilde{t} \mapsto (j' (\tilde{t}) , t (\tilde{t}))
\]
be an analytic map of the \( \tilde{t} \)-disk \( \tilde{\Delta} \) such that
\[
\gamma (0) = \{ Z_0 \}.
\]
Let
\[
\{ \tilde{t}^m+1 \} = \gamma^* (I').
\]
Then
\[
(\alpha \circ \gamma_* ) \left( \begin{array}{c}
\{ \tilde{t}^m+1 \} \\
\{ \tilde{t}^m+2 \}
\end{array} \right) = (\beta \circ \gamma_* ) \left( \begin{array}{c}
\{ \tilde{t}^m+1 \} \\
\{ \tilde{t}^m+2 \}
\end{array} \right).
\]

**Proof.** We consider the pull-back family of threefolds
\[
\begin{array}{ccc}
\tilde{X}_{\tilde{\Delta}} & \to & X_{\Delta} \\
\downarrow & & \downarrow \\
\tilde{\Delta} & \to & \Delta
\end{array}
\]
given by the zero-scheme of the section \( \tilde{\sigma} \) of the pull-back bundle
\[
(id., t (\tilde{t}))^* \pi^* \mathcal{E}
\]
on
\[
P \times \tilde{\Delta}.
\]
We have a family of curves
\[
\tilde{I}_n/\tilde{\Delta}_n \subseteq \tilde{X}_n/\tilde{\Delta}_n
\]
via pull-back of the family \((J/J') \times \Delta \) under \( \gamma \). Since
\[
R^1 p_* N_{J/J' \times P} = 0
\]
the family (24) extends to a family of curves
\[
\begin{array}{ccc}
\tilde{J} & \subseteq & P \times \tilde{\Delta} \\
\downarrow & \tilde{q} & \downarrow \bar{q} \\
\tilde{\Delta} & \to & P
\end{array}
\]
Then by Lemma 1.3 the zero-scheme of the section
\[ \tilde{s} = \tilde{p}_* \tilde{q}^* \tilde{\sigma} \]
of
\[ \tilde{E} = \tilde{p}_* \tilde{q}^* \tilde{\sigma} \]
is exactly
\[ \tilde{\Delta}_n. \]

Now construct a (transversely holomorphic) \( C^\infty \)-trivialization
\[ F : P \times \tilde{\Delta} \to P \times \tilde{\Delta} \]
over \( \tilde{\Delta} \) such that
i) 
\[ F(X) = X_0 \times \tilde{\Delta}, \]
ii) 
\[ F(\tilde{J}_n) \subseteq Z_0 \times \tilde{\Delta}. \]

Then as in §11 of [C1] the Kuranishi data
\[ \tilde{\xi} = \sum_{j>0} \tilde{\xi}_j \tilde{t}^j \]
associated to the trivialization \( F \) has the property that
\[ \left\{ \tilde{\xi}_{n+1} \big|_{Z_0} \right\} \in H^1(N_{Z_0 \setminus X_0}) \]
is the obstruction class to extending the family \( \tilde{J}_n \) to a family of curves in \( \tilde{X}_{n+1}/\tilde{\Delta}_{n+1} \). But by the construction of the trivialization \( F \) above, this class is given by the obstruction to extending the map
\[ \tilde{J}_n/\tilde{\Delta}_n \to \tilde{X}/\tilde{\Delta} \]
to a map
\[ \tilde{J}_{n+1}/\tilde{\Delta}_{n+1} \to \tilde{X}/\tilde{\Delta}. \]
But this last obstruction class, which lives in
\[ H^1(N_{Z_0 \setminus Z_0 \times X_0}) = H^1 \left( q^* T_{\tilde{X}_0} \big|_{Z_0} \right), \]
is the image of
\[ \tilde{s} \in \left\{ \tilde{t}^{n+1} \right\} \tilde{E} \]
under the map
\[ \tilde{E}_{\{Z_0\}} \to H^1(T_{X_0} \big|_{Z_0}) \]
induced by the exact sequence
\[ 0 \to q^* T_{X_0} \big|_{Z_0} \to q^* T_P \big|_{Z_0} \to \mathfrak{C} \big|_{Z_0} \to 0. \]
3.4. Obstructions as Kähler differentials.

**Lemma 3.2.** The map

\[(26) \quad R^1 p_* N_{Y_0 \setminus Y_0} \times X_0 \to \Omega^1_{Y'}\]

given in (3) in Theorem 1.1 is surjective.

**Proof.** Let \( \Gamma' \) denote the diagonal of \( S'_0 \times S'_0 \) and let \( \Gamma \) be the inverse image of \( \Gamma' \) under

\[ S'_0 \times S_0 \to S'_0 \times S'_0 \]

Then

\[
\Omega^1_{S'_0} = N^*_{\Gamma' \setminus S'_0} \otimes \omega_{S_0 / S'_0}
\]

\[
= R^1 p_* \left( N^*_{\Gamma' \setminus S_0} \otimes \omega_{S_0 / S'_0} \right)
\]

disjunctive isomorphism being often called “integration over the fiber.” Now consider the map

\[ N^*_{S_0 \setminus S'_0} \times X_0 \to N^*_{S'_0 \setminus S_0} \]

induced by

\[ S'_0 \times S_0 \to S'_0 \times X_0 \]

The induced map

\[
R^1 p_* \left( N^*_{S'_0 \setminus S_0} \otimes \omega_{S_0 / S'_0} \right) \to R^1 p_* \left( N^*_{\Gamma' \setminus S_0} \otimes \omega_{S_0 / S'_0} \right)
\]

is surjective because its dual via Verdier duality

\[ T_{S'_0} \to p_* N_{S'_0 \setminus S_0} \times X_0 \]

is injective. Composing (27) with integration over the fiber we obtain a surjection

\[
R^1 p_* \left( N^*_{S'_0 \setminus S_0} \otimes \omega_{S_0 / S'_0} \right) \to \Omega^1_{S'_0},
\]

Finally recall that \( S'_0 \) denotes the ideal of \( S_0 \subseteq S'_0 \times X_0 \). Since \( \omega_{X_0} \) is trivial and \( X_0 \) is a threefold we have

\[
\bigwedge^2 \left( S_0 / S'_0 \right) \otimes \omega_{S_0 / S'_0} = q^* \omega_{X_0} = \mathcal{O}_{S_0}.
\]

Thus

\[
N^*_{S'_0 \setminus S_0} \otimes \omega_{S_0 / S'_0} = \mathcal{O}_{S_0}.
\]

or, said otherwise,

\[
N^*_{S'_0 \setminus S_0} \otimes \omega_{S_0 / S'_0} \cong \mathcal{O}_{S_0}.
\]

where \( \alpha \) is a fixed generator of \( H^0(\omega_{X_0}) \). Composing with (28) we have the desired surjection

\[
R^1 p_* N_{S'_0 \setminus S_0} \times X_0 \to R^1 p_* \left( N^*_{S'_0 \setminus S_0} \otimes \omega_{S_0 / S'_0} \right) \to \Omega^1_{S'_0},
\]

\[ \nu \mapsto \langle \nu | q^* \alpha \rangle \]
3.5. **The crucial estimate.** Now the restriction of the map \((30)\) to \(Y'\) is the map \((3)\) in Theorem 1.1. So, in a situation in which the hypotheses of Theorem 1.1 are satisfied, the obstructions \((24)\) lie in the kernel of the surjection
\[ R^1 p_* N_0 \to \Omega^1_{S_0}. \]
Then applying Lemma 2.1 we can conclude
\[(31) \dim\{Z\}^I_{0} \geq \dim U^I_{0} (S_0) - (\text{rank} \ R^1 p_* N_0 - \dim S_0).\]

We now come to the main result of this paper, which treats a situation in which the hypotheses of Theorem 1.1 are indeed satisfied.

4. **The Hilbert scheme as gradient variety**

Suppose now that
\[ M_0 \subseteq J_1 (X_0) = \frac{(H^0 (\Omega^3_{X_0}) + H^1 (\Omega^2_{X_0}))^\vee}{H^3 (X_0; \mathbb{Z})} \]
is the maximal abelian subvariety of \(J_1 (X_0)\) in the annihilator of \(H^0 (\Omega^3_{X_0})\) and the relative intermediate Jacobian
\[ J_1 (X_{\Delta}/\Delta) = \frac{(H^0 (\Omega^3_{X_{\Delta}/\Delta}) + H^1 (\Omega^2_{X_{\Delta}/\Delta}))^\vee}{H^3 (X_{\Delta}/\Delta; \mathbb{Z})} \]
admits an isogeny
\[(32) \ J_1 (X_{\Delta}/\Delta) \to M \oplus M^\perp \]
extending the isogeny
\[ J_1 (X_0) \to M_0 \oplus M_0^\perp. \]

Let
\[ \overline{T} \]
do not the Zariski closure of \(I'\) in \(J' \times \Delta\). The composition of the Abel-Jacobi mapping
\[ \overline{T} \to J_1 (X_{\Delta}/\Delta) \]
with projection gives a map
\[ \overline{T} \to \overline{M} \]
whose image is proper over \(\Delta\). But by the maximality of \(M\), the image must therefore be finite over \(\Delta\).

It is at this point that the gradient variety construction in [2] enters. There we consider \(X'\), a versal local deformation of the \(K\)-trivial threefold \(X_0\), \(U'\) smooth over \(X'\) of minimal dimension containing the local (relative) Hilbert scheme \(Y'\), and
\[ \tilde{X}' = \{(x', \omega_{x'}) : x' \in X', \ \omega_{x'} \in (H^0 (\Omega^3_{X'/X}) - \{0\})\} \]
\[ \tilde{U}' = \tilde{U}' \times X', \tilde{X}' \xrightarrow{\tilde{\pi}} \tilde{X}'. \]
We choose a holomorphic product structure
\[ U' = U_0' \times X' \]
\[ \tilde{U}' = U_0' \times \tilde{X}' \]
and use it to define a map

\[ \kappa : U'_\Delta = U' \times_{X'} \Delta \to \tilde{U}' \]

which lifts the natural map

\[ U'_\Delta \to U'. \]

Theorem 6.1 of \[C2\] gives a holomorphic function Φ on \( \tilde{U}' \) such that, with respect to the exact sequence

\[
0 \to \tilde{\pi}^* \Omega^1_{\tilde{X}'}, \to \Omega^1_{U'}, \to \Omega^1_{U'/\tilde{X}'} \to 0,
\]

we have:

**Property 1:** The relative Hilbert scheme \( \tilde{Y}' \), considered as an analytic sub-scheme of \( U' \) is the zero-scheme of the section

\[ d_{U'/\tilde{X}'} \Phi \]

of

\[ \Omega^1_{U'/\tilde{X}'} . \]

**Property 2:** Under a natural isomorphism

\[
F^2H^3(\tilde{X}/\tilde{X}') \cong T_{\tilde{X}'} , \\
\Omega^1_{\tilde{X}'} \cong \left( F^2H^3(\tilde{X}/\tilde{X}') \right)^\vee
\]

given by Donagi-Markman, the section

\[ d\Phi|_{\tilde{Y}'} \]

of

\[ \tilde{\pi}^* \Omega^1_{\tilde{X}'} \]

is the normal function

\[
(33) \int_{r(L \times X', Y')/Y'} \tilde{Y}' \to \left( F^2H^3(\tilde{X}/\tilde{X}') \right)^\vee .
\]

Using the product structure on \( \tilde{U}' \) therefore, the holomorphic section \( \tilde{\pi}^* \pi^* \Phi \) of the bundle \( \left( F^2H^3(X \times X', U'/U') \right)^\vee \) extend the Abel-Jacobi mapping \([34]\). Thus the pullback \( \kappa^* (\tilde{\pi}^* \pi^* \Phi) \) gives a holomorphic mapping

\[
(34) U'_\Delta \to \left( F^2H^3(X_{U'_\Delta}/U'_\Delta) \right)^\vee
\]

extending the Abel-Jacobi mapping on \( Y'_\Delta \).

As above let

\[ V = Y'_\Delta \cap S'_0. \]

Suppose, for a mapping of disks

\[ \gamma : \tilde{\Delta} \to \Delta, \]

such that \( \gamma (0) = 0 \), we have a mapping

\[ \delta : V \times \tilde{\Delta} \to U'_{\Delta} \]

such that

\[ Y_{\Delta_n}/Y'_{\Delta_n} \]
is an extension to a family of curves in $X_{\Delta_n}$. Then there is a holomorphic map
\[ \delta : V \times \hat{\Delta} \rightarrow U'_{\Delta} \]
such that
1)\[ \delta (s'_0, 0) = (\{S_{s'_0}\}, \{X_0\}) \]
2) for all $\tilde{t} \in \hat{\Delta}$, $\delta$ immerses $V \times \{\tilde{t}\}$ into the fiber of $X'$ over $\{\gamma (t)\}$, 3)
\[ \delta \left( V \times \hat{\Delta}_n \right) = Y'_{\Delta_n}. \]

Pulling back by $\delta$, we have from (34) an extension
\[ (35) \quad V \times \hat{\Delta} \rightarrow \left( F^2 H^3 \left( X_{\hat{\Delta}}/\hat{\Delta} \right) \right)^\vee \]
of the Abel-Jacobi map on $V \times \hat{\Delta}_n$.

Now by hypothesis we have an isogeny
\[ (36) \quad J_1 \left( X_{\hat{\Delta}}/\hat{\Delta} \right) \rightarrow \hat{M} \oplus \hat{M}^\perp. \]
so
\[ \left( F^2 H^3 \left( X_{\hat{\Delta}}/\hat{\Delta} \right) \right)^\vee \cong \hat{\Delta} \times \mathbb{C}^{r_1} \times \mathbb{C}^{r_2} \]
where the vector spaces $\mathbb{C}^{r_1}$ and $\mathbb{C}^{r_2}$ correspond to the summands $\hat{M}$ and $\hat{M}^\perp$ respectively. So there is a projection of (36) onto the second factor given by $r_2$ holomorphic functions
\[ g_{ik} (s'_0, t) = \sum_{k=0}^{\infty} g_{ik} (s'_0) \tilde{t}^k. \]
where each $g_{ik}$ is a holomorphic function on $V$. Suppose for some $i$ that $g_{ik}$ is not a constant function for some $k \leq n$. Then the projection of the Abel-Jacobi map
\[ \varphi : V \times \hat{\Delta}_n \rightarrow J_1 \left( X_{\hat{\Delta}_n}/\hat{\Delta}_n \right) \]
into the factor $\hat{M}^\perp_n := \hat{\Delta}_n \times \hat{\Delta} M^\perp$ cannot lie inside an finite analytic subscheme of $\hat{M}^\perp_m$. However
\[ Y'_{\Delta_n} \subseteq I' \times \Delta A'_{\Delta_n} \]
and, as we have seen above, the image of the composition
\[ I' \times \Delta A'_{\Delta_n} \rightarrow J_1 \left( X_{\Delta_n}/\Delta \hat{\Delta} \right) \rightarrow \hat{M}^\perp_n \]
must be finite. Thus the functions $g_{ik} (s'_0)$ are constant for all $k \leq n$. Adjusting the isogeny (36) by a translation by these constants, we have
\[ \varphi \left( V \times \hat{\Delta}_n \right) \subseteq \hat{M}. \]
So by Corollary 1.2, the pairing
\[ R^1 p'_* \left( \Omega^1_\nu \right) \rightarrow \Omega^1_\nu \]
given by (3) is zero.
5. The main theorem

Continuing with the situation of §3, for

\[ I_0' := (I' \cap \{0\} \times J') \]

let \( S_0' \) be a component of

\[ (I_0')_{\text{red}} \]

and let \( \langle S_0' \rangle \) denote the generic point of \( S_0' \). We let \( \langle S_0 \rangle \) denote \( p^{-1}(\langle S_0' \rangle) \) where

\[ p : S_0 \rightarrow S_0' \]

is the universal curve. Then

\[ R^1 p_*(\text{Hom}_I(I/I^2, \mathcal{O}_I))|_{\langle S_0' \rangle} \]

is a vector space over \( \langle S_0' \rangle \) of dimension

\[ h^1(N_{Z_0\setminus X_0}) \]

where, referring to \( \{Z_0\} \in Y_0' \subset S_0 \). We consider the Abel-Jacobi map

\[ \varphi_0 : S_0' \rightarrow A_0 \subseteq J_1(X_0) = \left( \frac{H^0(\Omega^3_{X_0}) + H^1(\Omega^2_{X_0})}{H^3(X_0, \mathbb{Z})} \right) \]

where \( A_0 \) is an abelian variety spanned by \( \varphi_0(S_0') \). This map is determined by the choice of basepoint \( \{Z_0\} \in S_0' \). We write

\[ A_0 \subseteq M_0 \subseteq J_1(X_0) \]

where \( M_0 \) denotes the maximal abelian subvariety (lying in \( H^0(\Omega^3_{X_0}) \)).

**Theorem 5.1.** *Given (37), suppose that*

\[ M/\Delta \subseteq J_1(X/\Delta) = \left( \frac{H^0(\Omega^3_{X/\Delta}) + H^1(\Omega^2_{X/\Delta})}{H^3(X/\Delta, \mathbb{Z})} \right) \]

*is an abelian subvariety over \( \Delta \) extending \( M_0 \). Then (possibly after base extension) there is a family of curves on generic \( X_t \) generating an abelian subvariety \( A_t' \subseteq J_1(X_t) \) via the Abel-Jacobi mapping such that \( A_0 \subseteq A_0' \subseteq M_0 \).*

**Proof.** We first claim that the inequality (34) holds. To prove this, we must check that the hypotheses of Theorem 4.2 hold for one-parameter local analytic families extending \( Y_0/Y_0' \). But, since we are in a projective algebraic situation, these analytic families must come from analytic localizations as in §4 of maximal algebraic families. Using a small disk \( \bar{\Delta} \) with parameter \( \bar{t} \) as before, these algebraic families \( T'/\bar{\Delta} \) are of the following type. We consider a morphism

\[ T' \xrightarrow{\rho} \Delta \times J' \]

\[ \bar{\Delta} \rightarrow \Delta \]

such that \( T'/\bar{\Delta} \) is proper and flat and, if \( T_0' \) denotes the fiber over \( 0 \in \bar{\Delta} \), \( \dim_{\mathbb{C}} T_0' = \dim_{\mathbb{C}} S_0' \) and for some generic point \( \langle T_0' \rangle \) of \( T_0' \),

\[ \rho(T_0') = \langle S_0' \rangle. \]
For any such \( \rho \), let \( m \) be minimal such that
\[
\tilde{t}^{m+1} \in (\rho^* T')_{(T'_0)}.
\]
We let
\[
\tilde{X}/\tilde{\Delta}, \ M/\tilde{\Delta}
\]
be the pullbacks of \( X/\Delta \) and \( M/\Delta \) respectively,
\[
\tilde{\Delta}_m = \text{Spec} \mathbb{C} \left[ \tilde{t} \right] / \{ \tilde{t}^{m+1} \},
\]
and
\[
T'_m = T' \times_{\tilde{\Delta}} \tilde{\Delta}_m.
\]
We let
\[
\tilde{M}_m = \tilde{M} \times_{\tilde{\Delta}} \tilde{\Delta}_m.
\]
We have assumed that \( M_0 \) is the maximal abelian subvariety of \( J_1(X_0) \), \( \tilde{M}_m/\tilde{\Delta}_m \) is the maximal abelian subvariety of \( \tilde{J}_1(\tilde{X}_m/\tilde{\Delta}_m) \) for each \( m \). So the Abel-Jacobi mapping
\[
\tilde{\varphi} : T'_m/\tilde{\Delta}_m \to \tilde{J}_1(\tilde{X}/\tilde{\Delta})
\]
factors through the inclusion
\[
\tilde{M}_m/\tilde{\Delta}_m \subseteq \tilde{J}_1(\tilde{X}/\tilde{\Delta}).
\]
So we must show that, for each diagram (39) as above, the composition
\[
(\{ \tilde{t}^{m+1} \} / \{ \tilde{t}^{m+2} \}) \overset{\tilde{\varphi}}{\to} R^1 \hat{p}_* N_{(T_0) \setminus (T'_0)} \to \Omega^1_{(T'_0)}
\]
is zero, that is, the map
\[
H^{3,0}(\tilde{X}/\tilde{\Delta}) \overset{\tilde{\varphi}}{\to} H^{2,1}(\tilde{X}/\tilde{\Delta}) \overset{\text{restrict}}{\to} R^1 \hat{p}_* (\Omega^2_{(T_m)/\tilde{\Delta}_m}) = \Omega^1_{(T'_m)}
\]
given by the Gauss-Manin connection is zero.

But in the last section we established that
\[
\varphi((T'_0)) \subseteq \tilde{M}_0.
\]
So by Corollary 1.2, the hypotheses of Theorem 1.1 are satisfied and we have established that (39) is the zero map. Then by (31) we have at a general point \( \{ Z_0 \} \in S'_0 \) that
\[
\dim X' - \dim_{(Z_0)} I' \leq h^1 (N_{Z_0 \setminus X_0}) - \dim S'_0
\]
\[
= h^0 (N_{Z_0 \setminus X_0}) - \dim S'_0
\]
\[
= \dim (X' \cap \{ 0 \} \times J') - \dim S'_0.
\]
On the other hand, since codimension is lower semi-continuous,
\[
\dim (X' \cap \{ 0 \} \times J') - \dim S'_0
\]
\[
= \dim (X' \cap \{ 0 \} \times J') - \dim_{(Z_0)} (I' \cap \{ 0 \} \times J')
\]
\[
\leq \dim X' - \dim_{(Z_0)} I'.
\]
Thus the above inequalities are actually equalities and
\[
\dim_{(Z_0)} I' = \dim S'_0 = \dim X' - \dim (X' \cap \{ 0 \} \times J') = 1.
\]
So \( S'_0 \) is a codimension-one subvariety of some component of \( I'_{\text{red}} \). So, in particular, the general curve \( Z_0 \) parametrized by \( S'_0 \) is the specialization of curves on the nearby \( X_t \), which completes the proof of the theorem.

The weak point of Theorem 5.1, in addition to the fact that it is only a variational result, is the hypothesis that the curves \( Z \) of the family parametrized by \( S'_0 \) are unobstructed in \( P \). That this condition is necessary for the proof is shown by the following example due to C. Voisin. Let \( Z \) be the generic projection to \( \mathbb{P}^3 \) of a canonical curve of genus 5 and let \( X_0 \) be the most general quintic in \( \mathbb{P}^4 \) which contains \( Z \). Let \( A_0 \) be zero. In this case \( I'_{\text{red}} = S'_0 = \mathbb{P}^1 \), and \( Z \) moves in a basepoint-free pencil on a smooth hyperplane section of \( X_0 \). However a constant count shows that the generic quintic threefold contains only a finite number of curves which can specialize to curves in \( I' \). The problem is that \( Z \) lies in two components of the Hilbert scheme of curves (of genus 5 and degree 8) in \( \mathbb{P}^4 \), one corresponding to full canonical embeddings of curves of genus 5 and the other obtained by deforming the line bundle used to imbed \( Z \) to a general line bundle of degree 8. (One might be tempted to reimbed \( X_0 \) “more positively” in some “bigger” \( P \) to achieve the unobstructedness of \( Z \) in the new \( P \), but then, of course, one loses the vector bundle \( \mathcal{E} \).)

Notice that the strong unobstructedness condition

\[
H^1 (N_{Z/P}) = 0
\]

is always satisfied if

\[
Z \subseteq P = \mathbb{P}^n
\]

is a rational curve. Thus the final assertion in 1.4.

6. An example

A rather striking yet simple example of Theorem 5.1 is the following result on the mirror family to quintic threefolds, originally proved for the case of lines by van Geemen [AK]. Consider the pencil \( X_t \) of quintic threefolds given by

\[
F_t = \left( \sum_{j=0}^4 x_j^5 \right) - 5t \prod_{j=0}^4 x_j.
\]

Let \( \mu_5 \leq \mathbb{C}^* \) denote the fifth roots-of-unity and let \( G \) denote the kernel of the the group homomorphism

\[
(\mu_5) \times 5 \rightarrow \mu_5,
\]

\[
(\xi_0, \ldots, \xi_4) \mapsto \prod_{j=0}^4 \xi_j
\]

Then the natural action of \( G \) on each \( X_t \) gives a family

\[
Y_t = \frac{X_t}{G}
\]

with

\[
H^{3,0} (Y_t) = \mathbb{C} \text{-res} \left( \frac{\Omega}{F_t^2} \left( \prod_{j=0}^4 x_j \right) \right)
\]

and

\[
H^{2,1} (Y_t) = \mathbb{C} \text{-res} \left( \prod_{j=0}^4 x_j \right) \cdot \frac{\Omega}{F_t^2}.
\]
Let
\[ M_t = \ker \left( \mathcal{J}_1(X_t) \to \mathcal{J}_1(Y_t) \right). \]

To see that \( M_0 \) is maximal, we need only show that the 2-dimensional complex torus \( \mathcal{J}_1(Y_0) \) does not contain an elliptic curve in \( H^{3,0}(Y_0) \). Now by \([12]\) the action of the automorphism group
\[ \mu_5 = \left( \mu_5 \right)^{\times 5} \]
acts on \( H^3(Y_0) \) has 1-dimensional eigenspaces \( E_i = H^{4-i,i-1}(Y_0) \) for each of the non-trivial characters \( \xi^i i = 1, \ldots, 4 \), of \( \mu_5 \). So the existence of such an elliptic curve would imply that the eigenspace
\[ E_1 + E_4, \]
which is defined over \( \mathbb{R} \), is actually defined over \( \mathbb{Q} \). This would imply that the elliptic curve
\[ H^{3,0}(Y_0) \]
\[ H_3(Y_0, \mathbb{Z}) \cap (H_3(Y_0)_1 + H_5(Y_0)_4) \]
had a non-trivial automorphism of order 5 which is impossible.

Let \( S_0' \) be the Fermat quintic curve of lines on \( X_0 \) given by
\[ \alpha (\xi_0, -\xi_1, 0, 0, 0) + \beta (0, 0, x_2, x_3, x_4) \]
such that
\[ x_2^5 + x_3^5 + x_4^5 = 0 \]
(or any image of this curve under any automorphism given by permutation of the coordinates \( x_j \)). Thus
\[ S_0' \times S_0' \to M_0 \subseteq \mathcal{J}_1(X_0). \]

Thus by Theorem 5.1.

**Corollary 6.1.** Any continuous family \( S_0' \) of rational curves on the Fermat quintic \( X_0 \) whose generic member is smooth lies in the specialization at \( t = 0 \) of a continuous family \( T_0' \) on \( X_t \) for \( t \) generic.

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