SYNTHETIC NONLINEAR SECOND-ORDER OSCILLATORS ON Riemannian Manifolds AND THEIR NUMERICAL SIMULATION

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Abstract. The present paper outlines a general second-order dynamical system on manifolds and Lie groups that leads to defining a number of abstract non-linear oscillators. In particular, a number of classical non-linear oscillators, such as the simple pendulum model, the van der Pol circuitual model and the Duffing oscillator class are recalled from the dedicated literature and are extended to evolve on manifold-type state spaces. Also, this document outlines numerical techniques to implement these systems on a computing platform, derived from classical numerical schemes such as the Euler method and the Runge-Kutta class of methods, and illustrates their numerical behavior by a great deal of numerical examples and simulations.

1. Introduction. Nonlinear dynamical systems on manifolds constitute a challenging and exciting research topic at the cross-border of applied mathematics and theoretical sciences [7, 44]. Indeed, these dynamical systems have appeared in a number of subjects in pure mathematics and applied sciences [33, 41, 49]. Nonlinear dynamical systems on manifolds have been studied in the scientific literature because they arise naturally from the modeling of complex physical structures and because such dynamical systems constitute the basis for several modern applications. An interesting recent extension is model reduction of dynamical systems on nonlinear manifolds using deep convolutional autoencoders [28], where the authors proposed a framework for projecting dynamical systems onto nonlinear manifolds using minimum-residual formulations at the time-continuous level (which leads to manifold Galerkin projection) and at time-discrete level (which leads to manifold least-squares Petrov–Galerkin projection).

Nonlinear dynamical systems on manifolds are, in fact, currently being investigated upon in at least two different contexts, according to their realm of application:

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• **Synthetic (or algorithmic) systems**: These systems arise as extensions to manifolds of classical nonlinear oscillators, such as the Duffing and the van Der Pol systems, or are designed specifically to generate complicated trajectories. These nonlinear systems are used to test non-linear signal processing algorithms and nonlinear control strategies, such as in the case of secure transmission of information by signal masking.

• **Models of physical phenomena**: These systems arise upon modeling complicated physical phenomena, such as the rotational motion of a satellite or a drone and in general relativity.

The present paper focuses on synthetic, deterministic, nonlinear, second-order autonomous oscillators extended from flat Euclidean spaces to high-dimensional curved Riemannian manifolds and Lie groups. Deterministic nonlinear oscillators have been a subject of scientific investigation in virtually every area of research [38], starting from early observations and studies [26, 35, 46] to more recent endeavors and applications [3, 18, 23, 36]. As a curiosity, the paper [52] presents a method for generating strategic plans in the ancient oriental game of Go. Treating stones in play as continuous perturbations of a dynamical system composed of coupled non-linear oscillators, this paper exploits transitory synchrony in the wave patterns to make topological inferences about possible territories later in the game. Also, the paper [4] proposed a viable way to increase the bandwidth of a vibration-based energy harvester featuring non-linear oscillators.

Oscillatory systems may exhibit a number of conservation laws (or integrals of motion) [2, 5, 22, 32, 45], which may be represented by curved state manifolds. To exemplify the notion of conservation law, let $I_1$, $I_2$, $I_3$ be the principal moments of inertia of a rigid body. The angular momenta $I_1 x_1$, $I_2 x_2$, $I_3 x_3$ of this rigid body satisfy Euler’s equations of motion [15]:

$$
\frac{d}{dt} \begin{bmatrix} I_1 x_1 \\ I_2 x_2 \\ I_3 x_3 \end{bmatrix} = \begin{bmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{bmatrix} \begin{bmatrix} I_1 x_1 \\ I_2 x_2 \\ I_3 x_3 \end{bmatrix}.
$$

This system of differential equations has the property that the following functions:

$$
\mathcal{M} := I_1 x_1^2 + I_2 x_2^2 + I_3 x_3^2, 
$$

$$
\mathcal{E} := \frac{1}{2} I_1 x_1^2 + \frac{1}{2} I_2 x_2^2 + \frac{1}{2} I_3 x_3^2,
$$

are exactly preserved along its solutions, namely, $\mathcal{M}$ and $\mathcal{E}$ are first integrals of motion. Clearly an equation $\mathcal{M}(x_1, x_2, x_3) = \text{constant}$ and $\mathcal{E}(x_1, x_2, x_3) = \text{constant}$ define a ellipsoid in $\mathbb{R}^3$, hence the state space of such dynamical system is an ellipsoidal surface (a two-dimensional manifold). A further example is given by the Lotka-Volterra prey-predator model [30, 50]:

$$
\frac{d}{dt} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} A & -B x \\ C y & -D \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},
$$

where $x$ denotes the number of preys, $y$ denotes the number of predators and the 4-tuple $(A, B, C, D)$ of positive parameters represent characteristic constants of the prey-predator populations and of their interaction. Such dynamical system admits the invariant:

$$
\mathcal{H} := D \ln(x) - C x - B y + A \ln(y).
$$

The present contribution aims at exploring the rich dynamics of discrete-time nonlinear autonomous oscillators on manifolds and Lie groups and to exemplify
their dynamics on two Riemannian manifolds of interest in the literature, namely, the unit-hypersphere and the special orthogonal group. Trajectories generated by such nonlinear dynamical systems evolve over time in complicated, possibly non-repeating, deterministic patterns.

The motivation behind the present scientific endeavor was to gain a deeper understanding of non-linear phenomena represented by non-linear oscillatory models in the presence of non-linear constraints on their variables. In addition, we believe that non-linear multi-variable oscillatory models might find applications in engineering and applied sciences, in particular in the following fields:

- **Secure transmission of information**: Non-linear oscillatory systems, such as nonlinear electrical circuits, have been proposed to generate carrier signals in the context of signal masking. Useful information may get ‘buried’ into non-periodic carriers making such information hardly detectable or undecipherable by malicious interceptors [27, 40]. Such data-protection strategy relies on a robust synchronization mechanism to make sure the non-linear oscillator at the received is synchronized to the paired non-linear oscillator at the transmitter. Synchronization may take place only if the receiver gets sufficiently clear information across the transmission channel: we believe that manifold-type oscillators, being able to generate a number of carriers at the same time, which are non-linearly coupled by the manifold structure, may increase the robustness of synchronization.

- **Non-smooth non-convex optimization**: Non-conventional optimization in Euclidean spaces arise in several areas of engineering and applied sciences and concerns those optimization problems where the criterion to optimize is neither smooth nor convex. It has been proven by several experiments that non-linear oscillators may help in non-conventional optimization since they are able to explore the search space through non-repeating trajectories that allow the optimization algorithm to visit every location only once [13, 16, 53]. Non-conventional optimization problems are known on curved spaces: we believe that non-linear oscillators on manifolds might prove of use in such scientific challenges.

- **Control of artificial limbs**: Control of artificial limbs takes advantage of signals generated by specific non-linear oscillatory algorithms or biological circuitries. A common application is control of artificial legs to achieve gait [1, 51]. We believe that non-linear oscillators on manifolds, such as the unit hypersphere, might prove of use in such scientific challenges since they may generated non-linearly orchestrated oscillatory signals to be employed in synchronized artificial legs motion.

From the perspective of numerically simulating conservative oscillators on a computing platform, realizing numerical experiments is tied to the availability of adequate numerical methods to approximate their trajectories [17, 19, 21, 31, 32]. Classical numerical methods, such as the Euler method or the methods in the Runge-Kutta or Heun class, will fail if applied directly, as they are intrinsically designed to work on flat spaces and are not suitable to keep up with the non-flat shape of curved manifolds [9, 20]. These numerical methods are extended within the present paper so as to cope with curved state manifolds by means of an expanding mathematical theory known as numerical calculus on manifold. As a matter of fact, every dynamical system whose descriptive variables are bound to one another by nonlinear smooth constraints in such a way that their feasible states form
a smooth manifold may be simulated numerically by means of the rich variety of mathematical tools provided by manifold calculus.

The behavior of these second-order dynamical systems is characterized in terms of scalar functions of motions, such as the kinetic energy, the potential energy and the total energy. These systems are also characterized as being either dissipative (i.e., damped) or conservative (i.e., non-damped). A potential energy function gives rise to the discussed variety of dynamical systems, while the total energy, in the case of non-damped systems, helps characterizing the quality of numerical methods and simulations.

During the preparation of the present paper, a large number of numerical simulations were performed in order to explore the rich variety of extended non-linear oscillators on manifolds. A large portion of such numerical simulation results have been included in this paper with the aim of illustrating the behavior of non-linear oscillators on manifolds in a significant number of instances.

The present paper is organized as follows. Section 2 recalls dynamical systems on smooth manifolds and Lie groups. Section 3 illustrates the numerical resolution of second-order non-linear systems, examining in particular the forward Euler method, the Heun method and the Runge method. Section 4 shows potential energy functions of abstract oscillators extended to manifolds such as the simple pendulum, the Van der Pol oscillators, the Duffing-type oscillators and the Keplerian oscillators. Sections 5 and 6 illustrate the behavior of the introduced non-linear oscillators through numerical simulations on the unit hypersphere and on the special orthogonal group, respectively. Section 7 illustrates joint numerical simulations on both the unit hypersphere and the special orthogonal group. Section 8 concludes this document.

2. Dynamical systems on manifolds and Lie groups. The present section recalls some notation from manifold calculus and describes a class of dynamical systems on smooth manifolds and Lie groups.

2.1. Notation in manifold calculus. Let $\mathbb{M}$ denote a Riemannian manifold. At a point $x \in \mathbb{M}$, the tangent space to the manifold $\mathbb{M}$ is denoted as $T_x \mathbb{M}$. The symbol $T\mathbb{M}$ denotes the tangent bundle defined as $T\mathbb{M} := \{(x, v) \mid x \in \mathbb{M}, v \in T_x \mathbb{M}\}$.

A Riemannian manifold $\mathbb{M}$ is endowed with a bilinear, positive-definite form $\langle \cdot, \cdot \rangle_x : T_x \mathbb{M} \times T_x \mathbb{M} \to \mathbb{R}$. A local metric $\langle \cdot, \cdot \rangle_x$ also defines a local norm $\|v\|_x := \sqrt{\langle v, v \rangle_x}$, for $v \in T_x \mathbb{M}$. The Riemannian gradient of a function $\psi : \mathbb{R} \to \mathbb{M}$ evaluated at a point $x \in \mathbb{M}$ is denoted as $\text{grad}_x \psi$. The covariant derivative of a vector field $w \in T\mathbb{M}$ in the direction of a vector $v \in T_x \mathbb{M}$ is denoted as $\nabla_v w$. We assume $\mathbb{M}$ to be endowed with a metric connection (namely, that the covariant derivative of the metric tensor is identically zero).

The parallel transport operator $P^{x \to y}$ transports a tangent vector from $T_x \mathbb{M}$ to $T_y \mathbb{M}$. A manifold exponential map $\exp : T\mathbb{M} \to \mathbb{M}$ is applied as $\exp_x(v)$. Its inverse ‘log’ is defined only locally and is termed manifold logarithm. Given points $x, y \in \mathbb{M}$, a manifold logarithm computes a tangent vector $v = \log_x(y) \in T_x \mathbb{M}$ such that $\exp_x(v) = y$.

Given two points $x, y \in \mathbb{M}$ connectable by a geodesic arc, their Riemannian distance is denoted by $d(x, y)$. On a Riemannian manifold, the distance between two nearby points may be evaluated by $d(x, y) = \|\log_x(y)\|_x$. A fundamental result of the calculus on manifolds states that the Riemannian gradient of a squared distance function reads

$$\text{grad}_x d^2(x, y) = -2 \log_x(y),$$  \hspace{1cm} (6)
wherever the logarithm is defined.

2.2. Second-order dynamical systems on manifolds and Lie groups. In this subsection we are going to describe a fairly general second-order dynamical system on Riemannian manifolds $\mathbb{M}$ driven by a forcing term stemming from a potential energy function. The state of such a system is described by a manifold-valued variable $x$, whose time-evolution traces a trajectory (that is, a curve) $t \mapsto x(t)$ on the state space $\mathbb{M}$. The extended state of one such system is represented by a point on the tangent bundle $(x, \dot{x}) \in T\mathbb{M}$ (which indeed plays the role of phase-space for the system). An extended trajectory is a curve $t \mapsto (x(t), \dot{x}(t))$ on the state tangent bundle.

The kinetic energy function $\mathcal{K} : T\mathbb{M} \to \mathbb{R}$ for a dynamical system is defined by

$$\mathcal{K} = \frac{1}{2} \langle v, v \rangle_x$$

for $(x, v) \in T\mathbb{M}$. On a Riemannian manifold, the metric is positive-definite, hence, on every (extended) trajectory, it holds that $\mathcal{K} \geq 0$.

A potential energy function $\mathcal{V} : \mathbb{M} \to \mathbb{R}$ depends on the coordinate $x \in \mathbb{M}$ only.

The considered dynamical system is designed in such a way that, in the absence of any external solicitation, it generates a trajectory that follows the landscape of the potential energy function.

The total energy $\mathcal{H}$ of such a dynamical system is defined by:

$$\mathcal{H} = \mathcal{K} + \mathcal{V}.$$  \hfill (7)

Dynamical system on manifolds. A general second-order dynamical system on a manifold $\mathbb{M}$ is described by the equations

$$\begin{cases}
\dot{x} &= v, \\
\nabla_v v &= f(x, v, t),
\end{cases}$$

in the tangent-bundle variables $(x(t), v(t)) \in T\mathbb{M}$, where $f$ denotes a forcing term, which may be composed as the sum of several contributions, like:

- **Friction-type damping (dissipative force):** This kind of damping is expressed by the forcing term $-\mu \|v\|_x^{2(\epsilon-1)} v$, with $\mu \geq 0$. Since, by definition, $\|v\|_x^2 = 2\mathcal{K}$, this kind of damping takes values $-\mu(2\mathcal{K})^{\epsilon-1} v$. The coefficient $\epsilon \geq 1$ denotes a damping degree (the case $\epsilon = 1$ is known as viscous damping).
- **Conservative force:** It derives from the potential $\mathcal{V}$ and reads $-\nabla_v \mathcal{V}$.

Energy dissipation is an unavoidable phenomenon in physical systems. In an oscillatory system, it leads to a decay of the oscillation amplitude. The authors of the paper [6] discussed a novel dissipation engineering strategy that can support stable oscillations – without supplying external energy to compensate losses – based on fundamental intrinsic mechanism of resonant mode coupling to redistribute and store mechanical energy among vibrational modes.

We shall only consider the two terms above, thus the dynamical system (8) assumes the expression:

$$\begin{cases}
\dot{x} &= v, \\
\nabla_v v &= -\mu \|v\|_x^{2(\epsilon-1)} v - \nabla_v \mathcal{V}.
\end{cases}$$

\hfill (9)

It is straightforward to prove that the system (9) looses energy at a rate proportional to its kinetic energy. Only when $\mu = 0$ the dynamical system may retain its initial energy endlessly.

**Lemma 2.1.** Over any trajectory of the dynamical system (9) the total energy either decreases or stays constant if $\mu = 0$. 

\[\text{(Continued on next page)}\]
Proof. By manifold calculus, since the connection $\nabla$ was assumed to be metric, we get:

$$\dot{\mathcal{H}} = \dot{\mathcal{V}} + \dot{\mathcal{K}} = \langle \text{grad}_x \mathcal{V}, \dot{x} \rangle_x + \langle v, \nabla_x v \rangle_x.$$  \hfill (10)

From the equations (9) it follows that

$$\dot{\mathcal{H}} = \langle \text{grad}_x \mathcal{V}, v \rangle_x + \langle v, -\mu\|v\|^{2(\epsilon - 1)}v - \text{grad}_x \mathcal{V} \rangle_x.$$  \hfill (11)

Therefore, from the bilinearity of the inner product and from the definition of the kinetic energy, it is readily concluded that

$$\dot{\mathcal{H}} = -\mu\|v\|^{2\epsilon} = -\mu(2\mathcal{E})^\epsilon < 0,$$  \hfill (12)

which proves the assertion.

It is interesting to discuss about the meaning of the asymptotic vanishing of the total energy to zero, in a damped system, in relation to the sign of the potential energy:

- **Definite potential energy function** ($\mathcal{V} \geq 0$): Some dynamical systems are characterized by a potential energy function which takes exclusively non-negative values. Since the kinetic energy is always non-negative, in these systems, a vanishing total energy entails a vanishing kinetic and potential energy.

- **Indefinite potential energy function**: Those dynamical systems characterized by an indefinite potential energy function, which may take positive, null or negative values, exhibit a more interesting behavior. In fact, in these systems, a vanishing total energy does not necessarily correspond to a vanishing kinetic/potential energy. For instance, a non-damped system whose initial state is such that $\mathcal{H}(0) = 0$ will remain in a zero-energy state indefinitely, although the system may evolve while the potential energy of the system converts into kinetic energy, and vice-versa.

The indefiniteness of a continuous potential energy function entails the existence of a non-trivial critical set, defined as the point set $\mathbb{C} := \{x \in \mathbb{M} | \mathcal{V}(x) = 0\}$.

**Dynamical system on Lie groups.** A special case of manifold is a Lie group, that is a manifold which carries over additional structure. A general second-order dynamical system on a manifold $\mathbb{G}$ reads:

$$\begin{cases}
\dot{X} = X \cdot \Omega, \\
\dot{\Omega} = f(X, \Omega, t),
\end{cases}$$  \hfill (13)

in the trivialized tangent bundle variables $(X(t), \Omega(t)) \in \mathbb{G} \times \mathfrak{g}$. The notation $X$ is a short-hand for (inverse) left translation, which maps a tangent vector from the algebra $\mathfrak{g}$ to the tangent space $T_X \mathbb{G}$. Here, $f$ denotes again a forcing term (more like a mechanical torque, indeed), which may be composed by the sum of several contributions, like:

- **Friction-type damping (dissipative force)**: This kind of damping is expressed by the forcing term $-\mu\|\Omega\|^{2(\epsilon - 1)}\Omega$, with $\mu \geq 0$ and $\epsilon \geq 1$. Here, $\| \cdot \|$ denotes a norm in the Lie algebra $\mathfrak{g}$.

- **Conservative force**: It derives from a potential $\mathcal{V}$ and reads $-X^{-1} \cdot \text{grad}_X \mathcal{V}$. The notation $X^{-1}$ is a short-hand for left translation, which maps a tangent vector from the tangent space $T_X \mathbb{G}$ to the algebra $\mathfrak{g}$. 

In the following, we shall only consider the two terms above, thus the dynamical system (13) assumes the expression:

\[
\begin{align*}
\dot{X} &= X \cdot \Omega, \\
\dot{\Omega} &= -\mu \|\Omega\|^2(\epsilon - 1) \Omega - X^{-1} \cdot \text{grad}_X V.
\end{align*}
\] (14)

For more details, interested readers might consult [10, 11].

3. Numerical simulation of second-order non-linear systems. The numerical methods for the approximate resolution of the differential equations of Subsection 2.2 are based on classical numerical schemes developed to simulate dynamical systems evolving in \(\mathbb{R}^n\): we must therefore extend these equations to the case of a smooth manifold or a Lie group. The logic behind the implementation of these methods retraces the classical one, with particular care to the fact that in a curved manifold, to advance a solution from one step to the next, it is unfeasible to add to the starting point the displacement that must be taken. Subsection 3.1 recalls three classical numerical methods to solve initial-value problems, while Subsection 3.2 explains their extension to simulate second-order dynamical systems on manifolds.

3.1. Classical forward Euler, Heun and Runge methods. Let us recall that a first-order initial-value problem (IVP), to be solved numerically, reads:

\[
\begin{align*}
\dot{y}(t) &= f(y(t), t), \quad t \geq 0, \\
y(0) &= y_0.
\end{align*}
\] (15)

Let us recall and examine three numerical methods specifically:

- **Forward Euler method**: This is a first-order method, which means that the local error (error per step) is proportional to the square of the step size, and the global error (error at a given time) is proportional to the step size. Moreover, it can be said that for \(h \to 0\) the approximate solution and the analytic one (which cannot always be obtained with the rules of differential calculus) will be coincident. However, not being able to choose \(h\) infinitely small, it is customary to choose a small value of \(h\) below which the error, with respect to the analytic solution, is acceptable. From a mathematical point of view, the procedure carried out is the following: from \(\frac{dy}{dt} = f(y(t), t)\) and using an incremental rate approximation it can be written that:

\[
\frac{y_{k+1} - y_k}{t_{k+1} - t_k} \approx f(y_k, t_k),
\] (16)

where \(t_k := h \cdot k\) and \(y_k\) denotes a numerical approximation to the actual solution \(y(t_k)\). Solving for \(y_{k+1}\) gives the recurrence relation:

\[
y_{k+1} = y_k + hf(y_k, t_k),
\] (17)

where (17) will be called *forward Euler method* and \(k = 0, 1, 2, \ldots\).

- **Heun’s Method**: Heun’s method may refer to the improved or modified Euler’s method (that is, the explicit trapezoidal rule), or a similar two-stage Runge-Kutta method. The accuracy of the Euler method improves only linearly when the step size is decreased, whereas the Heun method improves accuracy quadratically. This scheme can be compared with the implicit trapezoidal method, but with \(f(y_{k+1}, t_{k+1})\) replaced by \(f(\hat{y}_{k+1}, t_{k+1})\) in order to make it explicit, where \(\hat{y}_{k+1}\) is the result of one step of forward Euler’s method on the same initial value problem. Therefore, Heun’s method is a predictor-corrector
method with forward Euler’s method as predictor and trapezoidal method as corrector. Let us start from the expression of the Crank-Nicolson method:

\[ y_{k+1} = y_k + h \frac{f(y_k, t_k) + f(y_{k+1}, t_{k+1})}{2}, \]

which is an implicit method that requires the knowledge of the function value at the point of arrival that is still unknown. Instead of \( y_{k+1} \) on the right-hand side, let us employ the approximation \( y_{k+1} = y_k + h f(y_k, t_k) \) on the right-hand side, hence it can be obtained that:

\[ y_{k+1} = y_k + h \frac{f(y_k, t_k)}{2} + h f(y_k + h f(y_k, t_k), t_{k+1}) \]

which can be rewritten more compactly by the classical Runge-Kutta notation as:

\[
\begin{cases}
\beta_{1,k} := f(y_k, t_k), \\
\beta_{2,k} := f(y_k + h \beta_{1,k}, t_{k+1}) \\
y_{k+1} = y_k + \frac{h}{2}(\beta_{1,k} + \beta_{2,k}),
\end{cases}
\]

where the third formula of (19) represents the Heun method, which belongs precisely to the class of second order methods.

- **Runge’s Method:** The RK2 method is also referred to as the midpoint method. Suppose to carry out a step by the explicit Euler method till half of the interval \( t_{k+\frac{1}{2}} := t_k + \frac{h}{2} \), to calculate the value of the function \( y_{k+\frac{1}{2}} \). Then, a complete step can be performed, in which the derivative is calculated in \( t_{k+\frac{1}{2}} \) (that is, in the middle of the interval). Thus the Runge method is obtained:

\[ y_{k+\frac{1}{2}} = y_k + h \frac{f(y_k, t_k)}{2}, \]

and by replacing the above expression in \( y_{k+1} = y_k + h f(y_{k+\left(\frac{1}{2}, \frac{1}{2}\right)}, t_{k+\frac{1}{2}}) \), it can be obtained that:

\[ y_{k+1} = y_k + h f(y_k + \frac{h}{2} f(y_k, t_k), t_k + \frac{h}{2}). \]

In the classical Runge-Kutta notation, the Runge method may be recast as a special case of the wide Runge-Kutta class of numerical methods by defining \( \beta_{1,k} := f(y_k, t_k) \) and \( \beta_{2,k} := f(y_k + c h \beta_{1,k}, t_k + bh) \). By this, Runge’s formula will be written as:

\[ y_{k+1} = y_k + h (a_1 \beta_{1,k} + a_2 \beta_{2,k}). \]

Setting \( a_1 := 0 \) and \( b, c := \frac{1}{2} \), it can be obtained that \( \beta_{2,k} = f(y_k + \frac{h}{2} \beta_{1,k}, t_k + \frac{h}{2}) \), therefore setting \( a_2 = 1 \), Runge’s formula can be rewritten as:

\[ y_{k+1} = y_k + h \beta_{2,k}. \]

### 3.2. Numerical simulation of second-order non-linear systems on manifolds by forward Euler, Heun and Runge techniques

This subsection extends the three numerical schemes recalled in the previous subsection to simulate second-order dynamical systems whose state space is a smooth manifold. In this subsection, we present general schemes on a smooth manifold to solve the system of first-order equations on a tangent bundle TM:

\[
\begin{cases}
\dot{x} = f_x(x, v), \ x(0) = x_0 \in M, \\
\nabla_x v = f_v(x, v), \ v(0) = v_0 \in T_{x_0}M,
\end{cases}
\]
where $f_x, f_v : TM \to TM$. Notice, in particular, that $f_x(x, v) \in T_xM$ and that $f_v(x, v) \in T_vM$, for $(x, v) \in TM$. In addition, notice that the dynamical system (8) is a special case of (24), except that the system (24) does not carry over a direct dependency from the time, namely, we shall consider only autonomous systems. Any extension of the previous numerical methods to smooth manifolds needs to be carried out according to two general principles: 1) the dynamics (24) is described by two, first-order, coupled equations, therefore the Euler, Heun and the Runge methods need to get spliced in two parts; 2) the dynamics (24) takes place on the tangent bundle $TM$ which is non-trivial (it trivializes to a Euclidean space when $M = \mathbb{R}^n$).

**Forward Euler method on manifold:** According to the expressions of the tangent-bundle-valued quantities of interest recalled above, a forward Euler-type method to achieve a discrete-time version of the dynamical system (24) reads:

$$
\begin{align*}
  x_{k+1} &= \exp_{x_k}(h f_x(x_k, v_k)), \\
v_{k+1} &= \mathbb{P}^{x_k\to x_{k+1}}(v_k + h f_v(x_k, v_k)).
\end{align*}
$$

**Heun method on manifold:** A Heun-type method to achieve a discrete-time version of the dynamical system (24) reads:

$$
\begin{align*}
  \beta_{1,k}^x &= f_x(x_k, v_k), \ (\text{Initial slope for } x) \\
  \hat{x}_k &= \exp_{x_k}(h \beta_{1,k}^x), \ (\text{Aimed-to } x) \\
  \beta_{1,k}^v &= f_v(x_k, v_k), \ (\text{Initial slope for } v) \\
  \hat{v}_k &= \mathbb{P}^{x_k\to x_{k+1}}(v_k + h \beta_{1,k}^v), \ (\text{Aimed-to } v) \\
  \beta_{2,k}^x &= f_x(\hat{x}_k, \hat{v}_k), \ (\text{Final slope for } x) \\
  \beta_{2,k}^v &= f_v(\hat{x}_k, \hat{v}_k), \ (\text{Final slope for } v) \\
x_{k+1} &= \exp_{x_k}(h \beta_{2,k}^x) + \mathbb{P}^{x_k\to x_{k+1}}(v_k + h \beta_{1,k}^v + \mathbb{P}^{x_k\to x_{k+1}}(\beta_{2,k}^v)), \\
v_{k+1} &= \mathbb{P}^{x_k\to x_{k+1}}(v_k + h \beta_{1,k}^v + \mathbb{P}^{x_k\to x_{k+1}}(\beta_{2,k}^v)).
\end{align*}
$$

Notice that the estimations of the slopes at the beginning/end of each interval are such that $\beta_{1,k}^x \in T_{x_k}M$, $\beta_{1,k}^v \in T_{v_k}M$, $\beta_{1,k}^x \in T_{x_k}M$ and $\beta_{2,k}^v \in T_{x_k}M$.

**Runge method on manifold:** A Runge-type method to achieve a discrete-time version of the dynamical system (24) reads:

$$
\begin{align*}
  \beta_{1,k}^x &= f_x(x_k, v_k), \ (\text{Initial slope for } x) \\
  \hat{x}_k &= \exp_{x_k}(h \frac{\beta_{1,k}^x}{2}), \ (\text{Aimed-to } x) \\
  \beta_{1,k}^v &= f_v(x_k, v_k), \ (\text{Initial slope for } v) \\
  \hat{v}_k &= \mathbb{P}^{x_k\to x_{k+1}}(v_k + h \beta_{1,k}^v), \ (\text{Aimed-to } v) \\
  \beta_{2,k}^x &= f_x(\hat{x}_k, \hat{v}_k), \ (\text{Midpoint slope for } x), \\
  \beta_{2,k}^v &= f_v(\hat{x}_k, \hat{v}_k), \ (\text{Midpoint slope for } v), \\
x_{k+1} &= \exp_{x_k}(h \beta_{2,k}^x) + \mathbb{P}^{x_k\to x_{k+1}}(v_k + h \beta_{1,k}^v + \mathbb{P}^{x_k\to x_{k+1}}(\beta_{2,k}^v)), \\
v_{k+1} &= \mathbb{P}^{x_k\to x_{k+1}}(v_k + h \beta_{2,k}^v). \\
\end{align*}
$$

The estimations of the slopes at the beginning/midpoint of each interval are such that $\beta_{1,k}^x, \beta_{1,k}^v \in T_{x_k}M$, while $\beta_{2,k}^x, \beta_{2,k}^v \in T_{x_k}M$.

Such numerical algorithms inherit the same structural properties of their predecessors, namely, Euler’s method is of order one (that is, its numerical error increases as $h^2$), although it is the least computationally demanding, while both Heun’s and
Runge’s methods are of order two (their numerical error increases as $h^3$) while Heun’s method is the most computationally burdensome.

4. Potential energy functions and associated abstract oscillators on manifolds. The present section retraces some classical potential energy functions and extends them to a general Riemannian manifold. A similar discussion might be carried out about extensions of classical abstract systems to Lie groups, although such case will be tackled directly through examples in Section 6.

The ultimate purpose of defining and extending potentials is to set up a consistent non-linear oscillator theory on manifolds. Non-linear oscillators find widespread applications in science and engineering. Examples are found in the design and implementation of a liquid mixing apparatus based on a stirring tank model [56], in the design of digital music synthesizers [47], in the design of haptic devices [55] and in secure transmission of information over digital networks by signal masking [12], where synchronization of non-linear oscillator is also a topic of prime interest.

Simple pendulum. A well-documented potential energy function arises in the study of a simple pendulum. For a recent review and generalization (by fractional calculus) see [37]. In the classical case $\mathbb{M} = \mathbb{R}$, such potential function reads $V \propto 1 - \cos x$, where $x$ denotes the angular displacement from the equilibrium state. Such potential energy function may be extended to a general Riemannian manifold $\mathbb{M}$, endowed with a Riemannian distance function $d(\cdot, \cdot)$, as:

$$V^{(pen)} := \kappa (1 - \cos d(x, r)),$$

with $\kappa > 0$ being a constant parameter and $r \in \mathbb{M}$ denoting a reference point, which plays the role of an equilibrium state. In fact, the potential $V^{(pen)}$ presents one of its minima in $x = r$. Rewrite the pendulum-type potential as

$$V^{(pen)} = \kappa - \kappa \cos(d^2(x, r))^{\frac{1}{2}}.$$  

According to the calculation rule (6), its Riemannian gradient reads:

$$\text{grad}_x V^{(pen)} = -\kappa \frac{\sin d(x, r)}{d(x, r)} \log_x(r).$$  

Notice that the above gradient is continuous in a neighborhood of $r$, in fact:

$$\lim_{x \to r} \frac{\sin d(x, r)}{d(x, r)} \log_x(r) = 0.$$  

Also, it is worth noticing that the simple pendulum potential is always bounded by $0 \leq V^{(pen)} \leq 2\kappa$ even in the case that the distance function is not bounded (as it is the case for non-compact manifolds).

The dynamical system associated to the simple-pendulum potential reads:

$$\begin{cases}
\dot{x} &= v, \\
\nabla_v v &= -\mu \|v\|^{2(\epsilon - 1)} v + \kappa \frac{\sin d(x, r)}{d(x, r)} \log_x(r).
\end{cases}$$  

Van der Pol oscillator. Balthazar Van der Pol was a Dutch electrical engineer who initiated modern experimental dynamics during the 1920’s and 1930’s. Van der Pol, first, introduced an equation to describe triode oscillations in electrical circuits [48]. The mathematical model for this system is a now well-known second-order ordinary differential equation with cubic nonlinearity – the Van der Pol equation. The Van der Pol oscillator is a classical example of self-oscillatory system and is now considered as an important mathematical model that can be used in much more complicated and modified systems. The potential energy function corresponding to the van der Pol dynamical system in the case $\mathbb{M} = \mathbb{R}$ is a quadratic function, namely
Numerical simulation of second-order oscillators on manifolds

Such potential energy function may be extended to a general Riemannian manifold $\mathbb{M}$, endowed with a Riemannian distance function $d(\cdot, \cdot)$, as:

$$V^{(\text{pol})} := \frac{1}{2} \kappa d^2(\cdot, \cdot),$$

(32)

with $\kappa > 0$ being a constant parameter and $r \in \mathbb{M}$ denoting a reference point. The potential $V^{(\text{pol})}$ presents its minimum in $x = r$. Its Riemannian gradient reads:

$$\nabla_x V^{(\text{pol})} = -\kappa \log_x(r).$$

(33)

Therefore, the associated dynamical system reads:

$$\begin{align*}
\dot{x} &= v, \\
\nabla_v v &= -\mu \|v\|^{2(\epsilon - 1)} \cdot v + \kappa \log_x(r).
\end{align*}$$

(34)

**Duffing-type oscillators.** The Duffing oscillator, named after Georg Duffing, is a non-linear second-order differential equation used to model the motion of a damped oscillator with a more complex potential than in simple harmonic motion [25]. In physical terms, it models, for example, a spring pendulum whose spring’s stiffness does not exactly obey Hooke’s law. An extension of the Duffing potential reads:

$$V^{(\text{duf})} := \pm \frac{1}{2} d^2(\cdot, \cdot) \pm \frac{1}{4} \kappa d^4(\cdot, \cdot),$$

(35)

where again $d(\cdot, \cdot)$ denotes the Riemannian (geodesic) distance on the manifold $\mathbb{M}$, $\kappa > 0$ is a free parameter and $r \in \mathbb{M}$ denotes a reference point. The signs $\pm$ were introduced to account for the hard, the soft and the double-well Duffing oscillator.

Let us distinguish the case that both addenda appear with the same sign from the case that the two addenda appear with opposite sign:

- If the Duffing potential is written as $V^{(\text{duf})}_{++} := \frac{1}{2} d^2(\cdot, \cdot) + \frac{1}{4} \kappa d^4(\cdot, \cdot)$, then $V^{(\text{duf})}_{++} \geq 0$, while if it is written as $V^{(\text{duf})}_{+-} := -V^{(\text{duf})}_{++}$, then $V^{(\text{duf})}_{+-} \leq 0$ for any value of $x \in \mathbb{M}$. In both cases, the point $x = r$ is the only point where the potential vanishes to zero and the only point of stationarity of the potential.

- If the Duffing potential is written as $V^{(\text{duf})}_{+-} := \frac{1}{2} d^2(\cdot, \cdot) - \frac{1}{4} \kappa d^4(\cdot, \cdot)$ or as $V^{(\text{duf})}_{++} := -V^{(\text{duf})}_{+-}$, then the Duffing potential may change sign according to the magnitude of the distance $d^2(\cdot, \cdot)$. In both cases, the point $x = r$ is not the only point where the potential vanishes to zero nor the only point of stationarity of the potential. In fact, by defining the critical distance $d_b := \sqrt{2/\kappa}$, it is immediate to see that at every point $x \in \mathbb{M}$ such that $d(x, r) = d_b$, the potential changes sign.

Examples of Duffing potentials corresponding to different combinations of signs and different values of $\kappa$ are illustrated in Figure 1.

The Riemannian gradient of Duffing potentials corresponding to different combinations of signs reads:

$$\nabla_x V^{(\text{duf})} = \pm 1 \mp \kappa d^2(\cdot, \cdot) \log_x(r).$$

(36)

Apparently, the potential vanishes when $x = r$ but, in case of mixed signs, it also vanishes at every point $x \in \mathbb{M}$ such that $d(x, r) = d_b$, therefore this system presents infinitely many critical points. Namely, we may define a set

$$C^{(\text{duf})} := \left\{ x \in \mathbb{M} \mid d(x, r) = \sqrt{2/\kappa} \right\}$$

(37)

of critical points in $\mathbb{M}$. 

\[ V \propto x^2. \]
Figure 1. Example of Duffing potentials corresponding to different combinations of signs and different values of $\kappa$. The $(++)$ combination is referred to as hard Duffing potential while the combination $(+-)$ is referred to as soft Duffing potential. (A combination $(-+)$, not shown in this figure, is referred to as double-well potential.)

Therefore, the dynamical system associated to a Duffing potential reads:

\[
\begin{align*}
\dot{x} &= v, \\
\nabla_v v &= -\mu v ||v||_2^{2(\epsilon-1)} v + \begin{cases} 
[1 + \kappa d^2(x,r)] \log_x (r), & \text{for the hard Duffing,} \\
[-1 - \kappa d^2(x,r)] \log_x (r), & \text{for the soft Duffing,} \\
[-1 + \kappa d^2(x,r)] \log_x (r), & \text{for the double-well Duffing.}
\end{cases}
\end{align*}
\]

(38)

On addition of a harmonic oscillating forcing term to Duffing’s equations, the soft-Duffing oscillatory system appears to be similar to the Halmholtz oscillator, except that the latter includes a second power rather than a third power. Moreover, the soft-Duffing oscillatory system appears to cover an oscillator type studied by J.W. Strutt, Third Baron Rayleigh [24].

Keplerian oscillator. Likewise, a Keplerian (attractive-repulsive) potential [34] may be extended to a general Riemannian manifold $M$ with Riemannian distance $d(\cdot, \cdot)$, by defining a Keplerian potential as:

\[
\Psi^{(\text{kep})} := -\frac{\rho}{d(x,r)} + \lambda d(x,r),
\]

(39)

with $\rho, \lambda > 0$. The Keplerian potential embodies a positive and a negative component. Indeed, upon defining a critical distance $d_c := \sqrt{\rho/\lambda}$, for which $\Psi^{(\text{kep})} = 0$, it is readily verified that if $d(x, r) < d_c$ then the negative component is prevailing, therefore $\Psi^{(\text{kep})} < 0$ and $|\Psi^{(\text{kep})}| \propto \frac{1}{d(x, r)}$, while if $d(x, r)$ is larger than the critical
distance, the positive component prevails and the Keplerian potential is approximately linear in $d(x,r)$. An example of Keplerian potential with critical distance $d_c = 1$ is shown in Figure 2.

![Figure 2. Example of Keplerian potential with critical distance $d_c = 1$.]

Rewriting the Keplerian potential as $-\rho[d^2(x,r)]^{-\frac{1}{2}} + \lambda[d^2(x,r)]^{\frac{1}{2}}$ and invoking again the calculation rule (6), its Riemannian gradient reads:

$$\text{grad}_x V^{(\text{kep})} = -\left[ -\frac{\rho}{d^3(x,r)} + \frac{\lambda}{d(x,r)} \right] \log_x (r).$$  \hspace{1cm} (40)

The Riemannian gradient of the Keplerian potential does not describe a smooth vector field, since it presents a singularity at $x = r$. We observe that the Keplerian forcing term $-\text{grad}_x V^{(\text{kep})}$ is always directed along the $\overrightarrow{rx}$ directrix and that its amplitude is always non-vanishing, in fact

$$\left\| -\text{grad}_x V^{(\text{kep})} \right\|_x = \left[ \frac{\rho}{d^3(x,r)} + \frac{\lambda}{d(x,r)} \right] \left\| \log_x (r) \right\|_x = \lambda + \frac{\rho}{d^2(x,r)} > 0.$$  \hspace{1cm} (41)

Moreover, for the Keplerian potential, we may define a set

$$C^{(\text{kep})} := \left\{ x \in \mathcal{M} \mid d(x,r) = \sqrt{\rho/\lambda} \right\}$$  \hspace{1cm} (42)

of critical points in $\mathcal{M}$.

The dynamical system associated to a Keplerian potential reads:

$$\begin{cases}
     \dot{x} = v, \\
     \nabla_v v = -\mu\|v\|^{2(\epsilon-1)} v + \left[ \frac{\rho}{d^3(x,r)} + \frac{\lambda}{d(x,r)} \right] \log_x (r).
\end{cases}$$  \hspace{1cm} (43)

It is straightforward to envisage how all potential functions known in the physics literature based on distance functions may be extended to Riemannian manifolds on the basis of the Riemannian distance function. A well known example is the Lennard-Jones potential [57] or the electric/gravitational potential. Other distance-based potential energy functions of possible interest are the Biswas-Hamann potential function [29], that take the form $V^{(\text{bhp})} := A_1 \exp(-\lambda_1 d(x,r)) + A_2 \exp(-\lambda_2 d(x,r))$, where $A_1, A_2, \lambda_1, \lambda_2$ denote specific coefficients, and the
Yukawa potential function for the nuclear force \( [42] \) \( \mathcal{Y}^{(\text{nuc})} := -g^2 e^{-\mu d(x,r)} \), where \( g \) denotes a magnitude scaling and \( \mu \) denotes the Yukawa particle mass.

5. **Numerical simulation on the unit hypersphere.** For the unit hypersphere \( \mathbb{S}^{n-1} \) endowed with the canonical metric \( \langle w, v \rangle := w^\top v \), it holds that:

\[
\begin{align*}
    d^2(x, y) &= \arccos^2(x^\top y), \\
    P^{x \rightarrow y}(w) &= \left[ I_n - \frac{(I_n - xx^\top) y y^\top}{1 + x^\top y} \right] w, \\
    \exp_x(v) &= x \cos(\|v\|) + v \sin(\|v\|)/\|v\|, \\
    \log_x(y) &= (I_n - xx^\top)y((\sin d(x, y))/d(x, y))^{-1},
\end{align*}
\]

(44)

where the symbol \( \| \cdot \| \) denotes a vector 2-norm, \( I_n \) denotes a \( n \times n \) identity matrix, the symbol \( ^\top \) denotes matrix transpose and \( x, y \in \mathbb{S}^{n-1}, w, v \in T_x \mathbb{S}^{n-1} \). It is assumed that \( x^\top y \neq -1 \) in the expression of the parallel transport and that \( v \neq 0 \) in the expression of the exponential map. Also, it is set \( \log_x(x) := 0 \) for any \( x \in \mathbb{S}^{n-1} \).

An important fact to take into account in the numerical simulations is that the unit hypersphere is a compact manifold, therefore the distance function is bounded from above. In the case of the canonical metric, the upper bound is reached when two points \( x, y \in \mathbb{S}^{n-1} \) are antipodal (namely, \( x^\top y = -1 \)), hence \( d_{\text{max}} = \pi \). This fact also implies bounds on the values of the potential energy functions, which are based on the Riemannian distance.

The unit sphere is the base manifold in a number of scientific and engineering problems. For examples, the unit sphere \( \mathbb{S}^2 \) plays an important role in robotics [14] as well as in quantum computation, where it represents the locus of pure quantum states \( |\psi\rangle \) in a 1-qubit register (termed Bloch sphere, where the north pole represents \( |0\rangle \) and the south pole represents the |1\rangle qubit) [39].

According to the expressions of the geometric quantities of interest recalled above, the discrete-time version of the dynamical system (9) reads:

\[
\begin{align*}
    x_{k+1} &= x_k \cos(h\|v_k\|) + v_k \sin(h\|v_k\|)/\|v_k\|, \\
    v_{k+1} &= P^{x_k \rightarrow x_{k+1}}[v_k - h\mu(v_k^\top v_k)^{-1}v_k - h\text{ grad}_{x_k}\mathcal{Y}],
\end{align*}
\]

(45)

with \( h > 0 \) being a discretization stepsize for the dynamical system and \( k = 0, 1, 2, \ldots \). Recall that \( \epsilon = 1 \) corresponds to linear (viscous) damping.

In all pictures in this section, the right-hand side panel shows the values taken by the kinetic energy, the potential energy and the total energy over the generated trajectory and the distance between the state \( x_k \) and the reference \( r \). The red dashed horizontal lines represent the values of the shown quantities at the beginning of the numerical simulation (corresponding to the initial values \( x_0 \) and \( v_0 \)).

5.1. **Numerical simulation on the simple pendulum.** The time-discretized version of the dynamical system (31) associated to the simple pendulum reads:

\[
\begin{align*}
    x_{k+1} &= x_k \cos(h\|v_k\|) + v_k \sin(h\|v_k\|)/\|v_k\|, \\
    v_{k+1} &= P^{x_k \rightarrow x_{k+1}}[v_k - h\mu(v_k^\top v_k)^{-1}v_k + h\kappa(sin d(x_k, r)) \frac{\text{grad}_{x_k} d(x_k, r)}{d(x_k, r)}],
\end{align*}
\]

(46)

with \( h > 0 \) being a discretization stepsize for the dynamical system and \( \kappa > 0 \) is the coefficient of the pendulum potential, while \( \epsilon \) is the exponential for nonlinear damping. Moreover, the system state is represented by the variable-pair \( (x_k, v_k) \in T\mathbb{S}^{n-1} \) for any \( k \in \mathbb{N} \). Notice that, as long as \( x_k \neq r \), a cancellation occurs within
the squared parentheses in the second equation, hence the discrete-time dynamical system (46) may be re-expressed as

\[
\begin{align*}
    x_{k+1} &= x_k \cos(h\|v_k\|) + v_k \sin(h\|v_k\|)/\|v_k\|, \\
    v_{k+1} &= P_{x_k \rightarrow x_{k+1}} \left[ v_k - h\mu(v_k^\top v_k)^{-1}v_k + h\kappa(r - x_k (x_k^\top r)) \right],
\end{align*}
\]

(47)

for \( k = 0, 1, 2, \ldots \). It is interesting to observe the emergence of an effective damping coefficient \( \mu^* := h\mu \) and of an effective potential coefficient \( \kappa^* := h\kappa \) which depend on the discretization stepsize. The values \( x_0 \) and \( v_0 \) denote the initial conditions of the systems. The simple pendulum on the sphere was simulated numerically in the presence as well as in the absence of a damping term.

The Figure 3 illustrates the behavior of a simple pendulum on the ordinary sphere \( \mathbb{S}^2 \) embedded in \( \mathbb{R}^3 \) in the absence of a damping term. This figure shows the trajectory generated by the dynamical system as well as the values of the potential, kinetic and total energy. In particular, the total energy \( \mathcal{H} \), sum of potential and kinetic energies, is supposed to keep preserved over time, while its value changes slightly over the course of this numerical simulation due to numerical errors intrinsic in the numerical scheme. The Figure 4 illustrates the behavior of a simple pendulum on the ordinary sphere \( \mathbb{S}^2 \) embedded in \( \mathbb{R}^3 \) in the presence of a damping term. In this case, as predicted by theoretical analysis, the total energy of the oscillator tends asymptotically to zero due to damping.

5.2. Numerical simulation on the hard Duffing oscillator. The time-discretized version of the dynamical system (38) associated to the hard Duffing potential reads:

\[
\begin{align*}
    x_{k+1} &= x_k \cos(h\|v_k\|) + v_k \sin(h\|v_k\|)/\|v_k\|, \\
    v_{k+1} &= P_{x_k \rightarrow x_{k+1}} \left[ v_k - h\mu(v_k^\top v_k)^{-1}v_k + h(1 + \kappa d^2(x_k, r)) \right. \\
    & \quad \left. \times \log_{x_k}(r) \right].
\end{align*}
\]

(48)
Figure 4. Behavior of the simple pendulum (46) in the presence of non-linear damping. The left-hand panel shows the trajectory in the space $S^2$, when the starting point is $x_0 = [0 \ 0 \ 1]^T$, the reference point for the oscillator is $r = [1 \ 0 \ 0]^T$ (denoted by a blue open circle) and the initial speed $v_0 = [0.5 \ -0.8 \ 0]^T$. The values of the parameters used in this simulation are $\kappa = 0.5$, $\mu = 0.5$, $\epsilon = 1.3$ and $h = 0.002$.

Similarly to the previous subsection, the hard Duffing oscillator on the sphere was simulated numerically in the presence as well as in the absence of a damping term.

Figure 5. Behavior of the hard Duffing oscillator (48) in the absence of non-linear damping (namely, $\mu = 0$). The left-hand panel shows the trajectory in the space $S^2$, when the starting point is $x_0 = [0 \ 0 \ 1]^T$, the reference point for the oscillator is $r = [1 \ 0 \ 0]^T$ (denoted by a blue open circle) and the initial speed $v_0 = [0 \ -0.4 \ 0]^T$. The values of the parameters used in this simulation are $\kappa = 0.5$ and $h = 0.0005$.

The Figure 5 illustrates the behavior of a hard Duffing oscillator on the ordinary sphere $S^2$ in the absence of a damping term. As expected, the state of this non-linear dynamical system oscillates around the reference point $r$. Moreover, the motion of the system continues endlessly due to conservation of its total energy. This oscillator generates an interesting ‘atom-like’ pattern on the state manifold $S^2$. The Figure 6 illustrates the behavior of a hard Duffing oscillator on the ordinary sphere $S^2$ in
Figure 6. Behavior of the hard Duffing oscillator (48) in the presence of non-linear damping. The left-hand panel shows the trajectory in the space \( S^2 \), when the starting point is \( x_0 = [0 \ 0 \ 1]^\top \), the reference point for the oscillator is \( r = [1 \ 0 \ 0]^\top \) (denoted by a blue open circle) and the initial speed \( v_0 = [0.5 \ -0.9 \ 0]^\top \). The values of the parameters used in this simulation are \( \kappa = 0.5, \mu = 0.5, \epsilon = 1.3 \) and \( h = 0.001 \).

The presence of a damping term. In this case, the total energy of the system tends asymptotically to zero due to damping, hence the trajectory of the system spirals into the reference point.

5.3. Numerical simulation on the soft Duffing oscillator. This particular dynamical system has been implemented numerically by three different methods to be able to observe a better oscillation of kinetic and potential energies, a better conservation of total energy and to better evaluate the effects of the critical distance.

**Euler method:** The time-discretized version of the dynamical system (38), implemented through an Euler-type method and associated to a soft Duffing potential reads:

\[
\begin{aligned}
    x_{k+1} &= x_k \cos(h \|v_k\|) + v_k \sin(h \|v_k\|)/\|v_k\|, \\
    v_{k+1} &= P_{x_k \to x_{k+1}} [v_k - h\mu(v_k^\top v_k)^{-1}v_k + h(1 + \kappa d^2(x_k, r)) \log_{x_k}(r)].
\end{aligned}
\]

(49)

The Figure 7 illustrates the behavior of a soft Duffing oscillator, implemented through a Euler-type method, on the ordinary sphere \( S^2 \) in the absence of a damping term. As expected, the state of this non-linear dynamical system oscillates around the reference point \( r \). Moreover, the motion of this synthetic system continues endlessly thanks to conservation of its total energy. The Figure 8 illustrates the behavior of a soft Duffing oscillator, implemented through an Euler-type method, on the ordinary sphere \( S^2 \) in the presence of a damping term. In this case, the total energy tends asymptotically to zero due to damping, hence the trajectory of the system spirals into the reference point.

**Heun method:** The time-discretized version of the dynamical system (38), implemented through the Heun-like method described in (26), associated to the soft
Figure 7. Behavior of the soft Duffing oscillator (49) in the absence of non-linear damping (namely, $\mu = 0$), implemented through the Euler method. The left-hand panel shows the trajectory in the space $S^2$, when the starting point is $x_0 = [0 \ 0 \ 1]^T$, the reference point for the oscillator is $r = [1 \ 0 \ 0]^T$ (denoted by a blue open circle) and the initial speed $v_0 = [-1 \ -1.5 \ 0]^T$. The values of the parameters used in this simulation are $\kappa = 0.5$ and $h = 0.0001$.

Duffing potential reads:

$$
\begin{align*}
\dot{x}_k &:= \exp_{x_k}(h v_k), \\
\beta^v_{1,k} &:= -\mu (v_k^Tv_k)^{-1} v_k + (1 + \kappa d^2(x_k, r)) \log_{x_k}(r), \\
\tilde{v}_k &:= P^{x_k \rightarrow x_{k+1}}(v_k + h \beta^v_{1,k}), \\
\beta^v_{2,k} &:= -\mu (\tilde{v}_k^T \tilde{v}_k)^{-1} \tilde{v}_k + (1 + \kappa d^2(\tilde{x}_k, r)) \log_{\tilde{x}_k}(r), \\
x_{k+1} &:= \exp_{\tilde{x}_k}(h v_k + h^2 \beta^v_{1,k}), \\
v_{k+1} &= P^{x_{k+1} \rightarrow x_{k+1}} \left[ v_k + h \left( \beta^v_{1,k} + P^{\tilde{x}_k \rightarrow x_k}(\beta^v_{2,k}) \right) \right].
\end{align*}
$$

(50)

Figure 8. Behavior of the soft Duffing oscillator (49) in the presence of non-linear damping, implemented through the Euler method. The left-hand panel shows the trajectory in the space $S^2$, when the starting point is $x_0 = [0 \ 0 \ 1]^T$, the reference point for the oscillator is $r = [1 \ 0 \ 0]^T$ (denoted by a blue open circle) and the initial speed $v_0 = [-1 \ -1.5 \ 0]^T$. The values of the parameters used in this simulation are $\kappa = 0.5$, $\mu = 0.2$, $\epsilon = 1.3$ and $h = 0.0001$. 
The Figure 9 illustrates the behavior of a soft Duffing oscillator, implemented through a Heun-like method, on the ordinary sphere \( S^2 \) in the absence of a damping term. As expected, the state of this non-linear dynamical system keeps oscillating around the reference point \( r \) indefinitely.

![Figure 9. Behavior of the soft Duffing oscillator (50) in the absence of non-linear damping (namely, \( \mu = 0 \)), implemented through the Heun method. The left-hand panel shows the trajectory in the space \( S^2 \), when the starting point is \( x_0 = [0 \ 0 \ 1]^\top \), the reference point for the oscillator is \( r = [1 \ 0 \ 0]^\top \) (denoted by a blue open circle) and the initial speed \( v_0 = [-1 \ -1.5 \ 0]^\top \). The values of the parameters used in this simulation are \( \kappa = 0.5 \) and \( h = 0.0001 \).](image)

**Runge (RK2) method:** The time-discretized version of the dynamical system (38), implemented through the Runge-like method (27), associated to the soft Duffing potential reads:

\[
\begin{align*}
\dot{x}_k &:= \exp_{x_k} \left( \frac{h}{2} v_k \right), \\
\beta^1_{1,k} &:= -\mu(v_k^\top v_k)^{-1} v_k + (1 + \kappa d^2(x_k, r))(x_k - r) \\
\hat{v}_k &:= P^x_{x_k \to \hat{x}_k} \left( \frac{v_k + h \beta^1_{1,k}}{2} \right), \\
\beta^2_{2,k} &:= -\mu(\hat{v}_k^\top \hat{v}_k)^{-1} \hat{v}_k + (1 + \kappa d^2(\hat{x}_k, r))(\hat{x}_k - r) \\
x_{k+1} &:= \exp_{x_k} \left( h v_k + \frac{h^2}{2} \beta^2_{2,k} \right), \\
v_{k+1} &:= P^x_{x_k \to x_{k+1}} \left( v_k + h P^{\hat{x}_k \to x_{k+1}}(\beta^2_{2,k}) \right),
\end{align*}
\]  

(51)

where \( k = 0, 1, 2, \ldots \). (It is worth underlining that, in general, the composition \( P^x_k \to x_{k+1} \circ P^{\hat{x}_k \to x_k} \) does not coincide with \( P^{\hat{x}_k \to x_{k+1}}(\beta^2_{2,k}) \), unless the points \( \hat{x}_k, x_k \) and \( x_{k+1} \) belong to the same geodesic arc. In practice, these points are pretty close to one another, therefore, in the context of a numerical stepping method, it would be feasible to replace the last step with \( v_{k+1} = P^x_{x_k \to x_{k+1}}(v_k + h P^{\hat{x}_k \to x_{k+1}}(\beta^2_{2,k})) \). Such approximation would save no computation, though.)

The Figure 10 illustrates the behavior of a soft Duffing oscillator, implemented through a Runge-like method, on the ordinary sphere \( S^2 \) in the absence of a damping term. As expected, the state of this non-linear dynamical system oscillates around the reference point \( r \).

The total energy is not exactly preserved by the Euler method. It can be seen that by increasing the order of the method, the error becomes smaller (in the Heun-based
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Figure 10. Behavior of the soft Duffing oscillator (51) in the absence of non-linear damping (namely, $\mu = 0$), implemented through the Runge-Kutta method. The left-hand panel shows the trajectory in the space $S^2$, when the starting point is $x_0 = [0 \ 0 \ 1]^T$, the reference point for the oscillator is $r = [1 \ 0 \ 0]^T$ (denoted by a blue open circle) and the initial speed $v_0 = [-1 \ -1.5 \ 0]^T$. The values of the parameters used in this simulation are $\kappa = 0.5$, $h = 0.0001$. 

and Runge-based simulations, it is present in the tenth decimal digit). Therefore, it can be concluded that the lack of conservation of the total energy in simulations conducted by an Euler method is due to errors implicit in the method itself. 

Moreover, Figures 7, 8, 9 and 10 show a relationship between the distance $d(x, r)$ and the potential energy $V(duf)$, that is also interesting in this case. In fact, one can observe stationary points (local minima and maxima) of the potential energy:

- A local maximum of the distance $d(x, r)$ corresponds to a local minimum of the potential energy $V(duf)$.
- To a local minimum of distance $d(x, r)$ corresponds a local minimum of the potential energy $V(duf)$.

The critical distance value is $d_0 = 2$. When the distance is smaller than the critical distance, the potential energy takes positive values, while it takes negative values when the distance is larger than the critical distance. It can also be noticed that in the soft Duffing case, the repulsive action wins over the attractive action and therefore the oscillator tends to stabilize in the sphere at the farthest point from the reference.

5.4. Numerical simulation on the double-well Duffing oscillator. The time-discretized version of the dynamical system (38), implemented through an Euler-like method and associated to the double-well Duffing potential reads:

$$
\begin{align*}
    x_{k+1} &= x_k \cos(h\|v_k\|) + v_k \sin(h\|v_k\|)/\|v_k\|, \\
    v_{k+1} &= P^{x_k \rightarrow x_{k+1}} \left[ v_k - h\mu(v_k^T v_k)^{-1}v_k + h(-1 + \kappa d^2(x_k, r)) \log_{x_k}(r) \right]. 
\end{align*}
\tag{52}
$$

The Figure 11 illustrates the behavior of the oscillator (52) on the ordinary sphere $S^2$ in the absence of a damping term. As expected, the state of this non-linear dynamical system oscillates around the reference point $r$ indefinitely. In addition, Figure 12 illustrates the behavior of a double-well Duffing oscillator, implemented through the same numerical method, on the ordinary sphere $S^2$ in the presence of a damping term.
Figure 11. Behavior of the double-well Duffing oscillator (52) in the absence of non-linear damping (namely, $\mu = 0$), implemented through the Euler method. The left-hand panel shows the trajectory in the space $S^2$, when the starting point is $x_0 = [0 \ 0 \ 1]^T$, the reference point for the oscillator is $r = [1 \ 0 \ 0]^T$ (denoted by a blue open circle) and the initial speed $v_0 = [-1 -1.5 \ 0]^T$. The values of the parameters used in this simulation are $\kappa = 0.8$ and $h = 0.0001$.

Figure 12. Behavior of the double-well Duffing oscillator (52) in the presence of non-linear damping, implemented through the Euler method. The left-hand panel shows the trajectory in the space $S^2$, when the starting point is $x_0 = [0 \ 0 \ 1]^T$, the reference point for the oscillator is $r = [1 \ 0 \ 0]^T$ (denoted by a blue open circle) and the initial speed $v_0 = [-1 -1.5 \ 0]^T$. The values of the parameters used in this simulation are $\kappa = 1.4$, $\mu = 0.2$, $\epsilon = 1.3$ and $h = 0.0001$.

In the numerical experiment illustrated in Figure 11, the critical distance is $d_b \approx 1.58$, while in the experiment illustrated in Figure 12, the critical distance is $d_b \approx 1.19$. We can observe a dual behavior with respect to the case of soft Duffing, since these oscillators have the same potential function and the same gradient field, with switched signs. In fact, in the double-well case, the attractive action wins, therefore the oscillator tends to ‘fall’ into the reference point.
5.5. Numerical simulation on the Var der Pol oscillator. On the basis of the general form (9) of a second-order dynamical system on a manifold \( \mathcal{M} \), the Euler-like time-discretized version of a dynamical system based on the van der Pol potential reads:

\[
\begin{aligned}
    x_{k+1} &= x_k \cos(h\|v_k\|) + v_k \sin(h\|v_k\|)/\|v_k\|, \\
    v_{k+1} &= P^{x_k \rightarrow x_{k+1}} \left[ v_k - h \mu(v_k^Tv_k)^{\epsilon-1} v_k + h \kappa \log_{x_k}(r) \right].
\end{aligned}
\] (53)

Even in this case, it may be observed the emergence of an effective damping coefficient \( \mu^* := h \mu \) and of an effective potential coefficient \( \kappa^* := h \kappa \) which depend on the discretization stepsize.

The Figure 13 illustrates the behavior of a Van der Pol oscillator on the ordinary sphere \( S^2 \) in the absence of a damping term. The total energy \( \mathcal{H} \) should keep constant, while a slight change is observable, which is solely due to numerical errors.

The potential energy function of the van der Pol oscillator on the unit hypersphere is bounded as \( 0 \leq \mathcal{V}^{(\text{pol})} \leq \frac{1}{2} \kappa \pi^2 \). Since, in this numerical simulation, \( \kappa = 0.5 \), the potential is bounded from above by \( \mathcal{V}^{(\text{pol})}_{\text{max}} \approx 2.47 \). In fact, the potential values encountered in the simulations stay well beyond such bound.

The Figure 14 illustrates the behavior of a Van der Pol oscillator on the ordinary sphere \( S^2 \) embedded in \( \mathbb{R}^3 \) in the presence of a damping term. In this case, the system tends to loose its energy over time due to damping.

5.6. Numerical simulation on the Kepler oscillator. The time-discretized version of the dynamical system (43) associated to a Kepler-like oscillator, again implemented by a Euler-like method, reads:

\[
\begin{aligned}
    x_{k+1} &= x_k \cos(h\|v_k\|) + v_k \sin(h\|v_k\|)/\|v_k\|, \\
    v_{k+1} &= P^{x_k \rightarrow x_{k+1}} \left[ v_k - h \mu(v_k^Tv_k)^{\epsilon-1} v_k + h \left( \frac{\rho}{d(x_k,r)} + \frac{\lambda}{d(x_k,r)} \right) \log_{x_k}(r) \right].
\end{aligned}
\] (54)

Even in this case, it may be observed the emergence of an effective damping coefficient \( \mu^* := h \mu \) and of effective potential coefficients \( \rho^* := h \rho \) and \( \lambda^* := h \lambda \) which
Figure 14. Behavior of Van der Pol oscillator (53) in the presence of non-linear damping. The left-hand panel shows the trajectory in the space $S^2$, when the starting point is $x_0 = [0 0 1]^T$, the reference point for the oscillator is $r = [1 0 0]^T$ (denoted by a blue open circle) and the initial speed $v_0 = [0.5 -0.9 0]^T$. The values of the parameters used in this simulation are $\kappa = 0.5$, $\mu = 0.5$, $\epsilon = 1.3$ and $h = 0.002$.

depend on the discretization stepsize. Notice that throwing out the expression of the logarithmic map and cancelling out an instance of distance, one gets the simplified expression:

$$
\begin{align*}
    x_{k+1} &= x_k \cos(h\|v_k\|) + v_k \sin(h\|v_k\|)/\|v_k\|, \\
    v_{k+1} &= P_{x_k \rightarrow x_{k+1}} \left[ v_k - h\mu(v_k^T v_k)^{-1}v_k + h \left( \lambda + \frac{\rho}{d^2(x_k, r)} \right) \frac{r-x_k(x_k^T r)}{\sin d(x_k, r)} \right].
\end{align*}$$

The Figure 15 illustrates the behavior of a Kepler oscillator on the ordinary sphere $S^2$ in the absence of any damping effect. In particular the total energy $H$, sum of the potential energy and of the kinetic energy, should keep constant, while it changes slightly due to the accumulation of numerical errors. The potential energy is proportional to the distance and, as it can be observed, always oscillates between a minimum and a maximum value that remain constant. The Figure 16 illustrates the behavior of a Kepler-like oscillator in the presence of a damping term. The energy curves illustrated in Figure 16 show a phenomenon that never occurred in the previous numerical simulations. In fact, a continuous-time Kepler oscillator is expected to drive the state-variable $x$ towards the reference point $r$, hence causing the value of the potential $V^{(kep)}$ to decrease indefinitely towards $-\infty$. In the numerical simulations performed by an Euler method, however, such expected behavior is not observed and the state-variable seems to stabilize in a circular orbit of radius $d(x, r) \approx 0.14$ around the reference point. We recognized this phenomenon as a purely numerical artifact that tends to disappear as the value of the stepsize $h$ decreases. For instance, taking $h = 0.00002$, the value of the radius decreases to approximately 0.01.

It can be seen that both in the damped and undamped case, if the distance is less than the critical distance, then the potential energy takes negative values, while if the distance is larger than the critical distance, then the potential energy takes positive values.
Figure 15. Behavior of the Kepler oscillator (54) in the absence of non-linear damping (namely, μ = 0). The left-hand panel shows the trajectory in the space $S^2$, when the starting point is $x_0 = [0 \ 0 \ 1]^\top$, the reference point for the oscillator is $r = [1 \ 0 \ 0]^\top$ (denoted by a blue open circle) and the initial speed $v_0 = [0 \ -0.3 \ 0]^\top$. The values of the parameters used in this simulation are $\lambda = 0.5$, $\rho = 0.08$ and $h = 0.001$ (hence, the critical distance is $d_c = 0.4$).

Figure 16. Behavior of the Kepler oscillator (54) in the presence of non-linear damping. The left-hand panel shows the trajectory in the space $S^2$, when the starting point is $x_0 = [0 \ 0 \ 1]^\top$, the reference point for the oscillator is $r = [1 \ 0 \ 0]^\top$ (denoted by a blue open circle) and the initial speed $v_0 = [0.5 \ -0.9 \ 0]^\top$. The values of the parameters used in this simulation are $\rho = 0.08$, $\mu = 0.1$, $\epsilon = 2$ and $\lambda = 0.5$ and $h = 0.002$ (hence, the critical distance is $d_c = 0.4$).

6. **Numerical simulation on the special orthogonal group** $SO(3)$. For the special orthogonal group $SO(3)$ endowed with the canonical metric $\langle W, V \rangle_x := \text{tr}(W^\top V)$, it holds that:

$$
\begin{align*}
\text{d}^2(X, Y) & = -\text{tr}[\text{Log}^2(X^\top Y)], \\
\exp_X(V) & = X\text{Exp}(X^\top V), \\
\log_X(Y) & = X\text{Log}(X^\top Y),
\end{align*}
$$

(56)
where \(X, Y \in \text{SO}(3), V \in T_X \text{SO}(3) := \{X\Omega | \Omega \in \mathfrak{so}(3)\}\). Moreover, the symbols \(\text{Exp}\) and \(\log\) denote matrix exponential and principal matrix logarithm, respectively. The matrix logarithm was implemented using a customized numerical recipe, based on a Schur decomposition, explained in [43]. The kinetic energy for the dynamical systems in this section takes the expression \(K = -\frac{1}{2} \text{tr}[\Omega^2]\), where \(\Omega \in \mathfrak{so}(3)\) denotes the system’s angular speed matrix.

A discrete-time version of the dynamical system (14) reads:

\[
\begin{align*}
X_{k+1} &= X_k \text{Exp}(h\Omega_k), \\
\Omega_{k+1} &= (1 - h\mu\|\Omega_k\|^2(\epsilon - 1))\Omega_k - hX_k^\top \text{grad}_{X_k} \mathcal{Y},
\end{align*}
\] (57)

with \(h > 0\) being a discretization stepsize for the dynamical system and \(\epsilon > 0\) being the exponential quantifying the degree of nonlinear damping. The system state is represented by the pair \((X_k, \Omega_k) \in \text{SO}(3) \times \mathfrak{so}(3)\) for \(k \in \mathbb{N}\). The first equation of the discrete-time dynamical system (57) represents a geodesic-based Euler-like step-forward numerical approximation of the flow associated with the first differential equation on the tangent bundle \(T\text{SO}(3)\). The second equation represents a direct Euler-like step-forward method that takes place on a linear space, namely, the Lie algebra \(\mathfrak{so}(3)\).

6.1. Visual rendering via a group action on a manifold. In order to achieve visual rendering of the \(\text{SO}(3)\)-trajectories generated by non-linear oscillators, we shall be exploiting the notion of group action on a manifold.

Given a group \(G\) and a smooth manifold \(M\), a group action \(\Lambda : G \times M \to M\) is denoted as \(y = \Lambda_g(x)\), where \(g \in G\) and \(x, y \in M\) [8]. It is worth recalling that an orbit of a point \(x \in M\) under a group action \(\Lambda\) is defined as \(G^\Lambda \cdot x := \{\Lambda_g(x) \mid g \in G\}\) [8]. In the case of interest in the present paper, a suitable group action is invoked in order to provide a 3-dimensional graphical rendering of trajectories generated in the Lie group of 3-dimensional rotations, which are represented by \(3 \times 3\) matrices. Therefore, we take \(G = \text{SO}(3)\) and \(M = \mathbb{S}^2\) and define \(\Lambda_X(p) := Xp\), with \(X \in \text{SO}(3)\) and \(p \in \mathbb{S}^2\). Clearly, a trajectory will be represented as a subset of \(\text{SO}(3)^\Lambda \cdot p \cong \mathbb{S}^2\).

In other words, in the numerical simulations, the trajectories generated by the oscillators will be represented in the hyper-sphere \(\mathbb{S}^2\). In particular, taken an arbitrary ‘seed’ on the hyper-sphere, it is multiplied by the initial rotation matrix, belonging to \(\text{SO}(3)\). Also the reference point is obtained multiplying the starting point by the reference rotation matrix.

Within the present section, in all figures the right-hand side panel shows the values taken by the kinetic energy, the potential energy and the total energy over the generated trajectory and the distance between the state and the reference. The red dashed lines represent the values of the shown quantities at the beginning of the numerical simulation.

6.2. Numerical simulation on the simple pendulum. In this instance of a dynamical system on the manifold \(\text{SO}(3)\), the potential energy function takes the expression \(\mathcal{Y}^(\text{pen}) := \kappa \left(1 - \cos \left(\sqrt{-\text{tr} \left[\text{Log}^2(X^\top R)\right]}\right)\right)\), where \(X \in \text{SO}(3)\) denotes the system’s orientation state matrix and \(R \in \text{SO}(3)\) denotes a reference point for the system’s oscillations.
The time-discretized version of the dynamical system (31) on the manifold of special orthogonal matrices, associated to the simple pendulum, reads:

\[
\begin{align*}
X_{k+1} &= X_k \text{Exp}(h\Omega_k), \\
\Omega_{k+1} &= (1 + h\mu \text{tr}[\Omega_k^2]^{-1})\Omega_k + h\kappa \frac{\sin(d(X_k, R))}{d(X_k, R)} \text{Log}(X_k^\top R),
\end{align*}
\]  
(58)

with \( h > 0 \) being a discretization stepsize for the dynamical system, \( k = 0, 1, 2, \ldots \), \( \mu, \epsilon > 0 \) being coefficients of the nonlinear damping term and \( \kappa > 0 \) being a coefficient for the potential.

The Figure 17 illustrates the behavior of a simple pendulum on the special orthogonal group \( \text{SO}(3) \) acting on the sphere \( \mathbb{S}^2 \) embedded in \( \mathbb{R}^3 \) in the absence of a damping term. Notice again how the total energy \( H \) is supposed to be preserved, although it changes slightly due to approximation errors introduced by the employed numerical method.

![Figure 17. Behavior of the classic pendulum oscillator (58) on the special orthogonal group SO(3), in the absence of non-linear damping (namely, \( \mu = 0 \)). The left-hand side panel shows the trajectory in the space \( \mathbb{S}^2 \), when the starting point and also the reference point are taken randomly, (denoted by a blue open circle) and the initial speed will be random, because related to the initial state. The values of the parameters used in this simulation are \( \kappa = 0.5 \) and \( h = 0.002 \).](image)

The Figure 18 illustrates the behavior of a single pendulum oscillator on the special orthogonal group \( \text{SO}(3) \) in the presence of a damping term. In this case, the total energy of the system tends asymptotically to zero due to damping.

6.3. **Numerical simulation on a hard Duffing oscillator.** In this instance of a dynamical system on the Lie group \( \text{SO}(3) \), the potential energy function takes the expression

\[
\mathcal{V}^{(\text{duff})} := \frac{1}{2} \text{tr} \left[ \text{Log}^2(X^\top R) \right] \pm \frac{1}{4} \kappa \text{tr} \left[ \text{Log}^2(X^\top R) \right]^2,
\]

where \( X \in \text{SO}(3) \) denotes the system’s orientation state matrix and \( R \in \text{SO}(3) \) denotes a reference orientation matrix for the system’s oscillations. Since the potential is a quartic form in the Riemannian distance (but not in the system’s state), the Duffing potential might be referred to as *quartic*. 

![Figure 18. Behavior of a single pendulum oscillator on the special orthogonal group SO(3) in the presence of a damping term.](image)
Figure 18. Behavior of the simple pendulum \((58)\) in the presence of non-linear damping. The left-hand panel shows the trajectory on the special orthogonal group \(SO(3)\), when the starting point and also the reference point are taken randomly (denoted by a blue open circle) the initial speed will be random, because related to the initial state. The values of the parameters used in this simulation are \(\kappa = 0.5, \mu = 0.3, \epsilon = 1.3\) and \(h = 0.002\).

The time-discretized version of the dynamical system \((38)\) on the manifold of special orthogonal matrices, associated to the hard Duffing oscillators, reads:

\[
\begin{align*}
X_{k+1} &= X_k \exp(h \Omega_k), \\
\Omega_{k+1} &= (1 + h \mu \text{tr} [\Omega_k^2]^{-1}) \Omega_k + h \left(1 + \kappa d^2(X_k, R)\right) \log(X_k^T R).
\end{align*}
\]  
\(59\)

The Figure 19 illustrates the behavior of a hard Duffing oscillator on the special orthogonal group \(SO(3)\) in the absence of a damping term via a Lie-group action-based rendering technique. Likewise, Figure 20 illustrates the behavior of a

Figure 19. Behavior of a hard Duffing oscillator \((59)\) on the special orthogonal group \(SO(3)\), in the absence of non-linear damping (namely, \(\mu = 0\)). The left-hand side panel shows the trajectory on the special orthogonal group \(SO(3)\), when the starting point and also the reference point are taken randomly, (denoted by a blue open circle) and the initial speed will be random, because related to the initial state. The values of the parameters used in this simulation are \(\kappa = 0.5\) and \(h = 0.0005\).
hard Duffing oscillator on the special orthogonal group SO(3) in the presence of a damping term.

Figure 20. Behavior of a hard Duffing oscillator (59) in the presence of non-linear damping. The left-hand panel shows the trajectory in the special orthogonal group SO(3), when the starting point and also the reference point are taken randomly (denoted by a blue open circle) the initial speed will be random, because related to the initial state. The values of the parameters used in this simulation are $\kappa = 0.5$, $\mu = 0.5$, $\epsilon = 1.3$ and $h = 0.0008$.

6.4. Numerical simulation on the Van der Pol oscillator. In this instance of a dynamical system on the manifold SO(3), the potential energy function takes the expression $\mathcal{V}_{\text{pol}} := -\frac{1}{2} \kappa \text{tr}[\log^2(X^\top R)]$, where $X \in \text{SO}(3)$ denotes the system’s orientation state matrix and $R \in \text{SO}(3)$ denotes a reference orientation matrix for the oscillations of this system. Since this potential is a quadratic form in the Riemannian distance (but not in the system’s state), this potential function potential might be referred to as quadratic. A quadratic function on a Euclidean space would give rise to a linear gradient (hence a linear forcing term, like a Hooke’s linear spring force). As opposed to this, a quadratic potential on a curved manifold (based on a squared distance function) gives rise to a non-linear forcing term (which is linear in the tangent log-vector).

The time-discretized version of the dynamical system (34) on the manifold of special orthogonal matrices, associated to the Van der Pol oscillator, reads:

$$
\begin{cases}
X_{k+1} = X_k \text{Exp}(h\Omega_k), \\
\Omega_{k+1} = (1 + h\mu \text{tr}[\Omega_k^2]^{-1})\Omega_k + h\kappa \log(X_k^\top R).
\end{cases}
$$

The Figure 21 illustrates the behavior of a Van der Pol oscillator on the special orthogonal group SO(3) in the absence of a damping term.

The Figure 22 illustrates the behavior of a Van der Pol oscillator on the special orthogonal group SO(3) in the presence of a damping term.

6.5. Numerical simulation on the Kepler oscillator. In the present instance of a dynamical system on the manifold SO(3), the potential energy function takes the expression $\mathcal{V}_{\text{kep}} := -\rho/d(X,R) + \lambda d(X,R)$, where $X \in \text{SO}(3)$ denotes the system’s orientation state matrix and $R \in \text{SO}(3)$ denotes a reference for the oscillations of this system.
Figure 21. Behavior of the Van der Pol oscillator (60) on the special orthogonal group $SO(3)$, in the absence of non-linear damping (namely, $\mu = 0$). The left-hand side panel shows the trajectory in the space $S^2$, when the initial point and also the reference point are taken randomly, (denoted by a blue open circle) and the initial speed will be random, because related to the initial state. The values of the parameters used in this simulation are $\kappa = 1$ and $h = 0.0005$.

Figure 22. Behavior of the Van der Pol (60) in the presence of non-linear damping. The left-hand panel shows the trajectory in the special orthogonal group $SO(3)$, when the initial point and also the reference point are taken randomly (denoted by a blue open circle) the initial speed will be random, because related to the initial state. The values of the parameters used in this simulation are $\kappa = 1$, $\mu = 0.5$, $\epsilon = 3$ and $h = 0.001$.

A time-discretized version of the dynamical system (43) on the group-manifold of special orthogonal matrices, associated to the Kepler oscillator, reads:

$$
\begin{aligned}
X_{k+1} &= X_k \text{Exp}(h\Omega_k), \\
\Omega_{k+1} &= (1 + h\mu \text{tr}([\Omega_k^2]^{-1}))\Omega_k + h(\mathbf{e} + \frac{\lambda}{d(X_k, R)}) \text{Log}(X_k^\top R).
\end{aligned}
$$

The Figure 23 illustrates the behavior of a Kepler oscillator on the special orthogonal group $SO(3)$ in the absence of a damping term. Likewise, Figure 24 illustrates the behavior of a Kepler oscillator on the special orthogonal group $SO(3)$ in the
Figure 23. Behavior of the Kepler oscillator (61) on the special orthogonal group SO(3), in the absence of non-linear damping (namely, \( \mu = 0 \)). The left-hand side panel shows the trajectory in the space \( S^2 \), when the starting point and also the reference point are taken randomly, (denoted by a blue open circle) and the initial speed will be random, because related to the initial state. The values of the parameters used in this simulation are \( \lambda = 0.2 \), \( \rho = 0.8 \) and \( h = 0.0003 \).

presence of a damping term. It is interesting to notice that the kinetic energy

Figure 24. Behavior of the Kepler oscillator (61) in the presence of non-linear damping. The left-hand panel shows the trajectory on the special orthogonal group SO(3), when the starting point and also the reference point are taken randomly (denoted by a blue open circle) the initial speed will be random, because related to the initial state. The values of the parameters used in this simulation are \( \lambda = 0.5 \), \( \mu = 0.3 \), \( \rho = 0.1 \), \( \epsilon = 1.5 \) and \( h = 0.003 \).

does not tend to zero, which indicates that this dynamical system keeps oscillating. From observations made after equation (39), let us recall that if \( d(x, r) < d_c \) (where \( d_c \approx 0.45 \)) the negative component becomes larger than the positive component of the potential \( V^{(\text{kep})} \), hence the potential energy \( V^{(\text{kep})} \leq 0 \), thus its value decreases. Therefore, the distance vanishes to zero, namely the reference point is reached, even though the kinetic energy and the total energy do not vanish to zero.
7. **Joint numerical simulation on the unit hypersphere and on the special orthogonal group.** This simulation shows the behavior of a rigid body (a ‘cube’) whose center of mass lays on the sphere $S^2$ and whose position changes according to a trajectory generated by a $S^2$-oscillator, while its instantaneous orientation changes according to a rotation sequence generated by a $SO(3)$-oscillator. The combination of the two motions will result in a body translating and rotating on the unit sphere according to a pre-defined oscillatory rule.

In this numerical simulation, a simple pendulum oscillator both for the position and for the orientation was chosen.

The Figure 25 illustrates the behavior of a simple pendulum that oscillates on the hyper-sphere $S^2$ and on the special orthogonal group $SO(3)$. The two oscillators are not damped, therefore they will oscillate around the reference point on the hypersphere and around their reference inclination indefinitely. The parameters used in the simulation are $\kappa = 0.5$ and $h = 0.001$. Moreover in $S^2$ the initial position point is $x_0 = [0 \ 0 \ 1]^\top$, the velocity is $v_0 = [0.5 \ -0.9 \ 0]^\top$ and the reference point is $r = [1 \ 0 \ 0]^\top$. In $SO(3)$, the initial rotation matrix is $X_0 = I_3$, the starting
angular velocity matrix is \( \Omega_0 = \begin{bmatrix} 0 & -0.5 & -1 \\ 0.5 & 0 & 0.7 \\ 1 & -0.7 & 0 \end{bmatrix} \) and the reference inclination matrix was chosen as \( R = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \).

Instead, Figure 26 illustrates a motion in a damped case. The parameters used in the simulation are the same as in the previous simulation, except that \( \mu = 0.5 \) and \( \epsilon = 1.3 \). The two oscillators are damped, therefore starting from their initial position and rotation (a) they will oscillate around the reference point and their reference inclination (d), until the references are reached.

8. Conclusion. The present document outlined the mathematical structure of a general second-order dynamical system on manifolds and on Lie groups expressed through the language of manifold calculus and Lie-group theory.

The studied dynamical systems are governed by a smooth vector field that describes the infinitesimal motion of a point under its flow. Compared to the familiar
case where $M = \mathbb{R}^n$, the manifold setting requires more conceptual work to properly define what is meant by a vector field at a point. In the latter case, the dynamics takes place in the tangent bundle, while when $M = \mathbb{R}^n$ any tangent plane at a point is isomorphic to the manifold itself and may be confused with it. A number of instances of the general dynamical systems were presented and discussed, leading to defining a number of abstract non-linear oscillators on manifolds and Lie groups.

A large portion of this document was devoted to developing numerical methods to implement these systems on a computing platform and in testing these methods through numerical simulations. Most simulations were carried out by means of an extension of the forward Euler method, although more accurate, second-order methods, such as the Heun scheme and the Runge (RK2) scheme were also presented and utilized to illustrate the benefits of higher-order numerical integration schemes. On account of observations made in Section 3.2 about the numerical precision and the computational burden of the presented numerical methods, we believe that Runge’s method appears as the most appealing numerical scheme, as it conjugates good precision with limited computational demand.

The numerical behavior of the studied systems was illustrated through a great deal of numerical examples and simulations, carried out on the sphere $S^2$ and on the rotation group $SO(3)$. The numerical results revealed the richness and beauty of the developed non-linear oscillator theory on smooth manifolds.

An interesting observation to be developed in a future research endeavor is that several classical oscillators (such as the Duffing oscillator) as well as newly developed oscillatory models (as the multi-scroll oscillators analyzed in [54]) rely on polynomial potentials, which may be readily extended to manifolds as $V := \sum_{p=0}^{P} c_p d^p(x, r)$.

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