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1. INTRODUCTION

In [16] Jucys observed that the primitive idempotents of the symmetric groups can be obtained by taking certain limit values of a rational function in several variables. A similar construction, now commonly referred to as the fusion procedure, was developed by Cherednik [5], while complete proofs relying on q-version of the Young symmetrizers were given by Nazarov [22]. This method has already been used by Nazarov (and Tarasov) [23–25] and Grime [8]. A simple version of the fusion procedure for the symmetric group was given by Molev in [21]. Here the idempotents are obtained by consecutive evaluations of a certain rational function. This version of the fusion procedure relies on the existence of a maximal commutative subalgebra generated by the Jucys-Murphy elements and was developed for various algebras and groups (see e.g. [11–14, 26, 27]).

Let m, n be positive integers and let Wm,n be the complex reflection group of type G(m, 1, n). By [28], Wm,n has a presentation with generators τ, s1, . . . , sn−1 where the defining relations are τm = 1, s2 1 = · · · = s2 n−1 = 1 and the homogeneous relations

\[ \tau s_1 \tau s_1 = s_1 t s_1 \tau, \quad \tau s_i = s_i t \quad \text{for} \ i \geq 2, \]

\[ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \quad \text{for} \ i \geq 1 \]

\[ s_i s_j = s_j s_i, \quad \text{if} \ |i - j| > 1. \]

It is well-known that Wm,n ∼ (Z/mZ)n × G n, where G n is the symmetric group of degree n (generated by s1, . . . , sn−1). The degenerate cyclotomic Hecke algebra Hm,n(Q) with parameters Q is a natural deformation of the group algebra KWm,n over a filed K (see Definition 2.2 below).

Let \( \mathcal{P}_{m,n} \) be the set of all m-multipartition of n. For \( \lambda \in \mathcal{P}_{m,n} \), let \( S^\lambda \) be the Specht module \( H_{m,n}(Q) \) of type \( \lambda \). Then \( \{ S^\lambda | \lambda \in \mathcal{P}_{m,n} \} \) is the set of irreducible representations of the generic cyclotomic Hecke algebras \( H_{m,n}(Q) \). Let \( E_t \) be the primitive idempotent of \( H_{m,n}(Q) \) corresponding to the standard \( \lambda \)-tableau \( t \). A complete system of pairwise orthogonal primitive idempotents of \( H_{m,n}(Q) \) is parameterized by the set of standard tableaux of the m-multipartitions of n.

This paper is concerned with the fusion procedure for \( H_{m,n}(Q) \). As in [21, 27], we use the Jucys-Murphy elements of \( H_{m,n}(Q) \). The main result of the paper is the following:

Theorem. Let \( \lambda \) be an m-multipartition of n and \( t \) a standard \( \lambda \)-tableau. Then the primitive idempotent \( E_t \) of \( H_{m,n}(Q) \) corresponding to \( t \) can be obtained by the following consecutive evaluations

\[ E_t = \Theta_\lambda(Q) \Phi(z_1, \cdots, z_n) \bigg| \begin{array}{c|c|c}
  z = \text{res}(1) & \cdots & z = \text{res}(n-1) \\
  z = \text{res}(n) & & z = \text{res}(n)
\end{array}, \]

where \( \Phi(z_1, \cdots, z_n) \) is a rational function in several variables with values in \( H_{m,n}(Q) \) and \( \Theta_\lambda(Q) \) is a rational functions in variables Q.

Remarkably, the coefficient \( \Theta_\lambda(Q) \) appearing in Theorem depends only on the m-multipartition \( \lambda \). In fact, the coefficient \( \Theta_\lambda(Q) \) is the weight of the Brundan-Kleshchev
trace on $H_{m,n}(Q)$ corresponding to the Specht modules $S^\lambda$, that is, $\Theta_\lambda(Q)$ is the inverse of the Schur element $s_\lambda(Q)$ of $H_{m,n}(Q)$ (see [30, Theorem 4.2]).

In additional, the degenerate cyclotomic Hecke algebra is a cellular algebra with Jucys-Murphy element (see [1, 20]). It may be surprising that the cellularity is not used in the construction of the rational function $\Phi(z_1, \cdots, z_n)$ appearing in Theorem. It seems reasonable that we may develop an abstract framework for the fusion procedure for those algebras equipped with a family of inductively defined Jucys-Murphy elements satisfying some certain conditions (cf. [20], [6]). We hope to return this issue in future work.

This paper is organized as follows. Section 2 contains definitions and notations about the multipartitions, the degenerate cyclotomic Hecke algebra, the Jucys-Murphy elements and the Baxterized elements and gives facts. The rational function $\Theta_\lambda(Q)$ is introduced and investigated in Section 3, in particular, a combinatorial formulation of $\Theta_\lambda(Q)$ is presented. Finally, we define the rational function $\Phi(z_1, \cdots, z_n)$ and prove the main theorem in the last section.

Acknowledgements. The authors are grateful to Professor Chengming Bai for his hospitality during their visits to the Chern Institute of Mathematics (CIM) in Nankai University. Part of this work was carried out while the first author was visiting CIM and the Kavli Institute for Theoretical Physics China (KITPC) at the Chinese Academy of Sciences in Beijing.

2. Preliminaries

2.1. A partition $\lambda = (\lambda_1, \lambda_2, \cdots)$ is a decreasing sequence of non-negative integers containing only finitely many non-zero terms. We define the length of $\lambda$ to be the smallest integer $\ell(\lambda)$ such that $\lambda_i = 0$ for all $i > \ell(\lambda)$ and set $|\lambda| := \sum_{i \geq 1} \lambda_i$. If $|\lambda| = n$ we say that $\lambda$ is a partition of $n$.

Let $m, n$ be positive integers. An $m$-multipartition of $n$ is an ordered $m$-tuple $\lambda = (\lambda^1 ; \cdots ; \lambda^m)$ of partitions $\lambda^i$ such that $n = \sum_{i=1}^m |\lambda^i|$. We denote by $P_{m,n}$ the set of all $m$-multipartitions of $n$.

The diagram of an $m$-multipartition $\lambda$ is the set

$$\lambda := \{(i, j, c) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbf{m} \mid 1 \leq j \leq \lambda_i^c\}, \quad \text{where } \mathbf{m} = \{1, \ldots, m\}.$$

The elements of $\lambda$ are the nodes of $\lambda$; more generally, a node is any element of $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbf{m}$. We may and will identify an $m$-multipartition with its diagram. A node $\alpha \notin \lambda$ is addable for $\lambda$ if $\lambda \cup \{\alpha\}$ is the diagram of an $m$-multipartition and a node $\beta$ of $\lambda$ is removable for $\lambda$ if $\lambda \setminus \{\beta\}$ is the diagram of an $m$-multipartition. We denote by $\mathcal{A}(\lambda)$ (resp. $\mathcal{R}(\lambda)$) the set of addable (resp. removable) nodes of $\lambda$.

A $\lambda$-tableau is a bijection $t : \lambda \rightarrow \{1, 2, \ldots, n\}$ and write $\text{Shape}(t) = \lambda$ if $t$ is a $\lambda$-tableau. We may and will identify a $\lambda$-tableau $t$ with an $m$-tuple of tableaux $t = (t^1; \cdots ; t^m)$, where the $c$-component $t^c$ is a $\lambda^c$-tableau for $1 \leq c \leq m$. A $\lambda$-tableau is standard if in each component the entries increase along the rows and down the columns. We denote by $\text{Std}(\lambda)$ the set of all standard $\lambda$-tableaux.

2.2. Definition. Let $K$ be a field and $Q = \{q_1, \ldots, q_m\} \subset K$. The degenerate cyclotomic Hecke algebra is the unital associative $K$-algebra $H := H_{m,n}(Q)$ generated by $t, t_1, \ldots, t_{n-1}$ and subjected to relations

- (i) $(t - q_1) \cdots (t - q_m) = 0$,
- (ii) $t(t_1 t_1 + t_1) = (t t_1 t_1 + t_1) t$ and $t t_1 = t_1 t$ for $i \geq 2$,
- (iii) $t_i^2 = 1$ for $1 \leq i < n$,
- (iv) $t_i t_{i+1} t_i = t_{i+1} t_{i+1} t_{i+1}$ for $1 \leq i < n - 1$,
- (v) $t_{i} t_{j} = t_{j} t_{i}$ for $|i - j| > 1$.

The Jucys-Murphy elements of the algebra $H$ are defined inductively as following:

$$J_1 := t \quad \text{and} \quad J_{i+1} := t_i J_i t_i + t_i, \quad i = 1, \cdots, n - 1.$$
Then $t_iJ_j = J_jt_i$ if $i \neq j - 1, j$ and $J_jJ_k = J_kJ_j$ if $1 \leq j, k \leq n$. Furthermore, the Jucys-Murphy elements $J_i$ ($i = 1, \ldots, n$) generate a maximal commutative subalgebra of $H$, furthermore, the center $Z(H)$ of $H$ is the algebras generated by the symmetric polynomials of the Jucys-Murphy elements $J_1, \ldots, J_n$ (ref. [2]).

For any distinct $i, j = 1, \ldots, n - 1$, we define the Baxterized elements with spectral parameters $x, y$:

$$t_{ij}(x, y) := \frac{1}{x - y} + t_{ij},$$

(2.4)

Here and in what follows $t_{ij}$ denotes the element $t_{ij} := t_it_{i+1}\cdots t_j$ for all $|i - j| > 1$.

These Baxterized elements take values in the algebra $K\mathfrak{S}_n$ and satisfy the following relations:

$$t_i(x, y)t_{i+1}(x, z)t_i(y, z) = t_{i+1}(y, z)t_i(x, z)t_{i+1}(x, y),$$

$$t_{ij}(x, y)t_{ji}(y, x) = 1 - (x - y)^{-2}.$$ (2.5)

From now on we let $f(z) = (z - q_1)(z - q_2)\cdots(z - q_m)$ and define the following rational function with values in $H$:

$$t(z) := \frac{f(z)}{z - t}.$$ (2.6)

Then $t(z)$ is a polynomial function in $z$. Moreover the elements $t(z)$ and $t_1(x, y)$ satisfy the following reflection equation with parameters $x, y$:

$$t(x)t_1(x, y)t(y)t_1 + t(x)t_1(x, y) = t_1(x, y)t(x) + t_1t(y)t_1(x, y)t(x).$$

2.7. Remark. The facts that Baxterized elements satisfy the reflection equation (2.6) will not be used in this paper. It seems likely that we may use this fact to construct the Bethe subalgebra of degenerate cyclotomic Hecke algebras (cf. [10]). We hope to return this issue in the future.

We shall work with a generic degenerated cyclotomic Hecke algebra, that is, $q_1, \ldots, q_m$ are indeterminates and the algebra $H$ over a certain localization of the ring $K[q_1, \ldots, q_m]$, or in a specialization such that the following separation condition is satisfied:

$$P_H(Q) = n! \prod_{1 \leq i < j \leq m} \prod_{|d| \leq n} (d + q_i - q_j) \neq 0.$$ (3.1)

3. Combinatorial formulae

3.1. Let $\lambda$ be an $m$-multipartition of $n$ and $t$ a standard $\lambda$-tableau. For $i = 1, \ldots, n$, we define the residue of $i$ in $t$ to be $\text{res}_t(i) = b - a + c$ if the number $i$ appears in the node $(a, b, c)$. More generally, if $x = (a, b, c)$ is in $\lambda \cup \mathcal{A}(\lambda)$ then we put $\text{res}(x) = b - a + c$.

The following Lemma is well-known (cf. [19, Lemma 3.34], [15, Lemma 3.12]), and easy verified by induction on $n$.

3.2. Lemma. Assume that $H$ is generic. Let $\lambda$ and $\mu$ be $m$-multipartitions of $n$ and suppose that $s \in \text{Std}(\lambda)$ and $t \in \text{Std}(\mu)$.

(i) $s = t$ (and $\lambda = \mu$) if and only if $\text{res}_s(k) = \text{res}_t(k)$ for $k = 1, \ldots, n$.

(ii) Suppose that $\lambda = \mu$ and there exists an $i$ such that $\text{res}_s(k) = \text{res}_t(k)$ for all $k \neq i, i + 1$. Then either $s = t$ or $s = t(i, i + 1)$.

3.3. The conjugate of a partition $\lambda$ is the partition $\hat{\lambda} = (\lambda_1 \geq \lambda_2 \geq \ldots)$ whose diagram is the transpose of the diagram of $\lambda$, i.e. $\hat{\lambda}_i$ is the number of nodes in the $i$-th column of the diagram of $\lambda$. Hence $\hat{\lambda}_1 = \ell(\lambda)$ and a node $(i, j)$ of $\lambda$ is removable if and only if $j = \lambda_i$ and $i = \lambda_j$. 

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Recall that the \((i, j)\)-th hook in \(\lambda\) is the collection of nodes to the right of and below the node \((i, j)\), including the node \((i, j)\) itself, and that the \((i, j)\)-th hook length of \(\lambda\) is
\[
h_{i,j}^\lambda = \lambda_i - i + \lambda_j - j + 1.
\]

Now let \(\lambda\) and \(\mu\) be partitions. If \((i, j)\) is a node of \(\lambda\) then the generalized hook length of the node \((i, j)\) with respect to \((\lambda, \mu)\) is
\[
h_{i,j}^{\lambda,\mu} = \lambda_i - i + \mu_j - j + 1.
\]

Observe that if \(\lambda = \mu\) then \(h_{i,j}^{\lambda,\mu} = h_{i,j}^{\lambda}\).

Let \(\lambda = (\lambda^1; \ldots; \lambda^m)\) be an \(m\)-multipartition of \(n\). We introduce the following function in variables \(Q\):
\[
\Theta_{\lambda}(Q) = \prod_{1 \leq s \leq m} \prod_{(i,j) \in \lambda_s} \left( \prod_{1 \leq t \leq m} \frac{1}{h_{i,j}^{\lambda_s,\lambda_t} + q_s - qt} \right).
\]

Thanks to Lemma 3.2, \(\Theta_{\lambda}(Q)\) is a rational function and can be reformulated as following:
\[
\Theta_{\lambda}(Q) = \prod_{1 \leq s \leq m} \prod_{(i,j) \in \lambda_s} \left( \frac{1}{h_{i,j}^{\lambda_s} + x - y} \frac{1}{h_{i,j}^{\lambda_s} + y - x} \right).
\]

3.5. **Remark.** The rational function \(\Theta_{\lambda}(Q)\) is the weight of the Brundan-Kleshchev trace on \(H_{m,n}(Q)\) corresponding to the Specht module \(S^\lambda\), that is, it is the inverse of the Schur element \(s_\lambda(Q)\) of \(H_{m,n}(Q)\) (ref. [29, Theorem 5.5] and [30, Theorem 4.2]).

3.6. **Lemma.** Let \(\lambda\) and \(\mu\) be partitions. Assume that \(\alpha = (i, j)\) is a removable node of \(\lambda\) and let \(\nu = \lambda - \{\alpha\}\). We define the following rational function in variables \(x, y\):
\[
\Theta_{\lambda,\mu}(x, y) := \prod_{(i,j) \in \mu} \frac{1}{h_{i,j}^{\mu} + x - y} \prod_{(i,j) \in \lambda} \frac{1}{h_{i,j}^{\lambda} + y - x}.
\]

Then
\[
\Theta_{\lambda,\mu}(x, y) = \prod_{\beta \in \mathcal{R}(\mu)} \left( \text{res}(\beta) - \text{res}(\alpha) + y - x \right) \prod_{\gamma \in \mathcal{S}(\mu)} \left( \text{res}(\gamma) - \text{res}(\alpha) + y - x \right)^{-1}.
\]

**Proof.** Let \(z := x - y\) and suppose that
\[
\mu = (\mu_1 = \cdots = \mu_i > \mu_{i+1} = \cdots = \mu_{i+2} > \cdots > \mu_{i+p+1} = \cdots = \mu_p > 0).
\]

We set \(i_0 = 0, \mu_{i+1} = 0\) and let \(z = q_c - q_t\). Then
\[
\mathcal{R}(\mu) = \{(i_k, \mu_{i_k}) \mid k = 1, \ldots, p\}; \quad \mathcal{S}(\mu) = \{(i_k + 1, \mu_{i_k+1} + 1) \mid k = 0, \ldots, p\}.
\]

Therefore, the right hand side of (3.7) is
\[
\text{RHS} = \prod_{\beta \in \mathcal{R}(\mu)} \left( \text{res}(\beta) - \text{res}(\alpha) - z \right) \prod_{\gamma \in \mathcal{S}(\mu)} \left( \text{res}(\gamma) - \text{res}(\alpha) - z \right)^{-1} = \frac{1}{\mu_1 + \mu - j - z} \prod_{k=1}^{p} \frac{\mu_{i_k} - i_k + \mu - j - z}{\mu_{i_{k+1}} - i_k + \mu - j - z}.
\]

Suppose that \(i_q < i \leq i_{q+1}\) for some \(0 \leq q \leq p + 1\) where \(i_{p+1} = +\infty\). Since \(\alpha = (i, j)\) is a removable node of \(\lambda\) and \(\nu = \lambda - \{\alpha\}\), we yield that
\[
\lambda_i = \nu_i \text{ for } i \neq i; \quad \lambda_j = \lambda_j \text{ for } j \neq j; \quad \lambda_i = \nu_i + 1 = j; \quad \lambda_j = \nu_j + 1 = i.
\]

Therefore
\[
\frac{\Theta_{\lambda,\mu}(x, y)}{\Theta_{\nu,\mu}(x, y)} = \frac{1}{h_{i,j}^{\mu,\lambda} - z} \prod_{(i,j) \in \mu} \frac{h_{i,j}^{\mu,\nu} + z}{h_{i,j}^{\mu,\lambda} + z} \prod_{(i,j) \in \nu} \frac{h_{i,j}^{\mu,\nu} - z}{h_{i,j}^{\mu,\lambda} - z}.
\]
As a consequence, we have completed the proof. \( \square \)

The following fact will be used in the sequence.

3.8. Lemma. Assume that \( \lambda \) is a partition and that \( \alpha \) is a removable node of \( \lambda \). Let \( \mu \) be the subpartition of \( \lambda \) by removing the node \( \alpha \). Then

\[
\frac{\prod_{(i,j) \in \mu} h_{i,j}^\mu}{\prod_{(i,j) \in \lambda} h_{i,j}^\lambda} = \prod_{\beta \in \mathcal{R}(\mu)} \left( \text{res}(\beta) - \text{res}(\alpha) \right) \prod_{\alpha \neq \gamma \in \mathcal{A}(\mu)} \left( \text{res}(\gamma) - \text{res}(\alpha) \right)^{-1} \]
\[
= \prod_{\beta \in \mathcal{A}(\mu)} \left( \text{res}(\alpha) - \text{res}(\beta) \right) \prod_{\alpha \neq \gamma \in \mathcal{A}(\mu)} \left( \text{res}(\alpha) - \text{res}(\gamma) \right)^{-1}.
\]

Proof. Suppose that \( \alpha = (i, j) \) and that

\( \mu = (\mu_1 = \cdots = \mu_{i_1} > \mu_{i_1+1} = \cdots = \mu_{i_2} > \cdots > \mu_{i_p+1} = \cdots = \mu_{i_p} > 0) \).

We set \( i_0 = 0, \mu_{i_p+1} = 0 \). Therefore we have

\( \mathcal{R}(\mu) = \{ (i_k, \mu_{i_k}) \mid k = 1, \cdots, p \} \);
\( \mathcal{A}(\mu) = \{ (i_k + 1, \mu_{i_k+1} + 1) \mid k = 0, \cdots, p \} . \)

Since \( \alpha \) is removable, \( i = i_q + 1 \) and \( j = \mu_{i_q+1} + 1 \) for some \( 0 \leq q \leq p + 1 \). Furthermore, we yield that

\( \lambda_i = \mu_i \) for \( i \neq i \);
\( \hat{\lambda}_j = \hat{\mu}_j \) for \( j \neq j \);
\( \lambda_i = \mu_i + 1 = j \);
\( \hat{\lambda}_j = \hat{\mu}_j + 1 = i \).

As a consequence,

\[
\frac{\prod_{(i,j) \in \mu} h_{i,j}^\mu}{\prod_{(i,j) \in \lambda} h_{i,j}^\lambda} = \frac{\prod_{(i,j) \in \mu} \mu_i + \hat{\mu}_j - i - j + 1}{\prod_{(i,j) \in \lambda} \lambda_i + \hat{\lambda}_j - i - j + 1}.
\]
\[
\begin{align*}
&= \prod_{(i,j) \in \mu} \frac{\mu_i - i + j - \lambda_i}{\mu_i - i + j - \lambda_i + 1} \frac{\mu_j - j + \lambda_j - \lambda}{\mu_j - j + \lambda_j - \lambda + 1} \\
&= \prod_{i=1}^{q} \frac{\mu_{i_k} - i_k + j - \lambda_i}{\mu_{i_k} - i_k + j - \lambda_i + 1} \prod_{k=q+1}^{p} \frac{\mu_{i_k} - i_k + j - \lambda_i}{\mu_{i_k} - i_k + j - \lambda_i + 1} \\
&= \left( \prod_{k=1}^{q} \frac{\mu_{i_k} - i_k + j - \lambda_i}{\mu_{i_k} - i_k + j - \lambda_i + 1} \right) \left( \prod_{k=q+1}^{p} \frac{\mu_{i_k} - i_k + j - \lambda_i}{\mu_{i_k} - i_k + j - \lambda_i + 1} \right) \\
&= \prod_{\beta \in \mathcal{R}(\mu)} (\text{res}(\beta) - \text{res}(\alpha)) \prod_{\alpha \neq \gamma \in \mathcal{A}(\mu)} (\text{res}(\gamma) - \text{res}(\alpha))^{-1}.
\end{align*}
\]

As a consequence, we have completed the proof. \qed

The following result gives another combinatorial reformulation for the rational function \( \Theta_{\lambda}(Q) \) defined in §3.4.

3.9. Proposition. Assume that \( \lambda \) is an \( m \)-multipartition of \( n \) and that \( t \) is a standard \( \lambda \)-tableau with the node \( \alpha \) containing the number \( n \). Let \( \mu \) be the shape of the subtableau of \( t \) by removing the node \( \alpha \). Then

\[
\Theta_{\lambda}(Q)\Theta_{\mu}(Q)^{-1} = \prod_{\beta \in \mathcal{R}(\mu)} (\text{res}(\beta) - \text{res}(n)) \prod_{\alpha \neq \gamma \in \mathcal{A}(\mu)} (\text{res}(\gamma) - \text{res}(n))^{-1}.
\]

Proof. Suppose that \( \lambda = (\lambda_1; \ldots; \lambda^m) \) and \( \alpha = (i, j, c) \). Then \( \alpha \) is removable and \( \mu = (\mu_1; \ldots; \mu^m) = (\lambda_1; \ldots; \lambda^c - 1, \mu^c; \lambda^{c+1}; \ldots; \lambda^m) \), where \( \mu^c = (\lambda_1, \ldots, \lambda_{i-1}, \lambda c - 1, \lambda^{c+1}, \ldots, \lambda^m) \). Observe that \( \mathcal{A}(\mu^t) = \mathcal{A}(\lambda^t) \) and \( \mathcal{R}(\mu^t) = \mathcal{R}(\lambda^t) \) for \( 1 \leq t \neq c \leq m \).

Applying the equality (3.4), we obtain that

\[
\frac{\Theta_{\lambda}(Q)}{\Theta_{\mu}(Q)} = \prod_{1 \leq s \leq m, (i,j) \in \lambda^s} \prod_{1 \leq t \leq m, (i,j) \in \lambda^t} \left( h_{i,j}^{\mu^s, \mu^t} + q_s - q_t \right) \]

\[
= \prod_{1 \leq s \leq m, (i,j) \in \lambda^s} \prod_{1 \leq t \leq m, (i,j) \in \lambda^t} \left( h_{i,j}^{\lambda^s, \lambda^t} + q_s - q_t \right) \]

\[
= \prod_{\beta \in \mathcal{R}(\mu^t)} (\text{res}(\beta) - \text{res}(\alpha)) \prod_{\alpha \neq \gamma \in \mathcal{A}(\mu^t)} (\text{res}(\beta) - \text{res}(\alpha))^{-1}.
\]
where the third equality follows by applying Lemmas 3.6 and 3.8. We have completed the proof. □

4. Fusion formulae for primitive idempotents

From now on, the following notations will be used throughout unless otherwise stated.

4.1. Notations. \( \lambda = (\lambda^1; \cdots ; \lambda^m) \) is an \( m \)-multipartition of \( n \) and \( t \) is a standard \( \lambda \)-tableau with the number \( n \) appearing in the node \( \alpha = (i, j, c) \). We let \( u \) be the subtableau of \( t \) by removing the node \( \alpha \) and let \( \mu = \text{Shape}(u) \). Denote by \( v \) the subtableau of \( u \) which contains the numbers \( 1, \cdots, n - 1 \) and denote by \( \nu = \text{Shape}(v) \).

Let \( \lambda \) be an \( m \)-multipartition of \( n \) and let \( S^\lambda \) be the Specht module corresponding to \( \lambda \). Then \( S^\lambda \) admits the following decomposition, as a vector space,

\[
S^\lambda = \bigoplus_{t \in \text{Std}(\lambda)} v_t,
\]
which is equipped with a Young seminorm form and the Jucys-Murphy elements act diagonally in this basis.

If \( H \) is generic then \( \{ S^\lambda \mid \lambda \in \mathcal{P}_{m,n} \} \) is a complete set of pairwise non-isomorphic irreducible \( H \)-modules. For a standard \( \lambda \)-tableau \( t \), we denote by \( E_t \) the corresponding primitive idempotent of \( H \). For all \( i = 1, \cdots , n \), we have

\[
J_i E_t = E_t J_i = \text{res}_t(i) E_t.
\]

Note that the idempotent \( E_t \) can be expressed in terms of the Jucys-Murphy elements. Indeed, the inductive formula for \( E_t \) in terms of Jucys-Murphy elements can be formulated as following:

\[
E_t = E_u \prod_{\alpha \neq \beta \in \mathcal{A}(\mu)} \frac{J_n - \text{res}(\beta)}{\text{res}(n) - \text{res}(\beta)},
\]

with the initial condition \( E_\emptyset = 1 \), which is well-defined thanks to Lemma 3.2.

On the other hand, let \( t_1, \cdots , t_a \) be the set of pairwise different standard \( \lambda \)-tableaux obtained for \( u \) by adding an node with number \( n \). Then the branching properties of the Young basis imply that

\[
E_u = \sum_{i=1}^{a} E_{t_i},
\]

and the rational function \( E_u \frac{z - \text{res}(n)}{z - J_n} \) in \( z \) is well-defined, which is non-singular at \( z = \text{res}(n) \) according to (4.2). Furthermore, we have

\[
E_t = E_u \frac{z - \text{res}(n)}{z - J_n} \bigg|_{z = \text{res}(n)}.
\]

We first define the following rational function in variable \( z \):

\[
\Theta_t(z) := \frac{z - \text{res}(n)}{f(z)} \prod_{i=1}^{n-1} \frac{(z - \text{res}(i))^2}{(z - \text{res}(i) + 1)(z - \text{res}(i) - 1)}.
\]

Clearly if \( n = 1 \) then \( \Theta_t(z) = \frac{z - \text{res}(1)}{f(z)} \).

4.5. Lemma. Keep notations as in §4.1. Then

\[
\Theta_t(z) = (z - \text{res}(n)) \prod_{\beta \in \mathcal{A}(\mu)} (z - \text{res}(\beta)) \prod_{\gamma \in \mathcal{A}(\mu)} (z - \text{res}(\gamma))^{-1}.
\]
Proof. We prove the lemma by induction on \( n \). If \( n = 1 \) then \( \mu = \emptyset \), \( R(\mu) = \emptyset \), and \( \mathcal{A}(\mu) = \{ (1,1,i) \mid 1 \leq i \leq m \} \). Thus the lemma follows directly by using the equality (4.4) for \( n = 1 \).

Now assume that the lemma holds for all standard tableaux \( t \) with \( n - 1 \geq 1 \) nodes. We show that it also holds for the standard tableaux with \( n \) nodes. Suppose that the node \( \alpha = (a,b,c) \) of \( t \) contains the number \( n - 1 \). Then we have the following cases:

(i) If \((a-1,b,c),(a,b-1,c) \notin R(\nu)\) then \( R(\mu) = R(\nu) \cup \{ \alpha \} \) and
\[
\mathcal{A}(\mu) = (\mathcal{A}(\nu) \cup \{ (a+1,b,c), (a,b+1,c) \}) \setminus \{ \alpha \}.
\]

(ii) If \((a-1,b,c) \in R(\nu) \) and \((a,b-1,c) \notin R(\nu)\), then
\[
R(\mu) = (R(\nu) \cup \{ (a,b,c) \}) \setminus \{ (a,b,c) \};
\]
\[
\mathcal{A}(\mu) = (\mathcal{A}(\nu) \cup \{ (a+1,b,c) \}) \setminus \{ \alpha \}.
\]

(iii) If \((a-1,b,c) \notin R(\nu) \) and \((a,b-1,c) \in R(\nu)\), then
\[
R(\mu) = (R(\nu) \cup \{ (a,b-1,c) \}) \setminus \{ (a,b-1,c) \};
\]
\[
\mathcal{A}(\mu) = (\mathcal{A}(\nu) \cup \{ (a+1,b,c) \}) \setminus \{ \alpha \}.
\]

(iv) If \((a-1,b,c),(a,b-1,c) \in R(\nu)\) then \( \mathcal{A}(\mu) = \mathcal{A}(\nu) \setminus \{ \alpha \} \) and
\[
R(\mu) = (R(\nu) \cup \{ (a,b,c) \}) \setminus \{ (a-1,b,c),(a,b-1,c) \}.
\]

Now applying the induction argument, we obtain
\[
\Theta_t(z) = \frac{(z - \text{res}_t(n))(z - \text{res}_t(n-1))^2}{(z - \text{res}_t(n) + 1)(z - \text{res}_t(n) - 1)} \prod_{\beta \in R(\nu)} (z - \text{res}(\beta)) \prod_{\gamma \in \mathcal{A}(\nu)} (z - \text{res}(\gamma))^{-1}
\]
\[
= (z - \text{res}_t(n)) \prod_{\beta \in R(\mu)} (z - \text{res}(\beta)) \prod_{\gamma \in \mathcal{A}(\mu)} (z - \text{res}(\gamma))^{-1}.
\]

We complete the proof.

The following lemma establishes the relationship between the rational function \( \Theta_\lambda(Q) \) and the rational function \( \Theta_t(z) \), which is crucial to the fusion formula.

4.6. Lemma. The rational function \( \Theta_t(z) \) is non-singular at \( z = \text{res}_t(n) \) and
\[
\Theta_t(\text{res}_t(n)) = \Theta_\lambda(Q) \theta_\mu(Q)^{-1}.
\]

Proof. Lemma 4.5 shows that the rational function \( \Theta_t(z) \) is non-singular at \( z = \text{res}_t(n) \). Noticing that the node \( \alpha \) is removable and applying Proposition 3.9, we obtain that
\[
\Theta_t(\text{res}_t(n)) = \prod_{\beta \in R(\mu)} (\text{res}_t(n) - \text{res}(\beta)) \prod_{\alpha \neq \gamma \in \mathcal{A}(\mu)} (\text{res}_t(n) - \text{res}(\gamma))^{-1}
\]
\[
= \Theta_\lambda(Q) \theta_\mu(Q)^{-1}.
\]

It completes the proof.

Let \( \phi_1(z) = t(z) \) and define
\[
\phi_{k+1}(z_1, \ldots, z_k; z) := t_k(z, z_k) \phi_k(z_1, \ldots, z_{k-1}; z) t_k
\]
\[
= t_k(z, z_k) t_{k-1}(z, z_{k-1}) \cdots t_1(z, z_1) t_1 \cdots t_k.
\]

4.7. Lemma. Keep notations as in §4.1. Then
\[
(4.8) \quad \Theta_t(z) \phi_n(\text{res}_t(1), \ldots, \text{res}_t(n-1), z) E_u = \frac{z - \text{res}_t(n)}{z - J_n} E_u.
\]
Proof. We prove the equality (4.8) by induction on \( n \). Using (2.5), (2.3) and (4.4), we obtain

\[
\frac{z - \text{res}(1)}{z - J_1} E_\emptyset = \frac{z - \text{res}(1)}{f(z)} \cdot \frac{f(z)}{z - t_0} E_\emptyset = \Theta_\ell(z) \phi_1(z) E_\emptyset.
\]

That is the equality holds for \( n = 1 \).

Note that \( E_\emptyset E_u = E_u \) and \( E_u t_{n-1} = t_{n-1} E_u \). Then, by induction hypothesis, the left hand side of the equality (4.8) can be rewritten as

\[
\text{LHS} = \Theta_\ell(z) t_{n-1} \left( z, \text{res}(n-1) \right) \phi_{n-1} \left( \text{res}(1), \ldots, \text{res}(n-2), z \right) t_{n-1} E_\emptyset E_u
\]

\[
= \Theta_\ell(z) t_{n-1} \left( z, \text{res}(n-1) \right) \phi_{n-1} \left( \text{res}(1), \ldots, \text{res}(n-2), z \right) E_u t_{n-1} E_u
\]

\[
= \Theta_\ell(z) \frac{t_{n-1} \left( z, \text{res}(n-1) \right) \left( \Theta_u(z) \phi_{n-1} \left( \text{res}(1), \ldots, \text{res}(n-2), z \right) E_u \right) t_{n-1} E_u}{t_{n-1} \left( z, \text{res}(n-1) \right) \phi_{n-1} \left( \text{res}(1), \ldots, \text{res}(n-2), z \right) E_u}
\]

\[
= \Theta_\ell(z) \Theta_u(z)^{-1} \frac{t_{n-1} \left( z, \text{res}(n-1) \right) \frac{z - \text{res}(n-1)}{z - J_{n-1}} E_u t_{n-1} E_u}{t_{n-1} \left( z, \text{res}(n-1) \right) \frac{z - \text{res}(n-1)}{z - J_{n-1}} E_u}
\]

Note that \( J_n \) commutes with \( E_u \). Therefore, to prove the equality (4.8), it suffices to show that

\[
t_{n-1} \left( z - J_n \right) E_u = \frac{z - \text{res}(n-1)}{z - J_{n-1}} t_{n-1} \left( \text{res}(n-1), z \right) E_u,
\]

which follows directly by using the equalities (2.4) and (2.3). It completes the proof. \( \square \)

Define the following rational function with values in the algebra \( H \):

\[
(4.9) \quad \Phi(z_1, \ldots, z_n) := \phi_n(z_1, \ldots, z_n) \phi_{n-1}(z_1, \ldots, z_n) \cdots \phi_1(z_1).
\]

Now we can prove the main result of this paper.

4.10. Theorem. Let \( \lambda \) be an \( m \)-multipartition of \( n \) and \( t \) a standard \( \lambda \)-tableau. Then the primitive idempotent \( E_t \) of \( H_{m,n}(Q) \) corresponding to \( t \) can be obtained by the following consecutive evaluations

\[
E_t = \Theta_\lambda(Q) \Phi(z_1, \ldots, z_n) \bigg|_{z = \text{res}(1)} \cdots \bigg|_{z = \text{res}(n-1)} \bigg|_{z = \text{res}(n)}.
\]

Proof. The theorem follows, by induction on \( n \), from (4.3) and Lemmas 4.7 and 4.6. \( \square \)

We close this paper with some remarks on the study of the fusion procedure for the degenerate cyclotomic Hecke algebras and related topics.

4.11. The fusion procedure for the Yang-Baxter equation was first introduced by Kulish et.al. in [18], which allows the construction of new solutions of the Yang-Baxter equation starting from a given fundamental solution. As it is suggested in [21], a “fused solution” of the Yang-Baxter equation can be investigated using the certain version of the Schur-Weyl duality (see [7]). Note that Brundan and Kleshchev have established the Schur-Weyl duality for higher levels in [3]. It may be interesting to use the fusion formula for the degenerate cyclotomic Hecke algebras to obtain a family of fused solutions of the Yang-Baxter equation acting on finite-dimensional irreducible representations of the finite \( W \)-algebras.

It is well-known that the (degenerate) cyclotomic Hecke algebras, the (degenerate) affine Hecke algebras are closely related (ref. [4, 17]). We would like to know whether
the fusion formulae in this paper are certain specializations of those ones in [23] and [26, 27]. On the other hand, it seems likely that we may use the reflection equation for the degenerate cyclotomic Hecke algebras to define and study the Bethe subalgebras of the degenerate cyclotomic Hecke algebras (cf. [10]). We hope that these Bethe subalgebras can be used to investigate the center of the (degenerate) cyclotomic Hecke algebras.

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