Caputo’s Finite Difference Solution of Fractional Two-Point BVPs Using AGE Iteration

R. Rahman, N. A. M. Ali, J. Sulaiman, and F. A. Muhiddin

Mathematics with Economics Programme, Faculty of Science and Natural Resources, Universiti Malaysia Sabah, 88400 Kota Kinabalu, Sabah, Malaysia.
Faculty of Computing and Mathematical Sciences, Universiti Teknologi MARA Sabah, Malaysia.

rostang054@gmail.com, afzamatali@yahoo.com, jumat@ums.edu.my, and fatih170@sabah.uitm.edu.my

Abstract. The aim of this paper deals with the performance of Alternating Group Explicit (AGE) iterative method for solving the linear system based on Caputo’s finite difference approximation equation. Actually, the approximation equation was be generated from the discretization process of the fractional two-point boundary value problems by using finite difference discretization scheme and Caputo’s operator. The formulation and implementation of AGE iterative method have been included. To investigate the efficiency of AGE iterative method, this paper considered four examples of numerical experiments. Based on numerical results, the findings showed that the AGE method is more efficient as compared with GS method.

1. Introduction

Fractional calculus becomes more attractive and popular in the recent years [1, 2]. Due to this matter, the numerical approaches of fractional differential equations have been discussed in numerous researches due to their wide applications such as electrochemistry, economics, physics, and engineering [3, 4, 5, 6]. Following to that, several numerical methods considered such as Quadrature Tau method [7], Reproducing Kernel method [8], Homotopy analysis method [9], and Extrapolation method [10].

However this paper deals with the finite difference method for solving two-point boundary value problems of fractional order in which the discretization process needs to be done to form the second-order Caputo’s approximation equation by considering the second-order finite difference scheme and Caputo’s operator. The performance of finite difference method by using Caputo’s operator for solving the fractional PDE’s has been pointed in previous studied [11, 12, 13, 14]. By considering the Caputo’s approximation equation, this approximation equation have been using to construct the linear system. Having a linear system, the numerical solution of this linear system has been can be calculated via iterative methods. Actually, there are several iterative methods from previous studies can be used especially to solve the large linear system [15, 16]. However, this paper focuses to investigate the performance of Alternating Group Explicit (AGE) iterative method [17] in order to solve fractional two-point boundary value problems. This method is modified from analogues of the classical method called Alternate Direction Implicit (ADI) scheme [18, 19, 20]. Besides that, this iterative method utilizes the fractional splitting strategy where applied alternately at each half time step and match for parallel computation to give advantageousness [21]. To analyse the performance AGE iterative method, this paper also implements the Gauss Seidel iterative method as a control method.
To investigate the performance of AGE iterative method, let us consider the fractional two-point boundary value problems defined as [22]

\[ d(x)D^\beta u(x) + a(x)u'(x) + b(x)u(x) + c(x)u(x) = f(x), x \in [\eta, \gamma] \]  

subject to the Dirichlet boundary conditions

\[ u(\eta) = \eta_0, \]
\[ u(\gamma) = \gamma_1. \]

where \( a(x), b(x) \) and \( c(x) \) are known functions or constants respectively and \( D^\beta \) is the fractional derivative operator and \( \beta \) is a parameter which refers to the fractional order.

To discretise problem (1) via finite difference scheme, let the solution domain \([\eta, \gamma]\) needs to divide with equal and sparse \( h \), in which its spacing is define as

\[ h = \Delta x = \frac{\gamma}{N} \]  

**Figure 1.** The finite grid network for the solution domain of problem 1.

Based on equation (2), the polar grid of the solution domain will be indicated as

\[ x_i = \eta + ih, \quad i = 0, 1, 2, K, N \]  

and the values of function \( u(x) \) at point \( x_i \) are donated as \( u(x_i) = u_i \).

2. Preliminaries

Based on problem (1), the following are some definitions and mathematical preliminaries related to fractional calculus in order to construct Caputo’s finite differences approximation equations via Caputo’s operator.

**Definition 1** [23, 24]. The fractional integral operator for Riemann-Liouville is define in the following form

\[ J^\beta f(x) = \frac{1}{\Gamma(\beta)} \int_0^x (x-t)^{\beta-1} f(t) dt, \quad \beta > 0, \quad x > 0 \]  

**Definition 2** [23, 24]. The fractional partial derivative operator for Caputo’s is define in the following form

\[ D^\beta f(x) = \frac{1}{\Gamma(m-\beta)} \int_0^x \frac{f^{(m)}(t)}{(x-t)^{\beta+m-1}} dt, \quad \beta > 0 \]  

Where \( m-1 < \beta \leq m, \quad m \in \mathbb{N}, \quad x > 0 \).

**Definition 3** [23, 24]. If \( m \) to be the smallest integer that exceeds \( \beta \), then

\[ D^\beta u(x,t) = \frac{d^\beta u(x,t)}{dx^\beta} = \left\{ \begin{array}{ll}
\frac{1}{\Gamma(m-\beta)} \int_0^x (x-s)^{m-\beta-1} \frac{\partial^n}{\partial s^n} u(s,t) ds, & \text{for } m-1 < \beta \leq m \\
\frac{\partial^n}{\partial x^n} u(x,t), & \text{for } \beta = m \in \mathbb{N} \end{array} \right\} \]  

To get the numerical solution from problem (1), the proposed finite difference scheme with Caputo’s operator were considered to construct the Caputo’s finite difference approximation equation with in the Section 3.
3. Derivation of Second-Order Caputo’s Finite Difference Approximation Equation

Before doing the discretization process, this section considers finite difference scheme and Caputo’s operator as mentioned in Section 2 to derive the Caputo’s finite difference approximation equation of problem (1). By referring definition 3 to discretize problem (1), the formulation of Caputo’s derivative operator can be given as

\[ D^\beta u_i = \sigma_{\beta,h} \sum_{j=0}^{N} g_j^\beta (u_{i-j+1} - 2u_{i-j} + u_{i-j-1}) \tag{7} \]

where

\[ \sigma_{\beta,h} = \frac{h^{-\beta}}{\Gamma(3 - \beta)} \tag{8} \]

and

\[ g_j^\beta = (j + 1)^{-\beta} - j^{-\beta} \tag{9} \]

Referring to the grid size as shown in Figure 1, we develop the uniformly finite grid network of the solution domain where the grid points are shown the numbers \( x_i = ih, i = 0,1,2,...,N \). Then the values of the function \( u(x) \) at the grid points are denoted mention in section 1 as \( u(x_i) = u_i \). By using formulation in equation (6) and the second order central difference discretization schemes, the Caputo’s finite difference approximation equation of problem (1) can be given as:

\[-\sigma_{\beta,h} \sum_{j=0}^{N} g_j^\beta (u_{i-j+1} - 2u_{i-j} + u_{i-j-1}) + a_i (u_{i+1} - 2u_i + u_{i-1}) + b_i (u_{i+1} - u_{i-1}) + c_i (u_i) = f_i \tag{10} \]

now by simplifying equation (10) at \( j = 0 \), we get:

\[ a_i^* u_{i+1} + b_i^* u_i + c_i^* - R_i = f_i, \quad i = 1, 2 \tag{11} \]

where

\[ a_i^* = a_i - \mu_i - \sigma_{\beta,h}, \]
\[ b_i^* = c_i - 2\alpha_i + 2\sigma_{\beta,h}, \]
\[ c_i^* = c_i + \mu_i - \sigma_{\beta,h}, \]
\[ R_i = \lambda \sum_{j=0}^{N} g_j^\beta (u_{i-j+1} - 2u_{i-j} + u_{i-j-1}). \]

Again, decide \( i = 2, 3 \), the derivation of approximation equation (10) need to be shown for \( i = 3, 4, 5, \ldots, N \). To do this, the second-order Caputo’s finite difference approximation equation can be stated as follows:

\[-R_i^* + p_i u_{i-3} + q_i u_{i-2} + r_i u_{i-1} + s_i u_i + z_i u_{i+1} = f_i, \quad i = 3, 4, 5, \ldots, N \tag{12} \]

where

\[ R_i^* = \lambda \sum_{j=0}^{N} g_j^\beta (u_{i-j+1} - 2u_{i-j} + u_{i-j-1}), \]
\[ \lambda = \sigma_{\beta,h}, \]
\[ p = -\lambda g_1^\beta, \]
\[ q = -\lambda g_1^\beta + 2\lambda g_2^\beta, \]
\[ r = a_i^* + 2\lambda g_1^\beta - \lambda g_2^\beta, \]
\[ s = b_i^* - \lambda g_1^\beta, \]
\[ z = c_i. \]

By considering approximation equations (11) and (12), the combination of these approximation equations can construct the following linear system.
where

\[ Au = f \]

(13)

\[
A = \begin{bmatrix}
    s_1 & z_1 & 0 & 0 & 0 & 0 & 0 \\
    r_2 & s_2 & z_2 & 0 & 0 & 0 & 0 \\
    0 & r_3 & s_3 & z_3 & 0 & 0 & 0 \\
    0 & 0 & r_4 & s_4 & z_4 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & r_{N-2} & s_{N-2} & z_{N-2} \\
    0 & 0 & 0 & 0 & 0 & r_{N-1} & s_{N-1} \\
\end{bmatrix}
\]

\[
u = \begin{bmatrix}
    u_1 \\
    u_2 \\
    u_3 \\
    \vdots \\
    u_{N-2} \\
    u_{N-1} \\
\end{bmatrix}
\]

\[ f = \begin{bmatrix}
    f_1 + r_0 u_0 \\
    f_2 + q_0 u_0 \\
    f_3 + p_0 u_0 + r_1 u_1 \\
    f_4 + R_1 - p_2 u_1 - q_2 u_2 \\
    \vdots \\
    f_{N-2} + R_{N-2} - p_{N-2} u_{N-5} - q_{N-2} u_{N-4} \\
    f_{N-1} + R_{N-1} - p_{N-4} u_{N-4} - q_{N-4} u_{N-3} - z_{N-4} u_{N-1} \\
\end{bmatrix}
\]

4. Formulation of AGE Iterative Method

Based on the linear system (13), can be seen that the characteristic of coefficient matrix for the linear system (13) matrix shows that the linear system has large sparse and scale. This means that the linear system is suitable to be solved using iterative methods other than direct method as we mentioned in Section 1. Because of that, this section considers the Alternating Group Explicit (AGE) iterative method as a linear solver proposed by Evans (1985). In order, to implementation this method needs the tridiagonal matrices of the linear split need to split into submatrices.

\[ A = G_1 + G_2 \]

(14)

if \( n \) is odd number so \( S_i = \frac{s}{2} (i = 1, 2, 3, K , N-1) \) then the submatrices \( G_1 \) and \( G_2 \) being given as:

\[
G_1 = \begin{bmatrix}
    S_1 & z_1 & 0 & 0 & 0 & 0 & 0 \\
    r_1 & S_2 & 0 & 0 & 0 & 0 & 0 \\
    0 & r_3 & S_3 & 0 & 0 & 0 & 0 \\
    0 & 0 & r_4 & S_4 & 0 & 0 & 0 \\
    0 & 0 & 0 & r_{N-3} & S_{N-3} & 0 & 0 \\
    0 & 0 & 0 & 0 & r_{N-2} & S_{N-2} & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & S_{N-1} \\
\end{bmatrix}
\]

\[
G_2 = \begin{bmatrix}
    0 & S_2 & z_2 & 0 & 0 & 0 & 0 \\
    0 & r_3 & S_3 & 0 & 0 & 0 & 0 \\
    0 & 0 & r_4 & S_4 & z_4 & 0 & 0 \\
    0 & 0 & 0 & r_{N-4} & S_{N-4} & 0 & 0 \\
    0 & 0 & 0 & 0 & r_{N-3} & S_{N-3} & 0 \\
    0 & 0 & 0 & 0 & 0 & r_{N-2} & S_{N-2} \\
    0 & 0 & 0 & 0 & 0 & 0 & r_{N-1} \\
\end{bmatrix}
\]

4
Based on Evans [17], the conditions matrices $G_1$ and $G_2$ need to satisfy when $dI + G_1$ and $dI + G_2$ are non-singular for $d > 0$ where $d$ is called the acceleration parameter. Besides that, any vectors $c$ and $e$ and for any $d > 0$ is practical to solve the linear systems explicitly

$$u^{(k+1/2)} = G_i^{-1} c$$

$$u^{(k+1)} = G_2^{-1} e$$

To get the approximate value of $u^{(k+1)}$ and $u^{(k+1/2)}$ respectively, the inverse of matrices $G_i$ and $G_2$ need to be determined. In order to construct the submatrices $G_i$ and $G_2$, concerned the situation are either small block systems (2x2) or can construct based on the suitable permutation row and corresponding column. This process is more helpful because the significantly less than it is required to solve the original linear system (13) directly. Then by substituting equation (14) into tridiagonal linear system (13), we rewrite the linear system as

$$\left( G_1 + G_2 \right) u = f$$

By using the same concept over the formulation of ADI method [17], $u^{(k+1)}$ and $u^{(k+1/2)}$ will get the implicitly and involve two steps to get numerical results as follows

$$\left( dI + G_i \right) u^{(k+1)} = f + \left( dI - G_i \right) u^{(k)}$$

$$\left( dI + G_2 \right) u^{(k+1/2)} = f + \left( dI - G_1 \right) u^{(k+1/2)}$$

As mentioned the Evans [17], $\left( dI + G_i \right)$ and $\left( dI + G_2 \right)$ are non-singular matrices. Therefore, if the respective inverses exist the formulation of the AGE method in form explicitly can be stated as [25]

$$u^{(k+1/2)} = (dI + G_i)^{-1} \left[ f + (dI - G_i) u^{(k)} \right]$$

$$u^{(k+1)} = (dI + G_2)^{-1} \left[ f + (dI - G_1) u^{(k+1/2)} \right]$$

where $d$ is the parameter for the iteration process [17]

$$d = (uv)^{1/2}$$

The minimum and maximum eigenvalue of the submatrices of $G_i$ and $G_2$ represent $u$ and $v$ . Besides that, the second smallest eigenvalue be considered when the submatrices are singular and the smallest eigenvalue is zero. Based on equation (18), then implementation of this method is present in Algorithm 1.

**Algorithm 1: AGE method**

i. Initialize $u^{(0)} = 0, \ v = 10^{-10}$.

ii. Calculate matrix $A$ and $f$.

iii. For first stage, calculate the approximate value of $u^{(k+1/2)}$,

$$u^{(k+1/2)} = (dI + G_i)^{-1} \left[ f + (dI - G_i) u^{(k)} \right]$$

iv. For second stage, calculate the approximate value of $u^{(k+1)}$,

$$u^{(k+1)} = (dI + G_2)^{-1} \left[ f + (dI - G_1) u^{(k+1/2)} \right]$$
v. Convergence test $\left| u^{(k+1)}_i - u^{(k)}_i \right| < \varepsilon = 10^{-10}$ If the convergence criterion is satisfied, go to step vi and otherwise go back to step iii.

vi. Display approximate solution.

5. Numerical Experiments

This section proposed four examples fractional differential equations where each example has three different values $\beta$ to demonstrate the efficiency of the proposed iterative methods such as GS and AGE. Beyond that point, three measurement parameters would be considered such as number of iterations, computational time in second (Time) and maximum absolute error. The convergence test considered the tolerance error is fixed as $\varepsilon = 10^{-10}$ for the performance of proposed iterative methods.

Example 1 [26]

In this example, we consider the following fractional two-point boundary value problem by taking the value $\beta = 1.25, 1.50, 1.75$ since the range $\beta$ is $1 < \beta \leq 2$.

$$D^\beta u(x) + u'(x) + u(x) = f(x), \quad 0 \leq x \leq 1$$

(20)

where the function

$$f(x) = x^2 + x + \frac{4}{\sqrt{\pi}} \sqrt{x} + 3$$

(21)

and the boundary condition given as

$$u(0) = 1, \quad u(1) = 3$$

The exact solution problem (20) is obtain as follows

$$u(x) = x^2 + x + 1$$

(22)

Example 2 [27]

Consider the following fractional two-point boundary value problem by taking the value $\beta = 1.25, 1.50, 1.75$ since the range $\beta$ is $1 < \beta \leq 2$.

$$u'(x) + D^\beta u(x) + u(x) = f(x), \quad 0 \leq x \leq 1$$

(23)

Where the function $f(x)$ is given by

$$f(x) = \frac{15}{4} x^{0.5} + \frac{15}{8} \sqrt{\pi} x + x^{2.5} + 1$$

(24)

and the boundary condition being given as

$$u(0) = 1, \quad u(1) = 2$$

The exact solution of problem (23) is obtain as follows

$$u(x) = x^{2.5} + 1$$

(25)

Example 3 [22]

In this example, we consider the following fractional two-point boundary value problem by taking the value $\beta = 0.25, 0.50, 0.75$ since the range $\beta$ is $0 < \beta \leq 1$.

$$u'(x) + x^2 u'(x) + D^\beta u(x) + u(x) = f(x)$$

(26)

Where the function $f(x)$ is given by

$$f(x) = 5x^6 - 3x^5 - x^4 + 20x^3 - 12x^2 - \frac{120}{\Gamma(5.3)} x^{4.3} + \frac{24}{\Gamma(4.3)} x^3$$

(27)

With the boundary condition given as
\[ u(0) = 0, \quad u(l) = 0 \]

The exact solution of problem (26) is obtained as follows
\[ u(x) = x^4(x - 1) \quad (28) \]

The numerical results were conducted in the C programming language. These results which obtained from the implementation of GS and AGE iterative methods have been recorded in Tables 1, 2, and 3 at different values of mesh sizes, \( m = 128, 256, 512, 1024, \) and 2048.

**Table 1.** Comparison of number iterations (K), execution time (seconds) and maximum errors for proposed Iterative methods using Examples 1 at \( \beta = 1.25, 1.50, 1.75. \)

| M  | METHOD | \( \beta = 1.25 \) | \( \beta = 1.50 \) | \( \beta = 1.75 \) |
|----|--------|-------------------|-------------------|-------------------|
|    | K      | Time             | Max Error        | K                | Time             | Max Error        | K                | Time             | Max Error        |
| 128| GS     | 18333            | 1.21230e-02      | 18198            | 1.03             | 1.3813e-05       | 20575            | 1.15             | 1.7118e-02       |
|    | AGE    | 1932             | 0.25             | 2.1230e-02       | 2048             | 0.25             | 1.3905e-05       | 2405             | 0.31             | 1.7118e-02       |
| 256| GS     | 67072            | 14.06            | 65479            | 22.28            | 1.3524e-05       | 72891            | 15.28            | 1.7081e-02       |
|    | AGE    | 8399             | 7.88             | 2.1228e-02       | 8125             | 7.63             | 1.3882e-05       | 9652             | 9.05             | 1.7082e-02       |
| 512| GS     | 243716           | 200.65           | 234899           | 283.34           | 1.2339e-05       | 257190           | 212.15           | 1.7061e-02       |
|    | AGE    | 30670            | 59.62            | 2.1229e-02       | 29131            | 56.88            | 1.3741e-05       | 34055            | 66.66            | 1.7063e-02       |
| 1024| GS    | 878044           | 3486.79          | 838194           | 3222.64          | 7.7771e-06       | 902430           | 2973.2           | 1.7047e-02       |
|    | AGE    | 103553           | 800.93           | 3.3317e-02       | 108593           | 839.86           | 1.4735e-05       | 127796           | 991.07           | 1.7053e-02       |
| 2048| GS    | 3128970          | 60236.70         | 2966072          | 40802.48         | 1.2307e-05       | 3141444          | 46972.44         | 1.7023e-02       |
|    | AGE    | 454434           | 13904.35         | 2.1234e-02       | 427897           | 13540.33         | 1.0830e-05       | 476480           | 14520.46         | 1.7046e-02       |

**Table 2.** Comparison of number iterations (K), execution time (seconds) and maximum errors for proposed Iterative methods using Examples 2 at \( \beta = 1.25, 1.50, 1.75. \)

| M  | METHOD | \( \beta = 1.25 \) | \( \beta = 1.50 \) | \( \beta = 1.75 \) |
|----|--------|-------------------|-------------------|-------------------|
|    | K      | Time             | Max Error        | K                | Time             | Max Error        | K                | Time             | Max Error        |
| 128| GS     | 17860            | 0.99             | 3.2820e-02       | 17756            | 0.98             | 8.8686e-04       | 2859540          | 1.74             | 3.2009e-02       |
|    | AGE    | 2006             | 0.27             | 3.2820e-02       | 2111             | 0.26             | 8.8698e-04       | 2426             | 0.33             | 3.2009e-02       |
| 256| GS     | 65189            | 13.68            | 3.3318e-02       | 63748            | 13.58            | 4.5977e-04       | 71089            | 23.2             | 3.1690e-02       |
|    | AGE    | 7912             | 3.81             | 3.3317e-02       | 8108             | 3.86             | 4.6014e-04       | 8831             | 4.18             | 3.1690e-02       |
| 512| GS     | 236202           | 194.51           | 3.3572e-02       | 228091           | 210.57           | 2.3923e-04       | 250227           | 86075.32         | 3.1519e-02       |
|    | AGE    | 30827            | 60.33            | 3.3571e-02       | 30203            | 59.11            | 2.4108e-04       | 31879            | 62.26            | 3.1521e-02       |
| 1024| GS    | 848050           | 2905.1           | 3.3705e-02       | 228091           | 3402.56          | 1.2280e-04       | 875406           | 4663.88          | 3.1426e-02       |
|    | AGE    | 118830           | 920.36           | 3.3698e-02       | 115961           | 890.04           | 1.2845e-04       | 118842           | 910.15           | 3.1432e-02       |
| 2048| GS    | 3009165          | 50524.12         | 3.3790e-02       | 2859540          | 12047.17         | 4.7423e-05       | 3036143          | 44792.83         | 3.1360e-02       |
|    | AGE    | 437048           | 13347.42         | 3.3767e-02       | 407471           | 11465.57         | 6.9608e-05       | 457373           | 13935.39         | 3.1384e-02       |

**Table 3.** Comparison of number iterations (K), execution time (seconds) and maximum errors for proposed Iterative methods using Examples 3 at \( \beta = 0.20, 0.50, 0.70. \)
| M     | METHOD | $\beta = 0.20$     | $\beta = 0.50$     | $\beta = 0.70.$ |
|-------|--------|-------------------|-------------------|----------------|
|       |        | K | Time | Max Error | K | Time | Max Error | K | Time | Max Error |
| 128   | GS     | 25261 | 2.14 | 5.2342e-03 | 27069 | 2.29 | 2.3890e-03 | 28849 | 2.57 | 8.8880e-05 |
|       | AGE    | 3532 | 0.61 | 5.2343e-03 | 3809 | 0.44 | 2.3891e-03 | 4077 | 0.50 | 8.8686e-05 |
| 256   | GS     | 89337 | 31.75 | 5.2798e-03 | 95456 | 31.13 | 2.4331e-03 | 101506 | 38.62 | 4.6529e-05 |
|       | AGE    | 12897 | 8.54 | 5.2806e-03 | 13849 | 6.37 | 2.4338e-03 | 14784 | 7.10 | 4.5747e-05 |
| 512   | GS     | 311465 | 506.93 | 5.2953e-03 | 331812 | 424.43 | 2.4476e-03 | 351938 | 586.15 | 3.2606e-05 |
|       | AGE    | 46526 | 82.38 | 5.2982e-03 | 49793 | 89.51 | 2.4507e-03 | 53022 | 102.67 | 2.9380e-05 |
| 1024  | GS     | 1064122 | 6333.14 | 5.2933e-03 | 1129596 | 5779.36 | 2.4444e-03 | 1194230 | 6247.41 | 3.7806e-05 |
|       | AGE    | 165083 | 2940.19 | 5.3047e-03 | 176140 | 1849.31 | 2.4568e-03 | 187101 | 1433.91 | 2.4466e-05 |
| 2048  | GS     | 3532806 | 67184.02 | 5.2578e-03 | 3732199 | 59956.29 | 2.4055e-03 | 3928076 | 72175.28 | 8.1794e-05 |
|       | AGE    | 581845 | 17352.25 | 5.3032e-03 | 24010 | 16160.24 | 2.4548e-03 | 645838 | 17246.37 | 2.8647e-05 |

Tables 1, 2, and 3 show that the numerical results for the three examples of the proposed problems were obtained from the proposed iterative methods which have been tested at different grid sizes, m=128, 256, 512, 1024, and 2048. Based on the numerical results obtained from GS and AGE iterative methods which have been demonstrated in Tables 1, 2, and 3, it can be pointed out that the number of iterations and execution time of the AGE method are smaller than GS method.

6. Conclusion
In this paper, the fractional two-point boundary value problems have been discretized by using discretization schemes and the Caputo’s derivative operation for the fractional operator, in which this matter can derive the Caputo’s finite approximation equation. In fact, this approximation equation leads by linear system. By doing the performance analysis of the GS and AGE iterative methods, it can be figured out that AGE method is superior to solve these problems than GS method. The number of iterations and the execution time of the AGE method is smaller than GS method. Apart from that, the observation on the accuracy of proposed iterative methods shows that their numerical solutions are satisfied in a good agreement.

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