Nordhaus-Gaddum bounds for locating domination

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Abstract

A dominating set $S$ of graph $G$ is called metric-locating-dominating if it is also locating, that is, if every vertex $v$ is uniquely determined by its vector of distances to the vertices in $S$. If moreover, every vertex $v$ not in $S$ is also uniquely determined by the set of neighbors of $v$ belonging to $S$, then it is said to be locating-dominating. Locating, metric-locating-dominating and locating-dominating sets of minimum cardinality are called $\beta$-codes, $\eta$-codes and $\lambda$-codes, respectively. A Nordhaus-Gaddum bound is a tight lower or upper bound on the sum or product of a parameter of a graph $G$ and its complement $\overline{G}$. In this paper, we present some Nordhaus-Gaddum bounds for the location number $\beta$, the metric-location-domination number $\eta$ and the location-domination number $\lambda$. Moreover, in each case, the graph family attaining the corresponding bound is fully characterized.

Keywords: Domination, Location, Locating domination, Nordhaus-Gaddum

1. Introduction

Given a graph $G = (V, E)$, the (open) neighborhood of a vertex $v \in V$ is $N_G(v) = N(v) = \{u \in V : uv \in E\}$. The distance between vertices $v, w \in V$ is denoted by $d_G(v, w)$, or $d(v, w)$ if the graph $G$ is clear from the context. The diameter $\text{diam}(G)$ is the maximum distance between any two vertices of $G$. Let $S = \{x_1, \ldots, x_k\}$ be a set of vertices and let $v \in V \setminus S$. The ordered $k$-tuple $c_j(v) = (d(v, x_1), \ldots, d(v, x_k))$ is called the vector of metric coordinates of $v$ with respect to $S$. For further notation see [4].

A set $D \subseteq V$ is a dominating set if for every vertex $v \in V \setminus D$, $N(v) \cap D \neq \emptyset$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of $G$. A dominating set of cardinality $\gamma(G)$ is called a $\gamma$-code [8].

A set $D = \{x_1, \ldots, x_k\} \subseteq V$ is a locating set if for every pair of distinct vertices $u, v \in V$, $c_D(u) \neq c_D(v)$. The location number (also called the metric dimension) $\beta(G)$ is the minimum cardinality of a locating set of $G$ [7, 14].

A locating set of cardinality $\beta(G)$ is called a $\beta$-code. A metric-locating-dominating set, a MLD-set for short, is any set of vertices that is both a dominating set and a locating set. The metric-location-domination number $\eta(G)$ is the minimum cardinality of a metric-locating-dominating set of $G$. A metric-locating-dominating set of cardinality $\eta(G)$ is called a $\eta$-code [10].

A set $D \subseteq V$ is a locating-dominating set, an LD-set for short, if for every two vertices $u, v \in V(G) \setminus D$, $\emptyset \neq N(u) \cap D \neq N(v) \cap D \neq \emptyset$. The location-domination number $\lambda(G)$ is the minimum cardinality of a locating-dominating set. A locating-dominating set of cardinality $\lambda(G)$ is called a $\lambda$-code [15]. A complete and regularly updated list of papers on locating dominating codes is to be found in [13].

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Clearly, every locating-dominating set is locating and also dominating. Moreover, both location and domination are hereditary properties. Particularly, if for two sets $S_1, S_2 \subseteq V$, $S_1$ is locating and $S_2$ is dominating, then $S_1 \cup S_2$ is both locating and dominating. Hence, for every graph $G$, $\max\{\gamma(G), \beta(G)\} \leq \eta(G) \leq \min\{\gamma(G) + \beta(G), \lambda(G)\}$ [2].

A Nordhaus-Gaddum bound is a tight lower or upper bound on the sum or product of a parameter of a graph $G$ and its complement $\overline{G}$ [11, 9, 12]. For example, in [5] it was shown that for any graph $G$ of order $n$, $\gamma(G) + \gamma(\overline{G}) \leq n + 1$, the equality being true only if $\{G, \overline{G}\} = \{K_n, K_n\}$. In this paper, we present some Nordhaus-Gaddum bounds on the sum of the location number $\beta$, the metric-location-domination number $\eta$ and the location-domination number $\lambda$. In all cases, the classes of graphs attaining both bounds are characterized.

2. Nordhaus-Gaddum bounds

Unless otherwise stated, along this section $G = (V; E)$ is a, not necessarily connected, nontrivial graph of order $n$. A graph $G$ is called doubly-connected if both $G$ and its complement $\overline{G}$ are connected. As usual, $K_n$, $C_n$ and $P_n$ denote respectively the complete graph, the cycle and the path on $n$ vertices.

2.1. Location number

**Theorem 1.** For every nontrivial graph $G$, $2 \leq \beta(G) + \beta(\overline{G}) \leq 2n - 1$. Moreover,

- $\beta(G) + \beta(\overline{G}) = 2$ if and only if $G = P_4$.
- $\beta(G) + \beta(\overline{G}) = 2n - 1$ if and only if $\{G, \overline{G}\} = \{K_n, \overline{K}_n\}$.

**Proof.** Every graph satisfies $1 \leq \beta(G)$, which means that $2 \leq \beta(G) + \beta(\overline{G})$. Moreover, the equality $\beta(G) + \beta(\overline{G}) = 2$ is only true for $G = P_4$, since paths $P_n$ are the only graphs with location number 1 [3], and $P_4 = \overline{P}_4$ is the only nontrivial path whose complement is also a path. The upper bound immediately follows from these facts: (1) the graph $\overline{K}_n$ is the only graph with location number $n$ and (2) $\beta(K_n) = n - 1$. Finally, claims (1) and (2) also allows us to derive that equality $\beta(G) + \beta(\overline{G}) = 2n - 1$ only holds when $\{G, \overline{G}\} = \{K_n, \overline{K}_n\}$.

![Figure 1: Solid lines are edges in $G$ and dashed lines are edges in $\overline{G}$.](image)
Lemma 1. Every doubly-connected graph $G$ of order $n \geq 6$ such that $\text{diam}(G) = \text{diam}(\overline{G}) = 2$ contains a locating set of cardinality $n - 4$.

Proof. Let $p$ be an induced path of order 4 in $G$, whose existence is guaranteed since, as was proved in [2], the complement of every nontrivial $P_4$-free graph is not connected. Assume that $V(p) = \{a, b, c, d\}$ and $E(p) = \{ab, bc, cd\}$. Since, $\text{diam}(G) = 2$, there exists a vertex $e \in V(G)$ such that $d_G(a, e) = d_G(e, d) = 1$. We distinguish three cases.

Case 1: $eb, ec \notin E(G)$ (see Figure 1 left). In this case, the set $\{a, b, c, d, e\}$ determines an induced cycle $\Gamma$ in $G$ and also an induced cycle $\overline{\Gamma}$ in $\overline{G}$. Let $f$ a vertex not belonging to $\{a, b, c, d, e\}$. Either in $G$ or in $\overline{G}$, $f$ has at most two neighbors in $\{a, b, c, d, e\}$. Without loss of generality we may suppose that $N_G(f) \cap \{a, b, c, d, e\} \subseteq 2$ (otherwise we interchange labels $G$ and $\overline{G}$), which means that there exist in $\{a, b, c, d, e\}$ a pair of non-consecutive vertices non-adjacent to $f$. Again w.l.o.g. we assume that $N_G(f) \cap \{a, c\} = \emptyset$. Certainly, the set $V(G) \setminus \{b, d, e, f\}$ is a locating set of $G$ since $c_{\{a,e\}}(b) = (1, 1), c_{\{a,e\}}(d) = (2, 1), c_{\{a,e\}}(e) = (1, 2)$ and $c_{\{a,e\}}(f) = (2, 2)$.

Case 2: $e$ is adjacent to exactly one vertex of $\{b, c\}$. Let us assume that $eb \in E(G)$ and $ec \notin E(G)$ (see Figure 1 center). In this case, $d_G(e, b) = 1$, which means that $\text{diam}(\overline{G}) = 2$. Therefore, there exists a vertex $f \notin \{a, b, c, d, e\}$ such that $d_G(e, f) = d_G(f, b) = 1$. This means that $d_G(e, f) = d_G(f, b) = 2$ as $\text{diam}(G) = 2$. Hence, the set $V(G) \setminus \{a, c, d, f\}$ is a locating set of $G$ since $c_{\{b,e\}}(a) = (1, 1), c_{\{b,e\}}(c) = (1, 2), c_{\{b,e\}}(d) = (2, 1)$ and $c_{\{b,e\}}(f) = (2, 2)$.

Case 3: $eb, ec \in E(G)$ (see Figure 1 right). Since $d_G(b, c) = 1$, we have $d_G(b, c) = 2$. Therefore, there exists a vertex $f \notin \{a, b, c, d, e\}$ such that $d_G(f, b) = d_G(f, c) = 1$. This means that $d_G(f, b) = d_G(f, c) = 2$. Hence, the set $V(G) \setminus \{a, d, e, f\}$ is a locating set of $G$ since $c_{\{b,e\}}(a) = (1, 2), c_{\{b,e\}}(d) = (2, 1), c_{\{b,e\}}(e) = (1, 1)$ and $c_{\{b,e\}}(f) = (2, 2)$. 

Take a connected graph $G$ of order $n$, and assume that $V(G) = \{1, \ldots, n\}$. Let $G[H_i^{(r)}]$ denote the graph obtained from $G$ by replacing vertex $i$ by a given graph $H$ and joining every vertex of $H$ to every neighbor of vertex $i$ in $G$. Similarly, $G[H_1^{(1)}, H_2^{(2)}]$ denotes the graph obtained from $G$ by replacing vertex $i$ by a graph $H_1$ and vertex $j$ by a graph $H_2$ and joining every vertex of $H_1$ (resp. vertex of $H_2$) to every neighbor of vertex $i$ (resp. $j$) in $G$ and, just if $ij \in E(G)$, also every vertex of $H_1$ to every vertex of $H_2$. Finally, $B$ denotes the bull graph shown in Figure 2.

Theorem 2. For any doubly-connected graph $G$ with $n \geq 4$, $2 \leq \beta(G) + \beta(\overline{G}) \leq 2n - 6$. Moreover,

- $\beta(G) + \beta(\overline{G}) = 2$ if and only if $G = P_4$.
- $\beta(G) + \beta(\overline{G}) = 2n - 6$ if and only if $G \in \Omega_1 \cup \Omega_2 \cup \Omega_3$, where
  \[ \begin{align*}
  &\Omega_1 = \{P_4, C_5, B\} \\
  &\Omega_2 = \{P_4[K^{(1)}_{n-3}], P_4[\overline{K}^{(1)}_{n-3}], P_4[K^{(2)}_{n-3}], P_4[\overline{K}^{(2)}_{n-3}]\} \\
  &\Omega_3 = \{P_4[K^{(1)}_{n-r}], K^{(2)}_{n-r-2} : 1 \leq r \leq n-3\} \cup \{P_4[\overline{K}^{(1)}_{r}, \overline{K}^{(3)}_{n-r-2} : 1 \leq r \leq n-3\}
  \end{align*} \]

Proof. In [3], it was proved that a connected graph $G$ satisfies $n - 2 \leq \beta(G) \leq n - 1$ if and only if, for some $1 \leq h \leq n - 1$, $G \in \{K_n, K_{n-h}, K_h + \overline{K}_{n-h}, K_h + (K_1 \cup K_{n-h-1})\}$. It is a routine exercise to check that the complement of any of these
graphs is not connected. Hence, every doubly-connected graph \( G \) of order \( n \geq 4 \) satisfies \( 1 \leq \beta(G) \leq n - 3 \), i.e., \( 2 \leq \beta(G) + \beta(\overline{G}) \leq 2n - 6 \). Moreover, according to Theorem [1] the lower bound 2 is attained only for \( G = P_4 \), since \( P_4 = P_4 \).

Let \( G \) be a doubly-connected graph of order \( n \geq 4 \) verifying \( \beta(G) + \beta(\overline{G}) = 2n - 6 \), i.e., such that \( \beta(G) = \beta(\overline{G}) = n - 3 \). In [3], it was proved that the order of a graph \( G \) of diameter \( D \) and location number \( \beta \) is at least \( \beta + D \). This means, that if \( \beta(G) = n - 3 \), then \( 2 \leq D \leq 3 \), since \( \beta(K_n) = n - 1 \). In [11], the set of graphs with \( n \) vertices, diameter \( D \) and location number \( n - D \) were characterized for all feasible values of \( n \) and \( D \). In particular, we have the set of graphs with \( n \geq 4 \) vertices, diameter \( diam(G) = D = 3 \) and location number \( n - 3 \), all of them being doubly-connected and verifying \( diam(\overline{G}) = 3 \). Among them, we are just interested in those graphs \( G \) for which \( \beta(G) = n - 3 \). It is a routine exercise to check that as well as the path \( P_4 \) and the bull graph \( B \), the only doubly-connected graphs of diameter 3 satisfying \( \beta(G) = \beta(\overline{G}) = n - 3 \) are those belonging to \( \Omega_3 \cup \Omega_3 \). Hence, according to Lemma [1] to finalize the proof it suffices to check that the only doubly-connected graph of order 4 \( \leq n \leq 5 \) having both itself and its complement diameter 2 is the cycle \( C_5 \). □

\[ \begin{align*}
\text{H} & \quad \text{B} & \quad \text{E} & \quad \text{F} \\
\text{Figure 2: House graph } H = \overline{P}_4, \text{ bull graph } B = \overline{K}_3, \text{ graph } E \text{ and graph } F = \overline{E}. \end{align*} \]

2.2. Metric-location-domination number

**Theorem 3.** For every nontrivial graph \( G \), \( 3 \leq \eta(G) + \eta(\overline{G}) \leq 2n - 1 \). Moreover,

- \( \eta(G) + \eta(\overline{G}) = 3 \) if and only if \( \{G, \overline{G}\} = \{K_2, \overline{K}_2\} \).

- \( \eta(G) + \eta(\overline{G}) = 2n - 1 \) if and only if \( \{G, \overline{G}\} = \{K_n, \overline{K}_n\} \).

**Proof.** The only nontrivial graph \( G \) such that \( \eta(G) = 1 \) is \( G = K_2 \), which means that for every graph \( G \), \( 3 \leq \eta(G) + \eta(\overline{G}) \). Moreover, the equality \( \eta(G) + \eta(\overline{G}) = 3 \) is only true when either \( G \) or \( \overline{G} \) is \( K_2 \), since \( \eta(K_2) = 2 \). The rest of the proof is similar to that of Theorem [1] □

Given two positive integers \( r, s \), let \( K_2(r, s) \) denote the so-called double star, obtained after joining the central vertices of the stars \( K_{1,r} \) and \( K_{1,s} \). If \( 2 \leq s \leq r - 1 \), let \( K^s_{1,r} \) represent the graph obtained by adding a new vertex adjacent to \( s \) leaves of the star \( K_{1,r} \). Finally, \( \overline{K}_2(r, s), \overline{K}^s_{1,r} \) denote the complements of \( K_2(r, s), K^s_{1,r} \), respectively, and graphs \( B, H, E \) and \( F \) are shown in Figure [2]
Theorem 4. For any doubly-connected graph $G$ with $n \geq 5$, $4 \leq \eta(G) + \eta(\overline{G}) \leq 2n - 5$. Moreover,

- $\eta(G) + \eta(\overline{G}) = 4$ if and only if $G \in \{P_5, C_5, B, H, E, F\}$.
- $\eta(G) + \eta(\overline{G}) = 2n - 5$ if and only if $G \in \{K_2(r, s), \overline{K_2(r, s)}, K^r_{1,s}, \overline{K^r_{1,s}}\}$.

Proof. Every doubly-connected graph $G$ of order at least 5 satisfies $2 \leq \eta(G)$, since the unique nontrivial graph such that $\eta(G) = 1$ is $G = P_2$. In other words, for every nontrivial doubly-connected graph $G$, $4 \leq \eta(G) + \eta(\overline{G})$. In [2], it was proved that there are exactly 51 connected graphs satisfying $\eta(G) = 2$, any of them having an order between 3 and 8. It is a routine exercise to check that the only doubly-connected graphs $G$ with order at least 5 of this family whose complement verify also $\eta(\overline{G}) = 2$ are exactly the graphs belonging to the set $\{P_5, C_5, B, H, E, F\}$.

In [10], it was proved that if $G$ is a connected graph such that $\eta(G) = n - 1$, then $G$ is either the complete graph $K_n$ or the star $K_{1,n-1}$. Hence, every doubly-connected graph $G$ of order $n \geq 4$ satisfies $\eta(G) \leq n - 2$, since both $K_n$ and $\overline{K_{1,n-1}}$ are not connected. Also in [10], all connected graphs $G$ for which $\eta(G) = n - 2$ were completely characterized. It is a routine exercise to check that the complement of any graph $G$ verifying $\eta(G) = n - 2$ is not connected unless $G$ is either a double star $K_2(r, s)$ or a graph $K^r_{1,s}$. As $\eta(\overline{K_2(r, s)}) = \eta(\overline{K^r_{1,s}}) = n - 3$, we conclude first, that every doubly-connected graph $G$ of order $n \geq 5$ satisfies $\eta(G) + \eta(\overline{G}) \leq 2n - 5$ and second, that these four families are the only ones attaining this upper bound.

$\square$

2.3. Location-domination number

Theorem 5. For every nontrivial graph $G$, $3 \leq \lambda(G) + \lambda(\overline{G}) \leq 2n - 1$. Moreover,

- $\lambda(G) + \lambda(\overline{G}) = 3$ if and only if $\{G, \overline{G}\} = \{K_2, K_\overline{2}\}$.
- $\lambda(G) + \lambda(\overline{G}) = 2n - 1$ if and only if $\{G, \overline{G}\} = \{K_n, \overline{K_n}\}$.

Proof. It is similar to that of Theorem 3.

$\square$

Theorem 6. For any doubly-connected graph $G$ with $n \geq 5$, $4 \leq \lambda(G) + \lambda(\overline{G}) \leq 2n - 5$. Moreover,

- $\lambda(G) + \lambda(\overline{G}) = 4$ if and only if $G \in \{P_5, C_5, B, H\}$.
- $\lambda(G) + \lambda(\overline{G}) = 2n - 5$ if and only if $G \in \{K_2(r, s), \overline{K_2(r, s)}, K^r_{1,s}, \overline{K^r_{1,s}}\}$.

Proof. Every doubly-connected graph $G$ of order at least 5 satisfies $2 \leq \lambda(G)$, since the unique nontrivial graph such that $\lambda(G) = 1$ is $G = P_2$. In other words, for every nontrivial doubly-connected graph $G$, $4 \leq \lambda(G) + \lambda(\overline{G})$. In [2], it was proved that there are exactly 16 connected graphs satisfying $\lambda(G) = 2$, any of them having an order between 3 and 5. It is a routine exercise to check that the only doubly-connected graphs $G$ of this family whose complement verify also $\lambda(\overline{G}) = 2$ are the 5-path $P_5$, the 5-cycle $C_5$, the bull graph $B$ and the house graph $H$ (see Figure 2). The rest of the proof is similar to that of Theorem 4 since for every graph $G$, if $\lambda(G) = n - 1$, then $G$ is either the complete graph $K_n$ or the star $K_{1,n-1}$ [15] and, $\lambda(G) = n - 2$ if and only if $\eta(G) = n - 2$ [2].

$\square$
Observe that the only doubly-connected graph of order at most 4 is \( P_4 \), and notice also that \( \overline{P}_4 = P_4 \) and \( \eta(P_4) = \lambda(P_4) \), which means that \( \eta(P_4) + \eta(\overline{P}_4) = \lambda(P_4) + \lambda(\overline{P}_4) = 4 \).

Finally, we present a further Nordhaus-Gaddum-type result for the parameter \( \lambda \), which is a direct consequence of the fact that LD-sets in a graph \( G \) are very strongly related to LD-sets in its complement \( \overline{G} \).

**Proposition 1.** If \( S \) is an LD-set of a graph \( G \) then \( S \) is also an LD-set of \( G \), unless there exists a vertex \( w \in V \setminus S \) such that \( S \subseteq N_G(w) \), in which case \( S \cup \{w\} \) is an LD-set of \( G \).

**Proof.** Take \( u, v \in V \setminus S \). Since \( S \) is an LD-set of \( G \), \( S \cap N_G(u) \neq \emptyset \) and \( S \cap N_G(v) \neq \emptyset \). Hence, \( S \cap N_G(u) = S \setminus S \cap N_G(u) \neq S \setminus N_G(v) = S \cap N_{\overline{G}}(v) \). At this point we distinguish two cases: if there exists a vertex \( w \in V \setminus S \) such that \( S \subseteq N_G(w) \), or equivalently, such that \( S \cap N_G(w) = \emptyset \), then it is unique as \( c_2(w) = (1 \ldots 1) \), and thus \( S \cup \{w\} \) is an LD-set. Otherwise, for every vertex \( w \), \( S \cap N_{\overline{G}}(w) \neq \emptyset \), which means that \( S \) is also an LD-set of \( G \).

**Theorem 7.** For every graph \( G \), \( |\lambda(G) - \lambda(\overline{G})| \leq 1 \).

**Proof.** According to Proposition 1, if \( S \) is an \( \lambda \)-code of \( G \), then there exists an LD-set of \( G \) of cardinality at most \( \lambda(G) + 1 \), which means that \( \lambda(\overline{G}) \leq \lambda(G) + 1 \). Similarly, it is derived that \( \lambda(G) \leq \lambda(\overline{G}) + 1 \), as \( G = \overline{\overline{G}} \).

**Corollary 1.** Every graph \( G \) satisfies: \( 2\lambda(G) - 1 \leq \lambda(G) + \lambda(\overline{G}) \leq 2\lambda(G) + 1 \).

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