A Note on Integer Parts of Real Closed Fields and the Axiom of Choice

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Abstract

An integer part $I$ of a real closed field $K$ is a discretely ordered subring with minimal element 1 such that, for every $x \in K$, there is $i \in I$ with $i \leq x < i + 1$. Mourgues and Ressayre showed in [MR] that every real closed field has an integer part. Their construction implicitely uses the axiom of choice. We show that $AC$ is actually necessary to obtain the result by constructing a transitive model of $ZF$ which contains a real closed field without an integer part. Then we analyze some cases where the axiom of choice is not necessary for obtaining an integer part. This also sheds some light on the possibility to effectivize constructions of integer parts.

1 Introduction

A real closed field ($RCF$) $K$ is a field in which $-1$ is not a sum of squares and every polynomial of odd degree has a root. We assume familiarity with the basic notions and theorems connected with $RCFs$ and refer the reader to [CK] otherwise. A field $K$ is formally real if $-1$ is not a sum of squares in $K$. $K$ is orderable if there is a linear ordering $\leq$ of $K$ that respects the addition and multiplication of $K$. If $K$ is formally real, then there is a real closed algebraic field extension $K'$ of $K$, called real closure of $K$, which we will denote by $K^\text{rc}$. The existence of real closures for formally real fields depends on the axiom of choice, while the existence of real closures for ordered fields is known to follow from $ZF$ alone. If $K$ is a real closed field, $X \subseteq K$ and $K_X$ is the smallest subfield of $K$ containing $X$ as a subset, then $K_X^\text{rc}$ is also a subfield of $K$. 
Definition 1. Let $K$ be an RCF. Then $I \subseteq K$ is an integer part of $K$ iff $I$ is a discretely ordered subring of $K$ such that 1 is the minimal positive element of $I$ and, for every $x \in K$, there is $i \in I$ with $i \leq x < i + 1$.

The idea here is that $I$ is in a relation to $K$ similar to the relation of $\mathbb{N}$ and $\mathbb{R}$. Integer parts of real closed fields are especially interesting as they are known to coincide with models of a certain natural fragment of Peano Arithmetic, namely Open Induction (see [S]).

In [MR], Mouruges and Ressayre showed that every real closed field has an integer part. Their construction uses the axiom of choice (in the form of Zorn’s lemma) implicitly at least in the proof of the crucial Corollary 4.2. There has recently been some interest in the complexity of such a construction, see e.g. [DKL] or [DKKL]. For this purpose, a well-ordering of the real closed field is assumed to be given. This motivates us to ask whether this ingredient is actually necessary, i.e. whether there is a way to ‘construct’ an integer part from the real closed field alone. It turns out that this is not the case: We construct below a transitive model of Zermelo-Fraenkel set theory without the axiom of choice ($ZF$) containing a real closed field $K$, but no integer part of $K$. Letting $\phi_{IP}$ denote the statement that every real closed field has an integer part, this shows that $\phi_{IP}$ is independent from $ZF$. Then, we give some extra conditions on real closed fields under which the axiom of choice is not necessary for obtaining an integer part.

2 Preliminaries

We briefly summarize some notions from set theory that are necessary to understand the central tool for constructing a choice-free universe with an RCF without an integer part, namely Lemma 3 below, originally used by Hodges to show the dependence of several algebraic constructions on the axiom of choice ([H1]). $ZFC$ is Zermelo-Fraenkel set theory with the axiom of choice, see e.g. Chapter 1 of [H1]. $ZF$ is $ZFC$ without the axiom of choice. A model of $ZF(C)$ is transitive iff $x \in M$ whenever $y \in M$ and $x \in y$. The rough idea is to start with a given countable transitive model $M$ of $ZFC$ (the existence of such models is known to be consistent with $ZFC$ unless $ZFC$ itself is not) containing a certain algebraic object $A$ and then build a model $N(A)$ of $ZF$ that contains an isomorphic copy $A'$ of $A$ (i.e. in the real world $V$, there is an isomorphism between $A$ and $A'$ - for our purposes, we can identify $A$ and $A'$), but only those subsets of $A$ that are respected by all automorphisms that fix some finite subset of $A$. It is important to note that ‘all automorphisms’ here is to be understood with respect to the
set-theoretical universe $V$, in particular not relativized to the model $N(A)$. Most of our notation is standard. If $f$ is a map and $X$ is a subset of its domain, then we denote by $f[X]$ the image of $X$ under $f$; if $\vec{a} = (a_1, ..., a_n)$ and $\vec{b} = (b_1, ..., b_n)$ are finite subsets of the domain and the range of $f$, respectively, then we write $f(\vec{a}) = \vec{b}$ to mean that $f(a_i) = b_i$ for all $i \in \{1, ..., n\}$. When we write $(ZF)$ before a theorem statement, we mean that the statement is provable in $ZF$ alone, i.e. without the axiom of choice.

**Definition 2.** Let $M \models ZF$, and let $R \in M$ be a ring. Then $R$ is $M$-symmetric iff, for every $X \in \mathcal{P}^M(R)$, there is a finite $s_X \subseteq R$ such that $\pi[X] = X$ for every automorphism $\pi$ of $R$ which fixes $s_X$ pointwise, i.e. $\pi_X(a) = a$ for every $a \in s$. In this case, $s$ is called a support for $X$.

**Remark:** Note that, if $R$ is an $RCF$, then $\pi$ must preserve the canonical ordering of $R$ so that for finite $s$, $\pi$ fixes $s$ pointwise iff $\pi(s) = s$.

**Lemma 3.** Let $L$ be a countable first-order language and $\mathfrak{A}$ be a countable $L$-structure. Then there is a transitive model $N(\mathfrak{A})$ of $ZF$ which contains a $N(\mathfrak{A})$-symmetric isomorphic copy of $\mathfrak{A}$.

**Proof.** This is Lemma 3 of [H1] and also Lemma 3.7 of [H2], where it is proved.

That $N(\mathfrak{A})$ contains an isomorphic copy $\mathfrak{A}'$ of $\mathfrak{A}$ is in general not enough to show an independence from $ZF$ - one must also know that the relevant properties of $\mathfrak{A}$ still hold for $\mathfrak{A}'$ in $N(\mathfrak{A})$. Some properties, as e.g. countability, will in general not be preserved. However, every notion relevant for our purposes is absolute between transitive models of $ZF$:

**Lemma 4.** Let $K$ and $R$ be sets. The following statements are absolute between all transitive models of $ZF$ containing $K$ (or $K$ and $R$, where relevant):

- (a) $K$ is a real closed field
- (b) $R$ is a subring of $K$
- (c) $R$ is a dense subset of $K$
- (d) $R$ is an integer part of $K$

**Proof.** Recall that a formula in the language of set theory is $\Delta_0$ if all of its quantifiers are bounded. By Lemma 12.9 of [J1], $\Delta_0$-formulas are absolute between transitive classes. It is easy to see that (b)-(d) are expressable with
quantifiers restricted to \( K \) and \( R \), so that (b)-(d) are in fact absolute between arbitrary transitive classes containing \( K \) and \( R \).

It remains to see that (d) is \( \Delta_0 \)-expressable in a transitive model of \( ZF \). This is obvious for the axioms of ordered fields. We need to say that \(-1\) is not a sum of squares in \( K \) and that every polynomial of odd degree with coefficients in \( K \) has a root in \( K \). Denoting by \( ^\omega X \) the set of finite sequences of elements of a set \( X \), this is easily expressable with quantifiers bounded by \( ^\omega (\omega \cup K) \). But \( ^\omega (\omega \cup K) \) exists and is absolute for every transitive model of \( ZF \) containing \( K \).

\[ \square \]

3 Real Closed Fields without Integer Parts

We now construct a transitive \( M \models ZF \) such that, for some \( K \in M \), \( M \models \exists K \) is a real closed field’ \( \wedge \) ’\( K \) has no integer part’. Our method is that used by Hodges in [H1] and [H2] to construct choicefree counterexamples to some algebraic theorems.

**Definition 5.** An RCF \( K \) is unbounded iff for every finite \( \vec{a} \subseteq K \), the real closure \( \text{RC}(\vec{a}) \) of \( \vec{a} \) in \( K \) is bounded in \( K \). \( K \) is \( \omega \)-homogenous iff, for all finite \( \vec{a}, \vec{b} \subseteq K \), \( \text{tp}(\vec{a}) = \text{tp}(\vec{b}) \) implies that for every \( c \in K \), there is \( d \in K \) such that \( \text{tp}(\vec{a}, c) = \text{tp}(\vec{b}, d) \).

The point behind the following lemma is that, if \( K \) is an \( RCF \), \( \vec{a} \subseteq K \) finite and \( x \in K \), then \( \text{tp}(x \mid \vec{a}) \) only depends on the place of \( x \) in the ordering of \( \text{RC}(\vec{a}) \). For a direct proof of this, see Lemma 5.4.3 of [CK].

**Lemma 6.** Let \( K \) be an \( RCF \), \( a \subseteq K \) finite such that \( \text{RC}(a) \) is bounded in \( K \), \( x > \text{RC}(a) \). Then there is \( \varepsilon > 0 \) in \( K \) such that \( \text{tp}(a, x) = \text{tp}(a, y) \) for all \( y \in (x - \varepsilon, x + \varepsilon) \).

In particular, if \( I \) is an integer part of \( K \), then there are \( i \in I \), \( r \in K \setminus I \) such that \( \text{tp}(a, i) = \text{tp}(a, r) \).

**Proof.** By the claim in the proof of Theorem 3.4 of [DKS], if \( r \notin \text{RC}(\vec{a}) \), then \( \text{tp}(r \mid \vec{a}) \) is realised by all elements of an interval \( C \) containing \( r \).

The other statement follows easily as \( \text{RC}(a) \) is bounded in \( K \), while \( I \) is unbounded and hence \( I \setminus \text{RC}(a) \neq \emptyset \).

\[ \square \]

**Lemma 7.** Let \( M \) be homogenous, \( n \in \mathbb{N} \), \( \vec{a}, \vec{b} \subseteq M^n \), \( \text{tp}(\vec{a}) = \text{tp}(\vec{b}) \). Then there is an automorphism \( \pi \) of \( M \) such that \( \pi(\vec{a}) = \vec{b} \). In particular, this holds when \( M \) is countable and \( \omega \)-homogenous.

**Proof.** See Proposition 4.2.13 of [Ma].

\[ \square \]
Lemma 8. Let $K$ be a countable, unbounded, $\omega$-homogenous real closed field, let $\vec{a} \subseteq K$ be finite, and let $I$ be an integer part of $K$. Then there is an automorphism $\pi$ of $K$ such that $\pi$ fixes $\vec{a}$ pointwise and there are $x \in I$, $y \in K \setminus I$ such that $\pi(x) = y$. In particular, $\vec{a}$ is not a support for $I$.

Proof. As $K$ is unbounded, $\text{RC}(\vec{a})$ is bounded in $K$. Hence, by Lemma 6, there are $i \in I$, $r \in K \setminus I$ such that $\text{tp}(\vec{a}, i) = \text{tp}(\vec{a}, r)$. By Lemma 7, there is an automorphism $\pi$ of $K$ such that $\pi$ is the identity on $\vec{a}$ and $\pi(i) = r$.

Hence $\pi$ fixes $\vec{a}$, but $I \neq \{ \pi(j) | j \in I \}$. Thus $\vec{a}$ is not a support for $I$.

Corollary 9. If $K$ is a countable, unbounded, $\omega$-homogenous real closed field, then no $I$ of $K$ has a support.

Proof. Immediate, as Lemma 8 is true for all integer parts $I$ of $K$ and all finite $\vec{a} \subseteq K$.

Theorem 10. Let $K$ be a countable, unbounded, $\omega$-homogenous RCF. Then there exists $M \models ZF$ containing an isomorphic copy of $K$ such that $K$ has no integer part in $M$.

Proof. By Lemma 8, no integer part of $K$ can have a support. Hence we can apply Lemma 3 to get a model $M$ of $ZF$ containing an isomorphic copy of $K$. Suppose that $M$ contains an integer part $I$ for $K$. Then $I$ is in particular a subset of $K$ contained in $M$ and hence has a support $\vec{a}$. But this contradicts Corollary 9 (mind our remark following Definition 2).

Corollary 11. There are transitive models of $ZF$ which contain an RCF without an integer part. Consequently, $\phi_{IP}$ is independent from $ZF$.

Proof. Let $I \models PA$ be countable and nonstandard, and let $K$ be the real closure of its fraction field. By Proposition 3.3 of [DKS], $K$ is unbounded. Certainly, $K$ is countable. By Theorem 5.1 of [DKS], $K$ is recursively saturated. By a Theorem of Barwise and Schlipf (see [BS]), countable recursively saturated structures are resplendent, and by Theorem 2.4 (ii) of the same paper, resplendent structures are $\omega$-homogenous. Hence, by Theorem 10 there is a transitive $M \models ZF$ such that $M$ contains an isomorphic copy $K'$ of $K$ without an integer part in $M$. By Lemma 3, $K'$ is an RCF in $M$. Assume for a contradiction that $M \models K'$ has an integer part', and let $I' \in M$ such that $M$ thinks that $I'$ is an integer part of $K'$. By Lemma 4 again, $I'$ is then an integer part of $K'$ in the real world contained in $M$, a contradiction. Hence $M$ believes that $K'$ is a real closed field without an integer part. Thus $\phi_{IP}$ is not provable in $ZF$.

On the other hand, $ZFC$ is shown to imply $\phi_{IP}$ in [MR]. As $ZFC$ is consistent relative to $ZF$ (see e.g. Theorem 3.5 of [J]), $\phi_{IP}$ is consistent with $ZF$. Thus $\phi_{IP}$ is independent from $ZF$. 

5
4 When supports suffice

In this section, we consider the converse question suggested by our results above: Namely conditions under which if, in the real world, $K$ is an $RCF$ with an integer part $I$ with a support $a$, there is an integer part of $K$ in any transitive model of $ZF$ containing $K$.

**Definition 12.** An $RCF$ $K$ is rigid iff there is a finite $a \subseteq K$ such that $K = RC(a)$.

We start with some easy observations:

**Proposition 13.** If $K$ is rigid, then every integer part of $K$ has a support. Furthermore, the only automorphism $\pi : K \to K$ with $\pi(a) = a$ is the identity.

**Proof.** Let $a \subseteq K$ be as in the definition of rigidity. Then $a$ is obviously a support for every subset of $K$, including every integer part. The second statement is also obvious. □

This excludes the above construction for eliminating integer parts from being applied to a rigid $K$. In fact, it follows from $ZF$ that every rigid real closed fields has an integer part:

**Lemma 14.** ($ZF$) Let $K$ be rigid. Then $K$ has an integer part.

**Proof.** Let $a \subseteq K$ be finite, $K = RC(a)$. Then $K$ is in itself the Skolem hull of $a$ (with respect to formulas in the language of ordered rings). As the formulas of the language of ordered rings are easily explicitly well-orderable, so is $K$. Hence it is provable in $ZF$ that every rigid $RCF$ is well-orderable. Furthermore, the proof of Mouruges and Ressayre shows in $ZF$ that every well-orderable $RCF$ has an integer part. □

**Lemma 15.** Let $K$ be an homogenous $RCF$, $I$ an integer part of $K$, $a \subseteq K$ a support for $I$. Then $K' := RC(a)$ is dense in $K$.

**Proof.** We start by observing that $K'$ must be unbounded in $K$. If not, then there are $i \in I$ with $i > K'$ and $j \in K \setminus I$ with $j > K'$ such that $tp(a, i) = tp(a, j)$ and we can proceed as in Lemma to show that $a$ is not a support for $I$, a contradiction.

Now assume that $K'$ is not dense in $K$ and let $(x, y)$ be an interval of $K$ such that $K' \cap (x, y) = \emptyset$. Withou loss of generality, we assume that $0 < x < y$. Now, as $K'$ is unbounded in $K$, there is $d \in K'$ such that $|dx - dy| > 1$. If $z \in K' \cap (dx, dy)$, then $zd^{-1} \in K' \cap (x, y)$, a contradiction - thus $k' \cap (dx, dy) = \emptyset$. 6
As $|dx - dy| > 1$, there is $i \in I \cap (dx, dy)$. Let $r \in (dx, dy) \setminus I$ be arbitrary. Then, as in the proof of Lemma 15 above, $\text{tp}(a, r) = \text{tp}(a, i)$. Hence, there is an automorphism $\pi$ of $K$ such that $\pi(a) = a$ and $\pi(i) = r$. So $a$ is not a support for $I$, a contradiction. \hfill \Box

**Corollary 16.** Under the assumptions of Lemma 15, every integer part of $\text{RC}(a)$ is also an integer part of $K$.

**Proof.** Let $I$ be an integer part of $\text{RC}(a)$. Clearly, $I \subseteq \text{RC}(a) \subseteq K$, $I$ is a subring of $K$ with minimal element 1 and hence discretely ordered. We need to show that each element of $K$ can be rounded down to some element of $I$. So let $x \in K$. By density of $\text{RC}(a)$, let $x' \in \text{RC}(a)$ such that $|x - x'| < 1$, and let $i + 1 \in I$ such that $i + 1 \leq x' < i + 2$. Then $i < x' - 1 < x < x' + 1 < i + 3$. Consequently, we have $x \in (i, i + 1] \cup (i + 1, i + 2] \cup (i + 2, i + 3)$, so there is $j \in I$ such that $j \leq x < j + 1$. As $x$ was arbitrary, $I$ is an integer part of $K$. \hfill \Box

**Remark:** Note that Lemma 15 and the Corollary were proved in $ZF$.

The following theorem shows that, in the homogenous case, supports are exactly what is needed to ensure that integer parts also exist in choice-free universes.

**Theorem 17.** Let $K$ be an homogenous $RCF$ with an integer part $I$ which has a support $a$. Then every transitive model of $ZF$ which contains $K$ also contains an integer part of $K$.

**Proof.** Let $M \models ZF$ be transitive, $K \in M$. Work in $M$, noting that, by our remark above, everything we use is provable in $ZF$ alone and hence holds in $M$. By Lemma 15, $\text{RC}(a)$ is dense in $K$. Obviously, $\text{RC}(a)$ is rigid. Hence, by Lemma 13, $\text{RC}(a)$ has an integer part $J$. By Lemma 16, $J$ is also an integer part of $K$. Hence $K$ has an integer part in $M$. \hfill \Box

The proof actually leads the following:

**Corollary 18.** (ZF) Let $K$ be an $RCF$, $a \subseteq K$ finite such that $\text{RC}(a)$ is dense in $K$. Then $K$ has an integer part. In particular, if there is a finite $a \subseteq K$ such that $\text{RC}(a)$ contains an integer part of $K$ in any transitive model of $ZFC$, then $K$ has an integer part in every model of $ZF$ containing $K$.

**Proof.** Let $M \models ZF$ be transitive such that $K \in M$. Then $\preceq K \in M$, and hence $a \in M$. The subfield $k$ of $K$ generated by $a$ exists and is ordered in $M$, so $\text{RC}(a) \in M$. By absoluteness of density (see Lemma 4), $\text{RC}(a)$ is dense in $K$ also in $M$. Hence, by Theorem 17, $M$ believes that $K$ has an integer part. \hfill \Box
In the countable case, we can summarize these results as follows:

**Theorem 19.** Let $K$ be a countable, $\omega$-homogenous $RCF$. Then $K$ has an integer part in any transitive $M \models ZF$ with $K \in M$ iff there is an integer part of $K$ which has a support.

**Proof.** The ‘if’ part is Theorem 17, the ‘only if’ part is a direct application of Lemma 3.

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## 5 Conclusion, Open Questions and Further Work

We have seen that there are rather natural examples of real closed fields that have no integer part in some transitive model of $ZF$. This shows in particular that in the analysis of ‘effective’ methods for constructing integer parts of given real closed fields as in [DKL] or [DKKL], it is indeed necessary in general, as is done there, to cancel out the use of $AC$ by fixing a well-ordering of the real closed field or an enumeration of the transcendence basis over the residue field. In contrast, we also show some examples where this additional assumption is unnecessary.

We do not know how strong a choice principle $\phi_{IP}$ is. In particular, we do not know whether $\phi_{IP}$ actually implies $AC$, (though we conjecture that it does not) or some weakening of $AC$ and how much of $AC$ is necessary for $\phi_{IP}$. Moreover, we want to consider the question whether $ZF + AD$ (i.e. $ZF$ with the axiom of determinacy, see e.g. [J1]) implies $\phi_{IP}$.

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**References**

[BS] J. Barwise, J. Schlipf. An Introduction to Recursively Saturated and Resplendent Models. J. Symbolic Logic, Vol. 41, No 2, pp. 531-536 (1976)

[CK] C.C. Chang, H. J. Keisler. Model Theory. North-Holland Publishing Company 1973

[DKL] P. D’Aquino, J. Knight, K. Lange. Limit computable integer parts. Archive for Mathematical Logic, November 2011, Volume 50, Issue 7-8, pp 681-695

[DKKL] P. D’Aquino, J. Knight, S. Kuhlmann, K. Lange. Real closed exponential fields. Fundamenta Mathematicae, 219, 163-190 (2012)
[DKS] D’Aquino, J. Knight, S. Starchenko. Real closed fields and models of Peano arithmetic. J. Symbolic Logic, Volume 75, Issue 1 (2010), 1-400

[H1] W. Hodges. Six impossible rings. Journal of Algebra, Vol. 31, No. 2. (August 1974), pp. 218-244

[H2] W. Hodges. On the effectivity of certain field constructions. Proc. London Math. Soc. (3) 32 (1976) 133-162

[J] T. Jech. The axiom of choice. Dover 2008

[J1] Jech. Set Theory - Third Millennium Edition. Springer 2003

[Ma] D. Marker. Model Theory: An Introduction. Springer 2002

[MR] M.H. Mourgues, J.P. Ressayre. Every real closed field has an integer part. J. Symbolic Logic, Vol. 58, No. 2 (1993), pp. 641-647

[S] J.C. Shepherdson. A nonstandard model for a free variable fragment of number theory. Bulletin de l’academie Polonaise des sciences, XII (1964), No. 2, pp. 79-86