Littlewood-Paley-Stein functions for Schrödinger operators

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Abstract

We study the boundedness on $L^p(\mathbb{R}^d)$ of the vertical Littlewood-Paley-Stein functions for Schrödinger operators $\Delta + V$ with non-negative potentials $V$. These functions are proved to be bounded on $L^p$ for all $p \in (1, 2)$. The situation for $p > 2$ is different. We prove for a class of potentials that the boundedness on $L^p$, for some $p > d$, holds if and only if $V = 0$.

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1 Introduction

Let $L := -\Delta + V$ be a Schrödinger operator with a non-negative potential $V$. It is the self-adjoint operator associated with the form

$$a(u, v) := \int_{\mathbb{R}^d} \nabla u \cdot \nabla v \, dx + \int_{\mathbb{R}^d} V uv \, dx$$
with domain

\[ D(a) = \{ u \in W^{1,2}(\mathbb{R}^d), \int_{\mathbb{R}^d} V|u|^2\,dx < \infty \}. \]

We denote by \((e^{-tL})_{t \geq 0}\) the semigroup generated by (minus) \(L\) on \(L^2(\mathbb{R}^d)\). Since \(V\) is nonnegative, it follows from the Trotter product formula that

\[ 0 \leq e^{-tL}f \leq e^{t\Delta f} \tag{1} \]

for all \(t \geq 0\) and \(0 \leq f \in L^2(\mathbb{R}^d)\) (all the inequalities are in the a.e. sense). It follows immediately from (1) that the semigroup \((e^{-tL})_{t \geq 0}\) is sub-Markovian and hence extends to a contraction \(C_0\)-semigroup on \(L^p(\mathbb{R}^d)\) for all \(p \in [1, \infty)\). We shall also denote by \((e^{-tL})_{t \geq 0}\) the corresponding semigroup on \(L^p(\mathbb{R}^d)\).

The domination property (1) implies in particular that the corresponding heat kernel of \(L\) is pointwise bounded by the Gaussian heat kernel. As a consequence, \(L\) has a bounded holomorphic functional calculus on \(L^p(\mathbb{R}^d)\) and even Hörmander type functional calculus (see [6]). This implies the boundedness on \(L^p(\mathbb{R}^d)\) for all \(p \in (1, \infty)\) of the horizontal Littlewood-Paley-Stein functions:

\[ g_L(f)(x) := \left( \int_0^\infty t|\nabla e^{-t\sqrt{V}}L f(x)|^2 \, dt \right)^{1/2} \]

and

\[ h_L(f)(x) := \left( \int_0^\infty t|Le^{-tL}f(x)|^2 \, dt \right)^{1/2}. \]

Indeed, these functions are of the form (up to a constant)

\[ S_L f(x) = \left( \int_0^\infty |\psi(tL)f(x)|^2 \, dt \right)^{1/2} \]

with \(\psi(z) = \sqrt{z}e^{-\sqrt{z}}\) for \(g_L\) and \(\psi(z) = ze^{-z}\) for \(h_L\). The boundedness of the holomorphic functional calculus implies the boundedness of \(S_L\) (see [8]). Thus, \(g_L\) and \(h_L\) are bounded on \(L^p(\mathbb{R}^d)\) for all \(p \in (1, \infty)\) and this holds for every nonnegative potential \(V \in L^1_{\text{loc}}(\mathbb{R}^d)\).

Now we define the so-called vertical Littlewood-Paley-Stein functions

\[ G_L(f)(x) := \left( \int_0^\infty t|\nabla e^{-t\sqrt{V}}L f(x)|^2 + t|\nabla e^{-t\sqrt{V}}L f(x)|^2 \, dt \right)^{1/2} \]

and

\[ H_L(f)(x) := \left( \int_0^\infty |\nabla e^{-tL}f(x)|^2 + |\nabla e^{-tL}f(x)|^2 \, dt \right)^{1/2}. \]

Note that usually, these two functions are defined without the additional terms \(t|\nabla e^{-t\sqrt{V}}L f(x)|^2\) and \(|\nabla e^{-tL}f(x)|^2\).
The functions $G_L$ and $H_L$ are very different from $g_L$ and $h_L$ as we shall see in the last section of this paper. If $V = 0$ and hence $L = -\Delta$ it is a very well known fact that $G_L$ and $H_L$ are bounded on $L^p(\mathbb{R}^d)$ for all $p \in (1, \infty)$. The Littlewood-Paley-Stein functions are crucial in the study of non-tangential limits of Fatou type and the boundedness of Riesz transforms. We refer to [13]-[15]. For Schrödinger operators, boundedness results on $L^p(\mathbb{R}^d)$ are proved in [10] for potentials $V$ which satisfy $|\nabla V| + \Delta V \in L^\infty(\mathbb{R}^d)$. This is a rather restrictive condition. For elliptic operators in divergence form (without a potential) boundedness results on $L^p(\mathbb{R}^d)$ for certain values of $p$ are proved in [2]. For the setting of Riemannian manifolds we refer to [4] and [5]. Again the last two papers do not deal with Schrödinger operators.

In this note we prove that $G_L$ and $H_L$ are bounded on $L^p(\mathbb{R}^d)$ for all $p \in (1, 2]$ for every nonnegative potential $V \in L^1_{\text{loc}}(\mathbb{R}^d)$. That is

$$\int_{\mathbb{R}^d} \left( \int_0^\infty t|\nabla e^{-t\sqrt{L}}f(x)|^2 + t|\sqrt{V}e^{-t\sqrt{L}}f(x)|^2 \, dt \right)^{p/2} \, dx \leq C \int_{\mathbb{R}^d} |f(x)|^p \, dx$$

and similarly,

$$\int_{\mathbb{R}^d} \left( \int_0^\infty |\nabla e^{-tL}f(x)|^2 + |\sqrt{V}e^{-tL}f(x)|^2 \, dt \right)^{p/2} \, dx \leq C \int_{\mathbb{R}^d} |f(x)|^p \, dx$$

for all $f \in L^p(\mathbb{R}^d)$.

Our arguments of the proof are borrowed from the paper [4] which we adapt to our case in order to take into account the terms with $\sqrt{V}$ in the definitions of $G_L$ and $H_L$. Second we consider the case $p > 2$ and $d \geq 3$. For a wide class of potentials, we prove that if $G_L$ (or $H_L$) is bounded on $L^p(\mathbb{R}^d)$ for some $p > d$ then $V = 0$. Here we use some ideas from [7] which deals with the Riesz transform on Riemannian manifolds. In this latter result we could replace $G_L$ by $\left( \int_0^\infty t|\nabla e^{-t\sqrt{L}}f(x)|^2 \, dt \right)^{1/2}$ and the conclusion remains valid.

Many questions of harmonic analysis have been studied for Schrödinger operators. For example, spectral multipliers and Bochner Riesz means [6] and [12] and Riesz transforms [12], [1], [13] and [3]. However little seems to be available in the literature concerning the associated Littlewood-Paley-Stein functions $G_L$ and $H_L$. Another reason which motivates the present paper is to understand the Littlewood-Paley-Stein functions for the Hodge de-Rham Laplacian on differential forms. Indeed, Bochner’s formula allows to write the Hodge de-Rham Laplacian on 1-differential forms as a Schrödinger operator (with a vector-valued potential). Hence, understanding the Littlewood-Paley-Stein functions for Schrödinger operators $L$ could be a first step in order to consider the Hodge de-Rham Laplacian. Note however that unlike the present case, if the manifold has a negative Ricci curvature part, then the semigroup of the Hodge de-Rham Laplacian does not necessarily act on all $L^p$ spaces. Hence the arguments presented in this paper have to be changed considerably. We shall address this problem in a forthcoming paper.
2 Boundedness on $L^p$, $1 < p \leq 2$

Recall that $L = -\Delta + V$ on $L^2(\mathbb{R}^d)$. We have

**Theorem 2.1.** For every $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^d)$, $\mathcal{G}_L$ and $\mathcal{H}_L$ are bounded on $L^p(\mathbb{R}^d)$ for all $p \in (1,2]$. 

**Proof.** By the subordination formula

$$e^{-t\sqrt{T}} = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{\frac{-s^2}{4t}}e^{-s^{-1/2}}ds$$

it follows easily that there exists a positive constant $C$ such that

$$\mathcal{G}_L(f)(x) \leq C\mathcal{H}_L(f)(x) \quad (2)$$

for all $f \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and a.e. $x \in \mathbb{R}^d$. See e.g. [4]. Therefore it is enough to prove boundedness of $\mathcal{H}_L$ on $L^p(\mathbb{R}^d)$. In order to do so, we may consider only nonnegative functions $f \in L^p(\mathbb{R}^d)$. Indeed, for a general $f$ we write $f = f^+ - f^-$ and since

$$|\nabla e^{-tL}(f^+ - f^-)|^2 \leq 2(|\nabla e^{-tL}f^+|^2 + |\nabla e^{-tL}f^-|^2)$$

and

$$|\sqrt{V}e^{-tL}(f^+ - f^-)|^2 \leq 2(|\sqrt{V}e^{-tL}f^+|^2 + |\sqrt{V}e^{-tL}f^-|^2)$$

we see that it is enough to prove

$$\|\mathcal{H}_L(f^+\|_p + \|\mathcal{H}_L(f^-)\|_p \leq C_p(\|f^+\|_p + \|f^-\|_p),$$

which in turn will imply $\|\mathcal{H}_L(f)\|_p \leq 2C_p\|f\|_p$.

Now we follow similar arguments as in [4]. Fix a non-trivial $0 \leq f \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and set $u(t,x) = e^{-tL}f(x)$. Note that the semigroup $(e^{-tL})_{t \geq 0}$ is irreducible (see [12], Chapter 4) which means that for each $t > 0$, $u(t,x) > 0$ (a.e.). Observe that

$$\left(\frac{\partial}{\partial t} + L\right)u^p = (1-p)Vu^p - p(p-1)u^{p-2}|\nabla u|^2.$$ 

This implies

$$p|\nabla u|^2 + V|u|^2 = -\frac{u^{2-p}}{p-1}\left(\frac{\partial}{\partial t} + L\right)u^p. \quad (3)$$

Hence, there exists a positive constant $c_p$ such that

$$\mathcal{H}_L(f)(x)^2 \leq -c_p \int_0^\infty u(t,x)^{2-p}\left(\frac{\partial}{\partial t} + L\right)u(t,x)^p dt$$

$$\leq c_p \sup_{t>0} u(t,x)^{2-p} J(x)$$

for all $f \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$.
where

\[
J(x) = - \int_0^\infty \left( \frac{\partial}{\partial t} + L \right) u(t, x)^p \, dt.
\]

The previous estimate uses the fact that \((\frac{\partial}{\partial t} + L)u(t, x)^p \leq 0\) which follows from (3). Since the semigroup \((e^{-tL})_{t \geq 0}\) is sub-Markovian it follows that

\[
\| \sup_{t>0} e^{-tL} f(x) \|_p \leq C \| f \|_p.
\]

The latter estimate is true for all \(p \in (1, \infty)\), see [15] (p. 73). Therefore, by Hölder’s inequality

\[
\int_{\mathbb{R}^d} |\mathcal{H}_L(f)(x)|^p \, dx \leq c_p \| f \|_p^{\frac{p}{2}} \left( \int_{\mathbb{R}^d} J(x) \, dx \right)^{p/2}.
\]

On the other hand,

\[
\int_{\mathbb{R}^d} J(x) \, dx = - \int_{\mathbb{R}^d} \int_0^\infty \left( \frac{\partial}{\partial t} + L \right) u(t, x)^p \, dt \, dx
\]

\[
= \| f \|_p^p - \int_0^\infty \int_{\mathbb{R}^d} Lu(t, x)^p \, dx \, dt
\]

\[
= \| f \|_p^p - \int_0^\infty \int_{\mathbb{R}^d} Vu(t, x)^p \, dx \, dt
\]

\[
\leq \| f \|_p^p.
\]

Inserting this in (5) gives

\[
\int_{\mathbb{R}^d} |\mathcal{H}_L(f)(x)|^p \, dx \leq c_p \| f \|_p^p
\]

which proves the theorem since this estimates extends by density to all \(f \in L^p(\mathbb{R}^d)\).

\section{Boundedness on \(L^p, p > 2\)}

We assume throughout this section that \(d \geq 3\). We start with the following result.

\begin{proposition}
Let \(0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^d)\). If \(G_L \) (or \(H_L\)) is bounded on \(L^p(\mathbb{R}^d)\) then there exists a constant \(C > 0\) such that

\[
\| \nabla e^{-tL} f \|_p \leq \frac{C}{\sqrt{t}} \| f \|_p
\]

for all \(t > 0\) and all \(f \in L^p(\mathbb{R}^d)\).
\end{proposition}
Proof. Remember that by (2), if \( \mathcal{H}_L \) is bounded on \( L^p(\mathbb{R}^d) \) then the same holds for \( \mathcal{G}_L \). Suppose that \( \mathcal{G}_L \) is bounded on \( L^p(\mathbb{R}^d) \). We prove that
\[
\| \nabla f \|_p \leq C \left[ \| L^{1/2} f \|_p + \| Lf \|_p^{1/2} \| f \|_p^{1/2} \right].
\]
(7)
The inequality here holds for \( f \) in the domain of \( L \), seen as an operator on \( L^p(\mathbb{R}^d) \). In order to do this we follow some arguments from [5]. Set \( P_t := e^{-t \sqrt{L}} \) and fix \( f \in L^2(\mathbb{R}^d) \). By integration by parts,
\[
\| \nabla P_t f \|_2^2 = (\Delta P_t f, P_t f) \leq (LP_t f, P_t f) = \| L^{1/2} P_t f \|_2^2.
\]
In particular,
\[
\| \nabla P_t f \|_2 \leq \frac{C}{t} \| f \|_2 \to 0 \quad \text{as} \quad t \to +\infty.
\]
The same arguments show that \( t \| \nabla L^{1/2} P_t f \|_2 \to 0 \) as \( t \to +\infty \). Therefore,
\[
| \nabla f |^2 = - \int_0^\infty \frac{d}{dt} | \nabla P_t f |^2 dt
\]
\[
= - \left[ t \frac{d}{dt} | \nabla P_t f |^2 \right]_0^\infty + \int_0^\infty \frac{d^2}{dt^2} | \nabla P_t f |^2 t dt
\]
\[
\leq \int_0^\infty \frac{d^2}{dt^2} | \nabla P_t f |^2 t dt
\]
\[
= 2 \int_0^\infty (| \nabla L^{1/2} P_t f |^2 + \nabla L P_t f \cdot \nabla P_t f) t dt
\]
\[
=: I_1 + I_2.
\]
Using the fact that \( \mathcal{G}_L \) is bounded on \( L^p(\mathbb{R}^d) \) it follows that
\[
\| I_1 \|_{p/2} \leq \| \mathcal{G}_L(L^{1/2} f) \|_p^2 \leq C \| L^{1/2} f \|_p^2.
\]
(8)
By the Cauchy-Schwartz inequality,
\[
|I_2| \leq \left( \int_0^\infty (| \nabla L P_t f |^2 t dt) \right)^{1/2} \left( \int_0^\infty (| \nabla P_t f |^2 t dt) \right)^{1/2}
\]
\[
\leq \mathcal{G}_L(Lf) \mathcal{G}_L(f).
\]
Integrating gives
\[
\| I_2 \|_{p/2}^{p/2} \leq \left( \int_{\mathbb{R}^d} | \mathcal{G}_L(Lf) |^p \right)^{1/2} \left( \int_{\mathbb{R}^d} | \mathcal{G}_L(f) |^p \right)^{1/2} \leq C \| Lf \|_p^{p/2} \| f \|_p^{p/2}.
\]
(9)
Combining (8) and (9) gives (11) for \( f \in D(L) \cap L^2(\mathbb{R}^d) \). In order to obtain (11) for all \( f \in D(L) \) we take a sequence \( f_n \in L^2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d) \) which converges in

\footnote{Since the semigroup \( e^{-tL} \) is sub-Markovian, it acts on \( L^p(\mathbb{R}^d) \) and hence the generator of this semigroup in \( L^p(\mathbb{R}^d) \) is well defined. This is the operator \( L \) we consider on \( L^p(\mathbb{R}^d) \).}
the $L^p$-norm to $f$. We apply (7) to $e^{-tL}f_n$ (for $t > 0$) and then let $n \to +\infty$ and $t \to 0$.

For $f \in L^p(\mathbb{R}^d)$ we apply (7) to $e^{-tL}f$ and we note that $\|L^{1/2}e^{-tL}f\|_p \leq \frac{C}{\sqrt{t}}\|f\|_p$ and $\|Le^{-tL}f\|_p \leq \frac{C}{t}\|f\|_p$. Both assertions here follow from the analyticity of the semigroup on $L^p(\mathbb{R}^d)$ (see [12], Chap. 7). This proves the proposition. \hfill \square

**Remark.** In the proof we did not use the boundedness of the function $G_L$ but only its gradient part, i.e. boundedness on $L^p(\mathbb{R}^d)$ of the Littlewood-Paley-Stein function:

$$G_L(f)(x) = \left( \int_0^\infty t|\nabla e^{-t\sqrt{L}}f(x)|^2 dt \right)^{1/2}. \quad (10)$$

In the next result we shall need the assumption that there exists $\varphi \in L^\infty(\mathbb{R}^d)$, $\varphi > 0$ such that

$$L\varphi = 0. \quad (11)$$

The meaning of (11) is $e^{-tL}\varphi = \varphi$ for all $t \geq 0$.

Note that (11) is satisfied for a wide class of potentials. This is the case for example if $V \in L^{d/2-\epsilon}(\mathbb{R}^d) \cap L^{d/2+\epsilon}(\mathbb{R}^d)$ for some $\epsilon > 0$, see [9]. See also [11] for more results in this direction.

**Theorem 3.2.** Suppose that there exists $0 < \varphi \in L^\infty(\mathbb{R}^d)$ which satisfies (11). Then $G_L$ (or $H_L$) is bounded on $L^p(\mathbb{R}^d)$ for some $p > d$ if and only if $V = 0$.

**Proof.** If $V = 0$ then $L = -\Delta$ and it is known that the Littlewood-Paley-Stein function $G_L$ (and also $H_L$) is bounded on $L^p(\mathbb{R}^d)$ for all $p \in (1, \infty)$.

Suppose now that $V$ is as in the theorem and $G_L$ is bounded on $L^p(\mathbb{R}^d)$ for some $p > d$.

Let $k_t(x,y)$ be the heat kernel of $L$, i.e.,

$$e^{-tL}f(x) = \int_{\mathbb{R}^d} k_t(x,y)f(y)dy$$

for all $f \in L^2(\mathbb{R}^d)$. As mentioned in the introduction, due to the positivity of $V$,

$$k_t(x,y) \leq \frac{1}{(4\pi t)^{d/2}}e^{-\frac{|x-y|^2}{4t}}. \quad (12)$$

On the other hand, using the Sobolev inequality (for $p > d$)

$$|f(x) - f(x')| \leq C|x - x'|^{1-\frac{d}{p}}\|\nabla f\|_p$$

we have

$$|k_t(x,y) - k_t(x',y)| \leq C|x - x'|^{1-\frac{d}{p}}\|\nabla k_t(\cdot,y)\|_p.$$
Using (12), Proposition 3.1 and the fact that

\[ k_t(x, y) = e^{-\frac{t}{2}L}k_{\frac{t}{2}}(\cdot, y)(x), \]

we have

\[ |k_t(x, y) - k_t(x', y)| \leq C|x - x'|^{1 - \frac{d}{p}}t^{-\frac{1}{2}}t^{-\frac{d}{2}(1 - \frac{1}{p})}. \] (13)

Thus, using again (12) we obtain

\[ |k_t(x, y) - k_t(x', y)| = |k_t(x, y) - k_t(x', y)|^{1/2}|k_t(x, y) - k_t(x', y)|^{1/2} \]
\[ \leq C|x - x'|^{\frac{3}{2} - \frac{d}{p}}t^{-\frac{1}{2} + \frac{d}{2p} - \frac{1}{4}} \left(e^{-\frac{|x-y|^2}{4t}} + e^{-\frac{|x'-y|^2}{4t}}\right). \]

Hence, for \( x, x' \in \mathbb{R}^d \)

\[ |\varphi(x) - \varphi(x')| = |e^{-tL}\varphi(x) - e^{-tL}\varphi(x')| \]
\[ = |\int_{\mathbb{R}^d}[k_t(x, y) - k_t(x', y)]\varphi(y)dy| \]
\[ \leq ||\varphi||_{\infty} \int_{\mathbb{R}^d}|k_t(x, y) - k_t(x', y)|dy \]
\[ \leq C|x - x'|^{\frac{1}{2} - \frac{d}{p} + \frac{d}{2p} - \frac{1}{4}}. \]

Letting \( t \to \infty \), the RHS converges to 0 since \( p > d \). This implies that \( \varphi = c > 0 \) is constant. The equality \( 0 = L\varphi = Lc = Vc \) and hence \( V = 0 \).

**Remark.** 1. The above proof is inspired from [7] in which it is proved that the boundedness of the Riesz transform \( \nabla L^{-1/2} \) on \( L^p(\mathbb{R}^d) \) for some \( p > d \) implies that \( V = 0 \).

2. According to a previous remark, we could replace in the last theorem the boundedness of \( G_L \) by the boundedness of \( G \) defined by (10).

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