Leray theorems in bounded cohomology theory

Nikolai V. Ivanov

Contents

1. Introduction 2
2. Cohomological Leray theorems 5
3. Homological Leray theorems 15
4. Extensions of coverings and bounded cohomology 19
5. A Leray theorem for $l_1$-homology 23
6. Uniqueness of Leray homomorphisms 26
7. Nerves of families and paracompact spaces 30
8. Closed subspaces and fundamental groups 38
9. Closed subspaces and homology groups 42

Appendix. Double complexes 49

References 57

© Nikolai V. Ivanov, 2020. Neither the work reported in the present paper, nor its preparation were supported by any corporate entity.
1. Introduction

Leray theory. Let $\mathcal{U}$ be a covering of a topological space $X$, and let $\mathcal{U}^\cap$ be the collection of all non-empty finite intersection of elements of $\mathcal{U}$. A classical theorem of Leray relates the cohomology of $X$ with cohomology of the sets $U \in \mathcal{U}^\cap$ and the combinatorial structure of the covering $\mathcal{U}$. The latter is encoded in a simplicial complex $N_\mathcal{U}$, the nerve of $\mathcal{U}$ (see Section 2 for the definition). If every element of $\mathcal{U}^\cap$ is acyclic, i.e. has the same cohomology as a point, Leray theorem implies that the cohomology of $X$ are equal to that of $N_\mathcal{U}$. Here “cohomology” are understood very broadly. Leray theorem applies to the cohomology of sheaves on $X$, as also to the singular cohomology theory (under moderate assumptions). Morally, Leray theorem applies to every cohomology theory which can be locally defined. For example, singular cohomology can be defined in terms of arbitrarily small simplices.

Bounded cohomology. Suppose that elements of $\mathcal{U}^\cap$ are in an appropriate sense “acyclic” with respect to the bounded cohomology. Gromov’s Vanishing theorem asserts that if $\mathcal{U}$ is also open, then the image of the canonical homomorphism $\tilde{\mathcal{H}}^*(X) \to H^*(X)$ vanishes in dimensions bigger than the dimension of $N$. See [Gro], Section 3.1.

As is well known, the bounded cohomology theory cannot be locally defined. In an attempt to find a proof and a conceptual framework for the Vanishing theorem, the author [I1], [I2] discovered that a part of Leray theory survives in the non-local setting of the bounded cohomology theory and leads to stronger results. Namely, under moderate assumptions about $X$ and $\mathcal{U}$, the canonical map $\tilde{\mathcal{H}}^*(X) \to H^*(X)$ factors through the natural homomorphism

$$l_\mathcal{U}: H^*(N_\mathcal{U}) \to H^*(X)$$

from Leray’s theory. We call such a result a Leray theorem, and call $l_\mathcal{U}$ a Leray homomorphism. The fact that $H^*(X)$ can be locally defined plays a crucial role. The proofs in [I1], [I2] were inspired by the sheaf theory and phrased in its language.

The goal of the present paper is to generalize these results from [I1], [I2] and at the same time provide elementary and transparent proofs. The sheaf theory is invoked only to deal with fairly bad spaces and coverings. In particular, no sheaf theory is needed to deal with open coverings of arbitrary spaces. Our main tools are the double complexes of coverings. See Section 2. In order to stress the elementary nature of these tools, direct and elementary proofs of required properties are included in Sections 2, 3, and an Appendix.

At the same time our proofs do not depend on any machinery specific to the bounded cohomology theory. We will use the fact that the bounded cohomology of a path-connected space depend only on its fundamental group, but not any ideas involved in its proofs. Also, in order to relate our main theorems to the Vanishing theorem, we will use the vanishing of the bounded cohomology of path-connected spaces with amenable fundamental group.
**Bounded acyclicity and amenability.** Let us call a topological space *boundedly acyclic* if its bounded cohomology are isomorphic to the bounded cohomology of a point. This is the most natural notion of “acyclicity” with respect to the bounded cohomology. Let us say that the covering \( \mathcal{U} \) is *boundedly acyclic* if every element of \( \mathcal{U} \cap \) is boundedly acyclic. Since the bounded cohomology of a path-connected space are equal to the bounded cohomology of its fundamental group, this means that for \( U \in \mathcal{U} \cap \) the fundamental group \( \pi_1(V) \) is *boundedly acyclic* in the obvious sense. Our main results are concerned with boundedly acyclic coverings. Moreover, using an argument from [I1], [I2], the condition of being boundedly acyclic can be relaxed. Namely, it is sufficient to assume that every \( U \in \mathcal{U} \cap \) is *weakly boundedly acyclic* in the sense that image of \( \pi_1(U) \) in \( \pi_1(X) \) are boundedly acyclic. See Section 4.

Starting with Gromov [Gro], the bounded acyclicity is almost always replaced by the stronger property of being boundedly acyclic “by the good reason”, namely, being a path-connected space with amenable fundamental group. In particular, Gromov’s proof of the Vanishing theorem depends on averaging over amenable groups. Amenable groups are boundedly acyclic, but the converse is not true. Sh. Matsumoto and Sh. Morita [MM] provided examples of groups which are boundedly acyclic by reasons completely different from being amenable. So, our results are stronger than Gromov’s ones in this respect also.

Gromov observed that in the context of the Vanishing theorem it is sufficient to assume that the images of the inclusion homomorphisms \( \pi_1(U) \to \pi_1(X) \) are amenable for every set \( U \in \mathcal{U} \). Cf. Theorem 4.4. Since a subgroup of an amenable group is amenable, under this assumption the image of \( \pi_1(U) \to \pi_1(X) \) is amenable also for every \( U \in \mathcal{U} \cap \). The notion of weakly boundedly acyclic subsets was suggested by this idea of Gromov.

**Open and closed coverings.** In order to deal with Leray maps \( l_\mathcal{U} : H^*(N) \to H^*(X) \) one needs to assume that the covering \( \mathcal{U} \) behaves sufficiently nicely with respect to the singular cohomology theory. A classical theorem of Eilenberg [E] ensures that all open coverings are sufficiently nice. See Theorem 2.3. Theorem 4.3 is our Leray theorem for open coverings. Gromov’s Vanishing theorem can be proved in the same way. See Theorem 4.4.

While Gromov’s Vanishing theorem is concerned only with open coverings, Leray theory suggests to consider also closed locally finite coverings. Suppose that \( \mathcal{U} \) is a closed locally finite covering, and that \( \mathcal{U} \) is weakly boundedly acyclic. We prove a Leray theorem for such coverings \( \mathcal{U} \) in two different situations. In both situations we need to assume that \( X \) is Hausdorff and paracompact, but further assumptions differ.

In Section 8 we assume that \( \mathcal{U} \) behaves nicely with respect to fundamental groups and covering spaces. More precisely, we assume that \( X \) is path connected, locally path connected, and semilocally simply connected, and that subsets \( U \in \mathcal{U} \) are path connected and locally path connected. It turns out that in this case \( \mathcal{U} \) can be replaced by an open covering with the same nerve. See Theorem 8.3. This theorem depends on subtle properties of paracompact spaces. It is hard to extract the full proofs of the needed results from the literature, so we presented them in Section 7. Theorem 8.4 is our Leray theorem in this situation.
In Section 9 we assume that $\mathcal{U}$ behaves nicely with respect to the singular homology theory. More precisely, we assume that the space $X$ and all elements of $\mathcal{U}$ are homologically locally connected. See Section 9 for the definition. Only in this section we resort to the sheaf theory. Probably, this can be avoided, but at the cost of obscuring the underlying ideas. Theorem 9.5 is our Leray theorem in this situation.

Abstract Leray theorems. As we already pointed out, the proofs do not rely on any tools from the bounded cohomology theory. In fact, the basic results hold for any cohomology theory arising from cochain complexes $A^\bullet(Z)$ functorially depending on subspaces $Z \subset X$ and equipped with a natural transformation $A^\bullet(Z) \to C^\bullet(Z)$ to the complexes of singular cochains. Theorems 2.5 and 9.4 are such abstract Leray theorems for open and closed coverings respectively. These results are stated and proved in such an abstract form not for the sake of generality, but in order to make their nature more transparent.

$l_1$-homology. These results and proofs admit a straightforward dualization, leading to Leray theorems for $l_1$-homology. Namely, under appropriate assumptions the natural homomorphism $H_\ast(X) \to H^{l_1}_\ast(X)$ can be factored through the map $H_\ast(X) \to H_\ast(N_U)$ from Leray theory. We limited ourselves by the case of open coverings. See Theorem 5.3. This theorem is deduced from an abstract homological Leray theorem, Theorem 3.2. Similar results for closed coverings also can be proved by dualization of cohomological proofs.

Theorem 5.3 is a strengthening of a recent result of R. Frigerio [F], who proved an analogue for $l_1$-homology of Gromov’s Vanishing theorem. Frigerio’s proofs are based on Gromov’s theories of multicomplexes and of the diffusion of chains, recently reconstructed by R. Frigerio and M. Moraschini [FMo], and are far from being elementary. A Leray theorem for $l_1$-homology was recently proved by Cl. Löh and R. Sauer [LS]. The methods of [LS] are based on author’s homological approach to bounded cohomology [I1], [I2] and assume amenability. It seems that the bounded acyclicity is not sufficient for methods of [F] and [LS].

Uniqueness of Leray maps. In this paper the Leray maps $l_U : H^\ast(N_U) \to H^\ast(X)$ for open coverings $\mathcal{U}$ are defined in terms of the double complex of $\mathcal{U}$. For closed locally finite coverings the definition is similar in spirit, but is more complicated.

If $\mathcal{U}$ is open and the space $X$ is paracompact, one can also use partitions of unity in order to define a natural map $H^\ast(N_\mathcal{U}) \to H^\ast(X)$. R. Frigerio and A. Maffei [FMa] recently proved that for open coverings of paracompact spaces the two definitions agree. In Section 6 we generalize their result by proving that every two reasonable definitions of Leray maps agree for open coverings of paracompact spaces. In more details, following M. Barr [Ba], we consider a category having as objects pairs $(X, \mathcal{U})$, where $\mathcal{U}$ is a covering of a topological space $X$, and define a Leray transformation as a natural transformation of functors on this category. See Section 6. It turns out that on the subcategory of open coverings of paracompact spaces all Leray transformations agree. See Theorem 6.3.
2. Cohomological Leray theorems

The nerve of a covering. Let $\mathcal{U}$ be a covering of a topological space $X$ by subsets of $X$. The nerve $N = N_\mathcal{U}$ of $\mathcal{U}$ is a simplicial complex in the sense, for example, of [Sp], Section 3.1, such that its set of vertices is in a one-to-one correspondence with $\mathcal{U}$ and its simplices are finite non-empty sets of vertices such that the intersection of the corresponding elements of $\mathcal{U}$ is non-empty. We will assume that a linear order $<\ $ on the set of vertices of $N$ is fixed. For each simplex $\sigma$ of $N$ we denote by $|\sigma|$ the intersection of the elements of $\mathcal{U}$ corresponding to the vertices of $\sigma$. We denote by $\mathcal{U} \cap$ the collection of all sets $|\sigma|$.

Classical Leray theorems. Let $\mathcal{U}$ be a covering of $X$. It is well known that under moderate "niceness" assumptions about $X$ and $\mathcal{U}$ there is a canonical homomorphism

$$l_\mathcal{U}: H^*(N) \longrightarrow H^*(X),$$

where $H^*(N)$ is the simplicial cohomology of $N$ and $H^*(X)$ is some appropriate cohomology of $X$. For example, if $\mathcal{U}$ is open and $H^*(X)$ is the Čech cohomology, then such a homomorphism exists by the definition of the latter. By a classical theorem of Leray $l_\mathcal{U}$ is an isomorphism if $\mathcal{U}$ is acyclic, i.e. if every non-empty element of $\mathcal{U} \cap$ has the same cohomology as a point. If $\mathcal{U}$ is not acyclic, then $H^*(N)$ and $H^*(X)$ are related by a Leray spectral sequence and $l_\mathcal{U}$ is one of its two edge homomorphisms. There are similar results in the homology theory. Cf. Section 3.

Leray theory applies to cohomology theory $X \longrightarrow H^*(X)$ which can be "locally defined". Originally, the cohomology $H^*(X)$ was the sheaf cohomology of $X$ with coefficients in a sheaf on $X$, and the acyclicity was understood in terms of cohomology with coefficients in the same sheaf. While the definition of singular cohomology of $X$ is not local, singular cohomology can be "localized" in a sense made precise in Theorem 2.3 below, and Leray theory applies to singular cohomology at least when the covering $\mathcal{U}$ is open.

Leray theorems for non-local cohomology. The goal of this section is to show that a part of Leray theory survives even when the cohomology groups involved cannot be "localized" at all. The motivating example is the bounded cohomology theory, but the results are more transparent if stated and proved in a general form.

Suppose that $Z \longrightarrow \tilde{H}^*(Z)$ is a "cohomology theory" defined in terms of natural cochain complexes, which are equipped with a natural transformation to the singular cochain complexes (all this will be made precise in a moment). This natural transformation leads to canonical maps $\tilde{H}^*(Z) \longrightarrow H^*(Z)$ from $\tilde{H}^*(Z)$ to the singular cohomology groups $H^*(Z)$.

Let us say that a covering $\mathcal{U}$ of $X$ is $\tilde{H}$-acyclic if for every non-empty $U \in \mathcal{U} \cap$ the cohomology $\tilde{H}^*(U)$ is naturally isomorphic to the singular cohomology of a point.
It turns out that if $\mathcal{U}$ is "nice" and $\tilde{H}$-acyclic, then the diagram of the solid arrows

\[
\begin{array}{ccc}
\tilde{H}^*(X) & \longrightarrow & H^*(X) \\
\downarrow & & \downarrow \\
H^*(N) & \longrightarrow & H^*(X)
\end{array}
\]

can be completed to a commutative triangle by a dashed arrow. In other words, the canonical map $\tilde{H}^*(X) \longrightarrow H^*(X)$ factors through the canonical homomorphism

$$l_\mathcal{U} : H^*(N) \longrightarrow H^*(X).$$

The existence of such factorization could be called a *Leray theorem* for the cohomology theory $Z \longrightarrow \tilde{H}^*(Z)$. In particular, there is a Leray theorem for the bounded cohomology theory $\tilde{H}^*(Z) = \hat{H}^*(Z)$, and this theorem implies *Vanishing theorem* of Gromov [Gro] (see Theorem 4.4 below) and is a vast generalization of the latter.

Naively, one could expect that $\tilde{H}^*(X)$ is isomorphic to $H^*(N)$ if the covering $\mathcal{U}$ is "nice" and $\tilde{H}$-acyclic. In fact, this is very unlikely unless the cohomology theory $Z \longrightarrow \tilde{H}^*(Z)$ is locally defined. In particular, this is not true for the bounded cohomology theory. A natural transformation from $\tilde{H}^*(Z)$ to a locally defined theory appears to be an inevitable feature.

**Generalized cochains.** In fact, for our purposes it is sufficient to have cohomology $\tilde{H}^*(Z)$ and the corresponding cochain complexes defined only for subspaces $Z \subset X$. Let us consider the category $\text{sub} X$ having subspaces of $X$ as objects and the inclusions $Y \subset Z$ as morphisms. Let $A^*$ be a contravariant functor from $\text{sub} X$ to augmented cochain complexes of modules over some ring $R$. In our applications $R = \mathbb{R}$, but assuming this does not simplifies anything. The functor $A^*$ assigns to a subspace $Z \subset X$ a complex

\[
\begin{array}{ccccccc}
0 & \longrightarrow & R & \longrightarrow & A^0(Z) & \longrightarrow & A^1(Z) & \longrightarrow & A^2(Z) & \longrightarrow & \cdots
\end{array}
\]

defined by $d_0 : A^0(Z) \longrightarrow A^1(Z)$, $d_1 : A^1(Z) \longrightarrow A^2(Z)$, etc.

Elements of $A^q(Z)$ are thought as *generalized q-cochains* of $Z$. The action of restriction morphisms will be denoted by

$$c \longrightarrow c|_Z.$$

We will need mostly the complexes $A^q(Z)$ for $Z = X$ or $Z \in \mathcal{U}^\cap$, but it is hard to imagine such a functor defined only for such $Z$. The examples are the functor $Z \longrightarrow C^*(Z)$ of singular cochains with coefficients in $R$ and the functor $Z \longrightarrow B^*(Z)$ of bounded real-valued singular cochains in the case of $R = \mathbb{R}$. We will say that the covering $\mathcal{U}$ is *$A^*$-acyclic* if $A^*(Z)$ is exact for every $Z \in \mathcal{U}^\cap$, and will say that $\mathcal{U}$ is *boundedly acyclic* if it is $B^*$-acyclic.
**The double complex of a covering.** Let \( \mathcal{U} \) be a covering of \( X \) and let \( N \) be its nerve. For \( p \geq 0 \) let \( N_p \) be the set of \( p \)-dimensional simplices of \( N \). For \( p, q \geq 0 \) let

\[
C^p(N, A^q) = \prod_{\sigma \in N_p} A^q(|\sigma|).
\]

An element \( c \in C^p(N, A^q) \) is a map assigning to each \( \sigma \in N_p \) a generalized \( q \)-cochain \( c_\sigma \in A^q(|\sigma|) \).

The functor \( A^\bullet \) assigns to each \( \sigma \in N_p \) the complex \( A^\bullet(|\sigma|) \), i.e. the complex

\[
\begin{array}{ccccccc}
0 & \longrightarrow & R & \longrightarrow & \Lambda^0(|\sigma|) & \longrightarrow & \Lambda^1(|\sigma|) & \longrightarrow & \ldots
\end{array}
\]

For each \( p \geq 0 \) the term-wise direct product of complexes (2.1) over \( \sigma \in N_p \) has the form

\[
\begin{array}{ccccccc}
0 & \longrightarrow & C^p(N) & \longrightarrow & C^p(N, A^0) & \longrightarrow & C^p(N, A^1) & \longrightarrow & \ldots
\end{array}
\]

where \( C^p(N) \) is the space of simplicial \( p \)-cochains of \( N \) with coefficients in \( R \). As every simplicial complex, the nerve \( N \) has one empty simplex. By the definition, the dimension of the simplex \( \emptyset \) is \( -1 \) and \( |\emptyset| = X \). So, the complex (2.2) is defined also for \( p = -1 \) and is equal to \( A^\bullet(X) \) in this case.

One can also set \( A^{-1}(Z) = R \) for every subspace \( Z \). Then one can replace the term \( R \) in (2.1) by \( A^{-1}(|\sigma|) \) and interpret the term \( C^p(N) \) in (2.2) as \( C^p(N, A^{-1}) \). We will denote the complex (2.2) by \( C^p(N, A^\bullet) \). For every \( p \geq -1 \) there is a canonical morphism

\[
\delta_p : C^p(N, A^\bullet) \longrightarrow C^{p+1}(N, A^\bullet)
\]

defined as follows. Suppose that \( \sigma \in N_{p+1} \) and let \( v_0 < v_1 < \ldots < v_{p+1} \) be the vertices of \( \sigma \) listed in the increasing order. For each \( i = 0, 1, \ldots, p+1 \) let \( \delta_i \sigma = \sigma \setminus \{v_i\} \) be the \( i \)th face of \( \sigma \). Clearly, \( |\sigma| \subset |\delta_i \sigma| \) and there is a restriction morphism

\[
A^\bullet(|\delta_i \sigma|) \longrightarrow A^\bullet(|\sigma|).
\]

If \( c \in C^p(N, A^q) \), then \( \delta_p(c) \) assigns to \( \sigma \) the generalized \( q \)-cochain

\[
(\delta_p(c))_\sigma = \sum_{i=0}^{p+1} (-1)^i c_{\delta_i \sigma} |_{|\sigma|} \in A^q(|\sigma|).
\]

The fact that each \( A^\bullet(|\delta_i \sigma|) \longrightarrow A^\bullet(|\sigma|) \) is a morphism of complexes implies that

\[
\delta_p : C^p(N, A^\bullet) \longrightarrow C^{p+1}(N, A^\bullet)
\]
is a morphism of complexes also. Equivalently, $\delta_p \circ d_q = d_q \circ \delta_p$ on $C^p(N, A^q)$ for every $p, q \geq -1$. Let us collect $C^p(N, A^q)$ and $d_q, \delta_p$ into a single diagram

$$
\begin{array}{cccccccc}
R & \rightarrow & A^0(X) & \rightarrow & A^1(X) & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
C^0(N) & \rightarrow & C^0(N, A^0) & \rightarrow & C^0(N, A^1) & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
C^1(N) & \rightarrow & C^1(N, A^0) & \rightarrow & C^1(N, A^1) & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
C^2(N) & \rightarrow & C^2(N, A^0) & \rightarrow & C^2(N, A^1) & \rightarrow & \cdots \\
\ldots & \downarrow & \ldots & \downarrow & \ldots & \downarrow & \ldots & \\
\end{array}
$$

(2.3)

Since $\delta_p \circ d_q = d_q \circ \delta_p$ on $C^p(N, A^q)$, this diagram is commutative. By the construction, $d_{q+1} \circ d_q = 0$ for every $q$, i.e. the rows of this diagram are complexes. A standard computation shows that $\delta_{p+1} \circ \delta_p = 0$ for every $p$, i.e. the columns of this diagram are also complexes. It turns out that the top row and the left column of this diagram should be treated differently from the rest. The part

$$
\begin{array}{cccccccc}
C^0(N, A^0) & \rightarrow & C^0(N, A^1) & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \\
C^1(N, A^0) & \rightarrow & C^1(N, A^1) & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \\
C^2(N, A^0) & \rightarrow & C^2(N, A^1) & \rightarrow & \cdots \\
\ldots & \downarrow & \ldots & \downarrow & \\
\end{array}
$$

(2.4)

of the diagram (2.3) is the double complex of the covering $\mathcal{U}$. See the Appendix for a dis-
cussion of double complexes and related notions. We will denote this double complex by \( C^\bullet(N, A^\bullet) \) and assume from now on that \( \bullet \) stands for non-negative integers. Let \( T^\bullet(N, A) \) be the total complex of the double complex \( C^\bullet(N, A^\bullet) \). By the definition,

\[
T^n(N, A) = \bigoplus_{p=0}^{n} C^p(N, A^{n-p})
\]

and the differential \( D : T^n(N, A) \to T^{n+1}(N, A) \) is given by the formula

\[
D|_{C^p(N, A^q)} = d_q + (-1)^p \delta_p .
\]

The homomorphisms \( d_{-1} \) from (2.2) lead to a morphism

\[
\lambda_A : C^\bullet(N) \to T^\bullet(N, A)
\]

Similarly, the morphism \( \delta_{-1} : A^\bullet(X) = C^{-1}(N, A^\bullet) \to C^0(N, A^\bullet) \) leads to a morphism

\[
\tau_A : A^\bullet(X) \to T^\bullet(N, A),
\]

where it is understood that the augmentation term \( A^{-1}(X) \) is removed from \( A^\bullet(X) \).

\textbf{2.1. Lemma.} If \( \mathcal{U} \) is \( A^\bullet \)-acyclic, then the morphism \( \lambda_A : C^\bullet(N) \to T^\bullet(N, A) \) induces an isomorphism of cohomology groups.

\textbf{Proof.} If \( \mathcal{U} \) is \( A^\bullet \)-acyclic, then for every simplex \( \sigma \neq \emptyset \) the complex (2.1) is exact. This implies that for every \( p \geq 0 \) the complex (2.2) is exact. It follows that the rows of the double complex \( C^\bullet(N, A^\bullet) \) are exact and \( d_{-1} \) induces an isomorphism of the complex \( C^\bullet(N) \) with the kernel of the morphism of complexes

\[
d_0 : C^\bullet(N, A^0) \longrightarrow C^\bullet(N, A^1).
\]

It remains to apply Theorem A.1 from the Appendix. ■

**Singular cochains.** Let \( Y \) be a topological space. Following Godement [Go], let us fix a topological space \( \Delta \) and call a continuous map \( s: \Delta \to Y \) a singular simplex in \( Y \). Cf. [Go], Chapter II, Example 3.9.1. The main examples are \( \Delta = \Delta^q \) for \( q \geq 0 \). Let \( S(Y) \) be the set of singular simplices in \( Y \). A singular cochain with coefficients in \( R \) is defined as a map \( S(Y) \to R \). Let \( C(Y) \) be the \( R \)-module of singular cochains in \( Y \). For \( p \geq 0 \) let

\[
C^p(N, C) = \prod_{\sigma \in N_p} C(|\sigma|).
\]
So, an element \( c \in C^p(N, C) \) is a map assigning to a simplex \( \sigma \in N_p \) a cochain

\[
c_\sigma : S(|\sigma|) \rightarrow R.
\]

The maps \( \delta_p : C^p(N, C) \rightarrow C^{p+1}(N, C) \) are defined as before. In more details,

\[
(2.5) \quad (\delta_p c)_\sigma (s) = \sum_{i=0}^{p+1} (-1)^i c \partial_i \sigma (s)
\]

if \( p \geq 0 \), \( \sigma \) is a \( p \)-simplex of \( N \), \( c \in C^p(N, C) \), and \( s : \Delta \rightarrow |\sigma| \) singular simplex.

A singular simplex \( s : \Delta \rightarrow X \) is said to be *small* if \( s(\Delta) \) is contained in some \( U \in \mathcal{U} \). Let \( S(X, \mathcal{U}) \) be the set of small singular simplices in \( X \). A *small singular cochain* is defined as a map \( S(X, \mathcal{U}) \rightarrow R \). Let \( C(X, \mathcal{U}) \) be the \( R \)-module of small singular cochains in \( X \). The restriction of singular cochains to \( S(X, \mathcal{U}) \) leads to a map \( C(X) \rightarrow C(X, \mathcal{U}) \). Similarly, restrictions of small singular cochains to \( S(U), U \in \mathcal{U} \), lead to a map

\[
\bar{\delta}_{-1} : C(X, \mathcal{U}) \rightarrow C^0(N, C).
\]

As before, \( \delta_{p+1} \circ \delta_p = 0 \) for every \( p \geq 0 \) and also \( \delta_0 \circ \bar{\delta}_{-1} = 0 \).

**2.2. Lemma.** The following sequence is exact:

\[
0 \rightarrow C(X, \mathcal{U}) \xrightarrow{\delta_{-1}} C^0(N, C) \xrightarrow{\delta_0} C^1(N, C) \xrightarrow{\delta_1} \cdots.
\]

**Proof.** Obviously, \( \bar{\delta}_{-1} \) is injective. Let us prove the exactness in the term \( C^0(N, C) \). An element \( c \in C^0(N, C) \) is a family of cochains \( c_U \in C(U), U \in \mathcal{U} \). It belongs to the kernel of \( \delta_0 \) if and only if for every \( U, V \in \mathcal{U} \) the cochains \( c_U \) and \( c_V \) are equal on singular simplices with the image in \( U \cap V \). If this is the case, then there exists a unique map \( \gamma : S(X, \mathcal{U}) \rightarrow R \) such that the restriction of \( \gamma \) to \( S(U) \) is equal to \( c_U \) for every \( U \in \mathcal{U} \). Clearly, \( \bar{\delta}_{-1}(\gamma) = c \). It follows that \( \bar{\delta}_{-1} \) is an isomorphism onto the kernel of \( \delta_0 \), i.e. the sequence is exact at the terms 0 and \( C^0(N, C) \). In order to prove the exactness at the terms \( C^p(N, C) \) with \( p \geq 1 \), it is sufficient to construct a contracting chain homotopy

\[
C^{p-1}(N, C) \xleftarrow{k_p} C^p(N, C), \quad p \geq 1.
\]

Let us choose for every small singular simplex \( s : \Delta \rightarrow X \) an element \( U_s \in \mathcal{U} \) such that \( s(\Delta) \subset U_s \) and denote by \( u_s \) the corresponding vertex of \( N \). Let \( c \in C^p(N, C) \). We need to define a cochain \( k_p(c)_\tau \in C^{p-1}(|\tau|) \) for every \( \tau \in N_{p-1} \). Suppose that

\[
r : \Delta \rightarrow |\tau|
\]
is a singular simplex and let \( \sigma = \tau \cup \{ u_r \} \). Clearly, \( r(\Delta) \subset |\tau| \cap U_r = |\sigma| \). In particular, \( \sigma \) is a simplex. If \( u_r \in \tau \), then \( \sigma \) is a \((p-1)\)-simplex and we set

\[
k_p(c)\tau(r) = 0.
\]

Otherwise, \( \sigma \) is a \( p \)-simplex, \( \tau = \partial_a \sigma \) for some \( a \), and we set

\[
k_p(c)\tau(r) = (-1)^a c_\sigma(r).
\]

Let \( c \in C^p(N, C) \), \( \sigma \in N_p \), and \( s: \Delta \to |\sigma| \) is a singular simplex. In order to prove that \( k_* \) is a contracting chain homotopy, we need to prove that

\[
d_{p-1}(k_p(c))_\sigma(s) + k_{p+1}(\delta_p(c))_\sigma(s) = c_\sigma(s)
\]

for every such \( c, \sigma, \) and \( s \). The formula (2.5) implies that

\[
\delta_{p-1}(k_p(c))_\sigma(s) = \sum_{i=0}^{p} (-1)^i k_p(c)\partial_i \sigma(s).
\]

Suppose first that \( u_s \in \sigma \) and let \( \tau = \sigma \setminus \{ u_s \} \). Then \( \tau = \partial_a \sigma \) for some \( a \). Clearly, \( u_s \in \partial_i \sigma \) if \( i \neq a \) and hence \( k_p(c)\partial_i \sigma(s) = 0 \) if \( i \neq a \). It follows that

\[
\delta_{p-1}(k_p(c))_\sigma(s) = (-1)^a k_p(c)\partial_a \sigma(s) = c_\sigma(s).
\]

Since also \( k_{p+1}(d)\sigma(s) = 0 \) for every \( d \), this implies (2.6) in the case when \( u_s \in \sigma \).

Suppose now that \( u_s \notin \sigma \). Let \( v_0 < v_1 < \ldots < v_p \) be the vertices of \( \sigma \) listed in the increasing order, and let \( a \) be such that \( v_{a-1} < u_s < v_a \) (for \( a = 0 \) or \( p \) one of these inequalities is vacuous). Let \( \rho = \sigma \cup \{ u_s \} \) and \( \rho_i = \partial_i \sigma \cup \{ u_s \} \). Then

\[
\rho_i = \partial_i \rho \quad \text{for} \quad i < a \quad \text{and} \quad \rho_i = \partial_{i+1} \rho \quad \text{for} \quad i \geq a.
\]

Also \( \sigma = \partial_a \rho \),

\[
\partial_i \sigma = \partial_{a-1} \rho_i \quad \text{for} \quad i < a, \quad \text{and} \quad \partial_i \sigma = \partial_a \rho_i \quad \text{for} \quad i \geq a.
\]

Similarly to the above, (2.5) implies that

\[
\delta_{p-1}(k_p(c))_\sigma(s) = \sum_{i=0}^{p} (-1)^i k_p(c)\partial_i \sigma(s)
\]

\[
= \sum_{i=0}^{a-1} (-1)^{i+a-1} c_{\rho_i}(s) + \sum_{i=a}^{p} (-1)^{i+a} c_{\rho_i}(s).
\]
At the same time

\[
k_{p+1}(\delta_p(c))_s = (-1)^a \delta_p(c)_s = (-1)^a \sum_{i=0}^{p+1} (-1)^i c_{\partial_i \rho}(s)
\]

\[
= \sum_{i=0}^{a-1} (-1)^{i+a} c_{\rho_i}(s) + c_s(s) + \sum_{i=a+1}^{p+1} (-1)^{i+a} c_{\rho_{i-1}}(s)
\]

By adding the results of these calculations we see that (2.6) holds when \( u_s \not\in \sigma \) also. ■

**The complex of small singular cochains.** Taking \( \Delta = \Delta^q \) leads to the notions of small singular \( q \)-simplices and small singular \( q \)-cochains. Let \( S_q(X, \mathcal{U}) \) be the set of small singular \( q \)-simplices and \( C^q(X, \mathcal{U}) \) be the \( \mathbb{R} \)-module of small singular \( q \)-cochains. Clearly, every face of a small singular \( q \)-simplex is small. Therefore, one can define the **coboundary maps**

\[
d : C^q(X, \mathcal{U}) \to C^{q+1}(X, \mathcal{U})
\]

in the usual way. As usual, \( d \circ d = 0 \) and \( C^q(X, \mathcal{U}) \) together with \( d \) form a complex, which we denote by \( C^*(X, \mathcal{U}) \). The restriction of singular \( q \)-cochains to \( S_q(X, \mathcal{U}) \) leads to a morphism \( C^*(X) \to C^*(X, \mathcal{U}) \). The following fundamental theorem is well known. It is due to Eilenberg [E], but this is hardly ever mentioned nowadays.

**2.3. Theorem.** The morphism \( C^*(X) \to C^*(X, \mathcal{U}) \) induces an isomorphism of cohomology groups if the interiors \( \operatorname{int} U \) of sets \( U \in \mathcal{U} \) cover \( X \). ■

**2.4. Lemma.** The homomorphisms \( \overline{\delta}_{-1} : C^q(X, \mathcal{U}) \to C^0(N, C^q) \) define a morphism

\[
\tau_{C, \mathcal{U}} : C^*(X, \mathcal{U}) \to T^*(N, C)
\]

inducing an isomorphism of cohomology groups. The morphism

\[
\tau_C : C^*(X) \to T^*(N, C)
\]

induces an isomorphism of cohomology groups if the interiors of the elements of \( \mathcal{U} \) cover \( X \).

**Proof.** Clearly, \( \overline{\delta}_{-1} \) commutes with the coboundary maps. Lemma 2.2 implies that \( \overline{\delta}_{-1} \) induces an isomorphism of the complex \( C^*(X, \mathcal{U}) \) with the kernel of the morphism

\[
\delta_0 : C^0(N, C^*) \to C^1(N, C^*)
\]

At the same time, Lemma 2.2 implies that the columns of the double complex \( C^*(N, C^*) \)
are exact. Therefore Theorem A.1 with the rows and columns interchanged implies the first statement of the lemma. The first statement implies the second in view of Lemma 2.3. ■

The homomorphism $H^*(N) \to H^*(X)$ for open coverings. The morphisms

$$C^*(N) \xrightarrow{\lambda_C} T^*(N, C) \xleftarrow{\tau_C} C^*(X)$$

lead to homomorphisms of cohomology groups,

$$H^*(N) \xrightarrow{\lambda_{C*}} H^*(N, C) \xleftarrow{\tau_{C*}} H^*(X),$$

where $H^*(N, C)$ denotes the cohomology of $T^*(N, C)$. If the interiors of the elements of $\mathcal{U}$ cover $X$, then $\tau_{C*}$ is an isomorphism by Lemma 2.4 and we take the composition

$$\tau_{C*}^{-1} \circ \lambda_{C*} : H^*(N) \to H^*(X),$$

as the canonical homomorphism $l_\mathcal{U} : H^*(N) \to H^*(X)$.

The $A^\ast$-cohomology and the singular cohomology. The $A^\ast$-cohomology groups

$$\tilde{H}^*(X) = H^*_{A}(X)$$

of $X$ are simply the cohomology groups of the complex $A^\ast(X)$ with the term $R$ omitted. Suppose that the functor $A^\ast$ is equipped with a natural transformation $\varphi^\ast : A^\ast \to C^\ast$. This natural transformation leads to a homomorphism

$$\tilde{H}^*(X) = H^*_{A}(X) \to H^*(X).$$

2.5. Theorem. If $\mathcal{U}$ is $A^\ast$-acyclic and the interiors of the elements of $\mathcal{U}$ cover $X$, then the homomorphism $\tilde{H}^*(X) \to H^*(X)$ can be factored through the canonical homomorphism $l_\mathcal{U} : H^*(N) \to H^*(X)$.

Proof. The homomorphisms

$$\varphi^q (|\sigma|) : A^q (|\sigma|) \to C^q (|\sigma|)$$

lead to a morphism

$$\varphi^{\ast\ast} : C^\ast(N, A^\ast) \to C^\ast(N, C^\ast)$$

of double complexes.
In turn, $\varphi^{**}$ leads to a morphism

$$\Phi^*: T^*(N, A) \rightarrow C^*(N, C)$$

Clearly, the diagram

$$
\begin{array}{ccc}
C^*(N) & \longrightarrow & T^*(N, A) \\
\downarrow & & \downarrow \\
C^*(N) & \longrightarrow & T^*(N, C) \\
\end{array}
\Rightarrow

\begin{array}{ccc}
\Lambda^*(X) & \longleftarrow & \Phi^* \\
\varphi^* & & \\
C^*(X) & \longleftarrow & \\
\end{array}
$$

is commutative and leads to the following commutative diagram of cohomology groups

$$
\begin{array}{ccc}
H^*(N) & \longrightarrow & H^*(N, A) \\
\downarrow & & \downarrow \\
H^*(N) & \longrightarrow & H^*(N, C) \\
\end{array}
\Rightarrow

\begin{array}{ccc}
\tilde{H}^*(X) & \longleftarrow & \Phi^* \\
\varphi^* & & \\
H^*(X) & \longleftarrow & \\
\end{array}
$$

where $H^*(N, A)$ denotes the cohomology of the total complex $T^*(N, A)$.

The red arrows are isomorphisms. Indeed, since the covering $\mathcal{U}$ is $\Lambda^*$-acyclic, the arrow $H^*(N) \rightarrow H^*(N, A)$ is an isomorphism by Lemma 2.1. Since the interiors of the elements of $\mathcal{U}$ cover $X$, the arrow $H^*(X) \rightarrow H^*(N, C)$ is an isomorphism by Lemma 2.4. By inverting these two arrows we get the commutative diagram

$$
\begin{array}{ccc}
H^*(N) & \longleftarrow & H^*(N, A) \\
\downarrow & & \downarrow \\
H^*(N) & \longleftarrow & H^*(N, C) \\
\end{array}
\Rightarrow

\begin{array}{ccc}
\tilde{H}^*(X) & \longleftarrow & \Phi^* \\
\varphi^* & & \\
H^*(X) & \longleftarrow & \\
\end{array}
$$

It follows that $\tilde{H}^*(X) \rightarrow H^*(X)$ factors through the canonical homomorphism

$$
\begin{array}{ccc}
H^*(N) & \longrightarrow & H^*(N, C) \\
\longrightarrow & & \longrightarrow \\
H^*(N) & \longrightarrow & H^*(X) \\
\end{array}
$$

This completes the proof. ■
3. Homological Leray theorems

Generalized chains. Let $e_\bullet$ be a covariant functor from $\text{sub} X$ to augmented chain complexes of modules over some ring $R$. The functor $e_\bullet$ assigns to a subspace $Z \subset X$ a complex

$$0 \longleftarrow R \longleftarrow e_0(Z) \longleftarrow e_1(Z) \longleftarrow e_2(Z) \longleftarrow \ldots,$$

For every $Y \subset Z$ there is an inclusion morphism $e_\bullet(Z) \rightarrow e_\bullet(Y)$. For $p, q \geq 0$ let

$$c_p(N, e_q) = \bigoplus_{\sigma \in N_p} e_q(|\sigma|).$$

So, an element $c \in c_p(N, e_q)$ is a direct sum of generalized $q$-chains

$$c = \bigoplus_{\sigma \in N_p} c_\sigma \in \bigoplus_{\sigma \in N_p} e_q(|\sigma|).$$

For every $p \geq 0$ there is a canonical morphism $\delta_p : c_p(N, e_\bullet) \rightarrow c_{p-1}(N, e_\bullet)$, defined as follows. Let $\sigma \in N_p$. For each face $\partial_i \sigma$ there is an inclusion morphism $\Delta_{\sigma, i} : e_\bullet(|\sigma|) \rightarrow e_\bullet(|\partial_i \sigma|)$.

For $c_\sigma \in e_q(|\sigma|)$ we set

$$\delta_p(c_\sigma) = \bigoplus_{i=0}^p (-1)^i \Delta_{\sigma, i}(c_\sigma) \in \bigoplus_{i=0}^p e_q(|\partial_i \sigma|)$$

and extend $\delta_p$ to the direct sum $c_p(N, e_q)$ by linearity. Since $\Delta_{\sigma, i}$ are morphisms of complexes, $\delta_p$ is a morphism also. The double complex $c_\bullet(N, e_\bullet)$ of the covering $\mathcal{U}$ is

$$c_0(N, e_0) \longleftarrow c_0(N, e_1) \longleftarrow \ldots$$

$$c_1(N, e_0) \longleftarrow c_1(N, e_1) \longleftarrow \ldots$$

$$c_2(N, e_0) \longleftarrow c_2(N, e_1) \longleftarrow \ldots$$

$$\ldots \ldots \ldots,$$

(3.1)
where the horizontal arrows are the direct sums of the maps $d_i$ and the vertical arrows are $\delta_i$. Let $t_*(N, e)$ be the total complex of $c_*(N, e_*)$ and let $C_*(N)$ be the complex of simplicial chains of $N$ with coefficients in $R$. The boundary maps $d_0$ and $\delta_0$ lead to morphisms

$$\lambda_e: t_*(N, e) \rightarrow C_*(N) \quad \text{and} \quad \tau_e: t_*(N, e) \rightarrow e_*(X),$$

where it is understood that the augmentation term is removed from $e_*(X)$. The covering $U$ is said to be $e_*$-acyclic if $e_*(Z)$ is exact for every $Z \in U \cap$. If the covering $U$ is $e_*$-acyclic, then $\lambda_e$ induces an isomorphism of homology groups. The proof is similar to the proof of Lemma 2.1, using Theorem A.2 instead of Theorem A.1.

**Singular chains.** Suppose that a space $\Delta$ is fixed and maps $s: \Delta \rightarrow Y$ are treated as singular simplices. A singular chain is a finite formal sum of singular simplices with coefficients in $R$. Let $c_*(Y)$ be the $R$-module of singular chains in $Y$ and let

$$C_p(N, c) = \bigoplus_{\sigma \in N_p} c(|\sigma|),$$

where $p \geq -1$. The maps $\delta_p: C_p(N, c) \rightarrow C_{p-1}(N, c), \ p \geq 0$, are defined as before. Similarly to the cohomological situation, $\delta_{p-1} \circ \delta_p = 0$ for every $p \geq 1$. A small singular chain is defined as a formal sum of small singular simplices with coefficients in $R$. Let $c_*(X, U)$ be the $R$-module of small singular chains in $X$. There is an obvious map $c_*(X, U) \rightarrow c_*(X)$. The inclusion maps $c_*(U) \rightarrow c_*(X), \ U \in U$, lead to a map

$$\bar{\delta}_0: C_0(N, c) \rightarrow c_*(X, U).$$

Clearly, $\delta_0$ is equal to the composition of $\bar{\delta}_0$ and the inclusion $c_*(X, U) \rightarrow c_*(X)$.

**3.1. Lemma.** The following sequence is exact:

$$0 \xleftarrow{k_{-1}} c_*(X, U) \xleftarrow{\bar{\delta}_0} C_0(N, c) \xleftarrow{\delta_1} C_1(N, c) \xleftarrow{\delta_2} \cdots .$$

**Proof.** It is sufficient to construct a contracting chain homotopy

$$k_{-1}: c_*(X, U) \rightarrow C_0(N, c), \ k_q: C_q(N, c) \rightarrow C_{q+1}(N, c),$$

where $q \geq 0$. For every small singular simplex $s: \Delta \rightarrow X$ let us choose a subset $U_s \in U$ such that $s(\Delta) \subset U_s$ and denote by $u_s$ be the corresponding vertex of $N$. If $s(\Delta) \subset |\sigma|$ for some $\sigma \in N_p$, we will denote by $s * \sigma$ the singular simplex $s$ considered as an element of $c(|\sigma|)$. In order to define $k_p$, it is sufficient to define $k_p$ on the chains of the form $s * \sigma$ and then extend $k_p$ first to $c(|\sigma|)$ and then to $C_p(N, c)$ by linearity. Suppose that $\sigma \in N_p$ and $s$ be a singular $q$-simplex in $X$ such that $s(\Delta^q) \subset |\sigma|$. Let $\rho = \sigma \cup \{u_s\}$. Then $s(\Delta) \subset |\sigma| \cap U_s = |\rho|$ and, in particular, $\rho$ is a simplex. If $u_s \in \sigma$, then $\rho$ is a...
In order to prove that $k_*$ is a contracting chain homotopy, it is sufficient to prove that

$\delta_{p+1}(k_p(s*\sigma)) + k_{p-1}(\delta_p(s*\sigma)) = s*\sigma$. \hspace{1cm} (3.2)

By rewriting the definition of $\delta_p$ in terms of the notations $s*\sigma$ we see that

$$\delta_p(s*\sigma) = \bigoplus_{i=0}^{p} (-1)^i s*\partial_i \sigma.$$ 

Suppose first that $u_s \in \sigma$. Let $\tau = \sigma \sim \{u_s\}$. Then $\sigma = \tau \cup \{u_s\}$ and $\tau = \partial_a \sigma$ for some $a$. Clearly, $u_s \in \partial_i \sigma$ if $i \neq a$ and hence $k_{p-1}(s*\partial_i \sigma) = 0$ if $i \neq a$. Therefore

$$k_{p-1}(\delta_p(s*\sigma)) = k_{p-1}((-1)^a s*\partial_a \sigma)$$

$$= (-1)^a k_{p-1}(s*\tau) = (-1)^a(-1)^a s*\sigma = s*\sigma.$$ 

Since also $k_p(s*\sigma) = 0$, this implies (3.2) in the case when $u_s \in \sigma$. Suppose now that $u_s \not\in \sigma$. Let $\rho = \sigma \cup \{u_s\}$ and let $a$ be such that $\sigma = \partial_a \rho$. Let $\rho_i = \partial_i \sigma \cup \{u_s\}$. Then

$$\rho_i = \partial_i \rho \text{ for } i < a \text{ and } \rho_i = \partial_{i+1} \rho \text{ for } i \geq a;$$

$$\partial_i \sigma = \partial_{a-1} \rho_i \text{ for } i < a \text{ and } \partial_i \sigma = \partial_a \rho_i \text{ for } i \geq a.$$ 

Therefore

$$\delta_{p+1}(k_p(s*\sigma)) = \delta_{p+1}((-1)^a s*\rho) = \bigoplus_{i=0}^{p+1} (-1)^{i+a} s*\partial_i \rho$$

$$= \bigoplus_{i=0}^{a-1} (-1)^{i+a} s*\rho_i \oplus s*\sigma \oplus \bigoplus_{i=a}^{p+1} (-1)^{i+a-1} s*\rho_i.$$ 

At the same time

$$k_{p-1}(\delta_p(s*\sigma)) = \bigoplus_{i=0}^{p} k_{p-1}((-1)^i s*\partial_i \sigma)$$

$$= \bigoplus_{i=0}^{a-1} (-1)^{i+a} s*\rho_i \oplus \bigoplus_{i=a}^{p+1} (-1)^{i+a-1} s*\rho_i.$$ 

By combining the last two calculations we see that (3.2) holds for $u_s \in \sigma$ also. □
The homomorphism $H_\ast(X) \to H_\ast(N)$ for open coverings. Let $C_q(X, \mathcal{U})$ be the $R$-module of small singular $q$-chains in $X$. Since the boundary of a small singular $q$-simplex is obviously a small chain, the modules $C_q(X, \mathcal{U})$ together with the restrictions of the usual boundary maps form a chain complex, which we denote by $C_\ast(X, \mathcal{U})$. Recall that $C_\ast(X)$ is the usual chain complex of $X$. The morphisms

$$C_\ast(N) \xleftarrow{\lambda} t_\ast(N, C) \xrightarrow{\tau} C_\ast(X)$$

lead to homomorphisms of cohomology groups,

$$H_\ast(N) \xleftarrow{\lambda} H_\ast(N, C) \xrightarrow{\tau} H_\ast(X)$$

where $H_\ast(N, C)$ is the homology of $t_\ast(N, C)$. If $\tau$ is an isomorphism, we can take

$$\lambda \circ \tau^{-1} : H_\ast(X) \to H_\ast(N)$$

as the canonical homomorphism $H_\ast(X) \to H_\ast(N)$. This is the case, for example, when $\mathcal{U}$ is open, or, at least, the interiors $\text{int} U$ of sets $U \in \mathcal{U}$ cover $X$. Indeed, Lemma 3.1 with $\Delta = \Delta^q$ implies that the columns of (3.1) are exact and that $\delta_0$ induces an isomorphism of the complex $C_\ast(X, \mathcal{U})$ with the cokernel of the morphism

$$\delta_1 : C_1(N, C_\ast) \to C_0(N, C_\ast).$$

Theorem A.2 with the rows and columns interchanged implies that the resulting morphism

$$t_\ast(N, C) \to C_\ast(X, \mathcal{U})$$

induces an isomorphism in homology. If the interiors $\text{int} U$ of sets $U \in \mathcal{U}$ cover $X$, then the homological version of Theorem 2.3 implies that the inclusion $C_\ast(X, \mathcal{U}) \to C_\ast(X)$ induces an isomorphism in homology and hence $\tau : t_\ast(N, C) \to C_\ast(X)$ induces an isomorphism in homology.

The $e_\ast$-homology and the singular homology. The $e_\ast$-homology groups $\tilde{H}_\ast(X) = H_\ast^e(X)$ of $X$ are simply the cohomology groups of the complex $e_\ast(X)$ with the term $R$ omitted. Suppose that the functor $e_\ast$ is equipped with a natural transformation $\varphi : C_\ast \to e_\ast$. This natural transformation leads to a homomorphism $H_\ast(X) \to \tilde{H}_\ast(X)$.

3.2. Theorem. If $\mathcal{U}$ is $e_\ast$-acyclic and the interiors of the elements of $\mathcal{U}$ cover $X$, then the homomorphism $H_\ast(X) \to \tilde{H}_\ast(X)$ can be factored through the canonical homomorphism $H_\ast(X) \to H_\ast(N)$.

Proof. It differs from the proof of Theorem 2.5 only by the directions of arrows. ■
4. Extensions of coverings and bounded cohomology

Weakly boundedly acyclic coverings. By applying Theorem 2.5 to the functor $B^*$ in the role of $A^*$, we see that if $\mathcal{U}$ is boundedly acyclic and the interiors of the elements of $\mathcal{U}$ cover $X$, then the map $\tilde{H}^*(X) \to H^*(X)$ can be factored through $l_\mathcal{U} : H^*(N) \to H^*(X)$. The fact that the bounded cohomology depend only on the fundamental groups allows to prove that the same conclusion holds under a weaker assumption. Namely, it holds for weakly boundedly acyclic coverings, to be defined in a moment. See Theorem 4.1 below.

A group $\pi$ is said to be boundedly acyclic if the bounded cohomology of Eilenberg-MacLane space $K(\pi, 1)$ are the same as the bounded cohomology of a point. A path connected subset $Z$ of $X$ is said to be weakly boundedly acyclic if the image of the inclusion homomorphism $\pi_1(Z, z) \to \pi_1(X, z)$ is boundedly acyclic, and amenable if this image is amenable. Since amenable groups are boundedly acyclic, an amenable subset is weakly boundedly acyclic. A covering $\mathcal{U}$ is said to be weakly boundedly acyclic if every element of $\mathcal{U} \cap$ is weakly boundedly acyclic, and amenable if every element of $\mathcal{U}$ is amenable and every element of $\mathcal{U} \cap$ is path-connected. Since every subgroup of an amenable group is amenable, in this case every element of $\mathcal{U} \cap$ is amenable and hence an amenable covering is weakly boundedly acyclic.

We would like to be able to turn a weakly boundedly acyclic covering into a boundedly acyclic one without affecting its nerve. This can be done after replacing $X$ by a larger space.

Extensions of coverings. Let $\mathcal{U}$ be a covering of $X$. Suppose that $X$ is a subspace of some other space $X'$. An extension of $\mathcal{U}$ to $X'$ is defined as a map $U \mapsto U'$ from $\mathcal{U}$ to the set of subsets of $X'$ such that $U' \cap X' = U$ for every $U \in \mathcal{U}$ and the collection

$$\mathcal{U}' = \{ U' \mid U \in \mathcal{U} \}$$

is a covering of $X'$. Since $U' \cap X' = U$ for $U \in \mathcal{U}$, the covering $\mathcal{U}'$ determines the extension $U \mapsto U'$ and we may identify this extension with $\mathcal{U}'$. We will say that an extension $\mathcal{U}'$ of $\mathcal{U}$ is nerve-preserving if the conditions

$$\bigcap_{U \in \mathcal{V}} U \neq \emptyset \quad \text{and} \quad \bigcap_{U' \in \mathcal{V}'} U' \neq \emptyset$$

are equivalent for every finite subset $\mathcal{V} \subset \mathcal{U}$. Clearly, if $\mathcal{U}'$ is a nerve-preserving extension of $\mathcal{U}$, then the nerves of $\mathcal{U}'$ and $\mathcal{U}$ are the same, or, rather, are canonically isomorphic.

Suppose that $\mathcal{U}'$ is a nerve-preserving extension of $\mathcal{U}$ and let $N$ be the common nerve of $\mathcal{U}$ and $\mathcal{U}'$. Recall that for a simplex $\sigma$ of $N$ we denote by $|\sigma|$ the intersection of the elements of $\mathcal{U}$ corresponding to the vertices of $\sigma$, and let us denote by $|\sigma|'$ the intersection of the elements of $\mathcal{U}'$ corresponding to the vertices of $\sigma$. Also, let $|\emptyset|' = X'$. 

19
4.1. Theorem. Let \( \mathcal{U} \) be a covering of \( X \) such that every element of \( \mathcal{U} \cap X \) is path-connected. Let \( N \) be the nerve of \( \mathcal{U} \). Then there exists a space \( X' \supset X \) and a nerve-preserving extension \( \mathcal{U}' \) of \( \mathcal{U} \) to \( X' \) such that every element of \( \mathcal{U}' \cap X' \) is path-connected, the inclusion map

\[
\pi_1(X, x) \to \pi_1(X', x),
\]

where \( x \in X \), is an isomorphism, and for every simplex \( \sigma \) of \( N \) the inclusion maps

\[
\pi_1(|\sigma|, z) \to \pi_1(|\sigma'|, z) \quad \text{and} \quad \pi_1(|\sigma'|, z) \to \pi_1(X', z),
\]

where \( z \in |\sigma| \), are, respectively, surjective and injective. If \( \mathcal{U} \) is an open (respectively, closed) covering, then \( \mathcal{U}' \) can be assumed to be open (respectively, closed).

Proof. Let us choose for every simplex \( \sigma \) of \( N \) a collection of loops in \( |\sigma| \) such that the homotopy classes of these loops normally generate the kernel of \( \pi_1(|\sigma|, z) \to \pi_1(X, z) \), where \( z \in |\sigma| \). For every \( U \in \mathcal{U} \) let \( U' \) be the result of attaching a two-dimensional disc to \( U \) along each loop from this collection contained in \( U \). A loop from this collection may (and usually will) be contained in several sets \( U \in \mathcal{U} \). In this case we attach the same disc to all sets \( U \in \mathcal{U} \) containing this loop. Let \( X' \) be the result of attaching all these discs to \( X \).

Clearly, the sets \( U' \) form a covering \( \mathcal{U}' \) of \( X' \). By the construction, \( \mathcal{U}' \) is an extension of \( \mathcal{U} \) to \( X' \). Moreover, several elements of \( \mathcal{U}' \) intersect if and only if the corresponding elements of \( \mathcal{U} \) intersect, i.e. \( \mathcal{U}' \) is a nerve-preserving extension of \( \mathcal{U} \). Since the discs are attached to \( X \) along loops contractible in \( X \), the inclusion map \( \pi_1(X, x) \to \pi_1(X', x) \) is an isomorphism. We will use it to identify the fundamental groups \( \pi_1(X, x) \) and \( \pi_1(X', x) \).

Let \( \sigma \) be a simplex of \( N \). Clearly, the inclusion map \( \pi_1(|\sigma|, z) \to \pi_1(|\sigma'|, z) \) is surjective. Among the loops used to attach discs there are loops in \( |\sigma| \) such that their homotopy classes generate the kernel of \( \pi_1(|\sigma|, z) \to \pi_1(X, z) \). The discs attached to these loops are contained in \( \mathcal{U}' \) for each \( U \in \mathcal{U} \) corresponding to a vertex of \( \sigma \). Therefore these discs are contained in \( |\sigma'| \). It follows that the kernel of the inclusion map

\[
\pi_1(|\sigma|, z) \to \pi_1(X, z) = \pi_1(X', z)
\]

is contained in the kernel of the inclusion map \( \pi_1(|\sigma|, z) \to \pi_1(|\sigma'|, z) \). Since the latter is surjective, this implies the injectivity of

\[
\pi_1(|\sigma'|, z) \to \pi_1(X', z).
\]

It remains to prove the last statement of the theorem. Clearly, if \( \mathcal{U} \) is a closed covering, then \( \mathcal{U}' \) is also closed. In general, \( \mathcal{U}' \) is not an open covering even if \( \mathcal{U} \) is open. But if \( \mathcal{U} \) is open, one can turn \( \mathcal{U}' \) into an open covering without affecting \( X' \) and the fundamental groups of intersections. In order to do this, let us for each attached disc remove from \( X' \) a
closed disc with the same center and smaller radius. Let $X''$ be the result. Clearly, $X''$ is open in $X'$ and there is a natural “radial” retraction $r : X'' \to X$. For every $U \in \mathcal{U}$ let

$$U'' = U' \cup r^{-1}(U).$$

Then $U''$ is open and $U'$ is a deformation retract of $U''$. Clearly, every finite intersection of sets $U''$ has the corresponding finite intersection of sets $U'$ as a deformation retract. Hence

$$\mathcal{U}'' = \{U'' | U \in \mathcal{U}\}$$

is an open covering of $X'$ and has all already established properties of $\mathcal{U}'$. ■

4.2. Corollary. If $\mathcal{U}$ be a weakly boundedly acyclic covering of $X$, then there exists $X' \supset X$ and a nerve-preserving extension $\mathcal{U}'$ of $\mathcal{U}$ to $X'$ such that $\mathcal{U}'$ is boundedly acyclic and

$$\pi_1(X, x) \to \pi_1(X', x),$$

where $x \in X$, is an isomorphism. If $\mathcal{U}$ is an open (respectively, closed) covering, then $\mathcal{U}'$ can be assumed to be open (respectively, closed).

Proof. Let $X'$ and $\mathcal{U}'$ be the space and the covering provided by Theorem 4.1. Then the inclusion map $\pi_1(X, x) \to \pi_1(X', x)$ is an isomorphism and we can use it to identify $\pi_1(X, x)$ and $\pi_1(X', x)$. Let $\sigma$ be an arbitrary simplex of the nerve of $\mathcal{U}$. Since

$$\pi_1(|\sigma|, z) \to \pi_1(|\sigma'|, z)$$

is surjective, the images of $\pi_1(|\sigma|, z)$ and $\pi_1(|\sigma'|, z)$ in $\pi_1(X', z)$ are equal. Since $\mathcal{U}$ is weakly boundedly acyclic, the image of $\pi_1(|\sigma|, z)$ is boundedly acyclic. It follows that the image $\pi_\sigma$ of $\pi_1(|\sigma'|, z)$ is boundedly acyclic. But since

$$\pi_1(|\sigma'|, z) \to \pi_1(X', z)$$

is injective, $\pi_\sigma$ is isomorphic to $\pi_1(|\sigma'|, z)$. Since $\hat{H}^\ast(|\sigma'|)$ is determined by $\pi_1(|\sigma'|, z)$, the subset $|\sigma'|$ is a boundedly acyclic. It follows that $\mathcal{U}'$ is boundedly acyclic. ■

4.3. Theorem. If $\mathcal{U}$ is an open weakly boundedly acyclic covering, then the canonical homomorphism $\hat{H}^\ast(X) \to H^\ast(X)$ can be factored through $H^\ast(N) \to H^\ast(X)$. In particular, this is true if $\mathcal{U}$ is an open amenable covering.

Proof. Theorem 2.5 this implies that the conclusion of the theorem holds if $\mathcal{U}$ is boundedly acyclic. If $\mathcal{U}$ is only weakly boundedly acyclic, Corollary 4.2 provides us with a space $X' \supset X$ and an open boundedly acyclic covering $\mathcal{U}'$ of $X'$ with the same nerve $N$. Moreover,
the inclusion \( i : X \to X' \) induces an isomorphism of fundamental groups and hence an isomorphism of bounded cohomology groups. The already proved case of the theorem implies that the canonical map \( \widehat{H}^*(X') \to H^*(X') \) is equal to the composition

\[
\widehat{H}^*(X') \to H^*(N) \to H^*(X') .
\]

Next, the construction of the horizontal arrows in the square

\[
\begin{array}{ccc}
H^*(N) & \to & H^*(X') \\
\downarrow & = & \downarrow \\
H^*(N) & \to & H^*(X)
\end{array}
\]

shows that this square is commutative. This leads to the diagram

\[
\begin{array}{ccc}
\widehat{H}^*(X') & \to & H^*(N) \to H^*(X') \\
\downarrow & \cong & \downarrow i^* \\
\widehat{H}^*(X) & \to & H^*(N) \to H^*(X)
\end{array}
\]

of solid arrows with commutative right square. Since \( \widehat{i}^* \) is an isomorphism, this diagram can be completed by a dashed arrow to a commutative diagram. The theorem follows. ■

**Coverings amenable in the sense of Gromov.** Gromov’s [Gro] notion of an amenable subset is different. Namely, he calls \( Z \) amenable if every path-connected component of \( Z \) is amenable in our sense. Let us call such subsets amenable in the sense of Gromov, and call \( U \) amenable in the sense of Gromov if elements of \( U \) are amenable in the sense of Gromov.

**4.4. Theorem (Vanishing theorem).** If \( U \) is an open covering amenable in the sense of Gromov, then the canonical homomorphism \( \widehat{H}^p(X) \to H^p(X) \) vanishes for \( p > \dim N \).

**Proof.** Now we know only that path components of sets \(|\sigma|\) are boundedly acyclic. Let us replace in the complex (2.1) with \( A^* = B^* \) the term \( R = R \) by the product of copies of \( R \) corresponding to path components of \(|\sigma|\). This forces us to replace the spaces \( C^p(N) \) in (2.2) and (2.3) by some other spaces which are still equal to 0 for \( p > \dim N \). Now the cohomology spaces \( H^p \) of the left column of (2.3) may be not equal to \( H^p(N) \), but, obviously, \( H^p = 0 \) if \( p > \dim N \). The rest of our arguments apply and show that \( \widehat{H}^p(X) \to H^p(X) \) factors through \( H^p \). The theorem follows. ■
5. A Leray theorem for $l_1$-homology

The $l_1$-norm of chains. The $l_1$-norm $\| c \|_1$ of a singular chain

$$c = \sum a c_\sigma \in C_m(X),$$

where $c_\sigma \in \mathbb{R}$ and the sum is taken over all singular $m$-simplices $\sigma$ in $X$, is defined as

$$\| c \|_1 = \sum |c_\sigma|$$

By the definition of singular chains, the sums above involve only a finite number of non-zero coefficients $c_\sigma$, and hence $\| c \|_1$ is well defined. The normed space $C_m(X)$ is not complete. Let $L_m(X)$ be its completion with respect to the $l_1$-norm. The boundary operators

$$\partial_m : C_m(X) \rightarrow C_{m-1}(X)$$

are obviously bounded and hence extend by continuity to bounded operators

$$d_m : L_m(X) \rightarrow L_{m-1}(X).$$

Since $C_m(X)$ is dense in $L_m(X)$, the identity $\partial_{m-1} \circ \partial_m = 0$ implies the identity

$$d_{m-1} \circ d_m = 0.$$

Therefore the sequence

$$0 \leftarrow L_0(X) \leftarrow \partial_1 L_1(X) \leftarrow \partial_2 L_2(X) \leftarrow \partial_3 \ldots$$

is a chain complex. The $l_1$-homology spaces of $X$ are defined as the homology spaces of this complex and are denoted by $H^{l_1}_*(X)$. In more details,

$$H^{l_1}_m(X) = \text{Ker } d_m / \text{Im } d_{m+1}.$$ 

The $l_1$-homology spaces $H^{l_1}_m(X)$ are real vector spaces carrying a canonical semi-norm induced by the $l_1$-norm on $L_m(X)$. If the image of the boundary operator

$$d_{m+1} : L_{m+1}(X) \rightarrow L_m(X)$$

is not closed, then $H^{l_1}_m(X)$ contains non-zero homology classes with the norm 0 and the canonical semi-norm on $H^{l_1}_m(X)$ is not a norm.
The duality between $L_m(X)$ and $B^m(X)$. The space $L_m(X)$ is nothing else but the space of real-valued $l_1$-functions on the set $S_m(X)$ of singular $m$-simplices in $X$. Similarly, $B^m(X)$ is the space of real-valued $l_\infty$-functions on $S_m(X)$. Let $L_m(X)^*$ be the Banach space dual to $L_m(X)$. The standard pairing between $m$-chains and $m$-cochains extends to a pairing

$$\langle \bullet, \bullet \rangle : L_m(X) \times B^m(X) \rightarrow \mathbb{R}$$

and leads to a map $L_m(X)^* \rightarrow B^m(X)$. By the well known duality between $l_1$ and $l_\infty$ spaces this map is an isometry of Banach spaces. Let us use this map to identify $B^m(X)$ with $L_m(X)^*$. By the very definition, this turns the coboundary operator

$$\partial^{m-1} : B^{m-1}(X) \rightarrow B^m(X)$$

into the Banach space adjoint of the boundary operator $\partial_m : L_m(X) \rightarrow L_{m-1}(X)$. Therefore the complex $B^\bullet(X)$ is the Banach dual of the complex $L^\bullet(X)$.

This duality between $L^\bullet(X)$ and $B^\bullet(X)$ leads to an imperfect duality between the $l_1$-homology and the bounded cohomology. In fact, already the duality between $l_1$ and $l_\infty$ spaces is not perfect. While $l_\infty$ spaces are the duals of the corresponding $l_1$ spaces, $l_1$ spaces are not the duals of $l_\infty$ spaces. When one passes to the cohomology spaces, an additional difficulty arises. The canonical pairing $\langle \bullet, \bullet \rangle$ between $L_m(X)$ and $B^m(X)$ leads to a pairing

$$H^{l_1}_m(X) \times \hat{H}^m(X) \rightarrow \mathbb{R},$$

but, in general, both $H^{l_1}_m(X)$ and $\hat{H}^m(X)$ contain non-zero elements with the canonical semi-norm equal to 0, and the pairing of such elements with every element of the other space is 0. Nevertheless, sometimes $H^{l_1}_m(X)$ and $\hat{H}^m(X)$ behave like a true duality between them exists. C. Löh [L] provided a systematic exploration of this duality. We will limit ourselves by the following observation of Sh. Matsumoto and Sh. Morita [MM].

5.1. Theorem. Let $m \geq 1$. If $\hat{H}^m(X) = \hat{H}^{m+1}(X) = 0$, then $H^{l_1}_m(X) = 0$. Also, if $H^{l_1}_m(X) = H^{l_1}_{m+1}(X) = 0$, then $\hat{H}^{m+1}(X) = 0$.

Proof. Given vector subspaces $U \subset L_m(X)$ and $V \subset B^m(X)$, let

$$U^\perp = \{ u \in B^m(X) \mid \langle c, u \rangle = 0 \text{ for every } c \in U \},$$

$$\perp V = \{ c \in L_m(X) \mid \langle c, u \rangle = 0 \text{ for every } u \in V \}$$

be their orthogonal complements with respect to $\langle \bullet, \bullet \rangle$. Recall that the complements $U^\perp$ and $\perp V$ are always closed. Also, the double complement $\perp(\perp V)^\perp$ is equal to the closure of $U$ with respect to the $l_1$-norm, but $(\perp V)^\perp$ is the closure of $V$ only in a weaker sense. Namely, $(\perp V)^\perp$ is the weak*-closure of $V$. See [R], Theorem 4.7.
Since boundary and coboundary operators are bounded operators, their kernels are closed. Moreover, since $\partial^m$ is the adjoint operator of $\partial_{m+1}$,

$$\text{Ker} \partial^m = \text{Ker} \partial_{m+1}^* = \left( \text{Im} \partial_{m+1} \right)^\perp$$

and

$$\text{Ker} \partial_{m+1} = \perp \left( \text{Im} \partial^m \right).$$

See [R], Theorem 4.12. Also, $\text{Im} \partial^m$ is closed with respect to the $l_\infty$-norm if and only if $\text{Im} \partial_{m+1}$ is closed with respect to the $l_1$-norm. In this case $\text{Im} \partial^m$ is also weak*-closed. This is a special case of the closed range theorem. See [R], Theorem 4.14.

Suppose that $H^1_{m}(X) = 0$. Then $\text{Im} \partial_{m+1} = \text{Ker} \partial_m$ and hence $\text{Im} \partial_{m+1}$ is closed. By the previous paragraph this implies that $\text{Im} \partial^m$ is also closed and weak*-closed. Suppose that also $H^1_{m+1}(X) = 0$. Then $\text{Im} \partial_{m+2} = \text{Ker} \partial_{m+1}$ and hence

$$\text{Ker} \partial^{m+1} = \left( \text{Im} \partial_{m+2} \right)^\perp = \left( \text{Ker} \partial_{m+1} \right)^\perp = \left( \perp \left( \text{Im} \partial^m \right) \right)^\perp.$$

The last space is equal to the weak*-closure of $\text{Im} \partial^m$. But $\text{Im} \partial^m$ is weak*-closed and hence $\text{Ker} \partial^{m+1} = \text{Im} \partial^m$. It follows that $\hat{H}^{m+1}(X) = 0$.

Suppose now that $\hat{H}^{m+1}(X) = 0$. Then $\text{Im} \partial^m = \text{Ker} \partial^{m+1}$ and hence $\text{Im} \partial^m$ is closed. As explained above, then $\text{Im} \partial_{m+1}$ is closed. If also $\hat{H}^m(X) = 0$, then $\text{Im} \partial^{m-1} = \text{Ker} \partial^m$ and hence $\text{Im} \partial^{m-1}$ is closed. It follows that

$$\text{Ker} \partial_m = \perp \left( \text{Im} \partial^{m-1} \right) = \perp \left( \text{Ker} \partial^m \right) = \perp \left( \left( \text{Im} \partial_{m+1} \right)^\perp \right).$$

The last space is equal to the closure of $\text{Im} \partial_{m+1}$. Since $\text{Im} \partial_{m+1}$ is closed, it follows that $\text{Ker} \partial_m = \text{Im} \partial_{m+1}$ and hence $H^1_{m}(X) = 0$. This completes the proof. 

5.2. Corollary. If $X$ is path-connected and the fundamental group $\pi_1(X, x)$ is boundedly acyclic, then $H^1_{m}(X) = 0$ for every $m \geq 1$.

Proof. Since $\hat{H}^m(X) = 0$ for every $m \geq 1$, this follows from Theorem 5.1. 

5.3. Theorem. If $\mathcal{U}$ be an open weakly boundedly acyclic covering of $X$, then the canonical homomorphism $H_*(X) \rightarrow H^1_*(X)$ can be factored through $H_*(X) \rightarrow H_*(N)$.

Proof. Clearly, $H^1_0(Z)$ is canonically isomorphic to $\mathbb{R}$ for every path-connected space $Z$. Together with Corollary 5.2 this implies that path-connected boundedly acyclic subsets are $L_*$-acyclic. The rest of the proof differs from the proof of Theorem 4.3 by using Theorem 3.2 instead of Theorem 2.5 and inverting the directions of arrows. 

25
6. Uniqueness of Leray homomorphisms

Leray homomorphisms. For every topological space $X$ and an open covering $\mathcal{U}$ of $X$ we constructed in Section 2 a canonical homomorphism

$$l_{\mathcal{U}} : H^*(N_{\mathcal{U}}) \rightarrow H^*(X),$$

where $N_{\mathcal{U}}$ is the nerve of $\mathcal{U}$, $H^*(N_{\mathcal{U}})$ is the simplicial cohomology of $N_{\mathcal{U}}$, and $H^*(X)$ is the singular cohomology of $X$. The homomorphisms $l_{\mathcal{U}}$ are called Leray homomorphisms. The goal of this section is to provide an axiomatic characterization of them.

The category of coverings. Following M. Barr [Ba], let us consider the category $\mathfrak{C}_\mathfrak{o}$ having as objects pairs $(X, \mathcal{U})$, where $\mathcal{U}$ is a covering of a topological space $X$, and as morphisms $(X, \mathcal{U}) \rightarrow (Y, \mathcal{V})$ continuous maps $f : X \rightarrow Y$ such that every $U \in \mathcal{U}$ is contained in the preimage $f^{-1}(V)$ for some $V \in \mathcal{V}$, i.e. the covering $\mathcal{U}$ is a refinement of the covering $f^{-1}(\mathcal{V}) = \{f^{-1}(V) \mid V \in \mathcal{V}\}$.

As before, $N_{\mathcal{U}}$ is the nerve of the covering $\mathcal{U}$. A morphism $f : (X, \mathcal{U}) \rightarrow (Y, \mathcal{V})$ induces a homomorphism $f^* : H^*(Y) \rightarrow H^*(X)$, turning $(X, \mathcal{U}) \rightarrow H^*(X)$ into a functor from $\mathfrak{C}_\mathfrak{o}$ to graded abelian groups. Such a morphism $f$ also induces a homomorphism $f^* : H^*(N_\mathcal{V}) \rightarrow H^*(N_{\mathcal{U}})$.

Indeed, since the covering $\mathcal{U}$ refines $f^{-1}(\mathcal{V})$, one can choose for each $U \in \mathcal{U}$ some $\varphi(U) \in \mathcal{V}$ such that $f(U) \subset \varphi(U)$. Clearly, if the intersection of several sets $U_i \in \mathcal{U}$ is non-empty, then the intersection of the images $\varphi(U_i)$ is also non-empty. Therefore $\varphi$ is a simplicial map $N_{\mathcal{U}} \rightarrow N_\mathcal{V}$ and hence defines an induced homomorphism

$$\varphi^* : H^*(N_\mathcal{V}) \rightarrow H^*(N_{\mathcal{U}}).$$

Suppose that $\varphi' : \mathcal{U} \rightarrow \mathcal{V}$ is another map such that $f(U) \subset \varphi'(U)$ for every $U \in \mathcal{U}$. If the intersection of several sets $U_i \in \mathcal{U}$ is non-empty, then the set

$$\bigcap_i \varphi(U_i) \cap \bigcap_i \varphi'(U_i) = f\left(\bigcap_i U_i\right)$$

is also non-empty. It follows that if $\sigma$ is a simplex of $N_{\mathcal{U}}$, then $\varphi(\sigma) \cup \varphi'(\sigma)$ is a simplex of $N_{\mathcal{V}}$. Therefore $\varphi$ and $\varphi'$ are connected by an elementary simplicial homotopy and hence $\varphi^* = \varphi'^*$. We see that the induced homomorphism $\varphi^*$ does not depend on the choice of $\varphi$ and hence we can take it as the induced map $f^* : H^*(N_\mathcal{V}) \rightarrow H^*(N_{\mathcal{U}})$. A trivial verification shows that these induced maps turn $(X, \mathcal{U}) \rightarrow H^*(N_{\mathcal{U}})$ into a functor from $\mathfrak{C}_\mathfrak{o}$ to graded abelian groups.
Leray transformations. For a space $X$ let $\mathcal{U}(X)$ be the covering of $X$ by the single set $X$. Clearly, $N_{\mathcal{U}(X)}$ consists of only one vertex. Therefore $H^m(N_{\mathcal{U}(X)}) = 0$ for $m \geq 1$ and $H^0(N_{\mathcal{U}(X)})$ is equal to the group of coefficients. If $X$ is path-connected, then $H^0(X)$ is also equal to the group of coefficients and there is a canonical isomorphism

$$i_X : H^0(N_{\mathcal{U}(X)}) \rightarrow H^0(X).$$

Let $\mathcal{C}$ be a full subcategory of $\mathcal{C}_o$ containing all coverings $(X, \mathcal{U}(X))$ with path-connected $X$. A natural transformation from the functor $(X, \mathcal{U}) \rightarrow H^*(N_{\mathcal{U}})$ on the category $\mathcal{C}$ to the functor $(X, \mathcal{U}) \rightarrow H^*(X)$ assigns to each object $(X, \mathcal{U})$ of $\mathcal{C}$ a homomorphism

$$l_{\mathcal{U}} : H^*(N_{\mathcal{U}}) \rightarrow H^*(X)$$

in such a way that for every morphism $f : (X, \mathcal{U}) \rightarrow (Y, \mathcal{V})$ the square

$$\begin{array}{ccc}
H^*(N_Y) & \xrightarrow{l_{\mathcal{V}}} & H^*(Y) \\
\downarrow f^* & & \downarrow f^* \\
H^*(N_{\mathcal{U}}) & \xrightarrow{l_{\mathcal{U}}} & H^*(X)
\end{array}$$

(6.1)

is commutative. Let us call such a natural transformation a Leray transformation on $\mathcal{C}$ if for every path-connected space $X$ the homomorphism $l_{\mathcal{U}(X)}$ is equal in dimension 0 to $i_X$.

6.1. Theorem. The canonical homomorphisms $l_{\mathcal{U}} : H^*(N_{\mathcal{U}}) \rightarrow H^*(X)$ from Section 2 form a Leray transformation on the category of open coverings.

Proof. The only not quite functorial element constructions of the construction of $l_{\mathcal{U}}$ is the choice of linear orders on sets of vertices. As is well known, the cohomology $H^*(N_{\mathcal{U}})$ is independent on this choice up to canonical isomorphisms, as also the induced maps. Given $f$ as above, one can choose the orders of vertices of $N_{\mathcal{U}}$ and $N_{\mathcal{V}}$ in such a way that $f$ is (non-strictly) order-preserving. A routine check shows that then (6.1) is commutative. Another routine check shows that $l_{\mathcal{U}(X)}$ is equal in dimension 0 to $i_X$. ■

Geometric realizations of simplicial complexes. Let $N$ is a simplicial complex, $|N|$ be its geometric realization, and $\mathcal{U}(N)$ be the covering of $|N|$ by the open stars of vertices. By a well known theorem of Eilenberg [E], $N_{\mathcal{U}(N)} = N$ and there is a canonical isomorphism

$$i_N : H^*(N) \rightarrow H^*(|N|).$$

For any reasonable Leray transformation one should have $l_{\mathcal{U}(N)} = i_N$. 27
6.2. **Theorem.** \( l_{\mathcal{U}(N)} = i_N \) for every simplicial complex \( N \).

**Proof.** The idea is to prove that \( N \to l_{\mathcal{U}(N)} \) is a natural transformation of cohomology theories and apply the Eilenberg–Steenrod uniqueness theorem. First of all, we need to extend the construction of \( l_{\mathcal{U}(N)} \) to pairs of simplicial complexes. Let

\[ T^\bullet(N) = T^\bullet(N_{\mathcal{U}(N)}, C) \]

be the total complex associated with the covering \( \mathcal{U}_N \) of \( |N| \) by open stars. For a simplicial subcomplex \( L \) of \( N \) let \( T^\bullet(N, L) \) be the kernel of morphism \( T^\bullet(N) \to T^\bullet(L) \) induced by the inclusion \( L \to N \). As we know, the morphisms

\[ \tau_C : C^\bullet(|N|) \to T^\bullet(N) \text{ and } C^\bullet(|L|) \to T^\bullet(L) \]

induce isomorphisms in cohomology groups. It follows that the natural morphism

\[ C^\bullet(|N|, |L|) \to T^\bullet(N, L) \]

also induces isomorphism in cohomology groups. This allows to define the map

\[ l_{\mathcal{U}(N),\mathcal{U}(L)} : H^\bullet(N, L) \to H^\bullet(|N|, |L|) \]

exactly as in the absolute case.

Next, we need to check the functoriality of the maps \( l_{\mathcal{U}(N)} \) and \( l_{\mathcal{U}(N),\mathcal{U}(L)} \). We will limit ourselves by the absolute case, the case of pairs being completely similar. For a vertex \( v \) of a simplicial complex we will denote by \( U_v \) the open star of \( v \). Let \( f : M \to N \) be a simplicial map and \( |f| : |M| \to |N| \) be its geometric realization. Then \( |f|(U_v) \subset U_{f(v)} \) for every vertex \( v \) of \( M \). It follows that \( |f| \) is a morphism of coverings

\[ (|N|, \mathcal{U}(N)) \to (|M|, \mathcal{U}(M)) \]

and hence induces a morphism \( |f|^* : T^\bullet(N) \to T^\bullet(M) \). The diagram

\[ \begin{array}{ccc}
C^\bullet(N) & \xrightarrow{\lambda_C} & T^\bullet(N) & \xleftarrow{\tau_C} & C^\bullet(|N|) \\
\downarrow{f^*} & & \downarrow{|f|^*} & & \downarrow{|f|^*} \\
C^\bullet(M) & \xrightarrow{\lambda_C} & T^\bullet(M) & \xleftarrow{\tau_C} & C^\bullet(|M|)
\end{array} \]

is commutative, as a routine check shows. This implies the commutativity of the correspond-
ing diagram of the (co)homology groups. In turn, this implies that the diagram

\[
\begin{array}{ccc}
H^*(N) & \xrightarrow{l_{\mathcal{U}(N)}} & H^*(|N|) \\
f^* & & |f|^* \\
\downarrow & & \downarrow \\
H^*(M) & \xrightarrow{l_{\mathcal{U}(M)}} & H^*(|M|)
\end{array}
\]

is commutative. This is the required functoriality of \( l_{\mathcal{U}(N)} \). We leave to the reader to check the functoriality of \( l_{\mathcal{U}(N), \mathcal{U}(L)} \), as also to check that the maps \( l_{\mathcal{U}(N)} \) and \( l_{\mathcal{U}(N), \mathcal{U}(L)} \) commute with the connecting homomorphisms \( \partial \) of the cohomological long exact sequence of the pair \((N, L)\). Clearly, in dimension 0 these maps are equal to the isomorphism \( i_N \) and its analogue for the pair \((N, L)\) respectively. Now the cohomological version of Eilenberg–Steenrod uniqueness theorem implies that \( l_{\mathcal{U}(N)} = i_N \). See [ES], Theorem VI.8.1.

6.3. Theorem. Let \( \mathcal{P} \) be the full subcategory of \( \mathcal{C} \) having open coverings of paracompact spaces as its objects. There exists exactly one Leray transformation on \( \mathcal{P} \).

Proof. We already constructed a Leray transformation on a bigger category. It remains to prove the uniqueness. Suppose that \( \mathcal{U} \) is an open covering of a paracompact space \( X \), and let \( N = N_{\mathcal{U}} \). Since \( X \) is paracompact, there exists a partition of unity \( t_U, U \in \mathcal{U} \) subordinate to \( \mathcal{U} \). It leads to a continuous map \( t: X \to |N| \). The open star of a vertex \( u \) corresponding to \( U \in \mathcal{U} \) consists of points with the barycentric coordinate \( t_u > 0 \). It follows that the image \( t(U) \) is contained in the open star of \( u \). Therefore \( t \) is a morphism

\[
(X, \mathcal{U}) \to (|N|, \mathcal{U}(N)).
\]

Clearly, \( N_{\mathcal{U}(N)} = N_{\mathcal{U}} \) and one can take the identity map as the map \( \varphi: N_{\mathcal{U}} \to N_{\mathcal{U}(N)} \) used to construct \( t^*: H^*(N_{\mathcal{U}(N)}) \to H^*(N_{\mathcal{U}}) \). Hence the latter map is equal to identity. If \( \mathcal{U} \to l_{\mathcal{U}} \) is a Leray transformation on \( \mathcal{P} \), the the diagram

\[
\begin{array}{ccc}
H^*(N) & \xrightarrow{l_{\mathcal{U}(N)}} & H^*(|N|) \\
\downarrow = & & \downarrow t^* \\
H^*(N) & \xrightarrow{l_{\mathcal{U}}} & H^*(X)
\end{array}
\]

is commutative. Together with Theorem 6.2 this implies that \( l_{\mathcal{U}} = t^* \circ i_N \). The uniqueness follows, as also the fact that \( t^* \) does not depend on the partition of unity.
7. Nerves of families and paracompact spaces

**Families of subsets.** Let $S$ be a set. A *family of subsets* $\mathcal{F} = \{F_i\}_{i \in I}$ of $S$ is a map

$$i \mapsto F_i \subset S$$

from a set $I$ to the set of all subsets of $S$. Usually the nature of the set $I$ is of no importance and $\mathcal{F}$ is treated almost as a collection of subsets of $S$. But our main interest is in the families indexed by the same set $I$. If $\mathcal{G} = \{G_i\}_{i \in I}$ is another family indexed by $I$ and $F_i \subset G_i$ for every $i \in I$, we say that $\mathcal{F}$ is *combinatorially refining* $\mathcal{G}$.

The *nerve* of a family $\mathcal{F} = \{F_i\}_{i \in I}$ is an abstract simplicial complex having the set $I$ as the set of vertices. Its simplices are finite non-empty subsets $\sigma \subset I$ such that

$$\bigcap_{i \in \sigma} F_i \neq \emptyset.$$

When two families $\mathcal{F} = \{F_i\}_{i \in I}$ and $\mathcal{G} = \{G_i\}_{i \in I}$ are indexed by the same set $I$, it make sense to say that their nerves are *equal*.

**Families of subspaces.** Suppose now that $S$ is a topological space. A family $\mathcal{F} = \{F_i\}_{i \in I}$ of subsets, or, what is the same, of subspaces of $S$ is said to be *open* if the sets $F_i$ are open, and *closed* if the sets $F_i$ are closed. The family $\mathcal{F} = \{F_i\}_{i \in I}$ is said to be *locally finite* if for every $x \in S$ there exists an open set $U$ such that $x \in U$ and $U \cap F_i \neq \emptyset$ for only a finite number of indices $i \in I$.

Given a closed family $\mathcal{F} = \{F_i\}_{i \in I}$, we would like to know when there exists an open family $\mathcal{U}$ such that $\mathcal{F}$ is combinatorially refining $\mathcal{G}$ and the nerves of $\mathcal{F}$ and $\mathcal{G}$ are equal. Since the nerves of families are involved, it is only natural to assume that $\mathcal{F}$ is locally finite. If $I$ is countable, then such a family $\mathcal{U}$ exists if $\mathcal{F}$ is locally finite and $S$ is a normal space. See Proposition 7.2. But the analogue of this result for general $I$ involves an additional assumption about $\mathcal{F}$ satisfied when $S$ is not only normal, but is, moreover, *paracompact* (see below). Now we turn to the key property of locally finite families.

**7.1. Lemma.** Let $\mathcal{F}$ be a locally finite family of subspaces of a topological space $S$. Then

$$\bigcup_{i \in I} F_i = \bigcup_{i \in I} \overline{F}_i,$$

where $\overline{A}$ denotes the closure of a subset $A \subset S$, and the family $\{\overline{F}_i\}_{i \in I}$ is locally finite.

**Proof.** If $U$ is an open set intersecting $F_i$ for only a finite number of $i \in I$, then $U$ intersects the closures $\overline{F}_i$ for only a finite number of $i \in I$. The lemma follows. ■
7.2. Proposition. Let $Z$ be a normal space. Let $\mathcal{F} = \{F_i\}_{i \in I}$ be a closed family combinatorially refining an open family $\mathcal{U} = \{U_i\}_{i \in I}$ of subsets of $Z$. If the family $\mathcal{F}$ is locally finite and $I$ is countable, then there exists an open family $\mathcal{G} = \{G_i\}_{i \in I}$ such that

$$F_i \subset G_i \subset \overline{G}_i \subset U_i$$

for every $i \in I$ and the nerves of $\mathcal{F}$ and $\mathcal{G}$ are equal.

Proof. We may assume that $I = \{0, 1, 2, \ldots \}$ is the set of non-negative integers. Let us first ignore the condition $\overline{G}_i \subset U_i$. Equivalently, let us assume that $U_i = Z$ for every $i$. Let us consider the family of intersections

$$\bigcap_{i \in K} F_i$$

such that $F_0 \cap \bigcap_{i \in K} F_i = \emptyset$ with finite $K \subset I$. Since $\mathcal{F}$ is locally finite, this family is also locally finite. By Lemma 7.1 the union $C_0$ of this family is closed. Clearly, $F_0 \cap C_0 = \emptyset$. Since $Z$ is normal, there exists an open set $G_0$ such that $F_0 \subset G_0$ and $G_0 \cap C_0 = \emptyset$. Let us consider the family

$$\mathcal{E}_1 = \{ \overline{G}_0, F_1, F_2, \ldots \}$$

Clearly, the nerve of $\mathcal{E}_1$ is equal to the nerve of $\mathcal{F}$. If an open set $U$ intersects an infinite number of sets from $\mathcal{E}_1$, then $U$ intersects an infinite number of the sets $F_i$ with $i \geq 1$, contrary to the assumption that $\mathcal{F}$ is locally finite. Hence $\mathcal{E}_1$ is locally finite. Arguing by induction, we may assume that the open sets $G_0, G_1, \ldots, G_{k-1}$ are already defined, that $F_i \subset G_i$ for $i = 0, 1, 2, \ldots, k-1$, the family

$$\mathcal{E}_k = \{ \overline{G}_0, \ldots, \overline{G}_{k-1}, F_k, F_{k+1}, \ldots \}$$

is locally finite, and the nerves of $\mathcal{E}_k$ and $\mathcal{F}$ are equal. By applying the above arguments to $\mathcal{E}_k$ and $F_k$ in the roles of $\mathcal{F}$ and $F_0$ respectively, we will get the next family

$$\mathcal{E}_{k+1} = \{ \overline{G}_0, \ldots, \overline{G}_{k-1}, \overline{G}_k, F_{k+1}, \ldots \}$$

with similar properties. By continuing this process indefinitely we define an open subset $G_i \supset F_i$ for every $i = 0, 1, 2, \ldots$. Let us consider the family

$$\mathcal{G} = \{ G_0, G_1, G_2, \ldots \}.$$ 

Suppose that $\bigcap_{i \in K} G_i \neq \emptyset$ for a finite subset $K \subset I$. Then $\bigcap_{i \in K} \overline{G}_i \neq \emptyset$ and if $k$ is the maximal element of $K$, then the sets $\overline{G}_i$ with $i \in K$ occur in the family $\mathcal{E}_{k+1}$. Since the nerves of the families $\mathcal{E}_{k+1}$ and $\mathcal{F}$ are equal, it follows that $\bigcap_{i \in K} F_i \neq \emptyset$. We see that $\bigcap_{i \in K} G_i \neq \emptyset$ implies $\bigcap_{i \in K} F_i \neq \emptyset$. The converse implication holds because $F_i \subset G_i$ for every $i$. Therefore the nerves of $\mathcal{F}$ and $\mathcal{G}$ are equal.
This proves the theorem when $U_i = Z$ for every $i$. The extension to the general case is easy. By the already proved part of the theorem there exists a family $G' = \{G'_i\}_{i \in I}$ of open sets such that $F_i \subset G'_i$ for every $i$ and the nerves of $\mathcal{F}$ and $\mathcal{G}'$ are equal. For every $i$ let $V_i = G'_i \cap U_i$. Clearly, $F_i \subset V_i$. Since $Z$ is normal, for every $i$ there exists an open set $G_i$ such that $F_i \subset G_i$ and $G_i \subset V_i \subset U_i$. Also, $G_i \subset G'_i$ for every $i$. Since the nerves of $\mathcal{F}$ and $\mathcal{G}'$ are equal, this implies that the nerves of $\mathcal{F}$ and $\mathcal{G}$ are also equal. ■

7.3. Theorem. Let $Z$ be a normal space. Let $\mathcal{F} = \{F_i\}_{i \in I}$ be a closed family combinatorially refining an open family $\mathcal{U} = \{U_i\}_{i \in I}$ of subsets of $Z$. If both families $\mathcal{F}$ and $\mathcal{U}$ are locally finite, then there exists an open locally finite family $\mathcal{G} = \{G_i\}_{i \in I}$ such that

$$F_i \subset G_i \subset \bar{G}_i \subset U_i$$

for every $i \in I$ and the nerves of $\mathcal{F}$ and $\mathcal{G}$ are equal.

Proof. When the set $I$ is uncountable, one needs to replace the usual induction by the transfinite induction or some equivalent tool. We will use Zorn lemma. More importantly, in this case there is no easy way to ensure that the families $\mathcal{E}_k$ from the proof of Proposition 7.2 are locally finite. By this reason one needs to assume that the family $\mathcal{U}$ is locally finite.

Let us consider the set $\mathcal{A}$ of all families $\mathcal{A} = \{A_i\}_{i \in I}$ of subspaces of $Z$ such that for every $i \in I$ either $A_i = F_i$, or $A_i$ is open and

$$F_i \subset A_i \subset \bar{A}_i \subset U_i.$$ 

In order to apply Zorn lemma, let us define a partial order $\preceq$ on $\mathcal{A}$ as follows. Suppose that $\mathcal{A} = \{A_i\}_{i \in I}$ and $\mathcal{B} = \{B_i\}_{i \in I}$ are two elements of $\mathcal{A}$. Then $\mathcal{A} \preceq \mathcal{B}$ if the nerves of $\mathcal{A}$ and $\mathcal{B}$ are equal and for every $i \in I$ either $A_i = B_i$, or $A_i = F_i$, or both. Clearly, $\preceq$ is a partial order on $\mathcal{A}$. We claim that $\preceq$ satisfies the assumptions of Zorn lemma.

Suppose that $\mathcal{B} \subset \mathcal{A}$ is a linearly ordered by $\preceq$ subset. Let $i \in I$. If $A_i \neq F_i$ for some family $\mathcal{A} = \{A_i\}_{i \in I}$, then $B_i = A_i$ for every family $\mathcal{B} = \{B_i\}_{i \in I}$ such that $\mathcal{A} \preceq \mathcal{B}$. Let $E_i = F_i$ if $A_i = F_i$ for every $\mathcal{A} \in \mathcal{B}$, and $E_i = A_i$ if $A_i \neq F_i$ for some $\mathcal{A} \in \mathcal{B}$. Since $\mathcal{B}$ is linearly ordered, this definition is correct. Let $\mathcal{E} = \{E_i\}_{i \in I}$. If $\mathcal{E} \in \mathcal{A}$, then, obviously, $\mathcal{A} \preceq \mathcal{B}$ for every $\mathcal{A} \in \mathcal{B}$, and hence $\mathcal{B}$ admits an upper bound.

Let us prove that $\mathcal{E} \in \mathcal{A}$. Clearly, for every $i \in I$ either $E_i = F_i$, or $E_i$ is open and

$$F_i \subset E_i \subset \bar{E}_i \subset U_i.$$ 

Suppose that $\bigcap_{i \in K} E_i \neq \emptyset$ for a finite subset $K \subset I$. Each set $E_i$ with $i \in K$ occurs in some family $\mathcal{B}_i \in \mathcal{B}$. Since $K$ is finite and $\mathcal{B}$ is linearly ordered, the set of families $\mathcal{B}_i$, where $i \in K$, has a maximal element. If $\mathcal{B} = \{B_i\}_{i \in I}$ is this maximal element, then $E_i =$
B_i for every i ∈ K. Since the nerves of $\mathcal{B}$ and $\mathcal{F}$ are equal, this implies that $\bigcap_{i \in K} F_i \neq \emptyset$. It follows that the nerves of $\mathcal{E}$ and $\mathcal{F}$ are equal. Therefore $\mathcal{E} \in \mathfrak{A}$.

We see that every linearly ordered subset of $\mathfrak{A}$ admits an upper bound. Therefore Zorn lemma applies and there exists a family $\mathcal{G} = \{G_i\}_{i \in I}$ in $\mathfrak{A}$ maximal with respect to $\leq$. As above, for every $i \in I$ either $G_i = F_i$, or $G_i$ is open and

$$F_i \subset G_i \subset \overline{G_i} \subset U_i.$$ 

The family $\mathcal{G}$ is locally finite because $\mathcal{U}$ is. The arguments of the previous paragraph show that the nerves of $\mathcal{B}$ and $\mathcal{F}$ are equal. It remains to show that $G_i$ is open for every $i \in I$. Suppose that $G_k$ is not open for some $k \in I$. Then $G_k = F_k$. In particular, $G_k$ is closed. Since the family $\mathcal{U}$ is locally finite and $\overline{G_i} \subset U_i$ for every $i \in I$, the family of closures

$$\{ \overline{G_i} \}_{i \in I}$$

is locally finite. It follows that the family of intersections

$$\bigcap_{i \in K} \overline{G_i}$$

such that $G_k \cap \bigcap_{i \in K} \overline{G_i} = \emptyset$ with finite $K \subset I$ is also locally finite. By Lemma 7.1 the union $C_k$ of this family is closed. Clearly, $G_k \cap C_k = \emptyset$. Since the space $Z$ is normal, there exists an open set $H_k$ such that

$$F_k \subset H_k, \quad \overline{H_k} \subset U_k, \quad \text{and} \quad \overline{H_k} \cap C_k = \emptyset.$$ 

Let us define a family $\mathcal{E} = \{E_i\}_{i \in I}$ as follows:

$$E_i = G_i \quad \text{if} \quad i \neq k \quad \text{and} \quad E_k = H_k.$$ 

Then the nerve of $\mathcal{E}$ is equal to the nerve of $\mathcal{G}$ and hence to the nerve of $\mathcal{F}$ and for every $i \in I$ either $E_i = F_i$, or $E_i$ is open and

$$F_i \subset E_i \subset \overline{E_i} \subset U_i.$$ 

Hence $\mathcal{E} \in \mathfrak{A}$. But since $G_k = F_k$ is assumed to be not open and $E_k = H_k$ is open, $E_k \neq F_k$ and $\mathcal{G} < \mathcal{E}$, contrary to the maximality of $\mathcal{G}$. The contradiction shows that $G_i$ is open for every $i \in I$. □

**Remarks.** Theorem 7.3 was proved by K. Morita [Mor]. See [Mor], Theorem 1.3. The above proof is a version of Morita's one, but, following J. van Mill [vM], we included a proof of its simplified version, Proposition 7.2. Cf. [vM], Proposition 3.2.1 and Corollary 3.2.2. The main idea is more transparent in the countable case, and the separation of the countable and the general cases shows why one needs to assume that $\mathcal{U}$ is locally finite.
Paracompactness. Suppose that \( F = \{ F_i \}_{i \in I} \) and \( U = \{ U_k \}_{k \in K} \) are two families of subsets of a set \( S \). If for every \( i \in I \) there exists \( k \in K \) such that \( F_i \subset U_k \), we say that \( F \) is refining \( U \). Obviously, if \( F \) is combinatorially refining \( U \), then \( F \) is refining \( U \). The family \( F \) is said to be a covering of \( S \) if the union of all sets \( F_i \) is equal to \( S \).

A topological space \( X \) is called paracompact if for every open covering \( F \) of \( X \) there exists a locally finite open covering refining \( F \). For the rest of this section we will assume that \( X \) is a paracompact Hausdorff space.

7.4. Lemma. The space \( X \) is normal.

Proof. Let \( A \subset X \) be a closed set and let \( b \in X \setminus A \). Since \( X \) is a Hausdorff space, for every \( a \in A \) there are open sets \( U_a \ni a \) and \( V_a \ni b \) such that \( U_a \cap V_a = \emptyset \). The family \( \{ U_a \}_{a \in A} \) together with \( X \setminus B \) forms an open covering of \( X \). Let \( \{ F_i \}_{i \in I} \) be a locally finite open covering refining this cover, and let \( I_A = \{ i \in I \mid F_i \cap A \neq \emptyset \} \). Since \( U_a \cap V_a = \emptyset \), the closure \( \overline{F_i} \) does not contain \( b \) if \( i \in I_A \). Together with Lemma 7.1 this implies that

\[
\bigcup_{i \in I_A} F_i \quad \text{and} \quad X \setminus \bigcup_{i \in I_A} F_i
\]

are disjoint open neighborhoods of \( A \) and \( b \) respectively. Therefore the space \( X \) is regular. Now a similar argument allows to prove that \( X \) is normal. Let \( A, B \subset X \) be closed sets such that \( A \cap B = \emptyset \). By the previous paragraph \( X \) is regular. Hence for every \( a \in A \) there are open sets \( U_a \) and \( V_a \) such that \( a \in U_a \), \( B \subset V_a \), and \( U_a \cap V_a = \emptyset \). The families \( \{ U_a \}_{a \in A} \) and \( \{ V_a \}_{a \in A} \) together with \( X \setminus B \) forms an open covering of \( X \). Let \( \{ F_i \}_{i \in I} \) be a locally finite open covering refining this covering, and let \( I_A = \{ i \in I \mid F_i \cap A \neq \emptyset \} \). Since \( U_a \cap V_a = \emptyset \), the closure \( \overline{F_i} \) is disjoint from \( B \) if \( i \in I_A \). Together with Lemma 7.1 this implies that

\[
\bigcup_{i \in I_A} F_i \quad \text{and} \quad X \setminus \bigcup_{i \in I_A} F_i
\]

are disjoint open neighborhoods of \( A \) and \( B \) respectively. \( \blacksquare \)

7.5. Theorem. If \( U = \{ U_i \}_{i \in I} \) is an open covering of \( X \), then there exists a locally finite closed covering \( F = \{ F_i \}_{i \in I} \) of \( X \) such that \( F \) is combinatorially refining \( U \).

Proof. Lemma 7.4 implies that the space \( X \) is regular. Hence if \( x \in U_i \), then \( x \in G \) for an open set \( G \) such that \( \overline{G} \subset U_i \). Such open sets \( G \) form an open covering \( \mathcal{G} = \{ G_k \}_{k \in K} \)
of $X$ such that the family of the closures
\[
\{ \overline{G_k} \}_{k \in K}
\]
is refining $\{U_i\}_{i \in I}$. Let $\{A_s\}_{s \in S}$ be a locally finite covering refining $\mathcal{G}$ (at this point we don’t even need it to be open). Then for every $s \in S$ there exists $i(s) \in I$ such that
\[
\overline{A_s} \subset U_{i(s)}.
\]
For every $i \in I$ let
\[
F_i = \bigcup_{i(s)=i} \overline{A_s}.
\]
By Lemma 7.1 the family $\{F_i\}_{i \in I}$ is locally finite. Since $\{A_s\}_{s \in S}$ is a covering, $\{F_i\}_{i \in I}$ is a covering also. Clearly, $F_i$ is closed and $F_i \subset U_i$ for every $i \in I$. ■

**Remark.** In the situation of Theorem 7.5 it may happen that $F_i = \emptyset$ for some $i \in I$. Note that adding empty subsets to a family does not affect the property of being locally finite.

**Stars.** Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a covering of a set $Z$. For $A \subset Z$ let
\[
\text{St}(A, \mathcal{U}) = \bigcup U_i,
\]
where the union is taken over $i \in I$ such that $A \cap U_i \neq \emptyset$. For $z \in Z$ we set
\[
\text{St}(z, \mathcal{U}) = \text{St}(\{z\}, \mathcal{U}).
\]
Clearly, for every $A \subset Z$
\[
\text{St}(A, \mathcal{U}) = \bigcup_{a \in A} \text{St}(a, \mathcal{U}).
\]
A covering $\mathcal{A} = \{A_s\}_{s \in S}$ is said to be a *barycentric refinement* of $\mathcal{U}$ if for every $z \in Z$ there exists $i \in I$ such that $\text{St}(z, \mathcal{A}) \subset U_i$, and is to be a *star refinement* of $\mathcal{U}$ if for every $s \in S$ there exists $i \in I$ such that $\text{St}(A_s, \mathcal{A}) \subset U_i$.

**7.6. Lemma.** Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open covering of $X$. Then there exists an open covering of $X$ barycentrically refining $\mathcal{U}$.

**Proof.** Let $\mathcal{U} = \{U_i\}_{i \in I}$. Then by Theorem 7.5 there exists a locally finite closed covering $\mathcal{F} = \{F_i\}_{i \in I}$ of $X$ such that $F_i \subset U_i$ for every $i \in I$. For every $x \in X$ let
\[
I(x) = \{ i \in I \mid x \in F_i \}.
\]
Since \( F \) is locally finite, \( I(x) \) is finite for every \( x \in X \). By Lemma 7.1 the union
\[
\bigcup_{k \in K} F_i
\]
is closed for every \( K \subset I \). It follows that the set
\[
V_x = \left( \bigcap_{i \in I(x)} U_i \right) \cap \left( X \setminus \bigcup_{i \not\in I(x)} F_i \right)
\]
is open for every \( x \in X \). Therefore \( V = \{ V_z \}_{z \in X} \) is an open covering of \( X \).

It is sufficient to prove that \( V \) is a barycentric refinement of \( \mathcal{U} \). Suppose that \( z \in X \) and let us choose some \( j \in I(z) \), so that \( z \in F_j \). By the definition of the sets \( V_x \), if \( z \in V_x \), then \( z \not\in F_i \) for every \( i \not\in I(x) \). Therefore in this case \( j \in I(x) \) and hence \( V_x \subset U_j \). Since \( z \in X \) was arbitrary, \( V \) is a barycentric refinement of \( \mathcal{U} \).

**7.7. Lemma.** Let \( \mathcal{A}, \mathcal{B}, \mathcal{C} \) be coverings of a set. If \( \mathcal{A} \) is a barycentric refinement of \( \mathcal{B} \) and \( \mathcal{B} \) is a barycentric refinement of \( \mathcal{C} \), then \( \mathcal{A} \) is a star refinement of \( \mathcal{C} \).

**Proof.** Let \( \mathcal{A} = \{ A_s \}_{s \in S} \) and \( \mathcal{B} = \{ B_t \}_{t \in T} \). Let us fix \( s \in S \) and for each \( a \in A_s \) choose \( t(a) \in T \) such that \( St(a, \mathcal{A}) \subset B_{t(a)} \). Then
\[
St(A_s, \mathcal{U}) = \bigcup_{a \in A_s} St(a, \mathcal{U}) \subset \bigcup_{a \in A_s} B_{t(a)}.
\]
Let us fix now some \( z \in A_s \). If also \( a \in A_s \), then
\[
z \in St(a, \mathcal{A}) \subset B_{t(a)}
\]
and hence \( z \in B_{t(a)} \). It follows that
\[
\bigcup_{a \in A_s} B_{t(a)} \subset St(z, \mathcal{B})
\]
and hence \( St(A_s, \mathcal{U}) \subset St(z, \mathcal{B}) \). Since \( s \in S \) was arbitrary and \( \mathcal{B} \) is a barycentric refinement of \( \mathcal{C} \), this implies that \( \mathcal{A} \) is a star refinement of \( \mathcal{C} \). ■

**7.8. Theorem.** Let \( \mathcal{U} \) be an open covering of \( X \). Then there exists an open covering of \( X \) star refining \( \mathcal{U} \).

**Proof.** It is sufficient to apply twice Lemma 7.6 and then apply Lemma 7.7. ■

**7.9. Theorem.** Let \( \mathcal{F} \) be a locally finite closed family of subsets of \( X \). Then there exists a locally finite open family \( \mathcal{U} \) of subsets of \( X \) such that \( \mathcal{F} \) is combinatorially refining \( \mathcal{U} \).
Proof. Let $\mathcal{F} = \{ F_i \}_{i \in I}$. For each $x \in X$ there is an open set $V_x$ such that $x \in V_x$ and $V_x \cap F_i \neq \emptyset$ for only a finite number of $i \in I$. Then $\mathcal{V} = \{ V_x \}_{x \in X}$ is an open covering of $X$. By Theorem 7.8 there is an open covering $\mathcal{W} = \{ W_s \}_{s \in S}$ star refining $\mathcal{V}$. For $i \in I$ let

$$U_i = \text{St}(F_i, \mathcal{W}).$$

Clearly, $F_i \subset U_i$ and $U_i$ is open for every $i$. Therefore, it is sufficient to prove that the family $\mathcal{U} = \{ U_i \}_{i \in I}$ is locally finite. Since $\mathcal{W}$ is an open covering, it is sufficient to prove that for every $s \in S$ the intersection $W_s \cap U_i$ is non-empty for no more than a finite number of $i \in I$. Let $s \in S$. Since $\mathcal{W}$ is star refining $\mathcal{V}$, there exists $x \in X$ such that

$$\text{St}(W_s, \mathcal{W}) \subset V_x.$$

Suppose that $W_s \cap U_i \neq \emptyset$. Then by the definition of $U_i$ there exists $t \in S$ such that $W_s \cap W_t \neq \emptyset$ and $W_t \cap F_i \neq \emptyset$. By the definition, $W_t \subset \text{St}(W_s, \mathcal{W})$. It follows that

$$\text{St}(W_s, \mathcal{W}) \cap F_i \neq \emptyset.$$

But $\text{St}(W_s, \mathcal{W}) \subset V_x$ and hence $V_x \cap F_i \neq \emptyset$. By the choice of $V_x$ this may happen only for a finite number of $i \in I$. This completes the proof. ■

Remark. This lemma is due to C.H. Dowker [Do]. See [Do], Lemma 1.

7.10. Theorem. Let $X$ be a paracompact space. Let $\mathcal{F} = \{ F_i \}_{i \in I}$ be a closed family combinatorially refining an open family $\mathcal{U} = \{ U_i \}_{i \in I}$ of subsets of $X$. If $\mathcal{F}$ is locally finite, then there exists an open locally finite family $\mathcal{G} = \{ G_i \}_{i \in I}$ such that

$$F_i \subset G_i \subset \overline{G_i} \subset U_i$$

for every $i \in I$ and the nerves of $\mathcal{F}$ and $\mathcal{G}$ are equal.

Proof. Theorem 7.9 implies that $\mathcal{F}$ is combinatorially refining a locally finite open family $\mathcal{V} = \{ V_i \}_{i \in I}$. Let $W_i = U_i \cap V_i$. Then $\mathcal{W} = \{ W_i \}_{i \in I}$ is a locally finite open family and $\mathcal{F}$ is combinatorially refining $\mathcal{W}$. Since $X$ is normal by Lemma 7.4, we can apply Theorem 7.3 to $Z = X$, the family $\mathcal{F}$, and the family $\mathcal{W}$ is the role of $\mathcal{U}$. Since $W_i \subset U_i$ for every $i \in I$, the family $\mathcal{G}$ from Theorem 7.3 has the required properties. ■

Remarks. The above presentation of a fragment of the theory of paracompact spaces is largely based on Chapter 5 of the book [En] of R. Engelking. Engelking's priorities are quite different from ours. For example, Theorem 7.5 is not stated, but only mentioned in Remark 5.1.7, and Theorem 7.9 appears only as Problem 5.5.17(a). Although Theorem 7.10 easily follows from Theorem 7.9, it may be new.
8. Closed subspaces and fundamental groups

Introduction. Let $X$ be a topological space. The main goal of this section is to prove an analogue of Theorem 4.3 for closed locally finite coverings of $X$. In this case we need to assume that $X$ is Hausdorff and paracompact. In addition, we need to assume that the main results of the theory of covering spaces apply to $X$, although we will not use this theory, at least not explicitly. Since the bounded cohomology theory is intimately related to fundamental groups and coverings, such an assumption seems to be only natural. Somewhat surprisingly, a weaker assumption about elements of coverings turns out to be sufficient.

Before stating our assumptions explicitly, let us review some standard definitions. By a neighborhood we will always understand an open set. So, a subset $U \subset Z$ of a topological space $Z$ is a neighborhood of $z \in Z$ if $z \in U$ and $U$ is open. A topological space $Z$ is called locally path-connected if every point $z \in Z$ has a path-connected neighborhood.

It is well known that the relation of being connected by a path is an equivalence relation on $Z$. The path components of $Z$ are the equivalence classes with respect to this equivalence relation. It is well known that $Z$ is locally path-connected if and only if for every open set $U \subset Z$ each path component of $U$ is open in $Z$. See [Mu], Theorem 25.4.

A space $Z$ is called semilocally simply-connected if every point $z \in Z$ has a path connected neighborhood $U$ such that the inclusion homomorphism $\pi_1(U, z) \to \pi_1(Z, z)$ is trivial. A space $Z$ has a universal covering space if and only if $Z$ is path connected, locally path connected, and semilocally simply connected. See [Mu], Corollary 82.2.

The assumptions. For the rest of this section $X$ will be a paracompact Hausdorff space which is path connected, locally path connected, and semilocally simply connected.

Simple subsets. A subset $U \subset X$ is said to be simple if $U$ is open and for every path component $V$ of $U$ the inclusion homomorphism $\pi_1(V, v) \to \pi_1(X, v)$ is trivial, where $v$ is an arbitrary point of $V$. Equivalently, $U$ is simple if $U$ is open and every loop in $U$ is contractible in $X$. Clearly, an open subset of a simple subset is simple.

8.1. Lemma. A closed subspace of a paracompact space is paracompact.

Proof. Let $Z$ be a paracompact space and let $F \subset Z$ a closed subset. Let $\mathcal{V} = \{V_i\}_{i \in I}$ be a covering of $F$ by subsets $V_i \subset F$ open in $F$. For every $i \in I$ there exists a subset $U_i \subset Z$ such that $V_i = F \cap U_i$ and $U_i$ is open in $Z$. The family $\{U_i\}_{i \in I}$ together with $Z \sim F$ is an open covering of $Z$. Since $Z$ is paracompact, there exists a locally finite open covering $\mathcal{W} = \{W_k\}_{k \in K}$ refining this covering. Clearly, the family $\{W_k \cap F\}_{k \in K}$ is a locally finite covering of $F$ by sets open in $F$ which is refining $\mathcal{V}$.

38
8.2. Theorem. Let $A \subset X$ be a path connected and locally path connected subset and $a \in A$. Then there exists a path connected open set $U \subset X$ such that $A \subset U$ and the images of the inclusion homomorphisms $\pi_1(A, a) \rightarrowtail \pi_1(X, a)$ and $\pi_1(U, a) \rightarrowtail \pi_1(X, a)$ are equal.

Proof. The proof deals with several families simultaneously. In order not to overburden the text with subscripts, let us pretend that families are collections of subsets. More precisely, given a family $F = \{ F_i \}_{i \in I}$, let us agree that $F \in F$ means that $F = F_i$ for some $i \in I$.

Since $X$ is semilocally simply-connected, there exists a covering $S$ of $X$ by path connected simple sets. Since $X$ is paracompact, by Theorem 7.8 there exists an open covering $T$ of $X$ star refining $S$. If $T, T' \in T$ and $T \cap T' \neq \emptyset$, then $T' \subset St(T, T')$ and hence $T \cup T' \subset S$ for some $S \in S$. It follows that the subset $T \cup T'$ is simple.

Let $A \cap T$ be the family of intersections $A \cap T$ with $T \in T$. Clearly, $A \cap T$ is a covering of $A$ be sets open in $A$. Since $A$ is locally path-connected, there exists a covering $P$ of $A$ by path-connected subsets open in $A$ such that $P$ is refining $A \cap T$. Then for every $P \in P$ there exists $T \in T$ such that $P \subset T$. Let us choose such a set $T = T(P)$ for each $P \in P$.

Since $X$ is locally path-connected, for every $P \in P$ there exists an open set $V(P)$ such that $P \subset V(P) \subset T(P)$ and every point in $V(P)$ is connected by a path with some point in $P$. Since $P$ is path connected, $V(P)$ is also path connected.

By Theorem 7.5 there is a closed locally finite covering $F$ of $A$ combinatorially refining $P$. For each $F \in F$ let $P(F) \in P$ be the subset corresponding to $F$, so that $F \subset P(F)$. Let

$$V(F) = V(P(F)) \quad \text{and} \quad T(F) = T(P(F))$$

for each $F \in F$. Then

$$F \subset V(F) \subset T(F)$$

for every $F \in F$. By Theorem 7.9 there exists a locally finite open family $W$ of subsets of $X$ such that $F$ is combinatorially refining $W$. For each $F \in F$ let $W(F) \in W$ be the subset corresponding to $F$, so that $F \subset W(F)$. Let $U(F) = V(F) \cap W(F)$. Then

$$F \subset U(F) \subset W(F)$$

and hence sets $U(F), F \in F$, form an open locally finite family $U$. By Theorem 7.10 there exists a family of open in $X$ sets $G(F)$, where $F \in F$, such that

$$F \subset G(F) \subset \overline{G(F)} \subset U(F)$$

for each $F \in F$ and the family $G$ of these sets has the nerve equal to the nerve of $F$. 39
In particular, \( G(F) \cap G(F') \neq \emptyset \) if and only if \( F \cap F' \neq \emptyset \). Since

\[
G(F) \subset U(F) \subset V(F) \subset T(F)
\]

for every \( F \in \mathcal{F} \), it follows that \( T(F) \cap T(F') \neq \emptyset \) if \( F \cap F' \neq \emptyset \). This, in turn, implies that the union \( T(F) \cup T(F') \) is simple if \( F \cap F' \neq \emptyset \).

Since \( \mathcal{F} \) is a covering of \( A \), i.e. the union of the sets \( F \in \mathcal{F} \) is equal to \( A \), the union of the sets \( G(F), F \in \mathcal{F}, \) contains \( A \). Let \( U \) be the path component of this union containing \( A \). It is sufficient to show that every loop in \( U \) based at \( a \) is homotopic in \( X \) relatively to \( a \) to a loop in \( A \). Let \( p : [0, 1] \to U \) be such a loop. Let us partition \( [0, 1] \) by several points

\[
0 = x_0 < x_1 < \ldots < x_n = 1
\]

into subintervals \( I_i = [x_{i-1}, x_i] \). As is well known, one can choose this partition in such a way that \( p(I_i) \subset G(F_i) \) for some \( F_i \in \mathcal{F} \) for every \( i \). Let \( y_1, y_2, \ldots, y_n \in [0, 1] \) be some numbers such that \( x_{i-1} < y_i < x_i \) for every \( i \geq 1 \). Since

\[
p(y_i) \in G(F_i) \subset V(F_i),
\]

\( V(F_i) \) is path connected, and \( F_i \subset V(F_i) \), for each \( i \geq 1 \) we can connect \( p(y_i) \) by a path \( q_i \) in \( V(F_i) \) with some point \( a_i \in F_i \). On the other hand, if \( i \leq n - 1 \), then

\[
p(x_i) \in G(F_i) \cap G(F_{i+1})
\]

and hence \( G(F_i) \cap G(F_{i+1}) \neq \emptyset \). As we saw, this implies that \( F_i \cap F_{i+1} \neq \emptyset \) and hence

\[
P(F_i) \cap P(F_{i+1}) \neq \emptyset.
\]

Since \( P(F_i) \) and \( P(F_{i+1}) \) are path connected, we can connect \( a_i \) with \( a_{i+1} \) by a path \( r_i \) in \( P(F_i) \cup P(F_{i+1}) \). Similarly, we can connect \( a \) with \( a_1 \) by a path \( r_0 \) in \( P(F_1) \) and connect \( a_n \) with \( a \) by a path \( r_n \) in \( P(F_n) \).

Let \( y_0 = 0, y_{n+1} = 1, \) and for \( i = 0, 1, \ldots, n \) let \( f_i : [0, 1] \to [y_i, y_{i+1}] \) be an increasing homeomorphism and \( p_i = p \circ f_i \). Then for every \( i \) the path \( p_i \) connects \( p(y_i) \) with \( p(y_{i+1}) \) in \( G(F_i) \cup G(F_{i+1}) \) and \( p \) is homotopic to the product \( p_0 \cdot p_1 \cdot \ldots \cdot p_n \). For every \( i = 1, 2, \ldots, n \) the path \( q_i^{-1} \cdot p_i \cdot q_{i+1} \) connects \( a_i \) with \( a_{i+1} \) in

\[
G(F_i) \cup G(F_{i+1}) \subset T(F_i) \cup T(F_{i+1})
\]

and the path \( r_i \) connects \( a_i \) with \( a_{i+1} \) in

\[
P(F_i) \cup P(F_{i+1}) \subset T(F_i) \cup T(F_{i+1}).
\]
Since $F_i \cap F_{i+1} \neq \emptyset$, the subset $T(F_i) \cup T(F_{i+1})$ is simple and hence $q_i^{-1} \cdot p_i \cdot q_{i+1}$ is homotopic to $r_i$ in $X$ relatively to the endpoints. Similarly, the paths $p_0 \cdot q_1$ and $r_0$ connect $a$ with $a_1$ in $G(F_1)$ and $P(F_1)$ respectively. Since both $G(F_1)$ and $P(F_1)$ are contained in $T(F_1)$ and $T(F_1)$ is simple, the path $p_0 \cdot q_1$ and $r_0$ are homotopic in $X$ relatively to the endpoints. Similarly, $q_n^{-1} \cdot p_n$ and $r_n$ are homotopic in $X$ relatively to the endpoints. Clearly, $p_0 \cdot p_1 \cdot \ldots \cdot p_n$ is homotopic in $X$ relatively to the endpoints to

$$
p_0 \cdot q_1 \cdot q_1^{-1} \cdot p_1 \cdot q_2 \cdot q_2^{-1} \cdot p_2 \cdot q_3 \cdot \ldots \cdot q_n^{-1} \cdot p_n \cdot q_n^{-1} \cdot p_n = (p_0 \cdot q_1) \cdot (q_1^{-1} \cdot p_1 \cdot q_2) \cdot (q_2^{-1} \cdot p_2 \cdot q_3) \cdot \ldots \cdot (q_n^{-1} \cdot p_n \cdot q_n) \cdot (q_n^{-1} \cdot p_n)
$$

and hence is homotopic in $X$ relatively to the endpoints to $r_0 \cdot r_1 \cdot \ldots \cdot r_n$. It follows that $p$ is homotopic in $X$ relatively to the endpoints to $r_0 \cdot r_1 \cdot \ldots \cdot r_n$. By the construction, $r_0 \cdot r_1 \cdot \ldots \cdot r_n$ is a loop in $A$ based at $a$. Hence every loop $p$ in $U$ based at $a$ is homotopic in $X$ relatively to the endpoints to a loop in $A$ based at $a$. ■

8.3. Theorem. Suppose that $\mathcal{F} = \{F_i\}_{i \in I}$ be a closed locally finite covering of $X$ by path connected and locally path connected subsets and that $a_i \in F_i$ for every $i \in I$. Then there exists an open locally finite covering $\mathcal{G} = \{G_i\}_{i \in I}$ of $X$ by path connected subsets such that $F_i \subset G_i$ for every $i \in I$, the nerves of $\mathcal{F}$ and $\mathcal{G}$ are equal, and the images of

$$
\pi_1(F_i, a_i) \rightarrow \pi_1(X, a_i) \quad \text{and} \quad \pi_1(G_i, a_i) \rightarrow \pi_1(X, a_i)
$$

are equal for every $i \in I$.

Proof. By Theorem 8.2 for every $i \in I$ there exists a path connected open set $U_i$ such that $F_i \subset U_i$ and the images of the inclusion homomorphisms

$$
\pi_1(F_i, a_i) \rightarrow \pi_1(X, a_i) \quad \text{and} \quad \pi_1(U_i, a_i) \rightarrow \pi_1(X, a_i)
$$

are equal. Clearly, if $U_i$ has this property, then every open path connected set $G_i$ such that $F_i \subset G_i \subset U_i$ also does. Hence the theorem follows from Theorem 7.10. ■

8.4. Theorem. Suppose that $\mathcal{F} = \{F_i\}_{i \in I}$ be a closed locally finite covering of $X$ by path connected and locally path connected subsets. Let $N$ be the nerve of the covering $\mathcal{F}$. If the covering $\mathcal{F}$ is weakly boundedly acyclic, then $\hat{H}^*(X) \rightarrow H^*(X)$ can be factored through $H^*(N) \rightarrow H^*(X)$.

Proof. Let $\mathcal{G}$ be the covering provided by Theorem 8.3. Since $\mathcal{F}$ is weakly boundedly acyclic, these properties imply that $\mathcal{G}$ is an open weakly boundedly acyclic covering with the same nerve $N$ as $\mathcal{F}$. Hence the theorem follows from Theorem 4.3. ■
9. Closed subspaces and homology groups

**Introduction.** Let $X$ be a Hausdorff space and $\mathcal{U}$ be a covering of $X$. The goal of this section is to prove an analogue of Theorem 2.5 when $\mathcal{U}$ is closed and locally finite. As in Section 4, this analogue implies an analogue of Theorem 4.3 for such $\mathcal{U}$. The latter is different from Theorem 8.4 and complements it; neither of them implies the other. In contrast with Theorem 8.4, now we will assume that the closed subsets covering $X$ behave nicely not with respect to fundamental groups, but with respect to the usual singular homology groups.

A space $Z$ is said to be *homologically locally connected* if for each $n \geq 0$, each $z \in Z$, and each neighborhood $U$ of $z$ there exists another neighborhood $V \subset U$ of $z$ such that the inclusion homomorphism $H_n(V, \{z\}) \rightarrow H_n(U, \{z\})$ is equal to zero. Here $H_n(\bullet)$ are the usual singular homology groups with integer coefficients. A homologically locally connected space is also called an HLC space, or, more precisely, an HLC$_Z^\infty$ space.

**Sheaves associated with singular cochains.** For every $q \geq 0$ let $C^q$ be the presheaf on $X$ assigning to an open subset $U \subset X$ the vector space of real-valued cochains $C^q(U)$ and to an inclusion $U \subset V$ the restriction homomorphism $C^q(V) \rightarrow C^q(U)$. Let $\mathcal{C}^{q-1}$ be the constant presheaf $\mathbb{R}$. For $q \geq -1$ let $\mathcal{C}^q$ be the sheaf associated with the presheaf $C^q$. The maps $d_q: C^q(U) \rightarrow C^{q+1}(V)$ lead to morphisms $d_q: \mathcal{C}^q \rightarrow \mathcal{C}^{q+1}$, which, in turn, lead to morphisms $d_q: \Gamma^q \rightarrow \Gamma^{q+1}$.

For every $Y \subset X$ let $\Gamma^q(Y)$ be the space of sections of the sheaf $\Gamma^q$ over $Y$. The morphisms $d_q: \Gamma^q \rightarrow \Gamma^{q+1}$ lead to homomorphisms $\Gamma^q(Y) \rightarrow \Gamma^{q+1}(Y)$ and hence to augmented cochain complexes $\Gamma^*(Y)$. When $Z \subset Y \subset X$, the restriction of sections leads to the restriction morphism $\Gamma^*(Y) \rightarrow \Gamma^*(Z)$. By considering only subsets $Y \in \mathcal{U}$ we get a functor from cat $\mathcal{U}$ to augmented cochain complexes, which we still denote by $\Gamma^*$. By applying constructions of Section 2 to $\Lambda^* = \Gamma^*$ we get a double complex $C^*(N, \Gamma^*)$ and morphisms

$$C^*(N) \xrightarrow{i_\Gamma} T^*(N, \Gamma) \xleftarrow{j_\Gamma} \Gamma^*(X),$$

where $T^*(N, \Gamma)$ is the total complex of the double complex $C^*(N, \Gamma^*)$.

By applying the same construction to open subsets of $X$ in the role of $X$, we will get a double complex of sheaves $C^*(N, \Gamma^*)$. In more details, for every open subset $U \subset X$ let

$$C^p(N, \Gamma^q)(U) = \prod_{\sigma \in N_p} \Gamma^q(|\sigma| \cap U).$$

If $V \subset U \subset X$, then the restriction maps $\Gamma^q(|\sigma| \cap U) \rightarrow \Gamma^q(|\sigma| \cap V)$ lead to a map

$$C^p(N, \Gamma^q)(U) \rightarrow C^p(N, \Gamma^q)(V).$$
These maps turn $C^p(N, \Gamma^q)$ into a presheaf. Since $\Gamma^q$ is a sheaf, $U \rightarrow \Gamma^q(|\sigma| \cap U)$ is also a sheaf, and hence $C^p(N, \Gamma^q)$, being a product of sheaves, is also a sheaf. The maps

$$\delta_p : C^p(N, \Gamma^*) \rightarrow C^{p+1}(N, \Gamma^*)$$

are defined as before.

9.1. Lemma. If $\mathcal{U}$ is closed and locally finite and $X$ is paracompact, then the sequence

$$0 \rightarrow \Gamma^q(X) \xrightarrow{\delta_{-1}} C^0(N, \Gamma^q)(X) \xrightarrow{\delta_0} C^1(N, \Gamma^q)(X) \xrightarrow{\delta_1} \ldots$$

is exact for every $q \geq 0$ and the canonical morphism $\tau_G : \Gamma^*(X) \rightarrow \Gamma^*(N, \Gamma)$ induces an isomorphism of cohomology groups.

Proof. The sheaf analogue of the sequence (9.1) is the sequence

$$0 \rightarrow \Gamma^q \xrightarrow{\delta_{-1}} C^0(N, \Gamma^q) \xrightarrow{\delta_0} C^1(N, \Gamma^q) \xrightarrow{\delta_1} \ldots$$

of sheaves. This sequence is exact. In fact, it is exact for every sheaf on $X$ in the role of $\Gamma^q$. The proof is similar to the proof of Lemma 2.2, but depends on the assumption that $\mathcal{U}$ is closed and locally finite (it works also for open coverings). See the classical book [Go] by R. Godement, Chapter II, Theorem 5.2.1. The sequence (9.1) results from (9.2) by taking the sections over $X$. It remains to prove that this operation preserves exactness.

The results about sections are usually stated for sections with supports in a family $\Phi$. Let $\Phi$ be the family of all closed subsets of $X$. Then the support of every section belongs to $\Phi$. Since $X$ is paracompact, the family of all closed subsets is a paracompactifying family. See [Go], Section II.3.2 or [Bre], Section I.6 for the definitions of paracompactifying families. This property is crucial for preserving exactness by taking sections.

The sheaf $\Gamma^q$ is soft because $X$ is paracompact. See [Go], Chapter II, Example 3.9.1. Since the family of all closed subsets is paracompactifying, this implies that taking the sections over $X$ preserves exactness. See [Go], Chapter II, Theorem 3.5.4. Hence the exactness of (9.2) implies exactness of (9.1). This proves the first statement of the lemma. The second one follows from the first and Theorem A.1 with the rows and columns interchanged. 

The functors of global sections. Let $Z$ be a topological space. By applying the above construction to $Z$ in the role of $X$ we get a sheaf of cochain complexes $\Gamma^* = \Gamma^*_Z$ on $Z$. Let

$$\gamma^*(Z) = \Gamma^*_Z(Z)$$

be the cochain complex of global sections of $\Gamma^*_Z$ on $Z$. Clearly, $Z \rightarrow \gamma^*(Z)$ is a functor
from topological spaces to cochain complexes. For \( Y \subseteq X \) the complex \( \gamma^\bullet(Y) \) in general differs from \( \Gamma^\bullet(Y) \) because \( \gamma^\bullet(Y) \) is defined inside of \( Y \), while \( \Gamma^\bullet(Y) \) is defined in terms of singular cochains on open subsets of \( X \) containing \( Y \). But if \( U \subseteq X \) is an open subset, then \( \gamma^\bullet(U) = \Gamma^\bullet(U) \). In particular, \( \gamma^\bullet(X) = \Gamma^\bullet(X) \). In general, the restriction of singular cochains on open sets \( U \subseteq X \) to intersections \( U \cap Y \) defines a map \( \Gamma^\bullet(Y) \rightarrow \gamma^\bullet(Y) \). By a classical theorem of the sheaf theory, if \( Z \) is paracompact, then the natural homomorphism \( C^\bullet(Z) \rightarrow \gamma^\bullet(Z) \) induces an isomorphism

\[ k: H^n(Z) \rightarrow H^n(\gamma^\bullet(Z)) \, . \]

This is more sophisticated form of Eilenberg's Theorem 2.3. See [Bre], Section 1.7.

**Comparing \( \Gamma^\bullet \) and \( \gamma^\bullet \).** Suppose that \( X \) is paracompact. Let \( F \) be a closed subset of \( X \). Then \( F \) is paracompact by Lemma 8.1 and hence the cohomology of \( \gamma^\bullet(Z) \) are (canonically isomorphic to) the singular cohomology of \( F \). In general, there are no reasons to expect that the map \( \Gamma^\bullet(F) \rightarrow \gamma^\bullet(F) \) induces isomorphisms in cohomology.

**9.2. Lemma.** If \( X \) is paracompact and both \( X \) and \( F \) are homologically locally connected, then the morphism \( \Gamma^\bullet(F) \rightarrow \gamma^\bullet(F) \) induces isomorphisms in cohomology.

**Proof.** See [Bre], Section III.1, the big diagram on p. 183. In more details, let \( \Phi \) be the family of all closed subsets of \( X \). Then for a sheaf \( B \) on \( X \) Bredon's cohomology \( \mathcal{S}H^\bullet(\Phi; B) \) is the cohomology of the complex of global sections of the sheaf \( \Gamma^\bullet \otimes B \). Our map

\[ H^n(\Gamma^\bullet(F)) \rightarrow H^n(\gamma^\bullet(F)) \]

corresponds to Bredon's map

\[ f^*: \mathcal{S}H^n(\Phi; A_F) \rightarrow \mathcal{S}H^n(\Phi|F; A|F) \]

with \( A \) being the sheaf associated with the constant presheaf \( R \) on \( X \). Here \( \Phi|F \) is the family of all closed subsets of \( F \) and \( A|F \) is the restriction of \( A \) to \( F \). Hence \( A|F \) is the sheaf associated with the constant presheaf \( R \) on \( F \). It follows that

\[ \mathcal{S}H^n(\Phi|F; A|F) = H^n(\gamma^\bullet(F)) \, . \]

The sheaf \( A_F \) is obtained by extension of the sheaf \( A|F \) from \( F \) to \( X \) by zero and hence sections of \( \Gamma^\bullet \otimes A_F \) over \( X \) can be identified with sections of \( \Gamma^\bullet \) over \( F \). It follows that

\[ \mathcal{S}H^n(\Phi; A_F) = H^n(\Gamma^\bullet(F)) \, . \]

According to [Bre], the map \( f^* \) is an isomorphism. The lemma follows. ■
9.3. Lemma. Suppose that the covering $\mathcal{U}$ is closed and that the space $X$ and elements of $\mathcal{U} \cap$ are locally homologically connected. Then the morphism $T^*(N, \Gamma) \to T^*(N, \gamma)$ of total complexes induced by the morphisms $\Gamma^*(F) \to \gamma^*(F)$, where $F \in \mathcal{U} \cap$, induces an isomorphism in cohomology.

**Proof.** By Lemma 9.2 for every simplex $\sigma$ of $N$ the morphism

$$\Gamma^*(|\sigma|) \to \gamma^*(|\sigma|)$$

induces an isomorphism in cohomology. It follows that for every $p \geq 0$ the product

$$C^p(N, \Gamma^*) \to C^p(N, \gamma^*)$$

of these morphisms over $\sigma \in N_p$ induces an isomorphism in cohomology. It remains to apply a well known comparison theorem about double complexes. See Theorem A.3. ■

The homomorphism $H^*(N) \to H^*(X)$ for closed locally finite coverings. Suppose that the covering $\mathcal{U}$ is closed and locally finite and $X$ is paracompact. The morphisms

$$C^*(N) \xrightarrow{\lambda \Gamma} T^*(N, \Gamma) \xleftarrow{\tau \Gamma} \Gamma^*(X) \xleftarrow{k} C^*(X)$$

lead to homomorphisms

$$H^*(N) \xrightarrow{\lambda \Gamma} H^*(N, \Gamma) \xleftarrow{\tau \Gamma} H^*(\Gamma^*(X)) \xleftarrow{k} H^*(X)$$

of cohomology groups, where we denoted by $H^*(N, \Gamma)$ the cohomology of $T^*(N, \Gamma)$. By Lemma 9.1 the homomorphism $j_{\Gamma^*}$ is an isomorphism. Since $X$ is paracompact and locally homologically connected, $k$ is also an isomorphism. Therefore, the homomorphism

$$k^{-1} \circ \tau_{\Gamma^*}^{-1} \circ \lambda \Gamma^* : H^*(N) \to H^*(X)$$

is well defined, and we take it as the canonical homomorphism $l_{\mathcal{U}} : H^*(N) \to H^*(X)$.

The $\Lambda^*$-cohomology and the singular cohomology. Suppose that $\Lambda^*$ is a functor of generalized cochains as in Section 2. Then $\Lambda^*$-cohomology groups $\widetilde{H}^*(X) = H^\Lambda_*(X)$ are defined. Suppose that the functor $\Lambda^*$ is equipped with a natural transformation $\Lambda^* \to C^*$, leading to a natural homomorphism $\widetilde{H}^*(X) \to H^*(X)$.

9.4. Theorem. Suppose that the covering $\mathcal{U}$ is closed and locally finite, and that the space $X$ and elements of $\mathcal{U} \cap$ are locally homologically connected. If $\mathcal{U}$ is $\Lambda^*$-acyclic, then the natural homomorphism $\widetilde{H}^*(X) \to H^*(X)$ can be factored through $l_{\mathcal{U}} : H^*(N) \to H^*(X)$.
**Proof.** Let us consider the following diagram of total complexes.

\[
\begin{array}{ccc}
C^\bullet(N) & \longrightarrow & T^\bullet(N, A) \\
\uparrow & & \downarrow \\
\downarrow & & \downarrow \\
C^\bullet(N) & \longrightarrow & T^\bullet(N, C) \\
\uparrow & & \downarrow \\
\downarrow & & \downarrow \\
C^\bullet(N) & \longrightarrow & T^\bullet(N, \gamma) \\
\uparrow & & \downarrow \\
\downarrow & & \downarrow \\
C^\bullet(N) & \longrightarrow & T^\bullet(N, \Gamma) \\
\end{array}
\]

This diagram leads to the diagram of cohomology groups

\[
\begin{array}{ccc}
H^\ast(N) & \longrightarrow & H^\ast(N, A) \\
\uparrow & & \downarrow & \downarrow & \downarrow & \downarrow \\
\downarrow & & \downarrow & \downarrow & \downarrow & \downarrow \\
H^\ast(N) & \longrightarrow & H^\ast(N, C) \\
\uparrow & & \downarrow & \downarrow & \downarrow & \downarrow \\
\downarrow & & \downarrow & \downarrow & \downarrow & \downarrow \\
H^\ast(N) & \longrightarrow & H^\ast(N, \gamma) \\
\uparrow & & \downarrow & \downarrow & \downarrow & \downarrow \\
\downarrow & & \downarrow & \downarrow & \downarrow & \downarrow \\
H^\ast(N) & \longrightarrow & H^\ast(N, \Gamma) \\
\end{array}
\]

Since \( \mathcal{U} \) is \( A^\ast \)-acyclic, the homomorphism \( \lambda_{A^\ast} \) is an isomorphism by Lemma 2.1. By Lemma 9.1 the homomorphism \( \tau_{\Gamma^\ast} \) is also an isomorphism. As we saw, \( k \) is an isomorphism too. Finally, by Lemma 9.3 the homomorphism \( H^\ast(N, \Gamma) \longrightarrow H^\ast(N, \gamma) \) is an iso-
morphism. The commutativity of the lower right square implies that $\tau_{\gamma^*}$ is also an isomorphism. By inverting $\tau_{\gamma^*}$ and $k$ we get the square

$$
\begin{array}{ccc}
H^*(N, C) & \xleftarrow{} & H^*(X) \\
\downarrow & & \uparrow \\
H^*(N, \gamma) & \longrightarrow & H^*(\gamma^*(X))
\end{array}
$$

commutative in the sense that the composition of its four arrows starting at $H^*(X)$ is equal to the identity. By inverting also $\lambda_{A^*}$ and $\tau_{\Gamma^*}$ we get the following commutative diagram.

$$
\begin{array}{ccc}
H^*(N) & \xleftarrow{} & H^*(N, A) & \xleftarrow{} & H_A^*(X) \\
\downarrow & & \downarrow & & \downarrow \\
H^*(N) & \longrightarrow & H^*(N, C) & \xleftarrow{} & H^*(X) \\
\downarrow & & \downarrow & & \downarrow \\
H^*(N) & \longrightarrow & H^*(N, \gamma) & \longrightarrow & H^*(\gamma^*(X)) \\
\downarrow & & \downarrow & & \downarrow \\
H^*(N) & \longrightarrow & H^*(N, \Gamma) & \longrightarrow & H^*(\Gamma^*(X)).
\end{array}
$$

The commutativity of this diagram implies that $\tilde{H}^*(X) = H_A^*(X) \longrightarrow H^*(X)$ is equal to the composition of the red arrows. But the composition of the last four red arrows is nothing else but the canonical homomorphism $\lambda_U : H^*(N) \longrightarrow H^*(X)$. The theorem follows.

Extensions of coverings. Suppose that the covering $\mathcal{U}$ is closed and locally finite, and that the space $X$ and elements of $\mathcal{U} \cap$ are locally homologically connected. Then the covering $\mathcal{U}'$ of a space $X' \supset X$ constructed in Theorem 4.1 is also closed and locally finite. Recall that $X'$ is obtained by attaching to $X$ some discs along their boundaries, and elements of $\mathcal{U}$ are obtained by from an element of $\mathcal{U}$ by attaching some of these discs. Moreover, elements of $\mathcal{U}' \cap$ can be obtained from elements of $\mathcal{U} \cap$ in the same way. Clearly, if a space $Z$ is locally homologically connected, then the result of attaching of a collection of discs to $Z$ has the same property. Therefore $X'$ and elements of $\mathcal{U}' \cap$ are locally homologically connected.
9.5. Theorem. Suppose that the covering $\mathcal{U}$ is closed and locally finite, and that the space $X$ and elements of $\mathcal{U}$ are locally homologically connected. If $\mathcal{U}$ is weakly boundedly acyclic, then $\tilde{H}^*(X) \rightarrow H^*(X)$ can be factored through $l_{\mathcal{U}} : H^*(N) \rightarrow H^*(X)$.

Proof. The main part of the work is already done. The rest is completely similar to the proof of Theorem 4.3. One only needs to refer to Theorem 9.4 instead of Theorem 2.5 and supplement Corollary 4.2 by the remarks preceding the theorem. ■
A. Double complexes

**Double complexes.** A double complex $K^{p,q}$ is a diagram of the form

$$
\begin{array}{cccccc}
K^{0,0} & \rightarrow & K^{0,1} & \rightarrow & K^{0,2} & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
K^{1,0} & \rightarrow & K^{1,1} & \rightarrow & K^{1,2} & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
K^{2,0} & \rightarrow & K^{2,1} & \rightarrow & K^{2,2} & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
K^{3,0} & \rightarrow & K^{3,1} & \rightarrow & K^{3,2} & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & & 
\end{array}
$$

The horizontal arrows $K^{p,q} \rightarrow K^{p,q+1}$ and the vertical arrows $K^{p,q} \rightarrow K^{p+1,q}$ are denoted by $d$ and $\delta$ respectively, and are called the *differentials* of the double complex $K^{p,q}$. Each row and each column of this diagram is assumed to be a complex. Equivalently, it is assumed that $d \circ d = 0$ and $\delta \circ \delta = 0$.

The diagram is assumed to be commutative in one of the following two senses. First, one may require that each square of the diagram is commutative, i.e. to require that the two differentials commute, $d \circ \delta = \delta \circ d$. Then the diagram is commutative in the usual sense. The double complexes used in Section 2 are commutative in this sense. Alternatively, one can require that the differentials anti-commute, i.e. that $d \circ \delta + \delta \circ d = 0$. The advantages of this condition will be clear in a moment. In order to pass from one version to the other it is sufficient to replace differentials $\delta = \delta^{p,q} : K^{p,q} \rightarrow K^{p+1,q}$ by differentials $(-1)^{p} \delta^{p,q}$.

**The total complex of a double complex.** Let

$$T^n = \bigoplus_{p+q=n} K^{p,q}
$$

and let $\partial : T^n \rightarrow T^{n+1}$ be the map equal to $d + (-1)^{p} \delta$ on $K^{p,q}$ if the differentials
 commute, and to $d + \delta$ if the differentials anti-commute. Since $d \circ d = 0$ and $\delta \circ \delta = 0$, a trivial computation shows that in both cases $\partial \circ \partial = 0$. Therefore $T^*$ together with $\delta$ is a complex. It is called the total complex of the double complex $K^{*,*}$.

Let $L^p$ be the kernel of the differential $d : K^{p,0} \rightarrow K^{p,1}$. Either of the commutativity assumptions implies that $\delta$ maps $L^p$ to $L^{p+1}$. Therefore $L^*$ together with the restriction of the differential $\delta$ to $L^*$ is a subcomplex of the total complex $T^*$.

A.1. Theorem. If each complex $(K_p^{*,*}, d)$ is exact, then the homomorphism

$$H^*(L^*) \rightarrow H^*(T^*)$$

induced by the inclusion $L^* \rightarrow T^*$ is an isomorphism.

Proof. This is a special case of Theorem 4.8.1 from Chapter I of Godement’s book [Go]. Its standard proof is based on the properties of spectral sequences associated with $K^{*,*}$.

Here is a direct proof. We may assume that the differentials anti-commute. Let us prove first the induced homomorphism is surjective. Let

$$z = \bigoplus_{i=0}^n z^i \in \bigoplus_{i=0}^n K^{i,n-i}.$$ 

Then

$$\partial z \in \bigoplus_{k=0}^n K^{k,n+1-k}$$

and the summand of $\partial z$ belonging to $K^{k,n+1-k}$ is equal to

$$dz^k + \delta z^{k-1} \quad \text{if} \quad 0 < k < n + 1,$$

$$dz^0 \quad \text{if} \quad k = 0,$$

$$\delta dz^d \quad \text{if} \quad k = n + 1.$$ 

If $z$ is a cocycle, i.e. if $\partial z = 0$, then $dz^0 = 0$. Since the complex $(K_0^{*,*}, d)$ is exact, this implies that $z^0 = \partial y^0$ for some $y^0 \in K^{0,n-1}$. Let $y \in T^{n-1}$ be the element having $y^0$ as the only non-zero summand. Then $z - \partial y$ is a cocycle representing the same cohomology class as $z$ and the summands of $z - \partial y$ belonging to $K^{0,n}$ is equal to 0.

Let us replace $z$ by $z - \partial y$ while keeping the notation $z$. Now the summand of $\partial z$ belonging to $K^{1,n}$ is equal to $dz^1$ and since $z$ is a cocycle, $dz^1 = 0$. Since $(K_1^{*,*}, d)$ is exact, this implies that $z^1 = \partial y^1$ for some $y^1 \in K^{1,n-2}$. Let the new $y \in T^{n-2}$ be the element having $y^1$ as the only non-zero summand. Then $z - \partial y$ represents the same cohomology class as $z$ and the summands of $z - \partial y$ belonging to $K^{0,n}$ and $K^{1,n-1}$ are equal to 0.
By continuing in this way we will eventually reach an element $z$ representing the same cohomology class as the original $z$ and having only one non-zero summand, namely, the summand $z^n \in K^{n,0}$. Since the new $z = z^n$ is a cocycle, $dz = \delta z = 0$. Therefore the new $z$ belongs to $L^n$ and is a cocycle of the complex $L^*$. Since the original $z$ was an arbitrary cocycle of the total complex, the surjectivity follows.

The proof of injectivity is similar. Suppose that $w^{n+1} \in L^{n+1}$ is a cocycle of the complex $L^*$. Let $w$ be the element of $T^{n+1}$ having $w^{n+1}$ as the only non-zero summand. Suppose that $w$ is a coboundary in $T^*$, i.e. $w = \delta z$ for some $z \in T^n$. Since the summands of $\delta z$ in $K^{i,n-i}$ with $i > 0$ are equal to 0, we can apply to $z$ the same process as above. At each step we replaced $z$ by an element of the form $z - \delta y$. Since $\delta \circ \delta = 0$, this does not affect the coboundary $\delta z$. At the last step we will get a new element $z$ such that $w = \delta z$ and $z$ has only one non-zero summand, namely, the summand $z^n \in K^{n,0}$. Since $w^{n+1} \in K^{n+1,0}$ is the only non-zero summand of $w$ and $w = \delta z = dz + \delta z$, we see that $dz^n = 0$ and $w^{n+1} = \delta z^n$. It follows that $z^n \in L^n$ and $w^{n+1}$ is a coboundary in $L^*$. Since $w^{n+1}$ was an arbitrary cocycle of $L^*$ turning into a coboundary in $T^*$, the injectivity follows.

**Homological double complexes.** One can also consider double complex $K_{\bullet, \bullet}$ of the form

\[
\begin{array}{cccccccc}
K_{0,0} & \leftarrow & K_{0,1} & \leftarrow & K_{0,2} & \leftarrow & K_{0,3} & \leftarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
K_{1,0} & \leftarrow & K_{1,1} & \leftarrow & K_{1,2} & \leftarrow & K_{1,3} & \leftarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
K_{2,0} & \leftarrow & K_{2,1} & \leftarrow & K_{2,2} & \leftarrow & K_{2,3} & \leftarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
K_{3,0} & \leftarrow & K_{3,1} & \leftarrow & K_{3,2} & \leftarrow & K_{3,3} & & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & & \cdots & & \cdots & & \cdots & & \cdots \\
\end{array}
\]

The horizontal arrows $K_{p,q} \rightarrow K_{p,q-1}$ and the vertical arrows $K_{p,q} \rightarrow K_{p-1,q}$ are denoted, as before, by $d$ and $\delta$ respectively, and are called the *differentials* of the double complex $K_{\bullet, \bullet}$. It is assumed that $d \circ d = 0$ and $\delta \circ \delta = 0$. Also, it is assumed that either each square is commutative, or each square is anti-commutative, i.e. that either $d \circ \delta = \delta \circ d$, or $d \circ \delta + \delta \circ d = 0$. The total complex $T_\bullet$ is defined as before.
Let $L_p$ be the cokernel of the differential $d : K_{p,1} \rightarrow K_{p,0}$, i.e.

$$L_p = K_{p,0}/d(K_{p,1}),$$

and let $\pi : K_{p,0} \rightarrow L_p$ be the quotient map. Either of the commutativity assumptions implies that $\delta$ maps $d(K_{p,1})$ to $d(K_{p-1,1})$ and hence induces homomorphisms

$$\delta_L : L_p \rightarrow L_{p-1}.$$

Therefore $L_\ast$ together with homomorphisms $\delta_L$ is a quotient complex of $T_\ast$.

**A.2. Theorem.** If each complex $(K_p, \cdot, d)$ is exact, then the homomorphism

$$H_\ast(T_\ast) \rightarrow H_\ast(L_\ast)$$

induced by the quotient map $T_\ast \rightarrow L_\ast$ is an isomorphism.

**Proof.** We may assume that the differentials commute. Let us prove first the induced homomorphism is surjective. Let

$$z = \bigoplus_{i=0}^{n} z_i \in \bigoplus_{i=0}^{n} K_{i,n-i}.$$ 

Then

$$\partial z \in \bigoplus_{k=0}^{n-1} K_{k,n-1-k}$$

and the summand of $\partial z$ belonging to $K_{k,n-1-k}$ is equal to

$$dz_k + \delta z^{k+1} \text{ if } 0 < k < n-1,$$

for every $k = 0, 1, \ldots, n-1$. Suppose that $w_n \in K_{n,0}$ and $\pi(w_n)$ is a cycle of $L_\ast$. Then

$$\delta(w_n) \in d(K_{n-1,1})$$

i.e. $\delta(w_n) = d(w_{n-1})$ for some $w_{n-1} \in K_{n-1,1}$. Let $x_{n-1} = \delta(w_{n-1})$. Then

$$d(x_{n-1}) = d \circ \delta(w_{n-1}) = \delta \circ d(w_{n-1}) = \delta \circ \delta(w_n) = 0.$$ 

Since $(K_{n-1,\ast}, d)$ is exact, it follows that $x_{n-1} = d(w_{n-2})$ for some $w_{n-2} \in K_{n-2,2}$. 

52
By continuing to argue in this way we will get elements \( w_{n-1}, w_{n-2}, \ldots, w_0 \) such that \( w_{n-i} \in K_{n-i, i} \) and \( \delta(w_{n-i}) = d(w_{n-i-1}) \) for every \( i \). Let

\[
w = \bigoplus_{i=0}^{n} w_i \in \bigoplus_{i=0}^{n} K_{i, n-i}.
\]

When the differentials commute, the differential \( \partial : T_n \longrightarrow T_{n-1} \) is equal to \( d + (-1)^p \delta \) on \( K_{p,q} \). It follows that \( \delta w = 0 \), i.e. \( w \) is a cycle of the total complex. Clearly, the quotient map \( T_* \longrightarrow L_* \) maps \( w \) to \( \pi(w_n) \). The surjectivity follows.

The proof of injectivity is similar. Suppose that

\[
z = \bigoplus_{i=0}^{n-1} z_i \in \bigoplus_{i=0}^{n} K_{i, n-i-1}
\]

is a cycle of the complex \( T_* \) such that \( \pi(z_{n-1}) \) is a boundary in \( L_* \). Then there exists an element \( w_n \in K_{n,0} \) such that \( \pi(\delta(w_n)) = \pi(z_{n-1}) \) and hence there exists an element \( w_{n-1} \in K_{n-1, 1} \) such that \( z_{n-1} = \delta(w_n) - d(w_{n-1}) \). Let \( y \in T_n \) be the element having only two non-zero summands, namely, \( w_n \) and \( w_{n-1} \). Then \( z + \delta y \) and \( z - \delta y \) are cycles and for an appropriate choice of sign the summand of \( z \pm \delta y \) belonging to \( K^{n-1,0} \) is equal to 0. Therefore, it is sufficient to prove that \( z \) is a boundary in \( T_* \) if \( z_{n-1} = 0 \).

If \( z_{n-1} = 0 \), then \( \delta z = 0 \) implies that \( d(z_{n-2}) = 0 \). By the assumption, this implies that \( z_{n-2} = d(w_{n-2}) \) for some \( w_{n-2} \in K_{n-2,2} \). Let us replace \( z \) by \( z \pm \delta(w_{n-2}) \). The new \( z \) is still a boundary, and with an appropriate choice of sign we have

\[
z_{n-1} = z_{n-2} = 0.
\]

By continuing to argue in this way we will eventually get \( z = 0 \). Since at each step we subtracted from \( z \) a boundary, it follows that the original element \( z \) is also a boundary. The injectivity follows. This completes the proof.

**Morphisms of double complexes.** Let \( K^{*,*} \) and \( L^{*,*} \) be two double complexes. A morphism \( f^{*,*} : K^{*,*} \longrightarrow L^{*,*} \) is a family of homomorphisms \( f^{p,q} : K^{p,q} \longrightarrow L^{p,q} \) commuting with the differentials \( d, \delta \) in an obvious sense. If \( T^{*,*}_K \) and \( T^{*,*}_L \) are the total complexes of \( K^{*,*} \) and \( L^{*,*} \) respectively, the \( f^{*,*} \) induces a morphism \( T^{*,*} f^{*,*} : T^{*,*}_K \longrightarrow T^{*,*}_L \). Also, \( f^{*,*} \) induces for every \( p \geq 0 \) a morphism of complexes \( f^{p,*} : K^{p,*} \longrightarrow L^{p,*} \) and for every \( q \geq 0 \) a morphism of complexes \( f^{*,q} : K^{*,q} \longrightarrow L^{*,q} \).

**A.3. Theorem.** Let \( f^{*,*} : K^{*,*} \longrightarrow L^{*,*} \) be a morphism of double complexes. If for every \( p \geq 0 \) the morphism of complexes \( f^{p,*} : K^{p,*} \longrightarrow L^{p,*} \) induces an isomorphism in cohomology, then \( T^{*,*} f^{*,*} : T^{*,*}_K \longrightarrow T^{*,*}_L \) also induces an isomorphism in cohomology.
**Proof.** The standard proof is based on comparing spectral sequences associated with $K^{*,*}$, $L^{*,*}$. Here is a direct elementary proof. We will assume that the differentials $d, \delta$ anticommute and the differentials of total complexes are defined as $d + \delta$. Let us prove first that the induced homomorphism is surjective. Let

$$x = \bigoplus_{i=0}^{n} x^i \in \bigoplus_{i=0}^{n} L^{i,n-i}$$

be a cocycle. Then $d x^0 = 0$, $\delta x^n = 0$, and

$$d x^i + \delta x^{i-1} = 0 \quad \text{for} \quad 0 < i < n.$$ 

In order to prove the surjectivity we need to find a cocycle

$$y = \bigoplus_{i=0}^{n} y^i \in \bigoplus_{i=0}^{n} K^{i,n-i}$$

such that $f(y) = x + \delta z$ for some

$$z = \bigoplus_{i=0}^{n} z^i \in \bigoplus_{i=0}^{n-1} L^{i,n-1-i},$$

or, equivalently, $x^0 = f(y^0) + d z^0$, $x^n = f(y^n) + \delta z^{n-1}$, and

$$x^i = f(y^i) + d z^i + \delta z^{i-1} \quad \text{for} \quad 0 < i < n.$$

Since $f^{0,*} : K^{0,*} \rightarrow L^{0,*}$ induces an isomorphism in cohomology, there exists $y^0 \in K^{0,n}$ and $z^0 \in K^{0,n-1}$ such that $f(y^0) = x^0 + d z^0$. Suppose that elements $y^0, \ldots, y^{i-1}$ and $z^0, \ldots, z^{i-1}$ with the required properties are already constructed. Then

$$d \delta y^{i-1} = -\delta d y^{i-1} = \delta d d y^{i-2} = 0$$

and hence $\delta y^{i-1}$ is a cocycle of the complex $K^{i,*}$. At the same time

$$x^{i-1} = f(y^{i-1}) + d z^{i-1} + \delta z^{i-2}$$

and hence

$$d x^i = -\delta x^{i-1} = -\delta f(y^{i-1}) - \delta d z^{i-1} = -f(\delta y^{i-1}) + d \delta z^{i-1}.$$

It follows that $f(\delta y^{i-1})$ is a coboundary in $L^{i,*}$. Since $f^{i,*} : K^{i,*} \rightarrow L^{i,*}$ induces an isomorphism in cohomology, this implies that $\delta y^{i-1}$ is a coboundary in $K^{i,*}$. Equivalently, $\delta y^{i-1} = d u^i$ for some $u^i \in K^{i,n-i}$. It follows that

$$f(\delta y^{i-1}) = f(d u^i) = d f(u^i)$$
and hence
\[ d \left( x^i - \delta z^{i-1} + f(u^i) \right) = dx^i - d \delta z^{i-1} + f(\delta y^{i-1}) = 0. \]

Since \( f^{i,\cdot} : K^{i,\cdot} \to L^{i,\cdot} \) induces an isomorphism in cohomology, this implies that
\[ x^i - \delta z^{i-1} + f(u^i) = f(v^i) + dz^i \]
for some \( z^i \in L^{i,n-i} \). Hence
\[ x^i = f(v^i - u^i) + dz^i + \delta z^{i-1} \]
and we can set \( y^i = v^i - u^i \). By continuing in this way we will find the sequences \( y \) and \( z \) with the required properties.

In order to prove the injectivity, suppose that \( y \) is an \( n \)-cocycle in the total complex of \( K^{\cdot,\cdot} \) such that \( f(y) = \partial z \). Then \( f(y^0) = dz^0, f(y^n) = \delta z^{n-1} \), and
\[ f(y^i) = dz^i + \delta z^{i-1} \quad \text{for} \quad 0 < i < n. \]

We will construct elements \( u^i \in K^{i,n-i} \) for \( 0 \leq i \leq n-1 \) and elements \( c^i \in L^{i,n-2-i} \) for \( 0 \leq i \leq n-2 \) such that \( y^0 = du^0, f(u^0) = z^0 + dc^0, y^n = \delta z^{n-1} \),
\[ y^i = du^i + \delta u^{i-1} \quad \text{for} \quad 0 < i < n, \quad \text{and} \]
\[ f(u^i) = z^i + dc^i + \delta c^{i-1} \quad \text{for} \quad 0 < i < n-1. \]

To begin with, note that since \( f^{0,\cdot} : K^{0,\cdot} \to L^{0,\cdot} \) induces an isomorphism in cohomology, there exists \( v^0 \in K^{0,n-1} \) such that \( y^0 = dv^0 \). Clearly,
\[ d \left( f(v^0) - z^0 \right) = f(dv^0) - dz^0 = f(y^0) - dz^0 = 0. \]

It follows that \( f(v^0) - z^0 = f(w^0) + dc^0 \) and hence
\[ f(v^0 - w^0) = z^0 + dc^0 \]
for some \( w^0 \in K^{0,n-1} \) such that \( dw^0 = 0 \) and some \( c^0 \in K^{0,n-2} \). Hence we can set \( u^0 = v^0 - w^0 \). Suppose that we already constructed \( u^0, \ldots, u^{i-1} \) and \( c^0, \ldots, c^{i-1} \) with the required properties. Then
\[ f(y^i) = dz^i + \delta z^{i-1} = dz^i + \delta f(u^{i-1}) - \delta dc^{i-1} \]
\[ = dz^i + f(\delta u^{i-1}) + d \delta c^{i-1}. \]

55
It follows that
\[ f(y^i - \delta u^{i-1}) = d(z^i + \delta c^{i-1}) . \]

Since \( f^{i,*} : K^{i,*} \rightarrow L^{i,*} \) induces an isomorphism in cohomology, there exists an element \( v^i \in K^{i,n-1-i} \) such that \( y^i - \delta u^{i-1} = dv^i \). Clearly,
\[ f(y^i) = f(\delta u^{i-1}) + f(dv^i) . \]

By comparing this with the previous expression for \( f(y^i) \) we conclude that
\[ dz^i + d\delta c^{i-1} = f(dv^i) = df(v^i) \]
and hence
\[ d\left(f(v^i) - z^i - \delta c^{i-1}\right) = 0 . \]

Since \( f^{i,*} : K^{i,*} \rightarrow L^{i,*} \) induces an isomorphism in cohomology, there exists elements \( w^i \in K^{i,n-1-i} \) and \( c^i \in L^{i,n-2-i} \) such that \( dw^i = 0 \) and
\[ f(v^i) - z^i - \delta c^{i-1} = f(w^i) + dc^i . \]

It follows that
\[ f(v^i - w^i) = z^i + dc^i + \delta c^{i-1} \]
and hence we can set \( u^i = v^i - w^i \). By arguing in this way we can construct elements \( u^i \) and \( c^i \) for \( i \leq n - 2 \). The last two steps corresponding to \( i = n - 1, n \) are similar and can be done by setting \( z^n = 0 \) and \( c^{n-1} = c^n = 0 \) in the above arguments.  ■
References

[Ba] M. Barr, *Acyclic models*, 2020, xii, 241 pp. First edition: AMS, 2002. Available at https://www.math.mcgill.ca/barr/.

[Bre] G. Bredon, *Sheaf theory*, Second edition, Springer, 1997, xi, 502 pp. The first edition: McGraw Hill Book Co., 1967.

[Do] C.H. Dowker, On a theorem of Hanner, *Arkiv för Math.*, V. 2 (1952), 307–313.

[E] S. Eilenberg, Singular homology theory, *Annals of Mathematics*, V. 45, No. 3 (1944), 407–447.

[ES] S. Eilenberg, N. Steenrod, *Foundations of algebraic topology*, Princeton University Press, 1952, xv, 328 pp.

[En] R. Engelking, *General topology*, Hendermann Verlag, Berlin, 1989, vii, 529 pp.

[F] R. Frigerio, Amenable covers and $l_1$-invisibility, 2019, 16 pp. ArXiv:1907.10547.

[FMa] R. Frigerio, A. Maffei, A remark on the Mayer–Vietoris double complex for singular cohomology, 2019, 9 pp. ArXiv:1912.07736v1.

[FMo] R. Frigerio, M. Moraschini, Gromov’s theory of multicomplexes with applications to bounded cohomology and simplicial volume, 2018, 150 pp. ArXiv:1808.07307v3.

[Go] R. Godement, *Topologie algébrique et théorie des faisceaux*, Actualités Sci. Ind., No. 1252, Publ. Math. Univ. Strasbourg, No. 13, Hermann, Paris, 1958, viii, 283 pp.

[Gro] M. Gromov, Volume and bounded cohomology, *Publicationes Mathematiques IHES*, V. 56, (1982), pp. 5–99.

[I1] N.V. Ivanov, Foundations of the bounded cohomology theory, *Research in topology, 5, Notes of LOMI scientific seminars*, V. 143 (1985), pp. 69–109.

[I2] N.V. Ivanov, Notes on bounded cohomology theory, 2017, 95 pp. ArXiv:1708.05150v2.

[L] C. Löh, Isomorphisms in $l^1$-homology, *Münster Journal of Mathematics*, V. 1 (2008), pp. 237–266.

[LS] C. Löh, R. Sauer, Bounded cohomology of amenable covers via classifying spaces, *L'Enseignement Mathematique*, V. 66 (2020), 151–172.

[MM] Sh. Matsumoto, Sh. Morita, Bounded cohomology of certain groups of homeomorphisms, *Proc. of the American Mathematical Society*, V. 94, No. 3 (1985), 539–544.
[Mor] K. Morita, On the dimension of normal spaces. II. *Journal of the Mathematical Society of Japan*, V. 2, Nos. 1-2 (1950), 16–33.

[Mu] J.R. Munkres, *Topology*, Second edition, Prentice Hall, 2000, xvi, 537 pp.

[R] W. Rudin, *Functional analysis*, McGraw-Hill Book Co., New York, 1973, xiii, 397 pp.

[Sp] E. Spanier, *Algebraic topology*, 2nd printing, Springer-Verlag Berlin Heidelberg New York 1989, xvi, 528 pp.

[vM] J. van Mill, *The infinite-dimensional topology of function spaces*, Elsevier, 2001, xii, 630 pp.

December 8, 2020

https://nikolaivivanov.com

E-mail: nikolai.v.ivanov@icloud.com