The effect of mirror surface distortions on higher order Laguerre-Gauss modes

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Abstract. Higher order Laguerre-Gauss (LG) beams have been proposed for use in future generation gravitational wave detectors for their potential to reduce the effects of the thermal noise of the test masses. However, it has been reported that due to the degeneracy of higher order modes using these beams will be extremely challenging. Our aim was to quantify these degeneracy effects. We present a new analytical approximation to compute the coupling between different LG modes, verified with simulation results of realistic arm cavities. This method is applied to Advanced LIGO mirror maps and used to derive requirements for mirrors for the use of the LG$_{33}$ beam.

1. Introduction

The sensitivities of second generation gravitational wave detectors are expected to be limited by thermal noise of the test masses, in particular coating brownian thermal noise, at a range of frequencies around 100 Hz \[1\]. The Brownian motion of atoms in the mirror surfaces leads to excitations of the elastic modes of the mirror, which is interpreted by the interferometer as a mirror displacement, mimicking the displacement caused by a gravitational wave. It has been proposed that the use of beams with a different intensity distribution than the currently used fundamental Gaussian beam could reduce the effects of mirror thermal noise \[2\,3\]. Beams whose intensity is more evenly distributed over a mirror surface can more effectively average over mirror surface distortions, such as those caused by thermal noise \[4\]. Several different beams have been suggested, including mesa beams \[5\], conical beams \[6\] and higher order Laguerre-Gauss (LG) beams. Higher order Laguerre-Gauss beams, unlike mesa and conical beams, are compatible with spherical mirrors, which are currently used in gravitational wave detectors. Higher order LG beams therefore seem the most appropriate candidate for thermal noise reduction.

The potential of using the particular higher order beam LG$_{33}$ in gravitational wave detectors has been investigated using simulations and table-top experiments \[7\,8\]. However, there is another aspect of higher order modes which must be considered. Higher-order gaussian modes are degenerate, or, in other words, there are multiple modes with the same order. The consequence of this is that an optical cavity resonant for a higher order LG mode will also be resonant for other modes of the same order. This is not the case for the currently used fundamental mode, which is unique in its mode order. This degeneracy can lead to additional optical losses. It has been shown that the LG$_{33}$ mode, compared to the fundamental mode, could
result in an unacceptably high contrast defect at the interferometer output due to coupling into other modes via mirror surface distortions [9, 10, 11, 12]. In this paper we summarise our investigations into this coupling, focusing on the effects of mirror surface distortions on beam purity when an LG_{33} beam is injected into a high finesse cavity. A more detailed account of these investigations can be found in [13].

2. Laguerre-Gauss beams
The shape of any paraxial beam can be described by a sum of Gaussian modes. Two complete and orthogonal sets are the Hermite-Gauss (HG) modes, with rectangular symmetry, and Laguerre-Gauss (LG) modes, with circular symmetry. The Laguerre-Gauss modes are defined by radial index \( p \geq 0 \) and azimuthal index \( l \), with the helical modes given by [14]:

\[
U_{p,l}(r,\phi,z) = \frac{1}{w(z)} \sqrt{\frac{2p!}{\pi(|l|+p)!}} \exp(i(2p+|l|+1)\Psi(z)) \\
\times \left( \sqrt{\frac{2r}{w(z)}} \right)^{|l|} L_{p}^{[|l|]} \left( \frac{2r^2}{w^2(z)} \right) \exp \left( - \frac{ikr^2}{2R_c(z)} - \frac{r^2}{w^2(z)} + il\phi \right)
\]  

where \( k \) is the wavenumber, \( w(z) \) is the beam radius, \( \Psi(z) \) is the Gouy phase and \( R_c(z) \) is the radius of curvature of the beam. \( L_{p}^{[|l|]} \) refer to the Laguerre polynomials. The quantity \( 2p+|l| \) is defined as the order of the mode. The displacement seen by an interferometer due to thermal noise depends on the intensity distribution of the beams incident on the mirrors. In general higher order beams more effectively average over the mirror surface and results in a smaller effective displacement. However, using higher order beams requires smaller beam spot sizes to achieve the same low power loss due to clipping by the mirrors. The particular LG beam suggested to reduce thermal noise is LG_{33}, which results in a good reduction of thermal noise for the low clipping losses required. A plot of the intensity distribution of this higher order beam, compared to the currently used fundamental beam, is shown in figure 1. This plot shows that the intensity of the fundamental beam is concentrated around the centre of the optical axis, whereas the intensity of the LG_{33} beam is more evenly distributed around the axis. In theory

![Figure 1](image_url)

**Figure 1.** Plots of the intensity of the fundamental gaussian mode (LG_{00}) and the LG_{33} mode as a function of distance from the optical axis.
2.1. Higher-order mode degeneracy

Laguerre-Gauss modes experience a phase shift of \((2p + |l| + 1)\Psi(z)\) compared to that of a plane wave. Modes of the same order \((2p + |l|)\) will accumulate the same round trip phase when circulating in the arm cavities of an interferometer and hence will be resonant for the same cavity tuning. Therefore, cavities designed for higher order modes, such as LG\(_{33}\), will be degenerate for modes of the same order. This can become a problem if there are processes which introduce new modes to the interferometer. The imperfect nature of the mirror surfaces results in distortions of the beams interacting with them. This can be described as coupling into other modes. If this coupling results in modes which are resonant for the cavity they will have large amplitudes in the cavity.

The fundamental Gaussian mode is currently used in gravitational wave detectors. LG\(_{00}\) is the only mode of order 0 so any coupling due to mirror surface distortions results in modes of a different order. These additional modes are suppressed in the cavities. LG\(_{33}\), on the other hand, is one of 10 order 9 modes; LG\(_{0,\pm 9}\), LG\(_{1,\pm 7}\), LG\(_{2,\pm 5}\), LG\(_{3,\pm 3}\) and LG\(_{4,\pm 1}\) (figure 2). Any coupling into these modes via mirror surface distortions could result in modes other than LG\(_{33}\) with large amplitudes in the cavity. As the surface distortions will be different in the mirrors in the different cavity arms the circulating fields will no longer have the same mode content and will not interfere as expected at the dark port. This could result in a large contrast defect.

3. Describing mirror surfaces

In an ideal world the mirrors used in gravitational wave detectors would be perfectly smooth, with a radius of curvature matching that of the incident beam. The effect of a distorted mirror surface \(Z(x, y)\), with reflectivity \(r\), on a beam \(U(x, y, z)\), reflected from the surface can be described by a phase shift:

\[
U_{\text{ref}}(x, y, z) = U(x, y, z) r \exp (i2kZ(x, y))
\]

(2)

where \(k\) is the wavenumber and the beam is propagating in the \(z\) direction. In terms of higher-order Laguerre-Gauss modes we can describe the beam reflected from the mirror surface as having a different mode composition to the incident beam. The coupling from a mode \(U_{p,l}\) in the incident beam into a mode \(U_{p',l'}\) in the reflected beam, considering a completely reflecting surface \(Z\) can be described by a coupling coefficient \([16, 17]\):

\[
k_{p,l,p',l'}^Z = \int_S U_{p,l} \exp (2ikZ(r, \phi)) U_{p',l'}
\]

(3)

\(Z\) describes the surface height of the mirror and \(S\) describes an infinite plane perpendicular to the optical axis.
Figure 3. Surface plot of the mirror map corresponding to the Advanced LIGO end test mass ETM08. The mirror map data refers to the difference in surface height from a perfect sphere.

3.1. Mirror maps
In order to investigate the effects of surface distortions in gravitational wave detectors we require methods of representing surface distortions in mathematical models and simulations. One such method is to use mirror surface maps measured from existing mirrors designed for gravitational wave detectors. Mirror maps refer to an array of data detailing particular properties of a mirror over its surface. In this case mirror maps refer to the surface height of the mirror. We can use these maps to represent realistic mirror surface distortions in simulations of gravitational wave detectors. The best mirror maps of this kind currently available are those produced from mirror substrates for Advanced LIGO. In our investigations we used a typical Advanced LIGO mirror map, ETM08 [18], produced from the curved surface of an end test mass. The surface map for this mirror is shown in figure 3, where the surface refers to the deviation from a perfect sphere with radius of curvature 2249.28 m. This surface has a RMS figure error of 0.523 nm.

3.2. Zernike polynomials
Zernike polynomials provide another method of representing mirror surface distortions. Zernike polynomials form a complete set of functions which are orthogonal over the unit disc [19, 20]. Each polynomial is defined by a radial index $n$ and azimuthal index $m$, with $0 \leq m \leq n$. For any index $m$ there is an even and odd polynomial:

$$Z_{n}^{+m}(\rho, \phi) = A_{n}^{+m} \cos(m\phi)R_{n}^{m}(\rho) \quad \text{even polynomial}$$

$$Z_{n}^{-m}(\rho, \phi) = A_{n}^{-m} \sin(m\phi)R_{n}^{m}(\rho) \quad \text{odd polynomial}$$

(4)

where $A_{n}^{\pm m}$ is the amplitude of the polynomial, $\rho$ is the normalised radius, $\phi$ is the azimuthal angle and $R_{n}^{m}(\rho)$ is the radial function. The radial function is given by:

$$R_{n}^{m}(\rho) = \sum_{h=0}^{\frac{1}{2}(n-m)} \frac{(-1)^{h}(n-h)!}{h!(\frac{1}{2}(n+m)-h)!} \frac{1}{2!(n-m)-h)!} \rho^{n-2h}$$

(5)

for even $n-m$ and 0 otherwise. Any surface defined over a disc, such as the mirrors used in gravitational wave detectors, can be described as a sum of Zernike polynomials, with higher order ($n$) Zernike polynomials representing higher spatial frequencies.
4. Analytical description of mode coupling via surface distortions

Using Zernike polynomials to describe mirror surface distortions we can analyse the coupling between LG modes. Coupling via a mirror described by an individual Zernike polynomial is given by:

\[ k_{p,l,p',l'}^{n,m} = \int_S U_{p,l} \exp (2ikZ_{n,m}) U_{p',l'}^* \]  

As the surface distortions are small compared to the wavelength we can simplify the integral using the approximation:

\[ \exp (2ikZ) \approx 1 + 2ikZ \]  

Solving the integral \[13\] it can be shown that the only non-zero solutions to the approximation occur when:

\[ m = |l - l'| \]  

This is a significant result as we can already make predictions about the types of surface distortions which could lead to significant coupling between modes of the same order. The final approximation of the coupling is given by:

\[ k_{p,l,p',l'}^{n,m} = A_n^m k \sqrt{p!p'!(p+|l|!(p'+|l'|)!)} \left| \sum_{i=0}^{p} \sum_{j=0}^{p'} \sum_{h=0}^{\frac{1}{2}(n-m)} \frac{(-1)^{i+j+h}}{(p-i)!(|l|+i)!i!} \times \frac{1}{(p'-j)!(|l'|+j)!j!} \gamma^2(i + j + h + \frac{1}{2}(|l| + |l'| + n) + 1, X) \right| \]

where \( A_n^m \) is the amplitude of the Zernike polynomial representing the mirror surface, \( X = \frac{2R^2}{w^2} \), \( R \) is the radius of the mirror, \( w \) is the beam radius and \( \gamma \) refers to the lower incomplete gamma function.

4.1. Coupling for order 9 modes

We are concerned with coupling from LG_{33} into the other order 9 modes, and want to identify the particular surface distortions which will cause such couplings. Table 1 shows the required azimuthal index required to couple significantly from LG_{33} into the other order 9 modes. We can predict which particular distortions, if present in the mirror surfaces, will cause coupling into particular order 9 modes. As we have \( m \leq n \) these \( m \) requirements also tell us the lowest order \( (n) \) Zernike polynomial required to couple from LG_{33} into each of the other order 9 modes.

| \( m \)   | 2  | 2  | 4  | 4  | 6  | 6  | 8  | 10 | 12 |
|----------|----|----|----|----|----|----|----|----|----|
| LG_{p,l} mode | 2, 5 | 4, 1 | 1, 7 | 4, -1 | 0, 9 | 3, -3 | 2, -5 | 1, -7 | 0, -9 |
5. Simulating cavities with mirror surface distortions

The problem of higher-order mode degeneracy originates in the arm cavities of the detectors. The analytical approach taken in section 4 applies to a single reflection, whereas inside a cavity multiple reflections and couplings of the beam occur. However, for small coupling coefficients it is expected that the direct coupling from the input beam will be the dominant process in determining the mode content of the circulating beam. In this section we compare our analytical approach with simulations of realistic arm cavities.

Simulations of arm cavities were carried out to assess the purity of the beams circulating in such cavities when $LG_{33}$ is injected into a cavity simulated with realistic mirror surface distortions. Here we refer to the purity in terms of the $LG_{33}$ mode content of the beam. Cavity simulations were carried out in Finesse [21, 22] based on Advanced LIGO arm cavities. The arm cavities in Advanced LIGO consist of two curved mirrors; the end test mass (ETM) and input test mass (ITM), separated by approximately 4 km. These cavities are designed for the fundamental Gaussian beam. As the $LG_{33}$ beam has a wider intensity distribution than $LG_{00}$ the Advanced LIGO cavity parameters result in a clipping loss of around 35%. In order to achieve smaller beam spots on the mirrors it was necessary to decrease the cavity length. A cavity length of 2802.9 m gives similar clipping to that experienced by the $LG_{00}$ beam in the Advanced LIGO cavities, for the 30 cm aperture which the mirror maps represent and using the original mirror curvatures of 1934 m for the ITM and 2245 m for the ETM. The simulations used the Advanced LIGO mirror parameters [23, 24].

5.1. Circulating purity with Advanced LIGO mirror maps

In our simulations the Advanced LIGO mirror map ETM08 (figure 3) was used to represent surface distortions on the end test mass. In order to relate these simulations with our previous analytical approach and predict the coupling at this mirror we look at the Zernike polynomials present in the map. The Zernike content of the ETM08 mirror map is summarised in table 2. The content is ordered according to the power coupled from $LG_{33}$ into the other order 9 modes by the individual polynomials, according to our approximation (equation 9). This shows that the astigmatism ($Z_2^2$) causes a large amount of coupling from $LG_{33}$ into the other order 9 modes. This suggests that the order 9 modes $LG_{25}$ and $LG_{41}$ will have significant amplitudes in the cavity, as polynomials with $m = 2$ should couple largely into $LG_{25}$ and $LG_{41}$ (equation 8). The polynomials $Z_2^2$ and $Z_4^4$ also couple relatively large powers into the other order 9 modes, which should add to $LG_{25}$ and $LG_{41}$ in the case of $Z_4^4$, and $LG_{17}$ and $LG_{4,1}$ in the case of $Z_4^4$.

The cavity defined in section 5 was simulated with a $LG_{33}$ input beam and the ETM08 mirror map. The field circulating in the cavity on resonance for $LG_{33}$ was detected and is shown in figure 4. Comparison with the $LG_{33}$ beam shown in figure 2 shows that the circulating beam is no longer pure $LG_{33}$. The mode content of the circulating beam is summarised in table 3. The purity of the $LG_{33}$ beam circulating in the simulated cavity was found to be 88.6% with most

Table 2. A summary of the Zernike polynomials representing the Advanced LIGO mirror map ETM08. The polynomials are ranked according to the power they individually subtract from $LG_{33}$ into the order 9 modes, according to the approximation (equation 9).

| $Z_m^n$ | 2, 2 | 4, 2 | 4, 4 | 6, 2 | 10, 8 | other |
|---------|------|------|------|------|------|-------|
| $A_m^n$ [nm] | 0.908 | 0.202 | 0.213 | 0.124 | 0.116 | - |
| Power [ppm] | 4.66 | 0.331 | 0.0431 | 0.0099 | 0.0059 | < 0.005 |
Table 3. A summary of the mode content of a beam circulating in a cavity simulated with a LG<sub>33</sub> input beam and the Advanced LIGO mirror map ETM08 applied to the end mirror.

| LG<sub>p,l</sub> mode | 3, 3 | 4, 1 | 2, 5 | 4, -1 | 1, 7 | other |
|----------------------|------|------|------|-------|------|-------|
| Power [%]            | 88.6 | 5.70 | 5.02 | 0.333 | 0.313 | < 0.05 |

of the coupled power in the other order 9 modes. We find that the two largest coupled modes in the cavity are LG<sub>25</sub> and LG<sub>41</sub>. This is in agreement with the predictions made from looking at the Zernike content of ETM08 mirror map (table 2). LG<sub>17</sub> and LG<sub>4,1</sub> also have relatively large amplitudes in the cavity. This could be due to direct coupling via \( m = 4 \) polynomials or coupling of LG<sub>25</sub> and LG<sub>41</sub> via \( m = 2 \) polynomials. The coupling process is complicated, but these simulation results appear to suggest that the direct coupling, as described in our mathematical model, is the dominant effect on the mode content in a cavity.

5.2. Circulating purity with improved mirrors

To provide mirror requirements for the purposes of using higher order LG beams we consider which distortions cause the most coupling. Firstly, looking at the results using the ETM08 mirror map, we suggest that the astigmatism is the dominant distortion, in terms of coupling between order 9 modes. In order to verify this the cavity was simulated again with the ETM08 mirror map but with the astigmatism removed from the map data. The circulating beam for this simulation is shown in figure 5. The LG<sub>33</sub> purity of the circulating beam was found to be 99.5%, a large improvement on the original purity of 88.6%. This supports the idea that astigmatism is a significant cause of coupling between order 9 modes. The mode content of the circulating beam is summarised in table 4. Although the power in modes LG<sub>41</sub> and LG<sub>25</sub> has been significantly reduced they are still the dominant modes after LG<sub>33</sub>. This suggests that astigmatism is not the only distortion present in this mirror which couples into these modes.

Figure 4. The field circulating on resonance in a simulated cavity with an LG<sub>33</sub> input beam and the ETM08 mirror map applied to the ETM.

Figure 5. The field circulating in a cavity simulated with an LG<sub>33</sub> input beam with the ETM08 mirror map. In this case the astigmatism has been removed from the mirror map.
Table 4. A summary of the modes circulating in a cavity simulated with a LG\textsubscript{33} input beam and the Advanced LIGO mirror map ETM08, with the astigmatism removed.

| \(LG_{p,l}\) mode | 3, 3 | 4, 1 | 2, 5 | 1, 7 | 4, -1 | 0, -9 | other |
|-------------------|------|------|------|------|-------|-------|-------|
| Power [%]         | 99.5 | 0.231| 0.208| 0.0524| 0.0165| 0.0137| < 0.05|

6. Mirror requirements for LG\textsubscript{33}

Our investigation has shown that mirror surface distortions lead to a large degradation of the purity of LG\textsubscript{33} circulating in the arm cavities. In order to use LG\textsubscript{33} in gravitational wave detectors we require specific Zernike polynomials to be smaller than in the current available mirrors. The direct coupling between order 9 modes was investigated for our example mirror map, ETM08. Figure 6 illustrates the coupling from LG\textsubscript{33} into the other order 9 modes from the ETM08 mirror map. In this plot the overall coupling from the Zernike polynomials representing the mirror map is approximated using equation 9, where \(n\) refers to the maximum order of the Zernike polynomials included in the approximation. This plot shows that the two modes LG\textsubscript{25} and LG\textsubscript{41} experience around 10 times the coupling than the other order 9 modes. The plot suggests that the coupling into these modes is not significantly increased by including higher order polynomials, i.e. the coupling into these modes is caused predominantly by low order polynomials. Therefore, reducing low order polynomials with \(m = 2\), the required index for these couplings, should lead to an improvement in LG\textsubscript{33} beam purity and should be the first requirement for the mirror surfaces.

Overall the ETM08 mirror map directly couples 31 ppm of the incident power from LG\textsubscript{33} into other modes. Of this 6.8 ppm is coupled into the other order 9 modes. The Zernike content of the ETM08 mirror map is summarised in table 2. In this case we consider individual polynomials coupling more than 0.01 ppm into order 9 modes as causing a large amount of coupling. For this mirror these are the \(Z_2^2\), \(Z_3^2\) and \(Z_4^4\) polynomials. Table 5 summarises the requirements for these
Table 5. Amplitude requirements for surfaces defined by Zernike polynomials to couple 0.01 ppm from LG\textsubscript{33} into the other order 9 modes.

| Z\textsuperscript{m/n} polynomial | 2, 2 | 4, 2 | 4, 4 |
|----------------------------------|------|------|------|
| Amplitude [nm]                   | 0.042| 0.035| 0.100|

3 polynomials to couple 0.01 ppm directly into order 9 modes. Applying these requirements to the mirror map ETM08 results in direct coupling of 19 ppm into other modes with just 0.043 ppm coupling into the other order 9 modes. The cavity defined in section 5 was simulated with this modified map, achieving a circulating beam impurity of 815 ppm. This is a very significant improvement on the original impurity of 114,000 ppm (88.6\% LG\textsubscript{33} purity) and a further improvement on the 5220 ppm impurity (99.5\% LG\textsubscript{33} purity) when the astigmatism was completely removed from the map.

We have demonstrated that a high beam purity is achievable by limiting the amplitudes of specific low order Zernike polynomials in the mirror surfaces. In order to achieve even higher beam purity the amplitudes of these distortions will have to be reduced further, as well as reducing other polynomials in the mirror surface.

7. Conclusion

We have presented an analytical approach to coupling between Laguerre-Gauss modes and used this approximation to investigate the problem of higher-order mode degeneracy in gravitational wave detectors, for the particular example of the LG\textsubscript{33} mode. The condition for significant coupling between two LG modes, \( m = |l - l'| \), allows us to identify which modes will have large amplitudes in the arm cavities of detectors and which polynomials should be limited to achieve high beam purity.

The performance of the LG\textsubscript{33} beam was investigated using simulations of arm cavities, with Advanced LIGO mirror maps describing mirror surface distortions. It was found that mirror surface distortions significantly reduce the purity of LG\textsubscript{33} circulating in a cavity, with relatively large amplitudes of LG\textsubscript{25} and LG\textsubscript{41} in the cavities. Our analytical approach suggested that the astigmatism of the mirror was largely responsible for the coupling into these modes, which was confirmed by the resulting increase in purity when the cavity was simulated with the astigmatism removed.

By analysing the coupling from the specific mirror map, ETM08, we derived mirror requirements (table 5) for the use of LG\textsubscript{33}. Applying these to the mirror map resulted in a reduction of circulating LG\textsubscript{33} impurity from 114,000 ppm to 815 ppm. This demonstrates that a high beam purity can be achieved by reducing specific low order polynomials in Advanced LIGO mirrors. However, implementing the LG\textsubscript{33} beam in gravitational wave detectors will be challenging as the amplitudes of the lower order Zernike polynomials must be very small. At the time of our investigation the mirrors had not been coated, which will further add to the distortions present. However, using this analytical approach we can suggest the requirements needed to design suitable mirrors for these higher order beams.
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