Holographic Beta functions for the Generalized Sine Gordon Theory

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Abstract
The Sine Gordon theory is generalized to include several cosine terms. This is similar to the world sheet description of a string propagating in a tachyon background. This model is studied as a (boundary) 2d euclidean field theory and also using an \(\text{AdS}_3\) holographic bulk dual. The beta functions for the cosine vertex of this modified theory are first computed in the boundary using techniques based on the exact RG. The beta functions are also computed holographically using position space and momentum space techniques. The results

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are in agreement with each other and with earlier computations. The 

beta functions of the field strength renormalization are computed in 

position space. They match with the earlier results in [8].

## Contents

1. Introduction  
2. The ERG beta function calculation of the generalized Sine-
   Gordon model.  
   2.1 The generalized Sine-Gordon model  
   2.2 The propagator, other preliminaries.  
   2.3 The ERG calculation.  
   2.4 Leading term in $\beta_F$.  
   2.5 The sub-leading term.  
   2.6 The beta function.  
3. Position space calculation of the beta function from the bulk.  
   3.1 A brief overview of AdS/CFT and the beta function calculation.  
   3.2 Leading order.  
   3.3 Order $\phi\gamma\chi$.  
   3.4 The Beta function.  
4. Beta function computation using momentum space techniques 
   from the bulk.  
   4.1 A brief summary of AdS/CFT from the momentum space 
      perspective[12].  
   4.2 Leading order.  
   4.3 Order $\phi\gamma\chi$.  
   4.4 The beta function.  
5. Beta function for $\delta$.  
   5.1 Overview of the calculation.  
   5.2 Fixing the coupling of the graviton-scalar-scalar vertex in the 
      bulk.  
   5.3 The 3-point correlator calculation.  
   5.4 The Beta function.  
6. Summary and Conclusions  

Appendices
Appendix A  Fixing relative normalization of the bulk and the boundary couplings and computing $\lambda_3$  
A.1  Fixing $\phi_0$, $\gamma_0$, $\chi_0$ and F, G, H relative normalizations . . . . . 24  
A.2  Computing $\lambda_3$ ......................................................... 25  
A.3  Relative normalization between $\sigma_0$ and $\xi_0$ . . . . . . . . . . . . . . . . . . . . . . 25  
A.4  Fixing $\lambda_\sigma$ ................................................................. 26  

Appendix B  The sub-leading term for $\beta_F$ using ERG on the boundary 28  

Appendix C  Position space calculation for $\beta_F$ from the bulk for the sub-leading term. 30  

Appendix D  Calculation for $\beta_6$. 34
1 Introduction

The Sine-Gordon model and it’s renormalization group evolution is interesting from many viewpoints. It is related to the X-Y model in two dimensions and gives the famous Kosterlitz-Thouless flow. A lot of work on both has been done before [3]–[8], [33]–[57]. It is also intimately related to string theory where a generalized version of the Sine-Gordon theory describes the bosonic string propagating in a tachyonic background. There also the RG flow is interesting because the $\beta$-function equations are related to the tachyon equation of motion.

The form of the $\beta$-functions for the Sine-Gordon model are well known and have been computed by both field theoretic methods[3] long back and more recently using the Exact Renormalization Group (ERG)[8]. An interesting computation would be to reproduce them holographically in order to understand better the holographic RG.

It has been shown in [4] that an ERG equation in a boundary theory can be mapped to a scalar field action in AdS space time. The main results are for a free theory. Some suggestions for how the interactions should work out were given there. To understand these issues better it is important to understand RG equations in the boundary theory and obtain them from some bulk computations. The precise connection between these equations and what is called “holographic RG” - which is really a radial evolution equation of the bulk theory - needs to be understood better. This paper is a step towards that goal. There is extensive literature on the AdS/CFT correspondence and holographic RG [1], [2], [9]–[28], [58]–[65].

The boundary theory is a free CFT perturbed by some composite (cosine) operators. The bulk theory that reproduces the leading two and three point correlators is a cubic theory. Of course there are any number of composite operators with definite scaling dimension and so the bulk theory should have a field of definite mass corresponding to each of these - we are assuming that an AdS dual exists for the free scalar theory in 2 dimensions. One can study the RG flow of this theory and one should be able to reproduce the $\beta$ - function of the cosine operator of the boundary theory. We do this calculation in this paper.

However, motivated by the string theory tachyon connection we consider a generalized Sine-Gordon theory. In string theory, instead of one scalar field, there are $D$ scalar fields ($D = 26$ for the bosonic string). The tachyon perturbation is of the form

$$
\int d^2 z \int_k \phi(k) e^{ik \cdot x}
$$
\( \phi(k) \) is the tachyon field in momentum space. We can consider a continuum of values of \( \vec{k} \). For each value of \( k \) it corresponds to a Sine-Gordon like theory. In [5] this theory was considered and shown to reproduce the leading non linear terms in the tachyon - dilaton system equations of motion in string theory.

Holographic techniques in position space are well suited for calculating correlation functions. In [6] a proper time method was used to evaluate the tachyon equation of motion starting from two point functions. We will use this technique here. For near marginal operators the two point function has the form

\[
\langle O_i(R)O_j(0) \rangle = \frac{G_{ij}}{R^4} + \frac{H_{ij}}{R^4} \ln \frac{R}{a}
\]

\( G_{ij} \) is the Zamoldchikov metric. A similar formula exists for the open string boundary CFT, with \( R^4 \) replaced by \( R^2 \). In [6] it was shown (in the context of the open string) that

\[
H_{ij} \phi^i = 0
\]

is the tachyon equation of motion to all orders in perturbation theory. Furthermore, it was argued by Polyakov [33] (for closed strings) that the equation of motion and \( \beta \)-function are related simply:

\[
\frac{\partial \Gamma[\phi]}{\partial \phi^i} = G_{ij} \beta^j
\]

This was also shown to all orders in perturbation theory in the open string context in [6]. Thus we can conclude [7] that

\[
H_{ij} \phi^i = G_{ij} \beta^j
\]

Thus to extract the beta function we can compute the two point function, corrected by interactions, and obtain the leading logarithmic deviation from the \( \frac{1}{R^4} \) scaling to obtain the \( \beta \) function. In the position space holographic calculation we employ this technique.

Once the perturbation is turned on it is no longer a CFT. This should reflect itself in the bulk deviations from AdS. This requires taking into account the gravitational back reaction. This back reaction in the bulk can be seen to manifest itself in the field strength renormalization of the boundary scalar fields. This gives us the beta function for the field strength renormalization. To compute this we look at the fluctuations of the graviton about the AdS. This contribution comes from another cubic vertex in the bulk. This is also equivalent to the dilaton equation in the string theory context.

In section 2 we start with a brief overview of the Sine-Gordon model. We fix propagators and other normalizations. Then we give a brief summary of
the calculation of the beta functions using ERG. Then we give the leading term of the beta function which has been computed in many ways in earlier papers as mentioned above. Next we motivate and detail the modification of the Sine-Gordon model. Then we compute the sub-leading term and extract the beta function. This concludes our boundary calculation.

In section 3 we give a brief overview of AdS-CFT computations using position space techniques. Then we compute the leading and sub-leading terms for the beta function from the bulk. In section 4 we start by briefly introducing computational techniques in AdS-CFT in momentum space. Then we calculate beta functions using momentum space techniques. All calculations are found to be in agreement.

In section 5 we compute the beta function for the running of the field strength renormalization. This calculation is done in position space. This is found to be in agreement with previous results.

In the appendix we determine relative normalizations between couplings in the bulk and boundary by comparing the generating functions for two-point functions computed from both sides. To fix the value of the coupling strength of the interaction vertex in the bulk we compare the generating function for the three point correlator on both sides. Doing this thus determines the bulk dual that reproduces correlations of the composite operators calculated in the free theory.
The ERG beta function calculation of the generalized Sine-Gordon model.

2.1 The generalized Sine-Gordon model

The action for the generalized theory is

\[ S_{\text{boundary}} = \frac{1}{4\pi} \int d^2 x \left[ (\partial_\mu \vec{X}).(\partial^\mu \vec{X}) + m^2 \vec{X}.\vec{X} + \frac{F}{a(0)^2} \cos (\vec{b}_1.\vec{X}) \right. \]
\[ \left. + \frac{G}{a(0)^2} \cos (\vec{b}_2.\vec{X}) + \frac{H}{a(0)^2} \cos (\vec{b}_3.\vec{X}) \right] \]

in euclidean \( d=2 \). Powers of \( a(0) \), the UV cutoff, have been added so that the engineering dimension of the action is zero. The mass term acts like an IR regulator in the propagator. In our calculations we cutoff all integrals in the IR by a moving scale, therefore we encounter no IR divergences. At marginality all \( b_i^2 = 4 \). This can be viewed as a world sheet action for a string in the presence of a background tachyon field with some definite momentum \[5]. The marginality condition is the “on-shell” condition for the tachyon. In that case the metric, \( g_{MN} \), for the dot product of \( b^M b^N g_{MN} \) has Minkowski signature. By doing this we have an additional freedom to tune the norm of the vector to the required value by modifying the individual components of the vector. This is important for the massless and higher string modes though it is not required for the tachyon. From here on all \( b_i \)’s and \( X \)’s are understood to be vectors - \( b_i^M, X^M \). We will drop all arrows on the top and suppress the vector index.

We want to calculate beta functions for the flow of \( F \) and \( b_1 \). \( b_1 \) multiplies \( X \) and therefore is modified by the field strength renormalization. \( F \) gets corrections from the self interaction of the cosine and corrections from higher order terms. From the string point of view, we are computing scattering amplitudes for the zeroeth mode of the closed strings with momenta \( b_i \) at position \( X(x) \), where \( \exp i b_i X(x) \) is the vertex operator for the tachyon. \( \exp i b_i X(x) \) is a tachyon vertex operator for a distinct closed string, each with momentum \( b_i \), where now instead of \( b \) being continuous, as in the introduction, we choose a discrete set of \( b_i \)’s. We will choose \( b_1 + b_2 + b_3 = 0 \) for reasons that will become clear.

We want to reproduce the beta functions of the Sine-Gordon model from the bulk. As explained in the introduction, to get beta functions we have to compute the appropriate multi-point correlation functions from the bulk. These correlators diverge when some or all of these points coincide. Thus,
to extract the beta function we compute the leading logarithmic deviation from the $1/R^4$ scaling of these correlators. In the position space holographic calculation we employ this technique.

When $b_1 + b_2 + b_3 = 0$, the first non vanishing higher point correlator is the cubic one involving all three cosines. From the point of view of the string theory tachyon, this constraint on the $b_i$ is just momentum conservation. From the CFT viewpoint, this comes from integrating over the zero mode of $X(x)$. In that case the beta function for $F$ starts at quadratic order

$$\beta_F \approx O(GH)$$

This is the cubic term in the tachyon equation of motion [5]. At higher orders the four point correlator is always non zero and there is a contribution of $O(F^3)$. (This is the first sub-leading term in the usual Sine-Gordon model.)

### 2.2 The propagator, other preliminaries.

We start with the kinetic term

$$S_{Kinetic} = \frac{1}{2\alpha'} \int d^2 x \partial_{\mu} X \partial^{\mu} X$$

(2.2.3)

$\alpha'$ is like the string tension. The propagator is

$$G^{MN}(x_1, x_2) = \langle X^M(x_1)X^N(x_2) \rangle = -g^{MN} \frac{\alpha'}{2\pi} \ln \frac{|\vec{x}_1 - \vec{x}_2|}{L}$$

(2.2.4)

Set $\alpha' = 2\pi$. $L$ is an arbitrary scale to make the argument of log dimensionless.

Therefore,

$$\langle : \cos b.X(x_1) :: \cos b.X(x_2) : \rangle = \frac{1}{2} \left( \frac{|\vec{x}_1 - \vec{x}_2|}{L} \right)^{-b^2}$$

(2.2.5)

The mass dimension of a marginal operator in $d=2$ is 2. Therefore, $b^2/2 = 2$. The beta functions are a power series expansion in the two couplings $F$ and $\vec{b}$ (the field strength renormaization). $F$ is a small number, $F \rightarrow 0$. $b^2 = 4$ is a large number, therefore we will look for a suitable expansion parameter which is small.
2.3 The ERG calculation.

The ERG can be described by

\[ \psi(X,t) = e^{-\frac{1}{2} \int d^2x_1d^2x_2 F_{x_1x_2t} \delta \frac{x_1}{\delta X_{x_1}} \delta \frac{x_2}{\delta X_{x_2}}} \psi(X,0) \quad (2.3.6) \]

Here

\[ F_{x_1x_2t} = -\frac{1}{2} \ln \frac{(x_1-x_2)^2+a(t)^2}{(x_1-x_2)^2+a(0)^2} \]

is the ERG high energy ”propagator”. \( t \) is the scale upto which you are doing the RG transformations, \( a(0) \) is the UV cutoff, \( a(t) \) is the IR cutoff. \( a(t) = a(0)e^t \). Which implies \( t = \ln(a(t)/a(0)) \), the log of the ratio of the scales whose coefficient is the beta function.

\[ \psi(X,0) = e^{-\int d^2x \frac{1}{2}\left[(\partial X)^2+(m X)^2+F \cos b X(x)+G \cos b_2 X(x)+H \cos b_3 X(x)\right]} \quad (2.3.7) \]

is the un-integrated ”partition function” of the theory and the evolution operator \( e^{-\frac{1}{2} \int d^2x_1d^2x_2 F_{x_1x_2t} \delta \frac{x_1}{\delta X_{x_1}} \delta \frac{x_2}{\delta X_{x_2}}} \) acting on \( \psi(X,0) \) upto some scale \( t \), gives \( \psi(X,t) \).

One can bring down appropriate powers of cosine from the exponential and act on it with the ERG operator. The calculation can then be organised as the ERG operator acting on a power series

\[ e^{-\frac{1}{2} \int d^2x_1d^2x_2 F_{x_1x_2t} \delta \frac{x_1}{\delta X_{x_1}} \delta \frac{x_2}{\delta X_{x_2}}} \left[ \int d^2x_1(a1) \cos b X(x_1) \right. \]

\[ + \int d^2x_1 \int d^2x_2(a2) \cos b X(x_1) \cos b X(x_2) \]

\[ + \int d^2x_1 \int d^2x_2 \int d^2x_3(a3) \cos b X(x_1) \cos b X(x_2) \cos b X(x_3) \]

the \( a_i \)'s, most generally, being the different corresponding coefficients. We look for corrections to \( \cos b X \) which is a term of the form

\[ (c1 + c2 + c3)t \cos b X(x_1) \quad (2.3.9) \]

where \( c1, c2 \) and \( c3 \) are the coefficients obtained after the ERG operator acts on the power series term by term. The final expression can be reorganized such that \( (c1 + c2 + c3)t \) is the correction to the coupling(the above equation) and the derivative w.r.t to \( t \) is the beta function. Further details can be found in [8].

For our case, to calculate the leading contribution we bring down one power of \( F \cos b_1 X \) and apply the ERG operator to it.
2.4 Leading term in $\beta_F$.

The ERG operator acting on the interaction term gives

$$\int \frac{d^2x_1}{a(0)^2} \frac{F}{4\pi} \exp\left(-\frac{1}{2} \int d^2x_1 d^2x_2 F_{x_1,x_2} \frac{\delta^2}{\delta X(x_1) \delta X(x_2)} \right) \cos(bX(x_1))$$

Simplifying we get,

$$\int \frac{d^2x_1}{a(t)^2} \frac{F}{4\pi} (1 - 2\delta t) \cos(bX(x_1))$$

The leading term in the $\beta$-function for $F$ is,

$$\beta_F = -2\delta F$$

$\delta = \frac{b^2}{4} - 1$ is the other small expansion parameter in terms of which we will calculate beta functions. We refer the reader to [8] for further details.

2.5 The sub-leading term.

To calculate the sub-leading contribution we will bring down one power of $\frac{G}{a(0)^2} \cos b_2 X(x)$ and $\frac{H}{a(0)^2} \cos b_3 X(x)$ each.

$$\frac{GH}{(4\pi)^2} \int \frac{d^2x_1 d^2x_2}{a^4} \exp\left(-\frac{1}{2} \int d^2x_1 d^2x_2 F_{x_1,x_2} \frac{\delta^2}{\delta X(x_1) \delta X(x_2)} \right) \cos b_2 X(x_1) \cos b_3 X(x_2)$$

$$= \frac{GH}{4} \int \frac{d^2x}{a(t)^2} \cos b_1 X(x)$$

Refer to Appendix (B) for further details.

2.6 The beta function.

We can organize the calculation as follows,

$$\left(F(1 - 2\delta t) + \frac{GH}{4} t\right) \frac{1}{4\pi} \int \frac{d^2x}{a(t)^2} \cos(bX(x_1))$$

Therefore, the full beta function, the t-derivative of the coefficient of the above expression, is,

$$\beta_F = -(2F\delta - \frac{GH}{4})$$
3 Position space calculation of the beta function from the bulk.

3.1 A brief overview of AdS/CFT and the beta function calculation.

The AdS metric in the Poincare patch is

\[ ds^2 = \frac{1}{z^2} [dz^2 + d\vec{x}^2] \] (3.1.16)

\( \vec{x} \) is euclidean.

According to the AdS-CFT correspondence

\[ \int D\Phi \exp(-S[\Phi]) = \langle \exp \left( - \int_{\partial AdS} \phi_0 O \right) \rangle \] (3.1.17)

which to leading order is

\[ S_{\text{bulk}}[\phi_0] = -W_{\text{QFT}}[\phi_0] \] (3.1.18)

where \( S_{\text{bulk}}[\phi_0] \) is the bulk action and \( W_{\text{QFT}}[\phi_0] \) is the connected generating functional of the boundary theory.

The correlation functions from the bulk are calculated by taking variations w.r.t \( \phi_0 \) on both sides. A general n point correlation function is given by

\[ \langle O_1(x_1) ... O_n(x_n) \rangle = (-1)^{n+1} \frac{\delta^n S_{\text{bulk}}}{\delta \phi_0(x_1) ... \delta \phi_0(x_n)} |_{\phi_0=0} \] (3.1.19)

To compute beta functions from the bulk we start with the bulk action with the term \( \phi \chi \gamma \) term,

\[ S_{\text{bulk}} = \int d^3x \sqrt{g} \left[ \frac{1}{2} (\partial \phi)^2 + \frac{1}{2} (m_\phi \phi)^2 + \frac{1}{2} (\partial \chi)^2 + \frac{1}{2} (m_\chi \chi)^2 \right. \]

\[ \left. + \frac{1}{2} (\partial \gamma)^2 + \frac{1}{2} (m_\gamma \gamma)^2 - \lambda_3 \phi \chi \gamma \right] \] (3.1.20)

The \( \phi, \gamma \) and \( \chi \) correspond to \( \cos b_1 X, \cos b_2 X, \cos b_3 X \) respectively.

The free equation of motion is

\[ \frac{1}{\sqrt{g}} \partial_\mu \left( \sqrt{g} g^{\mu \nu} \partial_\nu \phi \right) - m^2 \phi = 0 \] (3.1.21)

with boundary conditions

\( \phi(z_0, z) = 0 \) for \( z \to \infty \) and \( \phi(z_0, \tilde{z}) \to z_0^{d-\Delta} \phi_0(\tilde{z}) \) as \( z_0 \to 0 \)
The normalized bulk to boundary Green’s function is

\[ K_\Delta(z_0, \vec{z}, \vec{x}) = \frac{\Gamma(\Delta)}{\pi^{d/2} \Gamma(\Delta - d/2)} \left( \frac{z_0}{z_0^2 + (\vec{z} - \vec{x})^2} \right)^\Delta \]  

(3.1.22)

The solution to (3.1.21) is

\[ \phi(z_0, \vec{z}) = \frac{\Gamma(\Delta)}{\pi^{d/2} \Gamma(\Delta - d/2)} \int d^2x \left( \frac{z_0}{z_0^2 + (\vec{z} - \vec{x})^2} \right)^\Delta \phi_0(\vec{x}) \]  

(3.1.23)

To extract the beta function we first regulate the generating function for the correlators of the boundary theory to be calculated from the bulk by inserting \( x_0 \) which acts as the UV cutoff. Then we compute the generating function for the two point function with one particle offshell and then, as was described in the introduction, obtain the leading logarithmic deviation from the \( \frac{1}{R^4} \) scaling which comes, at the leading order, from taking the particle offshell. Then we extract terms which are logarithmically divergent in terms of \( x_0 \) which acts as the UV scale.

![Fig. 1. The leading order Witten diagram](image1.png)

![Fig. 2. Witten diagram for the sub-leading contribution](image2.png)

3.2 Leading order

The bulk action for the kinetic term is

\[ S = \int d^d x_1 d^d x_3 d^{d+1} y \sqrt{g} \left[ \frac{1}{2} g^{\mu \nu} \partial_\mu \phi(y) \partial_\nu \phi(y) + \frac{1}{2} m^2 \phi^2(y) \right] \]  

(3.2.24)
Integrate by parts

\[ \frac{1}{2} \phi_0^2 \int d^d x_1 d^d x_3 d^d y dy_0 \left[ \frac{\partial}{\partial y_0} \left( y_0^{-d+1} K_{\Delta}(y_0, \vec{y}; \vec{x}_1) \frac{\partial}{\partial y_0} K_{\Delta}(y_0, \vec{y}; \vec{x}_3) \right) \right. \right. \]

\[ \left. \left. - \partial_\mu y_0^{-d+1} K_{\Delta}(y_0, \vec{y}; \vec{x}_1) \frac{\partial}{\partial y^\mu} K_{\Delta}(y_0, \vec{y}; \vec{x}_3) - y_0^{-d+1} K_{\Delta}(y_0, \vec{y}; \vec{x}_1) \partial_\mu \partial^\mu K_{\Delta}(y_0, \vec{y}; \vec{x}_3) \right] \right) \]  

\[ + y_0^2 m^2 K_{\Delta}(y_0, \vec{y}; \vec{x}_1) K_{\Delta}(y_0, \vec{y}; \vec{x}_3) \right] \]  

The bulk equation of motion is (3.1.21). Substituting we get,

\[ \frac{1}{2} \phi_0^2 \int d^d x_1 d^d x_3 d^d y dy_0 \left[ \frac{\partial}{\partial y_0} \left( y_0^{-d+1} K_{\Delta}(y_0, \vec{y}; \vec{x}_1) \frac{\partial}{\partial y_0} K_{\Delta}(y_0, \vec{y}; \vec{x}_3) \right) \right. \]

\[ \left. \left. - \partial_\mu y_0^{-d+1} K_{\Delta}(y_0, \vec{y}; \vec{x}_1) \frac{\partial}{\partial y^\mu} K_{\Delta}(y_0, \vec{y}; \vec{x}_3) - y_0^{-d+1} K_{\Delta}(y_0, \vec{y}; \vec{x}_1) \partial_\mu \partial^\mu K_{\Delta}(y_0, \vec{y}; \vec{x}_3) \right] \right) \]  

\[ + y_0^2 m^2 K_{\Delta}(y_0, \vec{y}; \vec{x}_1) K_{\Delta}(y_0, \vec{y}; \vec{x}_3) \right] \]  

Choose the outward pointing normal along the radial direction and by the Gauss’s divergence theorem, do the surface integral.

\[ S = \left. \frac{1}{2} \phi_0^2 \int d^d x_1 d^d x_3 d^d y \left[ \frac{\partial}{\partial y_0} \left( y_0^{-d+1} K_{\Delta}(y_0, \vec{y}; \vec{x}_1) \frac{\partial}{\partial y_0} K_{\Delta}(y_0, \vec{y}; \vec{x}_3) \right) \right. \right. \]

\[ \left. \left. - \partial_\mu y_0^{-d+1} K_{\Delta}(y_0, \vec{y}; \vec{x}_1) \frac{\partial}{\partial y^\mu} K_{\Delta}(y_0, \vec{y}; \vec{x}_3) - y_0^{-d+1} K_{\Delta}(y_0, \vec{y}; \vec{x}_1) \partial_\mu \partial^\mu K_{\Delta}(y_0, \vec{y}; \vec{x}_3) \right] \right) \]

\[ + y_0^2 m^2 K_{\Delta}(y_0, \vec{y}; \vec{x}_1) K_{\Delta}(y_0, \vec{y}; \vec{x}_3) \right] \]  

\[ \bigg|_{y_0 = \epsilon} \]  

where \( \epsilon \to 0 \), therefore \( y_0 \) is close to the boundary, \( y_0 \to 0 \). We identify \( y_0 \) with \( x_0 \), the UV regulator. The minus sign comes from choosing the convention for the outward pointing normal \( n^\mu = (-\epsilon, 0) \). Therefore, using

\[ \lim_{x_0 \to 0} x_0^{\Delta-d} K_{\Delta}(x_0, \vec{y}; \vec{x}_1) \rightarrow \delta^{(d)}(\vec{y} - \vec{x}_1) \]  

\[ S = -\frac{1}{2} \phi_0^2 C_{\Delta} \int d^d x_1 d^d x_3 d^d y \left[ \frac{\Delta \delta^{(d)}(\vec{y} - \vec{x}_1)}{(x_0^2 + (\vec{y} - \vec{x}_3))^\Delta} - \frac{(2\Delta) x_0^2 \delta^{(d)}(\vec{y} - \vec{x}_1)}{(x_0^2 + (\vec{y} - \vec{x}_3))^{\Delta+1}} \right] \]  

This is the action for the kinetic term. A general n-point correlation function is given by

\[ \langle O_1(x_1)...O_n(x_n) \rangle = (-1)^{n+1} \frac{\delta^n S_{bulk}}{\delta \phi_0(x_1) ... \delta \phi_0(x_n)} \bigg|_{\phi_0 = 0} \]  

Therefore the generating function of a two point function will have another minus. The two point functions from the bulk and the boundary now match. The generating function for the two point function is,
\( S_2 = -(-1)^1 \frac{1}{2} \phi_0^2 C \Delta \int d^d x_1 d^d x_3 \left[ \frac{\Delta}{(x_0^2 + (x_1 - x_3))^{\Delta}} - \frac{(2\Delta)x_0^2}{(x_0^2 + (x_1 - x_3))^{\Delta + 1}} \right] \)  

The log divergent term comes from the first term and is retained. The second term is \( x_0^2 \) suppressed. In the limit \( x_0 \to 0 \), it vanishes. We drop this term. \( C_\Delta = \frac{\Gamma(\Delta)}{\pi^{d/2}\Gamma(\Delta - d/2)} \), therefore

\[
S_2 = \frac{\Gamma(\Delta + 1)}{\pi^{d/2}\Gamma(\Delta - d/2)} \int d^d x_1 d^d x_3 \frac{1}{(x_0^2 + (x_1 - x_3)^2)^\Delta} \frac{\phi_0^2}{2!} \tag{3.2.31}
\]

This is the generating functional for the two point function being computed slightly away from the boundary. By doing this we have introduced a bulk coordinate in the generating functional which is the UV regulator.

\( d = 2 \) and \( \Delta = 2(1 + \delta) \). We set \( x_1 \) to zero and \( x_3 \) to \( R \), \( R \to \infty \), the expression becomes,

\[
\frac{\phi_0^2}{2!} \pi \Delta(\Delta - 1) \int d \left( \frac{x_1^2}{x_0^2} \right) d \left( \frac{x_3^2}{x_0^2} \right) \frac{1}{((x_0^2 + R^2)/x_0^2)^{2(1+\delta)}} \tag{3.2.32}
\]

Multiply and divide by \( R \).

\[
\phi_0^2 \pi \int d \left( \frac{x_1^2}{R^2} \right) d \left( \frac{x_3^2}{R^2} \right) \left( 1 - 4\delta \log \frac{R}{x_0} \right) \tag{3.2.33}
\]

We will extract the leading term of the beta function from this.

### 3.3 Order \( \phi \gamma \chi \)

The beta function is the change in the couplings of the theory under scaling transformations. To determine the deviation from the canonical scaling dimension we look at the behaviour of the two point function slightly away from marginality and determine the leading term of the beta function. This was the calculation we did above. To calculate the sub-leading term we start with a two point function, insert another operator, therefore now we have a three point function, and look for the log deviation from \( 1/R^4 \) scaling for this object. We first calculate the generating function for the three point function. To do this we start with

\[
S_3 = -\lambda_3 \int d^{d+1} y \sqrt{g} \phi(y) \gamma(y) \chi(y) \tag{3.3.35}
\]
where

\[ \phi(z) = \phi(z_0, \vec{z}) = \int d^4 x K_\Delta(z_0, \vec{z}, \vec{x}) \phi_0(\vec{x}) \] (3.3.36)

and substitute this for \( \phi(y), \gamma(y), \chi(y) \) in \( S_3 \).

Here \( \sqrt{g} = y_0^{-(d+1)}, d = 2, C_\Delta = \frac{\Gamma(\Delta)}{\pi^{d/2} \Gamma(\Delta - d/2)} \).

\( \phi_0, \gamma_0 \) and \( \chi_0 \) are the couplings of the boundary theory (boundary values of the bulk fields, they have no coordinate dependence). This becomes (details in Appendix (C)),

\[ S_3 = -\lambda_3 \pi \phi_0 \gamma_0 \chi_0 \int d \left( \frac{x_1^2}{R^2} \right) d \left( \frac{x_3^2}{R^2} \right) \log \frac{R}{x_0} \] (3.3.37)

### 3.4 The Beta function

The full generating function can be organized as

\[ S = S_2 + S_3 \] (3.4.38)

\[ \begin{align*}
S &= \pi \int dx_1^2 dx_3^2 \left( \phi_0^2 - 2 \phi_0 (2 \phi_0 \delta + \frac{\lambda_3 \gamma_0 \chi_0}{2}) \right) \log \frac{R}{x_0} \\
&= \frac{1}{64} \int dx_1^2 dx_3^2 \left( F^2 - 2F(2\delta F - \frac{GH}{4}) \log \frac{R}{x_0} \right) 
\end{align*} \] (3.4.39)

Substituting the relations between \( \phi_0, \gamma_0, \chi_0 \) and \( F,G,H((A.1.95), (A.1.96), (A.1.97)) \) and the value of \( \lambda_3(A.2.101) \) we get,

\[ \begin{align*}
&= \frac{1}{64} \int dx_1^2 dx_3^2 \left( F^2 - 2F(2\delta F - \frac{GH}{4}) \log \frac{R}{x_0} \right) \\
&= \frac{1}{64} \int dx_1^2 dx_3^2 \left( F^2 - 2F(2\delta F - \frac{GH}{4}) \log \frac{R}{x_0} \right) 
\end{align*} \] (3.4.40)

We have calculated the correction to \( F^2 \). To get the beta function we want to isolate the change in \( F \). To do this we note,

\[ F^2 \to F'^2 = F^2 + \delta(F^2) \log x_0 = F^2 + 2F \delta(F) \log \frac{R}{x_0} \] (3.4.41)

\[ F^2 + (-4\delta F^2 + FGH/2) \log x_0 = F^2 - 2F(2\delta F - GH/4) \log \frac{R}{x_0} \] (3.4.42)
and comparing the above two expressions we see

$$\beta_F = \delta(F) = -\left(2\delta F - \frac{GH}{4}\right)$$

which matches our result from the boundary calculation (2.6.15).

4 Beta function computation using momentum space techniques from the bulk.

4.1 A brief summary of AdS/CFT from the momentum space perspective[12].

The bulk action with the term $\phi\chi\gamma$ is

$$S_{\text{bulk}} = \int d^3x \sqrt{|g|} \left[ \frac{1}{2} (\partial \phi)^2 + \frac{1}{2} (m_\phi \phi)^2 + \frac{1}{2} (m_\chi \chi)^2 + \frac{1}{2} (\partial \gamma)^2 + \frac{1}{2} (m_\gamma \gamma)^2 - \lambda_3 \phi \chi \gamma \right]$$

The equation of motion is

$$(-\Box_G + m^2) \Phi = \gamma \chi$$

$\Phi$ can be expanded in powers of $\lambda$

$$\Phi = \Phi_0 + \lambda \Phi_1 + ...$$

$$\Phi_0 = \phi_{00} + z^2 \phi_{02} + z^4 \phi_{04} + ...$$

$$\Phi_1 = \phi_{10} + z^2 \phi_{12} + z^4 \phi_{14} + ...$$

$\gamma$ and $\chi$ have similar expansions.

The equations of motion can be solved perturbatively

$$(-\Box_G + m^2) \Phi_0 = 0$$

and

$$(-\Box_G + m^2) \Phi_1 = \gamma \chi$$
We Fourier transform along all directions parallel to the boundary at \(z = 0\). We write the Fourier transform of \(\Phi(z, \vec{x})\) as \(\Phi(z, \vec{p})\). The free equation of motion becomes

\[
L_{d, \Delta}(z, p)\Phi(z, p) = 0 \tag{4.1.51}
\]

where

\[
L_{d, \Delta}(z, p) = -z^2 \partial_z^2 + (d - 1)z \partial_z + m^2 + z^2 p^2 \tag{4.1.52}
\]

The bulk to boundary propagator is given by

\[
L_{d, \Delta}(z, p)K_{d, \Delta}(z, p) = 0 \tag{4.1.53}
\]

with boundary conditions

\[
\lim_{z \to 0} [z^{-(d-\Delta)}K_{d, \Delta}(z, p)] = 1 \tag{4.1.54}
\]

and

\[
K_{d, \Delta}(\infty, p) = 0 \tag{4.1.55}
\]

The bulk to bulk propagator is the solution to

\[
L_{d, \Delta}(z, p)G_{d, \Delta}(z, p; \zeta) = \zeta^4 \delta(z - \zeta) \tag{4.1.56}
\]

with boundary conditions

\[
\lim_{z \to 0} [z^{-(d-\Delta)}G_{d, \Delta}(z, p; \zeta)] = 0 \tag{4.1.57}
\]

and

\[
G_{d, \Delta}(\infty, p; \zeta) = 0 \tag{4.1.58}
\]

The unique solutions to these equations are

\[
K_{d, \Delta}(z, p) = \frac{2^{\frac{d}{2}-\Delta+1}}{\Gamma(\Delta - \frac{d}{2})} p^{\Delta - \frac{d}{2}} K_{\Delta - \frac{d}{2}}(pz) \tag{4.1.59}
\]

and

\[
G_{d, \Delta}(z, p; \zeta) = (z\zeta)^{d/2} I_{\Delta - d/2}(pz) K_{\Delta - d/2}(p\zeta) \tag{4.1.60}
\]

for \(z \leq \zeta\)

and

\[
G_{d, \Delta}(z, p; \zeta) = (z\zeta)^{d/2} I_{\Delta - d/2}(p\zeta) K_{\Delta - d/2}(pz) \tag{4.1.61}
\]

17
for \( z \geq \zeta \)

The solutions to the equations of motion are

\[
\Phi_0(z, \vec{p}) = K_{d,\Delta}(z, p)\phi_0 \quad (4.1.62)
\]

\[
\phi_0 = \phi_{00} + \lambda_3 \phi_{10} + \lambda_3^2 \phi_{20} + \ldots \quad (4.1.63)
\]

should be considered the full source. To begin with we turn off the higher order terms. We only keep the leading term in \( \Phi_0 \) here. Later we will see that these will have to be turned on.

\[
\Phi_1 = \int \frac{d^4k_1d^4k_2}{(2\pi)^{2d}} \gamma_0 \delta^{(d)}(k_1 + k_2 + k_3) \int_0^\infty \frac{d\zeta}{\zeta^{d+1}} G_{d,\Delta}(z, k_1; \zeta) \mathcal{K}_{d,\Delta}(\zeta, k_2) \mathcal{K}_{d,\Delta}(k_3, \zeta) \quad (4.1.64)
\]

### 4.2 Leading order

For \( d = 2 \) and \( \Delta = 2 + 2\delta \), the solution to the free equation of motion is

\[
\Phi_0 = p^{1+2\delta} z K_{1+2\delta}(pz) \phi \quad (4.2.65)
\]

\[
= (1 + 2\delta \log z) \phi_0 \quad (4.2.66)
\]

At \( z = x_0 \) where \( x_0 \to 0 \), we can interpret \( 2\delta \phi_0 \log x_0 \) as the log divergent term whose coefficient gives the flow in \( \phi_0 \) because of the deviation from the canonical scaling dimension of \( \Phi \).

Therefore the correction to \( \phi_0 \) is

\[
2\delta \phi_0 \log x_0 \quad (4.2.67)
\]

The beta function to leading order is

\[
\frac{\partial}{\partial \log x_0} (2\delta \phi_0 \log x_0) = 2\delta \phi_0 \quad (4.2.68)
\]

### 4.3 Order \( \phi \gamma \chi \)

To get the contribution at this order we look at the solution of \( \Phi_1 \).

We want to solve the integral

\[
I_{d=2,\Delta=2}^{\delta<} = \int_{x_0}^z \frac{d\zeta}{\zeta^{d+1}} G_{d,\Delta}(z, p_1; \zeta) \mathcal{K}_{d,\Delta}(\zeta, p_2) \mathcal{K}_{d,\Delta}(p_3, \zeta) \quad (4.3.69)
\]
in the near boundary region $\zeta \leq z$. $x_0$ is the UV cut-off.

\[ \Phi_1 = \gamma_0 \chi_0 \int_{x_0}^{z} \frac{d\zeta}{\zeta^{d+1}} (z\zeta)^{d/2} I_{\Delta_1-d/2}(p_1\zeta) K_{\Delta_1-d/2}(p_1z) \]  

\[ \frac{2^{d/2-\Delta_2+1}}{\Gamma(\Delta_2 - d/2)} \frac{p_2^{\Delta_2-d/2} \zeta^{d/2} K_{\Delta_2-d/2}(p_2\zeta)}{\Gamma(\Delta_2 - d/2)} \]  

\[ \frac{2^{d/2-\Delta_3+1}}{\Gamma(\Delta_3 - d/2)} \frac{p_3^{\Delta_3-d/2} \zeta^{d/2} K_{\Delta_3-d/2}(p_3\zeta)}{\Gamma(\Delta_3 - d/2)} \]  

(4.3.70)

Since the $\delta$ has no effect at this order all $\Delta$'s are set to 2. The log divergent part of $\Phi_1$ is,

\[ \Phi_1 = \frac{1}{2} \gamma_0 \chi_0 p_1 z K_1(p_1 z) (-\log x_0) \]  

(4.3.71)

We can expand $\Phi$ in powers of $\lambda_3$,

\[ \Phi = \Phi_0 + \lambda_3 \Phi_1 + ... \]  

(4.3.72)

Therefore,

\[ \Phi = \Phi_0 + \lambda_3 \left( \frac{1}{2} \gamma_0 \chi_0 p_1 z K_1(p_1 z) (-\log x_0) \right) \]  

(4.3.73)

This diverges as $x_0 \to 0$. $p_1 z K_1(p_1 z)$ is the solution to $\Phi_0$, therefore to make the full $\Phi$ finite we can turn on a subleading $O(\lambda_3)$ term in the expansion of the source $\phi_0$ (as mentioned before we are turning on subleading coefficients).

\[ \phi_0 = \phi_{00} + \lambda_3 \phi_{10} + ... \]  

(4.3.74)

which should be regarded as the source for the boundary field, rather than just the leading piece $\phi_{00}$.

The modified source $\phi_0$ is

\[ \phi_0 = (1 + 2\delta \log x_0) \phi_{00} + \lambda_3 \phi_{10} \]  

(4.3.75)

set

\[ \phi_{10} = \frac{1}{2} \gamma_0 \chi_0 (\log x_0) \]  

(4.3.76)

The modified source becomes

\[ \phi_0 = (1 + 2\delta \log x_0) \phi_0 + \lambda_3 \frac{1}{2} \gamma_0 \chi_0 (\log x_0) \]  

(4.3.77)
4.4 The beta function.

Therefore the full beta function is

\[ \beta_{\phi_0} = \frac{\partial}{\partial \log x_0} \phi_0 = (2\delta \phi_0 + \frac{1}{2} \lambda_3 \gamma_0 \chi_0) \] (4.4.78)

Substituting the relations between \( \phi_0, \gamma_0, \chi_0 \) and \( F,G,H((A.1.95), (A.1.96), (A.1.97)) \) and the value of \( \lambda_3(A.2.101) \) we get,

\[ \beta_F = (2\delta F - \frac{GH}{4}) \] (4.4.79)

5 Beta function for \( \delta \).

5.1 Overview of the calculation.

As mentioned before \( b \) multiplies \( X \) inside the cosine and therefore can be interpreted as the field strength renormalization. \( b^2 \) is close to 4. This is large compared to \( F \) which is close to zero. It was mentioned that the beta functions of the Sine-Gordon are a power series expansion in two parameters, \( \delta = \frac{b^2}{4} - 1 \) is the other appropriate small parameter in which the expansion can be carried out. From the perspective of the boundary theory the corrections to this modified coupling \( \delta \) come from two cosines combining together to give \( (\partial X)^2 \) term whose coefficient is the running of the field strength renormalization and gives this beta function. This has been calculated before in an earlier paper [8].

From the bulk theory the leading contribution to the beta function has to come from a vertex of the type \( g_{\mu\nu} \partial^\mu \phi \partial^\nu \phi \) so that we have a structure similar to the boundary calculation and we can see that the two \( \phi \)'s correct the \( g_{\mu\nu} \) which is associated to the \( (\partial X)^2 \) term. Next we note that the boundary kinetic term involves only diagonal components and therefore we can attempt to model the graviton by a dilaton which takes into account only the diagonal degrees of freedom and is a scalar, thus simplifying the problem enormously.
5.2 Fixing the coupling of the graviton-scalar-scalar vertex in the bulk.

To compute the graviton-scalar-scalar vertex we want to look at the fluctuation about AdS

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$  \hspace{1cm} (5.2.80)

$\bar{g}_{\mu\nu}$ is AdS. We want to simplify this by modeling the graviton as a dilaton. Therefore,

$$g^{\mu\nu} = e^{-\lambda_\sigma} \bar{g}^{\mu\nu} = (1 - \lambda_\sigma) \bar{g}^{\mu\nu}$$  \hspace{1cm} (5.2.81)

$$h^{\mu\nu} = -\lambda_\sigma \bar{g}^{\mu\nu}$$  \hspace{1cm} (5.2.82)

Therefore the kinetic term \(\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi\) becomes

$$\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi = \frac{1}{2} (1 - \lambda_\sigma) \bar{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$$  \hspace{1cm} (5.2.83)

We treat the dilaton as a massive scalar with \(m_\sigma \rightarrow 0\), the full action therefore becomes,
\[ S_{\text{bulk}} = \int d^3x \sqrt{g} \left[ \frac{1}{2} (\partial \phi)^2 + \frac{1}{2} (m_\phi \phi)^2 + \frac{1}{2} (\partial \chi)^2 + \frac{1}{2} (m_\chi \chi)^2 + \frac{1}{2} (\partial \gamma)^2 + \frac{1}{2} (m_\gamma \gamma)^2 \right] \]

\[ + \frac{1}{2} \lambda_3 \phi \gamma \chi + \frac{1}{2} \lambda_\sigma \sigma \bar{\sigma} g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} (\partial \sigma)^2 + \frac{1}{2} m_\sigma^2 \sigma^2 \] (5.2.84)

The kinetic term in the boundary action is modified

\[ S_{\text{boundary}} = \frac{1}{4\pi} \int d^2x \left[ (1 + \xi_0)(\partial_\mu \vec{X})(\partial^\mu \vec{X}) + m^2 \vec{X} \cdot \vec{X} + \frac{F}{a(0)^2} \cos (\vec{b}_1 \cdot \vec{X}) \right. 
\[ + \frac{G}{a(0)^2} \cos (\vec{b}_2 \cdot \vec{X}) + \frac{H}{a(0)^2} \cos (\vec{b}_3 \cdot \vec{X}) \] (5.2.85)

\[ \xi_0 \] is related to the bulk field \( \sigma \) whose boundary value \( \sigma_0 \) is equal to \( \xi_0 \) up to normalization. We vary the action with respect to \( \xi_0 \) to compute various correlators.

### 5.3 The 3-point correlator calculation.

To compute the beta function we want to calculate the generating functional for the three point function as before, but this time for the vertex \( -\frac{1}{2} \lambda_\sigma \sigma \bar{\sigma} g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi \). To do this we again start with

\[ S_{\sigma^3} = -\frac{1}{2} \lambda_\sigma \int d^{d+1}y \sqrt{g} \bar{\phi} \partial_\mu \phi (y) \partial_\nu \phi (y) \sigma (y) \] (5.3.87)

\[ = -2\pi (1 + \delta) \lambda_\sigma \sigma_0 \phi^2 \int d \left( \frac{x_1^2}{R^2} \right) d \left( \frac{x_2^3}{R^2} \right) \log \frac{R}{x_0} \] (5.3.88)

\[ \left[ \frac{\Gamma(\Delta_2 + \Delta_3 - \Delta_1) \Gamma(\Delta_2 + \Delta_4 + \Delta_1 - d)}{\Gamma(\Delta_2) \Gamma(\Delta_3)} - \frac{\Gamma(\Delta_2 + \Delta_3 - \Delta_1) \Gamma(\Delta_2 + \Delta_4 + \Delta_1 - d + 2)}{\Gamma(\Delta_2) \Gamma(\Delta_3 + 1)} \right] \]

Details in Appendix (D).

\( S_{\sigma^3} \) is non-zero offshell but vanishes onshell where the square bracket is zero. This is resolved when \( \lambda_\sigma \) is fixed ((A.4.118) and the comment thereafter).

Putting in all the relative normalizations((A.1.95), (A.3.108)) and the value of \( \lambda_\sigma \) and we get,

\[ S_{\sigma^3} = -\frac{F^2 (1 + \delta) \xi_0}{8} \int d^2 \left( \frac{x_1}{R} \right) d \left( \frac{x_2^3}{R^2} \right) \log \frac{R}{x_0} \] (5.3.89)
5.4 The Beta function.

The kinetic term in the boundary theory, whose correction we are computing is

\[
\frac{1 + \xi_0}{4\pi} \int d^2 x \ (\partial X(x))^2
\]  

(5.4.90)

Comparing this with the expression for \( S_{\sigma 3} \) above we immediately see the beta function (the \( \frac{\partial}{\partial \log x_0} \) derivative) is,

\[
\beta_\delta = -\frac{F^2(1 + \delta)}{8}
\]  

(5.4.91)

which matches 5.2.46 in [8].

6 Summary and Conclusions

In this paper the beta function of a generalized Sine-Gordon theory has been calculated using a bulk holographic dual. The bulk theory is dual to a free field theory in the boundary. The calculations have been done both in position space as well as momentum space. The boundary calculation is also done and it is shown that the results agree.

The main motivation for doing this calculation is to understand the results of [4] better when interactions are involved. There the main example used was the free scalar theory and in this situation interactions are between composite operators. In 1+1 dimension, the cosine is one of the most interesting example of such operators and besides being related to string theory, has applications in 1+1 dimensional condensed matter systems, such as the X-Y model [3].

The model is also motivated by the first quantized description of a string propagating in a tachyon background. The beta function gives the equation of motion for the tachyon. The model also has a wave function renormalization which results in a beta function for the string theory dilaton coupling. The boundary calculation in this paper uses techniques derived from the exact RG, as used in [8] for the usual Sine-Gordon model. The main idea for the bulk calculation in position space is to identify the beta function with the coefficient of a logarithmic deviation from the canonical scaling of a two point function. This is based on the technique described in [6, 33] and is suitable in holographic computations. In the bulk momentum space calculation the technique is to first solve the fourier transformed equations of motion order by order in the coupling of the bulk interaction vertex and then identify the
log divergent terms in the solutions as described in [10]. All the results agree to the order calculated.

There are many further problems that need to be addressed. One technically interesting issue of course is to go to higher orders. This should constrain the bulk dual much more. The precise bulk dual of free scalar theory considered here, in particular the connection to higher spin theory in $AdS_3$ needs to be understood better. It would be interesting if one can say something about the IR fixed point of this theory by studying the bulk. One should remember that the underlying theory in the boundary is a free scalar theory. The interactions in the bulk involve fields dual to composite operators. There are an infinite number of them - they can be identified with the momentum modes of the string theory tachyon. One expects that there should be a corresponding simple way to package these in the bulk also. This needs to be understood better. Finally, the ERG description of composite operators and also the map to a holographic theory in the presence of these operators, there are many complications. These can be studied in a controlled way in this model. We hope to return to these questions.

**Appendix A** Fixing relative normalization of the bulk and the boundary couplings and computing $\lambda_3$

**A.1 Fixing $\phi_0$, $\gamma_0$, $\chi_0$ and $F$, $G$, $H$ relative normalizations**

To compute the relative normalization between the bulk and the boundary for the couplings $\phi_0$ anf $F$ we compare the generating functionals of the two point functions calculated for both sides.

The generating function for the two point function for the boundary theory is

$$GF_2 = \frac{A_2}{4} \frac{F^2}{(4\pi)^2}$$

(A.1.92)

where

$$A_2 = \int d^2x_1 d^2x_2 \frac{1}{(x_1 - x_2)^{2\Delta}}$$

(A.1.93)

The generating function for the two point function in the bulk is

$$S_2 = \frac{2}{\pi} A_2 \phi_0^2 / (2!)$$

(A.1.94)
Comparing we get

\[ \phi_0 = \frac{1}{8\sqrt{\pi}} F \] (A.1.95)

Similarly,

\[ \gamma_0 = \frac{1}{8\sqrt{\pi}} G \] (A.1.96)

\[ \chi_0 = \frac{1}{8\sqrt{\pi}} H \] (A.1.97)

**A.2 Computing \( \lambda_3 \)**

To compute \( \lambda_3 \) we compare the generating function for the three point function of the boundary theory and for the bulk theory.

The generating function for the three point function of the boundary theory is

\[ GF_3 = \frac{A_3 FGH}{4(4\pi)^3} \] (A.2.98)

\[ A_3 = \int d^2x_1 d^2x_2 d^2x_3 \frac{1}{(\vec{x}_1 - \vec{x}_2)\Delta_{123}(\vec{x}_2 - \vec{x}_3)\Delta_{231}(\vec{x}_3 - \vec{x}_1)\Delta_{312}} \] (A.2.99)

Here \( \Delta_{ijk} = \Delta_i + \Delta_j - \Delta_k \).

For the bulk theory

\[ S_3 = -\frac{\lambda_3}{2\pi^2} A_3 \phi_0 \gamma_0 \chi_0 \] (A.2.100)

Comparing we get

\[ \lambda_3 = -4(\pi)^{1/2} \] (A.2.101)

Determining this also determines the bulk dual to the Sine-Gordon on the boundary.

**A.3 Relative normalization between \( \sigma_0 \) and \( \xi_0 \)**

To fix this we calculate the generating function of \( (\partial X(x_1))^2(\partial X(x_2))^2 \) from the bulk and boundary and compare them.
Bulk:

\[
GF_{\sigma^2} = \frac{1}{2} \frac{\sigma^2}{\sigma_0^2} \frac{\Gamma(\Delta + 1)}{\Gamma(\Delta - d/2)} \int d^2x_1 d^2x_2 \frac{1}{x_{12}^\Delta} \tag{A.3.102}
\]

\[
GF_{\sigma^2} = \frac{\sigma_0^2}{\pi} A_2 \tag{A.3.103}
\]

Boundary:

\[
GF_{\xi^2} = \frac{1}{2!} \frac{\xi_0^2}{(4\pi)^2} \int d^2x_1 d^2x_2 \langle (\partial X(x_1))^2(\partial X(x_2))^2 \rangle \tag{A.3.104}
\]

\[
\langle (\partial X(x_1))^2(\partial X(x_2))^2 \rangle = \frac{2}{x_{12}^4} \tag{A.3.105}
\]

Therefore

\[
GF_{\xi^2} = \frac{1}{2!} \frac{\xi_0^2}{(4\pi)^2} \int d^2x_1 d^2x_2 \frac{2}{x_{12}^4} \tag{A.3.106}
\]

\[
GF_{\xi^2} = \frac{\xi_0^2}{(4\pi)^2} A_2 \tag{A.3.107}
\]

Comparing,

\[
\sigma_0 = \frac{\xi_0}{4\sqrt{\pi}} \tag{A.3.108}
\]

A.4 Fixing $\lambda_\sigma$

To fix $\lambda_\sigma$ we will compute $\langle (\partial X(x_1))^2 \cos bX(x_2) \cos bX(x_3) \rangle$ from the bulk and the boundary and compare them.

Bulk:

From (D.0.159)

\[
S_{\sigma^3} = -\frac{\lambda_\sigma}{\pi^2} \sigma_0 \phi_0^2 (1 + \delta) A_3 [ ] \tag{A.4.109}
\]

where $[ ] = \left[ \frac{\Gamma(\Delta_2 + \Delta_3 - \Delta_1)}{\Gamma(\Delta_2)\Gamma(\Delta_3)} - \frac{\Gamma(\Delta_2 + \Delta_3 - \Delta_1)}{\Gamma(\Delta_2)\Gamma(\Delta_3)} \right]

Boundary:

We want to compute $\langle (\partial X(x_1))^2 \cos bX(x_2) \cos bX(x_3) \rangle$

\[
\langle (\partial X(x_1))^2 \cos bX(x_2) \cos bX(x_3) \rangle_{\text{non-vanishing}} = 2/4 \langle (\partial X(x_1))^2 \exp ibX(x_2) \exp -ibX(x_3) \rangle \tag{A.4.110}
\]
We change to complex coordinates. \( (\partial X(x_1))^2 \rightarrow 4\partial X_1 \bar{\partial} X_1 \). We get,

\[
2 \langle \partial X_1 \bar{\partial} X_1 \exp ibX(x_2) \exp -ibX(x_3) \rangle \quad (A.4.111)
\]

We consider the product \( \exp i\alpha \partial X_1 \exp i\beta \bar{\partial} X_1 \), where \( \alpha \) and \( \beta \) are close to zero.

\[
\langle \exp i\alpha \partial X_1 \exp i\beta \bar{\partial} X_1 \exp ibX(x_2) \exp -ibX(x_3) \rangle = -\alpha \beta \langle \partial X_1 \bar{\partial} X_1 \exp ibX(x_2) \exp -ibX(x_3) \rangle \quad (A.4.112)
\]

is the part to linear order in \( \alpha \beta \). The coefficient of the \(-\alpha \beta /2\) term will give us the correlator \((A.4.110)\).

Now,

\[
\langle \exp i\alpha \partial X_1 \exp i\beta \bar{\partial} X_1 \exp ibX(x_2) \exp -ibX(x_3) \rangle = \exp \left[ \frac{1}{4} \int d^2 z d^2 z' \left( \delta(z-z_1)\delta(\bar{z}-\bar{z}_1)\alpha \partial + \delta(z-z_1)\delta(\bar{z}-\bar{z}_1)\beta \bar{\partial} \right. \right.
\]

\[
+ \delta(z-z_1)\delta(\bar{z}-\bar{z}_1)b + \delta(z-z_1)\delta(\bar{z}-\bar{z}_1)(-b) \left( \ln(z-z')(\bar{z}-\bar{z}') \right)
\]

\[
\left. \left( \delta(z'-z_1)\delta(\bar{z}'-\bar{z}_1)\alpha \partial + \delta(z'-z_1)\delta(\bar{z}'-\bar{z}_1)\beta \bar{\partial} \right. \right]
\]

\[
\left. + \delta(z'-z_1)\delta(\bar{z}'-\bar{z}_1)b + \delta(z'-z_1)\delta(\bar{z}'-\bar{z}_1)(-b) \right]\]

For \( b^2 = 4(1 + \delta) \) the expression becomes

\[
(-\alpha \beta /2) - \frac{2b^2}{4} \frac{1}{\sum_{12}^2 z_{13}^2 z_{23}^2} \quad (A.4.115)
\]

Therefore, the generating function from the boundary theory is,

\[
GF_{\sigma_3} = \frac{1}{2!} \frac{-2\xi_0 F^2(1 + \delta)}{(4\pi)^3} A_3 \quad (A.4.116)
\]

Comparing \((A.4.109)\) and \((A.4.117)\),

\[
\lambda_\sigma = \frac{4\sqrt{\pi}}{\Gamma(\Delta_2+\Delta_3-\Delta_1)\Gamma(\Delta_2+\Delta_3-\Delta_1-d)} \left[ \frac{\Gamma(\Delta_2+\Delta_3-\Delta_1-d+2)}{\Gamma(\Delta_2)\Gamma(\Delta_3)} - \frac{\Gamma(\Delta_2+\Delta_3-\Delta_1)}{\Gamma(\Delta_2)\Gamma(\Delta_3+1)} \right] \quad (A.4.117)
\]
Thus the square brackets cancel out in $S_{\sigma3}$. The correlator remains finite on-shell.

**Appendix B**  

The sub-leading term for $\beta_F$ using ERG on the boundary

The action of the evolution operator is

$$\int d^2x_1d^2x_2F_{x_1x_2t} \frac{\delta^2}{\delta X_1\delta X_2} \left[ \frac{\epsilon^{ib_3X_3} + \epsilon^{-ib_3X_3}}{2} \right] \left[ \frac{\epsilon^{ib_2X_4} + \epsilon^{-ib_2X_4}}{2} \right]$$ (B.0.119)

Here $X_i$ means $X(x_i)$. Keeping terms that contribute we get,

$$\int d^2x_1d^2x_2F_{x_1x_2t} \frac{\delta^2}{\delta X_1\delta X_2} \left[ \frac{\epsilon^{-ib_3X_3-bib_2X_4} + \epsilon^{ib_3X_3+ib_2X_4}}{4} \right]$$ (B.0.120)

In the last expression the two terms that conserve momenta have been retained. We look at the action of the evolution operator on the first term. The second term gives an identical contribution.

$$\int d^2x_1d^2x_2F_{x_1x_2t} \frac{\delta^2}{\delta X_1\delta X_2} \left[ \epsilon^{ib_3X_3+ib_2X_4} \right]$$ (B.0.121)

$$= (-b_2^2F_{33t} - b_2b_3F_{34t} - b_2b_3F_{34t} - b_2^2F_{44t})e^{i(b_3+b_2)X_4}$$ (B.0.122)

Here $X_3$ has been taylor expanded and brought to $X_4$. Therefore,

$$\left[ \frac{\epsilon^{ib_3X_3+ib_2X_4} + \epsilon^{-ib_3X_3-ib_2X_4}}{4} \right]$$ (B.0.123)

becomes

$$\frac{1}{2} \cos(b_2 + b_3)X_4 = \frac{1}{2} \cos b_1 X_4$$ (B.0.124)

Substituting (B.0.123) and (B.0.124) in (2.5.12) we get
\[
\begin{align*}
&\frac{GH}{(4\pi)^2} \int \frac{d^2x_1 d^2x_2}{a^4} \exp[-\frac{1}{2}(-b_3^2 - b_2^2) F_{11t} + b_2 b_3 F_{12t}] \\
&\frac{1}{2} \cos b_1 X(x_2)
\end{align*}
\]

where \( F_{12t} = -\frac{1}{2} \ln \frac{(x_1 - x_2)^2 + a(t)^2}{(x_1 - x_2)^2 + a(0)^2} \)

now we relable \( x_2 - x_1 \to y \) and \( x_2 \to x \) we get

\[
\begin{align*}
&\frac{GH}{8} \int \frac{dy^2}{a(t)^2} e^{4t} \frac{y^2 + a(t)^2}{a(0)^2} \left( y^2 + a(t)^2 \right)^{-\frac{b_2 b_3}{2}} \frac{1}{(4\pi)^2} \int \frac{d^2x}{a(t)^2} \cos b_1 X(x) \\
&\text{(B.0.126)}
\end{align*}
\]

\( a(t) \) is the IR cutoff therefore we drop \( y^2 \) from the numerator and integrate.

\[
\begin{align*}
&\frac{GH}{8} e^{4t} \frac{a(t)^2}{a(t)^2 - a(t)^2 + a(0)^2} \left( a(t)^2 + a(0)^2 \right)^{\frac{b_2 b_3}{2} + 1} \frac{a(t)^2 - a(0)^2}{b_2 b_3} \\
&\frac{1}{(4\pi)^2} \int \frac{d^2x}{a(t)^2} \cos b_1 X(x) \\
&\text{(B.0.127)}
\end{align*}
\]

dropping \( a(0)^2 \) from the first term.

\[
\begin{align*}
&\frac{GH}{8} e^{4t} \frac{a(t)^2}{a(t)^2 - a(t)^2 + a(0)^2} \left( a(t)^2 + a(0)^2 \right)^{\frac{b_2 b_3}{2} + 1} \frac{1}{(4\pi)^2} \int \frac{d^2x}{a(t)^2} \cos b_1 X(x) \\
&\text{(B.0.128)}
\end{align*}
\]

For \( b_2, b_3 \) close to \(-2\) and for \( b_2^2 = b_3^2 = 4 \) we get,

\[
\frac{GH}{4} \frac{1}{(4\pi)^2} \int \frac{d^2x}{a(t)^2} \cos b_1 X(x) \\
\text{(B.0.129)}
\]
Appendix C  Position space calculation for $\beta_F$ from the bulk for the sub-leading term.

\[ S_3 = -\lambda_3 \int d^d x_1 \int d^d x_2 \int d^d x_3 \int d^{d+1} y \, y_0^{-(d+1)} C_{\Delta_1} C_{\Delta_2} C_{\Delta_3} \]  
\[ \phi_0 \gamma_0 \chi_0 (y_0^\Delta_{1+\Delta_2+\Delta_3} - (y_0^2 + (\vec{y} - \vec{x}_1)^2)^{\Delta_1 - \Delta_2 - \Delta_3}) \]

After Feynman parameterization we get,

\[ = -\lambda_3 \int d^d x_1 \int d^d x_2 \int d^d x_3 \int d^{d+1} y \, C_{\Delta_1} C_{\Delta_2} C_{\Delta_3} \phi_0 \gamma_0 \chi_0 \]
\[ \int d\alpha_1 d\alpha_2 d\alpha_3 \alpha_1^{\Delta_1 - 1} \alpha_2^{\Delta_2 - 1} \alpha_3^{\Delta_3 - 1} \delta(\sum_{i=1}^3 \alpha_i - 1) \frac{\Gamma(\sum_{i=1}^3 \Delta_i)}{\Pi_{i=1}^3 \Gamma(\Delta_i)} \]
\[ (y_0^2 + (\vec{y} - \sum_{i=1}^3 \alpha_i \vec{x}_i)^2 + \sum_{i<j}^3 \alpha_i \alpha_j (\vec{x}_i - \vec{x}_j)^2)^{\Delta_1 + \Delta_2 + \Delta_3} \]

We do the $y_0$ and $\vec{y}$ integrals

\[ S_3 = -\lambda_3 \frac{\pi^{d/2} \Gamma\left(\frac{\sum_{i=1}^3 \Delta_i}{2}\right) \Gamma\left(\frac{\sum_{i=1}^3 \Delta_i - d}{2}\right)}{2 \Pi_{i=1}^3 \Gamma(\Delta_i)} \int d^d x_1 \int d^d x_2 \int d^d x_3 \int d^{d+1} y \, C_{\Delta_1} C_{\Delta_2} C_{\Delta_3} \phi_0 \gamma_0 \chi_0 \]  
\[ \int d\alpha_1 d\alpha_2 d\alpha_3 \frac{\alpha_1^{\Delta_1 - 1} \alpha_2^{\Delta_2 - 1} \alpha_3^{\Delta_3 - 1} \delta(\sum_{i=1}^3 \alpha_i - 1)}{(\sum_{i<j}^3 \alpha_i \alpha_j (\vec{x}_i - \vec{x}_j)^2)^{\Delta_1 + \Delta_2 + \Delta_3}} \]

Now we transform from $\alpha_i$'s to $\beta_i$'s. $\alpha_i = \beta_1 \beta_i$ for $i \geq 2$, $\alpha_1 = \beta_1$. The Jacobian for $n$ parameters is $\beta_1^{n-1}$. Here $n = 3$.

\[ S_3 = -\lambda_3 \frac{\pi^{d/2} \Gamma\left(\frac{\sum_{i=1}^3 \Delta_i}{2}\right) \Gamma\left(\frac{\sum_{i=1}^3 \Delta_i - d}{2}\right)}{2 \Pi_{i=1}^3 \Gamma(\Delta_i)} \int d^d x_1 \int d^d x_2 \int d^d x_3 \int d^{d+1} y \, C_{\Delta_1} C_{\Delta_2} C_{\Delta_3} \phi_0 \gamma_0 \chi_0 \]  
\[ \int d\beta_1 d\beta_2 d\beta_3 \frac{\beta_2^{\Delta_2 - 1} \beta_3^{\Delta_3 - 1} \beta_1^{\Delta_1 - 1} [\delta(\beta_1 - 1/(1 + \beta_2 + \beta_3))] [(1 + \beta_2 + \beta_3)]}{\beta_1 (\beta_2 x_{12}^2 + \beta_3 x_{13}^2 + \beta_2 \beta_3 x_{23}^2)^{\Delta_1 + \Delta_2 + \Delta_3}} \]
After doing the $\beta_1$ integral we get

$$S_3 = -\lambda_3 \pi^{d/2} \Gamma \left( \frac{\Delta_1}{2} \right) \Gamma \left( \frac{\Delta_1 - d}{2} \right) \frac{d}{2 \Gamma \left( \sum_{i=1}^{3} \Delta_i \right)} \int d^d x_1 \int d^d x_2 \int d^d x_3 \ C_{\Delta_1} C_{\Delta_2} C_{\Delta_3} \phi_0 \gamma_0 \chi_0$$

(C.0.134)

$$\int d\beta_2 d\beta_3 \frac{\beta_2^{\Delta_2-1} \beta_3^{\Delta_3-1}}{\beta_2 x_{12}^2 + \beta_3 x_{13}^2 + \beta_2 \beta_3 x_{23}^2}$$

$$S_3 = -\lambda_3 \pi^{d/2} \Gamma \left( \frac{\Delta_1}{2} \right) \Gamma \left( \frac{\Delta_1 - d}{2} \right) \frac{d}{2 \Gamma \left( \sum_{i=1}^{3} \Delta_i \right)} \int d^d x_1 \int d^d x_2 \int d^d x_3 \ C_{\Delta_1} C_{\Delta_2} C_{\Delta_3} \phi_0 \gamma_0 \chi_0$$

(C.0.135)

$$\int d\beta_2 d\beta_3 \frac{\beta_2^{\Delta_2-1} \beta_3^{\Delta_3-1}}{\beta_2 (x_0^2 + x_{12}^2) + \beta_3 (x_0^2 + x_{13}^2) + \beta_2 \beta_3 (x_0^2 + x_{23}^2)}$$

Here we have introduced $x_0^2$’s in the denominator ($x_0^2 \to 0$). These act as UV regulators. We do the $\beta_2$ and $\beta_3$ integrals. Any factors of $\delta$ coming from the two beta functions from the two beta integrals contribute at $O(\delta \phi_0 \gamma_0 \chi_0)$. Therefore they are dropped. We set $d = 2$ and $\Delta_2 = 2(1 + \delta)$, particle 2 is offshell. We substitute $C_{\Delta_i}$’s. We set $\vec{x}_1$ to zero using translation invariance, multiply and divide by $R$, the IR cut-off. Therefore, the integral simplifies to

$$S_3 = -\frac{\lambda_3}{2\pi^2} \int \frac{d^d x_1 d^d x_2 d^d x_3}{R^6} \frac{\phi_0 \gamma_0 \chi_0}{(x_0^2 + x_{12}^2)^{1+\delta} (x_0^2 + x_{13}^2)^{1-\delta} (x_0^2 + x_{23}^2)^{1+\delta}}$$

(C.0.136)

We will now calculate the log divergent term. The $x_2$ integral is,

$$\int \frac{d^d x_2}{R^2 (x_0^2 + x_{12}^2)^{1+\delta} (x_0^2 + x_{13}^2)^{1-\delta} (x_0^2 + x_{23}^2)^{1+\delta}}$$

(C.0.137)

The log divergent contributions come from the two regions, when (i) $\vec{x}_2 \to \vec{x}_3$, (ii) $\vec{x}_2 \to 0$.

(i) $\vec{x}_2 \to \vec{x}_3$.

Set $\vec{y} = \vec{x}_2 - \vec{x}_3$. At $\vec{x}_2 = \vec{x}_3$, $\vec{y} = 0$. 

31
\[ \int \frac{d^2y}{R^2} \frac{1}{\left( \frac{x_0^2 + (\bar{y} + x_3)^2}{R^2} \right)^{(1+\delta)}} \left( \frac{x_0^2 + x_3^2}{R^2} \right)^{(1-\delta)} \left( \frac{x_0^2 + y^2}{R^2} \right)^{(1+\delta)} \]  

(C.0.138)

We taylor expand the first term in the denominator. We get,

\[ \int \frac{d^2y}{R^2} \frac{1 - (1 + \delta) \left( \frac{\bar{y}^2 - 2\bar{x}_3 \bar{y}}{x_0^2 + x_3^2} \right)}{\left( \frac{x_0^2 + x_3^2}{R^2} \right)^{(1+\delta)}} \left( \frac{x_0^2 + y^2}{R^2} \right)^{(1+\delta)} \]  

(C.0.139)

We drop the \( \delta \) term. It is higher order. We look at,

\[ \int \frac{d^2y}{R^2} \frac{- \left( \frac{\bar{y}^2 - 2\bar{x}_3 \bar{y}}{x_0^2 + x_3^2} \right)}{\left( \frac{x_0^2 + x_3^2}{R^2} \right)^{(1+\delta)}} \left( \frac{x_0^2 + y^2}{R^2} \right)^{(1+\delta)} \]  

(C.0.140)

Add and subtract \( x_0^2 \).

\[ \int \frac{d^2y}{R^2} \frac{- \left( \frac{x_0^2 + \bar{y}^2 - x_0^2 - 2\bar{x}_3 \bar{y}}{x_0^2 + x_3^2} \right)}{\left( \frac{x_0^2 + x_3^2}{R^2} \right)^{(1+\delta)}} \left( \frac{x_0^2 + y^2}{R^2} \right)^{(1+\delta)} \]  

(C.0.141)

The \( x_0^2 + \bar{y}^2 \) term cancels in the numerator and the denominator. We drop that.

\[ \int \frac{d^2y}{R^2} \frac{- \left( \frac{-x_0^2 - 2\bar{x}_3 \bar{y}}{x_0^2 + x_3^2} \right)}{\left( \frac{x_0^2 + x_3^2}{R^2} \right)^{(1+\delta)}} \left( \frac{x_0^2 + y^2}{R^2} \right)^{(1+\delta)} \]  

(C.0.142)

\[ \int d^2y (-2\bar{x}_3 \bar{y}) = -2 \int_0^{2\pi} ydyd\theta_3y x_3y \cos \theta_3y = 0 \]  

(C.0.143)

Therefore we drop this term. We get

\[ \int \frac{d^2y}{R^2} \frac{- \left( \frac{-x_0^2}{x_0^2 + x_3^2} \right)}{\left( \frac{x_0^2 + x_3^2}{R^2} \right)^{(1+\delta)}} \left( \frac{x_0^2 + y^2}{R^2} \right)^{(1+\delta)} \]  

(C.0.144)
which in the limit \( x_0 \to 0 \) goes to zero.
We look at

\[
\left( \frac{x_0^2 + y^2}{R^2} \right)^{(1+\delta)} = \left( \frac{x_0^2 + y^2}{R^2} \right) \left( 1 + \delta \log \left( \frac{x_0^2 + y^2}{R^2} \right) \right)
\]

in the denominator. Again drop the \( \delta \) term. Set \( \vec{x}_3 = \vec{R} \),

\[
\int \frac{d^2 y}{R^2} \frac{1}{\left( \frac{x_0^2 + R^2}{R^2} \right)^{(1+\delta)}} \left( \frac{x_0^2 + R^2}{R^2} \right)^{(1-\delta)} \left( \frac{x_0^2 + y^2}{R^2} \right)^{(1+\delta)} = \pi \int \frac{d y^2}{x_0^2} \frac{1}{\left( \frac{x_0^2 + y^2}{R^2} \right)^{(1+\delta)}}
\]

The log divergent part is

\[
\pi \int \frac{d y^2}{x_0^2} \frac{1}{\left( \frac{x_0^2 + y^2}{R^2} \right)^{(1+\delta)}} = \pi \log R^2 \frac{R}{x_0^2}
\]

A similar computation for \( \vec{x}_2 \to 0 \) gives an identical contribution. The total contribution from both regions is,

\[
= 2\pi \log R^2 \frac{R}{x_0^2}
\]

The partition function becomes

\[
S_3 = -\frac{1}{2} 4\lambda_3 \pi \phi_0 \gamma_0 \chi_0 \int d \left( \frac{x_1^2}{R^2} \right) d \left( \frac{x_3^2}{R^2} \right) \log R \frac{R}{x_0}
\]

This expression corrects \( b_1 \) when \( x_2 \to x_3 \) and \( b_3 \) when \( x_2 \to x_1 \). These are both equal in magnitude. We only want the correction to \( b_1 \), therefore we divide the expression above by 2 to get the contribution of the generating functional to the beta function for \( \cos b_1 X \).

\[
S_3 = -\lambda_3 \pi \phi_0 \gamma_0 \chi_0 \int d \left( \frac{x_1^2}{R^2} \right) d \left( \frac{x_3^2}{R^2} \right) \log R \frac{R}{x_0}
\]
Appendix D  Calculation for $\beta_\delta$.

$$S_{\sigma 3} = -\frac{1}{2} \lambda_\sigma \int d^{d+1}y \sqrt{g} \bar{g}^\mu\nu \partial_\mu \phi(y) \partial_\nu \phi(y) \sigma(y)$$  \hspace{1cm} (D.0.152)

$$= -\frac{1}{2} \lambda_\sigma \sigma_0 \phi_0^2 C_{\Delta_1} C_{\Delta_2} C_{\Delta_3} \int d^{d+1}y \sqrt{g} \ y_0^2 \partial_\mu \left( \frac{y_0}{(y_0^2 + (y - x_1^2))^2} \right) \Delta_1$$  \hspace{1cm} (D.0.153)

$$\partial^\mu \left( \frac{y_0}{(y_0^2 + (y - x_2^2))^2} \right) \Delta_2 \left( \frac{y_0}{(y_0^2 + (y - x_3^2))^2} \right) \Delta_3$$

Set $\bar{x}_1 = 0$. Under inversion [9],

$$\frac{y_0}{y_0^2 + (\bar{x} - y)^2} \rightarrow \frac{y_0'}{y_0'^2 + (x' - y')^2}$$  \hspace{1cm} (D.0.154)

$$\partial^\mu y_0' \Delta = \partial^0 y_0' \Delta = \Delta y_0' \Delta^{-1}.$$

Thus, setting $\bar{x}_1$ to zero and using inversion we have reduced the number of factors in the denominator from 3 to 2. This simplifies Feynman parameter integrals significantly. Now we Feynman parameterize and do $y'$ integrals. Doing the integrals and dropping pre-factors we get,
\[
\frac{\pi}{2} \frac{\Gamma(\Delta_2+\Delta_3-\Delta_1) \Gamma(\Delta_2+\Delta_3+\Delta_1-\frac{d}{2})}{\Gamma(\Delta_2) \Gamma(\Delta_3)} \Gamma(\Delta_3+\Delta_2) \]  
(D.0.156)

\[
\int d\alpha_3 \, d\alpha_2 \, \delta(\alpha_3 + \alpha_2 - 1) \frac{1}{\alpha_2^{\Delta_3-1} \alpha_2^{\Delta_2-1} (\alpha_2 \alpha_3 x_{23}^2)^{\Delta_3+\Delta_2-\Delta_1}} = 1 \quad (D.0.157)
\]

The second,

\[
\int d\alpha_3 \, d\alpha_2 \, \delta(\alpha_3 + \alpha_2 - 1) \frac{1}{\alpha_3^{\Delta_3} \alpha_2^{\Delta_2} (\alpha_2 \alpha_3 x_{23}^2)^{\Delta_3+\Delta_2-\Delta_1}} = 1/2 \quad (D.0.158)
\]

Use \((x^2)^{2\Delta} = 1/x^{2\Delta}\) and
\((x^2-y^2)^{2(\Delta_2+\Delta_3-\Delta_1)/2} = (x^2-y^2)^{2(\Delta_1+\Delta_2-\Delta_3)/2} / (x^2(\Delta_1+\Delta_2-\Delta_3)/2)^{2(\Delta_1+\Delta_2-\Delta_3)/2}\).

\[
S_{\sigma^3} = -\frac{\pi^3}{4} \lambda \sigma_0^2 \phi_0^2 C_\Delta C_{\Delta_2} C_{\Delta_3} \Delta_1 \Delta_2 \int d^2x_1 d^2x_2 d^2x_3 \quad (D.0.159)
\]

\[
\left[ \frac{\Gamma(\Delta_2+\Delta_3-\Delta_1) \Gamma(\Delta_2+\Delta_3+\Delta_1-\frac{d}{2})}{\Gamma(\Delta_2) \Gamma(\Delta_3)} \right] - \left[ \frac{\Gamma(\Delta_2+\Delta_3-\Delta_1) \Gamma(\Delta_2+\Delta_3+\Delta_1-\frac{d+2}{2})}{\Gamma(\Delta_2) \Gamma(\Delta_3+1)} \right]
\]

Where now we have explicitly written integrals over the boundary coordinates (which were suppressed earlier). We insert a UV cutoff \(x_0^2\) as before and multiply and divide by powers of \(R^2\), we get.

35
\[ S_{\sigma_3} = -\frac{\pi}{4} \lambda \sigma_0 \phi_0^2 \Delta_1 \Delta_2 \Delta_2 \Delta_1 \int \frac{d x^1}{R^2} \frac{d x^2}{R^2} \frac{d x^3}{R^2} \] (D.0.160)

\[
\left( \frac{x^2 + x^3}{R^2} \right)^{\Delta_2 + \Delta_3 - \Delta_1} \left( \frac{x^1 + x^2}{R^2} \right)^{\Delta_2 - \Delta_1 + \Delta_3} \left( \frac{x^1 + x^3}{R^2} \right)^{-\Delta_2 + \Delta_1 + \Delta_3} \left[ \frac{\Gamma(\Delta_2 + \Delta_3 - \Delta_1) \Gamma(\Delta_2 + \Delta_3 + \Delta_1 - d)}{\Gamma(\Delta_2) \Gamma(\Delta_3)} - \frac{\Gamma(\Delta_2 + \Delta_3 - \Delta_1) \Gamma(\Delta_2 + \Delta_3 + \Delta_1 - d + 2)}{\Gamma(\Delta_2) \Gamma(\Delta_3 + 1)} \right] \]

\[ C_{\Delta_0} = 1/\pi. \] The square bracket vanishes. This gets renormalized when we fix \( \lambda \sigma_0 \). We have taken particle 2 offshell. Therefore \( \Delta_2 = 2(1+\delta) \), \( \Delta_1 = 2 \).

We get the same \( x_2 \) integral as before (C.0.137). The contribution from the \( x_2 \) integral from before is

\[ 4\pi \log \frac{R}{x_0} \] (D.0.161)

The contribution to the generating function is half of this as before.

\[ 2\pi \log \frac{R}{x_0} \] (D.0.162)

Therefore \( S_{\sigma_3} \) becomes

\[
S_{\sigma_3} = -2\pi(1+\delta) \lambda \sigma_0 \phi_0^2 \int d \left( \frac{x^1}{R^2} \right) d \left( \frac{x^2}{R^2} \right) log \frac{R}{x_0} \] (D.0.163)

\[
\left[ \frac{\Gamma(\Delta_2 + \Delta_3 - \Delta_1) \Gamma(\Delta_2 + \Delta_3 + \Delta_1 - d)}{\Gamma(\Delta_2) \Gamma(\Delta_3)} - \frac{\Gamma(\Delta_2 + \Delta_3 - \Delta_1) \Gamma(\Delta_2 + \Delta_3 + \Delta_1 - d + 2)}{\Gamma(\Delta_2) \Gamma(\Delta_3 + 1)} \right] \]

**Future directions**

We would like to compute the aforementioned four point Witten diagrams and reproduce the beta functions for the Sine-Gordon model.

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