We consider the following construction:

Given \( X_n(t) \): the simple random walk on \( G_n \).

\( \pi_{X_n}(x) \): the stationary density of \( X_n(t) \).

Our main result shows \( \pi_{X_n}(x) \) converges to an explicit function of \( p(x) \) and \( \tau(x) \). Combining with an estimate on the out-degree of points in \( G_n \) allows us to recover density and scale.

Let \( V_d \) be the volume of the unit d-ball and \( NB_n(x) \) the neighbors of \( x \) in \( G_n \).

**Theorem: Main result**

Given (\( \ast \)), a.s. in \( \mathcal{X} \), we have

\[
\pi_{X_n}(x) \to \frac{p(x)}{\tau(x)^2}
\]

for \( c^{-1} = \int p(x)^2 \tau(x)^{-2} dx \).

**Corollary: Density estimates**

Assuming (\( \ast \)), we have a.s. in \( \mathcal{X} \) that

\[
\frac{\pi_{X_n}(x)\|\nabla \pi_{X_n}(x)\|_{\nabla \tau(x)}}{\|\nabla \tau(x)\|_{\nabla \tau(x)}} \to p(x);
\]

\[
\frac{1}{\|\nabla \tau(x)\|_{\nabla \tau(x)}} \to \tau(x).
\]

The main results follow by proving that the process \( X_n(t) \) converges to an Itô process:

**Theorem: Continuum limit of the walk**

Under (\( \ast \)), as \( n \to \infty \) a.s. in the draw of \( \mathcal{X} \) the process \( X_n(t/n) \) converges in \( D([0, \infty), \mathcal{T}) \) to the isotropic \( \mathcal{T} \)-valued Itô process \( Y(t) \) with reflecting boundary condition defined by

\[
dY(t) = \frac{\partial p(Y(t))}{3\partial Y(t)} Y(t)^2 dt + \frac{\partial Y(t)}{\sqrt{3}} dW(t).
\]

**Empirical Results**

**Near perfect reconstruction**

We demonstrate near-perfect reconstruction performance on simulated data. Our estimator is nearly indistinguishable from the naive metric ball estimator and substantially outperforms prior work of [1].

**References**

[1] U. Von Luxburg and M. Alagir. Density estimation from unweighted k-nearest neighbor graphs: a roadmap. NIPS 2013.