SOME EXAMPLES OF DR-INDECOMPOSABLE SPECIAL FIBERS OF SEMI-STABLE REDUCTIONS OVER WITT RINGS

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ABSTRACT. We answer negatively an open problem of Illusie on the DR-decomposability of the log de Rham complex of the special fiber of a semi-stable reduction over the Witt ring. We also show that $E_1$ degeneration of the Hodge to log de Rham spectral sequence does not imply DR-decomposability of semi-stable varieties.

1. Introduction

The work of Deligne-Illusie [5] is fundamental in Hodge theory since it gives a new method to establish the $E_1$-degeneration property of the Hodge to de Rham spectral sequence. Let $k$ be a perfect field of positive characteristic and $X_0$ an algebraic variety over $k$. We have the following commutative diagram of Frobenius

$$
X_0 \xrightarrow{F=F_{X_0/k}} X'_0 \xrightarrow{\pi} X_0 \xrightarrow{\sigma} \text{Spec } k
$$

The variety $X_0$ is said to be DR-decomposable if the complex $\tau_{<p} F_\ast \Omega_{X_0}^\bullet$ is quasi-isomorphic to $\bigoplus_{i=0}^{\dim X} \Omega_{X_0/k}^i [-i]$, where $\Omega_{X_0}$ is the de Rham complex of $X_0/k$. The main result of Deligne-Illusie asserts that for smooth varieties, $X_0$ is $W_2 = W_2(k)$-liftable if and only if it is DR-decomposable. On the other hand, if $X_0$ is proper over $k$ and $\dim X_0 < p$, the DR-decomposability of $X_0$ implies the $E_1$-degeneration of the Hodge to de Rham spectral sequence (for $\dim X_0 = p$, the $E_1$-degeneration also holds by the Grothendieck duality). Properness on $X_0$ is required because of finite dimensionality of Hodge cohomologies. However, it is not clear whether one can remove the assumption on the dimension of $X_0$: this is exactly one of the two open problems posed by Illusie [6]. It is neither clear whether $E_1$-degeneration would imply the DR-decomposability.

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Problem 1.1 (Illusie, Problem 7.14 [6]). Is the complex \( \tau_{\leq p} F_\ast \Omega^\log_{X_0} \) decomposable in \( D(X'_0) \)?

Our answer to this problem is NO. Indeed, we constructed explicit examples of semi-stable reductions over \( W \) negating the problem, whose dimension can be arbitrary large (in the curve case the answer is affirmative for cohomological reason) and the characteristic of \( k \) can be arbitrary. See §3 for the construction. We also examined the \( E_1 \)-degeneration property of these examples. It turns out that all examples we constructed whose dimensions are less than or equal to the characteristic of the residue field have the \( E_1 \)-degeneration property. Therefore, the \( E_1 \)-degeneration property is NOT equivalent to the DR-decomposability in the semi-stable (non-smooth) case. We are not aware of similar results in the smooth case.

2. DR-decomposability and log deformation

We use the log geometry as developed in the work [3] to study Problem 1.1, and the construction of our examples is mainly based on a simple criterion of the DR-decomposability in terms of the existence of a log smooth deformation over the log scheme \( (W_2(k), 1 \mapsto 0) \) (Theorem 2.3).

Let \( X \) be a semi-stable reduction over \( W \). Let \( M_{X_0} \) (resp. \( M_{\text{Spec}(k)} \)) be the log structure on \( X \) (resp. \( \text{Spec}(W) \)) attached to the reduced normal crossing divisor \( X_0 \) (resp. \( \text{Spec}(k) \)) (Example (1.5) [3]). Then the extended morphism of log schemes \( f : (X, M_{X_0}) \rightarrow (\text{Spec}(W), M_{\text{Spec}(k)}) \) is smooth. Let \( (X_0, M_0) \rightarrow k := (k, 1 \mapsto 0) \) be the base change of \( f \) via the inclusion \( \text{Spec}(k) \rightarrow \text{Spec}(W) \). When the context is clear, we denote the log scheme \( (X_0, M_0) \) simply by \( X_0 \) (in some other occasion, we use \( X \) to denote the underlying scheme of a log scheme \( X \)). It is known that the morphism \( X_0 \rightarrow k \) is smooth, and the de Rham complex \( \Omega^\bullet_{X_0/k} \) of the log variety \( X_0/k \) is naturally isomorphic to the complex \( \Omega^\log_{X_0} \) considered in §1 (1.7 [3]). Moreover, it is known that the log structure \( M_0 \) of \( X_0 \) is of semi-stable type:

Definition 2.1. ([15]) A log variety \( X \) over \( k \) is called semi-stable type if étale locally over each closed point \( x \in X \) it is strict smooth over

\[
(\text{Spec}(k(x)[x_1, \cdots, x_r]/(x_1 \cdots x_r)), \bigoplus_{i=1}^r \mathbb{N} e_i, e_i \mapsto x_i),
\]

where the log structure is induced by the homomorphism of monoids \( \bigoplus_{i=1}^r \mathbb{N} e_i \rightarrow \mathcal{O}_X \) defined by \( e_i \mapsto x_i \).

Let \( F \) be the absolute Frobenius of the log scheme \( k \) which is given by the commutative diagram

\[
\begin{array}{ccc}
0 & \xrightarrow{p} & 0 \\
\downarrow & & \downarrow \\
\mathbb{N} & \xrightarrow{\times p} & \mathbb{N}.
\end{array}
\]

It is easy to verify that \( F \) is liftable to the log scheme \( W_2 := (W_2, 1 \mapsto 0) \) (but not to the log scheme \( (W_2, 1 \mapsto p) \)!), and an obvious lifting \( G \) over \( W_2 \) is given by the following
The following corollary ensures it is valid to assume \( k \) is algebraically closed in the study of Problem 1.1.

**Corollary 2.4.** Let \( f : X \to k \) be a smooth morphism of semistable type and \( k' \) be a perfect field containing \( k \). Denote by \( k' \) the field \( k' \) with the induced log structure from \( k \) and by \( X_{k'} \) the log base change. Then \( \tau_{<p} F_{X/k*} \Omega^\bullet_{X/k} \) is decomposable if and only if \( \tau_{<p} F_{X_{k'}/k'*} \Omega^\bullet_{X_{k'}/k'} \) is decomposable.
Proof. By Theorem 2.3, it is enough to show that a \((W_2(k'), \mathbb{N} \mapsto 0)\)-lifting of \(X_{k'}\) induces a \((W_2(k), \mathbb{N} \mapsto 0)\)-lifting of \(X\). By the flat base change, one has the isomorphism \(H^2(X, T_{X/k}) \otimes_k k' = H^2(X_{k'}, T_{X/k'})\) and hence the injection \(\alpha : H^2(X, T_{X/k}) \rightarrow H^2(X_{k'}, T_{X/k'})\). Then, by the same arguments in Theorem 2.3, the obstruction class \(ob_k\) to lifting \(X\) to \(W_2(k)\) is mapped to the obstruction class \(ob_{k'}\) of lifting \(X_{k'}\) to \((W_2(k'), \mathbb{N} \mapsto 0)\) via the map \(\alpha\). By the condition that \(\alpha(ob_k) = ob_{k'} = 0\), it follows that \(ob_k = 0\).

Remark 2.5. After presenting our results, Weizhe Zheng provided us a more conceptual proof of Theorem 2.3: Denote by \(\text{Lift}(X)\) (resp. \(\text{Lift}(X')\)) the groupoid of liftings of \(X\) (resp. \(X'\)) over \(W_2\). Let \(G : W_2 \rightarrow W_2\) be a lifting of the log Frobenius morphism \(F : k \rightarrow k\). Given a lifting \(X^{(1)} \in \text{Lift}(X)\), the pullback of \(X^{(1)}\) along \(G\) gives an object in \(\text{Lift}(X')\). With the obvious assignments on morphisms, one can get a functor

\[
A : \text{Lift}(X) \rightarrow \text{Lift}(X').
\]

Conversely, let \(X^{(1)} \in \text{Lift}(X')\) be a lifting of \(X'\). Denote by \(i : X' \hookrightarrow X^{(1)}\) the canonical strict closed immersion and by \(\sigma : X' \rightarrow X\) the base change of \(F : k \rightarrow k\). Recall that \(g : X' \rightarrow X\) is an isomorphism and \(\mathcal{M}_X \simeq \mathcal{M}_X \oplus \mathcal{M}_k\). One can construct the pushout \(X^{(1)} \amalg_{X'} X\) of the diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{\sigma} & X \\
\downarrow{i} & & \downarrow{} \\
X^{(1)} & & \\
\end{array}
\]

as follows:

- The underlying scheme \(X^{(1)} \amalg_{X'} X\) is defined to be \(X^{(1)}\).
- The log structure of \(X^{(1)} \amalg_{X'} X\) is defined to be \(\mathcal{M}_{X^{(1)}} \times_{\mathcal{M}_{X'}} \mathcal{M}_X\).

With the obvious assignments on morphisms, the pushout process along \(\sigma : X' \rightarrow X\) gives a functor

\[
B : \text{Lift}(X') \rightarrow \text{Lift}(X).
\]

It is straightforward to check the following proposition.

Proposition 2.6. The functor \(A\) gives an equivalence of groupoids, and the functor \(B\) is its quasi-inverse.

3. Examples

In this section, \(k\) is an algebraically closed field of characteristic \(p > 0\). We proceed to construct examples of semi-stable reductions over \(W\) whose special fibers do not admit log deformation to \(W_2\), which negate Problem 1.1 because of Theorem 2.3.

3.1. More preparations. The first lemma is another characterization of semi-stable reductions over \(W = W(k)\).

Lemma 3.1. Let \(K_0\) be the fractional field of \(W\). Then an \(W\)-scheme \(X\) is a semi-stable reduction over \(W\) if and only if the following two properties hold:

1. the generic fiber \(X_{K_0} = X \times_W K_0\) is smooth over \(K_0\),
2. the special fiber \(X_k = X \times_W k\) is a normal crossing variety over \(k\).

Proof. See [4], 2.16. \(\square\)

The second lemma is rather standard.
Lemma 3.2. Let $X/k$ be a log variety of semi-stable type. Assume the irreducible components \( \{X_i, i \in I\} \) of the underlying variety $X$ to be smooth. Let $\mathcal{X}$ be a smooth deformation $X$ over $W_2$. Then the underlying scheme of $\mathcal{X}$ is written into the schematic union of closed subschemes $\mathcal{X} = \bigcup_{i \in I} X_i$ with the property that, for each nonempty $J \subseteq I$, the schematic intersection $\bigcap_{j \in J} X_j$ is a $W_2$-lifting of $\bigcap_{j \in J} X_j$.

Proof. Set \( \mathcal{I}_i = I_i + pI_i, \)
where $I_i$ is the ideal sheaf of $X_i$ in $X$. Then, $\mathcal{I}_i$ is an ideal sheaf of $\mathcal{O}_X$. We claim that the closed subschemes $\mathcal{X}_i$'s defined by $\mathcal{I}_i$'s have the property in the lemma. To show this it suffices to prove the following properties:

1. $\mathcal{O}_X/\mathcal{I}_i$ is flat over $W_2$,
2. $\bigcap \mathcal{I}_i = 0$, and
3. for each nonempty $J \subseteq I$, $\mathcal{O}_X/\bigcup_{j \in J} \mathcal{I}_j$ is flat over $W_2$.

Since $\mathcal{O}_{\mathcal{X}, x}$ is faithfully flat over $\mathcal{O}_{\mathcal{X}, x}$ for each point $x \in \mathcal{X}$, it suffices to verify the above claim after tensoring with $\mathcal{O}_{\mathcal{X}, x}$ for every $x \in \mathcal{X}$. By ([3] Theorem 3.5, Proposition 3.14), there is an étale morphism $U \to \mathcal{X}$ such that we have

\[
U \xrightarrow{f} \text{Spec}(W_2[x_1, \cdots, x_n]/(x_1 \cdots x_r)),
\]

\[
\pi_1 : U \to \text{Spec}(W_2)
\]

where $f$ is an étale morphism. As a consequence, there is an isomorphism

\[
\alpha : \mathcal{O}_{\mathcal{X}, x} \cong W_2[[x_1, \cdots, x_n]]/(x_1 \cdots x_r)
\]

such that each $\mathcal{O}_{\mathcal{X}, x}$ (whenever it is nonempty) is generated by $\alpha^{-1}(\Pi_{j \in J} x_j)$ for some nonempty set $J_i \subseteq \{1, \cdots, r\}$. Moreover, \( \{1, \cdots, r\} \) is the disjoint union of $J_i$'s. Then the claim follows from direct calculations. \qed

By the above two lemmas, we can conclude the following

Proposition 3.3. Let $Z$ be a smooth scheme over $W$. Let $Y_0$ be a smooth closed subvariety of $Z_0 = Z \times_W k$. Set $X = \text{Bl}_{Y_0}Z$, the blowup of $Z$ along the closed subscheme $Y_0$. Then $X$ is a semi-stable reduction over $W$, whose special fiber $X_0$ is a simple normal crossing divisor consisting of two smooth components $\text{Bl}_{Y_0}Z_0$ and $\mathbb{P}(N_{Y_0/Z})$ (the projective normal bundle of $Y_0$ in $Z$) which intersect transversally along $\mathbb{P}(N_{Y_0/Z_0})$ (the projective normal bundle of $Y_0$ in $Z_0$). Furthermore, if the normal crossing variety $X_0$ over $k$ admits a smooth deformation over $W_2$, then both pairs $(\text{Bl}_{Y_0}Z_0, \mathbb{P}(N_{Y_0/Z_0}))$ and $(\mathbb{P}(N_{Y_0/Z}), \mathbb{P}(N_{Y_0/Z_0}))$ are $W_2(k)$-liftable.

Proof. The first statement follows from Lemma 3.1 (the remaining fact is fairly standard and therefore omitted, see [8]). The second statement follows from Lemma 3.2. \qed

Proposition 3.4 (Cynk-van Straten, [1] Theorem 3.1). Let $\pi : Y \to X$ be a morphism of schemes over $k$ and let $S = \text{Spec}(A)$, where $A$ is artinian with residue field $k$. Assume that $\mathcal{O}_X = \pi_* \mathcal{O}_Y$ and $R^1 \pi_*(\mathcal{O}_Y) = 0$. Then for every lifting $\mathcal{Y} \to S$ of $Y$ there exists a preferred...
lifting $\mathcal{X} \to S$ making a commutative diagram

\[
\begin{array}{c}
Y' \\
\downarrow \downarrow \\
X'
\end{array}
\begin{array}{c}
Y \\
\downarrow \downarrow \\
X
\end{array}
\]

**Corollary 3.5.** Notation as in Proposition 3.3. If $Y_0$ is not $W_2(k)$-liftable, then the special fiber $X_0$ of $X$ (regarded as a log variety over $k$) does not admit any smooth deformation over $W_2$.

**Proof.** Use Propositions 3.3 and 3.4 which assert that the $W_2$-liftability of $\mathbb{P}(N_{Y_0/Z})$ implies that of $Y_0$. \hfill \Box

3.2. **Example 1.** Corollary 3.5 provides direct examples: take a smooth projective variety $Y_0$ over $k$ which is non $W_2$-liftable, and take a closed embedding $Y_0 \hookrightarrow Z_0$ over $k$ into a smooth projective variety such that the codimension $\text{CodZ}_0 Y_0 \geq 2$ and $Z_0$ admits a smooth lifting $Z$ over $W$ (for example take $Z_0$ to be a projective space of high dimension). Set $X = \text{Bl}_Y Z$, the blowup of $Z$ along the closed subscheme $Y_0$. Then $X$ is a semi-stable reduction over $W$ whose special fiber $X_0/k$ does not admit $W_2$-deformation.

3.3. **Example 2.** Notice that Mukai [12] has obtained a nice generalization to higher dimension of Raynaud’s classical example [16] of non $W_2$-liftable smooth projective surface over $k$. His construction, together with an idea of Liedtke-Satriano (Theorem 1.1 (a) [10]), allows us to make concrete examples of all relative dimensions $\geq 2$.

Let us recall first the following

**Definition 3.6 ([12]).** A smooth curve $C$ over $k$ of genus $\geq 2$ is called a Tango-Raynaud curve if there exists a rational function $f$ on $C$ such that $df \neq 0$ and that $(df) = pD$ for some ample divisor $D$.

A typical example of Tango-Raynaud curve is the plane curve defined by the affine polynomial

$$G(x^p) - x = y^{p-1},$$

where $G$ is a polynomial of degree $e \geq 1$ in the variable $x$. The following lemma is well known.

**Lemma 3.7 ([12]).** Let $C$ be a Tango-Raynaud curve, then there exists a rank two vector bundle $E$ on $C$ together with a smooth curve $D$ in the projectification $\mathbb{P}_C(E)$ of $E/C$, such that the composite $D \to \mathbb{P}_C(E) \to C$ is the relative Frobenius $F_0: D \to D^{(p)} = C$.

**Proposition 3.8.** Notation as in Lemma 3.7. Let $\mathcal{E}$ be a $W$-lifting of $C$ and $\mathcal{E}'$ a lifting of $E$ over $\mathcal{E}$. For $d \geq 2$, set $Z_d = \mathbb{P}_C(\mathcal{E} \oplus \mathcal{E}'^d)$ and $X_d = \text{Bl}_D Z_d$. Then $X_d$ is a semi-stable reduction over $W$ of relative dimension $d$, whose special fiber, regarded as a log variety over $k$, is non $W_2$-liftable and therefore DR-indecomposable.

**Proof.** We prove the statement for $d = 2$ only (the proof for $d \geq 3$ is the same). Denote

$$C_0 = C, \quad Y_0 = D, \quad Z_0 = \mathbb{P}_C(E), \quad Z = Z_2.$$

Assume the contrary that the special fiber $X_0$ of $\text{Bl}_Y Z$, regarded as a log variety over $k$, admit a smooth deformation over $W_2$. It follows from Proposition 3.2 that the pair
implies that the projection is the following nonzero morphism over $D$ is an isomorphism in a closed regular center $Y$. Therefore, $X_0/k$ is indeed non $W_2$-liftable as claimed. □

4. An $E_1$-degeneration result

This section is devoted to prove the following

**Theorem 4.1.** Let $k$ be an algebraically closed field and $R$ a DVR with the residue field $k$. Let $Z/R$ be a smooth proper $R$-scheme and $X/R$ be a blow-up of $X$ along a closed regular center $Y_0$ supported in $Z_0 = Z \times_k Y_0$. If the Hodge to de Rham spectral sequence

$$E_1^{pq} = H^q(Z_0, \Omega^p_{Z_0}) \Rightarrow H^{p+q}(\Omega^\bullet_{Z_0})$$

degenerates at $E_1$ (e.g. when $\text{char}(k) = 0$ or $\dim Z_0 \leq \text{char}(k)$ and $R$ is of mixed characteristic), then the Hodge to log de Rham spectral sequence

$$E_1^{pq} = H^q(X_0, \wedge^p \Omega^\log_{X_0}) \Rightarrow H^{p+q}(\Omega^{\log \bullet}_{X_0})$$

degenerates at $E_1$.

Recall from Proposition 3.3 that $X_0$ is a simple normal crossing divisor consisting of two smooth components $X_1 = Bl_{Y_0} Z_0$ and $X_2 = \mathbb{P}(N_{Y_0/Z})$ which intersect transversally along $D = \mathbb{P}(N_{Y_0/Z_0})$. The blowdown morphism of the log pairs $(Z, Z_0) \to (X, X_0)$ restricts on the special fiber to a log morphism $\pi : X_0 \to (Z_0, 1 \mapsto 0)$ between log varieties over $(\text{Spec}(k), 1 \mapsto 0)$. This induces a canonical morphism

$$\pi^* : \Omega^i_{Z_0} \to R\pi_* \bigwedge^i \Omega^\log_{X_0}.$$ 

Our main technical step in proving Theorem 4.1 is the following

**Proposition 4.2.** Let $Z/R$ be a smooth proper $R$-scheme and $X/R$ be a blow-up of $X$ along a closed regular center $Y_0$ supported in $Z_0$. Denote by $\pi : X_0 \to Z_0$ the restriction morphism. Then for each $i$ the canonical morphism (defined in the proof)

$$\Omega^i_{Z_0} \to R\pi_* \bigwedge^i \Omega^\log_{X_0}$$

is an isomorphism in $D^b(Z_0)$.

From Proposition 4.2, we may derive the main result of the section.

**Proof of Theorem 4.1.** We actually prove that the two spectral sequences

(1) $$E_1^{pq} = H^q(Z_0, \Omega^p_{Z_0}) \Rightarrow H^{p+q}(\Omega^\bullet_{Z_0})$$

and

(2) $$E_1^{pq} = H^q(X_0, \wedge^p \Omega^\log_{X_0}) \Rightarrow H^{p+q}(\Omega^{\log \bullet}_{X_0})$$
are isomorphic. First recall that (1) is induced by the hypercohomology of the complex $\Omega^\bullet_{Z_0}$ with respect to the truncated filtration

$$F^i = \tau^\text{st}_{\geq i} \Omega^\bullet_{Z_0},$$

where $\tau^\text{st}$ is the stupid truncation. (2) is induced by the hypercohomology of the complex $\Omega^\text{log}\log^\bullet_{X_0}$ with respect to the truncated filtration

$$F^i = \tau^\text{st}_{\geq i} \Omega^\text{log}\log^\bullet_{X_0}.$$ By Proposition 4.2, there are natural quasi-isomorphisms

$$R\pi_* \Omega^\text{log}\log^\bullet_{X_0} \simeq \pi_* \Omega^\text{log}\log^\bullet_{X_0} \simeq \Omega^\bullet_{Z_0},$$

and the isomorphisms respect the filtration

$$F^i = R\pi_* \tau^\text{st}_{\geq i} \Omega^\text{log}\log^\bullet_{X_0} \simeq \pi_* \tau^\text{st}_{\geq i} \Omega^\text{log}\log^\bullet_{X_0}$$
in the left, middle and

$$F^i = \tau^\text{st}_{\geq i} \Omega^\bullet_{Z_0}$$
in the right. As a consequence, the two spectral sequences (1) and (2) are naturally isomorphic. \qed

To prove Proposition 4.2, we make some preparations. Let $X_0 = X_1 \cup_D X_2$ be a variety consisting of two smooth projective components $X_1$ and $X_2$ such that they intersect transversely along a smooth divisor $D$. Assume that $X_0$ has a log structure of semi-stable type (Definition 2.1). Then the normalization $X_1 \cup X_2 \to \overline{X}_0$ and the diagonal immersion $D \to X_1 \cup X_2$ lift to log morphisms

$$(X_1 \cup X_2, D_1 \cup D_2 \oplus (1 \mapsto 0)) \to X_0$$

and

$$(D, (1 \mapsto 0)^{\oplus 2}) \to (X_1 \cup X_2, D_1 \cup D_2)$$

over the base $(\text{Spec}(k), 1 \mapsto 0)$. These log morphisms induce morphisms of sheaves on $X_0$

$$(3) \quad \Omega^i_{X_0} \to \Omega^i_{X_1}(\log D) \oplus \Omega^i_{X_2}(\log D)$$

and

$$(4) \quad \Omega^i_{X_1}(\log D) \oplus \Omega^i_{X_2}(\log D) \to \Omega^k_{(D,(1\mapsto 0)^{\oplus 2})/(\text{Spec}(k),1\mapsto 0)},$$

for each $i$. By the definition of log cotangent sheaf,

$$\Omega_{(D,(1\mapsto 0)^{\oplus 2})/(\text{Spec}(k),1\mapsto 0)} \simeq \Omega_D \oplus (\mathbb{Z}^{\oplus 2}/\mathbb{Z} \otimes \mathcal{O}_D)/\alpha(m) \otimes m - d\alpha(m) \otimes 1.$$

Thanks to the log structure of $(D, (1 \mapsto 0)^{\oplus 2})$, $\alpha(m) \otimes m - d\alpha(m) \otimes 1$ are null relations. Therefore

$$\Omega_{(D,(1\mapsto 0)^{\oplus 2})/(\text{Spec}(k),1\mapsto 0)} \simeq \Omega_D \oplus \mathcal{O}_D.$$

This isomorphism induces the forgetful morphism

$$\Omega_{(D,(1\mapsto 0)^{\oplus 2})/(\text{Spec}(k),1\mapsto 0)} \to \Omega_D$$

and the log residue morphism

$$\Omega_{(D,(1\mapsto 0)^{\oplus 2})/(\text{Spec}(k),1\mapsto 0)} \to \mathcal{O}_D.$$

Therefore

$$\Omega^k_{(D,(1\mapsto 0)^{\oplus 2})/(\text{Spec}(k),1\mapsto 0)} \simeq \bigwedge^k(\Omega_D \oplus \mathcal{O}_D) \simeq \Omega^k_D \oplus \Omega^{k-1}_D.$$
and by local calculation the restriction morphism
\[ \Omega^k_{X_1}(\log D) \to \Omega^k_{(D, (1 \to 0)^{\oplus 2})/(\text{Spec}(k), 1 \to 0)} \]
is equivalent to
\[ (\iota, \text{res}_D) : \Omega^k_{X_1}(\log D) \to \Omega^k_D \oplus \Omega^{k-1}_D \]
\[ \beta + \gamma \frac{dz}{z} \mapsto (\beta, \gamma). \]
Here we use a local chart of \( X_1 \) where \( D = \{ z = 0 \} \) and \( \beta, \gamma \) does not contain \( dz \). This phenomenon is interesting in itself. The residue map \( \text{res}_D \) is a part of the restriction map of log cotangent sheaves. It makes log geometry a convenient and natural language in such a situation.

Assume locally \( X_0 \) is embedded into the affine space with a system of local coordinates \( (z_1, z_2, \cdots, z_n) \) such that \( X_1 = \{ z_1 = 0 \} \), \( X_2 = \{ z_2 = 0 \} \). Since
\[ \frac{dz_1}{z_1} + \frac{dz_2}{z_2} = 0 \]
on \( X_0 \), a log form on \( X_0 \) is of the form
\[ \beta + \gamma_1 \frac{dz_1}{z_1} = \beta - \gamma_1 \frac{dz_1}{z_1}. \]
Therefore two \( k \)-forms \( \beta_1 + \gamma_1 \frac{dz_1}{z_1} \) on \( X_2 \) and \( \beta_2 + \gamma_2 \frac{dz_2}{z_2} \) on \( X_2 \) glue to a log \( k \)-form on \( X_0 \) if and only if
\[ \beta_1|_D = \beta_2|_D \]
and
\[ \gamma_1|_D + \gamma_2|_D = 0. \]
This proves

**Lemma 4.3.** For each \( k \geq 0 \) there is a short exact sequence of sheaves
\[ 0 \to \Omega^k_{X_1 \log} \to \Omega^k_{X_1}(\log D) \oplus \Omega^k_{X_2}(\log D) \xrightarrow{\varphi} \Omega^k_D \oplus \Omega^{k-1}_D \to 0 \]
where \( \varphi \) is defined by
\[ \begin{pmatrix} \iota & \text{res}_D \\ -\iota & \text{res}_D \end{pmatrix}. \]
Here \( \Omega^{k-1}_D \) is defined to be 0.

The following well-known lemma will be used several times in the sequel.

**Lemma 4.4.** Let \( \mathbb{P}^n \) be the projective space over \( k \). The following vanishing results hold:

1. \( H^q(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n}) = 0, p \neq q \)
2. If \( i \neq 0 \), then
   \[ H^q(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n}(i)) = 0, \]
   for \( q = 0, i \leq p \) or \( q = n, i \geq p - n \) or \( q \neq 0, n \).
(3) Let $H$ be a hyperplane in $\mathbb{P}^n$, then

$$H^q(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^p(\log H)) = \begin{cases} k, & p = q = 0 \\ 0, & \text{otherwise} \end{cases}.$$ 

**Lemma 4.5.** Let $Z$ be a smooth variety and $\pi : P \to Z$ be a projective bundle of relative dimension $r$. Let $D \subset P$ be a relative hyperplane. Then for each $i \geq 0$ there is a canonical isomorphism

$$\Omega^i_Z \simeq R\pi_* \Omega^i_P(\log D)$$

in $D(Z)$.

**Proof.** The exact sequence

$$0 \to \pi^* \Omega_Z \to \Omega_P(\log D) \to \Omega_{P/Z}(\log D) \to 0$$

induces a decreasing filtration

$$F^p = \pi^* \Omega_Z^p \wedge \Omega_{P/Z}^{i-p}(\log D) \subset \Omega_P^i(\log D)$$

such that

$$F^p / F^{p+1} \simeq \pi^* \Omega_Z^p \otimes \Omega_{P/Z}^{i-p}(\log D).$$

Therefore we have a spectral sequence

$$E_1^{pq} = R^p \pi_* (\pi^* \Omega_Z^p \otimes \Omega_{P/Z}^{i-p}(\log D)) \Rightarrow R^{p+q} \pi_* (\Omega_P^i(\log D)).$$

By Lemma 4.4, we see that

$$E_1^{pq} \simeq \Omega_Z^p \otimes R^q \pi_* (\Omega_{P/Z}^{i-p}(\log D)) = \begin{cases} \Omega_Z^p, & p = i, q = 0 \\ 0, & \text{otherwise} \end{cases}.$$ 

This proves the lemma. \(\square\)

**Lemma 4.6.** Let $Z_0$ be a smooth projective variety and $Y_0$ be a smooth closed subvariety of $Z_0$. Denote $\pi : X_1 \to Z_0$ be the blowup along $Y_0$ with exceptional divisor $D$. Then for each $k \geq 0$, there is a distinguished triangle in $D^b(Z_0)$ induced by natural morphisms:

$$\Omega^k_{Z_0} \xrightarrow{\varphi} R\pi_* \Omega^k_{X_1} \oplus \Omega^k_{Y_0} \xrightarrow{\psi} R\pi_* \Omega^k_D \to \Omega^k_{Z_0}[1].$$

In other words, we have the short exact sequence

$$0 \to \Omega^k_{Z_0} \to \pi_* \Omega^k_{X_1} \oplus \Omega^k_{Y_0} \to \pi_* \Omega^k_D \to 0 \quad (5)$$

and the isomorphism

$$R^{i} \pi_* \Omega^k_{X_1} \to R^{i} \pi_* \Omega^k_D \quad (6)$$

for each $i > 0$.

**Proof.** Denote the following automorphism of $\pi_* \Omega^k_{X_1} \oplus \Omega^k_{Y_0}$ by $\phi$:

$$(a, b) \mapsto (a, a - b),$$

By composing with $\phi$, the exactness of the sequence $(5)$ is reduced to the following isomorphisms

$$\Omega^k_{Z_0} \cong \pi_* \Omega^k_{X_1}; \quad \Omega^k_{Y_0} \cong \pi_* \Omega^k_D.$$

For $k = 0$, these are obvious. For $k \geq 1$, their truth can be easily seen by considering the local model of a blow-up along a smooth center: we assume that $X_1$ is the blow up of
$Z_0 = \mathbb{A}^n$ along $Y_0 = \mathbb{A}^r$ defined by the intersection of some coordinate hyperplanes. Then the map

$$\pi : D \to Y_0$$

is the projection

$$\mathbb{A}^r \times \mathbb{P}^s \to \mathbb{A}^r.$$

Thus, it is trivial to get $\pi_*\Omega^k_D = \Omega^k_{Y_0}, k \geq 0$ by this description. For the first isomorphism, we use the following estimation:

$$\pi^*\Omega^k_{Z_0} \subset \Omega^k_{X_1} \subset \pi^*\Omega^k_{Z_0}(kD).$$

From this, it follows that

$$\Omega^k_{Z_0} \subset \pi_*\Omega^k_{X_1} \subset \Omega^k_{Z_0} \otimes \pi_*\mathcal{O}_{X_1}(kD) = \Omega^k_{Z_0},$$

and hence $\pi_*\Omega^k_{X_1} = \Omega^k_{Z_0}$.

The proof of (6) is divided into two parts. First we show that the natural map

$$R^i\pi_*\Omega^k_{X_1}|_D \to R^i\pi_*\Omega^k_D$$

is an isomorphism for each $i > 0$. Considering the long exact sequence associated to

$$0 \to \mathcal{O}_D(1) \otimes \Omega^{k-1}_D \to \Omega^k_{X_1}|_D \to \Omega^k_D \to 0$$

where $\mathcal{O}_D(1)$ is the tautological bundle of the projective bundle $D \to Y_0$, we see that it sufficient to prove that

$$R^i\pi_*(\mathcal{O}_D(1) \otimes \Omega^k_D) = 0, \quad i > 0.$$  \hspace{1cm} (7)

Notice that the short exact sequence

$$0 \to \pi^*\Omega_{Y_0} \to \Omega_D \to \Omega_{D/Y_0} \to 0$$

induces a decreasing filtration

$$F^p = \pi^*\Omega^p_{Y_0} \wedge \Omega^{k-p}_D \subset \Omega^k_D$$

such that

$$F^p / F^{p+1} \simeq \pi^*\Omega^p_{Y_0} \otimes \Omega^{k-p}_{D/Y_0}.$$  \hspace{1cm} (8)

Therefore we have a spectral sequence

$$E_1^{pq} = R^q\pi_*(\pi^*\Omega^p_{Y_0} \otimes \Omega^{k-p}_{D/Y_0} \otimes \mathcal{O}_D(1)) \Rightarrow R^{p+q}\pi_*(\mathcal{O}_D(1) \otimes \Omega^k_D).$$

Since $D \to Y_0$ is a projective bundle, we obtain that

$$E_1^{pq} = \Omega^p_{Y_0} \otimes R^q\pi_*(\Omega^{k-p}_{D/Y_0} \otimes \mathcal{O}_D(1))$$

for $p + q \geq 1$ and $p, q \geq 0$, thanks to the Lemma 4.4. This proves (7) and thus

$$R^i\pi_*\Omega^k_{X_1}|_D \to R^i\pi_*\Omega^k_D$$

is an isomorphism for each $i > 0$.

Next we show that the canonical morphism

$$R^i\pi_*\Omega^k_{X_1} \to R^i\pi_*(\Omega^k_{X_1}|_D)$$

is an isomorphism for each $i > 0$. By the long exact sequence associated to

$$0 \to \Omega^k_{X_1} \otimes \mathcal{O}_{X_1}(-D) \to \Omega^k_{X_1} \to \Omega^k_{X_1}|_D \to 0,$$
we see that it sufficient to show the vanishing

\[ R^i\pi_* (\Omega^k_{X_1} \otimes \mathcal{O}_{X_1}(-D)) = 0 \]

for each \( i > 0 \).

Notice that the short exact sequence

\[ 0 \to \pi^* \Omega_{Z_0} \to \Omega_{X_1} \to \Omega_{X_1/Z_0} \simeq \Omega_{D/Y_0} \to 0 \]

induces a decreasing filtration

\[ F^p = \pi^* \Omega^p_{Z_0} \wedge \Omega^{k-p}_{X_1} \subset \Omega^k_{X_1} \]

such that

\[ F^p / F^{p+1} \simeq \pi^* \Omega^p_{Z_0} \otimes \Omega^{k-p}_{D/Y_0}. \]

Therefore we have a spectral sequence

\[ E_1^{pq} = R^q\pi_* (\pi^* \Omega^p_{Z_0} \otimes \Omega^{k-p}_{D/Y_0} \otimes \mathcal{O}_{X_1}(-D)) \Rightarrow R^{p+q}\pi_* (\Omega^k_{X_1} \otimes \mathcal{O}_{X_1}(-D)). \]

Since \( D \to Y_0 \) is a projective bundle, we obtain that

\[ E_1^{pq} = \Omega^p_{Z_0} \otimes R^q\pi_* (\Omega^{k-p}_{D/Y_0} \otimes \mathcal{O}_D(1)) \]

for \( p + q \geq 1 \) and \( p, q \geq 0 \), thanks again to the Lemma 4.4. This proves (8) and thus

\[ R^i\pi_* \Omega_{X_1}^k \to R^i\pi_* (\Omega_{X_1}^k |_D) \]

is an isomorphism for each \( i > 0 \). So we finish the proof of (6).

Now we are ready to prove Proposition 4.2.

**Proof.** By Lemma 4.3 and 4.5, we have a distinguished triangle

\[ R\pi_* \left( \bigwedge^i \Omega^i_{X_1} \right) \to R\pi_* \Omega^i_{X_1} \otimes \Omega^{i_0}_{Y_0} \to \Omega^i_{Y_0} \to R\pi_* \left( \bigwedge^i \Omega^i_{X_1} \right) \]

in \( D^b(Z_0) \). This triangle fills in the following diagram in \( D^b(Z_0) \)

\[ \begin{array}{ccc}
R\pi_* \Omega^i_{X_1} & \to & 0 \\
\downarrow & & \downarrow \\
R\pi_* \Omega^i_{X_1} \otimes \Omega^{i_0}_{Y_0} & \to & R\pi_* \Omega^i_{X_1} \otimes \Omega^{i_0}_{Y_0} \\
\downarrow p \quad \downarrow \text{Id} & & \downarrow \text{Id} \\
R\pi_* \left( \bigwedge^i \Omega^i_{X_1} \right) & \to & R\pi_* \left( \bigwedge^i \Omega^i_{X_1} \right) \\
\downarrow & & \downarrow \\
R\pi_* \Omega^i_{X_1} \otimes \Omega^{i_0}_{Y_0} & \to & R\pi_* \left( \bigwedge^i \Omega^i_{X_1} \right) \otimes \Omega^{i_0}_{Y_0} \\
\downarrow & & \downarrow \\
R\pi_* \left( \bigwedge^i \Omega^i_{X_1} \right) & \to & R\pi_* \left( \bigwedge^i \Omega^i_{X_1} \right) \\
\downarrow & & \downarrow \\
R\pi_* \Omega^i_{D} & \to & R\pi_* \Omega^i_{D} \\
\downarrow & & \downarrow \\
R\pi_* \Omega^i_{D} & \to & R\pi_* \Omega^i_{D} \\
\end{array} \]
which is generated from the centered commutative square

\[
\begin{array}{ccc}
R\pi_*\Omega^i_{X_1}(\log D) \oplus \Omega^i_{Y_0} & \overset{\text{Id}}{\longrightarrow} & R\pi_*\Omega^i_D \\
R\pi_*\Omega^i_{X_1}(\log D) \oplus \Omega^i_{Y_0} & \overset{R\pi_*\varphi}{\longrightarrow} & R\pi_*\Omega^i_D \oplus R\pi_*\Omega^i_{D}^{-1}.
\end{array}
\]

In the diagram (9), \( p \) is induced (non-canonically) by the above commutative square. The second horizontal line is the direct sum of the distinguished triangles

\[
R\pi_*\Omega^i_{X_1} \rightarrow R\pi_*\Omega^i_{X_1}(\log D) \rightarrow R\pi_*\Omega^i_{D} \rightarrow R\pi_*\Omega^i_{X_1}[1]
\]

and

\[
\Omega^i_{Y_0} \overset{\text{Id}}{\longrightarrow} \Omega^i_{Y_0} \rightarrow 0 \rightarrow \Omega^i_{Y_0}[1].
\]

The horizontal lines of (9) are distinguished triangles. The second and third vertical lines are also distinguished. By the 3 \( \times \) 3 lemma of triangulated categories, the first vertical line induces a distinguished triangle

\[
R\pi_* \wedge^i \Omega^\log_{X_0} \rightarrow R\pi_* \Omega^k_{X_1} \oplus \Omega^i_{Y_0} \rightarrow R\pi_* \Omega^k_D \rightarrow R\pi_* \wedge^i \Omega^\log_{X_0}[1].
\]

Comparing with Lemma 4.6, we see that there is a quasi-isomorphism

\[
R\pi_* \wedge^i \Omega^\log_{X_0} \simeq \Omega^i_{Z_0}.
\]

Note that this isomorphism may not be the natural one induced by the morphism \( \pi \). However, we obtain as a consequence of the abstract quasi-isomorphism that

\[
R^k\pi_* \wedge^i \Omega^\log_{X_0} \simeq 0, \quad k > 0.
\]

It remains to show that the natural morphism of sheaves

\[
\Omega^i_{Z_0} \rightarrow \pi_* \wedge^i \Omega^\log_{X_0}
\]

is an isomorphism.

Let us consider the cohomologies at place 0 of the diagram (9),

\[
\begin{array}{ccc}
\pi_*\Omega^i_{X_1} \oplus \Omega^i_{Y_0} & \overset{p^0}{\longrightarrow} & \pi_*\Omega^i_{X_1}(\log D) \oplus \Omega^i_{Y_0} \\
& \overset{\text{Id}}{\longrightarrow} & \pi_*\Omega^i_D \\
\pi_* \wedge^i \Omega^\log_{X_0} & \rightarrow & \pi_*\Omega^i_{X_1}(\log D) \oplus \Omega^i_{Y_0} \overset{\pi_*\varphi}{\longrightarrow} \pi_*\Omega^i_D \oplus R\pi_*\Omega^i_{D}^{-1} \\
0 & \rightarrow & 0 \\
0 & \rightarrow & \pi_*\Omega^i_D \overset{\text{Id}}{\longrightarrow} \pi_*\Omega^i_D
\end{array}
\]

The two vertical sequences in the middle are short exact sequences. Therefore, by the snake lemma, there is an exact sequence

\[
0 \rightarrow \pi_* \wedge^i \Omega^\log_{X_0} \overset{p^0}{\rightarrow} \pi_*\Omega^i_{X_1} \oplus \Omega^i_{Y_0} \overset{\delta}{\rightarrow} \pi_*\Omega^i_D
\]

where \( \delta \) is the boundary map which is identical to the one in (5). Hence by (5) we see that the natural map (10) is an isomorphism. \( \square \)
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