Additive consistency of risk measures and its application to risk-averse routing in networks

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Abstract

This paper investigates the use of risk measures and theories of choice for modeling risk-averse route choice and traffic network equilibrium with random travel times. We interpret the postulates of these theories in the context of routing, and we identify additive consistency as a plausible and relevant condition that allows to reduce risk-averse route choice to a standard shortest path problem. Within the classical theories of choice under risk, we show that the only preferences that satisfy this consistency property are the ones induced by the entropic risk measures.

1 Introduction

Drivers are aware that travel time cannot be reliably predicted and is subject to random fluctuations arising from a multitude of factors such as congestion, weather conditions, accidents and traffic incidents, bottlenecks, traffic light disruptions, unexpected actions by pedestrians and other drivers, and so on. Even on a specific road segment at a specific time of the day, travel time exhibits a stochastic pattern that can be roughly approximated by the

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log-normal or Burr distributions \cite{18,49}. Thus, choosing a route to travel from a given origin to a destination is essentially a matter of comparing random variables. A basic question here is to understand the mechanisms by which these choices are made. While this calls for modeling the actual behavior of drivers, it can also be approached from a normative angle by asking which are the properties that characterize a rational route choice under risk. A related issue is to understand the consequences of risk-averse behavior upon congestion and the traffic equilibrium that is obtained. Answering these questions may change the way in which we model traffic and can be relevant for network design and traffic control.

Route preferences vary among individuals and also depending on trip purpose. Compare for instance a situation in which you must arrive on time to an important meeting, with that of a tourist strolling leisurely through the city, or still a fire truck heading towards an emergency. While a risk neutral driver may only care about the expected travel time, a risk-averse user will be more concerned with travel time reliability. Modeling such variety of behaviors has been approached with different tools. Mean-risk models—with risk quantified by the expected value plus the standard deviation—were considered by Nikolova and Stier-Moses \cite{35} to study both atomic and non-atomic equilibria. An algorithm to compute mean-stdev optimal paths was given by Nikolova \textit{et al.} \cite{33,34}. Route choice using $\alpha$-percentiles was investigated by Ordoñez and Stier-Moses \cite{39} and Nie \cite{31}, the former considering also an approach using robust optimization. Yet another proposal by Nie and Wu \cite{32} uses preferences based on the on-time arrival probability. An algorithm for this objective function was also given by Nikolova \textit{et al.} \cite{34}. Finally, Nie \textit{et al.} \cite{30,53} develop a model that uses stochastic dominance constraints. For a more detailed account of these and other relevant references we refer to \cite{35} and to the literature review included in \cite{35}.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{network.png}
\caption{A paradoxical route choice}
\end{figure}

In all the approaches just mentioned it may happen that a risk $X$ is preferred to $Y$ but the preference is reversed when we add an independent risk $Z$. In the simple network illustrated in figure 1 this means that if we go from $s$ to $j$ our best choice is the upper link, but if we extend our trip to $d$ then we must change our choice to the lower link. This may appear as paradoxical. For a concrete example, consider the mean-stdev map $\rho_{st}^{\text{std}}(X) = \mathbb{E}(X) + \gamma \sigma(X)$.
with $\gamma = 1$ and independent normal variables $X \sim N(11, 1)$, $Y \sim N(10, 5)$, $Z \sim N(10, 2)$, where $\rho_{\gamma}^{\text{std}}(X) = 12 < \rho_{\gamma}^{\text{std}}(Y) = 10 + \sqrt{5}$, but $\rho_{\gamma}^{\text{std}}(X + Z) = 21 + \sqrt{3} > \rho_{\gamma}^{\text{std}}(Y + Z) = 20 + \sqrt{7}$. In this paper we use the theories of choice and risk measures to characterize the so-called additive consistent preferences that are free from these paradoxes. We prove that these are exactly the preferences associated with the entropic risk measures.

The general theory of choice under risk is a well established field with a long history. In this setting, an agent is described by a preference relation over a set of random variables (or their distributions). Under suitable conditions these preferences can be represented by a scalar function. Representations by expected utilities were already considered by Bernoulli [5] and further developed by Kolmogorov [28], Nagumo [29], de Finetti [8], and von Neumann and Morgenstern [51] (see also [14, Fishburn]). Expected utilities were used by Arrow [2] and Pratt [41] to define a local index of absolute risk aversion that reflects the risk attitudes of an agent. However, empirical evidence shows that agents do not always conform to the postulates of expected utility theory, and the crucial independence axiom is sometimes violated (see [1, Allais], [12, Ellsberg], [25, Kahneman and Tversky]). By modifying the independence axiom, several alternative representations have emerged: the dual theory of choice by Yaari [54], the anticipated utility theory by Quiggin [42], the rank-dependent expected utility theory by Wakker [52] and Chateauneuf [6], and Schmeidler’s approach [47] based on subjective probabilities.

On the other hand, the extensive use of Value-at-Risk in finance gave birth to the notion of risk measure as an alternative tool for studying choice under risk. The axiomatic approach to risk measures was initiated by Artzner et al. [3], who also introduced the Average Value-at-Risk as a coherent risk measure that overcomes some limitations of Value-at-Risk. Mean-risk functionals have also been considered in this context, notably by Ogryczak and Ruszczynski [37] who studied a risk measure that combines the expected value and the standard semi-deviations. To some extent, risk measures can be unified with the theories of choice through the concept of premium principles (see Gerber [19], Goovaerts et al. [20, 21], Denut et al. [10], and Tsanakas and Desli [50]). A recent account of theories of choice and risk measures can be found in the book by Föllmer and Schied [17].

**Our contribution:** In this paper we investigate the use of risk measures and theories of choice to model risk-averse routing. The interpretation of the postulates in this context leads us to identify additive consistency as a plausible and relevant condition that extends the notion of translation invariance and reduces risk-averse route choice to a standard shortest path problem. We briefly discuss how this allows to formulate risk-averse equilibrium models for atomic and non-atomic network flows, which naturally fit
in the framework of congestion games. We then investigate additive consistency in some standard settings of theories of choice proving that, within the classes of distorted risk measures as well as rank dependent utilities, the only maps that satisfy additive consistency are the entropic risk measures. We also show that these are the only expected utility maps that are translation invariant, hence the only risk measures in this class. These results extend Gerber [19], Goovaerts et al. [22], Heilpern [23], and Luan [27].

Structure of the paper: In section §2 we recall the postulates of risk measures and their induced preferences, interpreting them in the context of route choice. We introduce the concept of additive consistency and discuss its application to risk-averse path choice and network equilibrium. In §3 we consider consecutively the classes of expected utility maps (§3.1), distorted risk measures (§3.2), and rank dependent utilities (§3.3), proving that within each of these classes the entropic risk measures are the only ones that satisfy translation invariance and/or additive consistency. In §4 we make some remarks on the use of dynamic risk measures as an alternative to model route choice, and we conclude in §5 with a brief discussion of related work.

2 Risk measures and additive consistency

Quantifying risk is an essential yet difficult task. Because of the subjective nature of risk perception, defining an appropriate measure remains controversial and several approaches have been proposed each one with its own advantages and limitations. A risk quantification attaches a scalar value to each random variable $X : \Omega \to \mathbb{R}$, where $\Omega$ is a set of events endowed with a $\sigma$-algebra $\mathcal{F}$ and a probability measure $\mathbb{P}$. More precisely, a risk measure is a map $\rho : \mathcal{X} \to \mathbb{R}$ defined over a prospect space $\mathcal{X}$ (a linear space of random variables containing the constants, usually a subspace of $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$) which satisfies the following postulates:

- **Normalization:** $\rho(0) = 0$,
- **Monotonicity:** if $X \leq Y$ almost surely then $\rho(X) \leq \rho(Y)$,
- **Translation Invariance:** $\rho(X + m) = \rho(X) + m$ for all $m \in \mathbb{R}$.

Such a map induces a preference relation $X \preceq Y \iff \rho(X) \leq \rho(Y)$ which defines a complete order. In this paper prospects are interpreted as costs or disutilities so that smaller values are preferred and, against common usage, $X \preceq Y$ is read as “$X$ is preferred to $Y$”. Naturally, the normal convention applies if $X$ represents a utility and larger values are better. We use $X \prec Y$ to denote strict preference and we write $X \sim Y$ when simultaneously $X \preceq Y$ and $Y \preceq X$. 
The normalization axiom is not restrictive as one can always take \( \rho(X) - \rho(0) \) instead of \( \rho(X) \). Monotonicity has a clear intuitive meaning: larger costs convey higher risk. In the context of routing, paths with larger travel times are riskier and less preferred. Translation invariance is equivalent (under normalization) to requiring simultaneously

- **NORMALIZATION ON CONSTANTS:** \( \rho(m) = m \) for all \( m \in \mathbb{R} \).
- **TRANSLATION CONSISTENCY:** \( \rho(X) \leq \rho(Y) \Rightarrow \rho(X + m) \leq \rho(Y + m) \).

The latter is a plausible condition stating that preferences between prospects are not altered when we add them a constant. While this postulate is not universally accepted in finance (attitudes towards risk might change after receiving a heritage), it seems very likely in the context of route choice (see §2.2). Finally, normalization on constants is also a mild requirement: it suffices to have \( m \mapsto \rho(m) \) strictly increasing and continuous, since then this function has an inverse \( \sigma \) and we may substitute \( \rho \) by \( \sigma \circ \rho \).

The axiomatic approach to risk measures was initiated by [3, Artzner et al.] who introduced the notion of a coherent risk measure, namely, a risk measure which is also sub-additive and positively homogeneous. Positive homogeneity translates the notion of scale invariance, while sub-additivity captures the idea that a merger of two risks cannot create additional risk. The validity of these axioms in finance has been thoroughly debated in the literature. In the context of route choice these assumptions seem less natural, specially positive homogeneity. A weaker property is convexity which still supports a useful dual representation for risk measures [15, 16, 17].

Preferences can also be modeled directly as a preorder, namely a reflexive and transitive relation \( \preceq \). We then say that \( \preceq \) is represented by \( \rho : \mathcal{X} \rightarrow \mathbb{R} \) if \( X \preceq Y \Leftrightarrow \rho(X) \leq \rho(Y) \). The following simple result clarifies when both modeling approaches coincide.

**Proposition 2.1.** Suppose \( \preceq \) is a preorder that satisfies

- **MONOTONICITY:** if \( X \preceq Y \) almost surely then \( X \preceq Y \),
- **TRANSLATION CONSISTENCY:** if \( X \preceq Y \) then \( X + m \preceq Y + m \) for \( m \in \mathbb{R} \),
- **REAL ORDERING:** for \( X, Y \) constant we have \( X \preceq Y \Leftrightarrow X \leq Y \),
- **SCALARIZATION:** for each \( X \in \mathcal{X} \) there is a unique \( \alpha \in \mathbb{R} \) with \( X \sim \alpha \).

Then the map \( X \mapsto \rho(X) = \alpha \) defined by the last condition is a risk measure and \( \preceq \) is represented by \( \rho \).

**Proof.** Reflexivity of \( \preceq \) gives \( X \sim X \) so that for \( X \equiv m \) constant we get \( \rho(m) = m \) which shows that \( \rho \) is normalized on constants. Next, by definition we have \( X \sim \rho(X) \) and translation consistency gives \( X + m \sim \rho(X) + m \)
so that $\rho(X + m) = \rho(X) + m$ proving the translation invariance of $\rho$. Since the monotonicity of $\preceq$ readily implies the monotonicity of $\rho$, it follows that $\rho$ is a risk measure. It remains to establish the representation property. By definition we have $X \sim \rho(X)$ and $Y \sim \rho(Y)$ so that transitivity gives $X \preceq Y$ iff $\rho(X) \preceq \rho(Y)$. According to the real ordering axiom, the latter is equivalent to $\rho(X) \leq \rho(Y)$.

As a corollary to Proposition 2.1 the preorder $\preceq$ must be complete, that is, all pairs are comparable. This also follows directly from the real ordering and scalarization axioms. Scalarization is a non-trivial condition. In section §3 we will revise this postulate in the light of the theories of choice.

2.1 Examples and counterexamples of risk measures

A first attempt to quantify risk was given in [28, Markowitz] by considering the mean-risk functional

$$\rho^{\text{var}}_{\gamma}(X) = \mu(X) + \gamma \sigma^2(X)$$

with $\mu(X)$ the mean of $X$, $\sigma^2(X)$ its variance, and $\gamma > 0$ a positive constant. Variations of this idea substitute the variance by the standard deviation

$$\rho^{\text{std}}_{\gamma}(X) = \mu(X) + \gamma \sigma(X)$$

or other variability measures such as the absolute semi-deviations [37, 38]. While these maps satisfy normalization and translation invariance, they are not risk measures since monotonicity might fail: take $X \sim U[0, 1]$ a uniform variable and $Y = (1 + X)/2$ so that $X \preceq Y$ almost surely, yet for $\gamma$ large we have $\rho^{\text{var}}_{\gamma}(Y) < \rho^{\text{var}}_{\gamma}(X)$ and the same for $\rho^{\text{std}}_{\gamma}$.

A popular measure is Value-at-Risk defined for $p \in (0, 1)$ as the percentile

$$\text{VaR}_p(X) = \inf \{m \in \mathbb{R} : P(X \leq m) \geq 1 - p\}.$$  

This is a risk measure which is also positively homogeneous, but not convex nor sub-additive (see [3]). The best known coherent risk measure is Average Value-at-Risk, introduced in [3] and defined for a level $p \in (0, 1)$ by

$$\text{AVaR}_p(X) = \frac{1}{p} \int_0^p \text{VaR}_q(X) dq,$$

which also has the following useful dual representation (cf. 15, 16, 43, 44)

$$\text{AVaR}_p(X) = \frac{1}{p} \inf_{z \in \mathbb{R}} \{\mathbb{E}((X - z)_+) + pz\}.$$
AVaR is also known by the names of Conditional Value-at-Risk, Tail Value-at-Risk, and Expected Shortfall. For continuous variables it coincides with the Tail Conditional Expectation
\[
\text{TCE}_p(X) = \mathbb{E}(X|X \geq \text{VaR}_p(X)).
\]
Note that when restricted to normal random variables both VaR$_p$ and AVaR$_p$ coincide with $\rho^\text{std}_\gamma$ for appropriate corresponding constants $\gamma$.

A family of convex (but not coherent) risk measures are the entropic measures defined as (cf. [15, 16, 17, 46])
\[
\rho^\text{ent}_\beta(X) = \frac{1}{\beta} \ln(\mathbb{E}(e^{\beta X})).
\]
These measures play a central role in our results. They can be derived from additive premium principles [19, Gerber], as well as from expected utilities with constant absolute risk aversion CARA ([2, Arrow], [41, Pratt]). The case $\beta > 0$ characterizes risk-averse behavior while $\beta < 0$ corresponds to a risk-prone agent. The limit $\beta \to 0$ gives $\rho^\text{ent}_0(X) = \mathbb{E}(X)$ which reflects risk neutrality, while $\beta \to \pm \infty$ yields extreme attitudes toward risk with $\rho^\text{ent}_\infty(X) = \text{ess sup } X$ and $\rho^\text{ent}_{-\infty}(X) = \text{ess inf } X$. In the sequel we only consider finite $\beta$’s and exclude the last two.

2.2 Risk measures and consistency in route choice

Consider a driver who must choose one among a finite set of routes, each of which has a random travel time in a suitable prospect space $X$. While a risk-neutral driver may prefer the route with smallest expected time (easily computed by any shortest path algorithm), a risk-averse user might be willing to trade some expected value against increased reliability.

We assume that the driver preferences $\preceq$ satisfy the axioms in Proposition 2.1, so that route choice is based on a risk measure $\rho : X \to \mathbb{R}$. As already mentioned, the scalarization postulate is a nontrivial assumption which will be discussed later. In contrast, the axioms of monotonicity and real ordering seem quite innocuous, while translation consistency is also very plausible in this context. Namely, consider the simple network illustrated in figure 2 with two paths from $s$ to $j$ with random times $X$ and $Y$, followed by a single path from $j$ to $d$ with constant time $Z \equiv m$. Translation consistency simply requires that a driver who prefers $X$ to $Y$ for moving from $s$ to $j$, should have the same preference when heading towards $d$.

A stronger consistency property requires the preservation of preferences when $Z$ is no longer constant but still independent from $X$ and $Y$, namely,
Figure 2: Translation invariance and additive consistency

if \( X \sqsubseteq Y \) then \( X + Z \sqsubseteq Y + Z \) for all \( Z \perp (X, Y) \). Intuitively, since the arc \((j, d)\) is compulsory and one must inevitably pass through it, the decision at \( s \) should not depend on \( Z \). This seems all the more plausible since, due to the independence, even if one observes \( Z \) this reveals no information that could affect the choice between \( X \) and \( Y \). This motivates the following definition.

**Definition 2.1 (Additive consistency)**. A map \( \rho : \mathcal{X} \to \mathbb{R} \) is called **additive consistent** if for all \( X, Y, Z \in \mathcal{X} \) with \( Z \perp (X, Y) \) we have

\[
\rho(X) \leq \rho(Y) \implies \rho(X + Z) \leq \rho(Y + Z).
\]

For risk measures this is equivalent to an apparently stronger requirement of additivity for sums of independent risks. The proof is elementary.

**Lemma 2.1.** Let \( \rho : \mathcal{X} \to \mathbb{R} \) be a risk measure. Then \( \rho \) is additive consistent if and only if \( \rho(X + Y) = \rho(X) + \rho(Y) \) for all \( X, Y \in \mathcal{X} \) with \( X \perp Y \). A map satisfying the latter is called **additive**.

**Proof.** The “if” part is obvious so we just prove the “only if”. Let \( X \perp Y \). Since \( X \sim \rho(X) \), from additive consistency we get \( X + Y \sim \rho(X) + Y \). Hence \( \rho(X + Y) = \rho(\rho(X) + Y) \) and the translation invariance of \( \rho \) yields \( \rho(X + Y) = \rho(X) + \rho(Y) \).

It is well known that the entropic risk measures \( \rho_\beta^{ent} \) are additive and hence additive consistent. The counterexample in the Introduction (see figure 1) shows that this is not the case for \( \rho_\gamma^{std} \). The example was for \( \gamma = 1 \) but it can be readily adapted to any \( \gamma > 0 \). Thus, in general \( \rho_\gamma^{std} \) is not additive consistent, and *a fortiori* neither VaR \( \rho_p \) nor AVaR \( \rho_p \) since they coincide with \( \rho_\gamma^{std} \) for normal variables. In section §3 we show that, among a wide class of risk measures, the entropic ones are the only that are additively consistent.

**Remark.** The use of normal distributions in the counterexample in the Introduction could raise some objections since these are unbounded and have positive mass on the negative reals, so they might not represent travel times.
However, the example is robust and can be modified to get distributions with bounded support on $\mathbb{R}_+$: it suffices to shift the variables by a large common constant so that the mass on $\mathbb{R}_-$ becomes negligible, and then truncate to a large interval $[0, M]$ and take conditional distributions.

### 2.3 Application to risk-averse network equilibrium

Additive consistency is a plausible assumption with interesting consequences for the computation of risk-minimizing routes and risk-averse network equilibrium. Consider a network $G = (V, A)$ in which every link $a \in A$ has a random travel time $\tau_a$ and assume that these variables are independent. Let $P$ be the set of paths connecting a given origin $s$ to a destination $d$, and for each $p \in P$ denote $T_p = \sum_{a \in p} \tau_a$ the corresponding travel time. Given a risk measure $\rho$ we consider the problem of finding a risk-minimizing path

$$\min_{p \in P} \rho(T_p). \quad (2.1)$$

When $\rho$ is additive the objective function separates as $\rho(T_p) = \sum_{a \in p} \rho(\tau_a)$ and (2.1) reduces to a standard shortest path problem with arc lengths $w_a = \rho(\tau_a)$. This can be efficiently solved using standard algorithms.

Consider now a non-atomic equilibrium problem with traffic demands $g_k \geq 0$ for a family of origin-destination pairs $(s_k, d_k)_{k \in K}$. The demands decompose into path-flows $x_p \geq 0$ so that $g_k = \sum_{p \in P_k} x_p$ where $P_k$ denotes the set of paths connecting $s_k$ to $d_k$. The cumulative flow on a link $a \in A$ is then $y_a = \sum_{p \ni a} x_p$ where the sum extends to all paths $p \in \bigcup_{k \in K} P_k$ containing $a$. Suppose that the distribution $\tau_a \sim F_a(y_a)$ depends on the total link flow $y_a$.

We may then define a risk-averse network equilibrium as a path-flow vector $x$ which uses only risk-minimizing paths, namely, for each OD pair $k \in K$ and every path $p \in P_k$ we must have

$$x_p > 0 \Rightarrow \rho(T_p) = \min_{r \in P_k} \rho(T_r).$$

If $\rho$ is additive and the function $\sigma_a(y_a) \triangleq \rho(\tau_a)$ increases with $y_a$, this reduces to a standard Wardrop equilibrium and equilibria are characterized as the optimal solutions of the convex program

$$\min_{(x, y) \in F} \sum_{a \in A} \int_0^{y_a} \sigma_a(z) \, dz \quad (2.2)$$

where $F$ stands for the set of all feasible flows satisfying flow conservation.

A similar model can be stated in the atomic case with finitely many players. Each player $i \in I$ choses a path $p_i$ from his origin to his destination and gets
\[ \rho(T_{p_i}) \text{ as payoff. Assuming that the distribution of } \tau_a \sim F_a(n_a) \text{ depends on the number of players that use the link } n_a = |\{ i \in I : a \in p_i \}|, \text{ and denoting } \sigma_a(n_a) = \rho(\tau_a), \text{ this yields a congestion game which falls in the framework of Rosenthal and admits the potential function} \]

\[ \Phi(p_i : i \in I) = \sum_{a \in A} \sum_{z=0}^{n_a} \sigma_a(z). \quad (2.3) \]

In both the atomic and non-atomic settings above, all drivers were assumed homogeneous with respect to their valuation of risk. In the next section we show that additive consistency limits the choice to entropic risk measures so that some similarity among users might be expected, nevertheless they can still differ in their absolute risk aversion index. The latter calls for an equilibrium model with multiple user classes, for which one can still establish the existence of equilibria but a simple variational characterization such as (2.2) or the existence of a potential function like (2.3) seems unlikely.

### 3 Theories of choice and additive consistency

The scalarization postulate in Proposition 2.1 is a nontrivial assumption that needs further justification. The theories of choice provide sufficient conditions for this property to be satisfied. In particular, a preorder \( \preceq \) on a topological space \( \mathcal{X} \) has a scalar representation \( C : \mathcal{X} \to \mathbb{R} \) if and only if there is a countable dense subset \( D \subset \mathcal{X} \) such that whenever \( X \preceq Y \) one can find \( Z \in D \) with \( X \preceq Z \preceq Y \) (see [17, Theorem 2.6] and references therein). Unfortunately this is not enough for our purposes and additional conditions are needed to get equivalence to a constant \( X \sim \alpha \). This can be achieved when \( \mathcal{X} \) is a prospect space of random variables, in which case more specific formulas for \( C(X) \) can be obtained.

Already in the 18th century, Daniel Bernoulli [5] observed that preferences on prospects could be represented by an expected utility \( C(X) = \mathbb{E}(c(X)) \). Axiomatic approaches for this type of representation were developed among others by Kolmogorov [28], Nagumo [29], de Finetti [8], and von Neumann & Morgenstern [51]. Here we consider a version given by Föllmer & Schied [17]. More recently, alternative representations have been obtained under different sets of axioms, including the dual theory of choice [51], Yaari and the rank-dependent utilities [42, Quiggin], [52, Wakker], [6, Chateauneuf].

Throughout this section we consider a preorder \( \preceq \) defined on the whole space \( \mathcal{X} = L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \) where \( (\Omega, \mathcal{F}, \mathbb{P}) \) is a standard atomless probability space. We denote \( \mathcal{D}_b \) the set of probability distributions on \( \mathbb{R} \) with bounded support so that each \( X \in \mathcal{X} \) has a distribution \( F_X \in \mathcal{D}_b \) and, by Skorohod’s
representation theorem \cite{48}, all distributions in $D_b$ are obtained in this way. On $\mathcal{X}$ we consider the convergence for the $L^1$-norm, denoted $X_n \to X$, as well as the convergence in distribution: $X_n \xrightarrow{D} X$ iff $F_{X_n}$ converges weakly to $F_X$, that is \( \int_{\mathbb{R}} \varphi(x) \, dF_{X_n}(x) \to \int_{\mathbb{R}} \varphi(x) \, dF_X(x) \) for all bounded continuous functions $\varphi : \mathbb{R} \to \mathbb{R}$.

3.1 Expected utility

According to \cite[Corollary 2.29]{17}, a preorder $\preceq$ over $\mathcal{X}$ admits an expected utility representation of the form

\[ C(X) = \mathbb{E}(c(X)) = \int_{\mathbb{R}} c(x) \, dF(X)(x), \]

with $c : \mathbb{R} \to \mathbb{R}$ strictly increasing and continuous (unique up to a positive affine transformation) if and only if the following axioms are satisfied

(A$_1$) LAW INVARIANCE: $F_X = F_Y \Rightarrow X \sim Y$.

(A$_2$) WEAK CONTINUITY: the sets \{ $Y \in \mathcal{X} : Y \preceq X$ \} and \{ $Y \in \mathcal{X} : X \preceq Y$ \} are closed for convergence in distribution.

(A$_3$) INDEPENDENCE: if $X \preceq Y$ then $\mathcal{L}(p; X; Z) \preceq \mathcal{L}(p; Y; Z)$ for all $Z \in \mathcal{X}$ and $p \in [0, 1]$. Here $\mathcal{L}(p; X; Z)$ denotes the lottery with distribution given by $\alpha F_X(x) + (1 - \alpha) F_Z(x)$ for all $x \in \mathbb{R}$.

This is a general version of the von Neumann and Morgenstern representation result \cite{51}, originally stated for lotteries over a finite event space. For further discussions see \cite{9, 13, 14, 17}.

In the context of route choice, expected utility preferences hold an intuitive appeal. Imagine for instance a fire truck rushing towards an emergency. Clearly enough, reaching the destination as quickly as possible is critical, all the more since the damage caused by fire increases non-linearly with time. A route with small expected time but affected by events of high congestion might be too risky, and a longer but more reliable route could be a better choice. Expected utility captures the nonlinear relation between “time” and “cost”, so that minimizing expected cost seems a reasonable model for the actual behavior of firemen.

The properties of the utility $c(\cdot)$ are naturally connected to those of $\preceq$. For instance, $c(\cdot)$ is convex iff $\preceq$ is risk-averse in the sense that the expected

\footnote{Since in our setting $X$ represents a cost, it might be more appropriate to call it expected disutility or expected cost, but we adhere to the standard terminology.}

\footnote{Note that in this case $\preceq$ induces a preorder $\leq$ on $D_b$ by $F_X \leq F_Y \Leftrightarrow X \preceq Y$.}
value of a prospect is always preferred to the prospect itself: $\mathbb{E}(X) \preceq X$. Also $c(\cdot)$ is increasing iff $\preceq$ is monotone, and strictly increasing if $X \triangleleft Y$ whenever $X < Y$ almost surely. In this latter case $c(\cdot)$ has an inverse and one can also represent $\preceq$ by the so-called certainty equivalent

$$
\rho_c(X) = c^{-1}(\mathbb{E}(c(X))).
$$

Note that taking $\alpha = \rho_c(X)$ we have $C(\alpha) = c(\alpha) = C(X)$ so that $X \sim \alpha$ and the scalarization postulate in Proposition 2.1 holds true. Moreover, note that if a preorder $\preceq$ is induced by a map $\rho : \mathcal{X} \to \mathbb{R}$ and satisfies (A1)-(A3), then $\rho$ is necessarily of the form $\rho_c$. For completeness we state this explicitly.

**Corollary 3.1.** Let $\rho : \mathcal{X} \to \mathbb{R}$ be such that it satisfies

- **Law Invariance:** $F_X = F_Y \Rightarrow \rho(X) = \rho(Y),$
- **Normalization on Constants:** $\rho(m) = m$ for all $m \in \mathbb{R}$,
- **Strict Monotonicity:** if $X < Y$ almost surely then $\rho(X) < \rho(Y)$,
- **Weak Continuity:** if $X_n \xrightarrow{D} X$ then $\rho(X_n) \to \rho(X),$
- **Independence:** if $\rho(X) \leq \rho(Y)$ then $\rho(\mathcal{L}(p; X; Z)) \leq \rho(\mathcal{L}(p; Y; Z))$.

Then $\rho = \rho_c$ for some $c : \mathbb{R} \to \mathbb{R}$ strictly increasing and continuous, and unique up to a positive affine transformation.

**Proof.** Let us consider the induced order $X \preceq Y \iff \rho(X) \leq \rho(Y)$. The assumptions imply that $\preceq$ is represented by an expected utility map $\rho_c$. Normalization on constants gives $\rho(X) = \rho(\rho(X))$ from which we deduce $X \sim \rho(X)$ and therefore $\rho_c(X) = \rho_c(\rho(X)) = \rho(X)$. \hfill \qed

In general $\rho_c$ is not a risk measure since translation invariance may fail. We show next that this only holds for the entropic risk measures. This result goes back to [19 Gerber] where it was proved under the stronger condition of additivity of $\rho_c$ and assuming $c(\cdot)$ concave non-decreasing and twice differentiable. Under translation invariance, but still assuming regularity of $c(\cdot)$, the result was proved in [27 Luan] (see also [23 Heilpern]). Our proof below relies exclusively on continuity and monotonicity. Regularity of $c(\cdot)$ as well as additivity of $\rho_c$ are obtained as a consequence.

**Theorem 3.1.** The only translation invariant maps of the form $\rho_c$ with $c : \mathbb{R} \to \mathbb{R}$ strictly increasing and continuous, are the entropic risk measures

$$
\rho^\text{ent}_\beta(X) = \begin{cases} 
\frac{1}{\beta} \ln(\mathbb{E}(e^{\beta X})) & \text{if } \beta \neq 0 \\
\mathbb{E}(X) & \text{if } \beta = 0.
\end{cases}
$$
Proof. Since $\rho_c$ does not change under affine transformations of $c(\cdot)$, we may assume $c(0) = 0$. The translation invariance of $\rho_c$ gives

$$c^{-1}(\mathbb{E}(c(X + m))) = c^{-1}(\mathbb{E}(c(X))) + m.$$ 

Take $X = zB_p$ with $z \in \mathbb{R}$ and $B_p$ a Bernoulli variable with parameter $p$. Developing the left and right hand sides, and using the fact that $c(0) = 0$, this equality becomes

$$c^{-1}(pc(z + m) + (1-p)c(m)) = c^{-1}(pc(z)) + m.$$ 

Defining $\varepsilon = c^{-1}(pc(z))$ this can be rewritten as

$$c(\varepsilon)[c(z + m) - c(m)] = c(z)[c(m + \varepsilon) - c(m)]. \quad (3.1)$$

Now, since $c(\cdot)$ is increasing it is differentiable almost everywhere. Take any point $m$ at which $c'(m)$ exists. By considering alternately $z > 0$ and $z < 0$ with $p \to 0^+$ we have respectively $\varepsilon \to 0^+$ and $\varepsilon \to 0^-$. Dividing (3.1) by $\varepsilon$ and noting that $c(z) \neq 0$ and $[c(z + m) - c(m)] \neq 0$, it follows that the lateral derivatives of $c(\cdot)$ at 0 exist and coincide, and moreover we have

$$c'(0)[c(z + m) - c(m)] = c(z)c'(m). \quad (3.2)$$

Now that we know that $c'(0)$ exists, we may reuse (3.1) and apply a similar argument at an arbitrary point $m$ to deduce that $c'(m)$ exists everywhere and satisfies (3.2). Moreover, since $c(\cdot)$ is strictly increasing the mean value theorem implies that $c'(m) > 0$ at some point $m$, and (3.2) yields $c'(0) > 0$. Using an affine transformation (which does not affect $\rho_c$) we may assume $c'(0) = 1$ and then rearranging (3.2) we get

$$c(z + m) - c(z) = c(m) + [c'(m) - 1]c(z). \quad (3.3)$$

Dividing by $m > 0$ and letting it to 0 it follows that $[c'(m) - 1]/m$ has a limit, which we denote by $\beta$, and $c(\cdot)$ satisfies the differential equation

$$c'(z) = 1 + \beta c(z).$$

This has a unique continuous solution with $c(0) = 0$, namely $c(x) = x$ if $\beta = 0$ and $c(x) = [e^{\beta x} - 1]/\beta$ otherwise. The conclusion follows.

### 3.1.1 The independence axiom and Allais’ paradox

Expected utility theory has not been without critics, mainly focusing on the independence axiom. The paradoxes of Allais [11] and Ellsberg [12] show specific contexts in which the independence axiom is violated and agents
do not behave consistently with the predictions of this theory. Further empirical evidence has been provided by Kahneman and Tversky [25].

To interpret the independence axiom, imagine a driver who has two options $X, Y$ to travel from $j$ to $d$ of which he prefers $X$ (see figure 3). Suppose now that he is actually at a point $s$ on the other side of a river, and to reach the intermediate node $j$ he must first cross a bridge which is open with probability $p$, and else take a long detour $Z$ to the destination $d$. Thus, the driver faces a choice between the lotteries $\mathcal{L}(p; X; Z)$ and $\mathcal{L}(p; Y; Z)$. The independence axiom postulates that the first should be preferred. While this seems a reasonable assumption in the route choice setting, Allais observed that it may fail in other contexts. Specifically, he considered

$$X = 50, \quad Y = \begin{cases} 35 & \text{with probability 0.8}, \\
100 & \text{with probability 0.2}. \end{cases}, \quad Z = 100.$$

and noted that while most people prefer $X$ to $Y$, for $p = 0.25$ they tend to choose $\mathcal{L}(p; Y; Z)$ over $\mathcal{L}(p; X; Z)$ where

$$\mathcal{L}(p; X; Z) = \begin{cases} 50 & \text{with probability 0.25}, \\
100 & \text{with probability 0.75}, \end{cases}$$

$$\mathcal{L}(p; Y; Z) = \begin{cases} 35 & \text{with probability 0.2}, \\
100 & \text{with probability 0.8}. \end{cases}$$

This points to a potential incongruence between the predictions based on the independence axiom and the actual choices made by agents. However, it is also true that “context matters” and decisions depend not only on the way the choice is formulated but even on the form in which information is communicated and processed. The lotteries above do not describe the route choice accurately since they obscure the fact that here we face a two-stage decision process with the possibility of recourse: the choice between $X$ or $Y$ — and $Z$ — can be postponed until we know whether the bridge is open. Stated in this way the observed inconsistency might disappear, though this should be contrasted with the actual choices made by drivers. In any case, Allais’ paradox and other empirical violations of the postulates of expected
utility theory have motivated alternative theories of choice. We consider two of them in the next subsections.

3.2 Dual theory of choice

While expected utility introduces risk-aversion by magnifying the effects of bad outcomes through a nonlinear transformation of the cost \(c(X)\), Yaari’s dual theory of choice [54] uses the idea that a risk-averse agent tends to overstate the probability of bad outcomes. An agent is then characterized by a continuous nondecreasing distortion map \(h : [0, 1] \rightarrow [0, 1]\) with \(h(0) = 0\) and \(h(1) = 1\), so that the probability \(P(X > x)\) is distorted as \(h(P(X > x))\). Risk-aversion corresponds to \(h(x) \geq x\) for all \(x \in [0, 1]\), while a risk-prone agent satisfies the reverse inequality.

The function \(x \mapsto h(P(X > x))\) is a decumulative distribution so we may find a random variable \(X_h\) such that \(P(X_h > x) = h(P(X > x))\), and we may describe the agent’s preferences \(\preceq\) by the functional

\[\rho^h(X) = E(X_h)\] (3.4)

or more explicitly in terms of the distribution of \(X\)

\[\rho^h(X) = \int_{-\infty}^{0} [h(P(X > x)) - 1]dx + \int_{0}^{\infty} h(P(X > x))dx.\] (3.5)

This is a law invariant risk measure which is also positively homogeneous and normalized on constants. It is called a distortion risk measure. In particular it is always translation invariant as opposed to the expected utility maps \(\rho_c\).

A characterization of the preferences that can be represented in this form is given in [54]. Namely, assuming that all prospects satisfy \(X(\omega) \in [0, 1]\) almost surely, a preorder \(\preceq\) on \(\mathcal{X}\) can be characterized by a distortion risk measure \(\rho^h\) if and only if it is law invariant \((A_1)\) and satisfies

\((A_2^*)\) \text{L}^1\text{-continuity:} the sets \(\{Y \in \mathcal{X} : Y \subseteq X\}\) and \(\{Y \in \mathcal{X} : X \subseteq Y\}\) are closed for convergence in the \(L^1\)-norm.

\((A_2^*)\) \text{Dual independence:} if \(X, Y, Z \in \mathcal{X}\) are pairwise comonotone and \(X \subseteq Y\) then \(\alpha X + (1-\alpha)Z \subseteq \alpha Y + (1-\alpha)Z\) for all \(\alpha \in [0, 1]\).

\((A_2^*)\) \text{Monotonicity under first-order stochastic dominance:} if \(F_X(t) \geq F_Y(t)\) for all \(t \in \mathbb{R}\) then \(X \preceq Y\).

The main difference with expected utility theory is the substitution of independence by dual independence which uses the concept of comonotonicity. For our purposes it suffices to say that \(X\) and \(Y\) are comonotone iff there is
a third variable $U$ and non-decreasing maps $f$ and $g$ such that $X = f(U)$ and $Y = g(U)$. An alternative set of axioms which ensure a representation by distortion risk measures is given in [9] by considering a prospect space $X$ of bounded random variables on a standard atomless probability space and continuity for the $L^\infty$ norm.

Although the maps $\rho^h$ are always translation invariant, they may fail to be additively consistent. As we show next the latter is a stringent condition which is only satisfied for $h(x) = x$ in which case $\rho^h(X) = \mathbb{E}(X)$. For $h$ concave and twice differentiable this result was established in [27, Luan] (see also [23, Heilpern] and [22, Goovaerts et al.]). Our proof does not require any a priori regularity on $h$ beyond continuity and monotonicity. It exploits the following elementary fact.

**Lemma 3.1.** Let $h : [0, 1] \to [0, 1]$ be continuous and suppose that for some $0 < p < 1$ the limit $L = \lim_{q \to 0^+} (h(q) - h(pq))/q$ exists. Then $h$ is right differentiable at 0 with $h'(0) = L/(1-p)$.

**Proof.** Take $\varepsilon > 0$ and choose $\delta > 0$ so that for all $q \in (0, \delta)$ we have

$$L - \varepsilon \leq \frac{h(q) - h(pq)}{q} \leq L + \varepsilon.$$ 

For each $x \in (0, \delta)$ we may take $q = p^j x$ in order to get

$$(L - \varepsilon)p^j \leq \frac{h(p^j x) - h(p^{j+1} x)}{x} \leq (L + \varepsilon)p^j.$$ 

Summing over all $j \geq 0$ we get a telescopic series that simplifies to

$$\frac{L - \varepsilon}{1 - p} \leq \frac{h(x) - h(0)}{x} \leq \frac{L + \varepsilon}{1 - p},$$

from which the conclusion follows since $\varepsilon$ was arbitrary. \hfill \Box

**Theorem 3.2.** The only distortion risk measure $\rho^h$ that is additive consistent is $\rho^h(X) = \mathbb{E}(X)$ which corresponds to $h(x) = x$.

**Proof.** Take $B_p$ and $B_q$ independent Bernoullis with success probabilities $p, q \in [0, 1]$. From Lemma 2.1 we know that $\rho^h$ is additive so that

$$\rho_h(B_p + B_q) = \rho_h(B_p) + \rho_h(B_q). \hspace{1cm} (3.6)$$

Denoting $\bar{p} = 1 - p$ and $\bar{q} = 1 - q$ we have

$$\mathbb{P}(B_p + B_q > x) = \begin{cases} 
1 & \text{if } x \leq 0 \\
1 - \bar{p}\bar{q} & \text{if } 0 \leq x < 1 \\
pq & \text{if } 1 \leq x < 2 \\
0 & \text{if } x \geq 2 
\end{cases} \hspace{1cm} (3.7)$$
and then using (3.5) we find $\rho_h(B_p + B_q) = h(1 - \bar{p}\bar{q}) + h(pq)$. Similarly we get $\rho_h(B_p) = h(p)$ and $\rho_h(B_q) = h(q)$ so that (3.6) becomes

$$h(1 - \bar{p}\bar{q}) + h(pq) = h(p) + h(q)$$

(3.8)

which can also be written as

$$h(p + q(1-p)) - h(p) = h(q) - h(pq).$$

(3.9)

Since $h$ is monotone we can find $\tilde{p} \in (0,1)$ such that $h'(\tilde{p})$ exists, so that (3.9) implies

$$\lim_{q \to 0^+} \frac{[h(q) - h(\tilde{p}q)]}{q} = h'(\tilde{p})(1 - \tilde{p})$$

and then Lemma 3.1 gives $h'_+(0) = h'(\tilde{p})$. Using this fact and dividing (3.9) by $q(1-p)$ with $q \to 0^+$, it then follows that $h$ has a right derivative at each point $p \in [0,1)$ and in fact $h'_+(p) = h'_+(0)$ is constant. It follows that $h$ is Lipschitz continuous and then absolutely continuous so that it can be recovered by integrating its derivative. Hence $h$ is affine and since $h(0) = 0$ and $h(1) = 1$ we conclude that $h(\cdot)$ must be the identity map.

### 3.3 Rank-dependent expected utilities

Expected utility theory and the dual theory of choice are complementary and can be combined by considering preference functionals of the form

$$\rho^h_c(X) = c^{-1}(E(c(X^h))) = c^{-1}(E(c(X)^h))$$

where $c$ is a utility function and $h$ is a distortion map. More explicitly

$$\rho^h_c(X) = c^{-1}\left(\int_{-\infty}^{0} h(\mathbb{P}(c(X) > x)) - 1]dx + \int_{0}^{\infty} h(\mathbb{P}(c(X) > x))dx\right).$$

(3.10)

Note that $\rho^h_c$ does not change under affine transformations of $c(\cdot)$, so we may assume $c(0) = 0$. For $h(x) = x$ we recover expected utilities, while $c(x) = x$ gives the distortion risk measures. The functionals $\rho^h_c$ are called rank-dependent expected utilities and have been considered by several authors including Quiggin [42], Wakker [52], and Chateauneuf [6], who provide axiomatic characterizations of the preorders $\succeq$ that can be represented in this form. Note that $\rho^h_c$ is normalized on constants but, just as for expected utilities, they need not be translation invariant nor additive consistent. The next result characterizes when these properties hold. This result was proved in [27, Luan] assuming $c(\cdot)$ and $h(\cdot)$ twice differentiable and increasing, with $h$ concave and $c$ convex. An alternative proof was given in [22, Goovaerts et al.] under the same hypothesis but assuming in addition that $c(\cdot)$ admits a McLaurin expansion. Our proof rests on the techniques developed in the previous sections and avoids such a priori regularity which is however obtained as a consequence.
Theorem 3.3. A rank dependent expected utility $\rho^h_c$ is translation invariant iff $c(\cdot)$ is an exponential function or the identity. Moreover, the only $\rho^h_c$ which are additive consistent are the entropic risk measures: $h$ is the identity and $c$ is either an exponential function or the identity.

Proof. Let us first assume that $\rho^h_c$ is translation invariant. Take $X = zB_p$ with $z > 0$ and $B_p$ a Bernoulli. We then have

$$P(c(X + m) > x) = \begin{cases} 1 & \text{if } x < c(m) \\ p & \text{if } c(m) \leq x < c(m + z) \\ 0 & \text{if } x \geq c(m + z) \end{cases}$$

so that using (3.10) and distinguishing cases according to the signs of $m$ and $m + z$, we get in all situations

$$\rho^h_c(X + m) = c^{-1}(h(p)c(m + z) + (1 - h(p))c(m)).$$

In particular for $m = 0$ we have $\rho^h_c(X) = c^{-1}(h(p)c(z))$ so that the translation invariance $\rho^h_c(X + m) = \rho^h_c(X) + m$ yields

$$h(p)c(m + z) + (1 - h(p))c(m) = c \left( c^{-1}(h(p)c(z)) + m \right).$$

Letting $\varepsilon = c^{-1}(h(p)c(z)) > 0$ we get the analog of (3.1)

$$c(\varepsilon)[c(m + z) - c(m)] = c(z)[c(m + \varepsilon) - c(m)]. \quad (3.11)$$

In the case $z < 0$, noting that $B_p \sim 1 - B_\bar{p}$ with $\bar{p} = 1 - p$, we may write $X + m = -zB_p + (m + z)$ so we get a similar formula for $\rho^h_c(X + m)$ by replacing $p$ by $\bar{p}$, $z$ by $-z$, and $m$ by $m + z$, namely

$$\rho^h_c(X + m) = c^{-1}(h(\bar{p})c(m) + (1 - h(\bar{p}))c(m + z))$$

from which we obtain again (3.11) this time with $\varepsilon = c^{-1}((1 - h(\bar{p}))c(z)) < 0$. Proceeding as in the proof of Theorem 3.1 we deduce that $c(\cdot)$ is either an exponential function or the identity map, which proves our first claim.

Let us assume next that $\rho^h_c$ satisfies the stronger condition of additive consistency, and let us show that in this case $h(x) = x$. The case when $c(\cdot)$ is the identity was settled in the previous section so we just consider the exponential case $c(x) = \frac{e^{\beta x} - 1}{\beta}$. Let us consider two independent Bernoullis $B_p$ and $B_q$ so that for all $z > 0$ we have

$$\rho^h_c(zB_p + zB_q) = \rho^h_c(zB_p) + \rho^h_c(zB_q). \quad (3.12)$$

The formulas given in the first part of the proof show that the right-hand side is equal to $c^{-1}(h(p)c(z)) + c^{-1}(h(q)c(z))$. To compute the expression
on the left we observe that $\mathbb{P}(c(zB_p + zB_q) > x) = \mathbb{P}(B_p + B_q > c^{-1}(x)/z)$
and we may use (3.7) to obtain
\[
\rho^h_c(zB_p + zB_q) = c^{-1} \left( \int_0^{c(z)} h(1 - \bar{p}q) \, dx + \int_{c(z)}^{c(2z)} h(pq) \, dx \right) 
= c^{-1} (c(z)h(1 - \bar{p}q) + [c(2z) - c(z)]h(pq)).
\]
Plugging these formulas into (3.12) we have
\[
c(z)h(1 - \bar{p}q) + (c(2z) - c(z))h(pq) = c \left( c^{-1}(h(p)c(z)) + c^{-1}(h(q)c(z)) \right).
\]
Using the exponential form of $c(\cdot)$ the left-hand side is given by
\[
\frac{e^{\beta z} - 1}{\beta} h(1 - \bar{p}q) + \frac{e^{2\beta z} - e^{\beta z}}{\beta} h(pq)
\]
whereas after some manipulation the right-hand side is seen to be
\[
\frac{e^{\beta z} - 1}{\beta} (h(p) + h(q) + [h(p)h(q)]e^{\beta z} - 1)
\]
so that the equation simplifies to
\[
h(1 - \bar{p}q) + h(pq) - h(p) - h(q) = e^{\beta z} (h(p)h(q) - h(pq)).
\]
Since this holds for all $z > 0$ we deduce that for all $p, q \in [0, 1]$
\[
h(p)h(q) = h(pq),
\]
\[
h(1 - \bar{p}q) + h(pq) = h(p) + h(q).
\]
The latter is the same as (3.8) so we may use the argument in Theorem 3.2 to conclude that $h$ is the identity.

4 A remark on dynamic risk measures

Time consistency is a central issue in multistage decision problems under risk where decisions are taken sequentially along periods $t = 0, 1, \ldots, T$. It has been thoroughly investigated using the concept of dynamic risk measures: a sequence of risk measures, one for each period, usually obtained by iterated composition of conditional risk maps that progressively incorporate the random information revealed along time (see for instance [7,11,40,45] and references therein). This approach structurally avoids the inconsistencies and allows to deal with non-additive risk measures such as VaR or AVaR. Moreover, the framework provides a dynamic programming recursion that allows to characterize and eventually compute optimal solutions.

Route choice can also be seen as a sequential decision process where at each step the driver is located at an intermediate node where he must chose
the next arc to follow. This view is actually the basis of most shortest path algorithms. It is then tempting to use dynamic risk measures to model route choice under risk. However, the notion of period is not obvious here. One option is to take the set of nodes in the network as state space and associate periods with link choice decisions so that time corresponds to the number of link choices that have been made so far. Naturally, one has to deal with the fact that the same node can be reached after different number of steps, depending on the number of links in the actual path followed. In particular it is unclear how to define the planning horizon $T$, maybe as the maximum number of links in all paths connecting the origin to the destination.

Although one can find ways to frame route choice as a multistage decision process, we do not pursue this goal here. Instead, we point out yet another difficulty in this approach which has to do with the network representation. We illustrate this with a very simple example on the network in figure 4 with independent normally distributed times $X \sim N(10,1)$, $Y \sim N(10,1)$ and $Z \sim N(20,3)$. Consider the non additive risk measure $\rho = \text{AVaR}_p$, with $p$ chosen so that for all normal variables we have $\rho(X) = \mathbb{E}(X) + \sigma(X)$. By independence, an iterated dynamic risk measure will evaluate the risk for the upper route as $\rho(X + \rho(Y|X)) = \rho(X + \rho(Y)) = \rho(X) + \rho(Y) = 22$. This is larger than $\rho(Z) = 20 + \sqrt{3}$ which is then the optimal choice. Suppose now that we merge both upper links into a single arc with time $U = X + Y \sim N(20,2)$, which is just a matter of how we decide to model the network. In this case the upper route has risk $\rho(U) = 20 + \sqrt{2}$ and has displaced $Z$ as the optimal solution. The conflict arises since there is no clear notion of period to guide our choice of the representation of the network. This will occur whenever one deals with non-additive risk measures, while for additive risk measures the conflict disappears.

5 Related work

Risk-sensitive route choice is a relatively new research area which has been growing steadily in the last decade or so. General discussions on risk eval-
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Evaluation in the context of route choice can be found in Bates et al. [4], Noland [36], and Hollander [24]. A mean-stdev risk model for atomic and non-atomic traffic equilibrium was investigated by Nikolova and Stier-Moses [35], distinguishing the case when only the expected values depend on the traffic intensity from the more difficult case where also the variance is flow-dependent. A similar traffic equilibrium model was considered by Ordoñez and Stier-Moses in [39], in which risk-aversion is treated by aggregating a variability index to the expected value. This is compared to a model based on $\alpha$-percentiles as well as a novel approach that uses ideas from robust optimization. Since computing an $\alpha$-percentile equilibrium is difficult, they investigate two classes of approximations which provide a better fit than a standard Wardrop model. Percentile equilibria in route choice were also investigated by Nie [31].

Algorithms to compute optimal routes for the mean-stdev objective were studied by Nikolova [33] and Nikolova et al. [34]. Despite the combinatorial nature of the problem and the nonlinear objective function, an exact algorithm with sub-exponential complexity $n^O(\log n)$ is found. The main difficulty here comes from the non-additivity of the standard deviation. As illustrated by the example in the Introduction, the optimality of a path is not inherited by its subpaths, which prevents the use of dynamic programming and makes the problem much more difficult to solve. In contrast, when using additive consistent risk measures this difficulty disappears and route choice reduces to a standard shortest path problem.

A different approach to risk-averse path choice considers user preferences based on the on-time arrival probability. This was studied by Nie and Wu [32], addressing the question of whether or not route optimality is inherited by subpaths. An algorithm for this objective function was also given by Nikolova et al. [34]. A related approach by Nie et al. [30, 53] reconsiders the route choice question under stochastic dominance constraints.

Our contribution partially differs from the previous ones as it uses risk measures to quantify route preferences. Exploiting the axiomatic frameworks provided by the theories of choice we showed that, in wide classes of risk functionals, the entropic risk measures emerge as the only ones that guarantee a form of consistency in route choice. In this light, all the models discussed above are susceptible to exhibit inconsistencies. While this raises a serious question about the capacity of these models to capture rational behavior, it does not invalidate them. From a practical viewpoint, all these models may plausibly describe the behavior of some drivers and —after all— no one has yet proved that drivers are actually consistent in their decisions! From a theoretical perspective our results require the preferences to be defined and satisfy additive consistency throughout the space $L^\infty(\Omega,\mathcal{F},\mathbb{P})$. 
This might be asking too much as one could argue that drivers are only able to make choices in a much narrower subset of random variables which might not be even a linear subspace (e.g. a set of uniformly bounded non-negative variables). In summary, while our contribution reveals some strong and interesting consequences of additive consistency, there is still work to be done before one can provide firm recommendations as to which is the most appropriate way to model route choice under risk.

References

[1] Allais, M. Le Comportement de l’Homme Rationnel devant le Risque: Critique des Postulats et Axiomes de l’Ecole Americaine. *Econometrica* 21, 4 (1953), 503–546.

[2] Arrow, K. J. *Aspects of the theory of risk bearing*. Yrjo Jahnssonin Saatio, Helsinki, 1965.

[3] Artzner, P., Delbaen, F., Eber, J.-M., and Heath, D. Coherent measures of risk. *Mathematical Finance* 9, 3 (1999), 203–228.

[4] Bates, J., Polak, J., Jones, P., and Cook, A. The valuation of reliability for personal travel. *Transportation Research Part E* 37 (2001), 191–229.

[5] Bernoulli, D. Specimen theoriae novae de mensura sortis. *Commentarii Academiae Scientiarum Imperialis Petropolitanae* 5 (1738), 175–192. [English translation: *Exposition of a new theory on the measurement of risk*, Econometrica 22, (1954), 23–35].

[6] Chateauneuf, A. Comonotonicity axioms and rank-dependent expected utility theory for arbitrary consequences. *Journal of Mathematical Economics* 32, 1 (1999), 21–45.

[7] Cheridito, P., Delbaen, F., and Kupper, M. Dynamic monetary risk measures for bounded discrete-time processes. *Electron. J. Probab.* 11 (2006), no. 3, 57–106.

[8] de Finetti, B. Sul concetto di media. *Giornale dell’Istituto Italiano degli Attuari* 2 (1931), 369–396.

[9] Dentcheva, D., and Ruszczyński, A. Common mathematical foundations of expected utility and dual theories. *Preprint* (2012).

[10] Denuit, M., Dhaene, J., Goovaerts, M., Kaas, R., and Laeven, R. Risk measurement with equivalent utility principles. Open Access publications from Katholieke Universiteit Leuven urn:hdl:123456789/200185, Katholieke Universiteit Leuven, 2006.
[11] Detlefsen, K., and Scandolo, G. Conditional and dynamic convex risk measures. *Finance and Stochastics* 9, 4 (2005), 539–561.

[12] Ellsberg, D. Risk, ambiguity and the Savage axioms. Levine’s Working Paper Archive 7605, David K. Levine, 2000.

[13] Fishburn, P. C. Utility theory. *Management Science* 14, 5 (1968), 335–378.

[14] Fishburn, P. C. *Utility theory for decision making*. Publications in Operations Research No. 18, John Wiley and Sons Inc., New York, 1970.

[15] Föllmer, H., and Schied, A. Convex measures of risk and trading constraints. *Finance and Stochastics* 6, 4 (2002), 429–447.

[16] Föllmer, H., and Schied, A. Robust preferences and convex measures of risk. In *Advances in Finance and Stochastics*. Springer Berlin Heidelberg, 2002, pp. 39–56.

[17] Föllmer, H., and Schied, A. *Stochastic finance: An introduction in discrete time*. Berlin, Gruyter Studies in Mathematics, 2002.

[18] Fosgerau, M., Hjorht, K., C., B., and Fukuda, D. Travel time variability: definition and valuation. Tech. rep., Danmarks Tekniske Universitet, Department of Transport, 2008.

[19] Gerber, H. On additive premium calculation principles. *ASTIN Bulletin* 7, 3 (1974), 215–222.

[20] Goovaerts, M., Kaas, R., Dhaene, J., and Tang, Q. A unified approach to generate risk measures. Tech. rep., Katholieke Universiteit Leuven, 2003.

[21] Goovaerts, M., Kaas, R., and Laeven, R. Decision principles derived from risk measures. Open Access publications from Katholieke Universiteit Leuven urn:hdl:123456789/278383, Katholieke Universiteit Leuven, 2010.

[22] Goovaerts, M. J., Kaas, R., and Laeven, R. J. A note on additive risk measures in rank-dependent utility. *Insurance: Mathematics and Economics* 47, 2 (2010), 187–189.

[23] Heilpern, S. A rank-dependent generalization of zero utility principle. *Insurance: Mathematics and Economics* 33, 1 (2003), 67–73.

[24] Hollander, Y. Direct versus indirect models for the effects of unreliability. *Transportation Research Part A* 40 (2006), 699–711.
[25] Kahneman, D., and Tversky, A. Prospect theory: An analysis of decision under risk. *Econometrica* 47, 2 (1979), 263–91.

[26] Kolmogorov, A. Sur la notion de la moyenne. *Atti Accad. Naz. Lincei* 9 (1930), 221–235.

[27] Luan, C. Insurance premium calculations with anticipated utility theory. *ASTIN Bulletin* 31, 1 (2001), 27–39.

[28] Markowitz, H. Portfolio selection. *The Journal of Finance* 7, 1 (1952), 77–91.

[29] Nagumo, M. On mean values. *Tokio Buturigakko-Zassi* 40 (1931), 19–21.

[30] Nie, Y., Wu, X., and Homem-de Mello, T. Optimal path problems with second-order stochastic dominance constraints. *Networks and Spatial Economics* 12, 4 (2012), 561.

[31] Nie, Y. M. Multi-class percentile user equilibrium with flow-dependent stochasticity. *Transportation Research Part B: Methodological* 45, 10 (2011), 1641–1659.

[32] Nie, Y. M., and Wu, X. Shortest path problem considering on-time arrival probability. *Transportation Research Part B: Methodological* 43, 6 (2009), 597–613.

[33] Nikolova, E. Approximation algorithms for reliable stochastic combinatorial optimization. In *Proceedings of the 13th international conference on Approximation, and 14 the International conference on Randomization, and combinatorial optimization: algorithms and techniques* (Berlin, Heidelberg, 2010), APPROX/RANDOM’10, Springer-Verlag, pp. 338–351.

[34] Nikolova, E., Kelner, J. A., Brand, M., and Mitzenmacher, M. Stochastic shortest paths via quasi-convex maximization. In *Proceedings of the 14th conference on Annual European Symposium - Volume 14* (London, UK, UK, 2006), ESA’06, Springer-Verlag, pp. 552–563.

[35] Nikolova, E., and Stier-Moses, N. A mean-risk model for the stochastic traffic assignment problem. *Columbia Working Paper DRO-2011-03* (2011).

[36] Noland, R., and Polak, J. Travel time variability: a review of theoretical and empirical issues. *Transport Reviews* 22 (2002), 39–54.
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[37] Ogryczak, W., and Ruszczyński, A. From stochastic dominance to mean-risk models: Semideviations as risk measures. *European Journal of Operational Research* 116, 1 (1999), 33–50.

[38] Ogryczak, W., and Ruszczyński, A. Dual stochastic dominance and related mean-risk models. *SIAM Journal on Optimization* 13, 1 (2002), 60–78.

[39] Ordóñez, F., and Stier-Moses, N. E. Wardrop equilibria with risk-averse users. *Transportation Science* 44, 1 (2010), 63–86.

[40] Pflug, G., and Pichler, A. On dynamic decomposition of multistage stochastic programs. *Preprint - Optimization Online* (2012), 1–24.

[41] Pratt, J. W. Risk aversion in the small and in the large. *Econometrica* 32, 1/2 (1964), 122–136.

[42] Quiggin, J. A theory of anticipated utility. *Journal of Economic Behavior & Organization* 3, 4 (1982), 323–343.

[43] Rockafellar, R. T., and Uryasev, S. Optimization of conditional value-at-risk. *Journal of Risk* 2 (2000), 21–41.

[44] Rockafellar, R. T., and Uryasev, S. Conditional value-at-risk for general loss distributions. *Journal of Banking and Finance* 26, 7 (2002), 1443–1471.

[45] Ruszczyński, A., and Shapiro, A. Conditional risk mappings. *Mathematics of Operations Research* 31, 3 (2006), 544–561.

[46] Ruszczyński, A., and Shapiro, A. Optimization of convex risk functions. *Mathematics of Operations Research* 31, 3 (2006), 433–452.

[47] Schmeidler, D. Subjective probability and expected utility without additivity. *Econometrica* 57, 3 (1989), 571–87.

[48] Skorohod, A. V. Limit theorems for stochastic processes. *Teor. Veroyatnost. i Primenen* 1 (1956), 289–319.

[49] Taylor, M., and Susilawati. Modeling travel time reliability with the burr distribution. In *Proceedings of the 15th Euro Working Group on Transportation* (2012), EWGT2012, pp. 1–9.

[50] Tsanakas, A., and Desli, E. Risk measures and theories of choice. *British Actuarial Journal* 9 (2003), 959–991.

[51] von Neumann, J., and Morgenstern, O. *Theory of Games and Economic Behavior*. Princeton University Press, Princeton, New Jersey, 1944.
[52] Wakker, P. Separating marginal utility and probabilistic risk aversion. *Theory and Decision* 36, 1 (1994), 1–44.

[53] Wu, X., and Nie, Y. M. Modeling heterogeneous risk-taking behavior in route choice: A stochastic dominance approach. *Transportation Research Part A: Policy and Practice* 45, 9 (2011), 896–915.

[54] Yaari, M. E. The dual theory of choice under risk. *Econometrica* 55, 1 (1987), 95–115.