IWAHORI-HECKE MODEL FOR MOD $p$ REPRESENTATIONS OF $GL_2(F)$

U. K. ANANDAVARDHANAN AND ARINDAM JANA

Abstract. For a $p$-adic field $F$, the space of pro-$p$-Iwahori invariants of a universal supersingular mod $p$ representation $\tau$ of $GL_2(F)$ is determined in the works of Breuil, Schein, and Hendel. The representation $\tau$ is introduced by Barthel and Livné and this is defined in terms of the spherical Hecke operator. In [ABL13, ABL15], an Iwahori-Hecke approach was introduced to study these universal supersingular representations in which they can be characterized via the Iwahori-Hecke operators. In this paper, we construct a certain quotient $\pi$ of $\tau$, making use of the Iwahori-Hecke operators. When $F$ is not totally ramified over $\mathbb{Q}_p$, the representation $\pi$ is a non-trivial quotient of $\tau$. We determine a basis for the space of invariants of $\pi$ under the pro-$p$ Iwahori subgroup. A pleasant feature of this “new” representation $\pi$ is that its space of pro-$p$-Iwahori invariants admits a more uniform description vis-à-vis the description of the space of pro-$p$-Iwahori invariants of $\tau$.

1. Introduction

For a $p$-adic field $F$, the study of irreducible smooth mod $p$ representations of $GL_2(F)$ started with the famous work of Barthel and Livné [BL94]. They showed that there exist irreducible smooth representations, called supersingular representations, which cannot be obtained as a subquotient of a parabolically induced representation.

It is shown in [BL94] that a supersingular representation can be realized as the quotient of a universal module constructed as follows. Let $G = GL_2(F)$ and let $K$ be its standard maximal compact subgroup. Let $Z$ denote the center of $G$. For an irreducible representation $\sigma$ of $KZ$, let $\text{ind}^G_{KZ}\sigma$ be the representation of $G$ compactly induced from $\sigma$. Its endomorphism algebra is a polynomial algebra in one variable:

$$\text{End}_G(\text{ind}^G_{KZ}\sigma) \cong \mathbb{F}_p[T],$$

where $T$ is the standard spherical Hecke operator and $\mathbb{F}_p$ denotes an algebraic closure of the finite field $\mathbb{F}_p$ of $p$ elements [BL94, Proposition 8]. The universal module in consideration is

$$\tau = \frac{\text{ind}^G_{KZ}\sigma}{(T)}$$

and a supersingular representation of $G$ is an irreducible quotient of the universal module for some $\sigma$ of $KZ$ up to a twist by a character [BL94].

Explicitly constructing a supersingular representation of $GL_2(F)$ is a challenging problem when $F \neq \mathbb{Q}_p$ [BP12]. When $F = \mathbb{Q}_p$, Breuil proved that the universal representation $\tau$ itself is irreducible [Bre03, Theorem 1.1]. The key step in Breuil’s proof of the irreducibility of $\tau$ is the explicit computation of its $I(1)$-invariant space, which is of dimension 2, where $I(1)$ is the pro-$p$-Iwahori subgroup of $K$ [Bre03, Theorem 3.2.4].

2020 Mathematics Subject Classification. Primary 20G05; Secondary 22E50, 11F70.
The space of $I(1)$-invariants of $\tau$ is infinite dimensional when $F \neq \mathbb{Q}_p$. An explicit basis for this infinite dimensional space is computed by Schein when $F$ is totally ramified over $\mathbb{Q}_p$ \cite{Schein} §2 and by Hendel more generally for any $p$-adic field $F$ \cite[Theorem 1.2]{Hendel}.

One can also construct a universal module from the perspective of the Iwahori-Hecke operators instead of the spherical Hecke operator $T$ \cite{AB13, AB15}. For this, instead of doing compact induction from an irreducible representation of $KZ$, we start with a regular character $\chi$ of $IZ$, where $I$ is the Iwahori subgroup $K$, and consider the compactly induced representation $\text{ind}_{IZ}^G \chi$. Its endomorphism algebra is \cite[Proposition 13]{BL94}:

$$\text{End}_G(\text{ind}_{IZ}^G \chi) \simeq \frac{\mathbb{F}_p[[T_{-1,0}, T_{1,2}]]}{(T_{-1,0}T_{1,2}T_{1,2}T_{-1,0})},$$

where $T_{-1,0}$ and $T_{1,2}$ are the Iwahori-Hecke operators. When $F$ is a totally ramified extension of $\mathbb{Q}_p$, it is proved in \cite[Proposition 3.1 & Remark 1]{AB15} that the image of one of these operators is equal to the kernel of the other; i.e.,

$$\text{Im } T_{-1,0} = \text{Ker } T_{1,2} \& \text{ Im } T_{1,2} = \text{Ker } T_{-1,0}.$$

Let $\mathbb{F}_q$ be the residue field of $F$ where $q = p^f$. Assume $0 < r < q - 1$ and write $r = r_0 + r_1 p + \cdots + r_{f-1} p^{f-1}$ with $0 \leq r_i \leq p - 1$ for $0 \leq i \leq f - 1$. Let

$$\sigma_r = \text{Sym}^0(\mathbb{F}_p^2) \otimes \text{Sym}^1(\mathbb{F}_p^2) \circ \text{Frob} \otimes \cdots \otimes \text{Sym}^{f-1}(\mathbb{F}_p^2) \circ \text{Frob}^{f-1}$$

be an irreducible representation of $GL_2(\mathbb{F}_q)$, where Frob is the Frobenius morphism. We continue to denote the corresponding irreducible representation of $K$, obtained via inflation, by $\sigma_r$. Similarly, let $\chi_r$ be the character of $I$, valued in $\mathbb{F}_p^\times$, obtained via the character of the Borel subgroup of $GL_2(\mathbb{F}_q)$ defined by

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto a^r.$$

We fix a uniformizing element $\omega$ of the ring of integers $\mathcal{O}$ of $F$. The representation $\sigma_r$ is treated as a representation of $KZ$ by making $\text{diag}(\omega, \omega)$ acting trivially and similarly the character $\chi_r$ is treated as a character of $IZ$.

For $g \in G$ and $v \in \sigma_r$, let $g \otimes v$ be the function in $\text{ind}_{KZ}^G \sigma_r$ supported on $KZg^{-1}$ that sends $g^{-1}$ to $\sigma_r(k)v$. Similarly, for $g \in G$, by $[g, 1]$ we define the function in $\text{ind}_{IZ}^G \chi_r$ which is supported on $IZg^{-1}$ and sending $g^{-1}$ to 1. It can be seen that every element of $\text{ind}_{KZ}^G \sigma_r$ (resp. $\text{ind}_{IZ}^G \chi_r$) is a finite sum of these type of functions $[g, 1]$ (resp. $g \otimes v$).

Now \cite[Theorem 1.1]{AB15} takes the form:

**Theorem 1.1.** Let $F$ be a finite extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$ and residue degree $f$. Let $0 < r < q - 1$ and $r = r_0 + r_1 p + \cdots + r_{f-1} p^{f-1}$ with $0 \leq r_i \leq p - 1$. Then

$$\tau_r = \frac{\text{ind}_{KZ}^G \sigma_r}{(T)} \simeq \frac{\text{ind}_{IZ}^G \chi_r}{(\text{Im } T_{1,2}, \text{Ker } T_{1,2})}.$$
Moreover, this isomorphism is determined by

\[
\text{Id} \otimes \bigotimes_{j=0}^{f-1} x_j^f \mod T \mapsto [\beta, 1] \mod (\text{Im } T_{1,2}, \text{Ker } T_{1,2}).
\]

Remark 1. Theorem 1.1 is stated and proved in [AB15, Theorem 4.1] when \(F\) is a totally ramified extension of \(\mathbb{Q}_p\) (see [AB15, Remark 3]) and exactly the same proof goes through in the general case as well.

Remark 2. As mentioned earlier, the space of \(I(1)\)-invariants of \(\tau_r\) is computed by Hendel [Hen19, Theorem 1.2]. Stating an explicit basis for this space involves four cases; (i) \(e = 1, f = 1\), (ii) \(e > 1, f = 1\), (iii) \(e = 1, f > 1\), and (iv) \(e > 1, f > 1\).

In this paper, we study a new universal representation given by

\[
\pi_r = \frac{\text{ind}^G_{I \mathcal{Z} \chi_r}}{(\text{Ker } T_{-1,0}, \text{Ker } T_{1,2})}
\]

which is a further quotient of \(\tau_r\). Note that this representation equals \(\tau_r\) when \(F\) is totally ramified over \(\mathbb{Q}_p\) by (1). We show that when \(F\) is not totally ramified over \(\mathbb{Q}_p\), we have strict containments

\[
\text{Im } T_{-1,0} \subset \text{Ker } T_{1,2} \& \text{Im } T_{1,2} \subset \text{Ker } T_{-1,0}.
\]

and thus we have a new representation to investigate for its properties (cf. Remark 4). At this stage, we also note that the representation \(\pi_r\) is indeed non-trivial (cf. Lemma 3.4).

The main result of this paper gives an explicit basis for the space of \(I(1)\)-invariants of \(\pi_r\). This space turns out to be infinite dimensional as well as in the case of [Hen19, Theorem 1.2]. However, in this case the basis can be written in a uniform manner whenever \(F \neq \mathbb{Q}_p\). Thus, the statement involves only two cases; (i) \(F = \mathbb{Q}_p\) and (ii) \(F \neq \mathbb{Q}_p\). It is interesting to compare our result with that of Hendel in this aspect (cf. Remark 2).

In order to state the theorem, we introduce a few more notations. Set \(I_0 = \{0\}\), and for \(n \in \mathbb{N}\), let

\[
I_n = \left\{ [\mu_0] + [\mu_1] \omega + \cdots + [\mu_{n-1}] \omega^{n-1} \mid \mu_i \in \mathbb{F}_q \right\} \subset \mathcal{O},
\]

where, for \(x \in \mathbb{F}_q\), we denote its multiplicative representative in \(\mathcal{O}\) by \([x]\). If \(0 \leq m \leq n\), let \([\cdot]_m : I_n \to I_m\) be the truncation map defined by

\[
\sum_{i=0}^{n-1} [\lambda_i] \omega^i \mapsto \sum_{i=0}^{m-1} [\lambda_i] \omega^i.
\]

Let us denote

\[
\alpha = \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 1 \\ \omega & 0 \end{pmatrix}, \quad w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]
and observe that $\beta = \alpha w$ normalizes $I(1)$. For any $n \in \mathbb{N}$, we denote

$$s^k_n = \sum_{\mu \in I_n} \left( \alpha^n 1 \mu \right),$$
$$t^s_n = \sum_{\mu \in I_n} \left( \alpha^{n-1} \mu \binom{n-1}{1} 1 \mu \right) w, 1,$$

where $0 \leq k, s \leq q - 1$. For $0 \leq l \leq f - 1$ and $m \geq 1$, we define the following sets

$$S^l_m = \{ s^{q-1-r+p^j}_n \}_{n \geq m} \cup \{ \beta s^{q-1-r+p^j}_n \}_{n \geq m},$$

$$S_m = \bigcup_{l=0}^{f-1} S^l_m,$$

$$T^l_m = \{ t^{r+p^j}_n \}_{n \geq m} \cup \{ \beta t^{r+p^j}_n \}_{n \geq m},$$

$$T_m = \bigcup_{l=0}^{f-1} T^l_m.$$

Now we state the main theorem of this paper.

**Theorem 1.2.** Let $F$ be a finite extension of $\mathbb{Q}_p$ with ramification index $e$. Let $\mathbb{F}_q$ be the residue field of $F$ with $q = p^f$. Let $0 < r < q - 1$ and $r = r_0 + r_1 p + \cdots + r_{f-1} p^{f-1}$ with $0 < r_j < p$ for all $0 \leq j \leq f - 1$. When $f = 1$, we assume $2 < r < p - 3$. Then a basis of the space of $I(1)$-invariants of the representation $\pi_r = \text{ind} G^{\text{I}Z} \chi_r (\text{Ker} T_{-1,0,0}, \text{Ker} T_{1,2})$ as an $\mathbb{F}_p$-vector space is given by the images of the following sets in $\pi_r$:

1. $\{ [\text{Id}, 1], [\beta, 1] \}$ when $F = \mathbb{Q}_p$
2. $S_2 \cup \{ [\text{Id}, 1], [\beta, 1] \} \cup T_2$ when $F \neq \mathbb{Q}_p$.

**Remark 3.** The representation $\pi_r$ that we construct and investigate in this paper is a quotient of the representation $\tau_r$ considered in [BL94, Bre03, Sch11, Hen19];

$$0 \to \text{Ker} T_{-1,0,0} \to \tau_r \to \pi_r \to 0.$$

When $F$ is totally ramified over $\mathbb{Q}_p$, the representations $\tau_r$ and $\pi_r$ are isomorphic by Theorem 1.1 together with the equality of spaces in (1). However, $\pi_r$ is a “new” representation when $F$ is not totally ramified over $\mathbb{Q}_p$. That there is no isomorphism between $\tau_r$ and $\pi_r$ can be checked, for instance, from the characterization of the space of $I(1)$-invariants of $\pi_r$ in Theorem 1.2 and that of $\tau_r$ in [Hen19, Theorem 1.2]. We give more details in §4.5.

Following the argument in [Hen19, Conclusion 3.10] word to word, we get the following corollary to Theorem 1.2.

**Corollary 1.3.** The representation $\pi_r$ is indecomposable; i.e., $\text{End}_G(\pi_r) \simeq \mathbb{F}_p$. 
The plan of the paper is as follows. We collect many results about the Iwahori-Hecke operators in Section 3. Several of these results are contained in some form in [AB13, AB15]. Theorem 1.2 and the key ideas in its proof are inspired by the work of Hendel [Hen19], though the Iwahori-Hecke approach which is employed in this paper as in [AB13, AB15] seems to be more amenable to carrying out the necessary calculations. We take up the proof in Section 4.

2. Two basic results

As in the work of Hendel [Hen19], we will need to frequently make use of the following two results in our computations.

Theorem 2.1 (Lucas). Let $n, r \in \mathbb{N}$ be such that $n = \sum_{i=0}^{k} n_i p^i$ and $r = \sum_{i=0}^{k} r_i p^i$, where $0 \leq n_i \leq p-1$ and $0 \leq r_i \leq p-1$. Then

$$\binom{n}{r} \equiv \prod_{i=0}^{k} \binom{n_i}{r_i} \mod p.$$  

Corollary 2.2. Let $n, r \in \mathbb{N}$. Then $p$ divides $\binom{n}{r}$ if and only if $n_i < r_i$ for some $0 \leq i \leq k$.

The next result gives a formula for adding multiplicative representatives in $O$ [Hen19, Lemma 1.7]. As in [Hen19], this formula will play a crucial role in the calculations to follow.

Lemma 2.3. Let $x, y \in \mathbb{F}_q$ with $q = p^f$. Then

$$[x] + [y] \equiv [x + y] + \omega^f[P_0(x, y)] \mod \omega^{f+1},$$

where $P_0(x, y) = \frac{x^f + y^f - (x+y)^f}{\omega^f}$.

3. Preliminaries on the Iwahori-Hecke operators

For $n \in \mathbb{N} \cup \{0\}$ and $\lambda \in I_n$, define

$$g_{n,\lambda}^0 = \begin{pmatrix} \omega^n & \lambda \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad g_{n,\lambda}^1 = \begin{pmatrix} 1 & 0 \\ \omega \lambda & \omega^{n+1} \end{pmatrix}.$$  

We have the relations

$$g_{0,0}^0 = \text{Id}, \quad g_{0,0}^1 = \alpha, \beta g_{n,\lambda}^0 = g_{n,\lambda}^1 \omega.$$  

Now $G$ acts transitively on the Bruhat-Tits tree of $SL_2(F)$, whose vertices are in a $G$-equivariant bijection with the cosets $G/KZ$ and whose oriented edges are in a $G$-equivariant bijection with the cosets $G/IZ$. We have the explicit Cartan decomposition given by

$$G = \prod_{i \in \{0,1\}, n \geq 0, \lambda \in I_n} g_{n,\lambda}^i KZ.$$
and an explicit set of coset representatives of $G/I\mathbb{Z}$ is given by
\begin{equation}
\left\{ g^{0}_{n,\lambda} w r g^{1}_{n,\lambda} w r g^{1}_{n,\lambda} w \left( \begin{array}{cc} 1 & \mu \\ 0 & 1 \end{array} \right) w \right\}_{n \geq 0, \lambda \in I_{n},}
\end{equation}
where $\mu \in I_{1}$.

Now we recall a few details about the Iwahori-Hecke algebra [BL94, §3.2]. By definition, this algebra, denoted by $H(I\mathbb{Z}, \chi_{r})$, is the endomorphism algebra of the compactly induced representation $\text{ind}^{G}_{I\mathbb{Z}} \chi_{r}$. For $n \in \mathbb{Z}$, let $\phi_{n,n+1}$ denote the convolution map supported on $I\mathbb{Z}a^{-n}I$ such that $\phi_{n,n+1}(a^{-n}) = 1$ [BL94, Lemma 9]. We denote by $T_{n,n+1}$ the corresponding element in $H(I\mathbb{Z}, \chi_{r})$. By [BL94, Proposition 13], for $0 < r < q - 1$, we have:
\begin{equation}
H(I\mathbb{Z}, \chi_{r}) \simeq \mathbb{F}_{p}[T_{-1,0}, T_{1,2}] / (T_{-1,0}T_{1,2}, T_{1,2}T_{-1,0}).
\end{equation}
Substituting $n = 1$ in [BL94, (16), (17)], we have the following explicit formulas for $T_{-1,0}$ and $T_{1,2}$:
\begin{equation}
T_{-1,0}([g, 1]) = \sum_{\lambda \in I_{1}} \left[ g^{0}_{\lambda} 1, 1 \right],
\end{equation}
\begin{equation}
T_{1,2}([g, 1]) = \sum_{\lambda \in I_{1}} \left[ g^{0}_{\lambda} \left( \begin{array}{cc} 1 & \lambda \\ 0 & 1 \end{array} \right) w, 1 \right].
\end{equation}

The following proposition characterizes the kernel of the Iwahori-Hecke operators $T_{-1,0}$ and $T_{1,2}$ [AB13, AB15].

**Proposition 3.1.** We have:

(1) Ker $T_{-1,0}$ is generated as a $G$-module by the vectors
   \begin{enumerate}
   \item[(a)] $(-1)^{q-1-r} s^{0}_{0} + t^{1}_{1}$,
   \item[(b)] $t^{1}_{s}$ where $0 \leq s \leq r - 1$,
   \item[(c)] $t^{1}_{s}$ where $s > r$ and $(q^{s-1-r}) \equiv 0 \mod p$.
\end{enumerate}

(2) Ker $T_{1,2}$ is generated as a $G$-module by the vectors
   \begin{enumerate}
   \item[(a)] $t^{0}_{k} + s^{q-1-r}_{1}$,
   \item[(b)] $s^{1}_{k}$ where $0 \leq k \leq q - 2 - r$,
   \item[(c)] $s^{1}_{k}$ where $k > q - 1 - r$ and $(q^{r-1-k}) \equiv 0 \mod p$.
\end{enumerate}

**Proof.** We indicate the proof for Ker $T_{1,2}$, with the other case being similar. An arbitrary vector in $\text{ind}^{G}_{I\mathbb{Z}} \chi_{r}$ is an $\mathbb{F}_{p}$-linear combination of vectors $[g, 1]$, where $g$ is in the set of coset representatives (3) of $G/I\mathbb{Z}$. Arguing as in the proof of [AB15 Proposition 3.1], we can restrict our attention to the vectors
\begin{equation}
\left\{ [\text{Id}, 1], [\beta, 1], [g^{0}_{\mu}, 1], \left[ \left( \begin{array}{cc} 1 & \mu \\ 0 & 1 \end{array} \right) w, 1 \right] \right\}
\end{equation}
for $\mu \in I_{1}$. Now the proof boils down to elementary linear algebra as in [AB15 Lemma 3.2], where one is led to analyse the indices $i$ for which
\begin{equation}
\sum_{\mu \in \mathbb{F}_{q}} \mu^{i}(\mu - \lambda)^{r} = 0,
\end{equation}
for \( \lambda \in \mathbb{F}_q \). Alternatively, this last step can be deduced directly from the explicit formulas for the Iwahori-Hecke operators in [AB13, p. 63-64].

\( \Box \)

Remark 4. We remarked in (2) in Section 1 that we have strict containments

\[ \text{Im } T_{-1,0} \subsetneq \text{Ker } T_{1,2} \text{ & Im } T_{1,2} \subsetneq \text{Ker } T_{-1,0}. \]

when \( F \) is not a totally ramified extension of \( \mathbb{Q}_p \). The reason for this is that the third type of vectors in both (1) and (2) in Proposition 3.1 do not belong to the images of the Iwahori-Hecke operators. Note that such vectors do not exist when \( f = 1 \); i.e., when \( q = p \). By the argument in [AB15, Lemma 3.2], it can be shown that the first two types of vectors are indeed in the image of the relevant Iwahori-Hecke operator.

**Corollary 3.2.** A basis of the space of \( I(1) \)-invariants of \( \text{Ker } T_{-1,0} \) is given by \( \{ t_{0}^{n}, \beta t_{0}^{n} \}_{n \geq 1} \) and that of \( \text{Ker } T_{1,2} \) is given by \( \{ s_{0}^{n}, \beta s_{0}^{n} \}_{n \geq 1} \). Moreover, the action of \( I \) is given by

\[
\left( \begin{array}{cc}
    a & b \\
    \omega c & d
\end{array} \right) \cdot v = \begin{cases}
    a'v & \text{or } t_{0}^{n}, \\
    d'v & \text{or } s_{0}^{n} or \beta t_{0}^{n}.
\end{cases}
\]

**Proof.** The first part of Proposition 3.1 together with the observation that the space of \( I(1) \)-invariants of the full induced representation is given by

\[
(\text{ind}_{I(Z)}^{G} \chi_{r})^{I(1)} = \langle s_{0}^{n}, t_{0}^{n}, \beta s_{0}^{n}, \beta t_{0}^{n} \rangle_{n \geq 0}.
\]

For the second part, observe that since

\[
I/I(1) = \left\{ \left( \begin{array}{cc}
    a & 0 \\
    0 & d
\end{array} \right) \mid a, d \in \mathbb{F}_q^\times \right\},
\]

it follows that

\[
\left( \begin{array}{cc}
    a & b \\
    \omega c & d
\end{array} \right) \& \left( \begin{array}{cc}
    a & 0 \\
    0 & d
\end{array} \right)
\]

have the same action on any \( I(1) \)-invariant vector. Now, for any \( k \geq 0 \), we have

\[
\left( \begin{array}{cc}
    a & 0 \\
    0 & d
\end{array} \right) s_{n}^{k} = \left( \begin{array}{cc}
    a & 0 \\
    0 & d
\end{array} \right) \sum_{\mu \in \mathbb{I}_{n}} \mu_{n-1}^{k} \left[ \left( \begin{array}{cc}
    \omega^{n} & \mu \\
    0 & 1
\end{array} \right), 1 \right]
\]

\[
= \sum_{\mu \in \mathbb{I}_{n}} \mu_{n-1}^{k} \left[ \left( \begin{array}{cc}
    \omega^{n} & ad^{-1} \mu \\
    0 & 1
\end{array} \right) \left( \begin{array}{cc}
    a & 0 \\
    0 & d
\end{array} \right), 1 \right]
\]

\[
= d^{r}(da^{-1})^{k}s_{n}^{k}.
\]

A similar computation gives

\[
\left( \begin{array}{cc}
    a & 0 \\
    0 & d
\end{array} \right) t_{n}^{k} = a^{r}(da^{-1})^{k}s_{n}^{k}.
\]

Similarly, we can check the action on \( \beta s_{n}^{k} \) and \( \beta t_{n}^{k} \). \( \Box \)

Next, we recall [AB15, Proposition 3.3], whose proof in [loc. cit.] is valid for any \( q \).

**Proposition 3.3.** We have

\[ \text{Ker } T_{-1,0} \cap \text{Ker } T_{1,2} = \{ 0 \}. \]

As a corollary to Proposition 3.3 we have the following lemma.
Remark 5. Which is related to the condition in (c) of Proposition 3.1 (2). Similarly, the condition in Lemma 3.5 (1) is precisely what gives, by Theorem 2.1, valid for any $q$

Lemma 3.5.

We end this section with two more results which immediately follow from considerations similar to Proposition 3.1. We state these in a ready to use format here (see also [AB13, p. 63-64]).

**Lemma 3.5.** Let $0 \leq i_j \leq q - 1$ for $0 \leq j \leq n - 1$ and $\mu = [\mu_0] + [\mu_1] \omega + \cdots + [\mu_{n-1}] \omega^{n-1} \in I_n$. Write $i_{n-1} = i_{n-1,0} + i_{n-1,1} p + \cdots + i_{n-1,f-1} p^{f-1}$. Then

1. $\sum_{\mu_0} \cdots \sum_{\mu_{n-1}} [g_{n,n+1}^0 \sum_{\mu_{n+1,1}^1} \cdots \mu_{n+1,n-1}^1] \in \text{Ker } T_{1,2}$ if and only if $0 \leq i_{n-1} \leq q - 2 - r$ or $i_{n-1} > q - 1 - r$ such that $i_{n-1,j} < p - 1 - r_j$ for some $0 \leq j \leq f - 2$.

2. $\sum_{\mu_0} \cdots \sum_{\mu_{n-1}} [g_{n,n+1}^0 \sum_{\mu_{n+1,1}^1} \cdots \mu_{n+1,n-1}^1] \in \text{Ker } T_{-1,0}$ if and only if $0 \leq i_{n-1} \leq r - 1$ or $i_{n-1} > r$ such that $i_{n-1,j} < r_j$ for some $0 \leq j \leq f - 2$.

**Remark 5.** Note that in Lemma 3.5 the range for $j$ is $0 \leq j \leq f - 2$ because

$i_{n-1} > q - 1 - r \implies i_{n-1,f-1} \geq p - 1 - r_{f-1}$.

**Remark 6.** Note that the condition

$i_{n-1} > q - 1 - r \& i_{n-1,j} < p - 1 - r_j$ for some $0 \leq j \leq f - 2$

in Lemma 3.5 (1) is precisely what gives, by Theorem 2.1

$$\left( q - 1 - i_{n-1} \right) \equiv 0 \mod p$$

which is related to the condition in (c) of Proposition 3.1 (2). Similarly, the condition

$i_{n-1} > r \& i_{n-1,j} < r_j$ for some $0 \leq j \leq f - 2$

in Lemma 3.5 (2) is related to (c) of Proposition 3.1 (1).

The following lemma is [AB13, Lemma 3.1]. We note that its proof in [loc. cit.] is valid for any $q$. 

**Lemma 3.4.** For the iwahori-hecke operators $T_{-1,0}$ and $T_{1,2}$, we have

$$\text{ind}_{I_n}^{G} \neq \text{Ker } T_{-1,0} \oplus \text{Ker } T_{1,2}.$$ 

**Proof.** If possible, let

$$\text{ind}_{I_n}^{G} = \text{Ker } T_{-1,0} \oplus \text{Ker } T_{1,2}.$$ 

Then we get $[\text{Id}, 1] = v_1 + v_2$ for some $v_1 \in \text{Ker } T_{-1,0}$ and $v_2 \in \text{Ker } T_{1,2}$. Then, for an element $g \in I$,

$$g(v_1 + v_2) = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) (v_1 + v_2) = d' [\text{Id}, 1] = d' (v_1 + v_2)$$

and this implies

$$g v_1 - d' v_1 = -g v_2 + d' v_2 = 0,$$

by Proposition 3.3. In particular, both $v_1$ and $v_2$ are $I(1)$-invariant. By Corollary 3.2, $v_1$ is a linear combination of vectors of the form $\{ \beta_n \}_{n \geq 1}$ and $v_2$ is a linear combination of vectors of the form $\{ s_n \}_{n \geq 1}$. But $[\text{Id}, 1]$ cannot be written as a linear combination of these types of vectors.

\[ \square \]
Lemma 3.6. Let \( \mu = [\mu_0] + \cdots + [\mu_{n-1}] \omega^{n-1} \in I_n \). Then modulo \((Ker T_{-1,0}, Ker T_{1,2})\), we have the identities

\[
\begin{align*}
(1) & \quad \sum_{\mu_{n-1} \in \mathbb{I}_1} \mu_{n-1}^{-r} \left[ g_{n-1,\mu_{n-1}}^0 \right] = -\left[ g_{n-2,\mu_{n-2}}^0 \left( \begin{array}{cc} 1 & [\mu_{n-2}] \\ 0 & 1 \end{array} \right) \right] w, 1, \\
(2) & \quad \sum_{\mu_{n-1} \in \mathbb{I}_1} \mu_{n-1}^{-r} \left[ g_{n-1,\mu_{n-1}}^0 \right] \left( \begin{array}{cc} 1 & [\mu_{n-1}] \\ 0 & 1 \end{array} \right) w, 1 = (-1)^{r-1} \left[ g_{n-1,\mu_{n-1}}^0 \right].
\end{align*}
\]

Remark. In fact, (1) is true modulo Ker \( T_{1,2} \) and (2) is true modulo Ker \( T_{-1,0} \) (cf. [AB13 (4) & (5) on p. 62]).

4. Proof of Theorem 1.2

In this section we take up the proof of Theorem 1.2. As mentioned in Section 1, several of the ideas of the proof here are already there in [Hen19].

4.1. A set of \( I(1) \)-invariants. First we make the following observation [Hen19 §2.1]. For \( a, b, c \in \mathcal{O} \), any matrix in \( I(1) \) can be written as

\[
\left( \begin{array}{cc} 1 + \omega a & b \\ \omega c & 1 + \omega d \end{array} \right) = \left( \begin{array}{cc} 1 & (1 + \omega d)^{-1} b \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ \omega c^{-1} & 1 \end{array} \right) \left( \begin{array}{cc} t & 0 \\ 0 & 1 + \omega d \end{array} \right),
\]

where \( t = 1 + \omega(a - bc(1 + \omega d)^{-1}) \). Hence to prove that a certain vector is \( I(1) \)-invariant modulo \((Ker T_{-1,0}, Ker T_{1,2})\), it is enough to check for invariance under

\[
\left( \begin{array}{cc} 1 & b \\ 0 & 1 \end{array} \right), \left( \begin{array}{cc} 1 & 0 \\ \omega c & 1 \end{array} \right), \left( \begin{array}{cc} 1 + \omega a & 0 \\ 0 & 1 \end{array} \right),
\]

where \( a, b, c \in \mathcal{O} \).

We first prove that the set of vectors \( S_2 \) and \( T_2 \) are \( I(1) \)-invariants when considered as vectors in \( \pi_r \); i.e., when we consider the images of these vectors modulo Ker \( T_{-1,0} \oplus Ker T_{1,2} \). The first step in achieving this is an inductive argument which reduces the general case to the case \( n = 2 \).

Lemma 4.1. If \( s_{n-1}^k \) (resp. \( t_{n-1}^k \)) is \( I(1) \)-invariant modulo \((Ker T_{-1,0}, Ker T_{1,2})\), then, for all \( n \geq 2 \), the vector \( s_n^k \) (resp. \( t_n^k \)) is also \( I(1) \)-invariant modulo \((Ker T_{-1,0}, Ker T_{1,2})\).

Proof. We prove the case of \( s_n^k \) and the case of \( t_n^k \) is similar. Assume that \( s_{n-1}^k \) is \( I(1) \)-invariant modulo \((Ker T_{-1,0}, Ker T_{1,2})\).
Now,
\[
\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} s_n^k
= \sum_{\mu \in I_{n-1}} \mu_{n-1}^k \left[ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \omega \\ \mu & 1 \end{pmatrix}, 1 \right]
= \sum_{\mu \in I_{n-1}} \mu_{n-1}^k \left[ \begin{pmatrix} \omega & [\mu_0] \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \omega^{-1} \sum_{i=1}^{n-1} [\mu_i] \omega^{i-1} \\ 0 & 1 \end{pmatrix}, 1 \right]
= \sum_{\mu \in I_{n-1}} \mu_{n-1}^k \left[ \begin{pmatrix} \omega & [\mu_0 + b_0] \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & B(\mu_0 + b_0) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \omega^{-1} \sum_{i=1}^{n-1} [\mu_i] \omega^{i-1}, 1 \end{pmatrix}, 1 \right]
\]
where
\[
\frac{B(\mu_0, b)}{\omega} = \omega^{e-1}[P_0(\mu_0, b_0)] + [b_1] + [b_2] \omega + \ldots
\]
and
\[
P_0(\mu_0, b_0) = \frac{\mu_0^{q_0} + b_0^{q_0} - (\mu_0 + b_0)^{q_0}}{\omega}
\]
is obtained from the formula in Lemma 2.3. Let
\[
\mu' = [\mu_1] + [\mu_2] \omega + \ldots + [\mu_{n-1}] \omega^{n-2}.
\]
We continue by making the substitution \( \mu_0 \rightarrow \mu_0 - b_0 \). Thus,
\[
\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} s_n^k
= \sum_{\mu_0 \in I_1} \begin{pmatrix} \omega & [\mu_0] \\ 0 & 1 \end{pmatrix} \left\{ \begin{pmatrix} 1 & B(\mu_0 - b_0, b) \\ 0 & 1 \end{pmatrix} \sum_{\mu' \in I_{n-1}} \mu_{n-1}^k \left[ \begin{pmatrix} \omega^{-1} & \mu' \\ 0 & 1 \end{pmatrix}, 1 \right] \right\}
= \sum_{\mu_0 \in I_1} \begin{pmatrix} \omega & [\mu_0] \\ 0 & 1 \end{pmatrix} \left\{ \sum_{\mu' \in I_{n-1}} \mu_{n-1}^k \left[ \begin{pmatrix} \omega^{-1} & \mu' \\ 0 & 1 \end{pmatrix}, 1 \right] + x_{\mu_0} \right\},
\]
by our assumption, where \( x_{\mu_0} \in (\text{Ker } T_{-1,0}, \text{Ker } T_{1,2}) \). Thus, we get
\[
\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} s_n^k = s_n^k + \sum_{\mu_0 \in I_1} \begin{pmatrix} \omega & [\mu_0] \\ 0 & 1 \end{pmatrix} x_{\mu_0},
\]
and hence
\[
\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} s_n^k - s_n^k \in (\text{Ker } T_{-1,0}, \text{Ker } T_{1,2}).
\]
Checking for invariance under
\[
\begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix} & \begin{pmatrix} 1 + \omega a & 0 \\ 0 & 1 \end{pmatrix}
\]
is even easier which we skip. \(\square\)
Now we take the case $n = 2$. Recall that $e$ (resp. $f$) is the ramification index (resp. residue degree) of $F$ over $\mathbb{Q}_p$. We write

$$r = r_0 + r_1p + \cdots + r_{f-1}p^{f-1}$$

where $0 \leq r_j \leq p - 1$ for $0 \leq j \leq f - 1$.

We first observe that for $a, b, c \in \mathcal{O}$, we have

$$1 + \omega a \begin{pmatrix} b \\ c \omega \end{pmatrix} \begin{pmatrix} \omega & [\mu] \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \omega & [\mu + b_0] \\ 0 & 1 \end{pmatrix},$$

for $k \in I(1)$. Indeed,

$$\text{LHS} = \begin{pmatrix} \omega(1 + \omega a) & [\mu + b_0] + \omega(*) \\ c\omega^2 & 1 + \omega(\Delta) \end{pmatrix} = \begin{pmatrix} \omega & [\mu + b_0] \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 + a\omega - [\mu + b_0]c\omega & (*) - (\mu + b_0)\Delta \\ c\omega^2 & 1 + \omega\Delta \end{pmatrix},$$

where $*, \Delta \in \mathcal{O}$. Similarly, one can show that

$$1 + \omega a \begin{pmatrix} b \\ c \omega \end{pmatrix} \begin{pmatrix} 1 & [\mu] \\ 0 & 1 \end{pmatrix} \omega = \begin{pmatrix} 1 & [\mu + b_0] \\ 0 & 1 \end{pmatrix},$$

for some $k' \in I(1)$.

**Lemma 4.2.** Assume $0 < r_j < p - 1$, and if $f = 1$, assume further that $2 < r < p - 3$. Then when $(e, f) \neq (1, 1)$, we have

$$g s_{2}^{q-1-r+p^l} - s_{2}^{q-1-r+p^l} \in (\text{Ker } T_{-1,0}, \text{Ker } T_{1,2})$$

and

$$g t_{2}^{r+p^l} - t_{2}^{r+p^l} \in (\text{Ker } T_{-1,0}, \text{Ker } T_{1,2})$$

for all $g \in I(1)$ and $0 \leq l \leq f - 1$.

**Proof.** We have

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} s_{2}^{q-1-r+p^l} = \sum_{\mu \in \mathbb{I}_2} \mu_{1}^{q-1-r+p^l} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \omega^2 & [\mu_0] + [\mu_1]\omega \\ 0 & 1 \end{pmatrix}, 1$$

$$= \sum_{\mu \in \mathbb{I}_2} \mu_{1}^{q-1-r+p^l} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \omega & [\mu_0] \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \omega & [\mu_1] \\ 0 & 1 \end{pmatrix}, 1$$

$$= \sum_{\mu \in \mathbb{I}_2} \mu_{1}^{q-1-r+p^l} \begin{pmatrix} \omega & [\mu_0 + b_0] \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & B(\mu_0, b) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \omega & [\mu_1] \\ 0 & 1 \end{pmatrix}, 1$$

where $B(\mu_0, b)$ is given by (7) in the proof of Lemma 4.1. Now write

$$B(\mu_0, b) = [b_1 + Z] + (*)\omega$$

where $Z = 0$ for $e > 1$ and $Z = P_0(\mu_0, b_0)$ for $e = 1$. 
To continue, the above expression equals
\[
\sum_{\mu \in \mathcal{I}_2} \mu_1^{q-1-r+p^l} \left[ \left( \begin{array}{c} \omega \\ 0 \\ 1 \end{array} \right) \left( \begin{array}{c} \mu_0 + b_0 \\ 1 \end{array} \right) \left( \begin{array}{c} 1 \\ 0 \\ 1 \end{array} \right) \left( \begin{array}{c} \omega \\ 0 \\ 1 \end{array} \right) \left( \begin{array}{c} [\mu_1] \\ 1 \end{array} \right) , 1 \right]
\]

which equals
\[
\sum_{\mu \in \mathcal{I}_2} \mu_1^{q-1-r+p^l} \left[ \left( \begin{array}{c} \omega \\ 0 \\ 1 \end{array} \right) \left( \begin{array}{c} \mu_0 + b_0 \\ 1 \end{array} \right) \left( \begin{array}{c} 1 \\ 0 \\ 1 \end{array} \right) \left( \begin{array}{c} \omega \\ 0 \\ 1 \end{array} \right) \left( \begin{array}{c} [\mu_1 + b_1 + Z] \\ 1 \end{array} \right) , k, 1 \right]
\]

for \( k \in I(1) \), by (9). We continue by making the change of variables
\[
\mu_1 \rightarrow \mu_1 - b_1 - Z \text{ & } \mu_0 \rightarrow \mu_0 - b_0,
\]
and we get
\[
\sum_{\mu \in \mathcal{I}_2} (\mu_1 - b_1 - Z)^{q-1-r+p^l} \left[ \left( \begin{array}{c} \omega^2 \\ 0 \\ 1 \end{array} \right) \left( \begin{array}{c} \mu \\ 1 \end{array} \right) , 1 \right]
\]
\[
= s_2^{q-1-r+p^l} + \sum_{\mu \in \mathcal{I}_2} \sum_{i=0}^{q-1-r+p^l-1} \binom{q-1-r+p^l}{i} (b_1 - Z)^{q-1-r+p^l-1} \mu_1^i \left[ \left( \begin{array}{c} \omega^2 \\ 0 \\ 1 \end{array} \right) \left( \begin{array}{c} \mu \\ 1 \end{array} \right) , 1 \right].
\]

Now we read the above expression modulo \( \text{Ker } T_{1,2} \). We claim that only the term corresponding to \( i = q-1-r \) remains amongst the \( q-1-r+p^l \) terms in the inner summation in the above expression. By Lemma 3.6 (1), we know that
\[
\sum_{\mu \in \mathcal{I}_2} \mu_1^i \left[ \left( \begin{array}{c} \omega^2 \\ 0 \\ 1 \end{array} \right) \left( \begin{array}{c} \mu \\ 1 \end{array} \right) , 1 \right] \in \text{Ker } T_{1,2}
\]

precisely when \( 0 \leq i \leq q-2-r \) or \( i > q-1-r \) such that \( i_j < p-1-r_j \) for some \( 0 \leq j \leq f-2 \). Note that if \( i > q-1-r \) and \( i_j > p-1-r_j \) for all \( 0 \leq j \leq f-1 \) (cf. Remark 5) then \( i_j > p-1-r_j \) for some \( 0 \leq j \leq l-1 \) (since \( i \leq q-1-r+p^l-1 \)). If this is the case then observe that
\[
\binom{q-1-r+p^l}{i} \equiv 0 \mod p
\]

by Corollary 2.2. Thus, modulo \( \text{Ker } T_{1,2} \), we get
\[
\binom{1 \ b}{0 \ 1} s_2^{q-1-r+p^l}
\]
\[
= s_2^{q-1-r+p^l} + \sum_{\mu \in \mathcal{I}_2} \binom{q-1-r+p^l}{q-1-r} (b_1 - Z)^{q-1-r} \mu_1^{q-1-r} \left[ \left( \begin{array}{c} \omega^2 \\ 0 \\ 1 \end{array} \right) \left( \begin{array}{c} \mu \\ 1 \end{array} \right) , 1 \right]
\]
\[
= s_2^{q-1-r+p^l} + \sum_{\mu_0 \in \mathcal{I}_1} (p-r_1) (b_1 - Z)^{p^l} \left[ \left( \begin{array}{c} 1 \\ 0 \\ 1 \end{array} \right) \mu_0 \right], 1 \right]
\]

by Lemma 3.6 (1) and the binomial coefficient here is computed via Theorem 2.1.

Now if \( e > 1 \) then we have \( Z = 0 \). Therefore, it follows, by Lemma 3.5 (2), that
\[
\binom{1 \ b}{0 \ 1} s_2^{q-1-r+p^l} = s_2^{q-1-r+p^l} \in \text{Ker } T_{-1,0,e}.
\]

and thus we have proved
\[
\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} s_2^{q-1-r+p'} s_2^{q-1-r+p'} \equiv \mod (\ker T_{-1,0}, \ker T_{1,2}).
\]

If \( e = 1 \) then \( Z = P_0(\mu_0, b_0) \). As \( F \) is unramified over \( \mathbb{Q}_p \), we have \( \omega = p \). Now by Corollary 2.2 it follows that
\[
Z = \frac{\mu_0^q e + b_0^q e - (\mu_0 + b_0)^q e}{\omega^e} \equiv - \sum_{i=1}^{p-1} \frac{1}{p} (\text{ip}^f) b_0^i - b_0^i \mu_0^i \mod p.
\]

In this case, if further \( f \neq 1 \) we have, modulo \( \ker T_{1,2} \),
\[
\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} s_2^{q-1-r+p'} s_2^{q-1-r+p'} = \sum_{\mu_0 \in l_1} (p - r_i)(-b_1 - Z)^p \left[ \begin{pmatrix} 1 & [\mu_0] \\ 0 & 1 \end{pmatrix} \right] w, 1
\]
\[
= \sum_{\mu_0 \in l_1} r_i (b_1^i + Z)^p \left[ \begin{pmatrix} 1 & [\mu_0] \\ 0 & 1 \end{pmatrix} \right] w, 1.
\]

Note that both
\[
\sum_{\mu_0 \in l_1} \left[ \begin{pmatrix} 1 & [\mu_0] \\ 0 & 1 \end{pmatrix} \right] w, 1 \] and \( \sum_{\mu_0 \in l_1} [\mu_0]^{ip-1} \left[ \begin{pmatrix} 1 & [\mu_0] \\ 0 & 1 \end{pmatrix} \right] w, 1 \)

are in \( \ker T_{-1,0} \) by Lemma 3.5 (2). Thus, once again we have proved
\[
\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} s_2^{q-1-r+p'} s_2^{q-1-r+p'} \equiv \mod (\ker T_{-1,0}, \ker T_{1,2}).
\]

Now we analyze invariance for the lower unipotent representative of \( I(1) \). We have
\[
\begin{pmatrix} 1 & 0 \\ \omega c & 1 \end{pmatrix} s_2^{q-1-r+p'} = \sum_{\mu \in l_2} \mu^{q-1-r+p'} \left[ \begin{pmatrix} 1 & 0 \\ \omega c & 1 \end{pmatrix} \right] \left[ \begin{pmatrix} 1 & [\mu_0] \\ 0 & 1 \end{pmatrix} \right] w, 1
\]
\[
= \sum_{\mu \in l_2} \mu^{q-1-r+p'} \left[ \begin{pmatrix} 1 & 0 \\ \omega c & 1 \end{pmatrix} \right] \left[ \begin{pmatrix} \omega & [\mu_0] \\ 0 & 1 \end{pmatrix} \right] \left[ \begin{pmatrix} \omega & [\mu_0] \\ 0 & 1 \end{pmatrix} \right] w, 1
\]

which we express as
\[
\sum_{\mu \in l_2} \mu^{q-1-r+p'} \left[ \begin{pmatrix} \omega & [\mu_0] \\ 0 & 1 \end{pmatrix} \right] \left( 1 - \omega c \right) \left[ \begin{pmatrix} \omega & [\mu_0] \\ 0 & 1 \end{pmatrix} \right] \left[ \begin{pmatrix} \omega & [\mu_0] \\ 0 & 1 \end{pmatrix} \right] w, 1
\]

and this equals
\[
\sum_{\mu \in l_2} \mu^{q-1-r+p'} \left[ \begin{pmatrix} \omega & [\mu_0] \\ 0 & 1 \end{pmatrix} \right] \left( \mu_1 - c_0 \mu_0^2 \right) \left( \begin{pmatrix} \omega & [\mu_0] \\ 0 & 1 \end{pmatrix} \right) k, 1
\]

for \( k \in I(1) \) by (9). Changing \( \mu_1 \to \mu_1 + c_0 \mu_0^2 \), we get
\[
\begin{pmatrix} 1 & 0 \\ \omega c & 1 \end{pmatrix} s_2^{q-1-r+p'} = \sum_{\mu \in l_2} (\mu_1 + c_0 \mu_0^2)^{q-1-r+p'} \left[ \begin{pmatrix} \omega & [\mu_0] \\ 0 & 1 \end{pmatrix} \right] w, 1
\]
which we read modulo $\text{Ker } T_{1,2}$ and get
\[
s_q^{a-1-r+p} + \sum_{\mu \in \mathcal{I}_2} \left( q - 1 - r + p \right)^{\mu} \mu_{q-1-r} \left[ \left( \begin{array}{cc} \omega^2 & \mu \\ 0 & 1 \end{array} \right), 1 \right]
\]
by Corollary 2.2 together with Lemma 3.5 (1), exactly as we have argued before. Now this equals, modulo $\text{Ker } T_{1,2}$,
\[
s_q^{a-1-r+p} + \sum_{\mu \in \mathcal{I}_1} (p - r) \mu_{q-1-r} \left[ \left( \begin{array}{c} 1 \\ \omega \end{array} \right), 1 \right]
\]
by Theorem 2.1 and Lemma 3.6 (1). By Lemma 3.5 (2), this vector belongs to $\text{Ker } T_{-1,0}$ (with the extra assumption that $3 \leq r$ when $f = 1$). Thus, we have proved
\[
\left( \begin{array}{c} 1 \\ \omega \end{array} \right) s_q^{a-1-r+p} \equiv s_q^{a-1-r+p} \mod (\text{Ker } T_{-1,0}, \text{Ker } T_{1,2}).
\]

The proof for showing that
\[
\left( \begin{array}{c} 1 + \omega a \\ 0 \end{array} \right) s_q^{a-1-r+p} - s_q^{a-1-r+p} \in (\text{Ker } T_{-1,0}, \text{Ker } T_{1,2})
\]
is similar and therefore we skip it.

The argument for
\[
g t_r^{a-1-r+p} - t_r^{a-1-r+p} \in (\text{Ker } T_{-1,0}, \text{Ker } T_{1,2})
\]
for all $g \in \mathcal{I}(1)$ is similar to the one for $s_q^{a-1-r+p}$.

4.2. Linear independence. The following lemma gives the action of the Iwahori subgroup $I$ on the $I(1)$-invariant vectors (cf. [Hen19, Lemma 3.6]).

**Lemma 4.3.** Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in I$. Let $s_n^k$ and $t_n^s$ be $I(1)$-invariant modulo $(\text{Ker } T_{-1,0}, \text{Ker } T_{1,2})$.

Then they are $I$-eigenvectors and those actions are given by

1. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot s_n^k = d'(da^{-1})^k s_n^k$,
2. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot t_n^s = a'(da^{-1})^s t_n^s$.

**Proof.** The proof is straightforward and we have already done it in the proof of the second part of Corollary 3.2.

Remark 8. Lemmas 4.1, 4.2 and 4.3 remain true for $\beta s_n^k$ and $\beta t_n^s$.

**Proposition 4.4.** The set of vectors in $\mathcal{S}_2 \cup \mathcal{T}_2$ of Theorem 1.2 are linearly independent.

**Proof.** Note that the vectors in $\mathcal{S}_2 \cup \mathcal{T}_2$ consist of vectors of the form
\[s_n^{a-1-r+p}, \beta s_n^{a-1-r+p}, r t_n^{r+p}, \beta r t_n^{r+p}\]
for \( n \geq 2 \) and \( 0 \leq l \leq f - 1 \). These are invariant under \( I(1) \) modulo (\( \text{Ker } T_{-1,0}, \text{Ker } T_{1,2} \)) except for the case when both \( e = 1 \) and \( f = 1 \) (cf. Lemmas 4.1, 4.2, and Remark 8).

For any vector \( v \in \text{ind}_{I Z}^G \), note that \( v \) and \( \beta v \) cannot cancel each other (pictorially they are on two different sides of the tree of \( \text{SL}_2(F) \)). Therefore, it is enough to show that the set \( \{ s_n^{q-1-r+p^l}, t_n^{r+p^l} \} \), for \( n \geq 2 \) and \( 0 \leq l \leq f - 1 \), is linearly independent. Since \( s_n^{q-1-r+p^l} \) and \( t_n^{r+p^l} \) have different \( I \)-eigenvalues, it is enough to show that \( \{ s_n^{q-1-r+p^l} \} \) and \( \{ t_n^{r+p^l} \} \), for \( n \geq 2 \) and \( 0 \leq l \leq f - 1 \), are linearly independent.

We show that the vectors in

\[
\{ s_n^{q-1-r+p^l} \}_{n \geq 2, 0 \leq l \leq f - 1}
\]

are linearly independent, and the proof for \( \{ t_n^{r+p^l} \} \) is similar. Suppose that

\[
\sum_{i=2}^{n} c_is_i^{q-1-r+p^l} \in (\text{Ker } T_{-1,0}, \text{Ker } T_{1,2})
\]

where \( c_i \in \mathbb{F}_p \) and \( n \in \mathbb{N} \). Since no reduction is possible in the above expression and also these vectors obviously cannot be in \( \text{Ker } T_{-1,0} \), it follows that

\[
\sum_{i=2}^{n} c_is_i^{q-1-r+p^l} \in \text{Ker } T_{1,2}.
\]

For \( i \neq j \) with \( 2 \leq i, j \leq n \), once again from the formula for \( T_{1,2} \), there cannot be any cancellation between \( T_{1,2}(c_is_i^{q-1-r+p^l}) \) and \( T_{1,2}(c_js_j^{q-1-r+p^l}) \), so we get

\[
c_is_i^{q-1-r+p^l} \in \text{Ker } T_{1,2}
\]

for all \( 2 \leq i \leq n \). By Lemma 3.5(1), it follows that \( c_i = 0 \) for all \( 2 \leq i \leq n \).

\[\square\]

Remark 9. It follows by eigenvalue considerations as in the proof of Proposition 4.4 that the set

\[
S_2 \cup \{ [\text{Id}, 1], [\beta, 1] \} \cup T_2
\]

is linearly independent.

4.3. Auxiliary lemmas. We will have to make use of the following elementary lemma [Hen19 Lemma 2.8].

Lemma 4.5. Let \( n \geq 1 \) and \( \phi : I_n \to \mathbb{F}_p \) be any set map. Then there exists a unique polynomial \( Q(x_0, \ldots, x_{n-1}) \in \mathbb{F}_p[x_0, x_1, \ldots, x_{n-1}] \) in which degree of each variable is at most \( q - 1 \) and \( \phi(\mu) = Q(\mu_0, \mu_1, \ldots, \mu_{n-1}) \) for all \( \mu \in I_n \).

The next two lemmas are the first steps towards the proof of Theorem 1.2.

Lemma 4.6. Let \( \mu = [\mu_0] + [\mu_1]a + \cdots + [\mu_{n-1}]a^{n-1} \in I_n \) and \( r = r_0 + r_1p + \cdots + r_{f-1}p^{f-1} \) with \( 0 < r_j < p - 1 \) for all \( 0 \leq j \leq f - 1 \). Let

\[
f_n = f'_n + f''_n
\]

be such that

\[
f'_n = \sum_{\mu \in I_n} a(\mu_0, \mu_1, \ldots, \mu_{n-1}) \left[ \begin{array}{cc} a^n & \mu \\ 0 & 1 \end{array} \right]
\]

\[
f''_n = \sum_{\mu \in I_n} a(\mu_0, \mu_1, \ldots, \mu_{n-1}) \left[ \begin{array}{cc} 1 & \mu \\ 0 & 1 \end{array} \right]
\]
and 
\[ f'''_n = \sum_{\mu \in I_n} b(\mu_0, \mu_1, \ldots, \mu_{n-1}) \left[ \begin{array}{c} \omega^{n-1} \\ \mu \\ 0 \\ 1 \end{array} \right] \left[ \begin{array}{c} \sum_{j=0}^{f-1} (p-1-r_j) n \\ 0 \\ 1 \end{array} \right] w, 1, \]

where \( a(\mu_0, \ldots, \mu_{n-1}) \) and \( b(\mu_0, \ldots, \mu_{n-1}) \) are polynomials in \( \mu_0, \ldots, \mu_{n-1} \). Suppose 
\[ \left( \begin{array}{c} 1 \\ -\omega^{n-1} \\ 0 \\ 1 \end{array} \right) f_n - f'_n \in (\text{Ker } T_{1,0}, \text{Ker } T_{1,2}). \]

Then

1. the possible powers of \( \mu_{n-1} \), say \( k = k_0 + k_1 p + \cdots + k_{f-1} p^{f-1} \), in \( a(\mu_0, \ldots, \mu_{n-1}) \) will satisfy one of the following three conditions:
   (a) there exists some \( 0 \leq j' \leq f - 1 \) such that \( k_{j'} < p - 1 - r_{j'} \),
   (b) \( k_j = p - 1 - r_j \) for all \( 0 \leq j \leq f - 1 \),
   (c) \( k_j = p - 1 - r_j \) for \( j \neq 1 \) and \( k_1 = p - r_l \) for some \( 0 \leq l \leq f - 1 \).

2. the possible powers of \( \mu_{n-1} \), say \( k = k_0 + k_1 p + \cdots + k_{f-1} p^{f-1} \), in \( b(\mu_0, \ldots, \mu_{n-1}) \) will satisfy one of the following three conditions:
   (a) there exists some \( 0 \leq j' \leq f - 1 \) such that \( k_{j'} < r_{j'} \),
   (b) \( k_j = r_j \) for all \( 0 \leq j \leq f - 1 \),
   (c) \( k_j = r_j \) for \( j \neq 1 \) and \( k_1 = r_l + 1 \) for some \( 0 \leq l \leq f - 1 \).

**Proof of Lemma 4.6** We will prove (1) and the proof of (2) is similar. Suppose (1) does not hold. Then there exists \( k \) such that \( k_j \geq p - 1 - r_j \) for all \( 0 \leq j \leq f - 1 \) with

\[ k_{j_0} > p - 1 - r_{j_0} \] for some \( 0 \leq j_0 \leq f - 1 \) & \( k \neq (p - r_{j_0}) p^{j_0} + \sum_{j_0 \neq j}^f (p - 1 - r_j) p^j. \]

Then either there exists \( j_1 \) with \( j_1 \neq j_0 \) such that \( k_{j_1} > p - 1 - r_{j_1} \) or

\[ k = k_{j_0} p^{j_0} + \sum_{j_0 \neq j}^f (p - 1 - r_j) p^j \]

with \( k_{j_0} > p - r_{j_0} \). Choose \( k \) with the above property such that there is no other monomial \( \mu_{n-1}^{k_j} \) in \( a(\mu_0, \ldots, \mu_{n-1}) \) with \( k_j \leq k_j' \) for all \( 0 \leq j \leq f - 1 \). Since a polynomial is of finite degree, such a \( k \) exists. Let

\[ g = \left( \begin{array}{c} 1 \\ -\omega^{n-1} \\ 0 \\ 1 \end{array} \right). \]

We have

\[ gf_n - f_n = (gf'_n - f'_n) + (gf'''_n - f'''_n) \in (\text{Ker } T_{1,0}, \text{Ker } T_{1,2}). \]

Note that,

\[ gf'_n - f'_n = \sum_{\mu \in I_n} [a(\mu)]_{n-1, \mu_{n-1} + 1} - a(\mu)]_{n-1, \mu_{n-1}} \left[ \begin{array}{c} 0 \\ \omega^n \\ \mu \\ 1 \end{array} \right], \]

and

\[ gf'''_n - f'''_n = \sum_{\mu \in I_n} [b(\mu)]_{n-1, \mu_{n-1} + 1} - b(\mu)]_{n-1, \mu_{n-1}} \left[ \begin{array}{c} 0 \\ \omega^{n-1} \\ \mu \\ 1 \end{array} \right], \]
Let
\[ \Delta a = a([\mu]_{n-1}, \mu_{n-1} + 1) - a([\mu]_{n-1}, \mu_{n-1}) \]
considered as a polynomial in \( \mu_{n-1} \) with coefficients in \( \mathbb{F}_p[\mu_0, \ldots, \mu_{n-2}] \). By Theorem 2.1 we have
\[ (\mu_{n-1} + 1)^k - \mu_{n-1}^k = \sum_{i=0}^{k-1} \prod_{j=0}^{f-1} \left( \begin{array}{c} k_j \\ i_j \end{array} \right) \mu_{n-1}^i \mod p. \]

Now if there exists \( j_1 \) with \( j_1 \neq j_0 \) such that \( k_{j_1} > p - 1 - r_{j_1} \), take
\[ k' = (k_{j_1} - 1)p^{j_1} + \sum_{j_1 \neq j = 0}^{f-1} k_j p^j. \]
The coefficient of \( \mu_{n-1}^{k'} \) in \( \Delta a \) is
\[ \left( \begin{array}{c} k \\ k' \end{array} \right) = \left( \begin{array}{c} k_{j_1} \\ k_{j_1} - 1 \end{array} \right) \neq 0 \mod p \]
by Theorem 2.1 and Corollary 2.2. Note that the term involving \( \mu_{n-1}^{k'} \) in \( g f'_n - f'_n \) cannot get cancelled by any other term in \( g f_n - f_n \). Indeed, it cannot get cancelled with any other term in \( g f'_n - f'_n \) because of the choice of \( k \) and anyway no term in \( g f_n - f_n \) can get cancelled with a term in \( g f'_n - f'_n \) (pictorially they represent edges of opposite orientation on the tree of \( SL_2(F) \)). So this term involving \( \mu_{n-1}^{k'} \) must be there in \( (\text{Ker } T_{-1,0}, \text{Ker } T_{1,2}) \), but then Lemma 3.5 (1) would imply that there exists some \( 0 \leq l \leq f - 1 \) such that \( k'_l < p - 1 - r_l \), which contradicts our assumption. So \( k \) must be of the form
\[ k = k_{j_0}p^{j_0} + \sum_{j_0 \neq j = 0}^{f-1} (p - 1 - r_j)p^j \]
with \( k_{j_0} > p - r_{j_0} \). Taking
\[ k' = (k_{j_0} - 1)p^{j_0} + \sum_{j_0 \neq j = 0}^{f-1} (p - 1 - r_j)p^j, \]
and using the same argument as in the previous case, we arrive at a contradiction. \( \square \)

**Remark 10.** The idea of choosing \( k \) as in Lemma 4.6 is already employed by Hendel in [Hen19, Lemma 3.13].

Now we state one more lemma whose main idea of proof also comes from [Hen19, Lemma 3.13]. In what follows, \( B(t) \) denotes the ball of radius \( m \) on the tree of \( SL_2(F) \) with center at the vertex representing the trivial coset \( G/KZ \). Explicitly it consists of linear combinations of vectors of the form
\[ B^0(t) = \left\{ \left[ \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \right], \left[ \begin{array}{c} b_1 \\ b_2 \\ b_3 \\ b_4 \end{array} \right] \mid a_1 = b_1, a_2 = b_2, a_3 = b_3, a_4 = b_4 \right\}_{n \leq t}, \]
and
\[ B^1(t) = \left\{ \left[ \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \right], \left[ \begin{array}{c} b_1 \\ b_2 \\ b_3 \\ b_4 \end{array} \right] \mid a_1 = b_1, a_2 = b_2, a_3 = b_3, a_4 = b_4 \right\}_{n \leq t}, \]
where \( \mu = [\mu_0] + [\mu_1] \omega + \cdots + [\mu_{n-1}] \omega^{n-1} \in I_n. \)
Lemma 4.7. Let

\[ f'_n = \sum_{\mu \in I_n} \sum_{l=0}^{f-1} P_l([\mu]_{n-1}) \mu_{n-1}^{q-1-r+p'} \left[ S_{n,\mu}, 1 \right] \]

and

\[ f''_n = \sum_{\mu \in I_n} \sum_{l=0}^{f-1} Q_l([\mu]_{n-1}) \mu_{n-1}^{r+p'} \left[ S_{n-1,\mu_{n-1}}, 1 \right] \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \]

where \( P_l([\mu]_{n-1}) \) and \( Q_l([\mu]_{n-1}) \) are polynomials in \( \mu_0, \ldots, \mu_{n-2} \). Let \( f_n = f'_n + f''_n \). Let \( f = f_n + f' \) be such that \( f' \in B(n-1) \) and

\[ \left( \begin{array}{c} 1 \\ -\omega^{n-m} \\ 0 \\ 1 \end{array} \right) f - f \in (\text{Ker } T_{-1,0}, \text{Ker } T_{1,2}), \]

for all \( 1 \leq m \leq n - 1 \). Then we have

\[ f'_n = \sum_{\mu \in I_n} \sum_{l=0}^{f-1} a_l \mu_{n-1}^{q-1-r+p'} \left[ S_{n,\mu}, 1 \right] \]

and

\[ f''_n = \sum_{\mu \in I_n} \sum_{l=0}^{f-1} b_l \mu_{n-1}^{r+p'} \left[ S_{n-1,\mu_{n-1}}, 1 \right] \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \]

where \( a_l \) and \( b_l \) are constants.

Proof of Lemma 4.7. We do the proof only for \( f'_n \), as the case of \( f''_n \) is similar. The proof is by induction on \( n \). Note that \( P_l([\mu]_{n-1}) \) is independent of \( \mu_{n-1} \). Suppose it is independent of \( \mu_{n-1}, \ldots, \mu_{n-m+1} \). Then

\[ f'_n = \sum_{\mu \in I_n} \sum_{l=0}^{f-1} P_l([\mu]_{n-m}, \mu_{n-m}) \mu_{n-1}^{q-1-r+p'} \left[ S_{n,\mu}, 1 \right] \]

We show that it is independent of \( \mu_{n-m} \). It is given to us that

\[ \left( \begin{array}{c} 1 \\ -\omega^{n-m} \\ 0 \\ 1 \end{array} \right) f - f = \left[ \left( \begin{array}{c} 1 \\ -\omega^{n-m} \\ 0 \\ 1 \end{array} \right) f_n - f_n \right] + \left[ \left( \begin{array}{c} 1 \\ -\omega^{n-m} \\ 0 \\ 1 \end{array} \right) f' - f' \right] \]

\[ \in (\text{Ker } T_{-1,0}, \text{Ker } T_{1,2}). \]

Now,

\[ \left( \begin{array}{c} 1 \\ -\omega^{n-m} \\ 1 \\ 0 \end{array} \right) \left( \omega^n \sum_{i=0}^{n-1} [\mu_i] \omega^i \right) = \left( \begin{array}{c} \omega^n [\mu_0] + \cdots + [\mu_{n-1}] \omega^{n-1} - \omega^{n-m} \\ 0 \\ 1 \end{array} \right) \]

and this equals

\[ \left( \begin{array}{c} \omega^n \sum_{i=0}^{n-m-1} [\mu_i] \omega^i + [\mu_{n-m-1}] \omega^{n-m} + [\mu'_{n-m+1}] \omega^{n-m+1} + \cdots + [\mu'_{n-1}] \omega^{n-1} \\ 0 \\ 1 \end{array} \right) \]

where \( \mu_k' = \mu_k + c_k(\mu_{n-m}, \ldots, \mu_{n-2}) \) for \( n-m+1 \leq k \leq n-1 \).
Note that the transformation \( \mu_k' \mapsto \mu_k - c_k(\mu_{n-m}, \ldots, \mu_{n-2}) \) does not affect the variables \( \mu_k \) for \( n - m + 1 \leq k \leq n - 1 \) in \( P_1([\mu]_{n-1}) \), as it is independent of these variables. This transformation together with \( \mu_{n-m} \mapsto \mu_{n-m} + 1 \) gives

\[
\begin{pmatrix}
1 & -\alpha^{n-m} \\
0 & 1 \\
\end{pmatrix} \begin{pmatrix}
f'_n \\
0 \\
\end{pmatrix} = \sum_{\mu \in I_n} \sum_{l=0}^{f-1} P_l([\mu]_{n-m}, \mu_{n-m} + 1) (\mu_{n-1} - c_{n-1})^{q-1-r+p'} [S^0_{n, \mu}, 1].
\]

In the above expression, by \( c_{n-1} \) we mean \( c_{n-1}(\mu_{n-m}, \ldots, \mu_{n-2}) \). Now,

\[
\begin{pmatrix}
1 & -\alpha^{n-m} \\
0 & 1 \\
\end{pmatrix} \begin{pmatrix}
f'_n \\
f''_n \\
\end{pmatrix} = \sum_{\mu \in I_n} \sum_{l=0}^{f-1} \alpha(\mu, l) [S^0_{n, \mu}, 1],
\]

where

\[
\alpha(\mu, l) = \left[ P_1([\mu]_{n-m}, \mu_{n-m} + 1) (\mu_{n-1} - c_{n-1})^{q-1-r+p'} - P_1([\mu]_{n-m}, \mu_{n-m}) \right]^{q-1-r+p'}.
\]

Thus,

\[
\begin{pmatrix}
1 & -\alpha^{n-m} \\
0 & 1 \\
\end{pmatrix} \begin{pmatrix}
f'_n \\
f''_n \\
\end{pmatrix} = \sum_{\mu \in I_n} \sum_{l=0}^{f-1} \left[ P_l([\mu]_{n-m}, \mu_{n-m} + 1) - P_l([\mu]_{n-m}, \mu_{n-m}) \right] \mu_{n-1}^{q-1-r+p'} [S^0_{n, \mu}, 1]
\]

\[+ \sum_{\mu \in I_n} \sum_{l=0}^{f-1} \sum_{i=0}^{q-1-r+p'-1} \beta(\mu, l, i) [S^0_{n, \mu}, 1],\]

where

\[
\beta(\mu, l, i) = P_l([\mu]_{n-m}, \mu_{n-m} + 1) (-1)^i (q-1-r+p')) (-c_{n-1})^{q-1-r+p'-i} \mu_{n-1}.
\]

Now we read this modulo \( \text{Ker } T_{1,2} \). Thus, we get

\[
\begin{pmatrix}
1 & -\alpha^{n-m} \\
0 & 1 \\
\end{pmatrix} \begin{pmatrix}
f'_n \\
f''_n \\
\end{pmatrix} = \sum_{\mu \in I_n} \sum_{l=0}^{f-1} \left[ P_l([\mu]_{n-m}, \mu_{n-m} + 1) - P_l([\mu]_{n-m}, \mu_{n-m}) \right] \mu_{n-1}^{q-1-r+p'} [S^0_{n, \mu}, 1]
\]

\[+ \sum_{\mu \in I_n} \sum_{l=0}^{f-1} P_l([\mu]_{n-m}, \mu_{n-m} + 1) \left( q-1-r+p' \right) \right] (-c_{n-1})^{q-1-r+p'i} \mu_{n-1}^{q-1-r+p'}.\]

by Corollary 22 and Lemma 3.5 (1), exactly as we have argued before in the proof of Lemma 4.2. Now by Lemma 3.6 (1), it follows that, modulo \( (\text{Ker } T_{1,0}, \text{Ker } T_{1,2}) \), we have

\[
\begin{pmatrix}
1 & -\alpha^{n-m} \\
0 & 1 \\
\end{pmatrix} \begin{pmatrix}
f'_n \\
f''_n \\
\end{pmatrix} = \sum_{\mu \in I_n} \sum_{l=0}^{f-1} \left[ P_l([\mu]_{n-m}, \mu_{n-m} + 1) - P_l([\mu]_{n-m}, \mu_{n-m}) \right] \mu_{n-1}^{q-1-r+p'} [S^0_{n, \mu}, 1] + \mathcal{G}_{n-1}
\]
where \( g_{n-1} \in B(n-1) \). As \( r_1 \neq 0 \), by Lemmas 3.5 (1) and 3.6 (1) we have
\[
\sum_{\mu \in I_n} \mu^q (r-1+r') \left[ g_{n-1}^0, 1 \right] \notin (\text{Ker } T_{-1,0}, \text{Ker } T_{1,2}) \text{.}
\]
Also the term involving \( \mu^q (r-1+r') \) cannot get cancelled by any other term in the expression
\[
\begin{pmatrix}
1 & -\omega^{n-m} \\
0 & 1
\end{pmatrix} f - f.
\]
So it follows that
\[
P_I([\mu]_{n-m}, [\mu]_{n-m} + 1) - P_I([\mu]_{n-m}, [\mu]_{n-m}) = 0.
\]
Hence \( P_I([\mu]_{n-1}) \) is independent of \( [\mu]_{n-m} \). Therefore, by induction \( P_I([\mu]_{n-1}) \) is a constant. \( \square \)

4.4. **Proof of Theorem 1.2** Clearly the vectors \([\text{Id}, 1]\) and \([\beta, 1]\) are fixed by \( I(1) \). By Lemmas 4.1 and 4.2 and Remark 8 the vectors in \( S_2 \) and \( T_2 \) are \( I(1) \)-invariant modulo \( (\text{Ker } T_{-1,0}, \text{Ker } T_{1,2}) \) except for the case when both \( e = 1 \) and \( f = 1 \). By Remark 9 the set \( S_2 \cup \{[\text{Id}, 1], [\beta, 1]\} \cup T_2 \) is linearly independent.

Now let \( f \in \text{ind}_{\mathbb{Z}\chi} G \) be an \( I(1) \)-invariant of
\[
\pi_r = \frac{\text{ind}_{\mathbb{Z}\chi} G}{(\text{Ker } T_{-1,0}, \text{Ker } T_{1,2})}.
\]
We write
\[
f = f^0 + f^1
\]
where \( f^0 \) (resp. \( f^1 \)) is a linear combination of vectors on the zero side (resp. one side) of the tree of \( \text{SL}_2(F) \). By this, we mean \( f^0 \) is a linear combination of vectors of the form
\[
\left[ g_{n-1, [\mu]}^0, 1 \right] \left[ g_{n-1, [\mu]}^0, 1 \right] \left( \begin{array}{c}
1 \\
0
\end{array} \right) \left( \begin{array}{c}
[\mu]_{n-1} \\
1
\end{array} \right) w, 1,
\]
and \( f^1 \) is a linear combination of vectors of the form
\[
\left[ g_{n-1, [\mu]}^1, w, 1 \right] \left[ g_{n-1, [\mu]}^1, w, 1 \right] \left( \begin{array}{c}
1 \\
0
\end{array} \right) \left( \begin{array}{c}
[\mu]_{n-2} \\
1
\end{array} \right) w, 1.
\]
Then,
\[
g f^i - f^i \in (\text{Ker } T_{-1,0}, \text{Ker } T_{1,2}),
\]
for all \( i \in \{0, 1\} \) and \( g \in I(1) \). Since \( \beta f^1 \) is a linear combination of vectors on the zero side and \( \beta \) normalizes \( I(1) \), without loss of generality, we may assume \( f = f^0 \). Write
\[
f = f_n + f',
\]
with \( f_n \neq 0, f' \in B(n-1) \), for \( n \) maximal. Now,
\[
f_n = \sum_{\mu \in I_n} a_{\mu} \left[ g_{n, [\mu]}^0, 1 \right] + \sum_{\mu \in I_n} b_{\mu} \left[ g_{n-1, [\mu]}^0, 1 \right] \left( \begin{array}{c}
1 \\
0
\end{array} \right) \left( \begin{array}{c}
[\mu]_{n-1} \\
1
\end{array} \right) w, 1.
\]
where \( \mu = [\mu_0] + [\mu_1] \omega + \cdots + [\mu_{n-1}] \omega^{n-1} \) and \( a_\mu, b_\mu \in \mathbb{P}_p \). By Lemma 4.5, the coefficients \( a_\mu \) and \( b_\mu \) can be replaced by the polynomials \( a(\mu_0, \ldots, \mu_{n-1}) \) and \( b(\mu_0, \ldots, \mu_{n-1}) \) respectively, where each \( \mu_i \) has maximum degree \( q - 1 \). Write

\[
f_n = f'_n + f''_n,
\]

where

\[
f'_n = \sum_{\mu \in \mathcal{I}_n} \sum_i a(i_0, i_1, \ldots, i_{n-1}) \mu_0^{i_0} \cdots \mu_{n-1}^{i_{n-1}} \left[ g_0^{0, \mu}, 1 \right],
\]

and

\[
f''_n = \sum_{\mu \in \mathcal{I}_n} \sum_j b(j_0, j_1, \ldots, j_{n-1}) \mu_0^{j_0} \cdots \mu_{n-1}^{j_{n-1}} \left[ g_{n-1, \mu}^{0}, 1 \right] \left( \begin{array}{c} 1 \\ [\mu_{n-1}] \\ 1 \end{array} \right) w, 1 \right].
\]

Let

\[
g' = \left( \begin{array}{cc} 1 & -\omega^{n-1} \\ 0 & 1 \end{array} \right) \in I(1).
\]

Since \( f' \) belongs in \( B(n - 1) \), it is easy to check that \( g' \) fixes \( f' \). This gives

\[
g' f_n - f_n \in (\text{Ker } T_{-1,0}, \text{Ker } T_{1,2}).
\]

Now Lemma 3.5 (1) together with Lemma 4.6 (1) gives

\[
f'_n = \sum_{\mu \in \mathcal{I}_n} \sum_i a(i_0, \ldots, i_{n-2}, q - 1 - r) \mu_0^{i_0} \cdots \mu_{n-1}^{i_{n-1}} \left[ g_0^{0, \mu}, 1 \right]
\]

\[
+ \sum_{\mu \in \mathcal{I}_n} \sum_{l=0}^{f-1} a_l([\mu]_{n-1}) \mu_{n-1}^{q-1-r+p'} \left[ g_0^{0, \mu}, 1 \right],
\]

which in turn implies that

\[
f'_n - \sum_{\mu \in \mathcal{I}_n} \sum_{l=0}^{f-1} a_l([\mu]_{n-1}) \mu_{n-1}^{q-1-r+p'} \left[ g_0^{0, \mu}, 1 \right] = \sum_{\mu_0, \ldots, \mu_{n-1} = 0}^{f-1} a(i_0, \ldots, q - 1 - r) \mu_0^{i_0} \cdots \mu_{n-1}^{i_{n-1}} \left[ g_0^{0, \mu}, 1 \right]
\]

which modulo \( \text{Ker } T_{1,2} \) equals

\[
\sum_{\mu_0, \ldots, \mu_{n-1} = 0}^{f-1} a(i_0, \ldots, q - 1 - r) \mu_0^{i_0} \cdots \mu_{n-2}^{i_{n-2}} \left[ g_{n-2, \mu}^{0, [\mu_{n-2}], 1} \right] \left( \begin{array}{c} 1 \\ [\mu_{n-2}] \\ 1 \end{array} \right) w, 1 \right],
\]

by Lemma 3.6 (1). This vector belongs to \( B(n - 1) \) which we call \( g'_{n-1} \). We get

\[
f'_n = \sum_{\mu \in \mathcal{I}_n} \sum_{l=0}^{f-1} a_l([\mu]_{n-1}) \mu_{n-1}^{q-1-r+p'} \left[ g_0^{0, \mu}, 1 \right] + g'_{n-1}.
\]

Similarly, working with \( f''_n \), we get

\[
f''_n = \sum_{\mu \in \mathcal{I}_n} \sum_{l=0}^{f-1} b_l([\mu]_{n-1}) \mu_{n-1}^{r+p'} \left[ g_0^{0, \mu}, 1 \right] \left( \begin{array}{c} 1 \\ [\mu_{n-1}] \\ 1 \end{array} \right) w, 1 \right] + g''_{n-1}
\]

for some \( g''_{n-1} \in B(n - 1) \), by Lemmas 3.5 (2), 4.6 (2) and 3.6 (2).
For $1 \leq m \leq n - 1$, we note that
\[
\begin{pmatrix}
1 & -\alpha^{n-m} \\
0 & 1
\end{pmatrix} \in I(1).
\]
Using the condition
\[
\begin{pmatrix}
1 & -\alpha^{n-m} \\
0 & 1
\end{pmatrix} f - f \in (\text{Ker } T_{-1,0}, \text{Ker } T_{1,2}),
\]
by Lemma 4.7, we have
\[
f' = \sum_{\mu \in I_n} \sum_{l=0}^{f-1} a_{l,n} \mu^{q-1-r+p^l} \left[ g_n^{0}, \mu, 1 \right] + g'_{n-1}
\]
and
\[
f'' = \sum_{\mu \in I_n} \sum_{l=0}^{f-1} b_{l,n} \mu^{r+p^l} \left[ g_n^{0}, \mu, 1 \right] + g''_{n-1},
\]
where $a_l$ and $b_l$ are constants.
Hence $f_n$ takes the form
\[
f_n = \sum_{l=0}^{f-1} a_{l,n} s_n^{q-1-r+p^l} + \sum_{l=0}^{f-1} b_{l,n} t_n^{r+p^l} + g_{n-1},
\]
where
\[
g_{n-1} = g'_n + g''_n \in B(n - 1).
\]
Thus it follows that
\[
f - \sum_{l=0}^{f-1} a_{l,n} s_n^{q-1-r+p^l} - \sum_{l=0}^{f-1} b_{l,n} t_n^{r+p^l} = g_{n-1} + f'
\]
is an $I(1)$-invariant vector modulo $(\text{Ker } T_{-1,0}, \text{Ker } T_{1,2})$ in $B(n - 1)$.
Applying this argument on vectors in $B(n - 1)$ and repeating this process, we get
\[
f = \sum_{l=0}^{f-1} a_{l,n} s_n^{q-1-r+p^l} + \sum_{l=0}^{f-1} b_{l,n} t_n^{r+p^l} + \cdots + \sum_{l=0}^{f-1} a_{l,2} s_2^{q-1-r+p^l} + \sum_{l=0}^{f-1} b_{l,2} t_2^{r+p^l} + f_1,
\]
where $f_1$ is an $I(1)$-invariant in $B(1)$. Write
\[
f_1 = f'_1 + f''_1,
\]
where
\[
f'_1 = \sum_{\mu \in I_1} \sum_{i} a_{i,\mu} \left[ g_1^{0}, \mu, 1 \right],
\]
and
\[
f''_1 = \sum_{\mu \in I_1} \sum_{j} b_{j,\mu} \left[ \left( \begin{array}{c} 1 \\ 0 \end{array} \right) w, 1 \right].
\]
Using the action of
\[
u = \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right)
\]
on $f_1$, by Lemma 3.6 (1), the possible powers $i$ of $\mu$ in $f'_1$ will satisfy either $0 \leq i \leq q - 1 - r$ or $i = q - 1 - r + p'$ for some $0 \leq l \leq f - 1$. If $i = q - 1 - r + p'$, then

$$(1 \ 1) f'_1 - f'_1 = \sum_{\mu \in l_1} \mu^{q - 1 - r} [g_{0,1,\mu}, 1] \in (\text{Ker } T_{-1,0}, \text{Ker } T_{1,2}).$$

This, by Lemma 3.6 (1), gives $[\beta, 1] \in (\text{Ker } T_{-1,0}, \text{Ker } T_{1,2})$, which is not possible. So we must have $0 \leq i \leq q - 1 - r$. Then, by Lemma 3.6 (1) and Lemma 3.6 (1), we have

$$f'_1 = [\beta, 1] \mod (\text{Ker } T_{-1,0}, \text{Ker } T_{1,2}).$$

Similarly, by Lemmas 3.5 (2) and 3.6 (2) and Lemma 3.6 (2), we can show that

$$f''_1 = [\text{Id}, 1] \mod (\text{Ker } T_{-1,0}, \text{Ker } T_{1,2}).$$

Thus, we have

$$f = \sum_{l=0}^{f-1} a_{1,n} s_n^{q - 1 - r + p'} + \sum_{l=0}^{f-1} b_{1,n} t_n^{r + p'} + \ldots$$

$$+ \sum_{l=0}^{f-1} a_{1,2} s_2^{q - 1 - r + p'} + \sum_{l=0}^{f-1} b_{1,2} t_2^{r + p'} + c [\beta, 1] + d [\text{Id}, 1].$$

Now assume $e = 1$ and $f = 1$. Let $f \in \text{ind}^G_{1,\chi_1}$ be an $(I)$-invariant vector modulo $(\text{Ker } T_{-1,0}, \text{Ker } T_{1,2})$. As in the previous case, we concentrate only on the zero side of the tree and assume that $f = f'^0$. We write $f = f_n + f'$ where $f_n \neq 0$ and $f' \in B(n - 1)$. We further write $f_n = f'_n + f''_n$ where $f'_n$ and $f''_n$ are same as in the previous case. Following the steps in the previous case, we have

$$f'_n = \sum_{\mu \in l_n} a_0 \mu^{p - r} [g_{0,1,\mu}, 1] + g'_n,$$

and

$$f''_n = \sum_{\mu \in l_n} b_0 \mu^{r + 1} [g_{0,1,\mu}, 1] + g''_n,$$

where $a_0$ and $b_0$ are constants and $g'_n, g''_n \in B(n - 1)$. Thus,

$$f_n = \sum_{\mu \in l_n} a_0 \mu^{p - r} [g_{0,1,\mu}, 1] + \sum_{\mu \in l_n} b_0 \mu^{r + 1} [g_{0,1,\mu}, 1] + g_n,$$

where $g_n = g'_n + g''_n \in B(n - 1)$. Write $f = f_n + f_{n-1} + f'$. We get

$$\left(\begin{array}{c} p^{n-2} \\ 1 \end{array}\right) f - f = \left[\left(\begin{array}{c} p^{n-2} \\ 1 \end{array}\right) f_n - f_n\right] + \left[\left(\begin{array}{c} p^{n-2} \\ 1 \end{array}\right) f_{n-1} - f_{n-1}\right]$$

$$(1 \ 1) \in (\text{Ker } T_{-1,0}, \text{Ker } T_{1,2}).$$

For $e = 1$, we have

$$\left(\begin{array}{c} p^{n-2} \\ 1 \end{array}\right) f_n - f' = \sum_{\mu \in l_n} a_0 \left[(\mu_{n-1} - (\ast))^{p - r} - \mu^{p - r}_{n-1}\right] [g_{0,1,\mu}, 1],$$
where
\[ (*) = \sum_{s=1}^{p-1} (-1)^{p-s} \binom{p}{s} \mu_{n-2}^s. \]

Then, by Lemmas 3.6 (1) and 3.5 (1), modulo \((\text{Ker } T_{-1,0}, \text{Ker } T_{1,2})\), the above expression becomes
\[ (11) \quad - \sum_{\mu \in I_{n-1}} a_0 \left( \frac{p-r}{p-1-r} \right) (*) \left[ \begin{array}{cc} \delta_{n-2,|\mu|n-2}^0 & \left( \begin{array}{c} 1 \\ 0 \\ 1 \end{array} \right) \end{array} \right] \left[ \begin{array}{c} \left[\mu_{n-2}\right] \\ 1 \end{array} \right] w, 1. \]

Writing \( f_{n-1} = f'_{n-1} + f''_{n-1} \), we have,
\[ \left( \begin{array}{c} 1 \\ 0 \\ 1 \end{array} \right) f_{n-1} - f_{n-1} = \left[ \left( \begin{array}{c} 1 \\ 0 \\ 1 \end{array} \right) f'_{n-1} - f'_{n-1} \right] + \left[ \left( \begin{array}{c} 1 \\ 0 \\ 1 \end{array} \right) f''_{n-1} - f''_{n-1} \right]. \]

No term in the first summand of the above equation can cancel a term in \((11)\). Also, by Lemma 4.6 (2), the possible powers, say \( k \), of \( \mu_{n-2} \) in \( f''_{n-1} \) must satisfy either \( 0 \leq k \leq r \) or \( k = r + 1 \). As \( r < p - 1 \), we have \( \max(r + 1) = p - 1 \). So the maximum power of \( \mu_{n-2} \) in the second summand of the above equation is \( p - 2 \). In both the cases, the term involving \( \mu_{n-2}^{p-1} \) in \((11)\) will not get cancelled. Since there is no reduction, this term must be in \( \text{Ker } T_{-1,0} \), which is not possible by Lemma 3.5 (2). Thus we arrive at a contradiction. So \( i_{n-1} \) cannot be \( p - r \). Thus one can always modify \( f'_{n} \) by a vector \( s'_{n-1} \) in \( B(n - 1) \). Similarly, working with \( f''_{n} \), we can modify it by a vector \( s''_{n-1} \) in \( B(n - 1) \). Thus \( f_{n} \) is congruent to a vector \( f_{n-1} \) in \( B(n - 1) \) modulo \((\text{Ker } T_{-1,0}, \text{Ker } T_{1,2})\) and hence by induction, \( f \) is congruent to a vector \( f_{1} \) in \( B(1) \) modulo \((\text{Ker } T_{-1,0}, \text{Ker } T_{1,2})\). Write \( f_{1} = f'_{1} + f''_{1} \), where
\[ f'_{1} = \sum_{i \in I_{1}} \sum_{\mu \in I_{1}} a_{i} \mu^{i} \left[ \begin{array}{c} 0 \\ \delta_{1,\mu}^0 \\ 1 \end{array} \right], \]
and
\[ f''_{1} = \sum_{j \in I_{1}} \sum_{\mu \in I_{1}} b_{j} \mu^{j} \left[ \begin{array}{c} 0 \\ \mu \end{array} \right] \left[ \begin{array}{c} 0 \\ 1 \end{array} \right] w, 1. \]

Considering the action of \( \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right) \) on \( f_{1} \) as in the previous case, we have \( 0 \leq i \leq p - 1 - r \) and \( 0 \leq j \leq r \), by Lemma 3.6 and Lemma 4.6. Then, by Lemma 3.6 and Lemma 3.5, modulo \((\text{Ker } T_{-1,0}, \text{Ker } T_{1,2})\), we get \( f'_{1} = [\beta, 1] \) and \( f''_{1} = [\text{Id}, 1] \). Thus we can conclude that
\[ f = c \left[ \text{Id}, 1 \right] + d \left[ \beta, 1 \right]. \]

This finishes the proof of Theorem 1.2.

4.5. A remark on \( \pi_{r} \). We show that there is no isomorphism between
\[ \tau_{r} = \frac{\text{ind}_{KZ}^{G} \sigma_{r}}{(T)} \]
and
\[ \pi_{r} = \frac{\text{ind}_{KZ}^{G} \chi_{r}}{(\text{Ker } T_{-1,0}, \text{Ker } T_{1,2})} \]
when \( f \neq 1 \); i.e., \( F \) is not a totally ramified extension of \( Q_{p} \) (cf. Remark 3).
Note that any $G$-linear isomorphism

$$\varphi : \pi_r \to \tau_r$$

must preserve $I(1)$-invariants and the corresponding $I$-eigenvalues.

Suppose $e = 1, f \neq 1$; i.e., $F / \mathbb{Q}_p$ is unramified. In this case, $s_n^{q-1-r+p}l$, for $n \geq 2$, is an $I(1)$-invariant in $\pi_r$ such that

$$\left( \begin{array}{cc} a & b \\ \alpha c & d \end{array} \right) \cdot s_n^{q-1-r+p}l = a^{r-p}l d^{p} \cdot s_n^{q-1-r+p}l$$

by Lemma 4.3. By [Hen19, Theorem 1.2], a basis of the $I(1)$-invariants in $\tau_r$ consists of the vectors

$$\text{Id} \otimes \bigotimes_{j=0}^{f-1} x_j^r, \alpha \otimes \bigotimes_{j=0}^{f-1} y_j^r, c_n^{p(r_l+1)}, \beta c_n^{p(r_l+1)}$$

for $n \geq 1$, where

$$c_n^k = \sum_{\mu \in I_n} \left( \begin{array}{cc} \alpha^n & \mu \\ 0 & 1 \end{array} \right) \otimes \mu_n \cdot \bigotimes_{j=0}^{f-1} x_j^r.$$

By [Hen19, Lemma 3.6],

$$\left( \begin{array}{cc} a & b \\ \alpha c & d \end{array} \right) \cdot c_n^k = a^{r-2k(ad)} \cdot c_n^k,$$

and it follows that there is no $I(1)$-invariant vector in $\tau_r$ with $I$-eigenvalue $a^{r-p}l d^{p}l$. Thus there is no vector in $\tau_r$ where $s_n^{q-1-r+p}l$ can be mapped under $\varphi$. This gives a contradiction.

Now, suppose $e > 1, f > 1$. In this case $t_n^{r+p}l, n \geq 2$, is an $I(1)$-invariant vector in $\pi_r$ with $I$-eigenvalue $a^{q-1-p}l d^{r+p}l$, by Lemma 4.3. A basis of the $I(1)$-invariants in $\tau_r$ consists of the vectors

$$\text{Id} \otimes \bigotimes_{j=0}^{f-1} x_j^r, \alpha \otimes \bigotimes_{j=0}^{f-1} y_j^r, c_n^{p(r_l+1)}, \beta c_n^{p(r_l+1)}, d_n^l, \beta d_n^l$$

for $n \geq 1$, where

$$d_n^l = \sum_{\mu \in I_n} \left( \begin{array}{cc} \alpha^n & \mu \\ 0 & 1 \end{array} \right) \otimes \bigotimes_{l \neq j=0}^{f-1} x_j^r \otimes x_j^{r-1} y_l^r,$$

by [Hen19, Theorem 1.2]. By [Hen19, Lemma 3.6],

$$\left( \begin{array}{cc} a & b \\ \alpha c & d \end{array} \right) \cdot d_n^l = a^{r-2p}l (ad)^p \cdot d_n^l,$$

and once again it can be checked that there is no $I(1)$-invariant vector in $\tau_r$ with $I$-eigenvalue $a^{q-1-p}l d^{r+p}l$, where $t_n^{r+p}l$ can be mapped under $\varphi$, giving a contradiction.
Acknowledgements

The second author would like to thank Council of Scientific and Industrial Research, Government of India (CSIR) and Industrial Research and Consultancy Centre, IIT Bombay (IRCC) for financial support.

References

[AB13] U. K. Anandavardhanan and Gautam H. Borisagar, On the $K(n)$-invariants of a supersingular representation of $GL_2(Q_p)$, The legacy of Srinivasa Ramanujan, Ramanujan Math. Soc. Lect. Notes Ser., vol. 20, Ramanujan Math. Soc., Mysore, 2013, pp. 55–75. MR 3221302

[AB15], Iwahori-Hecke model for supersingular representations of $GL_2(Q_p)$, J. Algebra 423 (2015), 1–27. MR 3283706

[BL94] L. Barthel and R. Livné, Irreducible modular representations of $GL_2$ of a local field, Duke Math. J. 75 (1994), no. 2, 261–292. MR 1290194

[BP12] Christophe Breuil and Vytautas Paškūnas, Towards a modulo $p$ Langlands correspondence for $GL_2$, Mem. Amer. Math. Soc. 216 (2012), no. 1016, vi+114. MR 2931521

[Bre03] Christophe Breuil, Sur quelques représentations modulaires et $p$-adiques de $GL_2(Q_p)$. I, Compositio Math. 138 (2003), no. 2, 165–188. MR 2018825

[Hen19] Yotam I. Hendel, On the universal mod $p$ supersingular quotients for $GL_2(F)$ over $\overline{\mathbb{F}}_p$ for a general $F/Q_p$, J. Algebra 519 (2019), 1–38. MR 3873949

[Sch11] Michael M. Schein, An irreducibility criterion for supersingular mod $p$ representations of $GL_2(F)$ for totally ramified extensions $F$ of $Q_p$, Trans. Amer. Math. Soc. 363 (2011), no. 12, 6269–6289. MR 2833554

Department of Mathematics, Indian Institute of Technology Bombay, Mumbai - 400076, India.
Email address: anand@math.iitb.ac.in

Department of Mathematics, Indian Institute of Technology Bombay, Mumbai - 400076, India.
Email address: arindam@math.iitb.ac.in