Higher dimensional Bondi energy with a globally specified background structure

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Abstract

A higher (even spacetime) dimensional generalization of the Bondi energy has recently been proposed (Hollands and Ishibashi 2005 J. Math. Phys. 46 022503) within the framework of conformal infinity and Hamiltonian formalism. The gauge condition employed in Hollands and Ishibashi to derive the Bondi-energy expression is, however, peculiar in the sense that cross sections of null infinity specified by that gauge are anisotropic and in fact non-compact. For this reason, that gauge is difficult to use for explicit computation of the Bondi energy in general, asymptotically flat radiative spacetimes. Also it is not clear, under that gauge condition, whether an apparent difference between the expressions of higher dimensional Bondi energy and the four-dimensional one is due to the choice of gauges or a qualitatively different nature of higher dimensional gravity from four-dimensional gravity. In this paper, we consider instead, the Gaussian null conformal gauge as one of the more natural gauge conditions that admit a global specification of background structure with compact, spherical cross sections of null infinity. Accordingly, we modify the previous definition of higher dimensional news tensor so that it becomes well defined in the Gaussian null conformal gauge and derive, for vacuum solutions, an expression for the Bondi energy–momentum in the new gauge choice, which takes a universal form in arbitrary (even spacetime) dimensions greater than or equal to 4.

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1. Introduction

There has recently been considerable interest in theories formulated in higher dimensional spacetimes, and therefore it is of great importance to define a precise notion of the total energy...
of an isolated gravitating system in higher dimensions. In four-dimensional general relativity, there are two distinguished notions of asymptotic flatness and an associated total energy: the ADM energy [2] defined at spatial infinity and the Bondi energy [3, 4] defined at a cross section of null infinity. The former is a constant in time, while the latter is in general, time dependent and can be used to measure energy loss of an isolated source by emission of radiation. In this paper, we are concerned with the latter energy in higher dimensions.

Hollands and the present author have previously proposed a higher (even spacetime) dimensional generalization of the Bondi energy [1]. We first studied conditions for asymptotic flatness at null infinity $I$ in higher dimensions within the framework of conformal infinity, analyzing the stability of conformal null infinity against linear gravitational perturbations. Then we derived an expression for the generator conjugate to asymptotic symmetries within the Hamiltonian framework of Wald and Zoupas [5] and proposed to take that generator conjugate to asymptotic time translation as the definition of a higher dimensional Bondi energy. In the derivation, we employed a particular gauge which demands that a conformal background geometry be locally, exactly Minkowskian in some neighborhood of conformal null infinity $I$. (We shall review below the gauge choice in [1], which we refer to as the Minkowskian conformal gauge in this paper.) This gauge choice simplifies relevant computations to a certain extent and thus is convenient for writing down the symplectic potential—whose integral over a segment of $I$ corresponds to radiation flux passing through the segment—as well as deriving the Bondi-energy expression in terms of curvature tensor with respect to unphysical conformal metric. However the Minkowskian conformal gauge is not globally well defined in the sense that under that gauge, cross sections of conformal null infinity $I$, where we evaluate the Bondi energy, become non-compact; they are spheres with a single point removed and do not naturally reflect the topology of $I$. For the purpose of computing the Bondi energy in general asymptotically flat radiative spacetimes and obtaining physical consequences, it would be much preferable that the Bondi energy is expressed in gauges that can be taken globally in a neighborhood of $I$ so that the Bondi energy is evaluated on a compact cross section of $I$. Also, with the Minkowskian conformal gauge, the definition of higher dimensional news tensor and the Bondi-energy expression [1] are slightly different from their four-dimensional counterparts obtained previously [5, 6]. It is not clear whether the differences are due to the choice of a peculiar gauge or a qualitatively different nature of higher dimensional gravity from the four-dimensional one.

The purpose of this paper is to obtain an expression for the Bondi energy in even-spacetime dimensions under a natural gauge choice that allows us to take compact, spherical cross sections of $I$. In even spacetime dimensions greater than or equal to 4, a conformal null infinity $I$ can exist as a smooth boundary null hypersurface of an unphysical, conformal spacetime $(\tilde{M}, \tilde{g}_{ab})$, being regular even in the presence of radiation [1], in contrast to the case of odd spacetime dimensions [7]. (For discussions on smoothness of null infinity in four dimensions, see e.g., [8–10] and references therein.) Following the standard arguments, one can then construct Gaussian null coordinates—which we refer to as the Gaussian null conformal gauge—in a neighborhood of $I$ on the unphysical, conformal spacetime. Therefore, for even dimensions, in addition to the drop-off conditions on the conformal metric, $\tilde{g}_{ab}$, derived in [1], one can naturally assume, as a part of the definition of asymptotic flatness, that the Gaussian null conformal gauge can be taken in some neighborhood of conformal null infinity. We discuss that the Gaussian null conformal gauge covers $I$ globally and naturally specifies compact, spherical cross sections of $I$.

In the next section, we briefly summarize the main results of the paper [1]. We first recapitulate our definition of asymptotic flatness at null infinity in higher (even) dimensions and our strategy for deriving a Bondi-type energy in higher dimensions within the Hamiltonian
framework. Then, we recall our higher dimensional Bondi-energy expression defined under the Minkowskian conformal gauge. In section 3, we introduce Gaussian null coordinates in a neighborhood of null infinity in an unphysical, conformal spacetime and express our asymptotic flatness conditions in terms of that coordinate system with respect to \( \mathcal{I} \). This Gaussian null coordinate system is useful for seeing differences between asymptotic behavior of higher dimensional gravity and that of four-dimensional gravity. For example, in four dimensions, it is well known that an asymptotic symmetry group at null infinity is not the Poincaré group but rather an infinite dimensional group, called the Bondi–Metzner–Sachs (BMS) group, which includes angle-dependent translations or supertranslations \([4, 6, 11]\). On the other hand, it has been shown in \([1]\) that a supertranslation does not exist in higher dimensions. This can be manifest and easy to compare with the four-dimensional case when one examines asymptotic translational symmetries in terms of Gaussian null coordinates.

Then, we provide a (modified) definition of the news tensor, which is slightly different from that given in \([1]\) and is now regular under the Gaussian null conformal gauge. We give our main theorem that indicates the existence of a Bondi energy–momentum vector in arbitrary even dimensions greater than or equal to 4, and then derive an expression of a higher dimensional Bondi energy in terms of our conjectured Bondi energy–momentum vector. The energy-loss formula is also given there. Summary and brief discussion on the regularity of our Bondi energy–momentum vector are given in section 4. The conventions of the metric signature and definitions of curvature tensors follow \([12]\). The spacetime dimension, denoted by \( d \), is assumed to be even throughout the paper, unless otherwise stated.

2. Asymptotic flatness in higher (even) dimensions

In this section, we shall recapitulate the results of the paper \([1]\) (see also \([13]\)). We first state our boundary conditions that define asymptotic flatness at null infinity in higher dimensions and then review a general strategy for defining conserved quantities in asymptotically flat spacetimes. After deriving the flux formula and the Bondi energy in higher dimensions under the Minkowskian conformal gauge, we see why that gauge specifies non-compact cross sections of null infinity.

2.1. Stable definition of asymptotic flatness

Let \((M, g_{ab})\) be a physical spacetime of dimension \( d \) which we wish to identify as an asymptotically flat spacetime in a certain sense. We are interested in, within the framework of conformal infinity, the question of how to specify the precise rate at which Minkowski spacetime, \((M', \eta_{ab})\), is approached \((M, g_{ab})\), at large distances toward null infinity. For this purpose it is convenient to introduce the following two fictitious spacetimes. Let \((\tilde{M}, \tilde{g}_{ab})\) and \((\bar{M}, \bar{g}_{ab})\) be an unphysical conformal spacetime and the background geometry associated with \((M, g_{ab})\). These two metrics, \(\tilde{g}_{ab}\) and \(\bar{g}_{ab}\), are related to the physical metric, \(g_{ab}\), and the Minkowski metric, \(\eta_{ab}\), via a smooth conformal factor, \(\Omega\), as \(\tilde{g}_{ab} = \Omega^2 g_{ab}\) and \(\bar{g}_{ab} = \Omega^2 \eta_{ab}\) so that one can attach a boundary \(\mathcal{I} \) at \(\Omega = 0\) to \(M\) such that there exists an open neighborhood of \(\mathcal{I}\) in \(\tilde{M} = M \cup \mathcal{I}\) which is diffeomorphic to an open subset of the manifold \(\tilde{M}\), and \(\mathcal{I}\) is mapped to a subset of the boundary of \(\bar{M}\). (See \([1]\) for more details.) As usual, indices on tensor fields on \(M\) with ‘tilde’ are lowered and raised with the unphysical metric, \(\tilde{g}_{ab}\), and its inverse, and a similar rule applies to tensor fields with ‘bar’ on \(\bar{M}\). Note that \(\mathcal{I}\) is divided into disjoint sets, \(\mathcal{I}^+\) and \(\mathcal{I}^-\), the future and past null infinity, respectively. In the following, by \(\mathcal{I}\) we mean the future null infinity \(\mathcal{I}^+\), unless otherwise stated. We assume that \(\mathcal{I}\) be
topologically $\mathbb{R} \times S^{d-2}$ so that it is consistent with the notion of a (higher dimensional version of) \emph{weakly asymptotically simple} spacetime.

We call $(M, g_{ab})$ of $d$ (even) dimensions \emph{asymptotically flat at null infinity} if the corresponding unphysical metric satisfies the following boundary (fall off) conditions near null infinity $\mathcal{I}$,

$$
\begin{align*}
\bar{g}_{ab} &= O(\Omega^{(d-2)/2}), \\
\bar{e}_{ab\cdot c} - \bar{e}_{ab\cdot c} &= O(\Omega^{d/2}),
\end{align*}
$$

(1)

$$
(\bar{g}^{ab} - \bar{g}^{ab})(d\Omega)_b = O(\Omega^{d/2}), \\
(\bar{g}^{ab} - \bar{g}^{ab})(d\Omega)_a(d\Omega)_b = O(\Omega^{(d+2)/2}),
$$

where $\bar{e}$ and $\bar{\epsilon}$ denote, respectively, the natural volume element of the unphysical spacetime and the background geometry.

The definition of asymptotic flatness above is arrived at through an analysis of stability of conformal null infinity against linear gravitational perturbations. More precisely, let $(M, g_{ab})$ be a globally hyperbolic solution to the vacuum Einstein equations and $\delta g_{ab}$ be a solution to the linearized vacuum Einstein equations with initial data of compact support on a Cauchy surface. Then, it is shown [1] that there exists a gauge for all even $d > 4$ so that the unphysical metric perturbations, $\delta \bar{g}_{ab} = \Omega^{d/2} \delta g_{ab}$, behave at $\mathcal{I}$ as

$$
\begin{align*}
\delta \bar{g}_{ab} &= O(\Omega^{(d-2)/2}), \\
\delta \bar{g}_{ab}\bar{n}^a &= O(\Omega^{d/2}), \\
\delta \bar{g}_{ab}\bar{\Omega} &= O(\Omega^{d/2}),
\end{align*}
$$

(2)

where $\bar{n}^a = \bar{g}^{ab}\bar{\nabla}_b \Omega$.

One can view equations (2) as a linearized version of the asymptotic flatness conditions above, equations (1). In four dimensions, linear stability of conformal null infinity was shown by Geroch and Xanthopoulos [14], in which the corresponding drop-off conditions are given by

$$
\begin{align*}
\delta \bar{g}_{ab} &= O(\Omega), \\
\delta \bar{g}_{ab}\bar{n}^b &= O(\Omega^2), \\
\delta \bar{g}_{ab}\bar{\Omega} &= O(\Omega^3), \\
\bar{g}^{ab}\delta \bar{g}_{ab} &= O(\Omega).
\end{align*}
$$

(3)

We note that in four dimensions, equations (3), the trace part of the perturbation falls off as fast as the other components of the metric perturbation, in contrast to the case of higher $d > 4$ dimensions, equations (2). We should also note that the transverse-traceless gauge works when $d > 4$, but does not when $d = 4$; one needs to use the Geroch–Xanthopoulos gauge [14] for $d = 4$. Under these gauge conditions, one can show that the vacuum Einstein equations with certain field variables that correspond to relevant unphysical metric perturbations—such as $\Omega^{-(d-2)/2}\delta \bar{g}_{ab}$—indeed form a hyperbolic system of partial differential equations that possesses a well-posed initial value formulation in the unphysical spacetime and thereby conclude that for compactly supported smooth initial data for perturbations, the field variables extend to smooth tensor fields at null infinity, $\mathcal{I}$.

It should be commented that the unphysical metric generically fails to be smooth at null infinity in odd-spacetime dimensions when gravitational radiation presents; it follows from half-integral powers of $\Omega$ in equations (2) that the unphysical metric appears to be at most $(d - 3)/2$ times differentiable at null infinity. This non-smoothness of the unphysical metric in odd dimensions has been shown more explicitly in [7] by examining the leading order behavior of the unphysical Weyl curvature perturbations off of Minkowski spacetime. Mainly for this reason, our conformal approach to defining stable notion of asymptotic flatness at null infinity does not apply to odd-dimensional spacetimes.

Note also that as mentioned above, the asymptotic flatness (fall-off) conditions, equation (1), are not derived from analysis of full nonlinear theory but rather postulated based on linear stability analysis of conformal null infinity against gravitational perturbations. Higher order perturbation effects—if taken into account—might possibly alter the asymptotic
fall-off behavior, but in the present paper, we are not going to investigate such nonlinear effects on the asymptotic flatness conditions.

Once boundary conditions for asymptotic flatness at null infinity are established, one can introduce the notion of asymptotic symmetries. Consider vector fields $\xi^a$ on $M$ which are smooth, complete on $\tilde{M}$, and in particular tangent to $I$ on $I$, and are such that

$$\Omega^2 \xi_{|g_{ab}} = \xi_{|\tilde{g}_{ab}} - 2\Omega^{-1} \xi^c \tilde{h}_{c|ab}$$

(4)

satisfies the (linearized version of the) asymptotic flatness conditions, equation (2) for $d > 4$ and equation (3) for $d = 4$. Since we are concerned with symmetry properties at null infinity $I$, we introduce an equivalence relation into the set of vector fields $\xi^a$ by viewing them as equivalent if they coincide on $I$. An equivalence class of such vector fields—we denote it by the same notation $\xi^a$ in the following—is called an infinitesimal asymptotic symmetry and generates an asymptotic symmetry group, which is a one-parameter group of diffeomorphisms $\phi$ which map any asymptotic flat metric $g_{ab}$ to an asymptotic flat metric $\phi^* g_{ab}$.

2.2. Strategy for defining Bondi energy

Based on what is sometimes called the covariant phase space method, Wald and Zoupas [5] developed a general strategy for defining conserved quantities associated with symmetries that preserve a given set of boundary conditions, which can apply to any theories derived from a diffeomorphism covariant Lagrangian. (For earlier work of the covariant phase space method, see e.g., [15–17] and references therein.) For Einstein’s gravity, our starting point is the Lagrangian density $d$-form, $L$, given in terms of the scalar curvature, $R$, and the natural $d$-volume element, $\epsilon$, with respect to the physical metric, $g_{ab}$, as

$$L = \frac{1}{16\pi G} R \epsilon,$$

(5)

with the boundary null infinity $I$ and the boundary conditions specified in the definition of asymptotic flatness above, equations (2) and (3), where associated are asymptotic symmetries $\xi^a$. Taking variation with respect to the metric $g_{ab}$, one has

$$\delta L = E + d \theta,$$

(6)

where $E$ denotes Einstein’s equations and $\theta$ is a $(d - 1)$-form given by

$$\theta_{\delta_1...\delta_{d-1}} = \frac{1}{16\pi G} \delta_{|g}^a \delta_{|g}^b (\nabla_a \delta g_{bd} - \nabla_d \delta g_{ab}) \epsilon_{ca_1...a_{d-1}}.$$

(7)

Consider a pair of variations $(\delta_1 g, \delta_2 g)$ and assume that the two variations commute, i.e., $\delta_1 \delta_2 g - \delta_2 \delta_1 g = 0$. Then, the symplectic current, $\omega$, can be defined as

$$\omega(g; \delta_1 g, \delta_2 g) = \delta_1 \Theta(g; \delta_2 g) - \delta_2 \Theta(g; \delta_1 g).$$

(8)

If $\omega$ has a vanishing extension to the boundary under consideration, then there exists a conserved Hamiltonian associated with $\xi^a$. However, this is, in general, not the case especially when radiation presents near the boundary $I$. However, even in that case, if (i) $\omega$ has a well-defined (finite) extension to $I$ for any asymptotically flat metric and in addition (ii) there exists a symplectic potential, $\Theta$, on $I$ such that

$$\xi^* \omega(g; \delta_1 g, \delta_2 g) = \delta_1 \Theta(g; \delta_2 g) - \delta_2 \Theta(g; \delta_1 g),$$

(9)

where here and in the following $\xi^*$ denotes the pull back of a tensor field to $I$, then one can define an associated charge $\mathcal{H}_\xi$ by

$$\delta \mathcal{H}_\xi = \int_B (\delta Q[\xi] - \xi \cdot \Theta) + \int_B \xi \cdot \Theta,$$

(10)
where ′·′ denotes the interior product, \( B \) is a given cross section at \( \mathcal{J} \), and where
\[
Q[\xi]_{a_1...a_{d-2}} = - \frac{1}{16\pi G} (\nabla^b \xi^c) \epsilon_{a_1...a_{d-2}b} \tag{11}
\]
is the Noether charge \( (d-2) \)-form. Note that the integrand, \( \xi \cdot \Theta \), of the second integral of the right-hand side of equation (10) is, by definition, a smooth quantity on \( \mathcal{J} \), whereas the integrand, \( \delta Q[\xi] - \xi \cdot \Theta \), of the first integral is defined only inside the spacetime. Therefore, here and in the following, an integral over a given cross section \( B \) of \( \mathcal{J} \) should be understood as follows: consider first an achronal hypersurface \( \Sigma \) in some neighborhood of \( \mathcal{J} \) which extends to \( B \) and a nested sequence of compact subsets \( K_i \) of \( \Sigma \) such that as \( i \to \infty \), the \( (d-2) \)-surface \( S_i = \partial K_i \) approaches \( \partial \Sigma = B \). Then, evaluate the integral over \( S_i \) and then take the limit \( i \to \infty \). The existence of the limit follows from the same argument below equation (45) in [1] when \( \omega \) has a smooth extension to \( \mathcal{J} \). Note also that field variations taken in equation (10) are assumed to satisfy the linearized Einstein equations.

In the four-dimensional case, it was shown [5] that under certain fall-off conditions and the choice of the Bondi gauge, the symplectic current 3-form, \( \omega \), extends smoothly to \( \mathcal{J} \) and, in general, is non-vanishing there (hence the assumptions (i) and (ii) hold) and the resultant \( \mathcal{H}_\xi \) indeed agrees with the Bondi-energy expression [6].

The assumptions (i) and (ii) were shown to hold also in higher (even) dimensions [1], under the boundary conditions, equation (1). Therefore, according to Wald and Zoupas [5], one is able to construct a charge \( \mathcal{H}_\xi \) associated with an asymptotic symmetry \( \xi^a \) also in higher dimensions. The proposal of [1] is that one can take \( \mathcal{H}_\xi \) as the definition of a higher dimensional Bondi energy. For this purpose, we seek a vector field \( P^a \) on \( \hat{M} \) that satisfies \( P^a \nabla_a \Theta = O(\Omega^2) \) and also
\[
\Theta = (d-1)\xi \cdot \hat{\nabla}_a P^a + O(\Omega), \tag{12}
\]
where \((d-1)\xi\) is the volume element on \( \mathcal{J} \) defined by \( \xi_{a_1...a_d} = d\hat{n}_{[a_1...a_d]}(d-1)\xi_{a_2...a_d]}. \) Note that above equation (12) determines \( P^a \) uniquely, up to the addition of a vector field \( X^a \) which satisfies \( X^a \nabla_a \Theta = O(\Omega^2) \) and \((d-1)\xi \cdot \hat{\nabla}_a X^a = d\mu + O(\Omega) \) with \( \mu \) being some \((d-2)\)-form. In other words, if \( P^a \) is a solution to equation (12), then
\[
P^a = P^a + \hat{\nabla}_b X^{ab} \tag{13}
\]
is also a solution to equation (12), provided that \( X^{ab} \) is an anti-symmetric tensor on \( \hat{M} \) such that \( \nabla_a X^{ab} = O(\Omega^2) \) in a neighborhood of \( \mathcal{J} \). Now let \( I \subset \mathcal{J} \) be a segment of \( \mathcal{J} \) bounded by two cross sections, \( B \) and \( B_0 \) (with \( B \) being in the future of \( B_0 \)). Integrating the above formula, equation (12), over \( I \subset \mathcal{J} \) and applying Stokes’ theorem, one formally obtains the following formula,
\[
F_\xi = - \int_I \Theta(g; \xi_g) = \int_B (d-2)\xi \cdot \hat{\nabla}_a P^a \hat{\xi}_a + \int_{B_0} (d-2)\xi \cdot \hat{\nabla}_a P^a \hat{\xi}_a, \tag{14}
\]
where \((d-2)\xi\) denotes the natural volume element on a cross section \( B \) induced from \( \hat{\xi} \), and where and hereafter \( \hat{\xi}_a \) denotes a null geodesic tangent in \( \hat{M} \) that satisfies \( \hat{\xi}_a \cdot \hat{\nabla}_a \Theta = 1 \), so that \( \hat{\xi}_a \) is transverse to \( \mathcal{J} \). Note that \( F_\xi \)—which corresponds to the flux through the segment \( I \) associated with an infinitesimal symmetry \( \xi^a \)—is well defined, and finite since by definition \( \Theta \) itself is a smooth, finite quantity on \( \mathcal{J} \) under our boundary conditions. Thus, if each integral of the right-hand side of equation (14) is well defined by itself (and independent of how \( S_i \) approaches \( B \)), one can identify the (higher dimensional) Bondi energy at \( B \) with
\[
\mathcal{H}_\xi = \int_B (d-2)\xi \cdot \hat{\nabla}_a P^a \hat{\xi}_a, \tag{15}
\]
and may view \( P^a \) as a (higher dimensional) Bondi energy–momentum integrand.
2.3. Bondi energy in the Minkowskian conformal gauge

Now we construct the symplectic potential, $\Theta$, for equation (9). For this purpose, let us introduce the following tensor field

$$\tilde{S}_{ab} = \frac{2}{d-2} \tilde{R}_{ab} - \frac{1}{(d-1)(d-2)} \tilde{R} \delta_{ab},$$

where $\tilde{R}_{ab}$ and $\tilde{R}$ are the Ricci tensor and the scalar curvature with respect to $\tilde{g}_{ab}$. Using the standard formulae for conformal transformation of a metric and associated curvature tensors (see e.g., [12]), one can rewrite the vacuum Einstein’s equation, $\tilde{R}_{ab} = 0$, as

$$\tilde{S}_{ab} = -2\Omega^{-1} \tilde{V}_a n_b + \Omega^{-2} \tilde{n}^c n_a \tilde{g}_{ab}.$$  \hfill (17)

Then, the pull back to $\mathcal{J}$ of the symplectic current $(d-1)$-form, $\omega$, can be expressed as

$$\xi^* \omega_{a_1 \ldots a_{d-1}} = -\frac{1}{32\pi G} \frac{\bar{\Omega}^{-(d-3)}}{\bar{\Omega}^{-(d-2)/2}} (\delta_{ab} \bar{S}_{bc} - \delta_{bc} \bar{S}_{ab}) \epsilon_{a_1 \ldots a_{d-1}}.$$  \hfill (18)

Note that from the asymptotic flatness conditions, equation (2), it follows that

$$\tilde{S}_{ab} - \tilde{S}_{ab} = O(\Omega^{d-4/2}), \quad \tilde{S}^{\rho}_\rho - \tilde{S}^{\rho}_\rho = O(\Omega^{d-2/2}), \quad \tilde{n}^a \tilde{n}_a = O(\Omega^{d+2/2}).$$  \hfill (19)

At this point, one can introduce a trace-free symmetric tensor, $N_{ab}$, defined on $\mathcal{J}$ so that $\xi^* \omega$ above can be expressed as

$$\xi^* \omega_{a_1 \ldots a_{d-1}} = -\frac{1}{32\pi G} \frac{\bar{\Omega}^{-(d-2)/2}}{\bar{\Omega}^{-(d-2)/2}} \delta_{ab} \bar{N}_{bc} \epsilon_{a_1 \ldots a_{d-1}}.$$  \hfill (20)

The symplectic potential is then expressed as

$$\Theta(g; \delta g)_{a_1 \ldots a_{d-1}} = -\frac{1}{32\pi G} \frac{\bar{\Omega}^{-(d-2)/2}}{\bar{\Omega}^{-(d-2)/2}} \delta_{ab} \bar{N}_{bc} \epsilon_{a_1 \ldots a_{d-1}}.$$  \hfill (21)

In order to define $N_{ab}$—called the news tensor—in terms of $\tilde{S}_{ab}$, the flat metric in a neighborhood of the boundary of $\mathcal{J}$, we impose gauge conditions. Perhaps the simplest gauge choice is to impose

$$\tilde{S}_{ab} = 0, \quad \tilde{n}^a \tilde{n}_a = 0, \quad \tilde{V}_a \tilde{n}^a = 0.$$  \hfill (22)

This gauge choice can be taken at least locally in some neighborhood of $\mathcal{J}$, where $\tilde{n}^a$ is $\tilde{g}^{ab} \nabla_b \Omega$. This gauge—the Minkowskian conformal gauge—employed in [1]—corresponds to requiring that the background metric, $\tilde{g}_{ab}$, be the flat metric in a neighborhood of the boundary of $M$, and the quantities on the unphysical manifold $\tilde{M}$ satisfy

$$\tilde{S}_{ab} = O(\Omega^{d-4/2}), \quad \tilde{n}^a \tilde{n}_a = O(\Omega^{d+2/2}), \quad \tilde{V}_a \tilde{n}^a = O(\Omega^{d-2/2}).$$  \hfill (23)

The news tensor is then defined by (equation (61) in [1])

$$N_{ab} = \xi^* (\Omega^{-(d-4)/2} q_a^m q_b^n \tilde{S}_{mn}),$$  \hfill (24)

where $q_{ab} = \tilde{q}_{ab} - 2 \tilde{\xi}_a n_b$, with $\tilde{\xi}_a$ being a covector on $\tilde{M}$ such that $\tilde{V}_a \tilde{\xi}_b = O(\Omega^{d+4/2})$, $\tilde{\xi}_a = O(\Omega^{d+2/2})$, $\tilde{n}^a \tilde{\xi}_a = 1 + O(\Omega^{d/2})$ near $\mathcal{J}$. Under the Minkowskian conformal gauge, expressing $\delta \tilde{g}_{ab}$ as $\Omega^2 \tilde{g}_a \tilde{g}_b$ in terms of $\tilde{S}_{ab}$ via the Einstein equations, equation (17), and then substituting into equation (21), one obtains, for the translation $\xi^a = \tau \tilde{n}^a$ with $\tau = \text{const.}$,

$$\Theta(g; \xi^a) = \frac{1}{32\pi G} \frac{\bar{\Omega}^{-(d-4)}}{\bar{\Omega}^{-(d-4)}} S_{ab} S_{cd} q_a^c q_b^d.$$  \hfill (25)

The Bondi energy–momentum, $P^a$, for $\xi^a = \tau \tilde{n}^a$ is then identified with

$$P^a = \frac{\tau}{8(d-3)\pi G} \frac{\bar{\Omega}^{-(d-4)}}{\bar{\Omega}^{-(d-4)}} (\tilde{n}^a \tilde{\xi}_b q_d^c \tilde{S}_{db} \tilde{V}_b \tilde{\xi}_c - \Omega^{-1} \tilde{C}^{abc} n_b \tilde{\xi}_c n_d),$$  \hfill (26)

$$= \frac{\tau}{8(d-3)\pi G} \frac{\bar{\Omega}^{-(d-4)}}{\bar{\Omega}^{-(d-4)}} (\tilde{n}^a \tilde{\xi}_b \tilde{g}_{cd} q_d^e \tilde{S}_{ab} \tilde{V}_b \tilde{\xi}_e - \Omega^{-1} \tilde{C}^{abc} n_b \tilde{\xi}_c n_d),$$  \hfill (27)

$$= \frac{\tau}{8(d-3)\pi G} \frac{\bar{\Omega}^{-(d-4)}}{\bar{\Omega}^{-(d-4)}} (\tilde{n}^a \tilde{\xi}_b \tilde{g}_{cd} q_d^e \tilde{S}_{ab} \tilde{V}_b \tilde{\xi}_e - \Omega^{-1} \tilde{C}^{abc} n_b \tilde{\xi}_c n_d).$$  \hfill (28)
and the Bondi energy for $\xi^\mu = \tau \tilde{n}^\mu$ is given by [1]

$$\mathcal{H}_\ell = \frac{1}{8(d-3)\pi G} \int_M \tau \sqrt{\Omega} \left( \frac{1}{(d-2)} \bar{R}_{ab} q^{ac} q^{bd} (\nabla_a \tilde{e}_d - \Omega^{-1} \tilde{c}_{abcd} n_b \tilde{e}_c n_d) \right) \tilde{e},$$

(27)

where $\tilde{c}_{abcd}$ denotes the Weyl tensor with respect to $\tilde{g}_{ab}$.

The Minkowskian conformal gauge can be achieved at least locally but does not seem to globally determine background structure with desired property. As a simple case, let us consider Minkowski spacetime $(M, \eta_{ab})$ as our physical spacetime. Then, the Minkowskian conformal gauge, equation (22), can be explicitly constructed as follows. Let $x^\mu = (x^0, x^i)$, $i = 2 \cdots (d-1)$ be Cartesian coordinates in $(M, \eta_{ab})$, and let $V^+ = \{ x|x_0, x^i < 0, \ x^0 > 0 \}$, i.e., the interior of the future light cone of the origin $x^\mu = 0$, and $W = \{ x|x^1 \geq |x^0| \}$, a Rindler wedge. Consider the map $\phi$ on $(M, \eta_{ab})$ defined by (see equation (B8) in [1]),

$$\phi^\mu(x) = a^\mu + b^\mu x_\nu x^\nu + 2(\eta^\mu\nu + a^\mu b^\nu)x_\nu,$$

(28)

where $a^\mu = (1, -1, 0, \ldots, 0)$ and $b^\mu = (1, 1, 0, \ldots, 0)$ in the above coordinates. One can observe that $\phi$ maps points in $V^+$ bijectively to points in $W$ and furthermore

$$(b_\sigma x^\sigma)^2 \eta_{\mu\nu} \partial \phi^\mu \partial \phi^\nu = \eta_{\mu\nu} \partial x^\mu \partial x^\nu.$$  

(29)

Thus, $\phi$ is a conformal isometry from $V^+$ to $W$. Therefore if we choose $M = W$ and $\tilde{g}_{ab} = \Omega^2 \phi_* \eta_{ab}$ with $\Omega = b_\sigma x^\sigma$, then our background geometry $(M, \tilde{g}_{ab})$ satisfies the Minkowskian conformal gauge conditions. Since defining a Rindler wedge $W$ needs to choose a particular spatial direction—the direction specified by the vector $b^\mu$ in the example above, the conformal isometry $\phi : V^+ \rightarrow W$ breaks the global isotropy of the original Minkowski spacetime possesses. As a result, all null generators of $\mathcal{I}$ are mapped to $\partial W$, except a single null generator of $\mathcal{I}$ in the direction specified by $b^\mu$. Consequently, a cross section $B$ on $\partial M$ specified by this gauge choice is a $(d-2)$-sphere with a single point removed, and hence non-compact. Although the above example of the Minkowskian conformal gauge is given in the special context that $(M, g_{ab})$ is Minkowski spacetime, it seems highly unlikely that the Minkowskian conformal gauge would admit a compact cross section in $\mathcal{I}$ for more generic, asymptotically flat spacetimes.

3. Gaussian null conformal gauge

In this section, instead of the Minkowskian conformal gauge discussed above, we shall consider Gaussian null conformal gauge, which allows us to take globally defined, compact spherical cross sections of $\mathcal{I}$. We shall discuss the asymptotic flatness conditions, the definition of news tensor and the Bondi energy–momentum under the Gaussian null conformal gauge condition.

3.1. Flatness conditions and asymptotic symmetries

As seen in the previous section, in even spacetime dimensions, a conformal null infinity, $\mathcal{I}$, can always exist as a smooth null hypersurface in an unphysical spacetime $(\tilde{M}, \tilde{g}_{ab})$, being stable against at least linear perturbations. Then, following the standard procedure (appendix A of [18]), one can construct a coordinate system $x^\mu = (u, \Omega, A)$ with $A = 1, \ldots, d-2$ on a (sufficiently small open) neighborhood, $\mathcal{O}$, of an arbitrary point $p \in \mathcal{I}$ in $\tilde{M}$ such that in $\mathcal{O}$, the unphysical metric takes the form

$$\mathrm{d} \tilde{s}^2 = \tilde{g}_{\mu\nu} \mathrm{d} x^\mu \mathrm{d} x^\nu = \tilde{\alpha} \mathrm{d} u^2 + 2\tilde{\beta}_A \mathrm{d} u \mathrm{d} \Omega + 2\tilde{\gamma}_{AB} \mathrm{d} x^A \mathrm{d} x^B,$$

(30)
where \( u \) parametrizes a congruence of null generators of \( \mathcal{I} \cap \mathcal{O} \) with \((\partial / \partial u)\)\(u\) being a tangent vector field of the congruence, and where \( \Omega \), chosen to be \( \Omega = 0 \) on \( \mathcal{I} \cap \mathcal{O} \), is an affine parameter of null geodesics which are orthogonal to each \((u, \Omega) = \text{const.} \) surfaces \( B(u, \Omega) \) in \( \mathcal{O} \) and transverse to \( \mathcal{I} \cap \mathcal{O} \) so that \( \hat{g}_{ab}(\partial / \partial \Omega)^{\mu}(\partial / \partial u)^{\nu} = 1 \), and where \( \hat{\alpha}, \hat{\beta}_A, \hat{\gamma}_{AB} \) are smooth functions with \( \hat{\alpha} = 0 = \hat{\beta}_A \) on \( \mathcal{I} \cap \mathcal{O} \). Note that \( x^A = (x^1, \ldots, x^{d-2}) \) may be regarded as local coordinates on \( B(u, \Omega) \) and also that \( \hat{\gamma}_{AB} dx^A dx^B \) is a Riemannian \((d-2)\)-metric, which does not necessarily coincide with the induced metric on \( B(u, \Omega) \) when \( \Omega \neq 0 \). The chart, \((\mathcal{O}, x^a)\), introduced in this way is called the Gaussian null coordinate system with respect to the null surface \( \mathcal{I} \cap \mathcal{O} \) because of its similarity to Gaussian normal coordinates with respect to a timelike or spacelike hypersurface in a spacetime.

It is, in general, not obvious when one can construct a Gaussian null coordinate system that covers the entire \( \mathcal{I} \) so that, in particular, the set of \( B(u) = B(u, \Omega) = 0 \) is a global foliation of \( \mathcal{I} \); each \( B(u) \) is, in general, an open subset of global cross sections of \( \mathcal{I} \) and one may need to patch together more than one coordinate chart to cover \( \mathcal{I} \). However, when, for example, a null hypersurface is ruled by some Killing vector field generating a one-parameter group of isometries, one can construct, by patching together local results, essentially a global Gaussian null coordinate system that covers the entire null hypersurface (see e.g., [19]). Although asymptotically flat spacetimes we are concerned with here do not necessarily have a Killing symmetry, they do admit asymptotic symmetries, \( \xi^a \), which are tangent to \( \mathcal{I} \) and play a similar role of a Killing symmetry on \( \mathcal{I} \). Therefore there seems to be no obstruction in assuming that one is always able to construct a desired, global Gaussian null coordinate system in some neighborhood \( \mathcal{O} \) of \( \mathcal{I} \) in \( M \) such that \( B(u) \) appropriately foliate \( \mathcal{I} \) as global cross sections of \( \mathcal{I} \) with topology \( B(u) \approx S^{d-2} \), reflecting \( \mathcal{I} \approx \mathbb{R} \times S^{d-2} \). In the following, we impose the existence of a global Gaussian null coordinate system, \((\mathcal{O}, x^a)\), with respect to \( \mathcal{I} \), as a part of our conditions for asymptotic flatness at null infinity.1

For our background geometry, \((\hat{M}, \hat{g}_{ab})\), the Gaussian null chart \((\mathcal{O}, x^a)\) yields

\[
\hat{\alpha} = -\Omega^2, \quad \hat{\beta}_A = 0, \quad \hat{\gamma}_{AB} = \sigma_{AB},
\]

and globally covers \( \partial \hat{M} \) (which is diffeomorphic to \( \mathcal{I} \)), where here and hereafter \( \sigma_{AB} \) denotes the metric of a \((d-2)\)-dimensional unit round sphere. We shall view \( \sigma_{AB} \) as a global specification of cross sections of \( \mathcal{I} \) by the choice of gauge, equation (30). In the following we call the gauge choice, equations (30) and (31), the Gaussian null conformal gauge.

Note that viewing \( \Omega \) as a conformal factor and relating it to the luminosity distance, \( r \), by \( r = 1 / \Omega \), one can find the associated physical Gaussian null coordinate system, \((u, r, x^A)\), with which the physical metric is written as

\[
r^2 \, ds^2 = dx^2 = \alpha \, du^2 - 2 \, du \, dr + 2 \beta_A \, du \, dx^A + \gamma_{AB} \, dx^A \, dx^B,
\]

on the corresponding region in the physical spacetime \((M, g_{ab})\). This coordinate system in \((M, g_{ab})\) is a sub-class of Bondi coordinates [4, 20].

The asymptotic flatness conditions given in equations (3) and (4) are now expressed in terms of the Gaussian null coordinates, equation (30) above, as

\[
\tilde{\gamma}_{AB} = \sigma_{AB} + O(\Omega^{(d-2)/2}), \quad \tilde{\gamma}_{AB} \frac{\partial}{\partial u} \tilde{\gamma}_{AB} = O(\Omega^{d/2}), \quad \tilde{\gamma}_{AB} \frac{\partial}{\partial \Omega} \tilde{\gamma}_{AB} = O(\Omega^{d-2)/2})
\]

\[
\tilde{\beta}_A, \tilde{\beta}^A = O(\Omega^{d/2}), \quad \tilde{\alpha} = -\Omega^2 + O(\Omega^{(d+2)/2}).
\]

It is worth re-examining asymptotic translational symmetries in terms of the Gaussian null coordinate system, as it helps to manifest the difference between symmetry aspects in four

1 We believe that a global Gaussian null coordinate system with respect to \( \mathcal{I} \) can always be constructed under the assumptions that have been made in the present paper, but we have not fully investigated this issue. A formal proof for the global existence of such a desired coordinate system needs to be given.
dimensions and in higher dimensions. Let $\xi^a$ be a generator of asymptotic symmetries at null infinity, and let $\tilde{\chi}_{ab} \equiv \Omega^2 \xi^a \xi^b$. Then, for any asymptotically flat metric $g_{ab}$, $\tilde{\chi}_{ab}$ must satisfy our asymptotic flatness conditions, equation (1). Thus, in particular, the relevant components, $\tilde{\chi}_{uu}$, of $\tilde{\chi}_{ab}$ in the Gaussian null conformal gauge must satisfy

$$\tilde{\chi}_{uu} = O(\Omega^{(d+2)/2}), \quad \tilde{\chi}_{uu} = O(\Omega^{d/2}), \quad \tilde{\chi}_{AB} = O(\Omega^{(d-2)/2}), \quad \tilde{g}^{\mu\nu} \tilde{\chi}_{\mu\nu} = O(\Omega^{d/2}).$$  \hspace{1cm} (34)

Note that for $d = 4$, the last condition should be replaced with $\tilde{g}^{\mu\nu} \tilde{\chi}_{\mu\nu} = O(\Omega)$.

Let us consider a vector field $\xi^a = \tilde{\xi}^a$ in $\tilde{M}$, $\tilde{g}_{ab}$ of the following form [21],

$$\xi^a = \tau \tilde{\nabla}^a \Omega - \Omega \tilde{\nabla}^a \tau,$$  \hspace{1cm} (35)

where $\tau$ is some function of the angular coordinates $x^A$. Then, we find

$$\tilde{\chi}_{uu} = O(\Omega^{(d+2)/2}), \quad \tilde{\chi}_{uu} = O(\Omega^{d/2}), \quad \tilde{\chi}_{AB} = -2\Omega(D_A D_B + \sigma_{AB})\tau + O(\Omega^{(d-2)/2}), \quad \tilde{g}^{\mu\nu} \tilde{\chi}_{\mu\nu} = \gamma^{AB} \tilde{\chi}_{AB} + O(\Omega^{d/2}),$$  \hspace{1cm} (36)

where $D_A$ denotes the derivative operator with respect to $\tilde{\gamma}_{AB}$.

It is clear that when $d = 4$, $\tilde{\chi}_{uu}$ satisfy the asymptotic flatness conditions, equation (34) with $\tilde{\gamma}^{\mu\nu} \tilde{\chi}_{\mu\nu} = O(\Omega)$. Therefore $\tilde{\xi}^a$ is an (infinitesimal) asymptotic symmetry. In fact, $\tau$ can be taken as an arbitrary function on a two-dimensional sphere, and thus there exist infinitely many, different (angle-dependent) symmetry generators, i.e., supertranslations [4, 11].

In contrast, when $d > 4$, $\tilde{\xi}^a$ fails, in general, to be an infinitesimal asymptotic symmetry, due to the first term of $\tilde{\chi}_{AB}$. However, if $\tau$ is a spherical harmonic function on $(d - 2)$-sphere with the second lowest angle quantum number, i.e., if $\tau$ is a solution to $(\sigma^{AB} D_A D_B + (d - 2))\tau = 0$, then it follows that $- (D_A D_B + \sigma_{AB})\tau = O(\Omega^{(d-2)/2})$, and therefore that $\tilde{\chi}_{AB}$ and $\tilde{g}^{\mu\nu} \tilde{\chi}_{\mu\nu}$ also satisfy the asymptotic flatness conditions, equation (34). Hence in this case, $\tilde{\xi}^a$ can become an infinitesimal asymptotic symmetry. Since $\tilde{\xi}^a$ cannot be an infinitesimal asymptotic symmetry for general $\tau$, there are no supertranslations in higher dimensions, as pointed out in [1] under the Minkowskian conformal gauge.

### 3.2. Bondi energy in Gaussian null conformal gauge

In the Gaussian null conformal gauge, the background curvature is non-vanishing, $\tilde{S}_{\mu\nu} \neq 0$, and instead of equation (23), we have

$$\tilde{S}_{\mu\nu} = (\Omega^2 + O(\Omega^{(d+2)/2})) \tilde{\nabla}_{[\mu} D_{\nu]} + (-2 + O(\Omega^{(d-2)/2})) \tilde{\nabla}_{[\mu} \tilde{\nabla}_{\nu]} \Omega$$

$$+ O(\Omega^{d/2}) \tilde{\nabla}_{[\mu|} \tilde{\nabla}_{\nu]} x^A + O(\Omega^{(d-4)/2}) (\tilde{\nabla}_{[\mu} \Omega \tilde{\nabla}_{\nu]} x^A + \tilde{\nabla}_{[\mu} A \tilde{\nabla}_{\nu]} x^B),$$  \hspace{1cm} (37)

$$\tilde{h}^{\mu\nu} \tilde{h}_{\mu\nu} = \Omega^2 + O(\Omega^{(d+2)/2}),$$  \hspace{1cm} (38)

$$\tilde{\nabla}_{\mu} \tilde{h}_{\mu} = (\Omega^3 + O(\Omega^{(d+2)/2})) \tilde{\nabla}_{[\mu} \tilde{\nabla}_{\nu]} + (2\Omega + O(\Omega^{d+2})) \tilde{\nabla}_{[\mu} \tilde{\nabla}_{\nu]} \Omega$$

$$+ O(\Omega^{(d+2)/2}) \tilde{\nabla}_{[\mu} \tilde{\nabla}_{\nu]} x^A + O(\Omega^{(d-2)/2}) (\tilde{\nabla}_{[\mu} \Omega \tilde{\nabla}_{\nu]} x^A + \tilde{\nabla}_{[\mu} A \tilde{\nabla}_{\nu]} x^B).$$  \hspace{1cm} (39)

It is immediately seen that the news tensor considered above, equation (24), becomes singular in $d > 4$ on this gauge choice. For example, when the physical spacetime is Minkowskian, $\tilde{S}_{\mu\nu} = \tilde{g}_{\mu\nu}$, and the news, equation (24), becomes $N_{\mu\nu} = \Omega^{-(d-4)/2} \sigma_{\mu\nu}$, which is singular for $d > 4$ and also (even in $d = 4$) fails to possess the desired property that news tensor should vanish for any stationary spacetime in the present gauge.

From the observations above, we define a *regularized* news tensor under the Gaussian null conformal gauge by subtracting $\sigma_{\mu\nu}$ from equation (24), so that

$$N_{\mu\nu} \equiv \xi^* (\Omega^{-(d-4)/2} q^m_{\mu} q^n_{\nu} \tilde{S}_{mn}) - \Omega^{-(d-4)/2} \sigma_{\mu\nu} = \xi^* (\Omega^{-(d-4)/2} \Delta S_{\mu\nu}).$$  \hspace{1cm} (40)
where $\Delta S_{ab} \equiv \bar{S}_{ab} - S_{ab}$. This is a global definition as it involves the $(d - 2)$-dimensional sphere metric $\sigma_{ab}$, which specifies a background structure at $\mathcal{F}$ and looks a natural higher dimensional generalization of the news tensor, $N_{ab} = \zeta^*(\bar{S}_{ab}) - \rho_{ab}$, in four dimensions, given by Geroch [6]. (In the four-dimensional case, the global specification, $\rho_{ab}$, is defined by equation (33) in [6] and can be taken such that $\rho_{ab} = \sigma_{ab}$ [5].) It follows from the asymptotic flatness conditions and the vacuum Einstein’s equations that $\Delta S_{aa} = O(\Omega^{d/2})$, $\Delta S_{AA} = O(\Omega^{d/2})$, and $\Delta S_{AB} = O(\Omega^{d-4})$, and therefore $N_{AA} = O(1)$. Also it can be checked under the Gaussian null conformal gauge that when the spacetime considered is stationary (i.e., $\delta/\partial u$ is a Killing field) the news, equation (40), is vanishing as desired. Furthermore, from the $(\Omega, \Omega)$-component of the vacuum Einstein equations and the asymptotic flatness conditions above, it can be shown that $\Omega^{-(d-3)} q^{bd} q^{ce} \sigma_{de} \nabla_b \tilde{\ell}_c = O(\Omega)$. Thus, one can anticipate that the first term in the right-hand side of equation (26), can simply be rewritten in terms of the regularized news tensor defined above, equation (40).

From the observations above, we expect the following vector field would be a possible candidate for higher dimensional Bondi energy–momentum with respect to asymptotic time-translations $\xi^a = \tau \tilde{n}^a$ (with $\tau = \text{const. on } \mathcal{F}$)

$$P^a[\xi] = \frac{1}{8(d-3)G} (\Omega^{-(d-4)/2} \xi^a \xi^b \bar{N}_{ab} q^{ce} C_{bc} f) = \Omega^{-(d-3)} \bar{C}^{abcd} \xi^a \xi^b n_c n_d),$$

where $C_{ab}^c$ is the connection defined by $(\hat{\nabla}_a - \nabla_a)\omega_b = C_{ab}^c \omega_c$, for an arbitrary 1-form $\omega_c$. Note that in four dimensions, the above formula, equation (41), agrees with the known, four-dimensional Bondi energy–momentum integrand. We now state our main results:

**Theorem.** Consider an even-dimensional $d \geq 4$ vacuum spacetime $(M, g_{ab})$ which is asymptotically flat at null infinity $\mathcal{F}$, satisfying the boundary conditions, equation (1) (or equivalently equation (2), and equation (3) for $d = 4$). Let $O \subset M$ be a neighborhood of $\mathcal{F}$ in which the Gaussian null conformal gauge, equation (30), can be taken. In $O$, the divergence, $\nabla_a P^a$, of $P^a$ introduced by equation (41) smoothly extends to $\mathcal{F}$ and $P^a$ solves equation (12).

**Proof.** We proceed along a similar line of appendix A of [1]. However, this time we need to treat background curvature quantities with more care since they are, different from the case of the Minkowskian conformal gauge, no longer vanishing upon the Gaussian null conformal gauge condition. In the following, for definiteness we choose $\tilde{\ell}_a = (du)_a$, so that $\tilde{\ell}^a = (\delta/\delta \Omega)^a (\equiv \tilde{\ell}^a)$ is a past directed (when future null infinity, $\mathcal{F}^*$, is concerned) null vector that satisfies

$$\hat{\nabla}_a \tilde{\ell}_b = -\Omega \tilde{\ell}_a \tilde{\ell}_b + O(\Omega^{-d/2}), \quad \tilde{\ell}^a \tilde{n}_a = 0, \quad \tilde{\ell}^a \tilde{n}_a = 1, \quad (42)$$

with $\Omega$ being an affine parameter. (Note that $q^{ab} \hat{\nabla}_a \tilde{\ell}_c = q^{ad} C_{bd} \tilde{\ell}_c + O(\Omega)$. So, when $C_{bd}^a$ appears with the contraction with $q^{bd}$ and $\tilde{\ell}_a$, we can use $C_{bd}^a$ and $\hat{\nabla}_a \tilde{\ell}_b$ interchangeably.) Let us consider the identity

$$2\Delta^a \hat{\nabla}_a \hat{\nabla}_b \tilde{\ell}_c = R_{abcd} \tilde{n}^d \tilde{\ell}^c = C_{abcd} \tilde{n}^d \tilde{\ell}^c + \frac{1}{2} \bar{S}_{bd} \tilde{n}^c \tilde{\ell}^d - \frac{1}{2} \bar{S}_{bc} - \frac{1}{2} \tilde{\ell}_b \hat{\nabla}_c \tilde{f} + \frac{1}{2} \hat{\nabla}_b \tilde{\ell}^d \hat{\nabla}_d \tilde{f}, \quad (43)$$

where $\tilde{f} = \Omega^{-1} \tilde{n}^a \omega_a$. Subtracting $2\Delta^a \hat{\nabla}_a \hat{\nabla}_b \tilde{\ell}_c$ from the above equation and using $\bar{C}_{abcd} = 0$ and $R_{abcd} = \bar{g}_{bd} \bar{g}_{ca} - \bar{g}_{bc} \bar{g}_{da}$, for our background geometry, equation (31), we have

$$2\Delta^a \hat{\nabla}_a (C_{bc} \hat{\nabla}_b \tilde{\ell}_c) = \bar{C}_{abcd} \tilde{n}^d \tilde{\ell}^c + \frac{1}{2} \Delta S_{bd} \tilde{n}^c \tilde{\ell}^d - \frac{1}{2} \Delta S_{bc} - \frac{1}{2} \tilde{\ell}_b \hat{\nabla}_c \Delta f + O(\Omega^{-(d-2)/2}), \quad (44)$$
with $\Delta f = \Omega^{-1}(\tilde{n}^a - \bar{n})^a n_a$. Multiplying $2\Omega^{-(d-4)}\Delta S_{de}q^{bd}q^{ce}$, which is $O(\Omega^{-(d-4)/2})$, to equation (44), we have

$$\Omega^{-(d-4)}\Delta S_{ab}\Delta S_{de}q^{bd}q^{ce} = -4\Omega^{-(d-4)}\Delta S_{de}q^{bd}q^{ce}n^n(\tilde{C}f_{b} \tilde{C}f_{e})$$

$$+ 2\Omega^{-(d-4)}\tilde{C}_{abc} \tilde{n}^a \tilde{n}^b \Delta S_{de} q^{bd} q^{ce} + O(\Omega).$$

(45)

Using the formulæ $\tilde{C}_{abc} \tilde{n}^a = \Omega \tilde{n}^c \tilde{S}_{fj} = \Omega \tilde{S}_{fj} \Delta S_{ef} + \Omega C_{b}^{c} \delta_{le} \tilde{S}_{f}, C_{b}^{c} \delta_{le} = O(\Omega^{(d-4)/2}) = \Delta S_{ab}$ and $\tilde{n}^a q^{ce} = O(\Omega)$, the second term in the right-hand side of equation (45) above is rewritten as

$$2\Omega^{-(d-4)}\tilde{C}_{abc} \tilde{n}^a \tilde{n}^b \Delta S_{de} q^{bd} q^{ce} = -\frac{d-4}{2} \Omega^{-(d-4)}\Delta S_{ab} \Delta S_{de} q^{bd} q^{ce} + O(\Omega).$$

(46)

Next, we rewrite the first term of the right-hand side of equation (45) using the following formula

$$\tilde{N}_{a}\Omega^{-(d-4)}\tilde{n}^a q^{bd} \Delta S_{de} q^{ce} C_{b}^{e} \tilde{f} e_{f}$$

$$= \Omega^{-(d-4)}\Delta S_{de} q^{bd} q^{ce} \tilde{n}^a(\tilde{C}f_{b} \tilde{C}f_{e}) + \tilde{N}_{a}\Omega^{-(d-4)}\tilde{n}^a q^{bd} \Delta S_{de} q^{ce} C_{b}^{e} \tilde{f} e_{f}$$

$$= \Omega^{-(d-4)}\Delta S_{de} q^{bd} q^{ce} \tilde{n}^a(\tilde{C}f_{b} \tilde{C}f_{e}) + \Omega^{-(d-4)}\tilde{n}^a q^{bd}(\tilde{N}\Delta S_{de}) q^{ce} C_{f}^{e} \tilde{f} e_{f} + O(\Omega),$$

(47)

where we have used $n^a n^a = \Omega(\tilde{C}^{2}) = \tilde{n}^a q^{ab}, \tilde{N} n^a = \Omega(\tilde{C}^{2}) = \tilde{N} q^{bc}$. Thus, we have

$$-4\Omega^{-(d-4)}\Delta S_{de} q^{bd} q^{ce} \tilde{n}^a(\tilde{C}f_{b} \tilde{C}f_{e}) = -4\tilde{N}_{a} \Omega^{-(d-4)}\tilde{n}^a q^{bd} \Delta S_{de} q^{ce} C_{f}^{e} \tilde{f} e_{f}$$

$$+ 4\Omega^{-(d-4)}\tilde{n}^a q^{bd}(\tilde{N}\Delta S_{de}) q^{ce} C_{f}^{e} \tilde{f} e_{f} + O(\Omega).$$

(48)

Also, from $\Delta S_{m}^{m} = O(\Omega^{(d-2)/2})$, $q^{bd} q^{ce} \tilde{n}^a \tilde{n}^c \Delta f = O(\Omega^{(d-2)})$, $\tilde{n}^{d} \tilde{\nabla}_{d} \Delta f = O(\Omega^{(d-2)})$, it follows that

$$-4\Omega^{-(d-4)}\tilde{n}^a q^{bd}(\tilde{N}_{a} \Delta S_{de}) q^{ce} C_{f}^{e} \tilde{f} e_{f} = O(\Omega).$$

(49)

Therefore, adding this to the right-hand side of equation (48), we can antisymmetrize the second term of equation (48) with respect to the indices $a$ and $e$ and thereby obtain

$$-4\Omega^{-(d-4)}\Delta S_{de} q^{bd} q^{ce} \tilde{n}^a(\tilde{C}f_{b} \tilde{C}f_{e}) = -4\tilde{N}_{a} \Omega^{-(d-4)}\tilde{n}^a q^{bd} \Delta S_{de} q^{ce} C_{f}^{e} \tilde{f} e_{f}$$

$$+ 8\Omega^{-(d-4)}\tilde{n}^a q^{bd}(\tilde{N}_{a} \Delta S_{de}) q^{ce} C_{f}^{e} \tilde{f} e_{f} + O(\Omega).$$

(50)

Then, inserting into the above equation

$$\tilde{N}_{a} \Delta S_{de} = -\Omega^{-1}\tilde{C}_{adef} \tilde{n} f - C_{f}^{d} \tilde{S}_{ef},$$

we have

$$-4\Omega^{-(d-4)}\Delta S_{de} q^{bd} q^{ce} \tilde{n}^a(\tilde{C}f_{b} \tilde{C}f_{e}) = -4\tilde{N}_{a} \Omega^{-(d-4)}\tilde{n}^a q^{bd} \Delta S_{de} q^{ce} C_{f}^{e} \tilde{f} e_{f}$$

$$- 8\tilde{N}_{a} \Omega^{-(d-4)}\tilde{n}^a q^{bd} \tilde{C}_{adef} \tilde{n} f q^{ce} C_{b}^{e} \tilde{f} e_{f}$$

$$- 8\Omega^{-(d-4)}\tilde{n}^a q^{bd} \tilde{C}_{adef} \tilde{n} f q^{ce} C_{b}^{e} \tilde{f} e_{f} + O(\Omega).$$

(52)

The third term of the right-hand side of equation (52) turns out to be $O(\Omega)$ as shown in the appendix. The second term of the right-hand side of equation (52) (see the appendix, for the derivation)

$$-8\Omega^{-(d-3)}\tilde{n}^a q^{bd} \tilde{C}_{adef} \tilde{n} f q^{ce} C_{b}^{e} \tilde{f} e_{f} = -4\tilde{N}_{d} \Omega^{-(d-3)}\tilde{\tilde{C}}_{adef} \tilde{n} f q^{ce} C_{b}^{e} \tilde{f} e_{f}$$

$$+ 4\Omega^{-(d-3)}\tilde{\tilde{C}}_{adef} \tilde{n} f q^{ce} C_{b}^{e} \tilde{f} e_{f} + 4\Omega^{-(d-3)}\tilde{\tilde{C}}_{adef} \tilde{n} f q^{ce} C_{b}^{e} \tilde{f} e_{f} + O(\Omega).$$

(53)

Thus, we find that the first term of the right-hand side of equation (45) becomes

$$-4\Omega^{-(d-4)}\Delta S_{de} q^{bd} q^{ce} \tilde{n}^a(\tilde{C}f_{b} \tilde{C}f_{e})$$

$$= -4\tilde{N}_{d} \Omega^{-(d-4)}\tilde{n}^a q^{bd} \Delta S_{de} q^{ce} C_{f}^{e} \tilde{f} e_{f} - 4\Omega^{-(d-3)}\tilde{C}_{ade} \tilde{n} f n_a$$

$$+ 4\Omega^{-(d-3)}\tilde{\tilde{C}}_{adef} \tilde{n} f q^{ce} C_{b}^{e} \tilde{f} e_{f} + O(\Omega).$$

(54)
Substituting equations (46) and (54) into equation (45), we have

\[
(d - 3)\Omega^{-(d-4)} \Delta S_{ab} \Delta S_{cd} q^{ac} q^{bd} = -4\tilde{\nabla}_a \left[ \Omega^{-(d-4)} n^a q^{bd} \Delta S_{bc} C_{ba} \tilde{\gamma}_{f} - \Omega^{-(d-3)} \tilde{C}_{abcd} n_b \tilde{\gamma}_{f} \right] \\
+ 2\Omega^{-(d-4)} \tilde{C}_{abc} \tilde{\nabla}_{b} \tilde{\gamma}_{f} g^{bd} g^{ce} \tilde{\nabla}_{c} + 4\Omega^{-(d-3)} \tilde{\nabla}_{b} \tilde{\gamma}_{f} \tilde{\nabla}_{d} \tilde{\gamma}_{b} \tilde{\gamma}_{c} \\
- 8\Omega^{-(d-4)} n^a q^{bd} C_{[a,b]} \tilde{\nabla}_{f} q^{ce} C_{b} \tilde{\gamma}_{f} \tilde{\gamma}_{g}.
\]

(55)

The last three terms are shown to be \( O(\Omega) \) in the appendix. Thus, substituting the defining equation (41) of \( P^a \) for \( \xi^a = \tau \tilde{n}^a \), we obtain

\[
\tau \Omega^{-(d-4)} \Delta S_{ab} \Delta S_{cd} q^{ac} q^{bd} = -32\pi G \tilde{\nabla}_a P^a + O(\Omega).
\]

(56)

Since \( \Delta S_{ab}, \Delta S_{cd} = \Omega(\Omega^{d-2}) \) and \( \Delta S_{AB} = \Omega(\Omega^{d-4}) \) under our boundary conditions, the left-hand side has a smooth extension to \( \mathcal{S} \). Hence the divergence, \( \nabla_a P^a \), also must smoothly extend to \( \mathcal{S} \).

We also find that the symplectic potential \((d - 1)\)-form with respect to \( \xi^a = \tau \tilde{n}^a \) is written, in terms of the regularized news tensor \( N_{ab} \), as

\[
\Theta(g, \xi_{ab}) = \frac{1}{32\pi G} \tau \Omega^{-(d-4)} \xi^a (\Delta S_{ab} \Delta S_{cd} q^{ac} q^{bd}) \cdot \tilde{\xi} = \frac{1}{32\pi G} \tau N_{ab} N_{ab} \cdot \tilde{\xi}.
\]

(57)

Thus, when \( \tau = \text{const.} \), comparing equation (56) with equation (57) we consequently obtain the formula, equation (41), for the translation symmetry \( \xi^a = \tau \tilde{n}^a \) with \( \tau = \text{const.} \), as a solution to equation (12).

The theorem above yields that the vector field, \( P^a \), defined by equation (41) is a good candidate for the higher dimensional Bondi energy–momentum integrand. It should be, however, noted that although the integrand of the flux formula, equation (12), is well defined, the limit to \( \mathcal{S} \) of \( P^a \) itself does not appear to exist under our boundary conditions. This can be seen by examining the relevant component, \( P^a = \tau \tilde{n}^a \), of equation (41),

\[
P^a[\xi] = \frac{-\tau}{16(d - 3)\pi G} \Omega^{-(d-4)} \left\{ \frac{1}{2} g^{AC} \partial_B g^{DB} \frac{\partial \tilde{\gamma}_{AB}}{\partial \Omega} \frac{\partial \tilde{\gamma}_{CD}}{\partial \Omega} - \frac{\partial}{\partial \Omega} \left[ \Omega^{2} \frac{\partial}{\partial \Omega} \left( \frac{\tilde{\gamma}}{\Omega^2} \right) \right] \right\} + O(\Omega).
\]

(58)

The first term of the right-hand side of above equation (58)—which comes from the news tensor—is \( O(1) \) and therefore well defined, whereas the second term of \( P^a \tilde{\gamma}_{b} \) behaves \( O(\Omega^{-4}) \) near \( \mathcal{S} \), according to the boundary condition equation (33). \( (P^A \) behaves in a similar way.) Therefore, except the \( d = 4 \) case, \( P^a \) itself would not appear to be regular at \( \mathcal{S} \) for general, asymptotically flat radiative spacetimes.

However, this apparent singular behavior of \( P^a \) does not necessarily imply that the integral of \( P^a \tilde{\gamma}_{b} \) over a compact cross section of \( \mathcal{S} \) would also be singular for asymptotically flat radiative spacetimes. Let \((M, g_{ab})\) be a vacuum spacetime which is asymptotically flat at future null infinity \( \mathcal{J}^+ \) and \( \Sigma_0 \) be a (partial) Cauchy surface which extends to spatial infinity, \( \bar{I}^0 \). Assume further that there is a compact subset \( K_0 \) of \( \Sigma_0 \) in \( M \) such that the region outside the causal future of \( K_0 \) is stationary, and gravitational radiation may present only in \( J^+(K_0) \).

Consider the flux formula, equation (14), which may be rewritten as

\[
\int_B (d - 2) \xi P^a \tilde{\gamma}_{b} = \int_{B_0} (d - 2) \xi P^a \tilde{\gamma}_{b} + F^a,
\]

(59)

where the integral over a given cross section \( B \) of \( \mathcal{S} \) in equation (59) above is defined by the limiting procedure, as described below equation (11); we first perform the integration over a sequence of \( (d - 2) \)-closed surfaces, \( S_j \), in some neighborhood of \( B \) and then take the limit \( S_j \rightarrow B \) in a certain manner. Since the Gaussian null coordinate chart, \((O, x^\mu)\), is
now available, one may naturally set \( S_\Omega = B(u, \Omega \neq 0) \) in \( O \) and take the limit \( S_\Omega \to B \) by \( \Omega \to 0 \) on a \( u = \text{const.} \) hypersurface. (Of course, the integrals in equation (59)—as defined on \( B \) in \( \mathcal{I} \), not inside the spacetime \( (M, g_{ab}) \)—should not depend on the way of taking \( S_\Omega \) in performing the limiting procedure.) Now let us take \( B \) in \( \mathcal{I}^* \cap J^* (K_0, \bar{M}) \) and \( B_0 \) in a sufficient past so that \( B_0 \subseteq (\mathcal{I}^* \cap D^+(\Sigma_0 \setminus K_0, \bar{M})) \). Since the spacetime region \( D^+(\Sigma_0 \setminus K_0, \bar{M}) \) is stationary, the integral of \( P^a \tilde{\ell}_a \) over \( B_0 \) is independent of the Killing time and should correspond to the ADM energy.\(^2\) This can be directly verified when the initial data on \( \Sigma_0 \) outside the compact region \( K_0 \) coincide with Schwarzschild data (so that \( D^+(\Sigma_0 \setminus K_0, \bar{M}) \) is isometric to Schwarzschild spacetime). Since \( \Theta \) is by definition smooth and finite on \( \mathcal{I} \), it follows from equation (57), together with the flux formula, equation (14), that the energy flux, \( F_\xi \), through the segment of \( \mathcal{I}^* \) with boundaries \( B \) and \( B_0 \) must be finite. Therefore, if the ADM energy evaluated on \( B_0 \) is finite, then the above formula, equation (59), yields that the integral of \( P^a \tilde{\ell}_a \) over the cross section \( B \) must also be finite. Furthermore, since the flux \( F_\xi \) through a segment of \( \mathcal{I}^* \) is always negative, as immediately seen from equations (14) and (57), the right-hand side of equation (59) may be viewed as the ADM energy minus radiation energy carried away from the spacetime by the flux through \( \mathcal{I}^* \). Therefore, at least under the setup described above, \( P^a \tilde{\ell}_a \) given by equation (41) is indeed our desired Bondi energy–momentum integrand. More generally, we would now like to make the following conjecture:

For any asymptotically flat spacetime (at null infinity \( \mathcal{I} \)) and any compact cross section \( B \) of \( \mathcal{I} \), the integral of \( P^a \tilde{\ell}_a \) given by equation (41) over a closed surface \( S \) always has a well-defined limit as \( S \) approaches \( B \), and the limit is independent of how \( S \) approaches \( B \).

Accordingly, we propose that \( \mathcal{H}_S \) defined by equation (15) with the vector \( P^a \) given by equation (41) can be taken as the definition for a higher dimensional generalization of the Bondi energy.

It also follows from equation (59) that the Bondi energy defined by equation (15) is a decreasing function in time. In Gaussian null coordinates, one finds that \( \nabla_u P^a = \partial P^a / \partial u + D_\xi P^A + O(\Omega) \). Then, via equations (12) and (15), the energy loss rate at time \( u \) (with the corresponding cross section \( B \)) is given by

\[
\frac{\partial \mathcal{H}_S}{\partial u} = -\frac{\tau}{32\pi G} \int_B \Omega^{-(d-2)} \epsilon^{CA} \sigma^{DB} \partial_\gamma \gamma^{AB} \frac{\partial \gamma^{CD}}{\partial u},
\]

which corresponds to, e.g., equation (5.12) in [20] obtained in four dimensions.

4. Summary and discussions

In this paper, we have considered asymptotic flatness at null infinity in higher (even spacetime) dimensions in terms of the Gaussian null conformal gauge, focusing on vacuum solutions of

\(^2\) For stationary spacetimes with \((\partial / \partial u)^\alpha\) being a timelike Killing vector field in a vicinity \( O \) of null infinity, the news tensor vanishes identically and the integral of \( P^a \tilde{\ell}_a \) over \( B_0 \) becomes the integral over \( \Omega^{d-2} \) of the Coulomb part of the Weyl tensor, \( \propto \Omega^{-(d-2)} \epsilon^{CA} \sigma^{DB} \partial_\gamma \gamma^{AB} \frac{\partial \gamma^{CD}}{\partial u} \). If one can, furthermore, set \( \bar{\beta}_A = 0 \) in \( O \), then it immediately follows from the vacuum Einstein equations, the \((u, A)\)-components of equation (17), that

\[
D_A \Omega^{d-2} \frac{\partial}{\partial u} \left( \bar{\alpha} + \Omega^2 \right) = O(\Omega^{d-4}),
\]

and hence that \( \bar{\alpha} = -\Omega^2 + O(\Omega^{d-4}) \). Thus, in this case, the second term of the right-hand side of equation (58)—which comes from the Coulomb part of the Weyl tensor—gives rise to a finite, constant value, which is to be identified with the ADM energy. We believe that even if \( \bar{\beta}_A \) is not set to be zero in \( O \), the Coulomb part of the Weyl tensor would give rise to the ADM energy whenever a cross section \( B_0 \) is taken in a portion of \( \mathcal{I}^* \) whose immediate vicinity \( O \) is stationary, but we have not fully investigated this case.
Einstein’s equations. As shown in the previous paper [1], when asymptotically flat spacetimes are even spacetime dimensional, one can define a smooth conformal null infinity $\mathcal{I}$ in the unphysical spacetime $(\tilde{M}, \tilde{g}_{ab})$ that is stable against, at least, linear perturbations. Then, we have discussed that a Gaussian null coordinate system $(u, \Omega, x^a)$ with respect to $\mathcal{I}$ can naturally be constructed on some neighborhood of $\mathcal{I}$ in $(\tilde{M}, \tilde{g}_{ab})$. By taking a luminosity distance as the inverse of $\Omega$ (viewed as a smooth, appropriately scaled conformal factor), one can define Gaussian null coordinates in the physical spacetime, which correspond to a restricted class of Bondi coordinates. In contrast to the Minkowskian conformal gauge condition employed in [1], the Gaussian null conformal gauge allows us to take compact, spherical cross sections at $\mathcal{I}$ with $(d-2)$-dimensional round sphere metric $\sigma_{ab}$, as a global specification of the background structure. Thus, one can directly compare the higher dimensional formulae with the four-dimensional ones obtained in the Bondi coordinates. The Gaussian null conformal gauge also helps to manifest the difference between asymptotic symmetry properties in four dimensions and those in higher dimensions, showing clearly the absence of supertranslations in higher dimensions.

We have modified the definition of the news tensor in higher (even spacetime) dimensions so that it becomes regular under the choice of the Gaussian null conformal gauge. The new definition of the news tensor involves a global specification of the background structure $\sigma_{ab}$ on null infinity $\mathcal{I}$, which is not included in the previous definition, equation (61) of [1]. The news tensor defined above, equation (40), looks natural as a higher dimensional generalization of the news tensor in four dimensions defined in [6]. Then, for the case of vacuum spacetimes, within the Hamiltonian framework of Wald and Zoupas [5], we have obtained the expression of the higher (even) dimensional generalization of the Bondi energy, $\mathcal{H}_\xi$, and energy–momentum, $P^a$, for the special asymptotic translation $\xi^a = \tau \hat{n}^a$ with $\tau = \text{const.}$ in terms of the regularized news tensor, equation (40).3

We have then pointed out the puzzling fact that although the flux formula—which is given in terms of the divergence of $P^a$ (see equation (12))—is well defined, $P^a$ itself would not appear to be regular at $\mathcal{I}$ for $d > 4$, as the second term of $P^a \ell_a$, equation (58), (and $P^A$, too) behaves near $\mathcal{I}$ as $O(\Omega^{-(d-4)/2})$ under our asymptotic flatness conditions. We have given an attempt to show that even if $P^a$ itself is singular, the integral of $P^a \ell_a$ over a compact cross section would be regular, by discussing a relationship between the flux formula, equation (14) (or equation (59)) and the ADM energy in the case in which an asymptotically flat, radiative spacetime has a stationary region in the past, at least in some neighborhood of spatial infinity $i^0$. It should be recalled here that the ADM energy is defined at spatial infinity, $i^0$, with respect to an asymptotic time-translation symmetry defined at $i^0$, whereas the Bondi energy is defined at a cross section of null infinity, $\mathcal{I}$, with respect to an asymptotic time-translation symmetry defined at $\mathcal{I}$. Therefore, in order to find a precise relationship between the two notions of gravitational total energies, one needs to treat asymptotic properties of gravitational fields and asymptotic symmetries at both $i^0$ and $\mathcal{I}$ in a unified manner. In the four-dimensional case, a framework for such a unified treatment of two infinities has been given by [22, 23], and the interpretation that the Bondi energy is the ADM energy minus the energy carried away by flux through null infinity has been justified [23]. However, the conformal framework of [22, 23] used to define asymptotic flatness at $i^0$ and $\mathcal{I}$ in a unified manner is different from the present framework for defining asymptotic flatness at null infinity under the Gaussian null conformal gauge. For this reason, although we believe that the unified treatment of $i^0$ and $\mathcal{I}$...
given by [22, 23] would be generalized to the higher dimensional case, it would not appear to be a straightforward task.

Another attempt to justify $P^a$ given by equation (41) as the ‘legitimate’ Bondi energy–momentum integrand may be given by considering gravitational perturbations off of Minkowski spacetime. The part of $P^a$ that involves the news tensor is $O(1)$, and the relevant part to be examined is the Weyl curvature part of $P^a \ell_a$ (the second term of the above equation (41)). Since the perturbation $\Omega^{-(d-3)} \delta C^{abcd} \tilde{\ell}_a \tilde{\ell}_b \tilde{\ell}_c \tilde{\ell}_d$ is a scalar quantity, it can be expanded in terms of spherical harmonics on a $(d - 2)$-sphere $B(u, \Omega)$. Then, given that the time dependence of perturbations is $\propto e^{-iu\omega}$, it turns out from the linearized Einstein equations and the Bianchi identities that general solutions of perturbations are given in terms of Bessel functions and that $\Omega^{-(d-3)} \delta C^{abcd} \tilde{\ell}_a \tilde{\ell}_b \tilde{\ell}_c \tilde{\ell}_d$ becomes near $\mathcal{J}$ as $\sim \Omega^{-(d-4)/2}$, which looks singular. However, contributions to $\Omega^{-(d-3)} \delta C^{abcd} \tilde{\ell}_a \tilde{\ell}_b \tilde{\ell}_c \tilde{\ell}_d$ from any modes of perturbations, except spherically symmetric mode (the $S$-wave) vanish identically, when integrated over a compact cross section $(d - 2)$-sphere, $B(u)$. Therefore only the $S$-wave becomes relevant. It turns out from the Bianchi identity that provided $\delta C^{abcd} \tilde{\ell}_a \tilde{\ell}_b \tilde{\ell}_c \tilde{\ell}_d$ is not divergent at $\mathcal{J}^+$, $\Omega^{-(d-3)} \delta C^{abcd} \tilde{\ell}_a \tilde{\ell}_b \tilde{\ell}_c \tilde{\ell}_d$ for the $S$-wave behaves $\sim O(1)$ [24], as expected from the uniqueness of static vacuum black holes in higher dimensions [25]. (Of course, the $S$-wave solutions in exact, flat spacetime are singular at the center of the spherical symmetry, but here we are not concerned with global regularity of perturbation solutions.) Therefore the linear perturbations produce only regular contributions to $H_\xi$. However, this does not seem to be the case for second-order perturbations. General solutions of second-order metric perturbations of a physical spacetime may be given in terms of Green’s functions, which is the ‘inverse’ of the wave operator appearing in the equation of motion for first-order metric perturbations, together with a source term, given as quadratics of the first-order perturbations (and their derivatives). It turns out from the linearized vacuum Einstein equations that the relevant Green’s function consists of powers of $\Omega$, spanning all values between $\Omega^{(d-2)/2}$ and $\Omega^{d-3}$ (see e.g., [26]). So, if second-order contributions from the second term of the right-hand side of equation (58) are determined in this way, then we would have a singular behavior in $H_\xi$.

In order to fully justify our definition of $P^a$ and the Bondi-energy expression, more work on nonlinear analysis of asymptotic flatness conditions needs to be done. For example, one may start with expanding the unphysical metric in powers of $\Omega$ at $\mathcal{J}$, and thereby expressing the Einstein equations in terms of the expansion coefficients in a similar manner performed for spacetimes with a different asymptotic structure (see [27] and also references therein), and then we read off at which order of $\Omega$ true dynamical information on gravitational radiation enters each component of the unphysical metric [28].

Concerning this regularity issue of the Bondi energy–momentum integrand, we also should keep in mind that the vector $P^a$ has degrees of freedom in the addition of a vector field of the form $\nabla_b X^{ab}$ with $X^{ab}$ being an arbitrary anti-symmetric tensor field on $\tilde{\mathcal{M}}$, as commented below equation (12); $P^a = P^a + \nabla_b X^{ab}$ also solves equation (12) if $P^a$ does. A relevant question is then whether there exists a $P^a$ for which $P^a \ell_a$ smoothly extends to $\mathcal{J}$ and yields an equivalent formula for the higher dimensional Bondi energy. We note that the main cause for the apparent singularity of $P^a \ell_a$ is the second term of the right-hand side of equation (58). For example, it is not so hard to find such $X^{ab}$ that removes only the second term from $P^a \ell_a$ of equation (58), but then the Bondi-energy expression with the new $P^a = P^a + \nabla_b X^{ab}$—which is now manifestly well defined at $\mathcal{J}$—would fail to reproduce the ADM energy for stationary spacetimes. This problem remains unsolved.

In some theories of higher dimensional gravity, our four-dimensional universe is modeled as an embedded (or a boundary) hypersurface—so called the braneworld—in a higher
dimensional bulk spacetime, in which only gravitational radiation can probe extra dimensions. Therefore, in this context, it is of considerable interest to define the notion of a higher dimensional Bondi-type energy that can be used to measure the energy flux of gravitational radiation from extra dimensions. For the case in which the bulk spacetime is higher, even dimensional and its curvature radius is sufficiently large compared to the typical scale of a system of interest (e.g., a mini black hole on the braneworld), one may be able to define, within the conformal framework, the notion of asymptotic flatness at null infinity \( \mathcal{I} \) for both the bulk spacetime and the braneworld, simultaneously. In that case, our Bondi-energy formula would apply to such a system almost as it stands. For example, the flux formula on the four-dimensional braneworld may simply be given by estimating the boundary integral of the formula, equation (59), on a two-dimensional closed sub-manifold which corresponds to the intersection of \( B \subset \mathcal{I} \) and the brane’s world-volume. However, when the bulk spacetime is five-(or higher, odd-)dimensional, our formula based on the conformal method would not work, as the unphysical metric becomes, in general, singular at null infinity. For odd-dimensional spacetime, however, it may still be possible to define the notion of Bondi-type energy by formulating an asymptotic structure in terms of the physical metric, instead of the conformal unphysical metric. Also for Kaluza–Klein-type models, the conformal framework would not appear to work. One may expect that the null infinity \( \mathcal{I} \) of a Kaluza–Klein spacetime should be \( \mathcal{I} \approx \mathbb{R} \times S^2 \times \mathcal{K} \), where \( \mathbb{R} \times S^2 \) corresponds to the conformal boundary of four-dimensional macroscopic spacetime and \( \mathcal{K} \) to the compactified extra dimensions. However, a conformal factor will make \( \mathcal{K} \) shrink to a point on \( \mathcal{I} \). Again one may be able to define the energy flux or the Bondi-type energy in Kaluza–Klein spacetime, using the physical metric, rather than the unphysical metric. The study of such cases with a non-trivial asymptotic structure is left open for future work.

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Appendix

We first show that the third term of the right-hand side of equation (52) is \( O(\Omega) \). Note first that

\[
C^a_{AB} \cdot C^b_{A\Omega B} = O(\Omega^{(d-4)/2}),
\]

and all the other components of the connection \( C^a_{bc} \) decay faster than the above two. Also note that \( q^{ab} = \sigma^{AB}(\partial/\partial x^A)^a(\partial/\partial x^B)^b + O(\Omega^{(d-2)/2}) \). From these, it immediately follows that

\[
-8\Omega^{-(d-4)/2}q^{ab}\partial^2C_{a\Omega B} q^{\alpha\beta} C^b_{\Omega c} \tilde{\ell}_g
= -2\Omega^{-(d-4)/2}q^{ab}\sigma^{BD}(C^f_{Da}\tilde{S}_{Ef} - C^a_{DE}\tilde{S}_{Da})\sigma^{CE} C^a_{BC} + O(\Omega)
= -2\Omega^{-(d-4)/2}\sigma^{BD} C^C_{Da} C^a_{BC} + O(\Omega)
= O(\Omega),
\]
where the capital letters describe coordinate components with respect to $x^A$, and where in the second line we have used $\tilde{n}^A = (\partial \tilde{f})^A + O(\tilde{\Omega}^2)$ and $\tilde{S}_{ab} = \sigma_{AB}(dx^A)_b(dx^B)_a + O(\Omega^2)$, and in the last line, $C^F_{Da} = O(\Omega^{(d-2)/2})$.

Next, we consider the second term of the right-hand side of equation (52),

$$-8\Omega^{-(d-3)}\tilde{n}^A q^{bc} C_{abcd\tilde{n}^f} q^{de} C_{de\tilde{n}^g} \tilde{\ell}_g = 4\Omega^{-(d-3)} q^{de} [\tilde{C}_d\tilde{n}^f \tilde{n}^g \tilde{\nabla}_b \tilde{\ell}_c + 4\Omega^{-(d-3)} q^{de} q^{cd} \tilde{C}_{abcd} \tilde{n}^b \tilde{n}^f \tilde{\nabla}_c \tilde{\ell}_d]$$

where we have used $\tilde{\nabla}_b \tilde{\ell}_c = -\Omega \tilde{\ell}_a \tilde{\ell}_b$ and $q^{ab} \tilde{\ell}_a = 0$, and where in the second line we have used

$$q^{de} q^{cd} \tilde{C}_{abcd} \tilde{n}^b \tilde{n}^f \tilde{\nabla}_c \tilde{\ell}_d = 2\tilde{C}_{abcd} \tilde{n}^b \tilde{n}^f \tilde{\nabla}_c \tilde{\ell}_d + \tilde{n}^a \tilde{n}^b \tilde{n}^c \tilde{\nabla} \tilde{\ell}^f \tilde{C}_{de\tilde{n}^f},$$

and

$$q^{de} q^{cd} \tilde{C}_{abcd} \tilde{n}^b \tilde{n}^f \tilde{\nabla}_c \tilde{\ell}_d = \tilde{C}_{abcd} \tilde{n}^b \tilde{n}^f \tilde{\nabla}_c \tilde{\ell}_d - \tilde{C}_{de\tilde{n}^f} \tilde{n}^a \tilde{n}^b \tilde{n}^c \tilde{\nabla} \tilde{\ell}^f \tilde{C}_{de\tilde{n}^f},$$

and the symmetry property of $C_{abcd}$. The first term in the third line of the right-hand side of equation (A.3) is rewritten as

$$-4\Omega^{-(d-3)} \tilde{C}_{abcd} \tilde{n}^b \tilde{n}^f \tilde{\nabla}_c \tilde{\ell}_d$$

Thus, combining all together, we have

$$-8\Omega^{-(d-3)}\tilde{n}^A q^{bc} C_{abcd\tilde{n}^f} q^{de} C_{de\tilde{n}^g} \tilde{\ell}_g = -4\tilde{\nabla}_d (\Omega^{-(d-3)} \tilde{C}_{abcde} \tilde{n}^b \tilde{n}^f \tilde{\nabla}_d \tilde{\ell}_c)$$

Finally we shall show below that the last three terms in the right-hand side of equation (55) are $O(\Omega)$. The first term is

$$2\Omega^{-(d-4)} \tilde{C}_{abcd} \tilde{n}^b \tilde{n}^f \tilde{\nabla}_c \tilde{\ell}_d = 2\Omega^{-(d-4)} \tilde{C}_{abcd} \tilde{n}^b \tilde{n}^f \tilde{\nabla}_c (\tilde{g}_{bc} - \tilde{g}_{bc}),$$

and the second term is

$$4\Omega^{-(d-3)} \tilde{C}_{abcd} \tilde{n}^b \tilde{n}^f \tilde{\nabla}_c \tilde{\ell}_d = 4\Omega^{-(d-4)} \tilde{C}_{abcd} \tilde{n}^b \tilde{n}^f \tilde{\nabla}_c \tilde{\ell}_d + O(\Omega),$$

where we have used $\tilde{n}^a \tilde{\nu}_d \tilde{\ell}_e = -\Omega \tilde{\ell}_d + O(\Omega^{(d-2)})$. Combining them, we have

$$2\Omega^{-(d-4)} \tilde{C}_{abcd} \tilde{n}^b \tilde{n}^f \tilde{\nabla}_c \tilde{\ell}_d = -2\Omega^{-(d-4)} \tilde{C}_{abcd} \tilde{n}^b \tilde{n}^f \tilde{\nabla}_c \tilde{\ell}_d (\tilde{g}_{bc} - \tilde{g}_{bc}),$$

where in the last line, we have used $\tilde{g}_{bc} = \tilde{g}_{bc} = O(\Omega^{(d-2)/2})$ and

$$\tilde{C}_{abcd} \tilde{n}^d = -\Omega (\tilde{\nabla}_d \tilde{S}_{bc} + C^e_{cde} \tilde{S}_{be}) = O(\Omega^{(d-4)/2}),$$

which is derived from $\tilde{C}_{abcd} \tilde{n}^d = -\Omega \tilde{\nabla}_d \tilde{S}_{bc}$ and $0 = \tilde{C}_{abcd} \tilde{n}^d = -\Omega \tilde{\nabla}_d \tilde{S}_{bc}$. The third term is

$$-8\Omega^{-(d-4)}\tilde{n}^A q^{bc} C_{ab\tilde{n}^f} q^{de} C_{de\tilde{n}^g} \tilde{\ell}_g = -4\Omega^{-(d-4)} \tilde{n}^A \tilde{n}^b \tilde{n}^f \tilde{\nabla}_c \tilde{\ell}_d + O(\Omega)$$

where the capital letters denote coordinate components with respect to $x^A$, and where we have used $C^F_{Da} = O(\Omega^{(d-2)/2})$ and $C^F_{Da} = 0$ in the first line, $\tilde{n}^a \tilde{\nu}_f = (d\Omega)_{\tilde{f}} + O(\Omega^{(d-2)})$, $C^F_{Da} = O(\Omega^{(d-2)/2})$ and $\tilde{S}_{EF} = \sigma_{EF}$ in the second line, and $n^A C^F_{Da} \sigma_{EF} = O(\Omega^{d-2})$ in the third line.
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