Analytical Results for Trapped Weakly Interacting Bosons in Two Dimensions

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Abstract

We consider a model of $N$ two-dimensional bosons in a harmonic trap with translational and rotational invariant, weak two-particle interaction. We present in configuration space a systematical recursive method for constructing all wave functions with angular momentum $L$ and corresponding energies and apply it to $L \leq 6$ for all $N$. The lower and the upper bounds for interaction energy are estimated. We analytically confirm the conjecture of Smith et al. that elementary symmetric polynomial is the ground state for repulsive delta interaction, for all $N \geq L$ up to $L \leq 6$. Additionally, we find that there exist vanishing-energy solutions for $L \geq N(N-1)$, signalizing the exclusive statistics. Finally, we consider briefly the case of attractive power-like potential $r^k$, $k > -2$, and prove that the lowest-energy state is still the one in which all angular momentum is absorbed by the center-of-mass motion.

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Considerable attention has been devoted to the phenomenon of Bose-Einstein (BE) condensation in atomic vapors in the past few years [1–3]. The main reason was the fact that such systems might form quantized vortices under rotation, like in superfluid $^4$He. Many authors considered the vortex problem in BE condensates theoretically [4–9]. In the meantime, the existence of vortex state has been experimentally confirmed [10,11].

Persuing an analogy with the fractional quantum Hall (FQH) effect [12] Wilkin, Gunn, and Smith introduced a model of weakly interacting bosons in a harmonic trap [5]. In the absence of interactions, there is a huge degeneracy corresponding to the number of ways to distribute $L$ units of angular momentum among $N$ bosons. The degeneracy can be lifted by diagonalizing the interaction in the restricted Hilbert space of the lowest-energy states with angular momentum $L$. Accordingly, Bertsch and Papenbrock used numerical computation to show that the ground-state energy depends linearly on angular momentum $L$ and that the corresponding ground state of the repulsive model for $L \leq N$ is the elementary symmetric polynomial of coordinates $z_i = x_i + iy_i$, relative to the center-of-mass, $R = \frac{1}{N} \sum_i z_i$ [7].

Recently, Smith and Wilkin proved analytically that these symmetric polynomials are indeed exact eigenstates [13]. Later, Huang [14] extended the work by Smith and Wilkin to the models interacting via arbitrary rotationally and translationally symmetric potentials. The main result was that the associated interaction energy still varies linearly with $L$. In addition, a lower bound for the attractive quartic interaction energy was derived. Finally, Bertsch and Papenbrock showed [15] that there is a subspace structure that explains why certain eigenstates turn out to be simple analytical functions. They succeeded in constructing basis states but only up to $L = 5$. The mentioned subspace structure is present in a rather wide class of two-body interactions. However, for general interaction, the problem of determining the ground state has not yet been solved analytically.

Our starting point is the two-dimensional Hamiltonian given by

$$H = \sum_{i=1}^{N} \left(-\frac{1}{2} \nabla_i^2 + \frac{1}{2} r_i^2 \right) + \sum_{i<j}^{N} v(|\mathbf{r}_i - \mathbf{r}_j|).$$

(1)

It describes $N$ Bose-Einstein particles in a harmonic trap, weakly interacting via the two-
particle potential $v$, possessing translational and rotational symmetries. Since the spacing between noninteracting energy levels is greater than the two-body interaction strength, we restrict the Hilbert space to the degenerate ground-state manifold of $N$ non-interacting bosons. The single particle state is of the form $z^m \exp(-\frac{1}{2}|z|^2)$ where $z = x + iy$ and $m$ is the angular momentum quantum number. The eigenfunctions of the many-body problem are

$$
\psi(z_1, z_2, \ldots, z_N) = \varphi(z_1, z_2, \ldots, z_N) \prod_{i=1}^{N} \exp(-\frac{1}{2}|z_i|^2),
$$

where $\varphi$ is a homogeneous polynomial of degree $L$ that is totally symmetric under permutation of particle indices. Suitable basis functions for such polynomials are given by

$$
B_\lambda(z_1, z_2, \ldots, z_N) = \frac{1}{n_1!n_2!\cdots n_p!} \sum_{i_1, i_2, \ldots, i_q} \prod_{i=1}^{q} z^{\lambda_{i_1}} z^{\lambda_{i_2}} \cdots z^{\lambda_{i_q}},
$$

where set $\{\lambda_1, \lambda_2, \ldots, \lambda_q\}$ denotes any partition of $L$ such that $\sum_{i=1}^{q} \lambda_i = L$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_q > 0$ for $q \leq N$. The prime denotes the summ over mutually different indices $i_1, i_2, \ldots, i_q$, while the numbers $n_1, n_2, \ldots, n_p$ denote the frequencies of appearances of equal $\lambda$’s (i.e., the number of particles carrying the same angular momentum). Note that the number of distinct monomial terms $z^{\lambda_{i_1}} z^{\lambda_{i_2}} \cdots z^{\lambda_{i_q}}$ in $B_\lambda$ is given by $N(N-1)\cdots(N-q+1)/(n_1!n_2!\cdots n_p!)$, where $n_1 + n_2 + \cdots + n_p = q$. For simplicity we omit the exponentials from wavefunctions throughout the paper. Owing to the translational and rotational symmetries of the two-particle interaction $v$, we find, for non-negative integers $n$ and $m$

$$
v(z_1 + z_2)^n (z_1 - z_2)^{2m} P(z_3, z_4, \ldots, z_N) = c_{2m}(z_1 + z_2)^n (z_1 - z_2)^{2m} P(z_3, z_4, \ldots, z_N),
$$

where $P$ denotes an arbitrary polynomial in $z_3, z_4, \ldots, z_N$ variables, and $c_n$ is given by

$$
c_n = \frac{\int_0^\infty dr r^{2n+1} v(r) \exp(-r^2/2)}{\int_0^\infty dr r^{2n+1} \exp(-r^2/2)}.
$$

It is evident that the coefficients $c_n$ represent the interaction energy $v(r)$ of the relative motion of two bosons in the single particle state $r^n \exp(-r^2/2)$ with angular momentum $n$. We are now in position to estimate the lower and the upper bound of the interaction
The total interaction energy lies between
\[
\frac{N}{2} c_{\text{min}} \leq V \leq \frac{N}{2} c_{\text{max}}.
\]  

In order to simplify the calculations concerning the action of the potential \( V \) on polynomials \( B_{\lambda} \), let us define symmetric functions of two variables:
\[
b_{ij}(z_1, z_2) = \frac{1}{2} (z_1^i z_2^j + z_1^j z_2^i), \quad i \geq j.
\]

The action of the potential \( v \) on \( b_{ij} \) is given by:
\[
v b_{ij} = \sum_{l=0}^{[\frac{n}{2}]} \alpha_{ijkl} b_{kl},
\]
where \( i + j = k + l = n \), and the coefficients \( \alpha_{ijkl} \) satisfy the summation rule \( \sum_{l=0}^{[n/2]} \alpha_{ijkl} = c_0 \).

The first constraint corresponds to the conservation of total angular momentum for a rotationally symmetric potential, while the summation rule reflects the presence of translational symmetry. By using the simple calculations it can be shown that
\[
\alpha_{ijkl} = 2 - \delta_{kl} \sum_{p=0}^{[\frac{n}{2}]} c_{2p} s_{l,j}^2 s_{n-2p,2p},
\]
where
\[
s_{l,j}^q = \sum_{r+s=q} (-)^s \binom{i}{r} \binom{j}{s}.
\]

The coefficients \( \alpha_{ijkl} \) represent the two-body matrix element \( V_{ijkl} \) (see Ref. [13]) of the interaction potential \( V \). The relation (9) expresses the coefficients \( \alpha_{ijkl} \) in terms of interaction energies \( c_n \) of a pair of bosons having the angular momentum \( n \). It can be verified that in the case of constant potential \( v = v_{\text{const}} = c_{2n} \) and \( \alpha_{ijkl} \) reduces to \( \alpha_{ijkl} = \delta_{i}^{k} \delta_{j}^{l} \). Applying potential \( V \) onto \( B_{1}^{n}(z_1, z_2, \ldots, z_N) \) and using the relation (9), we obtain
\[
V B_{1}^{n}(z_1, z_2, \ldots, z_N) = c_0 \binom{N}{2} B_{1}^{n}(z_1, z_2, \ldots, z_N),
\]
i. e., the polynomial $B_{1}^{n}(z_{1}, z_{2}, \ldots, z_{N})$ is an eigenvector with eigenvalue equal to the lowest (greatest) interaction energy, if $c_{0} = c_{\text{min(max)}}$. Furthermore, the action of the potential $V$ on the product $B_{1}^{n}B_{\lambda}$ reduces to

$$V B_{1}^{n}B_{\lambda} = B_{1}^{n}V B_{\lambda},$$

(12)

for any $n$ and partition $\lambda$. Owing to the property (12) it is convenient to choose the natural basis for a given total angular momentum $L$ as a set formed by $B_{L}^{1}$ and products $B_{1}^{L-K}B_{\lambda}$ where $2 \leq K \leq L$, and $\lambda$ stands for the special partition of $K$: $\lambda = \{1,1,\ldots,1\}$ $\equiv 1^{K}$ and all other partitions not containing 1’s i. e., $\lambda = \{\lambda_{1}, \lambda_{2}, \ldots, \lambda_{k}\}$, $\lambda_{1} \geq \lambda_{2} \geq \cdots \lambda_{k} \geq 2$, where $\sum_{i=1}^{k} \lambda_{i} = K$ and $2 \leq k \leq N$. For $K > N$ the partition $\lambda = 1^{K}$ is not present and for $K = N + 1, N + 2$ the remaining set of polynomials $B_{\lambda}$ is linearly independent. For $K > N + 2$ the above states become linearly dependent [16]. Let us call the corresponding subspace spanned by above defined $B_{\lambda}$ polynomials as $\mathcal{B}_{K}$. For example $\mathcal{B}_{2} = \text{span}\{B_{12}\}$, $\mathcal{B}_{3} = \text{span}\{B_{13}\}$, $\mathcal{B}_{4} = \text{span}\{B_{14}, B_{22}\}$, $\mathcal{B}_{5} = \text{span}\{B_{15}, B_{32}\}$, $\mathcal{B}_{6} = \text{span}\{B_{16}, B_{42}, B_{33}, B_{222}\}$, etc. The whole $L = 6$ space can be described as a summ of appropriate subspaces: $B_{1}^{0} \oplus B_{1}^{1}\mathcal{B}_{2} \oplus B_{1}^{2}\mathcal{B}_{3} \oplus B_{1}^{3}\mathcal{B}_{4} \oplus B_{1}^{4}\mathcal{B}_{5} \oplus \mathcal{B}_{6}$. We note in passing that these subspaces are not mutually orthogonal.

Let us now turn to the solution of the eigenvalue problem. Having in mind the translational and rotational invariance of the potential $V$ it is obvious that $VB_{1}^{L-K}\mathcal{B}_{K} = B_{1}^{L-K}V \mathcal{B}_{K}$ and $V \mathcal{B}_{K} \subset \text{span}\{V \mathcal{B}_{K}, B_{1}V \mathcal{B}_{K-1}, B_{1}^{2}V \mathcal{B}_{K-2}, \ldots, B_{1}^{K}\}$. These give rise to the following sequence of the subspace relations:

$$\begin{align*}
VB_{1}^{L} &= c_{0}\binom{N}{2} B_{1}^{L}, \\
VB_{1}^{L-2} \mathcal{B}_{2} &\subset \text{span}\{B_{1}^{L}, B_{1}^{L-2}\mathcal{B}_{2}\}, \\
VB_{1}^{L-3} \mathcal{B}_{3} &\subset \text{span}\{B_{1}^{L}, B_{1}^{L-2}\mathcal{B}_{2}, B_{1}^{L-3}\mathcal{B}_{3}\}, \\
\vdots &
\end{align*}$$

(13)

Consequently, the matrix of potential $V$ in the natural basis possesses the block triangular form which means that the original eigenvalue problem has been substantionally simplified.
i.e., reduced to the eigenvalue problem in the $B_K$ subspaces, for $2 \leq K \leq L$. Hence, we have to calculate $VB_{1L}$, $VB_{22}$, $VB_{32}$, $VB_{42}$, $VB_{33}$, $VB_{222}$, etc., then express the result in terms of our natural basis and finally project them onto the $B_K$ subspace. Applying the potential $V$ onto the state $B_{1L}$ yields:

$$VB_{1L}(z_1, z_2, \ldots, z_N) = V \sum_{i<j}^N z_i z_j \binom{N-2}{L-2} + (z_i + z_j) \binom{N-2}{L-1} + \binom{N-2}{L} t,$$

where the coefficients $\alpha$ are given by relation (9):

$$\alpha_{11} = \frac{c_0 + c_2}{2}, \quad \alpha_{20} = \frac{c_0 - c_2}{2}.$$

In (14) we have explicitly extracted the $(z_i, z_j)$ pair of coordinates to simplify the action of the potential $V$. Binomial coefficients with index $t$ symbolically denote the number of terms in remaining $N - 2$ variables. By using the identity:

$$B_{21L-2} = B_1 B_{1L-1} - LB_{1L},$$

we obtain

$$VB_{1L}(z_1, z_2, \ldots, z_N) = \frac{c_0 - c_2}{4} (N - L + 1) B_{1L-1} + \left[ c_0 \binom{N}{2} - \frac{c_0 - c_2}{4} NL \right] B_{1L}. \quad (17)$$

Hence, the eigenvalue is given by:

$$\Lambda_{1L} = c_0 \binom{N}{2} - \frac{c_0 - c_2}{4} NL. \quad (18)$$

Having find the eigenvalue we are now in the position to find the corresponding eigenfunction. By expanding the unknown eigenvector in terms of basis polynomials $B_{1L}$ and $B_{1L-k} B_k$ with unknown coefficients, we easily obtain the system of simple recursive relations. Solving these equations we finally get the expansion coefficients in which all dependence on the details of interaction $V$, namely on $c_0$ and $c_2$, simply drops out! The corresponding eigenvector is:

$$A_{1L} = \sum_{n=0}^{L} (-)^n B_{1L-n} \left( \frac{B_1}{N} \right)^n = B_{1L}(z_1 - \frac{B_1}{N}, z_2 - \frac{B_1}{N}, \ldots, z_N - \frac{B_1}{N}), \quad (19)$$
in agreement with Ref. [17]. In the same way we find:

\[ VB_{22} = \left[ \alpha_{22}^{22} + 2\alpha_{20}^{20}(N - 2) + c_0 \left( \frac{N - 2}{2} \right) \right] B_{22} + \frac{1}{2} \alpha_{22}^{31} B_{31} + \frac{N - 1}{2} \alpha_{22}^{40} B_4 + 2\alpha_{20}^{11} B_{211}, \tag{20} \]

\[ VB_{32} = \left[ \alpha_{32}^{32} + \alpha_{30}^{30}(N - 2) + \alpha_{20}^{20}(N - 2) + c_0 \left( \frac{N - 2}{2} \right) \right] B_{32} + \alpha_{32}^{41} B_{41} + (N - 1)\alpha_{32}^{50} B_5 + 2\alpha_{30}^{21} B_{221} + 2\alpha_{26}^{11} B_{311}, \tag{21} \]

\[ VB_{42} = \left[ \alpha_{42}^{42} + \alpha_{40}^{40}(N - 2) + \alpha_{20}^{20}(N - 2) + c_0 \left( \frac{N - 2}{2} \right) \right] B_{42} + 2\alpha_{42}^{33} B_{33} + \alpha_{42}^{51} B_{51} + (N - 1)\alpha_{42}^{60} B_6 + \alpha_{40}^{31} B_{321} + 6\alpha_{40}^{22} B_{222} + 2\alpha_{40}^{11} B_{411}, \tag{22} \]

\[ VB_{33} = \left[ \alpha_{33}^{33} + 2\alpha_{30}^{30}(N - 2) + c_0 \left( \frac{N - 2}{2} \right) \right] B_{33} + \frac{1}{2} \alpha_{33}^{42} B_{42} + \frac{1}{2} \alpha_{33}^{51} B_{51} + \frac{N - 1}{2} \alpha_{33}^{60} B_6 + \alpha_{30}^{21} B_{321}, \tag{23} \]

\[ VB_{222} = \left[ 3\alpha_{22}^{22} + 3\alpha_{20}^{20}(N - 3) + c_0 \left( \frac{N - 3}{2} \right) \right] B_{222} + \frac{1}{2} \alpha_{22}^{31} B_{321} + \frac{N - 2}{2} \alpha_{20}^{40} B_4 + 2\alpha_{20}^{11} B_{2211}. \tag{24} \]

In order to express the above results in terms of the vectors of our natural basis we multiply systematically all polynomials \( B_\lambda \) by \( B_1 \) in the following sequence: \( B_1^2; \ B_1 B_{11}, B_1 B_2; \ B_1 B_{13}, B_1 B_{21}, B_1 B_3; \ B_1 B_{14}, B_1 B_{22}, B_1 B_{211}, B_1 B_{31}, B_1 B_4; \) etc. These products can be easily evaluated by simple algebraic manipulations. Here, we give a few results of these manipulations:

\[ B_1^2 = B_2 + 2B_{11}, \]
\[ B_1 B_{11} = B_{21} + 3B_{13}, \]
\[ B_1 B_2 = B_3 + B_{21}, \]
\[ B_1 B_{13} = B_{211} + 4B_{14}, \]
\[ B_1 B_{211} = B_{312} + 2B_{221} + 3B_{213}, \]
\[ \vdots \]
It is obvious that any polynomial can be expressed in terms of natural basis by method of successive substitution. For example, from the first three relations we obtain $B_3 = B_1^3 - 3B_1B_{11} + 3B_{13}$.

Next, we turn to the final step. Projecting out the states $B_4, B_{31}$ and $B_{211}$ onto $B_{22}$, then $B_5, B_{41}, B_{312}, B_{213}$ and $B_{221}$ onto $B_{32}$, and $B_6, B_{51}, B_{321}, B_{2211}, B_{313}, B_{214}$ and $B_{412}$ onto $B_{42}, B_{33}$ and $B_{222}$, we obtain

\[
P(B_4) = -P(B_{31}) = 2B_{22},
\]
\[
P(B_{211}) = 0,
\]
\[
P(B_5) = -P(B_{41}) = 5B_{32},
\]
\[
P(B_{312}) = 2B_{32},
\]
\[
P(B_{213}) = 0,
\]
\[
P(B_{221}) = -B_{32},
\]
\[
P(B_6) = -P(B_{51}) = 9B_{42} + 16B_{33} - 18B_{222},
\]
\[
P(B_{321}) = -B_{42} - 2B_{33},
\]
\[
P(B_{2211}) = -\frac{1}{2}P(B_{313}) = \frac{1}{2}(B_{42} + 2B_{33} - 3B_{222}),
\]
\[
P(B_{214}) = 0,
\]
\[
P(B_{412}) = 4B_{42} + 8B_{33} - 9B_{222},
\]

where $P$ means the projection. Note that vectors $B_{42}, B_{33}$ and $B_{222}$ are linearly independent if $N \geq 4$, while for $N = 3$ we have: $P(B_{42}) = -2B_{33} + 3B_{222}$. Upon substitution of results (26) into relations (21-24) we find two more, yet not known, eigenvalues:

\[
\Lambda_{22} = c_0 \frac{N - 2}{2} \left( N - \frac{3}{4} \right) + c_2 \frac{3N - 6}{4} + c_4 \frac{N + 6}{8}, \quad N \geq 2, \ L = 4,
\]

\[
\Lambda_{32} = \frac{c_0}{16} \left[ 17N - 36 + 8 \binom{N - 2}{2} \right] + \frac{c_2}{8} (5N - 12) + \frac{c_4}{16} (5N + 2), \quad N \geq 3, \ L = 5.
\]

The first one in the case of repulsive $\delta$ potential ($c_0 \neq 0$, and $c_n = 0, \forall n > 0$) reduces to the result already obtain in Ref. [17]. The two corresponding eigenstates are obtained in the analogous way as $A_{1\nu}$:
\[ A_{22} = NB_4 - 3B_2^2, \quad (29) \]
\[ A_{32} = NB_5 + 2B_3B_2, \quad (30) \]

where we have used compact notation: \( B_n = \sum_{i=1}^{N} (z_i - B_1/N)^n \). Few comments are in order.

The two above eigenvectors have been already found in Ref. \[15\] for a wide class of two-body interactions. They do not depend on the details of interaction. But, starting with \( L = 6 \) the wave functions and eigenvalues in the subspace spanned by vectors \( B_{42}, B_{33} \) and \( B_{222} \) become dependent on the details of interaction. Generally, for any \( L \) it can be seen from relations \((22-24)\) that the action of the potential \( V \) on \( B_L \) reduces to \( VB_{\lambda} = \sum_{\mu} V_{\lambda\mu} B_{\mu} \), where the matrix elements have the structure \( V_{\lambda\mu} = \frac{c_0}{2} \delta_{\lambda\mu} N^2 + \beta_{\lambda\mu} N + \gamma_{\lambda\mu} \). All details of the interaction are incorporated in the constants \( \beta \) and \( \gamma \). For large \( N \), the eigenvalues of this matrix can be easily found by expansion in \( 1/N \). For example, for the weak \( \delta \) repulsive potential the corresponding \( 3 \times 3 \) \((N \geq 4, \ L = 6)\) matrix can be written as

\[
\begin{pmatrix}
\frac{N^2}{2} - \frac{51N}{32} + \frac{15}{4} & \frac{9N}{64} - \frac{3}{2} & \frac{N}{16} + \frac{1}{8} \\
\frac{N}{2} + \frac{33}{8} & \frac{N^2}{2} - \frac{7N}{4} - \frac{15}{16} & \frac{1}{2} \\
-\frac{9N}{16} - \frac{45}{16} & -\frac{9N}{32} + \frac{63}{32} & \frac{N^2}{2} - 2N + \frac{9}{8}
\end{pmatrix}.
\]

The corresponding eigenenergies can be easily expanded in powers of \( 1/N \) up to \( \mathcal{O}(1/N) \):

\[
\begin{align*}
\Lambda_6^{(1)} &= c_0 \left( \frac{N^2}{2} - 2N + \frac{27}{34} \right), \\
\Lambda_6^{(2)} &= c_0 \left( \frac{N^2}{2} - \frac{15}{8}N + \frac{30}{13} \right), \\
\Lambda_6^{(3)} &= c_0 \left( \frac{N^2}{2} - \frac{47}{32}N + \frac{2955}{3536} \right). 
\end{align*}
\]

For \( N \geq 10 \) the error is of order \( \lesssim 1\% \). Having in mind all obtained eigenvalues, it turns out that for \( L \leq 6 \) and arbitrary \( N \geq L \), the lowest energy is given by \( \Lambda_1L \). In this way we analytically support the conjecture of Smith et. al. \[13\], up to \( L = 6 \). Moreover, the first eigenvalue from the relation \((32)\) is in a perfect agreement with the numerical result for two excited octupole modes \[18\]. For \( L = N \) it is known that the ground state is a
vortex \( \psi = \prod_{i=1}^{N} (z_i - z_c) \) 1 around \( z_c = B_1/N \). For \( L > N \), we find that there exists lower eigenvalue then that given by (18). For example, for \( L = 6 \) and \( N = 4 \) eigenvalue \( \Lambda_6^{(1)} = 1.38 \) is lower than \( \Lambda_4 = 2 \). The same is true for \( L = 6 \) and \( N = 3 \), i.e. \( \Lambda_6^{(1)} = 0 \) and \( \Lambda_3 = 3/4 \). For \( N = 3 \) we find that \( V(A_1^2 A_2^3) \sim B_{2p+3q} \), and therefore, for a given \( L \) there exists only one eigenstate having positive eigenvalue \( V B_L = [1 - (-1/2)^{L-1}] B_L \). From the above examples, it seems that for sufficiently large \( L > N \) there exists the ground state with energy lower than \( \Lambda_1^N = c_0 N(N-2)/4 \). Moreover, we find vanishing eigenvalues for \( L \geq N(N-1) \). As \( \delta \) potential gives vanishing \( c_n \)'s for \( n > 0 \), from relation (4) the corresponding eigenvectors are given by

\[
\psi_0 = \prod_{i<j}^{N} (z_i - z_j)^2 \psi_{L-N(N-1)},
\]

where \( \psi_{L-N(N-1)} \) denotes any state in the space of total angular momentum equal to \( L - N(N-1) \). We note that these states saturate the lower bound in inequality (3). It is interesting to note that the lowest states (33) are of anyonic (fermionic) type in the sense that they vanish when two particles coincide [20]. In other words, particles behave like the hard-core bosons. This might signalize some sort of statistical transmutation and 0 of exclusion statistics [21].

We should finally point out that for the attractive, weak two-body potential \( v(r) = \eta r^k \) (for \( k > 0 \) and \( \eta > 0 \), respectively), the coefficient \( c_n \) is given by

\[
c_n = 2^{k/2} \eta \Gamma(n + 1 + \frac{k}{2}) \Gamma(n + 1),
\]

and is always greater then

\[
c_0 = 2^{k/2} \eta \Gamma(1 + \frac{k}{2}).
\]

This means that the state \( B_L^k \) is the lowest-energy state for fixed \( L \) in which all angular momentum is carried by the center-of-mass motion \( z_c \). For \( k = 4 \) and \( c_0 = 8\eta \) the corresponding energy equals \( 8 \binom{N}{2} \eta \) in agreement with the result of Ref. [14]. The \( B_L^k \) remains the lowest-energy eigenstate for the attractive \( \delta \) potential (\( c_0 = -\eta/2\pi < c_n = 0 \)) and for
all attractive two-particle potentials for which \( c_0 = c_{\text{min}} \). This generalization is a simple consequence of inequality (3). In all of these examples the presence of the uncondensed state \( B_1^L \) is not a surprise because of the attractive nature of the two-body interaction [5].

In conclusion, we have developed an analytical recursive method in configuration space for calculating all eigenstates and corresponding eigenvalues for trapped bosons interacting via weak, translationally and rotationally symmetrical pairwise potentials. We applied the method to states with \( L \leq 6 \), for all \( N \). We have derived the lower and the upper bounds for general interaction energy. For repulsive \( \delta \)–interaction we have proved the conjecture of Smith and Wilkin on the ground state structure up to \( L \leq 6 \), for \( N \geq L \). Moreover, we have demonstrated that there exist eigenstates with vanishing energy, for \( L \geq N(N - 1) \). In addition, we have considered the case of general attractive power-like two-particle potential \( r^k, \ k > -2 \), and found ground state and its energy.

Note added: After this work was completed a preprint by Kavoulakis et. al. [22] appeared. In the asymptotic limit \( N \rightarrow \infty \) our eigenvalue \( \Lambda_6^{(1)} \) from the relation (32) is in full agreement with their findings in the presence of two octupole excitations.

Acknowledgment

This work was supported by the Ministry of Science and Technology of the Republic of Croatia under contract No. 00980103.
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