Invisibility problem in acoustics, electromagnetism and heat transfer. Inverse design method

G Alekseev$^{1,2}$, A Tokhtina$^2$ and O Soboleva$^1$

$^1$Institute of Applied Mathematics, Vladivostok, Russia
$^2$Far Eastern Federal University, Vladivostok, Russia

E-mail: alekseev@iam.dvo.ru

Abstract. Two approaches (direct design and inverse design methods) for solving problems of designing devices providing invisibility of material bodies of detection using different physical fields – electromagnetic, acoustic and static are discussed. The second method is applied for solving problems of designing cloaking devices for the 3D stationary thermal scattering model. Based on this method the design problems under study are reduced to respective control problems. The material parameters (radial and tangential heat conductivities) of the inhomogeneous anisotropic medium filling the thermal cloak and the density of auxiliary heat sources play the role of controls. A unique solvability of direct thermal scattering problem in the Sobolev space is proved and the new estimates of solutions are established. Using these results, the solvability of control problem is proved and the optimality system is derived. Based on analysis of optimality system, the stability estimates of optimal solutions are established and numerical algorithms for solving particular thermal cloaking problem are proposed.

1. Introduction

In recent years significant research has focused on design of invisibility cloaking devices. Beginning with pioneering papers [1, 2, 3, 4] the large number of papers was devoted to developing different schemes of cloaking material objects. These schemes include metamaterial cloaking based on transformation optics (TO) proposed by Pendry et al. [2], conformal method proposed by Leonhardt [3], plasmonic cloaking method based on scattering cancellation proposed by Alú and Engheta [4], mantle cloaking [5], etc.

The first works in this field were focused on the electromagnetic cloaking. Then the main results of the electromagnetic cloaking theory were expanded to an acoustic cloaking [6] and to cloaking static (magnetic, electric and thermal) fields (see, e.g., [7, 8, 9]). Papers [10, 11] are devoted to studying invisibility problems in X-ray tomography.

Development of the above-mentioned approaches have opened up the opportunities for creation the invisibility cloaking design strategies. They obtained the name of direct design strategies as they were based on solving the forward electromagnetic (acoustic or static) problems. It should be noted that the invisibility devices (hereafter, cloaks) designed on the basis of direct strategies possess serious drawbacks. The main one is the difficulty of their technical realization. For example, the design of the TO-based cloaks involves extreme values of constitutive parameters and spatially varying distributions of the permittivity and permeability tensors which are very difficult to implement [12].
That is why the another cloak design strategy began develop recently. It obtained the name of inverse design as it is related with solving inverse electromagnetic (acoustic or static) problems (see [12, 13]). The optimization method forms the core of the inverse design methodology. This enables us to solve some substantial limitations of previous cloaking solutions. A growing number of papers is devoted to applying the inverse design methodology in various cloaking problems. Among them we mention [14, 15, 16] where numerical optimization algorithms are applied for finding the unknown material parameters of TO-based cloak and papers [17, 18, 19] devoted to theoretical analysis of cloaking problems using the optimization approach.

Optimization method is applied and in this paper for solving inverse problems for the 3D stationary thermal scattering model. These problems arise when designing thermal cloaking devices. We prove the solvability of direct and control problems for the thermal scattering model under study and derive the optimality system which describes necessary conditions of extremum. Based on it’s analysis we establish the stability estimates of optimal solutions and propose a numerical algorithm for solving the particular control problem.

2. Statement of direct thermal scattering problem

In this Section we formulate and study direct problem of thermal scattering in bounded domain $D$. As in brief note [19] we begin with introducing externally applied thermal field $T^e$. Considering for the simplicity the case of rectangular parallelepiped $D = \{(x, y, z) : |x| < x_0, |y| < y_0, |z| < z_0\}$ (see Figure 1), we assume that external field $T^e$ in $D$ is created by two horizontal boundaries $\Gamma_1 : z = -z_0$ and $\Gamma_2 : z = z_0$ which are kept at temperatures $T_1$ and $T_2 < T_1$, respectively, while the lateral boundaries are thermally insulated. Then external field $T^e$ satisfies equation $k_0 \Delta T^e = 0$ in $D$ and the following boundary conditions: $T^e|_{z = -z_0} = T_1$, $T^e|_{z = z_0} = T_2$, $\partial T^e/\partial x|_{x = \pm x_0} = 0$, $\partial T^e/\partial y|_{y = \pm y_0} = 0$. Here $k_0$ is a constant thermal conductivity of homogeneous isotropic medium (background) filling $D$.

![Figure 1](image1.png)  ![Figure 2](image2.png)

**Figure 1.** The geometry of the problem without cloak.

**Figure 2.** The geometry of the problem for the case when a cloak is placed in $D$.

We assume further that a material shell $(\Omega, \kappa)$ having the form of the spherical layer $a < r < b$ in spherical coordinates $(r, \theta, \phi)$ which is filled by anisotropic medium characterized by heat conductivity tensor $\kappa$ is placed into $D$ (see Figure 2). Placing the shell $(\Omega, \kappa)$ into $D$ leads to appearing the scattered thermal response $T^{sc}$ in $D \setminus \Omega$ which can be determined by solving the direct thermal scattering problem. In order to formulate the latter problem we denote by $\Omega_i$ (or $\Omega_e$) the interior (or exterior) of $\Omega$ in $D$. Then the mentioned thermal scattering problem
consists of finding a triple of functions: $T_i$ in the interior $\Omega_i$ of $\Omega$, $T_0$ in $\Omega$ and $T_e$ in the exterior $\Omega_e$ of $\Omega$ satisfying equations

$$k_0 \Delta T_i = 0 \text{ in } \Omega_i, \quad \text{div}(\kappa \text{grad} T_0) = 0 \text{ in } \Omega, \quad k_0 \Delta T_e = -f \text{ in } \Omega_e,$$

boundary conditions

$$T_e|_{z=-z_0} = T_1, \quad T_e|_{z=z_0} = T_2, \quad \partial T_e/\partial x|_{x=\pm x_0} = 0, \quad \partial T_e/\partial y|_{y=\pm y_0} = 0$$

and continuity conditions on internal and external components of boundary of $\Omega$. They have the form [13, ch.4]

$$T_i = T_0, \quad k_0 \partial T_i/\partial n = (\kappa \nabla T) \cdot \mathbf{n} \text{ at } r = a, \quad T_e = T_0, \quad k_0 \partial T_e/\partial n = (\kappa \nabla T) \cdot \mathbf{n} \text{ at } r = b.$$

Here $\mathbf{n}$ is the outward unit normal to the boundary of $\Omega$.

Let us define a weak solution of problem (1)–(3). Preliminarily, we introduce a number of function spaces to be used in the subsequent analysis. We will use the space $H^1(\Omega)$ where $\Omega$ is one of the domains $\Omega_i, \Omega, \Omega_e, \Omega_2$ and spaces $L^\infty(\Omega), H^s(\Omega), L^2(\Omega)$ where $Q \subset D$ is an arbitrary subset. The scalar products and norms in $\| \cdot \|_r$ and continuity conditions on internal and external components of boundary of $\Omega$. They have the form [13, ch.4]

(i) tensor $\kappa$ is diagonal in spherical coordinates $(r, \theta, \varphi)$ and its diagonal components (radial, polar and azimuthal conductivities) $k_r, k_\theta$ and $k_\varphi$ do not depend on $\varphi$ and satisfy

$$k_r \in L^\infty_k(\Omega), \quad k_\theta \in L^\infty_k(\Omega), \quad k_\varphi = \text{const} > 0, \quad k_\varphi = \text{const} > 0, \quad k_\varphi = k_\theta.$$

This assumption enables us to simplify substantially the study of the direct scattering problem (1)–(3) and to look for its solution in the subspace $X$ of the space $H^1(D)$ consisting of functions $T := (T_i, T_0, T_e)$ independent of $\varphi$. Using the general formula

$$\text{grad} S = \frac{\partial S}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial S}{\partial \theta} \mathbf{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial S}{\partial \varphi} \mathbf{e}_\varphi$$

for grad $S$ in spherical coordinates where $\mathbf{e}_r, \mathbf{e}_\theta$ and $\mathbf{e}_\varphi$ are unit basis vectors and taking into account the independence of elements of $X$ on angle $\varphi$ we define the norm in $X$ by

$$\|S\|_X^2 := \|S\|_D^2 + \|\nabla S\|_{H^1(\Omega)}^2 + \left\| \frac{\partial S}{\partial r} \right\|_{\Omega}^2 + \left\| \frac{1}{r} \frac{\partial S}{\partial \theta} \right\|_{\Omega}^2,$$

We introduce once more assumption concerning boundary functions $T_1$ and $T_2$: (ii) $T_1 \in H^{1/2}(\Gamma_1), T_2 \in H^{1/2}(\Gamma_2)$ and there exists a function $T^0 \in X$ such that

$$T^0|_{\Gamma_1} = T_1, \quad T^0|_{\Gamma_2} = T_2, \quad \|T^0\|_X \leq C_T := C_D \left( \|T_1\|_{1/2, \Gamma_1} + \|T_2\|_{1/2, \Gamma_2} \right).$$

Here $C_D$ is a constant which depends only on $D$.

Now we are able to define a weak solution of problem (1)–(3). Let $X_0 := S \in X : S|_{\Gamma_1} = S|_{\Gamma_2} = 0$. We multiply every equation in (1) by $S \in X_0$, integrate by part
and add the results obtained. Using Green formulae, assumption (i) and boundary conditions (2), (3), we arrive at the following relations for finding a triple $T = (T_1, T_0, T_e) \in X$:

$$a(k_0, k; T, S) := a_0(k_0; T, S) + a_1(k_r; T, S) + a_2(k_\theta; T, S) = \langle F, S \rangle \quad \forall S \in X_0, \quad T|_{\Gamma_1} = T_1, \quad T|_{\Gamma_2} = T_2.$$  

(5)

Here, $k := (k_r, k_\theta)$, $a_0(k_0; ; \cdot)$, $a_1(k_r; ; \cdot)$, $a_2(k_\theta; ; \cdot)$ or $F$ are bilinear or linear forms defined by

$$a_0(k_0; T, S) := k_0 \int_{\Omega, \Omega_0} \nabla T \cdot \nabla S \, dx,$$

$$a_1(k_r; T, S) := \int_{\Omega} \left( k_r \frac{\partial T}{\partial r} \frac{\partial S}{\partial r} \right) \, dx,$$

$$a_2(k_\theta; T, S) := \int_{\Omega} \left( k_\theta \frac{\partial T}{\partial r} \frac{\partial S}{\partial \theta} \right) \, dx,$$

$$\langle F, S \rangle = \langle f, S \rangle_{\Omega_e}.$$  

We call a triple $T := (T_1, T_0, T_e) \in X$ satisfying (5) a weak solution of problem (1)–(3) a weak solution of problem (1)–(3).

Using Hölder inequality, formula (4) for norm $\| \cdot \|_X$ and Poincaré inequality $\| \nabla T \|^2_2 \geq \delta \| T \|^2_X$ for $T \in X_0$ where the constant $\delta > 0$ depends on $D$ one can easily derive the following estimates:

$$|a(k_0, k; T, S)| \leq (k_0 + \| k_r \|_{L^\infty(\Omega)} + \| k_\theta \|_{L^\infty(\Omega)}) \| T \|_X \| S \|_X, \quad \langle F, S \rangle \leq \| f \|_{\Omega_e} \| S \|_X \quad \forall T, S \in X,$$

$$a(k_0, k; T, T) \geq k_0^0 \| \nabla T \|^2_D \geq \delta k^0 \| T \|^2_X \quad \forall T \in X_0, \quad k^0 = \min(k_0, k_r^0, k_\theta^0).$$  

These estimates mean that the linear form $F$ is continuous on $X$ while the bilinear form $a(k_0, k; \cdot, \cdot)$ is continuous on $X$ and is coercive on $X_0$. Based on the Lax-Milgram theorem one can prove then that under conditions (i), (ii) a weak solution $T := (T_1, T_0, T_e) \in X$ of problem (1)–(3) exists and is unique. More precisely, the following theorem holds.

**Theorem 2.1.** Let, under assumptions (i), (ii), $K_1 \subset L^2_{k_\theta}(\Omega)$, $K_2 \subset L^2_{k_r}(\Omega)$ are nonempty bounded sets. Then for any pair $(k_r, k_\theta) \in K_1 \times K_2$ and for any $f \in L^2(\Omega_e)$ direct problem (1)–(3) has a unique weak solution $T = (T_1, T_0, T_e) \in X$ which satisfies estimate

$$\| T \|_X \leq C_0 C_1 (C_T + \| f \|_{\Omega_e}), \quad C_0 = (\delta k^0)^{-1}.$$

Here, constant $C_1$ depends on $K_1$ and $K_2$ but is independent of $k_r$, $k_\theta$, $k_e$.

### 3. Statement of inverse problem. Using optimization method. Main results

We recall that our purpose is analysis of inverse problems arising when developing technologies of designing thermal cloaking devices. These problems consist of finding conductivities $k_r$, $k_\theta$ and the density $f$ of auxiliary heat sources placed in the domain $\Omega_e$ from the respective cloaking condition. For solving these problems we apply optimization method. This approach is based on introducing the cost functional to be minimized which adequately corresponds to the inverse problem of designing the device for approximate cloaking. As a result, the initial cloaking problem is reduced to study of the respective control problem using the well known methods of solving extremum problems. As a cost functional we choose one of the following:

$$I_1(T) = \| T - T_d \|^2_Q = \int_Q (T - T_d)^2 \, dx, \quad I_2(T) = \| T - T_d \|^2_{1, Q}.$$  

Here, function $T_d \in L^2(Q)$ (or $T_d \in H^1(Q)$) models the thermal field measured in some subset $Q \subset D \setminus \Omega$. As controls we choose variable conductivity coefficients $k_r(\cdot)$ and $k_\theta(\cdot)$ of the anisotropic inhomogeneous medium filling the shell $\Omega$ and the heat source density $f$ in $\Omega_e$. By definition, parameters $k_r$ and $k_\theta$ have the sense of passive controls while $f$ plays the role of active control. So our control problem corresponds to using mixed cloaking strategy (see [13]).
In the particular case when \( Q \subset \Omega_e \) and \( T_d = T^e \) functional \( I_1(T) \) has the meaning of the \( L^2(Q) \)-norm of the external scattered thermal response \( T^{sc} \) in the set \( Q \). Therefore parameter \( k^{opt} = (k_r^{opt}, k_\theta^{opt}) \) owing to which the minimum of functional \( I_1(T) \) is achieved corresponds to the thermal cloaking problem. In the general case when \( T_d = T^e + T_1^{sc} \) where \( T_1^{sc} \) is a thermal scattered response of a certain object, the minimizer \( k^{opt} \) of the cost functional \( I_1 \) describes an approximate solution of the respective thermal illusion problem [13, ch. 4].

Let \( K = K_1 \times K_2 \times K_3 \), \( u = (k_r, k_\theta, f) \). Define the operator \( G := (G_0, G_1, G_2) : X \times K \to X_0^* \times H^{1/2}(\Gamma_1) \times H^{1/2}(\Gamma_2) \) where \( (G_0(T, u), S) = (\alpha(T, \alpha, f), S) \) for all \( S \in X_0 \), \( G_1 T = T|_{\Gamma_1} - T_1 \), \( G_2 T = T|_{\Gamma_2} - T_2 \) and rewrite weak formulation (5) of problem (1)-(3) as operator equation \( G(T, u) = 0 \). It is assumed that controls \( k_r, k_\theta \) and \( f \) can change in sets \( K_1, K_2, K_3 \) and the following condition holds:

\[(j) \ K_1 \subset H_{K_0}^{1,2}(\Omega), \ K_2 \subset H_{K_0}^{1,2}(\Omega), \ K_3 \subset L^2(\Omega_e) \text{ are nonempty convex closed sets, where } s > 3/2, k_r^0 = \text{const} > 0, k_\theta^0 = \text{const} > 0; \alpha_0 > 0. \]

We consider the following control problem:

\[
J(T, u) := \frac{\alpha_0}{2} I(T) + \frac{\alpha_1}{2} \|k_r\|_{s, \Omega}^2 + \frac{\alpha_2}{2} \|k_\theta\|_{s, \Omega}^2 + \frac{\alpha_3}{2} \|f\|_{\Omega_e}^2 \to \inf, \quad G(T, u) = 0. \quad (6)
\]

Here, \( I(T) \) is a cost functional, \( \alpha_0, \alpha_1, \alpha_2 \text{ and } \alpha_3 \) are nonnegative parameters which serve to regulate the relative importance of each of the terms in (6). Denote by \( Z_{ad} = \{ (T, u) \in X \times K : G(T, u) = 0, I(T) < \infty \} \) the set of admissible pairs for problem (6).

We apply the mathematical procedure developed in [20] for studying control problems arising when optimization method is applied when solving inverse problems for linear diffusion-convection models. Based on this procedure, one can prove the following results.

**Theorem 3.1** Let, under assumptions (i), (ii), (j), \( I : X \to R \) be a weakly lower semicontinuous functional and \( Z_{ad} \) be a nonempty set. Let \( \alpha_1 \geq 0, \alpha_2 \geq 0, \alpha_3 \geq 0 \) and \( K_1, K_2, K_3 \) be bounded sets or \( \alpha_1 > 0, \alpha_2 > 0, \alpha_3 > 0 \) and functional \( I(T) \) is bounded below. Then, control problem (6) has at least one solution \( (T, u) \in (k_r, k_\theta, f) \in X \times K_1 \times K_2 \times K_3 \).

**Theorem 3.2** Let, under assumptions (i), (ii), (j), \( \alpha_1 > 0, \alpha_2 > 0, \alpha_3 > 0 \) or \( \alpha_1 \geq 0, \alpha_2 \geq 0, \alpha_3 \geq 0 \) and \( K_1, K_2, K_3 \) be bounded sets. Then control problem (6) for \( I = I_1(T) \), \( k = 1, 2 \), has at least one solution \( (T, k_r, k_\theta, f) \in X \times K_1 \times K_2 \times K_3 \).

**Theorem 3.3** Let, under assumptions (i), (ii), (j), the pair \( (T, u) \equiv (k_r, k_\theta, f) \in X \times K_1 \times K_2 \times K_3 \) be a solution of problem (6) and let a functional \( I(T) \) be continuously differentiable in the point \( T \). Then there exists a unique Lagrange multiplier \( (\hat{R}, \hat{\zeta}_1, \hat{\zeta}_2) \in X_0 \times H^{1/2}(\Gamma_1)^* \times H^{1/2}(\Gamma_2)^* \) which is the solution of the adjoint problem in the form of the Euler-Lagrange equation:

\[
a(k_0, \hat{k}_r, \hat{k}_\theta, \hat{R}) + \langle \hat{\zeta}_1, \Psi \rangle_{\Gamma_1} + \langle \hat{\zeta}_2, \Psi \rangle_{\Gamma_2} = -\langle \alpha_0/2, I_1'(\hat{T}), \Psi \rangle \quad \forall \Psi \in X, \quad (7)
\]

and the following variational inequalities hold:

\[
\alpha_1(k_r - \hat{k}_r)s,\Omega + a_1((k_r - \hat{k}_r)\hat{T}, \hat{R}) \geq 0 \quad \forall k_r \in K_1, \quad (8)
\]

\[
\alpha_2(k_\theta - \hat{k}_\theta)s,\Omega + a_2((k_\theta - \hat{k}_\theta)\hat{T}, \hat{R}) \geq 0 \quad \forall k_\theta \in K_2, \quad (9)
\]

\[
\alpha_3(f - \hat{f})\Omega_e - (f - \hat{f}, \hat{R})\Omega_e \geq 0 \quad \forall f \in K_3. \quad (10)
\]

Direct problem (5), the Euler-Lagrange equation (7) which has the meaning of the adjoint problem for the adjoint state \( (\hat{R}, \hat{\zeta}_1, \hat{\zeta}_2) \) and variational inequalities (8)-(10) with respect to controls \( k_r, k_\theta, f \) constitute the optimality system for control problem (6). Based on analysis of the optimality system, one can establish sufficient conditions on the data which provide the uniqueness and stability of solutions of particular control problems. More precisely, the following theorem holds for the case of cost functional \( I_1(T) = \|T - T_d\|^2_Q \).
Theorem 3.4 Let, in addition to assumptions (i), (ii), (j), $K_1, K_2$ and $K_3$ be bounded sets and let the quadruples $(T^i, k_r^i, k_\theta^i, f^i)$ be solutions of control problem (6) for $I = I_1(T)$ corresponding to given functions $T_0^d \in L^2(Q), i = 1, 2$. Let the following conditions take place:

\[
\begin{align*}
\alpha_1 (1 - \varepsilon) &> 5 \alpha_0 C_0^2 C_1^2 M_1^0 M_T, \\
\alpha_2 (1 - \varepsilon) &> 5 \alpha_0 C_0^2 C_2^2 M_2^0 M_T, \\
\alpha_3 (1 - \varepsilon) &> 5 \alpha_0 C_0^2 M_0^0 M_T,
\end{align*}
\]

where $\varepsilon \in (0, 1)$ is an arbitrary number, while constants $M_T$ and $M_0^0$ are defined by

\[
M_T := C_0 C_1 \sup_{f \in K_3} (C_T + \|f\|_{\Omega_e}), \quad M_0^0 := M_T + \max \left(\|T_0^d(1)\|_Q, \|T_0^d(2)\|_Q\right).
\]

Then the following stability estimates hold:

\[
\begin{align*}
\|T_1 - T_2\|_Q &\leq \|T_0^d(1) - T_0^d(2)\|_Q, \\
\|k_r^{(1)}(1) - k_r^{(2)}(2)\|_{s, \Omega} &\leq \sqrt{\alpha_0 / \varepsilon \alpha_1}\|T_0^d(1) - T_0^d(2)\|_Q, \\
\|k_\theta^{(1)} - k_\theta^{(2)}\|_{s, \Omega} &\leq \sqrt{\alpha_0 / \varepsilon \alpha_2}\|T_0^d(1) - T_0^d(2)\|_Q, \\
\|f_1 - f_2\|_\Omega &\leq \sqrt{\alpha_0 / \varepsilon \alpha_3}\|T_0^d(1) - T_0^d(2)\|_Q,
\end{align*}
\]

4. Numerical algorithms

The optimality system (5), (7)–(10) derived above can be used to design efficient numerical algorithms for solving control problem (6). The simplest one for the functional $I_1(T)$ can be obtained by applying the fixed point iteration method for solving the optimality system. The $m$-th iteration of this algorithm consists of finding unknown values $T^m, R^m, k_r^{m+1}, k_\theta^{m+1}$ and $f^{m+1}$ for given $k^m = (k_r^m, k_\theta^m), f^m, m = 0, 1, 2, \ldots$, beginning with given initial values $k_r^0, k_\theta^0, f^0$ by sequentially solving following problems:

\[
a(k^m; T^m, S) + \langle s_1^m, \Psi \rangle_{\Gamma_1} + \langle s_2^m, \Psi \rangle_{\Gamma_2} = (f^m, S)_{\Omega_e} \forall S \in X_0, \quad T^m|_{\Gamma_1} = T_1, \quad T^m|_{\Gamma_2} = T_2,
\]

\[
a(k^m; \Psi, R^m) = -\alpha_0 (T^m - T^d, \Psi)_Q \forall \Psi \in X,
\]

\[
\alpha_1 (k_r^{m+1}, k_r^m - k_r^m)_{s, \Omega} + a_1((k_r^m - k_r^{m+1})T^m, R^m) \geq 0 \quad \forall k_r \in K_1,
\]

\[
\alpha_2 (k_\theta^{m+1}, k_\theta^m - k_\theta^m)_{s, \Omega} + a_2((k_\theta^m - k_\theta^{m+1})T^m, R^m) \geq 0 \quad \forall k_\theta \in K_2,
\]

\[
\alpha_3 (f^{m+1}, f - f^m)_{\Omega_e} - ((f - f^{m+1}), R^m)_{\Omega_e} \geq 0 \quad \forall f \in K_3.
\]

For discretization and solving problems (11), (12) one can use open source software freeFEM++ (www.freefem.org) based on finite element method. Along with this algorithm one can apply alternative algorithm of global optimization and in particular the particle-swarm method used in [21] for solving problems of designing 2D layered thermal cloaking shells.

5. Conclusion

In this paper, we studied control problems for the 3D thermal scattering model. These problems arise when optimization method is applied for solving cloaking problems for a respective thermal scattering model. Radial and polar heat conductivities $k_r$ and $k_\theta$ of the inhomogeneous medium filling the cloaking shell and the density $f$ of auxiliary heat sources placed in the domain $\Omega_e$ play the role of controls. We studied some new properties of solutions of direct thermal scattering problem, proved the solvability of direct and control problems and derived the optimality system describing the necessary conditions of extremum. Based on analysis of the optimality system, we established the uniqueness and stability estimates of optimal solutions for particular control problems. Besides, we proposed numerical algorithms for solving particular cloaking problem. We plan to devote a forthcoming paper to studying the properties of the algorithms and to analysis of results of numerical experiments performed using these algorithms.
Acknowledgments
The first author was supported by the Russian Science Foundation (project no. 14-11-00079), the second and third authors were supported by the Russian Foundation for Basic Research (project no.16-01-00365-a).

References
[1] Dolin L S 1961 On a possibility of comparison of three-dimensional electromagnetic systems with nonuniform anisotropic filling Izv. Vuzov Radiofizika 4 964–7
[2] Pendry J B, Shurig D and Smith D R 2006 Controlling electromagnetic fields Science 312 1780–82
[3] Leonhardt U 2006 Optical conformal mapping Science 312 1777–80
[4] Alú A and Engheta N 2005 Achieving transparency with plasmonic and metamaterial coatings Phys. Rev. E 72 016623
[5] Alú 2009 A Mantle cloak: invisibility induced by a surface Phys. Rev. B 80 245115
[6] Cummer S A, Popa B I, Schurig D, Smith D R, Pendry J, Rahm M and Starr A 2008 Scattering theory derivation of a 3D acoustic cloaking shell Phys. Rev. Lett. 100 024301
[7] Sanchez A, Navau C, Prat-Camps J and Chen D X 2011 Antimagnets: controlling magnetic fields with superconductor metamaterial hybrids New J. Phys. 13 093034
[8] Yang F, Mei Z L and Jin T Y et al. 2012 DC electric invisibility cloak Phys. Rev. Lett. 109 053902
[9] Han T and Qiu C W 2016 Transformation Laplacian metamaterials: recent advances in manipulating thermal and dc fields J. Opt. 18 044003
[10] Anikonov D S, Nazarov V G and Prokhorov I V 1997 Visible and invisible media in tomography Dokl. Math. 56 955–8
[11] Anikonov D S, Nazarov V G and Prokhorov I V 2002 Poorly visible media in X-Ray Tomography (Utrecht: VSP)
[12] Xu S, Wang Y, Zhang B and Chen H 2013 Invisibility cloaks from forward design to inverse design Sci. China Inf. Sci. 56 120408
[13] Alekseev G V 2016 The Invisibility Problem in Acoustics, Optics and Heat Transfer (Vladivostok: Dalnauka)
[14] Xi S, Chen H, Zhang B, Wu B I and Kong J A 2009 Route to low-scattering cylindrical cloaks with finite permittivity and permeability Phys. Rev. B 79 155122
[15] Popa B I and Cummer S A 2009 Cloaking with optimized anisotropic layers Phys. Rev. A 79 023806
[16] Wang X and Semouchkina E 2013 A route for efficient non-resonance cloaking by using multilayer dielectric coating Appl. Phys. Lett. 102 113506
[17] Alekseev G V 2013 Control of boundary impedance in two-dimensional material-body cloaking by the wave flow method Comput. Math. and Math. Phys. 53 1853–69
[18] Alekseev G V 2013 Cloaking of material objects by controlling the impedance boundary condition for Maxwell’s equations Dokl. Phys. 58 482–6
[19] Alekseev G V and Levin V A 2016 An optimization method for the problems of thermal cloaking of material bodies Dokl. Phys. 61 546–50
[20] Alekseev G V, Vakhitov I S and Soboleva O V 2012 Stability estimates in identification problems for the convection-diffusion-reaction equation Comput. Math. and Math. Phys. 52 2190–205
[21] Alekseev G V, Levin V A and Tereshko D A 2017 Optimization analysis of the thermal-cloaking problem for a cylindrical body Dokl. Phys. 62 71–5