ABSTRACT APPROACH TO FINITE RAMSEY THEORY
AND A SELF-DUAL RAMSEY THEOREM

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Abstract. We give an abstract approach to finite Ramsey theory and prove a general Ramsey-type theorem. We deduce from it a self-dual Ramsey theorem, which is a new result naturally generalizing both the classical Ramsey theorem and the dual Ramsey theorem of Graham and Rothschild. In fact, we recover the pure finite Ramsey theory from our general Ramsey-type result in the sense that the classical Ramsey theorem, the Hales-Jewett theorem (with Shelah’s bounds), the Graham-Rothschild theorem, the versions of these results for partial rigid surjections due to Voigt, and the new self-dual Ramsey theorem are all obtained as iterative applications of the general result.

1. Introduction

1.1. Abstract approach to Ramsey theory and its applications. We give an abstract approach to pure (unstructured) finite Ramsey theory. The spirit of the undertaking is similar to Todorcevic’s approach to infinite Ramsey theory [18, Chapters 4 and 5], even though, on the technical level, the two approaches are different. There are perhaps two main points of the present paper. The first point is the existence of a single, relatively simple type of algebraic structures, called actoids of sets over backgrounds (or normed backgrounds), that underlies Ramsey theorems. It is interesting that an algebraic structure involving a partial action, a truncation operator, and a norm is invariably present in concrete Ramsey situations. The second main point is the existence of a single Ramsey theorem, which is a result about the algebraic structures just mentioned. Particular Ramsey theorems are instances, or iterative instances, of this general result for particular actoids of sets, much like theorems about, say, modules have particular instances for concrete modules. The latter point opens up a possibility of classifying concrete Ramsey situations, at least in limited contexts; see Subsection 7.5.

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Roughly speaking, a Ramsey-type theorem is a statement of the following form. We are given a set $S$ chosen arbitrarily from some fixed family $\mathcal{S}$ and a number of colors $d$. We find a set $F$ from another fixed family $\mathcal{F}$ with a “scrambling” function, usually a type of composition, defined on $F \times S$:

$$F \times S \ni (f, x) \rightarrow f . x \in F . S.$$ 

The arrangement is such that for each $d$-coloring of $F . S$ there is $f \in F$ with $f . S$ monochromatic. We find a general, algebraic framework in which we isolate an abstract pigeonhole principle and prove that it implies a precise version of the above abstract Ramsey-type statement. The abstract form of the pigeonhole principle is not a simple abstraction of the classical pigeonhole. It is a natural in the algebraic context condition that makes it possible to prove the general Ramsey theorem and that is flexible enough to accommodate in applications many concrete statements as special cases. For example, versions of the standard pigeonhole principle, the Hales–Jewett theorem, or the Graham–Rothschild theorem all serve as pigeonhole principles in different situations. It is somewhat surprising that a sparse algebraic setup can encompass a large variety of Ramsey-type results.

This algebraic framework, which is described precisely in Section 2, consists of a set $A$ with a partial binary operation, called multiplication, acting on another set $X$ by partial functions. The set $X$ comes equipped with additional structure—a unary operator $\partial: X \rightarrow X$, which we call truncation. The partial action of $A$ on $X$ is implemented by homomorphisms of this additional structure. The partial multiplication on $A$ is then lifted to a partial multiplication on a family $\mathcal{F}$ of subsets of $A$, and the partial action of $A$ on $X$ is lifted to a partial action of $\mathcal{F}$ on a family $\mathcal{S}$ of subsets of $X$. At this point, we state the abstract pigeonhole principle, Definition 2.8. Then, in Section 3 we give a precise incarnation of the abstract Ramsey theorem alluded to above and, in Corollary 3.3 of Theorem 3.1, we show that the abstract pigeonhole principle implies this Ramsey statement.

In Section 4 we prove Theorem 4.3 and Corollary 4.4 that, in many situations, makes it possible to check only a localized, and therefore easier, version of the abstract pigeonhole principle. Further, in Section 5 we show two results allowing us to propagate the pigeonhole principle. In the first one, Proposition 5.1, we get the pigeonhole principle for naturally defined products assuming it holds for the factors. The second result, Proposition 5.3, involves a notion of interpretability and establishes preservation of the pigeonhole principle under interpretability.

As a consequence of the general theory, in Section 7 we obtain a new self-dual Ramsey theorem. We give its statement and explain its relationship
with other results in Subsection 1.2 below. Let us only mention one interesting feature of the proof of this theorem: the role of the pigeonhole principle is played by the Graham–Rothschild theorem. Also in Section 7, we give other applications of the general theory to concrete examples. We show how to derive as iterative applications of the abstract Ramsey result the classical Ramsey theorem, the Hales–Jewett theorem, the Graham–Rothschild theorem as well as the versions of these results for partial rigid surjections due to Voigt. We note that in the proof of the Hales–Jewett theorem the bounds one obtains on the parameters involved in it turn out to be primitive recursive and are essentially the same as Shelah’s bounds from [12]. We will, however, leave it to the reader to check the details of this estimate. Finally, in Theorem 7.5, we give an interesting limiting example for which a Ramsey theorem fails. The objects that are being colored can be viewed as Lipschitz surjections with Lipschitz constant 1 between initial segments of natural numbers. This example is motivated by considerations in topological dynamics. More applications of the abstract approach to Ramsey theory involving finite trees can be found in [15].

As far as proofs of the known results are concerned, one advantage of the approach given here is its uniformity. Apart from it, however, the proofs that we obtain differ somewhat from the standard arguments. Moreover, our approach provides a hierarchy of the Ramsey results according to the number of times the abstract Ramsey theorem is applied in their proofs. For example, the classical Ramsey theorem requires one such application, the Hales–Jewett theorem requires two applications, the Graham–Rothschild theorem three, and the self-dual Ramsey theorem four.

Two comments about context. It may be worthwhile to point out that, on the conceptual level, the elegant approach of Graham, Leeb and Rothschild [1] and of Spencer [16] to finite Ramsey theorems for spaces is very much different from the approach presented here. The differences on the technical level are equally large. One main such difference is that, unlike here, the setting of [1] and [16] has a concrete pigeonhole principle built into it, which in that approach is the Hales–Jewett theorem.

The pure Ramsey theory, which is the subject matter of this paper, is a foundation on which the Ramsey theory for finite structures is built, but is not a part of it. Consequently, the methods of the present paper have nothing directly to say about the structural Ramsey theory as developed for relational structures by Nešetřil and Rödl in [7], [8], [9], and by Prömel in [10] and, more recently, for structures that incorporate both relations and functions by the author in [13], [14].
1.2. **Self-dual Ramsey theorem.** Here and in the rest of the paper we consistently use the language of rigid surjections and increasing injections rather than that of partitions and sets. (This language was proposed in [11].) In our opinion, this choice is more satisfying from the theoretical point of view and, unlike the other one, it easily accommodates objects coming from topology such as walks in Subsection 7.4. Note, however, that the abstract approach is also applicable to the partition and the set formalisms. A canonical way of translating statements in one language into the other is explained in Subsection 1.3. In particular, the self-dual Ramsey theorem is restated at the end of that subsection.

We consider 0 to be a natural number. As is usual, we adopt the convention that for a natural number \( N \),

\[ [N] = \{1, \ldots, N\}. \]

In particular, \([0] = \emptyset\).

As an application of the abstract approach outlined in Subsection 1.1, we prove a new, self-dual Ramsey theorem. As already mentioned, the Graham–Rothschild theorem plays the role of the pigeonhole principle in the proof of this theorem.

The classical Ramsey theorem can be stated as follows.

*Given the number of colors \( d \) and natural numbers \( K \) and \( L \), there exists a natural number \( M \) such that for each \( d \)-coloring of all increasing injections from \([K]\) to \([M]\) there exists an increasing injection \( j_0 : [L] \to [M] \) such that

\[ \{ j_0 \circ i : i : [K] \to [L] \text{ an increasing injection} \} \]

is monochromatic.*

The dual Ramsey theorem, that is, due to Graham and Rothschild, can be stated as follows.

*Given the number of colors \( d \) and natural numbers \( K \) and \( L \), there exists a natural number \( M \) such that for each \( d \)-coloring of all rigid surjections from \([M]\) to \([K]\) there exists a rigid surjection \( t_0 : [M] \to [L] \) such that

\[ \{ s \circ t_0 : s : [L] \to [K] \text{ a rigid surjection} \} \]

is monochromatic.*

Here a rigid surjection is a surjection with the additional property that images of initial segments of its domain are initial segments of its range, or, in other words, for each \( y \in [L] \), we have

\[ s(y) \leq 1 + \max_{1 \leq x < y} s(x) \]

with the convention that maximum over the empty set is 0. Note that the notion of rigid surjection is obtained simply by dualizing the notion of increasing injection: increasing injections are injections such that preimages...
of initial segments are initial segments, while rigid surjections are surjections such that images of initial segments are initial segments.

It is natural to ask if a “self-dual” Ramsey theorem exists that combines the two statements above. We formulate now such a self-dual theorem. We will be coloring pairs consisting of a rigid surjection and an increasing injection interacting with each other in a certain way. (A formulation using partitions and sets is given at the end of Subsection 1.3.) Let $K, L$ be natural numbers. We call a pair $(s, i)$ a connection between $L$ and $K$ if $s: [L] \to [K]$, $i: [K] \to [L]$ and for each $x \in [K]$

$$s(i(x)) = x \text{ and } \forall y < i(x) s(y) \leq x.$$ 

In other words, $i$ is a left inverse of $s$ with the additional property that at each $x \in [K]$ the value $i(x)$ is picked only from among those elements of $s^{-1}(x)$ that are “visible from $x$,” that is, from those $y' \in s^{-1}(x)$ for which

$$s \upharpoonright \{y : y < y'\} \leq x.$$

We write

$$(s, i): [L] \leftrightarrow [K].$$

It is easy to see that if $(s, i)$ is a connection, then $i$ is an increasing injection and $s$ is a rigid surjection. Also for each rigid surjection $s$ there is an increasing injection $i$ (usually many such injection) with $(s, i)$ a connection, and for each increasing injection $i$ there is a rigid surjection $s$ (again, usually many such surjections) with $(s, i)$ a connection.

Given connections $(s, i): [L] \leftrightarrow [K]$ and $(t, j): [M] \leftrightarrow [L]$, define

$$(t, j) \cdot (s, i): [M] \leftrightarrow [K]$$

as

$$(s \circ t, j \circ i).$$

Note that the orders of the compositions in the two coordinates are different from each other. One sees easily that the composition of two connections is a connection.

In Subsection 1.3 we show the following theorem; we rephrase it in terms of partitions and sets at the end of Subsection 1.3.

**Theorem 1.1.** Let $d > 0$ be a natural number. Let $K$ and $L$ be natural numbers. There exists a natural number $M$ such that for each $d$-coloring of all connections between $M$ and $K$ there is $(t_0, j_0): [M] \leftrightarrow [L]$ such that

$$\{(t_0, j_0) \cdot (s, i): (s, i): [L] \leftrightarrow [K]\}$$

is monochromatic.
The classical Ramsey theorem is just the theorem above for colorings that do not depend on the first coordinate; the Graham–Rothschild theorem is the above theorem for colorings that do not depend on the second coordinate.

1.3. Translation of rigid surjections into parameter sets or combinatorial cubes. We show here how to translate statements involving parameter sets (sometimes called combinatorial cubes) into statements about rigid surjections. This latter language was proposed by Prömel and Voigt [11]. It has been used in papers [13] and [14] in the context of structural Ramsey theory.

Let $A, l, n \in \mathbb{N}$. By an $A$-parameter set of dimension $l$ on $n$ we understand a pair of the form

$$(1.1) \quad V = (g, G),$$

where $G$ consists of $l$ non-empty, pairwise disjoint subsets of $[n]$ and $g: [n] \setminus \bigcup G \to [A]$. An $A$-parameter set of dimension $k$ on $n$ is a subobject of $V$ as in (1.1) if each set in $F$ is the union of some sets in $G$ included in its domain is constant. We say that $U$ is an $A$-parameter subset of $V$.

With each $A$-parameter set of dimension $l$ on $n$ as in (1.1), we associate a rigid surjection $s_V: [A + n] \to [A + l]$ as follows. We enumerate the sets in $G$ as $Y_1, \ldots, Y_l$ so that $\min Y_i < \min Y_j$ if $i < j$. We let

$s_V \upharpoonright [A] = \text{id}_{[A]}$

and

$s_V(A + x) = \begin{cases} A + i, & \text{if } x \in Y_i; \\ g(x), & \text{if } x \in [n] \setminus \bigcup_{i=1}^l Y_i. \end{cases}$

This association is a bijection between all $A$-parameter sets of dimension $l$ on $n$ and all rigid surjections $s: [A + n] \to [A + l]$ with the property $s \upharpoonright [A] = \text{id}_{[A]}$. Moreover, it is not difficult to check, and we leave it to the reader, that an $A$-parameter set $U$ of dimension $k$ on $n$ is a subobject of an $A$-parameter set $V$ of dimension $l$ on $n$ if and only if there is a rigid surjection $r: [A + l] \to [A + k]$ with $r \upharpoonright [A] = \text{id}_{[A]}$ and such that

$$(1.2) \quad s_U = r \circ s_V,$$

and for each rigid surjection $r: [A + l] \to [A + k]$ with $r \upharpoonright [A] = \text{id}_{[A]}$ there is an $A$-parameter set $U$ of dimension $k$ on $n$ that is a subobject of $V$ such that (1.2) holds. These remarks give the statements of the Hales–Jewett theorem and the Graham–Rothschild theorem in terms of parameter sets as consequences of the statements that we prove in Subsections 7.1 and 7.2.
To facilitate the translation process, we state now the Graham–Rothschild theorem in terms of parameter sets:

Fix $A \in \mathbb{N}$ and $d > 0$. For each $k, l \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that for each $d$-coloring of all $A$-parameter subsets of dimension $k$ of an $A$-parameter set $V$ of dimension $m$ there exists an $A$-parameter subset $U$ of $V$ of dimension $l$ such that all $A$-parameter subsets of $U$ of dimension $k$ get the same color.

In a similar manner, one gets statements due to Voigt [20] and corresponding to the partial rigid surjection statements of Sections 7.1 and 7.2.

To obtain statements involving partitions in place of parameter sets, one notes that if $(g, \mathcal{G})$ a 0-parameter set of dimension $l$ on $n$, then $g$ is necessarily the empty function and $\mathcal{G}$ is a partition of $[n]$ into $l$ non-empty sets. Thus, we can identify the 0-parameter set with the partition $\mathcal{G}$. This identification is bijective and such that subobjects of a 0-parameter set are identified with coarser partitions of the partition associated with the 0-parameter set.

Again, to give an instance of a translation, we rephrase the statement of the self-dual Ramsey theorem in terms of partitions and sets. Let $\mathcal{R}$ be a partition of $[n]$ and let $C$ be a subset of $[n]$. Let $m \in \mathbb{N}$. We say that $(\mathcal{R}, C)$ is an $m$-connection if $\mathcal{R}$ and $C$ have $m$ elements each and, upon listing $\mathcal{R}$ as $R_1, \ldots, R_m$ with $\min R_i < \min R_{i+1}$ and $C$ as $c_1, \ldots, c_m$ with $c_i < c_{i+1}$, we have $c_i \in R_i$ for $i \leq m$ and $c_i < \min R_{i+1}$ for $i < m$. We say that an $l$-connection $(\mathcal{Q}, B)$ is an $l$-subconnection of an $m$-connection $(\mathcal{R}, C)$ if $\mathcal{R}$ is a coarser partition than $\mathcal{Q}$ and $B \subseteq C$. Here is a reformulation if the self-dual Ramsey theorem (Theorem 1.1):

Let $d > 0$. For each $k, l \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that for each $d$-coloring of all $k$-subconnections of an $m$-connection $(\mathcal{R}, C)$ there exists an $l$-subconnection $(\mathcal{Q}, B)$ of $(\mathcal{R}, C)$ such that all $k$-subconnections of $(\mathcal{Q}, B)$ get the same color.

2. Algebraic and Ramsey notions

In this section, we introduce the new algebraic notions needed to phrase and prove the general Ramsey theorem. We illustrate the new notions with series of examples, one related to the classical Ramsey theorem, the other one to the Hales–Jewett theorem. We also make comments and state simple lemmas concerning the notions. The more impatient reader can go over the definitions only and after doing that skip ahead to Section 3. The progression of notions and results is as follows:

- local actoids, a most basic notion of action;
- actoids of sets over local actoids, a lift of the action on a local actoid to subsets;
— formulation of the \textit{Ramsey property} for actoids of sets over local actoids;
— \textit{backgrounds}, local actoids with an added operator of truncation;
— formulation of the \textit{pigeonhole principle} for actoids of sets over backgrounds.

In Section 3, we continue with
— a proof that the pigeonhole principle implies the Ramsey property for actoids of sets over backgrounds.

In Section 4, we go further as follows:
— formulation of the \textit{local pigeonhole principle} for actoids of sets over backgrounds;
— \textit{normed backgrounds}, backgrounds with al norm to a linear ordering;
— a proof that the local pigeonhole principle implies the Ramsey property for actoids of sets over normed backgrounds.

Recall that for \( N \in \mathbb{N} \), we let
\[ [N] = \{1, \ldots, N\}. \]

In the sequel, we use letters \( K, L, M, N, P, Q \), possibly with subscripts, to stand for natural numbers.

2.1. \textbf{Local actoids}. The notion of local actoid defined below is the most rudimentary version of a notion of action much like a semigroup action on a set.

\textbf{Definition 2.1.} By a \textit{local actoid} we understand two sets \( A \) and \( Z \), a partial binary function from \( A \times A \) to \( A \):
\[(a, b) \rightarrow a \cdot b,\]
and a partial binary function from \( A \times Z \) to \( Z \):
\[(a, z) \rightarrow a . z\]
such that for \( a, b \in A \) and \( z \in Z \) if \( a \cdot (b . z) \) and \( (a \cdot b) . z \) are both defined, then
\begin{equation}
(a \cdot (b . z)) = ((a \cdot b) . z). \tag{2.1}
\end{equation}

The binary operation \( \cdot \) on a local actoid as above will be called \textit{multiplication} and the binary operation \( . \) will be called \textit{action}. Unless otherwise stated, the multiplication will be denoted by \( a \cdot b \) and the action by \( a . z \).

With some abuse of notation, we denote a local actoid as in the definition above by \((A, Z)\).

To gain some intuitions about local actoids, one may think of both \( A \) and \( Z \) as sets of functions with multiplication \( a \cdot b \) on \( A \) corresponding to...
composition $a \circ b$ that is defined only when the range of $b$ is included in the domain of $a$. Similarly, the action of $A$ on $Z$, $a \cdot z$, corresponds to composition $a \circ z$ that is defined when the range of $z$ is included in the domain of $a$.

We will have two sequences of examples illustrating the main notions that are being introduced: sequence A leads to the classical Ramsey theorem, sequence B leads to the Hales–Jewett theorem.

**Example A1.** By an *increasing injection* we understand a strictly increasing function from $[K]$ to $\mathbb{N}$ for some $K \in \mathbb{N}$. Let

$$II = \{i : i \text{ is an increasing injection}\}.$$ 

We let $B = Y = II$ and we make $(B, Y)$ into a local actoid as follows. For $i, j \in II$, $j \cdot i$ and $j \cdot i$ are defined if $\text{range}(i) \subseteq \text{domain}(j)$ and then

$$j \cdot i = j \cdot i = j \circ i.$$ 

**Example B1.** Fix $K_0 \in \mathbb{N}$. Let

$$IS = \{p : \exists K, L \in \mathbb{N} (p : [L] \to [K] \text{ is a non-decreasing surjection})\}$$

and let

$$X_{K_0} = \{f : \exists L \in \mathbb{N} f : [L] \to \{0\} \cup [K_0]\}.$$ 

We adopt the convention that writing $p : [L] \to [K]$ for $p \in IS$ indicates that the image of $p$ is $[K]$, that is, $p$ is surjective onto $[K]$.

Let $p, q \in IS$ be such that $p : [L] \to [K]$ and $q : [N] \to [M]$. We declare $q \cdot p$ defined precisely when $M \geq L$ and let

$$q \cdot p = p \circ q,$$

where, naturally, the composition on the right is understood to be defined precisely for those $x \in [N]$ for which $q(x) \in [L]$. Note that the orders in which $p$ and $q$ appear on the two sides of the equation are reversed. Clearly, $p \circ q : [N'] \to [K]$ for some $N' \leq N$ and $p \circ q \in IS$.

For $p : [N] \to [M]$, $p \in IS$ and $f : [L] \to \{0\} \cup [K_0]$, $f \in X_{K_0}$, declare $p \cdot f$ to be defined precisely when $M \geq L$ and let

$$p \cdot f = f \circ p,$$

where $f \circ p$ is defined on those $x \in [N]$ for which $p(x) \in [L]$. Such elements $x$ form an initial segment of $[M]$. It is easy to see that $(IS, X_{K_0})$ is a local actoid.
2.2. **Actoids of sets.** Here we lift the operations on a given local actoid to its subsets.

Let \((A, Z)\) be a local actoid. For \(F, G \subseteq A\), we declare \(F \cdot G\) to be defined if \(f \cdot g\) is defined for all \(f \in F\) and \(g \in G\), and we let

\[
F \cdot G = \{ f \cdot g : f \in F, g \in G \}.
\]

For \(F \subseteq A\) and \(S \subseteq Z\), we say that \(F . S\) is defined if \(f . x\) is defined for all \(f \in F\) and \(x \in S\) and in that case we let

\[
F . S = \{ f . x : f \in F, x \in S \}.
\]

If \(F = \{f\}\) for some \(f \in A\), we write

\[
f . S
\]

for \(\{f\} . S\), if it is defined.

We record the following easy lemma.

**Lemma 2.2.** Let \((A, Z)\) be a local actoid. For \(F, G \subseteq A\) and \(S \subseteq Z\) if \((F \cdot G) . S\) and \(F . (G . S)\) are both defined, then they are equal and, moreover, for \(f \in F, g \in G, x \in S\)

\[
(f \cdot g) . x = f . (g . x).
\]

The lemma above says, in particular, that the pair consisting of the family of all subsets of \(A\) and the family of all subsets of \(Z\) with the operations defined by (2.3) and (2.4) is a local actoid. We define now actoids of sets over \((A, Z)\) essentially as substructures of this local actoid, but fulfilling an important additional condition (2.5). There is no harm, as far as applications go, in thinking about \(F\) and \(S\) in the definition below as consisting of finite non-empty sets.

**Definition 2.3.** Let \((A, Z)\) be a local actoid. Let \(F\) be a family of subsets of \(A\) and \(S\) a family of subsets of \(Z\). Let

\[
(F, G) \to F \bullet G
\]

be a partial function from \(F \times F\) to \(F\) and let

\[
(F, S) \to F \bullet S
\]

be a partial function from \(F \times S\) to \(S\). We say that \((F, S)\) with these two operations is an actoid of sets over \((A, Z)\) provided that whenever \(F \bullet G\) is defined, then so is \(F \cdot G\) and

\[
F \bullet G = F \cdot G,
\]

whenever \(F \bullet S\) is defined, then so is \(F . S\) and

\[
F \bullet S = F . S.
\]
and for all $F, G \in \mathcal{F}$ and $S \in \mathcal{S}$,

\begin{equation}
(2.5) \quad \text{if } F \bullet (G \bullet S) \text{ is defined, then so is } (F \bullet G) \bullet S.
\end{equation}

Because of (2.1), for $F, G, S$ as in condition (2.5), one has $F \bullet (G \bullet S) = (F \bullet G) \bullet S$.

As an immediate consequence of Lemma 2.2, one sees that an actoid of sets is a local actoid, that is, condition (2.1) holds for the two operations defined on a local actoid of sets. So operations $\bullet$ and $\cdot$ are a multiplication and an action whose values are computed by pointwise multiplication and pointwise action. Local actoids with condition (2.5) seem to be minimal action-like objects that naturally give rise to semigroupoids (for a definition and context see [17]) in the same way that group actions give rise to transformation groupoids (see [21]).

**Lemma 2.4.** Let $(\mathcal{F}, \mathcal{S})$ be an actoid of sets. Let $S \in \mathcal{S}$ and $F_1, \ldots, F_n \in \mathcal{F}$. Assume that $z_1 = F_n \bullet (F_{n-1} \bullet \cdots (F_2 \bullet (F_1 \bullet S)))$ is defined. Then $z_2 = (F_n \bullet (F_{n-1} \bullet \cdots (F_2 \bullet F_1))) \bullet S$ and $z_3 = (((F_n \bullet F_{n-1}) \cdots F_2) \bullet F_1) \bullet S$ are defined and $z_1 = z_2 = z_3$.

**Proof.** One proves the existence of $z_2$ and $z_1 = z_2$ and the existence of $z_3$ and $z_1 = z_3$ by separate inductions. To run the inductive argument for $z_1 = z_2$, note that by (2.1) and (2.5)

$$F_n \bullet (F_{n-1} \bullet \cdots (F_2 \bullet (F_1 \bullet S))) = F_n \bullet (F_{n-1} \bullet \cdots (F_3 \bullet ((F_2 \bullet F_1) \bullet S)))$$

and apply the inductive assumption. Similarly, to run the induction for $z_1 = z_3$, note that by (2.1) and (2.5)

$$F_n \bullet (F_{n-1} \bullet \cdots (F_2 \bullet (F_1 \bullet S))) = (F_n \bullet F_{n-1}) \bullet (F_{n-2} \bullet \cdots (F_2 \bullet (F_1 \bullet S))),$$

and apply the inductive assumption.

**Example A2.** We continue with Example A1 We let $\mathcal{F} = \mathcal{S}$ consist of all subsets of $\Pi$ of the form $(L^L_K)$, for $K, L \in \mathbb{N}, K \leq L$, where

$$\binom{L}{K} = \{i \in \Pi: i: [K] \rightarrow [L]\}.$$ 

Since an increasing injection from $[K]$ to $[L]$ is determined by, and of course determines, its image, that is, a $K$ element subset of $[L]$, the set $\binom{L}{K}$ defined above can be thought of as the set of all $K$ element subsets of $[L]$. 

For \( \binom{L}{K}, \binom{N}{M} \in \mathcal{F} = \mathcal{S} \), we let \( \binom{N}{L} \cdot \binom{L}{K} \) and \( \binom{N}{M} \cdot \binom{L}{K} \) be defined if and only if \( L = M \), and then we let
\[
\binom{N}{L} \cdot \binom{L}{K} = \binom{N}{L} = \binom{N}{K}.
\]

One easily checks that \((\mathcal{F}, \mathcal{S})\) with \( \cdot \) and \( \cdot \) is an actoid of sets over \((B, Y)\). Note that
\[
\binom{N}{M} \cdot \binom{L}{K} = \binom{N}{M}.
\]

Example B2. We continue with Example B1; in particular, \( K_0 \in \mathbb{N} \) remains fixed. We assume \( K_0 \geq 1 \). For \( K \geq 2 \), let \( S_K \subseteq X_{K_0} \) consist of all \( h: [K] \to \{0\} \cup [K_0] \) such that for some \( 1 \leq a \leq b < K \) and \( 0 \leq c \leq K_0 \), we have
\[
h(x) = \begin{cases} 
K_0, & \text{if } x \leq a; \\
c, & \text{if } a + 1 \leq x \leq b; \\
\max(1, K_0 - 1), & \text{if } b + 1 \leq x.
\end{cases}
\]

Formula (2.6) always gives \( h(1) = K_0 \) and \( h(K) = \max(1, K_0 - 1) \). Let
\[
S_{K_0} = \{S_K : K \geq 2\}.
\]

For \( 0 < K \leq L \), let
\[
F_{L,K} = \{p \in \text{IS} : p: [L] \to [K]\}.
\]

Let
\[
\mathcal{F}_0 = \{F_{L,K} : 0 < K \leq L\}.
\]

We let \( F_{N,M} \cdot F_{L,K} \) be defined if \( L = M \) and then we let
\[
F_{N,L} \cdot F_{L,K} = F_{N,K}.
\]

We let \( F_{M,L} \cdot S_K \) be defined precisely when \( K = L \) and, in this case, we let
\[
F_{M,K} \cdot S_K = S_M.
\]

It is now easy to check that \((\mathcal{F}_0, S_{K_0})\) with the operations defined above is an actoid of sets over \((\text{IS}, X_{K_0})\).
2.3. **Ramsey actoids of sets.** At this point, we have all the structure we need to precisely state the abstract Ramsey property alluded to in the introduction.

**Definition 2.5.** Let \((F, S)\) be an actoid of sets over a local actoid. We say that \((F, S)\) is Ramsey if for each \(d > 0\) and each \(S \in S\) there exists \(F \in F\) such that \(F \cdot S\) is defined and for each \(d\)-coloring of it there exists \(f \in F\) with \(f \cdot S\) monochromatic.

The condition of Ramseyness from the definition above, when interpreted for the actoid of sets from Example A2 becomes just the classical Ramsey theorem.

The aim now is to formulate a pigeonhole principle and show that it implies Ramseyness.

2.4. **Backgrounds.** To formulate the pigeonhole principle, we need additional structure on local actoids. We introduce a unary operation on an actoid that leads to the following definition.

**Definition 2.6.** A **background** is a local actoid \((A, Z)\) together with a unary function \(\partial: Z \to Z\) such that for \(a \in A\) and \(z \in Z\), if \(a \cdot z\) is defined, then \(a \cdot \partial z\) is defined and

\[
\partial a = \partial(a \cdot z).
\]

The additional function on \(Z\) will be called **truncation** and it will always be denoted by \(\partial\) possibly with various subscripts.

Condition (2.7) in the above definition states that in a background \((A, Z)\) the action of \(A\) on \(Z\) is done by partial homomorphisms of the structure \((Z, \partial)\). If we continue to think of a local actoid \((A, Z)\) as a family of functions \(A\) acting by composition on a family of functions \(Z\), then we can view truncation as a “restriction operator” on functions from \(Z\). So, condition (2.7) can be translated to say that if the composition of \(a\) and \(z\) is defined, then so is the composition of \(a\) and the restriction \(\partial z\) of \(z\) and its result is a restriction of the composition of \(a\) and \(z\), which we require to be given by the operator \(\partial\). Truncation can also be thought as producing out of an object \(z\) a simpler object \(\partial z\) of the same kind. In proofs, this point of view leads to inductive arguments.

We write

\[
\partial^t z
\]

for the element obtained from \(z\) after \(t \in \mathbb{N}\) applications of \(\partial\). For a subsets \(S \subseteq Z\), we write

\[
\partial S = \{\partial z : z \in S\}.
\]
Again, for $t \in \mathbb{N}$, we write

$$\partial^t S$$

for the result of applying the operation $\partial$ to $S$ $t$ times.

We record the following obvious lemma.

**Lemma 2.7.** Let $(A, Z)$ be a background. Then for $F \subseteq A$ and $S \subseteq Z$, if $F \cdot S$ is defined, then $F \cdot \partial S$ is defined and

$$\partial(F \cdot S) = F \cdot (\partial S).$$

It follows from the above lemma that if $(A, Z)$ is a background, then the pair consisting of the family of all subsets of $A$ and the family of all subsets of $Z$ becomes a background with the operations defined by (2.3), (2.4), and (2.8).

**Example A3.** We continue with Examples A1 and A2. For $i: [K] \to \mathbb{N}$ in $Y = \Pi$, let

$$\partial i = i \upharpoonright \max(0, K - 1).$$

It is easy to check that $(B, Y)$ with $\partial$ is a background.

**Example B3.** We continue with Examples B1 and B2. Fix $K_0 > 0$. On $X_{K_0}$, we define the following truncation. For $f \in X_{K_0}$, $f: [L] \to \{0\} \cup [K_0]$, let

$$(\partial f)(x) = \begin{cases} \max(1, K_0 - 1), & \text{if } f(x) = K_0; \\ f(x), & \text{if } f(x) \leq K_0 - 1. \end{cases}$$

It is now easy to see that for $p \in IS$, if $p \cdot f$ is defined, then

$$\partial(p \cdot f) = \partial(f \circ p) = (\partial f) \circ p = p \cdot (\partial f),$$

and, therefore, $(IS, X_{K_0})$ with $\partial$ defined above is a background.

### 2.5. Pigeonhole actoids of sets.

The following definition introduces the Ramsey-theoretic notion of pigeonhole actoid of sets. We formulate two pigeonhole principles: one here called (ph) and a localized version of it called (lph) in Section 4. They are not straightforward abstractions of the classical Dirichlet’s pigeonhole principle, rather they are conditions that make it possible to carry out inductive arguments proving the Ramsey property and that are easy to verify in concrete situations. This verification is done using the classical pigeonhole in the case of the classical Ramsey theorem; however, a variety of different principles are used in such verifications in other situations.

The pigeonhole principle (ph) below can be thought of in the following way. The Ramsey condition requires, upon coloring of $F \cdot S$, fixing of a color on $f \cdot S$ for some $f \in F$. In condition (ph), we consider the equivalence relation on $S$ that identifies $x_1$ and $x_2$ from $S$ if $\partial x_1 = \partial x_2$. The pigeonhole
principle (ph) requires fixing of a color on each equivalence class separately, rather than on the whole \( S \), after acting by an element of \( F \). We actually need to consider \( \partial^t S \) for \( t \in \mathbb{N} \), rather than just \( S \) in this condition (but on the first reading it may be helpful to take \( t = 0 \)). Here is the formal statement of the condition. Note that in it, \( F \cdot \partial^t S \) being defined follows from the assumption that \( F \cdot S \) is defined and from Lemma 2.7.

**Definition 2.8.** Let \((F, S)\) be an actoid of sets over a background \((A, Z)\). We call \((F, S)\) a pigeonhole actoid of sets if

\[(ph) \text{ for every } d > 0, t \geq 0, \text{ and } S \in S \text{ there exists } F \in F \text{ such that } F \cdot S \text{ is defined and for each } d\text{-coloring } c \text{ of } F \cdot (\partial^t S) \text{ there exists } f \in F \text{ such that for all } x_1, x_2 \in \partial^t S \text{ we have}
\]

\[\partial x_1 = \partial x_2 \implies c(f \cdot x_1) = c(f \cdot x_2).\]

It is convenient to illustrate the above definition by sequence B of examples. The localized pigeonhole principle from Section 4 will be illustrated by sequence A.

**Example B4.** Before we continue with Examples B1, B2, and B3, we state the most basic pigeonhole principle phrased here in a surjective form.

\[(*) \text{ For every } d > 0 \text{ and } K \geq 2 \text{ there exists } L \geq 2 \text{ such that for each } d\text{-coloring of all } q: [L] \to [2], q \in IS, \text{ there exists } q_0: [L] \to [K], q_0 \in IS, \text{ such that}
\]

\[\{p \circ q_0: p: [K] \to [2], p \in IS\}\]

is monochromatic.

The principle above is just a re-statement of the standard pigeonhole principle and one can take

\[L = d(K - 2) + 2.\]

We claim that the actoid of sets \((F_0, S_{K_0})\) defined in Example B2 is pigeonhole. Let \( S_K \in S_{K_0} \). Note first that in the background \((IS, X_{K_0})\), we have \( \partial^2 = \partial \). Thus, if \( t > 0 \), then for \( h_1, h_2 \in \partial^t S_K \), \( \partial h_1 = \partial h_2 \) implies \( h_1 = h_2 \). Therefore, we only need to check condition (ph) for \( t = 0 \). Note that if \( h \in S_K \), then \( \partial h \) uniquely determines \( h \) among functions in \( S_K \) unless \( h \) is of the following form: for some \( 0 < K_1 < K \),

\[(2.9) \quad h \upharpoonright [K_1] \equiv K_0 \text{ and } h \upharpoonright ([K] \setminus [K_1]) \equiv \max(1, K_0 - 1).\]

It follows that given \( d > 0 \), we need to find \( L \geq K \) so that for each \( d\text{-coloring } c \text{ of } F_{L,K} \cdot S_K \) there is \( p \in F_{L,K} \) such that the color \( c(h \circ p) \) is constant for \( h \in S_K \) of the form \( (2.9) \) as \( K_1 \) runs over \([K - 1]\). Such an \( L \) exists by the virtue of the basic pigeonhole principle \((*)\) stated above.
3. Pigeonhole implies Ramsey

We continue to adhere to the following convention: the three operations on a background \((A, Z)\) are denoted by \(\cdot\), \(\ast\), and \(\partial\), respectively, while the operations on a local actoid of sets over \((A, Z)\) are denoted by \(\bullet\) and \(\cdot\). We also use the notation set up in (2.3), (2.4), and (2.8).

Theorem 3.1 and Corollary 3.3 give general Ramsey statements derived from the pigeonhole principle (ph). Corollary 3.3 is simpler to state than Theorem 3.1 and is all that is needed from this theorem in most, but not all, situations.

**Theorem 3.1.** Let \((\mathcal{F}, \mathcal{S})\) be a pigeonhole actoid of sets over a background. For \(d > 0, t \geq 0,\) and \(S \in \mathcal{S}\), there exists \(F \in \mathcal{F}\) such that \(F \bullet S\) is defined and for each \(d\)-coloring \(c\) of \(F \bullet S\) there exists \(f \in F\) such that for \(x_1, x_2 \in S\)

\[
\partial^t x_1 = \partial^t x_2 \implies c(f \cdot x_1) = c(f \cdot x_2). 
\]

**Proof.** We derive the conclusion of the theorem assuming that for every \(d > 0, t \geq 0,\) and \(S \in \mathcal{S}\) there is \(F \in \mathcal{F}\) such that \(F \bullet S\) is defined and for every \(d\)-coloring \(c\) of \(\partial^t(F \bullet S)\) there is \(f \in F\) such that for \(x_1, x_2 \in S\)

\[
\partial^{t+1} x_1 = \partial^{t+1} x_2 \implies c(\partial^t(f \cdot x_1)) = c(\partial^t(f \cdot x_2)).
\]

This suffices since the assumption above follows from condition (ph) assumed of \((\mathcal{F}, \mathcal{S})\) as

\[
\partial^t(F \bullet S) = F \cdot (\partial^t S), \quad \text{and} \quad \partial^t(f \cdot x_1) = f \cdot (\partial^t x_1), \quad \partial^t(f \cdot x_2) = f \cdot (\partial^t x_2).
\]

Fix \(d > 0\). The argument is by induction on \(t \geq 0\) for all \(S \in \mathcal{S}\). For \(t = 0\), the conclusion is clear since it requires only that there be a non-empty \(F \in \mathcal{F}\) with \(F \bullet S\) defined, which is guaranteed by our assumption. Now we suppose that the conclusion of the theorem holds for \(t\) and we show it for \(t + 1\). Apply our assumption stated at the beginning of the proof to \(d, t,\) and \(S\) obtaining \(F_0 \in \mathcal{F}\). Note that \(F_0 \bullet S \in \mathcal{S}\). Apply the inductive assumption for \(t\) to \(F_0 \bullet S\) obtaining \(F_1 \in \mathcal{F}\). Note that \((F_1 \bullet F_0) \bullet S\) is defined, hence, since \((\mathcal{F}, \mathcal{S})\) fulfills condition (2.5), \((F_1 \bullet F_0) \bullet S\) is defined and, by Lemma 2.2

\[
(F_1 \bullet F_0) \bullet S = F_1 \bullet (F_0 \bullet S). 
\]

Note that \(F_1 \bullet F_0 \in \mathcal{F}\), and we claim that it works for \(t + 1\).

Let \(c\) be a \(d\)-coloring of \((F_1 \bullet F_0) \bullet S\). By (3.2), we can consider it to be a coloring of \(F_1 \bullet (F_0 \bullet S)\). By the choice of \(F_1\) there exists \(f_1 \in F_1\) such that for \(x, y \in S\) and \(f, g \in F_0\),

\[
\partial^t(f \cdot x) = \partial^t(g \cdot y) \implies c(f_1 \cdot (f \cdot x)) = c(f_1 \cdot (g \cdot y)).
\]
Define a $d$-coloring $\overline{c}$ of $\partial^d(F_0 \cdot S)$ by letting for $f \in F_0$ and $x \in S$

\begin{equation}
\overline{c}(\partial^d(f \cdot x)) = c(f_1 \cdot (f \cdot x)).
\end{equation}

The coloring $\overline{c}$ is well-defined by (3.3). By our choice of $F_0$, there exists $f_0 \in F_0$ such that for $x, y \in S$

\begin{equation}
\partial^{t+1} x = \partial^{t+1} y \implies \overline{c}(\partial^d(f_0 \cdot x)) = \overline{c}(\partial^d(f_0 \cdot y)).
\end{equation}

Combining (3.5) with (3.4), we see that for $x, y \in S$

\begin{equation}
\partial^{t+1} x = \partial^{t+1} y \implies c(f_1 \cdot (f_0 \cdot x)) = c(f_1 \cdot (f_0 \cdot y)).
\end{equation}

Now $f = f_1 \cdot f_0$ is as required since by (3.2) and Lemma 2.2 we have

\begin{align*}
f_1 \cdot (f_0 \cdot x) &= (f_1 \cdot f_0) \cdot x \quad \text{and} \quad f_1 \cdot (f_0 \cdot y) = (f_1 \cdot f_0) \cdot y,
\end{align*}

and the proof is completed. \hfill \Box

**Definition 3.2.** Let $(A, Z)$ be a background. A family $I$ of subsets of $Z$ is called vanishing if for every $F \in I$ there is $t \in \mathbb{N}$ such that $\partial^t F$ consists of at most one element.

**Corollary 3.3.** Let $(F, S)$ be a pigeonhole actoid of sets over a background. Assume $S$ is vanishing. Then $(F, S)$ is Ramsey.

**Proof.** The conclusion follows from Theorem 3.1 since for each $S \in S$ there is $t \in \mathbb{N}$ with $\partial^t S$ having at most one element. For this $t$, the left hand side in (3.1) holds for all $x_1, x_2 \in S$. \hfill \Box

4. Localizing the pigeonhole condition

We formulate here a localized version (lph) of condition (ph) and prove in Theorem 4.3 that, under mild assumptions, it implies (ph), making checking (ph) much easier. Even though condition (lph) can be stated for actoids of sets over backgrounds, the proof of Theorem 4.3 requires introduction in Subsection 4.2 of a new piece of structure on backgrounds, which is nevertheless found in all concrete situations.

4.1. **Localized version (lph) of (ph).** One can think of condition (lph) in the following way. In condition (ph), we are given a coloring of $F \cdot (\partial^d S)$ and are asked to find $f \in F$ making the coloring constant on each equivalence class of the equivalence relation on $\partial^d S$ given $\partial y_1 = \partial y_2$ for $y_1, y_2 \in \partial^d S$. Obviously, it is easier to fulfill the requirement of making the coloring constant, by multiplying by some $f \in F$, on a single, fixed equivalence class of this equivalence relation. Condition (lph) makes just such a requirement. The price for this weakening of the pigeonhole principle is paid by putting an additional restriction on the element $f \in F$ fixing the color. We will
comment on this restriction after the condition is stated. First, we introduce a piece of notation for equivalence classes of the equivalence relation mentioned above. For \( S \subseteq \mathbb{Z} \) and \( x_0 \in \mathbb{Z} \), put
\[
S_{x_0} = \{ y \in S : \partial y = x_0 \}.
\]
We will need the following notion.

**Definition 4.1.** Let \((A, \mathbb{Z})\) be a local actoid. For \( a, b \in A \), we say that \( b \) extends \( a \) if for each \( x \in \mathbb{Z} \) for which \( a \cdot x \) is defined, we have that \( b \cdot x \) is defined and \( a \cdot x = b \cdot x \).

For \( F \subseteq A \), let
\[
F_a = \{ b \in F : b \text{ extends } a \}.
\]

Let \((\mathcal{F}, \mathcal{S})\) be an actoid of sets over a background \((A, \mathbb{Z})\). The following criterion on \((\mathcal{F}, \mathcal{S})\) turns out to be the right formalization of the local version of (ph). Again, on the first reading, it may be helpful to take \( t = 0 \).

(lph) For \( d > 0 \), \( t \geq 0 \), \( S \in \mathcal{S} \), and \( x \in \partial^{t+1} S \), there is \( F \in \mathcal{F} \) and \( a \in A \) such that \( F \cdot S \) is defined, \( a \cdot x \) is defined, and for every \( d \)-coloring of \( F_a \cdot (\partial^t S)_x \) there is \( f \in F_a \) such that \( f \cdot (\partial^t S)_x \) is monochromatic.

The equivalence relation on \( \partial^t S \) given by \( \partial y_1 = \partial y_2 \) obviously has \( \partial^{t+1} S \) as its set of invariants, that is, two elements of \( \partial^t S \) are equivalent if and only if their images in \( \partial^{t+1} S \) under the function \( y \to \partial y \) are the same. In condition (lph), we consider the equivalence class given by \( x \in \partial^{t+1} S \) and we ask for \( a \in A \) that acts on a part of the set of invariants \( \partial^{t+1} S \) including \( x \) and is such that each \( d \)-coloring can be stabilized on \( (\partial^t S)_x \) by multiplication by some \( f \in F \) that acts in a manner compatible with \( a \).

### 4.2. Normed backgrounds.
We introduce here a new piece of structure on backgrounds.

**Definition 4.2.** Let \((A, \mathbb{Z})\) be a background. We say that \((A, \mathbb{Z})\) is normed if there is a function \(| \cdot | : \mathbb{Z} \to D\), where \((D, \leq)\) is a linear order, such that for \( x, y \in \mathbb{Z} \), \( |x| \leq |y| \) implies that for all \( a \in A \)
\[
a \cdot y \text{ defined } \Rightarrow (a \cdot x \text{ defined and } |a \cdot x| \leq |b \cdot y|).
\]

A function \(| \cdot | \) as in the above definition will be called a norm.

**Example A4.** We continue with Examples A1, A2, and A3. Define \(| \cdot | : \Pi \to \mathbb{N} \) by
\[
|i| = \max \text{range}(i),
\]
for \( i \in \Pi \). It is easy to see that the function defined above is a norm on \((B, Y)\); thus, \((B, Y)\) becomes a normed background.
Checking that \((\mathcal{F}, \mathcal{S})\) defined in Example A2 fulfills (lph) amounts to an application of the standard pigeonhole principle. It follows that \((\mathcal{F}, \mathcal{S})\) is a pigeonhole actoid of sets. Clearly \(\mathcal{S}\) is vanishing. Now an application of Corollary 4.4 below gives the classical Ramsey theorem.

### 4.3. Localized pigeonhole implies Ramsey

Here is the main theorem of this section.

**Theorem 4.3.** Let \((\mathcal{F}, \mathcal{S})\) be an actoid of sets over a normed background. Assume that \(\mathcal{S}\) consists of finite sets. If \((\mathcal{F}, \mathcal{S})\) fulfills criterion (lph), then \((\mathcal{F}, \mathcal{S})\) is pigeonhole, that is, it fulfills (ph).

**Proof.** Let \((A, Z)\) with \(\cdot\), \(\partial\), \(|\cdot|\) be the normed background over which \((\mathcal{F}, \mathcal{S})\) is defined. For the sake of clarity, in this proof, expressions of the form

\[
F_k F_{k-1} \cdots F_1 S \quad \text{and} \quad f k f_{k-1} \cdots f_1 x
\]

stand for

\[
F_k \bullet (F_{k-1} \bullet \cdots (F_1 \bullet S)) \quad \text{and} \quad f_k \bullet (f_{k-1} \bullet \cdots (f_1 \bullet x)),
\]

respectively. In particular, \(fx\) stands for \(f \cdot x\).

Fix \(d > 0\) and \(t \geq 0\). Let \(S \in \mathcal{S}\). Since \(\partial^{t+1} S\) is finite, we can list \(\partial^{t+1} S\) as \(x_1, x_2, \ldots, x_n\) with

\[
|x_n| \leq |x_{n-1}| \leq \cdots \leq |x_1|.
\]

We produce \(F_1, \ldots, F_n \in \mathcal{F}\) and \(b_1, \ldots, b_n \in A\) as follows. For \(1 \leq k \leq n + 1\), after \(k - 1\)-st step of the induction is completed, we have constructed \(F_1, \ldots, F_{k-1}, b_1, \ldots, b_{k-1}\). They have the following properties:

(a) \(F_{k-1} F_{k-2} \cdots F_1 S\) is defined;
(b) \(b_{k-1} b_{k-2} \cdots b_1 x_l\) is defined for \(k - 1 \leq l \leq n\);
(c) for \(1 \leq j \leq k - 1\), for every \(d\)-coloring of \(F_j \cdot (\partial^j (F_{j-1} \cdots F_1 S))\), there is \(f_j \in F_j\) extending \(b_j\) on \(\partial^{t+1} (F_{j-1} \cdots F_1 S)\) such that

\[
f_j \cdot (\partial^j (F_{j-1} \cdots F_1 S))
\]

is monochromatic;

(d) for \(k - 1 \leq l \leq n\), if \(f_j \in F_j\) extends \(b_j\) on \(\partial^{t+1} (F_{j-1} \cdots F_1 S)\), for \(1 \leq j \leq k - 1\), and \(\vec{x} \in S\) is such that \(\partial^{t+1} \vec{x} = x_l\), then

\[
\partial^{t+1} (f_{k-1} f_{k-2} \cdots f_1 \vec{x}) = b_{k-1} b_{k-2} \cdots b_1 x_l.
\]

We make step \(k \leq n\) of the recursion. With the fixed \(d\) and \(t\), we apply (lph) to \(F_{k-1} F_{k-2} \cdots F_1 S\), which exists by (a) and obviously is in \(\mathcal{S}\), and to \(b_{k-1} b_{k-2} \cdots b_1 x_k \in Z\), which exists by (b). This is permissible, since (c) and (d) taken with \(l = k\) imply

\[
b_{k-1} \cdots b_1 x_k \in \partial^{t+1} (F_{k-1} F_{k-2} \cdots F_1 S).
\]
This application of (lph) gives $F_k \in \mathcal{F}$ and $b_k \in A$. Now, (a), (b), and (c) follow immediately from our choice of $F_k$ and $b_k$ and the assumption $|x_l| \leq |x_k|$ for $l \geq k$. Point (d) is a consequence of (a) and (b) for $k$ and (d) for $k - 1$ by the following argument. Fix $k \leq l \leq n$. Let $f_j \in F_j$ extend $b_j$ on $\partial^{l+1}(F_{j-1} \cdots F_1 S)$, for each $1 \leq j \leq k$, and let $\tilde{x} \in S$ be such that $\partial^{l+1}\tilde{x} = x_l$. Note that using (d) for $k - 1$ and with the fixed $l$, we get

\begin{equation}
\partial^{l+1}(f_{k-1} \cdots f_1 \tilde{x}) = b_{k-1} \cdots b_1 x_l.
\end{equation}

Thus, since $b_kb_{k-1} \cdots b_1 x_l$ is defined, so is $b_k \partial^{l+1}(f_{k-1} \cdots f_1 \tilde{x})$. Now, since $f_k$ extends $b_k$ on $\partial^{l+1}(F_{k-1} \cdots F_1 S)$, we see that $f_k \partial^{l+1}(f_{k-1} \cdots f_1 \tilde{x})$ exists and

\begin{equation}
f_k \partial^{l+1}(f_{k-1} \cdots f_1 \tilde{x}) = b_k \partial^{l+1}(f_{k-1} \cdots f_1 \tilde{x}).
\end{equation}

Putting (4.1) and (4.2) together, we get (d) for $k$ since

\begin{align*}
\partial^{l+1}(f_k f_{k-1} \cdots f_1 \tilde{x}) &= f_k \partial^{l+1}(f_{k-1} \cdots f_1 \tilde{x}) \\
&= b_k \partial^{l+1}(f_{k-1} \cdots f_1 \tilde{x}) = b_k b_{k-1} \cdots b_1 x_l.
\end{align*}

So the recursive construction has been carried out. Note that by (a)

\begin{equation}
F_n F_{n-1} \cdots F_1 S
\end{equation}

is defined. We can apply Lemma 2.4 to the actoid $(\mathcal{F}, S)$ to see that the element $(F_n \bullet(F_{n-1} \cdots \bullet F_1)) \bullet S$ is defined as well. Now, $F_n \bullet(F_{n-1} \cdots \bullet F_1)$ is an element of $\mathcal{F}$, and we claim that for each $d$-coloring $c$ of

$$(F_n \bullet(F_{n-1} \cdots \bullet F_1)) \bullet (\partial^d S)$$

there are $f_1 \in F_1, \ldots, f_n \in F_n$ such that for $x_1, x_2 \in \partial^d S$ we have

\begin{equation}
\partial x_1 = \partial x_2 \implies c((f_n \cdot (f_{n-1} \cdots f_1)) \cdot x_1) = c((f_n \cdot (f_{n-1} \cdots f_1)) \cdot x_2).
\end{equation}

This will verify that $(\mathcal{F}, S)$ is pigeonhole.

Fix, therefore, a $d$-coloring $c$ of $(F_n \bullet(F_{n-1} \cdots \bullet F_1)) \bullet (\partial^d S)$. We recursively produce $f_n \in F_n, \ldots, f_1 \in F_1$. Note first that since (4.3) is defined, by Lemmas 2.7 and 2.4 we have that for each $1 \leq k \leq n$

$$(F_n \bullet(F_{n-1} \cdots \bullet F_1)) \bullet (\partial^d S)$$

\begin{align*}
&= \partial^d((F_n \bullet(F_{n-1} \cdots \bullet F_1)) \bullet S) \\
&= \partial^d(F_n F_{n-1} \cdots F_1 S) \\
&= F_n \cdot(F_{n-1} \cdots (F_k \cdot \partial^d(F_{k-1} F_{k-2} \cdots F_1 S))).
\end{align*}

Therefore, having produced $f_n, \ldots, f_{k+1}$, we can consider the $d$-coloring of $F_k \cdot \partial^d(F_{k-1} F_{k-2} \cdots F_1 S)$ given by

$$f y \rightarrow c(f_n \cdots f_{k+1} f y),$$

Putting (4.1) and (4.2) together, we get (d) for $k$ since

\begin{align*}
\partial^{l+1}(f_k f_{k-1} \cdots f_1 \tilde{x}) &= f_k \partial^{l+1}(f_{k-1} \cdots f_1 \tilde{x}) \\
&= b_k \partial^{l+1}(f_{k-1} \cdots f_1 \tilde{x}) = b_k b_{k-1} \cdots b_1 x_l.
\end{align*}

So the recursive construction has been carried out. Note that by (a)
for \( f \in F_k \) and \( y \in \partial^i(F_{k-1}F_{k-2} \cdots F_1 S) \). By the choice of \( F_k \), we get \( f_k \in F_k \) such that

\[
(4.5) \quad c(f_n \cdots f_k y) \text{ is constant for } y \in (\partial^i(F_{k-1}F_{k-2} \cdots F_1 S))_{b_k-1} b_{k-2} \cdots b_1 x_k
\]

and \( f_k \) extends \( b_k \) on \( \partial^{k+1}(F_{k-1} \cdots F_1 S) \).

We claim that \( f_n, \ldots, f_1 \) produced this way witness that (4.4) holds. Let \( y_1, y_2 \in \partial^i S \) be such that \( \partial y_1 = \partial y_2 \), and let this common value be \( x_k \) for some \( 1 \leq k \leq n \). For \( i = 1, 2 \), let \( \bar{x}_i \in S \) be such that \( y_i = \partial^i \bar{x}_i \). We have

\[
(4.6) \quad \partial^i(f_k f_{k-2} \cdots f_1 \bar{x}_i) = f_k f_{k-2} \cdots f_1 \partial^i \bar{x}_i = f_k f_{k-2} \cdots f_1 y_i
\]

and, therefore, applying condition (d) we get

\[
b_k-1 b_{k-2} \cdots b_1 x_k = \partial^{k+1}(f_k f_{k-2} \cdots f_1 \bar{x}_i) = \partial(f_k f_{k-2} \cdots f_1 y_i).
\]

The above equality and (4.6) give

\[
f_k f_{k-2} \cdots f_1 y_i \in (\partial^i(F_{k-1}F_{k-2} \cdots F_1 S))_{b_k-1} b_{k-2} \cdots b_1 x_k,
\]

which in light of (4.5) implies that

\[
(4.7) \quad c(f_n \cdots f_k y f_k f_{k-1} \cdots f_1 y_1) = c(f_n \cdots f_k y f_k f_{k-1} \cdots f_1 y_2)).
\]

Since (4.3) is defined, by Lemma 2.4, applied to the actoid \((F, S)\), and by inductively applied Lemma 2.2 we get that for \( i = 1, 2 \),

\[
f_n f_{n-1} \cdots f_1 \bar{x}_i = (f_n \cdot (f_{n-1} \cdots f_1)) \bar{x}_i
\]

and, therefore, using (4.6), we have

\[
f_n f_{n-1} \cdots f_1 y_i = \partial^i(f_n f_{n-1} \cdots f_1 \bar{x}_i)
= \partial^i((f_n \cdot (f_{n-1} \cdots f_1)) \bar{x}_i) = (f_n \cdot (f_{n-1} \cdots f_1)) y_i.
\]

From this and from (4.7) the conclusion follows. \( \square \)

The following corollary is an immediate consequence of Theorem 4.3 and Corollary 3.3.

**Corollary 4.4.** Let \((F, S)\) be an actoid of sets over a normed background. Assume that \( S \) consists of finite sets and is vanishing. If \((F, S)\) fulfills criterion (lph), then \((F, S)\) is Ramsey.

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4.4. **Remarks on normed backgrounds.** The main new algebraic notion introduced in the paper is the notion of normed background. Below, we give a list of conditions that are more symmetric than those defining normed backgrounds. We then prove in Lemma 4.5 that the new conditions define a structure that is essentially equivalent to a normed background. It is worth remarking that all the normed backgrounds in the present paper fulfill the conditions below.

Let \((A, Z, \cdot, \cdot, \partial, | \cdot |)\) be such that \(\cdot\) is a partial function from \(A \times A\) to \(A\), \(\cdot\) is a partial function from \(A \times Z\) to \(Z\), \(\partial\) is a function from \(Z\) to \(Z\) and \(| \cdot |\) is a function from \(Z\) to a set with a linear order \(\leq\). We consider the following set of conditions:

(a) if \(a \cdot (b \cdot z)\) and \((a \cdot b) \cdot z\) are defined for \(a, b \in A\) and \(z \in Z\), then
\[
a \cdot (b \cdot z) = (a \cdot b) \cdot z;
\]

(b) if \(a \cdot z\) and \(a \cdot \partial z\) are defined for \(a \in A\) and \(z \in Z\), then
\[
\partial(a \cdot z) = a \cdot \partial z;
\]

(c) \(|\partial z| \leq |z|\) for each \(z \in Z\);

(d) if \(|y| \leq |z|\) and \(a \cdot y\) and \(a \cdot z\) are defined for \(a \in A\) and \(y, z \in Z\), then
\[
|a \cdot y| \leq |a \cdot z|;
\]

(e) assuming \(|y| \leq |z|\) for \(y, z \in Z\), then for every \(a \in A\) if \(a \cdot z\) is defined, then so is \(a \cdot y\).

**Lemma 4.5.**

(i) If \((A, Z, \cdot, \cdot, \partial, | \cdot |)\) fulfills conditions (a)–(e), then \((A, Z)\) with \(\cdot, \cdot, \partial\) and \(| \cdot |\) is a normed background

(ii) If \((A, Z)\) with \(\cdot, \cdot, \partial\) and \(| \cdot |\) is a normed background, then there is a function \(| \cdot |_1\) on \(Z\) such that \((A, Z, \cdot, \cdot, \partial, | \cdot |_1)\) fulfills conditions (a)–(e).

**Proof.** (i) Almost all the properties defining a normed background are already explicit among (a)–(e). One only needs to check that for \(a \in A\) and \(z \in Z\) if \(a \cdot z\) is defined, then so is \(a \cdot \partial z\), and this property follows from (c) and (e).

(ii) Let \(L\) be the underlying set of the linear order that is the range of the norm \(| \cdot |\) on the normed background \((A, Z)\). By making \(L\) bigger and appropriately extending the linear order to the bigger set, we can assume that each non-empty subset of \(L\) has infimum. Now define \(| \cdot |_1: Z \to L\) by letting
\[
|z|_1 = \inf\{|y|: y \in Z \text{ and } z = \partial^t y \text{ for some } t \in \mathbb{N}\}.
\]

One checks without much difficulty that this definition works. \(\Box\)
5. Propagating the pigeonhole principle

In this section, we prove two results that make it possible to propagate condition (ph) to new examples. In the first result, we show how to obtain condition (ph) on appropriately defined finite products assuming it holds on the factors. The second result involves the notion of interpretation of sets from an actoid of sets in other actoids of sets. This result shows that if each set from an actoid of sets is interpretable in some pigeonhole actoid of sets then that actoid of sets is pigeonhole.

5.1. Products. We prove here a consequence of Theorem 3.1 that extends this theorem to products. First, we set up a general piece of notation. Let $X_i, 1 \leq i \leq l$, be sets, and let $U_i$ be a family of subsets of $X_i$. Let

$$
\prod_{i=1}^{l} U_i = \{ \prod_{i=1}^{l} U_i : U_i \in U_i \text{ for } i = 1, \ldots, l \}.
$$

When $U_i = U$ for all $1 \leq i \leq l$, we write $\prod_i U$ for $\prod_{i=1}^{l} U_i$. Note that $\prod_i U_i$ consists of subsets of $\prod_{i=1}^{l} X_i$.

Let $(A_i, Z_i), 1 \leq i \leq l$, be backgrounds. The multiplication on each of them is denoted by the same symbol $\cdot$; the truncation on $Z_i$ is denoted by $\partial_i$.

The product of $(A_i, Z_i), 1 \leq i \leq l$, is defined in the natural coordinatewise way. Its underlying sets are

$$
\prod_{i=1}^{l} A_i \quad \text{and} \quad \prod_{i=1}^{l} Z_i;
$$

the multiplication $(a_i) \cdot (b_i)$, for $(a_i), (b_i) \in \prod_{i=1}^{l} A_i$, is declared to be defined precisely when $a_i \cdot b_i$ is defined for each $i \leq l$ and then

$$(a_i) \cdot (b_i) = (a_i \cdot b_i);$$

the action $(a_i) \cdot (z_i)$, for $(a_i) \in \prod_{i=1}^{l} A_i$ and $(z_i) \in \prod_{i=1}^{l} Z_i$ is defined precisely when $a_i \cdot z_i$ is defined for each $i \leq l$ and then

$$(a_i) \cdot (z_i) = (a_i \cdot z_i);$$

the truncation $\partial_i$ of $(z_i)$ is given by

$$(\partial_i z_i).$$

The definitions above describe the product background $(\prod_{i=1}^{l} A_i, \prod_{i=1}^{l} Z_i)$. If $(A_i, Z_i) = (A, Z)$ for each $i \leq l$, we write $(A^l, Z^l)$ for $(\prod_{i=1}^{l} A_i, \prod_{i=1}^{l} Z_i)$.

Let now $(F_i, S_i)$ be actoids of sets over $(A_i, Z_i), 1 \leq i \leq l$. We define the operations on $\prod_i F_i$, $\prod_i S_i$ as follows. We declare

$$
(\prod_{i=1}^{l} F_i) \cdot (\prod_{i=1}^{l} G_i)
$$

to
be defined precisely when \( F_i \cdot G_i \) is defined for each \( 1 \leq i \leq l \) and then we let

\[
\left( \prod_{i=1}^{l} F_i \right) \cdot \left( \prod_{i=1}^{l} G_i \right) = \prod_{i=1}^{l} (F_i \cdot G_i),
\]

and \( \left( \prod_{i \leq l} F_i \right) \cdot \left( \prod_{i \leq l} S_i \right) \) is defined if \( F_i \cdot S_i \) is defined for each \( i \) and then

\[
\left( \prod_{i=1}^{l} F_i \right) \cdot \left( \prod_{i=1}^{l} S_i \right) = \prod_{i=1}^{l} (F_i \cdot S_i).
\]

It is easy to check that \( (\bigotimes F_i, \bigotimes S_i) \) with these operations is an actoid of sets over the background \( (\prod_{i \leq l} A_i, \prod_{i \leq l} Z_i) \).

The following proposition propagates the pigeonhole principle from factors to products. Note that in the proof of the proposition, Theorem 3.1 is used.

**Proposition 5.1.** Let \( (F_i, S_i), 1 \leq i \leq l, \) be pigeonhole actoids of sets. Assume that each \( S_i \) consists of finite sets. Then \( (\bigotimes S_i, \bigotimes F_i) \) is a pigeonhole actoid of sets over the background \( (\prod_{i \leq l} A_i, \prod_{i \leq l} Z_i) \).

**Proof.** We define a background structure on 

\[ A_* = \prod_{i=1}^{l} A_i, \quad Z_* = \{0, \ldots, l-1\} \times \prod_{i=1}^{l} Z_i \]

as follows. The multiplication on \( A_* \) is the same as in the product background \( (\prod_{i \leq l} A_i, \prod_{i \leq l} Z_i) \). For \( (a_i) \in A_* \) and \( (p, (z_i)) \in Z_* \), we make \( (a_i) \cdot (p, (z_i)) \) be defined if \( a_i \cdot z_i \) is defined for all \( i \) and

\[
(a_i) \cdot (p, (z_i)) = (p, (a_i \cdot z_i)).
\]

For \( (p, (z_i)) \in Z_* \), let

\[
\partial_*(p, (z_i)) = (p + 1(\text{mod} \ l), (y_i)),
\]

where \( y_i = z_i \) if \( i \neq p + 1 \) and

\[
y_{p+1} = \partial_{p+1} z_{p+1}.
\]

It is easy to see that \( (A_*, Z_*) \) is a background.

Define

\[
F_* = \bigotimes F_i
\]

and let \( S_* \) consist of all sets of the form

\[
\{p\} \times S,
\]

where \( p \in \{0, \ldots, l-1\} \), and \( S \in \bigotimes S_i \). Define \( \bullet \) on \( F_* \) to coincide with \( \bullet \) on \( \bigotimes F_i \). Declare \( F \cdot (\{p\} \times S) \) to be defined if and only if \( F \cdot S \) is defined in \( (\bigotimes F_i, \bigotimes S_i) \), and let

\[
F \cdot (\{p\} \times S) = \{p\} \times (F \cdot S).
\]
It is easy to check that \((F_*, S_*)\) is an actoid of sets over \((A_*, Z_*)\).

We claim that \((F_*, S_*)\) is a pigeonhole actoid of sets. To prove it, fix \(d > 0, \ t \geq 0\) and

\[
\{p\} \times \prod_{i=1}^{l} S_i \in S_*,
\]

for some \(S_i \in S_i\). Note that

\[
(5.1) \quad \partial_*^t(\{p\} \times \prod_{i=1}^{l} S_i) = \{q\} \times \prod_{i=1}^{l} \partial_i^t S_i,
\]

where \(q = p + t \pmod{l}\) and \(t_i\) are some natural numbers. For \(i \neq q + 1\), pick \(F_i \in F_i\) such that \(F_i \cdot S_i\) is defined. Such \(F_i\) exists by condition (ph) with the number of colors equal to 1 for the actoids \((F_i, S_i)\). Now, we apply condition (ph) to \((F_{q+1}, S_{q+1})\) with the following number of colors:

\[
(5.2) \quad d \prod_{i \neq q+1} |F_i \cdot \partial_i^t S_i|.
\]

(Note that the number defined above is finite since \(F_i \cdot S_i\) is finite, as it belongs to \(S_i\), and so \(\partial_i^t(F_i \cdot S_i) = F_i \cdot (\partial_i^t S_i)\) is finite.) This application gives us \(F_{q+1} \in F_{q+1}\) such that \(F_{q+1} \cdot S_{q+1}\) is defined and for each coloring of \(F_{q+1} \cdot (\partial_{q+1}^{q+1} S_{q+1})\) with the number of colors given by \((5.2)\) there is \(f \in F_{q+1}\) such that for any two \(x, y \in \partial_{q+1}^{q+1} S_{q+1}\) fulfilling

\[
(5.3) \quad \partial_{q+1}^t x = \partial_{q+1}^t y,
\]

\(f . x\) and \(f . y\) get the same color. Having defined \(F_i, 1 \leq i \leq l\), note that

\[
\prod_{i=1}^{l} F_i \in F_*,
\]

and that

\[
(\prod_{i=1}^{l} F_i) \cdot (\{p\} \times \prod_{i=1}^{l} S_i)
\]

is defined. Given a \(d\)-coloring \(c\) of

\[
(\prod_{i=1}^{l} F_i) \cdot (\partial_*^t(\{p\} \times \prod_{i=1}^{l} S_i)),
\]

which set is equal to

\[
\{q\} \times F_1 \cdot (\partial_1^t S_1) \times \cdots \times F_{q+1} \cdot (\partial_{q+1}^{q+1} S_{q+1}) \times \cdots \times F_l \cdot (\partial_l^t S_l)
\]

by \((5.1)\), consider the coloring of \(F_{q+1} \cdot (\partial_{q+1}^{q+1} G_{q+1})\) defined by

\[
(5.4) \quad h \to (c(q, h_1, \ldots, h_q, h, h_{q+2}, \ldots, h_l): (h_i)_{i \neq q+1} \in \prod_{i \neq q+1} F_i \cdot (\partial_i^t S_i)).
\]
This is a coloring with the number of colors equal to \([5.2]\). Therefore, there exists \(f_{q+1} \in \mathcal{F}_{q+1}\) such that for any two \(x, y \in \partial^{q+1} \mathcal{S}_{q+1}\) fulfilling \([5.3]\), \(f_{q+1} \cdot x\) and \(f_{q+1} \cdot y\) get the same color. Pick \(f_i \in \mathcal{F}_i\) for \(i \neq q+1\) arbitrarily. With these choices \((f_i)\) is an element of \(\prod_{i \leq l} \mathcal{F}_i\). Note now that for \((q, (x_i)), (q, (y_i))) \in \{q\} \times \prod_{i=1}^l \partial^i \mathcal{S}_i = \partial^q(\{p\} \times \prod_{i=1}^l \mathcal{S}_i)\) we have

\[
(5.5) \quad \partial^i(q, (x_i)) = \partial^i(q, (y_i))
\]

precisely when \(x_i = y_i\) for \(i \neq q+1\) and \([5.3]\) holds for \(x_{q+1}\) and \(y_{q+1}\). This observation allows us to say that the definition of the coloring in \([5.4]\) and our choice of \(f_{q+1}\) imply that if \((5.5)\) holds, then

\[
c((f_i) \cdot (q, (x_i))) = c((f_i) \cdot (q, (y_i))).
\]

Thus, indeed, \((\mathcal{F}_*, \mathcal{S}_*)\) is a pigeonhole actoid of sets.

Now apply Theorem \([3.1]\) (with \(t = l\)) to the pigeonhole actoid of sets \((\mathcal{F}_*, \mathcal{S}_*)\) while keeping in mind that \(\bigotimes_{i \leq l} \mathcal{F}_i = \mathcal{F}_*\) and that for \(t \geq 0\) and \(x \in \prod_{i \leq l} \mathcal{Z}_i\), we have

\[
(0, \partial^l_x x) = \partial^l_x (0, x).
\]

The proposition follows.

\[\square\]

**Example B5.** We continue with Examples B1, B2, B3, and B4. Let \(l \in \mathbb{N}\).

Consider the product background \((\mathbb{I}^l \mathbb{S}_l, \mathbb{X}_l \mathbb{K}_0)\). Then, by Proposition \([5.1]\), the actoid of sets \((\bigotimes_{i \leq l} \mathcal{F}_0, \bigotimes_{i \leq l} \mathcal{S}_0)\) is pigeonhole. Here is a consequence of being pigeonhole for this actoid that will be used later.

*Given \(d > 0\) and \(L_1, \ldots, L_l\) there exist \(M_1, \ldots, M_l\) such that for each \(d\)-coloring of

\[
(F_{M_1, L_1} \bullet S_{L_1}) \times \cdots \times (F_{M_l, L_l} \bullet S_{L_l}) = S_{M_1} \times \cdots \times S_{M_l}
\]

there are \(p_1 \in F_{M_1, L_1}, \ldots, p_l \in F_{M_l, L_l}\) such that

\[
p_1 \cdot S_{L_1} \times \cdots \times p_l \cdot S_{L_l} = S_{L_1} \circ p_1 \times \cdots \times S_{L_l} \circ p_l
\]

is monochromatic.*

### 5.2. Interpretations

We introduce here a notion of interpretability.

**Definition 5.2.** Let \((\mathcal{F}, \mathcal{S})\) be an actoid of sets over a background \((A, X)\), and let \((\mathcal{G}, T)\) be an actoid of sets over a background \((B, Y)\). Let \(T \in \mathcal{T}\) and \(t \in \mathbb{N}\). We say that \((T, t)\) is interpretable in \((\mathcal{F}, \mathcal{S})\) if there exists \(S \in \mathcal{S}\), \(s \in \mathbb{N}\) and a function \(\alpha : \partial^t T \to \partial^s \mathcal{S}\) such that
(i) for \( y_1, y_2 \in \partial^d T \),
\[
\partial y_1 = \partial y_2 \implies \partial \alpha(y_1) = \partial \alpha(y_2);
\]
(ii) if \( F \cdot S \) is defined for some \( F \in F \), then there exist \( G \in G \), with \( G \cdot T \) defined, and a function \( \phi : F \to G \) such that for \( f_1, f_2 \in F \) and \( y_1, y_2 \in \partial^d T \)
\[
(5.6) \quad f_1 \cdot \alpha(y_1) = f_2 \cdot \alpha(y_2) \implies \phi(f_1) \cdot y_1 = \phi(f_2) \cdot y_2.
\]

**Proposition 5.3.** Let \((G, T)\) be an actoid of sets. If each \((T, t)\), with \( T \in T \) and \( t \in \mathbb{N} \), is interpretable in a pigeonhole actoid of sets, then \((G, T)\) is a pigeonhole actoid of sets.

**Proof.** Let \( T \in T \), \( t \in \mathbb{N} \), and \( d > 0 \) be given. Let \((F, S)\) be a pigeonhole actoid of sets over \((A, X)\) in which \((T, t)\) is interpretable. Find \( S \in S \), \( s \in \mathbb{N} \), and \( \alpha : \partial^d T \to \partial^s S \) as in the definition of interpretability. Since \((F, S)\) is pigeonhole, we can find \( F \in F \) such that \( F \cdot S \) is defined and for each \( d \)-coloring \( c' \) of \( F \cdot (\partial^s S) \) there exists \( f \in F \) such that for all \( x_1, x_2 \in \partial^s S \) we have
\[
(5.7) \quad \partial x_1 = \partial x_2 \implies c'(f \cdot x_1) = c'(f \cdot x_2).
\]

For \( F \) given above, find \( G \in G \) such that \( G \cdot T \) is defined and \( \phi : F \to G \) for which \([5.6]\) holds. Assume we have a \( d \)-coloring \( c \) of \( G \cdot (\partial^d T) \). Define a \( d \)-coloring \( c' \) of \( F \cdot (\partial^s S) \) as an arbitrary extension to \( F \cdot (\partial^s S) \) of the function given by
\[
\alpha(f \cdot y) = c(f \cdot y),
\]
where \( f \in F \) and \( y \in \partial^d T \). Note that \( c' \) is well defined by \([5.6]\). For this \( c' \), find \( f \in F \) for which \([5.7]\) holds. Let now \( y_1, y_2 \in \partial^d T \) be such that
\[
(5.8) \quad \partial y_1 = \partial y_2.
\]
Since condition (i) in the definition of interpretability holds for \( \alpha \), \([5.8]\) gives
\[
\partial \alpha(y_1) = \partial \alpha(y_2).
\]
Therefore, by the definition of \( c' \), by the choice of \( f \) and since \( \alpha(y_1), \alpha(y_2) \in S \), we get
\[
c(\phi(f) \cdot y_1) = c'(f \cdot \alpha(y_1)) = c'(f \cdot \alpha(y_2)) = c(\phi(f) \cdot y_2).
\]
Thus, the above equality is implied by \([5.8]\). It follows that \( \phi(f) \in G \) is as required by condition (ph) for the coloring \( c \). \( \square \)

The proposition above will be applied in Section 7.
6. EXAMPLES OF BACKGROUNDS AND ACTOIDS OF SETS

The remainder of the paper has mainly illustrative purpose. It contains applications of the general results proved so far to particular cases. These applications essentially do not involve new arguments; they do involve formulating definitions and interpreting some statements as other statements. In particular, even though the material of these last two sections concerns natural numbers, induction is not used in them.

6.1. Basic notions and notation. We fix here some notation and some notions needed in the sequel.

Let

\[ S = \{ v: \exists K, L \ (K \leq L \text{ and } v: [L] \rightarrow [K] \text{ is a surjection}\} . \]

We adopt the convention that for \( v \in S \) writing \( v: [L] \rightarrow [K] \) signifies that \( v \) is onto \([K]\). A rigid surjection is a function \( s: [L] \rightarrow [K] \) that is surjective and such that for each \( y \in [L] \) there is \( x \in [K] \) with \( s([y]) = [x] \). Let

\[ RS = \{ s \in S: s \text{ is a rigid surjection}\} . \]

Recall the notion of increasing surjection from (2.2). An increasing surjection is a function \( p: [L] \rightarrow [K] \) that is surjective and such that for \( y_1, y_2 \in [L] \) if \( y_1 \leq y_2 \), then \( p(y_1) \leq p(y_2) \), so strictly speaking \( p \) is a non-decreasing surjection. Let

\[ IS = \{ p \in S: p \text{ is an increasing surjection}\} . \]

Finally, we need the notion of augmented surjection, which are ordered pairs whose elements are a rigid surjection and an increasing surjection with these elements appropriately interacting with each other. Let

\[ AS = \{ (s,p): \exists K, L \in \mathbb{N} \ (s,p: [L] \rightarrow [K], p \in IS, s \leq p, \]

\[ \forall x \in [K] \ s(\max p^{-1}(x)) = x) \} . \]

It is easy to see that \((s,p) \in AS\) implies that \(s\) is a rigid surjection. Elements of \( AS \) are called augmented surjections.

6.2. Rules for composing and for truncating. First, we present some rules that are used when composing surjections and rigid surjections and, second, we describe some ways of truncating such objects.

6.2.1. The canonical composition of a rigid surjection and a surjection. Let \( v: [L] \rightarrow [K] \) be a surjection and let \( s: [N] \rightarrow [M] \) be rigid surjections. The canonical composition of \( v \) and \( s \), which we denote by \( v \circ s \), is defined if and only if \( L \leq M \). In this case, let \( N_0 \leq N \) be the largest number such that \( s(y) \leq L \) for all \( y \leq N_0 \). Define

\[ (6.1) \quad v \circ s \]
to be the usual composition of $v$ with $s \upharpoonright [N_0]$. It is easy to see that $v \circ s$ is the restriction of the usual composition of $v$ and $s$ to the largest initial segment of $[N]$ on which this composition is defined. Note that $v \circ s: [N_0] \to [K]$ is a surjection. If $v$ is a rigid surjection, then $v \circ s$ is a rigid surjection as well. If $v$ and $s$ are increasing surjections, then $v \circ s$ is an increasing surjection. It is easy to verify that if $v$ is a surjection, $s$, $t$ are rigid surjections, and $(v \circ s) \circ t$ and $v \circ (s \circ t)$ are both defined, then

$$(v \circ s) \circ t = v \circ (s \circ t).$$

This observation will be frequently used in the sequel.

6.2.2. *Forgetful truncation of rigid surjections.* The first type of truncation we introduce is obtained by forgetting the largest value of a function. It is defined on rigid surjections. We call it the *forgetful truncation* and we define it as follows. Let $s: [L] \to [K]$ be a rigid surjection. If $K > 0$, then $L > 0$, and we let

$$L_0 = \min\{y \in [L]: s(y) = K\}.$$ 

Define

$$(6.2) \quad \partial_f s = s \upharpoonright [L_0 - 1].$$

If $K = 0$, then $L = 0$ and $s$ is the empty function, and we let

$$(6.3) \quad \partial_f \emptyset = \emptyset.$$

Thus, unless $s$ is empty, $\partial_f s$ forgets the largest value of $s$ while remaining a rigid surjection. Unless $s$ is empty, $\partial_f s$ is a proper restriction of $s$.

6.2.3. *Confused truncation of surjections.* Another way of truncating a surjection is obtained by confusing the largest value with the one directly below it. This type of truncation is defined on non-empty surjections. We define the *confused truncation* as follows. Let $v: [L] \to [K]$ be a surjection with $K > 0$. Define for $y \in [L]$

$$(6.4) \quad (\partial_c v)(y) = \begin{cases} v(y), & \text{if } v(y) < K; \\ \max(1, K - 1), & \text{if } v(y) = K. \end{cases}$$

Note that $\partial_c v: [L] \to [\max(1, K - 1)]$ is a surjection. The confused truncation when applied to a non-empty rigid surjection gives a rigid surjection.
6.3. **Examples of backgrounds.** Three backgrounds were defined in Examples A3, B3, and B5. In this section, we describe a number of new backgrounds. They are used in the proofs of Ramsey-type results later on. A couple more backgrounds are given in Subsections 7.4.

**Normed background** \((A_1, X_1)\). Let \(A_1 = \text{RS}\) and \(X_1 = \{v \in S : v \neq \emptyset\}\).

For \(s_1, s_2 \in A_1\) let \(s_1 \cdot s_2\) be defined when the canonical composition \(s_2 \circ s_1\) is defined, and let \(s_1 \cdot s_2 = s_2 \circ s_1\).

For \(s \in A_1\) and \(v \in X_1\) let \(s \cdot v\) be defined precisely when the canonical composition \(v \circ s\) is defined and let \(s \cdot v = v \circ s\).

We equip \(X_1\) with the confused truncation \(\partial_c\) given by (6.4). Define a norm \(|\cdot| : X_1 \rightarrow \mathbb{N}\) by letting \(|v| = L\) for \(v \in X_1\) with \(v : [L] \rightarrow [K]\).

The following lemma is straightforward to prove.

**Lemma 6.1.** \((A_1, X_1)\) with the operations defined above is a normed background.

**Normed background** \((A_2, X_2)\). Let \(A_2 = X_2 = \text{RS}\). We define the multiplication on \(A_2\) by the same formula \(s_2 \cdot s_1 = s_1 \circ s_2\), for \(s_1, s_2 \in A_2\), and the action of \(A_2\) on \(X_2\) by \(t \cdot s = s \circ t\), for \(t \in A_2\) and \(s \in X_2\), where all the compositions are canonical compositions of rigid surjections and they are taken under the assumptions under which canonical composition is defined. We equip \(X_2\) with the forgetful truncation \(\partial_f\) given by (6.2) and (6.3). Define \(|\cdot| : X_2 \rightarrow \mathbb{N}\) by letting for \(t : [L] \rightarrow [K]\), \(|t| = L\).

The following lemma is again straightforward to prove.

**Lemma 6.2.** \((A_2, X_2)\) with the operations defined above is a normed background.

**Normed background** \((A_3, X_3)\). Let \(A_3 = X_3 = \text{AS}\). If \((s, p), (t, q) \in \text{AS}, s, p : [L] \rightarrow [K]\) and \(t, q : [N] \rightarrow [M]\), we let \((t, q) \cdot (s, p)\) and \((t, q) \cdot (s, p)\) be defined if \(M \geq L\) and in that case we let \((t, q) \cdot (s, p) = (t, q) \cdot (s, p) = ((s \circ t) \upharpoonright \text{dom}(p \circ q), p \circ q).\)
where all $\circ$ on the right hand side are canonical compositions, and the left hand side is defined under the conditions under which the canonical compositions on the right hand side are defined. We also define a truncation $\partial$ on $X_3$ by

$$\partial(s, p) = (s \restriction \text{dom}(\partial f p), \partial f p),$$

where $\partial f$ is the forgetful truncation. Furthermore, we define $| \cdot | : AS \to \mathbb{N}$ by

$$|(s, p)| = L$$

if $s, p : [L] \to [K]$. We leave checking the following easy lemma to the reader.

**Lemma 6.3.** $(A_3, X_3)$ with the operations defined above is a normed background.

### 6.4. Examples of actoids of sets.

We give here examples of actoids of sets that are relevant to further considerations. Three actoids of sets were already defined in Examples A2, B2, and B5, and two more will be defined in Subsections 7.4.

**Actoid** $(F_1, S_1)$ over $(A_1, X_1)$. The family $F_1$ consists of all sets of the form

$$F_{N,M,L_0} = \{ s \in RS : s : [N] \to [M], s \restriction [L_0] = \text{id}_{[L_0]} \},$$

for $0 < L_0 \leq M \leq N$. The family $S_1$ consists of sets of the form $S_{L,v_0}$ that are defined as follows. Let $v_0 : [L_0] \to [K_0]$ be a surjection for some $0 < K_0 \leq L_0$ and let $L \geq L_0$. Put

$$S_{L,v_0} = \{ v \in S : v : [L'] \to [K_0] \text{ for some } L \geq L' \geq L_0, \text{ and } v \restriction [L_0] = v_0 \}.$$

We let $F_{Q,P,N_0} \cdot F_{M,L,K_0}$ be defined if $K_0 = N_0$ and $P = M$ and in that case we put

$$F_{Q,M,K_0} \cdot F_{M,L,K_0} = F_{Q,L,K_0}.$$

We let $F_{P,N,M_0} \cdot S_{L,v_0}$ be defined, where $v_0 : [L_0] \to [K_0]$, if $L_0 = M_0$ and $L = N$ and in that case we set

$$F_{P,L,L_0} \cdot S_{L,v_0} = S_{P,v_0}.$$

We leave checking the following easy lemma to the reader.

**Lemma 6.4.**

(i) $(F_1, S_1)$ is an actoid of sets over $(A_1, X_1)$.

(ii) $\partial_c S_{L,v_0} = S_{L,\partial_c v_0}$.

**Actoids** $(F_2, S_2)$ and $(G_2, T_2)$ over $(A_2, X_2)$. The definition of the actoid $(F_2, S_2)$ over $(A_2, X_2)$ is like the definition of $(F_1, S_1)$ except that elements of sets in the actoid are now rigid surjection and, importantly, the truncation on $(A_2, X_2)$ is the forgetful truncation and not the confused one.
Let the family $F_2 = F_1$, and let the multiplication $\cdot$ on it be taken from $F_1$. Define the family $S_2$ to consist of all sets of the following form. Fix a rigid surjection $s_0: [L_0] \to [K_0]$, for some $L_0 \geq K_0$, let $K \geq K_0$ and $L \geq L_0$, and put

$$S_{L,K,s_0} = \{ s \in RS: s: [L] \to [K] \text{ and } s \upharpoonright [L_0] = s_0 \}.$$ 

Now, let $F_{P,N,M_0} \cdot S_{L,K,s_0}$ be defined, where $s_0: [L_0] \to [K_0]$, if $L_0 = M_0$ and $L = N$ and, in that case, set

$$F_{P,L,L_0} \cdot S_{L,K,s_0} = S_{P,K,s_0}.$$

We define now the actoid $(G_2, \mathcal{T}_2)$ over $(A_2, X_2)$. For $L_0 \leq M \leq N$, set

$$G_{N,M,L_0} = \{ s \in RS: s: [N] \to [M] \text{ for } L_0 \leq N' \leq N, \ s \upharpoonright [L_0] = \id_{[L_0]} \}.$$ 

Fix a rigid surjection $s_0: [L_0] \to [K_0]$, for some $K_0 \leq L_0$, let $K \geq K_0$ and $L \geq L_0$, and put

$$T_{L,K,s_0} = \{ s \in RS: s: [L'] \to [K] \text{ for some } L \geq L' \geq L_0, \text{ and } s \upharpoonright [L_0] = s_0 \}.$$ 

Define $G_2$ to consist of all sets of the form $G_{N,M,L_0}$ and $\mathcal{T}_2$ to consist of all sets of the form $T_{L,K,s_0}$.

Essentially as before, let $G_{Q,P,N_0} \cdot G_{M,L,K_0}$ be defined if $K_0 = N_0$ and $P = M$ and in that case let

$$G_{Q,M,K_0} \cdot G_{M,L,K_0} = F_{Q,L,K_0}.$$ 

Let $G_{P,N,M_0} \cdot T_{L,K,s_0}$, where $s_0: [L_0] \to [K_0]$, be defined if $L_0 = M_0$ and $L = N$, and set

$$G_{P,L,L_0} \cdot T_{L,K,s_0} = T_{P,K,s_0}.$$ 

The proof of the following lemma amounts to easy checking. We leave it to the reader.

**Lemma 6.5.**

1. $(F_2, S_2)$ and $(G_2, \mathcal{T}_2)$ are actoids of sets over $(A_2, X_2)$.
2. Let $s_0: [L_0] \to [K_0]$, for some $K_0 \leq L_0$, be a rigid surjection, and let $L \geq L_0$ and $K \geq K_0$. Then

$$\partial f S_{L,K,s_0} = T_{L-1,K-1,s_0} \text{ if } K > K_0, \quad \text{and} \quad \partial f S_{L,K,s_0} = \{ \partial f s_0 \}.$$ 

3. Let $s_0: [L_0] \to [K_0]$, for some $K_0 \leq L_0$, be a rigid surjection, and let $L \geq L_0$ and $K \geq K_0$. Then

$$\partial f T_{L,K,s_0} = T_{L-1,K-1,s_0} \text{ if } K > K_0, \quad \text{and} \quad \partial f T_{L,K,s_0} = \{ \partial f s_0 \}.$$ 

**Actoids $(F_3, S_3)$ and $(G_3, \mathcal{T}_3)$ over $(A_3, X_3)$**. In the definitions below, we are slightly less general than in the definitions of actoids of sets defined so far. As before, the actoids of sets consist of sets of elements of $A_3$ and $X_3$ that map a given $[L]$ to a given $[K]$, but we refrain from considering such sets with the additional requirement that elements in them start with a fixed
augmented surjection. This additional generality can be easily achieved, but
it is not needed in applications.

For $K, L \in \mathbb{N}$, $L \geq K > 0$, let

$$F_{L,K} = \{(s,p) \in AS : s,p : [L] \to [K] \text{ and } s^{-1}(K) = \{L\}\},$$

and put

$$F_3 = S_3 = \{F_{K,L} : L \geq K > 0\}.$$ 

We let $F_{N,M} \cdot F_{L,K}$ and $F_{N,M} \cdot F_{L,K}$ be defined if and only if $L = M$ and

$$F_{N,L} \cdot F_{L,K} = F_{N,L} \cdot F_{L,K} = F_{N,K}.$$ 

For $K, L \in \mathbb{N}$, $L \geq K$, let

$$G_{L,K} = \{(s,p) \in AS : s,p : [L'] \to [K] \text{ for some } L' \leq L\},$$

and put

$$G_3 = T_3 = \{G_{K,L} : L \geq K\}.$$ 

As before $G_{N,M} \cdot G_{L,K}$ and $G_{N,M} \cdot G_{L,K}$ are defined if and only if $L = M$ and

$$G_{N,L} \cdot G_{L,K} = G_{N,L} \cdot G_{L,K} = G_{N,K}.$$ 

Note that $G_{L,0} = \{((\emptyset, \emptyset))\}$ for each $L \in \mathbb{N}$.

Recall definition (6.5) of the truncation $\partial$ on $(A_3, X_3)$. The following lemma is straightforward to check.

**Lemma 6.6.** (i) $(F_3, S_3)$ and $(G_3, T_3)$ with the operations defined above
are actoids of sets over $(A_3, X_3)$.

(ii) For $0 < K \leq L$,

$$\partial F_{L,K} = G_{L-1,K-1} \text{ and } \partial G_{L,K} = G_{L-1,K-1},$$

and, for every $L$, $\partial G_{L,0} = \{((\emptyset, \emptyset))\}$. 

7. Applications

In this section, we give applications of the methods developed in the paper.
We give two proofs in detail, that of the Hales–Jewett theorem, in Subsection 7.1, and that of the self-dual Ramsey theorem, in Subsection 1.2, as these two proofs are of more interest than the other ones. These two proofs illustrate how the results of Sections 4 and 5 can be applied: the proof of the Hales–Jewett theorem uses Propositions 5.1 and 5.3; the proof of the self-dual Ramsey theorem uses Theorem 4.3. Additionally, in Subsection 7.2, we sketch how to obtain the Graham–Rothschild theorem and, in Subsection 7.4, we describe a limiting case that is related to the considerations of 4.
Recall that in Subsection 1.3 we described a way of translating statements as the ones in this section into the terminology of parameter sets (combinatorial cubes) and partitions.

7.1. The Hales–Jewett theorem. We prove below the following statement that combines into one the usual Hales–Jewett theorem [3] and Voigt’s version of this theorem for partial functions [20, Theorem 2.7]. One gets the classical Hales–Jewett theorem from the statement below for $L = L_0 + 1$, $L_0 = K_0$, and $v_0 = \text{id}_{[K_0]}$ in the assumption and for $L' = L$ in the conclusion. One gets the Voigt version for the same values of the parameters in the assumption and for $L' < L$ in the conclusion.

Hales–Jewett, combined version. Given $d > 0$, $0 < K_0$, $0 < L_0 \leq L$ and a surjection $v_0: [L_0] \to [K_0]$, there exists $M \geq L_0$ with the following property. For each $d$-coloring $c$ of 

$$\{v: [M'] \to [K_0]: L_0 \leq M' \leq M \text{ and } v \upharpoonright [L_0] = v_0\}$$

there exists a rigid surjection $s_0: [M] \to [L]$ such that $s_0 \upharpoonright [L_0] = \text{id}_{[L_0]}$ and $c(v_1 \circ s_0) = c(v_2 \circ s_0)$ whenever $v_1, v_2: [L'] \to [K_0]$, for the same $L_0 \leq L' \leq L$, and $v_1 \upharpoonright [L_0] = v_2 \upharpoonright [L_0] = v_0$.

The proof of the following lemma is an application of the notion of interpretability.

Lemma 7.1. $(\mathcal{F}_1, \mathcal{S}_1)$ is a pigeonhole actoid of sets.

Proof. Recall first the conclusion of Example B5. In this example, we have a family of pigeonhole actoids of sets $(\bigotimes_l \mathcal{F}_0, \bigotimes_l \mathcal{S}_K)$ parametrized by natural numbers $K$ and $l$. We claim that $S_{L,v_0}$, with $v_0: [L_0] \to [K_0]$ for some $L_0 \leq L$, is interpretable in $(\bigotimes_{L-L_0} \mathcal{F}_0, \bigotimes_{L-L_0} \mathcal{S}_{K_0})$, which will prove the lemma by Proposition 5.3.

Set

$$l = L - L_0.$$

Take $(S_3)^l \in \bigotimes_l \mathcal{S}_K$ and define

$$\alpha: S_{L,v_0} \to (S_3)^l$$

as follows. For a natural number $0 \leq k \leq K_0$, let $\bar{k} \in S_3$ be the function

$$\bar{k}(x) = \begin{cases} 
K_0, & \text{if } x = 1; \\
k, & \text{if } x = 2; \\
\max(1, K_0 - 1), & \text{if } x = 3.
\end{cases}$$

Let $v \in S_{L,v_0}$ be such that $v : [L] \to [K_0]$. Define

$$\alpha(v) = (v(\widetilde{L_0} + 1), v(\widetilde{L_0} + 2), \ldots, v(\widetilde{L'}), \tilde{0}, \ldots, \tilde{0}),$$

where we put $L - L'$ entries $\tilde{0}$ at the end of the above formula. Now note that $(\prod_{i \leq l} F_{N_i, M_i}) \cdot (S_3)^l$ is defined precisely when $M_i = 3$ for all $i \leq l$. Fix therefore $\prod_{i \leq l} F_{N_i, 3} \in \bigotimes_l \mathcal{F}_0$. For $N = \sum_{i \leq l} N_i$, $F_{L_0 + N, L, 0} \cdot S_{L,v_0}$ is defined in $(\mathcal{F}_1, S_1)$. We need to describe a function

$$\phi : \prod_{i \leq l} F_{N_i, 3} \to F_{L_0 + N, L, 0}.$$

Let

$$\bar{p} = (p_1, \ldots, p_l) \in \prod_{i \leq l} F_{N_i, 3}.$$

Fix $l_0, l_1 \in [L_0]$ so that

$$v_0(l_0) = K_0 \quad \text{and} \quad v_0(l_1) = \max(1, K_0 - 1).$$

For $x \in [L_0]$, let

$$\phi(\bar{p})(x) = x,$$

for $x \in [L_0 + N_1 + \cdots + N_i] \setminus [L_0 + N_1 + \cdots + N_{i-1}]$, let

$$\phi(\bar{p})(x) = \begin{cases} 
    l_0, & \text{if } p_i(x) = 1; \\
    L_0 + i, & \text{if } p_i(x) = 2; \\
    l_1, & \text{if } p_i(x) = 3.
\end{cases}$$

We check that (5.6) of the definition of interpretability holds. Note that, for $v \in S_{L,v_0}$, with $v : [L'] \to [K_0]$, the sequence

$$\phi(\bar{p}) \cdot v = v \circ \phi(\bar{p})$$

is the concatenation of the sequences

$$v_0, \ v(\widetilde{L_0} + 1) \circ p_1, \ v(\widetilde{L_0} + 2) \circ p_2, \ldots, \ v(\widetilde{L'}) \circ p_{L'-L_0}, \ (K_0, \ldots, K_0),$$

where the sequence of $K_0$-s at the end has length equal to the size of $p_{L'-L_0+1}(1)$; in particular, it has length 0 if $L' = L$. On the other hand,

$$\bar{p} \cdot \alpha(v) =$$

$$(v(\widetilde{L_0} + 1) \circ p_1, v(\widetilde{L_0} + 2) \circ p_2, \ldots, v(\widetilde{L'}) \circ p_{L'-L_0}, \tilde{0} \circ p_{L'-L_0+1}, \ldots, \tilde{0} \circ p_l).$$

Thus, (5.6) follows since the size of $p_{L'-L_0+1}(1)$ is the number of entries equal to $K_0$ at the beginning of $\tilde{0} \circ p_{L'-L_0+1}$, and therefore the above formula determines $\phi(\bar{p}) \cdot v$. \hfill \Box

Theorem 3.1 applied to the pigeonhole actoid of sets $(\mathcal{F}_1, S_1)$ gives directly the Hales–Jewett theorem as stated at the beginning of this subsection; we apply Theorem 3.1 with $t = K_0 - 1$, to $S_{K,v_0} \in S_1$ with $v_0 : [L_0] \to [K_0]$.  

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7.2. The Graham–Rothschild theorem. We outline here a proof of the Graham–Rothschild theorem, both the original version \[2\] and the partial rigid surjection version isolated by Voigt \[20, \text{Theorem 2.9}\]. Here are the two statements.

**Graham–Rothschild.** Given \(d > 0\), \(K_0 \leq K\), \(L_0 \leq L\) and a rigid surjection \(s_0: [L_0] \to [K_0]\), there exists \(M \geq L_0\) with the following property. For each \(d\)-coloring of
\[
\{ s: [M] \to [K]: s \in \text{RS and } s \upharpoonright [L_0] = s_0 \}
\]
there is a rigid surjection \(t_0: [M] \to [L]\), with \(t_0 \upharpoonright [L_0] = \text{id}_{[L_0]}\), such that
\[
\{ s \circ t_0: s: [L] \to [K], s \in \text{RS and } s \upharpoonright [L_0] = s_0 \}
\]
is monochromatic.

**Graham–Rothschild, Voigt’s version.** Given \(d > 0\), \(K_0 \leq K\), \(L_0 \leq L\) and a rigid surjection \(s_0: [L_0] \to [K_0]\), there exists \(M \geq L_0\) with the following property. For each \(d\)-coloring \(c\) of
\[
\{ s: [M'] \to [K]: L_0 \leq M' \leq M \text{ and } s \upharpoonright [L_0] = s_0 \}
\]
there exist \(M'_0\) and a rigid surjection \(t_0: [M'_0] \to [L]\), with \(L_0 \leq M'_0 \leq M\) and \(t_0 \upharpoonright [L_0] = \text{id}_{[L_0]}\), such that
\[
\{ s \circ t_0: s: [L'] \to [K], L_0 \leq L' \leq L, s \in \text{RS, and } s \upharpoonright [L_0] = s_0 \}
\]
is monochromatic.

We use actoids of sets \((\mathcal{F}_2, \mathcal{S}_2)\) and \((\mathcal{G}_2, \mathcal{T}_2)\) over the background \((A_2, X_2)\) as defined in Subsection 6.4. It is not difficult to check from the Hales–Jewett theorem as stated in Subsection 7.1 that property (lph) holds for the actoids of sets \((\mathcal{F}_2, \mathcal{S}_2)\) and \((\mathcal{G}_2, \mathcal{T}_2)\) over \((A_2, X_2)\). So we have the following lemma.

**Lemma 7.2.** \((\mathcal{F}_2, \mathcal{S}_2)\) and \((\mathcal{G}_2, \mathcal{T}_2)\) are pigeonhole actoids of sets over the background \((A_2, X_2)\) fulfilling (lph).

Even tough the lemma above contains two statements, the checking that needs to be done for \((\mathcal{F}_2, \mathcal{S}_2)\) contains the checking that needs to be done for \((\mathcal{G}_2, \mathcal{T}_2)\). In performing this check, the following obvious observation plays a crucial role. If \(s\) and \(t\) are two rigid surjections with
\[
\partial f s = \partial f t: [L] \to [K],
\]
then
\[
s \upharpoonright [L + 1] = t \upharpoonright [L + 1]
\]
as \(s(L + 1) = t(L + 1) = K + 1\).
We note the obvious fact that $S_2$ and $T_2$ are vanishing. Now an application of Corollary 4.4 to $(F_2, S_2)$ (using Lemmas 6.2 and 6.5) yields the first version of the Graham–Rothschild theorem as stated at the beginning of this subsection and an application of this corollary to $(G_2, T_2)$ gives the second version.

7.3. The self-dual Ramsey theorem. We prove Theorem 1.1. Recall the definition of connections and their multiplication from Subsection 1.2. First we state a reformulation of Theorem 1.1 and the usual partial function version of this reformulation. We follow these reformulations with an explanation of how the first one implies Theorem 1.1. Then, we give arguments for the two statements.

Self-dual Ramsey theorem. Given $d > 0$, $0 < K \leq L$, there exists $M$ with the following property. For each $d$-coloring of

$$\{(s, p) \in AS: (s, p): [M] \to [K] \text{ and } s^{-1}(K) = \{M\}\}$$

there is an augmented surjection $(t_0, q_0): [M] \to [L]$ such that $t_0^{-1}(L) = \{M\}$ and

$$\{(t_0, q_0) \cdot (s, p): (s, p): [L] \to [K], (s, p) \in AS \text{ and } s^{-1}(K) = \{L\}\}$$

is monochromatic.

Self-dual Ramsey theorem; partial augmented surjection version.

Given $d > 0$, $K \leq L$, there exists $M$ with the following property. For each $d$-coloring of

$$\{(s, p) \in AS: (s, p): [M'] \to [K] \text{ for some } M' \leq M\}$$

there is an augmented surjection $(t_0, q_0): [M_0'] \to [L]$ for some $M_0' \leq M$ such that

$$\{(t_0, q_0) \cdot (s, p): (s, p) \in AS, (s, p): [L'] \to [K] \text{ for some } L' \leq L\}$$

is monochromatic.

To obtain Theorem 1.1 from the first of the above statements, associate with an increasing surjection $p: [L] \to [K]$ an increasing injection $i_p: [K] \to [L]$ given by $i_p(x) = \max p^{-1}(x)$. If $(s, p)$ is an augmented surjection with $s, p: [L] \to [K]$, for some $0 < K \leq L$, and $s^{-1}(K) = \{L\}$, then

$$(s \upharpoonright [L - 1], i_p \upharpoonright [K - 1])$$

is a connection between $[L - 1]$ and $[K - 1]$ and each connection between $[L - 1]$ and $[K - 1]$ is uniquely representable in this way. Moreover, if $(t, q)$ is
another augmented surjection with $t, q: [M] \to [L]$ and with $t^{-1}(L) = \{M\}$, then

$$(s \circ t) \upharpoonright [M - 1], i_{p_0q} \upharpoonright [K - 1])$$

$$= (t \upharpoonright [M - 1], i_q \upharpoonright [L - 1]) \cdot (s \upharpoonright [L - 1], i_p \upharpoonright [K - 1]),$$

where the multiplication on the right hand side is the multiplication of connections. These observation show that the first of the above statements implies Theorem 1.1.

The following lemma will turn out to be an immediate consequence of the Hales–Jewett theorem and the Voigt version of the Graham–Rothschild theorem.

**Lemma 7.3.** $(F_3, S_3)$ and $(G_3, T_3)$ are actoids of sets fulfilling (lph).

**Proof.** A moment of thought (using the way $\partial$ acts on subsets of $X_2$) convinces us that to see (lph) it suffices to show Conditions 1 and 2 below, for $L \geq K > 0$ and $d > 0$. To state these conditions, fix $(s_0, p_0) \in AS, s_0, p_0: [L_0] \to [K - 1]$ for some $L_0 < L$. The role of the elements $x$ and $a$ in (lph) is played by $(s_0, p_0)$ and $(id_{[L_0]}, id_{[L_0]})$, respectively.

Condition 1. There exists $M > L$ such that for each $d$-coloring of $F_{M, L} \cdot \{s, p \in F_{L, K}: \partial(s, p) = (s_0, p_0)\}$

there exists $(t_0, q_0) \in F_{M, L}$ such that

$$t_0 \upharpoonright [L_0] = id_{[L_0]}$$

and

$$q_0 \upharpoonright [L_0] = id_{[L_0]}$$

and

$$\{(t_0, q_0). (s, p): (s, p) \in F_{L, K}, \partial(s, p) = (s_0, p_0)\}$$

is monochromatic.

This statement amounts to proving the following result.

There exists $M > L_0$ such that for each $d$-coloring of all rigid surjections $t: [M - 1] \to [K - 1]$ with $t \upharpoonright [L_0] = s_0$ there exists a rigid surjection $t_0: [M - 1] \to [L - 1]$ such that $t_0 \upharpoonright [L_0] = id_{[L_0]}$ and

$$\{s \circ t_0: s: [L - 1] \to [K - 1] a rigid surjection and s \upharpoonright [L_0] = s_0\}$$

is monochromatic.

This is a special case of the Hales–Jewett theorem, as stated and proved in Subsection 7.1.

Condition 2. There exists $M \geq L$ such that for each $d$-coloring of

$$G_{M, L} \cdot \{s, p \in G_{L, K}: \partial(s, p) = (s_0, p_0)\}$$

there exists $(t_0, q_0) \in G_{M, L}$ such that

$$t_0 \upharpoonright [L_0] = id_{[L_0]}$$

and

$$q_0 \upharpoonright [L_0] = id_{[L_0]}$$
and
\[ \{(t_0, q_0), (s, p) : (s, p) \in G_{L,K}, \partial(s, p) = (s_0, p_0)\} \]
is monochromatic.

The above statement can be easily seen to be a consequence of the following result.

There exists \( M > L_0 \) such that for each \( d \)-coloring of all rigid surjections \( t : [M' - 1] \to [K] \) for some \( L_0 < M' \leq M \) with \( t \upharpoonright [L_0] = s_0 \) there exists a rigid surjection \( t_0 : [L_0] \to [L] \) for some \( L_0 < M' \leq M \) such that \( t_0 \upharpoonright [L_0] = \text{id}_{[L_0]} \) and
\[ \{s \circ t_0 : s : [L'] \to [K] \text{ for some } L_0 < L' \leq L, s(L') = K, \text{ and } s \upharpoonright [L_0] = s_0\} \]
is monochromatic.

This is a consequence of Voigt's version of the Graham–Rothschild theorem as stated in Subsection 7.2. Thus, (lph) holds and the lemma follows. \( \square \)

Since \( \mathcal{S}_3 \) and \( \mathcal{T}_3 \) are clearly vanishing, by Lemmas 6.3 and 6.6, Corollary 3.3 can be applied to \((F_3, \mathcal{S}_3)\) and \((G_3, \mathcal{T}_3)\) and yields the statements from the beginning of this subsection.

7.4. Walks, a limiting case. We define a normed background and an actoid of sets over it that is not pigeonhole. This provides a natural limiting example of the extent of the pigeonhole condition for actoids of sets. The motivation for this example comes from [4] and is related to a problem of Uspenskij [19].

By a walk we understand a function \( s : [L] \to [K] \) that is surjective and such that \( s(1) = 1 \) and
\[ |s(i) - s(j)| \leq 1 \text{ whenever } |i - j| \leq 1 \text{ for } i, j \in [L]. \]
Each walk is a rigid surjection. If both \( s \) and \( t \) are walks, then the canonical composition \( s \circ t \), as defined by (6.1), is also a walk. Let
\[ W = \{s : s \text{ is a walk}\}. \]

Let \( C = Z = W \). We note that \( C \subseteq A_2 \) and \( Z \subseteq X_2 \), as defined in Subsection 6.3. We equip \( C \) with the multiplication inherited from \( A_2 \) and we take the partial action of \( C \) on \( Z \) to be the one inherited from \((A_2, X_2)\). Note also that \( Z \) is closed under the forgetful truncation \( \partial_f \) with which \((A_2, X_2)\) is equipped. We take it as the truncation on \((C, Z)\). We also consider the function \(|·|\) defined on \( X_2 \), we restrict it to \( Z \), and denote it again by \(|·|\). The following lemma is an immediate consequence of Lemma 6.2.

**Lemma 7.4.** \((C, Z) \) with the operations defined above is a normed background.
Let $H = \mathcal{W}$ consist of all subsets of $W$ of the form $F_{K,L}$, for $K \leq L$, where
\[ F_{L,K} = \{ s \in W : s : [L] \to [K] \}. \]
For $F_{L,K}, F_{N,M} \in H = \mathcal{W}$, $F_{N,M} \bullet F_{L,K}$ and $F_{N,M} \bullet F_{L,K}$ are defined if and only if $L = M$ and then we let
\[ F_{N,L} \bullet F_{L,K} = F_{N,L} \bullet F_{L,K} = F_{N,K}. \]
One easily checks that $(H, \mathcal{W})$ with $\bullet$ and $\bullet$ is an actoid of sets over $(C, Z)$.

The question of whether this is a pigeonhole actoid of sets, or more precisely whether the Ramsey theorem that would be obtained from Corollary 4.4 if this actoid of sets were pigeonhole, was motivated by a question of Uspenskij [19] if the universal minimal flow of the homeomorphism group of the pseudo-arc is the pseudo-arc itself together with the natural action of the homeomorphism group. It would follow from this Ramsey theorem, from [4], and from a dualization of the techniques of [5] that the answer to Uspenskij’s question is positive. However, the theorem below implies that this Ramsey theorem is false, and therefore that the actoid of sets is not pigeonhole.

**Theorem 7.5.** For every $M \geq 3$ there exists a 2-coloring of all walks from $[M]$ to $[3]$ such that for each walk $t: [M] \to [6]$ the set
\[ \{ s \circ t : s : [6] \to [3] \text{ a walk} \} \]
is not monochromatic.

**Proof.** We show a bit more: to contradict monochromaticity we only need a set of walks $s : [6] \to [3]$ that vary at two elements of their common domain. Let
\[ A = \{ s : [6] \to [3] : s(1) = 1, s(2) = 2, s(5) = 2, s(6) = 3, s(3), s(4) \in \{1, 2\} \}. \]
Clearly each element of $A$ is a walk from $[6]$ to $[3]$. We claim that for each $M \geq 3$ there is a 2-coloring of all walks from $[M]$ to $[3]$ such that for each walk $t: [M] \to [6]$ the set $A \circ t$ is not monochromatic.

Let $M \geq 3$. For a walk $u : [M] \to [3]$ define
\[ a(u) = |\{ y \in [M] : u(x) \leq 2 \text{ for all } x \leq y, u(y) = 1, \text{ and } u(y+1) = 2 \}|. \]
Define a 2-coloring $c$ by letting
\[ c(u) = a(u) \mod 2. \]
We claim that this coloring is as required.

Let $t : [M] \to [6]$ be a walk. We analyze $t$ in order to compute $a(s \circ t)$ for $s \in A$ in terms of certain numbers associated with $t$. Let $M_0 \leq M$ be the smallest natural number with $t(M_0) = 6$. There exist unique, pairwise
disjoint intervals $I \subseteq [M_0]$ that are maximal with respect to the property $t(I) \subseteq \{3, 4\}$. For such an $I$, let $I^-$ and $I^+$ be $(\min I) - 1$ and $(\max I) + 1$, respectively. Note that $t(I^-), t(I^+) \in \{2, 5\}$. We distinguish four types of such intervals $I$:

\[
I \in P_1 \iff t(I^-) = 2 \text{ and } t(I^+) = 5;
I \in P_2 \iff t(I^-) = 5 \text{ and } t(I^+) = 2;
I \in Q_1 \iff t(I^-) = t(I^+) = 2;
I \in Q_2 \iff t(I^-) = t(I^+) = 5.
\]

Note right away that since $t$ is a walk, $t(1) = 1$, $t(M_0 - 1) = 5$, and $t(x) \in \{1, \ldots, 5\}$ for $x \in [M_0 - 1]$, it follows that $|P_1| - |P_2| = 1$ and therefore

\begin{equation}
|P_1| + |P_2| \text{ is odd.}
\end{equation}

For each $I \in P_1 \cup P_2 \cup Q_1 \cup Q_2$, define $a_t(I)$ as follows.

\[
a_t(I) = \begin{cases}
|x \in I : t(x) = 3, t(x + 1) = 4|, & \text{if } I \in P_1 \cup Q_1; \\
|x \in I : t(x) = 4, t(x + 1) = 3|, & \text{if } I \in P_2 \cup Q_2.
\end{cases}
\]

Note that for $I$ in $Q_1$ or $Q_2$, the two cases in the above definition give the same value for $a_t(I)$. Further, let

\[
a_t = |\{x \in [M_0] : t(x) = 1, t(x + 1) = 2\}|.
\]

We can write $A = \{s_1, s_2, s_3, s_4\}$, where $s_1, s_2, s_3$ and $s_4$ are determined by the conditions

\[
s_1(3) = s_1(4) = 1,
\]

\[
s_2(3) = 2, s_2(4) = 1,
\]

\[
s_3(3) = 1, s_3(4) = 2, \text{ and }
\]

\[
s_4(3) = s_4(4) = 2.
\]

An inspection convinces us that

\[
a(s_i \circ t) = \begin{cases}
a_t + \sum_{I \in P_1 \cup P_2 \cup Q_1 \cup Q_2} 1, & \text{if } i = 1; \\
a_t + \sum_{I \in P_1 \cup P_2} a_t(I) + \sum_{I \in Q_1} a_t(I) + \sum_{I \in Q_2}(a_t(I) + 1), & \text{if } i = 2; \\
a_t + \sum_{I \in P_1 \cup P_2} a_t(I) + \sum_{I \in Q_1}(a_t(I) + 1) + \sum_{I \in Q_2} a_t(I), & \text{if } i = 3; \\
a_t, & \text{if } i = 4.
\end{cases}
\]

Assume towards a contradiction that a walk $t : [M] \to [6]$ is such that $A \circ t$ is monochromatic. It follows from the above expressions for $a(s_i \circ t)$
that the numbers
\[
\sum_{I \in P_1 \cup P_2 \cup Q_1 \cup Q_2} 1, \\
\sum_{I \in P_1 \cup P_2} a_t(I) + \sum_{I \in Q_1} a_t(I) + \sum_{I \in Q_2} (a_t(I) + 1), \\
\sum_{I \in P_1 \cup P_2} a_t(I) + \sum_{I \in Q_1} (a_t(I) + 1) + \sum_{I \in Q_2} a_t(I),
\]
and 0
have the same parity, that is, since the last number is 0, they are all even.

Now it follows from the first line that
\[
(7.2) \quad |P_1| + |P_2| + |Q_1| + |Q_2| \text{ is even}
\]
and, by subtracting the second line from the third one, that \(|Q_1| - |Q_2|\) is even, and so
\[
(7.3) \quad |Q_1| + |Q_2| \text{ is even.}
\]
Equations (7.2) and (7.3) imply that the natural number \(|P_1| + |P_2|\) is even
contradicting (7.1). \qed

7.5. A problem. First, we point out that the failure of a Ramsey result
from Subsection 7.1, the classical Ramsey theorem, and the Graham–Rothschild theorem can be viewed in a uniform way as statements about backgrounds that are closely related to each other.

Fix \(A\) so that \(\text{IS} \subseteq A \subseteq \text{RS}\), where the families of increasing surjections \(\text{IS}\) and of rigid surjections \(\text{RS}\) are defined in Subsection 6.1. Assume that for \(s, t \in A\) with \(s: [L] \to [K]\) and \(t: [N] \to [M]\) with \(L \leq M\), the canonical composition \(s \circ t\), as defined in Subsection 6.2, is in \(A\). Assume further that \(\partial f s \in A\) for \(s \in A\), where the forgetful truncation \(\partial f\) is also defined in Subsection 6.2. Consider the background \((A, A)\) in which multiplication on \(A\) is the same as the action of \(A\) on \(A\), both are defined precisely when \(s \circ t\) is defined and are given by
\[
t \cdot s = t . s = s \circ t,
\]
and in which the truncation operator is given by \(\partial f\). Given \(K, L \in \mathbb{N}\), let
\[
H_{L,K}^A = \{s \in A: s: [L] \to [K]\}.
\]
Let \(\mathcal{F}_A = \mathcal{S}_A\) both consist of all \(H_{L,K}^A\) with \(K \leq L\). Declare \(H_{N,M}^A \cdot H_{L,K}^A\) and \(H_{N,M}^A \cdot H_{L,K}^A\) to be defined precisely when \(L = M\) and in that case let both of them be equal to \(H_{N,K}^A\).

Now note that if \(A = \text{RS}\) or \(A = \text{IS}\), then \((\mathcal{F}_A, \mathcal{S}_A)\) are pigeonhole actoids of sets over \((A, A)\). In the first case, the resulting Ramsey theorem
is the dual Ramsey theorem (see Subsection 7.2) and in the second case it is a reformulation of the classical Ramsey theorem (see Example A4 in Subsection 4.2). Note also that, by Theorem 7.5, if $A$ is the set of all walks $W$, then $(F^A, S^A)$ is not a pigeonhole actoid of sets. The following problem presents itself.

Find all sets $A$ as above for which $(F^A, S^A)$ is a pigeonhole actoid of sets.

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