Non decoupling ghosts in the light cone gauge.

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ABSTRACT

The gist of using the light cone gauge lies in the well known property of ghosts decoupling. But from the BRST point of view this is a stringency since for the construction of a nilpotent operator (from a Lie algebra) the presence of ghosts are mandatory. We will show that this is a foible which has its origins in the very fact of using just one light cone vector \( n_\mu \) instead of working with both light cone vectors \( n_\mu \) and \( m_\mu \) to fulfill the light cone base vectors. This will break out ghost decoupling from theory but allowing now a consistent BRST theory for the light cone gauge.

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BRST symmetry has a special delighting importance in theoretical physics. The light cone gauge has also a very strong relevance in theoretical physics. But meanwhile the former makes mandatory the use of ghosts [1], the later decouples them from theory [2] strangling the possibility of building up a consistent BRST operator.

There is an interesting example concerning the use of both ideas in the context of string theory quantization [3, 4]. The (super)string quantization for excellence is done making use the non covariant light cone gauge, this breaks the covariance of the theory but it is a consistent and also the quickest way of do that. Hence there is a well succeeded attempt of obtain a covariant quantization of superstring using the idea of mixing pure spinors with BRST operator [5], in order to obtain a covariant quantization of superstring theory building up a nilpotent operator.

The history of light cone gauge theories is shoved with really true degrees of freedom as was pointed out by Dirac [6]. The decoupling of ghosts in light cone gauge theories should be felt as a great bonus; but there are some big prices to pay for: a complicated structure of gauge propagator, the so called spurious poles appear in theory and the absence of consistent BRST operators (since ghosts are mandatory in constructing them [7]); just to cite some of them.

In order to show our ideas, we will use a non-abelian and also a non-covariant Yang-Mills theory in four dimensions since we want to fix the gauge freedom on the light-cone. This work is presented in four sections. The first one has a very briefly review of BRST symmetry. Section two shows the one base light cone vector reaches the ghost decoupling. Section three will explain how the completely two base light cone vectors do not reach the ghost decoupling. Conclusions are written in last section.
1 BRST operator gist.

The BRST symmetry origins remotes to quantum field theory where it was discovered. But this symmetry is more than a quantum field property. It is associated to gauge degrees of freedom of constrained systems. For a Lie algebra defined in terms of is generators $G_a$ and structures constant $f_{ab}^c$ by

$$[G_a, G_b] = i f_{ab}^c G_c,$$

an operator, called the BRST operator is defined as

$$Q = c^a G_a - \frac{i}{2} f_{ab}^c c^a c^b c^c,$$

where the $c^a$ and $b_a$ form a canonical conjugate pair of anti commuting variables that are known as ghosts since its statistical property and satisfy

$$\{b_a, c^b\} = \delta^b_a.$$

The BRST operator is nilpotent, that is

$$\{Q, Q\} \equiv 2Q^2 = 0,$$

which is easy to demonstrate making use of Jacobi identity. Also we could define the ghost number operator $J$ as

$$J = c^a b_a,$$

that implies following relation $[J, Q] = Q$. The very importance in defining the BRST operator is related to the fact that its cohomology give us the true physical states, say $|\phi\rangle$. If $|\phi'\rangle$ represents the same physical state, then it is related $|\phi\rangle$ throughout

$$|\phi'\rangle = |\phi\rangle + Q|\varphi\rangle,$$
where $|\varphi>$ is an arbitrary vector. In other words, they belong to the same cohomology since $Q|\varphi> = Q|\varphi'>$. This fact can be used to classified the space into equivalent classes. This condensate introduction on BRST operator is that we need to have in mind for next sections.

2 One light cone base vector.

The light cone gauge Yang-Mills theory \cite{8} is practically defined in terms of its gauge fixing Lagrangian part

$$\mathcal{L}_{\text{fix}} = -\frac{1}{2\alpha}(n^\mu A^a_\mu)^2, \quad \alpha \to 0.$$  

where $A^a_\mu$ and $n^\mu$ represent the gauge boson vector field and one light cone base vector respectively. There are many features concerning the use of this gauge in quantum field theory \cite{7}, but we are interested just in how ghosts fields decoupled from theory. We are going to show two ways of demonstrate this fact.

2.1 Ghost decoupling.

Ghost fields can be generated from the Fadeev-Popov mechanism for a gauge vector field $A_\mu$, this means that we have to analyze the following path integral

$$\int [dA_\mu] f(A_\mu) e^{i \int d^4 x \mathcal{L}}.$$  

In fact, $A_\mu$, means, $A^a_\mu$, where $a$ represents the gauge group index. Also the functional $f(A_\mu)$ is a gauge invariant quantity. It is assumed that $[dA_\mu] = [dA_\mu^a]$. And the ansatz for $[dA_\mu]$ is:

$$[dA_\mu] = \prod_{\mu,a,x} dA^a_\mu(x)$$

The path integral \cite{2} considers all possible configurations of $A_\mu(x)$, so there is an over counting in a gauge equivalence classes. This leads us to divide
the configuration space \{A_\mu(x)\} into classes of equivalence \{A_\mu^\omega(x)\} called
gauge group orbits. The gauge groups orbits have all of field’s configurations
which results from applying all transformations \omega of the gauge group \mathcal{G}
from an initial configuration \(A_\mu(x)\) of the field.
The gauge transformation is
\[
A'^a_\mu \equiv (A^\omega)^a_\mu = A^a_\mu + f^a_{bc} A^b_\mu \omega^c + \partial_\mu \omega^a. \tag{3}
\]

Now let’s allow \([d\omega]\) to represent an invariant measure over the gauge group
\(\mathcal{G}\), i.e. \([d\omega] = [d\omega\omega']\), where the ansatz is:
\[
[d\omega] = \prod_x d\omega(x).
\]

Introducing the functional \(\Delta[A_\mu(x)]\)
\[
1 = \Delta[A_\mu(x)]. \int [d\omega] \delta[F[A_\mu^\omega(x)]] \tag{4}
\]
Where \(\delta[f(x)]\) represents \(\prod_x \delta[f(x)]\). Also, for \(F[A_\mu^\omega]\) it is assumed:
\[
F[A_\mu^\omega] = 0.
\]

Which has exactly one solution, \(\omega_0\), whoever is \(A_\mu\). This last expression is
the constraint, that defines the hypersurface and also the “gauge”.

Observation: \(\Delta[A_\mu(x)]\) is a gauge invariant since:
\[
\Delta^{-1}[A_\mu^\omega] = \int [d\omega'] \delta[F[A_\mu^\omega\omega']] = \int [d\omega\omega'] \delta[F[A_\mu^\omega\omega'']], \int [d\omega\omega'] \delta[F[A_\mu^\omega\omega'']] = \Delta^{-1}[A_\mu],
\]

inserting (4) into (2), we obtain:
\[
\int [d\omega] \int [dA_\mu] f(A_\mu) \Delta[A_\mu(x)] \delta[F[A_\mu^\omega(x)]] e^{iS[A_\mu^\omega]},
\]

Observation: The expression inside of \(\int [d\omega]\) is also a gauge invariant; i.e.
it does not depend on \(\omega\).
\[
\left( \int [d\omega] \right) \int [dA_\mu] f(A_\mu) \Delta[A_\mu(x)] \delta[F[A_\mu^\omega(x)]] e^{iS[A_\mu^\omega]}. \quad (5)
\]

**Faddeev-Popov determinant.**

From eq. (4):

\[
\Delta^{-1}[A_\mu] = \int [dF] \left( \det \frac{\delta F[A_\mu^\omega]}{\delta \omega} \right)^{-1} \delta F,
\]

i.e.

\[
\Delta[A_\mu] = \det \left. \frac{\delta F[A_\mu^\omega]}{\delta \omega} \right|_{F[A_\mu^\omega]=0} = \det M,
\]

\(\Delta[A_\mu]\) is usually called as the Faddeev-Popov determinant. If we consider the gauge \(F[A_\mu^\omega] - C(x) = 0\), then our integral expression (5) shall looks like:

\[
\int [d\omega] \int [dA_\mu] \det M f(A_\mu) \delta[F[A_\mu^\omega(x)]] e^{iS[A_\mu^\omega]}.
\]

The expression for the functional generator \(Z[J=0]\) will be:

\[
Z[J=0] = N \int [d\omega] \int [dA_\mu] \det M f(A_\mu) \delta[F[A_\mu^\omega(x)]] e^{iS[A_\mu^\omega]}.
\]

Using the matrix identity from Grassmanian numbers:

\[
\det M = \int [d\bar{c}][dc] e^{i\bar{c}M c},
\]

shall made possible to write \(Z[J=0]\) as

\[
Z[J=0] = N \int [d\omega] \int [dA_\mu][d\bar{c}][dc] f(A_\mu) \delta[F[A_\mu^\omega(x)]] e^{i(S[A_\mu^\omega]-\bar{c} Ac)},
\]

where the new fields \(c\) and \(\bar{c}\) are called *ghost fields*, and since its statistical nature, they just appear as internal lines in Feynman graphs.

With the preliminaries given above now we are ready to show the ghosts decoupling. We will have two ways of demonstrate how the ghosts decouple in an axial type gauge:
2.1.1 Ghosts decoupling: way A.

Starting from the axial type gauge definition
\[ n^\mu A^a_\mu = 0, \]
\[ n^\mu n_\mu = \text{const}, \]
the fixing gauge term is \( F^a = n^\mu A^a_\mu \). Then, making use gauge transformation (3) we have
\[ \delta F^a \equiv n^\mu f^{abc} A^b_\mu \omega^c + n^\mu \partial_\mu \omega^a. \]
This means that
\[ \frac{\delta F^a}{\delta \omega^b} = \delta^{ab} n^\mu \partial_\mu. \]
Observe that the last expression represents the matrix \( M \) and for this case it does not involve the gauge field \( A^a_\mu \), this means that also \( \det M \) does not have \( A^a_\mu \) and this makes possible to put \( \det M \) out from the path integral of \([dA_\mu]\) in (4). Then the expression for \( Z[J = 0] \) should be written as
\[ Z[J = 0] = N \int [d\omega] \det M \int [dA_\mu] f(A_\mu) \delta[F[A^\omega_\mu(x)] - C(x)] e^{iS[A^\omega_\mu]}. \]
The term \( \int [d\omega] \det M \) could be absorbed into the constant, or if you prefer, we shall make a redefinition, nevertheless, arriving to
\[ Z[J = 0] = N \int [dA_\mu] f(A_\mu) \delta[F[A^\omega_\mu(x)] - C(x)] e^{iS[A^\omega_\mu]}. \]
As we can see, this last expression is ghosts fields free, that is the point we wanted to demonstrate.

2.1.2 Ghosts decoupling: way B.

Starting from the definition of \( \det M \) and using the property of the product of matrix determinants:
\[ \det M = \det(n^\mu D_\mu), \]
\[ = \det(n \partial) \cdot \det(1 - gG f^a A^{\alpha \mu} n_\mu), \]
where in this last expression we are using the adjoint representation. We also have that

\[ D_\mu = \partial_\mu - gf^a n_\mu \partial^\nu A_\nu^a, \]

\[ (\partial.n)G(x - y) = \delta^D(x - y), \]

so, working with the second determinant, using another well-known property of matrix determinants and then expanding the logarithmics in a sum over \( k \) \((k = 1 \cdots \infty)\):

\[ \det(1 - gGf^a A_\mu^a n_\mu) = e^{Tr[\ln(1 - gGf^a A_\mu^a n_\mu)]}, \]

\[ = e^{-Tr[\sum\frac{1}{k}G(x_0 - x_1)f^{a_1} A_{\mu_1}^a n_{\mu_1} \cdots G(x_{k-1} - x_k)f^{a_k} A_{\mu_k}^a n_{\mu_k}]} }. \]

Here we observe that we obtain a series of graphs, that are functions of \( k \)-points, in other words, each term of the sum is a Feynman graph that can be represented in the figure (1).

This implies that each process involves Feynman integrals, which can be computed (here we are using the Feynman parametrization) in the context of dimensional regularization. In this way, the integral that is associated to

\[ \text{Figure 1: } k\text{-points ghost function.} \]
the figure \( \figref{1} \) will have the form:

\[
 I_{\mu_1 \cdots \mu_k}^{a_1 \cdots a_k} = g^k \prod_{i=1}^{k} n_{\mu_i} T_r \left[ \prod_{i=1}^{k} f_{a_i} \right] (k-1)! \int_0^{1} \int_0^{x_{k-1}} \int_0^{x_k} \frac{d^D q}{(n.q + \cdots - n.p_2 x_{k-1})^k}.
\]

Making a change of variables,

\[
 I_{\mu_1 \cdots \mu_k}^{a_1 \cdots a_k} = g^k \prod_{i=1}^{k} n_{\mu_i} T_r \left[ \prod_{i=1}^{k} f_{a_i} \right] I.
\]

Where \( I \) (using Lorentz invariance)

\[
 I = \int \frac{d^D q}{(n.q)_k},
\]

\[
 = c(n, k) \frac{1}{n^k} \int \frac{d^D q}{q^k}.
\]

This last integral is zero in the context of dimensional regularization. With this in mind it is easy to see that \( \text{det}(1 - gG f^a A_{\mu}^a n_{\mu}) = 1 \), this means that \( \text{det}M \) again does not depend on ghost fields, so again we are able to say that the ghost fields decoupled from the theory.

3 Complete two light cone base vectors.

The four Minkowski space-time can be generated using a base of just two vectors: the light cone vector. Defining the dual base light-like four-vectors:

\[
 n_{\mu} = \frac{1}{\sqrt{2}} (1, 0, 0, 1),
\]

\[
 m_{\mu} = \frac{1}{\sqrt{2}} (1, 0, 0, -1),
\]

we observe that with the help of this base, the \( x^\pm \) coordinates can be expressed as

\[
 x^+ = x^\mu n_{\mu},
\]

\[
 x^- = x^\mu m_{\mu}.
\]
Using these two light-cone vectors, it is possible to build up a two degree
de light cone gauge theory as presented in [9]. In this case, the suitable gauge
fixing Lagrangian shall be

\[ \mathcal{L}_{fix} = -\frac{1}{2\alpha} (n.A)(m.A). \]  

(9)

this term has the disadvantage of generating a more complicated structure of
boson propagator in the theory, but allows us to discern some interesting
properties such as the possibilities of defining a light like planar gauge and
have a prescriptionless theory [10] for the so-called spurious poles. In this
work we will show another new feature. Let’s first see the ghost behavior for
this case.

3.1 Ghost non decoupling.

From eq. (3) we have that the variation of the gauge field is:

\[ \delta A^a_\mu = f^a_{bc} A^b_\mu \omega^c + \partial_\mu \omega^a, \]

and the Fadeev-Popov matrix is:

\[ M = \frac{\delta F[A^\omega]}{\delta \omega} \bigg|_{F[A^\omega]=0} \]

Since the gauge fixing Lagrangian is shown in eq. (9), we have that in this
case the gauge fixing term is

\[ F^a = [(n.A^a)(m.A^a)]^{1/2}, \]  

(10)

then is variation should take the form

\[ \delta F^a = \frac{1}{2[(n.A^a)(m.A^a)]^{1/2}} \left\{ (n^\mu f^a_{bc} A^b_\mu \omega^c + n^\mu \partial_\mu \omega^a)(m.A^a) \right. \\
+ (n.A^a)(m^\mu f^a_{bc} A^b_\mu \omega^c + m^\mu \partial_\mu \omega^a) \right\}, \]
that allows us to find

\[
\frac{\delta F^a}{\delta \omega_b} = \frac{\delta^{ab}}{2[(n.A^a)(m.A^a)]^{1/2}} \left\{ (n^\mu \partial_\mu)(m.A) + (n.A)(m^\mu \partial_\mu) \right\}.
\]

(11)

Actually eq. (11) represents the Fadeev-Popov matrix. It says that ghost does not decouple here, so they have a dynamical role in this axial type gauge.

4 Conclusions.

The fact that ghosts does not decouple from theory has a very close relation with the BRST operator, which now can be defined in the same manner as in eq. (1) as

\[
Q = c^a G_a - \frac{i}{2} f_{ab} c^a c^b b_c,
\]

where the \(G_a\) are generators of the Lie algebra associated to Yang-Mills theory on the light-cone gauge in four dimensions. The existence of light-like planar gauge is not an oldest one. In fact it was belived that there was not a consistent form to define it, until studies of some kind of axial type gauges with two degree of freedom \[\text{11}\]: we are talking about the use of both light-cone base vectors, which opened a small track in non covariant gauges studies. Besides this gauge, as we already have seen, takes care of the proper role of ghosts and its importance, in particular on the BRST-noncovariant gauges question. We are not boasted about this kind of gauge is better that the other one, we just want to point out this new feature (consistent BRST operator for the light cone gauge) when working with the full base of light cone vectors.
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