Existence of strong solutions to the steady Navier-Stokes equations for a compressible heat-conductive fluid with large forces

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Abstract

We prove that there exists a strong solution to the Dirichlet boundary value problem for the steady Navier-Stokes equations of a compressible heat-conductive fluid with large external forces in a bounded domain \( \Omega \subset \mathbb{R}^d \) \((d = 2, 3)\), provided that the Mach number is appropriately small. At the same time, the low Mach number limit is rigorously verified. The basic idea in the proof is to split the equations into two parts, one of which is similar to the steady incompressible Navier-Stokes equations with large forces, while another part corresponds to the steady compressible heat-conductive Navier-Stokes equations with small forces. The existence is then established by dealing with these two parts separately, establishing uniform in the Mach number a priori estimates and exploiting the known results on the steady incompressible Navier-Stokes equations.

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1 Introduction

This paper is mainly concerned with the existence of strong solutions to the steady Navier-Stokes equations of a compressible heat-conductive fluid in a bounded domain \( \Omega \subset \mathbb{R}^d \) \((d = 2, 3)\) with large external forces:

\[
\begin{align*}
\operatorname{div}(\rho u) &= 0, \\
\rho u \cdot \nabla u + \nabla p &= \operatorname{div} \mathbb{S}(\nabla u) + \rho f + g, \\
c_v \rho u \cdot \nabla \Theta + p \Delta u &= \kappa \Delta \Theta + \Psi. 
\end{align*}
\]

Here \( \rho \) denotes the density, \( u \in \mathbb{R}^d \) the velocity, \( \Theta \) the temperature, \( p = R \rho \Theta \) the pressure with \( R > 0 \) being the gas constant, \( c_v > 0 \) is the heat capacity at constant volume; \( f \) is the density of external body force and \( g \) is a given external force. The stress tensor \( \mathbb{S} \) and the dissipation function \( \Psi \) are defined by

\[
\mathbb{S} = 2\mu D(u) + \lambda \operatorname{div} u \mathbb{I}, \quad \Psi = 2\mu D(u) : D(u) + \lambda (\operatorname{div} u)^2 \geq 0,
\]

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\(D(u) = (\nabla u + \nabla u^t)/2\) is the deformation tensor. The viscosity coefficients \(\mu, \lambda\) satisfy \(2\mu + d\lambda \geq 0\) and \(\mu > 0, \kappa > 0\) is the heat conductivity coefficient. Moreover, the total mass is prescribed:

\[
\int_{\Omega} \rho dx = M > 0.
\]

We impose that the velocity \(u\) satisfies no-slip boundary condition and the temperature \(\Theta\) is a constant \(\vartheta_0\) on the boundary of \(\Omega\), i.e.,

\[
u = 0, \quad \Theta = \vartheta_0 \quad \text{on} \quad \partial \Omega. \quad (1.2)
\]

In the last decades, the steady compressible heat-conductive Navier-Stokes equations have been studied by many mathematicians and there are a lot of results on the existence in the literature, here we recall some of them for both small and large external forces which are related to our study in this paper, and we refer to the monograph \([20]\) for more details. When external forces are sufficiently small, Matsumura and Nishida in 1982/83 proved the existence of a solution with potential forces near a rest state \([14, 15]\), while Valli and Zajaczkowski \([23, 25]\) used the existence of global non-stationary solutions to get the existence of stationary solutions. Later, Valli \([24]\) showed the existence of stationary solutions in the general case by using an idea of Padula \([21]\) to decompose the equations into two parts that are governed by the Stokes equations and a transport equation, respectively. Beirão da Veiga \([1]\) obtained more general existence results in the \(L^p\)-setting by decomposing the equations into three parts that are governed by the Stokes equations, a transport equation and Laplace’s equation, respectively. In 1989, Farwig \([5]\) showed the existence of solutions to the steady compressible heat-conductive Navier-Stokes equations for small forces with slip boundary condition.

When external forces are of arbitrary size, the existence of strong solutions was proved in \([19, 16]\) for the case of potential forces. When the equations of state and the viscosity coefficients satisfy certain (growth) conditions, Novotný and Pokorný \([17, 18]\) showed that weak or strong solutions to the steady compressible heat-conductive Navier-Stokes equations exist. Unfortunately, their results exclude the case of ideal polytropic gases, for which the existence of strong solutions, to our best knowledge, still remains open.

The aim of the present paper is to establish the existence of strong solutions to the steady compressible heat-conductive Navier-Stokes system \((1.1)\) without any smallness assumption on the external forces \(f\) and \(g\), when the Mach number is small.

We mention that in the isentropic flow case, the existence of weak solutions or strong solutions under small Mach number for large external forces has been extensively investigated. Lions \([13]\) first proved the existence of weak solutions under the assumption that the specific heat ratio \(\gamma > 1\) in two dimensions and \(\gamma > 5/3\) in three dimensions. The restriction on \(\gamma\) actually comes from the integrability of the density \(\rho\) in \(L^p\), and in fact, the higher integrability of \(\rho\) has, the smaller \(\gamma\) can be allowed. In \([20]\) Novotný and Straškraba showed the existence of weak solutions for any \(\gamma > 3/2\) if \(f\) is potential and \(g = 0\). By deriving a new weighted estimate of the pressure, Frehse, Goj and Steinhauser \([6]\), Plotnikov and Sokolowski \([22]\) established an improved integrability for the density under the assumption of the \(L^1\)-boundedness of \(\rho u^2\) which was not shown to hold unfortunately. In 2008, Březina and Novotný \([8]\) was able to prove the existence of weak solution to the spatially periodic problem for any \(\gamma > (3 + \sqrt{41})/8\) when \(f\) is potential and \(g = 0\), or for any \(\gamma > (1 + \sqrt{13})/3 \approx 1.53\) when \(f, g \in L^\infty\), without assuming the \(L^1\)-boundedness of \(\rho u^2\), by combining the \(L^\infty\)-estimate of \(\Delta^{-1} P\) with the (usual) energy and density bounds. Then, in the framework of \([8]\), Frehse, Steinhauser and Weigant \([7, 8]\) established the existence of weak solutions to the Dirichlet boundary value problem for any \(\gamma > 4/3\) in three dimensions and to the spatially periodic or mixed boundary value problem for \(\gamma = 1\) (isothermal flow) in two dimensions. Recently, Jiang and Zhou \([10, 11]\) proved the existence of weak solutions to the spatially periodic or Dirichlet boundary value problem in \(\mathbb{R}^3\) for any \(\gamma > 1\). The existence of
strong solutions was shown by Choe and Jin [4] when the Mach number is small, by exploiting the known results for the incompressible steady Navier-Stokes equations.

Now, we rewrite (1.1) in the form of the Mach number. After scaling and a straightforward calculation we obtain the following dimensionless form of the steady full compressible Navier-Stokes equations:

\[
\begin{align*}
\text{div}(\rho u) &= 0, \\
\rho u \cdot \nabla u + \frac{\nabla p}{\epsilon^2} &= \text{div}(\nabla u) + \rho f + g, \\
\rho u \cdot \nabla \Theta + \nabla p &= \kappa \triangle \Theta + \epsilon^2 \Psi,
\end{align*}
\]  

(1.3)

where \(\epsilon\) is the Mach number.

Since the total mass of the fluid is given, we impose the condition

\[
\bar{\rho} := \frac{1}{|\Omega|} \int_{\Omega} \rho(x) dx > 0,
\]

which can be renormalized to \(\bar{\rho} = 1\) without loss of generality. Similarly, we also assume that \(\bar{\Theta} = 1\), \(R = c_v = 1\), \(\vartheta_0 = 1\).

To show the existence, we take the transformation

\[
\rho = 1 + \epsilon \rho, \quad \Theta = 1 + \epsilon \theta
\]  

(1.4)

to rewrite the system (1.3) in the form:

\[
\begin{align*}
\text{div}(u) + \epsilon \text{div}(\rho u) &= 0, \\
(1 + \epsilon \rho)(u \cdot \nabla u) + \frac{(1 + \epsilon \theta)\nabla \rho}{\epsilon} + \frac{(1 + \epsilon \rho)\nabla \theta}{\epsilon} &= \text{div}(\nabla u) + (1 + \epsilon \rho)f + g, \\
\epsilon(1 + \epsilon \rho)u \cdot \nabla \Theta + \text{div} u + (\epsilon \rho + \epsilon \theta + \epsilon^2 \rho \theta)\text{div} u &= \epsilon \kappa \triangle \Theta + \epsilon^2 \Psi,
\end{align*}
\]  

(1.5)

with boundary conditions

\[
u = 0, \quad \theta = 0 \quad \text{on } \partial \Omega.
\]  

(1.6)

Now, we state the main result of this paper.

**Theorem 1.1.** Let \(f, g \in H^2(\Omega)\). Then there is an \(\epsilon_0\) depending on \(\| (f, g) \|_{H^2}\) and \(\Omega\), such that for any \(\epsilon \in (0, \epsilon_0)\), there exists a solution \((\rho^\epsilon, u^\epsilon, \theta^\epsilon) \in \tilde{H}^2 \times (H^3 \cap H^1_0) \times (H^3 \cap H^1_0)\) to the boundary value problem (1.5), (1.6), satisfying

\[
\lim_{\epsilon \to 0} \inf_{U, P \in L} \| u^\epsilon - U \|_3 + \| \rho^\epsilon \|_2 + \| \theta^\epsilon \|_3 + \left\| \frac{\rho^\epsilon + \theta^\epsilon}{\epsilon} - P \right\|_2 = 0,
\]

where \((U, P) \in L := \{(U, P) \in (H^4 \cap H^1_0) \times \tilde{H}^3 \mid (U, P)\) is a solution of the incompressible steady Navier-Stokes equations (1.7) with external force \(f + g\), i.e.,

\[
\begin{align*}
U \cdot \nabla U - \mu \Delta U + \nabla P &= f + g, \\
\text{div} U &= 0, \\
U &= 0 \quad \text{on } \partial \Omega, \\
\int_{\Omega} P dx &= 0.
\end{align*}
\]  

(1.7)

**Remark 1.1.** If considering the existence of spatially periodic solutions to (1.5) in a periodic domain, we can also obtain an existence result similar to Theorem 1.1.

The system (1.5) is complicated mixed-type nonlinear equations containing such structures as elliptic and hyperbolic systems, for which the usual approach of the fixed point arguments used to prove the existence of classical solutions requires the smallness of data. To show Theorem 1.1 (the existence for large data), we split the system (1.5) into two parts, one of which is
similar to the steady incompressible Navier-Stokes equations with large force $f + g$, while another part corresponds to the steady compressible heat-conductive Navier-Stokes equations with small force $\mathbf{e} f$, provided the Mach number $\epsilon$ is small. Then, as noted in [4], we modify and combine elaborately the arguments in [2] where the existence of strong solutions to the incompressible Navier-Stokes equations for large forces was presented, and in [5] where strong solutions of the compressible viscous heat-conductive equations with small forces were dealt with, to establish Theorem 1.1.

Compared with the isentropic case studied in [4], due to presence of the energy equation, the main difficulties here lie in the existence of weak solutions to the approximate linearized system, dealing with the coupling terms between the velocity, density and temperature, and deriving the uniform estimates in a bounded domain, for example, how to control the energy norm $\|u - U\|_3 + \|\eta\|_2 + \|\theta\|_3$ uniformly in $\epsilon$ under the no-slip boundary condition. To circumvent such difficulties, we take the transform of $\varrho = 1 + \epsilon \rho, \Theta = 1 + \epsilon \theta$ for the system (1.3), instead of the transform $(\varrho = 1 + \epsilon^2 \rho, \Theta = 1 + \epsilon^2 \theta)$ used in [4], and utilize the lower order terms to control the higher order terms. More precisely, the main steps of the proof are the following: First, we apply Lax-Milgram’s theorem to get the existence of weak solutions to the regularized elliptic equations (2.14)–(2.16) of the transformed linearized equations. Then, we exploit the uniform estimates and a compactness argument to get the existence of a weak solution to the approximate linearized equations (2.10). Finally, we apply the Tikhonov fixed point theorem to obtain the existence of a strong solution.

The paper is organized as follows. In the next section, we will prove the existence of weak solutions and regularity to the linearized incompressible and compressible problems. Section 3 is devoted to establishing the existence for the nonlinear problem. Finally, the incompressible limit of solutions to the steady compressible heat-conductive Navier-Stokes equations is presented in Section 4.

Notations: We denote by $L^2$ the Lebesgue space $L^2(\Omega)$ with norm $\|\cdot\|_0$, by $H^m$ the Sobolev spaces $H^m(\Omega)$ with norm $\|\cdot\|_m$. Define the spaces

$$
\tilde{H}^m = \left\{ \rho \in H^m \mid \int_\Omega \rho(x) dx = 0 \right\}, \quad H^1_{0,\sigma} = \left\{ u \in H^1_0 \mid \text{div} u = 0 \right\}, \quad H^m_{0,\sigma} := H^m \cap H^1_{0,\sigma}.
$$

We denote by $H^{-1}$ the dual space of $H^1_0$ with the dual product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|_{-1} = \sup_{\|h\|_1 = 1} |\langle \cdot, h \rangle|$. We shall use the abbreviation:

$$
\int \cdot dx := \int_\Omega \cdot dx.
$$
2 Existence of solutions to the linearized problem

We first split the system (1.5) into two parts, so that one part looks like the incompressible Navier-Stokes equations, while the other part behaves like the compressible Navier-Stokes equations. More precisely, let \((U, P)\) and \((v, \eta)\) be the solutions to the following systems, respectively:

\[
\begin{align*}
U \cdot \nabla U + v \cdot \nabla U - \mu \Delta U + \nabla P &= f + g, \\
\text{div} U &= 0, \\
U &= 0 \quad \text{on} \; \partial \Omega \quad \text{and} \quad \int_\Omega P = 0; \\
\end{align*}
\]

and

\[
\begin{align*}
U \cdot \nabla \eta + \frac{\text{div} v}{\varepsilon} &= -v \cdot \nabla \eta - \eta \text{div} v - \varepsilon \text{div}(P(U + v)), \\
U \cdot \nabla v - \mu \Delta v - (\mu + \lambda)\nabla \text{div} v + \frac{\nabla \eta + \nabla \theta}{\varepsilon} &= \varepsilon F - v \cdot \nabla v - \theta \nabla \eta - \eta \nabla \theta, \\
U \cdot \nabla \theta - \kappa \Delta \theta + \frac{\text{div} v}{\varepsilon} &= \varepsilon G - v \cdot \nabla \theta - \eta \text{div} v - \theta \text{div} v, \\
v &= 0, \quad \theta = 0 \quad \text{on} \; \partial \Omega \quad \text{and} \quad \int_\Omega \eta = 0,
\end{align*}
\]

where the new force \(F\) and heat source \(G\) are defined by

\[
F = (\varepsilon P + \eta) f - (\varepsilon P + \eta)(U + v) \cdot \nabla (U + v) - \theta \nabla P - P \nabla \theta, \\
G = \Psi - (\varepsilon P + \eta)(U + v) \cdot \nabla \theta - (\varepsilon P + \eta) \theta \text{div} v - P \text{div} v.
\]

It is clear to observe that \(u := U + v, \rho := \varepsilon P + \eta\) and \(\theta\) are a solution to (1.5). Thus, we can obtain a solution of the system (1.5) if we can solve the systems (2.1) and (2.2). First, we will give the existence of weak solutions to the linearized incompressible problem (2.1) and derive a priori estimates of higher order derivatives of the unknowns \((U, P)\). Then, we shall show the existence of weak solutions to the linearized compressible problem (2.2) and establish uniform estimates of higher order derivatives of the unknowns \((\eta, v, \theta)\).

In what follows, we assume that \(\text{meas}(\Omega) = 1\) without loss of generality.

2.1 Linearized incompressible equations

Let \(\tilde{U}\) and \(\tilde{v}\) be given functions satisfying \(\tilde{U} \in H^4 \cap H^1_{0,\sigma}\) and \(\tilde{v} \in H^3 \cap H^1_0\). At first, we consider the linearized equations to (2.1) for given \(\tilde{U}\) and \(\tilde{v}\) as follows.

\[
\begin{align*}
(\tilde{U} + \tilde{v}) \cdot \nabla U - \mu \Delta U + \nabla P &= h, \\
\text{div} U &= 0, \\
U &= 0 \quad \text{on} \; \partial \Omega \quad \text{and} \quad \int_\Omega P = 0.
\end{align*}
\]

where \(h = f + g\).

The problem (2.3) is a Stokes problem which is solvable for arbitrarily large forces. In fact, (2.3) can be solved by using the Lax-Milgram theorem for small \(\tilde{v}\), and we can obtain the following existence result, the proof of which can be found, for example, in [4], and is therefore omitted here.

**Lemma 2.1.** Let \(h \in H^{-1}\). There exists a constant \(a_0\) depending on \(\mu\) and \(\Omega\), such that if \(\|\tilde{v}\|_3 < a_0\), then there exists a weak solution \((U, P) \in H^1_{0,\sigma} \times H^0\) of (2.3), satisfying

\[
\|U\|_1 \leq C_0\|h\|_{-1}, \quad \text{(2.4)}
\]
linearized system of (2.2) the existence of which will be shown below: Tikhonov fixed point theorem to show the existence of steady strong solutions to (1.5). To this derive some a priori estimates for solutions to the linearized equations of the system (2.2). For solution of (2.3) established in Section 2.1. Next, we give the existence of weak solutions and 2.2 Linearized compressible equations given ˜

\begin{align}
\|P\|_0 &\leq C_1\|h\|_{-1}(1 + \|h\|_{-1}), \\
(2.5)
\end{align}

where $C_0$ and $C_1$ are positive constants which depend only on $\Omega$, $\mu$ and $a_0$.

As for the regularity of solutions, we consider the Stokes equations:

$$
-\mu \Delta U + \nabla P = h - (\hat{U} + \hat{v}) \cdot \nabla U, \\
\text{div} U = 0.
$$

Then we can derive the following estimates by employing bootstrap arguments similar to those in [4].

**Lemma 2.2.** Let $h \in H^m$, $\hat{U} \in H^{m+1} \cap H^1_0$, $m = 0, 1, 2$, and $\hat{v}$ be the same as in Lemma 2.1. There are positive constants $C_2$, $C_3$ and $C_4$, depending only on $\Omega$, $\mu$ and $a_0$, such that if $\hat{U} \in H^1_0$ satisfies the inequality (2.4), then

$$
\|U\|_2 + \|\nabla P\|_0 \leq C_2\|h\|_0(\|h\|_0 + 1)^4.
$$

(2.6)

If $\hat{U} \in H^2 \cap H^1_0$ satisfies (2.6), then

$$
\|U\|_3 + \|\nabla P\|_1 \leq C_3\|h\|_1(\|h\|_1 + 1)^8,
$$

(2.7)

and if $\hat{U} \in H^3 \cap H^1_0$ satisfies (2.7), then

$$
\|U\|_4 + \|\nabla P\|_2 \leq C_4\|h\|_2(\|h\|_2 + 1)^{12}.
$$

(2.8)

Let $f, g \in H^2(\Omega)$, then it is obvious that $h \in H^2(\Omega)$. We define a function space $K_0$ by

$$
K_0 := \left\{ U \in H^2_{0,0}(\Omega) : \|U\|_1 \leq C_1\|h\|_1, \|U\|_2 \leq C_2\|h\|_0(\|h\|_0 + 1)^4, \right. \\
\left. \|U\|_3 \leq C_3\|h\|_1(\|h\|_1 + 1)^8, \|U\|_4 \leq C_4\|h\|_2(\|h\|_2 + 1)^{12} \right\}.
$$

(2.9)

Thus, by Lemma 2.2 we see that the solution $U$ of the system (2.3) also lies in $K_0$ for any given $\hat{U} \in K_0$, since the constants $C_1, \cdots, C_4$ do not depend on $\hat{U}$.

### 2.2 Linearized compressible equations

Let $(\hat{U}, \hat{v}, \hat{\theta}) \in (H^4 \cap H^1_0) \times (H^3 \cap H^1_0) \times (H^3 \cap H^1_0)$ be given functions, and $(U, P)$ be the solution of (2.3) established in Section 2.1. Next, we give the existence of weak solutions and derive some a priori estimates for solutions to the linearized equations of the system (2.2). For simplicity, we only consider the three-dimensional case. As aforementioned, we shall apply the Tikhonov fixed point theorem to show the existence of steady strong solutions to (1.5). To this end, for given $(\hat{v}, \hat{\theta}) \in (H^3 \cap H^1_0) \times (H^3 \cap H^1_0)$, let $(\eta, v, \theta)$ be the unique solution of the following linearized system of (2.2) the existence of which will be shown below:

$$
\begin{align}
U \cdot \nabla \eta + \frac{\text{div} v}{\epsilon} + \hat{v} \cdot \nabla \eta + \eta \text{div} \hat{v} &= -\epsilon \text{div}(P(U + \hat{v})), \\
U \cdot \nabla v - \mu \Delta v - \zeta \nabla \text{div} v + \frac{\nabla \eta + \nabla \theta}{\epsilon} + \hat{\theta} \nabla \eta + \eta \nabla \hat{\theta} &= \epsilon \tilde{F} - \hat{v} \cdot \nabla \hat{v}, \\
U \cdot \nabla \theta - \kappa \Delta \theta + \frac{\text{div} v}{\epsilon} + \eta \text{div} \hat{v} &= \epsilon \tilde{G} - \hat{v} \cdot \nabla \hat{\theta} - \hat{\theta} \text{div} \hat{v},
\end{align}
$$

(2.10)

where the new force $\tilde{F}$ and heat source $\tilde{G}$ are defined by

$$
\tilde{F} = (\epsilon P + \eta)f - (\epsilon P + \eta)(U + \hat{v}) \cdot \nabla(U + \hat{v}) - \hat{\theta} \nabla P - P \nabla \hat{\theta},
$$

$$
\tilde{G} = \tilde{\Psi} - (\epsilon P + \eta)(U + \hat{v}) \cdot \nabla \hat{\theta} - (\epsilon P + \eta) \hat{\theta} \text{div} \hat{v} - P \text{div} \hat{v},
$$

(2.11)

where $\tilde{\Psi}$ is a function depending only on $\hat{U}$.
2.2.1 Existence of weak solutions

Thus, for given $\tilde{U}, \tilde{v}$ and $\tilde{\theta}$, we can construct a map $N$:

$$N(\tilde{U}, \tilde{v}, \tilde{\theta}) := (U, v, \theta).$$

And, we have to show that $N$ maps some space into itself and is weak continuous to get a fixed point of the mapping $N$.

In order to obtain the existence of weak solution of (2.10), we set

$$\tilde{F}' = \epsilon Pf - \epsilon P(U + \tilde{v}) \cdot \nabla(U + \tilde{v}) - \tilde{\theta} \nabla P - \epsilon P \tilde{\theta} \text{div} \tilde{v} - P \text{div} \tilde{v},$$

$$\tilde{G}' = \tilde{\Psi} - \epsilon P(U + \tilde{v}) \cdot \nabla \tilde{\theta} - \epsilon P \tilde{\theta} \text{div} \tilde{v} - P \text{div} \tilde{v}.$$

So, $\tilde{F} = \tilde{F}' + \eta f + \eta(U + \tilde{v}) \cdot \nabla(U + \tilde{v})$, $\tilde{G} = \tilde{G}' - \eta(U + \tilde{v}) \cdot \nabla \tilde{\theta} - \eta \tilde{\theta} \text{div} \tilde{v}$.

2.2.1 Existence of weak solutions

Lemma 2.3. Let $\tilde{F}', \tilde{G}' \in H^{-1}$, $f \in H^2$, and $(U, P)$ be a solution of (2.3) established in Lemma 2.1. If $\|\tilde{v}\|_3 + \|\tilde{\theta}\|_3$ is sufficiently small, then there exists a unique weak solution $(\eta, v, \theta) \in H^0 \times H^2 \times H^1_0$ to the equations (2.10) with boundary condition (2.2).4.

Proof. First, the momentum equations (2.10)$_2$ can be rewritten as

$$\eta + \theta + \epsilon \nabla \tilde{\theta} + \epsilon \nabla (U \cdot \nabla v) - \epsilon (\mu + \zeta) \text{div} v + \epsilon^2 \Delta^{-1} \text{div} \left[\eta(U + \tilde{v}) \cdot \nabla(U + \tilde{v})\right]$$

$$= \epsilon \nabla \left(\epsilon (\tilde{F}' - \tilde{v} \cdot \nabla \tilde{\theta})\right).$$

We add (2.11) to (2.10)$_1$ and (2.10)$_3$, respectively, and rewrite the system (2.10) as the following equations:

$$\begin{cases}
U \cdot \nabla \eta + \frac{\text{div} \eta}{\epsilon} + \tilde{v} \cdot \nabla \eta + \eta \text{div} \tilde{v} + \eta + \theta + \epsilon \eta \tilde{\theta} + \epsilon \Delta^{-1} \text{div}(U \cdot \nabla v)
= \epsilon \text{div} \left(P(U + \tilde{v}) \cdot \nabla(U + \tilde{v})\right) + \epsilon \Delta^{-1} \text{div}(\epsilon \tilde{F}' - \tilde{v} \cdot \nabla \tilde{\theta}),

U \cdot \nabla \nabla \theta - \kappa \nabla \eta + \frac{\text{div} \eta}{\epsilon} + \eta \text{div} \tilde{v} + \epsilon \eta \tilde{\theta} \text{div} \tilde{v} + \eta + \theta + \epsilon \eta \tilde{\theta} + \epsilon \Delta^{-1} \text{div}(U \cdot \nabla v)
= \epsilon \text{div} \left(P(U + \tilde{v}) \cdot \nabla(U + \tilde{v})\right) + \epsilon \Delta^{-1} \text{div}(\epsilon \tilde{F}' - \tilde{v} \cdot \nabla \tilde{\theta}),

\end{cases}$$

with boundary conditions

$$v = \theta = 0 \quad \text{on} \quad \partial \Omega.$$ (2.13)

Consider the regularized elliptic equations of (2.12) as follows:

$$-\delta \Delta \eta^\delta + U \cdot \nabla \eta^\delta + \frac{\text{div} \eta^\delta}{\epsilon} + \tilde{v} \cdot \nabla \eta^\delta + \eta^\delta \text{div} \tilde{v} + \eta^\delta + \theta^\delta + \epsilon \eta^\delta \tilde{\theta} + \epsilon \Delta^{-1} \text{div}(U \cdot \nabla \eta^\delta)$$

$$= -\epsilon \text{div} \left(P(U + \tilde{v})\right) + \epsilon \Delta^{-1} \text{div}(\epsilon \tilde{F}' - \tilde{v} \cdot \nabla \tilde{\theta}),$$

(2.14)
\[ +\epsilon \eta^\delta (U + \tilde{v}) \cdot \nabla (U + \tilde{v}) = \epsilon \tilde{F}^\delta - \tilde{v}^\delta \cdot \nabla \tilde{v}, \quad (2.15) \]

\[
U \cdot \nabla \theta^\delta - \kappa \Delta \theta^\delta + \frac{\text{div} v^\delta}{\epsilon} + \eta^\delta \text{div} \tilde{v} + \epsilon \eta^\delta (U + \tilde{v}) \cdot \nabla \tilde{\theta} - \epsilon \eta^\delta \tilde{\theta} \text{div} \tilde{v} + \eta^\delta + \theta^\delta + \epsilon \eta^\delta \tilde{\theta} \nonumber \\
+ \epsilon \Delta^{-1} \text{div} (U \cdot \nabla \delta^\delta) - \epsilon (\mu + \zeta) \text{div} v^\delta + \epsilon^2 \Delta^{-1} \text{div} \left[ \eta^\delta (U + \tilde{v}) \cdot \nabla (U + \tilde{v}) \right] \\
- \epsilon \Delta^{-1} \text{div} (\eta^\delta f) = \epsilon \tilde{G}^\delta - \tilde{v}^\delta \cdot \nabla \tilde{\theta} - \tilde{\theta}^\delta \text{div} \tilde{v} + \epsilon \Delta^{-1} \text{div} (\epsilon \tilde{F}^\delta - \tilde{v} \cdot \nabla \tilde{v}), \quad (2.16) \]

with boundary conditions

\[
\frac{\partial \eta^\delta}{\partial n} = 0, \quad \tilde{v}^\delta = \theta^\delta = 0 \quad \text{on} \quad \partial \Omega. \quad (2.17) \]

Here \( n \) is the outer normal vector.

The system (2.14)–(2.17) in variational form reads as: Find \( (\eta^\delta, v^\delta, \theta^\delta) \in H^1 \times H^1_0 \times H^1_0 \), such that

\[
B(\eta^\delta, v^\delta, \theta^\delta; \eta, \nu, \theta) = \int [-\epsilon \text{div} (P(U + \tilde{v})) + \epsilon \Delta^{-1} \text{div} (\epsilon \tilde{F}^\delta - \tilde{v} \cdot \nabla \tilde{v})] \eta + (\epsilon \tilde{F}^\delta - \tilde{v} \cdot \nabla \tilde{v}) \nu + \epsilon \tilde{c} \cdot \nabla \delta^\delta \tilde{v} \theta \] \]

(2.18)

where

\[
B(\eta^\delta, v^\delta, \theta^\delta; \eta, \nu, \theta) := \delta \int \nabla \eta^\delta \cdot \nabla \nu dx - \int (U \cdot \nabla \eta) \eta^\delta + (U \cdot \nabla \nu) \cdot v^\delta + (U \cdot \nabla \theta) \theta^\delta dx \\
- \int (\tilde{v} \cdot \nabla \eta) \eta^\delta dx + \int (\eta^\delta + \theta^\delta) \eta + \mu \nabla \nu \cdot \nabla \zeta (\text{div} v^\delta) (\text{div} \nu) \\
+ \kappa \nabla \theta^\delta \cdot \nabla \theta dx + \int \left\{ \epsilon \eta^\delta \tilde{\theta} + \epsilon \Delta^{-1} \text{div} (U \cdot \nabla \delta^\delta) - \epsilon (\mu + \zeta) \text{div} v^\delta \\
- \epsilon^2 \Delta^{-1} \text{div} (\eta^\delta f) + \epsilon^2 \Delta^{-1} \text{div} \left[ \eta^\delta (U + \tilde{v}) \cdot \nabla (U + \tilde{v}) \right] \right\} \eta + \theta dx \\
+ \epsilon \int \left[ \left( \eta^\delta (U + \tilde{v}) \cdot \nabla (U + \tilde{v}) - \eta^\delta \tilde{\theta} \right) \cdot \nu - \eta^\delta \tilde{\theta} \text{div} v^\delta dx \\
+ \frac{1}{\epsilon} \int \left( \text{div} v^\delta \right) \eta - \nabla \nu (\eta^\delta + \theta^\delta) + \text{div} \theta^\delta dx, \quad (2.19) \]

for any \( (\eta, \nu, \theta) \in H^1 \times H^1_0 \times H^1_0 \).

We shall apply Lax-Milgram’s theorem to show the solvability of the variational problem (2.18). We first show the coerciveness of the bilinear form in (2.19). To this end, we take \( \eta^\delta = \eta, v^\delta = \nu, \theta^\delta = \theta \) in (2.19), integrate by parts and use Poincaré’s inequality \( \|\theta^\delta\|_0 \leq \sqrt{c_0} \|\nabla \theta^\delta\|_0 \) to see that

\[
B(\eta^\delta, v^\delta, \theta^\delta; \eta^\delta, v^\delta, \theta^\delta) \geq \int \left( \delta \|\nabla \eta^\delta\|^2 + (\eta^\delta + \theta^\delta)^2 + \mu \|\nabla v^\delta\|^2 + \zeta (\text{div} v^\delta)^2 + \kappa \|\nabla \theta^\delta\|^2 \right) dx - \int (\epsilon\|\theta\|_{L^\infty} + \|\eta^\delta\|_2) dx \\
- \epsilon \|U\|_2 \|\nabla v^\delta\|_0 - \|\tilde{\theta}\|_2 \|\eta^\delta\|_0 - \|\tilde{\theta}\|_2 \|\text{div} v^\delta\|_0 - \|\text{div} \nu\|_2 (\|\eta^\delta\|_2 + \|\theta^\delta\|_2) \]

8
where the constant $C'$ depends only on $\Omega$, $\mu$, $\lambda$, $\kappa$, $c_0$, $\|\tilde{\theta}\|_2$ and $\|\tilde{v}\|_3$. Notice that here the smallness of $\epsilon$, $\|\tilde{\theta}\|_2$ and $\|\tilde{v}\|_3$ are necessary. In fact, we have used the following estimate in the second inequality of (2.20):

$$
\int ((\eta^\delta + \theta^\delta)^2 + \frac{\kappa}{2} \nabla \theta^\delta)^2 dx \geq \int ((\eta^\delta + \theta^\delta)^2 + \frac{\kappa}{2c_0} \theta^\delta|^2) dx
$$

$$
= \int \left( (1 + \frac{\kappa}{2c_0})(\theta^\delta + \frac{2c_0}{2c_0 + \kappa} \eta^\delta)^2 + \frac{\kappa}{2c_0 + \kappa} \eta^\delta|^2 \right) dx
$$

$$
\geq \frac{\kappa}{2c_0 + \kappa} \|\eta^\delta\|_0^2.
$$

Obviously, the bilinear form in (2.19) is continuous on $\tilde{H}^1 \times H_0^1 \times H_0^1$. Thus, there is a unique solution $(\eta^\delta, \tilde{v}^\delta, \tilde{\theta}^\delta)$ of (2.14)–(2.17) satisfying the variation form (2.18) for all $(\eta, \tilde{v}, \tilde{\theta}) \in \tilde{H}^1 \times H_0^1 \times H_0^1$. Moreover, taking $\epsilon$ so small that $\epsilon < \frac{C'}{2}$, we obtain

$$
\delta \|\nabla \theta^\delta\|_0^2 + C'(\|\eta^\delta\|_0^2 + \|\tilde{v}^\delta\|_0^2 + \|\theta^\delta\|_0^2)
$$

$$
\leq \|\text{div}(P(U + \tilde{v})) + \epsilon \Delta^{-1}\text{div}(\epsilon \tilde{F}^\delta - \tilde{v} \cdot \nabla \tilde{v})\|_0^2 + \|\epsilon \tilde{F}^\delta - \tilde{v} \cdot \nabla \tilde{v}\|_0^2
$$

$$
\leq \|\epsilon \tilde{F}^\delta - \epsilon \tilde{v} \cdot \nabla \tilde{v} - \tilde{\theta}^\delta \text{div} \tilde{v} + \epsilon \Delta^{-1}\text{div}(\epsilon \tilde{F}^\delta - \tilde{v} \cdot \nabla \tilde{v})\|_0^2
$$

for some positive constant $C$.

By Rellich's compactness theorem, there is a subsequence $\eta^{\delta_n} \in \tilde{H}^0(\Omega)$, $\tilde{v}^{\delta_n} \in H_0^1(\Omega)$, $\tilde{\theta}^{\delta_n} \in H_0^1(\Omega)$, and $\eta \in \tilde{H}^0(\Omega)$, $\tilde{v} \in H_0^1(\Omega)$, $\tilde{\theta} \in H_0^1(\Omega)$ satisfying that

$$
\eta^{\delta_n} \text{ converges weakly to } \eta \text{ in } \tilde{H}^0(\Omega)
$$

$$
\tilde{v}^{\delta_n}, \tilde{\theta}^{\delta_n} \text{ converges weakly to } \tilde{v}, \tilde{\theta} \text{ in } H_0^1(\Omega)
$$

and

$$
\tilde{v}^{\delta_n}, \tilde{\theta}^{\delta_n} \text{ converges strongly to } \tilde{v}, \tilde{\theta} \text{ in } H_0^1(\Omega),
$$

respectively, as $n \to \infty$. And it can be easily validated that $(\eta, \tilde{v}, \tilde{\theta})$ is indeed the unique weak solution of (2.12). Due to the lower semicontinuity of the $\tilde{H}^m$-norm $(m = 0, 1)$ and the estimate

$$
\delta \|\nabla \eta^{\delta_n}\|_0^2 + (\|\eta^{\delta_n}\|_0^2 + \|\tilde{v}^{\delta_n}\|_0^2 + \|\tilde{\theta}^{\delta_n}\|_0^2)
$$

$$
\leq \|\epsilon \text{div}(P(U + \tilde{v})) + \epsilon \Delta^{-1}\text{div}(\epsilon \tilde{F}^\delta - \tilde{v} \cdot \nabla \tilde{v})\|_0^2 + \|\epsilon \tilde{F}^\delta - \tilde{v} \cdot \nabla \tilde{v}\|_0^2
$$

$$
\leq \|\epsilon \text{div}(P(U + \tilde{v})) + \epsilon \Delta^{-1}\text{div}(\epsilon \tilde{F}^\delta - \tilde{v} \cdot \nabla \tilde{v})\|_0^2 + \|\epsilon \tilde{F}^\delta - \tilde{v} \cdot \nabla \tilde{v}\|_0^2
$$

9
we find that
\[ ||v||_0^2 + ||v||_1^2 + ||\theta||_1^2 \]
\[ \leq ||\varepsilon \text{div}(\mathcal{P}(U + \bar{v})) + \varepsilon \text{div}(\mathcal{E}F - \bar{v} \cdot \nabla \bar{v})||_0^2 + \varepsilon \text{div}(\mathcal{E}F - \bar{v} \cdot \nabla \bar{v})||_0^2 \]
\[ + ||\varepsilon \mathcal{G} - \bar{v} \cdot \nabla \bar{\theta} - \bar{\theta} \text{div} \bar{v} + \varepsilon \Delta^{-1} \text{div}(\mathcal{E}F - \bar{v} \cdot \nabla \bar{v})||_0^2 \].

On the other hand, it is easy to verify that a weak solution of (2.12) is also a weak solution of (2.10). Moreover, a weak solution of (2.10) is unique. In fact, assuming that \((\eta_1, v_1, \theta_1), (\eta_2, v_2, \theta_2) \in \tilde{H}^0 \times H^1_0 \times H^1_0\) are two weak solutions to the problem (2.10) with boundary condition (2.24), and letting \(\bar{n} = \eta_1 - \eta_2, \bar{v} = v_1 - v_2, \bar{\theta} = \theta_1 - \theta_2\), we find that \((\bar{n}, \bar{v}, \bar{\theta})\) is a weak solution of the following boundary value problem:
\[
\begin{aligned}
U \cdot \nabla \eta + \frac{\text{div} v}{\varepsilon} + \bar{v} \cdot \nabla \bar{n} + \bar{n} \text{div} \bar{v} = 0,
U \cdot \nabla v - \mu \Delta v - \zeta \text{div} v + \frac{\nabla \eta + \nabla \bar{\theta}}{\varepsilon} + \bar{\theta} \nabla \eta + \eta \nabla \bar{\theta} = \varepsilon (\bar{n} f + \eta (U + \bar{v}) \cdot \nabla (U + \bar{v})),
U \cdot \nabla \bar{\theta} - \kappa \Delta \bar{n} + \frac{\text{div} v}{\varepsilon} + \frac{\text{div} \bar{n}}{\varepsilon} = \varepsilon (-\bar{n} (U + \bar{v}) \cdot \nabla \bar{\theta} - \bar{n} \text{div} \bar{v}),
\end{aligned}
\]
\[
\begin{array}{l}
\bar{v} = 0, \quad \bar{\theta} = 0 \text{ on } \partial \Omega, \quad \text{and } \int_{\Omega} \bar{n} = 0.
\end{array}
\]

Then, we test the equations (2.21) with \(\bar{n}, \bar{v}, \bar{\theta}\) respectively to deduce that
\[
\int \mu |\nabla v|^2 + \zeta |\nabla v|^2 dx + \int \kappa |\nabla \bar{\theta}|^2 dx
= \varepsilon \int \bar{n} v \left[ f + (U + \bar{v}) \cdot \nabla (U + \bar{v}) \right] - \bar{n} \bar{\theta} \left[ (U + \bar{v}) \cdot \nabla \bar{\theta} + \bar{n} \text{div} \bar{v} \right] dx
- \frac{1}{2} \text{div} \bar{v} |\bar{n}|^2 + \bar{n} \text{div} \bar{v} \bar{n} + \bar{n} \text{div} \bar{v} dx
\leq \left( ||\bar{n}||_0^2 + ||\bar{\theta}||_0^2 \right) \left( ||U||_2 + ||\bar{v}||_2 \right) ||\bar{n}||_3 + ||\bar{v}||_3 (1 + ||\bar{\theta}||_2)
+ \varepsilon C (||\bar{n}||_0^2 + ||\bar{v}||_3^2) (||f||_2 + ||U||_2 + ||\bar{v}||_2).
\]

And according to the inhomogeneous Stokes equations:
\[
\begin{aligned}
-\mu \Delta v + \frac{\nabla \eta + \nabla \bar{\theta}}{\varepsilon} &= -U \cdot \nabla v + \zeta \text{div} v + \varepsilon \left[ \bar{n} f + \eta (U + \bar{v}) \cdot \nabla (U + \bar{v}) \right] - (\bar{n} \nabla \eta + \eta \nabla \bar{\theta}),
\end{aligned}
\]
we have
\[
\varepsilon ||v||_0 + \varepsilon ||\eta||_0 + \varepsilon \text{div} v
\leq \varepsilon C \left( ||U||_2 ||v||_1 + ||\nabla v||_1 + ||\eta||_0 \left[ ||f||_2 + ||U||_2 + ||\bar{v}||_2 \right]^2 \right).
\]

By combining (2.22) and (2.23), making use of Poincaré’s inequality and the smallness of \(\varepsilon\), \(||v||_3\) and \(||\bar{\theta}||_3\), we see that
\[
||\eta||_0 + ||v||_1 + ||\eta||_0 \leq 0,
\]
which implies \(\eta_1 = \eta_2, v_1 = v_2, \theta_1 = \theta_2\). Hence, there is a unique weak solution \((\eta, v, \theta) \in \tilde{H}^0 \times H^1_0 \times H^1_0\) of the problem (2.10). \(\square\)
In order to get higher order uniform in $\epsilon$ estimates of the $(\eta, v, \theta)$, we give another bound about $\|v\|_1$ and $\|\theta\|_1$.

**Lemma 2.4.** Let $(U, P)$ be the solution of (2.3) given in Lemma 2.1. Let $\tilde{F}, \tilde{G} \in H^{-1}$. Then we have the following uniform in $\epsilon$ estimate:

\[
\begin{align*}
\|v\|_1^2 + \|\theta\|_1^2 & \leq C_5 \left[ \|\tilde{v}\|_2 \|\eta\|_0^2 + \epsilon^2 (\|\tilde{F}\|_2^2 + \|\tilde{G}\|_2^2) + \epsilon \|P\|_2 (\|U\|_2 + \|\tilde{v}\|_2) \|\eta\|_0 \\
& \quad + \|\tilde{v}\|_1^2 + \|\eta\|_1^2 \|\tilde{\theta}\|_1^2 + \|v\|_1^2 (\|\tilde{\theta}\|_1^2 + \|\eta\|_1^2) \right],
\end{align*}
\]

(2.25)

where the constant $C_5 > 0$ is independent of $\epsilon$.

**Proof.** Multiplying (2.10) by $\eta$, $v$ and $\theta$ in $L^2$ respectively, and summing up the resulting equations, we find that

\[
\frac{1}{2} \epsilon \int (\eta + \theta) \div v + v \cdot (\nabla \eta + \nabla \theta) \, dx - \int [(U + \tilde{v}) \cdot \nabla \eta + \eta^2 \div \tilde{v} + \epsilon \div (P(U + \tilde{v})\eta)] \, dx \\
+ \int (\epsilon \tilde{F} - \tilde{v} \cdot \nabla \tilde{v} - \tilde{\theta} \nabla \eta - \eta \nabla \tilde{\theta}) \cdot v \, dx + \int (\epsilon \tilde{G} - \tilde{v} \cdot \nabla \tilde{v} - (\eta + \tilde{\theta}) \div \tilde{v}) \theta \, dx \\
\leq \frac{1}{2} \|\tilde{v}\|_3 \|\eta\|_0 + \delta(\|v\|_1^2 + \|\theta\|_1^2) + C_{\delta} \epsilon^2 (\|\tilde{F}\|_2^2 + \|\tilde{G}\|_2^2) + \epsilon \|P\|_2 (\|U\|_2 + \|\tilde{v}\|_2) \|\eta\|_0 \\
+ \|\tilde{v}\|_1^2 + \|\eta\|_1^2 \|\tilde{\theta}\|_1^2 + \|\tilde{v}\|_1^2 (\|\tilde{\theta}\|_1^2 + \|\eta\|_1^2),
\]

(2.26)

where we have used integration by parts, Sobolev’s inequality and the fact that

\[
-\frac{1}{\epsilon} \int (\eta + \theta) \div v + v \cdot (\nabla \eta + \nabla \theta) \, dx = \frac{1}{\epsilon} \int [(\eta + \theta) \div v - (\eta + \theta) \div v] \, dx = 0.
\]

Finally, if we take $\delta$ in (2.26) suitably small and apply Poincaré’s inequality, we obtain the estimate (2.25). □

### 2.2.2 Stokes problem

We rewrite the momentum equations (2.10)$_2$ as an inhomogeneous Stokes problem to derive the desired bounds for $\|v\|_3$ and $\|\nabla (\eta + \theta)/\epsilon\|_1$:

\[
\begin{align*}
-\mu \Delta v + \nabla \eta + \nabla \theta & = \epsilon \tilde{F} - \tilde{v} \cdot \nabla \tilde{v} - \tilde{\theta} \nabla \eta - \eta \nabla \tilde{\theta} - U \cdot \nabla v + \zeta \nabla \div v, \\
\div v & = \div \tilde{v}, \\
v & = 0, \quad \text{on } \Omega.
\end{align*}
\]

(2.27)

By the usual estimates for the steady Stokes problem (cf. Galdi’s book [9] Chapter IV), Sobolev’s embedding $H^2 \hookrightarrow L^\infty$ and the inequality

\[
\|\div v\|_1^2 \leq \delta \|v\|_3^2 + C_{\delta} \|v\|_1^2,
\]

(2.28)

we have

\[
\begin{align*}
\|v\|_2 + \left\| \frac{\nabla \eta + \nabla \theta}{\epsilon} \right\|_0 & \leq C \left( \|\epsilon \tilde{F}\|_0 + \|\tilde{v} \cdot \nabla \tilde{v}\|_0 + \|\tilde{\theta} \nabla \eta\|_0 + \|\eta \nabla \tilde{\theta}\|_0 \\
& \quad + \|U \cdot \nabla v\|_0 + \|\div v\|_1 \right).
\end{align*}
\]

(2.29)
where
\[ C \]

which together with (2.25), (2.28) and (2.29) yields
\[
\|\mathbf{v}\|_3 + \left\| \frac{\nabla \eta + \nabla \theta}{\epsilon} \right\|_1 \leq C_6(1 + \|\mathbf{U}\|_2^2) \left\{ \epsilon(\|\tilde{F}\|_1 + \|\tilde{G}\|_1) + \|\tilde{v}\|_3^2 + \|\tilde{\theta}\|_3 \right\} + \|\nabla^2 \text{div} \mathbf{v}\|_0, \tag{2.30}
\]

where \( C_6 \) is a positive constant.

2.2.3 Estimate of \( \|\nabla^2 \text{div} \mathbf{v}\|_0 \)

As in \([23, 25]\), in order to control the term \( \|\nabla^2 \text{div} \mathbf{v}\|_0 \) we divide it into the interior part and the part near the boundary. We remark that here we have to carefully deal with the terms which involve with the large parameter \( 1/\epsilon \) in (2.10).

I. Interior estimate

First, we derive the interior estimate of \( \nabla^2 \text{div} \mathbf{v} \) by using the estimate (2.29). Let \( \chi_0 \) be a \( C_0^\infty \)-function, then we have

**Lemma 2.5.** *There is a positive constant \( C_7 \) independent of \( \epsilon \), such that*

\[
\begin{align*}
\mu \|\chi_0 \nabla^2 \mathbf{v}\|_0^2 + & \zeta \|\chi_0 \nabla \text{div} \mathbf{v}\|_0^2 + \kappa \|\chi_0 \nabla^2 \theta\|_0^2 \\
\leq & C_7 \left[ \|\mathbf{U}\|_3 + \|\nabla \mathbf{v}\|_3 \right]^2 + 2 \|\tilde{G}\|_0^2 + \|\tilde{F}\|_0^2 + \|\tilde{v}\|_2 \|\tilde{\theta}\|_2 \right] + C_7 \left[ \|\eta\|_2 \|\tilde{\theta}\|_2 + \|\mathbf{U}\|_3 \left( \|\nabla \eta\|_1^2 + \|\tilde{\theta}\|_1^2 + \|\tilde{v}\|_1^2 \right) + \|\nabla \mathbf{v}\|_1^2 \right] + \|\nabla \eta + \nabla \theta \|_0^2.
\end{align*}
\]

**Proof.** We differentiate (2.10) with respect to \( x \) to get that

\[
\begin{align*}
U^j \partial_{ij}^2 \eta + \frac{\partial_i \text{div} \mathbf{v}}{\epsilon} = & -\tilde{v} \partial_i \partial_{ij}^2 \eta - \partial_i (U^j + \tilde{v}^j) \partial_{ij} \eta - \partial_i (\eta \text{div} \mathbf{v}) - \epsilon \partial_i \text{div} (P \mathbf{U} + \tilde{v}), \\
U^j \partial_{ij}^2 \mathbf{v} + \partial_i U^j \partial_{ij} \mathbf{v} - \mu \partial_{ij}^2 \mathbf{v} - \zeta \partial_{ij}^2 \partial_{ij} \eta + \partial_i \text{div} \mathbf{v} + \frac{\partial_i \eta + \partial_i \theta}{\epsilon} = & \epsilon \partial_i \tilde{F}^k - \partial_i (\tilde{v} \partial_{ij} \partial_{ij} \tilde{v}^k) - \partial_{ik}^2 \tilde{\eta}, \\
U^j \partial_{ij} \eta + \partial_i U^j \partial_{ij} \eta - \kappa \partial_{ij} \theta + \frac{\partial_i \text{div} \mathbf{v}}{\epsilon} = & \epsilon \partial_i \tilde{G} - \partial_i (\tilde{v} \partial_{ij} \tilde{\theta}) - \partial_i (\eta \tilde{\theta}) \text{div} \mathbf{v}.
\end{align*}
\]

Multiplying (2.32) by \( \chi_0^2 \partial_i \eta, \chi_0^2 \partial_i \mathbf{v}^k \) and \( \chi_0^2 \partial_i \theta \) in \( L^2 \) respectively, and summing up the resulting equations, we find that

\[
\begin{align*}
\mu \|\chi_0 \partial_{ij}^2 \mathbf{v}^k\|_0^2 + & \zeta \|\chi_0 \partial_i \text{div} \mathbf{v}\|_0^2 + \kappa \|\chi_0 \partial_{ij}^2 \theta\|_0^2 \\
= & -\frac{1}{\epsilon} \int \chi_0^2 \partial_i \text{div} \mathbf{v} (\partial_i \eta + \partial_i \theta) + \chi_0^2 \partial_i \mathbf{v}^k \partial_k (\partial_i \eta + \partial_i \theta) dx \\
- & \int (2 \mu \chi_0 \partial_i \chi_0 \partial_{ij} \mathbf{v}^k \partial_i \mathbf{v}^k + 2 \zeta \chi_0 \partial_k \mathbf{v} \partial_i \text{div} \mathbf{v} \partial_i \mathbf{v}^k + 2 \kappa \chi_0 \partial_i \chi_0 \partial_{ij} \theta \partial_i \theta) dx \\
- & \int \chi_0^2 \left[ \partial_i (U^j + \tilde{v}^j) \partial_i \eta + (U^j + \tilde{v}^j) \partial_{ij}^2 \eta + \partial_i \eta \text{div} \mathbf{v} + \eta \partial_i \text{div} \mathbf{v} + \epsilon \partial_i \text{div} (P \mathbf{U} + \tilde{v}) \right] \partial_i \eta dx \\
+ & \int \chi_0^2 (\epsilon \partial_i \tilde{F}^k - \partial_i (\tilde{v} \cdot \nabla \mathbf{v}^k) - \partial_{ik} \tilde{\eta}) \partial_i \mathbf{v}^k dx \\
+ & \int \chi_0^2 (\epsilon \partial_i \tilde{G} - \partial_i (\tilde{v} \partial_{ij} \tilde{\theta}) - \partial_i ((\eta + \tilde{\theta}) \text{div} \mathbf{v})) \partial_i \theta dx.
\end{align*}
\]
If we apply partial integrations to the above identity, employ Sobolev’s and Young’s inequalities and the fact that
\[
- \frac{1}{\epsilon} \int \chi_0^2 \partial_t \text{div} v (\partial_t \eta + \partial_t \theta) + \chi_0^2 \partial_t v^k \partial_k (\partial_t \eta + \partial_t \theta) dx \\
= \frac{1}{\epsilon} \int 2 \chi_0 \partial_k \chi_0 \partial_t v^k (\partial_t \eta + \partial_t \theta) dx + \frac{1}{\epsilon} \int \chi_0^2 \partial_t v^k \partial_k (\partial_t \eta + \partial_t \theta) - \chi_0^2 \partial_t v^k \partial_k (\partial_t \eta + \partial_t \theta) dx \\
= \frac{1}{\epsilon} \int 2 \chi_0 \partial_k \chi_0 \partial_t v^k (\partial_t \eta + \partial_t \theta) dx \leq \delta \left\| \frac{\nabla \eta + \nabla \theta}{\epsilon} \right\|_0^2 + C_\delta \| v \|_1^2,
\]
we infer by summing up \(i, j, k\) that
\[
\mu \| \chi_0 \nabla^2 v \|_0^2 + \zeta \| \chi_0 \nabla^2 \text{div} v \|_0^2 + \kappa \| \chi_0 \nabla^2 \theta \|_0^2 \\
\leq (\| U \|_3 + \| \tilde{\nabla} \|_3) \| \eta \|_2^2 + \delta \left( \| v \|_2^2 + \| \theta \|_2^2 + \left\| \frac{\nabla \eta + \nabla \theta}{\epsilon} \right\|_0^2 \right) \\
+ \delta \epsilon \left[ \epsilon \left( \| U \|_3 + \| \tilde{\nabla} \|_3 \right) \| v \|_2^2 + \epsilon \| P \|_2 (\| U \|_2 + \| \tilde{\nabla} \|_2) \| \eta \|_1 + \| v \|_2^2 + \| \eta \|_2^2 \| \theta \|_2^2 \\
+ \| U \|_3 \left( \| v \|_2^2 + \| \theta \|_2^2 \right) + \| \tilde{\nabla} \|_2 (\| \tilde{\theta} \|_2^2 + \| \eta \|_2^2) + \| v \|_2^2 \right] + \delta \left\| \frac{\nabla \eta + \nabla \theta}{\epsilon} \right\|_1^2.
\]
which, by using Poincaré’s inequality and choosing \(\delta\) appropriately small, implies the lemma.

**Lemma 2.6.** There is a positive constant \(C_8\) independent of \(\epsilon\), such that
\[
\mu \| \chi_0 \nabla^3 v \|_0^2 + \zeta \| \chi_0 \nabla^2 \text{div} v \|_0^2 + \kappa \| \chi_0 \nabla^2 \theta \|_0^2 \\
\leq C_8 \left[ (\| U \|_3 + \| \tilde{\nabla} \|_3) \| \eta \|_2^2 + \epsilon \| P \|_3 (\| U \|_3 + \| \tilde{\nabla} \|_3) \| \eta \|_2 + \| \tilde{\nabla} \|_3 \| \eta \|_2^2 \\
+ \| \eta \|_2^2 \| \tilde{\theta} \|_2^2 + \| U \|_3 (\| \tilde{\theta} \|_2^2 + \| \eta \|_2^2) + \| v \|_2^2 \right] + \delta \left\| \frac{\nabla \eta + \nabla \theta}{\epsilon} \right\|_1^2.
\] (2.32)

**Proof.** We differentiate (2.10) twice with respect to \(x\) to get that
\[
\begin{align*}
U^j_i \partial^3_{ijl} \eta + \frac{\partial_i \text{div} v}{\epsilon} &= -\tilde{\partial}^3_{ijl} \eta - \partial^3_{i}(U^j + \tilde{v}^j) \partial_j \eta - \partial_i (U^j + \tilde{v}^j) \partial^3_{ijl} \\
&= \partial_i \left( U^j + \tilde{v}^j \right) \partial^2_{ijl} \eta - \partial_i (\eta \text{div} \tilde{v}) - \epsilon \partial^2_i \text{div} \left( P(U + \tilde{v}) \right), \\
U^j_i \partial^3_{ijl} v^k + \partial_i U^j \partial^3_{ijl} v^k + \partial^3_{i}(U^j + \tilde{v}^j) \partial_j v^k + \partial_i U^j \partial^2_{ijl} v^k - \mu \partial^3_{i} \eta \partial^3_{ijl} v^k - \zeta \partial^3_{i} \text{div} v + \frac{\partial^3_{i} \eta + \partial^3_{ijl} \theta}{\epsilon} \\
&= \epsilon \partial^2_{i} \tilde{\nabla} \tilde{v}^k - \partial^3_{i} \left( \tilde{v} \partial_j \tilde{\theta} \right) - \partial^3_{i} \left( \eta \partial_j \tilde{\theta} \right), \\
U^j_i \partial^3_{ijl} \theta + \partial_i U^j \partial^2_{ijl} \theta + \partial^2_i U^j \partial_j \theta + \partial_i U^j \partial^2_{ijl} \theta - \kappa \partial^3_{i} \theta + \frac{\partial^3_{ijl} \text{div} v}{\epsilon} \\
&= \epsilon \partial^2_{i} \tilde{\nabla} \tilde{\theta} - \partial^3_{i} \left( \tilde{v} \partial_j \tilde{\theta} \right) - \partial^3_{i} \left( \eta \partial_j \tilde{\theta} \right).
\end{align*}
\] (2.33)

Multiplying (2.33)\(_1\), (2.33)\(_2\) and (2.33)\(_3\) again by \(\chi_0^2 \partial_{ijl} \eta\), \(\chi_0^2 \partial_{ijl} v^k\) and \(\chi_0^2 \partial_{ijl} \theta\) respectively, and summing up the resulting equations, we deduce that
\[
\begin{align*}
\mu \| \chi_0 \partial^3_{ijl} v^k \|_0^2 + \zeta \| \chi_0 \partial^3_{ijl} \text{div} v \|_0^2 + \kappa \| \chi_0 \partial^3_{ijl} \theta \|_0^2 \\
= - \frac{1}{\epsilon} \int \chi_0^2 \partial_t \text{div} v (\partial_t \eta + \partial_t \theta) + \chi_0^2 \partial_t v^k \partial_k (\partial_t \eta + \partial_t \theta) dx \\
- \int 2 \chi_0 \partial_j \chi_0 (\mu \partial^3_{i} v^k \partial^3_{ijl} v^k + \kappa \partial^3_{i} \theta \partial^3_{ijl} \theta) + 2 \zeta \chi_0 \partial_k \chi_0 \partial_t \text{div} v \partial_t v^k dx \\
- \int \left[ U^j_i \partial^3_{ijl} \eta + \tilde{v}^j_i \partial^3_{ijl} \eta + \partial^3_{i} (U^j + \tilde{v}^j) \partial_j \eta + \partial_i (U^j + \tilde{v}^j) \partial^3_{ijl} \eta \\
+ \partial_i (U^j + \tilde{v}^j) \partial^2_{ijl} \eta + \partial^2_i (\eta \text{div} \tilde{v}) + \epsilon \partial^2_i \text{div} \left( P(U + \tilde{v}) \right) \right] \chi_0^2 \partial_{ijl} \eta dx
\end{align*}
\]
as follows. First, one construct the local coordinates by the isothermal coordinates boundary (also see [23, 25, 12]). For completeness, we briefly describe the local coordinates 

\[ \nabla \phi, \phi, \nabla \theta \] 

which, by employing Poincaré’s inequality and choosing \( \delta \) suitably small, gives the lemma. \( \square \)

**II. Boundary estimate**

Next, we shall use the method of local coordinates to bound \( \nabla^2 \text{div} \mathbf{v} \) in the vicinity of the boundary (also see \[ 23, 25, 12 \]). For completeness, we briefly describe the local coordinates as follows. First, one construct the local coordinates by the isothermal coordinates \( \lambda(\varphi, \phi) \) to derive an estimate near the boundary (see also \[ 23, 25 \]), where

\[ \lambda_\varphi \cdot \lambda_\varphi > 0, \quad \lambda_\phi \cdot \lambda_\phi > 0, \quad \lambda_\varphi \cdot \lambda_\phi = 0. \]

The boundary \( \partial \Omega \) can be covered by a finite number of bounded open sets \( W^k \subset \mathbb{R}^3, k = 1, 2, \cdots, L \), such that for any \( x \in W^k \cap \Omega \),

\[ x = \Lambda^k(\varphi, \phi, r) \equiv \lambda^k(\varphi, \phi) + r \mathbf{n}(\lambda^k(\varphi, \phi)), \]

where \( \lambda^k(\varphi, \phi) \) is the isothermal coordinates and \( \mathbf{n} \) is the unit outer normal to \( \partial \Omega \).

Without confusion, we will omit the superscript \( k \) in each \( W^k \) in the following. We construct the orthonormal system corresponding to the local coordinates by

\[ e_1 := \frac{\lambda_\varphi}{|\lambda_\varphi|}, \quad e_2 := \frac{\lambda_\phi}{|\lambda_\phi|}, \quad e_3 := e_1 \times e_2 \equiv \mathbf{n}(\lambda). \]

By a straightforward calculation, we see that for sufficiently small \( r \) and \( J \in C^2 \),

\[ J := \det \text{Jac} \Lambda = \det \frac{\partial x}{\partial (\varphi, \phi, r)} = \lambda_\varphi \times \lambda_\phi \cdot e_3 \]

\[ = |\lambda_\varphi| |\lambda_\phi| + r(|\lambda_\varphi| \mathbf{n}_\varphi \cdot e_2 + |\lambda_\phi| \mathbf{n}_\phi \cdot e_1) + r^2[(\mathbf{n}_\varphi \cdot e_1)(\mathbf{n}_\phi \cdot e_2) - (\mathbf{n}_\varphi \cdot e_2)(\mathbf{n}_\phi \cdot e_1)] > 0. \]
And, we can easily derive the following relations as \((\text{Jac} \Lambda^{-1}) \circ \Lambda = (\text{Jac} \Lambda)^{-1}\) (also see [23]):

\[
\begin{align*}
\nabla (\Lambda^{-1})^1 \circ \Lambda &= J^{-1}(\Lambda_\phi \times e_3), \\
\nabla (\Lambda^{-1})^2 \circ \Lambda &= J^{-1}(e_3 \times \Lambda_\phi), \\
\nabla (\Lambda^{-1})^3 \circ \Lambda &= J^{-1}(\Lambda_\phi \times \Lambda_\phi) = e_3,
\end{align*}
\]

where the symbol \(\circ\) stands for the composite of operators. Set \(y := (\varphi, \phi, r)\), and denote by \(D_i\) the partial derivative with respect to \(y_i\) in local coordinates. We set the unknowns in local coordinates

\[
\dot{\eta}(t, y) := \eta(t, \Lambda(y)), \quad \dot{\mathbf{v}}(t, y) := \mathbf{v}(t, \Lambda(y)), \quad \dot{\theta}(t, y) := \theta(t, \Lambda(y)),
\]

and the knowns

\[
\dot{U}(t, y) := U(t, \Lambda(y)), \quad \dot{\mathbf{v}}(t, y) := \dot{\mathbf{v}}(t, \Lambda(y)), \quad \dot{\theta}(t, y) := \dot{\theta}(t, \Lambda(y)).
\]

Then, we rewrite the system (2.10) in \([0, T] \times \tilde{\Omega}\), where \(\tilde{\Omega} := \Lambda^{-1}(W \cap \Omega)\), as follows.

\[
\begin{align*}
\dot{U}^j a_{kj} D_k \hat{\eta} + \frac{a_{kj} D_k \hat{v}^j}{\epsilon} &= -\dot{\eta} a_{kj} D_k \hat{\eta} - \eta a_{kj} D_k \hat{\eta} - \epsilon a_{kj} D_k (\hat{P}(\dot{U}^j + \hat{v}^j)), \\
\dot{U}^j a_{kj} D_k \hat{v}^j - \mu a_{kj} D_k (a_{ij} D_i \hat{v}^j) - \zeta a_{ki} D_k (a_{ij} D_i \hat{v}^j) + \frac{a_{kj} D_k (\hat{\eta} + \dot{\theta})}{\epsilon} &= \epsilon \hat{F} - \dot{\eta} a_{kj} D_k \hat{\theta} - \dot{\eta} a_{kj} D_k \hat{\theta} - \epsilon a_{kj} D_k \hat{\eta} - \dot{\eta} a_{kj} D_k \hat{\theta}, \\
\dot{U}^j a_{kj} D_k \hat{\theta} - \kappa a_{kj} D_k (a_{ij} D_i \hat{\theta}) + \frac{a_{kj} D_k \hat{v}^j}{\epsilon} &= \epsilon \hat{G} - \dot{\eta} a_{kj} D_k \hat{\theta} - \dot{\eta} a_{kj} D_k \hat{\theta} - \epsilon a_{kj} D_k \hat{\eta} - \dot{\eta} a_{kj} D_k \hat{\theta}
\end{align*}
\]

with boundary conditions

\[
\hat{\mathbf{v}}(t, y) = 0, \quad \dot{\theta}(t, y) = 0 \quad \text{on} \quad \partial \tilde{\Omega},
\]

where \(a_{ij}\) is the \((i, j)\)-th entry of the matrix \(\text{Jac}(\Lambda^{-1}) = \frac{\partial \eta}{\partial \mathbf{a}}\). Clearly, \(a_{ij}\) is a \(C^2\)-function, and it follows from (2.36)–(2.38) that

\[
\sum_{j=1}^{3} a_{3j} a_{3j} = |\mathbf{n}|^2 = 1, \quad \sum_{j=1}^{3} a_{1j} a_{3j} = \sum_{j=1}^{3} a_{2j} a_{3j} = 0.
\]

Moreover, this localized system has the following properties (see also [23]):

**Proposition 2.1.** \(D_i(Ja_{ij}) = 0\), for \(j = 1, 2, 3; \varsigma D_\tau \hat{\mathbf{v}} = 0, \varsigma D_\xi D_\tau \hat{\mathbf{v}} = 0\) on \(\partial \tilde{\Omega}\) \(\text{in the tangential directions} \, \tau, \xi = 1, 2,\) where \(\varsigma \in C^\infty_0(\Lambda^{-1}(W))\). Similarly, \(\varsigma D_\tau \dot{\theta} = 0, \varsigma D_\xi D_\tau \dot{\theta} = 0\) on \(\partial \tilde{\Omega}\).

Recalling \(D_j = \sum_{i=1}^{3} a_{ij} \partial_i\), we will frequently make use of the following relations without pointing out explicitly in subsequent calculations:

\[
\|D_y \hat{\mathbf{v}}\|_{L^p(\Omega)} \leq C \|\nabla_x v\|_{L^p(\Omega)}, \quad \|D_y^2 \hat{\mathbf{v}}\|_{L^p(\Omega)} \leq C \|\nabla_x v\|_{W^{1,p}(\Omega)}, \quad 1 \leq p \leq \infty.
\]

The above inequalities apply to \(\eta, \theta\) and \(U, \hat{\mathbf{v}}, \dot{\theta}\), too.

By virtue of the interpolation \(\|\cdot\|_{H^2} \leq \|\cdot\|_{H^3} + C_\delta \|\cdot\|_{H^1}\), the boundary estimate of \(\|\nabla^2 \text{div} u\|_{L^2(L^3)}\) can be reduced to the boundedness of

\[
\int_0^T \int_{\tilde{\Omega}} J \chi^2 |D_y^2 (a_{ji} D_j U^i)| dyds,
\]

where \(\chi\) is a \(C^\infty_0(\Lambda^{-1}(W))\)-function. So, we can split the estimate of derivatives on the boundary into two parts: the estimate of derivatives in the tangential directions and in the normal
direction.

Part 1. Estimate of derivatives in the tangential directions

First, we apply $D^2_{\tau\xi}$ to (2.39) with $\tau$, $\xi$ being the tangential directions to $\partial\tilde{\Omega}$ to get

$$
\begin{align*}
\hat{U}^j a_{kj} D^3_{k\tau\xi} \hat{\eta} + 1/D_{\tau\xi}[a_{kj} D_k \hat{v}^j] & = D_\xi (\hat{U}^j a_{kj} D^2_{k\tau\xi} \hat{\eta} + D^2_{\tau\xi}(\hat{U}^j a_{kj}) D_k \hat{\eta} + D_\tau (\hat{U}^j a_{kj}) D^2_{k\xi} \hat{\eta}) \\
& \quad - D^2_{\tau\xi}[\theta a_{kj} D_k \hat{v}^j + \eta a_{kj} D_k \hat{v}^j] + \epsilon a_{kj} D_k (\hat{P}(\hat{U}^j + \hat{v}^j)]),
\end{align*}
$$

$$
\hat{U}^j a_{kj} D^3_{k\tau\xi} \hat{\xi} - D^2_{\tau\xi}[\mu a_{kj} D_k (a_{lj} D_l \hat{v}^i) + \zeta a_{kj} D_k (a_{lj} D_l \hat{v}^j)] + 1/D_{\tau\xi}[a_{kj} D_k (\hat{\eta} + \hat{\theta})]
$$

(2.43)

$$
\hat{U}^j a_{kj} D^3_{k\tau\xi} \hat{\xi} - \kappa D^2_{\tau\xi}[a_{kj} D_k (a_{lj} D_l \hat{\theta})] + 1/D_{\tau\xi}[a_{kj} D_k \hat{v}^j]
$$

We multiply (2.43) 1, (2.43) 2 and (2.43) 3 by $J^2_{\chi} D^2_{\tau\xi} \hat{\eta}$, $J^2_{\chi} D^2_{\tau\xi} \hat{v}^j$ and $J^2_{\chi} D^2_{\tau\xi} \hat{\theta}$ respectively, and integrate the resulting identities to deduce that

$$
\begin{align*}
- \int_{\tilde{\Omega}} D^2_{\tau\xi} [\mu a_{kj} D_k (a_{lj} D_l \hat{v}^i) + \zeta a_{kj} D_k (a_{lj} D_l \hat{v}^j)] \cdot J^2_{\chi} D^2_{\tau\xi} \hat{v}^j \, dy \\
- \int_{\tilde{\Omega}} \kappa D^2_{\tau\xi}[a_{kj} D_k (a_{lj} D_l \hat{\theta})] \cdot J^2_{\chi} D^2_{\tau\xi} \hat{\theta} \, dy \\
+ \int_{\tilde{\Omega}} \hat{U}^j a_{kj} (D^2_{k\tau\xi} \hat{\eta} \cdot J^2_{\chi} D^2_{\tau\xi} \hat{v}^i + D^2_{k\tau\xi} \hat{\xi} \cdot J^2_{\chi} D^2_{\tau\xi} \hat{v}^i + D^2_{k\tau\xi} \hat{\xi} \cdot J^2_{\chi} D^2_{\tau\xi} \hat{\theta}) \, dy \\
+ \frac{1}{\epsilon} \int_{\tilde{\Omega}} \{D^2_{\tau\xi}[a_{kj} D_k \hat{v}^j] \cdot J^2_{\chi} D^2_{\tau\xi} \hat{\eta} + D^2_{\tau\xi}[a_{kj} D_k (\hat{\eta} + \hat{\theta})] \cdot J^2_{\chi} D^2_{\tau\xi} \hat{v}^j \} \cdot J^2_{\chi} D^2_{\tau\xi} \hat{\theta} \, dy \\
= \int_{\tilde{\Omega}} \{D_\xi (\hat{U}^j a_{kj}) D^2_{k\tau\xi} \hat{\eta} + D^2_{\tau\xi}(\hat{U}^j a_{kj}) D_k \hat{\eta} + D_\tau (\hat{U}^j a_{kj}) D^2_{k\xi} \hat{\eta}

-D^2_{\tau\xi}[\theta a_{kj} D_k \hat{v}^j + \eta a_{kj} D_k \hat{v}^j + \epsilon a_{kj} D_k (\hat{P}(\hat{U}^j + \hat{v}^j)]),
\end{align*}
$$

Now, we denote LHS of (2.44) := $L'_1 + L'_2 + L'_3 + L'_4$ and have to deal with each term due to integration by part and the boundary conditions.

$$
L'_1 = - \int_{\tilde{\Omega}} \{D^2_{\tau\xi}(\mu a_{kj}) D_k (a_{lj} D_l \hat{v}^i) + D_\tau (\mu a_{kj}) D^2_{k\xi} (a_{lj} D_l \hat{v}^i) + D_\xi (\mu a_{kj}) D_k D_\tau (a_{lj} D_l \hat{v}^i) + \mu a_{kj} D_k [D_\tau (a_{lj}) D_l \hat{v}^i + a_{lj} D^2_{l\tau} \hat{v}^i]

+ D^2_{\tau\xi}(\zeta a_{kj}) D_k (a_{lj} D_l \hat{v}^i) + D_\tau (\zeta a_{kj}) D^2_{k\xi} (a_{lj} D_l \hat{v}^i)
$$
we deduce that

\[
D_\xi (\zeta a_{ki}) D_k \left[ D_\tau (a_{lj}) D_\tau \dot{v}^j + a_{lj} D^2_\tau \dot{v}^j \right] + \zeta a_{ki} D^2_\kappa [D_\tau (a_{lj}) D_\tau \dot{v}^j + a_{lj} D_\tau \dot{v}^j] \right\} \cdot J^2 \hat{D}_\tau ^2 \dot{v}^i dy
\]

where

\[
- \int_\Omega [\mu a_{kj} D^2_\kappa (a_{lj} D_\tau \dot{v}^i) + \zeta a_{ki} D^2_\kappa (a_{lj} D^2_\tau \dot{v}^j)] \cdot J^2 \hat{D}_\tau ^2 \dot{v}^i dy
\]

\[
= \int_\Omega \mu J^2 \chi^2 a_{kj} \hat{D}_\kappa \hat{D}_\tau \dot{v}^i a_{lj} D^2_\tau \dot{v}^j + \zeta J^2 \chi^2 a_{ki} D^2_\kappa \hat{D}_\tau \dot{v}^i a_{lj} D^2_\tau \dot{v}^j dy
\]

\[
+ \int_\Omega [D_k (\mu J^2 \chi^2 a_{kj}) \hat{D}_\xi (a_{lj} D_\tau \dot{v}^i) \cdot D^2_\tau \dot{v}^i + \mu J^2 \chi^2 a_{kj} D^2_\kappa \hat{D}_\tau \dot{v}^i D_\xi (a_{lj}) D^2_\tau \dot{v}^i
+ D_k (\zeta J^2 \chi^2 a_{ki}) \hat{D}_\xi (a_{lj} D^2_\tau \dot{v}^i) \cdot D^2_\tau \dot{v}^i + \mu J^2 \chi^2 a_{ki} D^2_\kappa \hat{D}_\tau \dot{v}^i D_\xi (a_{lj}) D^2_\tau \dot{v}^i] dy,
\]

and

\[
L'_2 = \int_\Omega \kappa J^2 \chi^2 a_{kj} \hat{D}_\kappa \hat{D}_\tau \dot{\theta} a_{lj} D^2_\tau \dot{\theta} dy
\]

\[
+ \int_\Omega \left[ D_k (\kappa J^2 \chi^2 a_{kj}) \hat{D}_\xi (a_{lj} D^2_\tau \dot{\theta}) \cdot D^2_\tau \dot{\theta} + \kappa J^2 \chi^2 a_{kj} \hat{D}_\kappa \hat{D}_\tau \dot{\theta} D_\xi (a_{lj}) D^2_\tau \dot{\theta} \right] dy
\]

\[
- \int_\Omega \left\{ \hat{D}_\tau ^2 (\kappa a_{kj}) D_k (a_{lj} D_\tau \dot{\theta}) + D_\tau (\kappa a_{kj}) D^2_\kappa (a_{lj} D_\tau \dot{\theta})
+ D_\xi (\kappa a_{kj}) D_k \left[ D_\tau (a_{lj}) D_\tau \dot{\theta} + a_{lj} D^2_\tau \dot{\theta} \right] + \kappa a_{kj} \hat{D}_\kappa \hat{D}_\tau \dot{\theta} D_\tau (a_{lj}) D_\tau \dot{\theta} \right\} \cdot J^2 \hat{D}_\tau ^2 \dot{\theta} dy.
\]

On the other hand, recalling that \( a_{kj} D_k \hat{U}^j = 0 \), we have

\[
L'_3 = - \frac{1}{2} \int_\Omega J^2 \chi^2 a_{kj} D_k \hat{U}^j (\hat{D}^2_\tau \hat{v}^i)^2 + \hat{D}^2_\tau \dot{v}^i |^2 + \hat{D}^2_\tau \dot{\theta} |^2 dy
\]

\[
- \frac{1}{2} \int_\Omega D_k (J^2 \chi^2 a_{kj}) \hat{U}^j (\hat{D}^2_\tau \hat{v}^i)^2 + \hat{D}^2_\tau \dot{v}^i |^2 + \hat{D}^2_\tau \dot{\theta} |^2 dy
\]

\[
= - \frac{1}{2} \int_\Omega D_k (J^2 \chi^2 a_{kj}) \hat{U}^j (\hat{D}^2_\tau \hat{v}^i)^2 + \hat{D}^2_\tau \dot{v}^i |^2 + \hat{D}^2_\tau \dot{\theta} |^2 dy.
\]

As for \( L'_4 \), in view of the following identity

\[
\frac{1}{\epsilon} \int_\Omega \hat{D}^2_\tau \xi (a_{ki} D_k (\hat{\eta} + \hat{\theta})) \cdot J^2 \hat{D}^2_\tau \xi \dot{v}^i dy = - \frac{1}{\epsilon} \int_\Omega J^2 \chi^2 \hat{D}^2_\tau \xi (a_{ki} D_k \dot{v}^i) \hat{D}^2_\tau \xi (\hat{\eta} + \hat{\theta}) dy
\]

\[
+ \frac{1}{\epsilon} \int_\Omega \left[ \hat{D}^2_\tau \xi (a_{ki}) D_k (\hat{\eta} + \hat{\theta}) + D_\tau (a_{ki}) \hat{D}^2_\kappa \xi (\hat{\eta} + \hat{\theta}) + D_\xi (a_{ki}) D^2_\tau \xi (\hat{\eta} + \hat{\theta}) \right] \cdot J^2 \hat{D}^2_\tau \xi \dot{v}^i dy
\]

\[
+ \frac{1}{\epsilon} \int_\Omega \left[ D^2_\tau \xi (\hat{\eta} + \hat{\theta}) [\hat{D}^2_\tau (a_{ki}) D_k \dot{v}^i + D_\tau (a_{ki}) D^2_\kappa \xi \dot{v}^i + D_\xi (a_{ki}) D^2_\tau \xi \dot{v}^i
- D_k (J^2 \chi^2 a_{ki} D^2_\tau \xi \dot{v}^i - D_k (a_{ki}) J^2 \chi^2 \hat{D}^2_\tau \xi \dot{v}^i) dy,
\]

we deduce that

\[
L'_4 = \frac{1}{\epsilon} \int_\Omega \left[ \hat{D}^2_\tau \xi (a_{ki}) D_k (\hat{\eta} + \hat{\theta}) + D_\tau (a_{ki}) D^2_\kappa \xi (\hat{\eta} + \hat{\theta}) + D_\xi (a_{ki}) D^2_\tau \xi (\hat{\eta} + \hat{\theta}) \right] \cdot J^2 \hat{D}^2_\tau \dot{v}^i dy
\]

\[
+ \frac{1}{\epsilon} \int_\Omega \left[ D^2_\tau \xi (\hat{\eta} + \hat{\theta}) [\hat{D}^2_\tau (a_{ki}) D_k \dot{v}^i + D_\tau (a_{ki}) D^2_\kappa \xi \dot{v}^i + D_\xi (a_{ki}) D^2_\tau \xi \dot{v}^i
- D_k (J^2 \chi^2 a_{ki} D^2_\tau \xi \dot{v}^i - D_k (a_{ki}) J^2 \chi^2 D^2_\tau \xi \dot{v}^i) dy.
\]
Substituting the above estimates into (2.44), using Sobolev’s and Young’s inequalities and taking into account the property (2.42), we deduce that

\[
\int_{\Omega} \mu J^{2} a_{k} \kappa r \tilde{v}^{i} a_{ij} D_{r \xi}^{3} \tilde{v}^{j} d\Omega + \int_{\Omega} \zeta J^{2} a_{kl} D_{k \xi}^{3} \tilde{v}^{i} a_{ij} D_{l \xi}^{3} \tilde{v}^{j} d\Omega + \int_{\Omega} \kappa J^{2} a_{k} D_{k \xi}^{3} \hat{\theta} a_{ij} D_{r \xi}^{3} \hat{\theta} d\Omega \leq C_{9} \left[ \| U \|_{3}^{2} \| \eta \|_{2}^{2} + \| \tilde{\nu} \|_{3}^{2} \| \eta \|_{2}^{2} + \epsilon \| P \|_{3} \| (\| U \|_{3} + \| \tilde{\nu} \|_{3}) \| \eta \|_{2} \right. \\
+ \epsilon \| F \|_{3}^{2} + \| \tilde{G} \|_{3}^{2} + \| \tilde{\nu} \|_{3}^{2} \left( \| \eta \|_{2}^{2} + \| \tilde{\theta} \|_{2}^{2} \right) + \| \tilde{\nu} \|_{2}^{2} \| \tilde{\theta} \|_{2}^{2} \\
+ \left. \left( \| \tilde{\nu} \|_{2}^{2} + \| \theta \|_{2}^{2} \right) \right] + \delta \left( \| \tilde{\nu} \|_{3}^{2} + \| \theta \|_{3}^{2} + \| \nabla \eta \|_{1}^{2} \right)
\]

(2.45)

where \( C_{9} \) is a constant.

**Part 2. Estimate of derivatives in the normal direction**

We multiply (2.39) by \( a_{3i} \) to obtain that

\[
- (\mu + \zeta) D_{3} (a_{ij} D_{l} \tilde{v}^{j}) + \frac{1}{\epsilon} D_{3} (\hat{\eta} + \hat{\theta}) = - a_{3i} \hat{\nu}^{i} a_{ij} D_{k} \tilde{v}^{j} + \epsilon a_{3i} \hat{\nu}^{i} a_{ij} D_{k} \tilde{v}^{j} - \hat{\theta} D_{3} \hat{\eta} - \hat{\eta} D_{3} \hat{\theta} \\
+ \mu \left[ a_{3i} a_{kl} D_{k} (a_{ij} D_{l} \tilde{v}^{j}) - D_{3} (a_{ij} D_{l} \tilde{v}^{j}) \right],
\]

(2.46)

where the last term in RHS of (2.46) can be written as follows.

\[
\mu \left[ a_{3i} a_{kl} D_{k} (a_{ij} D_{l} \tilde{v}^{j}) - D_{3} (a_{ij} D_{l} \tilde{v}^{j}) \right] = \mu \left[ D_{3} (a_{3j}) D_{3} \tilde{v}^{j} + D_{3} (a_{rj}) D_{r} \tilde{v}^{j} + a_{rj} D_{3} ^{2} \tilde{v}^{j} - a_{3j} D_{3} (a_{3j}) a_{3i} D_{3} \tilde{v}^{i} \\
- a_{rj} a_{3i} D_{r} a_{ij} D_{l} \tilde{v}^{j} - a_{rj} a_{3j} D_{3} (a_{3j}) D_{r} \tilde{v}^{j} a_{3j} a_{3i} D_{3} (a_{rj}) D_{r} \tilde{v}^{j} \right], \quad \tau, \ \xi = 1, 2.
\]

(2.47)

which does not include the term \( D_{3} \tilde{v} \).

Step 1. To continue our estimate, we show the following lemma.

**Lemma 2.7.** There are a constant \( C_{10} \) and a small \( \delta > 0 \), such that

\[
\frac{\mu + \zeta}{2} \int_{\Omega} J^{2} | D_{3}^{2} (a_{ij} D_{l} \tilde{v}^{j}) |^{2} d\Omega + \frac{\kappa}{2} \int_{\Omega} J^{2} | D_{3} (a_{ij} D_{l} \tilde{v}^{j}) |^{2} d\Omega \leq C_{10} \left\{ \| U \|_{3}^{2} \| \tilde{\nu} \|_{2}^{2} + \epsilon^{2} \| \tilde{F} \|_{1}^{2} + \| \tilde{\nu} \|_{3}^{2} \| \eta \|_{2}^{2} \right. \\
+ \int_{\Omega} J^{2} | D_{3} \tilde{v}^{j} |^{2} d\Omega \\
+ \left. \int_{\Omega} J^{2} | D_{3} \tilde{\nu} |^{2} d\Omega \right\} + \delta (\| \eta \|_{3}^{2} + \| \theta \|_{3}^{2}).
\]

(2.48)

**Proof.** We differentiate (2.46) with respect to \( y_{\tau} \) (\( \tau = 1, 2 \)), then multiply \( - J^{2} D_{3} (a_{ij} D_{l} \tilde{v}^{j}) \) in \( L^{2} (\hat{\Omega}) \) to get

\[
\frac{\mu + \zeta}{2} \int_{\Omega} J^{2} | D_{3}^{2} (a_{ij} D_{l} \tilde{v}^{j}) |^{2} d\Omega - \frac{1}{\epsilon} \int_{\Omega} J^{2} | D_{3} (a_{ij} D_{l} \tilde{v}^{j}) |^{2} d\Omega \leq C \left\{ \| a_{3i} \hat{\nu}^{i} a_{kl} D_{k} \tilde{v}^{j} |^{2} + \epsilon^{2} \| \tilde{\nu} \|_{3}^{2} \| \eta \|_{2}^{2} \right. \\
+ \| \hat{\theta} D_{3} \hat{\eta} + \hat{\eta} D_{3} \hat{\theta} \|_{2}^{2} \right. \\
+ \left. \int_{\Omega} J^{2} | D_{3} \tilde{v}^{j} |^{2} d\Omega \right\}
\]

(2.49)

\[
\leq C \left\{ \| U \|_{3}^{2} \| \tilde{\nu} \|_{2}^{2} + \epsilon^{2} \| \tilde{F} \|_{1}^{2} + \| \tilde{\nu} \|_{3}^{2} \| \eta \|_{2}^{2} \right. \\
+ \int_{\Omega} J^{2} | D_{3} \tilde{v}^{j} |^{2} d\Omega \right\} + C \int_{\Omega} J^{2} | D_{3} \tilde{v}^{j} |^{2} d\Omega.
\]
In the meanwhile, we apply $D_r$ to (2.39)$_2$ and (2.39)$_1$, take the product of the resulting equations with $J\chi^2 D_{r3}^2 \hat{\theta}$ and $J\chi^2 D_{r3}^2 \hat{\eta}$ in $L^2(\Omega)$, and sum then two identities to get
\[
- \int_\Omega \kappa D_{r3}^2 \left[ a_{kj} D_k(a_{lj} D_l \hat{\theta}) \right] \cdot J\chi^2 D_{r3}^2 \hat{\theta} dy + \frac{1}{\epsilon} \int_\Omega D_{r3}^2(a_{kj} D_k \hat{\nu}^j) \cdot J\chi^2(D_{r3}^2 \hat{\theta} + D^2_{r3} \hat{\eta}) dy
+ \int_\Omega [D_{r3}^2(\hat{U}^j a_{kj} D_k \hat{\eta}) \cdot J\chi^2 D_{r3}^2 \hat{\eta} + D_{r3}^2(\hat{U}^j a_{kj} D_k \hat{\theta}) \cdot J\chi^2 D_{r3}^2 \hat{\theta}] dy
= - \int_\Omega D_{r3}^2 \left\{ \hat{\nu}^j a_{kj} D_k \hat{\eta} + \hat{\eta} a_{kj} D_k \hat{\nu}^j + \epsilon a_{kj} D_k \left[ \hat{P}(\hat{U}^j + \hat{\nu}^j) \right] \right\} \cdot J\chi^2 D_{r3}^2 \hat{\eta}
+ \int_\Omega D_{r3}^2 \left( \hat{\nu}^j - \hat{\nu}^j a_{kj} D_k \hat{\theta} - \hat{\eta} a_{kj} D_k \hat{\nu}^j - \hat{\theta} a_{kj} D_k \hat{\nu}^j \right) \cdot J\chi^2 D_{r3}^2 \hat{\theta} dy.
\] (2.50)

We denote LHS of (2.50) := $L''_1 + L''_2 + L''_3$. To control $L''_k$, we integrate by part to deduce that
\[
L''_1 = \kappa \int_\Omega J\chi^2 D_{r3}^2(a_{kj} D_k \hat{\theta}) D_{r3}^2(a_{lj} D_l \hat{\theta}) dy
- \int_\Omega \kappa J\chi^2 D_{r3}^2 \left( D_{r3}^2(a_{kj}) D_k(a_{lj} D_l \hat{\theta}) + D_r(a_{kj}) D_{r3}^2(a_{lj} D_l \hat{\theta}) + D_r(a_{kj}) D_{r3}^2(a_{lj} D_l \hat{\theta}) \right) dy
+ \kappa \int_\Omega D_k (J\chi^2 a_{kj}) D_{r3}^2(a_{lj} D_l \hat{\theta}) dy
- \kappa \int_\Omega J\chi^2 \left( D_{r3}^2(a_{kj}) D_k \hat{\theta} + D_r(a_{kj}) D_{r3}^2(a_{lj} D_l \hat{\theta}) + D_r(a_{kj}) D_{r3}^2(a_{lj} D_l \hat{\theta}) \right) \cdot D_{r3}^2(a_{lj} D_l \hat{\theta}) dy
\]
and
\[
L''_3 = \int_\Omega J\chi^2 D_{r3}^2 \hat{\eta} \cdot \left( D_{r3}^2(\hat{U}^j a_{kj}) D_k \hat{\eta} + D_r(\hat{U}^j a_{kj}) D_{r3}^2 \hat{\eta} + D_r(\hat{U}^j a_{kj}) D_{r3}^2 \hat{\eta} \right) dy
- \frac{1}{2} \int_\Omega D_k (J\chi^2 a_{kj}) \hat{U}^j |D_{r3}^2 \hat{\eta}|^2 + J\chi^2 a_{kj} D_k \hat{U}^j |D_{r3}^2 \hat{\eta}|^2 dy
+ \int_\Omega \left( D_{r3}^2(\hat{U}^j a_{kj}) D_k \hat{\theta} + D_r(\hat{U}^j a_{kj}) D_{r3}^2 \hat{\theta} + D_r(\hat{U}^j a_{kj}) D_{r3}^2 \hat{\theta} \right) \cdot J\chi^2 D_{r3}^2 \hat{\theta} dy
- \frac{1}{2} \int_\Omega D_k (J\chi^2 a_{kj}) \hat{U}^j |D_{r3}^2 \hat{\theta}|^2 + J\chi^2 a_{kj} D_k \hat{U}^j |D_{r3}^2 \hat{\theta}|^2 dy.
\]

Inserting the estimates for $L''_1$ and $L''_3$ into (2.50), we obtain
\[
\frac{\kappa}{2} \int_\Omega J\chi^2 |D_{r3}^2(a_{kj} D_k \hat{\nu}^j)|^2 dy + \frac{1}{\epsilon} \int_\Omega J\chi^2 D_{r3}^2 \left( \hat{\nu}^j + \hat{\theta} \right) D_{r3}^2(a_{lj} D_l \hat{\nu}^j) dy
\leq \|\theta\|_2^2 + \|\hat{\nu}^j\|_2^2 \left( \|D_{r3}^2(a_{lj} D_l \hat{\nu}^j)\|_0^2 + \|D_{r3}^2(a_{lj} D_l \hat{\nu}^j)\|_0^2 + \|\nu\|_0^2 \right)
+ C \left[ \|U\|_3 \|\eta\|_2^2 + \|U\|_3 \|\eta\|_2^2 + \|\hat{\nu}\|_2 \|\eta\|_2^2 + \epsilon \|U\|_3 (\|U\|_3 + \|\hat{\nu}\|_3) \|\eta\|_2 \right.
+ \epsilon^2 \|\hat{G}\|_2^2 + \|\hat{\nu}\|_2^2 \|\hat{\theta}\|_2^2 + \|\hat{\nu}\|_2^2 \|\eta\|_2^2 \right].
\] (2.51)

Thanks to Sobolev’s and Young’s inequalities, we take the sum of (2.49) and (2.51) to deduce the estimate (2.48).

\[
\square
\]

Step 2. Now, it suffices to bound $\|D_{r3}^2(a_{ij} D_i \hat{\nu}^j)\|_0$ in order to close the estimate for div\(v\). We apply $D_3$ to (2.46) to find that
\[
-(\mu + \zeta) D_{33}^2(a_{ij} D_i \hat{\nu}^j) + \frac{1}{\epsilon} D_{33}^2(\hat{\eta} + \hat{\theta})
\]
\[= D_3(a_{3i})(-\hat{U}^j a_{kj} D_k \hat{v}^i + \epsilon \hat{F}^i - \hat{v}^j a_{kj} D_k \hat{v}^i) - a_{3i} D_3(\hat{U}^j a_{kj} D_k \hat{v}^i - \epsilon \hat{F}^i + \hat{v}^j a_{kj} D_k \hat{v}^i) - D_3(\hat{\theta} D_3 \hat{\eta} + \hat{\eta} D_3 \hat{\theta}) + O(1)(D_{33r}^2 \hat{v}^i + D_{3}^2 \hat{v}^i + D_t \hat{v}^i). \tag{2.52} \]

Now, multiplying the above equality \((2.52)\) by \(-J \chi^2 D_{33r}^2(a_{ij} D_t \hat{v}^i)\) in \(L^2(\hat{\Omega})\), one infers that
\[
\frac{\kappa + \zeta}{2} \int \Omega J \chi^2 |D_{33r}^2(a_{ij} D_t \hat{v}^i)|^2 dy - \frac{1}{\epsilon} \int \Omega J \chi^2 D_{33r}^2(a_{ij} D_t \hat{v}^i) \cdot D_{33r}^2(\hat{\eta} + \hat{\theta}) dy
\leq C(\|U\|_3^2 \|v\|_2^2 + \epsilon^2 \|\hat{F}\|^2_2 + \|\hat{v}\|_4^4 + \|\hat{\theta}\|_2^2 \|\hat{\eta}\|_2^2) + \|v\|_2^2 + \|v\|^2_2 + \int \Omega J \chi^2 |D_{33r}^2 \hat{v}|^2 dy. \tag{2.53} \]

Correspondingly, applying \(D_{33r}^2\) to \((2.39)_1\) and \((2.39)_3\) and multiplying the resulting equations by \(J \chi^2 D_{33r}^2 \hat{\eta}\) and \(J \chi^2 D_{33r}^2 \hat{\theta}\) respectively, we get
\[
\frac{\kappa}{2} \int \Omega J \chi^2 |D_{33r}^2(a_{ij} D_t \hat{v}^i)|^2 dy + \frac{1}{\epsilon} \int \Omega J \chi^2 D_{33r}^2(a_{ij} D_t \hat{v}^i) \cdot D_{33r}^2(\hat{\eta} + \hat{\theta}) dy
\leq C(\|\theta\|_2^2 (1 + \|U\|_3) + \delta \|\theta\|_2^2 + \|\hat{F}\|_2^2 + \|\hat{\theta}\|_2 \|\hat{\eta}\|_2^2) + \epsilon \|P\|_3 (\|U\|_3 + \|\hat{v}\|_3) \|\eta\|_2^2
\]
\[
+ \epsilon^2 \|\tilde{G}\|_2^2 + \|\tilde{\eta}\|_2 \|\tilde{\theta}\|_2 + \|\tilde{\theta}\|_2 \|\tilde{\eta}\|_2^2. \tag{2.54} \]

Combining \((2.53)\) with \((2.54)\), we see that there is a constant \(C_{11}\) and a small \(\delta\), such that
\[
\frac{\mu + \zeta}{2} \int \Omega J \chi^2 |D_{33r}^2(a_{ij} D_t \hat{v}^i)|^2 dy + \frac{\kappa}{2} \int \Omega J \chi^2 |D_{33r}^2(a_{ij} D_t \hat{v}^i)|^2 dy
\leq \delta (\|\theta\|_2^2 + \|\hat{v}\|_3^2) + C_{11}\left\{ (\|U\|_3^2 \|v\|_2^2 + \epsilon^2 \|\hat{F}\|_2^2 + \|\hat{\theta}\|_2^2 \|\hat{\eta}\|_2^2)
\right.
\]
\[
+ \int \Omega J \chi^2 |D_{33r}^3, \tilde{v}|^2 dy + \|\theta\|_2^2 (1 + \|U\|_3) + \|U\|_3 \|\eta\|_2^2 + \|\hat{v}\|_3 \|\eta\|_2^2
\]
\[
+ \epsilon \|P\|_3 (\|U\|_3 + \|\hat{v}\|_3) \|\eta\|_2^2 + \epsilon^2 \|\tilde{G}\|_2^2 + \|\tilde{\eta}\|_2 \|\tilde{\theta}\|_2 + \|\tilde{\theta}\|_2 \|\tilde{\eta}\|_2^2 \right\}. \tag{2.55} \]

Step 3. To control the term \(D_{33r}^3 \tilde{v}\) on RHS of \((2.55)\), we introduce an auxiliary Stokes problem in the original coordinates in the region near the boundary:
\[
\left\{ \begin{array}{ll}
- \mu \Delta_x [(\chi D_r \tilde{v}) \circ \Lambda^{-1}] + \frac{1}{\epsilon} \nabla_x [(\chi D_r (\hat{\eta} + \hat{\theta}) \circ \Lambda^{-1})] = G_1 & \text{in } W \cap \Omega,
\text{div}_x [(\chi D_r \tilde{v}) \circ \Lambda^{-1}] = G_2 & \text{in } W \cap \Omega,
(\chi D_r \tilde{v}) \circ \Lambda^{-1} = 0 & \text{in } W \cap \Omega,
\end{array} \right.

where
\[
G_1 \equiv \chi D_r \left[ \tilde{\zeta} a_{ki} D_k (a_{ij} D_t \hat{v}^j) + \epsilon \hat{F}^i - \hat{v}^j a_{kj} D_k \hat{v}^i - \hat{\eta} a_{ki} D_k \hat{\eta} \right.
\]
\[
- \hat{U}^j a_{kj} D_k \hat{v}^i] + o(1) \left[ D_t \hat{v}^i + D_k \hat{v}^i + \frac{1}{\epsilon} D_k (\hat{\eta} + \hat{\theta}) \right]
\]
and
\[
G_2 = o(1)(D_r \hat{v}^i + D_k \hat{v}^i + D_{\tau k} \hat{v}^i),
\]
which can be bounded as follows.
\[
\|G_1\|_{L^2}^2 \leq C \left\| \frac{\nabla (\eta + \theta)}{\epsilon} \right\|_{L^2}^2 + \int \Omega J \chi^2 |D_{33r}^2 (a_{ij} D_t \hat{v}^i)|^2 dy + \delta \|v\|_3^2 + C \int \Omega J \chi^2 |D_{33r}^3 (a_{ij} D_t \hat{v}^i)|^2 dy \tag{2.56} \]
and
\[
\|G_2\|_{H^1}^2 \leq \delta \|v\|_3^2 + C_0 \|v\|_1^2 + C \int \Omega J \chi^2 |D_{33r}^3 (a_{ij} D_t \hat{v}^i)|^2 dy. \tag{2.57} \]
Due to the regularity theory of the Stokes problem (see [9]), one has
\[
\int_{\Omega} |\triangle x (\chi D_x \mathbf{v}) \circ \Lambda^{-1}(x)|^2 \, dx \leq C (\|G_1\|^2_{L^2(W\cap \Omega)} + \|G_2\|^2_{H^1(W\cap \Omega)}),
\]  
(2.58)

where the left-hand side of (2.58) is equal to
\[
\int_{\Omega} |\triangle x (\chi D_x \mathbf{v}) \circ \Lambda^{-1}(x)|^2 \, dx = \int_{\Omega} \left| \sum_{j=1}^{3} \sum_{k=1}^{3} \sum_{l=1}^{3} a_{jk} D_k (a_{lj} D_l (\chi D_x \mathbf{v})) \right|^2 \, dy
\]
\[
= \int_{\Omega} J \chi^2 \left\| \sum_{j,k,l=1}^{3} a_{kj} a_{ij} D_{k_{jr}} \mathbf{v} \right\|^2 \, dy + o(1) \int_{\Omega} (|D_{\tau r} \mathbf{v}|^2 + |D_{y r}^2 \mathbf{v}|^2) \, dy.
\]

And we use (2.41) to get
\[
D_{33\tau}^3 \mathbf{v} = \sum_{j,k,l=1}^{3} a_{kj} a_{ij} D_{k_{jr}} \mathbf{v} - \sum_{1 \leq k, l \leq 2}^{3} \sum_{j=1}^{3} a_{kj} a_{lj} D_{k_{jr}} \mathbf{v},
\]
from which, (2.56) and (2.57) it follows that the inequality (2.58) gives
\[
(C_{11} + 1) \int_{\Omega} J \chi^2 \|D_{33\tau}^3 \mathbf{v}\|^2 \, dy
\]
\[
\leq C (\|G_1\|^2_{L^2(W\cap \Omega)} + \|G_2\|^2_{H^1(W\cap \Omega)}) + C \int_{\Omega} J \chi^2 \|D_{33\tau}^3 \mathbf{v}\|^2 \, dy + C_\delta \|\nabla \mathbf{v}\|^2_{L^2} + \delta \|\mathbf{v}\|^2_3
\]
\[
\leq \delta \|\mathbf{v}\|^2_3 + C_{12} \left\| \frac{\nabla \eta + \theta}{\epsilon} \right\|^2_0 + \int_{\Omega} J \chi^2 \left( |D_{33\tau}^3 \mathbf{v}|^2 + |D_{33\tau}^3 (a_{ij} D_l \mathbf{v}^j)|^2 \right) \, dy
\]
\[
+ \epsilon^2 \|\bar{F}\|^2_1 + \|\nabla \mathbf{v}\|^4_2 + \|\bar{\theta}\|^2_2 \|\eta\|^2_2 + \|U\|^4_3 \|\mathbf{v}\|^4_1 \right].
\]  
(2.59)

Now, letting
\[
\Phi_\chi := \int_{\Omega} J \chi^2 \left( |D_{33\tau}^3 \mathbf{v}|^2 + |D_{33\tau}^3 (a_{ij} D_l \mathbf{v}^j)|^2 + |D_{33\tau}^3 (a_{ij} D_l \mathbf{v}^j)|^2 \right) \, dy
\]
and
\[
\Psi_\chi := \int_{\Omega} J \chi^2 \left( a_{kj} D_{k_{jr}} \mathbf{v} + |D_{33\tau}^3 (a_{kj} D_k \bar{\theta})|^2 + |D_{33\tau}^3 (a_{kj} D_k \bar{\theta})|^2 \right) \, dy,
\]
we can apply Cauchy-Schwarz’s and Young’s inequalities as well as the estimate (2.29) to deduce from (2.41), (2.48), (2.55) and (2.59) that
\[
\Phi_\chi + \Psi_\chi \leq \|U\|^2_3 \|\eta\|^2_2 + (\|\nabla \mathbf{v}\|^2_3 + \|\mathbf{v}\|^2_3) \|\eta\|^2_2 + \epsilon \|P\|_{L^2(\Omega)} (\|U\|^2_3 + \|\nabla \mathbf{v}\|^2_3) \|\eta\|^2_2
\]
\[
+ C (1 + \|U\|^4_3) (\|\mathbf{v}\|^2_1 + \|\theta\|^2_1) + \delta \left( \|\nabla \mathbf{v}\|^2_3 + \|\theta\|^2_3 + \left\| \frac{\nabla \eta + \theta}{\epsilon} \right\|^2_1 \right)
\]
\[
+ C_\delta \left[ 2 \left( \|\bar{F}\|^2_1 + \|G\|^2_2 \right) + \|\nabla \mathbf{v}\|^4_2 + (\|\bar{\theta}\|^2_2 + \|\mathbf{v}\|^2_2) \|\eta\|^2_2 + \|\mathbf{v}\|^4_2 \|\bar{\theta}\|^2_2 \right],
\]  
(2.60)

which, together with (2.30) – (2.32), results in
\[
\|\mathbf{v}\|^3_3 + \|\theta\|^3_3 + \left\| \frac{\nabla \eta + \nabla \theta}{\epsilon} \right\|^2_1
\]
\[
\leq C_{12} \left\{ \|U\|^2_3 \|\eta\|^2_2 + (1 + \|U\|^4_3) \left[ \epsilon \|P\|_{L^2(\Omega)} (\|U\|^2_3 + \|\nabla \mathbf{v}\|^2_3) + (\|\bar{\theta}\|^2_2 + \|\mathbf{v}\|^2_2 + \|\mathbf{v}\|^2_3) \right] \|\eta\|^2_2
\]
\[
+ (1 + \|U\|^4_3) (\|\mathbf{v}\|^2_1 + \|\theta\|^2_1) + (1 + \|U\|^4_3) \left[ \epsilon^2 (\|\bar{F}\|^2_1 + \|\mathbf{v}\|^4_2) + \|\nabla \mathbf{v}\|^4_2 + \|\mathbf{v}\|^4_2 \|\bar{\theta}\|^2_2 \right] \right\}.
\]  
(2.61)
2.2.4 Boundedness of $\eta$

In the next lemma, we derive upper bounds of $\|\eta\|_1$ and $\|\eta\|_2$.

**Lemma 2.8.** There are a small $\delta > 0$, and two positive constants $C_{13}$ and $C_{14}$ independent of $\epsilon$, such that

$$
\|\eta\|_1 \leq C_{13} \left[ \epsilon^2 \|\tilde{F}\|_0 + \epsilon (\|\tilde{v}\|_2 \|\tilde{v}\|_1 + \|\tilde{\theta}\|_2 \|\eta\|_2 + \|U\|_2 \|v\|_2 + \|\text{div}v\|_1 + (\|\tilde{F}\|_{-1} + \|\tilde{G}\|_{-1})) \right] + \epsilon \|P\|_2 (\|U\|_2 + \|\tilde{v}\|_2) + \|\tilde{v}\|_3 \|\eta\|_0 + \frac{1}{2} \|\tilde{v}\|_1 (\|\tilde{\theta}\|_1 + \|\eta\|_1) \right] \tag{2.62}
$$

and

$$
\|\eta\|_2 \leq C_{14} (1 + \epsilon)(1 + \|U\|_2^2) \left[ \epsilon (\|\tilde{F}\|_1 + \|\tilde{G}\|_{-1}) + \|\tilde{v}\|_3 \|\eta\|_2 + \|\tilde{v}\|_2 \|\eta\|_1 + \epsilon \|P\|_2 (\|U\|_2 + \|\tilde{v}\|_2) \|\eta\|_0 \right] + \left[ \epsilon \|P\|_2 (\|U\|_2 + \|\tilde{v}\|_2) \|\eta\|_0 \right]^{1/2} + \|\tilde{v}\|_1 (\|\tilde{\theta}\|_1 + \|\eta\|_1) \right] + \epsilon \|v\|_3 + \delta \|\theta\|_3. \tag{2.63}
$$

**Proof.** From (2.23) we get

$$
\epsilon \|v\|_2 + \|\nabla \eta + \nabla \theta\|_0 \leq C \left[ \epsilon^2 \|\tilde{F}\|_0 + \epsilon (\|\tilde{v}\|_2 \|\tilde{v}\|_1 + \|\tilde{\theta}\|_2 \|\eta\|_2 + \|U\|_2 \|v\|_2 + \|\text{div}v\|_1) \right],
$$

which together with Lemma 2.3 gives

$$
\|\nabla \eta\|_1 \leq \|\nabla \eta + \nabla \theta\|_0 + \|\nabla \theta\|_0 \\
\leq C \left[ \epsilon^2 \|\tilde{F}\|_0 + \epsilon (\|\tilde{v}\|_2 \|\tilde{v}\|_1 + \|\tilde{\theta}\|_2 \|\eta\|_2 + \|U\|_2 \|v\|_2 + \|\text{div}v\|_1) \right] + \epsilon \|P\|_2 (\|U\|_2 + \|\tilde{v}\|_2) \|\eta\|_0 \\
+ \|\tilde{v}\|_3 \|\eta\|_0^2 \|\tilde{\theta}\|_1^2 + \|\tilde{v}\|_2 \|\eta\|_1^2 \|\tilde{v}\|_2 (\|\tilde{\theta}\|_1^2 + \|\eta\|_1^2)^{1/2}
$$

If we apply Poincaré’s and Young’s inequalities to the above inequality, and use the fact that

$$(A_1 + A_2 + \cdots + A_n)^{1/2} \leq A_1^{1/2} + A_2^{1/2} + \cdots + A_n^{1/2} \quad \text{for} \; A_i \geq 0 \quad (i = 1, \cdots, n), \tag{2.64}$$

we obtain the estimate (2.62) immediately.

On the other hand, from the estimate (2.30) we conclude that

$$
\|\nabla \eta\|_1 \leq \|\nabla \eta + \nabla \theta\|_1 + \|\nabla \theta\|_1 \\
\leq C \epsilon (1 + \|U\|_2^2) \left[ \epsilon (\|\tilde{F}\|_1 + \|\tilde{G}\|_{-1}) + \|\tilde{v}\|_2 \|\tilde{v}\|_2 + \|\tilde{\theta}\|_3 \|\eta\|_2 + \|\tilde{v}\|_3 \|\eta\|_0 \right] \\
+ \left[ \epsilon \|P\|_2 (\|U\|_2 + \|\tilde{v}\|_2) \|\eta\|_0 \right]^{1/2} + \frac{1}{2} \|\tilde{v}\|_1 (\|\tilde{\theta}\|_1 + \|\eta\|_1) \right] + \epsilon \|v\|_3 + \delta \|\theta\|_3 + C_\delta \|\theta\|_1,
$$

which, together with Poincaré’s inequality, (2.62) and (2.64), implies (2.63). \hfill \Box

3 Existence of the nonlinear problem

In this section, we give the proof of the existence for the nonlinear problem (1.5) by using the Tikhonov Theorem which can be found in [20]. For completeness, we state the theorem in the following.

**Theorem 3.1.** *(Tikhonov Theorem, [20, P72, 1.2.11.6]*) Let $M$ be a nonempty bounded closed convex subset of a separable reflexive Banach space $X$ and let $F : M \to M$ be a weakly continuous mapping (i.e., if $x_n \in M$, $x_n \rightharpoonup x$ weakly in $X$, then $F(x_n) \rightharpoonup F(x)$ weakly in $X$ as well). Then $F$ has at least one fixed point in $M$. 22
Define a Banach space \( X \) by
\[
X = \bar{H}^1 \times H^1_0 \times H^1_0,
\]
which can be easily verified to be separable and reflexive.

A convex subset \( K_1(E) \) of \( X \) is defined by
\[
K_1(E) = \{(v, \theta) \in (H^3 \cap H^1_0) \times (H^3 \cap H^1_0) \mid \|v\|_3 + \|\theta\|_3 \leq E\},
\]
where \( E < 1 \) is a small positive constant. By the lower semi-continuity of norms, we easily see that the subset \( K_1(E) \) is also closed in \( X \).

We define a space \( K \) by
\[
K = K_0 \times K_1(E),
\]
where \( K_0 \) is defined by \([2.9]\). Note that \( K \) is a nonempty bounded closed convex subset of \( X \).

Now, we define a nonlinear operator \( N \) from \( K \) to \( X \) by
\[
N(\tilde{U}, \tilde{v}, \tilde{\theta}) := (U, v, \theta),
\]
where \( U \) and \((v, \theta)\) are the solutions of \([2.3]\) and \([2.10]\) for given \((\tilde{U}, \tilde{v}, \tilde{\theta})\), respectively.

Next, we want to find a fixed point \((U, v, \theta)\) of \( N \) in \( K \), such that \((U, v, \theta) = N(U, v, \theta)\), which, together with the existence of weak solution in Lemmas \([2.1]\) and \([2.4]\) gives that \((U, P)\) and \((\eta, v, \theta)\) are solutions of the boundary value problems \([2.1]\) and \([2.2]\), respectively. So \((U + v, \epsilon P + \eta, \theta)\) will be a solution to \([1.5]\). For this purpose, we have to show that \( N \) maps \( K \) into itself and \( N : K \to K \) is a weakly continuous mapping.

**Lemma 3.1.** There is a small constant \( \epsilon_0 > 0 \), depending only on \( \Omega, \mu, \lambda, \mathbf{f} \) and \( \mathbf{g} \), such that for any \( \epsilon \in (0, \epsilon_0) \), \( K \) is a nonempty bounded closed convex subset of \( X \) and \( N(K) \subset K \).

**Proof.** By virtue of the definition, it is obvious that \( K \subset X \) is a nonempty, bounded, closed convex set. Now, we will show that the operator \( N \) maps \( K \) into itself, i.e., \( N(K) \subset K \). To this end, let \((\tilde{U}, \tilde{v}, \tilde{\theta}) \subset K \) and \((U, v, \theta) = N(\tilde{U}, \tilde{v}, \tilde{\theta})\). By Lemmas \([2.1]\) and \([2.2]\), we see that \( U \in K_0 \) for all \( \tilde{U} \in K_0 \). Thus, it suffices to check that \((v, \theta) \in K_1(E) \) for \((\tilde{v}, \tilde{\theta}) \in K_1(E) \). By \([2.7]\) and \([2.3]\), we have
\[
\|U\|_3 + \|\nabla P\|_1 \leq M_1 \quad \text{and} \quad \|U\|_4 + \|\nabla P\|_2 \leq M_2, \quad (3.1)
\]
where \( M_1 = C_3\|h\|_1(\|h\|_1 + 1)^8 \) and \( M_2 = C_4\|h\|_2(\|h\|_2 + 1)^{12} \).

On the other hand, recalling the definition of \( \tilde{F} \) and \( \tilde{G} \), we get from \([3.1]\) that
\[
\|
\tilde{F}\|_1 = \|\epsilon P + \eta\| \mathbf{f} - (\epsilon P + \eta)(U + \tilde{v}) \cdot \nabla (U + \tilde{v}) - \tilde{\theta} \nabla P - P \nabla \tilde{\theta}\|_1 \\
\leq C[\|\eta\|_2(\|f\|_1 + \|U\|_2^2 + \|\tilde{v}\|_2^2) + \epsilon \|P\|_2(\|f\|_1 + \|U\|_2^2 + \|\tilde{v}\|_2^2 + \|\tilde{\theta}\|_2)] \\
\leq C\|\eta\|_2(\|f\|_1 + (M_1 + 1)^2) + \epsilon CM_1(\|f\|_1 + (M_1 + 1)^2), \quad (3.2)
\]
\[
\|
\tilde{G}\|_1 = \|\tilde{\mathbf{v}} - (\epsilon P + \eta)(U + \tilde{v}) \cdot \nabla \tilde{v} + (\epsilon P + \eta)\tilde{\theta} \nabla \tilde{v} + P \nabla \tilde{\theta}\|_1 \\
\leq C[\epsilon(\|U\|_2^2 + \|\tilde{v}\|_2^2) + (\|\eta\|_2 + \epsilon \|P\|_2)(\|U\|_2 + \|\tilde{v}\|_2)(\|\tilde{\theta}\|_2 + \|P\|_2)] \\
\leq C\|\eta\|_2(M_1 + 1) + C(\epsilon(M_1 + 1)^2 + (M_1 + 1)). \quad (3.3)
\]
As a result of Poincaré’s inequality and Lemma \([2.4]\) we have
\[
\|v\|_1 + \|\theta\|_1 \leq C\left\{ \left( E^1_1 + E \right) \|\eta\|_2 + \epsilon \left[ \|\eta\|_2(M_1 + 1) + C\left( \|\eta\|_2(\|f\|_1 + (M_1 + 1)^2) + \epsilon CM_1(\|f\|_1 + (M_1 + 1)^2) + \epsilon(M_1 + 1)^2 + (M_1 + 1) \right) \right] \right\} + \delta \|\eta\|_2
\]
\[ \leq C \left[ E^{\frac{1}{2}} + \epsilon \left( \| f \|_1 + (M_1 + 1)^2 \right) \right] \| \eta \|_2 + \epsilon^2 CM_1 (\| f \|_1 + (M_1 + 1)^2) + CE^2 \\
+ \delta \| \eta \|_2 + \epsilon C_5 (M_1 + 1)^2, \]  \tag{3.4}

where we have used the estimates (3.1), (3.2) and (3.3).

On the other hand, in view of Poincaré’s and Young’s inequalities, (3.4) and (2.63) in Lemma 2.8, we find that

\[ \| \eta \|_2 \leq C (1 + \epsilon) (1 + M_1^2) \left\{ \epsilon \left[ \| \eta \|_2 (\| f \|_1 + (M_1 + 1)^2) + M_1 (\| f \|_1 + (M_1 + 1)^2) \right] + \epsilon^2 \left[ E^2 + (E + E^{\frac{1}{2}}) \right] \| \eta \|_2 \\
+ \epsilon M_1^2 \| \eta \|_2 + E \| \eta \|_2 \right\} + \epsilon \| v \|_3 + \delta \| \theta \|_3 \]

\[ \leq C_{15} (1 + \epsilon) (1 + M_1^2) \left\{ \epsilon \left[ \| f \|_1 + (M_1 + 1)^2 + (M_1 + 1) \right] + (E + E^{\frac{1}{2}}) + \delta \right\} \| \eta \|_2 \\
+ \epsilon \| v \|_3 + \delta \| \theta \|_3 + C_{15} (1 + \epsilon) (1 + M_1^2) \left\{ \epsilon (M_1 + 1)^2 \\
+ \epsilon \left[ M_1 (\| f \|_1 + (M_1 + 1)^2) + (M_1 + 1)^2 \right] + E^2 \right\}, \]  \tag{3.5}

where \( C_{15} \) is a positive constant.

Combining (2.61) with (3.2)–(3.5), we conclude that there is a constant \( C_{16} \), such that

\[ \| v \|_3 + \| \theta \|_3 + \| \eta \|_2 \leq C_{16} (1 + M_1^5) \left\{ \epsilon \left[ \| f \|_1 + (M_1 + 1)^2 \right] + (E + E^{\frac{1}{2}}) + \delta \right\} \| \eta \|_2 \\
+ \epsilon M_1^2 \| v \|_3 + \delta M_1^2 \| \theta \|_3 + C_{16} (1 + M_1)^5 \left\{ \epsilon (M_1 + 1)^2 \\
+ \epsilon (1 + M_1) \left[ \| f \|_1 + (M_1 + 1)^2 \right] + E^2 \right\}. \]  \tag{3.6}

Thus, first taking \( \delta \) small enough and then choosing \( \epsilon_0 \) and \( E \) suitably small, such that

\[ C_{16} (1 + M_1)^5 \left\{ \epsilon_0 \left[ \| f \|_1 + (M_1 + 1)^2 \right] + (E + E^{\frac{1}{2}}) \right\} < 1, \quad \epsilon_0 M_1^2 < 1, \]

\[ C_{16} (1 + M_1)^5 \left\{ \epsilon_0 (M_1 + 1)^2 + \epsilon_0 (1 + M_1) \left[ \| f \|_1 + (M_1 + 1)^2 \right] + E^2 \right\} < E, \]

we deduce from (3.6) that for all \( \epsilon \in (0, \epsilon_0) \),

\[ \| v \|_3 + \| \theta \|_3 + \| \eta \|_2 \leq E, \]

which gives \( \| v \|_3 + \| \theta \|_3 \leq E \) immediately. This completes the proof. \( \Box \)

**Lemma 3.2.** Let \( N, X, K_0 \) and \( K_1(E) \) be the same as in Lemma 3.1. Then \( N : K \to K \) is a weakly continuous mapping.

**Proof.** By the definition of weakly continuous mapping (see, for example, [20] P72,1.4.11.6), it suffices to prove that \( N \) is continuous on \( K \) in the norm of \( X \).

Let \( (U_i, v_i, \theta_i) = N(\tilde{U}_i, \tilde{v}_i, \tilde{\theta}_i), \ i = 1, 2. \) In particular, let \( (U_i, P_i) \in (H^4 \cap H^1_{0,\sigma}) \times \bar{H}^3 \) and \( (\eta, v_i, \theta_i) \in \bar{H}^2 \times (H^3 \cap H^1) \times (H^3 \cap H^1) \) be the solutions of (2.3) and (2.10) for given \( (\tilde{U}_i, \tilde{v}_i, \tilde{\theta}_i) \) respectively, i.e.,

\[ \begin{cases}
(\tilde{U}_i + \tilde{v}_i) \cdot \nabla U_i - \mu \Delta U_i + \nabla P_i = \mathbf{h}, \\
div U_i = 0;
\end{cases} \tag{3.7} \]
and
\[
\begin{aligned}
&U_i \cdot \nabla \eta_i + \frac{\text{div} \psi}{\epsilon} = -\tilde{v}_i \cdot \nabla \eta_i - \eta_i \text{div} \tilde{v}_i - \epsilon \text{div}(P_i(U_i + \tilde{v}_i)), \\
&U_i \cdot \nabla \psi_i - \mu \Delta \psi_i - \zeta \nabla \text{div} \psi_i + \frac{\nabla \eta_i + \nabla \theta_i}{\epsilon} = \epsilon \tilde{F}_i - \tilde{v}_i \cdot \nabla \tilde{v}_i - \tilde{\theta}_i \nabla \eta_i - \eta_i \nabla \tilde{\theta}_i, \\
&U_i \cdot \nabla \theta_i - \kappa \Delta \theta_i + \frac{\text{div} \psi_i}{\epsilon} = \epsilon \tilde{G}_i - \tilde{v}_i \cdot \nabla \tilde{v}_i - \eta_i \text{div} \tilde{v}_i - \tilde{\theta}_i \text{div} \tilde{v}_i, \\
\end{aligned}
\tag{3.8}
\]
where the force $\tilde{F}_i$ and heat source $\tilde{G}_i$ are given by
\[
\begin{aligned}
\tilde{F}_i &= (\epsilon P_i + \eta_i) f - (\epsilon P_i + \eta_i)(U_i + \tilde{v}_i) \cdot \nabla (U_i + \tilde{v}_i) - \tilde{\theta}_i \nabla P_i - P_i \nabla \tilde{\theta}_i, \\
\tilde{G}_i &= \tilde{\Psi}_i - (\epsilon P_i + \eta_i)(U_i + \tilde{v}_i) \cdot \nabla \tilde{\theta}_i + (\epsilon P_i + \eta_i) \tilde{\theta}_i \text{div} \tilde{v}_i + P_i \text{div} \tilde{v}_i.
\end{aligned}
\]

Now, if we set
\[
\begin{aligned}
W &= U_2 - U_1, \\
\tilde{W} &= \tilde{U}_2 - \tilde{U}_1, \\
Q &= P_2 - P_1, \\
\xi &= \eta_2 - \eta_1, \\
w &= v_2 - v_1, \\
\tilde{w} &= \tilde{v}_2 - \tilde{v}_1, \\
\beta &= \theta_2 - \theta_1, \\
\tilde{\beta} &= \tilde{\theta}_2 - \tilde{\theta}_1,
\end{aligned}
\]
then, we can have the following systems:
\[
\begin{aligned}
\begin{cases}
(U_1 + \tilde{v}_1) \cdot \nabla W - \mu \Delta W + \nabla Q &= -(\tilde{W} + \tilde{w}) \cdot \nabla U_2, \\
\text{div} W &= 0,
\end{cases}
\tag{3.9}
\end{aligned}
\]
and
\[
\begin{aligned}
\begin{cases}
U_1 \cdot \nabla \xi + \frac{\text{div} w}{\epsilon} = -\text{div} (\tilde{v}_1 \xi + \tilde{w} \eta_2) - \epsilon \text{div}(P_1(W + \tilde{w}) + Q(U_2 + \tilde{v}_2)) - W \cdot \nabla \eta_2, \\
U_1 \cdot \nabla w - \mu \Delta w - \zeta \nabla \text{div} w + \frac{\nabla \xi + \nabla \beta}{\epsilon} = \epsilon J - \tilde{w} \cdot \nabla \tilde{v}_2 - \tilde{v}_1 \cdot \nabla \tilde{w} - W \cdot \nabla v_2 - \nabla (\tilde{\theta}_1 \xi + \tilde{\beta} \eta_2), \\
U_1 \cdot \nabla \beta - \kappa \Delta \beta + \frac{\text{div} w}{\epsilon} = \epsilon I - \tilde{w} \cdot \nabla \tilde{\theta}_2 - \tilde{v}_1 \cdot \nabla \tilde{\beta} - W \cdot \nabla \theta_2 - \xi \text{div} \tilde{v}_1 - \eta_2 \text{div} \tilde{w} - \tilde{\beta} \text{div} \tilde{v}_1 - \tilde{\theta}_2 \text{div} \tilde{w},
\end{cases}
\tag{3.10}
\end{aligned}
\]
where $J$ and $I$ read as
\[
\begin{aligned}
J &= (\epsilon Q + \xi) f - (\epsilon Q + \xi)(U_2 + \tilde{v}_2) \cdot \nabla (U_2 + \tilde{v}_2) \\
&- (\epsilon P_1 + \eta_1)((W + \tilde{w}) \cdot \nabla (U_2 + \tilde{v}_2) + (U_1 + \tilde{v}_1) \cdot \nabla (W + \tilde{w}))) - \nabla (\tilde{\beta} P_1 + Q \tilde{\theta}_2), \\
I &= 2\mu D(W + \tilde{w}) : D(U_2 + \tilde{v}_2) + 2\mu D(U_1 + \tilde{v}_1) : D(W + \tilde{w}) \\
&+ \lambda \text{div}(W + \tilde{w}) \cdot \text{div}(U_2 + \tilde{v}_2) + \lambda \text{div}(U_1 + \tilde{v}_1) \cdot \text{div}(W + \tilde{w}) \\
&- (\epsilon Q + \xi)(U_2 + \tilde{v}_2) \cdot \nabla \tilde{\theta}_2 - (\epsilon P_1 + \eta_1)((W + \tilde{w}) \cdot \nabla \tilde{\theta}_2 + (U_1 + \tilde{v}_1) \cdot \nabla \tilde{\beta}) \\
&+ (\epsilon Q + \xi) \tilde{\beta} \text{div} \tilde{v}_2 - (\epsilon P_1 + \eta_1)(\tilde{\beta} \text{div} \tilde{v}_2 + \tilde{\theta}_1 \text{div} \tilde{w}) + Q \text{div} \tilde{v}_2 + P_1 \text{div} \tilde{w},
\end{aligned}
\]
Note that $J$ and $I$ can be bounded as follows.
\[
\begin{aligned}
\|J\|_0 &\leq (\epsilon \|Q\|_1 + \|\xi\|_1)(\|f\|_2 + \|U_2\|_2^2 + \|\tilde{v}_2\|_2^2) + (\epsilon \|P_1\|_2 + \|\eta_1\|_2)(\|W\|_1 + \|\tilde{w}\|_1) \\
&\cdot (\|U_2\|_2 + \|\tilde{v}_2\|_2 + \|U_1\|_2 + \|\tilde{v}_1\|_2) + \|\tilde{\theta}_2\|_2 \|Q\|_1 + \|\tilde{\beta}\|_1 \|P_1\|_2 \\
\tag{3.11}
\end{aligned}
\]
\[ ||I||_0 \leq C \{ ||W||_1 + ||\hat{w}||_1 ||U_1||_2 + ||U_2||_2 + ||\hat{v}_1||_2 + ||\hat{v}_2||_2 + \epsilon ||Q||_1 + ||\xi||_1 \} (||U_2||_2 + ||\hat{v}_2||_2) \]
\[ \cdot ||\hat{\theta}_2||_2 + \epsilon ||P_1||_2 + ||\eta_1||_2 \} (||W||_1 + ||\hat{w}||_1 ||\hat{\theta}_2||_2 + ||U_1||_2 + ||\hat{v}_1||_2 ||\beta||_1) \]
\[ + \epsilon ||P_1||_2 + ||\eta_1||_2 \} (||\hat{\beta}||_1 ||\hat{v}_2||_2 + ||\hat{\theta}_1||_2 ||\hat{w}||_1) + ||Q||_1 ||\hat{v}_2||_2 + ||P_1||_2 ||\hat{w}||_1 \} \]

(3.12)

On the one hand, we multiply (3.9) by \( W \) and make use of Poincaré’s inequality to deduce that
\[ \left( \mu - \frac{C ||\hat{v}||_3}{2} \right) \| \nabla W \|^2_0 \leq C \{ ||\hat{W}||_1 + ||\hat{w}||_1 \} ||U_2||_3 ||W||_1, \]
where \( C \) is a positive constant depending only on \( \Omega \) and \( \mu \). Consequently,
\[ ||W||_1 \leq C (||\hat{W}||_1 + ||\hat{w}||_1) \] (3.13)
for some positive constant \( C \) depending only on \( \Omega, \mu, \lambda, f, E \) and \( \epsilon_0 \).

By the classical estimates for the Stokes equations
\[ \begin{cases}
\mu \Delta u + \nabla \xi = - \left( \hat{W} + \hat{w} \right) \cdot \nabla u - \left( \hat{U}_1 + \hat{v}_1 \right) \cdot \nabla \xi \\
div \hat{W} = 0,
\end{cases} \]
we obtain that
\[ ||W||_2 + ||\nabla \xi||_0 \leq C \{ ||\hat{W} + \hat{w}||_1 ||U_2||_2 + ||\hat{U}_1 + \hat{v}_1||_2 \} ||W||_1 \]
\[ \leq C (||\hat{W}||_1 + ||\hat{w}||_1), \] (3.14)
where the estimate (3.13) has been used.

On the other hand, if we multiply (3.10)_1, (3.10)_2 and (3.10)_3 by \( \xi, w \) and \( \beta \) in \( L^2 \) respectively, we find that
\[ \mu ||\nabla \xi||^2_0 + \zeta ||dive\xi||^2_0 + \kappa ||\nabla \beta||^2_0 \]
\[ = \int \left[ \xi div \hat{v} + \eta_2 div \hat{w} + \hat{v}_1 \cdot \nabla \xi + \hat{w} \cdot \nabla \eta_2 + \epsilon div (P_1 (W + \hat{w}) + Q (U_2 + \hat{v}_2)) + W \cdot \nabla \xi_1 \right] dx \]
\[ + \int \left[ \epsilon J - \hat{w} \cdot \nabla \hat{v}_2 - \hat{v}_1 \cdot \nabla \hat{w} - W \cdot \nabla v_2 - \nabla (\hat{\theta}_1 \xi + \hat{\beta} \eta_2) \right] \cdot w dx \]
\[ + \int \left[ \epsilon D - \hat{w} \cdot \nabla \hat{\theta}_2 - \hat{v}_1 \cdot \nabla \hat{\beta} - W \cdot \nabla \theta_2 - \xi div \hat{v}_1 - \eta_2 div \hat{w} - \hat{\beta} div \hat{v}_1 - \hat{\theta}_1 div \hat{w} \right] \beta dx \]
\[ \leq C \left\{ ||\xi||^2_0 ||\hat{v}||_3 + ||\xi||_1 ||\hat{w}||_1 ||\eta_2||_1 + ||\xi||_0 \left[ ||P_1||_2 (||W||_1 + ||\hat{w}||_1) + ||Q||_1 (||U_2||_2 + ||\hat{v}_2||_2) \right] \right\} \]
\[ + ||W||_1 ||\eta_2||_3 ||\xi||_1 \} \}
\[ + C \left\{ \epsilon ||J||^2_0 + ||\hat{w}||^2_1 ||\hat{v}_2||_2 + ||\hat{v}_1||^2_2 + ||W||^2_1 ||v_2||_2 + ||\hat{\theta}_2||^2_1 ||\xi||_1 \}
\[ + ||\xi||^2_0 ||\eta_2||^2_1 \} \}
\[ + C \left\{ \epsilon ||J||^2_0 + ||\hat{w}||^2_1 ||\hat{\theta}_2||_2 + ||\hat{v}_1||^2_2 ||\hat{\beta}||_2 + ||W||^2_1 ||\theta_2||_2 + ||\xi||^2_0 ||\hat{v}_1||_2 \}
\[ + ||\eta_2||^2_1 ||\hat{w}||^2_1 + ||\hat{v}_1||^2_2 + ||\hat{\theta}_1||^2_2 ||\hat{w}||_1 \} \}
\[ + \delta (||w||^2_0 + ||\beta||^2_0). \] (3.15)

Also, from the Stokes equations
\[ \begin{cases}
- \mu \Delta \xi + \nabla \xi = \epsilon \left( \mu J - \hat{w} \cdot \nabla \hat{v}_2 - \hat{v}_1 \cdot \nabla \hat{w} - W \cdot \nabla v_2 - \nabla (\hat{\theta}_1 \xi + \hat{\beta} \eta_2) + \xi \nabla div \xi \right) - \nabla \beta, \\
div \hat{w} = \hat{w},
\end{cases} \]
we get the following estimate
\[ ||w||_2 + ||\nabla \xi||_0 \leq C \left( ||div w||_2 + \epsilon ||J||_0 + ||\hat{w}||_1 ||\hat{v}_2||_2 + ||\hat{v}_1||_2 + ||W||_1 ||v_2||_2 \]
\[ + ||\hat{\theta}_1||_2 ||\xi||_1 + ||\hat{\beta}||_1 ||\eta_2||_2 \} \}
\[ + C ||\beta||_1. \] (3.16)
Applying Poincaré’s inequality and substituting (3.16) into (3.15), employing the estimates (3.11)–(3.14) and recalling the smallness of \( \epsilon_0 \) and \( E \), we conclude that

\[
\|W\|_1 + \|w\|_1 + \|\beta\|_1 \leq C(\|\bar{W}\|_1 + \|\bar{w}\|_1 + \|\bar{\beta}\|_1),
\]

where \( C \) is a positive constant depending only on \( \Omega, \mu, \lambda, f, E \) and \( \epsilon_0 \). This completes the proof.

Finally, having had Lemmas 3.1 and 3.2, we can apply the Tikhonov fixed point theorem to find a fixed point \((U, v, \theta) = N(U, v, \theta)\) in the set \( K \). Moreover, the pressure \( P \in \bar{H}^2 \) satisfies

\[
\nabla P = f + g + \mu \triangle U - (U + v) \cdot \nabla U,
\]

and \((U + v, \epsilon P + \eta, \theta)\) is a solution to (1.5). Thus, we have shown the following proposition.

**Proposition 3.1.** Let \( f, g \in H^2(\Omega) \). Then, there exists an \( \epsilon_0 \) depending only on \( \Omega, \mu, \lambda, f \) and \( g \), such that for all \( \epsilon \in (0, \epsilon_0) \), there is a solution \((U, P, v, \eta, \theta) \in (H^4 \cap H_0^1) \times \bar{H}^2 \times (H^3 \cap H_0^1) \times \bar{H}^2 \times (H^3 \cap H_0^1)\) of (2.1) and (2.2), satisfying

\[
\|v\|_3 + \|\eta\|_2 + \|\theta\|_3 \leq E,
\]

where \( E \) is a small positive constant depending only on \( \Omega, \mu, \lambda, f \) and \( g \). Moreover, \((U + v, \epsilon P + \eta, \theta)\) is a solution of the system (1.5) for any \( \epsilon \in (0, \epsilon_0) \).

## 4 Incompressible limit

Let \( \epsilon < \epsilon_0 \) and \((U^\epsilon, v^\epsilon, \theta^\epsilon) \in K\) be the solution established in Proposition 3.1. We take \( v = \tilde{v} = v^\epsilon \), \( \theta = \tilde{\theta} = \theta^\epsilon \) and \( \eta = \eta^\epsilon \) in (3.16) to get that

\[
\|v^\epsilon\|_3 + \|\theta^\epsilon\|_3 + \|\eta^\epsilon\|_2 \leq C_{16}(1 + M_1)^5 \left\{ \epsilon \left[ \|f\|_1 + (M_1 + 1)^2 \right] + (E + E^2) + \delta \right\} \|\eta^\epsilon\|_2
\]

\[
+ (\epsilon M_1^2 + E) \|v\|_3 + (\delta M_1^3 + E) \|\theta^\epsilon\|_3
\]

\[
+ C_{16}(1 + M_1)^5 \left\{ \epsilon(M_1 + 1)^2 + \epsilon(1 + M_1) \left[ \|f\|_1 + (M_1 + 1)^2 \right] \right\}.
\]

Thus, by taking \( \epsilon_0 \) and \( E \) so small that

\[
C_{16}(1 + M_1)^5 \left\{ \epsilon \left[ \|f\|_1 + (M_1 + 1)^2 \right] + (E + E^2) \right\} < 1, \quad \epsilon M_1^2 + E < 1,
\]

we obtain

\[
\|v^\epsilon\|_3 + \|\theta^\epsilon\|_3 + \|\eta^\epsilon\|_2 \leq C_{16}(1 + M_1)^5 \left\{ \epsilon(M_1 + 1)^2 + \epsilon(1 + M_1) \left[ \|f\|_1 + (M_1 + 1)^2 \right] \right\},
\]

whence,

\[
\|v^\epsilon\|_3 + \|\theta^\epsilon\|_3 + \|\eta^\epsilon\|_2 \to 0, \quad \text{as} \quad \epsilon \to 0.
\]

(4.1)

Furthermore, from (2.2), i.e.,

\[
U^\epsilon \cdot \nabla \eta^\epsilon + \frac{\text{div}v^\epsilon}{\epsilon} = -v^\epsilon \cdot \nabla \eta^\epsilon - \eta^\epsilon \text{div}v^\epsilon - \epsilon \text{div}(P^\epsilon(U^\epsilon + v^\epsilon))
\]

and (1.1) we get that as \( \epsilon \to 0 \),

\[
\left\| \frac{\text{div}v^\epsilon}{\epsilon} \right\|_1 \leq \|v^\epsilon \cdot \nabla \eta^\epsilon\|_1 + \|\eta \text{div}v^\epsilon\|_1 + \|\epsilon \text{div}(P^\epsilon(U^\epsilon + v^\epsilon))\|_1 + \|U^\epsilon \cdot \nabla \eta^\epsilon\|_1 \to 0.
\]

(4.2)
Due to (4.1) and

\[
\frac{\nabla \eta^\epsilon + \nabla \theta^\epsilon}{\epsilon} = \epsilon F^\epsilon - v^\epsilon \cdot \nabla v^\epsilon - \theta^\epsilon \nabla \eta^\epsilon - \eta^\epsilon \nabla \theta^\epsilon - U^\epsilon \cdot \nabla v^\epsilon + \mu \Delta v^\epsilon + (\mu + \lambda) \nabla \text{div} v^\epsilon
\]

with \( F^\epsilon = (\epsilon P^\epsilon + \eta^\epsilon) f - (\epsilon P^\epsilon + \eta^\epsilon) (U^\epsilon + v^\epsilon) \cdot \nabla (U^\epsilon + v^\epsilon) - \theta^\epsilon \nabla P^\epsilon - P^\epsilon \nabla \theta^\epsilon \), which comes from the transform of (2.2), one deduces, recalling Poincaré’s inequality, that

\[
\frac{\|\eta^\epsilon + \theta^\epsilon\|}{\epsilon} \rightarrow 0, \quad \text{as} \quad \epsilon \rightarrow 0. \tag{4.3}
\]

On the other hand, in view of Lemma 2.2, we observe that \((U^\epsilon, P^\epsilon)\) is a uniform-in-\(\epsilon\) bounded sequence in \((H^4 \cap H^1_0) \times \bar{H}^3\). Hence, there is a subsequence of \((U^{\epsilon_k}, P^{\epsilon_k})\), still denoted by \((U^{\epsilon_k}, P^{\epsilon_k})\) for simplicity, and \((\bar{U}, \bar{P}) \in (H^4 \cap H^1_0) \times \bar{H}^3\), such that as \(\epsilon \rightarrow 0\),

\[
(U^\epsilon, P^\epsilon) \rightarrow (\bar{U}, \bar{P}), \quad \text{weakly in} \quad (H^4 \cap H^1_0) \times \bar{H}^3,
\]

and

\[
(U^\epsilon, P^\epsilon) \rightarrow (\bar{U}, \bar{P}), \quad \text{strongly in} \quad (H^3 \cap H^1_0) \times \bar{H}^2.
\]

Thus, if we take to the limit as \(\epsilon \rightarrow 0\) in (2.1) and (2.2), we conclude that \((\bar{U}, \bar{P})\) is a solution of the steady incompressible Naiver-Stokes equations (1.7).

In conclusion, we have that

\[
\lim \inf_{\epsilon \rightarrow 0; U, P \in L} \|U^\epsilon + v^\epsilon - U\|_3 + \|P^\epsilon + \frac{\eta^\epsilon + \theta^\epsilon}{\epsilon} - P\|_2 + \|\theta^\epsilon\|_3 = 0,
\]

where \(L\) is the same as in Theorem 1.1. Thus, the proof of the low Mach number limit is completed.

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