ON SETS OF SINGULAR ROTATIONS FOR TRANSLATION INVARIANT BASES

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Abstract. It is studied the following problem: for a given function \( f \) what kind of may be a set of all rotations \( \gamma \) for which \( \int f \) is not differentiable with respect to \( \gamma \)-rotation of a given basis \( B \)? In particular, for translation invariant bases on the plane it is found the topological structure of possible sets of singular rotations.

A mapping \( B \) defined on \( \mathbb{R}^n \) is said to be a differentiation basis if for every \( x \in \mathbb{R}^n \), \( B(x) \) is a family of bounded measurable sets with positive measure and containing \( x \), such that there exists a sequence \( R_k \in B(x) \) \((k \in \mathbb{N})\) with \( \lim_{k \to \infty} \text{diam } R_k = 0 \).

For \( f \in L(\mathbb{R}^n) \), the numbers

\[
\overline{D}_B(\int f, x) = \lim_{\substack{R \in B(x) \\text{diam } R \to 0}} \frac{1}{|R|} \int_R f \quad \text{and} \quad D_B(\int f, x) = \lim_{\substack{R \in B(x) \\text{diam } R \to 0}} \frac{1}{|R|} \int_R f
\]

are called the upper and the lower derivative, respectively, of the integral of \( f \) at a point \( x \). If the upper and the lower derivative coincide, then their combined value is called the derivative of \( \int f \) at a point \( x \) and denoted by \( D_B(\int f, x) \).

We say that the basis \( B \) differentiates \( \int f \) (or \( \int f \) is differentiable with respect to \( B \)) if \( \overline{D}_B(\int f, x) = D_B(\int f, x) = f(x) \) for almost all \( x \in \mathbb{R}^n \). If this is true for each \( f \) in the class of functions \( X \) we say that \( B \) differentiates \( X \).

Denote by \( I = I(\mathbb{R}^n) \) the basis of intervals, i.e., the basis for which \( I(x) \) \((x \in \mathbb{R}^n)\) consists of all open \( n \)-dimensional intervals containing \( x \). Note that differentiation with respect to \( I \) is called strong differentiation.

For a basis \( B \) by \( F_B \) denote the class of all functions \( f \in L(\mathbb{R}^n) \) the integrals of which are differentiable with respect to \( B \).

A basis \( B \) is called translation invariant (briefly, TI-basis) if \( B(x) = \{x + R : R \in B(0)\} \) for every \( x \in \mathbb{R}^n \);

Denote by \( \Gamma_n \) the family of all rotations in the space \( \mathbb{R}^n \).

Let \( B \) be a basis in \( \mathbb{R}^n \) and \( \gamma \in \Gamma_n \). The \( \gamma \)-rotated basis \( B \) is defined as follows

\[
B(\gamma)(x) = \{x + \gamma(R - x) : R \in B(x)\} \quad (x \in \mathbb{R}^n).
\]

The set of two-dimensional rotations \( \Gamma_2 \) can be identified with the circumference \( \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\} \), if to a rotation \( \gamma \) we put into correspondence the complex number \( z(\gamma) \) from \( \mathbb{T} \), the argument of which is equal to the value of the angle by which the rotation about the origin takes place in the positive direction under the action of \( \gamma \).
The distance \( d(\gamma, \sigma) \) between points \( \gamma, \sigma \in \Gamma_2 \) is assumed to be equal to the length of the smallest arch of the circumference \( T \) connecting points \( z(\gamma) \) and \( z(\sigma) \).

Let \( B \) and \( H \) are bases in \( \mathbb{R}^n \) with \( B \subset H \) and \( E \subset \Gamma_n \). Let us call \( E \) a \( W_{B,H}-\)set (\( W_{B,H}^+-\)set), if there exists a function \( f \in L(\mathbb{R}^n) \) \( (f \in L(\mathbb{R}^n), f \geq 0) \) such that: 1) \( f \notin F_{B(\gamma)} \) for every \( \gamma \in E \); and 2) \( f \in F_{H(\gamma)} \) for every \( \gamma \notin E \).

Let \( B \) and \( H \) are bases in \( \mathbb{R}^n \) with \( B \subset H \) and \( E \subset \Gamma_n \). Let us call \( E \) an \( R_{B,H}-\)set (\( R_{B,H}^-\)set), if there exists a function \( f \in L(\mathbb{R}^n) \) \( (f \in L(\mathbb{R}^n), f \geq 0) \) such that: 1) \( \bar{D}_{B(\gamma)} (f, x) = \infty \) almost everywhere for every \( \gamma \in E \); and 2) \( f \in F_{H(\gamma)} \) for every \( \gamma \notin E \).

When \( B = H \) we will use terms \( W_B \) (\( W_B^+ \), \( R_B^+ \)) -set.

**Remark 1.** It is clear that:
1) each \( W_{B,H}^+(R_{B,H}^-) \)-set is \( W_{B,H} \) (\( R_{B,H} \))-set;
2) each \( W_{B,H} \) (\( W_{B,H}, R_{B,H}, R_{B,H}^+ \))-set is \( W_B \) (\( W_B^+, R_B, R_B^+ \))-set.

The definitions of \( R_{1(\mathbb{R}^2)} \), \( R_{1(\mathbb{R}^2)}^+ \) and \( W_{1(\mathbb{R}^2)} \)-sets were introduced in [5], [6] and [1], respectively.

Singularities of an integral of a fixed function with respect to the collection of rotated bases \( B(\gamma) \) were studied by various authors (see [1-9]). In particular, in [5] and [1], respectively, there were proved the following results about topological structure of \( R_{1(\mathbb{R}^2)} \) and \( W_{1(\mathbb{R}^2)} \).

**Theorem A.** Each \( R_{1(\mathbb{R}^2)} \)-set has \( G_\delta \) type.

**Theorem B.** Each \( W_{1(\mathbb{R}^2)} \)-set has \( G_{\delta \sigma} \) type.

There are true the following generalizations of Theorems A and B.

**Theorem 1.** For arbitrary translation invariant basis \( B \) in \( \mathbb{R}^2 \) each \( R_B \)-set has \( G_\delta \) type.

**Theorem 2.** For arbitrary translation invariant basis \( B \) in \( \mathbb{R}^2 \) each \( W_B \)-set has \( G_{\delta \sigma} \) type.

We will also prove the following result.

**Theorem 3.** For arbitrary bases \( B \) and \( H \) in \( \mathbb{R}^2 \) with \( B \subset H \) not more than countable union of \( R_{B,H} \)-sets (\( R_{B,H}^+ \)-sets) is \( W_{B,H} \)-set (\( W_{B,H}^+ \)-set).

**Proof of Theorem 1.** Let \( f \in L(\mathbb{R}^2) \). We must prove that the set

\[ W_B(f) = \{ \gamma \in \Gamma_2 : f \notin F_{B(\gamma)} \} \]

is of \( G_{\delta \sigma} \) type.

Without loss of generality let us assume that \( f \) is finite everywhere and \( \text{supp } f \subset (0, 1)^n \).

For a basis \( H \), \( x \in \mathbb{R}^2 \) and \( r > 0 \) set

\[ l_H(f)(x) = \lim_{\text{diam } R \to 0} \frac{1}{|R|} \left| \int_R f(x) \right|, \]

\[ l'_H(f)(x) = \sup_{\text{diam } R < r} \frac{1}{|R|} \left| \int_R f(x) \right|. \]
For numbers $\varepsilon > 0$, $\alpha \in (0, 1]$, $r > 0$ and $\beta \in (0, 1)$ denote
\[
W_B(f, \varepsilon, \alpha) = \{ \gamma \in \Gamma_2 : |\{ I_{B(\gamma)}(f) \geq \varepsilon \}| \geq \alpha \},
\]
\[
W_B^r(f, \varepsilon, \beta) = \{ \gamma \in \Gamma_2 : |\{ I_{B(\gamma)}^r(f) > \varepsilon \}| > \beta \}.
\]

First let us prove that $W_B^r(f, \varepsilon, \beta)$ is an open set for any $r > 0$, $\varepsilon > 0$ and $\beta \in (0, 1)$. Suppose $\gamma \in W_B^r(f, \varepsilon, \beta)$, i.e.
\[
|\{ I_{B(\gamma)}^r(f) > \varepsilon \}| > \beta.
\]

If $x \in \{ I_{B(\gamma)}^r(f) > \varepsilon \}$, then there is $R_x \in B(\gamma)(x)$ with $\text{diam} R_x < r$ such that
\[
\left| \frac{1}{|R_x|} \int_{R_x} f - f(x) \right| > \varepsilon.
\]

Taking into account absolute continuity of Lebesgue integral is easy to check that performing small enough rotation of $R_x$ around the point $x$ one derives the set $R'_x$ for which
\[
\left| \frac{1}{|R'_x|} \int_{R'_x} f - f(x) \right| > \varepsilon.
\]
Therefore for every $x \in \{ I_{B(\gamma)}^r(f) > \varepsilon \}$ we can find $k_x \in \mathbb{N}$ such that
\[
I_{B(\gamma')}^r(f)(x) > \varepsilon \quad \text{if} \quad \text{dist}(\gamma', \gamma) < 1/k_x.
\]

For every $m \in \mathbb{N}$ by $A_m$ denote the set of all points from $\{ I_{B(\gamma)}^r(f) > \varepsilon \}$ for which $k_x = m$. Obviously,
\[
A_1 \subset A_2 \subset \cdots \quad \text{and} \quad \bigcup_{m \in \mathbb{N}} A_m = \{ I_{B(\gamma)}^r(f) > \varepsilon \}.
\]

Now, using the property of continuity of outer measure from below we can find $m \in \mathbb{N}$ for which $|A_m|_* > \beta$. The last conclusion implies that
\[
|\{ I_{B(\gamma)}^r(f) > \varepsilon \}| > \beta \quad \text{if} \quad \text{dist}(\gamma', \gamma) < 1/m.
\]
Consequently, $W_B^r(f, \varepsilon, \beta)$ is an open set.

Now let us prove that $W_B(f, \varepsilon, \alpha)$ is of $G_\delta$ type for any $\varepsilon > 0$ and $\delta \in (0, 1]$. Let us consider strictly increasing sequences of positive numbers $(\varepsilon_k)$ and $(\alpha_k)$ such that $\varepsilon_k \to \varepsilon$ and $\alpha_k \to \alpha$. Taking into account openness of sets $W_B^r(f, \varepsilon, \beta)$ it is easy to see that for every $\gamma \in W_B(f, \varepsilon, \alpha)$ and $k \in \mathbb{N}$ there is a neighbourhood $V_{\gamma,k}$ of $\gamma$ such that
\[
|\{ I_{B(\gamma)}^{1/k}(f) > \varepsilon_k \}| > \alpha_k \quad \text{if} \quad \gamma' \in V_{\gamma,k}.
\]

Denote
\[
G_k = \bigcup_{\gamma \in W_B(f, \varepsilon, \alpha)} V_{\gamma,k} \quad (k \in \mathbb{N}).
\]

Since $W_B(f, \varepsilon, \alpha) \subset G_k$ ($k \in \mathbb{N}$), we have $W_B(f, \varepsilon, \alpha) \subset \bigcap_{k \in \mathbb{N}} G_k$. On the other hand, if $\gamma \in \bigcap_{k \in \mathbb{N}} G_k$, then
\[
|\{ I_{B(\gamma)}(f) \geq \varepsilon \}| = \left| \bigcap_{k \in \mathbb{N}} I_{B(\gamma)}^{1/k}(f) > \varepsilon_k \right| \geq \lim_{k \to \infty} \alpha_k = \alpha.
\]
Consequently, $\gamma \in W_B(f, \varepsilon, \alpha)$. Thus $W_B(f, \varepsilon, \alpha) \supset \bigcap_{k \in \mathbb{N}} G_k$. So we proved that $W_B(f, \varepsilon, \alpha) = \bigcap_{k \in \mathbb{N}} G_k$, wherefrom it follows the needed conclusion.

It is easy to check that

$$W_B(f) = \bigcup_{k \in \mathbb{N}} W_B(f, 1/k, 1/k),$$

wherefrom we conclude $W_B(f)$ to be of $G_{\delta \sigma}$ type. $\square$

**Proof of Theorem 2.** Let $f \in L(\mathbb{R}^2)$ and $\text{supp } f \subset (0, 1)^2$. Let us prove that the set

$$R_B(f) = \{ \gamma \in \Gamma_2 : \overline{D}_{B(\gamma)}(\int f, x) = \infty \text{ a.e. on } (0, 1)^2 \}$$

is of $G_{\delta}$ type. It is easy to check that this assertion implies the validity of the theorem.

For a basis $H$, $x \in \mathbb{R}^2$ and $r > 0$ set

$$N_H^r(f)(x) = \sup_{\text{diam } R < r} \frac{1}{|R|} \int_R f.$$

For numbers $\varepsilon > 0$, $r > 0$ and $\beta \in (0, 1)$ denote

$$R_B^r(f, \varepsilon, \beta) = \{ \gamma \in \Gamma_2 : |\{ N_{B(\gamma)}^r(f) > \varepsilon \} | > \beta \}.$$

First let us prove that $R_B^r(f, \varepsilon, \beta)$ is an open set for any $r > 0$, $\varepsilon > 0$ and $\beta \in (0, 1)$. Suppose $\gamma \in R_B^r(f, \varepsilon, \beta)$, i.e.

$$|\{ N_{B(\gamma)}^r(f) > \varepsilon \} | > \beta.$$

If $x \in \{ N_{B(\gamma)}^r(f) > \varepsilon \}$, then there is $R_x \in B(\gamma)(x)$ with $\text{diam } R_x < r$ such that

$$\frac{1}{|R_x|} \int_{R_x} f > \varepsilon.$$

Taking into account absolute continuity of Lebesgue integral is easy to check that performing small enough rotation of $R_x$ around the point $x$ one derives the set $R_x'$ for which

$$\frac{1}{|R_x'|} \int_{R_x'} f > \varepsilon.$$

Therefore for every $x \in \{ N_{B(\gamma)}^r(f) > \varepsilon \}$ we can find $k_x \in \mathbb{N}$ such that

$$N_{B(\gamma)}^r(f)(x) > \varepsilon \text{ if } \text{dist}(\gamma', \gamma) < 1/k_x.$$

For every $m \in \mathbb{N}$ by $A_m$ denote the set of all points from $\{ N_{B(\gamma)}^r(f) > \varepsilon \}$ for which $k_x = m$. Obviously,

$$A_1 \subset A_2 \subset \ldots \text{ and } \bigcup_{m \in \mathbb{N}} A_m = \{ N_{B(\gamma)}^r(f) > \varepsilon \}.$$

Now, using the property of continuity of outer measure from below we can find $m \in \mathbb{N}$ for which $|A_m| > \beta$. The last conclusion implies that

$$|\{ N_{B(\gamma)}^r(f) > \varepsilon \} | > \beta \text{ if } \text{dist}(\gamma', \gamma) < 1/m.$$

Consequently, $R_B^r(f, \varepsilon, \beta)$ is an open set.
Now let us prove that $R_B(f)$ is of $G_\delta$ type. Taking into account openness of sets $R^*_B(f, \varepsilon, \beta)$ it is easy to see that for every $\gamma \in R_B(f)$ and $k \in \mathbb{N}$ there is a neighbourhood $V_{\gamma,k}$ of $\gamma$ such that

$$|\{N^{1/k}_{B(\gamma)}(f) > k\}| > 1 - 1/k \text{ if } \gamma' \in V_{\gamma,k}.$$ 

Denote

$$G_k = \bigcup_{\gamma \in R_B(f)} V_{\gamma,k} \ (k \in \mathbb{N}).$$

Since $R_B(f) \subset G_k \ (k \in \mathbb{N})$, we have $R_B(f) \subset \bigcap_{k \in \mathbb{N}} G_k$. On the other hand, if $\gamma \in \bigcap_{k \in \mathbb{N}} G_k$, then

$$|\{D_{B(\gamma)}(\int f, x) = \infty \text{ a.e. on } (0,1)^2\}| = \left| \bigcap_{k \in \mathbb{N}} \{N^{1/k}_{B(\gamma)}(f) > k\} \right| \geq \lim_{k \to \infty} (1 - 1/k) = 1.$$ 

Consequently, $\gamma \in R_B(f)$. Thus $R_B(f) \supset \bigcap_{k \in \mathbb{N}} G_k$. So we proved that $R_B(f) = \bigcap_{k \in \mathbb{N}} G_k$. Wherefrom it follows the needed conclusion. \hfill $\Box$

**Proof of Theorem 3.** Let $N \subset \mathbb{N}$ be a not more than countable non-empty set and for each $k \in N$, $E_k$ be an $R_{B,H}$-set ($R^+_{B,H}$-set). For every $k \in N$ let us consider summable function $f_k$ with two properties from the definition of $R_{B,H}$-set ($R^+_{B,H}$-set): 1) $D_{B(\gamma)}(\int f_k, x) = \infty$ almost everywhere for every $\gamma \in E_k$; and 2) $f_k \in F_{H(\gamma)}$ for every $\gamma \notin E_k$. Let us consider also an arbitrary family of pairwise disjoint open squares $Q_k \ (k \in N)$.

Denote

$$g_k = \frac{f_k \chi_{Q_k}}{2^k \|f_k\|_L} \ (k \in N),$$

$$f = \sum_{k \in N} g_k.$$ 

Then we have

$$\|f\|_L = \sum_{k \in N} \|g_k\|_L \leq \sum_{k \in N} \frac{1}{2^k} < \infty.$$ 

Consequently, $f$ is summable function.

Using disjointness of squares $Q_k$ we have that for every $\gamma \in \Gamma_2$, $k \in N$ and $x \in Q_k$

$$D_{B(\gamma)}(\int f, x) = D_{B(\gamma)}(\int g_k, x).$$

Therefore for every $k \in N$ and $\gamma \in E_k$

$$D_{B(\gamma)}(\int f, x) = \infty \text{ for a.e. } x \in Q_k.$$ 

Thus,

$$f \notin F_{B(\gamma)} \text{ for every } \gamma \in \bigcup_{k \in N} E_k.$$
Now take arbitrary $\gamma \notin \bigcup_{k \in \mathbb{N}} E_k$. Then $g_k \in F_{H(\gamma)}$ for every $k \in \mathbb{N}$. Consequently, using disjointness of squares $Q_k$ we have that for every $k \in \mathbb{N}$

$$D_{H(\gamma)} \left( \int f, x \right) = D_{H(\gamma)} \left( \int g_k, x \right) = g_k(x) = f(x)$$

for a.e. $x \in Q_k$. Thus

$$D_{H(\gamma)} \left( \int f, x \right) = f(x) \text{ for a.e. } x \in \bigcup_{k \in \mathbb{N}} Q_k.$$

Now taking into account that $f(x) = 0$ for every $x \notin \bigcup_{k \in \mathbb{N}} Q_k$ we write

$$D_{H(\gamma)} \left( \int f, x \right) = f(x) \text{ for a.e. } x \in \mathbb{R}^2. \tag{0.2}$$

(1) and (2) implies that $\bigcup_{k \in \mathbb{N}} E_k$ is $W_{B,H}$-set ($W^+_{B,H}$-set).

□

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