PRESERVATION OF PRODUCT STRUCTURES UNDER THE RICCI FLOW WITH INSTANTANEOUS CURVATURE BOUNDS

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Abstract. In this note, we prove that there exists a constant \( \epsilon > 0 \), depending only on the dimension, such that if a complete solution to the Ricci flow splits as a product at time \( t = 0 \) and has curvature bounded by \( \frac{\epsilon}{t} \), then the solution splits for all time.

1. Introduction

In this note, we consider the problem of whether a solution to the Ricci flow

\[
\frac{\partial}{\partial t} g = -2Rc
\]

which splits as a product at \( t = 0 \) continues to do so for all time.

This problem is closely related, but not strictly equivalent, to the question of uniqueness of solutions to (1.1). For example, when \((\hat{M} \times \check{M}, \hat{g}_0 \oplus \check{g}_0)\), Shi’s existence theorem \([12]\) implies that there exist complete, bounded curvature solutions \((\hat{M}, \hat{g}(t))\) and \((\check{M}, \check{g}(t))\) with initial conditions \(\hat{g}_0\) and \(\check{g}_0\), respectively, which exist on some common time interval \([0, T]\). Then, \(\hat{g}(t) \oplus \check{g}(t)\) solves (1.1) on \(\hat{M} \times \check{M}\) for \(t \in [0, T]\) and is also complete and of bounded curvature. But, according to the uniqueness results of Hamilton \([5]\) and Chen-Zhu \([2]\), such a solution is unique among those which are complete and have bounded curvature. Thus, any solution in that class starting at \(\hat{g}_0 \oplus \check{g}_0\) continues to split as a product.

Outside of this class, less is known. While there are elementary examples which show that without completeness, a solution may instantaneously cease to be a product, the extent to which the uniform curvature bound can be relaxed is less well-understood. (One exception is in dimension two, where the work of Giesen and Topping \([3, 4]\) has established an essentially complete theory of existence and uniqueness for (1.1). In particular, in \([14]\), Topping shows that any two complete solutions with the same initial data must agree.)

One class of particular interest is that of solutions satisfying a curvature bound of the form \(c/t\) for some constant \(c\), which arise naturally as limits of exhaustions (see, e.g., \([1, 6, 13]\)). The purpose of this note is to prove the following.

**Theorem 1.1.** Let \((\hat{M}, \hat{g}_0)\) and \((\check{M}, \check{g}_0)\) be two connected Riemannian manifolds and let \(M = \hat{M} \times \check{M}\) and \(g_0 = \hat{g}_0 \oplus \check{g}_0\). Then there exists a constant \(\epsilon = \epsilon(n) > 0\), where \(n = \text{dim}(M)\), such that if \(g(t)\) is a complete solution to (1.1) on \(M \times [0, T]\) with \(g(0) = g_0\) satisfying

\[
|Rm| \leq \frac{\epsilon}{t},
\]

on \(M \times [0, T]\).
then \( g(t) \) splits as a product for all \( t \in [0,T] \), i.e., there exist \( \hat{g}(t) \), \( \tilde{g}(t) \) such that
\[
g(t) = \hat{g}(t) \oplus \tilde{g}(t),
\]
where \( \hat{g}(t) \) and \( \tilde{g}(t) \) are solutions to (1.1) on \( \hat{M} \) and \( \tilde{M} \), respectively, for \( t \in [0,T] \).

Lee [8] has already established the uniqueness of complete solutions satisfying the bound (1.2). However, his result does not directly imply Theorem 1.1: without any restrictions on the curvatures of \( \hat{g}_0 \) and \( \tilde{g}_0 \), we lack the short-time existence theory to guarantee that there are any solutions on \( \hat{M} \) and \( \tilde{M} \), respectively, with the given initial data, let alone solutions satisfying a bound of the form (1.2) for sufficiently small \( \epsilon \). Thus we are unable to construct a competing product solution on \( \hat{M} \times \tilde{M} \) to which we might apply Lee’s theorem.

Instead, we frame the problem as one of uniqueness for a related system, using a perspective similar to that of [9] and [11]. The key ingredient is a maximum principle closely based on one due to Huang-Tam [2] and modified by Liu-Székelyhidi [9]. These references establish, among other things, related results concerning the preservation of Kähler structures.

2. Tracking the product structure

Our first step toward proving Theorem 1 is to construct a system associated to a solution to Ricci flow which measures the degree to which a solution which initially splits as a product fails to remain a product. Consider a Riemannian product \( (M,g_0) = (\hat{M} \times \tilde{M}, \hat{g}_0 \oplus \tilde{g}_0) \), and let \( g(t) \) be a smooth solution to the Ricci flow on \( M \times [0,T] \) with \( g(0) = g_0 \). For the time being, we make no assumptions on the completeness of \( g(t) \) or bounds on its curvature.

2.1. Extending the projections. Let \( \hat{\pi} : M \to \hat{M} \) and \( \tilde{\pi} : M \to \tilde{M} \) be the projections on each factor, and let \( \hat{H} = \ker(d\hat{\pi}) \) and \( \tilde{H} = \ker(d\tilde{\pi}) \). We define \( \hat{P}_0, \tilde{P}_0 \in \text{End}(TM) \) to be the orthogonal projections onto \( \hat{H} \) and \( \tilde{H} \) determined by \( g_0 \).

Following [11], we extend each of them to a time-dependent family of projections for \( t \in [0,T] \) by solving the fiber-wise ODEs
\[
\begin{aligned}
\partial_t \hat{P}(t) &= \text{Rc} \circ \hat{P} - \hat{P} \circ \text{Rc}, \\
\hat{P}(0) &= \hat{P}_0.
\end{aligned}
\]

From \( \hat{P} \) and \( \tilde{P} \), we construct time-dependent endomorphisms \( \mathcal{P}, \tilde{\mathcal{P}} \in \text{End}(\Lambda^2 T^* M) \) by
\[
\mathcal{P}\omega(X,Y) = \omega(\hat{P}X, \tilde{P}Y), \\
\tilde{\mathcal{P}}\omega(X,Y) = \omega(\tilde{P}X, \hat{P}Y),
\]
\[
\mathcal{T} = (\nabla \text{Rm}) \circ (\text{Id} \times \mathcal{P}), \quad \mathcal{S} = (\nabla \nabla \text{Rm}) \circ (\text{Id} \times \text{Id} \times \mathcal{P}).
\]

In order to study the evolution of \( \mathcal{R} \), it will be convenient to introduce an operator \( \Lambda^a_b \) which acts algebraically on tensors via
\[
\Lambda^a_b A^{i_1 \ldots i_k}_{i_1 \ldots i_k} = \delta^a_{i_1} A^{i_1 \ldots i_k}_{i_2 \ldots i_k} + \cdots + \delta^a_{i_k} A^{i_1 \ldots i_k}_{i_1 \ldots i_{k-1}} - \delta^a_{i_1} A^{i_1 \ldots i_k}_{i_1 \ldots i_k} - \cdots - \delta^a_{i_k} A^{i_1 \ldots i_k}_{i_1 \ldots i_k}.
\]

We will also consider the operator
\[
D_t := \partial_t + R_{ab} g^{bc} \Lambda^a_c.
\]
This operator has the property that \( D_t g = 0 \), and for any time-dependent tensor fields \( A \) and \( B \),
\[
D_t \langle A, B \rangle = \langle D_t A, B \rangle + \langle A, D_t B \rangle,
\]
where \( \langle \cdot, \cdot \rangle \) is the metric induced by \( g(t) \). Note that by construction the projections satisfy
\[
D_t \hat{P} \equiv 0, \quad D_t \hat{P} \equiv 0, \quad D_t P \equiv 0, \quad D_t \bar{P} \equiv 0.
\]

2.2. Evolution equations. In order to determine how the components of \( X \) and \( Y \) evolve, we will make use of the following commutation formulas (see [11], Lemma 4.3):
\[
[D_t, \nabla_a] = \nabla_p R_{pabc} A^b_c + R_{ac} \nabla_c,
\]
\[
[D_t - \Delta, \nabla_a] = 2 R_{abde} A^d_b \nabla_e + 2 R_{ab} \nabla_b.
\]

Additionally, we will need to examine the sharp operator on endomorphisms of two forms. For any \( A, B \in \text{End}(\Lambda^2 T^* M) \),
\[
\langle A\# B(\varphi), \psi \rangle = \frac{1}{2} \sum_{\alpha, \beta} \langle [A(\omega_\alpha), B(\omega_\beta)], \varphi \rangle \cdot \langle [\omega_\alpha, \omega_\beta], \psi \rangle,
\]
where \( \varphi, \psi \in \Lambda^2 T^* M \) and \( \{\omega_\alpha\} \) is an orthonormal basis for \( \Lambda^2 T^* M \). Recall that the curvature operator evolves according to
\[
(D_t - \Delta) R = Q(R, R),
\]
under the Ricci flow, where \( Q(A, B) = \frac{1}{2}(AB + BA) + A\# B \).

**Proposition 2.1.** We have the following evolution equations for the projection \( \mathcal{P} \):
\[
D_t \nabla \mathcal{P} = Rm \ast \nabla \mathcal{P} + \mathcal{P} \ast S,
\]
\[
D_t \nabla^2 \mathcal{P} = Rm \ast \nabla^2 \mathcal{P} + \nabla Rm \ast \nabla \mathcal{P} + \mathcal{P} \ast T + \nabla Rm \ast \mathcal{P} \ast \nabla \mathcal{P}.
\]
In particular, there exists a constant \( C = C(n) \) such that
\[
|D_t \nabla \mathcal{P}| \leq C(|Rm| |\nabla \mathcal{P}| + |S|),
\]
\[
|D_t \nabla^2 \mathcal{P}| \leq C(|Rm| |\nabla^2 \mathcal{P}| + |\nabla Rm| |\nabla \mathcal{P}| + |T|).
\]

**Proof.** Using equation (2.2) and the fact that \( D_t \mathcal{P} = 0 \), we can see that \( D_t \nabla \mathcal{P} = [D_t, \nabla] \mathcal{P} \). With some additional computation, we can then see (as in Propositions 4.5 and 4.6 from [11]) that
\[
D_t \nabla \mathcal{P} = Rm \ast \nabla \mathcal{P} + \mathcal{P} \ast S.
\]
Similarly, using this equation together with (2.2) and the fact that \( \nabla S = T + \nabla Rm \ast \nabla \mathcal{P} \), we have
\[
D_t \nabla^2 \mathcal{P} = [D_t, \nabla] \nabla \mathcal{P} + \nabla (D_t \nabla \mathcal{P})
\]
\[
= Rm \ast \nabla^2 \mathcal{P} + \nabla Rm \ast \nabla \mathcal{P} + \mathcal{P} \ast T + \nabla Rm \ast \mathcal{P} \ast \nabla \mathcal{P},
\]
as claimed. \( \square \)

In order compute similar evolution equations for \( \mathcal{R}, \mathcal{S}, \) and \( \mathcal{T} \), we will need the following lemma.
Lemma 2.2. Let \( A, B \in \text{End}(\Lambda^2 T^* M) \) be self-adjoint operators. There exists \( C = C(n) > 0 \) such that
\[
|Q(A, B) \circ P| \leq C (|A \circ P|B| + |A||B \circ P|).
\]

Proof. Clearly,
\[
|(A \circ B + B \circ A) \circ P| \leq |A \circ P||B| + |A||B \circ P|.
\]

Furthermore, for \( \eta \in \Lambda^2 T^* M \),
\[
(\langle A \# B \rangle \circ P)(\eta) = \frac{1}{2} \sum_{\alpha, \beta} (\omega_\alpha, B \omega_\beta, \mathcal{P} \eta) \cdot (\omega_\alpha, \omega_\beta)
\]
\[
= \frac{1}{2} \sum_{\alpha, \beta} (\mathcal{P} \circ A \omega_\alpha, \mathcal{P} \circ B \omega_\beta, \mathcal{P} \eta) \cdot (\omega_\alpha, \omega_\beta)
\]
\[
+ \frac{1}{2} \sum_{\alpha, \beta} (\mathcal{P} \circ A \omega_\alpha, \mathcal{P} \circ B \omega_\beta, \mathcal{P} \eta) \cdot (\omega_\alpha, \omega_\beta)
\]
\[
+ \frac{1}{2} \sum_{\alpha, \beta} (\mathcal{P} \circ A \omega_\alpha, \mathcal{P} \circ B \omega_\beta, \mathcal{P} \eta) \cdot (\omega_\alpha, \omega_\beta),
\]
where \( \{\omega_\alpha\} \) is an orthonormal basis for \( \Lambda^2 T^* M \). The final term on the right hand side is zero (see [11], Lemma 3.5); the point is that the image of \( \mathcal{P} \) is closed under the bracket and is perpendicular to the image of \( \mathcal{P} \). Moreover, \( \mathcal{P} \circ A = (A \circ \mathcal{P})^* \) and \( \mathcal{P} \circ B = (B \circ \mathcal{P})^* \), so it follows that
\[
|(A \# B) \circ P| \leq C \left( |A \circ P||B| + |A||B \circ P| + |A \circ \mathcal{P}||B \circ \mathcal{P}| \right)
\]
\[
\leq C \left( |A \circ P||B| + |A||B \circ P| \right),
\]
completing the proof. \qed

Proposition 2.3. As defined above, \( \mathcal{R}, \mathcal{S}, \) and \( \mathcal{T} \) satisfy the inequalities
\[
\|D_t - \Delta \mathcal{R}\| \leq C (|\Delta \mathcal{R}| + |\nabla \mathcal{R}||\nabla \mathcal{P}| + |\Delta \mathcal{R}|^2),
\]
\[
\|D_t - \Delta \mathcal{S}\| \leq C (|\Delta \mathcal{R}||\mathcal{R}| + |\nabla \mathcal{R}||\nabla \mathcal{S}| + |\Delta \mathcal{R}||\nabla \mathcal{S}| + |\Delta \mathcal{R}|^2),
\]
\[
\|D_t - \Delta \mathcal{T}\| \leq C (|\Delta \mathcal{R}||\mathcal{R}| + |\nabla \mathcal{R}||\mathcal{S}| + |\Delta \mathcal{R}| |\mathcal{T}| + |\nabla \mathcal{R}||\nabla \mathcal{R}| + |\nabla \mathcal{R}||\nabla \mathcal{S}| + |\nabla \mathcal{R}||\nabla \mathcal{S}| + |\nabla \mathcal{R}||\nabla \mathcal{S}| + |\nabla \mathcal{R}||\nabla \mathcal{S}| + |\nabla \mathcal{R}||\nabla \mathcal{S}|) ,
\]
where \( C = C(n) > 0 \).

Proof. Using the evolution equation for \( \mathcal{R} \), we have
\[
(D_t - \Delta) \mathcal{R} = Q(\mathcal{R}, \mathcal{R}) \circ \mathcal{P} + \mathcal{R} \circ \Delta \mathcal{P} + 2 \nabla \mathcal{R} \mathcal{R} \mathcal{P} + \nabla \mathcal{P}.
\]
The first inequality then follows immediately from Lemma 2.2.

We now compute the evolution equation for \( \mathcal{S} \). First, note that
\[
(D_t - \Delta) \mathcal{S} = (D_t - \Delta, \nabla \mathcal{R}) \circ \mathcal{P} + \nabla ((D_t - \Delta) \mathcal{R}) \circ \mathcal{P}
\]
\[
+ \nabla \mathcal{R} \mathcal{R} \mathcal{P} + \nabla \mathcal{R} \mathcal{P} + \nabla \mathcal{R} \mathcal{P}.
\]
For the first term, using the commutator \([2.3]\), we have
\[
([D_t - \Delta], \nabla_a) R_{ijkl} = 2R_{abcd} \Lambda^a_{ij} \nabla_b R_{ijkl} + 2R_{ab} \nabla_b R_{ijkl}.
\]
As in the computation in Proposition 4.13 from \([11]\), we have
\[
R_{abcd} \Lambda^a_{ij} \nabla_b R_{mnkl} P_{ijmn} = Rm \ast S + \nabla Rm \ast R \ast P,
\]
which gives us
\[
(2.7) \quad ([D_t - \Delta, \nabla] Rm) \circ P = Rm \ast S + \nabla Rm \ast R \ast P.
\]
We then compute
\[
\nabla((D_t - \Delta) Rm) = \nabla Q(Rm, Rm)
= \nabla Rm \circ Rm + Rm \circ \nabla Rm + \nabla Rm \# Rm + Rm \# \nabla Rm
= 2Q(\nabla Rm, Rm),
\]
where we regard \(\nabla Rm\) as a one form with values in \(\text{Sym}(\Lambda^2 T^* M)\). Then, applying Lemma \(2.2\) and combining the result in \(2.7\) with \(2.6\), we obtain the second inequality.

For the third inequality, we begin with the identity
\[
(D_t - \Delta) T = ((D_t - \Delta) \nabla^2 Rm) \circ P + \nabla^2 Rm \ast \nabla^2 P + \nabla^3 Rm \ast \nabla P.
\]
The first term can be rewritten as
\[
((D_t - \Delta) \nabla^2 Rm) \circ P = ([D_t - \Delta, \nabla^2 Rm] \circ P + (\nabla^2 Rm \circ (D_t - \Delta, \nabla) Rm) \circ P.
\]
Applying equation \((2.3)\) once again gives us
\[
((D_t - \Delta) \nabla_a \nabla Rm) \circ P - (\nabla_a (D_t - \Delta) \nabla Rm) \circ P = (2R_{abcd} \Lambda^a_{ij} \nabla_b \nabla Rm + 2R_{ab} \nabla_b \nabla Rm) \circ P,
\]
and we have
\[
R_{abcd} \Lambda^a_{ij} \nabla_b \nabla Rm \circ P = Rm \ast T + \nabla^2 Rm \ast R \ast P
\]
(again see \([11]\), Proposition 4.13, also \([9]\)). We can see that
\[
(\nabla [D_t - \Delta, \nabla] Rm) \circ P = \nabla (([D_t - \Delta, \nabla] Rm) \circ P) + ([D_t - \Delta, \nabla] Rm) \ast \nabla P
= \nabla Rm \ast S + Rm \ast T + Rm \ast \nabla Rm \ast \nabla P + \nabla^2 Rm \ast R \ast P + \nabla Rm \ast S \ast P
+ \nabla Rm \ast Rm \ast \nabla P + \nabla Rm \ast R \ast \nabla P + \nabla Rm \ast \nabla Rm \ast \nabla P
\]
where we again use the facts that \(\nabla R = S + Rm \ast \nabla P\) and \(\nabla S = T + \nabla Rm \ast \nabla P\). Additionally,
\[
(\nabla \nabla (D_t - \Delta) Rm) \circ P = 2Q(\nabla^2 Rm, Rm) \circ P + 2Q(\nabla Rm, \nabla Rm) \circ P.
\]
Combining the above identities and again applying Lemma \(2.2\) to the last term, we obtain the third inequality.
2.3. Constructing a PDE-ODE system. With an eye toward Theorem 1, we now organize the tensors $\nabla P$, $\nabla^2 P$, $R$, $S$, and $T$ into groupings which satisfy a closed system of differential inequalities. Let

$$X = T^4(T^*M) \oplus T^5(T^*M) \oplus T^6(T^*M), \quad Y = T^5(T^*M) \oplus T^6(T^*M),$$

and define families of sections $X = X(t)$ of $X$ and $Y = Y(t)$ of $Y$ for $t \in (0, T]$ by

$$X = \left( \frac{R}{t}, \frac{S}{t^{3/2}}, T \right), \quad Y = \left( \frac{\nabla P}{\sqrt{t}}, \nabla^2 P \right).$$

**Proposition 2.4.** If $g(t)$ is a smooth solution to Ricci flow on $M \times [0, T]$ with $|\text{Rm}|(x, t) < a/t$ for some $a > 0$, then there exists a constant $C = C(a, n) > 0$ depending such that $X$ and $Y$ satisfy

$$|(D_t - \Delta)X| \leq C \left( \frac{1}{t} |X| + \frac{1}{t^2} |Y| \right), \quad |D_t Y| \leq C \left( |X| + \frac{1}{t} |Y| \right),$$

on $M \times (0, T]$.

**Remark 2.5.** Inspection of the proof reveals that the constant $C$ in fact has the form $C = aC$, where $C$ depends only on $n$ and $\max\{a, 1\}$.

This follows directly from Propositions 2.4 and 2.5 with the help of the following curvature bounds, which can be obtained from the classical estimates of Shi [12] with a simple rescaling argument.

**Proposition 2.6.** Suppose $(M, g(t))$ is a complete solution to Ricci flow for $t \in [0, T]$ which satisfies

$$|\text{Rm}|(x, t) \leq \frac{a}{t},$$

for some constant $a > 0$. Then for each $m > 0$, there exists a constant $C = C(m, n)$ such that

$$|\nabla^m \text{Rm}|(x, t) \leq \frac{aC}{m^{m/2}} (1 + a^{m/2}).$$

**Proof of Proposition 2.4.** Throughout this proof, $C$ will denote a constant which may change from line to line but depends only on $n$ and $a$. Using (2.4) in combination with the curvature estimates, we obtain

$$|D_t Y| \leq \frac{1}{2} t^{-3/2} |\nabla P| + t^{-1/2} |D_t \nabla P| + |D_t \nabla^2 P|$$

$$\leq Ct^{-1/2} |S| + C|T| + Ct^{-3/2} |\nabla P| + Ct^{-1} |\nabla^2 P|$$

$$\leq C |X| + \frac{C}{t} |Y|.$$

Applying the curvature estimates to the inequalities (2.4) for $R$, $S$, and $T$, we get

$$|(D_t - \Delta)R| \leq Ct^{-1} |R| + Ct^{-3/2} |\nabla P| + Ct^{-1} |\nabla^2 P|,$$

$$|(D_t - \Delta)S| \leq Ct^{-3/2} |R| + Ct^{-1} |S| + Ct^{-2} |\nabla P| + Ct^{-3/2} |\nabla^2 P|,$$

and

$$|(D_t - \Delta)T| \leq Ct^{-2} |R| + Ct^{-3/2} |S| + Ct^{-1} |T| + Ct^{-5/2} |\nabla P| + t^{-2} |\nabla^2 P|. $$
Combining these equations, we have
\[
| (D_t - \Delta)X | \leq t^{-1} | (D_t - \Delta)R | + t^{-2} | R | + t^{-1/2} | (D_t - \Delta)S | + \frac{1}{2} t^{-3/2} | S |
\]
\[
+ | (D_t - \Delta)T |
\]
\[
\leq Ct^{-2} | R | + Ct^{-3/2} | S | + Ct^{-1} | T | + Ct^{-5/2} | \nabla P | + Ct^{-2} | \nabla^2 P |
\]
\[
\leq Ct^{-1} | X | + Ct^{-2} | Y |,
\]
as desired.

3. A general uniqueness theorem for PDE-ODE systems

We now aim to show that \( X \) and \( Y \) vanish using a maximum principle from [7] by adapting it to apply to a general PDE-ODE system. The following theorem is essentially a reformulation of Lemma 2.3 in [7] and Lemma 2.1 in [9].

**Theorem 3.1.** Let \( M = M^n \) and \( \mathcal{X} \) and \( \mathcal{Y} \) be finite direct sums of \( T^k(M) \). There exists an \( \epsilon = \epsilon(n) > 0 \) with the following property: Whenever \( g(t) \) is a smooth, complete solution to the Ricci flow on \( M \) satisfying
\[
|R_m| \leq \frac{\epsilon}{t}
\]
on \( M \times (0, T] \), and \( X = X(t) \) and \( Y = Y(t) \) are families of smooth sections of \( \mathcal{X} \) and \( \mathcal{Y} \) satisfying
\[
| (D_t - \Delta)X | \leq C_t | X | + C_t^2 | Y |, \quad |D_tY| \leq C | X | + C_t | Y |,
\]
\[
D^k_t Y = 0, \quad D^k_t X = 0 \text{ for } k \geq 0 \text{ at } t = 0,
\]
and
\[
|X| \leq Ct^{-1},
\]
for some \( C > 0, l > 0 \), then \( X \equiv 0 \) and \( Y \equiv 0 \) on \( M \times [0, T] \).

The key ingredient in the proof of Theorem 3.1 is an the following scalar maximum principle due to Huang-Tam [7] (and its variant in [9]). Though the statement has been slightly changed from its appearance in [7], the proof is nearly identical. We detail here the modifications we make for completeness.

**Proposition 3.2** (c.f. [7], Lemma 2.3 and [9], Lemma 2.1). Let \( M \) be a smooth \( n \)-dimensional manifold. There exists an \( \epsilon = \epsilon(n) > 0 \) such that the following holds: Whenever \( g(t) \) is a smooth complete solution to the Ricci flow on \( M \times [0, T] \) such that the curvature satisfies \( |R_m| \leq \epsilon/t \) and \( f \geq 0 \) is a smooth function on \( M \times [0, T] \) satisfying
\[
(1) \quad (\partial_t - \Delta) f(x, t) \leq at^{-1} \max_{0 \leq s \leq t} f(x, s),
\]
\[
(2) \quad \partial^k_t |_{t=0} f = 0 \text{ for all } k \geq 0,
\]
\[
(3) \sup_{x \in M} f(x, t) \leq Ct^{-l} \text{ for some positive integer } l \text{ for some constant } C,
\]
then \( f \equiv 0 \) on \( M \times [0, T] \).

**Proof.** For the time-being, we will assume \( \epsilon > 0 \) is fixed and that \( g(t) \) is a smooth, complete solution to Ricci flow on \( M \times [0, T] \) satisfying \( |R_m| \leq \epsilon/t \). We will then specify \( \epsilon \) over the course of the proof.
As in [7] we may assume $T \leq 1$. We will first show that for any $k > a$, there exists a constant $B_k$ such that
\[
\sup_{x \in M} f(x, t) \leq B_k t^k.
\]

Let $\phi$ be a cutoff function as in [7], i.e., choose $\phi \in C^\infty([0, \infty))$ such that $0 \leq \phi \leq 1$ and
\[
\phi(s) = \begin{cases} 1 & 0 \leq s \leq 1, \\ 0 & 2 \leq s, \end{cases} \quad -C_0 \leq \phi' \leq 0, \quad |\phi''| \leq C_0,
\]
for some constant $C_0 > 0$. Then let $\Phi = \phi^m$ for $m > 2$ to be chosen later and define $q = 1 - \frac{2}{m}$. Then
\[
0 \geq \Phi' \geq -C(m)\Phi^q, \quad |\Phi''| \leq C(m)\Phi^q.
\]
where $C(m) > 0$ is a constant depending only on $m$ (and on $C_0$).

Fix a point $y \in M$. As in Lemma 2.2 of [7], there exists some $\rho \in C^\infty(M)$ such that
\[
d_g(T)(x, y) + 1 \leq \rho(x) \leq C'(d_g(T)(x, y) + 1), \quad |\nabla d_g(T)\rho|_g(T) + |\nabla^2 d_g(T)\rho|_g(T) \leq C',
\]
where $C'$ is a positive constant depending only on $n$ and $\frac{\partial}{\partial T}$. This function then also satisfies
\[
|\nabla \rho| \leq C_1 t^{-ce}, \quad |\Delta \rho| \leq C_2 t^{-1/2 - ce},
\]
where $C_1, C_2$ are constants depending only on $n$, $T$ and $\epsilon$, and $c > 0$ depends only on the dimension $n$. We may assume $\epsilon$ is small enough so that $cc < 1/4$. Let $\Psi(x) = \Psi(x) = \Phi(\rho(x)/r)$ for $r \gg 1$. Define also $\theta = \exp(-\alpha t^1 - \beta)$, where $\alpha > 0$ and $0 < \beta < 1$. By the estimates on the derivatives of $\rho$, we have
\[
|\nabla \Psi| = r^{-1}\Phi'(|\rho|)\nabla|\rho| \leq r^{-1}C(m)C_1 \Phi^q(\rho) t^{-ce} \leq C(m)\Psi q t^{-1/4}
\]
and
\[
|\Delta \Psi| = |r^{-2}\Phi''(\rho/\rho)\nabla|\rho|^2 + r^{-1}\Phi'(|\rho|)\Delta|\rho|
\leq r^{-2}C(m)\Phi^q(\rho/\rho) t^{-2ce} + r^{-1}C(m)\Phi^q(\rho) t^{-1/2 - ce}
\leq C(m)\Psi q t^{-3/4}.
\]

For $k > a$, let $F = t^{-k}f$. Then $F$ satisfies
\[
(\partial_t - \Delta) F = -kt^{-k-1}f + t^{-k}(\partial_t - \Delta)f
\leq -kt^{-k-1}f(x, t) + at^{-k-1}\max_{0 \leq s \leq t} f(x, s)
\]
and $F \leq Ct^{-l-k}$.

Let $H = \theta \Psi F$ and suppose that $H$ attains a positive maximum at the point $(x_0, t_0)$. Then, at this point, we have $\Psi > 0$ and both $(\partial_t - \Delta) H \geq 0$ and $\nabla H = 0$. Since $\nabla H = 0$, we have
\[
\nabla \Psi \cdot \nabla F = -F|\nabla \Psi|^2.
\]
Additionally, since $\Psi$ is independent of time,
\[
\theta(s)F(x_0, s) \leq \theta(t_0)F(x_0, t_0)
\]
for all $s \leq t_0$. Because $\theta$ is decreasing, we have
\[
s^{-k}f(x_0, s) = F(x_0, s) \leq F(x_0, t_0) = t_0^{-k}f(x_0, t_0)
\]
for $s \leq t_0$, which in turn implies
\[
\max_{0 \leq s \leq t_0} f(x, s) = f(x, t_0).
\]
Thus, at $(x_0, t_0)$ we have
\[
(\partial_t - \Delta) F \leq (-k + a)t_0^{-1} F \leq 0.
\]
Thus at $(x_0, t_0)$ we have
\[
\Delta H = \theta F \Delta \Psi + \theta \Psi \Delta F + 2\theta \nabla F \cdot \nabla \Psi
= \theta F \Delta \Psi + \theta \Psi \Delta F - 2\theta F |\nabla \Psi|^2
\geq -C(m)\theta F \Psi^q t_0^{-3/4} - C(m)\theta F \Psi^{2q-1} t_0^{-3/4} + \theta \Psi \Delta F
\]
and
\[
\partial_t H = -\alpha(1 - \beta)t_0^{-\beta} \theta F \Psi + \theta \Psi \partial_t F.
\]
We can then compute
\[
0 \leq (\partial_t - \Delta) H
\leq \theta \Psi(\partial_t - \Delta) F - \alpha(1 - \beta)t_0^{-\beta} \theta F + C(m)\theta \Psi^q F t_0^{-3/4} + C(m)\theta \Psi^{2q-1} F t_0^{-3/4}
\leq -\alpha(1 - \beta)t_0^{-\beta} \theta F + C(m)\theta ((\Psi F)^q t_0^{-3/4 - (1 - q)(l + k)}
+ C(m)\theta (\Psi F)^{2q-1} t_0^{-3/4 - (2 - 2q)(l + k)}.
\]
We now choose $m$ and $\beta$ so that the powers of $t_0$ in the denominators of the last two terms are less than $\beta$. We take $\beta$ to be $7/8$ (any $\beta \in (3/4, 1]$ will do). Recalling that $q = 1 - 2/m$, we choose $m$ large enough so that $7/8 > 3/4 + (1 - q)(l + k)$ and $7/8 > 3/4 + (2 - 2q)(l + k)$. Then
\[
\frac{\alpha}{8} \Psi F = \alpha(1 - \beta) \Psi F \leq C(m) \left((\Psi F)^q + (\Psi F)^{2q-1}\right).
\]
Finally, we choose $\alpha$ large enough so that $\alpha > 16C(m)$. Then
\[
2\Psi F \leq (\Psi F)^q + (\Psi F)^{2q-2},
\]
implying that $(\Psi F)(x_0, t_0) \leq 1$, and hence $H \leq 1$ everywhere. In particular, for any $x \in \{\rho \leq r\}$, $f(x, t) = t^k F(x, t) \leq e^\alpha t^k := B_k t^k$. Sending $r$ to infinity then proves that $f(x, t) \leq B_k t^k$.

Next, again as in (4), we define the function $\eta(x, t) = \rho(x) \exp \left(\frac{2c_2 k t^{1-b}}{1-t} \right)$ for $b > 1$. Since $|\Delta \rho| \leq C_2 t^{-b}$, we have
\[
(\partial_t - \Delta) \eta > 0, \quad \partial_t \eta > 0.
\]
Let $F = t^{-a} f$. Fix $\delta > 0$ and consider the function $F - \delta \eta - \delta t$. Note that by our previous argument, $F \leq Ct^2$, and in particular is bounded. For some $t_1 > 0$ depending on $\delta$ and $c$, $F - \delta t < 0$ for $t \leq t_1$ and for $t \geq t_1$, $F - \delta \eta < 0$ outside some compact set. So, if $F - \delta \eta - \delta t$ is ever positive, there must exist some $(x_0, t_0) \in M \times (0, T]$ at which it attains a positive maximum. Because $-\delta \eta - \delta t$ is decreasing in time, for any $s < t_0$ from the inequality
\[
F(x_0, s) - \delta \eta(x_0, s) - \delta s \leq F(x_0, t_0) - \delta \eta(x_0, t_0) - \delta t_0,
\]
we conclude
\[
F(x_0, s) \leq F(x_0, t_0).
\]
As in our previous argument, this implies that \( f(x_0, t_0) = \max_{0 \leq s \leq t_0} f(x_0, s) \), so that at \((x_0, t_0)\)
\[
(\partial_t - \Delta)(F - \delta \eta - \delta t) < 0,
\]
a contradiction. Thus, for any \( \delta > 0 \), \( F - \delta \eta - \delta t \leq 0 \). Taking \( \delta \to 0 \) then implies that \( F = 0 \). \( \square \)

We can now prove Theorem 3.1.

**Proof of Theorem 3.1.** For \( k > 0 \) to be determined later, define the functions \( F \) and \( G \) on \( M \times [0, T] \) by
\[
F = t - k |X|^2, \quad G = t - (k+1) |Y|^2,
\]
for \( t \in (0, T] \) and \( F(x, 0) = G(x, 0) = 0 \). From the assumption that \( D_l tX = D_l tY = 0 \) for all \( l \geq 0 \), it follows that both \( F \) and \( G \) are smooth on \( M \times [0, T] \) and that \( \partial_l t F = \partial_l t G = 0 \) for all \( l \geq 0 \).

We have
\[
(\partial_t - \Delta)F = -kt^{-(k+1)}|X|^2 + 2t^{-k}(\langle D_t - \Delta X, X \rangle - 2t^{-k}|\nabla X|^2
\]
\[
\leq -kt^{-(k+1)}|X|^2 + 2t^{-k}|(D_t - \Delta X)||X|
\]
\[
\leq t^{-(k+1)}(2C - k)|X|^2 + 2Ct^{-(k+2)}|X||Y|
\]
\[
\leq t^{-1}(3C - k)F + Ct^{-2}G
\]
and
\[
\partial_t G = -(k+1)t^{-(k+2)}|Y|^2 + 2t^{-(k+1)}(\langle D_t Y, Y \rangle
\]
\[
\leq (2C - k - 1)t^{-(k+2)}|Y|^2 + 2Ct^{-(k+1)}|X||Y|
\]
\[
\leq CF + t^{-1}(3C - k - 1)G.
\]
Choosing \( k > 3C \), this becomes
\[
(\partial_t - \Delta)F \leq t^{-2}CG, \quad \partial_t G \leq CF.
\]
In particular this implies that
\[
G(x, t) \leq C t \max_{0 \leq s \leq t} F(x, s),
\]
and therefore
\[
(\partial_t - \Delta)F \leq t^{-1}C^2 \max_{0 \leq s \leq t} F(x, s).
\]
By our assumption on \( X, F \leq Ct^{-2l-k} \). Thus \( F \) satisfies the hypotheses of Proposition 3.2 and must vanish identically. We then conclude that \( G \), hence \( Y \), vanishes as well. \( \square \)

4. **Proof of Theorem 1.1**

We are now almost ready to prove Theorem 1.1. We just need to first verify that \( X \) and \( Y \) satisfy the last major remaining hypothesis of Theorem 3.1, that is, that all time derivatives of \( X \) and \( Y \) vanish at \( t = 0 \).
4.1. Vanishing of time derivatives. We begin by recording a standard commutator formula, which is in fact valid (with obvious modifications) for any family of smooth metrics.

**Proposition 4.1.** Let $(M, g(t))$ be a smooth solution to the Ricci flow for $t \in [0, T]$. Then, for any $l \geq 1$, the formula

\[ [D_t, \nabla^{(l)}]A = \sum_{k=1}^{l} \nabla^{(k-1)}[D_t, \nabla^{(l-k)}]A \]

is valid for any smooth family of tensor fields $A$ on $M \times [0, T]$.

**Proof.** We proceed by induction on $l$. The base case, $l = 1$, is trivial. Now, suppose that (4.1) holds for $l \leq m$ for some $m \geq 1$. Then,

\[
[D_t, \nabla^{(m+1)}]A = D_t \nabla^{(m+1)}A - \nabla^{(m+1)}D_t A
\]

\[= [D_t, \nabla^{(m)}]\nabla A + \nabla^m D_t \nabla A - \nabla^{(m+1)}D_t A
\]

\[= [D_t, \nabla^{(m)}]\nabla A + \nabla^m [D_t, \nabla]A
\]

\[= \sum_{k=1}^{m} \nabla^{(k-1)}[D_t, \nabla^{(m-k)}(\nabla A)] + \nabla^{(m)}[D_t, \nabla]A
\]

\[= \sum_{k=1}^{m+1} \nabla^{(k-1)}[D_t, \nabla]A^{(m+1-k)}A,
\]

as desired. $\square$

Now we argue inductively that $D^k_t X = 0$ and $D^k_t Y = 0$ at $t = 0$.

**Proposition 4.2.** Let $M = \ddot{M} \times \ddot{M}$ be a smooth manifold and $g(t)$ be a smooth, complete solution to the Ricci flow such that $g(0)$ splits as a product. Define $\mathcal{P}$ and $\mathcal{R}$ as in Section 2. The following equations hold at $t = 0$ for all $k, l \geq 0$:

\[ D^k_t \nabla^{(l)}\mathcal{P} = 0, \quad D^k_t \nabla^{(l+1)}\mathcal{R} = 0. \]

**Proof.** We proceed by induction on $k$, beginning with the base case $k = 0$. Because the metric splits as a product initially, at $t = 0$ we have $\nabla^{(l)}\hat{P} = \nabla^{(l)}\hat{P} = 0$ for all $l \geq 0$ and $R(\hat{P}(\cdot), \hat{P}(\cdot), \cdot) = 0$.

From this we get that, for any $X, Y, Z, W \in TM$,

\[ R(X^* \wedge Y^*)(Z, W) = 2R(\hat{P}X, \hat{P}Y, W, Z) + 2R(\hat{P}X, \hat{P}Y, W, Z) = 0. \]

Combining these facts, we conclude

\[ \nabla^{(l+1)}\mathcal{P} = 0, \quad \nabla^{(l)}\mathcal{R} = 0, \quad \nabla^{(l)}\mathcal{R}^* = 0, \]

at $t = 0$, where $\mathcal{R}^* = \mathcal{P} \circ \text{Rm}$ denotes the adjoint of $\mathcal{R}$ with respect to $g$.

Now starting the induction step, suppose that for some $k \geq 0$, for all $l \geq 0$ and any $m \leq k$,

\[ D^m_t \nabla^{(l+1)}\mathcal{P} = 0, \quad D^m_t \nabla^{(l)}\mathcal{R} = 0, \]

hence also $D^m_t \nabla^{(l)}\mathcal{R}^* = 0$. Recall that

\[ (D_t - \Delta)\text{Rm} = \mathcal{Q}(\text{Rm}, \text{Rm}). \]
As in [11], Lemma 4.9, \(Q(Rm, Rm) \circ \mathcal{P} = R \ast U_1 + R \ast \ast U_2\), where \(U_1\) and \(U_2\) are smooth families of tensors on \(M\). Thus we can compute
\[
D_t \mathcal{R} = (D_t Rm) \circ \mathcal{P} + Rm \circ (D_t \mathcal{P})
\]
and thus
\[
\Delta \mathcal{R} = (\Delta Rm) \circ \mathcal{P} + Q(Rm, Rm) \circ \mathcal{P},
\]
and thus
\[
(4.2) \quad D_t^{k+1} \mathcal{R} = D_t^k ((\Delta Rm) \circ \mathcal{P}) + D_t^k (Q(Rm, Rm) \circ \mathcal{P}).
\]

Because
\[
\Delta \mathcal{R} = (\Delta Rm) \circ \mathcal{P} + Rm \circ \Delta \mathcal{P} + 2\nabla_t Rm \circ \nabla_t \mathcal{P},
\]
by the induction hypothesis \(D_t^k ((\Delta Rm) \circ \mathcal{P}) \equiv 0\) at \(t = 0\). Similarly,
\[
D_t^k (Q(Rm, Rm) \circ \mathcal{P}) = D_t^k (R \ast U_1) + D_t^k (R \ast \ast U_2) = 0.
\]
We conclude that \(D_t^{k+1} \mathcal{R} \equiv 0\), and thus \(D_t^{k+1} \mathcal{R} \ast \equiv 0\).

Now, using the commutator from equation (2.2) and Proposition 4.1 for any \(l > 0\) we have
\[
D_t \nabla^{(l)} \mathcal{R} = \sum_{m=1}^{l} \nabla^{(m-1)} [D_t, \nabla] \nabla^{(l-m)} \mathcal{R} + \nabla^{(l)} D_t \mathcal{R}
\]
and thus
\[
D_t^{k+1} \nabla^{(l)} \mathcal{R} = \sum_{m=1}^{l} D_t^k \nabla^{(m-1)} \left( \nabla Rm \ast \nabla^{(l-m)} \mathcal{R} + Rm \ast \nabla^{(l-m+1)} \mathcal{R} \right) + \nabla^{(l)} D_t \mathcal{R}.
\]
Expanding using the product rule and applying the induction hypothesis, all terms in the first sum vanish at \(t = 0\). For the remaining term, we again use the evolution equation for \(\mathcal{R}\). We have
\[
D_t^k \nabla^{(l)} D_t \mathcal{R} = D_t^k \nabla^{(l)} ((\Delta Rm) \circ \mathcal{P} + Q(Rm, Rm) \circ \mathcal{P}).
\]
As before, rewriting \(Q(Rm, Rm) \circ \mathcal{P}\) in terms of \(\mathcal{R}\) and \(\mathcal{R} \ast \) and expanding using the product rule, it follows that \(D_t^k \nabla^{(l)} D_t \mathcal{R} \equiv 0\) at \(t = 0\).

We now move on to the derivatives of \(\mathcal{P}\). Recall that
\[
D_t \nabla \mathcal{P} = [D_t, \nabla] \mathcal{P} = Rm \ast \nabla \mathcal{P} + \mathcal{P} \ast S.
\]
Applying this in combination with Proposition 4.1, we get, for any \(l \geq 1\),
\[
D_t^{k+1} \nabla^{(l)} \mathcal{P} = \sum_{m=1}^{l} D_t^k \nabla^{(m-1)} [D_t, \nabla] \nabla^{(l-m)} \mathcal{P} + D_t^k \nabla^{(l)} D_t \mathcal{P}
\]
and thus
\[
\sum_{m=1}^{l-1} D_t^k \nabla^{(m-1)} \left( \nabla Rm \ast \nabla^{(l-m)} \mathcal{P} + Rm \ast \nabla^{(l-m+1)} \mathcal{P} \right)
+ D_t^k \nabla^{(l-1)} [D_t, \nabla] \mathcal{P} + D_t^k \nabla^{(l)} D_t \mathcal{P}.
\]
As before, every term in the first sum vanishes by the induction hypothesis, while the final term vanishes because \(D_t \mathcal{P} \equiv 0\). Finally we can see that
\[
D_t^k \nabla^{(l-1)} [D_t, \nabla] \mathcal{P} = D_t^k \nabla^{(l-1)} (Rm \ast \nabla \mathcal{P} + \mathcal{P} \ast S),
\]
and because $S = (\nabla \text{Rm}) \circ \mathcal{P} = \nabla R + \text{Rm} \ast \nabla \mathcal{P}$, $D_t^k \nabla (t^{-1}) |D_t, \nabla| \mathcal{P} \equiv 0$ at $t = 0$. This completes the proof. \hfill \square

4.2. Preservation of product structures. In the proof of Theorem 1.1 we will use the operator $\mathcal{F} : \Lambda^2 T^* M \rightarrow \Lambda^2 T^* M$ defined by

$$\mathcal{F} \omega(X, Y) = \omega(\hat{\mathcal{P}} X, \hat{\mathcal{P}} Y) - \omega(\hat{\mathcal{P}} \hat{\mathcal{P}} X, \hat{\mathcal{P}} Y).$$

(See, for example, Section 2.2 of [10].) Observe that

$$\mathcal{P} \circ \mathcal{F}(X, Y) = \mathcal{F}(\mathcal{P} \hat{\mathcal{P}} X, \mathcal{P} \hat{\mathcal{P}} Y) + \mathcal{F}(\hat{\mathcal{P}} \hat{\mathcal{P}} X, \hat{\mathcal{P}} Y) = \omega(\hat{\mathcal{P}}^2 X, \hat{\mathcal{P}} Y) + \omega(\hat{\mathcal{P}} X, \hat{\mathcal{P}} \hat{\mathcal{P}} Y) - \omega(\hat{\mathcal{P}}^2 X, \hat{\mathcal{P}} \hat{\mathcal{P}} Y) = \omega(\hat{\mathcal{P}} X, \hat{\mathcal{P}} Y) - \omega(\hat{\mathcal{P}} X, \hat{\mathcal{P}} Y).$$

Therefore $\mathcal{P} \circ \mathcal{F} = \mathcal{F}$.

Proof of Theorem 1. We have shown in Propositions 2.1 and 2.2 that the system $X, Y$ satisfies the first two hypotheses of Theorem 3.1. Additionally, the curvature bounds from Proposition 2.3 imply that $|X| \leq Ct$. Thus, $X \equiv 0$ and $Y \equiv 0$ on $M \times [0, T]$. In particular, we know that $\mathcal{R} \equiv 0$ and $\nabla \mathcal{P} \equiv 0$ on $M \times [0, T]$.

We claim that $\nabla \hat{\mathcal{P}} \equiv \nabla \hat{\mathcal{P}} \equiv 0$ and $\partial_t \hat{\mathcal{P}} \equiv \partial_t \hat{\mathcal{P}} \equiv 0$. Similar to the proof of Lemma 7 in [10], if we define $W = \nabla \mathcal{P}$, then

$$D_t W^j_{ai} = [D_t, \nabla e_i] \hat{\mathcal{P}}^j = \nabla_p R_{pa} \hat{\mathcal{P}}^j_e - \nabla_p R_{pa}^j \hat{\mathcal{P}}^e + R^j_e W^j_{ei}.$$

Note that the first two terms combine to give

$$\langle \nabla e_p R(e_p, e_a) \hat{\mathcal{P}} e_i, \hat{\mathcal{P}} e_j \rangle = \frac{1}{2} \langle \nabla e_p \text{Rm}(e_i^* \wedge e_j^*)(e_p, e_a) - \nabla e_p \text{Rm}(e_i^* \wedge e_j^*)(e_p, e_a) \rangle = -\frac{1}{2} \nabla e_p \text{Rm} \circ \mathcal{F}(e_i^* \wedge e_j^*)(e_p, e_a).$$

But, since $\mathcal{P} \circ \mathcal{F} = \mathcal{F}$,

$$\nabla \text{Rm} \circ \mathcal{F} = \nabla \text{Rm} \circ \mathcal{P} \circ \mathcal{F} = \mathcal{S} \circ \mathcal{F} = 0,$$

so $D_t W^j_{ai} = R^j_e W^j_{ei}$. Thus, for any point $x \in M$, the function $f(t) = |\nabla \hat{\mathcal{P}}|^2(x, t)$ satisfies

$$f'(t) \leq Cf$$

for some $C$ depending on $x$. Since $f(0) = 0$, $f$ is identically zero. Thus $\hat{\mathcal{P}}$ (and similarly $\hat{\mathcal{P}}$) remain parallel.

Hence, we have

$$R(\cdot, \cdot, \hat{\mathcal{P}}(\cdot), \hat{\mathcal{P}}(\cdot)) = 0,$$

which implies that $\text{Re} \circ \hat{\mathcal{P}} = \hat{\mathcal{P}} \circ \text{Re}$ and $\text{Re} \circ \hat{\mathcal{P}} = \hat{\mathcal{P}} \circ \text{Re}$, and thus, from (2.1), $\partial_t \hat{\mathcal{P}} = \partial_t \hat{\mathcal{P}} = 0$ on $[0, T]$. Theorem 1.1 follows. \hfill \square
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