Deformed Quantum Mechanics and the Landau Problem

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A deformation of the Landau problem based on a modification of Fock algebra is considered. Systems with the Hamiltonians $\hat{H}$ where $\hat{H}$ is the Landau Hamiltonian in the lowest level are discussed. The case $f(\hat{H}) = \alpha_1 \hat{H} + \alpha_2 \hat{H}^2$ is studied and it is shown that in this particular example, parameters of the problem can be fixed by using the quadratic Zeeman effect data and the Breit-Rabi formula. The proposed approach allows to solve exactly Landau-like families of problems not previously discussed in the literature.

In the last years the possibility of deforming quantum mechanics has been widely discussed from different points of view as a way to find a solution to several fundamental problems \[1,2\].

In this regard, issues such as states evolution, algebras of observables and unitarity of the S-matrix have been re-analyzed and, as a consequence, new approaches and concepts in quantum theory have been introduced \[3\].

It is expected that these new concepts and approaches might shed light on different problems, ranging from the black hole physics (for example the evaporation process \[1,2\]) to solid state physics (such as an explanation to the high temperature superconductivity \[3\]).

Although a final answer to these questions has not been reached yet, the methods developed in this field bring different insights and perspectives and they also provide useful calculation techniques to tackle new and old problems. Some of these calculations methods were developed almost at the same time as quantum mechanics. For example, the non-commutativity of spacetime as an ultraviolet regulator \[3,10\] or the phase space quantum mechanics \[11\].

Others, however, such as deformed commutator structures, have lead to the development of new and important areas of mathematics such as non-commutative geometry \[12\], quantum groups \[13\], and deformed Poissonian geometry \[14\].

In the present manuscript, we use some of the results previously discussed in the literature. A deformation of the Landau problem based on a modification of Fock algebra is considered. The variables $\{x, y, p_x, p_y\}$ satisfy the Heisenberg algebra and $\theta$ is a parameter with dimensions of (energy)$^{-2}$. The operator $\mathcal{N}$ formally describes a charged two-dimensional harmonic oscillator with mass $m = 2/\theta^2$, frequency $\omega = \theta$, in an external magnetic field $B = \theta^2$.

The eigenvalues of $\mathcal{N}$ have been discussed in the context of non-commutative quantum mechanics \[15\] and the important fact is that in a two dimensional non-commutative space, any central field $V(|\mathbf{x}|^2)$ becomes $V(\mathcal{N})$ and therefore, general problems in two dimension can be explicitly addressed.

Let us discuss this last fact. Consider the potential $V(|\mathbf{x}|^2)$ in a space where coordinates $\{\hat{x}, \hat{y}\}$ satisfy $[\hat{x}, \hat{y}] = i \theta$. By changing the basis of the algebra, this potential satisfies

$$V(\hat{x}^2 + \hat{y}^2) = V(\frac{\theta^2}{4} \mathbf{p}^2 + x^2 + y^2 - \theta(x p_y - y p_x))$$

$$= V(H_{\text{HO}} - \theta L_z)$$

$$\equiv V(\mathcal{N}).$$

The variables $\{x, y, p_x, p_y\}$ satisfy the Heisenberg algebra, the operator $H_{\text{HO}}$ denotes a two-dimensional harmonic oscillator as in \[15\], while the operator $L_z = (x p_y - y p_x)$ is a conserved quantity in the sense $[L_z, H_{\text{HO}}] = 0$.

This system is an example of non-commutative quantum mechanics and the analysis can be extended to cases where the whole phase space is non-commutative – by introducing momentum variables $\{p_x, p_y\}$ satisfying $[p_x, p_y] = iB$ with $B$ a constant – giving rise to new interesting features such as a sort of phase transition for $\theta = \frac{1}{4}$.

The operator $\mathcal{N}$ can be diagonalized in terms of ladder operators. That is, by defining

$$a_\pm = \frac{1}{\sqrt{2}} (a_y \pm ia_x),$$

$$a_\pm^\dagger = \frac{1}{\sqrt{2}} (a_y^\dagger \mp ia_x^\dagger).$$

1 Through the paper we use natural units $c = 1, \hbar = 1$. 

with \( a_x = (\theta)^{-\frac{1}{2}}(x + i(\theta/2)p_x) \) and a similar definition for \( a_y \), being \( a^\dagger \) the conjugate transposed operator. The operators previously defined satisfy the following algebra

\[
[a_\pm, a_\mp^\dagger] = 1, \quad [a_\pm, a_\mp] = 0, \quad (5)
\]

and therefore, spaces + and – are orthogonal.

Then, \( \mathcal{N} \) operator is diagonal in the base \([n_-, n_+]\), that is

\[
\mathcal{N}|n_-, n_+\rangle = \Lambda_{n_-, n_+}|n_-, n_+\rangle, \quad (6)
\]

with

\[
\Lambda_{n_-, n_+} = \theta (2n_- + 1), \quad (7)
\]

thus, this is an infinitely degenerate system.

In (7) the \( \theta \) parameter emerges as an effective frequency and the factor 1 there is the zero point energy which can be removed by a normal order prescription of operators. The total Hamiltonian for a particle in this potential reads

\[
H = \frac{1}{2M} \mathbf{p}^2 + V(\mathcal{N}), \quad (8)
\]

where \( M \) is a mass scale. It has been shown that this Hamiltonian is equivalent to the Landau problem in the lowest level – in the strong magnetic field regime – in the limit \( M \to \infty \) for a linear potential \( V(\mathcal{N}) = \Omega \mathcal{N} \) with \( \Omega \) a constant with dimensions of (energy)\(^3\).

Therefore, we study the case in which \( M \to \infty \) and the Hamiltonian turn out to be

\[
H = V(\mathcal{N}). \quad (9)
\]

Since \( \mathcal{N} \) satisfies

\[
H|n\rangle = V(\mathcal{N})|n\rangle = V(\epsilon_n)|n\rangle, \quad (10)
\]

where \( n \) is a notation for the set of quantum numbers \( \{n_+, n_-\} \) and \( \epsilon_n \) denotes the eigenvalues of \( \mathcal{N} \).

For \( V(\mathcal{N}) \) expandable in a Taylor series around a small parameter \( \lambda \)

\[
V(\mathcal{N}) = \mathcal{N} + \lambda \mathcal{N}^2 + \cdots, \quad (11)
\]

an effective Hamiltonian can be defined

\[
H_{\text{eff}} =: V(\mathcal{N}) = \mathcal{N} + \lambda \mathcal{N}^2 + \cdots, \quad (12)
\]

and then

\[
H_{\text{eff}}|n\rangle = V(\epsilon_n)|n\rangle. \quad (13)
\]

An example of this system is the Euler-Heisenberg effective Hamiltonian density

\[
\mathcal{H}_{\text{eff}} = \frac{1}{2} (\mathbf{E}^2 + \mathbf{B}^2) + \xi (\mathbf{E}^2 + \mathbf{B}^2)^2 + \cdots, \quad (14)
\]

where \( \xi \) in this case in the fine structure constant.

The next step we will take is to make an extra deformation of the above non-commutative system. Indeed, the system previously analyzed admits also a different type of algebra deformation. That is, instead of considering the relation as the one in (15) we posit the relation

\[
[a_\pm, a_\mp^\dagger] = D \left[ \lambda_+ a_\pm^\dagger a_\mp \right], \quad (15)
\]

where \( D|x\rangle \) is a deformation operation which, in principle, can be a function of the operator \( x \) or an infinite series of \( x \) and \( \lambda_\pm \) is a dimensionless parameter which, in principle, can be chosen as \( \lambda_+ = \lambda_- = \lambda \), but we will keep both different in order to consider a more general scenario.

For example if we choose \( D|x\rangle = 1 - x \), then (15) becomes

\[
a_\pm a_\mp^\dagger - q_\pm a_\mp^\dagger a_\mp = 1, \quad (16)
\]

with \( q_\pm = 1 - \lambda_\pm \) which turn out to be a representation of the quantum group \( SU_q(2) \) for each sector \( '+' \) and \( '-' \).

The effect of the deformation is to change the spectrum of \( a_\pm^\dagger a_\mp \) and therefore \( \Lambda_{n_-, n_+} \) changes. Denoting by \( c_{n_-}(\lambda_-) \) the spectrum of the number operator in the \( '-' \) sector one gets the spectrum (normal ordered)

\[
\Lambda_{n_- n_+} = 2\theta c_{n_-}(\lambda_-), \quad (17)
\]

where, for the linear deformation \( D|x\rangle = 1 - x \) one finds

\[
c_{n_-}(\lambda) = \lambda^{-1} (1 - (1 - \lambda)^n). \quad (18)
\]

The infinite degeneracy in \( n_+ \) persists since it depends on the fact that \( x \) and \( y \) in (2) are decoupled sectors. The energy difference between two successive levels for this system turn out to be

\[
\Lambda_{n_-+1, n_+} - \Lambda_{n_- n_+} = 2\theta (1 - \lambda_-)^{n_-}. \quad (19)
\]

It follows that the spectrum is asymmetric, that is, the spectrum is not equally spaced by \( n_- \). This is true for any potential of the form \( V(\mathcal{N}) \).

With this in mind, let us use the fact that \( V \) is a function of \( p \) and \( x \) (or what is the same, \( a \) and \( a^\dagger \)) in order to study different physical systems. As a concrete example, consider

\[
V(\mathcal{N}) = \Omega_N \mathcal{N}^N, \quad (20)
\]

with \( \Omega_N \) a constant with dimensions of (energy)\(^{2N+1}\).

The diagonal basis of operator \( a_\pm^\dagger a_\pm \) is \( \{n_\pm\}\)\(\{n_\pm = 0, 1, 2, \ldots\} \) and due to the properties of the operator \( \mathcal{N} \), we consider the base \( \{n_-, m_+\} \), so that the matrix element (we omit the constant \( \Omega_N \))

\[
\langle n_-, n_+ | V | m_-, m_+ \rangle = \langle n_- | V | n_- \rangle \delta_{n_-, m_+} = V_{n_- m_-} \delta_{n_-, m_+} \quad \text{where the matrix element of the normal ordered operator } V \text{ is}
\]

\[
V_{n_- m_-} = \langle n_- | \mathcal{N} \mathcal{N} \cdots \mathcal{N} | m_- \rangle = \sum_{\{n_+\}} \langle n_- | n_1 \rangle \langle n_1 | n_2 \rangle \cdots \langle n_N | n_- \rangle \langle m_- \rangle \delta_{n_- m_-} = (2\theta)^N (c_{n_-}(\lambda_-))^N \delta_{n_- m_-}. \quad (21)
\]
For the particular case of the linear deformation $D[x] = 1 - \lambda x$ with $\lambda \equiv \lambda_-$ we obtain

$$V_{n-,m-} = \left(\frac{2\theta}{\lambda}\right)^N \left[1 - (1 - \lambda)^{-n}\right]^N \delta_{n-,m-}. \quad (22)$$

For the case of Hamiltonian $[\hat{N}]$, this result implies

$$\langle n-,n+|H|m-,m+\rangle = \langle n-,n+|\frac{\vec{p}^2}{2m}|m-,m+\rangle +$$

$$\Omega_N \left(\frac{2\theta}{\lambda}\right)^N \left[1 - (1 - \lambda)^{-n}\right]^N \delta_{n-,m-}\delta_{n+,m+}. \quad (23)$$

However, since we are considering the lowest Landau level limit, the spectrum (note that the change of notation $n_\pm \equiv n$) $E_n \sim V_{n,n}$ becomes

$$E_n = \Omega_N \left(\frac{2\theta}{\lambda}\right)^N \left[1 - (1 - \lambda)^n\right]^N. \quad (24)$$

Note that this energy is basically the energy of the turning points (which further highlights the non-perturbative character of this result).

In order to make contact with the Landau problem described before we take $\Omega_N = \Omega^N$ so that the identification $\Omega^N = e\hbar_0/2\mu$ holds. Here, $e$ is the electron charge, $\mu$ is a mass scale and $H_0$ is an external magnetic field.

For $N = 1$ (in the limit $M \to \infty$) this model defines a deformed Landau problem. The energy can be written, alternatively

$$E_n = 2\Omega^N \sum_{\ell=0}^{n-1} \frac{n(n-1)\cdots(n-\ell)}{( \ell + 1)!} (-\lambda)\ell$$

$$= 2\Omega^N n \left(1 - \frac{n - 1}{2}\lambda\right) + O(\lambda^2), \quad (25)$$

showing that corrections to the energy due to the deformation is of order $n^2$. Similar behavior is observed in the non-relativistic limit of the relativistic Landau problem, however, relativistic corrections there are also proportional to $(\Omega^2)^2$.

This suggests that a higher powers of $\lambda$ might be of physical interest. Consider, for example, the potential

$$V^{(2)} = \Omega N + \frac{\kappa^2}{4} \Omega^2 N^2, \quad (26)$$

where $\kappa^2$ is a length scale which is determined below by using the quadratic Zeeman effect data and the Breit-Rabi formula.

According to our previous analysis and also under same assumptions, the energy spectrum turn out to be

$$E_n^{(2)} = \left(\frac{eH_0}{2\mu}\right) n \left(2 + (1 - n)\lambda\right) +$$

$$\left(\frac{eH_0}{2\mu}\right)^2 \kappa^2 n^2 \left(1 + (1 - n)\lambda\right) + O(\lambda^2). \quad (27)$$

We observe here that linear terms in $\lambda$ has contributions from linear and quadratic terms of magnetic field with different powers of $n$ and

$$\Delta E_n^{(2)} = E_n^{(2)} - E_{n-1}^{(2)} = 2 \left(\frac{eH_0}{2\mu}\right) (1 + (1 - n)\lambda) +$$

$$\kappa^2 \left(\frac{eH_0}{2\mu}\right)^2 \left[2n - 1 + (5n - 2 - 3n^2)\lambda\right]. \quad (28)$$

For non-highly excited levels, the difference of energies becomes

$$\Delta E_n^{(2)} \approx 2 \left(\frac{eH_0}{2\mu}\right) (1 + (1 - n)\lambda) + \kappa^2 \left(\frac{eH_0}{2\mu}\right)^2 (2n - 1 - 3n^2) \lambda. \quad (29)$$

In the conventional Landau problem, the difference of frequencies is a constant proportional to the external applied magnetic field. In the case we are considering, the deformation takes an extra factor proportional to $\kappa^2 \left(\frac{eH_0}{2\mu}\right)^2$ and then, if the magnetic field is large enough, and using the condition that $\lambda$ can be properly adjusted, this correction of frequency could be relevant. To be precise, it is enough to have

$$H_0 \gg \left(\frac{2\mu}{e\kappa^2}\right) (1 + (1 - n)\lambda) \sim \left(\frac{2\mu}{e\kappa^2}\right) \frac{1}{3n^2}. \quad (30)$$

Actually the $H_0^2$ contribution due to the deformation plays a similar role of the quadratic Zeeman effect term. In our case, for strong magnetic field as described before, one have

$$|\Delta E^{(2)}| \approx H_0^2 \left(\frac{\kappa_0}{2\mu}\right)^2 3n^2 \lambda \quad (31)$$

$$\equiv \kappa_{eff} H_0^2 \quad (32)$$

Although the quadratic corrections of the Zeeman effect are not dominant in atomic physics in general, they are important, for example in the case of alkaline atoms where

$$|\Delta \omega| = \kappa_{eff} H_0^2. \quad (33)$$

These quadratic corrections have been measured in the last twenty years using different methods and great precision has been achieved with the development of cold atoms measurement techniques. For example for the $^{87}$Rb ground-state clock transition $\kappa$ is

$$\kappa_{eff} \sim 575,15 \times 10^8 Hz T^{-2}, \quad (34)$$

but from the the Breit-Rabi formula

$$\Delta \omega = \frac{(g_I - g_J)\mu_B^2}{2h\Delta H_{Hfs}} H_0^2 = \kappa_{eff} H_0^2, \quad (35)$$

where $g_I, g_J$ are the Landé factors, $\mu_B$ the Bohr magneton, $h$ the Planck constant and $\Delta H_{Hfs}$ is the hyperfine energy splitting.
In particular the experimental value [34] compared to the $\kappa$ calculated with the Breit-Rabi formula is in excellent agreement with standard measurements [23, 24].

Putting $\kappa$ in natural units then the dimensions of $\kappa$ are

$$\kappa \sim (\text{length})^3,$$

and replacing (34) we find

$$\kappa^{1/3} \sim 0.1 \text{Å} \quad (36)$$

which typically could be considered an x-ray regime effect.

I. CONCLUSIONS

In this work we have studied a linear deformation of the Fock algebra and we have applied it to the Landau problem which leads to a family of exactly soluble problems. We consider in detail the quadratic effective Hamiltonian and the length scale is fixed using data from the quadratic Zeeman effect.

A careful analysis of (29) shows that the validity of our arguments between neighboring levels as predicted by the Breit-Rabi formula [3].

The uncertainties of the Breit-Rabi formula are uncertainties in the physical and atomic constants so that the only possible sources of error may come from the g-factors, but the data considered (cold alkaline atoms [23]) seem to be well enough established.

The case of linear potential, on the other hand, can be solved completely and its extension to the relativistic case is straightforward and is equivalent to the formulation of non-commutative fields as it was discussed in [23].

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2 For a recent experimental Breit-Rabi formula verification see [24].