A New Construction of Codebooks Meeting the Levenshtein Bound

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ABSTRACT Codebooks with low coherence have extensive applications in many fields such as code division multiple access (CDMA) communication systems, MIMO communications, compressed sensing and so on. In this paper, based on additive characters over finite fields, we propose a construction of optimal codebook with respect to the Levenshtein bound and verify that it is a new construction. By shortening the length of the optimal codebooks, we present a construction of codebooks asymptotically meeting the Levenshtein bound. To the best of our knowledge, the parameters of the asymptotically optimal codebooks are new.

INDEX TERMS Codebook, asymptotic optimality, Levenshtein bound, character, finite field.

I. INTRODUCTION

Let \( D = \{d_0, d_1, \ldots, d_{N-1}\} \) be a set consisted of \( N \) unit norm \( 1 \times K \) complex vectors over an alphabet \( A \). The number of elements in the alphabet is said to be the alphabet size. The maximum inner-product correlation of \( D \) is given by

\[
I_{max}(D) = \max_{0 \leq i \neq j \leq N-1} |d_i^H d_j|,
\]

where \( d_i^H \) is the conjugate transpose of the complex vector \( d_i \). Then the set \( D \) is termed a codebook (also called signal set) with the parameters \((N, K)\) and maximum inner-product correlation \( I_{max}(D) \). In a code division multiple access system, the distinct vectors of a codebook are assigned to different users and the inner-product correlation is employed to distinguish among the signals of different users. As a result, the maximum inner-product correlation is an important indicator to judge the performance of a codebook. People would like to design a codebook \( D \) such that the parameter \( N \) is as great as possible and \( I_{max}(D) \) is as small as possible for a fixed \( K \). However, there exists a tradeoff between the parameters \( N, K \) and \( I_{max}(D) \), which is introduced by Welch [34].

Lemma 1 [34]: Let \( D \) be an \((N, K)\) codebook with \( I_{max}(D) \). Then we have

\[
I_{max}(D) \geq I_W = \sqrt{\frac{N - K}{(N - 1)K}}.
\]

Additionally, the equality holds if and only if it holds that

\[
|d_i^H d_j| = \sqrt{\frac{N - K}{(N - 1)K}}.
\]

for any pairs of \((i, j)\) with \( i \neq j \).

A codebook meeting the Welch bound with equality is called a maximum-Welch-bound-equality (MWBE) codebook [30]. The construction of MWBE codebooks has attracted a lot of attention (see [3], [5], [6], [10]–[13], [29]–[31], [35]) due to their wide utilization in MIMO communications [14], [15], CDMA communications systems [26], packing [3], compressed sensing [2], [22], combinatorial designs [5], [6], [35], coding theory (for instance, spherical codes [8], LDPC codes [9]) and quantum measurements [28]. In general, it is not easy to design an MWBE codebook.
Many researchers turned to the construction of codebooks asymptotically achieving the Welch bound, i.e., $I_{\max}(D)$ asymptotically meets the Welch bound for sufficiently large $N$. There has been extensive study of asymptotically optimal codebooks with respect to the Welch bound (see [4], [16]–[18], [20], [23]–[25], [33], [37]–[40]).

Strohmer and Heath [31] has pointed out that there are no $(N, K)$ real codebook meeting the Welch bound with equality if $N > \frac{K(K+1)}{2}$ and $(N, K)$ codebook achieving the Welch bound with equality if $N > K^2$. In other words, the Welch bound is not tight in these cases. When $N$ is much bigger than $K$, Levenshtein [21] deduced a new bound called Levenshtein bound which is turned out to be tighter than the Welch bound. Thus, for a codebook such that the number of vectors is much greater than the length of vectors, the Levenshtein bound is a preferable benchmark for the maximum inner-product correlation.

**Lemma 2** [21]: Let $D$ be an $(N, K)$ codebook with $I_{\max}(D)$. If $D$ is a real valued codebook and $N > \frac{K(K+1)}{2}$, then

$$I_{\max}(D) \geq I_L = \sqrt{\frac{3N - K^2 - 2K}{(N - K)(K + 2)}}.$$

If $D$ is a complex valued codebook and $N > K^2$, then

$$I_{\max}(D) \geq I_L = \sqrt{\frac{2N - K^2 - K}{(N - K)(K + 1)}}.$$

In fact, it is very hard to construct codebooks achieving the Levenshtein bound. Constructing codebooks (asymptotically) achieving the Levenshtein bound is harder than the design of (asymptotically) optimal codebooks with respect to the Welch bound. Limited work has been done in the construction of codebooks (asymptotically) achieving the Levenshtein bound. Until now, there are only four constructions of optimal codebooks with respect to the Levenshtein bound. One of the optimal construction was derived from Kerdock codes [1], [36] and the other constructions were constructed by planar functions [7], bent functions over the integer rings $\mathbb{Z}_4$, bent functions over finite fields [41], respectively. Besides optimal codebooks, there are a few constructions of asymptotically optimal codebooks. Tan et al. [32] proposed a construction of codebooks asymptotically achieving the Levenshtein bound by Gauss sums over finite fields. In [36], the authors presented a construction of codebooks asymptotically meeting the Levenshtein bound from binary codes and semi-bent functions. Based on multiplicative characters over finite fields, some asymptotically optimal codebooks with respect to the Levenshtein bound were obtained in [18], [20].

Codebooks (asymptotically) achieving the Levenshtein bound have several practical applications such as the construction of mutually unbiased bases [7] which are used in quantum physics, the construction of deterministic sensing matrices with low coherence [22], the design of spreading sequences for CDMA systems [26]. Hence, it is significant to construction (asymptotically) optimal codebooks with respect to the Levenshtein bound.

We are concerned in this paper with the following two objectives. The first objective is to provide a new construction of codebooks achieving the Levenshtein bound by additive characters over finite fields. Although the optimal codebooks have the same parameters as those in [7, Theorem 4], we prove that it is indeed a different construction. The other objective is to propose a new construction of asymptotically optimal codebooks with respect to the Levenshtein bound by considering the scalability issue regarding the length of the optimal codebooks. Notably, the parameters of the asymptotically optimal codebooks are flexible and not covered by the previous literatures.

This paper is built up as follows. Section 2 is devoted to the definitions and results for the characters and character sums over finite fields. The constructions of (asymptotically) optimal codebooks are summarized in Sections 3 and 4. In Section 5, we make a conclusion.

### II. Character and Character Sums Over Finite Fields

In this section, we recall the definitions of characters over finite fields and some results about character sums over finite fields.

Let $\mathbb{F}_q$ be a finite field with $q = p^n$ elements, where $n$ is a positive integer and $p$ is a prime. The trace function from $\mathbb{F}_q$ to $\mathbb{F}_p$ is defined as

$$\text{tr}_{q/p}(x) = x + x^p + \cdots + x^{p^{n-1}},$$

where $x \in \mathbb{F}_q$.

Let $m > 1$ be an integer and $\xi_m = e^{2\pi \sqrt{-1}/m}$. An additive character of $\mathbb{F}_q$ is defined to be the function $\chi_{a}(x) = \xi_m^{ax}$, where $a, x \in \mathbb{F}_q$. When $a$ runs around all the elements of $\mathbb{F}_q$, one can obtain all the additive characters of $\mathbb{F}_q$. Especially, the additive character $\chi_a$ of $\mathbb{F}_q$ is called the canonical additive character if $a = 1$. We write $\mathbb{F}_q^\ast = \mathbb{F}_q \setminus \{0\}$. Under the multiplication operation, $\mathbb{F}_q^\ast$ is a cyclic group of order $q - 1$ and a generator of it is said to be a primitive element of $\mathbb{F}_q$. Let $\omega$ be a primitive element of $\mathbb{F}_q$. For any integer $j$ with $0 \leq j \leq q - 2$, we define a multiplicative character of $\mathbb{F}_q$ to be the function $\varphi(j) = \xi_m^{j\omega^{-1}}$, where $0 \leq i \leq q - 2$.

If $j = \frac{q-1}{2}$, the multiplicative character $\varphi_{q+1}$ is termed the quadratic character of $\mathbb{F}_q$ and denoted by $\eta$ for simplicity.

Finite fields are special structures since they have both additive characters and multiplicative characters. Combining the canonical additive character $\chi_1$ and multiplicative character $\varphi$ of $\mathbb{F}_q$, the Gauss sum over $\mathbb{F}_q$ is defined as

$$G(\varphi) = \sum_{x \in \mathbb{F}_q^\ast} \varphi(x)\chi_1(x).$$

In most cases, it is very hard to determine the values of Gauss sums explicitly. Only in a few cases can Gauss sums be computed. Below, the explicit value of the Gauss sum over
\( \mathbb{F}_q \) is determined when the multiplicative character \( \varphi \) is the quadratic character of \( \mathbb{F}_q \).

**Lemma 3** [27, Theorem 5.15]: Let \( p \) be a prime and \( n \) a positive integer. Write \( q = p^n \). Suppose that \( \mathbb{F}_q \) is a finite field and \( \eta \) is the quadratic character of \( \mathbb{F}_q \). Then

\[
G(\eta) = (-1)^{n-1} \sqrt{p^n},
\]

where \( p^* = (-1)^{\frac{n-1}{2}} p \).

In the sequel, we need the following two lemmas.

**Lemma 4** [27, Theorem 5.33]: Assume that \( p \) is an odd prime. Let \( \chi_1 \) be the canonical additive character of \( \mathbb{F}_q \). Assume that \( f(x) = a_2 x^2 + a_1 x + a_0 \in \mathbb{F}_q[x] \) with \( a_2 \neq 0 \). Then, we have

\[
\sum_{x \in \mathbb{F}_q} \chi_1(f(x)) = \chi_1(a_0 - a_1^2(4a_2)^{-1}) \eta(a_2) G(\eta).
\]

**Lemma 5** [27, Theorem 5.38]: Let \( \chi_1 \) be the canonical additive character of \( \mathbb{F}_q \). Assume that \( f(x) \in \mathbb{F}_q[x] \) is of degree \( n > 0 \) with \( \gcd(n, q) = 1 \). Then

\[
\left| \sum_{x \in \mathbb{F}_q} \chi_1(f(x)) \right| \leq (n-1)q^{1/2}.
\]

### III. OPTIMAL CODEBOOKS

In this section, we present a new construction of optimal codebooks with respect to the Levenshtein bound and state the difference between our construction and the construction in [7].

**A. A NEW CONSTRUCTION**

Assume that \( p > 3 \) is a prime and \( n \) is a positive integer. Set \( q = p^n \) and \( \mathcal{E}_q \) to be the standard basis of the \( q \)-dimensional Hilbert space as follows:

\[
\begin{align*}
(1, 0, 0, \ldots, 0, 0), \\
(0, 1, 0, \ldots, 0, 0), \\
\vdots \\
(0, 0, 0, \ldots, 0, 1).
\end{align*}
\]

Let \( \chi_1 \) be the canonical additive character of \( \mathbb{F}_q \). For any \( a, b \in \mathbb{F}_q \), we can define a unit norm complex vector by

\[
d_{a,b} = \frac{1}{\sqrt{q}} \left( \chi_1((x+a)^3 + bx) \right)_{x \in \mathbb{F}_q}.
\]

After \( a, b \) walk along the finite field \( \mathbb{F}_q \), one can obtain a set of \( q^2 \) unit norm complex vectors as follows:

\[
\mathcal{D} = \{ d_{a,b} : a, b \in \mathbb{F}_q \}.
\]

**Theorem 6**: Let the symbols be the same as above. Then the set \( \mathcal{G} = \mathcal{D} \cup \mathcal{E}_q \) is a \((q^2 + q, q)\) codebook with \( I_{\max}(\mathcal{G}) = q^{-1/2} \) which meets the Levenshtein bound.

**Proof**: We divide our proof in three steps. First, we evaluate the inner-product correlation of \( \mathcal{G} \).

Case 1: If \( g_1 \neq g_2 \in \mathcal{E}_q \), it is easy to verify that

\[
|g_1 g_2^*| = 0.
\]

Case 2: If \( g_1 \in \mathcal{E}_q \) and \( g_2 \in \mathcal{D} \), it can be easily seen that

\[
|g_1 g_2^*| = \frac{1}{\sqrt{q}}.
\]

Case 3: If \( g_1 \neq g_2 \in \mathcal{D} \), then write \( g_1 = d_{a,b} \) and \( g_2 = d_{u,v} \), where \((a - u, b - v) \neq (0, 0)\). It follows from Eq. (1) that

\[
g_1 g_2^* = \frac{1}{q} \sum_{x \in \mathbb{F}_q} \chi_1 \left( (x+a)^3 + bx - (x+u)^3 - vx \right)
\]

\[
= \frac{1}{q} \sum_{x \in \mathbb{F}_q} \chi_1 \left( 3(a-u)x^2 + ex + d \right),
\]

where \( e = 3a^2 - 3u^2 + b - v \) and \( d = a^3 - u^3 \).

If \( a = u \), then by the fact \( b \neq v \), we derive that

\[
g_1 g_2^* = \frac{1}{q} \sum_{x \in \mathbb{F}_q} \chi_1 \left( (b-v)x \right)
\]

\[
= 0,
\]

where \( e = 3a^2 - 3u^2 + b - v \) and \( d = a^3 - u^3 \). According to the definition of additive and multiplicative characters over finite fields and Lemma 3, we obtain that \( |g_1 g_2^*| = \frac{1}{\sqrt{q}} \).

From the definition of the set \( \mathcal{G} \), it is easy to check that \( \mathcal{G} \) is a \((q^2 + q, q)\) codebook with \( I_{\max}(\mathcal{G}) = q^{-1/2} \). Clearly, \( I_{\max}(\mathcal{G}) \) coincides with the Levenshtein bound given in Lemma 2.

**Example 1**: Let \( p = 5 \) and \( n = 1 \). Then \( q = 5 \) and the set \( \mathcal{D} \) consists of the following 25 codewords of length 5, where every coordinate in a codeword stands for a power of \( \zeta_5 = e^{\frac{2\pi i}{5}} \):

\[
\begin{align*}
\sqrt[5]{5}d_0 &= (3, 4, 2, 1, 0), & \sqrt[5]{5}d_1 &= (0, 3, 0, 2, 0), \\
\sqrt[5]{5}d_2 &= (2, 2, 3, 3, 0), & \sqrt[5]{5}d_3 &= (4, 1, 1, 4, 0), \\
\sqrt[5]{5}d_4 &= (1, 0, 4, 0, 0), & \sqrt[5]{5}d_5 &= (2, 0, 4, 3, 1), \\
\sqrt[5]{5}d_6 &= (4, 4, 2, 4, 1), & \sqrt[5]{5}d_7 &= (1, 3, 0, 0, 1), \\
\sqrt[5]{5}d_8 &= (3, 2, 3, 1, 1), & \sqrt[5]{5}d_9 &= (0, 1, 1, 2, 1), \\
\sqrt[5]{5}d_{10} &= (4, 1, 0, 2, 3), & \sqrt[5]{5}d_{11} &= (1, 0, 3, 3, 3), \\
\sqrt[5]{5}d_{12} &= (3, 4, 1, 4, 3), & \sqrt[5]{5}d_{13} &= (0, 3, 4, 0, 3), \\
\sqrt[5]{5}d_{14} &= (2, 2, 2, 1, 3), & \sqrt[5]{5}d_{15} &= (0, 3, 1, 4, 2), \\
\sqrt[5]{5}d_{16} &= (2, 2, 4, 0, 2), & \sqrt[5]{5}d_{17} &= (4, 1, 2, 4, 2), \\
\sqrt[5]{5}d_{18} &= (1, 0, 0, 2, 2), & \sqrt[5]{5}d_{19} &= (3, 4, 3, 3, 2), \\
\sqrt[5]{5}d_{20} &= (1, 2, 3, 0, 4), & \sqrt[5]{5}d_{21} &= (3, 1, 1, 1, 4),
\end{align*}
\]
\[ \sqrt{5}d_{22} = (0, 0, 4, 2, 4), \quad \sqrt{5}d_{23} = (2, 4, 2, 3, 4), \]
\[ \sqrt{5}d_{24} = (4, 3, 0, 4, 4), \]

One can easily verify that the codebook \( G \) of Theorem 6 is a (30, 5) codebook with \( I_{\text{max}} = \frac{1}{\sqrt{5}} \). This is consistent with the conclusion of Theorem 6.

**B. COMPARISON**

In [7], the authors presented an optimal codebook \( C \) with parameters \((q^2 + q, q)\) and \( I_{\text{max}}(C) = q^{-1/2} \) as follows:

\[
C = \mathcal{E}_q \cup \left\{ \frac{1}{\sqrt{q}} \left( \chi_1 (x^2 + vx) \right)_{x \in \mathbb{F}_q} : u, v \in \mathbb{F}_q \right\}, \tag{2}
\]

where \( f(x) \) is a planar function from \( \mathbb{F}_q \) to \( \mathbb{F}_q \). Taking \( a = v = 0 \) in Eq. (2), one can obtain a vector \( e = \frac{1}{\sqrt{q}}(1, 1, \ldots, 1) \) of \( C \). If the vector \( e \) is contained in the codebook \( G \), then there exists a vector \( d \) in \( G \) such that \( |de^H| = 1 \). In fact, for any vector \( d \) of \( G \), we have

\[
|de^H| = \begin{cases} 
\frac{1}{\sqrt{q}}, & \text{if } d \in \mathcal{E}_q, \\
\frac{1}{q} \sum \chi_1 \left( (x + a)^3 + bx \right), & \text{if } d \in \mathcal{D}.
\end{cases}
\]

Thanks to Lemma 5, we have

\[
\left| \sum_{x \in \mathbb{F}_q} \chi_1 \left( (x + a)^3 + bx \right) \right| \leq 2\sqrt{q}.
\]

Thus, the vector \( e = \frac{1}{\sqrt{q}}(1, 1, \ldots, 1) \) is not contained in the codebook \( G \) which says that our construction of codebooks in Theorem 6 is not covered by the construction in [7, Theorem 4]. In a word, although the optimal codebooks generated by Theorem 6 have the same parameters as those in [7, Theorem 4], it is indeed a different construction.

**IV. ASYMPTOTICALLY OPTIMAL CODEBOOKS**

In this section, we consider the scalability issue regarding the length of the optimal codebooks obtained by Theorem 6. As a result, we present a new construction of asymptotically optimal codebooks with respect to the Levenshtein bound.

Suppose that \( p > 3 \) is a prime and \( m \) is a positive integer. Put \( q_1 = p^m \) and \( q = q_1^{kp} \), where \( k \) is a positive integer. Clearly, the finite field \( \mathbb{F}_{q_1} \) is a subfield of \( \mathbb{F}_q \). Assume that \( H \) is a subset of \( \mathbb{F}_{q_1} \) with \( h = \#H > 0 \), where \( \#H \) denotes the cardinality of the set \( H \). Denote by \( \mathcal{E}_{q-h} \) to be the standard basis of the \((q - h)\)-dimensional Hilbert space as follows:

\[
(1, 0, 0, \ldots, 0, 0),
(0, 1, 0, \ldots, 0, 0),
\vdots
(0, 0, 0, \ldots, 0, 1).
\]

Let \( \chi_1 \) be the canonical additive character of \( \mathbb{F}_q \). For any \( a, b \in \mathbb{F}_q \), define a unit norm complex vector by

\[
d_{a,b} = \frac{1}{\sqrt{q-h}} \left( \chi_1 \left( (x + a)^3 + bx \right) \right)_{x \in \mathbb{F}_q \setminus H},
\]

and then define a set

\[
\mathcal{D}(H) = \{ d_{a,b} : a, b \in \mathbb{F}_q \}.
\]

Thus one can obtain a codebook

\[
\mathcal{Q}(H) = \mathcal{D}(H) \cup \mathcal{E}_{q-h}. \tag{3}
\]

**Theorem 7:** Suppose that \( p > 3 \) is a prime and \( m, k \) are positive integers such that \( m \) is even. Let \( q_1 = p^m \) and \( q = q_1^{kp} \). Assume that \( h \) is an integer with \( 0 < h \leq q_1 \). Then the set \( \mathcal{Q}(H) \) defined by Eq. (3) is a \((q^2 + q, h)\) codebook with \( I_{\text{max}}(\mathcal{Q}(H)) = \frac{\sqrt{q-h}}{q-h} \).

**Proof:** Consider the codebook \( G \) constructed by Theorem 6. Clearly, \( G \) can be viewed as a \( q^2 + q \times q \) matrix. By deleting \( h \) columns of the codebook \( G \), one can obtain a new codebook \( \mathcal{Q}(H) \) with parameters \((q^2 + q, q, h)\). Due to \( I_{\text{max}}(G) = \frac{\sqrt{q}}{q} \), it is easy to check that \( I_{\text{max}}(\mathcal{Q}(H)) \leq \frac{\sqrt{q-h}}{q-h} \).

Below, we will show that \( I_{\text{max}}(\mathcal{Q}(H)) = \frac{\sqrt{q-h}}{q-h} \). According to Lemma 3, we get that \( G(H) = -\left((-1)^{p-1} \frac{k}{q} \right) \). Let \( \mu_1 \) be the canonical additive character of \( \mathbb{F}_{q_1} \). Put \( a, b, u, v \in \mathbb{F}_q \) such that \( a \neq u \in \mathbb{F}_{q_1} \) and \( \eta(3(a - u)) = \eta(-1)^{p-1} \frac{m}{q} \). Then

\[
d_{a,b}d_{u,v}^H = \begin{cases} 
\frac{1}{q-h} \sum_{x \in \mathbb{F}_q \setminus H} \chi_1 \left( (x + a)^3 + bx - (x + u)^3 - vx \right) \\
\frac{1}{q-h} \sum_{x \in \mathbb{F}_q \setminus H} \chi_1 \left( 3(a - u)x^2 + a^3 - u^3 - u \right) \\
\frac{1}{q-h} \chi_1(a^3 - u^3) \eta(3(a - u)G(H)) \\
\frac{1}{q-h} \sum_{x \in \mathbb{F}_q \setminus H} \chi_1 \left( 3(a - u)x^2 + a^3 - u^3 \right) \\
\frac{1}{q-h} \mu_1((a^3 - u^3) \tr_{q_1}(1)) \eta(3(a - u)G(H)) \\
\frac{1}{q-h} \sum_{x \in \mathbb{F}_q \setminus H} \mu_1 \left( (a^3 - u^3) \tr_{q_1}(1) + f \right) \\
\frac{1}{q-h} \mu_1((a^3 - u^3) \tr_{q_1}(1)) \sqrt{q} \\
\frac{1}{q-h} \sum_{x \in \mathbb{F}_q \setminus H} \mu_1 \left( 3(a - u)x^2 \tr_{q_1}(1) + f \right),
\end{cases}
\]

where \( f = (a^3 - u^3) \tr_{q_1}(1) \). Due to \( \tr_{q_1}(1) = kp = 0 \), we deduce that \( d_{a,b}d_{u,v}^H = -\frac{\sqrt{q}}{q-h} \). Consequently, \( I_{\text{max}}(\mathcal{Q}(H)) = \frac{\sqrt{q}}{q-h} \).
Notably, the codebook $Q(0)$ has a more flexible choice of the parameter $q$ than that in Theorem 7.

**Corollary 8:** Let $p > 3$ be a prime and $m > 0$ be an even integer. Put $q = p^m$. Then the set $Q(0)$ defined by Eq. (3) is a $(q^2 + q, q - 1)$ codebook with $I_{\text{max}}(Q(H)) = \frac{\sqrt{q+1}}{q-1}$. Additionally, $Q(0)$ asymptotically achieves the Levenshtein bound.

**Proof:** The proof is analogous to that in Theorem 7 and Remark 1. Thus, we omit the proof of the corollary. □

In Table 1, we list some explicit values of parameters of the codebooks in Corollary 8. Also, we compare $I_{\text{max}}(Q(H))$ with the Levenshtein bound $I_{L}$ in Table 1. It can be seen that $I_{\text{max}}(Q(H))$ is close to $I_{L}$ as $p$ and $m$ increase. This means that the codebooks defined in Corollary 8 are indeed asymptotically optimal with respect to the Levenshtein bound.

### TABLE 1. The parameters of the $(N,K)$ codebook in Corollary 8.

| $p$ | $m$ | $N$ | $K$ | $I_{\text{max}}$ | $I_{L}$ | $I_{\text{max}}$ |
|-----|-----|-----|-----|------------------|--------|------------------|
| 7   | 2   | 2450 | 48  | 0.1167           | 0.1471 | 1.1327           |
| 11  | 2   | 14762 | 120 | 0.1000           | 0.0921 | 1.0867           |
| 7   | 4   | 5767202 | 2400 | 0.2083        | 0.2042 | 1.0202           |
| 11  | 4   | 214373522 | 14640 | 0.8333        | 0.8265 | 1.0082           |
| 13  | 4   | 815759282 | 28560 | 0.5952        | 0.5917 | 1.0059           |
| 17  | 4   | 6975840962 | 83520 | 0.3472        | 0.3460 | 1.0035           |
| 19  | 4   | 16983693362 | 130321 | 0.2770        | 0.2770 | 1.0028           |
| 23  | 4   | 78311265122 | 279840 | 0.1894        | 0.1890 | 1.0019           |
| 29  | 4   | 500247120242 | 707280 | 0.1190        | 0.1189 | 1.0012           |
| 31  | 4   | 852891960962 | 923520 | 0.1042        | 0.1041 | 1.0010           |

According to the definition of the set $Q(H)$, it is easy to check that $Q(H)$ is a $(q^2 + q, q - h)$ codebook with $I_{\text{max}}(Q(H)) = \frac{\sqrt{q+h}}{q-h}$.

**Remark 1:** Note that the codebook $Q(H)$ produced by Theorem 7 is a complex codebook with parameters $(q^2 + q, q - h)$ and $I_{\text{max}}(Q(H)) = \frac{\sqrt{q+h}}{q-h}$. Since $q^2 + q > (q-h)^2$, the corresponding Levenshtein bound is

$$I_{L} = \sqrt{\frac{q^2 + q + 2qh - h^2 + h}{(q^2 + h)(q - h + 1)}}.$$  

It is easy to verify that $\lim_{q \rightarrow +\infty} \frac{I_{\text{max}}(Q(H))}{I_{L}} = 1$. Therefore, the codebook $Q(H)$ asymptotically meets the Levenshtein bound.

**Remark 2:** Some examples are used by Magma to verify the correctness of Theorem 7. The parameters $N$ and $K$ are very large even if for small $p$ and $m$. Hence, we omit these examples here.

In the following, By setting $H = 0$ in Eq. (3), we obtain a $(q^2 + q, q - 1)$ codebook with $I_{\text{max}}(Q(0)) = \frac{\sqrt{q+1}}{q-1}$.

### TABLE 2. The parameters of codebooks asymptotically meeting the Levenshtein bound.

| Parameters $(N, K)$                  | $I_{\text{max}}$ | References |
|-------------------------------------|------------------|------------|
| $(q^2 - 1, q - 1)$, where $q$ is a prime power | $\frac{\sqrt{q}}{q-1}$ | [32]         |
| $(q^2 - q - 1, q - 2)$, where $q$ is a prime power | $\frac{\sqrt{q}}{q-2}$ | [18]         |
| $(q^2 + q, q - 1)$, where $q$ is a prime power | $\frac{1}{\sqrt{q}}$ | [20], [40]   |
| $(2^m + 2^m, q)$, where $m > 0$ is an odd integer | $\frac{1}{\sqrt{2^m - 1}}$ | [36]         |
| $(p^km^p, p^km^p, p^km^p - h)$, where $p > 3$ is a prime, $k, m, h$ are positive integers such that $m$ is even and $h < p^m$ | $\frac{\sqrt{p^m-n^p+h}}{p^m-n^p-h}$ | Theorem IV 1 |
| $(q^2 + q, q - 1)$, where $q = p^m$, $p > 3$ is a prime and $m > 0$ is an even integer | $\frac{\sqrt{q+1}}{q-1}$ | Corollary IV 2 |

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