ASYMPTOTIC SPECTRAL MEASURES, QUANTUM MECHANICS, AND $E$-THEORY

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Abstract. We study the relationship between POV-measures in quantum theory and asymptotic morphisms in the operator algebra $E$-theory of Connes-Higson. This is done by introducing the theory of “asymptotic” PV-measures and their integral correspondence with positive asymptotic morphisms on locally compact spaces. Examples and applications involving various aspects of quantum physics, including quantum noise models, semiclassical limits, strong deformation quantizations, and pure half-spin particles, are also discussed.

1. Introduction

In the Hilbert space formulation of quantum mechanics by von Neumann [VN], an observable is modeled as a self-adjoint operator on the Hilbert space of states of the quantum system. The Spectral Theorem relates this theoretical view of a quantum observable to the more operational one of a projection-valued measure (PVM or spectral measure) which determines the probability distribution of the experimentally measurable values of the observable. To solve foundational problems with the concept of measurement and to better analyze unsharp results in experiments, this view was generalized to include positive operator-valued measures (POVMs). Since the work of Jauch and Piron [JP], POV-measures have played an ever increasing role in both the foundations and operational aspects of quantum physics [BGL,S]. See the Appendix for a quick review of POVMs, their use in quantum mechanics, and relation to the Spectral Theorem.

In this paper, we study the relationship between POVMs and asymptotic morphisms in the operator algebra $E$-theory of Connes and Higson [CH], which has already found many applications in mathematics [Bl,GHT,H,Tr], most notably to classification problems in operator $K$-theory, index theory, representation theory, geometry, and topology. The basic ingredients of $E$-theory are asymptotic morphisms, which are given by continuous families of functions $\{Q_h\}_{h>0} : \mathcal{A} \to \mathcal{B}$ from a $C^*$-algebra $\mathcal{A}$ to a $C^*$-algebra $\mathcal{B}$ that satisfy the axioms of a $*$-homomorphism in the limit as the parameter $h$ tends to 0. Asymptotic multiplicativity is a modern version of the Bohr-von Neumann correspondence principle [L] from quantization theory: For all $f, g \in \mathcal{A}$,

$$Q_h(fg) - Q_h(f)Q_h(g) \to 0 \text{ as } h \to 0.$$
It is then no surprise that quantization schemes may naturally define asymptotic morphisms, say, from the $C^*$-algebra $\mathcal{A}$ of classical observables to the $C^*$-algebra $\mathcal{B}$ of quantum observables. Hence, such quantizations can give cycles in the abelian group $E(\mathcal{A}, \mathcal{B})$, which was defined by Connes and Higson as a certain matrix-stable homotopy group of asymptotic morphisms from $\mathcal{A}$ to $\mathcal{B}$. For example, Guentner [G1] showed that Wick quantization on the Fock space $\mathcal{F}$ of $\mathbb{C}$ defines a positive asymptotic morphism $\{Q_\hbar\} : C_0(\mathbb{C}) \to K(\mathcal{F})$, whose $E$-theory class is equal to the class of the $\partial$-operator $[\partial] = [Q_\hbar] \in E(C_0(\mathbb{C}), \mathbb{C})$. (We will discuss Guentner’s work in our context in Example 5.5.) See the papers [N1,N2, Ro] and the books [C,GVF] for more on the connections between operator algebra $K$-theory, $E$-theory, and quantization.

We show that there is a fundamental quantum-$E$-theory relationship by introducing the concept of an asymptotic spectral measure (ASM or asymptotic PVM)

$$\{A_\hbar\}_{\hbar > 0} : \Sigma \to B(\mathcal{H})$$

associated to a measurable space $(X, \Sigma)$. (See Definition 3.1.) Roughly, this is a continuous family of POV-measures which are “asymptotically” projective (or quasiprojective) as $\hbar$ tends to 0:

$$A_\hbar(\Delta)^2 - A_\hbar(\Delta) \to 0 \text{ as } \hbar \to 0$$

for certain measurable sets $\Delta \in \Sigma$.

Let $X$ be a locally compact space with Borel $\sigma$-algebra $\Sigma_X$ and let $C_0(X)$ denote the $C^*$-algebra of continuous functions vanishing at infinity on $X$. One of our main results is an “asymptotic” Riesz representation theorem (Theorem 4.2) which gives a bijective correspondence between certain positive asymptotic morphisms

$$\{Q_\hbar\} : C_0(X) \to B$$

and Borel asymptotic spectral measures

$$\{A_\hbar\} : (\Sigma_X, C_X) \to (B(\mathcal{H}), B)$$

where $C_X$ denotes the open subsets of $X$ with compact closure and $B$ is a hereditary $C^*$-subalgebra of $B(\mathcal{H})$. This correspondence is given by operator integration

$$Q_\hbar(f) = \int_X f(x) \, dA_\hbar(x).$$

The associated asymptotic morphism $\{Q_\hbar\} : C_0(X) \to B$ then allows one to define an $E$-theory invariant for the asymptotic spectral measure $\{A_\hbar\}$,

$$[A_\hbar] = \text{def} [Q_\hbar] \in E(C_0(X), B) \cong E_0(X; B),$$

in the $E$-homology group of $X$ with coefficients in $B$.

It has been well-established that operator $K$-theory and the dual $K$-homology groups provide suitable receptacles for invariants of quantum systems, such as chiral anomalies in quantum field theory [N] and, more recently, as D-brane charges in
string theory and $M$-theory [P,W]. Since $E$-theory subsumes both $K$-theory and $K$-homology [Bl], it is reasonable to assume that $E$-theory elements of quantizations and asymptotic spectral measures may provide interesting topological invariants of the associated quantum systems. Although in this paper we will be more concerned with asymptotic morphisms and their relation to POV-measures than computing $E$-theory elements (but see Example 5.5), a long-range goal of this research project is to develop an $E$-theoretic calculus for computing these invariants directly from the asymptotic measure-theoretic data, e.g., by developing the appropriate notions of homotopy and suspension for ASMs, thus bypassing the technical functional-analytic aspects of asymptotic morphisms.

Another benefit of using this asymptotic measure-theoretic approach is operational in nature. Experimental data from position and momentum measurements on an elementary quantum system (via visibility data from interference experiments) is collected which is then used to construct the associated POVM. This method [S] is based on using frame manuals for the instrument state space and Sakai operators associated to localization operators on rectangles in the classical phase space $X$. The POVM $\{A_\hbar\}$ depends on Planck’s constant, of course, and generally satisfies the (unsharp) separation property

$$A_\hbar(\Delta_1 \cap \Delta_2) \neq A_\hbar(\Delta_1)A_\hbar(\Delta_2).$$

However, if letting $\hbar \to 0$ one then obtains an ASM, which is equivalent to

$$\lim_{\hbar \to 0} (A_\hbar(\Delta_1 \cap \Delta_2) - A_\hbar(\Delta_1)A_\hbar(\Delta_2)) = 0,$$

then one can directly associate an $E$-homological invariant $[A_\hbar] \in E_0(X; B)$ to the quantum system under experimental study using our theory.

The outline of this paper is as follows. In Section 2 we discuss positive asymptotic morphisms associated to hereditary and nuclear $C^*$-algebras. The basic definitions and properties of asymptotic spectral measures are developed in Section 3. Asymptotic Riesz representation theorems and some of their consequences are proven in Section 4. Examples and applications of ASMs associated to various aspects of quantum physics are discussed in Section 5, e.g., constructing ASMs from PVMs by quantum noise models, quasiprojectors and semiclassical limits, unsharp spin measurements of spin-$1/2$ particles (including an example from quantum cryptography), strong deformation quantizations, and Wick quantization on bosonic Fock space.

The authors would like to thank Navin Khaneja, Iain Raeburn, and Dana Williams for helpful conversations. Also, we would like to thank the referee for helpful comments. See Beggs [B] for a related method of obtaining asymptotic morphisms by an integration technique involving spectral measures.

2. Positive Asymptotic Morphisms and Hereditary $C^*$-subalgebras

Let $A$ and $B$ be $C^*$-algebras. Recall that a linear map $Q : A \to B$ is called positive [M] if $Q(f) \geq 0$ for all $f \geq 0$ in $A$. It is called completely positive if every inflation to $n \times n$ matrices $M_n(Q) : M_n(A) \to M_n(B)$ is also positive. Every $*$-homomorphism from $A$ to $B$ is clearly completely positive. The following definition interpolates between (completely) positive linear maps and $*$-homomorphisms.
Definition 2.1. A (completely) positive asymptotic morphism from $A$ to $B$ is a family of maps

$$\{Q_h\}_{h \in (0,1]} : A \to B$$

parameterized by $h \in (0,1]$ such that the following conditions hold:

a.) Each $Q_h$ is a (completely) positive linear map;

b.) The map $(0,1] \to B : h \to Q_h(f)$ is continuous for each $f \in A$;

c.) For all $f, g \in A$ we have

$$\lim_{h \to 0} \|Q_h(fg) - Q_h(f)Q_h(g)\| = 0.$$ 

For the basic theory of asymptotic morphisms see the books [GHT, C, Bl] and papers [CH, G2]. For the importance of positive asymptotic morphisms to $C^*$-algebra $K$-theory see [HLT]. Note that any $*$-homomorphism $Q : A \to B$ determines the constant completely positive asymptotic morphism $\{Q_h\} : A \to B$ defined by $Q_h = Q$ for all $h > 0$. Also, it follows that for any $f \in A$, a mild boundedness condition [Bl] always holds,

$$\limsup_{h \to 0} \|Q_h(f)\| \leq \|f\|.$$ 

Remark. In the $E$-theory literature, asymptotic morphisms are usually parameterized by $t \in [1, \infty)$. We chose to use the equivalent parameterization $h = 1/t \in (0,1]$ to make the connections to quantum physics more transparent. Note that other authors have used different parameter spaces, including discrete ones [L2, Th]. The results in this paper translate verbatim to these parameter spaces, and condition (b.) is obviously irrelevant in the discrete case.

Definition 2.2. Two asymptotic morphisms $\{Q_h\}, \{Q'_h\} : A \to B$ are called equivalent if for all $f \in A$ we have that

$$\lim_{h \to 0} \|Q_h(f) - Q'_h(f)\| = 0.$$ 

We will let $[A, B]_{a(c)}$ denote the collection of all asymptotic equivalence classes of (completely positive) asymptotic morphisms from $A$ to $B$.

A $C^*$-algebra $A$ is called nuclear [M] if the identity map $id : A \to A$ can be approximated pointwise in norm by completely positive finite rank contractions. This is equivalent to the condition that there is a unique $C^*$-tensor product $A \otimes B$ for any $C^*$-algebra $B$. If $\mathcal{H}$ is a separable Hilbert space, the $C^*$-algebra $K(\mathcal{H})$ of compact operators on $\mathcal{H}$ is nuclear. If $X$ is a locally compact space, then the $C^*$-algebra $C_0(X)$ of continuous complex-valued functions on $X$ vanishing at infinity is also nuclear.

If $A \cong C(X)$ is unital and commutative, then every positive linear map $Q : A \to B$ is completely positive by Stinespring’s Theorem. The following result is a consequence of the completely positive lifting theorem of Choi and Effros [CE] for nuclear $C^*$-algebras. (See also 25.1.5 of Blackadar [Bl] for a discussion.)

Lemma 2.3. Let $A$ be a nuclear $C^*$-algebra. Every asymptotic morphism from $A$ to any $C^*$-algebra $B$ is equivalent to a completely positive asymptotic morphism. That is, there is a bijection of sets $[A, B]_a \cong [A, B]_{a(c)}$. 

Definition 2.4. Let \( A_1 \subset A \) and \( B_1 \subset B \) be subalgebras of the \( C^*-\)algebras \( A \) and \( B \). If \( Q : A \to B \) is a linear map such that \( Q(A_1) \subset B_1 \), we will denote this by

\[
Q : (A, A_1) \to (B, B_1).
\]

The notation \( \{Q_h\} : (A, A_1) \to (B, B_1) \) then has the obvious meaning.

Lemma 2.5. Let \( A_1 \subset A \) and \( B_1 \subset B \) be non-closed \( * \)-subalgebras. Every positive linear map \( Q : (A, A_1) \to (B, B_1) \) also satisfies

\[
Q : (\overline{A_1}) \to (\overline{B_1}),
\]

where \( \overline{A_1} \) denotes the closure of \( A_1 \subset A \) (similarly for \( \overline{B_1} \)).

Proof. Follows from the fact that a positive linear map is automatically norm bounded. \( \Box \)

Let \( A \) be a \( * \)-subalgebra of a \( C^*-\)algebra \( B \). Recall that \( A \) is said to be hereditary [M] if 0 \( \leq b \leq a \) and \( a \in A \) implies that \( b \in A \). Every (closed two-sided \( * \)-invariant) ideal in a \( C^*-\)algebra is a hereditary \( * \)-subalgebra. In particular, if \( \mathcal{H} \) is a Hilbert space, the ideal of compact operators \( \mathcal{K}(\mathcal{H}) \) is a hereditary \( C^*-\)subalgebra of the \( C^*-\)algebra of bounded operators \( \mathcal{B}(\mathcal{H}) \). An important (non-closed) hereditary \( * \)-subalgebra for quantum theory is the (non-closed) ideal \( B_1(\mathcal{H}) \subset \mathcal{K}(\mathcal{H}) \) of trace-class operators:

\[
B_1(\mathcal{H}) = \{ \rho \in \mathcal{K}(\mathcal{H}) : \text{trace} |\rho| < \infty \}.
\]

We then have that \( \mathcal{K}(\mathcal{H})^* = B_1(\mathcal{H}) \) by the dual pairing \( \rho(T) = \text{trace}(\rho T) \), where \( \rho \in B_1(\mathcal{H}) \) and \( T \in \mathcal{K}(\mathcal{H}) \).

If \( X \) is a locally compact space, then the ideal \( C_0(X) \) is a hereditary \( C^*-\)subalgebra of the \( C^*-\)algebra \( C_b(X) \) of continuous bounded complex-valued functions on \( X \). Also, the (non-closed) ideal \( C_c(X) \) of compactly supported functions is a (non-closed) hereditary \( * \)-subalgebra of \( C_b(X) \). However, in general, \( C_\delta(X) \), for \( \delta = c, 0, b \), is not a hereditary subalgebra of the \( C^*-\)algebra \( B_b(X) \) of bounded Borel functions on \( X \).

3. Asymptotic Spectral Measures

In this section we give the basic definitions and properties of asymptotic spectral measures. See the Appendix for a review of POV and spectral measures. Let \( (X, \Sigma) \) be a measurable space and \( \mathcal{H} \) a separable Hilbert space. Let \( \mathcal{E} \subset \Sigma \) denote a fixed collection of measurable subsets.

Definition 3.1. A asymptotic spectral measure (ASM) on \( (X, \Sigma, \mathcal{E}) \) is a family of maps

\[
\{A_h\}_{h \in (0,1]} : \Sigma \to \mathcal{B}(\mathcal{H})
\]

parameterized by \( h \in (0,1] \) such that the following conditions hold:

a.) Each \( A_h \) is a POVM on \( (X, \Sigma) \) with \( \lim \sup_{h \to 0} \|A_h(X)\| \leq 1 \);

b.) The map \( (0,1] \to \mathcal{B}(\mathcal{H}) : h \to A_h(\Delta) \) is continuous for each \( \Delta \in \mathcal{E} \);

c.) For each \( \Delta_1, \Delta_2 \in \mathcal{E} \) we have that

\[
\lim_{h \to 0} \|A_h(\Delta_1 \cap \Delta_2) - A_h(\Delta_1)A_h(\Delta_2)\| = 0.
\]
The triple \((X, \Sigma, \mathcal{E})\) will be called an asymptotic measure space. The family \(\mathcal{E}\) will be called the asymptotic carrier of \(\{A_h\}\). Condition (c.) will be called asymptotic projectivity (or quasiprojectivity) and generalizes the projectivity condition (A.1) of a spectral measure. It is motivated by the quantum theory notion of quasiprojectors, as discussed in Example 5.2. If \(\mathcal{E} = \Sigma\) then we will call \(\{A_h\}\) a full ASM on \((X, \Sigma)\). If each \(A_h\) is normalized, i.e., \(A_h(X) = I_{\mathcal{H}}\), then we will say that \(\{A_h\}\) is normalized. The mild boundedness condition in (a.) is then redundant. (Also see the remark after Definition 2.1.)

A spectral (PV) measure \(E : \Sigma \to \mathcal{B}(\mathcal{H})\) determines a “constant” full asymptotic spectral measure \(\{A_h\}\) by the assignment \(A_h = E\) for all \(h\). Also, any continuous family \(\{E_h\}\) of spectral measures (in the sense of (b.)) determine an ASM on \((X, \Sigma, \mathcal{E})\). See [CHM] for an application of smooth families of spectral measures to the Quantum Hall Effect.

**Definition 3.2.** Two asymptotic spectral measures \(\{A_h\}, \{B_h\} : \Sigma \to \mathcal{B}(\mathcal{H})\) on \((X, \Sigma, \mathcal{E})\) are said to be (asymptotically) equivalent if for each measurable set \(\Delta \in \mathcal{E}\),

\[
\lim_{h \to 0} \|A_h(\Delta) - B_h(\Delta)\| = 0.
\]

This will be denoted \(\{A_h\} \sim_\mathcal{E} \{B_h\}\). If this holds for \(\mathcal{E} = \Sigma\) we will call them fully equivalent.

From now on, we let \(X\) denote a locally compact Hausdorff topological space with Borel \(\sigma\)-algebra \(\Sigma_X\). We will assume that \(\mathcal{E} = \mathcal{C}_X\) denotes the collection of all open subsets of \(X\) with compact closure, i.e., the pre-compact open subsets.

**Definition 3.3.** Let \(\mathcal{B} \subset \mathcal{B}(\mathcal{H})\) be a hereditary \(*\)-subalgebra. A Borel POV-measure \(A : \Sigma_X \to \mathcal{B}(\mathcal{H})\) will be called locally \(\mathcal{B}\)-valued if \(A(U) \in \mathcal{B}\) for all pre-compact open subsets \(U \in \mathcal{C}_X\) and this will be denoted by

\[
A : (\Sigma_X, \mathcal{C}_X) \to (\mathcal{B}(\mathcal{H}), \mathcal{B}).
\]

A family of Borel POV-measures \(\{A_h\}\) on \(X\) will be called locally \(\mathcal{B}\)-valued if each POVM \(A_h\) is locally \(\mathcal{B}\)-valued and will be denoted \(\{A_h\} : (\Sigma_X, \mathcal{C}_X) \to (\mathcal{B}(\mathcal{H}), \mathcal{B})\).

We will use the term locally compact-valued for locally \(\mathcal{K}(\mathcal{H})\)-valued. If \(\mathcal{B} = \mathcal{B}_1(\mathcal{H}) \subset \mathcal{K}(\mathcal{H})\) is the trace-class operators, then we will say that \(\{A_h\}\) has locally compact trace.

We will let \(\langle (X, \mathcal{B}) \rangle\) denote the set of all equivalence classes of locally \(\mathcal{B}\)-valued Borel asymptotic spectral measures on \((X, \Sigma_X, \mathcal{C}_X)\). The equivalence class of \(\{A_h\}\) will be denoted \(\langle \{A_h\} \rangle \in \langle (X, \mathcal{B}) \rangle\).

Given a Borel POV-measure \(A\) on \(X\), the cospectrum of \(A\) is defined as the set

\[
\text{cospec}(A) = \bigcup \{U \subset X : U \text{ is open and } A(U) = 0\}.
\]

The spectrum of \(A\) is the complement \(\text{spec}(A) = X \setminus \text{cospec}(A)\). The following definition is adapted from Berberian [Be]

**Definition 3.4.** A POVM \(A\) on \(X\) will be said to have compact support if the spectrum of \(A\) is a compact subset of \(X\). An ASM \(\{A_h\}\) on \(X\) will be said to have compact support if there is a compact subset \(K\) of \(X\) such that \(\text{spec}(A_h) \subset K\) for all \(h > 0\).

The relationship among these compactness notions is contained in the following.
Proposition 3.5. Let $X$ be second countable. Let $A$ be a Borel POVM on $X$ with compact support. Let $\mathcal{B}$ be the hereditary subalgebra of $\mathcal{B}(\mathcal{H})$ generated by $A(\text{spec}(A))$. Then $A$ is a locally $\mathcal{B}$-valued POVM, i.e., $A : (\Sigma_X, \mathcal{C}_X) \to (\mathcal{B}(\mathcal{H}), \mathcal{B})$.

Proof. Since $X$ is second countable, the $\sigma$-algebra $\mathcal{B}_X$ of Baire subsets equals the Borel $\sigma$-algebra $\mathcal{B}_X = \Sigma_X$. Thus, by Theorem 23 [Be] $A(\text{cospec}(A)) = 0$. Let $U \in \mathcal{C}_X$ be a pre-compact open subset of $X$. We then have that

$$0 \leq A(U \cap \text{cospec}(A)) \leq A(\text{cospec}(A)) = 0$$

and since $X$ is the disjoint union $X = \text{spec}(A) \sqcup \text{cospec}(A)$,

$$0 \leq A(U) = A(U \cap \text{spec}(A)) \leq A(\text{spec}(A)).$$

Since $\mathcal{B}$ is hereditary, $A(U) \in \mathcal{B}$ for all $U \in \mathcal{C}_X$ and so $A$ is locally $\mathcal{B}$-valued. □

4. Asymptotic Riesz Representation Theorems

Throughout this section, we let $X$ denote a locally compact Hausdorff space with Borel $\sigma$-algebra $\Sigma_X$. Let $\mathcal{C}_X \subset \Sigma_X$ denote the collection of all pre-compact open subsets of $X$. And we let $\mathcal{B} \subset \mathcal{B}(\mathcal{H})$ denote a hereditary $\ast$-subalgebra of the bounded operators on a fixed Hilbert space $\mathcal{H}$.

Lemma 4.1. There is a bijective correspondence between locally $\mathcal{B}$-valued Borel POVMs $A : (\Sigma_X, \mathcal{C}_X) \to (\mathcal{B}(\mathcal{H}), \mathcal{B})$ and positive linear maps $Q : C_0(X) \to \mathcal{B}$. This correspondence is given by

$$(4.1.1) \quad Q(f) = \int_X f(x) \, dA(x).$$

Proof. In view of Theorem A.3 we only need to check that the locally $\mathcal{B}$-valued condition corresponds to $Q(C_0(X)) \subset \mathcal{B}$. Suppose $A(\mathcal{C}_X) \subset \mathcal{B}$. Let $f \in C_c(X)$ be compactly supported. Since $Q$ is positive linear, it suffices to assume $f \geq 0$. Let $K = \text{supp}(f)$ which is a compact subset of $X$. By local compactness, there is an open subset $U \in \mathcal{C}_X$ such that $K \subset U$. By the Extreme Value Theorem there is a $C > 0$ such that $0 \leq f \leq C\chi_U$. Since $Q$ is positive,

$$0 \leq Q(f) = \int_X f \, dA \leq C \int_X \chi_U \, dA = CA(U) \in \mathcal{B}$$

by hypothesis. Since $\mathcal{B}$ is hereditary, $Q(f) \in \mathcal{B}$.

Conversely, suppose $Q : C_0(X) \to \mathcal{B}$ is positive linear and given by formula (4.1.1). Then $Q$ defines a positive map $Q : (B_b(X), C_0(X)) \to (\mathcal{B}(\mathcal{H}), \mathcal{B})$. Let $U \in \mathcal{C}_X$ be a pre-compact open subset. Since $X$ is completely regular, we have by Urysohn’s Lemma a continuous function $f \in C_c(X)$ with $0 \leq f \leq 1$ such that $\overline{U} = f^{-1}(1)$. Thus, $0 \leq \chi_U \leq f$ and so $0 \leq A(U) = Q(\chi_U) \leq Q(f) \in \mathcal{B}$. Thus, $A(U) \in \mathcal{B}$ and so $A$ is locally $\mathcal{B}$-valued as desired. □

Define $B_0(X)$ to be the $C^\ast$-subalgebra of $B_b(X)$ generated by $\{\chi_U : U \in \mathcal{C}_X\}$. If $X$ is also $\sigma$-compact, a paracompactness argument then shows that $C_0(X) \subset B_0(X)$ as a closed (but not necessarily hereditary) $\ast$-subalgebra. (Recall that if $f \in C_c(X)$ is compactly supported, then Interior(supp($f$)) $\in \mathcal{C}_X$.) The following is our main result.
Theorem 4.2. If $X$ is $\sigma$-compact, there is a bijective correspondence between positive asymptotic morphisms

$$\{Q_h\} : (B_0(X), C_0(X)) \to (B(H), B)$$

and locally $B$-valued Borel asymptotic spectral measures

$$\{A_h\} : (\Sigma_X, C_X) \to (B(H), B).$$

This correspondence is given by

$$(4.1) \quad Q_h(f) = \int_X f(x) \, dA_h(x)$$

Proof. Let $\{Q_h\} : (B_0(X), C_0(X)) \to (B(H), B)$ be a positive asymptotic morphism. By the lemma, there is a locally $B$-valued family of POVMs $\{A_h\} : (\Sigma_X, C_X) \to (B(H), B)$ such that (4.1) holds for all $h > 0$ and $f \in B_0(X)$. For each $U \in C_X$ we have that $h \mapsto A_h(U) = Q_h(\chi_U)$ is continuous by condition (2.1.b.). Also, we have that for any $U \neq \emptyset \in C_X$

$$\limsup_{h \to 0} \|A_h(U)\| = \limsup_{h \to 0} \|Q_h(\chi_U)\| \leq \|\chi_U\| = 1.$$ 

Since $X$ is $\sigma$-compact, there is an increasing sequence $\{U_n\} \subset C_X$ of pre-compact open subsets such that $X = \bigcup_{n=1}^\infty U_n$. By Theorem 18 [Be], for all $h > 0$, $A_h$ is a “regular” Borel POVM, so

$$A_h(X) = \text{LUB}\{A_h(U_n) : n \in \mathbb{N}\}$$

(in the sense of positive operators). It follows that $\limsup_{h \to 0} \|A_h(X)\| \leq 1$ as desired. Now let $U_1, U_2 \in C_X$. Since characteristic functions satisfy $\chi_{U_1 \cap U_2} = \chi_{U_1} \chi_{U_2}$ we then have by asymptotic multiplicativity (2.1.c) that

$$\lim_{h \to 0} \|A_h(U_1 \cap U_2) - A_h(U_1)A_h(U_2)\| = \lim_{h \to 0} \|Q_h(\chi_{U_1} \chi_{U_2}) - Q_h(\chi_{U_1})Q_h(\chi_{U_2})\| = 0.$$ 

Thus, the family $\{A_h\}$ is a locally $B$-valued ASM on $X$.

Conversely, let $\{A_h\} : (\Sigma_X, C_X) \to (B(H), B)$ be a locally $B$-valued ASM on $X$. Define the family of maps $\{Q_h\} : B_0(X) \to B(H)$ by equation (4.1). Hence, each $Q_h$ is positive linear and

$$Q_h : (B_0(X), C_0(X)) \to (B(H), B)$$

by Lemma 4.1. Let $S_0(X)$ denote the dense $*$-subalgebra of $B_0(X)$ consisting of simple functions $f = \sum_{i=1}^n a_i \chi_{U_i}$, where $U_i \in C_X$. Asymptotic projectivity (3.1.c) and the calculation above then show that for any simple functions $f, g \in S_0(X)$ we have

$$\lim_{h \to 0} \|Q_h(fg) - Q_h(f)Q_h(g)\| = 0.$$ 

Also, for any such simple function $f \in S_0(X)$,

$$h \mapsto Q_h(f) = \sum_{i=1}^n a_i A_h(U_i)$$
is continuous from $(0, 1] \to B$ by (3.1.b). To conclude that $\{Q_h\}$ is asymptotically multiplicative on the closure $B_0(X)$ we need to show that it is bounded. By (A.3.1) we have that for any $f \in B_0(X),$

$$\|Q_h(f)\| = \left\| \int_X f(x) dA_h(x) \right\| \leq 2\|f\|\|A_h(X)\|.$$  

By condition (3.1.a) we then have

$$\limsup_{h \to 0} \|Q_h(f)\| \leq 2\|f\| \limsup_{h \to 0} \|A_h(X)\| \leq 2\|f\|.$$  

The result now follows since every bounded asymptotic morphism on a dense $\ast$-subalgebra extends to the closure. □

**Corollary 4.3.** Under the above hypotheses, equivalent Borel asymptotic spectral measures correspond to equivalent positive asymptotic morphisms. Thus, there is a well-defined map $(X, B) \to [C_0(X), B]_{acp}$ which maps $(A_h) \to [Q_h]_{acp}.$

**Proof.** Follows from the fact that $A_h(U) = Q_h(\chi_U)$ and any two asymptotic morphisms equivalent on a dense subalgebra, are equivalent. Also, since $C_0(X)$ is nuclear, the second statement follows from Lemmas 2.3 and 2.5. □

Let $C_\delta(X)$ denote a unital $C^\ast$-subalgebra of $C_b(X)$ such that $C_0(X) \lhd C_\delta(X).$ By the Gelfand-Naimark Theorem [GN], $C_\delta(X) \cong C(\delta X)$ for some 'continuous' compactification $\delta X \supseteq X.$

**Corollary 4.4.** Let $\mathcal{I} \lhd B(\mathcal{H})$ be an ideal. Every locally $\mathcal{I}$-valued full Borel asymptotic spectral measure $\{A_h\}$ on $X$ determines a canonical relative asymptotic morphism (in the sense of Guentner [G2])

$$\{Q_h\} : (C_\delta(X), C_0(X)) \to (B(\mathcal{H}), \mathcal{I})$$

for any continuous compactification $\delta X$ of $X.$

**Definition 4.5.** A family $\{A_h\}_{h>0} : \Sigma \to B(\mathcal{H})$ of Borel POV-measures on $X$ will be called a $C_\delta$-asymptotic spectral measure if the family of maps $\{Q_h\}$ defined by equation (4.1) determines an asymptotic morphism $\{Q_h\} : C_\delta(X) \to B(\mathcal{H}).$

The following proposition is then easy to prove using Theorem A.3 and the results above.

**Proposition 4.6.** There is a one-one correspondence between locally $B$-valued $C_\delta$-asymptotic spectral measures

$$\{A_h\} : (\Sigma, C_X) \to (B(\mathcal{H}), B)$$

and positive asymptotic morphisms

$$\{Q_h\} : (C_\delta(X), C_0(X)) \to (B(\mathcal{H}), B).$$
5. Examples and Applications

5.1 Constructing ASMs via Quantum Noise Models.

We give a general method for constructing asymptotic spectral measures from spectral measures (on a possibly different measure space) by adapting a convolution technique used to model noise and uncertainty in quantum measuring devices. See Section II.2.3 of Busch et al [BGL] for the relevant background material.

Let \((X_1, \Sigma_1)\) and \((X_2, \Sigma_2)\) be measure spaces. Let \(\mathcal{E}_2 \subset \Sigma_2\). Consider a family of maps

\[
\{p_h\} : \Sigma_2 \times X_1 \rightarrow [0, 1]
\]

such that the following conditions hold:

a.) For every \(\omega \in X_1\), \(\Delta \mapsto p_h(\Delta, \omega)\) is a probability measure on \(X_2\);

b.) For each \(\Delta \in \mathcal{E}_2\), the map \(h \mapsto p_h(\Delta, \cdot)\) is continuous \([0, 1) \rightarrow B_0(X_1)\);

c.) For every \(\Delta_1, \Delta_2 \in \mathcal{E}_2\),

\[
\lim_{h \to 0} \|p_h(\Delta_1, \cdot)p_h(\Delta_2, \cdot) - p_h(\Delta_1 \cap \Delta_2, \cdot)\|_\infty = 0
\]

where \(\| \cdot \|_\infty\) denotes the sup-norm on \(B_0(X_1)\).

Let \(E : \Sigma_1 \rightarrow \mathcal{B}(\mathcal{H})\) be a spectral measure on \(X_1\). Define a family of maps \(\{A_h\} : \Sigma_2 \rightarrow \mathcal{B}(\mathcal{H})\) by the formula

\[
A_h(\Delta) = \int_{X_1} p_h(\Delta, \omega) \, dE(\omega)
\]

for any \(\Delta \in \Sigma_2\).

**Theorem 5.1.2.** The family \(\{A_h\} : \Sigma_2 \rightarrow \mathcal{B}(\mathcal{H})\) defines an ASM on \((X_2, \Sigma_2, \mathcal{E}_2)\). If \(E\) is normalized then \(\{A_h\}\) is also normalized.

**Proof.** The fact that each \(A_h\) is a POVM on \(X_2\) is easy. Continuity in \(h\) follows from condition (b.) and the following estimate for \(\Delta \in \mathcal{E}_2\),

\[
\|A_h(\Delta) - A_{h_0}(\Delta)\| = \left\| \int_{X_1} (p_h(\Delta, \omega) - p_{h_0}(\Delta, \omega)) \, dE(\omega) \right\| \leq \|p_h(\Delta, \cdot) - p_{h_0}(\Delta, \cdot)\|_\infty.
\]

Now we need to prove asymptotic projectivity. Let \(\Delta_1, \Delta_2 \in \mathcal{E}_2\). Consider the calculation

\[
\|A_h(\Delta_1)A_h(\Delta_2) - A_h(\Delta_1 \cap \Delta_2)\| = \\
= \left\| \int_{X_1} p_h(\Delta_1, \omega) \, dE(\omega) \int_{X_1} p_h(\Delta_2, \omega) \, dE(\omega) - \int_{X_1} p_h(\Delta_1 \cap \Delta_2, \omega) \, dE(\omega) \right\| \\
= \left\| \int_{X_1} p_h(\Delta_1, \omega)p_h(\Delta_2, \omega) \, dE(\omega) - \int_{X_1} p_h(\Delta_1 \cap \Delta_2, \omega) \, dE(\omega) \right\| \\
\leq \|p_h(\Delta_1, \cdot)p_h(\Delta_2, \cdot) - p_h(\Delta_1 \cap \Delta_2, \cdot)\|_\infty \to 0
\]

as \(h \to 0\) by (c.). We finish by showing that the mild normalization condition holds:

\[
A_h(X_2) = \int p_h(X_2, \omega) \, dE(\omega) = \int 1 \, dE(\omega) = E(X_1) \leq I
\]
by condition (a.) above and the fact that $E(X_1)$ is a projection. □

Note that the inequalities in the previous proof require that $E$ be a PVM. (See Theorems 15 and 16 [Be] and Theorem A.4.) See Example 5.3 below for a concrete example of this smearing technique.

The physical interpretation (for finite systems) is that $p_h$ models the noise or uncertainty in interpreting the readings of a measurement. For example, if $E$ has an eigenstate $\phi = E(\{\omega\})\phi$, then the expectation value of $A(h)(\Delta)$ when the system is in state $\phi$ is given by

$$\langle \phi | A_h(\Delta) | \phi \rangle = p_h(\Delta, \omega).$$

Thus, $p_h$ determines a (conditional) confidence measure of the system.

5.2 Quasiprojectors and Semiclassical Limits.

In this example, we show that the theory of ASMs can be used to study semiclassical limits. The relevant background for the material in this section can be found in Chapters 10 and 11 of Omnes book [O]. We first need the following well-known result which is an easy consequence of the functional calculus and spectral mapping theorem. (See also Lemma 5.1.6. [WO].) It gives a rigorous statement of the procedure used to “straighten out” quasiprojectors into projections.

**Lemma 5.2.1.** Let $\{a_h : h > 0\}$ be a continuous family of elements in a $C^*$-algebra $B$ such that $0 \leq a_h \leq 1$ for each $h > 0$ and

$$\lim_{h \to 0} \|a_h - a_h^2\| = 0.$$

There is a continuous family of projections $h \mapsto e_h = e_h^* = e_h^2$ such that

$$\lim_{h \to 0} \|a_h - e_h\| = 0.$$

Let $(X, \Sigma, \mathcal{E})$ be an asymptotic measure space.

**Proposition 5.2.2.** Let $\{A_h\}$ be a normalized ASM on $(X, \Sigma, \mathcal{E})$. For each subset $\Delta \in \mathcal{E}$ there is a continuous family of projections $h \mapsto E_h(\Delta)$ such that

$$\lim_{h \to 0} \|A_h(\Delta) - E_h(\Delta)\| = 0.$$

Moreover if $\Delta_1$ and $\Delta_2$ are disjoint measurable sets in $\mathcal{E}$ then

$$\lim_{h \to 0} \|E_h(\Delta_1)E_h(\Delta_2)\| = 0.$$

**Proof.** For each $\Delta \in \mathcal{E}$ we have by monotonicity and normalization that $0 \leq A_h(\Delta) \leq I$ for all $h > 0$. Setting $\Delta = \Delta_1 = \Delta_2$ in the asymptotic projectivity condition (3.1.c) we have that

$$\lim_{h \to 0} \|A_h(\Delta) - A_h(\Delta)^2\| = 0.$$

Now invoke the previous lemma to get the continuous family $\{E_h(\Delta)\}$ of projections. If $\Delta_1 \cap \Delta_2 = \emptyset$ then by condition (3.1.c) again, we have that

$$\lim_{h \to 0} \|A_h(\Delta_1)A_h(\Delta_2)\| = 0.$$

A simple triangle inequality argument plus normalization then shows that

$$\lim_{h \to 0} \|E_h(\Delta_1)E_h(\Delta_2)\| = 0$$

as was desired. □

The relation to semiclassical limits occurs when we take $X$ to be the locally compact phase space of a classical system and $B = \mathcal{B}_1(\mathcal{H})$ to be the algebra of trace-class operators.
Proposition 5.2.3. Let \{A_h\} be a Borel ASM on X with locally compact trace. Then for any subset \(\Delta \in C_X\) we have
\[
\lim_{h \to 0} \text{trace}(A_h(\Delta) - A_h(\Delta)^2) = 0
\]
and there is a unique integer \(N_{\Delta} \in \mathbb{N}\) such that
\[
N_{\Delta} = \lim_{h \to 0} \text{trace}(A_h(\Delta)).
\]
Moreover, this integer is constant on the asymptotic equivalence class of \(\{A_h\}\).

Proof. The first limit follows from the continuity of the trace. Let \(\{E_h(\Delta)\}\) be the projections from the previous result. Since
\[
A_h(\Delta) \in B_1(\mathcal{H}) \subset \overline{B_1(\mathcal{H})} = \mathcal{K}(\mathcal{H})
\]
it follows that \(E_h(\Delta) \in \mathcal{K}(\mathcal{H})\), i.e., \(h \mapsto E_h(\Delta)\) is a continuous family of compact (hence, finite rank) projections. Therefore, since the rank of a projection is a continuous invariant [D]
\[
\lim_{h \to 0} \text{trace}(A_h(\Delta)) = \lim_{h \to 0} \text{trace}(E_h(\Delta)) = \text{rank}(E_{h_0}(\Delta)) =_{\text{def}} N_{\Delta}
\]
for any \(h_0 > 0\). The last statement follows again by continuity of the trace. \(\square\)

Suppose \(X\) denotes the position-momentum phase space \((x, p)\) of a particle. Let \(\{A_h\}\) be a locally compact trace Borel ASM on \(X\). A bounded rectangle \(R\) in phase space with center \((x_0, p_0)\) and sides \(2\Delta x\) and \(2\Delta p\) can then be used to represent a classical property asserting the simultaneous existence of the position and momentum \((x_0, p_0)\) of the particle with given error bounds \((\Delta x, \Delta p)\) on measurement. The nonnegative integer \(N_R\) which satisfies
\[
N_R = \lim_{h \to 0} \text{trace}(A_h(R))
\]
can then be interpreted as the number of semiclassical states of the particle bound in the rectangular box \(R\), which is familiar from elementary statistical mechanics. We then have that
\[
\text{trace}(A_h(R) - A_h(R)^2) = N_R O(h)
\]
\[
\text{trace}(E_h(R) - A_h(R)) = N_R O(h).
\]
Thus, \(h\) represents a classicality parameter. When \(h \approx 0\) is small, the quantum representation of the classical property is essentially correct and when \(h \approx 1\) the classical property has essentially no meaning from the standpoint of quantum mechanics. Since these relations are preserved on equivalence classes, “a classical property corresponding to a sufficiently large a priori bounds \(\Delta x\) and \(\Delta p\) is represented by a set of equivalent quantum projectors” [O], i.e., equivalent locally compact trace ASMs. In addition, if \(R_1\) and \(R_2\) are disjoint rectangles, representing distinct classical properties, then we have that
\[
\|A_h(R_1)A_h(R_2)\| = O(h)
\]
and so “two clearly distinct classical properties are (asymptotically) mutually exclusive when considered as quantum properties” [O].
5.3 Unsharp Spin Measurements of Spin-$\frac{1}{2}$ Systems.

In this example, we give a geometric classification of certain asymptotic spectral measures associated to pure spin-$\frac{1}{2}$ particles.

Recall that pure spin systems are represented by the Hilbert space $\mathcal{H} = \mathbb{C}^2$ [BGL,S]. We then have $B(\mathcal{H}) \cong M_2(\mathbb{C})$. The Pauli spin operators $\sigma_1, \sigma_2, \sigma_3$ are the $2 \times 2$ matrices

$$
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
$$

which satisfy the relations:

- $\sigma_i^* = \sigma_i, \sigma_i^2 = I$
- $\sigma_i \sigma_j = -\sigma_j \sigma_i$ for $i \neq j$
- $\sigma_i \sigma_j = i \epsilon_{ijk} \sigma_k$ for $i \neq j$

where $I$ denotes the identity operator. A density operator (or state) on $\mathcal{H}$ is a positive matrix $\rho \geq 0$ with trace one. A fundamental result in the theory is the following.

**Lemma 5.3.1.** Any density operator $\rho$ on $\mathcal{H}$ can be written uniquely in the form

$$
\rho = \rho(\vec{x}) = \frac{1}{2} (I + \vec{x} \cdot \vec{\sigma}), \quad \vec{x} = (x_1, x_2, x_3) \in \mathbb{R}^3, \quad ||\vec{x}|| \leq 1,
$$

where $\vec{x} \cdot \vec{\sigma} = x_1 \sigma_1 + x_2 \sigma_2 + x_3 \sigma_3$ and $||\vec{x}||^2 = x_1^2 + x_2^2 + x_3^2$. Moreover, $\rho$ is a one-dimensional projection iff $\vec{x}$ is a unit vector $||\vec{x}|| = 1$.

**Definition 5.3.3.** A spin POVM on $X_2 = \{-\frac{1}{2}, +\frac{1}{2}\}$ is a normalized POVM $A = \{A^+, A^\pm\}$ such that trace($A^\pm$) = 1, where $A^\pm = A(\{\pm \frac{1}{2}\})$. Thus $A^\pm \geq 0$ is a density operator and $A^+ + A^- = I$. An asymptotic spectral measure $\{A_h\}$ on $X_2$ will be called spin if each $A_h$ is a spin POVM.

Let $B^3 = \{\vec{x} \in \mathbb{R}^3 : ||\vec{x}|| \leq 1\}$ denote the closed unit ball in $\mathbb{R}^3$. Let $S^2 = \partial B^3$ denote the unit sphere. For each $\vec{x} \in B^3$ we obtain a spin POVM $A_x$ on $X_2$ by defining

$$
A^\pm_x = \rho(\pm \vec{x}) = \frac{1}{2} (I \pm \vec{x} \cdot \vec{\sigma})
$$

which determines an “unsharp” spin observable. Let $\lambda = \lambda(\vec{x}) = ||\vec{x}||$ and define the quantities

$$
r_x = \frac{1 + \lambda(\vec{x})}{2} > \frac{1}{2}, \quad u_x = \frac{1 - \lambda(\vec{x})}{2} < \frac{1}{2}.
$$

The quantity $r_x$ is called the degree of reality and $u_x$ is the degree of unsharpness of the unsharp observable $A_x$ [BGL,RK].

**Lemma 5.3.5.** There is a bijective correspondence between spin POVMs $A = \{A^+, A^-\}$ and points $\vec{x} \in B^3$ in the closed unit ball of $\mathbb{R}^3$ given by (5.3.4). Moreover, $A$ is a spectral measure if and only if $\vec{x} \in S^2$ is a unit vector.

**Proof.** Follows from Lemma 5.3.1, Definition 5.3.3, and $A^- = I - A^+$. □
Theorem 5.3.6. There is a bijective correspondence between spin asymptotic spectral measures \( \{ A_h \} = \{ A_h^+, A_h^- \} \) and continuous maps \( \vec{A} : (0, 1] \to B^3 \) such that

\[
(5.3.6.1) \quad \lim_{h \to 0} \| \vec{A}(h) \| = 1.
\]

This correspondence is given by the formula

\[
A_h^\pm = \frac{1}{2} (I \pm \vec{A}(h) \cdot \vec{\sigma}).
\]

Proof. By the Lemma we only need to prove continuity in \( h \) and that asymptotic projectivity corresponds to condition (5.3.6.1) above. By the properties of the Pauli spin operators above, we can show by direct computations (see also formulas (66a) and (66b) in [S]) that

\[
4[(A_h^+)^2 - A_h^+] + (1 - \| \vec{A}(h) \|^2) I = 0
\]

and

\[
\| \vec{A}(h) - \vec{A}(h_0) \|^2 = -\det((\vec{A}(h) - \vec{A}(h_0)) \cdot \vec{\sigma}) = 4 \det(A_h^+ - A_{h_0}^+).
\]

The result now easily follows. \( \Box \)

Thus, we can geometrically realize the space of spin asymptotic spectral measures as the space of continuous paths in the closed unit ball of \( \mathbb{R}^3 \) which asymptotically approach the unit sphere, i.e. they are “asymptotically sharp.” Note that this provides nontrivial examples of asymptotic spectral measures which do not converge to a fixed spectral measure.

Let \( \vec{n} \) be a unit vector and define \( \vec{A}(h) = (1-h)\vec{n} \). The associated spin asymptotic spectral measure given by

\[
A_h^\pm = \frac{1}{2} (I \pm (1-h)\vec{n} \cdot \vec{\sigma})
\]

is used by Roy and Kar [RK] to analyze eavesdropping strategies in quantum cryptography using EPR pairs of correlated spin-\( \frac{1}{2} \) particles. Violations of Bell’s inequality occur when the parameter \( h > 1 - \sqrt{2(\sqrt{2} - 1)}^{\frac{1}{2}} \).

This spin ASM is also obtained by the asymptotic smearing construction in 5.1. Let \( E^\pm = A_0^\pm \) be the spectral measure associated to the unit vector \( \vec{n} \). Define the family \( \{ p_h \} : \mathcal{P}(X_2) \times X_2 \to [0, 1] \) by the formula \( p_h(\Delta, j) = \sum_{i \in \Delta} \lambda_{ij}^h \) where \( (\lambda_{ij}^h) \) is the stochastic matrix

\[
(\lambda_{ij}^h) = \begin{pmatrix}
1 - \frac{h}{2} & \frac{h}{2} \\
\frac{h}{2} & 1 - \frac{h}{2}
\end{pmatrix}.
\]

One can then verify that

\[
A_h^\pm = \sum_{l = \mp \frac{1}{2}} p_h(\{ \pm \frac{1}{2} \}, l) E^\mp.
\]
Corollary 5.3.7. Two spin asymptotic spectral measures \( \{A_h\} \) and \( \{B_h\} \) are equivalent if and only if their associated maps \( \vec{A}, \vec{B} : [0, 1] \to B^3 \) are asymptotic, i.e.,

\[
\lim_{\hbar \to 0} \| \vec{A}(\hbar) - \vec{B}(\hbar) \| = 0.
\]

Proof. \( \| \vec{A}(\hbar) - \vec{B}(\hbar) \|^2 = 4 \det(A^+_h - B^+_h) \). \( \square \)

5.4 Strong Deformation Quantization.

Let \( X \) be a locally compact space. Let \( B \) be a \( C^* \)-algebra. A strong deformation from \( X \) to \( B \) is a continuous field \( \{D\} \) of \( C^* \)-algebras \( \{B_\hbar : \hbar \in [0, 1]\} \) such that \( B_0 = C_0(X) \) and

\[
\{B_\hbar | \hbar > 0\} \cong B \times (0, 1).
\]

Here we give a measure-theoretic criterion, based on \( E \)-theory arguments, for when a locally \( B \)-valued Borel ASM \( \{A_h\} \) on \( X \) determines a strong deformation from \( X \) to \( B \), where \( B \) is a hereditary \( C^* \)-subalgebra of \( B(\mathcal{H}) \). First, we make the following general definition.

Definition 5.4.1. Let \( \{A_h\} \) be an ASM on \( (X, \Sigma, \mathcal{E}) \). We will call \( \{A_h\} \) injective if

\[
\lim_{h \to 0} \inf \| A_h(\Delta) \| > 0
\]

for all nonempty subsets \( \Delta \neq \emptyset \) in \( \mathcal{E} \).

Thus, if \( \{A_h\} \) is a locally \( B \)-valued Borel ASM on \( X \), then by local compactness and monotonicity, injectivity is equivalent to

\[
\lim_{h \to 0} \inf \| A_h(U) \| > 0
\]

for all nonempty open subsets \( U \neq \emptyset \) of \( X \). Let \( \{Q_h\} : C_0(X) \to B \) be the associated asymptotic morphism given by Theorem 4.2. Recall that \( \{Q_h\} \) is called injective \( [L1] \) if

\[
\lim_{h \to 0} \inf \| Q_h(f) \| > 0
\]

for all \( f \neq 0 \) in \( C_0(X) \). By the results in \( [CH,L1,DL] \), (weakly) injective asymptotic morphisms determine strong deformations from \( X \) to \( B \).

Theorem 5.4.2. Let \( \{A_h\} \) be an injective locally \( B \)-valued Borel ASM on \( X \). Then the associated asymptotic morphism \( \{Q_h\} : C_0(X) \to B \) is injective and so satisfies the continuity condition

\[
\| f \| = \lim_{h \to 0} \| Q_h(f) \|
\]

for all \( f \in C_0(X) \). Hence, there is an associated strong deformation from \( X \) to \( B \).

Proof. Let \( f \neq 0 \) be in \( C_0(X) \). Thus, there is an \( x_0 \in X \) such that \( |f(x_0)| > C > 0 \). Since \( \{Q_h\} \) is positive linear, without loss of generality, we may assume \( f \geq 0 \) and so \( f(x_0) > C > 0 \). Let \( U \subseteq C_X \) be the pre-compact open subset of \( X \) defined by \( U = \{x \in X : f(x) > C\} \). Then \( C_X U \leq f \) and so for all \( h > 0 \) we have that

\[
CA_h(U) = \int U C_X dA_h \leq \int f dA_h = Q_h(f).
\]
which implies that
\[ 0 < |C| \liminf_{h \to 0} \| A_h(U) \| \leq \liminf_{h \to 0} \| Q_h(f) \|. \]

It follows that \( \{Q_h\} : C_0(X) \to B \) is injective and so by Lemma 3 [L1]
\[ \|f\| = \lim_{h \to 0} \|Q_h(f)\| \]
for all \( f \in C_0(X) \). Thus, \( \{Q_h\} \) is the asymptotic morphism associated to a strong deformation from \( X \) to \( B \). □

The continuous sections of the field \( \{A_h\} \) are then determined by the equivalence class \([Q_h]_{a(cp)}\) of the associated asymptotic morphism \( \{Q_h\} : C_0(X) \to B \).

5.5 Wick Quantization on Bosonic Fock Space.

The background material for this section can be found in Guentner[G1]. Let \( \mathcal{H} = L^2(\mathbb{C}, d\mu(z)) \) denote the Hilbert space of measurable complex-valued functions on the complex plane \( X = \mathbb{C} \) which are square-integrable with respect to the normalized Gaussian measure \( d\mu(z) = \pi^{-1} e^{-|z|^2} d\lambda(z) = \pi^{-1} k(z, z) d\lambda(z) \), where \( k(z, w) = e^{z\overline{w}} \) denotes the Bergman kernel and \( d\lambda(z) \) denotes Lebesgue measure. The (bosonic) Fock space is the closed subspace \( \mathcal{F} \subset \mathcal{H} \) consisting of analytic functions. For any bounded Borel function \( f \in B_b(\mathbb{C}) \), the Wick operator \( T_f : \mathcal{F} \to \mathcal{F} \) of \( f \) is the integral operator defined by
\[ T_f(\phi) = \int_{\mathbb{C}} k(z, w) f(w) \phi(w) d\mu(z), \]
for all \( \phi \in \mathcal{F} \).

**Lemma 5.5.1.** For each \( f \in L^2(\mathbb{C}, d\lambda) \cap B_b(\mathbb{C}) \) the operator \( T_f \in \mathcal{K}(\mathcal{F}) \).

**Proof.** Follows from the calculations in the proof of Proposition 3.2 [G1] □

We define the Wick quantization map
\[ Q^W : C_b(\mathbb{C}) \to B(\mathcal{F}) : f \mapsto Q^W(f) = T_f. \]

Let \( P : \mathcal{H} \to \mathcal{F} \) denote the orthogonal projection. We can then define a POVM-measure \( A^W : \Sigma_\mathbb{C} \to B(\mathcal{F}) \) by
\[ A^W(\Delta) = P \circ \chi_\Delta \]
where \( \chi_\Delta \) denotes (the operator on \( \mathcal{H} \) of multiplication by) the characteristic function \( \chi_\Delta \). Note that it is the compression of the PVM \( \Delta \mapsto E(\Delta) = \chi_\Delta \).

**Lemma 5.5.2.** The POVM \( A^W : (\Sigma_\mathbb{C}, C_\mathbb{C}) \to (B(\mathcal{F}), \mathcal{K}(\mathcal{F})) \) is normalized and locally compact-valued. The associated positive linear map is the Wick-Toeplitz quantization
\[ Q^W : (C_b(\mathbb{C}), C_0(\mathbb{C})) \to (B(\mathcal{F}), \mathcal{K}(\mathcal{F})). \]

**Proof.** Follows from the fact that \( A^W(\Delta) = P \circ \chi_\Delta = T_{\chi_\Delta} = Q^W(\chi_\Delta) \). When \( U \in C_\mathbb{C} \) is pre-compact then \( \chi_U \in L^2(\mathbb{C}, d\lambda) \cap B_b(\mathbb{C}) \) and so \( A^W(U) = T_{\chi_U} \in \mathcal{K}(\mathcal{F}) \). Normalization follows from \( A^W(C_0) = P - I \). □
For each $\hbar > 0$ and $f \in B_{\hbar}(\C)$ define

$$\alpha_{\hbar}(f)(z) = f(\hbar z)$$

for all $z \in \C$. We can then define a family of positive linear maps

$$\{Q^W_{\hbar}\} : (C_{\hbar}(\C), C_0(\C)) \to (\mathcal{B}(\mathcal{F}), \mathcal{K}(\mathcal{F})) : f \mapsto Q^W_{\hbar}(f) = Q(\alpha_{\hbar}(f)).$$

Guentner [G1] realized that to obtain an asymptotic morphism from the Wick quantization we need to pass to a unital subalgebra of $C_{\hbar}(\C)$ that still contains $C_0(\C)$ as an ideal.

Let $\delta \C$ denote the compactification of the complex plane $\C$ by the circle at infinity. The continuous functions on $\delta \C$ are “flat at infinity” when restricted to $\C$. Let $C_\delta(\C) = C(\delta \C)$. We then have that $C_0(\C) \triangleleft C_\delta(\C) \subset C_{\hbar}(\C)$. The following result is a consequence of Proposition 4.6 above and Propositions 3.2 and 3.3 [G1].

**Proposition 5.5.4.** The family $\{Q^W_{\hbar}\}$ defines a relative positive asymptotic morphism

$$\{Q^W_{\hbar}\} : (C_\delta(\C), C_0(\C)) \to (\mathcal{B}(\mathcal{F}), \mathcal{K}(\mathcal{F}))$$

whose associated $C_\delta$-asymptotic spectral measure $\{A^W_{\hbar}\}$ is given by

$$A^W_{\hbar}(\Delta) = A^W(h^{-1}\Delta) = P \circ \alpha_{\hbar}(\chi\Delta).$$

The restricted asymptotic morphism $\{Q^W_{\hbar}\} : C_0(\C) \to \mathcal{K}(\mathcal{F})$ determines an $E$-theory class,

$$[A^W_{\hbar}] = \text{def} [Q^W_{\hbar}] \in E(C_0(\C), \mathcal{K}) \cong E(C_0(\C), \C),$$

where we have used the matrix-stability of $E$-theory.

Let $\bar{\partial} = \frac{1}{2}(\frac{\partial}{\partial x} + \sqrt{-1}\frac{\partial}{\partial y})$ be the $\bar{\partial}$-operator on $\C \cong \R^2$, considered as an unbounded elliptic differential operator on the Hilbert space $H = L^2(\C, d\lambda(z))$. The formal adjoint of $\bar{\partial}$ is the operator $-\partial$ where $\partial = \frac{1}{2}(\frac{\partial}{\partial x} - \sqrt{-1}\frac{\partial}{\partial y})$. It follows that the $2 \times 2$ matrix operator

$$D = \begin{pmatrix} 0 & -\partial \\ \bar{\partial} & 0 \end{pmatrix}$$

determines a symmetric unbounded operator on $H \oplus H$ with bounded propagation, and so is (essentially) self-adjoint.

By the results of Guentner [G1,G2] the operator $D$ determines an $E$-theory class denoted $[\bar{\partial}] \in E(C_0(\C), \C)$, which is the homotopy class of the asymptotic morphism determined by the formula

$$C_0(\R) \otimes C_0(\C) \to C_0(\R) \otimes \mathcal{K}(H \oplus H) : f \otimes \phi \mapsto M_{\phi} \circ f(hD + x\epsilon)$$

where $x \in \R$ and $\epsilon$ is the grading operator of the $\Z_2$-graded Hilbert space $H \oplus H$. A direct consequence of Proposition 5.5.4 above, Theorem 4.5 [G1], and the excision property of relative $E$-theory [G2] is that the $E$-theory classes of the Wick ASM above and the $\bar{\partial}$-operator are in fact equal.

**Theorem 5.5.5.** $[A^W_{\hbar}] = [\bar{\partial}] \in E(C_0(\C), \C) \cong \Z$. 

Appendix: POV-Measures and Quantum Mechanics

Let $X$ be a set equipped with a $\sigma$-algebra $\Sigma$ of subsets of $X$. Let $\mathcal{H}$ be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$. A positive operator-valued measure (POVM) on the measurable space $(X, \Sigma)$ is a mapping $A : \Sigma \to \mathcal{B}(\mathcal{H})$ which satisfies the following properties:

- $A(\emptyset) = 0$
- $A(\Delta) \geq 0$ for all $\Delta \in \Sigma$
- $A(\bigsqcup_{1}^{\infty} \Delta_n) = \sum_{1}^{\infty} A(\Delta_n)$ for disjoint measurable sets $\{\Delta_n\}_{1}^{\infty} \subset \Sigma$,

where the sum converges in the weak operator topology [Be,BGL,S]. Note that $0 \leq A(\Delta) \leq A(X) \leq \|A(X)\| < \infty$ for all $\Delta \in \Sigma$. We will say that $A$ is normalized if $A(X) = I_{\mathcal{H}}$. If each $A(\Delta)$ is a projection in $\mathcal{B}(\mathcal{H})$, i.e., $A(\Delta)^2 = A(\Delta)^* = A(\Delta)$, then we call $A$ a projection-valued measure (PVM or spectral measure) on $X$. This is equivalent to the condition that:

\[(A.1) \quad A(\Delta_1 \cap \Delta_2) = A(\Delta_1)A(\Delta_2)\]

Another reason for the importance of these operator-valued measures in quantization is the following generalized Riesz representation theorem for the dual of $C_0(X)$ (Compare Proposition 1.4.8 [L] and Theorem 19 [Be]):

**Theorem A.3.** Let $\Sigma_X$ be the Borel $\sigma$-algebra on the space $X$. There is a one-one correspondence between positive linear maps $Q : C_0(X) \to \mathcal{B}(\mathcal{H})$ and POVM-measures $A : \Sigma_X \to \mathcal{B}(\mathcal{H})$, given by

\[Q(f) = \int_{X} f(x) \, dA(x).\]

The map $Q$ is a general quantization if and only if $A$ is a normalized POVM. Moreover, $Q$ is a $*$-homomorphism if and only if $A$ is a spectral measure (PVM).

The above integral is to be interpreted in the weak sense: For all $v, w \in \mathcal{H}$,

\[\langle Q(f)v, w \rangle = \int_{X} f(x)\langle dA(x)v, w \rangle.\]
The map $Q$ then extends to $Q : B_b(X) \to \mathcal{B} \mathcal{H}$ and satisfies (Theorem 10 [Be]): For all $f \in C_0(X) \subset B_b(X)$,

\[(A.3.1) \quad \|Q(f)\| = \left\| \int_X f(x) \, dA(x) \right\| \leq 2 \|f\| \|A(X)\|.
\]

Thus, spectral measures (PVM’s) correspond to representations of abelian $C^*$-algebras on Hilbert space. The fundamental result in the von Neumann formulation of quantum theory is the following Spectral Theorem of Hilbert.

**Spectral Theorem A.4.** Let $X = \mathbb{R}$. There is a one-one correspondence between Borel spectral measures $A$ on $\mathbb{R}$ and self-adjoint operators $T$ on the associated Hilbert space. This correspondence is given by the formulas:

$$T = \int_{-\infty}^{\infty} \lambda \, dA(\lambda), \quad A(\Delta) = \chi_\Delta(T),$$

where $\chi_\Delta$ denotes the characteristic function of the Borel set $\Delta \subset \mathbb{R}$.

Let $A$ be a normalized POV-measure on the phase space $X$ of a quantum system. The physical interpretation of the map $\Delta \mapsto A(\Delta)$ is the probability that the physical system, in a state represented by a density operator $\rho$, is localized in the subset $\Delta$ of the phase space $X$ is given by the number

$$P_\rho(\Delta) = \text{trace}(\rho \circ A(\Delta)) = \text{trace} \left( \int_\Delta \rho \, dA \right).$$

The mean or vacuum expectation value of a quantum observable $T$ is then computed by the formula

$$\langle T \rangle = \text{trace}(\rho T) = \text{trace} \left( \int_{-\infty}^{\infty} \lambda \rho(\lambda) \, dA(\lambda) \right),$$

where $\rho$ is the (normalized) probability density operator of the physical system.

Note that according to the Naimark Extension Theorem [RS], every POVM $A$ is the compression of a PVM $E$ defined on a minimal extension $\mathcal{H}' \supset \mathcal{H}$. That is, $A(\Delta) = PE(\Delta)P$, where $P : \mathcal{H}' \to \mathcal{H}$ is the orthogonal projection. One could then try to compute the integrals $\int_X f(x) \, dA(x)$ by computing $\int_X f(x) \, dE(x)$ on $\mathcal{H}$ and then projecting back down to $\mathcal{H}$. There are two problems with this [S]. The first is that $\mathcal{H}'$ could have no physical meaning, thus making the analysis unsatisfying to the physicist. Also, the integration process may not commute with the projection process (e.g., when the associated operator is unbounded).
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