Turán-type problems for long cycles in random and pseudo-random graphs

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Abstract

We study the Turán number of long cycles in random graphs and in pseudo-random graphs. Denote by $ex(G(n, p), H)$ the random variable counting the number of edges in a largest subgraph of $G(n, p)$ without a copy of $H$. We determine the asymptotic value of $ex(G(n, p), C_t)$ where $C_t$ is a cycle of length $t$, for $p \geq \frac{C}{n}$ and $A \log n \leq t \leq (1-\varepsilon)n$. The typical behavior of $ex(G(n, p), C_t)$ depends substantially on the parity of $t$. In particular, our results match the classical result of Woodall on the Turán number of long cycles, and can be seen as its random version, showing that the transference principle holds here as well. In fact, our techniques apply in a more general sparse pseudo-random setting. We also prove a robustness-type result, showing the likely existence of cycles of prescribed lengths in a random subgraph of a graph with a nearly optimal density.

1 Introduction

One of the most central topics in extremal graph theory is the so-called Turán-type problems. Recall that $ex(n, H)$ denotes the maximum possible number of edges in a graph on $n$ vertices without having $H$ as a subgraph. Determining the value of $ex(n, H)$ for a fixed graph $H$ has become one of the most central problems in extremal combinatorics and there is a rich literature investigating it. Mantel [33] proved in 1907 that $ex(n, K_3) = \left\lfloor \frac{n^2}{4} \right\rfloor$; Turán [42] found the value of $ex(n, K_t)$ for $t \geq 3$ in 1941. In 1968, Simonovits [38] showed that the result of Mantel can be extended for an odd cycle of a fixed length, that is, $ex(n, C_{2t+1}) = \left\lfloor \frac{n^2}{4} \right\rfloor$, where the extremal example is the complete bipartite graph $K_{n/2, n/2}$. For the general case, it was proved in 1946 by Erdős and Stone [14] that $ex(n, H) = \left(1 - \frac{1}{\chi(H)-1} + o(1)\right) \binom{n}{2}$, where $\chi(H)$ is the chromatic number of the fixed graph $H$. Note that when $H$ is a graph with chromatic number 2, an even cycle for instance, then from...
the above result we can only obtain that \( ex(n, H) = o(n^2) \). Bondy and Simonovits [8] proved in 1974 that for even cycles we have \( ex(n, C_{2t}) = O(n^{1+1/t}) \). Unfortunately, a matching lower bound is known only for the cases where \( t = 2, 3, 5 \). For a survey see [39, 43].

In this paper we consider the case where \( H = C_t \) and \( t \) is not a fixed number. In this direction, it was proved by Erdős and Gallai [13] that if \( t = t(n) \) then \( ex(n, P_t) = \left(\frac{1}{2}(t - 1)n\right) \). For long cycles, it was shown by Woodall [44] that if \( t \geq \frac{1}{2}(n + 3) \) then \( ex(n, C_t) = \left(\frac{1}{3} - \frac{1}{2}\right) + \left(\frac{n - 4 - t}{2}\right) \), where the extremal example is given by two cliques intersecting in exactly one vertex. In the same paper, Woodall also showed that for odd cycles \( C_t \) shorter than \( \frac{1}{2}(n + 3) \), the trivial bound \( ex(n, C_t) \geq \left[\frac{n^2}{4}\right] \) is still tight.

In the past few decades several generalizations of the classical Turán number \( ex(n, H) \) were suggested and many results have been established in this area. Denote by \( ex(G, H) \) the number of edges in a largest subgraph of a graph \( G \) containing no copy of \( H \). Note that the value of \( ex(G, H) \) is bounded from below by the number of edges in \( G \) that are not contained in any copy of \( H \). As a consequence, if the number of copies of \( H \) in \( G \) is much smaller than the number of edges in \( G \), then we obtain that \( ex(G, H) \geq (1-o(1))e(G) \). Thus, it makes sense to restrict our attention to graphs \( G \) for which the number of copies of \( H \) is at least proportional to the number of edges.

We focus on the case where the host graph \( G \) is either a random graph or pseudo-random graph. Given a positive integer \( n \) and a real number \( p \in [0, 1] \), we let \( G(n, p) \) be the binomial random graph, that is, a graph sampled from the family of all labeled graphs on the vertex set \([n] := \{1, \ldots, n\}\), where each pair of elements of \([n]\) forms an edge with probability \( p := p(n) \), independently. We denote by \( ex(G(n, p), H) \) the number of edges in a largest subgraph of \( G(n, p) \) without a copy of \( H \) (note that \( ex(G(n, p), H) \) is a random variable). Clearly, in this case we want to consider only the values of \( p \) for which \( G(n, p) \) contains a copy of \( H \) with high probability (w.h.p., i.e., with probability tending to 1 as \( n \to \infty \)), and in fact, the number of copies of \( H \) in \( G \) is typically “large enough”.

For fixed-size graphs \( H \), this parameter has already been considered by various researchers. It is known that the threshold probability for a random graph to have the property that a typical edge is contained in a copy of \( H \), for a fixed graph \( H \), is \( n^{-1/m_2(H)} \), where \( m_2(H) \) is the maximum 2-density and defined to be \( m_2(H) = \max \left\{ \frac{e(H') - 1}{V(H')^2} \mid H' \subseteq H, \ v(H') \geq 3 \right\} \) (see [19] for more details). Therefore, it makes sense to consider graphs \( G(n, p) \) for the regime \( p = \Omega(n^{-1/m_2(H)}) \). The cases \( H = K_3 \), \( H = C_4 \), and \( H = K_4 \) were solved by Frankl and Rödl [15], Füredi [20], and by Kohayakawa, Łuczak, and Rödl [27], respectively. For fixed odd cycles, it was shown by Haxell, Kohayakawa, and Łuczak [22] that for \( p \geq Cn^{-(2t-1)/2t} \) we have that \( \frac{1}{2}e(G(n, p)) \leq ex(G(n, p), C_{2t+1}) \leq \left(\frac{1}{2} + \varepsilon\right)e(G(n, p)) \). For fixed even cycles, the same group of authors showed [23] that for \( p = \omega(n^{-(2t-2)/(2t-1)}) \) we have \( ex(G(n, p), C_{2t}) = o(e(G(n, p))) \) (for more precise bounds on the fixed even cycle case, see Kohayakawa, Kreuter, and Steger [26], and Morris and Saxton [35]). The authors of [22, 23, 27] conjectured that a similar behaviour should also hold for any fixed-size graph \( H \), that is, that the value of \( ex(G(n, p), H) \) should be asymptotically equal to \( \frac{ex(n, H)}{\binom{n}{2}} e(G(n, p)) \), for suitable values of \( p \). This conjecture was proved independently by Conlon and Gowers [10] (with certain constraints on \( H \)) and by Schacht [37], who showed that the Turán number of a fixed graph in \( G(n, p) \) is of the same proportion of edges as it is in the complete graph, where the latter has been determined by Erdős and Stone. More precisely, they proved that for \( p \geq Cn^{-1/m_2(H)} \), and for a fixed graph \( H \), w.h.p. \( ex(G(n, p), H) \leq \left(1 - \frac{\chi(H) - 1}{1 + 1} + \varepsilon\right)e(G(n, p)) \). A matching lower bound can be obtained by a random placement of the extremal example of \( ex(n, H) \). The phenomenon that we observe here is frequently called the transference principle, which in this context can be interpreted
as a random graph “inheriting” its (relative) extremal properties from the classical deterministic case, i.e., the complete graph. In their papers, Conlon and Gowers [10] and Schacht [37] discussed this principle and showed transference of several extremal results from the classical deterministic setting to the probabilistic setting.

In this paper we aim to study the transference principle in the context of long cycles. The first step is to understand what should be the relevant regime of $p$. It is easy to observe that if $p = o\left(\frac{1}{n}\right)$ then a typical $G(n, p)$ is a forest, that is, does not contain any cycle. Thus, when looking at the appearance of a cycle in $G(n, p)$, it is natural to restrict ourselves to the regime $p = \Omega\left(\frac{1}{n}\right)$. Furthermore, it is well known that cycles start to appear in $G(n, p)$ at probability $p = \Theta\left(\frac{1}{n}\right)$. We shall further recall what are the typical lengths of cycles one can expect to have in this regime. Note that for $p = \Theta\left(\frac{1}{n}\right)$ w.h.p. there are linearly many isolated vertices. Therefore, in this regime of $p$, we can hope to find in $G(n, p)$ cycles of length at most $(1 - \varepsilon)n$ for some constant $\varepsilon > 0$. Indeed, the typical appearance of nearly spanning cycles was shown in a series of papers by Ajtai, Komlós, and Szemerédi [1], de la Vega [11], Bollobás [6], Bollobás, Fenner and Frieze [7]. In 1986 Frieze [18] proved that if $G$ is a random graph “inheriting” its (relative) extremal properties from the classical deterministic setting to the probabilistic setting.

In 1991, Luczak showed [32] that for $p = \omega\left(\frac{1}{n}\right)$, w.h.p. $G(n, p)$ contains cycles of all lengths between 3 and $n - (1 + \varepsilon)v^1(n, p)$, where $v^1(n, p)$ is the number of vertices of degree at most 1 and $\varepsilon := \varepsilon(C)$ (and it was very recently improved even more by Anastos and Frieze [3]). In 1991, Frieze [18] proved that if $p \geq \frac{C}{n}$ then w.h.p. in $G(n, p)$ there exists a cycle of length at least $n - (1 + \varepsilon)v^1(n, p)$, where $v^1(n, p)$ is the number of vertices of degree at most 1 and $\varepsilon := \varepsilon(C)$ (and it was very recently improved even more by Anastos and Frieze [3]). In 1991, Luczak showed [32] that for $p = \omega\left(\frac{1}{n}\right)$, w.h.p. $G(n, p)$ contains cycles of all lengths between 3 and $n - (1 + \varepsilon)v^1(n, p)$. On the other hand, when looking at cycles of length $o(\log n)$ in the context of Turán-type problems, the regime $p = \Theta\left(\frac{1}{n}\right)$ is not quite relevant. It is easy to verify that for $p = \Theta\left(\frac{1}{n}\right)$ w.h.p. one expects $o(e(G(n, p)))$ cycles of such lengths, and hence they can be destroyed by deleting a negligible proportion of edges. Therefore, when requiring that the number of copies of $C_t$ will be w.h.p. at least proportional to the number of edges, combining it with the fact that $p = \Omega\left(\frac{1}{n}\right)$, we get that $t = \Omega(\log n)$.

Moving back to the extremal problem, it was shown by Dellamonica, Kohayakawa, Marciniak, and Steger [12] that if $p = \omega\left(\frac{1}{n}\right)$, then for all $\alpha > 0$, if $G'$ is a subgraph of $G(n, p)$ with $e(G') \geq (1 - (1 - w(\alpha))(\alpha + w(\alpha)) + o(1)) e(G(n, p))$, then w.h.p. $G'$ contains a cycle of length at least $(1 - \alpha)n$, where $w(\alpha) = 1 - (1 - \alpha)^{(1 - \alpha)^{-1}}$. This result is asymptotically tight by the classical result of Woodall [44] that guarantees a cycle of length at least $(1 - \alpha)n$ in any graph $G$ with $e(G) \geq (1 - (1 - w(\alpha))(\alpha + w(\alpha)) + o(1)) \binom{n}{2}$.

Very recently, Balogh, Dudek and Li [4] studied the asymptotic behavior of $e_x(G(n, p), P_t)$ for various ranges of $\ell = \ell(n)$.

In this paper we study the appearance of long cycles of a given length in subgraphs of pseudorandom graphs. As a direct consequence we get a result for $G(n, p)$. More precisely, we determine the asymptotic value $e_x((G(n, p)), C_t)$, where $p = \Omega\left(\frac{1}{n}\right)$ and $t$ is between $\Theta(\log n)$ and $(1 - \varepsilon)n$.

The more general statement deals with a class of graphs which is larger than the random graphs class. For this we use the following definition.

**Definition 1.1.** Let $G$ be a graph on $n$ vertices. Suppose $0 < \eta \leq 1$ and $0 < p \leq 1$. We say that $G$ is $(p, \eta)$-upper-uniform if for every $U, W \subseteq V(G)$ with $U \cap W = \emptyset$ and $|U|, |W| \geq \eta n$, we have $e_G(U, W) \leq (1 + \eta)p|U||W|$.

**Remark 1.2.** In a $(p, \eta)$-upper-uniform graph $G$ on $n$ vertices we have, for any $U \subseteq V(G)$ with $|U| \geq 2\eta n$, that $e(G[U]) \leq (1 + \eta)p\binom{|U|}{2}$.

Indeed, let $U \subseteq V(G)$ of size $u := |U| \geq 2\eta n$. We look at all possible partitions of $U$ into two
subsets $U_1, U_2$ such that $u_1 := |U_1| = \left\lfloor \frac{n}{2} \right\rfloor$ and $u_2 := |U_2| = \left\lceil \frac{n}{2} \right\rceil$, and we use it to count the number of edges in such cuts of $H$ in two ways. We have

$$e(U) \cdot 2 \left( \frac{u_2 - 2}{u_1 - 1} \right) = \sum_{U_1, U_2} e(G[U_1, U_2]).$$

By $(p, \eta)$-upper-uniformity of $G$ we have $e(G[U_1, U_2]) \leq (1 + \eta)pu_1u_2$, so we get

$$e(U) \leq \frac{1}{2} \left( \frac{u_2 - 2}{u_1 - 1} \right)^{-1} \left( \frac{u}{u_1} \right) (1 + \eta)pu_1u_2 = (1 + \eta)p \left( \frac{u}{2} \right).$$

The following notation is based on results by Erdős-Gallai [13] and Woodall [44] (see Theorem 2.1 and Theorem 2.2 for more information).

**Definition 1.3.** The functions $g_o, g_e$ are given as follows.

If $t$ is odd, then

$$g_o(t, n) \cdot \binom{n}{2} := ex(n, C_t) + 1 = \begin{cases} \frac{t-1}{2} + \binom{n-t+2}{2} + 1, & \text{if } t \geq \frac{1}{2}(n+3), \\ \left\lfloor \frac{1}{4}n^2 \right\rfloor + 1, & \text{if } t < \frac{1}{2}(n+3). \end{cases}$$

If $t$ is even and $\gamma > 0$ is a parameter,

$$g_e(t, n) \cdot \binom{n}{2} := \begin{cases} ex(n, C_t) + 1 = \frac{t-1}{2} + \binom{n-t+2}{2} + 1, & \text{if } t \geq \frac{1}{2}(n+3), \\ ex(n, P_t) + 1 = \left\lfloor \frac{1}{2}n(t-1) \right\rfloor + 1, & \text{if } \gamma n \leq t < \frac{1}{2}(n+3), \\ 0, & \text{if } t < \gamma n. \end{cases}$$

Furthermore, the function $g^\gamma : [0,1] \rightarrow [0,1]$ is defined as follows.

$$g^\gamma(t, n) = \begin{cases} g_o(t, n), & \text{if } t \text{ is odd} \\ g_e(t, n), & \text{if } t \text{ is even}. \end{cases}$$

Later we will set a specific value of the parameter $\gamma$ (see Remark 2.7).

We are now ready to state our main theorem. Here and later, $\log n$ refers to the natural logarithm.

**Theorem 1.4.** For every $0 < \beta < \frac{1}{4}$, there exist $\eta, n_0, \gamma > 0$ such that for every $n \geq n_0$, if $G$ is a $(p, \eta)$-upper-uniform graph on $n$ vertices with $e(G) \geq (1 - \beta/2)p(n)\binom{n}{2}$ for some $0 < p := p(n) \leq 1$, then for any $\frac{C}{\log(1/\beta)} \cdot \log n \leq t \leq (1 - C_2\beta)n$, where $C_1, C_2 > 0$ are absolute constants, if $G'$ is a subgraph of $G$ with

$$e(G') \geq (g^\gamma(t, n) + \beta) e(G)$$

edges, then $G'$ contains a cycle of length $t$.

Since we have $ex(n, P_t) = ex(n, P_{t-1}) + O(n)$, then $ex(n, P_t) - O(n) \leq ex(n, P_{t-1}) \leq ex(n, C_t)$, and we can deduce from our main result the following corollary.

**Corollary 1.5.** For every $0 < \beta < \frac{1}{4}$, there exist $\eta, n_0 > 0$ such that for every $n \geq n_0$, if $G$ is a $(p, \eta)$-upper-uniform graph on $n$ vertices with $e(G) \geq (1 - \beta/2)p(n)\binom{n}{2}$ for some $0 < p := p(n) \leq 1$, then for any $\frac{C}{\log(1/\beta)} \cdot \log n \leq t \leq (1 - C_2\beta)n$,

$$ex(G, C_t) \leq \left( \frac{ex(n, C_t)}{\binom{n}{2}} + \beta \right) e(G),$$

where $C_1, C_2 > 0$ are some absolute constants.
Remark 1.6. In both [Theorem 1.4] and [Corollary 1.5] we obtain, in fact, given \( t \), all cycles of length \( q \), where \( \frac{C_1}{\log(1/\beta)} \cdot \log n \leq q \leq t \), with the same parity as \( t \).

[Theorem 1.4] and [Corollary 1.5] are asymptotically optimal in a stronger form; a matching lower bound is true for any graph \( G \) on \( n \) vertices, not only for upper-uniform graphs. That is, for any graph \( G \) on the vertex set \([n]\) there exists a subgraph \( G_0 \) with \( \frac{ex(n,C)}{\binom{n}{2}} \cdot e(G) \) edges containing no cycle of length \( t \). Indeed, let \( W_t \) be a graph on \( n \) vertices with \( ex(n,C_t) \) edges containing no cycle of length \( t \). By averaging, there exists an assignment \( \sigma \) of the vertices of \( W_t \) into \([n]\) such that when intersecting with \( G \), we have \( e(G \cap W_t^\sigma) \geq \frac{ex(n,C_t)}{\binom{n}{2}} \cdot e(G) \). Clearly, the resulting graph \( G \cap W_t^\sigma \) contains no cycles of length \( t \). This gives the following.

**Fact 1.7.** For every graph \( G \) on \( n \geq 3 \) vertices and every integer \( t \in [3,n] \) we have

\[
ex(G, C_t) \geq \frac{ex(n, C_t)}{\binom{n}{2}} e(G).
\]

Remark 1.8. [Theorem 1.4] does not assume anything on the value of \( p \). However, graphs \( G \) satisfying the conditions of the statement exist only for a restricted spectrum of values for \( p \). More specifically, for \( p = o\left(\frac{1}{n}\right) \) there are no \((p,\eta)\)-upper-uniform graphs \( G \) with \( e(G) \geq (1 - \beta/2)p^{n/2} \), which makes the statement relevant only for \( p \geq \frac{C_1}{\log 4} \), where \( C > 0 \) is some constant. To see this, take \( G \) to be a \((p,\eta)\)-upper-uniform graph with \( p = o\left(\frac{1}{n}\right) \). Then by [Remark 1.2] \( e(G) \leq (1 + \eta)p^{n/2} = o(n) \). Thus, there is a subset \( I \) of isolated vertices in \( G \) of size \( \frac{n}{2} \). By the assumption \( e(G) \geq (1 - \beta)p^{n/2} \) we obtain that \( e(G[V \setminus I]) \geq (1 - \beta)p^{n/2} \). This contradicts the upper uniformity of \( G \) since \( e(G[V \setminus I]) \leq (1 + \eta)p^{n/2} \).

Probably the most natural application of [Theorem 1.4] is for the random graph case. It is not hard to see that for \( p = \Omega(1/n) \), \( G(n,p) \) w.h.p. satisfies the conditions of [Theorem 1.4]. Indeed, as the total number of edges in \( G(n,p) \) is distributed binomially with parameters \( \binom{n}{2} \) and \( p \), we have that \( e(G(n,p)) \geq (1 - \beta/2)p^{n/2} \) with probability \( 1 - e^{-\Omega(n)} \). In addition, for, say, \( p \geq \frac{\log 4}{n^{4}} \), we have that for every two disjoint subsets \( U_1, U_2 \) such that \(|U_1|, |U_2| \geq \eta n \), \( e(U_1, U_2) \leq (1 + \eta)p|U_1||U_2| \) with probability \( 1 - e^{-\Omega(n)} \). We obtain that, for \( p \geq \frac{C_1}{\log 4} \) and \( C := C(\eta) \) being large enough, the random graph \( G(n,p) \) is w.h.p. \((p,\eta)\)-upper-uniform. By the discussion regarding the expected cycle lengths in \( G(n,p) \), we easily get that the lower bound on \( t \) in [Theorem 1.4] is, in fact, necessary.

As a result, we obtain the following corollary.

**Corollary 1.9.** For every \( 0 < \beta < \frac{1}{2} \), there exist \( C, \gamma > 0 \) such that if \( G \sim G(n,p) \) where \( p \geq \frac{C_1}{n} \), then for any \( \frac{C_1}{\log(1/\beta)} \cdot \log n \leq t \leq (1 - C_2 \beta)n \), with probability \( 1 - e^{-\Omega(n)} \), where \( C_1, C_2 > 0 \) are absolute constants, if \( G' \) is a subgraph of \( G \) with

\[
e(G') \geq (g^\gamma(t,n) + \beta) e(G),
\]

then \( G' \) contains a cycle of length \( t \).

Similarly to [Corollary 1.5] we can write the upper bound on \( ex(G(n,p), C_t) \) only in terms of \( ex(n, C_t) \), as follows.

**Corollary 1.10.** For every \( 0 < \beta < \frac{1}{2} \), there exists \( C > 0 \) such that if \( G \sim G(n,p) \) where \( p \geq \frac{C_1}{n} \), then for any \( \frac{C_1}{\log(1/\beta)} \cdot \log n \leq t \leq (1 - C_2 \beta)n \), with probability \( 1 - e^{-\Omega(n)} \),

\[
ex(G(n,p), C_t) \leq \left( \frac{ex(n, C_t)}{\binom{n}{2}} + \beta \right) e(G(n,p)),
\]
where $C_1, C_2 > 0$ are absolute constants.

Thus, also here, we observe a manifestation of the transference principle, that is, the random graph $G(n, p)$ preserves the relative behavior of the Turán number of long cycles observed in the classical case, i.e., in the complete graph $K_n$.

As mentioned in Remark 1.6 given $t$, the statement holds for every $\frac{C_1}{\log(1/\beta)} \cdot \log n \leq q \leq t$ with the same parity as $t$.

Another natural application of the main theorem is for $(n, d, \lambda)$-graphs, which can be shown to be $(p, \eta)$-upper-uniform for suitable values of $d, \lambda$.

**Definition 1.11.** A graph $G$ is an $(n, d, \lambda)$-graph if $G$ has $n$ vertices, is $d$-regular, and the second largest (in absolute value) eigenvalue of its adjacency matrix is bounded from above by $\lambda$.

$(n, d, \lambda)$-graphs have been studied extensively, mainly due to their good pseudo-random properties. For a detailed background see [30]. Recently, it was shown in [16] that for a given $\beta > 0$, if $\frac{d}{\lambda} \geq C(\beta)$, then $(n, d, \lambda)$-graphs contain cycles of all lengths between $\frac{C_1}{\log(1/\beta)} \cdot \log n$ and $(1 - C_2 \beta)n$ (for some absolute constants $C_1, C_2 > 0$), improving the result in [24].

Using the Expander Mixing Lemma due to Alon and Chung [2], we can show that for suitable values of $d$ and $\lambda$, an $(n, d, \lambda)$-graph is also upper-uniform. Hence we obtain the following corollary.

**Corollary 1.12.** For every $0 < \beta < \frac{1}{4}$ there exist $n_0, \gamma, \eta > 0$ such that for every $n \geq n_0$ and for every $d, \lambda > 0$ satisfying $\frac{d}{\lambda} \geq \frac{1}{\eta}$, if $G$ is an $(n, d, \lambda)$-graph, then for any $\frac{C_1}{\log(1/\beta)} \cdot \log n \leq t \leq (1 - C_2 \beta)$, where $C_1, C_2 > 0$ are absolute constants, if $G'$ is a subgraph of $G$ with $e(G') \geq (\gamma^2(t, n) + \beta)e(G)$, edges, then $G'$ contains a cycle of length $t$.

Note that the lower bound on $t$ is tight due to the existence of $(n, d, \lambda)$-graphs with large girth. More explicitly, it was shown in [31, 34] that there exist infinitely many $(n, d, \lambda)$-graphs with girth $\Omega(\log n)$, such that $\frac{d}{\lambda}$ is larger than a given constant. Details of the proof and further discussion on this application can be found in Section 6.1.

Using very similar techniques, we can also obtain a robustness-type result (for a detailed survey on robustness problems see [40]). In this type of results, we consider a graph $G$ satisfying some extremal conditions that guarantee a graph property $P$ (in our case, containment of long cycles). The aim is to measure quantitatively the strength of these specific conditions. For this, we let $G$ be a graph satisfying these conditions, and let $G(p)$ be the random graph obtained by keeping each edge of $G$ independently with probability $p \in [0, 1]$. Note that if $G = K_n$ then $G(p) = G(n, p)$. In the next theorem we show that if $G$ has (slightly more than) the minimum number of edges that guarantees a long cycle of a given length, then with high probability $G(p)$ also contains such a cycle for $p = \Omega\left(\frac{1}{n}\right)$. This value of $p$ is best possible due to threshold of the existence of cycles in $G(n, p)$.

**Theorem 1.13.** For every $\beta > 0$ there exists $C > 0$ such that for $\frac{C}{\log(1/\beta)} \cdot \log n \leq t \leq (1 - C_2 \beta)n$ (where $C_1, C_2 > 0$ are absolute constants), for any $p \geq \frac{C_1}{\eta}$, and for any graph $G$ on $n$ vertices satisfying $e(G) \geq ex(n, C_1) + \beta \binom{n}{2}$,
then w.h.p. \( G(p) \) contains a copy of \( C_t \).

Note that starting with a graph with exactly \( ex(n,C_t) + 1 \) edges is not enough. Indeed, let \( G \) be an extremal example for \( ex(n,C_t) \) with an arbitrary edge \( e \) added to it. Then when taking \( G(p) \) with \( p = \omega(\frac{1}{n}) \) w.h.p. \( e \) is deleted. However, the above theorem shows that adding \( \beta(t,2) \) edges to the extremal number will be enough, and, in fact, for many values of \( t \) this number of edges is even tight. Therefore, adding \( \beta(t,2) \) edges to the extremal number is necessary in most cases as will be examined in Remark 6.6. Full details can be found in Section 6.2.

2 Notation and preliminaries

Our graph-theoretic notation is standard, in particular we use the following. For a graph \( G = (V,E) \) and a set \( U \subseteq V \), let \( G[U] \) denote the corresponding vertex-induced subgraph of \( G \). We also denote \( e(G) = |E(G)| \) and \( v(G) = |V(G)| \). For \( U \subseteq V \) we let \( \Gamma_G(U) = \{ v \in V \setminus U \mid \exists u \in U \text{ s.t. } \{u,v\} \in E \} \) be the neighborhood of \( U \) in \( G \). For an integer \( k \) and \( V_i \subseteq V \), \( i \in [k] \), we say that \( \Pi = (V_1, \ldots, V_k) \) is a partition of \( V \) if \( V = \bigcup_{i \in [k]} V_i \) and \( V_i \cap V_j = \emptyset \) for every \( i \neq j \).

2.1 Known extremal results

To prove our result, we use two classical theorems, one by Woodall [44] about cycles, and the other one by Erdős and Gallai [13] regarding paths.

Theorem 2.1 ([13], Theorem 2.6). Let \( G \) be an \( n \)-vertex graph with more than \( \left\lceil \frac{3}{4}n(t - 1) \right\rceil \) edges. Then \( G \) contains a path of length at least \( t \) (the number of edges).

By looking at a graph consisting of \( \left\lceil \frac{n}{2} \right\rceil \) vertex-disjoint cliques of size \( t \) and another clique on the remaining vertices, one can observe that the above result is tight.

Theorem 2.2 ([44], Corollary 11). Let \( G \) be a graph on \( n \geq 3 \) vertices and let \( 3 \leq t \leq n \). Assume that \( e(G) \geq w(t,n) \cdot \binom{n}{2} \) where

\[
w(t,n) \cdot \binom{n}{2} := \begin{cases} \binom{t-1}{2} + \binom{n-t+2}{2} + 1, & \text{if } t \geq \frac{1}{2}(n+3), \\ \frac{1}{4}n^2 + 1, & \text{if } t < \frac{1}{2}(n+3). \end{cases}
\]

Then \( G \) contains a cycle of length \( d \) for any \( 3 \leq d \leq t \).

The result in Theorem 2.2 is tight in the sense that there are graphs with \( w(t,n) \binom{n}{2} - 1 \) edges containing no cycle of some length between 3 and \( t \), and corresponding extremal examples are constructed explicitly (as was mentioned briefly in the beginning of the introduction). For \( t \geq \frac{1}{2}(n+3) \), the graph consisting of two cliques, one of size \( \binom{t-1}{2} \) and the other of size \( \binom{n-t+2}{2} \), sharing exactly one vertex, does not contain a cycle of length \( t \) or longer. For \( t < \frac{1}{2}(n+3) \), the complete bipartite graph with \( \frac{1}{4}n^2 \) edges does not contain a cycle of any odd length, and in particular of any odd length between 3 and \( t \).

Note that the function \( w(t,n) \) of Woodall is strongly related to the function \( g^2(t,n) \) given in Definition 1.3. In particular, for odd values of \( t \) we have \( w(t,n) = g_\circ(t,n) \), and furthermore, \( w(t,n) \geq \frac{1}{2} \) for any \( t \) and \( n \). In addition, \( w(t,n) \) is monotone increasing in \( t \) for this case. So for any odd \( t \), \( w(t,n) \binom{n}{2} = g_\circ(t,n) \binom{n}{2} = ex(n,C_t) - 1 \). As for even values of \( t \), we get \( w(t,n) = g_\circ(t,n) \) only when
t ≥ \frac{1}{2}(n+3)$. For an even $t < \frac{1}{2}(n+3)$, note that we do not necessarily have $w(t,n)\left(\frac{n}{2}\right) = ex(n,C_t)+1$ (although, as mentioned, Theorem 2.2 is still tight because of the requirement of having all cycles, also the odd ones, of length at most $t$.) For this reason, we also make use of $ex(n,P_t)$ in Definition 1.3 for even values of $t < \frac{1}{2}(n+3)$.

Remark 2.3. Note that if $0 < \varphi < \frac{1}{2}$ is constant and $t = (1 - \varphi + o_n(1))n$ then $w(t,n) = 1 - 2\varphi + 2\varphi^2 + o_n(1)$. In particular, if $e(G) \geq (1 - \varphi)\left(\frac{n}{2}\right)$, then $G$ contains a cycle of length $d$ for any $3 \leq d \leq (1 - \varphi)n$.

Another result to be used in this paper in a significant way is by Friedman and Pippenger [17], regarding the existence of large trees in expanding graphs.

**Theorem 2.4** ([17], Theorem 1). Let $T$ be a tree on $k$ vertices of maximum degree at most $d$. Let $H$ be a non-empty graph such that, for every $X \subseteq V(H)$ with $|X| \leq 2k - 2$ we have $|\Gamma_H(X)| \geq (d+1)|X|$. Let further $v \in V(H)$ be an arbitrary vertex of $H$. Then $H$ contains a copy of $T$, rooted at $v$.

### 2.2 Sparse Regularity Lemma

In order to prove Theorem 1.4, we make use of a variant of Szemerédi’s Regularity Lemma [41] for sparse graphs, the so-called Sparse Regularity Lemma due to Kohayakawa [25] and Rödl (see [9, 21, 28]). The sparse version of the Regularity Lemma is based on the following definition.

**Definition 2.5.** Let a graph $G = (V,E)$ and a real number $p \in (0,1]$ be given. We define the $p$-density of a pair of non-empty, disjoint sets $U,W \subseteq V$ in $G$ by

$$d_{G,p}(U,W) = \frac{e_G(U,W)}{p|U||W|}.$$  

For any $0 < \varepsilon \leq 1$, the pair $(U,W)$ is said to be $(\varepsilon,G,p)$-regular, or just $(\varepsilon,p)$-regular for short, if, for all $U' \subseteq U$ with $|U'| \geq \varepsilon|U|$ and all $W' \subseteq W$ with $|W'| \geq \varepsilon|W|$, we have

$$|d_{G,p}(U,W) - d_{G,p}(U',W')| \leq \varepsilon. \quad (1)$$

We say that a partition $\Pi = (V_1,\ldots,V_k)$ of $V$ is $(\varepsilon,p)$-regular if $||V_i| - |V_j|| \leq 1$ for all $i,j \in [k]$, and, furthermore, at least $(1 - \varepsilon)\left(\frac{k}{2}\right)$ pairs $(V_i,V_j)$ with $1 \leq i < j \leq k$ are $(\varepsilon,p)$-regular.

In the case $p = 1$ we say that the pair (or the partition) is $\varepsilon$-regular.

**Theorem 2.6** (Sparse Regularity Lemma [25]). For any given $\varepsilon > 0$ and $k_0 \geq 1$, there are constants $\eta = \eta(\varepsilon,k_0) > 0$ and $K_0 = K_0(\varepsilon,k_0) \geq k_0$ such that any $(p,\eta)$-upper-uniform graph $G$ with $0 < p \leq 1$ admits an $(\varepsilon,p)$-regular partition of its vertex set into $k$ parts, where $k_0 \leq k \leq K_0$.

**Remark 2.7.** In the main proof we make an extensive use of the Sparse Regularity Lemma (and, in fact, also of the Regularity Lemma in Section 6.2). As a result, we need to keep many parameters in mind. For simplicity, we present here some of the parameters and the relations between them. Unless mentioned otherwise, these are the values of the parameters during the proofs in the next sections, given here for future reference:
\[ \varepsilon \leq \frac{\beta}{10000}, \]  
regularity parameter

\[ \rho = 10\varepsilon \]  
density parameter

\[ k \geq k_0 \geq \frac{2}{\varepsilon^2} \]  
number of clusters

\[ \eta \leq \min\left(\frac{1}{3k_0}, \eta^*\right) \]  
parameter of upper uniformity

\[ \tau = \frac{\beta}{32} \]  
“extra” number of edges we have in the reduced graph

\[ \delta = 48\varepsilon \]  
proportion of number of vertices we are not able to use in each cluster

\[ \gamma \leq \frac{2(1-48\varepsilon)}{k} \]  
parameter of \( g(t, n) \) and \( g^*(t, n) \) that appears in Definition 1.3 and in Theorem 1.4

\[ m = \frac{n}{k} \]  
size of each cluster up to \( \pm 1 \)

where \( K_0 \) and \( \eta^* \) are as given in Theorem 2.6 (taking \( \eta^* \) to be \( \eta \)).

### 2.3 Organization

As was mentioned before, in the proof of Theorem 1.4 we rely heavily on the Sparse Regularity Lemma (see Section 2.2). Roughly speaking, we use the lemma to obtain a regular partition of our graph into clusters. Then, we define an auxiliary graph (the reduced graph) in which each vertex represents a cluster of the original graph, and show that if this auxiliary graph has enough edges, then the original graph contains the desired cycle. For this, in Section 3 we define the Reduced Graph and prove that it contains many edges. Then, in Section 4 we present the Key Lemma used in the paper to convert a cycle in the reduced graph to a cycle of an appropriate length in the original graph. In the same section we give the proof of Theorem 1.4 using the Key Lemma. Section 5 is devoted for the proof of the Key Lemma. In Section 6 we give some further related results.

### 3 The Reduced Graph

**Definition 3.1** (Reduced Graph). Let \( \varepsilon > 0, \ k \geq 1 \) an integer, \( 0 \leq p \leq 1 \), and \( 0 < \rho \leq 1 \). Let \( G_0 \) be a graph on \( n \) vertices, and \( \Pi = (V_1, \ldots, V_k) \) a partition of its vertices. We define the reduced graph \( R(G_0, \Pi, \rho, \varepsilon, p) \) to be the graph on the vertex set \( \{1, \ldots, k\} \), where vertices \( i \) and \( j \) are connected by an edge if and only if \( (V_i, V_j) \) is \( (\varepsilon, p) \)-regular and \( d_{G_0,p}(V_i, V_j) \geq \rho \). If we consider the reduced graph where \( p = 1 \), we omit this parameter from the notation.

**Lemma 3.2.** Let \( 0 < \beta < \frac{1}{4}, \ x \in [0, 1) \) such that \( x + \beta < 1 \). Let \( \varepsilon \leq \frac{\beta}{10000}, \ k \geq \frac{100}{\beta}, \ \tau = \frac{\beta}{32} \), and \( \eta < \frac{1}{3k} \) be positive. Assume that \( G \) is an \((p, \eta)\)-upper uniform graph, and \( e(G) \geq (1 - \beta/2)p(n/2) \), for some \( 0 < p := p(n) \leq 1 \). Let \( G' \) be the graph obtained from \( G \) by keeping at least \((x + \beta)\ e(G) \) edges, and assume that \( \Pi = (V_1, \ldots, V_k) \) is an \((\varepsilon, p)\)-regular partition of \( G' \). Let \( R := R(G', \Pi, \rho, \varepsilon, p) \) be the reduced graph as in Definition 3.1 for \( \rho = 10\varepsilon \). Then

\[ e(R) \geq (x + \tau) \left( \frac{k}{2} \right) \]

**Proof.** Denote \( m = \frac{n}{k} \), and recall that \( \lfloor m \rfloor \leq |V_i| \leq \lceil m \rceil \) for any \( i \in [k] \). Now we count the number of edges of \( G' \).

- The number of edges with endpoints in the same \( V_i \) for \( 1 \leq i \leq k \) is at most

\[ k(1 + \eta)p \left( \frac{\lfloor m \rfloor}{2} \right) \leq \frac{2}{k} (1 + \eta)^2 \frac{pm^2}{2} \]

9
• The number of edges in irregular pairs is at most
\[ \varepsilon \binom{k}{2}(1 + \eta)p[m]^2 \leq \varepsilon(1 + \eta)^2 \frac{pm^2}{2}. \]

• The number of edges in pairs that are of \( p \)-density less than \( \rho \) is at most
\[ \binom{k}{2}p[m]^2 \leq (1 + \eta)\frac{pn^2}{2}. \]

• The number of edges in \((\varepsilon, p)\)-regular pairs \((V_i, V_j)\) with \( p \)-density at least \( \rho \) is at most
\[ e(R)(1 + \eta)p[m]^2 \leq e(R)(1 + \eta)^2 \frac{2}{k^2} \frac{pm^2}{2}. \]

In total we get
\[ e(G') \leq (1 + \eta) \left( (1 + \eta) \left( \frac{2}{k^2} e(R) + \varepsilon + \frac{2}{k} \right) + \rho \right) \frac{pn^2}{2}. \]

On the other hand, recall that
\[ e(G') \geq (x + \beta) e(G) \geq (x + \beta) (1 - \beta/2) p \binom{n}{2} \]
\[ \geq (x + \beta) \left( 1 - \frac{2\beta}{3} \right) \frac{pn^2}{2}, \]
so we get
\[ (1 + \eta) \left( (1 + \eta) \left( \frac{2}{k^2} e(R) + \varepsilon + \frac{2}{k} \right) + \rho \right) \frac{pn^2}{2} \geq (x + \beta) \left( 1 - \frac{2\beta}{3} \right) \frac{pn^2}{2}, \]
and hence
\[ e(R) > \left( \frac{(x + \beta) \left( 1 - \frac{2\beta}{3} \right) - \rho}{(1 + \eta)^2} - \varepsilon - \frac{2}{k} \right) \frac{k^2}{2} \geq (x + \tau) \binom{k}{2}, \]
where the last inequality follows by the choice of the parameters \( \varepsilon, \eta, \rho, k, \tau \) combined with the fact that \( x \leq 1. \)

\[ \square \]

4 Key Lemma and proof of Theorem 1.4

In this section we state the Key Lemma and then use it to prove Theorem 1.4.

Definition 4.1. Let \( G \) be a graph and let \( V_1, V_2 \subseteq V(G) \) be two disjoint subsets of vertices with \(|V_1|, |V_2| \in \{[m], [m]\}\) for some positive number \( m \). Let \( \varepsilon > 0 \). We say that the pair \((V_1, V_2)\) satisfies the \( \varepsilon \)-property in \( G \) if for every two subsets \( U_1 \subseteq V_1 \) and \( U_2 \subseteq V_2 \) with \(|U_1|, |U_2| \geq \varepsilon m, G\) contains at least one edge between them, i.e., \( e(G[U_1, U_2]) > 0. \)
Corollary 4.4. Let $\varepsilon > 0$ and let $k$ be a positive integer. Let $G_0$ be a graph and let $\Pi = (V_1, \ldots, V_k)$ be a partition of its vertices into $k$ parts satisfying $\|V_i| - |V_j|\| \leq 1$ for all $i, j \in [k]$. We define the $\varepsilon$-graph $S := S(G_0, \Pi, \varepsilon)$ to be the graph with vertex set $[k]$ where $\{i, j\} \in E(S)$ if the pair $(V_i, V_j)$ satisfies the $\varepsilon$-property in $G_0$.

Lemma 4.3 (Key Lemma). Let $0 < \varepsilon < \frac{1}{20}$. Let $G_0$ be a graph on $n$ vertices, for large enough $n$, and let $\Pi = (V_1, \ldots, V_k)$ be a partition of its vertices satisfying $\|V_i| - |V_j|\| \leq 1$ for all $i, j \in [k]$, where $\frac{2}{\varepsilon} \leq k$ is a constant. Let $S := S(G_0, \Pi, \varepsilon)$ be the corresponding $\varepsilon$-graph as in Definition 4.2. Then for $\delta = 48\varepsilon$ and any absolute constant $C_1 > 2.1$ we have the following.

- If $S$ contains a path of an odd length $b$, $1 \leq b < k$, then $G_0$ contains cycles of all even lengths in $\left[\frac{C_1}{\log(1/\varepsilon)} \log n, (1 - \delta)an\right]$, with $a := \frac{b+1}{k}$.

- If $S$ contains a cycle of an odd length $b$, $3 \leq b < k$, then $G_0$ contains cycles of all odd lengths in $\left[\frac{(b-1)C_1}{2\log(1/\varepsilon)} \log n, (1 - \delta)an\right]$, with $a := \frac{b}{k}$.

The assumption in the first item that $b$ is odd is of technical nature and is in fact an artifact of our proof strategy. The proof of the Key Lemma can be found in Section 5.3.

Using this Key Lemma, we can deduce the existence of long cycles in a graph in cases where there are enough edges in a corresponding $\varepsilon$-graph.

Corollary 4.4. Let $0 < \beta < 1/3$, $\varepsilon = \frac{\beta}{10000}$, $k \geq \frac{2}{\varepsilon}$, $\gamma = \frac{2(1-48\varepsilon)}{k}$, and $\delta = 48\varepsilon$. Let $G_0$ be a graph on $n$ vertices, for large enough $n$, and let $C_1$ be an absolute constant and $C_1$ is the absolute constant from Lemma 4.3. Assume that there exists a partition $\Pi = (V_1, \ldots, V_k)$ of the vertices of $G_0$ such that the corresponding $\varepsilon$-graph $S := S(G_0, \Pi, \varepsilon)$ satisfies $e(S) \geq (g^\gamma(t, n) + \beta/32)(k)$, where $g^\gamma(t, n)$ is defined in Definition 1.3. Then $G_0$ contains a cycle of length $t$.

Proof. We split the proof into four cases by the parity and the value of $t$. Throughout all following cases we use the facts that $1 - C_2\beta \leq 1 - \delta$ and that $\varepsilon < \beta$.

Case 1: $t$ is even and $t < \gamma n$. In this case we have $g^\gamma(t, n) = 0$, and in particular $e(S) \geq \frac{\beta}{32}(k)$.

By Theorem 2.1 we get that $S$ contains a path of length at least $\frac{\beta}{32} \cdot k > 1$ and hence, by Lemma 4.3, $G_0$ contains a cycle of length $t$.

Case 2: $t$ is even and $\gamma n \leq t < \frac{1}{2}(n+3)$. In this case we have $g^\gamma(t, n) = \frac{1}{2} \binom{n(t-1)+1}{n}$, and in particular $e(S) \geq \left(\frac{1}{2} \binom{n(t-1)+1}{n} + \frac{\beta}{32}\right) \binom{k}{2} \geq \left(\frac{t}{n} + \frac{\beta}{32}\right) \binom{k}{2}$. By Theorem 2.1 we get that $S$ contains a path of length at least $\left(\frac{t}{n} + \frac{\beta}{32}\right) k$ and hence, by Lemma 4.3, and since $t < \left(\frac{t}{n} + \frac{\beta}{32}\right) (1 - \delta)n$, $G_0$ contains a cycle of length $t$.

Case 3: $t$ is odd and $t < \frac{1}{2}(n+3)$. In this case we have $g_0(t, n) = \frac{1}{2} \binom{n(t-2)+1}{n}$, and in particular $e(S) \geq \left(\frac{1}{2} \binom{n(t-2)+1}{n} + \frac{\beta}{32}\right) \binom{k}{2} > \left(\frac{t}{n} + \frac{\beta}{32}\right) \binom{k}{2}$. By Remark 2.3 and Theorem 2.2 we get that $S$ contains cycles of all lengths up to $\left(\frac{t}{n} + \frac{\beta}{32}\right) k > \left(\frac{n+3}{2n} + \frac{\beta}{32}\right) k > \left(\frac{t}{n} + \frac{\beta}{32}\right) k$ and hence, by Lemma 4.3, $G_0$ contains a cycle of length $t$. 

11
Proof of Theorem 1.4. In this case we have $g^n(t, n) = \frac{(\frac{t}{2^s} (n^2 + n + 1) + 1}{n}$, and thus we can rewrite $e(S) \geq \left(1 + \frac{\beta}{1 + \frac{\beta}{\sqrt{2}}(k)}\right)$ for some $\beta > 0$. By Remark 2.3 and Theorem 2.2 we get that $S$ contains all cycles of lengths up to $\frac{t}{n} \left(1 + \frac{\beta}{\sqrt{2}}\right) (1 - \frac{\beta}{\sqrt{2}}) t - 1$, we get that $G_0$ contains a cycle of length $t$, where if $t$ is odd then we look at the path in $S$ and if $t$ is even then we look at the path.

**Case 4:** $\frac{1}{2} (n + 3) \leq t \leq (1 - C_2 \beta)n$. In this case we have $g^n(t, n) = \frac{(\frac{t}{2^s} (n^2 + n + 1) + 1}{n}$, and thus $e(S) \geq \left(1 + \frac{\beta}{1 + \frac{\beta}{\sqrt{2}}(k)}\right)$ for some $\beta > 0$. By Remark 2.3 and Theorem 2.2 we get that $S$ contains all cycles of lengths up to $\frac{t}{n} \left(1 + \frac{\beta}{\sqrt{2}}\right) (1 - \frac{\beta}{\sqrt{2}}) t - 1$, we get that $G_0$ contains a cycle of length $t$, where if $t$ is odd then we look at the path in $S$ and if $t$ is even then we look at the path.

**Remark 4.5.** Given $\gamma$, note that the function $g^n(t, n)$ is monotone in the following sense. For any $0 < t < \frac{1}{2} (n + 3)$ we have $g^n(2t + 1, n) \geq g^n(2t, n)$, $g^n(2t + 1, n) \geq g^n(2t - 1, n)$, and $g^n(2t + 2, n) \geq g^n(2t, n)$. In addition, if $t \geq \frac{1}{2} (n + 3)$ then $g^n(t + 1, n) \geq g^n(t, n)$. Consequently, under the assumptions of Corollary 4.4 if $t$ is odd then $G$ contains all cycles of lengths between $\frac{C_1}{\log(1/\beta)} \log n$ and $t$, and if $t$ is even then $G$ contains all cycles of lengths between $\frac{C_1}{\log(1/\beta)} \log n$ and $t$. In addition, if $t \geq \frac{1}{2} (n + 3)$ then $G$ contains all cycles of lengths between $\frac{C_1}{\log(1/\beta)} \log n$ and $t$ (regardless of the parity of $t$).

We next show that $p$-regular pairs of subsets in our graph with non-negligible $p$-density satisfy the $\varepsilon$-property. Then by the Key Lemma we can deduce the main theorem.

**Claim 4.6.** Let $n$ be an integer, $\varepsilon > 0$, $\varepsilon < p < \frac{1}{2}$. Let $G_0$ be a graph on $n$ vertices and let $V_1, V_2 \subseteq V(G_0)$ be two subsets of vertices satisfying: $V_1 \cap V_2 = \emptyset, |V_1|, |V_2| \in \{[m], [n/2]\}$ for some $m$, and the pair $(V_1, V_2)$ is $(\varepsilon, p)$-regular in $G_0$ with $d_{G_0, p}(V_1, V_2) \geq \rho$, for some $0 < \rho \leq 1$. Then the pair $(V_1, V_2)$ satisfies the $\varepsilon$-property in $G_0$.

**Proof.** Let $U_1 \subseteq V_1$ and $U_2 \subseteq V_2$ be such that $|U_1|, |U_2| \geq \varepsilon m$. By regularity we have $|d_{G_0, p}(V_1, V_2) - d_{G_0, p}(U_1, U_2)| \leq \varepsilon$. Combining it with the assumption $d_{G_0, p}(V_1, V_2) \geq \rho$, we have that $e(U_1, U_2) \geq (\rho - \varepsilon) p |U_1| |U_2| > 0$.

Using Corollary 4.4 we can immediately prove our main theorem.

**Proof of Theorem 1.4.** Let $\varepsilon = \frac{\beta}{10000}, \rho = 10 \varepsilon$, and $k_0 = \frac{2}{\varepsilon^2}$. Let $\eta_0 := \eta_0(\varepsilon, k_0) > 0, K_0 := K_0(\varepsilon, k_0) \geq k_0$, and $k \in [k_0, K_0]$ be as given by the Sparse Regularity Lemma (Theorem 2.6) applied with $\varepsilon$ and $k_0$. Let $\eta := \min\{\eta_0, \frac{1}{3k_0}\}$ and let $\gamma = \frac{2(1 - 4\varepsilon)}{k}$. Recall that $G$ is a $(p, \eta)$-upper-uniform graph for some $0 < \rho := p(n) \leq 1$, with $e(G) \geq (1 - \frac{\beta}{2}) p(n)$. Let $G'$ be a graph obtained from $G$ by keeping at least $(g^n(t, n) + \beta) e(G)$ edges, and note that $G'$ is also $(p, \eta)$-upper-uniform. Let $\Pi = (V_1, \ldots, V_k)$ be an $(\varepsilon, p)$-regular partition of $G'$ guaranteed by the Sparse Regularity Lemma for the relevant parameters, for some $k_0 \leq k \leq K_0$. Let $R := R(G', \Pi, \varepsilon, p, \rho)$ be the reduced graph on $k$ vertices with parameters $\rho, \varepsilon, p$ and $k$, as in Definition 3.1. By Claim 4.6 if $\{i, j\}$ is an edge in $R$, then the pair $(V_i, V_j)$ satisfies the $\varepsilon$-property in $G'$. Hence, the reduced graph $R$ is a subgraph of the $\varepsilon$-graph $S := S(G', \Pi, \varepsilon)$, as defined in Definition 4.2. In particular $e(S) \geq e(R)$, and every path or cycle contained in $R$ is also contained in $S$.

Let $C_1 > 0$ be the constant from Lemma 4.3 and let $C_2 > 0$ be the constant from Corollary 4.4. Let $\frac{C_1}{\log(1/\beta)} \log n \leq t \leq (1 - C_2 \beta)n$. By Lemma 3.2 we have that $e(R) \geq (g^n(t, n) + \tau) \frac{n}{2}$, and thus $e(S) \geq (g^n(t, n) + \tau) \frac{n}{2}$. Applying Corollary 4.4 we get a cycle of length $t$ in $G$.\[\Box\]
Applying [Remark 4.5] to [Theorem 1.4], note that if \( t \) is odd then \( G \) contains all cycles of lengths between \( \frac{G_1 \log n}{\log(1/\beta)} \) and \( t \), and if \( t \) is even then \( G \) contains all even cycles of lengths between \( \frac{G_1 \log n}{\log(1/\beta)} \) and \( t \). In addition, if \( t > \frac{1}{2}(n+3) \) then \( G \) contains all cycles of lengths between \( \frac{G_1 \log n}{\log(1/\beta)} \) and \( t \) (regardless of the parity of \( t \)).

## 5 Proof of the Key Lemma

In this section we prove the Key Lemma (Lemma 4.3) using several claims and results regarding tree embeddings in expander graphs. The main idea is to show that every two vertices connected by an edge in the reduced graph represent a pair of clusters in the original graph that has “good expansion” properties (Section 5.1). Then, we show that the graph induced by any pair of such clusters contains a very specific tree (Section 5.2), which will later be used to embed the desired cycle (Section 5.3).

### 5.1 Expander graphs

**Definition 5.1.** A graph \( G = (V, E) \) is called a \((B, \ell)\)-expander if for every \( X \subseteq V \) with \( |X| \leq B \) we have \( |\Gamma_G(X)| \geq \ell |X| \).

For the proofs in this section we also need a somewhat more specific definition of expander graphs for the special case of bipartite graphs.

**Definition 5.2.** A bipartite graph \( G = (V_1 \cup V_2, E) \) is called a \((B,\ell)\)-bipartite-expander if for every \( X \subseteq V_i \), \( 1 \leq |X| \leq B \), we have \( |\Gamma_G(X)| \geq \ell |X| \).

**Remark 5.3.** If a bipartite graph \( G \) is an \((A, \ell+1)\)-bipartite-expander, then it is a \((2A, \frac{1}{2}\ell)\)-expander.

**Proposition 5.4.** Let \( \varepsilon > 0 \) and let \( a, b > 0 \) satisfy \((2b+2)(1-\varepsilon-ab) > 1 \) and \((2b+2)\varepsilon \geq 1 \). Let \( G \) be a bipartite graph with parts \( V_1, V_2 \) with \(|V_1|, |V_2| \geq (2b+2)\varepsilon m \) for some integer \( m \), and assume that every two subsets \( V_1^i, \subseteq V_1, V_2^i \subseteq V_2 \) with \(|V_1^i|, |V_2^i| \geq \varepsilon m \) span at least one edge in \( G \), i.e., \( e(G[V_1^i, V_2^i]) > 0 \). Then there exist \( U_1 \subseteq V_1 \) and \( U_2 \subseteq V_2 \) with \(|U_1| \geq (1-\varepsilon)|V_1| \) and \(|U_2| \geq (1-\varepsilon)|V_2| \) such that the bipartite graph \( G[U_1, U_2] \) is an \((a, b)\)-bipartite-expander, where \( x = \min(|V_1|, |V_2|) \).

**Proof.** If every subset of \( X_i \subseteq V_i \) of size at most \( ax \) satisfies \(|\Gamma_G(X_i)| \geq b|X_i| \) then we are done by setting \( U_1 = V_1 \) and \( U_2 = V_2 \). Otherwise, there are subsets violating the expansion condition. We iteratively remove such subsets of size at most \( \varepsilon m \), one by one, to create an \((\varepsilon m, b)\)-bipartite-expander. We then show that the expander we have created is, in fact, an \((a, b)\)-bipartite-expander. More formally, we define \( V_1^0 = V_1, V_2^0 = V_2 \) and \( W_0^0 = W_0^0 = \emptyset \). Let \( r \in \mathbb{N} \cup \{0\} \). If for \( 1 \leq i \neq j \leq 2 \), there exists \( V_i^r \subseteq V_i^r \) such that \(|W| \leq \varepsilon m \) and \(|\Gamma(W) \cap V_j^{r+1}| < b|W| \), then we define \( V_i^{r+1} = V_i^r \setminus W, V_j^{r+1} = V_j \), and \( W_i^{r+1} = W_i^r \cup W, W_j^{r+1} = W_j^r \). If at some point \( r \) there are no more subsets violating the \((\varepsilon m, b)\)-expansion condition in \( V_i^r, V_j^r \), and we have \(|W_1^r|, |W_2^r| < \varepsilon m \), then we define \( U_1 = V_1^r, U_2 = V_2^r \), which means that the graph \( G[U_1, U_2] \) is an \((\varepsilon m, b)\)-bipartite-expander. Otherwise, for some \( r \) we have, for the first time in this process, \(|W_i^r| \geq \varepsilon m \) for some \( i \in \{1, 2\} \). Since in each step \( r \) of the process we add to one of \( W_1^{r+1}, W_2^{r-1} \) at most \( \varepsilon m \) vertices, it follows that \( \varepsilon m \leq |W_i^r| \leq 2\varepsilon m \). By the definition of \( W_i^r \) we get \(|\Gamma(W_i^r) \cap V_j^r| < b|W_i^r| \), where \( i \neq j \in \{1, 2\} \). By the choice of \( r \) we know that \(|W_i^r| < \varepsilon m \) \((j \neq i)\), and thus

\[
|V_j^r \setminus \Gamma(W_i^r)| > |V_j| - |W_i^r| - b|W_i^r| \geq (2b+2)\varepsilon m - \varepsilon m - 2b\varepsilon m \geq \varepsilon m.
\]
It follows from our assumption that \( e_G(W_{i}^{m} \setminus \Gamma(W_{i}^{m})) > 0 \), which is a contradiction. Hence in the end of this vertex-removal process we are left with \( U_1 \subseteq V_1 \) and \( U_2 \subseteq V_2 \) of sizes \(|U_1| \geq (1 - \varepsilon)|V_1|\) and \(|U_2| \geq (1 - \varepsilon)|V_2|\) such that the bipartite graph \( G[U_1, U_2] \) is an \((\varepsilon m, b)\)-bipartite-expander.

We conclude by proving that \( G[U_1, U_2] \) is in fact an \((ax, b)\)-bipartite-expander. Assume, for contradiction, that for \( 1 \leq i \neq j \leq 2 \) there exists \( W \subseteq U_i \) with \( \varepsilon m < |W| \leq ax \) and such that \(|\Gamma(W) \cap U_j| < b|W|\). Recall that \( x = \min\{|V_1|, |V_2|\} \geq (2b + 2)\varepsilon m \), so it follows that

\[
|U_j \setminus \Gamma(W)| > (1 - \varepsilon)x - abx \geq (1 - \varepsilon - ab)x \geq (2b + 2)(1 - \varepsilon - ab)\varepsilon m > \varepsilon m,
\]

and by the assumption we get \( e(W, U_j \setminus \Gamma(W)) > 0 \), which is a contradiction. \( \square \)

**Corollary 5.5.** Let \( G \) be a bipartite graph with parts \( V_1, V_2 \), let \( m \) be some integer, let \( 0 < \varepsilon < \frac{1}{85} \), and denote \( x = \min(|V_1|, |V_2|) \). Assume that every two subsets \( V'_1 \subseteq V_1, V'_2 \subseteq V_2 \) with \(|V'_1|, |V'_2| \geq \varepsilon m \) span at least one edge in \( G \), i.e., \( e(G[V'_1, V'_2]) > 0 \). Then there exist \( U_1, W_1 \subseteq V_1 \) and \( U_2, W_2 \subseteq V_2 \) with \(|U_1|, |W_1| \geq (1 - \varepsilon)|V_1|\) and \(|U_2|, |W_2| \geq (1 - \varepsilon)|V_2|\) such that

1. If \(|V_1|, |V_2| \geq \frac{x}{2}m - 1\) then the bipartite graph \( G[W_1, W_2] \) is a \((6\varepsilon x, \frac{1}{85} + 1)\)-bipartite-expander, and hence a \((12\varepsilon x, \frac{1}{85})\)-expander.
2. If \(|V_1|, |V_2| \geq 20\varepsilon m\) then the bipartite graph \( G[U_1, U_2] \) is a \((\frac{4}{10}x, 9)\)-bipartite-expander, and hence a \((\frac{2}{5}x, 4)\)-expander.

### 5.2 Tree embeddings

We start by defining the following trees, playing a key role in our proofs.

**Definition 5.6.** Let \( T^{(r, h)} \) be the \( r \)-ary tree of depth \( h \) (that is, the tree where each vertex, but a leaf, has \( r \) children, and the distance, in edges, between the root and every leaf is exactly \( h \)). Let \( T^{(r, h)}_\ell \) be the tree consisting of two disjoint copies of \( T^{(r, h)} \) and a path of length \( \ell \) connecting their roots.

**Remark 5.7.** Note that a longest path in \( T^{(r, h)}_\ell \) is of length \( \ell + 2h \). Furthermore, the tree \( T^{(r, h)}_\ell \) has exactly \( \ell - 1 + 2 - \frac{r^h + 1}{r - 1} \) vertices.

The main ingredients in the proof of** Lemma 4.3** are the following claims regarding tree embeddings in bipartite-expander graphs.

**Proposition 5.8.** Let \( G \) be a bipartite graph with parts \( V_1, V_2 \) with \(|V_1|, |V_2| \in \{[m], [m]\}\) for some positive number \( m \). Let \( 0 < \varepsilon < \frac{1}{85} \) and assume that the pair \((V_1, V_2)\) satisfies the \( \varepsilon \)-property in \( G \). Then \( G \) contains every tree on at most \( 6\varepsilon m \) vertices with maximum degree at most \( \frac{1}{16\varepsilon} - 1 \). In particular, \( G \) contains a copy of \( T^{(r, h)}_\ell \) where \( r = \lceil \frac{1}{16\varepsilon} \rceil - 2 \), \( h = \lceil \frac{\log(\varepsilon m)}{\log r} \rceil \), and any integer \( \ell \in [1, 2\varepsilon m] \).

**Proof.** By** Corollary 5.5** there are subsets \( U_1 \subseteq V_1, U_2 \subseteq V_2 \) for which the graph \( G[U_1, U_2] \) is an \((12\varepsilon m, \frac{1}{16\varepsilon})\)-expander. By** Theorem 2.3** we get that \( G[U_1, U_2] \) contains a copy of any tree on at most \( 6\varepsilon m \) vertices with maximum degree at most \( \frac{1}{16\varepsilon} - 1 \). Set \( r = \lceil \frac{1}{16\varepsilon} \rceil - 2 \), \( h = \lceil \frac{\log(\varepsilon m)}{\log r} \rceil \), and \( \ell \in [1, 2\varepsilon m] \). By** Remark 5.7** the tree \( T^{(r, h)}_\ell \) has at most \( 6\varepsilon m \) vertices and maximum degree at most \( \frac{1}{16\varepsilon} - 1 \), so in particular \( G[U_1, U_2] \) contains a copy of it. \( \square \)
Proposition 5.9. Let $G$ be a bipartite graph on parts $V_1, V_2$ with $|V_1|, |V_2| \in \{[m], [m]\}$ for some positive number $m$. Let $0 < \varepsilon < \frac{1}{85}$ and assume that the pair $(V_1, V_2)$ satisfies the $\varepsilon$-property in $G$. Then $G[V_1, V_2]$ contains a copy of $T_{\ell}^{(2, h)}$ for $h = \left\lceil \frac{\log(m)}{\log 2} \right\rceil$, and any integer $\ell \in [1, 2(1 - 48\varepsilon)m]$. Moreover, if $\ell$ is even then we can embed a copy of $T_{\ell}^{(2, h)}$ with all leaves in $V_i$ for any $i \in \{1, 2\}$.

Proof. Assume first that $\ell$ is odd. Let $U_{11}, U_{12} \subseteq V_1$ be disjoint, and $U_{21}, U_{22} \subseteq V_2$ be also disjoint, such that $|U_{ij}| \geq [21\varepsilon m]$ for any $i, j \in \{1, 2\}$. By Corollary 5.5 (item 2) applied separately on $G[U_{11}, U_{21}]$ and on $G[U_{12}, U_{22}]$ we get four subsets $W_{ij} \subseteq U_{ij}$, $i, j \in \{1, 2\}$, all of size at least $20\varepsilon m$, such that each of the graphs $G[W_{11}, W_{21}]$ and $G[W_{12}, W_{22}]$ is a $(\frac{1}{2}\varepsilon m, 4)$-expander.

Let $X_1 \subseteq V_1 \setminus (W_{11} \cup W_{12})$ and let $X_2 \subseteq V_2 \setminus (W_{21} \cup W_{22})$ be such that $|X_1| = |X_2| = \lceil (1 - 43\varepsilon) m \rceil$. Let $\ell \in [1, 2(1 - 48\varepsilon)m]$ be odd, and let $q = 4\lceil \varepsilon m \rceil$. We now find a path of length exactly $\ell - 4 + q$. We do this using the following claim, implied by a standard DFS-based argument, stated implicitly in [5] and more explicitly in, e.g., [36]. For a more extensive discussion about the DFS (Depth First Search) algorithm in finding paths in expander graphs we refer the reader to [29].

Claim 5.10. For every graph $G$ there exists a partition of its vertices $V = S \cup T \cup U$ such that $|S| = |T|$, $G$ has no edges between $S$ and $T$, and $U$ spans a path in $G$.

Apply Claim 5.10 to the graph $G[X_1, X_2]$. Notice that $|U| = |X_1 \cup X_2| - |S| - |T| = 2|X_1| - 2|S|$ and in particular $|U|$ is even. $U$ spans a path in $G[X_1, X_2]$, which is a bipartite graph, so we get $|U \cap X_1| = |U \cap X_2|$. Assume w.l.o.g. that $|S \cap X_1| \geq |S \cap X_2|$, then $|T \cap X_2| \geq |T \cap X_1|$. If $|S| = |T| \geq 2\lceil \varepsilon m \rceil - 1$ then we get $|S \cap X_1|, |T \cap X_2| \geq \varepsilon m$. However, we know that $e(S \cap X_1, T \cap X_2) \leq e(S, T) = 0$, contradicting the $\varepsilon$-property of the pair $(V_1, V_2)$ in $G$. Hence we get that $|S| = |T| \leq 2\lceil \varepsilon m \rceil - 2$, which means that $|U| \geq 2\lceil (1 - 43\varepsilon) m \rceil - 4\lceil \varepsilon m \rceil + 4 \geq 2(1 - 45\varepsilon)m - 2$, and in particular $G[X_1, X_2]$ contains a path of length at least $2(1 - 45\varepsilon)m - 3$. Thus, let $P_0$ be a path of length $\ell - 4 + q \leq 2(1 - 45\varepsilon)m - 3$ and denote its endpoints by $u^* \in X_1$ and $v^* \in X_2$. Let $u_1, \ldots, u_q$ be the first $q$ vertices of $P_0$ when moving from $u^*$, that is $u^* = u_1$, and let $v_1, \ldots, v_q$ be the first $q$ vertices of $P_0$ when moving from $v^*$, that is $v^* = v_1$. Note that the vertices $\{u_1, \ldots, u_q\}$ are distributed equally between $X_1$ and $X_2$, having exactly $2\lceil \varepsilon m \rceil$ vertices in each set, and similarly the vertices $\{v_1, \ldots, v_q\}$. Consider now only the $2\lceil \varepsilon m \rceil$ vertices with odd indices, i.e., $\{u_1, u_3, \ldots, u_{q-1}\}$ and $\{v_1, v_3, \ldots, v_{q-1}\}$, and note that we have $\{u_1, u_3, \ldots, u_{q-1}\} \subseteq X_1$ and $\{v_1, v_3, \ldots, v_{q-1}\} \subseteq X_2$. Hence, by the $\varepsilon$-property of the pair $(V_1, V_2)$ in $G$, at least $\lceil \varepsilon m \rceil + 1$ of the vertices in $\{u_1, u_3, \ldots, u_{q-1}\}$ have some neighbor in $W_{21}$, and similarly, at least $\varepsilon m + 1$ of the vertices $\{v_1, v_3, \ldots, v_{q-1}\}$ have some neighbor in $W_{12}$. By the pigeonhole principle, there exists (an odd) $s \in \{1, \ldots, q - 1\}$ such that $u_s$ is connected to some vertex in $W_{21}$ and $v_{q-s}$ is connected to some vertex in $W_{12}$. Denote by $P$ the subpath of $P_0$ with endpoints $u_s$ and $v_{q-s}$, denoted by $u, v$, respectively, and note that it is of length exactly $\ell - 2$.

Now, let $w_1$ be a neighbor of $u$ in $W_{21}$ and $w_2$ be a neighbor of $v$ in $W_{12}$. Recall that by Theorem 2.4 there exists a copy of $T_{h}^{(2, h)}$ in $G[W_{11}, W_{21}]$, for $h = \left\lceil \frac{\log(cm)}{\log 2} \right\rceil$, rooted in any predetermined vertex of $W_{21}$. Similarly, there exists a copy of $T_{h}^{(2, h)}$ in $G[W_{12}, W_{22}]$, for the same value of $h$, rooted in any predetermined vertex of $W_{12}$. Let $T_{w_1}$ and $T_{w_2}$ be these copies of $T_{h}^{(2, h)}$ in $G[W_{11}, W_{21}]$ and in $G[W_{12}, W_{22}]$, respectively, rooted in $w_1 \in W_{21}$ and in $w_2 \in W_{12}$, respectively. Joining $T_{w_1}$ and $T_{w_2}$ to $P$, we get a copy of $T_{\ell}^{(2, h)}$, as required.

If $\ell$ is even then we repeat the same argument, with a minor change. Note first that if $\ell$ is even then any embedded copy of $T_{\ell}^{(2, h)}$ in $G[V_1, V_2]$ has all leaves in either $V_1$ or $V_2$. Assume that we wish to embed a copy of $T_{\ell}^{(2, h)}$ with all leaves in $V_i$ for some $i \in \{1, 2\}$. Note further that if $\ell$ is
even then $P_b$ is of an even length $\ell - 4 + q$, and hence both of its endpoints $u^*$ and $v^*$ are in $X_j$ for some $j \in \{1, 2\}$. Now, we look at $\{u_1, \ldots, u_q\}$ and $\{v_1, \ldots, v_q\}$ and split into two possible cases by the parity of $h$ and by the part in which the endpoints of $P_b$ are contained. If $h$ is even and $i \neq j$, or if $h$ is odd and $i = j$, then we consider only vertices of odd indices, i.e., $\{u_1, u_3, \ldots, u_{q-1}\}$ and $\{v_1, v_3, \ldots, v_{q-1}\}$. If $h$ is even and $i = j$, or if $h$ is odd and $i \neq j$, then we consider only vertices of even indices, i.e., $\{u_2, u_4, \ldots, u_q\}$ and $\{v_2, v_4, \ldots, v_q\}$. For simplicity we assume now that $h$ is even and $j = 1$, $i = 2$ (in particular $i \neq j$), where all other cases are handled similarly. This means that by the pigeon hole principle there exists (an odd) $s \in \{1, \ldots, q - 1\}$ such that $u_s$ is connected to some vertex in $W_{21}$ and $v_{q-s}$ is connected to some vertex in $W_{22}$, and equivalently to the odd $\ell$ case, we embed trees $T_{w_1}$ and $T_{w_2}$, having $w_1 \in W_{21}$ and $w_2 \in W_{22}$.

5.3 Proof of the Key Lemma

We are now ready to prove Lemma 4.3 using Proposition 5.8 and Proposition 5.9.

Proof of Lemma 4.3. Throughout the proof we denote $m := \frac{n}{2}$. Note that $k$ is constant, so $m = \Theta(n)$. Recall that $S$ is the $\varepsilon$-graph obtained from $G_0$ with respect to the partition $\Pi = (V_1, \ldots, V_k)$, that is, every edge $\{i, j\} \in E(S)$ represents a pair $(V_i, V_j)$ which satisfies the $\varepsilon$-property in $G_0$.

The general idea is to convert a cycle (or a path) from the graph $S$ to a cycle in $G_0$ of the desired length, by using tree embeddings between clusters of $G_0$. Assume that $(1, \ldots, b)$ is a cycle in $S$ and that $b$ is odd. Roughly speaking, we divide the cycle in $S$ into pairs of vertices that are connected with an edge $(2i, 2i + 1)$. We then embed in each pair of corresponding clusters $(V_{2i}, V_{2i+1})$ a tree $T^{(r,h)}_\ell$ with appropriate parameters such that the leaf sets are in different clusters. Since each of these leaf sets contains at least $\varepsilon m$ vertices, we can use the $\varepsilon$-property to connect some leaf from the leaves in $V_{2i+i}$ and some leaf from the leaves in $V_{2i+2}$ by an edge. This way, we are able to connect different copies of $T^{(r,h)}_\ell$ to a very large tree, containing a copy of $T^{(r,h)}_\ell$ for an appropriate $\ell^*$, where its leaf sets are in $V_2$ and $V_6$. We then use one vertex $v$ from $V_1$ and connect it to both leaf sets. This creates a cycle in $G_0$ of length exactly $t = \ell^* + 2h + 2$. For converting a path in $S$ to an even cycle in $G_0$ we use a similar argument, only this time we split each cluster into two clusters and use both endpoints of the path in $S$ to “close” the cycle in $G_0$. We give the full details below.

We start with the first item. Suppose that $S$ contains a path of an odd length $b$, where $1 \leq b < k$, and let $t \in \left[\frac{C_1}{\log(1/\varepsilon)} \log n, (1 - \delta)an\right]$ be even, $a := \frac{b+1}{k}$. Assume w.l.o.g. that this path is $(1, \ldots, b+1)$, and consider the sequence of corresponding clusters $V_1, \ldots, V_{b+1}$. We separate the case where $t$ is even into three parts. The first part deals with the case where $t \in \left[\frac{C_1}{\log(1/\varepsilon)} \log n, 2\varepsilon m\right]$, the second part deals with the case where $t \in \left[2\varepsilon m, (1 - \delta)an\right]$ and $b = 1$, and the third part deals with all other cases, i.e., $t \in \left[2\varepsilon m, (1 - \delta)an\right]$ and $b \geq 3$ (and is further separated into two subcases by the value of $b$). In each part we divide the vertices of the path into pairs, and embed a certain tree in the bipartite subgraph of the original graph induced by each pair. This is where we use the assumption of $b$ being odd, i.e., the path has an even number of vertices. A similar cluster pairing strategy was presented and used by Dellamonica et al. [12, Theorem 7].

If $t \in \left[\frac{C_1}{\log(1/\varepsilon)} \log n, 2\varepsilon m\right]$ is even, then we look at a single edge in the path, say, $(1, 2)$. The graph $G_0[V_1, V_2]$ is bipartite and the pair $(V_1, V_2)$ satisfies the $\varepsilon$-property in $G_0$. By Proposition 5.8 we know that $G_0[V_1, V_2]$ contains a copy of every tree with at most $6\varepsilon m$ vertices and maximum degree at most $\frac{1}{16\varepsilon} - 1$. In particular, $G_0[V_1, V_2]$ contains a copy of $T^{(r,h)}_\ell$ (as in Definition 5.6) for $r = \left\lfloor \frac{1}{16\varepsilon} \right\rfloor - 2$, $h = \left\lfloor \frac{\log(\varepsilon \log n)}{\log r} \right\rfloor$ and any odd $\ell \in [1, 2\varepsilon m]$ (as $T^{(r,h)}_\ell$ has at most $4\varepsilon m$ vertices for these
values of $r$ and $h$, and thus $T_{\ell}^{(r,h)}$ has at most $6\varepsilon m$. Note that a maximal path in $T_{\ell}^{(r,h)}$ is of length $2h + \ell$. Set $\ell = t - 2h - 1$ (note that it satisfies the constraints, as $1 \leq t - 2h - 1 \leq 2\varepsilon m$) and we get that a maximal path in a $T_{\ell}^{(r,h)}$-copy is of length exactly $t - 1$. Now, note that this copy of $T_{\ell}^{(r,h)}$ has at least $\varepsilon m$ leaves in $V_1$ and $\varepsilon m$ leaves in $V_2$, due to parity considerations. By the $\varepsilon$-property of the pair $(V_1, V_2)$ in $G_0$ there is an edge between these two sets of leaves, closing a cycle of length $\ell + 2h + 1 = t$, as required.

If $b = 1$ and $t \in [2\varepsilon m, (1 - \delta)an]$ is even, for $a := \frac{b+1}{k}$, then once again the graph $G_0[V_1, V_2]$ is bipartite and the pair $(V_1, V_2)$ satisfies the $\varepsilon$-property in $G_0$. We repeat the previous argument but with the only change of embedding a different tree in $G_0[V_1, V_2]$. By Proposition 5.9 we know that $G_0[V_1, V_2]$ contains a copy of $T_{\ell}^{(2,h)}$ for $h = \lceil \frac{\log(\varepsilon m)}{\log 2} \rceil$ and $\ell = t - 1 - 2h$. Also here, note that this copy of $T_{\ell}^{(2,h)}$ has at least $\varepsilon m$ leaves in $V_1$ and $\varepsilon m$ leaves in $V_2$, due to parity considerations. Again, by the $\varepsilon$-property of the pair $(V_1, V_2)$ in $G_0$ there is an edge between these two sets of leaves, closing a cycle of length $t$, as required.

If $b \geq 3$ and $t \in [2\varepsilon m, (1 - \delta)an]$ is even, for $a := \frac{b+1}{k}$, then we look at the full path $(1, \ldots, b+1)$ and the set of corresponding clusters $V_1, \ldots, V_{b+1}$. Informally, we embed two copies of $T_{\ell}^{(2,h)}$ for some carefully chosen values $h, \ell$, one in $G_0[V_1, V_2]$, and one in $G_0[V_b, V_{b+1}]$. Then, if we have used all the clusters already for tree embedding (i.e., $b = 3$), then we connect these two trees by two edges to create a cycle of the desired length. Otherwise, we keep embedding trees in all clusters we have not touched yet. Formally, we further separate this case into two subcases and argue as follows.

Assume first that $b = 3$. For following the arguments of this subcase Figure 1 can be helpful. Note that each of the pairs $(V_1, V_2)$ and $(V_3, V_4)$ satisfies the $\varepsilon$-property in $G_0$, and that we have $|V_j| \in \{|m|, |m|\}$ for any $j \in [4]$. Now let $j \in \{1, 3\}$. By Proposition 5.9 we know that $G_0[V_j, V_{j+1}]$ contains a copy of $T_{\ell_j}^{2,h}$ where $h = \lceil \frac{\log(\varepsilon m)}{\log 2} \rceil$ and $\ell_j$ is such that $\ell_1 + \ell_3 = t - 2h - 2, |\ell_1 - \ell_3| \leq 2$, and both are even. Note here that $\ell_j \leq \frac{1}{2}t - 2h \leq 2(1 - 48\varepsilon)m$. We embed two such $T_{\ell_j}^{(2,h)}$-copies, $j \in \{1, 3\}$, such that the leaf sets $L_2, L_2'$ and $L_3, L_3'$ are in $V_2$ and $V_3$, respectively (which is possible as $\ell_j$ is even). Having $|L_2|, |L_2'|, |L_3|, |L_3'| \geq \varepsilon m$, by the $\varepsilon$-property of the pair $(V_2, V_3)$ in $G_0$, there exist two edges, one between $L_2$ and $L_3$, and the other between $L_2'$ and $L_3'$. These two edges close a cycle of length exactly $t$.

Assume now that $b \geq 5$. When following the arguments of this subcase Figure 2 can be helpful. In this subcase too we embed two $T_{\ell}^{(2,h)}$-copies, for a suitable choice of $h, \ell$, in $G_0[V_1, V_2]$ and in $G_0[V_b, V_{b+1}]$. However, we do not connect them directly by two edges, but through other $T_{\ell}^{(2,h)}$-
copies we embed in the rest of clusters. More precisely, for each \(j \in \{3, \ldots, b-1\}\), arbitrarily split the vertex set \(V_j\) into two equally sized subsets (up to possibly one vertex), denoted by \(U_j, U'_j\). Set some \(i \in \{2, \frac{1}{2}(b-1)\}\) and look at the pair \((V_{2i-1}, V_{2i})\). Since \((V_{2i-1}, V_{2i})\) satisfies the \(\varepsilon\)-property in \(G_0\), it follows that each of the pairs \((U_{2i-1}, U_{2i})\) and \((U'_{2i-1}, U'_{2i})\) satisfies the \(\varepsilon'\)-property in \(G_0\) for \(\varepsilon' = \varepsilon (\frac{1}{2} m - 1) = \varepsilon m\) (namely, every two subsets, one from each set of the pair, of size at least \(\varepsilon' (\frac{1}{2} m - 1)\) each, span an edge in \(G_0\)). We have \(|U_{2i-1}|, |U_{2i}| \geq \frac{1}{2} m - 1\), so by Proposition 5.9 (taking \(\frac{1}{2} m - 1\) instead of \(m\)) we get that \(G_0[u_{2i-1}, u_{2i}]\) contains a copy of \(T^{(2, b)}_{\ell_0}\) for \(h = \left[\frac{\log(\varepsilon m)}{\log 2}\right]\) and \(\ell_0 = \left[\frac{b+1}{\varepsilon}\right] \text{odd} - 2h \leq 2(1 - 48\varepsilon)(\frac{1}{2} m - 1)\), where \([x]\) is the odd integer \(y\) such that \(y \leq x\) and \(x - y < 2\). Denote this copy by \(T_{2i-1, 2i}\) and its leaf sets in \(U_{2i-1}, U_{2i}\) by \(L_{2i-1}, L_{2i}\), respectively. We do the same for \(G_0[u'_{2i-1}, u'_{2i}]\) where we denote the embedded copy of \(T^{(2, b)}_{\ell_0}\) by \(T'_{2i-1, 2i}\), and its leaf sets in \(U'_{2i-1}, U'_{2i}\) by \(L'_{2i-1}, L'_{2i}\), respectively. Do this for every \(i \in \{2, \frac{1}{2}(b-1)\}\) with the same notations.

If \(b \geq 7\), then recall that for every \(i \in \{2, \frac{1}{2}(b-3)\}\) each of the pairs \((U_{2i}, U_{2i+1})\) and \((U'_{2i}, U'_{2i+1})\) satisfies the \(\varepsilon'\)-property in \(G_0\), and moreover, note that we have \(|L_{2i}|, |L_{2i+1}|, |L'_{2i}|, |L'_{2i+1}| \geq \varepsilon m\). Thus, for every \(i \in \{2, \frac{1}{2}(b-3)\}\) we have \(e_{G_0}(L_{2i}, L_{2i+1}), e_{G_0}(L'_{2i}, L'_{2i+1}) > 0\), so we add an edge between every such two leaf sets, summing up to total of \(b - 5\) new edges. If \(b = 5\) then there is only one pair of clusters we have splitted, \((V_3, V_4)\), so we not yet add any edges. This creates two disjoint copies of \(T^{(2, h)}_{\ell_0}\) in \(G_0\), where \(h = \left[\frac{\log(\varepsilon m)}{\log 2}\right]\), \(\ell^* = \frac{1}{2}(\ell_0 + 2h) - (b - 3) - 2h\), one contained in \(U := \bigcup_{j=1}^{b-1} U_j\) and the other in \(U' := \bigcup_{j=3}^{b-1} U'_{j}\). Moreover, the first \(T^{(2, h)}_{\ell_i}\)-copy, embedded in \(U\), has at least \(\varepsilon m\) leaves in \(U_0\) and at least \(\varepsilon m\) leaves in \(U_{b-1}\). Similarly, the other \(T^{(2, h)}_{\ell_i}\)-copy, embedded in \(U'\), has at least \(\varepsilon m\) leaves in \(U'_0\) and at least \(\varepsilon m\) leaves in \(U'_{b-1}\) (see Figure 2). Now, we treat the pairs \((V_j, V_{j+1})\) where \(j \in \{1, b\}\) almost similarly to how we treated them in the subcase \(b = 3\). More formally, we note that \((V_j, V_{j+1})\) also has the \(\varepsilon\)-property in \(G_0\) and that \(|V_j|, |V_{j+1}| \in \{[m], [m]\}\).

So by Proposition 5.9 we get that \(G_0[V_j, V_{j+1}]\) contains a copy of \(T^{(2, h)}_{\ell_i}\) for \(h = \left[\frac{\log(\varepsilon m)}{\log 2}\right]\) and the \(\ell_j\)'s are such that \(\ell_j + \ell_i = (t - 2\ell^* - 4h) - 4h - 4|\ell_j - \ell_i| \leq 2\), and both are even (which means that \(\ell_j \leq \frac{1}{2}(t - 2\ell^* - 4h) - 2h - 1 \leq 2(1 - 48\varepsilon) m\), where the leaf sets \(L_j, L'_j, L_b, L'_b\) are in \(V_2\) and \(V_b\), respectively (as \(\ell_j\) is even). Since \(|L_2|, |L'_{2i}|, |L_0|, |L'_{b-1}| \geq \varepsilon m\), once again, by the \(\varepsilon\)-property, we can connect some \(v_2 \in L_2\) with \(v_3 \in L_3\), some \(v'_2 \in L'_2\) with \(v'_3 \in L'_3\), some \(v_{b-1} \in L_{b-1}\) with \(v_b \in L_b\), and some \(v'_b \in L'_0\) (see Figure 2). By doing that we complete a cycle of length exactly \(\ell_1 + \ell_b + 2\ell^* + 8h + 4 = t\).

We now prove the second item. Suppose now that \(S\) contains an odd degree of length \(b\), where \(3 \leq b < k\), and let \(t = \left[\frac{b-1}{2}\log(1/\varepsilon)\right] \log n, (1 - \delta)an\) be odd, \(a = \frac{b}{k}\). Assume w.l.o.g. that this cycle is \((1, \ldots, b)\) and consider the set of corresponding clusters \(V_1, \ldots, V_b\). Let \(i \in \{1, \frac{1}{2}(b-1)\}\) and look at the pair \((V_{2i-1}, V_{2i})\). Using Proposition 5.8 and Proposition 5.9 we embed one of two different possible trees in \(G_0[V_{2i-1}, V_{2i}]\), depending on the value of \(t\), to eventually create a cycle of the required length. Recall that the pair \((V_{2i-1}, V_{2i})\) satisfies the \(\varepsilon\)-property in \(G_0\), and furthermore, that \(|V_{2i-1}|, |V_{2i}| \geq [m]\). Hence, by Proposition 5.8 and Proposition 5.9 \(G_0[V_{2i-1}, V_{2i}]\) contains a copy of \(T^{(r, h)}_{\ell_i}\) where \(h = \left[\frac{\log(\varepsilon m)}{\log r}\right]\) for both \(r = \left[\frac{1}{16}\right] - 2, \ell \in [1, 2\varepsilon m]\) and \(r = 2, \ell \in [1, 2(1 - 48 \varepsilon) m]\), respectively. Thus, we embed a copy of \(T^{(r, h)}_{\ell_i}\) in \(G_0[V_{2i-1}, V_{2i}]\) for \(h = \left[\frac{\log(\varepsilon m)}{\log r}\right]\), where \(r = \left[\frac{1}{16}\right] - 2\) if \(t \in \left[\frac{(b-1)}{2}\log(1/\varepsilon)\right] \log n, 2\varepsilon m\), and \(r = 2\) if \(t \in [2\varepsilon m, (1 - \delta)an]\). We choose the value of \(\ell_i\) as follows. For all \(i \in \{2, \ldots, \frac{1}{2}(b-1)\}\) we set \(\ell_i = \ell_0 = \left[\frac{2t - 2(1 + 2h)(b-1)}{b-1}\right] \text{odd}\), and \(\ell_1 = t - 1 - \frac{b-1}{2} - h(b-1) - \frac{1}{2}(b-3)\ell_0\). Note that \(\ell_1\) is also odd, and moreover, that \(\ell_0, \ell_1 \in [1, 2(1 - 48 \varepsilon) m]\). For every \(i \in \{1, \frac{1}{2}(b-1)\}\) we denote the embedded \(T^{(r, h)}_{\ell_i}\)-copy in \(G_0[V_{2i-1}, V_{2i}]\) by \(T_{2i-1, 2i}\), and further denote by \(L_{2i-1}, L_{2i}\).
its leaf sets in $V_{2i−1}$ and in $V_{2i}$, respectively. Note that for every $i ∈ \{1, \ldots, \frac{b−1}{2}\}$, a maximal path in $T_{2i−1,2i}$ is of length $2h+\ell_i$. Recall that for every $i ∈ \{1, \ldots, \frac{b−3}{2}\}$, also the pair $(V_{2i}, V_{2i+1})$ satisfies the $\varepsilon$-property in $G_0$, and note that we have $|L_2|, |L_{2i+1}| \geq \varepsilon m$. Thus we have $e_{G_0}(L_{2i}, L_{2i+1}) > 0$ for every $i ∈ \{1, \ldots, \frac{b−3}{2}\}$, so we add an edge between every such pair of leaf sets, summing up to $\frac{b−3}{2}$ new edges. Thus we get in $G_0$ a copy of the tree $T^{(r,h)}_\ell$, where

$$\ell^* = \sum_{i=1}^{\frac{b−1}{2}} \ell_i + (b−3)h + \frac{1}{2}(b−3) = t−2h−2,$$

with at least $\varepsilon m$ leaves in $V_1$ and at least $\varepsilon m$ leaves in $V_{b−1}$. Some maximal path inside this tree (connecting the mentioned two leaf sets) will be used to get a cycle of length $t$ along with extra two edges. Now, we note that there exists a vertex $v_b ∈ V_b$ which is adjacent both to a vertex in $L_1$ and a vertex in $L_{b−1}$. Indeed, otherwise one of $L_1, L_{b−1}$ would have fewer than $(1−\varepsilon)|m|$ neighbors in $V_b$, which contradicts the $\varepsilon$-property of the pairs $(V_1, V_b)$ and $(V_{b−1}, V_b)$ in $G_0$. Thus we can connect the vertex $v_b$ to a vertex in $L_1$ and to a vertex in $L_{b−1}$, adding two more edges and closing a cycle of length exactly $t$.



6 Further results

6.1 Applications

As mentioned in the introduction, Theorem 1.4 is also applicable to pseudo-random graphs.

For proving Corollary 1.12 we use the Expander Mixing Lemma due to Alon and Chung [2] cited below.

**Theorem 6.1 (Expander Mixing Lemma [2]).** Let $G$ be a $d$-regular graph on $n$ vertices where $\lambda \leq d$ is the second largest eigenvalue of its adjacency matrix, in absolute value. Then for any two disjoint subsets of vertices $A, B ⊆ V(G)$ we have

$$|e(A, B) − \frac{d}{n}|A||B|| \leq \lambda \sqrt{|A||B|}.$$

We now verify that for suitable values of $d$ and $\lambda$, an $(n, d, \lambda)$-graph is also upper-uniform. Concretely, an $(n, d, \lambda)$-graph is $(p, \eta)$-upper-uniform with $p = \frac{d}{n}$ and $\eta ≥ \frac{1}{2}$. proving Corollary 1.12.

**Proof of Corollary 1.12.** Take $\eta, n_0, \gamma$ as given by Theorem 1.4. Let $G$ be an $(n, d, \lambda)$-graph with $n ≥ n_0$ and $\frac{d}{n} ≥ \frac{1}{\eta}$. By the Expander Mixing Lemma (Theorem 6.1), any two subsets $A, B ⊆ V(G)$
satisfy
\[ e(A, B) \leq \frac{d}{n} |A||B| + \lambda \sqrt{|A||B|} . \]

Recalling that \( \frac{d}{\lambda} \geq \frac{1}{\eta} \) we get
\[ e(A, B) \leq \frac{d}{n} |A||B|(1 + \eta) \]
for any two subsets \( A, B \subseteq V(G) \) satisfying \(|A|, |B| \geq \eta n\). It follows that \( G \) is \((\frac{d}{n}, \eta)\)-upper-uniform. Note, in addition, that an \((n, d, \lambda)\)-graph \( G \) also satisfies \( e(G) = \frac{1}{2}dn \geq (1 - \beta/2)\frac{d}{n} \binom{n}{2} \). Hence, the statement follows by Theorem 1.4.

6.2 Robustness

For proving Theorem 1.13 we use Szemerédi’s celebrated Regularity Lemma [41].

**Theorem 6.2 (Szemerédi’s Regularity Lemma [41]).** For every positive real \( \varepsilon \) and for every positive integer \( k_0 \) there are positive integers \( n_0 \) and \( K \) with the following property: for every graph \( G \) on \( n \geq n_0 \) vertices there is an \( \varepsilon \)-regular partition \( \Pi = (V_1, \ldots, V_k) \) of \( V(G) \) such that \( |V_i|, |V_j| \leq 1 \) and \( k_0 \leq k \leq K \).

The following lemma bounds from below the number of edges in the reduced graph \( R \) of the graph \( G \) from Theorem 1.13, similarly to Lemma 3.2.

**Lemma 6.3.** Let \( \beta > 0 \) and \( \varepsilon \leq \frac{\beta}{100} \). Let \( G \) be a graph on \( n \geq n_0 \) vertices with an \( \varepsilon \)-regular partition \( \Pi = (V_1, \ldots, V_k) \) provided by the Regularity Lemma with parameters \( \varepsilon \) and \( k \geq \frac{5}{\beta} \). Assume that \( e(G) \geq (x + \beta)\binom{n}{2} \) for a constant \( 0 \leq x < 1 - \beta \). Let \( R := R(G, \Pi, \rho, \varepsilon) \) be the reduced graph as in Definition 3.1 (and as mentioned in Definition 3.1, here \( p = 1 \)) where \( \rho = 10\varepsilon \). Then \( e(R) \geq (x + \beta/2)\binom{k}{2} \).

**Proof.** Let \( G' \) by the subgraph of \( G \) obtained by keeping only the edges between the clusters \( V_i, V_j \) in which \( \{i, j\} \in E(R) \). We count the edges of \( G - G' \) as follows.

- Edges in non-regular pairs. There are at most \( \varepsilon \binom{k}{2} \frac{n^2}{k^2} \leq \frac{1}{200} \beta n^2 \) such edges.
- Edges in regular pairs with density less than \( \rho \). There are at most \( \rho \binom{k}{2} \frac{n^2}{k^2} \leq \frac{1}{20} \beta n^2 \) such edges.
- Edges inside clusters. There are at most \( k \cdot \binom{n/k}{2} \leq \frac{n^2}{2k} < \frac{1}{16} \beta n^2 \).

In total we kept all but at most \( \frac{31}{200} \beta n^2 < \frac{1}{4} \beta \binom{n}{2} \) edges, so \( G' \) has at least \( (x + 2\beta/3) \binom{n}{2} \geq (x + \beta/2)\binom{k}{2} \cdot \binom{n/k}{2} \) edges. Since any edge of \( R \) corresponds to at most \( \frac{n^2}{k^2} \) edges of \( G' \), we get \( e(R) \geq (x + \beta/2)\binom{k}{2} \) as required.

The following claim and corollary connect the reduced graph of \( G \) and the \( \varepsilon \)-graph of \( G(p) \), with respect to the same partition \( \Pi \).

**Claim 6.4.** Let \( \varepsilon > 0 \) and let \( G \) be a graph on \( n \) vertices. Then there exists \( C := C(\varepsilon) \) such that the following holds. Let \( \Pi = (V_1, V_2, \ldots, V_k) \) be an \( \varepsilon \)-regular partition of \( V(G) \) with \( |V_i|, |V_j| \leq 1 \), for some \( k := k(\varepsilon) \). Then for \( p \geq \frac{C}{n} \), for every \( i, j \) where \( (V_i, V_j) \) is an \( \varepsilon \)-regular pair with \( d(V_i, V_j) \geq \rho = 10\varepsilon \), we have that w.h.p. \( (V_i, V_j) \) satisfies the \( \varepsilon \)-property in the random graph \( G(p) \).
Proof. Denote \(|m| \leq |V_i| \leq |m|\), where \(m = \frac{n}{k}\). Let \(U_i \subseteq V_i\) and \(U_j \subseteq V_j\) be such that \(|U_i|, |U_j| \geq \varepsilon m\). By regularity we have \(|d(V_i, V_j) - d(U_i, U_j)| \leq \varepsilon\). Combining it with the assumption \(d(V_i, V_j) \geq \rho\), we have that

\[
e(U_i, U_j) \geq (\rho - \varepsilon)|U_i||U_j| \geq 9\varepsilon|U_i||U_j|.
\]

For two disjoint subsets \(U_i, U_j\) of \(V(G)\), denote by \(e_p(U_i, U_j)\) the random variable counting the number of edges between these sets in \(G(p)\). Then \(e_p(U_i, U_j)\) is distributed binomially with parameters \(e_G(U_i, U_j)\) and \(p\). Hence, the probability that there exist two such sets that do not satisfy the \(\varepsilon\)-property in \(G(p)\) is at most \((\frac{n}{m})^2 \Pr[e_p(U_i, U_j) = 0] \leq e^{-\Omega(n)}\), for, say, \(p \geq \frac{\log k}{3\varepsilon^3 m}\).

**Corollary 6.5.** Let \(0 < x < 1\), \(0 < \beta < 1 - x\) and let \(G\) be a graph on \(n\) vertices with \(e(G) \geq (x + \beta)\binom{n}{2}\) and an \(\varepsilon\)-regular partition \(\Pi = (V_1, \ldots, V_k)\) of its vertices with \(\varepsilon \leq \frac{\beta}{100}\) and \(k \geq \frac{2}{\varepsilon^2}\). Let \(R := R(G, \Pi, \rho, \varepsilon)\) be the reduced graph as in Definition 3.1. Let \(p \geq \frac{C}{n}\) where \(C\) is as in the previous claim, and let \(S := S(G(p), \Pi, \varepsilon)\) be the corresponding \(\varepsilon\)-graph as in Definition 4.2. Then w.h.p. \(R \subseteq S\), and therefore w.h.p. \(e(S) \geq (x + \beta/2)\binom{k}{2}\).

We can now prove Theorem 1.13.

**Proof of Theorem 1.13**. We can assume \(0 < \beta < 1/4\). Set \(\varepsilon = \frac{\beta}{10000}\) and \(k_0 = \frac{2}{\varepsilon^2}\). Take \(n_0, K\) as given in the Regularity Lemma (Theorem 6.2), and also set \(\gamma = \frac{2(1 - 48\varepsilon)}{k}\). Let \(\frac{C1}{\log(1/\beta)} \cdot \log n \leq t \leq (1 - C2\beta)n\), where \(C1, C2\) are the absolute constants from Corollary 4.4. Let \(G\) be a graph on \(n \geq n_0\) vertices with \(e(G) \geq ex(n, C1) + \beta\binom{n}{2} \geq (g^*(t, n) + \beta/2)\binom{n}{2}\) (recall that \(ex(n, C1) \geq g^*(t, n, \binom{n}{2} - 1)\)). Then by Theorem 6.2 there exists an \(\varepsilon\)-regular partition \(\Pi = (V_1, \ldots, V_k)\) of \(V(G)\) such that \(||V_i| - |V_j|| \leq 1\), with \(k_0 \leq k \leq K\).

We next look at the graph \(G(p)\) with the same partition and consider the \(\varepsilon\)-graph \(S := S(G(p), \Pi, \varepsilon)\). By Corollary 6.5 we have that w.h.p. \(e(S) \geq (g^*(t, n) + \beta/4)\binom{k}{2}\). Using Corollary 4.4 we get that w.h.p. \(G(p)\) contains a cycle of length \(t\).

**Remark 6.6.** As mentioned in the introduction, Theorem 1.13 is tight in the sense that for many values of \(t\) taking a graph \(G\) with \(\Theta(n^2)\) extra edges above the extremal number \(ex(n, C1)\) is in fact necessary for having w.h.p. a copy of \(C_t\) in \(G(p)\) where \(p = \frac{C}{n}\). As an example we show that it is tight for \(t \geq \frac{n}{3}\). First, it is known (and an easy exercise) that for any constant \(C > 0\), there exists some \(\alpha := \alpha(C) > 0\) such that w.h.p. for any \(\frac{n}{3} \leq t_0 \leq n\) the graph \(G(t_0, p)\) has w.h.p. at least \(\alpha n\) isolated vertices, where \(p = \frac{C}{n}\). Now, let \(\frac{n}{3} \leq t \leq n, \varepsilon > 0\) and \(\alpha n\) be some constant, and take \(G\) to be the graph on \(n\) vertices consisting of two cliques sharing exactly one vertex, one of size \((1 + \varepsilon)t\), denoted by \(K^1\), and the other of size \(n - (1 + \varepsilon)t + 1\), denoted by \(K^2\). Now take \(G(p)\) and look at a subgraph of it that is induced by the vertices of \(K^1\). This subgraph is exactly \(G((1 + \varepsilon)t, p)\) and thus w.h.p. \(G(p)[K^1]\) contains at least \(\alpha n\) isolated vertices. Therefore, w.h.p. \(G(p)[K^1]\) does not contain any cycle of length \((1 + \varepsilon)t - \alpha n < t\) or larger, and in particular \(G(p)\) does not contain any cycle of length \(t\) or larger. On the other hand, \(e(G) = \binom{(1 + \varepsilon)t}{2} + \binom{n - (1 + \varepsilon)t + 1}{2} \geq \binom{t-1}{2} + \binom{n-t+2}{2} + \frac{\varepsilon^2}{2} n^2 = ex(n, C_t) + \frac{\varepsilon^2}{2} n^2\).

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