INTRINSIC MIRRORS FOR MINIMAL ADJOINT ORBITS

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ABSTRACT. I discuss mirrors of Landau–Ginzburg models formed by a minimal semisimple adjoint orbit of $\mathfrak{sl}(n)$ together with a potential obtained via the Cartan–Killing form. I show that the Landau–Ginzburg models produced by the Gross–Siebert recipe give precisely the objects of the desired mirrors.

It is known that Landau–Ginzburg model $\text{LG}(2)$ over the semisimple adjoint orbit of $\mathfrak{sl}(2)$ does not have projective mirrors. I prove Homological Mirror Symmetry for $\text{LG}(2)$ by constructing a Landau–Ginzburg mirror and showing that its Orlov category of singularities is equivalent to $\text{Fuk}(\text{LG}(2))$.

 CONTENTS

1. LG models on adjoint orbits, Fukaya–Seidel categories and mirrors 1
2. Minimal orbits and extending the potential to the compactification 3
3. The Landau–Ginzburg model $\text{LG}(2)$ 3
   3.1. The intrinsic mirror of $\text{LG}(2)$ 4
   3.2. Critical points of the potential 5
   3.3. Mirror symmetry for $\text{LG}(2)$ 6
4. The Landau–Ginzburg model $\text{LG}(3)$ 8
   4.1. Monodromy and vanishing cycles 8
   4.2. The intrinsic mirror of $\text{LG}(3)$ 9
   4.3. Category of singularities of $\text{LG}^\vee(3)$ 12
5. The Landau–Ginzburg model $\text{LG}(n+1)$ 13
   5.1. $\text{LG}(n+1)$ fitted into the log geometry 13
   5.2. The intrinsic mirror of $\text{LG}(n+1)$ 13
   5.3. Singular fibres of of $\text{LG}^\vee(n+1)$ 15
   5.4. The mirror correspondence 16
   5.5. Category of singularities of $\text{LG}^\vee(n+1)$ 17
6. Acknowledgements 18
Appendix A. Nonperfect sheaves 18
Appendix B. Orlov category of $\Pi z_i$ 19
References 20

1. LG MODELS ON ADJOINT ORBITS, FUKAYA–SEIDEL CATEGORIES AND MIRRORS

Given a nicely behaved $A$-side Landau–Ginzburg model, that is, a complex potential function $f: X \to \mathbb{C}$ with at most Morse type singularities defining a symplectic Lefschetz fibration on a noncompact symplectic manifold $X$, there are 2 main conjectured ways to find a mirror.
**HMS1.** Compactify $X$ and extend its potential to a function with target $\mathbb{P}^1$, and then look for (a subvariety of) a complex variety $Y$ such that the bounded derived category of coherent sheaves $D^b\text{Coh}(Y)$ is equivalent to the Fukaya–Seidel category of the compactification of $(X, f)$. Hence, here we search an equivalence of categories

$$\text{Fuk}(X, f) \equiv D^b\text{Coh}(Y).$$

**HMS2.** find a mirror Landau–Ginzburg model $g : Y \to \mathbb{C}$ such that the Orlov’s category of singularities $D_{sg}(Y, g) := \bigoplus_i D^b\text{Coh}(Y_i)$, where $Y_i$ are the singular fibres of $g$, is equivalent to the Fukaya–Seidel category $\text{Fuk}(X, f)$ of Lagrangian thimbles of $f$. Hence, here we search for an equivalence of categories

$$\text{Fuk}(X, f) \equiv D_{sg}(Y, g).$$

We consider the case when $X$ is a minimal adjoint orbit of $\mathfrak{sl}(n, \mathbb{C})$, namely, the adjoint orbit $\mathcal{O}(H_0)$ of an element $H_0 = \text{Diag}(n-1, -1, \ldots, -1)$, which is diffeomorphic to the cotangent bundle of $\mathbb{P}^{n-1}$ [GGS2, Thm. 2.1]. Over such minimal adjoint orbits we have a preferred choice, denoted LG($n$), of Landau–Ginzburg model coming from Lie theory. Indeed, for each regular element $H \in h_{\mathbb{R}}$ the theorem [GGS1, Thm. 3.1] tells us that the potential $f_H : \mathcal{O}(H_0) \to \mathbb{C}$ defined by

$$f_H(A) = \langle H, A \rangle \quad A \in \mathcal{O}(H_0)$$

has a finite number of isolated singularities and defines a symplectic Lefschetz fibration, hence an $A$-side Landau–Ginzburg model; that is to say that $f_H$ admits only nondegenerate singularities; there exists a symplectic form $\Omega$ on $\mathcal{O}(H_0)$ such that the regular fibres of $f_H$ are pairwise isomorphic symplectic submanifolds; and at each critical point, the tangent cone of singular fibre can be written as a disjoint union of affine subspaces contained in $\mathcal{O}(H_0)$, each symplectic with respect to $\Omega$.

[BBGGS] showed that conjecture HMS1 totally fails to produce a mirror for the Landau–Ginzburg model LG(2) associated to the semisimple adjoint orbit of $\mathfrak{sl}(2, \mathbb{C})$, by proving that there does not exist (any subvariety of) a projective variety $Y$ whose bounded derived category of coherent sheaves $D^b\text{Coh}(Y)$ is equivalent to the Fukaya–Seidel category $\text{Fuk}\text{LG}(2)$.

I show that the Gross–Siebert intrinsic mirror symmetry algorithm produces a Landau–Ginzburg model $(\mathcal{M}_{n+1}, g)$ mirror candidate to LG($n$) for each $n$, in the sense that the Orlov’s category of singularities $D_{sg}(\mathcal{M}_{n+1}, g)$ corresponds to FukLG($n$). I prove the full categorical equivalence for the case of $\mathfrak{sl}(2)$.

**Theorem 1.** Homological Mirror Symmetry for LG(2) works as follows:

- **HMS 1** fails for LG(2), that is, for any (quasi) projective variety $Y$:

  $$\text{Fuk}(\text{LG}(2)) \not\equiv D^b\text{Coh}(Y)$$

- **HMS 2** holds true for LG(2), that is, we have an equivalence of categories:

  $$\text{Fuk}(\text{LG}(2)) \equiv D_{sg}(\mathcal{M}_2, g).$$

For the cases when $n > 2$, I discuss the equivalence $\text{Fuk}(\text{LG}(n+1)) \equiv D_{sg}(\mathcal{M}_{n+1}, g)$ at the level of objects and calculate the morphisms in the category $D_{sg}(\mathcal{M}_{n+1}, g)$. Since the morphisms of LG($n$) are not yet known for $n > 2$, at this point we can just say that $(\mathcal{M}_{n+1}, g)$ is a mirror candidate for LG($n$).
Theorem 2. The intrinsic mirror symmetry algorithm produces an LG-model \((\mathcal{Y}_{n+1}, g)\) for which
\[
\text{Fuk}(\text{LG}(n+1)) \cong D_{\text{sg}}(\mathcal{Y}_{n+1}, g)
\]
is a 1-1 correspondence.

I also completely calculate the category \(D_{\text{sg}}(\mathcal{Y}_{n+1}, g)\). Its objects are the nonperfect sheaves \(F(z_0), F(z_1), \ldots, F(z_n)\) defined in section 5.3, where \(F(z_0)\) is supported on the fibre over zero, and all the others are supported on the fibre over infinity. This shows that the general case is analogous to what happens for \(\text{LG}(3)\) as depicted in figure 2.

Theorem 3. The Orlov category of singularities of the Landau–Ginzburg model \((\mathcal{Z}_{n+1}, W)\) is generated by the nonperfect sheaves \(F(z_0), F(z_1), \ldots, F(z_n)\) with morphisms:
\[
\text{Hom}(F(z_i), F(z_j)) = \begin{cases} 
N_{ij} \oplus \tau = 2z_1 M_{ij}[t] & \text{if } i \cdot j \neq 0, \\
0 & \text{if } i \cdot j = 0.
\end{cases}
\]

The calculation of morphisms in \(D_{\text{sg}}(\mathcal{Y}_{n+1}, g)\) may be regarded as a prediction of how such morphisms in \(\text{LG}(n)\) ought to behave.

2. Minimal Orbits and Extending the Potential to the Compactification

We focus on the case of minimal semisimple orbits. Hence, for each \(n\), we discuss the orbit through \(H_0 = \text{Diag}(n, -1, \ldots, -1)\) which compactifies to \(\mathbb{P}^n \times \mathbb{P}^n\).

Let \(H = \text{Diag}(\lambda_1, \ldots, \lambda_{n+1}) \in h\), with \(\lambda_1 > \ldots > \lambda_{n+1} + \lambda_1 + \ldots + \lambda_{n+1} = 0\) (where \(h\) is the Cartan subalgebra of \(\mathfrak{sl}(n+1)\)).

Following [BGGS], we describe a rational map (factored through the Segre embedding) that coincides with the potential \(f_H\) on the adjoint orbit \(\Theta(H_0)\). Such a rational map is given by
\[
\psi : \mathbb{P}^n \times G_n(\mathbb{C}^{n+1}) \to \mathbb{P}^1,
\]
\[
\psi([\nu], [\varepsilon]) = \frac{\text{tr}(\nu \otimes \varepsilon) \rho(H)}{\text{tr}(\nu \otimes \varepsilon)} = \frac{\sum_{i=1}^{n+1} \lambda_i a_{i1} (\text{adj } g)_{1i}}{\sum_{i=1}^{n+1} a_{i1} (\text{adj } g)_{1i}},
\]
where the identification \(([\nu], [\varepsilon]) \to \nu \otimes \varepsilon\) is described in [GGS2, Sec. 4.2] and \(\rho\) is the canonical representation. Observe that if \(([\nu], [\varepsilon])\) belongs to the adjoint orbit, then \(\text{tr}(\nu \otimes \varepsilon) = 1\). Furthermore, the complement of the orbit in the compactification is the incidence correspondence variety \(\Sigma\), that is, the set of pairs \((\ell, \pi)\) such that \(0 < \ell < \pi < C^{n+1}\), where \(\pi\) is a hyperplane in \(C^{n+1}\) and \(\ell < \pi\) is a line. The variety \(\Sigma\) is the 2 step flag manifold classically denoted in Lie theory by \(\mathbb{F}(1, n)\), it is a divisor in \(\mathbb{P}^n \times \mathbb{P}^n\), and the fibre at infinity of the potential \(f_H\), that is, \(\psi = [f_H : 1]\) over the adjoint orbit, and \(\psi(\mathbb{F}(1, n)) = [1 : 0]\).

3. The Landau–Ginzburg Model LG(2)

We start with the Landau–Ginzburg model \(\text{LG}(2) = (\Theta_2, f_H)\) where \(\Theta_2\) is the semisimple adjoint orbit of \(\mathfrak{sl}(2, \mathbb{C})\), together with the potential \(f_H\) obtained by pairing with the regular element \(H\) via the Cartan–Killing form. We choose
\[
H = H_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]
and the adjoint orbit $\mathcal{O}_2 := \mathcal{O}(H_0)$ is then $\mathcal{O}_2 = \text{Ad}(SL(2))H_0 = \{ gH_0g^{-1} : g \in SL(2) \}$ with potential

$$f_H : \mathcal{O}_2 \longrightarrow \mathbb{C}$$

$$A \longrightarrow \langle H, A \rangle.$$  

The Fukaya–Seidel category of $LG(2)$ was calculated in [BBGGS] as follows:

**Theorem.** [BBGGS, Thm. 3.1] The Fukaya–Seidel category of $LG(2)$ is generated by 2 Lagrangians $L_0, L_1$ with morphisms:

$$\text{Hom}(L_i, L_j) = \begin{cases} 
Z \oplus Z[-1] & i < j \\
Z & i = j \\
0 & i > j 
\end{cases}$$

and the products $m_k$ all vanish except for $m_2(:, id)$ and $m_2(id, \_)$.

**Remark 4.** We observe that the 2 Lagrangians $L_0$ and $L_1$ have the same vanishing cycle, thus forming a matching cycle, as depicted in figure 1. The matching cycle formed by $L_0 \cup L_1$ is in fact the Lagrangian sphere $S^2 = \mathbb{P}^1$ identified with the zero section of $T^*\mathbb{P}^1$ (the smooth type of $\mathcal{O}_2$). However, there will be no matching cycles on $\mathcal{O}_n$ for $n > 2$, because the existence of a Lagrangian $n$-sphere is prohibited by the topological type $\mathcal{O}_n \simeq T^*\mathbb{P}^n$ when $n > 1$. This explains the absence of spheres in figure 2.

The goal is to calculate a Landau–Ginzburg mirror to $LG(2)$. For this, we use the compactification described in [BBGGS, Thm. 6.3] giving $\overline{LG}(2) = (\mathbb{P}^1 \times \mathbb{P}^1, R_H)$ where

$$R_H([x : y][z : w]) = [xw + yz : xw − yz]$$

is the extension of $f_H$ to a rational map $R_H : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1$. Let $\Delta$ be the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$, then we have that $\mathbb{P}^1 \times \mathbb{P}^1 \simeq \mathcal{O}_2 \cup \Delta$.

### 3.1. The intrinsic mirror of $LG(2)$

We start with the pair $(\mathbb{P}^1 \times \mathbb{P}^1, \Delta)$, but to fit into the hypothesis of log geometry we need to work relatively to an anticanonical divisor, in this case a divisor of type $(2,2)$. So, we choose an additional divisor $\Delta'$ of type $(1,1)$ such that $\Delta \cap \Delta' = \{p_1, p_2\}$ consists of 2 points.

Let $\tilde{X}$ denote the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ at these 2 points. We then study the pair

$$(\tilde{X}, D) \quad \text{with} \quad D = D_1 + D_2 + D_3 + D_4$$

where $D_2$ and $D_4$ are the proper transforms of $\Delta'$ and $\Delta$ respectively, and $D_1$ and $D_3$ are the exceptional curves obtained from blowing up the 2 points; thus $D_1^2 = D_3^2 = −1$.

To obtain the dual picture, following the Gross–Siebert intrinsic mirror recipe, we consider the vector space generated by the divisors $D_i$; then take a dual basis $D_1^*, D_2^*, D_3^*, D_4^*$, and theta functions $\theta_0, \theta_1, \theta_2, \theta_3, \theta_4$. Here $\theta_0$ is the theta function corresponding to the origin in the dual complex.

To discuss punctured Gromov–Witten invariants, we choose

$$p = D_2^*, \quad q = D_4^*, \quad r = 0.$$  

Following the steps described in the introduction of [GS], we obtain a dual surface $S$ described by the 2 equations:

$$\theta_2 \theta_4 = \theta_0(t^M + t^N) + \theta_1 t^{D_1} + \theta_3 t^{D_3} \tag{3}$$

$$\theta_1 \theta_3 = \theta_0 t^{D_1} \tag{4}$$
where $M$ and $N$ are the classes of lines of type $(1,0)$ and $(0,1)$ respectively. These equations arise as follows. In the product $\vartheta_2 \vartheta_4$, we are looking for curves which meet $D_2$ and $D_4$ to order 1 at one point each, but may have a negative tangency point with one of the $D_i$. First, in a ruling of $\mathbb{P}^1 \times \mathbb{P}^1$ either class $M$ or $N$ meets $D_2$ and $D_4$ transversally at one point each, and is disjoint from $D_1$ and $D_3$. This produces the contribution $\vartheta_0(t^M + t^N)$. Second, the curve $D_1$ meets $D_2$ and $D_4$ transversally at one point each, and has self-intersection $-1$ with $D_1$. Hence, after fixing a point $z \in D_1$, there is a unique punctured curve structure on $D_1$ giving the term $\vartheta_1 t^{D_1}$. Similarly, replacing 1 with 3, we get $\vartheta_1 t^{D_3}$. These are the only contributions to the product $\vartheta_2 \vartheta_4$. For $\vartheta_1 \vartheta_3$, a curve in the pencil $|D_2|$ meets $D_1$ and $D_3$ transversally, so after choosing a general point $z$, there is one such curve passing through $z$. This gives the term $\vartheta_0 t^{D_2}$.

Given that $\vartheta_0$ is the theta function corresponding to the origin in the dual complex, it is known that $\vartheta_0$ is the unit in the dual ring, so we may replace it with 1 in the equations that follow.

Let $P$ be the smallest monoid containing all classes of stable maps into $S$. We then obtain a mirror, by first taking

$$R = \mathbb{C}[P][\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4]/(3)(4)$$

and then the mirror variety is given by Spec $R$. This would give the mirror to $X \setminus D_2$, since we have added the extra divisor.

To continue analyzing the equations, we use the argument that $\vartheta_0$ is a unit, reducing the equations to:

$$\vartheta_2 \vartheta_4 = \vartheta_0(t^M + t^N) + \vartheta_1 t^{D_1} + \vartheta_3 t^{D_3},$$

where we have replaced $\vartheta_0$ with 1. We then note that the group of curve classes of the variety is generated by $M, N, D_1, D_3,$ and $D_2 \sim M + N - D_1 - D_3 \sim D_4$. Thus, setting $t^M, t^N, t^{D_1}, t^{D_3}$ to coefficients $a, b, c, d$ would give equations

$$\vartheta_2 \vartheta_4 = a + b + c \vartheta_1 + d \vartheta_3$$

$$\vartheta_1 \vartheta_3 = \frac{ab}{cd}$$

where we eliminate $\vartheta_3$, getting

$$\vartheta_2 \vartheta_4 = a + b + c \vartheta_1 + \frac{ab}{c} \vartheta_1^{-1}.$$ 

Then, for a sufficiently general choice of coefficients $a, \beta, \gamma$, we may rewrite the general equation as

$$\vartheta_2 \vartheta_4 = a \vartheta_1 + \beta \vartheta_1^{-1} + \gamma$$

which defines a surface in $\mathbb{C}^* \times \mathbb{C}^2$.

To obtain an LG model, we take as a potential the function $\vartheta_2$, linked to the compactification divisor, and we shall see that the choice of $\vartheta_2$ solves the problem at hand.

3.2. **Critical points of the potential.** To simplify calculations we set the constants to one: $\alpha = \beta = \gamma = 1$, and after one blow-up, we obtain the equation of the surface $u \vartheta_2 = v(\vartheta_1 + \vartheta_1^{-1} + 1)$. Changing to more standard algebraic coordinates $x := \vartheta_1, y := \vartheta_2$ and compactifying the direction $\vartheta_4$ to $\mathbb{P}^1$ with coordinates $[u : v]$, our problem is then to study the surface $\mathcal{Y}_2 \subset \mathbb{C}^* \times \mathbb{C} \times \mathbb{P}^1 = [(x, y), [u : v]]$ given by

$$\mathcal{Y}_2 := \{uy = v(x + 1 + 1/x)\}$$
with potential

\[ g := y = v(x + 1 + 1/x)/u. \]

Here the subscript 2 in the notation for surface \( \mathcal{M}_2 \) points to the fact that this will be the variety appearing in the mirror for the orbit of \( \mathfrak{sl}(2) \).

The critical points of the potential occur when \((x, y) \mid u : v \) takes the values:

\[ p_1 := (x_1, 0)[1 : 0], \quad \text{and} \quad p_2 := (x_2, 0)[1 : 0], \]

where \( x_1, x_2 \) are solutions of the equation \( x + 1 + 1/x = 0 \). In the affine chart where \( u = 1 \) we have the fiber over \( y = 0 \), given as the curve \( Y_o \) inside the affine chart \( \mathbb{C}^* \times \mathbb{C}^2 \) with coordinates \((x, y, v)\) cut out by the 2 equations:

\[ Y_o := \{(x, y, v) : y = 0 = v(x^2 + x + 1) \} \subset \mathbb{C}^* \times \mathbb{C}^2 \]

which contains 2 double points \( p_1 \) and \( p_2 \).

Taking \( v \to \infty \) we also obtain the point \([0 : 1] \in \mathbb{P}^1\), so we see that the critical fibre is \( Y := \{ y = 0 \} \) and has 3 irreducible components \( Y = Y_0 \cup Y_1 \cup Y_2 \) where

\[ Y := \begin{cases} Y_0 = \mathbb{C}^* \times \{ [1 : 0] \} \\ Y_1 = \{ x \} \times \mathbb{P}^1 \\ Y_2 = \{ x \} \times \mathbb{P}^1. \end{cases} \]

### 3.3. Mirror symmetry for \( \text{LG}(2) \)

We need to calculate the category of D-branes \( DB(\mathcal{M}_2, g) \) of the Landau–Ginzburg model \((\mathcal{M}_2, g)\). Given that the critical fibre \( Y \) has 3 irreducible components, following \[\text{KL}, \text{Example 8.2}\] we conclude that there are 3 D-branes to be considered, \( B_i \) corresponding to the components \( Y_i \) for \( i = 0, 1, 2 \). However, there is a symmetry of the LG-model, given by \( x \to x^{-1} \) which interchanges \( B_1 \) and \( B_2 \) and therefore these are equivalent in \( DB(\mathcal{M}_2, g) \).

We conclude that the category \( DB(\mathcal{M}_2, g) \) contains 2 objects, namely the 2 branes \( B_0 \) and \( B_1 \). To compute the morphisms we will pass to the category \( D_{\mathcal{I}_g}(\mathcal{M}_2) \) of singularities of \( \mathcal{M}_2 \) which is equivalent to the category of the pair \( DB(\mathcal{M}_2, g) \).

Let \( \pi : \mathcal{M}_2 \to \mathcal{M}_1 \) denote the smooth surface obtained by blowing up \( \mathcal{M}_2 \) at the double points \( p_i \) for \( i = 1, 2 \) with \( E_i \) the corresponding exceptional divisors, hence \( E_1 \approx \mathbb{P}^1 \). Denote by \( \mathcal{F} \) the sheaf on \( \mathcal{M}_2 \) supported on \( E_1 \) with rank 1 and degree \(-3\) on it, that is, it satisfies \( \mathcal{F}|_{E_1} = \mathcal{O}_{E_1}(-3) = \mathcal{O}_{\mathbb{P}^1}(-3) \) and it is extended by zero to the complement of \( E_1 \) in \( \mathcal{M}_2 \).

Let \( \mathcal{G} \) denote the sheaf on \( \mathcal{M}_2 \) supported on \( \pi^{-1}(Y_1) \) (more precisely on the proper transform \( \bar{Y}_1 \) of \( Y_1 \) in \( \mathcal{M}_2 \) and on the divisor \( E_1 \)) and of degree \(-1\) over each curve, that is \( \mathcal{G}|_{\bar{Y}_1} = \mathcal{O}_{\bar{Y}_1}(-1) = \mathcal{G}|_{E_1} \) supported on \( \pi^{-1}(Y_1) \) and extended by zero to the complement of \( \bar{Y}_1 \) in \( \mathcal{M}_2 \).

**Lemma 5.** Set \( \mathcal{L}_0 := \pi_* \mathcal{F} \) and \( \mathcal{L}_1 := \pi_* \mathcal{G} \). The equivalence \( DB(\mathcal{M}_2, g) \) to \( D_{\mathcal{I}_g}(\mathcal{M}_2) \) is given at the level of objects by \( B_0 \to \mathcal{L}_0 \) and \( B_1 \to \mathcal{L}_1 \).

**Proof.** Lemma 19 shows that the restriction of \( \mathcal{L}_0 \) generates \( D_{\mathcal{I}_g} \) at \( p_i \) for each of the singular points, and note that the 2 points appear together connected by \( B_0 \). \( \mathcal{G} \) generates the derived category of coherent sheaves on \( Y_1 \) and since the support of \( \mathcal{L}_1 \) contains the critical point \( p_1 \) it is not a perfect sheaf on \( \mathcal{M}_2 \), hence it is nontrivial in \( D_{\mathcal{I}_g}(\mathcal{M}_2) \). \( \square \)
Lemma 6. The morphisms are:
\[
\begin{cases}
\Hom(\mathcal{L}_0, \mathcal{L}_0) = \Hom(\mathcal{L}_1, \mathcal{L}_1) = \mathbb{Z} \\
\Hom(\mathcal{L}_0, \mathcal{L}_1) = \mathbb{Z} \oplus \mathbb{Z}[-1] \\
\Hom(\mathcal{L}_1, \mathcal{L}_0) = 0.
\end{cases}
\]

Proof. Note that the supports of $\mathcal{L}_0$ and $\mathcal{L}_1$ intersect only at the point $p_1$, therefore to calculate their morphisms it suffices to calculate morphisms between $\mathcal{F}$ and $\mathcal{G}$. To calculate their morphisms, note that $E_1$ is their common support on $\mathcal{F}_2$ and therefore their morphisms are trivial elsewhere. We have that $\Hom(\mathcal{L}_0, \mathcal{L}_1)$ come from their restriction to $E_1$, giving:
\[
\Hom(\mathcal{F}|_{E_1}, \mathcal{G}|_{E_1}) = \Hom(\mathcal{O}_{E_1}(-1), \mathcal{O}_{E_1}(-3)) = \Hom(\mathcal{O}_{E_1}(-1), \mathcal{O}_{E_1}(-3)) \oplus \Ext^1(\mathcal{O}_{E_1}(-1), \mathcal{O}_{E_1}(-3)) = H^0(\mathcal{O}_{P^1}(-2)) \oplus H^1(\mathcal{O}_{P^1}(-2)[-1]) = \mathbb{Z} \oplus \mathbb{Z}[-1].
\]

We have now obtained:

Theorem (1). Homological Mirror Symmetry for LG(2) works as follows:

- HMS 1 fails for LG(2), that is, for any (quasi) projective variety $Y$:
  \[
  \text{Fuk}(\text{LG}(2)) \not\equiv \text{D}^b\text{Coh}(Y).
  \]

- HMS 2 holds true for LG(2), that is, we have an equivalence of categories:
  \[
  \text{Fuk}(\text{LG}(2)) \equiv \text{D}^b\text{Coh}(Y, g).
  \]

Proof. The statement about the nonexistence of a suitable variety $Y$ to provide the categorical equivalence between Fuk(\text{LG}(2)) and $\text{D}^b\text{Coh}(Y)$ is just a rephrasing of [BBGGS, Thm. 4.1].

The equivalence between $\text{Fuk}(\text{LG}(2))$ and $\text{D}^b\text{Coh}(Y, g)$ is obtained from lemma 5 at the level of objects and lemma 6 at the level of morphisms.

Thus, the derived category obtained in Lemma 6 indeed coincides with the Fukaya category $\text{Fuk}(\text{LG}(2))$ described in [BBGGS, Thm. 3.1]. We observe that from the viewpoint of configuration of critical points the two LG models behave like as if $(Y_2, g)$ had been obtained from LG(2) by a $90^\circ$ rotation: the former configuration of 2 critical points on a single fibre can be achieved by changing the potential on the latter to a direction perpendicular to the original one.
4. The Landau–Ginzburg model LG(3)

The Landau–Ginzburg model LG(3) for the minimal adjoint orbit \( \mathfrak{o}_3 \) of \( \mathfrak{sl}(3) \) is described in detail in [BGRS, Sec. 4.1]. The adjoint orbit \( \mathfrak{o}_3 \) compactifies to a map onto \( \mathbb{P}^2 \times \mathbb{P}^2 \) where it is embedded as the open orbit of the diagonal action of \( \text{SL}(3, \mathbb{C}) \). This action has as a closed orbit the divisor \( D_1 = F(1, 2) \) and we have \( \mathbb{P}^2 \times \mathbb{P}^2 \setminus F(1, 2) \simeq \mathfrak{o}_3 \).

So, we study the pair \( (X, D_1) = (\mathbb{P}^2 \times \mathbb{P}^2, F(1, 2)) \) with a potential \( f_H \) obtained by choosing a regular element \( H = \text{Diag}(\lambda_1, \lambda_2, \lambda_3) \).

The rational map \( R_H \) extending \( f_H \) to \( \mathbb{P}^2 \times \mathbb{P}^2 \) is described in [BGGS] as

\[
R_H([x_1 : x_2 : x_3], [y_1 : y_2 : y_3]) = \frac{\lambda_1 x_1 y_1 + \lambda_2 x_2 y_2 + \lambda_3 x_3 y_3}{x_1 y_1 + x_2 y_2 + x_3 y_3}.
\]

The denominator vanishes precisely over the flag manifold, that is, over

\[ F(1, 2) = \{(x_1 : x_2 : x_3), (y_1 : y_2 : y_3) \in \mathbb{P}^2 \times \mathbb{P}^2 ; x_1 y_1 + x_2 y_2 + x_3 y_3 = 0 \}, \]

and the indeterminacy locus \( \mathcal{F} \) of \( R_H \), is the divisor in \( F(1, 2) \) cut out by

\[ \mathcal{F} = \{(x_1 : x_2 : x_3), (y_1 : y_2 : y_3) \in F(1, 2) ; \lambda_1 x_1 y_1 + \lambda_2 x_2 y_2 + \lambda_3 x_3 y_3 = 0 \}. \]

4.1. Monodromy and vanishing cycles. The potential of LG(3) has 3 critical points, happening at the 3 Weyl images of \( \text{Diag}(2, -1, -1) \). In the compactification they correspond to the 3 coordinate points \((e_i, e_i) \in \mathbb{P}^2 \times \mathbb{P}^2 \), where \( e_1 = [1 : 0 : 0] \) etc. Since LG(3) is a symplectic Lefschetz fibration, we know that the monodromy around each of the singular fibres is a classical Dehn twist.

We calculate the possible monodromy matrices fitting into this situation when we choose a regular fibre \( Y_b \). By [GGS1, Cor. 3.4] we know that \( H_3(Y_b) = \mathbb{Z} \oplus \mathbb{Z} \). We can write down the middle homology of the regular fibre as being generated by the vanishing cycles represented in homology as \( \sigma_1 = (1, 0) \) and \( \sigma_2 = (0, 1) \).

By definition of Lefschetz fibration, around each critical point we can choose coordinates to write the potential as \( z_1^2 + z_2^2 + z_3^2 + z_4^2 \), and each single monodromy may be expressed up to a change of coordinates as the action of \( T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \), with the vanishing cycle being fixed by the Dehn twist. Given that the potential was compactified to a map onto \( \mathbb{P}^1 \), we are looking for 3 monodromy matrices satisfying \( T_3 T_2 T_1 = I \), each representing a Dehn twist. In further generality this composition ought to equal the monodromy around the fibre at infinity; however, in our case, as proved in [BGRS, Lem. 17], there is no critical point at infinity. Let \( T_1 := T \) describe the monodromy around the first critical fibre, thus fixing vectors of the form \((a, 0)\) in \( H^0(Y_b) = \mathbb{Z} \oplus \mathbb{Z} \). The natural choice for \( T_2 \) is a Dehn twist that fixes the vectors \((0, b)\), represented as \( T_2 = T_1^t \), the transpose of \( T_1 \). Then \( T_3 = (T_1 T_2)^{-1} \) fixes only \((0, 0)\), which is geometrically clear, since it must invert both actions of \( T_1 \) and \( T_2 \). But the vanishing cycle corresponding to \( T_3 \) cannot be zero, so the only possibility left is that the vanishing cycle corresponding to \( T_3 \) occurs at infinity, at the limit of the unstable manifold corresponding to the third critical point in the open orbit \( \mathfrak{o}_3 \). Hence, a vanishing cycle placed at the compactifying divisor \( D_1 \). We conclude that 2 of the critical points have their corresponding vanishing cycles on the regular fibre, whereas the third critical point corresponds to a vanishing cycle at infinity. We state this fact as a lemma.

**Lemma 7.** The vanishing cycles of LG(3) can be depicted as the 2 generators of the middle homology of the regular fibre, together with one sphere at infinity.
By the end of the following section, we will verify that the mirror reverses this behaviour, having one critical point over the fibre at zero and 2 others at infinity.

4.2. The intrinsic mirror of LG(3). The minimal adjoint $\mathfrak{g}_3$ orbit is compactified to $\mathbb{P}^2 \times \mathbb{P}^2$ with fibre at infinity $D_1 \simeq \mathbb{F}(1,2)$ and potential of bidegree $(1,1)$ coming from 6. We take a second divisor $D_2$ as another fibre of the potential, so that $D_1 \cap D_2$ is the indeterminacy locus. To get to the log Calabi–Yau situation, we need additional divisors $D_3, D_4$ of bidegree $(1,0)$ and $(0,1)$ respectively, so that $D = \sum D_i$ is bidegree $(3,3)$, the anticanonical class. Their intersections satisfy:

- $D_1 \cap D_3$ and $D_2 \cap D_3$ are of type $(1,1)$ in $\mathbb{P}^1 \times \mathbb{P}^2$,
- $D_1 \cap D_4$ and $D_2 \cap D_4$ are of type $(1,1)$ in $\mathbb{P}^2 \times \mathbb{P}^1$,
- $D_3 \cap D_4 = \mathbb{P}^1 \times \mathbb{P}^1$ and $D_1 \cap D_2$ is a connected surface,
- $D_1 \cap D_3 \cap D_4$ and $D_2 \cap D_3 \cap D_4$ are of type $(1,1)$ in $\mathbb{P}^1 \times \mathbb{P}^2$,
- $D_1 \cap D_2 \cap D_3$ is an intersection of 2 type $(1,1)$ divisors in $\mathbb{P}^1 \times \mathbb{P}^2$,
- $D_1 \cap D_2 \cap D_3 \cap D_4$ consists of 2 connected components $(1,1) \cap (1,1)$ in $\mathbb{P}^1 \times \mathbb{P}^1$.

This gives a pillow-type picture similar to the one in [GS, Example 2.10] and our argument follows along the same lines. Using the Gross–Siebert recipe, we introduce functions corresponding to the barycenters of the two 4-dimensional cones. Writing $D = D_1 + D_2 + D_3 + D_4$ and variables $x_1, x_2, x_3, x_4, y_1, y_2$, the the corresponding equations for the mirror of $(\mathbb{P}^2 \times \mathbb{P}^2, D)$ are:

\begin{align*}
  x_1 x_2 x_3 x_4 &= y_1 + y_2 + t^{(1,0)} x_3 + t^{(0,1)} x_4, \quad \text{and} \\
  y_1 y_2 &= t^{(1,1)} x_3 x_4,
\end{align*}

where $(1,0), (0,1)$ and $(1,1)$ stand for homology classes of divisors with these degrees. In affine space of dimension 6, with coordinates $x_1, x_2, x_3, x_4, y_1, y_2$ over $\mathbb{C}[t_1, t_2]$, after calculation of the punctured Gromov–Witten invariants, we obtain:

\begin{align*}
  x_1 x_2 x_3 x_4 &= y_1 + y_2 + t_2 x_3 + t_1 x_4, \quad \text{and} \\
  y_1 y_2 &= t_1 t_2 x_3 x_4,
\end{align*}
where $t_1, t_2$ may be set to random values (we really only use here the fact that they are nonzero). We take as our potential $g = x_2$, corresponding to the first extra divisor $D_2$ that we added.

4.2.1. *The mirror variety as a hypersurface.* At first, the ring defining our mirror variety is $R = \mathbb{C}[t_1, t_2]|_{x_1, x_2, x_3, x_4, y_1, y_2]/ < r_1, r_2 >$ with relations

$$r_1 := x_1 x_2 x_3 x_4 = y_1 + y_2 + t_2 x_3 + t_1 x_4, \quad r_2 := y_1 y_2 = t_1 t_2 x_3 x_4.$$ 

Observing that $R$ is isomorphic to $\mathbb{C}[t_1, t_2]|_{x_1, x_2, x_3, x_4, y_1}/ < r >$ with the single relation $r := y_1 (x_1 x_2 x_3 x_4 - y_1 - t_2 x_3 - t_1 x_4) = t_1 t_2 x_3 x_4$, our mirror candidate variety $\mathcal{Y}_3$ may described by a single polynomial equation

$$y_1 x_1 x_2 x_3 x_4 - t_1 t_2 x_3 x_4 - y_1 t_2 x_3 - y_1 t_1 x_4 - y_1^2 = 0.$$ 

Equivalently, we take the variety defined by

$$y_1 x_1 x_2 x_3 x_4 = (y_1 + t_1 x_4)(y_1 + t_2 x_3),$$

and we have chosen the potential

$$x_2 = \frac{t_1 t_2 x_3 x_4 + y_1 t_2 x_3 + y_1 t_1 x_4 + y_1^2}{y_1 x_1 x_3 x_4}.$$ 

In further generality, we could have considered $x_2 + x_3 + x_4$ corresponding to all divisors added after compactifying, in order to arrive at the log geometry setup, but we will see that our simpler choice is enough for our purposes of finding a mirror.

4.2.2. *Singularities of the mirror ($\mathcal{Y}_3, x_2$) and rational extension of the potential.* Since the $t_i$’s are constants, we discuss the polynomials in $\mathbb{C}[x_1, x_2, x_3, x_4, y_1]$ giving the variety cut $\mathcal{Y}_3 \subset \mathbb{C}^5$ out by the equation

$$y_1 x_1 x_2 x_3 x_4 = (y_1 + t_1 x_4)(y_1 + t_2 x_3).$$

**Lemma 8.** The singularities of the variety $\mathcal{Y}_3$ occur when either $y_1 = x_3 = x_4 = 0$ or else $x_1 = x_2 = y_1 + t_2 x_3 = y_1 + t_1 x_4 = 0$.

**Proof.** The variety is given by

$$p = -x_1 x_2 x_3 x_4 y_1 + (y_1 + t_1 x_4)(y_1 + t_2 x_3)$$

and the partials are

$$\begin{cases}
\frac{\partial p}{\partial x_1} = -x_2 x_3 x_4 y_1 \\
\frac{\partial p}{\partial x_2} = -x_1 x_3 x_4 y_1 \\
\frac{\partial p}{\partial x_3} = -x_1 x_2 x_4 y_1 + t_2 y_1 + t_1 t_2 x_3 \\
\frac{\partial p}{\partial x_4} = -x_1 x_2 x_3 y_1 + t_1 y_1 + t_1 t_2 x_3 \\
\frac{\partial p}{\partial y_1} = -x_1 x_2 x_3 x_4 + 2 y_1 + t_1 x_4 + t_2 x_3.
\end{cases}$$

Vanishing of the equations occurs in 2 separate cases:

Case (1) $y_1 = 0$ which leads to $y_1 = x_3 = x_4 = 0$.
Case (2) $x_1 = x_2 = 0$ which leads to $x_1 = x_2 = y_1 + t_2 x_3 = y_1 + t_1 x_4 = 0.$

Now, we study the potential determined by $x_2$ over $\mathcal{Y}_3$. 

Lemma 9. Let $\mathcal{Y}' = \mathcal{Y} \cap \{y_1 \neq 0 \} \cap \{x_1 \neq 0 \} \cap \{x_3 \neq 0 \} \cap \{x_4 \neq 0 \}$, then the potential may be regarded as a smooth map $x_2 : \mathcal{Y}' \to \mathbb{C}$ given as a quotient

$$x_2 = \frac{(y_1 + t_1 x_1)(y_1 + t_2 x_3)}{y_1 x_1 x_3 x_4},$$

and the only finite critical value of the potential $x_2 : \mathcal{Y}' \to \mathbb{C}$ is $x_2 = 0$.

Proof. If the left hand side of $y_1 x_1 x_2 x_3 x_4 = (y_1 + t_1 x_1)(y_1 + t_2 x_3)$ is nonzero, then the equation defines a smooth variety, and since $x_1, x_3, x_4, y_1$ are all nonzero on $\mathcal{Y}'$ to obtain a finite critical value of the potential, we must have that $x_2 = 0$. (This may also be checked by standard calculations of the partial derivatives.)

Alternatively, we may regard $x_2$ defining as a rational map $\mathcal{Y} \dashrightarrow \mathbb{P}^1$, and homogenizing using an extra variable $y_2$ (the $t_i$'s are constants) we obtain the following expression:

$$R(x_1, x_2, x_3, x_4, y_1, y_2) = [(t_1 t_2 x_3 x_4 + t_1 t_2 x_3 + y_1 t_1 x_4 + y_1^2) y_2^2 : y_1 x_1 x_3 x_4],$$

which can be viewed as a rational map $\mathbb{P}^5 \dashrightarrow \mathbb{P}^1$. When $y_1 x_2 x_3 x_4 \neq 0$, we have

$$R(x_1, x_2, x_3, x_4, y_1, y_2) = \left[ \frac{(y_1 + t_1 x_1)(y_1 + t_2 x_3) y_2^2}{y_1 x_1 x_3 x_4} : 1 \right] = [x_2 : 1],$$

showing that $R$ is indeed an extension of the potential $x_2$. It is usual to call $\infty := [1 : 0]$ the point at infinity and refer to the divisor $D_\infty = \{y_1 x_1 x_3 x_4 = 0\}$ as the fibre at infinity. The indeterminacy locus of $R$ is

$$\mathcal{I} = \{(y_1 + t_1 x_1)(y_1 + t_2 x_3) y_2^2 = 0 = y_1 x_1 x_3 x_4\} \subset D_\infty.$$

4.2.3. The case $t_1 = t_2 = 1$. To match the notation of the general case in section 5, we rename $x_3 = w$ and $x_4 = z$ (in fact, it ought to be $w_1$ and $z_1$, but we omit the subscripts). We restate the results of lemmas 8 and 9 with $t_1 = t_2 = 1$:

Lemma 10. The singularities of the variety $\mathcal{Y} := \{x_1 x_2 y_1 z w = (y_1 + z)(y_1 + w)\}$ occur when either $y_1 = z = w = 0$ or else $x_1 = x_2 = y_1 + w = y_1 + z = 0$. The potential $x_2$ may be regarded as a smooth map on $\mathcal{Y}'$ given by $x_2 = (y_1 + z)(y_1 + w) x_1 x_2 y_1 z w$, and the map

$$R(x_1, x_2, y_1, y_2, z, w) = \left[ \frac{(y_1 + z)(y_1 + w) y_2^2}{x_1 x_2 y_1 z w} : 1 \right] = [x_2 : 1]$$

is a rational extension of $x_2$ to the compactification.

Considering $\mathbb{P}^5$ with coordinates $([x_1 : x_2 : y_1 : y_2 : z : w])$, we may write $\mathcal{Y}$ as the hypersurface (contained inside the affine chart $y_2 = 1$) cut out by the equation

$$x_1 x_2 y_1 z w = (y_1 + z)(y_1 + w).$$

There are 2 special points of the potential:

- $R = [0 : 1]$, we call this the zero of the potential $x_2$, happening when $(y_1 + z)(y_1 + w) y_2^2 = 0$. Note that $[0 : 1]$ is a critical value of the potential which contains singularities of the variety $\mathcal{Y}$; and
- $R = [1 : 0]$, we call this the infinity of the potential $x_2$, that is, when $x_1 y_1 z w = 0$. Note that $[1 : 0]$ also contains singularities of $\mathcal{Y}$. 


Now, to study finite values of the potential, the value of $x_1$ does not matter at all, as long as it is nonzero. To study the fibre at infinity, we either have $x_1 = 0$ or else, once again any nonzero value of $x_1$ has no effect. Hence, $x_1$ is only a placeholder marking the fibre at infinity, while $x_2$ is only used to name the map, therefore we may look at the restricted problem of a rational map $\mathbb{P}^3 \dashrightarrow \mathbb{P}^1$

$$R' = [(y_1 + z)(y_1 + w) y_2 : y_1 zw]$$

with corresponding indeterminacy locus

$$\mathcal{I}' = [(y_1 + z)(y_1 + w) y_2 = 0 = y_1 zw].$$

We will see that this dimensionally reduced problem gives our desired mirror.

4.2.4. Extension of the potential to the compactification. The next step is to consider a blow-up of $\mathbb{P}^3$ along the indeterminacy locus $\mathcal{I}'$ of the map $R'$. We denote by $\mathcal{Z}_3 := BL_{\mathcal{I}'}(\mathbb{P}^3)$, the result of this blowing up, obtained as follows.

Set $f = (y_1 + z)(y_1 + w) y_2$ and $g = y_1 zw$. Take $\mathbb{P}^1$ with homogeneous coordinates $[t : s]$. The pencil $\{tg + sf\}_{t,s \in \mathbb{C}}$ with base locus $\mathcal{I}'$ induces the rational map $R' : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1$. We denote by $\mathcal{Z}_3$ the closure of the graph of the blow-up map, so that

$$\begin{array}{ccc}
\mathcal{Z}_3 & \xrightarrow{\pi} & \mathbb{P}^1 \\
\downarrow & & \downarrow \\
\mathcal{Y}_3 & \xrightarrow{R'} & \mathbb{P}^3
\end{array}$$

and we get a map

$$W = R' \circ \pi : \mathcal{Z}_3 \subset \mathbb{P}^3 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$$

$$(y_1 : y_2 : z : w : t : s) \rightarrow [t : s].$$

Note that if $s \neq 0$ then $[t : s] = [\frac{t}{s} : 1] = [\frac{f}{g} : 1] = [f : g]$ where the middle equality holds since $tg = sf$. Thus, we have obtain an extension of $f/g$ to the compactification. We call $\infty := [1 : 0]$ the point at infinity, and refer to the divisor $D_\infty = [y_1 zw = 0]$ as the fibre at infinity.

**Lemma 11.** The potential $W$ has 2 critical values $[0 : 1]$ and $[1 : 0]$. We refer to them as zero and infinity respectively.

We observe that even though the variety $Y$ contains singular points, we do not consider them individually. Instead, we just take into account the critical fibres of the potential, and then we verify that this choice produces the correct category of singularities for the mirror.

4.3. Category of singularities of $LG^\vee(3)$. Considering $\mathbb{P}^3 \times \mathbb{P}^1$ with coordinates $[y_1 : y_2 : z : w], [t : s]$, we now analyse the critical fibres of the potential $W$ over $\mathcal{Z}_3$.

4.3.1. The fibre over 0 contributes a single object to $D_{sg}(\mathcal{Z}_3, g)$. The critical fibre at 0 happens when $[t : s] = [0 : 1] = [f : g]$, that is, when $[(y_1 + z)(y_1 + w) y_2 : y_1 zw] = [0 : 1]$.

If $y_2 = 0$ the equality is satisfied at all points of a divisor $D_{y_2} = \mathcal{Z}_3 \cap \mathbb{P}^2 \times \mathbb{P}^1$, and its contribution to the category of singularities of the compactified model would give us one object, which comes from the sheaf $\mathcal{F}(y_2)$ supported at the points of
intersection $D_{y_2} \cap (D_{w_0} + D_{w_0})$, where $D_{w_0}$ and $D_{w_0}$ are the divisors that appear below. However, we are interested only in the critical points of the potential in the mirror variety $\mathcal{Y}_3$ which lives inside the affine chart $y_2 = 1$, so that we ignore this object living outside $\mathcal{Y}_3$.

Suppose now $y_2 = 1$, then we obtain the subset of the fiber over zero cut out by equations $(y_1 + z)(y_1 + w) = 0$ with singularity in codimension 2 at the intersection of the 2 divisors $D_{y_0}$ given by $y_1 = -z$ and $D_{w_0}$ given by $y_1 = -w$. So, we obtain 2 nonperfect sheaves on $W^{-1}([0 : 1])$, these are the sheaves $\mathcal{F}(y_0) = i_{w*} \mathcal{O}_{D_{y_0}}(1)$ and $\mathcal{F}(w_0) = i_{w*} \mathcal{O}_{D_{w_0}}(1)$ supported on $D_{y_0}$ and $D_{w_0}$ included in $W^{-1}([0 : 1]) \cap \mathcal{Y}_3$ by $i_z$ and $i_w$ respectively. However, since there is a clear symmetry of this fibre given by exchanging $z$ and $w$ the sheaves $\mathcal{F}(y_0)$ and $\mathcal{F}(w_0)$ are quasi-isomorphic, so we obtain a unique object of $D_{sg}(W)$, say $\mathcal{F}(w_0)$ coming from the fibre over 0.

4.3.2. The fibre over $\infty$ contributes 2 objects to $D_{sg}(\mathcal{Y}_3, g)$. The critical fibre over $\infty$ happens when $\{f : s\} = [1 : 0] = \{f : g\}$, that is, when $\{(y_1 + z)(y_1 + w)y_2 : y_1 zw\} = [1 : 0]$. Here we may choose 3 extra sheaves depending on the 3 variables making $y_1 zw = 0$, once again, there is a clear symmetry of the space taking $z$ to $w$ so that we are left with 2 new objects $\mathcal{F}(y_1)$ and $\mathcal{F}(z)$; these are rank 1 sheaves supported on $y_1 = 0$ and $z = 0$ respectively.

Lemma 12. The category of singularities of $(\mathcal{Y}_3, g)$ is generated by 3 objects; namely one sheaf supported on the fibre over zero, and 2 sheaves supported on the fibre over infinity.

Proof. The generators are the sheaves $\mathcal{F}(y_0)$, $\mathcal{F}(y_1)$ and $\mathcal{F}(z)$ just defined. \qed

5. The Landau–Ginzburg model $LG(n + 1)$

$LG(n + 1) = (\mathcal{O}_n^{n+1}, f_H)$, embeds in $\mathbb{P}^n \times \mathbb{P}^n$ as the open orbit of the diagonal action of $SL(n + 1)$ with complement the diagonal, that is, the flag of lines and $n$-planes in $\mathbb{C}^{n+1}$.

5.1. $LG(n + 1)$ fitted into the log geometry. We have $\mathcal{O}_n^{n+1} \subset \mathbb{P}^n \times \mathbb{P}^n$ with compactifying divisor $F(1, n)$. We set $D_1 := F(1, n)$, and chose a second divisor $D_2$ of type $(1,1)$. We now describe the intrinsic mirror, generalising section 4.2.

We write the components of the boundary divisor as: $D_1, D_2$ of bidegree $(1,1)$; $E_1, ..., E_{n-1}$ of bidegree $(1,0)$; and $F_1, ..., F_{n-1}$ of bidegree $(0,1)$. Then we take variables $x_1, x_2$ corresponding to $D_1, D_2$; $z_1, ..., z_{n-1}$ corresponding to $E_1, ..., E_{n-1}$; and $w_1, ..., w_{n-1}$ corresponding to $F_1, ..., F_{n-1}$.

As in the n=2 case, we also need to take additional variables $y_1, y_2$ which come from the theta functions corresponding to the contact order information of meeting all divisors transversally; there are two zero-dimensional strata where this can happen.

5.2. The intrinsic mirror of $LG(n + 1)$. The equations coming from these two zero-dimensional strata are

$$x_1 x_2 \prod (w_i z_i) = y_1 + y_2 + t_1 \prod w_i + t_2 \prod z_i$$

$$y_1 y_2 = t_1 t_2 \prod (w_i z_i).$$

The first equation gives

$$y_2 = x_1 x_2 \prod (w_i z_i) - y_1 - t_1 \prod w_i - t_2 \prod z_i,$$
and plugging this into the second equation leads to:
\[ y_j(x_1 x_2 \prod w_i z_i) - y_i - t_1 \prod w_i - t_2 \prod z_i = t_1 t_2 \prod (w_i z_i). \]
So, we obtain the affine variety in \( \mathbb{C}[x_1, x_2, w_1, \ldots, w_{n-1}, z_1, \ldots, z_{n-1}, y_j] \) cut out by the single equation:
\[ y_j^2 + y_1(t_1 \prod w_i + t_2 \prod z_i - x_1 x_2 \prod (w_i z_i)) + t_1 t_2 \prod (w_i z_i) = 0, \]
or equivalently, we have the variety \( \mathcal{Y}_{n+1} \) defined by:
\[ (y_1 + t_1 \prod w_i)(y_1 + t_2 \prod z_i) = y_1 x_1 x_2 \prod (w_i z_i). \] (7)

Over the variety \( \mathcal{Y}_{n+1} \), we choose the potential \( g = x_2 \) corresponding to \( D_2 \), the first additional divisor which was added for adapting the pair \( (\mathbb{P}^n \times \mathbb{P}^n, [1, n]) \) to the log geometry setup. The Landau–Ginzburg model formed by the variety \( \mathcal{Y}_{n+1} \) together with the potential \( x_2 \) will give rise to an extended Landau–Ginzburg model \( (\mathcal{Z}_{n+1}, W) \) which we denoted by
\[ \text{LG}^y(n + 1) := (\mathcal{Y}_{n+1}, g = x_2) \]
and will turn out to be our desired mirror (although we only prove this at the level of objects).

**Example 13.** The mirror of LG(3) is obtained from the general formula 7 by making \( i = 1 \) so that \( w = w_1 = x_3 \) and \( z = z_1 = x_4 \).

We now must extend the potential to the compactification. As a first step we construct a rational extension. We have
\[ g := x_2 = \frac{(y_1 + t_1 \prod w_i)(y_1 + t_2 \prod z_i)}{y_1 x_1 \prod (w_i z_i)}. \]

Since \( x_1 \neq 0 \) at finite values of the expression for \( x_2 \), we may reduce the problem to looking at \( x_2 \) as the map
\[ x_2 = \frac{(y_1 + t_1 \prod w_i)(y_1 + t_2 \prod z_i)}{y_1 \prod (w_i z_i)}, \]
and then, as a computational tool, we homogenise, adding an extra variable \( y_2 \), thus obtaining
\[ \frac{(y_1 y_2^{n-1} + t_1 \prod w_i)(y_1 y_2^{n-1} + t_2 \prod z_i)}{y_1 \prod (w_i z_i)}. \]

Equivalently, we now have the rational map on \( \mathbb{P}^{2n-1} \) written with variables \( [y_1 : y_2 : z_1 : z_2 : \cdots : z_{n-1} : w_1 : w_2 : \cdots : w_{n-1}] \) defined by:
\[ R' = [(y_1 y_2^{n-1} + t_1 \prod w_i)(y_1 y_2^{n-1} + t_2 \prod z_i) y_2 : y_1 \prod (w_i z_i)]. \]

**Lemma 14.** The rational map \( R' : \mathbb{P}^{2n-1} \to \mathbb{P}^1 \) is an extension of the potential \( g : \mathcal{Y}_{n+1} \to \mathbb{C} \).

**Proof.** \( R' = [x_2 : 1] \) in the affine chart \( y_2 = 1 \), therefore \( R' \) is an extension of the potential \( g = x_2 \) defined over \( \mathcal{Y}_{n+1} \). \( \square \)

The map \( R' \) has as indeterminacy locus
\[ \mathcal{R}' = \{(y_1 y_2^{n-2} + t_1 \prod w_i)(y_1 y_2^{n-2} + t_2 \prod z_i) y_2 = y_1 \prod (w_i z_i) = 0\}. \]
5.2.1. **Extension of the potential to the compactification.** The next step is to consider a blow-up of \( \mathbb{P}^{2n-1} \) along the indeterminacy locus \( \mathcal{I} \) of the map \( R' \). We denote by \( Z_{n+1} := B\mathcal{I}R(\mathbb{P}^{2n-1}) \), the result of this blowing up, obtained as follows.

Set \( f_1 = (y_1y_2^{n-2} + t_1 \prod w_i)(y_1y_2^{n-2} + t_2 \prod z_i)y_2 \) and \( f_2 = y_1 \prod (w_iz_i) \). Take \( \mathbb{P}^1 \) with homogeneous coordinates \([t, s]\). The pencil \( \{tf_2 + sf_1\}_{t, s \in \mathbb{C}} \) with base locus \( \mathcal{I} \) induces the rational map \( R': \mathbb{P}^{2n-1} \rightarrow \mathbb{P}^1 \). Call \( Z_{n+1} \) the closure of the graph of the blow-up map, so that

![Diagram](attachment:image.png)

and we get a map

\[
W = R' \circ \pi: Z \subseteq \mathbb{P}^{2n-1} \times \mathbb{P}^1 \rightarrow \mathbb{P}^1 \\
(y_1 : y_2 : z_1 : \ldots : z_{n-1} : w_1 : \ldots : w_{n-1} : t : s) \mapsto [t : s].
\]

Note that if \( s \neq 0 \) then \([t : s] = [\frac{1}{s} : 1] = [\frac{t}{f_1} : 1] = [f_1 : f_2] \) where the middle equality holds since \( tf_2 = sf_1 \). We call \( \infty := [1 : 0] \) the point at infinity, and refer to the divisor \( D_{\infty} = \{y_1 \prod (w_iz_i) = 0\} \) as the fibre at infinity.

Thus, we obtain the desired holomorphic extension:

**Lemma 15.** The map \( W: Z_{n+1} \rightarrow \mathbb{P}^1 \) is a holomorphic extension of \( R' \). The critical points of \( W \) on \( Y_{n+1} \) come from critical points of \( R \) on \( Y_{n+1} \).

**Lemma 16.** The potential \( W \) has 2 critical values \([0 : 1]\) and \([1 : 0]\). We refer to them informally as zero and infinity respectively.

We now want to calculate the category of singularities of \((Z_{n+1}, W)\).

5.3. **Singular fibres of of** \( \mathcal{L}^g(n+1) \). We analyse the critical points of the potential over \( Z_{n+1} \subseteq \mathbb{P}^{2n-1} \times \mathbb{P}^1 \) with coordinates \([y_1 : y_2 : z_1 : \ldots : z_{n-1} : w_1 : \ldots : w_{n-1} : t : s]\), where \( Z_{n+1} \) is given by the equation

\[
t(y_1y_2^{n-2} + t_1 \prod w_i)(y_1y_2^{n-2} + t_2 \prod z_i)y_2 = sy_1 \prod (w_iz_i).
\]

5.3.1. **The fibre over 0.** This happens when \([t : s] = [0 : 1]\). If \( y_2 = 0 \) the equality is satisfied at all points of a divisor \( D_j = Z_{n+1} \cap (\mathbb{P}^{2n-1} \times \mathbb{P}^1) \), and its contribution to the category of singularities would give us one object, coming from the sheaf \( \mathcal{F}(y_2) \) supported at the points of intersection \( D_j \cap (\sum D_{z_i} + \sum D_{w_i}) \), where \( D_{z_i} \) and \( D_{w_i} \) are the divisors that appear below. However, this happens outside of the variety which interests us, \( Y_{n+1} \), contained in the affine chart \( y_2 = 1 \). So, for our purposes here, this contribution is ignored.

Assume now \( y_2 = 1 \), then we obtain the subset of the fibre over zero intersected with our variety \( Y_{n+1} \) is cut out by the equation

\[
W^{-1}([0 : 1]) \cap Y_{n+1} = \{(y_1 + \prod w_i)(y_1 + \prod z_i) = 0\}
\]

which is singular when either \( y_1 = -\prod z_i \) or \( y_1 = -\prod w_i \). Therefore, we obtain 2 nonperfect sheaves on \( W^{-1}([0 : 1]) \cap Y_{n+1} \). These are sheaves \( \mathcal{F}(z) \) supported on \( D_z = \{y_1 + \prod z_i = 0\} \) and \( \mathcal{F}(w) \) supported on and \( D_w = \{y_1 + \prod w_i = 0\} \) included in \( W^{-1}([0 : 1]) \cap Y_{n+1} \) by \( i_z \) and \( i_w \) respectively.
Since there is a clear symmetry of the variety $\mathcal{V}_{n+1}$ obtained by interchanging $z_i \leftrightarrow w_i$ the corresponding nonperfect sheaves are quasi-isomorphic. Hence, in total the fibre over zero contributes with a single sheaf, say $\mathcal{F}(z_0)$ corresponding to the singularity $y_1 = -\prod z_i$. Thus, the fibre over zero contributes one object to $D_{sg}(\mathcal{V}_{n+1}, x_2)$.

5.3.2. The fibre over $\infty$. This happens when $[t:s] = [1:0]$. Considering the fibre over $\infty$, that is, when

$$y_1w_1w_2\ldots w_{n-1}z_1z_2\ldots z_{n-1} = 0,$$

we see that the fibre at infinity has $2n - 1$ irreducible components, with sheaves supported on their corresponding individual components and of degree 1 over their support, as follows:

- $\mathcal{F}(w_i)$ supported on $w_i = 0$,
- $\mathcal{F}(z_i)$ supported on $z_i = 0$ for $i = 1, \ldots, n - 1$,
- $\mathcal{F}(z_n) := \mathcal{F}(y_1)$ supported on $y_1 = 0$.

Given the symmetry between $z$’s and $w$’s we have that for a fixed $i$ the sheaves $\mathcal{F}(w_i)$ are isomorphic; so that we obtain a total contribution $n$ objects from the fibre at infinity, $\mathcal{F}(z_i)$ with $i = 1, \ldots, n$.

We have now obtained all the $n + 1$ generators we needed to describe the category of singularities $D_{sg}(\mathcal{Z}_{n+1}, W)$ of the Landau–Ginzburg model $(\mathcal{Z}_{n+1}, W)$, which is our proposed mirror of $LG(n+1)$, they are $\mathcal{F}(z_0), \ldots, \mathcal{F}(z_n)$; where $\mathcal{F}(z_0)$ comes from the fibre over zero, and the others from the fibre over infinity.

In the following section we clarify some details about the mirror map.

5.4. The mirror correspondence. By [GGS1, Thm. 2.2] we know that the Landau–Ginzburg model $LG(n+1) = (\mathcal{O}(H_0), f_H)$ has $n + 1$ singularities corresponding to the $n + 1$ elements $[w H_0, w \in \mathcal{W}]$, where $\mathcal{W}$ is the Weyl group of $SL(n+1)$.

By [GGS1, Cor. 3.4] we know that the middle homology of the regular fibre $f^{-1}_H(c)$ of $LG(n+1)$ is

$$H_{\text{mid}}(f^{-1}_H(c)) = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}, \quad n \text{ times}.$$

The Weyl group $\mathcal{W}$ of $SL(n+1)$ is cyclic of order $n + 1$. Let $w_0$ be a generator of $\mathcal{W}$. Correspondingly, we denote by $L_i$ the Lagrangian sphere $S^{2n-1}$ that vanishes to the singularity $w_0^i H_0$. Then $L_1, \ldots, L_n$ generate the the middle homology of the regular fibre, that is,

$$H_{n-1}(f^{-1}_H(c)) = \mathbb{Z} < L_1, \ldots, L_n >.$$

The remaining vanishing cycle, the sphere $L_{n+1}$ lives at infinity. This can be shown by the same reasoning as the one we used to show lemma 7, namely, if the cycle $(n+1)$ were a finite linear combination of $L_1, \ldots, L_n$, then the corresponding monodromy matrices ought to compose to give the identity, however, $L_{n+1}$ would need to move all elements of the basis and this is not consistent with the monodromy of a Dehn twist. So, we have also in the general case a similar lemma:

**Lemma 17.** The vanishing cycles of $LG(n + 1)$ can be seen as the $n$ generators $L_i$ of the middle homology of the regular fibre, together with one sphere at infinity $L_{n+1}$.

**Lemma 18.** At the level of objects, the mirror map is given by the $1$-$1$ correspondence: $L_i \leftrightarrow \mathcal{F}(z_i)$, for $i = 1, \ldots, n$, $L_{n+1} \leftrightarrow \mathcal{F}(z_0)$. 
In other words, we have obtained:

**Theorem (2).** The intrinsic mirror symmetry algorithm produces an LG-model \((\mathcal{Y}_{n+1}, g)\) for which

\[
\text{Fuk}(\text{LG}(n+1)) \cong D_{sg}(\mathcal{Y}_{n+1}, g)
\]

is a 1-1 correspondence.

Note that on the symplectic side I have described the vanishing cycles, but not the morphisms between them in the Fukaya–Seidel category. This happens because even though I know existence of Lagrangian thimbles taking the given Lagrangian vanishing cycles to their corresponding critical points, I do not know their explicit expressions in the total space, which would be necessary for calculating the morphisms. Nevertheless, despite not knowing the morphisms on the symplectic (A) side, I can calculate morphisms on the algebraic (B) side using a simple Macaulay2 code presented in Appendix B code 9. We now continue to the description of the category \(D_{sg}(\mathcal{Z}_{n+1}, W)\).

5.5. **Category of singularities of** \(LG^\vee(n+1)\). The Orlov category of \(LG^\vee(n+1) = (\mathcal{Z}_{n+1} W)\) is by definition the sum of the categories of singularities of the fibres over zero and infinity, which are disjoint subvarieties of \(\mathcal{Z}_{n+1}\). The singular fibres were calculated in subsections 5.3.1 and 5.3.2. Their singularities categories are generated by \(\mathcal{F}(z_0)\) and \(\mathcal{F}(z_1), \ldots, \mathcal{F}(z_n)\) respectively. To express the morphisms, we first set some notation.

For calculations corresponding to the fibre at infinity we set the polynomial ring \(R = \mathbb{C}[z_1, \ldots, z_n]\) with ideal \(I = (z_1 \ldots z_n) \subset R\) and quotient ring \(S := R/I\). We consider modules \(J_i\) over \(S\) which correspond to the sheaves \(\mathcal{F}(z_i)\) with \(i = 1, \ldots, n\). Hence, we set \(S\)-ideals

\[
J_i := (z_i) \subset S \quad \text{for} \quad i = 1, \ldots, n.
\]

The required resolution of the module \(J_i\) is given by the unbounded, periodic resolution

\[
0 \to J_i \xrightarrow{f} P_0 \cong S^1 \xrightarrow{d_0 \bmod (z_1 \ldots z_n)} P_1 \cong S^1 \xrightarrow{d_1} P_2 \cong S^1 \xrightarrow{d_2 \bmod (z_1 \ldots z_n)} P_3 \cong S^1 \xrightarrow{d_3} \ldots.
\]

Then \(\text{Hom}_S(P_*, J_i)\) is the cochain complex

\[
0 \to \text{Hom}_S(P_0 = S^1, J_i) \xrightarrow{f^0} \text{Hom}_S(P_1 = S^1, J_i) \xrightarrow{g^0} \text{Hom}_S(P_2 = S^1, J_i) \xrightarrow{f^0} \ldots,
\]

and it represents \(R\text{Hom}(J_i, J_j)\) in the derived category. Since \(\text{Hom}_S(S, M) \cong M\) for any \(S\)-module \(M\), we have maps

\[
0 \to (z_j)S \xrightarrow{d_{k+1} = f_*} (z_j)S \xrightarrow{d_k = g_*} (z_j)S \xrightarrow{d_{k+1} = f_*} (z_j)S \xrightarrow{d_k = g_*} \ldots.
\]

We set the notation \(N_{ij} := \ker(d_{2k+1}) = (z_j z_k)S\) and \(M_{ij} = \ker(d_{(2k+1)*} - \text{im}d_{(2k)*})\) for all \(k \geq 0\). We then have:

\[
\text{Ext}^k(S_i, S_j) = \begin{cases} N_{ij} & \text{for } k = 0, \\ M_{ij} & \text{for odd } k, \text{ and} \\ 0 & \text{for even } k, k > 0. \end{cases}
\]
See Appendix B for further details. For $1 \leq i < j \leq n$, the corresponding sheaves $\mathcal{F}(z_1), \ldots, \mathcal{F}(z_n)$ therefore satisfy:

$$\text{Hom}(\mathcal{F}(z_i), \mathcal{F}(z_j)) = N_{ij} \bigoplus_{t=2s+1} M_{ij}[t]$$

and

$$\text{Hom}(\mathcal{F}(z_0), \mathcal{F}(z_j)) = 0.$$

Therefore, we have obtained the full description of the category of singularities, proving:

**Theorem (3).** The Orlov category of singularities of the Landau–Ginzburg model $(\mathcal{I}_{n+1}, W)$ is generated by the nonperfect sheaves $\mathcal{F}(z_0), \mathcal{F}(z_1), \ldots, \mathcal{F}(z_n)$ with morphisms:

$$\text{Hom}(\mathcal{F}(z_i), \mathcal{F}(z_j)) = \begin{cases} N_{ij} \bigoplus_{t=2s+1} M_{ij}[t] & \text{if } i \cdot j \neq 0, \\ 0 & \text{if } i \cdot j = 0. \end{cases}$$

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**Appendix A. Nonperfect sheaves**

Let $Z_2 := \text{Tot} \mathcal{O}_{\mathbb{P}^1}(-2)$ denote the total space of the canonical line bundle on $\mathbb{P}^1$ and let $X_2$ be the variety obtained from it by contracting the zero section to a point with $\pi : Z_2 \rightarrow X_2$ the contraction. Then $X_2$ is a concrete model for the ordinary double point surface singularity.

Consider the line bundle $L = \mathcal{O}_{Z_2}(-3)$; hence, the line bundle on the surface $Z_2$ obtained from pulling back $\mathcal{O}_{\mathbb{P}^1}(-3)$. Set $\mathcal{F} := \pi_* L$ as the rank 1 sheaf on $X_2$ obtained as direct image.

**Lemma 19.** $\mathcal{F}$ generates $D_{sg}(X_2)$.

**Proof.** [BGK, Example 2.14] computes the resolution of $\mathcal{F}$ showing it is infinite and periodic (see details below), so that $\mathcal{F}$ is a coherent sheaf which is not perfect and therefore nonzero in the Orlov category of singularities of $X_2$. Orlov [Or1, Sec. 3.3] showed that $D_{sg}(X_2)$ has a single generator and $\mathcal{F}$ is a concrete geometric description of it. □

We expand the calculations given in [BGK, Example 2.14] to show that $\mathcal{F}$ is not perfect. Let $M := \mathcal{F}_0$ be the completion of the stalk at the singular point. We will describe the structure of $M$ as a module over global functions and show that its resolution is infinite and periodic. In canonical coordinates $Z_2$ is given by charts $U = \{z, u\}, V = \{z^{-1}, z^2u\}$ and the transition matrix for the bundle $L = \mathcal{O}(-3)$ is $z^3$. The coordinate ring of $X_2$ is
$\mathbb{C}[x, y, z]/(y^2 - xw)$, and the lifting map induced by $\pi$ is (on the $U$-chart) $x \mapsto u$, $y \mapsto zu$ and $w \mapsto z^2 u$.

By the Theorem on Formal Functions $M = \lim_{\rightarrow} H^0(\ell_n, L_{\ell_n})$ where $\ell_n$ is the $n$-th infinitesimal neighborhood of $\ell$, and using [BGK, Lemma 2.5] to determine $M$ it suffices to compute $\sigma \in H^0(\ell_{\infty}, L_{\ell_{\infty}})$ for a large enough $N$ and direct computation of Cech cohomology gives the expression of such a global section as: $\sigma = \sum_{i=0}^N \frac{1}{i!} \alpha_i z^i u^i$. It then turns out that all terms of $\sigma$ are generated over

$$\hat{\mathcal{O}}_0 = \mathbb{C}[[x, y, z]]/(y^2 - xw)$$

by $\beta_0 = u^2 = x^2$ and $\beta_1 = zu^2 = xy$ and we obtain the structure of $M$ as an $\hat{\mathcal{O}}_0$-module:

$$M = \hat{\mathcal{O}}_0[\beta_0, \beta_1]/R$$

where $R$ is the set of relations

$$R = \left\{ \begin{array}{l}
R_1^1: \beta_0 y - \beta_1 x \\
R_2^1: \beta_0 w - \beta_1 y
\end{array} \right.$$

But then these relations themselves satisfy a second set of relations

$$R^2 = \left\{ \begin{array}{l}
R_1^2: R_1^1 y - R_1^1 x \\
R_2^2: R_1^1 w - R_1^1 y
\end{array} \right.$$

and so on

$$R^n = \left\{ \begin{array}{l}
R_1^n: R_1^{n-1} y - R_1^{n-1} x \\
R_2^n: R_1^{n-1} w - R_1^{n-1} y
\end{array} \right..$$

The corresponding resolution of $M$ therefore is

$$\cdots \to \hat{\mathcal{O}}_0 \otimes \hat{\mathcal{O}}_0 \otimes \hat{\mathcal{O}}_0 \otimes \hat{\mathcal{O}}_0 \otimes M \to 0.$$

**APPENDIX B. ORLOV CATEGORY OF $\Pi x_i$**

In the same style as [Or1, Sec. 3.3] we now calculate $D_{dg}(X_0)$ for the affine variety $X_0 = \text{Spec} \, \mathbb{C}[z_1, \ldots, z_n]/\Pi z_i$. Denote by $A$ the algebra $\mathbb{C}[z_1, \ldots, z_n]/\Pi z_i$. By [Or1, Prop. 1.23], any object of $D_{dg}(X_0)$ comes from a finite dimensional module $M$ over the algebra $A$.

We claim that the modules $M_i = <z_i>$ generated over $A$ by $z_i$ are indecomposable, and generate all modules over $A$. Furthermore, in the correspondence between sheaves on $X_0$ and modules over $A$, that is, $\mathcal{O}_X$-modules $= \text{module over } H^0(X, \mathcal{O}_X)$, they correspond to sheaves that are not perfect, since each of these modules possesses only infinite resolutions. This can be verified in Macaulay2 by a simple code, which for $n = 4$ is:

```
M2 code:
```

Macaulay2, version 1.17.2.1:
```
i1:k=ZZ/101;  
i2:R= k[z1,z2,z3,z4];  
i3:I=ideal(z1*z2*z3*z4);  
i4:N=R/I;  
i5:J= ideal(z1);  
i6:res J1  
o6:  
V^1 \overset{(z1)}{\longrightarrow} V^1 \overset{(z1)}{\longrightarrow} V^1 \overset{(z1)}{\longrightarrow} V^1 \overset{(z1)}{\longrightarrow} V^1 \overset{(z1)}{\longrightarrow} V^1 
```

The general resolution of $M_i$ takes the form:

$$M_i \longrightarrow M_i \longrightarrow M_i \longrightarrow M_i \longrightarrow M_i \longrightarrow \cdots .$$

Because any resolution of $M_i$ is infinite the corresponding sheaf will not be perfect. The following calculation of morphisms was written by Thomas Köppe.
Let $R = \mathbb{k}[z_1, z_2, z_3, z_4]$ be a polynomial ring, and $I = (z_1 z_2 z_3 z_4) \subset R$ a homogeneous ideal, and consider the quotient ring $S := R/I$. We consider modules over $S$. In particular, for the $S$-ideals

$$J_1 := (z_1) \subset S \quad \text{and} \quad J_2 := (z_2) \subset S$$

we consider $J_1$ and $J_2$ as $S$-modules in the natural way.

To compute $\text{Ext}^n(A, B)$ in the category of $S$-modules, we can replace $A$ with a projective resolution

$$0 \rightarrow A \xrightarrow{\epsilon} P_0 \xrightarrow{d_0} P_1 \xrightarrow{d_1} \ldots$$

where the augmentation map $\epsilon$ is the projection onto the cokernel of $d_0$, and then obtain $\text{Ext}^n(A, B)$ as the $n^{th}$ cohomology of the cochain complex $\text{Hom}(P_\bullet, B)$.

To compute $\text{Ext}^n_2(J_1, J_2)$, we first need a resolution of the module $J_1$, which is given by the unbounded, periodic resolution

$$0 \rightarrow J_1 \xrightarrow{\epsilon} P_0 \cong S^1 \xrightarrow{d_0} P_1 \cong S^1 \xrightarrow{d_1} P_2 \cong S^1 \xrightarrow{d_2} \ldots$$

In summary, the resolution $P_\bullet = (P_i, d_i)$ has $P_i \cong S^1$ for all $i = 0, 1, \ldots$, and

$$d_i : S^1 \rightarrow S^1, \quad d_i = \begin{cases} f := (z_2 z_3 z_4) & \text{for even } i, \\ g := (z_1) & \text{for odd } i, \end{cases}$$

and $J_1 = \text{coker}(f)$. (The module $J_1$ is not free, since $z_1$ is a zero divisor.) Then $\text{Hom}_S(P_\bullet, J_2)$ is the cochain complex

$$0 \rightarrow \text{Hom}_S(P_0 = S^1, J_2) \xrightarrow{f_{-\infty}} \text{Hom}_S(P_1 = S^1, J_2) \xrightarrow{g_{-\infty}} \text{Hom}_S(P_2 = S^1, J_2) \xrightarrow{f_{-\infty}} \ldots,$$

and it represents $R\text{Hom}(J_1, J_2)$ in the derived category. But note that $\text{Hom}_S(S, M) \cong M$ for any $S$-module $M$, so we have maps

$$0 \rightarrow (z_2) S \xrightarrow{d_{0, z_1}} (z_2) S \xrightarrow{d_{1, z_1}} (z_2) S \xrightarrow{d_{2, z_1}} (z_2) S \xrightarrow{d_{3, z_1}} \ldots \quad (10)$$

First, observe that $\ker(d_{2k+1}) = (z_1 z_2) S$ for all $k \geq 0$ (multiples of $z_1 z_2$ are those elements of $J_2$ that vanish when multiplied by $z_2 z_3 z_4$). In particular, $\text{Ext}^0(J_1, J_2) = \text{Hom}(J_1, J_2) = \ker(d_{0, z_1}) = (z_1 z_2) S$. Next, $\text{im}(d_{2k+1}) = (z_1 z_2) S$ for all $k \geq 0$ (the result of multiplying an element in $J_2$ by $z_1$ is a multiple of $z_1 z_2$). Finally,

$$\text{Ext}^k(S_1, S_2) = \begin{cases} (z_1 z_2) S & \text{for } k = 0, \\ M & \text{for odd } k, \text{and} \\ 0 & \text{for even } k, k > 0. \end{cases}$$

It remains to describe $M = \ker(d_{2k+1}) / \text{im}(d_{2k+1})$. We have that the kernel of $d_{2k+1}$ is $(z_2 z_3 z_4) S$, i.e. elements of $J_2$ that vanish when multiplied with $z_1$. The image of $d_{2k+1}$ consists of multiples of $z_2^2 z_3 z_4$. Thus the quotient consist of those elements of $(z_2 z_3 z_4) S$ that are linear in $z_2$. Abstractly, this can be presented as cokernel of the map $(z_2, z_1) : S^2 \rightarrow S^1$, i.e. as $S/(z_2 a + z_1 b)$: an element of $\text{Ext}^0(S_1, S_2)$ is $z_2$ times a function of $(z_2, z_3)$ only.

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