The Heun Equation and the Calogero-Mosser-Sutherland System I: The Bethe Ansatz Method

Kouichi Takemura

Department of Mathematical Sciences, Yokohama City University, 22-2 Seto, Kanazawa-ku, Yokohama 236-0027, Japan. E-mail: takemura@yokohama-cu.ac.jp

Received: 28 September 2001 / Accepted: 31 October 2002
Published online: 31 January 2003 – © Springer-Verlag 2003

Abstract: We propose and develop the Bethe Ansatz method for the Heun equation. As an application, holomorphy of the perturbation for the $BC_1$ Inozemtsev model from the trigonometric model is proved.

1. Introduction

Olshanetsky and Perelomov proposed a family of integrable quantum systems, which is called the Calogero-Moser-Sutherland system or the Olshanetsky-Perelomov system [6]. In the early 90’s, Ochiai, Oshima and Sekiguchi classified integrable models of quantum mechanics which are invariant under the action of a Weyl group with certain assumptions [5]. For the case of $B_N$ ($N \geq 3$), the generic model coincides with the $BC_N$ Inozemtsev model. Oshima and Sekiguchi found that the eigenvalue problem for the Hamiltonian of the $BC_1$ (one particle) Inozemtsev model is transformed to the Heun equation with full parameters [7]. Here, the Heun equation is a second-order Fuchsian differential equation with four regular singular points.

In this paper we will investigate solutions to the Heun equation motivated by the analysis of the Calogero-Moser-Sutherland systems.

In the papers [9, 4], holomorphy of the perturbation for the Calogero-Moser-Sutherland model of type $A_N$ from the trigonometric model was proved, and in [9] some results were obtained by applying the Bethe Ansatz method. Note that the Bethe Ansatz for the model of type $A_N$ was established by Felder and Varchenko [2] by investigating asymptotic behavior of an integral expression of a solution to the KZB equation.

For the case of type $A_1$, the Bethe Ansatz method was found in the textbook of Whittaker and Watson [12], although it was covered in the chapter entitled “Lamé’s equation”, and the phrase “Bethe Ansatz” was not used. Note that the Lamé equation is a special case of the Heun equation [8].
In this paper we will propose and develop the Bethe Ansatz method for the Heun equation, and holomorphy of the perturbation for the $BC_1$ Inozemtsev model from the trigonometric model will be obtained.

The method “Bethe Ansatz” appears frequently in physics. In particular the Bethe Ansatz method has great merits for investigating solvable lattice models [1]. In this paper, we use “Bethe Ansatz” in a somewhat restricted sense, similar to the usage in [2, 9]. The Bethe Ansatz method replaces the problem of finding eigenstates and eigenvalues of the Hamiltonian (for example Eq. (2.2)) with a problem of solving transcendental equations for a finite number of variables which are called the Bethe Ansatz equations (for example Eqs. (3.29)). In our case, we use elliptic functions to describe the Bethe Ansatz equation.

It would be very difficult to solve the transcendental equations explicitly. Instead, by applying the trigonometric limit the transcendental equations tend to algebraic equations, which may be more easily solvable.

On the other hand, the Hamiltonian of the $BC_1$ Calogero-Sutherland model appears after applying the trigonometric limit, and the eigenstates for the $BC_1$ Calogero-Sutherland model are described by Jacobi polynomials. Thus the Jacobi polynomials and the solutions to the Bethe Ansatz equation for the trigonometric case are closely connected, and they are related to the perturbation. In fact holomorphy of the perturbation for the $BC_1$ Inozemtsev model from the trigonometric model follows from holomorphy of the solutions to the Bethe Ansatz equation with respect to a parameter of a period of elliptic functions.

Now we examine the perturbation. Based on the Hamiltonian $H_{CS}$ of the $BC_1$ Calogero-Sutherland model, the Hamiltonian $H$ of the $BC_1$ Inozemtsev model can be regarded as a perturbed one, i.e. $H = H_{CS} + V_0 + \sum_{i=1}^{\infty} V_i(x) p^i$, where $V_0$ is a constant and $V_i(x)$ ($i \in \mathbb{Z}_{\geq 1}$) are functions. We can calculate eigenvalues and eigenstates of $H$ as a formal power series in $p$ by the perturbation. For our case, holomorphy of the perturbation can be shown and it implies convergence of the formal power series in $p$ if $|p|$ is sufficiently small.

This paper is organized as follows. In Sect. 2, we clarify the relationship between the Heun equation and the $BC_1$ Inozemtsev system [7]. In Sect. 3, we introduce the Bethe Ansatz method for the $BC_1$ Inozemtsev system, or equivalently for the Heun equation. In Sect. 3.1, invariant subspaces of doubly periodic functions are introduced. In Sect. 3.2, we obtain an integral representation of eigenfunctions of the $BC_1$ Inozemtsev system. In Sect. 3.3, we introduce the Bethe Ansatz method for the $BC_1$ Inozemtsev system and describe the Bethe Ansatz equation. In Sect. 3.4, we rewrite the Bethe Ansatz equation in terms of theta functions in order to consider the trigonometric limit. In Sect. 4, we consider the trigonometric limit and solve the trigonometric Bethe Ansatz equation. In Sect. 5, relationships between the Bethe Ansatz method and the spectral problem on $L^2$-space for the $BC_1$ Inozemtsev system are clarified. Then holomorphy of the perturbation for the $BC_1$ Inozemtsev model from the trigonometric model is obtained. A key observation is that holomorphy of the perturbation follows from holomorphy of the solutions to the Bethe Ansatz equation. In Sect. 6, we give comments, and in the appendix, we note definitions and formulae of elliptic functions.

### 2. The Heun Equation and the Hamiltonian of the $BC_1$ Inozemtsev System

The expression of the Heun equation in terms of elliptic functions was already known by Darboux in the 19th century, and this appeared in connection with the soliton theory by Verdier and Treibich [11]. In [7], the relationship between the eigenfunctions for the