A SHORT PROOF OF DE SHALIT’S CUP PRODUCT FORMULA

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ABSTRACT. We give a short proof of a formula of de Shalit, expressing the cup product of two vector valued one forms of the second kind on a Mumford curve in terms of Coleman integrals and residues. The proof uses the notion of double indices on curves and their reciprocity laws.

1. INTRODUCTION

In [3S88] de Shalit proved a formula for the cup product of two vector valued differential forms on a Mumford curve. This is based on an earlier partial result of his [3S89] for two holomorphic differentials. This formula was later reproved by Iovita and Spiess [IS03]. The goal of this short note is to give an alternative short proof of de Shalit’s formula, based on the theory of the double index [Bes00, Section 4].

Let us state de Shalit’s result. Let $K$ be a finite extension of $\mathbb{Q}_p$. Consider a Mumford curve $H/\Gamma$, where $\Gamma \subset \text{PGL}_2(K)$ is a Schottky group and $H \subset \mathbb{P}^1_K$ is the rigid analytic space obtained by removing the limit points of $\Gamma$. Let $V$ be a finite dimensional $K$-vector space with a representation of $\Gamma$. The group $\Gamma$ acts on the space of $V$-valued differential forms on $H$, $\Omega^1(H, V)$, by the rule

$$\gamma(\sum \omega_i v_i) = \sum (\gamma^{-1})^* \omega_i \gamma(v_i)$$

(compare [3S88 1.1]). We let it act by the same formula on spaces of functions. A $V$-valued differential one-form $\omega$ on $H$ with values in $V$ is $\Gamma$-invariant if $\gamma(\omega) = \omega$ for every $\gamma \in \Gamma$. It is of the second kind if its residues (with values in $V$, computed coordinatewise, in any basis), are 0 at any point $z \in H$. Let $\langle \rangle$ be a $\Gamma$-invariant bilinear form on $V$. The cup product of two $\Gamma$-invariant $V$-valued one forms of the second kind $\omega$ and $\eta$ can be described by the formula

$$\omega \cup \eta = \sum_{z \in \Gamma \setminus H} \text{Res}_z \langle F_\omega, \eta \rangle,$$

where $F_\omega$ is any primitive of $\omega$ locally near $z$, which exists (formally) because of the residue of $\omega$ at $z$ is 0, and is independent of the choice of the primitive because the residue of $\eta$ at $z$ is 0. Note that the expression to be summed indeed depends only on $z$ modulo $\Gamma$.

An open annulus is a rigid space isomorphic to the space $s < |z| < r$. An orientation on an annulus may be described as a choice of a parameter $z$ as above, with two parameters considered equivalent if they give the residue, as defined below. An annulus together with an orientation is called an oriented annulus. A differential form $\omega$ on an oriented annulus $\epsilon$ has a residue $\text{Res}_\epsilon \omega$ such that $\text{Res} \sum a_i z^i dz = a_{-1}$.

1991 Mathematics Subject Classification. Primary 14G22; Secondary 14F40, 11F85, 11G20.
Key words and phrases. Mumford curves, $p$-adic integration.
It can be shown that there are only two orientations, giving residue s differing by multiplication by $-1$. By choosing a basis for $V$ the residue extends easily to $V$ valued differential forms.

We now recall [dS89, Definition 2.5] that the action of $\Gamma$ on $H$ has a good fundamental domain in the following sense: There are pairwise disjoin t closed $K$-rational discs $B_i$ and $C_i$, $i = 1, \ldots, g$ and open annuli $b_i, c_i$, and elements $\gamma_i \in \Gamma$, such that the following holds:

1. The $\gamma_i$ freely generate $\Gamma$.
2. The unions $B_i \cup b_i$ and $C_i \cup c_i$ are open discs, still pairwise disjoint.
3. For each $i$, $\gamma_i$ maps $B_i$ isomorphically onto the complement of $C_i \cup c_i$ and $b_i$ isomorphically onto $c_i$.
4. The complement of $\bigcup_i (B_i \cup b_i \cup C_i)$ is a fundamental domain for $\Gamma$.

We give the annuli $c_i$ and $b_i$ the orientation given by the discs $C_i$ and $B_i$ respectively, i.e., one given by parameters extending to $C_i \cup c_i$ and taking the value 0 on $C_i$ (respectively with $b_i$ and $B_i$). Thus, $c_i$ is oriented in the same way as in [dS88, 1.5] while $b_i$ is oriented in the reversed direction to loc. cit. (the $b_i$’s do not show up in the formula). With this choice, $\gamma_i : b_i \to c_i$ is orientation reversing.

de Shalt’s formula involves Coleman integration of holomorphic $V$-valued $1$ forms. While this can be described in a completely elementary way since we are dealing with subdomains of the projective line [GvdP80, P. 41], we will use the more involved theory of Coleman [CdS88] and adapt it to our case by choosing a basis of $V$ and then integrate coordinate by coordinate. This is clearly independent of the choice of a basis because Coleman integration is linear (up to constant). The key property of Coleman integration is its functoriality. It immediately implies that from the property $\gamma \omega = \omega$ we may deduce that for any $\gamma \in \Gamma$ the function $\gamma(F_\omega) - F_\omega$ is constant. We can now state the main theorem.

**Theorem 1.1** (dS88, Theorem 1.6). With the data above we have

$$\omega \cup \eta = \sum_i \langle \gamma_i F_\omega - F_\omega, \text{Res}_{c_i} \eta \rangle - \langle \text{Res}_{c_i} \omega, \gamma_i F_\eta - F_\eta \rangle .$$

The main ingredient in the present proof is the theory of double indices and their reciprocity laws on curves [Bes00, Section 4]. We need a very easy extension of this theory to vector valued differential forms. Once this has been described, the proof is an easy computation.

We would like to thank Michael Spiess for suggesting this project. The author is supported by a grant from the Israel Science Foundation.

### 2. Double Indices of Vector Valued Differential Forms

In this section we describe a rather straightforward generalization of the theory of double indices [Bes00, Section 4] to the case of vector valued one forms. The extension is fairly trivial since we consider only constant coefficients. We work over $\mathbb{C}_p$ for convenience.

Let $A$ be either the field of meromorphic functions in the variable $z$ over $\mathbb{C}_p$ or the ring of rigid analytic functions on an annulus $\{ r < |z| < s \}$ over $\mathbb{C}_p$. Let $A_{\log} := A[\log(z)]$ and let $A_{\log,1} \subset A_{\log}$ be the subspace of $F \in A_{\log}$ which are linear in $\log(z)$, a condition which is equivalent to $dF \in Adz$. 

Definition 2.1. \cite[Proposition 4.5]{Bes00} The double index, $\text{ind}(\cdot) : A_{\log,1} \times A_{\log,1} \to \mathbb{C}_p$ is the unique antisymmetric bilinear pairing such that $\text{ind}(F,G) = \text{Res} FdG$, whenever $F \in A$.

Suppose now that $C$ is a proper smooth curve over $\mathbb{C}_p$ with good reduction, and that $U$ is a rigid analytic space obtained from $C$ by removing discs $D_i$ of the form $|z_i| \leq r$, with $r < 1$, where the reduction of $z_i$ is a local parameter near a point $x_i$ of the reduction. Let us call these domains simple domains. To the disc $D_i$ corresponds the annulus $e_i$ given by the equation $r < |z_i| < 1$, which is contained in $U$ and oriented by $z_i$.

Choose a branch of the $p$-adic logarithm. Given a rigid one form $\omega \in \Omega^1(U)$, Coleman’s theory provides us with a unique up to constant, locally analytic function $F_\omega$ on $U$ with the property that $dF_\omega = \omega$. Restricted to the annuli $e_i$ these clearly belong to $A_{\log,1} \eps{U}$, and one can therefore define, for two such functions $F_\omega$ and $F_\eta$, the double index $\text{ind}(e_i, (F_\omega, F_\eta))$. It follows from \cite[Lemma 4.6]{Bes00} that this index depends only on the orientation. One of the main technical results of \cite{Bes00} is the following.

Proposition 2.2 \cite[Proposition 4.10]{Bes00}). We have $\sum_i \text{ind}(e_i, (F_\omega, F_\eta)) = \Psi(\omega) \cup \Psi(\eta)$, where $\Psi : H^1_{\text{dR}}(U) \to H^1_{\text{dR}}(C)$ is a certain projection.

We will only need the following immediate Corollary, which follows because $H^1_{\text{dR}}(\mathbb{P}^1/\mathbb{C}_p) = 0$.

Corollary 2.3. Suppose that $C = \mathbb{P}^1$. Then, in the situation above, $\sum_i \text{ind}(e_i, (F_\omega, F_\eta)) = 0$.

We can now extend the theory to vector valued differential forms in a rather trivial way. Suppose we are given a finite dimensional $\mathbb{C}_p$-vector space with a bilinear form $\langle \cdot, \cdot \rangle$.

Definition 2.4. Chose bases $\{v_i\}$ and $\{u_i\}$ for $V$. Suppose that the $V$-valued Coleman functions $F_\omega$ and $F_\eta$ are written as

$$F_\omega = \sum F_{\omega,i} v_i,$$
$$F_\eta = \sum F_{\eta,i} v_i.$$

Then, the local index $\text{ind}(e_i, (F_\omega, F_\eta))$ is given by

$$\text{ind}(e_i, (F_\omega, F_\eta)) = \sum_{i,j} \text{ind}(e_i, (F_{\omega,i}, F_{\eta,j})) \langle v_i, u_j \rangle.$$ 

It is easy to check that this definition does not depend on the choice of bases. An easy consequence of the definitions is the following.

Proposition 2.5. Suppose that $\text{Res}_e \omega = 0$. Then $\text{ind}(e_i, (F_\omega, F_\eta)) = \text{Res}_e \langle F_\omega, \eta \rangle$ while $\text{ind}(e_i, (F_\eta, F_\omega)) = -\text{Res}_e \langle \eta, F_\omega \rangle$.

We now restrict to the case $C = \mathbb{P}^1$ but consider more general subdomains $U$, obtained by removing closed discs $D_i = |z - \alpha_i| = r_i$, including the case of removing a point when $r_i = 0$. For each $i$ we consider an annulus $e_i$ in $U$ surrounding $D_i$, in such a way that the open discs $D_i \cup e_i$ are still disjoint. We will call the $e_i$ the annuli ends of $U$. It is easy to see that $U$ can be obtained by gluing simple domains $U' \in \mathbb{P}^1$ along annuli. Note that the $U''$s are glued along annuli with reversed orientations.
Proposition 2.6. In the situation described above we have, for any rigid $V$-valued one-form on $U$, $\sum \text{ind}_e(F_\omega, F_\eta) = 0$.

Proof. The case $V$ trivial and $U$ simple is Corollary 2.3. We next consider the case $U = U'_1 \cup U'_2$ with $U'_1$ and $U'_2$ glued along an annulus $e$. Since $e$ has reversed orientations when considered in $U'_1$ and $U'_2$, the double index $\text{ind}_e$ has a reverse sign in these two cases by [Bes00, Lemma 4.6]. Thus, the result for $U$ follows from those for $U'_1$ and $U'_2$. Now, the case of a general $U$, still with trivial $V$, follows immediately. The general case follows by choosing bases. \qed

Proposition 2.7. Let $e$ be an annulus in $\mathcal{H}$ and let $\gamma \in \Gamma$. For $\omega \in \Omega^1(e, V)$ let $F_\omega$ be its integral. Then $\gamma F_\omega$ is a Coleman integral of $\gamma(\omega)$ on $\gamma(e)$, furthermore, if $\eta$ is another such form, then we have

$$\text{ind}_e(F_\omega, F_\eta) = \pm \text{ind}_{\gamma(e)}(\gamma(F_\omega), \gamma(F_\eta)),$$

depending on whether $\gamma$ is orientation reversing or saving.

Proof. We choose a basis $\{v_i\}$ of $V$ and we let $\{u_i\}$ be the dual basis with respect to $\langle , \rangle$. Then, since $\langle , \rangle$ is $\Gamma$-invariant, the bases $\{\gamma(v_i)\}$ and $\{\gamma(u_i)\}$ are also dual to each other. This implies that if $F_\omega = \sum f_i v_i$ while $F_\eta = \sum g_i u_i$, then

$$\text{ind}_e(F_\omega, F_\eta) = \sum_i \text{ind}_e(f_i, g_i)$$

and

$$\text{ind}_{\gamma(e)}(\gamma(F_\omega), \gamma(F_\eta)) = \sum_i \text{ind}_{\gamma(e)}((\gamma^{-1})^* f_i, (\gamma^{-1})^* g_i).$$

But by [Bes00, Lemma 4.6] we have, for each $i$,

$$\text{ind}_e(f_i, g_i) = \pm \text{ind}_{\gamma(e)}((\gamma^{-1})^* f_i, (\gamma^{-1})^* g_i),$$

depending on whether $\gamma^{-1}$ is orientation reversing or preserving, and the result follows immediately from this. \qed

3. The Proof

Proof of Theorem 1.1. By the remark following Equation (5) in [dSSS] we may assume that $b_i$ and $c_i$ contain no poles of $\omega$ and $\eta$. Consider the domain $\mathcal{F} = \mathbb{P}^1 - \bigcup_i (B_i \cup C_i)$, which is of the type considered in Section 2 and its annuli ends are the $b_i$ and $c_i$. It follows from the description of the fundamental domain for $\Gamma$ that $\mathcal{F} - \bigcup_i (c_i \cup b_i)$ contains exactly one out of every $\Gamma$ class of every singularity of either forms. It follows that

$$\omega \cup \eta = \sum_{x \in \mathcal{F}} \text{Res}_x(F_\omega, \eta) = \sum_{x \in \mathcal{F}} \text{ind}_x(F_\omega, F_\eta) = - \sum_i (\text{ind}_{b_i}(F_\omega, F_\eta) + \text{ind}_{c_i}(F_\omega, F_\eta))$$

where the last equality follows from Proposition 2.6. We now observe that since $\gamma_i$ is orientation reversing we have by Proposition 2.7 that $\text{ind}_{b_i}(F_\omega, F_\eta) = - \text{ind}_{c_i}(\gamma_i F_\omega, \gamma_i F_\eta)$. 

Given $\omega \in \Omega^1(U, V)$ one can define its Coleman integral $F_\omega$ first on each of the $U$'s as before and then by adjusting constants along the annuli. The intersection graph of the $U$'s is a tree so there is always a way of choosing an integral globally. This construction coincides with the definition of Coleman integrals in [GvdP 80].
Therefore
\[-(\text{ind}_b(F_\omega, F_\eta) + \text{ind}_c(F_\omega, F_\eta))\]
\[= \text{ind}_c(\gamma_i F_\omega, \gamma_i F_\eta) - \text{ind}_c(F_\omega, F_\eta)\]
\[= \text{ind}_c(\gamma_i F_\omega - F_\omega, \gamma_i F_\eta) + \text{ind}_c(F_\omega, \gamma_i F_\eta - F_\eta)\]
\[= \text{Res}_c(\gamma_i F_\omega - F_\omega, \gamma_i F_\eta) - \text{Res}_c(F_\omega, \gamma_i F_\eta - F_\eta)\]
by Proposition 2.5
\[= \langle \gamma_i F_\omega - F_\omega, \text{Res}_c F_\eta \rangle - \langle \text{Res}_c F_\omega, \gamma_i F_\eta - F_\eta \rangle.\]

The theorem follows immediately. \(\square\)

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