Spinon statistics in integrable spin-$S$ Heisenberg chains

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The spectrum of the integrable spin-$S$ Heisenberg chains is completely characterized in terms of spin-$1/2$ spinons. In the continuum limit they form a quasi-particle basis to the higher level SU(2) Wess-Zumino-Witten (WZW) models. Enumerating the spinon states in finite systems we obtain effective single particle distribution functions for these objects which generalize Haldane's generalized exclusion principle to quasi-particles with non-Abelian exchange statistics.

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Haldane's notion of fractional exclusion statistics [1] has been used successfully to characterize elementary excitations in low dimensional quantum systems. For the excitations in the lowest Landau level arising in the fractional quantum Hall effect (FQHE) this concept has been shown to coincide with the usual definition of fractional statistics, namely through their properties under exchange of anyons [2]. A complete description in terms of quasi particles obeying fractional statistics is possible for integrable quantum systems where detailed knowledge of the spectrum allow for an classification of all states in terms of the elementary excitations. Their statistical properties are described in terms of the statistical interaction parameters $g_{\alpha\beta}$ which determines the many particle Hilbert space dimension

$$W = \prod_{\alpha} \left( d_{\alpha} + N_{\alpha} - 1 \right).$$

Here

$$d_{\alpha} = d_{\alpha}^{(1)} - \sum_{\beta} g_{\alpha\beta} (N_{\beta} - \delta_{\alpha\beta})$$

is the dimension of the Hilbert space of a single particle of species $\alpha$ in the presence of $N_{\alpha} - 1$ other particles of that species at fixed coordinates. $d_{\alpha}^{(1)}$ are constants which can be interpreted as the dimensions of the single particle Hilbert spaces. This definition includes bosons ($g_{\alpha\beta} \equiv 0$) and fermions ($g_{\alpha\beta} = \delta_{\alpha\beta}$). An new feature in this picture are the possible off diagonal statistics parameters $g_{\alpha \neq \beta}$ encoding the mutual statistical interaction of different species of anyons. This is realized in spin-$1/2$ Heisenberg chains [1] and has recently been discussed in the context of the statistics of the quasiparticles and -holes in the FQHE at filling $\nu = 1/m$ with $m$ an odd integer [3,4].

The approach outlined above allows to introduce single particle distribution functions which already contain all information on the statistical properties of the elementary excitations of the system thereby giving an easy access to thermodynamic properties of the system such as the low temperature specific heat. Unfortunately, it is easily seen that Haldane statistics [1] fails to describe correctly the interesting case of quasiparticles obeying non-Abelian statistics as realized for example in the FQHE at half integer fillings [5]: the distribution functions obtained within this picture always lead to a low temperature specific heat characteristic of a Gaussian conformal field theory (CFT) with central charge $c = 1$ [6] while the quasi particles for the $\nu = 1/2$ FQHE are described by a SU(2)$_2$ WZW model with $c = 3/2$.

Recently, an extension of Haldane statistics to quasi particles obeying non-Abelian exchange statistics has been proposed based on recursion relations for truncated conformal spectra [8]: constructing the basis of the CFT explicitly in terms of quasi-particle operators simple distribution functions could be calculated that coincide with the ones found from [8] for Abelian statistics and correctly reproduce the central charge for the non-Abelian case.

In this letter we apply this method to characterize the spinon excitations of the Bethe Ansatz soluble spin-$S$ SU(2) Heisenberg chains [10,11]. The particular form of the Hamiltonian is not of interest here, it contains exchange interactions of nearest neighbour spins only that are given as polynomials of degree $2S$ in $S_j \cdot S_{j+1}$. The eigenstates for the spin $S$ chain of length $L$ are characterized by complex numbers $\lambda_j$, $j = 1, \ldots, M$ satisfying the system of Bethe Ansatz equations

$$\left( \frac{\lambda_j + iS}{\lambda_j - iS} \right)^L = \prod_{k \neq j} \frac{\lambda_j - \lambda_k + i}{\lambda_j - \lambda_k - i}.$$
It is well known that regular Bethe states (corresponding to solutions with $\lambda_j < \infty$) are highest weight states with spin $S_{\text{tot}} = LS - M$, limiting $M/L$ by $S$ \cite{12}. Moreover, in the thermodynamic limit $L \to \infty$ all solutions of (3) are built from so called strings $\lambda^{(n)} + \frac{1}{2} (n + 1 - 2 j)$ with real centers $\lambda^{(n)}$, and $j = 1, \ldots, n$.

Based on this classification the completeness of Bethe states has been proven for the periodic chain \cite{j3}: after rewriting (3) in terms of the real $\lambda^{(n)}$ any Bethe state is characterized uniquely by a collection of integer or half integer numbers $Q_j^{(n)}$. Now the number of possible states built from $\nu_n$ $n$-strings is just the number of admissible choices of quantum numbers $Q_j^{(n)} \neq Q_k^{(n)}$ for $j \neq k$. The latter are found to be restricted to the intervals $[-Q_{\text{max}}^{(n)}, Q_{\text{max}}^{(n)}]$ where

$$2Q_{\text{max}}^{(n)} + 1 = L \min(n, 2S) + \nu_n - \sum_{m=1}^{\infty} G_{mn} \nu_m$$

with $G_{mn} = 2 \min(m, n)$. In particular, the antiferromagnetic vacuum of a chain of even length is the unique singlet state in which all $L/2$ modes for the 2S-strings are occupied.

In a recent study \cite{14} of a related ideal system the “strings” appearing as solutions of a simplified version of the Bethe Ansatz equations (3) have been interpreted as the elementary objects spanning the Fock space of the system. In this picture one obtains the infinite dimensional matrix $G_{mn}$, for the statistical interaction of the system (see also \cite{15}). One should note, however, that the role of the various strings in building the excitations over the physical vacuum, namely the antiferromagnetic ground state of the system is quite different: in the thermodynamic limit, the energy and momentum of a configuration is completely determined by the distribution of holes in the sea of 2S strings, while the ones with $n \neq 2S$ are merely responsible for the correct counting of states \cite{10}. In particular, different configurations of the $n > 2S$-strings produce the $2^n$-fold degeneracy of the multiplet due to the spin $\frac{1}{2}$ degree of freedom carried by the $m$ holes (spinons) while the $n < 2S$ strings distinguish states with the same number of holes and total spin. Their presence is reflected in the $S$-dependence of the critical behaviour of the system \cite{11}: the low-energy effective field theory of the system has been identified as a $SU(2)_{2S}$ WZW model. Separating this field theory into a bosonic sector (with central charge $c = 1$) and a $Z_{2S}$ parafermionic sector \cite{17} the $n < 2S$ strings have been assigned to the latter. In a different approach they have been identified as spanning the state space of a restricted solid on solid (RSOS) model on the basis of the number of $m$-particle states \cite{17}.

First let us briefly recall the situation in the $S = \frac{1}{2}$ case where the ground state is a completely filled band of $\nu_1 = L/2$ (real) 1-strings. The elementary excitations above the antiferromagnetic vacuum can be identified with holes in their distribution \cite{12}. For chains of even (odd) length $L$ states with even (odd) number $n$ of holes form “Yangian” multiplets containing states with total spin $S = 0(\frac{1}{2}), \ldots, m/2$ which differ in the number of $n > 1$-strings fixing the total spin of the resulting state. Counting these states (including their $SU(2)$ multiplicity) the dimension of the $m$-spinon subspace is

$$d_m = \binom{L + 1}{m}.$$  

Summing over the admissible $m$ this gives the correct total number of states $2^L$. Here the statistics of the excitations is modeled by hard core lattice bosons reflecting the level–1 $SU(2)$ Kac-Moody symmetry of the critical theory. Taking into account the spin degree of freedom one finds that the spectrum of the $m$-spinon states decomposes exactly into the states present in the tensor product of $m$ spins $\frac{1}{2}$: in the $m = 2$ spinon sector there are $(L/2 + 1)$ triplets (built from 1-strings only) and $(L/2)$ singlets (containing a single 2-string). Thus we are led to distinguish spinons with different spin projection and to rewrite (3) as a sum over configurations with $n_\uparrow$ ($n_\downarrow$) spinons with spin $\uparrow$ ($\downarrow$) as

$$d_m = \sum_{\{n_\sigma\}} \delta_{n_\uparrow + n_\downarrow, m} \prod_\sigma \binom{d_\sigma + n_\sigma - 1}{n_\sigma}$$

where $d_\sigma (n_\uparrow, n_\downarrow) = \binom{1}{2} (L - n_\uparrow - n_\downarrow) + 1$. This gives $g_{\sigma \sigma'} = \frac{1}{2}$ establishing the semionic nature of the spinons just as in the ideal spinon gas realized in the Haldane-Shastry model \cite{18}. The difference between this model and the nearest neighbour Heisenberg model considered here is that a spinon-spinon interaction lifts the degeneracy of states with different total spin in the super multiplets when their density becomes finite. Following Ref. \cite{19} the distribution function of spinons is

$$n(\epsilon) = \frac{2}{1 + e^{\beta(\epsilon - \mu)}}$$

(6)
which gives the low temperature specific heat \( C/L = \pi T/3v_F \) of a \( c = 1 \) model (\( v_F \) is the Fermi velocity of the quasi particles).

For \( S > \frac{1}{2} \) one finds from (3) that there is a one to one mapping between the possible configurations of \( n \geq 2S \) strings between the spin-\( S \) model and the spin-\( \frac{1}{2} \) Heisenberg chain (see Table I): for any \( m \)-spinon Bethe state with total spin \( \leq m/2 \) in the \( S = \frac{1}{2} \) chain with string configuration \( \{\nu_n^{(1/2)}\} \) there exist Bethe states in the spin-\( S \) chain with string configuration \( \{\nu_n^{(S)}\} \) such that \( \nu_n^{(S)} = \nu_n^{(1/2)} \) for \( n > 1 \) with the same number \( m \) of holes in the distribution of \( 2S \)-strings. In addition, there are certain \( n < 2S \) strings present in these states. Note that this is seen only when considering holes, the number \( \nu_{2S} \) of \( 2S \)-strings differs for different configurations of the shorter strings. Proceeding as in the \( S = \frac{1}{2} \) case the number of \( 2m \) spinon states in a chain of even length \( L \) is found to be

\[
d_{2m} = \frac{r_{2S-1}^{2S-1}}{m} \left[ \frac{L + 1 + 2n_1}{2m} \right] \prod_{i=0}^{2S-2} \left( \frac{n_i + n_{i+2}}{2n_{i+1}} \right)
\]

with \( n_0 = m, n_{2S} = 0 \) in the product. Eq. (3) and a similar expression for odd \( L \) correctly reproduce the total number of states for the spin-\( S \) chain. However, an interpretation in terms of Haldane statistics, i.e. the ansatz (2) for spin-\( \frac{1}{2} \) quasi particles with an additional internal degree of freedom turns out to be impossible.

To proceed we apply the method introduced in [20]. We identify the Bethe states of the spin chain with the spinon basis of the chiral SU(2)\(_{k=2S}\) WZW model proposed in [21] (a realization of the chiral CFT is the open boundary version of the Takhtajan-Babujian models whose eigenstates are enumerated by the same relation (3)). In this basis the m-spinon states are of the form

\[
\phi^{\sigma_m} \left( \frac{1}{j_{m}j_{m-1}} \right) \cdots \phi^{\sigma_2} \left( \frac{1}{j_2j_1} \right) \phi^{\sigma_1} \left( \frac{1}{j_10} \right) \left| 0 \right>
\]

where each operator \( \phi^{\sigma_j} \) creates a spinon with spin projection \( \sigma_j = \uparrow, \downarrow \) and energy \( \epsilon(n_i + \Delta_i) = (2\pi v_F/L)(n_i + \Delta_i) \) (\( \left| 0 \right> \) is the Fock vacuum). The numbers \( j_i \) in (8) take values \( 0, \frac{1}{2}, \ldots, \frac{2S-1}{2} \) subject to the fusion rules \( \left| j_i - j_{i-1} \right| = \frac{1}{2} \) of the CFT and we have defined \( \Delta_i = \Delta(j_i) - \Delta(j_{i-1}) \) where \( \Delta(j) = j(j+1)/(k+2) \). A basis of the spinon Fock space is provided by the states (3) with \( \sigma_1 = \cdots = \sigma_{N_1} = \uparrow \), \( \sigma_{N_1} = \downarrow \). The modes \( n_i = n_{i+1} + \tilde{n}_i \) and non decreasing sequences of non negative integers \( \tilde{n}_i \) for \( i = 1 \) to \( i = N_1 + N_1 \). The minimal allowed mode sequence is constructed from \( n_{1\text{,min}} = 0 \) and

\[
n_{i+1\text{,min}} = \begin{cases} n_{i\text{,min}} + 1 & \text{if } j_{i+1} = j_i < j_i \\ n_{i\text{,min}} & \text{otherwise} \end{cases}
\]

To compute the distribution function for these modes we introduce truncated partition functions \( q^\Delta(j)X_{\ell}^{(j)}(q, x, h) \) of the chiral CFT for which the occupied spinon modes should satisfy \( n_N \leq \ell \) and \( j_N = j_N = j \geq 0 \). Here \( \mu + \alpha \) are chemical potentials for the spinons and we write \( q = e^{-\beta(2\pi v_F/L)} \), \( x = e^{\beta \mu} \). These partition functions satisfy recursion relations

\[
\left( \begin{array}{c} X_\ell^{(0)} \\ \vdots \\ X_\ell^{(k)} \end{array} \right) = \mathcal{R}_\ell(q, x, \beta h) \left( \begin{array}{c} X_{\ell-1}^{(0)} \\ \vdots \\ X_{\ell-1}^{(k)} \end{array} \right)
\]

with initial condition \( X_0^{(j)} = c_jx^j, c_j = \sinh((j+1)/2)/\sinh(\beta h/2) \). The matrix \( \mathcal{R}_\ell = R(xq^\ell, \beta h)D(x^2q^{2\ell-1}) \) is given in terms of \( D(y) = \text{diag}(1 - y, \ldots, 1 - y, 1) \) and the symmetric matrix

\[
R(y, \beta h) = \begin{pmatrix} 1 & c_1y & c_2y^2 & \cdots & c_{2S}y^{2S} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (1 + cy^2) & (1 + cy^2) & \cdots & (1 + cy^2) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_1y & c_2y^2 & \cdots & \cdots & \cdots \\ c_2y^2 & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \sum_{i=0}^{2S} c_iy^{2i} \end{pmatrix}
\]

Below it will be useful to factorize the matrix \( \mathcal{R}_\ell \) in (9) further as \( \tilde{\mathcal{R}}_\ell \mathcal{R}_{\ell-\frac{1}{2}} \). with \( \tilde{\mathcal{R}}_\ell = \tilde{R}(xq^\ell, \beta h)D(x^2q^{2\ell-1}) \). The non-zero matrix elements of this new matrix are \( (\tilde{R}(y, \beta h))_{i,k-j} = c_{i-j}y^{i-j} \) for \( 0 \leq i - j \leq k \).
Remarkably, the number of \( m \)-spinon states contributing to the partition function \( X_{\ell}^{(0)} \) coincides with \( \frac{1}{2} \) for the spin-\( S \) chain of even length \( L = 2\ell \). Since the finite size spectrum of low lying excitations of the spin chain is determined by the operator content of the \( SU(2)_k \) WZW model we can identify \( X_{\ell}^{(0)} \) with the partition function \( Z_L \) of the spin chain of length \( L = 2\ell \) and similarly \( q^{\lambda(k)}X_{\ell}^{(k)} = Z_{2\ell-1} \). In this sense the partition function of the integrable Heisenberg chain can be generated by a recursion relation similar to (9) with the matrix \( \bar{R}_{\ell} \). In addition, this observation provides further evidence for the equivalence of the spinons of the integrable chain with the ones forming the quasi particle basis \( \frac{1}{4} \) to higher level \( SU(2) \) WZW models of Ref. [21].

Viewing the \( k + 1 \) spinon modes added in the \( \ell \)th iteration of \( \frac{1}{4} \) as a single degenerate level in the single particle spectrum we can now approximate the partition function of the spin chain as \( Z_{2\ell} = \prod_{i=1}^{\ell} \lambda_i^2 \) with the maximal eigenvalues \( \lambda_i = \lambda^{(+)}(y = xq^i, \beta h) \) of the recursion matrix \( \bar{R}(y, \beta h)D(y^2) \). This leads to a spinon distribution function

\[
n(\epsilon) = 2y\partial_y \ln \lambda^{(+)}(y, \beta h), \quad y = e^{-\beta(\epsilon - \mu)}
\]

for the spin-\( S \) chain. For \( k = 1 \) this function has been studied in Ref. [8] where complete agreement with the corresponding results from Haldane exclusion statistics (e.g. \( \frac{1}{4} \)) for \( h = 0 \) could be established. In the general non-Abelian case \( k = 2S > 1 \) it is not possible to obtain a closed expression for \( n(\epsilon) \), for large negative \( \epsilon \) we find \( n(\epsilon) \sim 2k - (2 + 4\delta_{k,2})e^{2\beta(\epsilon - \mu)} \). The maximal occupation \( 4S \) of a level containing \( 2S + 1 \) spinon modes coincides with the result obtained from \( \frac{1}{4} \) using the fusion rules of the CFT. Furthermore we have checked numerically for spins up to \( S = 3 \) that the distribution function \( \frac{1}{4} \) gives the correct behaviour \( \frac{1}{4} \) of the low temperature specific heat [22].

For \( k = 2 \) the eigenvalues determining the distribution function for \( h = 0 \) and \( h \to \infty \) (leading to complete polarization of the spinons) are the largest real solutions of

\[
(\lambda - 1)^2 = y^2(\lambda + 1) \quad \text{for} \ h = 0,
\]

\[
(\lambda - 1)^2(\lambda + 1) = y^2 e^{\beta h/2}\lambda^2 \quad \text{for} \ h \to \infty.
\]

In Fig. 1 the distribution functions obtained from (12) are presented in comparison with the one for particles obeying \( g = \frac{1}{2} \) exclusion statistics.

To summarize, considering integrable lattice realizations of the \( SU(2)_k \) WZW models we have shown that the truncation scheme for chiral spectra of a CFT introduced in \( \frac{1}{4} \) can be interpreted as a recursion relation for partition functions of the corresponding lattice model. The truncated partition functions of the field theory show complete equivalence between the well established spinon picture for the integrable spin chain and the spinon formulation of the CFT. For the spin chains this method allows to compute distribution functions for spinons obeying non-Abelian exchange statistics. Frequently, the low-lying excitations in one-dimensional \( S = \frac{1}{2} \) systems have been interpreted by invoking a spinon-picture (see e.g. Ref. [23]). Similarly, we expect that the characterization of the spinons in higher \( S \) models given here will prove useful for a better understanding of massive perturbations of the latter such as the \( S = 1 \) Haldane system.

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FIG. 1. Distribution function obtained from (11) for the spinons of the spin-1 chain: shown are $n(\varepsilon)$ for vanishing magnetic field (bold curve) and for the fully polarized spinon system (dashed curve) in comparison with the result for $g = \frac{3}{4}$ exclusion statistics (thin curve).

TABLE I. String configurations for a Bethe state with given number $m$ of holes in the distribution of $2S$ strings and total spin $S_{tot}$: $[m^p]^q$ denotes a solution with $\nu_n = 1, \nu_p = 2$. For $S = 2$ the configuration of strings with $n > 2S$ ($< 2S$) are listed separately, for a Bethe state the $n > 2S$ configuration has to be supplemented by one of the possible $n < 2S$ configurations listed in the last column.

| $m$ | $S_{tot}$ | $S = \frac{1}{2}$ | $S = 2$ |
|-----|-----------|------------------|--------|
| 2   | 1         | [2]              | [3]    |
|     | 0         | [5]              | [3]    |
| 4   | 2         | [2]              | [3]    |
|     | 1         | [5]              | [3]    |
|     | 0         | [6],[5]          | [3]    |
| 6   | 3         | [3]              | [3], [3] |
|     | 2         | [2]              | [3], [3] |
|     | 1         | [5]              | [3], [3] |
|     | 0         | [6],[5],[3]      | [3], [3] |