SUGRA/Strings like to be bald

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Abstract

We explore the embedding of the phenomenological holographic models describing thermal relativistic ordered conformal phases in $\mathbb{R}^{2,1}$ in SUGRA/String theory. The dual black branes in a Poincare patch of asymptotically $AdS_4$ have “hair” — a condensate of the order parameter for the broken symmetry. In a gravitational dual the order parameter for a spontaneous symmetry breaking is represented by a bulk scalar field with a nontrivial potential. To construct the ordered conformal phases perturbatively we introduce a phenomenological deformation parameter in the scalar potential. We find that while the ordered phases exist for different values of the deformation parameter, they disappear before the deformation is removed, in one case once the potential is precisely as in the top-down holography. It appears that the holographic models with the conformal ordered phases are in the String theory swampland.

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Thermal relativistic ordered conformal phases are states of a $d$-dimensional conformal field theory, $CFT_d$, in $\mathbb{R}^{d-1,1}$ with spontaneously broken global symmetry. They are characterized by the energy density $\mathcal{E}$, the pressure $P$, the entropy density $s$, and an expectation value of an operator (one or more) of a conformal dimension $\Delta$, $\mathcal{O}_\Delta$, representing the condensate of the order parameter:

$$\mathcal{E} \propto T^d, \quad P = \frac{1}{d-1} \mathcal{E}, \quad s = \frac{d}{d-1} \frac{\mathcal{E}}{T}, \quad \mathcal{O}_\Delta \propto T^\Delta.$$  

Conformal ordered phases from the QFT perspective were recently discussed in [1], and their particular holographic realization was proposed in [2]. The holographic models of [2] are phenomenological, and in this paper we explore the question whether thermal conformal ordered phases can arise in top-down SUGRA/String theory holographic models.

In a gravitational dual, conformal ordered phases are represented by black branes in a Poincare patch of asymptotically $AdS_{d+1}$ bulk with a nonlinear scalar hair. This nonlinearity makes the construction rather subtle. We present a procedure where a scalar potential of the top-down holographic model is deformed with a parameter $b$, akin to the $b$-parameter in phenomenological models of [2], such that at $b = 1$ we restore the exact SUGRA scalar gravitational potential. In the limit $b \to +\infty$ the 'hairy' black branes can be constructed perturbatively as small deformations of the AdS-Schwarzschild black brane. As in [2], we find that the holographic conformal ordered phases exist for $b \in (b_{\text{crit}}, +\infty)$. In the limit $b \to b_{\text{crit}} + 0$ the order parameter diverges [2]. In all cases discussed, we find that $b_{\text{crit}} \geq 1$. Thus, it appears that holographic models realizing ordered conformal phases are in the String theory swampland [4].

We consider two specific holographic models in $\mathcal{N} = 8$ gauged supergravity with the bulk scalars dual to bosonic/fermionic bilinears in the M2 brane theory:

- **Model A**: is a consistent truncation of four-dimensional $\mathcal{N} = 8$ gauged supergravity to $U(1)^4$ invariant sector [3,6]. Here we have for the scalar potential

$$\mathcal{P}_A = -2g^2 \left[ 1 + 2 \cosh^2 \phi \right].$$  

Notice that $\mathcal{P}_A$ is unbounded from below.

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1See [3] for related nonconformal models.
2The method is very general and can be applied to any holographic model.
• Model B: is a consistent truncation of four-dimensional $\mathcal{N} = 8$ gauged supergravity to $SO(3) \times SO(3)$ invariant sector \cite{7}. Here we have for the scalar potential

$$\mathcal{P}_B = -\frac{g^2}{2} \left[ 3 + 10 \cosh^2 \phi - \cosh^4 \phi \right].$$

Unlike $\mathcal{P}_A$, the scalar potential $\mathcal{P}_B$ is bounded from below.

We take the Lagrangian of the scalars coupled to gravity to be:

$$\mathcal{L} = \frac{1}{2} R - (\partial \phi)^2 - \mathcal{P}.$$  \hspace{1cm} (4)

In both cases $g$ is a SUGRA coupling constant; we set $2g^2 = 1$, which sets the radius of the asymptotic $AdS_4$ geometry to one. Note that

$$\mathcal{P}_A = -3 - 2\phi^2 - \frac{2}{3} \phi^4 - O(\phi^6),$$  \hspace{1cm} (5)

$$\mathcal{P}_B = -3 - 2\phi^2 - \frac{5}{12} \phi^4 + O(\phi^6),$$

thus, the bulk scalar $\phi$ is dual to an operator of $\Delta = 1$ or $\Delta = 2$ — we will discuss both allowed quantizations. Both $\mathcal{P}_A$ and $\mathcal{P}_B$ are invariant under $\phi \leftrightarrow -\phi$; so a nonzero expectation value of $O_\Delta$ signals a spontaneous breaking$^4$ of this $\mathbb{Z}_2$ symmetry.

To study the thermal phases of (4), we take the following background ansatz

$$ds^2_4 = -c_1^2 dt^2 + c_2^2 \left[ dx_1^2 + dx_2^2 \right] + c_3^2 dr^2,$$  \hspace{1cm} (6)

where all the metric warp factors $c_i$ as well as the bulk scalar $\phi$ are functions of the radial coordinate $r$,

$$r \in \left[ r_0, +\infty \right),$$  \hspace{1cm} (7)

where $r_0$ is a location of a regular Schwarzschild horizon, and $r \to +\infty$ is the asymptotic $AdS_4$ boundary. Introducing a new radial coordinate

$$x \equiv \frac{r_0}{r}, \hspace{1cm} x \in (0, 1],$$  \hspace{1cm} (8)

and denoting

$$c_1 = r \left( 1 - \frac{r_0^3}{r^3} \right)^{1/2} a_1, \hspace{1cm} c_2 = r, \hspace{1cm} c_3 = \frac{1}{r} \left( 1 - \frac{r_0^3}{r^3} \right)^{-1/2} a_3,$$  \hspace{1cm} (9)

$^3$This fixes the typo in eq.(2.11) in \cite{7}. I would like to thank Nikolay Bobev for valuable correspondence.

$^4$We will always keep nonzero only one of the two normalizable modes of $\phi$ near the boundary.
we obtain the following system of ODEs (in a radial coordinate $x$, $\varphi = \frac{d}{dx}$, $\partial = \frac{\delta}{\delta \varphi}$):

\begin{align*}
0 &= a_1' + \frac{3 a_1}{2x(x^3 - 1)} + \frac{1}{2} x a_1 (\varphi')^2 + \frac{a_2 a_1}{2x(x^3 - 1)} \mathcal{P}, \\
0 &= a_3' + \frac{1}{2} x a_3 (\varphi')^2 - \frac{3 a_3}{2x(x^3 - 1)} - \frac{a_3^2}{2x(x^3 - 1)} \mathcal{P}, \\
0 &= \varphi'' + \left( \frac{1}{x} + \frac{a_2^2 \mathcal{P}}{x(1-x^2)} \right) \varphi' + \frac{a_2^2}{2(x^3 - 1)x^2} \partial \mathcal{P}.
\end{align*}

(10) \quad (11) \quad (12)

There is always a trivial solution to (10)-(12), i.e.,

\begin{align*}
a_1 &\equiv 1, \quad a_3 \equiv 1, \quad \varphi \equiv 0,
\end{align*}

(13) corresponding to a disordered, $\mathbb{Z}_2$ symmetric phase,

\begin{align*}
\mathcal{E} &= \frac{c}{96} (\pi T)^2, \quad \mathcal{O}_\Delta = 0,
\end{align*}

(14) where $c$ is a central charge of the boundary $CFT_3$.

We are after the nontrivial solution to (10)-(12), with $\varphi(x) \neq 0$ — in this phase the parity symmetry is spontaneously broken. Rather than to study ordered phases with a scalar potential $\mathcal{P}$, we introduce a deformed scalar potential $\mathcal{P}^b$, for which the existence of the broken phases, at least in the limit $b \to +\infty$, is physically well motivated:

\begin{align*}
\mathcal{P}^b[\varphi] &\equiv -3 - 2\varphi^2 + b \left( \mathcal{P}[\varphi] + 3 + 2\varphi^2 \right), \quad \mathcal{P}^{b=1}[\varphi] \equiv \mathcal{P}[\varphi].
\end{align*}

(15) Indeed, as $b \to \infty$ there is a systematic power series solution of (10)-(12):

\begin{align*}
\varphi &= \frac{1}{b^{1/2}} \varphi_1 + \frac{1}{b^{3/2}} \varphi_3 + \mathcal{O}(b^{-3/2}), \quad a_1 = 1 + \frac{1}{b} a_{1,1} + \frac{1}{b^2} a_{1,2} + \mathcal{O}(b^{-2}), \\
a_3 &= 1 + \frac{1}{b} a_{3,1} + \frac{1}{b^2} a_{3,2} + \mathcal{O}(b^{-2}).
\end{align*}

(16) Notice that to leading order, e.g.,

\begin{align*}
\mathcal{P}_A^b &= -3 - 2\varphi^2 - b \left( \frac{2}{3} \varphi^4 + b \mathcal{O}(\varphi^6) \right) = -3 + m_{eff}^2 \varphi^2 + \mathcal{O}(b^{-3}), \\
\mathcal{O}(b^{-1})
\end{align*}

(17) with

\begin{align*}
m_{eff}^2 &= \Delta(\Delta - 3) - b \left( \frac{2}{3} \varphi^2 \right) \mathcal{O}(b^{-1}),
\end{align*}

(18)
That is, to leading order in the limit $b \to \infty$, the effecting mass of the bulk scalar $\phi$ is shifted due to a nonlinear negative quartic term in $\mathcal{P}^b$. Potentially, when evaluated at the AdS-Schwarzschild horizon\textsuperscript{4}, it can dip below the Breitenlohner-Freedman bound, triggering the instability and leading to 'hair'.

To check whether or not the hair indeed arises, one needs to perform an explicit computation. We focus\textsuperscript{6} on the model with a scalar potential $\mathcal{P}^b_A$. To leading order as $b \to \infty$ we find

$$0 = \phi''_1 + \frac{x^3 + 2}{x(x^3 - 1)} \frac{\phi'_1}{x} - \frac{2\phi_1(2\phi_1^2 + 3)}{3(x^3 - 1)x^2}.$$ (19)

If we identify the bulk scalar as dual to the operator of $\Delta = 1$, we must impose the boundary asymptotics on $\phi_1$ as

$$\text{UV} : \quad \phi_1 = f_{1,1} x + \mathcal{O}(x^3), \quad x \to 0,$$
$$\text{IR} : \quad \phi_1 = f_{1,0}^b + \mathcal{O}((1 - x)^1), \quad (1 - x) \to 0.$$ (20)

A nontrivial solution of (19) with (20) can be used to construct perturbative hair, a $\mathbb{Z}_2$ broken phase in the limit $b \to +\infty$. In all models we considered, provided the quartic term in the scalar potential $\mathcal{P}$ is negative, such a nontrivial solution indeed exists\textsuperscript{7}. We find

$$f_{1,1} = \pm 0.6369, \quad f_{1,0}^b = \pm 1.10602.$$ (21)

Once the solution for $\phi_1$ is constructed, it will source the linear equations for $a_{1,1}$ and $a_{3,1}$. The conformal black brane with a perturbative hair thus constructed can be numerically extended to finite values of $b$, realizing the construction of the holographic dual for a conformal order in the $b$-deformed model. Details of the construction follow\textsuperscript{2} and will be omitted here. We parameterize the ordered conformal equation of state as

$$\mathcal{E} = \frac{c}{96} (\pi T)^2 \times \kappa_A^{\Delta}(b), \quad \frac{\mathcal{O}_{A}^{\Delta A}}{T^\Delta} \equiv \mathcal{O}_{A}^{\Delta A}(b).$$ (22)

In fig. 1 we present results for $\kappa_A^{\Delta=1}$ as a function of $b$ for a holographic model with a scalar potential $\mathcal{P}^b_A$. The ordered phase exists only for $b > 1$, and as $b \to 1$, the order parameter for the spontaneous symmetry breaking diverges. In practice we constructed conformal order with $b \geq 1.01$; extrapolation of the data for the order parameter predicts its divergence at $b \approx 0.999997$. We take it as a strong indication

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\textsuperscript{5}To leading order in $b$ the bulk geometry is not modified.

\textsuperscript{6}The other cases can be considered verbatim.

\textsuperscript{7}In fact, there is a whole ‘tower’ — a discrete spectrum — of solutions.
Figure 1: Holographic conformal order in Model A with a $b$-deformed scalar potential $\mathcal{P}_{A}^b$ and the bulk scalar quantization $\Delta = 1$. Parameter $\kappa_{A}^{\Delta=1}$ defines the equation of state, see (22) (left panel). Right panel: the order parameter $\hat{O}_{1,A}$ for the spontaneous $\mathbb{Z}_2$ symmetry breaking, see (22). Vertical dashed red line indicates extrapolated value of $b_{\text{crit}}^{A,\Delta=1} = 1$ at which the order parameter diverges. The ordered phase does not exist for $b \leq b_{\text{crit}}^{A,\Delta=1}$.

that the critical value of $b$, below which the ordered phase does not exist, is

$$b_{\text{crit}}^{A,\Delta=1} = 1.$$  \hspace{1cm} (23)

Remarkable, the critical value of $b$ is just what is needed for the model with $\mathcal{P}_{A}^b$ to become a top-down holography. We conclude that Model A with a bulk scalar field quantization $\Delta = 1$ does not admit a thermal phase with spontaneously broken $\mathbb{Z}_2$ symmetry.

As shown in fig. 2, Model A with a bulk scalar field quantization $\Delta = 2$ does not admit a thermal phase with spontaneously broken $\mathbb{Z}_2$ symmetry either, since $b_{\text{crit}}^{A,\Delta=2} > 1$.

In figs. 3-4 we report the corresponding results for Model B, for $\Delta = 1$ and $\Delta = 2$ quantizations of the bulk scalar field. Once again, since $b_{\text{crit}}^{B,\Delta=1} > 1$ and $b_{\text{crit}}^{B,\Delta=2} > 1$, there are no conformal ordered phases in Model B.

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Figure 2: Holographic conformal order in Model A with a $b$-deformed scalar potential $\mathcal{P}_A^b$ and the bulk scalar quantization $\Delta = 2$. Parameter $\kappa_{A}^{\Delta=2}$ defines the equation of state (left panel). Right panel: the order parameter $\hat{O}_{2,A}$ for the spontaneous $\mathbb{Z}_2$ symmetry breaking. Vertical dashed orange line indicates extrapolated value of $b_{\text{crit}}^{A,\Delta=2} = 3.4081$ at which the order parameter diverges. The ordered phase does not exist for $b \leq b_{\text{crit}}^{A,\Delta=2}$.

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Figure 3: Holographic conformal order in Model B with a $b$-deformed scalar potential $\mathcal{P}^b_B$ and the bulk scalar quantization $\Delta = 1$. Parameter $\kappa^\Delta=1_B$ defines the equation of state (left panel). Right panel: the order parameter $\mathring{O}_{1,B}$ for the spontaneous $\mathbb{Z}_2$ symmetry breaking. Vertical dashed red line indicates extrapolated value of $b^{B,\Delta=1}_{\text{crit}} = 2.13331$ at which the order parameter diverges. The ordered phase does not exist for $b \leq b^{B,\Delta=1}_{\text{crit}}$.

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Figure 4: Holographic conformal order in Model B with a $b$-deformed scalar potential $\mathcal{P}_B^b$ and the bulk scalar quantization $\Delta = 2$. Parameter $\kappa^{\Delta=2}_B$ defines the equation of state (left panel). Right panel: the order parameter $\hat{O}_{2,B}^2$ for the spontaneous $\mathbb{Z}_2$ symmetry breaking. Vertical dashed orange line indicates extrapolated value of $b^{B,\Delta=2}_{\text{crit}} = 11.933$ at which the order parameter diverges. The ordered phase does not exist for $b \leq b^{B,\Delta=2}_{\text{crit}}$. 