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Evaluation of Observables in the Gaussian
\( N = \infty \) Kazakov-Migdal Model

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ABSTRACT

We examine the properties of observables in the Kazakov-Migdal model. We present explicit formulae for the leading asymptotics of adjoint Wilson loops as well as some other observables for the model with a Gaussian potential. We discuss the phase transition in the large \( N \) limit of the \( d = 1 \) model. One of appendices is devoted to discussion of the \( N = \infty \) Itzykson-Zuber integrals for arbitrary eigenvalue densities.

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1 Introduction

The Kazakov-Migdal model (KMM) was introduced in [1] as an example of a gauge-invariant matrix model on a $D$-dimensional lattice. It consists of an $N \times N$ Hermitean matrix scalar field $\Phi(x)$ which lives on sites, $x$, and transforms in the adjoint representation of $SU(N)$ and a unitary $N \times N$ matrix gauge field (link variable) $U(xy)$ which lives on links, $\langle xy \rangle$. The partition function is

$$Z = \int d\Phi[dU] \exp \left( -N \sum_x \text{Tr}V[\Phi(x)] + N \sum_{<x,y>} \text{Tr}\Phi(x)U(xy)\Phi(y)U^\dagger(xy) \right)$$

where $V[\Phi]$ is a self-interaction potential for the scalar field. Due to the absence of a kinetic term (such as, for example, a Wilson term) for $U$, one can integrate out the gauge variables in (1) exactly. The resulting effective scalar theory contains $N$ degrees of freedom, the eigenvalues of the matrix $\Phi(x)$, whose quantum fluctuations are suppressed in the large $N$ limit. This makes it possible to employ large $N$ techniques. The field $\Phi(x)$ plays the role of a master field and the density of its eigenvalues, $\rho(\phi)$, can in principle be found by solving a saddle-point (master field) equation. In this sense the model is regarded as being solvable in the large $N$ limit. However, explicit solution of the master field equation remains one of the major technical obstacles in this program.

With the assumption of a translation invariant master field, Migdal [3] has reduced this problem to finding the solution of a singular highly nonlinear integral equation. Some further progress has also been made using the method of loop equations [4]. In both approaches, technical problems prevent one from obtaining very many analytic results. For instance, explicit solutions for the spectral density of the scalar field are thus far known for only two kinds of potentials: the Gaussian one ($V[\Phi] = \frac{m^2}{2} \Phi^2$) solved by Gross [5] and a potential related to the Penner model, which was investigated by Makeenko [6]. Furthermore, even for a Gaussian potential where the spectral density is known explicitly, only expectation values of products of scalar fields on a site [5] and one-link correlators of the gauge field [4] have been computed.

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1The exact integration of the gauge fields from (1) uses the Itzykson-Zuber formula [2] and can in some sense be regarded as a way to avoid solving the long-standing problem of summation of planar diagrams in lattice gauge theory.
so far. In this Paper, we shall use these previous results to present some explicit formulae for observables in the Gaussian KMM. In particular, we give a calculation of the expectation value of the adjoint Wilson loop as well as the gauge invariant two-point correlator of the scalar field. We regard this as an intermediate step in understanding the continuum limit of the KMM in the general case.

It was originally hoped that the Kazakov-Migdal model would possess a continuum limit which would be a gauge theory, such as QCD. It was also clear from the outset that this could not be realized in a straightforward way. Because the action of the model (1) does not contain the kinetic term for the gauge field (this was exactly the fact which allowed one to integrate out the gauge fields), the KMM possesses an additional local (gauge) $Z_N$ invariance [7] which implies that all of the usual Wilson loops vanish unless they have vanishing area. This can be interpreted as an infinite string tension and results in “ultraconfinement”, where no propagation of color is allowed at any distance scales. An ordinary continuum gauge theory does not have such a property.

Thus, if the KMM is to describe continuum gauge theories, then either there must exist a more sophisticated continuum limit [8], or the model itself should be modified in a way which would break the local $Z_N$ symmetry explicitly (yet preserving its solvability) [9, 10].

Irrespective of its relevance for the description of $D > 2$ Yang-Mills theories, the KMM has proven to be very interesting from a purely theoretical point of view as an intermediate step between generic matrix models which involve full unitary matrix integrals, and the well studied “$c < 1$” matrix models in which the unitary matrices (“angular variables”) completely decouple [11]. The KMM is furthermore directly relevant to a certain sector of $c = 1$ string theory [12] where it indeed has a phase transition and one can discuss its continuum limit.

The choice of physical observables in the KMM is not entirely obvious. Due to the existence of the additional $Z_N$ gauge invariance, all of the conventional Wilson loops vanish. Note, that in the large $N$ limit and in the mean field approximation, the unitary integrals in the KMM can be reduced to single link integrals of the type

$$C(n|\Phi, \bar{\Phi}) \propto \int_{N \times N} [dU]U_{i_1j_1} \ldots U_{i_nj_n}U_{j_1i_1}^\dagger \ldots U_{j_ni_n}^\dagger e^{\text{Tr}N\Phi U\bar{\Phi}U^\dagger},$$

(2)
They must involve the same number of $U$’s and $U^{\dagger}$’s in order to be invariant under the local $Z_N$ transformation. The simplest gauge invariant observables which can be constructed out of the $C$’s are the adjoint Wilson loop and the filled Wilson loop. Both of these involve only the averages of $|U_{ij}|^2$ i.e. $\mathcal{C}(1|\Phi, \bar{\Phi})$. (As we mentioned above, even for a soluble model such as the Gaussian KMM $\mathcal{C}(n|\Phi, \bar{\Phi})$ have been evaluated explicitly only for $n = 0$ and $n = 1$).

Before presenting these formulae we make some general comments about the integral (2) for arbitrary $n$. These integrals fall into the general framework of applicability of the Duistermaat-Heckmann theorem, and, in principle, they can be explicitly evaluated (see [8] and references therein for details). However, existing explicit expressions for $n \geq 1$ [13] are too complicated to be used effectively when $N \to \infty$, which is the limit of interest in the KMM. What is required is an expression for $\mathcal{C}(n|\Phi, \bar{\Phi})$ in terms of the densities $\rho(\phi) \equiv \frac{1}{N} \sum_{i=1}^{N} \delta(\phi - \phi_i)$ and $\bar{\rho}(\phi)$ of eigenvalues of the matrices $\Phi$ and $\bar{\Phi}$. This would allow taking a smooth large $N$ limit. In this limit, discrete sums over the indices become continuous integrals over the eigenvalues, $\frac{1}{N} \sum_i \to \int d\phi \rho(\phi)$. Expressing all quantities in terms of densities implies also relabelling of tensor structures like $\mathcal{C}(n)$ for $n \geq 1$. Indices $i, j, \ldots$ should be substituted by $\phi_i, \phi_j, \ldots$ according to the rule $C_{\phi_1 \ldots \phi_m} \equiv N^{m/2} C_{i_1 \ldots i_m}$, so that for example

$$\sum_{j=1}^{N} C_{ij} C_{jk} = \frac{1}{N} \sum_{\phi_j} C_{\phi_i \phi_j} C_{\phi_j \phi_k} = \int d\phi \rho(\phi) C_{\phi_i \phi} C_{\phi \phi_k}, \quad (3)$$

At the moment there exist only a few methods for the evaluation of $\mathcal{C}(n|\rho, \bar{\rho})$ with $n = 0$ [14] and $n = 1$ [4] (for details see Appendix B). Though they have the potential to work in a general case, these methods have not proven to be very efficient in practice. Until now the quantities $\mathcal{C}(n|\rho, \bar{\rho})$ have been computed only for two cases: for $n = 0, 1$ for the semi-circle [3, 4] and $n = 1$ for the “Penner-like” [3] distributions of the eigenvalues. (The latter is more complicated and will not be considered here.)

With the assumption that the mean field is spatially homogenous, the scalar field in the Gaussian KMM has the semi-circle spectral density

$$\rho(\phi)d\phi = \bar{\rho}(\phi)d\phi = \rho_\mu(\phi)d\phi \equiv \frac{1}{\pi} \sqrt{\mu - \frac{\mu^2 \phi^2}{4}} d\phi \quad (|\phi| \leq \frac{2}{\sqrt{\mu}}) \quad (4)$$
For this spectral density the explicit results for the Itzykson-Zuber integral \( C(0|\rho, \rho) \) and correlators \( C(1|\rho, \rho) \) are as follows

\[
C(\mu) = \lim_{N \to \infty} \frac{1}{N^2} \log \int_{N \times N} [dU] e^{\text{Tr} N \Phi U \Phi U^\dagger},
\]

\[
C(\mu) = \frac{\sqrt{\mu^2 + 4} - \mu}{2\mu} - \frac{1}{2} \log \frac{\sqrt{\mu^2 + 4} + \mu}{2\mu}; \tag{5}
\]

\[
\delta_{ij} C_{ij}(\mu) = \frac{\int_{N \times N} [dU] U^\dagger_{ij} U_{ji} e^{\text{Tr} N \Phi U \Phi U^\dagger}}{\int_{N \times N} [dU] e^{\text{Tr} N \Phi U \Phi U^\dagger}},
\]

\[
C_{\alpha\beta}(\mu) = \lim_{N \to \infty} \frac{1}{N} C_{ij}(\mu) = \frac{\mu + \sqrt{\mu^2 + 4}}{2 \alpha^2 + \beta^2 - \sqrt{\mu^2 + 4} \alpha \beta + \mu}. \tag{6}
\]

In practical calculations it is more convenient to use the “normalized” semicircle distribution, \( \hat{\rho}_s(\hat{\phi}) \), \( \hat{\phi} = \frac{2}{\sqrt{\pi}} \phi \), which has support \([-1, 1]\) and

\[
\rho_s(\phi) \equiv \frac{1}{\pi} \sqrt{1 - \phi^2}, \quad \rho(\phi) d\phi \equiv 2 \hat{\rho}_s(\hat{\phi}) d\hat{\phi}, \tag{7}
\]

so that

\[
C(\tau) = \frac{\cosh \tau - \sinh \tau}{2 \sinh \tau} - \frac{1}{2} \log \frac{\cosh \tau + \sinh \tau}{2 \sinh \tau},
\]

\[
\hat{C}_{\hat{\alpha}\hat{\beta}}(\mu) = 2C_{\alpha=\frac{\pi}{2}, \beta=\frac{\pi}{2}} = \frac{\sinh \tau (\cosh \tau + \sinh \tau)}{\alpha^2 + \beta^2 - 2\hat{\alpha}\hat{\beta} \cosh \tau + \sin^2 \tau}, \tag{8}
\]

where \( \sinh \tau = \frac{\mu}{2}, \quad \cosh \tau = \frac{\sqrt{\mu^2 + 4}}{2} \).

(sometimes we shall use notation \( \hat{C}_{\hat{\alpha}\hat{\beta}}(\tau) \) instead of \( \hat{C}_{\hat{\alpha}\hat{\beta}}(\mu) \), which should not cause any confusion).

## 2 Observables in the KMM

The set of observables in the KMM is restricted by the local \( Z_N \)-invariance, which, for example, forces averages of all the fundamental-representation Wilson loops to vanish and results in ultraconfinement. Non-vanishing observables contain an equal number of \( U \) and \( U^\dagger \) matrices at every link.\(^2\) Examples

\(^2\) Since we consider the large-\( N \) limit, we neglect “baryonic” observables, containing \( N \) or more \( U \)-matrices per link without any \( U^\dagger \)'s.
are shown in Fig.1. Dots can appear at every site and denote insertion of any number of $\Phi$ or $\bar{\Phi}$ operators. One can account for all possible insertions by using the generating function $\frac{1}{\lambda_x - \Phi}$ at every site. Occasionally some double-lines can become quadruple etc at some links, then the knowledge of higher-order correlators $C(n)$ with $n \geq 2$ is required for evaluation of averages. We restrict ourselves to consideration of the non-degenerate situations, where double lines never overlap. In such cases the answer is further simplified [8] by the occurrence of $\delta_{i,i} \hat{\delta}_{jj}$ in eq.(1). It allows us to consider every observable as associated with a graph $\Gamma$ which ascribe a single index $i_x$ to every site (i.e. effectively substituting double lines by single lines, with $C_{ij}$ playing the role of the propagator, and $(\lambda_x - \phi_{i_x})^{-1}$ - that of the vertex):

$$
\langle O_{\Gamma} \rangle = N^{-n_{\Gamma}} \sum_{\{i_x, x \in \Gamma\}} \left( \prod_{x \in \Gamma} \frac{1}{\lambda_x - \phi_{i_x}} \prod_{\langle xy \rangle \in \Gamma} C_{i_x i_y} \right) \xrightarrow{N \to \infty} \quad \rightarrow N^{-n_{\Gamma} + \#(x \in \Gamma) - \#(\langle xy \rangle \in \Gamma)} \prod_{x \in \Gamma} \int \frac{\rho(\phi_x)d\phi_x}{\lambda_x - \phi_{i_x}} \prod_{\langle xy \rangle \in \Gamma} C_{\phi_x \phi_y},
$$

(9)

where, as usual, $n_{\Gamma}$ is the number of traces, contained in the definition of $O_{\Gamma}$ (for discussion of different normalization prescriptions see ref.[8]). Note that, with our normalization of $C$ and $\rho$, each site contributes a factor of $N$ and each link a factor of $1/N$. It will turn out that the product of integrals in the right-hand-side of (9) is well-defined and $N$-independent. With this normalization of operators, the only ones which are non-zero in the infinite $N$ limit are tree-like configurations - these always have one index sum ($n_{\Gamma} = 1$) and also $\#(x \in \Gamma) - \#(\langle xy \rangle \in \Gamma) = 1$. Every loop in a configuration will only increase $\#(x \in \Gamma) - \#(\langle xy \rangle \in \Gamma)$, making it greater than one.

At least in principle, the integral on the r.h.s. can be evaluated for any graph, provided $\hat{C}_{\phi_x \phi_y}$ is known for the given $\rho$. At the moment, however, an explicit expression is available only for $\rho$ given by the semi-circle distribution. In $D = 1$ there are only two types of allowed graphs: segments and circles of length $L$ which, as we shall show, can be relatively easily computed for the semi-circle distribution[3]. This is the subject of the next section. In higher dimensions the full set of observables contains more objects, like filled Wilson loops [8], which are not considered in the present paper.

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3 For lattices which form “rare nets”, so that the theory remains essentially 1-dimensional, more sophisticated graphs are also of interest. We, however, ignore this possibility.
3 Evaluation of Observables

3.1 The Adjoint Wilson Loop

The Wilson Loop is obtained by considering the trace of the path-ordered product of the link operators for the links which occupy a contour $\Gamma$ on the lattice

$$W[\Gamma] \equiv \text{tr} \prod_{<xy> \in \Gamma} U(xy) \quad (10)$$

Because of the $Z_N$ symmetry, the expectation value of this operator vanishes unless $\Gamma$ has vanishing area. An example of an operator which is $Z_N$ invariant and also a physical interpretation is the adjoint Wilson loop,

$$W_A[\Gamma] = \frac{1}{N^2} \left( |W[\Gamma]|^2 - 1 \right) \quad (11)$$

The appearance of the constant term in this definition can be understood if one recalls that the product of two fundamental representation link operators contains both the adjoint and scalar representations, the latter of which must be subtracted: $(U_{ij}U_{kl}^\dagger)_A = U_{ij}U_{kl}^\dagger - \delta_{il}\delta_{jk}$.

In the mean field approximation of the KMM, the adjoint Wilson loop can be evaluated using the one-link expectation values of the the products $C_{ij} = \langle UU^\dagger \rangle$,

$$\langle W_A[\Gamma] \rangle = \frac{1}{N^2} \left( \text{tr} \left( C^{L[\Gamma]} \right) - 1 \right) = \frac{1}{N^2} \left( \sum_i c_i^{L[\Gamma]} - 1 \right) \quad (12)$$

where $c_i$ are the eigenvalues of the matrix $C_{ij}$ and $L[\Gamma]$ is the length of the contour $\Gamma$.

From their definition it is easy to see that the matrices $C_{ij}$ are real, have positive entries and, when the master field is homogeneous, they are also symmetric. Furthermore, they obey the sum rule

$$\sum_{i=1}^N C_{ij} = 1 \quad (13)$$

This sum rule implies that, for any $N$, $C_{ij}$ always has one eigenvalue which is one, with eigenvector $(1, 1, 1, \ldots, 1)$.

\footnote{We thank Yuri Makeenko for a discussion of this point.}
It is also easy to see that all other eigenvalues lie in the interval $[-1,1]$. Indeed, let us consider the set of real matrices with positive entries, satisfying the sum rule (13). Obviously, all matrices from this set have their traces bounded from above by $N$. Further, this set is closed under the matrix multiplication: if matrices $A$ and $B$ belong to this set, then so does the matrix $AB$. Take now any matrix $C$ from this set and consider its power $C^n$ with $n$ even. If there are eigenvalues of the matrix $C$, absolute value of which is greater than 1, then for big enough $n$ one can make the trace of $C^n$ arbitrary large, thus violating the mentioned upper bound. Q.E.D.

Let us recall that a rectangular adjoint Wilson loop (whose lengths in the space and time directions are $L$ and $T$, respectively) in the limit $T \to \infty$ can be interpreted as the energy of a pair of mesons with separation $L$,

$$E[L] = \lim_{T \to \infty} \frac{1}{T} \ln \left( N^2 < W_A[\Gamma] > \right)$$

(14)

Once 1 is subtracted in the sum on the right-hand-side of (12), the energy is dominated by the largest remaining eigenvalue of $C$.

In the following we shall compute $W_A$ in the large $N$ limit for the Gaussian Kazakov-Migdal model. First, however, let us consider the case of $N = 2$ which was previously analyzed in [8]. There, even though the mean field approximation for the scalar field is uncontrolled (since $N = 2$ is not large), it has been argued [15] that mean field theory can give accurate results. If the mean field eigenvalues of the $2 \times 2$ scalar matrix field $\phi$ (which can be taken as traceless) in this case are $\bar{\phi}$ and $-\bar{\phi}$, it was shown in [8] that

$$C_{11} = C_{22} = \frac{1 - 1/4 \bar{\phi}^2 + e^{-4\bar{\phi}^2}/4\bar{\phi}^2}{1 - e^{-4\bar{\phi}^2}}$$

$$C_{12} = C_{21} = \frac{1/4 \bar{\phi}^2 - e^{-4\bar{\phi}^2} (1 + 1/4\bar{\phi}^2)}{1 - e^{-2\bar{\phi}^2}}$$

(15)

The eigenvalues are 1 and $\coth(2\bar{\phi}^2) - 1/2 \bar{\phi}^2$. Note that the second eigenvalue varies between zero and one as $\bar{\phi}^2$ goes from zero to infinity. The energy of the meson pair is

$$E = -2 \ln \left( \coth(2\bar{\phi}^2) - 1/2 \bar{\phi}^2 \right)$$

(16)

which is positive and goes to zero at the “critical” value of $\bar{\phi}^2 \to \infty$. Note that, in this approximation the mesons have no interaction energy, i.e. (16) can be interpreted as twice the meson mass.

8
Now let us proceed to the large $N$ limit in the case of a Gaussian potential. We start with the basic relation, proven for the semi-circle distribution in Appendix A:

$$\int_{-1}^{+1} \hat{C}_{\hat{a}\hat{b}}(\tau_1)\hat{C}_{\hat{b}\hat{g}}(\tau_2)\hat{\rho}_s(\hat{b})d\hat{b} = \hat{C}_{\hat{a}\hat{g}}(\tau_1 + \tau_2).$$  \hspace{1cm} (17)

Therefore for the adjoint Wilson loop of length $L$ we find

$$\langle W_A[\Gamma] \rangle = \frac{1}{N^2} \left( \int d\alpha_1 \rho(\alpha_1) \ldots d\alpha_L \rho(\alpha_L)C_{\alpha_1\alpha_2}(\tau) \ldots C_{\alpha_{L-1}\alpha_L}(\tau) - 1 \right) = \frac{1}{N^2} \left( \int d\hat{\alpha}_1 \hat{\rho}(\hat{\alpha}_1) \ldots d\hat{\alpha}_L \hat{\rho}(\hat{\alpha}_L)\hat{C}_{\hat{\alpha}_1\hat{\alpha}_2}(\tau) \ldots \hat{C}_{\hat{\alpha}_{L-1}\hat{\alpha}_L}(\tau) - 1 \right) = \frac{1}{N^2} \left( \int d\hat{\alpha} \hat{\rho}(\hat{\alpha})\hat{C}_{\hat{\alpha}\hat{\alpha}}(L\tau) - 1 \right) = \frac{1}{N^2} e^{-L\tau}$$  \hspace{1cm} (18)

Note that, as noted in the previous section, the normalization of the adjoint Wilson loop is such that it vanishes in the large $N$ limit. We can regard $\langle W_A[\Gamma] \rangle$ as the leading asymptotics.

The interaction potential for a pair of mesons is obtained from the free energy,

$$E(L) = 2\tau$$  \hspace{1cm} (19)

which one can interpret as just the sum of the meson masses (each equal to $\tau$) without any interaction between mesons.

There is an apparent phase transition at $\tau = 0$. Recall that $m^2$, for the Gaussian potential $V[\phi] = \frac{m^2}{2}\phi^2$, so that the “bare mass” of $\phi$ is $m^2 - 2D$, and $\tau$ is given by

$$\sinh \tau = \frac{m^2(D-1) + D\sqrt{m^4 - 4(2D-1)}}{2(2D - 1)}$$  \hspace{1cm} (20)

or, equivalently,

$$m^2 = 2De^{-\tau} + 2\sinh \tau \approx 2D + 2(1 - D)\tau \ (\tau \sim 0)$$  \hspace{1cm} (21)

As has been observed by Gross $[\text{9}]$, the phase transition can be approached through the physical region, $m^2 \geq 2D$, only when $D \leq 1$.  

9
3.2 Other Observables

It is straightforward to compute the expectation values of a variety of other observables. Here, we shall present a few examples.

It is very easy to find the expectation value of powers of the scalar field $\Phi$ on a single site, using the generating function

$$
E_\lambda \equiv \langle \text{tr} \frac{1}{\lambda I - \Phi} \rangle = \sum_{n=0}^{\infty} \frac{\langle \text{tr} \Phi^n \rangle}{\lambda^{n+1}}.
$$

This generating function for the semi-circle spectral density was found by Gross [5] (in this case odd momenta are vanishing since the spectral density is even)

$$
E_\lambda \equiv \int \rho_s(\beta) d\beta = \frac{\mu \lambda}{2} - \sqrt{\frac{\mu^2 \lambda^2}{4} - \mu} = \frac{1}{\lambda} \left( 1 + \sum_{k \geq 1} \frac{(2k-1)!!}{(k+1)!} \left( \frac{2}{\mu \lambda^2} \right)^k \right),
$$

so that $\langle \text{tr} \Phi^2 \rangle = 1/\mu$ and so on.

For an adjoint loop of the length $L$ with the insertion of the operator $\frac{1}{\lambda I - \phi}$ into one of the products $\prod U$ at the single site we obtain:

$$
\langle O(\circ L) \rangle = \frac{1}{N^2} \int d\alpha_1 \rho(\alpha_1) \ldots d\alpha_L \rho(\alpha_L) \frac{1}{\lambda - \alpha_1} C_{\alpha_1 \alpha_2}(\tau) \ldots C_{\alpha_L \alpha_1}(\tau) = \\
= \frac{1}{N^2} \sqrt{\mu} \int d\hat{\alpha} \hat{\rho}(\hat{\alpha}) \frac{1}{\lambda - \hat{\alpha}} \hat{C}_{\hat{\alpha} \hat{\alpha}}(L \tau) = \\
= \frac{1}{N^2} \frac{e^{-L \tau}}{1 - e^{-L \tau} \lambda^2 - \frac{4}{\mu} \cosh^2(L \tau/2)} (\lambda - \frac{1}{\mu} (e^{L \tau} + 1) E_\lambda)
$$

where $E_\lambda$ is defined in the Appendix. The coefficient in front of $\lambda^{-1}$ in the large-$\lambda$ expansion of (24) reproduces (18).

Similarly, for a segment with $\phi$-insertions at its ends:

$$
\int d\alpha_1 \rho(\alpha_1) \ldots d\alpha_L \rho(\alpha_L) \frac{1}{\lambda - \alpha_1} \frac{1}{\nu - \alpha_L} C_{\alpha_1 \alpha_2}(\tau) \ldots C_{\alpha_{L-1} \alpha_L}(\tau) = \\
= \frac{\mu}{4} \int d\hat{\alpha} \frac{\hat{\rho}(\hat{\alpha})}{\lambda - \hat{\alpha}} d\hat{\beta} \frac{\hat{\rho}(\hat{\beta})}{\nu - \hat{\beta}} \hat{C}_{\hat{\alpha} \hat{\beta}}(L \tau) = \\
= \frac{\lambda E_\nu + \nu E_\lambda + 2 \sinh(L \tau) + (\cosh(L \tau) + \sinh(L \tau))(\frac{2 \sinh(L \tau)}{\mu} E_\nu E_\lambda - \lambda E_\lambda - \nu E_\nu)}{\nu^2 + \lambda^2 - 2 \cosh(L \tau) \nu \lambda + \frac{4}{\mu} \sinh^2(L \tau)}
$$

(25)
It is worth noting that, for real $\tau$, the expressions (24) and (25) are never singular for any $\lambda$ and $\nu$, as they should.

Also, note that this quantity is of order one, rather than $1/N^2$. This is a result of the fact that this observable is tree-like, whereas the previous two, which we considered, had loops (cf. the discussion after the equation (9)). The gauge invariant $\phi - \phi$ correlator, which is a flux tube with the field $\Phi$ at the edges, can be extracted from this result by taking the leading, order of $1/\hat{\lambda}^2$, $1/\hat{\nu}^2$, asymptotics of (25) as

$$\langle tr \phi \prod_{\Gamma} U \phi \prod_{-\Gamma} U^\dagger \rangle = \frac{1}{\mu} e^{-L\tau}$$

We see, that it has very simple form, with $\tau$ being the correlation length and agrees qualitatively with our interpretation of the result (19). When the length $L$ equals zero, so that the flux tube vanishes shrinking to a point, the expression (26) reproduces the result for the average of $tr\Phi^2$.

4 Conclusion

The effective field theory for the eigenvalues of the scalar field $\Phi$ is classical in the large $N$ limit, and the classical configurations are obtained by solving the saddle point equation. It is a remarkable feature of the Gaussian Kazakov-Migdal model that, in addition to determining the classical scalar field, one can take the fluctuations of the gauge fields into account exactly.

From the trivial counting of powers of $N$ we find that, with their conventional normalization, the only operators with non-zero expectation value in the limit $N \to \infty$ are those of tree-like configurations of tubes of glue with powers of the scalar fields inserted at the ends of the branches.

While the expectation value of the Wilson loops with non-zero area is always zero in this model, the expectation value of the adjoint Wilson loop is proportional to $1/N^2$ and obeys the perimeter law, with the correlation length equal to $1/\tau$, which is related to the bare scalar mass $m^2 - 2D$ and the coordination number of the lattice $2D$ according to (21),(22). In the continuum limit (which is possible only in $D = 1$), this corresponds to non-interacting mesons with the mass equal to $\tau$.

For a flux tube with scalar field at each end, the expectation value is proportional to the exponential of the length of the tube divided by the
correlation length, which is again given by $1/\tau$. We also obtained the general formula for the expectation value of the flux tube with arbitrary powers of the scalar fields at each end.

**APPENDIX A**

This appendix contains some formulas, relevant to the proof of eq. (17) as well as for other calculations with semi-circle distributions.

We repeat the definition of the “normalized” generating functional for the averages $\langle \text{Tr} \Phi^k \rangle$

$$\hat{E}_\gamma \equiv \int \hat{\rho}(\hat{\beta}) d\hat{\beta} = \hat{\gamma} - \sqrt{\hat{\gamma}^2 - 1} = \frac{1}{2\hat{\gamma}} \left( 1 + \sum_{k \geq 1} \frac{(2k - 1)!!}{(k + 1)!} \left( \frac{1}{2\hat{\gamma}^2} \right)^k \right),$$

$$\hat{E}_\gamma = \cosh \theta - \sinh \theta,$$

where $\hat{\gamma} \equiv \cosh \theta$.

Let us rewrite the correlators (8) for the “normalized” spectral density in the form

$$\hat{C}_{\hat{\alpha} \hat{\beta}} = \frac{\sinh \tau (\cosh \tau \pm \sinh \tau)}{\hat{\alpha}^2 + \hat{\beta}^2 - 2\hat{\alpha}\hat{\beta} \cosh \tau \pm \sinh^2 \tau} = \frac{\sinh \tau (\cosh \tau + \sinh \tau)}{(\beta - \alpha_+)(\beta - \alpha_-)},$$

(27)

where

$$\alpha_\pm = \hat{\alpha} \cosh \tau \pm \sqrt{\hat{\alpha}^2 - 1} \sinh \tau = \cosh(\tau \pm t), \quad \hat{\alpha} \equiv \cosh t$$

(28)

From (27) and (28) we get

$$E_{\alpha_\pm} = \cosh(\tau \pm t) - \sinh(\tau \pm t)$$

(29)

(here the choice of the signs corresponds to that of Eq.(23))

The first application of this convenient parametrization is to check explicitly the normalization condition (3) (we denote $\alpha \equiv \cosh \theta$)

$$\int d\beta \rho(\beta) C_{\alpha\beta} = 2 \int d\hat{\beta} \hat{\rho}(\hat{\beta}) C_{\hat{\alpha}\hat{\beta}} = \sinh \tau (\cosh \tau + \sinh \tau) \frac{E_{\alpha_+} - E_{\alpha_-}}{\alpha_- - \alpha_+} =$$

$$= \sinh \tau (\cosh \tau + \sinh \tau) \frac{\cosh(\tau + \theta) - \sinh(\tau + \theta) - \cosh(\tau - \theta) + \sinh(\tau - \theta)}{\cosh(\tau - \theta) - \cosh(\tau + \theta)} = 1$$

Note that since $\hat{\alpha}^2 \leq 1$, $t$ is complex
Let us now turn to the convolution of two $C$'s.

$$\int d\beta \rho(\beta) C_{\hat{\beta}_1 \hat{\beta}}(\tau) \hat{C}_{\hat{\beta}_2 \hat{\beta}}(\tau) = \frac{\sinh \tau_i (\cosh \tau_i - \sinh \tau_i)}{\beta_{+i} - \beta_{-i}} \frac{\sinh \tau_j (\cosh \tau_j - \sinh \tau_j)}{\beta_{+j} - \beta_{-j}}$$

$$\times \left\{ \frac{E_{\beta_{+i}} - E_{\beta_{+j}}}{\beta_{+i} - \beta_{+j}} + \frac{E_{\beta_{-i}} - E_{\beta_{-j}}}{\beta_{-i} - \beta_{-j}} - \frac{E_{\beta_{+i}} - E_{\beta_{-i}}}{\beta_{-i} - \beta_{+j}} - \frac{E_{\beta_{+j}} - E_{\beta_{-j}}}{\beta_{+i} - \beta_{-j}} \right\}$$

$$= \frac{\sinh \tau_i (\cosh \tau_i - \sinh \tau_i)}{\beta_{+i} - \beta_{-i}} \frac{\sinh \tau_j (\cosh \tau_j - \sinh \tau_j)}{\beta_{+j} - \beta_{-j}} \left\{ \frac{\coth \tau_i + \tau_j + \theta_1 + \theta_2}{2} + \frac{\coth \tau_i + \tau_j - \theta_1 - \theta_2}{2} - \frac{\coth \tau_i + \tau_j - \theta_1 + \theta_2}{2} - \frac{\coth \tau_i + \tau_j + \theta_1 - \theta_2}{2} \right\}$$

$$= \frac{\sinh(\tau_1 + \tau_2)(\cosh(\tau_1 + \tau_2) + \sinh(\tau_1 + \tau_2))}{\beta_{1}^2 + \beta_{2}^2 - 2\beta_1 \beta_2 \cosh(\tau_1 + \tau_2) + \sinh^2(\tau_1 + \tau_2)}$$

which proves Eq.(17).

**APPENDIX B**

**DMS-Matytsin theory of the $N = \infty$ Itzykson-Zuber integrals**

This appendix contains description of the currently available indirect methods to evaluate the (logarithm of the) Itzykson-Zuber integral $\hat{C}(0|\rho, \bar{\rho})$ and the first Itzykson-Zuber correlator $\hat{C}_{\alpha\beta}(1|\rho, \bar{\rho})$ for given eigenvalue densities $\rho(\phi)d\phi$ and $\bar{\rho}(\bar{\phi})d\bar{\phi}$.

As it often happens in the field theory, evaluation of the correlator $\hat{C}_{\alpha\beta}$ is somewhat simpler than that of the “free energy” $\hat{C}$, and the methods, used in the two cases, though similar in many respects, are still complementary rather than identical.

**Evaluation of $\hat{C}_{\alpha\beta}$**

Despite it produces the answers for individual (one-link) Itzykson-Zuber correlators the method of [4] actually works within the framework of the KMM and makes use of the approach of the loop equations.

---

6 In order to avoid confusion we emphasize that “bar” does not mean complex conjugation, $\rho(\phi)$ and $\bar{\rho}(\bar{\phi})$ are just independent functions. Complex conjugation will be denoted by “*” in what follows.
The main object of this approach is the following generating function \[ G_{\alpha\beta} = \frac{\langle \text{Tr} \frac{1}{\alpha I - \Phi} U \frac{1}{\beta I - \Phi} U^\dagger \rangle}{\langle 1 \rangle} = \int d\gamma \rho(\gamma) d\bar{\gamma} \bar{\rho}(\bar{\gamma}) \frac{C_{\gamma\bar{\gamma}}}{(\alpha - \gamma)(\beta - \bar{\gamma})}. \] (30)

It satisfies the following loop equation \[ (E_\alpha + \beta) G_{\alpha\beta} - E_\alpha \overset{\text{le}}{=} \oint_{2\pi i} d\omega L(\omega) G_{\omega\beta} \equiv L(\alpha) G_{\alpha\beta} - R_\beta(\alpha); \]

\[ (\bar{E}_\beta + \alpha) G_{\alpha\beta} - \bar{E}_\beta \overset{\text{le}}{=} \oint_{2\pi i} d\bar{\omega} \bar{L}(\bar{\omega}) G_{\alpha\bar{\omega}} \equiv \bar{L}(\beta) G_{\alpha\beta} - \bar{R}_\alpha(\beta) \] (31)

where

\[ E_\alpha = \langle \text{Tr} \frac{1}{\alpha I - \Phi} \rangle/\langle 1 \rangle, \quad \bar{E}_\beta = \langle \text{Tr} \frac{1}{\beta I - \Phi} \rangle/\langle 1 \rangle, \]

so that Disc\(_\alpha\)E\(_\alpha\) = 2πiρ(β), while Disc\(_\beta\)E\(_\beta\) = 2πi\bar{\rho}(\beta). Functions L(\omega) and \(\bar{L}(\bar{\omega})\) are some functions. On the support of the corresponding spectral density, these two functions are sums of the derivative of the logarithm of the Itzykson-Zuber integral and the real part of the generating function E(\(\bar{E}\)). Both L \(\text{and} \ \bar{L}\) are analytic in the vicinity of the supports of \(\rho \text{ and} \ \bar{\rho}\) and in principle can have cuts and singularities elsewhere. For the sake of simplicity we assume below that these singularities are located at \(\omega = \infty\) and \(\bar{\omega} = \infty\), and neither L nor \(\bar{L}\) have cuts. These two functions are a very natural starting point for the DMS procedure. If one assumes an ansatz for L \(\text{and} \ \bar{L}\) \(\text{then} \ \text{the} \ \text{search} \ \text{for} \ \text{the} \ \text{corresponding} \ E_\alpha \ \text{and} \ C_{\alpha\beta} \ \text{can} \ \text{be} \ \text{reduced} \ \text{to} \ \text{an} \ \text{algebraic} \ \text{problem [4, 6].}\)

Here, however, we will offer another variation of DMS approach, which is closer to the Matytsin’s ideas of computing the Itzykson-Zuber integral and considers instead of L, \(\bar{L}\) their combination with \(E_\alpha, \bar{E}_\alpha\) \(\text{[33].}\)

The functions R\(_\beta\)(\(\alpha\)) \(\text{and} \ \bar{R}_\alpha(\beta)\), entering right hand sides of Eq. (31), are defined as the contributions to the integrals \(\text{[31]}\) from the residues at infinity and possible singularities of the functions L \(\text{and} \ \bar{L}\). According to its definition R\(_\beta\)(\(\alpha\)) has no discontinuities as a function of \(\alpha\), Disc\(_\alpha\)R\(_\beta\)(\(\alpha\)) = 0.

\(^7\text{Note that the second scalar factor contains } \Phi.\)
Moreover, as a function of $\alpha$ it has its singularities only at those of $L(\alpha)$ (which are actually singularities of $g_R(\alpha) - \text{see eq.}(33)$). As a function of $\beta$, $R_\beta(\alpha)$ is analytic outside the support of $\bar{\rho}$, where its imaginary part has a finite jump. Under our simplifying assumption this means that $R_\beta(\alpha)$ is a polynomial in positive powers of $\alpha$ with $\beta$-dependent, everywhere finite, coefficients.

From (31) it follows that

$$G_{\alpha\beta} = \frac{E_\alpha - R_\beta(\alpha)}{\beta - g_R(\alpha)} = \frac{\bar{E}_\beta - \bar{R}_\alpha(\beta)}{\alpha - g_L(\beta)}, \quad (32)$$

where

$$g_R(\alpha) = -E_\alpha + L(\alpha), \quad g_L(\beta) = -\bar{E}_\beta + \bar{L}(\beta). \quad (33)$$

According to their definitions $E_\alpha \sim \frac{1}{\alpha}$, while $G_{\alpha\beta} \sim \frac{\bar{E}_\beta}{\alpha}$ as $\alpha \to \infty$.

Then $C_{\alpha\beta}$ is nothing but a double discontinuity of $G_{\alpha\beta}$:

$$C_{\alpha\beta} = -\frac{\text{Disc}_\alpha \text{Disc}_\beta}{2\pi i \rho(\alpha) 2\pi i \bar{\rho}(\beta)} G_{\alpha\beta} = -\frac{\text{Disc}_\beta R_\beta(\alpha) / 2\pi i \bar{\rho}(\beta)}{(\beta - g_R(\alpha))(\beta - g^*_R(\alpha))}, \quad (34)$$

where we have defined on the cut

$$g_R(\alpha) = L(\alpha) - \frac{1}{2} \bar{V}'(\alpha) + i\pi \rho(\alpha), \quad \frac{1}{2} \bar{V}'(\alpha) \equiv \text{Re}E_\alpha, \quad (35)$$

and "*" denotes complex conjugate. Alternative expression is

$$C_{\alpha\beta} = -\frac{\text{Disc}_\alpha \bar{R}_\alpha(\beta) / 2\pi i \rho(\alpha)}{(\alpha - g_L(\beta))(\alpha - g^*_L(\beta))}. \quad (36)$$

Assume that the correlator $C_{\alpha\beta}$ can be naively analytically continued (i.e. just by using instead of $\alpha$ and $\beta$ two arbitrary complex numbers) from the cut. Since for any given $\alpha$ the position of the poles of the expressions (34) and (36) should coincide we find that

$$g_L(g^*_R(\alpha)) = \alpha, \quad g_R(g^*_L(\beta)) = \beta. \quad (37)$$

\textsuperscript{8}If $L(\alpha)$ has extra singularities at some point $\nu$, $R_\beta(\alpha)$ will also contain a contribution, which is polynomial in $(\alpha - \nu)^{-1}$. This correction is easy to be accounted for in our reasoning below.
Thus we recover the functional form of the Master Field Equation in the Matytsin’s approach [14].

It remains to define the quantity in the numerator in (34). In order to do this let us return to eq. (32) and rewrite it as

\[ \frac{\text{Disc}_\beta R_\beta(\alpha)}{2\pi i \bar{\rho}(\beta)} = (\beta - g_R(\alpha)) \frac{\text{Disc}_\beta G_{\alpha\beta}}{2\pi i \bar{\rho}(\beta)} = \int \rho(\gamma) d\gamma C_{\gamma\beta} \left[ \frac{\beta - g_R(\alpha)}{\alpha - \gamma} \right]. \tag{38} \]

Now it is time to recall that \( R_\beta(\alpha) \) and thus \( \text{Disc}_\beta R_\beta(\alpha) \) is a polynomial in positive powers of \( \alpha \) (provided \( g_R \) and \( L \) are singular only at infinity). This allows to substitute \( \left[ \frac{\beta - g_R(\alpha)}{\alpha - \gamma} \right] \) at the r.h.s. by the part of its asymptotics in \( \alpha \), containing only its positive powers

\[ \left[ \frac{\beta - g_R(\alpha)}{\alpha - \gamma} \right]_+ = - \left[ \frac{g_R(\alpha)}{\alpha - \gamma} \right]_+ \equiv - \sum_{k,l \geq 0} \alpha^k \gamma^l \sigma_{kl} \tag{39} \]

i.e. by a finite polynomial in \( \alpha \) and \( \gamma \) with coefficients \( \sigma_{kl} = \sigma_{kl}(g_R) \), totally defined by the shape of the function \( g_R(\alpha) \).

Thus

\[ \frac{\text{Disc}_\beta R_\beta(\alpha)}{2\pi i \bar{\rho}(\beta)} = - \sum_{k,l \geq 0} \alpha^k \sigma_{kl} M_l(\beta), \tag{40} \]

where

\[ M_l(\beta) = \int \gamma^l \rho(\gamma) d\gamma C_{\gamma\beta}. \tag{41} \]

Note that it follows from the normalization condition for \( C_{\alpha\beta} \) that

\[ M_0(\beta) = 1 \tag{42} \]

Finally, from (34) we obtain:

\[ C_{\alpha\beta} = - \frac{\sum_{k,l \geq 0} \alpha^k M_l(\beta) \sigma_{kl}(g_R)}{(\beta - g_R(\alpha))(\beta - g_R^*(\alpha))}, \tag{43} \]

\[ \frac{1}{2\pi i} \text{Disc}_\beta G_{\alpha\beta} = \int \rho(\gamma) d\gamma C_{\gamma\beta} \bar{\rho}(\beta). \]

9 We use the fact, following from the definition (32), that

\[ \frac{1}{2\pi i} \text{Disc}_\beta G_{\alpha\beta} = \int \rho(\gamma) d\gamma C_{\gamma\beta} \bar{\rho}(\beta). \]
and \( M_l(\beta) \) in this formula can be obtained from solution of a finite system of equations, which arises after (13) is resubstituted into (11) with the condition (12).

This provides a complete solution for the problem of evaluation of \( C_{\alpha\beta} \), provided one starts from any adequate (i.e. satisfying (37)) pairs of functions \( g_{L,R}(\alpha) \). We did not prove here that any such pair of functions \( g_{L,R}(\alpha) \) provides an answer: we rather proved that any answer has this form with some \( g_{L,R}(\alpha) \). It is clear, however, that further restrictions on the shape of \( g_R \) should not be imposed: otherwise it would be impossible to suite arbitrary densities \( \rho \) and \( \bar{\rho} \). Therefore the algorithm is obliged to work.

Unfortunately, in all but the simplest cases the analytic structure of \( L(\alpha) \) and/or \( E_\alpha \) is very complicated, which hinders any practical calculations. One can see it by following. Take any function \( g^*_L \), which has negative imaginary part when its argument belongs to some real interval (support of \( \rho \)). Then consider the inverse of this function and call it \( g_R \) (it must have an interval on the real axis, the support of \( \rho \), where its imaginary part is positive). In most cases the analytic structure of \( g_R \) turns out very involved.

For illustrative purposes we turn to the bi-semi-circle example, where both \( \rho \) and \( \bar{\rho} \) are semi-circle. We start from the pair of functions

\[
\begin{align*}
g_L(\alpha) &= \sqrt{\mu^2 + 4\mu/\bar{\mu} + \mu^2 - \bar{\mu}} \\
g_R(\beta) &= \sqrt{\bar{\mu}^2 + 4\bar{\mu}/\mu + \bar{\mu}^2 - \mu}
\end{align*}
\tag{44}
\]

First let us find the coefficients \( \sigma_{kl} \) entering the power expansion (39):

\[
\sigma_{00} = -\frac{1}{2}(\sqrt{\mu^2 + 4\mu/\bar{\mu}} + \mu), \quad \sigma_{k>0,l>0} = 0
\tag{45}
\]

The general result for the correlator (13) then takes the form

\[
C_{\alpha\beta}(1|\rho_\mu, \rho_{\bar{\mu}}) = \frac{1}{2} \frac{\sqrt{\mu^2 \bar{\mu}^2 + 4\mu \bar{\mu} + \mu \bar{\mu}}}{\mu \alpha^2 + \bar{\mu} \beta^2 - \sqrt{\mu^2 \bar{\mu}^2 + 4\mu \bar{\mu} \alpha \beta + \mu \bar{\mu}}},
\tag{46}
\]

which coincides with the result of [4].

Knowledge of the correlator allows one in principle to compute the Itzykson-Zuber integral itself. To demonstrate that let us introduce an auxilary de-
dependence of $C(n)$ on some parameter $t$ in the following fashion

$$C_t(0) = \frac{1}{N^2} \log \int [dU] e^{N \text{Tr} \Phi U \Phi U^\dagger}.$$  (47)

Then for its derivative we find

$$t \frac{\partial C_t(0)}{\partial t} = \frac{\int [dU] \frac{t}{N} \text{Tr} \Phi U \Phi U^\dagger e^{N \text{Tr} \Phi U \Phi U^\dagger}}{\langle 1 \rangle} = \int d\alpha \rho_t(\alpha) d\beta \bar{\rho}_t(\beta) \beta C_{\alpha \beta}(t).$$  (48)

As an example let us consider again the case of the bi-semi-circle distribution. The dependence on $t$ can be absorbed into the redefinition $\mu$ and/or $\bar{\mu}$. It is convenient to consider the symmetric redefinition

$$\mu \rightarrow \frac{\mu}{\sqrt{t}}, \quad \bar{\mu} \rightarrow \frac{\bar{\mu}}{\sqrt{t}}.$$  (49)

Then

$$t \frac{\partial C_t(0)}{\partial t} = \frac{2t^2}{\sqrt{\mu \bar{\mu}}(\sqrt{\mu \bar{\mu}} + 4t^2 + \sqrt{\mu \bar{\mu}})}$$  (50)

Integrating the last line one obtains the generalization (51) of the Gross’ result

$$C(0|\rho, \bar{\rho}) = \frac{\sqrt{\mu \bar{\mu}} - \sqrt{\mu \bar{\mu}}}{2\sqrt{\mu \bar{\mu}}} - \frac{1}{2} \log \frac{\sqrt{\mu \bar{\mu}} + \sqrt{\mu \bar{\mu}}}{2\sqrt{\mu \bar{\mu}}}$$  (51)

**Matytsin’s theory**

A more direct way of calculating the Itzykson-Zuber integral $C(0)$, which does not require the knowledge of the correlators $C(1)$, was developed recently in [14]. It has been proved that

$$C(0|\rho, \bar{\rho}) = S(\rho, \bar{\rho}) + \frac{1}{2} \int \rho(\phi) \phi^2 d\phi + \frac{1}{2} \int \bar{\rho}(\phi) \phi^2 d\phi - \frac{1}{2} \int \rho(\phi) \rho(\phi') \log(\phi - \phi') d\phi d\phi' - \frac{1}{2} \int \bar{\rho}(\phi) \bar{\rho}(\phi') \log(\phi - \phi') d\phi d\phi'.$$  (52)
Integrals are over supports of $\rho(\phi)$ and $\bar{\rho}(\phi)$ and

$$S(\rho, \bar{\rho}) = \frac{1}{2} \int_0^1 dt \int d\phi [\text{Im} f(\phi, t)] \left( [\text{Re} f(\phi, t)]^2 + \frac{1}{3} [\text{Im} f(\phi, t)]^2 \right).$$

(53)

The function $f(\phi, t)$ is a solution to the “Hopf equation”

$$\frac{\partial f}{\partial t} = \phi \frac{\partial f}{\partial \phi}$$

(54)

with the initial condition

$$f(\phi, t = 0) = g_R(\phi) - \phi.$$  

(55)

It is then a consequence of the Hopf evolution that

$$f(\phi, t = 1) = -g_L(\phi) + \phi.$$  

(56)

The function $f(\phi, t)$ is in fact determined from an algebraic equation, because the differential Hopf equation (54) is in fact explicitly integrable. Its generic solution is given in the parametric form:

$$\phi = \alpha + tF(\alpha),$$

$$f(\phi, t) = F(\alpha),$$

(57)

with arbitrary function $F(\alpha)$. The shape of $F(\alpha)$ is dictated by initial condition (55):

$$F(\phi) = f(\phi, t = 0) = g_R(\phi) - \phi.$$  

(58)

Given this function, one finds $\alpha(\phi, t)$ from the algebraic equation

$$\phi = \alpha(\phi, t) + tF(\alpha(\phi, t)).$$  

(59)

\[\text{It is a remarkable equation, the first one in the “quasiclassical KdV hierarchy”, which plays an important in the theory of topological Landau-Ginsburg models and Generalized Kontsevich model. In Itzykson-Zuber theory } 1/t \text{ plays the role of the coefficient in front of the action, } \frac{1}{4} \text{Tr} \Phi U \Phi U^\dagger. \text{ It is interesting to understand the role of other KdV times in this context. The “action” } S(\rho, \bar{\rho}) \text{ in (53) is not an action for Hopf equation - it is rather one of the (quasiclassical-)KdV Hamiltonians. We note in passing that the action for the Hopf equation is not unique: see ref.[16] (where it is refered to as Bateman equation) for discussion of the corresponding universality structure.}\]
and then obtains

\[ f(\phi, t) = F(\alpha(\phi, t)) = g_R(\alpha(\phi, t)) - \alpha(\phi, t). \]  \hspace{1cm} (60)

In order to make this (exhaustive) description a little bit more transparent we turn to the semi-circle case.

\[ \rho(\alpha) = \bar{\rho}(\alpha) = \rho(\alpha) = \frac{1}{\pi} \sqrt{\mu - \frac{\mu^2 \alpha^2}{4}}; \]
\[ g_R(\alpha) = \alpha \sqrt{\frac{\mu^2 + 4}{2}} + i\pi\rho(\alpha) = \sqrt{\mu} \left( \frac{\alpha \cosh \tau}{\sinh \tau} + i\sqrt{1 - \alpha^2} \right), \]
\[ g_L(\beta) = \beta \sqrt{\frac{\mu^2 + 4}{2}} + i\pi\rho(\beta) = \sqrt{\mu} \left( \frac{\beta \cosh \tau}{\sinh \tau} - i\sqrt{1 - \beta^2} \right), \]
\[ \alpha = \frac{2}{\sqrt{\mu}} \hat{\alpha} = \sqrt{\mu} \frac{\hat{\alpha}}{\sinh \tau}, \quad \beta = \sqrt{\mu} \frac{\hat{\beta}}{\sinh \tau}. \]  \hspace{1cm} (61)

Then

\[ F(\phi) = f(\phi, t = 0) = g_R(\phi) - \phi = \sqrt{\mu} \left( \frac{\cosh \tau - 1}{\sinh \tau} \phi + i\sqrt{1 - \phi^2} \right). \]  \hspace{1cm} (62)

Given such \( F \) is actually a quadratic equation for \( \alpha(\phi, t) \), which is easily solved:

\[ \alpha(\phi, t) = \frac{\phi(1 + t(\cosh \tau - 1)) + i\sqrt{1 + 2t(1 - t)(\cosh \tau - 1)} - \phi^2}{1 + 2t(1 - t)(\cosh \tau - 1)}. \]  \hspace{1cm} (63)

and substituting this into (60) we get:

\[ f(\phi, t) = \sqrt{\mu} \frac{\cosh \tau - 1}{\sinh \tau} \phi + i\sqrt{1 + 2t(1 - t)(\cosh \tau - 1)} - \phi^2}{1 + 2t(1 - t)(\cosh \tau - 1)}. \]  \hspace{1cm} (64)

(One can now see that \( f(\phi, t = 1) = \sqrt{\mu} \left( -\frac{\cosh \tau - 1}{\sinh \tau} \phi + i\sqrt{1 - \phi^2} \right) \) is indeed equal to \(-g_L(\phi) + \phi\), in accordance with (56).)

Integral (53) for \( S(\rho_\mu, \rho_\mu) \) is now easy to evaluate and we get:

\[ S(\rho_\mu, \rho_\mu) = -\frac{\sqrt{\mu^2 + 4}}{2\mu} + \frac{1}{\mu} + \frac{1}{2} \log \frac{\sqrt{\mu^2 + 4} + \mu}{2}. \]  \hspace{1cm} (65)
To find the Van der Monde determinant in the case of the bi-semi-circle distribution, we first note that it is easy to calculate the following integral

\[
\int \int \rho_{s,\mu_1}(\alpha)\rho_{s,\mu_2}(\beta) \log(z\alpha - \beta)d\alpha d\beta = \\
\begin{cases}
    \frac{z^2}{\mu_1}, & z^2 \leq \frac{\mu_1}{\mu_2} \\
    \frac{1}{2}\left(1 + \log\frac{\mu_2}{\mu_1}\right), & z^2 \geq \frac{\mu_1}{\mu_2}
\end{cases}
\] (66)

Thus, the logarithm of the Van der Monde determinant for the semi-circle distribution is

\[
\int \int \rho_\mu(\phi)\rho_\mu(\phi') \log(\phi - \phi')d\phi d\phi' = -\frac{1}{2} \log \mu - \frac{1}{4}
\] (67)

Recalling that

\[
\int \rho_\mu(\phi)\phi^2 d\phi = \frac{1}{\mu}, \quad \int \bar{\rho}_\mu(\phi)\phi^2 d\phi = \frac{1}{\bar{\mu}}
\] (68)

we reproduce Gross's answer [5] for semi-circle distribution,

\[
C_\mu = \sqrt{\mu^2 + 4} - \mu - \frac{1}{2} \log \left(\sqrt{\mu^2 + 4} + \mu\right).
\] (69)

We also note in passing that the item \(S(\rho, \bar{\rho})\) at the r.h.s. of (52) can be distinguished from the other terms for the following reason. Exact answer for the Itzykson-Zuber integral is:

\[
\frac{1}{V_N} \int [dU] e^{Tr\Phi U\Phi U^\dagger} = \frac{\det_{ij} e^{\phi_i \bar{\phi}_j}}{\Delta(\phi)\Delta(\bar{\phi})} = \\
\frac{e^{\frac{1}{2}Tr\Phi^2 + \frac{1}{4}Tr\Phi^2}}{\Delta(\phi)\Delta(\bar{\phi})} \det_{ij} e^{-\frac{1}{2}(\phi_i - \bar{\phi}_j)^2}
\] (70)

It is the last determinant at the r.h.s. which corresponds to \(e^{S(\rho, \bar{\rho})}\) in the large \(N\) limit. This determinant tends to unity for very broad distributions when the average distance between adjacent eigenvalues is large. In the example of semi-circle distributions this corresponds to \(\mu, \bar{\mu} \to 0\) and indeed \(S(\rho, \bar{\rho}) \sim O(1)\), while the other terms are \(\sim O(1/\mu, \log \mu)\).
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