Semisimple characters for inner forms II: Quaternionic inner forms of classical groups

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December 2017

Abstract

In this article we consider a quaternionic inner form $G$ of a $p$-adic classical group defined over a non-archimedean local field of odd residue characteristic. We construct all full self-dual semisimple characters for $G$ and we classify their intertwining classes using endo-parameters. Further we prove an intertwining and conjugacy theorem for self-dual semisimple characters. We give the formulas for the set of intertwiners between self-dual semisimple characters. We count all $G$-intertwining classes of self-dual semisimple characters which lift to the same $\tilde{G}$-intertwining class of a semisimple character for the ambient general linear group $\tilde{G}$ for $G$.

1 Introduction

Self-dual Semisimple characters play an important role in the classification of smooth representations of $p$-adic classical groups in odd residue characteristic and for the explicit understanding of the Local Langlands correspondence and the Jaquet-Langlands correspondence. Throughout the introduction and the paper we only consider non-archimedian local fields of odd residue characteristic. One of them should be $F$. To start the history of the development of semisimple characters we start with work of Bushnell and Kutzko [5] who classified all irreducible complex representations of $GL_m(F)$ via types, the latter constructed using simple characters. This work which was generalized by Sécherre and Stevens to $GL_m(D)$, $D$ a non-split quaternion algebra of $F$, see [8], here as well using simple characters. Self-dual semisimple characters were used for the exhaustiveness proof for the classification of all cuspidal irreducible representations for $p$-adic classical groups in Stevens work [13] in introducing cuspidal types. A study of the intertwining of these characters was needed, see [6], to prove that Stevens’ construction leads to equivalent cuspidal irreducible representations only if the cuspidal types are conjugate up to equivalence. Motivated by that result the author generalized semisimple characters to $GL_m(D)$ in [10] and to quaternionic inner forms of $p$-adic classical groups in this paper.

Let $(D,\rho)$ be skew-field with an orthogonal anti-involution and $(V,h)$ be $\epsilon$-hermitian form with respect to $\rho$ on a finite dimensional vector space $V$. We consider the set $G = U(h)$ of isometries of $h$ in the ambient general linear group $\tilde{G}$. A semisimple character of $\tilde{G}$ is a character on a compact open subgroup of $\tilde{G}$ which is constructed from a datum $\Delta := [\Lambda, n, r, \beta]$ such that the following holds.

- $\beta$ is an element of $\text{End}_D(V)$ generating a product $E = \prod_{i \in I} E_i$ of fields over $F$. It also gives a direct sum decomposition of $V$ into $E_i \otimes D$-modules.
- $\Lambda$ is an $o_D$-lattice sequence in $V$, i.e. a point of the Bruhat-Tits building $B(\tilde{G})$ of $\tilde{G}$ with rational barycentric coordinates, which is in the image of the embedding
  \[ \prod_i B(\tilde{G}_i) \rightarrow B(\tilde{G}) \]
  where $\tilde{G}_i = \text{Aut}_{E_i \otimes D}(V^i)$. In particular $\Lambda$ splits into $\oplus_{i \in I} \Lambda^i$. 

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• The integers $n$ and $r$, $n > r$, are non-negative and indicate on which “level” the characters should be defined ($r$) and should be trivial ($n$).

• Some more conditions which simplify the calculation of the intertwining.

We call the set of semisimple characters defined by $\Delta$ by $C(\Delta)$. All the characters in $C(\Delta)$ have the same domain which we call $H(\Delta)$. To define self-dual semisimple characters we consider $\Lambda^1$ to be a point in the building of $(\prod_i \tilde{G}_i) \cap G$ and $\beta$ to be an element of the Lie-algebra of $G$. Now the self-dual semisimple characters for $G$ are the restrictions of elements of $C(\Delta)$ to $H(\Delta) \cap G$. We call the set of them by $C_{-}(\Delta)$.

The first steps in the study of self-dual semisimple characters give the following results:

• A nice intertwining formula, i.e. the set of elements in $G$ which intertwine $\theta_- \in C_{-}(\Delta)$ is up to multiplication from left and right by a compact subgroup of the 1-units of $\Lambda$ the centralizer of $\beta$ in $G$, see Theorem 6.12.

• Intertwining is an equivalence relation for full (this means $r = 0$) self-dual semisimple characters for $G$, see Corollary 6.16.

• We have an intertwining and conjugacy theorem, see Theorem 6.17, which we explain now.

The set of self-dual semisimple character $C_{-}(\Delta)$ comes along with an action of $\sigma$ on the index set which leads to a disjoint union $I = I_0 \cup I_+ \cup I_-$. where $I_0$ is the set of $\sigma$ fixed points and $I_+$ is a section through the $\sigma$-orbits of length 2. Given two full semisimple characters $\theta_- \in C_{-}(\Delta)$ and $\theta'_- \in C_{-}(\Delta')$ which intertwine by some element of $G$, they possess a bijection $\zeta : I \rightarrow I'$ between the index sets such that there is an element of $G \cap \prod_i \text{End}_D(V^i, V^{\Sigma(i)})$ which intertwines $\theta_-$ with $\theta'_-$. Further the intertwining gives a bijection $\zeta$ between the residue algebras of $F[\beta]$ and $F[\beta']$. Now Theorem 6.17 states

**Theorem 1.1** (see 6.17). Suppose there is an element $t \in G$ such that $t \Lambda^1$ is equal to $\Lambda^{(t)}$ and that the conjugation with $t$ verifies $\zeta$. Then there is an element $g \in G$ such that $g \cdot \theta_- = \theta'_-$. 

In the second part we parametrize the intertwining classes of self-dual semisimple characters using endo-parameters. The idea is to break up $\theta_-$ in elementary self-dual pieces, i.e. in self-dual semisimple characters where the index set is just one $\sigma$-orbit, $\theta_{i,-} = \theta_-|_{H(\Delta) \cap \tilde{G}_i}$, for $i \in I_0$, and $\theta_{i,-} = \theta_-|_{H(\Delta) \cap \tilde{G}_i, \ast(t)}$, for $i \in I_+$. To every elementary character we attach an endo-class, see after Definition 7.1. We denote the set of all elementary endo-classes by $E$. Further we can attach to any $h_i$, $i \in I_0$, an $\epsilon$-hermitian $\sigma_{E_i} \otimes \rho$-form $\tilde{h}_{\beta_i}$ which corresponds to some element $t_i$ of the Witt group $W_i(\sigma_{E_i} \otimes \rho)$ which we call Witt tower. We introduce an equivalence relation of the set these kind of pairs $(\gamma, t)$, see section 7.1 and we call the equivalence classes $(\rho, \epsilon)$-Witt types and the set of Witt types is denoted by $W_{\rho, \epsilon}$. An endo-parameter is a map of finite support

$$f_- = (f_1, f_2) : E_0 \rightarrow N_0 \times W_{\rho, \epsilon}$$

such that for simple $c_- \in E_0$ the value $f_1(c_-)$ essentially plays the role of a Witt index, and $f_2(c_-)$ is a Witt type which occurs in $c_-$. (note that in $c_-$ can occur several Witt types), and such that for non-simple $c_-$ we have that $f_2(c_-)$ is hyperbolic and $f_1(c_-)$ is a certain degree of $c_-$. These endo-parameters classify intertwining classes of self-dual semisimple characters, see Theorem 7.15.

At the end in the appendix we calculate the number of $G$-intertwining classes of self-dual semisimple characters whose semisimple lifts are in the same $G$-intertwining class.

I thank Shaun Stevens for his interest and remarks concerning this article.
2 Quaternionic inner Forms of $p$-adic classical groups

2.1 Fixing notation

At first, this article is the second in a series of articles where the first one is [10]. There will be only one difference in the notation, see Remark 2.1. Let $F$ be a non-archimedean local field of odd residual characteristic. We use the usual notation $o_F, p_F, \kappa_F$ and $\nu_F$ for the valuation ring, the valuation ideal, the residue field and the normalized valuation of $F$, the image of $\nu_F$ being $\mathbb{Z}$, and we use similar notation for other non-archimedean local skewfields. Further we fix a non-split quaternion algebra $D$ with centre $F$ together with an orthogonal anti-involution $\rho$ on $D$, i.e. an $F$-linear automorphism of $D$ which satisfies $\rho(xy) = \rho(y)\rho(x)$. We can choose $\rho$ such that there is an unramified field extension $L|F$ and a uniformizer $\pi_D$ in $D$ both point-wise fixed by $\rho$ such that $\pi_D$ normalizes $L$. We denote the non-trivial automorphism of $L|F$ by $\tau$. The square of $\pi_D$ is a uniformizer of $L$ and we denote it by $\pi_F$.

We further fix a finite dimensional non-zero right-$D$ vector space $V$ and an $\epsilon$-hermitian form $h$ on $V$, $\epsilon \in \{-1,1\}$, i.e. a $\mathbb{Z}$-bilinear form such that:

$$h(vx, wy) = \epsilon \rho(x)\rho(h(w, v)),$$

for all $x, y \in D$ and $v, w \in V$. We denote by $G$ the set

$$U(h) = \{ g \in \text{Aut}_D(V) \mid h(gv, gw) = h(v, w) \text{ for all } v, w \in V \}$$

of isometries of $h$. We write $\sigma_h$ for the adjoint anti-involution of $h$, $\tilde{G}$ for the ambient general linear group $\text{Aut}_D(V)$ and $A$ for the ring of $D$-linear endomorphisms of $V$.

**Remark 2.1.** There will be only one difference in the notation between [10] and this article. Precisely $\tilde{G}$ denotes the group $\text{GL}_D(V)$ and $G$ denotes the classical group in question.

2.2 $L$-rational points

The group $G$ is the set of $F$-rational points of an $F$-form of a symplectic or an orthogonal algebraic group $\mathbb{G}$. In this section we will describe the set $\mathbb{G}(L)$ by an hermitian $L$-form on $V$. The set of $L$-rational points of $\mathbb{G}$ is given by the anti-involution $\sigma_h \otimes_F \text{id}$ on $\text{End}_D(V) \otimes_F L$, and the latter $L$-algebra is canonically isomorphic to $\text{End}_L(V)$ via

$$\Phi: \text{End}_D(V) \otimes_F L \to \text{End}_L(V), \ f \otimes_F l \mapsto (v \mapsto f(vl)).$$

The group of $L$-rational points of $\mathbb{G}$ is algebraically isomorphic to

$$\{ g \in \text{End}_L(V) \mid \Phi((\sigma_h \otimes_F \text{id})(\Phi^{-1}(g)))g = 1 \}.$$ 

We identify this set with $\mathbb{G}(L)$. By [3] there is a unique $\epsilon$-hermitian $L$-form $h_L$ on $V$ such that

$$\text{tr}_{L|F} \circ h_L = \text{trd}_{D|F} \circ h.$$ 

**Proposition 2.2.** $\mathbb{G}(L)$ is equal to $U(h_L)$.

**Proof.** We need to show that $\sigma_h$ is equal to the push forward of $\sigma_h \otimes_F \text{id}$ via $\Phi$. Take $v, w \in V, f \in \text{End}_D(V)$ and $l \in L$. Then,

$$\text{tr}_{L|F}(h_L(\Phi(f \otimes_F l)v, w)) = \text{tr}_{L|F}(h_L(f(v)l, w)) = \text{tr}_{L|F}(h_L(f(v), wl)) = \text{trd}_{D|F}(h(f(v), wl)) = \text{trd}_{D|F}(h(v, \sigma_h(f)(wl))) = \text{tr}_{L|F}(h_L(v, \Phi((\sigma_h \otimes_F \text{id})(f \otimes l))(w)))$$

\[\Box\]
2.3 Witt group of a local non-split division algebra

Let $\tilde{D}$ be a central division algebra over $F$ of finite degree, with anti-involution $\tilde{\rho}$, such that $(\tilde{D},\tilde{\rho})$ is orthogonal or unitary. Then the Witt group of $\tilde{D}$ with respect to $\epsilon$ and $\tilde{\rho}$ is the set of equivalence classes of $\epsilon$-hermitian forms

$$h : V \times V \to (\tilde{D},\tilde{\rho})$$
onumber

on finite dimensional $\tilde{D}$-vector spaces $V$, where two forms are equivalent, if they have isomorphic anisotropic components in there Witt-decomposition. We write $h_\pi$ for the classes. We write $W_{\epsilon,\rho}(\tilde{D})$ for the Witt group. We call this Witt group

- **orthogonal** if $\epsilon = 1$ and $\tilde{\rho}|_F = \text{id}_F$,
- **symplectic** if $\epsilon = -1$ and $\tilde{\rho}|_F = \text{id}_F$ and
- **unitary** if $\tilde{\rho}|_F \neq \text{id}_F$.

If we have a Gram matrix $M$ for $h$ we also write $(M)$ for the form with Gram matrix $M$. If $f$ is a symmetric or skew-symmetric element of $\text{End}_F(V)$ then we write $h^f$ for the form which maps $(\bar{v},\bar{w}) \in V^2$ to $h(\bar{v},f(\bar{v}))$ and we call $h^f$ the *twist* of $h$ by $f$.

We refer to section 6 of [11] for the description of the Witt group in the case where $\tilde{D}$ is abelian. We write $C_n$ for the cyclic group of order $n$. In the case $(\tilde{D},\tilde{\rho}) = (D,\rho)$, see [2.4] we have the following result for the two Witt groups. Recall, that $D$ is not abelian.

**Proposition 2.3.** 
(i) The orthogonal Witt group of $(D,\rho)$ is isomorphic to an elementary 2-group of order 8. A set of generators is given by

$$\{ (1)_x, (\pi_D)_x, (\alpha)_x \},$$

where $\alpha$ is a non-square unit of $L$. We could take a skew-symmetric with respect to $\tau$ if $-1$ is not a square of $F$.

(ii) The symplectic Witt group is a cyclic group of order two. The non-hyperbolic class is $(\pi_D l_s)_x$,

where $l_s$ is a non-zero element of $L$ which is skew-symmetric with respect to $\tau$.

At first we need the next lemmas.

**Lemma 2.4.** For every symmetric element $d \in D^*$ and field extension $F'|F$ which is fixed point-wise by $\rho$ such that $F[d]|F$ is isomorphic to $F'|F$ there is an element $g$ of $D^*$ such that $gF[d]\rho(g) = F'$. In fact if $d \not\in F$ every element which conjugates the first field extension to the second works.

**Proof.** In the case $d \in F$ we take $g = 1$. So we only need to consider the case $d \not\in F$, i.e. where $F[d]|F$ has degree 2. By Skolem-Noether there is an element $g$ conjugating $F[d]|F$ to $F'|F$. The element $g d g^{-1}$ is symmetric by assumption. So, $\rho(g)g$ centralizes $F[d]$ and is thus an element of $F[d]$ and we obtain that $gF[d]g^{-1}$ is equal to $gF[d]\rho(g)$. □

**Lemma 2.5.** For every symmetric or skew-symmetric element $d \in D^*$ and $x \in F^x$ the Witt classes $(d)_x$ and $(dx)_x$ are equal.

**Proof.** We have to find an element $y$ of $D$ such that $\rho(y)dy$ is equal to $dx$. For the first case let us assume that $d$ is $\pi_D l_s$, $\pi_D$ or in $L^\times$. Using elements of $y \in L$ we can obtain all $x d$ for $x \in F^x$ with $2|\nu_F(x)$. If we use $y \in L^\times \pi_D$ we obtain every $x d$ for $x \in F^x$ with odd valuation. A general symmetric $d$ generates a field extension which is generated by an element $d'$ of $F l_s + F \pi_D$. If $d'$ has an odd $D$-valuation then $F[d]$ is isomorphic to $F[\pi_D]$ and if $d'$ has an even $D$-valuation then $F[d]|F$ is unramified. Thus Lemma 2.4 finishes the proof. □

We say that two elements $d$ and $d'$ of $D$ are congruent to each other modulo $\nu_D$ if $d - d'$ is an element of $p_D$. This is an equivalence relation, and two non-zero elements $d$ and $d'$ are congruent modulo $\nu_D$ if and only if $dd'^{-1}$ is an element of $1 + p_D$.  

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Lemma 2.6. Let $a_i, i = 1, 2, 3, 4$ be an $F$-splitting basis of $D$ and suppose that two non-zero elements $d = \sum_i x_i a_i$ and $d' = \sum_i y_i a_i$ satisfy $dd'^{-1} \in 1 + pF, x_i, y_i \in F$. Then, if $\nu_D(x_1 a_1) = \nu_D(d)$ then $x_1 y_1^{-1} \in 1 + pF$.

Proof. We have $\nu_D(d-d') > \nu_D(d)$ and therefore $\nu(x_1 a_1 - y_1 a_1) > \nu_D(d)$ for all $i$ because $(a_i)$ is a splitting basis for $D$. Thus, $\nu_D(x_1 a_1 - y_1 a_1) > \nu_D(d), i.e. \nu_F(x_1 - y_1) > \nu_F(x_1)$.

Lemma 2.7. Suppose $d, d' \in D^x$ are two symmetric or skew-symmetric elements of $D$ which are congruent to each other modulo $\nu_D$. Then

(i) If $d$ and $d'$ are not elements of $F^x(1 + pF)$ then $F[d][F$ and $F[d'][F$ are isomorphic and there is a $g \in D^x$ which conjugates the first field extension to the second such that $gdy^{-1}d'^{-1} \in 1 + pF[x]$.

(ii) The Witt classes $\langle d \rangle_x$ and $\langle d' \rangle_x$ are equal.

Proof. The residue characteristic is odd, so either both are skew-symmetric or both are symmetric.

Case 1: Let us first consider the case where $d$ and $d'$ commute. It is worth to remark that this is already implied if both elements are skew-symmetric, because $\rho$ is orthogonal. Now, then $d$ and $d'$ are elements of a $\rho$-invariant field $F^p$, and we have $d = ud'$ for some $\rho$-symmetric element $u$ of $1 + pF^p$, using the congruence of $d$ to $d'$. The residue characteristic is odd and thus $u$ is a square of a $\rho$-symmetric element $v$ of $1 + pF^p$ and thus $\langle d \rangle_x$ is equal to $\langle d' \rangle_x$. This proves (ii) in this case, and in (i) we can take $g = 1$.

Case 2: Suppose now that $d$ and $d'$ are symmetric and do not commute. We have two sub-cases.

Case 2.1: At first let us assume that $d$ is an element of $F^x(1 + pF_D)$, say $dx^{-1} \in (1 + pF_D)$, for some $x \in F^x$. The elements $d$ and $x$ commute and therefore $\langle d \rangle_x$ is equal to $\langle x \rangle_x$, by Case 1, and similar we have $\langle x \rangle_x = \langle d' \rangle_x$. So, the Witt classes of $\langle d \rangle_D$ and $\langle d' \rangle_D$ are the same. It proves (ii) The statement (i) is empty in this case.

Case 2.2: Let us assume that $d$ is not an element of $F^x(1 + pF_D)$, $d'$ is congruent to $d$ and thus it is not an element of $F^x(1 + pF_D)$ either.

We apply Lemma 2.6 on

$$d = x_1 + x_2 l_s + x_3 \pi_D, \quad d' = y_1 + y_2 l_s + y_3 \pi_D$$

to obtain $d - x_1$ and $d' - y_1$ are congruent modulo $\nu_D$. Indeed, either $\nu_D(d) = \nu_D(x_1)$ and thus the $x_1 - y_1$ is an element of $pDx_1$, by Lemma 2.6 or $\nu_D(d) < \nu_D(x_1)$ and all four elements $d, d', d - x_1$ and $d' - y_1$ are congruent to each other modulo $\nu_D$. Nevertheless, the difference of $d - x_1$ with $d' - y_1$ must be an element of $pD(d - x_1)$, because

$$pDd = pD(d - x_1) \geq pDx_1,$$

by $\nu_D(d) = \nu_D(d - x_1) \leq \nu_D(x_1)$. Thus, we can assume for the proof of (i) that $d$ and $d'$ are elements of $F\sigma + F\pi_D$. But then $d^2$ and $d'^2$ are congruent elements of $F$, i.e. there is a one-unit $v$ of $F$ such that $v^2d^2 = d'^2$, because the residue characteristic is odd. Both, $vd$ and $d'$ are not elements of $F$, and therefore $vd$ is conjugate to $d'$ by an element of $D^x$ by Skolem-Noether.

We know that there is an element $g$ satisfying the first assertion. The element $g$ is congruent to an element of $L$ or to an element of $L\pi_D$. Thus $\rho(g)g$, an element of $F[d]$, is congruent to an element of the form $\pi_F x^2$ with some unit $x$ of $L$. We claim that $\rho(g)g$ is of the form $a y^2$ for some $a \in F^x$ and $y \in \sigma_F(d)$. If $F[d][F$ is ramified then $\rho(g)g$ is congruent to an element of $F$ because $\nu_F[d](\rho(g)g)$ is even and Hensel’s Lemma implies the claim. If $F[d][F$ is unramified then

$$\pi_F \rho(g)g = \rho(\pi_D^{-1}g)\pi_D^{-1}g.$$ 

So, its residue class is a square and hence the claim. Thus we get

$$\langle d \rangle_x = (gd\rho(g)) = (gdg^{-1}y^{-1})_x = (gdd^{-1}y^{-1}ygg^{-1})_x.$$ 

The latter term is $(gdd^{-1})_x$, because $gg^{-1}$ is $\rho$-symmetric. Now, $gdd^{-1}$ and $d'$ are $\rho$-symmetric, congruent and commute, and hence we get (ii) by Case 1.
Remark 2.8. The proof of Lemma 2.7 shows that if two $\rho$-symmetric elements $d$ and $d'$ are conjugate by an element of $D^\times$ then they define the same Witt class.

Proof of Proposition 2.8. Using [4.1.14] every signed hermitian space $(h, V)$ over $D$ has a Witt basis, i.e. a basis which has a Gram matrix with two diagonal blocks:

(i) An anti-diagonal matrix having 1 and $\epsilon$ in the anti-diagonal (block $(1, 1)$),

(ii) A diagonal matrix which is the Gram matrix of an anisotropic subspace of $V$ (block $(2, 2)$).

(iii) The blocks $(2, 1)$ and $(1, 2)$ have only zero entries.

Thus to classify all equivalence classes of signed hermitian forms, one only needs to classify the possible diagonal blocks $(2, 2)$, and for them only the diagonal entries. The $F$-vector space of skew-symmetric elements of $D$ is $F\pi_D^\times$. Thus, from Lemma 2.5 follows that there is only one non-trivial Witt class in the symplectic case. The $F$-vector space of symmetric elements of $D$ is $L + \pi_DF$. By Lemma 2.7 and Lemma 2.5 the class $(d_z)$ is $(\pi_D)z$ or $(1_2z)$ or $(\alpha)z$. They are pairwise different because of valuation reasons and because $\alpha$ cannot be of the form $\rho(x)x$ because the residue class of a unit of the form $\rho(x)x$ is a square.

2.4 \( G \subseteq \text{SL}_D(V) \)

An element $g$ of $G$ satisfies $g\sigma_h(g) = 1$, and $\rho|_F = \text{id}$ leaves for $\text{Nrd}_{A|F}(g)$ only 1 or $-1$. It is remarkable that in fact all elements of $G$ have reduced norm 1. We are going to prove this fact in this section.

Proposition 2.9. Every element of $G$ has reduced norm 1.

Proof. We prove the assertion by induction on $m = \dim_F V$. We only need to consider orthogonal groups, because in the symplectic case the group $G(L)$ is a split symplectic group where every element has determinant 1. Thus let us assume that $G$ is an inner form of an orthogonal group.

Induction start ($m = 1$): Let $g$ be an element of $G$. We take the canonical isomorphism from $D$ to $\text{End}_D D$. Then $F[g]$ is a field in $D$, which is invariant under the action of $\sigma_h$. If $\sigma_h(g) = g$ then $g^2 = 1$, because $g \in G$ and thus $g \in \{1, -1\}$, i.e. the reduced norm of $g$ would be 1. If $\sigma_h(g) \neq g$ then $\sigma_h|_F[g]$ is the Galois generator of $F[g]/F$. Thus $\text{Nrd}(g) = N_{F[g]/F}(g) = \sigma_h(g)g = 1$.

Induction step ($m > 1$): We consider $F[g]$ generated by an element $g$ of $G$. The minimal polynomial $\mu_{g,F}$ of $g$ over $F$ has a prime factorization

$$\mu_{g,F} = P_1^{\nu_1} \cdots P_l^{\nu_l},$$

which gives a decomposition:

$$F[g] \cong F[X]/P_1^{\nu_1} \cdots F[X]/P_l^{\nu_l}$$

The factors are permuted by $\sigma_h$. For every orbit of this action we get a $\sigma_h$-fixed idempotent. In the case of at least two orbits we conclude that $g$ is an element of the product of at least two quaternionic inner forms of classical groups and the induction hypothesis implies that the reduced norm of the restrictions of $g$ to the factors is 1. Thus we are left with the case of one orbit.

Case $\mu_{g,F} = P_1^{\nu_1} P_2^{\nu_2}$ and $\sigma_h$ flips $F[X]/P_1^{\nu_1}$ and $F[X]/P_2^{\nu_2}$: The primitive idempotents $1^1$ and $1^2$ of $F[g]$ give rise to decompositions $V^1 + V^2$ of $V$ and $g_1 + g_2$ of $g$. From $g \in G$ follows $g_1^2 = \sigma_h(g_2)$. Thus

$$1^1 P_2(g_1^{-1})^{\nu_2} = 1^2 P_2(\sigma_h(g_2))^{\nu_2} = \sigma_h(1^2 P_2(g_2)^{\nu_2}) = \sigma_h(1^2 0) = 0.$$
multiplication by an element of $F^\times$. Thus $P_1$ and $P_2$ have the same multiplicity in $\chi$, and therefore $\chi(0) = 1$, taking $\frac{u_0 = P_1(0)}{P(0)} = \frac{1}{n_0}$ into account.

Case $\mu_{g,F} = P^\nu$: As in the case above we obtain that $X^{\deg P} P(X^{-1}) \frac{1}{P(0)}$ and $P$ coincide. We split $P$ in an algebraic closure of $F$:

$$P = (X - \lambda_1) \ldots (X - \lambda_k).$$

Then inverse map of $F^\times$ permutes the roots of $P$. Thus $P(0) = 1$ if every root of $P$ satisfies $\lambda^{-1} \neq \lambda$. If the latter is not the case then $P$ has a root $\lambda$ which is 1 or $-1$, and in this case $P = (X - \lambda)$ because it is irreducible. Then $\chi$ is an even power of $(X - \lambda)$ and thus $\chi(0) = 1$.

\section{Stevens’ cohomology argument on double cosets}

Let $p$ and $l$ be different primes and let $\Gamma$ be an $l$-group acting continuously on some topological Hausdorff group $Q$. Kurinczuk and Stevens proved the following result in [7, 2.7]: We denote by $S^\Gamma$ the set of $\Gamma$-fixed points of $S$ for any subset $S$ of $Q$.

\begin{thm}[7, 2.7(ii)(b)] Suppose $U_1$ and $U_2$ are two $\Gamma$-stable pro-$p$-subgroups of $Q$. Let $H$ be a further $\Gamma$-stable subgroup of $Q$ such that for every $h \in H$ the following identity holds:

$$(U_1 h U_2) \cap H = (U_1 \cap H) h (U_2 \cap H).$$

Then, we have the double coset decomposition:

$$(U_1 H U_2)^\Gamma = U_1^\Gamma H U_2^\Gamma.$$

In this section we are going to generalize this result in allowing $\Gamma$-stable cosets of $H$ instead of $H$.

\begin{prop}
Suppose $U_1$ and $U_2$ are two $\Gamma$-stable pro-$p$-subgroups of $Q$ and $H$ is a further $\Gamma$-stable subgroup of $Q$. Suppose $g H$ is a $\Gamma$-stable coset of $H$ in $Q$ such that for every $h \in H$ the following identity holds:

$$(U_1 g H U_2) \cap g H = (U_1 \cap g H g^{-1}) g h (U_2 \cap H).$$

Then, we have:

$$(U_1 g H U_2)^\Gamma = U_1^\Gamma (g H)^\Gamma U_2^\Gamma.$$

\end{prop}

\begin{rem}
The condition $U_1$, $g$, $H$, $U_2$ is equivalent to $U_1$, $g^{-1} H$, $g U_2$. So it is enough to establish $U_1$, $H'$, $U_2'$ for a big class of triples $U_1$, $H'$, $U_2'$.

The proof is literally the same, but we repeat the argument where minor changes occur. At first we need the following two statements from [7].

\begin{lem}[7, 2.7(ii)] Suppose that $U_1$ and $U_2$ are two subgroups of $Q$ such that the (non-abelian) cohomology $H^1(\Gamma, g U_1 g^{-1} \cap U_1)$ is trivial. Then we have for any $\Gamma$-fixed element $g$ of $Q$ the identity $(U_1 g U_2)^\Gamma = U_1^\Gamma g U_2^\Gamma$.

\end{lem}

\begin{lem}[7, 2.7(ii)(a)] For any two $\Gamma$-stable pro-$p$-subgroups $U_1$ and $U_2$ of $Q$ and any element $g$ of $Q$ the following assertions are equivalent:

(i) $(U_1 g U_2)^\Gamma \neq \emptyset$

(ii) $U_1 g U_2$ is $\Gamma$-stable.

\end{lem}

\begin{proof}[Proof of Proposition $3.3$]
There is nothing to prove for the inclusion $\supset$, so we continue with the other inclusion. Let $x$ be an element of $(U_1 g H U_2)^\Gamma$. Then there is an element $h$ of $H$ and there are elements $u_1 \in U_1$, $u_2 \in U_2$ such that $x = u_1 g h u_2$. We have to show that we could have taken $h$ such that $gh$ is $\Gamma$-fixed, because Lemma 3.6 would then imply that $x$ is an element of $U_1^\Gamma (g H)^\Gamma U_2^\Gamma$. Now, $U_1 g H U_2$ and $g H$ are $\Gamma$-stable, so, by $3.2$ and Lemma 3.7 there is a $\Gamma$-fixed point in $(U_1 \cap g H g^{-1}) g h (U_2 \cap H)$, which we could have chosen instead of $g h$ in the product decomposition of $x$. This finishes the proof.

\end{proof}
4 Strata for $G$

In this section we generalize the notion of self-duality for strata from $p$-adic classical groups, see [12] and [11], to its quaternionic inner forms. We generalize the usual statements about strata, as for example about the intertwining formula and a Skolem–Noether result, and recall endo–classes of semisimple strata. We take all definitions and results from [10].

4.1 First definitions

We refer to [12] and [11] for the non-quaternionic case. We use all notation and definitions in [10]. For example the definitions for strata (pure, simple, semi-pure, semisimple) can be found in section 4.

A stratum is denoted as a quadruple $[\Lambda, n, r, \beta]$ and if we write $\Delta'$ for a stratum, then the entries appear with a superscript ', i.e. $[\Lambda', n', r', \beta']$, and similar with subscripts: $\Delta_c := [\Lambda^c, n_c, r_c, \beta_c]$ (The superscript on $\Lambda$ is not a typo.) We write $E$ for $F[\beta]$ and similar $E'$, $E_c$ etc.. And we write $C_c(!)$ for the centralizer of $!$ in $\varphi$.

If $\Delta$ is a semi-pure stratum it has an associated splitting which we denote by $V = \oplus_{i=1}^l V^i$ coming from the decomposition of $E$ into a product of fields, and we call the corresponding idempotents $1^i$. Split means that $\Delta$ is the direct sum of its restrictions $\Delta|_{V^i}$, which is called the $i$th block of $\Delta$.

Given a full $o_F$-module $M$ in $V$ we define the dual for $M$ with respect to $h$ via

$$M^# = \{ v \in V \mid h(v, M) \subseteq pD \}.$$ 

The form $h$ defines an involution $\#$ on the set of lattice sequences for $V$, in defining $\Lambda^#$ as $(\Lambda, n^#)$. This depends on the choice of $h$. A lattice sequence $\Lambda$ is called self-dual if $\Lambda^#$ and $\Lambda$ differ by a translation, i.e. there is an integer $k$ such that $\Lambda - k$, which is defined as $(\Lambda_k)_{k \in \mathbb{Z}}$, coincides with $\Lambda^#$. This gives an action of $\#$ on the set of strata for $V$:

$$\Delta^# := [\Lambda^#, n^#, r^#, \sigma h(\beta^#)].$$ 

In our notation the latter says: $n^# = n$, $r^# = r$ and $\beta^# = -\sigma h(\beta)$.

**Definition 4.1.** A stratum $\Delta$ is called self-dual if $\Delta^#$ and $\Delta$ only differ by a translation of $\Lambda$, i.e. if $\Lambda$ is self-dual and $\sigma h(\beta) = -\beta$. Further if $\Delta$ is a self-dual semi-pure stratum, then $\sigma$ induces an involution on the index set $I$, which we also call $\sigma$. The action of $\{1, \sigma\}$ decomposes the index set $I$ into a set of fixed points $I_0$ and the set $I_-$ of elements which have an orbit of length 2. We usually choose a section $I_0 \subseteq I_-$ through all orbits of length 2. Given a union of $\sigma$-orbits $J \subseteq I$ we denote the restriction of $h$ to $V^J := \oplus_{i \in J} V^i$ by $h_J$. We also write $h_{i, \ldots, j}$ instead of $h_{(i, \ldots, j)}$. We call $\Delta$ skew if $I = I_0$.

4.2 Diagonalization for self-dual strata

One of the first important properties for strata is the diagonalization proposition for self-dual simple strata. Let us first state the diagonalization proposition for $GL_D(V)$.

**Proposition 4.2 (10) Theorem 4.30.** Let $V^i$, $i = 1, \ldots, l$, be sub-$D$-vector spaces of $V$ whose direct sum is $V$. Let $\Delta$ be a stratum which splits under $\oplus_i V^i$ into a direct sum of pure strata $\Delta_{V^i}$. Suppose further that $\Delta$ is equivalent to a simple strata. Then, there is a simple stratum which is equivalent to $\Delta$ and split by $\oplus_i V^i$.

We want to prove:

**Proposition 4.3** (see [11] 6.6 for the case over $F$). Let $V = \oplus_{i \neq j} V^j$ a decomposition of $V$ such that the projections $1^j : V \to V^j$ are permuted by $\sigma_h$. Let $\Delta$ be a self-dual stratum which is split under $(V^j)$ such that the restrictions $\Delta_{V^j}$ and $\Delta$ are equivalent to a simple strata. Then there is a self-dual simple stratum which is split by $(V^j)$ and equivalent to $\Delta$. 


For the non-quaternionic skew case, see \cite[Theorem 6.16]{11}. For the proof we need three further technical lemmas:

**Lemma 4.4** (\cite[4.28]{10}). Let $\Delta$ be a stratum. Then $\Delta$ is equivalent to a simple stratum if and only if $\operatorname{Res}_F(\Delta)$ is equivalent to a simple stratum.

**Lemma 4.5** (\cite[4.21]{10}). Let $\Delta$ be a pure stratum. Then $\Delta$ is a simple stratum if and only if $\operatorname{Res}_F(\Delta)$ is a simple stratum.

**Lemma 4.6** (\cite[1.9]{12}). Let $[\Lambda_F, n, r, \gamma_t]$ be a sequence of equivalent simple strata in $\operatorname{End}_F(V)$ such that $\gamma_t$ converges to some $\gamma$ in $\operatorname{End}_F(V)$. Then the stratum $[\Lambda_F, n, r, \gamma]$ is simple.

Here we consider $h_F := \operatorname{trd}_{D_F} \circ h$.

**Proof of Proposition 4.3**. This proof uses the strategy of \cite[1.10]{14}, where the author shows the existence of a sequence $\gamma_t$, $t \in \mathbb{N}_0$, satisfying:

(i) $[\Lambda_F, n, r, \gamma_t]$ is simple and equivalent to $\operatorname{Res}_F(\Delta)$ for all $t \in \mathbb{N}_0$,

(ii) $\gamma_t + \sigma h_F(\gamma_t) \in a_{\Lambda_F, r - rt}$ for all $t \in \mathbb{N}_0$.

(iii) $\gamma_t - \gamma_{t+1} \in a_{\Lambda_F, r - rt}$ for all $t \in \mathbb{N}_0$.

We show that we could have chosen $\gamma_t$ in $\prod_{i=1}^\ell \operatorname{End}_D V^i$. This follows from Proposition 4.2 for $\gamma_0$. So suppose that $\gamma_0, \ldots, \gamma_t$ are elements of $\prod_{i=1}^\ell \operatorname{End}_D V^i$ which satisfy (i)-(iii). $\sigma$ induces an involution on $J$. Let $J_0$ be the set of $\sigma$-fixed points in $J$. We denote the stratum $[\Lambda_F, n, r - t - 1, 2^{-\gamma_t}]$ by $\tilde{\Delta}$. Let $s$ be a tame corestriction with respect to $\gamma$. The derived stratum $\partial_s(\operatorname{Res}_F(\Delta))$ is equivalent to a stratum $[\Lambda, r - t, r - t - 1, \delta]$ for some $\delta \in F[\gamma_t]$, by the proof of \cite[1.10]{14}. Now, $\tilde{\beta}$ and $\delta$ commute with the projections $V^i$ and thus $\partial_{1, \gamma_t}(\operatorname{Res}_F(\tilde{\Delta}|_{V^i}))$ is equivalent to a simple stratum. Therefore the strata $\operatorname{Res}_F(\tilde{\Delta})$ and $\operatorname{Res}_F(\tilde{\Delta}|_{V^i})$ are equivalent to simple strata, by \cite[2.13]{8}. Thus $\tilde{\Delta}$ and its restrictions are equivalent to simple strata, by Lemma 4.4. Now, we can find by Proposition 4.2 a simple stratum $\tilde{\Delta}$ split under $(V^i)$ and equivalent to $\tilde{\Delta}$. We put $\gamma_{t+1} := \tilde{\beta}$.

The sequence $(\gamma_t)_{t \in \mathbb{N}_0}$ converges and we denote the limit by $\gamma$. Then $[\Lambda_F, n, r, \gamma]$ is simple, by Lemma 4.6 and therefore $[\Lambda, n, r, \gamma]$ is simple, by Lemma 4.3. This finishes the proof.

One consequence is:

**Corollary 4.7**. Let $\Delta$ be a semisimple stratum such that $\# \Delta$ is equivalent to $\Delta$. Then, $\Delta$ is equivalent to a self-dual semisimple stratum.

**Proof**. By Corollary 4.37 and Corollary 6.4 we can assume that the idempotents of the associate splitting of $\Delta$ are permuted by the action of $\sigma_h$. We apply Proposition 4.3 on $\Delta_i$, for $i \in I_0$, and we replace $\beta_i$ by $-\sigma_h(\beta_{\gamma(i)})$ for $i \in I_-$. \hfill $\square$

### 4.3 Intersection formulas

For the next sections we need some special cases of (3.2). For this we need to recall that on attaches to a semisimple stratum two further pro-$p$-subgroups of $G$: $S(\Delta)$ and $M(\Delta) := 1 + m(\Delta)$, see \cite[before 4.25, after 5.14]{10} for the definitions of $S(\Delta)$ and $m(\Delta)$. We need the technique of $\dagger$ construction for strata, see \cite[4.6]{10}, to attach to a stratum $\Delta$ a stratum $\Delta'$ where the lattice sequence is principal.

**Proposition 4.8**. Let $\Delta_1$ and $\Delta_2$ be two semisimple strata which share the associated splitting and the parameters $r$ and $n$. Let $H$ be one of the following groups:

- $\prod_{i \in I} \operatorname{Aut}_D V^i$ or
Let $h$ be an element of $H$. Then we have the following intersection formulas.

(i) \((S(\Delta_2)hS(\Delta_1)) \cap H = (S(\Delta_2) \cap H)h(\Delta_1) \cap H)\),

(ii) \((M(\Delta_2)hM(\Delta_1)) \cap H = (M(\Delta_2) \cap H)h(M(\Delta_1) \cap H)\),

Proof. We only show [1] because the proof of the second equation is obtained in replacing $S$ by $M$. We start with the case that $H$ is equal to $\prod\pi_1 \text{Aut}_D V_i$. Consider the group $\Gamma := \{ \pm 1 \}^{\#I_i}$ acting on $\tilde{G}$ by conjugation. Its fixed point set is $H$ and $\Gamma$ is contained in $P(\Lambda^1) \cap P(\Lambda^2)$. Lemma 3.6 implies (i).

Let us now assume $\beta_1 = \beta_2 = \beta$ and that $H$ is $C_A(\beta)^\ast$. Without loss of generality we can assume that both lattice sequences have the same $F$-period. We make the $\tilde{g}$-construction for $\Delta_1 \otimes L$ and $\Delta_2 \otimes L$, and we get $(\Delta_1 \otimes L)^\dagger$ and $(\Delta_2 \otimes L)^\dagger$ which are in fact equal to $\Delta_1^\dagger \otimes L$ and $\Delta_2^\dagger \otimes L$, respectively. Both latter strata are semisimple and the lattice sequences on the blocks are principal lattice chains. Thus both lattice sequences have the same $F$-period. We make the $\tilde{g}$-construction for $\Delta_1 \otimes L$ and $\Delta_2 \otimes L$, and we get $(\Delta_1 \otimes L)^\dagger$ and $(\Delta_2 \otimes L)^\dagger$ which are in fact equal to $\Delta_1^\dagger \otimes L$ and $\Delta_2^\dagger \otimes L$, respectively. Both latter strata are semisimple and the lattice sequences on the blocks are principal lattice chains. Thus there is an invertible element $g$ of $C := C_{\text{End}_L(V)}(\beta)^\dagger$ which sends $\Lambda_i^{\dagger\dagger}$ to $\Lambda_i^{\dagger\dagger}$. Formula (1) is true for the triple $(\Delta_2 \otimes L)^\dagger, (\Delta_2 \otimes L)^\dagger, C^\ast$, and for every element $c \in C^\ast$ by [5] (1.6.1). We conjugate back with $g$ to obtain (i) for $(\Delta_2 \otimes L)^\dagger, (\Delta_2 \otimes L)^\dagger, C^\ast$ and every $c \in C^\ast$. In particular the formula is valid for $\text{diag}(b_i, i \in I_1)$ with $b_i \in C_{\text{Aut}_L(V)}(\beta_i)^\dagger$, $i \in I_1$. Using a limit-argument we obtain for all $h \in C_{\text{Aut}_L(V)}(\beta)^\dagger$

\[ (S(\Delta_2 \otimes L)hS(\Delta_1 \otimes L)) \cap C_{\text{Aut}_L(V)}(\beta)^\dagger \subseteq (S(\Delta_2 \otimes L)) \cap C_{\text{Aut}_L(V)}(\beta)^\dagger h(S(\Delta_1 \otimes L) \cap C_{\text{Aut}_L(V)}(\beta)^\dagger). \]

The last inclusion is an equality. We consider $h \in C_A(\beta)^\ast$ and take $\tau$-fixed points. Then [5] implies the assertion.

### 4.4 Matching and intertwining

The intersection formulas of the last section allow us to determine the $G$-intertwining of two self-dual semisimple strata with the same $\beta$. Two semisimple strata with same parameters which intertwine by an element of $\tilde{G}$ possess a matching, i.e., they have a unique one-to-one correspondence between the block structures, in particular with coinciding dimensions. We prove in the self-dual case that this correspondence is given by isometries for $\sigma$-fixed blocks.

In this section we suppose that $\Delta$ and $\Delta'$ are two semisimple strata which satisfy $e(\Lambda_i F) = e(\Lambda_i' F)$, $n = n'$ and $r = r'$.

At first we recall the matching and the intertwining formulas for semisimple strata.

**Theorem 4.9** ([10] 4.41 (with 4.32)). Suppose $I(\Delta, \Delta') \neq \emptyset$. Then there is a unique bijection $\zeta: I \to I'$ such that the set $I(\Delta, \Delta') \cap \prod_{i \in I} \text{Hom}_D(V_i, V_{\zeta(i)})$ is not empty. The latter non-emptiness condition is equivalent in saying that for all $i \in I$ the $D$-dimensions of $V_i$ and $V_{\zeta(i)}$ coincide and the direct sum $\Delta_i \oplus \Delta'_{\zeta(i)}$ is equivalent to a simple stratum.

The map $\zeta$ is called the matching of $(\Delta, \Delta')$, and we denote it also as $\zeta_{\Delta, \Delta'}$.

**Theorem 4.10** ([10] 4.36). Suppose $I(\Delta, \Delta') \neq \emptyset$ with matching $\zeta$. Then the following holds.

(i) \(I(\Delta, \Delta') = M(\Delta')(I(\Delta, \Delta') \cap \prod_{i \in I} \text{Hom}_D(V_i, V_{\zeta(i)}))M(\Delta)\).

(ii) Suppose there is an element $\tilde{g}$ of $\tilde{G}$ such that $\tilde{g}\beta\tilde{g}^{-1} = \beta'$, then we have \(I(\Delta, \Delta') = M(\Delta')\tilde{g}C_A(\beta)^\ast M(\Delta)\).

Proof. The second assertion follows from [10] 4.36 and the first assertion follows from the second by diagonalization, Proposition 1.2 and the fact that $M(\ast)$ only depends on the equivalence class of the semisimple stratum.
We now can come to the self-dual case:

**Theorem 4.11** (see [11] 6.22 for the case over $F$). Suppose $\Delta$ and $\Delta'$ are intertwined by an element of $G$ with matching $\zeta$. Then the following holds.

(i) $I_G(\Delta, \Delta') = (G \cap M(\Delta'))(I_G(\Delta, \Delta') \cap \prod_{t \in I} \text{Hom}_D(V_i, V_{\zeta(t)}))(G \cap M(\Delta)).$

(ii) Suppose there is an element $\tilde{g}$ of $\tilde{G}$ such that $\tilde{g}\beta\tilde{g}^{-1} = \beta'$, then we have

$I_G(\Delta, \Delta') = (G \cap M(\Delta'))(G \cap \tilde{g}C_A(\beta')\tilde{g}^{-1})(G \cap M(\Delta)).$

**Proof.** This Theorem follows from Theorem 4.10, Proposition 4.8 and Proposition 3.3 for the group $\Gamma = \{\sigma, 1\}$. \hfill $\square$

The last Theorem has now consequences for the matching.

**Corollary 4.12** (see [6] 9.5 for the case over $F$). Suppose $\Delta$ and $\Delta'$ are self-dual and intertwined by an element of $G$ with matching $\zeta$. Then $I_G(\Delta, \Delta') \cap \prod_{t \in I} \text{Hom}_D(V_i, V_{\zeta(t)})$ is non-empty and $h_J$ is isometric to $h_{\zeta(J)}$ for all $\sigma$-invariant subsets $J$ of $I$. In particular $\zeta$ and $\sigma$ commute.

**Proof.** The non-emptiness of $I_G(\Delta, \Delta') \cap \prod_{t \in I} \text{Hom}_D(V_i, V_{\zeta(t)})$ follows from 4.11. Every element of this intersection restricts to an isometry from $h_J$ to $h_{\zeta(J)}$. If $\tilde{g}$ is an element of the above intersection then $\sigma(\tilde{g})$ too, and the uniqueness of the matching implies that $\zeta$ and $\sigma$ commute. \hfill $\square$

Later we are going to see that the non-emptiness of $I_G(\Delta, \Delta')$ is not needed in Corollary 4.12 for $\sigma$ and $\zeta$ to commute.

### 4.5 Skolem-Noether

We prove a version of Skolem–Noether for $G$.

**Theorem 4.13** (see [11] 5.2 for the case over $F$). Let $\Delta$ and $\Delta'$ be two pure strata such that $e(\Delta|F) = e(\Delta'|F)$ and $n = n' > r > r'$. Suppose that there is an element of $G$ which intertwines $\Delta$ with $\Delta'$ and that $\beta$ and $\beta'$ have the same minimal polynomial. Then the element $\beta$ is conjugate to $\beta'$ by an element of $G$.

**Proof.** We only consider pure strata, so without loss of generality we can assume $n = r - 1 = r' - 1$. We can find an element $\tilde{g} \in \tilde{G}$ such that $\tilde{g}\beta\tilde{g}^{-1} = \beta'$. The proof consists of two steps.

**Step 1** Let us at first assume that both strata are simple. The set $G \cap \tilde{g}B$ is non-empty by Theorem 4.11 because $I_G(\Delta, \Delta')$ is non-empty. This finishes Step 1.

**Step 2** There are simple strata $\Delta$ and $\Delta'$ equivalent to $\Delta$ and $\Delta'$, respectively. By Proposition 4.13 we can assume that $\beta$ and $\tilde{\beta}$ have the same minimal polynomial. Let $e_w$ be the wild ramification index of $E[F]$. There is an element $\gamma \in E$ which is congruent to $\tilde{\beta}e_w$ and generates the maximal tamely ramified sub-extension $E_{tr}$ in $E$. By [6] 2.15 there is an $F$-linear field monomorphism $\phi : E_{tr} \to E$ such that $\phi(\gamma)$ is congruent to $\beta e_w$ in $E$. We take $\gamma'$ to be the image of $\gamma$ under the isomorphism $E \cong E'$ which sends $\beta$ to $\beta'$. We repeat the argument to get an $F$-linear field monomorphism $\phi' : E_{tr}' \to E'$ such that $\phi'(\gamma')$ is congruent to $\beta e_w$. The elements $\phi(\gamma)$ and $\phi'(\gamma')$ are conjugate by an element of $G$ by Step 1, and we can assume that $\phi(\gamma)$ and $\phi'(\gamma')$ coincide and we denote $\phi(\gamma)$ by $t$. If $E[F[t]$ and $E'[F[t]$ are isomorphic by a map which sends $\beta$ to $\beta'$, then $\beta$ is conjugate to $\beta'$ by an element of $C_G(F[t])$ by [11] 5.1. Let $\psi$ be the $F$-linear isomorphism from $E$ to $E'$ which sends $\beta$ to $\beta'$. Then

\[ \psi(t) = \psi(\beta e_w) = \beta e_w \equiv t. \]  

(4.14)
Thus, \( \psi(x) \) is congruent to \( x \) in \( E' \) for all \( x \in F[t] \) because \( t \) is minimal over \( F \). We obtain at first that \( \psi ^{-1} \) is the identity on the maximal unramified sub-extension \( F[t]_{ur} \) of \( F[t]/F \). Let \( \pi_t \) be a uniformizer of \( F[t] \) whose \( e(F[t]/F) \)th power is an element of \( F[t]_{ur} \). We just denote the ramification index of \( F[t]/F \) by \( e_t \). From the congruence \( \psi(\pi_t) \equiv \pi_t \mod e_t \), we obtain an element \( y \in 1 + pE' \) such that \( y(\pi_t) = \psi(\pi_t) \). Thus \( y ^{-1} = 1 \). Therefore \( y \) and \( 1 \) are \( e_t \)th root of unity which are congruent in \( E' \). Thus \( y = 1 \) because \( e_t \) is not divisible by the residue characteristic of \( F \). We have established that \( \psi \) is the identity on \( F[t] \) which finishes the proof.

This has an immediate analogue consequence for semisimple strata.

**Corollary 4.15.** Suppose \( \Delta \) and \( \Delta' \) are self-dual semisimple strata with \( r = r' \), the same \( F \)-period and a bijection \( \zeta : I \to I' \) such that \( \beta_i \) and \( \beta'_i(\zeta) \) have the same minimal polynomial over \( F \). Suppose further there is an element of \( G \cap \prod \text{Hom}(V^i, V^{\zeta(i)}) \) which intertwines \( [\Lambda, n, n_i - 1, \beta_i] \) with \( [\Lambda^{\zeta(i)}, n'_i(\zeta), n'^{\zeta(i)} - 1, \beta'^{\zeta(i)}] \) for all \( i \in I \) with \( \beta_i \neq 0 \). Then \( \beta \) is conjugate to \( \beta' \) by an element of \( G \).

**Proof.** We take at first an index \( i \in I_0 \) with \( \beta_i \neq 0 \). Then we get \( n_i = n'_i(\zeta) \) from (4.16), and we get by (4.13) an isometry from \( h_i \) to \( h'_i(\zeta) \) which conjugates \( \beta_i \) to \( \beta'_i(\zeta) \). For \( i \in I \), we can take any \( D \)-linear isomorphism \( g_i \) from \( V^i \) to \( V'^i(\zeta) \) which conjugates \( \beta_i \) to \( \beta'_i(\zeta) \), and we obtain an element \( \sigma_b(\delta_i) \in \text{Hom}(V^{\zeta(\sigma_i)}, V^{\sigma_i}) \) which conjugates \( \beta'_i(\zeta(\sigma_i)) \) to \( \beta'_{\sigma(i)}(\zeta) \). This finishes the proof.

### 4.6 Conjugate semisimple strata

In this section we prove an “intertwining implies conjugacy”-kind of result for self-dual semisimple strata. It will lead directly to the analogue result for self-dual semisimple characters. In this section we fix two intertwining semisimple strata \( \Delta \) and \( \Delta' \) and assume \( n = n', r = r' \) and \( e(\Lambda/F) = e(\Lambda'/F) \).

Intertwining does not imply conjugacy in general, even if one would assume that \( \tilde{g}A \) is equal to \( \Lambda^{\zeta(i)} \) for some \( \tilde{g} \in \tilde{G} \) because one has to keep track of the embedding type. This is the reason why we have introduced the map \( \zeta \) in (4.46). It is the map from \( \kappa_E \) to \( \kappa_{E'} \) given by the diagram

\[
\kappa_E \to a_0/a_1 \to (ga_0g^{-1} + a'_0)/(ga_1g^{-1} + a'_1) \leftarrow a'_0/a'_1 \leftarrow \kappa_{E'}, \quad g \in I(\Delta, \Delta').
\]

(4.16)

This map does not depend on the choice of \( g \). We call the pair \( (\zeta, \tilde{\zeta}) \) the matching pair. In fact we do not need the whole map for the conjugacy. The restriction to \( \kappa_{E_{ur}} \) is enough. The algebra \( E_D \) is the product of the unramified field extensions \( E_iF \) of degree \( \text{gcd}(\sqrt{[D:F]}, f(E_iF)) \), \( i \in I \).

**Theorem 4.17** (4.48). Suppose there is an element \( \tilde{g} \in \prod \text{Hom}(V^i, V^{\zeta(i)}) \) such that \( \tilde{g}\Lambda = \Lambda' \) and such that the conjugation with \( \tilde{g} \) verifies \( \tilde{\zeta}|_{E_{ur}} \). Then there is an element \( u \in \prod \text{Hom}(V^i, V^{\zeta(i)}) \) such that \( u\Lambda = \Lambda' \) and \( u\Delta \) is equivalent to \( \Delta' \). We can choose \( u \) to further satisfy \( u\beta u^{-1} = \beta' \) if \( \beta \) and \( \beta' \) have the same characteristic polynomial over \( F \).

The second part of this Theorem follows directly from the proof of (4.48). We continue now with its self-dual analogue.

**Theorem 4.18.** Suppose both strata are self-dual and intertwine by an element of \( G \). Let \( g \) be an element of \( G \cap \prod \text{Hom}(V^i, V^{\zeta(i)}) \) which satisfies \( g\Lambda = \Lambda' \) and such that the conjugation with \( g \) verifies \( \tilde{\zeta}|_{E_{ur}} \). Then there is an element \( u \) of \( \prod \text{Hom}(V^i, V^{\zeta(i)}) \) such that \( u\Lambda = \Lambda' \) and \( u\Delta \) is equivalent to \( \Delta' \). We can choose \( u \) to further satisfy \( u\beta u^{-1} = \beta' \) if \( \beta \) and \( \beta' \) have the same characteristic polynomial over \( F \).

**Proof.** We apply \( g \) and can assume \( \Lambda^i = \Lambda^{\zeta(i)} \), for all \( i \), and \( \tilde{\zeta}|_{E_{ur}} = \text{id}|_{E_{ur}} \) without loss of generality. We can therefore restrict to the case of a single \( \sigma \)-orbit, i.e. \( I \) is either a singleton or consists of two non-fixed points of \( \sigma \). We apply in the latter case Theorem 4.17 for one index \( i \), to get an appropriate element \( u_i \), and \( u := \text{diag}(u_i, \sigma_h(u_i)^{-1}) \) satisfies the assertions. So we are left with the singleton case \( I = \{i_0\} \),
i.e. we assume that both strata are simple. By diagonalization, see [4,3] we can assume that $\beta$ and $\beta'$ have the same minimal polynomial over $F$. The element $\beta$ is conjugate to $\beta'$ by some element $g_0$ of $G$ by Theorem 4.7. The conjugation by $g_0$ induces $\zeta|_{E_0^F} = \text{id}_{E_0^F}$ because $g_0$ intertwines $\Delta$ with $\Delta'$. Thus, $g_0xg_0^{-1} + a_1$ is equal to $x + a_1$ in $a_0[a_1$ for all $x \in o_E$. By [10] 4.39 there is an element $v$ of $P(\Lambda)$ which conjugates $\beta$ to $\beta'$. We consider the $C_G(E_D)$-equivariant affine Broussous–Lemaire isomorphism

$$j_{E_D} : B_{red}(\tilde{G})^{E_D^F} \to B_{red}(C_G(E_D))$$

between the reduced Bruhat–Tits buildings, see [1]. The translation classes $[\Lambda]$ and $[g_0^{-1}\Lambda]$ are elements of $B_{red}(\tilde{G})^{E_D^F}$. They satisfy

$$v^{-1}g_0j_{E_D}([g_0^{-1}\Lambda]) = j_{E_D}(v^{-1}g_0[g_0^{-1}\Lambda]) = j_{E_D}([\Lambda]).$$

The element $v^{-1}g_0$ acts type-preserving on $B_{red}(C_G(E_D))$ because $\nu_{E_D}(\text{Nrd}_{E_D}(v^{-1}g_0))$ vanishes, i.e. the points $j_{E_D}([g_0^{-1}\Lambda])$ and $j_{E_D}([\Lambda])$ have the same simplicial type in $B_{red}(C_G(E_D))$. Note that both are points of the building $B(C_G(E_D))$ of the centralizer of $E_D$ in $G$. By [9] 5.2 there is an element $g_1 \in C_G(E_D)$ such that

$$g_1j_{E_D}([g_0^{-1}\Lambda]) = j_{E_D}([\Lambda]).$$

The $C_G(E_D)$-equivariance and the injectivity of $j_{E_D}$ imply $[g_1g_0^{-1}\Lambda] = [\Lambda]$. Thus $g_1g_0^{-1}\Lambda$ is a translate of $\Lambda$, i.e. $g_1g_0^{-1}$ lies in the normalizer of $\Lambda$ and hence in $d_0^\perp$, because it is an element of $G$. The element $u := g_1g_0^{-1}$ is an element of $P(\Lambda) \cap G$ which conjugates $\beta$ to $\beta'$. This finishes the proof. 

\section{Endo-classes of self-dual strata}

Now we compare self-dual semisimple strata which are allowed to correspond to different hermitian spaces where we also allow orthogonal and symplectic spaces over $F$. We further want to allow the strata to have different parameter $r$ and different $E_i$-periods, so we repeat the notion of group level and degree. The degree of a semi-pure stratum $\Delta$ is the $F$-dimension of $E$ and the group level of a semisimple stratum $\Delta$ is $\lceil\frac{\text{deg}(\Lambda)}{e(\Lambda|E)}\rceil$ where $e(\Lambda|E)$ is the lowest common multiple of $(e(\Lambda^i, E_i))_{i \in I}$ except for the null case where we set the group level to be infinity. Two semisimple strata $\Delta$ and $\Delta'$ (possibly given on different vector spaces and different skew-fields over $F$) of the same group level and the same degree are called endo-equivalent if there is a bijection $\zeta : I \to I'$ such that $\text{Res}_F(\Delta_i) \oplus \text{Res}_F(\Delta_{\zeta(i)})$ is equivalent to a simple stratum for all indexes $i \in I$, see [10] 6.6.6.7. We call $\zeta|_{\Delta, \Delta'}$ the matching from $\Delta$ to $\Delta'$. This is an equivalence relation by [10] 6.6.7, and note that the used direct sum is a slight generalization, see [10] after 6.3. The equivalence classes are called endo-classes and the endo-class of a semisimple stratum $\Delta$ is denoted by $\mathcal{E}(\Delta)$.

\textbf{Proposition 5.1} ([10] 6.7). Two endo-equivalent strata $\Delta$ and $\Delta'$ with matching $\zeta$ satisfy

$$e(\Lambda'|E_i) = e(\Lambda'^{\zeta(i)}|E'_{\zeta(i)}), \ f(\Lambda'|E_i) = f(\Lambda'^{\zeta(i)}|E'_{\zeta(i)}),$$

for all $i \in I$.

For endo-equivalence to make sense it is important that two intertwining semisimple strata of same degree and group level are endo-equivalent. This is true because the matching results remain true.

\textbf{Theorem 5.2.} Theorem 4.3 and Corollary 4.12 remain true if we replace the conditions: $n = n'$, $r = r'$ and $e(\Lambda|F) = e(\Lambda'|F)$ by the assumption that $\Delta$ and $\Delta'$ have the same degree and the same group level.

We come to the proof after some preparation.

\textbf{Remark 5.3.} There are two steps to reduce from same group level and same degree to equal parameters.

Step 1: The method of \textit{repeating}, sometimes also called doubling. Given a stratum $\Delta$ and a positive integer $k$ we can scale the $F$-period of $\Lambda$ to $k\Lambda|F$ in the following way. One constructs a new lattice sequence $k\Lambda$ by repeating $k$-times the lattices which occur in the image of $\Lambda$:

$$k\Lambda_j := \Lambda_{\lfloor \frac{j}{k} \rfloor}, \ j \in \mathbb{Z}.$$
Step 2: Raising or lowering the parameter $r$. Suppose $\Delta$ and $\Delta'$ are two intertwining semisimple strata of the same degree and the same group level, which share the $F$-period and! the parameter $n$. Suppose further $r \leq r'$. Then the stratum $\Delta'((r-r')+) = \text{the stratum obtained from } \Delta'$ in raising $r' \leq r$, is still semisimple, by the proof of [10, 5.48], and we have $\epsilon (\Lambda |E) = \epsilon (\Lambda |E')$ by [10, 4.2]. Thus $\Delta$ and $\Delta'((r-r')+)$ have the same group level and the degree. Instead of raising $r'$ we can lower $r$ in $\Delta$.

An endo-class is invariant under repeating, raising and lowering.

**Lemma 5.4.** Suppose $\Delta$ and $\Delta'$ are two intertwining semisimple strata sharing the $F$-period, the parameter $n$, the group level and the degree. Suppose $r \leq r'$. Then $\Delta((r-r')-)$ and $\Delta'$ intertwine. If further both strata are self-dual and intertwine by some element of $G$, then $\Delta((r-r')-)$ and $\Delta'$ intertwine by some element of $G$.

**Proof.** If the common group level is infinite then the identity is a possible intertwiner. So let us suppose that both strata have finite group level, i.e. are non-null. Then all strata

$$\Delta((r-r')-), \Delta', \Delta'((r-r')+), \Delta'$$

have the same group level, because $\epsilon (\Lambda |E) = \epsilon (\Lambda |E')$, see [5.3] Step 2 above, and two successive strata are endo-equivalent, the middle two by Theorem [4.9]. Thus, by transitivity, the strata $\Delta((r-r')-)$ and $\Delta'$ are endo-equivalent, and therefore intertwine by [10, 4.32].

In the self-dual case we apply diagonalization [4.3] on the endo-equivalent strata $\Delta((r-r')-)$ and $\Delta'$ to reduce to the case where $\beta$ and $\beta'$ have the same characteristic polynomial. So, if $\Delta$ and $\Delta'((r-r')+)$ intertwine by some element of $G$ then $\beta$ and $\beta'$ are conjugate by some element of $G$, by Corollary [4.16] and therefore $\Delta((r-r')-)$ and $\Delta'$ intertwine by some element of $G$.

**Proof of Theorem 5.2.** This Theorem follows now from Remark 5.3 and Lemma 5.4. We leave this to the reader as an exercise.

In the following, if we say “possibly different signed hermitian spaces” it includes possibly different vector spaces and possibly different skew-fields.

**Proposition 5.5.** Let $\Delta$ and $\Delta'$ be two endo-equivalent self-dual strata on possibly different signed hermitian spaces $h$ and $h'$ of the first kind and of the same type. Then the matching is equivariant with respect to the adjoint involutions.

**Proof.** We need to consider the restrictions of $\Delta$ and $\Delta'$ to $F$. So for this proof we just assume that both strata are strata over $F$. By repeating and raising we can assume that both strata share the $F$-period, and the parameters $n$ and $r$. By diagonalization, see Proposition 4.3 for $i \in I_0$ and Proposition 4.2 for $i \in I_+$, we can assume that $\beta$ and $\beta'$ have the same minimal polynomial. For $i \in I$ we have that $\beta_i$ and $\beta'_i$ have the same minimal polynomial, and therefore $-\sigma_h(\beta_i)$ and $-\sigma_{h'}(\beta_{i(\eta)})$, i.e. $\beta_{\sigma(i)}$ and $\beta'_{\sigma(i)}$, have the same minimal polynomial. Thus $\zeta \circ \sigma = \sigma' \circ \zeta$.

We denote the class of self-dual semisimple strata which are endo-equivalent to a given self-dual semisimple stratum $\Delta$ by $\mathcal{E}_-(\Delta)$.
6 Self-dual semisimple characters

In this section we study the first building block for explicit constructions of cuspidal irreducible representations of $G$. These building blocks are the self-dual semisimple characters. We are going to see that they correspond to the semisimple characters of $G$ which are fixed by the adjoint involution. After defining them we turn directly to transfers, followed by the matching theory and results on intertwining, diagonalization theory and conjugacy.

6.1 First definitions

Here we define self-dual semisimple characters. There is a big introduction about semisimple characters in section 5 in [10], and we use the notations and definitions from there. We fix an additive character $\psi_F$ from $F$ to $\mathbb{C}^*$. One attaches to semisimple stratum $\Delta$ a set $C(\Delta)$ of complex valued characters on a compact open subgroup $H(\Delta)$ of $\tilde{G}$. Given a character $\chi$ on a subgroup of $G$ we write $\sigma \cdot \chi$ for $\chi \circ \sigma$. Recall that $p$ is the odd residue characteristic of $F$.

**Definition 6.1.** Let $\Delta$ be a self-dual semisimple stratum. Then $H(\Delta)$ and $C(\Delta)$ are invariant under the action of $\sigma$. We define $C_-(\Delta)$ to be the set of all restrictions of the elements of $C(\Delta)\sigma$, i.e. the $\sigma$-fixed elements of $C(\Delta)$, to $H_-(\Delta) := H(\Delta) \cap G$. The restriction map is a bijection by Glaubermann-correspondence using that $H(\Delta)$ is a pro-$p$-group with odd $p$. We call an element $\theta \in C(\Delta)\sigma$ the lift of $\theta|_{H_-(\Delta)}$ to $H(\Delta)$. The elements of $C_-(\Delta)$ are called self-dual semisimple characters. For a stratum $\Delta'$ equivalent to $\Delta$ we define $C(\Delta') := C(\Delta)$ and $C_-(\Delta') := C_-(\Delta)$.

**Proposition 6.2.** Let $\Delta$ be a self-dual semisimple stratum and $\theta \in C(\Delta)$. Then $\theta|_{H_-(\Delta)}$ is an element of $C_-(\Delta)$.

**Proof.** If $\theta_1, \theta_2$ and $\theta_3$ are elements of $C(\Delta)$ then $\theta_1 \theta_2^{-1} \theta_3$ is an element of $C(\Delta)$ by the definition of $C(\Delta)$, see [10, 5.6]. Thus the number of elements of $C(\Delta)$ which have the same restriction as $\theta$ on $H_-(\Delta)$ is independent $\theta$. The number of elements of $C(\Delta)$ is a power of $p$ and thus there is a non-negative integer $s$ such that for every element of $\theta \in C(\Delta)$ there are exactly $p^s$ elements in $C(\Delta)$ with the same restriction as $\theta$. Thus the action of $\sigma$ on the set of those characters must have a fixed point. This finishes the proof.

The group $\tilde{G}$ acts by conjugation on the set of all complex characters $\chi$ on subgroups of $\tilde{G}$ which we denote by $(g, \chi) \mapsto g \cdot \chi$. An element $g \in \tilde{G}$ is said to intertwine a character $\chi : K \to \mathbb{C}$ with a second character $\chi' : K' \to \mathbb{C}$ if the restrictions of $\chi'$ and $g \cdot \chi$ coincide on $gKg^{-1} \cap K'$. We denote the set of all elements of $\tilde{G}$ which intertwine $\chi$ with $\chi'$ by $I(\chi, \chi')$. We adapt the notation $I_H(\chi, \chi')$ for the intersection of $I(\chi, \chi')$ with a subgroup $H$ of $\tilde{G}$. We say that $\chi$ and $\chi'$ intertwine by some element of $H$ if $I_H(\chi, \chi')$ is non-empty.

6.2 Transfers

We now recall the notion of transfer. For more detail consult section [10, section 6.2]. Transfers are defined between strata over different vectors spaces over different skew-fields central and of finite degree over $F$. Let $\Delta$ and $\Delta'$ be two endo-equivalent semisimple strata. Let us at first recall the transfer over the same skew-field. The canonically defined restriction maps

$$res_{\Delta \otimes \Delta'} : C(\Delta \otimes \Delta') \to C(\Delta)$$

define a bijection

$$\tau_{\Delta, \Delta'} := res_{\Delta \otimes \Delta'} \circ res_{\Delta \otimes \Delta'}^{-1} : C(\Delta) \to C(\Delta')$$

called the transfer map from $\Delta$ to $\Delta'$.

In the case of different skew-fields the transfer map is defined as follows:

$$\tau_{\Delta, \Delta'}(\theta) := \tau_{\Delta \otimes L, \Delta \otimes L}(\theta_L)|_{H(\Delta')}.$$
where $\theta_L$ is any extension of $\theta$ to $H(\Delta \otimes L)$. Here we use that $L$ splits the skew-fields. We call $\tau_{\Delta,\Delta'}(\theta)$ the transfer of $\theta$ from $\Delta$ to $\Delta'$. Let us recall that if $\Delta$ and $\Delta'$ are endo-equivalent and $V = V'$ then $\tau_{\Delta,\Delta'}(\theta) = \theta'$ if and only if $I(\Delta, \Delta') \subseteq I(\theta, \theta')$, see [10] 6.10, 5.9.

**Proposition 6.3.** Transfer commutes with the adjoint involutions, i.e. given two signed hermitian spaces $(h, V)$ and $(h', V')$ of the first kind and the same type with adjoint involutions $\sigma$ and $\sigma'$, respectively, and given two endo-equivalent semisimple strata $\Delta$ and $\Delta'$ in $V$ and $V'$, respectively, then $\sigma' \circ \tau_{\Delta,\Delta'} = \tau_{\#\Delta,\#\Delta'} \circ \sigma$.

**Proof.** Without loss of generality we can assume that we work over the same skew-field, because the transfer is defined by reducing to that case, i.e. from different skew-fields to $L$. Now the result follows, because the restriction maps commute with the adjoint involutions, i.e. $\text{res}_{\#\Delta \otimes H} \circ (\sigma \otimes \sigma') = \sigma' \circ \text{res}_{\Delta \otimes \Delta'}$.

The transfer map induces for self-dual semisimple strata $\Delta$ and $\Delta'$ via the Glauberman correspondence a bijection from $\text{C}_.(\Delta)$ to $\text{C}_.(\Delta')$ which maps $\theta_-$ to $\tau_{\Delta,\Delta'}(\theta)|_{H_.(\Delta')}$. We denote this map still by $\tau_{\Delta,\Delta'}$.

Let us recall:

**Definition 6.4** ([10] 6.3). Two semisimple characters $\theta \in C(\Delta)$ and $\theta' \in C(\Delta')$ on possibly different vector spaces over possibly different skew-fields over strata with the same degree and the same group level are called endo-equivalent if there are transfers which intertwine, i.e. if there are strata $\Delta \in \mathcal{C}(\Delta)$ and $\Delta' \in \mathcal{C}(\Delta')$ such that $\tau_{\Delta,\Delta'}(\theta)$ and $\tau_{\Delta',\Delta'}(\theta')$ intertwine.

### 6.3 Matching, diagonalization and intertwining

Here we repeat the result on matching and slightly generalize the diagonalization theorem from [10] and we further prove a diagonalization theorem for self-dual semisimple characters. We fix in this section two semisimple strata $\Delta$ and $\Delta'$ of the same group level and the same degree and two semisimple characters $\theta \in C(\Delta)$ and $\theta' \in C(\Delta')$. The restriction $\theta$ to $\text{Aut}_D(V') \cap H(\Delta)$ is denoted by $\theta_1$, and it is a simple character for $\Delta_1$.

#### 6.3.1 For $GL$

We recall the analogue of Theorem 4.9 for characters.

**Theorem 6.5** ([10] 5.48, 6.18,[2] 1.11). Suppose $I(\theta, \theta') \neq \emptyset$. Then there is a unique bijection $\zeta : I \to I'$ such that the set $I(\theta, \theta') \cap \prod_{i \in I} \text{Hom}_D(V^i, V^{\zeta(i)})$ is not empty. The latter non-emptiness condition is equivalent in saying that for all $i \in I$ the characters $\theta_i$ and $\theta'_{\zeta(i)}$ are endo-equivalent and the $D$-dimensions of $V^i$ and $V^{\zeta(i)}$ agree.

We call $\zeta$ the matching from $\theta \in C(\Delta)$ to $\theta' \in C(\Delta')$ and we write $\zeta_{\theta,\theta'}(\Delta,\Delta')$, but if there is no reason for confusion then we just skip $\Delta$ and $\Delta'$. Further we get a map $\bar{\zeta}_{\theta,\theta'}$ between the residue algebras defined in the similar way as it was done for strata, see [10] 5.52. $(\zeta, \bar{\zeta})$ is again called the matching pair of $\theta \in C(\Delta)$ and $\theta' \in C(\Delta')$.

Note that in the case of intertwining strata it is possible that $\zeta_{\theta,\theta}$ and $\zeta_{\Delta,\Delta'}$ do not coincide. For an example see [6] 9.7. But, if $\theta'$ is a transfer of $\theta$ from $\Delta$ to $\Delta'$, then both matching pairs coincide.

Diagonalization theorems are important to reduce a statement about semisimple characters to the case of transfers. Here is the GL-version.

**Theorem 6.6.** Suppose $\theta$ and $\theta'$ are endo-equivalent and suppose that the strata share the $F$-period and the parameters $r$ and $n$. Then there is a semisimple stratum $\Delta''$ on $V'$ with the same associate splitting as $\Delta'$, $\Lambda'' = \Lambda'$, $n'' = n$ and $r'' = r$ such that $\Delta \oplus \Delta''$ is semisimple and $\theta \oplus \theta' \in C(\Delta \oplus \Delta'')$. 

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The assertions of theorem imply that $\beta$ and $\beta''$ have the same minimal polynomial, because $\Delta \oplus \Delta''$ is semisimple, and $\theta$ and $\theta'$ are endo-equivalent.

**Proof.** See the proof of [10, 5.42] and use Theorem [10, 6.18] instead of [10, 5.35], to get $\Delta'' = [\Lambda', n, r, \beta'']$ such that $\beta''$ has the same minimal polynomial as $\beta$ and the same associate splitting as $\beta'$ and such that $\theta'$ is the transfer of $\theta$ from $\Delta$ to $\Delta''$. Then $\theta \otimes \theta' \in C(\Delta \oplus \Delta'')$. 

We are now able to state the result on intertwining for semisimple characters.

**Theorem 6.7.** Suppose are $\Delta$ and $\Delta'$ are semisimple strata which share the $F$-period and the parameters $n$ and $r$ and suppose that $\theta \in C(\Delta)$ and $\theta' \in C(\Delta')$ intertwine with matching $\zeta$. Then the following holds.

(i) $I(\theta, \theta') = S(\Delta')(I(\theta, \theta') \cap \prod_{i \in I}(V^i, V^{\zeta(i)})S(\Delta))$.

(ii) Suppose $I(\Delta, \Delta') \neq \emptyset$ and $\theta' = \tau_{\Delta, \Delta'}(\theta)$, then we have

$I(\theta, \theta') = S(\Delta')I(\Delta, \Delta')S(\Delta)$.

If further there is an element $\tilde{g}$ of $G$ such that $\tilde{g}\beta\tilde{g}^{-1} = \beta'$, then we have

$I(\theta, \theta') = S(\Delta')\tilde{g}C_A(\beta)'S(\Delta)$. (6.8)

**Proof.** The proof of Theorem 6.7 is the same as for Theorem 4.10 using [10, 5.15] and Proposition 6.6 instead of [10, 4.36] and Proposition 4.2. We use that the set $S(\Delta)$ only depends on $\theta$, $\Lambda$ and $r$ because it coincides with the intersection of $P_{\text{min}(-t_0 + r, \frac{1}{2}n)}(\Lambda)$ with $I(\theta)$. 

The next proposition is needed to establish the diagonalization theorem for self-dual characters later.

**Proposition 6.9.** Suppose that $\Delta$ is a stratum such that $C(\Delta) = C(\#\Delta)$, $\Lambda$ is self-dual and $-\sigma_h(\beta)$ has the same minimal polynomial and associate splitting as $\beta$. Suppose further that $\tau_{\Delta, \#\Delta}$ is the identity map. Then, there is an element $\gamma$ of $\Lambda$ with the same minimal polynomial and associate splitting as $\beta$ such that $[\Lambda, n, r, \gamma]$ is self-dual with the same set of semisimple characters as $\Delta$.

**Proof.** We write $\Delta'$ for $\#\Delta$. We assume $I = I'$ without loss of generality. The character $\theta \in C(\Delta)$ is intertwined by $1$. Thus $\zeta_{\theta, \Delta, \theta, \Delta'} = \text{id}_{\Delta}$. Further $\theta$ is its transfer from $\Delta$ to $\Delta'$ and thus

$\text{id}_{\Delta} = \zeta_{\theta, \Delta, \theta, \Delta'} = \zeta_{\Delta, \Delta'}$

and therefore $\beta_i$ and $\beta_i'$ have the same minimal polynomial, and

$\text{id}_{\Delta} = \tilde{\zeta}_{\theta, \Delta, \theta, \Delta'} = \tilde{\zeta}_{\theta, \tau_{\Delta, \Delta'}(\theta)} = \tilde{\zeta}_{\Delta, \Delta'}$.

Thus by Theorem 4.10 there is an element $u$ of $P_1(\Lambda)$ which conjugates $\beta$ to $\beta'$ and by (6.8) (for $\theta, \Delta, \Delta, \tilde{\gamma} = 1$) we can assume that $u$ is an element of $S(\Delta)$ and therefore the $S(\Delta)$-conjugacy class of $\beta$ is invariant under the map $-\sigma_h$. Using the latter action we use the pro-$p$-group property to show by induction that there is a sequence $(s_k)_{k \in \mathbb{N}}$ in $(\prod_i A^i) \cap S(\Delta)$ such that

$-\sigma_h(s_ks_{k-1}\cdots s_1\beta(s_ks_{k-1}\cdots s_1)^{-1}) \equiv s_ks_{k-1}\cdots s_1\beta(s_ks_{k-1}\cdots s_1)^{-1} \equiv \beta(k) \mod a_{1+k}$

and

$\beta(k) \equiv \beta(k-1) \mod a_k$.

But then $(\beta(k))_k$ converges to some element $\gamma$ and by compactness there is a convergent subsequence $s_{k_j}$, say with limit $s_0$. We obtain by continuity: $\gamma = s_0\beta s_0^{-1}$. This finishes the proof. 

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6.3.2 For $G$

In this section we assume that both strata $\Delta$ and $\Delta'$ are self-dual and $\theta \in C(\Delta)^\sigma$ and $\theta' \in C(\Delta')^{\sigma'}$. We also have a $G$-version of the diagonalization theorem. Note that there is no $G$ intertwining required in the assumption:

**Theorem 6.10.** Under the assumptions of Theorem 6.9 we further assume that $\Lambda \otimes \Lambda'$ is self-dual. Then there exists a stratum $\Delta''$ with $\Lambda'' = \Lambda'$, $n'' = n$ and $r'' = r$ such that $\Delta \otimes \Delta''$ is a self-dual semisimple stratum and $\theta \otimes \theta' \in C(\Delta \otimes \Delta'')$.

**Proof.** The proof is a complete analogue of the proof of Theorem 4.11 except for (6.13), which follows from diagonalization, see Proposition 4.3, (6.14) and Theorem 4.11.

We can choose a stratum $\Delta''$ which satisfies the assertions of Theorem 6.10. The character $\theta \otimes \theta'$ is fixed under $\tilde{\sigma}$ by (5.41), in particular we have $C(\Delta \otimes \Delta'') = C(\#(\Delta \otimes \Delta''))$ and therefore $C(\Delta'') = C(\#\Delta'')$. The elements $\beta''$, $\beta$ and $-\sigma_\delta(\beta'')$ have the same minimal polynomial. The character $\theta'$ is a restriction of $\theta \otimes \theta'$ to $\Delta''$ and to $\#\Delta''$, and thus the transfer map from $\Delta''$ to $\#\Delta''$ is the identity map. Now Lemma 6.9 finishes the proof. □

**Remark 6.11.** Starting with two self-dual lattice sequences $\Lambda$ and $\Lambda'$ of the same $F$-period we can always obtain $(2\Lambda_j)^{\#} = 2\Lambda_{1-j}$ for all $j \in \mathbb{Z}$ after a translation of the domain of $\Lambda$ and similar for $\Lambda'$. So one can establish that $2\Lambda \otimes 2\Lambda'$ is self-dual.

The same track to Theorem 4.11 leads to:

**Theorem 6.12** (see [4, 9.2, 9.3] for the case over $F$). Suppose $\Delta$ and $\Delta'$ are self-dual and share the $F$-period and the parameters $n$ and $r$ and suppose $\theta \in C(\Delta)^\sigma$ and $\theta' \in C(\Delta')^{\sigma'}$ intertwine by an element of $G$ with matching $\zeta$. Then the following holds.

1. $I_G(\theta, \theta') = (G \cap S(\Delta))(I_G(\theta, \theta') \cap \prod_{i \in I} \text{Hom}_D(V^i, V^{\zeta(i)}))(G \cap S(\Delta)).$

2. Suppose $\Delta$ and $\Delta'$ intertwine by an element of $\tilde{G}$ and $\theta' = \tau_{\Delta, \Delta'}(\theta)$, then we have $I_G(\theta, \theta') = (G \cap S(\Delta'))I_G(\Delta, \Delta')(G \cap S(\Delta)).$ (6.13)

If further there is an element $\tilde{g}$ of $\tilde{G}$ such that $\tilde{g}\beta\tilde{g}^{-1} = \beta'$ then $I_G(\theta, \theta') = (G \cap S(\Delta'))(G \cap \tilde{g}C_A(\beta)\tilde{g}^{-1}C_A(\beta'))(G \cap S(\Delta)).$ (6.14)

**Proof.** The proof is a complete analogue of the proof of Theorem 4.11 except for (6.13), which follows from diagonalization, see Proposition 4.3, (6.14) and Theorem 4.11. □

**Corollary 6.15.** Suppose $\Delta$ and $\Delta'$ are self-dual and $\theta \in C(\Delta)^\sigma$ and $\theta' \in C(\Delta')^{\sigma'}$. Suppose $I_G(\theta, \theta') \neq \emptyset$ with matching $\zeta$. Then $I_G(\theta, \theta') \cap \prod_{i \in I} \text{Hom}_D(V^i, V^{\zeta(i)})$ is non-empty and the $\epsilon$-hermitian spaces $h_i$ and $h_{i(\zeta)}$ are isometric for every $i \in I_0$.

**Proof.** By repeating and lowering, Remark 6.11 and Theorem 6.10 we can assume that $\beta$ and $\beta'$ have the same minimal polynomial and that $\theta' = \tau_{\Delta, \Delta'}(\theta)$. After raising, i.e. we consider the third parameter $\max(r, r')$, formula (6.14) implies that $\beta$ is conjugate to $\beta'$ over some element of $G$. Thus $I_G(\theta, \theta')$ contains an element of $\prod_{i \in I} \text{Hom}_D(V^i, V^{\zeta(i)})$. This finishes the proof. □

**Corollary 6.16.** Intertwining is an equivalence relation on the set of self-dual semisimple characters of $G$ of the same degree and the same group level.

**Proof.** Suppose $\theta \in \text{C}(-\Delta)$, $\theta', \text{C}(-\Delta')$ and $\theta'' \in \text{C}(\Delta'')$ are three self-dual semisimple characters such that $I_G(\theta, \theta') \neq \emptyset$ and $I_G(\theta', \theta'') \neq \emptyset$. Then by Glauberman correspondence $I_G(\theta, \theta')$ and $I_G(\theta', \theta'')$ are not empty. By Remark 6.11 and Theorem 6.10 (using repeating and lowering) we can assume without loss of generality that $\beta$, $\beta'$ and $\beta''$ have the same minimal polynomials and that the three characters are transfers of each other. From Theorem 6.12 follows now (after repeating and raising) that $\beta$, $\beta'$ and $\beta''$ are conjugate under some elements of $G$. Thus, as transfers, the characters $\theta$ and $\theta''$ intertwine by some element of $G$. □
6.4 Intertwining implies conjugacy for self-dual semisimple characters

Another important application of Theorem 6.10 is called intertwining and conjugacy theorem for self-dual semisimple characters.

**Theorem 6.17.** Suppose $\Delta$ and $\Delta'$ are self-dual semisimple strata sharing the $F$-period and the parameter $r$ and $\theta \in C(\Delta)\sigma$ and $\theta' \in C(\Delta')\sigma$ intertwine by an element of $G$. Let $(\zeta, \zeta')$ be their matching pair. Let $g$ be an element of $G \cap \prod_i \text{Hom}_D(V^i, V^C(V^i))$ which satisfies $g\Delta = \Lambda'$ and such that the conjugation with $g$ verifies $\zeta_{|\zeta'}$. Then there is an element $u \in G \cap \prod_i \text{Hom}_D(V^i, V^C(V^i))$ such that $u\Delta = \Lambda'$ and $u\theta = \theta'$. We can choose $u$ to further satisfy $u\beta u^{-1} = \beta'$ if $\beta$ and $\beta'$ have the same minimal polynomial over $F$ and $\tau_{\Delta, \Delta}(\theta) = \theta'$.

**Proof.** By diagonalization, see Theorem 6.10, we can assume that $\beta$ and $\beta'$ have the same minimal polynomial and $\theta' = \tau_{\Delta, \Delta}(\theta)$. The set $I_G(\Delta, \Delta')$ is not empty by Theorem 6.12 and from the transfer property follows that $(\theta, \theta')$ and $(\Delta, \Delta')$ have the same matching pair. Now the theorem follows from Theorem 4.18.

At the end of this section we show that $\sigma$-fixed semisimple character are always lifts of semisimple characters:

**Proposition 6.18.** Suppose $\Delta$ is a semisimple stratum such $C(\Delta)$ is invariant under the action of $\sigma$. Then there is a self-dual semisimple stratum $\Delta' = [\Lambda, n, r, \beta']$ such that $C(\Delta) = C(\Delta')$.

**Proof.** The proof is done by induction on $n-r$. If $n = r$, then $\beta = 0$ by the definition of semisimple strata, and $\Delta$ is therefore self-dual. Suppose $n > r$. By induction hypothesis there is a self-dual semisimple stratum $\Delta := [\Lambda, n, r + 1, \gamma]$ such that $C(\Delta) = C(\Delta(1+))$. By the translation principle [10, 5.43] we can assume without loss of generality that $\Delta(1+)$ is equivalent to $\Delta$. We take semisimple characters $\theta \in C(\Delta)\sigma$ and $\tilde{\theta} \in C(\Delta(1-))\sigma$.

Then there is an element $a \in \mathfrak{a}_{r-1}$ such that $\theta = \psi_a \tilde{\theta}$. Thus

$$\tilde{\theta} \psi_a = \theta = \sigma, \tilde{\theta} = \sigma \tilde{\theta} \psi_{-\sigma_a(a)} = \tilde{\theta} \psi_{-\sigma_a(a)}.$$

Thus $\psi_a$ is equal to $\psi_{a_\pm}$ for $a_\pm := \frac{a_\gamma}{2}$. We are going to prove that $\Delta'' := [\Lambda, n, r, \gamma + a_\pm]$ is equivalent to a semisimple stratum $\Delta'$, because then we can take $\Delta'$ to be self-dual by 4.14 and we get:

$$C(\Delta') = C(\tilde{\theta}) = C(\Delta).$$

We prove that the derived stratum $\partial_{\gamma}(\Delta'')$ is equivalent to a semisimple multi-stratum, see [10, 4.14, 4.15] for the definition and the usage of the derived multi-stratum. The stratum $\partial_{\gamma}(\Delta)$ is a semisimple multi-stratum, see Theorem [10, 4.15], because $\Delta$ is semisimple. Let $s_\gamma$ be the chosen corestriction for $\gamma$ (for forming the derived stratum). Then $s_\gamma(a_\pm + \gamma - \beta)$ is modulo $\mathfrak{a}_{r}$ congruent to an element of $F[\gamma]$ because it is intertwined by every element of $C_A(\gamma)$ because

$$C(\Delta(1-)) = C(\tilde{\theta}) \psi_{a_\pm + \gamma - \beta}.$$

Thus $\partial_{\gamma}(\Delta'')$ is equivalent to a semisimple multi-stratum. Thus $\Delta''$ is equivalent to a semisimple stratum by [10, 4.15].

7 Self-dual semisimple pss-characters

In this section we answer the question, when two transfers of endo-equivalent self-dual semisimple characters intertwine over an element of the corresponding classical group. We will introduce endo-parameters, as in [10].
7.1 Self-dual pss-characters

To introduce self-dual pss-characters we will take a slightly different path than in \cite{6} because they should not depend on \( \epsilon \). We consider all objects of the category \( \mathcal{H} \) of orthogonal and symplectic hermitian forms over \( (D, \rho) \) and \( (F, \text{id}) \). We start with what should be the domain of a self-dual semisimple character: Let \( (\Delta, h) \) be a pair consisting of \( h \in \mathcal{H} \) and a self-dual semisimple stratum with respect to \( h \), we also say \( h \)-self-dual. We define the self-dual endo-class of \( \Delta \) as the following class:

\[
\mathcal{E}_-.(\Delta) := \{ (\Delta', h') \in \mathcal{E}(\Delta) \times \mathcal{H} | \Delta' \text{ is } h' \text{-self-dual} \}.
\]

And we define now self-dual pss-character. For the definition of a pss-character (potentially semisimple character) we refer to \cite{10} 6.3], which is motivated by \cite{2}.

**Definition 7.1.** Let \( \mathcal{E}_- \) be a self-dual endo-class. A self-dual pss-character is a map

\[
\Theta_- : \mathcal{E}_- \to \bigcup_{(\Delta, h) \in \mathcal{E}_-} \mathcal{C}_-.(\Delta),
\]

such that \( \Theta_-(\Delta, h) \in \mathcal{C}_-.(\Delta) \) for all \((\Delta, h)\) and such that the values are related by transfer, i.e.

\[
\tau_{\Delta, \Delta'}(\Theta_-(\Delta, h)) = \Theta_-(\Delta, h'),
\]

for all \((\Delta, h), (\Delta, h') \in \mathcal{E}_-\).

We attach to a self-dual pss-character \( \Theta_- \) with domain \( \mathcal{E}_- \) the pss-character \( \Theta \) whose domain \( \mathcal{E} \) contains a stratum \( \Delta \), such that \((\Delta, h) \in \mathcal{E}_- \) for some \( h \in \mathcal{H} \) and such that \( \Theta(\Delta) \) is the lift of \( \Theta_-(\Delta, h) \). We call \( \Theta \) den lift of \( \Theta_- \). Two pss-characters \( \Theta \) and \( \Theta' \) (self-dual pss-characters \( \Theta_- \) and \( \Theta'_- \)) of the same degree and the same group level are called endo-equivalent if there are strata \( \Delta \in \mathcal{E} \) and \( \Delta' \in \mathcal{E}' \) (\( (\Delta, h) \in \mathcal{E}_- \) and \( (\Delta', h') \in \mathcal{E}'_- \)) such that \( \Theta(\Delta) \) and \( \Theta'(\Delta') \) intertwine (\( \Theta_-(\Delta, h) \) and \( \Theta'_-(\Delta', h) \) intertwine by an element of \( U(h) \)).

**Proposition 7.2** \cite{6} 6.18. Two self-dual pss-character of the same degree and the same group level are endo-equivalent if and only if their lifts are endo-equivalent.

The equivalence classes are called endo-classes of pss-characters (self-dual pss-characters). A pss-character is called ps-character (potentially simple) character if its values are simple characters. Similar we define self-dual ps-characters. A self-dual pss-character is called elementary if \( I_0 \) is a \( \sigma \)-orbit, i.e. either it is simple or \( I_0 = I_{0, \pm} \) consisting only of one element.

7.2 Idempotents and Witt groups

We recall the equivalence of categories of hermitian forms. Let \((E, \sigma_E)\) be a field extension of \( F \) together with an \( F \)-linear non-trivial involution on \( E \)

**Proposition 7.3.** Let \( \mathcal{H}_{\sigma_E \otimes \rho, \epsilon} \) be the category of \( \epsilon \)-hermitian \( E \otimes_F D \)-forms with respect to \( \sigma_E \otimes \rho \), and let \( \mathcal{H}_{\sigma_E, \epsilon} \) the category of \( \epsilon \)-hermitian \( E \)-forms with respect to \( \sigma_E \). Then \( \mathcal{H}_{\sigma_E \otimes \rho, \epsilon} \) is equivalent to \( \mathcal{H}_{\sigma_E, \epsilon} \).

**Proof.** We define a functor \( \mathcal{F} : \mathcal{H}_{\sigma_E \otimes \rho, \epsilon} \to \mathcal{H}_{\sigma_E, \epsilon} \) in the following way. We consider an \( E \)-algebra isomorphism \( \Phi : E \otimes_F D \cong M_2(E) \). The anti-involution \( \sigma_E \otimes \rho \) is pushed forward to a unitary anti-involution which can be interpreted as the adjoint anti-involution of a unitary form bilinear form. Such a form has a diagonal \( \sigma_E \)-symmetric Gram-matrix \( u = \text{diag}(u_1, u_2) \) because the characteristic of \( F \) is odd. So we can choose \( \Phi \) such that the push forward of \( \sigma_E \otimes \rho \) is \( u \sigma_E(\cdot)^T u^{-1} \). We identify \((E \otimes_F D, \sigma_E \otimes_F \rho)\) with \((M_2(E), u \sigma_E(\cdot)^T u^{-1})\). We consider

\[
1^1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad 1^2 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\]
We define 
\[ F(V, \tilde{h}) := (V^{11}, \text{tr}_E \circ \tilde{h}|_{V^{11} \times V^{11}}) \].

Exercise: Show that \( F(V, \tilde{h}) \) is non-degenerate.

The inverse functor \( G \) is given by
\[ G(W, h_E) := (W \oplus W, \tilde{h}), \quad \tilde{h}(w_1 + w_2, w'_1 + w'_2) = \left( \begin{array}{cc} h_E(w_1, w'_1) & h_E(w_1, w'_2) \\ w_2u_1^{-1}h_E(w_2, w'_1) & w_2u_1^{-1}h_E(w_2, w'_2) \end{array} \right). \]

The \( E \otimes F \)-action on \( W \oplus W \) is given by
\[ (w_1, w_2)E_{ij} := (\delta_{ij}w_1, \delta_{j2}w_i), \]
for \( 1 \leq i, j \leq 2 \), and by the \( E \) action on \( W \).

This Proposition shows that the objects \( H_{\sigma_E \otimes \rho, \epsilon} \) inherit the decomposition properties from the objects of \( H_{\sigma_E, \epsilon} \). In particular we have a Witt group of \( E \otimes F \) with respect to \( \sigma_E \otimes \rho \) and \( \epsilon \), we denote this Witt group by \( W_{\epsilon}(\sigma_E \otimes \rho) \).

**Proposition 7.4.** The Witt-group \( W_{\epsilon}(\sigma_E \otimes \rho) \) is independent of the choice of the isomorphism from \( E \otimes F \) to \( M_2(E) \).

**Proof.** Consider two \( E \)-algebra isomorphisms \( \Phi : (E \otimes F, \sigma_E \otimes \rho) \longrightarrow (M_2(E), u_i\sigma_E(\cdot)^T u_i^{-1}) \), such that \( u_i \) is diagonal and symmetric with respect to \( \sigma_E \). Let the upper functor be denoted by \( F \). Since \( F_2 \circ F_1^{-1} \) is an equivalence, in particular it sends isometric spaces to isometric spaces and respects orthogonal sums, we only need to prove that it sends hyperbolic spaces of dimension 2 to hyperbolic spaces of dimension 2. We take \( (V, \tilde{h}) \in H_{\sigma_E \otimes \rho, \epsilon} \) such that \( F_1(\tilde{h}) \) is a hyperbolic space of dimension 2. Then \( F_2(\tilde{h}) \) has a non-zero isotropic vector and thus \( \tilde{h} \) has a non-zero isotropic vector \( v \in V \). This vector generates a simple \( E \otimes F \)-module \( <v> \) on which \( \tilde{h} \) vanishes. Let \( 1^1_2 \) and \( 1^2_2 \) be the idempotents for \( u_2 \). There is a simple \( E \otimes F \)-module which does not vanish under \( 1^1_2 \); see for example \( E \times E \), so \( <v> \) does not vanish under \( 1^1_2 \), because all simple \( E \otimes F \)-modules are isomorphic to each other. Thus \( F_2(\tilde{h}) \) has a non-zero isotropic vector, which finishes the proof.

The functor \( F \) in the Proof of Proposition 7.4 is completely characterized by the idempotent \( e \) of \( E \otimes F \) which is mapped to \( 1^1 \) under \( \Phi \), more precisely \( F(\tilde{h}) = \text{tr}_E \circ \tilde{h}|_{V^{11}} \). We denote the functor associated to \( e \) by \( F_e \).

Let \( \text{Idemp}(\sigma_E \otimes \rho) \) be the set of rank 1 idempotents of \( E \otimes F \) which are fixed by \( \sigma_E \otimes \rho \). Two elements of \( \text{Idemp}(\sigma_E \otimes \rho) \) are conjugate by a similitude \( g \) of \( \sigma_E \otimes \rho \), i.e. \( g \) satisfies \( g(\sigma_E \otimes \rho)(g) = s \in E^* \).

**Proposition 7.5.** Let \( e, f \in \text{Idemp}(\sigma_E \otimes \rho) \) and let \( g \) be an element of \( (E \otimes F)^* \) such that \( \det g^{-1} = f \) and \( g(\sigma_E \otimes F \text{id}_D)(g) = s \in E^* \). Then \( F_f \circ F_e^{-1} \) is equivalent to the functor \( h_E \mapsto s^{-1}h_E \).

**Proof.** We take forms \( \tilde{h} \in H_{\sigma_E \otimes \rho, \epsilon} \) and \( h_E \in H_{\sigma_E, \epsilon} \) such that \( F_e(\tilde{h}) = h_E \). For \( v_1, v_2 \in Vf \) we have:
\[
\text{tr}_E(\tilde{h}(v_1, v_2)) = \text{tr}_E(f\tilde{h}(v_1, v_2)f) = \text{tr}_E(geg^{-1}\tilde{h}(v_1, v_2)geg^{-1}) = \text{tr}_E(g\tilde{h}(v_1gs^{-1}, v_2g)eg^{-1}) = s^{-1}h_E(v_1g, v_2g).
\]

Thus \( F_f(\tilde{h}) \) is isometric to \( s^{-1}h_E \) via \( g : Vf \rightarrow Vc \). Thus the maps \( g : F_f(\tilde{h}) \rightarrow F_c(\tilde{h}) \) form a natural transformation from \( F_f \) to \( s^{-1}F_c \) which is an equivalence. Thus \( F_f \circ F_c^{-1} \) is equivalent to \( s^{-1}F_c \circ F_c^{-1} \) and the latter is \( h_E \mapsto s^{-1}h_E \) by the definition of \( F_c^{-1} \) in the proof of Proposition 7.3.

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We are now able to describe the elements of $W_*(\sigma_E \otimes \rho)$ independent of the choice of an idempotent. We call a field extension $E = F[\beta]/F$ self-dual if there is an automorphism $\sigma_E$ of $E/F$ which maps $\beta$ to $-\beta$.

**Definition 7.6.** Suppose $E = F[\beta]$ is a self-dual field extension different from $F$. A Witt tower for $\sigma_E, \rho, \epsilon$ is a function

$$\text{wtower}_*: \text{Idemp}(\sigma_E \otimes \rho) \to W_*(\sigma_E)$$

which satisfies $\text{wtower}(f) = s^{-1}\text{wtower}(e)$ for all $e, f \in \text{Idemp}(\sigma_E \otimes \rho)$ using the similitude of Proposition 7.3. If there is no confusion with $\rho$ and $\epsilon$ then we also say Witt tower for $\sigma_E$ or for $\beta$.

We have a canonical bijection from $W_*(\sigma_E \otimes \rho)$ to the set of Witt towers for $\sigma_E, \rho, \epsilon$: An element $\beta_\sigma$ is mapped to $\text{wtower}_{\beta_\sigma}$ defined via $\text{wtower}_{\beta_\sigma}(\epsilon) = \mathcal{F}_\epsilon(\beta) \in W_*(\sigma_E)$. The map is well-defined by Proposition 7.4. We identify $W_*(\sigma_E \otimes \rho)$ with the set of Witt towers for $\sigma_E, \rho, \epsilon$. Given a field extension $(E, \sigma_E)[F]$ with non-trivial $\sigma_E$ and an $F$-linear $\sigma_E$-id$_F$-equivariant non-zero map $\lambda : E \to F$ there is a natural map

$$\text{Tr}_\lambda : W_*(\sigma_E \otimes \rho) \to W_*(\rho),$$

defined via $\text{Tr}_\lambda(\beta) := ((\lambda \otimes_F \text{id}_{D}) \circ \beta)_\sigma$.

### 7.3 Matching Witt towers

Given an $\epsilon$-hermitian form $h$ on $V$ with respect to $(D, \rho)$ and an element $\beta$ in the Lie algebra of $G$ such that $F[\beta]$ is a field, there are unique forms $h_\beta \in \mathcal{H}_{\sigma_E, \epsilon}$ and $\beta_\sigma \in \mathcal{H}_{\sigma_E \otimes \rho, \epsilon}$ such that $\lambda_\sigma \circ h_\sigma = \text{trd}_{DJF} \circ h$ and $(\lambda_\sigma \otimes_F \text{id}_D) \circ \beta_\sigma = h$, where $\lambda_\beta : E \to F$ is the $F$-linear extension of $\text{id}_F$ with kernel $\Sigma_{i=1}^{[E:F]-1} \beta^i F$. We have

$$h_\beta = \text{tr}_E \circ \beta_\sigma,$$

because:

**Lemma 7.7.** $\lambda_\sigma \circ \text{tr}_E = \text{trd}_{DJF} \circ (\lambda_\sigma \otimes_F \text{id}_D)$.

**Proof.** It is a direct consequence of $\text{trd}_{DJF}(x) = \text{tr}_E(x)$ for all $x \in D$. \qed

We call the function $\text{wtower}_{\beta, \sigma}$ the Witt tower of $(\beta, h)$ and we also denote it by $\text{wtower}_{\beta, \sigma}$.

We want to be able to compare Witt towers for different $\beta$ and $\beta'$. Let us recall that $\beta$ is called minimal over $F$, if $\nu_{[\beta]}(\beta)$ is co-prime to $e(F[\beta]/F)$ and $\beta^{e(F[\beta]/F) + p_F[\beta]}$ generates $\kappa_{F[\beta]}$ over $\kappa_F$. Define $\epsilon_{\beta}(\beta)$ as the fraction $\frac{e_{\text{wild}}(E[F])}{e_{\text{wild}}(E[F]) + p_F[\beta]}$, where $e_{\text{wild}}(E[F])$ is the wild ramification index. Among those field extensions $E[F]$ in $E = F[\beta]$ which are generated by an element congruent to $\beta_{\epsilon}(\beta)$ there is a smallest one. We write $F[\beta_{\min, tr}]$ where $\beta_{\min, tr}$ is a minimal element over $F$ and congruent to $\beta_{\epsilon}$. In saying $\beta_{\min, tr}$ we mean a choice. This choice is not canonical and can be changed to our purpose if necessary, for example if $\beta$ is skew-symmetric, we choose $\beta_{\min, tr}$ skew-symmetric (w.r.t. $\sigma_E$).

So let us assume that $E = F[\beta]$ and $E' = F[\beta']$ are self-dual field extensions with non-zero $\beta$ and $\beta'$. We are going to define a map from $W_*(\sigma_E \otimes \rho)$ to $W_*(\sigma_E' \otimes \rho)$. This is a generalization of [5]. Let us recall them:

$$w_{\beta, \beta'} : W_{-1}(\sigma_E) \to W_{-1}(\sigma_{E'}), \quad w_{\beta', \beta} : W_1(\sigma_E) \to W_1(\sigma_{E'})$$

are bijections which respect the anisotropic dimension and satisfy

$$w_{\beta, \beta'}((\beta)z) = (\beta')z, \quad w_{\beta', \beta}((\beta')z) = (\beta^2)z.$$

Let us write $w_{\beta, \beta', 1}$ for $w_{\beta', \beta}^2$ and $w_{\beta, \beta', -1}$ for $w_{\beta, \beta'}$. We need two further assumptions:

(A) Suppose there is an isomorphism from $F[\beta_{\min, tr}]\otimes F[\beta'_{\min, tr}]$ which sends $\beta_{\min, tr}$ to an element congruent to $\beta'_{\min, tr}$, we write “$\beta_{\min, tr} \mapsto \beta'_{\min, tr}$” for this unique isomorphism.
We define the bijection $w_{\beta,\beta',\epsilon} : W_\epsilon (\sigma_E \otimes \rho) \to W_\epsilon (\sigma_E' \otimes \rho)$ as follows: Let $e \in \text{Idemp}(\sigma_E [\beta_{\text{min,tr}}] \otimes_F \rho)$ and $e' \in \text{Idemp}(\sigma_E' [\beta'_{\text{min,tr}}] \otimes_F \rho)$ be idempotents such that $e \leftrightarrow e'$ under

$$F[\beta_{\text{min,tr}}] \otimes_F D \overset{\text{''} \beta_{\text{min,tr}} \leftrightarrow \text{''} \beta'_{\text{min,tr}}}{\longrightarrow} F[\beta'_{\text{min,tr}}] \otimes_F D.$$ 

We define $w_{\beta,\beta',\epsilon}(\text{wtower})$ to be the Witt tower of $\sigma_E' \otimes \rho$ which satisfies

$$w_{\beta,\beta',\epsilon}(\text{wtower})(e') := w_{\beta,\beta',\epsilon}(\text{wtower}(e)).$$

This map is well-defined, i.e. does not depend on the choice of the idempotent, by Proposition 7.5 and (B).

**Lemma 7.8.** Let $= F[\beta][F]$ be a field extension with a non-trivial involution $\sigma_E$ given by $\beta \mapsto -\beta$. Then $F[\beta_{\text{min,tr}}]$ is a subset of $N_{E/E_0}(E)$ if and only if $E[\beta_{\text{min,tr}}]$ has even ramification index and even inertia degree. If we do not have the above containment, then $x \in F[\beta_{\text{min,tr}}]$ is a norm of $F[\beta_{\text{min,tr}}][F][\beta_{\text{min,tr}}]$ if and only if it is a norm of $E|E_0$.

**Proof.** We just write $E'$ for $E[\beta_{\text{min,tr}}]$. At first we remark that a norm of $E'|E_0'$ is a norm of $E|E_0$ because both have degree two and $\sigma_E$ and $\sigma_E'|E$ are the Galois generators. Here we use that the residue characteristic is odd. Suppose now that $E|E_0'$ has even ramification index and even inertia degree. Then every element of $\sigma_E'$ is a square in $E$ and therefore a norm of $E|E_0$. If $E'|E_0'$ is ramified then $E_0'$ contains a uniformizer which is a norm of $E'|E_0'$, and we have the desired containment. In the case of an unramified $E'|E_0'$, since $\epsilon(E|E_0)$ is even there is a uniformizer of the maximal unramified extension in $E_0|E_0'$ which is a square in $E$, and since $-1$ is a norm of $E|E_0$ we obtain that the mentioned uniformizer is a norm of $E|E_0$. All elements of $\sigma_E'$ are norms of $E|E_0$ because $E|E_0$ is also unramified. Thus there is a uniformizer of $E_0'$ which is a norm of $E|E_0$. Suppose for the converse that all elements of $E_0'$ are norms of $E|E_0$. Then all elements of $\kappa_{E_0'}$ are squares in $\kappa_E$, i.e. $f(E|E_0')$ is even, and there is a uniformizer of $E_0'$ which is a norm of $\kappa_{E_0'}$, in particular the $v_E$-valuation of this uniformizer must be even and therefore $\epsilon(E|E_0')$ is even. In the case where $E_0'$ is not contained in $N_{E|E_0}(E)$, say $x \in E_0'$ is not a norm of $E|E_0$, the set of all non-norms of $E'|E_0'$ which is $x N_{E'|E_0'}(E')$ is disjoint to $N_{E|E_0}(E)$ which finishes the proof.

**Definition 7.9.** Let $h$ and $h'$ be two $\epsilon$-hermitian forms and suppose that $\beta \in \text{Lie}(U(h))$ and $\beta' \in \text{Lie}(U(h'))$ generate field extensions $E$ and $E'$ different from $F$, such that (A) and (B) hold. We say that the Witt towers of $(\beta, h)$ and $(\beta', h')$ match if $h$ is isometric to $h'$ and $w_{\beta,\beta',\epsilon}(h(\beta))_x = (h')_x$. Note that matching Witt towers is an equivalence relation on the pairs $(\beta, h)$ and not on Witt towers.

**Remark 7.10.** If $g : h \cong h'$ is an isometry then the Witt towers of $(\beta, h)$ and $(g \beta g^{-1}, h')$ match.

**Proof.** We consider $\tilde{h} := \tilde{h}_{\beta}$ and $\tilde{h}' := c_g \circ \tilde{h} \circ (g^{-1} \times g^{-1})$. We have to show $\tilde{h}' = \tilde{h}_{\beta'}$. We write $\phi(x)$ for $gxy^{-1}$, $x \in E$. For $v_1, v_2 \in V$ we have:

$$\tilde{h}'(v_1, v_2 \phi(x)) = g \circ \tilde{h}(g^{-1}(v_1), g^{-1}(v_2 \phi(x))) \circ g^{-1}$$

$$= g \circ \tilde{h}(g^{-1}(v_1), (g^{-1} \circ \phi(x))(v_2)) \circ g^{-1}$$

$$= g \circ \tilde{h}(g^{-1}(v_1), (x \circ g^{-1})(v_2)) \circ g^{-1}$$

$$= g \circ \tilde{h}(g^{-1}(v_1), g^{-1}(v_2) x) \circ g^{-1}$$

$$= g \circ \tilde{h}(g^{-1}(v_1), g^{-1}(v_2)) \circ x \circ g^{-1}$$

$$= \tilde{h}'(v_1, v_2) \phi(x)$$

Proceeding further this way we see that $\tilde{h}'$ is an $\epsilon$-hermitian form which satisfies $(\lambda_{\beta'} \otimes \text{id}_D) \circ \tilde{h}' = h'$ which finishes the proof.
Theorem 7.11. Let $\Delta$ and $\Delta'$ be two non-null self-dual simple strata on $(V,h)$ and $\theta_+ \in C_-(\Delta)$ and $\theta'_+ \in C_-(\Delta')$ be two endo-equivalent self-dual simple characters. Then $\theta_-$ and $\theta'_-$ intertwine by an element of $G$ if and only if the Witt towers of $(\beta,h)$ and $(\beta',h)$ match.

We need for the proof the twist $h'^\gamma$ of a signed hermitian form $h$ by a skew-symmetric or symmetric element $\gamma \in \hat{G}$, see subsection 2.3. If $\gamma$ is skew-symmetric and invertible then $h$ is symplectic if and only if $h'^\gamma$ is orthogonal.

Proof. We choose lifts $\theta$ and $\theta'$ for $\theta_-$ and $\theta'_-$. The field extensions $F[\beta_{min,tr},F]$ and $F[\beta'_{min,tr},F]$ are isomorphic by a $\sigma_n$-equivariant map which sends $\beta_{min,tr}$ to an element congruent to $\beta_{min,tr}'$ so we can assume without loss of generality that $\beta_{min,tr}'$ is the image of $\beta_{min,tr}$ under this isomorphism. Suppose at first $I_G(\theta,\theta') \neq \emptyset$. Then $\beta_{min,tr}$ and $\beta'_{min,tr}$ are conjugate by an element of $G$. Thus we can assume $\beta_{min,tr} = \beta_{min,tr}'$ without loss of generality. (Note that conjugation with an element of $G$ does not change the equivalence class of $(\beta,h)$ by Remark 7.10.) Now take any idempotent $e$ of $F[\beta_{min,tr}] \otimes_F D$ and choose $\tilde{\Delta},\text{trd}_D \circ h|_{V_E} \in \mathfrak{e}_-(\Delta,h)$ and $(\tilde{\Delta}',\text{trd}_D \circ h|_{V_E}) \in \mathfrak{e}_-(\Delta',h)$ such that $\tilde{\beta} = e\beta$ and $\tilde{\beta}' = e\beta'$. The transfers of $\theta$ and $\theta'$ to $\tilde{\Delta}$ and $\tilde{\Delta}'$, respectively, intertwine by an element of Aut$_F(V_E)$ by Theorem 6.1. Proposition 6.3 implies for the orthogonal case that the Witt towers of $(h,\beta_{min,tr})$ and $(\beta',\beta'_{min,tr})$ match and thus the Witt towers of $(\beta,h)$ and $(\beta',h)$ match. In the symplectic case we consider the twist $h'_{min,tr}$. The latter is orthogonal and the argument of part one shows that the Witt towers of $(h,\beta_{min,tr})$ and $(\beta',\beta'_{min,tr})$ match. Thus wtower$_{\beta,h_{min,tr}}(e)$ and wtower$_{\beta',h_{min,tr}}(e)$ are mapped to each other under $W_1(\sigma_E) \cong W_1(\sigma_{E'})$ via $(\beta')_z \mapsto (\beta'_z)$.

The twist with $\beta_{min,tr}$ induces the map

$$W_1(\sigma_E) \rightarrow W_1(\sigma_{E'}), \quad \langle \beta \rangle_z \mapsto \langle \beta_{\sigma_E}^\epsilon(\beta') \rangle_z = \left((-1)^{\frac{d_e(d_{\beta}^\epsilon)}{2}} \beta^2 \right)_z,$$

and we have the analogue formula for $\beta'$. Now $-1$ is a norm of $E|E_0$ if and only if it is a norm of $E'|E'_0$ because both $E|F$ and $E'|F$ have the same inertia degree. Thus the pull back of the map $W_1(\sigma_E) \cong W_1(\sigma_{E'})$ under the twist is the map $W_1(\sigma_E) \cong W_1(\sigma_{E'})$, $(\beta)_z \mapsto (\beta')_z$. We obtain that the Witt towers of $(\beta,h)$ and $(\beta',h)$ match.

We now consider the converse. So suppose that the Witt towers of $(\beta,h)$ and $(\beta',h)$ match. Without loss of generality we can assume that $\beta$ and $\beta'$ have the same minimal polynomial by Theorem 6.10 and that the characters are transfers of each other. Note that this did not leave the equivalence class of $(\beta,h)$ by the first direction of this proof. We have $e \in \text{Idemp}(\sigma_E \otimes \rho)$ and $e' \in \text{Idemp}(\sigma_{E'} \otimes \rho)$ which are mapped under “$\beta_{min,tr} \mapsto \beta'_{min,tr}$” to each other such that $F_E(\tilde{\beta})_z$ is mapped to $F_E(\tilde{\beta}')_z$ under $W_e(\sigma_E) \cong W_e(\sigma_{E'})$. We consider the pullback $\phi^*\tilde{h}_\beta$ of $\tilde{h}_\beta$ along the field-isomorphism $\phi : E \rightarrow E'$ with $\phi(\beta) = \beta'$. Then the matching Witt tower condition says that $F_E(\tilde{h}_\beta)$ and $F_E(\phi^*\tilde{h}_{\beta'})$ are isometric. Thus $\tilde{h}_\beta$ and $\phi^*\tilde{h}_{\beta'}$ are isometric and thus there is an element $g$ of $G$ which conjugates $\beta$ to $\beta'$. And this intertwines $\theta$ with $\theta'$ because the two characters are transfers of each other.

The last proof shows a nicer version for an equivalent criteria for $G$-intertwining.

Corollary 7.12. Let $\Delta$ and $\Delta'$ be two self-dual non-null simple strata for $(V,h)$ and $\theta \in C(\Delta)^o$ and $\theta' \in C(\Delta')^o$ be two endo-equivalent simple characters. Then the following conditions are equivalent:

(i) $I_G(\theta,\theta') \neq \emptyset$

(ii) $\Delta((n-r-1)+)$ and $\Delta'((n'-r'-1)+)$ intertwine under some element of $G$.

(iii) $(\beta_{min,tr},h)$ and $(\beta'_{min,tr},h)$ have matching Witt towers.

(iv) $(\beta,h)$ and $(\beta',h)$ have matching Witt towers.

Proof. The first and the last assertion are equivalent by Theorem 7.11 and [6, 7.1], [iv] implies [ii] and further the assertion [ii] implies [iii] by the implication [ii] then [iv] for strata (equivalently: characters which are transfers). So let us assume [iii]. We can assume without loss of generality that $\beta_{min,tr}$ is the image of $\beta_{min,tr}$ under $F[\beta_{min,tr}] \otimes_F F \cong F[\beta_{min,tr}] \otimes F$. Then matching Witt towers implies that $\beta_{min,tr}$
and \( \beta'_{\text{min,tr}} \) are conjugate by an element of \( G \), so we can assume without loss of generality that \( \beta_{\text{min,tr}} \) and \( \beta'_{\text{min,tr}} \) coincide. Now we proceed as in the proof of Theorem 7.11 to conclude from the endo-equivalence of \( \theta \) with \( \theta' \) that the Witt towers of \((\beta, h)\) and \((\beta', h)\) match. \( \square \)

We now generalize the formalism of [6] which leads to endo-parameters for quaternionic inner forms of classical groups.

For that we need to include the case \( \beta = 0 \) in the definition of matching Witt towers. We call \( h_\pm \in W_\epsilon(\text{id}_F) \) the Witt tower of \((0, h)\). We say that the Witt towers of \((0, h)\) and \((0, h')\) match if \( h \) is isometric to \( h' \).

Let us recall we can identify the index sets of strata of the domain of a pss-character to one set \( I_\Theta \) and endo-equivalent pss-characters \( \Theta \) and \( \Theta' \) determine a matching \( \zeta : I_\Theta \to I_{\Theta'} \), see Theorem 10.6.18.

Given lifts \( \Theta \) and \( \Theta' \) of self-dual pss-characters the index set decomposes into \( I_\Theta = I_{\Theta,0} \cup I_{\Theta,+} \), and we sometimes choose a disjoint union \( I_{\Theta,\pm} = I_{\Theta,0} \cup I_{\Theta,+} \) such that \( \sigma(I_{\Theta,\pm}) = I_{\Theta,\mp} \). Let us recall that given a bijection \( \zeta : I_\Theta \to I_{\Theta'} \), a \( \zeta \)-comparison pair is a pair \( (\Delta, \Delta') \in E \times E' \) such that both strata are defined over the same skew-field \( D \) and \( \dim_D V^i = \dim_D V'^i \) for all \( i \in I_\Theta \), see 10.6.17.

**Theorem 7.13.** Let \( \Theta_\pm \) on \( E_- \) and \( \Theta'_\pm \) on \( E'_\pm \) be two endo-equivalent self-dual pss-characters with lifts \( \Theta \) and \( \Theta' \) and matching \( \zeta = \zeta_{\Theta,\Theta'} \). Then for given pairs \((\Delta, h) \in E_- \) and \((\Delta', h') \in E_- \) are equivalent:

(i) \( \Theta_-(\Delta, h) \) and \( \Theta_-(\Delta', h) \) intertwine by an element of \( U(h) \).

(ii) \( (\Delta, \Delta') \) is a \( \zeta \)-comparison pair, and the Witt towers of \((\beta_i, h_i)\) and \((\beta_i', h_i')\) match, for all \( i \in I_{\Theta,0} \).

**Proof.** The direction \( (i) \Rightarrow (ii) \) follows from the Corollaries 6.1.13 and 7.1.2. Backwards follows block-wise from Theorem 10.6.18 and Corollary 7.1.2 \( \square \)

Finally we can introduce endo-parameters for quaternionic inner forms of classical groups.

## 7.4 Endo-parameters

At first we generalize the notion of Witt type from [3] Section 13.2. We fix \( \rho \) and \( \epsilon \). We consider pairs \((\beta, t)\) where \( F[\beta] \) is a self-dual field extension and \( t \in W_\epsilon(\sigma_\rho \otimes \rho) \). We call \((\beta, t)\) equivalent to \((\beta', t')\) if at least one of the following points hold:

- \( t \) and \( t' \) are hyperbolic.
- \( \beta \) and \( \beta' \) are non-zero, \( Tr_{\lambda_\beta}(t) = Tr_{\lambda_{\beta'}}(t') \) and the Witt towers match, i.e. \( w_{\beta,\beta'}(t) = t' \).
- \( \beta = \beta' = 0, t = t' \) and \( t \) is not hyperbolic.

The equivalence classes of these pairs are called \((\rho, \epsilon)\)-Witt types and the factor set is denoted by \( W_{\rho, \epsilon} \).

The Witt type of the pairs \((\beta, t)\) with \( t \) hyperbolic is denoted by \( 0 \).

The second data needed for endo-parameters are certain endo-classes: A semisimple character \( \theta \in C(\Delta) \) is called full if \( r = 0 \). A pss-character is called full if its domain contains a stratum with \( r = 0 \), these are the pss-characters of group level zero or infinity. Similar for self-dual pss-characters. We denote by \( E \) the set of all full endo-classes of pss-characters and by \( E_- \) the set of all elementary full endo-classes. We denote for a full semisimple character \( \theta \in C(\Delta) \) and a full endo-class \( c \in E \) by \( \theta_c \in \Delta_c \) the restriction of theta to the summand corresponding to \( c \), if all simple endo-classes in \( c \) occur in \( \theta \). Similar for self-dual semisimple characters. The degree of \( c \in E_- \) we define to be the degree of a simple block restriction \( c_1 \) of \( c \) where the degree of \( c \in E_- \) is the degree of any stratum in the domain an element of \( c \). We write \( \deg(c) \) and \( \deg(c) \).

We say that a Witt type \([\beta, t] \) is a Witt type for \( c_- \in E_- \) if either \( c_- \) is simple and there is a simple stratum \( \Delta = [\Lambda, n, 0, \beta] \) which occurs as a first coordinate in the domain of some element of \( c_- \) or if \( c_- \) is not simple and \([\beta, t] = 0 \).
Definition 7.14. A \((\rho, \epsilon)\)-endo-parameter is a map \(f_- = (f_1, f_2) : \mathcal{E}_- \rightarrow \mathbb{N} \times \mathcal{W}_{\rho, \epsilon}\) with finite support, such that \(f_2(c_-)\) is a Witt type for \(c_-\) and \(f_1(c_-)\) is divisible by \(\frac{\text{deg}(D)}{\gcd(\text{deg}(c_-), \text{deg}(D))}\) for every \(c_- \in \mathcal{E}_-\). (This divisibility condition is empty for the non-null simple elementary \(c_-\)). Recall that a GL-endo-parameter is just a map \(f : \mathcal{E} \rightarrow \mathbb{N}_0\) of finite support. See Definition [10] 7.1.

We can attach to a \((\rho, \epsilon)\)-endo-parameter \(f_-\) a GL-endo-parameter \(f\), also called its lift, where we define for a simple block restriction \(c_-\):

\[
f(c) = \begin{cases} f_1(c_-) - 2f_1(c_-) + \text{diman}(f_2(c_-)) & \text{if } c_- \text{ is not simple} \\ \frac{\text{deg}(D)}{\gcd(\text{deg}(c_-), \text{deg}(D))} & \text{else}
\end{cases}
\]

where \(\text{diman}(f_2(c_-))\) is the anisotropic \(F[\beta]\)-dimension of \(t(e)\) (for any idempotent \(e\)) in the Witt type \(f_2(c_-) = \{(\beta, t)\}\). We define

\[
\text{deg}(f_-) := \text{deg}(f) := \sum_{c \in \mathcal{E}} f(c) \text{deg}(c).
\]

We attach to any Witt type \([(\beta, t)]\) the Witt tower \(WT_D([(\beta, t)]) := \text{Tr}_{\lambda_D}(t)\).

Theorem 7.15 (see [3] 13.11, for the \(F\)-case). The set of intertwining classes of full semisimple characters for \(G = \text{U}(h)\) is in canonical bijection to the set of endo-parameters \(f_-\) which satisfy:

(i) \(\text{deg}(f_-) = \text{deg}(\text{End}_D(V))\).

(ii) \(\sum_{c \in \mathcal{E}} WT_D(f_2(c_-)) = h_\beta\).

The map is constructed as follows: Given an intertwining class of \(\theta_- \in C_-(\Delta)\) we define

\[
f_1(c_-) = \begin{cases} \text{Witt index of } \tilde{h}_{\beta_-} & \text{for simple } c_- \in \mathcal{E}_- \\ \text{deg}(\text{End}_{E_c \otimes_D V^c}) & \text{if } c \text{ is a block restriction of } c_- \in \mathcal{E}_- \text{ non-simple}. \end{cases}
\]

and

\[
f_2(c_-) = \begin{cases} \text{Witt type of } (\beta_-, h|_{V_-}) & \text{for simple } c_- \in \mathcal{E}_- \\ 0 & \text{for non-simple } c_- \in \mathcal{E}_-. \end{cases}
\]

One can see the non-simple \(c_-\) as GL-parts of the endo-parameter.

Proof. The map is well defined by Theorem 7.13. We show at first the injectivity of the map. We consider two full self-dual semisimple characters \(\theta_- \in C_-(\Delta)\) and \(\theta'_- \in C_-(\Delta')\) for \(U(h)\) with the same \((\rho, \epsilon)\)-endo-parameter \(f_-\). Then their lifts \(\theta\) and \(\theta'\) intertwine by Theorem [10] 7.2 because they have the same GL-endo-parameter. Now Theorem 7.14 implies that \(\theta_-\) and \(\theta'_-\) intertwine. Conversely, we have to show that any endo-parameter \(f_-\) of the form given in the theorem is attained by a self-dual semisimple character for \(U(h)\). We only consider \(c_-\) in the support of \(f_-\).

- For a simple \(c_-\) we take a full self-dual semisimple character \(\theta_{c_-} \in C(\Delta_-)\) whose self-dual ps-character is an element of \(c_-\) such that \(\tilde{h}_{\beta_-}\) has Witt index \(f_1(c_-) \frac{\text{deg}(D)}{\gcd(\text{deg}(c_-), \text{deg}(D))}\) and Witt type \(f_2(c_-)\).

- For a non-simple \(c_-\) we consider a simple block \(c_1\) of \(c_-\). We take a simple character \(\theta_{c_1} \in C(\Delta_{c_1})\) with endo-parameter supported in \(c_1\) which maps \(c_1\) to \(f_1(c_-)\). We construct a hyperbolic \(\epsilon\)-hermitian space \(h_{c_-}\) with Lagrangian \(V^{c_-}\). There is a full semisimple character with block restrictions \(\theta_{c_1}\) and \(\theta'_{c_1}\) by [10] 7.1. We can take the stratum to be self-dual by Proposition 6.13.

Again by [10] 7.1 and 6.13 there is a self-dual semisimple stratum \(\Delta\) for \(\Theta_{c_-} h_{c_-}\) such that \(\theta_- := \phi_{c_-} \theta_{c_-} \in C_-(\Delta)\). Now \(h\) is isometric to \(\Theta_{c_-} h_{c_-}\) and we can take an isometry to conjugate \(\theta_-\) to a character for \(h\). By construction and Remark 7.10 the character \(\theta_-\) has endo-parameter \(f_-\). \(\square\)
A  Intertwining classes of self-dual embeddings

In this section we answer the following question. Let us underline that in this section we use that $D$ is not a field. Say $θ ∈ C_{−}(Δ)$ is a full self-dual semisimple character with lift $θ ∈ C_{−}(Δ)^{o}$. How many $G$-intertwining classes of $σ$-fixed semisimple characters are contained the $G$-intertwining class of $θ$.

Let $f_{c}$ be the endo-parameter of $θ_{c}$ and $f$ its lift. By Theorem 7.15 we only need to find all endo-parameters $f'_{c}$ with lift $f$ and such that $Σ_{c}WT_{D}(f'_{c}(c_{c})) = h_{z}$. These endo-parameters only differ in their values for simple $c_{c} ∈ E_{c}$. We start the simple case. Let us recall: We call an equivariant-$F$-algebra homomorphism $φ : (E, σ_{E}) → (End_{D}(V), σ_{h})$ a self-dual embedding.

**Proposition A.1.** Let $E = F[β]$ is a self-dual field-extension different from $F$ and suppose there is a self-dual embedding into $(End_{D}(V), σ_{h})$. Then there are precisely two $G$-conjugacy classes of self-dual embeddings of $(E, σ_{E})$ into $(End_{D}(V), σ_{h})$.

Proposition 7.3 and 7.5 are true if one replaces $f'_{c}$ with lift $f$ and its lift. By Theorem 7.15 we only need to find all endo-parameters $f'_{c}$ with lift $f$ and such that $Σ_{c}WT_{D}(f'_{c}(c_{c})) = h_{z}$. These endo-parameters only differ in their values for simple $c_{c} ∈ E_{c}$. We start the simple case. Let us recall: We call an equivariant-$F$-algebra homomorphism $φ : (E, σ_{E}) → (End_{D}(V), σ_{h})$ a self-dual embedding.

**Proposition A.1.** Let $E = F[β]$ is a self-dual field-extension different from $F$ and suppose there is a self-dual embedding into $(End_{D}(V), σ_{h})$. Then there are precisely two $G$-conjugacy classes of self-dual embeddings of $(E, σ_{E})$ into $(End_{D}(V), σ_{h})$.

Proposition 7.3 and 7.5 are true if one replaces $σ_{E}$ with $id_{E}$. Note that $id_{E} ⊗ φ_{D}$ is orthogonal and thus the set $Idemp(id_{E} ⊗ φ_{D})$ of $id_{E} ⊗ φ_{D}$-fixed idempotents of rank 1 is non-empty. We can again identify the Witt-group $W_{c}(id_{E} ⊗ φ_{D})$ with the set of Witt towers, defined as in Definition 7.6, i.e. as maps

$$Idemp(id_{E} ⊗ φ_{D}) → W_{c}(id_{E}), \; wtower_{c}(e) := F_{c}(h)_{z}.$$  

For an extension $(E|σ_{E})(E'|σ_{E'})$, both of even degree over $F$ and an $E'$-linear non-zero map $λ$ we get a map

$$Tr_{c} : W_{c}(id_{E} ⊗ φ_{D}) → W_{c}(id_{E'} ⊗ φ_{D}),$$

which in terms of Witt towers is given by

$$wtower_{Tr_{c}(λ)}(e') = Tr_{c}(wtower_{c}(e')) \in Idemp(id_{E'} ⊗ φ_{D}).$$

where $Tr_{c} : W_{c}(σ_{E}) → W_{c}(σ_{E'})$ is given by $h_{z} ⇒ (λ ⊗ h)_{z}$. We are now able to prove Proposition.[A.1]

**Proof.** There are at most two conjugacy classes because the parity of the anisotropic dimension of $h$ is determined by the degree of $E|F$ and $dim_{D} V$ and there are exactly two Witt towers in $W_{c}(σ_{E} ⊗ φ_{D})$ with the same parity for the anisotropic dimension. It is enough to show that the maximal anisotropic Witt tower is mapped to the hyperbolic Witt tower under $Tr_{λ}$, $E'|F$ contains a $σ_{E}$-invariant quadratic extension $(E', σ_{E'}) = (E|σ_{E})$ of $F$. The image of the maximal anisotropic Witt tower under $Tr_{λ}$ does not depend on the choice of $λ : (E, σ_{E}) → F$, non-zero, equivariant and $F$-linear, by [6, 2.2]. So, we could replace $λ$ by $tr_{E'|F} ⊗ λ'$ for some non-zero, equivariant $E'$-linear $λ' : (E, σ_{E}) → (E', σ_{E'})$. The map $Tr_{λ'}$ sends the maximal anisotropic Witt tower to the maximal anisotropic one by [11, 4.4]. So, we have to show that $Tr_{E'|F}$ sends the maximal anisotropic Witt tower to the hyperbolic one. The maximal anisotropic Witt tower in $W_{c}(σ_{E'} ⊗ φ_{D})$ can be written in the form $X = (x)_{x} + h_{x}$ for some $h_{x} ∈ W_{c}(σ_{E'} ⊗ φ_{D})$ and $x ∈ F^{x}$. By [23] the forms $(tr_{E'|F} ⊗ id_{D}) ⊗ h_{x} + x((tr_{E'|F} ⊗ id_{D}) ⊗ h)_{x}$ are isometric and the Witt group of $(D, ρ)$ is an elementary 2-group. So $Tr_{E'|F}(X)$ is the hyperbolic Witt tower.

Two count the number of $(e, ρ)$-endo-parameters $f'_{c}$ with the same lift $f$ we only have to arrange Witt types such that $Σ_{c}WT_{D}(f'_{c}(c_{c})) = h_{z}$. The above proposition shows that for the two choices for $f'_{c}(c_{c})$ we have $WT_{D}(f'_{c}(c_{c})) = WT_{D}(f'_{c}(c_{c})).$ So for every non-null simple $c_{c} ∈ E_{c}$ we have two choices for the Witt type and for the others the Witt type is determined by $f$. So:

**Proposition A.2.** The number of $U(h)$-intertwining classes of $σ$-fixed semisimple characters in the $G$-intertwining class of $θ$ is equal $2^{#h_{0}}$ if there is no null block restriction for $θ$ and $2^{#h_{0} - 1}$ if $θ$ has a null block restriction.

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