Moufang loops and Lie algebras *

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Abstract

It is explicitly shown how the Lie algebras can be associated with the analytic Moufang loops. The resulting Lie algebra commutation relations are well known from the theory of alternative algebras and can be seen as a preliminary step to quantum Moufang loops.

1 Moufang loops

It is well known how the Lie algebras are connected with the Lie groups. In the present paper, it is explicitly shown how the Lie algebras can be associated with the analytic Moufang loops.

A Moufang loop \[1, 2\] is a quasigroup \(G\) with the identity element \(e \in G\) and the Moufang identity

\[(ag)(ha) = a(gh)a, \quad a, g, h \in G.\]

Here the multiplication is denoted by juxtaposition. In general, the multiplication need not be associative: \(gh \cdot a \neq g \cdot ha\). Inverse element \(g^{-1}\) of \(g\) is defined by

\[gg^{-1} = g^{-1}g = e.\]

The left (\(L\)) and right (\(R\)) translations are defined by

\[gh = L_g h = R_h g, \quad g, h \in G.\]

Both translations are invertible mappings and

\[L^{-1}_g = L_{g^{-1}}, \quad R^{-1}_g = R_{g^{-1}}.\]

2 Analytic Moufang loops and infinitesimal Moufang translations

Following the concept of the Lie group, the notion of an analytic Moufang loop can be easily formulated.

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A Moufang loop $G$ is said to be \textit{analytic} if $G$ is also a real analytic manifold and main operations - multiplication and the inversion map $g \mapsto g^{-1}$ - are analytic mappings.

Let $T_e(G)$ denote the tangent space of $G$ at the unit $e$. For $x \in T_e(G)$, infinitesimal Moufang translations are defined as vector fields on $G$ as follows:

$$L_x := L_x(g) := (dL_g)_e x \in T_g(G),$$
$$R_x := R_x(g) := (dR_g)_e x \in T_g(G).$$

Let the local coordinates of $g$ from the vicinity of $e \in G$ be denoted by $g^i (i = 1, \ldots, r := \text{dim} G)$. Define the auxiliary functions

$$L^i_j(g) := \left. \frac{\partial (gh) i}{\partial h^j} \right|_{h=e}, R^i_j(g) := \left. \frac{\partial (hg) i}{\partial g^j} \right|_{h=e}$$

Matrices $(L^i_j)$ and $(R^i_j)$ are invertible.

Let $b_j := \frac{\partial}{\partial g^j} |_e (j = 1, \ldots, r)$ be a base in $T_e(G)$. Then both

$$L_j := (dL_g)_e b_j = L^i_j(g) \frac{\partial}{\partial g^i} \in T_g(G),$$
$$R_j := (dR_g)_e b_j = R^i_j(g) \frac{\partial}{\partial g^i} \in T_g(G),$$

give base at $T_g(G)$. Thus one has two preferred base fields on $G$. When writing $T_e(G) \ni x = x^j b_j$, one can easily see that

$$L_x := L_x(g) = x^i L^i_j(g) \frac{\partial}{\partial g^i} \in T_g(G),$$
$$R_x := R_x(g) = x^i R^i_j(g) \frac{\partial}{\partial g^i} \in T_g(G).$$

3 Structure constants and tangent Mal’tsev algebra

As in the case of the Lie groups, the structure constants $c^i_{jk}$ of an analytic Moufang loop are defined by

$$c^i_{jk} := \left. \frac{\partial^2 (ghg^{-1}h^{-1})^i}{\partial g^j \partial h^k} \right|_{g=h=e} = -c^i_{kj}, \quad i, j, k = 1, \ldots, r.$$

For any $x, y \in T_e(G)$, their product $[x, y] \in T_e(G)$ is defined in component form by

$$[x, y]^i := c^i_{jk} x^j y^k = -[y, x]^i, \quad i = 1, \ldots, r.$$

The tangent space $T_e(G)$ being equipped with such an anti-commutative multiplication is called the \textit{tangent algebra} of the analytic Moufang loop $G$.

The tangent algebra of $G$ need not be a Lie algebra. There may exist a triple $x, y, z \in T_e(G)$ which does not satisfy the Jacobi identity:

$$J(x, y, z) := [x, [y, z]] + [y, [z, x]] + [z, [x, y]] \neq 0.$$

Instead, for any $x, y, z \in T_e(G)$ one has a more general \textit{Mal’tsev identity}

$$[J(x, y, z), x] = J(x, y, [x, z]).$$

Anti-commutative algebras with this identity are called the \textit{Mal’tsev algebras}.
4 Generalized Maurer-Cartan equations

Denote as above \( L_x := L_x(g) \) and \( R_x := R_x(g) \) for all \( x \in T_e(G) \).

It is well known that the infinitesimal translations of a Lie group obey the Maurer-Cartan equations

\[
[L_x, L_y] - L_{[x,y]} = [L_x, R_y] = [R_x, R_y] + R_{[x,y]} = 0.
\]

It turns out that for a non-associative analytic Moufang loop these equations are violated minimally. The algebra of infinitesimal Moufang translations reads as generalized Maurer-Cartan equations:

\[
[L_x, L_y] - L_{[x,y]} = -2[L_x, R_y] = [R_x, R_y] + R_{[x,y]} = 0.
\]

We outline a way of closing of this algebra (generalized Maurer-Cartan equations), which in fact means construction of a finite dimensional Lie algebra generated by infinitesimal Moufang translations.

Start by rewriting the generalized Maurer-Cartan equations as follows:

\[
[L_x, L_y] = 2Y(x; y) + \frac{1}{3}L_{[x,y]} + \frac{2}{3}R_{[x,y]} \tag{1}
\]

\[
[L_x, R_y] = -Y(x; y) + \frac{1}{3}L_{[x,y]} - \frac{1}{3}R_{[x,y]} \tag{2}
\]

\[
[R_x, R_y] = 2Y(x; y) - \frac{2}{3}L_{[x,y]} - \frac{1}{3}R_{[x,y]} \tag{3}
\]

Here (1) or (2) or (3) can be assumed as a definition of the Yamagutian \( Y \). It can be shown that

\[
Y(x; y) + Y(y; x) = 0, \tag{4}
\]

\[
Y([x, y]; z) + Y([y, z]; x) + Y([z, x]; y) = 0. \tag{5}
\]

The constraints (4) trivially descend from the anti-commutativity of the commutator bracketing, but the proof of (5) needs certain effort. Further, it turns out that the following reductivity conditions hold:

\[
6[Y(x; y), L_z] = L_{[x,y,z]}, \quad 6[Y(x; y), R_z] = R_{[x,y,z]} \tag{6}
\]

where the trilinear Yamaguti brackets \([\cdot, \cdot, \cdot]\) are defined \[6\] in \( T_e(G) \) by

\[
[x, y, z] := [x, [y, z]] - [y, [x, z]] + [[x, y], z].
\]

Finally, the Yamagutian obeys the Lie algebra

\[
6[Y(x; y), Y(z; w)] = Y([x, y, z]; w) + Y(z; [x, y, w]). \tag{7}
\]

The Lie algebra commutation relations (1)-(7) were proved in \[4\]. Dimension of this Lie algebra does not exceed \( 2r + r(r - 1)/2 \). The Jacobi identities are guaranteed by the defining identities of the Lie \[5\] and general Lie \[6\] triple systems associated with the tangent Mal’tsev algebra \( T_e(G) \) of \( G \).

The commutation relations of form (1)-(7) are well-known from the theory of alternative algebras \[7\] and can be seen as a preliminary step to construct quantum Moufang loops (QML). QML is a deformation of universal enveloping algebra of the Lie algebra (1)-(7).

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