A parametrically modulated Hénon map

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Abstract. In this work we consider a situation in which the two parameters of a Hénon map are linearly modulated by the output of another Hénon map, whose parameters are constant in time but can be adjusted. More specifically, here we numerically investigate modifications in basins of attraction of coexisting states, and shift in the location of critical points in bifurcation diagrams of a Hénon map, by virtue of periodic parametric modulation from another Hénon map.

1. Introduction
The Hénon map [1] is a well-known two-dimensional discrete-time dynamical system given by

\[ z_{t+1} = c - z_t^2 + dw_t, \]
\[ w_{t+1} = z_t, \]

where \( z_t, w_t \) represent dynamical variables, \( c, d \) are the nonlinearity and the dissipation parameters, and \( t = 0, 1, 2, \ldots \) is the discrete time. It was proposed in 1976 as a model to the Poincaré section of the continuous-time Lorenz system [2] governed by three autonomous nonlinear ordinary first-order differential equations, and have been extensively investigated in the last years. Apart from its theoretical importance, some applications are possible. It can be used as a model to a four-level CO\(_2\) laser with modulated losses in the limit of high dissipation [3], and its parameter-space can be isomorphic to a two-dimensional parameter-space of a two-level continuous-time model of a loss-modulated CO\(_2\) laser [4].

Dynamical systems as the Hénon map are usually characterized by a set of parameters that control their behavior and, consequently, give rise to changes in their dynamics. Most of the investigations consider these parameters constant, or allow to vary on a range of unchanging values. However, in the Nature the parameters governing the dynamics of a system frequently vary in time, and few are reports concerning it. Only in some previous investigations, some forms of modulation were applied on the parameters of the Hénon map. For example, in Ref. [5] a low-dissipative Hénon map (parameter dissipation \( d = 0.9 \)) with coexisting period-1 and period-3 orbits was considered, and a harmonic modulation was applied on the parameter \( c \). It was shown that, depending on the modulation, the period-3 branch can be expanded, retracted, or even suppressed. In Ref. [6] the parameter \( c \) of a high-dissipative Hénon map (\( d = 0.3 \)) is modulated by a parametric modulation based on deterministic pseudorandom dynamics. This time it was shown that the resulting dynamics can be aperiodic and not chaotic, i.e., aperiodic with a negative largest Lyapunov exponent.
In this paper we modulate both parameters, $c$ and $d$, in Hénon map (1), using the solution of another Hénon map. With this aim, we make

$$c = e + f x_t, \quad d = e + f y_t,$$

where $e, f$ are constant parameters, and $x_t, y_t$ are the solution of a Hénon map given by

$$x_{t+1} = a - x_t^2 + by_t, \quad y_{t+1} = x_t.$$  

(3)

The result of the use of the modulation (2) in Eqs. (1) is the four-dimensional system given by

$$x_{t+1} = a - x_t^2 + by_t, \quad y_{t+1} = x_t, \quad z_{t+1} = (e + f x_t) - z_t^2 + (e + f y_t)w_t, \quad w_{t+1} = z_t,$$

(4)

which can be regarded as a coupling of two maps of the plane: the Hénon map defined by the two first equations in (4), or by Eqs. (3), and the modulated Hénon map, defined by the two last equations in (4). Note that the two first equations do not depend on the two last equations. In other words, we have in (4) an unidirectional coupling of two Hénon maps, namely a master-slave system. System (4) maybe can be used to model the physical situation where we consider the coupling of two four-level CO$_2$ lasers with modulated losses in the limit of high dissipation. A similar modulation procedure was used in Ref. [7], to investigate the behavior of a logistic map modulated by the output of another logistic map.

Our purpose here is to verify possible modifications in the basins of attraction of coexisting states, and in the bifurcation points of the Hénon map (3), under the influence of the imposed modulation. The rest of this work is organized as follows. In the next section we discuss modifications introduced in bifurcation diagrams of the paradigmatic Hénon map, like shift of bifurcation points and changes in asymptotic behaviors in phase-space, as a function of the parametric modulation. Results involving basins of attraction evolution of the Hénon map when the parameters are modulated, are presented and discussed in Sec. 3. Finally, a summary is given in Sec. 4.

2. Bifurcation Diagrams

Figure 1(a) shows the bifurcation diagram (variable $x$) of the Hénon map (3) for $a = 1.45$ and $-1.0 \leq b \leq 0.3$, while in Figs. 1(b) and 1(c) are shown the bifurcation diagrams (variable $z$) of the modulated Hénon map in (4) for the same $a$ and $b$ parameters, with $e = 0.25$, $f = 0.2$, and $e = 0.3$, $f = 0.2$, respectively. If $a = 1.45$ and $-1.0 \leq b \leq -0.4$, then the Hénon map orbits are finally attracted to a fixed point, after a rapid transient, as can be seen in Fig. 1(a). Otherwise, if $a = 1.45$, $-1.0 \leq b \leq -0.4$, $e = 0.25$ or $e = 0.30$, and $f = 0.2$, then the modulated Hénon map orbits converge to a period-2 attractor, as shown in Figs. 1(b) and 1(c). Therefore, when the modulated Hénon map is perturbed by a Hénon map in a fixed point orbit, it converges to a period-2 orbit. The $1 \rightarrow 2$ bifurcation point at $b = -0.4$ in the Hénon map, is suppressed in the modulated Hénon map and, consequently, the fixed point disappears over the range $-1.0 \leq b \leq -0.4$. In other words, in this case the modulation procedure destroys the period-1 orbit, with consequent enlargement of the period-2 branch in the bifurcation diagrams.

Now we kept $b = 0.4$ fixed, and vary $0 \leq a \leq 1.25$. The bifurcation diagrams for the Hénon map and the modulated Hénon map are shown in Fig. 2(a) and Figs. 2(b) where $e = 0.3$, $f = 0.2$ and 2(c) where $e = 0.5$, $f = 0.2$, respectively. When $b = 0.4$, the $1 \rightarrow 2$ bifurcation point in
Figure 1. (a) Variable $x$ bifurcation diagram of the Hénon map (3), for $a = 1.45$, and $-1.0 \leq b \leq 0.3$. (b) Variable $z$ bifurcation diagram of the modulated Hénon map in (4), for $e = 0.25$ and $f = 0.2$. (c) Variable $z$ bifurcation diagram of the modulated Hénon map in (4), for $e = 0.3$ and $f = 0.2$.

The Hénon map is located near to $a = 0.25$, as shown in Fig. 2(a). If $b = 0.4$, $e = 0.3$, and $f = 0.2$, then the $1 \rightarrow 2$ bifurcation occurs close to point $a = 0.125$ for the modulated Hénon map, phenomenon that can be verified in Fig. 2(b). Therefore, depending on the modulation, the period-2 branch can be enlarged with a consequent retraction of the period-1 branch. In other words, the modulation can produces a shift in the location of the bifurcation points of the Hénon map. Figure 2(c) shows again that when the modulated Hénon map is perturbed by a Hénon map in a fixed point orbit, it converges to a period-2 orbit. The $1 \rightarrow 2$ bifurcation point close to $a = 0.25$ in Fig. 2(a) disappears, together with the fixed point orbit over the range $0 \leq a \leq 0.25$. Hence, this time a shift on the $1 \rightarrow 2$ bifurcation point, or a suppression of the period-1 orbit may be achieved with the modulation. For all diagrams in Figs.1 and 2, the horizontal axis was divided in 1000 points, the trajectories were initialized at $(x_0, y_0) = (0.48, 0.53)$ in Fig. 1(a), $(x_0, y_0) = (0.001, 0.05)$ in Fig. 2(a), $(x_0, y_0, z_0, w_0) = (0.48, 0.53, 0.48, 0.53)$ in Figs. 1(b) and 1(c), and $(x_0, y_0, z_0, w_0) = (0.001, 0.05, 0.001, 0.05)$ in Figs. 2(b) and 2(c).

3. Basins of Attraction
Figure 3 shows the basins of attraction of the Hénon map (3) for $b = -0.3$ and $a = 1.2$. For this set of parameters we determined the periodicities in phase-space, inside the window $-3 \leq x \leq 3$ and $-10 \leq y \leq 10$, which was discretized in a grid of $750 \times 750$ points. As can be seen in Fig. 3, we have two stable dynamic equilibrium states, namely period-1 (in black color) and period-3 (in red color) states. White represents the basin of infinity. This coexistence of more than one attractor for the same set of parameters is called multistability, phenomenon that was experimentally observed in various fields, for instance, in optics [8], in biology [9], and in chemical reactions [10].

Next we investigate the suppression of multistability in the Hénon map, as an effect of the
Figure 2. (a) Variable $x$ bifurcation diagram of the Hénon map (3), for $b = 0.4$ and $0 \leq a \leq 1.25$. (b) Variable $z$ bifurcation diagram of the modulated Hénon map in (4) for $e = 0.3$ and $f = 0.2$. (c) Variable $z$ bifurcation diagram of the modulated Hénon map in (4) for $e = 0.5$ and $f = 0.2$.

Figure 3. Basins of attraction of the Hénon map (3) for $b = -0.3$ and $a = 1.2$. Black and red colors indicate the basins of period-1 and period-3 states, respectively, and white represents the basin of the unbounded states.

modulation. As before, the periodicities in phase-space, inside the window $-3 \leq x \leq 3$ and $-10 \leq y \leq 10$, were determined in a grid of $750 \times 750$ points. Figure 4 displays two panels that refer to basins of attraction of the modulated Hénon map in (4). In the first panel, in
Figure 4. Basins of attraction of the modulated Hénon map in (4) for $e = 0.25$ and $f = 0.2$. (a) Initial condition $(x_0, y_0) = (0.1, 0.2)$ (point P in Fig. 3). The Hénon map in a period-1 orbit drives the modulated Hénon map to a period-2 orbit, in green. (b) Initial condition $(x_0, y_0) = (-0.9, 1.0)$ (point Q in Fig. 3). The Hénon map in a period-3 orbit drives the modulated Hénon map to a period-3 orbit, in red. As before, white represents the basin of the unbounded states.

Fig. 4(a), appears the result obtained when the modulated Hénon map with $e = 0.25$, $f = 0.2$ is perturbed by a Hénon map with $b = -0.3$ and $a = 1.2$, initialized at $(x_0, y_0) = (0.1, 0.2)$, the point P in Fig. 3, therefore a Hénon map in a period-1 orbit. We observe that this type of modulation, namely a fixed point modulation, annihilates the coexisting period-1 and period-3 attractors and their basins of attraction, and constructs a new period-2 attractor with its basin of attraction, shown in green in Fig. 4(a). In the second panel, in Fig. 4(b), is shown the result obtained when the modulated Hénon map with the same $e = 0.25$ and $f = 0.2$ parameters is perturbed by a Hénon map with the same $b = -0.3$ and $a = 1.2$ parameters, now initialized at
(x₀,y₀)=(-0.9,1.0), the point Q in Fig. 3, therefore a Hénon map in a period-3 orbit. This time we note that the modulation, namely a period-3 modulation, annihilates the coexisting period-1 and period-3 attractors with their basins of attraction, and constructs a new period-3 attractor with another basin of attraction, shown in red in Fig. 4(b).

Hence, a parameter modulation procedure of a Hénon map, by another Hénon map, is able to perform as a control, suppressing and/or generating attractors in the phase-space. In the two cases above presented two coexisting attractors were completely destroyed, while only one was generated, i.e., a bistable Hénon map became monostable on account of the modulation.

There are successful experiments on attractor destruction in multistable systems. By instance, in Ref. [11] was reported an experiment involving control of multistability in an erbium-doped fiber laser with pump modulation of the diode laser, where a harmonic modulation of the diode current annihilates one or two stable limit cycles in the laser, selecting a desired state.

Similar results as the above noticed by us, can be obtained for the Hénon map using other sets of parameters. For example, if we consider a = 1.3 and b = −0.24, the result is the coexistence of two other attractors in the phase-space of the Hénon map: a period-2 and a period-6 orbits. Then, if the modulated Hénon map with e = 0.25 and f = 0.2 is perturbed by this Hénon map initialized at (x₀,y₀)=(0.1,0.2), which oscillates in a period-2 orbit, the period-6 attractor is destroyed and the modulated Hénon map becomes monostable, in a period-2 state. Otherwise, if the modulated Hénon map with same e = 0.25 and f = 0.2 parameters as above is perturbed by this same Hénon map, now initialized at (x₀,y₀)=(-0.94,1.7), which oscillates in a period-6 orbit, the period-2 attractor is destroyed and the modulated Hénon map again becomes monostable, this time in a period-6 state.

4. Summary
In this work we have numerically investigated the consequence of using a solution arising from a Hénon map, to modulate the parameters of another Hénon map. With the help of bifurcation diagrams, we have shown that the modulation can produces a shift on the location of bifurcation points, including the suppression of periodic orbits. We also have shown that the basins of attraction of coexisting states in phase-space can be modified under the influence of periodic modulations. More specifically, we have shown that bistability can be controlled, and attractors can be annihilated or/and created depending on modulation, leading the system to monostability.

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