Isotropic submanifolds and the inverse problem for mechanical constrained systems

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Abstract
We give a new characterization of the inverse problem of the calculus of variations that is easily extended to constrained systems, both in the autonomous and non-autonomous cases. The transition from unconstrained to constrained systems is given by passing from Lagrangian submanifolds to isotropic ones. If the constrained system is variational we use symplectic techniques to extend these isotropic submanifolds to Lagrangian ones and describe the solutions of the constrained system as solutions of a variational problem without constraints. Mechanical examples such as the rolling disk are provided to illustrate the main results.

Keywords: inverse problem, Lagrangian and isotropic submanifolds, constrained variational calculus, nonholonomic systems, hamiltonization

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1. Introduction

The inverse problem of the calculus of variations consists in determining when a given system of second order ordinary differential equations (SODE)

\[ \ddot{q}^i = \Gamma^i \left( q, \dot{q}^j, \dot{q}^j \right), \quad i, j = 1, \ldots, n \]
is equivalent to Euler–Lagrange equations

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0, \]

for a regular Lagrangian \( L \) to be determined. To prove the equivalence of these two systems is the same as to find a non-singular matrix \( (g_{ij}(t, q, \dot{q})) \) such that the following system is satisfied

\[ g_{ij}(\dot{q}^i - \dot{r}^i) = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i}. \]

When such a matrix exists, the system of second order ordinary differential equations is called variational. In 1886 Sonin [38] proved that a single second order ordinary differential equation is always variational. This problem was also studied in 1887 by Helmholtz [45] for general systems of second order ordinary differential equations in implicit form.

History has shown this is an extremely difficult problem because only the full solution for at most two dimensional systems of second order ordinary differential equations is known [19]. Douglas’ solution consists of an exhaustive classification in different cases using Riquier–Janet theory. Variational and nonvariational SODEs are included in his work. The techniques used by Douglas turned out to be very difficult to generalize to higher dimension.

Since 1980, the inverse problem has been considered by many authors [12, 25, 33, 36, 39] giving a geometric interpretation of Douglas’ classification and generalizing some of the results to higher dimensions. In particular, a free coordinate characterization of the inverse problem is given in [16]. As a result, it has been proved that cases I and IIa1 in Douglas [19] are always variational for arbitrary dimension ([37] and [14], respectively). Case I was also studied in [2] and [23] using different approaches. Other extensions of the inverse problem include partial differential equation [3], field theory [26], nonholonomic mechanics [35], driven SODEs [27], jet bundles [28], etc.

In our paper we will follow a symplectic approach working with Lagrangian submanifolds of symplectic manifolds [46] associated to the geometry of the tangent bundle, which is the space where a SODE is geometrically defined. In terms of the closedness of a suitable one-form, constructed from the given SODE and a transformation between the tangent bundle and its dual, the cotangent bundle, we provide a new characterization of a variational second order differential equation. The use of other distinguished submanifolds of symplectic manifolds, isotropic submanifolds, turns out to be suitable to characterize the inverse problem for constrained systems. Moreover, using a standard construction in symplectic geometry we can extend these isotropic submanifolds to Lagrangian ones, allowing us to describe the constrained solutions as solutions of a variational problem now without constraints such that the solutions of the new variational problem with initial conditions verifying the constraints are precisely real solutions of the original constrained system. Our techniques are also related to classical results about the comparison of solutions of nonholonomic systems and constrained variational problems (see [5, 10, 20] and references therein).

The paper is organized as follows: section 2 contains examples of Lagrangian submanifolds and the geometric definition of SODE (see [24, 29, 40, 41] for full background). In section 3 we introduce some relevant examples of constrained Lagrangian systems: nonholonomic systems and constrained variational systems. In section 4 we briefly describe the inverse problem of the calculus of variations from the geometric approach given in [12]. Then the new geometric characterization of the inverse problem is introduced: a system of second
order differential equations on a manifold $Q$ is variational if it can be associated to a Lagrangian submanifold of the symplectic manifold $(T^*\!\! Q, \omega_{TQ})$, where $\omega_{TQ}$ is the natural symplectic structure of $T^*\!\! Q$. We relate our results to the so-called Chaplygin hamiltonization for a special type of nonholonomic system [6]. The novel and key results appear in sections 5 and 6. The problem for constrained systems, in particular, the nonholonomic mechanics, is described in section 5 by linking the notion of being variational to isotropic submanifolds of $(T^*\!\! Q, \omega_{TQ})$. Applying standard techniques in symplectic geometry we can extend (not uniquely) this isotropic submanifold to a Lagrangian one, in particular, to Lagrangian submanifolds which are described by a Lagrangian function defined on $TQ$. Assuming the regularity of this Lagrangian, we can always find an associated Hamiltonian by means of the Legendre transformation and the dynamics is described by a Hamiltonian system without constraints. This implies strong qualitative restrictions on the behavior of the solutions of the original constrained SODE (see remark 5.10). The rolling disk is considered as an example and regular and singular Lagrangians associated to it are given. Section 6 focuses on holonomic dynamics where the system evolves on a submanifold $TN$ of $TQ$. In some cases it is easier to study the problem in the manifold with greater dimension, instead of working on $TN$ as if there were no constraints. In section 7 it is shown how to adapt the previous description to the non-autonomous case by using the notion of Lagrangian and isotropic submanifolds in Poisson geometry. Finally, some future research lines are discussed.

2. Geometric preliminaries

The basic definitions and results from differential geometry, in particular symplectic geometry, that are necessary in the sequel can be found in [1] and [29]. In this section we briefly introduce the objects that will be most relevant.

2.1. Isotropic and Lagrangian submanifolds

Lagrangian and isotropic submanifolds play an important role in the present work. We list some results that provide examples of Lagrangian submanifolds that will appear in the following sections.

**Proposition 2.1** ([29]). Let $\gamma$ be a one-form on $Q$ and $L = \text{Im} \gamma \subset T^*Q$. The submanifold $L$ of $T^*Q$ is Lagrangian if and only if $\gamma$ is closed.

**Proposition 2.2** ([24]). Let: $N \rightarrow TQ$ be an immersion. For each Lagrangian submanifold $S \subset T^*N$ we can define a Lagrangian submanifold $\hat{S} \subset T^*TQ$ by $\hat{S} = \{ \mu \in T^*TQ: \iota^*\mu \in S \}$.

In the above proposition, if $N$ is a submanifold and $S = \text{Im} (df)$ for some $f: N \rightarrow \mathbb{R}$, then we recover the following result by Tulczyjew:

**Proposition 2.3** ([40, 41]). Let $Q$ be a smooth manifold, $\tau_Q: TQ \rightarrow Q$ its tangent bundle projection, $\pi_Q: T^*Q \rightarrow Q$ the cotangent bundle projection, $N \subset Q$ a submanifold, and $f: N \rightarrow \mathbb{R}$ a smooth map. Then
is a Lagrangian submanifold of $T^*Q$.

**Example 2.4.** Let $(M, \omega)$ be a symplectic manifold. Given a function $H: M \to \mathbb{R}$, and its associated Hamiltonian vector field $X_H$, that is, $i_{X_H}\omega = dH$, then the image of $X_H$ is a Lagrangian submanifold of $(TM, d_T\omega)$, where $d_T\omega$ denotes the tangent lift of $\omega$ to $TM$, which is also a symplectic structure.

Now we introduce a way to extend isotropic submanifolds to Lagrangian ones. This construction can be found in [43] and will be very useful in section 5. Assume we have a submanifold $N$ of a symplectic manifold $(M, \omega)$ such that for a neighborhood $U_p$ of a point $p$ in $M$ we can write

$$\bigcap_{k=1}^{\dim(N_0)} \phi_\tau(p, N_0) = \emptyset.$$  

If we have an isotropic submanifold $N_0 \subset N$ with $p \in N_0$, $\dim(N_0) = \frac{\dim(N) - k}{2}$ and the Hamiltonian vector fields $X_{\phi_1}, \ldots, X_{\phi_k}$ of $\phi_1, \ldots, \phi_k$ satisfy that

- $\exists \epsilon > 0$ such that the flows of $X_{\phi_1}$ are defined for all $|t| < \epsilon$,
- $X_{\phi_i}(p) \not\in T_pN_0$, for all $i = 1, \ldots, k$ and $p \in N_0$,
- $X_{\phi_i}(p)$ are linearly independent for all $p \in N_0$,

then we can extend it to a Lagrangian submanifold transporting $N_0$ along the flows of the Hamiltonian vector fields $X_{\phi_1}, \ldots, X_{\phi_k}$.

We will illustrate the construction for the case $k = 1$ and rename $\phi_1$ by $\phi$. Since $X_{\phi}$ is transverse to $N_0$, there exists an open interval $I$ about 0 in $\mathbb{R}$ such that $\exp(tX_{\phi}(\tilde{p}))$ is defined for all $t \in I$ and $\tilde{p} \in N_0 \cap U_p$. Therefore the map

$$j: N_0 \times I \to M,$$

$$(p, t) \mapsto \exp(tX_{\phi}(\tilde{p}))$$

allows us to realize locally $N_0 \times I$ as a submanifold $Z$ of $M$ whose tangent space is

$$T_{(p, t)}(N_0 \times I) \subset (\exp(tX_{\phi}(\tilde{p})))_{*}(T_pN_0) \oplus \text{span} \left\{ X_{\phi}\left( \exp(tX_{\phi}(\tilde{p})) \right) \right\},$$

where $(\exp(tX_{\phi}))_{*}$ is the pushforward of $\exp(tX_{\phi})$. Obviously $\dim(Z) = \dim(N_0) + 1$ and $Z$ is also isotropic because, first, for any two vectors in $(\exp(tX_{\phi}))_{*}(T_pN_0)$ we have that

$$\omega\left( \left( \exp(tX_{\phi}) \right)_{*} v_1, \left( \exp(tX_{\phi}) \right)_{*} v_2 \right) = \left( \left( \exp(tX_{\phi}) \right)_{*} \omega \right)(v_1, v_2) = \omega(v_1, v_2) = 0$$

since $\exp(tX_{\phi})$ is a symplectomorphism and $v_1, v_2 \in T_pN_0$.

Second, it must be checked that the two-form $\omega$ also vanishes for a vector in $(\exp(tX_{\phi}))_{*}(T_pN_0)$ and one in $X_{\phi}(\exp(tX_{\phi}(\tilde{p})))$. Note that

$$\omega\left( \left( \exp(tX_{\phi}) \right)_{*} v, X_{\phi}\left( \exp(tX_{\phi}(\tilde{p})) \right) \right) = d\phi(\tilde{p})(v) = 0,$$

because $\phi$ vanishes on $N_0$ and $v \in T_pN_0$.  

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2.2. Second order differential equations

Consider the tangent bundle $\tau: TQ \rightarrow Q, (q^i, \dot{q}^i)$ canonical coordinates on $TQ$ and $(q^i)$ on $Q$. In $TQ$ we can define the following geometric objects: the Liouville or dilation vector field $\Delta \in \mathfrak{X}(TQ)$ and a type $(1, 1)$ tensor field $S$ called the vertical endomorphism (see [18]). In canonical coordinates

$$\Delta = \dot{q}^i \frac{\partial}{\partial \dot{q}^i} \quad \text{and} \quad S = dq^i \otimes \frac{\partial}{\partial q^i}.$$  

A SODE (second order differential equation) $\Gamma$ is a vector field on $TQ$ satisfying $S(\Gamma) = \Delta$. In coordinates

$$\Gamma = \dot{q}^i \frac{\partial}{\partial \dot{q}^i} + \Gamma^i(q, \dot{q}) \frac{\partial}{\partial q^i}.$$  

The solutions of the SODE $\Gamma$ are precisely the solutions of the system of second order differential equations $\ddot{q}^i = \Gamma^i(q, \dot{q})$. As shown in the following section, SODEs are key elements to describe intrinsically Lagrangian mechanics.

3. Lagrangian mechanics

The Euler–Lagrange equations arise from Hamilton’s principle using standard arguments from variational calculus. Here we recall how to derive intrinsically the Euler–Lagrange equations using the geometry of the tangent bundle. Given $L: TQ \rightarrow \mathbb{R}$ we define the Poincaré–Cartan one-form $\Theta_L = S^*(dL)$, the associated Poincaré–Cartan two-form $\Omega_L = -d\Theta_L$ and the energy function $E_L: TQ \rightarrow \mathbb{R}$ by $E_L = \Delta(L) - L$. Locally

$$\Theta_L = \frac{\partial L}{\partial \dot{q}^i} dq^i \quad \text{and} \quad E_L = \dot{q}^i \frac{\partial L}{\partial \dot{q}^i} - L.$$  

When the Lagrangian $L$ is regular, that is, $\Omega_L$ is a symplectic two-form, or locally when the $n \times n$-Hessian matrix $(\partial^2 L/\partial \dot{q}^i \partial \dot{q}^j)$ is regular, then there exists a unique SODE $\Gamma_L$ solution of the equation

$$i_{\Gamma_L} \Omega_L = dE_L, \quad (1)$$

or alternatively

$$\mathcal{L}_{\Gamma_L} \Theta_L = dL, \quad (2)$$

where $\mathcal{L}_{\Gamma_L} \Theta_L$ is the Lie derivative of $\Theta_L$ along $\Gamma_L$. The integral curves of $\Gamma_L$ are precisely the solutions to the Euler–Lagrange equations for $L$.

3.1. Constrained Lagrangian mechanics: nonholonomic systems

In this section we will see one of the main examples where second order differential equations along submanifolds arise: the case of nonholonomic Lagrangian systems. To do so, we introduce constraints to a given Lagrangian system $L: TQ \rightarrow \mathbb{R}$.

Definition 3.1. A nonholonomic Lagrangian system on a manifold $Q$ consists of a pair $(L, M)$ where $L: TQ \rightarrow \mathbb{R}$ is a Lagrangian function and $M$ is a submanifold of $TQ$. 

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Let $\tau_0: TQ \to Q$ be the canonical projection. In the sequel we will assume that $\tau_0(M) = Q$ avoiding the existence of holonomic constraints (see section 6 for more details). In mechanical and real examples, $M$ is typically a vector subbundle $D$ of $\tau_0$, that is, the constraints are linear on velocities; in other examples, $M$ is an affine subbundle modelled on a vector bundle $D$. From now on, we assume that $M = D$ is a vector subbundle or an affine subbundle modelled on $D$.

The existence of the constraints prescribed by $M$ induces the introduction of reaction forces which restrict the motion to $M$. These forces are determined by the Lagrange–d’Alembert principle.

Define the set of admissible curves by

$$C^2_M(q_0, q_1, [a, b]) = \left\{ c: [a, b] \subset \mathbb{R} \to Q \bigg| c \in C^2, \right.$$

$$c(a) = q_0, \quad c(b) = q_1, \quad c'(t) \in M_{c(t)} \quad \forall t \in [a, b] \right\},$$

and the set of possible virtual variations along $c$ by

$$V_c = \left\{ X: [a, b] \to TQ \bigg| X \in C^1, X(t) \in D_{c(t)} \quad \forall t \in [a, b] \text{ and } X(a) = X(b) = 0 \right\}.$$

**Definition 3.2. (Lagrange–d’Alembert’s principle)** Let $c \in C^2_M(q_0, q_1, [a, b])$, then $c$ is a solution of the nonholonomic Lagrangian system $(L, M)$ if

$$\langle dJ(c), X \rangle = 0 \text{ for all } X \in V_c, \quad \text{where } J(c) = \int_a^b L(c'(t)) \, dt.$$

Locally, if the submanifold $M$ is determined by the vanishing of constraints $\phi^\alpha(q', q^\dot) = 0$ (either linear or affine constraints), then the equations of motion of a nonholonomic Lagrangian system are:

$$\frac{d}{dt} \left( \frac{\partial L}{\partial q^\dot} \right) - \frac{\partial L}{\partial q} = \lambda^\alpha \frac{\partial \phi^\alpha}{\partial q^\dot},$$

$$\phi^\alpha(q', q^\dot) = 0. \quad (3)$$

If the constraints are written as $\phi^\alpha(q', q^\dot) = \mu^\alpha_i(q)q^i + \mu^\alpha_0(q)$, then the previous equations reduce to:

$$\frac{d}{dt} \left( \frac{\partial L}{\partial q^\dot} \right) - \frac{\partial L}{\partial q} = \lambda^\alpha \mu^\alpha_i(q),$$

$$\mu^\alpha_i(q)q^i + \mu^\alpha_0(q) = 0.$$

If the Hessian matrix $W$ of $L$ with respect to the velocities is definite, then the matrix

$$C = \left( C^{\alpha\beta} \right) \text{ with } C^{\alpha\beta} = \mu^\alpha_i W^{ij} \mu^\beta_j$$

is regular, where $(W^{ij})$ is the inverse of the Hessian matrix $W_{ij} = \left( \frac{\partial^2 L}{\partial q^i \partial q^j} \right)$ and analogously $(C^{\alpha\beta})$ is the inverse of $C$.

Observe that the definiteness condition is automatically satisfied in mechanics when $L = T - V$, $T$ being the kinetic energy associated to a Riemannian metric on $Q$ and $V$ being...
the potential energy. It is easy to show that under this condition, we can write the equations of motion of a nonholonomic system as a system of explicit second order differential equations on the constraint submanifold $M$. In fact, the Lagrange multipliers are determined univocally as

$$\lambda_\alpha(q, q) = -C_{\alpha\beta} \left( \frac{\partial \mu_\beta}{\partial q^i} \dot{q}^i \dot{q}^j + \frac{\partial \mu_\beta}{\partial \dot{q}^i} \ddot{q}^i + \mu_\beta \frac{\partial L}{\partial \dot{q}^i} - \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j \partial q^k} \ddot{q}^i \right),$$

and given an initial condition on $M, \dot{c}(0) \in M_{c(0)}$, the unique solution of the second order differential equation

$$\dot{q}^i = W^i \left( \lambda_\alpha(q, q) \mu_\beta(q) + \frac{\partial L}{\partial \dot{q}^i} - \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \ddot{q}^i \right)$$

evolves on the constraint submanifold $M$, that is, $\dot{c}(t) \in M_{c(t)}$.

3.1.1. Nonholonomic Chaplygin systems. Chaplygin systems are a particular type of nonholonomic systems with symmetry (see, for instance, [7]). The configuration space is a principal $G$-bundle $\pi: Q \to Q/G$, associated with a free and proper action $\Psi: G \times Q \to Q$ such that $L$ is $G$-invariant and $D$ is determined by the horizontal distribution of a principal connection $\gamma: TQ \to g$. Note that $T\pi_0^0 = D_q \oplus T_q \text{Orb}(q)$.

The projection map $\pi: Q \to Q/G$ induces an isomorphism from $D_q$ to $T\pi_q(Q/G)$, and the inverse map is called the horizontal lift. Thus for any vector field $X \in \mathfrak{X}(Q/G)$ on the base space, we have a unique vector field $X^h$ (the horizontal lift of $X$) that is horizontal and $\pi$-related to $X$.

Consider a local trivialization $U \times G$ of $\pi$ where now the action of $G$ is given by left translation on the second factor and $U$ is a neighborhood of $Q/G$. Take coordinates $r^a$ on $U$ and a basis $\alpha e^a$ of $g$. In this local trivialization we can write the connection $\gamma$ as

$$\gamma_{\alpha} = \gamma_{\alpha}^a e^a(r^a, \dot{r}^a)$$

and the coefficients of the curvature, $B(X, Y) = -\gamma([X^h, Y^h])$, are $B_{ab} = \gamma_{abc}^b - \gamma_{abc}^a \gamma_{bca}^c$.

In this case, the Lagrangian $L: TQ \to \mathbb{R}$ induces a Lagrangian $L^*: T(Q/G) \to \mathbb{R}$ by

$$L^*(X(q)) = L(X^h(q)).$$

Locally, $L^*(r^a, \dot{r}^a) = l(r^a, \dot{r}^a, -A_{a\alpha}^b \dot{r}^\alpha e_a)$, where $l: TU \times g \to \mathbb{R}$ represents the reduction of $L: T(U \times G) \to \mathbb{R}$ to $TQ/G$.

After some computations, we can see that the reduced dynamics are given by the following system of equations on $T(Q/G)$:

$$\frac{d}{dt} \left( \frac{\partial L^*}{\partial \dot{r}^a} \right) - \frac{\partial L^*}{\partial r^a} = \Lambda_a, \quad \text{where} \quad \Lambda_a = -\left( \frac{\partial}{\partial q^a} \right) \frac{\partial L^*}{\partial \dot{q}^a} B_{ab}^h \dot{r}^b.$$

and the subindex ‘$c$’ on the right-hand side indicates that, after computing the derivative of $l$ with respect to $\dot{q}^a$, one evaluates this partial derivative on $(r^a, \dot{r}^a, -A_{a\alpha}^b \dot{r}^\alpha e_a)$.

Moreover, if $L$ is regular, we have that $L^*$ is also regular and we obtain the following system of second-order differential equations now defined on the full space $T(Q/G)$:
\[
\frac{d^2 r^a}{dt^2} = \tilde{W}^{ab} \left( \frac{\partial L^b}{\partial q^b} - \dot{r}^c \frac{\partial^2 L^b}{\partial \dot{q}^c \partial q^b} + \Lambda_b \right),
\]
where \((\tilde{W}^{ab})^{-1}\) is the inverse of the Hessian matrix \(\tilde{W}_{ab} = \left( \frac{\partial^2 L^a}{\partial q^b \partial \dot{q}^c} \right)\).

### 3.2. Variational constrained equations

Now we study a dynamical system given by the same pair \((L, M)\) but using purely variational
techniques. As above, let us consider a regular Lagrangian \(L: TQ \rightarrow \mathbb{R}\), and a set of constraints \(\psi^\alpha(q^i, \dot{q}^i) = 0\), \(1 \leq \alpha \leq m\) that determine a \(2n - m\) dimensional submanifold \(M \subset TQ\). Take the extended Lagrangian \(\mathcal{L} = L + \lambda _\alpha \psi ^\alpha\) which includes the Lagrange multipliers \(\lambda _\alpha \) as new extra variables. The equations of motion for the constrained variational problem are the Euler–Lagrange equations for \(\mathcal{L}\), that is:

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i} = -\lambda _\alpha \frac{\partial \psi ^\alpha}{\partial q^i} - \lambda _\alpha \left[ \frac{d}{dt} \left( \frac{\partial \psi ^\alpha}{\partial \dot{q}^i} \right) - \frac{\partial \psi ^\alpha}{\partial q^i} \right],
\]

\[
\psi ^\alpha(q^i, \dot{q}^i) = 0, \quad 1 \leq \alpha \leq m.
\]

Observe that the equations of a variational constrained system are different from the
equations of a nonholonomic system given in (3) (see [5, 10]).

### 3.3. Lagrangian mechanics using the Tulczyjew’s triple

The theory of Lagrangian submanifolds gives an intrinsic geometric description of Lagran-
gian and Hamiltonian dynamics [40, 41]. Moreover, it allows us to relate Lagrangian and
Hamiltonian formalisms using as a main tool the so-called Tulczyjew’s triple

\[
T^*TQ \rightarrow \alpha_Q TT^*Q \rightarrow \beta_Q T^*T^*Q.
\]

The Tulczyjew map \(\alpha _Q\) is an isomorphism between \(TT^*Q\) and \(T^*TQ\). Besides, it is also a
symplectomorphism between these vector bundles considered as symplectic manifolds, i.e. \((TT^*Q, d_T \omega _Q)\), where \(d_T \omega _Q\) is the tangent lift of \(\omega _Q\), and \((TT^*Q, \omega _{TTQ})\). An intrinsic
definition can be given using the canonical involution and the tangent pairing. In local
coordinates \((q^i, p^i)\) for \(TT^*Q\) and \((q^i, \dot{q}^i, \dot{p}^i)\) for \(T^*T^*Q\), we have

\[
\alpha _Q(q^i, \dot{q}^i, \dot{p}^i) = (q^i, \dot{q}^i, \dot{p}^i, p^i).
\]

The isomorphism \(\beta _Q: TT^*Q \rightarrow T^*T^*Q\) is just given by \(\beta _Q = b_{\omega _Q}\), where \(b_{\omega _Q}\) is the isomorphism defined by \(\omega _Q\), that is, \(b_{\omega _Q}(\nu) = i, \omega _Q\).

The Lagrangian dynamics is described by the Lagrangian submanifold \(dL(TQ)\) of \(T^*TQ\)\nwhere \(L: TQ \rightarrow \mathbb{R}\) is the Lagrangian function, while the Hamiltonian formalism is described by
the Lagrangian submanifold \(dH(T^*Q)\) of \(T^*T^*Q\) where \(H: T^*Q \rightarrow \mathbb{R}\) is the cor-
responding Hamiltonian energy. The solutions of the dynamics are curves \(\gamma: I \subset \mathbb{R} \rightarrow T^*Q\)
such that \(\frac{d\gamma}{dt}(I) \subset \alpha _Q^{-1}(dL(TQ))\) in the Lagrangian description and \(\frac{d\gamma}{dt}(I) \subset \beta _Q^{-1}(dH(T^*Q))\) in the Hamiltonian case.

Variationally constrained problems described in section 3.2 are determined by a pair
\((M, l)\) where \(M\) is a submanifold of \(TQ\), with inclusion \(i_M: M \hookrightarrow TQ\), and \(l: M \rightarrow \mathbb{R}\) is a
Lagrangian function restricted to \(M\). The submanifold \(\Sigma _l\) is a Lagrangian submanifold of
Using the Tulczyjew’s symplectomorphism \( \alpha_Q \), we induce a new Lagrangian submanifold \( \alpha_Q^{-1}(\Sigma_I) \) of \( (TT^*Q, d\omega_Q) \), which completely determines the constrained variational dynamics. We check now that this procedure gives the correct equations for the constrained variational dynamics. Take an arbitrary extension \( L: TQ \rightarrow \mathbb{R} \) of \( l: M \rightarrow \mathbb{R} \), that is, \( L \circ i_M = l \). Locally

\[
\Sigma_I = \left\{ (q^i, \dot{q}^i, \mu_i, \dot{\mu}_i) \in T^*TQ \mid \mu_i = \frac{\partial L}{\partial q^i} + \lambda_a \frac{\partial \phi^a}{\partial q^i}, \right. \\
\left. \dot{\mu}_i = \frac{\partial L}{\partial \dot{q}^i} + \lambda_a \frac{\partial \phi^a}{\partial \dot{q}^i}, \quad \phi^a(q, \dot{q}) = 0, \quad 1 \leq a \leq m \right\}.
\]

Therefore

\[
\alpha_Q^{-1}(\Sigma_I) = \left\{ (q^i, p_i, \dot{q}^i, \dot{p}_i) \in TT^*Q \mid p_i = \frac{\partial L}{\partial q^i} + \lambda_a \frac{\partial \phi^a}{\partial q^i}, \right. \\
\left. \dot{p}_i = \frac{\partial L}{\partial \dot{q}^i} + \lambda_a \frac{\partial \phi^a}{\partial \dot{q}^i}, \quad \phi^a(q, \dot{q}) = 0, \quad 1 \leq a \leq m \right\}.
\]

The solutions for the dynamics given by \( \alpha_Q^{-1}(\Sigma_I) \subset TT^*Q \) are curves \( \gamma: I \subset \mathbb{R} \rightarrow TT^*Q \) such that \( \frac{d\gamma}{dt} \subset TT^*Q \) verifies that \( \frac{d\gamma}{dt} (t) \subset \alpha_Q^{-1}(\Sigma_I) \). Locally, if \( \gamma(t) = (q^i(t), p_i(t)) \) then it must verify the following set of differential equations:

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial q^j} + \lambda_a \frac{\partial \phi^a}{\partial q^j} \right) = \frac{\partial L}{\partial \dot{q}^j} - \lambda_a \frac{\partial \phi^a}{\partial \dot{q}^j} = 0, \\
\phi^a(q^i, \dot{q}^i) = 0,
\]

which coincide with equation (5).

**4. The inverse problem of the calculus of variations**

In the previous section it is shown that given a regular Lagrangian function \( L: TQ \rightarrow \mathbb{R} \) we can always associate a unique SODE \( \Gamma_L \), see equation (1). The (autonomous) inverse problem of the calculus of variations studies when a prescribed SODE \( \Gamma \) is equivalent to the Euler-Lagrange equations for a regular Lagrangian \( L: TQ \rightarrow \mathbb{R} \), in the sense of searching a non-singular multiplier matrix \( (g_{ij}(q, \dot{q})) \) such that

\[
g_{ij}(\dot{q}^i - \Gamma^i) = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^j} \right) - \frac{\partial L}{\partial q^j}, \quad i, j = 1, \ldots, n = \dim Q \tag{6}
\]

has a regular solution \( L \). Note that in the affirmative case we have that \( g_{ij} = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \) and the solutions to \( \Gamma \) are exactly the same as the solutions to the Euler-Lagrange equations for \( L \).

Geometrically, condition (6) can be captured into the requirement of the existence of a function \( L: TQ \rightarrow \mathbb{R} \) such that \( L_{\Gamma} \Theta_L = dL \) (see (2)). When the condition is satisfied, the SODE \( \Gamma \) is called variational.

The existence of a regular Lagrangian for \( \Gamma \) is equivalent to the existence of multipliers \( (g_{ij}(q, \dot{q})) \) satisfying the Helmholtz conditions (see [45] for a more general version):
\[
\det \left( g_{ij} \right) \neq 0, \quad g_{\mu} = g_{ij}, \quad \frac{\partial g_{ij}}{\partial q^k} = \frac{\partial g_{ik}}{\partial q^j},
\]

(7)

\[
\Gamma \left( g_{ij} \right) - \nabla_j g_{ik} - \nabla_i g_{kj} = 0,
\]

(8)

\[
g_{ik} \Phi_j^k = g_{jk} \Phi_i^j,
\]

(9)

where \( \Gamma = q^l \frac{\partial}{\partial q^l} + \Gamma^i \left( q, \dot{q} \right) \frac{\partial}{\partial \dot{q}^i} \), \( \nabla_j = -\frac{1}{2} \frac{\partial \Gamma^r}{\partial q^j} \) and \( \Phi_j^k = \Gamma \left( \frac{\partial \Gamma^i}{\partial \dot{q}^j} \right) - \frac{1}{2} \frac{\partial \Gamma^i}{\partial q^j} \frac{\partial \Gamma^r}{\partial q^k} \). The problem is specially difficult since Helmholtz conditions are a mixed set of algebraic equations and partial differential equations for the multipliers \( g_{ij} \). There are many characterizations of the inverse problem of the calculus of variations in the literature, but not much is known about the complete solution. For instance, \( n = 1 \) is always variational [38] and \( n = 2 \) was solved by Douglas in [19], but for \( n > 2 \) no complete classification exists. Some partial results exist, more precisely, some cases in Douglas’ classification have been generalized to arbitrary \( n \). See for instance [14, 16, 37].

The following characterization of being variational will be very useful in the sequel.

**Theorem 4.1.** [12] A SODE \( \Gamma \) on \( TQ \) is variational if and only if there exists a two-form \( \Omega \) on \( TQ \) of maximal rank such that

1. \( d \Omega = 0 \),
2. \( \Omega \left( v_1, v_2 \right) = 0 \ \forall \ v_1, v_2 \in V \left( TQ \right) \), where \( V \left( TQ \right) \) denotes the set of all vertical vector fields for \( \tau_Q: TQ \rightarrow Q \), that is, \( V \left( TQ \right) = \text{Ker} T_\tau Q \),
3. \( \mathcal{L}_F \Omega = 0 \).

4.1. A new geometric characterization for the inverse problem of the calculus of variations

In this section we introduce a characterization of the inverse problem that will be very helpful in the next section to define the variantionality of a SODE on a submanifold of \( TQ \). We regard this result as a previous step towards constrained systems.

For a given SODE \( \Gamma: TQ \rightarrow TTQ \) and a local diffeomorphism \( F: TQ \rightarrow T^*Q \) of fibre bundles over \( Q \) (that is, \( \pi_Q \circ F = \tau_Q \)), we define a submanifold \( \Sigma_{F, F} := \text{Im} \left( \mu_{F, F} \right) \subset T^*TQ \), where \( \mu_{F, F} = \alpha_Q \circ TF \circ \Gamma \) is a one-form on \( TQ \):

\[
\begin{array}{ccc}
TTQ & \xrightarrow{TF} & TT^*Q \\
\downarrow F & & \downarrow \alpha_Q \circ \left( \mu_{F, F} \right) \\
TQ & \xrightarrow{\Gamma} & T^*Q
\end{array}
\]

Let \( (q^i, \dot{q}^i) \) denote fibered coordinates on \( TQ \) and we write \( F \) and \( \Gamma \) in these coordinates as

\[
F \left( q^i, \dot{q}^i \right) = \left( q^i, F_i \left( q, \dot{q} \right) \right), \quad \Gamma \left( q^i, \dot{q}^i \right) = \left( q^i, q^i', \dot{q}^i, \Gamma^i \left( q, \dot{q} \right) \right).
\]
Then the above diagram in coordinates becomes

\[ (q^i, q^i, \Gamma^i(q, q)) \xrightarrow{T_F} (q^i, F_i, q^i, \Gamma^i(q, q)) \xrightarrow{\alpha_Q} (q^i, q^i, \frac{\partial F_i}{\partial q^i} \dot{q}^j + \frac{\partial F_i}{\partial q^j} \Gamma^i, F_i) \]

Note that \( \mu_{\Gamma, F} \) is a one-form on \( TQ \) locally given by \( \frac{\partial F_i}{\partial q^i} \dot{q}^j + \frac{\partial F_i}{\partial q^j} \Gamma^i \). From this last expression it is easy to deduce that

\[ \mu_{\Gamma, F} = \mathcal{L}_F \Gamma^* \theta_Q, \quad (10) \]

where \( \theta_Q \) denotes the Liouville one-form on \( T^*Q \).

In this section we will show that the inverse problem of the calculus of variations for a SODE \( \Gamma \) is equivalent to see whether or not it is possible to find a local diffeomorphism \( F: TQ \rightarrow T^*Q \) of fibre bundles over \( Q \) such that \( \Sigma_{\Gamma, F} = \text{Im} (\mu_{\Gamma, F}) \) is a Lagrangian submanifold of \( (T^*Q, \omega_{TQ}) \). This characterization will be useful for our approach to the inverse problem for constrained systems.

Observe that since \( \Sigma_{\Gamma, F} \) is the image of the one-form \( \mu_{\Gamma, F} \) on \( TQ \), \( \Sigma_{\Gamma, F} \) is a Lagrangian submanifold of \( (T^*Q, \omega_{TQ}) \) if and only if \( \mu_{\Gamma, F} \) is closed, i.e. \( d\mu_{\Gamma, F} = 0 \). Therefore, using Poincaré lemma we deduce the local existence of a function \( L \) on \( TQ \) such that

\[ \mu_{\Gamma, F} = \mathcal{L}_F^* L \theta_Q. \]

**Theorem 4.2.** A SODE \( \Gamma \) on \( TQ \) is variational if and only if there exists a local diffeomorphism \( F: TQ \rightarrow T^*Q \) of fibre bundles over \( Q \) such that \( \Sigma_{\Gamma, F} = \text{Im} (\mu_{\Gamma, F}) \) is a Lagrangian submanifold of \( (T^*Q, \omega_{TQ}) \).

**Proof.** We use the characterization in theorem 4.1 to prove this result.

\[ \Rightarrow \ \] Define \( \Omega = -d(F^* \theta_Q) \) and note that if \( F(q^i, \dot{q}^i) = (q^i, F_i(q, \dot{q})) \), then

\[ \mathcal{L}_F F^* \theta_Q = \mathcal{L}_F (F_i \dot{q}^i) = F_i \dot{q}^i + F_i d\dot{q}^i = \left( \frac{\partial F_i}{\partial q^i} \dot{q}^j + \frac{\partial F_i}{\partial q^j} \Gamma^i \right) \dot{q}^j + F_i d\dot{q}^i = \mu_{\Gamma, F}. \]

Then \( \Omega \) trivially satisfies all the conditions in theorem 4.1.

\[ \Leftarrow \ \] From theorem 4.1 we have that \( \Gamma \) is variational if and only if there exists a non-degenerate two-form \( \Omega \) on \( TQ \) satisfying \( \mathcal{L}_\Gamma \Omega = 0 \), \( \Omega(v, w) = 0 \) for all \( v, w \in V(TQ) \) and \( d\Omega = 0 \). From the last condition we deduce that locally \( \Omega = d\Theta \) on a neighborhood \( U \subseteq TQ \), where \( \Theta \) is a one-form on \( U \). The restriction of \( d\Theta \) to vertical subspaces is zero. Thus the restriction of \( \Theta \) to each fiber is exact, then there is a function \( f: U \rightarrow \mathbb{R} \) such that \( \Theta(v) = (df, v) \) for any \( v \in V(TQ) \). Therefore, \( \tilde{\Theta} = \Theta - df \) verifies \( \tilde{\Theta}(v) = 0 \) for all \( v \in V(TQ) \) and \( d\tilde{\Theta} = \Omega \). Using \( \tilde{\Theta} \) we construct the map \( F: U \subseteq TQ \rightarrow T^*Q \) as follows:

\[ \{ F(v_q), w_q \} = \{ \tilde{\Theta}(v_q), W_q \}. \]
where $v_q \in TQ$, $w_q \in TQ$ and $W_q \in TTQ$ satisfies $T\Theta_q(W_q) = w_q$. This definition does not depend on the choice of $W_q$ since $\tilde{\Theta}$ vanishes on vertical vector fields. Then, it is easy to show that $\tilde{\Theta} = F^*\theta_Q$ and from equation (10), $\mu_{\Gamma,F} = \mathcal{L}_F \tilde{\Theta}$ verifies
\[
\mu_{\Gamma,F} = d\mathcal{L}_F \tilde{\Theta} = \mathcal{L}_F d\tilde{\Theta} = \mathcal{L}_F \Omega = 0.
\]
Hence $\text{Im} \ (\mu_{\Gamma,F})$ is a Lagrangian submanifold of $(T^*TQ, \omega_{TQ})$. Note that the non-degeneracy of $\Omega$ implies that $\det \left( \frac{\partial F_i}{\partial q^j} \right) \neq 0$ which is precisely the condition for $F$ to be a local diffeomorphism.

Remark 4.3. Since $\alpha_Q: TTQ \to T^*TQ$ is a symplectomorphism (see section 3.3) then we can alternatively characterize the inverse problem of the calculus of variations for a SODE $\Gamma$, seeing whether the submanifold $S_{\Gamma,F}$ defined by
\[
S_{\Gamma,F} = TF(\Gamma(Q)) = \alpha_Q^{-1} \left( \mu_{\Gamma,F}(Q) \right)
\]
is a Lagrangian submanifold of the symplectic manifold $(TTQ, \omega_{TQ})$.

Remark 4.4. The submanifold $\Sigma_{\Gamma,F}$ will be Lagrangian if and only if
\[
d \left[ \left( \frac{\partial F_i}{\partial q^j} \dot{q}^j + \frac{\partial F_i}{\partial \dot{q}^j} \dot{\Gamma}^j \right) dq^i + F_i dq^i \right] = 0.
\]
Equivalently, we get the following conditions:
\[
\frac{\partial F_i}{\partial q^k} = \frac{\partial F_k}{\partial q^i}, \quad (11)
\]
\[
\frac{\partial^2 F_i}{\partial q^j \partial q^l} \ddot{q}^j + \frac{\partial^2 F_i}{\partial q^j \partial q^k} \ddot{q}^j + \frac{\partial F_i}{\partial q^j} \dot{q}^j + \frac{\partial^2 F_k}{\partial q^j \partial q^l} \ddot{q}^j + \frac{\partial F_k}{\partial q^j} \dot{q}^j = \frac{\partial^2 F_k}{\partial q^j \partial q^l} \ddot{q}^j + \frac{\partial F_k}{\partial q^j} \dot{q}^j + \frac{\partial^2 F_i}{\partial q^j \partial q^k} \ddot{q}^j + \frac{\partial F_i}{\partial q^j} \dot{q}^j.
\]

(12)

\[
\frac{\partial F_k}{\partial q^l} = \frac{\partial^2 F_i}{\partial q^j \partial q^l} \ddot{q}^j + \frac{\partial F_i}{\partial q^k} + \frac{\partial^2 F_i}{\partial q^j \partial q^k} \ddot{q}^j + \frac{\partial F_i}{\partial q^j} \dot{q}^j + \frac{\partial^2 F_i}{\partial q^j \partial q^k} \ddot{q}^j + \frac{\partial F_i}{\partial q^j} \dot{q}^j.
\]

(13)

Direct computations showing the equivalence between the equations (11)–(13) and the Helmholtz conditions (7)–(9) can be carried out.

Note that the conditions (11)–(13) are given in the standard basis. We can also easily recover the usual Helmholtz conditions using the basis \( V_{IH} \), \( V_{H} \), \( V_{V} \)

\[
\dot{\mu} \Gamma \delta \quad \text{on pairs} \quad (H_i, H_j), (H_i, V_j), (V_i, V_j)
\]

vanishes and also use the condition \( \mu \Gamma = \mu \Gamma = 0 \) which is the same as (13)\( _{ik} \)–(13)\( _{ki} = 0 \).

**Remark 4.5.** In theorem 4.2 we are asking for the existence of a Legendre transformation for \( \Gamma \). In [8, theorem 5.3] the characterization is given in terms of the existence of a Poincaré–Cartan one-form, so the semi-basic one-form they seek and the local diffeomorphism we seek are simply related by \( \theta = F^\theta \theta Q \).

**Remark 4.6.** If we admit that the matrix \( (g_{ij}) \) is degenerate, then we get conditions for the existence of a singular Lagrangian \( L \) such that

\[
g_{ij} \left( \dot{q}^j - \Gamma^j \right) = \frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{q}^j} \right) - \frac{\partial L}{\partial q^j},
\]

which implies that the solutions of the SODE are also solutions to the Euler–Lagrange equations for \( L \).

**Example 4.7.** Let \( Q = \mathbb{R}^2 \), \( \Gamma \) be given by \( \bar{x} = f(x, y), \bar{y} = f(x, y) \), that is, \( \Gamma^3 = \Gamma^2 = f(x, y) \). Then \( L = \frac{1}{2} (\bar{x} - y)^2 \) is a singular Lagrangian that gives the dynamics \( \bar{x} = \bar{y} \), which includes the solutions to \( \Gamma \), and satisfies

\[
g_{ij} \left( \dot{q}^j - \Gamma^j \right) = \frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{q}^j} \right) - \frac{\partial L}{\partial q^j}
\]

with \( g_{11} = g_{22} = 1 \) and \( g_{12} = g_{21} = -1 \). For some choices of \( f(x, y) \), the SODE will fall into one of the cases in [19] which do not admit a regular Lagrangian. For instance if we take \( f(x, y) = xy \), then, in the notation of [19] (except for the coordinates which we denote as \( t, x, y, \bar{x}, \bar{y} \)), we get

\[
A = -2x, \quad B = (y - x), \quad C = 2y;
\]

\[
A_1 = -2\bar{x}, \quad B_1 = 2(\bar{y} - \bar{x}), \quad C_1 = 2y;
\]

\[
A_2 = -2xy, \quad B_2 = 0, \quad C_2 = 2xy.
\]

Then the determinant of

\[
\begin{pmatrix}
A & B & C \\
A_1 & B_1 & C_1 \\
A_2 & B_2 & C_2 \\
\end{pmatrix}
\]

is nonzero and the example falls into the nonvariational case IV of Douglas [19].

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4.2. Chaplygin hamiltonization

As we have seen in section 3.1.1, the equations of motion of a noholonomic Chaplygin system can be reduced to a second-order differential equation on \( Q \). Then, we can apply the inverse problem of the calculus of variations in an attempt to find a Lagrangian \( \mathcal{L} : T(Q/G) \to \mathbb{R} \) such that equations (4) are equivalent to the Euler–Lagrange equations for the Lagrangian \( \mathcal{L} \).

Denote by \( \Gamma \) the SODE on \( T(Q/G) \) in equations (4). By theorem 4.2, \( \Gamma \) is equivalent to the Euler–Lagrange equations of a regular Lagrangian if there exists a fiber diffeomorphism \( F : T(Q/G) \to T^*(Q/G) \) such that \( \text{Im}(\mu_{F,\mathcal{L}}) \) is a Lagrangian submanifold of \( (T^*T(Q/G), \omega_{F(Q/G)}) \).

Equivalently, in the case of Chaplygin systems we can use the reduced Lagrangian \( \mathcal{L}^\ast : T(Q/G) \to \mathbb{R} \) defined in section 3.1.1 and its associated Legendre transformation \( \text{Leg}_L^* : T^*(Q/G) \to T^*(Q/G) \).

Then we can define the vector field \( \vec{\Gamma} = (\text{Leg}_L^*)_\# \Gamma \) on \( T^*(Q/G) \) representing the nonholonomic dynamics, now on the Hamiltonian side. But if there exists a solution \( F : T(Q/G) \to T^*(Q/G) \) of the inverse problem of the calculus of variations then the vector field \( F_\ast \Gamma \) is locally Hamiltonian. That is, locally there exists a function \( \hat{H} : T^*(Q/G) \to \mathbb{R} \) such that

\[
i_{F_\ast \Gamma} \omega_{Q/G} = d\hat{H}.
\]

Therefore, if we consider the diffeomorphism \( G : T^*(Q/G) \to T^*(Q/G) \) given by \( G = F \circ (\text{Leg}_L^*)^{-1} \) then it is clear by construction that \( G_\ast \vec{\Gamma} = F_\ast \Gamma \) and

\[
i_{\vec{\Gamma}} \Omega = d\hat{H}, \tag{14}
\]

where \( \Omega = G^\ast(\omega_{Q/G}) \) and \( \hat{H} = H \circ G \). Equation (14) corresponds to the standard notion of hamiltonization of a Chaplygin system [4, 6]

\[
\begin{array}{cccc}
TT^*(Q/G) & \xrightarrow{\mathcal{T}G} & TT^*(Q/G) \\
\mathcal{T}_G & \mathcal{L}_G^* & \mathcal{F}
\end{array}
\]

5. The inverse problem for constrained systems

In this section we will study the extension of the inverse problem of the calculus of variations to the case of constrained systems. Consider a submanifold \( M \) of \( TQ \) and a vector field \( \Gamma \) on \( M \) verifying the SODE condition, that is

\[
S_\epsilon(\Gamma(x)) = \Delta(x), \quad \forall \ x \in M.
\]

Nonholonomic mechanics is an example of this situation, as we will see later.
From now on, we assume that $M$ projects over the whole configuration manifold $Q$. Inspired by theorem 4.2 we give the following definition:

**Definition 5.1.** A SODE $\Gamma$ on the submanifold $M$ of $TQ$ is variational if there exists an immersion $F: M \rightarrow T^*Q$ over $Q$ such that $\Sigma_{F,\Gamma} := \text{Im} (\mu_{F,\Gamma})$ is an isotropic submanifold of $(T^*TQ, \omega_{TQ})$, where $\mu_{F,\Gamma} = \alpha_Q \circ TF \circ \Gamma$

\[ TM \xrightarrow{TF} TT^*Q \xrightarrow{\alpha_Q} T^*TQ \]
\[ M \xrightarrow{F} T^*Q \]

Assume that $M$ is determined by the constraints $\dot{q}^a = \psi^a(q^i, \dot{q}^b)$, $1 \leq a \leq m$, so $(q^i, \dot{q}^b)$ are local coordinates on $M$, $1 \leq i \leq n - m$, $n = \text{dim } Q$. Then the solutions of the SODE $\Gamma$ are now represented by the following system of differential equations

\[ \ddot{q}^a = \Gamma^a(q^i, \dot{q}^b), \]
\[ \dot{q}^a = \psi^a(q^i, \dot{q}^b) \]

For each map

\[ F: M \rightarrow T^*Q \]
\[ \left( q^i, \dot{q}^b \right) \rightarrow \left( q^i, F_j(q^i, \dot{q}^b) \right) \]

satisfying that $\text{rank} \left( \frac{\partial F_j}{\partial q^i} \right) = n - m$, the submanifold $\text{Im} (\alpha_Q \circ TF \circ \Gamma) = \text{Im} (\mu_{F,\Gamma})$ is given in coordinates by

\[ \left( q^i, \dot{q}^b, \psi^a, \frac{\partial F_i}{\partial q^a} q^b + \frac{\partial F_j}{\partial q^b} \dot{q}^a + \frac{\partial F_i}{\partial q^a} \psi^b + \frac{\partial F_j}{\partial q^b} \dot{q}^a \right) \]

We look for an immersion $F: M \rightarrow T^*Q$ such that $\text{Im} (\mu_{F,\Gamma})$ is isotropic in $(T^*TQ, \omega_{TQ})$, that is, such that the following conditions are satisfied:

\[ 0 = \frac{\partial E_a}{\partial q^b} + \frac{\partial \psi^a}{\partial q^b} \frac{\partial E_a}{\partial \dot{q}^b} - \frac{\partial E_b}{\partial q^a} \frac{\partial \psi^a}{\partial \dot{q}^b} - \frac{\partial E_a}{\partial \dot{q}^b} \frac{\partial \psi^a}{\partial q^b}, \quad (15) \]
\[ 0 = \frac{\partial^2 F_i}{\partial q^j \partial q^b} q^b + \frac{\partial^2 F_i}{\partial q^j \partial \dot{q}^b} \dot{q}^b + \frac{\partial F_i}{\partial q^j} \frac{\partial \psi^a}{\partial \dot{q}^b} \frac{\partial E_a}{\partial q^b} + \frac{\partial F_i}{\partial \dot{q}^b} \frac{\partial \psi^a}{\partial q^b} \frac{\partial E_a}{\partial \dot{q}^b} \frac{\partial \psi^a}{\partial q^b} - \frac{\partial \psi^a}{\partial \dot{q}^b} \frac{\partial E_a}{\partial q^b} \frac{\partial \psi^a}{\partial \dot{q}^b} \frac{\partial E_a}{\partial q^b}, \quad (16) \]
\[ 0 = \frac{\partial^2 F_i}{\partial q^j \partial \dot{q}^b} q^b + \frac{\partial^2 F_i}{\partial q^j \partial q^b} \dot{q}^b + \frac{\partial^2 F_i}{\partial q^j} \frac{\partial \psi^a}{\partial \dot{q}^b} \frac{\partial E_a}{\partial q^b} + \frac{\partial^2 F_i}{\partial q^j} \frac{\partial \psi^a}{\partial \dot{q}^b} \frac{\partial E_a}{\partial q^b} \frac{\partial \psi^a}{\partial \dot{q}^b} \frac{\partial E_a}{\partial q^b} + \frac{\partial F_i}{\partial \dot{q}^b} \frac{\partial \psi^a}{\partial q^b} \frac{\partial E_a}{\partial \dot{q}^b} \frac{\partial \psi^a}{\partial q^b} - \frac{\partial \psi^a}{\partial \dot{q}^b} \frac{\partial E_a}{\partial \dot{q}^b} \frac{\partial \psi^a}{\partial q^b} \frac{\partial E_a}{\partial \dot{q}^b} \frac{\partial \psi^a}{\partial q^b} \frac{\partial E_a}{\partial \dot{q}^b} \frac{\partial \psi^a}{\partial q^b}, \quad (17) \]

We will refer to them as constrained Helmholtz conditions.
Now we will see the relationship between \( \text{Im}(\mu_{\Gamma,F}) \) and the dynamics given by the SODE \( \Gamma \) on \( M \). Take the submanifold \( \alpha_Q^{-1}(\text{Im}(\mu_{\Gamma,F})) = TF(\Gamma(M)) \) of \( TT^*Q \). Since \( TT^*Q \) is a tangent bundle, we have dynamics related to any submanifold. In our case \( TF(\Gamma(M)) \) is given by

\[
\left( q^i, F_i(q^j, \dot{q}^j), \dot{q}^a, \psi^a(q^j, \dot{q}^j), \frac{\partial F_i}{\partial q^a} \dot{q}^a + \frac{\partial F_i}{\partial \dot{q}^a} \psi^a + \frac{\partial F_i}{\partial q^a} \Gamma^a \right)
\]

in the typical coordinates in \( TT^*Q \). Tangent curves to this submanifold satisfy the equations

\[
\dot{q}^a = \psi^a(q^j, \dot{q}^j) \quad \text{and} \quad \frac{d}{dt} F_i = \frac{\partial F_i}{\partial q^a} \dot{q}^a + \frac{\partial F_i}{\partial \dot{q}^a} \psi^a + \frac{\partial F_i}{\partial q^a} \Gamma^a.
\]

Then

\[
\frac{\partial F_i}{\partial q^a} \left( \dot{q}^a - \Gamma^a(q^j, \dot{q}^j) \right) = 0.
\]

Since \( \left( \frac{\partial F_i}{\partial q^a} \right) \) is assumed to have maximal rank, we get \( \ddot{q}^a = \Gamma^a(q^j, \dot{q}^j) \) and \( \dot{q}^a = \psi^a(q^j, \dot{q}^j) \).

In this case we have seen that the isotropic submanifold \( \text{TF}(\Gamma(M)) = \alpha_Q^{-1}(\Sigma_{\Gamma,F}) \) on \( TT^*Q \) carries the original dynamics defined by the SODE \( \Gamma \) on \( M \).

Now we generalize the characterization of theorem 4.1 to constrained systems:

**Theorem 5.2.** A SODE \( \Gamma \) on \( M \) is variational if and only if there exists a two-form \( \Omega \) on \( M \) satisfying

(i) \( d\Omega = 0 \),
(ii) \( \Omega(v_1, v_2) = 0 \) for all \( v_1, v_2 \in V(M) \),
(iii) \( \mathcal{L}_F \Omega = 0 \),
(iv) \( \partial_{\Omega}|_{V(M)} \) is injective.

**Proof.** \( \Rightarrow \) Assume that \( \Gamma \) is variational, that is, there exists an immersion \( F : M \longrightarrow T^*Q \) such that \( \Sigma_{\Gamma,F} = \text{Im}(\mu_{\Gamma,F}) \) is isotropic in \( (T^*TQ, \omega_{TQ}) \). Then we define \( \Omega = dF^*\theta_Q \in \bigwedge^2(M) \).

We first prove that \( \Sigma_{\Gamma,F} \) is isotropic if and only if \( \text{Im}(\mathcal{L}_F F^*\theta_Q) \) is Lagrangian in \( (T^*M, \omega_M) \), that is, \( d(\mathcal{L}_F F^*\theta_Q) = \mathcal{L}_F \Omega = 0 \). In local coordinates \( (q^i, \dot{q}^a) \) on \( M \)

\[
\mathcal{L}_F F^*\theta_Q = \left( \Gamma(F_i) + \frac{\partial \psi^a}{\partial q^j} F_a \right) dq^j + \left( F_a + \frac{\partial \psi^a}{\partial q^a} F_i \right) dq^a.
\]

On the other hand, \( \Sigma_{\Gamma,F} \) is given by the following set of points of \( T^*TQ \):

\[
\left( q^i, \dot{q}^a, \psi^a, \frac{\partial F_i}{\partial q^j} \dot{q}^a + \frac{\partial F_i}{\partial \dot{q}^a} \psi^a + \frac{\partial F_i}{\partial q^a} \Gamma^a, F_i \right).
\]

If we denote by \( i_{\Sigma_{\Gamma,F}} : \Sigma_{\Gamma,F} \longrightarrow T^*TQ \) the inclusion, then

\[
i_{\Sigma_{\Gamma,F}}^* \theta_{TQ} = \Gamma(F_i) dq^i + F_a dq^a + F_i dq^j = \left( \Gamma(F_i) + \frac{\partial \psi^a}{\partial q^j} F_a \right) dq^j + \left( F_a + \frac{\partial \psi^a}{\partial q^a} F_i \right) dq^a.
\]

Now it is clear that the condition of isotropy, \( i_{\Sigma_{\Gamma,F}}^* \omega_{TQ} = 0 \), is equivalent to the condition of \( \text{Im}(\mathcal{L}_F F^*\theta_Q) \) being Lagrangian in \( (T^*M, \omega_M) \), in other words, \( d\mathcal{L}_F F^*\theta_Q = \mathcal{L}_F \Omega = 0 \).
The first two properties in the statement of the theorem follow directly from the definition of $\Omega$ and the last one from $F$ being an immersion. Indeed, $\Omega = \left( \frac{\partial F}{\partial q^j} \right) dq^j \wedge dq^i + \left( \frac{\partial F}{\partial q^i} \right) dq^i \wedge dq^j$, and for any $v_1, v_2$ in $V(M)$, $i_{v_1} \Omega - i_{v_2} \Omega = \left( v_1^i - v_2^i \right) \left( \frac{\partial F}{\partial q^i} \right) dq^i = 0$ for all $i = 1, \ldots, n$. As $\left( \frac{\partial F}{\partial q^i} \right)$ has maximal rank, $v_1 = v_2$.

Next, given a two-form on $M$ satisfying the conditions in the statement, we construct an immersion that provides an isotropic submanifold $\Sigma_{F,F}$ of $(T^*TQ, \omega_{TQ})$. Since $d\Omega = 0$, locally we can write $\Omega = d\theta$. Then using the second condition we get that there exists a locally defined function $f$ on $M$ such that $\theta(v) = df(v)$ for each vertical vector $v \in V(M)$. We can define $\tilde{\theta} = \theta - df$ which is a semi-basic one-form on $M$, that is, it vanishes on vertical vectors and can be written in coordinates as $\tilde{\theta} = \mu_1 dq^1, \mu_i$ being functions on $M$.

Moreover $d\tilde{\theta} = \Omega$. Then we define $F: M \to T^*Q$ by

$$\{F(m), v_i\} = \{\tilde{\theta}(m), w_m\},$$

where $m \in M$ and $w_m$ is any vector in $T_mM$ satisfying $T_mTQ|_{\Sigma_m} = v_i$. This definition does not depend on the choice of $w_m$ since $\tilde{\theta}$ vanishes on vertical vectors and it gives $\tilde{\theta} = F^*\theta_Q$.

Since the one-form $L_F F^*\theta_Q \in \wedge(M)$ is closed, then $\text{Im} (L_F F^*\theta_Q)$ is a Lagrangian submanifold of $(T^*M, \omega_M)$. Having proposition 2.2 in mind, we obtain from it a Lagrangian submanifold of $(T^*TQ, \omega_{TQ})$

$$\text{Im} (L_F F^*\theta_Q) = \left\{ \mu \in T^*TQ \mid i_{t^*\mu} \in \text{Im} (L_F F^*\theta_Q) \right\},$$

where $i_{t^*\mu}: M \to TQ$ is the canonical inclusion. In coordinates, $\text{Im} (L_F F^*\theta_Q)$ is expressed as

$$\left\{ q^i, \dot{q}^a, \eta, \Gamma (F_i) + \frac{\partial \mu^a}{\partial q^i} F_a - \frac{\partial \eta}{\partial q^i} \tilde{p}_a, F_a + \frac{\partial \mu^a}{\partial q^i} \tilde{p}_a - \frac{\partial \eta}{\partial q^i} \tilde{p}_a \right\}.$$

In particular for $\tilde{p}_a = F_a$ we have

$$\text{Im} \left( \mu_{F,F} \right) \subset \text{Im} (L_F F^*\theta_Q).$$

As $L_F \Omega = 0$, we get that both $\text{Im} (L_F F^*\theta_Q)$ and $\text{Im} (\mu_{F,F})$ need to be Lagrangian and therefore $\text{Im} (\mu_{F,F})$ is isotropic in $(T^*TQ, \omega_{TQ})$.

Finally since $\text{Im} \left( \frac{\partial F}{\partial q^i} \right)$ is injective and $d\tilde{\theta} = \Omega$, $\left( \frac{\partial F}{\partial q^i} \right)$ has maximal rank and $F$ is an immersion. Now we conclude that $F$ is variational according to definition 5.1.

**Remark 5.3.** Note that in the proof above we have described a way to assign to each isotropic submanifold $\Sigma_{F,F}$ a Lagrangian submanifold that contains it and projects over the constraint submanifold, see proposition 2.2. From $L_F \Omega = 0$ we obtain a locally defined function $l: M \to \mathbb{R}$ such that $L_F F^*\theta_Q = dl$. Since $\text{Im} (L_F F^*\theta_Q)$ coincides with $\Sigma_l = \{ \mu \in T^*TQ \mid i^*\mu = dl \} \subset T^*TQ$, the construction from proposition 2.3, it gives the constrained variational dynamics associated to $l$ (see section 3.2). Summing up, given a variational SODE $\Gamma$ on $M$, we can always find a local Lagrangian $l$ on $M$ such that the solutions of $\Gamma$ are constrained variational trajectories for $l$.
Note that in this case we were not addressing the question of finding a Lagrangian \( L: TQ \rightarrow \mathbb{R} \) such that the solutions of the nonholonomic equations for \( L \) coincide with the solutions of \( \Gamma \), but asking when the nonholonomic dynamics can be seen as constrained variational dynamics, see sections 3.1 and 3.2.

In the next, we will study the problem of how to derive a description of the constrained dynamics in terms of a variational problem without constraints (see [6, 32]). We will need the following lemma.

**Lemma 5.4.** Let \( P \) be a smooth manifold, \( C \subset P \) a submanifold and \( \gamma \) a section of \( T^* P \to C \), where \( T^* P \mid_C = \{ \mu \in T^* P : \pi_\mu(\mu) \in C \} \) and \( \pi_\mu : T^* P \to P \) denotes the projection over \( P \). If \( \gamma (C) \) is isotropic in \( (T^* P, \alpha_\mu) \), then there is a one-form \( \tilde{\gamma} \) defined in a neighborhood of \( C \) such that

- \( \tilde{\gamma}_C = \gamma \),
- \( \partial \gamma = 0 \).

**Proof.** Take adapted coordinates \((x^i, y^a)\), \( i = 1, \ldots, n - m \), \( a = 1, \ldots, m \), on \( P \) such that \( C \) is given by \( y^a = 0 \) and denote the corresponding momenta coordinates by \( p_i \) and \( \tilde{p}_a \). Then \( \gamma (C) \) is given by

\[
( x^i, 0, \gamma_i(x), \tilde{p}_a(x)),
\]

and it projects over \( C \). The isotropy condition gives \( \frac{\partial \gamma_i}{\partial x^i} = \frac{\partial \gamma_i}{\partial y^a} \). We want to see \( \gamma (C) \) inside some submanifold \( N \) of \( T^* P \) of dimension \( 2n - m \) and then apply the construction at the end of section 2.1 to extend it to a Lagrangian submanifold via the Hamiltonian vector fields corresponding to the constraints defining \( N \). For that we have many options, for instance we can choose among the constraints

\[
y^a = 0, \quad p_i - \gamma_i = 0, \quad \tilde{p}_a - \tilde{\gamma}_a = 0
\]

and linear combinations of them. If we consider \( \phi_\mu = \tilde{p}_a - \tilde{\gamma}_a \) the Hamiltonian vector field is given by

\[
X_\phi = \frac{\partial}{\partial y^a} + \frac{\partial \gamma_i}{\partial x^i} \frac{\partial}{\partial p_j},
\]

which satisfies \( X_\phi(\gamma_i) = 1 \), so it is not tangent to \( \gamma (C) \). Extending \( \gamma (C) \) along the flows of \( X_\phi \) we obtain

\[
( x^i, y^a, \gamma_i(x) + \frac{\partial \gamma_i}{\partial x^i} y^a, \tilde{p}_a(x)),
\]

which is the image of \( \tilde{\gamma} = dL \) with \( L: P \to \mathbb{R}, \ L(x, y) = \tilde{y}_a(x)y^a + f(x) \), not necessarily regular, and \( \frac{\partial \gamma_i}{\partial y^a} = \gamma_i \). The existence of such a function \( f \) on \( C \) is guaranteed by the isotropy condition.

**Remark 5.5.** Note that there are many possible ways to choose the constraints and construct Lagrangians. For instance taking \( \phi_\mu = y^a + \tilde{p}_a - \gamma_a \) we obtain \( L = \gamma_i y^a + \gamma_i x^i - \frac{\partial \gamma_i}{\partial y^a} y^a \tilde{\gamma} \). On the other hand, if we take \( \phi_\mu = y^a \) then we obtain a Lagrangian submanifold projecting over \( M \) which corresponds to the constrained variational description.
As a consequence of lemma 5.4, taking $\gamma(C) = \Sigma_{F,F}$ we obtain the following important result.

**Theorem 5.6.** If a SODE $\Gamma$ on $M$ is variational, then there exists a Lagrangian $\mathcal{L} : TQ \to \mathbb{R}$ such that the integral curves of $\Gamma$ are the restriction of the solutions of the Euler–Lagrange equations for $\mathcal{L}$ to $M$.

**Example 5.7.** Let $\mathcal{Q} = \mathbb{R}^2$ with coordinates $(x, y)$ and denote fibered coordinates on $TQ$ and $T^*TQ$ by $(x, y, \dot{x}, \dot{y})$ and $(x, y, \dot{x}, \dot{y}, \mu, \rho)$ respectively. Let $N = \{ (x, y, f(x, y, \dot{x})) \} \subset TQ$ be the constraint submanifold and the SODE $\Gamma$ on $N$ be given by $\dot{x} = 0$, $\dot{y} = f(x, y, \dot{x})$. That is, we have the dynamics given by

$$\dot{x} = 0, \quad \dot{y} = f(x, y, \dot{x}).$$

We define $F : N \to T^*Q$ by $F(x, y, \dot{x}) = (x, y, \dot{x} + y, x)$, which is an immersion. Then $\Sigma_{F,F} \subset T^*TQ$ is locally described by $(x, y, \dot{x}, f, \dot{x}, \dot{y}, \dot{x} + y, \dot{y})$ and is an isotropic submanifold of dimension 3, for $dx \wedge df + dy \wedge d\dot{x} + d\dot{y} \wedge d(\dot{y} + y) + df \wedge dx = 0$. Note that

$$\gamma = \Sigma_{F,F} \Gamma F,$$

is recovered. Therefore $\text{Im} \Sigma_{F,F} \subset T^*TQ$ is locally described by

$$\left( x, y, \dot{x}, f, f + x \frac{\partial f}{\partial x} - \frac{\partial f}{\partial \rho}, \dot{x} + x \frac{\partial f}{\partial y} - \frac{\partial f}{\partial \dot{\rho}}, \dot{x} + y + x \frac{\partial f}{\partial \dot{x}} - \frac{\partial f}{\partial \dot{\rho}}, \dot{x}, \dot{y} \right).$$

When $\rho = x$, $\Sigma_{F,F}$ is recovered. Since $d\mathcal{L}_F F^*\theta_Q = 0$, we have a local Lagrangian $l : N \to \mathbb{R}$,

$$l = \frac{x^2}{2} + \dot{x}y + x\dot{f}(x, y, \dot{x}),$$

satisfying

$$\frac{\partial l}{\partial x} = f + x \frac{\partial f}{\partial x}, \quad \frac{\partial l}{\partial y} = \dot{x} + x \frac{\partial f}{\partial y}, \quad \frac{\partial l}{\partial \dot{x}} = \dot{x} + y + x \frac{\partial f}{\partial \dot{x}}.$$

Note that $l$ is the restriction of the singular Lagrangian $L_4 = \frac{x^2}{2} + \dot{x}y + x\dot{f}$ to $\dot{y} = f$.

Consider the constraint $\phi = \dot{y} - f + \dot{\rho} - x$ and the corresponding Hamiltonian vector field for the symplectic structure $\omega_{TQ}$:

$$X_\phi = -\frac{\partial}{\partial \dot{\rho}} + \frac{\partial}{\partial \dot{x}} \frac{\partial}{\partial \rho} + \frac{\partial}{\partial \dot{y}} \frac{\partial}{\partial \dot{\rho}} + \frac{\partial}{\partial \dot{x}} \frac{\partial}{\partial \dot{\rho}} + \frac{\partial}{\partial \dot{y}} + \frac{\partial}{\partial \rho}.$$

If we extend the isotropic submanifold $\Sigma_{F,F}$ along its flow we obtain the Lagrangian submanifold

$$\left( x, y, \dot{x}, \dot{y}, \dot{y} + x \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}, \dot{x} + \frac{\partial f}{\partial y} - \frac{\partial f}{\partial \dot{y}}, \dot{x} + \dot{y} + x \frac{\partial f}{\partial \dot{x}} - \frac{\partial f}{\partial \dot{y}}, \dot{x}, \dot{y} + f \right),$$

which is the image of $dL_2$ with $L_2 = x\dot{y} - \frac{x^2}{2} + f\dot{y} + \frac{y^2}{2} + \dot{y} - \frac{f^2}{2}$, another extension of $l$. However, this is a regular Lagrangian since $\det \left( \frac{\partial^2 l}{\partial \dot{x} \partial \dot{y}} \right) = -1 - y \frac{\partial \dot{f}}{\partial \dot{x}} + f \frac{\dot{y}}{\dot{x}}$, which does not vanish in a neighborhood of $\Sigma_{F,F}$. It is possible to recover $\Gamma$ by computing the corresponding Euler–Lagrange equations and restricting them to $M$. 19
Example 5.8 Vertical rolling disk. Consider the configuration space \( Q = S^1 \times S^1 \times \mathbb{R}^2 \) with coordinates \((\theta, \varphi, x, y)\), where \(\theta\) denotes the angle of rotation, \(\varphi\) the angle between the direction in which the disk moves and the x-axis and \((x, y)\) are the coordinates of the contact point. We consider the Lagrangian \( L = \frac{1}{2}(\dot{\theta}^2 + \dot{\varphi}^2 + \dot{x}^2 + \dot{y}^2) \) and the constraints given by the condition of rolling without sliding are \( \dot{x} = \cos(\varphi)\dot{\theta} \) and \( \dot{y} = \sin(\varphi)\dot{\theta} \).

We know that for the rolling disk the nonholonomic equations are
\[
\dot{\theta} = 0, \quad \dot{\varphi} = 0, \quad \dot{x} = \cos(\varphi)\dot{\theta}, \quad \dot{y} = \sin(\varphi)\dot{\theta},
\]
and the variational constrained ones are
\[
2\dot{\theta} = \dot{\varphi}(-A \sin(\varphi) + B \cos(\varphi)), \quad \dot{\varphi} = \dot{\theta}(A \sin(\varphi) - B \cos(\varphi)),
\]
where \(A\) and \(B\) are constants (see [5]). Taking \(A = B = 0\) we see that the set of nonholonomic solutions is contained in the set of variational constrained ones. Now consider the constrained Lagrangian \( L(\theta, \varphi, x, y, \dot{\theta}, \dot{\varphi}, \dot{x}, \dot{y}) = \dot{\theta}^2 + \frac{\dot{\varphi}^2}{2} \) and define \( F \) as the Legendre transformation associated to the following extension \( L(\theta, \varphi, x, y, \dot{\theta}, \dot{\varphi}, \dot{x}, \dot{y}) = \dot{\theta}^2 + \frac{\dot{\varphi}^2}{2} \), that is
\[
F \equiv \text{Leg}_L: \quad M \longrightarrow T^*Q
\]
\[
(\theta, \varphi, x, y, \dot{\theta}, \dot{\varphi}) \longmapsto (\theta, \varphi, x, y, 2\dot{\theta}, \dot{\varphi}, 0, 0).
\]
As \( \Gamma^1 = \Gamma^2 = 0 \), the submanifold \( \Sigma_{\Gamma,F} \subset T^*TQ \) can be locally described by
\[
(\theta, \varphi, x, y, \dot{\theta}, \dot{\varphi}, \cos(\varphi)\dot{\theta}, \sin(\varphi)\dot{\theta}, 0, 0, 0, 0, 0, 0).
\]
It is isotropic and has dimension 6, so we want to choose two constraint functions on \( T^*TQ \) satisfied by \( \Sigma_{\Gamma,F} \) and extend it in the corresponding directions. First we take the constraints
\[
\phi_1 = \dot{x} - \cos(\varphi)\dot{\theta} + \bar{\mu}_1, \quad \phi_2 = \dot{y} - \sin(\varphi)\dot{\theta} + \bar{\mu}_2,
\]
with corresponding Hamiltonian vector fields
\[
X_{\phi_1} = -\frac{\partial}{\partial \bar{\mu}_1} + \cos(\varphi)\frac{\partial}{\partial \bar{\mu}_\theta} - \sin(\varphi)\frac{\partial}{\partial \bar{\mu}_\varphi} + \frac{\partial}{\partial \bar{\mu}_x},
\]
\[
X_{\phi_2} = -\frac{\partial}{\partial \bar{\mu}_2} + \sin(\varphi)\frac{\partial}{\partial \bar{\mu}_\theta} + \cos(\varphi)\frac{\partial}{\partial \bar{\mu}_\varphi} + \frac{\partial}{\partial \bar{\mu}_y}.
\]
Extending \( \Sigma_{\Gamma,F} \) along the flows of \( X_{\phi_1} \) and \( X_{\phi_2} \) we obtain the Lagrangian submanifold with local expression
\[
(\theta, \varphi, x, y, \dot{\theta}, \dot{\varphi}, \dot{x}, \dot{y}, 0, \dot{\theta}(\cos(\varphi)\dot{y} - \sin(\varphi)\dot{x}), 0, 0,
\]
\[
\dot{\theta} + \cos(\varphi)\dot{x} + \sin(\varphi)\dot{y}, \dot{\varphi}, -\dot{x} + \cos(\varphi)\dot{\theta}, -\dot{y} + \sin(\varphi)\dot{\theta})
\]
which is the image of \( \delta\bar{L} \) with \( \bar{L} = \frac{1}{2}(\dot{\theta}^2 + \dot{\varphi}^2 - \dot{x}^2 - \dot{y}^2) + \dot{\theta}(\cos(\varphi)\dot{x} + \sin(\varphi)\dot{y}) \). So we have obtained a regular Lagrangian whose unconstrained trajectories include the nonholonomic trajectories of the first Lagrangian. This is the same Lagrangian as the one obtained in [20].

If we take \( \phi_1 = \bar{\mu}_1, \quad \phi_2 = \bar{\mu}_2 \) then we obtain the Lagrangian submanifold
\[
(\theta, \varphi, x, y, \dot{\theta}, \dot{\varphi}, \dot{x}, \dot{y}, 0, 0, 0, 0, 0, 2\dot{\theta}, \dot{\varphi}, 0, 0)
\]
and recover the singular Lagrangian function \( L = \dot{\theta}^2 + \frac{\dot{\varphi}^2}{2} \).
For $\phi_1 = \dot{x} - \cos(\varphi)\dot{\varphi}$, $\phi_2 = \dot{y} - \sin(\varphi)\dot{\varphi}$ we get the Lagrangian submanifold

\[
\left( \theta, \varphi, x, y, \dot{\theta}, \dot{\varphi}, \cos(\varphi)\dot{\theta}, \sin(\varphi)\dot{\theta}, 0, \dot{\theta} \left( \mu_1 \sin(\varphi) - \mu_2 \cos(\varphi) \right), 0, 0, 2\dot{\theta} - \mu_1 \cos(\varphi) - \mu_2 \sin(\varphi), \dot{\varphi}, \mu_1, \mu_2 \right)
\]

which coincides with $\text{Im}(\mathcal{L}_F \mathcal{F}^* \theta_Q)$, for $\frac{\partial \varphi^1}{\partial \theta} = \cos(\varphi)$ and $\frac{\partial \varphi^2}{\partial \theta} = \sin(\varphi)$, where $\varphi^1 = \cos(\varphi)\dot{\theta}$, $\varphi^2 = \sin(\varphi)\dot{\theta}$. Therefore, we obtain the variational constrained equations for the constrained Lagrangian $I: M \rightarrow \mathbb{R}$.

Now we find another immersion $F: M \rightarrow T^*Q$ that makes $\Sigma_{F,F}$ isotropic. After extending it we get new Lagrangian functions defined on $TQ$.

We make the following assumptions on the dependence of coordinates of

\[
F_\rho(\theta, \varphi) = F_\xi(\theta, \varphi) = F_\gamma(\theta, \varphi). \quad F_\rho(\varphi, \theta, \varphi).
\]

Then the only constrained Helmholtz equations (15)–(17) that do not vanish identically are

\[
\frac{\partial F_\rho}{\partial \theta} = (1 + \cos(\varphi) + \sin(\varphi)) \frac{\partial F_\varphi}{\partial \varphi}, \quad (18)
\]

\[
0 = \dot{\varphi} \frac{\partial^2 F_\rho}{\partial \theta \partial \varphi} + \ddot{\theta} (\cos(\varphi) - \sin(\varphi)) \frac{\partial F_\varphi}{\partial \theta}, \quad (19)
\]

\[
\frac{\partial F_\varphi}{\partial \varphi} = \frac{\partial}{\partial \varphi} \left( \frac{\partial F_\varphi}{\partial \varphi} \dot{\varphi} \right) + \ddot{\theta} (\cos(\varphi) - \sin(\varphi)) \frac{\partial F_\varphi}{\partial \varphi}, \quad (20)
\]

and $F_\theta = F_\xi = F_\gamma = \frac{\dot{\theta}}{\dot{\varphi}}$, $F_\varphi = \rho(\varphi) - \frac{\dot{\theta}^2}{2\dot{\varphi}}(1 + \cos(\varphi) + \sin(\varphi))$ is a solution, where $\rho: \mathbb{R} \rightarrow \mathbb{R}$ is arbitrary.

Setting $\rho(\varphi) = \dot{\varphi}$, define

\[
F: \quad M \quad \rightarrow \quad T^*Q
\]

\[
(\theta, \varphi, x, y, \dot{\theta}, \dot{\varphi}) \quad \mapsto \quad \left( \theta, \varphi, x, y, \frac{\dot{\theta}}{\dot{\varphi}}, \varphi - \frac{\dot{\theta}^2}{2\dot{\varphi}^2} (1 + \cos(\varphi) + \sin(\varphi)), \frac{\dot{\theta}}{\dot{\varphi}}, \frac{\dot{\varphi}}{\dot{\varphi}} \right)
\]

to get $\Sigma_{F,F}$ given by

\[
\left\{ \theta, \varphi, x, y, \cos(\varphi)\dot{\theta}, \sin(\varphi)\dot{\theta}, \dot{\theta}, \dot{\varphi}, 0, \frac{1}{2} \frac{\dot{\theta}^2}{\dot{\varphi}} (\sin(\varphi) - \cos(\varphi)), \right. \left. 0, 0, \frac{\dot{\theta}}{\dot{\varphi}}, \frac{\dot{\varphi}}{\dot{\varphi}} - \frac{\dot{\theta}^2}{2\dot{\varphi}^2} (1 + \cos(\varphi) + \sin(\varphi)), \frac{\dot{\theta}}{\dot{\varphi}}, \frac{\dot{\varphi}}{\dot{\varphi}} \right\}
\]

which is isotropic of dimension 6 on $(T^*TQ, \omega_{TQ})$. 

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If we take $\phi_\mu = -\frac{\theta}{\varphi}$ and $\phi_\nu = -\frac{\theta}{\varphi}$, the corresponding Hamiltonian vector fields are

$$
X_{\phi_\mu} = \frac{\partial}{\partial x} + \frac{1}{\varphi} \frac{\partial}{\partial \rho} - \frac{\partial}{\varphi^2} \frac{\partial}{\partial \rho}, \\
X_{\phi_\nu} = \frac{\partial}{\partial y} + \frac{1}{\varphi} \frac{\partial}{\partial \rho} - \frac{\partial}{\varphi^2} \frac{\partial}{\partial \rho}.
$$

Extending $\Sigma_{\Gamma,F}$ along the flows of $X_{\phi_\mu}$ and $X_{\phi_\nu}$ we get

$$
\begin{align*}
\left( \theta, \varphi, x, y, \dot{\theta}, \dot{\varphi}, \dot{x}, \dot{y}, 0, 0, \frac{1}{\varphi} \frac{\partial^2}{\partial \varphi} (\sin(\varphi) - \cos(\varphi)), 0, 0, \frac{\dot{\theta}}{\varphi} (1 - \sin(\varphi) - \cos(\varphi)) \right) \\
+ \left( \frac{\dot{x} + \dot{y}}{\varphi}, \dot{\varphi} - \frac{\dot{\theta}^2}{2\varphi^2} (1 - \sin(\varphi) - \cos(\varphi)) - \frac{\dot{\theta}}{\varphi^2} (\dot{x} + \dot{y}), \frac{\dot{\theta}}{\varphi}, \frac{\dot{\theta}}{\varphi}, \frac{\dot{\theta}}{\varphi} \right),
\end{align*}
$$

which is the image of $dL$ for the singular Lagrangian

$$
L = \frac{\dot{\varphi}^2}{2} + \frac{\dot{\theta}^2}{\varphi} \left( \frac{1}{2} - \cos(\varphi) - \sin(\varphi) \right) + \frac{\dot{\theta}}{\varphi} (\dot{x} + \dot{y}).
$$

Now we choose constraints $\phi_1 = \dot{x} - \cos(\varphi) \dot{\theta} + \mu_1 - \frac{\theta}{\varphi}$, $\phi_2 = \dot{y} - \sin(\varphi) \dot{\theta} + \mu_2 - \frac{\theta}{\varphi}$ with Hamiltonian vector fields

$$
X_{\phi_1} = -\frac{\partial}{\partial \mu_1} + \left( \cos(\varphi) + \frac{1}{\varphi} \frac{\partial}{\partial \rho} \right) \frac{\partial}{\partial \theta} - \frac{\partial}{\varphi^2} \frac{\partial}{\partial \rho} - \dot{\theta} \sin(\varphi) \frac{\partial}{\partial \rho} + \frac{\partial}{\partial x}, \\
X_{\phi_2} = -\frac{\partial}{\partial \mu_2} + \left( \sin(\varphi) + \frac{1}{\varphi} \frac{\partial}{\partial \rho} \right) \frac{\partial}{\partial \theta} - \frac{\partial}{\varphi^2} \frac{\partial}{\partial \rho} + \dot{\theta} \cos(\varphi) \frac{\partial}{\partial \rho} + \frac{\partial}{\partial y}.
$$

Extending $\Sigma_{\Gamma,F}$ along their flows we obtain

$$
\begin{align*}
\left( \theta, \varphi, x, y, \dot{\theta}, \dot{\varphi}, \dot{x}, \dot{y}, 0, 0, \frac{1}{\varphi} \frac{\partial^2}{\partial \varphi} (\sin(\varphi) - \cos(\varphi)) - \dot{x} \dot{\theta} \sin(\varphi) + \dot{\theta} \cos(\varphi), 0, 0, \\
\frac{\dot{\theta}}{\varphi} (1 - \cos(\varphi) - \sin(\varphi)) + \dot{x} \cos(\varphi) - \dot{x} + \frac{\dot{x} + \dot{y}}{\varphi} + \dot{y} \sin(\varphi), \\
\dot{\varphi} - \frac{\dot{\theta}^2}{2\varphi^2} (1 - \sin(\varphi) - \cos(\varphi)) - \frac{\dot{\theta}}{\varphi^2} (\dot{x} + \dot{y}), \\
\frac{\dot{\theta}}{\varphi} = \dot{x} + \cos(\varphi) \dot{\theta}, \frac{\dot{\theta}}{\varphi} = \dot{y} + \sin(\varphi) \dot{\theta},
\end{align*}
$$

which is the image of $d\bar{L}$ for

$$
\bar{L} = \frac{1}{2} \left( \dot{\varphi}^2 - \dot{\theta}^2 - \dot{x}^2 - \dot{y}^2 \right) + \frac{\dot{\theta}^2}{2\varphi} (1 - \cos(\varphi) - \sin(\varphi)) \\
+ \dot{x} \left( \cos(\varphi) + \frac{1}{\varphi} \right) + \dot{y} \left( \sin(\varphi) + \frac{1}{\varphi} \right).
$$
As
\[
\det\left( \frac{\partial^2 L}{\partial q^i \partial q^j} \right) = \frac{1}{\phi^3} \left( -2\phi^2 (1 - \sin(\phi) - \cos(\phi)) - \phi^2 \phi + 2\phi^3 + \phi^4 (1 + \sin(\phi) + \cos(\phi)) \right).
\]
observe that this Lagrangian is regular except at a hypersurface of singular points.

**Example 5.9 (Nonholonomic particle).** Consider the system defined by \( Q = \mathbb{R}^3 \), \( L = \frac{1}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \) and constraint \( \dot{z} = -x\dot{y} \). The nonholonomic SODE is given by \( \Gamma^1 = 0, \Gamma^2 = -\frac{1}{1+x^2} \). This SODE is variational as a constrained system as we will see. Indeed, in [6] the authors show that this system can be represented as the restriction of the Euler–Lagrange vector field associated to a Lagrangian defined on the full space \( TQ \). In our framework, we define the map
\[
F: \quad M \rightarrow T^*Q \quad \text{by} \quad (x, y, z, \dot{x}, \dot{y}) \mapsto \left( x, y, z, \dot{x} - \frac{\dot{y}^2}{2\dot{x}} \sqrt{1 + x^2} (1 + x), \frac{\sqrt{1 + x^2} \dot{y}}{\dot{x}}, -\frac{\sqrt{1 + x^2} \dot{y}}{\dot{x}} \right)
\]
then \( \Sigma_{\Gamma,F} \) is given by
\[
\left\{ x, y, z, \dot{x}, \dot{y}, -x\dot{y}, -\frac{\dot{y}^2 (1 - x)}{2\dot{x} \sqrt{1 + x^2}}, 0, 0, \right.
\]
\[
\left. \dot{x} - \frac{\dot{y}^2}{2\dot{x} \sqrt{1 + x^2}} \sqrt{1 + x^2} (1 + x), \frac{\sqrt{1 + x^2} \dot{y}}{\dot{x}}, -\frac{\sqrt{1 + x^2} \dot{y}}{\dot{x}} \right\}
\]
and is isotropic in \((T^*TQ, \omega_{TQ})\). Also \( L_F F^*\theta_Q = dl \) for
\[
l = \frac{x^2}{2} + \frac{\dot{y}^2 \sqrt{1 + x^2}}{\dot{x}} (1 + x).
\]
Note that \( l \not\equiv L \big|_M = \frac{1}{2} (x^2 + \dot{y}^2 (1 + x^2)) \). Since \( \Sigma_{\Gamma,F} \subset \Sigma_l \), the solutions of \( \Gamma \) can be seen as constrained variational for \( l \), although not for \( L \big|_M \) (see [20]).

Now we look for a Lagrangian on \( TQ \). Taking \( \phi = \mu + \frac{\sqrt{1 + x^2}}{x} \) as constraint and extending \( \Sigma_{\Gamma,F} \) along the flow of
\[
X_\phi = \frac{\partial}{\partial z} - \frac{\dot{y}^2 (1 - x)}{2\dot{x} \sqrt{1 + x^2}} \frac{\partial}{\partial \dot{x}} + \frac{\sqrt{1 + x^2} \dot{y}}{\dot{x}^2} \frac{\partial}{\partial \dot{y}} - \frac{\dot{y}}{\dot{x}} \frac{x}{\sqrt{1 + x^2}} \frac{\partial}{\partial \mu}
\]
we get
\[
\left\{ x, y, z, \dot{x}, \dot{y}, \dot{z}, -\frac{\dot{y}^2 (1 - x)}{2\dot{x} \sqrt{1 + x^2}} - \frac{\dot{y}}{\dot{x} \sqrt{1 + x^2}} (z + xy), 0, 0, x + \frac{\sqrt{1 + x^2}}{2\dot{x} \sqrt{1 + x^2}} \dot{y}^2 (x - 1), \right.
\]
\[
\left. \frac{\sqrt{1 + x^2} \dot{y}}{\dot{x}} (1 - x) - \frac{\sqrt{1 + x^2}}{\dot{x}} \dot{z}, -\frac{\sqrt{1 + x^2}}{\dot{x}} \dot{y} \right\}
\]
generated by the regular Lagrangian
\[ L = \frac{\dot{x}^2}{2} + \frac{(1 - x)\sqrt{1 + x^2}}{2\dot{x}}y' - \frac{\sqrt{1 + x^2}}{\dot{x}}\dot{y}. \]

**Remark 5.10.** If a SODE \( \Gamma \) on \( M \) is variational, from theorem 5.6 we know that there exists a Lagrangian function such that its associated Euler–Lagrange vector field \( \Gamma_L \) verifies \( (\Gamma_L) |_M = \Gamma \). Since \( \Omega_L = \text{d}(\dot{E}_L) \) and \( \Gamma_M = \text{d}(E_L) \) then \( i_{\Gamma_L}\Omega_M = d(E_L) \) \( |_M \). As a result, the flow of \( \Gamma \) preserves the two-form \( \Omega_M \) (this result is also a direct consequence of theorem 5.2). Hence, \( \mathcal{L}_\Gamma \Omega_M^k = 0 \), for all \( k \), giving information about the qualitative behavior of the flow of \( \Gamma \).

Additionally, if we derive a constant of motion \( \mathcal{I} : \Omega \to \mathbb{R} \) for \( \Gamma_L \), then the restriction of \( \mathcal{I} \) to \( M \) is also a constant of motion of \( \Gamma \). Thus, \( \mathcal{L}_\Gamma \mathcal{I} = 0 \) for all \( k \), giving information about the qualitative behavior of the flow of \( \Gamma \).

6. The inverse problem for holonomic constraints

A particular case of constrained systems is given by a submanifold \( M \) of \( TQ \) which is precisely a tangent bundle of a submanifold \( N \) of \( Q \), this is the case of holonomic constraints. In other words, \( M = TN \). In many cases of interest it is useful to work extrinsically, that is, on the manifold \( Q \) instead of intrinsically, that is, on \( N \). As a result, the system on \( N \) is described in terms of a system on \( Q \). Assume that \( TN \) is locally described by the vanishing of the constraints
\[ \psi^\alpha(q^a, \dot{q}^a) = 0 \quad \text{and} \quad \frac{\partial \psi^\alpha}{\partial q^a} \dot{q}^a + \frac{\partial \psi^\alpha}{\partial \dot{q}^a} \ddot{q}^a = 0, \quad 1 \leq \alpha \leq m. \]

For simplicity and without loss of generality, we consider the local coordinates on \( Q \) adapted to \( N \) and the corresponding local coordinates on \( TQ \) adapted to \( TN \), so that \( N = \{(q^a, \dot{q}^a) \in Q \mid q^a = 0\} \), and \( TN = \{(q^a, \dot{q}^a, \ddot{q}^a, 
\dddot{q}^a) \in TQ \mid q^a = 0, \dot{q}^a = 0\} \), where \( a = 1, \ldots, n - m \). The SODE \( \Gamma \) on \( TN \) is locally described by
\[ \Gamma(q^a, \dot{q}^a) = (q^a, \dot{q}^a, \ddot{q}^a, \dddot{q}^a). \]

The difference between holonomic dynamics and the nonholonomic one considered in section 5 is that \( M = TN \) does not project over the entire \( Q \). Thus, the notion of variational SODE for constrained systems in definition 4.1 must be adapted, because if \( M \) does not project over the entire \( Q \), \( F : M \to T^*Q \) might not be an immersion.

**Definition 6.1.** Let \( \Gamma \) be a SODE along \( M \) and assume that \( N = \tau_Q(M) \) is a submanifold so that we have the canonical inclusion \( i_{TN} : TN \to TQ \). The SODE \( \Gamma \) is variational if there exists a function \( F : M \to T^*Q \) such that the map \( \tau_{TN} \circ F |_M : M \cap TN \to T^*N \) is an immersion and \( \Sigma_\Gamma = \text{Im}(\tau_Q \circ T^*F \circ \Gamma) \) is an isotropic submanifold of \( (T^*TQ, \omega_{TQ}) \), where \( i_{TN} \) is the transpose map of \( i_{TN} \) as defined below in (21).
With this adapted notion of a variational SODE for holonomic constraints theorem 5.2 can be also proved similarly as the proof in section 5 when M projects onto the entire Q.

Our interest now is to establish a relationship between the inverse problem without constraints when we work intrinsically on TN and the inverse problem with the holonomic constraints, when we work extrinsically on TQ.

\[ T^*TN \xrightarrow{\alpha_N} TT^* N \xrightarrow{Tf} TTN \xrightarrow{T\mu} TT^* Q \xrightarrow{\alpha_Q} T^*TQ \]

\[ T^*N \xleftarrow{\gamma} TN \xrightarrow{F} T^*Q \]

**Theorem 6.2.** A SODE \( \Gamma \) on TN is variational for the inverse problem of the calculus of variations without constraints if and only if it is variational along the submanifold TN of TQ in the inverse problem for constrained systems.

**Proof.** \( \Rightarrow \) If \( \Gamma \) is variational for the unconstrained system on TN, then there exists a regular Lagrangian \( l : TN \rightarrow \mathbb{R} \) whose solutions of the Euler–Lagrange equations are also integral curves of the SODE \( \Gamma \) and vice versa. The function \( f : TN \rightarrow T^*N \) in the above diagram is the Legendre transformation of \( l \), that is, \( f(q, \dot{q}) = \text{Leg}_l(q, \dot{q}) = (q, \partial l/\partial \dot{q}) \). Moreover, \( \text{Im} (\mu_{\Gamma,f}) \) is a Lagrangian submanifold of \( (T^*TN, \omega_N) \).

Let \( i_{TN} : TN \rightarrow TQ \) be the inclusion and consider an arbitrary fiber function \( F : TN \rightarrow T^*Q \) such that the following diagram is commutative:

\[ TN \xrightarrow{F} T^*_N Q \]

where \( i^*_TN \) is the transpose map of \( i_{TN} \) defined by

\[ \left\{ i^*_TN(p_q), v_q \right\} = \left\{ p_q, \ i_{TN}(v_q) \right\}, \quad (21) \]

where \( p_q \in T_q^*Q, v_q \in TN, \ (\tau_Q \circ i_{TN})(v_q) = \pi_Q(p_q), \ q \in N, \ \dot{q} \in Q, \ \pi_N(q) = \dot{q}. \)

Since \( l : TN \rightarrow \mathbb{R} \) is regular (that is, \( \text{Leg}_l : TN \rightarrow T^*N \) is a local diffeomorphism), it is easy to deduce that \( F : TN \rightarrow T^*Q \) is an immersion. In local coordinates, the function \( F \) looks like

\[ F : TN \rightarrow T^*Q \]

\[ (q^a, \dot{q}^a) \mapsto \left( q^a, 0, \frac{\partial l}{\partial q^a} \ F_a \left( q^b, \dot{q}^b \right) \right) \]

where \( F_a \) are arbitrary functions on TN.
The local expression in adapted coordinates of the submanifold $\text{Im}(\mu_{\Gamma,F})$ of $T^*TQ$ is

$$\left(q^a, 0, \dot{q}^a, 0; \frac{\partial^2 l}{\partial q^a \partial \dot{q}^b} \dot{q}^b + \frac{\partial^2 l}{\partial q^a \partial \dot{q}^b} \dot{\Gamma}^b + \frac{\partial F_a}{\partial q^b} \dot{q}^b + \frac{\partial F_a}{\partial \dot{q}^b} \dot{\Gamma}^b, \dot{\Gamma}^b, \dot{\Gamma}^b, F_a \right).$$

This submanifold is isotropic if $(\mu_{\Gamma,F})^*(\omega_{TQ})$ is equal to zero, equivalently

$$d\left(\frac{\partial^2 l}{\partial q^a \partial \dot{q}^b} \dot{q}^b + \frac{\partial^2 l}{\partial q^a \partial \dot{q}^b} \dot{\Gamma}^b\right) \wedge d\dot{q}^a + d\left(\frac{\partial l}{\partial q^a}\right) \wedge d\dot{q}^a = d^2 l = 0$$

because $\Gamma$ is the Euler–Lagrange vector field for $l: T\mathcal{N} \to \mathbb{R}$, that is, locally

$$\frac{d}{dt} \left(\frac{\partial l}{\partial \dot{q}^a}\right) = \frac{\partial^2 l}{\partial q^a \partial \dot{q}^b} \dot{q}^b + \frac{\partial^2 l}{\partial q^a \partial \dot{q}^b} \dot{\Gamma}^b = \frac{\partial l}{\partial q^a}.$$

Assuming now that $\Gamma$ is variational for the inverse problem with constraints, then there exists $F: T\mathcal{N} \to T^*\mathcal{N}$ such that the map $(i_{\mathcal{N}} \circ F): T\mathcal{N} \to T^*\mathcal{N}$ is an immersion and $\text{Im}(\mu_{\Gamma,F})$ is isotropic in $(T^*TQ, \omega_{TQ})$. Now we find a solution of the inverse problem of the calculus of variations (without constraints) by taking $f = i_{\mathcal{N}} \circ F: T\mathcal{N} \to T^*\mathcal{N}$. In coordinates, $f(q^a, \dot{q}^a) = (q^a, F_a(q^b, \dot{q}^b))$. Obviously, $\text{Im}(\mu_{\Gamma,F})$ is Lagrangian in $(T^*\mathcal{N}, \omega_{\mathcal{N}})$ and $f$ is a local diffeomorphism.

This result can be also proved intrinsically because $f$ and $F$ must make the following diagram commutative:

$$\begin{array}{ccc}
TN & \xrightarrow{F} & T^*_N Q \\
\downarrow & & \downarrow f \\
T^* \mathcal{N} & \xrightarrow{i_{\mathcal{N}}} & T^*_N \mathcal{N}
\end{array}$$

Note that the diagram is commutative if $F_a = f_a$, but the remaining $F_a$ are arbitrary. It can be easily proved that $f^* \theta_{\mathcal{N}} = F^* \theta_{\mathcal{N}}$. Then the two-form characterizing the inverse problem for the calculus of variations, theorem 3.1, and the one characterizing the inverse problem for the constrained systems, theorem 4.2, coincide. This concludes the proof.

Let $\Gamma$ be a SODE on $T\mathcal{N}$ which is the Euler–Lagrange vector field corresponding to a regular Lagrangian $l: T\mathcal{N} \to \mathbb{R}$. Applying theorem 6.2 we obtain an isotropic submanifold of $(T^*TQ, \omega_{TQ})$ by simply taking $\text{Im}(\mu_{\Gamma,F})$ for any map $F: M \to T^*\mathcal{N}$ verifying

$$i_{\mathcal{N}} \circ F = \text{Leg}_l,$$

where Leg: $TN \to T^*\mathcal{N}$ is the Legendre transformation associated to $l: T\mathcal{N} \to \mathbb{R}$.

Recall that in section 5 for the case of a submanifold projecting over the entire $\mathcal{N}$, we saw that a constrained variational SODE could be seen as the restriction of a variational SODE on $T\mathcal{N}$, theorem 5.6. In order to do this we just need to find a Lagrangian submanifold projecting over the entire $T\mathcal{N}$ and containing $\text{Im}(\mu_{\Gamma,F})$ which in this case has the expression

$$\left(q^a, 0, \dot{q}^a, 0; \frac{\partial^2 l}{\partial q^a \partial \dot{q}^b} \dot{q}^b + \frac{\partial^2 l}{\partial q^a \partial \dot{q}^b} \dot{\Gamma}^b + \frac{\partial F_a}{\partial q^b} \dot{q}^b + \frac{\partial F_a}{\partial \dot{q}^b} \dot{\Gamma}^b, \dot{\Gamma}^b, \dot{\Gamma}^b, F_a \right).$$
If we take a Lagrangian \( L: TQ \to \mathbb{R} \) such that \( L_{TN} = l \) and verifying
\[
\frac{\partial L}{\partial q^a} = \frac{\partial^2 L}{\partial q^a \partial \dot{q}^a} \dot{q}^a + \frac{\partial^2 L}{\partial q^a \partial \dot{q}^a} \Gamma^a
\]
on \( TN \), then we can define \( F = L_{TN}: TN \to T^*Q \) and get \( \text{Im}(\mu_{F,T}) \subset \delta L \).

For instance, in adapted coordinates to \( TN \), we can take any Lagrangian \( L: TQ \to \mathbb{R} \) of the form
\[
L(q, \dot{q}) = l(q^a, \dot{q}^a) + \frac{1}{2}(\dot{q}^a)^2 A_a(q, \dot{q}) + \frac{1}{2}(q^a)^2 B_a(q, \dot{q}),
\]
where \( A_a, B_a \in C^\infty(TQ) \). Obvioulsy
\[
F(q^a, \dot{q}^a) = \left( q^a, 0, \frac{\partial l}{\partial q^a}, 0 \right).
\]

Therefore, we conclude that the solutions of the holonomic problem given by \( l \) are included in the solutions of \( L \) with initial conditions given on \( TN \).

**Example 6.3.** Planar pendulum of length \( h \) with a particle of mass \( m \). In this case \( TN = TS^1 \) and \( TQ = T\mathbb{R}^2 \). The local adapted coordinates are \((q^1, q^2) = (\theta, r - h)\). We consider the SODE \( \Gamma \) on \( TS^1 \) coming from the Lagrangian \( l : TS^1 \to \mathbb{R} \)
\[
l(\theta, \dot{\theta}) = \frac{1}{2}m\dot{\theta}^2 - mg\cos \theta.
\]
In this case \( f(\theta, \dot{\theta}) = (\theta, m\dot{\theta}) \) and we could take \( F(\theta, \dot{\theta}) = \left( \theta, 0, m\dot{\theta}, F_2(\theta, 0, \dot{\theta}, 0) \right) \).

Proposition 6.2 guarantees that \( \text{Im}(\mu_{F,T}) \) is isotropic in \((T^*TQ, \omega_{TQ})\). A choice of Lagrangian \( L: TQ \to \mathbb{R} \) associated with that \( F \) is
\[
L = \frac{1}{2}m\dot{\theta}^2 - mg\cos \theta + \frac{1}{2}r^2 A(q, \dot{q}) + \frac{1}{2}(r - h)^2 B(q, \dot{q}),
\]
and a regular one is, for instance
\[
L = l + \frac{1}{2}r^2 + B(q, \dot{q})(r - h)^2.
\]

### 7. Time-dependent inverse problem for unconstrained and constrained systems

In this section we show how, for time-dependent systems, we can recover theorems 4.2 and 5.2 by considering Lagrangian and isotropic submanifolds of a Poisson manifold, more precisely \((T(R \times T^*Q), \{\cdot, \cdot\}_T)\), where \(\{\cdot, \cdot\}_T\) denotes the tangent lift of the projected Poisson structure of \( T^*(R \times Q) \) onto \( R \times T^*Q \). We recall these definitions:

**Definition 7.1** ([11, 22]). Let \((P, \{\cdot, \cdot\})\) be a Poisson manifold. The tangent Poisson bracket is given by
\[
\{f^c, g^c\}^T = \{f, g\}^c, \quad \{f^c, g^v\}^T = \{f, g\}^v, \quad \{f^v, g^v\}^T = 0,
\]
where \(f^c\) and \(f^v\) denote respectively the complete and vertical lift of \(f \in C^\infty(P)\) (see [47]).

If \((x^i)\) denote local coordinates in \( P \) and the Poisson bivector is given by \( \Lambda = \frac{1}{2}(A^a(x) \frac{\partial}{\partial x^a} \wedge \frac{\partial}{\partial x^a} \), then
\[ d_T \Lambda = A^T := A^0(x) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} + \frac{1}{2} \frac{\partial A^i}{\partial x^j} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} \]

is the Poisson bivector corresponding to the bracket \( \{ \cdot, \cdot \} \).

**Definition 7.2** ([42, 44]). Let \((P, \{ \cdot, \cdot \})\) be a Poisson manifold and \(N\) be a submanifold of \(P\). Denote by \(\Lambda\) the Poisson bivector and by \(\# : T^*P \to TP\) the induced morphism of vector bundles. The submanifold \(N\) is called

- Lagrangian if \(\#(TN^+) = TN \cap C\),
- isotropic if \(\#(TN^+) \supseteq TN \cap C\),

where \(TN^+\) is the annihilator of \(TN\) and \(C := \text{Im}(\#)\) is the characteristic distribution.

### 7.1. A new geometric characterization for the time-dependent inverse problem

Now we consider a non-autonomous second order differential system of the form

\[ \ddot{q}^i = \Gamma^i \left( t, q^j, \dot{q}^j \right). \tag{22} \]

We want to characterize when a regular time-dependent Lagrangian \(L(t, q, \dot{q})\) exists such that the solutions of the corresponding Euler–Lagrange equations coincide with the solutions of the system (22). Finding a regular Lagrangian is equivalent to finding a multiplier matrix \((g_i(t, q, \dot{q}))\) satisfying the Helmholtz conditions for time-dependent SODEs, which can be written as in section 4, (7)–(9), but now \(\Gamma = \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i} + \Gamma^i \frac{\partial}{\partial q^i}.\) In [19], Douglas solved this problem for the two dimensional case. He thoroughly analyzed the Helmholtz conditions using Riquier theory to give a classification of variational and nonvariational SODE’s in terms of conditions that depend only on \(\Gamma^j\) and some of its partial derivatives.

**Definition 7.3.** A vector field \(\Gamma\) on \(\mathbb{R} \times TQ\) is a SODE if \((\Gamma, \theta^0) = 0\) and \((\Gamma, dt) = 1\), where \(\theta^0 = dq^i - \dot{q}^i dt\) are the usual contact one-forms. In local coordinates \((t, q, \dot{q})\) for \(\mathbb{R} \times TQ\)

\[ \Gamma = \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i} + \Gamma^i \frac{\partial}{\partial q^i}. \]

The integral curves of \(\Gamma\) are the ones satisfying the system of explicit second order differential equations \(\dot{q}^i = \Gamma^i(t, q, \dot{q}).\)

**Remark 7.4.** An example of SODE on \(\mathbb{R} \times TQ\) is the Euler–Lagrange vector field associated to a regular Lagrangian function \(L: \mathbb{R} \times TQ \to \mathbb{R}\), which is defined as the unique vector field \(\Gamma\) satisfying \(i_{\Gamma} \Omega_L = 0\) and \((\Gamma, dt) = 1\), where \(\Omega_L = -d\theta_L\) is the Cartan two-form, \(\theta_L = dt + dL\circ S\) is the Cartan one-form and \(S = \frac{\partial}{\partial \dot{q}} \otimes \theta^0\). Note that \((\Omega_L, dt)\) provides \(\mathbb{R} \times TQ\) with a cosymplectic structure if \(L\) is regular [9].

In [15] a characterization analogous to the one in [12] is given for the time-dependent case:

**Theorem 7.5.** [15] A SODE \(\Gamma\) on \(\mathbb{R} \times TQ\) is variational if and only if there exists a two-form \(\Omega\) on \(\mathbb{R} \times TQ\) of maximal rank such that

(i) \(d\Omega = 0\),

(ii) \(\Omega(v_1, v_2) = 0\), \(\forall v_1, v_2 \in V(\mathbb{R} \times TQ)\).
Consider now the following diagram, where $F: \mathbb{R} \times TQ \rightarrow \mathbb{R} \times T^*Q$ is a local diffeomorphism over $\mathbb{R} \times TQ$:

\[
\begin{array}{c}
T(\mathbb{R} \times TQ) \xrightarrow{T^F} T(\mathbb{R} \times T^*Q) \cong T\mathbb{R} \times TT^*Q \\
\Gamma \\
\mathbb{R} \times TQ \xrightarrow{F} \mathbb{R} \times T^*Q.
\end{array}
\]

In local coordinates, if we write $\Gamma(t, q^i, \dot{q}^i) = (t, q^i, \dot{q}^i, 1, \dot{q}^i, \Gamma^i(t, q^i, \dot{q}^i))$ and $F(t, q^i, \dot{q}^i) = (t, q^i, F_i(t, q, \dot{q}))$, then we get

\[
\gamma_{\Gamma^F}(t, q^i, \dot{q}^i) = \left(t, q^i, F_i(t, q, \dot{q}), 1, \dot{q}^i, \frac{\partial F_i}{\partial t} + \dot{q}^i \frac{\partial F_i}{\partial q^i} + \Gamma^i \frac{\partial F_i}{\partial \dot{q}^i}\right).
\]

The standard Poisson bracket in $T^*(\mathbb{R} \times Q)$ induces a Poisson structure on $\mathbb{R} \times T^*Q$ such that $\pi = (\pi_{\mathbb{R}}, id_{T^*Q}): T^*(\mathbb{R} \times Q) \equiv T^*\mathbb{R} \times T^*Q \rightarrow \mathbb{R} \times T^*Q$ is a Poisson morphism. Locally, in coordinates $(t, q^i, p_i)$ for $\mathbb{R} \times T^*Q$ we have that the induced bracket $\{ , \}$ is defined by

\[
\{ t, q^i \} = \{ t, p_i \} = \{ q^i, q^j \} = \{ p_i, p_j \} = 0 \text{ and } \{ q^i, p_i \} = 1.
\]

Then we take its tangent lift to $T(\mathbb{R} \times T^*Q)$, which is defined on the induced coordinate functions $(t, q, p, v, \dot{q}, \dot{p})$ by

\[
\{ \dot{q}^i, \dot{p}_i \}^T = 1, \quad \{ \dot{q}^i, p_i \}^T = 1,
\]

and the remaining Poisson brackets vanish. Now we write the conditions that arise when forcing $\text{Im}(\mu_{F,F})$ to be Lagrangian. In local coordinates $(t, q, p, v, \dot{q}, \dot{p})$ for $T(\mathbb{R} \times T^*Q)$ we have

\[
T(\text{Im}(\gamma_{F,F})) = \text{span} \left\{ \frac{\partial}{\partial t} + \frac{\partial F_j}{\partial t} \frac{\partial}{\partial p_j}, \frac{\partial}{\partial q^i} + \frac{\partial F_j}{\partial q^i} \frac{\partial}{\partial p_j}, \frac{\partial}{\partial \dot{q}^i} + \frac{\partial F_j}{\partial \dot{q}^i} \frac{\partial}{\partial p_j}, \frac{\partial}{\partial \dot{p}_j} \frac{\partial}{\partial \dot{q}^i} + \frac{\partial}{\partial \dot{p}_j} \frac{\partial}{\partial \dot{q}^i} + \frac{\partial}{\partial \dot{p}_j} \frac{\partial}{\partial \dot{q}^i} + \frac{\partial}{\partial \dot{p}_j} \frac{\partial}{\partial \dot{q}^i} + \frac{\partial}{\partial \dot{p}_j} \frac{\partial}{\partial \dot{q}^i} + \frac{\partial}{\partial \dot{p}_j} \frac{\partial}{\partial \dot{q}^i} + \frac{\partial}{\partial \dot{p}_j} \frac{\partial}{\partial \dot{q}^i} + \frac{\partial}{\partial \dot{p}_j} \frac{\partial}{\partial \dot{q}^i} + \frac{\partial}{\partial \dot{p}_j} \frac{\partial}{\partial \dot{q}^i} + \frac{\partial}{\partial \dot{p}_j} \frac{\partial}{\partial \dot{q}^i} \right\},
\]

\[
T(\text{Im}(\gamma_{F,F}))^o = \text{span} \left\{ dv_i \frac{\partial F_j}{\partial q^i} dq^i - dp_j + \frac{\partial F_j}{\partial q^i} dt + \frac{\partial F_j}{\partial \dot{q}^i} d\dot{q}^i, \frac{\partial}{\partial \dot{q}^i} dq^i - dp_j + \frac{\partial}{\partial \dot{q}^i} dt + \frac{\partial}{\partial \dot{q}^i} d\dot{q}^i \right\}.
\]
and
\[ \mathcal{H}(T(\text{Im}(\gamma_{T,F}))) = \text{span} \left\{ \frac{\partial F}{\partial q^i}, \frac{\partial}{\partial \dot{q}^i}, \frac{\partial}{\partial \dot{p}^i}, \frac{\partial \Gamma(F)}{\partial q^i}, \frac{\partial \Gamma(F)}{\partial \dot{q}^i}, \frac{\partial \Gamma(F)}{\partial \dot{p}^i} \right\}. \]

As \( C = \text{span} \left\{ \frac{\partial}{\partial q^i}, \frac{\partial}{\partial \dot{q}^i}, \frac{\partial}{\partial \dot{p}^i} \right\} \), the equality \( \mathcal{H}(T(\text{Im}(\gamma_{T,F}))) = T(\text{Im}(\gamma_{T,F})) \cap C \) holds if the following conditions are satisfied
\[ \frac{\partial F}{\partial q^i} = \frac{\partial F}{\partial \dot{q}^i}, \frac{\partial F}{\partial \dot{p}^i} = \frac{\partial \Gamma(F)}{\partial \dot{q}^i}, \frac{\partial \Gamma(F)}{\partial \dot{p}^i} = \frac{\partial \Gamma(F)}{\partial \dot{p}^i}. \] (23)

**Remark 7.6.** Note that the above conditions are the same that arise if we require that the natural projection of \( \text{Im}(\gamma_{T,F}) \subset T(\mathbb{R} \times T^*Q) \) onto \( TT^*Q \) be a Lagrangian submanifold for each time coordinate with the symplectic structure \( \omega_{TQ} \).

Finally, the analog of theorem 4.2 reads as follows:

**Theorem 7.7.** A SODE \( \Gamma \) on \( \mathbb{R} \times TQ \) is variational if and only if there is a local diffeomorphism \( \mathbb{R} \times \pi : \mathcal{E} \rightarrow \mathbb{R} \times \pi \) such that \( \text{Im}(\gamma_{T,F}) \) is a Lagrangian submanifold of \( (T(\mathbb{R} \times T^*Q)), \{, \} \).

**Proof.** If \( \Gamma \) is variational, define \( F \) as the corresponding Legendre transformation. On the other hand, given \( F \) satisfying equation (23), we can define the two-form
\[ \Omega = -dF^*\theta_Q - i_F dF^*\theta_Q \wedge dt = -dF^*\theta_Q + \left( di_F F^*\theta_Q - L_F F^*\theta_Q \right) \wedge dt, \]
which satisfies all the conditions in theorem 7.5 and then \( \Gamma \) is variational. The details are left to the reader. Note that the Cartan two-form \( \Omega_{\mathcal{L}} \) (see remark 7.4) can be alternatively rewritten as \( \Omega_{\mathcal{L}} = \omega + de_{\mathcal{L}} + dt \), with \( \omega = -d\left( dl \circ S - (i_S dl \circ S) dt \right) \), \( e_{\mathcal{L}} = \Delta(L) - L \) and this motivates the definition of \( \Omega \). \( \square \)

**Remark 7.8.** If we replace the trivial bundle \( \mathbb{R} \times Q \rightarrow \mathbb{R} \) by an arbitrary fiber bundle \( \pi : E \rightarrow \mathbb{R} \), then the first jet manifold, denoted by \( J^1E \) is the generalization of \( \mathbb{R} \times TQ \). The generalization of \( \mathbb{R} \times T^*Q \) is \( V^*\pi \), the dual bundle of the vertical bundle to \( \pi \). \( V^*\pi \) is also equipped with a Poisson structure that can be lifted to \( TV^*\pi \), so we could copy the same scheme to study the variationality of a SODE on \( J^1E \) [21].

### 7.2. Time-dependent inverse problem for constrained systems

Let \( M \subset TQ \) be a submanifold projecting over the whole configuration manifold \( Q \), and \( \Gamma \) a SODE on \( \mathbb{R} \times M \). If \( (t, q^i, \dot{q}^a) \) denote coordinates on \( \mathbb{R} \times M, \ i = 1, \ldots, n = \dim \ Q, \ a = 1, \ldots, m = n, \) then the solutions of \( \Gamma \) are given by
\[ \dot{q}^a = \Gamma^a(t, q^i, \dot{q}^b), \quad \dot{q}^\alpha = \psi^\alpha(t, q^i, \dot{q}^b). \quad \alpha = 1, \ldots, n - m. \]

**Definition 7.9.** We say that a SODE \( \Gamma \) on \( \mathbb{R} \times M \) is variational if there is an immersion \( F: \mathbb{R} \times M \rightarrow \mathbb{R} \times T^*Q \) over \( \mathbb{R} \times Q \) such that \( \text{Im}(TF \circ \Gamma) \) is an isotropic submanifold of \( (T(\mathbb{R} \times T^*Q), \langle \cdot, \cdot \rangle^T) \).

\[
\begin{align*}
T(\mathbb{R} \times M) & \xrightarrow{T \circ F} T(\mathbb{R} \times T^*Q) \cong T\mathbb{R} \times TT^*Q \xrightarrow{\text{pr}_{TT^*Q}} TT^*Q \\
\mathbb{R} \times M & \xrightarrow{F} \mathbb{R} \times T^*Q
\end{align*}
\]

In local coordinates \( \gamma_{F,F} \) is given by

\[
\gamma_{F,F}(t, q^i, \dot{q}^a) = \left( t, q^i, F_i, 1, \dot{q}^a, \psi^a, \Gamma(F_i) = \frac{\partial F_i}{\partial t} + \dot{q}^a \frac{\partial F_i}{\partial q^a} + \psi^a \frac{\partial F_i}{\partial \psi^a} + \Gamma^a \frac{\partial F_i}{\partial \dot{q}^a} \right).
\]

Then we have

\[
T\left( \text{Im}\left( \gamma_{F,F} \right) \right) \cap C = \text{span}\left\{ V_i := \frac{\partial}{\partial q^i} + \frac{\partial F_i}{\partial q^a} \frac{\partial}{\partial \dot{q}^a} + \frac{\partial \psi^a}{\partial q^a} \frac{\partial}{\partial \dot{\psi}^a} + \frac{\partial \Gamma(F_i)}{\partial q^a} \frac{\partial}{\partial \dot{\psi}^a} \right\}
\]

\[
W_i := \frac{\partial}{\partial q^a} + \frac{\partial F_i}{\partial q^a} \frac{\partial}{\partial \dot{q}^a} + \frac{\partial \psi^a}{\partial q^a} \frac{\partial}{\partial \dot{\psi}^a} + \frac{\partial \Gamma(F_i)}{\partial q^a} \frac{\partial}{\partial \dot{\psi}^a} \right\}
\]

\[ \#\left( T\left( \text{Im}\left( \gamma_{F,F} \right) \right) \right) = \text{span}\left\{ A_i := \frac{\partial}{\partial q^i} + \frac{\partial F_i}{\partial q^a} \frac{\partial}{\partial \dot{q}^a} + \frac{\partial F_i}{\partial \psi^a} \frac{\partial}{\partial \dot{\psi}^a} + \frac{\partial \Gamma(F_i)}{\partial q^a} \frac{\partial}{\partial \dot{\psi}^a} \right\}
\]

\[ B_i := \frac{\partial \Gamma(F_i)}{\partial q^a} \frac{\partial}{\partial \dot{q}^a} + \frac{\partial \Gamma(F_i)}{\partial \psi^a} \frac{\partial}{\partial \dot{\psi}^a} \right\}
\]

\[ C^a := -\frac{\partial}{\partial \dot{\psi}^a} + \frac{\partial \psi^a}{\partial q^a} \frac{\partial}{\partial \dot{q}^a} + \frac{\partial \psi^a}{\partial \dot{q}^a} \frac{\partial}{\partial \dot{\psi}^a} \right\}
\]

Imposing the isotropy condition on \( \text{Im}(\gamma_{F,F}) \), that is,

\[ T\left( \text{Im}\left( \gamma_{F,F} \right) \right) \cap C \subset \#\left( T\left( \text{Im}\left( \gamma_{F,F} \right) \right) \right) \]

we obtain the time-dependent Helmholtz conditions for constrained systems:

\[ \frac{\partial F_a}{\partial q^b} + \frac{\partial \psi^a}{\partial q^a} \frac{\partial F_b}{\partial q^a} = \frac{\partial F_b}{\partial q^a} + \frac{\partial \psi^a}{\partial q^a} \frac{\partial F_a}{\partial q^a}, \quad (24) \]

\[ \frac{\partial \Gamma(F_i)}{\partial q^k} + \frac{\partial F_a}{\partial q^k} \frac{\partial \psi^a}{\partial q^a} = \frac{\partial \Gamma(F_i)}{\partial q^a} + \frac{\partial F_a}{\partial q^a} \frac{\partial \psi^a}{\partial q^a}, \quad (25) \]

\[ \frac{\partial F_a}{\partial q^a} = \frac{\partial \psi^a}{\partial q^a} \frac{\partial F_a}{\partial q^a} = \frac{\partial \Gamma(F_i)}{\partial q^a} + \frac{\partial F_a}{\partial q^a} \frac{\partial \psi^a}{\partial q^a}, \quad (26) \]
Equations (24) and (26) are obtained by imposing that $W_a$ be in $\mathcal{I}(T \mathrm{Im}(\gamma_{T,F}))$, while (25) and (26) are the conditions that arise when imposing that $V_i$ be in $\mathcal{I}(T \mathrm{Im}(\gamma_{T,F}))$.

**Theorem 7.10.** A SODE $\Gamma$ on $\mathbb{R} \times M$ is variational if and only if there is a two-form $\Omega$ on $\mathbb{R} \times M$ such that

(i) $d\Omega = 0,$
(ii) $\Omega(v_1, v_2) = 0$, for all vertical vectors $v_1, v_2 \in V(\mathbb{R} \times M),$
(iii) $i_T \Omega = 0,$
(iv) $\frac{\partial \Omega}{\partial (\mathbb{R} \times M)}$ is injective.

**Proof.** We can prove this result using theorem 7.5.

$\Rightarrow$ If $\Gamma$ is variational in the sense given in definition 7.9, then we define a two-form on $\mathbb{R} \times M$ by

$$\Omega = -dF^*\theta_Q + d\hat{F}^*\theta_Q \wedge dt - \mathcal{L}_F F^*\theta_Q \wedge dt$$

and check that conditions (i), (ii), (iii) and (iv) are satisfied.

$\Leftarrow$ We proceed as in the proofs of theorems 4.2 and 5.2 to get a local one-form $\Theta$ on $\mathbb{R} \times M$ such that $d\Theta = \Omega$ and $\Theta(v) = 0$ for all vertical vector fields $v$. We define

$$F: \mathbb{R} \times M \rightarrow \mathbb{R} \times T^*Q$$

by

$$\left( t, v_q \right) \mapsto \left( t, \tilde{F}(t, v_q) \right),$$

where $v_q \in M$, $w_q \in TQ$, $W_q \in TM$, $T\tau_Q|_M$, $W_q = v_q$ and $p_{T^*}(\mathbb{R} \times M) \rightarrow T^*M$ is the projection onto the second factor. Then check that $F$ is an immersion and $\mathrm{Im}(\gamma_{T,F})$ is isotropic, that is, equations (24)-(26) hold. \qed

**8. Conclusions and future developments**

The contributions of this paper include a characterization of the inverse problem of the calculus of variations in terms of special submanifolds in symplectic geometry; precisely, Lagrangian and isotropic submanifolds. Our approximation is flexible enough to take into account systems of second order differential equations with constraints, in particular, non-holonomic systems and their hamiltonization. Moreover, using symplectic techniques, we can prove that if a constrained explicit second order differential equation is variational then it can always be represented by a Lagrangian system without constraints. This last system agrees with the constrained SODE along the submanifold of $TQ$ which gives the constraints (see theorem 5.6). We adapt our techniques to the case of explicit time-dependent SODE’s now using Poisson techniques instead of the symplectic ones.

As we said before, one of the advantages of our approach is the easy adaptability to different cases. In particular, in future work we will study the following extensions:

- The inverse problem for reduced systems; in particular, Euler–Poincaré equations and Lagrange–Poincaré equations. In this case, we need to work with a notion of SODE over more general spaces than tangent bundles (for instance, $TQ/G$ where $G$ is a Lie group
acting free and properly on the configuration manifold). To study this problem, we will use the Lie algebroid formalism developed in [17]. Note that this problem has been addressed for Lie algebras in [13]. Besides, the Helmholtz conditions for SODEs on Lie algebroids are given in [34], but their relationship to the solution of the inverse problem on a Lie algebroid is not discussed.

• We will carefully study the relationship between our techniques and hamiltonization of nonholonomic systems. This is useful to study invariance properties of the nonholonomic flow (preservation of a volume form, symmetries...).

• Another interesting possibility is to extend our technique, always using Lagrangian and isotropic submanifolds, now for the symplectic cotangent bundle \((T^*Q \times T^*Q, \Omega)\), where \(\Omega = pr_1^*\omega Q - pr_2^*\omega Q\). This case will be useful to study the inverse problem for discrete systems, that is, when a second-order difference equation can be derived as the flow associated to the discrete Euler–Lagrange equations for a discrete Lagrangian \(L_d: Q \times Q \to \mathbb{R}\) (see [31]). Of course, we will have a version for reduced systems using similar techniques to the ones in the previous paragraph [30].

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