ARZELÀ-ASCOLI’S THEOREM IN UNIFORM SPACES

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Abstract. In the paper, we generalize the Arzelà-Ascoli’s theorem in the setting of uniform spaces. At first, we recall the Arzelà-Ascoli theorem for functions with locally compact domains and images in uniform spaces, coming from monographs of Kelley and Willard. The main part of the paper introduces the notion of the extension property which, similarly as equicontinuity, equates different topologies on $C(X, Y)$. This property enables us to prove the Arzelà-Ascoli’s theorem for uniform convergence. The paper culminates with applications, which are motivated by Schwartz’s distribution theory. Using the Banach-Alaoglu-Bourbaki’s theorem, we establish the relative compactness of subfamily of $C(\mathbb{R}, D'(\mathbb{R}^n))$.

1. Introduction. Around 1883, Cesare Arzelà and Giulio Ascoli provided the necessary and sufficient conditions under which every sequence of a given family of real-valued continuous functions, defined on a closed and bounded interval, has a uniformly convergent subsequence. Since then, numerous generalizations of this result have been obtained. For instance, in [10], p. 278 the compact families of $C(X, \mathbb{R})$, with $X$ a compact space, are exactly those which are equibounded and equicontinuous. The space $C(X, \mathbb{R})$ is given with the standard norm

$$
\|f\| := \sup_{x \in X} |f(x)|,
$$

where $f \in C(X, \mathbb{R})$.

Changing the norm topology to the topology of uniform convergence on compacta, one can obtain a version of Arzelà-Ascoli’s theorem on $C(X, Y)$, where $X$ is a locally compact Hausdorff space and $Y$ is a metric space ([10], p. 290). The author, together with Bogdan Przeradzki endeavoured to retain the topology of uniform convergence (cf. [7]). The idea of using the extension property goes back to [11], where B. Przeradzki studied the existence of bounded solutions to the equation

$$
x' = A(t)x + r(x, t),
$$

where $A$ is a continuous function taking values in the space of bounded linear operators in a Hilbert space and $r$ is a nonlinear continuous mapping. The paper gives a characterization of relatively compact sets $K \subset C^b(\mathbb{R}, E)$, where $E$ is a Banach space. In addition to pointwise relative compactness and equicontinuity, the following condition was introduced:

(P): For any $\varepsilon > 0$, there exist $T > 0$ and $\delta > 0$ such that if $\|x(T) - y(T)\| \leq \delta$, then $\|x(t) - y(t)\| \leq \varepsilon$ for $t \geq T$ and if $\|x(-T) - y(-T)\| \leq \delta$, then $\|x(t) - y(t)\| \leq \varepsilon$ for $t \leq -T$, where $x$ and $y$ are arbitrary functions in $K$.

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R. Stańczy developed this approach, while investigating the existence of solutions to Hammerstein equations in the space of bounded and continuous functions (cf. [14]). The author, together with B. Przeradzki rewrote (P) for σ-locally compact Hausdorff space X and metric space Y (cf. [7]).

In this paper, we improve the result obtained in [7] by considering the uniform spaces (more precisely, Y will be a Hausdorff uniform space), which are thoroughly described e.g. in [4] Chapter 7 and 8. The concept of uniform spaces generalizes the concept of metric spaces and provides a convenient setting for studying the uniform continuity and the uniform convergence, as well. A brief recap on the uniformity of uniform convergence and the uniformity of uniform convergence on compacta is presented in section 2. We highlight the importance of equicontinuity, which equates the topology of pointwise convergence, the compact-open topology and the topology of uniform convergence on compacta. This property of ‘bringing the topologies together’ is a motivation for the extension property, whose objective is to equate the topology of uniform convergence with the aforementioned ones.

The main part of the paper is in Section 3, where we build on the Arzelà-Ascoli’s theorem in [5], p. 236 (Theorem 3.1 in this paper). We introduce a new concept of extension property and investigate how it relates to the $C^b(X,Y)$-extension property studied in [7]. The culminating point is Theorem 3.2, which characterizes compact subsets of $C(X,Y)$ with the topology of uniform convergence (rather than uniform convergence on compacta). The Hausdorff space X is assumed to be locally compact, while Y is assumed to be a Hausdorff uniform space. Finally, in Section 4, we present two possible applications of Theorem 3.2 in the theory of distributions. Let us recall that the space $D'(\mathbb{R}^n)$ is known to be nonmetrizable ([2], p. 81), yet it can be considered as a uniform structure.

2. Pointwise convergence, uniform convergence and uniform convergence on compacta. We begin with the theorem, which describes the uniformity of pointwise convergence. This result summarizes the discussions in [4], p. 279 and [5], p. 182.

**Theorem 2.1.** (Uniformity of pointwise convergence) Let $X$ be a set and let $(Y,V)$ be a uniform space and $F \subset Y^X$. The uniformity of pointwise convergence on $F$, which is denoted by $W_{pc}$, is the product uniformity on $F$. The subbase set of this uniformity is of the form

$$\left\{ (f,g) \in F \times F : (f(x),g(x)) \in V \right\}$$

where $x \in X$ and $V \in \mathcal{V}$.

We recall the concept of the uniformity of uniform convergence. A comprehensive study of this notion can be found in [5], p. 226 or [16], p. 280. For convenience of the reader, we extract the essence of these discussions in the following:

**Theorem 2.2.** (Uniformity of uniform convergence) Let $X$ be a set, $(Y,\mathcal{V})$ be a uniform space, $F \subset Y^X$ and let us define a mapping $\dagger : \mathcal{V} \to 2^{F \times F}$ by the formula

$$\forall V \in \mathcal{V} \quad V^\dagger := \left\{ (f,g) \in F \times F : \forall x \in X \quad (f(x),g(x)) \in V \right\}. \quad \text{(1)}$$

Then the family $\{ V^\dagger : V \in \mathcal{V} \}$ is a base for the uniformity $W_{uc}$ on $F$, which we call the uniformity of uniform convergence.
The topology induced by the uniformity of uniform convergence is called the topology of uniform convergence, which we denote by \( \tau_{uc} \). The base sets for this topology are of the form

\[
V^+[f] = \left\{ g \in F : \forall x \in X \ g(x) \in V[f(x)] \right\},
\]

where \( V[y] = \{ z \in Y : (y, z) \in V \} \) as in [5], p. 176. A concept, which is closely related to the uniformity of uniform convergence is the uniformity of uniform convergence on compacta. It appears for example in [5], p. 229 or [16], p. 283.

Theorem 2.3. (Uniformity of uniform convergence on compacta) Let \((X, \tau_X)\) be a topological space, \((Y, V)\) be a uniform space and let \( F \subset Y^X \). The family

\[
\left\{ (f, g) \in F \times F : \forall x \in D \ (f(x), g(x)) \in V \right\}_{V \in V, \ D \subset X}
\]

is a subbase for the uniformity \( W_{ucc} \) on \( F \), which we call the uniformity of uniform convergence on compacta.

Remark 1. The notation \( D \subset X \) stands for ‘\( D \) is a compact subset of \( X \)’.

The topology induced by the uniformity of uniform convergence on compacta is called the topology of uniform convergence on compacta, which we denote by \( \tau_{ucc} \). It is an easy observation that \( \tau_{ucc} \) is weaker than \( \tau_{uc} \). In fact, if we restrict our attention to \( C(X, Y) \) rather than to \( Y^X \), which we are going to do in the sequel, \( \tau_{ucc} \) becomes a familiar compact-open topology (see the proof in [5], p. 230 or [16], p. 284).

Another important concept is that of equicontinuity (see [5], p. 232 or [16], p. 286). For a topological space \((X, \tau_X)\) and a uniform space \((Y, V)\), we say that the family \( F \subset Y^X \) is equicontinuous at \( x \in X \) if

\[
\forall V \in V \ \exists U_x \in \tau_X \ \forall f \in F \ f(U_x) \subset V[f(x)].
\]

The family \( F \) is said to be equicontinuous if it is equicontinuous at every \( x \in X \). If one considers such a family, then the topology of pointwise convergence coincides with the compact-open topology, i.e. \( \tau_{pc}|F = \tau_{co}|F \) (see the proof in [5], p. 232 or [16], p. 286). Together with what we said above and the fact that an equicontinuous family consists of continuous functions, we infer that the topology of pointwise convergence coincides with the topology of uniform convergence on compacta on \( F \), i.e. \( \tau_{pc}|F = \tau_{ucc}|F \).

3. **Arzelà-Ascoli’s theorem.** Our objective in this section is to present the Arzelà-Ascoli’s theorem for the topology of uniform convergence in the setting of uniform spaces. Our starting point is the theorem provided in [5], p. 236.

**Theorem 3.1. (Arzelà-Ascoli’s theorem for uniform convergence on compacta)** Let \((X, \tau_X)\) be a locally compact Hausdorff space, \((Y, V)\) be a Hausdorff uniform space and let \( C(X, Y) \) be endowed with the topology of uniform convergence on compacta \( \tau_{ucc} \). A family \( F \subset C(X, Y) \) is relatively \( \tau_{ucc} \)-compact if and only if the following two conditions are satisfied:

- **(AAucc1):** \( F \) is pointwise relatively compact, i.e. \( \{ f(x) : f \in F \} \) is relatively \( \tau_Y \)-compact at every \( x \in X \);
- **(AAucc2):** \( F \) is equicontinuous.
Our line of attack is motivated by the equicontinuity, which equates three topologies, namely the topology of pointwise convergence, the compact-open topology and the topology of uniform convergence on compacta. We look for a property that will do the same thing with the topology of uniform convergence.

For a topological space \((X, \tau_X)\), uniform space \((Y, \mathcal{V})\) and \(F \subset Y^X\) suppose that \(\mathcal{W}_{pc}\) and \(\mathcal{W}_{ucc}\) are uniformities of pointwise convergence and uniform convergence on compacta on \(F\), respectively. We say that \(F\) satisfies the finite extension property if

\[
\forall V \in \mathcal{V} \exists W \in \mathcal{W}_{pc} \ W \subset V, \tag{5}
\]

and that it satisfies the compact extension property if

\[
\forall V \in \mathcal{V} \exists W \in \mathcal{W}_{ucc} \ W \subset V. \tag{6}
\]

Intuitively, condition (5) means that the topology \(\tau_{pc}\) coincides with \(\tau_{uc}\) on \(F\) (the inclusion \(\tau_{pc} \subset \tau_{uc}\) always holds, so (5) guarantees the reverse inclusion). Condition (6) ‘merely’ implies that \(\tau_{ucc}\) coincides with \(\tau_{uc}\) on \(F\). Obviously if \(F\) satisfies the finite extension property then it satisfies the compact extension property.

In order to strengthen our intuition and provide further motivation, we compare the above conditions with the one appearing in [7]. Condition (6) can be written explicitly as

\[
\forall \varepsilon > 0 \exists D \subset X \forall f, g \in F \left( \forall x \in D \ g(x) \in U[f(x)] \implies \forall x \in X \ g(x) \in V[f(x)] \right). \tag{7}
\]

If we suppose that \(Y\) is a metric space, then \(V[f(x)]\) becomes a ball \(B(f(x), \varepsilon)\) and \(U[f(x)]\) becomes a ball \(B(f(x), \delta)\). Hence (7) turns into

\[
\forall \varepsilon > 0 \exists D \subset X \forall f, g \in F \left( \sup_{x \in D} d_Y(g(x), f(x)) < \delta \implies \sup_{x \in X} d_Y(g(x), f(x)) < \varepsilon \right) \tag{8}
\]

which is the compact extension property studied in [7]. Reasoning analogously for (5), we obtain

\[
\forall \varepsilon > 0 \exists D \text{ finite} \forall f, g \in F \left( \max_{x \in D} d_Y(g(x), f(x)) < \delta \implies \sup_{x \in X} d_Y(g(x), f(x)) < \varepsilon \right). \tag{9}
\]

Equipped with these tools, we are ready to prove the following theorem.

**Theorem 3.2.** (Arzelà-Ascoli’s theorem for uniform convergence) Let \((X, \tau_X)\) be a locally compact Hausdorff space, \((Y, \mathcal{V})\) be a Hausdorff uniform space and let \(C(X, Y)\) be endowed with the topology of uniform convergence \(\tau_{uc}\). A family \(F \subset C(X, Y)\) is relatively \(\tau_{uc}\)-compact if and only if

(AA1): \(F\) is pointwise relatively compact and equicontinuous;

(AA2): \(F\) satisfies the finite extension property.

**Proof.** The ‘if’ part is easy. By (AA1) and Theorem 3.1 we obtain that \(F\) is relatively \(\tau_{ucc}\)-compact, consequently (AA2) implies that the topologies \(\tau_{pc}\) and \(\tau_{ucc}\) coincide with \(\tau_{uc}\) on \(F\) and we are done.

We focus on ‘only if’ part. We already noted that \(\tau_{uc}\) is stronger than \(\tau_{ucc}\), which implies that \(F\) is relatively \(\tau_{ucc}\)-compact. Hence, by Theorem 3.1, we obtain (AA1).
Suppose that (AA2) is not satisfied, which means that there exists $V \in \mathcal{V}$ such that $W \setminus V \neq \emptyset$ for every $W \in \mathcal{W}_{pc}$. In what follows, $V^\frac{1}{2}$ will mean a symmetric set such that $V^\frac{1}{2} \circ V^\frac{1}{2} \circ V^\frac{1}{2} \subset V$ and likewise, by $V^\frac{1}{3}$ we denote a symmetric set such that $V^\frac{1}{3} \circ V^\frac{1}{3} \subset V^\frac{1}{3}$, where

$$A \circ B = \{(a, b) : \exists c \ (a, c) \in B, \ (c, b) \in A\}$$

as in [5], p. 176. The existence of sets $V^\frac{1}{2}$ and $V^\frac{1}{3}$ follows from the axioms of uniformity (see [4], p. 103). The negation of (AA2) means that

$$\left\{(f, g) \in \mathcal{F} \times \mathcal{F} : \forall x \in D \ (f(x), g(x)) \in V^\frac{1}{2}, \ \exists x \in X \ (f(x), g(x)) \not\in V\right\} \neq \emptyset \quad (10)$$

for every finite set $D \subset X$. The family

$$\left\{g \in C(X, Y) : \forall x \in X \ (f(x), g(x)) \in V^\frac{1}{2} \right\}_{f \in \mathcal{F}} \quad (11)$$

is a $\tau_{uc}$-open cover of $\mathcal{F}$, the closure of $\mathcal{F}$ in $\tau_{uc}$. Indeed, if $\overline{f} \in \mathcal{F}$, then by the characterization of belonging to a closure, using (2), we get

$$\left(V^\frac{1}{2}\right)^\dagger \mathcal{F} \cap \mathcal{F} \neq \emptyset \iff \exists f \in \mathcal{F} \ \forall x \in X \ (f(x), \overline{f}(x)) \in V^\frac{1}{2}.$$  

Since we assume that $\mathcal{F}$ is $\tau_{uc}$-compact (and we aim to reach a contradiction), we can choose a finite subcover from (11), which means that there is a sequence $(f_k)_{k=1}^n, f_k \in \mathcal{F}$ for $k = 1, \ldots, n$, such that

$$\forall g \in \mathcal{F} \ \exists k = 1, \ldots, n \ \forall x \in X \ (f_k(x), g(x)) \in V^\frac{1}{2} \quad (12).$$

For every $k, l = 1, \ldots, n, k \neq l$, let us define a set

$$D_{k,l} := \left\{x \in X : (f_k(x), f_l(x)) \not\in V^\frac{1}{3}\right\}.$$  

Let $D$ be a set which consists of one element from each nonempty $D_{k,l}$. This set is finite and serves as a guard, watching whether each pair $(f_k, f_l)$ ‘drifts apart’. More precisely, the following implication

$$\forall x \in D \ (f_k(x), f_l(x)) \in V^\frac{1}{3} \implies \forall x \in X \ (f_k(x), f_l(x)) \in V^\frac{1}{3} \quad (13)$$

holds for every $k, l = 1, \ldots, n$. Indeed, if for some $k, l = 1, \ldots, n$ there exists $x \in X$ such that $(f_k(x), f_l(x)) \not\in V^\frac{1}{3}$, then $D_{k,l} \neq \emptyset$. Hence $D \cap D_{k,l} \neq \emptyset$ and there exists $x_D \in D$ such that $(f_k(x_D), f_l(x_D)) \not\in V^\frac{1}{3}$. This proves the implication (13).

By (10), we pick $(f_*, g_*)$ such that

$$\forall x \in D \ (f_*(x), g_*(x)) \in V^\frac{1}{2} \quad \text{and} \quad \exists x, \in X \ (f_*(x), g_*(x)) \not\in V. \quad (14)$$

Let $k_f, k_g \in \mathbb{N}$ be constants chosen as in (12) for $f_*$ and $g_*$, respectively. We have

$$\forall x \in D \ (f_k(x), f_*(x)), (f_*(x), g_*(x)), (g_*(x), f_k(x)) \in V^\frac{1}{2} \implies \forall x \in D \ (f_k(x), f_k(x)) \in V^\frac{1}{2}. \quad (15)$$

By (13) we know that $(f_k(x), f_k(x)) \in V^\frac{1}{3}$ for every $x \in X$. Finally,

$$\forall x \in X \ (f_*(x), f_k(x)), (f_k(x), f_k(x)), (f_k(x), g_*(x)) \in V^\frac{1}{4} \implies \forall x \in X \ (f_*(x), g_*(x)) \in V.
which is a contradiction with (14). Hence, we conclude that the finite extension property must hold.

The next corollary characterizes the relation between the finite and the compact extension property.

**Corollary 1.** Let \((X, \tau_X)\) be a locally compact Hausdorff space, \((Y, V)\) be a Hausdorff uniform space and let \(C(X, Y)\) be endowed with the topology of uniform convergence \(\tau_{uc}\). For a family \(\mathcal{F} \subset C(X, Y)\) satisfying (AA1), the following conditions are equivalent:

- \((C1): \) \(\mathcal{F}\) is relatively \(\tau_{uc}\)-compact;
- \((C2): \) \(\mathcal{F}\) satisfies the finite extension property;
- \((C3): \) \(\mathcal{F}\) satisfies the compact extension property.

**Proof.** Theorem 3.2 claims the equivalence of \((C1)\) and \((C2)\). Moreover, we already observed that the finite extension property implies the compact extension property \((C2)\) implies \((C3)\). Finally, if \((C3)\) is satisfied, then the topologies \(\tau_{uc}\) and \(\tau_{ac}\) coincide on \(\mathcal{F}\) and relative \(\tau_{ac}\)-compactness follows from Theorem 3.1.

Before presenting the applications of Theorem 3.2, we compare it with the result from the paper [7]. Let \(X\) be a \(\sigma\)-locally compact Hausdorff space and let \((Y, d_Y)\) be a metric space. By \(C^b(X, Y)\) we denote the metric space of bounded and continuous functions with the distance

\[ d_{C^b(X,Y)}(f, g) = \sup_{x \in X} d_Y(f(x), g(x)). \]

We recall that theorem without the proof, because it is an evident corollary of Theorem 3.2.

**Theorem 3.3.** (Arzelà-Ascoli’s theorem for \(\sigma\)-locally compact Hausdorff space) Let \((X, \tau_X)\) be a \(\sigma\)-locally compact Hausdorff space and \((Y, d_Y)\) be a metric space. A family \(\mathcal{F} \subset C^b(X, Y)\) is relatively \(d_{C^b(X,Y)}\)-compact if and only if

- \((KP1): \) \(\mathcal{F}\) is pointwise relatively compact;
- \((KP2): \) \(\mathcal{F}\) is equicontinuous;
- \((KP3): \) \(\mathcal{F}\) satisfies the compact extension property.

The first apparent difference is that Theorem 3.2 does not require \(X\) to be a countable union of compact sets. This is due to the fact, that the proof of Theorem 3.3 focuses on sequences (and choosing convergent subsequences) and the authors needed countability in their argument. However, the proof of Theorem 3.2 has rather a topological character, focusing on open covers. Hence, no countability is assumed.

The most striking difference is that Theorem 3.2 replaces a metric space \((Y, d_Y)\) by a Hausdorff uniform structure \((Y, V)\). This is because uniform spaces serve perfectly as generalizations of metric spaces.

Finally, in [7], assuming that \(Y\) is a Banach space, we are dealing with a Banach space \(C^b(X, Y)\). Hence the functions needed to be bounded. However, Theorem 3.2 showed that we can consider \(C(X, Y)\) which need not be a metric space. Consequently, in theorem 3.3 we could replace \(C^b(X, Y)\) by \(C(X, Y)\), accepting the loss of metricity.
4. Applications in distribution theory. In the final section of the paper, we present two possible applications of Theorem 3.2. They were motivated by Schwartz’s distribution theory, described e.g. [13] Chapter 2 or [17] Chapter 1. We briefly recall the essentials that will be used in the sequel.

After [17], p. 28 by $D_K(\mathbb{R}^n)$ we denote the set of all real-valued functions $f \in C_0^\infty(\mathbb{R}^n)$ with $\text{supp}(\varphi) \subset K \subset \mathbb{R}^n$. This is a locally convex space with a family of seminorms

$$
\|\varphi\|_{K,m} = \sup_{|s| \leq m, x \in K} \left| \frac{\partial^{|s|}}{\partial x_1^{s_1} \cdots \partial x_n^{s_n}} \varphi(x) \right| .
$$

Moreover, if $K_1 \subset K_2$ then the topology of $D_{K_1}(\mathbb{R}^n)$ coincides with the relative topology of $D_K(\mathbb{R}^n)$ as a subset of $D_{K_2}(\mathbb{R}^n)$. Hence, we may construct the inductive limit, namely the space of test functions $D(\mathbb{R}^n)$. A set $U_0$ is an open neighbourhood of 0 in $D(\mathbb{R}^n)$ if it is absorbing, balanced, convex and $U_0 \cap D_K(\mathbb{R}^n)$ is open in $D_K(\mathbb{R}^n)$ for every $K \subset \mathbb{R}^n$.

The space of distributions, denoted by $D'(\mathbb{R}^n)$, comes with the weak* topology ([13], p. 94 or [3], p. 51). As such, it can be defined by the pseudometrics

$$
p^{-1}_T(0, \varepsilon) = \left\{ (T,S) \in D'(\mathbb{R}^n) \times D'(\mathbb{R}^n) : |T(\varphi) - S(\varphi)| < \varepsilon \right\} ,
$$

for $\varphi \in D(\mathbb{R}^n)$ and $\varepsilon > 0$, is a subbase for the uniformity on $D'(\mathbb{R}^n)$. Let $\mathcal{F}$ be a family of functions from $\mathbb{R} \times \mathbb{R}^n$ to $\mathbb{R}$ such that the following conditions are satisfied:

- **(EQ):** For every $t_* \in \mathbb{R}$, $\varepsilon > 0$ and $K \subset \mathbb{R}^n$, there exists $\delta > 0$ such that
  $$
  \forall_{t - t_* | < \delta} \forall_{x \in K} \forall_{f \in \mathcal{F}} \forall_{f \in \mathcal{F}} \forall_{f \in \mathcal{F}} \forall_{f \in \mathcal{F}} \forall_{f \in \mathcal{F}} \forall_{f \in \mathcal{F}} |f(t, x) - f(t_*, x)| < \varepsilon .
  $$

- **(PRC):** For every $t_* \in \mathbb{R}$, the set $\left\{ f(t_*, \cdot) : f \in \mathcal{F} \right\}$ is $L^1_{loc}(\mathbb{R}^n)$-relatively compact.

- **(EP):** For every $\varepsilon > 0$ and $K \subset \mathbb{R}^n$, there exists $R > 0$ such that for every $f, g \in \mathcal{F}$ we have
  $$
  \sup_{|t| > R} \int_K |f(t, x) - g(t, x)| \, dx < \varepsilon .
  $$

We consider a family of distributions defined by

$$
\forall_{\varphi \in D(\mathbb{R}^n)} T_{f,t}(\varphi) := \int_{\mathbb{R}^n} f(t, x) \varphi(x) \, dx ,
$$

where $f \in \mathcal{F}$ and moreover, we denote $F_f : t \mapsto T_{f,t}$ (a distribution-valued function). With Theorem 3.2 at our disposal, we are able to prove the following:

**Theorem 4.1.** Under the assumptions (EQ), (PRC) and (EP), the family $F_{\mathcal{F}} := (F_f)_{f \in \mathcal{F}}$ is relatively $\tau_{uc}$-compact in $C(\mathbb{R}, D'(\mathbb{R}^n))$.

**Proof.** The first two conditions are stronger than the Carathéodory conditions (cf. [1], p. 153). Condition (EQ) will turn out to be very useful when proving the equicontinuity of $F_{\mathcal{F}}$. (PRC) will be used to establish the pointwise relative
compactness in the space $\mathcal{D}'(\mathbb{R}^n)$. Finally, with the use of (EP), we will justify (AA2).

First of all, we check that $F_\mathcal{F}$ is equicontinuous, so in particular $F_f \in C(\mathbb{R}, \mathcal{D}'(\mathbb{R}^n))$ for every $f \in \mathcal{F}$. Condition (4) at a fixed point $t_\ast \in \mathbb{R}$ reads as
\begin{equation}
\forall f \in \mathcal{D}(\mathbb{R}^n) \ \exists \delta > 0 \ \forall |t - t_\ast| \leq \delta \left| \int_{\mathbb{R}^n} \left( f(t, x) - f(t_\ast, x) \right) \varphi(x) \, dx \right| < \varepsilon. \tag{15}
\end{equation}
For a fixed $\varphi_\ast \in \mathcal{D}(\mathbb{R}^n)$ and $\varepsilon > 0$, we use (EQ) to choose $\delta > 0$ such that
\begin{equation}
\forall |t - t_\ast| \leq \delta \ |f(t, x) - f(t_\ast, x)| < \frac{\varepsilon}{\lambda(\text{supp } \varphi_\ast) \| \varphi_\ast \|_{C^n(\mathbb{R}^n)}},
\end{equation}
where $\lambda$ is the Lebesgue measure on $\mathbb{R}^n$. Hence,
\begin{equation}
\forall |t - t_\ast| \leq \delta \ |f(t, x) - f(t_\ast, x)| \leq \int_{\text{supp } \varphi_\ast} |f(t, x) - f(t_\ast, x)| \, dx \leq \varepsilon \int_{\text{supp } \varphi_\ast} \frac{\varphi_\ast(x)}{\lambda(\text{supp } \varphi_\ast) \| \varphi_\ast \|_{C^n(\mathbb{R}^n)}} \, dx \leq \varepsilon
\end{equation}
and we infer that $F_\mathcal{F}$ is equicontinuous.

Next, we prove the relative compactness of $F_\mathcal{F}$ at $t_\ast$, i.e. we show that
\begin{equation}
\left\{ T_{f, t_\ast} : f \in \mathcal{F} \right\} = \left\{ \varphi \mapsto \int_{\mathbb{R}^n} f(t_\ast, x) \, \varphi(x) \, dx : f \in \mathcal{F} \right\}
\end{equation}
is relatively compact in $\mathcal{D}'(\mathbb{R}^n)$. By theorem 39.9 in [16], p. 262 it is sufficient and necessary that for a fixed $\varepsilon > 0$ and $\varphi_\ast \in \mathcal{D}(\mathbb{R}^n)$, we find $(T_k)_{k=1}^m \subset \mathcal{D}'(\mathbb{R}^n)$ such that the family $p^{-1}_\varphi([0, \varepsilon])T_k$ for $k = 1, \ldots, m$, covers the set (16). Without loss of generality, we may assume that $\varphi_\ast \neq 0$.

Since $\left\{ f(t_\ast, \cdot) : f \in \mathcal{F} \right\}$ is relatively compact in $L^1_{\text{loc}}(\mathbb{R}^n)$, which is a complete metric space (cf. [8], p. 5), we can find $(f_k)_{k=1}^m \subset L^1_{\text{loc}}(\mathbb{R}^n)$ such that
\begin{equation}
\forall f \in \mathcal{F} \ \exists k = 1, \ldots, m \ |f_k - f(t_\ast, \cdot)|_{L^1(\text{supp } \varphi_\ast)} < \frac{\varepsilon}{\lambda(\text{supp } \varphi_\ast) \| \varphi_\ast \|_{C^n(\mathbb{R}^n)}}.
\end{equation}
This implies that
\begin{equation}
\forall f \in \mathcal{F} \ \exists k = 1, \ldots, m \ |T_k(\varphi_\ast) - \int_{\mathbb{R}^n} f(t_\ast, x) \, \varphi_\ast(x) \, dx| < \varepsilon,
\end{equation}
where
\begin{equation}
\forall \varphi \in \mathcal{D}(\mathbb{R}^n) \ T_k(\varphi) := \int_{\mathbb{R}^n} f_k(x) \, \varphi(x) \, dx.
\end{equation}
We conclude that each distribution in set (16) is in $p^{-1}_\varphi([0, \varepsilon])T_k$ for some $k = 1, \ldots, m$.

Lastly, we need to verify (AA2). The uniformity of uniform convergence on $F_\mathcal{F} \subset C(\mathbb{R}, \mathcal{D}'(\mathbb{R}^n))$ has the base sets of the form
\begin{equation}
(p^{-1}_\varphi([0, \varepsilon]))^\dagger = \left\{ (F_f, F_g) \in F_\mathcal{F} \times F_\mathcal{F} : \forall t \in \mathbb{R} \ (F_f(t), F_g(t)) \in p^{-1}_\varphi([0, \varepsilon]) \right\}
\end{equation}
Using the well-known estimate
\[
\parallel g \parallel _\infty = \sup _{t \in \mathbb{R}} |g(t)|
\]
by the relative compactness of \(G \subseteq \mathbb{R}\), we consider \(\mathcal{F}\) to be compact if there exists a base set of the form
\[
\phi, \quad \epsilon > 0\] such that
\[
\exists \tau \in (0, \infty) \text{ s.t. } V \subseteq \tau \quad \text{implies} \quad \parallel f \parallel _\infty < \epsilon/2\]
where \(\phi \in \mathcal{D}(\mathbb{R}^n)\), \(R > 0\), \(\delta > 0\). Fix \(\epsilon > 0\), \(\phi_\ast \in \mathcal{D}(\mathbb{R}^n)\) and by (EP) choose \(R > 0\) such that
\[
\forall f, g \in \mathcal{F} \quad \sup _{|t| > R} \int _{\operatorname{supp} \phi_\ast} |f(t, x) - g(t, x)| \, dx < \frac{\epsilon}{2 \parallel \phi_\ast \parallel _{C^1(\mathbb{R}^n)}}.
\]
Since
\[
\sup _{t \in \mathbb{R}} \int _{\mathbb{R}^n} \left( f(t, x) - g(t, x) \right) \phi_\ast (x) \, dx \\
\leq \sup _{|t| \leq R} \int _{\mathbb{R}^n} \left( f(t, x) - g(t, x) \right) \phi_\ast (x) \, dx \\
+ \sup _{|t| > R} \int _{\operatorname{supp} \phi_\ast} |f(t, x) - g(t, x)| \, dx \parallel \phi_\ast \parallel _{C^1(\mathbb{R}^n)} \\
\overset{(17)}{\leq} \sup _{|t| \leq R} \left| \int _{\mathbb{R}^n} \left( f(t, x) - g(t, x) \right) \phi_\ast (x) \, dx \right| + \frac{\epsilon}{2},
\]
we can conclude that for \(\delta < \frac{\epsilon}{2}\) we have \(W_{\phi, R, \delta} \subseteq (p_{\phi_\ast}^{-1}(0, \delta))^\dagger\), i.e. condition (AA2). Thus we proved, using Theorem 3.2, that \(\mathcal{F} \subseteq \mathcal{C}(\mathbb{R}, \mathcal{D}'(\mathbb{R}^n))\) is relatively \(\tau_{cc}\)-compact. \(\square\)

In order to provide an example of a family \(\mathcal{F}\) satisfying (EQ), (PRC) and (EP), we consider
\[
\mathcal{F} := \left\{ f(x) e^{-|c(t) - x|} : f \in \mathcal{G} \right\},
\]
where \(\mathcal{G} \subseteq L^\infty(\mathbb{R}^n)\) is relatively compact and \(c \in C(\mathbb{R}, \mathbb{R}^n)\) satisfies \(\lim _{|t| \to \infty} |c(t)| = \infty\). (PRC) is obviously satisfied as the relative compactness in \(L^\infty(\mathbb{R}^n)\) implies the relative compactness in \(L_{loc}^\infty(\mathbb{R}^n)\). For the remaining two conditions, fix \(t_\ast \in \mathbb{R}\), \(\varepsilon > 0\), \(K \subseteq \mathbb{R}^n\) and put \(M := \sup _{f \in \mathcal{G}} \parallel f \parallel _{L^\infty(\mathbb{R}^n)}\). The last constant is finite by the relative compactness of \(\mathcal{G}\).

By continuity, choose \(\delta > 0\) such that
\[
\forall |t - t_\ast| < \delta \quad M \left( e^{|c(t) - c(t_\ast)|} - 1 \right) < \varepsilon.
\]
Using the well-known estimate
\[
\forall u, v \geq 0 \quad e^{-u} - e^{-v} \leq e^{-v} |e^{-(u-v)} - 1| \leq e^{-v} \left( e^{u-v} - 1 \right),
\]
we can verify (EQ) as follows:
\[
\forall_{|t-t_*|<\delta} \forall_{x \in K} \forall_{f \in \mathcal{F}} \left| f(x) e^{-|c(t)-x|} - f(x) e^{-|c(t_*)-x|} \right| \\
\leq M e^{-|c(t_*)-x|} \left( e^{\left| c(t)-x \right| - |c(t_*)-x|} - 1 \right) \\
\leq M \left( e^{\left| c(t)-c(t_*) \right|} - 1 \right) \leq M \left( e^{R} - 1 \right) \leq 2M e^{R} \leq 2M e^{R}. 
\]

It remains to prove (EP). Without loss of generality, we assume that \( \lambda(K) > 0 \), because for \( \lambda(K) = 0 \), the condition (EP) is trivial. Since \( \lim_{|t|\to\infty} |c(t)| = \infty \), we can choose \( R > 0 \) such that
\[
\forall_{|t|\geq R} \forall_{x \in K} |c(t) - x| > -\ln \left( \frac{\varepsilon}{2M \lambda(K)} \right). 
\]
For every \( f, g \in \mathcal{G} \) we have
\[
\sup_{|t|<R} \int_K \left| f(x) e^{-|c(t)-x|} - g(x) e^{-|c(t)-x|} \right| dx \\
\leq 2M \sup_{|t|<R} \int_K e^{-|c(t)-x|} dx < \varepsilon, 
\]
which proves that \( \mathcal{F} \) satisfies (EP).

The second application of Theorem 3.2 concerns the family of distribution-valued functions \( \{\delta_{f(t)}\}_{f \in \mathcal{F}} \), where \( \delta_p \in \mathcal{D}'(\mathbb{R}^n) \) is the Dirac delta function (cf. [15], p. 14) at point \( p \in \mathbb{R}^n \).

**Theorem 4.2.** If the family \( \mathcal{F} \subset C(\mathbb{R},\mathbb{R}^n) \) is equicontinuous and satisfies
\[
\forall_{\varphi \in \mathcal{D}'(\mathbb{R}^n)} \forall_{|t|>R} \forall_{f \in \mathcal{F}} \left| \varphi \circ f(t) - \varphi \circ g(t) \right| < \varepsilon, \quad (20)
\]
then the family \( \{\delta_{f(t)}\}_{f \in \mathcal{F}} \subset C(\mathbb{R},\mathcal{D}'(\mathbb{R}^n)) \) is relatively \( \tau_{uc} \)-compact.

**Proof.** At first, we check (AA1), i.e. equicontinuity and relative pointwise compactness at a fixed point \( t_* \in \mathbb{R} \). The condition (4) reads as
\[
\forall_{\varphi \in \mathcal{D}'(\mathbb{R}^n)} \forall_{|t-t_*|<\varepsilon} \forall_{f \in \mathcal{F}} \left| \varphi \circ f(t) - \varphi \circ f(t_*) \right| < \varepsilon. \quad (21)
\]
Fix \( \varepsilon > 0 \) and \( \varphi_* \in \mathcal{D}(\mathbb{R}^n) \). By the uniform continuity of \( \varphi_* \), we obtain
\[
\exists_{\rho>0} \forall_{u,v \in \mathbb{R}^n} \left| u - v \right| < \rho \Rightarrow \left| \varphi(u) - \varphi(v) \right| < \varepsilon. \quad (22)
\]
By the equicontinuity of \( \mathcal{F} \), we find that
\[
\exists_{\rho>0} \forall_{|t-t_*|<\rho} \forall_{f \in \mathcal{F}} \left| f(t) - f(t_*) \right| < \rho. \quad (23)
\]
By (22) and (23) we easily obtain (21).

In order to prove the pointwise relative compactness at a fixed point \( t_* \), we need to show that
\[
\left\{ \delta_{f(t_*)} : f \in \mathcal{F} \right\} \quad (24)
\]
is relatively compact in \( \mathcal{D}'(\mathbb{R}^n) \). Putting
\[
U_0 = \left\{ \varphi \in \mathcal{D}(\mathbb{R}^n) : \|\varphi\|_{C^0(\mathbb{R}^n)} < 1 \right\}
\]
satisfies (20). Indeed, fix \( \varepsilon > 0 \) and \( B \)

To see this, fix \( \varepsilon > 0 \) and \( B \) such that

Condition (20) may seem ugly due to the appearance of the test functions. However, it turns out to be rather flexible. In fact, any family \( F \subset (\delta_{f(t)}, \delta_{g(t)}) : \forall t \in \mathbb{R} | \varphi \circ f(t) - \varphi \circ g(t) | < \varepsilon \},

where \( \varepsilon > 0, \varphi \in \mathcal{D}(\mathbb{R}^n) \), is a base for the uniformity of uniform convergence on \( (\delta_{f(t)})_{f \in F} \) and the uniformity of uniform convergence on compacta has the base sets of the form

\( W_{\varphi, R, r} = \left\{ (\delta_{f(t)}, \delta_{g(t)}) : \forall |t| \leq R | \varphi \circ f(t) - \varphi \circ g(t) | < r \right\}, \)

where \( \varphi \in \mathcal{D}(\mathbb{R}^n), R > 0, r > 0 \). If we fix \( \varepsilon > 0 \) and \( \varphi_* \in \mathcal{D}(\mathbb{R}^n) \), then by (20) there exists \( R > 0 \) such that

\[ \forall |t| > R | \varphi_* \circ f(t) - \varphi_* \circ g(t) | < \varepsilon \]

Finally, since

\[ \forall t \in \mathbb{R} | \varphi_* \circ f(t) - \varphi_* \circ g(t) | \]

\[ \leq \sup_{|t| \leq R} | \varphi_* \circ f(t) - \varphi_* \circ g(t) | + \sup_{|t| > R} | \varphi_* \circ f(t) - \varphi_* \circ g(t) | \]

\[ \leq \sup_{|t| \leq R} | \varphi_* \circ f(t) - \varphi_* \circ g(t) | + \frac{\varepsilon}{2}, \]

it becomes evident that \( W_{\varphi, R, \varepsilon} \subset (p^{-1}_\varphi([0, \varepsilon]))^\perp \), meaning (AA2) is satisfied. We conclude, via Theorem 3.2, that \( (\delta_{f(t)})_{f \in F} \) is relatively \( \tau_{uc} \)-compact.

Condition (20) may seem ugly due to the appearance of the test functions. However, it turns out to be rather flexible. In fact, any family \( F \subset C(\mathbb{R}, \mathbb{R}^n) \) such that

\[ \exists f_* \in F \lim_{|t| \to \infty} |f_*(t)| = \infty \]

and

\[ \forall f \in F |f(t)| \geq |f_*(t)|, \]

satisfies (20). Indeed, fix \( \varepsilon > 0, \varphi_* \in \mathcal{D}(\mathbb{R}) \) and choose \( R > 0 \) such that

\[ \forall |t| > R \text{ supp}(\varphi_*) \subset B(0, f_*(t)). \]

It is easy to see that for \( |t| > R \) we have \( \varphi_* \circ f(t) = 0 \) for every \( f \in F \). Consequently, condition (20) is satisfied.

Another condition which implies (20) is the following:

\[ \forall \varepsilon > 0 \exists M > 0 \forall f, g \in F \left( |f(t)| < M \lor |g(t)| < M \right) \implies |f(t) - g(t)| < \varepsilon. \quad (25) \]

To see this, fix \( \varepsilon > 0 \) and \( \varphi_* \in \mathcal{D}(\mathbb{R}^n) \). Pick \( M > 0 \) large enough so that \( \text{supp}(\varphi) \subset B(0, M) \). Again, by the uniform continuity of \( \varphi_* \) we have

\[ \exists \varepsilon > 0 \forall \varphi, \in \mathcal{D}(\mathbb{R}^n) |u - v| < \varepsilon \implies |\varphi_*(u) - \varphi_*(v)| < \varepsilon. \quad (26) \]
For the above choice of $M, r > 0$, by (25) we choose a suitable $R > 0$. Observe that for every $|t| > R$, $f, g \in \mathcal{F}$, we have the implication
$$(|f(t)| \geq M \land |g(t)| \geq M) \implies |\varphi \ast f(t) - \varphi \ast g(t)| = 0,$$
so (20) is satisfied. However, if it turns out that either $|f(t)| < M$ or $|g(t)| < M$, then by (25) we have $|f(t) - g(t)| < r$, which together with (26) implies (20).

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