Meson Mass Spectrum of Heavy-Light Quarks Combinations with Dirac Equation

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Abstract

We use the Dirac equation to study the mass spectrum of mesons with heavy-light quark combinations. First we study the Dirac equation with spherically symmetry and funnel potential, and apply them on the hydrogen-like atom problem to check the correctness of our numerical program. Then we test the parameters in Olsson’s paper [1]. We show that Olsson’s parameters are good in fitting the averaged central mass, but fail to get correct energy fine splitting. Finally we fit the mass spectrum data of $D, D_s, B$ and $B_s$ mesons with our parameters by solve the Dirac equation and funnel potential, calculate the energy splitting of the $S$ and $P$ states. Our parameters can fit the mass and fine splitting with errors in less than $7 MeV$.

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1 Meson Mass Spectrum Question

In the standard model, a meson is composed of a quark and an anti-quark, bound together by the strong interaction. Through the studying of the mass spectrum of mesons, we can demonstrate the correctness of the quantum fields theory, and predict the particle’s mass that has not been found yet in the experiments.

Many articles had studied the meson mass spectrum problems and got many very good results [1] [2] [3] [4]. Most authors used the Schrödinger equation to solve the problem. Their treatment is accurate enough for the heavy mesons, which are composed by two heavy quarks and moving slowly, thus may be treated non-relativistically. Let’s take an estimate. If we think the mass of a meson is the combination of the total mass, kinetic and potential energies of the two composition quarks, the binding energy (kinetic + potential) is calculated and listed in Table 1. The constituent mass [5] values we used are: $m_u = 0.30$

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GeV, \( m_d = 0.30 \text{ GeV}, \ m_s = 0.45 \text{ GeV}, \ m_c = 1.3 \text{ GeV}, \ m_b = 4.5 \text{ GeV} \) and \( m_t = 180 \text{ GeV} \). The "ratio" column is defined as

\[
\text{Ratio} = \frac{\text{binding energy}}{\text{light quark's mass}}
\]  

Compared to the light quark’s mass in the \( J/\psi, \Upsilon \) and \( B_c \) mesons, their kinetic and potential energies are not very large. But in the \( D, D_s, B \) and \( B_s \) mesons, the ratio is very large. So the light quarks in these mesons, \( D, D_s, B \) and \( B_s \), are moving with relativistic energies. Thus requires us to use relativistic equation, the Dirac equation, to solve the spectrum problem.

**Table 1: The composition quarks and mass of some mesons [1] [2] [5].**

| Meson’s name | Meson mass (GeV) | Composition quarks | Binding energy (GeV) | Ratio |
|--------------|------------------|--------------------|---------------------|-------|
| \( D \)     | 1.975            | \( \bar{c}u \)     | 0.37                | 1.23  |
| \( D_s \)   | 2.075            | \( \bar{c}d \)     | 0.32                | 0.75  |
| \( B \)     | 5.314            | \( \bar{u}b \)     | 0.37                | 1.23  |
| \( B_s \)   | 5.410            | \( sb \)           | 0.32                | 0.75  |
| \( J/\psi \)| 3.097            | \( \bar{c}c \)     | 0.40                | 0.28  |
| \( \Upsilon \)| 9.464          | \( \bar{b}b \)     | 0.40                | 0.09  |
| \( B_c \)   | 6.264            | \( \bar{c}b \)     | 0.40                | 0.28  |

We should point out that the quark’s mass we used here is the constituent quark mass, which is the quark’s current mass plus the mass of the gluon fields and sea-quarks, as the effective quark mass of the valence quark. The current quark mass means the mass of a quark itself only. Values of the composition mass and current mass differ greatly. For example, proton’s mass is about 0.938 GeV. The rest current masses of its three valence quarks are only about 0.011 GeV each. But we can treat the mass of each up or down quark with constituent mass as large as 0.30 GeV.

In a charm meson, there are two quarks, with \( m_c \gg m_{u,d} \). So we may simplify the problem by treat the heavier quark’s mass as infinite. Then the problem is reduced to the light quark moving in the funnel potential that created by the heavier quark. Solving the Dirac equation with the funnel potential, we can get the eigenenergy of the light quark in the funnel potential. Because of the Dirac equation includes the light quark’s spin-orbit interaction, the eigenenergy will reflect the splitting of the \( S \) and \( P \) states. The influence of the heavier quark is treated by adding its mass in the spin dependant force. The total mass of the meson can be expressed as:

\[
M_{\tilde{q}_1\tilde{q}_2} = m_{\tilde{q}_1} + m_{\tilde{q}_2} + E + \Delta,
\]

that’s the sum of the masses of the two quarks \( m_{\tilde{q}_1} \) and \( m_{\tilde{q}_2} \), eigenenergy of the system \( E \), and the hyper-fine energy splitting \( \Delta \).
M. G. Olsson [1] shows that the Dirac equation can be used to get a good fitting in the average mass spectrum. But he did not calculate the fine structure splitting.

D. Ebert, V. O. Galkin and R. N. Faustov [2] use Schrödinger equation with relativistic potential to solve the mass spectrum and calculate the fine splitting. They point out that in the spin dependent potentials, we should use $\frac{1}{m_q}$ instead of use $\frac{1}{\text{eigenenergy}}$ in Eichten and Quigg’s paper [3] to get energy splitting.

In this paper, we will solve the Dirac equation numerically to fit the meson’s mass spectrum of the heavy-light quarks combination system, such as $D$, $D_s$, $B$ and $B_s$, and calculate the spin-orbit energy hyper fine splitting. Our result will be compared to the experimental data.

2 Dirac Equation with Spherically Symmetry

The Dirac equation for free particle is

$$\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + m\right)\psi = 0. \quad (3)$$

If the potential has spherical symmetry, and can be written as a combination of Lorentz scalar part $V_s(r)$ and vector Coulomb potential $V_v(r)$, then the Dirac equation can be written as [6] [7] [8]:

$$i\frac{\partial \psi}{\partial t} = \left[-i\alpha_i \frac{\partial}{\partial x_i} + \beta(m + V_s(r)) + V_v(r) \right] \psi, \quad (4)$$

with the solution has the form like

$$\psi(r) = \begin{pmatrix} F(r) \\ G(r) \end{pmatrix} = \begin{pmatrix} f^\pm(r) y_{jjz}^\pm(\hat{r}) \\ i g^\pm(r) y_{jjz}^+(\hat{r}) \end{pmatrix}, \quad (5)$$

in which $y_{jjz}$ is a two component generalization of the spherical harmonic functions $Y_{l,j}^z$,

$$y_{jjz}^k(\hat{r}) = -\frac{k}{|k|} \sqrt{\frac{k + \frac{3}{2} - j_z}{2k + 1}} \alpha Y_{l,jz} - \frac{j_z}{2} (\hat{r}) + \sqrt{\frac{k + \frac{3}{2} + j_z}{2k + 1}} \beta Y_{l,jz} + \frac{j_z}{2} (\hat{r}). \quad (6)$$

The number $k$ is

$$k = \pm \left(j + \frac{1}{2}\right), \quad (7)$$

in which the $\pm$ sign is

$$\left\{ \begin{array}{ll} \text{if} & l = j + \frac{1}{2} \Rightarrow k = (j + \frac{1}{2}) = l, \\
\text{if} & l = j - \frac{1}{2} \Rightarrow k = -(j + \frac{1}{2}) = -(l + 1). \end{array} \right. \quad (8)$$

Thus the coupled equations for the radial functions are

$$\left[E - m - V_s(r) - V_v(r)\right] f(r) = -\frac{dg(r)}{dr} - \frac{1 - k}{r} g(r), \quad (9)$$
\[
E + m + V_s(r) - V_v(r) g(r) = \frac{df(r)}{dr} + \frac{1 + k}{r} f(r).
\]

Notes that the angular dependent variables have been removed, and only two radial coordinate dependent functions \(f\) and \(g\) are left in the equation. By solving the equations (10) and (11), we will get the energies and eigenfunctions of a relativistic particle that moves in the central potential.

### 3 Potential inside Meson

#### 3.1 QCD One Gluon Exchange Coulomb-like Potential

Quantum Chromodynamics (QCD) describes the strong interaction among quarks and gluons. The Lagrangian density in QCD can be written as:

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + \sum_{f=1}^{n_f} \bar{q}_f(i\gamma^\mu \mathcal{D}_\mu - m_f)q_f,
\]

in which

\[
F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g_s \varepsilon_{abc} A_\mu^b A_\nu^c,
\]

\[
\mathcal{D}_\mu = \partial_\mu - ig_s T_a A_\mu^a,
\]

where \(q_f\) is the quark field, \(g_s\) is the strong interaction coupling strength, \(T_a\) are the generators of the color \(SU(3)\) group, \(\varepsilon_{abc}\) is the structure constant of the \(su(3)\) Lie algebra, and \(A_\mu\) is the \(SU(3)\) color gauge field.

Effective potential is used to study the bound states of the quarks in mesons. The one gluon exchange Coulomb-like potential \([10]\) between two quarks is

\[
V_v(r) = -\frac{4 \alpha_s}{3 r},
\]

in which

\[
\alpha_s = \frac{g_s^2}{4\pi}.
\]

#### 3.2 Funnel Potential

At large distances, there should exist a confine potential that describes the color confinement. Unfortunately, till now, the color confinement cannot be derived from the QCD first principle. So we may add it in "by hand". The commonly used Cornell confinement potential \([11]\) \([12]\) is

\[
V_s(r) = V_{\text{confine}}(r) = ar,
\]

which will make the combination of the scalar \(V_s(r)\) and vector \(V_v(r)\) potential has a funnel shape. The spectrum obtained with the funnel potential is in good agreement with experimental data for the light and heavy mesons \([1] [2] [3]\). Typical values for the parameters are \(a \approx 0.2\, GeV^{-2}\), and \(\alpha_s \approx 0.2 \sim 0.3\).
4 Spin Dependant Hyper-fine Energy Splitting

In QED, the relativistic Dirac equation describes two or more massive spin 1/2 particles interacting electromagnetically. The perturbation QED formula of the fine splitting includes spin-orbit interaction \( (L \cdot S) \), spin-spin interaction \( (s_i \cdot s_j) \), and Breit spin-spin tensor interaction \( (s_i \cdot s_j - \frac{(s_i \cdot \vec{r}_{ij})(s_j \cdot \vec{r}_{ij})}{r_{ij}^2}) \) terms.

Within the framework of QCD, there are similar potentials. If the Schrödinger equation is used to solve the mass spectrum problem, the spin-dependent hyperfine energy splitting to the first order of \( \alpha_s \) can be written as [3]:

\[
\Delta = \sum_{k=1}^{4} T_k
\]

with

\[
T_1 = \frac{\langle L \cdot s_i \rangle}{2m_i^2} \tilde{T}_1(m_i, m_j) + \frac{\langle L \cdot s_j \rangle}{2m_j^2} \tilde{T}_1(m_j, m_i),
\]

\[
T_2 = \frac{\langle L \cdot s_i \rangle}{m_i m_j} \tilde{T}_2(m_i, m_j) + \frac{\langle L \cdot s_j \rangle}{m_i m_j} \tilde{T}_2(m_j, m_i),
\]

\[
T_3 = \frac{\langle s_i \cdot s_j \rangle}{m_i m_j} \tilde{T}_3(m_i, m_j),
\]

\[
T_4 = \frac{\langle S_{ij} \rangle}{m_i m_j} \tilde{T}_4(m_i, m_j),
\]

and

\[
\tilde{T}_1(m_i, m_j) = -\left( \frac{1}{r} \frac{dV}{dr} \right) + 2\tilde{T}_2(m_i, m_j),
\]

\[
\tilde{T}_2(m_i, m_j) = \frac{4\alpha_s}{3} (r^{-3}),
\]

\[
\tilde{T}_3(m_i, m_j) = \frac{32\pi\alpha_s}{9} |\psi(0)|^2,
\]

\[
\tilde{T}_4(m_i, m_j) = \frac{\alpha_s}{3} (r^{-3}),
\]

in which

\[
S_{ij} = 4[3(s_i \cdot \hat{n})(s_j \cdot \hat{n}) - s_i \cdot s_j].
\]

Now we use the Dirac equation to solve the mass spectrum. The energy splitting terms in eq. (16) and (17) should be modified a little bit. In the non-relativistic limit, \( p = mv << mc \). If we let \( V_s = 0 \) and \( V = V_v \), the Dirac equation (4) and wave function (5) will be reduced to:

\[
\left[ \frac{\hat{p}^2}{2m} + V \right] F(r) = \frac{\hbar^2}{4m^2 c^2} \frac{dV}{dr} \frac{dF}{dr} + \frac{\hbar}{2m^2 c^2} \frac{1}{r} \frac{dV}{dr} \hat{s} \cdot \hat{L} F(r) = (E - mc^2) \cdot F(r).
\]

(19)
Compare to the Schrödinger equation, the non-relativistic limited Dirac equation (19) has already included the \( \frac{1}{2m} \frac{d^2}{dr^2} \mathbf{s} \cdot \mathbf{L} \) spin-orbit interaction term. So when we use the Dirac equation to study energy splitting, the \(-\langle \frac{1}{r} \frac{d}{dr} \rangle\) term should be removed from \( \hat{T}_1(m_i, m_j) \) in eq.(17). Since we treat the heavier quark’s mass as infinite in the Dirac equation, the fracture \( \frac{1}{m_i} \) will be replaced by a meaningful limited number: over energy eigenvalue, \( \frac{1}{E} \). Now the terms in the energy splitting eqn.(15) will be

\[
\begin{align*}
T_1 &= \frac{\langle \mathbf{L} \cdot \mathbf{s}_i \rangle}{2E^2} \hat{T}_1(m_i, m_j) + \frac{\langle \mathbf{L} \cdot \mathbf{s}_j \rangle}{2Em_j} \hat{T}_1(m_j, m_i), \\
T_2 &= \frac{\langle \mathbf{L} \cdot \mathbf{s}_i \rangle}{Em_j} \hat{T}_2 + \frac{\langle \mathbf{L} \cdot \mathbf{s}_j \rangle}{Em_j} \hat{T}_2, \\
T_3 &= \frac{\langle \mathbf{s}_i \cdot \mathbf{s}_j \rangle}{Em_j} \hat{T}_3, \\
T_4 &= \frac{\langle \mathbf{S}_{ij} \rangle}{Em_j} \hat{T}_4, 
\end{align*}
\]

(20)

and

\[
\begin{align*}
\hat{T}_1(m_i, m_j) &= 2\hat{T}_2(m_i, m_j), \\
\hat{T}_2(m_i, m_j) &= \frac{4\alpha_s}{3} (r^{-3}), \\
\hat{T}_3(m_i, m_j) &= \frac{32\pi \alpha_s}{9} |\psi(0)|^2, \\
\hat{T}_4(m_i, m_j) &= \frac{\alpha_s}{3} (r^{-3}). 
\end{align*}
\]

(21)

There is a trick in calculating the \( \psi(0)^2 \) term in eqn.(21). It can be replaced by the production of the wave function and a \( \delta \) function: \( \delta(r) \psi(r)^2 \). The \( \delta \) function can be defined as

\[
\delta(r) \rightarrow \lim_{b \rightarrow 0} \frac{1}{b^{3\pi^2}} e^{-\frac{r^2}{b^2}}.
\]

(22)

So

\[
\begin{align*}
\hat{T}_3(m_i, m_j) &= \frac{32\pi \alpha_s}{9} |\psi(0)|^2 \\
&= \frac{32\pi \alpha_s}{9} \int_0^\infty \psi^2(r) \delta(r)r^2 dr \\
&= \lim_{b \rightarrow 0} \frac{32\pi \alpha_s}{9b^{3\pi^2}} \int_0^\infty \psi^2(r)e^{-\frac{r^2}{b^2}}r^2 dr,
\end{align*}
\]

(23)

which can be easily integrated numerically.

By adding the spin dependant part (15) with (20) and (21), into the solution of eqn.(9) and eqn.(10), we will get the meson spectrum with fine structure energy splitting.
5 Numerical Results for the Mesons

We will use the double shooting method and Runge-Kutta 4th method to solve the Dirac equation. Because the Dirac equation for the hydrogen-like atom has exact analytical solution, we first run our program on the hydrogen-like atom case to test whether our code works or not. The test results are shown in Appendix A, that we can get up to $10^{-5}$ accuracy. Next we run with Olsson’s parameters and the funnel potential to compare to Olsson’s [1] results, which are shown in Appendix B.

Finally with the funnel potential

$$V_s(r) = ar,$$  \hspace{1cm} (24)

$$V_v(r) = -\frac{\kappa}{r},$$  \hspace{1cm} (25)

we solve the Dirac equation numerically with our own parameters. Because meson has a confining part of potential, the wave function tends to contract to the center. So the system will be smaller in scale than the hydrogen-like atom, which does not have a confining part. According to our test, we choose the boundary condition as

$$\psi(r)|_{r=0} = 0, \hspace{1cm} (26)$$

and

$$\psi(r)|_{r\geq 20} = 0. \hspace{1cm} (27)$$

The radial part of the wave functions will be normalized with the Simpson’s integration rule to one,

$$\int_0^\infty (f^2+g^2)r^2dr = 1. \hspace{1cm} (28)$$

The fitting method is done by inputting two mesons masses as initial values to determine the parameters in the expression (15), equ (10) and (11). Then use the trial parameters, input configuration parameters of an unknown meson, to get the mass of that unknown meson.

After many trials, we find out the following set of parameters can fit the meson’s average mass and splitting very well.

$$m_{ud} = 0.280 \text{ GeV},$$

$$m_s = 0.429 \text{ GeV},$$

$$m_c = 1.095 \text{ GeV},$$

$$m_b = 4.435 \text{ GeV},$$

$$a = 0.368 \text{ GeV}^2,$$

$$\kappa = 0.400. \hspace{1cm} (29)$$

Our fitting results are listed in Table 2, in which the particle’s experimental mass are from the PDG [5] book. Spin averaged mass is calculated by taking $\frac{3}{4}$ of the triplet mass, and $\frac{1}{4}$ of the singlet mass for the s(p) states [1]. The column ”Numerical center mass” are the numerical result of the central mass of
Table 2: Using Dirac equation to fit the meson spectrum. Particle’s experimental mass are from the PDG [5] book. Mass values are in MeV.

| States | Spin averaged mass | Numerical mass | Numerical center mass | Parameter b |
|--------|--------------------|----------------|-----------------------|-------------|
| $D(1867)$ | $1S_0$ | 1S(1974) | -1 | 1975 | 1867 |
| $D^*(2010)$ | $3S_1$ | 2010 | b=1.07 |
| $D_1(2423)$ | $1P_1$ | 2P(2444) | -2 | 2407 | 2423 |
| $D_2(2457)$ | $3P_2$ | 2457 | N/A |
| $D_s(1969)$ | $1S_0$ | 1S(2075) | -1 | 2074 | 1968 |
| $D^*_s(2110)$ | $3S_1$ | 2109 | b=1.08 |
| $D_{s1}(2535)$ | $1P_1$ | 2P(2559) | -2 | 2515 | 2528 |
| $D_{sJ}(2573)$ | $3P_2$ | 2570 | N/A |
| $B(5279)$ | $1S_0$ | 1S(5314) | -1 | 5314 | 5279 |
| $B^*(5325)$ | $3S_1$ | 5325 | b=0.87 |
| $B_s(5375)$ | $1S_0$ | 1S(5410) | -1 | 5412 | 5376 |
| $B^*_s(5421)$ | $3S_1$ | 5422 | b=0.88 |

the $S$ and $P$ states. Then we use the fine structure formula (15) to calculate the energy fine splitting that are listed in the column "Numerical splitting mass". For the $P$ states, by intentionally choosing parameters that let the spin averaged mass does not sit between the $1P_1$ and $3P_2$ states, but let the average mass lower than both of the $1P_1$ and $3P_2$ states, we can get good fittings for their splitting. The errors for $S$ states are about 1 MeV, while the $P$ states errors are less than 7 MeV.

We also calculate the average values of $r$, $r_2$, $r^{-1}$ and $r^{-2}$, which are listed in Table 3. In our model, the wave functions are related to the light quark’s mass, but not to the heavy quark’s mass.

6 Discussion

By using our set of parameters [29], we can fit the $D$, $D_s$, $B$ and $B_s$ mesons spectrum with the errors within $\sim 7$ MeV. The parameters for the mass of the $u$, $d$, $s$, $c$ and $b$ quarks, $\kappa$ are all in reasonable range. We use $a = 0.368 \text{ GeV}^2$. But in other people’s paper [2] [3], $a \approx 0.18 \sim 0.20 \text{ GeV}^2$, in which they used with the Schrödinger equation. The reason may be explained as the difference between the Dirac equation and the Schrödinger equation. The one-gluon
Table 3: Average value of $r$, $r^2$, $r^{-1}$ and $r^{-2}$, with values are in GeV\(^n\).

| Light quark | \(2^{S+1}L_J\) | \(<r>\) | \(<r^2>\) | \(<r^{-1}>\) | \(<r^{-2}>\) |
|-------------|----------------|--------|--------|-----------|--------|
| u/d         | \(^1S^0\)      | 1.520  | 2.811  | 0.887     | 1.412  |
|             | \(^3S_1\)      |        |        |           |        |
| u/d         | \(^1P_1\)      | 2.205  | 5.423  | 0.521     | 0.326  |
|             | \(^3P_2\)      |        |        |           |        |
| s           | \(^1S^0\)      | 1.437  | 2.521  | 0.944     | 1.617  |
|             | \(^3S_1\)      |        |        |           |        |
| s           | \(^1P_1\)      | 2.120  | 5.026  | 0.543     | 0.355  |
|             | \(^3P_2\)      |        |        |           |        |

The exchange process plus the confining potential is

$$V(p, q) = \sqrt{\frac{m}{E_p}} \overline{u}(p) \left[ -\frac{4\alpha_s 4\pi}{3} \gamma^0 + V_{\text{conf}} \right] \sqrt{\frac{m}{E_q}} u(q),$$

(30)

in which

$$u(p) = \sqrt{\frac{E_p + m}{2m}} \left( \frac{1}{E_p + m} \right) \chi$$

(31)

and

$$u^+ u = \sqrt{\frac{E}{m}}.$$  

(32)

In the non-relativistic limit, assume the exchanging gluon’s energy is small, then \(E_p \approx E_q\). By using \(E^2 = p^2 + m^2\), we can get

$$V \to -\frac{4\alpha_s 4\pi}{3} \frac{k^2}{k^2} + \frac{m}{E} V_{\text{conf}}.$$  

(33)

That means when we use confining potential \(V_{\text{conf}}\) in the Dirac equation, it is equivalent to the confining potential \(\frac{m}{E} V_{\text{conf}}\) in the Schrödinger equation,

$$V_{\text{conf}} = ar \quad \rightarrow \quad \frac{m}{E} V_{\text{conf}} = \frac{m}{E} ar.$$  

(34)

So the relation between the parameters of "a" in the Dirac equation and the Schrödinger equation is

Dirac: \(a \leftrightarrow\) Schrödinger: \(\frac{m}{E} a\).  

(35)

Let’s take an estimate. For the D meson, using our parameters \(m_{\text{average}} = 1.975\ \text{GeV}, m_u = 0.28\ \text{GeV}, m_c = 1.095\ \text{GeV}\), so the eigenenergy is

$$E = M_{\text{average}} - m_c - m_u$$
Wave functions of the D meson with Our parameters

\[ n = 1, k = -1 \]
\[ j = 0.5, l = 0 \]
\[ \kappa = 0.400 \]

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig1}
\caption{Wave functions of the 1S state with our parameters.}
\end{figure}

\[ m = 1.975 - 1.095 - 0.28 \]
\[ = 0.60 \text{ GeV}. \]  
\((36)\)

That means the parameters of "a" in the Schrödinger equation is

\[ \frac{m}{E} a = \frac{0.3}{0.60} \times 0.368 = 0.184 (\text{GeV})^2, \]  
\((37)\)

which is in the range that people used with the Schrödinger equation.

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Appendix A: Numerical Results for the Hydrogen-like Atoms

The hydrogen-like atom is defined as a particle moves in the central Coulomb potential. We can use its analytical solution results to test the correctness of our numerical program.

For the hydrogen-like atom with a Coulomb central potential

\[ V = -\frac{\kappa}{r}, \]

the exact Dirac solution [7][8] is

\[
E_{n,k} = m \left[ 1 - \frac{\kappa^2}{(N + |k|)^2 + 2N(\sqrt{K^2 - \kappa^2} - |k|)} \right]^{\frac{1}{2}},
\]

\[ (39) \]

where

\[ n = N + |k| \geq 1, \quad -n \leq k < n. \]

\[ (40) \]

We use the following parameters in our numerical code.

\[ m = 0.3 \text{GeV}, \]

\[ \kappa = 0.4 \text{GeV}^{-1} \]

\[ j = 1.5, l = 1 \]

Wave functions of the D meson with Our parameters

\[ n = 2, k = -2 \]

\[ j = 1.5, l = 1 \]

\[ \kappa = 0.4 \text{GeV}^{-1} \]

\[ f \]

\[ g \]

Figure 2: Wave functions of the 2P state with our parameters.
\[ Z \alpha = \kappa = 0.579, \]

\[ V = -\frac{\kappa}{r}. \]  \hspace{1cm} (41)

Because the hydrogen-like atom does not have a confining part of potential, the wave functions tend to extend far away from the center. So the system will be large in scale. In our numerical program, we should set a large region to solve the problem. According to our test, we choose the boundary condition as

\[ \psi(r)|_{r=0 \ (GeV^{-1})} = 0, \]  \hspace{1cm} (42)

and

\[ \psi(r)|_{r \geq 200 \ (GeV^{-1})} = 0. \]  \hspace{1cm} (43)

Both of the numerical and analytical results are listed in Table 4. It shows that our numerical program works very well, that there are no differences among the numerical and analytical results of the eigenenergy to the precision of \(10^{-5}\).

We also calculate the average value of \(r, r^2, r^{-1}\) and \(r^{-2}\), which are listed in Table 5, 6 and 7. Our numerical results agree with the exact Dirac solution’s expect values. The Schrödinger exact solution’s expect values are also listed for comparison. In the non-relativistic limit, Dirac equation’s \(f\) part wave function will reduce to the Schrödinger wave function. The difference in our results between the Dirac and Schrödinger equations indicate that for a system with quark’s mass and interaction strength, we should use relativistic Dirac equation, instead of the non-relativistic Schrödinger equation.

**Appendix B: Numerical Results of with Olsson’s Parameters**

The parameters in Olsson’s paper are

\[ m_{ud} = 0.300\ GeV \]
\[ m_s = 0.463\ GeV \]
\[ m_c = 1.301\ GeV \]
\[ m_b = 4.639\ GeV \]
\[ a = 0.308\ GeV^2 \]
\[ \kappa = 0.579, \]  \hspace{1cm} (44)

with the potential

\[ V_s(r) = ar, \]  \hspace{1cm} (45)
\[ V_v(r) = -\frac{\kappa}{r}. \]  \hspace{1cm} (46)

We solve the Dirac equation with Olsson’s parameters. Our fitting results of the meson’s mass spectrum for both center average mass and energy
Table 4: Exact Dirac solutions and our numerical results of hydrogen-like atom.

| n  | k  | j  | l  | name   | Analytical energy (GeV) | Numerical result (GeV) |
|----|----|----|----|--------|-------------------------|------------------------|
| 1  | -1 | 1/2| 0  | 1$S_{1/2}$ | $m\sqrt{1 - Z\alpha^2} = 0.24460$ | 0.24460                |
| 2  | -2 | 3/2| 1  | 2$P_{3/2}$ | $m\sqrt{1 - \left(\frac{Z\alpha}{2}\right)^2} = 0.28715$ | 0.28715                |
|    | 1  | 1/2| 1  | 2$P_{1/2}$ | $m\sqrt{1 - \left(\frac{Z\alpha}{2}\right)^2} = 0.28581$ | 0.28581                |
|    | -1 | 1/2| 0  | 2$S_{1/2}$ | $m\sqrt{1 - \left(\frac{Z\alpha}{2}\right)^2} = 0.29436$ | 0.29436                |
| 3  | -3 | 5/2| 2  | 3$D_{5/2}$ | $m\sqrt{1 - \left(\frac{Z\alpha}{5}\right)^2} = 0.29436$ | 0.29436                |
|    | 2  | 3/2| 2  | 3$D_{3/2}$ | $m\sqrt{1 - \left(\frac{Z\alpha}{5+2\sqrt{4-(Z\alpha)^2}}\right)^2} = 0.29425$ | 0.29425                |
|    | -2 | 3/2| 1  | 3$P_{3/2}$ | $m\sqrt{1 - \left(\frac{Z\alpha}{5+2\sqrt{4-(Z\alpha)^2}}\right)^2} = 0.29385$ | 0.29385                |
|    | 1  | 1/2| 1  | 3$P_{1/2}$ | $m\sqrt{1 - \left(\frac{Z\alpha}{5+4\sqrt{1-(Z\alpha)^2}}\right)^2} = 0.29385$ | 0.29385                |
|    | -1 | 1/2| 0  | 3$S_{1/2}$ | $m\sqrt{1 - \left(\frac{Z\alpha}{5+4\sqrt{1-(Z\alpha)^2}}\right)^2} = 0.29385$ | 0.29385                |

splitting are listed in Table 8 with Olsson’s parameters. The results show that we can reproduce the spin average center mass values, which are listed in Osson’s paper. Because we have the parameter $b$, which is the width of the $\delta$ function, we can get good results for the fine splitting of the $S$ states by adjusting the values of $b$. On the other hand, the calculated values of the splitting for the $P$ states are unfortunately not so good, with the errors are around 90 MeV.

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Table 5: Average value of $r$ and $r^2$ of the hydrogen-like atom. Exact Dirac (column "Exact(D)"") and Schrödinger (column "Exact(S)"") solutions are also listed for comparison to our numerical results.

| n  | k   | j   | l   | $< r >$ ($GeV^{-1}$) | $< r^2 >$ ($GeV^{-2}$) |
|----|-----|-----|-----|----------------------|------------------------|
| 1  | -1  | $\frac{1}{2}$ | 0   | 7.572                | 7.572                  |
|    |     |                   |     | 8.635                | 99.43                  |
| 2  | -2  | $\frac{3}{2}$   | 1   | 27.80                | 27.80                  |
|    |     |                   |     | 28.79                | 932.9                  |
|    |     |                   |     | 932.8                | 994.3                  |
|    |     |                   |     | 994.3                |                        |
| 2  | 1   | $\frac{1}{2}$   | 1   | 24.24                | 24.25                  |
|    |     |                   |     | 28.79                | 729.7                  |
|    |     |                   |     | 730.4                | 994.3                  |
|    |     |                   |     | 994.3                |                        |
| 2  | -1  | $\frac{1}{2}$   | 0   | 29.99                | 30.01                  |
|    |     |                   |     | 34.54                | 1074.                  |
|    |     |                   |     | 1074.                | 1392.                  |
|    |     |                   |     | 1392.                |                        |
| 3  | -3  | $\frac{5}{2}$   | 2   | 59.44                | 59.47                  |
|    |     |                   |     | 60.45                | 4043.                  |
|    |     |                   |     | 4050.                | 4176.                  |
|    |     |                   |     | 4176.                |                        |
| 3  | 2   | $\frac{3}{2}$   | 2   | 57.71                | 57.73                  |
|    |     |                   |     | 60.45                | 3832.                  |
|    |     |                   |     | 3836.                | 4176.                  |
|    |     |                   |     | 4176.                |                        |
| 3  | -2  | $\frac{3}{2}$   | 1   | 69.16                | 69.24                  |
|    |     |                   |     | 71.96                | 5540.                  |
|    |     |                   |     | 5558.                | 5965.                  |
|    |     |                   |     | 5965.                |                        |
| 3  | 1   | $\frac{1}{2}$   | 1   | 64.14                | 64.18                  |
|    |     |                   |     | 71.96                | 4798.                  |
|    |     |                   |     | 4806.                | 5966.                  |
|    |     |                   |     | 5966.                |                        |
| 3  | -1  | $\frac{1}{2}$   | 0   | 69.87                | 69.94                  |
|    |     |                   |     | 77.72                | 5597.                  |
|    |     |                   |     | 5611.                | 6861.                  |
|     |     |                   |     |                      |                        |

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Table 6: Average values of $r^{-1}$ in GeV of the hydrogen-like atom. Exact Dirac (column "Exact(D)") and Schrödinger (column "Exact(S)") solutions are also listed for comparison to our numerical results.

| n  | k  | j | l | Numeric | Exact(D) | Exact(S) |
|----|----|---|---|---------|----------|----------|
| 1  | -1 | $\frac{1}{7}$ | 0 | $2.13 \times 10^{-1}$ | $2.13 \times 10^{-1}$ | $1.74 \times 10^{-1}$ |
| 2  | -2 | $\frac{3}{2}$ | 1 | $4.54 \times 10^{-2}$ | $4.54 \times 10^{-2}$ | $4.34 \times 10^{-2}$ |
| 2  | 1  | $\frac{1}{2}$ | 1 | $5.59 \times 10^{-2}$ | $5.59 \times 10^{-2}$ | $4.34 \times 10^{-2}$ |
| 2  | -1 | $\frac{1}{2}$ | 0 | $5.59 \times 10^{-2}$ | $5.59 \times 10^{-2}$ | $4.34 \times 10^{-2}$ |
| 3  | -3 | $\frac{5}{2}$ | 2 | $1.97 \times 10^{-2}$ | $1.97 \times 10^{-2}$ | $1.93 \times 10^{-2}$ |
| 3  | 2  | $\frac{3}{2}$ | 2 | $2.05 \times 10^{-2}$ | $2.05 \times 10^{-2}$ | $1.93 \times 10^{-2}$ |
| 3  | -2 | $\frac{3}{2}$ | 1 | $2.05 \times 10^{-2}$ | $2.05 \times 10^{-2}$ | $1.93 \times 10^{-2}$ |
| 3  | 1  | $\frac{1}{2}$ | 1 | $2.36 \times 10^{-2}$ | $2.36 \times 10^{-2}$ | $1.93 \times 10^{-2}$ |
| 3  | -1 | $\frac{1}{2}$ | 0 | $2.36 \times 10^{-2}$ | $2.36 \times 10^{-2}$ | $1.93 \times 10^{-2}$ |
Table 7: Average value of $r^{-2}$ in $GeV^2$ of the hydrogen-like atom. Exact Dirac (column "Exact(D)") and Schrödinger (column "Exact(S)") solutions are also listed for comparison to our numerical results.

| n  | k  | j  | l  | Numeric       | Exact(D)       | Exact(S)       |
|----|----|----|----|---------------|----------------|----------------|
| 1  | -1 | 1/2| 0  | $1.15 \times 10^{-1}$ | $1.17 \times 10^{-1}$ | $6.03 \times 10^{-2}$ |
| 2  | -2 | 3/2| 1  | $2.79 \times 10^{-3}$  | $2.79 \times 10^{-3}$ | $2.51 \times 10^{-3}$ |
| 2  |  1 | 1/2| 1  | $5.82 \times 10^{-3}$  | $5.84 \times 10^{-3}$ | $2.51 \times 10^{-3}$ |
| 2  | -1 | 1/2| 0  | $1.83 \times 10^{-2}$  | $1.87 \times 10^{-2}$ | $7.54 \times 10^{-3}$ |
| 3  | -3 | 5/2| 2  | $1.97 \times 10^{-2}$  | $1.97 \times 10^{-2}$ | $1.93 \times 10^{-2}$ |
| 3  |  2 | 3/2| 2  | $5.14 \times 10^{-4}$  | $5.14 \times 10^{-4}$ | $4.47 \times 10^{-4}$ |
| 3  | -2 | 3/2| 1  | $8.67 \times 10^{-4}$  | $8.66 \times 10^{-4}$ | $7.45 \times 10^{-4}$ |
| 3  |  1 | 1/2| 1  | $1.79 \times 10^{-3}$  | $1.80 \times 10^{-3}$ | $7.45 \times 10^{-4}$ |
| 3  | -1 | 1/2| 0  | $5.44 \times 10^{-3}$  | $5.56 \times 10^{-3}$ | $2.23 \times 10^{-3}$ |
Figure 3: Numerical results of the Dirac wave function of the hydrogen-like atom’s $1S_{1/2}$ state.

Figure 4: Numerical results of the Dirac wave function of the hydrogen-like atom’s $2P_{1/2}$ state.
Figure 5: Numerical results of the Dirac wave function of the hydrogen-like atom's $2S_{1/2}$ state.

Figure 6: Numerical results of the Dirac wave function of the hydrogen-like atom's $2P_{3/2}$ state.
Figure 7: Numerical results of the Dirac wave function of the hydrogen-like atom’s $3S_{1/2}$ state.

Figure 8: Numerical results of the Dirac wave function of the hydrogen-like atom’s $3P_{1/2}$ state.
Figure 9: Numerical results of the Dirac wave function of the hydrogen-like atom's $3P_{3/2}$ state.

Figure 10: Numerical results of the Dirac wave function of the hydrogen-like atom's $3D_{3/2}$ state.
Figure 11: Numerical results of the Dirac wave function of the hydrogen-like atom’s $3D_{5/2}$ state.

Table 8: Using Dirac equation to fit the meson spectrum with Olsson’s parameters \[21\]. The values of mass are in $MeV$.

| States       | $^{2S+1}L_J$ | Spin averaged mass | $k$ | Numerical center mass | Numerical splitting mass | Parameter $b$ |
|--------------|--------------|-------------------|-----|-----------------------|-------------------------|---------------|
| $D(1867)$    | $^1S_0$      | 1S(1974)          | -1  | 1975                  | 1867                    | 2010          |
| $D^*(2010)$  | $^3S_1$      |                   |     |                       |                         | b=1.64        |
| $D_1(2423)$  | $^1P_1$      | 2P(2444)          | -2  | 2444                  | 2535                    | 2428          |
| $D_2^*(2457)$| $^3P_2$      |                   |     |                       |                         | N/A           |
| $D_s(1969)$  | $^1S_0$      | 1S(2075)          | -1  | 2074                  | 1968                    | 2109          |
| $D_s^*(2110)$| $^3S_1$      |                   |     |                       |                         | b=1.61        |
| $D_{s1}(2535)$| $^1P_1$     | 2P(2559)          | -2  | 2559                  | 2537                    | 2660          |
| $D_{sJ}(2573)$| $^3P_2$     |                   |     |                       |                         | N/A           |
| $B(5279)$    | $^1S_0$      | 1S(5314)          | -1  | 5314                  | 5279                    | 5325          |
| $B^*(5325)$  | $^3S_1$      |                   |     |                       |                         | b=1.43        |
| $B_s(5375)$  | $^1S_0$      | 1S(5410)          | -1  | 5412                  | 5376                    | 5422          |
| $B_s^*(5421)$| $^3S_1$      |                   |     |                       |                         | b=1.43        |
Table 9: Average value of $r$, $r^2$, $r^{-1}$ and $r^{-2}$, with values are in $GeV^n$, by using Olsson’s parameters.

| Light quark | $^{2S+1}L_J$ | $<r>$ | $<r^2>$ | $<r^{-1}>$ | $<r^{-2}>$ |
|-------------|-------------|-------|---------|-----------|-----------|
| u/d         | $^1S_0$    | 1.482 | 2.736   | 0.956     | 1.899     |
|             | $^3S_1$    |       |         |           |           |
| u/d         | $^1P_1$    | 2.286 | 5.867   | 0.507     | 0.313     |
|             | $^3P_2$    |       |         |           |           |
| s           | $^1S_0$    | 1.369 | 2.353   | 1.046     | 2.306     |
|             | $^3S_1$    |       |         |           |           |
| s           | $^1P_1$    | 2.177 | 5.333   | 0.534     | 0.349     |
|             | $^3P_2$    |       |         |           |           |

Figure 12: Wave functions of the $1S$ state with Olsson’s parameters.
Figure 13: Wave functions of the $2P$ state with Olsson’s parameters.