A distance measure for non-trace-preserving quantum operations

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(Dated: January 25, 2022)

We propose a distance measure for non-trace-preserving quantum operations, such as the measurement and post-selection procedure. The distance measures the maximal trace distance between the normalized output states from two non-trace-preserving quantum operations given the same input states, which is a generalization of the diamond distance. We also present an efficient algorithm to bound the distance by renormalization. In certain instances, the bound coincides with the exact distance. As an example of its applications, we analyze the lossy beam splitter and the nondeterministic conditional sign flip gate, which are two primary non-trace-preserving operations in the Knill-Laflamme-Milburn (KLM) protocol. The proposed distance measure broadens the class of physical imperfections we can analyze in the quantum circuit model to non-trace-preserving operations, such as qubit loss, measurement error, and imperfect feed-forward control. As a result, it has a wide range of applications in analyzing real systems in quantum information processing.

Quantifying the errors of an imperfect quantum operation from the ideal one is important in quantum information processing. Applications, such as quantum certification [1] and determining thresholds for fault-tolerant quantum computing [2, 3], depend on properly measuring the distance between two quantum operations. Many measures have been proposed to quantify the distance for completely positive trace-preserving maps [4]. However, these approaches cannot be used to analyze non-trace-preserving operations, which play an important role in a broad class of quantum information applications. For example, the measurement and post-selection procedure and errors due to qubit loss are all non-trace-preserving quantum processes.

Researchers have developed an ansatz to quantify the error for non-trace-preserving operations by calculating the fidelity of their normalized process matrices [5, 6] using channel-state duality [4]. But the normalized process matrix used in this approach is nonphysical as it can have negative eigenvalues, while a valid process matrix must have nonnegative eigenvalues. Additionally, this ansatz can only estimate the average-case error of a quantum operation. In fault-tolerant quantum computing, we need the worst-case error to calculate the threshold error for fault tolerance [7, 8]. Therefore, we still lack a proper error measure for non-trace-preserving quantum processes [4, 9].

We propose a general distance measure for non-trace-preserving quantum operations, which measures the maximal trace distance between the output states from two non-trace-preserving operations given the same input states. Additionally, we present an efficient algorithm to bound the distance by combining a diamond distance and a normalizing distance. As a demonstration of its application, we analyze the lossy beam splitter and the nondeterministic conditional sign flip gate, which are two primary non-trace-preserving operations in the Knill-Laflamme-Milburn (KLM) protocol [10]. The proposed distance broadens the class of physical imperfections we can analyze in the quantum circuit model. Therefore, it has a wide range of applications in analyzing real systems in quantum information processing.

We first review the required properties of a distance measure for quantum operations. A quantum operation $E$ is a map between the input and output states, described by $\rho_{\text{out}} = E(\rho_{\text{in}})$. The distance between two quantum operations measures the difference of their output states given the same input states. Any candidate measure of distance, $\Delta(E, F)$ between quantum operations $E$ and $F$, should possess the following mathematical properties [4]:

1. Metric. To be a metric, the distance must satisfy three requirements: (a) $\Delta(E, F) \geq 0$, with $\Delta(E, F) = 0$ if and only if $E = F$; (b) symmetry, $\Delta(E, F) = \Delta(F, E)$; and (c) triangle inequality, $\Delta(E, F) \leq \Delta(E, G) + \Delta(G, F)$.

2. Chaining. $\Delta(E_2 \circ E_1, F_2 \circ F_1) \leq \Delta(E_1, F_1) + \Delta(E_2, F_2)$, where $E_2 \circ E_1$ denotes a composite process in which operation $E_1$ is followed by operation $E_2$ (similar for $F$). This property ensures the total distance of the composite operations should be less than the sum of the distance in their steps.

3. Stability. $\Delta(E \otimes I, F \otimes I) = \Delta(E, F)$, where $I$ represents the identity operation in an additional state space. This property ensures that the distance is unchanged when the operations act on a subsystem, which can be entangled with other systems.

Practically, we are typically interested in the distance between an experimentally implemented quantum operation and the ideal one, for which the distance measure
is also an error measure. The chaining and stability are useful for analyzing large and complex quantum information processing tasks composed of sequential applications of quantum operations. These properties allow us to bound the error of the whole process using the sum of the distances of the sequential operations.

One example of a distance measure for completely positive trace-preserving maps is the diamond distance [11], which measures the maximum trace distance of the output states from two quantum channels given the same input states. The diamond distance is given by

\[ d_d (\mathcal{E}, \mathcal{F}) = \frac{1}{2} \max_{\rho} \left\| (\mathcal{E} \otimes \mathcal{I}_R) (\rho) - (\mathcal{F} \otimes \mathcal{I}_R) (\rho) \right\|_1 , \]  

where \( \| \cdot \|_1 \) denotes the trace norm, \( \mathcal{E} \) and \( \mathcal{F} \) are two quantum channels mapping quantum states from input space \( X \) to output space \( Y \). \( \mathcal{I}_R \) is the identity operation on an auxiliary space \( R \), and \( \rho \) is a quantum state on the extended space \( X \otimes R \). The diamond distance satisfies the three properties required by a distance measure for quantum operations. The metric property is a direct property of the trace norm. The chaining property can be proven using the triangle inequality of the trace norm. And the stability property is satisfied as long as \( \text{dim} (X) \leq \text{dim} (R) \) [12]. Currently, there is no explicit formula to calculate the diamond distance for general quantum channels, but we can compute it using convex optimization [13].

Our proposed distance generalizes the diamond distance for non-trace-preserving quantum operations. We define the general distance \( d_g (\mathcal{E}, \mathcal{F}) \), as

\[ d_g (\mathcal{E}, \mathcal{F}) = \frac{1}{2} \max_{\rho} \left\| \frac{(\mathcal{E} \otimes \mathcal{I}_R) (\rho)}{\text{Tr} [(\mathcal{E} \otimes \mathcal{I}_R) (\rho)]} - \frac{(\mathcal{F} \otimes \mathcal{I}_R) (\rho)}{\text{Tr} [(\mathcal{F} \otimes \mathcal{I}_R) (\rho)]]_1 . \]  

In the above equation, \( \mathcal{E} \) and \( \mathcal{F} \) are non-trace-preserving operations represented by \( \mathcal{E} (\rho) = \sum_i a_i \rho a_i^\dagger \) and \( \mathcal{F} (\rho) = \sum_j b_j \rho b_j^\dagger \), where \( a_i \) and \( b_j \) are Kraus operators satisfying \( \sum_i a_i a_i^\dagger \leq I \) and \( \sum_j b_j b_j^\dagger \leq I \). The general distance maximizes the trace distance of the normalized output states from two non-trace-preserving quantum operations given the same input states. Thus, the metric and chaining criteria are satisfied directly by the properties of the trace norm. In the Supplemental Material [14], we show that the general distance satisfies the stability if \( \text{dim} (X) \leq \text{dim} (R) \), and is maximized over a pure state on the extended space \( X \otimes R \). However, unlike the diamond distance, it cannot be computed by convex optimization as far as we know, because the objective function is neither convex nor concave generally.

We can bound the general distance by renormalization. First, we represent the two quantum operations in Stinespring form as \( \mathcal{E} = \text{Tr}_Z \left\{ A \rho A^\dagger \right\} \) and \( \mathcal{F} = \text{Tr}_Z \left\{ B \rho B^\dagger \right\} \), where \( A = \sum_j a_j \otimes a_i \), \( B = \sum_j b_j \otimes b_i \), and \( e_i \) and \( e_j \) represent the basis on a reference space \( Z \) [12]. The general distance between \( \mathcal{E} \) and \( \mathcal{F} \) is given by

\[ d_g (\mathcal{E}, \mathcal{F}) = \frac{1}{2} \max_{\psi} \left\| \text{Tr}_Z \left\{ \frac{A \psi \psi^\dagger A^\dagger}{(\psi | A \psi)} - \frac{B \psi \psi^\dagger B^\dagger}{(\psi | B \psi)} \right\} \right\|_1 . \]  

where \( \psi \) is a pure state on the extended input space \( X \otimes R \), and we abbreviate \( A \otimes \mathcal{I}_R \) as \( A \) and \( B \otimes \mathcal{I}_R \) as \( B \). In the Supplemental Material [14], we show a renormalization method to bound Equation 3. The essential step of the renormalization is to apply singular value decomposition to \( BA^{-1} \), resulting in \( BA^{-1} = UMV^{-1} \). Here, \( U \) and \( V \) are isometries with dimensions of \( k' m \times n \) and \( km \times n \), and

\( M \) is a nonnegative diagonal operator with a dimension of \( n \times n \), where \( n \) and \( m \) are the dimensions of the input and output spaces of the quantum operations, and \( k \) and \( k' \) are the number of Kraus operators \( a_i \) and \( b_j \). We define two renormalized quantum channels

\[ \mathcal{U} (\rho) = \text{Tr}_Z \left\{ U \rho U^\dagger \right\} , \]  

\[ \mathcal{V} (\rho) = \text{Tr}_Z \left\{ V \rho V^\dagger \right\} , \]  

and a normalizing operation

\[ \mathcal{M} (\rho) = \frac{M \rho M}{\text{Tr} [M \rho M]} . \]  

The general distance between \( \mathcal{E} \) and \( \mathcal{F} \) is bounded by

\[ d_g (\mathcal{E}, \mathcal{F}) \leq d_o (\mathcal{U}, \mathcal{V}) + d_g (\mathcal{I}, \mathcal{M}) , \]  

where \( d_o (\mathcal{U}, \mathcal{V}) \) is a diamond distance, which can be computed by convex optimization; \( d_g (\mathcal{I}, \mathcal{M}) \) is a general distance between an identity operation and a normalizing operation, which we refer to as the normalizing distance. We can calculate \( d_g (\mathcal{I}, \mathcal{M}) \) by

\[ d_g (\mathcal{I}, \mathcal{M}) = \lambda_{\text{max}} - \lambda_{\text{min}} \]  

where \( \lambda_{\text{max}} \) and \( \lambda_{\text{min}} \) are the maximum and minimum eigenvalues of \( M \) (see the Supplemental Material [14]). Combining \( d_o (\mathcal{U}, \mathcal{V}) \) and \( d_g (\mathcal{I}, \mathcal{M}) \), we can compute an upper bound of the distance between two non-trace-preserving quantum operations.

To demonstrate the application of the general distance, we analyze the two primary fundamental building blocks
of the KLM protocol [10] for linear optical quantum computing: the single-qubit rotation and the nondeterministic conditional sign flip (NS) gate. We analyze these operations in the presence of photon loss and detector dark counts, the two primary non-trace-preserving mechanisms. These results can combine with trace-preserving mechanisms, such as phase instability, to provide a complete error analysis.

We first consider the single-qubit rotation, which is implemented using a beam splitter. However, faulty beam splitters with state-dependent photon loss not only decrease the success probability of the KLM protocol, but also implement erroneous rotations to the processed quantum state. The operator associated with a beam splitter \( B_{\theta,\phi} \) is given by

\[
U (B_{\theta,\phi}) = \begin{pmatrix}
\gamma_{r} \cos \theta & -\gamma_{r} \sin \theta e^{i\phi} \\
\gamma_{r} \sin \theta e^{-i\phi} & \gamma_{t} \cos \theta
\end{pmatrix},
\]

(8)

where \( \gamma_{r} \) and \( \gamma_{t} \) are the photon loss rates for reflection and transmission, respectively. Given \( \gamma_{r}, \gamma_{t} = 1 \), the operator corresponds to an ideal beam splitter. We denote the operator of an ideal beam splitter as \( U_{id} \) and the operator of a faulty beam splitter as \( U_{r} \). The distance between the ideal and faulty beam splitters follows by post-selecting the non-loss instances is given by

\[
d_{g}(U_{id}, U_{r}) = \frac{1}{2} \max_{\rho} \left\| \rho \left( U_{id} \otimes I \right) - \left\{ \frac{1}{2} \left( U_{id} \otimes I \right) - \frac{1}{2} \right\} \rho \left( U_{id} \otimes I \right) \right\|_1.
\]

Using Equation 6, we compute this distance to be

\[
d_{g}(U_{id}, U_{r}) = \frac{\left| \gamma_{r} - \gamma_{t} \right| \cos \theta \sin \theta}{\sqrt{\gamma_{r}^2 \cos^2 \theta + \gamma_{t}^2 \sin^2 \theta}}.
\]

(9)

The above solution is exact, not an upper bound. It is because the normalizing operator \( M = \sqrt{\gamma_{r}^2 + \gamma_{t}^2} / 2 \) (see Equation 5), and the normalizing distance is zero.

The red curve in Figure 1 shows the general distance calculated by Equation 9 between the ideal and faulty beam splitters, where we set \( \theta = \pi/4 \). This condition corresponds to a 50/50 beam splitter in the ideal case. We plot the general distance as a function of \( \Gamma = \gamma_{r}/\gamma_{t} \), which is the ratio of the photon loss between reflection and transmission. The distance achieves an ideal value of zero when the loss is balanced \( (\gamma_{r} = \gamma_{t}) \), as expected. To show that the calculated distance gives a tight upper bound, we compare it to a Monte Carlo simulation, where we apply the operator \( U (B_{\pi/4,0}) \) to randomly generated input states and calculate the trace distance between the normalized output states from the ideal and faulty beam splitters. The blue violin plots in Figure 1 show the statistics of the trace distance between the output states from the Monte Carlo simulation, where the vertical segments of the violin plots represent the range of the trace distances, and their width indicates the probability distribution. Most of the simulated trace distances locate near their maximum. The theoretical calculation is tight, in which the maximal trace distance from the Monte Carlo simulation corresponds to the same value calculated by Equation 9.

We next analyze the NS gate, which applies a nonlinear phase shift on one photonic mode using two ancilla photonic modes. The initial state of the system is prepared in \( |\psi_{in}\rangle \otimes |10\rangle_{A} \), where \( |\psi_{in}\rangle = (\alpha_{0}|0\rangle + \alpha_{1}|1\rangle + \alpha_{2}|2\rangle \) is an input state represented in the photon number bases, and \( |10\rangle_{A} \) is the state of the two ancilla modes. Upon linear optical transformations and detection of the ancilla modes in state \( |10\rangle_{A} \), the NS gate produces the output state of \( |\psi_{out}\rangle = (\alpha_{0}|0\rangle + \alpha_{1}|1\rangle - \alpha_{2}|2\rangle \). However, detector dark counts can cause erroneous detection events in the first ancilla mode, which create errors in the gate operation. To measure the error, we denote the state space for \( |\psi_{in}\rangle \) as \( X = \{|0\rangle, |1\rangle, |2\rangle\} \), while the state space for \( |\psi_{out}\rangle \) is \( Y = \{|0\rangle, |1\rangle, |2\rangle, |3\rangle\} \). We need to include one additional photon in the output space because it is possible that the ancilla photon may also exit into the output port, adding an additional photon to the system. In an ideal case, these events would not trigger a detection event in the first ancilla mode, but dark counts can cause these events to be erroneously included. The ideal NS gate transforms the input state as \( NS_{id}(\rho) = \frac{E_{10} \rho E_{10}^{\dagger}}{Tr[E_{10} \rho E_{10}^{\dagger}]} \), and the faulty NS gate transforms the state as \( NS_{f}(\rho) = \frac{E_{10} \rho E_{10}^{\dagger} + \mu E_{00} \rho E_{10}^{\dagger}}{Tr[E_{10} \rho E_{10}^{\dagger} + \mu E_{00} \rho E_{10}^{\dagger}]} \), where \( \mu \) is the dark count rate. In these equations, \( E_{10} \) and \( E_{00} \) are the operators conditioned on the measurement of the ancilla modes with outcomes \( |10\rangle_{A} \) and \( |00\rangle_{A} \),

![FIG. 1. The distance as a function of the ratio of the photon loss between reflection and transmission \( \Gamma = \gamma_{r}/\gamma_{t} \) for the ideal and faulty beam splitters. The red curve shows the theoretical calculation. The blue violin plots show the statistics of the trace distance between the normalized output states from the Monte Carlo simulation.](image-url)
and are given by \[10\]
\[
E_{10} = \begin{pmatrix}
\frac{1}{2} & 0 & 0 \\
0 & \frac{1}{2} & 0 \\
0 & 0 & -\frac{1}{2}
\end{pmatrix},
\]
\[
E_{00} = \begin{pmatrix}
0 & 0 & 0 \\
\frac{1}{2\sqrt{2}} & 0 & 0 \\
0 & -\frac{2+\sqrt{2}}{2\sqrt{2}} & \sqrt{6} \left(\frac{1}{2\sqrt{2}} - 2^{3/4}\right)
\end{pmatrix}.
\]

Within these parameters, we can calculate the general distance between an ideal NS gate and a faulty NS gate with dark counts by \(d_{\text{NS}}(\rho_{\text{id}}, \rho_{\text{NS}}) = \frac{1}{2} \max_{\rho} ||(N\rho_{\text{id}} \otimes I)(\rho) - (N\rho_{\text{NS}} \otimes I)(\rho)||_{1}\).

Figure 2 shows the distance as a function of the dark count rate. The red curve shows the theoretical bound for the quantum operation distance between the ideal and faulty NS gates calculated by Equation 6. The blue violin plots show the statistics of the trace distance between the output states from a Monte Carlo simulation, where we apply the ideal and faulty NS gate to randomly generated input states and calculate the trace distance between the normalized output states. We find that most probability locates around the median of the distribution. The blue curve represents the maximal trace distance from the Monte Carlo simulation. In this case, Equation 6 still represents an upper bound of the distance. But the bound is no longer tight, as can be seen from the gap between the blue and red curves.

Even though the upper bound is not tight generally, as shown in Figure 2, we still consider it a good error measure for non-trace-preserving quantum operations for the following reasons: (a) It gives an upper bound of the worst-case error for the quantum operations, which is useful in calculating the threshold in fault-tolerant quantum computations and analyzing the performance of quantum systems; (b) The bound and exact distances are within the same order of magnitude, and the gap goes to zero when the physical imperfection vanishes; (c) The gap arises from the triangle inequality of combining two operation distances, i.e., the diamond distance and the normalizing distance, which complies with the concept of the chaining property: to analyze a complex quantum system, we can bound the total distance by the sum of its steps.

In the above analysis we considered two important examples of non-trace preserving errors. However, our formalism is extremely general and can be applied to a much broader class of problems. For example, a broad range of quantum information processing tasks use measurement and feed-forward control, including quantum teleportation, quantum distillation [15, 16], and cluster state computation [17]. In these schemes, the measurement error can introduce faulty transformations, which are non-trace preserving. Such errors have been traditionally difficult to analyze but can be easily studied using the proposed distance in this work. Another example of non-trace preserving errors are leakage [18, 19] and loss [20], which arise when the quantum state leaks out of the encoding subspace. These types of errors are particularly important in superconducting and optical quantum computing.

In conclusion, we propose a distance measure for non-trace-preserving operations, which generalizes the diamond distance. We can bound the distance by the sum of a diamond distance and a normalizing distance. The proposed distance provides a computationally tractable approach for performing error analysis on non-trace-preserving quantum operations, which broadens the types of error we can analyze in the quantum circuit model. As a result, it may have broad application for analyzing real systems in quantum information processing.

The authors would like to acknowledge financial support from the National Science Foundation (grant number OMA1936314 and ECCS1933546), the Air Force Office of Scientific Research (grant number UWSC12985 and FA23862014072), and the Army Research Laboratory (grant W911NF1920181).

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A distance measure for non-trace-preserving quantum operations: Supplemental Materials

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S1. STABILITY OF THE GENERAL DISTANCE

In the main text, we define the general distance between two non-trace-preserving quantum operations as

\[ d_g (\mathcal{E}, \mathcal{F}) = \frac{1}{2} \max_{\rho \in X \otimes R} \left\| \frac{1}{\text{Tr} \left( \mathcal{I} \otimes \mathcal{I}_R \right)(\rho)} \left[ \text{Tr} \left( (\mathcal{E} \otimes \mathcal{I}_R)(\rho) \right) - \text{Tr} \left( (\mathcal{F} \otimes \mathcal{I}_R)(\rho) \right) \right] \right\|_1, \tag{S1} \]

where \( \mathcal{E} \) and \( \mathcal{F} \) are non-trace-preserving operations mapping states from the input space \( X \) to the output space \( Y \), \( \mathcal{I}_R \) is the identity operation on the auxiliary space \( R \), and \( \rho \) is a quantum state on the extended space \( X \otimes R \). Here, we show that the distance satisfies the stability, as \( d_g (\mathcal{E} \otimes \mathcal{I}, \mathcal{F} \otimes \mathcal{I}) = d_g (\mathcal{E}, \mathcal{F}) \), if \( \text{dim} (X) \leq \text{dim} (R) \), where \( \mathcal{I} \) represents the identity operation in an additional state space. Moreover, the distance is optimized over a pure state on the space \( X \otimes R \). The proof is a generalization of Lemma 3.45 in [S1].

We first purify quantum state \( \rho \) by introducing another auxiliary space \( R' \), such that \( \rho = \text{Tr}_{R'} \left[ uu^\dagger \right] \), where \( u \) is a pure state on space \( X \otimes R \otimes R' \). The general distance between two non-trace-preserving quantum operations \( \mathcal{E} \) and \( \mathcal{F} \) is given by

\[ d_g (\mathcal{E}, \mathcal{F}) = \frac{1}{2} \max_{u \in X \otimes R \otimes R'} \left\| \text{Tr}_{R'} \left\{ \frac{\sum_i a_i uu^\dagger a_i^\dagger}{\text{Tr} \left[ \sum_i a_i uu^\dagger a_i^\dagger \right]} - \frac{\sum_j b_j uu^\dagger b_j^\dagger}{\text{Tr} \left[ \sum_j b_j uu^\dagger b_j^\dagger \right]} \right\} \right\|_1, \tag{S2} \]

In the above equation, we write the operations in operator-sum representations, as \( \mathcal{E} (\rho) = \sum_i a_i \rho a_i^\dagger \) and \( \mathcal{F} (\rho) = \sum_j b_j \rho b_j^\dagger \), where \( a_i \) and \( b_j \) are Kraus operators satisfying \( \sum_i a_i^\dagger a_i \leq I \) and \( \sum_j b_j^\dagger b_j \leq I \), and we abbreviate \( a_i \otimes I \) as \( a_i \) and \( b_j \otimes I \) as \( b_j \). Using the contractility of trace distance \([S2]\), as

\[ \frac{1}{2} \left\| \text{Tr}_{R'} (\sigma_1 - \sigma_2) \right\|_1 \leq \frac{1}{2} \left\| \sigma_1 - \sigma_2 \right\|_1, \tag{S3} \]

where \( \sigma_1 \) and \( \sigma_2 \) are two quantum states, we can bound Equation S2 by

\[ d_g (\mathcal{E}, \mathcal{F}) \leq \frac{1}{2} \max_{u \in X \otimes R \otimes R'} \left\| \frac{\sum_i a_i uu^\dagger a_i^\dagger}{\text{Tr} \left[ \sum_i a_i uu^\dagger a_i^\dagger \right]} - \frac{\sum_j b_j uu^\dagger b_j^\dagger}{\text{Tr} \left[ \sum_j b_j uu^\dagger b_j^\dagger \right]} \right\|_1. \tag{S4} \]

We then prove that the bound in Equation S4 is tight. Applying Schmidt decomposition on state \( u \), we get \( u = \sum_{i=1}^n \sqrt{p_i} x_i \otimes z_i \), where \( p_i \) is the Schmidt coefficient, \( x_i \) and \( z_i \) are bases on spaces \( X \) and \( R \otimes R' \), respectively, and \( n \) is the dimension of space \( X \). If \( \text{dim} (X) \leq \text{dim} (R) \), we can choose a pure state \( v \in X \otimes R \), such that \( v = \sum_{i=1}^n \sqrt{p_i} x_i \otimes e_i \), where \( e_i \) represents a basis in space \( R \). We can construct an isometry \( U (R, R \otimes R') \), as \( U = \sum_{i=1}^n z_i e_i^\dagger \). It holds that \( u = (I_X \otimes U) v \), and

\[ \frac{\sum_i a_i uu^\dagger a_i^\dagger}{\text{Tr} \left[ \sum_i a_i uu^\dagger a_i^\dagger \right]} = \frac{\sum_i (a_i \otimes U) vv^\dagger (a_i^\dagger \otimes U^\dagger)}{\text{Tr} \left[ \sum_i (a_i \otimes U) vv^\dagger (a_i^\dagger \otimes U^\dagger) \right]} = \frac{\sum_i (I \otimes U) a_i vv^\dagger a_i^\dagger (I \otimes U^\dagger)}{\text{Tr} \left[ \sum_i (I \otimes U) a_i vv^\dagger a_i^\dagger (I \otimes U^\dagger) \right]} = (I \otimes U) \frac{\sum_i a_i vv^\dagger a_i^\dagger}{\text{Tr} \left[ \sum_i a_i vv^\dagger a_i^\dagger \right]} \left( I \otimes U^\dagger \right), \tag{S5} \]
and similar for quantum operation $\mathcal{F}$, as
\[
\frac{\sum_j b_j u u^\dagger b_j^\dagger}{Tr \left[ \sum_j b_j u u^\dagger b_j^\dagger \right]} = (I \otimes U) \frac{\sum_j b_j v v^\dagger b_j^\dagger}{Tr \left[ \sum_j b_j v v^\dagger b_j^\dagger \right]} (I \otimes U^\dagger). \tag{S6}
\]

The trace distance is invariant under isometry, so from Equation S4, we derive
\[
d_g(\mathcal{E}, \mathcal{F}) \leq \frac{1}{2} \max_{v \in X \otimes R} \left\| \frac{\sum_i a_i u u^\dagger a_i^\dagger}{Tr \left[ \sum_i a_i u u^\dagger a_i^\dagger \right]} - \frac{\sum_j b_j v v^\dagger b_j^\dagger}{Tr \left[ \sum_j b_j v v^\dagger b_j^\dagger \right]} \right\|_1
= \frac{1}{2} \max_{v \in X \otimes R} \left\| \left( I \otimes U \right) \frac{\sum_i a_i v v^\dagger a_i^\dagger}{Tr \left[ \sum_i a_i v v^\dagger a_i^\dagger \right]} - \frac{\sum_j b_j v v^\dagger b_j^\dagger}{Tr \left[ \sum_j b_j v v^\dagger b_j^\dagger \right]} \right\|_1
= \frac{1}{2} \max_{v \in X \otimes R} \left\| \frac{\left( \mathcal{E} \otimes I_R \right) (v v^\dagger)}{Tr \left[ \mathcal{E} \otimes I_R \right] (v v^\dagger)} - \frac{\left( \mathcal{F} \otimes I_R \right) (v v^\dagger)}{Tr \left[ \mathcal{F} \otimes I_R \right] (v v^\dagger)} \right\|_1. \tag{S7}
\]

Comparing Equation S7 with Equation S1, the pure state $v v^\dagger$ is on a subset of the domain of optimization variable $\rho$ on space $X \otimes R$. Therefore, we conclude that $\dim(X) \leq \dim(R)$ guarantees the stability of the general distance, and the distance is optimized over a pure state $v$ on space $X \otimes R$, such that
\[
d_g(\mathcal{E}, \mathcal{F}) = \frac{1}{2} \max_{v \in X \otimes R} \left\| \frac{\left( \mathcal{E} \otimes I_R \right) (v v^\dagger)}{Tr \left[ \mathcal{E} \otimes I_R \right] (v v^\dagger)} - \frac{\left( \mathcal{F} \otimes I_R \right) (v v^\dagger)}{Tr \left[ \mathcal{F} \otimes I_R \right] (v v^\dagger)} \right\|_1. \tag{S8}
\]

**S2. Renormalization of Quantum Operations**

We can bound the general distance in Equation S1 by renormalization. We first represent the two quantum operations in Stinespring form [S1], as $\mathcal{E}(\rho) = Tr_Z \{ A \rho A^\dagger \}$ and $\mathcal{F}(\rho) = Tr_Z \{ B \rho B^\dagger \}$, where $A = \sum_i e_i \otimes a_i$, $B = \sum_j e_j \otimes b_j$, and $e_i$ and $e_j$ represent bases on a reference space $Z$. The general distance between $\mathcal{E}$ and $\mathcal{F}$ is given by
\[
d_g(\mathcal{E}, \mathcal{F}) = \frac{1}{2} \max_v \left\| Tr_Z \left\{ \frac{A v v^\dagger A^\dagger}{\langle v | A^\dagger A | v \rangle} - \frac{B v v^\dagger B^\dagger}{\langle v | B^\dagger B | v \rangle} \right\} \right\|_1, \tag{S9}
\]

In the above equation, $v$ is a pure state on the extended state space $X \otimes R$, and we abbreviate $A \otimes I_R$ as $A$ and $B \otimes I_R$ as $B$, where $I_R$ is the identity operator on the auxiliary space $R$. By defining a new state $\psi = \frac{A v}{\langle A v \rangle}$, we reformulate Equation S9 by
\[
d_g(\mathcal{E}, \mathcal{F}) = \frac{1}{2} \max_\psi \left\| Tr_Z \left\{ \psi \psi^\dagger - \frac{(BA^{-1}) \psi \psi^\dagger (BA^{-1})^\dagger}{\langle \psi | (BA^{-1})^\dagger BA^{-1} | \psi \rangle} \right\} \right\|_1, \tag{S10}
\]

where $A^{-1}$ is the pseudoinverse of $A$. Applying singular value decomposition to $BA^{-1}$ results $BA^{-1} = UMV^\dagger$. Here, $U$ and $V$ are isometries with dimensions of $k' m \times n$ and $km \times n$, and $M$ is a nonnegative diagonal operator with a dimension of $n \times n$, where $n$ and $m$ are the dimensions of the input and output spaces of the quantum operations, and $k$ and $k'$ are the number of Kraus operators $a_i$ and $b_j$. Thus, we can write Equation S10 by
\[
d_g(\mathcal{E}, \mathcal{F}) = \frac{1}{2} \max_\psi \left\| Tr_Z \left\{ \psi \psi^\dagger - \frac{UMV^\dagger \psi \psi^\dagger VMU^\dagger}{\langle \psi | VM^2 V^\dagger | \psi \rangle} \right\} \right\|_1. \tag{S11}
\]
By defining a new state $\phi = V^\dagger \psi$ and using the triangular inequality and the contractility of trace distance, we can bound Equation S11 by

$$
\begin{align*}
    d_g(\mathcal{E}, \mathcal{F}) &= \frac{1}{2} \max_{\phi} \left\| Tr_{Z} \left\{ V \phi^\dagger V^\dagger - \frac{U M \phi^\dagger M U^\dagger}{\langle \phi | M^2 | \phi \rangle} \right\} \right\|_1 \\
    &= \frac{1}{2} \max_{\phi} \left\| Tr_{Z} \left\{ V \phi^\dagger V^\dagger - U \phi^\dagger U^\dagger + U \phi^\dagger U^\dagger - \frac{U M \phi^\dagger M U^\dagger}{\langle \phi | M^2 | \phi \rangle} \right\} \right\|_1 \\
    &\leq \frac{1}{2} \max_{\phi} \left\| Tr_{Z} \left\{ V \phi^\dagger V^\dagger - U \phi^\dagger U^\dagger \right\} \right\|_1 + \frac{1}{2} \max_{\phi} \left\| Tr_{Z} \left\{ U \phi^\dagger U^\dagger - \frac{U M \phi^\dagger M U^\dagger}{\langle \phi | M^2 | \phi \rangle} \right\} \right\|_1 \\
    &\leq \frac{1}{2} \max_{\phi} \left\| Tr_{Z} \left\{ V \phi^\dagger V^\dagger - U \phi^\dagger U^\dagger \right\} \right\|_1 + \frac{1}{2} \max_{\phi} \left\| \phi \phi^\dagger - \frac{M \phi^\dagger M}{\langle \phi | M^2 | \phi \rangle} \right\|_1.
\end{align*}
$$

(S12)

We define two renormalized quantum channels

$$
\begin{align*}
    \mathcal{U}(\rho) &= Tr_{Z} \left\{ U \rho U^\dagger \right\}, \\
    \mathcal{V}(\rho) &= Tr_{Z} \left\{ V \rho V^\dagger \right\},
\end{align*}
$$

(S13)

and a normalizing operation

$$
\mathcal{M}(\rho) = \frac{M \rho M}{Tr[M \rho M]}.
$$

(S14)

The general distance between $\mathcal{E}$ and $\mathcal{F}$ is bounded by Equation S12, as

$$
\begin{align*}
    d_g(\mathcal{E}, \mathcal{F}) &\leq d_g(\mathcal{U}, \mathcal{V}) + d_g(\mathcal{I}, \mathcal{M}),
\end{align*}
$$

(S15)

where $d_g(\mathcal{U}, \mathcal{V})$ is a diamond distance, which can be computed by convex optimization; $d_g(\mathcal{I}, \mathcal{M})$ is a general distance between an identity operation and a normalizing operation, which we refer to as the normalizing distance.

S3. CALCULATION OF THE NORMALIZING DISTANCE

We can directly calculate the normalizing distance

$$
\begin{align*}
    d_g(\mathcal{I}, \mathcal{M}) &= \frac{1}{2} \max_{\psi} \left\| \psi \psi^\dagger - \frac{M \psi^\dagger M}{\langle \psi | M^2 | \psi \rangle} \right\|_1,
\end{align*}
$$

(S16)

where $\psi$ is a pure quantum state, and $M$ is a nonnegative diagonal operator with a dimension of $n \times n$. We can denote the normalized output state as $|\phi\rangle = \frac{M |\psi\rangle}{|M |\psi| \rangle}$, and the normalizing distance is given by

$$
\begin{align*}
    d_g(\mathcal{I}, \mathcal{M}) &= \frac{1}{2} \max_{\psi} \left\| \psi \psi^\dagger - \phi \phi^\dagger \right\|_1;
\end{align*}
$$

(S17)

The trace distance between two pure states is related to their inner product by $\frac{1}{2} \left\| \psi \psi^\dagger - \phi \phi^\dagger \right\|_1 = \sqrt{1 - |\langle \psi | \phi \rangle|^2}$ [S2], so that maximizing $\left\| \psi \psi^\dagger - \phi \phi^\dagger \right\|_1$ is equivalent to minimizing $\cos \theta = |\langle \psi | \phi \rangle|$, where we refer to $\theta$ as the rotating angle between the input and output states.

We can compute $\cos \theta$ by induction. We denote the normalizing operator as $M = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. We first consider the projection of $M$ on a $l$-dimensional subspace as $M_l = diag(\lambda_1, \lambda_2, \cdots, \lambda_l)$. A vector $|u\rangle$ on the subspace can be decomposed as $|u\rangle = \sqrt{x} |l\rangle + \sqrt{1-x} |v\rangle$, where $x \in [0, 1]$, $|l\rangle$ is the basis of $M$ corresponding to the eigenvalue $\lambda_l$, and $|v\rangle$ is a unit vector on the $(l-1)$-dimensional subspace orthogonal to $|l\rangle$. The assumption of positive coefficients loses no generality because we can always redefine the basis with any phase. Applying $M_l$ to $|u\rangle$ gives

$$
M_l |u\rangle = \lambda_l \sqrt{x} |l\rangle + \sqrt{1-x} M_{l-1} |v\rangle,
$$

(S18)

where $M_{l-1}$ is the projection of $M$ on the $(l-1)$-dimensional subspace. Applying $M_{l-1}$ to $|v\rangle$ results

$$
M_{l-1} |v\rangle = r_{l-1} \cos \theta_{l-1} |v\rangle + r_{l-1} \sin \theta_{l-1} |v\rangle,
$$

(S19)
where \( r_{l-1} \) and \( \theta_{l-1} \) are the contracting factor and rotating angle, respectively, and \(|v_{\perp}\rangle\) represents a unit vector orthogonal to \(|v\rangle\). A specific rotating axis does not affect the following calculation because the axis is always orthogonal to \(|l\rangle\). The contracting factor and rotating angle of \( M_l |u\rangle \) are given by

\[
\begin{align*}
  r_l^2 &= x\lambda^2_l + (1-x) r_{l-1}^2, \\
  r_l \cos \theta_l &= x\lambda_l + (1-x) r_{l-1} \cos \theta_{l-1},
\end{align*}
\]

i.e., \((r_l^2, r_l \cos \theta_l)\) is a convex combination of \((\lambda^2_l, \lambda_l)\) and \((r_{l-1}^2, r_{l-1} \cos \theta_{l-1})\). By induction, the feasible domain of \((r^2, r \cos \theta)\) for the normalizing operation \( \mathcal{M} \) is given by

\[
\begin{align*}
  r^2 &= \sum_i x_i \lambda_i^2, \\
  r \cos \theta &= \sum_i x_i \lambda_i,
\end{align*}
\]

where \( x_i \) represents the coefficient of convex combination satisfying \( \sum_i x_i = 1 \), and \( \lambda_i \) is the eigenvalue of \( \mathcal{M} \). The minimum of \( \cos \theta \) is on the boundary of the feasible domain depending on the combination two eigenvalues. We can calculate the minimum of \( \cos \theta \) easily by calculus, as

\[
\min \cos \theta = \frac{2\sqrt{\lambda_{\text{max}}\lambda_{\text{min}}}}{\lambda_{\text{max}} + \lambda_{\text{min}}},
\]

where \( \lambda_{\text{max}} \) and \( \lambda_{\text{min}} \) are the maximum and minimum eigenvalues of \( \mathcal{M} \). Therefore, the distance between the identity and normalizing operations is given by

\[
d_g (\mathcal{I}, \mathcal{M}) = \frac{\lambda_{\text{max}} - \lambda_{\text{min}}}{\lambda_{\text{max}} + \lambda_{\text{min}}}. \tag{S23}
\]

Figure S1 shows the feasible domain of \((r, \cos \theta)\) for a normalizing operator \( \mathcal{M} = \text{diag}(0.3, 0.5, 0.6, 0.8, 0.9) \). The red curve represents its boundary computed by Equation S21. Each blue point represents an instance calculated by a Monte Carlo simulation, where we apply \( \mathcal{M} \) to randomly generated input states and calculate the contracting factor and rotating angle of the output states.

FIG. S1. The feasible domain of contracting factor and rotating angle \((r, \cos \theta)\) for applying \( \mathcal{M} \) to a vector, given \( \mathcal{M} = \text{diag}(0.3, 0.5, 0.6, 0.8, 0.9) \).

[S1] J. Watrous, *The Theory of Quantum Information* (2018).
[S2] A. N. Michael and L. C. Isaac, *Quantum Computation and Quantum Information 10th Anniversary Edition* (2010).