STANLEY–REISNER RINGS WITH LARGE MULTICIPICITIES ARE COHEN–MACAULAY

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ABSTRACT. We prove that certain class of Stanley–Reisner rings having sufficiently large multiplicities are Cohen–Macaulay using Alexander duality.

1. Introduction

Throughout this paper, let $S = k[X_1, \ldots, X_n]$ be a homogeneous polynomial ring over a field $k$ with deg $X_i = 1$. For a simplicial complex $\Delta$ on vertex set $[n] = \{1, \ldots, n\}$ (note that $\{i\} \in \Delta$ for all $i$), $k[\Delta] = k[X_1, \ldots, X_n]/I_\Delta$ is called the Stanley–Reisner ring of $\Delta$, where $I_\Delta$ is an ideal generated by all square-free monomials $X_{i_1} \cdots X_{i_p}$ such that $\{i_1, \ldots, i_p\} \notin \Delta$. The ring $A = k[\Delta]$ is a homogeneous reduced ring with the unique homogeneous maximal ideal $m = (X_1, \ldots, X_n)k[\Delta]$ and the Krull dimension $d = \dim \Delta + 1$. Let $e(A)$ denote the multiplicity $e_0(mA_m, A_m)$ of $A$, which is equal to the number of facets (i.e., maximal faces) $F$ of $\Delta$ with $\dim F = d - 1$. Also, we frequently call it the multiplicity of $\Delta$. Note that $\Delta$ is called pure if all facets of $\Delta$ have the same dimension. See [1, 9] for more details.

Take a graded minimal free resolution of a homogeneous $k$-algebra $A = S/I$ over $S$:

$$0 \rightarrow \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{1,j}(A)} \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_2} \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{p,j}(A)} \xrightarrow{\varphi_p} S \rightarrow A \rightarrow 0.$$ 

Then the initial degree $\text{indeg} A$ (resp. the relation type $\text{rt}(A)$) of $A$ is defined by $\text{indeg} A = \min\{j \in \mathbb{Z} : \beta_{1,j}(A) \neq 0\}$ (resp. $\text{rt}(A) = \max\{j \in \mathbb{Z} : \beta_{1,j}(A) \neq 0\}$).

Also, $\text{reg} A = \max\{j - i \in \mathbb{Z} : \beta_{i,j}(A) \neq 0\}$ is called the Castelnuovo–Mumford regularity of $A$. It is easy to see that $\text{reg} A \geq \text{indeg} A - 1$, and $A$ has linear resolution if equality holds.

The main purpose of this paper is to prove the following theorems:

**Theorem 2.1** Let $A = k[\Delta]$ be a Stanley–Reisner ring of Krull dimension $d \geq 2$. Put $\text{codim} A = c$. If $e(A) \geq \binom{n}{c} - c$, then $A$ is Cohen–Macaulay.

**Theorem 3.1** Let $A = k[\Delta]$ be a Stanley–Reisner ring of Krull dimension $d \geq 2$. Put $\text{codim} A = c$. Suppose that $\Delta$ is pure (i.e., $A$ is equidimensional). If $e(A) \geq \binom{n}{c} - 2c + 1$, then $A$ is Cohen–Macaulay.
It is easy to prove the above theorems in the case of \( d = 2 \). When \( d = 2 \), \( A \) is Cohen–Macaulay if and only if \( \Delta \) is connected. In fact, a disconnected graph has at most \( \binom{n-2}{2} = \binom{n}{2} - (n - 2) - 1 \) edges. This shows that Theorem 2.1 is true in this case. Similarly, a disconnected graph without an isolated point has at most \( \binom{n-2}{2} + 1 = \binom{n}{2} - 2(n - 2) \) edges. Indeed, such a graph is contained in a disjoint union of an \((n - i)\)-complete graph and an \(i\)-complete graph for some \( 2 \leq i \leq n - 2 \). When \( i = 2 \), the number of edges of the above union is just \( \binom{n-2}{2} + 1 \). Thus we also get Theorem 3.1 in this case.

The case \( \text{indeg} \ A = d \) and \( c \geq 2 \) is essential in the above two theorems. In order to prove Theorems 2.1 and 3.1 in this case, we consider their Alexander dual versions:

**Theorem 2.7.** Let \( A = k[\Delta] \) be a Stanley–Reisner ring of Krull dimension \( d \geq 2 \). Suppose that \( \text{indeg} \ A = d \). If \( e(A) \leq d \), then \( A \) has \( d \)-linear resolution. In particular, \( \text{rt} (A) = d \).

**Theorem 3.3.** Let \( A = k[\Delta] \) be a Stanley–Reisner ring of Krull dimension \( d \geq 2 \). Suppose that \( \text{indeg} \ A = \text{rt} (A) = d \). If \( e(A) \leq 2d - 1 \), then \( A \) has \( d \)-linear resolution. In particular, \( a(A) < 0 \).

For a Stanley–Reisner ring \( A \) with \( \text{indeg} \ A = \dim A = d \), it has \( d \)-linear resolution if and only if \( a(R) < 0 \). Thus the assertion of Theorem 3.3 could be seen as an analogy of the following: Let \( R \) be a homogeneous integral domain over an algebraically closed field of characteristic 0. If \( e(R) \leq 2 \dim R - 1 \) and \( \text{codim} R \geq 2 \), then \( a(R) < 0 \).

In the last section, we will provide several examples related to the above results.
Lemma 2.3. Under the above notation, the following conditions are equivalent:

1. \( \text{indeg} A = d + 1 \).
2. \( e(A) = \binom{n}{d} \).
3. \( I_\Delta = (X_{i_1} \cdots X_{i_{d+1}} : 1 \leq i_1 < \cdots < i_{d+1} \leq n) \).
4. \( A \) has \((d + 1)\)-linear resolution.

When this is the case, \( A \) is Cohen–Macaulay with \( \text{rt} (A) = d + 1 \).

Proof. See, e.g., [11, Proposition 1.2]. \( \square \)

Therefore we may assume that \( \text{indeg} A = d \) to prove Theorem 2.1.

Lemma 2.4. Suppose \( n = d + 1 \). If \( e(A) \geq d \), then \( A \) is a hypersurface.

Proof. Suppose that \( A \) is not a hypersurface. Then we can write

\( I_\Delta = X_{i_1} \cdots X_{i_p} J \)

for some monomial ideal \( J(\not\in R) \) with height \( J \geq 2 \) since height \( I_\Delta = 1 \). In particular, \( A \) is not Cohen–Macaulay. Thus \( \text{indeg} A \leq d \) by Lemma 2.3. Then \( e(A) = p \leq d - 1 \). This contradicts the assumption. \( \square \)

Thus we may also assume that \( c = \text{codim} A \geq 2 \). then let \( \Delta^* \) be the Alexander dual of \( \Delta \):

\( \Delta^* = \{ F \in 2^V : V \setminus F \not\in \Delta \} \).

Then \( \Delta^* \) is a simplicial complex on the same vertex set \( V \) of \( \Delta \) for which the following properties are satisfied:

Proposition 2.5. Under the above notation, we have

1. \( \text{indeg} k[\Delta^*] + \dim k[\Delta] = n. \)
2. \( \text{rt} (k[\Delta^*]) = \text{bight } I_\Delta, \) where

\( \text{bight } I = \max \{ \text{height } p : p \text{ is a minimal prime divisor of } I \} \).

In particular, \( \Delta \) is pure if and only if \( \text{rt} (k[\Delta^*]) = \text{indeg} k[\Delta^*] \).
3. \( \beta_{0,c}(I_{\Delta^*}) = e(k[\Delta]), \) where \( q^* = \text{indeg} k[\Delta^*] \).
4. \( (\Delta^*)^* = \Delta. \)

Also, the following theorem is fundamental. See [3] for more details.

Theorem 2.6 (Eagon–Reiner [3]). \( k[\Delta] \) is Cohen–Macaulay if and only if \( k[\Delta^*] \) has linear resolution.

We want to reduce Theorem 2.6 to its Alexander dual version. Let \( \Delta^* \) be the Alexander dual of \( \Delta \). Then \( \text{indeg} k[\Delta^*] = n - \dim k[\Delta] = c \) and \( \dim k[\Delta^*] = n - \text{indeg} k[\Delta] = n - d = c. \) Also, since \( \text{indeg} k[\Delta^*] = \dim k[\Delta] = c \), we have

\( e(k[\Delta^*]) = \binom{n}{c} - \beta_{0,c}(I_{\Delta^*}) = \binom{n}{c} - e(A) \leq c \)

Therefore, it is enough to prove the following theorem.

Theorem 2.7 (Alexander dual version of Theorem 2.1). Let \( A = k[\Delta] \) be a Stanley–Reisner ring of Krull dimension \( d \geq 2 \). Suppose that \( \text{indeg} A = d \). If \( e(A) \leq d \), then \( A \) has \( d \)-linear resolution. In particular, \( \text{rt} (A) = d. \)
Proof. (1) Put \( a(A) = \sup \{ p \in \mathbb{Z} : [H^d_m(A)]_p \neq 0 \} \), the \( a \)-invariant of \( A \). From the assumption we obtain that

\[
a(A) + d \leq e(A) - 1 \leq d - 1,
\]

where the first inequality follows from e.g. [3, Lemma 3.1]. Hence \( a(A) < 0 \). On the other hand, we have that \( [H^i_m(A)]_j = 0 \) for all \( i \) and \( j \geq 1 \) since \( A \) is a Stanley–Reisner ring. Then

\[
\text{reg } A = \inf \{ p \in \mathbb{Z} : [H^i_m(A)]_j = 0 \text{ for all } i + j > p \} \leq d - 1 = \text{indeg } A - 1.
\]

This means that \( A \) has \( d \)-linear resolution, as required. \( \square \)

Now let us discuss a generalization of Theorem 2.7. Let \( A = S/I \) be an arbitrary homogeneous reduced \( k \)-algebra over a field \( k \) of characteristic \( p > 0 \). The ring \( A \) is called \( F \)-pure if the Frobenius map \( F: A \to A \ (a \mapsto a^p) \) is pure. It is known that a Stanley–Reisner ring is \( F \)-pure, and that if \( A \) is \( F \)-pure then \( [H^i_m(A)]_j = 0 \) for all \( j \geq 1 \). Thus the proof of Theorem 2.7 involves that of the following proposition.

**Proposition 2.8.** Let \( A = S/I \) be a homogeneous \( F \)-pure \( k \)-algebra. Put \( \dim A = \text{indeg } A = d \geq 2 \). If \( e(A) \leq d \), then \( A \) has \( d \)-linear resolution. In particular, \( \text{rt } (A) = d \) and \( a(A) < 0 \).

3. Complexes \( \Delta \) with \( e(k[\Delta]) \geq \binom{n}{c} - 2c + 1 \)

We use the same notation as in the previous section. For a face \( G \) in \( \Delta \) and \( v \in V \), we put

\[
\Delta_{\setminus \{v\}} = \{ F \in \Delta : v \notin F \},
\]

\[
\text{star}_\Delta G = \{ F \in \Delta : F \cup G \in \Delta \},
\]

\[
\text{link}_\Delta G = \{ F \in \Delta : F \cup G \in \Delta, F \cap G = \emptyset \}.
\]

The main purpose of this section is to prove the following theorem.

**Theorem 3.1.** Let \( A = k[\Delta] \) be a Stanley–Reisner ring of Krull dimension \( d \geq 2 \). Put \( c = \text{codim } A \). Suppose that \( \Delta \) is pure. If \( e(A) \geq \binom{n}{c} - 2c + 1 \), then \( A \) is Cohen–Macaulay.

Now suppose that \( c = 1 \) (resp. \( \text{indeg } A \geq d + 1 \)). Then the assertion follows from Lemma 2.4 (resp. Lemma 2.3). Thus we may assume that \( c \geq 2 \) and \( q = \text{indeg } A \leq d \). The following lemma corresponds to Lemma 2.2.

**Lemma 3.2.** If \( e(k[\Delta]) \geq \binom{n}{c} - 2c + 1 \), then \( \text{indeg } k[\Delta] \geq d - 1 \), i.e., (1) \( \text{indeg } k[\Delta] = d \) or (2) \( \text{indeg } k[\Delta] = d - 1 \).

**Proof.** Suppose that \( \text{indeg } k[\Delta] < d - 1 \). Take a squarefree monomial \( M \in I_{\Delta} \) with \( \deg M = d - 2 \). Then there are \( \binom{n-d+2}{2} \) squarefree monomials in degree \( d \) in \( I_{\Delta} \). Note \( \binom{n-d+2}{2} = \binom{c+2}{2} \geq 2c \). Hence

\[
e(k[\Delta]) \leq \binom{n}{c} - 2c.
\]

This contradicts the assumption. \( \square \)

First, we consider the Alexander dual version of Theorem 3.1 in the case of \( \text{indeg } k[\Delta] = d \). Namely, we will prove the following theorem.
Theorem 3.3 (Alexander dual version of Theorem 3.1). Case (1)). Let $A = k[\Delta]$ be a Stanley–Reisner ring of Krull dimension $d \geq 2$. Suppose that $\indeg A = \reg (A) = d$. If $e(A) \leq 2d - 1$, then $A$ has $d$-linear resolution. In particular, $a(A) < 0$.

The proof of the above theorem can be reduced to that of the following theorem, which is a key result in this paper.

Theorem 3.4. Let $A = k[\Delta]$ be a Stanley–Reisner ring of Krull dimension $d \geq 2$. Suppose that $\reg (A) \leq d$. If $e(A) \leq 2d - 1$, then $\reg A \leq d - 1$, equivalently, $\tilde{H}_{d-1}(\Delta) = 0$.

Proof. Put $e = e(A)$. Let $\Delta'$ be the subcomplex that is spanned by all facets of dimension $d - 1$. Replacing $\Delta$ with $\Delta'$, we may assume that $\Delta$ is pure.

We use induction on $d = \dim A \geq 2$. First suppose $d = 2$. The assumption shows that $\Delta$ does not contain the boundary complex of a triangle. Hence $\tilde{H}_1(\Delta) = 0$ since $e(A) \leq 3$.

Next suppose that $d \geq 3$, and that the assertion holds for any complex the dimension of which is less than $d - 1$. Assume that $\tilde{H}_{d-1}(\Delta) \neq 0$. Take one $\Delta$ whose multiplicity is minimal among the multiplicities of those complexes. Then $\Delta$ does not contain any free face (see [7]). That is, every face that is not a facet is contained in at least two facets. Indeed, suppose that $\Delta$ contains a free face (say, $G$) and put $\Delta' = \Delta \setminus \{ F \in \Delta : F \supseteq G \}$. Then since $G$ is a free face of $\Delta$, $\Delta'$ is homotopy equivalent to $\Delta$ and $e(k[\Delta']) = e(k[\Delta]) - 1$. In particular, $\tilde{H}_{d-1}(\Delta') \approx \tilde{H}_{d-1}(\Delta) \neq 0$. This contradicts the minimality of $e(k[\Delta])$.

First consider the case of $\reg (A) = d$. Take a generator $X_{i_1} \cdots X_{i_d}$ of $I_\Delta$. For every $j = 1, \ldots, d$, each $G_j = \{ i_1, \ldots, \widehat{i_j}, \ldots, i_d \}$ is contained in at least two facets as mentioned above. Then $e(A) \geq 2d$ since those facets are different from each other. This is a contradiction.

Next we consider the case of $\reg (A) < d$. Take a Mayer–Vietoris sequence with respect to $\Delta = V \setminus \{ n \} \cup \text{star}_\Delta \{ n \}$ as follows:

$$
\tilde{H}_{d-1}(\Delta \setminus \{ n \}) \oplus \tilde{H}_{d-1}(\text{star}_\Delta \{ n \}) \to \tilde{H}_{d-1}(\Delta) \to \tilde{H}_{d-2}(\text{link}_\Delta \{ n \}).
$$

The minimality of $e(k[\Delta \setminus \{ n \}])$ yields that $\tilde{H}_{d-1}(\Delta \setminus \{ n \}) = 0$ since $e(k[\Delta \setminus \{ n \}]) < e(k[\Delta])$. On the other hand, it is known that $\tilde{H}_i(\text{star}_\Delta \{ n \}) = 0$ for all $i$. Hence $\tilde{H}_{d-1}(\Delta) \to \tilde{H}_{d-2}(\text{link}_\Delta \{ n \})$. In particular, $\tilde{H}_{d-2}(\text{link}_\Delta \{ n \}) \neq 0$.

Set $\Delta' = \text{link}_\Delta \{ n \}$. Then $\Delta'$ is a complex on $V \setminus \{ n \}$ such that $\dim k[\Delta'] = d - 1$ and $\reg (k[\Delta']) \leq \reg (k[\Delta]) \leq d - 1$. In order to apply the induction hypothesis to $\Delta'$, we want to see that $e(k[\Delta']) \leq 2d - 3$. In order to do that, we consider $e(k[\Delta \setminus \{ n \}])$. As $\Delta \neq \text{star}_\Delta \{ n \}$, one can take $F = \{ i_1, \ldots, i_p, n \} \notin \Delta$ for some $p \leq d - 2$ such that $X_{i_1} \cdots X_{i_p} X_n$ is a generator of $I_\Delta$. Then $G := \{ i_1, \ldots, i_p \} \in \Delta$, but it is not a facet of $\Delta$. Thus it is contained in at least two facets of $\Delta$, each of which does not contain $n$. Hence $e(k[\Delta \setminus \{ n \}]) \geq 2$. Thus we get

$$
eq e(k[\text{star}_\Delta \{ n \}]) = e(k[\Delta]) - e(k[\Delta \setminus \{ n \}]) \leq 2d - 3.$$

By induction hypothesis, we have $\tilde{H}_{d-2}(\text{link}_\Delta \{ n \}) = 0$. This is a contradiction. □

Next, we consider the Alexander dual version of Theorem 3.1 in the case of $\indeg k[\Delta] = d - 1$. Namely, we must prove the following proposition.
Proposition 3.5 (Alexander dual version of Theorem 3.1, Case (2)). Let \( A = k[\Delta] \) be a Stanley–Reisner ring of Krull dimension \( d \geq 2 \). Suppose that \( \text{indeg} \ A = \text{rt} \ (A) = d - 1 \). If \( \mu(I_{\Delta}) \geq \left( \frac{n}{d - 1} \right) - 2d + 3 \), then \( A \) has \((d - 1)\)-linear resolution with \( e(A) = 1 \).

Proof. First we show that \( e(A) = 1 \). Now suppose that \( e(A) \geq 2 \). Then there exist at least two facets \( F_1 \) and \( F_2 \) with \( \#(F_1) = \#(F_2) = d \). This implies that \( f_{d-2}(\Delta) \geq 2d - 1 \). However, by the assumption, we have

\[
f_{d-2}(\Delta) = \left( \frac{n}{d - 1} \right) - \beta_{0,d-1}(I_{\Delta}) = \left( \frac{n}{d - 1} \right) - \mu(I_{\Delta}) \leq 2d - 3.
\]

This is a contradiction. Hence we get \( e(A) = 1 \).

In order to prove that \( A \) has \((d - 1)\)-linear resolution, it is enough to show that \( \beta_{i,j}(A) = 0 \) for all \( i \geq c \) and \( j \geq i + d - 1 \) by [5, Theorem 5.2]. Also, it suffices to show that \( H_{d-1}(\Delta) = H_{d-2}(\Delta) = H_{d-2}(\Delta_W) = 0 \) for all subsets \( W \subset V \) with \( \#(W) = n - 1 \) by virtue of Hochster’s formula on the Betti numbers:

\[
\beta_{i,j}(A) = \sum_{W \subset V, \#(W)=j} \dim_k H_{j-i-1}(\Delta_W; k).
\]

Claim 1. \( H_{d-1}(\Delta) = H_{d-2}(\Delta) = 0 \).

Since \( \text{rt} \ (A) \leq d - 1 \leq d \) and \( e(A) = 1 \leq 2d - 1 \), we have \( H_{d-1}(\Delta) = 0 \) by Theorem 3.4. Now let \( F = \{1, 2, \ldots, d\} \) be the unique facet with \( \#(F) = d \). Consider a simplicial subcomplex \( \Delta' := \Delta \setminus \{F, G\} \) where \( G = \{1, 2, \ldots, d - 1\} \). Then \( \dim k[\Delta'] = d - 1 \) and \( e(k[\Delta']) \leq 2d - 4 \leq (2d - 1) - 1 \). Also, since \( \text{rt} \ (k[\Delta']) \leq \text{rt} \ (k[\Delta]) \leq d - 1 \), applying Theorem 3.4 to \( \Delta' \), we obtain that \( H_{d-2}(\Delta) \cong H_{d-2}(\Delta') = 0 \), as required.

Claim 2. \( H_{d-2}(\Delta_W) = 0 \) for all subsets \( W \subset V \) with \( \#(W) = n - 1 \).

Let \( W \) be a subset of \( V \) such that \( \#(W) = n - 1 \). Put \( \{a\} = V \setminus W \). If \( a \) is not contained in \( F \), then \( H_{d-2}(\Delta_W) = 0 \) by the similar argument as in the proof of the previous claim. So we may assume that \( a \in F \). Then \( \dim k[\Delta_W] = d - 1 \) and \( e(k[\Delta_W]) \leq (d - 3) + 1 = d - 2 \leq (d - 1) - 1 \). Also, since \( \text{rt} \ (k[\Delta_W]) \leq d - 1 \), we have \( H_{d-2}(\Delta_W) = 0 \) by Theorem 3.4 again.

Hence \( k[\Delta] \) has \((d - 1)\)-linear resolution, as required.

Example 3.6. Let \( \rho, d \) be an integers with \( 0 \leq \rho \leq d - 3 \). Let \( \Delta \) be a simplicial complex on \( V = [n] \) spanned by \( F = \{1, 2, \ldots, d\} \), any distinct \( \rho \) elements from \( \binom{[n]}{d-1} \setminus \binom{[d-1]}{d-1} \) and all elements of \( \binom{[n]}{d-2} \). Then \( \dim k[\Delta] = d \), \( \text{indeg} k[\Delta] = \text{rt} (k[\Delta]) = d - 1 \). Also, we have

\[
\mu(I_{\Delta}) = \beta_{0,d-1}(I_{\Delta}) = \left( \frac{n}{d - 1} \right) - \rho - d \geq \left( \frac{n}{d - 1} \right) - 2d + 3.
\]

Hence \( \Delta \) satisfies the assumption of the above proposition.

On the other hand, we have no results for F-pure \( k \)-algebras corresponding to Theorem 3.8. But we remark the following.
Remark 3.7. As mentioned in the introduction, if \( A \) is a homogeneous integral domain over an algebraically closed field of char \( k = 0 \) with \( \text{codim} \, A \geq 2 \) and \( e(A) \leq 2d - 1 \) then one has \( a(A) < 0 \). In fact, it is known that an inequality

\[
a(A) + d \leq \left\lceil \frac{e(A) - 1}{\text{codim} \, A} \right\rceil
\]

holds; see e.g., the remark after Theorem 3.2 in [6]. Moreover, Professor Chikashi Miyazaki told us that this inequality is also true in positive characteristic.

Question 3.8. Let \( A = k[A_1] \) be a homogeneous \( F \)-pure, equidimensional \( k \)-algebra. Put \( \text{dim} \, A = \text{indeg} \, A = d \geq 2 \). If \( e(A) \leq 2d - 1 \), then does \( a(A) < 0 \) hold?

4. Buchsbaumness

A Stanley–Reisner ring \( A = k[\Delta] \) is Buchsbaum if and only if \( \Delta \) is pure and \( k[\text{link}_A \{ i \}] \) is Cohen–Macaulay for every \( i \in [n] \). As an application of Theorem 3.1, we can provide sufficient conditions for \( k[\Delta] \) to be Buchsbaum.

Proposition 4.1. Let \( A = k[\Delta] \) be a Stanley–Reisner ring of Krull dimension \( d \geq 3 \). Suppose that \( \Delta \) is pure, \( \text{indeg} \, A = d \) and \( e(A) \geq \binom{n}{c} - 2c \). Then

1. \( e(k[\text{link}_A \{ i \}]) \geq \binom{n-1}{c} - 2c \) for all \( i \).
2. If \( \text{height} \, [I_{\Delta}]_d S \geq 2 \), then \( A \) is Buchsbaum.
3. If \( \text{rt} \, (A) = d \), then \( A \) is Buchsbaum.

Proof. We may assume that \( c \geq 2 \), \( e(A) = \binom{n}{c} - 2c \), and that \( \Delta \neq \text{star}_A \{ i \} \) for every \( i \in [n] \). Put \( \Gamma_i = \text{link}_A \{ i \} \) for each \( i \in [n] \).

1. We first show the following claim.

Claim: \( e(A) \leq \binom{n}{d} - \left\{ \binom{n-1}{d-1} - e(k[\Gamma_i]) \right\} \) for every \( i \in [n] \). Also, equality holds if and only if \( i \in F \) holds for all \( F \in \binom{[n]}{d} \setminus \Delta \).

Put \( W_i = \left\{ F \in \binom{[n]}{d} : i \in F \notin \text{star}_A \{ i \} \right\} \). Then \( \#(W_i) \leq \#(\bigcup_{i=1}^{n} W_i) \) implies that

\[
\binom{n-1}{d-1} - e(k[\Gamma_j]) \leq \binom{n}{d} - e(A),
\]
as required. Also, equality holds if and only if \( W_i = \bigcup_{i=1}^{n} W_i \), that is, \( i \in F \) holds for all \( F \in \binom{[n]}{d} \setminus \Delta \).

Now suppose that \( e(k[\Gamma_i]) \leq \binom{n-1}{d-1} - 2c - 1 \) for some \( i \in [n] \). Then the claim implies that \( e(A) \leq \binom{n}{d} - 2c - 1 \), which contradicts the assumption. Thus we get (1).

2. Suppose that \( \text{height} \, [I_{\Delta}]_d S \geq 2 \). Then there is no element \( i \in [n] \) for which \( i \in F \) holds for all \( F \in \binom{[n]}{d} \setminus \Delta \). Thus the claim yields that

\[
\binom{n}{d} - 2c = e(A) \leq \binom{n}{d} - \left[ \binom{n-1}{d-1} - e(k[\Gamma_i]) \right] - 1,
\]
that is, \( e(k[\Gamma_i]) \geq \binom{n-1}{d-1} - 2c + 1 \) for every \( i \in [n] \). Also, we note that \( \Gamma_i \) is pure and \( \text{indeg} \, k[\Gamma_i] = \text{dim} \, k[\Gamma_i] = d - 1 \). Applying Theorem 3.1 to \( k[\Gamma_i] \), we obtain that \( k[\Gamma_i] \) is Cohen–Macaulay. Therefore \( A \) is Buchsbaum since \( \Delta \) is pure.
(3) Now suppose that $A$ is not Buchsbaum. Then since height $[I_\Delta]_d S = 1$, one can take $i \in [n]$ for which $i \in F$ holds for all $F \in \binom{n}{[i]} \setminus \Delta$. We may assume $i = n$. Then $\{1, \ldots, \widehat{i}, \ldots, d + 1\} \in \Delta$ for all $i \in [d + 1]$ because $n - 1 \geq d + 1$. This means that $X_1 \cdots X_{d+1}$ is a generator of $I_\Delta$; thus $\text{rt} (A) = d + 1$. \hfill \Box

5. Examples

Throughout this section, let $c$, $d$ be given integers with $c, d \geq 2$. Set $n = c + d$.

**Example 5.1.** Put $F_{i,j} = \{1, 2, \ldots, \widehat{i}, \ldots, d, j\}$ for each $i = 1, \ldots, d; j = d + 1, \ldots, n$. For a given integers $e$ with $1 \leq e \leq cd$, we choose $e$ faces (say, $F_1, \ldots, F_e$) from $\{F_{i,j} : 1 \leq i \leq d, d + 1 \leq j \leq n\}$, which is a simplicial join of $2^d \setminus \{d\}$ and $c$ points.

Let $\Delta$ be a simplicial complex spanned by $F_1, \ldots, F_e$ and all elements of $\binom{n}{d-1}$. Then $k[\Delta]$ is a $d$-dimensional Stanley–Reisner ring with $\text{indeg} k[\Delta] = \text{rt} (k[\Delta]) = d$ and $e(k[\Delta]) = e$.

In particular, when $e \leq 2d - 1$, $k[\Delta]$ has $d$-linear resolution by Theorem 4.2. Thus, the Alexander dual complexes of them provide examples satisfying hypothesis of Theorem 3.1.

The following example shows that the assumption “$e(A) \leq 2d - 1$” is optimal in Theorem 3.1.

**Example 5.2.** There exists a complex $\Delta$ on $V = [n]$ ($n = d + 2$) for which $k[\Delta]$ does not have $d$-linear resolution with $\dim k[\Delta] = \text{indeg} k[\Delta] = \text{rt} (k[\Delta]) = d$ and $e(k[\Delta]) = 2d$.

In fact, put $n = d + 2$. Let $\Delta_0$ be a complex on $V = [n]$ such that $k[\Delta_0]$ is a complete intersection defined by $(X_1 \cdots X_d, X_{d+1}X_{d+2})$. Also, let $\Delta$ be a complex on $V$ that is spanned by all facets of $\Delta_0$ and all elements of $\binom{n}{d-1}$:

$I_\Delta = (X_1 \cdots X_d) S + (X_{i_1} \cdots X_{i_{d-2}}X_{d+1}X_{d+2} : 1 \leq i_1 < \cdots < i_{d-2} \leq d) S.$

Then $\tilde{H}_{d-1}(k[\Delta]) \cong \tilde{H}_{d-1}(k[\Delta_0]) \neq 0$ since $a(k[\Delta_0]) = 0$. Hence $k[\Delta]$ does not have linear resolution.

**Remark 5.3.** The above example is obtained by considering the case $c = 2, e = 2d$ in Example 4.1.

The next example shows that the assumption “$\text{rt} (A) = d$” is not superfluous in Theorem 3.1.

**Example 5.4.** Suppose that $d + 1 \leq e \leq \binom{n}{d} - 1$. There exists a simplicial complex $\Delta$ on $V = [n]$ such that $\dim k[\Delta] = \text{indeg} k[\Delta] = d$, $\text{rt} (k[\Delta]) = d + 1$ and $e(k[\Delta]) = e$. In particular, $k[\Delta]$ does not have $d$-linear resolution.

In fact, put $F = \binom{n}{d} \setminus \binom{d+1}{d}$. Let $\Delta_0$ be a simplicial complex on $V$ such that

$I_{\Delta_0} = (X_1 \cdots X_dX_{d+1}) S + (X_{i_1} \cdots X_{i_d} : \{i_1, \ldots, i_d\} \in F) S.$

Then $\dim k[\Delta_0] = \text{indeg} k[\Delta_0] = d$, $\text{rt} (k[\Delta_0]) = d + 1$, and $e(k[\Delta_0]) = d + 1$.

For a given integer $e$ which satisfies above condition, one obtains the required simplicial complex by adding any $(e - d - 1)$ distinct $d$-subsets of $2^n$ that is not contained in $\binom{d+1}{d}$ to $\Delta_0$.  

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Remark 5.5. Now let $\Delta$ be a simplicial complex on $V = [n]$. Set $A = k[\Delta]$. Suppose that $\dim A = \text{indeg } A = d \geq 2$. Then one can easily see that $d \leq \text{rt}(A) \leq d+1$; $\text{rt}(A) = d$ (resp. $d+1$) if $1 \leq e(A) \leq d$ (resp. $e(A) = \binom{n}{d}$). So we put

$$f(n, d) = \min \left\{ m \in \mathbb{Z} : \begin{array}{l} \text{rt } k[\Delta] = d+1 \text{ for all } (d-1)\text{-dimensional} \smallskip \text{complexes } \Delta \text{ on } V \text{ with } \text{indeg } k[\Delta] = d \\ \text{and } e(k[\Delta]) \geq m \end{array} \right\}$$

Then $f(n, d) \geq cd+1$ by Example 5.1. From the definition of $f(n, d)$, one can easily see that there exists a simplicial complex $\Delta$ which satisfies $\text{rt}(k[\Delta]) = d$ and $e(k[\Delta]) = e$ for each $e$ with $d \leq e \leq f(n, d) - 1$. On the other hand, by virtue of Example 5.4, one can also find a simplicial complex $\Delta$ which satisfies $\text{rt}(k[\Delta]) = d+1$ and $e(k[\Delta]) = e$ for each $e$ with $d + 1 \leq e \leq f(n, d) - 1$.

It seems to be difficult to determine $f(n, d)$ in general. Let $T(n, p, k)$ be the so-called Turan number. Then we have

$$f(n, d) = \binom{n}{d} - T(n, d+1, d).$$

In particular, we get

\begin{equation}
(5.1) \quad f(n, 2) = \begin{cases} 
\frac{n^2}{4} + 1, & \text{if } n \text{ is even} \\
\frac{n^2}{4} + 1, & \text{otherwise}
\end{cases}
\end{equation}

by Turan’s theorem (e.g., [2, Theorem 7.1.1]) However, no formula is known for $T(n, 4, 3)$; see [3, pp. 1320].

In the rest of this section, we show that the purity of $\Delta$ is very strong condition in Theorem 3.3.

**Proposition 5.6.** Then the following conditions are equivalent:

1. There exists a $d$-dimensional Stanley–Reisner ring $k[\Delta]$ such that $\Delta$ is pure, $\text{indeg } k[\Delta] = d$ and $e(k[\Delta]) = e \leq 2d - 1$.
2. $n = d + 2$, $d \leq 5$ and $(d, e)$ is one of the following pairs:
   - $(2, 2)$, $(2, 3)$, $(3, 4)$, $(3, 5)$, $(4, 6)$, $(4, 7)$, $(5, 9)$.

To prove the proposition, we need the following lemma.

**Lemma 5.7.** Let $A = k[\Delta]$ be a $d$-dimensional Stanley–Reisner ring which is not a hypersurface. Suppose that $\Delta$ is pure and $\text{indeg } A = d \geq 3$. Then there exists a vertex $i \in [n]$ such that $e(k[\Delta \setminus \{i\}]) \geq 2$.

**Proof.** Note that $n \geq d + 2$ by the assumption. Put $e = e(A)$. Suppose that $e(k[\Delta \setminus \{i\}]) = 1$ for all $i$. Then since there exist $(e-1)$ facets containing $i$ for each $i \in [n]$, we have

$$(d+2)(e-1) \leq n(e-1) \leq de;$$

hence $e \leq \frac{d+2}{2}$.

On the other hand, by counting the number of subfacets (i.e., the maximal faces among all faces except facets) of $\Delta$ we get

$$de \geq \binom{n}{d-1}.$$
since \( \text{indeg} A = d \) and \( \Delta \) is pure. It follows from these inequalities that
\[
\frac{d(d+2)}{2} \geq de \geq \binom{n}{d-1} \geq \binom{d+2}{d-1} = \binom{d+2}{3}.
\]
Hence \( d \leq 2 \). This is a contradiction.

Proof of Proposition 5.6. We first show \((1) \implies (2)\). Let \( A = k[\Delta] \) be a \( d \)-dimensional Stanley–Reisner ring for which \( \Delta \) is pure, \( \text{indeg} A = d \), and \( e = e(A) \leq 2d - 1 \). We may assume that \( d \geq 3 \). Since \( \Delta \) is pure, any subfacet is contained in some \( d \)-subset of \( \Delta \). By counting the number of subfacets that contain \( n \), we obtain that
\[
\binom{n-1}{d-2} \leq (e - e(k[\Delta\setminus\{n\}]))(d - 1) \leq (e - 2)(d - 1),
\]
where the last inequality follows from Lemma 5.7.

Now let us see that \( n = d + 2 \). Suppose that \( n \geq d + 3 \). Then we get
\[
\binom{d+2}{4} \leq \binom{n-1}{d-2} \leq (e - 2)(d - 1) \leq (2d-3)(d-1)
\]
by the assumption. This implies that \( d \leq 4 \).

First we consider the case of \( d = 4 \). Then \( n = d + 3 = 7 \), \( e = 2d - 1 = 7 \). Let \( \{F_1, \ldots, F_7\} \) be the set of facets of \( \Delta \). Since \( e(k[\Delta\setminus\{i\}]) = 2 \), we may assume that \( 7 \in F \) if and only if \( 1 \leq i \leq 5 \). Note that \( F_i \) contains only one subfacet that does not contain 7 for each \( 1 \leq i \leq 5 \). On the other hand, one can find at most \( 4 \times 2 \) subfacets as faces of \( F_6 \) or \( F_7 \). Therefore the total number of subfacets that do not contain 7 is at most 13. However the number of all subfacets which do not contain 7 is \( \binom{n-1}{d-1} = 20 \) since \( \text{indeg} A = 4 \). This is a contradiction.

By the similar observation as in the case of \( d = 4 \), one can prove that the case of \( d = 3 \) does not occur. Therefore we conclude that \( n = d + 2 \).

Under the assumption that \( n = d + 2 \), let us determine \((d, e)\). Let \( \Delta^* \) be the Alexander dual of \( \Delta \) and put \( R = k[\Delta^*] \). Then \( R \) is a two-dimensional Stanley–Reisner ring with \( \text{indeg} R = 2 \). Also, \( \text{rt}(R) = \text{indeg} R = 2 \) since \( \Delta \) is pure. Thus by virtue of Turan’s theorem (see Eq. 5.1), we have
\[
\binom{d+2}{2} - e = e(R) \leq f(d+2, 2) - 1 = \left\lfloor \frac{(d+2)^2}{4} \right\rfloor,
\]
where \( \lfloor a \rfloor \) denotes the maximum integer that does not exceed \( a \). Namely, we have
\[
2d - 1 \geq e \geq \left\lfloor \frac{(d+1)^2}{4} \right\rfloor.
\]
It immediately follows from here that \((d, e)\) is one of the pairs listed above.

Conversely, in order to prove \((2) \implies (1)\), it is enough to find \((n, e')\)-graphs (i.e., 1-dimensional simplicial complexes \( \Gamma \) on \([n]\) with \( e' \) edges) which does not contain any triangle for each \((n, e') = (4, 4), (4, 3), (5, 6), (5, 5), (6, 9), (6, 8), (7, 12)\). Those complexes will be given in the following example.

Example 5.8. There exists a 1-dimensional simplicial connected complex \( \Gamma \) on \([n]\) with with \( e(k[\Gamma]) = e' \) and \( \text{rt}(k[\Gamma]) = 2 \) for each \((n, e') = (4, 4), (4, 3), (5, 6), (5, 5), (6, 9), (6, 8), (7, 12)\).
(6, 9), (6, 8), (7, 12). Put
\[
S_{4,4} = \{[12], [14], [23], [34]\}, \\
S_{4,3} = \{[12], [23], [34]\}, \\
S_{5,6} = \{[12], [14], [23], [25], [34], [45]\}, \\
S_{5,5} = \{[12], [14], [23], [34], [45]\}, \\
S_{6,9} = \{[14], [15], [16], [24], [25], [26], [34], [35], [36]\}, \\
S_{6,8} = \{[12], [14], [23], [25], [34], [36], [45], [56]\}, \\
S_{7,12} = \{[15], [16], [17], [25], [26], [27], [35], [36], [37], [45], [46], [47]\},
\]
where \([i_1 i_2 \cdots i_p]\) means \([i_1, i_2, \ldots, i_p]\).

Let \(\Gamma_{n,e'}\) be a simplicial complex spanned by \(S_{n,e'}\). Then \(k[\Gamma_{n,e'}]\) is a two-dimensional Cohen–Macaulay Stanley–Reisner ring with \(e(\Gamma) = e'\) and \(rt(k[\Gamma]) = 2\). Note that when \(e' = f(n, 2) - 1\), \(\Gamma_{n,e'}\) is the so-called Turan graph \(T^2(n)\), that is, it is the unique complete bipartite graph on \([n]\) whose two partition sets differ in size by at most 1.

Let \(\Delta_{d,e}\) be the Alexander dual complex of \(\Gamma_{n,e'}\) where \(d = n - 2\) and \(e = \binom{d+1}{2} - e'\). Namely, \(\Delta_{d,e}\) is the complex spanned by \(T_{d,e}\), respectively:
\[
T_{2,2} = \{[13], [24]\}, \\
T_{2,3} = \{[13], [23], [24]\}, \\
T_{3,3} = \{[124], [135], [234], [245]\}, \\
T_{3,4} = \{[124], [134], [135], [234], [245]\}, \\
T_{4,6} = \{[1234], [2345], [3456], [4561], [5612], [6123]\}, \\
T_{4,7} = \{[1235], [1246], [1345], [1356], [2345], [2346], [2456]\}, \\
T_{5,9} = \{[12345], [12346], [12347], [12357], [12567], [13567], [14567], [23567], [24567], [34567]\}.
\]

Then \(A = k[\Delta_{d,e}]\) is a \(d\)-dimensional equidimensional Stanley–Reisner ring with \(\text{indeg } A = d\) and \(e(A) = e\) for each \((d, e) = (2, 2), (2, 3), (3, 4), (3, 5), (4, 6), (4, 7), (5, 9)\).

**Corollary 5.9.** Let \(A = k[\Delta]\) be a \(d\)-dimensional Buchsbaum Stanley–Reisner ring which is not a hypersurface. Suppose that \(\text{indeg } A = d \geq 3\). Then \(d = 3\) and \(\Delta\) is isomorphic to a simplicial complex spanned by \([124], [134], [135], [235], [245]\).

**Proof.** Since \(A\) is Buchsbaum and \(\text{indeg } A = d\) we have
\[
e = e(A) \geq \frac{c + d}{d} \left(\frac{c + d - 2}{d - 2}\right)
\]
by [11] Proposition 2.1. Also, \(n = d + 2\) by Proposition 5.4 since \(\Delta\) is pure. Thus
\[
2d - 1 \geq e \geq \frac{d + 2}{d} \left(\frac{d}{d - 2}\right) = \frac{(d + 2)(d - 1)}{2}.
\]
This implies that \(d \leq 3\), and thus \(d = 3\) and \(e = 5\). Then one can easily see that \(\Delta\) is isomorphic to the complex spanned by \([124], [134], [135], [235], [245]\), which is the Alexander dual complex of a 1-dimensional connected complex spanned by \([12], [23], [34], [45]\). \(\square\)
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