A nine-point sampling point scheme for twenty node brick element

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Abstract: A new sampling point scheme with nine evaluation points were introduced in this research study for twenty noded brick elements. The new sampling points were located inside the brick element at the corners and center point of the 20 node brick element. This integration scheme can be assumed to be an imitation of Gaussian integration scheme. Standard benchmark problems were chosen from the different research works and compared with our proposed scheme. Finally, the proposed integration scheme achieves good results for twenty node brick element on different performance parameters of finite element analysis.

Keywords: Quadrature, Brick Element, Stiffness Matrix

1. Introduction

Finite element analysis is a tool originated for computing complex engineering problems. The finite element analysis is considered as the best approximation method in the field of engineering for resolving problems in engineering models. The Engineering problems need to handle large amount of differing computational data which in turn lead take more computational time in solving the problems thus the number of computational points plays a major role in the computation of stiffness matrix [1]. Stability issues of one-point sampling schemes were studied various researchers and the results of such schemes lead to singularity matrix. Singularity issue of one-point quadrature was has studied and proposed a method to remove singularity issue [2]. A quadrature scheme with 14 sampling points for 20 node brick elements was developed [3]. A group super matrix scheme for the derivation of stiffness matrix for two dimensional and three dimensional elements was developed [4]. He also proposed monomial shape function in the G-invariant sub spaces. [5] has attempted to modify the standard Gauss numerical integration scheme on 6 node and 16 node elements. He has tested the scheme successfully on beams using different bending conditions. [6] has introduced a midpoint quadrature scheme for quadrilateral element with five point quadrature scheme for evaluating the stiffness matrix. Later [7] has proposed a scheme with five sampling points for the quadrilateral element. [8] has developed a lower order brick element using the principle of three field variational with high accuracy 14 node brick element.

This research paper focus to propose a new sampling point scheme with nine evaluation points located at the corners and center point of the standard problems defined by the various researchers were taken for the evaluating the results of the proposed scheme.

2. Sampling Point formulation

The evaluating points in a problem are the locations where the problem is evaluated and these points are assumed to be located within the element. Using the method of undetermined coefficients, the new nine-point sampling scheme were developed. The focus of this research paper is to develop a new sampling points scheme for evaluating the element stiffness matrix for the 20 node brick element.
element. The proposed scheme corner center point Method takes eight sampling points at the corners of the assumed brick element and one sampling point located at the center of the assumed brick element. The standard quadrature scheme such as Gauss two-point quadrature scheme is shown in Figure 1(a) and figure 1(b) shows the standard three-point gauss quadrature Scheme. In the proposed scheme itself there are two set of sampling points based on the location of sampling points in the standard 20 node brick element. The proposed Corner Center Point Method at location of 0.75 is shown in figure 1(c) and the second scheme Corner Center Point Method at the sampling points location at 0.5 is shown in figure 1(d). A polynomial was chosen for deriving the sampling points in such a way that it has the terms in p, q and r. The assumptions mentioned above shows that there will be two unknown weights and two points will be there which are named as a, Wa, b and Wb.

Figure 1 (a). Gauss Two-point scheme, (b). Gauss Three-points scheme, (c). Corner Center Point Method with point location at 0.75 (d). Corner Center Point Method with point location at 0.5

The Assumed polynomial for integrating is shown below

$$
\Phi(p, q, r) = d_1 + d_2 p + d_3 q + d_4 r + d_5 p^2 + d_6 q^2 + d_7 r^2 + ... 
$$

Upon integrating the equation (1) the following equation will be there

$$
\int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \Phi(p, q, r) dp dq dr = d_1 + d_2 p + d_3 q + d_4 r + d_5 p^2 + ... 
$$

(2)

On integrating Equation (2) we will get the following equation

$$
= d_1 \left(8\right) + d_5 \left(\frac{8}{3}\right) + d_6 \left(\frac{8}{3}\right) + d_7 \left(\frac{8}{3}\right) + ...
$$

(3)

Using the following numerical form function \(\Phi\) can be evaluated

$$
\Phi = \sum_{i=1}^{9} W_i \Phi(f, g, h)
$$

(4)

Again substituting (1) in (4) we get

$$
=W_1 \left(1 + d_1 p + d_2 q + d_3 r + ... \right) + W_2 \left(1 + d_1 p + d_2 q + d_3 r + ... \right) + ... + W_9 \left(1 + d_1 p + d_2 q + d_3 r + ... \right)
$$

(5)

As shown in figure 1(c) and 1(d) we can infer that on the basis locations of evaluation points there can be only two weights such as Wa and Wb. On substituting the values of coordinates and weights on each evaluation points simultaneously.
\[ 8W_a + W_h = 8 \]
\[ 8W_a a^2 = \frac{8}{3} \]  

(6)

On substituting the points and solving the (6) the set of points are found and shown in table I.

For three dimensional elements the generalized reduced integration scheme is given as follows

\[ I = \int \int \int f(p, q, r)dpdqdr = \sum_{i=1}^{q} f(p_i, q_i, r_i)W_i \]  

(7)

| Integration point location | Weighting function (at a =0.75) | Weighting function (at a =0.5) |
|----------------------------|---------------------------------|--------------------------------|
| Corner points              | 0.59259                         | 1.3333                         |
| center points              | 3.23768                         | -2.6664                        |

Table 1. New sampling points for 20 node brick element

3. Numerical results

Several research studies have given standard problems for verification of results which are considered to be the benchmark problems. Few standard benchmarked problems were chosen from the standard literature to analyze the performance of results by using Corner Center Point Method on comparison with Gauss Quadrature Scheme [9] [10]. Accuracy of results plays a major role in Finite element analysis because finite element analysis method is the method of approximation.

3.1. Problem 1: 5 beam element

This problem was suggested in order to investigate the bending performance of irregular meshes. This test consists of considering a beam with five irregular elements and end point loads were applied. The results were compared with standard methods and shown in Table 2. Figure 3 shows the cantilever beam with five elements

![Cantilever beam with five elements](image)

Table 2. Normalized numerical results of 5 beam element

| Quadrature Method | Displacement |
|-------------------|--------------|
| H8                | 44.39        |
| H11               | 98.26        |
| Hm11              | 95.72        |
| NH11              | 95.80        |
| HBHEX8R           | 83.47        |
| SBB               | 100          |
3.2. Problem 2: Cooke’s beam problem

Cooke’s skew beam problem was defined by [12][13]. This problem is considered as a benchmark problem to study the distortion of elements and convergence of results in 3D finite element analysis. The trapezoidal plate is chosen as shown in figure 3. The plate is evaluated by increasing the number of meshes. The problem is evaluated and compared using the various finite element results and Corner Center Point method. The results of normalized displacement are shown in tables 3.

![Cooke's Problem (E= 1500, \(\mu=0.3\), Thickness =1)](image)

**Table 3: Normalized displacement results of Cooke’s Skew problem**

| Element Method | 2x2x1     | 4x4x1     | 8x8x1     | 16x16x1   | 32x32x1   |
|----------------|-----------|-----------|-----------|-----------|-----------|
| HVCC8          | 100       | 100       | 100       | 100       | 100       |
| TH8            | 100       | 100       | 100       | 100       | 100       |
| US-ATFH8       | 100       | 100       | 100       | 100       | 100       |
| Gauss 2x2x2    | 104.2981  | 104.2981  | 104.2981  | 104.2981  | 104.2981  |
| Gauss 3x3x3    | 103.5892  | 103.5892  | 103.5892  | 103.5892  | 103.5892  |
| Gauss 4x4x4    | 129.7613  | 129.7613  | 129.7613  | 129.7613  | 129.7613  |
| Ansys          | 104.3011  | 104.3011  | 104.3011  | 104.3011  | 104.3011  |
| CCPM 0.75      | 103.4879  | 103.4879  | 103.4879  | 103.4879  | 103.4879  |
| CCPM 0.5       | 109.9364  | 109.9364  | 109.9364  | 109.9364  | 109.9364  |
5.3. Problem 3: CPU time comparison

Calculation time to resolve problems is an important factor in finite element method. On increasing the number of evaluation points it should not affect the computational time factor. The code for evaluation of stiffness matrix were written in MATLAB [14] [15]. Here the time required for the evaluation of element stiffness matrix for 20 node brick element is calculated using the Corner Center Point Method and Gaussian quadrature scheme. The CPU time comparison for evaluating is done using a computer with configuration specification of processor Intel(R), i3 @ 3.30 GHz with 4 GB RAM. The coded program was run through MATLAB for 100,000 elements to analyze the variation in CPU execution time for evaluating the element stiffness matrix using the sampling points schemes. The results of CPU execution time between the Corner Center Point method and Gauss quadrature scheme is shown in figure 4 and the time comparison is shown in table 4.

Table 4. Number of elements (Vs) CPU computation time

| No. of Elements | 2x2x2   | CCPM 0.75 | CCPM 0.5 | 3X3X3 | 4X4X4 |
|-----------------|---------|-----------|----------|-------|-------|
| 100             | 0.603949| 0.624368  | 0.678845 | 0.814086 | 1.21275 |
| 1000            | 0.999651| 1.069872  | 1.04723  | 1.843114 | 3.422405 |
| 10000           | 4.339085| 4.680622  | 4.661274 | 12.29757| 28.15522 |
| 25000           | 9.85606 | 10.43624  | 10.99847 | 29.41343 | 69.69278 |
| 50000           | 18.94927| 20.21863  | 20.73633 | 59.03794 | 140.3082 |
| 75000           | 28.32831| 30.14671  | 30.64282 | 89.24203 | 208.0734 |
| 100000          | 37.66819| 40.62356  | 40.30534 | 116.0759 | 278.6164 |

Figure 4. CPU execution time comparison for the computation of stiffness matrix

3.4. Problem 4: distortion test

This problem is to investigate the sensitivity of element using a parameter “e”. Here this parameter “e” varying as aspect ratio which will fall to distortion of meshes thus this test focus study while improving the parameter “e” whether the change is affecting the results or not. The figure 5 shows a beam with two brick elements and load is applied at the end point. The end point deflection results are shown in table 5.
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Figure 5. Distortion test (E=1500, $\mu=0.3$)

Table 5. Numerical results normalized end point result of distortion test

| Element / Method | e=0.5 | e=1   | e=2   | e=3   | e=4   |
|------------------|-------|-------|-------|-------|-------|
| CAO              | 103.5 | 108.5 | 114.5 | 121.5 |       |
| HMIX1            | 100   | 100   | 100   | 100   |       |
| TH8              | 92.65 | 85.93 | 91.78 | 101.8 | 110.1 |
| H8i9             | 82.89 | 65.80 | 60.86 | 63.97 | 65.05 |
| H11              | 82.89 | 65.81 | 60.90 | 64.07 | 65.23 |
| rHm11(-2)        | 100.0 | 99.78 | 96.78 | 90.49 | 81.95 |
| rNH11(-2)        | 100.1 | 100.2 | 99.82 | 99.66 | 100.7 |
| ANSYS            | 100.3181 | 101.0089 | 101.5863 | 98.11446 | 90.56831 |
| Gauss 2x2x2      | 100.3185 | 101.0085 | 101.5867 | 98.11463 | 90.56837 |
| Gauss 3x3x3      | 100.0192 | 100.2303 | 98.01779 | 85.673 | 59.14453 |
| CCPM 0.75        | 100.4445 | 100.9972 | 101.7847 | 99.85715 | 93.33193 |
| CCPM 0.5         | 98.51656 | 101.2729 | 99.1175 | 95.10914 | 82.75415 |
| Gauss 4x4x4      | 106.516 | 112.7454 | 113.0135 | 109.9407 | 98.08234 |

3.5. Example 5: patch test problem

Distortion of meshes was conducted using standard patch test benchmarked problem defined by the researchers [10]. The test is conducted by taking a linear elastic material with material property such as Poisson’s ratio ($\nu$) of 0.25 and Young’s modulus (E) is taken as 1000000. The standard problem is shown in figure 7. The objective of patch test is to study the error on comparing the element stiffness matrix of different quadrature schemes. Here patch test error in stiffness matrix is determined using the equation (3.1) where comparing stiffness matrix of different quadrature’s were done.

The following equation is used to study error in stiffness matrix of the brick elements.

Element Stiffness matrix error ($E_s$) is defined as

$$
E_s = \sqrt{\frac{\sum_{i=1}^{n} \sum_{j=1}^{n} \left( K_s^{Gauss} - K_s^{Proposed} \right)^2}{\sum_{i=1}^{n} \sum_{j=1}^{n} K_s^{Gauss}}}.
$$

Using the equation (8) the stiffness matrix error was calculated and tabulated in Table 6. The results show that the error of stiffness matrix of Corner Center Pont Method and conventional gauss quadrature is found to negligible thus we can infer the new scheme has passed the patch test.
Figure 6. Discretized Patch Test model of 20-node brick element

Table 6. Error in Stiffness matrix of various quadrature schemes

| Quadrature scheme | Stiffness matrix Error |
|-------------------|------------------------|
| Gauss 2x2x2 Vs CCPM 0.75 | 0.0139 |
| Gauss 3x3x3 Vs CCPM 0.75 | 0.0102 |
| Gauss 4x4x4 Vs CCPM 0.75 | 0.0093 |
| Gauss 2x2x2 Vs CCPM 0.5 | 0.0113 |
| Gauss 3x3x3 Vs CCPM 0.5 | 0.0093 |
| Gauss 4x4x4 Vs CCPM 0.5 | 0.0111 |

4. Discussion and Conclusion

A new nine-point sampling scheme is introduced with sampling points at the corners and center point of the brick element. The proposed scheme results were compared with standard ANSYS results, Gauss numerical results and standard benchmark problem results. It is found that though it has nine sampling points the results from various problems were found to be accurate and comparable with standard literature results. Large varying data in computation of stiffness matrix and calculations in post processing part may lead affect the computation time and accuracy of results but for the proposed scheme there is no such issues. The formulation of new sampling points is derived using the method of undetermined coefficients. Benchmarked problems defined by various researchers were taken and displacement results were compared. Based on the results from benchmarked problems the following points were inferred:

- The five beam element test was used to analyze the bending performance and the results show that the new scheme is comparable with other methods and element results.
- The convergence of results was studied using the Cooke’s membrane problem and it is found that initially the displacement results were found to be high but on increasing the mesh size the results are found to be getting converged to standard results.
- On increasing the number of sampling points normally the CPU execution time also will be increased similarly the proposed Corner Center Point method also reached almost equal time with gauss scheme without compromising in the accuracy of values.
- Distortion test is test used to analyze the distortion of meshes using a factor “e”. The displacement results significantly show that on increasing the factor “e” all quadrature results were found to be varying thus large variation in factor “e” will affect the displacement results.
• Significantly the patch test results show that the error between stiffness matrix of Corner Centre Point scheme and Gaussian quadrature scheme can be negligible and also using proposed Corner Center Point scheme can solve complex problems.

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