Endomorphism Semigroups and Lightlike Translations

D. R. Davidson*

Dipartimento di Matematica
Università di Roma “La Sapienza” 00185 Roma, Italy
Email: davidson@mat.uniroma1.it

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Abstract

Borchers and Wiesbrock have studied the one-parameter semigroups of endomorphisms of von Neumann algebras that appear as lightlike translations in the theory of algebras of local observables, showing that they automatically transform under the appropriate modular automorphisms as under velocity transformations. These results are here abstracted and analyzed as essentially operator-theoretic. Criteria are then established for a spatial derivation of a von Neumann algebra to generate a one-parameter semigroup of endomorphisms, and all of this is combined to establish a von Neumann-algebraic converse to the Borchers and Wiesbrock results. This sort of analysis is then applied to questions of isotony and covariance for local algebras, to show that Poincaré covariance together with a domain condition for the translations can imply isotony.

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I Introduction

The standard situation for a pair of complementary spacetime regions in the theory of algebras of local observables, under the assumption of duality in the vacuum sector, is just that termed standard in the theory of von Neumann algebras: we have a von Neumann algebra $\mathcal{M}$ and its commutant $\mathcal{M}'$ acting on a Hilbert space $\mathcal{H}$, with a common cyclic and separating unit vector $\Omega$, the vacuum vector. In the particular situation in which $\mathcal{M}$ and $\mathcal{M}'$ correspond to the algebras of observables for a pair of complementary wedge regions (for definiteness let us take them to be $W_R = \{ x \mid x_1 > |t| \}$ and $W_L = \{ x \mid x_1 < -|t| \}$ respectively) it is expected that the modular automorphism group $\sigma_t(A) = \Delta^{it} A \Delta^{-it}$ will correspond to the Lorentz velocity transformations $V_1(2\pi t)$ in the direction $\hat{x}_1$ orthogonal to the vertex $x_1 = t = 0$ of the wedges, and that the modular conjugation $J$ will be a slight variant of the TCP operator $\Theta$, namely $J = \Theta R$ where $R$ is a rotation by the angle $\pi$ about the direction $\hat{x}_1$. In that case the lightlike translations $U(a) = T(a(\hat{x}_0 + \hat{x}_1))$ will be a strongly continuous one-parameter group of unitary operators on $\mathcal{H}$, which should have the following four properties:

(a) By Lorentz and TCP covariance, $\Delta^{it} U(a) \Delta^{-it} = U(e^{-2\pi t} a)$ and $J U(a) J = U(-a)$ for all real $a$ and $t$;

(b) By the spectral condition, $U(a)$ should have a positive generator $H$;

(c) By isotony, for $a \geq 0$ the corresponding adjoint action $A \to U(a) A U(-a)$ should give a one-parameter semigroup of endomorphisms of $\mathcal{M}$ (and thus for $a \leq 0$ likewise of $\mathcal{M}'$); and, finally,

(d) The vacuum vector $\Omega$ should be fixed by all $U(a)$, and thus annihilated by $H$.

In this connection Borchers has shown that these four conditions are not all independent: in particular, if the last three hold, then the covariance conditions follow automatically \[2\]. Wiesbrock then proved conversely that if (a), (c), and (d) hold, $U(a)$ automatically has a positive generator \[3\]. In this note we first analyze the results of Borchers and of Wiesbrock, showing that they are essentially operator-theoretic statements about relations between $J$, $\Delta$, and $H$; we then demonstrate that they are part of a larger chain of converses, in which we separate out the von Neumann-algebraic
content, and in the process perhaps shed some further light on these remarkable theorems. Specifically, we show that if the generator \( H \) gives a derivation \( \delta \) of \( \mathcal{M} \) satisfying certain additional conditions, then any three of the conditions listed above for \( U(a) \) together imply the fourth. Note that in the local algebra context, it can be shown that \( \mathcal{M} \) and \( \mathcal{M}' \) must be Type III\(_1\) factors [4], but this will not be used in the following; the results will simply be stated in terms of arbitrary von Neumann algebras. We then proceed to apply these methods to questions of isotony and covariance in the context of the theory of local algebras.

The situation here is analogous to, but in some respects altogether different from, the case of spatial derivations that generate automorphism groups of von Neumann algebras, which has been extensively studied ([5], Section 3.2.5, and references therein). We will develop the analogy more specifically in Section III, but the relevant condition in the automorphism case is that \( U(a) \) should commute with \( J \) and with all \( \Delta^\omega \); then the key question is to determine precisely what additional conditions on a derivation \( \delta \) suffice to show that it generates an automorphism group. The best result in this direction is that of [6], in which the only additional assumption is that the derivation has a domain \( D(\delta) \) such that \( D(\delta)\Omega \) is a core for \( H \). However, the proof of this result is rather difficult, and does not generalize to the endomorphism case. We will generally make do with more restrictive conditions here, but it would be interesting to determine precisely what conditions suffice to guarantee that \( \delta \) generates an endomorphism semigroup; in Section V we discuss a case in which a condition like that of [6] suffices.

In what follows, Section II presents the results of Borchers and Wiesbrock, simplified and reduced to their operator-theoretic essence; Section III discusses criteria for a spatial derivation to generate a one-parameter semigroup of endomorphisms; Section IV collects the resulting information about lightlike translations; and Section V discusses applications of this analysis to certain systems of local algebras, showing that isotony relations hold given only covariance and a core condition as above for the translations.
II Commutation Relations

The following is of course based heavily on [2], with contributions from [3] and [7]. We present these results in what would seem to be their natural setting: that of commutation relations for one-parameter unitary groups, and their unbounded generators.

**Theorem 1:** Let \( V(\lambda) = \Delta^{i\lambda/2\pi} \) and \( U(a) = e^{iaH} \) be two strongly continuous one-parameter unitary groups. Then any two of the following conditions imply the third:

(a) \( V(\lambda)U(a)V(-\lambda) = U(e^{-\lambda}a) \) for all real \( a \) and \( \lambda \);

(b) \( H \) is positive;

(c) \( \Delta^{1/2}U(a) \supset U(-a)\Delta^{1/2} \) for all \( a \geq 0 \).

**Proof of Theorem 1:** Here \( \Delta \) and \( H \) are unbounded operators with their natural domains of definition, \( \Delta \) positive and \( H \) self-adjoint by Stone’s Theorem; there is therefore a dense set \( D_\omega \) of vectors \( \psi \) for which \( \Delta z^\psi = V(2\pi z)\psi \) is entire analytic. Let us take two fixed vectors \( \psi, \phi \in D_\omega \), and define two functions

\[
F(z, w) = \langle \Delta^{iz} \psi \mid U(e^{2\pi w})\Delta^{iz} \phi \rangle \quad \text{and} \quad G(z, w) = \langle \Delta^{iz} \psi \mid U(-e^{2\pi w})\Delta^{iz} \phi \rangle, \tag{1}
\]

both entire analytic functions of \( z \) for real \( w \). If \( H \) is positive, then \( F \) and \( G \) will have jointly analytic continuations, continuous at the boundary, to \( 0 \leq \text{Im} \, w \leq 1/2 \) and \(-1/2 \leq \text{Im} \, w \leq 0 \) respectively, satisfying

\[
|F(z, w)| \leq \|\Delta^{\text{Im} \, z} \psi\| \|\Delta^{-\text{Im} \, z} \phi\| \quad \text{and} \quad |G(z, w)| \leq \|\Delta^{\text{Im} \, z} \psi\| \|\Delta^{-\text{Im} \, z} \phi\| \tag{2}
\]

independent of \( w \) and \( \text{Re} \, z \) over the whole region of definition; these bounds hold for \( \text{Im} \, w = 0 \) whether \( H \) is positive or not.

If (a) and (b) hold, then \( F \) satisfies the complex identity \( F(z, w) = F(0, z + w) \). Taking \( w \) real, \( z = i/2 \), this implies that \( \langle \psi \mid U(-a)\phi \rangle = \langle \Delta^{1/2} \psi \mid U(a)\Delta^{-1/2} \phi \rangle \) for all \( a \geq 0 \). Since \( \phi' = \Delta^{-1/2} \phi \in D_\omega \) if and only if \( \phi \in D_\omega \), we may equally write

\[
\langle \psi \mid U(-a)\Delta^{1/2} \phi' \rangle = \langle \Delta^{1/2} \psi \mid U(a)\phi' \rangle. \tag{3}
\]

Since \( \psi \) may be any vector in a dense set, this in fact holds for any \( \psi \in D(\Delta^{1/2}) \). The right-hand side is a bounded function of \( \phi' \), so that the left-hand side is also; this
implies that \( U(a)\psi \in D(\Delta^{1/2}) \), and we may write
\[
\langle \Delta^{1/2}U(a)\psi \mid \phi' \rangle = \langle U(-a)\Delta^{1/2}\psi \mid \phi' \rangle .
\] (4)

Since \( \phi' \) may be any vector in a dense set, \( \Delta^{1/2}U(a)\psi = U(-a)\Delta^{1/2}\psi \) for every \( \psi \in D(\Delta^{1/2}) \) and any \( a \geq 0 \); but this is just the statement of (c).

On the other hand, if we are given (c), then we have for real \( t, s \)
\[
F(t + i/2, s) = \langle \Delta^{1/2}\Delta^{it}\psi \mid U(e^{2\pi s})\Delta^{-1/2}\Delta^{it}\phi \rangle
= \langle \psi \mid \Delta^{-it}U(-e^{2\pi s})\Delta^{1/2}\Delta^{-1/2}\Delta^{it}\phi \rangle = G(t, s).
\] (5)

Not only is \(|F(t, s)| \leq ||\psi|| ||\phi|| \) for real \( t, s \), but also \(|F(t + i/2, s)| \leq ||\psi|| ||\phi|| \). In addition, the bound given above for \( \text{Im} \, w = 0 \), depending only on \( \text{Im} \, z \), implies that \( F(z, w) \) is bounded on the strip \( 0 \leq \text{Im} \, z \leq 1/2 \), \( \text{Im} \, w = 0 \); then by Hadamard’s three-line theorem, \(|F(z, w)| \leq ||\psi|| ||\phi|| \) on this strip. If in addition (a) holds, then the identity \( F(z, w) = F(z+w, 0) \) shows that \(|F(z, w)| \leq ||\psi|| ||\phi|| \) for \( 0 \leq \text{Im} \, z + \text{Im} \, w \leq 1/2 \). Thus \( |\langle \psi \mid e^{-H}\phi \rangle| = |F(0, i/4)| \leq ||\psi|| ||\phi|| \); but \( \psi, \phi \) may be any vectors in a dense set, so \( ||e^{-H}|| \leq 1 \) and \( H \) is positive.

If instead (b) and (c) hold, then also \( F(t, s + i/2) = G(t, s) \) for real \( t, s \). From
\[
F(t, s + i/2) = G(t, s) = F(t + i/2, s)
\] (6)

it follows that \( F(z, w) \) and \( G(z-i/2, w) \) have a common jointly entire analytic continuation \( \tilde{F}(z, w) \) satisfying \( \tilde{F}(z, w+i/2) = \tilde{F}(z+i/2, w) \) for all \( z, w \). Consider \( \tilde{F}(t+\zeta, s-\zeta) \) as a function of \( \zeta \): it is periodic with period \( i/2 \), and its modulus has an upper bound depending only on \( \text{Im} \, \zeta \); therefore it is bounded, hence constant. It follows that \( \tilde{F}(z, w + \zeta) = \tilde{F}(z + \zeta, w) \) for all complex \( \zeta \). In particular, \( F(t, s) = F(0, t + s) \), so that \( \langle \psi \mid V(\lambda)U(a)V(-\lambda)\phi \rangle = \langle \psi \mid U(e^{-\lambda}a)\phi \rangle \) for all real \( \lambda \) and all \( a \geq 0 \). Since \( \psi, \phi \) may be any vectors in a dense set, \( V(\lambda)U(a)V(-\lambda) = U(e^{-\lambda}a) \) for all real \( \lambda \) and all \( a \geq 0 \), and taking adjoints we obtain the result for all real \( a \).

Examining the proof, we see that (c) is in fact equivalent to a number of other statements, a suitably weak one being for example that for all \( a \geq 0 \), \( \Delta^{1/2}U(a) \) is equal to \( U(-a)\Delta^{1/2} \) in the sense of quadratic forms, with form domain \( D_\omega \).
The conditions of Theorem 1 are trivially satisfied if $H = 0$ no matter what $V(\lambda)$ may be; furthermore, if $\psi$ is a vector invariant under all $U(a)$, then so also is $V(\lambda)\psi$ for any $\lambda$. The Hilbert space can thus be decomposed into a direct sum of the subspace of such vectors, and its orthogonal complement, on which $V(\lambda)$ and $U(a)$ are non-trivial. Then the following is a simple computation:

**Proposition 2:** Given a representation of the relations of Theorem 1, if $H$ is restricted to the orthogonal complement of its null space, then it has a self-adjoint logarithm $P$, such that $H$ so restricted equals $T(i)$ where $T(x) = e^{-ixP}$. This satisfies

$$V(\lambda)T(x) = e^{i\lambda x}T(x)V(\lambda)$$

(7)

for all real $x$ and $\lambda$, so that $V(\lambda)$ and $T(x)$ give a representation of the canonical commutation relations in Weyl form. Conversely, any representation of these commutation relations gives a representation of the relations of Theorem 1 in this manner.

Thus the Stone-von Neumann classification of representations of the canonical commutation relations provides a classification of representations of the relations in Theorem 1: up to multiplicity, the only representations are either trivial (with $V(\lambda)$ arbitrary), or else given by $\Delta = e^{2\pi X}$ and $H = e^P$, where $X$ and $P$ have the familiar Schrödinger form for the commutation relations $[X, P] = i$.

To adapt these results to modular theory, we need to extend them slightly so as to include the modular conjugation.

**Theorem 3:** In addition to the premises of Theorem 1, let $J$ be a complex conjugation (an antiunitary involution) commuting with all $V(\lambda)$. Then any two of the following conditions imply the third:

(a') $V(\lambda)U(a)V(-\lambda) = U(e^{-\lambda}a)$ and $JU(a)J = U(-a)$ for all real $a$ and $\lambda$;

(b) $H$ is positive;

(c') $J\Delta^{1/2}U(a) \supset U(a)J\Delta^{1/2}$ for all $a \geq 0$.

To show (b) it is not necessary to assume that $JU(a)J = U(-a)$. 
Proof of Theorem 3: The proof is quite similar to that of Theorem 1. If \((a')\) and \((b)\) hold, then by Theorem 1, we have \(J\Delta^{1/2}U(a) \supset JU(-a)\Delta^{1/2} = U(a)J\Delta^{1/2}\) for all \(a \geq 0\), and we are finished immediately. For the remaining parts we define two additional functions with properties like those of \(G\) and \(F\),

\[
H(z, w) = \langle \Delta^2 \psi | JU(e^{2\pi w})J\Delta \Delta^2 \phi \rangle \quad \text{and} \quad K(z, w) = \langle \Delta^2 \psi | JU(-e^{2\pi w})J\Delta \Delta^2 \phi \rangle . \tag{8}
\]

If \((c')\) holds, then we also have the adjoint statement \(\Delta^{1/2}JU(-a) \supset U(-a)\Delta^{1/2}J\) for all \(a \geq 0\). Instead of one identity, we now have two:

\[
F(t + i/2, s) = \langle \Delta^{1/2}J^2 \Delta \Delta^2 \psi | U(e^{2\pi s})\Delta^{-1/2} \Delta \Delta^2 \phi \rangle
\]

\[
= \langle \psi | \Delta^{-1/2}JU(e^{2\pi s})J\Delta^{1/2} \Delta^{-1/2} \Delta \Delta^2 \phi \rangle = H(t, s); \tag{9}
\]

\[
K(t + i/2, s) = \langle \Delta^{1/2} \Delta \Delta^2 \psi | JU(-e^{2\pi s})J\Delta^{-1/2} \Delta \Delta^2 \phi \rangle
\]

\[
= \langle \psi | \Delta^{-1/2}U(-e^{2\pi s})\Delta^{1/2} \Delta^{-1/2} \Delta \Delta^2 \phi \rangle = G(t, s). \tag{10}
\]

If in addition \((a)\) of Theorem 1 holds, then the argument of Theorem 1 applies (with the substitution of \(H(z, w)\) for \(G(z, w)\)) to show that \(H\) is positive.

If instead \((b)\) and \((c')\) hold, then the analytic continuation argument is only slightly more complicated. We have already \(F(t + i/2, s) = H(t, s)\) and \(K(t + i/2, s) = G(t, s)\); in addition now \(F(t, s + i/2) = G(t, s)\) and \(K(t, s + i/2) = H(t, s)\). It follows that \(F(z, w), G(z, w - i/2), H(z - i/2, w)\) and \(K(z - i/2, w + i/2)\) have a common jointly entire analytic continuation \(\tilde{F}(z, w)\) satisfying \(\tilde{F}(z, w + i) = \tilde{F}(z + i, w)\) for all \(z, w\).

For real \(t, s\), the function \(\tilde{F}(t + \zeta, s - \zeta)\) of \(\zeta\) again has a bound independent of \(\text{Re} \zeta\), and now it is periodic with period \(i\); therefore it is again bounded and constant. Thus \(\tilde{F}(z, w + \zeta) = \tilde{F}(z + \zeta, w)\) for all complex \(\zeta\), and as in Theorem 1, \(V(\lambda)U(a)V(-\lambda) = U(e^{-\lambda})\) for all real \(\lambda\) and \(a\). Furthermore \(K(t, s) = F(t, s)\), so that \(\langle \psi | JU(-a)J\phi \rangle = \langle \psi | U(a)\phi \rangle\) for all \(a \geq 0\). Since \(\psi, \phi\) may be any vectors in a dense set, \(JU(-a)J = U(a)\) for all \(a \geq 0\), and taking adjoints we obtain the result for all real \(a\).

Proposition 2 still holds for the representations of the relations of Theorem 3, but now \(J\) is a complex conjugation such that \(JV(\lambda)J = V(\lambda)\) and \(JT(x)J = T(-x)\). Thus in an irreducible Schrödinger representation \(J\) is the complex conjugation in \(P\)-space (up to multiplication or conjugation by a complex phase).
III Endomorphism Semigroups

If we have a von Neumann algebra $\mathcal{M}$ and its commutant $\mathcal{M}'$ acting on a Hilbert space $\mathcal{H}$, with a common cyclic and separating vector $\Omega$, we may define real linear spaces $R = M_{sa}\Omega$ and $R' = M'_{sa}\Omega$. Then $\langle \psi | \phi \rangle$ is real for all $\psi \in R, \phi \in R'$, and furthermore $R'$ is precisely the set of all $\psi$ such that $\langle \psi | \phi \rangle$ is real for all $\phi \in R$. Also, both $D(\Delta^{1/2}) = R + iR$ and $D(\Delta^{-1/2}) = R' + iR'$ are dense in $\mathcal{H}$, and we have $R = \{ \psi \in D(\Delta^{1/2}), J\Delta^{1/2}\psi = \psi \}$ and $R' = \{ \psi \in D(\Delta^{-1/2}), J\Delta^{-1/2}\psi = \psi \}$. Thus the condition $J\Delta^{1/2}U(a) \supset U(a)J\Delta^{1/2}$ corresponds to $U(a)R \subset R$.

For any $\psi \in R$, there is a sequence $X_n \in M_{sa}$ such that $X_n\Omega \to \psi$, but there need not be a bounded operator $X \in M_{sa}$ such that $X\Omega = \psi$; in general there is only a closed symmetric operator $\tilde{X}$ affiliated with $\mathcal{M}$ such that $\tilde{X}\Omega = \psi$, to which the $X_n$ converge on the common core $\mathcal{M}'\Omega$, so that $\tilde{X}Y\Omega = Y\psi$ for every $Y \in \mathcal{M}'$.

If we are to have $U(a)\mathcal{M}U(-a) \subset \mathcal{M}$ for all $a \geq 0$, then we must have $U(a)R \subset R$ for all $a \geq 0$. In addition, the generator $H$ of the unitary group $U(a)$ must give a derivation $\delta$ of $\mathcal{M}$ by $\delta(X) = i[H,X]$; however, this derivation will be unbounded, hence defined only on a dense set, and the problem is to give sufficient conditions for $\delta$ to generate a semigroup of endomorphisms of $\mathcal{M}$.

In the automorphism case, the relevant result of [6] can be expressed as follows:

**Theorem 4:** Suppose that $U(a)\Omega = \Omega$ and $U(a)R = R$ for all real $a$, and that the set $D(\delta) = \{ X | X \in \mathcal{M}, i[H,X] \in \mathcal{M} \}$ is such that $D(\delta)\Omega$ is a core for $H$. Then $U(a)\mathcal{M}U(-a) = \mathcal{M}$ for all real $a$.

The replacement for $U(a)R = R$ in the endomorphism case is clearly $U(a)R \subset R$ for all $a \geq 0$; we must also find a sufficient replacement for the core condition. Let

$$\mathcal{M}_\epsilon = \{ X | U(a)XU(-a) \in \mathcal{M} \text{ for all } 0 \leq a \leq \epsilon \},$$

and let $R_\epsilon = M_{sa}\epsilon\Omega$; then $\mathcal{M}_\epsilon \supset \mathcal{M}_{\epsilon'}$ and $R_\epsilon \supset R_{\epsilon'}$ whenever $\epsilon' \geq \epsilon$. In addition, let

$$\mathcal{M}_+ = \bigcup_{\epsilon > 0} \mathcal{M}_\epsilon \quad \text{and} \quad R_+ = \bigcup_{\epsilon > 0} R_\epsilon.$$
Then $\mathcal{M}_\epsilon$ contains those elements $X$ of $\mathcal{M}$ for which the differential equation $X(t)' = \delta(X(t))$, $X(0) = X$ in the Banach space $\mathcal{M}$ has a solution curve of length at least $\epsilon$; likewise, $\mathcal{M}_+$ contains those for which there is a solution curve of any positive length. Conditions on $\mathcal{M}_\epsilon$ and $\mathcal{M}_+$ can thus be regarded as local existence conditions for this differential equation; we will use criteria of this sort to control the behavior of $\delta$.

**Theorem 5:** With the above notation and assumptions, suppose that $U(a)\Omega = \Omega$ and $U(a)R \subset R$ for all $a \geq 0$, and that for some $\epsilon > 0$, $\Omega$ is cyclic for $\mathcal{M}_\epsilon$, i.e., $R_\epsilon + iR_\epsilon$ is dense in $\mathcal{H}$. Then $U(a)\mathcal{M}U(-a) \subset \mathcal{M}$ for all $a \geq 0$.

**Proof of Theorem 5:** It will suffice to show that $U(a)\mathcal{M}'U(-a) \supset \mathcal{M}'$ for all $a \geq 0$; we have from our assumptions that $U(a)R' \supset R'$ for all $a \geq 0$. Let us pick $a$ such that $0 \leq a \leq \epsilon$, so that $\mathcal{M}_\epsilon \subset \mathcal{M} \cap U(-a)\mathcal{M}U(a)$. Let $Y$ be a self-adjoint element of $\mathcal{M}'$; then $Y\Omega \in R' \subset U(a)R'$, so that $U(-a)Y\Omega \in R'$. Then there is a closed symmetric operator $\tilde{Y}$ affiliated with $\mathcal{M}'$ such that $\tilde{Y}\Omega = U(-a)Y\Omega$, defined on the core $\mathcal{M}\Omega$ by $\tilde{Y}X\Omega = Xu(-a)Y\Omega$ for every $X \in \mathcal{M}$. $\tilde{Y}$ therefore agrees with the bounded operator $U(-a)YU(a)$ on the dense set $\mathcal{M}_\epsilon\Omega$, from which it follows that $\tilde{Y}$ is in fact bounded and equal to $U(-a)YU(a)$, so that $Y \in U(a)\mathcal{M}'U(-a)$ and $U(a)\mathcal{M}U(-a) \supset \mathcal{M}'$. This is so for all $0 \leq a \leq \epsilon$, hence by the semigroup property for all $a \geq 0$.

**Theorem 6:** With the above notation and assumptions, suppose that $U(a)\Omega = \Omega$ and $U(a)R \subset R$ for all $a \geq 0$, that $\Delta^iU(a)\Delta^{-i} = U(e^{-2\pi t}a)$ for all real $a,t$, and that $\Omega$ is cyclic for $\mathcal{M}_+$, i.e., $R_+ + iR_+$ is dense in $\mathcal{H}$. Then $U(a)\mathcal{M}U(-a) \subset \mathcal{M}$ for all $a \geq 0$.

**Proof of Theorem 6:** Notice that $\Delta^iR_\epsilon = R_{e^{-2\pi t}\epsilon}$, so for any $\epsilon > 0$, we have $R_+ = \cup_{t \geq 0}\{\Delta^iR_\epsilon\}$. By assumption, for any $\psi \in \mathcal{H}$, there is some $\epsilon > 0$ and some $\phi \in R_\epsilon + iR_\epsilon$ such that $\langle \psi, \phi \rangle \neq 0$. Thus given any $\epsilon > 0$, there is some $\phi \in R_\epsilon + iR_\epsilon$ and some $t \geq 0$ such that $\langle \psi, \Delta^i\phi \rangle \neq 0$. But $\phi \in R + iR = D(\Delta^{1/2})$, so that $\phi$ is an analytic vector for $\Delta^i$ in the strip $-1/2 \leq \Im t \leq 0$. Thus $\langle \psi, \Delta^i\phi \rangle$ is the boundary value of a function analytic in $t$ on that strip, and cannot vanish for all $t \leq 0$. So $R_\epsilon + iR_\epsilon = \cup_{t \leq 0}\{\Delta^i(R_\epsilon + iR_\epsilon)\}$ is dense in $\mathcal{H}$ already, and Theorem 5 applies.
The condition that \( R_+ + iR_+ \) be dense will be referred to as the local existence condition of Theorem 6; the condition of Theorem 5 is a uniform version of it. In specific cases, for example those involving perturbations of known endomorphism semigroups, we might hope to establish local existence conditions of these sorts by means of fixed point theorems and other standard methods for differential equations.

At this point, it seems worthwhile to present the motivating example for this discussion, in the simple form of a massive scalar free field in 1+1 spacetime dimensions. Let \( \mathfrak{h} \) be a Hilbert space, the one-particle space, and let \( \mathcal{H} = \exp(\mathfrak{h}) \) be a symmetric Fock space constructed over it, whose \( n \)-particle subspace \( \mathcal{H}^n \) is the \( n \)-fold symmetric tensor product of \( \mathfrak{h} \) with itself; \( \mathcal{H}^0 \) is a one-dimensional Hilbert space, identified with the complex multiples of the vacuum \( \Omega \). The vectors of \( \mathcal{H} \) we will index by the exponential map for vectors

\[
\exp(f) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} f^{(n)} \quad \text{for } f \in \mathfrak{h},
\]

where \( f^{(n)} \in \mathcal{H}^n \) is the \( n \)-fold tensor product of \( f \) with itself, so that \( \langle \exp(f) | \exp(g) \rangle = \exp(\langle f | g \rangle) \); then for any \( f \in \mathfrak{h} \) we can define the unitary Weyl operator \( w(f) \) by

\[
w(f) \exp(g) = e^{-\frac{1}{2} \|f\|^2 - \langle f | g \rangle} \exp(f + g).
\]

If \( u \) is a unitary operator on \( \mathfrak{h} \), then its multiplicative second quantization \( U \) given by \( U \exp(f) = \exp(uf) \) will be a unitary operator on \( \mathcal{H} \); if \( a \) is a self-adjoint operator on \( \mathfrak{h} \), then its additive second quantization \( A \), the generator of the multiplicative second quantization of \( u(t) = e^{ita} \), will be a self-adjoint operator on \( \mathcal{H} \). The additive second quantization of the identity is an operator \( N \), the number operator, which has the eigenvalue \( n \) on \( \mathcal{H}^n \). Then for any \( f \in \mathfrak{h} \), \( D_S = \cap_{n=1}^\infty D(N^n) \) will be a core for the generator \( \phi(f) \) of \( w(tf) \), such that \( \phi(f)D_S \subset D_S \).

Let \( \mathfrak{h} \), realized as \( \mathfrak{h} = L^2(\mathbb{R}, d\kappa) \) with \( \nu = -i\partial_\kappa \), carry the simplest non-trivial representation of the relations of Theorem 3, realized by \( \delta^it = e^{2\pi i\kappa} \), \( u(a) = e^{i\kappa a} \), and \( jf(\kappa) = \bar{f}(-\kappa) \). The multiplicative second quantizations \( \Delta^it, U(a) \), and \( J \) of \( \delta^it, u(a) \), and \( j \) respectively also satisfy the relations of Theorem 3, with a one-dimensional trivial space (the vacuum space \( \mathcal{H}^0 \)) and a non-trivial representation of infinite multiplicity.
Then we will let \( M \) (intended to correspond to the right wedge \( W_R \)) be the von Neumann algebra generated by \( w(f) \) for all \( f \) in the real linear space
\[
 r = \left\{ f \mid j\delta^{1/2} f = f \right\} = \left\{ f(\kappa) = g(\kappa) + e^{-\pi\kappa \hat{g}(-\kappa)} \mid g \in D(e^{\pi\kappa}) \right\}.
\]
It can be shown \[8\] that \( M' \) is generated by \( w(f) \) for all \( f \in r' \), where
\[
 r' = \left\{ f \mid j\delta^{-1/2} f = f \right\} = \left\{ f(\kappa) = g(\kappa) + e^{\pi\kappa \hat{g}(\kappa)} \mid g \in D(e^{-\pi\kappa}) \right\},
\]
and that \( J \) and \( \Delta^2 \) give the modular conjugation and automorphisms for \( M \) with respect to \( \Omega \). Then the unbounded operators \( \phi(f) \) for \( f \in r \) will be self-adjoint and affiliated with \( M \), and in fact will generate \( M \). It is easy to see that \( u(a)r \subset r \) for \( a \geq 0 \), and thus that \( U(a)M \subset M \) for \( a \geq 0 \). Let us ignore this for the moment, however, and proceed to apply Theorems 5 and 6.

Clearly \( h_{\lambda,\rho} = \lambda e^\nu + \rho e^{-\nu} \) is an unbounded self-adjoint operator on \( h \) for each \( (\lambda, \rho) \in \mathbb{R}^2 \setminus \{ (0, 0) \} \); we may then define the self-adjoint operator \( H_{\lambda,\rho} \) as the additive second quantization of \( h_{\lambda,\rho} \), or alternatively by \( i[H_{\lambda,\rho}, \phi(f)] = \phi(ih_{\lambda,\rho}f) \). Then let
\[
 r_1 = \left\{ f(\kappa) = (\hat{g}(\sinh \nu) + i \cosh \nu \hat{h}(\sinh \nu))^{-1} \mid g, h \text{ real and supported in } [1, \infty) \right\},
\]
where \(^{-1}\) and \(^{-\dagger}\) represent direct and inverse Fourier transforms. It can be shown that for every \( (\lambda, \rho) \in \mathbb{R}^2 \), there is some \( \epsilon \) such that for every \( f \in r_1 \), \( \phi(f) \) is affiliated with \( M_\epsilon \) with respect to \( H_{\lambda,\rho} \). Furthermore \( r_1 + ir_1 \) is dense in \( h \). It follows that for every \( H_{\lambda,\rho} \), the local existence condition of Theorem 6 and the uniform local existence condition of Theorem 5 both hold.

Then for \( (\lambda, \rho) \in \mathbb{R}^2 \setminus (0, 0) \), we have the following:

(i) \( H_{\lambda,\rho} \) is positive if and only if \( \lambda \) and \( \rho \) are both non-negative;
(ii) \( \Delta^2 H_{\lambda,\rho} \Delta^{-2} = H_{e^{-2\pi\nu}, e^{2\pi\nu}} \), and \( JH_{\lambda,\rho}J = H_{-\lambda, -\rho} \);
(iii) \( H_{\lambda,\rho} \) generates a one-parameter semigroup of endomorphisms of \( M \) if and only if \( \lambda \) and \( -\rho \) are both non-negative; and
(iv) \( H_{\lambda,\rho} \) generates a one-parameter semigroup of endomorphisms of \( M' \) if and only if \( -\lambda \) and \( \rho \) are both non-negative.

For mass \( m \), \( H_{m/2,m/2} \) is the Hamiltonian, \( H_{m/2,-m/2} \) the momentum; \( U(a) = e^{iaH_{m,0}} \).
Of course, this is a very simple example, in which it is easy to compute the effects of the $U(a)$. In more complicated cases, Theorems 5 and 6 could perhaps be applied to greater effect. However, their conditions may well be more restrictive than is necessary; one might conjecture that the local existence conditions could be replaced by conditions purely on $D(\delta)$—for example, as in [3], by the condition that $D(\delta)\Omega$ be a core for $H$.

IV Lightlike Translations

Let us return to the situation described in the introduction, and consider again the conditions (a)–(d). We know already that (a) and (b) each follow from the remaining three conditions; we have now to consider (c) and (d). One branch is available immediately: suppose that (a) is satisfied, but $U(a)\Omega$ is not known. Then

$$\langle \Omega | U(a)\Omega \rangle = \langle \Omega | \Delta^itU(a)\Delta^{-it}\Omega \rangle = \langle \Omega | U(e^{-2\pi t}a)\Omega \rangle$$

(18)

is independent of $t$, and hence must be a constant for all $a > 0$ and for all $a < 0$. Taking the limit as $t \to \infty$, these constants must both be 1; but since $U(a)\Omega$ is a unit vector, it must therefore equal $\Omega$ for all $a$. Thus (a) alone implies (d). With this out of the way, we proceed to combine the results of Sections II and III:

**Theorem 7:** If $H$, the generator of $U(a)$, is positive and annihilates the vacuum, and if the local existence condition of Theorem 6 holds, then the isotony relation $U(a)M U(-a) \subset M$ for all $a \geq 0$ (and thus $U(a)M' U(-a) \subset M'$ for all $a \leq 0$) holds if and only if the covariance relations hold in the form

$$\Delta^itU(a)\Delta^{-it} = U(e^{-2\pi t}a) \quad \text{and} \quad J U(a)J = U(-a)$$

(19)

for all real $a$ and $t$. Likewise the covariance and isotony relations together imply the positivity of $H$.

**Remarks:** Although this is in some respects similar to the case of automorphisms, there are a number of significant differences. For example, if $H$ were positive in the automorphism case, then it would be affiliated with $M$, and since it annihilates $\Omega$, it would have to vanish; here, however, $H$ can be positive and non-trivial.
Proof of Theorem 7: Theorem 6 allows us to reduce this to a question about the relations between $U(a)$ and $R$: it will suffice for the first part of the theorem to show that $U(a)R \subset R$ for all $a \geq 0$ if and only if covariance holds. But this follows from Theorem 3, since $(a')$ of Theorem 3 corresponds to the covariance relations, and $(c')$ of Theorem 3 is equivalent to the condition that $U(a)R \subset R$ for all $a \geq 0$. The second part of the theorem also follows from Theorem 3, without the aid of Theorem 6.

Corresponding results for the backward lightlike translations $W(a) = T(a(\hat{x}_1 - \hat{x}_0))$ can be derived by exchanging $\mathcal{M}$ and $\mathcal{M}'$, and replacing $a$ by $-a$ in the above. $W(a)$ should have a negative generator, and should satisfy Lorentz and TCP covariance in the form $\Delta^itW(a)\Delta^{-it} = W(e^{2\pi t}a)$ and $JW(a)J = W(-a)$. With these substitutions, the corresponding theorem obtains. The situation for the intermediate case, the spacelike translations $T(a\hat{x})$ taking $W_R$ into itself, is somewhat more complicated, although not essentially different: Theorem 5 still holds, but now the relations between the generator (which now in some frame of reference is the momentum) and the modular operators are no longer so simple. We must just show that $J\Delta^{1/2}T(a\hat{x}) \supset T(a\hat{x})J\Delta^{1/2}$ for all $a \geq 0$. For example, if $U(a)$ satisfies the conditions of Theorem 7, and $W(a)$ the corresponding requirements for a backward lightlike translation, and if $U(a)$ and $W(b)$ commute for all $a$ and $b$, then $U(\lambda a)W(-\rho a)$ gives an endomorphism semigroup of this intermediate type for any $\lambda, -\rho > 0$. This is just the situation described in the example at the end of Section III. Combining Theorem 7 with these remarks, we have the following omnibus theorem, as advertised:

Theorem 8: Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$, which with its commutant $\mathcal{M}'$ has a separating and cyclic vector $\Omega$. Given a strongly continuous one-parameter group $U(a)$ of unitary operators on $\mathcal{H}$, for which the local existence condition of Theorem 6 holds, any three of the following conditions imply the fourth:

(a) $\Delta^itU(a)\Delta^{-it} = U(e^{-2\pi t}a)$ and $JU(a)J = U(-a)$ for all real $a$ and $t$;
(b) the generator $H$ of the $U(a)$ is positive;
(c) $U(a)\mathcal{M}U(-a) \subset \mathcal{M}$ for all $a \geq 0$;
(d) $U(a)\Omega = \Omega$ for all $a$. 

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Likewise, any three of the following conditions imply the fourth:

(a') $\Delta^t U(a) \Delta^{-t} = U(e^{2\pi t} a)$ and $JU(a)J = U(-a)$ for all real $a$ and $t$;
(b') the generator $H$ of the $U(a)$ is negative;
(c) $U(a) \mathcal{M} U(-a) \subset \mathcal{M}$ for all $a \geq 0$;
(d) $U(a) \Omega = \Omega$ for all $a$.

In addition, the first part of either (a) or (a') implies (d); if the first part of (a) holds, then (c) implies (b), or if all of (a) holds, then (b) and (c) are equivalent. Likewise if the first part of (a') holds, then (c) implies (b'), or if all of (a') holds, then (b') and (c) are equivalent. Otherwise, no two of these conditions imply any other.

V Local Algebras

Let us now extend the example of Section III to a massive scalar free field in $d + 1$ spacetime dimensions where $d > 1$. This will provide us with an opportunity to prove a result comparable to that of Theorem 8, but without the assistance of Theorem 6.

Since the $H_{\lambda, \rho}$ are intended to correspond to momenta, and since the different components of momentum must commute, the proper way to extend to higher dimensions is as follows: take $h = L^2(\mathbb{R}, p_0^{-1} dp)$, where $p_0 = \sqrt{m^2 + p^2}$.

On it represent the $d + 1$-dimensional Poincaré group and TCP transformation: the translations by $t(x) = e^{i(x_0 p_0 - x \cdot p)}$; the rotations by rotations of $p$; the velocity transformations in the $\hat{x}_1$ direction by $p_i \rightarrow p_i \cosh \lambda + p_0 \sinh \lambda$; and the PCT transformation by $\theta f(p) = \overline{f(p)}$.

Then these all have multiplicative second quantizations on the symmetric Fock space $\mathcal{H}$, which give a representation of the Poincaré group for which the vacuum is the only invariant vector.

Define $\nu_i$ by $p_i = \sqrt{p_0^2 - p_i^2} \sinh \nu_i$ (so that $p_0 = \sqrt{p_0^2 - p_i^2} \cosh \nu_i$ and $e^{\nu_i} = p_0 + p_i$) and $\kappa_i$ by $\kappa_i = i \partial_{\nu_i}$ (the derivative being taken with $p_{i'}$ fixed for $i' \neq i$, so that the $\kappa_i$ so defined do not commute). Take $j = \theta \rho$, where $\rho$ is a rotation by the angle $\pi$ about the $\hat{x}_1$ axis, and let $\delta = e^{2\pi \kappa_1}$, where $\kappa_1$ is to be the generator of the velocity transformations in the $\hat{x}_1$ direction. Then again let the von Neumann algebra $\mathcal{A}(W_R)$
be that generated by \( w(f) \) for all \( f \) in the real linear space

\[
\rho = \left\{ f \mid j \delta^{1/2} f = f \right\} = \left\{ f(\kappa_1, p_i) = g(\kappa_1, p_i) + e^{-\pi \kappa_1} g(-\kappa_1, -p_i) \mid g \in D(e^{\pi \kappa_1}) \right\},
\]

where \( i \) runs from 2 to \( d \). From the earlier discussion we see that the modular conjugation and automorphisms will have the geometric form described in the introduction.

Any wedge \( W \) whose vertex contains the origin is produced from \( W_R \) by some Lorentz transformation; for such wedges we can define corresponding von Neumann algebras \( \mathcal{A}(W) \) by the corresponding transformation of \( \mathcal{A}(W_R) \), and see directly (for example) that \( \mathcal{A}(W_L) = \mathcal{A}(W_R)' \). It is also not difficult to show that if \( W \) is a rotation of \( W_R \), then \( \mathcal{A}(W) \) is generated by the two von Neumann algebras \( \mathcal{A}(W) \cap \mathcal{A}(W_R) \) and \( \mathcal{A}(W) \cap \mathcal{A}(W_L) \), each of which has the vacuum for a cyclic and separating vector.

In fact, given any family of such wedges \( W_i \) with a nonempty intersection, the \( \mathcal{A}(W_i) \) also have a nonempty intersection, sufficiently large that it has the vacuum as a cyclic vector (these regions are spacelike cones). No one of these wedges contains any other, so there are no isotony relations to establish. Once we seek to add the translations, however, we must establish the isotony relation \( T(x) \mathcal{A}(W) T(-x) \subset \mathcal{A}(W) \) for all \( x \) such that \( x + W \subset W \). For the translations given as above, this is immediate; however, as before we wish to ignore this in the interests of generality.

**Theorem 9:** Let \( \mathcal{H} \) be a Hilbert space with a representation of the \( d+1 \)-dimensional Lorentz group, \( d > 1 \), and the TCP operator \( \Theta \), for which the vacuum \( \Omega \) is the only invariant vector; let \( \mathcal{A}(W) \) be a family of von Neumann algebras on \( \mathcal{H} \) for all \( W \) whose vertices contain the origin, covariant under the Lorentz group, with \( \mathcal{A}(W_L) = \mathcal{A}(W_R)' \), and such that the modular operators with respect to the vacuum are given geometrically from \( \Theta \) and the representation of the Lorentz group, as described above. Suppose that if \( W(\neq W_R, W_L) \) is a rotation of \( W_R \), then \( \mathcal{A}(W) \) is generated by \( \mathcal{A}(W) \cap \mathcal{A}(W_R) \) and \( \mathcal{A}(W) \cap \mathcal{A}(W_L) \), each of which has the vacuum for a cyclic and separating vector.

Let \( H_\mu, \mu = 0 \ldots d \), be a Lorentz \( d+1 \)-vector of unbounded strongly commuting self-adjoint operators on \( \mathcal{H} \); suppose they are odd under \( \Theta \), annihilate the vacuum, and
have their joint spectrum contained in the closed forward light cone. Let
\[ \Gamma = \{ X | X \in \mathcal{A}(W_R), i[H_\mu, X] \in \mathcal{A}(W_R) \text{ for all } \mu \}, \tag{21} \]
and let us suppose further that \( \Gamma \Omega \) is a common core for all the \( H_\mu \).

Then the \( H_\mu \) are the generators of a representation \( T(x) \) of the \( d + 1 \)-dimensional translation group, which together with the given representation of the Lorentz group forms a representation of the Poincaré group under which the vacuum is the only invariant vector. We may consistently define \( \mathcal{A}(W) \) for any wedge as a Poincaré transformation of \( \mathcal{A}(W_R) \), and the \( \mathcal{A}(W) \) so defined satisfy isotony. If the wedges whose vertices contain the origin possess the property that for any family \( W_i \) with a nonempty intersection, the intersection of the \( \mathcal{A}(W_i) \) has the vacuum for a cyclic vector, then the same may be said for any family of wedges whose vertices contain a common point.

**Proof of Theorem 9:** The assumptions immediately imply that the \( H_\mu \) exponentiate to a representation \( T(x) = e^{ix^aH_\mu} \) of the \( d + 1 \)-dimensional translation group. Since they transform covariantly under the given representation of the Lorentz group, they must combine with it to produce a representation of the Poincaré group for which the vacuum is the unique invariant vector. Let \( R(W) = \mathcal{A}(W)^{\text{as}} \Omega \); then by Lorentz and TCP covariance \( T(x)R(W) = R(W) \) whenever \( x \) is parallel to the vertex of \( W \), and by Theorem 3 we have \( T(a(\hat{x}_1 + \hat{x}_0))R(W_R) \subset R(W_R) \) and \( T(a(\hat{x}_1 - \hat{x}_0))R(W_R) \subset R(W_R) \) for all \( a \geq 0 \). Thus \( T(x)R(W_R) \subset R(W_R) \) for every \( x \in \overline{W} \).

Let us then show that \( T(x)\mathcal{A}(W_R)T(-x) = \mathcal{A}(W_R) \) for all \( x \) parallel to the vertex of \( W_R \). Let \( \hat{x} \) be some unit vector thus parallel, and \( U(a) = T(a\hat{x}) \). Since \( \Gamma \Omega \) is a core for the generator of \( U(a) \), and \( U(a)R(W_R) = R(W_R) \) for all \( a \), by Theorem 4 we know that \( U(a)\mathcal{A}(W_R)U(-a) = \mathcal{A}(W_R) \) for all \( a \). Then covariance implies that \( T(x)\mathcal{A}(W)T(-x) = \mathcal{A}(W) \) for every wedge \( W \) and every \( x \) parallel to the vertex of \( W \).

Next let us show that \( T(x)\mathcal{A}(W)T(-x) \subset \mathcal{A}(W) \) for every \( x \) such that \( x + \overline{W} \subset W \). Without loss of generality we may suppose \( W \) to be a rotation of \( W_R \), let us say the wedge in the \( \hat{x}_2 \) direction. By making a velocity transformation in the \( \hat{x}_2 \) direction, and a translation in an orthogonal direction, it suffices to show this for the translations \( U(a) = T(a\hat{x}_2) \). Now, \( \mathcal{M}_1 = \mathcal{A}(W) \cap \mathcal{A}(W_R) \) and \( \mathcal{M}_2 = \mathcal{A}(W) \cap \mathcal{A}(W_L) \) together
generate $\mathcal{A}(W)$; the vacuum is cyclic and separating for $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3 = \mathcal{A}(W') \cap \mathcal{A}(W_R)$, and $\mathcal{M}_4 = \mathcal{A}(W') \cap \mathcal{A}(W_L)$, where $W'$ is the wedge in the $-\hat{x}_2$ direction. Furthermore by the result of the preceding paragraph $U(a)\mathcal{A}(W_R)U(-a) = \mathcal{A}(W_R)$ and $U(a)\mathcal{A}(W_L)U(-a) = \mathcal{A}(W_L)$ for all $a$. Take $X \in \mathcal{M}_1^{sa}$; then for any $a \geq 0$ we have $U(a)XU(-a) \in \mathcal{A}(W_R)$, but also since $U(a)R(W) \subset R(W)$, we have $U(a)X\Omega \in R(W)$. Thus there is a closed symmetric operator $\tilde{X}$ affiliated with $\mathcal{A}(W)$, and such that $\tilde{X}\Omega = U(a)X\Omega$. But $\tilde{X}$ and $U(a)XU(-a)$ agree on the dense set $\mathcal{M}_4\Omega$, from which it follows that $\tilde{X}$ is in fact bounded and equal to $U(a)XU(-a)$, and that $U(a)XU(-a) \in \mathcal{A}(W)$. The same argument applies if $X \in \mathcal{M}_2^{sa}$, using the dense set $\mathcal{M}_3\Omega$. Thus $U(a)\mathcal{M}_1U(-a) \subset \mathcal{M}_1$ and $U(a)\mathcal{M}_2U(-a) \subset \mathcal{M}_2$ for all $a \geq 0$, from which it follows that $U(a)\mathcal{A}(W)U(-a) \subset \mathcal{A}(W)$ for all $a \geq 0$.

Then weak closure implies that $T(x)\mathcal{A}(W)T(-x) \subset \mathcal{A}(W)$ for all $x$ such that $x + \bar{W} \subset \bar{W}$. This is equivalent to isotony; the remaining condition follows by translation from the corresponding condition for wedges whose common point is the origin.

More work is required to show that the resulting family is in fact local: for example, that for any family of wedges $W_i$ with a nonempty intersection, the intersection of the $\mathcal{A}(W_i)$ is nonempty. Of course, the free field is a very simple example, in which it is easy to compute the effects of the $T(x)$. In more complicated cases, Theorem 9 could perhaps be applied to greater effect.
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References

[1] Bisognano, J. J. and Wichmann, E. H.: On the Duality Condition for a Hermi-
tian Scalar Field, *J. Math. Phys.* **16**, 985 (1975); On the Duality Condition for
Quantum Fields, *J. Math. Phys.* **17**, 303 (1976).

[2] Borchers, H. J.: The CPT-Theorem in Two-Dimensional Theories of Local Ob-
servables, *Commun. Math. Phys.* **143**, 315 (1992).

[3] Wiesbrock, H. W.: A Comment on a Recent Work of Borchers, *Lett. Math. Phys.*
**25**, 157 (1992).

[4] Driessler, W.: Comments on Lightlike Translations and Applications in Relativis-
tic Quantum Field Theory, *Commun. Math. Phys.* **44**, 133 (1975).

[5] Bratteli, O. and Robinson, D. W.: *Operator Algebras and Quantum Statistical
Mechanics*, Springer-Verlag, New York, 1979, Vol. I.

[6] Bratteli, O. and Haagerup, U.: Unbounded Derivations and Invariant States, *Com-
mun. Math. Phys.* **59**, 79 (1978).

[7] Brunetti, R., Guido, D., and Longo, R.: Group Cohomology, Modular Theory,
and Spacetime Symmetries, *Rev. Math. Phys.* **7**, 57 (1995).

[8] Rieffel, M.: A Commutation Theorem and Duality for Free Bose Fields, *Commun.
Math. Phys.* **39**, 153 (1974).