NOTES ON 2D CONFORMAL FIELD THEORY AND STRING THEORY

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Introduction

1. CONTENTS OF THESE NOTES

1.1. These notes appeared out of an attempt to write down explanations and solutions of home assignments for the course on String Theory which was given by E. D’Hoker in Spring 1997 at IAS.

Our original plan was to deal only with questions that admitted a rigorous mathematical formulation or axiomatization. A variety of such questions arises in the most basic ingredient of String Theory, namely in 2-dimensional conformal field theory (CFT).

We shall restrict our attention to a tiny part of 2D CFT: our goal is to explain the mathematical formalism that describes the operator product expansion (OPE) of quantum fields. To make things as simple as possible (and since we wish to stay within the realm of algebraic geometry) we will deal only with fields from the so-called holomorphic sector.

Fortunately, appropriate mathematical objects that imitate the OPE operation on quantum fields have been found several years ago by A. Beilinson and V. Drinfeld (they called them “chiral algebras”). We shall adopt their point of view completely and the present paper (and especially Chapters 1, 3 and 4) can be regarded as an example-oriented digest of the long-awaited text of Beilinson and Drinfeld.

The language of chiral algebras is in many ways equivalent to the language of vertex operator algebras (VOA’s). More precisely, a vertex operator algebra is the same as a “universal chiral algebra”, i.e. one that is defined naturally over an arbitrary Riemann surface. Since we will be interested in chiral algebras that are universal in the above sense, our treatment of such matters as the Sugawara construction (Sect. 9), fermion algebras and boson-fermion correspondence (Sect. 11), the BRST complex (Sect. 12) and of many others, is nothing more than a reinterpretation in terms of algebraic geometry of constructions that have been known to the founders of the theory of vertex operator algebras for many years (cf. [10],[9],[13],[14],[15]).

Therefore, we wish to emphasize that hardly any part of the contents of these notes is the invention of their author. Besides the classical sources mentioned just above we have borrowed the material from the partially written manuscripts [2] and [3].

Date: May 1999. This paper will appear in Quantum Fields and Strings: A course for Mathematicians (P. Deligne, P. Etingof, D.S. Freed, L. Jeffrey, D. Kazhdan, J. Morgan, D.R. Morrison and E. Witten eds.), Amer. Math. Soc., Providence (1999).
1.2. Let us discuss in some detail the mathematical contents of the paper.

Chapter 1 deals with the basics of the theory of chiral algebras.

In Section 3 we introduce the main character, namely the notion of a chiral algebra. A chiral algebra on a curve $X$ is a D-module with a binary operation that satisfies the axioms of a Lie-* bracket. One of the features of chiral algebras is that these objects are not so easy to construct. The more elementary constructions go through the procedure of taking the chiral universal enveloping algebra of a more accessible object – a Lie-* algebra (the latter notion is introduced in Section 4). Finally, in Section 5 we introduce the notion of the space of conformal blocks of a chiral algebra (the dual to the space of conformal blocks should be thought of as a set of possible correlation functions of a quantum field theory (QFT) with a given OPE of the fields).

In Chapter 2 we make an attempt to define axiomatically what a 2D CFT is. However, it should be clear from the very beginning that our list of axioms is far from being complete: we are dealing with the short-distance singularities of quantum fields only and we do not mention at all what happens when a Riemann surface approaches the boundary of the moduli space. We hope to be able to address this more subtle and interesting issue in the future.

In Section 6 we introduce the notion of a local $\mathcal{O}$-module over the universal curve $\mathcal{X}$ over the moduli space of smooth curves. A local $\mathcal{O}$-module is a quasi-coherent sheaf which is “universal” (in the same sense as a VOA is a “universal” chiral algebra). We need this notion since fields of a CFT are clearly local in the above sense, i.e. the fields and the OPE structure on them over a given region of a Riemann surface do not “feel” a change of a metric at a remote point. In Section 7 we give the axioms of a CFT data in the simplest case of central charge 0 and in Section 8 we generalize the discussion to the case of an arbitrary central charge. In particular, we show (what should be a standard thing by now) that the space of conformal blocks of a CFT chiral algebra acquires a (projective) connection along the moduli space of smooth curves.

Chapter 3 is devoted to the examples. (Unfortunately, all our examples are trivial or almost trivial from the point of view of physics. They correspond to either non-existing physical theories or to theories that are free.)

We discuss three types of examples here: Kac-Moody chiral algebras, i.e. those associated to Lie algebras with an invariant quadratic form, in particular, the Heisenberg chiral algebra which corresponds to a Lie algebra with a zero bracket (all this in Section 9); the dilaton twist of the Heisenberg chiral algebra (Section 10) and the fermionic chiral algebra, which is referred to as the $bc$-system in the physics literature (Section 11). In all these examples we write down explicitly the energy-momentum tensor (or equivalently, the Sugawara construction) and perform explicitly the corresponding calculations. We should mention that there are many more ways to make the same calculations and ours are not always the optimal ones: we just wanted to exhibit as many techniques as possible of manipulating with chiral algebras. Also, the reader should notice that the way we present these calculations is very close to how a physicist would have done it.

In Chapter 4 we discuss the BRST complex and related matters.

In Section 12, given a Lie-* algebra $\mathcal{B}$, we construct (following Beilinson) its central extension $\mathcal{B}'$ and define a BRST differential on a chiral algebra obtained by
tensoring the twisted chiral universal enveloping algebra of \( \mathcal{B}' \) and the corresponding fermionic algebra. This enables one to write a BRST reduction of any chiral algebra endowed with an action of \( \mathcal{B} \) (as usual in such situations, this action must be of “Harish-Chandra” type). In Section 13 we apply the discussion of Sect. 12 in the case of the BRST reduction of a chiral algebra corresponding to a CFT of central charge 26 with respect to the Virasoro algebra. This procedure is used by physicists for what can be mathematically reformulated as writing down “explicitly” the connection along \( \mathfrak{M} \) on the space of conformal blocks. However, we do not pretend to having really understood why the physicists really need the BRST procedure.

Starting from Chapter 5, this paper becomes much more involved mathematically and sometimes we are forced to omit proofs of certain important statements. Our goals in Chapters 5 and 6 are the construction of the bosonic chiral algebra (in the VOA terminology it is sometimes called the lattice VOA) and the proof of Theorem 18.2 that computes its space of conformal blocks on a given curve in terms of the canonical line bundle over the Jacobian (or, rather a direct product of several copies of the Jacobian) of this curve. Usually, in the theory of VOA’s, the sought-for bosonic chiral algebra is constructed by an explicit formula (cf. [13]) (which is the most classical example of a “vertex operator”) and the fact that this formula works remains for us a miracle.

The construction of the bosonic chiral algebra described in this paper is quite different: it is very geometric and uses a sophisticated algebro-geometric object called the Beilinson-Drinfeld Grassmannian (cf. [16], [11]). In fact, our exposition covers a part of a baby (=abelian) version of a new construction of the geometric Langlands correspondence due again to Beilinson and Drinfeld.

In Section 14 we describe a new way of thinking about chiral algebras (due to Beilinson): now a chiral algebra is interpreted as a system of \( \mathcal{O} \)-modules on various powers of our curve endowed with what we shall call “factorization isomorphisms”. In Section 16 (after some geometric preliminaries in Section 15) we demonstrate in a relatively simple situation an interplay between the two pictures.

In Section 17 we describe the canonical line bundle over the B.-D. Grassmannian corresponding to a torus, which arises from an even quadratic form on the lattice of 1-parameter subgroups of this torus. Finally, in Section 18 we introduce the bosonic chiral algebra and prove Theorem 18.2.

2. Some background on D-modules

2.1. First, some notation:

For a scheme (or a stack) \( Z, \mathcal{O}_Z \) (resp., \( \Theta_Z, \Omega_Z, D_Z \)) will denote the structure sheaf (resp., the tangent sheaf, the sheaf of 1-forms, the sheaf of differential operators) on \( Z \). The terms “an \( \mathcal{O} \)-module on \( Z \)” and “a quasi-coherent sheaf on \( Z \)” will mean for us the same thing (with a few exceptions in Sections 6-7, where we will be dealing with projective limits of coherent sheaves). If \( f: Z_1 \to Z_2 \) is a morphism of algebraic varieties, we shall denote by \( f_* \) and \( f^* \) the direct and the inverse image functors respectively, in the category of quasi-coherent sheaves. The functor of the inverse image in the category of sheaves (and not \( \mathcal{O} \)-modules) will be used only once (in Section 8.3) and will be denoted by \( f^{-1} \).
For an algebraic curve $X$, $\Delta$ will denote the closed embedding of the diagonal divisor in $X^n$ and $j : X^n \setminus \Delta \to X^n$ will denote the open embedding of its complement. A symbol like $\Delta_{x_i = x_j}$ will designate an embedding of the closed subset of the diagonal divisor corresponding to points $(x_1, \ldots, x_n) \in X^n$ with $x_i = x_j$. By $p_i : X^n \to X$ we shall denote the projection on the $i$-th factor.

$\mathcal{M}$ will denote the moduli stack of smooth complete curves and $\pi : X \to \mathcal{M}$ will denote the universal curve over $\mathcal{M}$. $X^n$ will denote the $n$-fold Cartesian product of $X$ with itself over $\mathcal{M}$; $\Delta$ will as above denote the embedding of the divisor of diagonals into $X^n$ and $p_i : X^n \to X$ will stand for the corresponding projections. If $L_1$ and $L_2$ are quasi-coherent sheaves on $X^{k_1}$ and $X^{k_2}$ respectively we let $L_1 \boxtimes L_2$ denote (by slightly abusing the notation) the sheaf on $X^{k_1+k_2}$ obtained by restricting the corresponding sheaf on $X^{k_1} \times X^{k_2}$.

2.2. In this paper we will be extensively using the language of D-modules. Unfortunately, there are very few sources with an appropriate exposition of the theory, the standard reference being the book [7].

For a smooth variety $Z$ a (right) D-module is a (right) module over the sheaf $D_Z$ of differential operators on $Z$. By definition, there exists a natural forgetful functor $D$-modules $\to \mathcal{O}$-modules.

When $Z$ is a singular scheme, one proceeds as follows: we embed $Z$ (locally) as a closed subscheme into a non-singular variety $Z'$ and consider D-modules on $Z'$ that are set-theoretically supported on $Z$. (The correctness of this definition follows from the well-known Kashiwara theorem). In this case, there still is a forgetful functor $D$-modules $\to \mathcal{O}$-modules which sends a D-module $M$ over $Z'$ as above to the $\mathcal{O}'_{Z'}$-module of its sections that are scheme-theoretically supported on $Z$. This enables one to make sense of right D-modules over a strict ind-scheme that are set-theoretically supported on a closed finite-dimensional subscheme (cf. Section 6).

For a smooth variety $Z$ there exists a functor

$$h : \{ \text{D-modules on } Z \} \to \{ \text{Sheaves in the étale topology on } Z \}$$

defined as follows:

$$M \mapsto M/M \cdot \Theta_Z$$

(we shall often abuse the notation and denote by $h$ also the projection $M \to h(M)$.)

The functor $h$ is right-exact and we let $Lh$ denote the corresponding left derived functor. (the functor $Lh$ applied to holonomic D-modules with regular singularities realizes the celebrated Riemann-Hilbert correspondence). The functor

$$R\Gamma \circ Lh : \{ \text{D-modules } \} \to \{ \text{Vector spaces } \}$$

is called the De Rham cohomology functor, denoted $M \to DR^\bullet(Z, M)$.

When $f : Z_1 \to Z_2$ is a map between smooth algebraic varieties, one defines the direct image functor

$$f_* : \{ \text{D-modules on } Z_1 \} \to \{ \text{D-modules on } Z_2 \}$$

(to distinguish it from the corresponding functor on the category of $\mathcal{O}$-modules; in cases where confusion can occur, we shall specify explicitly which one we are using.)

When $f$ is a projection $Z_1 = Z_2 \times Y \to Z_2$ the definition of $f_*$ is a straightforward

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1 One of the best readings on D-modules is an unpublished series of lectures by J. Bernstein ([6]) (obtainable by e-mail from the author of these notes).
generalization of that of $DR^*\big)$ (the latter corresponds to the case $Z_2 = \text{pt}$). When $f$ is a closed embedding, $f_*$ is defined using Kashiwara’s theorem (see above); in this case we shall replace the notation $f_*$ by $f_!$ in order to avoid confusion with the functor $f_!$ for $\mathcal{O}$-modules.

For instance, when $f$ is affine, the functor $f_*$ is right-exact and when $f$ is an open embedding, $f_*$ is left exact.

For $(f, Z_1, Z_2)$ as above one defines also the inverse image functor

$$f^! : \{\text{D-modules on } Z_2\} \rightarrow \{\text{D-modules on } Z_1\}.$$ 

We refer the reader to [7] for the definition.

When $Z$ is a smooth variety, in addition to right $\mathcal{D}$-modules one can consider left $\mathcal{D}$-modules (it may seem surprising, but one cannot define left $\mathcal{D}$-modules for non-smooth schemes). The categories of left and right $\mathcal{D}$-modules are equivalent and the functor that realizes this equivalence sends a left $\mathcal{D}$-module $\mathcal{M}$ to $\mathcal{M} := \mathcal{M} \otimes \Omega^{\text{top}}_Z$; in particular, the basic left $\mathcal{D}$-module $\mathcal{O}_Z$ goes over to the basic right $\mathcal{D}$-module $\Omega^{\text{top}}_Z$.

If $\mathcal{M}$ is a left $\mathcal{D}$-module on $Z$ and $\mathcal{N}$ is another left (resp., right) $\mathcal{D}$-module, the tensor product $\mathcal{M} \otimes \mathcal{N}$ is naturally a left (resp., right) $\mathcal{D}$-module. If now $\mathcal{M}_1$ and $\mathcal{M}_2$ are two right $\mathcal{D}$-modules, we shall denote by $\mathcal{M}_1 \otimes \mathcal{M}_2$ the right $\mathcal{D}$-module

$$(\mathcal{M}_1 \otimes \mathcal{M}_2) \otimes \Omega^{\text{top}}_Z.$$ 

The operation $\otimes$ makes the category of right $\mathcal{D}$-modules over a smooth variety into a symmetric tensor category. For instance, one can talk about Lie algebras in the category of right $\mathcal{D}$-modules and of their universal enveloping algebras (this will be important in Section 4.4).

In addition, in this paper we will be dealing with $\mathcal{D}$-modules on algebraic stacks in the smooth topology ($\mathcal{M}$ is an example of such). It follows from the definition of a stack that these objects are perfectly suitable for having $\mathcal{D}$-modules on them. For example, if a stack has the form $Z/G$, where $Z$ is a scheme and $G$ is a group acting on $Z$, a $\mathcal{D}$-module on $Z/G$ is the same as a strongly $G$-equivariant $\mathcal{D}$-module on $Z$ (cf. [1], Section 1 for the discussion of equivariant $\mathcal{D}$-modules.)

The following observation will be used several times in this paper:

For a smooth variety $Z$ consider the co-induction functor

$$\{\mathcal{O}\text{-modules on } Z\} \rightarrow \{\mathcal{D}\text{-modules on } Z\}$$ 

which sends an $\mathcal{O}$-module $\mathcal{L}$ to $\mathcal{L} \otimes_{\mathcal{O}_Z} \mathcal{D}_Z$.

**Lemma 2.1.** We have:

(a) For a closed embedding $f : Z_1 \rightarrow Z_2$ and an $\mathcal{O}$-module $\mathcal{L}$ over $Z_1$ there is a canonical isomorphism:

$$f_!(\mathcal{L} \otimes_{\mathcal{O}_Z} \mathcal{D}_{Z_1}) \simeq f_*(\mathcal{L}) \otimes_{\mathcal{O}_Z} \mathcal{D}_{Z_2}.$$ 

(b) For two $\mathcal{O}$-modules $\mathcal{L}_1$ and $\mathcal{L}_2$ over $Z$ the space

$$\text{Hom}_{\mathcal{O}\text{-mod}}(\mathcal{L}_1, \mathcal{L}_2 \otimes \mathcal{D}_Z) \simeq \text{Hom}_{\mathcal{D}\text{-mod}}(\mathcal{L}_1 \otimes \mathcal{D}_Z, \mathcal{L}_2 \otimes \mathcal{D}_Z)$$ 

is canonically isomorphic to the space of differential operators from $\mathcal{L}_1$ to $\mathcal{L}_2$. 

2.3. Another notion that will be important for us is that of a Lie algebroid.

A Lie algebroid \( a \) over a smooth variety \( Z \) is a quasi-coherent sheaf \( a \) over \( Z \) with the following additional structure:

- An \( \mathcal{O} \)-module map \( \phi : a \to \Theta Z \).
- A bi-differential operator \([ , , ] : a \otimes \mathbb{C} a \to a\) satisfying the axioms of a Lie-* bracket.

The above structures should be compatible in the following way: if \( a_1 \) and \( a_2 \) are sections of \( a \) and \( f \) is a function over \( Z \), we must have:

\[
[f \cdot a_1, a_2] = f \cdot [a_1, a_2] + a_1 \cdot \text{Lie}_\phi(a_2)(f).
\]

The most basic example of a Lie algebroid is \( a = \Theta Z \).

In a natural way one defines a notion of a (left) module over a Lie algebroid. In particular, if \( a \) is as above, any left D-module on \( Z \) is automatically an \( a \)-module.

Let now \( f : Z_1 \to Z_2 \) be a map between smooth varieties and let \( a \) be a Lie algebroid on \( Z_2 \). Consider the \( \mathcal{O} \)-module \( a' \) over \( Z_1 \) given by

\[
\ker[f^*(a) \oplus \Theta_{Z_1} \to f^*(\Theta_{Z_2})],
\]

where the maps \( f^*(a) \to f^*(\Theta_{Z_2}) \) and \( \Theta_{Z_1} \to f^*(\Theta_{Z_2}) \) are \( f^*(\phi) \) and \( df \), respectively.

**Lemma 2.2.** The \( \mathcal{O} \)-module \( a' \) on \( Z_1 \) has a canonical structure of Lie algebroid. Moreover, the functor \( f^* \) maps naturally left \( a \)-modules to left \( a' \)-modules.

We shall call \( a' \) the pull-back of \( a \) with respect to \( f \).

There are several other notions and constructions connected to Lie algebroids that will be used in this paper (such as Picard Lie algebroids and their connection with rings of twisted differential operators; Lie algebroids in the equivariant setting, in particular, Harish-Chandra Lie algebroids). They all are discussed in a clear and self-contained way in the first two sections of \cite{1}.

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Chapter I. Chiral Algebras

In this chapter we will be working over a fixed smooth curve $X$ (in Section 5 we will assume, in addition, that $X$ is complete). The symbols $O$ (resp., $\Omega$, $\Theta$) with an omitted subscript will mean the corresponding objects for $X$. However, all the results of this section are valid in the case when $X$ is in fact a family of curves of the above type over an arbitrary base; in particular, $X$ can be taken to be the universal curve $\mathcal{X}$ over the moduli stack $\mathcal{M}$ of smooth complete curves.

3. Definition of chiral algebras

3.1. Let $\mathcal{A}$ be a (left) D-module on $X$, which we shall think of as consisting of “fields” of our CFT. Consider the corresponding right D-module $\mathcal{A} = \mathcal{A} \otimes \Omega$; its sections should be viewed as fields with values in 1-forms, i.e. as currents.

A chiral algebra structure on $\mathcal{A}$ is a D-module map (called “chiral bracket”):

$$j_*j^*(\mathcal{A} \boxtimes \mathcal{A}) \xrightarrow{(\cdot)} \Delta_!(\mathcal{A}),$$

that satisfies the following two conditions:

- Antisymmetry:

  If $f(x, y) \cdot a \boxtimes b$ is a section of $j_*j^*(\mathcal{A} \boxtimes \mathcal{A})$, then
  $$\{f(x, y) \cdot a \boxtimes b\} = -\sigma_{1,2}(\{f(y, x) \cdot b \boxtimes a\}),$$

  where $\sigma_{1,2}$ is the transposition acting on $\Delta_!(\mathcal{A})$.

- Jacobi identity:

  If $a \boxtimes b \boxtimes c \cdot f(x, y, z)$ is a section of the restriction of $\mathcal{A} \boxtimes \mathcal{A} \boxtimes \mathcal{A}$ to the complement of the divisor of diagonals in $X \times X \times X$, then the element

$$\{\{f(x, y, z) \cdot a \boxtimes b\} \boxtimes c\} + \sigma_{1,2,3}(\{\{f(z, x) \cdot b \boxtimes c\} \boxtimes a\}) + \sigma_{1,2,3}(\{\{f(y, z) \cdot c \boxtimes a\} \boxtimes b\})$$

of $\Delta_{x=y=z}!(\mathcal{A})$ vanishes. (Here $\sigma_{1,2,3}$ denotes the lift of the cyclic automorphism of $X \times X \times X$: $(x, y, z) \rightarrow (y, z, x)$ to $\Delta_{x=y=z}!(\mathcal{A})$.)

A basic example of a chiral algebra is $\mathcal{A} = \Omega$. The bracket operation is given by the canonical map $\text{can}_\Omega : j_*j^*(\Omega \boxtimes \Omega) \rightarrow \Delta_!(\Omega)$ (see below).

In most examples we shall deal with chiral algebras that possess a unit: a unit in a chiral algebra $\mathcal{A}$ is a map $\text{unit} : \Omega \rightarrow \mathcal{A}$ such that the square below becomes commutative.

$$\begin{array}{ccc}
\mathcal{A} \otimes \Omega & \xrightarrow{\text{unit} \otimes \text{id}} & j_*j^*(\mathcal{A} \boxtimes \mathcal{A}) \\
\downarrow\text{can} & & \downarrow\text{id} \\
\Delta_!(\mathcal{A}) & \xrightarrow{\text{id}} & \Delta_!(\mathcal{A}),
\end{array}$$

where the left vertical arrow is the canonical map that comes from the short exact sequence

$$0 \rightarrow \Omega \rightarrow \mathcal{A} \rightarrow j_*j^*(\Omega \boxtimes \mathcal{A}) \rightarrow \Delta_!(\mathcal{A}) \rightarrow 0.$$

Along with ordinary chiral algebras one can consider the corresponding superobjects. In this case, the above antisymmetry and Jacobi identity conditions get transformed according to the sign rules of the super world.
3.2. Now let $\mathcal{A}$ be a chiral algebra and let $\mathcal{M}$ be a (right) D-module on $X$.

A chiral $\mathcal{A}$-module structure on $\mathcal{M}$ is a D-module map

$$
\rho : j_* j^*(\mathcal{A} \boxtimes \mathcal{M}) \to \Delta_1(\mathcal{M})
$$

such that if $f(x, y, z) \cdot a \boxtimes b \boxtimes m$ is a section of the restriction of $\mathcal{A} \boxtimes \mathcal{A} \boxtimes \mathcal{M}$ to the complement of the divisor of diagonals in $X \times X \times X$, we have:

$$
\rho(\{ f(x, y, z) \cdot a \boxtimes b \}, m) = \rho(a, \rho(f(x, y, z) \cdot b, m)) - \sigma_{1,2}(\rho(b, \rho(f(y, x, z) \cdot a, m)),
$$

as sections of $\Delta_{x=y=z}(\mathcal{M})$.

When $\mathcal{A}$ has a unit we require moreover that the induced map $j_* j^*(\Omega \boxtimes \mathcal{M}) \to \Delta_1(\mathcal{M})$ coincide with the map $\text{can}_\mathcal{M}$.

It is easy to check that the chiral bracket makes $\mathcal{A}$ into a chiral module over itself.

3.3. Let us now explain the concept of the “mode expansion” of a field.

Let $x$ be a point of $X$ and let $\mathcal{M}$ be a chiral $\mathcal{A}$-module supported at $x$; we shall denote by $\mathcal{M}$ the underlying vector space (i.e. $M \simeq h(\mathcal{M}) \simeq DR^0(\mathcal{M})$).

A typical example of such an $\mathcal{A}$-module is $\mathcal{M} := \Omega_{x!}^l(A)[1]$, called the vacuum module at $x$, in this case $A_x \simeq i^l_x(A)[1]$.

Let $\mathfrak{z}$ be a coordinate near $x$, i.e. $\mathfrak{z}$ is a regular function on some $U$ with $\mathfrak{z}(x) = 0$, $d\mathfrak{z}(x) \neq 0$. Then any section $a \in \Gamma(U, \mathcal{A})$ gives rise to a collection $a_n$, $n \in \mathbb{Z}$, of elements in $\text{End}(\mathcal{M})$.

By passing to the De Rham cohomology, the structure map $\rho$ gives rise to a map $DR^0(U \setminus x, \mathcal{A}) \boxtimes \mathcal{M} \to \Delta_1(\mathcal{M})$ and we take $a_n$ to be the endomorphism of $\mathcal{M}$ corresponding to the image of $a \cdot \mathfrak{z}^n$ under the projection

$$
\Gamma(U \setminus x, \mathcal{A}) \to DR^0(U \setminus x, \mathcal{A}).
$$

Remark 3.1. We shall see shortly that for any $(x \in U)$ as above, $DR^0(U \setminus x, \mathcal{A})$ has a natural structure of Lie algebra. The vector space $A_x$ together with the action of $DR^0(U \setminus x, \mathcal{A})$ on it is a prototype of the Hilbert space of our CFT.

Lemma 3.1. For any $m' \in M$ and $a$ as above we have

$$
a_n \cdot m' = 0 \text{ for } n >> 0.
$$

Now let $a$ and $b$ be two sections of $\mathcal{A}$ over $U$; we can produce a third section $c \in \Gamma(U, \mathcal{A})$ by setting

$$
c = (\text{id} \boxtimes h)(\{ \frac{1}{z_1 - z_2} \cdot a \boxtimes b \}) \in (h \boxtimes \text{id})(\Delta(\mathcal{A})) \simeq \mathcal{A},
$$

where $z_1 = \mathfrak{z} \boxtimes 1$ and $z_2 = 1 \boxtimes \mathfrak{z}$ are the corresponding functions on $X \times X$.

Proposition 3.1. Modes of $c$ can be expressed through the modes of $a$ and $b$ by the “normal ordering” formula:

$$
c_k =: a \cdot b \cdot \mathfrak{z}^k = \sum_{n \geq 0} a_{k-n} \cdot b_n + \sum_{n \geq 0} b_{-1-n} \cdot a_{k+n}.
$$

Note that both sides of the above formula depend on the choice of the coordinate $\mathfrak{z}$.
Proof. Choose a section \( m \in M \) which goes over to \( m' \in M \) under the identification \( DR^0(X,M) \simeq M \).

The element \( c_k \cdot m' \) is the image of 
\[
\rho\left(\frac{z_k^1}{z_1} \cdot a \boxtimes b\right), m) \in \Delta_1(M)
\]
under the identification \( DR^0(X \times X \times X, \Delta_{1,2,3!}(M)) \simeq DR^0(X,M) \simeq M \).

By definition, the above expression can be rewritten as 
\[
\rho(\frac{1}{1 - \frac{z_2}{z_1}}, a, \rho(\frac{1}{1 - \frac{z_2}{z_1}}, \sigma_{1,2}, \Delta_{1,2,3!}(M))) + \rho(\frac{1}{1 - \frac{z_2}{z_1}}, \sigma_{1,2}, \Delta_{1,2,3!}(M)).
\]
(In this formula we have omitted \( \sigma_{1,2} \) since it does not affect the image of our element in De Rham cohomology.)

Now, Lemma \( \boxed{1} \) implies that we can expand \( \frac{1}{1 - \frac{z_2}{z_1}} \) into a power series \( 1 + \frac{z_2}{z_1} + \frac{z_2^2}{z_1^2} + \ldots \) and our assertion follows.

4. Lie-* algebras and construction of chiral algebras

We shall now make a brief detour and discuss the auxiliary notion of a Lie-* algebra. These are objects that do not appear naturally in QFT; however, they are useful (and even necessary) for a construction of any non-trivial chiral algebra.

4.1. A Lie-* algebra on \( X \) is a (right) D-module \( \mathcal{B} \) with a map (called “Lie-* bracket”):
\[
\mathcal{B} \boxtimes \mathcal{B} \xrightarrow{\Delta_1} \Delta_1(\mathcal{B}),
\]
which is antisymmetric and satisfies the Jacobi identity in the sense similar to what we had in the definition of chiral algebras.

If \( \mathcal{B} \) is a Lie-* algebra, it follows from the definition that \( h(\mathcal{B}) \) is a sheaf (in the étale topology) of ordinary Lie algebras; moreover it acts on \( \mathcal{B} \) by endomorphisms of the D-module structure that are derivations of the Lie-* structure.

In particular, for an affine subset \( U \subset X \), \( DR^0(U, \mathcal{B}) \) is a Lie algebra.

4.2. As for modules over a Lie-* algebra, there are two types of these:
A Lie-* module over a Lie-* algebra \( \mathcal{B} \) is a D-module \( \mathcal{M} \) with a map
\[
\rho: \mathcal{B} \boxtimes \mathcal{M} \to \Delta_1(\mathcal{M})
\]
such that for a section \( a \boxtimes b \boxtimes m \) of \( \mathcal{B} \boxtimes \mathcal{B} \boxtimes \mathcal{M} \) the two sections
\[
\rho(a \boxtimes b), m) \text{ and } \rho(a, \rho(b, m)) - \sigma_{1,2}(\rho(b, \rho(a, m)))
\]
of \( \Delta_{1,2,3}(\mathcal{B}) \) coincide.

A chiral module over a Lie-* algebra is again a D-module \( \mathcal{M} \), but with an operation
\[
\rho: j_* j^*(\mathcal{B} \boxtimes \mathcal{M}) \to \Delta_1(\mathcal{M})
\]
such that every section
\[
f(x, y, z) \cdot a \boxtimes b \boxtimes m \in \Gamma((X \times X \times X \setminus (\Delta_{x=z} \cup \Delta_{y=z})), \mathcal{B} \boxtimes \mathcal{B} \boxtimes \mathcal{M})
\]
satisfies an identity similar to the one in the definition of chiral modules over a chiral algebra.
For example, if $M$ is a Lie-* (resp., chiral) module over a Lie-* algebra $B$ which is supported at $x \in U \subset X$, the Lie algebra $DR^0(U, B)$ (resp., $DR^0(U \setminus x, B)$) acts on the underlying vector space $M = DR^0(U, M)$.

We have an obvious forgetful functor from the category of chiral modules over a Lie-* algebra $B$ to the category of Lie-* modules.

In addition, there is a forgetful functor from the category of chiral algebras to the category of Lie-* algebras. If $M$ is a chiral module over a chiral algebra $A$ it is automatically a chiral module over the corresponding Lie-* algebra.

4.3. Let us now consider a few examples of Lie-* algebras.

**Example 1**
Let $g$ be a Lie algebra with an ad-invariant quadratic form $Q$ on it. Consider the D-module $B(g, Q) := g \otimes DX \oplus \Omega$. We define a Lie-* algebra structure on $B(g, Q)$ in the following way:

The Lie-* bracket $\{,\}$ factors through $B(g, Q) \boxtimes B(g, Q) \to g \otimes DX \boxtimes g \otimes DX$ and

$$\{\xi_1 \otimes 1 \boxtimes \xi_2 \otimes 1\} = [\xi_1, \xi_2] \otimes 1 \oplus Q(\xi_1, \xi_2) \otimes 1^\prime,$$

where $1$ is the unit section of $DX \subset \Delta(DX)$ and $1^\prime$ is the canonical antisymmetric section of $\Omega \boxtimes \Omega(2 \cdot \Delta)/\Omega \boxtimes \Omega$.

If Spec($\hat{O}_x$) (resp., Spec($\hat{K}_x$)) is the formal disc (resp., formal punctured disc) around $x \in X$, we can naturally identify the Lie algebras $DR^0(\text{Spec}(\hat{O}_x), B(g, Q))$ and $DR^0(\text{Spec}(\hat{K}_x), B(g, Q))$ with $g \otimes \hat{O}_x$ and $g \otimes \hat{K}_x \oplus \mathbb{C}$. For instance, when $g$ is semisimple, $DR^0(\text{Spec}(\hat{K}_x), B(g, Q))$ is the corresponding Kac-Moody algebra.

**Example 2**
Let $\mathcal{I}$ denote the D-module $\Theta \otimes DX$. The action of $\Theta$ on itself by Lie derivations is a bi-differential operator $\Theta \otimes \Theta \to \Theta$; hence, according to Section 2, it gives rise to a D-module map

$$\mathcal{I} \boxtimes \mathcal{I} \simeq \Theta \otimes DX \boxtimes \Theta \otimes DX \to \Theta \otimes DX \otimes DX \simeq \Delta_!(\mathcal{I}),$$

which we take to be minus the structure map for the Lie-* algebra $\mathcal{I}$.

It is easy to check that the resulting map satisfies the axioms of a Lie-* bracket. The Lie algebras $DR^0(\text{Spec}(\hat{O}_x), \mathcal{I})$ and $DR^0(\text{Spec}(\hat{K}_x), \mathcal{I})$ identify with $\text{Vir}^+ := \Theta \otimes \hat{O}_x$ and $\text{Vir} := \Theta \otimes \hat{K}_x$, respectively (the bracket being minus the usual one).

**Example 3**
Let $\Theta'$ be an extension $0 \to \Omega \to \Theta' \to \Theta \to 0$. Assume that the sheaf $\Theta$ of vector fields on $X$ acts on $\Theta'$ "by Lie derivations" in a way compatible with the above filtration and with its action on $\Omega$ and on $\Theta$. We shall endow the D-module

$$\mathcal{I}' := \Theta' \otimes DX / \ker(\Omega \otimes DX \to \Omega)$$

\footnote{The minus sign is chosen in order to be consistent with the physics literature.}

\footnote{We say that $\Theta$ acts on an $O$-module $\mathcal{L}$ by Lie derivations if the action map $\Theta \otimes \mathcal{L} \to \mathcal{L}$ is a bi-differential operator that satisfies

$$\xi(f \cdot l) = f \cdot \xi(l) + \text{Lie}_\xi(f) \cdot l,$$

for $\xi \in \Theta$, $l \in \Theta'$ and $f \in O$.}

We have an obvious forgetful functor from the category of chiral modules over a Lie-* algebra $B$ to the category of Lie-* modules.
with a Lie-* algebra structure. As a D-module $\mathcal{T}'$ is an extension
$$0 \rightarrow \Omega \rightarrow \mathcal{T}' \rightarrow \mathcal{T} \rightarrow 0$$
and the Lie-* algebra structure on it will be such that $\Omega$ is central (i.e. annihilated by the Lie-* bracket) and $\mathcal{T}'$ will be a central extension of the Lie-* algebra $\mathcal{T}$ considered in Ex.2.

According to Section 2, the action of $\Theta$ on $\Theta'$ can be viewed as a map
$$\Theta \boxtimes \Theta' \rightarrow \Delta_t(\Theta') \otimes D_{X \times X} \simeq \Delta_t(\Theta' \otimes D_X).$$

Therefore, we obtain a D-module map
$$(\Theta \otimes D_X) \boxtimes (\Theta' \otimes D_X) \rightarrow \Delta_t(\Theta' \otimes D_X) \rightarrow \Delta_t(\mathcal{T}').$$

It is easy to see that $\Theta \otimes D_X \boxtimes \Omega \otimes D_X$ lies in the kernel of the above map, which means that it factors as $\mathcal{T} \boxtimes \mathcal{T} \rightarrow \Delta_t(\mathcal{T}')$. We define now the Lie-* bracket
$$\mathcal{T}' \boxtimes \mathcal{T}' \rightarrow \mathcal{T} \boxtimes \mathcal{T} \rightarrow \Delta_t(\mathcal{T'}).$$

We shall see in Section 8.1 that the set of isomorphism classes of extensions $\Theta'$ as above is in bijection with $\mathbb{C}$. For any $\Theta'$ of this form, the Lie algebra $DR^0(\text{Spec}(\hat{O}_x), \mathcal{T}')$ is the corresponding Virasoro extension
$$0 \rightarrow \mathbb{C} \rightarrow \text{Vir}' \rightarrow \text{Vir} \rightarrow 0.$$

4.4. A fact of crucial importance for us is that the forgetful functor from chiral algebras to Lie-* algebras admits a left adjoint, i.e. that for every Lie-* algebra $\mathcal{B}$ there exists a (unital) chiral algebra $\mathcal{U}(\mathcal{B})$ (called the universal enveloping algebra of $\mathcal{B}$) such that
$$\text{Hom}_{\text{Lie-*}}(\mathcal{B}, A) \simeq \text{Hom}_{\text{ch}}(\mathcal{U}(\mathcal{B}), A),$$
functorially in $A$.

Let us sketch the construction of $\mathcal{U}(\mathcal{B})$ (we shall assume for simplicity that $\mathcal{B}$ is torsion-free as an $\mathcal{O}$-module).

The construction will be local and we can assume that $X$ is affine. Consider the D-module direct images $B_1 := p_1^*(\Omega \boxtimes \mathcal{B})$ and $B_2 := p_1^*(j_* j^*(\Omega \boxtimes \mathcal{B}))$ on $X$ (here $p_1$ is the projection $(x, y) \in X \times X \rightarrow x \in X$). There is an obvious map $B_1 \rightarrow B_2$ and from the fact that $\mathcal{B}$ is a Lie-* algebra we infer that both $B_1$ and $B_2$ are Lie algebras in the category of D-modules.

We define $\mathcal{U}(\mathcal{B})$ to be the “vacuum representation”, i.e.
$$\mathcal{U}(\mathcal{B}) := \mathcal{U}(B_2) \otimes_{\mathcal{U}(B_1)} \mathbb{C},$$
where “$\mathcal{U}$” is the universal enveloping algebra in the ordinary sense. By definition, for $x \in X$, the fiber of $\mathcal{U}(\mathcal{B})$ at $x$ is identified with $\mathcal{U}(DR^0(X \setminus x, \mathcal{B})) \otimes_{\mathcal{U}(DR^0(X, \mathcal{B}))} \mathbb{C}$.

Since $X$ is affine, the cokernel
$$\text{coker}(DR^0(X, \mathcal{B}) \rightarrow DR^0(X \setminus x, \mathcal{B}))$$
is identified with
$$\text{coker}(DR^0(\text{Spec}(\hat{O}_x), \mathcal{B}) \rightarrow DR^0(\text{Spec}(\hat{K}_x), \mathcal{B})) \simeq B_x := i^*_x(\mathcal{B})[1].$$

4The category of right D-modules is a tensor category under the operation $\otimes$, cf. Section 2.
This description implies that, first of all, \( \mathcal{U}(\mathcal{B}) \) constructed in the above way will not change if we shrink \( X \) to a smaller open affine subset, and secondly, that \( \mathcal{U}(\mathcal{B}) \) is naturally a chiral module over the Lie-* algebra \( \mathcal{B} \).

Note, that by the construction, \( \mathcal{U}(\mathcal{B}) \) carries a filtration \( \mathcal{U}(\mathcal{B}) \simeq \bigcup_{n \geq 0} \mathcal{U}(\mathcal{B})_n \) induced from the filtration on \( U(B_2) \). We have \( \mathcal{U}(\mathcal{B})_0 \simeq \Omega, \mathcal{U}(\mathcal{B})_1 \simeq \mathcal{U}(\mathcal{B})_0 \oplus \mathcal{B} \).

**Lemma 4.1.** Consider the chiral action of \( j_*j^*(\mathcal{B} \boxtimes \mathcal{U}(\mathcal{B})) \to \Delta_!(\mathcal{U}(\mathcal{B})) \). We have:

(a) \( \text{Im}(\mathcal{B} \boxtimes \mathcal{U}(\mathcal{B})_n) = \Delta_i(\mathcal{U}(\mathcal{B})_n) \).

(b) \( \text{Im}(j_*j^*(\mathcal{B} \boxtimes \mathcal{U}(\mathcal{B})_n)) = \Delta_i(\mathcal{U}(\mathcal{B})_{n+1}) \).

(c) \( \mathcal{U}(\mathcal{B})_n/\mathcal{U}(\mathcal{B})_{n-1} \simeq \text{Sym}^n(\mathcal{B}) \).

4.5. We shall now endow \( \mathcal{U}(\mathcal{B}) \) with a chiral algebra structure. If fact, we will show that whenever \( \mathcal{M} \) is a chiral \( \mathcal{B} \)-module, there exists a canonical map

\[
j_*j^*(\mathcal{U}(\mathcal{B}) \boxtimes \mathcal{M}) \to \Delta_!(\mathcal{M})
\]

Consider the D-modules:

\[
j_y \neq z, j_y^*(\Omega \boxtimes \Omega \boxtimes \mathcal{B}) \quad \text{and} \quad j_{x \neq y \neq z} j_{x \neq y \neq y} j_{x \neq z \neq y}(\Omega \boxtimes \Omega \boxtimes \mathcal{B})
\]

on \( X \times X \times X \) and let \( B'_1 \) and \( B'_2 \) denote their direct (D-module) images with respect to the projection \((x, y, z) \to (x, y)\).

We have an embedding \( B'_1 \to B'_2 \) and the Lie-* algebra structure on \( \mathcal{B} \) makes \( B'_1 \) and \( B'_2 \) into Lie algebras in the category of D-modules on \( X \times X \). For any \((x, y) \in X \times X\), the fiber of \( B'_1 \) (resp., of \( B'_2 \)) at this point is identified with \( DR^0(X \setminus \{y, \mathcal{B}\}) \) (resp., with \( DR^0(X \setminus \{x, \mathcal{B}\}) \)). In particular, for a chiral \( \mathcal{B} \)-module \( \mathcal{M} \) and for \((x, y)\) as above, the action of \( DR^0(X \setminus \{y\}, \mathcal{B}) \) on \( M_y := i^!_y(\mathcal{M})[1] \) defines on \( \Omega \boxtimes \mathcal{M} \) a structure of module over \( B'_1 \). Consider now the induced module \( \mathcal{M}' := U(B'_2) \otimes_{U(B'_1)} (\Omega \boxtimes \mathcal{M}) \). It is clear that \( j^!(\mathcal{M}') \simeq j^!(\mathcal{U}(\mathcal{B}) \boxtimes \mathcal{M}) \) and that \( \Delta_!(\mathcal{M'})[1] \simeq \mathcal{M} \). The canonical map

\[
j_*j^!(\mathcal{M}') \to \Delta_!(\mathcal{M'})[1]
\]

yields, therefore, a map \( j_*j^!(\mathcal{U}(\mathcal{B}) \boxtimes \mathcal{M}) \to \Delta_!(\mathcal{M}) \).

It is easy to show now that if we set \( \mathcal{M} \) to be \( \mathcal{U}(\mathcal{B}) \), then the resulting operation on \( \mathcal{U}(\mathcal{B}) \) will satisfy the axioms of a chiral bracket. Moreover, the above construction shows that any \( \mathcal{M} \) which is chiral module over \( \mathcal{B} \) acquires a structure of chiral module over \( \mathcal{U}(\mathcal{B}) \).

5. **Conformal blocks, correlation functions**

From now on we shall assume that \( \mathcal{A} \) is a chiral algebra possessing a unit.

---

5 Here \( j_y \neq z \) and \( j_{x \neq y \neq z} \) denote the open embeddings of the complements to the divisors \( \Delta_{y=z} \) and \( \Delta_{x=z} \cup \Delta_{y=z} \) in \( X \times X \times X \) respectively.
5.1. We have a surjection of D-modules $j_\ast j^\ast(A \boxtimes A) \to \Delta_t(A)$; let $A^{(2)}$ denote its kernel.

For a complete curve $X$ the space of conformal blocks $H_\nabla(X, A)$ of $A$ is by definition the vector space $DR^2(X \times X, A^{(2)})$. From the long exact cohomology sequence we obtain that

$$H_\nabla(X, A) \simeq \text{coker}(DR^1(X \times X \setminus \Delta(X), j_\ast j^\ast(A \boxtimes A)) \to DR^1(X, A)).$$

Let $x$ be a point of $X$ and recall that $A_x$ is a $DR^0(X \setminus x, A)$-module.

**Proposition 5.1.** The space $H_\nabla(X, A)$ is identified naturally with the space of coinvariants

$$\text{coker}(DR^0(X \setminus x, A) \otimes A_x \to A_x).$$

**Proof.** We have a commutative square:

$$\begin{array}{ccc}
DR^0(X \setminus x, A) \otimes A_x & \longrightarrow & A_x \\
\downarrow & & \downarrow \\
DR^1(X \times X, j_\ast j^\ast(A \boxtimes A)) & \longrightarrow & DR^1(X, A)
\end{array}$$

such that the vertical arrows are surjective. To prove the statement it remains to show that the kernel of the map $A_x \to DR^1(X, A)$ contains the image of $DR^0(X \setminus x, B) \otimes A_x$.

The unit in $A$ provides a canonical element $\text{unit}_x \in A_x$ and the map $DR^0(X \setminus x, A) \simeq DR^0(X \setminus x, A) \otimes \text{unit}_x \to A_x$ coincides with the canonical map $DR^0(X \setminus x, A) \to A_x$.

Since the image of the latter is precisely the kernel of the map $A_x \to DR^1(X, A)$ our assertion follows.

The following observation is useful for actual computations of the space of conformal blocks:

**Proposition 5.2.** Let $A$ be a chiral universal enveloping algebra of a Lie-* algebra $B$. Then the natural map

$$A_x/(DR^0(X \setminus x, B) \otimes A_x) \to A_x/(DR^0(X \setminus x, A) \otimes A_x) \simeq H_\nabla(X, A)$$

is an isomorphism.

**Proof.** We will prove a slightly more general assertion: we can replace $A_x$ in the formulation of the proposition by the underlying space $M$ of any chiral $A$-module $M$ supported at $x$.

Let $A_n$ be the filtration on $A$ as in Section 4.4. We must show that the surjection

$$M/(DR^0(X \setminus x, B) \otimes M) \to M/(DR^0(X \setminus x, A_n) \otimes M)$$

is an isomorphism for any $n \geq 1$. We shall argue by induction.

For $a \in \Gamma(X \setminus x, A_{n+1})$ and $m \in M$ consider the element $\rho(h(a), m) \in M$. By the construction, it belongs to $\text{Im}(DR^0(X \setminus x, A_{n+1}) \otimes M)$ and it would suffice to show that it also belongs to $\text{Im}(DR^0(X \setminus x, A_n) \otimes M)$. 


Using the Lemma 4.1 and the fact that $X \setminus x$ is affine, for $a \in \Gamma(X \setminus x, A_{n+1})$ as above, we can find a section $b \boxtimes a' \cdot f(x, y) \in \Gamma(X \setminus x \times X \setminus x, j_x j^*(\mathcal{B} \boxtimes A_n))$ such that

$$(h \boxtimes \text{id})(\{b \boxtimes a' \cdot f(x, y)\}) = a.$$  

Using the Jacobi identity, the element $\rho(h(a), m) \in M$ is the image under the projection

$$h \boxtimes h \boxtimes h : \Delta_{1,2,3}(M) \rightarrow M$$  

of the element

$$\rho(a', \rho(b, m \cdot f(x, y))) - \rho(b, \rho(a', m \cdot f(y, x))).$$  

Now, the first term obviously belongs to $\text{Im}(\text{DR}^0(X \setminus x, A_n) \otimes M)$ and the second term even belongs to

$$\text{Im}(\text{DR}^0(X \setminus x, \mathcal{B}) \otimes M) \subset \text{Im}(\text{DR}^0(X \setminus x, A_n) \otimes M).$$

5.2. Now let $x_1, \ldots, x_k$ be a non-empty collection of distinct points of $X$. Each $A_{x_i}$ is a module over the Lie algebra $\text{DR}^0(X \setminus \{x_1, \ldots, x_k\}, \mathcal{A})$; hence, so is the tensor product $A_{x_1} \otimes \ldots \otimes A_{x_k}$.

**Proposition 5.3.** The space $H_\nabla(X, \mathcal{A})$ is canonically isomorphic to the space of coinvariants

$$\text{coker}(\text{DR}^0(X \setminus \{x_1, \ldots, x_k\}, \mathcal{A}) \otimes (A_{x_1} \otimes \ldots \otimes A_{x_k}) \rightarrow A_{x_1} \otimes \ldots \otimes A_{x_k}).$$

**Proof.** When $k = 1$ our assertion coincides with that of Proposition 5.1 above. To treat the case $k \geq 2$ we shall proceed by induction, so we can assume that the assertion is true for $k - 1$.

We shall construct two mutually inverse maps $\phi$ and $\psi$ between

$$(A_{x_1} \otimes \ldots \otimes A_{x_{k-1}}) \otimes (A_{x_1} \otimes \ldots \otimes A_{x_k}) \rightarrow (A_{x_1} \otimes \ldots \otimes A_{x_k}) \otimes (A_{x_1} \otimes \ldots \otimes A_{x_{k-1}}).$$

For $a_1 \otimes \ldots \otimes a_{k-1}$ in $A_{x_1} \otimes \ldots \otimes A_{x_{k-1}}$ we set $\phi(a_1 \otimes \ldots \otimes a_{k-1}) \in A_{x_1} \otimes \ldots \otimes A_{x_k}$ to be $a_1 \otimes \ldots \otimes a_{k-1} \otimes \text{unit}_{x_k}$. It is clear that $\phi$ induces a well-defined map between the spaces of coinvariants.

Now let $a_1 \otimes \ldots \otimes a_{k-1} \otimes b'$ be an element of $A_{x_1} \otimes \ldots \otimes A_{x_{k-1}} \otimes A_{x_k}$. Since $X \setminus \{x_1, \ldots, x_{k-1}\}$ is affine, we can find an element $b \in \text{DR}^0(X \setminus \{x_1, \ldots, x_k\}, \mathcal{A})$ that maps to $b'$ under the canonical map $\text{DR}^0(X \setminus \{x_1, \ldots, x_k\}, \mathcal{A}) \rightarrow A_{x_k}$.

We set $\psi(a_1 \otimes \ldots \otimes a_{k-1} \otimes b') \in A_{x_1} \otimes \ldots \otimes A_{x_{k-1}}$ to be $-b(a_1 \otimes \ldots \otimes a_{k-1})$. It is again easy to see that $\psi$ is well-defined as a map between the coinvariants and that $\phi$ and $\psi$ are mutually inverse.

It remains to show that the isomorphism

$$\text{coker}(\text{DR}^0(X \setminus \{x_1, \ldots, x_k\}, \mathcal{A}) \otimes (A_{x_1} \otimes \ldots \otimes A_{x_k}) \rightarrow A_{x_1} \otimes \ldots \otimes A_{x_k}) \simeq H_\nabla(X, \mathcal{A})$$

does not depend on the ordering of the points $x_1, \ldots, x_k$. Obviously, it is enough to do this when $k = 2$.

However, for two distinct points $x_1, x_2 \in X$, our map $A_{x_1} \otimes A_{x_2} \rightarrow H_\nabla(X, \mathcal{A})$ coincides with the canonical map

$$A_{x_1} \otimes A_{x_2} \simeq i_{x_1 \times x_2}^*(\mathcal{A} \boxtimes \mathcal{A})[2] \simeq i_{x_1 \times x_2}^*(\mathcal{A}^{(2)})[2] \rightarrow \text{DR}^2(X \times X, \mathcal{A}^{(2)}) \simeq H_\nabla(X, \mathcal{A})$$.
and our assertion follows, since the fact that \{,\} is antisymmetric implies that the isomorphism $DR^2(X \times X, \mathcal{A}^{(2)}) \simeq H_\nabla(X, \mathcal{A})$ is invariant with respect to the transposition.

\[\square\]

5.3. By globalizing the argument of Proposition $\PageIndex{3}$ we obtain that there exists a canonical map of D-modules on $X^n$:

\[\langle \ldots \rangle : j_* j^*(\mathcal{A} \boxtimes \ldots \boxtimes \mathcal{A}) \to j_* j^*(\Omega \boxtimes \ldots \boxtimes \Omega) \otimes H_\nabla(X, \mathcal{A})\]

with the following properties:

- For $x_1, \ldots, x_n$ distinct points on $X$ the induced map $\langle \ldots \rangle : A_{x_1} \otimes \ldots \otimes A_{x_n} \to H_\nabla(X, \mathcal{A})$ coincides with the composition

\[A_{x_1} \otimes \ldots \otimes A_{x_1} \to (A_{x_1} \otimes \ldots \otimes A_{x_n})\text{DR}^n(X \setminus \{x_1, \ldots, x_n\}, \mathcal{A}) \simeq H_\nabla(X, \mathcal{A}).\]

In particular, the map $\langle \ldots \rangle$ is equivariant with respect to the action of the symmetric group $S^n$ on $j_* j^*(\mathcal{A}^\otimes n)$ and on $j_* j^*(\Omega^\otimes n)$.

- The square

\[
\begin{array}{c}
j_* j^*(\mathcal{A}^\otimes n) \xrightarrow{\langle \ldots \rangle} j_* j^*(\Omega^\otimes n) \otimes H_\nabla(X, \mathcal{A}) \\
id^\otimes_{\mathcal{A}} \otimes \text{id} \downarrow \quad \downarrow \text{id}^\otimes_{\Omega}
j_* j^* (\mathcal{A}^\otimes n) \xrightarrow{\langle \ldots \rangle} j_* j^*(\Omega^\otimes n) \otimes H_\nabla(X, \mathcal{A})
\end{array}
\]

is commutative.

- The square

\[
\begin{array}{c}
\langle \ldots \rangle j_* j^*(\mathcal{A}^\otimes n) \xrightarrow{\langle \ldots \rangle} j_* j^*(\Omega^\otimes n) \otimes H_\nabla(X, \mathcal{A}) \\
\Delta_{x_1=x_2} (j_* j^*(\mathcal{A}^\otimes n)) \xrightarrow{\langle \ldots \rangle} \Delta_{x_1=x_2} (j_* j^*(\Omega^\otimes n)) \otimes H_\nabla(X, \mathcal{A}).
\end{array}
\]

commutes as well.

(The first two of the above properties are evident from the definitions. The third one can be easily deduced from the second one using the fact that the upper horizontal arrow in the first commutative diagram is surjective).

By passing to the corresponding left D-modules, we obtain a canonical $S^n$-invariant map

\[\langle \ldots \rangle : j_* j^*(\mathcal{L} \boxtimes \ldots \boxtimes \mathcal{L}) \to j_* j^*(\Omega \boxtimes \ldots \boxtimes \Omega) \otimes H_\nabla(X, \mathcal{A}).\]

\textbf{Remark 5.1.} For a functional $\chi$ on $H_\nabla(X, \mathcal{A})$ and a section $a_1 \boxtimes \ldots \boxtimes a_n \in \mathcal{L}^\otimes n$ we can produce, therefore, a function $\langle a_1, \ldots, a_n \rangle_\chi$ on $X^n$ with poles on the diagonal divisor. This function should be thought of as an analog of the $n$-point correlation function of the fields $a_1, \ldots, a_n$. 
Chapter II. CFT Data (Algebraic Version)

Recall that $\mathcal{X}$ denotes the universal curve over the moduli stack $\mathcal{M}$ of smooth complete curves. In this section the default meaning of $\mathcal{O}$ (resp., $\Omega$, $\Theta$, $D$) is the structure sheaf (resp., the sheaf of relative 1-forms, the sheaf of vertical vector fields, vertical differential operators) on $\mathcal{X}$; $\mathcal{D}$-modules (unless specified otherwise) are right modules over the sheaf of rings $D$.

6. Local $\mathcal{O}$-modules on $\mathcal{X}$

On the intuitive level, we want to call a sheaf $\mathcal{L}$ on $\mathcal{X}$ local if the following holds: whenever $\mathcal{X}$ and $\mathcal{X}'$ are two curves such that their (analytic) open subsets $U \subset \mathcal{X}$ and $U' \subset \mathcal{X}'$ are identified, the sheaves $(\mathcal{L}|X)/U$ and $(\mathcal{L}|X')/U'$ are identified as well. For instance, if we are dealing with a 2-dimensional CFT which is defined for every Riemann surface, its fields form naturally a local sheaf over $\mathcal{X}$.

6.1. Let $\mathcal{L}$ be an $\mathcal{O}$-module on $\mathcal{X}$, i.e. for a scheme $S$ with a map to $\mathcal{M}$ we have an $\mathcal{O}$-module $\mathcal{L}|S$ on the corresponding family of curves $\mathcal{X}|S$. We say that $\mathcal{L}$ is local if it possesses the following additional structure:

Let $I$ be a local Artinian scheme and let $X^{S \times I}$ be a family of curves over $X \times I$ extending $X^S$. Let, in addition, $x_1^{S \times I}, \ldots, x_n^{S \times I}$ be sections $S \times I \to X^{S \times I}$ (we shall denote by $x_1^S, \ldots, x_n^S$ their restrictions to $S \to S \times I$). Assume now that $\phi^{S,I}$ is an isomorphism:

$$X^{S \times I} \setminus \{x_1^{S \times I}, \ldots, x_n^{S \times I}\} \simeq (X^S \setminus \{x_1^S, \ldots, x_n^S\}) \times I.$$ 

A local structure on $\mathcal{L}$ is lifting of $\phi^{S,I}$ to an isomorphism of sheaves

$$\beta^{S,I} : \phi^{S,I*(\mathcal{L}|X^S| \setminus \{x_1^S, \ldots, x_n^S\} \otimes \mathcal{O}(I))} \to \mathcal{L}^{S \times I}|X^{S \times I} \setminus \{x_1^{S \times I}, \ldots, x_n^{S \times I}\}.$$ 

These $\beta^{S,I}$s for various $X^{S \times I}, x_1^{S \times I}, \ldots, x_n^{S \times I}$, $\phi^I$ must satisfy two conditions. The first condition is a compatibility in the obvious sense in the situation when a map $S \times I \to \mathcal{M}$ factors as $S \times I \to S' \times I' \to \mathcal{M}$. To formulate the second condition we need to introduce some notation.

Let $X^S$ be again a curve over the base $S$ and let $y_1^S, \ldots, y_n^S$ and $z_1^S, \ldots, z_k^S$ be two disjoint collections of sections $S \to X^S$. Let $I$ and $J$ be two local Artinian schemes and let $(X^{S \times I}, y_1^{S \times I}, \ldots, y_n^{S \times I}, \phi^I)$ and $(X^{S \times J}, z_1^{S \times J}, \ldots, z_k^{S \times J}, \phi^J)$ be two pieces of data as above. (In what follows, to simplify the notation, we shall abbreviate $\{y_1, \ldots, y_n\}$ and $\{z_1, \ldots, z_m\}$ to $y$ and $z$ respectively.)

We have sections $\tilde{y}^I : S \times J \to X^{S \times J}$ and $\tilde{z}^I : S \times I \to X^{S \times I}$ and we can construct a curve $X^{S \times I \times J}$ over $S \times I \times J$ equipped with sections $y^{S \times I \times J}, z^{S \times I \times J}$ such that:

\begin{align*}
(1) & \quad X^{S \times I \times J} \setminus y^{S \times I \times J} \phi^{S \times I \times J,I} \simeq (X^I \setminus \tilde{y}^I) \times I \\
(2) & \quad X^{S \times I \times J} \setminus z^{S \times I \times J} \phi^{S \times I \times J,J} \simeq (X^I \setminus \tilde{z}^I) \times J \\
(3) & \quad X^{S \times I \times J} \setminus \{y^{S \times I \times J}, z^{S \times I \times J}\} \phi^{S \times I \times J} \simeq (X^S \setminus \{y^S, z^S\}) \times I \times J.
\end{align*}

Note that the composition:

$$\phi^{S,J} \circ \phi^{S,I} : X^{S \times I \times J} \setminus \{y^{S \times I \times J}, z^{S \times I \times J}\} \to (X^S \setminus \{y^S, z^S\}) \times I \times J$$

implies a well-defined map $X^S \to (X^S \otimes \mathcal{O}(I)) \times (X^S \otimes \mathcal{O}(J))$.


coincides with $\phi^{S,I \times J}$.

The second condition on $\beta$ reads as follows:

We need that the two isomorphisms between

$$\phi^{S,I \times J *}(\mathcal{L}^S | X \setminus \{y^S, z^S\} \otimes \mathcal{O}(I) \otimes \mathcal{O}(J))$$

and

$$(\mathcal{L}^{S \times I \times J}) | X^{S \times I \times J} \setminus \{y^{S \times I \times J}, z^{S \times I \times J}\},$$

namely, $\beta^{S,I \times J}$ and $(\beta^{S,J \times I} \circ \text{id}_I) \circ ((\phi^{S,J} \times \text{id}_I)^*(\beta^{S,J,I})))$, coincide.

The most basic example of a local $\mathcal{O}$-module on $X$ is the sheaf $\mathcal{O}$ itself. Local $\mathcal{O}$-modules on $X$ form an abelian category (morphisms in this category are, by definition, maps of $\mathcal{O}$-modules compatible with the $\beta^{S,I}$'s). It is easy to see that the sheaves $\Omega^i$, $D$, etc. are all local $\mathcal{O}$-modules in a natural way.

In a similar way one defines local $\mathcal{O}$-modules on $X^n$.

An $\mathcal{O}$-module $\mathcal{L}$ on $X^n$ assigns, by definition, to every scheme $S$ mapping to $\mathfrak{M}$ an $\mathcal{O}$-module $\mathcal{L}^S$ over

$$X^S \times_S \ldots \times_S X^S,$$

where $X^S$ is as before. A local structure on $\mathcal{L}$ attaches to each data $X^{S,I}, x^S_k, \phi^{S,I}$ as above an isomorphism

$$\beta^{S,I} : \phi^{S,I *}(\mathcal{L}^S | (X^S \setminus \{x^S_1, \ldots, x^S_n\}) \times_S \ldots \times_S (X^S \setminus \{x^S_1, \ldots, x^S_n\}) \otimes \mathcal{O}(I)) \simeq$$

$$(\mathcal{L}^{S \times I}) | (X^{S \times I} \setminus \{x^{S \times I}_1, \ldots, x^{S \times I}_n\}, x^{S \times I}_1 \times_S \ldots \times_S x^{S \times I}_n).$$

These isomorphisms should satisfy two conditions analogous to what we had in the case $n = 1$.

For example, if $\mathcal{L}_1$ and $\mathcal{L}_2$ are local $\mathcal{O}$-modules on $X$, then $\mathcal{L}_1 \boxtimes \mathcal{L}_2$ is a local $\mathcal{O}$-module on $X^2$. In addition, if $\mathcal{L}$ is a local $\mathcal{O}$-module on $X$, so is $j_* j^*(\mathcal{L})$, and $\Delta^*(\mathcal{L})$ is a local $\mathcal{O}$-module back on $X$.

In particular, we have the following assertion:

**Lemma 6.1.** Let $\mathcal{B}$ be a local $\mathcal{O}$-module on $X$ with the following additional structure: $\mathcal{B}$ is a $D$-module and a Lie-* algebra such that:

(a) The map $\mathcal{B} \otimes D \to \mathcal{B}$ is a map of local $\mathcal{O}$-modules on $X$.

(b) The Lie-* bracket $\mathcal{B} \boxtimes \mathcal{B} \to \Delta_!(\mathcal{B})$ is a map of local $\mathcal{O}$-modules on $X^2$ (the RHS is a local $\mathcal{O}$-module on $X^2$ in a natural way).

Then the chiral universal enveloping algebra $\mathcal{U}(\mathcal{B})$ is a local $\mathcal{O}$-module on $X$ and conditions (a) and (b) hold for $\mathcal{U}(\mathcal{B})$ as well.
6.2. We shall now describe a typical way of producing local \(\mathcal{O}\)-modules.

Consider the standard 1-dimensional local ring \(\hat{\mathcal{O}} := \mathbb{C}[[t]]\) and the corresponding local field \(\hat{\mathbb{C}} := \mathbb{C}(t)\). Let \(\text{Aut}_0^+\) (resp., \(\text{Aut}^+, \text{Aut}\)) be the group scheme of automorphisms of \(\hat{\mathcal{O}}\) that preserve the maximal ideal (resp., the group ind-scheme of all automorphisms of \(\hat{\mathcal{O}}\), the group ind-scheme of all automorphisms of \(\hat{\mathbb{C}}\)). We shall denote by \(\text{Vir}_0^+ \subset \text{Vir}^+ \subset \text{Vir}\) the corresponding Lie algebras.

Consider the scheme \(\hat{\mathbf{X}}\) that classifies the following data:

\[
(X \in \mathfrak{M}, x \in X, \alpha : \hat{\mathcal{O}}_x \to \hat{\mathcal{O}}),
\]

where \(\alpha\) is an isomorphism that preserves the maximal ideal. The group \(\text{Aut}_0^+\) acts on this scheme by changing \(\alpha\) and the quotient is identified with \(\hat{\mathbf{X}}\). Let \(q\) denote the projection of \(\hat{\mathbf{X}}\) to \(\mathbf{X}\).

If now \(V\) is a representation of the group \(\text{Aut}_0^+\), the \(\mathcal{O}_\mathbf{X}\)-module \(V \otimes \mathcal{O}_{\hat{\mathbf{X}}}\) is \(\text{Aut}_0^+\)-equivariant and hence descends to an \(\mathcal{O}\)-module \(\text{Ass}(V)\) over \(\mathbf{X}\). The next assertion is evident from the definitions:

**Lemma 6.2.** For \(V\) as above, the \(\mathcal{O}\)-module \(\text{Ass}(V)\) has a natural structure of local \(\mathcal{O}\)-module.

For instance, for \(i \in \mathbb{Z}\) let \(L_i\) be the 1-dimensional \(\text{Aut}_0^+\)-module of Section 8.1. \((\text{Aut}_0^+\) acts on \(L_i\) via the \(i\)-th power of the standard character.) We have:

\[
\text{Ass}(L_i) \simeq \Omega^i.
\]

6.3. Let \(\mathcal{L}\) be a local \(\mathcal{O}\)-module on \(\mathbf{X}\). We shall now construct a map\(^6\)

\[
\nabla_{\text{loc}} : \mathcal{L} \to p_{1*} (\mathcal{L} \boxtimes \Omega^2(\infty \cdot \Delta)).
\]

Let \(x \in X \in \mathfrak{M}\) be a point of \(\mathbf{X}\), let \(l\) be a local section of \(\mathcal{L}\) on \(\mathbf{X}\) and let \(\xi\) be an element of \(\mathcal{K}_x / \mathcal{O}_x \otimes \Theta\).

Constructing \(\nabla_{\text{loc}}\) amounts to our being able to evaluate \((\text{id} \boxtimes \text{Res}_x) (\nabla_{\text{loc}}(l) \otimes (1 \boxtimes \xi))\) as a local section of \(\mathcal{L}|X\) defined outside of \(x\). (The triple \((X, x, \xi)\) should be thought of as living over a base \(S\), where \(S\) is a scheme mapping to \(\mathfrak{M}\); we shall denote by \(x|X\) the \(\mathcal{O}\)-module \(\mathcal{L}^S\) over \(X^S\).)

However, \(\xi\) as above is the same as a deformation of \(X\) over the scheme \(I = \text{Spec}(\mathbb{C}[t]/t^2)\) which is trivialized outside of \(x\). As \(\mathcal{L}\) is local, we have a section \(\beta^{l-1}(l)\) of \(((\mathcal{L}|X)|X \setminus x) \otimes \mathbb{C}[t]/t^2\) and we set

\[
(id \times \text{Res}_x)(\nabla_{\text{loc}}(l) \otimes (1 \boxtimes \xi)) := \partial_l(\beta^{l-1}(l)).
\]

Since the isomorphism \(\beta^{l}\) above respects the \(\mathcal{O}\)-module structure, the map \(\nabla_{\text{loc}}\) satisfies an analog of the Leibnitz rule: for a function \(f\) on \(\mathbf{X}\) and a local section \(l\) of \(\mathcal{L}\) we have:

\[
\nabla_{\text{loc}}(f \cdot l) = (f \boxtimes 1) \cdot \nabla_{\text{loc}}(l) + \nabla_{\text{loc}}(f) \otimes (l \boxtimes 1) \in p_{1*} (\mathcal{L} \boxtimes \Omega^2(\infty \cdot \Delta)),
\]

(here \(\nabla_{\text{loc}}(f)\) is the corresponding operation for \(\mathcal{L} = 0\)).

Analogously, when \(\mathcal{L}\) is a local \(\mathcal{O}\)-module on \(\mathbf{X}^n\) we have a map

\[
\nabla_{\text{loc}} : \mathcal{L} \to p_{1, \ldots, n*} (\mathcal{L} \boxtimes \Omega^{2}(\infty \cdot \Delta_{x_{n+1} \neq x_k})),
\]

\(^6\)For two \(\mathcal{O}\)-modules \(\mathcal{L}_1\) and \(\mathcal{L}_2\) on \(\mathbf{X}\), it will be sometimes more convenient to use the notation \(\mathcal{L}_1 \boxtimes \mathcal{L}_2(\infty \cdot \Delta)\) for \(j_*j^*(\mathcal{L}_1 \boxtimes \mathcal{L}_2)\).
that satisfies the Leibnitz rule in the same sense as above.

In particular, for a local $\mathcal{O}$-module $\mathcal{L}$ on $\mathfrak{X}$, the $\mathcal{O}$-module $\mathcal{L} \boxtimes \Omega^2(\infty \cdot \Delta)$ on $\mathfrak{X}$

\[ \nabla_{\text{loc}} : \mathcal{L} \boxtimes \Omega^2(\infty \cdot \Delta) \to p_1^*(\mathcal{L} \boxtimes \Omega^2(\infty \cdot \Delta)) \]

Lemma 6.3. Let $\sigma_{2,3}$ be the transposition of the last 2 coordinates acting on the sheaf $p_1^*(\mathcal{L} \boxtimes \Omega^2(\infty \cdot \Delta))$. We have:

\[ \sigma_{2,3} \circ \nabla_{\text{loc}}^2 = \nabla_{\text{loc}}^2, \]

i.e. the image of $\nabla_{\text{loc}}^2$ is symmetric in the last two variables.

The proof follows immediately from the second condition on the $\beta$’s in the definition of local $\mathcal{O}$-modules.

Let us consider a few examples:

Note that the cotangent bundle of $\hat{\mathfrak{X}}$ is identified in an $\text{Aut}^+_0$-equivariant way

\[ q^*p_1^*(\mathcal{O} \boxtimes \Omega^2(\infty \cdot \Delta)). \]

Therefore, the De Rham differential on $\hat{\mathfrak{X}}$ yields maps

\[ d^0 : \mathcal{O} \to \Omega_{\hat{\mathfrak{X}}} \simeq p_1^*(\mathcal{O} \boxtimes \Omega^2(\Delta)) \to p_1^*(\mathcal{O} \boxtimes \Omega^2(\infty \cdot \Delta)) \text{ and} \]

\[ d^1 : p_1^*(\mathcal{O} \boxtimes \Omega^2(\infty \cdot \Delta)) \to p_1^*(\mathcal{O} \boxtimes \Omega^2 \boxtimes \Omega^2(\infty \cdot \Delta)). \]

We leave it to the reader to check that $d^0$ is the map $\nabla_{\text{loc}}$ for $\mathcal{L} = \mathcal{O}$ and that $d^1$ is the map $\nabla_{\text{loc}}$ for the local $\mathcal{O}$-module $\mathcal{O} \boxtimes \Omega^2(\infty \cdot \Delta)$ on $\mathfrak{X}$.

We claim, in fact, that for an $\mathcal{O}$-module $\mathcal{L}$ on $\mathfrak{X}$, a structure of local $\mathcal{O}$-module on it is equivalent to the data of $\nabla_{\text{loc}}$:

Theorem 6.1. The following three categories are equivalent:

- The category of local $\mathcal{O}$-modules on $\mathfrak{X}$.
- The category of weakly $\text{Aut}^+_0$-equivariant left $D$-modules on $\hat{\mathfrak{X}}$.
- The category of $\mathcal{O}$-modules $\mathcal{L}$ on $\mathfrak{X}$ endowed with a map $\nabla_{\text{loc}} : \mathcal{L} \to p_1^*(\mathcal{L} \boxtimes \Omega^2(\infty \cdot \Delta))$ such that:
  
  (a) For any function $f$ on $\hat{\mathfrak{X}}$ and a local section $l$ of $\mathcal{L}$
  
  \[ \nabla_{\text{loc}}(f \cdot l) = (f \boxtimes 1) \cdot \nabla_{\text{loc}}(l) + \nabla_{\text{loc}}(f) \otimes (l \boxtimes 1) \in p_1^*(\mathcal{L} \boxtimes \Omega^2(\infty \cdot \Delta)). \]

  (b) If we define (assuming (a))
  
  \[ \nabla_{\text{loc}} : \mathcal{L} \boxtimes \Omega^2(\infty \cdot \Delta) \to p_1^*(\mathcal{L} \boxtimes \Omega^2 \boxtimes \Omega^2(\infty \cdot \Delta)) \]

  according to the Leibnitz rule applied to $\nabla_{\text{loc}}$ acting on $\mathcal{L}$ and on $\Omega^2$, then the composition

  \[ \nabla_{\text{loc}}^2 := \nabla_{\text{loc}} \circ \nabla_{\text{loc}} : \mathcal{L} \to p_1^*(\mathcal{L} \boxtimes \Omega^2 \boxtimes \Omega^2(\infty \cdot \Delta)) \]

  satisfies $\sigma_{2,3} \circ \nabla_{\text{loc}}^2 = \nabla_{\text{loc}}^2$.

  Morphisms in this category are $\mathcal{O}$-module maps $\mathcal{L}_1 \to \mathcal{L}_2$ compatible with the $\nabla_{\text{loc}}$’s.
The proof of this theorem is quite straightforward and we shall omit it, especially since this statement will not be used in the sequel.

We shall conclude this subsection by showing that if \( \mathcal{L} \) is a local \( \mathcal{O} \)-module on \( \mathfrak{X} \), there is a natural action of \( \Theta \) on it “by Lie derivations” (cf. Section 4.3).

Consider the composition

\[
\nabla'_\text{loc} : \mathcal{L} \overset{\nabla_\text{loc}}{\longrightarrow} p_1*(\mathcal{L} \boxtimes \Omega^2(\infty \cdot \Delta)) \twoheadrightarrow p_1*(\mathcal{L} \boxtimes \Omega^2(\infty \cdot \Delta)/\mathcal{L} \boxtimes \Omega^2).
\]

This map is \( \mathcal{O}_\mathfrak{M} \)-linear and it satisfies:

\[
\nabla'_\text{loc}(f \cdot l) = (f \boxtimes 1) \cdot \nabla'_\text{loc}(l) + (l \boxtimes 1) \otimes df,
\]

where \( df \in \Omega \) is viewed as an element of \( \mathcal{O} \boxtimes \Omega^2(\infty \cdot \Delta)/\mathcal{O} \boxtimes \Omega^2 \) via

\[
\Omega \simeq \mathcal{O} \boxtimes \Omega^2(\Delta)/\mathcal{O} \boxtimes \Omega^2 \subset \mathcal{O} \boxtimes \Omega^2(\infty \cdot \Delta)/\mathcal{O} \boxtimes \Omega^2.
\]

If \( l \) is a local section of \( \mathcal{L} \) and \( \xi \) is a vertical vector field on \( \mathfrak{X} \) (i.e. a local section of \( \Theta \)), we set

\[
\text{Lie}_\xi(l) := (\text{id} \boxtimes \text{Res}_\Delta)(\nabla'_\text{loc}(l) \otimes (1 \boxtimes \xi)) \in \mathcal{L}.
\]

The properties of \( \nabla'_\text{loc} \) imply that the map \( (l, \xi) \to \text{Lie}_\xi(l) \) is a bi-differential operator and is \( \mathcal{O}_\mathfrak{M} \)-linear in both variables. In addition, we have:

\[
\text{Lie}_\xi(f \cdot l) = f \cdot \text{Lie}_\xi(l) + \text{Lie}_\xi(f) \cdot l.
\]

7. A formulation of CFT (central charge 0)

7.1. A CFT data (of central charge 0) for us will consist of

(1) A local \( \mathcal{O} \)-module \( A \) on \( \mathfrak{X} \) endowed with a structure of a chiral algebra (in particular, \( A \) is a right \( \mathcal{D} \)-module).

(2) A map of local \( \mathcal{O} \)-modules \( T : \Theta \to A \) (called the energy-momentum tensor).

These data must satisfy the following conditions:

(a) The chiral bracket on \( A \):

\[
j_* j^*(A \boxtimes A) \overset{(\text{id})_\Delta}{\longrightarrow} \Delta_(A)
\]

is a map of local \( \mathcal{O} \)-modules on \( \mathfrak{X}^2 \). Moreover, the unit map \( \text{unit} : \Omega \to A \) is also a map of local \( \mathcal{O} \)-modules.

(b) Let \( \xi \) be a vertical vector field on \( \mathfrak{X} \) and consider the map \( A \to A \) given by

\[
l \to (\text{id} \boxtimes h)((l, T(\xi))) - \text{Lie}_\xi(l).
\]

We need this map to coincide with the action of \( \xi \) on \( A \) coming from the right \( \mathcal{D} \)-module structure on \( A \).

It follows from (a) and (b) that the map \( A \boxtimes D \to A \) defining the right \( \mathcal{D} \)-module structure on \( A \) is a map of local \( \mathcal{O} \)-modules.

Recall the Lie-* algebra \( \mathcal{T} = \Theta \boxtimes D \) that we constructed in Section 4.3. The map \( T \) gives rise to a map \( \mathcal{T} \to A \) and conditions (a) and (b) above guarantee that this is a homomorphism of Lie-* algebras.

Remark 7.1. In the physics literature one often encounters the term “primary field of weight \( i \)” In our language, this is the same as a map of local \( \mathcal{O} \)-modules \( \Omega^{1-i} \to A \).
7.2. Let $\mathcal{A}$ be the corresponding left D-module $\mathcal{A} := \mathcal{A} \otimes \Theta$ and let $\mathcal{T}$ denote the corresponding map $\Omega^{-2} \rightarrow \mathcal{A}$. We will show now that $\mathcal{A}$ acquires a (left) module structure over some canonical Lie algebroid on $\hat{X}$ (cf. Section 3 for the notion of a Lie algebroid). This will later on enable us to produce a connection along $\mathcal{M}$ on the space of conformal blocks of $\mathcal{A}$.

Recall the scheme $\hat{X}$ that we introduced in the previous subsection. It is known that the natural $\text{Aut}^+_\mathcal{M}$-action on it extends to an action of the whole of $\text{Aut}$. Let us recall how this action is constructed.

For a point $(x \in X \in \mathcal{M}, \alpha)$ of $\hat{X}$ and a point $g \in \text{Aut}$ with values in a local Artinian scheme $\text{Spec}(I)$, we must produce a $\text{Spec}(I)$-valued point $(x_I \in X_I, \alpha_I)$ of $\hat{X}$.

We let $X_I$ be unchanged as a topological space; moreover, we set $X_I \setminus x_I$ to be $(X \setminus x) \times \text{Spec}(I)$. To fix the triple $(x_I \in X_I, \alpha_I)$ it remains to specify which of the meromorphic functions on $(X \setminus x) \times \text{Spec}(I)$ are regular at $x_I \in X_I$.

The set of meromorphic functions on $X \times \text{Spec}(I)$ embeds into $\hat{X} \otimes I$ by means of $\alpha$. We declare a meromorphic function on $(X \setminus x) \times \text{Spec}(I)$ to be regular for $x_I$ at $x_I$ if its image in $\hat{X} \otimes I$ is transformed by $g : \hat{X} \otimes I \rightarrow \hat{X} \otimes I$ into an element of $\hat{\Delta} \otimes I$.

In particular, the Lie algebra $\text{Vir}$ acts on $\hat{X}$ by vector fields and we can form the Lie algebroid $\tilde{\mathcal{V}} = \text{Vir} \otimes \mathcal{O}_X$ (resp., $\tilde{\mathcal{V}}^+ = \text{Vir}^+ \otimes \mathcal{O}_X$, $\tilde{\mathcal{V}}^+_0 = \text{Vir}^+_0 \otimes \mathcal{O}_X$) on $\hat{X}$; all the three are Harish-Chandra Lie algebroids with respect to $\text{Aut}^+$. Let $\mathcal{V}$, $\mathcal{V}^+$ and $\mathcal{V}^+_0$ denote the corresponding Lie algebroids on $X$, i.e. $\tilde{\mathcal{V}} \simeq q^*(\mathcal{V})$ and similarly for $\mathcal{V}^+$ and $\mathcal{V}^+_0$; the Harish-Chandra property implies that $\mathcal{V}^+_0$ is an ideal in the other two.

We have:

$$ \mathcal{V} \simeq p_{1*}(\mathcal{O} \otimes \Theta(\infty \cdot \Delta)) \quad \text{and} \quad \mathcal{V}/\mathcal{V}^+_0 \simeq p_{1*}(\mathcal{O} \otimes \Theta(\infty \cdot \Delta)/\mathcal{O} \otimes \Theta(-\Delta)), $$

where the notation $\mathcal{O}\text{-}\text{mod}$ means completion along the diagonal divisor.

Moreover $\mathcal{V}^+ / \mathcal{V}^+_0 \hookrightarrow \mathcal{V} / \mathcal{V}^+_0$ is identified with

$$\Theta \simeq \mathcal{O} \otimes \Theta / \mathcal{O} \otimes \Theta(-\Delta) \hookrightarrow p_{1*}(\mathcal{O} \otimes \Theta(\infty \cdot \Delta)/\mathcal{O} \otimes \Theta(-\Delta)).$$

There are two maps $\mathcal{V} \otimes \mathcal{A} \rightarrow \mathcal{A}$:

The first map $\tau_{\text{loc}} : \mathcal{V} \otimes \mathcal{A} \rightarrow \mathcal{A}$ is defined as follows:

A typical local section of $\mathcal{V} \otimes \mathcal{A}$ can be written as $(f \cdot 1 \otimes \xi) \otimes a$, for $f \in p_{1*}(\mathcal{O} \otimes \Theta(\infty \cdot \Delta)), \xi \in \Theta$ and $a \in \mathcal{A}$. We set

$$\tau_{\text{loc}}((f \cdot 1 \otimes \xi), a) = (\text{id} \otimes \text{Res}_{\Delta})(f \cdot \nabla_{\text{loc}}(a) \otimes (1 \otimes \xi)).$$

To define the second map $\tau_{\text{em}}$ (the subscript “em” here stands for “energy-momentum”), let $(f \cdot 1 \otimes \xi) \otimes a \in \mathcal{V} \otimes \mathcal{A}$ be as above and we set

$$\tau_{\text{em}}((f \cdot 1 \otimes \xi), a) = (\text{id} \otimes h)((f \cdot a \otimes T(\xi)) \in \mathcal{A}.$$

\footnote{Using Proposition 3, one can rephrase the definition of $\tau_{\text{loc}}$ as follows: since $q^*(\mathcal{A})$ is a weakly $\text{Aut}^+\text{-equivariant}$ D-module on $\hat{X}$, it is also a weakly equivariant module over the algebroid $\mathcal{V}$; this defines on $\mathcal{A}$ a structure of $\mathcal{V}$-module, which is the same as a map $\tau_{\text{loc}}$.}

\footnote{Here we have denoted by $\text{id} \otimes h((\cdot))$ the map $\mathcal{A} \otimes \mathcal{A}(\infty \cdot \Delta) \rightarrow \mathcal{A}$ that comes from the chiral bracket on $\mathcal{A}$.}
Let $\tau$ be the difference:
\[
\tau = \tau_{\text{loc}} - \tau_{\text{em}} : v \otimes \mathcal{I}A \rightarrow \mathcal{I}A.
\]

**Lemma 7.1.** The map $\tau$ is a left action of the algebroid $v$ on $\mathcal{I}A$. Moreover, $\tau$ annihilates the ideal $v_0^+ \subset v$ and the restriction of $\tau$ to $\Theta \simeq v^+/v_0^+$ coincides with the action of $\Theta$ on $\mathcal{I}A$ coming from the structure of a left $D$-module on $\mathcal{I}A$.

The proof follows from conditions (a) and (b) in Section 7.1 and we leave the verification to the reader.

7.3. Let $A$ be as in Section 7.1. For each $X \in \mathcal{M}$ we can construct a vector space $H_\nabla(X, A)$ and globally over $\mathcal{M}$ this construction yields an $\mathcal{O}_{\mathcal{M}}$-module $H_\nabla(X, A)$.

According to the discussion in Section 5.3, we have a map of left $D$-modules on $\mathcal{X}$:
\[
\langle \ldots \rangle : \mathcal{I}A \rightarrow \pi^*(H_\nabla(X, A)).
\]

**Proposition 7.1.** There exists a (unique) left $D_{\mathcal{M}}$-module structure on $H_\nabla(X, A)$ such that when $\pi^*(H_\nabla(X, A))$ is viewed as a $v/v_0^+$-module via $v/v_0^+ \rightarrow \Theta_X$, the map $\langle \ldots \rangle$ is compatible with the $v/v_0^+$-module structure.

**Proof.** Since the map $\langle \ldots \rangle$ is surjective, there is at most one $v/v_0^+$-module structure on $\pi^*(H_\nabla(X, A))$ compatible with that on $\mathcal{I}A$.

To see that it is correctly defined, we must show that if $c$ is a local section of $\mathcal{I}A$ of the form
\[
c = (h \otimes \text{id})(\{f_0 \cdot a \boxtimes b\}),
\]
where $f_0 \cdot a \boxtimes b \in p_{2*}(\mathcal{A} \boxtimes \mathcal{I}A(\infty \cdot \Delta))$ and if $f \cdot 1 \boxtimes \xi \in p_{1*}(\mathcal{O} \boxtimes \Theta(\infty \cdot \Delta))$ is a local section of $v/v_0^+$, the section $c' := \tau(f \cdot 1 \boxtimes \xi, c)$ is a sum of sections of the form
\[
(h \otimes \text{id})(\{f_0' \cdot a' \boxtimes b'\}),
\]
for $f_0' \cdot a' \boxtimes b'$ a local section of $p_{2*}(\mathcal{A} \boxtimes \mathcal{I}A(\infty \cdot \Delta))$.

This, however, follows from the next lemma.

**Lemma 7.2.** Let $f_0 \cdot a \boxtimes b$ and $f \cdot 1 \boxtimes \xi$ and $c'$ be as above and let $a''$ be the section of $p_{2*}(\mathcal{A} \boxtimes \mathcal{O}(\infty \cdot \Delta))$ defined as
\[
a'' = (\text{id} \otimes \text{id} \otimes \text{Res}_{x_2 = x_1})(\nabla_{\text{loc}}(f_0 \cdot a \boxtimes 1) \otimes (1 \boxtimes f \boxtimes \xi)).
\]
Then
\[
c' = (h \otimes \text{id})(\{f_0 \cdot a \boxtimes \tau(f \cdot 1 \boxtimes \xi, b)\}) + (h \otimes \text{id})(\{a'' \otimes (1 \boxtimes b)\}).
\]

Next, we must show that the action of $v/v_0^+$ on $\pi^*(H_\nabla(X, A))$ factors through $\Theta_X$. The map $v/v_0^+ \rightarrow \Theta_X$ is surjective and its kernel is identified with
\[
p_{1*}(\mathcal{O} \boxtimes \Theta(\infty \cdot \Delta)) \subset p_{1*}(\mathcal{O} \boxtimes \Theta(\infty \cdot \Delta)),
\]
in (other words, the fiber of the above kernel at $x \in X \in \mathcal{M}$ is identified with the subspace $H^0(X \setminus x, \Theta_X) \subset \Theta \otimes \mathcal{K}_x$).

For $f \cdot 1 \boxtimes \xi \in p_{1*}(\mathcal{O} \boxtimes \Theta(\infty \cdot \Delta))$ and for a local section $a$ of $\mathcal{I}A$ we have by definition:
\[
\langle \tau(f \cdot 1 \boxtimes \xi, a) \rangle = (\langle \text{id} \boxtimes \text{Res}_\Delta \rangle(f \cdot \nabla_{\text{loc}}(a) \otimes (1 \boxtimes \xi))) - (\langle \text{id} \boxtimes h \rangle(\{f \cdot a \boxtimes T(\xi)\})).
\]
However, the first term vanishes by the residue formula, since \( f \cdot \nabla_{\text{loc}}(a) \otimes (1 \otimes \xi) \) is a section of \( p_1^*(\mathcal{A} \boxtimes \Omega(\infty \cdot \Delta)) \) and the second term vanishes by the very definition of \( H_{\nabla}(X,A) \).

We have shown, therefore, that \( \pi^*(H_{\nabla}(X,A)) \) is a \( D \)-module on \( X \) such that the action of vertical vector fields on it is the “tautological” one. Therefore, the \( \mathcal{O} \)-module \( H_{\nabla}(X,A) \) on \( \mathcal{M} \) is naturally a \( D_{\mathcal{M}} \)-module.

Here is another characterization of the connection on \( H_{\nabla}(X,A) \):

Let \( \nabla \) denote the covariant derivative acting on \( \pi^*_k(H_{\nabla}(X,A)) \); in particular, there is a map\(^9\)

\[
\nabla : j_*j^*(\pi_k^*(H_{\nabla}(X,A))) \to p_{1,...,k*}(j_*j^*(\pi_k^*(H_{\nabla}(X,A)) \boxtimes \Omega^2)).
\]

**Proposition 7.2.** Let \( f \cdot a_1 \boxtimes \ldots \boxtimes a_k \) be a section of \( j_*j^*(\mathcal{A}^{\boxtimes k}) \) on \( X^k \setminus \Delta \). Then \( \nabla((f \cdot a_1 \boxtimes \ldots \boxtimes a_k)) = ((\ldots) \boxtimes \text{id})(\nabla_{\text{loc}}(f \cdot a_1 \boxtimes \ldots \boxtimes a_k)) + (f \cdot a_1 \boxtimes \ldots \boxtimes a_k \boxtimes T) \).

**Proof.** When \( k = 1 \), the statement follows immediately from the definition of the action \( \tau \).

To treat the case of arbitrary \( k \) one proceeds as follows:

**Step 1:**
Consider the diagram

\[
\begin{array}{ccc}
j_*j^*(\mathcal{A}^{\boxtimes k}) & \xrightarrow{(\ldots)} & p_{1,...,k*}(j_*j^*(\mathcal{A}^{\boxtimes k} \boxtimes \Omega^2)) \\
\downarrow(\ldots) \boxtimes \text{id} & & \downarrow(\ldots) \boxtimes \text{id} \\
j_*j^*(\pi_k^*(H_{\nabla}(X,A))) & \xrightarrow{\nabla} & p_{1,...,k*}(j_*j^*(\pi_k^*(H_{\nabla}(X,A)) \boxtimes \Omega^2)),
\end{array}
\]

where the upper horizontal arrow sends a section \( f \cdot a_1 \boxtimes \ldots \boxtimes a_k \) of \( j_*j^*(\mathcal{A}^{\boxtimes k}) \) to \( \nabla_{\text{loc}}(f \cdot a_1 \boxtimes \ldots \boxtimes a_k) + f \cdot a_1 \boxtimes \ldots \boxtimes a_k \boxtimes T \).

**Lemma 7.3.** There exists a unique map

\[
\nabla' : j_*j^*(\pi_k^*(H_{\nabla}(X,A))) \to p_{1,...,k*}(j_*j^*(\pi_k^*(H_{\nabla}(X,A)) \boxtimes \Omega^2))
\]

which completes the above diagram to a commutative diagram.

**Step 2:**
It remains to show that the map \( \nabla' \) constructed above coincides with the covariant derivative \( \nabla \). Consider the diagram

\[
\begin{array}{ccc}
\quad & \xrightarrow{(\ldots) \boxtimes \text{id}} & p_{1,...,k*}(j_*j^*(\mathcal{A} \boxtimes \Omega^{k-1} \boxtimes \Omega^2)) \\
\downarrow(\ldots) \boxtimes \text{id} & & \downarrow(\ldots) \boxtimes \text{id} \\
j_*j^*(\pi_k^*(H_{\nabla}(X,A))) & \xrightarrow{\nabla' = \nabla} & p_{1,...,k*}(j_*j^*(\pi_k^*(H_{\nabla}(X,A)) \boxtimes \Omega^2)),
\end{array}
\]

where the upper horizontal arrow is given by the same formula as above.

\(^9\)In the formula below, \( (f \cdot a_1 \boxtimes \ldots \boxtimes a_k \boxtimes T)^k \) will denote a section of \( j_*j^*(\mathcal{O}^{\boxtimes k} \boxtimes \Omega^2) \) characterized by the property that for \( \eta \in \Omega^{-2} \):

\[
\langle f \cdot a_1 \boxtimes \ldots \boxtimes a_k \boxtimes T \rangle \otimes (1 \boxtimes \ldots \boxtimes 1 \boxtimes \eta) := (f \cdot a_1 \boxtimes \ldots \boxtimes a_k \boxtimes T(\eta)) \in j_*j^*(\mathcal{O}^{\boxtimes k}).
\]
On the one hand, according to Step 1 and Section 5.3, this diagram becomes commutative if we set the lower horizontal arrow to be $\nabla'$. On the other hand, the assertion of our proposition for $k = 1$ implies that this diagram becomes commutative if we set the lower horizontal arrow to be $\nabla$. Since the left vertical arrow is a surjection (by Proposition 5.3), $\nabla' = \nabla$. 

8. Introducing the central charge

8.1. We shall start with some linear algebra preliminaries.

Consider the canonical decreasing sequence of ideals in the Lie algebra $\text{Vir}^+_0$:

$$0 \subset \ldots \subset \text{Vir}^+_n \subset \ldots \subset \text{Vir}^+_1 \subset \text{Vir}^+_0.$$ 

For $k \in \mathbb{N}$ let $L_k$ denote the $\text{Aut}^+_0$-module $\text{Vir}^+_k / \text{Vir}^+_{k+1}$. We have canonical isomorphisms $L_k \otimes L_{k'} \cong L_{k+k'}$ and for $k \in \mathbb{N}$ we define $L_{-k}$ as $L_k^*$. 

**Lemma 8.1.** The following two categories are equivalent:

(a) The category $\text{Ext}_{\text{Aut}^+_0}(L_{-1}, L_1)$ of extensions $0 \to L_1 \to E \to L_{-1} \to 0$ of $\text{Aut}^+_0$-modules.

(b) The category of Harish-Chandra pairs $(\text{Vir}'_+, \text{Aut}^+_0)$, where $\text{Vir}'_+$ is a central extension of $\text{Vir}$:

$$0 \to \mathbb{C} \to \text{Vir}' \to \text{Vir} \to 0.$$ 

In fact, both categories are groupoids such that every object has no non-trivial automorphisms; moreover the set of isomorphism classes of objects in each of them is canonically identified with $\mathbb{C}^{10}$. This implies the assertion of the lemma.

The above equivalence of categories has the following additional property:

To an object $0 \to L_1 \to E \to L_{-1} \to 0$ of $\text{Ext}_{\text{Aut}^+_0}(L_{-1}, L_1)$ one can attach a local $\mathcal{O}$-module $\Theta'$ on $\mathfrak{X}$ (cf. Section 6.2). By the construction, $\Theta'$ is an extension

$$0 \to \mathcal{O} \to \Theta' \to \Theta \to 0.$$ 

Now let $(x \in X)$ be a point of $\mathfrak{X}$. Consider the corresponding extension of $\mathcal{K}_x$-modules:

$$0 \to \mathcal{O} \otimes \mathcal{K}_x \to \Theta' \otimes \mathcal{K}_x \to \Theta \otimes \mathcal{K}_x \to 0$$

and consider the vector space

$$\Theta' \otimes \mathcal{K}_x / \ker(\text{Res} : \mathcal{O} \otimes \mathcal{K}_x \to \mathbb{C}).$$

The action of $\Theta \otimes \mathcal{K}_x$ on $\Theta' \otimes \mathcal{K}_x$ by Lie derivations (cf. Section 6.3) makes the above vector space into a Lie algebra.

---

10These identifications are chosen as follows:

If $0 \to L_1 \to E \to L_{-1} \to 0$ is an object of $\text{Ext}_{\text{Aut}^+_0}(L_{-1}, L_1)$, the action of $\text{Vir}^+_0$ on $E$ produces a map $L_2 \otimes L_{-1} \to L_1$, i.e. a scalar $c \in \mathbb{C}$, since $L_2 \otimes L_{-1}$ is apriori identified with $L_1$. We set the class of this $E$ in $\mathbb{C}$ to be $-2c$.

If $\text{Vir}'$ is an object of the second category, the adjoint action of $\mathbb{C}^* \subset \text{Aut}^+_0$ on $\text{Vir}'$ defines a decomposition of the latter as a vector space: $\text{Vir}' = L_i \oplus \mathbb{C}$. The Lie bracket on $\text{Vir}'$ induces a map $L_2 \otimes L_{-2} \to \mathbb{C}$, i.e. a scalar $c \in \mathbb{C}$. We set the class of this $\text{Vir}'$ to be equal to $-2c$. 


Lemma 8.2. Every choice of an identification $\alpha : \hat{O} \simeq \hat{O}_x$ gives rise to an isomorphism

$$\text{Vir}' \simeq \Theta'(\hat{O}_x)/\ker(\text{Res} : \Omega \otimes \hat{O}_x \to \mathbb{C}),$$

where $\text{Vir}'$ is the central extension of $\text{Vir}$ corresponding to $E$. A modification of $\alpha$ by means of $g \in \text{Aut}_{\hat{O}}^+$ corresponds to the adjoint action of $g$ on $\text{Vir}'$.

Let us now describe explicitly the extension $0 \to \Omega \to \Theta'_2 \to \Theta \to 0$ that corresponds to the object $E_2 \in \text{Ext}_{\text{Aut}_{\hat{O}}^+}(L_{-1}, L_1)$ whose class in $\mathbb{C}$ equals 2:

Consider the local $\mathcal{O}$-module $\Theta'_2 \simeq (\mathcal{O} \otimes \Theta(\Delta) \to \mathcal{O}(\Omega \otimes \Theta(\Delta)))$.

Lemma 8.3. The local $\mathcal{O}$-module $\Theta'_2$ is identified with

$$(\mathcal{O} \otimes \Theta(\Delta) \to \mathcal{O}(\Omega \otimes \Theta(\Delta)))/\mathcal{O}.$$

In what follows, for $c \in \mathbb{C}$ we shall denote by $\Theta'_c$ (resp., $\text{Vir}'_c$) the corresponding extension of $\Theta$ (resp., the corresponding central extension of $\text{Vir}$). The construction of Example 2 of Section 4.3 produces from $\Theta'_c$ a Lie-* algebra $\mathcal{T}'_c$ on $\mathfrak{X}$ that satisfies the conditions of Lemma 6.1.

Let $\mathcal{P}_c$ denote the Serre dual to $\Theta'_c$. This is again a local $\mathcal{O}$-module on $\mathfrak{X}$, which is an extension of $\mathcal{O}$ by $\Omega^2$. It is easy to see that $\mathcal{P}_c$ identifies with the $\Omega^2$-torsor of $c$-projective connections. In particular, for a fixed $X \in \mathfrak{X}$, the extensions

$$0 \to \Omega \to \Theta'_c \to \Theta \to 0$$

and

$$0 \to \Omega^2 \to \mathcal{P}_c \to \mathcal{O} \to 0$$

are (non-canonically) split.

8.2. A CFT of central charge $c$ consists of:

1. A local $\mathcal{O}$-module $\mathcal{A}$ on $\mathfrak{X}$ endowed with a structure of chiral algebra.
2. A map of local $\mathcal{O}$-modules $T : \Theta'_c \to \mathcal{A}$ (called the energy-momentum tensor), such that the composition $\Omega \to \Theta'_c \to \mathcal{A}$ coincides with the map $\text{unit}$.

The pair $(\mathcal{A}, T)$ must satisfy the following conditions:

(a) The chiral bracket on $\mathcal{A}$:

$$j_*j^*(\mathcal{A} \otimes \mathcal{A}) \xrightarrow{\text{Lie}(\xi)} \mathfrak{X}(\mathcal{A})$$

is a map of local $\mathcal{O}$-modules on $\mathfrak{X}^2$.

(b) Let $\xi$ be a vertical vector field on $\mathfrak{X}$ and let $\xi'$ be some lift of $\xi$ to a section of $\Theta'_c$. Consider the map $\mathcal{A} \to \mathcal{A}$ given by

$$l \mapsto (\text{id} \otimes \text{h})(\{l, T(\xi')\}) - \text{Lie}_\xi(l).$$

We need this map to coincide with the action of $\xi$ on $\mathcal{A}$ coming from the right $D$-module structure on $\mathcal{A}$. (Note that the above expression a priori does not depend on the choice of $\xi'$.)

As in the case of central charge 0, it follows that the map $\mathcal{A} \otimes D \to \mathcal{A}$ defining the $D$-module structure on $\mathcal{A}$ is a map of local $\mathcal{O}$-modules and that the $D$-module map $\mathcal{T}'_c \to \mathcal{A}$ induced by $T : \Theta'_c \to \mathcal{A}$ is a Lie-* algebra homomorphism.
8.3. Let $\tilde{v}_c'$ denote the Lie algebroid $\text{Vir}_c' \otimes \hat{O}_X$ on $\hat{X}$. Let $\tilde{v}_{\text{glob}}$ denote the kernel $\tilde{v} \to \Theta_{\hat{X}}'$; obviously $\tilde{v}_{\text{glob}}$ is an ideal in $\tilde{v}$ and its fiber at a point $(X, x, \alpha) \in \hat{X}$ is identified with $H^0(X \setminus x, \Theta)$.

**Proposition 8.1.** There is a canonical lifting:

$$\tilde{v}_{\text{glob}} \to \tilde{v}_c'.$$

Moreover, $\tilde{v}_{\text{glob}}$ becomes an ideal in $\tilde{v}_c'$.

**Proof.** To construct the embedding $\tilde{v}_{\text{glob}} \to \tilde{v}_c'$ we must exhibit for each $(X, x, \alpha) \in \hat{X}$ as above a map

$$H^0(X \setminus x, \Theta) \to \Theta_{c'} \otimes \hat{K}_x / \ker(\text{Res} : \Omega \otimes \hat{K}_x \to \mathbb{C}).$$

We obviously have a map $H^0(X \setminus x, \Theta') \to \Theta' \otimes \hat{K}_x$, and the composition

$$H^0(X \setminus x, \Omega) \to \Theta' \otimes \hat{K}_x \to \Theta' \otimes \hat{K}_x / \ker(\text{Res} : \Omega \otimes \hat{K}_x \to \mathbb{C})$$

vanishes by the residue formula.

Since the extension $0 \to \Omega \to \Theta' \to \Theta \to 0$ is (non-canonically) split, the projection $H^0(X \setminus x, \Theta') \to H^0(X \setminus x, \Theta)$ is surjective and we obtain a well-defined map

$$H^0(X \setminus x, \Theta) \to \Theta' \otimes \hat{K}_x / \ker(\text{Res} : \Omega \otimes \hat{K}_x \to \mathbb{C})$$

as needed.

To prove that $\tilde{v}_{\text{glob}}$ is an ideal in $\tilde{v}_c'$ we must show that it is stable under the action of the formal group $\text{Aut}$ on $\hat{X}$. This follows from the description of this action that was given in Section 7.2.

\[ \Box \]

Obviously, $\tilde{v}_c'$ is a Harish-Chandra Lie algebroid with respect to the group $\text{Aut}_0^+; \text{let } v_c'$ and $v_c'/v_0^+$ denote the corresponding Lie algebroids on $\hat{X}$. Clearly, we have short exact sequences:

$$0 \to \Theta_c' \to \hat{v}_c' \to v \to 0 \quad \text{and} \quad 0 \to \Theta_c' \otimes v_0^+ \to v / v_0^+ \to 0$$

with $\Theta$ being an ideal and the adjoint action of $v$ on $\Theta$ being the canonical one. As an $\Theta$-module, $v_c'$ is identified with

$$p_{1*}(\Theta_c'(\infty \cdot \Delta)) / \ker(p_{1*}(\Theta_c'(\infty \cdot \Delta)) \to \Theta)$$

and similarly for $v_c'/v_0^+$.

Now let $v_{\text{glob}}$ denote the kernel of $\Theta \to \Theta_{\hat{X}}$. It follows from the proposition above that $v_{\text{glob}}$ embeds naturally into $v_c'$ and is an ideal thereof. Moreover, the quotient

$$\Theta_{\hat{X}, c} := v_c' / (v_0^+ + v_{\text{glob}})$$

is a Picard Lie algebroid on $\hat{X}$.

Note that we have a canonical embedding $\text{Vir}^+ \to \text{Vir}_c'$, which gives rise to an embedding of Lie algebroids $\Theta \to v_c'/v_0^+$. In particular, there is a homomorphism of Lie algebroids $\Theta \to \Theta_{\hat{X}, c}$.

**Proposition 8.2.** There exists a canonical Picard Lie algebroid $\Theta_{\mathfrak{M}_c}$ on $\mathfrak{M}$ such that $\Theta_{\hat{X}, c}$ is its pull-back (in the sense of Section 3) under the map $\pi$. As an $\mathfrak{O}_{\mathfrak{M}}$-module, $\Theta_{\mathfrak{M}_c}$ is identified with $R^1\pi_* (\Theta_c')$. 

Proof. As an $\mathcal{O}$-module, the Picard Lie algebroid $\Theta'_X$ is identified with
\[ R^1p_1\ast(\mathcal{O} \boxtimes \Theta'(-\Delta))/\ker(R^1p_1\ast(\mathcal{O} \boxtimes \Omega(-\Delta)) \to R^1p_1\ast(\mathcal{O} \boxtimes \Omega)) \]
and the quotient $\Theta'_X/\Theta$ is therefore identified with $\pi^\ast(R^1\pi\ast(\Theta'_X))$.

Let $N(\Theta)$ denote the normalizer of $\Theta$ in $\Theta'_X$.

Lemma 8.4. $N(\Theta)/\Theta \subset \Theta'_X/\Theta$ corresponds to $\pi\ast(R^1\pi\ast(\Theta'_X)) \subset \pi\ast(R^1\pi\ast(\Theta'_X))$.

Therefore, $\Theta'_M := R^1\pi\ast(\Theta'_X)$ has a canonical structure of Lie algebroid on $\mathfrak{M}$ and we have a commutative square:
\[
\begin{array}{ccc}
\Theta & \longrightarrow & \Theta'_X \\
\downarrow & & \downarrow \\
\Theta & \longrightarrow & \Theta_X
\end{array}
\]
and $\pi\ast(\Theta'_M)$.

8.4. Similarly to the case $c = 0$, we shall show now that our CFT chiral algebra $\mathcal{L}^1A$ carries a natural action of the Lie algebroid $\mathfrak{g}_c/\mathfrak{g}_0$. To do this, we must construct a map $\tau : \mathfrak{g}_c/\mathfrak{g}_0 \otimes \mathcal{L}^1A \to \mathcal{L}^1A$ such that the composition $\mathcal{O} \otimes \mathcal{L}^1A \hookrightarrow \mathfrak{g}_c/\mathfrak{g}_0 \otimes \mathcal{L}^1A \to \mathcal{L}^1A$ coincides with the tautological map $\mathcal{O} \otimes \mathcal{L}^1A \to \mathcal{L}^1A$.

We have a map $\tau_{loc} : \mathfrak{g}_c/\mathfrak{g}_0 \otimes \mathcal{L}^1A \to \mathcal{L}^1A$ that factors through $\mathfrak{g}_c/\mathfrak{g}_0 \otimes \mathcal{L}^1A \to \mathcal{L}^1A$, the second arrow being defined by the same formula as in Section 7.2 (it comes from the structure on $\mathcal{L}^1A$ of a local $\mathcal{O}$-module).

We have a map $\tau_{em} : \mathfrak{g}_c/\mathfrak{g}_0 \otimes \mathcal{L}^1A \to \mathcal{L}^1A$ that attaches to a local section $f \cdot \mathcal{L}^1\xi^\prime \in p_1\ast(\mathcal{O} \boxtimes \Theta'_X(\infty \cdot \Delta)) \simeq \mathfrak{g}_c$ and a local section $a \in \mathcal{L}^1A$ the section $(\text{id} \boxtimes h)(\{f \cdot a \boxtimes T(\xi^\prime)\}) \in \mathcal{L}^1A$.

The difference $\tau := \tau_{loc} - \tau_{em}$ is therefore a map $\mathfrak{g}_c/\mathfrak{g}_0 \otimes \mathcal{L}^1A \to \mathcal{L}^1A$ which factors (due to condition (b) of Section 8.2) as
\[ \mathfrak{g}_c/\mathfrak{g}_0 \otimes \mathcal{L}^1A \to \mathcal{L}^1A. \]

As in the previous subsection we have the following assertion:

Lemma 8.5. The map $\tau : \mathfrak{g}_c/\mathfrak{g}_0 \otimes \mathcal{L}^1A \to \mathcal{L}^1A$ is a left action of the Lie algebroid $\mathfrak{g}_c/\mathfrak{g}_0$ on $\mathcal{L}^1A$.

Consider now the $\mathcal{O}_\mathfrak{M}$-module $H_\mathcal{V}(\mathfrak{X}, A)$. 
Proposition 8.3. The $\mathcal{O}_\mathcal{M}$-module $H\nabla(\mathfrak{X},\mathcal{A})$ on $\mathcal{M}$ has a (unique) structure of left $\Theta'_\mathcal{M}$-module such that when we view $\pi^*(H\nabla(\mathfrak{X},\mathcal{A}))$ as a $\mathcal{V}'_\pi/\mathcal{V}^+_\pi$-module via $\mathcal{V}'_\pi/\mathcal{V}^+_\pi \rightarrow \Theta'_\mathcal{X}_c$, the projection $\langle \ldots \rangle : \mathcal{V}'_\pi \rightarrow \pi^*(H\nabla(\mathfrak{X},\mathcal{A}))$ is compatible with the $\mathcal{V}'_\pi/\mathcal{V}^+_\pi$-action.

The proof of this assertion is the same as in the case $c = 0$.

For $k \in \mathbb{N}$ let $\Theta'_{\mathfrak{X}}$ denote the pull-back of the Picard Lie algebroid $\Theta'_{\mathcal{M}}$ to $\mathfrak{X}$ (in the sense of Section 2); by definition, for $k = 1$ we re-obtain on $\mathfrak{X}$ the Picard Lie-algebroid $\Theta'_{\mathfrak{X}}$. Note that the corresponding twisted cotangent bundle on $\mathfrak{X}$ is identified with $\ker(p_1,\ldots,k^*(\mathcal{O}^{\otimes k} \boxtimes \mathcal{P}_c(\Delta)))$.

In particular, the $\Theta'_{\mathfrak{X}}$-action on $\pi^*_k(H\nabla(\mathfrak{X},\mathcal{A}))$ yields a covariant derivative map $\nabla : j^*j^*(\pi^*_k(H\nabla(\mathfrak{X},\mathcal{A}))) \rightarrow p_1,\ldots,k^*(j^*(\pi^*_k(H\nabla(\mathfrak{X},\mathcal{A}) \boxtimes \mathcal{P}_c)))$.

The following is a generalization to the case $c \neq 0$ of Proposition 7.2:

Proposition 8.4. Let $f \cdot a_1 \boxtimes \ldots \boxtimes a_k$ be a local section of $j^*j^*(\mathcal{A}^{\otimes k})$ on $\mathfrak{X} \setminus \Delta$. Then the section $\nabla(f \cdot a_1 \boxtimes \ldots \boxtimes a_k)$ of $j^*j^*(\pi^*_k(H\nabla(\mathfrak{X},\mathcal{A}) \boxtimes \mathcal{P}_c))$ equals

$\langle \ldots \rangle \boxtimes \text{id}(\nabla_{loc}(f \cdot a_1 \boxtimes \ldots \boxtimes a_k)) + (f \cdot a_1 \boxtimes \ldots \boxtimes a_k \boxtimes \langle \ldots \rangle)$,

where the insertion of $\langle \ldots \rangle$ in the correlation function has the same meaning as in Proposition 7.2.
Chapter III. Examples

9. Heisenberg and Kac-Moody algebras

9.1. Let $g$ and $Q$ be as in Example 1 of Section 4.3 and let $B(g, Q)$ be the corresponding Lie-* algebra; consider its chiral universal enveloping algebra $\mathcal{U}(B(g, Q))$.

There are two maps of $\Omega$ into $\mathcal{U}(B(g, Q))$: one is the map $\text{unit} : \Omega \to \mathcal{U}(B(g, Q))$ and the other is the composition $\tilde{\text{unit}} : \Omega \to B(g, Q) \to \mathcal{U}(B(g, Q))$.

We define the chiral algebra $A(g, Q)$ as a quotient of $\mathcal{U}(B(g, Q))/I$, where $I$ is the ideal generated by the image of $\text{unit} - \tilde{\text{unit}} : \Omega \to \mathcal{U}(B(g, Q))$.

We are going to show now that under certain conditions, $A(g, Q)$ defines a CFT data.

It is clear that $B(g, Q)$ has a natural structure of local $\mathcal{O}$-module on $X$, that satisfies the conditions of Lemma 6.1. Hence, $A(g, Q)$ is a chiral algebra on $X$ which has a structure of local $\mathcal{O}$-module. Therefore, in order for $A(g, Q)$ to define a CFT structure, we must construct a map of local $\mathcal{O}$-modules

$$T_{g,Q} : \Theta'_c \to A(g, Q)$$

for some $c \in \mathbb{C}$ that satisfies condition (b) of Section 8.2.

All our constructions will be local, therefore, we will be working on a fixed curve $X$.

Let $B \in g \otimes g$ be an $\text{ad}_g$-invariant symmetric tensor that satisfies the following conditions:

- If $B = \sum b_{i,j} \cdot u_i \otimes u_j$, then the endomorphism of $g$ given by
  $$v \to \sum b_{i,j} \cdot [u_i, [u_j, v]]$$
  is a scalar operator $\kappa(B) \cdot \text{Id}_g$.
- The map $B \circ Q : g \to g^* \to g$ is a scalar operator that satisfies
  $$2(B \circ Q) + \kappa(B) = \text{Id}_g.$$

First, we define a map $\bar{T}_{g,Q} : \Theta'_2 \to A(g, Q)$ as follows:

Recall that in Section 8.2, $\Theta'_2$ was constructed as a quotient of the $\mathcal{O}$-module $p_{2*}(\mathcal{O} \boxtimes \mathcal{O}(\Delta))$. Let $\bar{T}_{g,Q}$ be the map

$$p_{2*}(\mathcal{O} \boxtimes \mathcal{O}(\Delta)) \to A(g, Q)$$

that sends a section $f(x,y) \in \mathcal{O} \boxtimes \mathcal{O}(\Delta)$ to

$$(h \boxtimes \text{id}) \left( \sum b_{i,j} \cdot \{u_i \boxtimes u_j \cdot f(x,y)\} \right) \in A(g, Q).$$

---

11 An ideal in a chiral algebra $A$ is by definition a chiral submodule of $A$, when the latter is viewed as a module over itself. For a $D$-submodule of $A$ there exists a minimal ideal that contains it, called the “ideal generated by the given $D$-sub-module”. If $I \subset A$ is an ideal, the quotient $D$-module $A/I$ has a natural structure of chiral algebra.
Lemma 9.1. For any symmetric $B \in \mathfrak{g} \otimes \mathfrak{g}$, the map $\tilde{T}_{B, Q} : p_{2*}(\mathcal{O} \boxtimes \mathcal{O}(\Delta)) \to \mathcal{A}(\mathfrak{g}, Q)$ factors as

$$p_{2*}(\mathcal{O} \boxtimes \mathcal{O}(\Delta)) \to \Theta_2^{\tilde{T}_{B, Q}} \mathcal{A}(\mathfrak{g}, Q).$$

Moreover, the composition

$$\Omega \to \Theta_2^{\tilde{T}_{B, Q}} \mathcal{A}(\mathfrak{g}, Q)$$

coincides with $\text{Tr}(B \circ Q) \cdot \text{unit}$.

We now set $c = 2 \text{Tr}(B \circ Q)$ and define the map $T_{B, Q} : \Theta_2^{\tilde{T}_{B, Q}} \mathcal{A}(\mathfrak{g}, Q)$ as a composition

$$\Theta_2^{\tilde{T}_{B, Q}} : \Theta_2 \xrightarrow{\tilde{T}_{B, Q}} \mathcal{A}(\mathfrak{g}, Q),$$

where the first arrow is the unique isomorphism that fits into the commutative diagram

$$0 \to \Omega \xrightarrow{\Theta_2^{\tilde{T}_{B, Q}}} \Theta_2 \xrightarrow{\text{id}} \Theta \to 0.$$

Clearly, the map $T_{B, Q}$ is compatible with the local $\mathcal{O}$-module structure.

Proposition 9.1. For $B$ and $c$ as above, the map $T_{B, Q} : \Theta_c^{\tilde{T}_{B, Q}} \mathcal{A}(\mathfrak{g}, Q)$ satisfies condition (b) of the CFT data.

Before giving a proof of this proposition, let us consider two examples:

Example 1

Let $\mathfrak{g} = \mathfrak{h}$ be abelian; in this case $B$ exists if and only if $Q$ is non-degenerate and $B = (2 \cdot Q)^{-1}$. We have $c = \dim(\mathfrak{h})$.

Example 2

Now let $\mathfrak{g}$ be semisimple. The condition for the existence of $B$ is that $\kappa(Q^{-1}) + 2 \neq 0$, i.e. that $Q$ is not equal to $-\frac{1}{2} K$, where $K$ is the Killing form.

Therefore, in this case the construction we described above is just an invariant way to write down Sugawara’s construction. Moreover, if $Q = q \cdot K$ we see that the central charge in this case equals $q \cdot \dim(\mathfrak{g})/(q + 1/2)$.

9.2. Let us now prove Proposition 9.1.

Proof. We have to show that for a section $f(x, y)$ of $\mathcal{O} \boxtimes \mathcal{O}(\Delta)$ and for any section $l \in \mathcal{A}(\mathfrak{g}, Q)$, we have:

$$(h \boxtimes \text{id})(\{T_{B, Q}(\xi_f), l\}) = -\text{Lie}_{\xi_f}(l) - l \cdot \xi_f,$$

where $\xi_f$ is a vector field on $X$ corresponding to $f(x, y)$ via the identification

$$\mathcal{O} \boxtimes \mathcal{O}(\Delta)/\mathcal{O} \boxtimes \mathcal{O} \simeq \Theta.$$

First of all, since $\mathcal{B}(\mathfrak{g}, Q)$ generates $\mathcal{A}(\mathfrak{g}, Q)$, the Jacobi identity implies that it is enough to prove the above equality for $l \in \mathcal{B}(\mathfrak{g}, Q) \subset \mathcal{A}(\mathfrak{g}, Q)$. Moreover, we can assume that $l$ is of the form $l = v \otimes 1 \in \mathfrak{g} \otimes D_X$. Therefore, we have to prove the equality

$$(h \boxtimes h \boxtimes \text{id})(\sum_{i,j} b_{i,j} \{u_i \boxtimes u_j \cdot f(x, y), v\}) = -v \otimes \xi_f \in \mathcal{B}(\mathfrak{g}, Q) \subset \mathcal{A}(\mathfrak{g}, Q).$$
It is easy to see that the assertion holds for \( f(x, y) \in \mathcal{O} \boxtimes \mathcal{O} \), therefore, we can assume that \( f(x, y) = -f(y, x) \). Using the fact that the tensor \( \sum_{i,j} b_{i,j} \cdot u_i \otimes u_j \) is symmetric and the Jacobi identity, the above expression can be rewritten as:

\[
\sum_{i,j} 2b_{i,j} \cdot (h \boxtimes h \boxtimes \text{id})(\{u_i \boxtimes \{f(x, y) \cdot u_j \boxtimes v\}\}).
\]

By the definition of the Lie-* bracket on \( \mathcal{B}(\mathfrak{g}, Q) \), the last expression equals

\[
\sum_{i,j} 2b_{i,j} \cdot (h \boxtimes \text{id})(\{f(x, y) \cdot u_i \boxtimes [u_j, v]\}) + \sum_{i,j} 2b_{i,j} \cdot Q(u_j, v) \cdot (h \boxtimes h \boxtimes \text{id})(\{f(x, y) \cdot u_i \boxtimes 1'_{y,z}\}),
\]

where \( 1'_{y,z} \) denotes the canonical antisymmetric section of \( \Delta_!(\Omega) \subset \Delta_!(\mathcal{A}(\mathfrak{g}, Q)) \).

To prove the proposition we must show that

a) \( \sum_{i,j} 2b_{i,j} \cdot (h \boxtimes \text{id})(\{f(x, y) \cdot u_i \boxtimes [u_j, v]\}) = -\sum_{i,j} b_{i,j} \cdot [u_i, [u_j, v]] \otimes \xi_f \) and

b) that for \( u \in \mathfrak{g} \), \( (h \boxtimes h \boxtimes \text{id})(\{f(x, y) \cdot u \boxtimes 1'_{y,z}\}) = -u \otimes \xi_f \).

Using the fact that the tensor \( \sum_{i,j} b_{i,j} \cdot u_i \otimes u_j \) is \( \text{ad}_q \)-invariant, the expression

\[
\sum_{i,j} 2b_{i,j} \cdot (h \boxtimes \text{id})(\{f(x, y) \cdot u_i \boxtimes [u_j, v]\})
\]

can be rewritten as

\[
\sum_{i,j} b_{i,j} \cdot (h \boxtimes \text{id})(\{f(x, y) \cdot u_i \boxtimes [u_j, v]\} - \{f(x, y) \cdot [u_j, v] \boxtimes u_i\})
\]

and assertion a) follows from the next lemma:

**Lemma 9.2.** Let \( \mathcal{A} \) be a chiral algebra and let \( \sum_{i,j} a_i \boxtimes a_j \) be a section of \( \mathcal{A} \boxtimes \mathcal{A} \). Assume that \( \sum_{i,j} a_i \boxtimes a_j \in \mathcal{A} \subset \Delta_!(\mathcal{A}) \). Now let \( f(x, y) \) be an antisymmetric section of \( \mathcal{O} \boxtimes \mathcal{O}(\Delta) \). We have:

\[
\sum_{i,j} (\{a_i \boxtimes a_j \cdot f(x, y)\} - \{a_j \boxtimes a_i \cdot f(x, y)\}) = \sum_{i,j} (a_i \boxtimes a_j) \cdot (\xi_f, -\xi_f) \in \Delta_!(\mathcal{A}),
\]

where \( \xi_f \) has the same meaning as above.

The expression \( (h \boxtimes h \boxtimes \text{id})(\{f(x, y) \cdot u \boxtimes 1'_{y,z}\}) \) equals

\[
(h \boxtimes \text{id})(\{u \boxtimes d_u(f(x, y))\})
\]

and assertion b) follows from the following general fact:

**Lemma 9.3.** Let \( \mathcal{M} \) be a right \( D \)-module on a curve \( X \), let \( m \in \mathcal{M} \) be its local section and let \( f(x, y) \) be a section of \( \mathcal{O} \boxtimes \mathcal{O}(\Delta) \). Then under the map

\[
(h \boxtimes \text{id}) \circ \text{can}_{\mathcal{M}} : j_{x,y}^*(\mathcal{M} \boxtimes \Omega) \to \Delta_!(\mathcal{M}) \to \mathcal{M}
\]

the section \( m \boxtimes d_u(f(x, y)) \) goes over to \( -m \cdot \xi_f \). \( \square \)
10. The linear dilaton

10.1. Now let $g = \mathbb{C}$. We are going to consider a twist of the CFT corresponding to $A(\mathbb{C}, Q)$, called the linear dilaton.

We start with the following observation:

**Lemma 10.1.** Consider the category $\text{Ext}_{\text{Aut}^+_0}(L_0, L_1)$ of extensions

$$0 \to L_1 \to E \to L_0 \to 0$$

of $\text{Aut}^+_0$-modules. We have a canonical isomorphism $\pi_0(\text{Ext}_{\text{Aut}^+_0}(L_0, L_1)) \simeq \mathbb{C}$.

Thus, for $\lambda \in \mathbb{C}$ one obtains a canonical extension $F_\lambda$:

$$0 \to \Omega \to F_\lambda \to \emptyset \to 0$$
in the category of local $\mathcal{O}$-modules on $X$. For instance, if $\lambda = -2n$, where $n$ is an integer, we have a canonical isomorphism $F_\lambda \simeq \text{Diff}_1(\Omega^n, \Omega^{n+1})$.

For a pair of complex numbers $(\lambda, Q)$ we now define a Lie-* algebra structure on the D-module

$$\mathcal{B}(\lambda, Q) := F_\lambda \otimes D/\ker(\Omega \otimes D \to \Omega)$$

by the rule that the Lie-* bracket is the composition

$$\mathcal{B}(\lambda, Q) \boxtimes \mathcal{B}(\lambda, Q) \to D \boxtimes D \to \Delta!(\Omega),$$

where the last arrow sends $1 \boxtimes 1 \in D \boxtimes D$ to $Q \cdot 1' \in \Delta!(\Omega)$.

We define the chiral algebra $A(\lambda, Q)$ as the quotient of $U(\mathcal{B}(\lambda, Q))$ by the standard relation $\text{unit} = \text{unit}$.

Here is another useful point of view on the chiral algebra $A(\lambda, Q)$:

**Proposition 10.1.** Consider the chiral algebra $A(\mathbb{C}, Q)$ ($Q$ is viewed as a quadratic form on $\mathbb{C}$).

(a) There is a natural map of sheaves $\Omega \to \text{Aut}(A(\mathbb{C}, Q))$.

(b) The chiral algebra $A(\lambda, Q)$ is identified naturally with a twist of $A(\mathbb{C}, Q)$ with respect to the $\Omega$-torsor $F_\lambda$.

**Proof:** For a section $\omega \in \Omega$ we define the corresponding automorphism of $\mathcal{B}(\mathbb{C}, Q)$ by the rule that it acts as the identity on $\Omega \subset \mathcal{B}(\mathbb{C}, Q)$ and that it sends the section $1 \in D \subset \mathcal{B}(\mathbb{C}, Q)$ to $\omega \in \Omega \subset \mathcal{B}(\mathbb{C}, Q)$; clearly, it extends to an automorphism of $A(\mathbb{C}, Q)$. This establishes point (a) of the proposition, whereas point (b) follows immediately from the definitions.

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12We fix this isomorphism in such a way that if $0 \to L_1 \to E \to L_0 \to 0$ is an object of $\text{Ext}_{\text{Aut}^+_0}(L_0, L_1)$ corresponding to $\lambda \in \mathbb{C}$, the map $L_1 \simeq L_1 \otimes L_0 \to L_1$ induced by the action of $L_1 \subset \text{Vir}^+_0$ on $E$ is multiplication by $\lambda$. 
10.2. We shall now show that the chiral algebra $\mathcal{A}(\lambda, Q)$ defines a CFT with $c = 1 + 3\lambda^2/Q$.

Obviously, $\mathcal{A}(\lambda, Q)$ carries a structure of local $\mathcal{O}$-module on $X$ which is compatible with the chiral bracket. Thus, to define the corresponding CFT data we must exhibit a map of local $\mathcal{O}$-modules $T_{\lambda, Q} : \Theta_e \to \mathcal{A}(\lambda, Q)$ with $c$ as above which will satisfy condition (b) of Section 8.3.

**Proposition 10.2.** There exists a unique map $\overline{T}_{\lambda, Q} : \Theta \to \mathcal{A}(\lambda, Q)/\Omega$ such that the following holds:

For any local section $l \in \mathcal{A}(\lambda, Q)$, any vector field $\xi$ on $X$ and any section $\xi'$ of $\mathcal{A}(\lambda, Q)$ that projects to $\overline{T}_{\lambda, Q}(\xi)$ we have:

$$(h \boxtimes \text{id})(\{\xi', l\}) = -\text{Lie}_\xi(l) - l \cdot \xi.$$ 

**Proof.** Note first of all that the left hand side of the above expression is a priori independent of the choice of $\xi'$. Moreover, the uniqueness statement is clear, since $\Omega$ is the only $\mathcal{O}$-submodule in $\mathcal{A}(\lambda, Q)$ that annihilates the Lie-* bracket (this follows, for example, from the corresponding fact in the untwisted case (i.e. when $\lambda = 0$) in [Proposition 10.1(b)]).

The existence assertion is local and let us choose a coordinate $z$ on some open sub-set $U \subset X$; let $\partial_z$ denote the corresponding vector field on $U$. It is clear that the choice of $\partial_z$ as above trivializes the $\Omega$-torsor $\mathcal{F}_\lambda$ over $U$; let $\phi_3$ denote the corresponding isomorphism

$$\phi_3 : \mathcal{A}(\mathbb{C}, Q) \to \mathcal{A}(\lambda, Q)$$

de fined over $U$.

We define the map $\overline{T}_{\lambda, Q}$ by the condition that

$$\overline{T}_{\lambda, Q}(\partial_3) = \phi_3(T_{\mathbb{C}, Q}(\partial_3') - \lambda/2Q \cdot \partial_3),$$

where $\partial_3'$ in the first term is any lifting of $\partial_3$ to a section of $\Theta'_1$ and $\partial_3$ in the second term is viewed as a section of $D_X \subset \mathcal{B}(\mathbb{C}, Q)$.

Let us check now that the map $\overline{T}_{\lambda, Q}$ defined in the above way satisfies the condition of the proposition.

Let $f$ be a function on $U$. It is enough to show that

$$(h \boxtimes \text{id})(\{\phi_3(T_{\mathbb{C}, Q}(\partial_3') \cdot f - \lambda/2Q \cdot \partial_3 f) \boxtimes \phi_3(1)\}) = -\phi_3(1) \cdot f \partial_3 - \text{Lie}_f \phi_3(\phi_3(1)),$$

Since $\phi_3$ is a homomorphism of chiral algebras and since $T_{\mathbb{C}, Q}$ is the energy momentum tensor for $\mathcal{A}(\mathbb{C}, Q)$, we have:

$$(h \boxtimes \text{id})(\{\phi_3(T_{\mathbb{C}, Q}(\partial_3') \cdot f) \boxtimes \phi_3(1)\}) = -\phi_3(1) \cdot f \partial_3$$

and the above assertion reduces to the equality

$$\lambda \cdot d(\partial_3 f) = 2 \text{Lie}_f \phi_3(\phi_3(1)),$$

which follows from the definition of $F_\lambda$.

The map $\overline{T}_{\lambda, Q}$ constructed above is compatible with the local $\mathcal{O}$-module structure due to the uniqueness part of the assertion of Proposition 10.2. We now define $\Theta'$ to be the extension $0 \to \Omega \to \Theta' \to \Theta \to 0$ induced from the extension $0 \to \Omega \to \mathcal{A}(\lambda, Q) \to \mathcal{A}(\lambda, Q)/\Omega \to 0$ by means of the map $\overline{T}_{\lambda, Q}$. 


By the construction, \( \Theta' \) carries in a natural way a structure of local \( \mathcal{O} \)-module on \( \mathcal{X} \) and we have a map \( T_{\lambda,Q} : \Theta' \to \mathcal{A}(\lambda, Q) \) that satisfies condition (b) of Section 8.2.

**Proposition 10.3.** The local \( \mathcal{O} \)-module \( \Theta' \) constructed above is identified with \( \Theta_c \) for \( c = 1 + 3\lambda^2 / Q \).

**Proof.** The proof of the proposition will be based on the following lemma:

**Lemma 10.2.** Let \( 0 \to \Omega \to \Theta' \to \Theta \to 0 \) be an arbitrary extension in the category of local \( \mathcal{O} \)-modules on \( \mathcal{X} \). Assume that for any pair \( (x \in X) \) the following holds: if \( \mathfrak{z} \) is a coordinate around \( x \) and if \( \tilde{\partial}_\mathfrak{z} \) is a local section of \( \Theta' \) that projects to \( \partial_\mathfrak{z} \), the value of \( \text{Lie}_{\mathfrak{z}}(\partial_\mathfrak{z}') \) at \( x \) equals \( -c/2 \cdot d_3 \). Then \( \Theta' \) is canonically isomorphic to \( \Theta'_c \).

Let \( U, \mathfrak{z} \) and \( \phi_\mathfrak{z} \) be as in Proposition 10.2. For a function \( f \) on \( U \) consider the map from \( \mathcal{A}(\mathbb{C}, Q) \) to itself given by

\[
l \mapsto \phi_\mathfrak{z}^{-1} \circ \text{Lie}_f \partial_\mathfrak{z} \circ \phi_\mathfrak{z}(l) - \text{Lie}_f \partial_\mathfrak{z}(l).
\]

We claim that it coincides with the map

\[
l \mapsto \lambda/2Q(h \boxtimes \text{id})(\{ \partial_\mathfrak{z} \cdot f \boxtimes l \}).
\]

Indeed, both maps are derivations of the chiral algebra structure on \( \mathcal{A}(\mathbb{C}, Q) \) (in particular, they commute with the action of \( D_X \)) and coincide on the generators.

Let us now apply Lemma 10.2 to the extension \( \Theta' \) as in the formulation of the proposition. By the construction, we can take \( \tilde{\partial}_\mathfrak{z} \) to be

\[
\phi_\mathfrak{z}(T_{C,Q}(\partial_\mathfrak{z}') - \lambda/2Q \cdot \partial_\mathfrak{z}) \in \mathcal{A}(\lambda, Q),
\]

where \( \partial_\mathfrak{z}' \) is the corresponding section of \( \Theta'_c \).

We have:

\[
\phi_\mathfrak{z}^{-1}(\text{Lie}_f \partial_\mathfrak{z}(\tilde{\partial}_\mathfrak{z})) = \lambda/2Q(h \boxtimes \text{id})(\{ \partial_\mathfrak{z} \cdot \mathfrak{z}^3 \boxtimes T_{C,Q}(\partial_\mathfrak{z}') \}) -
\]

\[
- \lambda/2Q(h \boxtimes \text{id})(\{ \partial_\mathfrak{z} \cdot \mathfrak{z}^3 \boxtimes \lambda/2Q \cdot \partial_\mathfrak{z} \}) + \text{Lie}_{\mathfrak{z}}(T_{C,Q}(\partial_\mathfrak{z}')) - \text{Lie}_{\mathfrak{z}}(\lambda/2Q \cdot \partial_\mathfrak{z}).
\]

It is easy to see that the values at \( x \) of the first and of the fourth terms of the above formula are 0. The value at \( x \) of the second term is \(-3\lambda^2/2Q \cdot d_3 \), by definition. Since the chiral algebra \( \mathcal{A}(\mathbb{C}, Q) \) defines a CFT of central charge 1, the value at \( x \) of the third term is \(-1/2 d_3 \).

\[\boxdot\]

## 11. The BC-system

### 11.1. Let \( \mathcal{M} \) be a locally free finitely generated \( D \)-module on a curve \( X \) (e.g. \( \mathcal{M} = \mathcal{L} \otimes D_X \), where \( \mathcal{L} \) is a torsion-free quasi-coherent sheaf). Let \( \mathcal{M}^* \) denote the Verdier-dual \( D \)-module, i.e.

\[
\mathcal{M}^* := \text{Hom}_{D_X}(\mathcal{M}, D_X \otimes \Omega).
\]

Consider the \( D \)-module \( \mathcal{B}_{bc}(\mathcal{M}) := \Omega \oplus \mathcal{M} \oplus \mathcal{M}^* \). We define a \( \mathbb{Z} \)-grading (and hence a \( \mathbb{Z}_2 \)-grading) on \( \mathcal{B}_{bc}(\mathcal{M}) \) by declaring that \( \Omega \) is homogeneous of degree 0 and \( \mathcal{M} \) and \( \mathcal{M}^* \) are of degrees \(-1 \) and \( 1 \) respectively. We introduce a (super) Lie-* algebra structure on \( \mathcal{B}_{bc}(\mathcal{M}) \) as follows:

\[
\Omega \text{ is central and the map } \mathcal{M} \boxtimes \mathcal{M}^* \to \Delta_1(\mathcal{B}_{bc}(\mathcal{M})) \text{ is the composition}
\]

\[
\mathcal{M} \boxtimes \mathcal{M}^* \to D_X \otimes \Omega \cong D_0(\Omega) \to \Delta_1(\mathcal{B}_{bc}(\mathcal{M})).
\]
The (super) chiral algebra $\mathcal{A}_{bc}(\mathcal{M})$ is by definition the quotient of $\mathcal{U}(\mathcal{B}_{bc}(\mathcal{M}))$ modulo the standard relation $\text{unit} - \text{unit} = 0$.

The chiral algebra $\mathcal{A}_{bc}(\mathcal{M})$ carries a $\mathbb{Z}$-grading $\mathcal{A}_{bc}(\mathcal{M}) = \bigoplus_{i} \mathcal{A}_{bc}(\mathcal{M})^i$ that comes from the $\mathbb{Z}$-grading on $\mathcal{B}_{bc}(\mathcal{M})$ and a filtration $\mathcal{A}_{bc}(\mathcal{M}) = \bigcup_{i \in \mathbb{Z}^+} \mathcal{A}_{bc}(\mathcal{M})^i$ that comes from the canonical filtration on $\mathcal{U}(\mathcal{B}_{bc}(\mathcal{M}))$ (cf. Section 4.4).

In this subsection we will compute the space of conformal blocks of $\mathcal{A}_{bc}(\mathcal{M})$ (assuming that $X$ is complete).

Note that $DR^{-1}(X, \mathcal{M}) = 0$, since $\mathcal{M}$ is locally free as a D-module. Let us denote by $\text{det} DR(\mathcal{M})$ the 1-dimensional vector space $\Lambda^{top}(DR^0(X, \mathcal{M})) \otimes (\Lambda^{top}(DR^1(X, \mathcal{M})))^{-1}$.

**Proposition 11.1.** The space $H_{\nabla}(X, \mathcal{A}_{bc}(\mathcal{M}))$ is 1-dimensional. It is concentrated in the homogeneity degree $\dim(DR^0(X, \mathcal{M})) - \dim(DR^1(X, \mathcal{M}))$ and is canonically isomorphic to $(\text{det} DR(\mathcal{M}))^{-1}$.

**Proof.** Let $x$ be a point of $X$. We shall compute the space $H_{\nabla}(X, \mathcal{A}_{bc}(\mathcal{M}))$ using Proposition 5.2, which implies that

$$H_{\nabla}(X, \mathcal{A}_{bc}(\mathcal{M})) = \text{coker}(DR^0(X \setminus x, \mathcal{M} \oplus \mathcal{M}^*) \otimes \mathcal{A}_{bc}(\mathcal{M}) \otimes \mathcal{A}_{bc}(\mathcal{M})_x) \rightarrow \mathcal{A}_{bc}(\mathcal{M})_x).$$

The space $DR^0(\text{Spec}(\hat{\mathcal{M}}_x), \mathcal{M})$ is a topological vector space of Tate type (cf. [8]) and we have a canonical isomorphism

$$DR^0(\text{Spec}(\hat{\mathcal{M}}_x), \mathcal{M}^*) \simeq (DR^0(\text{Spec}(\hat{\mathcal{M}}_x), \mathcal{M}))^*.$$

Therefore, $V := DR^0(\text{Spec}(\hat{\mathcal{M}}_x), \mathcal{M} \oplus \mathcal{M}^*)$ acquires a non-degenerate quadratic form such that

$$V_1 := DR^0(\text{Spec}(\hat{\mathcal{M}}_x), \mathcal{M} \oplus \mathcal{M}^*)$$

and $V_2 := DR^0(X \setminus x, \mathcal{M} \oplus \mathcal{M}^*)$ are two Lagrangian subspaces of $V$, which are compact and co-compact, respectively.

We have the following general statement:

**Lemma 11.1.** Let $V$ be a Tate topological vector space endowed with a symmetric nondegenerate pairing $Q : V \otimes V \to \mathbb{C}$, let $V_1$ be a Lagrangian compact subspace of $V$ and let $S$ be the corresponding representation of the Clifford algebra. Let $V_2 \subset V$ be a Lagrangian co-compact subspace of $V$. We have a canonical isomorphism

$$\text{coker}(V_2 \otimes S \rightarrow S) \simeq \Lambda^{top}(V/V_1 + V_2).$$

This implies the existence of an isomorphism as in the statement of the proposition using Verdier duality:

$$DR^1(X, \mathcal{M}^*) \simeq (DR^0(X, \mathcal{M}))^*.$$

The fact that the isomorphism

$$H_{\nabla}(X, \mathcal{A}_{bc}(\mathcal{M})) \simeq (\text{det} DR(\mathcal{M}))^{-1}$$

does not depend on the choice of $x \in X$ is easy to show by generalizing the above argument to the case of several points $x_1, \ldots, x_n \in X$ and by establishing the appropriate compatibility when one of them is deleted.
Lemma [11.1] above implies also the following statement:

Let \( x_1, \ldots, x_l, y_1, \ldots, y_k \) be distinct points of \( X \) and consider the composition:

\[
\kappa_1 \otimes M_{x_1} \otimes \cdots \otimes M_{x_l} \otimes \kappa_2 \otimes M_{y_1} \otimes \cdots \otimes M_{y_k} \to \ker A_{bc}(M)_{x_1} \otimes \kappa_2 \otimes A_{bc}(M^*)_{y_j} \to H\nu(X, A_{bc}(M)) \simeq (\det DR(M))^{-1}.
\]

(Here \( M_x \) denotes the fiber of \( M \) at \( x \in X \) and similarly for \( M^* \).)

**Corollary 11.1.** The above map \( M_{x_1} \otimes \cdots \otimes M_{x_l} \otimes M^*_y \otimes \cdots \otimes M^*_y \to (\det DR(M))^{-1} \) is zero unless \( k_1 = \dim(\det R^1(X, M)) \) and \( k_2 = \dim(\det R^0(X, M)) = \dim(\det R^0(X, M^*)) \) and in the latter case it coincides with the composition

\[
\kappa_1 \otimes M_{x_1} \otimes \cdots \otimes M_{x_l} \to DR^1(X, M) \otimes \cdots \otimes DR^1(X, M) \otimes DR^1(X, M^*) \otimes \cdots \otimes DR^1(X, M^*) \to \Lambda^{k_1} (DR^1(X, M)) \otimes \Lambda^{k_2} (DR^1(X, M^*)) \simeq (\det DR(M))^{-1}
\]

11.2. Let \( A_{bc}(n) \) denote the chiral algebra \( A_{bc}(M) \) for \( M = \Omega^n \otimes D \). In this subsection we shall show that it defines a CFT with central charge equal to \( c_n := -2(6n^2 - 6n + 1) \). In the physics literature this CFT is often referred to as the bc-system.

According to Lemma [6.4] \( A_{bc}(n) \) carries a structure of local \( \mathcal{O} \)-module on \( X \) that satisfies condition (a) of Section [5.2]. Therefore, to define the corresponding CFT we must construct the energy-momentum tensor \( T_{bc,n} : \Theta \to A_{bc}(n) \) that satisfies condition (b) of Section [5.2].

**Proposition 11.2.** There exists a unique map \( T_{bc,n} : \Theta \to A_{bc}(n)/\Omega \) such that for any section \( l \in A_{bc}(n) \), any vector field \( \xi \) and a section \( \xi' \) of \( A_{bc}(n) \) that projects to \( T_{bc,n}(\xi) \) we have:

\[
(h \boxtimes \text{id})([\xi', l]) = -l \cdot \xi - \text{Lie}_\xi(l).
\]

**Proof.** It is easy to see that \( \Omega \) is the unique \( \mathcal{O} \)-sub-module of \( A_{bc}(n) \) which annihilates the Lie\(^*\) bracket. This implies the uniqueness part in the assertion of Proposition 11.2.

Therefore, the existence part is local; let \( \xi \) be a coordinate over some \( U \subset X \).

For \( f(x, y) \in \mathcal{O} \otimes \mathcal{O}(\Delta) \), let \( \xi_f \) denote the corresponding vector field and set \( T_{bc,n}(\xi_f) \) to be equal to the image of

\[
\xi'_f := -(h \boxtimes \text{id})(n\{d_\xi \otimes n, \partial \partial \xi \otimes l - n - 1\}) - (n - 1)\{d_\xi \otimes l, d_\xi \otimes l - n - 1\} \partial \partial \xi \cdot f(x, y)
\]

under the projection \( A_{bc}(n) \to A_{bc}(n)/\Omega \).

It is easy to see that the above expression is well-defined (i.e. does not depend on the choice of \( f(x, y) \) that gives rise to \( \xi_f \)) and that, moreover, it equals the projection to \( A_{bc}(n) / \Omega \) of

\[
-(\text{id} \boxtimes h)(n\{d_\xi \otimes n, \partial \partial \xi \otimes l - n - 1\} - (n - 1)\{d_\xi \otimes l, d_\xi \otimes l - n - 1\} \partial \partial \xi \cdot f(x, y))
\]

Let us check that the map \( T_{bc,n} \) constructed above satisfies the condition of the proposition. It is enough to check that for \( \xi_f \) and \( \xi'_f \) as above

\[
(h \boxtimes \text{id})([\xi'_f, l]) = -l \cdot \xi_f - \text{Lie}_\xi(\xi_f)(l)
\]

for \( l \) being a nonvanishing section of either \( \Omega^n \) or \( \Omega^{1-n} \). As the situation is symmetric in \( n \) and \( 1 - n \) we shall treat the case of \( l = d_\xi \otimes n \in \Omega^n \subset A_{bc}(n) \).
By applying the Jacobi identity we obtain:
\[(h \otimes \text{id})(\{\xi_f, l\}) = -(n(h \otimes h \otimes \text{id})((d_3 \otimes \{d_3 \otimes \{d_3 \otimes (\{d_3 \otimes f(x, y)\})\}) + (n - 1)(h \otimes h \otimes \text{id})((d_3 \otimes \{d_3 \otimes \{d_3 \otimes f(y, x)\})\)).\]

An easy computation shows that the first term in the above expression equals 
\[-nd\otimes \xi_f - \text{Lie}_{\xi_f}(d\otimes n).\] The second term equals 
\[(n - 1)d\otimes \xi_f\], according to Lemma 1[1.3]. This implies the assertion of the proposition.

The uniqueness assertion in Proposition 1[1.2] guarantees that the map \(\tilde{T}_{bc,n} : \Theta \to \mathcal{A}_{bc}(n)/\Omega\) is compatible with the \(\mathcal{O}\)-module structure. We define the local \(\mathcal{O}\)-module \(\Theta'(n)\) to be the extension
\[0 \to \Omega \to \Theta'(n) \to \Theta \to 0\]
induced from the extension \(0 \to \Omega \to \mathcal{A}_{bc} \to \mathcal{A}_{bc}(n)/\Omega \to 0\) by means of the map \(\tilde{T}_{bc,n}\). By definition, we have the energy-momentum tensor map
\[T_{bc,n} : \Theta'(n) \to \mathcal{A}_{bc}(n)\].

It can be shown by a direct computation (as in Proposition 1[1.3]), that \(\Theta'(n) \simeq \Theta'_{c,\nu}\). In the next subsection we shall show a way to establish such an isomorphism without making any computations: we shall deduce it from the so-called Bose-Fermi correspondence.

**Corollary 11.2.** The line bundle on \(\mathcal{M}\) whose fiber at \(X \in \mathcal{M}\) is \(\text{det}(R\Gamma(X, \Omega^n))\) carries a canonical structure of a \(\Theta'_{\mathcal{M}_{\mathcal{C},\nu}}\)-module.

**Proof.** This assertion is simply a combination of Proposition 1[1.1] and Proposition 1[1.3].

**Remark 11.1.** Take \(n = -1\) (we have \(c_{-1} = -26\)) and consider the union of the connected components of \(\mathcal{M}\) that correspond to curves of genus \(g \geq 2\). In this case the line bundle \(X \to \text{det}(R\Gamma(X, \Omega^n))\) is identified with \(\Omega^{top}_{\mathcal{M}}\). Therefore, the ring of twisted differential operators on \(\mathcal{M}\) corresponding to the Picard Lie algebroid \(\Theta_{\mathcal{M}_{\mathcal{C},\nu}}\) is identified in this case with \(D^{op}_{\mathcal{M}}\) in such a way that \((H_X(X, \mathcal{A}_{bc}(-1)))^*\) becomes the canonical right \(D\)-module over \(\mathcal{M}\).

11.3. Consider the local \(\mathcal{O}\)-module \(p_{2*}(\Omega^n \otimes \Omega^{1-n}(\Delta)/\Omega^n \otimes \Omega^{1-n}(-\Delta))\). We have a short exact sequence:
\[0 \to \Omega \to p_{2*}(\Omega^n \otimes \Omega^{1-n}(\Delta)/\Omega^n \otimes \Omega^{1-n}(-\Delta) \to 0 \to 0\]
and it is easy to show that we have in fact a canonical isomorphism
\[p_{2*}(\Omega^n \otimes \Omega^{1-n}(\Delta)/\Omega^n \otimes \Omega^{1-n}(-\Delta) \simeq F_{2n-1},\]
(c.f. Section 1[0.1]).

We define a map \(\tilde{J} : F_{2n-1} \to \mathcal{A}_{bc}(n)\) by setting
\[\tilde{J}(\omega^n \otimes \omega^{1-n} \cdot f(x, y)) = (h \otimes \text{id})(\{\omega^n \otimes \omega^{1-n} \cdot f(x, y)\}).\]

Clearly, \(\tilde{J}\) is a map of local \(\mathcal{O}\)-modules. It is commonly referred to as the “fermion number current” due to the following property:
Theorem 11.1. We have a canonical isomorphism $T$ coincides under this identification with the map $J$.

Proof. It enough to check the assertion on the generators, i.e. for $l \in \Omega^n$ and for $l \in \Omega^{1-n}$, in which case it follows immediately from the Jacobi identity.

The following result is a part of the twisted Bose-Fermi correspondence:

**Theorem 11.1.** (a) The map $J$ gives rise to a map of chiral algebras

$J : A(2n - 1, -1) \rightarrow A_{bc}(n)$.

The map $J$ is compatible with the local $\mathcal{O}$-module structure.

(b) We have a canonical isomorphism $\Theta'(n) \simeq \Theta'_{bc,n}$ and the composition

$\Theta'_{bc,n} \circ T_{2n-1} \rightarrow A(2n - 1, -1) \xrightarrow{J} A_{bc}(n)$

coincides under this identification with the map $T_{bc,n}$.

Proof. To prove point (a) of the theorem it is enough to show that if $g \in F_{2n-2}$ is a section that maps to $\Omega$ under the projection $F_{2n-1} \rightarrow \Omega$, then

$\Theta'(n)\Omega \simeq (\{J(f_g), l_1\}) = -dg.$

Set $j_g = J(f_g)$. As in Proposition 11.3 we have:

$\Theta'(n)\Omega \simeq (\{j_g, l\}) = g \cdot l$ for $l \in \Omega^{1-n}$ and $\Theta'(n)\Omega \simeq (\{j_g, l\}) = -g \cdot l$ for $l \in \Omega^n$.

The assertion of point (a) now follows from the Jacobi identity.

In order to prove point (b) observe that it is enough to show that the map

$J \circ T_{2n-1,-1} : \Theta \rightarrow A(2n - 1, -1) / \Omega \rightarrow A_{bc}(n) / \Omega$

coincides with the map $T_{bc,n}$. This will be done in two steps:

**Step 1:** We first show that $J \circ T_{2n-1,-1}$ is proportional to $T_{bc,n}$.

Indeed, $T_{bc,n}$ is obviously a map $\Theta \rightarrow A_{bc}(n) \cap A_{bc}(n)^0$. We claim that the same is true for $J \circ T_{2n-1,-1}$. A priori, the latter is a map $\Theta \rightarrow A_{bc}(n) \cap A_{bc}(n)^0$.

However, it is easy to see that its symbol, i.e. the composition

$\Theta \rightarrow A_{bc}(n) \cap A_{bc}(n)^0 / A_{bc}(n)^0 \cap A_{bc}(n)^0$

vanishes. Therefore, the image of $J \circ T_{2n-1,-1}$ lies in $\Theta \rightarrow A_{bc}(n) \cap A_{bc}(n)^0$. However, by parity considerations, $A_{bc}(n) \cap A_{bc}(n)^0 = A_{bc}(n)^0 \cap A_{bc}(n)^0$.

Observe now that as an $\mathcal{O}$-module, $A_{bc}(n) \cap A_{bc}(n)^0$ is identified with

$(\Omega^n \otimes D) \otimes (\Omega^{1-n} \otimes D)$,

according to Lemma 4.1(c).

The assertion of Step 1 now follows from the next lemma:

**Lemma 11.2.** The space $\text{Hom}(\Theta, (\Omega^n \otimes D) \otimes (\Omega^{1-n} \otimes D))$ in the category of local $\mathcal{O}$-modules is 1-dimensional.

---

13 Of course, $c_n := -2(6n^2 - 6n + 1) = 1 - 3(2n - 1)^2$. 
Step 2:

We know by now that \( J \circ T_{2n-1, -1} = \epsilon \cdot \tilde{T}_{bc,n} \) for some \( \epsilon \in \mathbb{C} \) and it remains to show that \( \epsilon = 1 \).

For \( \xi \in \Theta \) let \( \xi' \) be any section of \( \mathcal{A}_{bc}(n) \) that projects to \( T_{bc,n}(\xi) \). Since \( T_{bc,n} \) is an energy-momentum tensor for \( \mathcal{A}_{bc}(n) \), on the one hand we have
\[
(h \boxtimes \text{id})(\{ \epsilon \cdot \xi' \boxtimes J(l) \}) = \epsilon (-J(l) \cdot \xi - J(\text{Lie}_\xi(l))) \quad \forall l \in \mathcal{A}_{bc}(n).
\]

On the other hand, since \( J \) is a homomorphism of chiral algebras and \( T_{2n-1, -1} \) is an energy-momentum tensor for \( \mathcal{A}(2n - 1, -1) \), we have
\[
(h \boxtimes \text{id})(\{ \epsilon \cdot \xi' \boxtimes J(l) \}) = -J(l) \cdot \xi - J(\text{Lie}_\xi(l)) \quad \forall l \in \mathcal{A}(2n - 1, -1).
\]

The two equations are compatible only when \( \epsilon = 1 \).
Chapter IV. BRST and String Amplitudes

12. The BRST complex

12.1. Let $\mathcal{B}$ be a Lie-* algebra over a curve $X$, which is locally free and finitely generated as a D-module. We shall now attach to $\mathcal{B}$ a canonical central extension

$$0 \to \Omega \to \mathcal{B}'' \to \mathcal{B} \to 0$$

of Lie-* algebras.

The construction will be based on the following general assertion:

**Lemma 12.1.** Let $\mathcal{M}_1$, $\mathcal{M}_2$ be two D-modules on $X$ with $\mathcal{M}_1$ being locally free and finitely generated. Then:

(a) For a third D-module $\mathcal{N}$ on $X$, there is a canonical isomorphism:

$$\text{Hom}_{D(X)}(\mathcal{M} \boxtimes \mathcal{M}_1, \Delta_!(\mathcal{M}_2)) \cong \text{Hom}_D(\mathcal{M}, \mathcal{M}_1^* \otimes \mathcal{M}_2).$$

(b) The canonical map (from point (a)) $\mathcal{M}_1^* \otimes \mathcal{M}_2 \boxtimes \mathcal{M}_1 \to \Delta_!(\mathcal{M}_2)$ induces an isomorphism $h(\mathcal{M}_1^* \otimes \mathcal{M}_2) \to \text{Hom}_D(\mathcal{M}, \mathcal{M}_2)$.

**Corollary 12.1.** For a Lie-* algebra $\mathcal{B}$ we have canonical maps

$$\text{co-ad : } \mathcal{B} \boxtimes \mathcal{B}^* \to \Delta_!(\mathcal{B}^*) \text{ and } \text{co-br : } \mathcal{B}^* \to \mathcal{B}^* \boxtimes \mathcal{B}^*.$$ 

We shall call the two maps of the above corollary “the co-adjoint action” and “the co-bracket”, respectively.

Let us for a moment view $\mathcal{B}$ as a plain D-module on $X$ (i.e. forget the Lie-* algebra structure) and consider the chiral algebra $A_{bc}(\mathcal{B})$ of Section 11.1. Let $i_\mathcal{B}$ (resp., $i^*_\mathcal{B}$) denote the canonical embedding of $\mathcal{B}$ (resp., of $\mathcal{B}^*$) into $A_{bc}(\mathcal{B})$.

It follows from Lemma 11.1(c) that the intersection $A_{bc}(\mathcal{B})_2 \cap A_{bc}(\mathcal{B})^0$ is an extension of D-modules

$$0 \to \Omega \to A_{bc}(\mathcal{B})_2 \cap A_{bc}(\mathcal{B})^0 \to \mathcal{B} \otimes \mathcal{B}^* \to 0.$$

According to Lemma 12.1 above, the Lie-* bracket on $\mathcal{B}$ gives rise to a map

$$S_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}^* \boxtimes \mathcal{B} \hookrightarrow A_{bc}(\mathcal{B})/\Omega.$$

**Proposition 12.1.** For two local sections $b_1$ and $b_2$ of $\mathcal{B}$ let $S_{\mathcal{B}}(b_1)'$ and $S_{\mathcal{B}}(b_2)'$ be local sections of $A_{bc}(\mathcal{B})$ that project to $S_{\mathcal{B}}(b_1)$ and $S_{\mathcal{B}}(b_2)$, respectively. Then the section $\{S_{\mathcal{B}}(b_1)' \boxtimes S_{\mathcal{B}}(b_2)\}'$ of $\Delta_!(A_{bc}(\mathcal{B}))$ projects to the section $S_{\mathcal{B}}(\{b_1, b_2\})$ of $\Delta_!(A_{bc}(\mathcal{B}))/\Omega$.

**Proof.** It is enough to prove that

$$(h \boxtimes h)(\{S_{\mathcal{B}}(b_1)' \boxtimes S_{\mathcal{B}}(b_2)\}') = h(S_{\mathcal{B}}(\{b_1, b_2\})) \in h(A_{bc}(\mathcal{B}))/\Omega)$$

for any $b_1$ and $b_2$ as above.

For a section $l$ of $h(\mathcal{B}^* \boxtimes \mathcal{B})$ and for some lift of $l$ to a section $l'$ of $h(A_{bc}(\mathcal{B})_2 \cap A_{bc}(\mathcal{B})^0)$ consider the map $\mathcal{B} \to A_{bc}(\mathcal{B})$ given by

$$b \to (h \boxtimes \text{id})(\{l' \boxtimes i_{\mathcal{B}}(b)\}).$$
It follows from the definition of the chiral bracket on $A_{bc}(\mathcal{B})$ that this map takes values in $\mathcal{B} \subset A_{bc}(\mathcal{B})$ and coincides with the canonical map $h(\mathcal{B}^* \otimes \mathcal{B}) \otimes \mathcal{B} \to \mathcal{B}$ of point (b) of Lemma 12.1.

The assertion of the proposition now follows from the Jacobi identity combined with Lemma 12.1(b), since both $(h \boxtimes h)((\mathcal{S}_\mathcal{B}(b_1) \boxtimes \mathcal{S}_\mathcal{B}(b_2)')$ and $h(\mathcal{S}_\mathcal{B}((b_1, b_2)))$ belong to $h(\mathcal{B}^* \otimes \mathcal{B}) \subset h(A_{bc}(\mathcal{B})/\Omega)$.

We now define $\mathcal{B}'$ to be the induced extension:

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \Omega & \longrightarrow & \mathcal{B}' & \longrightarrow & \mathcal{B} & \longrightarrow & 0 \\
\downarrow & & \downarrow \text{id} & & \downarrow \mathcal{S}_\mathcal{B} & & \downarrow \mathcal{S}_\mathcal{B} & & \\
0 & \longrightarrow & \Omega & \longrightarrow & \mathcal{A}_{bc}(\mathcal{B})_2 \cap \mathcal{A}_{bc}(\mathcal{B})^0 & \longrightarrow & \mathcal{B}^* \otimes \mathcal{B} & \longrightarrow & 0.
\end{array}
$$

Proposition 12.1 above implies that $\mathcal{B}'$ acquires a canonical Lie-* algebra structure which is a central extension of that of $\mathcal{B}$.

12.2. Let $\mathcal{B}'$ denote the extension $0 \to \Omega \to \mathcal{B}' \to \mathcal{B} \to 0$ obtained from $\mathcal{B}'$ by acting by $-1$ on $\Omega$. Let $\mathcal{U}(\mathcal{B}')'$ denote the quotient of $\mathcal{U}(\mathcal{B}')$ by the standard relation $\text{unit} - \text{unit} = 0$ and consider now the tensor product chiral algebra $\mathcal{A}_{BRST}(\mathcal{B}) := \mathcal{U}(\mathcal{B}')' \otimes \mathcal{A}_{bc}(\mathcal{B})$. The chiral algebra $\mathcal{A}_{BRST}(\mathcal{B})$ inherits from $\mathcal{A}_{bc}(\mathcal{B})$ a $\mathbb{Z}$- and a $\mathbb{Z}_2$-grading.

From the construction of $\mathcal{B}'$ it follows that we have a canonical map of Lie-* algebras $S_{\mathcal{B}, BRST} : \mathcal{B} \to \mathcal{A}_{BRST}(\mathcal{B})$.

Our next goal will be to construct a canonical element $\delta_\mathcal{B} \in h(\mathcal{A}_{BRST}(\mathcal{B}))$ with the following properties:

- $\delta_\mathcal{B}$ has degree 1 with respect to the $\mathbb{Z}$-grading on $\mathcal{A}_{BRST}(\mathcal{B})$ and
  $$(h \boxtimes h)((\delta_\mathcal{B}, \delta_\mathcal{B})) = 0.$$  

- $(h \boxtimes \text{id})((\delta_\mathcal{B}, S_{\mathcal{B}, BRST}(b))) = 0$ for any $b \in \mathcal{B}$.
- $(h \boxtimes \text{id})((\delta_\mathcal{B}, i_B(b))) = -S_{\mathcal{B}, BRST}(b)$ for any $b \in \mathcal{B}$.

In particular, for any chiral $\mathcal{B}'$-module $\mathcal{L}$, the map $(h \boxtimes \text{id})((\delta_\mathcal{B}, \cdot))$ would define a differential on the tensor product $\mathcal{L} \otimes \mathcal{A}_{bc}(\mathcal{B})$, which is homogeneous of degree 1 and whose square equals 0.

We have two maps $G_1, G_2 : j_\ast j^*(\mathcal{B} \boxtimes \mathcal{B}^*) \to \Delta_l(\mathcal{A}_{BRST}(\mathcal{B}))$:

The map $G_1$ is defined as minus the composition
$$j_\ast j^*(\mathcal{B} \boxtimes \mathcal{B}^*) \xrightarrow{S_{\mathcal{B}, BRST} \boxtimes \mathcal{I}_B} j_\ast j^*(\mathcal{A}_{BRST}(\mathcal{B}) \boxtimes \mathcal{A}_{BRST}(\mathcal{B})) \xrightarrow{\{,\}} \Delta_l(\mathcal{A}_{BRST}(\mathcal{B})).$$

To define the map $G_2$ note that the chiral bracket on $\mathcal{A}_{bc}(\mathcal{B})$ gives rise to a map $\mathcal{B}^* \boxtimes \mathcal{B}^* \to \mathcal{A}_{bc}(\mathcal{B})$ and let $i_{2, 2}$ denote the composition $\mathcal{B}^* \xrightarrow{\text{co-br}} \mathcal{B}^* \otimes \mathcal{B}^* \to \mathcal{A}_{bc}(\mathcal{B})$.

We set the map $G_2$ to be the composition:

$$j_\ast j^*(\mathcal{B} \boxtimes \mathcal{B}^*) \xrightarrow{i_{2, 2} \boxtimes \mathcal{I}_B} j_\ast j^*(\mathcal{A}_{bc}(\mathcal{B}) \boxtimes \mathcal{A}_{bc}(\mathcal{B})) \xrightarrow{\{,\}} \Delta_l(\mathcal{A}_{bc}(\mathcal{B})) \to \Delta_l(\mathcal{A}_{BRST}(\mathcal{B})).$$

\[14\text{It follows from the definitions that, for two chiral algebras } A_1 \text{ and } A_2, \text{ the tensor product } A_1 \boxtimes A_2 \text{ acquires a natural chiral algebra structure.}\]
Lemma 12.2. The restriction of $G_1$ (resp., of $G_2$) to $\mathcal{B} \otimes \mathcal{B}^* \subset j_*j^*(\mathcal{B} \otimes \mathcal{B}^*)$ coincides with the map

$$\mathcal{B} \otimes \mathcal{B}^* \xrightarrow{\text{can-ad}} \Delta_l(\mathcal{B}^*) \xrightarrow{i_2^\dag} \Delta_l(\mathcal{A}_{BRST}(\mathcal{B})).$$

In particular, the difference $G_1 - G_2$ is a well-defined map

$$\mathcal{B} \otimes \mathcal{B}^* \to \mathcal{A}_{BRST}(\mathcal{B}).$$

We set $\delta_\mathcal{B}$ to be the image under $G_1 - G_2$ of the canonical element $\text{id}_\mathcal{B} \in \text{End}_D(\mathcal{B}) \simeq h(\mathcal{B} \otimes \mathcal{B}^*)$

Proposition 12.2. The element $\delta_{BRST}^2$ satisfies conditions (1)–(3) above.

Proof. Property (3) is a straightforward calculation.

To prove property (2), it is enough to show that

$$(h \boxtimes h)(\{\delta_{BRST}^2, S_{\mathcal{B}_{,BRST}}(b)\}) = 0 \in h(\mathcal{A}_{BRST}(\mathcal{B}))$$

for any $b \in \mathcal{B}$.

Consider the action of $h(\mathcal{B})$ on the $\mathcal{D}$-module $\mathcal{B} \boxtimes \mathcal{B}^*$ given by the formula:

$$b' \cdot (b \boxtimes b^*) \to (h \boxtimes \text{id})(\{b', b\}) \otimes b^* + b \boxtimes (h \boxtimes \text{id})(\text{co-ad}(b' \boxtimes b^*)).$$

This action extends to an action of $h(\mathcal{B})$ on $j_*j^*(\mathcal{B} \otimes \mathcal{B}^*)$.

Consider also the action of $\mathcal{B}$ on $\mathcal{A}_{BRST}(\mathcal{B})$ given by

$$b \cdot l \to (h \boxtimes \text{id})(\{S_{\mathcal{B}_{,BRST}}(b), l\}).$$

Lemma 12.3. The maps $G_1$ and $G_2$ commute with the above $h(\mathcal{B})$–actions.

Property (2) now follows from the fact that the section $\text{id}_\mathcal{B} \in h(\mathcal{B} \otimes \mathcal{B}^*)$ is $h(\mathcal{B})$–invariant.

To establish property (1) note that the element $\delta_{BRST}^2$ satisfies $\{\delta_{BRST}^2, i_\mathcal{B}(b)\} = 0$ for all $b \in \mathcal{B}$. The assertion now follows from the next lemma:

Lemma 12.4. Let $\mathcal{M}$ be as in Section 11.1 and let $l \in h(\mathcal{A}_{bc}(\mathcal{M}))$ be an element of positive degree. Assume that $(h \boxtimes \text{id})(\{l, i_\mathcal{M}(m)\}) = 0 \forall m \in h(\mathcal{M})$. Then $l = 0$.

13. String amplitudes

13.1. Let us now apply the discussion of the previous subsection to the case $\mathcal{B} = \mathcal{F} = \Theta \otimes D_X$.

In this case the chiral algebra $\mathcal{A}_{bc}(\mathcal{B})$ is identified with the $bc$-system chiral algebra $\mathcal{A}_{bc}(n)$ for $n = -1$.

Lemma 13.1. The canonical central extension $\mathcal{B}''$ of Section 12.2 is identified for $\mathcal{B} = \mathcal{F}$ with $\mathcal{F}''_{-26}$. Moreover, under this identification, the map $S''_\mathcal{B}$ goes over to the energy-momentum tensor $T_{bc,-1}$.
This assertion follows immediately from the definitions (using the equality $c_{-1} := -2(6 \cdot (-1)^2 - 6 \cdot (-1) + 1 = -26)$.

Now let $(\mathcal{A}, T)$ be a CFT data of central charge 26. Consider the chiral algebra $\mathcal{A}_{\text{comp}} := \mathcal{A} \otimes \mathcal{A}_{bc}(-1)$. The energy momentum tensor for $\mathcal{A}$ gives rise to a homomorphism of chiral algebras:

$$\varphi : \mathcal{A}_{\text{BRST}}(\mathcal{T}) \rightarrow \mathcal{A}_{\text{comp}},$$

in particular, by composing it with the map $S_{\mathcal{T},\text{BRST}}$ we obtain a map $T_{\text{comp}} : \Theta \rightarrow \mathcal{T} \rightarrow \mathcal{A}_{\text{comp}}$.

**Proposition 13.1.** The pair $\mathcal{A}_{\text{comp}}, T_{\text{comp}}$ defines a CFT data of central charge 0.

**Proof.** Since both $\mathcal{A}$ and $\mathcal{A}_{bc}(-1)$ carry a structure of local $\mathcal{O}$-module over $\mathfrak{X}$, so does $\mathcal{A}_{\text{comp}}$.

The fact that the pair $(\mathcal{A}_{\text{comp}}, T_{\text{comp}})$ satisfies conditions (a) and (b) of Section 7.1 follows from the corresponding facts for $\mathcal{A}$ and for $\mathcal{A}_{bc}(-1)$. \hfill \Box

13.2. Let $\delta_{\text{comp}} \in h(\mathcal{A}_{\text{comp}})$ denote the image of $\delta_{\text{BRST}} \in h(\mathcal{A}_{\text{BRST}}(\Theta))$ under the homomorphism $\varphi$.

Note that the $\mathcal{O}_{\mathfrak{M}}$-module $X \rightarrow H_{\mathfrak{X}}(X, \mathcal{A}_{\text{comp}})$ carries a canonical structure of left $D_{\mathfrak{M}}$-module, according to Proposition 7.1. The next proposition describes the connection between the differential $\delta_{\text{comp}}$ and the De Rham differential on $H_{\mathfrak{X}}(\mathfrak{X}, \mathcal{A}_{\text{comp}})$:

**Proposition 13.2.** Let $f \cdot a_1 \mathfrak{X} \cdots \mathfrak{X} a_k$ be a $\mathbb{Z}_2$-homogeneous section of $j_* j^*(\mathcal{A}_{\text{comp}}^{\mathbb{Z}_2})$ on $\mathfrak{X}^k \setminus \Delta$. Then

$$\nabla((f \cdot a_1, \ldots, a_k)) = (\langle \ldots \mathfrak{X} \rangle)(\nabla_{\text{loc}}(f \cdot a_1, \ldots, a_k)) +$$

$$+ \sum_{i=1}^{k} (-1)^{\text{deg}(a_i)} f \cdot a_1, \ldots, \delta_{\text{BRST}}(a_i), \ldots, a_k, i_{\mathfrak{T}}).$$

**Proof.** Using Proposition 7.2, the assertion of the proposition reduces to the fact that

$$\sum_{i=1}^{k} (-1)^{\text{deg}(a_i)} f \cdot a_1, \ldots, \delta_{\text{BRST}}(a_i), \ldots, a_k, i_{\mathfrak{T}} = f \cdot a_1, \ldots, a_k, i_{\mathfrak{T}}.$$

This, however, follows from the equality

$$\sum_{i=1}^{k} (-1)^{\text{deg}(a_i)} f \cdot a_1, \ldots, \delta_{\text{BRST}}(a_i), \ldots, a_k, i_{\mathfrak{T}} +$$

$$+ (-1)^{\text{deg}(a_1)} f \cdot a_1, \ldots, a_k, \delta_{\text{BRST}}(i_{\mathfrak{T}}) = 0$$

and property (3) of the element $\delta_{\text{BRST}}$ (cf. Section 12.2). \hfill \Box

Since $c = 26$, the space of conformal blocks $H_{\mathfrak{X}}(\mathfrak{X}, \mathcal{A})$ carries a canonical right $\mathcal{D}$-module structure over $\mathfrak{M}$ (cf. Remark 11.3). Moreover, over the connected components of $\mathfrak{M}$ corresponding to curves of genus $g \geq 2$, we have an isomorphism

$$H_{\mathfrak{X}}(\mathfrak{X}, \mathcal{A}) \simeq H_{\mathfrak{X}}(\mathfrak{X}, \mathcal{A}_{\text{comp}}) \otimes \Omega^{\text{top}}_{\mathfrak{M}};$$

\[\delta_{\text{BRST}}(i_{\mathfrak{T}})\] denotes the map $\Omega^{\mathfrak{X}} \rightarrow \mathcal{A}_{\text{comp}}$ obtained from the canonical map $i_{\mathfrak{T}} : \Theta \rightarrow \mathcal{A}_{\text{comp}}$ and its insertion into the correlation functions has the same meaning as in Proposition 7.2.
in view of the following lemma:

**Lemma 13.2.** For two chiral algebras $A_1$ and $A_2$ and a complete curve $X$ we have a canonical isomorphism

$$H^\nabla(X, A_1 \otimes A_2) \simeq H^\nabla(X, A_1) \otimes H^\nabla(X, A_2).$$

Therefore, Proposition 13.2 expresses the right D-module structure on $H^\nabla(X, A)$ in terms of the fields of the composite theory (i.e. of $A_{\text{comp}}$).
Chapter V. Further Constructions

14. Chiral algebras via the Ran space

The definition of chiral algebras that we used above was a rather straightforward axiomatization of the OPE operation on quantum fields. We shall now discuss a different approach to chiral algebras, which is much less obvious from the point of view of QFT.

In what follows we shall use the following conventions:

For a curve $X$ and a surjection of finite sets $\phi : J \to J$, $\Delta_\phi$ will denote the corresponding diagonal embedding $X^J \to X^3$.

If $J = J_1 \cup \ldots \cup J_k$ is a decomposition of $I$ into a disjoint union of finite subsets we shall denote by $j_1, \ldots, j_k$ the embedding of the open subset $X^{j_1} \times \ldots \times X^{j_k}$ of $X^J$ that consists of points

$$x_1^1, \ldots, x_1^{j_1}, \ldots, x_k^1, \ldots, x_k^{j_k} \in X^{j_1} \times \ldots \times X^{j_k} \simeq X^J$$

with $x_i^j \neq x_i^{j'}$ whenever $j \neq j'$.

When $k = |I|$, we shall replace the notation $j_{j_1, \ldots, j_k}$ by $j_I$ or simply by $j$.

14.1. For a chiral algebra $A$ consider the corresponding left D-module $^lA$ and for a finite set $J$ consider the left D-module $^lA^J$ over $X^J$.

Now let $\phi : J \to J$ be a surjection of finite sets with $|J| = |J| - 1$. The chiral bracket on $A$ yields a well-defined map

$$\{, \} : j_2, j_3 (^lA^J) \to \Delta_\phi (j_3, j_2 (^lA^J)).$$

The fact that the chiral bracket on $A$ satisfies the Jacobi identity implies that we can form a complex $C^*$ of left D-modules on $X^n$, with $C^{-k}$ being

$$\phi : \{1, \ldots, n\} \to \{1, |J| = k\}$$

Let $^lA^{(n)}$ denote the $-n$-th cohomology of this complex, i.e. $^lA^{(n)}$ is a sub-D-module of $^lA^n$ obtained by intersecting the kernels of the maps $\{, \}_\phi$ for all possible pairs (where $\phi : \{1, \ldots, n\} \to \{1, |J| = n - 1\}$. For example, $^lA^{(2)}$ is the left D-module corresponding to $A^{(2)}$ of Section 13.

Since $^lA^{(n)}$ is equivariant with respect to the action of the symmetric group on $X^n$, we can form a D-module $^lA^{(2)}$ for any finite non-empty set $J$.

**Proposition 14.1.** $H^i(C^*) = 0$ for $i \neq -n$.

**Proof.** To simplify the notation, we shall give a proof in the case $n = 3$. In the general case, the proof is completely analogous.

Our complex (for the corresponding right D-modules) looks as follows:

$$j_{(1,2,3)} (A^3) \xrightarrow{d-3} \Delta_{x_1 = x_2} (j_2 j^* (A^2)) \oplus \Delta_{x_1 = x_3} (j_3 j^* (A^2))$$

and we have to prove that it is exact at the $-2$-nd term (the fact that it is exact at the $-1$-st term is obvious, since $A$ has a unit).
We claim, first of all, that any section of $C^{-2}$ is equivalent modulo the image of $d_{-3}$ to a section that belongs to $\Delta_{x_2=x_3}(j_*j^*(A^2))$. This again follows from the fact that $\mathcal{A}$ has a unit.

Now let $l$ be a section
\[ l \in \Gamma(X \times X, j_*j^*(A^2)) \subset \Gamma(X \times X \times X, \Delta_{x_2=x_3}(j_*j^*(A^2))) \]
with $d_{-2}(l) = 0$; it remains to show that $l$ lies in the image of $d_{-3}$.

Choose (locally) a 1-form $\omega$ on $X$ and a function $f(x_1, x_2) \in \mathcal{O} \boxtimes \mathcal{O}(\Delta)$ such that $f(x_1, x_2) \cdot \omega = 1 \mod \mathcal{O} \boxtimes \mathcal{O}(-\Delta)$ (we have used here the identification $\Omega \simeq \mathcal{O} \boxtimes \mathcal{O}(-\Delta)/\mathcal{O} \boxtimes \mathcal{O}(-2 \cdot \Delta)$).

It now follows from the Jacobi identity that $l = d_{-3}(l \boxtimes \omega \cdot f(x_2, x_3))$.

\[ \square \]

**Corollary 14.1.** For a surjection $\phi : \mathcal{I} \to \mathcal{J}$, we have a canonical isomorphism
\[ \Delta^\phi_{l^i(\mathcal{A}(3))} \simeq l^i(\mathcal{A}(3)) \]
and for a decomposition $\mathcal{I} = \mathcal{I}_1 \cup \ldots \cup \mathcal{I}_k$ we have a canonical isomorphism
\[ j^*_{i_1, \ldots, i_k}(l^i(\mathcal{A}(3))) \simeq j^*_{i_1, \ldots, i_k}(l^i(\mathcal{A}(3_1)) \boxtimes \ldots \boxtimes l^i(\mathcal{A}(3_k))). \]
Moreover, the above isomorphisms are compatible in the obvious sense.

We shall call the isomorphisms of Corollary 14.1 “factorization isomorphisms”.

**Remark 14.1.** Consider the topological space $\text{Ran}(X)$ that consists of all finite non-empty subsets of $X$. The above corollary implies that the collection of $l^i(\mathcal{A}(3))$'s can be viewed as “$D$-module” over $\text{Ran}(X)$: its fiber over a finite subset of $X$ that corresponds to a map $\mathcal{J} \to X$ is, by definition, the stalk of $l^i(\mathcal{A}(3))$ at the corresponding point of $X^2$.

14.2. Next we want to show that the chiral algebra structure can be completely recovered from the system of the $l^i(\mathcal{A}(n))$'s viewed as plain $\mathcal{O}_{X^2}$-modules.

Let $l^i(\mathcal{A})$ be an $\mathcal{O}$-module on $X$. Assume now that for every finite set $\mathcal{J}$ we are given an $\mathcal{O}_{X^2}$-module $l^i(\mathcal{A}(3))$ (such that $l^i(\mathcal{A}(3)) = l^i(\mathcal{A}(3_1))$) together with a compatible system of factorization isomorphisms:
\[ \Delta^\phi_{l^i(\mathcal{A}(3))} \simeq l^i(\mathcal{A}(3)) \]
for a surjection $\phi : \mathcal{I} \to \mathcal{J}$ and
\[ j^*_{i_1, \ldots, i_k}(l^i(\mathcal{A}(3))) \simeq j^*_{i_1, \ldots, i_k}(l^i(\mathcal{A}(3_1)) \boxtimes \ldots \boxtimes l^i(\mathcal{A}(3_k))) \]
for a decomposition $\mathcal{I} = \mathcal{I}_1 \cup \ldots \cup \mathcal{I}_k$.

Assume also that there exists a map $\text{id} \boxtimes \text{unit} : \mathcal{O} \to l^i(\mathcal{A})$ with the following properties:

- For $\mathcal{J} = \mathcal{I}_0 \cup i$ the map
  \[ \text{id} \boxtimes \text{unit} : j^*_{\mathcal{I}_0, (i)}(l^{(3_0)} \boxtimes \mathcal{O}) \to j^*_{\mathcal{I}_0, (i)}(l^{(3_0)} \boxtimes l^i(\mathcal{A})) \simeq j^*_{\mathcal{I}_0, (i)}(l^{(3)}) \]
  extends to a map $\text{id} \boxtimes \text{unit} : l^{(3_0)} \boxtimes \mathcal{O} \to l^{(3)}$.
- Take $\mathcal{J} = \mathcal{I}_0 \cup i_1 \cup i_2$ and $\mathcal{J} = \mathcal{I}_0 \cup i_1$ and let $\phi : \mathcal{I} \to \mathcal{J}$ be a map that contracts $\{i_1, i_2\} \to i_1$. We need that the composition
  \[ l^{(3)} \simeq \Delta^\phi_{l^{(3)} \boxtimes \mathcal{O}} \text{id} \boxtimes \text{unit} \Delta^\phi_{l^{(3)}} \simeq l^{(3)} \]
  is the identity map.
Theorem 14.1. Let \( \mathcal{A}, \mathcal{A}' \) be data as above. Then there exists a (canonical) chiral algebra structure on \( \mathcal{A} := \mathcal{A} \otimes \Omega^{-1} \) such that the corresponding D-modules \( \mathcal{A}' \) are identified as \( \mathcal{O}_{X^2} \)-modules with the \( \mathcal{A}' \)’s.

We shall not give here a complete proof of this theorem. Instead, we will explain the key points:

Proof. (sketch)
Let \( \mathcal{A}'(n) \) denote the \( \mathcal{O}_{X^n} \)-module corresponding to \( J = \{1, \ldots, n\} \).

We shall first show how to endow \( \mathcal{A} \) with a left D-module structure:

The embedding of \( \mathcal{O}_{X^2} \)-modules \( \text{id} : \mathcal{O} \hookrightarrow \mathcal{A} \) induces an isomorphism between their restrictions to the diagonal. This fact implies that the restrictions of the above two \( \mathcal{O}_{X^2} \)-modules to the second infinitesimal neighborhood of \( \Delta(X) \subset X \times X \) are isomorphic as well, i.e. that

\[
\mathcal{O} \otimes \mathcal{A}/\mathcal{O} \otimes \mathcal{A}(-2 \cdot \Delta) \simeq \mathcal{A}'(2) \otimes \mathcal{A}'(2)(-2 \cdot \Delta).
\]

Symmetrically, we have an isomorphism

\[
\mathcal{A} \otimes \mathcal{O} \otimes \mathcal{A}(-2 \cdot \Delta) \simeq \mathcal{A}'(2) \otimes \mathcal{A}'(2)(-2 \cdot \Delta)
\]

and by transitivity, we obtain an isomorphism of \( \mathcal{O}_{X^2} \)-modules:

\[
\varphi : \mathcal{O} \otimes \mathcal{A}/\mathcal{O} \otimes \mathcal{A}(-2 \cdot \Delta) \rightarrow \mathcal{A} \otimes \mathcal{O} \otimes \mathcal{A}(-2 \cdot \Delta).
\]

This enables us to define a connection on \( \mathcal{A} \): to a section \( l \in \mathcal{A} \) we associate a section \( \nabla(l) \in \mathcal{A} \otimes \Omega \) as follows:

Note that \( l \otimes 1 - \varphi(1 \otimes l) \) is a section of \( \mathcal{A} \otimes \mathcal{O} \otimes \mathcal{A}(-2 \cdot \Delta) \) which vanishes modulo \( \mathcal{A} \otimes \mathcal{O}(-\Delta) \). We set \( \nabla(l) \) to be the image of \( l \otimes 1 - \varphi(1 \otimes l) \) under the identification

\[
\mathcal{A} \otimes \mathcal{O}(-\Delta)/\mathcal{A} \otimes \mathcal{O}(-2 \cdot \Delta) \simeq \mathcal{A} \otimes \Omega.
\]

We leave it to the reader to check the correctness of this definition (this can be done using \( \mathcal{A}'(3) \)). Moreover, the above construction can be generalized to produce left D-module structures on all the \( \mathcal{A}'(3) \)'s in a way compatible with the factorization isomorphisms.

Let us now show how \( \mathcal{A} \) acquires a chiral algebra structure:

The short exact sequence of D-modules

\[
0 \rightarrow \mathcal{A}'(2) \rightarrow j* j^* (\mathcal{A}'(2)) \rightarrow \Delta_! \Delta^! (\mathcal{A}'(2))[1] \rightarrow 0
\]

gives rise under our identifications to a map \( j* j^* (\mathcal{A} \otimes \mathcal{A}) \rightarrow \Delta_!(\mathcal{A}) \).

Using the D-module \( \mathcal{A}'(3) \), it can be checked that this operation satisfies the Jacobi identity when we pass to the corresponding right D-modules.

Finally, we have to identify \( \mathcal{A}'(3) \) with \( \mathcal{A}(3) \).

It follows from the construction that there is a natural embedding \( \mathcal{A}'(2) \hookrightarrow \mathcal{A}(3) \). Moreover, for every \( \phi : J \rightarrow \beta \) this map induces an isomorphism

\[
j^*_\phi (\mathcal{A}'(3)) \simeq j^*_\beta (\mathcal{A}'(3)) \simeq j^*_\beta (\mathcal{A}(3)) \simeq j^*_\phi (\mathcal{A}(3)).
\]

This implies that the embedding \( \mathcal{A}'(2) \hookrightarrow \mathcal{A}(3) \) is an isomorphism, too.
Remark 14.2. One may have observed that the construction of the chiral universal enveloping algebra discussed in Section 14.1 was also in the spirit of Theorem 14.1. In fact, it can be completely reformulated in terms of the system of $\mathcal{O}_{X^n}$-modules $\mathcal{U}(\mathcal{B})^{(n)}$, which we leave as an exercise to the reader.

14.3. Let $\mathcal{A}$ be a chiral algebra over a complete curve $X$. It follows from Section 5.3 that the map

$$\langle \ldots \rangle : j_* j^* ((\mathcal{A} \boxtimes \ldots \boxtimes \mathcal{A}) \rightarrow j_* j^* (\mathcal{O} \boxtimes \ldots \boxtimes \mathcal{O}) \otimes H_{\mathcal{F}}(X, \mathcal{A})$$

restricts to a D-module map

$$j_! \mathcal{A}^{(n)} \rightarrow \mathcal{O}_{X^n} \otimes H_{\mathcal{F}}(X, \mathcal{A}).$$

Hence, we obtain a map of vector spaces $DR^n(X^n, j_! \mathcal{A}^{(n)}) \rightarrow H_{\mathcal{F}}(X, \mathcal{A})$.

Proposition 14.2. For $n \geq 2$, the map $DR^n(X^n, j_! \mathcal{A}^{(n)}) \rightarrow H_{\mathcal{F}}(X, \mathcal{A})$ is an isomorphism.

Proof. We can compute $DR^n(X^n, j_! \mathcal{A}^{(n)})$ using the resolution $C^\bullet$ of Section 14.1. We have:

$$DR^n(X^n, j_! \mathcal{A}^{(n)}) \simeq \text{coker}(DR^1(X^n, C^{-2}) \rightarrow DR^1(X^n, C^{-1})).$$

However, $DR^1(X^n, C^{-1})$ is just $DR^1(X, \mathcal{A})$ and $DR^1(X^n, C^{-2})$ is a direct sum of several copies of $DR^1(X \times X, j_* j^*(\mathcal{A} \boxtimes \mathcal{A}))$. The assertion of the proposition follows from the fact that

$$H_{\mathcal{F}}(X, \mathcal{A}) \simeq DR^1(X, \mathcal{A}^{(2)}) \simeq \text{coker}(DR^1(X^2, j_* j^*(\mathcal{A} \boxtimes \mathcal{A})) \rightarrow DR^1(X, \mathcal{A})).$$

\[ \square \]

15. Geometry of the affine Grassmannian

Let $G$ be an algebraic group and let $\mathfrak{g}$ be the corresponding Lie algebra. Consider the Lie-* algebra $\mathcal{B}(\mathfrak{g}, 0)$ of Ex.1 in Section 13. In the next two subsections we shall describe a geometric construction of the corresponding chiral universal enveloping algebra $\mathcal{A}(\mathfrak{g}, 0)$ via the so-called affine Grassmannian of the group $G$. This description will be used later on for the construction of the free bosonic theory.

15.1. Consider the group $G(\hat{0})$ and the ind-group $G(\hat{K})$ that classify maps

$$\text{Spec}(\hat{0}) \rightarrow G \text{ and Spec}(\hat{K}) \rightarrow G,$$

respectively. Consider now the affine Grassmannian $\mathcal{G}_G := G(\hat{K})/G(\hat{0})$ corresponding to the group $G$. The ind-scheme $\mathcal{G}_G$ may be highly non-reduced; for instance, when $G = H$ is a torus, the corresponding reduced scheme is identified with the discrete set of co-characters of $H$, while $\mathcal{G}_H$ is infinite-dimensional.

Note that the group $\text{Aut}_G^+$ of automorphisms of $\hat{0}$ acts naturally on $\mathcal{G}_G$.

For a curve $X$ consider the group scheme $G_O(X)$ over $X$ whose fiber at $x \in X$ is $G(\hat{0}_x)$ and a group ind-scheme $G_X(X)$, whose fiber at $x \in X$ is $G(\hat{K}_x)$. The quotient $\mathcal{G}_G(X) := G_X(X)/G_O(X)$ is a global version of the affine Grassmannian $\mathcal{G}_G$ considered above. Let $r : \mathcal{G}_G(X) \rightarrow X$ denote the natural projection and let $\text{unit}_{\mathcal{G}_G}$ denote the unit section $X \rightarrow \mathcal{G}_G(X)$.
Lemma 15.1. The ind-scheme $\mathfrak{Gr}_G(X)$ represents the functor whose value on a scheme $S$ is the data of a map $f : S \to X$, a $G$-torsor $P$ on $S \times X$ and a trivialization

$$\alpha : P|_{S \times X \setminus \Gamma_f} \simeq P_0|_{S \times X \setminus \Gamma_f},$$

where $\Gamma_f$ is the graph of the map $f$ and $P_0$ is the trivial $G$-torsor.

This is a version of the Beauville-Laszlo theorem ([3]).

Proposition 15.1. The ind-scheme $\mathfrak{Gr}_G(X)$ possesses a natural connection along $X$. The section $\text{unit}_{\mathfrak{Gr}_G(X)} : X \to \mathfrak{Gr}_G(X)$ is preserved by this connection.

Proof. Let $I$ be a local Artinian scheme and let $f$ be a map $I \times S \to X$, where $S$ is an arbitrary scheme. Let $\text{Spec}(\mathbb{C}) \simeq I_0 \subset I$ be the reduced (point) scheme and let $\tilde{f}_0$ be a map $I_0 \times S \to \mathfrak{Gr}_G(X)$ such that the map $r \circ \tilde{f}_0 : I_0 \times S \to X$ coincides with the composition $I_0 \times S \to I \times S \to X$.

To define a connection on $\mathfrak{Gr}_G(X)$ along $X$, we must associate to a triple $(I,f,\tilde{f}_0)$ as above a map $\tilde{f} : I \times S \to \mathfrak{Gr}_G(X)$ such that $r \circ \tilde{f} = f$ that extends $\tilde{f}_0$.

This can be done as follows:

The corresponding map $I \times S \to X$ has already been given to us (this is our $f$) and we set the $G$-torsor $P^I$ over $I \times S \times X$ that would correspond to $\tilde{f}$ to be pulled back under $I \times S \times X \to S \times X$ from the $G$-torsor $P$ that corresponds to $\tilde{f}_0$. However, $I \times S \times X \setminus \Gamma_f \simeq (S \times X \setminus \Gamma_{\tilde{f}_0}) \times I$ and the data of trivialization for $P^I$ comes from the corresponding data for $P$.

When $X$ is complete let $\text{Bun}_G(X)$ denote the moduli stack of $G$-bundles over $X$. We have a natural projection $k : \mathfrak{Gr}_G(X) \to \text{Bun}_G(X)$.

Lemma 15.2. The projection $k : \mathfrak{Gr}_G(X) \to \text{Bun}_G(X)$ is preserved by the connection on $\mathfrak{Gr}_G(X)$ along $X$.

It follows from the definitions that the dependence of $\mathfrak{Gr}_G(X)$ on the curve $X$ is local (in the sense of Section 3.1):

We have the ind-stack $\mathfrak{Gr}_G(X)$ fibered over $X$, that is, for a family $X^S$ of curves over a base $S$ we can form a scheme $\mathfrak{Gr}_G(X^S)$ over $X^S$. Now let $X^S,I, x^{S,I}, \phi^{S,I}$ be as in Section 3.1. We have a canonical isomorphism:

$$\mathfrak{Gr}_G^I(X^{S,I}) \setminus r^{-1}(x^{S \times I}) \simeq (\mathfrak{Gr}_G(X^S) \setminus r^{-1}(x^S)) \times I.$$

The ind-scheme $\mathfrak{Gr}_G(X)$ carries a connection along the fibers of the projection $X \to \mathcal{M}$ and it is easy to see that the above isomorphism is compatible with this connection.

5.2. For a finite set $J$ we define the Beilinson-Drinfeld Grassmannian $\mathfrak{Gr}_G^J(X)$ to be the ind-scheme representing the following functor:

For a scheme $S$, $\text{Hom}(S, \mathfrak{Gr}_G^J(X))$ is the data consisting of a map $f^J : S \to X^J$, a $G$-torsor $P$ on $S \times X$ and a trivialization

$$\alpha : P|_{S \times X \setminus \{\Gamma_{f_{J_1}} \ldots, \Gamma_{f_{J_n}}\}} \simeq P_0|_{S \times X \setminus \{\Gamma_{f_{J_1}} \ldots, \Gamma_{f_{J_n}}\}},$$

where for $i_k \in J$, $f^J_{i_k}$ is the composition of $f^J$ with the projection on the $i_k$-th factor $X^J \to X$. 
Let \( r^j \) denote the natural projection \( \mathfrak{G} r_h^j(X) \to X^j \) and let \( \text{unit}^j_{\mathfrak{G} r_h} : X^j \to \mathfrak{G} r_h^j(X) \) denote the corresponding unit section.

When \( J = \{1, \ldots, n\} \) we shall denote \( \mathfrak{G} r_h^j(X) \) by \( \mathfrak{G} r^n_h(X) \) and the corresponding maps \( r^j \) and \( \text{unit}^j_{\mathfrak{G} r_h} \) by \( r^n \) and \( \text{unit}^n_{\mathfrak{G} r_h} \), respectively.

A remarkable feature of the ind-scheme \( \mathfrak{G} r_h^j(X) \) is the following factorization property:

**Proposition 15.2.** For a surjection of finite sets \( J \to \beta \), we have a natural isomorphism:

\[
X^\beta \times \mathfrak{G} r_G(X)^\beta \simeq \mathfrak{G} r_G(X)^\beta
\]

and for a decomposition \( J = J_1 \cup \ldots \cup J_k \) we have a natural isomorphism

\[
X^{3_i \neq j_i} \times \mathfrak{G} r_G(X)^{3_i} \simeq X^{3_i \neq j_i} \times \mathfrak{G} r_G(X)^{3_i} \times \cdots \times \mathfrak{G} r_G(X)^{3_k}.
\]

**Proof.** The first assertion follows immediately from the definitions. To prove the second one let us assume for simplicity that \( J = \{1, 2\} \), \( J_1 = \{1\} \) and \( J_2 = \{2\} \).

Given an object \((x_1, x_2, P, \alpha)\) of \( \mathfrak{G} r_h^2(X) \) we construct an object of

\[
(x_1, P_1, \alpha_1) \times (x_2, P_2, \alpha_2) \in \mathfrak{G} r_G(X) \times \mathfrak{G} r_G(X)
\]

as follows:

The \( G \)-torsor \( P_1 \) (resp., \( P_2 \)) is set to be isomorphic to \( P_0 \) over \( X \setminus x_1 \) (resp., over \( X \setminus x_2 \)) and to be isomorphic to \( P \) over \( X \setminus x_2 \) (resp., over \( X \setminus x_1 \)). The datum of \( \alpha \) for \( P \) provides the gluing data for \( P_1 \) and \( P_2 \). The data of \( \alpha_1 \) and \( \alpha_2 \) follow from the construction.

Vice versa, given an object \((x_1, P_1, \alpha_1, x_2, P_2, \alpha_2) \in \mathfrak{G} r_G(X) \times \mathfrak{G} r_G(X) \) we define \((x_1, x_2, P, \alpha) \in \mathfrak{G} r_h^2(X)\) by setting \( P \) to equal \( P_1 \) over \( X \setminus x_2 \) and to equal \( P_2 \) over \( X \setminus x_1 \). The isomorphisms \( \alpha_1 \) and \( \alpha_2 \) provide the gluing data for \( P \) together with the trivialization \( \alpha \) over \( X \setminus \{x_1, x_2\} \).

\(\square\)

**Remark 15.1.** It follows from Proposition 15.2 above that the fiber of \( \mathfrak{G} r_h^2(X) \) over a point \((x_1, x_2) \in X \times X\) is identified with \( \mathfrak{G} r_G \times \mathfrak{G} r_G \) whenever \( x_1 \neq x_2 \) and with \( \mathfrak{G} r_G \) when \( x_1 = x_2 \). This is a purely infinite-dimensional phenomenon (in the finite-dimensional situation the dimension of fibers cannot drop under a specialization). Moreover, it is not difficult to prove that the map \( r^3 \) is formally smooth.

As in the case \(|\beta| = 1\), it is easy to show that \( \mathfrak{G} r_h^j(X) \) possesses a natural connection along \( X^j \) and that the isomorphisms constructed in Proposition 15.2 are compatible with these connections. When \( X \) is complete we have a projection \( k^j : \mathfrak{G} r_h^j(X) \to \text{Bun}_G(X) \) which (as in the case of \(|\beta| = 1\)) is preserved by the connection on \( \mathfrak{G} r_h^j(X) \) along \( X^j \). (When \( J = \{1, \ldots, n\} \) we shall replace the notation \( k^j \) simply by \( k^n \).)

Again, as in the case of \( J = \{1\} \), there exists an ind-stack \( \mathfrak{G} r_h^j(Y) \) fibered over \( X^f \) which has a locality property in the sense that was specified at the end of Section 15.1. It follows from the definitions that the local structure on the \( \mathfrak{G} r_h^j(Y) \)'s is compatible with the connections along the \( X^j \)'s and with the factorization isomorphisms of Proposition 15.2.
15.3. We shall now introduce a twisted version of the Beilinson-Drinfeld Grassmannian, which will be used for the construction of the bosonic chiral algebra in the next section.

For an integer $n$ we introduce the iterated Beilinson-Drinfeld Grassmannian $\tilde{\mathcal{G}}_{r_G}(X)$ as follows:

$\tilde{\mathcal{G}}_{r_G}(X)$ is the ind-scheme representing the functor whose value on a scheme $S$ is the data consisting of a map $f^n : S \to X^n$ and a collection of $n$ $G$-torsors $\mathcal{P}_i$ over $S \times X$ together with isomorphisms

$$\alpha_i : \mathcal{P}_i \mid_{S \times X \setminus r^n_{i-1}} \cong \mathcal{P}_{i-1} \mid_{S \times X \setminus r^n_{i-1}},$$

(here $f^n_i$ is the composition of the map $f^n$ with the projection on the $i$-th factor $X^n \to X$.)

Let $\tilde{r}^n$ denote the projection $\tilde{\mathcal{G}}_{r_G}^n(X) \to X^n$. It is easy to see that $\tilde{\mathcal{G}}_{r_G}^n(X)$ is a fibration over $\tilde{\mathcal{G}}_{r_G}^{n-1}(X) \times X$ with typical fiber $\mathcal{G}_{r_G}$.

More precisely, consider the direct product $\tilde{\mathcal{G}}_{r_G}^{n-1}(X) \times X$ and let $\tilde{G}_{r_G}^n(X)$ denote the pull-back of the group scheme $G_{r_G}(X)$ with respect to the projection

$$\tilde{\mathcal{G}}_{r_G}^{n-1}(X) \times X \to X.$$

On the one hand, there is a canonical $\tilde{G}_{r_G}^0(X)$-torsor over $\tilde{\mathcal{G}}_{r_G}^{n-1}(X) \times X$, whose fiber at a point $(x_1, \ldots, x_{n-1}, \mathcal{P}_1, \ldots, \mathcal{P}_{n-1}, x_n)$ is the restriction of $\mathcal{P}_{n-1}$ to the formal disc around $x_n$. On the other hand, $\tilde{G}_{r_G}^0(X)$ acts on $\tilde{\mathcal{G}}_{r_G}^{n-1}(X) \times X$-scheme $\tilde{\mathcal{G}}_{r_G}^{n-1}(X) \times \mathcal{G}_{r_G}(X)$. It is easy to see that the $\tilde{\mathcal{G}}_{r_G}^{n-1}(X) \times X$-scheme $\tilde{\mathcal{G}}_{r_G}^{n-1}(X)$ is a twist of $\tilde{\mathcal{G}}_{r_G}^{n-1}(X) \times \mathcal{G}_{r_G}(X)$ with respect to the above-mentioned $\tilde{G}_{r_G}^0(X)$-torsor.

We have a canonical projection $u_n : \tilde{\mathcal{G}}_{r_G}^n(X) \to \mathcal{G}_{r_G}^n(X)$ that sends the data of $(x_1, \ldots, x_n, \mathcal{P}_1, \ldots, \mathcal{P}_n, \alpha_1, \ldots, \alpha_n)$ to $(x_1, \ldots, x_n, \mathcal{P}_n, \alpha)$, where $\alpha$ is obtained by composing $\alpha_1, \ldots, \alpha_n$. By repeating the proof of Proposition 15.2 we see that the map $u_n$ is an isomorphism over $X^n \setminus \Delta$.

As in the case of $\mathcal{G}_{r_G}^n(X)$, there exists a scheme $\tilde{\mathcal{G}}_{r_G}^n(X)$ over $X^n$ which is local in the sense of Section 15.1.

16. Chiral algebra attached to the affine Grassmannian

16.1. Consider the right D-module $\text{unit}_{\mathcal{G}_{r_G}}(\Omega)$ on $\mathcal{G}_{r_G}(X)$.

[16] As an $\mathcal{O}_{\mathcal{G}_{r_G}(X)}$-module, $\text{unit}_{\mathcal{G}_{r_G}}(\Omega)$ is a union of $\mathcal{O}$-modules on $\mathcal{G}_{r_G}(X)$ that are supported scheme-theoretically on an increasing family of finite-dimensional infinitesimal neighborhoods of $\text{unit}_{\mathcal{G}_{r_G}}(X)$ in $\mathcal{G}_{r_G}(X)$.

Consider now the $\mathcal{O}$-module direct image

$$\mathcal{A}_{\mathcal{G}_{r_G}} := r_*(\text{unit}_{\mathcal{G}_{r_G}}(\Omega)).$$

The connection on $\mathcal{G}_{r_G}(X)$ along $X$ defines a right D-module structure on $\mathcal{A}_{\mathcal{G}_{r_G}}$ and the section $\text{unit}_{\mathcal{G}_{r_G}}$ defines an embedding $\Omega \hookrightarrow \mathcal{A}_{\mathcal{G}_{r_G}}$. Let $\mathcal{A}_{\mathcal{G}_{r_G}}^{-}$ be the corresponding left D-module.

---

[16] When $Z$ is a strict ind-scheme (i.e. $Z$ is a union of closed finite dimensional subschemes) and $Z' \subset Z$ is a closed finite-dimensional subscheme of $Z$ it makes perfect sense to talk about right D-modules on $Z$ supported on $Z'$ (cf. Section 4).
More generally, for a finite set $I$ let $A^{(3)}_{\mathfrak{g}^I}$ be the right D-module on $X^I$ defined as $r_2^{3}(\text{unit}_{\mathfrak{g}^I}(\Omega_{X^I}))$ and let $A^{(3)}_{\mathfrak{g}^I}$ denote the corresponding left D-module.

Let us now view the $A^{(3)}_{\mathfrak{g}^I}$'s as plain $\mathcal{O}$-modules on $X^I$. The next assertion follows from Proposition 15.2.

Lemma 16.1. The system $I \mapsto A^{(3)}_{\mathfrak{g}^I}$ satisfies the conditions of Theorem 14.1.

By applying Theorem 14.1 we obtain, therefore, a chiral algebra structure on $A_{\mathfrak{g}^I}$ with the $A^{(3)}_{\mathfrak{g}^I}$'s being identified with the corresponding $A^{(3)}_{\mathfrak{g}^I}$'s as $\mathcal{O}$-modules.

However, since the factorization isomorphisms of Proposition 15.2 are compatible with the connections along $X^I$, it is easy to see that the isomorphisms $A^{(3)}_{\mathfrak{g}^I} \simeq A^{(3)}_{\mathfrak{g}^I}$ are compatible with the D-module structure.

16.2. Consider the fiber $A_{\mathfrak{g}^I,x}$ of the chiral algebra $A_{\mathfrak{g}^I}$ at a point $x \in X$. The action of the group $G(\tilde{K}_x)$ on $\mathfrak{g}^I = G(\tilde{K}_x)/G(\tilde{O})$ defines a structure of $(\mathfrak{g} \otimes \tilde{K}_x, G(\tilde{O}))$-module on $A_{\mathfrak{g}^I,x}$. We shall denote by

$$\text{Act}_x : (\mathfrak{g} \otimes \tilde{K}_x) \otimes A_{\mathfrak{g}^I,x} \rightarrow A_{\mathfrak{g}^I,x}$$

the corresponding action map.

It is easy to see, moreover, that $A_{\mathfrak{g}^I,x}$ is in fact canonically isomorphic to the vacuum module $\text{Ind}_{\mathfrak{g} \otimes \tilde{O}_x}^\mathfrak{g} \tilde{K}_x(C)$.

Our goal now is to prove the following theorem:

Theorem 16.1. There exists a natural isomorphism of chiral algebras $A_{\mathfrak{g}^I} \simeq A(\mathfrak{g},0)$. Moreover, for every $x \in X$ the map $\text{Act}_x$ goes over to the action of $\mathfrak{g} \otimes \tilde{K}_x \simeq DR^0(\text{Spec}(\tilde{K}_x), \mathcal{B}(\mathfrak{g},0)_{x})$ on $A(\mathfrak{g},0)_{x}$.

Remark 16.1. A similar description of the chiral algebra $A(\mathfrak{g},Q)$ can be given when $Q \neq 0$. This will be done explicitly in Section 18.1 when $G = H$ is a torus.

We shall deduce the assertion of the theorem from the following result:

Proposition 16.1. There is a natural map

$$\text{Act} : j_* j^*(\mathcal{B}(\mathfrak{g},0) \boxtimes A_{\mathfrak{g}^I}) \rightarrow \Delta_! (A_{\mathfrak{g}^I})$$

such that

(a) $\mathcal{B}(\mathfrak{g},0)$ acts on $A_{\mathfrak{g}^I}$ by derivations of the chiral algebra structure.

(b) The induced map

$$(h \boxtimes \text{id})(\text{Act}) : DR^0(\text{Spec}(\tilde{K}_x), \mathcal{B}(\mathfrak{g},0)) \otimes A_{\mathfrak{g}^I,x} \rightarrow A_{\mathfrak{g}^I,x}$$

coincides with the map $\text{Act}_x$.

Proof. (Of Theorem 16.1)

According to property (b) of the map $\text{Act}$, the composition

$$j_* j^*(\mathcal{B}(\mathfrak{g},0) \boxtimes \Omega) \xrightarrow{id \times \text{unit}} j_* j^*(\mathcal{B}(\mathfrak{g},0) \boxtimes A_{\mathfrak{g}^I}) \rightarrow \Delta_! (A_{\mathfrak{g}^I})$$

factors as

$$j_* j^*(\mathcal{B}(\mathfrak{g},0) \boxtimes \Omega) \rightarrow \mathcal{B}(\mathfrak{g},0) \rightarrow A_{\mathfrak{g}^I},$$
i.e. we obtain a map of D-modules

\[
\text{emb} : \mathcal{B}(\mathfrak{g}, 0) \to \mathcal{A}_{\mathfrak{rep}}.
\]

On the level of fibers, this map corresponds to the embedding \(\hat{\mathcal{K}}_x/\hat{\mathcal{O}}_x \otimes \mathfrak{g} \hookrightarrow \text{Ind}_{\hat{\mathcal{K}}_y/\hat{\mathcal{O}}_y} \otimes (\mathbb{C})\).

By again applying point (b) of Proposition [16.1], we obtain that \(\text{emb} : \mathcal{B}(\mathfrak{g}, 0) \to \mathcal{A}_{\mathfrak{rep}}\) is a map of chiral \(\mathcal{B}(\mathfrak{g}, 0)\)-modules.

Therefore, to prove the theorem it remains to show that the two maps

\[
j_*j^*(\mathcal{B}(\mathfrak{g}, 0) \boxtimes \mathcal{A}_{\mathfrak{rep}}) \to \Delta_!(\mathcal{A}_{\mathfrak{rep}}),
\]

namely \(\text{Act and } \{\{\},\} \circ (\text{emb} \times \text{id})\), coincide. This is, however, an almost immediate consequence of point (a) of Proposition [16.1].

Let \(b \boxtimes a \cdot f(x, y)\) be a section of \(j_*j^*(\mathcal{B}(\mathfrak{g}, 0) \boxtimes \mathcal{A}_{\mathfrak{rep}})\). It is enough to show that

\[
(\text{id} \times h) (\text{Act}(b \boxtimes a \cdot f(x, y))) = (\text{id} \times h)(\{\text{emb}(b) \boxtimes a \cdot f(x, y)\})
\]

for any \(a, b\) and \(f(x, y)\) as above.

Let \(\omega\) be a non-vanishing 1-form on \(X\) and let \(h(x, y)\) be a function on \(X \times X \setminus \Delta\) with a simple pole along the diagonal and with \(\text{Res}_\Delta (\omega \otimes h(x, y)) = 1\). Consider the section

\[
b \boxtimes \omega \boxtimes a \cdot f(x, z) \cdot h(x, y) \in j_*j^*(\mathcal{B}(\mathfrak{g}, 0) \boxtimes \mathcal{A}_{\mathfrak{rep}} \boxtimes \mathcal{A}_{\mathfrak{rep}}).
\]

The fact that \(\mathcal{B}(\mathfrak{g}, 0)\) acts on \(\mathcal{A}_{\mathfrak{rep}}\) by derivations of the chiral algebra structure implies that

\[
\{\text{Act}(b \boxtimes \omega \cdot h(x, y)) \boxtimes a \cdot f(x, z)\} = \\
\text{Act}(b \boxtimes \{\omega \boxtimes a \cdot h(x, y) \cdot f(x, z)\}) - \sigma_{1,2}\{\omega, \text{Act}(b \boxtimes a \cdot h(y, x) \cdot f(y, z))\}.
\]

Note, first of all, that the expression \(\text{Act}(b \boxtimes \omega \cdot h(x, y))\) is equal by definition to \(\text{emb}(b)\) and that the first term on the RHS vanishes.

By applying \((\text{id} \boxtimes h) \boxtimes h\) to both sides of the above formula we obtain the needed result.

\[\square\]

16.3. The proof of Proposition [16.1] will be based on the following construction:

Assume that \(X\) is affine and consider the formal group \(\text{Maps}(X, G)\) that classifies maps \(X \to G\). In addition, given a finite set \(\mathcal{J}\) we shall consider a formal groupscheme \(\overline{\text{Maps}}^3(X, G)\) over \(X^2\) that corresponds to the functor whose value on a pair \((S, f^3 : S \to X^3)\) is the group of regular maps \((S \times X \setminus \{\Gamma(f^3_{i_1}), \ldots, \Gamma(f^3_{i_s})\}) \to G\).

It follows from the definitions that \(\overline{\text{Maps}}^3(X, G)\) carries a natural connection along \(X^3\). The corresponding sheaf of Lie algebras \(\text{Lie}(\overline{\text{Maps}}^3(X, G))\) is a left D-module on \(X^3\) which is identified with \((\mathcal{P}\circ j_{(1)}, \mathfrak{g} \boxtimes O_{X^3(\mathcal{J})})\)\footnote{In this formula, \((\mathcal{P}\circ j_{(1)})_*\) denotes the direct image in the category of \(\mathcal{O}\)-modules.} where \(\mathcal{P}\) denotes the projection \(X^3_{\mathcal{J}^{(1)}} \to X^2\).

For a surjection of finite sets \(\phi : \mathcal{J} \to \mathcal{J}\) we have a natural map

\[
X^3 \times \overline{\text{Maps}}^3(X, G) \to \overline{\text{Maps}}^3(X, G)
\]
and for a decomposition $I = I_1 \cup \ldots \cup I_k$, we have a map

$$\widehat{\text{Maps}}^j(X, G)((X^j, x^j)) \to \widehat{\text{Maps}}^{j_1}(X, G) \times \ldots \times \widehat{\text{Maps}}^{j_k}(X, G)((X^j, x^j)).$$

Observe now that there is a natural action

$$\text{Act}^j : \widehat{\text{Maps}}^j(X, G) \times \mathfrak{S}_G^j(X) \to \mathfrak{S}_G^j(X):$$

Take

$$(x^j \in X^j, g : X \setminus x^j \to G) \in \widehat{\text{Maps}}^j(X, G)$$

and for a decomposition $I \subset \{1, 2\}$,

$$\Delta_1((\text{Maps}^{(1,2)}(X, G)) \otimes A_{\mathfrak{S}_G^1}) \xrightarrow{\text{Act}^{(1,2)}} j_* j^* (A_{\mathfrak{S}_G^1} \boxtimes A_{\mathfrak{S}_G^1})$$

commutes.

In the proof of Proposition [16.4] we shall use the following observation:

Let $\text{Lie}(\text{Maps}^{(1)}(X, G))_{x}$ denote the fiber of the sheaf $\text{Lie}(\text{Maps}^{(1)}(X, G))$ at $x \in X$.

**Lemma 16.2.** The action $\text{Act}^{(1)} : \text{Lie}(\text{Maps}^{(1)}(X, G))_{x} \otimes A_{\mathfrak{S}_G^{(1)}} \to A_{\mathfrak{S}_G^{(1)}}$ coincides with the one that comes from the embedding

$$\text{Lie}(\text{Maps}^{(1)}(X, G))_{x} \simeq \mathfrak{g} \otimes \mathfrak{O}(X \setminus x) \to \mathfrak{g} \otimes \hat{\mathfrak{K}}_{x}$$

and the canonical action $\text{Act}_x : (\mathfrak{g} \otimes \hat{\mathfrak{K}}_{x}) \otimes A_{\mathfrak{S}_G^{(1)}} \to A_{\mathfrak{S}_G^{(1)}}$.

**Proof.** (Of Proposition [16.4])

Consider the completion $p_{1*}(\mathfrak{g} \otimes \mathfrak{O}(\mathfrak{S}_G))$ as a left D-module on $X$.

To specify a map $j_* j^* (\mathfrak{B}(\mathfrak{g}, \mathfrak{O}) \boxtimes A_{\mathfrak{S}_G^{(1)}}) \to \Delta_1(A_{\mathfrak{S}_G^{(1)}})$ is the same as to specify a continuous map $p_{1*}(\mathfrak{g} \otimes \mathfrak{O}(\mathfrak{S}_G) \otimes A_{\mathfrak{S}_G^{(1)}} \to A_{\mathfrak{S}_G^{(1)}}$ compatible with the right D-module structure.
Without restricting the generality, we can assume that $X$ is affine. In this case $\text{Lie}(\text{Maps}^{(1)}(X, G)) \simeq p_{1*}(\mathfrak{g} \otimes \mathcal{O} \boxtimes \mathcal{O})$ is dense in $p_{1*}(\mathfrak{g} \otimes \mathcal{O} \boxtimes \mathcal{O})$ and it is enough to construct a continuous map

$$p_{1*}(\mathfrak{g} \otimes \mathcal{O} \boxtimes \mathcal{O}) \otimes A_{\mathfrak{g} \tau_y} \to A_{\mathfrak{g} \tau_y}.$$ 

However, such a map has been already constructed: this is the map $\text{Act}^{(1)}$. The fact that property (a) of Proposition 16.1 holds follows from the fact that the map $\text{Act}^{(1,2)}$ commutes with the chiral bracket (cf. the commutative diagram above). Property (b) of Proposition 16.1 follows from Lemma 16.2.

$\square$
Chapter VI. The Free Bosonic Theory

17. The canonical line bundle

Let $H$ be a torus, let $\Lambda$ denote the lattice of 1-parameter subgroups in $H$ and let $\mathfrak{h}$ be the Lie algebra of $H$. Consider the affine Grassmannians $G_{\mathfrak{h}}$, $G_{\mathfrak{h}}(X)$ and $G_{\mathfrak{h}}^I(X)$ correspondingly onto $X$ identified with $\Lambda$ and $\Lambda \times X$ shall denote by $\text{unit}_{\mathfrak{h}}^\Lambda$, the canonical section $X \simeq_{\text{red}} G_{\mathfrak{h}}^\Lambda \rightarrow G_{\mathfrak{h}}^\Lambda$.

The reduced schemes $\text{red} G_{\mathfrak{h}}$ and $\text{red} G_{\mathfrak{h}}(X)$ are finite dimensional and are identified with $\Lambda$ and $\Lambda \times X$, respectively. For $\lambda \in \Lambda$ we shall denote by $G_{\mathfrak{h}}^{\lambda}$ (resp., $G_{\mathfrak{h}}^{\lambda}(X)$) the corresponding connected component of $G_{\mathfrak{h}}$ (resp., of $G_{\mathfrak{h}}(X)$); we shall denote by $\text{unit}_{\mathfrak{h}}^\lambda$, the corresponding section $X \simeq_{\text{red}} G_{\mathfrak{h}}^{\lambda} \rightarrow G_{\mathfrak{h}}^{\lambda}$.

As for $G_{\mathfrak{h}}^{\lambda}(X)$, its set of connected (resp., irreducible) components is identified with $\Lambda$ (resp., with $\Lambda^I$). The reduced scheme corresponding to each irreducible component projects isomorphically onto $X^I$. We leave it to the reader to work out the intersection pattern of various irreducible components of $\text{red} G_{\mathfrak{h}}^I(X)$. For $\lambda \in \Lambda^I$ we shall denote by $\text{unit}_{\mathfrak{h}}^{\lambda}$, the corresponding section $X^I \rightarrow \text{red} G_{\mathfrak{h}}^I(X) \rightarrow G_{\mathfrak{h}}^I(X)$.

17.1. Recall the groups $H(\hat{\mathcal{O}})$ and $H(\hat{\mathcal{K}})$ of Section [5.3]. Now let now $Q$ be an integral-valued even symmetric form on $\Lambda$. We shall denote by the same character $Q$ the corresponding quadratic form on $\Lambda$.

It is known (cf. [5, 6]) that to the data of $(\Lambda, Q, \epsilon)$ one can associate in a canonical way a central extension $H(\hat{\mathcal{K}})'$ of the ind-algebraic group $H(\hat{\mathcal{K}})$:

$$1 \rightarrow \mathbb{C}^* \rightarrow H(\hat{\mathcal{K}})' \rightarrow H(\hat{\mathcal{K}}) \rightarrow 1.$$  

The group $H(\hat{\mathcal{K}})'$ has the following properties:

- We have a canonical lifting of the embedding $H(\hat{\mathcal{O}}) \rightarrow H(\hat{\mathcal{K}})$ to an embedding $H(\hat{\mathcal{O}}) \rightarrow H(\hat{\mathcal{K}})'$. In particular, we obtain an $H(\hat{\mathcal{K}})'$-equivariant line bundle $\mathcal{R}_Q$ on $G_{\mathfrak{h}} = H(\hat{\mathcal{K}})/H(\hat{\mathcal{O}})$.
- The Lie algebra of $H(\hat{\mathcal{K}})'$ can be identified with the Heisenberg algebra $\text{Heis}(\mathfrak{h}, Q) := \mathfrak{h} \otimes \hat{\mathcal{K}} \oplus \mathbb{C}$.
- For a complete curve curve $X$ and $n$ points $x_1, \ldots, x_n \in X$ consider the corresponding central extension

$$1 \rightarrow \mathbb{C}^* \rightarrow (H(\hat{\mathcal{K}}_{x_1}) \times \ldots \times H(\hat{\mathcal{K}}_{x_n}))' \rightarrow H(\hat{\mathcal{K}}_{x_1}) \times \ldots \times H(\hat{\mathcal{K}}_{x_n}) \rightarrow 1.$$

We have a canonical lifting of the embedding

$$\text{Maps}(X \setminus \{x_1, \ldots, x_n\}, H) \subset H(\hat{\mathcal{K}}_{x_1}) \times \ldots \times H(\hat{\mathcal{K}}_{x_n})$$

to a homomorphism of formal group schemes $\text{Maps}(X \setminus \{x_1, \ldots, x_n\}, H) \rightarrow H(\hat{\mathcal{K}})'_n$.

To state the additional property of $H(\hat{\mathcal{K}})'$ we need to introduce some notation. For a character $\mu \in \mathfrak{h}^*$, let $V^\mu$ denote the Weyl module over $\text{Heis}(\mathfrak{h}, Q)$, i.e.

$$V^\mu \simeq \text{Ind}_{\mathfrak{h} \otimes \hat{\mathcal{O}} \otimes \mathbb{C}}^{\text{Heis}(\mathfrak{h}, Q)} (\mathbb{C}^\mu),$$
where $\mathbb{C}^\mu$ denotes the 1-dimensional representation of $\mathfrak{h} \otimes \hat{\mathfrak{h}} \oplus \mathbb{C}$ corresponding to $\mu : \hat{\mathfrak{h}} \to \mathbb{C}$. When $\mu = Q(\lambda, \cdot)$ for $\lambda \in \Lambda \subset \Lambda \otimes \mathbb{C} \simeq \hat{\mathfrak{h}}$, we endow $V^\mu$ with an action of the group $\text{Aut}_0^+$ by requiring that it be compatible with the $\text{Aut}_0^+$-action on $\text{Heis}(\mathfrak{h}, Q)$ and by setting $\mathbb{C}^\mu \simeq L_{-Q(\lambda, \lambda)/2}$ as an $\text{Aut}_0^+$-module (cf. Section 8.1).

Now note that the vector space $\Gamma(\mathfrak{h}, R_Q)$ carries an $\text{Aut}_0^+$-action (this is due to the fact that the construction of $H(\hat{\mathfrak{h}})'$ and hence of $R_Q$ is canonical).

The fourth property of $H(\hat{\mathfrak{h}})'$ reads as follows:

- We have an isomorphism of $\text{Heis}(\mathfrak{h}, Q)$-modules: $\Gamma(\mathfrak{h}, (R_Q)^{-1})^* \simeq V^\mu$, where $\mu = Q(\lambda, \cdot)$. This isomorphism is compatible with the $\text{Aut}_0^+$-action.

17.2. Globally over a (not necessarily complete) curve $X$, the form $Q$ gives rise to a central extension

$$1 \to \mathbb{C}^\mu \to H^*_X(V) \to H_X(V) \to 1$$

of the ind-group scheme $H_X(V)$. Let $R_Q(V)$ denote the line bundle

$$R_Q(V) := H^*_X(V)/H_0(X)$$

over $\mathfrak{g}_H(V) = H_X(V)/H_0(X)$. Property (2) of the extension $H(\hat{\mathfrak{h}})'$ implies that $R_Q(V)$ is equivariant with respect to the action of the group scheme $H_0(X)$ on $\mathfrak{g}_H(V)$.

Property (4) of the group $H(\hat{\mathfrak{h}})'$ implies the following assertion:

**Lemma 17.1.** We have a canonical isomorphism

$$\text{unit}^\lambda_\mathfrak{g}_H^* (R_Q(V)) \simeq \Omega^{-Q(\lambda, \lambda)/2}.$$

A generalization of the above construction yields a line bundle $R^*_0(V)$ over the pre-image in $\mathfrak{g}_H^0(V)$ of $X^\lambda \setminus \Delta$. Our present goal is to extend this line bundle to the whole of $\mathfrak{g}_H^0(V)$, i.e. to the locus where the base points $x_{i_1}, \ldots, x_{i_n} \in X$ collide with one another.

Assume now that $X$ is complete. Property (3) of the extension $H(\hat{\mathfrak{h}})'$ implies the following assertion:

**Lemma 17.2.** For a complete curve $X$ there exists a canonical line bundle $R_Q(V)$ over $\text{Bun}_H(V)$ such that for every $J$, the restriction of the line bundle $k^J_\mathfrak{g}_H^* (R_Q(V))$ to $\mathfrak{g}_H^J(V) \setminus (r^J)^{-1}(\Delta)$ is identified with $R^J_0(V)$.

We now set $R^J_0(V)$ to be the line bundle over $\mathfrak{g}_H^J(V)$ equal to $k^J_\mathfrak{g}_H^* (R_Q(V))$. (When $J = \{1, \ldots, n\}$, we shall replace the notation $R^J_0(V)$ simply by $R^0_0(V)$.)

Since the projection $k^J : \mathfrak{g}_H^J(V) \to \text{Bun}_H(V)$ respects the connection on $\mathfrak{g}_H^J(V)$ along $X^J$, we obtain a connection along $X^J$ on the line bundle $R^J_0(V)$.

**Proposition 17.1.** Under the isomorphisms of Proposition 17.2, the system of line bundles $3 \to R^J_0(V)$ satisfies the following factorization property:

(a) For a surjection of finite sets $\phi : J \to 3$, the first isomorphism of Proposition 17.1 underlies an isomorphism of line bundles: $R^J_0(V)|_{\mathfrak{g}_H^J(V)} \simeq R^J_0(V)$.
(b) For a decomposition $J = J_1 \cup \ldots \cup J_k$ the second isomorphism of Proposition 15.3 underlies an isomorphism of line bundles:

$$R^j_Q(X)|_{X^{j_1} \times \ldots \times j_k} \times \phi^j \simeq R^j_Q(X) \times \ldots \times R^j_Q(X)|_{X^{j_1} \times \ldots \times j_k} \times \phi^j \times \ldots \times \phi^j.$$ 

Moreover, the above isomorphisms are compatible with the connections on the line bundles $R^j_Q(X)$'s along the $X^j$'s.

**Proof.** The isomorphism of point (a) follows from the definitions. To prove point (b) note that the two line bundles are obviously isomorphic over the pre-image of $X^j \setminus \Delta$ in $\tilde{\mathfrak{r}}^j_H(X)$ and we only have to show that this isomorphism extends to the whole of $\tilde{\mathfrak{r}}^j_H(X)$. This follows easily (by degree considerations) from the following assertion:

**Lemma 17.3.** For $\lambda = \lambda_1, \ldots, \lambda_n$ the line bundle $\operatorname{unit}_{\tilde{\mathfrak{r}}^j}^*(R^j_Q(X))$ over $X^n$ is identified canonically with

$$\Omega^{-Q(\lambda_1, \lambda_1)/2} \otimes \ldots \otimes \Omega^{-Q(\lambda_n, \lambda_n)/2}(\Sigma_{i<j} Q(\lambda_i, \lambda_j) \cdot \Delta_{x_i = x_j}).$$

17.3. We now want to show that the dependence of the line bundles $R^j_Q(X)$ on $X$ is local in the sense of Section 15.1. This fact is not immediately obvious, since in the definition of $R^j_Q(X)$ we gave above we used the fact that $X$ is complete. A way to overcome this difficulty will be via the ind-schemes $\tilde{\mathfrak{r}}^j_H(X)$ introduced in Section 15.3.

Using the iterative description of the ind-scheme $\tilde{\mathfrak{r}}^n_H(X)$ given in Section 15.3 we can produce a line bundle $\tilde{R}^n_Q(X)$ over $\tilde{\mathfrak{r}}^n_H(X)$. This construction works for curves that are not necessarily complete; therefore we can form line bundle $\tilde{R}^n_Q(X)$ over $\tilde{\mathfrak{r}}^n_H(X)$ which is local in the sense of Section 15.1. In more detail, let $X^{S, J}, X^{S, J}, \varphi^{S, J}$ be as in Section 15.1 and let $\tilde{\mathfrak{r}}^n_H(X^S) \setminus x^S$ denote the open subscheme of $\tilde{\mathfrak{r}}^n_H(X^S)$ equal to the pre-image in $\tilde{\mathfrak{r}}^n_H(X^S)$ of

$$(X^S \setminus x^S) \times \ldots \times (X^S \setminus x^S) \subset X \times \ldots \times X$$

under the projection $r^\circ w_n$.

Then the isomorphism of ind-schemes

$$\tilde{\mathfrak{r}}^n_H(X^S \times I) \setminus x^S \times I \simeq (\tilde{\mathfrak{r}}^n_H(X^S) \setminus x^S) \times I$$

underlies an isomorphism of line bundles:

$$\tilde{R}^n_Q(X^S \times I)|_{\tilde{\mathfrak{r}}^n_H(X^S \times I) \setminus x^S \times I} \simeq (\tilde{R}^n_Q(X^S)|_{\tilde{\mathfrak{r}}^n_H(X^S) \setminus x^S}) \times I.$$ 

Note that over the pre-image of $X^n \setminus \Delta$ in $\tilde{\mathfrak{r}}^n_H(X)$ (which is identified with the pre-image of $X^n \setminus \Delta$ in $\mathfrak{r}^n_H(X)$) the line bundle $\tilde{R}^n_Q(X)$ is identified naturally with $R^j_Q(X)$.
Proposition 17.2. The line bundle $R^n_Q(X)$ is uniquely characterized by the following property:

There exists a canonical isomorphism

$$w^*_n(R^n_Q) \simeq \widetilde{R^n}_Q$$

which reduces to the identity isomorphism over $X^n \setminus \Delta$.

The proof of this assertion is not difficult to deduce from the construction of the line bundle $\widetilde{R^n}_Q(X)$ and we will omit it.

Corollary 17.1. The line bundle $R^n_Q(X)$ over $\mathfrak{S}_H^I(X)$ is local (in the above sense). Moreover, the factorization isomorphisms of Lemma 17.1 are compatible with the local structure.

18. Construction of the bosonic chiral algebra

18.1. For $(H, Q)$ as above consider the chiral algebra $A(h, Q)$ of Example 1 of Section 9.1. Our goal in this subsection is to construct a bigger chiral algebra $A(H, Q)$ that will contain $A(h, Q)$ as a subalgebra.

Recall that if $Z$ is a strict ind-scheme and $R$ is a line bundle over $Z$ it makes sense to talk about $R$-twisted right D-modules on $Z$ which are set-theoretically supported on finite-dimensional subschemes of $Z$.

Thus, consider the $R_Q(X)$-twisted right D-module

$$\text{unit}_{\tilde{\mathfrak{S}}_H^I}(\Omega) \otimes R_Q(X)$$
on $\mathfrak{S}_H(X)$. Let $A(H, Q)^{\lambda}$ denote its $\mathcal{O}$-module direct image onto $X$, i.e.

$$A(H, Q)^{\lambda} := r_* (\text{unit}_{\tilde{\mathfrak{S}}_H^I}(\Omega) \otimes R_Q(X)).$$

The connection on the pair $(\tilde{\mathfrak{S}}_H(X), R_Q(X))$ along $X$ defines on $A(H, Q)^{\lambda}$ a structure of right D-module on $X$. Finally, let $A(H, Q)$ be the direct sum $\bigoplus_{\lambda \in \Lambda} A(H, Q)^{\lambda}$. The sections $\text{unit}_{\tilde{\mathfrak{S}}_H^I} : X \to \tilde{\mathfrak{S}}_H(X)^{\lambda}$ give rise to $\mathcal{O}$-module maps

$$\text{unit}^{\lambda} : \Omega^{-Q_{\lambda}^{(\lambda, \lambda)/2+1}} \to A(H, Q);$$

for $\lambda = 0$ the map

$$\text{unit} : \Omega \to A(H, Q)^0 \to A(H, Q)$$

is in fact a map of D-modules.

More generally, for a finite set $\mathcal{I}$ and for an element $\mathcal{T} \in \Lambda^\mathcal{I}$ consider on $X^\mathcal{I}$ the right D-module

$$A(H, Q)^{\mathcal{T}} := r^*_\mathcal{T} (\text{unit}_{\tilde{\mathfrak{S}}_H^I}(\Omega_{X^\mathcal{I}}) \otimes R_Q(X)).$$

Let $A(H, Q)^{\mathcal{T}}$ denote the direct sum $\bigoplus_{\mathcal{T} \in \Lambda^\mathcal{I}} A(H, Q)^{\mathcal{T}}$ and let $\mathcal{T} A(H, Q)^{\mathcal{T}}$ be the corresponding left D-module on $\mathcal{X}^\mathcal{I}$.

Consider the sheaf $r^*_\mathcal{T} (\tilde{R}_Q^\mathcal{T}(X)^{-1})$ over $X^\mathcal{I}$. Since $\tilde{\mathfrak{S}}_H^I(X)$ is an inductive limit of schemes finite over $X^\mathcal{I}$, the above direct image is a projective limit of coherent sheaves on $X^\mathcal{I}$; moreover, it carries a natural left D-module structure due to the connection on $R^n_Q(X)$ along $X^\mathcal{I}$. It is easy to see that we have in fact a canonical isomorphism of left D-modules:

$$\mathcal{T} A(H, Q)^{\mathcal{T}} \simeq \text{Hom}_{\mathcal{O}_{X^\mathcal{I}}}(r^*_\mathcal{T} (\tilde{R}_Q^\mathcal{T}(X)^{-1}), \mathcal{O}_{X^\mathcal{I}}).$$
The following assertion follows from Proposition 17.1.

**Lemma 18.1.** The system of \( \mathcal{O} \)-modules \( \mathcal{J} \rightarrow {}^!\mathcal{A}(H,Q)\) satisfies the conditions of Theorem 14.4.

Thus, \( \mathcal{A}(H,Q) \) has a natural structure of chiral algebra such that for every \( \mathcal{J} \)

\[ {}^!\mathcal{A}(H,Q)^{\mathcal{J}} \approx {}^!\mathcal{A}(H,Q)^{\mathcal{J}}(3), \]

as \( \mathcal{O} \)-modules. The last assertion of Proposition 17.1 implies that the above isomorphisms preserve the D-module structure as well.

For \( x \in X \), let \( \mathfrak{g}_H(X)_x \) denote the fiber of \( \mathfrak{g}_H(X) \) over \( x \) and let \( R_Q(X)_x \) denote the restriction of \( R_Q(X) \) to this fiber. (Once we choose an isomorphism \( \hat{\mathcal{O}} \cong \hat{\mathcal{O}}_x \), the pair \( (\mathfrak{g}_H(X)_x, R_Q(X)_x) \) can be identified with \( (\mathfrak{g}_H, R_Q) \).) By the construction, the fiber \( A(H,Q)_x^\lambda \) of \( A(H,Q)^\lambda \) at \( x \) is identified with

\[ \Gamma(\mathfrak{g}_H(X)_x, (R_Q(X)_x)^{-1}). \]

Therefore, the Harish-Chandra pair \( (\mathfrak{h} \otimes \hat{\mathcal{K}}_x \oplus \mathbb{C}, H(\hat{\mathcal{O}}_x)) \) acts on \( A(H,Q)_x \); we shall denote by

\[ \text{Act} : (\mathfrak{h} \otimes \hat{\mathcal{K}}_x \oplus \mathbb{C}) \otimes A(H,Q)_x \rightarrow A(H,Q)_x \]

the corresponding action map.

Property (4) from Section 17.1 implies that as a \( (\mathfrak{h} \otimes \hat{\mathcal{K}}_x \oplus \mathbb{C}, H(\hat{\mathcal{O}}_x)) \)-module, \( A(H,Q)_x^\lambda \) is canonically isomorphic to the Weyl module \( V^\mu \) with \( \mu = Q(\lambda, \cdot) \).

**Theorem 18.1.** We have:

(a) The chiral bracket on \( A(H,Q) \) preserves the \( \Lambda \)-grading, i.e.

\[ j_* j^* (A(H,Q)^{\lambda} \boxtimes A(H,Q)^{\mu}) \rightarrow \Delta_!(A(H,Q)^{\lambda+\mu}). \]

In particular, \( A(H,Q)^0 \) is a chiral subalgebra in \( A(H,Q) \) and each \( A(H,Q)^\lambda \) is a chiral module over \( A(H,Q)^0 \).

(b) There exists a canonical isomorphism of chiral algebras \( A(H,Q)^0 \approx A(\mathfrak{h},Q) \).

(c) For \( x \in X \), the map \( (\mathfrak{h} \otimes \hat{\mathcal{K}}_x \oplus \mathbb{C}) \otimes A(H,Q)_x \rightarrow A(H,Q)_x \) induced by the chiral action of \( \mathbb{B}(\mathfrak{h},Q) \) on \( A(H,Q) \) coincides with the map \( \text{Act} \) (cf. above).

The proof of this theorem goes along the same lines as the proof of Theorem 16.1 and we will omit it.

18.2. We shall now show that the chiral algebra \( A(H,Q) \) defines a CFT of central charge \( c = \dim(H) \).

It follows from Corollary 17.1 that the \( \mathcal{O}_X \)-modules \( {}^!\mathcal{A}(H,Q)^{\mathcal{J}}(3) \) carry a structure of local \( \mathcal{O} \)-modules over \( \mathcal{X} \). Moreover, the factorization isomorphisms are morphisms of local \( \mathcal{O} \)-modules. Hence, according to Proposition 14.1, the chiral algebra \( A(H,Q) \) satisfies condition (a) of Section 8.2 and it remains to endow it with an energy-momentum tensor such that condition (b) of Section 8.2 holds.

Consider the map \( T_{H,Q} : \Theta_{\dim(h)} \rightarrow A(H,Q) \) obtained as a composition:

\[ \Theta_{\dim(h)} \xrightarrow{T_{H,Q}} A(h,Q) \approx A(H,Q)^0 \subset A(H,Q). \]

**Proposition 18.1.** The map \( T_{H,Q} \) satisfies condition (b) of Section 8.2.
Proving. For a vector field $\xi$ on a curve $X$ consider the two maps $A(H,Q)^{\lambda} \to A(H,Q)^{\lambda}$ given by $l \mapsto l \cdot \xi + \text{Lie}_\xi(l)$ and $(\text{id} \otimes h)\{l, T_{H,Q}(\xi')\}$, where $\xi'$ is some lifting of $\xi$ to a section of $\Theta^\lambda_{\dim(h)}$. Both these maps are endomorphisms of the D-module structure on $A(H,Q)^{\lambda}$; therefore, they induce endomorphisms of each fiber $A(H,Q)^{\lambda}_x$ for $x \in X$.

Now choose an identification $\hat{\Theta}_x \simeq \hat{0}$. Since $T_{h,Q}$ is an energy-momentum tensor for $A(h,Q)$, the above construction yields two actions of the Lie algebra $\text{Vir}^+$ on $V^\mu$ (here $\mu = Q(\lambda, \cdot)$) that are compatible with the $\text{Vir}^+$-action on $\text{Heis}(h, Q)$.

Moreover, we claim that the restrictions of these two actions to $\text{Vir}_0^+ \subset \text{Vir}^+$ coincide. Indeed, it is enough to check this fact on the generating space $\mathbb{C}^\mu \subset V^\mu$ and in both cases $\text{Vir}_0^+$ acts on $\mathbb{C}^\mu$ as on $L_{-Q(\lambda, \lambda)/2}$ (this is an easy computation using Lemma 17.1 and Proposition 3.1).

The assertion of the proposition now follows from the following general fact:

**Lemma 18.2.** There is at most one extension of the $\text{Vir}_0^+$-action on $V^\mu$ to an action of $\text{Vir}^+$ which is compatible with the action of the latter Lie algebra on $\text{Heis}(h, Q)$. 

18.3. Assume now that $X$ is complete. Our present goal is to compute the space of conformal blocks of the chiral algebra $A(H,Q)$.

**Theorem 18.2.** For a complete curve $X$ we have a canonical isomorphism

$$H_{\nabla}(X, A(H,Q)) \simeq H^0(\text{Bun}_H(X), (\mathcal{R}_Q(X))^{-1})^*.$$ 

Before proving this theorem let us make a few observations:

**Observation 1:**

The stack $\text{Bun}_H(X)$ splits into connected components

$$\text{Bun}_H(X) = \bigcup_{\lambda \in A} \text{Bun}_H(X)^\lambda$$

and each $\text{Bun}_H(X)^\lambda$ is isomorphic to the quotient of the Jacobian of $X$ by the trivial action of the group $H$; this corresponds to the fact that $\forall \mathcal{P} \in \text{Bun}_H(X)$, $\text{Aut}(\mathcal{P}) \simeq H$.

**Observation 2:**

It is easy to infer from the properties of the extension $H(\hat{X})'$ in Section 17.1 that for every $\mathcal{P} \in \text{Bun}_H(X)^\lambda$ the group $H$ of automorphisms of $\mathcal{P}$ acts on the fiber of $\mathcal{R}_Q(X)$ at $\mathcal{P}$ via the character $Q(\lambda, \cdot)$. This implies, in particular, that the contribution to $H^0(\text{Bun}_H(X), (\mathcal{R}_Q(X))^{-1})$ comes only from the 0-th component of $\text{Bun}_H(X)$. This shows that the RHS of the isomorphism of Theorem 18.2 is finite-dimensional.

It is equally easy to see that if $x_1, \ldots, x_n$ are distinct points of $X$, then the canonical map $A(H,Q)^{\lambda_1}_x \otimes \ldots \otimes A(H,Q)^{\lambda_n}_x \to H_{\nabla}(X, A(H,Q))$ is non-zero only if $\lambda_1 + \ldots + \lambda_n = 0$. (This is due to the fact that the subspace of “constant currents”

$$\mathfrak{h} \subset DR^0(X, A(h, Q)) \subset DR^0(X, A(H, Q))$$

acts on each $A(H,Q)^{\lambda_i}_x \simeq V^\mu_i$ by the character $\mu_i$.)

**Observation 3:** The assertion of the theorem can be reformulated as an isomorphism between $H^0(\text{Bun}_H(X), (\mathcal{R}_Q(X))^{-1})$ and $H_{\nabla}(X, A(H,Q))^*$.
According to Proposition 14.2
\[ H_{\nabla}(X, \mathcal{A}(H, Q))^* \simeq \text{Hom}_{D(X^n)} \left( \mathcal{A}(H, Q)^{(n)}, \mathcal{O}_{X^n} \right) \]
for any \( n \geq 2 \). Since
\[ \mathcal{A}(H, Q)^{(n)} \simeq \text{Hom}_{\mathcal{O}_{X^n}} \left( r^n_* \left( (R^n_Q(X))^{-1} \right), \mathcal{O}_{X^n} \right), \]
we obtain that the space \( H_{\nabla}(X, \mathcal{A}(H, Q))^* \) is identified with the space
\[ \Gamma(\mathfrak{g}^n_H(X), (R^n_Q(X))^{-1})^{D_X^n} \]
of \( D_X^n \)-invariant sections of \( (R^n_Q(X))^{-1} \) for \( n \) as above.

Let us write out the above isomorphism
\[ H_{\nabla}(X, \mathcal{A}(H, Q))^* \simeq \Gamma(\mathfrak{g}^n_H(X), (R^n_Q(X))^{-1})^{D_X^n} \]
more explicitly:

Let \( \chi \) be an element of \( H_{\nabla}(X, \mathcal{A}(H, Q))^* \). Let us on the one hand consider the composition
\[ \Omega^{-Q(\lambda_1, \lambda_1)/2} \boxtimes \ldots \boxtimes \Omega^{-Q(\lambda_n, \lambda_n)/2} \xrightarrow{\text{unit}^\lambda_1 \otimes \ldots \otimes \text{unit}^\lambda_n} \mathcal{A}(H, Q) \boxtimes X \rightarrow j_* j^* (\mathcal{O}_{X^n}) \xrightarrow{\chi} j_* j^* (\mathcal{O}_{X^n}). \]
The above map can be regarded as a section of \( \Omega^{Q(\lambda_1, \lambda_1)/2} \boxtimes \ldots \boxtimes \Omega^{Q(\lambda_n, \lambda_n)/2} \) over \( X^n \setminus \Delta \).

On the other hand, an element \( \chi' \in \Gamma(\mathfrak{g}^n_H(X), (R^n_Q(X))^{-1})^{D_X^n} \) can be regarded as a map from \( \mathfrak{g}^n_H(X) \) to the total space of the line bundle \( (\mathcal{R}_Q(X))^{-1} \) over \( \text{Bun}_H(X) \). According to Lemma 17.1, the pull-back of the line bundle \( (\mathcal{R}_Q(X))^{-1} \) under the composition
\[ k^n \circ \text{unit}^\lambda_{\mathfrak{g}^n} \circ j : X^n \setminus \Delta \hookrightarrow X^n \rightarrow \text{Bun}_H(X) \]
is the line bundle \( \Omega^{Q(\lambda_1, \lambda_1)/2} \boxtimes \ldots \boxtimes \Omega^{Q(\lambda_n, \lambda_n)/2} \) over \( X^n \setminus \Delta \). Therefore, the composition
\[ \chi' \circ \text{unit}^\lambda_{\mathfrak{g}^n} \circ j : X^n \setminus \Delta \hookrightarrow X^n \rightarrow \text{Tot}(\mathcal{R}_Q(X)^{-1}) \]
is again a section of \( \Omega^{Q(\lambda_1, \lambda_1)/2} \boxtimes \ldots \boxtimes \Omega^{Q(\lambda_n, \lambda_n)/2} \) over \( X^n \setminus \Delta \).

It follows from the definitions that the above two sections of \( \Omega^{Q(\lambda_1, \lambda_1)/2} \boxtimes \ldots \boxtimes \Omega^{Q(\lambda_n, \lambda_n)/2} \) coincide if the elements \( \chi \) and \( \chi' \) correspond to one another under the isomorphism
\[ H_{\nabla}(X, \mathcal{A}(H, Q))^* \simeq \Gamma(\mathfrak{g}^n_H(X), (R^n_Q(X))^{-1})^{D_X^n}. \]

Proof. (Of Theorem 18.2)

By the very definition of the connection on the line bundle \( R^n_Q(X) \), we have a map
\[ H^0(\text{Bun}_H(X), (\mathcal{R}_Q(X))^{-1}) \rightarrow \Gamma(\mathfrak{g}^n_H(X), (R^n_Q(X))^{-1}) \]
whose image belongs to \( \Gamma(\mathfrak{g}^n_H(X), (R^n_Q(X))^{-1})^{D_X^n} \). Moreover, the assertion of the above Observation 3 combined with Proposition 5.3 imply that the composition
\[ H^0(\text{Bun}_H(X), (\mathcal{R}_Q(X))^{-1}) \rightarrow \Gamma(\mathfrak{g}^n_H(X), (R^n_Q(X))^{-1})^{D_X^n} \rightarrow H_{\nabla}(X, \mathcal{A}(H, Q))^* \]
is the same for any \( n \).

Moreover, the map \( k^n \) is dominant for any \( n \). Therefore, the map
\[ H^0(\text{Bun}_H(X), (\mathcal{R}_Q(X))^{-1}) \rightarrow H_{\nabla}(X, \mathcal{A}(H, Q))^* \]
is injective and it remains to show that it is surjective. This will be done in two steps.

**Step 1**

Let

\[ \chi' \in \Gamma(\mathfrak{S}^n_H(X), (R^0_Q(X))^{-1}) \]

be as above. We shall first show that it is locally constant along the fibers of the projection \( k^n \).

Let \((x_1, \ldots, x_n, \mathcal{P}, \alpha : \mathcal{P} \to \mathcal{P}_0, X \setminus \{x_1, \ldots, x_n\})\) be a \( \mathbb{C} \)-point of \( \mathfrak{S}^n_H(X) \) and for a local Artinian scheme \( I \) let \((x'_1, \ldots, x'_n, \mathcal{P}, \alpha' : \mathcal{P} \to \mathcal{P}_0, X \setminus \{x'_1, \ldots, x'_n\})\) be its extension to an \( I \)-valued point of \( \mathfrak{S}^n_H(X) \), such that \( \mathcal{P}' = \mathcal{P} \times I \).

We must show that the composition

\[ I \to \mathfrak{S}^n_H(X) \xrightarrow{\chi'} \text{Tot}(R^0_Q(X)^{-1}) \]

is a constant map

\[ I \to \text{Spec}(\mathbb{C}) \to \mathfrak{S}^n_H(X) \xrightarrow{\chi'} \text{Tot}(R^0_Q(X)^{-1}). \]

Obviously, it is enough to check this property when \( I = \text{Spec}(\mathbb{C}[t]/t^2) \).

First of all, using the connection along \( X^n \) on \( \mathfrak{S}^n_H(X) \) we can reduce the assertion to the case when the \( x_i \) are constant maps \( I \to X \) and without loss of generality we can assume that the points \( x_1, \ldots, x_n \) are distinct.

In this case the difference between the trivializations \( \alpha \times I \) and \( \alpha' \cdot I \) of \( \mathcal{P} \times I \) over \((X \setminus \{x_1, \ldots, x_n\}) \times I\) is given by an infinitesimal “gauge transformation”, i.e. by an action on the fiber of \( \mathfrak{S}^n_H(X) \) over \((x_1, \ldots, x_n)\) by an element of

\[ H^0(X \setminus \{x_1, \ldots, x_n\}, \mathfrak{h} \otimes \mathfrak{O}) \subset \mathfrak{h} \otimes \mathfrak{K}_{x_1} \oplus \cdots \otimes \mathfrak{h} \otimes \mathfrak{K}_{x_n}. \]

Therefore, in terms of the corresponding functional \( \chi : H^0(V, A(H, Q)) \to \mathbb{C} \), we must prove that the functional on \( A(H, Q)_{x_1} \otimes \cdots \otimes A(H, Q)_{x_n} \) obtained as a composition

\[ A(H, Q)_{x_1} \otimes \cdots \otimes A(H, Q)_{x_n} \to H^0(V, A(H, Q)) \xrightarrow{\chi} \mathbb{C} \]

is \( H^0(X \setminus \{x_1, \ldots, x_n\}, \mathfrak{h} \otimes \mathfrak{O}) \)-invariant. However, this follows from the definition of the space of conformal blocks in view of point (c) of Theorem [18.1].

**Step 2**

Let us view \( \chi' \) as a map from \( \mathfrak{S}^n_H(X) \) to the total space of the line bundle \((R^0_Q(X))^{-1}\) (cf. Observation 3). To prove the theorem it remains to show that for two \( \mathbb{C} \)-points \((x_1, \ldots, x_n, \mathcal{P}, \alpha)\) and \((x'_1, \ldots, x'_n, \mathcal{P}', \alpha')\) of \( \mathfrak{S}^n_H(X) \) with \( \mathcal{P} = \mathcal{P}' \), the map \( \chi' \) has the same value on these points.

Let now \( \nu_1, \ldots, \nu_k \) be a basis of \( \Lambda \), let \( m \) be an integer satisfying \( m > 2g(X) - 2 \) and let \( n = 2m \cdot k \). Take an element \( \vec{\lambda}(m) \in \Lambda^n \) equal to

\[ \vec{\lambda}(m) = \lambda_1, \ldots, \lambda_1, \ldots, -\lambda_1, \ldots, -\lambda_1, \ldots, \lambda_k, \ldots, \lambda_k, \ldots, -\lambda_k, \ldots, -\lambda_k. \]

First of all, using Observations 2 and 3 and by increasing \( n \) if necessary, we can arrange that our two points \((x_1, \ldots, x_n, \mathcal{P}, \alpha)\) and \((x'_1, \ldots, x'_n, \mathcal{P}', \alpha')\) belong to the same irreducible component \( \mathfrak{S}^n_H(X) \), where \( m \) and \( \vec{\lambda}(m) \) are as above. Recall
that \( \text{unit}_{\text{et}}(X) : X^n \to \mathfrak{g}r_H^n(X) \) defines an isomorphism between \( X^n \) and the reduced scheme of the corresponding irreducible component of \( \mathfrak{g}r_H^n(X) \).

The composition

\[
X^n \hookrightarrow \text{red} \mathfrak{g}r_H^n(X) \hookrightarrow \mathfrak{g}r_H^n(X) \xrightarrow{k_n} \text{Bun}_H(X)
\]

factors in this case via

\[
X^n \to X^{(m)} \times \ldots \times X^{(m)} \to \text{Pic}^m(X) \times \ldots \times \text{Pic}^m(X) \simeq \text{Bun}_H(X),
\]

where \( X^{(m)} \) denotes the \( m \)-th symmetric power of \( X \) and the second arrow is the product of the Abel-Jacobi maps \( X^{(m)} \to \text{Pic}^m(X) \).

We claim that it is enough to show that the map \( \chi' : X^n \to \text{Tot}(\mathbb{R}Q(X)^{-1}) \) factors via

\[
X^n \to X^{(m)} \times \ldots \times X^{(m)} \to \text{Tot}(\mathbb{R}Q(X)^{-1}):
\]

Indeed, the map

\[
X^{(m)} \times \ldots \times X^{(m)} \to \text{Bun}_H(X)
\]

is smooth, proper and has connected fibers, therefore any section of the pull-back of \( (\mathbb{R}Q(X))^{-1} \) to \( X^{(m)} \times \ldots \times X^{(m)} \) comes from a section of this line bundle over \( \text{Bun}_H(X) \).

To prove the required factorization property of \( \chi' \), we can restrict our attention to \( X^n \setminus \Delta \subset X^n \). Using Observation 3, all we need to show is that the composition

\[
(\Omega^{-Q(\lambda_1,\lambda_1)/2})^{2k}\otimes \ldots \otimes (\Omega^{-Q(\lambda_n,\lambda_n)/2})^{2k} \to \text{Tot}(\mathbb{A}(H,Q))^{2k} \to j_*j^*(\mathcal{O}_{X^n})
\]

is invariant under the group \( S^m \times \ldots \times S^m \) of permutations. However, this follows from the fact that any correlation function is symmetric in the insertions (cf. Section 5.3).
NOTES ON 2D CONFORMAL FIELD THEORY AND STRING THEORY

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