Abstract. We present a formalization of the well-known thesis that, in the case of independent identically distributed random variables $X_1, \ldots, X_n$ with power-like tails of index $\alpha \in (0, 2)$, large deviations of the sum $X_1 + \cdots + X_n$ are primarily due to just one of the summands.

1. INTRODUCTION, SUMMARY, AND DISCUSSION. Let $X_1, X_2, \ldots$ be independent identically distributed random variables. For each natural $n$, let $S_n := \sum_{i=1}^n X_i$.

Heyde [3] showed the following: Suppose that, for some sequence $(B_n)$ of positive real numbers, $S_n/B_n$ converges in distribution to a stable law of index $\alpha \in (0, 2) \setminus \{1\}$, whose support is the entire real line $\mathbb{R}$. (For a definition and basic properties of stable laws, see e.g. [6, section IV.3].) Then, for any sequence $(x_n)$ going to $\infty$,

$$P(|S_n| > x_nB_n) \sim P(\max_{1 \leq i \leq n} |X_i| > x_nB_n).$$  \hspace{1cm} (1)

As indicated in [3], one-sided analogs of (1) could also be obtained, even in the case $\alpha = 1$. However, such a task would involve additional technical difficulties.

The conditions in [3] for (1) imply that the tail of the distribution of each $X_i$ is power-like—more specifically,

$$P(|X_1| > u) = u^{-\alpha+o(1)} \text{ as } u \to \infty.$$  \hspace{1cm} (2)

This work by Heyde was followed by a large number of publications, including [1, 4, 5, 7].

The asymptotic equivalence (1) and, especially, its proof suggest the well-known interpretation that, in the cases of power-like tails as in (2), large deviations of the sum $S_n$ are mainly due to just one of the summands $X_1, \ldots, X_n$.

In this note, we present a formal version of this interpretation:

**Theorem 1.** Take any $\alpha \in (0, 2)$. Let $X_1, X_2, \ldots$ and $S_n$ be as in the first paragraph of this note. To avoid technicalities, suppose that the distribution of $X_1$ is symmetric about 0 and has a probability density function $f$ such that

$$f(u) \asymp u^{-1-\alpha} \text{ as } u \to \infty.$$  \hspace{1cm} (3)

(cf. (2)). Then

$$P(S_n > x) \sim P \left( S_n > x, \bigcup_{i \in [n]} \{ X_i > x, |S_n - X_i| \leq bx, \max_{j \in [n] \setminus \{i\}} |X_j| \leq cx \} \right)$$  \hspace{1cm} (4)

whenever $n \in \mathbb{N}$, $x \in (0, \infty)$, $c \in (0, 1)$, and $b \in (0, 1)$ vary in such a way that

$$n \ll x^\alpha.$$  \hspace{1cm} (5)
\[nx^{-\alpha} < c^{2\alpha}, \quad (6)\]
\[nx^{-\alpha} < b^2 c^{\alpha - 2}. \quad (7)\]

Here, as usual, \(\mathbb{N} := \{1, 2, \ldots\}\) and \([n] := \{1, \ldots, n\}\) for \(n \in \mathbb{N}\). For positive expressions \(E\) and \(F\) (in terms of \(x, n, c, b\)), we write (i) \(E \sim F\) if \(E/F \to 1\); (ii) \(E \ll F\) or, equivalently, \(F \gg E\) if \(E = o(F)\)—that is, if \(E/F \to 0\); (iii) \(E \preccurlyeq F\) or, equivalently, \(F \succeq E\) if \(\limsup E/F < \infty\); and (iv) \(E \asymp F\) if \(E \preccurlyeq F \preccurlyeq E\). The “much smaller than” sign \(\ll\) should not be confused with Vinogradov’s symbol \(\ll\) (the latter is usually taken to mean the same as \(\lesssim\)).

**Proposition 2.** For \(S_n\) as in Theorem 1 and for all \(n \in \mathbb{N}\) and \(x \in (0, \infty)\), we have \(P(S_n > x) \to 0\) if and only if condition (5) holds. Moreover, if either (5) holds or \(P(S_n > x) \to 0\), then \(P(S_n > x) \asymp nx^{-\alpha}\).

**Remark 3.** Condition \(P(S_n > x) \to 0\) means precisely that \(P(S_n > x)\) is a large-deviation probability for \(S_n\). So, in view of Proposition 2, Theorem 1 concerns all the large deviations of \(S_n\). \(\blacksquare\)

**Remark 4.** Given (6), for (7) to hold it is enough that \(b \asymp c\) or even \(b > \asymp c^{1+\alpha/2}\). Therefore and because the probability on the right-hand side of (4) is non-decreasing in \(c\) and in \(b\), without loss of generality
\[c \ll 1 \quad \text{and} \quad b \ll 1. \quad (8)\]

So, (4) shows that the large deviation event \(\{S_n > x\}\) is mainly due to just one of the summands \(X_1, \ldots, X_n\). More specifically, (4) tells us that, given \(S_n > x\), the conditional probability that exactly one of the \(X_i\)'s is \(> x\) while the absolute values of the other \(X_i\)'s and of the sum of the other \(X_i\)'s are all \(o(x)\) is close to 1. \(\blacksquare\)

**Remark 5.** In contrast with (1), the condition \(n \to \infty\) is not required in Theorem 1; in particular, \(n\) may be fixed there. However, it is clear that condition (5) in Theorem 1 necessarily implies that \(x \to \infty\). In another distinction from (1), in Theorem 1 the common distribution of the \(X_i\)'s is not required to be in the domain of attraction of a stable law. \(\blacksquare\)

**2. PROOFS.**

*Proof of Theorem 1.* This proof is based on two lemmas. To state the lemmas, let us introduce the following notations:

\[p_0(n, x) := P\left(S_n > x, \max_{j \in [n]} |X_j| \leq cx\right), \quad (9)\]
\[p_{\geq 2}(n, x) := P\left(S_n > x, \bigcup_{i \in [n]} \bigcup_{j \in [n] \setminus \{i\}} \{|X_i| > cx, |X_j| > cx\}\right), \quad (10)\]
\[p_{1,0}(n, x) := P\left(S_n > x, \bigcup_{i \in [n]} \{cx < |X_i| \leq x, \max_{j \in [n] \setminus \{i\}} |X_j| \leq cx\}\right), \quad (11)\]
\[p_{1,-}(n, x) := P\left(S_n > x, \bigcup_{i \in [n]} \{X_i < -x, \max_{j \in [n] \setminus \{i\}} |X_j| \leq cx\}\right), \quad (12)\]
\[p_{1,+}(n, x) := P\left(S_n > x, \bigcup_{i \in [n]} \{X_i > x, \max_{j \in [n] \setminus \{i\}} |X_j| \leq cx\}\right). \quad (13)\]
Lemma 6. For \( n \) and \( x \) as in the conditions of Theorem 1 (that is, for \( n \in \mathbb{N} \) and \( x \in (0, \infty) \) such that (5) holds), we have
\[
P(S_n > x) \gtrsim n P(X_1 > x) \asymp nx^{-\alpha}.
\] (14)

Proof. By (3),
\[
P(X_1 > u) \asymp u^{-\alpha} \quad \text{as} \quad u \to \infty.
\] (15)
So, in view of (5), \( n P(X_1 > x) \asymp nx^{-\alpha} << 1 \). Now (14) follows from \([2, \text{ inequality V, (5.10)}]\), which immediately implies \( P(S_n > x) \geq \frac{1}{4} (1 - e^{-2n P(X_1 > x)}) \) (since the distribution of \( X_1 \) is symmetric and absolutely continuous). \( \blacksquare \)

Lemma 7. For \( n, x, c \) as in the conditions of Theorem 1,
\[
p_0(n, x) \ll nx^{-\alpha},
\] (16)
\[
p_{\geq 2}(n, x) \ll nx^{-\alpha},
\] (17)
\[
p_{1,0}(n, x) \ll nx^{-\alpha},
\] (18)
\[
p_{1,1,-}(n, x) \ll nx^{-\alpha}.
\] (19)

Proof. For all natural \( i \), let
\[
Y_i := X_i \mathbf{1}(|X_i| \leq cx),
\]
where \( \mathbf{1}(A) \) denotes the indicator of an assertion \( A \), so that \( \mathbf{1}(A) = 1 \) if \( A \) is true, and \( \mathbf{1}(A) = 0 \) if \( A \) is false. Then the \( Y_i \)'s are independent identically distributed symmetric random variables. Also, by (6) and (5), \( (cx)^{2\alpha} \gg nx^a \gg 1 \), so that \( cx \gg 1 \). Therefore, in view of (3), for some real \( A > 0 \) we have
\[
E Y_1^2 \lesssim \int_0^A u^2 f(u) \, du + \int_{cx}^{\infty} u^2 u^{-1-\alpha} \, du \asymp (cx)^{2-\alpha}.
\]
Therefore, with
\[
T_n := \sum_i Y_i,
\]
by (9), Markov’s inequality, and (8),
\[
p_0(n, x) \leq P(T_n > x) \leq \frac{E T_n^2}{x^2} = \frac{n E Y_1^2}{x^2} \lesssim \frac{c^{2-\alpha} n x^{-\alpha}}{x^a} \ll nx^{-\alpha}.
\] (20)
So, (16) is proved.
Next, by (10), (15), and (6),
\[
p_{\geq 2}(n, x) \leq \left(\frac{n}{2}\right) P(|X_1| > cx, |X_2| > cx) \leq n^2 (cx)^{-2a} \ll nx^{-a},
\]
which proves (17).
Further, using (11), (3), and Markov’s inequality as in (20), we have
\[
p_{1,0}(n, x) = n P(S_n > x, cx < |X_1| \leq x, |X_2| \leq cx, \ldots, |X_n| \leq cx)
\leq n P(cx < |X_1| \leq x, Y_2 + \cdots + Y_n > x - X_1)
\asymp n \int_{cx}^{x} u^{-1-\alpha} P(Y_2 + \cdots + Y_n > x-u) \, du \lesssim I,
\] (21)
where
\[ I := \int_{cx}^{x} g(u) \, du, \quad g(u) := nu^{-1-a} \min\left(1, \frac{n(cx)^{2-a}}{(x-u)^2}\right). \]

Next,
\[ u_x := x - n^{1/2}(cx)^{1-a/2} \sim x, \quad (22) \]
by conditions (6) and (8) on \( c \). It follows that
\[ I = I_1 + I_2 + I_3, \]
where
\[ I_1 := \int_{cx}^{x/2} g(u) \, du \leq \int_{cx}^{\infty} nu^{-1-a} \frac{n(cx)^{2-a}}{(x/2)^2} \, du \propto \left(\frac{n}{x^a}\right)^2 c^{2-2a} \ll nx^{-a}, \]
again by the mentioned conditions on \( c \);
\[ I_2 := \int_{x/2}^{u_x} g(u) \, du \leq \int_{-\infty}^{u_x} n(x/2)^{-1-a} \frac{n(cx)^{2-a}}{(x-u)^2} \, du \]
\[ \propto \left(\frac{n}{x^a}\right)^{3/2} c^{1-a/2} \ll nx^{-a}, \]
once again by the conditions on \( c \); and, in view of the definition of \( u_x \) in (22),
\[ I_3 := \int_{u_x}^{x} g(u) \, du \leq (x-u_x)nu_x^{-1-a} \propto \left(\frac{n}{x^a}\right)^{3/2} c^{1-a/2} \ll nx^{-a}, \]
as in the bounding of \( I_2 \). So, the bound on \( p_{1,0}(n, x) \) in (18) follows immediately from (21) and the bounds on the integrals \( I_1, I_2, I_3 \).

Finally, in view of the definition of \( p_{1,1,\ldots}(n, x) \) in (12),
\[ p_{1,1,\ldots}(n, x) = n P\left(S_n > x, X_1 < -x, \max_{j \in [n], \{1\}} |X_j| \leq cx\right) \]
\[ \leq n P\left(S_n - X_1 > x, X_1 < -x, \max_{j \in [n], \{1\}} |X_j| \leq cx\right) \]
\[ \leq n P(T_n - Y_1 > x, X_1 < -x) \]
\[ = P(T_n - Y_1 > x) n P(X_1 < -x) \propto \frac{nc^{2-a}}{x^a} \frac{n}{x^a} \ll nx^{-a}. \]
The \( \ll \) comparison here is obtained by bounding \( P(T_n - Y_1 > x) \) similarly to the bounding of \( P(T_n > x) \) in (20) and using the symmetry of the distribution of \( X_1 \), the condition \( x \to \infty \), and the relation (15); the \( \ll \) comparison in the above multiline display follows, yet again, by the conditions on \( c \). So, (19) is proved as well.

This completes the proof of Lemma 7.

Now we can complete the proof of Theorem 1. Note that
\[ P(S_n > x) = p_0(n, x) + p_{\geq 2}(n, x) + p_{1,0}(n, x) + p_{1,1,\ldots}(n, x) + p_{1,1,\ldots}(n, x). \]
So, by Lemmas 7 and 6,
\[ P(S_n > x) \sim p_{1,1,+}(n, x). \]  

(23)

Finally, the difference between \( p_{1,1,1}(n, x) \) and the probability on the right-hand side of (4) is
\[
\begin{align*}
&\leq n \mathbb{P}\left( X_1 > x, |S_n - X_1| > bx, \max_{j \in [n]\setminus\{1\}} |X_j| \leq cx \right) \\
&\leq n \mathbb{P}(X_1 > x) \mathbb{P}(|T_n - Y_1| > bx) \\
&\leq \mathbb{P}(S_n > x) \frac{n(cx)^{2-\alpha}}{(bx)^2} \ll \mathbb{P}(S_n > x);
\end{align*}
\]

the \( \leq \) comparison here is obtained using the \( \geq \) comparison in (14) and bounding \( \mathbb{P}(T_n - Y_1 > bx) \) similarly to the bounding of \( \mathbb{P}(T_n > x) \) in (20); and the latter \( \ll \) comparison follows by (7). Now (4) follows from (23).

The proof of Theorem 1 is complete.

Proof of Proposition 2. Suppose first that condition (5) holds. Then, by (23), (13), and (14), \( \mathbb{P}(S_n > x) \sim p_{1,1,1}(n, x) \leq n \mathbb{P}(X_1 > x) \asymp nx^{-\alpha} \leq \mathbb{P}(S_n > x) \), so that \( \mathbb{P}(S_n > x) \asymp nx^{-\alpha} \rightarrow 0 \).

On the other hand, if \( \mathbb{P}(S_n > x) \rightarrow 0 \), then, by the inequality \( \mathbb{P}(S_n > x) \geq \frac{1}{3} (1 - e^{-2n \mathbb{P}(X_1 > x)}) \) in the proof of Lemma 6, we have \( n \mathbb{P}(X_1 > x) \rightarrow 0 \), and hence \( \mathbb{P}(X_1 > x) \rightarrow 0 \) and \( x \rightarrow \infty \). So, \( \mathbb{P}(S_n > x) \geq n \mathbb{P}(X_1 > x) \asymp nx^{-\alpha} \), by (15). Thus, \( \mathbb{P}(S_n > x) \rightarrow 0 \) implies (5), which in turn implies \( \mathbb{P}(S_n > x) \asymp nx^{-\alpha} \rightarrow 0 \).  

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Odds Inversion Problem With Replacement

In the recent piece [1] in this Monthly, Moniot worked toward determining which probabilities \( \frac{p}{q} \) were achievable when drawing two balls from a jar of \( x \) red balls and \( y \) blue balls, without replacement, where a successful trial has the two drawn balls of different colors. This was done by reducing the problem to solving the Pell-like equation \( u^2 - Dv^2 = p^2 \), where \( D = q(q - 2p) \). Solutions to this, however, sometimes give only extraneous \( x, y \): e.g., \( p = 4, q = 9, D = 9 \), has no possible nonnegative integers \( x, y \) with \( x + y \geq 2 \) giving probability \( \frac{4}{9} \).

We completely answer the simpler question where the draws are with replacement, by reducing the problem to finding nontrivial solutions to the similar Diophantine equation \( u^2 - Dv^2 = 0 \).

**Theorem.** Probability \( \frac{p}{q} \) is achievable as the probability of two drawn balls (with replacement, from \( x \) red and \( y \) blue balls) being different, if and only if \( D = q(q - 2p) \) is a perfect square.

**Proof.** Probability \( \frac{p}{q} \) is achievable, if and only if there are nonnegative integers \( x, y \), not both zero, with \( \frac{p}{q} = \frac{2xy}{(x+y)^2} \). This rearranges to \( px^2 - 2(q-p)xy + py^2 = 0 \).

Taking the substitution \( v = y + x, t = y - x \), this rearranges to \( (2p-q)v^2 + qt^2 = 0 \), which in turn rearranges to \( (qt)^2 - Dv^2 = 0 \). Lastly, taking \( u = qt \), this becomes \( u^2 - Dv^2 = 0 \) (where \( u, v \) are not both zero).

If \( \frac{p}{q} \) is achievable, then \( D \) must be a perfect square (by considering unique factorization of integers \( u, v, D \) into primes). Suppose now that \( D = m^2 \). We take \( u = 2qm, v = 2q \), which satisfy \( u^2 - Dv^2 = 0 \). These \( u, v \) correspond to \( t = 2m, x = q - m, y = q + m \), where \( \frac{2xy}{(x+y)^2} = \frac{p}{q} \). Each of \( x, y \) are nonnegative integers, since \( m^2 = D = q^2 - 2qmp \leq q^2 \). Hence, \( \frac{p}{q} \) is achievable.

In particular, no probabilities greater than \( \frac{1}{2} \) (i.e., \( D < 0 \)) are achievable. Compare to the “elliptical case” in [1], where infinitely many such probabilities are achieved without replacement. Note also that achievable probabilities are dense in \([0, \frac{1}{2}]\), because a brief calculation shows that achievable \( \frac{p}{q} \) are exactly those rationals equal to \( \frac{1}{2}(1 - r^2) \) for some rational \( r \in [0, 1] \).

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doi.org/10.1080/00029890.2022.2104078

MSC: Primary 11D09