Abstract

This Festschrift in honour of J. A. de Azcárraga\footnote{To appear in the proceedings of Symmetries in Gravity and Field Theory, Workshop in honour of Prof. J.A. de Azcárraga, June 9–11, 2003, Salamanca, Spain.} gives an introduction to the concept of duality, i.e., to the relativity of the notion of a quantum, in the context of the quantum mechanics of a finite number of degrees of freedom. Although the concept of duality arises in string and M–theory, Vafa has argued that it should also have a counterpart in quantum mechanics, before moving on to second quantisation, fields, strings and branes. We illustrate our analysis with the case when classical phase space is complex projective space, but our conclusions can be generalised to other complex, symplectic phase spaces, both compact and noncompact.

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Fibre bundles \[\text{[1]}\] are powerful tools to formulate the gauge theories of fundamental interactions and gravity. The question arises whether or not quantum mechanics may also be formulated using fibre bundles. Important physical motivations call for such a formulation.

In quantum mechanics one aims at constructing a Hilbert–space vector bundle over classical phase space. In geometric quantisation this goal is achieved in a two–step process that can be very succinctly summarised as follows. One first constructs a certain holomorphic line bundle (the quantum line bundle) over classical phase space. Next one identifies certain sections of this line bundle as defining the Hilbert space of quantum states. Alternatively one may skip the quantum line bundle and consider the one–step process of directly constructing a Hilbert–space vector bundle over classical phase space. Associated with this vector bundle there is a principal bundle whose fibre is the unitary group of Hilbert space.

Textbooks on quantum mechanics \[\text{[2]}\] usually deal with the case when this Hilbert–space vector bundle is trivial. Such is the case, \textit{e.g.}, when classical phase space is contractible to a point. However, it seems natural to consider the case of a nontrivial bundle as well. Beyond a purely mathematical interest, important physical issues that go by the generic name of dualities \[\text{[3]}\] motivate the study of nontrivial bundles.

Triviality of the Hilbert–space vector bundle implies that the transition functions all equal the identity of the structure group. In passing from one coordinate chart to another on classical phase space, vectors on the fibre are acted on by the identity. Since these vectors are quantum states, we can say that all observers on classical phase space are quantised in the same way. This is no longer the case on a nontrivial vector bundle, where the transition functions are different from the identity. As opposed to the previous case, different neighbourhoods on classical phase space are quantised independently and, possibly, differently. The resulting quantisation is only local on classical phase space, instead of global. This reflects the property of local triviality satisfied by all fibre bundles.

Given a certain base manifold and a certain fibre, the trivial bundle over the given base with the given fibre is unique. This may mislead one to conclude that quantisation is also unique, or independent of the observer on classical phase space. In fact the notion of duality points precisely to the opposite conclusion, \textit{i.e.}, to the nonuniqueness of the quantisation procedure and to its dependence on the observer \[\text{[3]}\].

Clearly a framework is required in order to accommodate dualities within quantum mechanics \[\text{[3]}\]. Nontrivial Hilbert–space vector bundles over classical phase space provide one such framework. They allow for the possibility of having different, nonequiv-
alent quantisations, as opposed to the uniqueness of the trivial bundle. However, although nontriviality is a necessary condition, it is by no means sufficient. A flat connection on a nontrivial bundle would still allow, by parallel transport, to canonically identify the Hilbert–space fibres above different points on classical phase space. This identification would depend only on the homotopy class of the curve joining the basepoints, but not on the curve itself. Now flat connections are characterised by constant transition functions \[1\], this constant being always the identity in the case of the trivial bundle. Hence, in order to accommodate dualities, we will be looking for nonflat connections. We will see presently what connections we need on these bundles.

This article is devoted to constructing nonflat Hilbert–space vector bundles over classical phase space. Our notations are as follows. \(C\) will denote a complex \(n\)-dimensional, connected, compact classical phase space, endowed with a symplectic form \(\omega\) and a complex structure \(J\). We will assume that \(\omega\) and \(J\) are compatible, so holomorphic coordinate charts on \(C\) will also be Darboux charts. We will primarily concentrate on the case when \(C\) is complex projective space \(\mathbb{CP}^n\). Its holomorphic tangent bundle will be denoted \(T(\mathbb{CP}^n)\). The tautological line bundle \(\tau^{-1}\) over \(\mathbb{CP}^n\) and its dual \(\tau\) will also be considered. The Picard group of \(C\) will be denoted \(\text{Pic}(C)\).

Towards the end of this article we will also consider the infinite–dimensional projective space \(\mathbb{CP}(\mathcal{H})\), corresponding to complex, separable, infinite–dimensional Hilbert space \(\mathcal{H}\).

Finally we would like to draw attention to refs. [4, 5, 6, 7, 8, 9, 10], where issues partially overlapping with ours are studied.

\section*{2 \(\mathbb{CP}^n\) as a classical phase space}

We will first consider a classical mechanics whose phase space is complex, projective \(n\)-dimensional space \(\mathbb{CP}^n\). The following properties are well known \[11\].

Let \(Z^1, \ldots, Z^{n+1}\) denote homogeneous coordinates on \(\mathbb{CP}^n\). The chart defined by \(Z^k \neq 0\) covers one copy of the open set \(U_k = \mathbb{C}^n\). On the latter we have the holomorphic coordinates \(z_j^{(k)} = Z_j/Z^k, j \neq k\); there are \(n + 1\) such coordinate charts. \(\mathbb{CP}^n\) is a Kähler manifold with respect to the Fubini–Study metric. On the chart \((U_k, z_j^{(k)})\) the Kähler potential reads

\[ K(z_j^{(k)}, \bar{z}_j^{(k)}) = \log \left(1 + \sum_{j=1}^{n} z_j^{(k)} \bar{z}_j^{(k)} \right). \tag{1} \]

The singular homology ring \(H_* (\mathbb{CP}^n, \mathbb{Z})\) contains the nonzero subgroups

\[ H_{2k} (\mathbb{CP}^n, \mathbb{Z}) = \mathbb{Z}, \quad k = 0, 1, \ldots, n, \tag{2} \]

while

\[ H_{2k+1} (\mathbb{CP}^n, \mathbb{Z}) = 0, \quad k = 0, 1, \ldots, n - 1. \tag{3} \]

We have \(\mathbb{CP}^n = \mathbb{C}^n \cup \mathbb{CP}^{n-1}\), with \(\mathbb{CP}^{n-1}\) a hyperplane at infinity. Topologically, \(\mathbb{CP}^n\) is obtained by attaching a (real) \(2n\)–dimensional cell to \(\mathbb{CP}^{n-1}\). \(\mathbb{CP}^n\) is simply
connected,
\[ \pi_1(\mathbb{C}P^n) = 0, \]  
\[ (4) \]

it is compact, and inherits its complex structure from that on \( \mathbb{C}^{n+1} \).

Let \( \tau^{-1} \) denote the tautological bundle on \( \mathbb{C}P^n \). We recall that \( \tau^{-1} \) is defined as the subbundle of the trivial bundle \( \mathbb{C}P^n \times \mathbb{C}^{n+1} \) whose fibre at \( p \in \mathbb{C}P^n \) is the line in \( \mathbb{C}^{n+1} \) represented by \( p \). Then \( \tau^{-1} \) is a holomorphic line bundle over \( \mathbb{C}P^n \). Its dual, denoted \( \tau \), is called the hyperplane bundle. For any \( l \in \mathbb{Z} \), the \( l \)-th power \( \tau^l \) is also a holomorphic line bundle over \( \mathbb{C}P^n \). In fact every holomorphic line bundle \( L \) over \( \mathbb{C}P^n \) is isomorphic to \( \tau^l \) for some \( l \in \mathbb{Z} \); this integer is the first Chern class of \( L \).

3 The quantum line bundle

In the framework of geometric quantisation [12] it is customary to consider the case when \( C \) is a compact Kähler manifold. In this context one introduces the notion of a quantisable, compact, Kähler phase space \( C \), of which \( \mathbb{C}P^n \) is an example. This means that there exists a quantum line bundle \( (L, g, \nabla) \) on \( C \), where \( L \) is a holomorphic line bundle, \( g \) a Hermitian metric on \( L \), and \( \nabla \) a covariant derivative compatible with the complex structure and \( g \). Furthermore, the curvature \( F \) of \( \nabla \) and the symplectic 2–form \( \omega \) are required to satisfy
\[ F = -2\pi i \omega. \]  
\[ (5) \]

It turns out that quantisable, compact Kähler manifolds are projective algebraic manifolds and viceversa [13]. After introducing a polarisation, the Hilbert space of quantum states is given by the global holomorphic sections of \( L \).

Recalling that, on \( \mathbb{C}P^n \), \( L \) is isomorphic to \( \tau^l \) for some \( l \in \mathbb{Z} \), let \( \mathcal{O}(l) \) denote the sheaf of holomorphic sections of \( L \) over \( \mathbb{C}P^n \). The vector space of holomorphic sections of \( L = \tau^l \) is the sheaf cohomology space \( H^0(\mathbb{C}P^n, \mathcal{O}(l)) \). The latter is zero for \( l < 0 \), while for \( l \geq 0 \) it can be canonically identified with the set of homogeneous polynomials of degree \( l \) on \( \mathbb{C}^{n+1} \). This set is a vector space of dimension \( \binom{n+l}{n} \):
\[ \dim H^0(\mathbb{C}P^n, \mathcal{O}(l)) = \binom{n+l}{n}. \]  
\[ (6) \]

We will give a quantum–mechanical derivation of eqn. (6) in section 4.

Equivalence classes of holomorphic line bundles over a complex manifold \( C \) are classified by the Picard group \( \text{Pic}(C) \). The latter is defined [14] as the sheaf cohomology group \( H^1_{\text{sheaf}}(C, \mathcal{O}^*) \), where \( \mathcal{O}^* \) is the sheaf of nonzero holomorphic functions on \( C \). When \( C = \mathbb{C}P^n \) things simplify because the above sheaf cohomology group is in fact isomorphic to a singular homology group,
\[ H^1_{\text{sheaf}}(\mathbb{C}P^n, \mathcal{O}^*) = H^2_{\text{sing}}(\mathbb{C}P^n, \mathbb{Z}), \]  
\[ (7) \]

and the latter is given in eqn. (2). Thus
\[ \text{Pic}(\mathbb{C}P^n) = \mathbb{Z}. \]  
\[ (8) \]
The zero class corresponds to the trivial line bundle; all other classes correspond to nontrivial line bundles. As the equivalence class of \( \mathcal{L} \) varies, so does the space \( \mathcal{H} \) of its holomorphic sections vary.

4 Quantum Hilbert–space bundles over \( \mathbb{C}P^n \)

In order to quantise \( \mathbb{C}P^n \) we will construct a family of vector bundles over \( \mathbb{C}P^n \), all of which will have a Hilbert space \( \mathcal{H} \) as fibre. We will analyse such bundles, that we will call quantum Hilbert–space bundles, or just \( \mathcal{QH} \)–bundles for short. Our aim is to demonstrate that there are different, nonequivalent choices for the \( \mathcal{QH} \)–bundles, to classify them, and to study how the corresponding quantum mechanics varies with each choice.

Compactness of \( \mathbb{C}P^n \) implies that, upon quantisation, the Hilbert space \( \mathcal{H} \) is finite–dimensional, and hence isomorphic to \( \mathbb{C}^{N+1} \) for some \( N \). This property follows from the fact that the number of quantum states grows monotonically with the symplectic volume of \( \mathcal{L} \); the latter is finite when \( \mathcal{L} \) is compact. We are thus led to considering principal \( U(N+1) \)–bundles over \( \mathbb{C}P^n \) and to their classification. Equivalently, we will consider the associated holomorphic vector bundles with fibre \( \mathbb{C}^{N+1} \). The corresponding projective bundles are \( \mathbb{C}P^N \)–bundles and principal \( PU(N) \)–bundles. Each choice of a different equivalence class of bundles will give rise to a different quantisation.

So far we have left \( N \) undetermined. In order to fix it we first pick the symplectic volume form \( \omega_n \) on \( \mathbb{C}P^n \) such that

\[
\int_{\mathbb{C}P^n} \omega^n = n + 1.
\]

Next we set \( N = n \), so \( \dim \mathcal{H} = n + 1 \). This normalisation corresponds to 1 quantum state per unit of symplectic volume on \( \mathbb{C}P^n \). Thus, \( e.g. \), when \( n = 1 \) we have the Riemann sphere \( \mathbb{C}P^1 \) and \( \mathcal{H} = \mathbb{C}^2 \). The latter is the Hilbert space of a spin \( s = 1/2 \) system, and the counting of states is correct. There are a number of further advantages to this normalisation. In fact eqn. 9 is more than just a normalisation, in the sense that the dependence of the right–hand side on \( n \) is determined by physical consistency arguments. This will be explained in section 4.1. Normalisation arguments can enter eqn. 9 only through overall numerical factors such as \( 2\pi, i\hbar \), or similar. It is these latter factors that we fix by hand in eqn. 9.

The right–hand of our normalisation differs from that corresponding to eqn. 5. Up to numerical factors such as \( 2\pi, i\hbar \), etc, it is standard to set \( \int_{\mathbb{C}P^n} F^n = n \). There is also an alternative normalisation developed in ref. 15. However we will find our normalisation more convenient.

4.1 Computation of \( \dim H^0(\mathbb{C}P^n, \mathcal{O}(1)) \)

Next we present a quantum–mechanical computation of \( \dim H^0(\mathbb{C}P^n, \mathcal{O}(1)) \) without resorting to sheaf cohomology. That is, we compute \( \dim \mathcal{H} \) when \( l = 1 \) and prove that it coincides with the right–hand side of eqn. 9. The case \( l > 1 \) will be treated in section 4.3.
Starting with $\mathcal{C} = \mathbb{CP}^0$, i.e., a point $p$ as classical phase space, the space of quantum rays must also reduce to a point. Then the corresponding Hilbert space is $\mathcal{H}_1 = \mathcal{C}$. The only state in $\mathcal{H}_1$ is the vacuum $|0\rangle_{t=1}$, henceforth denoted $|0\rangle$ for brevity.

Next we pass from $\mathcal{C} = \mathbb{CP}^0$ to $\mathcal{C} = \mathbb{CP}^1$. Regard $p$, henceforth denoted $p_1$, as the point at infinity with respect to a coordinate chart $(\mathcal{U}_1, z_{(1)})$ on $\mathbb{CP}^1$ that does not contain $p_1$. This chart is biholomorphic to $\mathbb{C}$ and supports a representation of the Heisenberg algebra in terms of creation and annihilation operators $A^\dagger(1)$, $A(1)$. This process adds the new state $A^\dagger(1)|0(1)\rangle$ to the spectrum. The new Hilbert space $\mathcal{H}_2 = \mathbb{C}^2$ is the linear span of $|0(1)\rangle$ and $A^\dagger(1)|0(1)\rangle$.

On $\mathbb{CP}^1$ we have the charts $(\mathcal{U}_1, z_{(1)})$ and $(\mathcal{U}_2, z_{(2)})$. Point $p_1$ is at infinity with respect to $(\mathcal{U}_1, z_{(1)})$, while it belongs to $(\mathcal{U}_2, z_{(2)})$. Similarly, the point at infinity with respect to $(\mathcal{U}_2, z_{(2)})$, call it $p_2$, belongs to $(\mathcal{U}_1, z_{(1)})$ but not to $(\mathcal{U}_2, z_{(2)})$. Above we have proved that the Hilbert–space bundle $Q\mathcal{H}_2$ has a fibre $\mathcal{H}_2 = \mathbb{C}^2$ which, on the chart $\mathcal{U}_1$, is the linear span of $|0(1)\rangle$ and $A^\dagger(1)|0(1)\rangle$. On the chart $\mathcal{U}_2$, the fibre is the linear span of $|0(2)\rangle$ and $A^\dagger(2)|0(2)\rangle$, $A(2)$ being the creation operator on $\mathcal{U}_2$. On the common overlap $\mathcal{U}_1 \cap \mathcal{U}_2$, the coordinate transformation between $z_{(1)}$ and $z_{(2)}$ is holomorphic. This implies that, on $\mathcal{U}_1 \cap \mathcal{U}_2$, the fibre $\mathbb{C}^2$ can be taken in either of two equivalent ways: either as the linear span of $|0(1)\rangle$ and $A^\dagger(1)|0(1)\rangle$, or as that of $|0(2)\rangle$ and $A^\dagger(2)|0(2)\rangle$.

The general construction is now clear. Topologically we have $\mathbb{CP}^n = \mathbb{C}^n \cup \mathbb{CP}^{n-1}$, with $\mathbb{CP}^{n-1}$ a hyperplane at infinity, but we also need to describe the coordinate charts and their overlaps. There are coordinate charts $(\mathcal{U}_j, z_{(j)})$, $j = 1, \ldots, n+1$ and nonempty $f$–fold overlaps $\cap_{j=1}^f \mathcal{U}_j$ for $f = 2, 3, \ldots, n+1$. Each chart $(\mathcal{U}_j, z_{(j)})$ is biholomorphic with $\mathbb{C}^n$ and has a $\mathbb{CP}^{n-1}$–hyperplane at infinity; the latter is charted by the remaining charts $(\mathcal{U}_k, z_{(k)})$, $k \neq j$. Over $(\mathcal{U}_j, z_{(j)})$ the Hilbert–space bundle $Q\mathcal{H}_{n+1}$ has a fibre $\mathcal{H}_{n+1} = \mathbb{C}^{n+1}$ spanned by

$$|0(j)\rangle, \quad A^\dagger(j)|0(j)\rangle, \quad i = 1, 2, \ldots, n. \quad (10)$$

Analyticity arguments similar to those above prove that, on every nonempty $f$–fold overlap $\cap_{j=1}^f \mathcal{U}_j$, the fibre $\mathbb{C}^{n+1}$ can be taken in $f$ different, but equivalent ways, as the linear span of $|0(j)\rangle$ and $A^\dagger(j)|0(j)\rangle$, $i = 1, 2, \ldots, n$, for every choice of $j = 1, \ldots, f$.

A complete description of this bundle requires the specification of the transition functions. We take the excited states $A^\dagger(j)|0(j)\rangle$ to transform according to the jacobian matrices $t(T\mathbb{CP}^n)$ corresponding to coordinate changes on $\mathbb{CP}^n$, while the vacuum $|0\rangle$ will transform with the transition functions $t(\tau)$ of the line bundle $\tau$. Thus the complete transition functions are the direct sum

$$t(Q\mathcal{H}(\mathbb{CP}^n)) = t(T\mathbb{CP}^n) \oplus t(\tau), \quad (11)$$

and the $Q\mathcal{H}$–bundle itself decomposes as the direct sum of a holomorphic line bundle $N(\mathbb{CP}^n) = \tau$, plus the holomorphic tangent bundle $T(\mathbb{CP}^n)$,

$$Q\mathcal{H}(\mathbb{CP}^n) = T(\mathbb{CP}^n) \oplus N(\mathbb{CP}^n). \quad (12)$$

It follows that tangent vectors to $\mathbb{CP}^n$ are quantum states in (the defining representation of) Hilbert space. In eqn. (10) we have given a basis for these states in terms of
creation operators acting on the vacuum $|0\rangle$. The latter can be regarded as the basis vector for the fibre $\mathbb{C}$ of the line bundle $N(\mathbb{C}P^n)$.

4.2 Representations

The $(n+1)$–dimensional Hilbert space of eqn. (11) may be regarded as a kind of defining representation, in the sense of the representation theory of $SU(n+1)$. The latter is the structure group of the bundle (13). Comparing our results with those of section 3 we conclude that $L = \tau$, because $l = 1$. This is the smallest value of $l$ that produces a nontrivial $H$, as eqn. (6) gives a 1–dimensional Hilbert space when $l = 0$. So our $H$ spans an $(n+1)$–dimensional representation of $SU(n+1)$, that we can identify with the defining representation. There is some ambiguity here since the dual of the defining representation of $SU(n+1)$ is also $(n+1)$–dimensional. This ambiguity is resolved by convening that the latter is generated by the holomorphic sections of the dual quantum line bundle $L^* = \tau^{-1}$. On the chart $U_j, j = 1, \ldots, n+1$, the dual of the defining representation is the linear span of the covectors

$$\langle j|0\rangle, \quad \langle j|0\rangle \alpha_i(j), \quad i = 1, 2, \ldots, n. \quad (13)$$

Taking higher representations is equivalent to considering the principal $SU(n+1)$-bundle (associated with the vector $C^{n+1}$-bundle) in a representation higher than the defining one. We will see next that this corresponds to having $l > 1$ in our choice of the line bundle $\tau^l$.

4.3 Computation of $\dim H^0(\mathbb{C}P^n, \mathcal{O}(l))$

We extend now our quantum–mechanical computation of $\dim H^0(\mathbb{C}P^n, \mathcal{O}(l))$ to the case $l > 1$. As in section 4.1 we do not resort to sheaf cohomology. The values $l = 0, 1$ respectively correspond to the trivial and the defining representation of $SU(n+1)$. The restriction to nonnegative $l$ follows from our convention of assigning the defining representation to $\tau$ and its dual to $\tau^{-1}$. Higher values $l > 1$ correspond to higher representations and can be accounted for as follows. We have

$$\mathbb{C}P^{n+l} = SU(n+l+1)/(SU(n+l) \times U(1)), \quad (14)$$

where now $SU(n+l+1)$ and $SU(n+l)$ act on $C^{n+l+1}$. Now $SU(n+l)$ admits \binom{n+l}{n}-dimensional representations (Young tableaux with a single column of $n$ boxes) that, by restriction, are also representations of $SU(n+1)$. Letting $l > 1$ vary for fixed $n$, this reproduces the dimension of eqn. (6).

By itself, the existence of $SU(n+1)$ representations with the dimension of eqn. (6) does not prove that, picking $l > 1$, the corresponding quantum states lie in those \binom{n+l}{n}-dimensional representations. We have to prove that no other value of the dimension fits the given data. In order to prove it the idea is, roughly speaking, that a value of $l > 1$ on $\mathbb{C}P^n$ can be traded for $l' = 1$ on $\mathbb{C}P^{n+l}$. That is, an $SU(n+1)$ representation higher than the defining one can be traded for the defining representation of $SU(n+l+1)$. In this way the $\mathcal{O}H$–bundle on $\mathbb{C}P^n$ with the Picard class
$l' = l$ equals the $\mathcal{Q}H$–bundle on $\mathbb{CP}^{n+l}$ with the Picard class $l' = 1$. On the latter we have $n + l$ excited states (i.e., other than the vacuum), one for each complex dimension of $\mathbb{CP}^{n+l}$. We can sort them into unordered sets of $n$, which is the number of excited states on $\mathbb{CP}^n$, in $\left(\begin{array}{c} n+l \\ n \end{array}\right)$ different ways. This selects a specific dimension for the $SU(n + 1)$ representations and rules out the rest. More precisely, it is only when $n > 1$ that some representations are ruled out. When $n = 1$, i.e. for $SU(2)$, all representations are allowed, since their dimension is $l + 1 = \left(1 + \frac{l}{1}\right)$. However already for $SU(3)$ some representations are thrown out. The number $\left(\begin{array}{c} 2+l \\ 2 \end{array}\right)$ matches the dimension $d(p, q) = (p + 1)(q + 1)(p + q + 2)/2$ of the $(p, q)$ irreducible representation if $p = 0$ and $l = q$ or $q = 0$ and $l = p$, but arbitrary values of $(p, q)$ are in general not allowed.

To complete our reasoning we have to prove that the quantum line bundle $\mathcal{L} = \tau$ on $\mathbb{CP}^{n+l}$ descends to $\mathbb{CP}^n$ as the $l$–th power $\tau^l$. For this we resort to the natural embedding of $\mathbb{CP}^n$ into $\mathbb{CP}^{n+l}$. Let $(U_1, z(1)), \ldots, (U_{n+1}, z(n+1))$ be the coordinate charts on $\mathbb{CP}^n$ described in section 2 and let $(\tilde{U}_1, \tilde{z}(1)), \ldots, (\tilde{U}_{n+1}, \tilde{z}(n+1)), (\tilde{U}_{n+2}, \tilde{z}(n+2)), \ldots, (\tilde{U}_{n+l+1}, \tilde{z}(n+l+1))$ be charts on $\mathbb{CP}^{n+l}$ relative to this embedding. This means that the first $n + 1$ charts on $\mathbb{CP}^{n+l}$, duly restricted, are also charts on $\mathbb{CP}^n$; in fact every chart on $\mathbb{CP}^n$ is contained $l$ times within $\mathbb{CP}^{n+l}$. Let $t_{jk}(\tau)$, with $j, k = 1, \ldots, n + l + 1$, be the transition function for $\tau$ on the overlap $\tilde{U}_j \cap \tilde{U}_k$ of $\mathbb{CP}^{n+l}$. In passing from $\tilde{U}_j$ to $\tilde{U}_k$, points on the fibre are acted on by $t_{jk}(\tau)$. Due to our choice of embedding, the overlap $\tilde{U}_j \cap \tilde{U}_k$ on $\mathbb{CP}^{n+l}$ contains $l$ copies of the overlap $U_j \cap U_k$ on $\mathbb{CP}^n$. Thus points on the fibre over $\mathbb{CP}^n$ are acted on by $(t_{jk}(\tau))^l$, where now $j, k$ are restricted to $1, \ldots, n + 1$. This means that the line bundle on $\mathbb{CP}^n$ is $\tau^l$ as stated, and the vacuum $|0\rangle_{\nu=1}$ on $\mathbb{CP}^n$ equals the vacuum $|0\rangle_{\nu=1}$ on $\mathbb{CP}^{n+l}$. Hence there are on $\mathbb{CP}^n$ as many inequivalent vacua as there are elements in $\mathbb{Z} = \text{Pic}(\mathbb{CP}^n)$ (remember that sign reversal $l \rightarrow -l$ within $\text{Pic}(\mathbb{CP}^n)$ is the operation of taking the dual representation, i.e., $\tau \rightarrow \tau^{-1}$).

### 4.4 Classification of $\mathcal{Q}H$–bundles

As a holomorphic line bundle, $N(\mathbb{CP}^n)$ is isomorphic to $\tau^l$ for some $l \in \text{Pic}(\mathbb{CP}^n) = \mathbb{Z}$. Now the bundle $T(\mathbb{CP}^n) \oplus N(\mathbb{CP}^n)$ has $SU(n + 1)$ as its structure group, which we consider in the representation $\rho_l$ corresponding to the Picard class $l \in \mathbb{Z}$:

$$\mathcal{Q}H_l(\mathbb{CP}^n) = \rho_l(T(\mathbb{CP}^n)) \oplus \tau^l, \quad l \in \mathbb{Z}. \tag{15}$$

The above generalises eqn. (12) to the case $l > 1$. The importance of eqn. (15) is that it classifies $\mathcal{Q}H$–bundles over $\mathbb{CP}^n$: holomorphic equivalence classes of such bundles are in 1–to–1 correspondence with the elements of $\mathbb{Z} = \text{Pic}(\mathbb{CP}^n)$. The class $l = 1$ corresponds to the defining representation of $SU(n + 1)$,

$$\mathcal{Q}H_{l=1}(\mathbb{CP}^n) = T(\mathbb{CP}^n) \oplus \tau, \tag{16}$$

and $l = -1$ to its dual. The quantum Hilbert–space bundle over $\mathbb{CP}^n$ is generally nontrivial, although particular values of $l$ may render the direct sum trivial. The
separate summands \( T(\mathbb{CP}^n) \) and \( N(\mathbb{CP}^n) \) are both nontrivial bundles. Nontriviality of \( N(\mathbb{CP}^n) \) means that, when \( l \neq 0 \), the state \( |0\rangle \) transforms nontrivially (albeit as multiplication by a phase factor) between different local trivialisations of the bundle. When \( l = 0 \) the vacuum transforms trivially.

According to eqn. (15), the transition functions \( t(QH_l) \) for \( QH_l \) decompose as a direct sum of two transition functions, one for \( \rho_l(T(\mathbb{CP}^n)) \), another one for \( \tau_l \):

\[
t(QH_l(\mathbb{CP}^n)) = t(\rho_l(T(\mathbb{CP}^n))) \oplus t(\tau_l).
\]  

(17)

If the transition functions for \( \tau \) are \( t(\tau) \), those for \( \tau_l \) are \( (t(\tau))^l \). On the other hand, the transition functions \( t(\rho_l(T(\mathbb{CP}^n))) \) are the jacobian matrices (in representation \( \rho_l \)) corresponding to coordinate changes on \( \mathbb{CP}^n \). Then all the \( QH_l(\mathbb{CP}^n) \)–bundles of eqn. (15) are nonflat because the tangent bundle \( T(\mathbb{CP}^n) \) itself is nonflat. Eqn. (17) generalises eqn. (11) to the case \( l > 1 \).

### 4.5 Diagonalisation of the projective Hamiltonian

Deleting from \( \mathbb{CP}^n \) the \( \mathbb{CP}^{n-1} \)–hyperplane at infinity produces the noncompact space \( \mathbb{C}^n \). The latter is the classical phase space of the \( n \)–dimensional harmonic oscillator (now no longer projective, but linear). The corresponding Hilbert space \( \mathcal{H} \) is infinite–dimensional because the symplectic volume of \( \mathbb{C}^n \) is infinite.

The deletion of the hyperplane at infinity may also be understood from the viewpoint of the Kähler potential (1) corresponding to the Fubini–Study metric. No longer being able to pass holomorphically from a point at finite distance to a point at infinity implies that, on the conjugate chart \( (U_k, z^{(k)}) \), the squared modulus \( |z^{(k)}|^2 \) is always small and we can Taylor–expand eqn. (1) as

\[
\log \left(1 + \sum_{j=1}^{n} z_j^{(k)} \bar{z}_j^{(k)}\right) \simeq \sum_{j=1}^{n} z_j^{(k)} \bar{z}_j^{(k)}.
\]  

(18)

The right–hand side of eqn. (18) is the Kähler potential for the usual Hermitean metric on \( \mathbb{C}^n \). As such, \( \sum_{j=1}^{n} z_j^{(k)} \bar{z}_j^{(k)} \) equals the classical Hamiltonian for the \( n \)–dimensional linear harmonic oscillator. Observers on this coordinate chart effectively see \( \mathbb{C}^n \) as their classical phase space. The corresponding Hilbert space is the (closure of the) linear span of the states \( |m_1, \ldots, m_n\rangle \), where

\[
H_{\text{lin}}|m_1, \ldots, m_n\rangle = \sum_{j=1}^{n} \left(m_j + \frac{1}{2}\right) |m_1, \ldots, m_n\rangle, \quad m_j = 0, 1, 2, \ldots, \quad (19)
\]

and

\[
H_{\text{lin}} = \sum_{j=1}^{n} \left(A_j^{\dagger}(k)A_j(k) + \frac{1}{2}\right)
\]  

(20)

is the quantum Hamiltonian operator corresponding to the classical Hamiltonian function on the right–hand side of eqn. (18). Then the stationary Schrödinger equation for
the projective oscillator reads

\[ H_{\text{proj}}|m_1, \ldots, m_n\rangle = \log \left( 1 + \sum_{j=1}^{n} \left( m_j + \frac{1}{2} \right) \right) |m_1, \ldots, m_n\rangle, \tag{21} \]

where

\[ H_{\text{proj}} = \log \left( 1 + \sum_{j=1}^{n} \left( A_j^\dagger(k) A_j(k) + \frac{1}{2} \right) \right) \tag{22} \]

is the quantum Hamiltonian operator corresponding to the classical Hamiltonian function on the left-hand side of eqn. (18).

The same states \(|m_1, \ldots, m_n\rangle\) that diagonalise \(H_{\text{lin}}\) also diagonalise \(H_{\text{proj}}\). However, eqns. (19)–(22) above in fact only hold locally on the chart \(U_k\), which does not cover all of \(\mathbb{CP}^n\). Bearing in mind that there is one hyperplane at infinity with respect to this chart, we conclude that the arguments of section 4.1 apply in order to ensure that the projective oscillator only has \(n\) excited states. Then the occupation numbers \(m_j\) are either all 0 (for the vacuum state) or all zero but for one of them, where \(m_j = 1\) (for the excited states), and \(\dim \mathcal{H} = n + 1\) as it should. Moreover, the eigenvalues of eqn. (21) provide an alternative proof of the fact, demonstrated in section 4.3, that the Picard group class \(l' = l > 1\) on \(\mathbb{CP}^n\) can be traded for \(l' = 1\) on \(\mathbb{CP}^{n+1}\).

5 \(\mathbb{CP}(\mathcal{H})\) as a classical phase space

Realise \(\mathcal{H}\) as the space of infinite sequences of complex numbers \(Z^1, Z^2, \ldots\) that are square-summable, \(\sum_{j=1}^{\infty} |Z^j|^2 < \infty\). The \(Z^j\) provide a set of holomorphic coordinates on \(\mathcal{H}\). The space of rays \(\mathbb{CP}(\mathcal{H})\) is

\[ \mathbb{CP}(\mathcal{H}) = (\mathcal{H} \setminus \{0\})/ (\mathbb{R}^+ \times U(1)). \tag{23} \]

The \(Z^j\) provide a set of projective coordinates on \(\mathbb{CP}(\mathcal{H})\). Now assume that \(Z^k \neq 0\), and define \(z^j(k) = Z^j/Z^k\) for \(j \neq k\). Then \(\sum_{j \neq k} |z^j(k)|^2 < \infty\) for every fixed value of \(k\). As \(j \neq k\) varies, these \(z^j(k)\) cover one copy of \(\mathcal{H}\) that we denote by \(U_k\). The open set \(U_k\), endowed with the coordinate functions \(z^j(k), j = 1, 2, \ldots k, \ldots\), where a check over an index indicates omission, provides a holomorphic coordinate chart on \(\mathbb{CP}(\mathcal{H})\) for every fixed \(k\). A holomorphic atlas is obtained as the collection of all pairs \((U_k, z(k))\), for \(k = 1, 2, \ldots\). There are nonempty \(f\)-fold overlaps \(\cap_{m=1}^{f} U_m\) for all values of \(f = 1, 2, \ldots\) When \(f = 2\), tangent vectors transform according to an (infinite-dimensional) jacobian matrix.

\(\mathbb{CP}(\mathcal{H})\) is a Kähler manifold. On the coordinate chart \((U_k, z(k))\), the Kähler potential reads

\[ K(z(k), \bar{z}(k)) = \log \left( 1 + \sum_{j \neq k}^{\infty} z^j(k) \bar{z}^j(k) \right), \tag{24} \]
and the corresponding metric $ds_K^2$ reads on this chart

$$ds_K^2 = \sum_{m,n\neq k}^{\infty} \frac{\partial^2 K(z(k), \bar{z}(k))}{\partial z_m(k) \partial \bar{z}_n(k)} dz_m(k) d\bar{z}_n(k).$$

(25)

Being infinite–dimensional, $\mathbb{C}P(H)$ is noncompact. It is simply connected:

$$\pi_1(\mathbb{C}P(H)) = 0.$$  

(26)

Its Picard group is the group of integers:

$$\text{Pic}(\mathbb{C}P(H)) = \mathbb{Z}.$$  

(27)

It has trivial homology in odd real dimension,

$$H_{2k+1}(\mathbb{C}P(H), \mathbb{Z}) = 0, \quad k = 0, 1, \ldots, $$

(28)

while it is nontrivial in even dimension,

$$H_{2k}(\mathbb{C}P(H), \mathbb{Z}) = \mathbb{Z}, \quad k = 0, 1, \ldots $$

(29)

6 Quantum Hilbert–space bundles over $\mathbb{C}P(H)$

By eqn. (27), for each integer $l \in \mathbb{Z}$ there exists one equivalence class $N_l(\mathbb{C}P(H))$ of holomorphic line bundles over $\mathbb{C}P(H)$. For $l \neq 0$ this bundle is nontrivial; its fibre $C$ is generated by the vacuum state $\langle 0 \rangle_l$. Let $A^j_j(k), A_j(k), j \neq k$, be creation and annihilation operators on the chart $U_k$, for $k$ fixed. We can now construct the $Q\mathcal{H}_l$–bundle over $\mathbb{C}P(H)$. To this end we will describe the fibre over each coordinate chart $U_k$, plus the transition functions on the 2–fold overlaps $U_k \cap U_m$, for all $k \neq m$.

The Hilbert–space fibre over $U_k$ is $\mathcal{H}$ itself, the latter being the $C$–linear span of the infinite set of linearly independent vectors

$$\langle 0(k) \rangle_t, \quad A^j_j(k)\langle 0(k) \rangle_t, \quad j = 1, 2, \ldots, \hat{k}, \ldots$$

(30)

Reasoning as in section 4 one proves that, on the 2–fold overlaps $U_k \cap U_m$, the fibre $\mathcal{H}$ can be chosen in either of two equivalent ways. $\mathcal{H}$ is either the $C$–linear span of the vectors $\langle 0(k) \rangle_t, A^j_j(k)\langle 0(k) \rangle_t$, for $j = 1, 2, \ldots, \hat{k}, \ldots$, or the $C$–linear span of the vectors $\langle 0(m) \rangle_t, A^j_j(m)\langle 0(m) \rangle_t$, for $j = 1, 2, \ldots, \hat{m}, \ldots$

As in section 4 we have that the vacuum $\langle 0(k) \rangle_t$ is the fibrewise generator of a holomorphic line bundle $N_l(\mathbb{C}P(H))$. Its excitations $A^j_j(k)\langle 0(k) \rangle_t$ are tangent vectors to $\mathbb{C}P(H)$ on the chart $U_k$, and thus transition functions are the sum of two parts. One is a phase factor accounting for the transformation of $\langle 0(k) \rangle_t$; the other one is a jacobian matrix. The complete $Q\mathcal{H}_l$–bundle splits as

$$Q\mathcal{H}_l(\mathbb{C}P(H)) = T(\mathbb{C}P(H)) \oplus N_l(\mathbb{C}P(H)).$$

(31)
7 Quantum Hilbert–space bundles over \( C \)

Next we present a summary, drawn from ref. \[16\], on how to holomorphically embed a noncompact \( C \) within \( \mathbb{CP}(\mathcal{H}) \). This procedure is applied in section 7.3 in order to quantise \( C \).

7.1 The Bergman metric on \( C \)

Denote by \( F \) the set of holomorphic, square–integrable \( n \)-forms on \( C \). \( F \) is a separable, complex Hilbert space (finite–dimensional when \( C \) is compact). Let \( h_1, h_2, \ldots \) denote a complete orthonormal basis for \( F \), and let \( z \) be (local) holomorphic coordinates on \( C \). Then

\[
K(z, \bar{w}) = \sum_{j=1}^{\infty} h_j(z) \wedge \bar{h}_j(\bar{w})
\]

is a holomorphic \( 2n \)-form on \( C \times \bar{C} \), where \( \bar{C} \) is complex manifold conjugate to \( C \). The form \( K(z, \bar{z}) \) is independent of the choice of an orthonormal basis for \( F \); it is called the kernel form of \( C \). If \( \bar{z} \) is the point of \( \bar{C} \) corresponding to a point \( z \in C \), the set of pairs \( (z, \bar{z}) \in C \times \bar{C} \) is naturally identified with \( M \). In this way \( K(z, \bar{z}) \) can be considered as a \( 2n \)-form on \( C \). One can prove that \( K(z, \bar{z}) \) is invariant under the group of holomorphic transformations of \( C \).

Next assume that, given any point \( z \in C \), there exists an \( f \in F \) such that \( f(z) \neq 0 \). That is, the kernel form \( K(z, \bar{z}) \) of \( C \) is everywhere nonzero on \( C \):

\[
K(z, \bar{z}) \neq 0, \quad \forall z \in C.
\]

Let us write, in local holomorphic coordinates \( z^j \) on \( C \), \( j = 1, \ldots, n \),

\[
K(z, \bar{z}) = k(z, \bar{z}) \, dz^1 \wedge \ldots \wedge dz^n \wedge d\bar{z}^1 \wedge \ldots \wedge d\bar{z}^n,
\]

for a certain everywhere nonzero function \( k(z, \bar{z}) \). Define a hermitean form \( ds_B^2 \)

\[
ds_B^2 = \sum_{j,k=1}^{n} \frac{\partial^2 \log k}{\partial z^j \partial \bar{z}^k} \, dz^j \, d\bar{z}^k.
\]

One can prove that \( ds_B^2 \) is independent of the choice of coordinates on \( C \). Moreover, it is positive semidefinite and invariant under the holomorphic transformations of \( C \).

Let us make the additional assumption that \( C \) is such that \( ds_B^2 \) is positive definite,

\[
ds_B^2 > 0.
\]

Then \( ds_B^2 \) defines a (Kähler) metric called the Bergman metric on \( C \) \[17\].

7.2 Embedding \( C \) within \( \mathbb{CP}(\mathcal{H}) \)

Let \( \mathcal{H} \) be the Hilbert space dual to \( \mathcal{F} \). Given \( f \in \mathcal{F} \), let its expansion in local coordinates be

\[
f = f \, dz^1 \wedge \ldots \wedge dz^n,
\]
for a certain function \( f \). Let \( \iota' \) denote the mapping that sends \( z \in \mathcal{C} \) into \( \iota'(z) \in \mathcal{H} \) defined by

\[
\langle \iota'(z)|f \rangle = f(z).
\]

(38)

Then \( \iota'(z) \neq 0 \) for all \( z \in \mathcal{C} \) if and only if property (33) holds. Assuming that the latter is satisfied, and denoting by \( p' \) the natural projection from \( \mathcal{H} - \{0\} \) onto \( \mathbb{C}P(\mathcal{H}) \), the composite map \( \iota = p' \circ \iota' \)

\[
\iota: \mathcal{C} \rightarrow \mathbb{C}P(\mathcal{H})
\]

(39)

is well defined on \( \mathcal{C} \), independent of the coordinates, and holomorphic.

One can prove the following results. When property (33) is true, the quadratic differential form \( ds_B^2 \) of eqn. (35) is the pullback, by \( \iota \), of the canonical Kähler metric \( ds_K^2 \) of eqn. (25):

\[
ds_B^2 = \iota^*(ds_K^2).
\]

(40)

Moreover, the differential of \( \iota \) is nonsingular at every point of \( \mathcal{C} \) if and only if property (36) is satisfied. These two results give us a geometric interpretation of the Bergman metric. Namely, if properties (33) and (36) hold, then \( \iota \) is an isometric immersion of \( \mathcal{C} \) into \( \mathbb{C}P(\mathcal{H}) \).

The map \( \iota \) is locally one–to–one in the sense that every point of \( \mathcal{C} \) has a neighbourhood that is mapped injectively into \( \mathbb{C}P(\mathcal{H}) \). However, \( \iota \) is not necessarily injective in the large. Conditions can be found that ensure injectivity of \( \iota \) in the large. Assume that, if \( z, z' \) are any two distinct points of \( \mathcal{C} \), an \( f \in \mathcal{F} \) can be found such that

\[
f(z) \neq 0, \quad f(z') = 0.
\]

(41)

Then \( \iota \) is injective. Therefore, if \( \mathcal{C} \) satisfies assumptions (33), (36) and (41), it can be holomorphically and isometrically embedded into \( \mathbb{C}P(\mathcal{H}) \).

### 7.3 Quantisation of \( \mathcal{C} \) as a submanifold of \( \mathbb{C}P(\mathcal{H}) \)

Finally we quantise a noncompact \( \mathcal{C} \) with infinite symplectic volume,

\[
\int_{\mathcal{C}} \omega^n = \infty,
\]

(42)

so \( \mathcal{H} \) will be infinite–dimensional. On the other hand, \( \mathcal{C} \) admits only \( n \) linearly independent, holomorphic tangent vectors, so the technique of section 4 must be modified.

We need an infinite–dimensional \( \mathcal{QH} \)–bundle over \( \mathcal{C} \). For this purpose we assume embedding \( \mathcal{C} \) holomorphically and injectively within \( \mathbb{C}P(\mathcal{H}) \) as in eqn. (39). Then the bundle \( \mathcal{QH}_l(\mathbb{C}P(\mathcal{H})) \) of eqn. (31) can be pulled back to \( \mathcal{C} \) by the embedding \( \iota \). We take this to define the bundle \( \mathbb{QH}_l(\mathcal{C}) \):

\[
\mathbb{QH}_l(\mathcal{C}) = \iota^* \mathcal{QH}_l(\mathbb{C}P(\mathcal{H})).
\]

(43)

Even if \( \mathcal{QH}_l(\mathbb{C}P(\mathcal{H})) \) were trivial (which it is not for \( l \neq 0 \)), it might contain nonflat (hence nontrivial) subbundles, thus allowing for nontrivial dualities.
A detailed analysis of $QH_i(\mathcal{C})$ requires specifying $\mathcal{C}$ explicitly. However some properties can be stated in general. Thus, e.g., the kernel form is the quantum–mechanical propagator. On $\mathbb{C}^n$ it reads

$$K_{\mathbb{C}^n}(z, \bar{z}) = N \exp \left( i \sum_{j=1}^{n} \bar{z}^j z^j \right) dz^1 \wedge \ldots \wedge dz^n \wedge d\bar{z}^1 \wedge \ldots \wedge d\bar{z}^n,$$

(44)

where $N$ is some normalisation. The Bergman metric (35) derived from this kernel is the standard Hermitean metric on $\mathbb{C}^n$. The embedding $\iota$ naturally relates physical information (the propagator) and geometric information (the metric on $\mathcal{C}$). In retrospective, this justifies our quantisation of $\mathcal{C}$ by embedding it within $\mathbb{C}P(\mathcal{H})$.

8 Summary

Our analysis has dealt primarily with the case when $\mathcal{C} = \mathbb{C}P^n$. In section 3 we have recalled some well–known facts from geometric quantisation. They concern the dimension of the space of holomorphic sections of the quantum line bundle on a compact, quantisable Kähler manifold. This dimension has been rederived in section 4 using purely quantum–mechanical arguments, by constructing the Hilbert–space bundle of quantum states over $\mathbb{C}P^n$. For brevity, the following summary deals only with the case when the Hilbert space is $\mathbb{C}^{n+1}$ (see sections 4.2, 4.3 for the general case). The fibre $\mathbb{C}^{n+1}$ over a given coordinate chart on $\mathbb{C}P^n$ is spanned by the vacuum state $|0(j)\rangle_l$, plus $n$ states $A^j_l|0(j)\rangle_l$, $j = 1, \ldots, n$, obtained by the action of creation operators. We have identified the transition functions of this bundle as jacobian matrices plus a phase factor. The jacobian matrices account for the transformation (under coordinate changes on $\mathbb{C}P^n$) of the states $A^j_l|0(j)\rangle_l$, while the phase factor corresponds to $|0(j)\rangle_l$. This means that all quantum states (except the vacuum) are tangent vectors to $\mathbb{C}P^n$. In this way the Hilbert–space bundle over $\mathbb{C}P^n$ splits as the direct sum of two holomorphic vector bundles: the tangent bundle $T(\mathbb{C}P^n)$, plus a line bundle $N(\mathbb{C}P^n)$ whose fibrewise generator is the vacuum.

All complex manifolds admit a Hermitian metric, so having tangent vectors as quantum states suggests using the Hermitian connection and the corresponding curvature tensor to measure flatness. Now $T(\mathbb{C}P^n)$ is nonflat, so it fits our purposes. The freedom in having different nonflat Hilbert–space bundles over $\mathbb{C}P^n$ resides in the different possible choices for the complex line bundle $N(\mathbb{C}P^n)$. Such choices are 1–to–1 with the elements of the Picard group $\text{Pic}(\mathbb{C}P^n) = \mathbb{Z}$. The latter appears as the parameter space for physically inequivalent choices of the vacuum state. Every choice of a vacuum leads to a different set of excitations and thus to a different quantum mechanics. Moreover, the $QH$–bundles constructed here are nonflat. This implies that, even after fixing a vacuum, there is still room for duality transformations between different observers on classical phase space. These two facts provide an explicit implementation of quantum–mechanical dualities.

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