Thouless pumping provides one of the simplest manifestations of topology in quantum systems and has attracted a lot of recent interest, both theoretically and experimentally. Since the seminal works by David Thouless and Qian Niu in 1983 and 1984, it has been argued that the quantization of the pumped charge is robust against weak disorder, but a clear characterization of the localization properties of the relevant states, and the breakdown of quantized transport in the presence of interaction or out of the adiabatic approximation, has long been debated. Thouless pumping is also the first example of a topological phase emerging in a periodically driven system. Driven systems can exhibit exotic topological phases without any static analogue and have been the subject of many recent proposals both in fermionic and in bosonic systems. Recent experimental studies have been performed in diverse platforms ranging from cold atoms to photonics and condensed-matter systems. This Review serves as a basis to understand the robustness of the topology of slowly driven systems and also highlights the rich properties of topological pumps and their diverse range of applications. Examples include systems with synthetic dimensions or work towards understanding higher-order topological phases, which underline the relevance of topological pumping for the fast-growing field of topological quantum matter.

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**Key points**

- A direct current is usually associated with a dissipative flow of electrons in response to an applied bias voltage. In quantum systems, however, coherent transport can be induced via adiabatic cyclic variation of at least two system parameters in the absence of any external bias.
- A Thouless pump is a 'quantum' pump, in which the amount of 'charge' pumped during one cycle is quantized according to the Chern number — a topological invariant.
- Quantized charge pumps are robust to perturbations, such as disorder or weak interactions, as long as they do not change the topology of the pump.
- One-dimensional topological pumps can be seen as dynamical versions of the 2D integer quantum Hall effect in (1+1)-D, in which time plays the role of one spatial dimension and can be understood as a synthetic dimension.
- The concept of synthetic dimensions opens the door towards realizations of exotic higher-dimensional systems, such as the 4D quantum Hall effect.

**Introduction**

A quantum pump is a device able to generate a particle current via slow and periodic modulation of at least two system parameters, in the absence of any external bias. For this reason, it is considered as one of the most intriguing effects in quantum mechanics. In a conductor, a direct current (d.c.) is usually associated with a dissipative flow of electrons in response to an applied bias voltage. In systems of mesoscopic scale, a d.c. can be generated even at zero bias (such as in semiconductor nanostructures of nanometre size and tens of atoms) in the presence of slow periodic perturbations. In the adiabatic limit, when the applied perturbations are slow in comparison to the escape rate to external contacts, the electronic state of the quantum system is the same after a period, but a net charge has been transferred owing to a squeezing of the wavefunction in the central region. As the quantum state of the system remains coherent, this effect is known as quantum charge pumping. Although different in nature, quantum pumping shares some similarity with other fascinating phenomena such as persistent currents and superconductivity and allows exploring fundamental issues regarding the role of topology and symmetries, finding its maximum expression in the Thouless pump (which is topological in nature), in which transport is quantized.

Quantum pumping has received much attention in mesoscopic electronic systems, mainly owing to its potential of reducing the dissipation of energy as wasteful heat, defining a better current standard for metrological purpose or even being used for quantum computing. Recent experimental realizations of Thouless pumps have been observed in photonics, magneto-mechanical and electro-mechanical systems and ultracold atoms, including pumps with interactions and pumps in open systems.

In this Review, we introduce the fundamentals of topological pumping, discuss its limitations in the presence of interactions, disorder and non-adiabaticity and connect its properties to the physics of higher-dimensional topological systems via the concept of synthetic dimensions and higher-order pumps.

**Topological quantized pumping**

Thouless pumping entails the transport of charge, in the absence of a net external electric or magnetic field, through an adiabatic cyclic evolution of the underlying Hamiltonian. In contrast to classical transport, the transported charge in a Thouless pump is quantized and purely determined by the topology of the pump cycle, making it robust against perturbations, such as interaction effects or disorder. Intuitively, the Thouless pump can be thought of as the quantum version of the famous Archimedes screw (Fig. 1a), in which a directional motion of water is generated by slow and periodic movement of the handle. In the quantum case, the turning handle corresponds to a moving potential. The key difference being that in the classical Archimedes screw the amount of water pumped per cycle can be continuously tuned by changing the tilt angle of the screw, whereas for the topological Thouless pump, the amount of charge pumped per cycle cannot be continuously tuned, rather, it is quantized according to the topology of the pump cycle.

Consider spinless electrons in 1D subject to a potential $U(x)$ with periodicity $a$, such that $U(x + a) = U(x)$. Provided the number of electrons $n_a$ per period $a$ equals an integer $N$, the lowest $N$ bands of the energy spectrum are occupied, whereas the higher bands are empty. Now let the potential slowly vary periodically in time, such that $U(x, t) = U_0(x) + U_1(x - vt)$, where $U_0$ and $U_1$ share the same periodicity $a$ and $v$ is some small velocity with $a/v = T$ (Fig. 1b). If the electrons follow the variation of the potential adiabatically, a quantized charge $Q$ is transferred per period $T$, in which the quantization is determined by the topology of the energy bands that are occupied with electrons and by the topology of the pump cycle. Each energy band including

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**Fig. 1 | Classical and quantum pumping.**  
**a,** Illustration of a classical Archimedes screw. **b,** Schematic drawing of two periodic potentials, $U_0(x)$ and $U_1(x, t)$, that share the same periodicity. One of the two potentials, $U_1(x, t)$, is moving with a small velocity $v$ with respect to the other to realize a quantum pump. Panel **a** illustrated by C. Hohmann (MCQST).
the time \( t \) as a second dimension, hence realizing a \((1+1)\)-D parameter space, is associated with an integer number, a topological invariant called the Chern number (Box 1). The response of the whole system is given by the sum of all Chern numbers of the occupied energy bands, which can be viewed as a dynamical realization of the integer quantum Hall effect (IQHE) (Box 2).

Intuitively, the time evolution of the potential can be described in terms of a ‘trajectory’ \( \gamma \) of the system in a 2D plane, whose parameters are determined by the time-dependent potential \( U(x, t) \). If \( \gamma \) encircles a degeneracy point of the Hamiltonian, the pumped charge is quantized and the response will remain quantized, as long as the trajectory encircles the degeneracy point. This illustrates the robustness of topological pumping to disorder and interactions, as small deformations of the trajectory cannot alter the pumped charge. However, the quantization of the pumped charge is only valid as long as the potential is varied adiabatically. Studies away from the adiabatic limit show that, despite its topological nature, this phenomenon is not generally robust to non-adiabatic effects. Indeed, the mean value of the pumped charge shows a deviation from the topologically quantized limit, which is quadratic in the driving frequency for a sudden switch-on of the drive. Interestingly, even at fast driving, one can realize an ideal pump under a family of protocols, which contain the adiabatic one as a limiting case\(^{2}\).

**Charge pumping**

Topological charge pumping has remained out of reach in most electron-based condensed matter experiments because of challenges in realizing the adiabatic regime. Owing to the versatility and control of synthetic quantum systems, experimental demonstrations of adiabatic quantum pumps have been achieved with cold atoms\(^{14,12,13}\) and photons\(^{14,15}\). Specifically, ultracold atoms in optical superlattices (Fig. 2) have emerged as an ideal platform for the implementation of quantized topological charge pumps\(^{12,14}\). In these experiments, the Thouless potential, a sliding superlattice, is formed by superimposing two lattices with different periodicities \( d_s \) and \( d_l = a d_s \), \( a < 1 \). This superposition generates a potential \( V_s = \sin^2\left(\frac{\pi x}{d_s} + \frac{\pi y}{2}\right) + V_l \sin^2\left(\frac{\pi x}{d_l} - \varphi/2\right) \) in which \( V_s(V_l) \) denotes the depth of the short-(long-)wavelength lattice, respectively (Fig. 2a). Changing the relative phase \( \varphi \) between the two lattices results in a periodically modulated superlattice potential. Varying the phase adiabatically by an amount of \( 2\pi \) realizes one pump cycle, in which the long lattice has moved by \( d_l \) with respect to the short lattice. As the change is performed adiabatically, a particle initially in a Bloch eigenstate \( |\psi_n(k_x, \varphi(t))\rangle \) of the Hamiltonian \( H(k_x, \varphi(t)) \) follows the instantaneous eigenstate \( |\psi_0(k_x, \varphi(t))\rangle \) and remains in an eigenstate at all times; here \( n \) denotes the \( n \)th Bloch band and \( k_x \) is the quasi-momentum. However, after one cycle, the state has acquired a geometric phase proportional to \( \varphi \). This phase, known as the Berry phase, is proportional to the integral over the Berry curvature \( \Omega \) of the occupied bands along the pump path (Fig. 2c). Owing to the non-commutativity between position and momentum, this phase generates an anomalous velocity given by \( \mathbf{v}_\varphi = \Omega \partial \varphi \). The displacement of the cloud after one cycle can be obtained by integrating the anomalous velocity of the occupied Bloch states over one pump cycle. For a uniformly filled band, the displacement is quantized according to the topological (2D) invariant, the Chern number, \( v_\varphi \), which is an integer. It is defined as the integral over the closed surface in \((k_x, \varphi)\) space:

\[
v_\varphi = \frac{1}{2\pi} \int_{BZ} \int_{0}^{2\pi} \Omega_\varphi(k_x, \varphi) \, dk_x \, d\varphi,
\]

where BZ denotes the first Brillouin zone of the superlattice. During one pump cycle, the centre of mass (CoM) changes by \( v_\varphi d_l \) and is quantized in units of \( d_l \). If more than one energy band is completely filled, the CoM response is given by the sum over all energy bands \( v_\varphi \sum_i d_{y_i} \). Intriguingly, the induced motion can occur either in the same or in the opposite direction as the moving lattice, depending on the sign of \( v_\varphi \). This counterintuitive behaviour has been demonstrated by preparing ultracold bosonic atoms in the first excited band of an optical superlattice, where quantized transport in the opposite direction of the moving lattice was found\(^{12}\).

We emphasize that quantized topological charge pumps require uniformly filled bands. In contrast, if a Bose–Einstein condensate occupies just a single quasi-momentum state, the system exhibits non-quantized charge pumping set by the local geometric properties of the band\(^{14}\). This is known as a geometric charge pump. Similar to topological charge pumps, there is an overall displacement per pump cycle, which in this case is, however, not quantized. Near-perfect quantized pumping can be restored, however, by adding a linear tilt. Intuitively, the added potential assists in the uniform sampling of all momenta owing to the Bloch oscillations induced by the tilt\(^{14}\).

**Box 1**

**The Chern number**

Quantized pumping can be understood as the result of geometric phases. The most well-known example is the Aharonov–Bohm phase. In brief, charged particles in the presence of a magnetic field that evolve along a closed trajectory acquire a phase that is equal to the magnetic flux piercing the area that is enclosed by the trajectory in units of \( h/e \), where \( h \) is Planck’s constant and \( e \) is the charge of an electron. This phase can be measured directly and manifests itself, such as in the anomalous magnetoresistance\(^{13,19}\).

Berry\(^{12}\) demonstrated that the Aharonov–Bohm phase is an example of a more general phenomenon. Suppose the Hamiltonian of a closed quantum system depends on some parameters, which are changed adiabatically in time, such that after a period \( T \), the system returns to the original values, thereby encircling a closed path \( \gamma \) in the parameter space. Simultaneously, the wavefunction of the system may acquire an additional geometric phase, the so-called Berry phase, which depends on \( \gamma \). Similar to the Aharonov–Bohm phase, the Berry phase can be thought of as the result of a flux of some effective ‘magnetic field’, known as Berry curvature in the mathematical literature, through the contour \( \gamma \). The difference is that both the contour and the magnetic field exist in an abstract parameter space rather than in real space. The single-valuedness property of the wavefunction requires that the Berry curvature integrated over a closed manifold is quantized in units of \( 2\pi \). For a 1D time-periodic system with Bloch bands, the pumped charge results from a fictitious ‘magnetic flux’ in an abstract \((1+1)\)-D parameter space formed by quasi-momentum \( k \) and time \( t \), which are both periodic. The charge pumped after one cycle can be expressed as an integral of the effective magnetic field over the area enclosed by the pump path \( \gamma \). For a uniformly filled Bloch band, the pumped charge after one period \( T \) is quantized according to the Chern number \( v_\varphi \), which is an integer.
Box 2

Connection to the quantum Hall effect

To illustrate the connection between the Thouless pump and the integer quantum Hall effect (IQHE), let us consider a 1D topological charge pump described by the generalized Rice–Mele model:

$$\hat{H}(\phi) = -\sum_{m} [J_{m}(\phi)\hat{a}_{m}^{\dagger}\hat{a}_{m} + \text{h.c.}] + \sum_{m} \Delta_{m}(\phi)\hat{a}_{m,\uparrow}^{\dagger}\hat{a}_{m,\downarrow}.$$  

(16)

This Hamiltonian is periodic in the site index $m$ and the pump parameter $\phi \in [0, 2\pi]$. Therefore, the eigenstates of the pump are parametrized by $k_{\phi}, \phi$. For implementations based on cold atoms in bichromatic superlattices\(^{131}\), the local site-dependent potential can be expressed as $V_m(\phi) = -\Delta \cos(2\pi m l - \phi)$, where $\alpha = d_\phi/d_\psi$ is the ratio of the two lattice constants. Assuming $\Delta = 0$, it is apparent that Eq. (16) corresponds to the 1D Harper equation, that is, the 1D eigenvalue equation for states with well-defined transverse quasi-momentum $k_{\phi}$ of the 2D Harper–Hofstadter model\(^{140–142}\) that describes the dynamics of charged particles on a square lattice with homogeneous magnetic field. Hence, we can identify the phase $\phi$ with the transverse quasi-momentum $k_{\phi}$, and the topological charge pump can be interpreted as a dynamical version of the IQHE. The BZ spanned by $k_{\phi}, \phi$ in 1D is then equivalent to the BZ spanned by $k_{\phi}, k_{\phi}$ in the corresponding 2D model. Note that in the context of localization Hamiltonian (16) is known as the Aubry–André model\(^ {143}\), if $\alpha$ is irrational.

The quantization of the transverse Hall conductance in the 2D IQHE is one of the hallmark phenomena of topological condensed matter systems. The plateau values are uniquely defined by the topological invariant characterizing the electron bands, that is, the (first) Chern number. Similarly, quantized charge transport in 1D Thouless pumps is determined by the first Chern number of the Rice–Mele model, and can be interpreted as the dynamical version of the quantum Hall effect. In general, a 1D pump Hamiltonian can be extended to a 2D Hamiltonian via dimensional extension\(^ {112,144}\):

$$\hat{H}_{2D} = \frac{1}{2m} \int_{0}^{2\pi} \hat{H}(\phi) d\phi.$$  

(17)

For $\Delta = 0$, this yields a square-lattice tight-binding Hamiltonian with uniform flux $2\pi\alpha$, given by the phase factor in

$$\hat{H}_{HH} = -\sum_{m,n} (J_{m+1,n}\hat{a}_{m,n}^{\dagger}\hat{a}_{m,n} + \Delta/4 \hat{a}_{m+1,n}^{\dagger}\hat{a}_{m,n}^{\dagger}\hat{a}_{m,n+1}\hat{a}_{m,n} + \text{h.c.}),$$  

(18)

where $\alpha$ corresponds to the spatial periodicity of the potential, which for the superlattice potential is determined by the value $\alpha = d_\phi/d_\psi$. Intriguingly, this paves the way towards studying various different topological pumps, in which the Chern number of the lowest band can, in principle, take arbitrary integer or fractional integer values\(^ {115}\), which can be experimentally realized by simply adjusting the ratio of the lattice constants $\alpha = d_\phi/d_\psi$. This can lead to counterintuitive cases of charge pumping, in which the atoms move faster than the sliding lattice\(^ {116}\).

The mechanism underlying the quantum pump can also be described on a microscopic level by considering the Wannier tunnelling regime $V_{\psi} \ll (4E_{\psi}/E_{\psi})$, where $E_{\psi} = h^2/(8md_{\gamma})$ denotes the recoil energy and $m$ is the mass of an atom. In the tight-binding limit, the superlattice model with $d_{\gamma} = 2d_{\psi}$ and two sites per unit cell (Fig. 2a) is described by the Rice–Mele (RM) Hamiltonian\(^ {135}\):

$$\hat{H}(\phi) = -\sum_{l} \left( J_{l}(\phi)\hat{b}_{l}^{\dagger}\hat{b}_{l} + J_{l}(\phi)\hat{b}_{l}^{\dagger}\hat{b}_{l} + \text{h.c.} \right) + \frac{\Delta(\phi)}{2} \sum_{l} (\hat{d}_{l}^{\dagger}\hat{d}_{l} - \hat{d}_{l}\hat{d}_{l}^{\dagger}).$$  

(2)

where $\hat{d}_{l}^{\dagger}$ and $\hat{d}_{l}$ are the creation (annihilation) operators acting on the left and right sites of the $l$th unit cell, $J_{l}$ and $J_{l}$ denote the intra-unit and inter-unit cell hopping, respectively, and $\Delta$ is the energy offset between neighbouring sites.

The pumping mechanism can be understood from the point of view of a cycle in the parameters space $(J_{l}, -J_{l}, \Delta)$, whose winding number is directly related to the Chern number of the pump\(^ {25}\). For $J_{l} - J_{l} = \Delta = 0$, the ground state at half filling is gapless, defining the degeneracy point in the centre of the $(J_{l}, -J_{l}, \Delta)$ parameter space (Fig. 2b). The winding number $w$ is an integer that characterizes the chirality as well as the number of times the pump cycle winds around the degeneracy point. The winding number for the cycle shown in Fig. 2b is $w = +1$, which is equal to the Chern number of the pump, and states that this cycle realizes quantized particle transport of one unit cell in the direction of the moving potential.

Intuitively, the mechanism of the quantized pump can be described as follows: changing $\phi$ is equivalent to changing the shape of the superlattice potential (Fig. 2d). Starting at $\phi = 0$ $(\Delta = 0$ and $J_{l} > J_{l})$, the ground state at half filling for non-interacting fermions\(^ {13}\) or equivalently hardcore bosons\(^ {13}\) consists of localized particles in the symmetric superposition between the two sites of the unit cell. For increasing $\phi$, the long lattice is shifted to the right and the double wells are tilted $(\Delta > 0)$ in such a way that the atom tunnels towards the lower-lying site on the right. The tilt is maximum at $\phi = \pi/2$ and changes its sign afterwards until the lattice forms symmetric double wells again at $\phi = \pi$, with a shift of one short lattice constant $d_{\gamma}$ to the right. The atom that was on the lower site for large $\Delta$ delocalizes on the double well and moves by $d_{\gamma}$ during the first half of the pumping cycle. After one full cycle, the lattice configuration becomes identical to the initial one, but the atom has moved to the neighbouring double well. Intriguingly, the lattice minima do not move in real space; hence, a classical particle would not move. In contrast, quantum mechanically, the particle tunnels to the neighbouring sites during one pump cycle. Experimentally, the resulting motion of the atoms is observed by measuring the CoM position of the atoms (Fig. 2d). The displacement occurs in steps and is indeed quantized\(^ {25}\) in units of the long-lattice constant $d_{\gamma}$ for $\nu = +1$.

Before topological Thouless pumps have been realized in cold atoms and photonics, a closely related quantized, however, non-topological, transport of particles in the absence of a bias voltage has been observed in quantum dots\(^ {26}\) and in 1D channels in the presence of acoustic waves\(^ {27}\). In particular, the basic idea for the realization of...
an adiabatic quantum dot is that a d.c. can be pumped through a quantum dot by periodically varying two independent parameters \(X_1\) and \(X_2\), for example, a gate voltage or a magnetic field. In this case, one can relate the pumped current to the parametric derivatives of the scattering matrix \(S(X_1, X_2)\) of the system. The charge pumped over one cycle is given by:

\[
Q(m) = \frac{e}{h} \int \sum_{\alpha} \sum_{a=m} S_{\alpha a} \frac{\partial S_{\alpha a}}{\partial X_1} \, dX_1 \, dX_2,
\]

where \(m\) labels the contact, \(X_1\) and \(X_2\) are two external parameters whose trace encloses the area \(A\) in the parameter space, \(a\) and \(b\) label the conducting channels and \(3\) stands for the imaginary part. Although the physical description of open systems is markedly different from closed ones, the concept of a geometric phase can still be applied. The integrand in Eq. (3) can be thought as the Berry curvature \(\Omega\), and the pumped charge is essentially the Abelian geometric phase of the scattering matrix. Note that the term geometric pump is used in a different context here. It is non-topological in nature and therefore its properties should not be confused with the geometric pump introduced earlier, which was realized in ref. 14.

Spin pumping

The idea of topological charge pumps has been generalized to spin pumps, which represent 1D dynamical versions of 2D topological insulators (TIs). The simplest example of a 2D TI consists of two uncoupled integer quantum Hall insulators, in which the Chern number for the two spins is opposite. A quantum spin pump was realized experimentally with two-component ultracold bosonic atoms in an optical superlattice\(^{12}\). To this aim, ultracold atoms in two different hyperfine states are prepared in a spin-dependent dynamically controlled optical superlattice. In the tight-binding limit, the dynamics of hyperfine states are prepared in a spin-dependent dynamically controlled optical superlattice. The closed path encircles the degeneracy point at the origin where \(\Delta = 0\) and \(J_1 = J_2\) in quantized transport. The Berry curvature \(\Omega\) of the lowest band as a function of \(\varphi\) and quasi-momentum \(k_x\) for a lattice with \(V_x = 10E_c\) and \(V_y = 5E_c\). The panel below shows the Berry curvature averaged over \(k_x\), whereas the panel on the right shows \(\Omega(\varphi)\) for \(\varphi = 0, d\). Centre-of-mass motion during one pump cycle. The solid black line depicts the calculated centre-of-mass motion of a localized Wannier function. Inset: evolution of the ground-state wavefunction illustrated for one double well. Figure adapted with permission from ref. 12, Springer Nature Ltd.

\[
\hat{H} = -\frac{1}{4} \sum \left( J_{ex} + (-1)^m \delta J_{ex} \right) (\hat{S}_m^+ \hat{S}_m^- + \text{h.c.}) + \frac{\Delta}{2} \sum (-1)^m \hat{S}_m^z.
\]
pump was studied in ref. 37. Moreover, for spin pumps with highly
degenerate many-body ground states, a fractional transport is pre-
dicted38. Away from the hardcore constraints for bosonic atoms, the
effect of a finite interaction can be taken into account via a bosoniza-
tion approach49. In this limit, the topological classification of the spin
pump still remains valid, with one important difference: the topological
excitations are solitons and antisolitons, which carry a spin of 1/2.

Beyond the adiabatic approximation

As we have discussed earlier, having a topological nature, the quanti-
tization of the transported charge in a quantum pump shows robustness to
various factors such as interactions or disorder44 with deviations that
depend on the pumping protocol. Similarly, non-adiabatic effects owing
to finite pumping frequencies have always been believed to be unim-
portant, that is, exponentially small in the driving frequency \( \omega \) (refs. 3.40)
in analogy with the IQHE, in which the Hall plateaux show corrections that
are exponentially small in the longitudinal electric field49. Theoretically,
this follows from the fact that the quantized Chern number expression
for the Hall conductivity, usually obtained through a Kubo formula in
linear response, is valid at all orders in perturbation theory43.

However, deviations from this behaviour have been recently found
in Thouless pumping out of the perfect adiabatic limit \( \omega \to 0 \) (ref. 43).
Consider a closed, clean, non-interacting system—the driven RM model—in
the thermodynamic limit. The system starts from the initial ground-
state Slater determinant and the driving is switched on suddenly.
Such a driven system can be analysed through using the Floquet theory.
Owing to discrete time-translation invariance of a Hamiltonian with
periodicity \( T \), there exists a basis of solutions of the time-dependent
Schrödinger equation, which are periodic up to a phase, the so-called
Floquet states \( \langle \phi_\alpha(t) \rangle = e^{-i \varepsilon_\alpha / \hbar} \phi_\alpha(t) \) (refs. 44,45). The \( T \)-periodic states
\( |\phi_\alpha(t)\rangle \) are the so-called Floquet modes and \( \varepsilon_\alpha \) are the corresponding
quasigaps for \( \varepsilon_\alpha \), \( \varepsilon_\alpha \) is a wavevector of the first BZ.

A careful Floquet analysis40,42,46–48 of a non-interacting system
in the thermodynamic limit reveals that the charge pumped after
many cycles shows a deviation from perfect quantization that is always
polynomial in the driving frequency \( \omega \), contradicting the expected
topological robustness. This quadratic deviation is present also after
a finite number of pumping cycles, even if apparently hidden under a
highly oscillatory non-analytic behaviour44 in \( \omega \). An exponentially
small deviation would only be obtained, if the system is prepared in
a specific Floquet state, which requires a suitable switch-on of the
driving. Recently, it was demonstrated that even at a fast driving fre-
quency, one can realize an ideal pump under a family of protocols
which contains the adiabatic one as a limiting case41. Note that the
question about the breakdown of quantized pumping for increasing
pumping speeds has also been addressed experimentally in refs. 13,17,23.
These studies, however, consider frequencies \( \omega \) that are rather far from the adiabatic limit. The fundamental question
about robustness of pumping owing to non-adiabatic effects discussed
here cannot be easily studied in experiments, as small deviations of
a few percent are extremely challenging to resolve owing to other
experimental imperfections.

In a PBC ring geometry, the total current operator \( I(t) \) is obtained
as a derivative of \( \hat{H}(t) \) with respect to a flux \( \Phi \) threading the ring,
\( I = \partial_t \hat{H}/\hbar \), where \( \kappa = (2\pi/L) (\Phi/\Phi_0) \), \( L \) is the length of the system and
\( \Phi_0 \) is the magnetic flux quantum. As a consequence, the charge
pumped in one period \( T \) by a single Floquet state \( |\psi_\alpha(t)\rangle \) is
\[
Q_\alpha(T) = \int_0^T \frac{\partial}{\partial t} \langle \psi_\alpha(t) | J(t) | \psi_\alpha(t) \rangle dt = \frac{\hbar}{e} \varepsilon_{\alpha k}.
\]
where \( \varepsilon_{\alpha k} \) is the \( \alpha \)-derivative has been replaced with a
\( \kappa \)-derivative, as \( \varepsilon_{\alpha k} \) depends on \( k \). This, if \( \varepsilon_{\alpha k} \) wraps around the FBZ in a continuous way
as a function of \( k \), \( Q_\alpha(T) \) is equivalent to the winding number of the
band, that is, the number of times \( \varepsilon_{\alpha k} \) goes around the FBZ,
\( \varepsilon_{\alpha k} \sim \varepsilon_{\alpha k} + n \hbar \omega \), and \( Q_\alpha(T) \) is therefore quantized: \( Q_\alpha(T) = n \).

This result applies in the adiabatic limit, that is, \( \omega \to 0 \); if \( |\psi_\alpha(t)\rangle \) is a Slater
determinant made up of the instantaneous Hamiltonian Bloch eigen-
states \( e^{i\hat{u}_{\alpha k}(\mathbf{k}, t)} \), the adiabatic theorem guarantees that such a state
returns onto itself after a period \( T \) by acquiring a geometric (Berry)
phase \( \gamma_{\alpha k} = \int_0^T dt \langle \hat{u}_{\alpha k}(\mathbf{k}, t) | \hat{u}_{\alpha k}(\mathbf{k}, t) \rangle \). The dynamical phase \( \gamma_{\alpha k} = \int_0^T dt \epsilon_{\alpha k} \).

Thus \( |\psi_\alpha(t)\rangle \) is a Floquet state with quasi-energy \( \varepsilon_{\alpha k} = \hbar \omega \gamma_{\alpha k} \). Substituting
this into Eq. (5) only the geometric phase survives, leading to the formula of Thouless40:
\[
Q_\alpha(T) = \frac{\hbar}{e} \int \frac{d\mathbf{k}}{2\pi} \int_0^T dt \langle \hat{u}_{\alpha k}(\mathbf{k}, t) | \hat{u}_{\alpha k}(\mathbf{k}, t) \rangle \gamma_{\alpha k} \cdot \mathbf{c}.c.\).
\]
This formula permits to identify the pumped charge with a Chern
number41.

Away from the adiabatic limit \( \omega \to 0 \), deviations from the perfect
quantization strongly depend on how the system is brought away from
the lowest-energy Floquet band. In fact, in the experiment, one would
be able to prepare an initial state coinciding with the lowest-energy
Floquet band, such that the deviation from perfect quantization would
be exponentially small. This comes from the fact that, as noticed in
ref. 42, the quasi-energy spectrum contains some crossings giving a
non-vanishing winding number. Generally, that crossings turn into
avoided crossings with opening of gaps for any finite \( T \) — in
the present case (Fig. 3a) at the border of the FBZ — implying a deviation
from perfect quantization of the pumped charge for the Floquet band
under consideration42. Owing to the presence of an avoided crossing,
an exponentially small \( \hbar/\omega \) gap appears43. This implies that the pumped
charge deviates from an integer by terms proportional to the sum of the
gaps for \( \omega > 0 \). This deviation is therefore exponentially small in \( \omega \).
However, this is more an artefact coming from the avoided crossings.

In contrast, one knows that, independently from the initial state,
any local observable reaches a periodic steady state with the same
periodicity as the driving40. This asymptotic regime is described by
the Floquet diagonal density matrix40,41. If \( \langle m|t\rangle \) is the total charge pumped in the first \( m \) periods starting from the initial ground
state \( |\Psi(0)\rangle \) of \( \hat{H}(0) \), the asymptotic charge pumped is given by the Floquet diagonal
eigenstate41:
\[
Q^a_m = \lim_{m \to \infty} \frac{1}{\hbar \omega} \sum \frac{1}{2\pi} \int \frac{d\mathbf{k}}{2\pi} \langle \hat{u}_{\alpha k}(\mathbf{k}, m) | \hat{u}_{\alpha k}(\mathbf{k}, m) \rangle \gamma_{\alpha k} \cdot \mathbf{c}.c.\).
\]
where \( n_{\alpha k} \) is the initial ground-state occupation of the Floquet–Bloch
mode labelled by \( (\alpha, k) \). The occupations \( n_{\alpha k} \) can give rise to a stronger
deviation from quantization than the gaps, as numerically shown in ref. 43.
Starting from the lowest-energy Floquet band \( \varepsilon_{1 \pm k} \) and the associated
occupations $n_{\ell,k}$, one can develop a perturbation theory in $\omega$ for the Floquet modes, along the lines of ref. 52, to show that:

$$n_{\ell,k} = 1 - f(k, \phi)\omega^2 + \ldots$$

(8)

where $f$ is a function of the pump trajectory parametrized by the pump parameter $\phi$ of the RM model defined in Eq. (2), leading to quadratic corrections to $Q_d$ (Fig. 3b). Indeed, one finds that the deviation from quantization increases as $r$, where $r$ is the dimensionless radius of the closed trajectory in the $(\Delta, J_1 - J_2)$-parameter space.

Figure 3b shows the charge pumped after a single cycle, $Q(T)$, as a function of the frequency $\omega$, for the RM model with a suddenly switched-on driving (smooth blue line). The red dotted line is the Rice–Mele (RM) model. The thick band is the lowest-energy Floquet band of $Q_d$ plotted against disorder strength $W$. The transition between the quantized regime and the trivial one $Q_d = 0$ is clearly linked to the closing of the minimum energy gap owing to the disorder, shown in the inset. Here, $L$ is the system size, $J_0 = (J_1 + J_2)/2$ the mean tunnel coupling, $J = J_1 - J_2 = 2\delta_0\cos\phi$ the dimerization and $\Delta(\phi) = 2\delta_0\sin\phi$ the staggered offset in the RM model defined in Eq. (2). Inset: minimal many-body gap as a function of disorder strength $W$. Panels a and b adapted with permission from ref. 43, APS. Panel c adapted with permission from ref. 53, APS.

**The role of disorder**

A key feature of topological states of matter is their robustness to disorder. The robustness of topological pumps to disorder has been studied theoretically for different distributions ranging from true-random to quasi-periodic. As discussed earlier, as the Hamiltonian is time-periodic, one can exploit the Floquet representation of the evolution operator. For non-interacting fermions, it suffices to know the single-particle Floquet states $|\psi_{\ell,k}(t)\rangle$ and their occupation number $n_{\ell,k}$ to explicitly calculate the diagonal ensemble pumped charge $Q_d$ (Eq. (7)).

Let us now consider on-site disorder of the form $\hat{H}_{\text{dis}} = \sum_i W \xi_i \hat{b}_i^\dagger \hat{b}_i$, where $W$ denotes the strength of the disorder and $\xi$ are uniformly distributed random numbers $\xi \in [-1/2, 1/2]$. Figure 3c shows the disorder-averaged charge pumped per cycle as a function of the disorder strength $W$. Although topological pumping persists for sufficiently small $W \lesssim 3J_0$, it breaks down in the regime of large $W \gtrsim 8J_0$, where $Q_d = 0$. The intermediate region $W/J_0 = 4$ exhibits large sample-to-sample fluctuations. The inset shows a correlation between the drop of $Q_d$ and the closing of the minimal many-body instantaneous gap $\Lambda_N = \min_{t\in[0,T]} |E_N(t) - E_0(t)|$, where $E_N(t)$ is the $N$-particle ground-state energy at time $t$.

The puzzling question is how a disordered 1D system that shows Anderson-localized instantaneous energy eigenstates and a pure-point spectrum can transport charge. It has been found that topological pumping only takes place, if there is a significant fraction of delocalized single-particle Floquet states, which seems to be in
contradiction with the adiabatic limit, in which all states are necessarily Anderson-localized. Indeed, it was found that the dynamics is adiabatic only at the many-body level. The driving mixtures localized single-particle states, which results in extended Floquet modes. This can be clearly shown by looking at the real-space inverse participation ratio (IPR)\textsuperscript{31}. For a finite system, IPR, $I = \frac{1}{L^2} \sum_i |\phi_i|^2$, where $\mathbf{IPR} = \frac{1}{L^2} \sum_i |\phi_i|^2$ signals a completely delocalized (plane-wave-like) state, whereas $\mathbf{IPR} = 1$ corresponds to a perfect localization on a single site. As shown in ref. 33, the IPR shows a localization→delocalization transition at crossover disorder strength $W - L^2$ separating the two regimes, vanishing in the thermodynamic limit. Delocalization renders quantized pumping robust, until extended Floquet states with opposite winding coalesce for large disorder. Even though the physics of quantum pumping in clean systems is the same as the 2D quantum Hall effect, this analogy is not trivial in the presence of disorder. The competition between (quasi-periodic) disorder and topology has recently been studied in cold atoms\textsuperscript{34} and photonics\textsuperscript{35}. Both studies identify a clear connection between the closing of the gap in the instantaneous energy spectrum and a breakdown of quantized pumping, which can be understood as the result of Landau–Zener transitions induced by the periodic modulation. Interestingly, disorder can also induce topological transport in an otherwise topologically trivial regime, as recently investigated in a cold-atom experiment with quasi-periodic disorder\textsuperscript{41}. This establishes exciting connections to 2D topological Anderson insulators\textsuperscript{66,67} and anomalous Floquet–Anderson insulators\textsuperscript{68}, which are 2D topological phases in the presence of disorder.

**The role of interactions**

The interplay between interactions and topology in 1D charge pumps gives rise to a rich variety of topological many-body phenomena and has been studied both for fermionic\textsuperscript{68–72} and for bosonic atoms\textsuperscript{73–77}. The formalism presented earlier has been based on a description using the instantaneous single-particle eigenstates of the time-dependent Hamiltonian. Here we present the generalized formalism for many-body systems\textsuperscript{24}, before discussing a few selected examples in more detail.

**Generalized many-body formalism**

Originally, it was shown that quantization is unaffected by weak interactions under fairly broad assumptions\textsuperscript{21}. One can argue that the charge pump is robust to disorder and interaction as long as the system remains in its ground state during the pump cycle. The demonstration is based on the concept of twisted boundary conditions for the many-particle wavefunction:

\[
\Phi(x_1, \ldots, x_{L-1}, \ldots, x_N) = e^{iKt} \Phi(x_1, \ldots, x_{L-1}, \ldots, x_N),
\]

where $L$ is the size of the system. The corresponding Hamiltonian $H(K, t)$ in the presence of a slow time-varying potential together with the boundary condition (9) describes a 1D system placed on a ring of length $L$ threaded by a magnetic flux $2\pi a/L$\textsuperscript{78}, where $\Phi_0$ is the magnetic flux quantum. Thus, the current operator is given by $\partial H(K, t)/\partial K$ and one obtains:

\[
j(K) = \frac{\partial \varepsilon(K)}{\partial K} - \Omega_{K,t},
\]

where $\Omega_{K,t}$ is the Berry curvature in the many-body manifold $\Omega_{K,t} = i(\langle \partial_{\mathbf{K}} \Phi_0 \partial_{\mathbf{K}} \Phi_0 - \langle \partial_{\mathbf{K}} \Phi_0 \partial_{\mathbf{K}} \Phi_0 \rangle_0)$, where $\langle \Phi_0 \rangle$ is the many-body ground state. The key step for deriving the quantized transport for a many-body wavefunction is achieved by realizing that if the Fermi energy lies in a gap, then the current $j(K)$ should be insensitive to the boundary condition specified by Eq. (9)\textsuperscript{32,79}. Consequently, one can take the thermodynamic limit and average $j(k)$ over different boundary conditions. Let us note that $K$ and $K + 2\pi a/L$ describe the same boundary condition in Eq. (9). Therefore, the parameter space for $K$ and $t$ is a torus in 2D and the particle transport is given by the Chern number of the occupied band. According to the previous discussion, it is quantized and can vary only in a discontinuous way. This is a general outcome of topological invariance. However, deviations from the exact quantization can be observed for intermediate interactions, in which numerical simulations have shown that interactions can lead to a breakdown of the quantized pumping by closing the many-body gap\textsuperscript{32,76}.

**Nonlinear Thouless pumping**

One of the first experimental realizations of mean-field-type interacting topological pumps was realized in an optical waveguide array with Kerr nonlinearity\textsuperscript{80}. In these experiments, the propagation of monochromatic light is described by a nonlinear Schrödinger equation that is equivalent to an attractive Gross–Pitaevskii equation describing interacting bosonic atoms in the mean-field limit. In the experiment, an off-diagonal implementation of the Aubry–André–Harper model\textsuperscript{81} with three sites per unit cell labelled as A, B and C is realized (Fig. 4a). The corresponding tight-binding Hamiltonian is characterized by uniform on-site potentials and real off-diagonal nearest-neighbour couplings $J_{ij}(z)$ that are periodic functions along the propagation direction $z$, which plays the role of time $t$. In the linear (non-interacting) regime, one pumping cycle (of the lowest band) can be understood intuitively using the following simplified description. Suppose there is always only a single coupling switched on, such that the intensity couples completely from one site to the next. Starting with a single occupation on site A, after switching on $J_{AC}$, the occupation shifts to site C. Subsequently, coupling $J_{CA}$ is turned on and finally $J_{AB}$ is turned on, so that the wavefunction is pumped by one unit cell. The periodic modulation of the couplings is realized by changing the distance between the waveguides (Fig. 4b), which in turn modifies the spatial overlap of the neighbouring waveguide modes\textsuperscript{84}.

In the linear regime, quantized pumping is observed for an initial state with uniform occupation of the lowest band. This distribution however strongly depends on the strength of the nonlinearity. Experimentally, this is controlled by changing the amount of power that is injected into the system. Surprisingly, quantized pumping is also observed for large nonlinearity\textsuperscript{86}, despite non-uniform band occupation (Fig. 4c). This is explained by stable soliton solutions that exist at each instant in time and are identical after one pump cycle, up to translation invariance. Transport is dictated by the Chern number of the band the soliton bifurcates from. However, this constitutes a new mechanism as it does not rely on uniform band occupation, in contrast to Thouless pumping of fermions or hardcore bosons. These results highlight that nonlinearity or mean-field interactions can induce quantized transport and topological behaviour in regimes where the linear limit is topologically trivial. Above a certain threshold, nonlinearity matters transfer is completely arrested.

Quantized pumping of solitons was derived more formally by expressing the soliton solutions on the basis of linear Wannier orbitals\textsuperscript{81,83}. While in the moderate nonlinearity regime, it is sufficient to consider a single band, and excited bands need to be included for larger nonlinearities. The topology of these excited bands then determines the direction and magnitude of the average velocity of the solitons. The motion remains quantized, admits fractional values and can even be
after one pump cycle.

\[ \frac{\partial}{\partial t} \Psi(t) = i\left[ H(t), \Psi(t) \right] \]

where \( H(t) = H(t - \Delta t) \) describes the evolution of the many-body polarization

\[ \Psi(t) \]

for the time-periodic pump Hamiltonian

\[ H = H_0 + \Delta H(t) \]

is the bosonic number operator.

To investigate charge pumps in interacting systems, it is convenient to consider the evolution of the many-body polarization \( P(t) \) of a state \( |\Psi(t)\rangle \) for the time-periodic pump Hamiltonian \( H(t + T) = H(t) \).

In general, the total transported charge \( \Delta Q \) can be related to the polarization \( P(\Box 2) \) via the current density \( J(t) \):

\[ \Delta Q = \int_0^\Delta t dt J(t) = \int_0^\Delta t dt \cdot \partial P(t) \]  

For charge pumps in the adiabatic limit, the polarization is also cyclic in \( T \), with \( [P(T) \mod qa = P(0)] \), where \( q \) denotes the charge of the particle (for cold atoms \( q = 1 \)) and \( a \) is the size of the unit cell. For \( \Delta t = T \), the transported charge \( \Delta Q(T) \) then corresponds to the winding number of the many-body polarization, which implies quantization.

It is interesting to address the question, for which parameters a single-particle interpretation of the quantized response in the strongly interacting regime remains valid. To this end, one can consider the momentum-weighted single-particle Berry curvature, defined as \( \Omega^{\phi}_{\alpha}(k, \phi) = \Omega_{\alpha}(k, \phi) n_{\phi}(k, \phi) \), where \( n_{\phi}(k, \phi) \) denotes the momentum distribution of the interacting system expressed in the single-particle basis and \( \phi \) is the pump parameter. In analogy to the single-particle formalism, the transported charge would then be given by the integral of the weighted Berry curvature \( \Omega^{\phi}_{\alpha}(k, \phi) \):

\[ \Delta Q_{\alpha} = q \int_{-\pi/a}^{\pi/a} dq \int_0^{2\pi/qa} d\phi \Delta \Omega^{\phi}_{\alpha}(k, \phi) \]
Synthetic dimensions and higher-dimensional systems

Fig. 5 | Synthetic dimensions and higher-dimensional systems. a, Schematic of synthetic dimensions realized using internal states of fermionic $^{173}$Yb atoms that are coupled by complex two-photon Raman transitions. b, Observation of skipping orbits in a synthetic quantum Hall ribbons with $^{173}$Yb atoms. c, Schematic drawing of Laughlin’s charge pump. d, Schematic of a 4D quantum Hall system, which can be composed of two 2D quantum Hall systems in the $x$–$z$ and $y$–$w$ planes. e, Schematic drawing of the experimental realization of the 2D pump using optical superlattices. f, Topological edge transport in a coupled waveguide array realizing a 2D topological charge pump. Panel a adapted with permission from ref. 96, AAAS. Panel b adapted with permission from ref. 97, AAAS. Panel c adapted with permission from ref. 107, APS. Panels d and e adapted with permission from ref. 113, Springer Nature Ltd. Panel f adapted with permission from ref. 114, Springer Nature Ltd.

$\Delta Q'$ will, in general, deviate from the exactly quantized $\Delta Q_0$ as $n_a(k, \varphi)$ is not flat. The connection between quantized transport in interacting pumps and the single-particle interpretation has been studied numerically in ref. 86 using matrix–product-state methods in an infinite-system size formulation. As shown in Fig. 4e, clear deviation of $\Delta Q_a/q$ is found for most pump trajectories (Fig. 4d). This is expected, as, in general, the interacting ground-state wavefunction is far from a product state and therefore, applying a single-particle picture breaks down. Interestingly, for the experimental trajectory used in ref. 12, a single-particle interpretation remains valid even down to very small interaction strength. The reason is that for this particular parametrization of the pump, the atoms remain essentially localized to individual sites or double wells during the entire evolution along the pump cycle. Recently, breakdown of quantized particle transport was studied experimentally for an interacting fermionic RM model in a dynamical superlattice and in a two-component Fermi system. Moreover, interaction-induced topological pumping has been predicted in generalized RM models as well as for quasi steady states that exist in a Floquet-prethermal regime.

Beyond the many-particle framework discussed earlier, the few-body problem can be treated within a generalized Wannier-state formalism. The general understanding is that the interparticle interaction breaks the translational symmetry of individual particles and the Wannier states cannot be constructed in the usual way. Despite this, it is possible to introduce multiparticle Wannier states for interacting systems, which provide an orthogonal basis for constructing effective Hamiltonians. The shift of multiparticle Wannier state relates to the Chern number of the multiparticle Bloch band allowing for the Thouless pumping of bound states when two or more particles move unidirectionally as a whole. In general, it is possible to perform topological pumping of cluster and kink excitations which can be related to spin flips. This offers a paradigm for multiparticle pumping.

**Synthetic dimensions and higher-dimensional systems**

Topological pumping can be viewed as a dynamical version of the IQHE, in which time plays the role of a second real-space dimension (Box 2). More generally, the pump parameter can be seen as a synthetic dimension, in which conventional real-space degrees of freedom are replaced by any other degrees of freedom that are available in an experimental platform. This concept is very general and has found applications in a number of experiments ranging from cold atoms to photonics, in which synthetic dimensions have been realized. For example, synthetic dimensions can take the form of internal levels in an atom as illustrated in Fig. 5a, discrete momentum–space degrees of freedom or different eigenmodes in photonic waveguide arrays. This opens the door to various different implementations of 2D quantum Hall physics in engineered, well-controlled quantum systems.

The first cold-atom experiments have realized 2D synthetic quantum Hall ribbons by replacing one spatial direction with internal atomic states that are coupled by additional laser beams. Owing to the finite number of internal states, this setting naturally realizes open boundary
conditions and has facilitated the observation of chiral edge modes\(^{(96,97)}\) (Fig. 5b). The experimental realization of Laughlin’s charge pump, however, requires PBCs and an adiabatic change of the axial magnetic flux. In his ‘Gedankenexperiment’ (‘thought experiment’), Laughlin considered a quantum Hall state on a cylinder (Fig. 5c), in which the perpendicular magnetic field is pointing radially outwards. When the additional axial magnetic flux is varied adiabatically in time, a quantized axial transport is induced between the two ends of the cylinder. In contrast to real-space lattices, synthetic dimensions facilitate the realization of PBCs\(^{(100–102)}\). Moreover, by controlling the phases of the laser-induced spin–orbit couplings, an additional varying axial magnetic flux was recently realized in synthetic quantum Hall cylinders using Dy atoms, which facilitated the first experimental realization of Laughlin’s charge pump\(^{(107)}\).

### 2D topological pumps — exploring 4D quantum Hall physics

Many parameter spaces provide an exciting path towards studying higher-dimensional systems that cannot be accessed with experiments in three real-space dimensions, such as 4D quantum Hall physics\(^{(103–105)}\). Using similar arguments as described in Box 2, the concept of dimensional extension offers a direct path towards studying higher-dimensional topological systems\(^{(11)}\) (Fig. 5d).

Indeed, the quantized nonlinear response in 4D arises in the presence of two perturbing external fields: an electric field and a magnetic field that couples the two 2D quantum Hall systems. The resulting response is then orthogonal to both perturbing fields and is proportional to the so-called second Chern number. The corresponding 2D topological pump was realized with cold atoms in a 2D superlattice\(^{(111)}\) with phases \(\phi_x\) and \(\phi_y\), where the phase \(\phi_y\) depends on the position along \(x\) (Fig. 5e). The pump parameter \(\phi_y\) is implemented by varying \(\phi_y\) whereas the magnetic perturbation is realized by the spatial dependence of \(\phi_x\) which couples transport in the two 1D charge pumps. This induces a nonlinear response along \(y\), which is determined by the second Chern number \(n_2 = (1/4\pi^2) \oint \Omega^x \Omega^y dk_x dk_y d\phi_x d\phi_y\), where \(B_2\) denotes the 4D Brillouin zone and \(\Omega^x(k_x,\phi_x)\) and \(\Omega^y(k_y,\phi_y)\) are the respective Berry curvatures along \(x\) and \(y\).

At the same time, a similar implementation has been reported with coupled photonic waveguide arrays (Fig. 3f), where topological edge pumping in a 2D array is supported by a non-zero second Chern number\(^{(114)}\). The second Chern number has further been measured in an artificial parameter space, which was realized by a cyclic coupling of four internal levels of bosonic \(^{87}\text{Rb}\) atoms\(^{(115)}\). Beyond cold atoms, higher-dimensional pumping has been achieved by replicating a 2D-Chern insulator in one spatial and one temporal dimension using a 1D metamaterial composed of magnetically coupled mechanical resonators\(^{(10)}\).

### Higher-order symmetry-protected topological phases

Intriguingly, topological charge pumps can also be used to characterize the topological properties of higher-order TIs and higher-order symmetry-protected topological phases (HOSPTs)\(^{(116,117)}\), in which an \(n\)-dimensional bulk with topology of order \(m\) can exhibit \((n-m)\)-dimensional boundary modes. Such systems have been realized, for instance, in solids and classical meta-materials\(^{(118–120)}\). To characterize their topological properties, efforts have been made to define higher-order Thouless pumps\(^{(121,122)}\). The concept of higher-order topological pumps developed in ref. 132 based on the specific example of interacting bosonic \(C_4 \times Z_2\)-symmetric HOSPTs can be summarized as follows: by introducing corner-periodic boundary conditions (CPBCs), the argument of Resta on polarization\(^{(113)}\) (Box 3) can be extended to higher-order systems. CPBCs are realized by adding links that connect the corners of the underlying 2D system. On the basis of these CPBCs, a tuple of Zak (Berry) phases can be defined that act as topological invariants for HOSPTs and govern charge transport through the corners during Thouless pumping in a direction-dependent manner.

The generalization of the Zak (Berry) phases for higher-order systems\(^{(123,124)}\) relies on the idea of magnetic flux insertion through the two ‘super-cells’, which are delimited by the edge and the corner-connecting links and meet at the corner labelled by the index \(i\). This process is associated with an induced electric field pointing along a diagonal and can be formally described by gauge transformations \(U_i\), \(i \in \{1, 2, 3, 4\}\), applied only to the corner parts of a Hamiltonian \(\hat{H}\):

\[
\hat{H}_i^\gamma(\theta) = U_i(\theta) \hat{H} U_i(\theta)^\dagger,
\]

with \(U_i(\theta) = e^{i\theta n_a}a_\omega\) and \(a_\omega\) is the particle number operator at the \(i\)th corner. Separately for each gauge choice \(U_i\), one can define a higher-order Zak (Berry) phase \(\gamma\), as the geometric phase picked up by the ground-state wavefunction \(|\psi_i(\theta)\rangle\) of \(\hat{H}_i(\theta)\), when changing \(\theta\) from 0 to \(2\pi\):

\[
\gamma_i = \oint_{\theta_0} d\theta \langle \psi_i(\theta) | i\partial_\theta |\psi_i(\theta)\rangle.
\]

We note that, similar to the 1D Zak phase\(^{(10)}\), the higher-order version \(\gamma\) is gauge-dependent, whereas the difference \(\Delta \gamma\) is gauge-invariant.

### Box 3

#### Polarization theory

The modern theory of polarization associates \(P\) with the mean position of the charge distribution per unit cell in the case of translationally invariant states\(^{(138)}\). For a system of length \(L_a\) with periodic boundary conditions, one can define the polarization for a many-body wavefunction \(|\Psi(t)\rangle\) via\(^{(139)}\):

\[
\hat{P}(t) = \frac{\hbar q}{2\pi L_a} \int_0^{2\pi} d\theta \langle \Psi(t) | e^{-i\gamma_l^{\text{perc}}} |\Psi(t)\rangle \left(\text{mod } qa\right),
\]

where \(\hat{X} = \sum_{x=0}^{L-1} a(x-x_0)\), \(a(x)\) is the position operator, \(\hat{n}_x\) is the local density operator, \(q\) is the charge per particle (\(q=1\) for neutral atoms), \(a\) is the length of the unit cell, \(L_a\) is the number of sites and \(x_0\) is the unit-cell centre. Let us note that \(P\) is only defined modulo \(qa\) and it has the units of a dipole moment.

It is insightful to see that for the case of non-interacting fermions for a filled band, the expression for the many-body polarization indeed reduces to the usual form\(^{(138,147,148)}\):

\[
\hat{P}_m(t) = -\frac{i q}{2\pi L_a} \int_{-L_a/2}^{L_a/2} dk \langle \Psi_0(k, t) | \hat{P}_m |\Psi_0(k, t)\rangle,
\]

where \(|\Psi_0(k, t)\rangle\) are single-particle momentum eigenstates of the instantaneous Hamiltonian and \(k\) denotes the quasi-momentum. In fact, one can see that the polarization is determined by the Zak phase associated with a homogeneously filled band, which is defined modulo \(2\pi\) (ref. 136).
The final step is to relate the higher-order Zak (Berry) phase to charge transport. Indeed, the adiabatic flux insertion in Eq. (13) can be directly related to the current passing diagonally through a corner, \( I = \partial_\theta \mathcal{H}_i(\theta) |_{\theta = 0} \) for \( i = 1, \ldots, 4 \). Integrating these currents along an adiabatic path that connects two HOSPTs yields the change of the charge \( \Delta q_i \) in corner \( i \):

\[
\Delta q_i = \frac{\Delta \gamma_i}{2\pi}.
\]

Since \( \gamma_i \) is quantized by \( C_i \times Z_2 \) symmetry, it follows that the corner charge \( \Delta q_i \) is also quantized and represents an intrinsic topological invariant. The total amount of charge \( \Delta Q = \sum_i \Delta q_i \) transported during one full pumping cycle, or equivalently, the amount of charge piling up at the corners in a system with open boundary conditions, can be measured by four Chern numbers \( \nu_i \) with \( i \in \{1, 2, 3, 4\} \), which are obtained as winding numbers of the higher-order Zak (Berry) phase. As the Zak (Berry) phase is defined mod \( 2\pi \), it follows directly that the Chern numbers, \( \nu_i \), and the associated bulk charge transport are integer-quantized.

Outlook

Quantum pumps are transport mechanisms by which a d.c. results from a cyclic evolution of the potential. Starting from the Thouless pump, we have discussed the topological origin of the pumping process, when considering the motion of quantum particles in spatially and temporally periodic potentials. Different generalizations of the concept of Thouless pumping have recently appeared. One of this is the concept of non-Abelian Thouless pumping. It yields a displacement across the lattice but it also generates a holonomic transformation among the different bands, extending the known results of Resta and Vanderbilt. Non-Abelian Thouless pumping hints at the possibility to detect the role of non-Abelian holonomies also in quantum Hall experiments. Moreover, in a standard Thouless pump, the drive is usually imparted from outside, as was the case in the experimental systems studied so far. However, recently, an emergent mechanism for geometric pumping in a quantum gas coupled to an optical resonator has been reported, in which a particle current has been observed without applying a periodic drive. The pumping potential experienced by the atoms is formed by the self-consistent field interfering with the static laser field driving the atoms. Last but not the, an exciting new direction has been opened with the realization of nonlinear Thouless pumps in optical waveguide arrays that combines topology and the dynamics of solitons. Despite its longstanding discovery, Thouless pumps still deserve special interests, and the recent developments within cold atoms and photonics raise interesting questions that call for future investigations. Natural extensions would be realizing topological time crystal pumps with temporally and spatially periodic potentials, useful for quantum memories, or topological dissipative pumps with spin degrees of freedom or synthetic dimensions.

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Competing interests
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