Deformation of $p$-adic String Amplitudes in a Magnetic Field

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ABSTRACT

A new term in the $p$-adic world-sheet action is proposed, which couples a constant $B$-field to the boundary of the world-sheet at disk level. The induced deformation of tachyon scattering amplitudes by star-products is derived. This is in agreement with the deformation of effective action postulated in recent investigations of noncommutative solitons in $p$-adic string theory.

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1 Introduction

It has been appreciated that allowing non-Archimedean number fields (for a review, see [1]) leads to modifications of string theory, thereby yielding exactly solvable models. Given a prime number \( p \), one may consider the field \( \mathbb{Q}_p \) of \( p \)-adic numbers, and modify the theory by making the coordinates \( p \)-adic on the boundary of the world-sheet. Other quantities such as actions, amplitudes, target-space coordinates and coordinates in the interior of the world-sheet remain usual complex numbers. Moreover, as shown by Zabrodin [2], realization of the boundary as the projective set of leaves of an infinite tree allows for a new picture of \( p \)-adic string theory at disk level. The action of the bosonic string must be rewritten using some discrete scheme, which couples the space-time metric to the tree:

\[
\int_{\Sigma} d^2 \sigma \, \partial_{\alpha} X^\mu \partial^\alpha X_\mu \longrightarrow \sum_{i} \sum_{j(i)} \left( X^\mu(j(i)) - X^\mu(i) \right) \left( X_{\mu}(j(i)) - X_{\mu}(i) \right).
\]

Degrees of freedom corresponding to the interior of the world-sheet can furthermore be integrated out, leading to a non-local action on the \( p \)-adic boundary [3, 4, 5]. Scattering amplitudes of tachyons can be recovered either by computing path integrals with Neumann boundary conditions in the discretized scheme, or by saddle-point method with the non-local action. The output of both methods is the tachyon Lagrangian [6, 7, 8, 9, 10]

\[
\mathcal{L} = -\frac{1}{2} \phi p^{-1/2} \Delta \phi + \frac{1}{p+1} \phi^{p+1},
\]

whose exactness gives appeal to this whole approach by non-Archimedean number fields. It provides a toy model for string theory, allowing to explicitly check the scenario of tachyon condensation [11].

The recent work by Ghoshal [12] investigated noncommutative \( p \)-adic solitons, postulating a deformation of the effective action by star-products coming from the \( B \)-field:

\[
\mathcal{L} = -\frac{1}{2} \phi \ast p^{-1/2} \Delta \phi + \frac{1}{p+1} (\ast \phi)^{p+1}.
\]

This natural guess begs for a world-sheet action. First of all, one may ask how the \( B \)-field couples to the \( p \)-adic boundary of the disk, and subsequently compute the modifications induced by this coupling in the scattering amplitudes. To this end, I shall consider the \( B \)-field as a magnetic field coupled to the boundary of the disk. This will allow for quicker comparison to the previously known non-local term involving only the metric. Since I only consider a constant

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\( p \)-adic norm of a rational number \( x \) is defined as \( |x|_p = p^{-n} \), where \( p^n \) is the largest power of \( p \) dividing \( x \). The field \( \mathbb{Q}_p \) is defined as the Cauchy completion of \( \mathbb{Q} \) with the \( p \)-adic norm. Measure, integration, changes of variables and Fourier transform can be defined on \( \mathbb{Q}_p \), whereas derivatives do not find a straightforward analog. This lack of analogy makes the difference between the exhibition of a world-sheet Lagrangian and a translation work.
$B$-field, inspiration from usual string theory indeed allows for an investigation of mere boundary couplings, through

$$
\int_{\Sigma} B = \int_{\partial\Sigma} B_{\mu\nu} X^\mu \frac{dX^\nu}{dt} dt.
$$

In this letter, I shall propose a natural $p$-adic analog of this coupling. Using special functions on extensions of $\mathbb{Q}_p$, I shall incorporate this additional term in the path integral, and read off star-products on scattering amplitudes.

## 2 Coupling the $B$-field to the boundary

When no $B$-field is turned on, the following non-local action on the $p$-adic boundary of the world-sheet was shown by Zhang \cite{3} to lead to the scattering amplitudes previously derived by the means of discretized Laplacian operator on the whole world-sheet:

$$
S[X] = \int_{\mathbb{Q}_p} du \left( g_{\mu\nu} X^\mu(-u) |u|_p X^\nu(u) \right),
$$

$$
A_N(k_1, \ldots, k_N) = \int DX \exp \left( -S[X] + i \sum_{i=1}^N k_i \delta_{\mu\nu} X^\mu(u_i) \right) = \prod_{1 \leq i < j \leq N} |u_i - u_j|_p^{k_i \delta_{\mu\nu} k_j \nu}.
$$

On dimensional grounds, we expect the same structure as in the symmetric case, as far as powers of $u$ are concerned, because the same number of derivatives are present in usual string theory. The essential difference comes from the index structure. We need an antisymmetric kernel in order to write down a non-zero coupling to $B$. Fortunately, a $p$-adic notion of sign function does exist, though it is not unique \cite{13, 14} and depends on the choice of a quadratic extension of the field of $p$-adic numbers. Consider the $\epsilon_\tau$-function, defined on the quadratic extension of $\mathbb{Q}_p$ by a non-square $\tau$ as follows: $\epsilon_\tau(x) = 1$ if $x$ is the product of two conjugate numbers in $\mathbb{Q}_p(\sqrt{\tau})$, and $\epsilon_\tau(x) = -1$ otherwise. It is a multiplicative character, and for it to contribute a minus sign, we need the property

$$
\epsilon_\tau(x - y) = -\epsilon_\tau(y - x),
$$

which reduces to

$$
\epsilon_\tau(-1) = -1.
$$

This property is however not verified if we take $\tau$ to be a $(p - 1)$-th root of unity, but can be achieved by choosing $\tau$ to be $p$ or a multiple of it by a $(p - 1)$-th root of unity. From now on I shall just write $\epsilon$ for this sign function, and $| |$ for the $p$-adic norm. I therefore write the following non-local action on the $p$-adic boundary in flat space-time, coupled to a constant $B$-field:

$$
S[X] = \int_{\mathbb{Q}_p} du \left( \delta_{\mu\nu} X^\mu(-u) |u|_p X^\nu(u) + B_{\mu\nu} X^\mu(-u) |u| \epsilon(u) X^\nu(u) \right).
$$
We are now instructed to use this action to compute saddle-point approximation to the path integral corresponding to the scattering of \( N \) tachyons. To this end we need the Green function of the integration kernel we have written down. It is computed by splitting the action between symmetric and antisymmetric parts, as far as tensors are concerned, because the flat metric tensor commutes with all the tensors at hand and their inverses. We denote by \( G \) and \( \Theta \) the symmetric and antisymmetric parts of the inverse of \( \delta + B \):

\[
G^{\mu\nu} := \left( \frac{1}{\delta + B} \right)^{\alpha\mu} g_{\alpha\beta} \left( \frac{1}{\delta - B} \right)^{\beta\nu},
\]

\[
\Theta^{\mu\nu} := -\left( \frac{1}{\delta + B} \right)^{\alpha\mu} B_{\alpha\beta} \left( \frac{1}{\delta - B} \right)^{\beta\nu}.
\]

The Green function is obtained by Fourier transform, taking the same regularization scheme by a small parameter \( s \) as in [3]:

\[
\Delta^{\mu\nu}_{\text{reg}}(x - y) = \int_{\mathbb{Q}_p} du e^{i(x - y)u} \left( G^{\mu\nu} \frac{1}{|u|^{1+s}} + \Theta^{\mu\nu} \frac{\epsilon(u)}{|u|^{1+s}} \right).
\]

The problem of the singularity at \( s = 0 \) was dealt with in [3] by extracting the finite part in an expansion in powers of \( s \). This problem does not show up in the second term we have just induced, and we do not need to remove any singularity from it at \( s = 0 \). The sign function we use can be integrated against functions over \( p \)-adic numbers, just as in Fourier transform. This procedure provides a generalization of Gamma-functions:

\[
\hat{\Gamma}_p(s) := \int_{\mathbb{Q}_p} du e^{i2\pi[u]} \epsilon(u) |u|^{s-1}.
\]

These generalized functions nicely parallel the role played by Gamma-functions in [3]. Changing integration variables to \( v = (x - y)u \), we obtain

\[
\Delta^{\mu\nu}_{\text{reg}}(x - y) = |x - y|^{-s} \int_{\mathbb{Q}_p} dv e^{iv} \left( G^{\mu\nu} \frac{1}{|v|^{1+s}} + \Theta^{\mu\nu} \frac{\epsilon(v)}{|v|^{1+s}} \right),
\]

which reads in terms of \( p \)-adic Gamma-functions and generalizations thereof:

\[
\Delta^{\mu\nu}_{\text{reg}}(x - y) = |x - y|^{-s} \left( G^{\mu\nu} \Gamma_p(s) + \Theta^{\mu\nu} \epsilon(x - y) \hat{\Gamma}_p(s) \right).
\]

On extensions of \( \mathbb{Q}_p \) where the sign of \(-1\) can be negative, generalized Gamma-functions obey the following up to a sign:

\[
\hat{\Gamma}_p(s) = \sqrt{\epsilon(-1)p^{s-1/2}},
\]

so that, with \( \epsilon(-1) = -1 \), the regular part of the propagator reads

\[
\Delta^{\mu\nu}(x - y) = G^{\mu\nu} \log |x - y| + \frac{i}{\sqrt{p}} \Theta^{\mu\nu} \epsilon(x - y).
\]

This completes the derivation of the Green function needed to evaluate scattering amplitudes. The guess of the coupling resides essentially in symmetry properties of the kernel. The choice of a quadratic extension of \( \mathbb{Q}_p \) with negative sign for \(-1\) is crucial to obtain a non-zero coupling.
3 Modification of the scattering amplitudes

We are now able to compute scattering amplitudes at disk level, thereby observing the modifications induced by the $B$-field. Without the $B$-field, the non-local action leads to amplitudes that are formally the same as in ordinary string theory, except for integration on $Q_p$ and prefactors depending on $p$. Checking whether our action leads to the deformation postulated in [12] will amount to checking whether the modification of the scattering amplitudes by the antisymmetric piece of the propagator neatly arranges into phases that can be recognized as Fourier transforms of star-products. This is the lesson of Feynman rules of noncommutative field theory [15] [16].

Let us write the saddle-point approximation to the following path integral corresponding to scattering of $N$ tachyons:

$$A_N(k_1, \ldots, k_N) = \int DX \exp \left( -S[X] + i \sum_{i=1}^{N} k_{i,\mu} X^\mu(u_i) \right).$$

Contractions between pairs of scalars using the propagator derived above yield phases reminiscent of the Moyal vertex in ordinary string theory [15]:

$$A_N(k_1, \ldots, k_N) = \prod_{1 \leq i < j \leq N} e^\frac{i}{2\sqrt{p}} k_{i,\mu} \Theta^{\mu\nu} k_{j,\nu} \epsilon(u_i - u_j) k_{i,\mu} G^{\mu} k_{j,\nu}.$$

The influence of the $B$-field is completely encoded into phases, which read as Fourier transform of star-products. We have therefore established the relationship:

$$\int DX \exp \left( -S_{G,\Theta}[X] + i \sum_{i=1}^{N} k_{i,\mu} X^\mu(u_i) \right) = \prod_{1 \leq i < j \leq N} e^\frac{i}{2\sqrt{p}} k_{i,\mu} \Theta^{\mu\nu} k_{j,\nu} \epsilon(u_i - u_j) \int DX \exp \left( -S_{G,\Theta=0}[X] + i \sum_{i=1}^{N} k_{i,\mu} X^\mu(u_i) \right).$$

We notice that amplitudes involving more general vertex operators (namely waves weighted by a polynomial of derivatives of $X$), would not allow for contractions coming from the new term in the propagator. This is the very same situation as in ordinary string theory, where noncommutativity induces star-products between the vertex operators, without parturbing the inner structure of any of them. From these amplitudes we read off the relevant star-product in position space, that actually depends on $p$ as well as on the $B$-field:

$$* = \exp \left( \frac{i}{2\sqrt{p}} \partial^\mu \Theta^{\mu\nu} \partial_{\nu} \right) \big|_{x'=x}.$$

4 Conclusions

We have established that deformations of the $p$-adic tachyon Lagrangian by star-products can actually be derived using a prescription for the coupling of a $p$-adic boundary to a magnetic...
field. Moreover, the term in the world-sheet Lagrangian that we proposed consists of a minimal modification of the action written by Zhang, that is essentially dictated by the antisymmetry of the $B$-field. We observe that the formal limit in which $p$ goes to 1 yields the usual Moyal product. This provides a sense of naturality, that is independent from the requirement that noncommutativity still appear in $p$-adic string theory. The way computations are organized is very similar to the symmetric case, with generalized $p$-adic Gamma-functions entering the scalar two-point functions, as the close relatives of $p$-adic Gamma-functions previously used to regularize the propagator.

The non-locality of the effective action, which is manifest even without any deformation, made plausible the expectation that noncommutativity could still be felt by $p$-adic boundaries. On the other hand, the non-locality of the world-sheet action is more familiar for the coupling to the $B$-field than it was when the symmetric term was exhibited.

The issue of the quadratic extension of $\mathbb{Q}_p$, that has to be chosen so that $\epsilon(-1) = -1$, thus ensuring a non-zero coupling to the $B$-field, is a genuinely $p$-adic feature of the construction. In usual string theory, the open-string world-sheet indeed looks like the complex upper-half plane, the normal direction being naturally associated with extension of the field $\mathbb{R}$ of the boundary coordinates by square root of unity. Maybe the relevance of the extension by $p$ instead of roots of unity is related to the fact that the boundary of the $p$-adic world-sheet actually consists of leaves of a tree, which come in bouquets of $p$ leaves, leading to $p$ links coming from the boundary into the interior of the world-sheet, at fixed $p$-adic norm. Making the argument more precise would require to obtain our full non-local action by integrating out degrees of freedom on the interior of the world-sheet. This would compel us to consider $B$ as a genuine closed-string field coupling to the whole world-sheet, instead of taking the gauge-equivalent picture of a magnetic field coupled to the boundary.

**Note added.** After this letter had been completed, the work [17] appeared, where the same boundary coupling is proposed. It furthermore addresses the problem of invariance of correlation functions under $\text{GL}(2, \mathbb{Q}_p)$. This issue is motivated by the decoupling between ordering and algebraic properties occurring in the $p$-adic number field.

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