The Role of Symmetry in Non-Hermitian Scattering

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Abstract. We review recent work on asymmetric scattering by Non-Hermitian (NH) Hamiltonians. Quantum devices with an asymmetric scattering response to particles incident from right or left in effective 1D waveguides will be important to develop quantum technologies. They act as microscopic equivalents of familiar macroscopic devices such as diodes, rectifiers, or valves. The symmetry of the underlying NH Hamiltonian leads to selection rules which restrict or allow asymmetric response. NH-symmetry operations may be organized into group structures that determine equivalences among operations once a symmetry is satisfied. The NH Hamiltonian possesses a particular symmetry if it is invariant with respect to the corresponding symmetry operation, which can be conveniently expressed by a unitary or antiunitary superoperator. A simple group is formed by eight symmetry operations, which include the ones for Parity-Time symmetry and Hermiticity as specific cases. The symmetries also determine the structure of poles and zeros of the $S$ matrix. The ground-state potentials for two-level atoms crossing properly designed laser beams realize different NH symmetries to achieve transmission or reflection asymmetries.

1. Introduction

It is hardly necessary to emphasize the importance of devices with asymmetric response, such as valves, diodes, or rectifiers, which appear in biology and various technologies to manage and control macroscopic currents and flows. We expect that these asymmetries will similarly play a key role in the microscopic realm to develop quantum technologies.

Among the microscopic asymmetric devices, the Maxwell demon is possibly the best known, [1, 2, 3]. The demon was first discussed theoretically by prominent physicists such as Maxwell, Smoluchowski, Szilard, Feynman, or Landauer [2]. A frequent attitude has been to exorcise the (malignant) demon, either to save the second law, if the demon does not work; or to show that, if it works, the 2nd law is not violated. A modern perspective that we embrace here is more pragmatic, and instead aims at exercising the demon, namely, building up demonic devices, i.e., one-way barriers, for practical applications [3], such as particle trapping or cooling. Skordos and Zurek [4] made an important observation: Two gas chambers of equal volume were connected via an asymmetric trapdoor that could slide on rails on one of the chambers only. When the door motion was cooled a differential of pressure between the two parts occurred, but not otherwise. In general automated demons like this function thanks to some dissipation. In a 1D barrier, if an incident particle dissipates its energy only on one side, right and left transmission probabilities

\textsuperscript{1} This paper is largely based on the talks by J. G. Muga and A. Ruschhaupt within the PHHQP seminar series.
may differ. Otherwise, in a conventional process with a Hermitian potential the probabilities are equal.

One-way barriers ("atom diodes") have been designed using lasers interacting with two- or three-level atoms [5, 6, 7, 3, 8]. These ideas have been used in conjunction with the group of Mark Raizen, which has implemented them experientially [9, 10, 11, 12]. This work evolved into an isotope separation technique for medical use [13].

In an ideal one-way barrier, particles are fully transmitted if they are incident from one side (say the left), and fully reflected from the other side. We could encode this asymmetric response as $T/R$, where $T$ stands for full transmission and $R$ stands for full reflection. In fact there are several other asymmetric devices. For example, an ideal one-way detector may function as $R/A$, absorbing -i.e., detecting- all particles impinging from one side only [14] ($A$ stands for “absorption”). Taking as a reference maximal all-or-nothing responses (1 or 0 scattering coefficients), six basic asymmetric responses were identified [15], namely $TR/A, T/R, T/A, TR/R, R/A, TR/T$. All these responses would macroscopically correspond to different rectifier types and may have useful applications. Can we construct them? It turns out that the different asymmetric responses may or may not be realized depending on scattering selection rules that follow from Hamiltonian symmetries. In particular, Hermiticity, which may be understood as one of these symmetries, forbids any of the asymmetric responses, whereas PT-symmetry forbids responses that are transmission-asymmetric. However, other Hamiltonian symmetries applicable to Non-Hermitian Hamiltonians are able to produce the different scattering asymmetries. In order to fully understand and exploit the connection between Hamiltonian symmetry and scattering properties, we first review NH scattering in 1D. Furthermore, we also outline NH symmetries, how they are most conveniently formulated, and their consequences.

2. A pinch of 1D scattering theory for Non-Hermitian potentials

We consider scattering Hamiltonians that can be written as the sum of a kinetic energy operator $H_0 = p^2/(2m)$ for a particle of mass $m$ and a potential energy operator $V$,

$$H = H_0 + V. \quad (1)$$

The potential function in position representation $V(q, q') = \langle q | V | q' \rangle$ is assumed to decay fast enough to 0 when $q$ or $q'$ go to infinity so that the usual operators of scattering theory are well defined and the Hilbert space is biorthogonally decomposed into a continuum part with real eigenvalues and a discrete part, see [16]. Notably, the potential is generally considered to be non-local, in the sense that elements $\langle q | V | q' \rangle$ may be nonzero for $q \neq q'$. These non-local interactions can arise as effective interactions for some subspace even if underlying interactions for the larger space are local [17, 18]. A simple example that we shall use later on is a two-level atom interacting with a semiclassical laser field [19, 20].

We use for scattering transmission and reflection amplitudes from left and right incidence, the symbols $T^l, R^l, T^r, R^r$. As well, $T^t, R^t, T^{t'}, R^{t'}$ denote corresponding amplitudes for $H^t$ [16]. The $S$ matrix, related to the scattering operator $S$ that connects asymptotic states, is nothing but the scattering amplitudes for $H$ in matrix form

$$S = \begin{pmatrix} T^l(p) & R^l(p) \\ R^l(p) & T^r(p) \end{pmatrix}, \quad (2)$$

We adopt the convention of calling the squared moduli of the amplitudes “coefficients”. In NH Physics these are not necessarily “probabilities”, since they may not add to one.

$S$ is the on-shell part of the matrix for $S$ in momentum representation factoring out an energy Dirac delta and a $p/m$ factor, see [16] for further details and precise definitions.
which, for NH Hamiltonians, is generically not unitary. However, generalized unitarity relations exist that relate \( S \) and the corresponding operator for \( H^{\dagger} \), \( \hat{S} \), as \( \hat{S}^\dagger S = SS^\dagger \) [16]. Generalized unitarity relates hatted and unhatted amplitudes and helps to set the “scattering selection rules” [15]. These are recipes which use the Hamiltonian symmetries to determine whether scattering asymmetries in transmission and reflection are attainable or unattainable. Our next task will be to define and study these symmetries.

3. Non-Hermitian symmetries

We adopt here the perspective that defines symmetries by the simple prescription “change without changing” [21], namely, if an operation is performed on an object -in our case the Hamiltonian- and the object looks the same before and after the operation, that object possesses the symmetry associated with that symmetry operation (a technical refinement, following Wigner, will be discussed below). For example, rotating a square by \( \pi/2 \) about a perpendicular axis across its center gives the same square, so it is symmetrical with respect to \( \pi/2 \)-rotations. Note that the operation (rotating by \( \pi/2 \)) makes sense and can be carried out irrespective of whether or not the object possesses the corresponding symmetry. Some of the results we shall discuss, specifically group structures, refer to the symmetry operations, rather than to the symmetries.

Coming back to Hamiltonians, consider the following operations on \( H \): \( A^{-1}HA \), \( A^{-1}H^\dagger A \), where \( A \) is unitary or antiunitary. The first operation could be phrased as: “multiply \( H \) by \( A^{-1} \) from the left and by \( A \) from the right”. The second operation would start with the instruction “take the adjoint of \( H \)”, and then perform the multiplications. If one of these operations leaves the Hamiltonian invariant, \( H \) possesses that particular symmetry, i.e.,

\[
\text{Operation on } H : A^{-1}HA. \quad \text{Symmetry : } A^{-1}HA = H \text{ or } AH = HA, \quad (3)
\]

\[
\text{Operation on } H : A^{-1}H^\dagger A. \quad \text{Symmetry : } A^{-1}H^\dagger A = H \text{ or } AH = H^\dagger A. \quad (4)
\]

The first type of symmetry is the “standard” one. The second one is termed \( A \)-pseudohermiticity [22, 15]. For a physical discussion of the two possibilities and their relation to dynamical invariants see [23]. In practice we shall limit \( A \) to the following set of operators that form Klein’s four-group

\[
A \in K_4 = \{1, \Pi, \Theta, \Pi\Theta\}, \quad (5)
\]

where \( \Pi \) is parity, and \( \Theta \) time reversal. The action on a position basis state multiplied by a complex constant \( c \) is \( \Pi c|q\rangle = c| -q\rangle \), \( \Theta c|q\rangle = c^*|q\rangle \). Klein’s group is an Abelian group with two generators. It has many applications, among others it is the basis for dodecaphonic music [24]. Fascinating as this is, the reason for us to restrict the possible \( A \) to this group is that, when combined with the two operations in (3) and (4), an Abelian group\(^4\) \( \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \) of eight symmetry operations is formed that may be considered a complete set of operations combining transposition, inversion, and complex conjugation. Table 1 gives the corresponding eight symmetry operations, which may be expressed as a superoperator (column 3) acting on \( H \), generically \( \mathcal{S} \), see the next section, which also discusses the relation to Wigner’s characterization of quantum symmetries. Column 5 gives \( \langle q|\mathcal{S}H|q'\rangle \). \( H \) will posses the symmetry associated with a particular \( \mathcal{S} \) if \( \langle q|H|q'\rangle = \langle q|\mathcal{S}H|q'\rangle \) for all \( q, q' \). In practice this condition must be

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\(^4\) This “elementary Abelian group” is sometimes denoted as \( E_8 \), not to be confused with the exceptional Lie group.

\(^5\) The roman numeral is used to design both the symmetry operation or the corresponding symmetry, the context should clarify its exact meaning.
Table 1. Group elements (transformations) in different notations. 1: Roman number code; 2: Group theory notation of group elements in terms of generators $x$, $y$, $z$; 3: Superoperators $S$; 4: Explicit action of the superoperators on $H$; 5: Matrix element $\langle q | (SH) | q' \rangle$.

|   | 1  | 2  | 3  | 4  | 5  |
|---|----|----|----|----|----|
| I | $e$ | $L_1$ | $H$ | $\langle q | H | q' \rangle$ |
| II | $x$ | $D$ | $H^\dagger$ | $\langle q' | H | q \rangle^*$ |
| III | $z$ | $L_{II}$ | $\Pi H \Pi$ | $\langle -q | H | -q' \rangle$ |
| IV | $xz$ | $L_{II}D$ | $\Pi H \Pi^\dagger$ | $\langle -q' | H | -q \rangle^*$ |
| V | $y$ | $L_{\theta}$ | $\theta H \theta$ | $\langle q | H | q' \rangle^*$ |
| VI | $xy$ | $L_{\theta}D$ | $\theta H^\dagger \theta$ | $\langle q' | H | q \rangle$ |
| VII | $zy$ | $L_{II\theta}$ | $\Pi \theta H \Pi \theta$ | $\langle -q' | H | -q \rangle^*$ |
| VIII | $zxy$ | $L_{II\theta}D$ | $\Pi \theta H^\dagger \Pi \theta$ | $\langle -q' | H | -q \rangle$ |

satisfied by the potential since the kinetic energy is symmetrical with respect to all operations. The multiplication table of the group of operations and its “presentation” (set of relations among the generators) is given in Table 2.

4. Symmetry transformations as superoperators
Let us define first superoperators $L_A$ acting on $H$ as

$$L_A H = A^{-1} H A,$$  \hspace{1cm} (6)

and a “dagger” superoperator $D$ as $[23]$, $DH = H^\dagger$. Hermiticity is the invariance upon this superoperator. $L_A$ and $D$ commute when $A^{-1} = A^\dagger$, as it happens for our basic operators $1$, $\Pi$, $\theta$ and $\theta \Pi$ in Klein’s group. Note that $L_A$ alone, with $A \in K_4 = \{1, \Pi, \theta, \Pi \theta\}$, constitute a Klein group with two generators, whereas including $D$ provides the third generator needed for $Z_2 \times Z_2 \times Z_2$.

Superoperators, just like ordinary operators, are linear if they leave complex constants invariant and antilinear if they transform them to their complex conjugates. In particular $[23]$,

$$L_A(cH) = cA^\dagger HA, \quad \text{A unitary},$$
$$L_A(cH) = c^* A^\dagger HA, \quad \text{A antiunitary},$$
$$D(cH) = c^* H^\dagger,$$
$$L_A D(cH) = D L_A(cH) = c^* A^\dagger H^\dagger A, \quad \text{A unitary},$$
$$L_A D(cH) = D L_A(cH) = cA^\dagger H^\dagger A, \quad \text{A antiunitary}.$$  \hspace{1cm} (7)

The superoperators in Table 1 form a group $Z_2 \times Z_2 \times Z_2$. As noted before, we denote a superoperator in this group by a generic notation $S$. In formal manipulations we shall later on use distinguishing subscripts, e.g. $S_j$, where $j = 1, 2, \ldots, 8$ mapping $I \rightarrow 1$, $II \rightarrow 2$, etc...

4.1. Valid symmetry operations.
The transformations that in quantum physics lead to possible symmetries, i.e. symmetry transformations (operations), are not arbitrary. Wigner stated that the transformations that act on wave functions should leave the modulus of the scalar product of two states, their “transition probability”, invariant, which restricts the corresponding operators to be unitary or antiunitary
Our superoperators imply a mild generalization of Wigner’s definition, as they leave the scalar product of two density operators, the most general way of expressing a state, invariant [23].

We will denote a scalar product of two given (linear) operators \( F \) and \( G \) as the standard Hilbert-space inner product \( \langle\langle F, G \rangle\rangle = \text{Tr}\left[F^\dagger G\right] \). Expectation values for an observable \( F \) and a density operator \( \rho \), both Hermitian, take the form \( \langle\langle F \rangle\rangle = \text{Tr}\left[F \rho\right] = \text{Tr}\left[F^\dagger \rho\right] = \langle\langle F, \rho \rangle\rangle \).

We define the adjoint of a given superoperator \( S \) as the superoperator \( S^\dagger \) which fulfills

\[
\langle\langle G, S F \rangle\rangle = \langle\langle F, S^\dagger G \rangle\rangle \quad \text{for } S \text{ linear,} \tag{8}
\]

\[
\langle\langle G, SF \rangle\rangle = \langle\langle F, S^\dagger G \rangle\rangle \quad \text{for } S \text{ antilinear.} \tag{9}
\]

All eight superoperators satisfy \( S^\dagger = S^{-1} \), so these superoperators are unitary or antiunitary. Thus they keep invariant the scalar product of two density operators,

\[
\langle\langle \rho_1, \rho_2 \rangle\rangle = \langle\langle S \rho_1, S \rho_2 \rangle\rangle. \tag{11}
\]

Our definition of symmetry operation is indeed broader than Wigner’s, which, for example, does not include relations of the form (4).

| Table 2. Multiplication table of \( \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \) which may be summarized in the “presentation” \( \langle x, y, z \mid x^2 = y^2 = z^2 = e, xy = yx, zx = xz, zy = yz \rangle \). |
|---|---|---|---|---|---|---|
| e | x | z | zx | y | xy | zy |
| e | e | x | z | zx | y | xy | zy |
| x | x | e | z | xz | y | xy | zy |
| z | z | xz | e | x | zy | xz | y |
| zx | zx | e | x | z | xy | y | x |
| y | y | xy | zy | xz | e | x | z |
| xy | xy | zy | xz | e | z | x |
| zy | zy | xy | zy | xz | e | x | z |
| xz | xz | y | xy | zy | e | z | x |
| e | e | x | z | zx | y | xy | zy |

5. Equivalences among symmetry operations

From the group multiplication table, it follows that once one symmetry is satisfied, automatically the symmetry operations are equivalent by pairs. More precisely, we will say that, conditioned to an existing symmetry, \( S_i H = H \), two symmetry operations represented by \( S_k \) and \( S_l \) are equivalent, \( S_k \sim S_l \), if \( S_k H = S_l H \). As \( S_i \) represents a symmetry by hypothesis, the operations represented by the pair \( S_i \) and \( S_k = S_l S_i \) (found from the multiplication table) are equivalent.

An example of equivalence that may be familiar to some is that, conditioned on \( xy H = H \) which

\[6\] In [15] the stronger assumption \( S_k H = S_l H = H \) was made, namely, that the equivalent pairs were also symmetries. Note that equivalence can be formulated more generally as done here or in [26] without that assumption.
Table 3. Equivalences among symmetry operations for the potential elements. Given the symmetry of the upper row, the table provides the equivalent symmetry operations. The trivial identity (I) is excluded as this symmetry is satisfied by all potentials.

| II~IV | II~IV | II~III | II~VI | II~V | II~VII | II~VIII | II~VII |
|-------|-------|--------|-------|------|--------|---------|--------|
| III~VI | V~VII | V~VIII | III~VII | III~VIII | III~V | III~VI |        |
| V~VII | V~VIII | IV~VII | IV~VIII | IV~VII | IV~VI | IV~V   |

is satisfied in particular by all local potentials in coordinate representation; more generally this symmetry implies that the matrix is complex-symmetric or, equivalently, self-transpose, then \( z^* H = z H \). In the alternative operator language this means that, conditioned on \( \Theta^\dagger \Theta = H \), we have that \( \Pi \Theta H \Pi \Theta = \Pi H \Pi \). In words, with the proper conditioning (\( \Theta \)-pseudohermiticity, i.e., the symmetry of complex-symmetric matrices), the symmetry transformations for PT-symmetry and for parity-pseudohermiticity give the same result when acting on the Hamiltonian. If it happens that \( H \) is indeed PT-symmetrical (i.e., \( \Pi \Theta H \Pi \Theta = H \)), then it will also be parity-pseudo-Hermitian (\( \Pi H \Pi \dagger = H \)) and vice versa \([22, 27]\). These pairs were worked out systematically in \([15]\) conditioned on a given (primary) symmetry, see Table 3, and follow directly from the group multiplication table.

6. Selection rules

Putting together the effect of the symmetries on \( S \)-matrix elements and generalized unitarity relations, selection rules may be found, see Table 4. It turns out that symmetries II (Hermiticity) and III (parity) are not capable of any, reflection or transmission, scattering asymmetry; symmetries VI (time reversal pseudohermiticity) and VII (PT symmetry) allow for reflection asymmetry but not for transmission asymmetry; symmetries V (time-reversal symmetry) and VIII (PT pseudohermiticity) allow for transmission asymmetry but not for reflection asymmetry, whereas I (trivial symmetry) and IV (parity pseudohermiticity) allow for both scattering asymmetries. Note also the importance on non-locality for asymmetric transmission: All local potentials do satisfy automatically symmetry VI, and are therefore necessarily transmission reciprocal. All these results are for linear (Schrödinger) dynamics. Nonlinearity could break down these selection rules \([28, 29, 30]\).

Table 5 connects these results with the six asymmetry types mentioned in the Introduction.

7. Pole symmetries

The symmetries which imply real or complex-conjugate pairs of energy eigenvalues for bound eigenstates are II, IV, V and VII. These complex-conjugate pairs have been previously discussed in \([31, 32, 33, 34]\) for a general class of diagonalizable Hamiltonians that posses discrete spectrum. A novel aspect uncovered in \([35]\) is that whenever one of the above-mentioned four symmetries are satisfied, not only the complex eigenvalues representing the bound states come in conjugate-complex pairs, but all the complex poles of the \( S \)-matrix eigenvalues have this property. This applies in particular to antibound or virtual states, resonances, and antiresonances. The four Hamiltonian symmetries could also be formulated as the commutation of the Hamiltonian with specific antilinear operators, which are evident for V (where the antilinear operator is \( \Theta \)) and VII (\( \Theta \Pi \)), and can be constructed for II and IV \([35]\). In fact, the results for discrete Hamiltonians in \([31, 32, 33]\) generalize to scattering Hamiltonians also in the sense that the pseudohermitian symmetries in \( \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \) can be expressed as ordinary commutations with operators \( B \) (linear for \( A \) antilinear and antilinear for \( A \) linear) that may be explicitly constructed \([35]\).
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Table 4. Hamiltonian symmetries, relation between matrix elements of \( S \) and/or \( \hat{S} \) matrices (\( \hat{S} \) is for \( H \) and \( \hat{S} \) for \( H^\dagger \)). From them the next four columns relate the scattering amplitudes. Together with generalized unitarity relations they also imply relations for the moduli and phases (not shown). The last two columns summarize if perfect asymmetric transmission or reflection are possible: “✓” means possible (but not necessary), “✗” means impossible. In some cases “✓” is accompanied by a condition that must be satisfied.

| Code | Symmetry | \( \langle p|S|p'\rangle = \rangle^\dagger \Gamma \Gamma = U \lambda^\dagger \lambda = R \lambda = R \lambda \rangle \rangle | Moduli relations | \( |\Gamma|^2 = 1 \) | \( |R|^2 = 1 \) |
|------|----------|---------------------------------|----------------|------------------|------------------|
| I    | 1H = H1  | \( \langle p|S|p'\rangle \) \( T = T^\prime \) \( R = R^\prime \) \( R^\prime \) | No condition | ✓                | ✓                |
| II   | 1H = H'1 | \( \langle p|S|p'\rangle \) \( T^\prime \) \( R^\prime \) \( R^\prime \) \( T^\prime \) | \( |T^\prime|^2 \) \( |R^\prime|^2 \) |✗               |│                 |
| III  | \( \Pi H = HI \) | \( \langle -p|S - p'|\rangle \) \( T^\prime \) \( R^\prime \) \( R^\prime \) \( T^\prime \) | \( |T^\prime|^2 \) \( |R^\prime|^2 \) | ✓               | ✓               |
| IV   | \( \Pi H = H'I \) | \( \langle -p|S - p'|\rangle \) \( T^\prime \) \( R^\prime \) \( R^\prime \) \( T^\prime \) | ✓ \( |R^\prime|^2 \) \( |T^\prime|^2 \) | ✓               | ✓               |
| V    | \( \Theta H = H\Theta \) | \( \langle -p|\bar{S} - p'|\rangle \) \( T^\prime \) \( R^\prime \) \( R^\prime \) \( T^\prime \) | ✓ \( |R|^2 \) \( |T|^2 \) | ✓               | ✓               |
| VI   | \( \Theta H = H'\Theta \) | \( \langle -p|\bar{S} - p'|\rangle \) \( T^\prime \) \( R^\prime \) \( R^\prime \) \( T^\prime \) | ✓ \( |R|^2 \) \( |T|^2 \) | ✓               | ✓               |
| VII  | \( \Theta I'H = H'\Theta I \) | \( \langle p'|\bar{S}|p \rangle \) \( T^\prime \) \( R^\prime \) \( R^\prime \) \( T^\prime \) | ✓ \( |R|^2 \) \( |T|^2 \) | ✓               | ✓               |
| VIII | \( \Theta I'H = H'\Theta I \) | \( \langle p'|\bar{S}|p \rangle \) \( T^\prime \) \( R^\prime \) \( R^\prime \) \( T^\prime \) | ✓ \( |R|^2 \) \( |T|^2 \) | ✓               | ✓               |

Table 5. Device types for transmission and/or reflection asymmetry in the first row (see main text for nomenclature, binary values (0 or 1) for the transmission and reflection coefficients are considered here as an ideal case). The second row gives the corresponding symmetries that allow each device.

| TABLE 5 | | | | | | | |
|---------|---------|---------|---------|---------|---------|---------|---------|
| \( TR/A \) | \( TR/R \) | \( TR/A \) | \( TR/R \) | \( R/A \) | \( TR/T \) |
| I       | I       | I,III   | I,III   | I,VI    | I,IV,VI,VI, VII |

For Hermitian Hamiltonians with scattering (and possibly bound) eigenstates, their \( S \)-matrix poles are symmetric in the complex momentum plane with respect to the imaginary axis. In the upper half-plane, which corresponds to the first energy Riemann sheet, the poles are on the imaginary axis and represent bound states in the point spectrum of the Hamiltonian. In the lower half-plane they come in symmetrical resonance and antiresonance pairs, and may also lie on the imaginary axis as “virtual states”. A further symmetry is the occurrence of a zero at the complex-conjugate momentum of a given pole. These properties are well known for partial wave scattering by spherical potentials but also hold for the \( S \)-matrix eigenvalues in one dimensional scattering [16]. For NH-Hamiltonians the above pole- and pole-zero-symmetries do not hold in general. The point spectrum of \( H \) may include, apart from “ordinary bound states” on the imaginary axis, also eigenvalues (\( S \)-matrix poles) in the first and second quadrant. However the symmetries of zeros and poles, which are characteristic of Hermitian potentials, also hold for the symmetries IV, V, and VII. These relations had only been shown for PT-symmetry (symmetry VII) in [16].

The configuration with poles located on the imaginary axis or as symmetrical pairs implies important consequences. In particular, it provides stability of the real energy eigenvalues with respect to parameter variations of the potential. While a simple pole on the imaginary axis can move along that axis when a parameter is changed, it cannot move off this axis (since this would violate the pole-pair symmetry) or bifurcate. The formation of pole pairs occurs near special parameter values for which two poles on the imaginary axis collide.
8. Physical realization

Whereas Hermiticity does not allow for any asymmetry in transmission and reflection probabilities, PT symmetry or the symmetry of local potentials, technically “pseudohermiticity with respect to time reversal” [15], do not allow for asymmetric transmission [16, 36], see symmetries II (Hermiticity), VII (PT symmetry), and VI (time-reversal pseudohermiticity) in Tables 4 and 5. Thus non-local, non-PT, and non-Hermitian potentials are needed to implement a rich set of scattering asymmetries, and in particular asymmetric transmission.

In [20] potentials with proper symmetries were inverse engineered formally from the wanted scattering responses using polynomial ansatzes. Similarly in [35] we designed separable potentials with different symmetries to study and illustrate pole motions and structure. While the mathematical modelling was clearly possible, the question of a possible physical realization was open. Finally [20] put forward a physical realization of effective NH, non-local Hamiltonians which do not posses PT symmetry.

The system considered in [20] is a two-level atom with ground level $|1\rangle$ and excited state $|2\rangle$ impinging onto a laser illuminated region [19, 14]. The motion is assumed to be one dimensional, and the wavefunction accounts for atoms before the first spontaneous emission, i.e. it is assumed that no resetting to the ground state occurs. The (two-component) state $\Phi_k = (\phi_k^{(1)} \phi_k^{(2)})^T$ for the atom impinging with wavenumber $k$ obeys, in an interaction picture, an effective stationary Schrödinger equation with a time-independent Hamiltonian [19, 14] $H\Phi_k(x) = E\Phi_k(x)$, where

$$\mathcal{H} = H_0\mathbf{1} + V = \frac{1}{2m} \begin{pmatrix} p^2 & 0 \\ 0 & p^2 \end{pmatrix} + \mathcal{V}(x),$$

$$\mathcal{V}(x) = \frac{\hbar}{2} \begin{pmatrix} 0 & \Omega(x) \\ \Omega(x)^* & -(2\Delta + i\gamma) \end{pmatrix}. \tag{13}$$

$E = \hbar^2 k^2 / 2m$ is the energy, and $\Omega(x)$ is the position-dependent, on-resonance Rabi frequency, where real and imaginary parts may be controlled independently using two laser field quadratures; $\gamma$ is the inverse of the life time of the excited state; $\Delta = \omega_L - \omega_{12}$ is the detuning (laser angular frequency minus the atomic transition angular frequency $\omega_{12}$); $p = -i\hbar \partial / \partial x$ is the momentum operator; and $\mathbf{1} = |1\rangle\langle 1| + |2\rangle\langle 2|$ is the unit operator for the internal-state space. Complementary projectors $P = |1\rangle\langle 1|$ and $Q = |2\rangle\langle 2|$ are defined to select ground and excited state components. Using the partitioning technique [17, 18], it is found that the ground state amplitude $\phi_k^{(1)}$ satisfies the equation

$$E\phi_k^{(1)}(x) = H_0\phi_k^{(1)}(x) + \int dy \langle x, 1|\mathcal{W}(E)|y, 1\rangle \phi_k^{(1)}(y), \tag{14}$$

where $\mathcal{W}(E) = PVQ + PVQ(E + i0 - QHQ)^{-1}QVP$, is non local and energy dependent. This provides a a physical realization of an effective (in general) non-local, non-Hermitian potentials

$$V(x, y) = \langle x, 1|\mathcal{W}(E)|y, 1\rangle = \frac{m}{4} \frac{e^{i|x-y|q}}{iq} \Omega(x)\Omega(y)^*, \tag{15}$$

where $q = \sqrt{2mE(1 + \mu^{1/2})}/\hbar$, $\text{Im} \ q \geq 0$, and $\mu = \hbar(2\Delta + i\gamma)/(2E)$. The reflection and transmission amplitudes $R^{r,l}$ and $T^{r,l}$ may be calculated directly using the potential (15) or as corresponding amplitudes for transitions from ground state to ground state in the full two-level theory.

Imposing symmetries on $\Omega(x)$ leads to different Hamiltonian symmetries. We are mostly interested in the ones leading to asymmetric scattering which excludes II and III. The structure in (15) implies that symmetries IV, V and VII also imply Hermiticity. We are thus left with
symmetries I, VI and VIII as the interesting ones in this physical setting. Rabi frequencies for transmission asymmetric devices of one-way filter of types $T/A$ and $R/A$ are worked out in [20] making use of simply and realistic Gaussian shapes. A $TR/A$ device with equal $1/2$ transmission and reflection coefficients is also designed. An optical analog would be a darkish one-way mirror, where an observer on the left $O_L$ sees himself/herself reflected, and an observer on the right $O_R$ would see the person standing on the left $O_L$ through the mirror but not himself/herself reflected in the mirror, and $O_R$ cannot be seen by the observer on the left $O_L$.

9. Discussion
This review focuses on our recent results on how Non-Hermitian symmetries determine the possible asymmetric scattering of single particles by 1D potentials. While PT-symmetry [37] has got most of the attention in NH Physics, in particular because its applications in optics [38, 39, 40], Non-Hermitian symmetries beyond PT-symmetry have been considered recently in different contexts [41, 42, 43, 15, 23, 35, 26, 44, 45], offering a richer set of properties and applications. In particular the group structure described here for scattering systems can be generalized for finite matrices, where a larger non-Abelian group of symmetry operations naturally emerges [26]. Symmetry has always played a prominent role in physics and we expect it to be very relevant in Non-Hermitian Physics as well, for which some interesting symmetry-operation groups, relations and consequences are being found.

Acknowledgments
We are grateful to A. Ala˜na, S. Mart´inez Garaot, A. Buendía, S. Longhi, A. Mostafazadeh, M. Znojil, and V. V. Konotop for useful comments. This work was supported by the Basque Country Government (Grant No. IT986-16) and PGC2018-101355-B-I00 (MCIU/AEI/FEDER,UE). M.A.S. acknowledges financial support by the Basque Government predoctoral program (Grant No. PRE-2018-2-0177).

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