Integer Quantum Hall Effect in Graphene

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(Dated: April 27, 2015)

We study the quantum Hall effect in a monolayer graphene by using an approach based on thermodynamical properties. This can be done by considering a system of Dirac particles in an electromagnetic field and taking into account of the edges effect as a pseudo-potential varying continuously along the x direction. At low temperature and in the weak electric field limit, we explicitly determine the thermodynamical potential. With this, we derive the particle numbers in terms of the quantized flux and therefore the Hall conductivity immediately follows.

PACS numbers: 71.15.-m, 02.40.Gh

INTRODUCTION

Graphene is a two dimensional (2D) monolayer of graphite atoms [1, 2]. It has a honeycomb lattice structure of carbon atoms packed in a 2D system. Its single layer has a band structure analogous to massless relativistic particle, where the valence and the conduction bands meet in two in-equivalent points K and K′, called Dirac points, at the corners of Brillouin zone. The quantum Hall effect (QHE) in graphene is one of most remarkable phenomena, not only because of the Hall conductivity is quantized on plateaus and magnetoeconductance vanishing in magnetic field but also provides a bridge between condensed matter physics and quantum electrodynamics [3].

The successful experimental works [3, 4] and several theoretical attempts [5–8, 11] established the Hall conductivity is quantized on plateaus and magnetoconductance vanishing in magnetic field but also provides a bridge between condensed matter physics and quantum electrodynamics [12].

As an interesting result, we end up with the quantized plateaux characterizing the integer quantum Hall effect in graphene.

This letter is organized as follows. In section 2, we formulate our problem by setting the Hamiltonian describing Dirac particle in the presence of the electromagnetic fields and involving a pseudo-potential along the z-axis. After some algebra, we diagonalize our Hamiltonian to get the solutions of the energy spectrum. In section 3, using Fermi-Dirac statistics and Mellin transformation to explicitly evaluate the grand thermodynamical potential. In section 4, we calculate the particle number to end up with the Hall conductivity and therefore the corresponding filling factors. Finally, we conclude in last section.

SOLUTIONS OF THE ENERGY SPECTRUM

We consider a rectangular sheet of graphene parameterized by two sides (Lx, Ly) and subjected to an electromagnetic field (\vec{E}, \vec{B}). To deal with our task, we describe the present system by the Hamiltonian

\[ H = v_F \sigma \frac{p_y}{eB} + \sigma_y E_y + \Delta \hat{p} + g \mu_B \vec{B} \cdot \vec{S} \]

(1)

where the first term is the Dirac operator in the presence of \( \vec{B} \) and second is resulting from an applied electric field along y-direction, i.e. \( \vec{E} = E_y \hat{e}_y \). The continuum pseudo-potential \( \Delta \hat{p} \) is reflecting the edges effect contribution and the last one is the magnetic coupling. \( \sigma \) are Pauli matrices, \( g \) is the Landé factor, \( v_F \approx \frac{100}{100} \) is the Fermi velocity and \( \mu_B \) is the Bohr magneton.

It is convenient to consider the Landau gauge \( \vec{A} = (-B_y, 0) \) where the momentum operators read as \( \pi_x = p_x - \frac{eB}{2} y \) and \( \pi_y = p_y \). For simplicity, we decompose [11] into three parts. These are

\[ H = H_0 + \Delta \hat{p} + g \mu_B \vec{B} \cdot \vec{S} \]

(2)

where \( H_0 \) is corresponding to the two first terms in [11]. This decomposition is helpful in sense that we can treat...
each part separately and therefore derive easily the spectrum of \( \Psi \).

Now solving the eigenvalue equation to end up with the eigenvalues

\[
E_{\rho n} = \sgn(n) \sqrt{\left( \frac{\hbar v_F}{l_B} \right)^2 |n| + \Delta \bar{\rho} + g\mu_B B m_s} \tag{3}
\]
as well as the corresponding eigenfunctions

\[
\Psi_{n \neq 0, k, m_s} = \frac{1}{\sqrt{2}} \left( -\sgn(n) \phi_{|n| - 1} \right) e^{ikx} \otimes \alpha_{m_s} \tag{4}
\]
and eigenfunction are

\[
\phi_n = \frac{1}{\sqrt{2^n \pi^{n/2} n! l_B}} e^{-\frac{(y - y_0)^2}{2l_B^2}} H_n \left( \frac{y - y_0}{l_B} \right) \tag{5}
\]
where \( |n| = 0, 1, 2, \cdots \) is the LL index, \( y_0 = -kl_B^2 \), the magnetic length \( l_B = \sqrt{\frac{\hbar c}{eB}} \) and \( H_n \) being the Hermite polynomial. The zero-energy mode is

\[
\Psi_{n = 0, k, m_s} = \begin{pmatrix} 0 \\ \phi_0 \end{pmatrix} e^{ikx} \otimes \alpha_{m_s} \tag{6}
\]
where \( m_s = \pm \frac{1}{2} \) is the azimuthal number of spin operator \( S_z \) whose associated states are

\[
\alpha_{\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad \alpha_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{7}
\]
and by convention we choose \( \sgn(0) = 0 \).

**GRAND THERMODYNAMICAL POTENTIAL**

To achieve our goal we start by determining the grand thermodynamical potential (GTP)

\[
\Omega = -k_B T \ln (Z) \tag{8}
\]
such that the partition function associated to our system is given by the Fermi-Dirac Distribution

\[
Z = \prod_{\tau, \tau n, \tilde{\rho}, m_s} \left[ 1 + e^{\beta (\mu - E_{\tau, \tau n, \tilde{\rho}, m_s})} \right] \tag{9}
\]
where \( \tilde{\rho} \) is the chemical potential of particles, \( \tau \) takes plus one when \( n > 0 \) and minus one otherwise. \( \beta = \frac{1}{k_B T} \) with \( K_B \) is Bolzaman constant and \( T \) is the temperature. We define the shorthand notation \( \{ l \} = \tau, \tau n, \tilde{\rho}, m_s \) to be used in the next. Using \( \Omega \) to write

\[
\Omega = -\frac{1}{\beta} \sum_{\{ l \}} \ln \left[ 1 + e^{\beta (\mu - E_{n \tilde{\rho}})} \right]. \tag{10}
\]

It is convenient to adopt the dimensionless variable \( \mu = \frac{\mu}{\hbar c}, \quad \varepsilon_{n \tilde{\rho}} = \frac{E_{n \tilde{\rho}}}{\hbar c}, \quad \theta = \frac{1}{\beta} \hbar c \). Requiring \( \Delta \tilde{\rho} = -\frac{c E_{n \tilde{\rho}}}{\hbar c} \) and assuming that \( |\tilde{\rho}| \leq \frac{E_{n \tilde{\rho}}}{\hbar c} \) is fulfilled, we write GTP as

\[
\Omega = -mc^2 \theta N_\phi \int_{b/2}^{b/2} d\tilde{\rho} \sum_{\{ l \}} \ln \left[ 1 + e^{\frac{\varepsilon_{n \tilde{\rho}}}{\theta}} \right]. \tag{11}
\]

where \( b = \frac{cBL_c}{mc^2} \) and \( N_\phi = \frac{eB S}{\hbar c} \) is the number of quantum electron states in the magnetic field for a given \( n \) in an area \( S = L_x L_y \). To evaluate GTP, we use the Mellin transformation method with respect to the variable \( e^{\frac{x}{\theta}} \).

After calculation, we obtain

\[
\Omega = \mp 2e\theta \sum_{s = -\infty}^{\infty} \text{Res} \left[ \sum_{\{ l \}} \frac{\pi e^{\frac{\varepsilon_{n \tilde{\rho}}}{\theta}}}{s \sin (\pi s)} e^{-\frac{\varepsilon_{n \tilde{\rho}}}{\theta}} \right] \tag{12}
\]

where the minus (plus) sign refers the closing sense of the counter to the left (right) of the imaginary axis for \( \mu > 0 \) (\( \mu < 0 \)). Now we show

\[
\Omega = \mp 2e\theta N_\phi \sum_s \text{Res} \left[ \frac{\pi e^{\frac{s}{\theta}} \sum_{n} \left( -1 + 2 \sum_{(\tau n) = 0}^{\infty} e^{-\frac{\pi}{2} \sqrt{2|\nu F|}} \sqrt{\nu F} \right) (e^{-\frac{s}{\theta}} \sum_{m_s = \pm \frac{1}{2}} (e^{-\frac{s}{2} \nu F}) m_s \int_{-b/2}^{b/2} e^{e^{\frac{s}{2} \nu F} d\tilde{\rho}} / b \right) \right]. \tag{13}
\]

where \( g^* = \frac{e\mu B}{\hbar c}, \quad \kappa = \frac{eB}{c}, \quad \epsilon = mc^2 \). After integration, we end up with

\[
\Omega = \mp 2e\theta N_\phi \sum_s \text{Res} \left[ \frac{\pi e^{\frac{s}{\theta}} \sum_{n} \left( -1 + 2 \sum_{(\tau n) = 0}^{\infty} e^{-\frac{\pi}{2} \sqrt{2|\nu F|}} \sqrt{\nu F} \right) (e^{-\frac{s}{2} \nu F}) m_s \int_{-b/2}^{b/2} e^{e^{\frac{s}{2} \nu F} d\tilde{\rho}} / b \right]. \tag{14}
\]
where \( z = \frac{2\pi}{\lambda} \). For the residue calculations, we distinguish two special parts of GTP \( \Omega = \Omega_{\text{mon}} + \Omega_{\text{osc}} \). The first concerning the real poles called the monotonic part \( (\Omega_{\text{mon}}) \). But the second is related to the imaginary poles called the oscillating part \( (\Omega_{\text{osc}}) \). In our analysis, we restrict the calculation of \( \Omega_{\text{mon}} \) and \( \Omega_{\text{osc}} \) only for the minus sign, i.e for \( \mu > 0 \). We calculate \( \Omega_{\text{mon}} \) in \( s = 0 \) and we neglect the contribution of other real poles. This gives

\[
\Omega_{\text{mon}} \approx -2\epsilon N_\phi \beta \left[ \frac{1}{3} + \frac{g^*}{8} \left( \frac{\kappa}{\lambda} \right)^2 + \frac{z^2}{8\pi^2} \left( \frac{\kappa}{\lambda} \right)^2 + \frac{\alpha^2}{24} \left( \frac{\kappa}{\lambda} \right)^2 + \frac{\alpha\pi^2}{6} \left( \frac{\theta}{\lambda} \right)^2 \right].
\]  

(15)

Let us evaluate \( \Omega_{\text{osc}} \) in the poles \( s_l = \frac{i\pi l\theta}{\lambda} \) with \( l = 1, 2, 3, \ldots \). Indeed, at low temperature and strong magnetic field, \( i.e \theta \ll \kappa \), we obtain

\[
\Omega_{\text{osc}} \approx -4\epsilon N_\phi \lambda \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{l^2} \cos \left( \frac{2\kappa}{\lambda} l \right) \cos \left( \frac{\kappa g^* \pi}{\lambda} l \right) \sin \left( \frac{\alpha \pi \kappa}{2\lambda} l \right).
\]  

(16)

with \( \alpha = \frac{eE_L}{\kappa c} \) and \( \lambda = \sqrt{\frac{\kappa^2}{2}} \). Now combining all and using the assumption of very weak electric field \( (\alpha \ll 1) \) to write \( \Omega \) as

\[
\Omega \approx -\epsilon N_\phi \lambda \left[ \frac{2}{3} + \left( \frac{g^*}{2\lambda} \right)^2 + \left( \frac{z\kappa}{\pi\lambda} \right)^2 + \frac{4}{\pi^2} \Gamma(z) \right].
\]  

(17)

where \( \Gamma(z) \) is a periodic function of \( \frac{2\kappa}{\lambda} \) defined as

\[
\Gamma(z) = \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{l^2} \cos \left( \frac{2\kappa}{\lambda} l \right) \cos \left( \frac{\kappa g^* \pi}{\lambda} l \right). \quad \text{(18)}
\]

In what follows, the above function will play a crucial role in getting the quantized Hall plateaux for Dirac particles in graphene.

\[
\Gamma(z) = \frac{1}{2} \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{l^2} \left\{ \cos \left( \frac{k}{\lambda} \left( \frac{z}{2} \right) + \frac{g^* \pi}{2} \right) l \right\} + \cos \left( \frac{k}{\lambda} \left( \frac{z}{2} - g^* \pi \right) l \right). \quad \text{(21)}
\]

\[
\sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{l^2} \cos(lx) = \frac{\pi^2}{12} - \frac{x^2}{4}, \quad -\pi \leq x \leq \pi \quad \text{(22)}
\]

\[
\Gamma(z) = \begin{cases} \frac{-5\pi^2}{12} - \frac{1}{2} \left( \frac{\kappa}{\lambda} \right)^2 \left( \frac{5}{2} \right)^2 + \frac{z^2}{4} \left( \frac{\kappa}{\lambda} + \frac{\kappa g^* \pi}{\lambda} \right) & \text{if } z \in \mathbf{I}_1 \cap \mathbf{I}_2 \\ \frac{z^2}{4} - \frac{1}{4} \left( \frac{\kappa}{\lambda} \right)^2 \left( \frac{5}{2} \right)^2 - \left( \frac{i^2 + (i+2)^2}{8} + \frac{1}{2} \left( \frac{\kappa}{\lambda} \right)^2 \right) g^* \pi^2 + \frac{1}{4} (i+1) \left( \frac{\kappa}{\lambda} \right) & \text{if } z \in \mathbf{I}_3 \cap \mathbf{I}_4 \end{cases}
\]  

(23)
where $i$ is an even integer and $I_j$, $j = 1, 2, 3, 4$ are intervals defined as

\[
I_1 = \left[ 2 \left( \frac{\lambda}{\kappa} - g^* \right) \pi, 2 \left( \frac{3\lambda}{\kappa} - g^* \right) \pi \right], \quad I_3 = \left[ 2 \left( \frac{\beta}{\kappa} (i + 1) - g^* \right) \pi, 2 \left( \frac{\lambda}{\kappa} (i + 1) - g^* \right) \pi \right]
\]
\[
I_2 = \left[ 2 \left( g^* - \frac{\lambda}{\kappa} \right) \pi, 2 \left( g^* + \frac{\lambda}{\kappa} \right) \pi \right], \quad I_4 = \left[ 2 \left( \frac{\lambda}{\kappa} (i - 1) + g^* \right) \pi, 2 \left( \frac{\lambda}{\kappa} (i + 1) + g^* \right) \pi \right]
\]

To describe the quantum Hall effect, it is essential to evaluate the Hall conductivity $\sigma_H$. Hence, using the Drude model, $\sigma_H$ is

\[
\sigma_H = -\frac{\rho e^2}{B}
\]

where $\rho$ is the particle number per unit area. In function of the degree of the degeneracy of each LL $N_\phi$ and the particle number, $\sigma_H$ is expressed in terms of von Klitzing conductance $\frac{e^2}{h}$, as

\[
\sigma_H = \frac{N e^2}{N_\phi h} = -\nu \frac{e^2}{h}
\]

where $\nu$ is the filling factor of LL. Using (20), to obtain

\[
\sigma_H = -\frac{1}{\pi} \left( \frac{\kappa}{\lambda} \right) \left[ z + 8 \left( \frac{\lambda}{\kappa} \right)^2 \frac{d\Gamma(z)}{dz} \right] \frac{e^2}{h}
\]

This compared to (25) gives

\[
\nu = \frac{1}{\pi} \left( \frac{\kappa}{\lambda} \right) \left[ z + 8 \left( \frac{\lambda}{\kappa} \right)^2 \frac{d\Gamma(z)}{dz} \right].
\]

Now using (23) to find

\[
\nu = \begin{cases} 
2 & \text{if } z \in I_1 \cap I_2 \\
2(i + 1) & \text{if } z \in I_3 \cap I_4 
\end{cases}
\]

Recall that $i$ is taking even value and then we can write $i = 2n$ to recover the famous result $\nu = 4 \left( n + \frac{1}{2} \right)$, with $n$ is an integer. This clearly shows how one can describe the integer quantum Hall effect in graphene based on the thermodynamical properties of our system.

**CONCLUSION**

By taking into account of the edges effects in terms of a pseudo-potential in a monolayer graphene, we have shown that the corresponding Hall conductivity undergoes to a sequence of plateaux. Its quantization is performed by make using of the Fermi-Dirac statistical.

This has been done by considering a system of Dirac fermions in graphene submitted to an electromagnetic field and evaluating the grand thermodynamical potential as well as related physical quantities like number of particles.

**ACKNOWLEDGMENT**

The generous support provided by the Saudi Center for Theoretical Physics (SCTP) is highly appreciated by the author.

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