A SEARS-TYPE SELF-ADJOINTNESS RESULT FOR DISCRETE MAGNETIC SCHröDINGER OPERATORS

Abstract. In the context of a weighted graph with vertex set $V$ and bounded vertex degree, we give a sufficient condition for the essential self-adjointness of the operator $\Delta_\sigma + W$, where $\Delta_\sigma$ is the magnetic Laplacian and $W: V \to \mathbb{R}$ is a function satisfying $W(x) \geq -q(x)$ for all $x \in V$, with $q: V \to [1, \infty)$. The condition is expressed in terms of completeness of a metric that depends on $q$ and the weights of the graph. The main result is a discrete analogue of the results of I. Oleinik and M. A. Shubin in the setting of non-compact Riemannian manifolds.

1. Introduction and the main result

1.1. The setting. Let $G = (V, E)$ be an infinite graph without loops and multiple edges between vertices. By $V = V(G)$ and $E = E(G)$ we denote the set of vertices and the set of unoriented edges of $G$ respectively. In what follows, the notation $m(x)$ indicates the degree of a vertex $x$, that is, the number of edges that meet at $x$. We assume that $G$ has bounded vertex degree: there exists a constant $N > 0$ such that

$$m(x) \leq N, \quad \text{for all } x \in V. \quad (1.1)$$

In what follows, $x \sim y$ indicates that there is an edge that connects $x$ and $y$. We will also need a set of oriented edges

$$E_0 := \{[x, y], [y, x] : x, y \in V \text{ and } x \sim y\}. \quad (1.2)$$

The notation $e = [x, y]$ indicates an oriented edge $e$ with starting vertex $o(e) = x$ and terminal vertex $t(e) = y$. The definition (1.2) means that every unoriented edge in $E$ is represented by two oriented edges in $E_0$. Thus, there is a two-to-one map $p: E_0 \to E$. For $e = [x, y] \in E_0$, we denote the corresponding reverse edge by $\hat{e} = [y, x]$. This gives rise to an involution $e \mapsto \hat{e}$ on $E_0$.

To help us write formulas in unambiguous way, we fix an orientation on each edge by specifying a subset $E_s$ of $E_0$ such that $E_0 = E_s \cup \hat{E}_s$ (disjoint union), where $\hat{E}_s$ denotes the image of $E_s$ under the involution $e \mapsto \hat{e}$. Thus, we may identify $E_s$ with $E$ by the map $p$.

In the sequel, we assume that $G$ is connected, that is, for any $x, y \in V$ there exists a path $\gamma$ joining $x$ and $y$. Here, $\gamma$ is a sequence $x_1, x_2, \ldots, x_n \in V$ such that $x = x_1$, $y = x_n$, and $x_j \sim x_{j+1}$ for all $1 \leq j \leq n - 1$.

In what follows, $C(V)$ is the set of complex-valued functions on $V$, and $C(E_s)$ is the set of functions $Y: E_0 \to \mathbb{C}$ such that $Y(e) = -Y(\hat{e})$. The notations $C_c(V)$ and $C_c(E_s)$ denote the sets of finitely supported elements of $C(V)$ and $C(E_s)$ respectively.

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In the sequel, we assume that $V$ is equipped with a weight $w: V \to \mathbb{R}^+$. By $\ell_2^w(V)$ we denote the space of functions $f \in C(V)$ such that $\|f\| < \infty$, where $\|f\|$ is the norm corresponding to the inner product

$$(f, g) := \sum_{x \in V} w(x)f(x)\overline{g(x)}.$$  

(1.3)

Additionally, we assume that $E$ is equipped with a weight $a: E_0 \to \mathbb{R}^+$ such that $a(e) = a(\hat{e})$ for all $e \in E_0$. This makes $G = (G, w, a)$ a weighted graph with weights $w$ and $a$.

1.2. Magnetic Schrödinger operator. Let $U(1) := \{z \in \mathbb{C}:|z| = 1\}$ and $\sigma: E_0 \to U(1)$ with $\sigma(\hat{e}) = \overline{\sigma(e)}$ for all $e \in E_0$, where $\overline{\sigma}$ denotes the complex conjugate of $z \in \mathbb{C}$.

We define the magnetic Laplacian $\Delta_\sigma: C(V) \to C(V)$ on the graph $(G, w, a)$ by the formula

$$(\Delta_\sigma u)(x) = \frac{1}{w(x)} \sum_{e \in O_x} a(e)(u(x) - \sigma(\hat{e})u(t(e))),$$

(1.4)

where $x \in V$ and

$$O_x := \{e \in E_0: o(e) = x\}.$$  

(1.5)

For the case $a \equiv 1$ and $w \equiv 1$, the definition (1.4) is the same as in [9]. For the case $\sigma \equiv 1$, see [30] and [32].

Let $W: V \to \mathbb{R}$, and consider a Schrödinger-type expression

$$Hu := \Delta_\sigma u + Wu.$$  

(1.6)

Let $q: V \to [1, \infty)$, and assume that $W$ satisfies

$$W(x) \geq -q(x), \quad \text{for all } x \in V.$$  

(1.7)

In the sequel, we will need the notion of weighted distance on $G$. Let $w$ and $a$ be as in (1.4) and let $q$ be as in (1.7). We define the weighted distance $d_{w,a,q}$ on $G$ as follows:

$$d_{w,a,q}(x, y) := \inf_{\gamma \in \Gamma_{x,y}} L_{w,a,q}(\gamma),$$

(1.8)

where $\Gamma_{x,y}$ is the set of all paths $\gamma: x = x_1, x_2, \ldots, x_n = y$ such that $x_j \sim x_{j+1}$ for all $1 \leq j \leq n - 1$, and the length $L_{w,a,q}(\gamma)$ is computed as follows:

$$L_{w,a,q}(\gamma) = \sum_{j=1}^{n-1} \frac{\min\{w^{1/2}(x_j), w^{1/2}(x_{j+1})\} \cdot \min\{q^{-1/2}(x_j), q^{-1/2}(x_{j+1})\}}{\sqrt{a([x_j,x_{j+1}])}}.$$  

In the case $q \equiv 1$, the weighted distance (1.8) was defined in [4].

We say that the metric space $(G, d_{w,a,q})$ is complete if every Cauchy sequence of vertices has a limit in $V$.  

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1.3. Statement of the main result. We now state the main result.

**Theorem 1.4.** Assume that \((G, w, a)\) is an infinite, connected, oriented, and weighted graph. Assume that \(G\) has bounded vertex degree. Assume that \(W\) satisfies (1.7) and \(q : V \to [1, \infty)\) satisfies

\[
|q^{-1/2}(t(e)) - q^{-1/2}(o(e))| \leq C \left( \frac{\min\{w(t(e)), w(o(e))\}}{a(e)} \right)^{1/2},
\]

for all \(e \in E_s\), where \(C\) is a constant.

Additionally, assume that \((G, d_{w,a,q})\) is a complete metric space. Then, the operator \(H|_{C_c(V)}\) is essentially self-adjoint in \(\ell^2_w(V)\).

**Remark 1.5.** The origin of the result presented in Theorem 1.4 can be traced back to the paper [25] by D. B. Sears concerning the essential self-adjointness of \((-\Delta + W)|_{C_c^\infty(\mathbb{R}^n)}\) in \(L^2(\mathbb{R}^n)\). Here, \(\Delta\) is the standard Laplacian on \(\mathbb{R}^n\) and \(-q \leq W \leq L^\infty(\mathbb{R}^n)\), where \(q\) is a radially symmetric function on \(\mathbb{R}^n\) satisfying properties analogous to those of Theorem 1 in the present paper (with “completeness” replaced by the divergence of \(\int_0^\infty q^{-1/2}(r) dr\), where \(r = r(x)\) is the Euclidean distance between \(x \in \mathbb{R}^n\) and \(0 \in \mathbb{R}^n\)). We should mention that the paper [25] followed an idea of E. C. Titchmarsh [31]. More recently, I. Oleinik [23, 24] gave a sufficient condition for the essential self-adjointness of \((\Delta_M + W)|_{C_c^\infty(M)}\) in \(L^2(M)\), where \(\Delta_M\) is the scalar Laplacian on a Riemannian manifold \(M\) and \(-q \leq W \in L^\infty_{\text{loc}}(M)\). Here, \(q\) is a function on \(M\) satisfying properties analogous to those of Theorem 1 in the present paper. Oleinik’s proof was simplified by M. A. Shubin [26], and the result was extended to magnetic Schrödinger operators in [27].

Theorem 1.4 of the present paper is a discrete analogue of the mentioned results of Oleinik and Shubin.

**Remark 1.6.** Assuming (1.1), the completeness of \((G, d_{w,a,1})\), and

\[
(Hu, u) \geq k\|u\|^2,
\]

for all \(u \in C_c(V)\), (1.10)

where \(k\) is a constant independent of \(u\), the essential self-adjointness of \(H|_{C_c(V)}\) was established in [21] Theorem 1.3. If \(q(x) \equiv c_0\), where \(c_0\) is a constant, then the operator \(H|_{C_c(V)}\), with \(W\) as in (1.7), satisfies (1.10). However, there are operators \(H\) that satisfy the hypotheses of Theorem 1.4 but do not satisfy (1.10), as illustrated by the example below.

**Example.** Consider \(G = (V, E)\) with \(V = \{1, 2, 3, \ldots\}\) and \(E = \{[n, n+1] : n \in V\}\). Define \(a([n, n+1]) = 1\) and \(w(n) = 1\), for all \(n \in V\). Let \(H\) be as in (1.6) with \(\sigma([n, n+1]) = 1\) and \(W(n) = -n^2\), for all \(n \in V\). It is easy to see that for every \(k \in \mathbb{R}\), there exists a function \(u \in C_c(V)\) such that the inequality (1.10) is not satisfied. Thus, the operator \(H\) is not semi-bounded from below, and we cannot use [21] Theorem 1.3. Turning to hypotheses of Theorem 1.4, note that \(W\) satisfies (1.7) with \(q(n) = n^2\). It is easy to see that \(q^{-1/2} = n^{-1}\) satisfies (1.9) with \(C = 1\). Fix \(K_1 \in V\), and let \(K > K_1\). For \(x_1 = K_1\) and \(x = K\), by (1.8) we have

\[
d_{w,a,q}(x_1, x) = \sum_{n=K_1}^{K-1} \frac{1}{n+1} \to \infty, \quad \text{as } K \to \infty.
\]
Thus, the metric \( d_{w,a,q} \) is complete, and by Theorem 1.4 the operator \( H|_{C_c(V)} \) is essentially self-adjoint in \( L^2_w(V) \).

Remark 1.7. Thanks to assumption (1.10), the proof of [21, Theorem 1.3] reduced to showing that if \( u \in \text{Dom}(H_{\text{max}}) \), with \( H_{\text{max}} \) as in Section 2 below, and \( (H + \lambda)u = 0 \) with sufficiently large \( \lambda > 0 \), then \( u = 0 \). To this end, a sequence of cut-off functions was constructed and a “summation by parts” method was used. In the absence of assumption (1.10), the essential self-adjointness can be established by showing that \( H_{\text{max}} \) is symmetric. This requires an approach different from [21]: in the present paper, we consider the sum \( J_s \) that incorporates the metric \( d_{w,a,q} \) (see (3.20) below) and show that \( J_s \to 0 \) as \( s \to +\infty \). A key ingredient in this endeavor, not present in [21], is the estimate (3.2) for \( d_\sigma u \), where \( u \in \text{Dom}(H_{\text{max}}) \). The estimate (3.2) is a discrete analogue of [27, Lemma 4.3].

Remark 1.8. For studies of the operator (1.4) with \( a \equiv 1, \sigma \equiv 1, \) and \( w \equiv m \), see, for instance, [3] and [22]. For general information concerning magnetic Laplacian on graphs, see [20] and [29]. For a proof the discrete analogue of Kato’s inequality, see [9].

For the problem of self-adjoint realization of the operator (1.6) and its special cases \( (a \equiv 1, \sigma \equiv 1, w \equiv 1, \) and \( W \equiv 0) \), see, for instance, [4], [5], [11], [12], [15], [17], [18], [32], [33], and [35]. We should mention that the authors of [12] and [17, 18] worked in the setting of discrete sets, a more general context than locally finite graphs. For a study of the essential self-adjointness of discrete Laplace operator on forms, see [19].

The problem of stochastic completeness of graphs is considered in [7], [33], [35], and [36]. In the setting of Dirichlet forms on discrete sets, stochastic completeness is studied in [12], [17, and [18]. For another approach to stochastic completeness on discrete sets, see [13]. For a study of random walks on infinite graphs, see [6], [8], [34], and references therein.

For studies of essential self-adjointness of Schrödinger operators in the context of non-compact Riemannian manifolds, see, for instance, [1], [2], [10], [23], [24], [29], [27], and [28].

2. Preliminaries

In what follows, the deformed differential \( d_\sigma : C(V) \to C(E_s) \) is defined as

\[
(d_\sigma u)(e) := \overline{\sigma(e)u(t(e))} - u(o(e)), \quad \text{for all } u \in C(V),
\]

where \( \sigma \) is as in (1.4).

The deformed co-differential \( \delta_\sigma : C(E_s) \to C(V) \) is defined as

\[
(\delta_\sigma Y)(x) := \frac{1}{w(x)} \sum_{e \in E_s, t(e) = x} \sigma(e)a(e)Y(e) - \frac{1}{w(x)} \sum_{e \in E_s, o(e) = x} a(e)Y(e),
\]

for all \( Y \in C(E_s) \), where \( \sigma, w, \) and \( a \) are as in (1.4).

In the case \( \sigma \equiv 1 \), the definitions (2.1) and (2.2) give us the standard differential \( d \) and standard co-differential \( \delta \), respectively.
Let $\sigma$ be as in (1.4). For a function $u \in C(V)$, we define $u^\# \in C(E_s)$ by the formula
\[
  u^\#(e) := \frac{\sigma(e)u(t(e)) + u(o(e))}{2}, \quad \text{for all } e \in E_s.
\]
(2.3)

For $\sigma \equiv 1$ in (2.3), we define $u^\sharp := u^\#_1$.

In what follows, for $x \in V$, we define
\[
  S_x := \{e \in E_s : o(e) = x \text{ or } t(e) = x\}. \quad (2.4)
\]

The proofs of the following two lemmas are straightforward computations based on the definitions of $d$, $d_\sigma$, $\delta$ and $\delta_\sigma$. For detailed proofs in the case $\sigma \equiv 1$ see [19, Lemma 3.1].

**Lemma 2.1.** For all $u \in C(V)$ and all $v \in C(V)$, the following equality holds:
\[
  d_\sigma(uv) = (d_\sigma u)v^\# + u^\# \sigma dv, \quad (2.5)
\]
where $d_\sigma$ is as in (2.1) with $\sigma(e)$ replaced by $\sigma(e)$, $u^\#_\sigma$ is as in (2.3), and $v^\#$ is as in (2.3) with $\sigma \equiv 1$.

**Lemma 2.2.** For all $u \in C(V)$ and all $Y \in C(E_s)$, the following equality holds:
\[
  (\delta(u^\# Y))(x) = u(x)(\delta_\sigma Y)(x) - \frac{1}{2w(x)} \sum_{e \in S_x} a(e)Y(e)d_\sigma u(e), \quad (2.6)
\]
where $d_\sigma$ is as in (2.1) with $\sigma(e)$ replaced by $\sigma(e)$, $u^\#_\sigma$ is as in (2.3), and $S_x$ is as in (2.4).

**Lemma 2.3.** Assume that $\phi \in C(V)$ is real-valued. Then
\[
  (\phi^\#(e))^2 \leq (\phi^2)^\#(e), \quad \text{for all } e \in E_s. \quad (2.7)
\]

**Proof** By (2.3) with $\sigma \equiv 1$, for all $e \in E_s$ we have
\[
  (\phi^\#(e))^2 - (\phi^2)^\#(e) = \left(\frac{\phi(t(e)) - \phi(o(e))}{2}\right)^2 \geq 0,
\]
which gives (2.7). $\Box$

Let $\ell^2_a(E_s)$ denote the space of functions $F \in C(E_s)$ such that $\|F\| < \infty$, where $\|F\|$ is the norm corresponding to the inner product
\[
  (F, G) := \sum_{e \in E_s} a(e)F(e)\overline{G(e)}.
\]

It is easy to check the following equality:
\[
  (d_\sigma u, Y) = (u, \delta_\sigma Y), \quad \text{for all } u \in \ell^2_a(V), Y \in C_c(E_s), \quad (2.8)
\]
where $(\cdot, \cdot)$ on the left-hand side (right-hand side) denotes the inner product in $\ell^2_a(E_s)$ (in $\ell^2_a(V)$).

A computation shows that the following equality holds:
\[
  \delta_\sigma d_\sigma u = \Delta_\sigma u, \quad \text{for all } u \in C(V). \quad (2.9)
\]

For the proofs of (2.8) and (2.9), see, for instance, [21, Section 3]. The following lemma follows easily from (2.9) and (2.8).
Lemma 2.4. The operator $\Delta_\sigma|_{C_c(V)}$ is symmetric in $\ell_w^2(V)$:

$$(\Delta_\sigma u, v) = (u, \Delta_\sigma v), \quad \text{for all } u, v \in C_c(V).$$

We now give the definitions of minimal and maximal operators associated with the expression (1.6). We define the operator $H_{\text{min}}$ by the formula

$$H_{\text{min}} u := Hu, \quad \text{Dom}(H_{\text{min}}) := C_c(V).$$

(2.10)

Since $W$ is real-valued, the following lemma follows easily from Lemma 2.4.

Lemma 2.5. The operator $H_{\text{min}}$ is symmetric in $\ell_w^2(V)$.

We define $H_{\text{max}} := (H_{\text{min}})^*$, where $T^*$ denotes the adjoint of operator $T$. We also define $\mathcal{D} := \{u \in \ell_w^2(V) : Hu \in \ell_w^2(V)\}$.

For a proof of the following lemma, see, for instance, [21, Lemma 3.7].

Lemma 2.6. The following hold:

$$\text{Dom}(H_{\text{max}}) = \mathcal{D}$$

and

$$H_{\text{max}} u = Hu \quad \text{for all } u \in \mathcal{D}.$$
\[
\lim_{n \to \infty} \chi_n(x) = 1; \text{ (iv) the functions } \chi_n \text{ have finite support; and (v) the functions } d\chi_n \text{ satisfy the inequality }
\]
\[
|d\chi_n(e)| \leq \frac{d_{w,a,1}(\phi(e), t(e))}{n}.
\]
It is easy to see that the properties (i)–(iii) and (v) hold. By hypothesis, we know that \((G, d_{w,a,q})\) is a complete metric space and, thus, balls with respect to \(d_{w,a,q}\) are finite; see, for instance, [21, Section 6.1]. Let \(B_{2n}^{w,a,q}(x_0)\) be as in (3.4) with \(d_{w,a,1}\) replaced by \(d_{w,a,q}\). Since \(q \geq 1\) it follows that \(B_{2n}^{w,a}(x_0) \subseteq B_{2n}^{w,a,q}(x_0)\). Thus, property (iv) is a consequence of property (ii) and the finiteness of \(B_{2n}^{w,a}(x_0)\).

**Proof of Proposition 3.1**

Let \(u \in \text{Dom}(H_{\text{max}})\) and let \(\phi \in C_c(V)\) be a real-valued function. Define
\[
I := \left( \sum_{e \in E_s} a(e)|(d_{\sigma}u)(e)|^2(\phi^2)^\sharp(e) \right)^{1/2},
\]
(3.5)

where \(f^\sharp(e)\) is as in (2.3) with \(\sigma \equiv 1\).

We will first show that
\[
I^2 \leq |(\phi^2 Hu, u)| + (\phi^2 qu, u) + 2I \left( \sum_{e \in E_s} a(e)|(d\phi)(e)|^2|E_\sigma^\sharp(e)|^2 \right)^{1/2},
\]
(3.6)

where \(f^\sharp_\sigma(e)\) is as in (2.3), and \(E_\sigma\) is the conjugate of \(z \in \mathbb{C}\).

Using (2.6), the equality \(\Delta_\sigma u = Hu - W u\), and
\[
(d_\sigma(\phi^2)) = (d_\sigma u)(\phi^2)^\sharp(e) + 2(E_\sigma^\sharp(e)\phi^\sharp(e)(d\phi)(e),
\]
we have
\[
\delta \left( (\phi^2)^\sharp_\sigma d_\sigma u \right)(x) = \phi^2(x)u(x)(Hu - W u)(x)
\]
\[
- \frac{1}{2w(x)} \sum_{e \in S_x} a(e)|(d_\sigma u)(e)|^2(\phi^2)^\sharp(e)
\]
\[
- \frac{1}{w(x)} \sum_{e \in S_x} a(e)(d_\sigma u)(e)(E_\sigma^\sharp(e)\phi^\sharp(e)(d\phi)(e). \quad (3.7)
\]

Since \(\phi\) has finite support, using the definition of \(\delta\) it follows that
\[
\sum_{x \in V} w(x)\delta \left( (\phi^2)^\sharp_\sigma d_\sigma u \right)(x) = 0. \quad (3.8)
\]

Multiplying both sides of (3.7) by \(w(x)\), summing over \(x \in V\), and using (3.8), we get
\[
\frac{1}{2} \sum_{x \in V} \sum_{e \in S_x} a(e)|(d_\sigma u)(e)|^2(\phi^2)^\sharp(e) = (\phi^2 Hu, u) - (\phi^2 W u, u)
\]
\[
- \sum_{x \in V} \sum_{e \in S_x} a(e)(d_\sigma u)(e)(E_\sigma^\sharp(e)\phi^\sharp(e)(d\phi)(e). \quad (3.9)
\]
Rewriting the double sum on the left-hand side of (3.9) as the sum over $E_s$, taking real parts on both sides of (3.9), and using (1.7), we have
\[
\sum_{e \in E_s} a(e)(d_e u)(e)^2(\phi^2)(e) = \text{Re} \left( \phi^2 H u, u \right) - (\phi^2 W u, u) \\
- \text{Re} \sum_{x \in V} \sum_{e \in S_x} a(e)(d_e u)(e)(\overline{\sigma}(e)\phi^2(e)(d\phi))(e) \\
\leq |(\phi^2 H u, u)| + (\phi^2 W u, u) \\
+ 2 \sum_{e \in E_s} a(e)(d_e u)(e)\left| (\overline{\sigma}(e)\phi^2(e)(d\phi))(e) \right|,
\]
which, after applying Cauchy–Schwarz inequality and (2.7), gives (3.6).

Let $\chi_n$ be as in (3.3) and let $q$ be as in (1.7). Define
\[
\phi_n(x) := \chi_n(x)q^{-1/2}(x). \tag{3.10}
\]
By property (iv) of $\chi_n$ it follows that $\phi_n$ has finite support. By property (i) of $\chi_n$ and since $q \geq 1$, we have
\[
0 \leq \phi_n(x) \leq q^{-1/2}(x) \leq 1, \quad \text{for all } x \in V. \tag{3.11}
\]
By property (iii) of $\chi_n$ we have
\[
\lim_{n \to \infty} \phi_n(x) = q^{-1/2}(x), \quad \text{for all } x \in V. \tag{3.12}
\]
By (2.5), (1.9), properties (i) and (v) of $\chi_n$, the inequality $q \geq 1$, and (1.8), we have
\[
|\left(\overline{\sigma}(e)\phi^2(e)(d\phi)(e)\right)| \leq \left(\frac{1}{n} + C\right) \min\left\{w^{1/2}(o(e)), w^{1/2}(t(e))\right\}, \tag{3.13}
\]
where $C$ is as in (1.9).

We also have
\[
|\left(\overline{\sigma}(e)\phi^2(e)(d\phi)(e)\right)| \leq \frac{|u(o(e))|^2 + |u(t(e))|^2}{2}. \tag{3.14}
\]
By (3.13), (3.14), and (1.1) we get
\[
\left(\sum_{e \in E_s} a(e)(d_e u)(e)^2\left| (\overline{\sigma}(e)\phi^2(e)(d\phi))(e) \right|^2\right)^{1/2} \\
\leq \frac{1}{\sqrt{2}} \left(\frac{1}{n} + C\right) \left(\sum_{e \in E_s} |u(o(e))|^2 w(o(e)) + \sum_{e \in E_s} |u(t(e))|^2 w(t(e))\right)^{1/2} \\
\leq \frac{1}{\sqrt{2}} \left(\frac{1}{n} + C\right) \left(2N\|u\|^2\right)^{1/2} = \left(\frac{1}{n} + C\right) \sqrt{N}\|u\|. \tag{3.15}
\]
By (3.6) with \( \phi = \phi_n \), (3.15), and (3.11), we obtain
\[
I_n^2 \leq \|Hu\|\|u\| + \|u\|^2 + 2I_n \left( \frac{1}{n} + C \right) \sqrt{N} \|u\|,
\] (3.16)
for all \( u \in \text{Dom}(H_{\text{max}}) \), where \( I_n \) is as in (3.5) with \( \phi = \phi_n \).

Using the inequality \( ab \leq a^2 + b^2 \) in the third term on the right-hand side of (3.16) and rearranging, we obtain
\[
I_n^2 \leq 2 \left( \|Hu\|\|u\| + \left( \frac{2}{N} \left( \frac{1}{n} + C \right)^2 + 1 \right) \|u\|^2 \right).
\] (3.17)

Letting \( n \to \infty \) in (3.17) and using (3.12) together with Fatou’s lemma, we get
\[
\sum_{e \in E_s} a(e)\|d_e u(e)\|^2(q^{-1})^2(e) \leq 2 \left( \|Hu\|\|u\| + \left( \frac{2}{NC^2} + 1 \right) \|u\|^2 \right).
\] (3.18)

Since \( \min\{q^{-1}(o(e)), q^{-1}(t(e))\} \leq (q^{-1})^2(e) \), for all \( e \in E_s \), the inequality (3.2) follows directly from (3.18). \( \square \)

In the sequel, we will prove (3.1). Let \( d_{w,a,q} \) be as in (1.8). Fix \( x_0 \in V \) and define
\[
P(x) := d_{w,a,q}(x_0, x), \quad x \in V.
\] (3.19)

In what follows, for a function \( f : V \to \mathbb{R} \) we define \( f^+(x) := \max\{f(x), 0\} \).

Let \( u, v \in \text{Dom}(H_{\text{max}}) \) and let \( s > 0 \). Define
\[
J_s := \sum_{x \in V} \left( 1 - \frac{P(x)}{s} \right)^+ \left( (Hu)(x)v(x) - u(x)(Hv)(x) \right) w(x),
\] (3.20)
where \( P \) is as in (3.19), \( H \) is as in (1.6), and \( \overline{z} \) denotes the conjugate of \( z \in \mathbb{C} \).

Since \( (G, d_{w,a,q}) \) is a complete metric space, by [21 Section 6.1] it follows that the set
\[
U_s := \{ x \in V : P(x) \leq s \}
\] is finite. Thus, for all \( s > 0 \), the summation in (3.20) is performed over finitely many vertices.

**Lemma 3.2.** Let \( J_s \) be as in (3.20). Then
\[
\lim_{s \to +\infty} J_s = (Hu, v) - (u, Hv).
\] (3.21)

**Proof** For all \( x \in V \), as \( s \to +\infty \), the summand in (3.20) converges to
\[
((Hu)(x)v(x) - u(x)(Hv)(x))w(x).
\]
Additionally, for all \( x \in V \) and \( s > 0 \), the summand in (3.20) is estimated from above by
\[
|(Hu)(x)||v(x)||w(x)| + |u(x)|||(Hv)(x)||w(x)|.
\]
Since \( u, v \in \text{Dom}(H_{\text{max}}) \), by Lemma 2.6 we have \( Hu \in \ell_w^2(V) \) and \( Hv \in \ell_w^2(V) \). Hence, by Cauchy–Schwarz inequality it follows that

\[
\sum_{x \in V} |(Hu)(x)||v(x)|w(x) < +\infty \quad \text{and} \quad \sum_{x \in V} |u(x)||(Hv)(x)|w(x) < +\infty.
\]

Thus, by dominated convergence theorem we obtain (3.21). \( \square \)

**Lemma 3.3.** Let \( J_s \) be as in (3.20) and let \( N \) be as in (1.1). Then

\[
|J_s| \leq \frac{\sqrt{N}}{s} \|v\| \left( \sum_{e \in E_s} a(e) \min\{q^{-1}(o(e)), q^{-1}(t(e))\} |(d_\sigma u)(e)|^2 \right)^{1/2} + \frac{\sqrt{N}}{s} \|u\| \left( \sum_{e \in E_s} a(e) \min\{q^{-1}(o(e)), q^{-1}(t(e))\} |(d_\sigma v)(e)|^2 \right)^{1/2}.
\]

(3.22)

**Proof** Using (1.4), (1.6), and the property \( \sigma(\tilde{e}) = \overline{\sigma(e)} \), and recalling that \( W \) is real-valued, we can rewrite (3.20) as

\[
J_s = \sum_{x \in V} \sum_{e \in E_x} \left( 1 - \frac{P(x)}{s} \right)^+ a(e) \left( \sigma(e)u(x)v(t(e)) - \sigma(\tilde{e})u(t(e))v(x) \right). \tag{3.23}
\]

An edge \( e = [x, y] \in E_0 \) occurs twice in (3.23): once as \([x, y]\) and once as \([y, x]\). Since \( a([x, y]) = a([y, x]) \), it follows that the contribution of \( e = [x, y] \) and \( \tilde{e} = [y, x] \) together in (3.23) is

\[
\left( \left( 1 - \frac{P(x)}{s} \right)^+ - \left( 1 - \frac{P(t(e))}{s} \right)^+ \right) a(e) \left( \sigma(e)u(x)v(t(e)) - \sigma(\tilde{e})u(t(e))v(x) \right). \tag{3.24}
\]

Using (3.24) and the definition of \( d_\sigma \), we can rewrite (3.23) as

\[
J_s = \sum_{e \in E_s} \left( \left( 1 - \frac{P(o(e))}{s} \right)^+ - \left( 1 - \frac{P(t(e))}{s} \right)^+ \right) a(e) \left( (d_\sigma v)(e)|v(o(e))| - (d_\sigma u)(e)|v(o(e))| \right). \tag{3.25}
\]

Using triangle inequality and property

\[
|f^+(x) - g^+(x)| \leq |f(x) - g(x)|,
\]

from (3.25) we obtain

\[
|J_s| \leq \frac{1}{s} \sum_{e \in E_s} a(e) |P(t(e)) - P(o(e))|[|(d_\sigma v)(e)||u(o(e))| + |(d_\sigma u)(e)||v(o(e))|]. \tag{3.26}
\]
By (3.19) and (1.8) we get
\[
|P(t(e)) - P(o(e))| \leq d_{w,a,q}(t(e), o(e)) \\
\leq \frac{w^{1/2}(o(e)) \min\{q^{-1/2}(o(e)), q^{-1/2}(t(e))\}}{a(e)}.
\] (3.27)
Combining (3.26) and (3.27), and using Cauchy–Schwarz inequality together with assumption (1.1), we obtain (3.22).

\[\square\]

Continuation of the proof of Theorem 1.4

Let \( u \in \text{Dom}(H_{\text{max}}) \) and \( v \in \text{Dom}(H_{\text{max}}) \). By Lemma 2.6 it follows that \( H u \in L^2_V(V) \) and \( H v \in L^2_V(V) \). Letting \( s \to +\infty \) in (3.22) and using (3.2), it follows that \( J_s \to 0 \) as \( s \to +\infty \).

This, together with (3.21), shows (3.1).

\[\square\]

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