LBM modeling of wave-current interaction with a square cylinder based on Shallow Water Equations

J J Li¹, T Wei¹, L C Qiu¹,² and Y Han¹
¹College of Water Resources & Civil Engineering, China Agricultural University, Beijing 100083, China
E-mail: qiuliuchao@cau.edu.cn

Abstract. The interaction between a square cylinder with wave-current is numerically simulated using the Lattice Boltzmann Method (LBM) based on Shallow Water Equations. The velocity and water level around the pile are presented and discussed. The feasibility of the method is verified by comparing the free surface with the literature, and the use range of the Lattice Boltzmann Method is broadened. The simulation process mainly includes three different scenarios, namely wave, current and wave-current. By simulating the three operating conditions, it is analyzed that the interaction of wave-current is not a linear wave and current accumulation, but a complex coupling behaviour. The paper shows that the wave-current interaction with a square cylinder leads to complex phenomena. As a consequence, the wave surface elevation cannot be obtained by a linear overlapping of wave and current profiles.

1. Introduction

In the study of coastal and offshore engineering, the interaction of wave structures has been studied in depth for several decades. The hydrodynamic elements of offshore areas mainly include: waves, tides and river runoff. A variety of hydrodynamic factors co-exist in the offshore area, and various dynamic factors interact with each other to form an extremely complex dynamic environment. For example, in the shallow sea area, waves have an impact on the coastal buildings. However, in some particularly trendy areas, the trend may also be the dominant factor causing damage to coastal structures. Therefore, the interaction of waves and currents on buildings is not a mere wave or tidal effect.

Although several laboratory studies have analysed the interaction between waves and slender cylinders [1,2], very few investigations have been focused on square cylinders so far. And in recent decades, numerical simulation is a more mainstream research method. Jin and Meng [3] developed a two-dimensional potential flow model to calculate the wave load of the superstructure of the bridge completely submerged in water, and verified the results through large-scale laboratory measurements. Considering the linear elastic formula of structure and the nonlinear velocity potential formula of wave current, Sudharsan et al [4] solved structure behaviour and the non-linear fluid in the time domain. For overcome the difficulties of the lattice gas model, the lattice Boltzmann model is evolved from the lattice gas model [5,6]. By using the the shallow water equations, Zhou [7] developed the lattice Boltzmann model (LABSWE). It solves equations such as bed slope, bed friction and other source terms. The model was verified to solve the problem of stable and unstable flow. Ding [8] used Reynolds’ average Navier-Stokes (N-S) equation to study the wave force and the flow field around cylinders in the periodic wave field with numerical model.

In this paper, we performed numerical simulations by using LBM for the shallow water equations
[7] to analyse the interaction between wave-current and a square cylinder.

2. Methodology
In this section, LBM is constructed for solving the 2D shallow water equations [9,10]. Deriving from the incompressible N-S equations, a variety of hydrodynamic process can be represented by the shallow water equation. The momentum equations and depth-averaged continuity can be written as flows:

\[
\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} = 0, \tag{1}
\]

\[
\frac{\partial (hu)}{\partial t} + \frac{\partial (huu)}{\partial x} = -g \frac{\partial (h^2)}{\partial x} + \frac{\partial \left( hv \frac{\partial u}{\partial x} \right)}{\partial x} + F_i, \tag{2}
\]

in which the acceleration of gravity to 9.81 m/s\(^2\); \(t\) is time; \(u\) is flow velocity; \(h\) is water depth; \(F_i\) is force term in \(i\) direction, which is given by the following equation:

\[
F_i = -gh \frac{\tau_{wi}}{\rho} + \frac{\tau_{bi}}{\rho}, \tag{3}
\]

in which \(z_b\) is the bed elevation; \(\tau_{wi}\) is the wind shear stress, defined as \(\tau_{wi} = \rho_u C_u u_{wi}^2\), where \(C_u\) is the drag coefficient, \(\rho_u\) is the air density and \(u_{wi}\) is the wind velocity in the \(i\) direction; \(\rho\) of water is 1000 kg/m\(^3\); \(\tau_{bi}\) is the bed shear stress, calculated by \(\tau_{bi} = \rho C_b u_{wi} \sqrt{u_i u_j}\), where \(C_b\) is the bed friction coefficient calculated by \(C_b = \frac{g n_b^2}{h^{(0)}}\) where \(n_b\) is the Manning roughness coefficient.

LBM has two main steps: a streaming step and a collision step. The particles move to new sites according to the previous positions and the velocities. Streaming look like this:

\[
f'_a(x + e_a \Delta t + \Delta t) = f_a(x,t) + \frac{\Delta t}{N_a e^2} e_a F_i(x,t), \tag{4}
\]

where \(f_a\) is the distribution function of particles; \(f'_a\) is the value of \(f_a\) after the streaming; \(e^2 = \Delta x/\Delta t\); \(\Delta t\) is the time step; \(\Delta x\) is the lattice size; \(F_i\) is the component of the force in \(i\) direction; \(e_a\) is the velocity vector of a particle in the \(a\) link and \(N_a\) is a constant. \(N_a\) can be calculated by the lattice pattern as follows:

\[
N_a = \frac{1}{e^2} \sum_a e_{ai} e_{ai}. \tag{5}
\]

In the collision step, the scattering rule is used to calculate the velocity change after particle collision, which can be calculated by

\[
f_a(x,t) = f'_a(x,t) - \Omega_a \left[ f'_a(x,t) - f^{eq}_a(x,t) \right], \tag{6}
\]

where \(f^{eq}_a\) is the local equilibrium distribution function and \(\Omega_a\) the collision operator which controls the speed of change in \(f_a\) from collision.

Usually, with the BGK(BhatnagarGrossKrook) approximation [11] under which the collision operator \(\Omega_a\) is replaced with the single relaxation time \(\tau\), these two steps can be combined into the following lattice Boltzmann equation:

\[
f_a(x + e_a \Delta t + \Delta t) - f_a(x,t) = -\frac{1}{\tau} \left( f_a - f^{eq}_a \right) + \frac{\Delta t}{N_a e^2} e_a F_i(x,t), \tag{7}
\]
which is the common form of LBM. The distribution function of the water depth \( h \), the physical

variables and the velocity \( u_i \), are defined as follows:

\[
\begin{align*}
  h &= \sum_{a} f_a, \\
  u_i &= \frac{1}{h} \sum_{a} e_{ai} f_a.
\end{align*}
\]  

(8)

There are two main lattice patterns (hexagonal lattice and square lattice) to the lattice Boltzmann

model, and the 9-speed square lattice shown is frequently used as figure 1. It usually gives more

accurate results and the boundary conditions can be provided by an easy way [12]. In the present study,

the 9-speed square lattice is used. The velocity vector of particles is defined as follows:

\[
\begin{align*}
  e_{\alpha} &= \begin{cases} 
  0, & \alpha = 0 \\
  i e \cos \left( \frac{\pi (\alpha - 1)}{4} \right) + j e \sin \left( \frac{\pi (\alpha - 1)}{4} \right), & \alpha = 1, 3, 5, 7 \\
  i \sqrt{2} e \cos \left( \frac{\pi (\alpha - 1)}{4} \right) + j \sqrt{2} e \sin \left( \frac{\pi (\alpha - 1)}{4} \right), & \alpha = 2, 4, 6, 8
\end{cases}
\end{align*}
\]  

(9)

Figure 1. 9-Speed square lattice.

With the square lattice it can be shown \( N_s = 6 \).

The equations (1) and (2) is solved by the equation (7) which need a defined and suitable local

equilibrium function \( f_{eq} \). In the lattice Boltzmann dynamics, \( f_{eq} \) is generally expressed as a power

series in the bulk velocity:

\[
\begin{align*}
  f_{eq} &= A + B e_{ai} u_i + C e_{aj} u_j u_j + D u_i u_j \delta_{ij},
\end{align*}
\]  

(10)

where the Kronecker delta function \( \delta_{ij} \) is

\[
\delta_{ij} = \begin{cases} 
  0, & \text{if } i \neq j \\
  1, & \text{if } i = j
\end{cases}
\]  

(11)

The local equilibrium function (10) must satisfy the following conditions:

\[
\begin{align*}
  \sum_{a} f_{eq}^a &= h, \\
  \sum_{a} e_{ai} f_{eq}^a &= hu_i,
\end{align*}
\]  

(12)  (13)
\begin{equation}
\sum_{\alpha} e_{\alpha} e_{ij} f_{\alpha}^{eq} = \frac{1}{2} gh^2 \delta_{ij} + hu_i u_j,
\end{equation}

so that the solution of the lattice Boltzmann equation (7) approaches the solution of the 2D shallow water equations (1) and (2). With the conditions (12)-(14) the coefficients A, B, C and D are determined in a similar way to that of the lattice Boltzmann equation for the Navier–Stokes equations detailed by Rothman and Zaleski [13], hence it is not shown here. After the coefficients are defined, the local equilibrium function assumes the following form:

\begin{align}
    f_{\alpha}^{eq} &= \begin{cases} 
        h - \frac{5gh^2}{6e^2} - \frac{2h^2}{3e^2} u_i u_j \delta_{ij} & \alpha = 0 \\
        \frac{gh^2}{6e^2} + \frac{h}{3e^2} e_{ia} u_i + \frac{h}{2e^4} e_{ij} e_{ai} u_j - \frac{h}{6e^2} u_i u_j \delta_{ij} & \alpha = 1, 3, 5, 7 \\
        \frac{gh^2}{24e^2} + \frac{h}{12e^2} e_{ia} u_i + \frac{h}{8e^4} e_{ij} e_{ai} u_j - \frac{h}{24e^2} u_i u_j \delta_{ij} & \alpha = 2, 4, 6, 8
    \end{cases}
\end{align}

By applying the Chapman–Engelkog procedure, it can be shown that the solution of the shallow water equations (1) and (2) can be solved by the solution of the lattice Boltzmann equation (7) with the equilibrium function (15), which is given by the following equation:

\begin{equation}
    F_i (x, t) = -gh \frac{\partial e}{\partial x_i} + \frac{\tau_{hi}}{\rho}, \quad v = \frac{e^2 \Delta t}{6(2\tau - 1)}
\end{equation}

It must be noted that the lattice Boltzmann equation (7) via equation (16) can add more source terms such as Coriolis force and wind shear stress easily.

The boundary conditions in the solution of the shallow water equations are different. These conditions must be correctly transformed to suitable boundary conditions for the LABSWE. This is briefly described as follows:

- **Outflow**, inflow boundary conditions: if the depth and the velocity are known, using the method described by Zou and He [14] to describe the unknown distribution function \( f_{\alpha} \). For example, if the velocity at the inflow boundary is known, the unknown \( f_1 \), \( f_2 \) and \( f_8 \) (see figure 1) after streaming can be determined as

\begin{align}
    f_1 &= f_5 + \frac{2hu}{3e}, \\
    f_2 &= \frac{hu}{6e} + f_6 + \frac{f_1 - f_3}{2}, \\
    f_8 &= \frac{hu}{6e} + f_4 + \frac{f_3 - f_7}{2},
\end{align}

- The velocity and depth to 0 at the either outflow or inflow boundary can be achieved by setting the gradient of the distribution function normal as 0 at the boundary.
- There are two main forms of the solid boundary, which are slip or no-slip boundary conditions respectively. The implementation of no-slip conditions is through the use of the standard bounce-back scheme, and slip conditions can be achieved by using a zero gradient of the distribution function normal.

3. **Validation of the model**
In this section, the LBM is used to study the interaction between the wave and a vertical rectangular pile. And through comparing numerical predictions with literature results, the accuracy is demonstrated.

3.1. Introduction of the problem

In this section, the computational domain (30 m×10 m) is deployed a vertical pile (1 m×1 m) in the horizontal plane as figure 2 [15]. The water depth in the computational domain is 1 m. The pile is placed on the left side of the 10m and located in the middle y-direction. The left border as the inflow boundary, and produce a linear wave train with a wave period of 4s and a wave height of 0.05 m. Such hydrodynamic conditions lead to non-breaking waves. The upper and lower boundary is set as a non-slip boundary condition. The whole domain is set to a square grid of 600*200.

3.2. Free surface profile within one wave period

When the upper boundary of the problem is set to solid, the average flow is usually two-dimensional. This type of problem has been deeply studied by predecessors through numerical simulation and experimental methods [16]. However, when there is a free surface, the problem becomes more complex. Defining \( n=t/T \) as a number of cycles. Figure 3 shows the free surface profile of the Central Linec of the basin in a wave period. The existence of the pile cause the flow separation, and the effect of waves causes pressure difference between the upstream and downstream of the column. This also makes the sudden changes of the free surface at the pile upstream and downstream.

![Figure 2. Model size measures in m.](image-url)
3.3. Velocity on the horizontal plane

Figure 4 shows the velocity field within one wave period around the pile. Vortices produced at the four corners of the pile, and these vortices are restricted in close proximity to the structure. However, when vertical piles are exposed to the uniform flow, the vortices generated at the four corners are removed from the structure, and this phenomenon is a regular vortex shedding with a certain frequency [17]. In this study the detachment of eddy currents is not be considered.
Figures 5 and 6 show the flow rate in the x, y direction of the flow chart, through the x-direction chart can be clearly observed the impact of the wave. It is also consistent that the direction of the water flow at the undulations is changing. The above results show that the numerical simulation results are consistent with the real situation, and the LBM is accurate.
4. Results and discussions

4.1. Interaction between flow and a square cylinder
This section establishes the same model shown in figure 2 and sets up a condition driven only by water flow. The horizontal velocity of the initial flow of water is set at 0.5 m/s, the vertical velocity at 0. The water flows from the left upstream to the right downstream outlet.

Figures 7 and 8 shows the surface velocity in x and y directions only driven by flow. On the downstream side of the square column, its flow rate is also slow. When the water encounters the middle square cylinder during the movement, the water flows around both sides of the square cylinder. Due to the viscosity of the liquid, the water particles at the boundary surface of the column will adhere to the solid surface and the flow velocity is 0. Due to the fact that the liquid particle is stagnant, the subsequent liquid particles change the direction of flow in front of the square cylinder. In this way, the mainstream is separated from the square cylinder, and the liquid immediately fills the left area of the mainstream and creates a vortex after the square cylinder.

![Figure 7. Velocity (m/s) through the y-direction (number of lattices).](image1)

![Figure 8. Velocity (m/s) through the x-direction (number of lattices).](image2)

4.2. Interaction between wave-current and a square cylinder
This section establishes the same model shown in figure 2 and it sets up a condition driven by wave-current. Figures 9 and 10 shows the surface velocity in x and y directions only driven by wave-current.
4.3. Comparison of different conditions
At T = 7, the water level elevation under different conditions is shown in figure 11.

In incoming flow generate flow around the square cylinder. Compared with the flow-only condition, the addition of wave increases the velocity of vortex around the square cylinder, but reduces the influence range of the vortex. In addition, the wave-current interaction cannot be obtained by a linear superposition.

Figure 9. Velocity (m/s) through the x-direction (number of lattices).

Figure 10. Velocity (m/s) through the y-direction (number of lattices).

Figure 11. Free surface profile at the centerline in different conditions.

In the case of flow-only driving, the depth and velocity of flow change due to the presence of the center square. The depth of water at the downstream side of the square column is significantly lower than that at the upstream side. Due to the constant velocity of the injected water, however, the water
surface is generally flat. Water flow velocity has a role in promoting the wave when wave and flow are considered together. In addition, the wavelength becomes longer when compared to the condition that take into account only the wave. The comparison of wave height in three working conditions shows that the height of wave crest increases in the wave-current scenarios. At the same time, the position of the wave crest is not at the same location if the two different scenarios (wave-current and wave) are compared, and the flow velocity does not affect the periodicity of the wave.

5. Conclusion
In this paper, the accuracy of LBM is assessed. The interaction between water flow and buildings under different working conditions has been investigated by analysing the interaction between a square cylinder with wave, current, and the simultaneous combination of wave-current. The wave-current interaction with a square cylinder leads to complex phenomena. As a consequence, the wave surface elevation cannot be obtained by a linear overlapping of wave and current profiles.

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