On 2D Harmonic Extensions of Vector Fields and Stellarator Coils

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Abstract

We consider a problem relating to magnetic confinement devices known as stellarators. Plasma is confined by magnetic fields generated by current-carrying coils, and here we investigate how closely to the plasma they need to be positioned. Current-carrying coils are represented as singularities within the magnetic field and therefore this problem can be modelled mathematically as finding how far we can harmonically extend a vector field from the boundary of a domain.

For this paper we consider two-dimensional domains with real analytic boundary, and prove that a harmonic extension exists if and only if the boundary data satisfies a combined compatibility and regularity condition. Our method of proof uses a generalisation of a result of Hadamard on the Cauchy problem for the Laplacian. We then provide a lower bound on how far we can harmonically extend the vector field from the boundary via the Cauchy–Kovalevskaya Theorem.

1 Introduction

The motivation for the results in this paper arises from the study of magnetic confinement devices and magnetohydrodynamic equilibrium. A magnetic confinement device uses magnetic fields to confine charged particles that make up a plasma. The magnetic fields are typically generated by current-carrying coils that are located in the vacuum surrounding the plasma. One type of magnetic confinement device is the stellarator, which has its plasma configured to be topologically a solid torus. This paper considers a problem arising in the study of such devices. Let $\Omega$ be a region of toroidal plasma with a magnetic field $B$ tangent to the plasma boundary $\partial \Omega$ generated by a collection of external current-carrying coils. How close to the plasma boundary does the nearest coil need to be? Alternatively, this problem can be posed as trying to find how far the magnetic field $B$ can be externally extended from the plasma boundary, subject to the vacuum field equations (curl and divergence free), before arriving at a singularity. The singularity indicates where a current-carrying coil is expected to be located. This is because in the
magnetic fields generated by current-carrying coils, the coils appear as singularities in the magnetic field.

In this paper we focus on a two-dimensional version of this problem. More precisely, let \( \Omega \subset \mathbb{R}^2 \) be an open bounded and simply connected set with real analytic boundary \( \partial \Omega \). We use \( t \) and \( n \) to denote the unit tangent and outward normal vectors on \( \partial \Omega \) respectively. Given real-valued functions \( f \) and \( h \) on \( \partial \Omega \), we are interested in finding an open connected \( U \subset \mathbb{R}^2 \) satisfying \( \Omega \subset U \) such that there exists a vector field \( B = (B_1, B_2) \) solving the following Cauchy problem

\[
\begin{align*}
\text{div } B &:= \frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} = 0 \quad \text{in } U \setminus \overline{\Omega} \\
\text{curl } B &:= \frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y} = 0 \quad \text{in } U \setminus \overline{\Omega} \\
B \cdot t &= f \quad \text{on } \partial \Omega \\
B \cdot n &= h \quad \text{on } \partial \Omega.
\end{align*}
\]

We are particularly interested in finding how large we can make the distance between \( \partial U \) and \( \partial \Omega \). We note that \( \partial U \) is external to \( \Omega \) since \( \Omega \subset U \).

We call \( B \) harmonic if it satisfies equations (1a) and (1b). Therefore, this problem is equivalent to finding how far we can harmonically extend \( B \) outwards from \( \partial \Omega \). Equations (1a) and (1b) are the vacuum field equations for a magnetic field, and equations (1c) and (1d) are the boundary conditions. The physically relevant boundary conditions require that \( B \) is tangent to the plasma boundary \( \partial \Omega \) which in our set up means taking \( h = 0 \) and leaving \( f \) arbitrary, but we treat the case of a general \( h \) for its interesting mathematics.

We now give a couple of useful remarks. Given open connected \( U \subset \mathbb{R}^2 \) satisfying \( \Omega \subset U \), the uniqueness of solutions to the Cauchy problem (1) in the class \( B \in C^1(U \setminus \overline{\Omega}; \mathbb{R}^2) \cap C(U \setminus \Omega; \mathbb{R}^2) \) follows from Holmgren’s Uniqueness Theorem [1 §2.3]. This theorem will also be useful later on in Section 2 when we have to combine together solutions and make sure they coincide on overlaps. The equations \( \text{div } B = \text{curl } B = 0 \) are the Cauchy–Riemann equations for the complex function \( B := B_1 - iB_2 \) with respect to \( z = x + iy \). This fact will also come in use later on in section 3. Note that \( B \) being complex analytic implies that \( B_1 \) and \( B_2 \) are harmonic.

We introduce some required definitions. By identifying the unit circle \( T \) with the interval \([0, 2\pi]\), the boundary \( \partial \Omega \) being real analytic means there exists a parameterisation \( \gamma = (\gamma_1, \gamma_2): T \to \partial \Omega \) that is both real analytic (each component is real analytic) and regular (\( \gamma'(t) \neq 0 \) for all \( t \in T \)). Given \( t_0 \in T \) such that \( \gamma(t_0) = v_0 \), we say that \( f \) on \( \partial \Omega \) is real analytic at \( v_0 \) if \( f(\gamma(t)) \) is real analytic at \( t_0 \). It is straightforward to check this definition is independent of the parameterisation chosen. We use \( C^\omega(\partial \Omega) \) to denote the set of functions that are real analytic at every point in \( \partial \Omega \).

We now provide a summary of the results in this paper. In Section 2 we prove the boundary data has to satisfy a certain degree of regularity in order for a solution to the Cauchy problem (1) to exist. We prove that, for \( f, h \in C^1(\partial \Omega) \), there exists an open connected \( U \subset \mathbb{R}^2 \) satisfying \( \Omega \subset U \) and a solution \( B \in C^1(U \setminus \overline{\Omega}; \mathbb{R}^2) \cap C(U \setminus \Omega; \mathbb{R}^2) \) to the Cauchy problem (1) if and only if the boundary data satisfies a combined regularity and compatibility condition. The condition is that \( f - \mathcal{H}h \) is real analytic on \( \partial \Omega \) where \( \mathcal{H} \) is the operator given by

\[
\mathcal{H}h(v) := \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{\partial \Omega \setminus B_{\varepsilon}(v)} h(w) \frac{t(v) \cdot (v - w)}{|v - w|^2} \, dw
\]

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for $v \in \partial \Omega$ and $B_\varepsilon(v)$ is the ball in $\mathbb{R}^2$ centred at $v$ with radius $\varepsilon$. We note that if $\partial \Omega$ is taken to be a straight line, then the operator $\mathcal{H}$ is identical to the standard Hilbert transform. In the case $h = 0$, the condition of $f - \mathcal{H}h$ being real analytic simplifies to $f$ being real analytic on $\partial \Omega$; and in this case the proof can be considerably shortened.

Our method of proof involves generalising a similar type of result due to Hadamard on the Cauchy problem for the Laplacian [2]. The relevance of the Cauchy problem for the Laplacian comes from the fact that on simply connected domains every harmonic vector field can be written as the gradient of a harmonic scalar potential. A survey of results on the Cauchy problem for the Laplacian is presented in [3]. Hadamard’s result considers the case of a flat boundary. We generalise this result to the case where the boundary data lies on an analytic curve. A detailed version of the proof of Hadamard’s result but with the Laplacian replaced by the equation $\partial_{y^2}u + c\partial_{xx}u = 0$, can be found in [4].

Section 2 shows us that it is not unreasonable to assume our boundary data is real analytic. Therefore, in Section 3 we assume $f, h \in C^\omega(\partial \Omega)$ and use the Cauchy–Kovalevskaya Theorem to find a lower bound on how far we can solve the Cauchy problem (1). The lower bound depends on the two functions $\Theta$ and $\Lambda$, which will come to be defined by (21) and (25) respectively. $\Theta$ depends on the parameterisation $\gamma$ and boundary data $f$ and $h$, whereas $\Lambda$ only depends on $\gamma$. We find that we can solve at least a distance $d^*$ away from $\partial \Omega$ where $d^*$ depends on the the Taylor series coefficients of $\Lambda$ and radius of convergence of the Taylor series of $\Lambda \Theta'$. We show that the distance $d^*$ is no more than half the minimum radius of curvature,

$$d^* \leq \frac{1}{2} \inf_{\gamma} \left( \frac{1}{\kappa} \right),$$

where $\kappa$ is the curvature of $\gamma$. We then conclude with some examples on computing and estimating $d^*$.

2 Boundary Data Regularity

Given a function $\Psi: [-1, 1] \to \mathbb{R}$ that has a real analytic extension to an open neighbourhood of $[-1, 1]$, we let

$$\Gamma := \{(x, \Psi(x)) : x \in (-1, 1)\} \subset \mathbb{R}^2$$

be the curve that is the graph of $\Psi$. Since every analytic curve can locally be written as the graph of an analytic function, we initially consider a local version of the Cauchy problem (11) where $\partial \Omega$ is replaced by $\Gamma$. Let $\Omega = \{(x, y) \in (-1, 1) \times \mathbb{R} : y < \Psi(x)\}$, and $n$ be the unit normal to the curve $\Gamma$ facing away from $\Omega$.

Since on simply connected domains every harmonic vector field can be written as the gradient of a scalar potential, we can locally find a harmonic scalar potential $u$ satisfying $\mathcal{B} = \nabla u$. In this notation the boundary condition (11) becomes

$$\nabla u(x, \Psi(x)) \cdot (1, \Psi'(x)) = f(x, \Psi(x))\sqrt{1 + \Psi'(x)^2},$$

which by the Fundamental Theorem of Calculus for Line Integrals can be integrated to obtain $u(x, \Psi(x)) = g(x)$ where $g'(x) = f(x, \Psi(x))\sqrt{1 + \Psi'(x)^2}$. Furthermore, the boundary condition (11) becomes $\frac{\partial u}{\partial n}(x, \Psi(x)) = h(x, \Psi(x))$. To simplify notation we replace $h(x, \Psi(x))$ with $h(x).$
This shows that the Cauchy problem (1) is in a local sense equivalent to the Cauchy problem for the Laplacian given by (2). We now introduce the following theorem on the existence of the Cauchy problem for the Laplacian.

**Theorem 2.1.** Let \( g, h \in C^1([-1, 1]) \). There exists \( U \subset \mathbb{R}^2 \), an open connected neighbourhood of \( \Gamma \), and \( u \in C^2(U \setminus \overline{\Omega}) \cap C^1(U \setminus \Omega) \) that solves

\[
\begin{align*}
\Delta u(x, y) &= 0 \quad \text{for } (x, y) \in U \setminus \overline{\Omega} \\
u(x, \Psi(x)) &= g(x) \quad \text{for } x \in (-1, 1) \\
\frac{\partial u}{\partial n}(x, \Psi(x)) &= h(x) \quad \text{for } x \in (-1, 1)
\end{align*}
\]

if and only if

\[
H(x) := g(x) - \frac{1}{\pi} \int_{-1}^{1} h(t) \sqrt{1 + \Psi'(t)^2} \log \sqrt{(x-t)^2 + (\Psi(x) - \Psi(t))^2} \, dt
\]

is real analytic on \((-1, 1)\).

If we were to take \( \Gamma \) to be flat (\( \Psi = 0 \)), then this theorem recovers a result due to Hadamard [2]. In the proof of Theorem 2.1 we will require the following lemma on the analyticity of functions defined by integrals.

**Lemma 2.2.** Suppose that \( U \subset \mathbb{C} \) is open and \( I \subset \mathbb{R} \) is a compact interval. If the continuous function \( A: U \times I \to \mathbb{C} \) is complex analytic in \( z \in U \) for each \( x \in I \), then \( \int_I A(z, x) \, dx \) is complex analytic on \( U \).

**Proof.** Let \( \Gamma \) be a triangle in \( U \). The continuity of \( A \) implies that \( \int_I |A(z, x)| \, dx \) is bounded on \( \Gamma \) and so by Fubini’s Theorem

\[
\int_{\Gamma} \int_I A(z, x) \, dx \, dz = \int_I \int_{\Gamma} A(z, x) \, dz \, dx = 0.
\]

Hence Morera’s Theorem [5, §5.1] implies that \( \int_I A(z, x) \, dx \) is complex analytic on \( U \). \( \square \)

The proof of Theorem 2.1 follows.

**Proof.** We begin by proving that \( H \) is necessarily real analytic. Suppose that there exists \( u \in C^2(U \setminus \overline{\Omega}) \cap C^1(U \setminus \Omega) \) that solves the Cauchy problem (2). We aim to show that \( H \) is real analytic at a fixed \( x_0 \in (-1, 1) \). To achieve this we construct a region \( V_{\delta, \varepsilon} \), shaded in Figure 1 below, and apply a Green’s identity (3) over this region to the function \( u \) and a constant multiple of the fundamental solution of the Laplacian centred at \((x, \Psi(x)) \in \Gamma\). Then we will proceed with analysing the boundary terms of the Green’s identity.

To define the region \( V_{\delta, \varepsilon} \) we choose \( \delta > 0 \) small enough such that

\[
\delta < \min\{|x_0 + 1|, |x_0 - 1|\}
\]

and the open ball \( B_\delta(x_0, \Psi(x_0)) \) centred at \((x_0, \Psi(x_0))\) with radius \( \delta \) is contained within \( U \). If we were to travel anti-clockwise along the circle \( \partial B_\delta(x_0, \Psi(x_0)) \) starting from the highest point \((x_0, \Psi(x_0) + \delta)\), then eventually we would come into contact with the curve \( \Gamma \). Let \( x_{0, \delta}^- \) be the \( x \)-coordinate of the first point of contact. A straightforward compactness argument can be used to prove the existence of such a point. Define \( x_{0, \delta}^+ \) similarly.
when the circle is traversed clockwise. We denote by $C_\delta^+$ the arc of $\partial B_\delta(x_0, \Psi(x_0))$ from $(x_{0,\delta}^-, \Psi(x_{0,\delta}^-))$ to $(x_{0,\delta}^+, \Psi(x_{0,\delta}^+))$ that passes through the point $(x_0, \Psi(x_0) + \delta)$.

Figure 1: Diagram of the region $V_{\delta,\varepsilon}$.

Let $x \in (x_{0,\delta}^-, x_{0,\delta}^+)$ and define $\Phi_x: \mathbb{R}^2 \setminus \{(x, \Psi(x))\} \to \mathbb{R}$ to be the following constant multiple of the fundamental solution of the Laplacian,

$$\Phi_x(\xi, \eta) := \log |(x, \Psi(x)) - (\xi, \eta)| = \log \sqrt{(x - \xi)^2 + (\Psi(x) - \eta)^2}.$$

To avoid the singularity of $\Phi_x$ we need to cut out a region around the point $(x, \Psi(x))$. Given $0 < \varepsilon < \min\{\|x - x_{0,\delta}^-\|, \|x - x_{0,\delta}^+\|\}$ we find $x_{\varepsilon}^-, x_{\varepsilon}^+, C_{\varepsilon}^-$ from $x$ in the same way as we constructed $x_{0,\delta}^-, x_{0,\delta}^+, C_\delta^+$ from $x_0$. We now define the curves

$$L_{\delta,\varepsilon}^- := \{(t, \Psi(t)) : t \in [x_{0,\delta}^-, x_{\varepsilon}^-]\}$$

and

$$L_{\delta,\varepsilon}^+ := \{(t, \Psi(t)) : t \in [x_{\varepsilon}^+, x_{0,\delta}^+]\},$$

which are the segments of $\Gamma$ connecting $C_{\varepsilon}^-$ and $C_{\delta}^+$. Then we define the open set $V_{\delta,\varepsilon}$, shaded in Figure 1, as the region contained in $U \setminus \Omega$ bounded by the curves $L_{\delta,\varepsilon}^-, C_{\varepsilon}^-, L_{\delta,\varepsilon}^+$, and $C_{\delta}^+$.

Due to the way we have constructed $V_{\delta,\varepsilon}$, it has a piecewise $C^2$ boundary $\partial V_{\delta,\varepsilon}$. To apply a Green’s identity over $V_{\delta,\varepsilon}$, we will need to translate the set slightly upwards since we are not assuming that $u$ is $C^2$ up to $\Gamma$. Let $\rho > 0$ be small enough such that $V_{\delta,\varepsilon,\rho} := V_{\delta,\varepsilon} + (0, \rho)$ is contained within $U \setminus \Omega$, which means that $u, \Phi_x \in C^2(V_{\delta,\varepsilon,\rho})$. We can then apply a Green’s identity to obtain

$$\int_{\partial V_{\delta,\varepsilon,\rho}} \left( u \frac{\partial \Phi_x}{\partial \nu} - \Phi_x \frac{\partial u}{\partial \nu} \right) \, ds = \int_{V_{\delta,\varepsilon,\rho}} (u \Delta \Phi_x - \Phi_x \Delta u) \, dA, \quad (3)$$
where \( \nu \) is the unit outward normal to \( V_{\delta,\epsilon,\rho} \). Since both \( u \) and \( \Phi_x \) are harmonic in \( V_{\delta,\epsilon,\rho} \), the right-hand side of equation (3) vanishes. Hence as the integrand on the left-hand side is continuous up to \( \Gamma \), we can take the limit as \( \rho \) tends to zero to obtain

\[
\int_{\partial V_{\delta,\epsilon}} \left( u \frac{\partial \Phi_x}{\partial \nu} - \Phi_x \frac{\partial u}{\partial \nu} \right) \, ds = 0. \tag{4}
\]

We now split the above integral into four, using the different pieces of boundary \( \partial V_{\delta,\epsilon} \), and separately evaluate their limiting values as \( \epsilon \to 0 \). When analysing the integrals over \( C_{\epsilon}^- \), we will need to know the limiting value of \( |C_{\epsilon}^-|/\epsilon \) as \( \epsilon \to 0 \) where \( |C_{\epsilon}^-| \) is the length of \( C_{\epsilon}^- \). Notice that the manner in which we constructed the arc \( C_{\epsilon}^- \), implies that

\[
\frac{|C_{\epsilon}^-|}{\epsilon} = \pi - \tan^{-1}\left( \frac{\Psi(x_{x_{\epsilon}}) - \Psi(x)}{x_{x_{\epsilon}} - x} \right) + \tan^{-1}\left( \frac{\Psi(x_{x_{\epsilon}}) - \Psi(x)}{x_{x_{\epsilon}} - x} \right). \]

It follows \( |C_{\epsilon}^-|/\epsilon \to \pi \) as \( \epsilon \to 0 \) because \( x_{x_{\epsilon}} \to x \) and \( x_{x_{\epsilon}} \to x \) as \( \epsilon \to 0 \). Therefore, since \( \frac{\partial \Phi_x}{\partial \nu} = -\frac{1}{\epsilon} \) on \( C_{\epsilon}^- \), we have

\[
\int_{C_{\epsilon}^-} u \frac{\partial \Phi_x}{\partial \nu} \, ds = -\frac{1}{\epsilon} \int_{C_{\epsilon}^-} u \, ds = -\frac{|C_{\epsilon}^-|}{\epsilon} \int_{C_{\epsilon}^-} u \, ds \to -\pi g(x)
\]

as \( \epsilon \to 0 \) using the boundary condition (2). Furthermore, if we let \( K \) be a compact neighbourhood of \((x, \Psi(x))\) in \( U \setminus \Omega \), then once \( \epsilon \) is small enough we have

\[
\left| \int_{C_{\epsilon}^-} \Phi_x \frac{\partial u}{\partial \nu} \, ds \right| \leq \left( \sup_K |\nabla u| \right) \int_{C_{\epsilon}^-} |\Phi_x| \, ds \\
= \left( \sup_K |\nabla u| \right) \frac{|C_{\epsilon}^-|}{\epsilon} |\log(\epsilon)| \\
\to 0
\]

as \( \epsilon \to 0 \). As a result, this term will not contribute to the limiting value of equation (4).

Next, we evaluate the integrals over \( L_{\delta,\epsilon}^- \) and \( L_{\delta,\epsilon}^+ \). Firstly,

\[
\int_{L_{\delta,\epsilon}^-} \Phi_x \frac{\partial u}{\partial \nu} \, ds = \int_{x_{0,\delta}}^{x_{x_{\epsilon}}} \Phi_x(t, \Psi(t)) \frac{\partial u}{\partial \nu}(t, \Psi(t)) \sqrt{1 + \Psi'(t)^2} \, dt \\
- \int_{x_{0,\delta}}^{x_{x_{\epsilon}}} h(t) \sqrt{1 + \Psi'(t)^2} \log \sqrt{(x - t)^2 + \Psi(x) - \Psi(t)^2} \, dt.
\]

We now show that when taking the limit of this expression as \( \epsilon \to 0 \), the upper limit of the integral changes to \( x \). Using the existence of \( \phi \in C([-1,1]^2) \) satisfying

\[
\Psi(x) - \Psi(t) = \phi(x, t)(x - t),
\]

we have

\[
\left| \int_{x_{x_{\epsilon}}}^{x} h(t) \sqrt{1 + \Psi'(t)^2} \log \sqrt{(x - t)^2 + \Psi(x) - \Psi(t)^2} \, dt \right| \\
\leq \int_{x_{x_{\epsilon}}}^{x} |h(t)| \sqrt{1 + \Psi'(t)^2} \left| \log |x - t| + \log \sqrt{1 + \phi(x, t)^2} \right| \, dt \\
\leq \sup_{[-1,1]} \left( |h| \sqrt{1 + \Psi'(t)^2} \right) \int_{x_{x_{\epsilon}}}^{x} |\log |x - t|| + \log \sqrt{1 + \phi(x, t)^2} \, dt \\
\to 0
\]
as \( \varepsilon \to 0 \) where we have used the fact that \( \log |x| \in L^1([-1, 1]) \). Therefore,

\[
\int_{L_{x,\varepsilon}^-} \frac{\partial u}{\partial \nu} \, ds \to - \int_{x_{0,\varepsilon}^-}^x h(t) \sqrt{1 + \Psi'(t)^2} \log \sqrt{(x-t)^2 + (\Psi(x) - \Psi(t))^2} \, dt \]

as \( \varepsilon \to 0 \). Similarly we have

\[
\int_{L_{x,\varepsilon}^+} \frac{\partial u}{\partial \nu} \, ds \to - \int_{x_{0,\varepsilon}^+}^x h(t) \sqrt{1 + \Psi'(t)^2} \log \sqrt{(x-t)^2 + (\Psi(x) - \Psi(t))^2} \, dt
\]

as \( \varepsilon \to 0 \).

We also have

\[
\int_{L_{x,\varepsilon}^-}^x u \frac{\partial \Phi}{\partial \nu} \, ds = \int_{x_{0,\varepsilon}^-}^x u(t, \Psi(t)) \left[ \frac{\Psi(x) - \Psi(t) - \Psi'(t)(x-t)}{(x-t)^2 + (\Psi(x) - \Psi(t))^2} \right] \, dt.
\]

Notice that by using the fact that \( \Psi \) is twice continuously differentiable, the integrand has a continuous extension to \( t = x \). Consequently, since \( u = g \) on \( \Gamma \), we have

\[
\int_{L_{x,\varepsilon}^-}^x u \frac{\partial \Phi}{\partial \nu} \, ds \to \int_{x_{0,\varepsilon}^-}^x g(t) \left[ \frac{\Psi(x) - \Psi(t) - \Psi'(t)(x-t)}{(x-t)^2 + (\Psi(x) - \Psi(t))^2} \right] \, dt
\]

as \( \varepsilon \to 0 \), and similarly

\[
\int_{L_{x,\varepsilon}^+}^x u \frac{\partial \Phi}{\partial \nu} \, ds \to \int_{x_{0,\varepsilon}^+}^x g(t) \left[ \frac{\Psi(x) - \Psi(t) - \Psi'(t)(x-t)}{(x-t)^2 + (\Psi(x) - \Psi(t))^2} \right] \, dt
\]

as \( \varepsilon \to 0 \). Substituting everything into equation \( \text{(4)} \) and then taking the limit as \( \varepsilon \to 0 \) results in

\[
g(x) = \frac{1}{\pi} \int_{x_{0,\varepsilon}^-}^{x_{0,\varepsilon}^+} h(t) \sqrt{1 + \Psi'(t)^2} \log \sqrt{(x-t)^2 + (\Psi(x) - \Psi(t))^2} \, dt
\]

\[
= \frac{1}{\pi} \int_{L_{x,\varepsilon}^-}^{x_{0,\varepsilon}^+} \left( \frac{\partial \Phi}{\partial \nu} - \frac{\partial \Phi}{\partial \nu} \frac{\partial u}{\partial \nu} \right) \, ds + \frac{1}{\pi} \int_{x_{0,\varepsilon}^-}^{x_{0,\varepsilon}^+} g(t) \left[ \frac{\Psi(x) - \Psi(t) - \Psi'(t)(x-t)}{(x-t)^2 + (\Psi(x) - \Psi(t))^2} \right] \, dt. \tag{5}
\]

We would like to show that the right-hand side of this equation is real analytic at \( x = x_0 \).

We start by noting that Taylor’s Theorem with integral remainder yields

\[
\Psi(x) = \Psi(t) + \int_0^1 \Psi'((1 - \tau)x + \tau t) \, d\tau (x-t)
\]

and

\[
\Psi(x) = \Psi(t) + \Psi'(t)(x-t) + \int_0^1 \Psi''((1 - \tau)x + \tau t) \tau \, d\tau (x-t)^2.
\]

Therefore, by setting

\[
A(x, t) := \int_0^1 \Psi'((1 - \tau)x + \tau t) \, d\tau,
\]

\[
B(x, t) := \int_0^1 \Psi''((1 - \tau)x + \tau t) \tau \, d\tau,
\]

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we can write

\[ \frac{\Psi(x) - \Psi(t) - \Psi'(t)(x-t)}{(x-t)^2 + (\Psi(x) - \Psi(t))^2} = \frac{B(x,t)}{1 + A(x,t)^2}. \]

Since $\Psi$ is real analytic, there exists an open neighbourhood of $[-1,1]$ in the complex plane $\mathbb{C}$ where $\Psi$ is complex analytic. We can then choose $a > 0$ small enough such that the closure of

\[ R := \{ z \in \mathbb{C} : \text{Re}(z) \in (-1,1), \text{Im}(z) \in (-a,a) \} \]

lies within the region where $\Psi$ is complex analytic. Thus the expressions $\Psi'((1-\tau)z + \tau t)$ and $\Psi''((1-\tau)z + \tau t)\tau$ are complex analytic in $z \in R$ for all $t \in [-1,1], \tau \in [0,1]$. Hence, we can use Lemma 2.2 to guarantee that both $A(z,t)$ and $B(z,t)$ are complex analytic on $R$ for all $t \in [-1,1]$. Now choose $a$ small enough such that $|\text{Im}(\Psi'(z))| < 1/2$ for all $z \in R$, to acquire the bound

\[ |1 + A(z,t)^2| \geq |\text{Re}(1 + A(z,t)^2)| \]

\[ = 1 + (\text{Re}A(z,t))^2 - (\text{Im}A(z,t))^2 \]

\[ \geq \frac{3}{4} \]

for $z \in R$ and $t \in [-1,1]$. Then $\frac{B(z,t)}{1 + A(z,t)^2}$ is complex analytic on $R$ for each $t \in [-1,1]$. Once again we can apply Lemma 2.2 this time to justify the complex analyticity of

\[ \int_{x_0,\delta}^{x_0,\delta} g(t) \frac{B(z,t)}{1 + A(z,t)^2} \ dt \]

on $R$. It follows that

\[ \int_{x_0,\delta}^{x_0,\delta} g(t) \left[ \frac{\Psi(x) - \Psi(t) - \Psi'(t)(x-t)}{(x-t)^2 + (\Psi(x) - \Psi(t))^2} \right] \ dt \]  

(6)

is real analytic at $x = x_0$.

We still need to show that

\[ \int_{C_\delta^+} \left( u \frac{\partial \Phi}{\partial \nu} - \Phi x \frac{\partial u}{\partial \nu} \right) \ ds \]

is real analytic at $x = x_0$. For some $\theta_1 < \theta_2$ depending on $\delta$, we have

\[ \int_{C_\delta^+} \Phi x \frac{\partial u}{\partial \nu} \ ds = \int_{\theta_1}^{\theta_2} \delta \frac{\partial}{\partial \nu} \log(\mathcal{E}(x,t)) \frac{\partial u}{\partial \nu} (x_0 + \delta \cos t, \Psi(x_0) + \delta \sin t) \ dt \]

where $ds = \delta \ dt$ and

\[ \mathcal{E}(x,t) = (x - x_0 - \delta \cos t)^2 + (\Psi(x) - \Psi(x_0) - \delta \sin t)^2. \]

Let $0 < r < \frac{\delta}{3\sqrt{2}}$ be small enough such that the complex disc

\[ D_r(x_0) = \{ z \in \mathbb{C} : |z - x_0| < r \} \]
lies within the region of complex analyticity of \( \Psi \) and \( |\Psi(z) - \Psi(x_0)| < \frac{\delta}{3\sqrt{2}} \) for all \( z \in D_r(x_0) \). Then, because for all \( t \in [\theta_1, \theta_2] \) either \( |\cos t| > \frac{1}{\sqrt{2}} \) or \( |\sin t| > \frac{1}{\sqrt{2}} \), it follows that

\[
\begin{align*}
\Re \mathcal{E}(z, t) &= (\Re(z - x_0 - \delta \cos t))^2 - (\Im(z - x_0 - \delta \cos t))^2 \\
&\quad + (\Re(\Psi(z) - \Psi(x_0) - \delta \sin t))^2 - (\Im(\Psi(z) - \Psi(x_0) - \delta \sin t))^2 \\
&\geq \left( \frac{\delta}{\sqrt{2}} - \frac{\delta}{3\sqrt{2}} \right)^2 - (\Im(z - x_0))^2 - (\Im(\Psi(z) - \Psi(x_0)))^2 \\
&\geq \frac{2\delta^2}{9} - 2 \left( \frac{\delta}{3\sqrt{2}} \right)^2 \\
&= \frac{\delta^2}{9}.
\end{align*}
\]

Now if we take \( \log \) to be the principle value complex logarithm defined away from the negative real axis, then \( \log(\mathcal{E}(z, t)) \) is complex analytic on \( D_r(x_0) \) for all \( t \in [\theta_1, \theta_2] \). Hence by Lemma \[2.2\], \( \int_{C^+_{\delta}} \Phi_x \frac{\partial u}{\partial \nu} \, ds \) is complex analytic for \( z \in D_r(x_0) \) and so real analytic at the point of interest \( z = x_0 \). Similar arguments can be employed to show that

\[
\int_{C^+_{\delta}} u \frac{\partial \Phi_x}{\partial \nu} \, ds,
\]

\[
\int_{-1}^{x_0,\delta} h(t) \sqrt{1 + \Psi'(t)^2} \log \sqrt{(x - t)^2 + (\Psi(x) - \Psi(t))^2} \, dt,
\]

\[
\int_{x_0,\delta}^{1} h(t) \sqrt{1 + \Psi'(t)^2} \log \sqrt{(x - t)^2 + (\Psi(x) - \Psi(t))^2} \, dt
\]

are all real analytic at \( x = x_0 \). The above together with equation (5) conclude our proof that \( H \) is real analytic at \( x_0 \) and thus the entirety of \((-1, 1)\), since \( x_0 \) was arbitrary.

We now prove the sufficiency of \( H \) being real analytic on \((-1, 1)\). Let \( U_1 := (-1, 1) \times \mathbb{R} \) and consider \( G : U_1 \setminus \Omega \to \mathbb{R} \) defined by

\[
G(x, y) := \frac{1}{\pi} \int_{-1}^{1} h(t) \sqrt{1 + \Psi'(t)^2} \log \sqrt{(x - t)^2 + (y - \Psi(t))^2} \, dt.
\]

Notice that the integrand

\[
h(t) \sqrt{1 + \Psi'(t)^2} \log \sqrt{(x - t)^2 + (y - \Psi(t))^2}
\]

and all its partial derivatives with respect to \( x \) and \( y \) are continuous in \( (x, y, t) \in (U_1 \setminus \overline{\Omega}) \times [-1, 1] \). We can thus interchange integral and partial derivative to justify \( G \) belonging to \( C^\infty(U_1 \setminus \overline{\Omega}) \). Furthermore, \( \Delta G = 0 \) which suggests, as we will come to discover, that \( G \) can be used to construct a solution \( u \) to the Cauchy problem \([2]\). We remark that if \( h = 0 \), then this step can be skipped as \( G = 0 \). Therefore, for this part of the proof we can assume \( h \neq 0 \), and we will find useful to do so.

We will now show that \( G \) is continuous up to \( \Gamma \). Let \( x_0 \in (-1, 1) \) and \( (x, y) \in U_1 \setminus \overline{\Omega} \).
We use \( f \lesssim g \) to denote the existence of a constant \( C \) such that \( f \leq C g \). Observe

\[
|G(x, y) - G(x_0, \Psi(x_0))| = \frac{1}{\pi} \int_{-1}^{1} h(t) \sqrt{1 + \Psi'(t)^2} \log \sqrt{(x-t)^2 + (y - \Psi(t))^2} \, dt
\]

\[
- \frac{1}{\pi} \int_{-1}^{1} h(t) \sqrt{1 + \Psi'(t)^2} \log \sqrt{(x_0-t)^2 + (\Psi(x_0) - \Psi(t))^2} \, dt
\]

\[
\lesssim \int_{-1}^{1} \left| \log((x-t)^2 + (y - \Psi(t))^2) - \log((x_0-t)^2 + (\Psi(x_0) - \Psi(t))^2) \right| \, dt
\]

\[
\leq \int_{-1}^{1} \left| \log((x-t)^2) - \log((x_0-t)^2) \right| \, dt + \int_{-1}^{1} \left| \log \left(1 + \frac{(y - \Psi(t))^2}{(x-t)^2}\right) - \log \left(1 + \frac{(\Psi(x_0) - \Psi(t))^2}{(x_0-t)^2}\right) \right| \, dt. \tag{7}
\]

The integral in (7) converges to zero as \((x, y) \to (x_0, \Psi(x_0))\) because the \(L^1\) norm is continuous with respect to translations. It remains to prove the integral in (8) also tends to zero. We carry out the substitution \( s = x - t \), and by supposing \((x, y)\) is sufficiently close to \((x_0, \Psi(x_0))\), there exists constants \( C_1, C_2 > 1 \) such that

\[
1 + \frac{(y - \Psi(x-s))^2}{s^2} \leq \frac{C_1}{s^2} \quad \text{and} \quad 1 + \frac{(\Psi(x_0) - \Psi(x-s))^2}{(x_0 - (x-s))^2} \leq C_2
\]

for almost every \( s \in [x-1, x+1] \). We then dominate the integrand as follows

\[
\chi_{[x-1, x+1]}(s) \left| \log \left(1 + \frac{(y - \Psi(x-s))^2}{s^2}\right) - \log \left(1 + \frac{(\Psi(x_0) - \Psi(x-s))^2}{(x_0 - (x-s))^2}\right) \right| \\
\leq |\log(C_1/s^2)| + \log C_2 \\
\leq \log C_1 + \log C_2 + 2 |\log s|,
\]

which lies within \(L^1([-2, 2])\). By applying the Dominated Converge Theorem we finish our proof that \( |G(x, y) - G(x_0, \Psi(x_0))| \to 0 \) as \((x, y) \to (x_0, \Psi(x_0))\). Notice that we have shown \( G(x, \Psi(x)) = g(x) - H(x) \) on \((-1, 1)\).

We will also show that the first order derivatives of \( G \) can be continuously extended to \( \Gamma \). We start by observing that

\[
\nabla G(x, y) \cdot \mathbf{n}(x) = \left( \frac{\partial G}{\partial x}(x, y), \frac{\partial G}{\partial y}(x, y) \right) \cdot \frac{1}{\sqrt{1 + \Psi'(x)^2}} (-\Psi'(x), 1)
\]

\[
= \frac{1}{\sqrt{1 + \Psi'(x)^2}} \left( \frac{\partial G}{\partial y}(x, y) - \Psi'(x) \frac{\partial G}{\partial x}(x, y) \right). \tag{9}
\]

for \((x, y) \in U_1 \setminus \overline{\Omega}\). We aim to show the right-hand side of equation (9) is continuous up to \( \Gamma \). We can write

\[
\frac{\partial G}{\partial y}(x, y) - \Psi'(x) \frac{\partial G}{\partial x}(x, y) = \frac{1}{\pi} \int_{-1}^{1} h(t) \sqrt{1 + \Psi'(t)^2} \left[ \frac{y - \Psi(t) - \Psi'(x)(x-t)}{(x-t)^2 + (y - \Psi(t))^2} \right] \, dt.
\]
Note
\[
\frac{y - \Psi(t) - \Psi'(x)(x-t)}{(x-t)^2 + (y - \Psi(t))^2} = \frac{y - \Psi(x)}{(x-t)^2 + (y - \Psi(t))^2} + \frac{\Psi(x) - \Psi(t) - \Psi'(x)(x-t)}{(x-t)^2 + (y - \Psi(t))^2},
\]
where
\[
\phi(x,t) = \Psi'(x) - \int_0^1 \Psi''((1-\tau)t + \tau x)\tau \, d\tau (x-t).
\]
We therefore define
\[
\mathcal{I}_1(x,y,t) := \frac{y - \Psi(x)}{(x-t)^2 + [y - \Psi(x) + \phi(x,t)(x-t)]^2}
\]
and
\[
\mathcal{I}_2(x,y,t) := \frac{\Psi(x) - \Psi(t) - \Psi'(x)(x-t)}{(x-t)^2 + (y - \Psi(t))^2}.
\]
We first investigate the limit of the integral \(\frac{1}{\pi} \int_{-1}^{1} h(t) \sqrt{1 + \Psi'(t)^2} \mathcal{I}_1(x,y,t) \, dt\) as \((x,y) \to (x_0, \Psi(x_0))\). The difference between \(\mathcal{I}_1\) and
\[
\tilde{\mathcal{I}}_1(x,y,t) := \frac{y - \Psi(x)}{(x-t)^2 + [y - \Psi(x) + \phi(x,x)(x-t)]^2}
\]
is
\[
\begin{align*}
|\mathcal{I}_1(x,y,t) - \tilde{\mathcal{I}}_1(x,y,t)| &
\leq \frac{|(y - \Psi(x))(\phi(x,x) - \phi(x,t))(x-t)^2 + 2(\phi(x,x) - \phi(x,t))(y - \Psi(x))^2(x-t)|}{(x-t)^2 + [y - \Psi(x) + \phi(x,t)(x-t)]^2}\] 
\[ \leq \frac{\left|\int_0^1 \Psi''((1-\tau)t + \tau x)\tau \, d\tau (\phi(x,x) + \phi(x,t))(y - \Psi(x))(x-t)^3\right|}{(x-t)^2 + [y - \Psi(x) + \phi(x,t)(x-t)]^2}
\[ + \frac{2\left|\int_0^1 \Psi''((1-\tau)t + \tau x)\tau \, d\tau (y - \Psi(x))^2(x-t)^2\right|}{(x-t)^2 + [y - \Psi(x) + \phi(x,x)(x-t)]^2}\]
\[\leq \frac{|y - \Psi(x)||x-t|^2}{(x-t)^2 + [y - \Psi(x) + \phi(x,x)(x-t)]^2}
\[+ \frac{|y - \Psi(x)|^2|x-t|^2}{(x-t)^2 + [y - \Psi(x) + \phi(x,x)(x-t)]^2}.
\]
We can bound this above using
\[
\frac{|y - \Psi(x)||x-t|}{(x-t)^2 + [y - \Psi(x) + \phi(x,x)(x-t)]^2}
\leq \frac{|y - \Psi(x) + \phi(x,x)(x-t)||x-t|}{(x-t)^2 + [y - \Psi(x) + \phi(x,x)(x-t)]^2} + \frac{\phi(x,x)||x-t|^2}{(x-t)^2 + [y - \Psi(x) + \phi(x,x)(x-t)]^2}
\leq 1 + \sup_{x \in [-1,1]} |\phi(x,x)|.
\]
and the similar estimate
\[
\frac{|y - Ψ(x)||x - t|}{(x - t)^2 + |y - Ψ(x) + φ(x, t)(x - t)|^2} \leq 1 + \sup_{(x,t)\in[-1,1]^2} |φ(x, t)|.
\]

Overall, \(|I_1(x, y, t) - \tilde{I}_1(x, y, t)|\) is bounded on \((U_1 \setminus \overline{B}) \times [-1, 1]\). Seeing that \(I_1(x, y, t) \to 0\) and \(\tilde{I}_1(x, y, t) \to 0\) as \((x, y) \to (x_0, Ψ(x_0))\) for almost every \(t \in [-1, 1]\), justifies being able to apply the Dominated Convergence Theorem to obtain
\[
\int_{-1}^{1} h(t)\sqrt{1 + Ψ'(t)^2} \left(I_1(x, y, t) - \tilde{I}_1(x, y, t)\right) dt \to 0
\]
as \((x, y) \to (x_0, Ψ(x_0))\).

Hence we can now focus our attention on the integral of \(\tilde{I}_1\). After substituting \(φ(x, x) = Ψ'(x)\) and some rearrangement we find
\[
\tilde{I}_1(x, y, t) = \frac{1}{y - Ψ(x)} \cdot \frac{1 + Ψ'(x)^2}{\left((1 + Ψ'(x)^2)\frac{x - t}{y - Ψ(x)} + Ψ'(x)\right)^2 + 1},
\]
which when integrated over \(\mathbb{R}\) yields
\[
\int_{-∞}^{∞} \tilde{I}_1(x, y, t) dt = \int_{-∞}^{∞} \frac{1 + Ψ'(x)^2}{\left(1 + Ψ'(x)^2\right)t + Ψ'(x))^2 + 1} dt
\]
\[= \tan^{-1} \left(\frac{1 + Ψ'(x)^2}{t + Ψ'(x)}\right) \bigg|_{t=-∞}^{t=∞}
\]
\[= \pi.
\]
We can use this fact to show \(\tilde{I}_1\) behaves like an approximation to the identity as \((x, y) \to (x_0, Ψ(x_0))\). Given \(ε > 0\), there exists \(0 < η < \min\{|x_0 - 1|, |x_0 + 1|\}\) small enough such that
\[
|h(t)\sqrt{1 + Ψ'(t)^2} - h(x_0)\sqrt{1 + Ψ'(x_0)^2}| < \frac{ε}{2}
\]
whenever \(|t - x_0| < η\). Now suppose \((x, y) \in U_1 \setminus \overline{B}\) satisfies
\[
|(x, y) - (x_0, Ψ(x_0))| < η/2.
\]
Then
\[
\int_{\mathbb{R} \setminus B_{η}(x_0)} \tilde{I}_1(x, y, t) dt \leq \int_{\mathbb{R} \setminus B_{η/2}(x)} \tilde{I}_1(x, y, t) dt
\]
\[= \int_{\mathbb{R} \setminus B_{\frac{η}{2(y - Ψ(x))}}(0)} \frac{1 + Ψ'(x)^2}{\left((1 + Ψ'(x)^2)t + Ψ'(x))^2 + 1} dt
\]
\[= \pi - \tan^{-1} \left(\frac{η}{2(y - Ψ(x))} + Ψ'(x)\right) + \tan^{-1} \left(-\frac{(1 + Ψ'(x)^2)η}{2(y - Ψ(x))} + Ψ'(x)\right)
\]
\[\to 0
\]
as \((x, y) \to (x_0, \Psi(x_0))\) so there exists \(\delta \in (0, \eta/2)\) such that

\[
\int_{\mathbb{R} \setminus B_\eta(x_0)} \widetilde{I}_4(x, y, t) \, dt < \frac{\pi \varepsilon}{4 \sup_{[-1,1]} |h\sqrt{1 + \Psi^2}|}
\]

for \(|(x, y) - (x_0, \Psi(x_0))| < \delta\). The right hand side is well defined since we are assuming \(h \neq 0\). When putting these inequalities together, we have

\[
\left| \frac{1}{\pi} \int_{-1}^{1} h(t) \sqrt{1 + \Psi'(t)^2} \widetilde{I}_1(x, y, t) \, dt - h(x_0) \sqrt{1 + \Psi'(x_0)^2} \right|
\]

\[
= \frac{1}{\pi} \int_{-\infty}^{\infty} \left( \chi_{[-1,1]}(t) h(t) \sqrt{1 + \Psi'(t)^2} - h(x_0) \sqrt{1 + \Psi'(x_0)^2} \right) \widetilde{I}_1(x, y, t) \, dt
\]

\[
\leq \frac{1}{\pi} \int_{B_\eta(x_0)} \left| h(t) \sqrt{1 + \Psi'(t)^2} - h(x_0) \sqrt{1 + \Psi'(x_0)^2} \right| \widetilde{I}_1(x, y, t) \, dt
\]

\[
+ \frac{2}{\pi} \sup_{[-1,1]} |h\sqrt{1 + \Psi^2}| \int_{\mathbb{R} \setminus B_\eta(x_0)} \widetilde{I}_1(x, y, t) \, dt
\]

\[
< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\]

for \(|(x, y) - (x_0, \Psi(x_0))| < \delta\). This proves

\[
\frac{1}{\pi} \int_{-1}^{1} h(t) \sqrt{1 + \Psi'(t)^2} \widetilde{I}_1(x, y, t) \, dt \to h(x_0) \sqrt{1 + \Psi'(x_0)^2}
\]

(11)

as \((x, y) \to (x_0, \Psi(x_0))\). Combining the limits (10) and (11) gives us

\[
\frac{1}{\pi} \int_{-1}^{1} h(t) \sqrt{1 + \Psi'(t)^2} \mathcal{I}_2(x, y, t) \, dt \to h(x_0) \sqrt{1 + \Psi'(x_0)^2}
\]

as \((x, y) \to (x_0, \Psi(x_0))\).

We can now look at our second integral \(\int_{-1}^{1} h(t) \sqrt{1 + \Psi'(t)^2} \mathcal{I}_2(x, y, t) \, dt\) as \((x, y) \to (x_0, \Psi(x_0))\). The upper bound

\[
|\mathcal{I}_2(x, y, t)| = \left| \frac{\Psi(x) - \Psi(t) - \Psi'(x)(x - t)}{(x - t)^2 + (y - \Psi(t))^2} \right|
\]

\[
= \left| \int_{0}^{1} \Psi''((1 - \tau)t + \tau x) \tau \, d\tau (x - t)^2 \right|
\]

\[
\leq \int_{0}^{1} |\Psi''((1 - \tau)t + \tau x)| \, d\tau
\]

\[
\leq \sup_{[-1,1]} |\Psi''|
\]

enables us to apply the Dominated Convergence Theorem resulting in

\[
\int_{-1}^{1} h(t) \sqrt{1 + \Psi'(t)^2} \mathcal{I}_2(x, y, t) \, dt \to \int_{-1}^{1} h(t) \sqrt{1 + \Psi'(t)^2} \mathcal{I}_2(x_0, \Psi(x_0), t) \, dt
\]
as \((x, y) \rightarrow (x_0, \Psi(x_0))\). Altogether we have
\[
\frac{\partial G}{\partial y}(x, y) - \Psi'(x) \frac{\partial G}{\partial x}(x, y)
= \frac{1}{\pi} \int_{-1}^{1} h(t) \sqrt{1 + \Psi'(t)^2} \left( I_1(x, y, t) + I_2(x, y, t) \right) dt
\]
\[
\rightarrow h(x_0) \sqrt{1 + \Psi'(x_0)^2} + \frac{1}{\pi} \int_{-1}^{1} h(t) \sqrt{1 + \Psi'(t)^2} I_2(x_0, \Psi(x_0), t) dt
\]
as \((x, y) \rightarrow (x_0, \Psi(x_0))\) and so
\[
\frac{\partial G}{\partial \mathbf{n}}(x, \Psi(x)) = h(x) + F(x),
\]
where
\[
F(x) := \frac{1}{\pi \sqrt{1 + \Psi'(x)^2}} \int_{-1}^{1} h(t) \sqrt{1 + \Psi'(t)^2} \left[ \frac{\Psi(x) - \Psi(t) - \Psi'(x)(y - \Psi(t))}{(x - t)^2 + (y - \Psi(t))^2} \right] dt
\]
for \(x \in (-1, 1)\).

We will now attempt to show
\[
\sqrt{1 + \Psi'(x)^2}(\nabla G(x, y) \cdot \mathbf{t}(x)) = \frac{\partial G}{\partial x}(x, y) + \Psi'(x) \frac{\partial G}{\partial y}(x, y)
= \frac{1}{\pi} \int_{-1}^{1} h(t) \sqrt{1 + \Psi'(t)^2} \left[ \frac{(x - t) + \Psi'(x)(y - \Psi(t))}{(x - t)^2 + (y - \Psi(t))^2} \right] dt
\]
has a continuous extension up to \(\Gamma\). We start by noticing
\[
\frac{(x - t) + \Psi'(x)(y - \Psi(t))}{(x - t)^2 + (y - \Psi(t))^2}
= \frac{(\Psi'(x) - \Psi'(t))(y - \Psi(t))}{(x - t)^2 + (y - \Psi(t))^2}
- \frac{\partial}{\partial t} \left( \log \sqrt{(x - t)^2 + (y - \Psi(t))^2} \right)
\]
\[
= \frac{(\Psi'(x) - \Psi'(t))(y - \Psi(x))}{(x - t)^2 + (y - \Psi(t))^2}
+ \frac{(\Psi'(x) - \Psi'(t))(\Psi(x) - \Psi(t))}{(x - t)^2 + (y - \Psi(t))^2}
- \frac{\partial}{\partial t} \left( \log \sqrt{(x - t)^2 + (y - \Psi(t))^2} \right)
\]
and as a result define
\[
I_3(x, y, t) := \frac{(\Psi'(x) - \Psi'(t))(y - \Psi(x))}{(x - t)^2 + (y - \Psi(t))^2},
\]
\[
I_4(x, y, t) := \frac{(\Psi'(x) - \Psi'(t))(\Psi(x) - \Psi(t))}{(x - t)^2 + (y - \Psi(t))^2},
\]
and
\[
I_5(x, y, t) := \frac{\partial}{\partial t} \left( \log \sqrt{(x - t)^2 + (y - \Psi(t))^2} \right).
\]
Since \(I_3(x, y, t) = (\Psi'(x) - \Psi'(t))I_4(x, y, t)\), our previous work shows
\[
\int_{-1}^{1} h(t) \sqrt{1 + \Psi'(t)^2} I_3(x, y, t) dt \rightarrow 0
\]
as \((x, y) \to (x_0, \Psi(x_0))\). For \(\int_{-1}^{1} h(t) \sqrt{1 + \Psi'(t)^2} I_4(x, y, t) \, dt\) we have

\[
\left| I_4(x, y, t) \right| = \frac{\left| \int_0^1 \Psi''((1 - \tau)x + \tau t) \, d\tau \right| \left| \int_0^1 \Psi'((1 - \tau)x + \tau t) \, d\tau \right| (x - t)^2}{(x - t)^2 + (y - \Psi(t))^2}
\]

implying

\[
\int_{-1}^{1} h(t) \sqrt{1 + \Psi'(t)^2} I_4(x, y, t) \, dt \to \int_{-1}^{1} h(t) \sqrt{1 + \Psi'(t)^2} I_4(x_0, \Psi(x_0), t) \, dt
\]
as \((x, y) \to (x_0, \Psi(x_0))\) by the Dominated Convergence Theorem. Lastly, using integration by parts

\[
\int_{-1}^{1} h(t) \sqrt{1 + \Psi'(t)^2} I_5(x, y, t) \, dt
\]

which has a limit as \((x, y) \to (x_0, \psi(x_0))\) since the above integral has the same form as the integral given by \(G\). Overall, we have shown that

\[
\frac{\partial G}{\partial x}(x, y) + \Psi'(x) \frac{\partial G}{\partial y}(x, y)
\]

has a continuous extension to \(\Gamma\). Since both

\[
\frac{\partial G}{\partial y}(x, y) - \Psi'(x) \frac{\partial G}{\partial x}(x, y) \quad \text{and} \quad \frac{\partial G}{\partial x}(x, y) + \Psi'(x) \frac{\partial G}{\partial y}(x, y)
\]

have continuous extensions to \(\Gamma\), it implies that \(\frac{\partial G}{\partial x}\) and \(\frac{\partial G}{\partial y}\) also have continuous extensions to \(\Gamma\).

Altogether, we have shown that \(G \in C^2(U_1 \setminus \overline{\Omega}) \cap C^1(U_1 \setminus \Omega)\) solves

\[
\begin{cases}
\Delta G(x, y) = 0 & \text{for } (x, y) \in U_1 \setminus \overline{\Omega} \\
G(x, \Psi(x)) = g(x) - H(x) & \text{for } x \in (-1, 1) \\
\frac{\partial G}{\partial n}(x, \Psi(x)) = h(x) + F(x) & \text{for } x \in (-1, 1).
\end{cases}
\]

Now to construct a solution to the Cauchy problem \((\mathbb{Q})\), it is enough to find an open connected neighbourhood \(U_2\) of \(\Gamma\) and \(W \in C^2(U_2 \setminus \overline{\Omega}) \cap C^1(U_2 \setminus \Omega)\) solving

\[
\begin{cases}
\Delta W(x, y) = 0 & \text{for } (x, y) \in U_2 \setminus \overline{\Omega} \\
W(x, \Psi(x)) = H(x) & \text{for } x \in (-1, 1) \\
\frac{\partial W}{\partial n}(x, \Psi(x)) = -F(x) & \text{for } x \in (-1, 1).
\end{cases}
\]
Note that $F$ is real analytic by the same reasoning we used to justify integral (6) is real analytic. Since $F$ and $H$ are real analytic, the existence of such a $W$ is guaranteed by the Cauchy–Kovalevskaya Theorem. Finally, we have that $u := G + W$ solves the Cauchy problem (2) for $U = U_1 \cap U_2$.

Using this result, we return to the Cauchy problem (1). As in the introduction, we take $\Omega \subset \mathbb{R}^2$ to be an open bounded and simply connected set with real analytic boundary. We finish this section with a proof of the following theorem relating the existence of solutions to the Cauchy problem (1) to the boundary regularity.

**Theorem 2.3.** Let $f, h \in C^1(\partial \Omega)$. There exists an open connected set $U \subset \mathbb{R}^2$ satisfying $\Omega \subset U$ and a vector field $B \in C^1(U \setminus \Omega; \mathbb{R}^2) \cap C(U \setminus \Omega; \mathbb{R}^2)$ that solves the Cauchy problem (1) if and only if $f - \mathcal{H} h$ is real analytic on $\partial \Omega$.

**Proof.** We first prove that $f - \mathcal{H} h$ is necessarily real analytic. Suppose that $B \in C^1(U \setminus \Omega; \mathbb{R}^2) \cap C(U \setminus \Omega; \mathbb{R}^2)$ solves the Cauchy problem (1). We will show $f - \mathcal{H} h$ is real analytic at $v_0 \in \partial \Omega$. We start by noticing that there exists an open neighbourhood $V \subset U$ of $v_0$ such that $V \setminus \Omega$ is simply connected. Since on simply connected domains every harmonic vector field has a harmonic scalar potential, there exists $u \in C^2(V \setminus \Omega) \cap C^1(V \setminus \Omega)$ satisfying $B = \nabla u$. Furthermore, since every real analytic curve is locally the graph of a real analytic function, there exists $a > 0$ and $\Psi : [-a, a] \rightarrow \mathbb{R}$ such that

$$\Gamma := \{ v_0 + \bar{x} \mathbf{t}(v_0) + \Psi(\bar{x}) \mathbf{n}(v_0) : \bar{x} \in (-a, a) \}$$

is a segment of $\partial \Omega$ containing $v_0$, $\Gamma$ lies within $V$, and $\Psi$ is has a real analytic extension to an open neighbourhood of $[-a, a]$. That the vector field $B$ is a solution to the Cauchy problem (1) implies that $u$ solves

$$\begin{cases}
\Delta u = 0 & \text{in } V \setminus \Omega \\
\nabla u \cdot \mathbf{t} = f & \text{on } \Gamma \\
\frac{\partial u}{\partial \mathbf{n}} = h & \text{on } \Gamma.
\end{cases}$$

We now perform the coordinate transformation

$$(x, y) = T(\bar{x}, \bar{y}) := v_0 + \bar{x} \mathbf{t}(v_0) + \bar{y} \mathbf{n}(v_0),$$

with $\bar{u}(\bar{x}, \bar{y}) := u(T(\bar{x}, \bar{y}))$ to system (13). Note that $T$ is an isometry and the Laplacian is invariant under isometries. $T$ transforms the equation $\nabla u \cdot \mathbf{t} = f$ to

$$\nabla \bar{u}(\bar{x}, \Psi(\bar{x})) \cdot (1, \Psi(\bar{x})) = \tilde{f}(\bar{x}) \sqrt{1 + \Psi(\bar{x})^2},$$

where $\tilde{f}(\bar{x}) := f(T(\bar{x}, \Psi(\bar{x})))$. By the Fundamental Theorem of Calculus for Line Integrals, equation (15) can be integrated to obtain $\bar{u}(\bar{x}, \Psi(\bar{x})) = \tilde{g}(\bar{x})$ where $\tilde{g}'(\bar{x}) = \tilde{f}(\bar{x}) \sqrt{1 + \Psi(\bar{x})^2}$. Overall, by letting

$$\tilde{V} := T^{-1} V, \quad \tilde{\Omega} := T^{-1} \Omega, \quad \tilde{\mathbf{n}} := T^{-1} \mathbf{n}, \quad \tilde{h}(\bar{x}) := h(T(\bar{x}, \Psi(\bar{x}))),$$

we have a solution $\tilde{u}$ to

$$\begin{cases}
\Delta \bar{u}(\bar{x}, \bar{y}) = 0 & \text{for } (\bar{x}, \bar{y}) \in \tilde{V} \setminus \tilde{\Omega} \\
\bar{u}(\bar{x}, \Psi(\bar{x})) = \tilde{g}(\bar{x}) & \text{for } \bar{x} \in (-a, a) \\
\frac{\partial \bar{u}}{\partial \tilde{\mathbf{n}}}(\bar{x}, \Psi(\bar{x})) = \tilde{h}(\bar{x}) & \text{for } \bar{x} \in (-a, a).
\end{cases}$$

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Theorem 2.1 tells us the existence of a solution to this system implies that

$$H(x) := \tilde{g}(x) - \frac{1}{\pi} \int_{-a}^{a} \tilde{h}(t) \sqrt{1 + \Psi'(t)^2} \log \sqrt{(x - t)^2 + (\Psi(x) - \Psi(t))^2} dt$$  \hspace{1cm} (17)$$

is real analytic on $(-a, a)$. It follows that the derivative of $H$ is also real analytic on $(-a, a)$. We will now show that in a Cauchy principle value sense we can interchange the integral in $H$ with a derivative. Care has to be taken around the singularity of the integrand. For small positive $\varepsilon$ we define $I_\varepsilon(x) = (x - \varepsilon, x + \varepsilon)$ and

$$J_\varepsilon(x) := \frac{1}{\pi} \int_{-a}^{a} \tilde{h}(t) \sqrt{1 + \Psi'(t)^2} \log \sqrt{(x - t)^2 + (\Psi(x) - \Psi(t))^2} dt,$$

which satisfies $H(x) = \tilde{g}(x) - \lim_{\varepsilon \to 0} J_\varepsilon(x)$. Using the Leibniz integral rule we have

$$J_\varepsilon'(x) = \int_{[-a, a] \setminus I_\varepsilon(x)} I(x, t) dt + R_\varepsilon(x),$$

where

$$I(x, t) := \tilde{h}(t) \sqrt{1 + \Psi'(t)^2} \frac{(x - t + \Psi'(x)(\Psi(x) - \Psi(t))}{(x - t)^2 + (\Psi(x) - \Psi(t))^2}$$

and

$$R_\varepsilon(x) := \tilde{h}(x - \varepsilon) \sqrt{1 + \Psi'(x - \varepsilon)^2} \log(\sqrt{\varepsilon^2 + (\Psi(x) - \Psi(x - \varepsilon))^2}) - \tilde{h}(x + \varepsilon) \sqrt{1 + \Psi'(x + \varepsilon)^2} \log(\sqrt{\varepsilon^2 + (\Psi(x) - \Psi(x + \varepsilon))^2}).$$

Our next step is to prove $J_\varepsilon'$ converges uniformly as $\varepsilon$ goes to zero. Firstly, $|R_\varepsilon(x)| \lesssim \varepsilon \log \varepsilon + \varepsilon$ and so $R_\varepsilon \to 0$ uniformly as $\varepsilon \to 0$. Secondly, observe that

$$I(x, t) = \frac{\psi(x, t)}{x - t},$$

where

$$\psi(x, t) := \tilde{h}(t) \sqrt{1 + \Psi'(t)^2} \frac{1 + \Psi'(x)\phi(x, t)}{1 + \phi(x, t)^2}$$

and $\phi \in C^2([-a, a]^2)$ satisfies

$$\Psi(x) - \Psi(t) = \phi(x, t)(x - t).$$

Note that $\psi \in C^1([-a, a]^2)$ and so by letting $0 < \delta < \varepsilon$ we have

$$\left| \int_{[-a, a] \setminus I_\varepsilon(x)} I(x, t) dt - \int_{[-a, a] \setminus I_\delta(x)} I(x, t) dt \right|$$

$$= \left| \int_{I_\varepsilon(x) \setminus I_\delta(x)} \frac{\psi(x, t)}{x - t} dt \right|$$

$$= \left| \int_{I_\varepsilon(x) \setminus I_\delta(x)} \frac{\psi(x, t)}{x - t} dt - \psi(x, x) \int_{I_\delta(x) \setminus I_\delta(x)} \frac{1}{x - t} dt \right|$$

$$\leq \int_{I_\varepsilon(x) \setminus I_\delta(x)} \left| \psi(x, x) - \psi(x, t) \right| \frac{1}{x - t} dt$$

$$\leq \int_{I_\varepsilon(x) \setminus I_\delta(x)} \left| \psi(x, x) \right| \frac{1}{x - t} dt$$

$$\leq \varepsilon \sup_{[-a, a]^2} |\partial_2 \psi|.$$
This proves that \( \int_{[-a,a] \setminus I_{\epsilon}(\bar{x})} I(\bar{x}, t) \, dt \) is uniformly Cauchy and so converges uniformly as \( \epsilon \to 0 \). Therefore, we have shown \( J'_\epsilon \) converges uniformly to \( \lim_{\epsilon \to 0} \int_{[-a,a] \setminus I_{\epsilon}(\bar{x})} I(\bar{x}, t) \, dt \) as \( \epsilon \to 0 \) which in turn proves

\[
H'(\bar{x}) = g'(\bar{x}) - \frac{1}{\pi} \lim_{\epsilon \to 0} \int_{[-a,a] \setminus I_{\epsilon}(\bar{x})} I(\bar{x}, t) \, dt.
\]

We will show that we can replace \( I_{\epsilon}(\bar{x}) \) with the set

\[
S_{\epsilon}(\bar{x}) := \{ t \in [-a,a] : |(\bar{x}, \Psi(\bar{x})) - (t, \Psi(t))| < \epsilon \}
\]

to make the integral independent on the parameterisation of \( \Gamma \). This independence will come from the fact that

\[
\Gamma \cap B_{\epsilon}(T(\bar{x}, \Psi(\bar{x}))) = \{ T(t, \Psi(t)) \in \mathbb{R}^2 : t \in S_{\epsilon}(\bar{x}) \}.
\]

Note that if \( t \in S_{\epsilon}(\bar{x}) \), then

\[
(\bar{x} - t)^2 + (\Psi(\bar{x}) - \Psi(t))^2 < \epsilon^2,
\]

which implies

\[
|\bar{x} - t| < \frac{\epsilon}{\sqrt{1 + \phi(\bar{x}, t)^2}}.
\]

For ease of notation it will be useful to define \( \varphi \in C^2([-a,a]^2) \) by

\[
\varphi(\bar{x}, t) := \frac{1}{\sqrt{1 + \phi(\bar{x}, t)^2}}.
\]

We now wish to show,

\[
\lim_{\epsilon \to 0} \int_{[-a,a] \setminus I_{\epsilon}(\bar{x})} I(\bar{x}, t) \, dt = \lim_{\epsilon \to 0} \int_{[-a,a] \setminus S_{\epsilon}(\bar{x})} I(\bar{x}, t) \, dt.
\]

Fix \( \zeta > 0 \). Let \( 0 < \eta < \varphi(\bar{x}, \bar{x}) \) be small enough such that

\[
\log \left( \frac{\varphi(\bar{x}, \bar{x}) + \eta}{\varphi(\bar{x}, \bar{x}) - \eta} \right) < \frac{\zeta}{2 \sup_{[-a,a]^2} |\psi|}.
\]

Since \( \varphi \) is continuous there exists \( \epsilon > 0 \) small enough such that if \( t \in [-a,a] \) and \( |\bar{x} - t| < \epsilon \), then \( |\varphi(\bar{x}, t) - \varphi(\bar{x}, \bar{x})| < \eta \). Now that we have chosen an \( \epsilon \), let \( t \in S_{\epsilon}(\bar{x}) \). It follows that \( |\bar{x} - t| < \epsilon \) and \( |\bar{x} - t| < \epsilon \varphi(\bar{x}, t) \). From this it is evident that

\[
I_{\epsilon(\varphi(\bar{x}, \bar{x}) - \eta)}(\bar{x}) \subset S_{\epsilon}(\bar{x}) \subset I_{\epsilon(\varphi(\bar{x}, \bar{x}) + \eta)}(\bar{x}),
\]

and obviously

\[
I_{\epsilon(\varphi(\bar{x}, \bar{x}) - \eta)}(\bar{x}) \subset I_{\epsilon(\varphi(\bar{x}, \bar{x}) + \eta)}(\bar{x}) \subset I_{\epsilon(\varphi(\bar{x}, \bar{x}) + \eta)}(\bar{x}).
\]
These inclusions guarantee
\[
\left| \int_{[-a,a]\setminus I_{x'}(\bar{x}_{x'})} \mathcal{I}(\bar{x}, t) \, dt - \int_{[-a,a]\setminus S_c(\bar{x})} \mathcal{I}(\bar{x}, t) \, dt \right|
\]
\[
= \left| \int_{S_c(\bar{x})\setminus I_{x'}(\bar{x}_{x'})} \mathcal{I}(\bar{x}, t) \, dt - \int_{I_{x'}(\bar{x}_{x'})\setminus S_c(\bar{x})} \mathcal{I}(\bar{x}, t) \, dt \right|
\]
\[
\leq \int_{S_c(\bar{x})\setminus I_{x'}(\bar{x}_{x'})} |\mathcal{I}(\bar{x}, t)| \, dt + \int_{I_{x'}(\bar{x}_{x'})\setminus S_c(\bar{x})} |\mathcal{I}(\bar{x}, t)| \, dt
\]
\[
\leq \int_{I_{x'}(\bar{x}_{x'})} |\mathcal{I}(\bar{x}, t)| \, dt
\]
\[
\leq \sup_{[-a,a]^2} |\psi| \int_{I_{x'}(\bar{x}_{x'})} \left| \frac{1}{\bar{x} - t} \right| \, dt
\]
\[
= 2 \sup_{[-a,a]^2} |\psi| \int_{I_{x'}(\bar{x}_{x'})} \frac{1}{t} \, dt
\]
\[
= 2 \sup_{[-a,a]^2} |\psi| \log \left( \frac{\varphi(\bar{x}, \bar{x}_{x'}) + \eta}{\varphi(\bar{x}, \bar{x}_{x'}) - \eta} \right)
\]
\[
< \zeta
\]
as required. We can conclude that
\[
H'(\bar{x}) = \tilde{g}'(\bar{x}) - \frac{1}{\pi} \lim_{t \to 0} \int_{[-a,a]\setminus S_c(\bar{x})} \mathcal{I}(\bar{x}, t) \, dt.
\]
Observe that we can rewrite \( \mathcal{I}(\bar{x}, t)/\sqrt{1 + \Psi'(\bar{x})^2} \) as
\[
\frac{\mathcal{I}(\bar{x}, t)}{\sqrt{1 + \Psi'(\bar{x})^2}} = \tilde{h}(t) |(t, \Psi'(t))| \left[ (1, \Psi'(t)) \cdot (\bar{x}, \Psi(\bar{x})) - (t, \Psi(t)) \right]
\]
\[
= h(T(t, \Psi(t))) |T(t, \Psi'(t))| \left[ T(1, \Psi'(\bar{x})) \cdot (T(\bar{x}, \Psi(\bar{x})) - T(t, \Psi(t))) \right]
\]
Recall that \( T(\bar{x}, \Psi(\bar{x})) \) is a parameterisation of \( \Gamma \). Hence we define the parameterisation \( \gamma(s) := T(s, \Psi(s)) \) for \( s \in [-a, a] \). Since \( T \) is an isometry, it holds that \( \gamma'(s) = T(1, \Psi'(s)) \). Therefore,
\[
\frac{\mathcal{I}(s, t)}{\sqrt{1 + \Psi'(s)^2}} = h(\gamma(t)) |\gamma'(t)| \left[ \frac{\gamma'(s) \cdot (\gamma(s) - \gamma(t))}{\gamma'(s) |\gamma(s) - \gamma(t)|^2} \right]
\]
\[
= h(\gamma(t)) |\gamma'(t)| \left[ \frac{t(\gamma(s) \cdot (\gamma(s) - \gamma(t))}{|\gamma(s) - \gamma(t)|^2} \right].
\]
Furthermore, by recalling the formula for \( \tilde{g}' \), it follows that
\[
\tilde{g}'(s) = \tilde{f}(s) \sqrt{1 + \Psi'(s)^2} = f(\gamma(s)) \sqrt{1 + \Psi'(s)^2}
\]
By substituting these expressions into our equation for \( H' \) gives us
\[
\frac{H'(s)}{\sqrt{1 + \Psi'(s)^2}} = f(\gamma(s)) - \frac{1}{\pi} \lim_{t \to 0} \int_{[-a,a]\setminus S_c(s)} h(\gamma(t)) |\gamma'(t)| \left[ \frac{t(\gamma(s) \cdot (\gamma(s) - \gamma(t))}{|\gamma(s) - \gamma(t)|^2} \right] \, dt
\]
\[
= f(\gamma(s)) - \frac{1}{\pi} \lim_{t \to 0} \int_{\Gamma \setminus B_s(\gamma(s))} h(\omega) \left[ \frac{t(\gamma(s) \cdot (\gamma(s) - \omega))}{|\gamma(s) - \omega|^2} \right] \, d\omega.
\]
Since the left-hand side is real analytic on \((-a,a)\), and \(\gamma(0) = v_0\), it follows by definition that
\[
f(v) - \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{\Gamma \setminus B_r(v)} h(w) \frac{t(v) \cdot (v - w)}{|v - w|^2} \, dw \tag{18}
\]
is real analytic at \(v = v_0\).

It remains to show
\[
\int_{\partial \Gamma \setminus \Gamma} h(w) \frac{t(v) \cdot (v - w)}{|v - w|^2} \, dw \tag{19}
\]
is real analytic at \(v = v_0\). Let \(\sigma : I \to \mathbb{R}^2\) be a real analytic parameterisation of \(\partial \Omega \setminus \Gamma\). Since there is a positive distance between \(\partial \Omega \setminus \Gamma\) and \(v_0\), there exists \(r > 0\) small enough such that \(\gamma\) has a complex analytic extension to the complex disc \(D_r(0)\) and \(|\gamma(z) - \sigma(t)| > 0\) for all \(z \in D_r(0)\) and \(t \in I\). Consequently,
\[
h(\sigma(t)) |\sigma'(t)| \left[ \frac{t(\gamma(z)) \cdot (\gamma(z) - \sigma(t))}{|\gamma(z) - \sigma(t)|^2} \right]
\]
is complex analytic on \(D_r(0)\) for all \(t \in I\). We can now apply Lemma 2.2 to justify
\[
\int_I h(\sigma(t)) |\sigma'(t)| \left[ \frac{t(\gamma(z)) \cdot (\gamma(z) - \sigma(t))}{|\gamma(z) - \sigma(t)|^2} \right] \, dt
\]
being complex analytic on \(D_r(0)\). Thus the integral (19) is real analytic at \(v = v_0\).

By combining the real analyticity of expressions (18) and (19), we obtain the real analyticity of \(f - \mathcal{H}h\) at \(v_0\). As \(v_0\) was chosen arbitrarily, \(f - \mathcal{H}h\) is real analytic on \(\partial \Omega\). This concludes the necessity section of the proof.

For the sufficiency section of the proof, we start by assuming \(f - \mathcal{H}h\) is real analytic on \(\partial \Omega\). We begin by reversing the arguments used in the necessity part. Afterwards, we will have to make sure solutions over different regions coincide on their overlap.

Given \(v_0 \in \partial \Omega\), let the boundary segment \(\Gamma\) in (12) and coordinate transform \(T\) in (14) be defined as before. By reversing previous arguments, the function \(f - \mathcal{H}h\) being real analytic on \(\Gamma\) implies \(H\), defined in (17), is real analytic on \((-a,a)\). Therefore, by Theorem 2.1 there exists \(\tilde{V} \subset \mathbb{R}^2\), an open neighbourhood of \(T\Gamma\), and \(\tilde{u} \in C^2(\tilde{V} \setminus \Omega) \cap C^1(\tilde{V} \setminus \Omega)\) that solves the scalar system (19). If we now perform the coordinate transform \(T^{-1}\) on system (16), then \(u(x,y) := \tilde{u}(T^{-1}(x,y))\) solves system (13) with \(V := TV\). Therefore, the vector field \(B := \nabla u\) satisfies \(B \in C^1(V \setminus \Omega; \mathbb{R}^2) \cap C(V \setminus \Omega; \mathbb{R}^2)\) and solves
\[
\begin{align*}
\text{div } B &= 0 \quad \text{in } V \setminus \Omega \tag{20a} \\
\text{curl } B &= 0 \quad \text{in } V \setminus \Omega \tag{20b} \\
B \cdot t &= f \quad \text{on } \Gamma \tag{20c} \\
B \cdot n &= h \quad \text{on } \Gamma. \tag{20d}
\end{align*}
\]

For all \(v \in \partial \Omega\) we can find a boundary segment \(\Gamma_v \subset \partial \Omega\) that is the graph of a real analytic function and contains \(v\). We can apply the above method to obtain, for every \(v \in \partial \Omega\), an open neighbourhood \(V_v \subset \mathbb{R}^2\) of \(\Gamma_v\), and vector field \(B_v \in C^1(V_v \setminus \Omega; \mathbb{R}^2) \cap C(V_v \setminus \Omega; \mathbb{R}^2)\) solving system (20) with \(V = V_v\) and \(\Gamma = \Gamma_v\). To show that the \(\{B_v\}_{v \in \partial \Omega}\) can be combined to form a solution to the Cauchy problem (11), we need to make sure the \(B_v\) coincide on the regions where they overlap. We will do this by restricting our vector fields to regions which we call exterior collar neighbourhoods.
Given boundary segment \( \Gamma \subseteq \partial \Omega \) and continuous function \( l : \Gamma \rightarrow (0, \infty) \), we define the fibre \( F(w) := \{ w + \varepsilon n(w) : \varepsilon \in [0, l(w)) \} \) for \( w \in \Gamma \). If the collection of fibres \( \{ F(w) \}_{w \in \Gamma} \) is pairwise disjoint, we call \( N := \bigcup_{w \in \Gamma} F(w) \) an exterior collar neighbourhood of \( \Gamma \). We also say \( N \) has width \( l(w) \) at \( w \in \Gamma \). An example of an exterior collar neighbourhood is given by the shaded region in Figure 2. Note, the existence of an exterior collar neighbourhood of a curve is guaranteed if the curve is \( C^2 \). Furthermore, as \( \partial \Omega \) is compact and sufficiently regular there exists a constant \( l^* \in (0, \infty] \) such that \( N^* := \bigcup_{w \in \partial \Omega} \{ w + \varepsilon n(w) : \varepsilon \in [0, l^*) \} \) is an exterior collar neighbourhood of \( \partial \Omega \) with constant width \( l^* \).

![Figure 2: Exterior collar neighbourhood \( N \).](image)

For each \( v \in \partial \Omega \) it is easy to construct an exterior collar neighbourhood \( N_v \) of \( \Gamma_v \) that is contained within \( V_v \cap N^* \). We restrict the local solutions \( B_v \) to \( N_v \) in order to avoid overlaps where the \( B_v \) do not coincide. Choosing the \( N_v \) to be within \( N^* \) guarantees that for distinct \( v, w \in \partial \Omega \), the intersection \( N_v \cap N_w \) is connected and in particular an exterior collar neighbourhood of \( \Gamma_v \cap \Gamma_w \). This is trivially satisfied if \( N_v \cap N_w \) and \( \Gamma_v \cap \Gamma_w \) are empty.

![Figure 3: Intersection of exterior collar neighbourhoods \( N_v \) and \( N_w \).](image)

Now \( B_v \) and \( B_w \) solve system (20) with \( V \setminus \Omega = N_v \cap N_w \) and \( \Gamma = \Gamma_v \cap \Gamma_w \). Holmgren’s Uniqueness Theorem [1] §2.3] tells us that for open connected neighbourhoods \( V \) of \( \Gamma \), solutions to system (20) are unique. Therefore, \( B_v \) and \( B_w \) must coincide on \( N_v \cap N_w \). Hence, the vector field \( B^* \), defined pointwise by \( B^*(w) := B_v(w) \) for \( w \in N_v \), is well defined on the exterior collar neighbourhood \( U^* := \bigcup_{v \in \partial \Omega} N_v \) of \( \partial \Omega \). The vector field \( B^* \) also solves system (20) with \( V = U^* \cup \Omega \) and \( \Gamma = \partial \Omega \). As a result, \( U = U^* \cup \Omega \) and \( B^* \) solve the Cauchy problem (1).

3 Cauchy–Kovalevskaya Theorem and Distance from Boundary

We recall from the introduction that the equations \( \text{div} \, B = \text{curl} \, B = 0 \) can be viewed as the Cauchy–Riemann equations of \( B = B_1 - iB_2 \) with respect to \( z = x + iy \). Furthermore,
the equations can be combined into the single complex equation

\[ \frac{\partial B}{\partial y} = i \frac{\partial B}{\partial x}. \]

We think of \( B \) as a function from \( \mathbb{R}^2 \) to \( \mathbb{C} \). Let \( \gamma : \mathbb{T} \to \partial \Omega \) be a real analytic parameterisation for \( \partial \Omega \), oriented such that \( \gamma' = (-\gamma_2',\gamma_1') \) is an outward normal to \( \Omega \). The boundary conditions \( B \cdot t = f \) and \( B \cdot n = h \) are equivalent to \( B(\gamma(t)) = \Theta(t) \) where

\[ \Theta(t) := (\gamma_1'(t) - i\gamma_2'(t)) \left( \frac{f(\gamma(t)) - ih(\gamma(t))}{|\gamma'(t)|} \right) \tag{21} \]

and \( |\gamma'(t)| = \sqrt{\gamma_1'(t)^2 + \gamma_2'(t)^2} \).

We can therefore rewrite the Cauchy problem (1) as

\[
\begin{cases}
\frac{\partial B}{\partial y} = i \frac{\partial B}{\partial x} \\ B(\gamma) = \Theta,
\end{cases}
\tag{22a}
\]

\[
\begin{cases}
\frac{\partial B}{\partial \tilde{x}} = \frac{\partial B}{\partial \tilde{y}} \\ B(\gamma(t)) = \Theta(t),
\end{cases}
\tag{22b}
\]

where it is understood that equation (22a) is being solved in a neighbourhood of \( \partial \Omega \subset \mathbb{R}^2 \). Given \( f, h \in C^\omega(\partial \Omega) \), we can solve this system using the Cauchy–Kovalevskaya Theorem. Note that the Cauchy–Kovalevskaya Theorem solves the system on both sides of \( \partial \Omega \) simultaneously. Unfortunately, this means any result provided by Cauchy–Kovalevskaya Theorem on the external distance at which we can solve, may be affected by singularities that arise on the inside of \( \Omega \).

We would like to gain quantitative information on the size of the region on which we can solve the first-order system (22). To do this, we will follow the proof of the Cauchy–Kovalevskaya Theorem given in [1, §2.2], and then find the domains where the Taylor series converges. We will focus our attention on finding how far we can solve system (22) from the boundary point \( \gamma(t_0) \) for some arbitrary \( t_0 \in \mathbb{T} \). Then, we use the same procedure in patching together solutions as in the end of the proof of Theorem 2.3.

The first step is to transform our system so that the boundary is flat. We start by considering the variables \((\tilde{x}, \tilde{y})\) defined according to

\[ (x, y) = \gamma(\tilde{x}) + \tilde{y}\gamma'(t_0) \tag{23a} \]

This change of variables from \((x, y)\) to \((\tilde{x}, \tilde{y})\) has the effect of flattening the boundary since the curve \( \gamma \) is mapped to the line \( \tilde{y} = 0 \). Furthermore, the variables can be described as follows: As \( \tilde{x} \) varies we travel along the curve \( \gamma \) whereas as \( \tilde{y} \) varies we travel in the fixed direction \( \gamma'(t_0) \), not in the normal direction to the curve. By changing our variables to \((\tilde{x}, \tilde{y})\), system (22) becomes

\[
\begin{cases}
\frac{\partial B}{\partial \tilde{y}} = i \left( \frac{\gamma_1'(t_0) + i\gamma_2'(t_0)}{\gamma_1' + i\gamma_2'} \right) \frac{\partial B}{\partial \tilde{x}} \\ B(\gamma(t)) = \Theta(t),
\end{cases}
\tag{23b}
\]

This change of variables is well defined since there exists \( \delta > 0 \) such that it is a diffeomorphism on \((\tilde{x}, \tilde{y}) \in (t_0 - \delta, t_0 + \delta) \times \mathbb{R}\). It is important to note that our change of variables has been chosen such that the coefficient in the partial differential equation (23a) does not depend on \( \tilde{y} \).
Now by setting \( \phi(\bar{x}, \bar{y}) = B(\bar{x}, \bar{y}) - \Theta(\bar{x}) \),
we obtain
\[
\begin{align*}
\frac{\partial \phi}{\partial \bar{y}} &= \Lambda \frac{\partial \phi}{\partial \bar{x}} + \Lambda \Theta' & \text{(24a)} \\
\phi(\bar{x}, 0) &= 0, & \text{(24b)}
\end{align*}
\]
where
\[
\Lambda(\bar{x}) := i \left( \frac{\gamma_1'(t_0) + i \gamma_2'(t_0)}{\gamma_1'(\bar{x}) + i \gamma_2'(\bar{x})} \right). & \text{(25)}
\]
By rewriting the system in this form, we can determine all the partial derivatives of \( \phi \) at \((t_0, 0)\) in terms of the derivatives of \( \Lambda \) and \( \Lambda \Theta' \) at \( t_0 \). The derivatives with respect to \( \bar{x} \)
are zero due to the boundary condition \((24b)\), and the mixed
derivatives can be deduced from equation \((24a)\) to be polynomials
with positive coefficients and whose variables are the derivatives of \( \Lambda \) and \( \Lambda \Theta' \) at \( t_0 \).
Therefore, a Taylor series for \( \phi \) in terms of \( \bar{x} \) and \( \bar{y} \)
which is based at \((t_0, 0)\) can be constructed. If it can be shown that
this Taylor series converges, then it solves system \((24)\) within its domain of convergence. This is achieved
by replacing \( \Lambda \) and \( \Lambda \Theta' \) in equation \((24a)\) with functions whose derivatives at \( t_0 \) have a
larger magnitude, and then showing that this new system has an explicit solution with convergent
Taylor series.

Let \( R_1(t_0), R_2(t_0) > 0 \) be the radii of convergence for the Taylor series
\[
\Lambda(\bar{x}) = \sum_{n=0}^{\infty} b_n(t_0)(\bar{x} - t_0)^n \quad \text{and} \quad \Lambda(\bar{x})\Theta'(\bar{x}) = \sum_{n=0}^{\infty} c_n(t_0)(\bar{x} - t_0)^n
\]
respectively. Note that \( b_0 = \Lambda(t_0) = i \). For \( r \in (0, \min\{R_1, R_2\}) \) let
\[
M_1(r) := \sup\{1, |b_1| r, |b_2| r^2, \ldots\}
\]
and
\[
M_2(r) := \sup\{|c_0|, |c_1| r, |c_2| r^2, \ldots\}.
\]
We have defined the \( M_i \) in such a way that the absolute value of the \( k \)th
derivatives of \( \Lambda \) and \( \Lambda \Theta' \) at \( t_0 \) are bounded above by
\( M_1 k! r^{-k} \) and \( M_2 k! r^{-k} \) respectively.

A key step within the proof of the Cauchy–Kovalevskaya Theorem is to observe that the function
\[
m_i(\bar{x}) = \frac{M_i r}{r - (\bar{x} - t_0)}
\]
has derivatives
\[
\frac{d^k m_i}{d\bar{x}^k}(t_0) = M_i k! r^{-k}
\]
for \( i = 1, 2 \). By replacing \( \Lambda \) and \( \Lambda \Theta' \) with \( m_1 \) and \( m_2 \) in equation \((23a)\) we obtain the new system
\[
\begin{align*}
\frac{\partial \tilde{\phi}}{\partial \bar{y}} &= \left( \frac{M_1 r}{r - (\bar{x} - t_0)} \right) \frac{\partial \tilde{\phi}}{\partial \bar{x}} + \frac{M_2 r}{r - (\bar{x} - t_0)} & \text{(26a)} \\
\tilde{\phi}(\bar{x}, 0) &= 0. & \text{(26b)}
\end{align*}
\]
Since
\[
\left| \frac{d^k \Lambda}{d\bar{x}^k}(t_0) \right| \leq \frac{d^k m_1}{d\bar{x}^k}(t_0) \quad \text{and} \quad \left| \frac{d^k (\Lambda \Theta')}{d\bar{x}^k}(t_0) \right| \leq \frac{d^k m_2}{d\bar{x}^k}(t_0),
\]

it follows that if $\phi$ and $\tilde{\phi}$ solve systems (24) and (26) respectively, then
\[
\left| \frac{\partial^{k+l}\phi}{\partial z^k \partial \bar{y}^l}(t_0,0) \right| \leq \frac{\partial^{k+l}\tilde{\phi}}{\partial z^k \partial \bar{y}^l}(t_0,0),
\] (27)
for $k, l \geq 0$. Using the method of characteristics, the system (26) has an explicit solution of the form $\phi(x, \bar{y}) = \frac{M_2}{M_1} V(x - t_0, \bar{y})$ in a neighbourhood of $(t_0,0)$ where
\[V(x, \bar{y}) := r - \bar{x} - \sqrt{(r - \bar{x})^2 - 2M_1 r \bar{y}}.
\]
This solution is analytic at $(t_0,0)$ and so by inequality (27) the function $\phi$ has a convergent Taylor series at $(t_0,0)$ that solves system (24). This is usually where the proof of the Cauchy–Kovalevskaya Theorem ends, but we continue as we wish to find where the Taylor series of $\phi$ converges.

We will now attempt to find where the Taylor series for $V$ based at $(0,0)$ converges absolutely. Let $r_1, r_2 > 0$ and use $D(0, r_1)$ to denote the disc in the complex plane $C$ centred at the origin with radius $r_1$. From the theory of complex analysis on several variables [6, §2], if we can show that $V$ is complex analytic in each variable separately on $D(0, r_1) \times D(0, r_2)$, then the Taylor series of $V$ based at $(0,0)$ converges absolutely on $D(0, r_1) \times D(0, r_2)$. We can use this result to find out where in $\mathbb{R}^2$ the Taylor series of $V$ at $(0,0)$ converges absolutely.

It is enough to find where the Taylor series of $\sqrt{(r - \bar{x})^2 - 2M_1 r \bar{y}}$ converges absolutely since it differs from $V$ by a linear term. Note that the square root function can be extended to $C$ whilst being complex analytic away from the negative real axis. Take $b \in \mathbb{R}$ with $b \leq 0$ to be a point on the negative real axis. Let $a \in (0, r)$ and $(z_1, z_2) \in D(0, a) \times D(0, (r - a)^2/2M_1 r)$. We plan to show $(r - z_1)^2 - 2M_1 r z_2$ remains away from the negative real axis so that $\sqrt{(r - z_1)^2 - 2M_1 r z_2}$ is complex analytic in each variable separately on $D(0, a) \times D(0, (r - a)^2/2M_1 r)$. We do this by considering two cases. Firstly, if $(r - \text{Re}(z_1))^2 \geq \text{Im}(z_1)^2$, then $\text{Re}((r - z_1)^2) \geq 0$ and so we have
\[
|(r - z_1)^2 - 2M_1 r z_2 - b| \geq |(r - z_1)^2 - b| - 2M_1 r |z_2| \\
\geq |r - z_1|^2 - 2M_1 r |z_2| \\
> (r - a)^2 - (r - a)^2 \\
= 0.
\]
Secondly, if instead $(r - \text{Re}(z_1))^2 < \text{Im}(z_1)^2$, then
\[
|(r - z_1)^2 - 2M_1 r z_2 - b| \geq |\text{Im}((r - z_1)^2 - 2M_1 r z_2)| \\
= |2(r - \text{Re}(z_1)) \text{Im}(z_1) - 2M_1 r \text{Im}(z_2)| \\
\geq 2(r - \text{Re}(z_1))|\text{Im}(z_1)| - 2M_1 r |\text{Im}(z_2)| \\
> 2(r - \text{Re}(z_1))^2 - (r - a)^2 \\
> (r - a)^2 \\
> 0.
\]
Altogether this implies the Taylor series of $\sqrt{(r - z_1)^2 - 2M_1 r z_2}$ at $(0,0)$ converges absolutely on $D(0, a) \times D(0, (r - a)^2/2M_1 r)$.

Thus the Taylor series for $V$ at $(0,0)$ converges absolutely on $D(0, a) \times D(0, (r - a)^2/2M_1 r)$. Therefore, the Taylor series of $\tilde{\phi}$ at $(t_0,0)$ converges absolutely in the rectangle
\[
\left\{ (\bar{x}, \bar{y}) \in \mathbb{R}^2 : |\bar{x} - t_0| < a, \ |\bar{y}| < \frac{(r - a)^2}{2M_1 r} \right\}.
\]
We can take the union of these rectangles over \( a \in (0, r) \) to obtain convergence within
\[
\tilde{P}_r := \left\{ (\tilde{x}, \tilde{y}) \in \mathbb{R}^2 : |\tilde{x} - t_0| < r, |\tilde{y}| < \frac{(r - |\tilde{x} - t_0|)^2}{2M_1 r} \right\}.
\]
Since this holds for all \( r \in (0, \min\{R_1, R_2\}) \) we can also take the union over \( r \) to obtain convergence within
\[
\tilde{P} := \bigcup_{r \in (0, \min\{R_1, R_2\})} \tilde{P}_r,
\]
which by inequalities (27) implies system (24) has a solution \( \phi \) on \( \tilde{P} \).

Before we change our variables back to \((x, y)\), we are interested in finding how far \( \tilde{P} \) extends in the \( \tilde{y} \) direction from \((t_0, 0)\). Note that \((t_0, \tilde{y}) \in \tilde{P}_r \) for \(|\tilde{y}| < r^2 M_1^2 \) and since \( M_1 \) is dependent on \( r \), the quantity of interest is \( \sup_{r \in (0, \min\{R_1, R_2\})} r^2 M_1^2(r) \). This quantity can be expressed as follows.

**Lemma 3.1.** By defining
\[
r_0 := \min \left\{ \frac{1}{\sup_{n \geq 1} |b_n|^\frac{1}{n}}, R_2 \right\},
\]
it holds that
\[
\sup_{r \in (0, \min\{R_1, R_2\})} \frac{r}{2M_1(r)} = \frac{r_0}{2}. \tag{28}
\]

**Proof.** We begin by showing \( 0 < r_0 \leq \min\{R_1, R_2\} \). The inequality \( r_0 \leq \min\{R_1, R_2\} \) is a consequence of
\[
\frac{1}{\sup_{n \geq 1} |b_n|^\frac{1}{n}} \leq \limsup_{n \to \infty} |b_n|^\frac{1}{n} = R_1.
\]
For \( r \in (0, R_1) \) there exists some constant \( C > 1 \) such that
\[
|b_n|r^n \leq \sum_{n=0}^{\infty} |b_n|r^n \leq C,
\]
which implies \( \sup_{n \geq 1} |b_n|^\frac{1}{n} \leq \sup_{n \geq 1} C^n/r \leq C/r < \infty \). It follows that \( r_0 > 0 \).

To prove the equality (28) it is enough to show that
\[
\inf_{r \in (0, \min\{R_1, R_2\})} \frac{M_1(r)}{r} = \frac{1}{r_0}.
\]

By recalling the definition of \( M_1(r) \), we have
\[
\frac{M_1(r)}{r} = \sup \left\{ \frac{1}{r} \cdot |b_1|, |b_2|r, |b_3|r^2, \ldots \right\}.
\]
If \( r \in (0, r_0) \), then \( r \leq 1/\sup_{n \geq 1} |b_n|^\frac{1}{n} \) which implies \( |b_n|r^{n-1} \leq 1/r \) for all \( n \geq 1 \). Therefore \( M_1(r)/r = 1/r \) and so
\[
\frac{1}{r_0} = \inf_{r \in (0, r_0)} \frac{1}{r} = \inf_{r \in (0, r_0)} \frac{M_1(r)}{r} \geq \inf_{r \in (0, \min\{R_1, R_2\})} \frac{M_1(r)}{r}.
\]
It remains to prove
\[
\frac{1}{r_0} \leq \inf_{r \in (0, \min\{R_1, R_2\})} \frac{M_1(r)}{r},
\]
which is equivalent to showing \(1/r_0 \leq M_1(r)/r\) for all \(r \in (0, \min\{R_1, R_2\})\). We have already shown \(1/r_0 \leq 1/r = M_1(r)/r\) for \(r \in (0, r_0]\), so we only need to consider \(r \in (r_0, \min\{R_1, R_2\})\). Of course if \(r_0 = \min\{R_1, R_2\}\), then we are done as no such \(r\) exist. Therefore, we assume \(r_0 < \min\{R_1, R_2\}\), which must mean that \(r_0 = 1/\sup_{n>0} |b_n|^{1/n}\).

Suppose for a contradiction that there exists some \(r \in (r_0, \min\{R_1, R_2\})\) such that \(1/r_0 > M_1(r)/r\). As a result, there exists an \(\varepsilon \in (0, r - r_0)\) such that \(1/(r_0 + \varepsilon) > |b_n|^{n-1}\) for all \(n \geq 1\). However, \(r_0 + \varepsilon > r_0 = 1/\sup_{n>0} |b_n|^{1/n}\) which implies there exists \(n \geq 1\) such that
\[
1/(r_0 + \varepsilon) < |b_n|^{(r_0 + \varepsilon)^{n-1}} < |b_n|r^{n-1}.
\]
This provides us with a contradiction and proves that \(1/r_0 \leq M_1(r)/r\) for all \(r \in (r_0, \min\{R_1, R_2\})\) which concludes the proof.

We are now ready to change back to our original variables \((x, y)\). We are only interested in the points that lie above the \(\tilde{x}\)-axis and that are within the region where our change of variables is a diffeomorphism, so we define \(\tilde{Q} := \tilde{P} \cap ((t_0 - \delta, t_0 + \delta) \times [0, \infty))\). When changing back to the \((x, y)\) variables, the region \(\tilde{Q}\) is mapped to a region \(Q\) as depicted in the following figure.

![Diagram](image.png)

Figure 4: Mapping \(\tilde{Q}\) to \(Q\) via the change of variables from \((\tilde{x}, \tilde{y})\) to \((x, y)\).

Altogether, we have shown system (22) has a solution \(B\) on \(Q\). It is clear from Figure 4 and Lemma 5.1 that we can find an exterior collar neighbourhood within \(Q\) that has the form
\[
N_{t_0} := \bigcup_{t \in (t_0 - \delta, t_0 + \delta)} \{ \gamma(t) + \varepsilon\gamma^\perp(t) : \varepsilon \in [0, l(t)) \},
\]
with \(l: (t_0 - \delta, t_0 + \delta) \to (0, \infty)\) satisfying \(l(t_0) = \frac{r_0}{2}\). By recalling the definition of \(r_0\) and that the \(b_n\) and \(R_2\) are dependent on \(t_0\), we can express the width of \(N_{t_0}\) at \(\gamma(t_0)\) as
\[
d(t_0) := \frac{r_0}{2} |\gamma(t_0)^\perp| = \frac{|\gamma'(t_0)|^2}{2} \min\left\{ \frac{1}{\sup_{n \geq 1} |b_n(t_0)|^{1/n}}, R_2(t_0) \right\}.
\]

Now by letting \(N^*\) be an exterior collar neighbourhood of \(\partial\Omega\) with constant width \(l^*\), we can patch together the exterior collar neighbourhoods \(N_t \cap N^*\) over \(t \in \mathbb{T}\) in the same way as in the end of the proof of Theorem 2.3. This generates a solution to the Cauchy problem (1) on the exterior collar neighbourhood of constant width
\[
d^* := \min\left\{ \inf_{t \in \mathbb{T}} d(t), l^* \right\},
\]
26
where we take \( l^* \) to be the maximum width of all possible constant width exterior collar neighbourhoods of \( \partial \Omega \). Altogether, we have proven the following theorem.

**Theorem 3.2.** For \( f, h \in C^\omega(\partial \Omega) \) there exists \( B \in C^1(U \setminus \overline{\Omega}; \mathbb{R}^2) \cap C(U \setminus \Omega; \mathbb{R}^2) \) that solves the Cauchy problem \( \square \) on the exterior collar neighbourhood of constant width \( d^* \) which has the form

\[
U \setminus \Omega = \bigcup_{w \in \partial \Omega} \{ w + \varepsilon n(w) : \varepsilon \in [0, d^*) \}.
\]

We have shown that there exists an external harmonic extension to at least a distance \( d^* \) away from \( \partial \Omega \). We now remark on how \( d^* \) relates to the curvature of \( \partial \Omega \).

**Remark 3.3.** Observe that

\[
\text{Re} \Lambda = \frac{\gamma_1'(t_0)\gamma_2''(t_0) - \gamma_2'(t_0)\gamma_1''(t_0)}{|\gamma'|^2},
\]

and hence

\[
(\text{Re} \Lambda)'(t_0) = \frac{\gamma_1'(t_0)\gamma_2''(t_0) - \gamma_2'(t_0)\gamma_1''(t_0)}{|\gamma'(t_0)|^2}.
\]

The curvature of \( \gamma \) is

\[
\kappa = \frac{|\gamma_1''\gamma_2 - \gamma_2''\gamma_1|}{|\gamma'|^3}
\]

which implies

\[
\kappa(t_0) = \frac{|(\text{Re} \Lambda)'(t_0)|}{|\gamma'(t_0)|} = \frac{|\text{Re} b_1(t_0)|}{|\gamma'(t_0)|}.
\]

Therefore,

\[
d^* \leq d(t_0) \leq \frac{|\gamma'(t_0)|}{2|b_1(t_0)|} \leq \frac{|\gamma'(t_0)|}{2|\text{Re} b_1(t_0)|} = \frac{1}{2\kappa(t_0)}
\]

and so

\[
d^* \leq \frac{1}{2} \inf_{\frac{1}{\kappa}} \left( \frac{1}{\kappa} \right).
\]

This shows that our lower bound on how far we can harmonically extend is no more than half the minimum radius of curvature.

We now go about finding \( d^* \) for some simple examples where we can compute the quantity \( \sup_{n \geq 1} |b_n(t_0)||^\frac{1}{n} \) explicitly. Note that two of our examples are for boundaries \( \partial \Omega \) that are not closed curves, however, our workings can easily be adapted to these settings.

**Example 3.4.** Let \( \partial \Omega \) be the circle of radius \( R > 0 \) parameterised clockwise by \( \gamma(t) = R(\cos t, -\sin t) \), and suppose the boundary data \( f(\gamma(t)), h(\gamma(t)) \) has an analytic continuation to \( \mathbb{C} \). We have

\[
\Lambda(\bar{x}) = ie^{i(\bar{x}-t_0)} = \sum_{n=0}^{\infty} \frac{i^{(n+1)}}{n!} (\bar{x} - t_0)^n.
\]

It follows that \( |b_n(t_0)| = 1/n! \), which implies \( \sup_{n \geq 1} |b_n(t_0)| \leq 1 \). Both \( \Lambda \) and \( \Theta \) have an analytic continuation to \( \mathbb{C} \) and so \( R_2(t) = \infty \) for all \( t \in \mathbb{T} \). Therefore, \( d(t) = R/2 \). In the case of a circle we have \( l^* = \infty \). Overall, \( d^* = R/2 \), which shows that in this setting \( d^* \) is dependent on the curvature of \( \partial \Omega \).
Example 3.5. Let \( \partial \Omega \) be the flat boundary of the form \( \gamma(t) = (t,0) \), and suppose the boundary data \( f(\gamma(t)), h(\gamma(t)) \) has an analytic continuation to the complex strip \( \{ z \in \mathbb{C} : |\text{Im}(z)| < a \} \) for some \( a > 0 \) and no further. We have \( \Lambda(\bar{x}) = i \), which implies \( |b_n(t_0)| = 0 \) for \( n \geq 1 \) and \( \sup_{n \geq 1} |b_n(t_0)|^{\frac{1}{n}} = 0 \). The function \( \Lambda \Theta' \) has an analytic continuation to \( \{ z \in \mathbb{C} : |\text{Im}(z)| < a \} \) and no further, which implies \( \inf_{t \in \mathbb{R}} R_2(t) = a \). Hence \( \inf_{t \in \mathbb{R}} d(t) = a/2 \). In the case of a flat boundary \( t^* = \infty \) and thus \( d^* = a/2 \), which shows that in this setting \( d^* \) is dependent on the extent to which the boundary data can be analytically continued.

Example 3.6. Let \( \partial \Omega \) be the parabola of the form \( \gamma(t) = (t,t^2) \), and suppose the boundary data is such that \( f(\gamma(t)), h(\gamma(t)) \) has an analytic continuation to \( \mathbb{C} \). We have

\[
\Lambda(\bar{x}) = i \left( \frac{1 + 2t_0^2}{1 + 2\bar{x}t} \right) = \sum_{n=0}^{\infty} i \left( \frac{-2i}{1 + 2t_0^2} \right)^n (\bar{x} - t_0)^n,
\]

which implies \( |b_n(t_0)|^{\frac{1}{n}} = 2/\sqrt{1 + 4t_0^2} \) and \( \sup_{n \geq 1} |b_n(t_0)|^{\frac{1}{n}} = 2/\sqrt{1 + 4t_0^2} \). The functions \( \Lambda(t) \) and \( |\gamma'(t)| = \sqrt{1 + 4t^2} \) both have an analytic continuation to the complex strip \( \{ z \in \mathbb{C} : |\text{Im}(z)| < 1/2 \} \), which implies \( \Lambda \Theta' \) also has an analytic continuation there. Consequently, \( R_2(t) \geq 1/2 \) for all \( t \in \mathbb{R} \). It follows that \( \inf_{t \in \mathbb{R}} d(t) = d(0) = 1/4 \) since \( 1/\sup_{n \geq 1} |b_n(t)|^{\frac{1}{n}} \geq 1/2 \) and \( 1/\sup_{n \geq 1} |b_n(t)|^{\frac{1}{n}} = 1/2 \). The quantity \( t^* \) for this parabola is the smallest radius of curvature of \( \gamma \), which turns out to be \( 1/2 \). We therefore conclude \( d^* = 1/4 \).

In the case where \( \sup_{n \geq 1} |b_n(t_0)|^{\frac{1}{n}} \) can not computed explicitly, we can approximate it using the following method. For \( t_0 \in \mathbb{T} \), let \( a_n(t_0) \in \mathbb{C} \) be the Taylor coefficients satisfying

\[
\gamma_1'(\bar{x}) + i \gamma_2'(\bar{x}) = \sum_{n=0}^{\infty} a_n(t_0)(\bar{x} - t_0)^n.
\]

Then the Taylor coefficients of \( \Lambda \) at \( t_0 \) can be expressed in terms of the \( a_n \) as

\[
b_n = \frac{i}{(\gamma_1'(t_0) + i \gamma_2'(t_0))^n} \det A_n
\]

for \( n \geq 1 \) where

\[
A_n = \begin{pmatrix}
0 & a_1 & a_2 & \cdots & a_{n-1} \\
0 & a_0 & a_1 & \cdots & a_{n-2} \\
& \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & \cdots & a_0
\end{pmatrix}
\]

To prove this it is enough to show that

\[
b_n = -\frac{1}{a_0} \sum_{k=1}^{n} a_k b_{n-k} \tag{29}
\]

which comes from multiplying the Taylor series of \( \gamma_1' + i \gamma_2' \) and \( \Lambda \). To show that the expression for \( b_n \) satisfies \( \text{(29)} \), expand \( \det A_n \) by the first row and then keep expanding the determinants of the minors by the columns consisting only of \( a_0 \) until the result is obtained.

We bound the expression for the Taylor coefficients \( b_n \) using Hadamard’s Inequality [7, §14.1].
Theorem 3.7 (Hadamard’s Inequality). Let \( M = (m_{j,k}) \) be a real or complex \( n \times n \) matrix. Then
\[
| \det M | \leq \prod_{k=1}^{n} \left( \sum_{j=1}^{n} |m_{j,k}|^2 \right)^{\frac{1}{2}}.
\]

Using this inequality we have
\[
|b_n| = \frac{1}{|\gamma'(t_0)|} |\det A_n| \leq \frac{1}{|\gamma'(t_0)|} \prod_{k=1}^{n} \left( \sum_{j=0}^{k} |a_j|^2 \right)^{\frac{1}{2}} \leq \frac{1}{|\gamma'(t_0)|} \left( \sum_{j=0}^{n} |a_j|^2 \right)^{\frac{1}{2}} \leq \frac{1}{|\gamma'(t_0)|} \left( \sum_{j=0}^{n} |a_j| \right)^{n},
\]
which implies
\[
\sup_{n \geq 1} |b_n(t_0)|^{\frac{1}{n}} \leq \frac{1}{|\gamma'(t_0)|} \sum_{n=0}^{\infty} |a_n(t_0)|. \quad (30)
\]

We can use this result to find an approximate of \( d^* \) for more complicated boundaries as shown in the following example.

Example 3.8. Suppose the boundary \( \partial \Omega \) can be parameterised by \( \gamma \) that has the form of the finite Fourier series
\[
\gamma_1(t) + i\gamma_2(t) = \sum_{k=-N}^{N} c_k e^{ikt}
\]
for some \( N \geq 1 \) and \( c_k \in \mathbb{C} \). Thus
\[
a_n(t_0) = \sum_{k=-N}^{N} c_k e^{ikt_0} \frac{(ik)^{n+1}}{n!}
\]
and so by substituting this into inequality (30), we obtain
\[
\sup_{n \geq 1} |b_n(t_0)|^{\frac{1}{n}} \leq \frac{1}{|\gamma'(t_0)|} \sum_{n=0}^{\infty} \sum_{k=-N}^{N} \frac{|c_k||k|^{n+1}}{n!} = \frac{1}{|\gamma'(t_0)|} \sum_{k=-N}^{N} |c_k||k| e^{i|k|}. \]

If we suppose the boundary data is such that \( f(\gamma(t))/|\gamma'(t)| \) and \( h(\gamma(t))/|\gamma'(t)| \) have an analytic continuation to \( \mathbb{C} \), then \( R_2(t) = \infty \) for all \( t \in \mathbb{T} \). Therefore,
\[
d(t) \geq \frac{|\gamma'(t)|^2}{2 \sum_{k=-N}^{N} |c_k||k| e^{i|k|}}
\]
which provides us with
\[
d^* \geq \min \left\{ \frac{\inf_{t \in \mathbb{T}} |\gamma'(t)|^2}{2 \sum_{k=-N}^{N} |c_k||k| e^{i|k|}}, t^* \right\}.
\]

The right hand side is a lower bound for \( d^* \) and therefore a lower bound on how far we can harmonically extend from a boundary that is represented by a finite Fourier series.
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References

[1] Michael Renardy and Robert C. Rogers. *An introduction to partial differential equations*, volume 13 of *Texts in Applied Mathematics*. Springer-Verlag, New York, second edition, 2004.

[2] J. Hadamard. Sur les problèmes aux dérivées partielles et leur signification physique. *Princeton University Bulletin*, 13:49–52, 1902.

[3] Dinh Nho Hào, Tran Duc Van, and Rudolf Gorenflo. Towards the Cauchy problem for the Laplace equation. In *Partial differential equations, Part 1, 2 (Warsaw, 1990)*, volume 2 of *Banach Center Publ.*, 27, Part 1, pages 111–128. Polish Acad. Sci. Inst. Math., Warsaw, 1992.

[4] L. E. Payne and D. Sather. On an initial-boundary value problem for a class of degenerate elliptic operators. *Ann. Mat. Pura Appl. (4)*, 78:323–337, 1968.

[5] Elias M. Stein and Rami Shakarchi. *Complex analysis*, volume 2 of *Princeton Lectures in Analysis*. Princeton University Press, Princeton, NJ, 2003.

[6] Lars Hörmander. *An introduction to complex analysis in several variables*, volume 7 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, third edition, 1990.

[7] D. J. H. Garling. *Inequalities: A Journey into Linear Analysis*. Cambridge University Press, 2007.