Inequalities for the generalized numerical radius

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Abstract
In the present paper, we provide several inequalities for the generalized numerical radius of operator matrices as introduced by A. Abu-omar and F. Kittaneh in [3]. We generalize the convexity and the log-convexity results obtained by M. Sababheh in [12] for the case of the numerical radius to the case of the generalized numerical radius. We illustrate our work by providing a positive answer for the question addressed in [12] for the convexity of a certain matrix operator function. Moreover, and motivated by the results of A. Aldalabih and F. Kittaneh in [2] for the case of Hilbert-Schmidt numerical radius norm, we use some Schatten p-norm inequalities for partitioned $2 \times 2$ block-matrices to provide several Schatten p-norm numerical radius inequalities.

1 Introduction
It is well known that the numerical radius plays an important role in various fields of operator theory and matrix analysis (cf. [1, 9, 10]). Based on some operator theory studies on Hilbert spaces, several generalizations for the concept of numerical radius have recently been introduced [7, 2, 14]. In a paper of Abou-omar and F. Kittaneh [2] the authors introduced the so-called generalized numerical radius. If $H$ is a separable Hilbert space and $N$ is any norm on the space of bounded operators $\mathcal{B}(H)$, the generalized numerical radius for $A \in \mathcal{B}(H)$, denoted by $w_N(A)$, is obtained via the supremum of the norm over the real parts of all rotations of $A$ i.e.

$$w_N(A) = \sup_{\theta \in \mathbb{R}} N(\text{Re}(e^{i\theta} A)).$$

Simple computation shows that when $N$ is the usual operator norm inherited from the inner product on $H$ then $w_N$ coincides with the usual numerical radius.
norm $w(\cdot)$. We refer the reader to [3] and [12] for intermediate properties and inequalities of the norm $w_N(\cdot)$.

In the present paper, we restrict our attention to operator matrices i.e. $A \in M_n(\mathbb{C})$ and we provide several inequalities for the matrix norm $w_N(\cdot)$. Some results are obtained via convexity whenever $N$ is unitarily invariant norm. Other results are restricted to the case when $N$ is the Schatten $p$-norm and are obtained via some norm inequalities of $2 \times 2$ partitioned block-matrices.

On the one hand, we follow up the work of M. Sababheh in [12] for the case of the numerical radius, to establish a new Young-type inequality for $w_N(\cdot)$. Addressing to an open question proposed by the author in [12] about the convexity of the function $t \mapsto w(A^t X A^t)$, here $A > 0$ denotes a positive definite matrix, on $\mathbb{R}$ we prove that the convexity is not only true for $w$ but remains true for $w_N$. On the other hand, we use the results of R. Bhatia and F. Kittaneh in [5] to establish upper bounds for the Schatten $p$-generalized numerical radius of a partitioned $2 \times 2$ block matrix with respect to the Schatten $p$-generalized numerical radius of the diagonal part and the Schatten $p$-norm of the off-diagonal parts.

Below we state the first result of this note which generalizes Theorem 2.2 and Theorem 2.3 in [12].

**Theorem 1.1** Let $N(\cdot)$ be a unitarily invariant norm on $M_n(\mathbb{C})$. Given a positive definite matrix $A > 0$ and an arbitrary matrix $X \in M_n(\mathbb{C})$. Each of the following functions

$\begin{align*}
- f(t) &= w_N(A^t X A^{1-t} + A^{1-t} X A^t), \\
- g(t) &= w_N(A^t X A^{1-t}).
\end{align*}$

are convex over $\mathbb{R}$.

In connection to the work in [2] for the Hilbert-Schmidt generalized numerical radius norm, we use the following notation: given $p \in [1, \infty[$ we consider $N = \|\cdot\|_p$, the Schatten $p$-norm, and we denote by $w_p$ the corresponding generalized numerical radius. We obtained the following estimation:

**Theorem 1.2** Let $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in M_{2n}(\mathbb{C})$ then for any $p \in [2, \infty[$ we have

$$w^p_p(T) \leq \frac{1}{2^{2-p}} \left( w^p_p(A) + w^p_p(D) + \frac{1}{2p-1} \left[ \| B \|_p + \| C \|_p \right]^p \right).$$

Moreover, if $p \in [1, 2]$ then

$$w^p_p(T) \leq \left( w^p_p(A) + w^p_p(D) + \frac{1}{2p-1} \left[ \| B \|_p + \| C \|_p \right]^p \right).$$

We point out that a lower bound for the Schatten $p$-generalized numerical radius of an operator with respect to the Schatten $p$-norm of the operator has
already been established by T. Bottazi and C. Conde in [6]. Indeed, using a Clarkson inequality obtained by O. Hirzalla and F. Kittaneh in [11] it follows directly that inequality (1) is bounded below by \( \frac{1}{2} \| T \|_p^p \) and (2) is bounded below by \( \frac{1}{2} \| T \|_p^p \).

2 Sketch of the proof of Theorem 1.1

Throughout this section, \( N \) denotes a unitarily invariant norm on \( M_n(\mathbb{C}) \). We aim to establish the convexity in Theorem 1.1 using some type of Hölder and Heinz inequalities. It should be mentioned that \( w_N \) is in general not unitarily invariant but rather weakly unitarily invariant. For such reason, we need to investigate such types of inequalities in a weaker condition (cf. Ch. IV in [4] for the case of unitarily invariant norms).

Lemma 2.1 Let \( A, X \in M_n(\mathbb{C}) \) such that \( A > 0 \). Given \( t \in [0, 1] \), the following estimates hold

\[
\begin{align*}
&w_N(A^t X A^t) \leq w_N(A X A) w_N^{1-t}(X), \\
&2w_N(A^{\frac{1}{2}} X A^{\frac{1}{2}}) \leq w_N(A^t X A^{1-t} + A^{1-t} X A^t) \leq w_N(A X + X A).
\end{align*}
\]

Proof. As \( A^t \) is Hermitian then for any \( \theta \in \mathbb{R} \), it follows that

\[
\text{Re}(e^{i\theta} A^t X A^t) = A^t \text{Re}(e^{i\theta} X) A^t.
\]

Now applying Hölder inequality for the unitarily invariant norm \( N \) (cf. for example Corollary IV.5.4 in [4]), we get

\[
N(\text{Re}(e^{i\theta} A^t X A^t)) \leq N^t(A \text{Re}(e^{i\theta} X)A) N^{1-t}(\text{Re}(e^{i\theta} X)).
\]

Taking the supremum over all \( \theta \in \mathbb{R} \), the inequality in (3) follows. A similar proof holds for (4) by using Corollary IV.4.9 in [4]. \( \square \)

We point out that replacing \( A \) by \( A^2 \) and for \( t = 1 \) in (4) we get

\[
w_N(A X A) \leq \frac{1}{2} w_N(A^2 X + X A^2).
\]

Hence, for \( t, s \in \mathbb{R} \) we obtain

\[
f\left(\frac{t + s}{2}\right) = w_N(A^{t+s} (A^s X A^{1-t} + A^{1-t} X A^s) A^{t+s})
\]

\[
\leq \frac{1}{2} w_N(A^{t-s} (A^s X A^{1-t} + A^{1-t} X A^s) + (A^s X A^{1-t} + A^{1-t} X A^s) A^{t-s})
\]

\[
\leq \frac{1}{2} w_N(A^t X A^{1-t} + A^{1-t} X A^t) + \frac{1}{2} w_N(A^s X A^{1-s} + A^{1-s} X A^s) = \frac{1}{2} f(t) + \frac{1}{2} f(s).
\]

We note that the first inequality of (4) ensures that \( f \) admits a global minimum at \( t = \frac{1}{2} \). The proof for the convexity of \( g \) follows by a similar argument.

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3 Sketch of proof of Theorem 1.2

Following the notation in [5], we recall the following Schatten p-norm estimates of an operator to that of its $2 \times 2$ block matrix entries.

**Lemma 3.1** Let $T = \begin{pmatrix} T_{ij} \end{pmatrix}$, $1 \leq i, j \leq 2$ be a block matrix. Then for any $p \in [2, \infty[$ the following holds

$$
\| T \|_p^p \leq \frac{1}{2^\frac{2}{p} - 1} \left( \sum_{i,j} \| T_{i,j} \|_p^p \right).
$$

(6)

Let $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in M_{2n}(\mathbb{C})$, then for any $\theta \in \mathbb{R}$ we have $Re(e^{i\theta}T) = \begin{bmatrix} Re(e^{i\theta}A) & F \\ F^* & Re(e^{i\theta}D) \end{bmatrix}$, where $F = \frac{1}{2}(e^{i\theta}B + e^{-i\theta}C^*)$. Applying (6) we obtain

$$
\| Re(e^{i\theta}T) \|_p^p \leq \frac{1}{2^{2-p}} \left( \| Re(e^{i\theta}A) \|_p^p + \| Re(e^{i\theta}D) \|_p^p + 2 \| F \|_p^p \right)
$$

$$
\leq \frac{1}{2^{2-p}} \left( \| Re(e^{i\theta}A) \|_p^p + \| Re(e^{i\theta}D) \|_p^p + \frac{1}{2p-1} \left( \| B \|_p + \| C \|_p \right)^p \right)
$$

$$
\leq \frac{1}{2^{2-p}} \left( w^p(A) + w^p(D) + \frac{1}{2p-1} \left( \| B \|_p + \| C \|_p \right)^p \right).
$$

Taking the supremum over all $\theta \in \mathbb{R}$ inequality (1) is obtained. Using the corresponding results in [2], for the case $p \in [1, 2[$, inequality (2) is established in a similar way.

4 Final remarks

In this section, we provide further results with some applications of Theorems 1.1 and 1.2.

Motivated by the work of Shabebh for the numerical radius, the convexity of the function $f$, together with the convexity of $\ell(t) := w_N(A^tXA^{1-t} - A^{1-t}XA^t)$, as defined in Theorem 1.1 provides the following reversed inequalities for the generalized numerical radius.

**Corollaire 4.1** Let $N$ be a unitarily invariant matrix norm and let $A > 0$. For any $X \in M_n(\mathbb{C})$ the following inequalities hold

$$
\begin{cases}
w_N(A^tXA^{1-t} \pm A^{1-t}XA^t) \leq w_N(AX \pm XA), & \text{if } t \in [0, 1] \\
w_N(A^tXA^{1-t} \pm A^{1-t}XA^t) \geq w_N(AX \pm XA), & \text{if } t \notin [0, 1].
\end{cases}
$$

The convexity of the function $g$ defined in Theorem 1.1 provides the following Young-type inequality which generalizes Corollary 2.11 in [13].
Corollaire 4.2 Let $N$ be a unitarily invariant matrix norm. Given $A > 0$ and $X \in M_n(\mathbb{C})$ then

$$
\begin{align*}
w_N(A^t X A^{1-t}) &\leq t.w_N(AX) + (1-t).w_N(XA) \quad \text{if } t \in [0,1], \\
w_N(A^t X A^{1-t}) &\geq t.w_N(AX) + (1-t).w_N(XA) \quad \text{if } t \not\in [0,1].
\end{align*}
$$

Given $A > 0$, and $X \in M_n(\mathbb{C})$, the author in [13] proposed a question on the convexity of $t \mapsto w(A^t X A^t)$ for the numerical radius and indicated that he has no answer whether this is true or not. Below, we provide a positive answer for the aforementioned questioned not only for the numerical radius but also for the generalized numerical radius whenever $N$ is unitarily invariant.

Indeed, by Theorem IX.4. 8 in [4] the inequality $2N(AXB) \leq N(A^2X + XB^2)$ holds true for all positive matrices $A$ and $B$. Following a similar argument to that in Lemma 2.1, we get $2w_N(AXB) \leq w_N(A^2X + XB^2)$. In particular, if $h(t) = w_N(A^t X A^t)$ then

$$
h\left(\frac{t+s}{2}\right) = w_N\left(A^\frac{t-s}{2} (A^t X A^t) A^\frac{s-t}{2}\right) \leq \frac{1}{2} w_N(A^t X A^t + A^s X A^s) = \frac{1}{2} h(t) + \frac{1}{2} h(s).
$$

We point out that by choosing $t = \frac{1}{2}$ in (3), the log-convexity of $h$ is also obtained. Moreover, and by repeating a similar argument we get

Corollaire 4.3 Let $N$ be a unitarily invariant matrix norm. Given $A > 0$ and $X \in M_n(\mathbb{C})$ the function

$$k(t) = w_N(A^t X A^{1-t} + A^{-1} X A^{1+t})$$

is convex on $\mathbb{R}$, with minimum attained at $t = 0$.

Addressing to the consequences of Theorem 1.2 and following a similar logic to that in [2] we obtain the following results

Corollaire 4.4 Let $p \in [2, \infty[$ then the following estimates holds true

1. $w_p\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right) \leq \frac{1}{2^\frac{1}{p-1}} \left( w_p^p(A) + w_p^p(D) \right)^\frac{1}{p}$, and $w_p\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right) = (w_p^p(A) + w_p^p(D))^\frac{1}{p}$ if $A$ and $D$ are Hermitian.

2. $w_p\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\right) \leq \frac{1}{2^\frac{1}{p-1}} \left( w_p^p(A) + \frac{1}{2^\frac{1}{p-1}} \| B \|_p^p \right)^\frac{1}{p}$.

Using the unitarily invariance of Schatten $p$-norms together with the proof of Theorem 1.1 we get a generalization for Lemma 2 in [2]:

Corollaire 4.5 Let $p \in [2, \infty[$ then the following are true

1. $w_p\left(\begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix}\right) = 2^{\frac{1}{p}} w_p^p(B)$. 

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2. \( w_p \left( \begin{bmatrix} A & B \\ B & A \end{bmatrix} \right) \leq \frac{1}{2^{p-1}} \left( w_p^p(A+B) + w_p^p(A-B) \right)^{\frac{1}{p}} \), and \( w_p \left( \begin{bmatrix} A & B \\ B & A \end{bmatrix} \right) = \left( w_p^p(A+B) + w_p^p(A-B) \right)^{\frac{1}{p}} \) if \( A \) and \( B \) are Hermitian.

Finally, we note that similar inequalities to those obtained in the previous two corollaries can be derived for the case \( p \in [1,2] \) using (2). Moreover, such results remain true for the generalized numerical radius on bounded operators acting on separable Hilbert spaces.

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