Distribution of particles which produces a ”smart” material

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Abstract

If \( A_q(\beta, \alpha, k) \) is the scattering amplitude, corresponding to a potential \( q \in L^2(D) \), where \( D \subset \mathbb{R}^3 \) is a bounded domain, and \( e^{ik\alpha \cdot x} \) is the incident plane wave, then we call the radiation pattern the function \( A(\beta) := A_q(\beta, \alpha, k) \), where the unit vector \( \alpha \), the incident direction, is fixed, and \( k > 0 \), the wavenumber, is fixed. It is shown that any function \( f(\beta) \in L^2(S^2) \), where \( S^2 \) is the unit sphere in \( \mathbb{R}^3 \), can be approximated with any desired accuracy by a radiation pattern: \( ||f(\beta) - A(\beta)||_{L^2(S^2)} < \epsilon \), where \( \epsilon > 0 \) is an arbitrary small fixed number. The potential \( q \), corresponding to \( A(\beta) \), depends on \( f \) and \( \epsilon \), and can be calculated analytically. There is a one-to-one correspondence between the above potential and the density of the number of small acoustically soft particles \( D_m \subset D \), \( 1 \leq m \leq M \), distributed in an a priori given bounded domain \( D \subset \mathbb{R}^3 \). The geometrical shape of a small particle \( D_m \) is arbitrary, the boundary \( S_m \) of \( D_m \) is Lipschitz uniformly with respect to \( m \). The wave number \( k \) and the direction \( \alpha \) of the incident upon \( D \) plane wave are fixed.

It is shown that a suitable distribution of the above particles in \( D \) can produce the scattering amplitude \( A(\alpha', \alpha) \), \( \alpha', \alpha \in S^2 \), at a fixed \( k > 0 \), arbitrarily close in the norm of \( L^2(S^2 \times S^2) \) to an arbitrary given scattering amplitude \( f(\alpha', \alpha) \), corresponding to a real-valued potential \( q \in L^2(D) \), i.e., corresponding to an arbitrary refraction coefficient in \( D \).

1 Introduction

Let \( D \subset \mathbb{R}^3 \) be a bounded connected domain with Lipschitz boundary \( S \).

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The scattering of an acoustic plane wave $u_0 = u_0(x) = e^{ika \cdot x}$, incident upon $D$, is described by the problem:

$$\left(\nabla^2 + k^2 n_0(x)\right) u = 0 \quad \text{in} \quad \mathbb{R}^3,$$

$$u = u_0(x) + v,$$

$$v = A(\alpha', \alpha) \frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right), \quad r := |x| \to \infty, \quad \frac{x}{r} := \alpha'.$$

The coefficient $A(\alpha', \alpha)$ is called the scattering amplitude, $k > 0$ is the wave number, which is assumed fixed throughout the paper, and the dependence of $A$ on $k$ is not shown by this reason, $\alpha \in S^2$ is the direction of the incident plane wave, $\alpha'$ is the direction of the scattered wave, $n_0(x)$ is the known refraction coefficient in $D$, $n_0(x) = 1$ in $D' := \mathbb{R}^3 \setminus D$, and $v$ is the scattered field.

Let $D_m, 1 \leq m \leq M$, be a small particle, i.e.,

$$k_0 a \ll 1, \text{ where } a = \frac{1}{2} \max_{1 \leq m \leq M} \text{diam } D_m, \quad k_0 = k \max_{x \in D} |n_0(x)|.$$

The geometrical shape of $D_m$ is arbitrary. We assume that $D_m$ is a Lipschitz domain uniformly with respect to $m$. This is a technical assumption which can be relaxed. It allows one to use the properties of the electrostatic potentials. Denote

$$d := \min_{m \neq j} \text{dist}(D_m, D_j).$$

Assume that

$$a \ll d.$$

We do not assume that $d \gg \lambda_0$, that is, that the distance between the particles is much larger than the wavelength. Under our assumptions, it is possible that there are many small particles on the distances of the order of the wavelength.

The particles are assumed soft, i.e.,

$$u|_{S_m} = 0 \quad 1 \leq m \leq M.$$

As a result of the distribution of many small particles in $D$, one obtains a new material, which we want to be a ”smart” material, that is, a material which has some desired properties. Specifically, we want this material to scatter the incident plane wave according to an a priori given desired radiation pattern. Is this possible? If yes, how does one distribute the small particles in order to create such a material?

We study this problem and solve the following two problems, which can be considered as problems of nanotechnology.

The first problem is:
Given an arbitrary function \( f(\beta) \in L^2(S^2) \), can one distribute small particles in \( D \) so that the resulting medium generates the radiation pattern \( A(\beta) := A(\beta, \alpha) \), at a fixed \( k > 0 \) and a fixed \( \alpha \in S^2 \), such that

\[
\|f(\beta) - A(\beta)\|_{L^2(S^2)} \leq \varepsilon,
\]

where \( \varepsilon > 0 \) is an arbitrary small fixed number?

The answer is yes, and we give an algorithm for calculating such a distribution. This distribution is not uniquely defined by the function \( f(\beta) \) and the number \( \varepsilon > 0 \).

The second problem is:

Given a scattering amplitude \( f(\alpha', \alpha) \), corresponding to some refraction coefficient \( n(x) \) in a bounded domain \( D \), can one distribute small particles in \( D \) so that the resulting medium generates the scattering amplitude \( A(\alpha', \alpha) \) such that

\[
\|f(\alpha', \alpha) - A(\alpha', \alpha)\|_{L^2(S^2 \times S^2)} \leq \varepsilon,
\]

where \( \varepsilon > 0 \) is an arbitrary small fixed number?

The answer is yes, and we give an algorithm for calculating the density of the desired distribution of small particles given \( f(\alpha', \alpha) \), \( \forall \alpha', \alpha \in S^2 \), \( k > 0 \) being fixed.

To our knowledge the above two problems have not been studied in the literature. Our solution to these problems is based on some new results concerning the properties of the scattering amplitudes, on our earlier results on wave scattering by small bodies of arbitrary shapes (see [8]), and on our solution of the 3D inverse Schrödinger scattering problem with fixed-energy data [7], [5], [6].

In Section 2 we derive some new approximation properties of the scattering amplitudes. Essentially, we prove the existence of a potential \( q \in L^2(D) \) such that the corresponding to this \( q \) scattering amplitude \( A(\beta) \), \( \beta = \alpha' \), at an arbitrary fixed \( \alpha \in S^2 \) and an arbitrary fixed \( k > 0 \), approximates with any desired accuracy any given function \( f(\beta) \in L^2(S^2) \) with a small norm. Moreover, we give formulas for calculating this \( q \), and these formulas often work numerically for \( f \) which are not small. The potential \( q \) is related explicitly to a certain distribution of small particles in \( D \). Consequently, we give formulas for calculating this distribution.

In Section 3 we derive an equation describing the self-consistent field in the medium consisting of the small particles distributed in \( D \). This equation is equivalent to a Schrödinger equation with a potential \( q(x) \) supported in the bounded domain \( D \) and related in a simple way to the density of the distribution of the small particles.

The author has solved the 3D inverse scattering problem of finding a compactly supported potential \( q \) from the knowledge of noisy fixed-energy scattering amplitude [6]. This algorithm allows one to calculate \( q_\delta(x) \) from the knowledge of noisy data \( f_\delta(\alpha', \alpha) \), \( \sup_{\alpha', \alpha \in S^2} |f_\delta - f| \leq \delta \), such that

\[
\sup_{x \in D} |q_\delta(x) - q(x)| \leq \eta(\delta) \xrightarrow{\delta \to 0} 0,
\]

(10)
where \( q(x) \) is the exact potential, generating the exact scattering amplitude \( f(\alpha', \alpha) \) at a fixed \( k > 0 \).

Applying this algorithm to the exact data \( f(\alpha', \alpha) \) or to the noisy data \( f_\delta(\alpha', \alpha) \), one obtains a stable (in the sense (10)) approximation of \( q \), and, consequently, of the density of the distribution of small particles, which generates the scattering amplitude arbitrarily close to the a priori given scattering amplitude.

The author’s solution of the 3D inverse scattering problem with the error estimates is presented in Section 4.

2 Approximation properties of the scattering amplitudes

If \( k > 0 \) is fixed, then the scattering problem (11)–(13) is equivalent to the Schrödinger scattering problem on the potential \( q_0(x) \):

\[
|\nabla^2 + k^2 - q_0(x)|u = 0 \quad \text{in } \mathbb{R}^3,
\]

\[
q_0(x) = \begin{cases} 
0 \text{ in } D', & D' := \mathbb{R}^3 \setminus D \\
k^2[n_0(x) - 1] \text{ in } D.
\end{cases}
\]

The scattering solution \( u = u_{q_0} \) solves (uniquely) the equation

\[
u_{q_0} = u_0 - \int_D g(x, y)q_0(y)u_{q_0}(y)dy, \quad g(x, y) := \frac{e^{ik|x-y|}}{4\pi|x-y|}.
\]

The corresponding scattering amplitude is:

\[
A_0(\alpha', \alpha) = -\frac{1}{4\pi} \int_D e^{-ik\alpha' \cdot x}q_0(x)u_{q_0}(x, \alpha)dx,
\]

where the dependence on \( k \) is dropped since \( k > 0 \) is fixed.

If \( q_0 \) is known, then \( A_0 := A_{q_0} \) is known. Let \( q \in L^2(D) \) be a potential and \( A_q(\alpha', \alpha) \) be the corresponding scattering amplitude. Fix \( \alpha \in S^2 \) and denote

\[
A(\beta) := A_q(\alpha', \alpha), \quad \alpha' = \beta.
\]

Then

\[
A(\beta) = -\frac{1}{4\pi} \int_D e^{-ik\beta \cdot x}h(x)dx, \quad h(x) := q(x)u_q(x, \alpha).
\]

**Theorem 1.** Let \( f(\beta) \in L^2(S^2) \) be arbitrary. Then

\[
\inf_{h \in L^2(D)} \left\| f(\beta) - \left(-\frac{1}{4\pi} \int_D e^{-ik\beta \cdot x}h(x)dx\right) \right\| = 0.
\]
Proof of Theorem 1

If (17) fails, then there is a function \( f(\beta) \in L^2(S^2), f \neq 0, \) such that
\[
\int_{S^2} d\beta f(\beta) \int_D e^{-ik\beta \cdot x} h(x) dx = 0 \quad \forall h \in L^2(D).
\]
(18)

This implies
\[
\varphi(x) := \int_{S^2} d\beta f(\beta) e^{-ik\beta \cdot x} = 0 \quad \forall x \in D.
\]
(19)

The function \( \varphi(x) \) is an entire function of \( x \). Therefore \( \varphi(x) = 0 \quad \forall x \in \mathbb{R}^3. \)
(20)

This and the injectivity of the Fourier transform imply \( f(\beta) = 0. \) Note that \( \varphi(x) \) is the Fourier transform of the distribution \( f(\beta) \delta(k - \lambda) \lambda^{-2} \), where \( \delta(k - \lambda) \) is the delta-function and \( \lambda \beta \) is the Fourier transform variable. The injectivity of the Fourier transform implies \( f(\beta) \delta(k - \lambda) = 0, \) so \( f(\beta) = 0. \)

Theorem 1 is proved. \( \square \)

Let us give an algorithm for calculating \( h(x) \) in (17) such that the left-hand side of (17) does not exceed \( \varepsilon, \) where \( \varepsilon > 0 \) is an arbitrary small given number.

Let \( \{Y_\ell(\beta)\}_{\ell=0}^\infty, Y_\ell = Y_{\ell,m}, -\ell \leq m \leq \ell, \) be the orthonormal in \( L^2(S^2) \) spherical harmonics,
\[
Y_{\ell,m}(\beta) = (-1)^\ell Y_{\ell,m}(\beta), \quad \tilde{Y}_{\ell,m}(\beta) = (-1)^{\ell+m} Y_{\ell,m}(\beta),
\]
(21)

\[
 j_\ell(r) := \left( \frac{\pi}{2r} \right)^{1/2} J_{\ell+\frac{1}{2}}(r),
\]
(22)

where \( J_\ell \) are the Bessel functions. It is known that
\[
e^{-ik\beta \cdot x} = \sum_{\ell=0, -\ell \leq m \leq \ell} 4\pi(-i)^\ell j_\ell(kr) Y_{\ell,m}(x^0) Y_{\ell,m}(\beta), \quad x^0 := \frac{x}{|x|}.
\]
(23)

Let us expand \( f \) into the Fourier series with respect to spherical harmonics:
\[
f(\beta) = \sum_{\ell=0, -\ell \leq m \leq \ell} f_{\ell,m} Y_{\ell,m}(\beta).
\]
(24)

Choose \( L \) such that
\[
\sum_{\ell>L} |f_{\ell,m}|^2 \leq \varepsilon^2.
\]
(25)

With so fixed \( L, \) take \( h_{\ell,m}(r), 0 \leq \ell \leq L, -\ell \leq m \leq \ell, \) such that
\[
f_{\ell,m} = -(-i)^\ell \left( \frac{\pi}{2k} \right)^{1/2} \int_0^b r^{3/2} J_{\ell+\frac{1}{2}}(kr) h_{\ell,m}(r) dr,
\]
(26)
where \( b > 0 \), the origin \( O \) is inside \( D \), the ball centered at the origin and of radius \( b \) belongs to \( D \), and \( h_{\ell,m}(r) = 0 \) for \( r > b \). There are many choices of \( h_{\ell,m}(r) \) which satisfy (26). If (25) and (26) hold, then the norm on the left-hand side of (17) is \( \leq \varepsilon \).

A possible explicit choice of \( h_{\ell,m}(r) \) is

\[
h_{\ell,m} = \begin{cases} 
\frac{f_{\ell,m}}{-(-i)^{\ell} \sqrt{\pi} b_1,\ell + \frac{1}{2}}, & \ell \leq L, \\
0, & \ell > L,
\end{cases}
\]  

(27)

where we have assumed that \( b = 1 \) in (26), and used the following formula (see [1, formula 8.5.8]):

\[
\int_0^1 x^{\mu+\frac{1}{2}} J_\nu(kx) dx = k^{-\mu-\frac{3}{2}} \left[ \left( \nu + \mu - \frac{1}{2} \right) k J_\nu(r) S_{\mu-\frac{1}{2},\nu-1}(k) 
- k J_{\nu-1}(k) S_{\mu+\frac{1}{2},\nu}(k) + 2^{\mu+\frac{1}{2}} \frac{\Gamma\left(\frac{\mu+\nu+1}{2}\right)}{\Gamma\left(\frac{\mu-\nu+1}{2}\right)} \right] := g_{\mu,\nu}(k),
\]  

(28)

where \( S_{\mu,\nu}(k) \) are Lommel’s functions, \( \Gamma(x) \) is the Gamma-function, \( h_{\ell,m}(r) \) in (27) do not depend on \( r \), and we assume that \( h(x) = 0 \) for \( r := |x| > 1 \).

Let us prove that for any \( q \in L^2 \) there exists a \( q \in L^2(D) \) such that \( q(x) u_q \) approximates \( h(x) \) in \( L^2(D) \)-norm with arbitrary accuracy.

**Theorem 2.** Let \( h \in L^2(D) \) be arbitrary. Then

\[
\inf_{q \in L^2(D)} \|h - qu_q(x, \alpha)\| = 0.
\]  

(29)

Here \( \alpha \in S^2 \) and \( k > 0 \) are arbitrary, fixed. There exists a potential \( q \) such that \( h = qu \), if \( \|h\|_{L^2(D)} \) is sufficiently small.

**Proof of Theorem 2** In this proof we first assume that the norm of \( f \) is small, and then we drop the ”smallness” assumption. If the norm of \( f \) is sufficiently small, the the norm of \( h \) is small, so that the condition

\[
\inf_{x \in D} |u_0(x) - \int_D g(x, y) h(y) dy| > 0
\]  

(30)

is satisfied. Here \( g \) is defined in formula (13). If this condition is satisfied, then the formula

\[
q(x) = h(x)[u_0(x) - \int_D g(x, y) h(y) dy]^{-1}
\]  

(31)

yields the desired potential \( q \). The function \( h \) generates the function \( u := \frac{h}{q} \), where \( u \) is the scattering solution, corresponding to the potential \( q \), constructed by formula (31).

Therefore, the infimum in (29) is attained if condition (30) is satisfied by the given \( h \).

If \( f \) is arbitrary, not necessarily small, then \( h \) is not necessarily small. If, nevertheless, condition (30) holds for this \( h \), then the potential \( q \), given by formula (31), belongs to \( L^2(D) \) and yields the scattering amplitude \( A_q(\beta) \) which satisfies (8).
On the other hand, if condition (30) does not hold, then formula (31) may yield a potential which is not locally integrable. In this case, as was proved in [10], one can perturb $h$ slightly, so that the perturbed $h$, denoted by $h_\varepsilon$, $||h - h_\varepsilon||_{L^2(D)} < \varepsilon$, would yield by formula (31), with $h_\varepsilon$ in place of $h$, a potential $q_\varepsilon \in L^2(D)$. This potential generates the scattering amplitude $A_{q_\varepsilon}(\beta)$ which satisfies estimate (3), possibly with $c\varepsilon$ in place of $\varepsilon$, where the positive constant $c$ does not depend on $\varepsilon$. Theorem 2 is proved.

Let us give a different point of view on the role of the "smallness" assumption. If $||q||_{L^2(D)} \to 0$, then the set of functions $qu$ becomes a linear set. Thus, if (29) fails, then there exists an $h \neq 0$, $h \in L^2(D)$ such that

$$\int_D h(x)q(x)u_q(x,\alpha)dx = 0 \quad \forall q \in L^2(D) \quad ||q|| << 1. \quad (32)$$

Condition (32) holds in the limit $||q|| \to 0$ because in this limit the set of functions $qu$ becomes linear.

Let $c = \text{const} > 0$ be small. We will take $c \to 0$ eventually. Choose

$$q = c\overline{e}^{ik\alpha \cdot x}.$$ 

For sufficiently small $c > 0$ the equation

$$u_q = e^{ik\alpha \cdot x} - \int_D g(x,y)\overline{e}^{-ik\alpha \cdot y}u_qdy := e^{ik\alpha \cdot x} - Tu_q$$

is uniquely solvable for $u_q$ in $C(D)$ because $||T|| < 1$ if $c > 0$ is sufficiently small. We have

$$qu_q = qe^{ik\alpha \cdot x} - qTu_q = c\overline{e} + O(c^2), \quad c \to 0. \quad (33)$$

Substitute (33) into (32), divide by $c$, and take $c \to 0$. The result is:

$$\int_D |h|^2dx = 0. \quad (34)$$

This implies $h = 0$.

We describe the relation between $q(x)$ and the density distribution of small particles in Section 3. This relation makes it clear that a suitable distribution of small particles will produce any desirable potential $q \in L^2(D)$, and, consequently, any desirable scattering amplitude (radiation pattern) at an arbitrary fixed $\alpha \in S^2$ and $k > 0$.

We describe the algorithm for calculating the above distribution of small particles, given $f(\beta) \in L^2(S^2)$, in Section 3.

3 Scattering by many small particles.

The exact statement of the problem is:

$$[\nabla^2 + k^2 - q_0(x)]u = 0 \quad \text{in } \mathbb{R}^3 \setminus \bigcup_{m=1}^M D_m, \quad (35)$$
\( u = 0 \) on \( \bigcup_{m=1}^{M} S_m, \quad S_m := \partial D_m. \)

\( u = e^{ikx} + v = u_0 + v, \)

\( v = A(\alpha', \alpha) \frac{e^{ikr}}{r} + o \left( \frac{1}{r} \right), \quad r := |\alpha| \to \infty, \quad \alpha' = \frac{x}{r}. \)

We look for the solution of the form

\[ u(x) = u_0(x) + \sum_{m=1}^{M} \int_{S_m} G(x, s) \sigma_m(s) ds, \tag{39} \]

where \( G(x, s) \) is the Green function which solves the scattering problem in the absence of small particles, i.e.:

\[ [\nabla^2 + k^2 - q_0(x)] G = -\delta(x - y) \text{ in } \mathbb{R}^3, \tag{40} \]

\[ \lim_{|x| \to \infty} |x| \left( \frac{\partial G}{\partial |x|} - ikG \right) = 0, \tag{41} \]

and \( u_0 \) is the corresponding scattering solution. It was proved in [3], [7], that

\[ G(x, y) = \frac{e^{ik|x|}}{4\pi|x|} u_0(y, \alpha) + o \left( \frac{1}{|x|} \right), \quad |x| \to \infty, \quad \alpha = -\frac{x}{|x|}, \tag{42} \]

where \( u_0 \) is the scattering solution corresponding to \( q_0. \)

The function (39) solves equation (35), satisfies the radiation condition (38), because

\[ u_0 = u_0 + v_0, \tag{43} \]

where \( v_0 \) satisfies the radiation condition (11). Therefore (39) solves the problem (35)–(38) if \( \sigma_m \) are such that the boundary condition (35) is satisfied. All the above did not use the smallness of the particles.

Let us now use assumptions (11) and (10). Let \( x_j \in D_j \) be an arbitrary point inside \( D_j. \) Then

\[ \sup_{S \in S_j} |G(x, s) - G(x, x_j)| = O \left( ka + \frac{a}{d} \right), \quad |x - x_j| > d. \tag{44} \]

This follows from the integral equation, relating \( G \) and \( g: \)

\[ G(x, y) = g(x, y) - \int_{\mathbb{R}^3} g(x, z) q_0(z) G(z, y) dz, \]
and from the estimates:

\[
e^{-\frac{ik|x-s|}{4\pi|x-s|}} - e^{-\frac{ik|x-x_j|}{4\pi|x-x_j|}} = \frac{1}{4\pi|x-x_j|} \frac{e^{ik|x-s|/2} - e^{ik|x-x_j|/2}}{|x-s|} - 1,
\]

\[
k|x-s| - |x-x_j| = k|x-x_j| \left(1 + O\left(\frac{a}{d}\right) + O(ka)\right),
\]

\[
|x-s| = |x-x_j - (s-x_j)| = |x-x_j| \left(1 + O\left(\frac{a}{d}\right)\right).
\]

From the integral equation for \( G \) it follows that

\[
G(x, y) \sim g(x, y) \quad \text{as} \quad x \to y.
\]

Therefore one may approximate (39) as

\[
u(x) = u_0(x) + \sum_{m=1}^{M} G(x, x_j) Q_m \left[ 1 + O\left(ka + \frac{a}{d}\right)\right],
\]

(45)

where \(|x-x_j_m| \geq d\) for all \(m, 1 \leq m \leq M\), and

\[
Q_m = \int_{S_m} \sigma_m(s) ds.
\]

(46)

Therefore, if one knows the numbers \(Q_m, 1 \leq m \leq M\), then one knows the scattering solution \(u(x)\) at any point which is at a distance \(\geq d\) from the nearest to \(x\) small body.

Let us derive a linear algebraic system for calculating \(Q_m\). To do this, let us use the boundary condition (36). We have:

\[
\int_{S_m} G(s, t) \sigma_m dt = - \left[u_0(x_m) + \sum_{j \neq m} G(x_m, x_j) Q_j\right].
\]

(47)

Since \(k|s-t| \leq 2a \ll 1\), one has

\[
G(s, t) \approx \frac{e^{ik|s-t|}}{4\pi|s-t|} = \frac{1}{4\pi|s-t|} \left(1 + O(ka)\right).
\]

(48)

Consequently, equation (47) can be written as

\[
\int_{S_m} \frac{\sigma_m(t)}{4\pi|s-t|} dt = - \left[u_0(x_m) + \sum_{j \neq m} G(x_m, x_j) Q_j\right].
\]

(49)

This is an equation for the electrostatic charge distribution \(\sigma_m\) on the surface \(S_m\) of the perfect conductor \(D_m\), charged to the potential which is given by the right-hand side of (49). The total charge on the surface of the conductor is given by the formula:

\[
\int_{S_m} \sigma_m dt = Q_m = -C_m \left[u_0(x_m) + \sum_{j \neq m} G(x_m, x_j) Q_j\right], \quad 1 \leq m \leq M.
\]

(50)
where \( C_m \) is the electrical capacitance of the conductor \( D_m \). Equation (50) is a linear algebraic system for the unknown \( Q_j \).

Assume that the distribution of small bodies \( D_m \) in \( D \) is such that

\[
\lim_{M \to \infty} \sum_{D_m \subset \tilde{D}} C_m = \int_{\tilde{D}} C(x) dx, \tag{51}
\]

where \( \tilde{D} \) is an arbitrary subdomain of \( D \). This means that \( C(x) \) is the limiting density of the capacitance per unit volume around an arbitrary point \( x \in D \). Then the relation (45) in the limit \( M \to \infty, \ k a \to 0, \ a \to 0 \), takes the form

\[
u(x) = \nu_0(x) - \int_{D} G(x,y) C(y) u(y) dy, \tag{52}
\]

where \( C(x) \) is defined in (51). An equation which is similar to (52), with \( g(x_j, y) \) in place of \( G(x_j, y) \), has been derived in [2] by a different argument.

Equation (52) is equivalent to the Schrödinger equation

\[
[\nabla^2 + k^2 - q_0(x) - C(x)] u = 0, \tag{53}
\]

and \( u(x) \) is the scattering solution corresponding to the potential

\[
q(x) = q_0(x) + C(x). \tag{54}
\]

If \( q_0(x) \) is known (which we assume), then \( q(x) \) and \( C(x) \) are in one-to-one correspondence.

If the small particles \( D_m \) are identical, and \( C_0 \) is the electrical capacitance of a single particle, then

\[
C(x) = N(x) C_0, \tag{55}
\]

where \( N(x) \) is the density of the number of particles in a neighborhood of the point \( x \), that is, the number of particles per unit volume around point \( x \).

Therefore, given \( f(\beta) \in L^2(S) \), one finds \( q(x) \), such that \( \| A_q(\beta) - f(\beta) \| \leq \varepsilon \), where \( A_q(\beta) \) is the scattering amplitude, corresponding to the potential \( q \), the energy \( k^2 > 0 \) and the incident direction \( \alpha \) being fixed, and \( \beta = \alpha' \) is the direction of the scattered wave.

Let us describe the steps of our algorithm.

**Step 1.** Given \( f(\beta) \), find \( h \in L^2(D) \).

**Step 2.** Given \( h \in L^2(D) \), find \( q \) such that \( \| h - q(x) u_q(x) \|_{L^2(D)} \leq \varepsilon \).

Let us elaborate on Step 2. First, assume the existence of a potential \( q \), such that \( h = q u \). Consider the equation

\[
u = \nu_0 - \int g qu_q dy = \nu_0 - \int_D ghdy. \tag{56}
\]
We have
\[ qu_q := h. \]
Thus,
\[ A_q(\beta) = -\frac{1}{4\pi} \int_D e^{-ik\beta \cdot x} h(x) \, dx. \]  \hspace{1cm} (57)
Multiply (56) by \( q \). Then
\[ h = u_0 q - q \int_D gh \, dy. \]
Therefore, if
\[ \inf_{x \in D} |u_0(x) - \int_D g(x, s) h(y) \, dy| > 0, \]
then the solution of the equation \( qu_q = h \) is unique and is given by the formula:
\[ q(x) = \frac{h(x)}{u_0(x) - \int_D gh \, dy}. \]  \hspace{1cm} (58)
Formula (58) yields a potential for which \( A_q(\beta) \) is given by formula (57), and the corresponding scattering solution is given by formula (56). All this is true provided, for example, that
\[ \sup_{x \in D} \left| \int_D g(x, y) h(y) \, dy \right| < 1. \]  \hspace{1cm} (59)
Inequality (59) holds if \( h \) is fixed and \( \text{diam} \, D \) is sufficiently small, because of the following estimate:
\[ \sup_{x \in D} \left| \int_D g(x, y) h(y) \, dy \right| \leq (4\pi)^{-\frac{1}{2}} \| h \|_{L^2(D)} (\text{diam} \, D)^{\frac{1}{2}}. \]
Inequality (59) also holds if \( \| h \|_{L^2(D)} \) is sufficiently small and \( D \) is fixed. The norm \( \| h \|_{L^2(D)} \) is small if \( \| f \|_{L^2(S^2)} \) is sufficiently small. For the formula (58) to yield the desired potential, the inequality (59) is not necessary. If one can find a potential \( q(x) \) from the given \( h \) by formula (58), then this \( q \) generates the scattering solution by the formula
\[ u_q = u_0 - \int_D gh \, dy \]  \hspace{1cm} (60)
and
\[ h = q(x)u_q(x). \]  \hspace{1cm} (61)
The potential \( q \) can be found by formula (58), provided that \( f(\beta) \) is sufficiently small, because then \( h \) will be sufficiently small as follows, e.g., from (27).
If \( q \) is found, then
\[ N(x) = \frac{q(x) - q_0(x)}{C_0}. \]  \hspace{1cm} (62)
Thus, the corresponding distribution density of small particles is given analytically.

Analytical formulas, which allow one to calculate \( C_0 \) with any desired accuracy, are derived in [8], see also formula (91) below.
Remark. If \( f(\beta) \) corresponds to a real-valued \( q(x) \), then formula (58) yields a real-valued potential. In general, formula (58) yields a complex-valued potential. To get a complex-valued potential by a formula, similar to (55), one has to replace the boundary condition (36) by the condition

\[
    u_N = \zeta u \quad \text{on} \quad S_m, 
\]

where \( N \) is the exterior unit normal to the boundary \( S \), and \( \zeta \) is a complex constant, the impedance. Then \( C_0 \) in (55) should be replaced by the quantity:

\[
    C_\zeta = \frac{C_0}{1 + \frac{C_0}{\zeta S}},
\]

(see [8]), and, therefore, formula (55) yields a complex-valued \( C_\zeta(x) \) if \( \zeta \) is a complex number.

Suppose that a given \( h \) corresponds to a potential \( q(x) \in L^2(D) \) in the sense that \( h = q(x)u(x) \), where \( u(x) \) is the scattering solution corresponding to this \( q(x) \) at the wavenumber \( k > 0 \) and with the incident direction \( \alpha \). Then formula (58) defines \( q(x) \), and the corresponding scattering solution is \( u = \frac{h(x)}{q(x)} \).

If formula (58) does not produce a \( p \in L^2(D) \), then one can replace \( h \) in (58) by an \( h_\epsilon, ||h - h_\epsilon||_{L^2(D)} < \varepsilon \), and get a square-integrable potential \( q_\epsilon \) by formula (58) with \( h \) replaced by \( h_\epsilon \). If \( \varepsilon \) is sufficiently small, this potential \( q_\epsilon \) generates the radiation pattern, which differs by \( O(\varepsilon) \) from the desired \( f \).

4 Ramm’s solution of the 3D inverse scattering problem with fixed-energy data

We follow [6], [7]. Consider first the inversion of the exact data \( A_q(\alpha',\alpha) \). Let

\[
    A_q(\alpha',\alpha) = \sum_{\ell,\ell'=0}^{\infty} A_{\ell,m,\ell',m'} Y_{\ell',m'}(\alpha') Y_{\ell,m}(\alpha). \tag{65}
\]

It is proved in [7] that

\[
    |Y_{\ell}(\theta)| \leq \frac{1}{\sqrt{4\pi |J_\ell(r)|}}, \quad \forall r > 0, \quad \theta \in \mathcal{M}, \tag{66}
\]

where

\[
    \mathcal{M} = \{z : z \in \mathbb{C}^3, z \cdot z = k^2\}, \quad z \cdot \zeta := \sum_{j=1}^{3} z_j \zeta_j.
\]

Estimate (66) allows one to prove ( [6]) that the series (65) converges absolutely for \( \alpha' = \theta' \in \mathcal{M} \), so that the exact data \( A_q(\alpha',\alpha) \) allow one to calculate the values \( A_q(\theta',\alpha), \theta' \in \mathcal{M} \). These values are used below in the inversion formula (68).

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One can prove [7], that any $\xi \in \mathbb{R}^3$ can be written (nonuniquely) as
\[ \xi = \theta' - \theta, \quad \theta', \theta \in \mathcal{M}, \quad |\theta| \to \infty. \tag{67} \]
In [7] explicit analytical formulas are given for $\theta'$ and $\theta$ satisfying (67).

The exact data $A(\alpha', \alpha)$ admit an analytic continuation from $S^2 \times S^2$ onto $\mathcal{M} \times S^2$. Let
\[ \tilde{q}(\xi) := \int_D q(x) e^{-i\xi \cdot x} dx. \]
The inversion formula, proved in [7], is
\[ \tilde{q}(\xi) = \lim_{|\theta| \to \infty} \left[ -4\pi \int_{S^2} A(\theta', \alpha) \nu(\alpha, \theta)d\alpha \right], \tag{68} \]
where (67) holds and $\nu(\alpha, \theta)$ is an arbitrary approximate solution to the problem
\[ \mathcal{F}(\nu) := \int_{a_1 \leq |x| \leq b} |\rho(x)|^2 dx = \inf f := d(\theta). \tag{69} \]
Here
\[ \rho(x) := e^{-i\theta \cdot x} \int_{S^2} u(x, \alpha) \nu(\alpha, \theta)d\alpha - 1, \tag{70} \]
$a_1 > 0$ is a radius of a ball which contains $D$ as a strictly inner-subdomain, and $b > a_1$ is an arbitrary fixed number. The approximate solution $\nu$ to (69) is understood in the following sense:
\[ \mathcal{F}(\nu) \leq 2d(\theta). \tag{71} \]
This means that it is not necessary to find a very accurate approximation of the infimum in problem (69). It is sufficient, for example, to find any function $\nu(\alpha, \theta)$ for which the functional (69) takes the value not more than $2d(\theta)$, and the inversion formula (68) holds with such $\nu(\alpha, \theta)$. Also, formula (73) below, with the error term, holds as well.

It is proved in [7] that
\[ d(\theta) \leq \frac{c}{|\theta|}, \quad \theta \in \mathcal{M}, \tag{72} \]
where $c = c(||q||) > 0$ is a constant depending on an $L^\infty(D)$ norm of $q$. Therefore, given the exact data $A_q(\alpha', \alpha)$, one recovers the potential $q(x)$ by formula (68).

The error estimate of formula (68) is given by the formula:
\[ \tilde{q}(\xi) = -4\pi \int_{S^2} A(\theta', \alpha) \nu(\alpha, \theta)d\alpha + O\left( \frac{1}{|\theta|} \right), \quad |\theta| \to \infty, \tag{73} \]
where (67) holds.

If $q(x)$ is found, then
\[ N(x)C_0 = q(x) - q_0(x), \tag{74} \]
so that the density of distributions of small particles is found explicitly.

Consider now inversion of noisy data $A_\delta(\alpha', \alpha)$,

$$\sup_{\alpha', \alpha \in S^2} |A_\delta(\alpha', \alpha) - A(\alpha', \alpha)| \leq \delta.$$  \tag{75}

Here $A(\alpha', \alpha)$ corresponds to an exact potential, and $A(\alpha', \alpha)$ is not known. Instead, its noisy measurements $A_\delta(\alpha', \alpha)$ are known.

Define

$$N(\delta) = \left[ \frac{\ln \delta}{\ln \left| \ln \delta \right|} \right],$$  \tag{76}

where $[x]$ is the integer nearest to $x > 0$, and

$$\hat{A}_\delta(\theta', \alpha) = \sum_{\ell=0}^{N(\delta)} A_{\delta \ell}(\alpha) Y_\ell(\theta'), \quad \sum_{\ell} = \sum_{-\ell \leq m \leq \ell},$$  \tag{77}

$$u_\delta(x, \alpha) = e^{ik\alpha \cdot x} + \sum_{\ell=0}^{N(\delta)} A_{\delta \ell}(\alpha) Y_\ell(\alpha') h_\ell(kr), \quad \alpha' := \frac{x}{r}, \quad r = |x|,$$  \tag{78}

$$\rho_\delta(x; \nu) = e^{-i\theta \cdot x} \int_{S^2} u_\delta(x, \alpha) \nu(\alpha) d\alpha - 1, \quad \theta \in \mathcal{M},$$  \tag{79}

$$\mu(\delta) = e^{-\gamma N(\delta)}, \quad \gamma = \ln \frac{a_1}{b_0} > 0,$$  \tag{80}

$$b_0 := \frac{1}{2} \text{diam } D, \quad \kappa = |Im\theta|.$$  \tag{81}

Let

$$b_0 < a_1 < b,$$  \tag{82}

where $a_1$ and $b$ are arbitrary positive fixed numbers. Consider the problem:

$$|\theta| = \sup := \vartheta(\delta)$$  \tag{83}

under the constraints

$$|\theta| \left[ \left\| \rho_\delta(\nu) \right\|_{L^2(x:a_1 \leq |x| \leq b)} + \left\| \nu \right\|_{L^2(S^2)} e^{\kappa b} \mu(\delta) \right] \leq c,$$  \tag{84}

$$\theta \in \mathcal{M}, \quad \theta' - \theta = \xi, \quad \theta', \theta \in \mathcal{M},$$  \tag{85}

where $c > 0$ is a sufficiently large constant, and $b_0 < a_1 < b$.

It is proved in [7] that

$$\vartheta(\delta) = O \left( \frac{|\ln \delta|}{\ln |\ln \delta|^2} \right) \quad \delta \to 0.$$  \tag{86}
Let $\theta(\delta)$ and $\nu_\delta(\alpha)$ be any approximate solution to (83)–(85) in the sense that
\[
|\theta(\delta)| \geq \frac{1}{2} \vartheta(\delta).
\] (87)

Define
\[
\hat{q}_\delta := -4\pi \int_{S^2} A_\delta(\theta', \alpha) \nu_\delta(\alpha) d\alpha.
\] (88)

The following result is proved in [7]
Theorem (Ramm). One has
\[
\sup_{\xi \in \mathbb{R}^3} |\hat{q}_\delta - \tilde{q}(\xi)| = O\left(\frac{\ln|\ln \delta|}{|\ln \delta|}\right), \quad \delta \to 0.
\] (89)

This result gives an inversion formula for finding the potential from noisy fixed-energy scattering data.

Thus, the algorithm for finding the density of the distribution of small particles from the fixed-energy scattering data $A(\alpha', \alpha)$ can be formulated as follows:

**Step 1.** Given $A(\alpha', \alpha)$, find $q(x)$ using the inversion formulas (68) in the case of the exact data or (88) in the case of noisy data.

**Step 2.** Find the density of the distribution of the small particles by formula (62), where formulas for $C_0$ are given in [7].

\[
|C_0 - C^{(n)}| = O(Q^n), \quad 0 < Q < 1,
\] (90)

where $Q$ depends only on the geometry of the surface,

\[
C^{(n)} = 4\pi |S|^2 \left\{ \left(\frac{-1}{2\pi}\right)^n \int_S \int_S \frac{ds dt}{r_{st}} \left[ \int_S \psi(t, t_1) \cdots \psi(t_{n_1}, t_{n}) dt_1 \cdot dt_n \right] \right\}
\] (91)

\[
\psi(t, s) = \frac{\partial}{\partial N_t} \frac{1}{r_{st}}, \quad r_{st} = |s - t|, \quad |S| = \text{meas } S,
\] (92)

$S$ is the surface of the conductor, $C_0$ is the electrical capacitance of this conductor, and $N_t$ is the exterior normal to $S$ at the point $t$.

In particular, for $n = 0$ one gets

\[
C^{(0)} = \frac{4\pi |S|^2}{J}, \quad J := \int_S \int_S \frac{ds dt}{r_{st}}.
\] (93)

It is proved in [8] that

\[
C^{(0)} \leq C_0.
\] (94)

Formula (91) given an approximate value $C^{(n)}$ of the electrical capacitance of a perfect conductor placed in the space with dielectric permittivity $\varepsilon_0 = 1$. If $\varepsilon_0 \neq 1$, then one has to multiply the right-hand side of (91) by $\varepsilon_0$. 

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