On a Residue Representation of 
Deformation, Koszul and Chiral Rings

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ABSTRACT
A residue-theoretic representation is given for massless matter fields in 
(quotients) of (weighted) Calabi-Yau complete intersection models and 
the corresponding chiral operators in Landau-Ginzburg orbifolds. The 
well known polynomial deformations are thus generalized and the uni-
versal but somewhat abstract Koszul computations acquire a concrete 
realization and a general but more heuristic reinterpretation. A direct 
correspondence with a BRST-type analysis of constrained systems also 
emerges naturally.

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1. Introduction, Results and Summary

Two rather general techniques for studying the complete massless spectra of superstring compactifications have been developed over the past several years. One of these relies on the ‘large radius’ description in terms of the geometry of Calabi-Yau spaces, wherein low energy particles correspond to elements of \( H^1(T) \), \( H^1(T^*) \) and \( H^1(\text{End}T) \). For Calabi-Yau complete intersections in products of projective spaces, these cohomology groups may be calculated through vigorous application of Koszul spectral sequences and the Bott-Borel-Weil theorem. In fact, complete intersections in products of any generalized flag spaces can be analyzed in this manner; further details of these models and the details of this technique may be found in Ref. [3].

Complementing these results, the massless spectra of Landau-Ginzburg orbifolds with worldsheet \((2,2)\)-supersymmetry have been analyzed from the 2-dimensional supersymmetric quantum field theory point of view [4,5]. Interestingly, these two complementing approaches—the former focusing on special features of target space geometry while the latter centering on exceptional characteristics of the underlying world-sheet field theory—actually produce rather closely related descriptions of the massless spectra for related models [6,7,8]. Indeed, both Landau-Ginzburg and Calabi-Yau models may be regarded as different ‘phases’ or regimes of a more general underlying 2-dimensional quantum field theory; they occur at two opposite special points in a complexified radial moduli space [9], and may be connected by a sort of analytic continuation. The remarkably detailed similarity in the results of these two approaches then follows from the fact that many of the relevant observables, in particular properties of the complex structure moduli space, are independent of this complex radial modulus. More precisely, a general Calabi-Yau complete intersection does not have a pure Landau-Ginzburg model as the radially counterpoint ‘phase’, but rather a so-called gauged Landau-Ginzburg model. Nevertheless, the \( E_6 \) \( 27 \)'s and \( 27^* \)'s of the low-energy effective particle theory correspond on the geometrical side to elements of \( H^1(T) \) and \( H^1(T^*) \) respectively, while in the Landau-Ginzburg orbifolds framework the corresponding states generate the \((c,c)\)- and the \((a,c)\)-ring, respectively [10]. Besides the complex structure and Kähler moduli fields, the low-energy effective particle theory also abounds in matter \( 1 \)'s—chargeless with respect to the \( E_6 \times E_8 \) Yang-Mills gauge interaction. These states correspond to elements of the \( H^1(\text{End}T) \) cohomology group and are not chiral-primary, but nevertheless admit a similar analysis based on the \((0,2)\)-subgroup of the \((2,2)\)-supersymmetry.

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1. By complete we mean including \( E_6 \) gauge singlets in addition to the moduli fields associated to deformations of the Kähler and complex structure.

2. In principle, the technique also applies to (quotients) of complete intersections in products of any weighted flag varieties and, in particular, weighted projective spaces. While we are not aware of any readily available weighted generalization of the Bott-Borel-Weil theorem, we will show that the method which we propose herein naturally applies to these latter models as well.
Generalizing the analysis of Ref. [11], a general machinery has recently been developed to calculate via Special Geometry [12,13,14] both the Yukawa couplings and the kinetic terms for the ‘27’s—and then also the ‘27*-s by using mirror symmetry—as functions over the entire moduli space, for many families of models [11,15,16,17,18]. This reveals many global properties of the Yukawa couplings—and with it certain global properties of the parameter spaces [19,20,21,22,23,24]. In addition, the analysis of Refs. [15,17] is well adapted to the fact that twisted (c,c)- and (a,c)-states may (at least sometimes) be represented by certain radical polynomials; that is, polynomials which also involve roots of certain special and in a sense universal polynomials [8]. Related to these, the analysis of the so-called Picard-Fuchs equations also reveals a great deal of information about the periods and so the Yukawa couplings [25,16,18].

In the present article we introduce another representation for the massless states, based on certain residue integrals. Besides a possible application in its own right, we find this residue map bridging the unpleasant chasm between the well-accepted polynomial deformation method of Ref. [20], together with the very closely related construction of marginal operators in a Landau-Ginzburg orbifold [10,3,5] on one side, and the universal but rather abstract technique of Koszul spectral sequences [1,3] on the other. Although we are not at this point able to propose a direct generalization of the Koszul calculation for (quotients) of complete intersections in weighted projective spaces, the residue recipe does extend naturally to these models also. The calculations based on Koszul spectral sequences (where applicable) and the residue mapping also seem to acquire a natural interpretation in terms of a BRST analysis of the underlying 2-dimensional (2,2)-supersymmetric constrained σ-model [27] and the more general linear quantum field theory [9].

Each representative obtained through the Koszul computation will be shown to have a precisely corresponding residue integral, which turns out to be a straightforward generalization of a well-known result. Indeed, our starting point is provided by the well-known Atiyah-Bott-Gårding-Candelas residue formula [28,26]:

\[
\Omega \overset{\text{def}}{=} \text{Res}_M \left[ \frac{(xd^n x)}{P} \right].
\] (1.1)

This defines the “nowhere vanishing holomorphic (n−1)-form” Ω on a complex (n−1)-dimensional Calabi-Yau hypersurface \(M \overset{\text{def}}{=} \{ P=0 \} \subset \mathbb{P}^n\) as a residue at \(M\) of the rational differential form \((xd^n x)/P\) where

\[
(xd^n x) \overset{\text{def}}{=} \frac{1}{(n+1)!} \epsilon_{\mu_0 \cdots \mu_n} x^{\mu_0} dx^{\mu_1} \cdots dx^{\mu_n}.
\] (1.2)

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3 Pedantry: the techniques we mention rely either on the geometrical or on the 2-dimensional field theory interpretation of a given model, whence ‘universal’ means ‘universal within the scope of interpretation’. Since there exist models for which both interpretations are not known, the two categories overlap significantly but neither contains the other.
Note the appearance of the antisymmetric tensor $\epsilon \ldots$ in the above expression, which will have a crucial role in understanding the exceptional residues. Since each such top differential transforms at most with an overall factor, we will refer to them as ‘covariant’. $\Omega$ is also called the “holomorphic volume-form”, as $\Omega \wedge \overline{\Omega}$ is a (perhaps non-standard) volume form on $\mathcal{M}$. Explicitly, by means of a contour integral,

$$\Omega \overset{\text{def}}{=} \oint_{\Gamma(P)} \frac{(xd^n x)}{P} . \quad (1.3)$$

Here $\Gamma(P)$ is a contour encircling $\mathcal{M} = \{ P=0 \}$. That is, $\Gamma(P)$ may be identified with a small circle centered at some point $x \in \mathcal{M}$ and which lies in a complex plane in $\mathbb{P}^n$, locally transversal to $\mathcal{M}$ at $x$. Provided $\mathcal{M}$ is smooth, i.e., $P$ is transversal, the integrand has a simple pole and the contour integral picks out the simple residue at $x$. For the integral to be well defined as an element of $H^3(\mathcal{M}, \mathbb{C})$, it must be of homogeneity zero, which induces the Calabi-Yau condition

$$\deg(P) = \sum_{\mu=0}^{n} \deg(x^\mu) , \quad \text{over } \mathbb{P}^n . \quad (1.4)$$

Note immediately that away from $\mathcal{M} \in \mathbb{P}^n$, where $P \neq 0$, the integrand is analytic and the value of the contour integral is zero. In other words, $\Omega$ in (1.1) is supported precisely on the hypersurface $\mathcal{M}$, where it is nonzero and invariant under holonomy.

While an invariant under holonomy over the given $\mathcal{M}$, the $\Omega$ of Eq. (1.1) however does depend on all the complex structure moduli and is therefore the key object for the analysis in Ref. [11,15,17], where it appears through its periods:

$$\varpi_k \overset{\text{def}}{=} \oint_{\gamma_k} \Omega , \quad (1.5)$$

where the cycles $\gamma^k$ form a basis for $H_{n-1}(\mathcal{M})$. Alternatively, one notes that the choice of some particular 3-form to be the holomorphic 3-form is equivalent to having chosen a particular complex structure; varying this choice then is equivalent to varying the complex structure. Therefore

$$\frac{\partial \Omega}{\partial t^\alpha} = K_\alpha \Omega + \varphi_\alpha , \quad \text{or } \quad \varphi_\alpha \overset{\text{def}}{=} \nabla_\alpha \Omega , \quad (1.6)$$

\[ \quad \text{4} \quad \text{The holonomy group of } \mathcal{M} \text{ is generated by parallel transport around closed loops in } \mathcal{M}, \text{ so holonomy-invariance generalizes single-valuedness and is essential in compactification on } \mathcal{M} \text{ [3]. This holonomy-invariance also coincides with the invariance with respect to a gauge-transformation discussed in § 2.1.} \]
where $\nabla_\alpha \Omega \overset{\text{def}}{=} (\partial_\alpha - K_\alpha)\Omega$, and $K_\alpha dt^\alpha$ is identified as the connection 1-form. In particular, for deformations of the complex structure which may be realized as deformations of the defining polynomial, $P(t) \overset{\text{def}}{=} P_0 + t^\alpha \delta P_\alpha$ and

$$\left. \frac{\partial \Omega}{\partial t^\alpha} \right|_{t=0} = - \oint_{\Gamma(P_0)} \frac{(xd^{n+1}x)}{P_0} \left( \frac{\delta P_\alpha}{P_0} \right) . \quad (1.7)$$

So, up to terms which merely reproduce a multiple of $\Omega_0$, the homogeneity-0 quantities $(\frac{\delta P_\alpha}{P_0})$, and so also the polynomials $\delta P_\alpha$ (modulo the defining polynomials’ gradients), represent the deformations of the complex structure around $P_0$.

In certain special “flat” local coordinates $t^\alpha$, the connection 1-form

$$K_\alpha dt^\alpha = \frac{\int_M \overline{\Omega} \wedge d_t \Omega}{\int_M \overline{\Omega} \wedge \Omega} \quad (1.8)$$

is zero. However, for purposes of calculating the Yukawa couplings, the connection terms may freely be omitted and partial derivatives suffice \[12,13,14\] even if the $t^\alpha$ are not the “flat” local coordinates:

$$\kappa_{\alpha\beta\gamma} = \int_M \Omega \wedge \frac{\partial^3 \Omega}{\partial t^\alpha \partial t^\beta \partial t^\gamma} \quad (1.9)$$

is the (unnormalized) Yukawa coupling. The normalization derives from the Weil-Petersson-Zamolodchikov metric, for which

$$K = - \ln \left( i \int_M \overline{\Omega} \wedge \Omega \right) \quad (1.10)$$

is the Kähler potential. Thus, in principle, $\Omega$ completely determines the special geometry on the space of complex structures and so also the complete dynamics of the corresponding low-energy physics matter fields modulo higher loop corrections to the Kähler potential.

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Our main goal is to generalize the result (1.7) so as to obtain representatives of “twisted” massless states in the Landau-Ginzburg orbifold and of the “higher cohomology” contributions in the Koszul calculations, these two being in a partial but very detailed

\[5\] This does exhaust all deformations of complex structure for all homogeneous hypersurfaces, but not so for their quasihomogeneous (weighted) cousins, nor for the homogeneous complete intersections \[25\].
correspondence wherever both ‘phases’ of a model are well understood. Notably, these residue representatives turn out to be of the general form

$$\sum_{\vec{r}} \Omega_{(q)}^{\vec{r}} f_{\vec{r}}(x) \tag{1.11}$$

where \(\vec{r}\) is a suitable multi-index and \(\Omega_{(q)}^{\vec{r}}\) are the nowhere vanishing holomorphic volume-forms on certain ‘intermediate’ Calabi-Yau q-dimensional spaces and the \(f_{\vec{r}}(x)\) are holomorphic on the complementary factor of the embedding space. These turn out to provide a universal generalization of the Jacobian ring structure of Ref. [26]—the well known “polynomials modulo the defining polynomials’ gradients” ring structure.

The knowledge of periods (1.5) of the holomorphic volume-form \(\Omega\) and certain monodromy information [11,30,20,24] suffices to calculate both the Yukawa couplings and the kinetic terms, and no further generalization is in principle necessary. However, a variety of simply technical or perhaps more essential obstacles may thwart such a program. For example, a complete set of cycles \(\gamma^k\) in (1.5) may be very difficult to find, and the action of the modular groups may not be known sufficiently well to generate all the periods. Certain deformations may not be representable as polynomial deformations of the defining polynomial (which in fact is typical of complete intersections in products of projective spaces [29]).

Finally, recall that both the Koszul machinery [1,4] and also the Landau-Ginzburg orbifold analysis of Ref. [5], each enables a systematic and complete calculation, covering not only \(H^1(\mathcal{T})\) but also \(H^1(\mathcal{T}^*)\) and \(H^1(\text{End}\mathcal{T})\), and within the same framework. By establishing a 1–1 correspondence between the universally valid Koszul calculation for Calabi-Yau models and the residue integral representations provided here, we prove that the residue calculations also enjoy the corresponding generality and completeness. A detailed correspondence with the universally valid Landau-Ginzburg orbifold analysis of Ref. [5]—for models where both the Calabi-Yau and the Landau-Ginzburg orbifold ‘phases’ are known—seems inviting, but will require a study on its own. Herein, we content ourselves with some cursory remarks in this regard and focus more on comparison with the (2,2)-supersymmetric Landau-Ginzburg orbifold analysis of Refs. [4], as facilitated by known results [1,8]. Throughout the article, we also indicate another detailed 1–1 correspondence: that with the BRST analysis of constraint systems and the associated ghost,- ghost-for- ghost,- etc. degrees of freedom. Ultimately, this correspondence should provide a fully

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6 Since the Calabi-Yau, Landau-Ginzburg and the various ‘hybrid’ phases are connected through variations of the complexified radial moduli, results pertaining to the complex structure will agree in any two phases. However, not so for the Kähler structure: although the number of massless states remains the same, the large radius limit does not have the ‘quantum symmetry’ selection rule which is a feature of the Landau-Ginzburg orbifold and so the Kähler structure Yukawa couplings are different in the two phases.
developed quantum field theory generalization of our present results and we hope to return to that in the future.

On the mathematical side, we show that the abstract and generally quite difficult to realize cohomology elements obtained in such Koszul spectral sequences have explicit class representatives in terms of residues such as (1.1). Also, the various mappings—the so-called differentials in the spectral sequences—will also be realized rather easily and are amenable for calculation.

The paper is organized as follows. In §2 we motivate and present the basic paradigm for constructing the residue representatives, illustrated by several examples in §3. In §4, we present a rather natural generalization of the residue representation to weighted hypersurfaces and quotients thereof. The main properties of the resulting ring structure are discussed in §5, while §6 presents an alternative derivation of the residue representatives and the ‘radical deformations’.

2. Reaping Residues

It seems most natural to introduce the residue representation in the context of complete intersections of $K$ hypersurfaces in a product of $N$ projective spaces:

$$\mathcal{M} \subset \prod_{i=1}^{N} \mathbb{P}^{n_i}_i \left\{ \begin{array}{c} d_{11} \ldots d_{1K} \\ \vdots \quad \ddots \quad \vdots \\ d_{N1} \ldots d_{NK} \end{array} \right\},$$

defined as the simultaneous zero-set of the system of $K$ homogeneous polynomials

$$f_j^i(x_i) = 0, \quad d_{ij} \overset{\text{def}}{=} \deg_{x_i}(f_j^i), \quad j = 1, \ldots, K,$$

where $d_{ij} \overset{\text{def}}{=} \deg_{x_i}(f_j^i)$ is the degree of the $j^{th}$ defining polynomial with respect to $x_i$, the array of homogeneous coordinates on $\mathbb{P}^{n_i}_i$. The matrix on the r.h.s. of (2.1) specifies the degrees of homogeneity $d_{ij}$ and suffices to specify the Chern classes of $\mathcal{M}$. The vanishing of the first Chern class (for $\mathcal{M}$ to be a Calabi-Yau manifold) is ensured by requiring that

$$\sum_{j=1}^{K} d_{ij} = n_i + 1, \quad i = 1, \ldots, N.$$  

For the generic model, $\dim \mathcal{M} = \sum_{i=1}^{N} n_i - K$; we obtain a three-dimensional Calabi-Yau complete intersection for $K = \sum_{i=1}^{N} n_i - 3$. It is merely for reasons of preserving a modicum of sanity with the already unwieldy notation heavily beset with indices that
we refrain from allowing generalized (unitary) flag spaces\footnote{Generalized flag spaces are quotients $G/H$, where $H$ is a maximal regular subgroup of a finite-dimensional Lie group $G$ \cite{31}; all of these can be utilized.} $\{U(n_1+\ldots+n_F)/\prod_{f=1}^F U(n_f)\}$ from appearing as factors in the embedding space \cite{3}. As the projective space is the simplest flag space, $\mathbb{P}^n = \frac{U(n+1)}{U(1)\times U(n)}$, the adventurous reader should have no problems other than notational in extending our results to this even more general class. Note that at least some of these models (involving Grassmannians $G_{n,k} = \frac{U(n+k)}{U(n)\times U(k)}$) turn out to be equivalent to certain gauged Landau-Ginzburg orbifolds \cite{9}. For most of the time, we furthermore restrict to ordinary (isotropic) projective spaces and will discuss the (anisotropic) weighted projective spaces \S 4; accordingly, the defining polynomials $f^j(x_i)$ are for the time being all homogeneous rather than quasihomogeneous.

Note that, apart from the degrees $d_{ij}$, the matrix in the r.h.s. of (2.1) does not specify the defining polynomials $f^j(x_i)$; the coefficients in $f^j(x_i)$ are therefore free and serve to parametrize the deformation family of models represented by the configuration matrix (2.1). A generic member of this family, $\mathcal{M}$, is smooth and we write $b_{2,1}$ and $b_{1,1}$ for the number of its independent $(2,1)$- and $(1,1)$-forms, respectively ($\dim \mathcal{M}=3$).

We start by considering several equivalent expressions for the residue (1.1), adapted here from \cite{26},

\[ \Omega \overset{\text{def}}{=} \text{Res}_\mathcal{M} \left[ \prod_{i=1}^N (x_i^1 d^{n_i} x_i) \right]. \tag{2.4a} \]

The residue may be calculated by means of a suitable $K$-fold contour integration:

\[ \Omega = \frac{1}{(2\pi i)^K} \oint_{\Gamma(f^1)} \cdots \oint_{\Gamma(f^K)} \frac{\prod_{i=1}^N (x_i^1 d^{n_i} x_i)}{f^1 f^2 \cdots f^K}, \tag{2.4b} \]

where $\Gamma(f^j)$ is a small loop encircling the complex hypersurface defined by $f^j$. As usual, the residue integrals are in fact independent of any specific choice of these poly-contours, in part owing to Eq. (2.3). In practice, the residues are obtained by taking the limit $\varepsilon_j \to 0$ and are independent of the radii $\varepsilon_j$.

A somewhat tedious but completely straightforward calculation produces the result of such integrations. For example, work in the coordinate patch where $x_i^{\mu_0} \neq 0$, for $i=1,\ldots,N$. Noting that $\sum_i n_i = K+3$, one performs a change of variables

\[ (x^{\mu_1},\ldots,x^{\mu_1},\ldots,x^{\mu_N},\ldots,x^{\mu_N}) \longrightarrow (x^\nu, x^\rho, x^\sigma, f^1, \ldots, f^K), \tag{2.5} \]

whence (2.4b) becomes

\[ \Omega = \frac{1}{(2\pi i)^K} \oint_{\Gamma(f^1)} \cdots \oint_{\Gamma(f^K)} \left( \prod_{i=1}^N x_i^{\mu_0} \right) \frac{dx^\nu dx^\rho dx^\sigma}{J_{\mu_0,\ldots,\mu_0}} \prod_{j=1}^K \frac{df^j}{f^j}. \tag{2.4c} \]
The $K$-fold residue integral is now easily completed to produce

$$\Omega = \left[ \prod_{i=1}^{N} x_{i}^{\mu_{i}} \right] \frac{dx^{\nu} dx^{\rho} dx^{\sigma}}{J_{(\mu_{1},...,\mu_{N})}^{\nu,\rho,\sigma}} f^{j} = 0, \quad \begin{cases} x_{i}^{\mu_{i}} \neq 0, & i = 1, \ldots, N, \\ \text{no sum on } \nu, \rho, \sigma. \end{cases}$$ (2.4d)

Of course, $J_{(\mu_{0},...,\mu_{N})}^{\nu,\rho,\sigma}$ is the Jacobian of the inverse of the change of coordinates (2.3). That is,

$$J_{(\mu_{0},...,\mu_{N})}^{\nu,\rho,\sigma} \overset{\text{def}}{=} \det \left[ \frac{\partial (x^{\nu}, x^{\rho}, x^{\sigma}, f^{1}, \ldots, f^{K})}{\partial (x^{\mu_{1}}, \ldots, x^{\mu_{N}})} \right]_{x_{i}^{\mu_{i}} \neq 0}.$$ (2.6)

Most notably, this residue sports several quite remarkable properties.

1. Despite appearances, $\Omega$ is independent of any particular choice of $x^{\nu}, x^{\rho}, x^{\sigma}$. This easily follows from the observation that, under a change of this choice of coordinates, both numerator and denominator in (2.4d) transform as tensor densities and with the same multiplicative factor which then cancels out.

2. Division by the Jacobian in (2.4b) and (2.4c) produces no singularity on $M$. This follows from the (assumed) smoothness of $M$, whence the $K$ polynomials $f^{j}$ form a non-degenerate set and there always exist precisely three independent coordinates $x^{\nu}, x^{\rho}, x^{\sigma}$ (as $\dim M=3$) to complete this set; the Jacobian is then non-zero. That is, wherever $J_{(\mu_{0},...,\mu_{N})}^{\nu,\rho,\sigma}$ may vanish, the numerator of (2.4c) must vanish also, and in such a way that the ratio remains non-zero and may be calculated upon a change to a more suitable set of coordinates. This may be easier to see, from Eq. (2.4b), as follows. The hypersurfaces $\{f^{j}=0\}$ must intersect transversely for $M$ to be smooth. That means that at least at the common zero-set $M = \bigcap_{j=1}^{K} \{f^{j}=0\}$, each of the hypersurfaces $\{f^{j}=0\}$ must be transversal, i.e., vanish linearly. Division by each $f^{j}$ therefore creates a simple pole and the corresponding integration over $\Gamma(f^{j})$ then picks out the (finite) residue.

3. Since the residues are non-zero only where the defining polynomials $f^{j}$ vanish, $\Omega$ vanishes identically everywhere on $X$ $\overset{\text{def}}{=} \mathbb{P}^{n_{1}} \times \cdots \times \mathbb{P}^{n_{N}}$—except on the complete intersection, $M$. That is, $\Omega$ is supported precisely and exclusively on $M$—it looks somewhat like a finite Dirac $\delta$-function, being non-zero (but then finite) only on the subspace $M \subset X$.

4. By construction, $\Omega$ is homogeneous and of degree 0 with respect to each $\mathbb{P}^{n_{i}}$. Therefore, $\Omega$ is the $SU(3)$ holonomy invariant $\epsilon_{\mu\nu\rho} dz^{\mu} dz^{\nu} dz^{\rho}$, i.e. $\Omega$ is a $O(0)$-valued 3-form on $M$. All of this is equally valid if the isotropic $\mathbb{P}^{n_{i}}$ are replaced by general flag spaces and/or their weighted cousins; upon replacing the top differentials $(x_{i} dz^{n_{i}}, x_{i})$ accordingly, the construction applies verbatim.

These properties will be crucial in the subsequent analysis: massless states will be assigned representatives ((1.11), in the form of an expansion in terms of ‘nowhere zero holomorphic $q$-forms’ (over certain intermediate $q$-dimensional subspaces) and with holomorphic coefficients. To that end, however, we need a telegraphic summary and a reinterpretation of the Koszul spectral sequence computation.
2.1. Rhyme and reason for residues

The basic fact underlying the Koszul spectral sequences is related to the BRST symmetry induced by enforcing the constraints which define \( \mathcal{M} \) as a submanifold of the embedding space \( \mathcal{X} \) where the defining polynomials \( f^j(x_i) \) simultaneously vanish. Typically, the embedding space is chosen to be a product \( \mathcal{X} = \prod_{i=1}^{N} \mathbb{P}^n_i \) of projective spaces. The BRST symmetry derives from the fact that all functions (and hence physical observables in particular) on \( \mathcal{M} \) then become equivalence classes of functions over \( \mathcal{X} \), modulo suitable multiples of the \( f^j(x_i) \). Thus, on \( \mathcal{M} \), a polynomial \( \phi(x) \) of degree \( \vec{d}_\phi \) is well defined only up to the equivalence

\[
\phi(x_i) \, \cong \, \phi(x_i) + \sum_{j=1}^{K} \lambda_j(x_i) f^j(x_i) ; \quad (2.7)
\]

since the \( f^j(x_i) \) vanish on \( \mathcal{M} \). Clearly, the \( \lambda_j(x) \) must be complex-analytic and have degree \( (\vec{d}_\phi - \vec{d}_j) \) for the above sum to make sense.

The corresponding inhomogeneous gauge transformation is generated by

\[
\delta \phi(x_i) = \sum_{j=1}^{K} \lambda_j(x_i) f^j(x_i) , \quad (2.8)
\]

whereupon the defining polynomials \( f^j(x_i) \) may be regarded as the generators, and the \( \lambda_j(x_i) \) as the parameters of the associated symmetry; in the corresponding BRST analysis, the \( \lambda_j(x_i) \) are assigned ghost variables (anticommutativity will follow naturally, see below).

We note that the gauge transformation \((2.8)\) may be used to ‘gauge away’ a function \( \phi(x_i) \) at any particular point of \( \mathcal{X} \)—except on \( \mathcal{M} \), where \( f^j(x_i) = 0 \) and where the gauge transformation \((2.8)\) becomes vacuous. Notably, the \( \lambda_j(x_i) \)'s must be chosen to vary over \( \mathcal{X} \) in just the right way for this to happen. For the product \( \lambda_j f^j(x) \) to be in general non-zero except at \( \{ f^j = 0 \} \), the only place where \( \lambda_j \) may possibly diverge is the zero set \( \{ f^j = 0 \} \) itself, whence the \( \lambda_j \) may be regarded as local gauge parameters which are holomorphic on \( \mathcal{X} - \{ f^j = 0 \} \). Now, if \( \mathcal{M} \) is smooth then \( f^j \) vanish to first order in their local Taylor series at \( \mathcal{M} \). Around any point \( \hat{x} \in \mathcal{M} \), we have \( f(x) \approx (x-\hat{x}) f'(\hat{x}) \) with \( f'(\hat{x}) \neq 0 \) and near \( \hat{x} \), \( \lambda_j(x) \sim const./(x-\hat{x})^{1-\epsilon} \), with \( \epsilon > 0 \). (\( \epsilon < 0 \) would spoil the holomorphicity of the polynomials \( \phi(x_i) \)). Allowing \( \epsilon = 0 \) would cause the defining polynomials \( f^j(x_i) \) to be equivalent to polynomials that do not necessarily vanish at \( \mathcal{M} \)—contradicting the very

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8 Notation: By the degree \( \vec{d}_\phi \) of a polynomial \( \phi(x_i) \) we mean the array of degrees \( (d_1, \ldots, d_N) \), where \( d_i \) is the degree of homogeneity of \( \phi(x_i) \) with respect to the homogeneous coordinates \( x_i \) of \( \mathbb{P}^n_i \). Similarly, by \( \vec{d}_j \) we mean the analogous array of degrees \( (d_{ij}, \ldots, d_{Nj}) \) of \( f^j(x_i) \). We forego the pedantry of calling this a multi-degree and emphasizing this hereafter.
definition of $\mathcal{M}$.) For the $\lambda_j(x_i)$ to be single-valued, $\epsilon$ must be an integer and the $\lambda_j$ must in fact be holomorphic on all of $\mathcal{X}$, rather than just on the complement of $\mathcal{M}$. In particular, only functions complex-analytically proportional to some $f^j(x_i)$ are equivalent to zero modulo this ‘globally holomorphic’ transformation (2.8).

Focus now on the first two constraint polynomials, $f^1(x_i)$ and $f^2(x_i)$ and set temporarily $\lambda_j(x_i) = 0$, $j = 3, \ldots, K$. Note that the special choice of the first two ‘ghost’ variables (assuming it permitted by the degrees of homogeneity)

$$\lambda_1(x_i) = \lambda_{12}(x_i) f^2(x_i) , \quad \lambda_2(x_i) = \lambda_{21}(x_i) f^1(x_i) ,$$ (2.9)

corresponds to a trivial component of the equivalence relation (2.7)

$$\phi(x_i) \cong \phi(x_i) + (\lambda_{12}(x_i) f^2(x_i) f^1(x_i) + \lambda_{21}(x_i) f^1(x_i) f^2(x_i)) ,$$

$$= \phi(x_i) + [\lambda_{12}(x_i) + \lambda_{21}(x_i)] f^1(x_i) f^2(x_i) ,$$ (2.10)

$$\equiv \phi(x_i) , \quad \text{precisely if} \quad \lambda_{21} = -\lambda_{12} .$$

This implies that the two-parameter equivalence relation (2.7) gauges away ‘too much’ and the one-parameter degree of freedom $\lambda_{12} = \frac{1}{2}(\lambda_{12} - \lambda_{21})$ compensates for this. Therefore, the equivalence relations (2.7) must themselves be taken modulo the equivalence between the two ‘ghost’ variables (2.9), as generated by $\lambda_{12}(x_i)$. Such second order equivalence relations are familiar from the BRST analysis and $\lambda_{12}$ corresponds to a ‘ghost-for-ghost’ variable. For the full set of $K$ first order ‘ghost’ variables $\lambda_j$, there exist $\binom{K}{2}$ second order, ‘ghost-for-ghost’ variables $\lambda_{[jl]}(x_i)$. Clearly, this produces an avalanche of such higher order equivalence relations, and a corresponding hierarchy of ‘ghost-for-ghost-for-...’ variables $\lambda_{[ji\ldots l]}(x_i)$. Equally obvious should be the fact that this hierarchy stops at the $K^{th}$ order, with a single ‘ghost-for-ghost-for-...’ variable $\lambda_{[ji\ldots jK]}$. The antisymmetry here reflects (although is not identical to) the graded anticommutivity of the actual BRST ghost fields.

For all but the few simplest cases, an explicit listing of this hierarchy of equivalence relations and ‘ghost’ variables is plagued by a proliferation of indices, variables and confusion. Let us therefore introduce a diagrammatic representation, where an equivalence generated by

$$\phi(x_i) \cong \phi(x_i) + \lambda_1(x_i) f^1(x)$$ (2.11)

---

9 On a singular $\mathcal{M}$, some of the $f_j$ vanish to higher order. This then admits certain meromorphic $\lambda_j$’s, enlarging the space of ‘ghosts’, and then also ‘ghost-for-ghosts’, etc., allowing for possible additional operators and corresponding low-energy fields.
is represented by
\[ \lambda_1(x_i) \xrightarrow{f^1} \phi(x_i) , \] (2.12)
meaning simply that \( f^1 \)-multiples of \( \lambda_1 \) may be added to \( \phi \) at will. This notation also
reminds of the BRST and gauge-theoretic interpretation, where the \( f^1(x_i) \) may be regarded
as the generator and \( \lambda_1 \) the gauge parameter for \( \delta \phi(x_i) = \lambda_1(x_i)f^1(x) \). For two constraint
polynomials, we obtain the diagram
\[ \lambda_{[12]}(x_i) \xrightarrow{\lambda_2(x_i)} \phi(x_i) \Rightarrow \phi_M(x_i) , \] (2.13)
where \( \rightarrow \) stands for multiplication by \( f^1 \) and \( \\backslash \) by \( f^2 \) and \( \Rightarrow \phi_M \) says that the resulting
equivalence class may be regarded as the corresponding function on \( M \). Stacked in the
first column to the left of \( \phi(x_i) \) are the ‘ghost’ variables, and in the (here, a single entry)
second column to the left—the ‘ghost-for-ghost’ variables. The diagram (2.13) stands for
the equivalence relations
\[ \phi(x_i) \cong \phi(x_i) + \lambda_1(x_i)f^1(x) + \lambda_2(x_i)f^2(x) , \]
where the \( \lambda_j \) are themselves taken modulo the equivalence
\[ \lambda_j(x_i) \cong \lambda_j(x_i) + \lambda_{[j]}(x_i)f^j(x_i) . \]

The full hierarchy of these gauge equivalence relations may then be represented dia-
grammatically as
\[ \mathcal{O}(\tilde{d}_\phi - \sum_{j \neq 1} \tilde{d}_j) \rightarrow \cdots \rightarrow \mathcal{O}(\tilde{d}_\phi - \tilde{d}_K) \backslash \rightarrow \mathcal{O}(\tilde{d}_\phi) \Rightarrow \mathcal{O}_M(\tilde{d}_\phi) , \] (2.15)
where \( \mathcal{O}(\tilde{d}_\phi) \) denotes (the sheaf of) functions of degree \( \tilde{d}_\phi \) over \( X \) and the arrows represent
multiplication with the \( f^j(x_i) \). The subscript on \( \mathcal{O}_M(\tilde{d}_\phi) \) of course denotes restriction
to the submanifold \( M \)—which is what we are after. The process summarized in the
sequence (2.15) is in fact the underlying one in the Koszul calculations and we will refer
to it throughout. Of course, the various arrows represent multiplication by the \( f^j(x_i) \); as
the degrees typically specify this fairly well, we forego labeling the arrows.

---

\[^{10}\] Such pictures seem quite worth the \( 2^K \) equivalence relations employing \( 2^K - 1 \) ghost vari-
bles which they represent; at \( K = 8 \) for example, already the number of equivalence relations
reaches truly proverbial proportions, a corresponding word count having surpassed it long ago.

---
The first task in analyzing the function ring on a submanifold is the determination of scalars—the degree-$\vec{0}$ objects. To that end, set $\vec{d}_\phi = \vec{0}$ in the sequence (2.15). Note that whereas functions of negative degree clearly cannot by themselves provide complex-analytic functions, they will be useful in constructing differential forms. This in fact is not at all novel: the best known but often overlooked example is the combination of $x^2 + y^2$ and $x^2 + y^2$, multiplied respectively by $dy$ and $-dx$, to produce the differential of the polar angle $d\phi$—perfectly regular on a submanifold encircling but not including the origin.

A systematic and complete listing of non-zero such forms is possible owing to the ‘master’ theorem by Bott, Borel and Weil [1,3]. A straightforward application of this theorem yields the resulting cohomology (on $X$) displayed in the lower left quadrant of the chart below:

$$
\begin{array}{cccccc}
(\otimes_j E_j^*) & \to & \cdots & \to & E_K^* & \Rightarrow O_X \\
(\otimes_j \not\equiv K E_j^*) & \to & \cdots & \to & E_1^* & \Rightarrow O_M \\
0 & 0 & \ldots & 0 & H^0 \approx \mathbb{C} & \Rightarrow H^0(M) \\
0 & 0 & \ldots & 0 & 0 & \Rightarrow H^1(M) \\
0 & 0 & \ldots & 0 & 0 & \Rightarrow H^2(M) \\
0 & 0 & \ldots & 0 & 0 & \Rightarrow H^3(M) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
H^X \approx \mathbb{C} & 0 & \ldots & 0 & 0 & \Rightarrow H^X(M) \equiv 0
\end{array}
$$

where $E_j^* \simeq O(-\vec{d}_j)$ is the normal bundle associated to $f_j$. This chart is typical of Koszul calculations and it contains the cohomology groups on $X$, $H^0, \ldots, H^X$ (with $X = \dim X$), valued in the bundle under which they are stacked. Through a process called ‘filtration’ (see below), the cohomology groups on $X$ give rise to the cohomology groups on $M$, stacked in the lower right quadrant.

The above chart thus presents the result that there are only two types of degree-$\vec{0}$ holomorphic forms on the Calabi-Yau manifold (2.1):

1. the restriction of degree-$\vec{0}$ 0-forms on the embedding space $X$ to the subspace $M$,

2. the restriction of $X$-forms with coefficients of degree $(-\sum_j \vec{d}_j)$.

Back in the upper left quadrant, this degree-$(-\sum_j \vec{d}_j)$ in 2. function must become a complex-analytic degree-$\vec{0}$ function upon multiplication by the $f_j$ as indicated by the arrows in the sequence on the top. Therefore, this degree-$(-\sum_j \vec{d}_j)$ function must be a
constant multiple of $[\prod_{i=1}^N f^j]^{-1}$. Note now that the condition for $\mathcal{M}$ to be a Calabi-Yau 3-fold is $\sum_j d_j = (n_1 + 1, \ldots, n_N + 1) \equiv (\bar{n}_i + 1)$.

Precisely because the Calabi-Yau condition is satisfied, we can make a degree-$\bar{0}$ differential from such a function through multiplying by the top differential $\prod_{i=1}^N (x_i d^{n_i} x_i)$, very much in analogy with $d \phi = \frac{vdz - x \, dv}{x^2 + y^2}$. This degree-$\bar{0}$ $X$-form appears in the bottom left of the above chart. Finally, this will yield a holomorphic 3-form on $\mathcal{M}$ upon the $\mathcal{K}$-fold contour integration:

$$\Omega \overset{\text{def}}{=} \frac{1}{(2\pi i)^K} \int_{\Gamma(f^1)} \cdots \int_{\Gamma(f^K)} \prod_{i=1}^N (x_i d^{n_i} x_i),$$

as discussed for Eqs. (2.4). Each contour integration may be identified, in the chart (2.16), as taking one step upwards (each one integration ‘cancels’ one differential) and one step to the right (the residue evaluation ‘cancels’ the pole which produced the residue). The concatenation of such diagonal steps leads to the cohomology on $\mathcal{M}$ where the contribution eventually ends up and represents what is called ‘filtration’ in the general theory of spectral sequences. So,

$$(\otimes i \notin \mathcal{E}_1^*) \overset{\text{…}}{\rightarrow} \mathcal{E}_K^* \overset{\text{…}}{\rightarrow} \mathcal{E}_1^* \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_\mathcal{M}$$

where the contributions are placed as before the ‘filtration’: the $\Omega$ here ‘filters’ along the SW–NE diagonal to the fourth element in the right-most column of the lower left quadrant, thereby contributing to $H^3(\mathcal{M})$.

As the subsequent discussions and examples will hopefully clarify, the identification of the content of charts (2.16) and (2.17) defines the ‘residue map’ (‘residue operator’ or ‘residue symbol’), which assigns residue representatives in $H^*(\mathcal{M}, \ldots)$ to certain rational
forms on the embedding space. Indeed, our analysis pertains to generalizations\textsuperscript{11} of the so-called Poincaré residue symbol\textsuperscript{32,33}, of which the formula (1.1) is a sample application.

Admittedly, we have not derived anything new so far. Instead, we have related the major ingredients in Koszul calculations with the simple residue recipe (2.4). Our aim now is to use this insight and obtain all other cohomology straightforwardly from such residue considerations. As a byproduct, this will effectively rederive the required results of the Bott-Borel-Weil theorem and should also better specify the residue map.

2.2. A residue refinement

Motivated by Eqs. (2.4), a natural generalization comes to mind, and which will turn out to be essential in the future:

\[
\text{Res}_S^Q[\phi] \overset{\text{def}}{=} \frac{1}{(2\pi i)^{|Q|}} \int \cdots \int \prod_{j \in Q} \Gamma(f^j) \prod_{i \in S} (x_i d^{n_i} x_i) \phi(x_1, \ldots, x_N), \tag{2.18a}
\]

where \(\phi(x_i)\) is chosen so that

\[
\text{deg}_i(\phi) = \sum_{j \in Q} \text{deg}_i(f^j) - \sum_{\mu=0}^{n_i} \text{deg}_i(x_i^\mu) \geq 0, \quad \text{for all } i \in S, \tag{2.18b}
\]

\[
\text{deg}_i(\phi) \geq \sum_{j \in Q} \text{deg}_i(f^j) \geq 0, \quad \text{for all } i \notin S, \tag{2.18c}
\]

and where \(Q\) labels a subset of the defining polynomials which occur in the denominator of the integrand in Eq. (2.18a), \(|Q|\) is the number of these polynomials, \(S\) labels the subset of projective spaces over which the integral is performed, \(\text{deg}_i\) denotes degrees of homogeneity with respect to \(\mathbb{P}^{n_i}_i\). By definition, this ‘partial’, or ‘intermediate’ residue will be understood to vanish if the conditions (2.18b,c) are not satisfied. Henceforth, \(|Q|\) will be referred to as the level of the residue, and the rational differential forms appearing in these integrals will be called the kernel of the residue.

\[
\text{Res}_Q^S[\phi] \text{ is a 'nowhere zero holomorphic } q\text{-form' as in (2.4), on the } q\text{-dimensional Calabi-Yau complete intersection } Q \overset{\text{def}}{=} \cap_{j \in Q} \{f^j=0\} \subset \prod_{i \in S} \mathbb{P}^{n_i}_i, \text{ and is parametrized by its dependence on the complementary factor } \prod_{i \notin S} \mathbb{P}^{n_i}_i. \text{ Also, } \text{Res}_Q^S[\phi] \text{ vanishes identically on the complement of } Q \text{ within } \prod_{i \in S} \mathbb{P}^{n_i}_i.
\]

\textsuperscript{11} While perhaps straightforward in principle, the generalizations presented here have not been reported heretofore, to the best of our knowledge. Owing to an easy corollary of Murphy’s Law, the results most dearly sought are not readily available in the literature.
Owing to its definition, the residue (2.18) shares all the salient features of $\Omega$ in (2.4). That is, with regard to the integrated $\mathbb{P}^n_i$, $i \in S$, $\text{Res}^S_Q[\phi]$ is non-zero and invariant under holonomy precisely and exclusively over the subspace $Q \subset \prod_{i \in S} \mathbb{P}^n_i$. With regard to the un-integrated $\mathbb{P}^n_i$, $i \not\in S$, $\phi$ can be chosen so that $\text{Res}^S_Q[\phi]$ is holomorphic and of degree $\deg_i(\phi) - \sum_{j \in Q} \deg_i(f^j)$. Thereby, it is $\tilde{\partial}$-closed and (obviously) not exact. The well-known formula (2.4) is then simply the special case with $\phi = 1$, and when $Q$ and $S$ include all polynomials and all $\mathbb{P}^n$'s, respectively. Also, the well-known polynomial deformations correspond to the special case when $\phi$ ranges over the polynomials and $Q = \emptyset = S$. Similar generalizations of (1.1), however to non-compact Calabi-Yau spaces, appear in constructing spacetime variable superstring vacua [34]. We now turn to the details of this.

The residue is evaluated much the same as (2.4), employing a change of variables such as (2.5), and direct contour integration. In view of the requirement (2.18b), the degree of the residue is $\vec{0}$ for all the $i \in S$, and it is a $q$-form, where

$$q = \sum_{i \in S} n_i - |Q|.$$  \hspace{1cm} (2.19)

Consider first the properties of (2.18a) over $\mathbb{P}^n_i$, $i \in S$. If $q \geq 0$ and in a coordinate neighborhood where $x_{i \mu_i}^b \neq 0$, for fixed $\mu_0$ and all $i \in S$, the residue will evaluate to

$$\text{Res}^S_Q[\phi] = \left( \prod_{i \in S} x_{i \mu_i}^b \right) \frac{dx_{\nu_1} \cdots dx_{\nu_q}}{J_{(\mu_0, i \in S)}} \phi(x_i),$$  \hspace{1cm} (2.20a)

with $J_{(\mu_0, i \in S)}$ the Jacobian of the coordinate transformation

$$(x^{\mu_1}, \ldots, x^{\mu_i}; i \in S) \rightarrow (x^{\nu_1}, \ldots, x^{\nu_q}, f^j; j \in Q),$$  \hspace{1cm} (2.21)

which has been used to single out $x^{\nu_1}, \ldots, x^{\nu_q}$.

In the event that $q < 0$, i.e., $\sum_{i \in S} n_i < |Q|$, there are not enough differentials to perform all the $|Q|$ integrations, and we can of course perform only $\sum_{i \in S} n_i$ of them. The resulting rational function will then be holomorphic only if the function $\phi$ can be chosen

\[12\] The erudite Reader will have noticed the similarity between ‘un-integrated’ here, and ‘un-projected’ in [4]. Indeed, this is not an accident: upon performing a contour integral over any one of the $n+1$ coordinates of a $\mathbb{P}^n = U(n+1)/[U(n) \otimes U(1)]$, the result must, by $U(n+1)$-covariance, also be independent (up to an overall scale) of the remaining $n$ coordinates of that $\mathbb{P}^n$—implying a projection along that $\mathbb{P}^n$. This relation will become even more manifest in § 4.
so as to cancel the remaining poles. Such applications of the L’Hospital theorem will be understood as an extension of the standard residue calculations.

On the other hand, the degree of the residue \((2.18a)\) with respect to the un-integrated coordinates of \(\prod_{i \notin S} \mathbb{P}^{n_i}\) is the same as that of the \(m_i^{th}\) derivative of \(\phi(x_i)\) with respect to \(x_i, i \notin S\), written \(\partial^{m_i} \phi(x_i)\), where (see \((2.18a)\))

\[
m_i \overset{\text{def}}{=} \sum_{k \in Q} d_{ik}, \quad i \notin S, \quad \partial^\mb \overset{\text{def}}{=} \left( \prod_{i \notin S} \partial^\otimes m_i \right).
\]

The result of the contour integration \((2.18a)\) is finite over \(Q \overset{\text{def}}{=} \bigcap_{k \in Q} \{f^k = 0\}\) since the intersection of hypersurfaces is smooth by assumption. (Actually, we only care about the smoothness of the subspace \(\mathcal{M}\) where all \(K\) hypersurfaces meet.) Choosing \(\phi\) to have a sufficiently positive degree over the un-integrated \(\prod_{i \notin S} \mathbb{P}^{n_i}\), implies that \(\text{Res}^{S}_Q[\phi]\) can be represented as a polynomial over \(\prod_{i \notin S} \mathbb{P}^{n_i}\). Therefore, \(\text{Res}^{S}_Q[\phi]\) can be written as a linear combination of the \(\mb^{th}\) multi-derivatives of \(\phi(x)\):

\[
\text{Res}^{S}_Q[\phi] = \oint \cdots \oint_{\prod_{k \in Q} \Gamma(f^k)} \frac{\prod_{i \in S}(x_i \partial_{d_{i}^{n_i}x_i})}{\prod_{k \in Q} f^k} \phi,
\]

\[
= \left( \prod_{i \in S} x_i^{\mu_i} \right) \frac{dx^{\nu_1} \cdots dx^{\nu_q}}{\partial^\otimes J_{(\mu_i, i \in S)}} \partial^\mb \phi(x_i),
\]

\[
\overset{\text{def}}{=} \sum_\mb \Omega^{\mb}_{(q)} \phi^{\mb}(x).
\]

The multi-index \(\mb\) contains one index for each of the \(m_i\) derivatives with respect to the un-integrated \(x_i (i \notin S)\). Notably, the underbraced term, \(\Omega^{\mb}_{(q)}\), is a ‘nowhere zero holomorphic \(q\)-form’ on the \(q\)-dimensional Calabi-Yau space \(Q\) embedded in \(\prod_{i \in S} \mathbb{P}^{n_i}\) as the common zero-set of \(f^j = 0, j \in Q\), and parametrized by \(x_i \in \mathbb{P}^{n_i}, i \notin S\), and any other parameter that the \(f^j\) depend upon; \(Q\) is not Calabi-Yau over the un-integrated \(\mathbb{P}^{n_i}, i \notin S\).

Alternatively, the multi-derivatives \(\partial^\mb f^j, j \in Q\) and indexed by \(\mb\), may be regarded the defining equations of a Calabi-Yau \(q\)-fold \(Q_\mb \subset \prod_{i \in S} \mathbb{P}^{n_i}\), for each of which the

\[
\Omega^{\mb}_{(q)} \overset{\text{def}}{=} \left( \prod_{i \in S} x_i^{\mu_i} \right) \frac{dx^{\nu_1} \cdots dx^{\nu_q}}{\partial^\otimes J_{(\mu_i, i \in S)}}.
\]

\(^{13}\) With a little forethought, we note that non-analytic terms would fail to be \(\bar{\partial}\)-closed and so would not contribute to \(H^*_\partial(\mathcal{M}, \ldots )\). The Reader may instead wish to ignore this motivation and regard this as simply weeding out non-holomorphic contributions.
are the holomorphic volume-forms. They are calculated much the same as \( (2.4) \), for which the Jacobian \( \text{à la Eqs. } (2.4) \) and \( (2.18) \) is indeed \( \partial \tilde{m} J_{\nu_1 \ldots \nu_q}^{\alpha_1 \ldots \alpha_t} \) since the Jacobian \( J_{\nu_1 \ldots \nu_q}^{\alpha_1 \ldots \alpha_t} \) is multi-linear in the \( \{ f^j, j \in Q \} \), and the multi-order of the multi-derivative \( \partial \tilde{m} \) over the un-integrated \( \mathbb{P}^{n_i} \), \( i \not\in S \) exactly equals the multi-degree of \( J_{\nu_1 \ldots \nu_q}^{\alpha_1 \ldots \alpha_t} \) over the un-integrated space. Therefore, \( \Omega_{(q)}^m \) are constant over the un-integrated factors \( \prod_{i \not\in S} \mathbb{P}^{n_i} \) of the embedding space \( \mathcal{X} = \prod_{i} \mathbb{P}^{n_i} \), while being nowhere-zero and invariant under holonomy over \( Q \).

Yet another way to think about these is along the discussion of the \( q < 0 \) case; that is, for the residue to be holomorphic, we seek suitable functions \( \phi \) in a factorized form, such that one factor precisely cancels the contribution of negative degree over \( \mathbb{P}^{n_i} \), \( i \not\in S \), from the residue kernel. The remaining factor may then be written as in \( (2.20) \), and is precisely of the promised general form \( (1.11) \). In fact, when \( q < 0 \), the \( \Omega_{(q)}^m \) is formally a differential form of negative order, which is naturally identified with a vector or higher rank (contravariant) tensor field, \textit{i.e.}, a “reparametrization” or “gauge” degree of freedom.

---

As a quick (and well known) example, consider the Calabi-Yau 3-fold intersection of a degree-(3,0) and a degree-(1,3) hypersurface in \( \mathbb{P}^3 \times \mathbb{P}^2 \); \( \mathcal{M} \in \mathbb{P}^3 \times \mathcal{M} \), for short \( [3] \). Now note that the constraint \( g(x,y) \), of degree-(1,3), produces a 1-dimensional Calabi-Yau space (a torus) in \( \mathbb{P}^2 \). Its defining equation depends non-trivially (linearly) on \( x \in \mathbb{P}^3 \), so that these tori are fibered over the \( \mathbb{P}^3 \) linearly. Writing \( y \) for coordinates on \( \mathbb{P}^2 \) and for a suitable function \( \phi \) with \( \text{deg}_x(\phi) \geq 1 \) and \( \text{deg}_y(\phi) = 0 \), the partial residue is

\[
\text{Res}_{2}^{y}[\phi] = \frac{1}{(2\pi i)} \oint_{\Gamma(g)} \frac{(yd^2y)}{g(x,y)} \phi = \sum_{a=0}^{3} \Omega_{(1)}^{a}(\partial_{a} \phi) , \tag{2.24a} \]

\[
\Omega_{(1)}^{a} = \frac{1}{(2\pi i)} \oint_{\Gamma(g)} \frac{(yd^2y)}{\partial_{a} g(x,y)} = y^0 \frac{dy^{\hat{\alpha}}}{\partial_{a} J_{(0)}^{\hat{\alpha}}} = y^{\alpha} \frac{\epsilon_{\alpha\beta\gamma} dy^{\beta}}{\partial_{a} J_{(\alpha)}^{\beta}} , \tag{2.24b} \]

\[
J_{(0)}^{\hat{\alpha}} = \left| \frac{\partial(y^{\hat{\alpha}}, g)}{\partial(y^{1}, y^{2})} \right|_{y^{0} \neq 0} , \quad \hat{\alpha}, \hat{\beta} = 1, 2 , \quad \alpha, \beta, \gamma = 0, 1, 2 ; \tag{2.24c} \]

there is no summation on any of the repeated indices in \( (2.24) \), where the last expression is valid in all coordinate patches. Note that the free index \( \beta=0, 1, 2 \) on the far right of \( (2.24) \) stems from the coordinate over which the contour integral was performed and it simply labels different \( U(3) \)-covariant coordinate choices for writing the same.

It is easy to verify that \( \text{Res}_{2}^{y}[\phi] \) is \( (1) \) independent of the choice of the coordinate patch (here \( y^0 \neq 0 \)), as should be manifest from the last expression in \( (2.24) \), \( (2) \) holomorphic over \( \mathbb{P}^3 \), \( (3) \) identically zero outside \( \{ g=0 \} \subset \mathbb{P}^2 \), \( (4) \) non-zero and finite on the cubic tori \( \{ g(x,y)=0 \} \subset \mathbb{P}^2 \).
2.3. Racks and racks of residues

Instead of determining the degree-$0$ forms ($\mathcal{O}_\mathcal{M}$-valued cohomology) as done in the previous subsection, we now turn to degree-$d_j$ forms, for any $j = 1, \ldots, K$. To this end, we shift the degree labels in (2.17) so as to produce $\mathcal{O}(\vec{d}_j)$ at the far right. The degree-$d_j$ ‘ghost-for-ghost...’ sequence is then

$$\mathcal{O}(\sum_{l \neq j} \vec{d}_l) \rightarrow \mathcal{O}(\sum_{l \neq j, K} \vec{d}_l) \rightarrow \cdots \rightarrow \mathcal{O}(\vec{d}_j - \vec{d}_K) \rightarrow \mathcal{O}((\vec{d}_j - \vec{d}_1) \mathcal{O}(\vec{d}_j) \Rightarrow \mathcal{O}_\mathcal{M}(\vec{d}_j),$$

or,

$$(\bigotimes_{l \neq j} \mathcal{E}_{f_l}^*) \rightarrow \cdots \rightarrow \mathcal{E}_{f_1}^* \mathcal{O}(\vec{d}_j) \Rightarrow \mathcal{E}_{f_j}^* \Rightarrow \mathcal{O}_\mathcal{M}(\vec{d}_j),$$

where $j$ is being omitted from the above products because $\mathcal{E}_{f_j}^* \otimes \mathcal{E}_j = \mathcal{C}$. Remember that these sequences denote the entire process of listing all $\frac{K}{2}2^K$ equivalence relations and assigning the $2^K - 1$ ‘ghost’, ‘ghost-for-ghost’ etc. variables, as sketched in §2.1. Also, $\mathcal{E}_{f_j}$ denotes (the sheaf of) functions of degree $\vec{d}_j$ and $\mathcal{E}_{f_j}^*$ is its dual. The duals stem from division by $f^j(x_i)$, so that the multiplication indicated by arrows would produce holomorphic functions on $\mathcal{X} = \prod_i \mathbb{P}^{n_i}$ of degree $\vec{d}_j$, that is, $\mathcal{O}_\mathcal{X}(\vec{d}_j)$.

Whenever the degrees are non-negative, there clearly is a straightforward restriction. Moreover, if there is a mapping in the sequence, there will also be a mapping between the restrictions of the corresponding functions. For example, in (2.25) there exists a sequence

$$[\mathcal{O}(\vec{d}_j - \vec{d}_j) = \mathcal{O}(\vec{0})] \quad \cdot f^j(x_i) \quad \mathcal{O}(\vec{d}_j) \Rightarrow \mathcal{O}_\mathcal{M}(\vec{d}_j).$$

As discussed in §2.1, the degree-$\vec{0}$ function (constant) here acts as one complex variable worth of a ‘ghost’ degree of freedom, so that degree-$\vec{d}_j$ functions on $\mathcal{M}$ are obtained as degree-$\vec{d}_j$ functions on $\mathcal{X}$, taken however, modulo a complex multiple of $f^j(x_i)$:

$$\delta f^j(x_i) \triangleq \delta f^j(x_i) + \lambda f^j(x_i),$$

where $\delta f^j(x_i)$ denotes a general variation of $f^j(x_i)$—thus a general function with the same degrees. In the light of the definition (2.18), these equivalence classes may be considered as the ‘zeroth level’ contributions to $H^0(\mathcal{M}, \mathcal{E}_{f_i})$ and thereby to $H^1(\mathcal{T})$.

However, there are in general further non-vanishing contributions to the cohomology classes $H^*(\mathcal{M}, \mathcal{E}_{f_i})$, i.e., to $H^*(\mathcal{T})$. In order to check our arguments and results below, we recall some of facts from the Bott-Borel-Weil theorem. In particular,

$$\dim H^q(\mathbb{P}^n, \mathcal{O}(k)) = \begin{cases} \delta_{q,0}(\frac{n+k}{n}) & \text{if } 0 \leq k, \\ 0 & \text{if } -(n+1) < k < 0, \\ \delta_{q,n}(\frac{k}{n}) & \text{if } k \leq -(n+1), \end{cases}$$

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where the middle case may be subsumed under the last one, since

\[ (-k-1)^{\text{def}} = 0, \quad \text{for } -(n+1) < k < 0. \]  

(2.29)

Consider then any \( O(q) \) in the sequence (2.25). This will give rise to a non-zero result in the Koszul complex if \( q_i \), the components of \( \vec{q} \), satisfy either \( q_i \geq 0 \) or \( q_i < -n_i \), that is, if no \( q_i \) lies within \( -1, \ldots, -n_i \). Let \( R \) denote the subrange of the index \( i \) for which \( q_i \) lies within \( 0 \), and let \( \bar{R} \) denote the complementary subrange of \( i \) for which \( q_i < -n_i \). Applying Eq. (2.28) for each \( \mathbb{P}_i^{n_i} \) separately and then putting it all together, we obtain

\[
\dim H^q(\mathcal{X}, O(\vec{q})) = \prod_{i \in R} \binom{n_i + q_i}{q_i} \prod_{i \in \bar{R}} \binom{-q_i - 1}{n_i}, \quad q = \sum_{i \in R} n_i,
\]

(2.30)

and this contributes \( q \) steps below \( O(\vec{q}) \) in the chart (2.17). Note that \( q_i \geq -(n_i+1) \) owing to the Calabi-Yau condition (2.3), so that the product over \( i \in \bar{R} \) is non-zero and then equal to one only if \( q_i = -(n_i+1) \), for all \( i \in \bar{R} \). In fact, the Bott-Borel-Weil theorem says more: the nonzero \( O(k) \)-valued cohomology groups are generated by degree-\( k \) monomials for \( k \geq 0 \); the \( O(\vec{q}) \)-valued cohomology is then simply the product of such factors, or zero if \( -n_i \leq q_i \leq -1 \) for some \( i \).

---

Our task now is to demonstrate that exactly the same result is obtained by collecting all non-zero and holomorphic partial residues of the type (2.18)!

Consider therefore the rational polynomials appearing in the sequence (2.25), and search for the cohomology on \( \mathcal{M} \) valued in \( \delta f^j(x_i) \)—polynomials of the homogeneity of \( f^j(x_i) \). Starting in (2.25) from \( O(\vec{d}_j) \) and going to the left, we divide one-by-one by the defining polynomials \( f^j(x_i) \), to obtain

\[
\frac{\delta f^j(x_i)}{\prod_{k \in Q} f^k(x_i)},
\]

(2.31)

where \( Q \) is the subset labeling the polynomials with which we have divided. As \( |Q| \) is the number of defining polynomials in the denominator, this contribution is in the \( |Q|^{\text{th}} \) column of a chart like (2.17). The degrees of homogeneity of (2.31) are \( \vec{d}_j - \sum_{k \in Q} \vec{d}_k \).

In order for (2.31) to produce an analytic contribution to \( H^*(\mathcal{M}, \ldots) \), the degree with respect to each \( \mathbb{P}_i^{n_i} \) must be made non-negative. To that end, we multiply (2.31) by the top differentials of those \( \mathbb{P}_i^{n_i} \) (labeled by the \( i \in S \)) with respect to which (2.31) has negative degrees. Now, owing to the homogeneity of the \( \mathbb{P}_i^{n_i} \), including only a proper factor of \( (x_i d^{n_i} x_i) \) makes no sense: no such factor is invariant under (projective) coordinate
reparametrizations, i.e., with respect to \( \text{PGL}(n_i+1; \mathbb{C}) \approx \text{SU}(n_i+1; \mathbb{C}) \). This then defines the residue

\[
\text{Res}^S_Q[\delta f^j] = \frac{1}{(2\pi i)^{|Q|}} \oint \cdots \oint \prod_{k \in Q} \Gamma(f^k) \prod_{i \in S} \frac{\prod_{d_i} x_i d_{n_i} x_i}{\prod_{k \in Q} f^k(x_i)} \delta f^j(x_i) .
\]

(2.32a)

By construction,

\[
\text{deg}_i (\text{Res}^S_Q[\delta f^j]) = \begin{cases} 
  d_{ij} + (n_i+1) - \sum_{k \in Q} d_{ik} & , \quad i \in S, \\
  d_{ij} - \sum_{k \in Q} d_{ik} & , \quad i \notin S,
\end{cases}
\]

(2.33)

so that the degrees are non-negative over the un-integrated \( \mathbb{P}^{n_i}_i, i \notin S \). As for the integrated \( \mathbb{P}^{n_i}_i, i \in S \), only the residues with the strict equality for all \( i \in S \) produce well-defined \( (U(n_i+1))-\text{covariant}, i \in S \) forms for \( H^*(\mathcal{M}, \mathcal{E}_j) \), and we require

\[
\text{deg}_i (\text{Res}^S_Q[\delta f^j]) = d_{ij} + (n_i+1) - \sum_{k \in Q} d_{ik} , \quad i \in S ,
\]

(2.33)′

to vanish, whereupon the requirements (2.18b, c) are met. This recovers the ‘vanishing’ part of the Bott-Borel-Weil theorem (upon the ‘filtration’ into \( H^*(\mathcal{M}, \ldots) \)) for \( \mathcal{O}(k) \) bundles \( (2.28) \). That is, the contribution vanishes if some of the degrees ends up in the \(-1, \ldots, -n_i \) region; the subset of indices \( S \) in (2.32a) was labeled \( \bar{R} \) for the Bott-Borel-Weil theorem. The Bott-Borel-Weil theorem also provides for the degrees to be more negative than \(-(n_i+1)\), but this cannot occur if we restrict to Calabi-Yau subspaces of \( \mathcal{X} = \prod_i \mathbb{P}^{n_i}_i \).

Now, if \( q = 0, 1, 2, 3 \) and in a coordinate neighborhood where \( x_i^{\mu_i} \neq 0 \), the residue will evaluate, as in (2.20a), to

\[
\text{Res}^S_Q[\delta f^j] = \left( \prod_{i \in S} x_i^{\mu_i} \right) \frac{dx^{\nu_1} \cdots dx^{\nu_q}}{J_{(\mu_1, \cdots, \nu_q)_{i \in S}}} \delta f^j(x_i) ,
\]

(2.32b)

with \( J_{(\mu_1, \cdots, \nu_q)_{i \in S}} \) the appropriate Jacobian.

If \( q < 0 \), only \( \sum_{i \in S} n_i \) contour integrals can be performed, and the result is a rational function with poles of total order \( |q| \), placed \( |q| \) steps to the left from the the top row of the lower left quadrant of a chart such as (2.17). This contributes to \( H^*(\mathcal{M}, \mathcal{E}_j) \) only if the function \( \delta f^j \) can be chosen so as to cancel the remaining \( |q| \) poles and provide a non-zero and complex-analytic result via the application of the L’Hospital theorem as discussed above. This application of the L’Hospital theorem does not reduce the number of differentials (the order of the form) and cancels a pole merely through judicious choice of \( \delta f(x_i) \). Therefore, the evaluation via L’Hospital theorem will not affect the ‘filtration’ and will not move the contribution within the lower left quadrant of the corresponding
chart, such as (2.17). The contribution stays \(|q|\) places to the left of the top right-most element in the lower left quadrant, and represents \(|q|^{th}\) order ‘ghost-for-ghost...’ variables. Indeed, as differentials of negative order, these may be identified with vector or higher rank (contravariant) tensor variables and therefore represent reparametrizations, see §2.4.

Finally, if \(q > 3\), the residue (2.32a) would seem to contribute to non-existent cohomology groups on the 3-fold \(\mathcal{M}\). However, such residues either occur in pairs and together with maps of the type (2.26) such that both the domain and the image ‘gauged each other away’, or such residues end up being ‘gauged away’ through coordinate reparametrizations (see below). Moreover, it can be shown as in Ref. [29], that even the case \(q = 3\) can be avoided, except of course for the holomorphic 3-form (1.1) in the degree-0 case. That is, complete intersections where there exist ‘additional’ non-zero residues with \(q = 3\) turn out to be completely equivalent to others where no such residue occurs.

The residue is evaluated as in §2.2, and takes the final form

\[
\text{Res}^S_{\delta f} = \Omega^{(q)}(\delta f)^j(\bar{m})(x),
\]

where \(\Omega^{(q)}\) is defined in (2.23), and \(m_i\) and \(\partial\bar{m}\) are as in (2.22). Note that \(m_i > 0\) for at least one \(P_i\) (otherwise, the complete intersection configuration (2.1) would become block-diagonal and so correspond to either \(T^2 \times K3\) or \(T^6\), whence the variation of the \(\bar{m}^{th}\) gradient, \(\delta f^j(\bar{m})\), is of too small a degree to contribute by itself to \(H^\star(\mathcal{M}, \mathcal{E}_f)\). A ‘missing piece’, i.e., a certain universal quantity of the appropriate ‘missing’ degree is needed for constructing a proper contribution to \(H^\star(\mathcal{M}, \mathcal{E}_f)\). When comparing with Landau-Ginzburg orbifold results, this ‘missing piece’ is identified with the twisted vacuum, the charges of which indeed complement the charges of the monomials \(\delta f^j(\bar{m})(x)\) so as to provide marginal, charge-\((\pm 1, 1)\) states. When comparing with the Koszul calculation, this ‘missing piece’ is identified with (a product of) Levi-Civita alternating symbols \(\epsilon\), which in turn precisely corresponds to the twisted vacua [8]. In some cases at least, this ‘missing piece’ may be identified with a radical monomial [8,15]; more about this below.

To summarize, all the residue kernels are determined completely from the ‘ghost-for-ghost...’ sequence such as the one in the upper left quadrant of the chart (2.17): each step to the left implies division by one of the defining polynomials; these reciprocals of polynomials are then multiplied by the (covariant) top-differentials of the embedding space factors so as to enable a residue integral to calculate the residue. Integrating variations of the defining polynomials using these kernels, we form a rack of polynomial-valued residues [placed into the lower left quadrant of (2.17)], contributing thus to \(H^\star(\mathcal{M}, \oplus_j \mathcal{E}_f)\), and thereby to \(H^\star(\mathcal{M}, \mathcal{T})\); see below.
2.4. Reparametrizing residues

The erudite reader will undoubtedly have protested by now that the contributions obtained thus far are overabundant, simply on account of not having discounted the coordinate reparametrization degrees of freedom. Of course, the situation with the polynomial deformations (2.27) is quite well understood [26,29,3]. The same Jacobian ring structure (=polynomials modulo the ideal of gradients) shows up again in the Landau-Ginzburg orbifold analysis [4,10,6,5] and then in a somewhat modified form also for the $E_6$'s [5]. Let us therefore turn to the action of coordinate reparametrizations on the residues discussed above.

Coordinate reparametrizations. The action of coordinate reparametrizations of $\mathbb{P}^n$ on functions over $\mathbb{P}^n$ is generated by the operators $\ell^\mu(x)\partial_\mu$, where $\ell^\mu(x)$ is an $(n+1)$-vector of linear combinations of the homogeneous coordinates on $\mathbb{P}^n$; write $\ell^\mu(x) = x^\nu\lambda_\nu^\mu$. The trace part of the matrix $\lambda_\nu^\mu$ is easily seen to correspond to (a complex multiple of) the Euler homogeneity operator; when acting on homogeneous functions, the trace part of $\lambda_\nu^\mu$ then merely duplicates the overall rescalings of the kind already accounted for by imposing equivalence relations such as (2.27). Therefore—when acting on homogeneous functions—the matrix $\lambda_\nu^\mu$ will be required to be traceless. On the product $\mathcal{X} = \bigoplus_i \mathbb{P}^n_i$, the reparametrization group is the direct product of the individual reparametrization groups, and so is generated by $\bigoplus_i \ell^{\nu_i}(x_i)\partial_{\nu_i}$. Notably, all these reparametrization operators (tangent vectors) are homogeneous and of degree zero.

For the polynomial deformations $\delta f^j(x_i)$, i.e., the zeroth level residues, the resulting equivalence may be written (on $\mathcal{M}$) as

$$
\delta f^j(x) \simeq \delta f^j(x) + \sum_{\nu_i} N \sum_{\ell^{\nu_i}(x_i)\partial_{\nu_i}} f^j(x) .
$$

These equivalence relations may be used to eliminate (‘gauge away’) $\sum_{i=1}^N n_i(n_i + 2)$ parameters from (2.27), the remaining ones then representing the reparametrization class.

On comparing this equivalence relation (2.34) with the earlier (2.7) and (2.11), a mapping notation akin to (2.12) immediately suggests itself:

$$
\bigoplus_{i=1}^N \ell^{\nu_i}(x_i)\partial_{\nu_i} \overset{df^j}{\longrightarrow} \delta f^k(x_i) ,
$$

where the $\ell\cdot\partial$ is regarded as a (tangent) vector field [14], and is mapped to holomorphic functions of degree $\text{deg}(f^j)$. The two groups of mappings, (2.12) and (2.35), provide

\[14\] The contraction $\text{d}x^\mu,\partial_\nu = \delta_\nu^\mu$ cancels the differential against the derivative, so that $\ell^{\nu_i}\partial_{\nu_i}\text{d}x^\mu_j\partial_{\mu_j}f^k = \ell^{\nu_i}\partial_{\nu_i}f^k = \delta f^k$ produces a linear reparametrization of $f^k$. 

\[22\]
the very core of the Koszul calculations. Also, the equivalence relation (2.11) may be regarded as a special case of (2.34), corresponding to the trace part of $\lambda^\mu_i x^\nu_i$. Therefore, both equivalence relations may be regarded as generated by the gradients of the defining polynomials, whence the immediate connection to the results of Ref. [26,4,10,6]. Indeed, the effective identity of the ideals was observed in Ref. [7]; here we see the source of this.

The natural analogue of this action of coordinate reparametrizations to the more general residues such as those in (2.32c) is straightforward upon the following realization. The previous subsection showed that the residue operator entails the natural restriction of holomorphic functions (polynomials) to the subspace $M \subset X$. The same then better be true also of vector fields. Note that $\partial_{\nu_i}$ may continue to serve as a basis for tangent vectors; it is the coefficients $\ell^{\nu_i}(x_i)$ which will be restricted to the submanifold through residue integrals. Without much ado, then,

$$\oint \ldots \oint \prod_{k \in Q} \Gamma(f^k) \frac{\prod_{i \in S} (x_i d^{n_i} x_i)}{\prod_{k \in Q} f^k} \sum_{\nu_i} \ell^{\nu_i}(x) \partial_{\nu_i} = \sum_{\nu_i} \text{Res}^S_Q[\ell^{\nu_i}] \partial_{\nu_i} , \tag{2.36}$$

with the $\text{Res}^S_Q[\ell^{\nu_i}]$ holomorphic or zero, is the appropriate restriction of the tangent vectors to the subspace $M$, and so the natural reparametrization operator for the residue (2.32c).

Indeed, this operator is obtained with the same residue kernel as (2.32c), whereby the coordinate reparametrization action is perfectly analogous to (2.34), except that all terms are placed within the residue symbol. This also means that the operations of linear reparametrization and evaluation of the residue commute. In fact, requiring that this always be so prevents the residue reparametrization operators from acting on (residue) functions unless they are both of the same residue level and moreover have the same residue kernel. This ‘selection rule’ is the residue analogue of the $j$-twisting selection rule in Landau-Ginzburg orbifolds. The effective identity of these ‘selection rules’ of rather disparate origins seems to be borne out in practice, although we are aware neither of a rigorous proof nor of a counter-example. (See however § 4.)

To summarize, we have the residue analogue of the coordinate reparametrization equivalence relation (2.34):

$$\text{Res}^S_Q[\delta f^j(x)] \cong \text{Res}^S_Q[\delta f^j(x)] + \sum_{k, \nu_i} \text{Res}^S_Q[\ell^{\nu_i}](\partial_{\nu_i} f^j(x)) , \tag{2.37}$$

where all the terms have their integral expressions, given above. In the ‘mapping’ notation:

$$\bigoplus_{i=1}^{N} \text{Res}^S_Q[\ell^{\nu_i}] \partial_{\nu_i} \xrightarrow{\delta f^j} \text{Res}^S_Q[\delta f^k(x_i)] , \tag{2.38}$$

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Thus, the holomorphic (tangent) vector fields $\text{Res}_0^S[\ell^{\nu_i}]\partial_{\nu_i}$ serve to eliminate (‘gauge away’) some of the parameters in (2.32), the residue symbol again accounting for the proper restriction to the subspace $\mathcal{M}$. The mapping (2.35) and the equivalence relations (2.34) are now clearly the special ‘zeroth level’ case ($|Q| = 0 = |S|$) of (2.33) and (2.37), respectively; the maps themselves, i.e., generators of the equivalence relations remain the same: $f^j$ and $df^j$, i.e., (linear combinations of) the gradients of $f^j$.

This, however, is not the whole reparametrization story. It is possible to list all the residue kernels, racking them in a chart such as (2.17), just as was done above for the polynomial-valued residues. Recall that the degree of homogeneity of the linear reparametrization operators is $\tilde{\nu}$, and note that the original paradigm (1.1) may be regarded as the residue of the constant 1, with the residue kernel being $\prod_{i=1}^{N} (x_i d^{n_i} x_i) / \prod_{j=1}^{K} f^j$. We thus immediately turn to

\[
\oint_{\Gamma_{f^1}} \ldots \oint_{\Gamma_{f^K}} \prod_{i=1}^{N} (x_i d^{n_i} x_i) / \prod_{j=1}^{K} f^j (x) \sum_{\mu_j, \nu_j} x^{\mu_j}_j \lambda^{\nu_j}_j \partial_{\nu_j} ,
\]

and note that the canonical contractions

\[
\partial_{\nu_j} dx^{\mu_j}_i = \delta^{j}_j \delta^{\mu_j}_j ,
\]

lower the overall degree of the differential by one and produce (upon taking the residues) $N$ independent 2-forms of homogeneity $\tilde{\nu}$; one corresponding to each of $\mathbb{P}^{n_i}$. Since $\sum_{\mu_j, \nu_j} x^{\mu_j}_j \lambda^{\nu_j}_j \partial_{\nu_j}$ are local tangent vectors, these 2-forms are tangent bundle valued. The (Serre-) duals of these are then cotangent bundle valued 1-forms, that is, (1,1)-forms.

The trace-part of each matrix $\lambda^{\nu_j}_j$ contributes precisely nothing, in virtue of the skew-symmetry of $(x_i d^{n_i} x_i)$, with which $x^{\mu_j}_j \lambda^{\nu_j}_j \partial_{\nu_j}$ become contracted via (2.40):

\[
(x d^n x) \cdot x^\mu_\mu \lambda^{\nu_j}_j \partial_\nu = \frac{1}{(n+1)!} \epsilon_{\rho_0 \ldots \rho_n} x^{\rho_0} \ldots x^{\rho_n} \cdot x^\mu \left( \frac{1}{n} \delta^{\nu}_\mu \text{Tr}[\lambda] + \hat{\lambda}^{\nu}_\mu \right) \partial_\nu ,
\]

\[
= \frac{\text{Tr}[\lambda]}{n (n+1)!} \epsilon_{\rho_0 \ldots \rho_j \ldots \rho_n} x^{\rho_0} x^{\rho_j} \ldots x^{\rho_n} \cdot \underbrace{x^{\rho_1} \ldots \hat{x}^{\rho_j} \ldots x^{\rho_n}}_{=0} ,
\]

\[
+ \frac{1}{(n+1)!} \epsilon_{\rho_0 \ldots \rho_j \ldots \rho_n} x^{\rho_0} \hat{\lambda}^{\rho_j}_\mu x^\mu \cdot \underbrace{x^{\rho_1} \ldots \hat{x}^{\rho_j} \ldots x^{\rho_n}}_{\neq 0} ,
\]

where we have suppressed the subscript $i$ for clarity and the ‘hat’ labels omitted factors; the traceless part of matrix $\hat{\lambda}^{\nu_j}_j$ does contribute. However, unlike the other $\mathcal{T}_X$-valued residues which act as reparametrizations of some polynomial-valued residue or another, these residues typically ‘stand by themselves’ as there is typically nothing they could act on (see however below). All of them are $\mathcal{T}_X$-valued 2-forms on the 3-fold $\mathcal{M}$ — one set, parametrized by $\hat{\lambda}^{\nu_j}_j$, for each factor $\mathbb{P}^{n_i}$. It may seem as each of these would depend on
each of the off-diagonal matrix elements, but this is not so: each one such element is merely a coordinate-reparametrized copy of another. That is, modulo coordinate reparametrizations themselves, the \( \binom{n_i+1}{2} \) various possible choices are all equivalent, whence such residues produce precisely one such 2-form for each factor \( \mathbb{P}^{n_i} \subset X \). Tangent-bundle valued 2-forms being dual on a Calabi-Yau 3-fold to (1,1)-forms, and there being precisely one per each factor \( \mathbb{P}^{n_i} \)— each of the residues (2.39) corresponds to (the dual of the pullback of) the Kähler form on each of the \( \mathbb{P}^{n_i} \)'s!

We employ here the standard results [32,3] which enable us to identify the kernel and the cokernel of the map (2.38):

\[
H^q(Q, T_Q) \xrightarrow{\iota} \bigoplus_{i=1}^N \text{Res}^S_{Q}[\ell^\nu_i] \partial_{\nu_i} \xrightarrow{df^j} \text{Res}^S_{Q} [\delta f^k(x_i)] \xrightarrow{\Delta} H^{q+1}(Q, T_Q),
\]

where \( q = \sum_{i\in S} n_i - |Q| \), and \( Q \) is the simultaneous zero-set \( f^j=0, j \in Q \); ultimately, once \( Q \) includes all constraints, \( Q \) becomes the desired complete intersection \( \mathcal{M} \). In this extension of (2.38), \( i \) identifies with elements of \( H^q(Q, T_Q) \) those elements of \( \text{Res}^S_{Q}[\ell^\nu_i] \partial_{\nu_i} \) which are annihilated by \( df^j \) (a.k.a. the kernel of \( df^j \)). Also, the equivalence class of \( \text{Res}^S_{Q} [\delta f^k(x_i)] \) modulo the \( df^j \)-multiples of \( \text{Res}^S_{Q}[\ell^\nu_i] \partial_{\nu_i} \) (a.k.a. the cokernel of \( df^j \)) contributes, via the ‘degree-changing map’ \( \Delta \), to \( H^{q+1}(Q, T_Q) \).

Equivalently, we may consider, dually, the cotangent bundle valued residues, replacing \( \sum_{\nu_j} x_j^{\mu_j} \lambda_{\mu_j} \partial_{\nu_j} \) in the above calculations with

\[
\sum_{\nu_i} (\gamma_i)_{\nu_i}(x) dx_i^{\nu_i}, \quad i = 1, \ldots, N.
\]

(2.43)

For these to be of homogeneity \( \vec{0} \), the function \( (\gamma_i)_{\nu_i}(x) \) and must have degree \(-1\) over \( \mathbb{P}^{n_i} \), but be constant over the remaining factors in the embedding space. It is easy to see that only the ‘diagonal’ rational differentials have a residue along corresponding linear hypersurfaces in \( \mathbb{P}^{n_i} \). On each factor \( \mathbb{P}^{n_i} \) and upon coordinate reparametrizations, such ‘diagonal’ rational (in fact, logarithmic) differential forms may be written as, say, \( \frac{dx_i^0}{x_i^0} = d\log(x_i^0) \) and the corresponding linear hypersurface \( (\approx \mathbb{P}^{n_i-1}) \) where the residue is supported is defined by \( x_i^0 = 0 \). The \( \frac{dx_i^0}{x_i^0} \) are then the de Rham duals of hyperplanes in \( \mathbb{P}^{n_i} \), i.e., representatives of the Kähler classes on \( \mathbb{P}^{n_i} \), and may also serve as their pull-backs on \( \mathcal{M} \). We will not pursue this alternative calculation here any further.

Back to the \( T_X \)-valued residues, it is a tedious but straightforward exercise to show in a case by case analysis, that no additional non-zero residue contribution to the cohomology on \( \mathcal{M} \) can occur. In the dual calculation with (2.43), the explicit appearance of some particular \( dx_i^{\nu_i} \) precludes multiplication by \( (x_i d^{n_i} x_i) \), whereas multiplication by some \( (x_j d^{n_j} x_j) \),
$j \neq i$, does not help produce a contour integral to pick out the residue — except for the cases which are duals to the $\mathcal{T}_X$-valued residues listed above.

A detailed comparison then with the Koszul computation is again straightforward, but rather laborious and will not be presented here. It however reinforces that the residue map (symbol) naturally provides a concrete realization of the Koszul calculations. Simple examples such as this one certainly can be analyzed by simply listing all possible residue kernels, all relevant rational/radical polynomials, and then determining the final list of effective deformations. In general, however, this approach quickly gets out of hand and the application of the standard Koszul machinery seems difficult to avoid. The residue map can then be used to provide a concrete realization of any particular cohomology element.

Finally, although they may be combined, note that the gauge-equivalence in (2.34) is different from that one in (2.7). The former is a consequence of the global symmetry $\text{PGL}(n+1, \mathbb{C}) \approx \text{GL}(n+1, \mathbb{C})/\mathbb{C}^*$ of projective spaces. That is, in the underlying 2-dimensional field theory as studied by Witten [4], there is a $\text{GL}(n+1, \mathbb{C})$ global field reparametrization symmetry, of which the diagonal (projective) $\mathbb{C}^*$ is gauged. The equivalence relation (2.7), however, stems from the imposition of constraints $f^j(x_i) = 0$, from which the whole construction of the Koszul sequence (2.15) and the subsequent calculations (2.17) are developed. Where appropriate, we have indicated the corresponding BRST-type treatment of such constraints, but are not specifying here the details of this approach any further.

Too many Kähler forms. Finally, it remains to discuss a type of ‘reparametrizations’ which can occur only when two or more hypersurface is being intersected to define $\mathcal{M}$, the submanifold under study, and then only for $H^2(\mathcal{M}, \mathcal{T})$. Such models do not have a straightforward Landau-Ginzburg orbifold analogue la Refs. [4,6,10].

In certain configurations (2.1) 

\[ M \in \mathbb{P}^3 \left[ \begin{array}{cc} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{array} \right], \quad \left\{ \begin{array}{l} f(x, z) = 0, \text{ of degree } (4,0,1), \\ g(y, z) = 0, \text{ of degree } (0,2,1), \end{array} \right. \]  

(2.44)

where the generic manifold has $b_{1,1} = 2$, $b_{2,1} = 86$ and $\chi_E = -168$.

The ghost-for-ghost- sequence is

\[ O(p-4, q-2, r-2) \Rightarrow O(p, q-2, r-1) \Rightarrow O(p, q, r) \Rightarrow O_M(p, q, r), \]  

(2.45)

\[ ^{15} \text{These are configurations which have a “decomposing 2-leg”, see [29] or § 2.1.2 of [3].} \]
and the full rack of $\delta f$- and $\delta g$-valued residues and similarly the $T_x$, $T_y$, and $T_z$-valued residues are easy to calculate along the lines described above. The novel feature occurs when accounting for all the equivalence relations of the type (2.37), that is, of all the maps of the type (2.38). In particular, there are four contributions to $H^2(M, T_X)$ for the above example:

$$
\text{Res}^x_{f,g} \left[ \vartheta_x \right], \quad \text{Res}^x_{f,g} \left[ \vartheta_y \right], \quad \text{Res}^x_{f,g} \left[ \vartheta_z \right], \quad \text{Res}^x_{f,g} \left[ \vartheta_s \right],
$$

(2.46)

and

$$
\text{Res}_{f}^{x} \left[ \vartheta_z \right] = \text{Res}_{f}^{x} \left[ z^r \lambda^s \right] \partial_s = \Lambda^s \partial_s,
$$

(2.47)

where

$$
\vartheta_x \overset{\text{def}}{=} x^a \lambda^a \partial_b, \quad \vartheta_y \overset{\text{def}}{=} y^a \lambda^a \partial_b, \quad \vartheta_z \overset{\text{def}}{=} z^r \lambda^s \partial_s.
$$

(2.48)

This last contribution (2.47) provides a straightforward (2-parameter) reparametrization of $\text{Res}_f^{z} \left[ \delta g \right]$, just as in (2.37). Simply on account of the degrees, we can fix this to be

$$
\text{Res}_{f}^{x} \left[ \delta g(y, z) \right] \equiv \text{Res}_{f}^{x} \left[ \delta g(y, z) \right] + \text{Res}_{f}^{x} \left[ z^r \lambda^s \right] \left( \partial_s g(y, z) \right),
$$

(2.49)

and note that $\text{Res}_{f}^{x} \left[ \delta g \right]$ is a quadric over $\mathbb{P}^1_y$, which we may write as $\delta g'_s$. Modulo the two gradients $\lambda^s(\partial_s g)$, this produces a 1-parameter equivalence class, which we may write as $\{\delta g'_s/g'_s\}$. More precisely, as in (2.32d), we can write this in more detail as

$$
\begin{align*}
\text{Res}_{f}^{x} \left[ \delta g(y, z) \right] & = \left[ \frac{1}{\pi^x} \epsilon_{abcd} \frac{dx^c dx^d}{\partial_s \partial_b f(x, z)} \right] \delta(\partial_s g(y, z)) \overset{\text{def}}{=} \Delta^s \delta(\partial_s g(y, z)) , \\
\text{Res}_{f}^{x} \left[ z^r \lambda^s \right] & = \left[ \frac{1}{\pi^x} \epsilon_{abcd} \frac{dx^c dx^d}{\partial_r \partial_b f(x, z)} \right] \lambda^s \overset{\text{def}}{=} \Lambda^s,
\end{align*}
$$

(2.50a, 2.50b)

where the quantities in the square brackets, $\Delta^s$ and $\Lambda^s$, are nowhere-zero 2-forms on the Calabi-Yau quartics in $\mathbb{P}^3_x$, $\{\partial_s f(x, z) = 0\}$, one for each $s = 1, 2$. $\delta(g'_s) = \delta(\Delta^r \partial_r g(y, z))$ being a quadric over $\mathbb{P}^2_y$, it depends on $\binom{3}{2} = 3$ parameters and $g'_s = (\Lambda^r \partial_r g(y, z))$ providing two gauge degrees of freedom, $\Lambda^r$, the quotient $\{\delta(g'_s)/g'_s\}$ is 1-dimensional.

Having resolved this equivalence relation, we would seem to remain with three elements in $H^2(M, T_X)$—the duals of (the pullbacks of) the Kähler forms (2.40). Also, the 1-dimensional cohomology group of polynomial-valued 2-forms $\{\delta(g'_s)/g'_s\}$, should then produce a 1-dimensional contribution to $H^3(M, T) = H^{2,3}(M)$ much as polynomial-valued 0-forms contribute to $H^1(M, T)$; see (2.12). However, on general grounds, we know that $H^{2,3}(M) = 0$, whence precisely one linear combination of the three residues (2.46) must provide one last equivalence relation so as to gauge away $\{\delta(g'_s)/g'_s\}$ completely. In the process, we will remain with only two independent elements for $H^2(M, T_X)$, and will therefore have also $\dim H^2(M, T_M) = 2$, i.e., $b_{2,2} = b_{1,1} = 2$. 

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To see from where such a map is induced, note that the $\vartheta_x$- and $\vartheta_z$-valued residues may be written in a ‘cascade’ fashion:

\[
\text{Res}^x_{f,g} [\vartheta_x \oplus \vartheta_z] = \text{Res}^y_f [\text{Res}^z_g [(zd\vartheta_x \oplus zd\vartheta_z)]].
\] (2.51)

That is, the $\Gamma(g)$-contour integral may be calculated first, over $\mathbb{P}^1_y$, which produces a quantity of degree $(0,0,1)$ and so is a well-defined ‘partial residue’ of the type (2.18). This would not be possible for $\text{Res}^x_{f,g} [\vartheta_y]$, since the $\partial_y$ contracts with $d\vartheta$ and we cannot integrate over $\mathbb{P}^1_y$ to define the ‘inside’ residue. Therefore, the equivalence relation (2.49) must be enlarged, so as to correspond to the combined map

\[
\text{Res}^x_f [\text{Res}^y_g [(zd\vartheta_x \oplus zd\vartheta_z)]] \xrightarrow{\text{d}g} \text{Res}^x_f [\delta g(y,z)],
\] (2.52)

where the maps have been determined solely from degrees of homogeneity. Note that all these contributions happen within the $\text{Res}^x_f [\ ]$ operator, and we can therefore determine the equivalence relation corresponding to the upper map from considering the ‘inner’ $\text{Res}^y_g [\ ]$. Using that

\[
(zd\vartheta) \cdot \vartheta_z = \frac{1}{2} \varepsilon_{pq} z^p d\vartheta_q \cdot z^r \lambda_s^r \partial_s = \frac{1}{2} \varepsilon_{ps} z^p z^r \lambda_s^r,
\] (2.53)

we note that, using (2.32),

\[
\text{Res}^y_g [(zd\vartheta) \cdot \vartheta_z] = \left[ \frac{\varepsilon_{\alpha \beta} y^\beta}{\partial_x \partial_\alpha \vartheta_0(y,z)} \right]_{g=0} (\lambda_s^r \varepsilon_{q}s z^q) = Z(z),
\] (2.54)

is constant over $\mathbb{P}^3_x$, nowhere-zero over $\{\partial_x, \partial_\alpha \vartheta=0\} \subset \mathbb{P}^1_y$, and linear over $\mathbb{P}^1_z$, so that upon multiplication with $\partial_z g$, it can be added to $\delta g$ within the $\text{Res}^x_f [\ ]$ symbol in the target of the maps in (2.52). The equivalence relation (2.49) therefore becomes enlarged to

\[
\text{Res}^x_f [\delta g(y,z)] \cong \text{Res}^x_f [\delta g(y,z)] + \text{Res}^x_f [z^r \lambda_s^r + Z^s(z)] (\partial_s g(y,z)),
\] (2.55)

where $Z^s = \varepsilon^{rs} \lambda^p_{(r} \varepsilon_q s z^q = z^r (\lambda^p - \frac{1}{2} \delta^p_s \text{Tr}[\lambda]), s = 1, 2$, are two linearly independent terms in the linear function obtained as the residue (2.54). This then provides for completely ‘gauging away’ the contribution to polynomial-valued 2-forms, whence we recover $H^3(\mathcal{M}, T) = H^{2,3}(\mathcal{M}) = 0$ and also $b_{2,2} = b_{1,1} = 2$, since only $\text{Res}^x_{f,g} [\vartheta_x]$ and $\text{Res}^x_{f,g} [\vartheta_y]$ remain from (2.49).
This residue calculation is in complete agreement with the Koszul calculation, even to the extent that there is a formal 1–1 correspondence in the form of the representatives. For example, the fact that all the representatives in (2.52) occur within \( \text{Res}_x \left[ \right] \) is paralleled by the fact that all Koszul representatives corresponding to those in (2.52), occur with an overall factor of \( \epsilon^{abcd} \), the Levi-Civita alternating symbol on \( \mathbb{P}_3 \). Indeed, \( \text{Res}_x \left[ \right] \) involves integration over the 3-form \((xd^3x)\), which in turn includes \( \epsilon^{abcd} \) in the definition (1.2).

Moreover, a similarly detailed 1–1 correspondence can be established between all of the residue and Koszul representatives, whereby we believe to have demonstrated the effective identity between these two methods. Our goal will now be to see if the residue calculations might be extended beyond what is known about the Koszul calculations and also to explore whether the similarly detailed agreement with the Landau-Ginzburg orbifold methods persists beyond the overlap with the Koszul calculations, where the identity follows owing to earlier results \([7,8]\). Before that, however, a few examples are perhaps in order, to illustrate the various types of ‘higher order’ contributions.

3. A Representative Ragout

3.1. A reconnoissance residue raffle

Consider the ‘warped’ model of Ref. \([7]\):

\[
\mathcal{M} \in \mathbb{P}^3 \left[ \begin{array}{ccc} 3 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{array} \right] : \begin{cases} f(x) = 0 , \text{ of degree (3,0,0),} \\ g(x,y) = 0 , \text{ of degree (1,2,0),} \\ h(x,y) = 0 , \text{ of degree (0,1,2),} \end{cases} \tag{3.1}
\]

where the generic manifold \( \mathcal{M} \) has \( b_{1,1} = 9 \), \( b_{2,1} = 33 \) and \( \chi_E = -48 \). The ghost-for-ghost-... sequence is

\[
\begin{align*}
\mathcal{O} \left( \frac{p-4}{q-3} \right) \quad &\Rightarrow \quad \mathcal{O} \left( \frac{p-1}{q-2} \right) \\
\mathcal{O} \left( \frac{p-3}{q-3} \right) \quad &\Rightarrow \quad \mathcal{O} \left( \frac{p-1}{q-2} \right) \\
\mathcal{O} \left( \frac{p-4}{q-2} \right) \quad &\Rightarrow \quad \mathcal{O} \left( \frac{p-3}{q} \right)
\end{align*}
\]

\[
\Rightarrow \mathcal{O}_\mathcal{M} \left( \frac{p}{q} \right) , \tag{3.2}
\]

where we listed the degrees vertically, akin to (3.1).

Among the polynomial-valued residues, at zero level, there are

\[
\delta f(x) , \quad \delta g(x,y) , \quad \delta h(y,z) , \tag{3.3a}
\]
which are the usual polynomial deformations, and
\[
\frac{\delta f(x)}{f(x)} \to \lambda_f, \quad \frac{\delta g(x, y)}{g(x, y)} \to \lambda_g, \quad \frac{\delta h(y, z)}{h(y, z)} \to \lambda_h, \quad (3.3b)
\]
each of which is holomorphic only if the variation is chosen to be proportional to the polynomial in the denominator, whence the ratios are constants: \( \lambda_f, \lambda_g, \lambda_h \). From the \( \mathcal{T}_X \)-valued residues, at zero level, there are
\[
\theta_b \overset{\text{def}}{=} x^\alpha \lambda_0^b \partial_b, \quad \theta_y \overset{\text{def}}{=} y^\alpha \lambda_\alpha \partial_\beta, \quad \theta_z \overset{\text{def}}{=} z^r \lambda_r \partial_s, \quad (3.3c)
\]
which generate the usual coordinate reparametrizations. The matrices \( \lambda \) being traceless, they combine with (3.3b) to produce the familiar equivalence class [26]
\[
\left[ \begin{array}{c} \delta f(x) \\ \delta g(x, y) \\ \delta h(y, z) \end{array} \right] \cong \left[ \begin{array}{c} \delta f(x) \\ \delta g(x, y) \\ \delta h(y, z) \end{array} \right] + \left( x^\alpha \lambda_0^b \partial_b + y^\alpha \lambda_\alpha \partial_\beta + z^r \lambda_r \partial_s \right) \left[ \begin{array}{c} f(x) \\ g(x, y) \\ h(y, z) \end{array} \right], \quad (3.4)
\]
where now the \( \lambda \) matrices are no longer traceless. These produce the well-known polynomial deformation contributions to \( H^1(M, \mathcal{T}_M) \).

Among higher-level polynomial-valued residues, we find nonzero only:
\[
\frac{\delta g(x, y)}{h(y, z)} \to \text{Res}^y_h[\delta g] = \Delta^\alpha \delta(\partial_\alpha g_y) \sim \delta g'_y, \quad \deg = (1, 1, 0), \quad (3.6a)
\]
\[
\frac{\delta f(x)}{g(x, y) h(y, z)} \to \text{Res}^{y,z}_{g,h}[\delta f] = \Delta^\alpha \delta(\partial_\alpha f), \quad \deg = (2, 0, 0), \quad (3.6b)
\]
where \( \Delta^\alpha \) and \( \Delta^a \) are the nowhere-zero 0- and 1-forms calculated from the respective residues in the coordinate patch where, e.g., \( y^0, z^0 = \text{const.} \)
\[
\Delta^\alpha \overset{\text{def}}{=} \left[ \begin{array}{c} z^0 \\ \partial_\alpha J(0) \end{array} \right]_{h=0}, \quad J(0) \overset{\text{def}}{=} \left[ \begin{array}{c} \partial h \\ \partial z^1 \end{array} \right]_{z^0 \neq 0}, \quad (3.7a)
\]
\[
\Delta^a \overset{\text{def}}{=} \left[ \begin{array}{c} y^0 z^0 \partial_\eta \\ \partial_\alpha J(0,0) \end{array} \right]_{h=0}, \quad J(0,0) \overset{\text{def}}{=} \left[ \begin{array}{c} \partial(\eta, g, h) \\ \partial(y^1, y^2, z^1) \end{array} \right]_{y^0, z^0 \neq 0}. \quad (3.7b)
\]
Among higher-level \( \mathcal{T}_X \)-valued residues, we find nonzero only:
\[
\frac{y^\alpha \lambda_0^\beta \partial_\beta}{h(y, z)} \to \text{Res}^y_h[\partial_y] = A^\beta \partial_\beta, \quad \deg(A^\beta) = (0, 0, 0), \quad (3.8a)
\]
\[
\frac{x^\alpha \lambda_0^b \partial_b}{g(x, y) h(y, z)} \to \text{Res}^{y,z}_{g,h}[\partial_x] = A^b \partial_b, \quad \deg(A^b) = (0, 0, 0), \quad (3.8b)
\]
and the three duals of (the pullbacks of) the Kähler forms:

\[
\text{Res}_{f,g,h}^x, y, z [\vartheta_x] , \quad \text{Res}_{f,g,h}^x, y, z [\vartheta_y] , \quad \text{Res}_{f,g,h}^x, y, z [\vartheta_z] ,
\]

(3.8c)

where

\[
\vartheta_x = \def{}{x^\alpha \lambda^\beta \partial_{\beta} ,} \quad \vartheta_y = \def{}{y^\alpha \lambda^\beta \partial_{\beta} ,} \quad \vartheta_z = \def{}{z^r \lambda^s \partial_{s} ,}
\]

(3.9)

and the matrices \( \lambda \) are traceless. The \( A^\beta \) and \( A^b \) are the nowhere-zero 0- and 1-forms calculated from the respective residues in the coordinate patch where, e.g., \( y^0, z^0 = \text{const} \).

\[
A^\beta = \def{}{\left[ z^0 \frac{\partial J(0)}{\partial \alpha} \lambda^\beta \right]_{h=0} ,} \quad A^b = \def{}{\left[ y^0 z^0 \frac{d \eta}{\partial \alpha} \lambda^b \right]_{q=0,h=0} ,}
\]

(3.10)

and where \( \partial_{\alpha} J(0) \) and \( \partial_{\alpha} J^n(0,0) \) are the same as in (3.7). For example, the 1-forms \( \Delta^a \) and \( \Lambda^a \) are both nowhere zero holomorphic 1-forms on the torus embedded as the simultaneous zero-set \( \partial_{a} g(x,y) = 0 = h(y,z) \) in \( \mathbb{P}^2 \times \mathbb{P}^1 \), i.e., on a member of \( \mathbb{P}^2[2:1] \).

To summarize, the degree-(3,0,0) polynomial-valued cohomology is obtained from:

\[
\begin{array}{cccc|c}
\mathcal{O} \left( \begin{array}{c}
-2 \\
-3 \\
-2
\end{array} \right) & \rightarrow & \mathcal{O} \left( \begin{array}{c}
3 \\
1 \\
-2
\end{array} \right) \\
\mathcal{O} \left( \begin{array}{c}
-1 \\
-3 \\
-2
\end{array} \right) & \rightarrow & \mathcal{O} \left( \begin{array}{c}
0 \\
-1 \\
-2
\end{array} \right) & \rightarrow & \mathcal{O}_x \left( \begin{array}{c}
3 \\
0 \\
0
\end{array} \right) \\
\mathcal{O} \left( \begin{array}{c}
-1 \\
-2 \\
0
\end{array} \right) & \rightarrow & \mathcal{O} \left( \begin{array}{c}
0 \\
0 \\
0
\end{array} \right)
\end{array}
\]

\[
\Rightarrow \mathcal{O}_M \left( \begin{array}{c}
3 \\
0 \\
0
\end{array} \right)
\]

(3.11)

\[
0 \quad 0 \quad \lambda_f \quad \delta f \quad \Rightarrow \{ \delta f / f \cdot \lambda_f \} \in H^0
\]

\[
0 \quad 0 \quad 0 \quad 0 \quad \Rightarrow \Delta^a \delta (\partial_a f) \in H^1
\]

\[
0 \quad 0 \quad 0 \quad 0 \quad \Rightarrow H^2 = 0
\]

\[
0 \quad \Delta^a \delta (\partial_a f) \quad 0 \quad 0 \quad \Rightarrow H^3 = 0
\]

\[
0 \quad 0 \quad 0 \quad 0 \quad \Rightarrow H^4(\mathcal{M}) \equiv 0
\]

\[
0 \quad 0 \quad 0 \quad 0 \quad \Rightarrow H^5(\mathcal{M}) \equiv 0
\]

\[
0 \quad 0 \quad 0 \quad 0 \quad \Rightarrow H^6(\mathcal{M}) \equiv 0
\]

The residue \( \Delta^a \delta (\partial_a f) \) is defined in Eq. (3.62).
the degree-$(1,2,0)$ polynomial-valued cohomology from:

\[
\begin{array}{c|c}
\mathcal{O} \left( \begin{array}{c} 0 \\ -1 \\ -2 \end{array} \right) & \mathcal{O} \left( \begin{array}{c} 1 \\ -1 \\ -2 \end{array} \right) \\
\mathcal{O} \left( \begin{array}{c} -3 \\ -1 \\ -2 \end{array} \right) & \mathcal{O} \left( \begin{array}{c} -2 \\ 0 \\ -2 \end{array} \right) \\
\mathcal{O} \left( \begin{array}{c} -3 \\ 0 \\ 0 \end{array} \right) & \mathcal{O} \left( \begin{array}{c} -2 \\ 0 \\ 0 \end{array} \right)
\end{array}
\Rightarrow \mathcal{O}_M \left( \begin{array}{c} 1 \\ 2 \end{array} \right)
\]

(3.12)

\[
\begin{array}{c|c}
0 & \lambda_g \\
0 & \delta g \\
0 & \delta g_y' \\
\vdots & \vdots
\end{array}
\Rightarrow \{\delta g/(g \lambda_g)\} + \delta g_y' \in H^0
\]

\[
\begin{array}{c|c}
0 & \delta g_y' \\
0 & 0 \\
0 & 0 \\
\vdots & \vdots
\end{array}
\Rightarrow H^1 = 0
\]

\[
\begin{array}{c|c}
0 & 0 \\
\vdots & \vdots
\end{array}
\Rightarrow H^2 = 0
\]

The contribution \(\delta g_y'\) is defined in Eq. (3.6a).

and the degree-$(0,1,2)$ polynomial-valued cohomology from:

\[
\begin{array}{c|c}
\mathcal{O} \left( \begin{array}{c} -1 \\ 0 \\ 0 \end{array} \right) & \mathcal{O} \left( \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right) \\
\mathcal{O} \left( \begin{array}{c} -4 \\ 0 \\ 0 \end{array} \right) & \mathcal{O} \left( \begin{array}{c} -3 \\ 0 \\ 0 \end{array} \right) \\
\mathcal{O} \left( \begin{array}{c} -1 \\ 0 \\ 0 \end{array} \right) & \mathcal{O} \left( \begin{array}{c} -3 \\ 1 \\ 2 \end{array} \right)
\end{array}
\Rightarrow \mathcal{O}_M \left( \begin{array}{c} 0 \\ 1 \\ 2 \end{array} \right)
\]

(3.13)

\[
\begin{array}{c|c}
0 & \lambda_h \\
0 & \delta h \\
0 & 0 \\
\vdots & \vdots
\end{array}
\Rightarrow \{\delta h/(h \lambda_h)\} \in H^0
\]

\[
\begin{array}{c|c}
0 & \delta h \\
0 & 0 \\
\vdots & \vdots
\end{array}
\Rightarrow H^1 = 0
\]

This produces only ‘polynomial deformations’.

The Reader should now have no difficulty reproducing the analogous charts for the \(\vartheta_x\)-, \(\vartheta_y\)- and \(\vartheta_x\)-valued residues. Note that the various contributions listed in (3.3a), (3.8a) and (3.8b) occur in the same place as the polynomials and residues on which they act; this illustrates the selection rule encoded in (2.37), wherein all residues must be constructed with the same residue kernel, and so the same index-sets \(S, Q\). So, the (3.8a) act on (3.6a), and the (3.8b) act on (3.6b) as reparametrizations:

\[
\text{Res}^{h}_{\vartheta} [\delta g] \cong \text{Res}^{z}_{\vartheta} [\delta g] + \text{Res}^{z}_{\vartheta} [\vartheta_y] \cdot d_y g ,
\]

(3.14a)

\[
\text{Res}^{y,z}_{g,h} [\delta f] \cong \text{Res}^{y,z}_{g,h} [\delta f] + \text{Res}^{y,z}_{g,h} [\vartheta_x] \cdot d_x f ,
\]

(3.14b)

which evaluate to

\[
\delta g_y' \cong \delta g_y' + \lambda^\alpha (\partial_\alpha g) ,
\]

(3.14a’)

\[
\Delta^\alpha \delta (\partial_\alpha f) \cong \Delta^\alpha \delta (\partial_\alpha f) + \Lambda^\alpha (\partial_\alpha f) .
\]

(3.14b’)

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The former of these two equivalence classes are bilinear in \(x, y\), and taken modulo \(\partial_x g\), leaving \(4 \cdot 3 - 3 = 9\) elements for \(H^1(M, T_M)\). The latter of these are quadratic in \(x\), taken modulo \(\partial_a f\), leaving \((5^2) - 4 = 6\) elements for \(H^2(M, T_M)\); see (2.42).

Note also that the \(\vartheta_x, \vartheta_y\) and \(\vartheta_x\)-valued residues in (3.8c) occur in positions which are void in the polynomial-valued charts (3.11)–(3.13). By the selection rule of (2.37), the residues (3.8c) cannot act on anything and ‘stand on their own’, as three elements of \(H^2(M, T_M)\). Under the obvious embedding map \(T_M \to T_X\), these are isomorphic to three corresponding elements in \(H^2(M, T_M) = H^{2,2}(M)\), which are (the duals of the pullbacks of) the Kähler forms on \(\mathbb{P}^3\), \(\mathbb{P}^2\) and \(\mathbb{P}^1\).

Putting all these together, we have obtained:
- all the \(T_M\)-valued 1-forms: 24 zeroth-level and 9 first-level;
- all the \(T_M\)-valued 2-forms: 6 second-level and 3 third-level;
- that all contributions are in precise 1–1 correspondence with the Koszul computation and also with the Landau-Ginzburg orbifold analysis, except that the higher-level residues (3.14) represent only the ‘monomial’ part of the massless modes [4,5], excluding the ‘twisted vacuum’ part.

To remedy this last observation and increase the degree of the representatives without changing the number of parameters, as in Ref. [8], we seek a “universal” scalar multiplier, that is, a scalar which may only depend on the defining equations (3.1), and determinants of their derivative matrices. The representatives (3.14c) clearly have degree \((1,1,0)\) and the scalar multiplier must have degree \((0,1,0)\). Without much ado,

\[
\sqrt{\det \left[ \partial_r \partial_s h(y, z) \right]} \quad (3.15)
\]

precisely fits the bill. In a concrete example for a Landau-Ginzburg potential à la Refs. [4,6] \(W = f(X) + g(X, Y) + h(Y, Z)\), we may choose:

\[
f(X) = \sum_{r=0}^{3} X_r^3, \quad g(X, Y) = \sum_{\alpha=0}^{2} X_{\alpha} Y_{\alpha}^2, \quad h(Y, Z) = \sum_{r=0}^{1} Y_r Z_r^2. \quad (3.16)
\]

Then, \(\sqrt{\det \left[ \partial_r \partial_s h(Y, Z) \right]} = \sqrt{Y_0 Y_1}\). It is straightforward to verify that not only is the scaling weight of this object equal to the scaling weight of \(|0\rangle^{(6)}_{NS}\), the 6th twisted Neveu-Schwarz-vacuum in the Landau-Ginzburg picture, but the ‘warp’ symmetry charges [7] agree as well.

The representatives (3.14d) clearly have degree \((2,0,0)\) and the scalar multiplier must have degree \((1,0,0)\). This object is in fact a little more difficult to spot, as it relies on the
fact that the $X, Y, Z$ fields are coupled. That is, the superpotential $W = f(X) + g(X, Y) + h(Y, Z)$ may be decoupled into four irreducible models:

$$W = \sum_{r=0}^{1} \left[ X_r^3 + X_r Y_r^2 + Y_r Z_r^2 \right] + \left[ X_2^3 + X_2 Y_2^2 \right] + \left[ X_3^3 \right]. \quad (3.17)$$

Then, it makes perfect sense to consider $\frac{1}{6} \sqrt{|\det \left[ \partial_r \partial_s f(X) \right]|} = \sqrt{X_0 X_1}$ which again perfectly fits the bill; both the scaling weight (degree) and the warp charge equal those of the twisted vacuum.

The translation of these radicals into the Koszul language is again fairly straightforward. We note that the Koszul representatives corresponding to (3.6a) and (3.6b) carry an additional factor: $\epsilon^{rs}$ and $\epsilon^{\alpha\beta\gamma} \epsilon^{rs}$, respectively. These being skew-symmetric, no contraction with the defining polynomial coefficient tensors is possible directly (as those are symmetric). However, their (direct) square can; the product

$$\epsilon^{pq} \epsilon^{rs} h_{\alpha pr} h_{\beta qs} y^\alpha y^\beta$$

is in fact a scalar. Since $h_{\alpha pr} y^\alpha = \partial_p \partial_r h(y, z)$, this product is simply the determinant of the hessian (with respect to $z$) of $h(y, z)$. This shows that $\epsilon^{pq} \epsilon^{rs}$ is dual to $h_{\alpha pr} h_{\beta qs} y^\alpha y^\beta$, whence $\epsilon^{rs}$ is, rather formally, dual to $\sqrt{h_{\alpha pr} h_{\beta qs} y^\alpha y^\beta}$. The second “universal” scalar multiplier follows in the same vein, noting that

$$\epsilon^{pq} \epsilon^{rs} h_{\alpha pr} h_{\delta qs} \epsilon^{\alpha\beta\gamma} \epsilon^{\delta\lambda\kappa} g_{\alpha\beta} g_{\gamma\delta} x^a x^b$$

is a scalar, whence $\sqrt{\epsilon^{pq} \epsilon^{rs} \epsilon^{\alpha\beta\gamma} \epsilon^{\delta\lambda\kappa}}$ is dual to $\sqrt{h_{\alpha pr} h_{\delta qs} g_{\alpha\beta} g_{\gamma\delta} x^a x^b}$. Amusingly, with the above specific choice of polynomials (3.17), this contraction yields zero, unless for example $h(Y, Z)$ is shifted by a cross-coupling term $Y_2 Z_0 Z_1$ (for which the above square-root does produce $\sqrt{X_0 X_1}$, as above). For a generic choice of superpotential, the radical is of course nonzero.

### 3.2. Reaming, refining and reducing residues

Of course, the wealth of complete intersections also features models where somewhat unusual representatives or relations amongst those occur. The following examples are meant to provide further practical guidance.

Consider for example the family of complete intersections

$$\mathcal{M} \in \mathbb{P}^5 \left[ \begin{array}{ccc} 4 & 1 & 1 \\ 0 & 1 & 1 \end{array} \right], \quad \left\{ \begin{array}{l} f(x) \overset{\text{def}}{=} f_{abcd} x^a x^b x^c x^d = 0, \\
g(x, y) \overset{\text{def}}{=} g_{\alpha\beta} x^\alpha y^\beta = 0, \\
h(x, y) \overset{\text{def}}{=} h_{\alpha\beta} x^\alpha y^\beta = 0, \end{array} \right. \quad (3.20)$$
where \( b_{1,1} = 2 \) and \( b_{2,1} = 86 \). The Koszul (ghost-for-ghost-...) sequence for degree-\((p, q)\) functions here becomes

\[
\mathcal{O}(p-2, q-2) \to \mathcal{O}(p-1, q-1) \to \mathcal{O}(p-5, q-1) \to \mathcal{O}(p-1, q-1) \to \mathcal{O}(p, q) \to \mathcal{O}_M(p, q) .
\]  

(3.21)

Owing to the commensurate degrees of \( g(x, y) \) and \( h(x, y) \), there occur several atypical cohomology representatives which may be regarded as residues in the above generalized sense. For example, variations of \( h(x, y) \) are listed from (B.21) with \((p, q) = (1, 1)\):

\[
\mathcal{O}(-1, -1) \to \mathcal{O}(0, 0) \to \mathcal{O}(1, 1) .
\]  

(3.22)

The two \( \mathcal{O}(0, 0) \) sheaves of degree-\((0, 0)\) functions correspond to the two residue kernels: \( \delta h(x, y) / h(x, y) = \lambda_h \) and \( \delta h(x, y) / g(x, y) \). The former is obviously of zeroth level and provides for the usual one-parameter equivalence as discussed above

\[
\delta h(x, y) \equiv \delta h(x, y) + \lambda_h h(x, y) .
\]  

(3.23)

The other kernel \( \delta h(x, y) / g(x, y) \), however, is also of zeroth level and makes perfect sense—on \( M \)—as a holomorphic object, if this \( \delta h(x, y) \) is again chosen to be proportional to \( h(x, y) \). While the rational function \( h/g \) is not holomorphic on \( \mathbb{P}^5 \times \mathbb{P}^1 \), it is on \( M \). To see this, note that the intersection of hypersurfaces \( \{g=0\} \cap \{h=0\} \) contains \( M \) as the locus of \( f=0 \) therein. However, by definition, both \( g(x, y) \) and \( h(x, y) \) vanish in \( \{g=0\} \cap \{h=0\} \), so that their (limiting) ratio may be evaluated using the L’Hospital theorem. Such ratios are also considered as residues and we define

\[
\text{Res}_M \left[ \frac{h(x, y)}{g(x, y)} \right] \overset{\text{def}}{=} \lim_M \left[ \frac{h(x, y)}{g(x, y)} \right] = \text{const.} ,
\]  

(3.24)

noting that this is a holomorphic degree-\((0, 0)\) function on the compact manifold \( M \). This of course provides for another one-parameter equivalence, and combining with (3.23),

\[
\delta h(x, y) \equiv \delta h(x, y) + \lambda_h h(x, y) + \lambda_g g(x, y) .
\]  

(3.25)

The analogous is then true of variations of \( g(x, y) \):

\[
\delta g(x, y) \equiv \delta g(x, y) + \kappa_g g(x, y) + \kappa_h h(x, y) .
\]  

(3.26)

Note that this is almost obvious from the fact that Eq. (2.27) is not written in a covariant fashion. Instead, on writing

\[
\delta f^j(x_i) \equiv \delta f^j(x_i) + \lambda_k f^k(x_i) ,
\]  

(3.27)
where $\lambda^j_k$ is understood to be zero if it cannot be holomorphic owing to the relative degrees, such cross-reparametrizations become obvious. However, merely ‘covariantizing’ Eq. (2.27) will not suffice in general and examples of further generalizations through higher level residues will occur below.

Next, there is also an unusual first level residue from the degree-(4,0) sequence

$$\mapsto \mathcal{O}(2, -2) \to \mathcal{O}(3, -1) \mapsto \mathcal{O}(3, -1) \to \mathcal{O}(0, 0) \mapsto \mathcal{O}(4, 0) \Rightarrow \mathcal{O}_\mathcal{M}(4, 0). \tag{3.28}$$

From $\mathcal{O}(2, -2) \sim \delta f_{\mathcal{M}} g h$, the residue integrand $\frac{(y dy) \delta f(x)}{g(x, y) h(x, y)}$ is formed which has degree-(2,0). This residue may be evaluated by contour-integration about one of the hypersurfaces, either $\{g = 0\}$ or $\{h = 0\}$:

$$\int_{\Gamma_g \, \text{or} \, \Gamma_h} \biggl| \frac{(y dy) \delta f(x)}{g(x, y) h(x, y)} \biggr|_{\text{on } \mathcal{M}} = \frac{\delta f(x)}{\det[\partial_\alpha g \partial_\beta h]} \biggl|_{\text{on } \mathcal{M}}$$

$$= \Omega^{ab}_{(-1)} \delta(\partial_\alpha \partial_\beta f) = \delta f''(x). \tag{3.29}$$

The second equality follows on noting that the determinant $\det[\partial_\alpha g \partial_\beta h]$ does not vanish on $\mathcal{M}$, owing to smoothness. Alternatively, this is a straightforward consequence of the general formulae in §2.2. The subscript $-1$ reminds us that this contribution ends up in the next to the right-most place in the top line of the lower left quadrant of the chart à la (2.17), and so is a ‘ghost’ variable. This in turn produces a further equivalence relation to which the variations of $f(x)$ must be subject:

$$\delta f(x) \cong \delta f(x) + \mu \delta f(x) + \det[\partial_\alpha g \partial_\beta h] \delta f''(x), \tag{3.30}$$

where again the $\det[\partial_\alpha g \partial_\beta h]$ factor has been introduced for correct degree of homogeneity. This exemplifies additional reparametrizations from $\text{Aut}(\oplus_j \mathcal{O}(d_j))$ which go beyond a mere ‘covariantization’ of (2.27) into (3.27). Note also that the copy of $\det[\partial_\alpha g \partial_\beta h]$ in (3.29) is not canceled by the one in (3.30); the former produces the (double) derivative $\delta f''(x)$.

The alert Reader will have noticed how “following the filtration” and the resulting index $q$ (here $-1$) unambiguously determined the fate of this contribution as generating a 21-parameter equivalence class of variations of $f(x)$ rather than, say, an independent source of polynomial-valued 0-forms. Recall that $q$, as defined in (2.19), counts the order of the differential form minus the order of the pole, subtracting the latter on account of contour integrations with which to evaluate the residue. The present case then illustrates the $q < 0$ case of our general result in §2.2.
Finally, as a double-check, we use the Bott-Borel-Weil theorem to calculate the cohomology corresponding to the sequence (3.28):

\[
\begin{array}{cccc}
\mathcal{O}(2, -2) & \to & \mathcal{O}(3, -1) & \leftarrow \\
\mathcal{O}(-2, -2) & \to & \mathcal{O}(-1, -1) & \leftarrow \\
\mathcal{O}(-1, -1) & \to & \mathcal{O}(0, 0) & \\
\end{array}
\]

Indeed, the non-zero cohomology as obtained here from the Bott-Borel-Weil theorem and also the general features of spectral sequences (filtering and induced ‘differential’ maps) are precisely reflected in the residue calculations above.

\[\delta f(x) \cong \delta f(x) + \mu f(x) + \det[\partial_\alpha g \partial_\beta h](\delta f'')\]

A remark about (possibly) related Landau-Ginzburg orbifolds is in order. The model (3.20) naively allows the construction of a Landau-Ginzburg orbifold à la Refs. [4,6,10].
for which \( f(x) + g(x, y) + h(x, y) \) serves as the superpotential. However, the chiral superfields \( X^a \) and \( Y^\alpha \) having scaling charges \( \frac{1}{4} \) and \( \frac{3}{4} \), respectively, this Landau-Ginzburg orbifold would seem to have central charge 6, which does not correspond correctly to the fact that (3.20) describes a complex 3-dimensional Calabi-Yau manifold. Worse yet, the model (3.32) does not even allow a consistent (non-zero) scaling charge assignment for a Landau-Ginzburg orbifold as described in [9]. Both of these fall in the category of ‘split models’, for which Landau-Ginzburg orbifolds are ill-defined [36]. Hopefully, the framework of Ref. [9] may provide a resolution in such no-go situations.

4. Reflected and Ramified Residues

The above results are applicable to all complete intersections in products of flag-spaces — the simplest of which, \( \mathbb{P}^n \)'s, were explicitly studied above. However, most of the Landau-Ginzburg orbifolds [4,10,6,37,5], or their gauged generalizations [9], naturally apply to weighted projective hypersurfaces. In many situations it is also of interest to study models realized in a quotient of a (weighted) CICY. It is therefore interesting to see if the above residue calculations admit a ‘weighted’ generalization.

4.1. Ratifying the residue recipe

For simplicity let us consider a hypersurface \( \mathcal{M} \), defined by \( P(x)=0 \) in a quotient of a single weighted projective space \( \mathbb{P}^4(k_0,...,k_4) \) by the group \( H \). The general case of complete intersections in products of weighted projective spaces follows straightforwardly. Since \( \mathbb{P}^4(k_0,...,k_4) = \mathbb{P}^4/j \) where \( j \simeq (\mathbb{Z}_d : k_0,...,k_4) \), it is natural to consider a quotient by \( G = H \times j \); the notation implies the action:

\[
j(x^0, x^1, x^2, x^3, x^4) = (\lambda^{k_0}x^0, \lambda^{k_1}x^1, \lambda^{k_2}x^2, \lambda^{k_3}x^3, \lambda^{k_4}x^4), \quad \lambda^d = 1 \quad (4.1)
\]

The essential novelty with weighted projective hypersurfaces, and then necessarily also the complete intersections in products of weighted flag-spaces, owes to the inherent singularity of these spaces. This stems from the unequal scaling weights of the quasihomogeneous coordinates (and more generally, the twist-charges with respect to \( G = H \times j \)); each singular subspace is fixed by at least one element of the quotient group \( G \).

We then find the fixed-point sets of \( G \) in the usual manner, i.e. for each element \( g \in G \), the fixed-point set is \( \Sigma_g = \{ x^\mu \mid P=0, (g^{-1})x^\mu=0 \} \). Let \( N_g \) label the coordinates

\[\text{We continue indexing coordinates with superscripts, as appropriate for contravariance, hoping that the Reader will have little if any difficulty in distinguishing superscripts from exponents.}\]
Thus, now there does exist a natural way to split the top differential form,

\[(xd^4x) \overset{\text{def}}{=} \sum_{\mu, \nu, \rho, \sigma, \tau} \frac{k_{\mu}}{5!} \epsilon_{\mu \nu \rho \sigma \tau} x^\mu dx^\nu dx^\rho dx^\sigma dx^\tau. \quad (4.2)\]

However, due to the unequal scaling weights, the coordinate differentials in the various permutations in (4.2) are, in general, of uneven order over the $\Sigma$. Therefore, we choose to work instead on the affine space $\mathcal{C}^5(k_0, \ldots, k_4)$, where the top differential may be written as

\[d^5x \overset{\text{def}}{=} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4, \quad (4.3)\]

and remember later to reduce the order of the obtained differentials by one and also to return the explicit $\epsilon_{\cdot \cdot \cdot \cdot \cdot}$’s. Then,

\[d_{\parallel |N_g|}^5x \overset{\text{def}}{=} \bigwedge_{\mu \notin N_g} dx^\mu, \quad x^\mu \in \Sigma_g \quad (4.4)\]

is the affine version of the top differential on $\Sigma$, and

\[d_{\perp |N_g|}^5x \overset{\text{def}}{=} \bigwedge_{\mu \in N_g} dx^\mu, \quad x^\mu \notin \Sigma_g \quad (4.5)\]

is the affine version of the top differential on the normal bundle to $\Sigma \subset \mathbb{P}^4(k_0, \ldots, k_4)$. Note that both top differentials are invariant under $g$. In this way the $x^\mu$ are not all at the same footing and hence we can try to write down residue expressions where it is not necessary to include all $x^\mu$ in the differential as was the case for the homogeneous projective spaces.

In fact, we are facing a situation not at all unlike the one with the complete intersections in products of homogeneous projective spaces, in §2.2. There, we were restricting the residue integrals to a proper factor $\prod_{i \in S} \mathbb{P}^4_i \subset \prod_{\text{all } i} \mathbb{P}^4_i$. Now, we restrict the residue integrals to a coordinate subset within a given weighted projective space. That is, in the affine version we again restrict to a proper factor $\mathcal{C}^5(k_0, \ldots, k_4)$. Upon re-projectivization, however, this factorization is no longer global: the subset $\Sigma_g$ and its normal bundle no longer form a global holomorphic tensor product, although locally this factorization prevails. The natural generalization of (2.18) therefore becomes

\[
\text{Res}^N_{\mathcal{P}}[\delta P] \overset{\text{def}}{=} \frac{1}{(2\pi i)} \oint_{\gamma(P)} \frac{d^{\mid N_g \mid}x}{P} \delta P, \quad (4.6)
\]

17 The fixed-point sets must be counted with appropriate multiplicity, one for each element of the symmetry group separately: if $g^n = 1$ for $n$ a prime, then there will be a total of $n-1$ coinciding fixed point sets. If $n$ is not a prime, one must take special care of the fixed-point sets of $g^m$ when $m$ divides $n$. 

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The residue is evaluated as before, using a change of variables as outlined in §2.2, and we obtain a holomorphic $q + 1 = |N_g| - 1$ form, (before re-projectivization!)

$$\text{Res}_{P}^{N_g}[\delta P] = \frac{dx^\nu_1 \cdots dx^\nu_{q+1}}{J^\nu_1 \cdots ^\nu_{q+1}} \delta P , \quad (4.7a)$$

with $J^{\nu_1 \cdots \nu_{q+1}}$ the Jacobian for the coordinate transformation

$$(x^\mu_1, \ldots, x^\mu_{N_g}; \mu_i \in N_g) \longrightarrow (x^\nu_1, \ldots, x^\nu_{q+1}, P) . \quad (4.8)$$

However, we also note that the degree of the residue (4.6) with respect to the coordinates $x^\mu$, $\mu \notin N_g$ is the same as that of a multi-derivative of $P$ with respect to $x^\mu$, which we again denote by $\partial^\bar{m} P$ where now $\bar{m} = (m_\mu, \mu = 0, \ldots, 4)$ satisfy $(\bar{m} \cdot \bar{k}) = \sum_{\mu \notin N_g} k_\mu$ with $\bar{k} = (k_\mu, \mu = 0, \ldots, 4)$. Thus, we can again write $\text{Res}_{P}^{N_g}[\delta P]$, formally, as a general variation of the $\bar{m}^{th}$ gradient of $P$:

$$\text{Res}_{P}^{N_g}[\delta P] = \frac{1}{(2\pi i)} \oint_{\Gamma(P)} \frac{d|N|}{P} \delta P$$

$$= \frac{dx^\nu_1 \cdots dx^\nu_{q+1}}{\partial^\bar{m} J^\nu_1 \cdots ^\nu_{q+1}} \delta (\partial^\bar{m} P(x_i)) , \quad (4.7c)$$

$$\text{nowhere-zero}$$

$$\equiv \sum_{\bar{\vec{r}}} \Omega_{(q+1)}^{\bar{\vec{r}}} P^{(\bar{\vec{m}})}(\bar{x}) . \quad (4.7d)$$

The multi-index $\bar{\vec{r}}$ again contains one index for each of the $\bar{m}$ derivatives. Each $\Omega_{(q+1)}^{\bar{\vec{r}}}$ will become, upon re-projectivization, the ‘nowhere zero holomorphic $q$-form’ on the $q$-dimensional Calabi-Yau space $Q_{\bar{\vec{r}}}$ (given below), and is again parametrized by the $x^\mu$, $\mu \notin N$, and any other parameter that $P$ depends upon. This is indeed very similar to the situation with Eqs. (2.20)–(2.23). Here however the actual number of partial derivatives in the multi-derivative $\partial^\bar{m}$ depends on the weights, $k_\mu$, of the coordinates; the weight of $\partial_\mu$ clearly being $-k_\mu$.

Alternatively, the multi-derivatives $\partial^\bar{m} P$, indexed by $\bar{\vec{r}}$, again may be regarded the defining equations of a Calabi-Yau $q$-fold $Q_{\bar{\vec{r}}}$, embedded in $\mathbb{P}^{(|N_g| - 1)}_{(k_{\nu_i} \cdot \nu_i \in N_g)}$, the re-projectivized normal bundle of $\Sigma_g$. For each $Q_{\bar{\vec{r}}}$,

$$\Omega_{(q+1)}^{\bar{\vec{r}}} \equiv \frac{dx^\nu_1 \cdots dx^\nu_{q+1}}{\partial^\bar{m} J^\nu_1 \cdots ^\nu_{q+1}} \quad (4.9)$$

becomes the holomorphic volume-form upon projectivization. Indeed, each $Q_{\bar{\vec{r}}}$ has vanishing first Chern class, since

$$\deg(\partial^\bar{m} P) = \deg(P) - (\bar{m} \cdot \bar{k}) = d - \sum_{\nu \notin N_g} k_\nu = \sum_{\mu \in N_g} k_\mu . \quad (4.10)$$
To find the dimension of this contribution one would first have to find the number of monomials in $x_\mu, \mu \not\in N_g$ of degree (4.10), and then project onto those residue representatives which are invariant under the action of all elements of $G$, not just $g$. It is then important to remember to take into account the non-trivial transformation of $\epsilon \cdots$ which appears in the differential in (4.3), which is equivalent to that of $\prod_{\mu \not\in N_g} dx^\mu$. Finally, depending on whether $|N_g|$ is two or three this will contribute either to the complex structure or to the Kähler deformations in perfect analogy with the discussion in §2.

Next we consider the reparametrization operator-valued residues. Apart from the contribution from the original projective space in the form of

$$\vartheta_{\mathbf{P}^4} \overset{\text{def}}{=} x^\mu \lambda^\nu_{\mu} \partial_{\nu} , \quad \mu, \nu = 0, \ldots, 4 .$$

(4.11)

there will be contributions from each of the fixed point sets $N_g$, associated to the action of the element $g \in G$ which takes the form

$$\vartheta_{\Sigma_g} \overset{\text{def}}{=} x^\mu \lambda^\nu_{\mu} \partial_{\nu} , \quad \mu, \nu \not\in N_g .$$

(4.12)

Equivalently, and writing $|g|$ for the order of $g$

$$\vartheta_{\Sigma_g} = \mathcal{P}_g(x^\mu) \lambda^\nu_{\mu} \partial_{\nu} , \quad \mathcal{P}_g \overset{\text{def}}{=} \frac{1}{|g|}(g + g^2 + \ldots + g^{|g|}) , \quad g^{|g|} = 1 ,$$

(4.13)

that is, $\mathcal{P}_g$ projects on the $g$-invariant set. The $\vartheta_{\Sigma_g}$ are in fact the Kähler forms inherited from the fixed-point sets themselves, much the same as the Kähler forms of each $\mathbb{P}^n_i$ factor in the discussion of the homogeneous complete intersections; see §2.4. Therefore, in place of a single Kähler form with homogeneous projective spaces, we now obtain rather naturally the multi-component residue class

$$\oint_{\Gamma(P)} \frac{d^5 x}{\mathcal{P}} \left( \bigoplus_{g \not\equiv 1} \vartheta_{\Sigma_g} \oplus \vartheta_{\mathbf{P}^4} \right) = \oint_{\Gamma(P)} \frac{d^5 x}{\mathcal{P}} \left( \bigoplus_{\text{all } g} \vartheta_{\Sigma_g} \right) ,$$

(4.14)

since $\mathbb{P}^4_{(k_0, \ldots, k_4)}$ is the fixed-point set of the identity. The differential order is decreased by one each owing to: (1) contraction between a $dx^\mu$ and a derivative in $\bigoplus_{g \not\equiv 1} \vartheta_{\Sigma_g} \oplus \vartheta_{\mathbf{P}^4}$, (2) evaluation of the contour-integral, (3) re-projectivization.

In case $|g|$ is not prime one would have study the fixed point set more carefully in which case not all of the $\vartheta_{\Sigma_g}$ may be independent.
4.2. A reassembling rally

As an illustration, we consider the simple and well-studied model, the family of quasihomogeneous octics in $\mathbb{P}^4_{(1,1,2,2,2)}$, denoted as $\mathbb{P}^4_{(1,1,2,2,2)}[8]$. The projectivization symmetry is

$$j = (\mathbb{Z}_8 : 1, 1, 2, 2, 2),$$

and we consider no additional quotient. The embedding projective space, $\mathbb{P}^4_{(1,1,2,2,2)}$, is singular at the subspace $\Sigma \approx \mathbb{P}^2$, found at $x^0 = 0 = x^1$ and parametrized by the weight-2 coordinates $x^2, x^3, x^4$, is fixed under the action of $j^4 = (\mathbb{Z}_2 : 1, 1, 0, 0, 0)$, and each point is a local $\mathbb{Z}_2$-quotient singularity. An octic quasihomogeneous hypersurface, $\mathcal{M} = \{P=0\}$, in $\mathbb{P}^4_{(1,1,2,2,2)}$ cannot, in general, avoid meeting this singular plane and will intersect it in a curve $C$; thus, $\mathcal{M}$ is said to have inherited a curve of local $\mathbb{Z}_2$-quotient singularities.

There exists a blow-up of $\mathcal{M}$ along the curve, sometimes denoted $\tilde{\mathcal{M}}$, in which the singular curve $C$ is replaced by a “ruled surface” $E$, obtained by fibering a $\mathbb{P}^1$ over the curve $C$. This complex 2-fold, $E$, is a divisor in $\tilde{\mathcal{M}}$ and contributes a new and non-trivial class to $H^4(\tilde{\mathcal{M}})$. It is also isomorphic to a class in $H^{1,1}(\tilde{\mathcal{M}})$, both having a common dual in $H_2(\tilde{\mathcal{M}})$. Together with the (pull-back of the) Kähler class of $\mathbb{P}^4_{(1,1,2,2,2)}$, this provides for dim $H^{1,1}(\tilde{\mathcal{M}}) = 2$. Next, note that the octic when restricted to $\Sigma$ becomes a quartic in $x^2, x^3, x^4$, and so has genus 3; the three handles provide a dual pair of $S^1$'s each, together with 1-forms supported on each of these, and so dim $H^{1,0}(C) = 3$. In $E$, therefore, there are three dual pairs of 3-cycles of the form $S^1 \times \mathbb{P}^1$, and produce three new and non-trivial elements for $H^{2,1}(\tilde{\mathcal{M}})$ [the duals being in $H^{1,2}(\tilde{\mathcal{M}})$]. The remaining 83 elements in $H^{2,1}(\tilde{\mathcal{M}}) \approx H^1(\tilde{\mathcal{M}}, T_{\tilde{\mathcal{M}}})$ are easy to find as linear reparametrization classes of octic quasihomogeneous polynomials. Therefore, the blow-up $\tilde{\mathcal{M}}$ is a smooth Calabi-Yau space with $b_{1,1} = 2$, and $b_{2,1} = 86$, and so $\chi_E = -168$. This fully agrees with the Landau-Ginzburg orbifold calculation à la Refs. [4,10,6] and also [5]: there are two twisted $(a, c)$ vacua—matching $b_{1,1} = 2$, and 83 untwisted and 3 twisted $(c, c)$ states—matching $b_{2,1} = 86$ and also the ‘twistedness’ of three of these.

We now turn to the residues. In the case at hand, the factors of (4.3)

$$d^3 x \equiv dx^2 \wedge dx^3 \wedge dx^4,$$

$$d^2 x \equiv dx^0 \wedge dx^1,$$

are the affine top differential on $\Sigma$, and the affine top differential on the normal bundle to $\Sigma \subset \mathbb{P}^4_{(1,1,2,2,2)}$, respectively.
Thus, in addition to the residues that the above analysis covers, we now also have to consider
\[ \oint_{\Gamma(P)} \frac{d^2 x}{P} \delta P , \quad \text{and} \quad \oint_{\Gamma(P)} \frac{d^3 x}{P} \delta P, \] (4.18)
providing a degree-2 and a degree-6 contribution. As before, these may be written as
\[ \left[ \sum_{i,j,k=0}^{1} \epsilon_{ij} \frac{d^2 x^j}{\partial_i \partial^\bar{m} P} \right] \delta (\partial^{\bar{m}} P) \|, \quad \bar{m} \cdot \bar{k} = 6, \] (4.19)
and
\[ \left[ \sum_{i,j,k,l=2}^{4} \epsilon_{ijk} \frac{d^3 x^j d^3 x^k}{\partial_i \partial^\bar{m} P} \right] \delta (\partial^{\bar{m}} P) \perp, \quad \bar{m} \cdot \bar{k} = 2, \] (4.20)
respectively. Here, the subscript “\( \| \)” denotes restriction to local coordinates on \( \Sigma \) while “\( \perp \) \( \Sigma \)” labels a restriction to local fibre coordinates of the normal bundle of \( \Sigma \subset \mathbb{P}^4_{(1,1,2,2,2)} \). The manifest factorization is very similar to that in §2, as discussed in 4.1. In both cases, the general variation of the multi-derivative of \( P \) simply becomes a general polynomial of degree 2 and 6, respectively.

However, only the former of these contributes. To see this, note that \( \delta (\partial^{\bar{m}} P) \| \Sigma \) is a degree-2 polynomial in coordinates of the exceptional set \( \Sigma \), and which are also fixed by \( j^4 \). Clearly, any linear combination of \( x^2, x^3, x^4 \) will do, whence a 3-parameter family of such contributions. By contrast, \( \delta (\partial^{\bar{m}} P) \perp \Sigma \) is a degree-6 polynomial in coordinates which are normal to the exceptional set, \( x^0, x^1 \), both of which are however ‘projected out’ by \( j^4 \). Finally, note that the former contribution is a 1-form before returning from the affine \( \mathcal{C}^5 \) to the projective \( \mathbb{P}^4_{(1,1,2,2,2)} \), whence we conclude that (4.19) supplies, upon re-projectivization, three polynomial-valued 0-forms. As usual, these then contribute to the \( H^1(\hat{\mathcal{M}}, \mathcal{T}_{\hat{\mathcal{M}}}) \approx H^{2,1}(\hat{\mathcal{M}}) \). Note that there is no action of coordinate reparametrizations on this; there are no inappropriate residues of \( x^i \lambda^i_j \partial_j \). These three contributions exactly correspond to the massless twisted \((c,c)\) states of the form \( X_i \mid \frac{3}{4}, \frac{3}{4} \rangle_{\text{NS}}, \quad i = 2, 3, 4 \), and also correspond to the three \((2,1)\)-forms we have described above.

Next we reconsider the reparametrization operator-valued residues, and see that, owing to the unequal scaling weights, there are two separate classes of reparametrization (degree-0 and \( j \)-invariant) operators:
\[ \vartheta_{\mathbb{P}^4} \overset{\text{def}}{=} x^i \lambda^i_j \partial_j, \quad i, j = 0, \ldots, 4, \] (4.21)
and
\[ \vartheta_{\Sigma} \overset{\text{def}}{=} x^i \lambda^i_j \partial_j, \quad i, j = 2, 3, 4. \] (4.22)
The mixed operators
\[ \vartheta_{\text{mixed}} \overset{\text{def}}{=} x^i x^j \lambda^k_{ij} \partial_k, \quad i, j = 0,1 \quad k = 2, 3, 4, \] (4.23)
merely contribute to the reparametrization of (4.22), and are anyway ‘projected out’ by $j^4$. Therefore, in place of a single Kähler form with homogeneous projective spaces, we now obtain a 2-component residue class

$$\int_{\Gamma(P)} \frac{d^5x}{P} \left( \partial_{\mathbb{P}^4} \oplus \partial_{\Sigma} \right)$$

and so two 2-forms: the differential order is decreased by three, as in the general case considered above. These two 2-forms again match the massless twisted $(a, c)$ states $|−1, 1\rangle_{NS}^4$ and $|−1, 1\rangle_{NS}^7$, and also the two $(1,1)$-forms described above.

Together with the usual residues obtained as described in the preceding sections, this then completes the complete residue representation of the massless $27$ and $27^*$ states for $\mathbb{P}^{4}_{(1,1,2,2,2)}$[8]. It is in complete and precise 1–1 correspondence with both the geometrical description given above, the Landau-Ginzburg orbifold description (when restricted to the complex deformation moduli space) à la Refs. [4,10,6] and so also with the results à la Ref. [3]. The non-exceptional part of the analysis is in a similarly detailed 1–1 correspondence with the Koszul computation, which is then generalized through the inclusion of the exceptional residues (4.18) and the second contribution in (4.24). Note however that this still does not provide a weighted Koszul calculation by itself: the exceptional contributions are found via the exceptional residues. A ‘purely’ weighted Koszul calculation (and so also a weighted Bott-Borel-Weil Theorem) can hopefully be developed in the context of equivariant cohomology, but this is well beyond the scope of this article.

5. Residue Rings

The foregoing has established a 1–1 correspondence between the above residue calculations and the Koszul calculations of Refs. [1,3]. It should be clear that this correspondence provides a residue representation not only for the $E_6$ $27$’s and $27^*$’s, but also to the $E_6$ 1’s. Calculations of the 1-spectrum is typically rather more involved [1,5] and will not be detailed here.

On the other hand, the ring structure of the moduli fields for the complex structure as determined from the Koszul calculations and from the Landau-Ginzburg orbifold analysis turns out to be remarkably similar [7], and in fact has to be the same; changing the Kähler structure in going from the Landau-Ginzburg phase to the large radius limit cannot affect the ring structure of the complex structure moduli. In particular it means that we can use selection rules based on “quantum symmetries” obtained at the Landau-Ginzburg point, in the Calabi-Yau phase. That is, as long as this symmetry is not broken by deforming in a direction given by a moduli from one of the fixed point sets. In other words we can blow up the fixed point set in a way that keeps the shape of the blow up and only affects the size of it, by varying the toric divisor which came from the reparametrization of the fixed
point set. Of course, since the model is no longer at the specially symmetric point in the Kähler moduli space, we can no longer use the quantum symmetry there and hence there will be no straightforward selection rules for the Yukawa couplings among the (1, 1) forms.

The present residue representation is clearly simply extending the polynomial deformation analysis of Ref. [26], and so inherits the same ring structure. That is, the Yukawa couplings for the non-polynomial deformations, such as those specified in (2.37), are easily calculated—by exactly the same method as in Ref. [26]. The only difference being that the radical factors will have to occur in such products that the Yukawa coupling product would become (modulo the reparametrization ideals (2.34) and (2.37)) proportional to the ‘top-degree’ polynomial. For example, in the model (3.1), this ‘top-degree’ polynomial is proportional to

$$\det \left[ \partial^2 (f(x) \oplus g(x,y) \oplus h(y,z)) \right] \cong \left| \begin{array}{c} \partial^2 f(x) \\ \partial x^a \partial x^b \\ \partial x^c \partial x^d \end{array} \right| \left| \begin{array}{c} \partial^2 g(x,y) \\ \partial y^\alpha \partial y^\beta \\ \partial y^\gamma \partial y^\delta \\
\partial z^r \partial z^s \end{array} \right|$$

(5.1)

modulo the Jacobian ideal (generated by gradients of $f(x)$, $g(x,y)$ and $h(y,z)$).

Notice that the additional 9 representatives (3.14a) were multiplied by the radical (3.15). It is then straightforward that only couplings with an even (including zero) number of such ‘radical deformations’ may be non-zero. This produces a straightforward selection rule which, matches the effect of the selection rule based on the quantum symmetry in the corresponding Landau-Ginzburg orbifold. However, this is not at all surprising but should be expected, since on general grounds a variation in the Kähler moduli space does not affect the (c,c)-ring. In fact, following the analysis in Ref. [8], it can be shown that this extension of the by now standard Yukawa coupling calculation perfectly agrees with the corresponding Landau-Ginzburg orbifold calculations. It also matches the general Yukawa coupling formula obtained for the Koszul calculation [1,3]. Moreover, as in Ref. [8], the calculations can be performed both for the model (3.1) “as is”, and also for its ‘ineffectively split’ variant in which the additional 9 representatives (3.14a) become ordinary polynomial deformations whereupon the standard calculations apply straightforwardly.

The 27* Yukawa couplings, on the other hand, are easiest to determine using the ‘dual’ description (2.43), where they become 2-forms and the usual (‘topological’) Yukawa coupling may be calculated straightforwardly. Of course, the instanton-corrected Yukawa couplings are best calculated using mirror symmetry, for which techniques are being vigorously developed [11,15,20,16,24,17,18].
6. Resolvents and Radicals

We wish to present an alternative and perhaps even more heuristic derivation of the above results. Let us therefore focus, for the moment, on the deformations of the complex structure which may be realized as deformations of the defining polynomial system. This does exhaust all deformations of complex structure for all homogeneous hypersurfaces, but not so for their quasihomogeneous (weighted) cousins. Consider then, for the moment, a simple \( n \)-dimensional hypersurface \( \mathcal{M} \) defined by \( \{ P = 0 \} \subset X \), where \( X \) is some homogeneous space (or product thereof).

As is well known, the choice of \( P \) determines the complex structure of its zero-set, \( \mathcal{M} \). \( P \) also determines the holomorphic volume-form \( \Omega \) on \( \mathcal{M} \) via Eq. (2.4) and, indeed, the choice of \( \Omega \) among all elements of \( \bigoplus_{p+q=n} H^{p,q}(\mathcal{M}) \) is equivalent to the choice of the complex structure. The variations of \( \Omega \) also correspond to variations of the complex structure (see Eq. (1.6)) and we presently examine this by direct calculation.

To that end, deform \( P \rightarrow P - t^\alpha \delta P_\alpha \), and calculate

\[
\frac{\partial \Omega}{\partial t^\alpha} = \int_{\Gamma(P)} \frac{(xd^{n+1}x)}{P} \left( \frac{\delta P_\alpha}{P} \right).
\]

(1.7)

Iterating this \( n \) times,

\[
\frac{\partial^n \Omega}{\partial t^{\alpha_1} \cdots \partial t^{\alpha_n}} = \frac{1}{n!} \int_{\Gamma(P)} \frac{(xd^{n+1}x)}{P} \frac{\delta P_{\alpha_1}}{P} \cdots \frac{\delta P_{\alpha_n}}{P},
\]

(6.1)

which becomes the “Yukawa” \( n \)-point coupling upon multiplying \( \Omega \) and integrating over the manifold \( \mathbb{P}^4 \). Thus, the homogeneity degree-\( \vec{0} \) rational polynomials \( \delta P_\alpha \), taken however modulo terms which merely rescale \( \Omega \), may be identified with elements of \( H^1(\mathcal{T}) \), i.e., with tangent vectors \( \nabla_\alpha \Omega \) to the moduli space. Such polynomial deformations of the complex structure have been studied in great detail so far both from the geometrical point of view where \( P - t^\alpha \delta P \) is the (deformed) defining equation of a manifold, and also from the Landau-Ginzburg orbifold point of view, where \( P - t^\alpha \delta P \) is the (deformed) superpotential. An iteration of (1.7) and integration by parts leads to the Picard-Fuchs equations \([25,48]\), which provide additional information for the Special Geometry calculations \([20,24]\).

We will not pursue these considerations here, but instead turn to a little more involved example: \( \mathcal{M} \in \mathbb{P}^4 \times \mathbb{P}^1 \), defined as the common zero-set of \( f(x) \) and \( g(x, y) \), which have degree \((4, 0)\) and \((1, 2)\), respectively, over the embedding space \( \mathbb{P}^4 \times \mathbb{P}^1 \). Write \( f_0, g_0 \) for the reference choice of polynomials and consider a deformation of the holomorphic volume-form \( \Omega \):

\[
\Omega \overset{\text{def}}{=} \int_{\Gamma(f) \times \Gamma(g)} \frac{(xd^4x)(ydy)}{(f_0 - \delta f)(g_0 - \delta g)},
\]

(6.2)
so that $\Omega_0$ refers back to the reference choice of the complex structure, where $\delta f = 0 = \delta g$.

Now expand to first order in $\delta f, \delta g$:

$$
\Omega = \Omega_0 + \oint\int_{\Gamma(f) \times \Gamma(g)} \frac{(xd^4x)(ydy)}{f_0 \, g_0} \left( \frac{\delta f}{f_0} \right) + \oint\int_{\Gamma(f) \times \Gamma(g)} \frac{(xd^4x)(ydy)}{f_0 \, g_0} \left( \frac{\delta g}{g_0} \right). 
$$

(6.3)

On the face of it, these two (double) residues are simply the polynomial deformations of complex structure, as discussed above.

However, note that the first double residue also produces another contribution when calculated in the following stepwise fashion:

$$
\oint\int_{\Gamma(f) \times \Gamma(g)} \frac{(xd^4x)(ydy)}{f_0 \, g_0} \left( \frac{\delta f}{f_0} \right) = \oint \frac{xd^4x}{f_0} \left[ \oint \frac{(ydy)}{g_0} \delta f \right] 
$$

(6.4a)

$$
= \oint \frac{xd^4x}{f_0} \left( \frac{\delta f'}{f_0} \right) 
$$

(6.4b)

where

$$
\delta f' \defeq \oint \frac{ydy}{g_0} \delta f = \Omega_a^{(0)} \partial_a f(x) 
$$

(6.5)

and

$$
\Omega_a^{(0)} \defeq \oint \frac{ydy}{\partial_a g_0}, 
$$

(6.6)

are exactly as defined in §2.2, and used throughout the foregoing analysis! We emphasize, however, that this occurrence of the partial residue (2.18) owes to our insisting that the intermediate residue integral within the square brackets in (6.4a) should have an independent meaning. On comparison with the twisted states in Landau-Ginzburg orbifold, we note that, formally at least, $\oint \frac{xd^4x}{f_0} \Omega_a^{(0)}$ plays the rôle of the twisted vacuum, while $\partial_a f(x)$ is exactly the monomial part.

In fact, it should be clear that all the polynomial-valued residues can be recovered by simply expanding the “main” residue in such a fashion; the Reader should encounter no difficulty in recovering all of the above results in this alternate manner.

A posteriori at least, this focus on the independent rôle of the intermediate residues may be argued by direct comparison with the results from other methods. However, considerable further work seems to be required to recover all the reparametrization relations and degrees of freedom and in this respect, this approach is presently lacking. Nevertheless, this approach has the virtue of being a straightforward and direct study of the deformations of the complex structure, by studying directly the deformations of the holomorphic volume-form. The Koszul computations, with all the charts and maps and ‘filtration’ . . . , may then be viewed as a bookkeeping device for racking all the myriad of possible contributions to the deformations of $\Omega$ and their various relations.

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The form of Eq. (6.44), perhaps more forcibly than any argument before, suggests reinterpreting \( \delta f' \) as a deformation of the original defining polynomial \( f_0(x) \). Of course, it is not possible simply to deform \( f_0(x) \rightarrow f_0(x) + \delta f'(x) \). From the Landau-Ginzburg orbifold side, the difference in the degrees of (quasi)homogeneity of \( f_0(x) \) and of \( \delta f'(x) \) implies that their linear combination as the superpotential explicitly breaks the quantum symmetry; the IR fixed point would be determined by \( \delta f' \), as that one has a lower degree. This is precisely not what we are after; it says nothing about marginal deformations of the model with \( f_0(x) \) in the superpotential. From the geometrical side, the deformation \( f_0(x) \rightarrow f_0(x) + \delta f'(x) \) simply makes no sense! It is nowhere well-defined on the projective embedding space, in view of the different degrees of homogeneity in \( f' \) and \( f \). Furthermore, it should be clear that the \( \delta f' \) cannot possibly provide bona fide deformations of the defining polynomial, since \( \delta f' \) corresponds to deformations of the complex structure which precisely are not deformations of the embedding!

Thus — if the expansion (6.3), eventually containing \( \delta f'(x) \) — is to be collapsed back somehow differently, so that \( \delta f' \) would appear explicitly in a variation of \( f_0(x) \), \( \delta f' \) must come multiplied by a factor \( \Delta \) of compensating degrees of homogeneity:

\[
f_0(x) \rightarrow f_0(x) + \Delta \delta f'(x)
\]

(6.7)

In addition, this \( \Delta \) must be ‘universal’, that is, it must be independent of the deformation parameters. We now observe that for all complete intersections in products of homogeneous (rather than quasihomogeneous) flag spaces, the Koszul calculation obtains certain Levi-Civita alternating symbols in place of \( \Delta \). These can, in turn, always be identified with square-roots of certain precisely corresponding determinants [8].

For the quasihomogeneous models, this identification of \( \Delta \) with radicals is neither straightforward nor is it clear that such radicals would always exist. In fact, the three exceptional contributions to \( H^1(\mathcal{M}, \mathcal{T}_\mathcal{M}) \) in (4.19), for the case \( \mathcal{M} \in \mathbb{P}^4_{(1,2,2,2)}[8] \) have degree 2, and need a \( \Delta \) of degree 6. Straightforwardly, and following the experience from the homogeneous cases, one tries

\[
\sqrt{\det [\partial_i \partial_j P(x)]_{\parallel \Sigma}}.
\]

(6.8)

That is, \( i, j = 2, 3, 4 \) are restricted to the coordinates ‘along’ \( \Sigma \). This indeed has the correct degree and has the virtue of being in the same 1–1 correspondence with the \( \Sigma \)-restriction of the Levi-Civita symbol which persists for all homogeneous cases [8]. However, strange things may happen: if \( P(x) \) is chosen to be the most popular of all, the Fermat polynomial,

\[
P_F(x) = (x^0)^8 + (x^1)^8 + (x^2)^4 + (x^3)^4 + (x^4)^4,
\]

(6.9)

then

\[
\sqrt{\det [\partial_i \partial_j P(x)]_{\parallel \Sigma}} = \sqrt{\det [\delta_{ij} \cdot 3(x^i)^2]} , \quad i, j = 2, 3, 4,
\]

\[
= 24 \sqrt{3} x^2 x^3 x^4 ,
\]

(6.10)
is not radical at all! Moreover, the product of this and the \( \delta(\partial_\perp^6 P) \) from (6.13) becomes a linear combination

\[
(x^2 \oplus x^3 \oplus x^4) x^2 x^3 x^4,
\]

(6.11)

which were already accounted for among the `plain’ polynomial deformations. Of course, for a generic reference defining polynomial \( P(x) \), no such degeneration occurs and the radical (6.9) is indeed radical. This is not unlike the situation encountered in §3.3, where the analogous radical vanished for a simple choice of defining equations. The conditions for the required radical to either vanish or degenerate into a non-radical polynomial seem to be independent of well-definedness; that means that such degenerations are to be expected in every family of well behaved Calabi-Yau and/or Landau-Ginzburg orbifold models. Until it is understood precisely why such degenerations occur at innocuous albeit special reference defining polynomials, the identification of the factor \( \Delta \) with such radicals certainly cannot be regarded \textit{universal}.

We emphasize again that the deformation (6.7) is \textit{formal} and does not correspond to actual deformations of the embedding. By the same token, the introduction of this universal factor \( \Delta \) and then its identification with the radicals of the general type (6.8) is at best only equally formal and may serve for Yukawa coupling calculations [8], and so also for the calculation of periods of \( \Omega \) by direct integration [13][17].

Eventually, the residues representing these ‘higher cohomology’ and ‘twisted’ deformations of the complex structure should be obtainable in the framework of Ref. [9]. Suffice it here to note the following. In the 2-dimensional field theory, the representatives we have been studying should correspond to marginal operators. In a correlation function (the Yukawa coupling), these would appear something like

\[
\langle 0| \cdots (\Delta \delta f') \cdots |0 \rangle = Z_0^{-1} \int D[\phi] \cdots (\Delta \delta f') \cdots e^{-S_0},
\]

(6.12)

where \( \phi \) denote the scalar field zero-modes (the contribution from path-integration over nonzero-modes canceling between fermions and bosons) and \( \delta f' \) is the polynomial part of such a radical deformation. The second equality follows on noting that the additional factor \( \Delta \) (at least in all above examples and certainly all homogeneous complete intersections in projective spaces [8]) turns out to be a square-root of a determinant such as (6.8), and so can be ‘re-exponentiated’ by Gaussian integration over anticommuting \( \psi \)’s. The so re-exponentiated term is then

\[
S_2 = \int d^2\sigma \left( \psi i \partial_i \partial_j P(\phi) \psi^j \right),
\]

(6.13)

where \( \Delta = \sqrt{\det[\partial_i \partial_j P(\phi)]} \) and \( P \) is typically one of the defining polynomials or some part thereof, as in (6.8). Clearly, such a term is routinely present in the supersymmetric
completion of the action, and will contribute in the correlation function if the $\psi$'s are not paired with commuting modes, i.e., if the $\psi$'s are Fermionic zero-modes. Even in purely bosonic $\sigma$-models, such terms can arise in a BRST-type treatment of constraints, where the $\psi$'s would then be (odd order) ghost variables. In any case, this would seem to provide a straightforward field-theoretic explanation of the radical deformations.

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