Effects of chiral helimagnets on vortex states in a superconductor

Saoto Fukui\textsuperscript{1}, Masaru Kato\textsuperscript{1} and Yoshihiko Togawa\textsuperscript{2}

\textsuperscript{1} Department of Mathematical Sciences, Osaka Prefecture University, 1-1, Gakuencho, Nakaku, Sakai, Osaka 599-8531, Japan
\textsuperscript{2} Department of Physics and Electronics, Osaka Prefecture University, 1-1, Gakuencho, Nakaku, Sakai, Osaka 599-8531, Japan

E-mail: st110035@edu.osakafu-u.ac.jp (S Fukui)

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Abstract
We have investigated vortex states in chiral helimagnet/superconductor bilayer systems under an applied external magnetic field $H_{\text{appl}}$, using the Ginzburg–Landau equations. Effect of the chiral helimagnet on the superconductor is taken as a magnetic field $H_{\text{CHM}}$, which is perpendicular to the superconductor and oscillates spatially. For $H_{\text{appl}} = 0$ and weak $H_{\text{CHM}}$, there appear pairs of up- and down-vortices. Increasing $H_{\text{appl}}$, down-vortices gradually disappear, and the number of up-vortices increases in the large magnetic field region. Then, up-vortices form parallel, triangular, or square structures.

Keywords: chiral helimagnet, superconductor/magnet bilayer, vortex, Ginzburg–Landau equations, finite element method

(Some figures may appear in colour only in the online journal)

1. Introduction
Vortices are key features for type-II superconductors in determining a critical magnetic field and a critical current. In general, when a homogeneous magnetic field is applied to a superconductor, vortices form a triangular lattice called the Abrikosov lattice \cite{1–3}. Under an external current, vortices may move, giving rise to electric resistivity. Also, configurations of vortices affect motions of vortices. So, controlling vortex states is important for applications of superconductivity.

Recently, ferromagnet (FM)/superconductor (SC) systems have been studied \cite{4–6}. FM magnetic structures cause a magnetic fields in superconductors, and this magnetic field affects the superconductivity. It was found that the vortices appear and, because of a pinning effect on vortices, transport properties of the SC such as critical current are changed.

FMs are the most effective magnetic materials in SCs. However, there are other magnetic materials that have a large effect on the vortex state.

Recently, chiral helimagnets (CHMs) have been actively studied in the field of magnetism \cite{7–10}. The chiral helimagnet consists of spins that form a helical rotation along some direction, as shown in figure 1(a). This helical spin ordering comes from a competition between two interactions: the Dzyaloshinsky–Moriya interaction and the ferromagnetic exchange interaction. The Dzyaloshinsky–Moriya interaction is expressed as $-D \cdot (S_1 \times S_2)$, where $S_1$ and $S_2$ are localized nearest neighbor spins along one direction \cite{11, 12}. $D$ is the Dzyaloshinsky–Moriya interaction vector (DM vector). This interaction causes directions of $S_1$ and $S_2$ to be perpendicular. The direction of this vector is determined by a crystal structure in the CHM. On the other hand, the ferromagnetic exchange interaction is expressed as $-J S_1 \cdot S_2$, where $J$ is an exchange coefficient ($J > 0$). This interaction causes all spins to be parallel. When $|D|$ is much smaller than $J$, the directions of $S_1$ and $S_2$ are slightly deviated from the ferromagnetic structure because of the competition between two interactions. This deviation leads to the helically clockwise rotated structure of spins along the direction of the vector $D$.

Under an applied magnetic field, the magnetic structure of the CHM transforms into an incommensuate magnetic structure, which is called a chiral soliton lattice (CSL); see figure 1(b). The period of solitons can be controlled by the magnetic field. In \cite{7}, magnetic structures of CHMs and CSLs were observed using Lorentz microscopy analysis and small-
angle electron diffraction. This formation of CSL causes peculiar properties [8]. CHMs are expected to be used in controlling the spin current in the field of spintronics [9] and a novel magnetic processor [10].

We expect that the CHM affects the SC strongly. These influences may be unlike those of FMs.

In this paper, we investigate effects of CHMs on a SC in a CHM/SC bilayer system, using the Ginzburg–Landau (GL) equations. In particular, we focus on the effect on vortex states in the SC. In section 2, we introduce a model and equations. In particular, we focus on the effect on vortex states in the SC. In section 2, we introduce a model and numerical methods. In section 3, we show results about distributions of order parameters (A) and vortex states in the CHM/SC bilayer system (B, C). In section 4, we discuss results and summarize this paper. In the appendix, we describe some coefficients used in the GL equations in section 2.

2. Methods

We consider a CHM/SC bilayer system, as shown in figure 2. The effect of the CHM on the SC is taken as an external magnetic field $H_{\text{CHM}}$. In this study, the SC layer is much thinner than the CHM layer. So, this SC layer is considered as a two-dimensional system. In addition, the CHM layer is thick and thus $H_{\text{CHM}}$ is assumed to have only components perpendicular to the interface between the CHM and the SC. On the other hand, effects of the SC on the CHM (for example, the Meissner effect) are neglected. Then, we investigate vortex states in the two-dimensional superconducting systems under $H_{\text{CHM}}$ [13].

To obtain vortex states in the two-dimensional superconducting systems under $H_{\text{CHM}}$, we solve the GL equations [14]. Under a magnetic field, the GL free energy $G(\psi, A)$ is written as,

$$ G(\psi, A) = \int_{\Omega} \left[ f_n + \alpha |\psi|^2 + \frac{\beta}{2} |\psi|^4 \right] d\Omega + \int_{\Omega} \left\{ \frac{1}{2m_s} \left( -i\hbar \nabla - \frac{eA}{\epsilon} \right) \psi \right\}^2 d\Omega + \left[ \frac{|H|^2}{8\pi} - \frac{\mathbf{h} \cdot \mathbf{H}}{4\pi} \right] d\Omega, \quad (1) $$

where $\psi$ is a superconducting order parameter and $f_n$ is the Helmholtz free energy of the normal state. $\alpha$ is a coefficient which depends on temperature $T$ as $\alpha(T) = \alpha'(T - T_c)$, where $\alpha'$ is a constant ($\alpha' > 0$) and $T_c$ is the critical temperature of the SC. $\beta$ is a positive constant, $m_s$ is the effective mass of the SC, and $e_s$ is the effective charge of electrons in the SC. $\mathbf{H}$ is an external field, and $\mathbf{h} = \nabla \times \mathbf{A}$ is the local magnetic field, where $\mathbf{A}$ is a magnetic vector potential. We use an alternative form,

$$ \epsilon(\psi, A) = G + \int_{\Omega} \left\{ \frac{\alpha^2}{2\beta} + \frac{\mathbf{H} \cdot \mathbf{H}}{8\pi} - f_n + \left( \text{div} A \right)^2 \right\} d\Omega. \quad (2) $$

We add the term $(\text{div} A)^2$ to equation (2) to ensure the Coulomb gauge $\text{div} A = 0$. For minimizing $\epsilon$ with respect $\psi$, $\psi^*$, and $A$, we obtain following equations,

$$ \int_{\Omega} \left[ i(\nabla \psi - \tilde{\mathbf{A}} \tilde{\psi})(-i
abla \psi^* - \tilde{\mathbf{A}} (\nabla \tilde{\psi}^*)) + (i\nabla (\nabla \tilde{\psi}) - \tilde{\mathbf{A}} (\nabla \tilde{\psi}^* - \tilde{\psi}^* \nabla \tilde{\psi})) + \frac{1}{\xi^2}(|\psi|^2 - 1)(\psi (\nabla \tilde{\psi}^*) + \tilde{\psi}^* (\nabla \psi)) \right] d\Omega = 0, \quad (3) $$

$$ \int_{\Omega} \left[ \kappa^2 \xi^2 \left| \text{div} \tilde{\mathbf{A}} \cdot \text{div} (\delta \tilde{\mathbf{A}}) + \nabla \times \tilde{\mathbf{A}} \cdot \nabla \times (\delta \tilde{\mathbf{A}}) \right| \right] d\Omega = \kappa^2 \xi^2 \int_{\Omega} \frac{2\pi}{\Phi_0} \mathbf{H} \cdot \nabla \times (\delta \mathbf{A}) d\Omega, \quad (4) $$

where $\delta \tilde{\psi}$, $\delta \tilde{\mathbf{A}}$ are variations of the order parameter and the vector potential, $\tilde{\psi}$ and $\tilde{\mathbf{A}}$ are the normalized order parameter and vector potential respectively:

$$ \tilde{\psi} = \frac{\psi}{\sqrt{\alpha^2 / \beta}}, \quad (5) $$

$$ \tilde{\mathbf{A}} = \frac{2\pi}{\Phi_0} \mathbf{A}. \quad (6) $$

$\Phi_0$ is the quantum flux, $\Phi_0 = \hbar c/2e$, where $e$ is an electron, $e_s = 2e$. $\xi$ is the coherence length, which depends on the temperature, $\xi^2(T) = h^2/(4\pi \kappa(T))$, $\kappa$ is the GL parameter, $\kappa = \lambda/\xi$, where $\lambda$ is the penetration length. In this study, we use the following boundary conditions: $\mathbf{j} \cdot \mathbf{n} = 0$, where $\mathbf{n}$ is the normal vector to the surface, and $(i\nabla \tilde{\psi} - \tilde{\mathbf{A}} \tilde{\psi}) \cdot \mathbf{n} = 0$ at the boundary.

To solve these two equations (3) and (4), we use the two-dimensional finite element method. $\psi$ and $\tilde{\mathbf{A}}$ are expanded...
using area coordinates \( N_j^e \) \((j = 1, 2, \text{and} 3)\) for \(e\)th element,

\[
\hat{\psi}(x) = \sum_{j=1}^{3} \sum_{e=1}^{3} N_j^e(x) \hat{\psi}_j^e
\]

\[
\hat{A}(x) = \sum_{j=1}^{3} \sum_{e=1}^{3} N_j^e(x) \hat{A}_j^e,
\]

where \( \hat{\psi}_j^e \) and \( \hat{A}_j^e \) are values of order parameter and vector potential at the \( j \)th node in the \( e \)th element. We set test functions \( \delta \hat{\psi} \) and \( \delta \hat{A} \) as

\[
\delta \hat{\psi} = N_j^e(x) \quad (i = 1, 2, 3),
\]

\[
\delta \hat{A} = N_j^e(x) e_\gamma \quad (i = 1, 2, 3, \ \gamma = x, y, z).
\]

Then, equations (3) and (4) become

\[
\sum_j \left[ P_0^e(\{\hat{A}\}, \{\hat{\psi}\}) + P_0^{2R}(\{\hat{\psi}\}) \right] \text{Re} \ \hat{\psi}_j^e + \sum_j \left[ Q_0^e(\{\hat{A}\}) + Q_0^{2R}(\{\hat{\psi}\}) \right] \text{Im} \ \hat{\psi}_j^e = V_i^{2R}(\{\hat{\psi}\}),
\]

\[
\sum_j \left[ -Q_0^e(\{\hat{A}\}) + Q_0^{2R}(\{\hat{\psi}\}) \right] \text{Re} \ \hat{\psi}_j^e + \sum_j \left[ P_0^e(\{\hat{A}\}, \{\hat{\psi}\}) + P_0^{2E}(\{\hat{\psi}\}) \right] \text{Im} \ \hat{\psi}_j^e = V_i^{2E}(\{\hat{\psi}\}),
\]

\[
\sum_j R_0^e(\{\hat{\psi}\}) \hat{A}_j^e + \sum_j S_0^e \hat{\psi}_j^e = T_i^{ex} - U_i^{co},
\]

\[
-\sum_j S_0^e \hat{A}_j^e + \sum_j R_0^e(\{\hat{\psi}\}) \hat{\psi}_j^e = T_i^{ex} + U_i^{co}.
\]

Coefficients are given in the appendix.

The magnetic field from the CHM, \( H_{CHM} \) is included in the external magnetic field \( H \) in equation (4). \( H_{CHM} \) is obtained from the Hamiltonian,

\[
H = -2J \sum_n \mathbf{S}_n \cdot \mathbf{S}_{n+1} + D \cdot \sum_n \mathbf{S}_n \times \mathbf{S}_{n+1} + \mu_B H_{appl} \sum_n S_n^z
\]

where \( H_{appl} \) is the homogeneous applied magnetic field. \( \mu_B \) is the Bohr magneton. We represent \( \mathbf{S}_n \) in terms of the polar coordinates as

\[
\mathbf{S}_n = S(\sin \theta_n \cos \varphi, \sin \theta_n \sin \varphi, \cos \theta_n)
\]

and minimize the energy with respect to \( \theta_n \) [15]. We obtain the sine-Gordon equation,

\[
\frac{d^2 \theta}{dx^2} - H^* \sin \theta = 0,
\]

where \( H^* \) is a normalized applied magnetic field,

\[
H^* = \frac{2 \mu_B H_{appl}}{\xi_0^2 \sqrt{J^2 + D^2}}.
\]

We assume \( \xi_0 = a \), where \( a \) and \( \xi_0 \) are the lattice constant and coherence length, respectively, for the SC at \( T = 0 \). The solution is

\[
\theta(x) = 2 \sin^{-1} \left[ \frac{\sqrt{H^*}}{k} \right] + \pi,
\]

where \( k \) is the modulus of the Jacobi’s elliptic function \( sn( \pi, k) \) and is determined by,

\[
\frac{\pi \phi}{4 \sqrt{H^*}} = E(k),
\]

where \( \phi = \tan^{-1}(D/J) \) and \( E(k) \) is the complete elliptic integral of the second kind. In equation (20), when the applied magnetic field \( H^* \) increases, the modulus \( k \) also increases monotonically from 0 to 1, as shown in figure 3.

The period of the CHM \( L' \) is given as,

\[
\frac{L'}{\xi_0} = \frac{2kK(k)}{\sqrt{H^*}},
\]

where \( 2K(k) \) is the period of the \( sn(\pi, k) \) function and \( K(k) \) is the complete elliptic integral of the first kind. This relationship (equation (21)) is shown in figure 4. Increasing the applied magnetic field, the period becomes longer and spins form the CSL. The period increases rapidly before transition to the ferromagnetic.

Using the solution of the sine-Gordon equation in equation (19) to the GL equations, we can obtain vortex states in the magnetic field from the CHM.
3. Results and discussions

We solve the Ginzburg–Landau equations and study the distributions of the order parameter and vortex states. We take the Ginzburg–Landau parameter \( \kappa = \lambda / \xi = 10 \), the temperature \( T = 0.3T_c \). The ratio of the strength of the Dzyaloshinsky–Moriya interaction to that of the ferromagnetic exchange interaction is taken from \( \text{Cr}_2/\text{NbS}_2 \) as \( D/J = 0.16 \) [16]. The external magnetic field \( H_{\text{ext}} \) is given by the sum of the magnetic field from the CHM \( H_{\text{CHM}} \) and the applied magnetic field \( H_{\text{appl}} \). We only consider the \( z \)-component of the external magnetic field \( H_{\text{ext}} \); using equation (15), it is given by

\[
(H_{\text{ext}})_z(x) = H_0 \cos \left( 2 \sin^{-1} \left( \frac{\sqrt{H_0^2 - s^2}}{s} \right) \right) + H_{\text{appl}}.
\]  

Here, we assume that the distribution of the magnetic field from the CHM (the first term in equation (22)) is proportional to the distribution of spins in the CHM (equation (16)). The factor \( H_0 \) represents the magnitude of the magnetic field from the CHM. In the following, the system size is set as \( 7.0L' \times 20\xi_{\text{sp}} \), where \( L' \) is the period of the CHM in equation (21). When \( H_{\text{appl}}/(\Phi_0/\xi_0^2) = 0.00 \), \( L'/\xi_0 \) becomes approximately 39.2699.

3.1. Effect of CHMs on the distribution of the order parameter

First, we show effect of the CHM on the distribution of the order parameter without the external applied magnetic field \( (H_{\text{appl}} = 0) \). Results for \( H_0/(\Phi_0/\xi_0^2) = 0.01 \) are shown in figure 5. From these results, the order parameter can be seen to oscillate spatially, although the order parameter is uniform under the weak homogeneous magnetic field. The period of the order parameter is half that of the magnetic field \( (H_{\text{ext}})_z \).

3.2. Effect of CHMs on vortex states

Next, we show the effect of the CHM on vortex states without the external applied magnetic field \( (H_{\text{appl}} = 0) \). We show vortex configurations for \( H_0/(\Phi_0/\xi_0^2) = 0.012 \) (figure 6), 0.013 (figure 7), 0.019 (figure 8), and 0.025 (figure 9).

When \( H_0/(\Phi_0/\xi_0^2) = 0.010 \) (figure 5) and 0.012 (figure 6), vortices do not appear. However, at \( H_0/(\Phi_0/\xi_0^2) = 0.013 \) (figure 7), four vortices appear. From figures 7(b) and (c), we find two kinds of vortices whose quantum flux directions are parallel to the directions of their magnetic fields. In this paper, we call these vortices up- \((B_u > 0)\) and down-vortices \((B_d < 0)\) respectively. Here, \( B_i \) is magnetic flux density. In figure 7(a), up- and down-vortices appear next to each other. They do not appear separately. In addition, the pair annihilation of up- and down-vortices does not occur, despite the short distance between up- and down-vortices. This behavior can be explained by considering two magnetic fields. Magnetic fields from the CHM \( H_0/(\Phi_0/\xi_0^2) = 0.012 \) and the applied magnetic fields \( H_{\text{appl}}/(\Phi_0/\xi_0^2) = 0.000 \).

Figure 5. (a) Distribution of the order parameter. (b) Amplitude of the order parameter. (c) Distribution of the magnetic field from the CHM. In equation (22), \( H_0/(\Phi_0/\xi_0^2) = 0.01 \) and \( H_{\text{appl}}/(\Phi_0/\xi_0^2) = 0.00 \).

Figure 6. Distributions of (a) order parameter, (b) phase, and (c) magnetic fields. Magnetic fields from the CHM \( H_0/(\Phi_0/\xi_0^2) = 0.012 \) and the applied magnetic fields \( H_{\text{appl}}/(\Phi_0/\xi_0^2) = 0.000 \).
interactions for vortices. One of these interactions is an attractive interaction between up- and down-vortices. Due to this attractive interaction, these vortices tend to approach each other. The other is the interaction between the vortex and the magnetic field: because of this interaction, vortices tend to appear in the large magnetic field region. Due to competition between these two interactions, up- and down-vortices approach each other, but remain in the stronger field region. Therefore, pair annihilation does not occur and up- and down-vortices appear next to each other.

For larger fields from the CHM (figures 8 and 9), the number of vortices increases. From the distributions of phases and magnetic fields, up- and down-vortices appear alternately. These configurations can be explained by the same discussion about interactions of vortices. Generally, up- and down-vortices appear in the parallel magnetic field region.

3.3. Effect of CHMs and applied magnetic fields on vortex states

So far, we have discussed vortex states under the magnetic field from the CHM without homogeneous applied magnetic field. Next, we show combined effects of the CHM and the applied magnetic field on vortex states. In the following, \( H_0/\langle \Phi_0/\xi_0^2 \rangle \) is fixed to 0.019. The vortex state under zero applied magnetic field has already been shown in figure 8. We now show vortex configurations for \( H_{\text{app}}/\langle \Phi_0/\xi_0^2 \rangle \) 0.0005 (figure 10), 0.001 (figure 11), 0.002 (figure 12) and 0.005 (figure 13).

In figure 10, the number of up-vortices increases due to the applied magnetic field. For \( H_{\text{app}}/\langle \Phi_0/\xi_0^2 \rangle = 0.001 \) (figure 11), down-vortices approach the edges. When the applied magnetic field increases, for \( H_{\text{app}}/\langle \Phi_0/\xi_0^2 \rangle = 0.002 \) (figure 12) and 0.005 (figure 13) down-vortices disappear. In figure 12, the total external magnetic field \( H_{\text{ext}}/\langle \Phi_0/\xi_0^2 \rangle \) oscillates between \(-0.017\) and \(0.021\). Because the absolute value of the negative magnetic field becomes small (\(-0.017\)), the interaction between a down-vortex and the external magnetic field becomes weaker than that between up- and down-vortices. So, down-vortices gradually disappear with increasing applied magnetic field.

For larger applied magnetic field, the number of up-vortices increases, and they form parallel, triangular, or...
square structures. These structures of up-vortices are stable in the positive magnetic field region, and do not prefer to appear in the negative magnetic field region. If we apply an external current along the $y$-axis, up-vortices cannot move through the negative magnetic field region. Therefore, we expect the pinning effect of the vortex due to the helical magnetic structure, which leads to the increase of the critical current.

4. Summary and conclusions

We have investigated vortex states in the CHM/SC bilayer systems using two-dimensional Ginzburg–Landau equations. We found vortex configurations under the oscillating magnetic field from the CHM $H_{\text{CHM}}$. Both up- and down-vortices appear alternately under the $H_{\text{CHM}}$. Under the applied magnetic field $H_{\text{app}}$, down-vortices disappear and the average number of up-vortices changes periodically in space.
From these vortex configurations, we expect the following applications. These vortex configurations may affect the dynamics of vortices. If the current flows along the \(y\) direction, vortices move along the \(x\) direction. Usually, in a homogeneous magnetic field, all vortices move along the same direction. However, in the \(H_{\text{CHM}}\), up- and down-vortices move along opposite directions. Furthermore, because \(H_{\text{ext}}\), changes spatially, vortices experience a force from the magnetic field, as mentioned in section 3.2. This situation is different from that in the FM/SC bilayer system. In the FM/SC bilayer system, the \(z\)-component of the magnetic field changes only in the domain walls. So the force from the magnetic field appears only in the domain wall. In the CHM/SC bilayer system, because of this force from the magnetic field, vortices do not prefer to appear under magnetic fields of the opposite direction. Thus, vortices may not move through the region where magnetic field is opposite to vortices. Therefore, we expect a pinning effect on the vortex due to the oscillating magnetic field. This pinning effect may lead to increase of the critical current. So, we expect to use our CHM/SC as a superconducting material with large critical current.

In this study, we have solved two-dimensional Ginzburg–Landau equations. However, the effect of the CHM has been taken as only the \(z\)-component of the magnetic field—so we have still not completely treated the magnetic structure of the CHM. In the future, we will solve GL equations for the three-dimensional bilayer system in order to investigate the effect of the chirality of the CHM on the bilayer system. Moreover, a pinning effect on vortices is expected from the above considerations, leading to increase of the critical current. Thus investigating the dynamics of vortices in the CHM/SC bilayer system by solving time-dependence Ginzburg–Landau equations remains a problem for future study.

**Appendix**

In this appendix, we give the coefficients used in equations (11)–(14). They are defined as,

\[
P_g^2(|\psi|) \equiv \frac{1}{\xi^2(T)} \sum_{i,j} I_{i,j}(\text{Re} \, \psi^*_i \, \text{Re} \, \psi^*_j + 3 \text{Im} \, \psi^*_i \, \text{Im} \, \psi^*_j) \tag{A.3}
\]

\[
Q(|\psi|) \equiv \sum_{i,j} (J_{i,j}^x A_i^x + J_{i,j}^y A_i^y - J_{i,j}^x A_i^x - J_{i,j}^y A_i^y) \tag{A.4}
\]

\[
R_g(|\psi|) \equiv \kappa^2 \xi^2(T) \left(K_{ij}^{x} + K_{ij}^{y}\right) + \sum_{i,j} I_{i,j}(\psi^*_i \psi^*_j) \tag{A.5}
\]

\[
S_i \equiv \kappa^2 \xi^2(T) \left(K_{ij}^{x} - K_{ij}^{y}\right) \tag{A.6}
\]

\[
T_{ij} \equiv \sum_{i,j} I_{i,j}(\text{Im} \, \psi^*_i \psi^*_j) \tag{A.7}
\]

\[
U_{ij} \equiv \kappa^2 \xi^2(T) \frac{2\pi}{\Phi_0} H_{ij} \tag{A.8}
\]

\[
V_{ij}^R(|\psi|) \equiv \frac{2}{\xi^2(T)} \sum_{i,j} \lambda_{i,j} \psi^*_i \psi^*_j \text{Re} \psi^*_i \tag{A.9}
\]

\[
V_{ij}^I(|\psi|) \equiv \frac{2}{\xi^2(T)} \sum_{i,j} \lambda_{i,j} \psi^*_i \psi^*_j \text{Im} \psi^*_i \tag{A.10}
\]

These coefficients can be calculated by integrals \(I_{ij}^R, I_{ij}^I, I_{ij}^{R'}, I_{ij}^{I'}, J_{ij}^R, J_{ij}^I, J_{ij}^{R'}, J_{ij}^{I'}, K_{ij}^{x,y}\). The integrals are given by,

\[
I_{ij}^R \equiv \int_{\Omega} N_{i}^x N_{j}^x \, d\Omega \tag{A.11}
\]

\[
I_{ij}^I \equiv \int_{\Omega} N_{i}^x N_{j}^y \, d\Omega \tag{A.12}
\]

\[
I_{ij}^{R'} \equiv \int_{\Omega} N_{i}^x N_{j}^y \, d\Omega \tag{A.13}
\]

\[
J_{ij}^R \equiv \int_{\Omega} \frac{\partial N_{i}^x}{\partial x} N_{j}^x \, d\Omega \tag{A.14}
\]

\[
J_{ij}^I \equiv \int_{\Omega} \frac{\partial N_{i}^x}{\partial y} N_{j}^y \, d\Omega \tag{A.15}
\]

\[
J_{ij}^{R'} \equiv \int_{\Omega} \frac{\partial N_{i}^x}{\partial y} N_{j}^y \, d\Omega \tag{A.16}
\]

\[
J_{ij}^{I'} \equiv \int_{\Omega} \frac{\partial N_{i}^x}{\partial x} N_{j}^x \, d\Omega \tag{A.17}
\]

\[
K_{ij}^{x,y} \equiv \int_{\Omega} \frac{\partial N_{i}^x}{\partial y} \frac{\partial N_{j}^y}{\partial x} d\Omega \tag{A.18}
\]

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