Non-trivial ground state for gravitational perturbation in quadratic gravity

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Abstract

We consider gravitational perturbation around maximally symmetric background of a theory of gravity involving quadratic curvature correction. This leads to a decoupled model of the standard transverse, traceless graviton mode and an additional scalar degree of freedom. Evolution of the latter is governed by a Lagrangian, which depends on higher derivative terms, inherited from the quadratic curvature correction in the action. It turns out that stability of the gravitational theory allows the scalar sector to choose between possible lowest energy states (or, ground states), that can sustain periodic form of inhomogeneity, either in space or in time. The latter, with a restricted form of invariance in time translation, is possibly a variant form of the time crystal. Existence of these novel ground states are solely related to the presence of higher curvature terms in the action and thus these non-trivial condensate-like ground states describe the spacetime fabric itself by introducing fundamental length or time scales in the theory. Possible implications are also discussed.

1 Introduction

Recent detection of gravitational waves from the collisions between binary black holes and neutron stars as well as direct measurement of the black hole shadow have tested general relativity to an unprecedented accuracy \cite{1–7}. This have further cemented the place of general relativity as the theory describing gravitational interaction. Besides such astonishing success of general relativity, it also suffers from several shortcomings. In the small length scale, the appearance of black hole singularity breaks the predictability of Einstein’s equations and hence renders the theory useless. In an identical manner, at large length scale one has to invoke some exotic matter, e.g., dark energy, for the theory to be consistent with recent observations \cite{8–17}. Despite several attempts, till date, there are no satisfactory resolution to the black hole singularity, but in the cosmological context, certain modifications over and above general relativity may lead to late time cosmic acceleration without invoking any exotic matter \cite{18–20}. Such modifications
can not be arbitrary, since the resulting field equations must admit Schwarzschild like solutions, so that local physics remains identical to that of general relativity. Among various such modifications, $f(R)$ gravity \[21\text{–}29\], Lanczos-Lovelock models of gravity \[30\text{–}34\] as well as a certain class of scalar-tensor theories of gravity, known as Horndeski theories and beyond \[35\text{–}39\], looks promising. In the context of $f(R)$ gravity, one adds various powers of the Ricci scalar (preferably positive) with appropriate coefficients to the Einstein-Hilbert term. The first such non-trivial correction to the Einstein-Hilbert action corresponds to the addition of a term, which is quadratic in the curvature and is believed to encode basic features of the $f(R)$ gravity \[40\]. Even though generic higher derivative gravity theories may possibly have better UV behaviour, they are often plagued with existence of ghost modes \[41,42\], while this particular model with quadratic correction is ghost free for certain choice of parameters \[43\].

Although the theories beyond general relativity appear promising to address some of the well-known issues in general relativity, they must also be stable under gravitational perturbation. In other words, the Hamiltonian constructed out of the dynamical degrees of freedom, associated with any gravitational theory beyond general relativity, must be bounded from below. This should not only hold for perturbations around flat spacetime, but also for perturbation around de Sitter spacetime. This is because one of the main motivation of these models is to explain either inflation or late time cosmic acceleration. Thus study of stability of the dynamical degrees of freedom around maximally symmetric spacetime is of vital importance. Further, the configuration of the minima of the Hamiltonian associated with the dynamical degrees of freedom is also of interest, since any non-trivial temporal structure may lead to a time crystal-like condensate \[44,45\], which has created an enormous amount of interest both in theoretical as well as experimental physics communities (see \[46,47\] for review).

Keeping these motivations in mind, we have used one of the simplest modification to general relativity, namely, the addition of a quadratic curvature term to the gravitational action and have studied the gravitational perturbation thereof \[40,43\]. Subsequently, we have exploited the conventional Hamiltonian physics involving higher derivatives and have determined the structure of the minima in the momentum space. This is partially motivated by the idea of spontaneous symmetry breaking in the momentum space, suggested by one of us in \[48\]. It is worthwhile to point out the basic difference between our approach and that of \[44\], leading to ground state with partial time (or, space) translation symmetry. The proposal of \[44\] requires at least quartic terms in field derivatives, i.e., terms of the form $\sim (\partial \phi)^4$ in the action whereas in our approach higher derivative terms in the action play the crucial role \[48\]. There exist several other works exploring the idea of time crystal \[44\] in the cosmological context, see e.g., \[49\text{–}53\]. Among these works, those presented in \[49,50\] deal with the time crystal structure in the matter sector of FRW universe, \[51\] studied time crystal in non-commutative extended gravity, \[52\] consider a quadratic gravity model with matter and lastly \[53\] studied thermodynamic aspect of quadratic gravity in mini-superspace framework. While the present work has revealed the form of the time crystal presented in \[48\], but starting from quadratic corrections to the Einstein-Hilbert action and employing an approach hitherto unemployed.

The present work finds its basis on the following steps: (i) expanding the quadratic gravity action to the second order in the gravitational perturbation around a maximally symmetric background; (ii) starting from the Lagrangian for the perturbation and hence determining the field equations, paving way for the associated dispersion relations; (iii) constructing the Hamiltonian following \[54\text{–}56\], which requires special care, since the Lagrangian depends on higher derivative terms and finally (iv) minimizing the Hamiltonian subjected to the dispersion relation, leading to a non-trivial ground state. It needs to be stressed that no matter fields are introduced from outside and hence these non-trivial condensate-like ground states, if they exist, are a byproduct of the spacetime geometry itself.

The paper is organized as follows: In Section 2 we have presented the gravitational action and its perturbation around a maximally symmetric background upto quadratic order. Using the Lagrangian
derived from the perturbation, we have determined the Hamiltonian and the associated dispersion relation in Section 3. Subsequently, we have used the previously derived Hamiltonian, in order to determine the structure of the minima and stability of the theory in Section 4. We end in Section 5 with a discussion on the results obtained and possible future lines of research.

Notations and Conventions: We have set the fundamental constants \( c = 1 = \hbar \). Roman letters \( a, b, c, \ldots \) denote spacetime indices, while Greek letters \( \mu, \nu, \rho, \ldots \) denote spatial indices.

2 Quadratic action and its expansion around maximally symmetric background

In this section, we will work with the Einstein-Hilbert action with quadratic correction term, such that the gravitational action takes the following form

\[
A_{\text{grav}}^{\text{quad}} = \int d^4 x \sqrt{-g} \left\{ \frac{R}{16\pi G} + \alpha R^2 - \frac{\Lambda}{8\pi G} \right\},
\]

where \( G \) is the Newton’s constant, \( \Lambda \) is the cosmological constant and \( \alpha \) is a dimensionless constant that couples the higher curvature term. The gravitational field equations are obtained by varying the above action with respect to the spacetime metric \( g_{ab} \), yielding,

\[
(1 + 32\pi G\alpha R) R_{ab} - \frac{1}{2} (R + 16\pi G\alpha R^2) g_{ab} + (g_{ab} \nabla^a \nabla^b - 1/2 \nabla_a h_{bc} h^{bc}) = 8\pi G \left( -\frac{\Lambda}{8\pi G} \right) g_{ab}.
\]

Taking trace of the above equation we immediately obtain, \(-R + 3\Box (1 + 32\pi G\alpha R) = -4\Lambda\). A solution of the above equation corresponds to \( R = 4\Lambda = \text{constant} \). Note that we have not yet specified the sign of the cosmological constant and hence the above solution encompasses both the de Sitter and the anti-de Sitter spacetime including Minkowski spacetime. Thus the background spacetime is maximally symmetric in nature.

Having described the background spacetime and the gravitational action, it is now time to consider metric perturbations around the maximally symmetric background in the context of quadratic gravity. The aim is to rewrite the gravitational Lagrangian up to quadratic order in the metric perturbation and decompose the same into transverse traceless part and an additional scalar part. The origin of this scalar can be traced back to the fact that any higher curvature theory involving Ricci scalar alone has a scalar-tensor representation. In other words, the Lagrangian written down in Eq. (1) can be re-expressed as the Einstein-Hilbert action with an additional scalar field, which plays a crucial role in the stability of the theory, which we will explicitly demonstrate and is one of the main aims of this work. Following this motivation we express the metric and its inverse in terms of the perturbation \( h_{ab} \) as,

\[
g_{ab} = \bar{g}_{ab} + h_{ab} , \quad g^{ab} = \bar{g}^{ab} - h^{ab} + h^{ac} h^b_c + O(h^3).
\]

where \( \bar{g}_{ab} \) is the background maximally symmetric spacetime, i.e., de Sitter or anti-de Sitter for the present problem, and \( h_{ab} \) is the perturbation. Given the above expansion one can subsequently compute the Christoffel connection, the Riemann tensor and its descendants, i.e., Ricci tensor and Ricci scalar keeping terms quadratic in the perturbation \( h_{ab} \). The detailed expression for these quantities have been derived in Appendix A and the Ricci scalar to quadratic order has the following expression

\[
(\sqrt{-g}R)^{(2)} = \sqrt{-g} \left( \bar{R} \left( -\frac{1}{4} h_{ab} h^{ab} + \frac{1}{8} h^2 \right) + \bar{R}_{ab} \left( h^a_c h^{bc} - \frac{1}{2} h h^{ab} \right) + \frac{1}{4} \nabla_a h \nabla^a h - \frac{1}{2} \nabla_a h \nabla_b h \right)
\]
\[-\frac{1}{4} \nabla_c h^{ab} \nabla^c h_{ab} + \frac{1}{2} \nabla_c h_{ab} \nabla^a h^{cb}\] + Total Derivative, \hspace{1cm} (4)

where the superscript (2) denotes the fact that this action is correct up to quadratic terms of the perturbation \(h_{ab}\) and \(\bar{R}_{ab}\) is the background Ricci tensor. Further, the above expansion for the Einstein-Hilbert Lagrangian is around an arbitrary background spacetime. However, we are interested in the corresponding expression for maximally symmetric background. Such an expansion has been performed in Appendix B and thus to quadratic order in the perturbation \(h_{ab}\), the Einstein-Hilbert action with the cosmological constant term around dS/AdS background becomes \([43]\) (for a first principle derivation, see Appendix A and Appendix B),

\[A^{(2)}_{\text{EH}} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[ \frac{1}{4} h^{ab} \left( \Box - \frac{\bar{R}}{6} \right) h_{ab} - \frac{3}{32} \Phi \left( \Box + \frac{\bar{R}}{3} \right) \Phi \right], \hspace{1cm} (5)\]

where we have used the following decomposition for the perturbation \(h_{ab}\) into its irreducible components, which reads \([43]\),

\[h_{ab} = h^\perp_{ab} + \nabla_a a_b + \nabla_b a_a + \left( \nabla_a \nabla_b - \frac{1}{4} g_{ab} \Box \right) a + \frac{1}{4} g_{ab} h \hspace{1cm} (6)\]

where, \(h^\perp_{ab}\) is the transverse traceless part such that \(\nabla_a h^\perp_{ab} = 0 = g^{ab} h^\perp_{ab}\), \(a\) is the vector part satisfying \(\nabla_i a^i = 0\) and \(a\) is the scalar part with \(h\) being the trace of \(h_{ab}\). As evident from Eq. (5) the Einstein-Hilbert action along with the cosmological constant term when expanded up to quadratic order in the perturbation depends only on \(h^\perp_{ab}\) as well as \(a\) and \(h\) through the following combination \(\Phi \equiv h - \Box a\) \([43]\). Note that the (background) cosmological constant is appearing in this action through the identification \(\bar{R} = 4\Lambda\), where \(\bar{R}\) is the Ricci scalar of the background de Sitter or anti-de Sitter metric.

We can follow an identical path for computing the contribution of the quadratic term in the gravitational Lagrangian in the perturbation. First of all, one can determine the action to quadratic order in \(h_{ab}\) for an arbitrary background and then specialize to the maximally symmetric case under consideration. After a long algebra along these lines, the full action including the quadratic correction term \(\sqrt{-g}R^2\), reduces to the following form (for a derivation, see Appendix C and Appendix D),

\[A^{(2)}_{\text{grav}} = \int d^4x \sqrt{-g} \left[ \frac{1}{16\pi G} \left\{ \frac{1}{4} h^{ab} \left( \Box - \frac{\bar{R}}{6} \right) h_{ab} \right\} + \alpha \bar{R} \left\{ \frac{3}{4} h^{ab} \left( \Box - \frac{\bar{R}}{6} \right) h_{ab} \right\} \right. \]
\[\left. + \frac{1}{16\pi G} \left( - \frac{3}{32} \Phi \left( \Box + \frac{\bar{R}}{3} \right) \Phi \right) \right] + \alpha \left\{ \frac{9}{16} \left( \Box \Phi + \frac{\bar{R}}{3} \Phi \right)^2 - \frac{3\bar{R}}{16} \Phi \left( \Box + \frac{\bar{R}}{3} \right) \Phi \right\} \right]. \hspace{1cm} (7)\]

It is interesting to note that the higher derivatives of the dynamical variable appear only in the \(\Phi\)-sector, while the evolution of the transverse and traceless sector \(h^\perp_{ab}\) is governed by second order field equations. In particular, the Lagrangian for \(h^\perp_{ab}\) remains identical to that in general relativity, with a change in the overall factor, which now depends on both Newton’s constant \(G\), as well as \(\alpha\). Thus the propagation of the transverse traceless mode is unaffected by the presence of the quadratic curvature term in the Lagrangian. Following which, in our subsequent analysis the transverse traceless sector characterized by \(h^\perp_{ab}\) will not play any role and we will exclusively concentrate on the scalar sector \(\Phi\). Extracting and combining the relevant expressions from Eq. (7) associated with the scalar sector, the action for the scalar field \(\Phi\), with \(16\pi G\alpha \equiv \tilde{\alpha}\), yields

\[A_{\text{scalar}} = \frac{1}{16\pi G} \frac{3}{16} \int d^4x \sqrt{-g} \left[ \left( \frac{1}{2} + \tilde{\alpha} \bar{R} \right) \left\{ \Phi \left( \Box + \frac{\bar{R}}{3} \right) \Phi \right\} + 3\tilde{\alpha} \left\{ \left( \Box \Phi + \frac{\bar{R}}{3} \Phi \right)^2 \right\} \right]. \hspace{1cm} (8)\]
As evident from the above expression, the Lagrangian for \( \Phi \) involves \( \Box^2 \Phi \), i.e., inhibits higher derivative terms originating due to inclusion of the \( \alpha R^2 \) term in the gravitational Lagrangian. In what follows we will determine the dispersion relation associated with the above Lagrangian as well as the Hamiltonian, which we will subsequently minimize in order to determine any non-zero ground state for the scalar sector, having diverse implications.

3 Dispersion relation and structure of the Hamiltonian

In the previous section, we have determined the Lagrangian associated with the scalar sector of the gravitational perturbation starting from the perturbation of the Einstein-Hilbert Lagrangian with quadratic correction. We will now vary the Lagrangian and determine the associated field equations and the associated dispersion relations by transforming to the Fourier space. Earlier, we have denoted quantities in the background spacetime with a ‘bar’, while for notational convenience and since no confusion is likely to arise, we will remove the ‘bar’ from all the relevant quantities of the background spacetime.

To proceed further, it is important to decompose the Lagrangian written down in Eq. (8) in terms of spatial and temporal derivatives of \( \Phi \). For that purpose, it will be convenient to express the background metric in a suitable coordinate system. First of all, as emphasized earlier our primary interest is to study the early universe physics and the implications of the higher curvature term in the Lagrangian. Thus we will discuss the evolution of the scalar part of the gravitational perturbation, i.e., \( \Phi \) for dS metric in the conformal time coordinate, \( \eta \), defined as, \( d\eta \equiv \left\{ \frac{dt}{a(t)} \right\} \), where \( t \) is the cosmic time. Therefore, the line element can be expressed in the following form,

\[
ds^2 = a^2(\eta) \left( -d\eta^2 + dx^2 + dy^2 + dz^2 \right),
\]

which can be used in order to write down the Lagrangian for the scalar perturbation \( \Phi \), as expressed in Eq. (8), in terms of the derivatives with respect to the conformal time and the spatial derivatives as,

\[
L = -\left( \frac{1}{2} + \tilde{\alpha} R \right) a^2 \Phi \left\{ \nabla^2 \Phi - \Phi'' - 2\frac{a'}{a} \Phi' + \frac{R}{3} a^2 \Phi \right\} + 3\tilde{\alpha} \left\{ \nabla^2 \Phi - \Phi'' - 2\frac{a'}{a} \Phi' + \frac{R}{3} a^2 \Phi \right\}^2
\]

where ‘prime’ denotes derivative with respect to the conformal time coordinate, e.g., \( a' = (da/d\eta) \). Starting from the above Lagrangian it is straightforward to determine the field equation for \( \Phi \), by varying the above Lagrangian with respect to \( \Phi \). After discarding several total derivatives, which requires the assumption that both \( \Phi \) and its time derivatives are fixed on the two end point hypersurfaces. Thus we finally obtain, the following differential equation for \( \Phi \),

\[
\left[ 6\tilde{\alpha} \left\{ \nabla^2 - \partial_\eta^2 + 2\frac{a'}{a} \partial_\eta + 2 \frac{d}{d\eta} \left( \frac{a'}{a} \right) \right\} - a^2 \right] \left( \nabla^2 \Phi - \Phi'' - 2\frac{a'}{a} \Phi' + \frac{R}{3} a^2 \Phi \right) = 0.
\]

As evident the field equation involves four derivatives acting on \( \Phi \), which is simply a consequence of the higher derivative nature of the Lagrangian. It is also possible to read off the Hamiltonian starting from the above Lagrangian. Since the Lagrangian involves higher derivative, special care must be taken in deriving the Hamiltonian. In particular, one can employ two possible techniques for this purpose. The first one obviously corresponds to the Ostrogradsky’s method [54,55] of determining the Hamiltonian out of higher derivative Lagrangian. In this scheme, one treats \( \Phi \) and \( \Phi' \) as two independent variables and determine the momentum conjugate to each one of them. These momenta are used to eliminate \( \Phi'' \) from
the Lagrangian and hence one determines the Hamiltonian using standard procedure, in terms of $\Phi$, $\Phi'$ and their conjugate momenta. The second procedure is due to [56], where one modifies the Lagrangian by adding some linear function of $\Phi'$ which in turn introduces a new variable, which replaces $\Phi'$. Then one determines the conjugate momentum and the associated Hamiltonian. It follows that the Hamiltonian derived by these two procedure are related by canonical transformations. For a derivation of both these Hamiltonians in the present situation along with the relation between them, see Appendix E. Here we state the final result for the Hamiltonian, expressed in the coordinate space, which takes the following form,

$$H = \frac{a^2}{2} \left( -\Phi'^2 - |\nabla \Phi|^2 + \frac{1}{3} Ra^2 \Phi'^2 - 2 \frac{a'}{a} \Phi \Phi' \right) + \bar{\alpha} \left\{ 9 (\Phi'')^2 - 3 (\nabla^2 \Phi)^2 + Ra^2 \Phi'^2 - 6 \Phi'' \nabla^2 \Phi \\
+ Ra^2 (\nabla \Phi)^2 + 24 \left( \frac{a'}{a} \right)^2 \Phi'^2 - 12 \frac{a''}{a} \Phi'^2 + 2 Ra a' \Phi \Phi' \right\}.$$  

(12)

The terms in the first line constitutes the Hamiltonian arising out of general relativity, while the expression in the second line signifies the contribution from the higher curvature correction. As evident from the above expression, the Hamiltonian depends on $(\Phi'')^2$ as well as on $(\nabla^2 \Phi)^2$, originating from the $R^2$ term in the gravitational Lagrangian. Having derived the Lagrangian as well as the Hamiltonian, we would like to transform the same in the momentum space. This will not only help us to determine the dispersion relations but also the structure of the Hamiltonian and its ground state.

However, in a time dependent situation it is difficult to write down the mode functions necessary for Fourier decomposition, which will certainly affect the present scenario as well. Thus in order to proceed further, we will choose the mode functions to be consistent with the instantaneous vacuum prescription. We will just sketch one particular way of doing the same, which is achieved by transforming to constant frequency oscillator. Briefly speaking, each Fourier mode satisfies a differential equation which resembles a harmonic oscillator with time dependent frequency and mass. Each of these harmonic oscillators can be transformed to another oscillator, but with constant mass and frequency, thus defining an instantaneous vacuum state [57]. We will assume such is the case here, such that for the scalar perturbation $\Phi$, we have the following Fourier decomposition,

$$\Phi(\vec{x}, \eta) = \frac{1}{(2\pi)^2} \int d^3k d\omega \ e^{i(\vec{k} \cdot \vec{x} - \omega \eta)} \Phi(\vec{k}, \omega).$$

(13)

Since the field $\Phi$ is originating from the gravitational perturbation, it is manifestly real and hence we have $\Phi^*(\vec{k}, \omega) = \Phi(-\vec{k}, -\omega)$. Note that here both $\omega$ and $\vec{k}$ depends on the conformal time, however we will assume that throughout the evolution the adiabaticity conditions are maintained, such that derivatives of $\omega$ and $\vec{k}$ can be neglected. Further complications arise, as the terms involving $(\Phi'/\Phi)$ are considered, since these terms will lead to imaginary contribution to the equation of motion, which are not desirable. Interestingly all these terms are multiplied by $(a'/a)$ and hence we will also assume in accordance with the adiabaticity condition that $(a'/a)$ can be neglected. Thus substituting the above Fourier decomposition in the field equation derived from the Lagrangian, the following dispersion relations are obtained,

$$\text{dispersion relation I: } \omega^2 = \vec{k}^2 - \frac{R}{3} a^2; \quad \text{dispersion relation II: } \omega^2 = \vec{k}^2 + \frac{a^2}{9\alpha}.$$  

(14)

It is worth emphasizing that, even though the original Lagrangian has higher derivative terms, both the dispersion relations involve terms depending on $\omega$ and $\vec{k}$ in a quadratic manner. Also note that, the first
dispersion relation depends on the de Sitter radius \((1/\sqrt{R})\), while the second dispersion relation depends on the coupling coefficient \(\tilde{\alpha}\). Thus, both the de Sitter nature of the background spacetime and the presence of higher curvature term are essential to arrive at such non-trivial dispersion relations.

Having derived the dispersion relations, let us now determine the Hamiltonian in the Fourier space, which following the expression already derived in \(\text{Eq. (12)}\), takes the following form,

\[
H(|\vec{k}|, \omega) = |\Phi(\vec{k}, \omega)|^2 \left\{ \frac{a^2}{2} \left( \omega^2 + |\vec{k}|^2 + \frac{R}{3}a^2 \right) + \tilde{\alpha} \left( 9\omega^4 - 3|\vec{k}|^4 - a^2 R |\vec{k}|^2 - Ra^2 \omega^2 - 6\omega^2 |\vec{k}|^2 \right) \right\}
\]

\[
\equiv f(\omega, |\vec{k}|)|\Phi(\omega, \vec{k})|^2 .
\]

Here the last expression defines the function \(f(\omega, |\vec{k}|)\). Further, the first term in the Hamiltonian is originating from general relativity, while the second term has its origin from the quadratic correction term in curvature. Thus we see that for each mode, characterized by \((\omega, \vec{k})\), the Hamiltonian can be expressed as \(f(\omega, |\vec{k}|)|\Phi(\omega, \vec{k})|^2\) and it is interesting to look for the ground state by minimizing the function \(f(\omega, |\vec{k}|)\). This is because the existence of a non-trivial ground state may require a non-zero \(\tilde{\alpha}\), which can be related to a time crystal-like behaviour and emergence of an effective cosmological constant.

4 Minimum of the Hamiltonian and an effective cosmological constant

In the previous section we have worked with the higher derivative Lagrangian originating from the scalar perturbation of the Einstein-Hilbert action along with a quadratic correction. In particular, we have derived the equation of motion as well as the Hamiltonian out of the above higher derivative Lagrangian. Subsequently, we have determined the dispersion relation by transforming to the Fourier space as well as obtained the Hamiltonian in the Fourier space, which has been presented in \(\text{Eq. (15)}\). In this section we will minimize the Hamiltonian, or to be precise we will minimize \(f(\omega, |\vec{k}|)\) in order to determine any non-trivial ground state for the Fourier modes. It is important to realize a subtlety in the minimization procedure. Notice that \(f(\omega, |\vec{k}|)\) depend explicitly on \(a(\eta)\) which is (conformal) time dependent, see e.g., \(\text{Eq. (9)}\). Thus we are in fact minimizing on an arbitrary but fixed time slice. This approach was pursued in [57], though in a different context. For this purpose, we will adopt the following procedure: (a) we will express \(f(\omega, |\vec{k}|)\) solely in terms of either \(|\vec{k}|\) or \(\omega\) using the respective dispersion relations; (b) we will minimize \(f(\omega, |\vec{k}|)\) in terms of either \(|\vec{k}|\) or \(\omega\) and (c) finally we shall look for any non-trivial expression for \(f(\omega, |\vec{k}|)\) at the minimum.

Let us start with the following dispersion relation: \(\omega^2 = |\vec{k}|^2 - (R/3)a^2\). Then the Hamiltonian can be expressed solely in terms of either \(|\vec{k}|\) or \(\omega\), such that,

\[
f_1(|\vec{k}|) = (1 - 6\tilde{\alpha}R) a^2 |\vec{k}|^2 + \frac{4}{3}\tilde{\alpha}Ra^4 ,
\]

\[
f_1(\omega) = (1 - 6\tilde{\alpha}R) a^2 \omega^2 + (1 - 2\tilde{\alpha}R) \frac{Ra^4}{3} .
\]

Minimizing the first form of the function \(f_1(|\vec{k}|)\) with respect to \(|\vec{k}|\), i.e., imposing \(\{\partial f_1/\partial |\vec{k}|\} = 0\) and \(\{\partial^2 f_1/\partial |\vec{k}|^2\} > 0\), we obtain: \(|\vec{k}| = 0\) and \(\omega^2 = -(R/3)a^2\), along with \((1 - 6\tilde{\alpha}R) > 0\). However, \(R\) has to be negative (for real value for \(\omega\)) which is not consistent with the de Sitter nature of the background
spacetime. Thus this particular choice is not relevant for our discussion. On the other hand, by minimizing the function $f_I(\omega)$ with respect to $\omega$, i.e., demanding $\{\partial f_I/\partial \omega\} = 0$ and $\{\partial^2 f_I/\partial \omega^2\} > 0$ we obtain, the following conditions: $\omega = 0$, $|\vec{k}|^2 = (R/3)\omega^2$ and $(1 - 6\tilde{\alpha}R) > 0$. Note that the point where $\tilde{\alpha}R = (1/6)$ denotes the point of inflection, since both $\{\partial f_I/\partial \omega\}$ and $\{\partial^2 f_I/\partial \omega^2\}$ vanishes there. Thus at this minima we have $f_I = (1 - 2\tilde{\alpha}R)(Ra^4/3)$, which is positive, since $\tilde{\alpha}R < (1/6)$. It follows from Eq. (13) that the time translation symmetry of the ground state is intact but $\Phi(\vec{x})$ is periodic in $x$, indicating that the spatial translation symmetry is partially lost.

On the other hand, for the other dispersion relation, i.e., $\omega^2 = |\vec{k}|^2 + (a^2/6\tilde{\alpha})$, we obtain the following two expressions for $f(\omega, |\vec{k}|)$, expressed solely in terms of either $|\vec{k}|$ or in terms of $\omega$, as

$$f_{II}(|\vec{k}|) = (3 - 2\tilde{\alpha}R)a^2|\vec{k}|^2 + \frac{a^4}{3\tilde{\alpha}},$$

$$f_{II}(\omega) = (3 - 2\tilde{\alpha}R)a^2\omega^2 + \left(1 - \frac{1}{2\tilde{\alpha}R}\right) \frac{Ra^4}{3}.$$  

Minimizing $f_{II}(\omega)$, we obtain, $\omega = 0$ and $|\vec{k}|^2 = -(a^2/6\tilde{\alpha})$. Since $\tilde{\alpha} > 0$ is necessary for the stability of the theory, it follows that this particular minimization condition will lead to instability and hence is discarded. On the other hand, minimizing $f_{II}(|\vec{k}|)$ with respect to $|\vec{k}|$, we obtain: $|\vec{k}| = 0$, $\omega^2 = (a^2/6\tilde{\alpha})$ along with $(3 - 2\tilde{\alpha}R) > 0$. Imposing these minimization conditions, we obtain the Hamiltonian to have the following structure, $H_{II} = (a^4/3\tilde{\alpha})[\Phi(\omega, \vec{k})]^2$. Note that the positivity of the Hamiltonian along with the existence of a minima demands $\tilde{\alpha} > 0$, which is necessary for the stability of the gravitational action.

Again, following from Eq. (13) we find that the space translation symmetry of the ground state is intact but $\Phi(\eta)$ has acquired a periodicity in $\eta$ indicating that the time translation symmetry is partially lost. This is our cherished form of the time crystal. Thus we have two possible choices for the wave modes, originating from the minimization of the Hamiltonian, which read,

$$|\vec{k}|_I = a \sqrt{\frac{R}{3}}; \quad \omega_I = 0; \quad H_I = (1 - 2\tilde{\alpha}R) \left(\frac{Ra^4}{3}\right)|\Phi(\omega, \vec{k})|^2; \quad \tilde{\alpha}R < \frac{1}{6},$$

$$|\vec{k}|_{II} = 0; \quad \omega_{II} = \frac{a^2}{6\tilde{\alpha}}; \quad H_{II} = \frac{a^4}{3\tilde{\alpha}}|\Phi(\omega, \vec{k})|^2; \quad \tilde{\alpha}R < \frac{3}{2}.\tag{21}$$

Thus our analysis clearly demonstrates that the minimization of the Hamiltonian is intimately connected with the stability of the theory, by ensuring $\tilde{\alpha} > 0$. Hence the above provides another alternative route to assess the stability of the theory. Further, the first minimum configuration is associated with vanishing frequency, but non-zero wave vector. Hence this depicts a situation, where there is a preferred length scale in the theory, which is governed by the background cosmological constant. The same feature is governed by the energy density of the ground state Hamiltonian, $(H/a^4)$, scaling as the background cosmological constant modulated by $(1 - 2\tilde{\alpha}R)$. For small $\tilde{\alpha}$ it follows that the effective cosmological constant is of the same order as the background cosmological constant. Thus one may argue that the perturbation itself is capable of generating an effective cosmological constant, induced by the background spacetime.

While, for the second scenario, the modes have vanishing wave vector but a non-trivial frequency $\omega$, which is explicitly dependent on $\tilde{\alpha}$, the coupling of the quadratic correction term. In particular, this shows that the ground state of the Hamiltonian of the scalar perturbation depicts a preferred time scale in the theory $\sim \tilde{\alpha}^{-1}$. The same trend is seen in the Hamiltonian density as well, since $(H/a^4) \sim \tilde{\alpha}^{-1}$. This is very much akin to the structure of the time crystal, where there is a preferred time scale in the ground state of the theory. Thus presence of higher curvature term results into scalar perturbations to behave as time
crystalline state. Furthermore, the positivity of the minimum of the Hamiltonian is intimately connected with the stability of the theory. This also leads to an effective cosmological constant which depends on $\tilde{\alpha}$.

In addition, the above relation between the Hamiltonian of the ground state and the curvature coupling $\tilde{\alpha}$ also enables one to provide an estimation for $\alpha$. This can be obtained by noting that the above scenario is most likely to appear in the context of early universe, when the universe may have experienced an inflationary expansion. During the initial stages of such an expansion, the universe had an almost de Sitter expansion and higher curvature terms are likely to be important. Thus the minimum value of the Hamiltonian can be expressed into the following form, $(\rho/M_{pl}^4) = (1/6\alpha)\{\Phi(\omega, \vec{k})|^2/M_{pl}^2\}$, where we have used the result, $\tilde{\alpha} = (\alpha/M_{pl}^2)$.

During the inflationary epoch, if we want our analysis to hold, then all the energy scales must be sub-Planckian. Thus it follows that $(\rho/M_{pl}^4) \sim 1$ as well as $\{\Phi(\omega, \vec{k})|^2/M_{pl}^2\} \sim 10^{-4}$, which suggests, $\alpha \sim 10^{-4}$.

This is consistent with our expectation as well. Thus using inflationary dynamics, it is indeed possible to determine the coefficient of the quadratic correction $\alpha$ and thus one may impose interesting bounds on the inflationary energy scale. Therefore, the presence of higher curvature terms in the gravitational action, leads to non-trivial ground state for the Hamiltonian associated with scalar perturbation, introducing non-trivial length and time scale into the problem and an effective cosmological constant. This in turn results into an estimation of the curvature coupling through inflationary dynamics.

5 Discussion and concluding remarks

We have started with an action which includes the Einstein-Hilbert term as well as a term involving quadratic correction in the curvature. This being the simplest choice among the $f(R)$ theories of gravity. Such a higher curvature correction term brings in an additional length scale in the problem, namely $(16\pi G \alpha)^{-1/2}$ and is expected to modify the nature of the propagating gravitational degrees of freedom, which must be intimately connected with the stability of the corresponding gravitational theory. Following which, we have considered the “effective” action for the gravitational perturbation, obtained by expanding the gravitational action around an arbitrary background and keeping terms quadratic in the perturbation. Surprisingly, for perturbation around a maximally symmetric background it follows that the transverse, traceless part of the gravitational perturbation satisfies a second order differential equation, even though the original action involves higher derivative terms. However, in addition to the transverse, traceless part there is also a contribution from the scalar part of the gravitational perturbations, whose evolution depends on the higher derivative terms. Thus we can conclude that, the effect of quadratic correction to the Einstein-Hilbert action is not to generate a higher derivative evolution equation for the transverse traceless part of the gravitational perturbation, rather to have an extra scalar degree of freedom, whose evolution is modified by the presence of higher derivative terms. Hence stability of the theory seemingly depends on the stability of this additional scalar degree of freedom, and following such a line of thought we have determined the existence of the minima of the Hamiltonian associated with the scalar field dynamics.

For this purpose, we have used the de Sitter spacetime as the background and it turns out that the stability of the Hamiltonian, i.e., existence of a minima holds true if and only if the additional coupling parameter $\alpha$ is positive. Since for negative values of $\alpha$ the Hamiltonian becomes unbounded from below, which is a manifestation of the Ostrogradsky instability associated with higher derivative theories. Further,
the structure of the minima also provides us an interesting insight about the nature of the ground state. In particular, there are two possible configurations associated with the minima of the Hamiltonian — (a) involving non-trivial spatial wavelength $\sqrt{3/R}$, associated with the cosmological length scale and (b) involving non-trivial frequency $\sim 1/(6\tilde{\alpha})$, associated with the presence of higher curvature degrees of freedom. The presence of a non-trivial temporal behaviour is reminiscent of the idea of time crystal developed in the context of many body quantum mechanical system. The above results hint at possible emergence of a time crystal-like scenario in an expanding universe with higher curvature correction. Note that the existence of time crystal-like behaviour is solely due to the presence of the higher curvature term. Moreover, the non-zero minimum value of the Hamiltonian can act as a proxy for the effective cosmological constant, since it will contribute over and above the de Sitter background. To summarize, we have demonstrated the existence of non-trivial ground state configuration for the scalar part of the gravitational perturbation in the presence of higher curvature correction. Existence of such ground states are intimately related with stability of the higher curvature gravity theory and leads to non-trivial time crystal-like structure and possibly an effective cosmological constant.

This opens up to us several future directions, in particular, existence of a time crystal-like structure in the context of quantum fields in cosmological spacetime will be a very interesting avenue to explore. Further, in this work we have discussed the quadratic correction to the Einstein-Hilbert action, thus it will be interesting to ask, whether the transverse traceless part of the gravitational perturbation remains free of higher derivatives even in the context of generic $f(R)$ theory of gravity or in higher order perturbation. Also possible connection with the scalar tensor representation can be explored. In a different perspective, it will be interesting to look at possible imprints of the new length and time scales in the propagation of gravitational waves in these non-trivial space or time dependent backgrounds. These we leave for the future.

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A Quadratic action for Einstein gravity around an arbitrary background

In this section we will consider the gravitational action perturbed up to second order with respect to an arbitrary background. We will first work with the Einstein-Hilbert Lagrangian, before discussing implications of higher curvature corrections. Thus we start with,

$$g_{ab} = \bar{g}_{ab} + h_{ab}, \quad (23)$$

where $g_{ab}$ is the background spacetime and $h_{ab}$ is the perturbation. The inverse metric takes the following form,

$$g^{ab} = \bar{g}^{ab} - h^{ab} + h^{ac}h^b_c + O(h^3). \quad (24)$$
Hence we obtain,

\[
\sqrt{-g} = \sqrt{-\bar{g}} \left\{ 1 + \frac{1}{2} h - \frac{1}{4} h_{ab} h^{ab} + \frac{1}{8} h^2 \right\},
\]

(25)

where \( h = h_{a}^{a} \) is the trace of the perturbed metric. The Christoffel symbols upon perturbation till quadratic order can be expressed as,

\[
\Gamma_{bc}^{a} = \bar{\Gamma}_{bc}^{a} + \hat{\Gamma}_{bc}^{a} + \tilde{\Gamma}_{bc}^{a}.
\]

(26)

In the above expression \( \bar{\Gamma}_{bc}^{a} \) is the Christoffel symbol associated with the background metric, \( \hat{\Gamma}_{bc}^{a} \) depends on \( h_{ab} \) linearly while \( \tilde{\Gamma}_{bc}^{a} \) depends on the quadratic powers of the perturbed metric. Expanding out we get,

\[
\bar{\Gamma}_{bc}^{a} = \frac{1}{2} \bar{g}^{ad} \left( -\partial_{d} \bar{g}_{bc} + \partial_{b} \bar{g}_{cd} + \partial_{c} \bar{g}_{bd} \right),
\]

(27)

\[
\hat{\Gamma}_{bc}^{a} = \frac{1}{2} \bar{g}^{ad} \left( -\nabla_{d} h_{bc} + \nabla_{b} h_{cd} + \nabla_{c} h_{bd} \right),
\]

(28)

\[
\tilde{\Gamma}_{bc}^{a} = -\frac{1}{2} h^{ad} \left( -\nabla_{d} h_{bc} + \nabla_{b} h_{cd} + \nabla_{c} h_{bd} \right).
\]

(29)

Using these expressions we finally obtain,

\[
\hat{\Gamma}_{ab}^{a} = \frac{1}{2} \nabla_{b} h; \quad \tilde{\Gamma}_{ab}^{a} = -\frac{1}{2} h^{cd} \nabla_{b} h_{cd}.
\]

(30)

The Riemann tensor can also be expanded as,

\[
\bar{R}_{bcd}^{a} = \bar{\bar{R}}_{bcd}^{a} + \bar{\hat{R}}_{bcd}^{a} + \bar{\tilde{R}}_{bcd}^{a},
\]

(31)

where various terms, dependent on zeroth, first and second order in \( h_{ab} \) takes the form,

\[
\bar{R}_{bcd}^{a} = \partial_{c} \bar{\Gamma}_{bd}^{a} - \partial_{d} \bar{\Gamma}_{bc}^{a} + \bar{\Gamma}_{cp}^{a} \bar{\Gamma}_{bd}^{p} - \bar{\Gamma}_{dp}^{a} \bar{\Gamma}_{bc}^{p},
\]

(32)

\[
\bar{R}_{bcd}^{a} = \nabla_{c} \bar{\Gamma}_{bd}^{a} - \nabla_{d} \bar{\Gamma}_{bc}^{a},
\]

(33)

\[
\bar{R}_{bcd}^{a} = \nabla_{c} \bar{\Gamma}_{bd}^{a} - \nabla_{d} \bar{\Gamma}_{bc}^{a} + \Gamma_{cp}^{a} \bar{\Gamma}_{bd}^{p} - \Gamma_{dp}^{a} \bar{\Gamma}_{bc}^{p}.
\]

(34)

Similar expansion holds for Ricci scalar as well,

\[
\bar{R}_{ab} = \bar{R}_{ab} + \bar{\hat{R}}_{ab} + \bar{\tilde{R}}_{ab},
\]

(35)

where also various terms of different orders of \( h_{ab} \) take the following form,

\[
\bar{R}_{ab} = \delta_{b}^{c} \bar{\Gamma}_{ac}^{d},
\]

(36)

\[
\bar{R}_{ab} = \nabla_{c} \bar{\Gamma}_{ab}^{c} - \nabla_{a} \bar{\Gamma}_{bc}^{c},
\]

\[
= \nabla_{c} \left[ \frac{1}{2} \bar{g}^{cd} \left( -\nabla_{d} h_{ba} + \nabla_{b} h_{ad} + \nabla_{a} h_{bd} \right) \right] - \nabla_{a} \left[ \frac{1}{2} \bar{\nabla}_{b} h \right]
\]

\[
= \frac{1}{2} \left[ -\nabla h_{ab} + \nabla_{c} \nabla_{a} h_{b}^{c} + \nabla_{c} \nabla_{b} h_{a}^{c} - \nabla_{a} \nabla_{b} h \right]
\]

(37)

\[
\bar{R}_{ab} = \nabla_{c} \bar{\Gamma}_{ba}^{c} - \nabla_{b} \bar{\Gamma}_{ac}^{c} + \bar{\Gamma}_{cp}^{a} \bar{\Gamma}_{bd}^{p} - \bar{\Gamma}_{dp}^{a} \bar{\Gamma}_{bc}^{p},
\]

\[
\bar{R}_{ab} = \nabla_{c} \bar{\Gamma}_{ba}^{c} - \nabla_{b} \bar{\Gamma}_{ac}^{c} + \bar{\Gamma}_{cp}^{a} \bar{\Gamma}_{bd}^{p} - \bar{\Gamma}_{dp}^{a} \bar{\Gamma}_{bc}^{p}.
\]
\[
\begin{align*}
&\nabla_c \left[ -\frac{1}{2} h^{cd} (-\nabla_d h_{ba} + \nabla_b h_{ad} + \nabla_a h_{bd}) \right] - \nabla_b \left[ -\frac{1}{2} h^{cd} \nabla_a h_{cd} \right] + \frac{1}{2} \nabla_p h \left[ \frac{1}{2} \hat{g}^{pq} (-\nabla_d h_{ba} + \nabla_b h_{ad} + \nabla_a h_{bd}) \right] \\
&- \left[ \frac{1}{2} \hat{g}^{cd} (-\nabla_d h_{pa} + \nabla_p h_{ad} + \nabla_a h_{pd}) \right] \left[ \frac{1}{2} \hat{g}^{pq} (-\nabla_q h_{bc} + \nabla_b h_{qc} + \nabla_c h_{qb}) \right] \\
&= \frac{1}{2} \nabla_b \left[ h^{cd} \nabla_a h_{cd} \right] - \frac{1}{2} \nabla_c \left[ h^{cd} (-\nabla_d h_{ba} + \nabla_b h_{ad} + \nabla_a h_{bd}) \right] \\
&+ \frac{1}{4} \nabla^d h (-\nabla_d h_{ba} + \nabla_b h_{ad} + \nabla_a h_{bd}) - \frac{1}{4} \left(-\nabla^c h_{pa} + \nabla_p h^c_a + \nabla_a h^p_c \right) \left(-\nabla^p h_{bc} + \nabla_b h^p_c + \nabla_c h^p_b \right).
\end{align*}
\]

The Ricci scalar becomes,

\[
R = \bar{R} + \bar{R} + \bar{R}
\]

where, \(\bar{R}\) is the background Ricci scalar and we also have,

\[
\bar{R} = -h^{ab} \bar{R}_{ab} + \tilde{g}^{ab} \tilde{R}_{ab}
\]

\[
= -h^{ab} \bar{R}_{ab} + \tilde{g}^{ab} \frac{1}{2} \left[ -\Box h_{ab} + \nabla_c \nabla_a h^c_b + \nabla_c \nabla_b h^c_a - \nabla_a \nabla_b h \right]
\]

\[
= -h^{ab} \bar{R}_{ab} + \left(-\Box h + \nabla_a \nabla_b h^{ab} \right)
\]

To simplify the last expression, let us consider the following identity

\[
\tilde{g}^{ab} \tilde{R}_{ab} = \frac{1}{2} \nabla^a \left[ h^{cd} \nabla_a h_{cd} \right] - \frac{1}{2} \nabla_c \left[ h^{cd} (-\nabla_d h_{ba} + 2 \nabla_b h^a_d) \right] \\
+ \frac{1}{4} \nabla^d h (-\nabla_d h_{ba} + 2 \nabla_b h^a_d) - \frac{1}{4} \left(-\nabla^c h_{pa} + \nabla_p h^c_a + \nabla_a h^p_c \right) \left(-\nabla^p h_{bc} + \nabla_b h^p_c + \nabla_c h^p_b \right)
\]

\[
= \frac{1}{2} \nabla^a \left[ h^{cd} \nabla_a h_{cd} \right] - \frac{1}{2} \nabla_c \left[ h^{cd} (-\nabla_d h_{ba} + 2 \nabla_b h^a_d) \right] - \frac{1}{4} \nabla^d h \nabla_d h + \frac{1}{2} \nabla^d h \nabla_a h^a_d \\
- \frac{1}{4} \left[-\nabla^c h_{pa} \nabla_c h^p_a - \nabla_p h^c_a \left(-\nabla^a h^p_c + \nabla_c h^p_a \right) + \nabla_a h^p_c \nabla^c h^p_c \right] \\
= \frac{1}{2} \nabla^a \left[ h^{cd} \nabla_a h_{cd} \right] - \frac{1}{2} \nabla_c \left[ h^{cd} (-\nabla_d h_{ba} + 2 \nabla_b h^a_d) \right] - \frac{1}{4} \nabla^d h \nabla_d h + \frac{1}{2} \nabla^d h \nabla_a h^a_d \\
- \frac{1}{4} \left[-\nabla^c h_{pa} \nabla_c h^p_a - \nabla_p h^c_a \left(-\nabla^a h^p_c + \nabla_c h^p_a \right) \right] \\
= \frac{1}{2} \nabla^a \left[ h^{cd} \nabla_a h_{cd} \right] - \frac{1}{2} \nabla_c \left[ h^{cd} (-\nabla_d h_{ba} + 2 \nabla_b h^a_d) \right] - \frac{1}{4} \nabla^d h \nabla_d h + \frac{1}{2} \nabla^d h \nabla_a h^a_d \\
+ \frac{1}{4} \nabla^c h_{pa} \nabla_c h^p_a - \frac{1}{2} \nabla_p h^c_a \nabla_a h^p_c 
\]

as well as,

\[
h^{ab} \bar{R}_{ab} = \frac{1}{2} h^{ab} \left[ -\Box h_{ab} + \nabla_c \nabla_a h^c_b + \nabla_c \nabla_b h^c_a - \nabla_a \nabla_b h \right]
\]

\[
= \frac{1}{2} \left(-h^{ab} \Box h_{ab} + 2 h^{ab} \nabla_c \nabla_a h^c_b - h^{ab} \nabla_a \nabla_b h \right)
\]
Hence we immediately obtain the second order correction to the Ricci scalar to read,

\[
\tilde{R} = h^a h^{bc} \tilde{R}_{ab} - \frac{1}{2} \left\{ -h^{ab} \Box h_{ab} + 2h^{ab} \nabla_c \nabla_a h_b^c - h^{ab} \nabla_a \nabla_b h \right\} + \frac{1}{2} \nabla^a \left[ h^{cd} \nabla_a h_{cd} \right] - \frac{1}{2} \nabla_c \left[ h^{cd} \left( -\nabla_d h + 2\nabla_a h_d^a \right) \right] 
\]

\[
- \frac{1}{4} \nabla^d h \nabla_d h + \frac{1}{2} \nabla^d h \nabla_a h_d^a + \frac{1}{4} \nabla^c h_p a \nabla_c h^{pa} - \frac{1}{2} \nabla_p h^{ca} \nabla a h_c^p 
\]

(44)

Thus the Lagrangian density becomes,

\[
\sqrt{-g} R = \sqrt{-g} \left\{ 1 + \frac{1}{2} \frac{1}{2} h - \frac{1}{4} h^{ab} h_{ab} + \frac{1}{8} h^2 \right\} \left( \tilde{R} + \tilde{R} + \tilde{R} \right) = \tilde{\mathcal{L}} + \tilde{\mathcal{L}} + \tilde{\mathcal{L}} 
\]

(45)

where \( \tilde{\mathcal{L}} = \sqrt{-g} R \) and the other higher order terms are,

\[
\tilde{\mathcal{L}} = \sqrt{-g} \left( \tilde{R} + \frac{1}{2} h \tilde{R} \right) = \sqrt{-g} \left\{ -h^{ab} \tilde{R}_{ab} + \left( -\Box h + \nabla_a \nabla_b h^{ab} \right) + \frac{1}{2} h \tilde{R} \right\} 
\]

\[
= \sqrt{-g} \left\{ -h^{ab} \left( \tilde{R}_{ab} - \frac{1}{2} \tilde{g}_{ab} \tilde{R} \right) + \left( -\Box h + \nabla_a \nabla_b h^{ab} \right) \right\} 
\]

(46)

as well as,

\[
\tilde{\mathcal{L}} = \sqrt{-g} \left\{ \tilde{R} \left( -\frac{1}{4} h^{ab} h_{ab} + \frac{1}{8} h^2 \right) + \frac{1}{2} h \tilde{R} + \tilde{R} \right\} 
\]

\[
= \sqrt{-g} \left\{ \tilde{R} \left( -\frac{1}{4} h^{ab} h_{ab} + \frac{1}{8} h^2 \right) + \frac{1}{2} h \left\{ -h^{ab} \tilde{R}_{ab} + \left( -\Box h + \nabla_a \nabla_b h^{ab} \right) \right\} 
\]

\[
+ h^a h^{bc} \tilde{R}_{ab} - \frac{1}{2} \left\{ -h^{ab} \Box h_{ab} + 2h^{ab} \nabla_c \nabla_a h_b^c - h^{ab} \nabla_a \nabla_b h \right\} + \frac{1}{2} \nabla^a \left[ h^{cd} \nabla_a h_{cd} \right] - \frac{1}{2} \nabla_c \left[ h^{cd} \left( -\nabla_d h + 2\nabla_a h_d^a \right) \right] 
\]

\[
- \frac{1}{4} \nabla^d h \nabla_d h + \frac{1}{2} \nabla^d h \nabla_a h_d^a + \frac{1}{4} \nabla^c h_p a \nabla_c h^{pa} - \frac{1}{2} \nabla_p h^{ca} \nabla a h_c^p 
\]

(47)

The above expression can be further simplified to yield the final expression for the Einstein-Hilbert Lagrangian till quadratic order in the perturbation \( \tilde{g}_{ab} \) when expanded around an arbitrary background \( \tilde{g}_{ab} \).

The corresponding expression has been presented in Eq. (4). As emphasized earlier, in the above expression we have kept the background arbitrary, while in the next section we consider the quadratic action around dS/AdS background. These results have been used in Section 2.
\section*{B Decomposition of the quadratic action for Einstein gravity around dS/AdS background}

In this section we will explicitly write down the action for the perturbation $h_{ab}$ around dS/AdS background. For that purpose we express the Lagrangian in this background as,

$$\mathcal{L} = \sqrt{-g} \left[ R \left( -\frac{1}{4} h_{ab} h^{ab} + \frac{1}{8} h^2 \right) + \frac{R}{4} g_{ab} \left( h^a h^{bc} - \frac{1}{2} h h^{ab} \right) + \frac{1}{4} \nabla_a h \nabla^a h - \frac{1}{2} \nabla_a h \nabla_b h^{ab} \right]$$

we now have the following identity,

$$\frac{1}{2} \nabla_c h_{ab} \nabla^c h^{cb} = -\frac{1}{2} h^{cb} \nabla_d \nabla_c h_{ab} + \text{Total Derivative}$$

Thus the Lagrangian becomes,

$$\tilde{\mathcal{L}} = \sqrt{-g} \left[ \frac{1}{4} \nabla_a h \nabla^a h - \frac{1}{2} \nabla_a h \nabla_b h^{ab} - \frac{1}{4} \nabla_c h_{ab} \nabla^c h^{ab} + \frac{1}{2} \nabla_c h_5 h \nabla^c h_{ab} - \frac{R}{24} \left( 4 h_{ab} h^{ab} - h^2 \right) \right]$$

Let us now use the following decomposition,

$$h_{ab} = h_{ab}^+ + \nabla_a a_b + \nabla_b a_a + \left( \nabla_a \nabla_b - \frac{1}{4} g_{ab} \Box \right) a + \frac{1}{4} g_{ab} \dot{h}$$

where, $\nabla_a h_{ab}^+ = 0 = g_{ab} h_{ab}^+$ and $\nabla_i a^i = 0$. We have the following expressions, already worked out in a later section.

$$\frac{1}{4} h^{ab} \Box h_{ab} = \frac{1}{4} \text{Term E} = \frac{1}{16} h \Box h + \frac{1}{4} h^{ab} \Box h_{ab}$$

$$- \frac{1}{2} \Box a^b a_b - \frac{R}{3} a_b \Box a_b - \frac{5 R^2}{96} a^a a_b + \frac{3}{16} \Box a^2 a + \frac{3 R}{16} \Box a^2 a + \frac{R^2}{24} a \Box a$$

$$= \sqrt{-g} \left[ \frac{1}{4} \nabla_a h \nabla^a h - \frac{1}{2} \nabla_a h \nabla_b h^{ab} - \frac{1}{4} \nabla_c h_{ab} \nabla^c h^{ab} + \frac{1}{2} \nabla_c h_5 h \nabla^c h_{ab} - \frac{R}{24} \left( 4 h_{ab} h^{ab} - h^2 \right) \right]$$
Similarly,

$$\frac{1}{2} h \nabla_a \nabla_b h^{ab} = \frac{R}{8} h \Box a + \frac{3}{8} h \Box^2 a + \frac{1}{8} h \Box h $$  \hspace{1cm} (53)$$

as well as,

$$\frac{1}{2} \nabla_c h^c_b \nabla_a h^{ab} = \frac{1}{2} \left[ a^b \Box a_b + \frac{R}{2} \left( a^b + \nabla^b a \right) \Box a_b + \frac{3}{2} \left( \nabla^b \Box a \right) \Box a_b + \frac{1}{2} \nabla^b h \Box a_b + \frac{R^2}{16} \left( a^b + \nabla^b a \right) \left( a_b + \nabla_b a \right) 
+ \frac{3R}{8} \nabla^b a \left( a_b + \nabla_b a \right) + \frac{R}{8} \left( a^b + \nabla^b a \right) \nabla_b h + \frac{9}{16} \nabla_b \Box a \nabla_b h + \frac{3}{8} \nabla^b \Box a \nabla_b h + \frac{1}{16} \nabla_b h \nabla^b h \right]$$

Finally,

$$h_{ab} h^{ab} = h_{ab}^i h_{ab}^i - 2a_b \Box a^b - \frac{R}{2} a^b a_b + \frac{R}{4} a \Box a + \frac{3}{4} (\Box a)^2 + \frac{1}{4} h^2 $$  \hspace{1cm} (55)$$

Thus we have terms independent of Ricci scalar,

$$\mathcal{L} \text{ (independent of } R) = \sqrt{-g} \left[ -\frac{1}{4} h \Box h + \frac{3}{8} h \Box^2 a + \frac{1}{8} \nabla h + \frac{1}{16} h \Box h + \frac{1}{4} h^a b \Box h + \frac{1}{2} \Box a^b \Box a_b + \frac{3}{16} a \Box^2 a 
+ \frac{1}{2} \Box a^b \Box a_b + \frac{3}{4} \left( \nabla^b \Box a \right) \Box a_b + \frac{1}{4} \nabla b h \Box a_b + \frac{9}{32} \nabla b \Box a \nabla b \Box a + \frac{3}{16} \nabla^b \Box a \nabla b h + \frac{1}{32} \nabla h \nabla^b h \right]$$

$$= \sqrt{-g} \left[ -\frac{1}{4} h \Box h + \frac{3}{8} h \Box^2 a + \frac{1}{8} \nabla h + \frac{1}{16} h \Box h + \frac{1}{4} h^a b \Box h + \frac{1}{2} \Box a^b \Box a_b + \frac{3}{16} a \Box^2 a 
+ \frac{1}{2} \Box a^b \Box a_b - \frac{9}{32} a \Box^2 a - \frac{3}{16} \Box a h - \frac{1}{32} h \Box h \right]$$

$$= \sqrt{-g} \left[ \frac{1}{4} h^a b \Box h + \frac{3}{32} h \Box h + \frac{3}{16} \Box a^2 a - \frac{3}{32} \Box a \Box a - \frac{3}{16} \Box a \Box a \right]$$

$$= \sqrt{-g} \left[ \frac{1}{4} h^a b \Box h + \frac{3}{32} h \Box h + \frac{3}{16} \Box a^2 a - \frac{3}{32} \Box a \Box a \right]$$

$$= \sqrt{-g} \left[ \frac{1}{4} h^a b \Box h + \frac{3}{32} h \Box h + \frac{3}{16} \Box a^2 a - \frac{3}{32} \Box a \Box a \right]$$

Proceeding Further, we have terms linear in the background Ricci scalar,

$$\mathcal{L} \text{ (linear in } R) = \sqrt{-g} \left[ \frac{R}{8} h \Box a - \frac{R}{3} a^b \Box a_b + \frac{3R}{16} a \Box^2 a + \frac{R}{4} \left( a^b + \nabla^b a \right) \Box a_b + \frac{3R}{16} \nabla^b \Box a \left( a_b + \nabla_b a \right) 
+ \frac{R}{16} \left( a^b + \nabla^b a \right) \nabla h - \frac{R}{24} \left( 3 \{ R \text{ independent part of } (h_{ab} h^{ab}) \} + h_{ab}^a h_{ab}^b - 2a_b \Box a^b + \frac{3}{4} (\Box a)^2 + \frac{1}{4} h^2 \right) \right]$$

$$= \sqrt{-g} \left[ \frac{R}{8} h \Box a - \frac{R}{3} a^b \Box a_b + \frac{3R}{16} a \Box^2 a + \frac{R}{4} a^b \Box a_b - \frac{3R}{16} a \Box a \Box a \right]$$

$$= \sqrt{-g} \left[ \frac{R}{16} h \Box a - \frac{R}{12} a^b \Box a_b - \frac{R}{8} \left( R \text{ independent part of } (h_{ab} h^{ab}) \right) 
- \frac{R}{24} \left( h_{ab}^a h_{ab}^b - 2a_b \Box a^b + \frac{3}{4} (\Box a)^2 - \frac{3}{4} h^2 \right) \right]$$

$$= \sqrt{-g} \left[ -\frac{R}{24} h_{ab}^a h_{ab}^b - \frac{R}{16} h \Box a - \frac{R}{8} \left( R \text{ independent part of } (h_{ab} h^{ab}) \right) - \frac{R}{32} (\Box a)^2 - \frac{R}{32} h^2 + \frac{R}{16} h^2 \right]$$

$$= \sqrt{-g} \left[ \right]$$
Finally, the quadratic piece becomes,

\[
\mathcal{L}_{\text{quadratic in } R} = \sqrt{-g} \left[ -\frac{5R^2}{96} a^b a_b + \frac{R^2}{24} a \Box a + \frac{R^2}{32} (a^b + \nabla^b a) (a_b + \nabla_b a) \right] - \frac{R}{8} \left\{ \text{linear in } R \text{ part of} (h_{ab} h^{ab}) \right\} - \frac{R}{8} \left( -\frac{R}{2} a_b a^b + \frac{R}{4} a \Box a \right) \]

Thus combining all these results together, we obtain,

\[
= \sqrt{-g} \left[ -\frac{5R^2}{96} a^b a_b + \frac{R^2}{24} a \Box a + \frac{R^2}{32} (a^b + \nabla^b a) (a_b + \nabla_b a) \right] - \frac{R}{8} \left\{ \text{linear in } R \text{ part of} (h_{ab} h^{ab}) \right\} \]

Thus the total Lagrangian becomes,

\[
\mathcal{L} = \sqrt{-g} \left[ \frac{1}{4} h_{ab}^{\perp} h^{ab} - \frac{3}{32} \Phi \Box \Phi \right] + \sqrt{-g} \left[ -\frac{R}{24} h_{ab}^{\perp} h_b^{ab} - \frac{R}{32} \Phi^2 \right] - \frac{R}{8} \left\{ \text{R independent part of} (h_{ab} h^{ab}) \right\} + \frac{R}{16} h^2 \] + \sqrt{-g} \left[ -\frac{R}{8} \left( h_{ab} h^{ab} - \frac{1}{2} h^2 \right) \right] \] (59)

Finally we have,

\[
\mathcal{L}_\Lambda = -2\Lambda \sqrt{-g} = \frac{\Lambda}{2} \sqrt{-g} \left\{ h_{ab} h^{ab} - \frac{1}{2} h^2 \right\} = \frac{R}{8} \sqrt{-g} \left\{ h_{ab} h^{ab} - \frac{1}{2} h^2 \right\} \] (60)

where we have used the result that, $R = 4\Lambda$.

Thus combining all these results together, we obtain,

\[
\mathcal{A} = \int d^4 x \sqrt{-g} \left\{ (R - 2\Lambda) \right\} = \frac{1}{16\pi G} \int d^4 x \sqrt{-g} \left[ \frac{1}{4} h_{ab}^{\perp} \left( \Box - \frac{R}{6} \right) h_b^{ab} - \frac{3}{32} \Phi \left( \Box + \frac{R}{3} \right) \Phi \right] \] (61)

This is the expression for the quadratic action of the Einstein-Hilbert action used in the main text. In the next section we consider the action with a $R^2$ term included and write down the associated quadratic action. This result is necessary to determine the action derived in Section 2.

## C Quadratic action for $R^2$ gravity around an AdS/dS background

Let us now consider the term involving $R^2$ in the gravitational action and the associated second order Lagrangian it produces. For that let us decompose the Lagrangian $R^2$, yielding,

\[
R^2 = \left( \mathring{R} + \mathring{\mathring{R}} \right)^2 = R^2 + \mathring{R}^2 + \mathring{\mathring{R}}^2 \] (62)
where, $\overline{R^2} = \overline{R^2}$ and the other terms yield,
$$\overline{R^2} = 2\overline{R} \overline{R} = 2\overline{R} \left\{ -h^{ab} \overline{R}_{ab} + \left( -\square h + \nabla_a \nabla_b h^{ab} \right) \right\}$$

$$\text{AdS/dS} \equiv -\frac{\overline{R^2}}{2} h + 2\overline{R} \left( -\square h + \nabla_a \nabla_b h^{ab} \right)$$  \hspace{1cm} (63)

where in the last line we have used the result, $\overline{R}_{ab} = \left( 1/4 \right) \overline{R} g_{ab}$ as well as $\overline{R}_{abcd} = \left( R/12 \right) (g_{ac} g_{bd} - g_{ad} g_{bc})$ for maximally symmetric spacetime. Thus following an identical route as the previous one, we have for the second order term in the $\overline{R^2}$ action to be,
$$\overline{R^2} = \overline{R^2} + 2\overline{R} \overline{R}$$

$$= \{ -h^{ab} \overline{R}_{ab} + \left( -\square h + \nabla_a \nabla_b h^{ab} \right) \}^2 + 2\overline{R} \left[ h^{ab} h^{bc} \overline{R}_{ab} - \frac{1}{2} \left\{ -h^{ab} \square h_{ab} + 2h^{ab} \nabla_a \nabla_b h^{ab} - h^{ab} \nabla_a \nabla_b \overline{R} \right\} \right]$$

$$+ \frac{1}{2} \nabla_a \left[ h^{cd} \nabla_a h_{cd} - \frac{1}{2} \nabla_c \left[ h^{cd} \left( -\nabla_d h + 2\nabla_a h^{ab} \right) \right] - \frac{1}{4} \nabla^d h \nabla_d h + \frac{1}{2} \nabla^d h \nabla_a h^{ab} + \frac{1}{4} \nabla_c p a \nabla_c h^{pa} - \frac{1}{2} \nabla p h^{ca} \nabla_a h^{b}_c \right]$$

$$\text{AdS/dS} \equiv -\frac{\overline{R^2}}{4} h + 2\overline{R} \left( -\square h + \nabla_a \nabla_b h^{ab} \right)$$

$$+ \frac{1}{4} \nabla^d h \nabla_d h + \frac{1}{2} \nabla^d h \nabla_a h^{ab} + \frac{1}{4} \nabla^c p a \nabla_c h^{pa} - \frac{1}{2} \nabla p h^{ca} \nabla_a h^{b}_c \right\} + \text{Total Derivative}$$

$$= \{ -h^{ab} \overline{R}_{ab} + \left( -\square h + \nabla_a \nabla_b h^{ab} \right) \}^2 + h^{ab} h_{ab} \overline{R^2} - \overline{R} \left\{ -h^{ab} \square h_{ab} + 2h^{ab} \nabla_a \nabla_b h^{ab} - h^{ab} \nabla_a \nabla_b \overline{R} \right\}$$

$$+ 2\overline{R} \left[ \frac{1}{4} h \nabla^d h / 2 - \frac{1}{2} \nabla^d h \nabla_a h^{ab} - \frac{1}{4} \nabla p a \nabla_c h^{pa} + \frac{1}{2} \nabla h^{ca} \nabla_a h^{b}_c \right\} + \text{Total Derivative}$$

$$= \{ -h^{ab} \overline{R}_{ab} + \left( -\square h + \nabla_a \nabla_b h^{ab} \right) \}^2 + h^{ab} h_{ab} \overline{R^2} - \overline{R} \left[ \frac{1}{2} h^{ab} \square h_{ab} + \nabla_a h^{ab} \nabla_b h^{ab} + \frac{1}{2} h \nabla^d h \right] + \text{Total Derivative}$$  \hspace{1cm} (64)

Thus the Lagrangian density becomes, $\mathcal{L}_2 = \sqrt{-g} \overline{R^2} = \mathcal{L}_2 + \overline{\mathcal{L}}_2 + \overline{\mathcal{L}}_2$, where $\overline{\mathcal{L}}_2 = \sqrt{-g} \overline{R^2}$. Further, we obtain,
$$\mathcal{L}_2 = \sqrt{-g} \left\{ \overline{R^2} + \frac{1}{2} h \overline{R^2} \right\}$$

$$\text{AdS/dS} \equiv \sqrt{-g} \left\{ -\frac{\overline{R^2}}{2} h + 2\overline{R} \left( -\square h + \nabla_a \nabla_b h^{ab} \right) + \frac{1}{2} h \overline{R^2} \right\}$$

$$= \sqrt{-g} \overline{R} \left( -\square h + 2\nabla_a \nabla_b h^{ab} \right) = \text{Total Derivative}$$  \hspace{1cm} (65)

Finally, we obtain,
$$\overline{\mathcal{L}}_2 = \sqrt{-g} \left\{ \overline{R^2} \left( -\frac{1}{4} h h^{ab} \overline{R}_{ab} + \frac{1}{8} h^2 \right) + \frac{1}{2} h \overline{R^2} + \overline{R^2} \right\}$$

$$\text{AdS/dS} \equiv \sqrt{-g} \left[ \overline{R^2} \left( -\frac{1}{4} h h^{ab} \overline{R}_{ab} + \frac{1}{8} h^2 \right) + \frac{1}{2} h \left\{ -\frac{\overline{R^2}}{2} h + 2\overline{R} \left( -\square h + \nabla_a \nabla_b h^{ab} \right) \right\} \right]$$

$$+ \left\{ -h \left( \frac{\overline{R}}{4} + \left( -\square h + \nabla_a \nabla_b h^{ab} \right) \right)^2 + h^{ab} h_{ab} \overline{R^2} \right\} + \overline{R} \left[ \frac{1}{2} h^{ab} \square h_{ab} + \nabla_a h^{ab} \nabla_a h^{ab} + \frac{1}{2} h \overline{R^2} \right\]$$
The above Lagrangian depicts the \( \nabla h_{ab} \) term, which we have, \( \nabla \nabla h_{ab} - h_{ab} \nabla^2 h + \text{Total Derivative} \). Therefore the second order correction becomes,

\[
\bar{\mathcal{L}}_2^{\text{AdS/AdS}} = \sqrt{-g} \left\{ -h \frac{\bar{R}}{4} - \square h + \nabla_a \nabla_b h^{ab} \right\} + \bar{R} \left\{ \frac{1}{2} h_{ab} \nabla_a h^{ab} + \nabla_c h^{ab} \nabla_c h^{ab} - \frac{1}{2} \left( h^{2} - h_{ab} h^{ab} \right) \right\}
\]

(68)

The above Lagrangian depicts the \( R^2 \) term, expanded till quadratic order in the perturbation \( h_{ab} \) around dS/AdS background. In what follows, we will first decompose the perturbation in its irreducible parts and then rewrite the quadratic action. This result will be used to determine the action for the gravitational perturbation around maximally symmetric background in the next section.

### D Decomposition of the quadratic action to irreducible parts of the perturbation

Thus in the previous section we have expressed the quadratic Lagrangian, namely the \( R^2 \) term, upto quadratic order in the perturbation around a dS/AdS background. Here we will decompose the perturbation in its irreducible parts. Further, the background spacetime has been denoted with a ‘bar’ in our earlier calculations, however due to cumbersome expressions resulting from the same we will refrain from using ‘bar’ in this section. The background geometry will be denoted by \( g_{ab} \) alone, which we will take to be dS/AdS. With this preamble, we break the perturbation in its irreducible parts as,

\[
h_{ab} = h_{ab}^\perp + \nabla_a a_b + \nabla_b a_a + \left( \nabla_a \nabla_b - \frac{1}{4} g_{ab} \square \right) a + \frac{1}{4} g_{ab} \tilde{h}
\]

(69)
where, $\nabla_a h_{ab} = 0 = g^{ab} h_{ab}$ and $\nabla_a a^i = 0$.

Let us try to express the Lagrangian in terms of these reduced quantities individually. For that purpose, let us start with the following term,

$$\text{Term 1} = \nabla_a \nabla_b h_{ab} = \nabla a \nabla b \left[ h_{ab} + \nabla_a a_b + \nabla_b a_a + \left( \nabla_a \nabla_b - \frac{1}{4} g_{ab} \Box \right) a + \frac{1}{4} g_{ab} h \right]$$

$$= \nabla a \nabla b \left( \nabla a a_b + \nabla b a_a + \nabla a \nabla b - \frac{1}{4} g_{ab} \Box \right) a + \frac{1}{4} g_{ab} h$$

(70)

Let us evaluate the above expression term by term, yielding,

$$\text{Term 1a} = \nabla a \nabla b \nabla a a_b = \nabla a \nabla b \nabla a a_{b} + \nabla a \nabla b a_b + \nabla a \nabla b a_b + \left( \nabla a \nabla b - \frac{1}{4} g_{ab} \Box \right) a + \frac{1}{4} g_{ab} h$$

(71)

$$\text{Term 1b} = \nabla a \nabla b \nabla b a_a = \nabla a \nabla b \nabla b a_a + \nabla a \nabla b a_a + \nabla a \nabla b a_a + \left( \nabla a \nabla b - \frac{1}{4} g_{ab} \Box \right) a + \frac{1}{4} g_{ab} h$$

(72)

$$\text{Term 1c} = \nabla a \nabla b \nabla a a_b = \nabla a \nabla b \nabla a a_b + \nabla a \nabla b a_b + \nabla a \nabla b a_b + \left( \nabla a \nabla b - \frac{1}{4} g_{ab} \Box \right) a + \frac{1}{4} g_{ab} h$$

(73)

Therefore combining all these results we obtain,

$$\text{Term 1} = \text{Term 1a} + \text{Term 1b} + \text{Term 1c} - \frac{1}{4} \Box^2 a + \frac{1}{4} h$$

(74)

Therefore we obtain,

$$\text{Term A} = -R \frac{\Box}{4} a + \frac{3}{4} \Box^2 a + \frac{1}{4} h$$

(75)

This suggests to define,

$$\Phi = h - \Box a$$

(76)

Hence we obtain,

$$\text{Term A} = -R \frac{\Phi}{4} - \frac{3}{4} \Box \Phi = -\frac{3}{4} \left( \Box \Phi + R \frac{\Phi}{3} \right)$$

(77)
Next, let us compute another term, which yields,

\[
\text{Term 2} = \nabla_a h^{ab} = \nabla_a \left( h^{ab}_{\perp} + \nabla^a a^b + \nabla^b a^a + \left( \nabla^a b^b - \frac{1}{4} g^{ab} \Box \right) a + \frac{1}{4} g^{ab} h \right)
\]

\[
= \Box a^b + \nabla_a \nabla^a a^b + \Box b^b - \frac{1}{4} \nabla^b \Box a + \frac{1}{4} \nabla^b h
\]

we can now use the following results,

\[
\text{Term 2a} = \nabla_a \nabla^a a^a = \nabla^a a^a = R_{\alpha \beta}^{\text{AdS/dS} \alpha \beta} \equiv \frac{R}{4} a^b
\]

\[
\text{Term 2b} = \Box \nabla^b a = \nabla^i \nabla^i \nabla^b a = \nabla^b \nabla^i a = \frac{1}{4} \nabla^b \Box a + [\nabla_i, \nabla^b] \nabla^i a
\]

\[
= \nabla^b \Box a + R_{ji}^{\text{AdS/dS}} \nabla^b a + R_{4}^{\text{AdS/dS}} \nabla^b a
\]

which leads to,

\[
\text{Term 2} \equiv \Box a^b + \frac{R}{4} (a^b + \nabla^b a) + \frac{3}{4} \nabla^b \Box a + \frac{1}{4} \nabla^b h
\]

Thus using the above expression for “Term 2” we obtain,

\[
\text{Term B} \equiv \nabla_a h^{ab} \nabla_b h^c = \text{AdS/dS} \left\{ \right. \Box a^b + \frac{R}{4} (a^b + \nabla^b a) + \frac{3}{4} \nabla^b \Box a + \frac{1}{4} \nabla^b h \left. \right\}
\]

\[
= \Box a^b \nabla_b a + \frac{R}{2} (a^b + \nabla^b a) \nabla_b a + \frac{3}{2} \left( \nabla^b \Box a \right) \nabla_b a + \frac{1}{2} \nabla^b h \nabla_b a + \frac{R^2}{16} (a^b + \nabla^b a) (a_b + \nabla_b a)
\]

\[
+ \frac{3R}{8} \nabla^b \nabla_a (a_b + \nabla_b a) + R \left( a^b + \nabla^b a \right) \nabla_b h + \frac{9}{16} \nabla_b \nabla_a \nabla_b a + \frac{3}{8} \nabla^b \nabla_b a \nabla_b h + \frac{1}{16} \nabla_b h \nabla^b h
\]

The above can be broken into three pieces,

\[
\text{Term B1 (independent of R)} \equiv \text{AdS/dS} \left\{ \right. \Box a^b \nabla_b a + \frac{3}{2} \left( \nabla^b \Box a \right) \nabla_b a + \frac{1}{2} \nabla^b h \nabla_b a + \frac{9}{16} \nabla_b \nabla_a \nabla_b a + \frac{3}{8} \nabla^b \nabla_a \nabla_b h + \frac{1}{16} \nabla_b h \nabla^b h \left. \right\}
\]

\[
\text{Term B2 (linear in R)} \equiv \text{AdS/dS} \left\{ \right. \frac{R}{2} (a^b + \nabla^b a) \nabla_b a + \frac{3R}{8} \nabla^b \nabla_a (a_b + \nabla_b a) + \frac{R}{8} (a^b + \nabla^b a) \nabla_b h \left. \right\}
\]

\[
\text{Term B3 (quadratic in R)} \equiv \text{AdS/dS} \frac{R^2}{16} (a^b + \nabla^b a) (a_b + \nabla_b a)
\]

Then we have,

\[
\text{Term C} \equiv \frac{1}{2} \nabla^2 h
\]

as well as,

\[
\text{Term D} \equiv \nabla_a h^{ab} \nabla_b h
\]
\[
\text{AdS}/dS \left( \Box_{ab} + \frac{R}{4} (a^b + \nabla^b a) + \frac{3}{4} \nabla^b \Box a + \frac{1}{4} \nabla^b h \right) \nabla_b h = - \frac{1}{4} h \Box h - \frac{3}{4} \Box a \Box h + \frac{R}{4} (a^b + \nabla^b a) \nabla_b h + \nabla_b h \Box a^b + \text{Total Derivative} \quad (87)
\]

The final and most complicated term correspond to,

\[
\text{Term E} \equiv h_{ab} \Box h_{ab} = \left( h_{ab}^+ + \nabla^a a^b + \nabla^b a^a + \left( \nabla^a \nabla^b - \frac{1}{4} g^{ab} \Box \right) a + \frac{1}{4} g^{ab} h \right) \Box h_{ab}
\]

\[
= \underbrace{h_{ab}^+ \Box h_{ab}}_{\text{Term 3}} + \underbrace{2 \nabla^a a^b \Box h_{ab} + \left( \nabla^a \nabla^b - \frac{1}{4} g^{ab} \Box \right) a \Box h_{ab} + \frac{1}{4} h \Box h}_{\text{Term 4}} \quad (88)
\]

Expanding out the first term we obtain,

\[
\text{Term 3} = \underbrace{h_{ab}^+ \Box h_{ab}}_{\text{Term 3a}} + \underbrace{2 \nabla^a a^b \Box h_{ab} + h_{ab}^+ \Box \nabla_a a_b}_{\text{Term 3b}} \quad (89)
\]

where we have used the result, \( g^{ab} h_{ab}^+ = 0 \). Thus each of these terms individually can be expressed as,

\[
\text{Term 3a} = 2 h_{ab}^+ \Box \nabla_a a_b = 2 h_{ab}^+ \Box [\nabla_a, \nabla_b] a_b + 2 h_{ab}^+ \Box \nabla_a \nabla_c a_b = -2 h_{ab}^+ \Box \nabla_c \left( R_{bca}^i a_i \right) + 2 h_{ab}^+ \Box \nabla_c \left( \nabla_a a_b + 2 h_{ab}^+ \Box \nabla_a \nabla_c a_b \right) \quad (90)
\]

where we have used the result, \( \nabla_a h_{ab}^+ = 0 \). Further,

\[
\text{Term 3b} = h_{ab}^+ \Box \nabla_a \nabla_b a = h_{ab}^+ \Box \nabla_i [\nabla_i, \nabla_b] a_b + h_{ab}^+ \Box \nabla_i \nabla_a \nabla_i \nabla_b a
\]

where the last line follows from the result, \( \nabla_a h_{ab}^+ = 0 \). Thus neglecting total derivative terms we obtain,

\[
\text{Term 3} = h_{ab}^+ \Box h_{ab} \quad (92)
\]
Let us now consider the “Term 5”. This yields,

\[
\text{Term 5} = \left( \nabla^a \nabla_b - \frac{1}{4} g^{ab} \square \right) h_{ab}
\]

\[
= \left( \nabla^a \nabla_b - \frac{1}{4} g^{ab} \square \right) \left( h_{ab} + \nabla_a a_b + \nabla_b a_a + \left( \nabla_a \nabla_b - \frac{1}{4} g_{ab} \square \right) a + \frac{1}{4} g_{ab} h \right)
\]

\[
= -\frac{1}{4} \square h + (\nabla^a \nabla_b a) \square h_{ab} + 2 (\nabla^a \nabla_b a) \nabla_a a_b + (\nabla^a \nabla_b a) \left( \nabla_a \nabla_b - \frac{1}{4} g_{ab} \square \right) a + \frac{1}{4} \square a \square h
\]

\[
= \left( \nabla^a \nabla_b a \right) \square h_{ab} + 2 \left( \nabla^a \nabla_b a \right) \nabla_a a_b + \left( \nabla^a \nabla_b a \right) \left( \nabla_a \nabla_b - \frac{1}{4} g_{ab} \square \right) a
\]

\[\text{Term 5a}\]

\[
\text{Term 5b}
\]

\[
\text{Term 5c}
\]

Individual terms yield,

\[
\text{Term 5a} = (\nabla^a \nabla_b a) \square h_{ab} = \text{Total Derivative} - (\nabla^b a) \nabla^a \nabla^i \nabla_i h_{ab} = - (\nabla^b a) [\nabla^a, \nabla^i] \nabla_i h_{ab} - (\nabla^b a) \nabla^a \nabla^i \nabla_i h_{ab}
\]

\[
= - (\nabla^b a) \left\{ -R^k_i a \nabla_k h_{ab} - R^k_i a \nabla_k h_{ab} - R^k_i a \nabla_k h_{ab} \right\} - (\nabla^b a) \nabla^i (\nabla^a, \nabla_i) [h_{ab}, - (\nabla^b a) \nabla^i \nabla_i h_{ab}
\]

\[
= - (\nabla^b a) \left\{ -R^k_i a \nabla_k h_{ab} + R^k_i a \nabla_k h_{ab} - R^k_i a \nabla_k h_{ab} \right\} - (\nabla^b a) \nabla^i (\nabla^a, \nabla_i) [h_{ab}, - (\nabla^b a) \nabla^i \nabla_i h_{ab}
\]

\[
= \frac{\text{AdS/dS}}{\text{AdS/dS}} \left\{ - \frac{R^k_i a \nabla_k h_{ab}}{4} + \frac{R^k_i a \nabla_k h_{ab}}{12} \left( g^{ka} g_{ab} - g^{ka} \delta^b_a \right) h_{ab} \right\} = 0
\]

where the results \( \nabla^i h^{\perp}_{ik} = 0 \) as well as \( g^{ik} h^{- \perp}_{ik} = 0 \) have been used repeatedly. Subsequently,

\[
\text{Term 5b} = 2 (\nabla^a \nabla^b a) \square \nabla_a a_b = 2 (\nabla^b \nabla^a a) \square \nabla_a a_b = -2 (\nabla^a \nabla^b a) \nabla^a \nabla^i \nabla_i \nabla_a a_b + \text{Total Derivative}
\]

\[
= -2 (\nabla^a \nabla^b a) [\nabla^i, \nabla^i] \nabla_a a_b - 2 (\nabla^a \nabla^b a) [\nabla^i, \nabla^i] \nabla_a a_b
\]

\[
= -2 (\nabla^a \nabla^b a) \left\{ -R^k_i a \nabla_k \nabla_a a_b - R^k_i a \nabla_k \nabla_a a_b - R^k_i a \nabla_k \nabla_a a_b \right\}
\]

\[
= -2 (\nabla^a \nabla^b a) \nabla_i \left\{ -R^k_i a \nabla_k \nabla_a a_b - R^k_i a \nabla_k \nabla_a a_b - R^k_i a \nabla_k \nabla_a a_b \right\}
\]

\[
= -2 (\nabla^a \nabla^b a) \nabla_i \left\{ -R^k_i a \nabla_k \nabla_a a_b - R^k_i a \nabla_k \nabla_a a_b - R^k_i a \nabla_k \nabla_a a_b \right\}
\]

\[
= -2 (\nabla^a \nabla^b a) \nabla_i \left\{ -R^k_i a \nabla_k \nabla_a a_b - R^k_i a \nabla_k \nabla_a a_b - R^k_i a \nabla_k \nabla_a a_b \right\}
\]

\[
= \frac{\text{AdS/dS}}{\text{AdS/dS}} \left\{ - \frac{R}{4} g^{ka} \nabla_k \nabla_a a_b + \frac{R}{12} g^{ka} \nabla_k \nabla_a a_b \right\}
\]

\[
- 2 (\nabla^a \nabla^b a) \nabla_i \left\{ -\frac{R}{12} g^{ka} \nabla_k g_{ab} + \frac{R}{4} g^{ka} \nabla_k a_b \right\} + 2 (\nabla^a \nabla^b a) \nabla^i R^k_{ab a k}
\]

\[
= -\frac{R}{6} (\nabla^a \nabla^b a) \square a_a - \frac{R}{6} (\nabla^a \nabla^b a) \square a_a - \frac{R}{2} (\nabla^a \nabla^b a) \nabla_i a_b - \frac{R}{2} (\nabla^a \nabla^b a) \nabla_i a_b
\]

\[
= -\frac{R}{6} (\nabla^a \nabla^b a) \square a_a - \frac{R}{2} (\nabla^a \nabla^b a) \nabla_i a_b - \frac{R}{2} (\nabla^a \nabla^b a) \nabla_i a_b
\]

\[
= -\frac{R}{6} (\nabla^a \nabla^b a) \square a_a - \frac{R}{2} (\nabla^a \nabla^b a) \nabla_i a_b - \frac{R}{2} (\nabla^a \nabla^b a) \nabla_i a_b + \text{Total Derivative}
\]

(95)
We now have,

\[(\nabla^a a) \square a = - a \nabla \square a^i + \text{Total Derivative} = - a [\nabla_i, \nabla_j] \nabla^j a^i - a \nabla^j \nabla_i \nabla_j a^i \]

\[= -a \left( R^j_{\ kij} \nabla^k a^i + R^i_{\ kij} \nabla^j a^k \right) - a \nabla^j [\nabla_i, \nabla_j] a^i \]

\[= -a \left( -R^k_{\ kij} \nabla^k a^i + R^i_{\ kij} \nabla^j a^k \right) - a \nabla^j \left( R^i_{\ kij} a^k \right) = -a \nabla^j \left( R^i_{\ kij} a^k \right) = 0 \]  \hspace{1cm} (96)

Thus except for total derivatives,

\[\text{Term 5b} = 0 \]  \hspace{1cm} (97)

Also note that there is another identity,

\[\square a^i \nabla_i a = -\Box a \nabla_i \square a^i = -\Box a [\nabla_i, \nabla_j] \nabla^j a^i - \Box a \nabla_j \nabla_i \nabla^j a^i \]

\[= - \Box a \left( R^j_{\ kij} \nabla^k a^i + R^i_{\ kij} \nabla^j a^k \right) - \Box a \nabla^j [\nabla_i, \nabla_j] a^i \]

\[= -\Box a \left( -R^k_{\ kij} \nabla^k a^i + R^i_{\ kij} \nabla^j a^k \right) - \Box a \nabla^j \left( R^i_{\ kij} a^k \right) = 0 \]  \hspace{1cm} (98)

where the result \( \nabla_i a^i = 0 \) has been used repeatedly.

Thus finally we obtain,

\[
\text{Term 5c} = (\nabla^a \nabla^b a) \Box \left( \nabla a \nabla_b - \frac{1}{4} g_{ab} \Box \right) a = (\nabla^a \nabla^b a) \Box \nabla a \nabla_b a - \frac{1}{4} \Box a \Box^2 a \\
= - \frac{1}{4} \Box a \Box^2 a + (\nabla^a \nabla^b a) \Box [\nabla_i, \nabla_a] \nabla_b a + (\nabla^a \nabla^b a) \Box \nabla_i \nabla_a \nabla_b a \\
= - \frac{1}{4} \Box a \Box^2 a + (\nabla^a \nabla^b a) \Box \left( -R^k_{\ bia} \nabla_k a + (\nabla^a \nabla^b a) \Box \nabla_i \nabla_a \nabla_b a + (\nabla^a \nabla^b a) \Box \nabla_i \nabla_a \nabla_b a \\
= \frac{1}{4} \Box a \Box^2 a - R^R_{\ 12} (\nabla^a \nabla^b a) \Box \left( \delta^i_{\ gab} - \delta^i_{\ gba} \right) \nabla_k a + \left( \nabla^a \nabla^b a \right) \left( R^i_{\ kia} \nabla^k \nabla_i a - R^i_{\ bia} \nabla^i \nabla_k a \right) \\
+ (\nabla^a \nabla^b a) \Box \nabla_i \nabla_a \nabla_i a + \left( \nabla^a \nabla^b a \right) \Box \nabla_i \nabla_k a \\
= \frac{1}{4} \Box a \Box^2 a - R^R_{\ 12} \left( \Box a \right)^2 \left( -a \nabla^b a \right) \Box \nabla_k a + \left( \nabla^a \nabla^b a \right) \left( \Box a \nabla_i a \right) \\
+ (\nabla^a \nabla^b a) \Box \left( R^i_{\ kib} \nabla^b a \right) - \Box a \left( \nabla^b \nabla^a a \right) \Box \nabla_k a + \text{Total Derivative} \\
= \frac{1}{4} \Box a \Box^2 a - R^R_{\ 6} \left( \Box a \right)^2 + \frac{5R}{12} (\nabla^a \nabla^b a) (\nabla_a \nabla_b a) + \frac{R}{4} (\nabla^a \nabla^b a) (\nabla_b \nabla_a a) - ([\nabla_a, \nabla_b] \nabla^a a) \nabla_b \Box a - \nabla_b \nabla_a \nabla^b a \Box a + \text{Total Derivative} \\
= \frac{3}{4} \Box a \Box^2 a - \frac{R}{6} (\Box a)^2 + \frac{2R}{3} (\nabla^a \nabla^b a) (\nabla_a \nabla_b a) + \frac{R}{4} (\Box a)^2 \\
= \frac{3}{4} \Box a \Box^2 a + \frac{R}{12} (\Box a)^2 - \frac{2R}{3} ([\nabla_a, \nabla_b] \nabla^a a) \nabla_b a - \frac{2R}{3} \nabla_b \nabla_a \nabla^b a + \text{Total Derivative} \\
= \frac{3}{4} \Box a \Box^2 a + \frac{R}{12} (\Box a)^2 - \frac{2R}{3} (R^a_{\ kab} \nabla^k a) \nabla^b a + \frac{2R}{3} \Box a \Box^2 a + \text{Total Derivative} \\
= 23
Thus combining together, ignoring total derivative terms we obtain,

\[
\text{Term 5} = \text{Term 5a} + \text{Term 5b} + \text{Term 5c} = \frac{3}{4} a \Box a^2 a + \frac{3R}{4} a \Box a^2 a + \frac{R^2}{6} a \Box a + \text{Total Derivative}
\]

(99)

The only remaining term corresponds to "Term 4", which has the following decomposition,

\[
\text{Term 4} = 2 \nabla^a b^\Box h_{ab} = 2 \nabla^a b^\Box (\nabla_a a_b + \nabla_b a_a) + \left( \nabla_a \nabla_b - \frac{1}{4} g_{ab} \Box \right) a + \frac{1}{4} g_{ab} h
\]

(100)

Let us start with the first term in the above expression, which can be expressed as,

\[
\text{Term 4a} = 2 \nabla^a b^\Box h_{ab} = -2a^b \nabla^a \nabla^c h_{ac} + \text{Total Derivative}
\]

(101)

Let us now compute the second term, which reads,

\[
\text{Term 4b} = 2 \nabla^a b^\Box (\nabla_a a_b + \nabla_b a_a) = 2 \left( \nabla_a a_b + \nabla_b a_a \right) \Box \nabla^a b^b
\]

(102)

Let us now compute the third term, which reads,

\[
\text{Term 4c} = 2 \nabla^a b^\Box (\nabla_a a_b + \nabla_b a_a) = 2 \left( \nabla_a a_b + \nabla_b a_a \right) \Box \nabla^a b^b
\]
Thus we obtain

\[
\text{Term E} = -2\Box a^b \Box a_b - \frac{4R^2}{3} a_b \Box a_b - \frac{5R^2}{24} a_b^b a_b \tag{103}
\]

Finally, we obtain

\[
\text{Term 4c} = 2\nabla^a a^b \Box (\Box a \nabla_b a) = 2\nabla^a a^b \Box (\nabla_c, \nabla_b) (\nabla_a a) + 2\nabla^a a^b \nabla_c \nabla_b \nabla_c (\nabla_a a)
\]

\[
= 2\nabla^a a^b \nabla^c (\nabla_a a) + 2\nabla^a a^b (\nabla_c, \nabla_b) (\nabla^c \nabla_a a) + 2\nabla^a a^b \nabla_b \Box \nabla_a a
\]

\[
= -\frac{R}{6} \nabla^a a^b \nabla_c \nabla_d a + \frac{R}{2} \nabla^a a^b \nabla_c \nabla_b a - \frac{R}{6} \nabla^a a^b \nabla_c \nabla^d \nabla_b a \nabla_a a + \text{Total Derivative}
\]

\[
= -\frac{5R}{6} \nabla^a a^b \nabla_c \nabla_b a - 2[(\nabla_b, \nabla_a) a^b \Box \nabla a] = -\frac{5R}{6} \nabla^a a^b \nabla_c \nabla_b a - 2R \nabla_c \nabla^a a \nabla^a a \nabla a + \text{Total Derivative}
\]

\[
= -\frac{5R^2}{24} a^p \nabla p a - \frac{R}{2} \nabla_c \nabla^a a \nabla^a a = -\frac{5R^2}{24} a^p \nabla p a - \frac{R}{2} \nabla_c \nabla^a a \nabla^a a = \frac{5R}{3} a^b a_{a b}
\]

Thus combining together we obtain,

\[
\text{Term 4} = -\frac{R^2}{3} a^p \nabla p a - 2\Box a^b \Box a_b - \frac{4R}{3} a_b \Box a_b - \frac{5R^2}{24} a_b^b a_b \tag{105}
\]

Thus we obtain,

\[
\text{Term E} = \frac{1}{4} \Box a \Box a + \frac{5R^2}{24} a^b a_{a b} - \frac{R^2}{3} a^p \nabla p a - 2\Box a^b \Box a_b - \frac{4R}{3} a_b \Box a_b - \frac{5R^2}{24} a_b^b a_b + \frac{3R}{4} a^2 a + \frac{4R}{3} a^2 a + \frac{R^2}{6} a \Box a \tag{106}
\]

Thus the term in the action of quadratic gravity becomes,

\[
\text{Term F} = \frac{1}{2} h^b h \Box a + \nabla_c h^b \nabla_c h^a - \frac{1}{2} \Box a h - \nabla a h \nabla h^a
\]

\[
= \frac{1}{2} (\text{Term E} + \text{Term B} - \frac{1}{2} \Box a h - \text{Term D})
\]

\[
= \frac{1}{2} \left[ \frac{1}{4} \Box a h + \frac{5R^2}{24} a^b a_{a b} - \frac{5R^2}{24} a^p \nabla p a - 2\Box a^b \Box a_b - \frac{4R}{3} a_b \Box a_b - \frac{5R^2}{24} a_b^b a_b + \frac{3R}{4} a^2 a + \frac{4R}{3} a^2 a + \frac{R^2}{6} a \Box a \right]
\]

\[
= \frac{1}{2} \Box a h + \frac{3}{4} \Box a h - \frac{R}{4} (a^b + \nabla h a) \nabla h - \nabla b h a^b + \text{Term B1} + \text{Term B2} + \text{Term B3}
\]

Thus we have terms proportional to $R^2$ to yield,

\[
\text{Term Quadratic} = \text{Term B3} - \frac{R^2}{6} a^p \nabla p a - \frac{5R^2}{48} a_b^b a_b + \frac{R^2}{12} a \Box a
\]

\[
= \frac{R^2}{16} (a^b + \nabla h a) (a_b + \nabla h a) - \frac{R^2}{6} a^p \nabla p a - \frac{R^2}{48} a_b^b a_b + \frac{R^2}{12} a \Box a
\]

\[
\text{Total Derivative}
\]

25
\[ \text{Term Independent} = \text{Term B1} = \frac{R}{8} a_b \Box a_b - \frac{3}{8} a \Box^2 a - \frac{R}{8} \phi \Box^2 a - \frac{1}{4} (a^b + \nabla^b \phi) \nabla_b \phi \]

and the term linear in \( R \) becomes,

\[ \text{Term Linear} = \text{Term B2} - \frac{2R}{3} a_b \Box a_b + \frac{3R}{8} a \Box^2 a - \frac{R}{4} (a^b + \nabla^b \phi) \nabla_b \phi \]

Further the term independent of Ricci scalar yields,

\[ \text{Term Independent} = \text{Term B1} + \frac{1}{8} h \Box h + \frac{1}{2} h^a_b h^b_a - \frac{1}{8} a^a \Box a_b + \frac{3}{8} a \Box^2 a - \frac{1}{2} h \Box h + \frac{1}{4} h \Box h + \frac{3}{4} a \Box h \]

The action reads,

\[ \mathcal{L}_2 \equiv \sqrt{-g} \left[ \text{Term A} \right]^2 + R^2 \left( -\frac{1}{12} h_{ab} h^{ab} - \frac{1}{24} h^2 \right) + R \left\{ \text{Term F} \right\} \]

\[ = \sqrt{-g} \left[ \left\{ -\frac{3}{4} \left( \Box \phi + \frac{R}{3} \phi \right) \right\}^2 + R^2 \left( -\frac{1}{12} h_{ab} h^{ab} - \frac{1}{24} h^2 \right) \right. \]

\[ + R \left\{ \frac{1}{2} h^a_b \Box h^b_a - \frac{3}{16} \phi \Box \phi + \text{Term Linear} + \text{Term Quadratic} \right\} \]

\[ = \sqrt{-g} \left[ \frac{9}{16} \left( \Box \phi + \frac{R}{3} \phi \right)^2 + R^2 \left( -\frac{1}{12} h_{ab} h^{ab} - \frac{1}{24} h^2 \right) \right] \]
Now we have,

\[
h_{ab} h^{ab} = h_{ab} h^\perp_{ab} + \nabla_a a_b + \nabla_b a_a + \left( \nabla_a \nabla_b - \frac{1}{4} g_{ab} \Box \right) a + \frac{1}{4} g_{ab} h
\]

\[
= h_{ab} h^\perp_{ab} + 2 h_{ab} \nabla_a a_b + h^{ab} \nabla_a \nabla_b a - \frac{1}{4} h \Box a + \frac{1}{4} h^2
\]

\[
= \left[ h^\perp_{ab} + \nabla^a a^b + \nabla^b a^a + \left( \nabla^a \nabla^b - \frac{1}{4} g^{ab} \Box \right) a + \frac{1}{4} g^{ab} h \right] h^\perp_{ab}
\]

\[
+ 2 \left[ h^\perp_{ab} + \nabla^a a^b + \nabla^b a^a + \left( \nabla^a \nabla^b - \frac{1}{4} g^{ab} \Box \right) a + \frac{1}{4} g^{ab} h \right] \nabla_a a_b
\]

\[
+ \left[ h^\perp_{ab} + \nabla^a a^b + \nabla^b a^a + \left( \nabla^a \nabla^b - \frac{1}{4} g^{ab} \Box \right) a + \frac{1}{4} g^{ab} h \right] \nabla_a \nabla_b a - \frac{1}{4} h \Box a + \frac{1}{4} h^2
\]

\[
= h^\perp_{ab} \nabla_a a_b + 2 \left( \nabla^a \nabla^b a^a + \nabla_a \nabla_b a + 2 \nabla^a \nabla^b a^a \nabla_a a_b + 2 \nabla^a a^b \nabla^b a + \nabla^a \nabla^b a \nabla a \nabla b a - \frac{1}{4} (\Box a)^2 + \frac{1}{4} h^2
\]

\[
= h^\perp_{ab} - 2 a_b \Box a^b - 2 a_b \nabla_a \nabla b a - (\nabla^a a) (\nabla \nabla \nabla \nabla b a) - \frac{1}{4} (\Box a)^2 + \frac{1}{4} h^2 + \text{Total Derivative}
\]

\[
= h^\perp_{ab} - 2 a_b \Box a^b - 2 a_b \nabla_a \nabla b a - (\nabla^a a) (\nabla b, \nabla b) a^a - (\nabla^a a) (\nabla b, \nabla a) - (\nabla^a a) (\nabla a) a - \frac{1}{4} (\Box a)^2 + \frac{1}{4} h^2
\]

\[
= h^\perp_{ab} - 2 a_b \Box a^b - 2 a_b R_{pab} a^p - (\nabla^a a) (R^b_{pba} \nabla b a) + (\Box a)^2 - \frac{1}{4} (\Box a)^2 + \frac{1}{4} h^2
\]

\[
= h^\perp_{ab} - 2 a_b \Box a^b - R^2 a^b a_b + R \frac{a^b a^b}{4} a + 3 \frac{a^b a^b}{4} (\Box a)^2 + \frac{1}{4} h^2
\]  

(111)

Thus action becomes,

\[
\tilde{\mathcal{L}}_2^{\text{AdS/D5}} = \sqrt{-g} \left[ \frac{9}{16} (\Box a + \frac{R}{3} \frac{\phi}{\phi})^2 + R^2 \left( - \frac{1}{12} (h^a b h^\perp_{ab} - a_b \Box a^b - R^2 a^b a_b + R \frac{a^b a^b}{4} a + 3 \frac{a^b a^b}{4} (\Box a)^2 + \frac{1}{4} h^2 - \frac{1}{24} h^2) \right) \right.
\]

\[
+ R \left\{ \frac{1}{2} h^\perp_{ab} \nabla^\perp a_b - \frac{3}{16} \Box a^b + \left( - \frac{R}{6} a^b a_b + R \frac{a^b a^b}{8} a + \frac{R}{48} (\Box a)^2 + \frac{1}{4} h^2 \right) \right\} \right]
\]

\[
= \sqrt{-g} \left[ \frac{9}{16} (\Box a + \frac{R}{3} \frac{\phi}{\phi})^2 + R^2 \left( - \frac{1}{12} h^\perp_{ab} h^\perp_{ab} + \frac{1}{6} a_b \Box a^b + \frac{R^2 a^b a_b}{24} - \frac{R}{48} a^b a^b \right) \right. - \frac{1}{12} \left( \frac{3}{4} (\Box a)^2 + \frac{1}{4} h^2 \right) - \frac{1}{24} h^2\right]
\]

\[
+ R \left\{ \frac{1}{2} h^\perp_{ab} \nabla^\perp a_b - \frac{3}{16} \Box a^b + \left( - \frac{R}{6} a^b a_b + R \frac{a^b a^b}{8} a + \frac{R}{48} (\Box a)^2 + \frac{1}{4} h^2 \right) \right\} \right]
\]

\[
= \sqrt{-g} \left[ \frac{9}{16} (\Box a + \frac{R}{3} \frac{\phi}{\phi})^2 + R^2 \left( - \frac{1}{12} h^\perp_{ab} h^\perp_{ab} + \frac{3}{16} \Box a^b + \left( - \frac{R}{6} a^b a_b + R \frac{a^b a^b}{8} a + \frac{R}{48} (\Box a)^2 + \frac{1}{4} h^2 \right) \right) \right.\right]
\]

\[
+ R \left\{ \frac{1}{2} h^\perp_{ab} h^\perp_{ab} + \frac{1}{6} a_b \Box a^b + R \frac{a^b a^b}{24} - \frac{R}{48} a^b a^b - \frac{1}{12} \left( \frac{3}{4} (\Box a)^2 + \frac{1}{4} h^2 \right) - \frac{1}{24} h^2 \right\} \right]
\]

\[
+ R \left\{ \frac{1}{2} h^\perp_{ab} h^\perp_{ab} + \frac{3}{16} \Box a^b + \left( - \frac{R}{6} a^b a_b + R \frac{a^b a^b}{8} a + \frac{R}{48} (\Box a)^2 + \frac{1}{4} h^2 \right) \right\} \right]
\]

\[
+ \left\{ \frac{1}{2} h^\perp_{ab} h^\perp_{ab} + \frac{3}{16} \Box a^b + \left( - \frac{R}{6} a^b a_b + R \frac{a^b a^b}{8} a + \frac{R}{48} (\Box a)^2 + \frac{1}{4} h^2 \right) \right\} \right]
\]

\[
+ \left\{ \frac{1}{2} h^\perp_{ab} h^\perp_{ab} + \frac{3}{16} \Box a^b + \left( - \frac{R}{6} a^b a_b + R \frac{a^b a^b}{8} a + \frac{R}{48} (\Box a)^2 + \frac{1}{4} h^2 \right) \right\} \right]
\]
and is given by,

\[ L = \sqrt{-g} \left[ \frac{9}{16} (\square \Phi + \frac{R}{3} \Phi)^2 + R \left\{ \frac{1}{2} h_{ab} \square h_{ab} - \frac{3}{16} \Phi \square \Phi - \frac{R}{12} h_{ab} h_{ab} + \frac{R}{8} a \square h + R \left( -\frac{1}{16} (\square a)^2 - \frac{1}{16} h^2 \right) \right\} \right] \]

This finishes our discussion. This is the result we have used in order to arrive at the action for quadratic gravity expanded about dS/AdS background till quadratic order in Section 2.

E Hamiltonian Formulation

In this appendix, we have worked out the steps for the derivation of the Hamiltonian for \( \Phi \), which have been used in the main text. Since the Lagrangian associated with the same involves higher time derivatives, the standard prescription will not work and hence the Hamiltonian formulation of a higher time-derivative Lagrangian requires an extension of the conventional scheme. There are several frameworks in this context, firstly, the well known Ostrogradsky scheme \([54]\), while the other one we will use to verify the same was developed in \([56]\). We will demonstrate that both of these approaches lead to the same expression for the Hamiltonian.

For this purpose we start with the Lagrangian for the scalar \( \Phi \), obtained from the perturbed action for quadratic gravity, i.e., Eq. (8) and is given by,

\[ L = - \left( \frac{1}{2} + \dot{\alpha} R \right) a^2 \Phi \left\{ \nabla^2 \Phi - \Phi'' - 2 \frac{\alpha'}{a} \Phi' + \frac{R}{3} a^2 \Phi \right\} + 3 \dot{\alpha} \left[ \nabla^2 \Phi - \Phi'' - 2 \frac{\alpha'}{a} \Phi' + \frac{R}{3} a^2 \Phi \right]^2 , \]  

where \( \alpha' = (\partial a/\partial \eta) \), with \( \eta \) being the conformal time. Starting from the Lagrangian, we get the following expressions for the momentum conjugate to the dynamical variables \( \Phi \) and \( \Phi' \) following Ostrogradsky’s procedure, as,

\[ P_2 = \left( \frac{\partial L}{\partial \Phi''} \right) = \left( \frac{1}{2} + \dot{\alpha} R \right) a^2 \Phi - 6 \dot{\alpha} \left[ \nabla^2 \Phi - \Phi'' - 2 \frac{\alpha'}{a} \Phi' + \frac{R}{3} a^2 \Phi \right] \]
\[ = \left( \frac{1}{2} + \dot{\alpha} R \right) a^2 X_1 - 6 \dot{\alpha} \left[ \nabla^2 X_1 - \Phi'' - 2 \frac{\alpha'}{a} X_2 + \frac{R}{3} a^2 X_1 \right] , \]  

where, we have taken \( \Phi = X_1 \) as one of the dynamical variable. Following similar lines of argument we obtain,

\[ P_1 = \frac{\partial L}{\partial \Phi'} - \frac{d}{d\eta} \left( \frac{\partial L}{\partial \Phi''} \right) = 2 \frac{\alpha'}{a} \left( \frac{1}{2} + \dot{\alpha} R \right) a^2 \Phi - 12 \frac{\alpha'}{a} \dot{\alpha} \left[ \nabla^2 \Phi - \Phi'' - 2 \frac{\alpha'}{a} \Phi' + \frac{R}{3} a^2 \Phi \right] \]
\[ - \frac{d}{d\eta} \left\{ \left( \frac{1}{2} + \dot{\alpha} R \right) a^2 \Phi - 6 \dot{\alpha} \left[ \nabla^2 \Phi - \Phi'' - 2 \frac{\alpha'}{a} \Phi' + \frac{R}{3} a^2 \Phi \right] \right\} . \]  


Here, $\Phi' = X_2$ is considered as another dynamical variable. The above expression for $P_2$ can be inverted yielding, $\Phi''$ as a function of $(X_1, X_2, P_2)$, such that,

$$
\Phi'' = \frac{1}{6\alpha} \left\{ P_2 - \left( \frac{1}{2} + \tilde{a}R \right) a^2 X_1 \right\} + \nabla^2 X_1 - 2 \frac{a'}{a} X_2 + \frac{R}{3} a^2 X_1 .
$$

(116)

Therefore, the momentum space Hamiltonian can be expressed as,

$$
H = P_1 X_2 + P_2 \Phi'' - L
$$

$$
= P_1 X_2 + \frac{1}{6\alpha} P_2 \left\{ P_2 - \left( \frac{1}{2} + \tilde{a}R \right) a^2 X_1 \right\} + P_2 \nabla^2 X_1 - 2 P_2 \frac{a'}{a} X_2 + \frac{R}{3} P_2 a^2 X_1
$$

$$
+ \left( \frac{1}{2} + \tilde{a}R \right) a^2 \Phi \left\{ \nabla^2 \Phi - \Phi'' - 2 \frac{a'}{a} \Phi' + \frac{R}{3 a^2} \Phi \right\} - 3\alpha \left[ \nabla^2 \Phi - \Phi'' - 2 \frac{a'}{a} \Phi' + \frac{R}{3 a^2} \Phi \right]^2
$$

$$
= P_1 X_2 + \frac{1}{6\alpha} P_2 \left\{ P_2 - \left( \frac{1}{2} + \tilde{a}R \right) a^2 X_1 \right\} + P_2 \nabla^2 X_1 - 2 P_2 \frac{a'}{a} X_2 + \frac{R}{3} P_2 a^2 X_1
$$

$$
+ \left( \frac{1}{2} + \tilde{a}R \right) (a^2 X_1) \frac{1}{6\alpha} \left\{ P_2 - \left( \frac{1}{2} + \tilde{a}R \right) a^2 X_1 \right\} - \frac{1}{12\alpha} \left[ P_2 - \left( \frac{1}{2} + \tilde{a}R \right) a^2 X_1 \right]^2 .
$$

(117)

To see how the above formalism and the Hamiltonian derived using Ostrogradsky’s procedure connects with the result presented in [56], we also discuss the derivation of the Hamiltonian following [56]. In this approach we can introduce the following set of variables,

$$
q \equiv \Phi ; \quad \chi \equiv \Phi'' ; \quad F \equiv \beta q' Q .
$$

(118)

such that $q' = \Phi'$. The quantity $Q$ defined above, is determined by solving the following equation, $(\partial L/\partial \chi) + (\partial F/\partial q') = 0$, which yields,

$$
\beta Q = - \left( \frac{1}{2} + \tilde{a}R \right) a^2 \Phi + 6\alpha \left\{ \nabla^2 \Phi - \Phi'' - 2 \left( \frac{a'}{a} \right) \Phi' + R/3a^2 \Phi \right\} = 0 .
$$

(119)

Thus the old Lagrangian, in terms of the new variables, takes the following form,

$$
L = \frac{1}{6\alpha} \left( \frac{1}{2} + \tilde{a}R \right) a^2 q \left\{ \beta Q + \left( \frac{1}{2} + \tilde{a}R \right) a^2 q \right\} + \frac{1}{12\alpha} \left\{ \beta Q + \left( \frac{1}{2} + \tilde{a}R \right) a^2 q \right\}^2 .
$$

(120)

In order to define a Hamiltonian one needs to introduce a modified Lagrangian $L_{\text{mod}}$, which can be expressed in terms of the old Lagrangian as,

$$
L_{\text{mod}} = L + \frac{\partial F}{\partial q} q' + \frac{\partial F}{\partial Q} Q' + \frac{\partial F}{\partial \chi} \chi = L + \beta q' Q' + \beta Q \chi
$$

$$
= - \frac{1}{6\alpha} \left( \frac{1}{2} + \tilde{a}R \right) a^2 q \left\{ \beta Q + \left( \frac{1}{2} + \tilde{a}R \right) a^2 q \right\} + \frac{1}{12\alpha} \left\{ \beta Q + \left( \frac{1}{2} + \tilde{a}R \right) a^2 q \right\}^2 + \beta q' Q'
$$

$$
+ \beta Q (\nabla^2 \Phi - \Phi'' - 2 (a'/a) \Phi' + R/3a^2 \Phi) - \frac{1}{6\alpha} \beta Q (\beta Q + (1/2 + \alpha R) a^2 \Phi) ,
$$

(121)

Now the conjugate momenta associated with the variables $q$ and $Q$ originating from the above Lagrangian are defined by,

$$
p = \frac{\partial L_{\text{mod}}}{\partial q'} = \beta Q' - 2 \beta \frac{a'}{a} Q ; \quad P = \frac{\partial L_{\text{mod}}}{\partial Q'} = \beta q' .
$$

(122)
Hence the Hamiltonian takes the following form,
\[
H = pq' + PQ' - L_{mod} = \frac{2pP}{\beta} + 2 \left( \frac{a'}{a} \right) PQ + \frac{1}{6\alpha} \left( \frac{1}{2} + \alpha R \right) a^2 q \left\{ \beta Q + \left( \frac{1}{2} + \alpha R \right) a^2 q \right\} - \frac{pP}{\beta} - 2 \left( \frac{a'}{a} \right) PQ \\
- \frac{1}{12\alpha} \left( \beta Q + \left( \frac{1}{2} + \alpha R \right) a^2 q \right)^2 - \beta Q \left\{ \nabla^2 q - 2 \left( \frac{a'}{a} \right) \left( \frac{P}{\beta} \right) + \frac{R}{3} a^2 q \right\} - \frac{\beta Q}{6\alpha} \left\{ \beta Q + \left( \frac{1}{2} + \alpha R \right) a^2 q \right\}.
\]

(123)

Using the following canonical transformation between \((X_1, X_2, P_1, P_2)\) and \((q, Q, p, P)\),
\[
q = X_1, \quad P/\beta = X_2, \quad p = P_1, \quad -\beta Q = P_2,
\]

(124)
once can show that Eq. (123) is identical to Eq. (117). Thus we have established the equivalence between the two procedures of constructing the Hamiltonian in the present context.

Finally, converting the above Hamiltonian to coordinate space we obtain,
\[
H = \left\{ \frac{2a'}{a} \left( \frac{1}{2} + \alpha R \right) a^2 \Phi - 12 \frac{a'}{a} \alpha \left[ \nabla^2 \Phi - \Phi'' - 2 \frac{a'}{a} \Phi' + \frac{R}{3} a^2 \Phi \right] \right\} \Phi' \\
- \Phi' \frac{d}{d\eta} \left\{ \left( \frac{1}{2} + \alpha R \right) a^2 \Phi - 6\alpha \left[ \nabla^2 \Phi - \Phi'' - 2 \frac{a'}{a} \Phi' + \frac{R}{3} a^2 \Phi \right] \right\} \\
+ \left\{ \left( \frac{1}{2} + \alpha R \right) a^2 \Phi - 6\alpha \left[ \nabla^2 \Phi - \Phi'' - 2 \frac{a'}{a} \Phi' + \frac{R}{3} a^2 \Phi \right] \right\} \Phi'' \\
+ \left( \frac{1}{2} + \alpha R \right) a^2 \Phi \left\{ \nabla^2 \Phi - \Phi'' - 2 \frac{a'}{a} \Phi' + \frac{R}{3} a^2 \Phi \right\} - 3\alpha \left[ \nabla^2 \Phi - \Phi'' - 2 \frac{a'}{a} \Phi' + \frac{R}{3} a^2 \Phi \right]^2
\]

(125)

This can be combined together to yield,
\[
H = \left\{ \left( \frac{1}{2} + \alpha R \right) a^2 \Phi - 6\alpha \left[ \nabla^2 \Phi - \Phi'' - 2 \frac{a'}{a} \Phi' + \frac{R}{3} a^2 \Phi \right] \right\} \left( \frac{2a'}{a} \Phi' + \Phi'' \right) \\
- \Phi' \frac{d}{d\eta} \left\{ \left( \frac{1}{2} + \alpha R \right) a^2 \Phi - 6\alpha \left[ \nabla^2 \Phi - \Phi'' - 2 \frac{a'}{a} \Phi' + \frac{R}{3} a^2 \Phi \right] \right\} \\
+ \left( \frac{1}{2} + \alpha R \right) a^2 \Phi \left\{ \nabla^2 \Phi - \Phi'' - 2 \frac{a'}{a} \Phi' + \frac{R}{3} a^2 \Phi \right\} - 3\alpha \left[ \nabla^2 \Phi - \Phi'' - 2 \frac{a'}{a} \Phi' + \frac{R}{3} a^2 \Phi \right]^2
\]

(126)

Further simplification finally yields,
\[
H = -\Phi' \frac{d}{d\eta} \left\{ \left( \frac{1}{2} + \alpha R \right) a^2 \Phi - 6\alpha \left[ \nabla^2 \Phi - \Phi'' - 2 \frac{a'}{a} \Phi' + \frac{R}{3} a^2 \Phi \right] \right\} \\
+ \left( \frac{1}{2} + \alpha R \right) a^2 \Phi \left( \nabla^2 \Phi + \frac{R}{3} a^2 \Phi \right) - 3\alpha \left[ \left( \nabla^2 \Phi + \frac{R}{3} a^2 \Phi \right)^2 - \left( \Phi'' + 2 \frac{a'}{a} \Phi' \right)^2 \right]
\]

(127)

This is the expression we have used in Section 3 to determine the Hamiltonian and in Section 4 we have carried out the minimization of the Hamiltonian.
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