Left-Covariant Differential Calculi on $SL_q(2)$ and $SL_q(3)$

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Abstract:
We study $N^2 - 1$ dimensional left-covariant differential calculi on the quantum group $SL_q(N)$ for which the generators of the quantum Lie algebras annihilate the quantum trace. In this way we obtain one distinguished calculus on $SL_q(2)$ (which corresponds to Woronowicz’ 3D-calculus on $SU_q(2)$) and two distinguished calculi on $SL_q(3)$ such that the higher order calculi give the ordinary differential calculi on $SL(2)$ and $SL(3)$, respectively, in the limit $q \to 1$. Two new differential calculi on $SL_q(3)$ are introduced and developed in detail.

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0. Introduction
After the seminal work [18] of S.L. Woronowicz, bicovariant differential calculi on quantum groups (Hopf algebras) have been extensively studied in the literature. There is a well developed general theory of such calculi. Bicovariant differential calculi on the quantum group $SL_q(N)$, $N \geq 3$, have been recently classified in [12]. The case of $SL_q(2)$ has been treated before in [15] und in [11]. All calculi occurring in this classification have dimension $N^2$, i.e. their dimension does not coincide with the dimension $N^2 - 1$ of the corresponding classical Lie group. On the other hand, the first example of a non-commutative differential calculus on a quantum group was Woronowicz’ 3D-calculus on $SU_q(2)$ [17]. This is a three dimensional left-covariant calculus which is not bicovariant. The 3D-calculus is algebraically much simpler and in many respects nearer to the classical differential calculus on $SU_q(2)$ than the four dimensional bicovariant calculi on $SU_q(2)$. This motivates to look for $N^2 - 1$ dimensional left-covariant differential calculi on $SL_q(N)$. The purpose of this paper is to study the cases $N = 2$ and $N = 3$ in detail. The main aim of our approach is to follow the classical situation as close as possible.

Let us briefly explain the basic idea of the approach given in this paper. As in [12], we assume that the differentials $du^i_j$ of the matrix entries $u^i_j$ generate the left module of 1-forms. Hence the differential $d$ can be expressed as $dz = \sum (X_{ij} * x) \omega(u^i_j)$ for $x \in SL_q(N)$, where $\omega(u^i_j) := \sum_n \kappa(u^i_n) du^j_n$ are the left-invariant Maurer-Cartan forms and $X_{ij}$ are linear functionals on $SL_q(N)$ such that $X_{ij}(1) = 0$. In our approach for $N = 2, 3$ the...
functionals $X_{ij}$ will be chosen. From the quantized universal enveloping algebra $U_q(sl_N)$. For the functionals $X_{ij}$ with $i \neq j$ we take quantum analogues of the corresponding root vectors of $sl_N$ multiplied by some polynomials in the diagonal generators of $U_q(sl_N)$. We assume that the vector space of left-invariant 1-forms has dimension $N^2 - 1$ and that $X_{ij}(u^r_s) = \delta_{ir}\delta_{js}$ for $i \neq j$. In case of the ordinary differential calculus on $SL(N)$ we have $\sum_i \omega_{ii} = 0$, so it seems to be natural to suppose that $\sum q^{-2i}\omega_{ii} = 0$ in the quantum case. Then all functionals of the corresponding quantum Lie algebra annihilate the quantum trace $U := \sum q^{-2i}u^i_i$. Note that Woronowicz’ 3D-calculus on $SU_q(2)$ fits into this scheme, see Section 2 for details.

This paper is organized as follows. Section 1 contains some general results about left-covariant differential calculi on quantum groups which will be needed later. In particular, we describe the construction of the universal higher order differential calculus associated with a given left-covariant first order calculus on a quantum group.

In Section 2 we develop four left-covariant differential calculi $(\Gamma_r, d), r = 1, 2, 3, 4$, on the quantum group $SL_q(2)$ which satisfy the above requirements. All four first order calculi and quantum Lie algebras give the ordinary differential calculus on $SL(2)$ and the Lie algebra $sl_2$ when $q \to 1$. However, this changes if we look at the associated higher order calculi. For only one of these calculi the higher order calculus yields the classical calculus on $SL(2)$ in the limit $q \to 1$. As might be expected, this is Woronowicz’ 3D-calculus on $SU_q(2)$ or more precisely its analogue for $SL_q(2)$. For the other three calculi the 2-form $\omega_2 \wedge \omega_0$ is zero and all 3-forms vanish.

In Sections 3 and 4 we are concerned with left-covariant differential calculi on the quantum group $SL_q(3)$. The functionals $X_n$ and $X_{ij}$ with $|i - j| = 1$ are defined completely similar to the corresponding formulas for the 3D-calculus in Section 2. For the functionals $X_{13}$ and $X_{31}$ we use the Ansatz $X_{13} = X_{12}X_{23} - \alpha X_{23}X_{12}$ and $X_{31} = X_{32}X_{21} - \beta X_{21}X_{23}$ with $\alpha$ and $\beta$ complex. For arbitrary complex parameters $\alpha$ and $\beta$, we obtain a first order differential calculus on $SL_q(3)$ which fits into the above scheme. It turns out that if $(\alpha, \beta) \neq (q^{-1}, q)$ and $(\alpha, \beta) \neq (q^{-1}, q)$, then the 2-form $\omega_{31} \wedge \omega_{13}$ vanishes and hence the space of 2-forms does not yield the corresponding space for the classical differential calculus on $SL(3)$ when $q \to 1$. The differential calculi obtained in the two remaining cases $(\alpha, \beta) = (q^{-1}, q^{-1})$ and $(\alpha, \beta) = (q^{-1}, q)$ for the parameters $\alpha$ and $\beta$ are studied in Section 4. In both cases the higher order calculi give the ordinary higher order calculus on $SL(3)$ in the limit $q \to 1$. The corresponding formulas show that these two calculi are very close to the classical differential calculi on $SL(3)$ in many respects.

In Section 5 we generalize the two differential calculi on $SL_q(3)$ from Section 4 to $SL_q(N)$. We define two $N^2 - 1$ dimensional left-covariant first order differential calculi $(\Gamma_r, d), r = 1, 2$, over $SL_q(N)$ for which all quantum Lie algebra generators annihilate the quantum trace. Both first order calculi give the ordinary first order calculus on $SL(N)$ when $q \to 1$. If $N \geq 4$, this is no longer true for the higher order calculi.

Throughout this paper $q$ is a nonzero complex number such that $q^2 \neq 1$ and we abbreviate $\lambda := q - q^{-1}$ and $\lambda_+ := q + q^{-1}$.

We recall the definition of the quantized universal enveloping algebra $U_q(sl_N)$, see [4],[7]. We shall need it only for $N = 2$ and $N = 3$. The algebra $U_q(sl_N)$ has $4(N - 1)$ generators $k_i, k_i^{-1}, e_i, f_i, i = 1, \ldots, N - 1$, with defining relations
Lemma 1. A vector space 

\[ k_i k_j^{-1} = k_j^{-1} k_i = 1, k_i k_j = k_j k_i, k_i e_i = q e_i k_i, k_i f_i = q^{-1} f_i k_i, \]

\[ e_i f_j - f_j e_i = \delta_{ij} \lambda^{-1}(k_i^2 - k_j^{-2}), \]

\[ k_i e_j = q^{-1/2} e_j k_i \text{ and } k_i f_j = q^{1/2} f_j k_i \text{ if } |i - j| = 1, \]

\[ e_i^2 e_j - \lambda_+ e_i e_j + e_j e_i^2 = f_i f_j - \lambda_+ f_i f_j f_i + f_j f_i^2 = 0 \text{ if } |i - j| = 1, \]

\[ k_i e_j = e_j k_i, k_i f_j = f_j k_i, e_i e_j = e_j e_i \text{ and } f_i f_j = f_j f_i \text{ if } |i - j| \geq 2. \]

The Hopf algebra structure of \( \mathcal{U}_q(\text{sl}_N) \) is given by the comultiplication \( \Delta \) with \( \Delta(k_i) = k_i \otimes k_i, \Delta(e_i) = k_i \otimes e_i + e_i \otimes k_i^{-1}, \Delta(f_i) = k_i \otimes f_i + f_i \otimes k_i^{-1} \) and the counit \( \varepsilon \) with \( \varepsilon(k_i) = 1, \varepsilon(e_i) = \varepsilon(f_i) = 0 \). There is a pairing between the Hopf algebras \( \mathcal{U}_q(\text{sl}_N) \) and \( SL_q(N) \) such that \( (k_i, u_m^a) = \delta_m^a \text{ if } n \neq m \text{ or } n = m \neq i, i + 1, (k_i, u_i^i) = q^{1/2}, (k_i, u_i^{i+1}) = q^{-1/2}, (e_i, u_m^m) = \delta_{ni} \delta_{m,i+1} \text{ and } (f_i, u_m^m) = \delta_n \delta_{i,m+1}, \) where \( u_m^a \) are the matrix entries of the fundamental matrix of \( SL_q(N) \).

1. Left - Covariant Differential Calculi on Quantum Groups

Our basic reference concerning differential calculi on quantum groups is [18]. Let \( \mathcal{A} \) be a fixed Hopf algebra with comultiplication \( \Delta \), counit \( \varepsilon \), antipode \( \kappa \) and unit element 1. Sometimes we use Sweedler’s notation \( \Delta^{(n)}(a) = a_{(1)} \otimes a_{(2)} \otimes \cdots \otimes a_{(n+1)} \). For \( a \in \mathcal{A} \) we put \( \bar{a} := a - \varepsilon(a)1 \).

A first order differential calculus (briefly, a FODC) over \( \mathcal{A} \) is a pair \((\Gamma, d)\) of an \( \mathcal{A} \)-bimodule \( \Gamma \) and a linear mapping \( d: \mathcal{A} \to \Gamma \) such that \( d(ab) = da \cdot b + a \cdot db \) for \( a, b \in \mathcal{A} \) and \( \Gamma = \text{Lin} \{ a \cdot db : a, b \in \mathcal{A} \} \). A FODC \((\Gamma, d)\) is called left covariant if there is a linear mapping \( \Delta _L : \Gamma \to \mathcal{A} \otimes \Gamma \) for which \( \Delta _L(ab) = \Delta (a)(d \otimes d)\Delta (b), a, b \in \mathcal{A} \).

Suppose that \((\Gamma, d)\) is a left-covariant FODC over \( \mathcal{A} \). Recall that the canonical projection of \( \Gamma \) into \( \Gamma_{\text{inv}} := \{ \omega \in \Gamma : \Delta _L(\omega) = 1 \otimes \omega \} \) is defined by \( P_{\text{inv}}(da) = \kappa(a_{(1)})da_{(2)}, \) cf. [18]. We abbreviate \( \omega(a) = P_{\text{inv}}(da) \). Then \( \mathcal{R} := \{ x \in \ker \varepsilon : \omega(x) = 0 \} \) is the right ideal of \( \ker \varepsilon \) associated with \((\Gamma, d)\). The vector space \( \mathcal{X} := \{ X \in \mathcal{A}' : X(1) = 0 \text{ and } X(x) = 0 \text{ for } x \in \mathcal{R} \} \) is called the quantum Lie algebra of the FODC \((\Gamma, d)\).

Lemma 1. A vector space \( \mathcal{X} \) of linear functionals on \( \mathcal{A} \) is the quantum Lie algebra of a left-covariant FODC \((\Gamma, d)\) if and only if \( X(1) = 0 \) and \( \Delta X - \varepsilon \otimes X \in \mathcal{X} \otimes \mathcal{A}' \) for all \( X \in \mathcal{X} \).

Proof. The necessity of the condition \( \Delta X - \varepsilon \otimes X \in \mathcal{X} \otimes \mathcal{A}' \) follows at once from formula (5.20) in [18]. To prove the sufficiency part, let us note that the above conditions imply that \( \mathcal{R} := \{ x \in \ker \varepsilon : X(x) = 0 \text{ for } X \in \mathcal{X} \} \) is a right ideal of \( \ker \varepsilon \). From the general theory (cf. Theorem 1.5 in [18]) we conclude easily that \( \mathcal{R} \) is the right ideal associated with some left-covariant FODC over \( \mathcal{A} \).

The calculus \((\Gamma, d)\) is uniquely determined by \( \mathcal{X} \) (because \( \mathcal{R} \) is so) and can be described as follows. Let \( \{ X_i : i \in I \} \) be a basis of the vector space \( \mathcal{X} \) and \( \{ x_i : i \in I \} \) a set of elements of \( \mathcal{A} \) such that \( X_i(x_j) = \delta_{ij} \). Then, letting \( \omega_i = \omega(x_i) \), we have

\[ da = \sum (X_i * a) \omega_i, a \in \mathcal{A}. \]
For notational simplicity we shall write \( \eta \otimes \zeta \) instead of \( \eta \otimes_A \zeta \), where \( \eta, \zeta \in \Gamma \). We set for \( x \in \mathcal{R} \)
\[
S(x) := \sum_{i,j} (X_iX_j)(x) \omega_i \otimes \omega_j.
\]

Some properties of the mapping \( S : \mathcal{R} \to \Gamma \otimes_A \Gamma \) are collected in the following

**Lemma 2.** Let \( P_{inv} \) denote the canonical projection of the left-covariant bimodule \( \Gamma \otimes_A \Gamma \) into \( (\Gamma \otimes_A \Gamma)_{inv} \). For \( x \in \mathcal{R} \) and \( a \in A \), we have:

(i) \( S(x)a = a_1^{(1)} S xa_{(2)} \).

(ii) \( S xa = \kappa(a_1^{(1)}) S(x)a_{(2)} = P_{inv}(S(x)a) \).

**Proof.** We show the first equality of (ii). Recall that \( \Delta X_i = \varepsilon \otimes X_i + X_k \otimes f_i^k, i \in I \), by formula (5.20) in [18], where \( f_i^k \) are functionals on \( A \) such that \( \omega_k a = (f_i^k * a) \omega_i, a \in A \). Hence we get
\[
S(x)a = \sum_{i,j} (X_iX_j)(x) a \omega_i \otimes \omega_j = \sum_{i,j} (\Delta X_i)(\Delta X_j)(x \otimes a) \omega_i \otimes \omega_j
\]
\[
= \sum_{i,j,k,l} (X_kX_l)(x)f_i^k(a_1)(f_j^l(a_2)) \omega_i \otimes \omega_j .
\]

On the other hand, we have
\[
\kappa(a_1^{(1)}) S(x)a_{(2)} = \sum_{k,l} \kappa(a_1^{(1)})(X_kX_l)(x)\omega_k \otimes \omega_l a_{(2)}
\]
\[
= \sum_{k,l} \kappa(a_1^{(1)})(X_kX_l)(x)f_i^k(a_1)(f_j^l(a_2)) \omega_i \otimes \omega_j
\]
\[
= \sum_{i,j,k,l} \kappa(a_1^{(1)}) a_{(2)} f_i^k(a_3)(f_j^l(a_4))(X_kX_l)(x) \omega_i \otimes \omega_j .
\]

By the Hopf algebra axioms both expressions are equal.

The preceding calculation yields \( a_1^{(1)} S xa_{(2)} = a_1^{(1)} \kappa(a_2^{(1)}) S(x)a_{(3)} = S(x)a \) which proves (i). Applying \( P_{inv} \) to (i) and using the fact that \( P_{inv}(a\eta) = \varepsilon(a)P_{inv}(\eta) \) we get the second equality of (ii). □

A **differential calculus** over \( \mathcal{A} \) is a pair \((\Gamma^\wedge, d)\) of a graded algebra \( \Gamma^\wedge = \bigoplus_{n=0}^{\infty} \Gamma_n^\wedge \) with product \( \wedge : \Gamma_n^\wedge \times \Gamma_m^\wedge \to \Gamma_{n+m}^\wedge \) and a linear mapping \( d : \Gamma^\wedge \to \Gamma^\wedge \) of degree one such that \( d^2 = 0, d : \Gamma_n^\wedge \to \Gamma_{n+1}^\wedge, d(\eta \wedge \zeta) = d\eta \wedge \zeta + (-1)^n \eta \wedge d\zeta \) for \( \eta \in \Gamma_n^\wedge, \zeta \in \Gamma_m^\wedge, \Gamma_0^\wedge = \mathcal{A} \) and \( \Gamma_n^\wedge = \text{Lin} \{a da_1 \wedge \ldots \wedge da_n : a, a_1, \ldots, a_n \in \mathcal{A} \} \) for \( n \in \mathbb{N} \). The definition of left-covariance of a differential calculus is similar to the case of first order calculi. Sometimes we simply write \( \eta \zeta \) for \( \eta \wedge \zeta \).

For each left-covariant FODC \((\Gamma, d_1)\) over \( \mathcal{A} \) there exists a unique (up to isomorphism) universal left-covariant differential calculus \((\Gamma^\wedge, d)\) over \( \mathcal{A} \) such that \( \Gamma_1^\wedge = \Gamma \) and \( d[\mathcal{A} = d_1 \). We briefly describe the construction of \((\Gamma^\wedge, d)\).
Let $\Omega = \bigoplus_{n=0}^{\infty} \Omega^n$ be the universal differential envelope of the algebra $A$, see e.g. [2] or [3]. We have $\Omega^0 = A$ and $\Omega^n = A \otimes (\ker \varepsilon)^{\otimes n}$ by identifying $a \otimes a_1 \otimes \ldots \otimes a_n$ and $a \circ a_1 \ldots a_n$. The differential $d$ of $\Omega$ is given by $d(a \circ a_1 \ldots a_n) = d(a) \circ a_1 \ldots a_n$. Clearly, $(\Omega, d)$ is a differential calculus over the Hopf algebra $A$. Let $\mathcal{J}(\mathcal{R}) := \Omega \omega(\mathcal{R})\Omega + \Omega \circ d(\mathcal{R})\Omega$ be the differential ideal generated by the set $\omega(\mathcal{R})$. Here $\mathcal{R}$ is the right ideal of $\ker \varepsilon$ associated with the given FODC $(\Gamma, d_1)$. We have $\mathcal{J}(\mathcal{R}) = \sum_n \mathcal{J}_n(\mathcal{R})$, where $\mathcal{J}_n(\mathcal{R}) := \mathcal{J}(\mathcal{R}) \cap \Omega^n$. Obviously, the quotient algebra $\Omega/\mathcal{J}(\mathcal{R}) = \sum_n \Omega^n/\mathcal{J}_n(\mathcal{R})$ endowed with the quotient map of $d$ is also differential calculus over $A$. By formula (1.23) in [18], $\mathcal{J}_1(\mathcal{R}) = A \omega(\mathcal{R})$ coincides with the submodule $N$ occurring in Theorem 1.5 of [18]. Therefore, the first order calculus $(\Omega/\mathcal{J}_1(\mathcal{R}), d)$ of $(\Omega/\mathcal{J}(\mathcal{R}), d)$ is isomorphic to $(\Gamma, d_1)$. Moreover, it is not difficult to verify that $(\Omega/\mathcal{J}(\mathcal{R}), d)$ is left-covariant. From the preceding construction it is clear that $(\Omega/\mathcal{J}(\mathcal{R}), d)$ has the following universal property: If $(\Omega', d')$ is another differential calculus over $A$ such that $(\Omega'_1, d'_1)$ is isomorphic to the FODC $(\Gamma, d_1)$, then $(\Omega', d')$ is (isomorphic to) a quotient of $(\Omega/\mathcal{J}(\mathcal{R}), d)$ by some differential ideal.

In order to obtain a more explicit description of the calculus $(\Omega/\mathcal{J}, d)$, we shall use a construction of the differential envelope $\Omega = \bigoplus_{n=0}^{\infty} \Omega^n$ of the Hopf algebra $A$ developed in [14]. More details and proofs of all unproven assertions in the following discussion can be found in [14].

Let $\Omega^0 := A$. For $n \in \mathbb{N}$, we set $\Omega^n := A \otimes (\ker \varepsilon)^{\otimes n}$ and we write $a \omega(a_1) \ldots a(a_n)$ instead of $a \otimes a_1 \otimes \ldots \otimes a_n$. We put $\omega(\lambda a) = \omega(a)$ for $\lambda \in \mathbb{R}, a \in \ker \varepsilon$. The product of $\Omega = \bigoplus_{n=0}^{\infty} \Omega^n$ and the differential $d$ are defined by

\[
\begin{align*}
\omega(a_1) \ldots \omega(a_n) b & \omega(b_1) \ldots \omega(b_m) := \\
ab_1 & \omega(a_1 b_2) \ldots \omega(a_n b_{n-1}) \omega(a_n b_{n+1}) \omega(b_1) \ldots \omega(b_m)
\end{align*}
\]

and $d(a \omega(a_1) \ldots a(a_n)) := a_1 \omega(a_2) \omega(a_1) \ldots a(a_n) + \sum_{i=1}^{n} (-1)^i a \omega(a_1) \ldots \omega(a_{i-1}) \omega(a_{i+1}) \omega(a_{i+2}) \omega(a_{i+1}) \ldots \omega(a_n),$

where $a, b \in \ker \varepsilon$ and $a, b \in A$. (In order to motivate these formulas, we recall that for any left-covariant differential calculus over $A$ we have $da = d(a_1) \omega(a_2)$, $d\omega(a) = -\omega(a_1) \circ \omega(a_2)$ and $\omega(b) = c_1 \omega(b \circ c_2)$ for $a, c \in A$ and $b \in \ker \varepsilon$.)

It can be shown that the pair $(\Omega, d)$ endowed with the above definitions becomes a left-covariant differential calculus over $A$ which is (isomorphic to) the differential envelope of $A$. For $a \in \mathcal{R}$, the element $\omega(a)$ of $\Omega$ is equal to $\kappa(a_1) da_2$ which justifies to use the notation $\omega(a)$. Obviously, the kernel of the map $\omega : A \to \Omega$ is $\Phi \cdot 1$.

Let $\mathcal{R}, \{X_i\}, \{x_i\}, \{\omega_i\}$ and $\mathcal{S}$ be as defined above for the left-covariant FODC $(\Gamma, d_1)$. We put

\[
\mathcal{S}_u(x) := \sum_{i,j} (X_i X_j)(x) \omega_i \omega_j \quad \text{for} \quad x \in \mathcal{R},
\]

where the product $\omega_i \omega_j$ is taken in the algebra $\Omega$. For $a \in A$, the element $\tilde{a} - \sum_i X_i(a) \tilde{x}_i$ is annihilated by $\mathcal{X}$ and by $\varepsilon$, so it belongs to $\mathcal{R}$ and hence $\omega(a) - \sum_i X_i(a) \omega_i \in \omega(\mathcal{R})$. 

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Therefore, since $d\omega(x) = -\omega(x_{(1)})\omega(x_{(2)})$, we obtain
\[
d\omega(x) + S_u(x) = \\
- \left(\omega(x_{(1)}) - \sum_i X_i(x_{(1)})\omega_i\right) \omega(x_{(2)}) + \sum_i X_i(x_{(1)})\omega_i \left(\sum_j X_j(x_{(2)})\omega_j - \omega(x_{(2)})\right) \\
\in \omega(\mathcal{R})\Omega^1 + \Omega^1\omega(\mathcal{R}) \text{ for } x \in \mathcal{R}.
\]

Hence the differential ideal $\mathcal{J}(\mathcal{R})$ is generated by the sets $\omega(\mathcal{R})$ and $S_u(\mathcal{R})$.

Next we define the exterior algebra for the FODC $(\Gamma, d_1)$. Let $\Gamma^{\otimes n} := \mathcal{A}_{\otimes A} \cdots \otimes \mathcal{A}$ $\Gamma$ ($n$ times) for $n \in \mathbb{N}$ and let $\Gamma^0 := \oplus_{n=0}^\infty \Gamma^{\otimes n}$ be the tensor algebra of $\Gamma$ over $\mathcal{A}$. We denote by $S = \oplus_{n=2}^\infty S_n$ the two-sided ideal of the algebra $\Gamma^0$ generated by the set $S(\mathcal{R})$. The quotient algebra $\Gamma^\wedge = \Gamma^0/S$ is called the exterior algebra over $\mathcal{A}$ for the FODC $(\Gamma, d_1)$. Clearly, $\Gamma^\wedge$ is also a graded algebra $\Gamma^\wedge = \oplus_{n=0}^\infty \Gamma^\wedge_n$ with $\Gamma^\wedge_0 = \mathcal{A}, \Gamma^\wedge_1 = \Gamma$ and $\Gamma^\wedge_n = \Gamma^{\otimes n}/S_n$ for $n \in \mathbb{N}$. The product of $\Gamma^\wedge$ is denoted by $\wedge$.

For $x \in \ker \varepsilon$, let $[x]$ denote the coset $x + \mathcal{R}$. Let $\pi$ be the product of the mapping $\pi_1 : \Omega \to \Gamma^0$ defined by $\pi_1(a) = a, \pi_1(aw(a_1) \ldots \omega(a_n)) = a\omega([a_1]) \ldots \omega([a_n])$ and the quotient map From $\Omega$ onto $\Gamma^\wedge$. Then $\pi$ is an algebra homomorphism of $\Omega$ onto $\Gamma^\wedge$. It can be shown that the kernel of $\pi$ is the two-sided ideal in $\Omega$ generated by $\omega(\mathcal{R})$ and $S_u(\mathcal{R})$. Therefore, by the paragraph before last, $\ker \pi = \mathcal{J}(\mathcal{R})$. Hence the quotient algebra $\Omega/\mathcal{J}(\mathcal{R})$ and the exterior algebra $\Gamma^\wedge$ are isomorphic. We define $d\pi(a) := da$ for $a \in \Gamma$. Then $(\Gamma^1, d)$ is a left-covariant differential calculus over $\mathcal{A}$ which is isomorphic to the calculus $(\Omega/\mathcal{J}(\mathcal{R}), d)$. By construction, the first order calculus $(\Gamma^1, d)$ of $(\Gamma^\wedge, d)$ is the given FODC $(\Gamma, d_1)$. We call $(\Gamma^\wedge, d)$ (or likewise $(\Omega/\mathcal{J}(\mathcal{R}), d)$) the universal differential calculus associated with the FODC $(\Gamma, d_1)$.

By definition, $S_2 = \mathcal{A}\mathcal{S}(\mathcal{R})\mathcal{A}$. From Lemma 2, (i), we see that $S_2 = \mathcal{A}\mathcal{S}(\mathcal{R})$ is an $\mathcal{A}$-submodule of $\Gamma \otimes_\mathcal{A} \Gamma$ and hence a left-covariant bimodule over $\mathcal{A}$. Therefore, each basis $(\zeta_n)$ of the vector space $\mathcal{S}(\mathcal{R})$ of symmetric elements is a free left module basis for $S_2$, i.e. any element $\zeta$ of $S_2$ can be written as $\zeta = \sum_n a_n\zeta_n$ with elements $a_n \in \mathcal{A}$ uniquely determined by $\zeta$.

2. Left-covariant differential calculi on $SL_q(2)$

In this section we denote the matrix entries $u_1^1, u_1^2, u_2^1, u_2^2$ for the quantum group $SL_q(2)$ by $a, b, c, d$, respectively, and the generators of $\mathcal{U}_q(sl_2)$ by $k, k^{-1}, e, f$. Our aim is to study left-covariant differential calculi $(\Gamma, d)$ on $\mathcal{A} := SL_q(2)$ of the form
\[
dx = \sum_{i=0}^2 (X_i \ast x) \omega_i, \ x \in SL_q(2), \tag{2.1}
\]
where $\omega_0, \omega_1$ and $\omega_2$ are left-invariant 1-forms and $X_0, X_1$ and $X_2$ are linear functionals From $\mathcal{U}_q(sl(2))$ satisfying
\[
X_1(a) = X_0(b) = X_2(c) = 1, \ X_1(b) = X_1(c) = X_0(a) = X_0(c) = X_2(a) = X_2(b) = 0 \tag{2.2}
\]
and $X_i(1) = X_i(q^{-2}a + q^{-4}d) = 0$ for $i = 0, 1, 2$. \tag{2.3}
From the pairing between \( U_q(sl_2) \) and \( SL_q(2) \) it follows that arbitrary linear functionals \( X_i \in U_q(sl(2)), i = 0, 1, 2 \), satisfying (2.2) and (2.3) can be written as \( X_1 = ep_{11}(k) + p_{12}(k) + X'_1, X_0 = ep_{0}(k) + X'_0 \) and \( X_2 = fp_{2}(k) + X'_2 \), where \( p_{11}, p_{12}, p_0, p_2 \) are Laurent polynomials in \( k \) and \( X'_1, X'_0, X'_2 \) annihilate all four matrix entries \( a, b, c, d \). This suggests to consider the following Ansatz

\[
X_1 = ep_{11}(k) + p_{12}(k), X_0 = ep_{0}(k), X_2 = fp_{2}(k) \tag{2.4}
\]

with polynomials \( p_{11}, p_{12}, p_0 \) and \( p_2 \) in \( k \) and \( k^{-1} \).

**Theorem 1.** There are precisely 4 non-isomorphic three dimensional left-covariant differential calculi \( (\Gamma_r, d) \), \( r = 1, 2, 3, 4 \), satisfying (2.1)-(2.3) obtained by the Ansatz (2.4). The right ideal \( R_r \) of \( \ker \varepsilon \) associated with the calculus \( (\Gamma_r, d) \) is generated by the six elements

\[
a + q^{-2}d - (1 + q^{-2})1, b^2, c^2, bc, (a - \gamma_{0r})b, (a - \gamma_{2r})c,
\]

where \( \gamma_{0r} \) and \( \gamma_{2r} \) are the coefficients given by \( \gamma_{01} = \gamma_{02} = 1, \gamma_{03} = \gamma_{04} = q^{-2}, \gamma_{21} = \gamma_{23} = 1 \) and \( \gamma_{22} = \gamma_{24} = q^{-2} \).

**Proof.** We first suppose that \( (\Gamma, d) \) is a differential calculus such that (2.1)-(2.4) are fulfilled. Let \( R \) be its associated right ideal of \( \ker \varepsilon \). From the formulas (2.1)-(2.4) and From the pairing between \( U_q(sl_2) \) and \( SL_q(2) \) we compute easily that \( b^2 \) and \( c^2 \) are in \( R \) and that there are complex numbers \( \gamma_1, \gamma_0, \gamma_2 \) such that \( bc - \gamma_1(a - 1), ab - \gamma_0b \) and \( ac - \gamma_2c \) are in \( R \). As usual, we shall write \( x \equiv y \) if \( x - y \in R \). By (2.3), \( a - 1 \equiv -q^{-2}(d - 1) \). Thus

\[
0 \equiv (bc - \gamma_1(a - 1))(a - 1) = q^{-2}abc - bc - \gamma_1(a - 1)^2
= \gamma_0q^{-2}bc - bc - \gamma_1(a - 1)^2 = (\gamma_0q^{-2} - 1)\gamma_1(a - 1) - \gamma_1(a - 1)^2
= -\gamma_1(a - 1)(a - \gamma_0q^{-2}1) = \gamma_1q^{-2}(d - 1)(a - \gamma_0q^{-2}1)
= \gamma_1q^{-2}(da - a - \gamma_0q^{-2}(d - 1)) = \gamma_1q^{-2}(q^{-1}bc + 1 - a - \gamma_0q^{-2}(d - 1))
\equiv \gamma_1q^{-2}(\gamma_1q^{-1}(a - 1) + 1 - a + \gamma_0(a - 1)) = \gamma_1q^{-2}(\gamma_1q^{-1} + \gamma_0 - 1)(a - 1)
\]

and

\[
0 \equiv (bc - \gamma_1(a - 1))b = b^2c - \gamma_1ab + \gamma_1b \equiv \gamma_1(1 - \gamma_0)b.
\]

Since \( a - 1, b \) and \( c \) are not in \( R \) by (2.2), we get \( \gamma_1(\gamma_1q^{-1} + \gamma_0 - 1) = 0 \) and \( \gamma_1(1 - \gamma_0) = 0 \), so that \( \gamma_1 = 0 \). This implies that \( da - 1 \equiv 0 \) and hence \( q^2(a - 1)(a - q^{-2}) = (d - 1)(q^{-2} - a) \equiv -1 + a + q^{-2}(d - 1) \equiv 0 \), so \( 0 \equiv (a - 1)(a - q^{-2})b = qaba - (1 + q^{-2})ab + q^{-2}b = (\gamma_0 - 1)(\gamma_0 - q^{-2})b \) which yields \( (\gamma_0 - 1)(\gamma_0 - q^{-2}) = 0 \). Similarly we obtain \( (\gamma_2 - 1)(\gamma_2 - q^{-2}) = 0 \). The two latter equations imply that \( R \) contains one of the right ideals \( R_r, r = 1, 2, 3, 4 \). From (2.3), \( \omega_0, \omega_1 \) and \( \omega_2 \) are linearly independent. Hence we have \( \text{codim } R = \dim \Gamma_{inv} \geq 3 \).

Since \( \text{codim } R_r \leq 3 \) by the definition of \( R_r \), we conclude that \( R = R_r \).

To complete the proof of Theorem 1, we have to construct the first order differential calculi \( (\Gamma_r, d) \) having the desired properties. For this let \( \mathcal{X}_r \) denote the linear span of functionals
$X_1, X_0, X_2 \in \mathcal{U}_q(sl_2)$, where

$$X_0 := q^{-1/2}ek^{-1} \text{ for } r = 1, 2, \quad X_0 := q^{-5/2}ek^{-5} \text{ for } r = 3, 4,$$

$$X_2 := q^{1/2}fk^{-1} \text{ for } r = 1, 3, \quad X_2 := q^{5/2}fk^{-5} \text{ for } r = 2, 4,$$

and $$X_1 := q\lambda^{-1}(\varepsilon - k^{-4}) \text{ for } r = 1, 2, 3, 4.$$

From these definitions and From the comultiplication in \(\mathcal{U}_q(sl_2)\) we obtain

$$\Delta X_j = \varepsilon \otimes X_j + X_j \otimes k^{-2} \text{ for } r = 1, 2, j = 0 \text{ and } r = 1, j = 2,$$

$$\Delta X_j = \varepsilon \otimes X_j + X_j \otimes k^{-6} + (q^{-2} - 1)X_1 \otimes X_j \text{ for } r = 3, 4, j = 0 \text{ and } r = 2, 4, j = 2,$$

and $$\Delta X_1 = \varepsilon \otimes X_1 + X_1 \otimes k^{-4} \text{ for } r = 1, 2, 3, 4.$$

This shows that \(\Delta X - \varepsilon \otimes X \in \mathcal{X}_r \otimes A'\) for all \(X \in \mathcal{X}_r\). Therefore, by Lemma 1, each vector space \(\mathcal{X}_r\) defines a left-covariant first order differential calculus \((\Gamma_r, d)\) over \(SL_q(2)\). Let \(\mathcal{R}'_r\) denote the right ideal of \(\ker \varepsilon\) associated with the calculus \((\Gamma_r, d)\). One verifies that the functionals \(X_0, X_1, X_2\) for the calculus \((\Gamma_r, d)\) have the properties (2.2)-(2.3) and that they annihilate the six generators of the right ideal \(\mathcal{R}_r\). Hence \(\mathcal{R}'_r \subseteq \mathcal{R}_r\). Since \(\text{codim } \mathcal{R}'_r = \dim (\Gamma_r)_{inv} = 3\) and \(\text{codim } \mathcal{R}_r \leq 3\), we have \(\mathcal{R}'_r = \mathcal{R}_r\). This completes the proof of Theorem 1. \(\blacksquare\)

We now describe the structure of the four differential calculi \((\Gamma_r, d)\), \(r = 1, 2, 3, 4\), more in detail. By the general theory \([18]\), the above formulas for the comultiplication of \(X_j\) lead to the following commutation rules between matrix entries and 1-forms:

$$\omega_j a = q^{-1}a\omega_j, \quad \omega_j b = qb\omega_j, \quad \omega_j c = q^{-1}c\omega_j, \quad \omega_j d = qd\omega_j \quad \text{for } r = 1, 2, j = 0 \text{ and } r = 1, 3, j = 2,$$

$$\omega_j a = q^{-3}a\omega_j, \quad \omega_j b = q^3b\omega_j, \quad \omega_j c = q^{-3}c\omega_j, \quad \omega_j d = q^3d\omega_j \quad \text{for } r = 3, 4, j = 0 \text{ and } r = 2, 4, j = 2,$$

$$\omega_1 a = q^{-2}a\omega_1, \quad \omega_1 c = q^{-2}c\omega_1 \quad \text{for } r = 1, 3,$$

$$\omega_1 b = q^2b\omega_1, \quad \omega_1 d = q^2d\omega_1 \quad \text{for } r = 1, 2,$$

$$\omega_1 a = q^{-2}a\omega_1 + (q^{-2} - 1)b\omega_2, \quad \omega_1 c = q^{-2}c\omega_1 + (q^{-2} - 1)d\omega_2 \quad \text{for } r = 2, 4,$$

$$\omega_1 b = q^2b\omega_1 + (q^{-2} - 1)a\omega_0, \quad \omega_1 d = q^2d\omega_1 + (q^{-2} - 1)c\omega_0 \quad \text{for } r = 3, 4.$$

Recall that \(\omega(a) = P_{inv}(da) = \kappa(a_{(1)})da_{(2)}\) if \(\Delta(a) = a_{(1)} \otimes a_{(2)}\). From the formulas (2.2) we compute that for all four calculi

$$\omega_1 = \omega(a), \quad \omega_0 = \omega(b), \quad \omega_2 = \omega(c),$$

and $$da = b\omega_0 + a\omega_1, \quad db = a\omega_2 - q^2b\omega_1, \quad dc = c\omega_1 + d\omega_0, \quad dd = -q^2d\omega_1 + c\omega_2.$$

According to the general theory \([18]\), the calculus \((\Gamma_r, d)\) is a \(\ast\)-calculus for an algebra involution \(x \rightarrow x^*\) on \(SL_q(2)\) if and only if \(\kappa(x)^* \in \mathcal{R}_r\) for all \(x \in \mathcal{R}_r\). Obviously, it suffices to check this condition for the six generators of the right ideal \(\mathcal{R}_r\). The four calculi \((\Gamma_r, d), r = 1, 2, 3, 4\), are \(\ast\)-calculi for the Hopf \(\ast\)-algebra \(SL_q(2, \mathbb{R}), |q| = 1\), while only \((\Gamma_1, d)\) and \((\Gamma_4, d)\) are \(\ast\)-calculi for the real forms \(SU_q(2)\) and \(SU_q(1, 1), q \in \mathbb{R}\), of the quantum group \(SL_q(2)\). However, for the Hopf \(\ast\)-algebras \(SU_q(2)\) and \(SU_q(1, 1)\) we have
Next we consider the commutation rules between the generators $X_0, X_1, X_2$ of the quantum Lie algebra $\mathcal{X}_r$ of the calculus $(\Gamma_r, d)$. We have

$$q^2X_1X_0 - q^{-2}X_0X_1 = (1 + q^2)X_0 \quad \text{for } r = 1, 2, 3, 4,$$

$$q^2X_2X_1 - q^{-2}X_1X_2 = (1 + q^2)X_2 \quad \text{for } r = 1, 2, 3, 4,$$

$$qX_2X_0 - q^{-1}X_0X_2 = -q^{-1}X_1 \quad \text{for } r = 1,$$

$$q^3X_2X_0 - q^{-3}X_0X_2 = -q^{-1}X_1 + q^{-2}\lambda X_1^2 \quad \text{for } r = 2, 3,$$

$$q^5X_2X_0 - q^{-5}X_0X_2 = -q^{-1}X_1 + 2q^{-2}\lambda X_1^2 - q^{-3}\lambda^2 X_1^3 \quad \text{for } r = 4,$$

What about the classical limits $q \to 1$ of the calculi $(\Gamma_r, d)$? If we keep the basis $\{\omega_0, \omega_1, \omega_2\}$ of $(\Gamma_r)_{\text{inv}}$ fixed, all above equations make sense in the limit $q \to 1$ and we obtain the classical first order differential calculus on the Lie group $SL(2)$. That is, all four calculi $(\Gamma_r, d), r = 1, 2, 3, 4$, can be considered as deformations of the classical first order differential calculus on $SL(2)$. In particular, the preceding equations for $r = 1, r = 2, 3$ and $r = 4$ define three deformations of the Lie algebra $sl_2$. Note that the quadratic and cubic terms of $X_1$ in the two last equations vanish in the limit $q \to 1$. It might be worth to mention that the quantum Lie algebras $\mathcal{X}_2$ and $\mathcal{X}_3$ are isomorphic (because the commutation rules of the generators $X_0, X_1, X_2$ for $r = 2$ and $r = 3$ are the same), but the right ideals $\mathcal{R}_2$ and $\mathcal{R}_3$ are different and hence the differential calculi $(\Gamma_2, d)$ and $(\Gamma_3, d)$ are not isomorphic.

Clearly, for the quantum group $SU_q(2)$ the differential calculus $(\Gamma_1, d)$ is nothing but the 3D-calculus discovered by S.L. Woronowicz [17], because the right ideal $\mathcal{R}_1$ coincides with the right ideal of the 3D-calculus, cf. formula (2.27) in [17]. (The slight differences between some of our formulas stated above and the corresponding formulas in [17] stem from the fact that we assumed $X_2(c) = 1$ by (2.2), while $X_2(c) = -q$ by formula (2.3) in [17].)

Now we turn to the higher order differential calculi.

**Lemma 3.** If $q^{12} \neq 1$, then $\omega_0 \otimes \omega_2 \in \mathcal{S}(\mathcal{R}_r)$ for $r = 2, 3, 4$.

**Proof.** Obviously, $a^2 - (1 + q^{-2})a + q^{-2}1 \in \mathcal{R}_r$. Easy computations yield

$$\mathcal{S}(a^2-(1+q^{-2})a+q^{-2}1) = (1+q^{-2})(q^{-2}\omega_1 \otimes \omega_1 + (\gamma_0 r \gamma_2 r - 1)\omega_0 \otimes \omega_2)$$

and

$$\mathcal{S}(a + q^{-2}d - (1 + q^{-2})1) = (1+q^2)\omega_1 \otimes \omega_1 + \omega_0 \otimes \omega_2 + q^{-2}\omega_2 \otimes \omega_0$$

for $r = 2, 3, 4$, so that

$$q^6\omega_0 \otimes \omega_2 + \omega_2 \otimes \omega_0 \in \mathcal{S}(\mathcal{R}_4) \text{ and } (q^6 + q^4 - 1)\omega_0 \otimes \omega_2 + \omega_2 \otimes \omega_0 \in \mathcal{S}(\mathcal{R}_2) \cap \mathcal{S}(\mathcal{R}_3).$$ (2.5)

Let $r = 2$. Then we have $ab - b \in \mathcal{R}_2$ and $\mathcal{S}(ab-b) = \omega_0 \otimes \omega_1 + q^{-2}\omega_1 \otimes \omega_0$. Using Lemma 2, (ii), we compute

$$\mathcal{S}((ab-b)c) = P_{\text{inv}}(\mathcal{S}(ab-b)c) = P_{\text{inv}}(\omega_2 \otimes \omega_1 c + q^{-2}\omega_1 \otimes \omega_0 c)$$

$$= P_{\text{inv}}(q^{-3}\omega_0 \otimes \omega_1 + q(q^{-2} - 1)d\omega_0 \otimes \omega_2 + q^{-5}\omega_1 \otimes \omega_0 + q^{-3}(q^{-2} - 1)d\omega_2 \otimes \omega_0)$$

$$= (q^{-2} - 1)(q\omega_0 \otimes \omega_2 + q^{-3}\omega_2 \otimes \omega_0) \in \mathcal{S}(\mathcal{R}_2).$$ (2.6)
Since \( q^6 \neq 1 \), (2.5) and (2.6) imply that \( \omega_0 \otimes \omega_2 \in S(\mathcal{R}_2) \). Interchanging the role of \( b \) and \( c \) we get \( \omega_0 \otimes \omega_2 \in S(\mathcal{R}_3) \). For \( r = 4 \), we have \( (ac - q^{-2}c)b \in \mathcal{R}_4 \) and
\[
S((ac - q^{-2}c)b) = P_{inv}((\omega_1 \otimes \omega_2 + q^{-4}\omega_2 \otimes \omega_1)b)
\]
\[
P_{inv}(q^5b\omega_1 \otimes \omega_2 + q^3(q^{-2} - 1)a\omega_0 \otimes \omega_2 + qb\omega_2 \otimes \omega_1 + q^{-7}(q^{-2} - 1)a\omega_2 \otimes \omega_0)
\]
\[
= (q^{-2} - 1)(q^3\omega_0 \otimes \omega_2 + q^{-7}\omega_2 \otimes \omega_0).
\] (2.7)

From (2.5) and (2.7) we obtain \( \omega_0 \otimes \omega_2 \in S(\mathcal{R}_4) \), because we assumed that \( q^{12} \neq 1 \). □

Therefore, in contrast to the classical case, the 2-form \( \omega_0 \wedge \omega_2 \) is zero for all three calculi \((\Gamma_r, d), r = 2, 3, 4\). In particular, this implies that all 3-forms vanish and that the differential of \( \omega_1 \) is zero. Hence the higher order calculi of \((\Gamma_r, d), r = 2, 3, 4\), do not give the ordinary differential calculus on \( SL(2) \) when \( q \to 1 \). Recall that the calculus \((\Gamma_1, d)\) is 3D-calculus of S.L. Woronowicz [17]. As shown in [17], the higher order calculus of the 3D-calculus yields the ordinary differential calculus on \( SU(2) \) (and on \( SL(2) \)) in the limit \( q \to 1 \). Thus the considerations in this section emphasize the distinguished role of Woronowicz’ 3D-calculus on \( SU_q(2) \) resp. of the calculus \((\Gamma_1, d)\) on \( SL_q(2) \) among all three dimensional left-covariant differential calculi on \( SU_q(2) \) resp. \( SL_q(2) \).

**Remark.** The functionals \( \tilde{X}_0 := q^{1/2}ek, \tilde{X}_2 := q^{-1/2}fk \) and \( \tilde{X}_1 := q\lambda^{-1}(\varepsilon - k^4) \) also satisfy the commutation relations of the quantum Lie algebra \( \mathcal{X}_1 \). This presentation has been found by A. Sudbery [16], see e.g. [10]. The functionals \( \{\tilde{X}_0, \tilde{X}_1, \tilde{X}_2\} \) define another left covariant FODC \((\tilde{\Gamma}, d)\) on \( SL_q(2) \). Since \( \tilde{X}_1 \) does not annihilate the quantum trace \( q^{-2}a + q^{-4}d \), the right ideal of \( ker \varepsilon \) associated with \((\tilde{\Gamma}, d)\) is different from \( \mathcal{R}_1 \), so that the first order calculi \((\tilde{\Gamma}, d)\) and \((\Gamma_1, d)\) are not isomorphic.

**3. Left-covariant differential calculi on \( SL_q(3) \)**

In this and the following section we consider left-covariant differential calculi \((\Gamma, d)\) over \( \mathcal{A} = SL_q(3) \) of the form
\[
dx = \sum_{i,j=1;i\neq j}^3 (X_{ij} \ast x) \omega_{ij} + \sum_{n=1}^2 (X_n \ast x) \omega_n, x \in SL_q(3).
\] (3.1)

Here \( \omega_{ij} \) and \( \omega_n \) are left-invariant 1-forms and \( X_{ij} \) and \( X_n \) are linear functionals of \( U_q(sl_3) \) such that
\[
X_{ij}(1) = 0 \text{ and } X_{ij}(u_s^r) = \delta_{ir}\delta_{js} \text{ for } i \neq j,
\]
\[
X_n(u_s^r) = 0 \text{ for } r \neq s \text{ and } X_n(1) = X_n(U) = 0.
\] (3.2)

Recall that \( U := \sum_{i=1}^3 q^{-2i}u_i^i \) is the quantum trace. We define
\[
X_i := q\lambda^{-1}(\varepsilon - k_i^{-4}), \quad X_{i,i+1} := q^{-1/2}e_i k_i^{-1} \text{ and } X_{i+1,i} := q^{1/2} f_i k_i^{-1} \text{ for } i = 1, 2.
\] (3.3)
Let $\alpha$ and $\beta$ be complex numbers. We set

$$X_{13} = X_{12}X_{23} - \alpha X_{23}X_{12} \text{ and } X_{31} = X_{32}X_{21} - \beta X_{21}X_{32}.$$  

Then all linear functionals $X_{ij}$ and $X_n$ satisfy the conditions (3.2). Let $\mathcal{X}$ denote the vector space generated by these functionals. We compute

$$\Delta X_n = \varepsilon \otimes X_n + X_n \otimes k_n^{-4}, \quad \Delta X_{ij} = \varepsilon \otimes X_{ij} + X_{ij} \otimes k_j^{-2} \text{ for } |i - j| = 1,$$

and

$$\Delta X_{13} = \varepsilon \otimes X_{13} + X_{13} \otimes (k_1k_2)^{-2} + (q-\alpha)X_{12} \otimes X_{23}k_1^{-2} + (1-\alpha q)X_{23} \otimes X_{12}k_2^{-2},$$

and

$$\Delta X_{31} = \varepsilon \otimes X_{31} + X_{31} \otimes (k_1k_2)^{-2} + (1-\beta q^{-1})X_{21} \otimes X_{32}k_1^{-2} + (q^{-1}-\beta)X_{32} \otimes X_{21}k_2^{-2}.$$  

That is, we have $\Delta X - \varepsilon \otimes X \in \mathcal{X} \otimes \mathcal{A}'$ for all $X \in \mathcal{X}$. Therefore, by Lemma 1, the above Ansatz gives a left-covariant differential calculus $(\Gamma, \delta)$ over $\mathcal{A} = SL_q(3).$  

From the formulas for $\Delta_{ij}$ and $\Delta_n$ we see that the corresponding homomorphism $f$ of the algebra $SL_q(3)$ (as defined in Theorem 2.1, 3., in [18]) decomposes into a direct sum of an upper triangular part, a lower triangular part and a diagonal part. Using the pairing between $U_q(sl_3)$ and $SL_q(3)$ we obtain the following commutation relations between matrix entries and 1-forms:

$$\omega_{12}u_{ij} = q^{\delta_{j1} + \delta_{j2}}u_{ij}^{\omega_{12}} + \delta_{j3}(q - \alpha)u_{ij}^{\omega_{13}},$$

$$\omega_{23}u_{ij} = q^{\delta_{j2} + \delta_{j3}}u_{ij}^{\omega_{23}} + \delta_{j2}(q^{-1} - \alpha)u_{ij}^{\omega_{13}},$$

$$\omega_{13}u_{ij} = q^{\delta_{j3} + \delta_{j1}}u_{ij}^{\omega_{13}},$$

$$\omega_{21}u_{ij} = q^{\delta_{j1} + \delta_{j2}}u_{ij}^{\omega_{21}} + \delta_{j2}(q - \beta)u_{ij}^{\omega_{23}},$$

$$\omega_{32}u_{ij} = q^{\delta_{j2} + \delta_{j3}}u_{ij}^{\omega_{23}} + \delta_{j1}(q^{-1} - \beta)u_{ij}^{\omega_{21}},$$

$$\omega_{31}u_{ij} = q^{\delta_{j3} + \delta_{j1}}u_{ij}^{\omega_{21}},$$

and

$$\omega_nu_{ij} = q^{-2\delta_{nj} + 2\delta_{nj+1}}u_{ij}^{\omega_n}.$$  

In particular, if $\alpha = \alpha(q)$ and $\beta = \beta(q)$ are functions of $q$ such that their limits are equal to 1 as $q \to 1$, then the calculus $(\Gamma, \delta)$ gives the ordinary differential calculus on $SL(2)$ in the limit $q \to 1.$

From the general theory [18] we know that the right ideal $\mathcal{R}$ of $\ker \varepsilon$ associated with a FODC plays a crucial role. For the calculus $(\Gamma, \delta)$ defined above the right ideal $\mathcal{R}$ is generated by the following elements:

$$u_s^ru_j^i \text{ for } r \neq s, r \neq j, i \neq j, i \neq s; u_r^ru_j^i - u_j^i \text{ for } i \neq j;$$

$$u_j^iu_i^j \text{ for } i \neq j; u_1^2u_3^1, u_3^1u_2^1, u_3^1u_3^1, u_3^2u_3^3, u_2^1u_3^2, u_2^1u_3^3;$$

$$u_2^2u_1^1 - (q^{-1} - \alpha)u_1^1u_2^2 - (q - \beta)u_3^1;$$

$$u_1^3u_3^1 - u_1^1 - u_3^3 + 1, u_2^3u_1^1 + q^{-2}u_3^3 - (q^{-2} + 1)1; u_3^2u_2^1 - u_2^2 - q^{-2}u_3^3 + q^{-2}1;$$

$$u_1^1u_1^1 - (1 + q^{-2})u_1^1 + q^{-2}1; u_2^1u_2^1 - (1 + q^{-2})u_2^2 + (q^{-1} - 1)u_1^1 - (q^4 - 1 - q^{-2})1;$$

$$u_3^3u_3^3 - (1 + q^{-2})u_3^3 + q^21; U - \varepsilon(U)1.$$
To prove this, we first verify by direct computations that the functionals $X_{ij}$ and $X_n$ annihilate all these elements. We omit the (boring) details of these computations. Thus $\mathcal{R}$ is contained in the right ideal of $\ker \varepsilon$ associated with $(\Gamma, d)$. Let $\mathcal{E}$ be the eight dimensional vector space spanned by the elements $u_i^j$ for $i \neq j$, $u_1^2 - 1$ and $u_2^2 - 1$. From the above list of generators of $\mathcal{R}$ we conclude that each quadratic term $u_i^j u_j^i - \delta_{rs} \delta_{ij} 1$ belongs to $\mathcal{E} + \mathcal{R}$. (For two missing terms $u_1^2 u_3^2$ and $u_2^2 u_2^2$ we get $u_1^2 u_3^2 - (q - \alpha)u_4^3 \in \mathcal{R}$ and $u_2^2 u_2^2 - (q - 1 - \beta)u_3^3 \in \mathcal{R}$.) Hence we have $\text{codim } \mathcal{R} \leq 8$. Since $\dim (\Gamma, d) = 8$, $\mathcal{R}$ is the right ideal of $\ker \varepsilon$ associated with the calculus $(\Gamma, d)$.

Now we turn to the higher order calculus of $(\Gamma, d)$. Our first aim is to prove the following

**Lemma 4.** If $(\alpha, \beta) \neq (q, q^{-1})$ and $(\alpha, \beta) \neq (q^{-1}, q)$, then we have $\omega_{13} \otimes \omega_{31} \in S(\mathcal{R})$.

**Proof.** Recall that the elements $u_1^2 u_1^2, u_2^2 u_2^2$ and $u_3^3 u_3^3$ belong to the right ideal $\mathcal{R}$. We determine the corresponding symmetric elements. All functionals $X_{ij} X_{rs}$ with $i < j, r < s$ or $i > j, r > s$ and all functionals $X_n X_{rs}, X_{rs} X_n, X_n X_m$ annihilate these elements. We have

$$S(u_1^2 u_1^2) = X_{12}(u_1^2 u_1^2) X_{21}(u_2^2 u_1^2) \omega_{12} \otimes \omega_{21} + X_{13}(u_1^2 u_1^2) X_{31}(u_2^2 u_1^2) \omega_{13} \otimes \omega_{31} + X_{21}(u_1^2 u_1^2) X_{12}(u_2^2 u_1^2) \omega_{21} \otimes \omega_{12} = q \omega_{12} \otimes \omega_{21} + (q^{-1} - \beta) \omega_{13} \otimes \omega_{31} + (q - \alpha) \omega_{13} \otimes \omega_{31} + q^{-1} \omega_{21} \otimes \omega_{12} = q \omega_{12} \otimes \omega_{21} + q^{-1} \omega_{21} \otimes \omega_{12} + (\lambda_+ - \alpha - \beta) \omega_{13} \otimes \omega_{31}.$$

Similarly, we obtain

$$S(u_2^2 u_2^2) = q \omega_{23} \otimes \omega_{32} + q^{-1} \omega_{32} \otimes \omega_{23} + (\lambda_+ - \alpha - \beta) \omega_{31} \otimes \omega_{13},$$

and

$$S(u_3^3 u_3^3) = q \omega_{13} \otimes \omega_{31} + q^{-1} \omega_{31} \otimes \omega_{13}.$$

Using two of these formulas and the facts that $S(xy) = P_{inv}(S(x)y), x \in \mathcal{R}$, by Lemma 1, and $P_{inv}(yn) = \varepsilon(y) P_{inv}(n)$ for $n \in \Gamma \otimes \Lambda$, $\varepsilon, \lambda \in \Lambda$, we compute

$$S(u_1^2 u_1^2) = P_{inv}(S(u_2^2 u_1^2) u_3^3) = P_{inv}(q \omega_{12} \otimes \omega_{21} u_3^3 + q^{-1} \omega_{21} \otimes \omega_{12} u_3^3 + (\lambda_+ - \alpha - \beta) \omega_{13} \otimes \omega_{31} u_3^3) = P_{inv}(q \omega_{12} \otimes \omega_{21} + q(q - \alpha) u_3^3 \omega_{12} \otimes \omega_{21} + q^{-1} u_3^3 \omega_{21} \otimes \omega_{12} + (\lambda_+ - \alpha - \beta) q^2 u_3^3 \omega_{13} \otimes \omega_{31}) = q \omega_{12} \otimes \omega_{21} + q^{-1} \omega_{21} \otimes \omega_{12} + q^{-1} (q - \alpha)(q - \beta) \omega_{31} \otimes \omega_{13} + q^2 (\lambda_+ - \alpha - \beta) \omega_{13} \otimes \omega_{31},$$

so that $\omega_{13} \otimes \omega_{31} \in S(\mathcal{R})$ provided that $\alpha \neq q^{-1}$ and $\beta \neq q^{-1}$.

Similarly, we get

$$S(u_2^2 u_2^2) = S(u_3^3 u_3^3) + (q^{-1} - \alpha)(q^{-1} - \beta) S(u_3^3 u_3^3) + (\alpha - q)(\beta - q) \omega_{13} \otimes \omega_{31},$$

hence $\omega_{13} \otimes \omega_{31} \in S(\mathcal{R})$ when $\alpha \neq q$ and $\beta \neq q$. ■
Therefore, if $(\alpha, \beta) \neq (q^{-1}, q)$ and $(\alpha, \beta) \neq (q, q^{-1})$, then the 2-form $\omega_{13} \wedge \omega_{31}$ vanishes in $\Gamma_2^\wedge$ and hence the space of 2-forms does not give the corresponding space for the ordinary differential calculus on $SL(3)$ when $q \to 1$. The two remaining distinguished cases $(\alpha, \beta) = (q^{-1}, q)$ and $(\alpha, \beta) = (q, q^{-1})$ will be treated in the following section. For this let $(\Gamma_1, d)$ and $(\Gamma_2, d)$ denote the first order differential calculus $(\Gamma, d)$ on $SL_q(3)$ with $(\alpha, \beta) = (q^{-1}, q)$ and $(\alpha, \beta) = (q, q^{-1})$, respectively.

4. The differential calculi $(\Gamma_1, d)$ and $(\Gamma_2, d)$ on $SL_q(3)$

Let $\mathcal{R}_r$ be the right ideal of $\ker \epsilon$ and let $\mathcal{X}_r$ be the linear span of linear functionals $X_{ij}, i \neq j, i, j = 1, 2, 3$, and $X_n, n = 1, 2$, for the calculus $(\Gamma_r, d), r = 1, 2$. From the definitions of functionals $X_{i,j}, X_n$ and the commutation rules in the algebra $\mathcal{U}_q(\mathfrak{sl}_3)$ we obtain the following commutation relations for the generators of the quantum Lie algebra $\mathcal{X}_r$.

$$\mathcal{X}_1$$

\[ X_{12}X_{32}-q^{-1}X_{32}X_{12}=0, X_{23}X_{21}-q^{-1}X_{21}X_{23}=0, \]
\[ X_{12}X_{21}-q^2X_{21}X_{12}=X_1, X_{13}X_{31}-q^2X_{31}X_{13}+q^{-1}\lambda X_1X_2=X_1+X_2, \]
\[ X_{23}X_{32}-q^2X_{32}X_{23}=X_2, X_1X_2-X_2X_1=0, \]
\[ X_1X_{12}-q^{-4}X_{12}X_1=(1+q^{-2})X_{12}, X_1X_{21}-q^4X_{21}X_1=-(q^2+q^4)X_{21}, \]
\[ X_2X_{23}-q^{-4}X_{23}X_2=(1+q^{-2})X_{23}, X_2X_{32}-q^4X_{32}X_2=-(q^2+q^4)X_{32}, \]
\[ X_1X_{13}-q^{-2}X_{13}X_1=X_{13}, X_1X_{31}-q^2X_{31}X_1=-q^2X_{31}, \]
\[ X_2X_{13}-q^{-2}X_{13}X_2=X_{13}, X_2X_{31}-q^2X_{31}X_2=-q^2X_{31}, \]
\[ X_2X_{12}-q^2X_{12}X_2=-q^2X_{12}, X_2X_{21}-q^{-2}X_{21}X_2=X_{21}, \]

$$\mathcal{X}_2$$

\[ X_{13}X_{23}-qX_{23}X_{13}=0, X_{32}X_{31}-q^{-1}X_{31}X_{32}=0, \]
\[ X_{12}X_{13}-qX_{13}X_{12}=0, X_{31}X_{21}-q^{-1}X_{21}X_{31}=0, \]
\[ X_{12}X_{23}-q^{-1}X_{23}X_{12}=X_{13}, X_{32}X_{21}-qX_{21}X_{32}=X_{31}, \]
\[ X_{12}X_{31}-qX_{31}X_{12}=-qX_{32}, X_{13}X_{21}-qX_{21}X_{13}=-qX_{23}, \]
\[ X_{13}X_{32}-qX_{32}X_{13}=0, X_{23}X_{31}-qX_{31}X_{23}+q^{-1}\lambda X_{21}X_2=X_{21}. \]

Next we shall describe the corresponding higher order differential calculi. For this we need to know the vector space $\mathcal{S}(\mathcal{R}_r)$. Let $I$ denote the ordered index set $\{1, 2, 21, 31, 32, 12, 13, 23\}$. The right ideal $\mathcal{R}_r, r=1, 2$, has 37 generators which have been listed in the preceding section.

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(recall that $\alpha = q^{-1}, \beta = q$ for $\mathcal{R}_1$ and $\alpha = q, \beta = q^{-1}$ for $\mathcal{R}_2$). Let $\mathcal{B}_r$ be the linear span of these generators. Using the formulas for the comultiplications of $X_{ij}$ and $X_n$ and for the pairing between $U_q(sl_3)$ and $SL_q(3)$ one can compute the symmetric elements $S(x)$ for the generators of $\mathcal{R}_r$. We state only the result of these (long) computations. The following 36 elements of $\Gamma \otimes_{\mathcal{A}} \Gamma$ belong to $S(\mathcal{R}_r)$ and form a basis of the vector space $S(\mathcal{B}_r)$.

$$S(\mathcal{R}_1) \text{ and } S(\mathcal{R}_2) : \omega_{ij} \otimes \omega_{ij} \text{ for } i \neq j, i=1, 2, 3; \omega_n \otimes \omega_n \text{ for } n=1, 2;$$

$$\omega_{12} \otimes \omega_{32} + q \omega_{32} \otimes \omega_{12}, \omega_{23} \otimes \omega_{21} + q \omega_{21} \otimes \omega_{23},$$

$$\omega_{12} \otimes \omega_{21} + q^{-2} \omega_{21} \otimes \omega_{12}, \omega_{13} \otimes \omega_{31} + q^{-2} \omega_{31} \otimes \omega_{13},$$

$$\omega_{23} \otimes \omega_{32} + q^{-2} \omega_{32} \otimes \omega_{23}, \omega_1 \otimes \omega_2 + \omega_2 \otimes \omega_1 - q^{-1} \lambda \omega_{13} \otimes \omega_{31},$$

$$\omega_1 \otimes \omega_{12} + q^4 \omega_{12} \otimes \omega_1, \omega_1 \otimes \omega_{21} + q^{-4} \omega_{21} \otimes \omega_1,$$

$$\omega_2 \otimes \omega_{23} + q^4 \omega_{23} \otimes \omega_2, \omega_2 \otimes \omega_{32} + q^{-4} \omega_{32} \otimes \omega_2,$$

$$\omega_1 \otimes \omega_{23} + q^{-2} \omega_{23} \otimes \omega_1, \omega_1 \otimes \omega_{32} + q^2 \omega_{32} \otimes \omega_1,$$

$$\omega_1 \otimes \omega_{13} + q^2 \omega_{13} \otimes \omega_1, \omega_1 \otimes \omega_{31} + q^{-2} \omega_{31} \otimes \omega_1,$$

$$\omega_2 \otimes \omega_{13} + q^2 \omega_{13} \otimes \omega_2, \omega_2 \otimes \omega_{31} + q^{-2} \omega_{31} \otimes \omega_2,$$

$$\omega_2 \otimes \omega_{12} + q^{-2} \omega_{12} \otimes \omega_2, \omega_2 \otimes \omega_{21} + q^2 \omega_{21} \otimes \omega_2,$$

$$S(\mathcal{R}_1) :$$

$$\omega_{13} \otimes \omega_{23} + q^{-1} \omega_{23} \otimes \omega_{13}, \omega_{32} \otimes \omega_{31} + q \omega_{31} \otimes \omega_{32},$$

$$\omega_{12} \otimes \omega_{13} + q^{-1} \omega_{13} \otimes \omega_{12}, \omega_{31} \otimes \omega_{21} + q \omega_{21} \otimes \omega_{31},$$

$$\omega_{12} \otimes \omega_{23} + q \omega_{23} \otimes \omega_{12}, \omega_{32} \otimes \omega_{31} + q^{-1} \omega_{31} \otimes \omega_{32},$$

$$\omega_{12} \otimes \omega_{31} + q^{-1} \omega_{31} \otimes \omega_{12}, \omega_{23} \otimes \omega_{13} + q \omega_{13} \otimes \omega_{23} + \lambda \omega_{13} \otimes \omega_{21},$$

$$\omega_{13} \otimes \omega_{32} + q^{-1} \omega_{32} \otimes \omega_{13}, \omega_2 \otimes \omega_{21} + q^2 \omega_{21} \otimes \omega_2 + \lambda \omega_{31} \otimes \omega_{23}.$$}

$$S(\mathcal{R}_2) :$$

$$\omega_{13} \otimes \omega_{23} + q \omega_{23} \otimes \omega_{13}, \omega_{32} \otimes \omega_{31} + q^{-1} \omega_{31} \otimes \omega_{32},$$

$$\omega_{12} \otimes \omega_{13} + q \omega_{13} \otimes \omega_{12}, \omega_{31} \otimes \omega_{21} + q^{-1} \omega_{21} \otimes \omega_{31},$$

$$\omega_{12} \otimes \omega_{23} + q^{-1} \omega_{23} \otimes \omega_{12}, \omega_{32} \otimes \omega_{31} + q \omega_{31} \otimes \omega_{32},$$

$$\omega_1 \otimes \omega_{32} + q^2 \omega_{32} \otimes \omega_1 + \lambda \omega_{12} \otimes \omega_{31}, \omega_{13} \otimes \omega_{21} + q^{-1} \omega_{21} \otimes \omega_{21},$$

$$\omega_{12} \otimes \omega_{13} + q \omega_{13} \otimes \omega_{12}, \omega_{31} \otimes \omega_{21} + q^{-1} \omega_{21} \otimes \omega_{31},$$

$$\omega_{12} \otimes \omega_{23} + q^{-1} \omega_{23} \otimes \omega_{12}, \omega_{32} \otimes \omega_{31} + q \omega_{31} \otimes \omega_{32},$$

$$\omega_1 \otimes \omega_{32} + q^2 \omega_{32} \otimes \omega_1 + \lambda \omega_{12} \otimes \omega_{31}, \omega_{13} \otimes \omega_{21} + q^{-1} \omega_{21} \otimes \omega_{21},$$

$$\omega_{12} \otimes \omega_{13} + q \omega_{13} \otimes \omega_{12}, \omega_{31} \otimes \omega_{21} + q^{-1} \omega_{21} \otimes \omega_{31},$$

$$\omega_{12} \otimes \omega_{23} + q^{-1} \omega_{23} \otimes \omega_{12}, \omega_{32} \otimes \omega_{31} + q \omega_{31} \otimes \omega_{32},$$

The preceding formulas showed that both FODC $(\Gamma_1d)$ and $(\Gamma_2d)$ are very close to the classical differential calculus on $SL(3)$. For instance, except for two cases (that is, $\omega_{12}u_{3}^n, \omega_{32}u_{1}^n$ for $r=1$ and $\omega_{23}u_{2}^n, \omega_{21}u_{2}^n$ for $r=2$), each 1-form $\omega_i u_{m}^n$ is equal to $u_{m}^n \omega_i$ multiplied by some power of $q$. Except for three cases, the commutation relations of the quantum Lie algebra $X_r$ and the elements of the left module basis of $S_2$ contain only two quadratic terms as in the classical case.

**Lemma 5:** For $r=1, 2$, we have $S(\mathcal{B}_r)$=S(\mathcal{R}_r).

**Proof.** By Lemma 2, (ii), it suffices to show that $P_{inv}(\zeta u_j^i) \in S(\mathcal{B}_r)$ for $i, j=1, 2, 3$ and for all 36 basis elements $\zeta$ of listed above. This is obviously fulfilled if $\zeta u_j^i$ is a scalar multiple
of $u^i_j \zeta$, since $P_{inv}(a \zeta) = \varepsilon(a) P_{inv}(\zeta)$ by Lemma 2.2 in [18]. From the commutation relations between matrix entries and 1-forms we conclude that it remains to check the condition $P_{inv}(\zeta u^i_j) \in S(B_r)$ for all basis elements $\zeta$ of $S(B_r)$ which are sums of three terms or contain the 1-forms $\omega_{12}$ or $\omega_{32}$ in case $r=1$ resp. $\omega_{23}$ or $\omega_{21}$ in case $r=2$. These are 14 elements $\zeta$ in either case $r=1$ and $r=2$. We carry out this verification for the two elements $\zeta_1 := \omega_{12} \otimes \omega_{32} + q \omega_{32} \otimes \omega_{12}$ and $\zeta_2 := \omega_{1} \otimes \omega_{2} + \omega_{2} \otimes \omega_{1} - q^{-1} \lambda \omega_{13} \otimes \omega_{31}$ in case $r=1$. From $\omega_{12} \otimes \omega_{32} u^i_j = q u^i_j \omega_{12} \otimes \omega_{32} + q \lambda u^i_j \omega_{13} \otimes \omega_{32}$ and $\omega_{32} \otimes \omega_{12} u^i_j = q u^i_j \omega_{32} \otimes \omega_{12} + q^{-1} \lambda u^i_j \omega_{32} \otimes \omega_{13}$ it follows that $P_{inv}(\zeta_1 u^i_j) = q \zeta_1$ and $P_{inv}(\zeta_2 u^i_j) = q \lambda (\omega_{13} \otimes \omega_{32} + q^{-1} \omega_{32} \otimes \omega_{13}) \in S(B_r)$. Moreover, $\zeta_1 u^i_j$ is a multiple of $u^i_j \zeta_1$ for all elements $u^i_j$ other than $u^i_3$ and $u^i_3$. For $\zeta_2$ we obtain $\zeta_2 u^i_j = q^{-2} \delta_{j1} + 2 \delta_{j3} u^i_j \zeta_2$, so the condition is also valid for $\zeta_2$. The other 26 cases are treated in a similar way.

Thus, by Lemma 5, the 36 basis elements for the vector space $S(B_r)$ of the above list form a free left module basis for $S_2 = A S(\mathcal{R}_r) A$. This gives a precise description of the $A$-submodule $S_2$ of $\Gamma \otimes_A \Gamma$.

From the definitions of functionals $X_{ij}, X_n$ and the pairing between $U_q(sl_3)$ and $SL_q(3)$ it follows that for both calculi

$$\omega(u^i_j) = \omega(1) \otimes \omega(u^i_j) = \omega(1) \otimes \zeta$$

Recall that for any left-covariant differential calculus over $A$ we have $d\omega(a) = -\omega(a_{(1)}) \otimes \omega(a_{(2)})$, $a \in A$. From the preceding we obtain the following Maurer-Cartan formulas for our differential calculi.

$$(\Gamma_1, d) \text{ and } (\Gamma_2, d) : \text{d} \omega_1 = -\omega_{12} \otimes \omega_{21} - \omega_{13} \otimes \omega_{31}, \text{d} \omega_2 = -\omega_{13} \otimes \omega_{31} - \omega_{23} \otimes \omega_{32},$$

$$\text{d} \omega_{13} = -\omega_{1} \otimes \omega_{13} - \omega_{2} \otimes \omega_{13} - \omega_{12} \otimes \omega_{23},$$

$$\text{d} \omega_{23} = q^2 \omega_{1} \otimes \omega_{23} - (1+q^{-2}) \omega_{2} \otimes \omega_{23} + q \omega_{13} \otimes \omega_{21},$$

$$\text{d} \omega_{21} = (q^2 + q^4) \omega_{1} \otimes \omega_{21} - \omega_{2} \otimes \omega_{21} - (q^2 + q^4) \omega_{23},$$

$$(\Gamma_1, d) : \text{d} \omega_{12} = -(1+q^{-2}) \omega_{1} \otimes \omega_{12} + q^2 \omega_{2} \otimes \omega_{12} - \omega_{13} \otimes \omega_{23},$$

$$\text{d} \omega_{31} = q^2 \omega_{1} \otimes \omega_{31} + q^2 \omega_{2} \otimes \omega_{31} + q^{-1} \omega_{12} \otimes \omega_{32},$$

$$\text{d} \omega_{32} = -\omega_{1} \otimes \omega_{32} + (q^2 + q^4) \omega_{2} \otimes \omega_{32} + \omega_{12} \otimes \omega_{32},$$

$$(\Gamma_2, d) : \text{d} \omega_{12} = -(1+q^{-2}) \omega_{1} \otimes \omega_{12} + q^2 \omega_{2} \otimes \omega_{12} - q^2 \omega_{13} \otimes \omega_{32},$$

$$\text{d} \omega_{31} = q^2 \omega_{1} \otimes \omega_{31} + q^2 \omega_{2} \otimes \omega_{31} + q \omega_{21} \otimes \omega_{32},$$

$$\text{d} \omega_{32} = -\omega_{1} \otimes \omega_{32} + (q^2 + q^4) \omega_{2} \otimes \omega_{32} + q^{-1} \omega_{12} \otimes \omega_{31}.$$
of \( \mathcal{S}(\mathcal{R}) \) are \( \omega_i \otimes \omega_j + \sum_{n,m} \sigma_{ij}^{nm} \omega_n \otimes \omega_m, i, j \in I, i \leq j \). From the above formulas we see that the commutation relations of the quantum Lie algebra generators can be expressed as

\[
X_i X_j - \sum_{n,m} \sigma_{ij}^{nm} X_n X_m = \sum C_{ij}^k X_k, i \neq j, \text{ with certain coefficients } C_{ij}^k \in \mathcal{F}. \]

The linear transformation \( \sigma \) of the 64-dimensional vector space \( \Gamma_{\text{inv}} \otimes \Gamma_{\text{inv}} \) has eigenvalues 1 and -1 with multiplicities 36 and 28, respectively. Obviously, \( \sigma^2 \) is the identity. However, \( \sigma \) does not satisfy the braid relation \( \sigma_{12} \sigma_{23} \sigma_{12} = \sigma_{23} \sigma_{12} \sigma_{23} \) on \( \Gamma_{\text{inv}} \otimes \Gamma_{\text{inv}} \otimes \Gamma_{\text{inv}} \). (For instance, we have \( \sigma_{12} \sigma_{23} \sigma_{12}(\omega_1 \otimes \omega_2 \otimes \omega_3) = q \omega_3 \otimes \omega_2 \otimes \omega_1 - q^{-2} \lambda \omega_3 \otimes \omega_2 \otimes \omega_1 \) and \( \sigma_{23} \sigma_{12} \sigma_{23}(\omega_1 \otimes \omega_2 \otimes \omega_3) = q \omega_3 \otimes \omega_2 \otimes \omega_1 - \lambda \omega_3 \otimes \omega_2 \otimes \omega_1 \) for the calculus (\( \Gamma_1, d \)).

Let \( J(\mathcal{R}) = \oplus_n J_n(\mathcal{R}) \) be the two-sided ideal of the tensor algebra \( \Gamma_{\text{inv}}^\otimes / J_{\text{inv}} \) of the vector space \( \Gamma_{\text{inv}} \) which is generated by the set \( \mathcal{S}(\mathcal{R}) \). Clearly, \( (\Gamma^\wedge)_{\text{inv}} \) is (isomorphic to) the quotient algebra \( \Gamma_{\text{inv}}^\otimes / J(\mathcal{R}) = \oplus_n \Gamma_{\text{inv}}^{\otimes n} / J_n(\mathcal{R}) \). The above basis elements of the vector space \( \mathcal{S}(\mathcal{R}) \) form a Gröbner basis (see, e.g. [6] for this concept) of the ideal \( J(\mathcal{R}) \) with respect to the above ordering of the index set \( I \). We have checked this by using the computer algebra system FELIX [1]. (One may also verify this assertion by performing explicit calculations.) Therefore, we get a vector space basis for the quotient space \( \Gamma_{\text{inv}}^{\otimes n} / J_n(\mathcal{R}) \) by taking all monomials in the 1-forms \( \omega_i, i \in I \), which are not multiples of the leading term of one of these Gröbner basis elements. From the special form of these elements we see that this set of monomials is the same as in the case \( q = 1 \). Hence the dimension of the vector space \( \Gamma_{\text{inv}}^{\otimes n} / J_n(\mathcal{R}) \) is equal to the corresponding dimension \( \left( \begin{array}{c} \frac{s}{n} \\ n \end{array} \right) \) in the classical case. Moreover, it follows that the associated higher order calculi of both calculi \( (\Gamma_1, d) \) and \( (\Gamma_2, d) \) give the ordinary higher order calculus on \( SL(3) \) in the limit \( q \to 1 \).

5. Left-covariant differential calculi on \( SL_q(N) \)

Let \( L^+ = (+l_j^i) \) and \( L^- = (-l_j^i) \) be the \( N \times N \) matrices of linear functionals \( +l_j^i \) and \( -l_j^i \) on \( \mathcal{A} := SL_q(N) \) as defined in [5]. Recall that \( L^\pm \) is uniquely determined by the properties that \( L^\pm : \mathcal{A} \to M_N(\mathcal{F}) \) is a unital algebra homomorphism and \( \pm l_j^i(u^n_{im}) = p^{\mp 1} (R_{ij})^n_{mj} \) for \( i, j, n, m = 1, \ldots, N \), where \( p \) is an \( N \)-th root of \( q \) and \( R \) is the \( R \)-matrix of the quantum group \( SL_q(N) \).

In this section we define \( N^2 - 1 \) dimensional left-covariant FODC \( (\Gamma_1, d) \) and \( (\Gamma_2, d) \) over \( \mathcal{A} = SL_q(N) \). They generalize the two calculi over \( SL_q(N) \) studied in the preceding section. For this we set

\[
X_{ij} = \lambda^{-1} \kappa(-l_j^i) - l_i^i \text{ and } X_{ji} = -\lambda^{-1} \kappa(+l_j^i) + l_i^i \text{ for } i < j \text{ and } r = 1,
\]

\[
X_{ij} := -\lambda^{-1} l_j^i - l_i^i \text{ and } X_{ji} = \lambda^{-1} - l_j^i + l_i^i \text{ for } i < j \text{ and } r = 2.
\]

\[
X_n = q \lambda^{-1} (\varepsilon - (-l_{n+1}^n + l_{n+1}^{n+1})^2) \text{ for } n = 1, \ldots, N - 1 \text{ and } r = 1, 2.
\]

Let \( \mathcal{X}_r \) denote the linear span of functionals \( X_{ij}, i \neq j \), \( i, j = 1, \ldots, N \), and \( X_n, n = 1, \ldots, N - 1 \).

Computing the comultiplications of these generators, we obtain for \( r = 1 \) and \( i < j \),

\[
\Delta X_{ij} = \varepsilon \otimes X_{ij} + \sum_{m=i+1}^j X_{im} \otimes -l_i^i \kappa(-l_m^m) \text{ and } \Delta X_{ji} = \varepsilon \otimes X_{ji} + \sum_{m=i}^{j-1} X_{jm} \otimes +l_j^j \kappa(+l_i^i)
\]
for \( r = 2 \) and \( i < j \),

\[
\Delta X_{ij} = \varepsilon \otimes X_{ij} + \sum_{m=i}^{j-1} X_{mj} \otimes j^j - l_i^m \text{ and } \Delta X_{ji} = \varepsilon \otimes X_{ji} + \sum_{m=i}^{j} X_{mi} \otimes -l_i^j + l_i^m
\]

and for \( r = 1, 2 \),

\[
\Delta X_n = \varepsilon \otimes X_n + X_n \otimes (-l_i^n + l_i^{n+1})^2
\]

In particular, these formulas show that \( \Delta X - \varepsilon \otimes X \in \mathcal{X}' \otimes A' \) for all \( X \in \mathcal{X}' \) and \( r = 1, 2 \). Therefore, by Lemma 1, the vector space \( \mathcal{X}' \) defines a left-covariant FODC over \( SL_q(N) \). Furthermore, we verify that \( X_{ij}(u^r_s) = \delta_{ir}\delta_{js} \) for \( i \neq j \) and \( X_n(u^r_s) = \delta_{rs}(\delta_{nr} - q^2\delta_{n+1,r}) \) for \( n = 1, \ldots, N - 1 \). Thus all elements of the quantum Lie algebra \( \mathcal{X}' \) annihilate the quantum trace \( U = \sum q^{-2i}u^r_i. \) Since \( \dim \mathcal{X}' = N^2 - 1 \), the FODC \( (\Gamma, \mathcal{X}) \) has dimension \( N^2 - 1 \).

It is not difficult to check that in the classical limit \( q \to 1 \) both FODC \( (\Gamma, \mathcal{X}) \) give the ordinary FODC on \( SL(N) \). (As in the preceding sections, we define the limit \( q \to 1 \) of the calculus \( (\Gamma, \mathcal{X}) \) by keeping the Maurer-Cartan basis \( \omega(u^r_s), (r, s) \neq (N, N), \) of \( (\Gamma, \mathcal{X}) \) fixed.) Some computations show that for \( |i-j| \geq 3 \) and \( r = 1, 2 \) the 2-form \( \omega_{ij} \wedge \omega_{ji} \) vanishes in \( (\Gamma, \mathcal{X}) \). That is, if \( N \geq 4 \), both associated higher order calculi do not have the classical higher order calculus on \( SL(N) \) as their limits when \( q \to 1 \). To overcome this disadvantage, we have taken up another approach in [14].

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