Ahmed E. Abouelregal

A comparative study of a thermoelastic problem for an infinite rigid cylinder with thermal properties using a new heat conduction model including fractional operators without non-singular kernels

Abstract In this research, two alternative approaches to fractional derivatives, namely Caputo–Fabrizio (CF) and Atangana–Baleanu (AB) fractional operators, are used to propose a generalized model of thermoelastic heat transfer of a rigid cylinder with thermal characteristics. The proposed model can be constructed by combining the DPL model with phase delays and the two temperature theories. It will be taken into account that the solid cylinder has variable physical properties. It was also assumed that the surface of the cylinder is penetrated by a continuous magnetic field and is regularly exposed to thermal loading from a continuous heat source. The numerical solutions of the studied physical fields in the AB and CF fractional derivative cases were derived using the Laplace transform method and are compared visually and tabularly and discussed in detail.

Keywords Fractional DPL model · Two-temperature · Exponentially heat source · Caputo-Fabrizio · Atangana-Baleanu

1 Introduction

Several decades ago, the field of fractional calculus, previously thought to be a part of pure mathematics, was shown to have enormous value and multiple applications in a variety of fields of study and applied research. It is used and applied to represent a wide range of physical and natural phenomena, such as neural network models, electrodes, biomaterials, mathematical biology and ramifications, business and finance, mechanical and electrical engineering, fluid mechanics, electronic systems, and plant genetics, as well as many other areas [1, 2].

Until recently, fractional calculus was just a mathematical tool with no obvious uses. Currently, dynamic fractional equations are a large part of how we model the effects of strange behavior and memory, which are common in nature. Fractional derivative models are used in the design of polymer models in the glass state, biology, wire and fiber coatings, fluid, chemically treated equipment, and a variety of internal combustion engines, COVID-19 and heat transfer, among other applications. A variety of methods are being used by scholars to extend classical models to fractional ones [3–9].

The Riemann–Liouville fractional integral, which is a simple and effective modification of the Cauchy model of repeated integrals in classical calculus, is the most frequently utilized technique for constructing a fractional-order integral [10]. One of the main reasons for this interesting subject’s rapid rise and popularity is its ability to describe dynamic systems with historical influences (memory) and anomalous behavior, which is often seen in most known natural and physical systems. However, the traditional integer-order computation...
lacks such possibilities because of the finite degrees of freedom of the integer-order parameter. Different approaches for fractional-order derivatives and integrals have been presented in the latter half of the current decade. The non-singularity displayed at one end of the period of the Riemann–Liouville integration is believed to be a catalyst for the development of these new technologies. The proposed new methods attempt to avoid such singularities [11–13].

The problems of the locality singular kernel and the situation of the non-singular kernel with non-locality have caused fractional calculus to become a hot topic of study in recent years. In this context, Caputo and Fabrizio (CF) [14] made the first effort to develop the concept of fractional calculus by introducing a non-singular integral (kernel) based on a decreasing smooth exponential function. In fact, they did not confirm that the fractional derivative operator had a singular kernel based on the data collected; instead, they stated that the fractional derivative factor is appropriate for a variety of physical issues. To improve this method, Atangana–Baleanu (AB) proposed a technique in [15, 16], replacing a smooth exponential function using the extended Mittag–Leffler function with a single parameter. They stated that, as the anti-derivative of their operators, their fractional derivative had a fractional integral. The kernel of a Atangana–Baleanu fractional-order derivative is non-local and non-singular. Atangana and Koca [17] apply the notion of the AB fractional derivative to a simple nonlinear system, demonstrating the uniqueness and existence of the fractional-order system solution. Algahtani [18] introduced the Allen–Cahn fractional model with both CF and AB fractional derivatives to evaluate the variations in real-world issues. Recent work on fractional Atangana–Baleanu and Caputo–Fabrizio derivatives, as well as their engineering and industrial applications, can be found in [19–26].

The conventional Fourier law of heat conduction is one of the basic system equations for classical thermoelasticity that describes the thermal field. It also becomes of the hyperbolic-parabolic type when reduced to the displacement temperature field equations. This means that the response of a conventional thermally elastic body to mechanical convection spreads at an unlimited rate as this result contradicts physical phenomena [27]. A variety of models have been developed since 1967, and they are collectively known as generalized thermoelastic concepts. The following are the most significant theories:

1- The Lord and Shulman theory (LS) [28]. In this model, the conventional Fourier law of heat transfer is exchanged by the Maxwell–Cattaneo law to include one relaxation time compared to the system equations in the classical thermoelastic theory.

2- Green and Lindsay model (GL) [29]. The dissipation inequality and the constitutive relationships are expanded to include two relaxation periods in this model in contrast to the conventional system of thermoelastic equations.

3- The thermoelastic model without energy dissipation presented by Green and Naghdi [30, 31], also known as the second (II) type GN theory (GN-II), in which a heat flow rate–temperature gradient relationship substitutes the conventional Fourier law, is the third modification. This model permits undamped thermoelastic waves in thermoelastic material because the thermal transfer equation does not include a temperature-rate factor.

4- In the fourth extension of the theory of thermoelasticity given by Green and Naghdi [32], heat transfer law is also included, which includes the classical law and uses a gradient of thermal displacement between constitutive variables. The name given to this model is the GN-III model, and it generally includes energy dissipation and allows for damped elastic heat waves.

5- The dual-phase delay thermoelastic theory established by Tzou [33–35] is the fifth extension of the theory of thermoelasticity. An approximation changes the Fourier law to a version of the Fourier law with two alternative time translations of the heat flow and the gradient of temperature compared to conventional thermoelastic theory.

6- Roychoudhuri [36] developed an extended mathematical thermoelastic model of the conventional thermoelastic theory with three-phase delays in the heat flow vector and gradient of temperature as well as the gradient of thermal displacement.

7- Abouelregal [37–41] introduced some generalized thermoelastic models with higher-order time derivatives using the thermoelastic models provided by Green and Naghdi [31, 32].

Chen and Gurtin [42] and Chen et al. [43, 44] suggested the model of thermal conductivity in deformable objects that includes two temperatures: the conduction and thermodynamic temperatures, since the distinction between the two temperatures is proportional to the heat source in time-independent conditions; the two temperatures are identical in the absence of any heat supply [43]. According to Gurtin and William [45, 46], there are no generalized reasons to believe that the second law of thermodynamics for continuous objects only includes one temperature and that it is more reasonable to assume a second law in which the entropy contribution due to thermal conductivity is controlled by one temperature and is due to the flow of heat.
with other temperatures. When it comes to the notion of two temperatures (2TT), Quintanilla [47] explored resolution potential, structural stability, and spatial behaviors. The linearized form of the 2TT theory has been examined by several researchers [48–51].

In the current analysis, a new generalized fractional thermoelastic heat conduction model of thermoelasticity with two temperature (2TT) and two phase lags has been constructed. The proposed model includes Riemann–Liouville, Caputo–Fabrizio, and Atangana–Baleanu fractional derivative operators. In contrast to various physical problems, it will be assumed that the cylinder’s physical properties change with the temperature change. We think this fractional model is important for the theoretical analysis and real-world testing that are needed to study thermoelastic problems.

Using the developed model, the thermomagnetic responses of a circular solid cylinder due to heat flux to its boundary and the space surrounding the cylinder immersed in a constant axial magnetic field are illustrated. The proposed method of the Laplace transform, as well as the inversion algorithm for the Laplace calculus approximation, has solved the issue. The numerical results of the thermodynamical temperature, stress, displacement, magnetic, and induced electric fields were represented graphically. In the case of the non-simple medium, the effects of different domains on the simple medium were compared and contrasted.

The structure of the present paper is as follows: In Sect. 2, the fractional derivation of the thermoelastic model includes the three types of integrals under study. The third section and its subsections give some special cases of the proposed model. The fourth and fifth sections are devoted to the Laplace transform technique and the Laplace numerical inverse to solve the system of equations. The sixth section and its subsections are dedicated to summarizing the results obtained and showing the analysis of numerical data. Finally, in the seventh part, the study concluded by discussing some of the most important findings and main conclusions.

1.1 Abreviations and symbols

$S_{ij}$ stress tensor, $e_{ij}$ strain tensor, $\gamma = (3\lambda + 2\mu)\alpha_1$ coupling parameter, $\alpha_1$ thermal expansion, $\lambda, \mu$ Lamé’s constants, $\delta_{ij}$ Kronecker's delta function, $e_{kk}$ cubical dilatation, $\theta = T - T_0$ temperature increment, $T$ absolute temperature, $T_0$ environmental temperature, $\varphi$ conductive temperature, $U_i$ displacement components, $R_i$ the body force, $\rho$ material density, $\tau_q$ phase lag of heat flow, $\tau_0$ phase lag of the temperature gradient, $K$ thermal conductivity, $C_s$ specific heat, $Q$ heat source, $\alpha$ fractional order, $\vec{F}$ heat flux vector, $\Gamma(\alpha)$ Gamma function. Also, $\vec{h}$ induced magnetic, $\vec{E}$ electric fields, $\vec{J}$ current density, $\vec{B}$ magnetic field, and $\mu_0$ magnetic permeability.

2 Modeling of the fractional thermoelastic problem

In this section, a novel mathematical model of fractional thermoelastic theory with two temperatures, including three different types of fractional integrals, will be presented. In the current study, three different fractional operators of Riemann–Liouville (RL), Caputo–Fabrizio (CF), and Atangana–Baleanu (AB) are considered. In the basic equations, a homogeneous solid will be considered.

In the context of thermoelasticity with two temperatures, the fundamental governing system equations are as follows:

The relationships between stress, displacement, and temperature are provided by:

$$S_{ij} = 2\mu e_{ij} + \left[ \lambda e_{kk} - \gamma \theta \right] \delta_{ij},$$

(1)

The following are the strain tensor components:

$$2e_{ij} = U_{i,j} + U_{j,i},$$

(2)

In the presence of an external force $\vec{a}$, the equation of motion may be stated as:

$$S_{ij,j} + R_i = \rho \frac{\partial^2 U_i}{\partial t^2},$$

(3)

The coupled energy equation is given by

$$\rho C_s \frac{\partial \theta}{\partial t} + T_0 \gamma \frac{\partial e_{kk}}{\partial t} = -\nabla \cdot \vec{F} + Q.$$
Using Eqs. (1) and (3), equation of motion (3) can be rewritten as

\[(\lambda + \mu) U_{j,j} + \mu U_{i,j} - \gamma \theta_1 + R_i = \rho \frac{\partial^2 U_i}{\partial t^2}.\]  

(5)

Tzou [33–35] proposed a modified Fourier law with phase delays, which is called the DPL model, as follows

\[\overrightarrow{F} + \tau_q \frac{\partial \overrightarrow{F}}{\partial t} = -K \left(1 + \tau_0 \frac{\partial}{\partial t}\right) \nabla \theta.\]  

(6)

In recent decades, fractional calculus has been applied to various physical issues. Many systems display memory, date, or non-local effects depending on the main explanations for using fractional derivative models, which are usually difficult to interpret using integer derivatives. Many researchers have devised different definitions for a non-integer-order integral or derivative throughout the decades, each utilizing their formulas and techniques. The fractional-order Riemann–Liouville derivative or the fractional Caputo derivative is used in most of the previous literature on the subject.

For a real locally integrative function \(Y(t)\), the Riemann–Liouville (RL) fractional integral of the order \(\alpha \in \mathbb{R}^+\) is described by [52]:

\[\text{RLD}^{(\alpha)}_t Y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \xi)^{\alpha - 1} Y(t) d\xi.\]  

(7)

For any arbitrary numbers \(\alpha_1, \alpha_2 > 0\), this implies that \(D^{(\alpha_1)}_t Y(t) \cdot D^{(\alpha_2)}_t Y(t) = D^{(\alpha_1 + \alpha_2)}_t Y(t)\) (convolution property). The Laplace transformation of \(D^{(\alpha)}_t Y(t)\) in integer-order procedures is provided by

\[\mathcal{L}\left[\text{RLD}^{(\alpha)}_t Y(t)\right] = s^\alpha \mathcal{L}[Y(t)] - \sum_{k=0}^{\alpha-1} s^{\alpha-k-1} Y^{(k)}(0).\]  

(8)

It was recently pointed out that the kernels of these derivatives include a singularity at the conclusion of a specific time, as defined by their definitions. As a result, various new fractional derivative notions and definitions have been proposed in order to tackle this difficulty. The kernels that may be employed to fulfill the demands of different applications are the primary variations between the definitions of different fractional derivatives. Some researchers have suggested amendments to the RL and Caputo derivatives that are focused on the substitution of their weakly singular kernels by some non-singular function that is continuous over the closed interval \([0, \xi]\), \(\xi > 0\) in order to circumvent the challenges created by singularities.

The derivative in the sense of Caputo of fractional order \(\alpha \in (0, 1)\) is given by [52]:

\[\text{CD}^{(\alpha)}_t Y(t) = \int_0^t \frac{\dot{Y}(\tau)}{\Gamma(1 - \alpha)} (t - \xi)^{\alpha - 1} d\xi.\]  

(9)

Caputo and Fabrizio have recently defined the fractional derivative with an exponential kernel (CF fractional derivative) for \(\alpha \in (0, 1)\) for smooth function \(Y(t)\) in the following form [14]:

\[\text{CFD}^{(\alpha)}_t Y(t) = \frac{1}{1 - \alpha} \int_0^t \dot{Y}(\xi) e^{-\frac{\alpha}{\alpha - 1}(t - \xi)} d\xi, \quad t \geq 0, \quad 0 < \alpha < 1.\]  

(10)

The Laplace transform of the Caputo and Fabrizio fractional (CF) derivative is provided by

\[\mathcal{L}\left[\text{CFD}^{(\alpha)}_t Y(t)\right] = \frac{1}{s + \alpha(1 - s)} [s \mathcal{L}[Y(t)] - Y(0)].\]  

(11)

In reality, the fractional derivative operator of CF fails to capture a non-singular kernel. They showed that their derivative operator might solve various physical difficulties [53]. On the other hand, the CF fractional derivative had some complications since the kernel in the integral was non-singular but still non-local. In addition, the related integral in the CF fractional derivative is not a fractional operator. Atangana and Baleanu [15, 16] have developed two fractional derivatives in the Caputo and Riemann–Liouville senses based on the extended Mittag–Leffler function to address the non-singularity and non-localization of the kernel.
The Atangana–Baleanu fractional integral of order \( \alpha \in (0, 1) \) for any function \( Y \in H^1(0, b) \) with \( b > 0 \) can be defined in the following form [15, 16]

\[
ABD_t^{(\alpha)}Y(t) = \frac{1}{1-\alpha} \int_0^t \dot{Y}(\xi) E_\alpha \left( -\frac{\alpha}{(1-\alpha)(t-\xi)^\alpha} \right) d\xi.
\] (12)

The function \( E_\alpha \) is the Mittag–Leffler function. For \( 0 < \alpha \leq 1 \), by applying the Laplace transform technique to Eq. (12), we have

\[
\mathcal{L}\left[ABD_t^{(\alpha)}f(t)\right] = \frac{1}{1-\alpha} \frac{s^\alpha \mathcal{L}[f(t)] - s^{\alpha-1} f(0)}{s^\alpha + \frac{\alpha}{1-\alpha}}, \quad s > 0.
\] (13)

The new fractional formulation, like the usual Caputo derivative, makes it clear that if the function \( Y(t) \) is a constant, then \( D_t^{(\alpha)}Y(t) = 0 \). The difference between the present and previous definitions of the fractional derivative is that the current kernel, unlike the one mentioned above, does not have a singularity when \( t = \xi \).

It can be observed that when \( \alpha = 1 \), we get to the traditional first-order derivative, which is

\[
\lim_{\alpha \to 1} \left[ D_t^{(\alpha)}Y(t) \right] = \dot{Y}(t).
\] (14)

To create a modified generalized fractional thermoelastic model, we substitute partial derivatives \( \left( \frac{\partial}{\partial t} \right)^\alpha \) with respect to time \( t \) in modified Fourier law (6) by a fractional operator \( \left( D_t^{(\alpha)} \right) \) of order \( \alpha \), \( (0 < \alpha \leq 1) \), resulting in:

\[
\dot{F} + \tau_q D_t^{(\alpha)} \dot{F} = -K \left( \nabla \dot{\theta} + \tau_0 D_t^{(\alpha)} \nabla \theta \right).
\] (15)

The symbol \( D_t^{(\alpha)} \) denotes one of the three fractional operators (RL, CF, and AB). Combining Eqs. (5) and (11) yields a linear form of the generalized heat conduction theory with a fractional operator as:

\[
\left( 1 + \tau_q D_t^{(\alpha)} \right) \left[ \rho C_E \frac{\partial \theta}{\partial t} + T_0 \gamma \frac{\partial e_{kk}}{\partial t} - Q \right] = \left( 1 + \tau_0 D_t^{(\alpha)} \right) \left( \nabla \cdot (K \nabla \theta) \right).
\] (16)

Chen and Gurtin [43, 44] developed the theory of heat transfer on a non-simple deformable body that is based on two different temperatures: the conductive temperature \( \varphi \) and the dynamic temperature \( \theta \). They suggested that there is a relationship between the two temperatures given by the following formula:

\[
\varphi - b \nabla^2 \varphi = \theta.
\] (17)

where \( b > 0 \) indicates a temperature difference (temperature discrepancy). Consequently, if \( b = 0 \) is present, the two temperatures will coincide. Based on Chen and Gurtin’s study [43, 44], a different approach to the Fourier law will be proposed as follows

\[
\left( 1 + \tau_q \frac{\partial}{\partial t} \right) \dot{F} = -K \left( 1 + \tau_0 \frac{\partial}{\partial t} \right) \nabla \varphi.
\] (18)

Based on the fractional derivative, Fourier law (18) can be reformulated based on the concept of conductive temperature \( \varphi \) as:

\[
\left( 1 + \tau_q D_t^{(\alpha)} \right) \dot{F} = -K \left( 1 + \tau_0 D_t^{(\alpha)} \right) \nabla \varphi.
\] (19)

By eliminating \( \dot{F} \) from Eqs. (4) and (19), we can get a modified fractional two-temperature heat equation with phase delays and fractional operators as

\[
\left( 1 + \tau_q D_t^{(\alpha)} \right) \left[ \rho C_E \frac{\partial \theta}{\partial t} + T_0 \gamma \frac{\partial e_{kk}}{\partial t} - Q \right] = \left( 1 + \tau_0 D_t^{(\alpha)} \right) \left( \nabla \cdot (K \nabla \varphi) \right).
\] (20)

The Lorentz force law and Maxwell’s equations constitute the cornerstone of conventional electromagnetism, traditional optics, and electric circuits. The equations serve as mathematical formulas for electrical, optical, and radio systems such as power production, electric motors, communication systems, prescription
glasses, sensors, etc. They explain how the domains’ charges, currents, and variations cause electric and magnetic fields to form. Maxwell proposed that light is an electromagnetic phenomenon using these equations. Maxwell’s equations describe the universe of electromagnetics, a set of four complicated equations. These equations explain the propagation and interaction of electric and magnetic fields and how they are affected by bodies. The electromagnetic governing system equations of Maxwell may be represented as [54, 55]

\[
\begin{align*}
\nabla \times \vec{h} &= \mu_0 \frac{\partial}{\partial t} \left( \frac{\vec{E}}{\mu_0} \right), \\
\nabla \times \vec{E} &= J - \mu_0 \left( \nabla \times \vec{B} \times \nabla \right), \\
\nabla \cdot \vec{h} &= 0.
\end{align*}
\]

(21)

In addition to the previous equations, the Maxwell stress \( M_{ij} \) can be determined by [54, 55]

\[
M_{ij} = \mu_0 \left[ B_i h_j + B_j h_i - B_k h_k \delta_{ij} \right].
\]

(22)

The Lorentz force \( \vec{L} \) can also be calculated by

\[
\vec{L} = \mu_0 \left( \vec{J} \times \vec{B} \right).
\]

(23)

3 Special cases

Many different thermoelasticity theories can be classified as special cases of the two-temperature thermal conductivity equation with modified fractional derivatives (20) as follows:

3.1 Thermoelasticity theories

By ignoring the fractional differential \( \alpha = 1 \) as well as the temperature difference \( b = 0 \) in the previous equations, many of the following thermoelastic models can be obtained based on the phase lag \( \tau \Theta \text{ and } \tau \eta \) values.

- Thermoelastic coupled model (CTE): \( \tau \Theta = \tau \eta \).
- Thermoelasticity model with one-phase lag (LS): \( \tau \Theta = 0 = b, \tau \eta > 0 \).
- The thermoelastic dual-phase-lag theory (DPL): \( b, \tau \eta, \tau \Theta > 0 \).

3.2 Fractional thermoelastic models

When the parameter responsible for the temperature discrepancy is absent \( b = 0 \) and fractional differential operators \( D_t^{(\alpha)} \) are used \( 0 < \alpha \leq 1 \), the following fractional two models of thermoelasticity can be drawn:

- The fractional thermoelastic model with one-phase lag (FLS): \( 0 < \alpha \leq 1, \tau \Theta = 0, \tau \eta > 0, b > 0 \).
- The thermoelastic fractional dual-phase-lag theory (FDPL): \( 0 < \alpha \leq 1, \tau \eta, \tau \Theta, b > 0 \).

3.3 Fractional thermoelastic models with two temperatures

When applying a different fractional differential operator \( D_t^{(\alpha)} \) and in the presence of the parameter responsible for the temperature difference \( b > 0 \), the following models can be derived.

- The two-temperature fractional thermoelastic model with one-phase lag (2TFLS): \( 0 < \alpha \leq 1, \tau \Theta = 0, \tau \eta, b > 0 \).
- The thermoelastic fractional dual-phase-lag model with two temperature (2TFDPL): \( \tau \eta, \tau \Theta, b > 0, 0 < \alpha \leq 1 \).
4 Applicable problem formulation

As can be seen from Fig. 1, a problem of a long isotropic solid cylinder of radius \( a \) will be considered as an application to the suggested fractional thermoelastic model. In the absence of an external electric field, it was assumed that there was a uniform axial magnetic field \( \vec{B} = (0, 0, B_0) \) penetrating the elastic medium. The boundary of the solid cylinder is constrained due to an axial line heat supply. Due to the nature of the problem, the cylindrical \((r, \xi, z)\) coordinate system will be used so that the origin of the coordinates is at the center of the cylinder and the \(z\)-axis is along the cylinder axis. It will be assumed that the thermoelastic responses are symmetric about the axis. As a result of this symmetry, the functions of the studied case will depend only on the radial coordinate \( r \) and time \( t \). As a condition of regularity, it will be taken into account that all state variables are limited.

As a result, for a one-dimensional problem, the governing equations can be written as

\[
U_r = U(r, t), \quad U_\xi(r, t) = U_z(r, t) = 0,
\]

\[
e = \frac{\partial U}{\partial r} + \frac{U}{r} = \frac{1}{r} \frac{\partial (rU)}{\partial r},
\]

\[
S_{rr} = 2\mu \frac{\partial U}{\partial r} + \lambda e - \gamma \theta,
\]

\[
S_{\xi\xi} = 2\mu \frac{U}{r} + \lambda e - \gamma \theta,
\]

\[
S_{zz} = \lambda e - \gamma \theta,
\]

\[
\frac{\partial S_{rr}}{\partial r} + \frac{1}{r} (S_{rr} - S_{\xi\xi}) + L_r = \rho \frac{\partial^2 U}{\partial t^2}.
\]

Applying the constant magnetic field vector \( \vec{B} = (0, 0, B_0) \) results in an induced magnetic field \( \vec{h} \) that will have one component in the \( z \)-direction and an induced electric field \( \vec{E} \) and electric current density \( \vec{J} \) that will have one component in the \( \xi \)-direction as

\[
\vec{h} = -B_0 \left( 0, 0, \frac{1}{r} \frac{\partial (rU)}{\partial r} \right), \quad \vec{J} = B_0 \left( 0, \frac{\partial e}{\partial r}, 0 \right), \quad \vec{E} = \mu_0 B_0^2 \left( 0, \frac{\partial U}{\partial t}, 0 \right).
\]

By substituting Eq. (30) into Eq. (13), the radial Lorentz force \( L_r \) and the stress Maxwell \( M_{rr} \) components will be in the form

\[
L_r = \mu_0 \left( \vec{J} \times \vec{B} \right)_r = \mu_0 B_0^2 \frac{\partial e}{\partial r}, \quad M_{rr} = \mu_0 B_0^2 e.
\]
Temperature change is among the many factors affecting elastic materials’ thermophysical properties and is the most crucial influence. The deformation of materials and all thermal and mechanical processes are affected by surface temperature. The properties of thermoplastic materials, such as thermal conductivity, diffusion, and specific heat capacity, are essential in determining the temperature of these objects.

In the present work, specific heat $C_e$ and thermal conductivity $K$ are considered to be directly proportional to temperature change $\theta$. Ignoring the density variance $\rho$ due to thermal expansion, we will assume that [56]:

$$
K(\theta) = k_0 (1 + K_1 \theta),
$$

$$
C_e(\theta) = C_{e0} (1 + K_1 \theta),
$$

where $k_0$ and $C_{e0}$ are, respectively, thermal conductivity and specific heat at $T_0$, and $K_1$ is the rate of increase (deviation) in thermal conductivity and specific heat. Thermal conductivity $K(\theta)$ and specific heat $C_e(\theta)$ must be highly positive in the sense that $K(\theta), C_e(\theta) > 0$ from a physical standpoint.

The thermal conductivity and specific heat of materials as a function of the conductive temperature can be set using Eqs. (17) and (32) as follows:

$$
K = k_0 (1 + K_1 (\varphi - b \nabla^2 \varphi)) = k_0 (1 + K_1 \varphi) - k_0 K_1 b \nabla^2 \varphi \approx k_0 (1 + K_1 \varphi),
$$

$$
C_e = C_{e0} (1 + K_1 (\varphi - b \nabla^2 \varphi)) = C_{e0} (1 + K_1 \varphi) - k_0 K_1 b \nabla^2 \varphi \approx C_{e0} (1 + K_1 \varphi).
$$

(33)

We will consider the following Kirchhoff transform [57]

$$
\{\Theta, \Phi\} = \left\{ \frac{1}{k_0}, \frac{1}{C_{e0}} \right\} \int_0^\theta \int_0^{\varphi} (1 + K_1 \zeta) d\zeta.
$$

(34)

We obtain when we apply the operator $\nabla$ to both sides of Eq. (30):

$$
C_{e0} \nabla \Phi = K \nabla \varphi, k_0 \nabla \Theta = K \nabla \theta.
$$

(35)

Using the div operator on Eq. (35) once more, we get

$$
C_{e0} \nabla^2 \Phi = \nabla \cdot (K \nabla \varphi), k_0 \nabla^2 \Theta = \nabla \cdot (K \nabla \theta).
$$

(36)

When differentiating Eq. (34) with respect to time $t$, we can get

$$
K_0 \frac{\partial \Theta}{\partial t} = K \frac{\partial \theta}{\partial t}.
$$

(37)

The modified linear heat equation with fractional order for non-simple elastic material can be obtained by substituting Eqs. (35) and (36) into Eq. (20) as follows:

$$
\nabla^2 \Phi + \tau_0 D_t^{(\alpha)} (\nabla^2 \Phi) = \frac{1}{k} \left(1 + \tau_0 D_t^{(\alpha)}\right) \frac{\partial \Theta}{\partial t} + \frac{\gamma T_0}{K} \left(1 + \tau_0 D_t^{(\alpha)}\right) \frac{\partial \theta}{\partial t},
$$

where $\frac{K(\varphi)}{\rho C_e(\varphi)} = \frac{K_0}{\rho C_{e0}} = k$ is the diffusivity of the material. For linearity and substitution from Eqs. (34) and (36) into Eq. (27), we can approximate Eq. (27) as

$$
\Phi - b \nabla^2 \Phi = \Theta.
$$

(39)

Equation of motion (29), after using Eqs. (34) and (36), can be written as

$$
\left(\frac{\lambda + 2\mu}{\rho} + \frac{\mu_0 H_0^2}{\rho}\right) \frac{\partial e}{\partial r} = \frac{\gamma}{\rho} \frac{\partial \Theta}{\partial r} + \frac{\partial^2 U}{\partial r^2}.
$$

(40)

Introducing Eq. (33) into Eq. (40), we get

$$
\rho (c_0^2 + a_0^2) \frac{\partial e}{\partial r} = \frac{\gamma}{(1 + K_1 \theta)} \frac{\partial \Theta}{\partial r} + \rho \frac{\partial^2 U}{\partial r^2}.
$$

(41)

where $c_0 = \sqrt{\frac{\lambda + 2\mu}{\rho}}$ and $a_0 = \sqrt{\frac{\mu_0 H_0^2}{\rho}}$. 
Finally, by ignoring the nonlinear terms in the previous equation and assuming \(|\theta/T_0|=1\), we can get
\[
\rho \left(c_0^2 + a_0^2\right) \frac{\partial e}{\partial r} = \gamma \frac{\partial \theta}{\partial t} + \rho \frac{\partial^2 U}{\partial t^2}.
\] (42)

To transform the governing equations to non-dimensional forms, we now employ the following non-dimensional field variables:
\[
\{r', U'\} = \frac{c_0}{k} \{r, U\}, \quad t' = \frac{c_0^2}{k} t, \quad \{\theta', \varphi', \Theta', \Phi'\} = \frac{\gamma}{\rho c_0^2} \{\theta, \varphi, \Theta, \Phi\},
\]
\[
K' = T_0 K, \quad b' = \frac{c_0^2}{k^2} b, \quad \{S'_{ij}, M'_{ij}\} = \frac{1}{\rho c_0^2} \{S_{ij}, M_{rr}\}.
\] (43)

After removing the primes, the resulting non-dimensional basic equations are as follows:
\[
\left(1 + \tau_0 D_t^{(\alpha)}\right) \left(\nabla^2 \Phi\right) = \left(1 + \tau_q D_t^{(\alpha)}\right) \left(\frac{\partial \Phi}{\partial t} + \varepsilon \frac{\partial \Phi}{\partial t}\right),
\] (44)
\[
\frac{\partial S}{\partial r} - a_2 \frac{\partial^2 e}{\partial r^2} = a_1 \frac{\partial \Theta}{\partial r},
\] (45)
\[
\Phi - \Theta = b \nabla^2 \Phi,
\] (46)
\[
S_{rr} = \frac{\partial U}{\partial r} + \left(1 - 2\beta^2\right) \frac{U}{r} - \theta,
\] (47)
\[
S_{\varphi\varphi} = \frac{U}{r} + \left(1 - 2\beta^2\right) \frac{\partial U}{\partial r} - \theta,
\] (48)
\[
S_{zz} = \left(1 - 2\beta^2\right) e - \theta,
\] (49)
in which
\[
\beta^2 = \frac{\mu}{\lambda + 2\mu}, \quad \varepsilon = \frac{T_0 \gamma^2}{\rho^2 c_0^2 C_e}, \quad a_1 = \frac{g}{c_0^2 + a_0^2}, \quad a_2 = \frac{\rho c_0^2}{c_0^2 + a_0^2}, \quad g = \frac{\gamma T_0}{\rho}.
\] (50)

The displacement can be expressed as
\[
U(r, t) = \frac{\partial \Psi(r, t)}{\partial r},
\] (51)

When one plugs the proposed function \(\Psi\) into Eqs. (44) and (45) one gets:
\[
\left(1 + \tau_0 D_t^{(\alpha)}\right) \left(\nabla^2 \Phi\right) = \left(1 + \tau_q D_t^{(\alpha)}\right) \left(\frac{\partial \Phi}{\partial t} + \varepsilon \nabla^2 \Phi\right),
\] (52)
\[
\left(\nabla^2 - a_2 \frac{\partial^2 \Phi}{\partial r^2}\right) \Psi = a_1 \theta.
\] (53)

At time \(t = 0\), the initial conditions are assumed to be:
\[
U(r, t) = 0 = \frac{\partial U(r, t)}{\partial r}, \quad \theta(r, t) = 0 = \frac{\partial \theta(r, t)}{\partial r}, \quad \Phi(r, t) = 0 = \frac{\partial \Phi(r, t)}{\partial r},
\] (54)

We also considered that the variables \(S_{ij}, M_{rr}, U, \theta, \Theta, \varphi, \) and \(\Phi\) are limited when \(r \to 0\) for regularity criteria.

We will assume that the ambient plane \(r = a\) of the cylinder is exposed to a moving heat flux \(F\). When \(t > 0\), the modified Fourier law with fractional order (19) will be considered as follows:
\[
\left(1 + \tau_0 D_t^{(\alpha)}\right) \left(K(\varphi) \frac{\partial \varphi}{\partial r}\right) = - \left(F + \tau_q D_t^{(\alpha)} F\right) at \quad r = a.
\] (55)

Then, if we use Eq. (36), we get
\[
\left(1 + \tau_0 D_t^{(\alpha)}\right) \left(k_0 \frac{\partial \Phi}{\partial r}\right) = - \left(F + \tau_q D_t^{(\alpha)} F\right), \quad \text{when} \ t > 0, \ r = a.
\] (56)
It was assumed that the heat flux $F$ moves at a constant speed $\vartheta$ in the direction of the axis of the radial cylinder and decreases exponentially with increasing time as:

$$F = Q_0 e^{-\omega t} \delta(r - \vartheta t), \quad \omega, \nu > 0,$$

where the parameters $\omega$ and $Q_0$ are assumed to be constants, and $\delta(.)$ is the Dirac delta.

When we use dimensionless variables (43) and substitute them in (57), we get

$$\left(1 + \tau_\vartheta D_1^{(\alpha)}\right) \frac{\partial \Phi(r, t)}{\partial r} = -q_1 \left(1 + \tau_q D_1^{(\alpha)}\right) e^{-\alpha t} \delta(r - \vartheta t), \quad q_1 = \frac{Q_0 \rho k c_0}{\gamma}.$$  \hspace{1cm} (58)

In addition to the above, we will assume that the mechanical limits ensure that the surface displacement is constrained and mathematically described by the formula:

$$U(r, t) = 0 \text{ at } r = a.$$  \hspace{1cm} (59)

5 Problem solution

The Laplace transform of any function $g(r, t)$ can be calculated using the following formula:

$$\overline{g}(r, s) = \int_0^\infty g(r, t) e^{-st} dt, \quad s > 0.$$  \hspace{1cm} (60)

In the case of initial conditions (54), the following equations are obtained by applying Laplace to the basic equations:

$$\begin{align*}
(\nabla^2 - \alpha_1) \Phi &= \alpha_2 \nabla^2 \Psi, \\
(\nabla^2 - a_2 s^2) \Psi &= a_1 \overline{\theta},
\end{align*}$$  \hspace{1cm} (61)

$$\overline{\theta} = (1 - b \nabla^2) \Phi,$$  \hspace{1cm} (62)

$$\overline{S}_{rr} = 2 \beta^2 \frac{\partial U}{\partial r} + (1 - 2 \beta^2) \overline{\vartheta} - \overline{\vartheta},$$  \hspace{1cm} (63)

$$\overline{S}_{\xi\xi} = 2 \beta^2 \frac{U}{r} + (1 - 2 \beta^2) \overline{\vartheta} - \overline{\vartheta},$$  \hspace{1cm} (64)

$$\overline{S}_{zz} = (1 - 2 \beta^2) \overline{\vartheta} - \overline{\vartheta},$$  \hspace{1cm} (65)

where

$$\alpha_0 = \begin{cases} 
\frac{s(1+s\tau_\vartheta)}{(1+s\tau_\theta)} & \text{for RL fractional operator}, \\
\frac{s(1+s\tau_q)}{(1+s\tau_\theta)} & \text{for CF fractional operator}, \\
\frac{s(1+s\tau_q(1-s))}{s(1+s\tau_\theta(1-s))} & \text{for AB fractional operator},
\end{cases}$$  \hspace{1cm} (66)

with $\alpha_1 = \frac{\alpha_0}{1+\alpha_0 b}$ and $\alpha_2 = \frac{\alpha_0 b}{1+\alpha_0 b}$.

We can get this by removing the function $\overline{\theta}$ from Eqs. (61)–(63):

$$\begin{align*}
(\nabla^4 - \delta_1 \nabla^2 + \delta_2) \Phi &= 0, \\
(\nabla^4 - \delta_1 \nabla^2 + \delta_2) \Psi &= 0,
\end{align*}$$  \hspace{1cm} (67)

where

$$\begin{align*}
\delta_1 &= \frac{\alpha_1 + s^2 a_2 + \alpha_2 a_1}{1 + \alpha_2 a_1 b}, \\
\delta_2 &= \frac{\alpha_1 s^2 a_2}{1 + \alpha_2 a_1 b}. 
\end{align*}$$  \hspace{1cm} (68)
When the parameters $\mu_i$, $i = 1, 2$ are presented in Eq. (68), and the following results are obtained:

$$
\left( \nabla^2 - \mu_1^2 \right) \left( \nabla^2 - \mu_2^2 \right) \Phi = 0,
$$

$$
\left( \nabla^2 - \mu_1^2 \right) \left( \nabla^2 - \mu_2^2 \right) \Psi = 0,
$$

(70)

where the parameters $\mu_1^2$ and $\mu_2^2$ denote two roots of a given equation

$$
\mu^4 - \delta_1 \mu^2 + \delta_2 = 0.
$$

(71)

The roots of $\mu_1^2$ can be obtained by solving (71) as:

$$
\mu_1^2 = \frac{\delta_1 + \sqrt{\delta_1^2 - 4\delta_2}}{2}, \quad \mu_2^2 = \frac{\delta_1 - \sqrt{\delta_1^2 - 4\delta_2}}{2}.
$$

(72)

Equation (70) has a general solution that may be stated as:

$$
\Phi = \alpha_2 A_1 \mu_1^2 I_0(\mu_1 r) + \alpha_2 A_2 \mu_2^2 I_0(\mu_2 r),
$$

(73)

$$
\Psi = (\mu_1^2 - \alpha_1) A_1 I_0(\mu_1 r) + (\mu_2^2 - \alpha_1) A_2 I_0(\mu_2 r),
$$

(74)

where parameters $A_i$, ($i = 1, 2, 3$) are the integral coefficients and $I_0$ is the first class of zero-order modified Bessel functions. One may derive the following from Eqs. (51) and (74):

$$
U = A_1 \mu_1 (\mu_1^2 - \alpha_1) I_1(\mu_1 r) + \eta_2 (\mu_2^2 - \alpha_1) A_2 I_1(\mu_2 r).
$$

(75)

We may get the following result after differentiating Eq. (72):

$$
\frac{\partial U}{\partial r} = A_1 \mu_1^2 (\mu_1^2 - \alpha_1) \left[ I_0(\mu_1 r) - \frac{1}{r \eta_1} I_1(\mu_1 r) \right]
+ A_2 \mu_2^2 (\mu_2^2 - \alpha_1) \left[ I_0(\mu_2 r) - \frac{1}{r \eta_2} I_1(\mu_2 r) \right].
$$

(76)

When Eq. (73) is introduced into Eq. (63), we can get

$$
\bar{\varphi} = \alpha_2 \mu_1^2 (1 - b \mu_1^2) A_1 I_0(\mu_1 r) + \alpha_2 \mu_2^2 (1 - b \mu_2^2) A_2 I_0(\mu_2 r).
$$

(77)

We may derive the following result by substituting Eq. (33) into Eq. (34) and integrating:

$$
\Theta = \theta + \frac{1}{2} K_1 \phi^2, \quad \Phi = \phi + \frac{1}{2} K_1 \varphi^2.
$$

(78)

By solving the abovementioned equations, we can find the solution for $\bar{\varphi}$ and $\bar{\phi}$ as

$$
\bar{\varphi}(r, s) = \frac{-1 + \sqrt{1 + 2K_1 \bar{\varphi}}}{K_1},
$$

(79)

$$
\bar{\phi}(r, s) = \frac{1}{K_1} \left( -1 + \sqrt{1 + 2K_1 \bar{\phi}} \right).
$$

(80)

As a result, the thermal stresses may be calculated using the following formula:

$$
\bar{S}_{rr} = A_1 \mu_1^2 (\mu_1^2 - \alpha_1) \left[ I_0(\mu_1 r) - \frac{2\beta^2}{r \mu_1} I_1(\mu_1 r) \right] - \frac{1 + \sqrt{1 + 2K_1 \Theta}}{K_1},
$$

(81)

$$
\bar{S}_{\theta\theta} = A_1 \mu_1^2 (\mu_1^2 - \alpha_1) \left[ (1 - 2\beta^2) I_0(\mu_1 r) + \frac{2\beta^2}{r \mu_1} I_1(\mu_1 r) \right] - \frac{1 + \sqrt{1 + 2K_1 \Theta}}{K_1},
$$

(82)
The solution to Maxwell’s stress $M_{rr}$ is given by:

$$M_{rr} = \frac{a_0^2}{\alpha_2 c_0} \left( A_1 \mu_1^2 (\mu_1^2 - \alpha_1)(1 - 2\beta^2)I_0(\mu_1 r) + A_2 \mu_2^2 (\mu_2^2 - \alpha_1)(1 - 2\beta^2)I_0(\mu_2 r) \right).$$

Boundary conditions (58) and (59) in the converted domain are written in the following form:

$$\frac{\partial \overline{\Phi}(r, s)}{\partial r} = -\frac{\alpha_0 Q_0}{\nu} e^{-\Omega r}, \quad \Omega = \frac{\omega + s}{\nu}, \quad r = a,$$

$$U(r, s) = 0, \quad r = a.\quad (85)$$

From Eqs. (73) and (75) and boundary conditions (85) and (86), we obtain

$$\eta_1 A_1 I_1(\mu_1 a) + \eta_2 A_2 I_1(\mu_2 a) + \frac{\alpha_0 Q_0}{\nu} e^{-\Omega a} = 0,\quad (87)$$

$$\eta_2 (\eta_1^2 - \alpha_1) A_1 I_1(\mu_1 a) + \mu_1 (\mu_2^2 - \alpha_1) A_2 I_1(\mu_2 a) = 0.\quad (88)$$

Solving system Eqs. (87) and (88) yields the parameters $A_i$ ($i = 1, 2$).

### 6 Numerical Laplace Inversions

In the previous section, solutions for the domains of temperature change, thermal stress, and thermal displacement were produced using the Laplace transform. These solutions must then be transformed into the time domain ($r, t$) by inverting the Laplace transforms of these studied domains. The present study will obtain an inversion of the Laplace inversion for various fields using a well-established, practical, and accurate numerical approach. The numerical inversion approach based on the Fourier series extension [35] was used in this remarkable method. The following procedure may be used to convert any field $\overline{H}(r, s)$ in Laplace space domain to time and space domain:

$$H(r, t) = \frac{e^{\xi t}}{\tau_1} \left( \frac{H(\xi)}{2} + Re \sum_{k=1}^{N_0} e^{i k \pi t / \tau_1} \overline{H}(\xi + i k \pi / \tau_1) \right), \quad 0 \leq t \leq 2 \tau_1$$

The truncated unlimited Fourier series has $N_0$ terms. The parameter $N_0$ is the number of terms that must be selected in order to satisfy the given formula:

$$e^{\xi t} Re(e^{i N_0 \pi t / \tau_1} \overline{H}(\xi + i N_0 \pi / \tau_1)) \leq \epsilon$$

where $\epsilon$ is a minor disturbed positive integer equal to the required accuracy level. The constant is a free positive component that must equal or exceed the real portions of all $\overline{H}(r, s)$ singularities. The parameter is set to fulfill the requirements of [35]. Using the programming language Mathematica, numerical code was created.

### 7 Results and Discussion

An infinite thermal flexible solid cylinder immersed in a magnetic field and under the influence of a moving heat source will be studied. The numerical discussion will be carried out in this investigation using the new thermoelastic model of the heat transfer equation, which is based on fractional operators of fractional order. The copper material’s mechanical properties will be considered while presenting and calculating the numerical results. Here are the physical parameters of the copper:

$$\lambda, \mu = (7.76, 3.86) \times 10^{10} \text{ kg m}^{-1} \text{ s}^{-2}, \quad \rho = 8954 \text{ kg m}^{-3}, \quad H_0 = 10^7 \text{ Am}^{-1}, \quad k_0 = 386 \text{ W m}^{-1} \text{ K}^{-1}, \quad C_v = 383.1 \text{ J kg}^{-1} \text{ K}^{-1}, \quad \alpha_l = 0.5 \times 10^{-6} \text{ K}^{-1}, \quad \mu_0 = 126 \times 10^{-8} \text{ H m}^{-1}, \quad T_0 = 298 \text{ K}. $$
We calculated the numerical values of the physical field variables based on the physical data provided above. The calculations were carried out with the distance \( r(0 \leq r \leq 1) \) and using \( t = 0.12, Q_0 = 1, \) and \( \omega = 1. \) We employed the computational methods outlined in Eq. (89) to get the distributions of Maxwell’s stress \( M_{rr}, \) radial and hoop thermal stresses \( S_{rr} \) and \( S_{\theta \theta}, \) deformation \( U, \) dynamical temperature \( \theta, \) and conductive temperature \( \varphi \) under the impact of several influential factors.

### 7.1 Comparison and analysis of different fractional derivative operators

The purpose of this subsection is to show how modified fractional operators (Caputo–Fabrizio (CF) and Atangana–Baleanu (AB)), traditional fractional operators (Riemann–Liouville (RL)), and conventional derivatives (\( \alpha = 1 \)) may be compared. We compare these fractional operators to traditional derivatives to see which fractional derivative gets closer to the classical derivative first. The table is used to describe the behavior of each derivative of the non-integer order. Furthermore, the primary objective is to use graphic examples to illustrate and explain the main results of the work presented.

To do this, some appropriate values of the fractional order \( \alpha \) will be set, and the natural behavior of the different physical fields will be observed in the case of the proposed model. A comparison will be made between generalized AB and extended CF as well as their conventional counterparts. With the help of various fractional parameter values, the given graphs revealed the fundamental difference between the determinant non-local fraction factors and the traditional model.

Tables 2 through 7 show the findings at various positions \( r. \) The investigations have been carried out for \( \nu = 5, b = 0.01, K_1 = -0.5, \tau_q = 0.2, \) and \( \tau_0 = 1.0. \) The simulation results of a fractional differential heat transport model that focuses on the AB, CF, and RL fractional operator were examined in the instance of \( 0 < \alpha < 1. \) In the usual case \( (\alpha = 1), \) the computational results are consistent with the previous generalized non-fractional two-temperature thermoelastic theory with phase delay (2TDPL).

The numerical findings show that, in general, the fractional differential operators significantly influence the distributions of all the thermophysical domains being explored. It is also noted that thermal and mechanical waves are also stable, and they reach their steady state according to the fractional order’s values and the fractional operator’s type. It was also discovered that raising the value of the fractional parameter raises the propagation velocity of the waves being probed near the boundary of the cylinder, where the heat flow is present at the beginning and fades out faster as they travel deeper into the body.

Table 1 demonstrates the incremental changes in dynamic temperature \( \theta \) with the radial coordinate variable \( r \) for several fractional operators and two fractional parameter values \( \alpha (\alpha = 0.5, 0.8). \) It is noted from Table 1 that the variance of dynamic temperature \( \theta \) is strongly influenced by the fractional order \( \alpha. \) Also, by checking the movement of heat across the surface, the dynamic temperature on the surface of the cylinder takes its most significant value and then gradually decreases until it finally approaches zero. This means that the dynamic temperature \( \theta \) spread is limited. Some other researchers who study the equation of heat transfer without fractional differentiation are often unaware of the problem of heat wave propagation at infinite velocities.

Table 1 shows a comparison of the dynamic temperatures \( \theta \) in the case of the AB, CF, and RL fractional operators. It is observed that at \( t = 0.12, \) the numerical findings obtained for the dynamic temperature \( \theta \) by the CF operator approach are more significant than if the fractional AB operator approach is aided. It is also shown

| \( r \) | Classical derivative | Atangana–Baleanu (AB) | Caputo–Fabrizio (CF) | Riemann–Liouville (RL) |
|------|---------------------|------------------------|----------------------|------------------------|
| \( \alpha = 1 \) | \( \alpha = 0.5 \) | \( \alpha = 0.8 \) | \( \alpha = 0.5 \) | \( \alpha = 0.8 \) |
| 0.0  | 0.00180507          | 0.00263982            | 0.000203916          | 0.00240736            |
| 0.1  | 0.00204895          | 0.00274536            | 0.000220707          | 0.00267368            |
| 0.2  | 0.00287273          | 0.00307567            | 0.00027507           | 0.00354937            |
| 0.3  | 0.00439062          | 0.00838072            | 0.00573903           | 0.00528923            |
| 0.4  | 0.0078736           | 0.00474712            | 0.00912052           | 0.00481686            |
| 0.5  | 0.0140632           | 0.00767548            | 0.01912392           | 0.0139553             |
| 0.6  | 0.035941            | 0.010382              | 0.0215205            | 0.0240276             |
| 0.7  | 0.0392677           | 0.015785              | 0.03024881           | 0.035177              |
| 0.8  | 0.0597343           | 0.0493558             | 0.0643784            | 0.0435177             |
| 0.9  | 0.025722            | 0.124826              | 0.150814             | 0.181444              |
| 1.0  | 0.463117            | 0.331585              | 0.378655             | 0.42688               |
that the numerical results obtained through the RL operator approach are greater than those obtained through the fractional AB and CF approaches. Finally, compared to the traditional case (no fractional differentiation), it can be deduced from the tables that the presence of fractional differential reduces the propagation of heat waves. In other words, the thermodynamic temperature distribution decreases with the decreasing value of the fractional order and vice versa.

The conductive temperature change $\varphi$ is shown in Table 2 with the radial distance $r$. It is clear from the table that the existence of the fractional parameter $\alpha$ has a considerable influence on the numerical results of the conductive temperature $\varphi$ profile. It is also observed that the existence of the fractional differentiation parameter in the different cases of the fractional operators (RL, CF, AB) works to reduce the conductive temperature values. By comparing the results in Tables 1 and 2, the conductive temperature $\varphi$ responds to the dynamic temperature $\theta$, but in a more flexible manner and without a point of intersection between the results.

From the table, it is noticed that the conductive temperature $\varphi$ takes its maximum value at the surface of the solid cylinder ($r = 1$), where the moveable heat flux is present. It gradually decreases away from the turbulence area until it vanishes inside the medium. It was also found from the numerical results that the conductive temperature $\varphi$ values are lower in the case of the fractional AB operator compared to the CF operator and the traditional RL operator.

By evaluating some values assigned to the fractional differential parameter $\alpha$, Table 3 shows the variance of the displacement $U$ as a function of the variable $r$. As shown in this table, as the distance between the two points within the center increases, the numerical values of the displacement distribution $U$ decrease until eventually they reach zero. The table also showed that raising the values of the fractional differential parameter $\alpha$ leads to a rise in the displacement values. This may be because the heat source changes regularly over time. It can be noticed that the displacement $U$ values on the surface of the plane are always equal to zero in all the different cases, in accordance with the mechanical boundary conditions of problem (59), where the surface of the cylinder is assumed to be constrained. This confirms the validity and accuracy of the numerical results obtained in addition to applied numerical method (89). Table 2 shows that the fractional AB model produces the lowest absolute values of displacement, while the conventional fractional RL model produces the most significant absolute values of displacement.

Tables 4 and 5 are presented to explain the influence of the fractional differentiation parameter $\alpha$ as well as the different fractional operators (RL, CF, and AB) on the radial stress $S_{rr}$ distributions and the hoop stress $S_{\xi\xi}$ in the case of the radial coordinate change $r$. The numerical values in the tables show that due to the change in the fractional-order parameters, the magnitudes of the thermal stresses change. The pressures are also likely to grow or decrease in size depending on the type of fractional actuator. The numerical results also show that stresses remain in the negative region across the surface of the cylinder, rapidly increasing in size near the cylinder surface before gradually decreasing and eventually fading away. It is also noticed that the behavior of the stresses is always in a compressive form near the surface of the cylinder in the turbulence region. It can also be concluded that the presence of a fractional differential in the thermal conductivity equation reduces the spread of mechanical waves.

The effect of the fractional parameter $\alpha$ on the fluctuation of Maxwell’s stress $M_{rr}$ is depicted in Table 6. The table shows that the tension disappears quickly as the distance between them grows, indicating that the magnetic field’s influence is immediate and restricted. The table also demonstrates that the fractional
parameter $\alpha$ has a small impact on Maxwell’s stress. Moreover, the comparative analysis between the presence and absence of the fractional operators determines the nature and behavior of the increasing and decreasing numerical values of Maxwell stress, as shown in Table 6.

Moreover, we are sure that the new fractional derivatives will be crucial in studying the macroscopic activity of various simple and non-simple materials, which is mostly related to partial exchanges such as thermoelectricity, fluidity, and others. The theory discussed in this study is closely related to the challenges of contemporary aerodynamic engineering using thermoplastic cylinders. By introducing phase delays into the modified thermal conductivity equation in the recommended model, this work also demonstrates that heat waves propagate naturally with a finite frequency.
In this study, the two-temperature thermoplastics model (2TFDPL) will be applied in the case of using the Caputo–Fabrizio integral in a linear combination of the original function and its normal integral. On the other hand, it can be concluded that the Atangana–Baleanu fractional coefficient is also a linear combination of the original function and the Riemann–Liouville fractional integral itself [13]. Despite being expressed in linear combination with the RL integral, the Atangana–Baleanu is a significant operator in the solution of various dynamical systems that could not be solved using only the RL operator. Finally, the fractional-order parameter may seem a new criterion for classifying materials based on their ability to conduct heat.

### 7.2 The effect of variable thermal conductivity

In addition to the issue of fractional differentiation, the current research is concerned with the issue of heat transfer through elastic materials, whose thermal conductivity and specific heat vary with temperature. Often, a series of procedures are performed at different temperatures to determine the quality of this property (the thermal conductivity coefficient, specific heat, etc.) as a function of temperature. Thus, the form of the appropriate function can be determined at that temperature (polynomial, periodic, exponential, etc.). In this subsection, the results of the distribution of different thermophysical domains will be presented and plotted with the change of radial distance to consider the effect of temperature change on the behavior of elastic materials. In this case, the other physical parameters are assumed to be \( \nu = 5, \tau_0 = 0.2, \tau_\phi = 0.1, \alpha = 0.8, \) and \( b = 0.05. \)

In this study, the two-temperature thermoplastics model (2TFDPL) will be applied in the case of using the Atangana–Baleanu fractional operator (AB).

The change in the value of the parameter \( K_1 \) will be studied to explore the effects of changing the thermal conductivity and specific heat with the change in temperature on the studied physical fields. The \( K_1 \) parameter value for the thermophysical fields covers the effects of changing the thermal conductivity and specific heat in three scenarios \((K_1 = 0.0, -0.5, -1)\). It is noticed that when the parameter \( K_1 \) is neglected \((K_1 = 0)\), the old case appears where the physical properties are not affected by temperature changes (the thermal conductivity coefficient and specific heat constants).

Figures 2 and 3 describe the changes in conductive temperature \( \varphi \) and thermodynamic temperature \( \theta \) versus the distance \( r \) when the parameter values responsible for the change of physical properties \( K_1 \) are changed. It is noted that the temperature distributions \((\theta \text{ and } \varphi)\) are affected by the change in some physical properties. The two figures show that the temperatures \((\theta \text{ and } \varphi)\) decrease with a high coefficient of change in thermal conductivity \( K_1 \). This observation shows that the thermal conductivity factor and the specific heat are closely related to the temperature change and positively affect the temperature profiles. In comparison, with the increase of the thermal conductivity change parameter \( K_1 \), the wavefront effect is observed.

The thermal conductivity can be presumed constant for a relatively small temperature difference. Still, a different thermal conductivity must be used or established for the same difference in temperature moved up or down the temperature measurement. This is because all-natural and manufactured materials have temperature-dependent features, the amount of which varies depending on the substance and the temperature range in question. Thermal conductivity under steady-state circumstances may be determined using a variety of instruments. The change of physical properties with temperature change (when \( K_1 \) is a constant) is often overlooked.

### Table 6 The variation of Maxwell’s stress \( M_{\varphi r} \) for different fractional operator

| \( \alpha \) | Classical derivative | Atangana–Baleanu (AB) | Caputo–Fabrizio (CF) | Riemann–Liouville (RL) |
|-------------|----------------------|------------------------|----------------------|------------------------|
| \( 0.0 \)   | \(-0.00269425\)     | \(-0.000285464\)     | \(-0.00362155\)     | \(-0.00459597\)       |
| \( 0.1 \)   | \(-0.00269425\)     | \(-0.00285464\)     | \(-0.00362155\)     | \(-0.00459597\)       |
| \( 0.2 \)   | \(-0.00269425\)     | \(-0.00285464\)     | \(-0.00362155\)     | \(-0.00459597\)       |
| \( 0.3 \)   | \(-0.0057497\)      | \(-0.00347568\)     | \(-0.00437463\)     | \(-0.00551415\)       |
| \( 0.4 \)   | \(-0.00991095\)    | \(-0.00450224\)     | \(-0.00560385\)     | \(-0.00699522\)       |
| \( 0.5 \)   | \(-0.0178437\)     | \(-0.00646732\)     | \(-0.00789096\)     | \(-0.00967533\)       |
| \( 0.6 \)   | \(-0.0330889\)     | \(-0.0106395\)      | \(-0.0126086\)      | \(-0.0150404\)        |
| \( 0.7 \)   | \(-0.0630843\)     | \(-0.0205236\)      | \(-0.0235485\)      | \(-0.0271934\)        |
| \( 0.8 \)   | \(-0.124408\)      | \(-0.0457793\)      | \(-0.0511612\)      | \(-0.0574427\)        |
| \( 0.9 \)   | \(-0.200111\)      | \(-0.102666\)       | \(-0.111093\)       | \(-0.120248\)         |
| \( 1.0 \)   | \(-0.390101\)      | \(-0.272138\)       | \(-0.291253\)       | \(-0.311285\)         |

An important observation that can be drawn from the results is that the Caputo–Fabrizio integral is a linear combination of the original function and its normal integral. On the other hand, it can be concluded that the Atangana–Baleanu fractional coefficient is also a linear combination of the original function and the Riemann–Liouville fractional integral itself [13]. Despite being expressed in linear combination with the RL integral, the Atangana–Baleanu is a significant operator in the solution of various dynamical systems that could not be solved using only the RL operator. Finally, the fractional-order parameter may seem a new criterion for classifying materials based on their ability to conduct heat.
A comparative study of a thermoelastic problem

Fig. 2 Thermodynamical temperature $\theta$ variation vs. the variability parameter $K_1$

Fig. 3 The conductive temperature $\varphi$ variation vs. the variability parameter $K_1$

by many other researchers examining thermal conductivity regardless of temperature. From the numerical results, it is noted that this phenomenon cannot be ignored, but rather it must be considered, especially in the process of thermal insulation and the design of some devices.

Figure 4 depicts the influence of thermal conductivity variability on displacement field fluctuations. We can observe from the graph that the variability parameter $K_1$ has a small influence on the displacement $U$ distribution. As the parameter $K_1$ is changed, the radial and hoop stresses $S_{rr}$ and $S_{\xi\xi}$ fields fluctuate in space, as shown in Figs. 5 and 6. As seen in the figures, the presence of the parameter $K_1$ leads the stresses to have a notable propensity to go higher or lower in amplitude. The dependence of Maxwell’s stress on parameter $K_1$ is graphically depicted in Fig. 7. Figure 7 further indicates that, like in the case of displacement, parameters $K_1$ have a minor influence on Maxwell’s stress $M_{rr}$, as described in [58].

According to the study, in this case, the effects of the features of the variable content on the thermodynamic response are primarily illustrated by two temperatures and two thermal stress patterns. Furthermore, as seen in these numerical studies, all curves converge when $r$ approaches 0. Put another way, the resultant solution is constrained to a specific region of space (the perturbation region) and does not spread indefinitely. On the other hand, the thermodynamic concept of doublets has a solution that extends to infinity, implying that heat waves propagate at an endless pace. The relationship between thermal conductivity and temperature is an empirical relationship based only on experimental data. Extrapolation outside the empirical parameters should be considered speculative.
Fig. 4 The displacement $U$ variation vs. the variability parameter $K_1$

Fig. 5 The radial stress $S_{rr}$ variation vs. the variability parameter $K_1$

Fig. 6 The hoop stress $S_{\xi\xi}$ variation vs. the variability parameter $K_1$
8 Conclusions

In the present work, a mathematical equation for the thermal conductivity of homogeneous solids with specific heat and conductivity depending on the temperature change in the direction of heat flow was developed. In the proposed model, the fractional differential operators (Riemann–Liouville, Caputo–Fabrizio, and Atangana–Baleanu), the dual-phase delay, and the concept of two temperatures were taken into consideration. According to the current research evidence, all three integral operators are also important in various scientific domains, including computational fluid dynamics, chaos theory, epidemiological studies, controller design systems, and applied physics. As special cases of the new model, several fractional and non-fractional thermoelasticity models can be derived, in addition to the one-temperature thermal conductivity model. The governing equations for the problem are technically solved by applying the Laplace transform strategy. The numerical results and discussions showed that all the studied field variables are greatly influenced by the change of the fractional operators and the order of fractional differentiation, in addition to the variable thermal conductivity based on the temperature.

The theory discussed in this study is closely related to the challenges of contemporary aerodynamic engineering using thermoplastic cylinders. According to results and discussion, it is noted that the fractional-order parameter is a new indicator of elastic materials’ ability to transfer and conduct heat, such as some physical properties such as the thermal conductivity factor. It also adopts how to improve the efficiency of the thermoelastic material value by determining the estimates of the order of fractional derivatives. By introducing phase delays into the modified thermal conductivity equation in the recommended model, this work also demonstrates that heat waves propagate naturally with a finite frequency. The success of both Caputo, Caputo–Fabrizio, and Atangana–Baleanu operators can be shown on a physical model such as thermoelasticity, which indicates that they can be reliably applied to physical problems such as viscoelasticity and others. This approach may be extended in the future to other forms of fractional and fractional factor groups, such as Caputo, Caputo–Fabrizio, and Atangana–Baleanu derivatives.

Finally, it is evident to concluded that choosing proper fractional derivatives and fractional order is critical in the modeling development. Based on the numerical and analytical results in this paper, it is possible to generalize the new fractional calculus and operators in possible future works, as well as generalize previous models in the fields of mathematics, physics, biology, and medicine.

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Declarations

Conflict of interest The authors declared no potential conflicts of interest with respect to the research, authorship, and publication of this article.
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