The Riemann-Hilbert approach to focusing Kundu-Eckhaus equation with nonzero boundary conditions

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Abstract

In this article, we focus on investigating the focusing Kundu-Eckhaus equation with nonzero boundary condition. A appropriate two-sheeted Riemann surface is introduced to map the spectral parameter $k$ into a single-valued parameter $z$. Starting from the Lax pair of Kundu-Eckhaus equation, two kind of Jost solutions are construed. Further their asymptotic, analyticity, symmetries as well as spectral matrix are detailed analyzed. It is shown that the solution of Kundu-Eckhaus equation with nonzero boundary condition can characterized with a matrix Riemann-Hilbert problem. Then a formula of $N$-soliton solutions is derived by solving Riemann-Hilbert problem. As applications, the first-order explicit soliton solution is obtained.

Keywors: the focusing Kundu-Eckhaus equation; nonzero boundary conditions; Riemann-Hilbert problem; soliton solution.

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1 Introduction

It is well-known that the nonlinear Schrödinger equation plays an important role and achieved great success in physical fields such as nonlinear optics, nonlinear water waves, plasma physics and so on. But Schrödinger equation with higher-order nonlinear terms, such as the self-steepening and self-frequency shift, has an significant effect in specific physical background, for example, optic fiber communication and Bose-Einstein condensates [1, 2].

The modified Gross-Pitaevskii equation [3]

\[ iu_t + u_{xx} + 2f(t)|u|^2u + \frac{1}{2}\delta^2 x^2 u + \kappa_1 |u|^4 u + i\kappa_2 (|u|^2)_x u = 0 \]  \hspace{1cm} (1)

was firstly proposed by Wadati for describing the interactions between two-body and three-body on the condensates. The variable parameters \( \kappa_1 \) and \( \kappa_2 \) are constants. The two-body scattering lengths \( f(t) \) can be adjusted by Feshbach resonance. Neither \( \kappa_1 = 0 \) nor \( \kappa_2 = 0 \), equation (1) is suitable for Bose-Einstein condensates with higher densities. Reversely, if \( \kappa_1 = 0 \) and \( \kappa_2 = 0 \), equation (2) only for lower densities in Bose-Einstein condensates. This situation can be described by Gross-Pitaevskii equation [4]

\[ iu_t + u_{xx} + 2f(t)|u|^2 u + \frac{1}{2}\delta^2 x^2 u = 0. \]  \hspace{1cm} (2)

If \( f(t) \in \mathbb{R} \) and independent of \( t \), the equation (1) reduce to the Kundu-Eckhaus (KE) equation [5]

\[ iu_t + u_{xx} - 2\sigma |u|^2 u + 4\beta^2 |u|^4 u + 4i\beta \sigma (|u|^2)_x u = 0, \]  \hspace{1cm} (3)

with \( u(x,t) \) being the complex potential function of spatial \( x \) and temporal \( t \) \( u(x,t) : \mathbb{R}^2 \rightarrow \mathbb{C} \). This equation is called the defocusing KE equation as \( \sigma = 1 \) and focusing KE equation as \( \sigma = -1 \). The KE equation (3) was firstly put forward by Kundu to investigate the Landau-Lifshitz equations and derivative nonlinear Schrödinger type equations [5]. For the special case \( \beta = 0 \), the KE
equation (3) reduces to nonlinear Schrödinger equation. In recent years, the KE equation (3) has been investigated via different methods, for example, the long time asymptotic [6–9], the higher-order rogue wave solutions [10], rogue waves in a chaotic wave field [11], the Darboux transformation [12, 13], integrable discretizations [14]. Recently, the KE equation with zero boundary conditions was investigated by using Riemann-Hilbert method [8].

To our knowledge, there is still no work on investigating KE equation (3) with nonzero boundary conditions by using inverse scattering transformation or Riemann-Hilbert method. In this paper, we would like to investigate the soliton solution of focusing KE equation (3) with the following nonzero boundary conditions

$$\lim_{x \to \pm \infty} u(x, t) = u_\pm e^{2it(q_\mp^2 - 2\beta^2 q_\mp^4) + i\theta_\pm},$$

where $|u_\pm| = u_0 > 0$ and $u_0$ and $\theta_\pm$ are constants, $\theta_\pm$ is the arguments of $q_\pm$.

As we all known, the inverse scattering transform method plays an important role for finding the exact solutions of completely integrable systems [15]. The Riemann-Hilbert method as a new version of inverse scattering transform method streamline the research process and preferred by researchers. And Riemann-Hilbert method can be used to investigate the soliton solutions [16] and the long-time asymptotic of integrable systems [7]. Especially, in recent years, it has become a hot topic to investigate integrable systems with nonzero boundary conditions [17–28].

This work is organized as follows. In section 2, we analyze the spectral problem via introduce a transformation. And we introduced a appropriate Riemann surface for the single-valued function of the spectral parameter, that is $k$-plane mapped into $z$-plane. Section 3-5, we obtained the asymptotic, analyticity and symmetries of Jost solution and scattering matrix, which are used to get a Riemann-Hilbert problem in section 6. Section 7, we analyze the dis-
crete spectrum and residue conditions which are used to solve the Riemann-Hilbert problem. Section 8, we establish the connection between the solution of KE equation and the solution of Riemann-Hilbert problem. A formula of the \( N \)-soliton solution of KE equation is obtained by using the Riemann-Hilbert problem. A conclusion is given in section 9.

2 Spectral Analysis

It is well-known that the focusing KE equation (\( \sigma = -1 \)) admits Lax pair

\[
(\partial_x - U)\phi = 0, \quad (\partial_t - V)\phi = 0, \quad (5)
\]

where

\[
U = -ik\sigma_3 + i\beta|q|^2\sigma_3 + U, \quad U = \begin{pmatrix} 0 & u \\ -\pi & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

\[
V = -2ik^2\sigma_3 + 2kU + 2\beta|u|^2U + i \left(|u|^2 + 4\beta^2|u|\right)\sigma_3 - iU_x\sigma_3
\]

\[-\beta(UU_x - U_xU),
\]

where \( k \) is the spectral parameter and \( \pi \) denotes the complex conjugation of \( u \).

For convenience of using Riemann-Hilbert method, we first deal with Lax par (5) and the boundary condition (4). By making transformation

\[
q(x, t) = u(x, t)e^{-2i(\theta_0 - \beta^2\theta_0^4)},
\]

the focusing KE equation can be reduced to the form

\[
iq_t + q_{xx} + 2(|q|^2 + 2\beta^2|q|^4 - q_0^2 - 2\beta^2q_0^4)q - 4i\beta(|q|^2)_{x}q = 0, \quad \beta \in \mathbb{R}, \quad (6)
\]

and corresponding boundary condition (4) becomes

\[
\lim_{x \to \pm\infty} q(x, t) = q_{\pm} = q_0e^{i\beta x},
\]

where we have denote \( u_{\pm} \) and \( u_0 \) as \( q_{\pm} \) and \( q_0 \) respectively.
The equation (6) admits the following Lax pair

\begin{align}
(\partial_x - U)\phi &= 0, \\
(\partial_t - V)\phi &= 0,
\end{align}

where

\[
U = -ik\sigma_3 + i\beta |q|^2 \sigma_3 + Q, \quad Q = \begin{pmatrix} 0 & q \\ -q & 0 \end{pmatrix},
\]

\[
V = -2ik^2\sigma_3 + 2kQ + 2\beta |q|^2Q + i \left(|q|^2 + 4\beta^2 |q|^4 - q_0^2 - 2\beta^2 q_0^4\right) \sigma_3
\]

\[-iQ_x\sigma_3 - \beta (QQ_x - Q_x Q).
\]

Let \( x \to \pm \infty \) and by using boundary condition (4), we obtain the following asymptotic Lax pair

\begin{align}
(\partial_x - U_\pm)\phi &= 0, \quad U_\pm = \lim_{x \to \pm \infty} U = (-ik + i\beta q_0^2) \sigma_3 + Q_\pm, \\
(\partial_t - V_\pm)\phi &= 0, \quad V_\pm = \lim_{x \to \pm \infty} V = (-2ik^2 + 2i\beta^2 q_0^4) \sigma_3 + (2k + 2\beta q_0^2) Q_\pm,
\end{align}

where

\[
Q_\pm = \lim_{x \to \pm \infty} Q = \begin{pmatrix} 0 & q_\pm \\ -q_\pm & 0 \end{pmatrix}.
\]

It is obvious that \( V_\pm \) and \( U_\pm \) possess the linear relationship

\[
V_\pm = (2k + 2\beta q_0^2) U_\pm.
\]

The fundamental matrix solution of asymptotic spectral problem (8) can be obtained as

\[
\phi_\pm(x, t, k) = \Xi_\pm(k)e^{-i\theta(x, t, k)\sigma_3}, \quad k \neq \beta q_0^2 \pm iq_0,
\]

where

\[
\theta(x, t, k) = \lambda \left[x + (2k + 2\beta q_0^2)t\right], \quad \lambda = \sqrt{(k - \beta q_0^2)^2 + q_0^2},
\]

\[
\Xi_\pm = \begin{pmatrix} 1 & \frac{-iq_\pm}{\lambda + (k - \beta q_0^2)} \\ \frac{-iq_\pm}{\lambda + (k - \beta q_0^2)} & 1 \end{pmatrix}.
\]
Since $\lambda$ is doubly branched with branch points are $k = \beta q^2_0 \pm iq_0$, it is necessary to introduce a two-sheeted Riemann surface to such that $\lambda$ is a single-valued function on each sheet.

Denote $h = k - \beta q^2_0$ and let

$$h + iq_0 = \rho_1 e^{i\theta_1}, \quad h - iq_0 = \rho_2 e^{i\theta_2}, \quad -\frac{\pi}{2} \leq \theta_1, \theta_2 \leq \frac{3\pi}{2},$$

one can rewrite $\lambda$ on each sheet as

$$\lambda_I = \sqrt{\rho_1 \rho_2} e^{i(\frac{\theta_1 + \theta_2}{2})}, \quad \lambda_{II} = -\lambda_I = \sqrt{\rho_1 \rho_2} e^{i(\frac{\theta_1 + \theta_2}{2}) + i\pi}.$$

And the branch cut of the Riemann surface is the segment $[-iq_0, iq_0]$ in the complex $h$-plane.

Now we introduce a uniformization variable

$$z = \lambda + h,$$  \hspace{1cm} (10)

then its inverse transformation gives

$$\lambda = \frac{1}{2}(z + q^2_0/z), \quad h = \frac{1}{2}(z - q^2_0/z), \quad k = \frac{1}{2}(z - q^2_0/z) + \beta q^2_0.$$ \hspace{1cm} (11)

Further we can show the following relations between the Riemann surface, the $h$-plane and the $z$-plane.

- The region where $\text{Im}\lambda > 0$ come from the upper-half plane of the sheet-I and the lower-half plane of the sheet-II. The region where $\text{Im}\lambda < 0$ come from the upper-half plane of the sheet-II and the lower-half plane of the sheet-I.

- On the sheet-I, $z \to \infty$ as $h \to \infty$, and on the sheet-II, $z \to 0$ as $h \to \infty$.

- The real $\lambda$ (real $k$) axes is mapped into the real $z$ axes.

- The branch cut $[-iq_0, iq_0]$ is mapped into the circle $C_0$ of the radius $q_0$ in $z$-plane.
The sheet-I and sheet-II, except for the branch cut, are mapped into the exterior and the interior of \(C_0\), respectively.

The jump contour in the complex \(z\)-plane is denoted by \(\Sigma = \mathbb{R} \cup C_0\). The gray and white regions in Fig.1 denote \(D^+\) and \(D^-\), respectively

\[
D^+ = \{ z \in \mathbb{C} | \text{Im}\lambda = (|z|^2 - q_0^2) \text{Im}z > 0 \},
\]

\[
D^- = \{ z \in \mathbb{C} | \text{Im}\lambda = (|z|^2 - q_0^2) \text{Im}z < 0 \}.
\]

**Fig. 1.** (a) The complex \(k\)-plane, the branch cut, \(\text{Im}k > 0\)(gray) and \(\text{Im}k < 0\)(white).

(b) Sheet-I of Riemann surface, showing the branch cut (red) and the region where \(\text{Im}\lambda > 0\)(gray) and \(\text{Im}\lambda < 0\)(white). (c) Sheet-II of Riemann surface, showing the branch cut
(red) and the region where $\text{Im}\lambda > 0$ (gray) and $\text{Im}\lambda < 0$ (white). (d) The complex $z$-plane, showing the region $D^+$ where $\text{Im}\lambda > 0$ (gray), the region $D^-$ where $\text{Im}\lambda < 0$ (white), the orientation of the contours for the Riemann-Hilbert problem and the discrete spectrums (zero points of $s_{11}(z)$ and $s_{22}(z)$).

Based on the above results, we can rewrite the fundamental matrix solution (9) as

$$
\phi_\pm(x, t, z) = \Xi_\pm(z) e^{-i\theta(x, t, z)\sigma_3}, \quad z \neq iq_0
$$

where

$$
\begin{align*}
\Xi_\pm(z) &= \left( \frac{1}{z - iq_\pm} \right) = 1 - i\frac{z}{z} \sigma_3 Q_\pm, \\
\theta(x, t, z) &= \frac{1}{2} \left( z + \frac{q_0^2}{z} \right) \left( x + \left( z - \frac{q_0^2}{z} \right) + 4\beta q_0^2 \right) t.
\end{align*}
$$

Direct calculation shows that

$$
\det \Xi_\pm = 1 + \frac{q_0^2}{z^2} \neq 0, \quad \Xi_\pm^{-1} = \frac{1}{\gamma} (1 + i\frac{z}{z} \sigma_3 Q_\pm).
$$

### 3 Jost Solutions

The Lax pair (7) can be rewrite as

$$
\begin{align*}
(\partial_x - U_\pm - \Delta \hat{Q}_\pm) \phi &= 0, \\
(\partial_t - V_\pm - \Delta \hat{R}_\pm) \phi &= 0,
\end{align*}
$$

where

$$
\begin{align*}
\Delta \hat{Q}_\pm &= i\beta \left( |q|^2 - q_0^2 \right) \sigma_3 + \Delta Q_\pm, \quad \Delta \hat{R}_\pm = \hat{R} - \hat{R}_\pm, \\
\Delta Q_\pm &= Q - Q_\pm, \quad \hat{R}_\pm = 2kQ_\pm + 2\beta q_0^2 Q_\pm + 2i\beta^2 q_0^4 \sigma_3, \\
\hat{R} &= 2kQ + 2\beta |q|^2 Q + i \left( |q|^2 + 4\beta^2 |q|^4 - q_0^2 - 2\beta^2 q_0^4 \right) \sigma_3 - iQ_x \sigma_3 \\
&\quad - \beta \left( QQ_x - Q_x Q \right).
\end{align*}
$$

Now, one can define the Jost solutions as the simultaneous solutions of Lax pair (7) such that

$$
\Psi_\pm(x, t, z) \sim \Xi_\pm(z) e^{-i\theta(x, t, z)\sigma_3}, \quad z \in \Sigma, \quad x \to \pm\infty,
$$

8
and the modified Jost solutions
\[ e^{-\int_{-\infty}^{x} i\beta(|q|^2-q_0^2)dy} \xi_{\pm}(x, t, z) e^{-\int_{x}^{\infty} i\beta(|q|^2-q_0^2)dy} \sigma_3 = \Psi_{\pm}(x, t, z) e^{i\beta(x, t, z)\sigma_3} = \eta_{\pm}(x, t, z), \] (14)

Then the Lax pair (12) can be rewritten as
\[ \left( \Xi_{\pm}^{-1} \eta_{\pm}(x, t, z) \right)_x = -i\lambda \left[ \sigma_3, \Xi_{\pm}^{-1} \eta_{\pm}(x, t, z) \right] + \Xi_{\pm}^{-1} \Delta \hat{Q}_{\pm} \eta_{\pm}(x, t, z), \]
\[ \left( \Xi_{\pm}^{-1} \eta_{\pm}(x, t, z) \right)_t = -i\lambda \left( 2k + 2\beta q_0^2 \right) \left[ \sigma_3, \Xi_{\pm}^{-1} \eta_{\pm}(x, t, z) \right] + \Xi_{\pm}^{-1} \Delta \hat{R}_{\pm} \eta_{\pm}(x, t, z). \] (15)

It is easily known that Lax pair (15) can be written in full derivative form
\[ d \left( e^{i\beta(x, t, z)\sigma_3} \Xi_{\pm}^{-1} \eta_{\pm}(x, t, z) \right) = e^{i\beta(x, t, z)\sigma_3} \left( V_1 dx + V_2 dt \right), \] (16)
where
\[ V_1 = \Xi_{\pm}^{-1} \Delta \hat{Q}_{\pm} \eta_{\pm}(x, t, z), \]
\[ V_2 = \Xi_{\pm}^{-1} \Delta \hat{R}_{\pm} \eta_{\pm}(x, t, z). \]

We can obtain the Volterra integral equations
\[ \mu_{-}(x, t, z) = \Xi_{-} + \int_{-\infty}^{x} e^{-\int_{-\infty}^{x'} i\beta(|q|^2-q_0^2)dy} \Xi_{-} e^{i\lambda(x'-x)\sigma_3} dx', \]
\[ \Xi_{-}^{-1} \Delta \hat{Q}_{-} e^{-\int_{x}^{\infty} i\beta(|q|^2-q_0^2)dy} \mu_{-} dx', \]
\[ \mu_{+}(x, t, z) = \Xi_{+} - \int_{x}^{\infty} e^{-\int_{x}^{x'} i\beta(|q|^2-q_0^2)dy} \Xi_{+} e^{i\lambda(x'-x)\sigma_3} dx', \]
\[ \Xi_{+}^{-1} \Delta \hat{Q}_{+} e^{i\beta(x'-(|q|^2-q_0^2)dy)} \mu_{+} dx'. \] (17)

It can be shown that the first column of \( \mu_{-} \) is analytically extended to \( D^+ \) and the second column of \( \mu_{-} \) is analytically extended to \( D^- \). Similarly, the first column of \( \mu_{+} \) can be analytically extended to \( D^- \) and the second column of \( \mu_{+} \) can be analytically extended to \( D^+ \). It can be summarized as follows
\[ D^+: \mu_{-, 1}, \mu_{+, 2}; \]
\[ D^-: \mu_{-, 2}, \mu_{+, 1}, \]
where the subscripts ‘1’ and ‘2’ identify the columns of matrix.
Consider the Laurent expansion of $\Xi^{-1}_\pm \eta_{\pm}(x, t, z)$ in system (15)

$$
\Xi^{-1}_\pm \eta_{\pm}(x, t, z) = \alpha^{(0)} + \frac{\alpha^{(1)}}{z} + \frac{\alpha^{(2)}}{z^2} + O\left(\frac{1}{z^3}\right), \quad z \to \infty, \quad \text{(18a)}
$$

$$
\Xi^{-1}_\pm \eta_{\pm}(x, t, z) = \eta^{(0)} + \eta^{(1)} z + \eta^{(2)} z^2 + O\left(z^3\right), \quad z \to 0. \quad \text{(18b)}
$$

where $\alpha^{(0)}, \alpha^{(1)}, \alpha^{(2)}, \ldots$ and $\eta^{(0)}, \eta^{(1)}, \eta^{(2)}, \ldots$ are independent of $z$. Substituting the above expansion into (15) and comparing the coefficients of $z^n$ ($n = 0, \pm 1, \pm 2, \ldots$), we obtain the following equations

- $\left[\sigma_3, \alpha^{(0)}\right] = 0,$
- $\alpha^{(0)} = \frac{i}{2} \left[\sigma_3, \alpha^{(1)}\right] + i \beta \left(|q|^2 - q_0^2\right) \sigma_3 \alpha^{(0)} + \Delta Q_{\pm} \alpha^{(0)},$
- $\alpha^{(1)} = -\frac{i}{2} q_0^2 \left[\sigma_3, \alpha^{(0)}\right] - \frac{i}{2} q_0^2 \left[\sigma_3, \alpha^{(1)}\right] + \beta \left(|q|^2 - q_0^2\right) \left[\sigma_3, \sigma_3 Q_{\pm}\right] \alpha^{(0)} + i \left[\sigma_3 Q_{\pm}, \Delta Q_{\pm}\right] \alpha^{(0)} + i \beta \left(|q|^2 - q_0^2\right) \sigma_3 \alpha^{(1)} + \Delta Q_{\pm} \alpha^{(1)},$

- $\left[\sigma_3, \eta^{(0)}\right] = 0,$
- $\eta^{(0)} = -\frac{i}{2} q_0^2 \left[\sigma_3, \eta^{(1)}\right] + \frac{1}{q_0^2} \beta \left(|q|^2 - q_0^2\right) \sigma_3 \eta^{(0)} + \frac{1}{q_0^2} \sigma_3 Q_{\pm} \Delta Q_{\pm} \sigma_3 Q_{\pm} \eta^{(0)},$
- $\eta^{(1)} = -\frac{i}{2} q_0^2 \left[\sigma_3, \eta^{(0)}\right] - \frac{i}{2} q_0^2 \left[\sigma_3, \eta^{(1)}\right] + \frac{1}{q_0^2} \beta \left(|q|^2 - q_0^2\right) \left[\sigma_3, \sigma_3 Q_{\pm}\right] \eta^{(0)} + \frac{i}{q_0^2} \left[\sigma_3 Q_{\pm}, \Delta Q_{\pm}\right] \eta^{(0)} + \frac{1}{q_0^2} \sigma_3 Q_{\pm} \Delta Q_{\pm} \sigma_3 Q_{\pm} \eta^{(1)}$

from which, we can derive the following results

- $\alpha^{(0)} = e^{-\int_{-\infty}^{t_0} \beta \left(|q|^2 - q_0^2\right) dz_{\sigma_3}},$
- $\alpha^{(1)} = -i \sigma_3 \Delta Q_{\pm} \alpha^{(0)}$
- $\alpha^{(1)} = i \left[\sigma_3 Q_{\pm}, \Delta Q_{\pm}\right] \alpha^{(0)} + i \beta \left(|q|^2 - q_0^2\right) \sigma_3 \alpha^{(1)} - i \Delta Q_{\pm} \sigma_3 \Delta Q_{\pm} \alpha^{(0)}$

and

- $\eta^{(0)} = e^{-\int_{-\infty}^{t_0} \beta \left(|q|^2 - q_0^2\right) dz_{\sigma_3}},$
- $\eta^{(1)} = \frac{i}{q_0^2} \left[\sigma_3 Q_{\pm}, \Delta Q_{\pm}\right] \eta^{(0)} - i \beta \left(|q|^2 - q_0^2\right) \sigma_3 \eta^{(1)} + \frac{1}{q_0^2} \sigma_3 Q_{\pm} \Delta Q_{\pm} \sigma_3 Q_{\pm} \eta^{(0)}.$
And the asymptotic of the modified Jost solutions for $z \to \infty$ and $z \to 0$ are respectively derived as

$$
\mu_{\pm} = I - \frac{i}{z} e^{-\int_{-\infty}^{x} \beta (|q|^2 - q_0^2) dy \sigma_3} \sigma_3 Q_{\pm} + \frac{1}{z} e^{\int_{-\infty}^{x} \beta (|q|^2 - q_0^2) dy \sigma_3} \sigma_3 \hat{Q}_{\pm} + O\left(\frac{1}{z^2}\right), \quad z \to \infty, \quad (19)
$$

$$
\mu_{\pm} = -\frac{i}{z} \sigma_3 Q_{\pm} + O\left(1\right), \quad z \to 0.
$$

4 Scattering Matrix and Asymptotic

It is easy to check that $\text{tr} \mathcal{U} = \text{tr} \mathcal{V} = 0$, then by using Abel formula, we have

$$(\det \Psi_{\pm})_x = (\det \Psi_{\pm})_t = 0,$$

which implies that

$$
(\det \Psi_{\pm})_x = (\det \Psi_{\pm})_t = 0,
$$

by using asymptotic at $x \to \pm \infty$. Since $\Psi_+$ and $\Psi_-$ are the fundamental solutions of the spectral problem $[7]$, they satisfy the following linear relationship

$$
\Psi_+(z) = \Psi_-(z) S(z), \quad z \in \Sigma \setminus \{\pm i q_0\}, \quad (21)
$$

where $S(z) = (s_{ij}(z))_{2 \times 2}$ is called spectral matrix. From $[20]$, we know that $\det S = 1$. The relation formula $[21]$ can be expanded as

$$
\Psi_{+,1} = s_{11} \Psi_{-,1} + s_{21} \Psi_{-,2}, \quad \Psi_{+,2} = s_{12} \Psi_{-,1} + s_{22} \Psi_{-,2}. \quad (22)
$$

The reflection coefficients are defined as

$$
\rho(z) = \frac{s_{21}}{s_{11}}, \quad \tilde{\rho}(z) = \frac{s_{12}}{s_{22}}. \quad (23)
$$

According to $[22]$ the scattering coefficients have the following Wronskian representations

$$
\begin{align*}
{s_{11}} & = \frac{\text{Wr} (\Psi_{+,1}, \Psi_{-,2})}{\gamma}, \quad \gamma = \text{Wr} (\Psi_{+,2}, \Psi_{-,2}), \quad (24a) \\
s_{21} & = \frac{\text{Wr} (\Psi_{-,1}, \Psi_{+,1})}{\gamma}, \quad s_{22} = \text{Wr} (\Psi_{-,2}, \Psi_{+,2}). \quad (24b)
\end{align*}
$$
The equation (24) implies that $s_{11}$ is analytic in $D^-$ and $s_{22}$ is analytic in $D^+$. However, $s_{12}$ and $s_{22}$ are just continuous on $\Sigma$.

**Proposition 1.** The asymptotic behaviors of the scattering matrix $S(z)$ are given as follows

$$S(z) = I + O\left(\frac{1}{z}\right), \quad z \to \infty, \quad (25a)$$

$$S(z) = \text{diag} \left( \frac{q_-}{q_+}, q_+/q_- \right) + O\left(z\right), \quad z \to 0. \quad (25b)$$

**Proof.** By using (14), (24) and the asymptotic behaviors of $\mu_{\pm}$, we can prove that

As $z \to \infty$,

$$s_{11} = \frac{\text{Wr} \left( \Psi_+, \Psi_- \right)}{\gamma} = \frac{1}{1 + \frac{q_0}{z^2}} \det \begin{pmatrix} \mu_{+,11} & \mu_{-,12} \\ \mu_{+,21} & \mu_{-,22} \end{pmatrix}$$

$$= \det \left( 1 + O\left(\frac{1}{z}\right) \quad O\left(\frac{1}{z}\right) \quad O\left(\frac{1}{z}\right) \right) \left(1 - \frac{q_0^2}{z^2} + \frac{q_0^4}{z^4} - \cdots\right)$$

$$= (1 + O\left(\frac{1}{z}\right))(1 - \frac{q_0^2}{z^2} + \cdots)$$

$$= 1 + O\left(\frac{1}{z}\right).$$

As $z \to 0$,

$$s_{11} = \frac{\text{Wr} \left( \Psi_+, \Psi_- \right)}{\gamma} = \frac{1}{1 + \frac{q_0}{z^2}} \det \begin{pmatrix} \mu_{+,11} & \mu_{-,12} \\ \mu_{+,21} & \mu_{-,22} \end{pmatrix}$$

$$= \det \left( O\left(1\right) \quad \frac{z}{z^2 q_+} \quad O\left(1\right) \right) \left(\frac{z}{q_0}^2 \left(1 - \frac{z^2}{q_0^2} + \cdots\right)\right)$$

$$= O\left(1\right) + \frac{z}{z^2} \frac{q_+}{q_0} \left(\frac{z}{q_0}^2 - \frac{z^4}{q_0^4} + \cdots\right)$$

$$= \frac{q_+}{q_0} + \frac{q_-}{q_+} + O\left(z\right).$$

The asymptotic behaviors of $s_{22}$, $s_{12}$, and $s_{21}$ can also be derived by the similar way. \qed
5 Symmetries

For the focusing KE equation with nonzero boundary conditions, the Jost functions $\Psi_\pm(z)$ and spectral matrix $S(z)$ possess two kinds of symmetries.

5.1 First Symmetry

Here we consider the symmetries between two points $z \mapsto \overline{z}$ (upper/lower half plane).

**Proposition 2.** For $z \in \Sigma$,

(1) The Jost solutions satisfy the symmetries

\[
\Psi_\pm(z) = \sigma_2 \overline{\Psi_\pm(\overline{z})} \sigma_2, \\
\Psi_{\pm,1}(z) = i \sigma_2 \overline{\Psi_{\pm,2}(\overline{z})}, \\
\Psi_{\pm,2} = -i \sigma_2 \overline{\Psi_{\pm}(\overline{z})}.
\]

(2) The scattering matrix satisfy the symmetries

\[
S(z) = \sigma_2 \overline{S(\overline{z})} \sigma_2, \\
s_{11}(z) = \overline{s_{22}(\overline{z})}, \\
s_{12}(z) = -\overline{s_{21}(\overline{z})}.
\]

(3) The reflection coefficient satisfy the symmetries

\[
\rho(z) = -\overline{\rho(\overline{z})}.
\]

**Proof.** (1) The $U$ and $V$ in the Lax pair with the following symmetries on $z$-plane

\[
\overline{U(\overline{z})} = \sigma_2 U(z) \sigma_2, \quad \overline{V(\overline{z})} = \sigma_2 V(z) \sigma_2,
\]

by which, we can show that

\[
(\sigma_2 \overline{\Psi_\pm(\overline{z})} \sigma_2)_x = U(z) (\sigma_2 \overline{\Psi_\pm(\overline{z})} \sigma_2), \\
(\sigma_2 \overline{\Psi_\pm(\overline{z})} \sigma_2)_t = V(z) (\sigma_2 \overline{\Psi_\pm(\overline{z})} \sigma_2),
\]

13
Further, by suing asymptotic

$$
\Psi_{\pm}(z), \sigma_2 \overline{\Psi}_{\pm}(z) \sigma_2 \sim \Xi_{\pm}(z) e^{-i\lambda[z + (2k + 2)q_0]} \sigma_1, \ x \to \infty,
$$

we obtain the symmetry [26].

(2) On the basis of [21],

$$
S(z) = \Psi^{-1}_{-}(z) \Psi_{+}(z)
= \sigma_2 \overline{\Psi}_{-}(z)^{-1} \sigma_2 \overline{\Psi}_{+}(z) \sigma_2,
= \sigma_2 \overline{\Psi}_{-}(z)^{-1} \Psi_{+}(z) \sigma_2,
= \sigma_2 \overline{S}(z) \sigma_2.
$$

Then (30) can be obtained.

(3)

$$
\rho(z) = \frac{s_{21}(z)}{s_{11}(z)} = -\frac{s_{12}(z)}{s_{22}(z)} = -\rho(z).
$$

\[\square\]

5.2 Second Symmetry

Here we consider the symmetries between two points $z \mapsto -\frac{q_2^2}{2}$ (outside/inside of the circle $C_0$).

**Proposition 3.** For $z \in \Sigma$,

(1) The Jost solutions satisfy the following symmetries

$$
\Psi_{\pm}(z) = -\frac{i}{z} \Psi_{\pm} \left( -\frac{q_0^2}{z} \right) \sigma_3 Q_{\pm}, \quad (29a)
$$

$$
\Psi_{\pm,1}(z) = -\frac{i}{z} \eta_{\pm} \Psi_{\pm,2} \left( -\frac{q_0^2}{z} \right), \quad (29b)
$$

$$
\Psi_{\pm,2}(z) = -\frac{i}{z} \eta_{\pm} \Psi_{\pm,1} \left( -\frac{q_0^2}{z} \right), \quad (29c)
$$
(2) The scattering matrix satisfies the symmetries

\[
S(z) = (\sigma_3 Q_-)^{-1} S\left(-\frac{q_0^2}{z}\right) \sigma_3 Q_+,
\]

(30a)

\[
s_{11}\left(-\frac{q_0^2}{z}\right) = \frac{q_-}{q_+} s_{22}(z),
\]

(30b)

\[
s_{12}\left(-\frac{q_0^2}{z}\right) = \frac{q_+}{q_-} s_{21}(z).
\]

(30c)

(3) The reflection coefficient satisfies the symmetries

\[
\rho\left(-\frac{q_0^2}{z}\right) = -\frac{q_-}{q_+} \rho(z).
\]

(31)

**Proof.** The process of proof is similar with the Proposition 2. \qed

6 Riemann-Hilbert Problem

Based on the analytical and asymmetry properties of eigenfunctions \(\mu_\pm\) and \(S(z)\), we derive Riemann-Hilbert problem associated with KE equation with nonzero boundary conditions.

**Proposition 4.** Define sectionally meromorphic matrix

\[
M(x,t,z) = \begin{cases} 
M^+(x,t,z) = \left( \begin{array}{cc} \mu_{-,1} & \mu_{+,2} \\ \frac{s_{22}}{s_{11}} & \mu_{-,2} \end{array} \right), & z \in D^+, \\
M^-(x,t,z) = \left( \begin{array}{cc} \mu_{+,1} & \mu_{-,2} \\ \frac{s_{11}}{s_{22}} & \mu_{-,1} \end{array} \right), & z \in D^-
\end{cases}
\]

(32)

then we have the following Riemann-Hilbert problem

- Analyticity: \(M(x,t,z)\) is meromorphic in \(D^+ \cup D^-\) and has simple poles.

- Jump condition:

\[
M^-(x,t,z) = M^+(x,t,z) (I - G(x,t,z)), \quad z \in \Sigma,
\]

where

\[
G(x,t,z) = e^{-i\theta \sigma_3} \left( \begin{array}{cc} \rho \hat{\rho} & \hat{\rho} \\ -\rho & 0 \end{array} \right) e^{i\theta \sigma_3}.
\]
- Asymptotic behavior:

\[ M^\pm(x, t, z) = I + O\left(\frac{1}{z}\right), \quad z \to \infty \]

\[ M^\pm(x, t, z) = -\frac{i}{z} \sigma_3 Q_- + O(1), \quad z \to 0 \]

**Proof.** The analyticity can be derived from (22) and the analyticity of \( \mu^\pm \).

From (22), we can known that

\[
\begin{aligned}
\Psi_{+, 1}^{s_1} &= (1 - \rho \tilde{\rho}) \Psi_{-, 1} + \rho \Psi_{+, 2}^{s_2}, \\
\Psi_{-, 2} &= -\rho \Psi_{+, 2}^{s_2},
\end{aligned}
\]

which lead to

\[
\begin{pmatrix}
\Psi_{+, 1}^{s_1} \\
\Psi_{-, 2}
\end{pmatrix} = 
\begin{pmatrix}
\Psi_{-, 1} \\
\Psi_{+, 2}^{s_2}
\end{pmatrix} 
\begin{pmatrix}
1 - \rho \tilde{\rho} & -\rho \\
\rho & 1
\end{pmatrix}.
\]

Then the jump condition can be derived as

\[
M^-(x, t, z) = \begin{pmatrix} \mu_{+, 1}^{s_1} & \mu_{-, 2} \\ \mu_{-, 1} & \mu_{+, 2}^{s_2} \end{pmatrix} 
= \begin{pmatrix} \mu_{-, 1} & \mu_{+, 2}^{s_2} \end{pmatrix} e^{-i\theta \sigma_3} \begin{pmatrix} 1 - \rho \tilde{\rho} & -\rho \\ \rho & 1 \end{pmatrix} e^{i\theta \sigma_3}
= M^+(x, t, z) (I - G(x, t, z)).
\]

Now we proof the asymptotic behavior. From (32), \( M^+(x, t, z) \) with the following form

\[
M^+(x, t, z) = \begin{pmatrix} \mu_{-, 1} & \mu_{+, 2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{s_{22}} \end{pmatrix}.
\]

As \( z \to \infty \),

\[
M^+(x, t, z) = \left( I + O\left(\frac{1}{z}\right)\right) \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{1 + O(\frac{1}{z})} \end{pmatrix}
= I + O\left(\frac{1}{z}\right).
\]

As \( z \to 0 \),

\[
M^+(x, t, z) = \begin{pmatrix} O(1) & -\frac{i}{z} q_+ + O(1) \\ -\frac{i}{z} q_- + O(1) & O(1) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\frac{1}{z} + O(z)} \end{pmatrix}
= -\frac{i}{z} \sigma_3 Q_- + O(1).
\]

The asymptotic behavior of \( M^-(x, t, z) \) can be derived in the similar way. \( \square \)
7 Discrete Spectrum and Residue Conditions

If \( s_{22}(z_n) = 0 \), we can derive from the forth equation of (24) that the eigenfunctions \( \Psi_{+2}(x, t, z_n) \) and \( \Psi_{-1}(x, t, z_n) \) must be proportional

\[
\Psi_{+2}(x, t, z_n) = b_n \Psi_{-1}(x, t, z_n),
\]

where \( b_n \) is independent of \( x, t \) and \( z \). Let \( s_{22} \) has a finite number of simple zeros \( z_1, z_2, \cdots, z_N \) in \( D^+ \cap \{ z \in \mathbb{C} \mid \text{Im} z > 0 \} \). According to the symmetry properties of \( S(z) \), we have that

\[
s_{22}(z_n) = 0 \iff s_{11}(\overline{z_n}) = 0 \iff s_{11}(\frac{-q_0^2}{\overline{z_n}}) = 0 \iff s_{22}(\frac{-q_0^2}{\overline{z_n}}) = 0.
\]

So the discrete spectrum accumulate the set

\[
\mathcal{Z} = \{ z_n, \overline{z_n}, \frac{-q_0^2}{z_n}, \frac{-q_0^2}{\overline{z_n}} \}_{n=1}^N.
\]

Now we study the residue condition that is very important to solve the Riemann-Hilbert problem. The equation (33) can be rewritten as

\[
\mu_{+2}(z_n) = b_n \mu_{-1}(z_n) e^{-2i\theta(z_n)}.
\]

Then

\[
\text{Res}_{z=z_n} \left[ \frac{\mu_{+2}(z)}{s_{22}(z)} \right] = \frac{b_n \mu_{-1}(z_n) e^{-2i\theta(z_n)} / s_{22}(z_n)}{s_{22}(z_n)} = A_n \mu_{-1}(z_n) e^{-2i\theta(z_n)},
\]

where \( A_n = \frac{b_n}{s_{22}(z_n)} \).

If \( s_{11}(\overline{z_n}) = 0 \), similarly, the first equation of (24) implies that the eigenfunctions \( \Psi_{+1}(x, t, \overline{z_n}) \) and \( \Psi_{-2}(x, t, \overline{z_n}) \) must be proportional

\[
\Psi_{+1}(x, t, \overline{z_n}) = \tilde{b}_n \Psi_{-2}(x, t, \overline{z_n}),
\]

which also can be rewritten as

\[
\mu_{+1}(\overline{z_n}) = \tilde{b}_n \mu_{-2}(\overline{z_n}) e^{2i\theta(\overline{z_n})}.
\]
And derived that

\[
\text{Res}_{z=\tau_n} \frac{\mu_{+1}(z)}{s_{11}(z)} = \frac{\tilde{b}_n \mu_{-2}(\tau_n) e^{-2i\theta(\tau_n)}}{s_{11}(\tau_n)} = \tilde{A}_n \mu_{-2}(\tau_n) e^{2i\theta(\tau_n)},
\]

where \( \tilde{A}_n = \frac{\tilde{b}_n}{s_{11}(\tau_n)} \). And it is easy to show that

\[
\tilde{A}_n = -\tilde{\mathcal{A}}_n.
\]

Combining (29), (33) and (35) we have the following relations

\[
\begin{align*}
\Psi_{+,2}( -\frac{q_0^2}{z_n}) &= \frac{q-\tilde{b}_n}{q_+} \Psi_{-,1}( -\frac{q_0^2}{z_n}), \quad (38a) \\
\Psi_{+,1}( -\frac{q_0^2}{z_n}) &= \frac{\tilde{q}-q_+}{q_+} \Psi_{-,2}( -\frac{q_0^2}{z_n}). \quad (38b)
\end{align*}
\]

Using the symmetries of \( s_{ij} \), we have results

\[
\begin{align*}
\text{Res}_{z=-\frac{q_0^2}{z_n}} \frac{\mu_{+2}(z)}{s_{22}(z)} &= A_{N+n} \mu_{-1}( -\frac{q_0^2}{z_n}) e^{-2i\theta( -\frac{q_0^2}{z_n})}, \quad (39a) \\
\text{Res}_{z=-\frac{q_0^2}{z_n}} \frac{\mu_{+1}(z)}{s_{11}(z)} &= \tilde{A}_{N+n} \mu_{-2}( -\frac{q_0^2}{z_n}) e^{2i\theta( -\frac{q_0^2}{z_n})}. \quad (39b)
\end{align*}
\]

where

\[
A_{N+n} = \frac{q-\tilde{q}}{q_+} \tilde{A}_n, \quad \tilde{A}_{N+n} = \frac{\tilde{q}-q_+}{q_-} A_n,
\]

and \( \tilde{A}_{N+n} = -\tilde{A}_{N+n} \).

### 8 N-soliton solutions of the KE equation

To obtain solution of KE equation with nonzero boundary conditions, we should establish the connection between the solution of KE equation and the Riemann-Hilbert problem.

#### 8.1 Reconstruction formula

To solve the above Riemann-Hilbert problem, it is necessary to regularize it by subtracting out the asymptotic and the pole contributions. For convenient, we
define $\zeta_n = z_n$ and $\zeta_{N+n} = -\frac{z_0}{z_n}$, $(n = 1, \cdots, N)$ and

$$M^- - I + \frac{i}{z} \sigma_3 Q_- = -\sum_{n=1}^{2N} \frac{\text{Res } M^-}{z - \zeta_n} - \sum_{n=1}^{2N} \frac{\zeta_n}{z - \zeta_n}$$

$$= M^+ - I + \frac{i}{z} \sigma_3 Q_- - \sum_{n=1}^{2N} \frac{\text{Res } M^+}{z - \zeta_n} - \sum_{n=1}^{2N} \frac{\zeta_n}{z - \zeta_n} - M^+ G \tag{41}$$

The left-hand side of this equation is analytic in $D^-$ and is $O\left(\frac{1}{z}\right)$ as $z \to \infty$, and the sum of the first four terms of the right-hand side is analytic in $D^+$ and is $O\left(\frac{1}{z}\right)$ as $z \to \infty$. The asymptotic behavior of the off-diagonal scattering coefficients implies that $G(x, t, z)$ is $O\left(\frac{1}{z}\right)$ as $z \to \pm \infty$ and $O(z)$ as $z \to 0$ along the real axis.

Now we introduce the Cauchy projectors $P_\pm$ over $\Sigma$

$$P_\pm[f](z) = \frac{1}{2\pi i} \int_{\Sigma} \frac{f(\zeta)}{\zeta - (z \pm i0)} d\zeta,$$

where $\int_{\Sigma}$ denotes the integral along the oriented contour shown in Fig.1, and the notation $z \pm i0$ indicates that when $z \in \Sigma$, the limit is taken from the left(right) of it. Now recall Plemelj’s formulae: if $f^\pm$ are analytic in $D^\pm$ and are $O\left(\frac{1}{z}\right)$ as $z \to \infty$, one has $P^\pm f^\pm = \pm f^\pm$ and $P^+ f^- = P^- f^+ = 0$. Applying $P^+$ and $P^-$ to (41), then we have

$$M(x, t, z) = I - \frac{i}{z} \sigma_3 Q_- + \sum_{n=1}^{2N} \frac{\text{Res } M^-}{z - \zeta_n} + \sum_{n=1}^{2N} \frac{\zeta_n}{z - \zeta_n}$$

$$+ \frac{1}{2\pi i} \int_{\Sigma} \frac{M^+(x, t, \zeta)G(x, t, \zeta)}{\zeta - z} d\zeta, \quad z \in \mathbb{C}\backslash\Sigma. \tag{42}$$

From (42), we deriving the asymptotic behaviors of $M^\pm(x, t, z)$ as $z \to \infty$ as

$$M(x, t, z) = I + \frac{i}{z} \left\{ -\sigma_3 Q_- + \sum_{n=1}^{2N} \text{Res } M^- + \sum_{n=1}^{2N} \frac{\zeta_n}{z - \zeta_n} \right\}$$

$$+ \frac{1}{2\pi i} \int_{\Sigma} \frac{M^+(x, t, \zeta)G(x, t, \zeta) d\zeta}{\zeta - z} + O\left(\frac{1}{z}\right), \tag{43}$$

Comparing the (1,2) element of (43) with (19). Then we obtain the formula of
the potential \( q(x, t) \)

\[
q(x, t) = e^{2 \int_{-\infty}^{x} i \beta (|q|^2 - q_0^2) dy} \left[ q_+ + i \left( \sum_{n=1}^{2N} \text{Res} M^+_{\zeta_n} \right) - \frac{1}{2\pi} \int_\Sigma (M^+(x, t, \zeta) G(x, t, \zeta))_{1,2} d\zeta \right].
\] (44)

### 8.2 N-soliton solutions

In this subsection we consider the reflectionless potentials. In this case, the reflection coefficient \( \rho(z) = 0 \), the potential formula (44) reduced as

\[
q(x, t) = e^{2 \int_{-\infty}^{x} i \beta (|q|^2 - q_0^2) dy} \left[ q_+ + i \sum_{n=1}^{2N} A_n e^{-2i\theta(\zeta_n)} \mu_{-,11}(\zeta_n) \right].
\] (45)

For convenient, we introduce the quantities

\[
a_j(x, t, \zeta) = \tilde{A}_j \frac{e^{2i\theta(x, t, \zeta_j)}}{z - \zeta_j}, \quad j = 1, \cdots, 2N.
\]

Then from (42), we can derive the results

\[
\mu_{-,12}(\zeta_j) = -\frac{i q_-}{\zeta_j} + \sum_{k=1}^{2N} \frac{A_k e^{-2i\theta(\zeta_k)}}{\zeta_j - \zeta_k} \mu_{-,11}(\zeta_k) = -\frac{i q_-}{\zeta_j} - \sum_{k=1}^{2N} \bar{\pi}_k(\zeta_j) \mu_{-,11}(\zeta_k)
\] (46a)

\[
\mu_{-,11}(\zeta_n) = 1 + \sum_{j=1}^{2N} \frac{\tilde{A}_j e^{2i\theta(\zeta_j)}}{\zeta_n - \zeta_j} \mu_{-,12}(\zeta_j) = 1 + \sum_{j=1}^{2N} a_j(\zeta_n) \mu_{-,12}(\zeta_j).
\] (46b)

Substituting the equation into the second one gives

\[
\mu_{-,11}(\zeta_n) = 1 - i q_- \sum_{j=1}^{2N} \frac{a_j(\zeta_n)}{\zeta_j} - \frac{2N}{2} \sum_{j=1}^{2N} a_j(\zeta_n) \bar{\pi}_k(\zeta_j) \mu_{-,11}(\zeta_k).
\] (47)

We write this system in matrix form. Let

\[
X = (X_1, \cdots, X_{2N})^T, \quad B = (B_1, \cdots, B_{2N})^T,
\]

where

\[
X_N = \mu_{-,11}(\zeta_n), \quad B_n = 1 - i q_- \sum_{j=1}^{2N} \frac{a_j(\zeta_n)}{\zeta_j}, \quad n = 1, \cdots, 2N,
\]
and defining the $2N \times 2N$ matrix $A = (A_{n,k})$, where

$$A_{n,k} = \sum_{j=1}^{2N} a_j(\zeta_n)\overline{a}_k(\zeta_j), \quad n,k = 1, \cdots, 2N, \quad (48)$$

the system can be rewritten as $MX = B$, where $M = I + A = (M_1, \cdots, M_{2N})$.

The system is simply

$$X_n = \frac{\det M_n^{ext}}{\det M}, \quad n = 1, \cdots, 2N, \quad (49)$$

Finally, upon substituting $X_1, \cdots, X_{2N}$ into the reconstruction formula (45), the $N$-soliton solution of KE equation can be written compactly as

$$q(x,t) = e^{2 \int_{-\infty}^{x} i\beta(|r|^2 - q_0^2)dy}r(x,t), \quad (50)$$

where

$$r(x,t) = q_- - i\frac{\det M^{aug}}{\det M}. \quad (51)$$

and the augmented $(2N+1) \times (2N+1)$ matrix $M^{aug}$ is given by

$$M^{aug} = \begin{pmatrix} 0 & \mathbf{Y}^t \\ \mathbf{B} & M \end{pmatrix}, \quad \mathbf{Y} = (Y_1, \cdots, Y_{2N})^T,$$

$$Y_n = A_n e^{-2i\theta(x,t,\zeta_n)}, \quad n = 1, \cdots, 2N. \quad (52)$$

**8.3 one-Soliton solutions**

In this subsection we mainly consider the one-soliton solution for which the reflection coefficient $\rho = 0$. Take $q_0 = 1$, eigenvalue $z_1 = i\chi$ ($\chi=$constant value and $\chi > 1$) and $\tilde{A}_1 = e^{\alpha - iy}$ $(\alpha, \gamma \in \mathbb{R})$. From the $N$-soliton solutions formula, we can obtain the soliton solution

$$q_1(x,t) = e^{2 \int_{-\infty}^{x} i\beta(|r_1|^2 - q_0^2)dy}r_1(x,t), \quad (53)$$
where \( r_1(x, t) \) with the following trigonometric function form

\[
r_1(x, t) = \frac{\cosh \nu + \frac{1}{2} a_1 (1 + \frac{a_2^2}{a_1^2}) \sin(\gamma + a_1 a_2 t) - i a_2 \cos(\gamma + a_1 a_2 t)}{\cosh \nu + \frac{2}{a_1} \sin(\gamma + a_1 a_2 t)},
\]

and

\[
\nu = (x + 4 \beta t) a_2 + \alpha + a_0,
\]

\[
e^{a_0} = \frac{a_1}{2 \chi a_2},
\]

\[
e^{-a_0} = \frac{2 \chi a_2}{a_1}.
\]

For \( \beta = 0 \), soliton solution (53) convert into the one-soliton solution of focusing nonlinear Schrödinger equation Fig.2 (a). Fig.2 (b) displays the \( \beta \neq 0 \) case and Fig.2 (c) displays the \( \beta \neq 0 \) and \( \chi \rightarrow 1 \) case. The Fig.2 (d) displays the more general situation that is the zero point \( z_1 = i \chi e^{i \alpha} \) where \( \chi = 2 \) and \( \alpha = \frac{\pi}{4} \).

![Fig. 2. (a) The one-soliton solution of equation (6) as \( \beta = 0, \chi = 2 \). (b) The one-soliton solution of equation (6) as \( \beta = 1, \chi = 2 \). (c) The one-soliton solution of equation (6) as \( \beta = \frac{1}{2}, \chi = \frac{11}{10} \). (d) The one-soliton solution of equation (6) as \( \beta = \frac{1}{4}, z_1 = 2i e^{i \frac{\pi}{4}} \).](image-url)
9 Conclusion

In this work, we investigated the focusing Kundu-Eckhaus equation with nonzero boundary condition at infinity. For $\beta = 0$, the soliton solutions are reduced as the soliton solutions of focusing NLS with nonzero boundary conditions. We introduced a transformation, such that the asymptotic spectral problem with the linear relationship $V_\pm = (2k + 2\beta q_0^2)U_\pm$. A appropriate Riemann surface for the single-valued function of the spectral parameter was introduced. Then, the complex $k$-plane transformed into the complex $z$-plane. Unlike the focusing nonlinear Schrödinger equation, the branch cut not on the Im$k$ axis, but shift $\beta q_0^2$ along the $x$ axis. The orientation is up to $\beta$. The nonzero boundary conditions is different from zero boundary conditions mainly reflected in the analytic region and the zero points of $s_{11}$ and $s_{22}$. The analytic region not only involved with the upper-half/lower-half plane, but also involved with the inside/outside of the circle $C_0$.

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