Stopping strategies and gambler’s ruin problem

Theo van Uem

School of Technology, Amsterdam University of Applied Sciences,
Weesperzijde 190, 1097 DZ Amsterdam, The Netherlands.

Email: t.j.van.uem@hva.nl

Abstract
We obtain absorption probabilities and expected time until absorption for different stopping strategies in gambler’s ruin problem using the concept of multiple function barriers.

2000 Mathematics Subject Classification: Primary 60G50; Secondary 60J05

1. Introduction

In gambler’s ruin problem a well known stopping strategy is: go on until ruin or reaching a predefined boundary. In this paper we use a more flexible concept: a player may stop in a set of predefined barriers.

We use the concept of a multiple function barrier (mfb) to model our stopping strategies. In a mfb there is a positive probability s that the player will immediately stop the game and probabilities p(1-s) and q(1-s) that the player will go one step forward or backward (p+q=1). In all other states (except the absorption barrier 0) the player will go one step forward or backward with probabilities p and q.

We consider three players, A, B and C, with different stopping strategies. All players start with initial capital $i_0$ and their bet is always one unit of capital. Arriving at state 0 will always give immediate absorption in that state: ruin has happened. $M_A = \{ki_0 | k \geq 1\}$ is the set of multiple function barriers for player A. We notice that player A will stop at the very beginning of the game with probability s.

Player B has the same strategy as player A, except at time 0; the state $i_0$ is a delayed multiple function barrier for t>0. At t=0 it is a normal state (player B will always start the game, player A is a fictive player which is easier to analyze and there is a simple relationship between A and B).

Player C is more risk seeking than B: mfb’s are $M_C = \{ki_0 | k \geq 2\}$; $i_0$ is a normal state, also for t>0. In the appendix we consider the special cases s=0 and s=1.

2. Expected number of arrivals before absorption

We define:

\[
p_{ij}^{(n)} = \Pr(\text{system is in state } j \text{ after } n \text{ steps without absorption } | \text{ start in } i).
\]

The moment generating function for player X:

\[
X_j = X_{i_0,j}(z) = \sum_{m=0}^{\infty} p_{i_0,j}^{(m)} z^m \quad (0 < z \leq 1).
\]

\[
\omega = \frac{p}{q}
\]

In the main text we suppose 0<s<1.
Theorem 1

The moment generating functions $U$ for player A on the set $\{ki_0 | k \geq 0\}$ are:

CASE [0<z<1]:

$$U_{ki_0} = U_{ki_0}(z, \omega, s) = \omega^{-i_0} \varphi_2$$

$$U_{ki_0} = U_{ki_0}(z, \omega, s) = \frac{(\tau_2 - \tau_1 \omega_i)}{q(1-s)z(\tau_2 - \tau_1 \omega_i)} (k \geq 1)$$

where $\varphi_1, \varphi_2$ are solutions of $\varphi^2 - \theta \varphi + \omega^{i_0} = 0 \ (\varphi_1 > 1 > \varphi_2 > 0)$

$$\theta = \theta(z, \omega, s) = \frac{(\tau_2 - \tau_1 \omega_i) - 2pz(\tau_{i_0-1} - \tau_{i_0-1})}{qz(\tau_2 - \tau_1)}$$

and $\tau_1, \tau_2$ are solutions of $qz\tau^2 - \tau + pz = 0 \ (\tau_1 > 1 > \tau_2 > 0)$

CASE [z=1 and $\omega \neq 1$]

$$\tau_1 = \max(1, \omega), \tau_2 = \min(1, \omega).$$

CASE [z=1 and $\omega = 1$]

$$U_0 = U_0(1,1,s) = \varphi_2$$

$$U_{ki_0} = U_{ki_0}(1,1,s) = \frac{2i_0 \varphi_2}{1-s} (k \geq 1)$$

$$\varphi^2 - \theta \varphi + 1 = 0 \ (\varphi_1 > 1 > \varphi_2 > 0)$$

$$\theta = \theta(1,1,s) = 2 \left( \frac{i_0}{1-s} + 1 - i_0 \right)$$

Proof

We prove the results of CASE [0<z<1] in 5 steps:

Step 1

We determine the structure of the random walk between the multiple function barriers:

$$U_{ki_0+n} = \sum_{m=1}^{\infty} p_{i_0, ki_0+n}^{(m)} \sum_{m=1}^{\infty} \left[ p_{i_0, ki_0+n-1, p}^{(m-1)} + p_{i_0, ki_0+n+1, q}^{(m-1)} \right] z^m = pz U_{ki_0+n-1} + qz U_{ki_0+n+1}$$

where at first glance $k \geq 0, \ n = 2, 3, ..., i_0 - 2$, but $n=1$ and $n=i_0 - 1$ also satisfy the pattern.

We get:

$$U_{ki_0+n} = a_{k+1} \tau_1^n + b_{k+1} \tau_2^n \ (k \geq 0, \ n = 1, 2, ..., i_0 - 1)$$

and $\tau_1, \tau_2$ are solutions of $qz\tau^2 - \tau + pz = 0 \ (\tau_1 > 1 > \tau_2 > 0)$

Step 2

We express the constants of step 1 in the U values of the mfb’s.

On the interval $[ki_0, (k+1)i_0)$, $k \geq 1$ we have

$$U_{ki_0+1} = p(1-s)zU_{ki_0} + qzU_{ki_0+2}$$

$$U_{(k+1)i_0-1} = pzU_{(k+1)i_0-2} + q(1-s)zU_{(k+1)i_0}$$
Stopping strategies and gambler’s ruin problem

\[ a_{k+1} = \frac{(1 - s) \left( \tau_2^{i_0} U_{k+1} - U_{(k+1)i_0} \right) \tau_2^{i_0} \tau_1^{i_0} - \tau_1^{i_0} \tau_2^{i_0}}{\tau_2^{i_0} - \tau_1^{i_0}} \]

\[ b_{k+1} = \frac{(1 - s) \left( U_{(k+1)i_0} - \tau_1^{i_0} U_{ki_0} \right) \tau_1^{i_0} - \tau_1^{i_0} \tau_2^{i_0}}{\tau_1^{i_0} - \tau_2^{i_0}} \]

Step 3:
Focussing on the mfb in \( ki_0 \) we get:

\[ U_{ki_0} = pzU_{k+1} + qzU_{ki_0+1} \quad (k > 1) \]

and after some calculation:

\[ U_{(k+1)i_0} - \theta U_{ki_0} + \omega^{i_0} U_{(ki_0+1)i_0} = 0 \quad (k > 1) \]

where

\[ \theta = \theta(z, \omega, s) = \frac{i_0 \tau_2^{i_0} - i_0 \tau_1^{i_0} - 2z (i_0 \tau_2^{i_0 - 1} - i_0 \tau_1^{i_0 - 1})}{qz (\tau_2 - \tau_1)} \]

and \( \tau_1, \tau_2 \) are solutions of

\[ qz \tau^2 - \tau + pz = 0 \quad (\tau_1 > 1 > \tau_2 > 0) \]

So we have:

\[ U_{ki_0} = C_2 \phi_2^k \quad (k \geq 1), \]

where \( \phi_1, \phi_2 \) are solutions of \( \varphi^2 - \theta \varphi + \omega^{i_0} = 0 \quad (\varphi_1 > 1 > \varphi_2 > 0) \)

We remark that the last equation includes \( k=1 \) because \( U_{i_0} \) satisfies the pattern.

Step 4
Our focus is now the interval \([0, i_0]\).

\[ U_1 = qz U_2 \]
\[ U_{i_0-1} = pz U_{i_0-2} + qz (1 - s) U_{i_0} \]

so:

\[ b_1 = -a_1 = \frac{(1 - s) U_{i_0}}{\tau_2^{i_0} - \tau_1^{i_0}} \]

Using \( U_0 = qz U_1 \) we get:

\[ (\tau_2^{i_0} - \tau_1^{i_0}) U_0 = q (1 - s) z (\tau_2 - \tau_1) U_{i_0} \]

Step 5
The last step uses the starting point:

\[ U_{i_0} = 1 + pz U_{i_0-1} + qz U_{i_0+1} \]

and after some calculation, using the preceding steps, we get:

\[ U_{ki_0} = \frac{(\tau_2^{i_0} - \tau_1^{i_0}) \phi_2^k}{q (1 - s) z (\tau_2 - \tau_1) \omega^{i_0}} \quad (k \geq 1) \]

\[ U_0 = \omega^{-i_0} \varphi_2 \]

CASE \([z=1 \text{ and } \omega \neq 1] \)
We can follow the 5 steps above, the only difference is: \( \tau_1 = \max(1, \omega), \tau_2 = \min(1, \omega) \).
CASE \([z=1 \text{ and } \omega = 1]\)

The strategy is the same; we give the result of each step.

Step 1
Because of \(\tau_1 = \tau_2 = 1\) we have:
\[
U_{ki_0+n} = a_{k+1}n + b_{k+1} \quad (k \geq 0, \quad n = 1, 2, \ldots, i_0 - 1)
\]

Step 2
\[
a_{k+1} = (1-s)\frac{[U_{(k+1)i_0} - U_{ki_0}]}{i_0} \quad (k>0)
\]
\[
b_{k+1} = (1 - s)U_{ki_0} \quad (k>0)
\]

Step 3
\[
\phi^2 - \theta \phi + 1 = 0 \quad (\phi_1 > 1 > \phi_2 > 0)
\]
\[
\theta = 2 \left( \frac{i_0}{1-s} + 1 - i_0 \right)
\]
\[
U_{ki_0} = C_2 \phi_2^k \quad (k \geq 1),
\]

Step 4
\[
a_1 = \frac{(1-s)U_{i_0}}{i_0}; \quad b_1 = 0
\]

Step 5
\[
U_0 = U_0(1,1,s) = \phi_2
\]
\[
U_{ki_0} = U_{ki_0}(1,1,s) = \frac{2i_0\phi_2^k}{1-s} \quad (k \geq 1)
\]

Remark: The same result can be obtained by applying l’Hospital’s rule in the case \(0<z<1\)

**Theorem 2**
The moment generating functions \(V\) for player B on the set \(\{ki_0 \mid k \geq 0\}\) are:
\[
V_{i_0} = \frac{U_{i_0}-1}{1-s}
\]
\[
V_{ki_0} = \frac{U_{ki_0}}{1-s} \quad (k \neq 1)
\]

Proof

We use the following notation:

\(p_{i_0,ki_0}^{[X|m]} = \text{P(player X arrives in } ki_0 \text{ in } m \text{ steps before absorption has taken place, when starting in } i_0)\).

The difference of player A and B is the first step: player A starts at \(t=0\) with probabilities \(p(1-s)\) and \(q(1-s)\), while B is starting with \(p\) and \(q\). The difference is a factor \((1-s)\).

\[
U_{ki_0} = \sum_{m=0}^{\infty} p_{i_0,ki_0}^{[A](m)} z^m = \delta(k,1) + \sum_{m=1}^{\infty} p_{i_0,ki_0}^{[A](m)} z^m = \delta(k,1) + (1-s) V_{ki_0}
\]
**Theorem 3**
The moment generating functions $W$ for player C on the set $\{ki_0 \mid k \geq 0\}$ are:

CASE $[0 < z < 1]$

$$W_0 = W_0(z, \omega, s) = \frac{1}{\tau_1^0 + \tau_2^0 - \varphi_2}$$

$$W_{i_0}^0(z, \omega, s) = \frac{(\tau_2^0 - \tau_1^0)}{qs(\tau_2 - \tau_1)(\tau_1^0 + \tau_2^0 - \varphi_2)}$$

$$W_{ki_0}^0(z, \omega, s) = \frac{(\tau_2^0 - \tau_1^0)^{\varphi_2 - 1}}{q(1-s)z(\tau_2 - \tau_1)(\tau_1^0 + \tau_2^0 - \varphi_2)} \quad (k \geq 2)$$

where $\varphi_1, \varphi_2$ are solutions of $\varphi^2 - \theta \varphi + \omega^i_0 = 0 \quad (\varphi_1 > 1 > \varphi_2 > 0)$

$$\theta = \theta(z, \omega, s) = \frac{i_0^i - i_1^i - 2pz(i_0^i - 1 - i_1^i)}{qz(\tau_2 - \tau_1)}$$

and $\tau_1, \tau_2$ are solutions of $qz\tau^2 - \tau + pz = 0 \quad (\tau_1 > 1 > \tau_2 > 0)$

CASE $[z = 1 \text{ and } \omega \neq 1]$

$$\tau_1 = \max(1, \omega), \tau_2 = \min(1, \omega).$$

CASE $[z = 1 \text{ and } \omega = 1]$

$$W_0 = \frac{1}{2 - \varphi_2}$$

$$W_{i_0} = \frac{2i_0}{2 - \varphi_2}$$

$$W_{ki_0} = \frac{2i_0^{\varphi_2 - 1}}{(1-s)(2-\varphi_2)} \quad (k \geq 2)$$

$$\varphi^2 - \theta \varphi + 1 = 0 \quad (\varphi_1 > 1 > \varphi_2 > 0)$$

$$\theta = \theta(1, 1, s) = 2 \left( \frac{i_0^i + 1 - i_0^i}{1-s} \right)$$

Proof

First we prove the results of CASE $[0 < z < 1]$ in 8 steps:

**Step 1**
As in proof of theorem 1:

$$W_{ki_0+n} = a_{k+1}^n \tau_1^n + b_{k+1}^n \tau_2^n \quad (k \geq 0, \quad n = 1, 2, \ldots, i_0 - 1)$$

**Step 2**
As in theorem 1, now with $k \geq 2$:

$$a_{k+1} = \frac{(1-s)\left[\tau_2^0 W_{ki_0} - W_{(k+1)i_0}\right]}{\tau_2^0 - \tau_1^0}$$

$$b_{k+1} = \frac{(1-s)\left[W_{(k+1)i_0} - \tau_1^0 W_{ki_0}\right]}{\tau_2^0 - \tau_1^0}$$

**Step 3**
The same result as in theorem 1, but now $k \geq 2$:

$$W_{ki_0} = c_2 \varphi_2^k$$
Step 4
Our focus is the interval \([0, i_0]\).

\[ W_1 = qzW_2; \quad W_{i_0-1} = pzW_{i_0-2} + qzW_{i_0} \]

\[ b_1 = -a_1 = \frac{W_{i_0}}{\tau_1^{i_0} - \tau_2^{i_0}} \]

\[ W_0 = qzW_1 \]

\[ (\tau_2^{i_0} - \tau_1^{i_0})W_0 = qz(\tau_2 - \tau_1)W_{i_0} \]

Step 5
Focus on \([i_0, 2i_0]\).

\[ W_{i_0+1} = pzW_{i_0} + qzW_{i_0+2} \]
\[ W_{2i_0-1} = pzW_{2i_0-2} + qz(1 - s)W_{2i_0} \]

\[ a_2 = \frac{\tau_2^{i_0}W_{i_0} - (1 - s)W_{2i_0}}{\tau_2^{i_0} - \tau_1^{i_0}} \]
\[ b_2 = \frac{[(1 - s)W_{2i_0} - \tau_1^{i_0}W_{i_0}]}{\tau_2^{i_0} - \tau_1^{i_0}} \]

Step 6
This step uses the starting point:

\[ W_{i_0} = 1 + pzW_{i_0-1} + qzW_{i_0+1} \]

\[ qz(\tau_2 - \tau_1)\left[(\tau_2^{i_0} + \tau_1^{i_0})W_{i_0} - (1 - s)W_{2i_0}\right] = (\tau_2^{i_0} - \tau_1^{i_0}) \]

Step 7
Connection in \(2i_0\):

\[ W_{2i_0} = pzW_{2i_0-1} + qzW_{2i_0+1} \]

Using \(\varphi^2 - \theta \varphi + \omega^{i_0} = 0\) we get:

\[ \omega^{i_0}W_{i_0} - (1 - s)\varphi_1W_{2i_0} = 0 \]

Step 8
Combining previous steps:

\[ W_0 = \frac{1}{\tau_1^{i_0} + \tau_2^{i_0} - \varphi_2} \]

\[ W_{i_0} = \frac{(\tau_2^{i_0} - \tau_1^{i_0})}{qz(\tau_2 - \tau_1)(\tau_1^{i_0} + \tau_2^{i_0} - \varphi_2)} \]

\[ W_{k,i_0} = \frac{(\tau_2^{i_0} - \tau_1^{i_0})\varphi_2^{-1}}{q(1 - s)z(\tau_2 - \tau_1)(\tau_1^{i_0} + \tau_2^{i_0} - \varphi_2)} \quad (k \geq 2) \]

CASE \([z=1 \text{ and } \omega \neq 1] \)
We can follow the 8 steps above, the only difference is : \(\tau_1 = \max(1, \omega), \tau_2 = \min(1, \omega). \)
CASE \([z=1 \text{ and } \omega = 1]\)
The strategy is the same; we give the result of each step.

Step 1
Because of \(\tau_1 = \tau_2 = 1\) we have:
\[
W_{ki_0+n} = a_{k+1}n + b_{k+1} \quad (k \geq 0, \quad n = 1, 2, \ldots, i_0 - 1)
\]

Step 2
\[
a_{k+1} = \frac{(1-s)(W_{k+1;i_0} - W_{k+1;0})}{i_0} \quad (k>1)
\]
\[
b_{k+1} = (1-s)W_{k;i_0} \quad (k>1)
\]

Step 3
\[
\varphi^2 - \theta \varphi + 1 = 0 \quad (\varphi_1 > 1 > \varphi_2 > 0)
\]
\[
\theta = 2 \left( \frac{i_0}{1-s} + 1 - i_0 \right)
\]
\[
W_{ki_0} = C \varphi_2^k \quad (k \geq 1),
\]

Step 4
\[
a_1 = \frac{W_{i_0}}{i_0}; \quad b_1 = 0
\]

Step 5
\[
a_2 = \frac{(1-s)W_{2i_0} - W_{i_0}}{i_0} \quad ; \quad b_2 = W_{i_0}
\]

Step 6
\[
2W_{i_0} - (1-s)W_{2i_0} = 2i_0
\]

Step 7
\[
W_{i_0} - (1-s)\varphi_1 W_{2i_0} = 0
\]

Step 8
\[
W_0 = \frac{1}{2-\varphi_2}
\]
\[
W_{i_0} = \frac{2i_0}{2-\varphi_2}
\]
\[
W_{ki_0} = \frac{2i_0 \varphi_2^{k-1}}{(1-s)(2-\varphi_2)} \quad (k \geq 2)
\]

Remark: Again the last results can be obtained by applying l’Hospital’s rule in the CASE \([0<z<1]\).
3. Absorption probabilities

It is easy to obtain the absorption probabilities in the absorption barrier 0 and in the mfb’s.

We use the following notation: \( P_X (Y) = P(\text{absorption in state } Y \text{ for player } X) \).

**Theorem 4**

CASE \( \omega \neq 1 \)

Absorption probabilities player B:

\[
P_B(0) = V_0(1, \omega, s) = \frac{\omega^{-i_0} \varphi_2}{1-s}
\]

\[
P_B(i_0) = sV_{i_0}(1, \omega, s) = s \frac{U_{i_0}(1, \omega, s) - 1}{1-s} = \frac{s}{1-s} \left[ (\frac{\omega^{-i_0} - 1}{(1-s)(q-p)} - 1 \right]
\]

\[
P_B(ki_0) = sV_{ki_0}(1, \omega, s) = \frac{s(\omega^{-i_0} - 1) \varphi_2^k}{(1-s)^2(q-p)} \quad (k \geq 2)
\]

Player C:

\[
P_C(0) = W_0(1, \omega, s) = \frac{1}{1+\omega^6-\varphi_2}
\]

\[
P_C(ki_0) = sW_{ki_0}(1, \omega, s) = \frac{s(1-\omega^{i_0}) \varphi_2^{k-1}}{(1-s)(q-p)(1+\omega^6-\varphi_2)} \quad (k \geq 2)
\]

CASE \( \omega = 1 \)

Player B:

\[
P_B(0) = V_0(1,1,s) = \frac{\varphi_2}{1-s}
\]

\[
P_B(i_0) = sV_{i_0}(1,1,s) = s \frac{U_{i_0}(1,1,s) - 1}{1-s} = \frac{s}{1-s} \left[ \frac{2i_0 \varphi_2}{(1-s)} - 1 \right]
\]

\[
P_B(ki_0) = sV_{ki_0}(1,1,s) = \frac{2i_0 \varphi_2^k}{(1-s)^2} \quad (k \geq 2)
\]

Player C:

\[
P_C(0) = W_0(1,1,s) = \frac{1}{2-\varphi_2}
\]

\[
P_C(ki_0) = sW_{ki_0}(1,1,s) = \frac{2i_0 \varphi_2^{k-1}}{(1-s)(2-\varphi_2)} \quad (k \geq 2)
\]
We notice that there is a constant ratio between the B and C absorption probabilities:

**CASE $\omega \neq 1$**

\[
\frac{P_B(0)}{P_C(0)} = \frac{P_B(ki_0)}{P_C(ki_0)} = \frac{\varphi_2(1 + \omega^i_0 - \varphi_2)}{(1 - s)\omega^i_0} < 1 \quad (k \geq 2)
\]

**CASE $\omega = 1$**

\[
\frac{P_B(0)}{P_C(0)} = \frac{P_B(ki_0)}{P_C(ki_0)} = \frac{\varphi_2(2 - \varphi_2)}{(1 - s)} < 1 \quad (k \geq 2)
\]

**4. Mean absorption time**

**4.1 Mean time before absorption in any mfb**

We define the mean time until absorption in any mfb for player X, when starting in i as:

\[
m_i^\mathcal{X} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} np_i^{\mathcal{X}^{(n)}} s_k
\]

where $s_k$ is the (immediate) absorption probability in state $ki_0 \quad (k \geq 0)$.

We have:

\[
m_i^\mathcal{X} = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} np_i^{\mathcal{X}^{(n)}} s_k = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} (n - 1)p_i^{\mathcal{X}^{(n)}} s_k + \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} p_i^{\mathcal{X}^{(n)}} s_k =
\]

\[
\sum_{n=1}^{\infty} \sum_{k=0}^{\infty} (n - 1)\left[p_{i+1,k}^{\mathcal{X}^{(n-1)}} + q_{i-1,k}^{\mathcal{X}^{(n-1)}}\right] s_k + 1 = pm_i^{\mathcal{X}^{(n-1)}} + qm_i^{\mathcal{X}^{(n-1)}} + 1
\]

**Theorem 5**

The mean absorption time in any mfb for player A is:

**CASE $\omega \neq 1$**

\[
m_i^\mathcal{A} = i_0 \frac{(1-s)}{s} (1 - \varphi_2 \omega^{-i_0})
\]

**CASE $\omega = 1$**

\[
m_i^\mathcal{A} = i_0 \frac{(1-s)}{s} (1 - \varphi_2)
\]

**Proof**

**CASE $\omega \neq 1$**

**Step 1**

We describe the behaviour of $m_i^\mathcal{A}$ between the multiple function barriers:

\[
m_i^{\mathcal{A}} = pm_i^{\mathcal{A}} + qm_i^{\mathcal{A}} + 1
\]

where at first glance $k \geq 0, \quad n = 2,3,\ldots,i_0 - 2$, but $n=1$ and $n=i_0 - 1$ also satisfy the pattern.

We get:

\[
m_i^{\mathcal{A}} = \frac{n}{q-p} + \omega^{-n}a_k + b_k \quad (k \geq 0, \quad n = 1,2,\ldots,i_0 - 1)
\]
Step 2
We express the constants of step 1 in the m values of the mfb’s.
On the interval $[k_{i_0}, (k+1)i_0]$, $k \geq 0$ we have

$$m_{k+i_0+1}^A = pm_{k+i_0+2}^A + qm_{k+i_0}^A + 1$$
$$m_{(k+1)i_0-1}^A = pm_{(k+1)i_0}^A + qm_{(k+1)i_0-2}^A + 1$$

$$a_{k+1} = \frac{m_{k+i_0}^A - m_{(k+1)i_0}^A - i_0}{1-\omega^{-i_0}}$$
$$b_{k+1} = \frac{m_{(k+1)i_0}^A + i_0 - \omega^{-i_0}m_{k+i_0}^A}{1-\omega^{-i_0}}$$

where $m_{i_0}^A = 0$.

Step 3:
Focussing on the mfb in $k_{i_0}$ ($k \geq 1$) we get:

$$m_{k+i_0}^A = p(1-s)m_{k+i_0+1}^A + q(1-s)m_{k+i_0-1}^A + (1-s)$$

and after some calculation:

$$\omega^i m_{(k+1)i_0}^A - \theta m_{k+i_0}^A + m_{(k-1)i_0}^A = \frac{i_0(1-\omega^{-i_0})}{p-q} \quad (k \geq 1)$$

so we have:

$$m_{k+i_0}^A = i_0 \left( 1 - s \right) + C_1 \varphi_1^{-k} \quad (k \geq 0),$$

and, using $m_{i_0}^A = 0$:

$$m_{i_0}^A = i_0 \left( 1 - s \right) \left( 1 - \varphi_1^{-1} \right)$$

CASE $\omega = 1$

The strategy is the same; now starting with $m_{k+i_0+n}^A = a_{k+1} + nb_{k+1} - n^2$.

**Theorem 6**
The mean absorption time in any mfb for player B is:

$$m_{i_0}^B = i_0 \left( 1 - \varphi_2 \omega^{-i_0} \right)$$

Proof
The only difference of player A and B is the first step; player A starts at $t=0$ with probabilities $p(1-s)$ and $q(1-s)$, while B is starting with $p$ an $q$. The difference in $m_{i_0}$ is a factor $(1-s)$:

$$m_{i_0}^A = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} np_{i_0,k+i_0}^A s_k = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} n(1-s)p_{i_0,k+i_0}^B s_k = (1-s)m_{i_0}^B$$
**Theorem 7**
The mean absorption time in any mfb for player C is:

**CASE \( \omega \neq 1 \)**

\[
m_{i_0}^{[C]} = i_0 \left\{ \frac{1-s}{s} \left( l - \varphi^{-1}_l \right) + \frac{l - \omega^{-i_0}}{p - q} \right\} \frac{1}{1 + \omega^{-i_0} - \varphi^{-1}_l}
\]

**CASE \( \omega = 1 \)**

\[
m_{i_0}^{[C]} = i_0 \left[ 2i_0 + \frac{1-s}{s} (1 - \varphi_2) \right] \frac{1}{2 - \varphi_2}
\]

**Proof**

**CASE \( \omega \neq 1 \)**

Steps 1,2 as in theorem 5

Step 3, now with \( k \geq 1 \):

\[
m_{ki_0}^{[C]} = i_0 \left( \frac{1-s}{s} \right) + C_i \varphi^{-k}_l \quad (k \geq 1)
\]

Step 4

\[
m_{i_0}^{[C]} = pm_{i_0+1}^{[C]} + qm_{i_0-1}^{[C]} + 1
\]

and using the result of step 3 we get:

\[
m_{i_0}^{[C]} = \frac{i_0 (l - \omega^{-i_0})}{q - p} \quad (k \geq 0)
\]

**CASE \( \omega = 1 \)**

We give the results:

Step 1

\[
m_{k_i0+n}^{[C]} = a_{k+l} + nb_{k+l}n^2 \quad (k \geq 0)
\]

Step 2

\[
a_{k+l} = m_{ki_0}^{[C]} \quad (k \geq 0)
\]

\[
b_{k+l} = i_0 + \frac{m_{(k+l)i_0}^{[C]} - m_{ki_0}^{[C]}}{i_0} \quad (k \geq 0)
\]

Step 3

\[
m_{(k+l)i_0}^{[C]} - \theta (1,l,s) m_{ki_0}^{[C]} + m_{(k-l)i_0}^{[A]} = -2i_0^2 \quad (k \geq 1)
\]

\[
m_{ki_0}^{[A]} = i_0 \left( \frac{1-s}{s} \right) + C_2 \varphi^{-k}_l \quad (k \geq 1)
\]

Step 4

\[
m_{2i_0}^{[C]} - 2m_{i_0}^{[C]} = -2i_0^2
\]

\[
m_{i_0}^{[C]} = \frac{i_0 \left[ 2i_0 + \frac{1-s}{s} (1 - \varphi_2) \right]}{2 - \varphi_2}
\]
4.2 Mean time before absorption in a specific mfb

We define:
\[ E \left[ T_{ki_0}^{|X|} \right] = \text{mean time until absorption in mfb } ki_0, \text{ player } X, \text{ when starting in } i_0. \]

We have, if \( \omega \neq 1 \):
\[ E \left[ T_{ki_0}^{|X|} \right] = \sum_{n=0}^{\infty} np_{ki_0}^{\left| X \right| (n)} \cdot s_k = s_k \left( \frac{dM_k}{dz} \right) \bigg|_{z=1} \]
where \( s_k \) is the (immediate) absorption probability in state \( ki_0 \) \((k \geq 0)\) and \( M_k \) is the moment generating function of player \( X \).

The connection with section 4.1 is:
\[ m_{ki_0}^{\left| X \right|} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} np_{ki_0}^{\left| X \right| (n)} s_k = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} np_{ki_0}^{\left| X \right| (n)} s_k = \sum_{k=0}^{\infty} E \left[ T_{ki_0}^{|X|} \right] \]

We discuss the CASE \( \omega \neq 1 \).

From theorems 1, 2, and 3 we see that the mean time before absorption in the case \( \omega \neq 1 \) is closely related to differentiating the function \( \varphi_2 \).

Implicit differentiation of \( qz\tau^2 - \tau + pz = 0 \) gives
\[ \frac{d\tau_i}{dz} = (-1)^i h(z)z^{-1} \tau_i \quad (i=1,2) \]
with
\[ h(z) = (1 - 4pqz^2)^{-\frac{1}{2}} \]
\[ \frac{dh(z)}{dz} = 4pqzh^3(z) \]
\[ h(1) = \frac{1}{|p-q|} \]

Implicit differentiation of \( \varphi^2 - \theta \varphi + \omega^4 = 0 \) gives
\[ \frac{d\varphi_i}{dz} = \left[ \frac{\varphi_i}{2\varphi_i - \theta} \right] \frac{d\theta}{dz} = (-1)^i \left[ \frac{\varphi_i}{\varphi_2 - \varphi_1} \right] \frac{d\theta}{dz} \quad (i=1,2) \]
with
\[ \theta(z, \omega, s) = \frac{\frac{i_0}{s} - \frac{i_0}{1-s}}{2ps\tau_1 - \tau_j} - \frac{i_0}{1-s} \]
and
\[ \left( \frac{d\theta}{dz} \right)_{z=1} = \frac{4pq\theta(1, \omega, s) - \frac{i_0(1+\omega)}{1-s} + 2p[\omega(1-\omega)(1-\omega^4) + (\omega^4 - 1)(1+\omega^4)]}{(p-q)^2} \]

**Theorem 8**

\[ \left( \frac{d\varphi_i}{dz} \right)_{z=1} = (-1)^i \left[ \frac{\varphi_i}{\varphi_2 - \varphi_1} \right] \left[ \frac{4pq\theta(1, \omega, s) - \frac{i_0(1+\omega)}{1-s} + 2p[\omega(1-\omega)(1-\omega^4) + (\omega^4 - 1)(1+\omega^4)]}{(p-q)^2} \right] \quad (i=1,2) \]
Appendix

In this appendix we study two special cases: s=0 and s=1.

A1. The case s=0

There are no mfb’s. Player’s A, B and C now have the same strategy: play until ruin.

If s=0 then

$\nu_0 = \max(1, \omega_i)$, $\nu_2 = \min(1, \omega_i)$.

Using theorem 4, we get:

$P_{A,B,C}(0) = \omega^{-i_0} \nu_2 = \begin{cases} 1 & \text{if } \omega \leq 1 \\ \omega^{-i_0} & \text{if } \omega > 1 \end{cases}$

Using theorem 8:

$m_0^{[A,B,C]} = E\left[T_0^{A,B,C}\right] = \left(\frac{d \nu_2}{dz}\right)_{z=1} = \begin{cases} \frac{i_0 \omega_i}{q-p} & \text{if } \omega \leq 1 \\ \frac{i_0}{p-q} & \text{if } \omega > 1 \end{cases}$

(Well known results, see [1]).

A2. The case s=1

Player B

Now we can’t use player A as a reference.

We first calculate the moment generating function $V$ on $(0, i_0)$ when starting in $i_0 - 1$:

$V_n = \frac{(\tau_2^n - \tau_1^n)}{qz(\tau_2^{i_0} - \tau_1^{i_0})}; V_0 = qzV_1 = \frac{\tau_2 - \tau_1}{(\tau_2^{i_0} - \tau_1^{i_0})}$; $V_i = pzV_{i-1} = \frac{\omega(\tau_2^{i-1} - \tau_1^{i-1})}{(\tau_2^{i_0} - \tau_1^{i_0})}$

Next the calculate $V$ on $(i_0, 2i_0)$ when starting in $i_0 + 1$:

$V_{i_0+n} = \frac{(\tau_2^n - \tau_1^n)}{pz(\tau_2^{i_0} - \tau_1^{i_0})}$ (0 < n < i_0); $V_{2i_0} = pzV_{2i_0-1} = \frac{\omega^{i_0-1}(\tau_2 - \tau_1)}{(\tau_2^{i_0} - \tau_1^{i_0})}$; $V_i = qzV_{i+1} = \frac{(\tau_2^{i-1} - \tau_1^{i-1})}{(\tau_2^{i_0} - \tau_1^{i_0})}$

Absorption in 0 can only be obtained by moving one step backward on t=0, so we have:

$p_0^{[B]} = qV_0(1) = \frac{q-p}{1-\omega_i}$

$p_i^{[B]} = q(pV_{i-1}) + p(qV_{i+1}) = \frac{2p(1-\omega_i)}{(1-\omega_i)}$

$p_{2i_0}^{[B]} = pV_{2i_0-1}(1) = \frac{(q-p)\omega_i}{1-\omega_i}$

If $\omega = 1$ then:

$p_0^{[B]} = p_{2i_0}^{[B]} = \frac{1}{2i_0}$

$p_i^{[B]} = \frac{i_0 - 1}{i_0}$
Define \( m_{i_0}^{[B]} \) as the mean absorption time in any mfb, starting in \( i \), for \( t > 0 \): state \( i_0 \) has become an absorption barrier.

We have:
\[
m_{i_0}^{[B]} = pm_{i_0+1}^{[B]} + qm_{i_0-1}^{[B]} + 1 = 1 + p \left[ \frac{1}{q-p} + \frac{i_0}{p(1-\omega^{i_0})} \right] + q \left[ \frac{1}{p-q} + \frac{i_0}{q(1-\omega^{i_0})} \right] = i_0
\]

We define \( T_{i,j}^{[B]} \) as the time needed for player B for absorption in state \( j \), when starting in \( i \).

We have:
\[
E[T_{i_0,0}^{[B]}] = \sum_n n p_{i_0,0}^{(n)} = \sum (n - 1) p_{i_0,0}^{(n)} + \sum p_{i_0,0}^{(n)} = \sum (n - 1) q p_{i_0-1,0}^{(n-1)} + \sum p_{i_0,0}^{(n)} = q E[T_{i_0-1,0}^{[B]}] + p_i^{[B]}
\]

where
\[
E[T_{i_0,0}^{[B]}] = \left( \frac{dv_i}{dx} \right)_{x=1}
\]
so:
\[
E[T_{i_0,0}^{[B]}] = \frac{1}{(q-p)(\omega^{i_0}-1)} + \frac{i_0(\omega^{i_0}+1)}{(\omega^{i_0}-1)^2} + p_i^{[B]}
\]

\[
E[T_{i_0,2i_0}^{[B]}] = \omega^{i_0} \left[ \frac{1}{(q-p)(\omega^{i_0}-1)} + \frac{i_0(\omega^{i_0}+1)}{(\omega^{i_0}-1)^2} \right] + p_i^{[B]}
\]

\[
E[T_{i_0,i_0}^{[B]}] = -\frac{2p(\omega^{i_0}-1)}{(q-p)(\omega^{i_0}-1)} + \frac{4i_0\omega^{i_0}}{(\omega^{i_0}-1)^2} + p_i^{[B]}
\]

and
\[
E[T_{i_0,0}^{[B]}] + E[T_{i_0,2i_0}^{[B]}] + E[T_{i_0,i_0}^{[B]}] = i_0 = m_{i_0}^{[B]}
\]

Player C
Using the same methods as in our main paper:
\[
p_0^{[C]} = \frac{1}{(1 + \omega^{i_0})}
\]

\[
p_{2i_0}^{[C]} = \frac{\omega^{i_0}}{(1 + \omega^{i_0})}
\]

\[
m_{i_0}^{[C]} = \frac{i_0(1 - \omega^{i_0})}{(q-p)(1 + \omega^{i_0})}
\]

\[
E[T_0^{[C]}] = \frac{i_0(1 - \omega^{i_0})}{(q-p)(1 + \omega^{i_0})^2}
\]

\[
E[T_{2i_0}^{[C]}] = \frac{i_0(1 - \omega^{i_0})\omega^{i_0}}{(q-p)(1 + \omega^{i_0})^2}
\]

We can simply verify that
\[
m_{i_0}^{[C]} = E[T_0^{[C]}] + E[T_{2i_0}^{[C]}]
\]
References

[1] Feller W 1968 *An Introduction to probability theory and its applications*, (third edition) Vol. 1, John Wiley, New York.

[2] Bolina O 2001 The gambler's ruin problem in path representation form, arXiv:math/0111242

[3] Yamamoto, K 2012 Exact solution to a gambler’s ruin problem with a nonzero halting probability, arXiv:1211.4314

[4] Uem, van T 2013 Maximum and minimum of modified gambler’s ruin problem, arXiv:1301.2702