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Proof of a basic hypergeometric supercongruence modulo the fifth power of a cyclotomic polynomial

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Abstract

By means of the $q$-Zeilberger algorithm, we prove a basic hypergeometric supercongruence modulo the fifth power of the cyclotomic polynomial $\Phi_n(q)$. This result appears to be quite unique, as in the existing literature so far no basic hypergeometric supercongruences modulo a power greater than the fourth of a cyclotomic polynomial have been proved. We also establish a couple of related results, including a parametric supercongruence.

1. Introduction

In 1997, Van Hamme [27] conjectured that 13 Ramanujan-type series including

$$\sum_{k=0}^{\infty} (-1)^k (4k+1) \frac{\frac{1}{2}^3}{k!^3} = \frac{2}{\pi}$$

admit nice $p$-adic analogues, such as

$$\sum_{k=0}^{p-1} (-1)^k (4k+1) \frac{\frac{1}{2}^3}{k!^3} \equiv p (-1)^{p-1} \pmod{p^3},$$

where $(a)_n = a(a+1) \ldots (a+n-1)$ denotes the Pochhammer symbol and $p$ is an odd prime. Up to present, all of the 13 supercongruences have been confirmed. See [21,24] for historic remarks on these supercongruences. Recently, $q$-analogues of congruences and supercongruences have caught the interests of many authors [1–7,8–20,23,25,26,29]. In particular, the first author and Zudilin [16] devised a method, called ‘creative microscoping’, to prove quite a few $q$-supercongruences by introducing an additional parameter $a$. In [13], the authors of this paper proved many additional $q$-supercongruences by the creative microscoping method. Supercongruences modulo a higher integer power of a prime, or, in...
Let $n > 1$ be a positive odd integer. Then
\[
\sum_{k=0}^{n-1} [4k - 1] \frac{(q^{-1}; q^2)_k}{(q^2; q^2)_k} q^{4k} = -q^n(1 - q + q^2)[n]_q^2 \quad (\text{mod } [n]_q^2 \Phi_n(q^2)) \quad (3a)
\]
and
\[
\sum_{k=0}^{n-1} [4k + 1] \frac{(q^{-1}; q^2)_k}{(q^2; q^2)_k} q^{4k} \equiv -(1 - q + q^2)[n]_q^2 \quad (\text{mod } [n]_q^2 \Phi_n(q^2)). \quad (3b)
\]
In the above $q$-supercongruences and in what follows:

$$(a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1})$$

is the $q$-shifted factorial,

$$[n] = [n]_q = 1 + q + \cdots + q^{n-1}$$

is the $q$-number,

$$\binom{n}{k} = \binom{n}{k}_q := \frac{(q; q)_n}{(q; q)_k(q; q)_{n-k}}$$

is the $q$-binomial coefficient and $\Phi_n(q)$ is the $n$th cyclotomic polynomial of $q$. Note that the

congruences in Theorem 1.1 modulo $[n]_q\Phi_n(q)^2$ and the congruences in Theorem 1.2 modulo $[n](1 - aq^n)(a - q^n)$ have already been proved by the authors in [13, Equations (5.5) and (5.10)].

2. Proof of Theorem 1.1 by the Zeilberger algorithm

The Zeilberger algorithm [cf. 22] can be used to find that the functions

$$f(n, k) = (-1)^k \frac{(4n - 1)(-\frac{1}{2})^n(-\frac{1}{2})^{n+k}}{(1)_n'(1)_{n-k}(-\frac{1}{2})^k},$$

$$g(n, k) = (-1)^{k-1} \frac{4(-\frac{1}{2})^n(-\frac{1}{2})^{n+k-1}}{(1)^3_{n-1}(1)_{n-k}(-\frac{1}{2})^k}$$

satisfy the relation

$$(2k - 3)f(n, k - 1) - (2k - 4)f(n, k) = g(n + 1, k) - g(n, k).$$

Of course, given this relation, it is not difficult to verify by hand that it is satisfied by the above pair of doubly indexed sequences $f(n, k)$ and $g(n, k)$.

Here we use the convention $1/(1)_m = 0$ for all negative integers $m$. We now define the $q$-analogues of $f(n, k)$ and $g(n, k)$ as follows:

$$F(n, k) = (-1)^k q^{(k-2)(k-2n+1)} \frac{[4n - 1](q^{-1}; q^2)_n(q^{-1}; q^2)_n(q^{-1}; q^2)_n}{(q^2; q^2)_n(q^2; q^2)_n(q^{-1}; q^2)_n},$$

$$G(n, k) = (-1)^{k-1} q^{(k-2)(k-2n+3)} \frac{(q^{-1}; q^2)_n(q^{-1}; q^2)_n(q^{-1}; q^2)_n}{(1 - q^2; q^2)_n(q^{-1}; q^2)_n(q^{-1}; q^2)_n},$$

where we have used the convention that $1/(q^2; q^2)_m = 0$ for $m = -1, -2, \ldots$. Then the functions $F(n, k)$ and $G(n, k)$ satisfy the relation

$$[2k - 3]F(n, k - 1) - [2k - 4]F(n, k) = G(n + 1, k) - G(n, k). \quad (4)$$
Indeed, it is straightforward to obtain the following expressions:

\[
\frac{F(n, k - 1)}{G(n, k)} = \frac{q^{2n-4k+6}(1 - q)(1 - q^{4n-1})(1 - q^{2k-3})^2}{(1 - q^{2n-2k+2})(1 - q^{2n})^3},
\]

\[
\frac{F(n, k)}{G(n, k)} = -\frac{q^{4-2k}(1 - q)(1 - q^{4n-1})(1 - q^{2n+2k-3})}{(1 - q^{2n})^3},
\]

\[
\frac{G(n + 1, k)}{G(n, k)} = \frac{q^{4-2k}(1 - q^{2n-1})^3(1 - q^{2n+2k-3})}{(1 - q^{2n})^3(1 - q^{2n-2k+2})}.
\]

It is easy to verify the identity

\[
\frac{q^{2n-4k+6}(1 - q^{4n-1})(1 - q^{2k-3})^3}{(1 - q^{2n-2k+2})(1 - q^{2n})^3} + \frac{q^{4-2k}(1 - q^{2k-4})(1 - q^{4n-1})(1 - q^{2n+2k-3})}{(1 - q^{2n})^3} = \frac{q^{4-2k}(1 - q^{2n-1})^3(1 - q^{2n+2k-3})}{(1 - q^{2n})^3(1 - q^{2n-2k+2})} - 1,
\]

which is equivalent to (4). (Alternatively, we could have established (4) by only guessing \(F(n, k)\) and invoking the \(q\)-Zeilberger algorithm [28].)

Let \(m > 1\) be an odd integer. Summing (4) over \(n\) from 0 to \((m + 1)/2\), we get

\[
[2k - 3] \sum_{n=0}^{m+1/2} F(n, k - 1) - [2k - 4] \sum_{n=0}^{m+1/2} F(n, k) = G\left(\frac{m + 3}{2}, k\right) - G(0, k)
\]

\[
= G\left(\frac{m + 3}{2}, k\right). \tag{5}
\]

We readily compute

\[
G\left(\frac{m + 3}{2}, 1\right) = \frac{q^{m-1}(q^{-1}; q^2)^{4}(m+3)/2}{(1 - q)^2(q^2; q^2)^{4}(m+1)/2(1 - q^{-1})^2}
\]

\[
= \frac{q^{m-3}[m]^{4}}{[m+1]^{4}(-q; q)^{8}(m-1)/2} \left[\frac{m - 1}{(m - 1)/2}\right]^{4} \tag{6a}
\]

and

\[
G\left(\frac{m + 3}{2}, 2\right) = -\frac{(q^{-1}; q^2)^{3}(m+3)/2(q^{-1}; q^2)^{(m+5)/2}}{(1 - q)^2(q^2; q^2)^{3}(m+1)/2(q^2; q^2)(m-1)/2(q^{-1}; q^2)^2}
\]

\[
= -\frac{q^{-2}[m]^{4}[m + 2]}{[m + 1]^{3}(-q; q)^{8}(m-1)/2} \left[\frac{m - 1}{(m - 1)/2}\right]^{4}. \tag{6b}
\]
Combining (5) and (6), we have
\[
\sum_{n=0}^{m+1} F(n, 0) = \left[ \frac{-2}{-1} \right] \sum_{n=0}^{m+1} F(n, 1) + \frac{1}{[-1]} G \left( \frac{m + 3}{2}, 1 \right)
\]
\[
= \frac{1 + q}{q} G \left( \frac{m + 3}{2}, 2 \right) - qG \left( \frac{m + 3}{2}, 1 \right)
\]
\[
= - \frac{(1 + q)[m]^{4}[m + 1][m + 2] + q^{m+1}[m]^{4}}{q^{3}[m + 1]^{4}(-q; q)^{8 \left( m - 1 \right) / 2}} \left[ \frac{m - 1}{(m - 1) / 2} \right]^{4},
\]
i.e.
\[
\sum_{n=0}^{m+1} [4n - 1] \frac{(q^{-1}; q^{2})_{n}^{4} q^{2n}}{(q^{2}; q^{2})_{n}^{4}} = - \frac{(1 + q)[m]^{4}[m + 1][m + 2] + q^{m+1}[m]^{4}}{q[m + 1]^{4}(-q; q)^{8 \left( m - 1 \right) / 2}} \left[ \frac{m - 1}{(m - 1) / 2} \right]^{4}.
\]
(7)

By [4, Lemma 2.1] (or [3, Lemma 2.1]), we have \((-q; q)^{2 \left( m - 1 \right) / 2} \equiv q^{(m^{2} - 1) / 8} \pmod{\Phi_{m}(q)}\). Moreover, it is easy to see that
\[
\left[ \frac{m - 1}{(m - 1) / 2} \right] = \prod_{k=1}^{(m-1)/2} \frac{1 - q^{m-k}}{1 - q^{k}}
\]
\[
\equiv \prod_{k=1}^{(m-1)/2} \frac{1 - q^{-k}}{1 - q^{k}} = (-1)^{(m-1)/2} q^{(1-m^{2})/8} \pmod{\Phi_{m}(q)},
\]
and \([m]\) is relatively prime to \((-q; q)_{(m-1)/2}\). It follows from (7) that
\[
\sum_{n=0}^{m+1} [4n - 1] \frac{(q^{-1}; q^{2})_{n}^{4} q^{2n}}{(q^{2}; q^{2})_{n}^{4}} q^{4n} \equiv -((1 + q)^{2} + q)[m]^{4} \pmod{[m]^{4} \Phi_{m}(q)}.
\]

Concluding, the congruence (2a) holds.

Similarly, summing (4) over \(n\) from 0 to \(m - 1\), we get
\[
[2k - 3] \sum_{n=0}^{m-1} F(n, k - 1) - [2k - 4] \sum_{n=0}^{m-1} F(n, k) = G(m, k),
\]
and so
\[
\sum_{n=0}^{m-1} [4n - 1] \frac{(q^{-1}; q^{2})_{n}^{4} q^{2n}}{(q^{2}; q^{2})_{n}^{4}} q^{4n} = \frac{1 + q}{q} G(m, 2) - qG(m, 1)
\]
\[
= - \frac{(1 + q)[2m - 2][2m - 1] + q^{2m-2}}{q(-q; q)^{8 \left( m - 1 \right) / 2}} \left[ \frac{2m - 2}{m - 1} \right]^{4}. \quad (8)
\]
It is easy to see that
\[
\frac{1}{m} \left[ \frac{2m - 2}{m - 1} \right] = \frac{1}{m - 1} \left[ \frac{2m - 2}{m - 2} \right] \equiv (-1)^{m-2}q^{2-\binom{m-1}{2}} \pmod{\Phi_m(q)},
\]
and \((-q; q)_{m-1} \equiv 1 \pmod{\Phi_m(q)} [4]. The proof of (2b) then follows easily from (8).

3. Proof of Theorems 1.2 and 1.3

Proof of Theorem 1.2: It is easy to see by induction on \(N\) that
\[
\sum_{k=0}^{N} (4k - 1) \left( \frac{aq^{-1}, q^2}{k} (aq^{-1}/a; q^2)_k (q^{-1}; q^2)_k^2 q^{4k} \right)
\]
\[
= \frac{(aq; q^2)_N(q/a; q^2)_N((a + 1)^2 q^{2N+1} - a(1 + q)(1 + q^{4N+1}))}{q(a - q)(1 - aq)(aq^2; q^2)_N(q^2/a; q^2)_N(-q; q)^4_N} \left[ \frac{2N^2}{N} \right].
\]
(9)

For \(N = (n + 1)/2\) or \(N = n - 1\), we see that \((aq; q^2)_N(q/a; q^2)_N\) contains the factor \((1 - aq^n)(1 - q^n/a)\). Moreover,
\[
\left[ \begin{array}{c} (n + 1)/2 \\ (n - 1)/2 \end{array} \right] \left[ \begin{array}{c} n \\ (n - 1)/2 \end{array} \right] = \left[ \begin{array}{c} n - 1 \\ (n - 1)/2 \end{array} \right]
\]
is a polynomial in \(q\). Since \([n + 1]/2\) and \([n]\) are relatively prime, we conclude that \(\left[ \begin{array}{c} n + 1 \\ n \end{array} \right]\) is divisible by \([n]\). Therefore, \(\left[ \begin{array}{c} n + 1 \\ (n + 1)/2 \end{array} \right] \left[ \begin{array}{c} n \\ (n - 1)/2 \end{array} \right]\) is also divisible by \([n]\). It is also well known that \(\left[ \begin{array}{c} 2n - 2 \\ n - 1 \end{array} \right]\) is divisible by \([n]\). Moreover, it is easy to see that \([n]\) is relatively prime to \(1 + q^m\) for any non-negative integer \(m\). The proof then follows from (9) by taking \(N = (n + 1)/2\) and \(N = n - 1\).

Proof of Theorem 1.3: For \(a = -1\), the identity (9) reduces to
\[
\sum_{k=0}^{N} (4k - 1) \left( \frac{q^{-2}, q^4}{k} q^{4k} \right)
\]
\[
= -\frac{(-q; q^2)_N^2 (1 + q^{4N+1})}{q(1 + q)(-q^2; q^2)_N^2} \left[ \frac{2N^2}{N} \right]
\]
\[
= -\frac{(1 + q^{4N+1})}{q(1 + q)(-q^2; q^2)_N^2} \left[ \frac{2N^2}{N} \right].
\]
(10)

Note that, in the proof of Theorem 1.2, we have proved that \(\left[ \frac{2N}{N} \right]_{q^2}\) is divisible by \([n]_{q^2}\) for both \(N = (n + 1)/2\) and \(N = n - 1\). Moreover, \([n]_{q^2}\) is relatively prime to \((-q^2; q^2)_m\) for \(m \geq 0\). Hence the right-hand side of (10) is congruent to 0 modulo \([n]_{q^2}\) for \(N = (n + 1)/2\) or \(N = n - 1\). To further determine the right-hand side of (10) modulo \([n]_{q^2}\), \(\Phi_n(q^2)\), we need only to use the same congruences (with \(q \mapsto q^2\)) used in the proof of Theorem 1.1.

4. Immediate consequences

Notice that for \(n = p^r\) being an odd prime power, \(\Phi_{p^r}(q) = [p]_{q^{p^r-1}}\) holds. This observation was used in [15] to extend (1) to a supercongruence modulo \([p^r][p^3]_{q^{p^r-1}}\). In the same vein, we immediately deduce from Theorem 1.1 the following result:
Corollary 4.1: Let $p$ be an odd prime and $r$ a positive integer. Then

$$\sum_{k=0}^{p^r+1} (4k-1) \left( \frac{q^{-1}; q^2}_k q^{4k} \right) \equiv -(1+3q+q^2) \left( \begin{array}{c} p^r \\ q \end{array} \right)^4 \pmod {p^r q^p q^{-1}} \quad (11a)$$

and

$$\sum_{k=0}^{p^r-1} (4k-1) \left( \frac{q^{-1}; q^2}_k q^{4k} \right) \equiv -(1+3q+q^2) \left( \begin{array}{c} p^r \\ q \end{array} \right)^4 \pmod {p^r q^p q^{-1}}. \quad (11b)$$

The $q \to 1$ limiting cases of these two identities yield the following supercongruences:

Corollary 4.2: Let $p$ be an odd prime and $r$ a positive integer. Then

$$\sum_{k=0}^{p^r+1} \frac{4k+3}{16(k+1)^4 256^k} \left( \begin{array}{c} 2k \\ k \end{array} \right)^4 \equiv 1 - 5p^{4r} \pmod {p^{4r+1}} \quad (12a)$$

and

$$\sum_{k=0}^{p^r-1} \frac{4k+3}{16(k+1)^4 256^k} \left( \begin{array}{c} 2k \\ k \end{array} \right)^4 \equiv 1 - 5p^{4r} \pmod {p^{4r+1}}. \quad (12b)$$

Similarly, we deduce from Theorem 1.3 the following result:

Corollary 4.3: Let $p$ be an odd prime and $r$ a positive integer. Then

$$\sum_{k=0}^{p^r+1} \frac{(q^{-2}; q^4)_k^2}{(q^4; q^4)_k^2} q^{4k} \equiv -q^{p^r} (1-q+q^2) \left( \begin{array}{c} p^r \\ q \end{array} \right)^2 q^{r-1} \pmod {p^{2r+1}} \quad (13a)$$

and

$$\sum_{k=0}^{p^r-1} \frac{(q^{-2}; q^4)_k^2}{(q^4; q^4)_k^2} q^{4k} \equiv -(1-q+q^2) \left( \begin{array}{c} p^r \\ q \end{array} \right)^2 q^{r-1} \pmod {p^{2r+1}}. \quad (13b)$$

The $q \to 1$ limiting cases of these two identities yield the following supercongruences:

Corollary 4.4: Let $p$ be an odd prime and $r$ a positive integer. Then

$$\sum_{k=0}^{p^r+1} \frac{4k+3}{4(k+1)^2 256^k} (2k^2)^2 \equiv 1 - p^{2r} \pmod {p^{2r+1}} \quad (14a)$$

and

$$\sum_{k=0}^{p^r-1} \frac{4k+3}{4(k+1)^2 256^k} (2k^2)^2 \equiv 1 - p^{2r} \pmod {p^{2r+1}}. \quad (14b)$$

The supercongruences in Corollaries 4.2 and 4.4 are remarkable since they are valid for arbitrarily high prime powers. Swisher [24] had empirically observed several similar but different hypergeometric supercongruences and stated them without proof.
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