Rate of convergence to Barenblatt profiles for the fast diffusion equation with a critical exponent

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Abstract

We study the asymptotic behaviour near extinction of positive solutions of the Cauchy problem for the fast diffusion equation with a critical exponent. After a suitable rescaling that yields a nonlinear Fokker–Planck equation, we find a continuum of algebraic rates of convergence to a self-similar profile. These rates depend explicitly on the spatial decay rates of initial data. This improves a previous result on slow convergence for the critical fast diffusion equation and provides answers to some open problems.

1. Introduction

We consider the Cauchy problem for the fast diffusion equation,

\[
\begin{align*}
  u_{\tau} &= \nabla \cdot (u^{m-1}\nabla u), \quad y \in \mathbb{R}^n, \quad \tau \in (0, T), \\
  u(y, 0) &= u_0(y) \geq 0, \quad y \in \mathbb{R}^n,
\end{align*}
\]

where \( n \geq 3, \ T > 0 \) and \( m = (n-4)/(n-2) \). It is known that, for \( m < m_c := (n-2)/n \), all solutions with initial data in some suitable space, such as \( L^p(\mathbb{R}^n) \) with \( p = n(1-m)/2 \), extinguish in finite time. We shall consider solutions that vanish in a finite time \( \tau = T \) and study their behaviour near \( \tau = T \).

For the extinction range \( m < m_c \) there are (infinite-mass) solutions of the self-similar form

\[
U_{D,T}(y, \tau) := \frac{1}{R(\tau)^n} \left( D + \beta(1-m) \frac{|y|}{R(\tau)} \right)^{-1/(1-m)},
\]

where \( D \geq 0 \) and

\[
R(\tau) := (T-\tau)^{-\beta}, \quad \beta := \frac{1}{n(1-m) - 2} = \frac{1}{n(m_c - m)} > 0.
\]

We will call these solutions Barenblatt solutions.

Many papers ([3–7], for example) are concerned with the convergence of solutions of (1.1) to the Barenblatt solutions as \( \tau \to T \). More precisely, the decay rates of

\[
R(\tau)^n(u(\tau, y) - U_{D,T}(y, \tau))
\]

as \( \tau \to T \) are discussed there.

The reasons why the critical exponent

\[
m_* := \frac{n-4}{n-2} < m_c,
\]

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plays a very important role in the results of [3–7] will be explained below. If \( n = 3,4 \), then \( m_* \leq 0 \), which is a case treated in some more detail in [4].

To study the asymptotic profile as \( \tau \to T \), it is convenient to rewrite \((1.1)\) in similarity variables:

\[
 t := \frac{1}{\mu} \ln \left( \frac{R(\tau)}{R(0)} \right) \quad \text{and} \quad x := \sqrt{\frac{\beta}{\mu} R(\tau)}, \quad \mu := \frac{2}{1 - m},
\]

with \( R \) as above, and the rescaled function

\[
v(x, t) := R(\tau)^m u(y, \tau)
\]

satisfies then the nonlinear Fokker–Planck equation

\[
v_t = \nabla \cdot (v^{m-1} \nabla v) + \mu \nabla \cdot (xv), \quad x \in \mathbb{R}^n, \quad t > 0.
\]

\[
(1.3)
\]

The Barenblatt solutions \( U_{D,T}(y, \tau) \) are thereby transformed into Barenblatt profiles \( V_D(\tau) \), which have the advantage of being stationary:

\[
V_D(x) := (D + |x|^2)^{-1/(1-m)}, \quad x \in \mathbb{R}^n.
\]

\[
(1.4)
\]

In the new variables, the convergence of solutions of \((1.1)\) to \( U_{D,T} \) takes the form of stabilization of solutions of \((1.3)\) to non-trivial equilibria \( V_D \).

The critical exponent \( m_* \) has the property that the difference of two Barenblatt profiles is integrable for \( m \in (m_*, m_c) \), while it is not integrable for \( m \leq m_* \). Furthermore, \( m_* \) is the unique value of \( m \) such that the linearization of the operator \( \nabla \cdot (v^{m-1} \nabla v) + \mu \nabla \cdot (xv) \) around \( V_D \) (on a natural weighted \( L^2 \)-space) has no spectral gap, see [4]. This is why one can expect that the rate of convergence to \( V_D \) is exponential for \( m \neq m_* \) and algebraic for \( m = m_* \).

In [3, 4, 6, 7], one can find several sufficient conditions under which \( v(\cdot, t) \) converges to \( V_D \) exponentially if \( m < m_c, m \neq m_* \). The case \( m = m_c \) was treated in [5] by functional analytic methods. A suitable linearization of the nonlinear Fokker–Planck equation \((1.3)\) was viewed as the plain heat flow on a suitable Riemannian manifold, and then nonlinear stability was studied by entropy methods. Theorem 3.1 in [5] (which can be viewed as the main result of [5]) gives algebraic upper bounds for the decay rate of the entropy functional and for the convergence rate to \( V_D \). One can expect the rates to be sharp since the linearization decays at those rates, but in [5] there is no rigorous proof of optimality. In fact, no lower bounds for the rates are established in [5]. One of the main aims of the present paper is to prove optimal lower bounds for the convergence rates for a large class of initial data. Our first main result says that convergence to \( V_D \) from below cannot occur at any rate faster than \( t^{-1/2} \), which is the fastest decay rate of positive solutions of the linear one-dimensional heat equation.

**Theorem 1.1.** Let \( n > 2, m = m_* \) and \( D > 0 \). Assume that \( v_0 \) is continuous and non-negative on \( \mathbb{R}^n \), \( v_0 \leq V_D, v_0 \neq V_D \), with \( V_D \) given by \((1.4)\). Then there exists \( c > 0 \) such that the solution \( v \) of \((1.3)\) with the initial condition \( v(\cdot, 0) = v_0 \) satisfies

\[
v(0, t) \leq V_D(0) - ct^{-1/2} \quad \text{for} \ t > 1.
\]

If \( v_0 \) intersects \( V_D \), then we expect that a faster rate of convergence may occur, similarly as for sign-changing solutions of the linear heat equation.

Next, we discuss upper bounds for the convergence rate. Corollary 3.2 in [5] says (among other things) that if \( 0 < D_1 < D_0, D \in [D_1, D_0] \) and

\[
V_{D_0}(x) \leq v_0(x) \leq V_{D_1}(x), \quad x \in \mathbb{R}^n,
\]

\[
(1.5)
\]

\[
|v_0(x) - V_D(x)| \leq f(|x|), \quad x \in \mathbb{R}^n, \quad f(|\cdot|) \in L^1(\mathbb{R}^n),
\]

\[
(1.6)
\]
then, for the solution $v$ of (1.3) with the initial condition $v(x, 0) = v_0(x)$, it holds that
\[ ||v(\cdot, t) - V_D||_{L^\infty(\mathbb{R}^n)} \leq Kt^{-1/4}, \quad t \geq 1, \] (1.7)
for some $K > 0$.

The question of whether the rates obtained in [5] are optimal for a class of data was posed in [5] as an open problem together with the question of whether one can prove convergence, maybe with worse rates or without rates, for more general initial data (see [5, Subsection 8.2]).

Our first step in answering these questions is the following:

**Theorem 1.2.** Assume that $n > 2$, $m = m_*$ and $D > 0$, and that $V_D$ is as defined in (1.4). Let $v$ be the solution of (1.3) with the initial condition
\[ v(x, 0) = v_0(x) := (|x|^2 + D + \psi_0(x))^{-(n-2)/2}, \quad x \in \mathbb{R}^n, \] (1.8)
where $\psi_0$ is continuous and non-negative on $\mathbb{R}^n$, $\psi_0 \neq 0$. If there are $B > 0$ and $\gamma \in (0, 1)$ such that
\[ \psi_0(x) \leq B \ln^{-\gamma} |x|, \quad |x| > 2, \] (1.9)
then there exists $C > 0$ such that
\[ V_D(x)(1 - CV_D^{2/(n-2)}(x)t^{-\gamma/2}) \leq v(x, t) \leq V_D(x), \quad x \in \mathbb{R}^n, \quad t \geq 1. \]

This theorem yields convergence with rates for a class of data that do not satisfy (1.6), but also for data that belong to the class considered in [5]. Namely, if $\psi_0$ satisfies (1.9) with $\gamma > 1$, then (1.9) also holds with $\gamma = 1 - \varepsilon$, $\varepsilon \in (0, 1)$, and some $B = B(\varepsilon) > 0$.

As an immediate consequence of Theorems 1.1 and 1.2 we obtain:

**Corollary 1.3.** Let $n > 2$, $m = m_*$ and $D > 0$. Assume that $\psi_0$ is continuous and non-negative on $\mathbb{R}^n$, $\psi_0 \neq 0$. Let $v$ be the solution of (1.3) with the initial condition (1.8). If there are $B > 0$ and $\gamma \geq 1$ such that (1.9) holds, then there is $c > 0$ and, for any $\varepsilon \in (0, 1)$, there exists $C_\varepsilon > 0$ such that
\[ ct^{-1/2} \leq ||V_D - v(\cdot, t)||_{L^\infty(\mathbb{R}^n)} \leq C_\varepsilon t^{-(1-\varepsilon)/2}, \quad t \geq 1. \]

If $\gamma > 1$, then the initial data from Corollary 1.3 satisfy (1.5) and (1.6), and fill a large part of the range of applicability of the entropy method from [5]. The wrong power of time appearing in (1.7) is due to interpolation. It was shown in [5] that, in the linearized situation, the heat kernel decay has a one-dimensional behaviour in the sense that its rate is $t^{-1/2}$ (see [5, Corollaries 4.4 and 4.5]), but the consequent smoothing effect between $L^1$ and $L^2$ yields a $t^{-1/4}$ decay only (see [5, Section 4.4]). The $L^1 - L^2$ bounds allow one to recover the correct $L^1 - L^\infty$ decay in the linear situation, but the lack of such functional analytic tools in the nonlinear situation causes the appearance of the wrong power of time for the $L^\infty$-norm.

In this paper, we work with the PDE directly, without any use of functional analysis. Our next result implies that Theorem 1.2 is sharp.

**Theorem 1.4.** Assume that $n > 2$, $m = m_*$ and $D > 0$. Let $V_D$ be as defined in (1.4) and let $v$ be the solution of (1.3) with the initial condition (1.8). If there are $b > 0$ and $\gamma \in (0, 1)$ such that
\[ \psi_0(x) \geq b \ln^{-\gamma} |x|, \quad |x| > 2, \]
then there exists \( c > 0 \) such that
\[
v(0, t) \leq V_D(0) - ct^{-\gamma/2}, \quad t > 1.
\]

Theorems 1.2 and 1.4 yield that if \( V_D(x) - v_0(x) \) behaves like \(|x|^{-n} \ln^{-\gamma} |x| \) for \(|x| \) large and some \( \gamma \in (0, 1) \), then \( \|v(\cdot, t) - V_D\|_{L^\infty(\mathbb{R}^n)} \) behaves like \( t^{-\gamma/2} \) for \( t \) large. Hence, we obtain a continuum of algebraic rates for initial data that do not satisfy (1.6). These rates are the same as for \( u_t = u_{xx}, \ x \in \mathbb{R} \), with positive initial data decaying as \(|x|^{-\gamma}\). Hence, the long-time behaviour of solutions of (1.3) is one-dimensional, while the short-time behaviour of solutions of the linearized equation is \( n \)-dimensional (cf. [5, Corollary 4.4]).

We prove our results by constructing suitable sub- and super-solutions. In order not to make the paper unnecessarily long, we consider only initial data below \( V_D \), but one can modify the arguments to prove analogous results for initial data above \( V_D \).

In Section 2, we establish the lower bound from Theorem 1.2, and in Section 3 the upper bound from Theorem 1.4. Section 4 is devoted to the proof of Theorem 1.1.

2. Lower bound. Proof of Theorem 1.2

To construct a suitable super-solution, we shall use the following:

**Lemma 2.1.** Let \( \gamma \in (0, 1) \). Then the solution of the problem
\[
\begin{align*}
\Phi''(z) + \frac{z}{2} \Phi'(z) + \frac{\gamma}{2} \Phi(z) &= 0, \quad z > 0, \\
\Phi(0) &= 1, \quad \Phi'(0) = 0,
\end{align*}
\]
(2.1)
is positive and decreasing on \([0, \infty)\), and there exist \( c > 0 \) and \( C > 0 \) such that
\[
\begin{align*}
c z^{-\gamma} \leq \Phi(z) \leq C z^{-\gamma} & \text{ for all } z \geq 1 \\
- C z^{-\gamma-1} \leq \Phi'(z) \leq -c z^{-\gamma-1} & \text{ for all } z \geq 1
\end{align*}
\]
(2.2)
and
\[
|\Phi''(z)| \leq C z^{-\gamma-2} \text{ for all } z \geq 1.
\]
(2.4)

**Proof.** The solution \( \Phi \) of (2.1) can be written explicitly in the form
\[
\Phi(z) = e^{-\zeta} \mathcal{M}
\left(
\frac{1 - \gamma}{2}, \frac{1}{2}, \zeta
\right), \quad \zeta := \frac{z^2}{4},
\]
where \( \mathcal{M} \) is Kummer’s function (see [1])
\[
\mathcal{M}(a, b, \zeta) := 1 + \frac{a}{b} \zeta + \cdots + \frac{a(a+1)\cdots(a+k)}{b(b+1)\cdots(b+k)k!} \zeta^k + \cdots
\]
and
\[
\zeta^{b-a} e^{-\zeta} \mathcal{M}(a, b, \zeta) \to \frac{\Gamma(b)}{\Gamma(a)} \quad \text{as } \zeta \to \infty,
\]
(2.5)
which yields (2.2).

If we now rewrite the equation in (2.1) as
\[
\Phi''(z) + \frac{1}{2} z^{1-\gamma} (z^\gamma \Phi(z))' = 0
\]
and use the identity
\[ \zeta \frac{d}{d\zeta} (\zeta^{b-a} e^{-\zeta} M(a, b, \zeta)) = (b-a) \zeta^{b-a} e^{-\zeta} M(a-1, b, \zeta) \]
(see [1]) together with (2.5), then we obtain that
\[ \left| \frac{1}{2} z^{1-\gamma} (z^{\gamma} \Phi(z))' \right| \leq C z^{-\gamma - 2}, \quad z \geq 1, \]
which implies (2.4).
Since \( \Phi \) cannot have any local minimum, one can see that \( \Phi' \) is negative and (2.3) follows from (2.4).

For \( m = m_* \) and radial solutions \( v = v(r, t) \), (1.3) becomes
\[ v_t = (v^{-2/(n-2)} v_r)_r + \frac{n-1}{r} v^{-2/(n-2)} v_r + (n-2) (r v_r + n v), \quad r > 0, \; t > 0. \]
If we further transform \( v \) via
\[ v(r, t) = (r^2 + D + \varphi(r, t))^{-(n-2)/2}, \quad r > 0, \; t > 0, \]
then, after some computation, it can be checked that \( \varphi \) satisfies, for \( r > 0 \) and \( t > 0 \), the equation
\[ P \varphi := \varphi_t - (r^2 + D + \varphi) \left( \varphi_{rr} + \frac{n-1}{r} \varphi_r \right) + (n-2) r \varphi_r + \frac{n-2}{2} \varphi_r^2 = 0. \] (2.6)
The change of variables
\[ \chi(\xi, t) := \varphi(r, t), \quad \xi := \ln r, \; r > 0, \; t > 0, \]
yields that
\[ Q \chi := \chi_t - \chi_{\xi \xi} - e^{-2\xi} \left\{ (D + \chi)[\chi_{\xi \xi} + (n-2) \chi_\xi] - \frac{n-2}{2} \chi_\xi^2 \right\} = 0 \] (2.7)
for \( \xi \in \mathbb{R} \) and \( t > 0 \).

In a region where \( r \) is appropriately large, we shall use functions of the form
\[ \chi^{(\xi_0, t_0, A)}(\xi, t) := A(t + t_0)^{-\gamma/2} \Phi((\xi + \xi_0) (t + t_0)^{-1/2}), \quad \xi \geq 0, \; t \geq 0, \] (2.8)
as (upper) comparison functions. For clarity of notation, we consider \( \xi_0 > 0, t_0 \geq 1 \) and \( A > 0 \) as free parameters here. We shall fix \( \xi_0, t_0 \) in Lemma 2.7 and \( A > 0 \) in the proof of Lemma 2.8.

**Lemma 2.2.** Let \( \gamma \in (0, 1) \). For \( t_0 \geq 1, \; \xi_0 \in \mathbb{R} \) and \( A > 0 \), the function \( \chi = \chi^{(\xi_0, t_0, A)} \) defined in (2.8) satisfies
\[ \chi_t = \chi_{\xi \xi} \quad \text{for} \; \xi > 0 \; \text{and} \; t > 0. \] (2.9)
Moreover, there exists \( t_* > 1 \) with the property that, whenever \( t_0 > t_* \), for any choice of \( \xi_0 > 0 \) and \( A > 0 \) we have
\[ \chi_{\xi \xi} + (n-2) \chi_\xi \leq 0 \quad \text{for all} \; \xi > 0 \; \text{and} \; t > 0. \] (2.10)

**Proof.** Since
\[ \chi_\xi = A(t + t_0)^{-\gamma/2 - 1/2} \Phi'(z), \quad \chi_{\xi \xi} = A(t + t_0)^{-\gamma/2 - 1} \Phi''(z) \] (2.11)
and
\[ \chi_t = -\frac{1}{2} A(t + t_0)^{-\gamma/2 - 1} z \Phi'(z) - \frac{\gamma}{2} A(t + t_0)^{-\gamma/2 - 1} \Phi(z) \]
with \( z := (\xi + \xi_0)(t + t_0)^{-1/2} \), the identity (2.9) is immediate from (2.1).

To verify (2.10), we observe that, since \( \Phi''(0) < 0 \) by (2.1), there exists \( z_0 > 0 \) such that
\[ \Phi''(z) \leq 0 \quad \text{for all} \quad z \in [0, z_0]. \] (2.12)

Then (2.3) and (2.4) ensure that, with some \( c_1 > 0 \) and \( c_2 > 0 \), we have
\[ \Phi'(z) \leq -c_1 z^{-\gamma - 1} \quad \text{for all} \quad z > z_0 \] (2.13)
and
\[ \Phi''(z) \leq c_2 z^{-\gamma - 2} \quad \text{for all} \quad z > z_0. \] (2.14)

We now let \( t_* > 1 \) be large enough such that
\[ t_* \geq \left( \frac{c_2}{(n-2)c_1 z_0} \right)^2, \] (2.15)
and claim that (2.10) holds whenever \( t_0 > t_* \), \( \xi_0 > 0 \) and \( A > 0 \). Indeed, recalling (2.11), (2.12) and the monotonicity of \( \Phi \), we easily see that in the region where \( z = (\xi + \xi_0)(t + t_0)^{-1/2} \leq z_0 \), both \( \chi_{\xi \xi} \) and \( \chi_\xi \) are non-positive, and hence clearly \( \chi_{\xi \xi} + (n-2)\chi_\xi \leq 0 \). On the other hand, if \( z > z_0 \), then from (2.11), (2.13) and (2.14) it follows that
\[ \frac{\chi_{\xi \xi}(\xi, t)}{-(n-2)\chi_\xi(\xi, t)} = \frac{\Phi''(z)}{-(n-2)\sqrt{t + t_0}\Phi'(z)} \leq \frac{c_2}{(n-2)c_1(\xi + \xi_0)}. \]

Since \( \xi + \xi_0 > z_0\sqrt{1 + 1/2} \), (2.15) implies that
\[ \frac{\chi_{\xi \xi}(\xi, t)}{-(n-2)\chi_\xi(\xi, t)} < \frac{c_2}{(n-2)c_1 z_0\sqrt{t + t_0}} < \frac{c_2}{(n-2)c_1 z_0 t_*} \leq 1 \]
holds at any such point, as claimed.

**Lemma 2.3.** Let \( D > 0 \) and \( \gamma \in (0, 1) \). Then there exists \( t_* > 1 \) such that, for any choice of \( t_0 > t_* \), \( \xi_0 > 0 \) and \( A > 0 \), the function \( \chi_{(\xi_0, t_0, A)} \) in (2.8) satisfies
\[ \mathcal{Q} \chi_{(\xi_0, t_0, A)} \geq 0 \quad \text{for all} \quad \xi > 0 \quad \text{and} \quad t > 0, \] (2.16)
where \( \mathcal{Q} \) is the operator defined in (2.7).

**Proof.** We take \( t_* \) as given by Lemma 2.2 and assume that \( t_0 > t_* \). Then, writing \( \chi := \chi_{(\xi_0, t_0, A)} \) and using (2.9) and (2.10), we obtain
\[ \mathcal{Q} \chi = -e^{-2\xi} \left( (D + \chi)[\chi_{\xi \xi} + (n-2)\chi_\xi] - \frac{n-2}{2} \chi_\xi^2 \right) \geq e^{-2\xi} \frac{n-2}{2} \chi_\xi^2 \geq 0 \quad \text{for all} \quad \xi > 0 \quad \text{and} \quad t > 0, \]
because \( D + \chi \geq 0 \) according to the non-negativity of \( \chi \) asserted by Lemma 2.1, and because \( n \geq 3 \).

The function we shall use as a super-solution near the origin (cf. (2.22) below) will have a certain self-similar structure. As a preparation, let us state the following lemma.
Lemma 2.4. Let $D > 0$ and $\gamma > 0$. For $\lambda := (1/D)(\gamma/2 + 1)$, let $\rho$ denote the solution of

\[
\begin{cases}
\rho''(\sigma) + \frac{1}{\sigma}\rho'(\sigma) + \lambda \rho(\sigma) = 0, & \sigma > 0, \\
\rho(0) = 1, & \rho'(0) = 0.
\end{cases}
\] (2.17)

Then there exists $\sigma_0 \in (0, 1)$ such that $\rho$ is positive and decreasing on $[0, \sigma_0]$.

Proof. Both statements are obvious from (2.17). \hfill \Box

In order to match inner and outer functions appropriately, we shall need a correcting factor that is time-dependent, but approaches one in the large time limit.

Lemma 2.5. Given $D > 0$ and $\gamma \in (0, 1)$, let $\Phi$, $\rho$ and $\sigma_0$ be as in Lemmas 2.1 and 2.4. Then, for $\xi_0 > 0$ and $t_0 > \sigma_0^{-2}$, the function $f^{(\xi_0, t_0)}$ defined by

\[
f^{(\xi_0, t_0)}(t) := \frac{\Phi(\xi_0(t + t_0)^{-1/2})}{\rho((t + t_0)^{-1/2})}, \quad t \geq 0,
\] (2.18)

satisfies

\[
f^{(\xi_0, t_0)}(t) \to 1 \quad \text{as} \quad t \to \infty.
\] (2.19)

Furthermore, for any $\xi_0 > 0$ there exists $C(\xi_0) > 0$ such that whenever $t_0 > 1$, we have

\[
|f^{(\xi_0, t_0)}'(t)| \leq \frac{C(\xi_0)}{(t + t_0)^2} \quad \text{for all} \quad t > 0.
\] (2.20)

Proof. Since $\Phi(0) = \rho(0) = 1$, (2.19) is obvious. As for (2.20), we for $t > 0$ compute

\[
(f^{(\xi_0, t_0)}(t))' = \frac{1}{2}(t + t_0)^{-3/2} \left(-\xi_0 \Phi'(\xi_0(t + t_0)^{-1/2}) \frac{\Phi(\xi_0(t + t_0)^{-1/2})}{\rho((t + t_0)^{-1/2})} + \frac{\Phi(\xi_0(t + t_0)^{-1/2})\rho'((t + t_0)^{-1/2})}{\rho^2((t + t_0)^{-1/2})}\right).
\] (2.21)

Since $\rho$ is positive on $[0, \sigma_0]$ and $\Phi'(0) = \rho'(0) = 0$, we can choose $c_1 > 0, c_2 > 0$ and $c_3 > 0$ such that

\[
\rho(\sigma) \geq c_1 \quad \text{for all} \quad \sigma \in [0, \sigma_0]
\]
as well as

\[
|\Phi'(z)| \leq c_2 z \quad \text{for all} \quad z \in [0, \xi_0] \quad \text{and} \quad |\rho'(\sigma)| \leq c_3 \sigma \quad \text{for all} \quad \sigma \in [0, \sigma_0].
\]

We thereby obtain from (2.21) that, for any choice of $t_0 > \sigma_0^{-2}$, one has

\[
|f^{(\xi_0, t_0)}'(t)| \leq \frac{c_2 \xi_0^2}{2c_1} + \frac{c_3}{2c_1^2} \quad t + t_0)^{-2} \quad \text{for all} \quad t > 0,
\]

because $\Phi \leq 1$ on $[0, \infty)$ by Lemma 2.1. \hfill \Box

We can now introduce a family of functions, one of which will serve as a super-solution in the region where $r < 1$. To this end, for $D > 0$ and $\gamma \in (0, 1)$ we let $\rho, \sigma_0$ and $f^{(\xi_0, t_0)}$ as in Lemmas 2.4 and 2.5, and given $\xi_0 > 0$, $t_0 > \sigma_0^{-2}$ and $A > 0$, we define, for $r \in [0, 1]$ and $t \geq 0$, the function

\[
\phi^{(\xi_0, t_0, A)}(r, t) := A f^{(\xi_0, t_0)}(t)(t + t_0)^{-\gamma/2} \rho(t + t_0)^{-1/2}).
\] (2.22)

We then have the following lemma.
Lemma 2.6. Let $D > 0$ and $\gamma \in (0, 1)$, and let $\rho$ and $\sigma_0$ be as in Lemma 2.4. Then, for each $\xi_0 > 0$ there exists $t^* > \sigma_0^{-2}$ such that, for any choice of $t_0 > t^*$ and any $A > 0$, the function $\varphi^{(\xi_0, t_0, A)}$ given by (2.22) satisfies
\[
P_{r}(\varphi^{(\xi_0, t_0, A)}) \geq 0 \quad \text{for all } r \in (0, 1) \text{ and } t > 0. \tag{2.23}
\]

Proof. Given $\xi_0 > 0$, we take $C(\xi_0)$ as provided by Lemma 2.5, and claim that (2.23) is valid whenever $t_0 > t^*$ and
\[
t^* = \max \left\{ \frac{1}{\sigma_0}, \frac{C(\xi_0)}{\Phi(\xi_0)} \right\}, \tag{2.24}
\]
where $\Phi$ is from Lemma 2.1.

To see this, we fix any such $t_0$ and, writing $\varphi = \varphi^{(\xi_0, t_0, A)}$, $f = f^{(\xi_0, t_0)}$ and $\sigma = r(t + t_0)^{-1/2}$, compute
\[
\varphi_r = Af(t)(t + t_0)^{-\gamma/2 - 1/2} \rho'(\sigma), \quad \varphi_{rr} = Af(t)(t + t_0)^{-\gamma/2 - 1} \rho''(\sigma) \tag{2.25}
\]
and
\[
\varphi_t = -\frac{1}{2} Af(t)(t + t_0)^{-\gamma/2 - 1} (\sigma \rho'(\sigma) + \gamma \rho(\sigma)) - Af'(t)(t + t_0)^{-\gamma/2} \rho(\sigma)
\]
for $r \in (0, 1)$ and $t > 0$. Now, since $t_0 > t^* > \sigma_0^{-2}$, in the region where $r < 1$ and $t > 0$ we have $\sigma < \sigma_0^{-1/2} < \sigma_0$, so that Lemma 2.4 guarantees that $\rho(\sigma) > 0$ and $\rho'(\sigma) \leq 0$, and hence $\rho''(\sigma) + (1/\sigma) \rho'(\sigma) = -\lambda \rho(\sigma) < 0$. In particular, if we write (2.6) as
\[
P_{r}(\varphi) = \varphi_t - (D + \varphi) \left( \varphi_{rr} + \frac{n - 1}{r} \varphi_r \right) - r^2 \varphi_{rr} - r \varphi_r + \frac{n - 2}{2} \varphi_r,
\]
and use (2.25), we obtain
\[
-(D + \varphi) \left( \varphi_{rr} + \frac{n - 1}{r} \varphi_r \right) = -(D + \varphi) Af(t)(t + t_0)^{-\gamma/2 - 1} \left( \rho''(\sigma) + \frac{n - 1}{\sigma} \rho'(\sigma) \right)
\]
\[
= -(D + \varphi) Af(t)(t + t_0)^{-\gamma/2 - 1} \left( -\lambda \rho(\sigma) + \frac{n - 2}{\sigma} \rho'(\sigma) \right)
\]
\[
\geq \lambda(D + \varphi) Af(t)(t + t_0)^{-\gamma/2 - 1} \rho(\sigma)
\]
\[
\geq \lambda DAf(t)(t + t_0)^{-\gamma/2 - 1} \rho(\sigma),
\]
because $n \geq 3$. Moreover,
\[
-r^2 \varphi_{rr} - r \varphi_r = -Af(t)(t + t_0)^{-\gamma/2 - 1}r^2 \left( \rho''(\sigma) + \frac{1}{\sigma} \rho'(\sigma) \right)
\]
\[
= \lambda Af(t)(t + t_0)^{-\gamma/2 - 1}r^2 \geq 0
\]
for $r < 1$ and $t > 0$. Since $((n - 2)/2) \varphi_r^2 \geq 0$, we therefore have
\[
P_{r}(\varphi) \geq \varphi_t + \lambda DAf(t)(t + t_0)^{-\gamma/2 - 1} \rho(\sigma)
\]
\[
= Af(t)(t + t_0)^{-\gamma/2 - 1} \left\{ -\frac{\sigma}{2} \rho'(\sigma) - \frac{\gamma}{2} \rho(\sigma) - \frac{f'(t)}{f(t)}(t + t_0) \rho(\sigma) + \lambda D \rho(\sigma) \right\}
\]
for $r \in (0, 1)$ and $t > 0$. \tag{2.26}

Now, the monotonicity properties of $\Phi$ and $\rho$ imply that, since $t_0 > 1$, we obtain
\[
f(t) \geq \frac{1}{\rho(0)} \Phi(\xi_0 t_0^{-1/2}) \geq \Phi(\xi_0) \quad \text{for all } t > 0,
\]
so that, using (2.20), we obtain
\[
\left| \frac{f'(t)}{f(t)}(t + t_0) \right| \leq \frac{C(\xi_0)}{\Phi(\xi_0)(t + t_0)} \leq \frac{C(\xi_0)}{\Phi(\xi_0) t_0} \quad \text{for all } t > 0.
\]
Thus, according to the fact that \( t^* > C(\xi_0)/\Phi(\xi_0) \) by (2.24), we have
\[
\left| \frac{f'(t)}{f(t)}(t + t_0) \right| \leq 1 \quad \text{for all } t > 0.
\]
Hence, (2.26) entails that
\[
\mathcal{P}\varphi \geq Af(t)(t + t_0)^{-\gamma/2 - 1}\left\{ -\frac{\sigma}{2} \rho'(\sigma) + \left( \lambda D - \frac{\gamma}{2} - 1 \right) \rho(\sigma) \right\}
\]
\[
= -Af(t)(t + t_0)^{-\gamma/2 - 1}\frac{\sigma}{2} \rho'(\sigma) \geq 0 \quad \text{for } r \in (0, 1) \text{ and } t > 0,
\]
because of our choice of \( \lambda \) in (2.17) and, again, the monotonicity of \( \rho \) on \((0, \sigma_0)\). This completes the proof.

**Lemma 2.7.** Let \( D > 0 \) and \( \gamma \in (0, 1) \). Then, with \( \sigma_0 \) as in Lemma 2.4, there exist \( \xi_0 > 0 \) and \( t_0 > \sigma_0^{-2} \) such that, for any \( A > 0 \), the function \( \varphi^{(A)} \) defined by
\[
\varphi^{(A)}(r, t) := \begin{cases} 
\varphi^{(\xi_0, t_0, A)}(r, t), & r \in [0, 1], \ t \geq 0, \\
\chi^{(\xi_0, t_0, A)}(\ln r, t), & r > 1, \ t \geq 0,
\end{cases}
\]
is continuous in \([0, \infty)^2\) and satisfies
\[
\mathcal{P}\varphi^{(A)} \geq 0 \quad \text{for all } r \in (0, \infty) \setminus \{1\} \text{ and } t > 0,
\]
where \( \mathcal{P} \) is as in (2.6), and such that
\[
\lim_{r^1 \uparrow 1} \varphi^{(A)}(r, t) > \lim_{r \downarrow 1} \varphi^{(A)}(r, t) \quad \text{for all } t > 0.
\]

**Proof.** Given \( D > 0 \) and \( \gamma \in (0, 1) \), we let \( \rho \) and \( \Phi \) be as defined by (2.17) and (2.1). Then, since \( \rho'(0) = \Phi'(0) = 0 \) and \( \Phi''(0) = -\gamma/2 < 0 \), we can find \( c_1 > 0 \) and \( c_2 > 0 \) fulfilling
\[
\rho'(\sigma) \geq -c_1 \sigma \quad \text{for all } \sigma \in (0, \sigma_0)
\]
and
\[
\Phi'(z) \leq -c_2 z \quad \text{for all } z \in (0, 1).
\]
We now first fix \( \xi_0 > 0 \) large such that
\[
\xi_0 > \frac{c_1}{c_2 \rho(\sigma_0)}
\]
and then take \( t_\star \) and \( t^\star \) as provided by Lemmas 2.3 and 2.6, respectively, when applied to this particular choice of \( \xi_0 \). We finally pick some \( t_0 > \sigma_0^{-2} \) satisfying
\[
t_0 > \max \{ t_\star, t^\star, \xi_0^2 \}
\]
and claim that these choices ensure that \( \varphi^{(A)} \) is continuous, and that (2.28) and (2.29) are valid whenever \( A > 0 \).

In fact, (2.28) is an immediate consequence of Lemmas 2.3 and 2.6, while the continuity of \( \varphi^{(A)} \) directly results from the definitions of \( \varphi^{(\xi_0, t_0, A)} \), \( \chi^{(\xi_0, t_0, A)} \) and the function \( f^{(\xi_0, t_0)} \) introduced in Lemma 2.5. To verify (2.29), we recall (2.22) and (2.8) in computing
\[
I_1(t) := \lim_{r^1 \uparrow 1} \varphi^{(A)}(r, t) = \varphi^{(\xi_0, t_0, A)}(1, t)
\]
\[
= Af^{(\xi_0, t_0)}(t)(t + t_0)^{-\gamma/2 - 1/2} \rho'((t + t_0)^{-1/2}), \quad t > 0.
\]
and

\[ I_2(t) := \limsup_{r \searrow 1} \varphi^{(A)}(r, t) = \chi^{(\xi_0, t_0), A}(0, t) = A(t + t_0)^{-\gamma/2-1/2} \Phi'(\xi_0(t + t_0)^{-1/2}), \quad t > 0. \]  \hspace{1cm} (2.35)

Here, we note that, by (2.18) and the monotonicity of \( \Phi \) and \( \rho \),

\[ f(\xi_0, t_0)(t) \leq \Phi(0)\rho^{-1}(t_0^{-1/2}) = \rho^{-1}(t_0^{-1/2}) \leq \rho^{-1}(\sigma_0) \quad \text{for all } t > 0, \]  \hspace{1cm} (2.36)

because \( t_0 > \sigma_0^{-2} \). Furthermore, (2.30) and (2.31) assert that

\[ \rho'(t_0^{-1/2}) \geq -c_1(t + t_0)^{-1/2} \quad \text{for all } t > 0 \]  \hspace{1cm} (2.37)

and

\[ \Phi'((\xi_0(t + t_0)^{-1/2}) \leq -c_2\xi_0(t + t_0)^{-1/2} \quad \text{for all } t > 0, \]  \hspace{1cm} (2.38)

again since \( (t + t_0)^{-1/2} < \sigma_0 \), and since \( \xi_0(t + t_0)^{-1/2} < 1 \) due to (2.33). Using (2.36)-(2.38), we obtain from (2.34) and (2.35) that

\[ I_1(t) - I_2(t) \geq A(t + t_0)^{-\gamma/2-1}(-c_1\rho^{-1}(\sigma_0) + c_2\xi_0) \quad \text{for all } t > 0, \]

so that our requirement (2.32) guarantees that (2.29) holds. 

\[ \square \]

**Lemma 2.8.** Let \( D > 0 \). Assume that \( \varphi_0 \) is continuous and non-negative on \([0, \infty)\), and there exist \( \gamma \in (0, 1) \) and \( B > 0 \) such that

\[ \varphi_0(r) \leq B \ln^{-\gamma} r \quad \text{for all } r > 2. \]  \hspace{1cm} (2.39)

Then there exists \( C > 0 \) such that the solution \( \varphi \) of (2.6) with \( \varphi(\cdot, 0) = \varphi_0 \) satisfies

\[ \varphi(r, t) \leq C(t + 1)^{-\gamma/2} \quad \text{for all } r \geq 0 \text{ and } t \geq 0. \]  \hspace{1cm} (2.40)

**Proof.** Given \( D > 0 \) and \( \gamma \in (0, 1) \), we fix \( \sigma_0 \in (0, 1), \xi_0 > 0 \) and \( t_0 > \sigma_0^{-2} \) as in Lemmas 2.4 and 2.7, and take \( f = f^{(\xi_0, t_0)} \) from Lemma 2.5. In order to define, with some specific \( A > 0 \), a super-solution of the form (2.27) which initially dominates \( \varphi \), we set \( z_0 := (\ln 2 + \xi_0)t_0^{-1/2} \) and then obtain from Lemma 2.1 that, for some \( c_1 > 0 \), the function \( \Phi \) in (2.1) satisfies

\[ \Phi(z) \geq c_1z^{-\gamma} \quad \text{for all } z \geq z_0. \]  \hspace{1cm} (2.41)

Moreover, since \( \varphi_0 \) is bounded, we can pick \( c_2 > 0 \) such that

\[ \varphi_0(r) \leq c_2 \quad \text{for all } r \in [0, 2]. \]  \hspace{1cm} (2.42)

We now fix any \( A > 0 \) fulfilling

\[ A > \max \left\{ \frac{c_2t_0^{\gamma/2}}{f(0)\rho(t_0^{-1/2})}, \frac{c_2t_0^{\gamma/2}}{\Phi(z_0)}, \frac{B}{c_1} \left( 1 + \frac{\xi_0}{\ln 2} \right)^{\gamma} \right\} \]  \hspace{1cm} (2.43)

and claim that then the function \( \varphi^{(A)} \) in (2.27) has the property

\[ \varphi^{(A)}(r, 0) > \varphi_0(r) \quad \text{for all } r \geq 0. \]  \hspace{1cm} (2.44)

To prove this, we first observe that, for small \( r \), by (2.42) and (2.43) it holds that

\[ \frac{\varphi^{(A)}(r, 0)}{\varphi_0(r)} \geq \frac{\varphi^{(A)}(r, 0)}{c_2} = \frac{1}{c_2} A f(0)t_0^{-\gamma/2}\rho(r t_0^{-1/2}) \geq \frac{1}{c_2} A f(0)t_0^{-\gamma/2}\rho(r t_0^{-1/2}) > 1, \quad r \in [0, 1], \]
because $\rho' \leq 0$ on $(0, \sigma_0)$ and $t_0^{-1/2} < \sigma_0$. Similarly, in the intermediate region where $1 < r \leq 2$, (2.42), (2.43) and the monotonicity of $\Phi$ yield
\[
\frac{\varphi^{(A)}(r, 0)}{\varphi_0(r)} \geq \frac{1}{c_2} A t_0^{-\gamma/2} \Phi((\ln 2 + \xi_0)(t_0^{-1/2})) > 1, \quad r \in (1, 2].
\]
Finally, for large $r$ we apply (2.41) to estimate
\[
\varphi^{(A)}(r, 0) = A t_0^{-\gamma/2} \Phi((\ln r + \xi_0)(r_0^{-1/2})) \geq c_1 A (\ln r + \xi_0)^{-\gamma}, \quad r > 2,
\]
because $\ln r + \xi_0 \leq \ln r + \xi_0 \ln r / \ln 2$ for such $r$. Along with (2.43) and (2.39), this guarantees that also
\[
\varphi^{(A)}(r, 0) > \varphi_0(r), \quad r > 2.
\]
Having thus found that (2.44) is true, we may invoke Lemma 2.7 combined with the comparison principle to infer that
\[
\varphi \leq \varphi^{(A)}(r, 0) = A t_0^{-\gamma/2} \Phi((\ln r + \xi_0)(t_0^{-1/2})) \geq c_1 A \left(1 + \frac{\xi_0}{\ln 2}\right)^{-\gamma} (\ln r)^{-\gamma}, \quad r \in (0, 1],
\]
as well as
\[
\varphi(r, t) \leq A(t + t_0)^{-\gamma/2} \quad \text{for all } r > 1 \text{ and } t \geq 0,
\]
from which (2.40) clearly follows.

**Proof of Theorem 1.1.** If we choose $\varphi_0$ satisfying (2.39) such that $\psi_0(x) \leq \varphi_0(|x|)$ for $x \in \mathbb{R}^n$, then we obtain by comparison that
\[
(|x|^2 + D + \varphi(|x|, t))^{-(n-2)/2} \leq v(x, t) \leq V_D(x), \quad x \in \mathbb{R}^n, \quad t \geq 0.
\]
Lemma 2.8 and the Mean Value Theorem yield then the result.

---

3. Upper bound. Proof of Theorem 1.4

**Lemma 3.1.** For $\gamma > 0$, let
\[
\hat{\Phi}(z) := \left(1 + \frac{z^2}{4}\right)^{-\gamma/2}, \quad z \geq 0.
\]
Then,
\[
\hat{\Phi}''(z) + \frac{\gamma}{2} \hat{\Phi}'(z) + \frac{\gamma}{2} \hat{\Phi}(z) \geq \frac{\gamma}{4} \left(1 + \frac{z^2}{4}\right)^{-\gamma/2 - 1} \quad \text{for all } z > 0.
\]
Moreover,
\[
\hat{\Phi}'(z) = -\frac{\gamma}{4} z \left(1 + \frac{z^2}{4}\right)^{-\gamma/2 - 1} \quad \text{for all } z > 0
\]
and
\[
|\hat{\Phi}''(z)| \leq \frac{\gamma (\gamma + 1)}{4} \left(1 + \frac{z^2}{4}\right)^{-\gamma/2 - 1} \quad \text{for all } z > 0.
\]

**Proof.** The statements can be verified by straightforward computations.
Lemma 3.2. Let $D > 0$ and $\gamma > 0$. Then there exists $\xi_0 > 1$ such that, for any choice of $a \in (0, 1)$, the function $\chi^{(a)}$ given by

$$
\chi^{(a)}(\xi, t) := a(t + 1)^{-\gamma/2} \Phi \left( \frac{\xi - \xi_0}{\sqrt{t + 1}} \right), \quad \xi \geq \xi_0, \ t \geq 0,
$$

satisfies

$$
Q \chi^{(a)} \leq 0 \quad \text{for all } \xi > \xi_0 \text{ and } t > 0,
$$

(3.6)

where $Q$ is as defined in (2.7).

Proof. We abbreviate

$$
z := \frac{\xi - \xi_0}{\sqrt{t + 1}}
$$

and calculate

$$
\dot{\chi}_\xi^{(a)} = a(t + 1)^{-\gamma/2 - 1/2} \dot{\Phi}(z), \quad \dot{\chi}_{\xi \xi}^{(a)} = a(t + 1)^{-\gamma/2 - 1} \ddot{\Phi}(z)
$$

and

$$
\dot{\chi}_t^{(a)} = -\frac{1}{2} a(t + 1)^{-\gamma/2 - 1} \dot{\Phi}'(z) - \frac{\gamma}{2} a(t + 1)^{-\gamma/2 - 1} \dot{\Phi}(z).
$$

Therefore,

$$
Q \chi^{(a)} = a(t + 1)^{-\gamma/2 - 1} \left\{ -\frac{z}{2} \dot{\Phi}'(z) - \frac{\gamma}{2} \dot{\Phi}(z) - \ddot{\Phi}(z) - e^{-2\xi} [D + a(t + 1)^{-\gamma/2} \dot{\Phi}(z)] [\ddot{\Phi}(z) + (n - 2) \sqrt{t + 1} \dot{\Phi}'(z)]
\right.
\left. + \frac{n - 2}{2} e^{-2\xi} (t + 1)^{-\gamma/2} \dot{\Phi}''(z) \right\} \quad \text{for } \xi > \xi_0 \text{ and } t > 0.
$$

Here, we recall (3.1) and our assumption $a < 1$ in estimating

$$
|D + a(t + 1)^{-\gamma/2} \dot{\Phi}(z)| \leq D + 1
$$

and use (3.2) to see that

$$
-\frac{z}{2} \dot{\Phi}'(z) - \frac{\gamma}{2} \dot{\Phi}(z) - \ddot{\Phi}(z) \leq -\gamma \left( 1 + \frac{z^2}{4} \right)^{-\gamma/2 - 1}
$$

at any point $(\xi, t) \in (\xi_0, \infty) \times (0, \infty)$. Thus,

$$
\frac{Q \chi^{(a)}}{a(t + 1)^{-\gamma/2 - 1}} \leq -\gamma \left( 1 + \frac{z^2}{4} \right)^{-\gamma/2 - 1} + (D + 1) e^{-2\xi} |\dot{\Phi}''(z)|
\left. + (n - 2) e^{-2\xi} \sqrt{t + 1} |\dot{\Phi}'(z)| + \frac{n - 2}{2} e^{-2\xi} \dot{\Phi}''(z) \right\} =: -I_1 + I_2 + I_3 + I_4 \quad \text{for } \xi > \xi_0 \text{ and } t > 0,
$$

and we claim that this implies (3.6) if we pick $\xi_0 > 1$ large enough such that

$$
(\gamma + 1)(D + 1) e^{-2\xi} < \frac{1}{3} \quad \text{for all } \xi > \xi_0
$$

(3.7)

and

$$
(n - 2)(D + 1)(\xi - \xi_0) e^{-2\xi} < \frac{1}{3} \quad \text{for all } \xi > \xi_0
$$

(3.8)

as well as

$$
\frac{(n - 2)\gamma}{2} e^{-2\xi} < \frac{1}{3} \quad \text{for all } \xi > \xi_0.
$$

(3.9)
Indeed, in conjunction with (3.4), (3.7) implies that
\[
\frac{I_2}{I_1} = \frac{4(D+1)}{\gamma} e^{-2\xi} \left(1 + \frac{z^2}{4}\right)^{\gamma/2+1} |\Phi''(z)| \\
\leq \frac{4(D+1)}{\gamma} e^{-2\xi} (\gamma + 1) = (\gamma + 1)(D+1) e^{-2\xi} \\
< \frac{1}{3} \quad \text{for all } \xi > \xi_0 \text{ and } t > 0.
\] (3.10)

Since \(\sqrt{t+1} = (\xi - \xi_0)/z\), (3.3) and (3.8) next guarantee that
\[
\frac{I_3}{I_1} = \frac{4(n-2)(D+1)}{\gamma} e^{-2\xi} \sqrt{t+1} \left(1 + \frac{z^2}{4}\right)^{\gamma/2+1} |\Phi'(z)| \\
= \frac{4(n-2)(D+1)}{\gamma} (\xi - \xi_0) e^{-2\xi} (1 + z^2/4)^{\gamma/2+1} |\Phi'(z)| \\
< \frac{1}{3} \quad \text{for all } \xi > \xi_0 \text{ and } t > 0.
\] (3.11)

Finally, again by (3.3),
\[
\frac{I_4}{I_1} = \frac{2(n-2)}{\gamma} e^{-2\xi} (t+1)^{\gamma/2+1} \Phi'(z) \\
= \frac{2(n-2)}{8} e^{-2\xi} (t+1)^{\gamma/2+1} \Phi'(z) \\
= \frac{(n-2)\gamma}{8} e^{-2\xi} (1 + z^2/4)^{-\gamma/2-1},
\]
so that, since clearly \(z^2(1 + z^2/4)^{-\gamma/2-1} \leq 4\), from (3.9) we infer that
\[
\frac{I_4}{I_1} \leq \frac{(n-2)\gamma}{2} e^{-2\xi} < \frac{1}{3} \quad \text{for all } \xi > \xi_0 \text{ and } t > 0.
\]

Combined with (3.10) and (3.11), this establishes (3.6).

In view of the explicit definition (3.1) of \(\Phi\), the above function \(\tilde{\chi}(a)\) can alternatively be written in the fully explicit form
\[
\tilde{\chi}(a)(\xi, t) = a \left(t + 1 + \frac{(\xi - \xi_0)^2}{4}\right)^{-\gamma/2}, \quad \xi \geq \xi_0, \ t \geq 0.
\]

**Lemma 3.3.** Let \(D > 0\) and \(\gamma > 0\). Then there exists \(r_0 > e\) such that, for all \(a \in (0,1)\),
\[
\varphi(a)(r,t) := \begin{cases}
\frac{a(t+1)^{-\gamma/2}}{r}, & r \in [0,r_0], \ t \geq 0; \\
\chi(a)(\ln r, t), & r > r_0, \ t \geq 0,
\end{cases}
\] (3.12)
defines a continuous function \(\varphi(a)\) on \([0,\infty)^2\) such that also \(\varphi(a)\) is continuous on \([0,\infty)^2\), and such that
\[
\mathcal{P}\varphi(a) \leq 0 \quad \text{for all } r \in (0,\infty) \setminus \{r_0\} \text{ and } t > 0.
\] (3.13)

Here, \(\tilde{\chi}(a)\) is as defined in Lemma 3.2 with \(\xi_0 := \ln r_0\), and \(\mathcal{P}\) is as in (2.6).

**Proof.** With \(\xi_0 > 1\) as provided by Lemma 3.2, we let \(r_0 := e^{\xi_0} > e\) and thereupon obtain that (3.6) precisely yields \(\mathcal{P}\varphi(a) \leq 0\) for \(r > r_0\) and \(t > 0\). To see the same for small \(r\), we only
need to note that clearly
\[ \varphi_r(a) = \varphi_r(a) = 0 \quad \text{for } r < r_0 \text{ and } t > 0, \]
so that
\[ \mathcal{P} \varphi_r = \varphi_t = -\frac{a\gamma}{2} (t + 1)^{-\gamma/2 - 1} < 0 \quad \text{for } r < r_0 \text{ and } t > 0. \]
Having thus established (3.13), we are left with proving the continuity of \( \varphi_r(a) \). In view of (3.14), however, this immediately follows from the observation that
\[ \lim_{r \searrow r_0} \varphi_r(a)(r, t) = \frac{1}{r_0} \varphi_0(\xi_0, t) = \frac{a}{r_0} (t + 1)^{-\gamma/2} \Phi'(0) = 0 \quad \text{for all } t > 0, \]
whereby the proof is completed.

**Lemma 3.4.** Let \( D > 0 \). Suppose that \( \varphi_0 \in C^0([0, \infty)) \) is positive and such that
\[ \varphi_0(r) \geq b \ln^{-\gamma} r \quad \text{for all } r > 2 \]
with some positive constants \( b \) and \( \gamma \). Then there exists \( c > 0 \) such that the solution \( \varphi \) of (2.6) fulfilling \( \varphi(r, 0) = \varphi_0 \) satisfies
\[ \varphi(0, t) \geq c(t + 1)^{-\gamma/2} \quad \text{for all } t > 0. \]

**Proof.** We let \( r_0 > e \) be as given by Lemma 3.3. Then, since \( \varphi_0 \) is continuous and positive, we can find \( c_1 > 0 \) such that
\[ \varphi_0(r) \geq c_1 \quad \text{for all } r \in [0, r_0], \]
and fix \( a \in (0, 1) \) small enough fulfilling
\[ a < \min\{c_1, bc_2^{\gamma/2}\}, \]
where
\[ c_2 := \min\left\{ \frac{1}{16}, \frac{1}{4\xi_0^2} \right\} \]
with \( \xi_0 := \ln r_0 > 1 \). We claim that this choice ensures that, with \( \varphi_r(a) \) defined by (3.12), we have
\[ \varphi_0(r) \geq \varphi_r(a)(r, 0) \quad \text{for all } r \geq 0. \]
In fact, if \( r \) is small, then by (3.17) and (3.18),
\[ \varphi_0(r) \geq c_1 > a = \varphi_r(a)(r, 0) \quad \text{for all } r \in [0, r_0]. \]
In order to show (3.19) for large \( r \), we observe that, by (3.12), (3.5) and (3.1),
\[ \varphi_r(a)(r, 0) = a \Phi(\ln r - \xi_0) = a \left( 1 + \frac{\ln r - \xi_0^2}{4} \right)^{-\gamma/2}, \quad r > r_0, \]
because \( r_0 > e > 2 \). Here, we estimate
\[ 1 + \frac{\ln r - \xi_0^2}{4} \geq \frac{\ln r - \xi_0^2}{4} \geq \frac{(\ln r)^2}{16} \quad \text{if } \ln r \geq 2\xi_0 \]
and
\[ 1 + \frac{\ln r - \xi_0^2}{4} \geq 1 \geq \left( \frac{\ln r}{2\xi_0} \right)^2 \quad \text{if } \ln r < 2\xi_0, \]
whence, by definition of \( c_2 \), it follows that
\[ \varphi_r(a)(r, 0) \leq a(c_2(\ln r)^2)^{-\gamma/2} < b(\ln r)^{-\gamma} \leq \varphi_0(r) \quad \text{for all } r > 2. \]
We have thereby verified (3.19), which in turn, on an application of the comparison principle, entails that \( \varphi \geq \varphi^{(a)} \) in \([0, \infty)^2\). Evaluated at \( r = 0 \), this implies in particular that
\[
\varphi(0, t) \geq \varphi^{(a)}(0, t) = a(t + 1)^{-\gamma/2} \quad \text{for all } t \geq 0,
\]
and hence proves (3.16).

Proof of Theorem 1.4. We choose \( \varphi_0 \) satisfying (3.15) such that \( \psi_0(x) \geq \varphi_0(|x|) \) for \( x \in \mathbb{R}^n \). Then, we obtain by comparison that
\[
(|x|^2 + D + \varphi(|x|, t))^{-(n-2)/2} \geq v(x, t), \quad x \in \mathbb{R}^n, \ t \geq 0.
\]
Lemma 3.4 and the Mean Value Theorem yield then the result.

4. Universal upper bound. Proof of Theorem 1.1

Lemma 4.1. Let \( \xi_1 \in \mathbb{R} \), and suppose that \( \alpha \) and \( \beta \) are smooth functions on \((\xi_1, \infty) \times (0, \infty)\), for which there exist \( k > 0 \) and \( K > 0 \) such that
\[
k \leq \alpha(\xi, t) \leq K \quad \text{and} \quad |\beta(\xi, t)| \leq K \quad \text{for all } \xi > \xi_1 \text{ and } t > 0.
\]
Then, for any non-negative solution
\[
0 \neq w \in C^{2,1}((\xi_1, \infty) \times (0, \infty)) \cap C^0((\xi_1, \infty) \times [0, \infty))
\]
of
\[
w_t = \alpha(\xi, t)w_{\xi\xi} + \beta(\xi, t), \quad \xi > \xi_1, \ t > 0,
\]
one can find \( c > 0 \) such that
\[
\sup_{\xi > \xi_1} w(\xi, t) \geq c(t + 1)^{-1/2} \quad \text{for all } t > 0.
\]

Proof. This lower bound follows from [2], for example.

Lemma 4.2. Let \( D > 0 \) and assume that \( \varphi_0 \) is continuous and non-negative on \([0, \infty)\), \( \varphi_0 \neq 0 \). Then there exists \( c > 0 \) such that the solution \( \varphi \) of (2.6) with \( \varphi(\cdot, 0) = \varphi_0 \) satisfies
\[
\sup_{r > 0} \varphi(r, t) \geq c(t + 1)^{-1/2} \quad \text{for all } t > 0.
\]

Proof. Passing to a suitable minorant of \( \varphi_0 \) if necessary, in view of the comparison principle we may assume that, for some \( r_0 > 0 \), we have \( 0 \neq \varphi_0 \in C^\infty_0((r_0, \infty)) \) with \( 0 \leq \varphi_0 \leq 1 \). Now, conveniently rewritten in terms of \( \chi(\xi, t) = \varphi(t, t), \ \xi = \ln r \), (2.6) becomes (cf. also (2.7))
\[
\chi_t = \chi\xi + e^{-2\xi} \left\{ (D + \chi)\chi_\xi + (n - 2)\chi_\xi - \frac{n - 2}{2} \chi_\xi^2 \right\}
\]
\[
= [1 + (D + \chi)e^{-2\xi}]\chi_\xi + e^{-2\xi} \left\{ (n - 2)(D + \chi) - \frac{n - 2}{2} \chi_\xi \right\} \chi_\xi
\]
\[
=: \alpha(\xi, t)\chi_\xi + \beta(\xi, t), \quad \xi \in \mathbb{R}, \ t > 0.
\]
Let us next choose \( \xi_0 \in \mathbb{R} \) such that \( \xi_0 < \ln r_0 - 2 \). Then, since \( 0 \leq \chi \leq 1 \) in \( \mathbb{R} \times (0, \infty) \), we have
\[
1 \leq \alpha(\xi, t) \leq 1 + (D + 1)e^{-2\xi_0} \quad \text{for all } \xi > \xi_0 \text{ and } t > 0.
\]
Therefore, due to the fact that $\varphi_0$ is smooth with compact support, interior parabolic Schauder estimates [8] provide $c_1 > 0$ such that
\[ |\chi_\xi(\xi, t)| \leq c_1 \quad \text{for all } \xi > \xi_1 \text{ and } t > 0, \]
so that
\[ |\beta(\xi, t)| \leq e^{-2(\xi_0+1)} \left[ (n - 2)(D + 1) + \frac{n-2}{2} c_1 \right] \quad \text{for all } \xi > \xi_1 \text{ and } t > 0. \]
Since we already know that $\chi \geq 0$ and that $\chi(\cdot, 0) \neq 0$ in $(\xi_0 + 1, \infty)$ according to our choices of $\tau_0$ and $\xi_0$, we may now invoke Lemma 4.1 to conclude that there exists $c_2 > 0$ such that
\[ \sup_{\xi > \xi_0+1} \chi(\xi, t) \geq c_2(t+1)^{-1/2} \quad \text{for all } t > 0. \]
Restated using the variable $\varphi$, this immediately yields (4.1).

Proof of Theorem 1.1. We write the initial function $v_0$ as
\[ v_0(x) = (|x|^2 + D + \psi_0(x))^{-(n-2)/2}, \quad x \in \mathbb{R}^n, \]
where $\psi_0$ is continuous and non-negative on $\mathbb{R}^n$, $\psi_0 \neq 0$. We can assume, without loss of generality, that $\psi_0(0) > 0$. We choose $\varphi_0$ such that $\psi_0(x) \geq \varphi_0(|x|)$ for $x \in \mathbb{R}^n$ and $\varphi_0 \neq 0$ is non-increasing. We then obtain by comparison that
\[ (|x|^2 + D + \varphi(|x|, t))^{-(n-2)/2} \geq v(x, t), \quad x \in \mathbb{R}^n, \quad t \geq 0. \]
Since $\sup_{r>0} \varphi(r, t) = \varphi(0, t)$, the result follows from Lemma 4.2 and the Mean Value Theorem.

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