The SO(6) Scalar Product and Three-Point Functions from Integrability

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Abstract

In 1012.2475 Escobedo, Gromov, Sever and Vieira suggested a formula for an $SU(2)$ three-point correlation function at weak coupling based on integrability techniques. We conjecture a generalization of it to the $SO(6)$ sector, thus including all possible single-trace scalar operators in $\mathcal{N} = 4$ super Yang–Mills, and prove, by direct comparison to a perturbative $SO(6)$ calculations, that our generalization is valid.
1 Introduction

Three-point functions of single-trace operators in $\mathcal{N} = 4$ supersymmetric Yang-Mills have become a foremost research topic during the last couple of years. The main techniques for calculating three-point functions may be classified into four domains: direct perturbative calculation, integrability, semiclassics and string field theory. Perturbative calculations and integrability are realized for small 't Hooft coupling $\lambda$, semiclassics and string field theory for large $\lambda$. It is generally believed and has been proved “experimentally” for some special cases \[1, 2, 3, 4\] that the Frolov–Tseytlin limit \[5\] gives the regime where weak and strong coupling results may be compared against each other.

Perturbative calculations were historically the earliest technique to be used \[6, 7\] for computing three-point correlators of non-BPS operators. The direct perturbative calculations for three-point functions in the spirit of \[6, 7\] were performed for one $\frac{1}{2}$ BPS and two BMN $SU(2)$ states in \[8\], for two BPS and one BMN $SU(2)$ states in \[9\], for two $SL(2)$ BMN states and one BPS state in \[10\], for two BPS states and one twist-two state \[11\], for three short $SO(6)$ operators at one loop in \[12, 13\], for three long $SO(6)$ operators up to one loop in \[3, 4\]. The advantage of this technique is its straightforwardness, yet dealing with states with more than two magnons is rather cumbersome. It should be also noted that perturbative treatment at higher loops is burdened by extra problems of mixing between bosonic and bifermionic operators \[9\], mixing between BMN operators of different momentum \[16\] and non-trivial fudge-factors in the wave function \[17\]. For special kinematic cases (extremal correlators) the $1/N$ mixing \[7\] starts playing its role as well, already at tree level.

A systematic combinatorial improvement of the perturbative technique has been developed by means of integrability \[18\]. The general idea of the method is to operate in the basis of Bethe states rather than in the field-theoretical basis. Although this application of integrability techniques that has revolutionized the three-point function calculations is quite new, the ingredients needed to construct the correlators in this way have a long story. The most non-trivial part of the Bethe Ansatz calculation of a three-point function is the scalar product of two arbitrary Bethe states. In 1982 Korepin gave a recursive relation for a scalar product of an $SU(2)$ Bethe eigenstate with an arbitrary $SU(2)$ Bethe state \[19\]. This expression was represented in a concise explicit fashion by Nikita Slavnov in \[20\]. Later the Korepin-Slavnov scalar products, represented as domain wall partition functions and determinant expressions, were applied to tree-level three-point correlators in the $SU(2)$ sector in \[21, 22, 23, 24\]. The scalar products of Bethe states were used to calculate OPE of non-supersymmetric QCD field-strength operators in \[25\]. An integral representation of the scalar product for a very general class of Bethe Ansätze was obtained in \[26\]. The scalar product of the Korepin-Slavnov type was derived for the $SU(3)$ generalizations of the Bethe Ansatz \[27, 28, 29\].

The integrability-assisted combinatorics for three-point functions was originally suggested for tree-level three multi-BMN $SU(2)$ states by Escobedo, Gromov, Sever and Vieira \[18\] (EGSV) at weak coupling. It was extended to the semiclassical domain where two of the operators are heavy and described by semiclassical string states, whereas the remaining operator is a light mode \[1\], and also to the light-light-heavy case \[30\]. Quite recently the one-loop three-point correlators of multi-BMN $SU(2)$ states were described in terms of the Bethe Ansatz in \[31\]. Similar techniques were developed subsequently by Kostov in \[32\], where the
semiclassical result of [1] was generalized beyond the semiclassical approximation to an exact
determinant expression for three non-BPS states. In [33] a factorized operator expression for
a scalar product of two multimagnon states was derived. Another modification of this limit
was suggested by Serban [34].

A lot of interest in understanding the three-point functions from the point of view of inte-
grability has recently been shown. Kostov and Matsuo [35] have demonstrated that the inner
product of an on-shell state with $N$ Bethe roots with a generic $N$-root state is equivalent to
a scalar product of the $2N$-root state with a vacuum descendant state. Three-point functions
for GKP states were found from semiclassical integrability algebraic curve technique in [36].

In this paper we propose a conjecture for the $SO(6)$ sector scalar products, thus general-
izing the formulation of the three point function of [18]. We then calculate the three-point
correlation function for three states which cannot be embedded into smaller sectors ($SU(2)$
or $SU(3)$) and show that this structure constant is identical to the one previously found from
string field theory and perturbation theory independently [3].

2 The $SO(6)$ Conjecture

The starting point for our discussion is eq. (1.5) for the three-point function and eq. (A.2)
for the scalar product in [18]. We generalize these three-point function and the scalar product
to the $SO(6)$ sector and calculate the correlator of the same states that are used in [3] [4], to
enable comparison between these results. This is yet another step towards the construction
of a general formula for the three point function valid for any set of local gauge invariant
operator in any sector at weak coupling. For the $SU(2)$ case the expression for the scalar
product has been found in [18, 21]. The novelty of these techniques have required direct
analytic and numeric tests to ensure that the formula has been reproduced and interpreted
correctly. The $SO(6)$ generalization of the $SU(2)$ formula that we propose in this paper is
a sensible conjecture and it also requires some tests [1]. Therefore, we then specialize our
conjecture to the explicit case of three BMN operators with two impurities, since in this case
one can check the $SO(6)$ formula analytically thanks to the results of [3].

Consider a set of operators $\mathcal{O}_A$ normalized to unity

$$
\langle \mathcal{O}_A(x)\mathcal{O}_A(0) \rangle = \frac{1}{x^{2\Delta_A}} .
$$

(1)

The space-time dependence of any three-point function of local gauge invariant operators is
prescribed by conformal symmetry to be

$$
\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3) \rangle = \frac{C_{123}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}|x_{31}|^{\Delta_3+\Delta_1-\Delta_2}} .
$$

(2)

The procedure proposed by EGSV [18] to determine $C_{123}$ in (2) is the following:

1. Represent each of the operators as a Bethe vector.

2. Split the rapidities of the three Bethe vectors in all possible partitions “cutting” each
   of the Bethe vectors into two subvectors.

\[\text{Recently doubts have been raised, for example, as whether a determinant formula for a scalar product of two Bethe states can be written down for the $SU(3)$ sector [28, 29].}\]
3. For each of the partitions calculate the three scalar products of the states corresponding to the subvectors. Note that in order to have well defined scalar products, one of the two subvectors belonging to each operator has to be “flipped”, which means that one of the bra is mapped in to a ket by reversing all the spin chain sites (leaving the same charges as the original substate).

4. Sum over all the partitions, taking into account phase factors due to cutting of the states and conjugating half of them.

5. Normalize the results to comply with (1).

The general expression for tree level and planar structure constant arising from the EGSV procedure then is

$$N_c C_{123} = \sum_{\text{Root partitions}} \text{Cut} \times \text{Flip} \times \text{Norm} \times \text{Scalar products}.$$ (3)

The structure of the $SO(6)$ formula is conjectured by us to be analogous to (3) and directly generalizable from it apart from two subtle issues: the norms and the scalar products. In the $SO(6)$ sector the Bethe ansatz equations take this form

$$\left( \frac{u_j - i V_{a_j} / 2}{u_j + i V_{a_j} / 2} \right)^L = \prod_{k=1 \atop k \neq j}^K \frac{u_j - u_k - \frac{i}{2} M_{a_j a_k}}{u_j - u_k + \frac{i}{2} M_{a_j a_k}}$$ (4)

where $L$ is the length of the chain, $K$ the number of roots, $M$ is the Cartan matrix and $V$ are the Dynkin labels of the spin representation. For each of the Bethe roots $u_j$ one needs to specify which of the simple roots is excited by $a_j$, which is the number of simple roots and runs from 1 to 3 for the $SO(6)$ sector. The Cartan matrix for the $SO(6)$ sector is

$$M_{ab} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix},$$ (5)

and the Cartan weights are

$$V_a = (0, 1, 0).$$ (6)

We can see that now we have three sectors instead of one as in the $SU(2)$ sector, thus we represent the magnon $u_i$ by a vector containing its rapidity $u_i$ and its level index $a_i$:

$$u_i = \{u_i, a_i\}$$ (7)

Let us consider each of the four elements of the procedure, cutting, flipping, taking a scalar product and dividing over the norm of the state, separately. The general idea of EGSV has been to use integrability techniques and get an analytic expression for (3) in terms of the rapidities of the three operators. It is well known since [14] that the $SO(6)$ spin chain is integrable, so starting from the R-matrix, which acts on the tensor product of the physical spin chain site vector space and the auxiliary space, and using its properties, as Yang Baxter algebra, crossing symmetry and unitarity, it is possible to build a monodromy and a transfer
The off diagonal terms of the monodromy matrix can be used as lowering and raising operators in the sense that acting with them on a reference state, which is a state with all spins up or down, one obtains all the possible states. In this context, the Bethe equations \((4)\) play a central role ensuring that the states that we obtain are eigenstates of the transfer matrix. As explained in details in \([18]\) the building blocks used to express the final result of \((3)\) are the functions \(f(u)\) and \(g(u)\) (or combination of them) which appear in the commutation relations of the elements of the monodromy matrix (see Table 1 of \([18]\) for the complete algebra) as well as \(a(u)\) and \(d(u)\) which instead can be read from the action on a reference state of the diagonal elements of the monodromy matrix. In the following, based on these observations, we propose a conjecture to extend to the \(SO(6)\) sector the result for the three point function of EGSV. The most natural generalization of the EGSV formula to a general group with Cartan matrix \(M_{ab}\) would follow from replacing the factors \(f, g\) in their expressions \((2.29-2.33)\) by their analogs in higher sectors. Thus we postulate:

\[
\begin{align*}
    f(u_i, u_j) &= 1 + \frac{iM_{a_i a_j}}{2(u_i - u_j)}, \\
    g(u_i, u_j) &= \frac{iM_{a_i a_j}}{2(u_i - u_j)}. \\
\end{align*}
\]

The indices \(a_i, a_j\) are exactly the level indices of the \(i^{th}\) magnon just defined above. The \(S\)-matrix everywhere remains defined as

\[
S(u, v) = \frac{f(u, v)}{f(v, u)}. \tag{9}
\]

The holonomy factors \(a(u), d(u)\) retain their standard definitions for higher levels

\[
\begin{align*}
    a(u_j) &= u_j + iV_{a_j}/2, \\
    d(u_j) &= u_j - iV_{a_j}/2, \\
    e(u) &= \frac{a(u)}{d(u)} \tag{10}
\end{align*}
\]

so that the Bethe equations have the form \((4)\). Following EGSV we introduce useful shorthand notation for products of functions: for an arbitrary function \(F(u, v)\) of two variables and for arbitrary sets \(\alpha, \tilde{\alpha}\) of lengths \(K, \tilde{K}\), \(\alpha = \{\alpha_i\}_K\), \(\tilde{\alpha} = \{\tilde{\alpha}_i\}_{\tilde{K}}\)

\[
\begin{align*}
    F^{\alpha, \tilde{\alpha}} &= \prod_{i,j} F(\alpha_i, \tilde{\alpha}_j), \\
    F^{<, \alpha} &= \prod_{i<j} F(\alpha_i, \alpha_j), \\
    F^{>, \alpha} &= \prod_{i>j} F(\alpha_i, \alpha_j). \tag{11}
\end{align*}
\]

For functions \(G(u)\) of a single variable let us define

\[
\begin{align*}
    G^{\alpha} &= \prod_j F(\alpha_j), \\
    G^{\alpha \pm i/2} &= \prod_j F(\alpha_j \pm i/2). \tag{12}
\end{align*}
\]

Let us take three Bethe vectors \(u, v, w\) of lengths \(L_1, L_2, L_3\), corresponding to the operators \(O_1, O_2, O_3\) and split each of them into two pieces so that the rapidities are such that \(\alpha \cup \tilde{\alpha} = ...\)
$u, \beta \cup \bar{\beta} = v, \gamma \cup \bar{\gamma} = w$. The lengths $L_{\bar{\alpha}}, L_{\alpha}, L_{\bar{\beta}}, L_{\beta}, L_{\bar{\gamma}}, L_{\gamma}$ of these pieces are uniquely defined by the possible contraction structures:

$$
L_{\alpha} = L_{\bar{\beta}} = L_1 + L_2 - L_3,
L_{\beta} = L_{\bar{\gamma}} = L_2 + L_3 - L_1,
L_{\gamma} = L_{\bar{\alpha}} = L_3 + L_1 - L_2.
$$

(13)

The expression (3) will look like

$$
N_c C_{123} = \sum_{\alpha \cup \bar{\alpha} = u} \sqrt{L_1 L_2 L_3} \text{Cut}(\alpha, \bar{\alpha}) \text{Cut}(\beta, \bar{\beta}) \text{Cut}(\gamma, \bar{\gamma}) \times \text{Flip}(\bar{\alpha}) \text{Flip}(\bar{\beta}) \text{Flip}(\bar{\gamma}) \times \frac{1}{\sqrt{\text{Norm}(u) \text{Norm}(v) \text{Norm}(w)}} \times \langle \alpha \beta \rangle \langle \beta \gamma \rangle \langle \gamma \alpha \rangle.
$$

(14)

We work in the “coordinate” normalization, where the $\text{Cut}(\alpha, \bar{\alpha})$ factor is organized as

$$
\text{Cut}(\alpha, \bar{\alpha}) = \left( a_{\bar{\alpha}} d_{\bar{\alpha}} f_{\alpha} f_{\bar{\alpha}} \right) L_{\alpha} \left( a_{\alpha} \right),
$$

(15)

the factors $\text{Cut}(\beta, \bar{\beta})$ and $\text{Cut}(\gamma, \bar{\gamma})$ being analogous to (15). The $a, d, f, g$ factors are all defined in terms of Bethe Ansatz with higher-level states taken into account as well. In similar terms the flip factor may now be rewritten as

$$
\text{Flip}(\bar{\alpha}) = (e^{\bar{\alpha}}) \left( g_{\bar{\alpha} - i/2} f_{\alpha} \right) L_{\bar{\alpha}} g_{\bar{\alpha} + i/2} f_{\alpha},
$$

(16)

analogous expressions work for $\text{Flip}(\bar{\beta})$ and $\text{Flip}(\bar{\gamma})$. The norm can also be generalized directly from eq. (5.2) in [18] and in the coordinate normalization we get

$$
\text{Norm}(u) = d^u a^u f^u g^u \frac{1}{g^{u-i/2} g^{u+i/2} \det(\partial_j \phi_k)},
$$

(17)

here $\partial_j = \frac{\partial}{\partial u_j}$ and the phases are the ratio of the left and right sides of the Bethe equations

$$
e^{i\phi_j} = e^{(u_j)L_u} \prod_{k \neq j} S^{-1}(u_j, u_k),
$$

(18)

with $a, d, S$ satisfying the multi-level definitions above.

The remaining factor necessary to construct the correlator is the scalar product. Considering the expression for the scalar product of [18]

$$
\langle v | u \rangle = \frac{1}{d^w a^w g^w f^w g^v f^v} \times \sum_{\alpha \cup \bar{\alpha} = u} (-1)^{P_{\alpha} + P_{\gamma}} (a^{\alpha})^{L_v} (a^{\bar{\alpha}})^{L_v} (a^{\gamma})^{L_v} (d^{\bar{\gamma}})^{L_v} \times \prod h^{\alpha \gamma} h^{\bar{\bar{\gamma}} \bar{\alpha}} h^{\gamma \bar{\alpha}} \det f^\alpha f^{\bar{\alpha}},
$$

(19)
where \( t(u) = g^2(u)/f(u) \), and trying to extend the definition towards the \( SO(6) \) sector of the factor \( h \) as defined in [18]

\[
h(u) = \frac{f(u)}{g(u)},
\]

(20)

the result one gets is not well defined. In fact \( h \) does not have a direct physical meaning unlike \( f \) and \( g \). A factor \( h \) defined as in (20) would be meaningless since it would then contain division by zero.

To circumvent this problem we formulate the \( SO(6) \) norm conjecture via the recursive relation proposed in [18], eq.(A.5). This expression is completely regular and is formulated in terms of physically meaningful objects \( f, g, a, d, S \), thus it makes full sense to conjecture that its validity extends towards a broader sector. The meaning of this formula goes beyond the original \( SU(2) \) and is supposed to cover the full \( SO(6) \)

\[
\langle v_1 \ldots v_N | u_1 \ldots u_N \rangle_N = \sum_n b_n \langle v_1 \ldots \hat{v}_n \ldots v_N | u_1 \ldots u_N \rangle_{N-1} - 
\]

(21)

\[
- \sum_{n<m} c_{n,m} \langle u - 1 v_1 \ldots \hat{v}_n \ldots \hat{v}_m \ldots v_N | u_1 \ldots u_N \rangle_{N-1},
\]

where

\[
b_n = \frac{g(u_1, v_n) \left( \prod_{j=1}^N f(u_1, v_j) \prod_{j<n} S(v_j, v_n) - \frac{e(u_1)}{e(v_n)} \prod_{j<n} f(v_j, u_1) \prod_{j>n} S(v_n, v_j) \right)}{g(u_1 + i/2)g(v_n - i/2)\prod_{j\neq 1} f(u_1, u_j)},
\]

(22)

and

\[
c_{n,m} = \frac{e(u_1)g(u_1 - i/2)g(u_1, v_n)g(u_1, v_m)\prod_{j\neq n,m} f(v_j, u_1)}{g(u_1 + i/2)g(v_n - i/2)g(v_m - i/2)\prod_{j \neq 1} f(u_1, u_j)} \times
\]

\[
\times \left( \frac{S(v_m, v_n)}{e(v_n)} \prod_{j>n} S(v_n, v_j) \prod_{j<m} S(v_j, v_m) + \frac{d(v_m)}{a(v_n)} \prod_{j>m} S(v_m, v_j) \prod_{j<n} S(v_j, v_n) \right).
\]

(23)

This will be our working proposal, which shall be checked in a specific example in the next section. Whenever \( g(u_i) \) is used here as a function of a single argument, it is meant to be \( g(u_i) = \frac{i M_m a_i}{2n} \).

3 Test of integrability against perturbation theory

Let us introduce our states as Bethe states. We shall denote an \( N \)-root state as

\[
\langle u \rangle = \{u_1, l_1, \ldots, u_N, l_N\}
\]

(24)

where \( u_i \) denotes the value of the rapidity and \( l_i \) the level of Bethe Ansatz it belongs to. The states corresponding to those studied in [3,4] are

\[
O_1 \sim \langle u \rangle = \{0, 1, \frac{1}{2} \cot \frac{\pi n}{J_1+2}, 2\}; \{-\frac{1}{2} \cot \frac{\pi n}{J_1+2}, 2\};
\]

\[
O_2 \sim \langle v \rangle = \{0, 3, \frac{1}{2} \cot \frac{\pi n}{J_2+2}, 2\}; \{-\frac{1}{2} \cot \frac{\pi n}{J_2+2}, 2\};
\]

(25)

\[
O_3 \sim \langle w \rangle = \{\frac{1}{2} \cot \frac{\pi n}{J_3+2}, 2\}; \{-\frac{1}{2} \cot \frac{\pi n}{J_3+2}, 2\}.
\]
The lengths of the states are $L_1 = J_1 + 2$, $L_2 = J_2 + 2$, $L_3 = J_3 + 2$. The lengths of substates (or, alternatively, the number of contractions between each $i$th and $j$th states) are $L_{12} = J_1$, $L_{23} = J_2 + 1$, $L_{31} = J_1 + 1$; following [3,4] we introduce the parameter $r$: $J_1 = r J, J_2 = (1-r) J$.

Using the definitions of the $SO(6)$ $a, d, f, g, S$ given above we find all the necessary factors. Expansion in $1/J$ is presumed everywhere below. We use one of the possible four choices of the partitions contributing at the leading order in $1/J$

$$\alpha = \{\{0, 1\}, \{\frac{1}{2} \cot \frac{\pi n_1}{J_1 + 1}, 2\}\}, \quad \bar{\alpha} = \{\{-\frac{1}{2} \cot \frac{\pi n_1}{J_1 + 1}, 2\}\},$$

$$\beta = \{\{-\frac{1}{2} \cot \frac{\pi n_2}{J_2 + 2}, 2\}\}, \quad \bar{\beta} = \{\{0, 3\}, \{\frac{1}{2} \cot \frac{\pi n_2}{J_2 + 2}, 2\}\},$$

$$\gamma = \{\{-\frac{1}{2} \cot \frac{\pi n_1}{J_1 + 1}, 2\}\}, \quad \bar{\gamma} = \{\{\frac{1}{2} \cot \frac{\pi n_1}{J_1 + 1}, 2\}\},$$

The flip and cut factors together are

$$\text{Cut}(\alpha, \bar{\alpha})\text{Cut}(\beta, \bar{\beta})\text{Cut}(\gamma, \bar{\gamma}) \times \text{Flip}(\bar{\alpha})\text{Flip}(\bar{\beta})\text{Flip}(\bar{\gamma}) = -1,$$

the norms yield

$$\text{Norm}(u)\text{Norm}(v)\text{Norm}(w) = 4 J^2 n_1^2 n_2^2 \pi^4,$$

and the scalar products read

$$\langle \alpha \bar{\beta} \rangle \langle \beta \gamma \rangle \langle \gamma \bar{\alpha} \rangle = \frac{n_1 n_2 \sin^2(\pi n_3 r)}{2(n_1 - r n_3)(n_2 + (1-r) n_3)}.$$

The other contributing partitions in the leading order are realized by simple transformations $n_1 \to -n_1, n_2 \to -n_2$. There are also partitions that contribute at higher orders in $1/J$, which we do not list here.

Taking all the pieces together we get

$$N_c C_{123} = -\frac{n_3^2 J^{1/2} (r(1-r))^{3/2} \sin^2(\pi n_3 r)}{(n_2^2 - n_3^2 (1-r)^2)^2(n_1^2 - n_3^2 (1-r)^2)},$$

which corresponds exactly to the results of [3] obtained both from perturbation theory and string field theory.

### 4 Discussion

Equation (30) constitutes the main result of the paper: the conjectured $SO(6)$ scalar product works non-trivially and yields precise agreement with perturbative and string-theoretical calculations from [3]. This is a stringent test for the validity of our proposal for the extension of the EGSV method to a sector with rank greater than one. Only $SL(2)$ and $SU(1|1)$ sectors have been realized so far, and even for those no comparison to direct analytic perturbative calculations have been made. Here we stress again that such a comparison is crucial, although no such thing as “integrability calculation of a three-point function” exists so far, since it is likely that while using a complicated combination of $f$’s, $g$’s and $S$’s some factors may go astray. Surely the extension of the EGSV method must go further, to the loop corrections and towards the whole $SU(2,2|4)$. 
Our result is obtained from a recursive relation for the scalar product, it is an important step towards its realization in a more compact determinant form as was done for the $SU(2)$ case in [18,37,21,22]. Following the derivation in [37,21] most likely it is possible to generalize the scalar products to a determinant form, but for the norms a regularization prescription must be provided. Then, it should be possible to write down also the structure constants in terms of a product of a domain wall partition function and Slavnov scalar products, which are both expressed in terms of determinants. This would be extremely important since it might then lead to a generalization to all loops and to any group.

Strong coupling regime must also be tested against integrability predictions, perhaps with the forthcoming all-loop version. Recent strong-coupling tests of three-point correlation functions may seem to have yielded a disagreement for some particular cases. Namely, it has been shown by Gromov and Vieira in [38] that semiclassical $SU(2)$ folded string solutions agree with the $SU(2)$ “integrability calculations” only in the strict thermodynamic limit. However this seeming “disagreement” does not invalidate any calculations on either side of the correspondence, since a one-loop weak coupling result cannot be directly compared to the strongly-coupled result.

Another crucial example was provided in [2] where it was shown that the one-loop correction disagrees with that for semiclassical calculations of heavy-heavy-light type. This disagreement may be due to the approximation of Bethe states with coherent states. A mismatch between weak and strong coupling results were also observed in extremal heavy-heavy-light correlators for two giant gravitons and one point-like graviton [39]. Apart from giving general arguments on the causes of such (seeming or true) disagreements, they must be resolved in an exact quantitative way, and therefore more tests of the “integrability three-point functions conjecture” must be suggested.

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