Ergodicity and Mixing for Stochastic Partial Differential Equations

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Abstract

Recently, a number of authors have investigated the conditions under which a stochastic perturbation acting on an infinite dimensional dynamical system, e.g. a partial differential equation, makes the system ergodic and mixing. In particular, one is interested in finding minimal and physically natural conditions on the nature of the stochastic perturbation. I shall review recent results on this question; in particular, I shall discuss the Navier-Stokes equation on a two dimensional torus with a random force which is white noise in time, and excites only a finite number of modes. The number of excited modes depends on the viscosity \( \nu \), and grows like \( \nu^{-3} \) when \( \nu \) goes to zero. This Markov process has a unique invariant measure and is exponentially mixing in time.

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1. Introduction

The goal of this paper is to consider stochastic partial differential equations and to study conditions on the random perturbation that imply exponential convergence to a stationary state. In fact, one wants ‘minimal’ conditions, in the following sense: by expanding the solution in a basis of eigenfunctions of a linear operator associated with the PDE, one can write the latter as an infinite dimensional system of coupled differential equations. The question, then, is: to how many such equations do we need to add noise in order to make the system ergodic and mixing?

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The physical motivation for this question comes from the fact that isotropic turbulence is often mathematically modelled by the Navier-Stokes equation subjected to an external stochastic driving force which is stationary in space and time. If the solution is expanded into Fourier modes, the driving force, which, in the language of physicists, acts on “large scale”, should not perturb, or perturb very weakly, the high modes which represent the small scale properties of the system. So, one would like to show that the system becomes ergodic and mixing by adding noise to as few modes as possible. Obviously, this requires some detailed understanding of the nonlinear dynamics of the deterministic PDE.

This problem is interesting from another point of view. As we shall see below, one can show that all but a finite number of modes converge to equilibrium provided the remaining ones do. So, we can reduce ourselves to a finite dimensional problem, which would be standard, except for the fact that the discarded modes produce a memory effect on the remaining ones, so that the problem is no longer Markovian. At this point, one introduces techniques coming from the study of the statistical mechanics of one-dimensional systems (where the unique dimension corresponds physically to space rather than time) with “long range, exponentially decaying, interactions” which have already been very useful in the study of SRB measures in dynamical systems (see [28]).

At present, the best results require that the number of modes to which noise must be added depends on the parameters of the system, although a stronger result is likely to hold (see Remark 4 after Theorem 1.1).

The type of question discussed here (for the Navier-Stokes equation but also for other equations) has been at the center of attention of several groups of people (see Remark 3 after Theorem 1.1 below). In this paper, I shall try to explain, in a simplified form, the approach followed by A. Kupiainen, R. Lefevere and myself in [3] (see also [1, 2] for previous results).

To be concrete, consider the stochastic Navier-Stokes equation for the velocity field \( u(t, x) \in \mathbb{R}^2 \) defined on the torus \( T = (\mathbb{R}/2\pi\mathbb{Z})^2 \):

\[
du + ((u \cdot \nabla)u - \nu \nabla^2 u + \nabla p) dt = df
\]

where \( f(t, x) \) is a Wiener process with covariance

\[
Ef_\alpha(t, x)f_\beta(t', y) = \min\{t, t'\}C_{\alpha\beta}(x - y)
\]

and \( C_{\alpha\beta} \) is a smooth function satisfying \( \sum_\alpha \partial_\alpha C_{\alpha\beta} = 0 \). Equation (1.1) is supplemented with the incompressibility condition \( \nabla \cdot u = 0 = \nabla \cdot f \), and we will also assume that the averages over the torus vanish: \( \int_T u(0, x) = 0 = \int_T f(t, x) \), which imply that \( \int_T u(t, x) = 0 \) for all times \( t \).

It is convenient to change (1.1) to dimensionless variables so that \( \nu \) becomes equal to one. This is achieved by setting \( u(t, x) = \nu u'(\nu t, x) \). Then \( u' \) satisfies (1.1), (1.2) with \( \nu \) replaced by 1, and \( C' \) by

\[
C' = \nu^{-3} C.
\]
From now on, we work with such variables and drop the primes. The dimensionless control parameter in the problem is the (rescaled) energy injection rate \( \frac{1}{2} \text{tr} C(0) \), customarily written as \((\text{Re})^3\) where \( \text{Re} \) is the Reynolds number:

\[
\text{Re} = \epsilon^4 \nu^{-1},
\]

and \( \epsilon = \frac{1}{2} \text{tr} C(0) \) is the energy injection rate in the original units (for explanations of the terminology see [10]).

In two dimensions, the incompressibility condition can be conveniently solved by expressing the velocity field in terms of the vorticity \( \omega = \partial_1 u_2 - \partial_2 u_1 \). First (1.1) implies the transport equation

\[
d\omega + ((u \cdot \nabla)\omega - \nabla^2 \omega) dt = db,
\]

where \( b = \partial_1 f_2 - \partial_2 f_1 \) has the covariance

\[
E b(t, x) b(t', y) = \min\{t, t'\} (2\pi)^{-1} \gamma(x - y)
\]

with \( \gamma = -2\pi \nu^{-3} \Delta \text{tr} C \).

Next, going to the Fourier transform, \( \omega_k(t) = \frac{1}{2\pi} \int_T e^{ik \cdot x} \omega(t, x) dx \), with \( k \in \mathbb{Z}^2 \); we may express \( u \) as \( u_k = i(\frac{k_1 k_2}{2\pi}) \omega_k \), and write the vorticity equation as

\[
d\omega(t) = F(\omega(t)) dt + db(t),
\]

where the drift is given by

\[
F(\omega)_k = -k^2 \omega_k + \frac{1}{2\pi} \sum_{l \in \mathbb{Z}^2 \setminus \{0, k\}} \frac{k_1 l_2 - k_2 l_1}{|l|^2} \omega_{k-l} \omega_l
\]

and \( \{b_k\} \) are Brownian motions with \( \bar{b}_k = b_{-k} \) and

\[
E b_k(t) b_l(t') = \min\{t, t'\} \delta_{k, -l} \gamma_k.
\]

The dimensionless control parameter for the vorticity equation is

\[
R = \sum_{k \in \mathbb{Z}^2} \gamma_k = 2\pi \gamma(0)
\]

which is proportional to the \( \omega \) injection rate, and also to the third power of the Reynolds number. One is interested in the turbulent region, where \( R \) is large; therefore, we will always assume below, when it is convenient, that \( R \) is sufficiently large.

For turbulence one is interested in the properties of stationary state of the stochastic equation (1.4) in the case of smooth forcing (see [1] for some discussion of this issue) and, ideally, one would like to consider the case where one excites only a finite number of modes,

\[
\gamma_k \neq 0, k^2 \leq N,
\]
with \( N \) of order of one (for that, see Remark 4 below). In this paper we assume that \( N \) scales as

\[ N = \kappa R, \tag{1.7} \]

with \( \kappa \) a constant, taken large enough. We set all the other \( \gamma_k = 0 \), although this condition can easily be relaxed. Let us denote the minimum of the covariance by

\[ \rho = \min \{|\gamma_k| \mid |k|^2 \leq N\}. \]

Before stating our result, we need some definitions. Let \( P \) be the orthogonal projection in \( H = L^2(\mathbb{T}) \) to the subspace \( H_s \) of functions having zero Fourier components for \( |k|^2 > N \). We will write \( \omega = s + l \) with \( s = P\omega \), \( l = (1 - P)\omega \) (respectively, the small \( k \) and large \( k \) parts of \( \omega \)). Denote also by \( H_l \) the complementary subspace (containing the nonzero components of \( l \)). \( H \) is our probability space, equipped with \( \mathcal{B} \), the Borel \( \sigma \)-algebra.

The stochastic equation (1.4) gives rise to a Markov process \( \omega(t) \) and we denote by \( P^t(E|\omega) \) the transition probability of this process. The main result of [3] is the

**Theorem 1.1.** The stochastic Navier-Stokes equation (1.4) defines a Markov process with state space \( (H, \mathcal{B}) \) and for all \( R < \infty \), \( \rho > 0 \) it has a unique invariant measure \( \mu \) there. Moreover, \( \forall \omega \in H \), for all Borel sets \( E \in H_s \) and for all bounded Hölder continuous functions \( F \) on \( H_l \), we have,

\[
| \int P^t(d\omega \mid \omega)1_E(s')F(l') - \int \mu(d\omega')1_E(s')F(l') | \leq C\|F\|_{\alpha}e^{-mt} \tag{1.8}
\]

where \( C = C(\|\omega\|, R, \rho) < \infty \), \( m = m(R, \rho, \alpha) > 0 \), and \( \|F\|_{\alpha} \) is the Hölder norm of exponent \( \alpha \).

**Remark 1.** In [3], we stated, for convenience, Theorem 1.1 by saying that the constant \( C \) in (1.8) was a function of \( \omega \) which was almost surely finite. Since this was stressed e.g. in [20], it is worth remarking that \( C \) is simply a function of \( \|\omega\| \) (depending also on the parameters \( R \) and \( \rho \)), which is finite \( \forall \omega \in H \). To check this, we refer the reader to equations (86) and (97) in [3]. The main reason why this bound holds, however, lies in the fact that the only dependence of our estimates on \( \omega \) appears in Lemma 4.1 below and occurs through \( \|\omega\| \).

**Remark 2.** In [1] it was proven that, with probability 1, the functions on the support of the measure constructed here are real analytic. In particular all correlation functions of the form

\[
\int \mu(d\omega) \prod_i \nabla^{n_i} u(x_i)
\]

exist. For further results on analyticity, see [20, 23].
Remark 3. While the existence of the invariant measure follows with soft methods \cite{29}, its uniqueness and the ergodic and mixing properties of the process has been harder to establish. With a nonsmooth forcing (meaning that the strength of the noise, $\gamma_k$, decays only polynomially with $|k|$) this was established in \cite{9} and for large viscosity in \cite{23}. However, those results did not cover the most physically interesting situations. The first result for a smooth forcing was by Kuksin and Shirikyan \cite{13} who considered a periodically kicked system with bounded kicks (for results on exponential convergence in that model, see \cite{14,15,19,22}). In particular they could deal with the case where only a finite number of modes are excited by the noise (the number of modes depends both on the viscosity and the size of the kicks). In \cite{2}, we proved uniqueness and exponential mixing for such a kicked system where the kicks have a Gaussian distribution, but we required that there be a nonzero noise for each mode. An essential ingredient in analysis of \cite{13}, which was used in \cite{8} and by other authors, is the Lyapunov-Schmidt type reduction that allows to transform the original Markov process with infinite dimensional state space to a non-Markovian process with finite dimensional state space. While the analysis of \cite{13} was limited to bounded noise acting at discrete times, it was extended in \cite{16,17,20,21} to cover unbounded noise and continuous time, as well as to obtain results on the strong law of large number and the central limit theorem. The first results on ergodicity of the system with unbounded noise and finitely many excited modes were obtained in \cite{7,3} (see also \cite{6} for applications to other equations) and, for exponential convergence, in \cite{3}, which was also proved in \cite{24}. For results on related problems, see \cite{3,11,12,22}.

Remark 4. What one would like to obtain is a result similar to Theorem 1.1, but with $N$ finite, independently of $R$. An interesting result in that direction was obtained by Weinan E and Mattingly \cite{8} who showed that, if one adds noise to only 2 (suitably chosen) modes, ergodicity holds, provided one truncates the system \cite{1.4,1.3}, by keeping only a finite, but arbitrarily large, number of modes. This of course suggests that the 2 stochastically perturbed modes produce an “effective noise” on any finite number of modes, in particular on all those with $|k| \geq \kappa R$; then, one could hope to combine this with the results in \cite{7,3} to obtain ergodicity and mixing for the full system. This, however, has not been done.

Remark 5. The parameters in our problem are $R$ and $\rho$. All constants that do not depend on them will be generically denoted by $C$ or $c$. These constants can vary from place to place.

Let me now explain the connection with ideas coming from statistical mechanics.

First, observe that, if one neglects the nonlinear term in \cite{1.4,1.3}, one expects $\|\omega\|$ to be of order $R^\frac{1}{2}$, for typical realizations of the noise ($R^\frac{1}{2}$ is the typical size of the noise, and the $-k^2\omega_k$ term will dominate in eq. \cite{1.4} for larger values of $\|\omega\|$). It turns out that similar probabilistic estimates hold for the full equation \cite{1.4} as shown in Section 4. Now, if $\|\omega\|$ is of size $R^\frac{1}{2}$, the $-k^2\omega_k$ term will dominate the nonlinear term (which is roughly of size $\|\omega\|^2$) in eq. \cite{1.4}, for $|k| \geq \kappa R^\frac{1}{2}$, and one
can expect that those modes (corresponding to \( l \) above) will behave somewhat like the solution of the heat equation and, in particular, that they will converge to a stationary state.

Thus, the first step is to express the \( l \)-modes in terms of the \( s \)-modes at previous times. This is done in Section 2 and produces a process for the \( s \)-modes that is no longer Markovian but has an infinite memory. In statistical mechanics, this would correspond to a system of unbounded spins (the \( s \)-modes) with infinite range interactions, with the added complications that, here, the measure is not given in a Gibbsian form, but only through a Girsanov formula, i.e. (2.9) below, and that time is continuous. Hence, we have to solve several problems: the possibility that \( \omega \) be atypically large, the long range “interactions”, and finally, showing that a version of the \( s \)-process with a suitable cutoff is ergodic and mixing.

In Section 3, I introduce a “toy model”, namely a process with infinite memory, but with bounded variables, so that the problems caused by the unprobably large values of \( \|\omega\| \) does not occur. In that model, I explain how the statistical mechanical techniques, developed to study systems on one dimensional lattices, can be adapted to our setting.

The large \( \omega \) problem is treated in Section 4, using probabilistic estimates developed in [1], which, in statistical mechanics, would be called stability estimates. In Section 5, I sketch how the remaining problems are handled: showing that the techniques explained in Section 3 can be applied here. However, this is where several technical complications enter, for the treatment of which I refer to [3]. The problem is that, even though for typical noise, hence for typical \( \omega \)'s, the \( l \)-modes depend exponentially weakly on their past (see Section 2), thus producing, typically, “interactions” that decay exponentially fast, they may depend sensitively on their past when the noise is large. In the language of statistical mechanics, atypically large noise produces long range correlations, and that is the source of many technical difficulties. My goal here is to present the main conceptual tools used in [3], putting aside those difficulties.

2. Finite dimensional reduction

Using an idea of [13], one can reduce the problem of the study of a Markov process with infinite dimensional state space to that of a non-Markovian process with finite dimensional state space.

For this purpose, write the equation (1.4) for the small and large components of \( \omega \) separately:

\[
\begin{align*}
    ds(t) &= PF(s(t) + l(t))dt + db(t) \\
    \frac{d}{dt}l(t) &= (1 - P)F(s(t) + l(t)).
\end{align*}
\]

(2.1)  \hspace{1cm} (2.2)

The idea of [13] is to solve the \( l \) equation for a given function \( s \), thereby defining \( l(t) \) as a function of the entire history of \( s(t') \), \( t' \leq t \). Then, the \( s \) equation will have a drift with memory. Let us fix some notation. For a time interval \( I \), we denote the restriction of \( \omega \) (or \( s, l \) respectively) to \( I \) by \( \omega(I) \), and use the boldface
notation \( s(I) \), to contrast it with \( s(t) \), the value of \( s \) at a single time. \( \| \cdot \| \) will denote the \( L^2 \) norm. In \( \mathbb{R} \) it was proven that, for any \( \tau < \infty \), there exists a set \( \mathcal{B}_\tau \) of Brownian paths \( b \in C([0, \tau], H_s) \) of full measure such that, for \( b \in \mathcal{B}_\tau \), \( \mathbb{E} \left[ \int_0^\tau \| \nabla \omega(t) \|^2 dt \right] < \infty \). In particular, the projections \( s \) and \( l \) of this solution are in \( C([0, \tau], H_s(t)) \) respectively.

On the other hand, let us denote, given any \( s \in C([0, \tau], H_s) \), the solution — whose existence will be discussed below — of \( \mathcal{P} \), with initial condition \( l(0) \) by \( l(t, s([0, t]), l(0)) \). More generally, given initial data \( l(t') \) at time \( t' < \tau \) and \( s([t', \tau]) \), the solution of \( \mathcal{P} \) is denoted, for \( \tau \leq \sigma \leq \tau \), by \( l(\sigma, s([t', \sigma]), l(t')) \) and its corresponding \( \omega \) by \( \omega(\sigma, s([t', \sigma]), l(t')) \). The existence and key properties of those functions are given by:

**Proposition 2.1.** Let \( l(0) \in H_1 \) and \( s \in C([0, \tau], H_s) \). Then \( l(\cdot, s([0, t]), l(0)) \in C([0, \tau], H_1) \cap L^2([0, \tau], H^1_1) \), where \( H^1_1 = H_1 \cap H^1_1 \). Moreover, given two initial conditions \( l_1, l_2 \) and \( t \leq \tau \)

\[
\|l(t, s([0, t]), l_1) - l(t, s([0, t]), l_2)\| \leq \exp \left[ -\kappa R t + a \int_0^t \| \nabla \omega_1 \|^2 \right] \|l_1 - l_2\| \quad (2.3)
\]

where \( a = (2\pi)^{-2} \sum |k|^{-4} \) and \( \omega_1(t) = s(t) + l_1(t, s([0, t]), l_1) \). The solution also satisfies

\[
l(t, s([0, t]), l(0)) = l(t, s([\tau, t]), l(\tau, s([0, \tau]), l(0))). \quad (2.4)
\]

**Remark.** What this Proposition shows is that the dependence of the function \( l \) upon its initial condition \( l_i \), \( i = 1, 2 \), decays exponentially in time (i.e. like the solution of the heat equation), provided \( \omega \) is not too large, in the sense that \( \int_0^\tau \| \nabla \omega_1 \|^2 \leq c R t \), for a suitable constant \( c \). As we will see in Section 4, this event is highly probable.

Now, if \( s = P \omega \) with \( \omega \) as above being the solution of \( \mathcal{P} \) with noise \( b \in \mathcal{B}_\tau \), then the \( l(s) \) constructed in the Proposition equals \( (1 - P) \omega \) and the stochastic process \( s(t) \) satisfies the reduced equation

\[
ds(t) = f(t) dt + db(t) \quad (2.5)
\]

with

\[
f(t) = PF(\omega(t)). \quad (2.6)
\]

where \( \omega(t) \) is the function on \( C([0, t], H_s) \times H_1 \) given by

\[
\omega(t) = s(t) + l(t, s([0, t]), l(0)). \quad (2.7)
\]

\( \mathcal{P} \) has almost surely bounded paths and we have a Girsanov representation for the transition probability of the \( \omega \)-process in terms of the \( s \)-variables

\[
P^t(F|\omega(0)) = \int \mu^t_\omega(ds)F(\omega(t)) \quad (2.8)
\]
with
\[ \mu_t^s(\omega) = e^{\int_0^t (f(\tau), \gamma^{-1}(ds(\tau) - \frac{1}{2} f(\tau)d\tau)) \nu_s^t(\omega) d\tau} \] \hspace{1cm} (2.9)

where \( \nu_s^t(\omega) \) is the Wiener measure with covariance \( \gamma \) on paths \( s = s([0, t]) \) with starting point \( s(0) \) and \( (\cdot, \cdot) \) the \( \ell^2 \) scalar product. Define the operator \( \gamma^{-1} \) in terms of its action on the Fourier coefficients:
\[ (f, \gamma^{-1} f) = \sum_{|k|^2 \leq N} |f_k|^2 \gamma^{-1}_k. \] \hspace{1cm} (2.10)

The Girsanov representation (2.8) is convenient since the problem of a stochastic PDE has been reduced to that of a stochastic process with finite dimensional state space. The drawback is that this process has infinite memory. In the next section, I will show how to deal with this problem in a simplified situation.

3. A Toy Model

In order to explain the main ideas in the proof, I will consider first a ‘toy model’ and then explain the steps needed to control the full model.

Let us consider variables \( x_t \in [0, 1], t \in \mathbb{Z} \) about which a set of (consistent) conditional probability densities \( p(x_t|x_{[-\infty, t-1]}) \) is given, i.e. one is given the probability densities of the variables \( x_t \), at time \( t \), given a ‘past history’ \( x_{[-\infty, t-1]} \), where we write, for \( I \subset \mathbb{Z}, x_I = (x_t)_{t \in I} \in [0, 1]^I \).

Before stating precise assumptions on \( p \), here is what one wants to prove: \( \exists C < \infty, m > 0 \) and a probability \( p \) on \([0, 1]\) such that \( \forall E \subset [0, 1], E \) measurable,
\[ |p(x_T \in E|x_{[-\infty, 0]}) - \overline{p}(E)| \leq Ce^{-mT} \] \hspace{1cm} (3.1)

for all \( T > 0 \) and all \( x_{[-\infty, 0]} \), where
\[ p(x_T|x_{[-\infty, 0]}) = \prod_{t=1}^T \prod_{t=1}^{T-1} dx_t p(x_t|x_{[1, t-1]} \vee x_{[-\infty, 0]}) \] \hspace{1cm} (3.2)

and \( x_{[1, t-1]} \vee x_{[-\infty, 0]} \) denotes the obvious configuration on \([-\infty, t-1]\).

Now let us state the assumptions on \( p \) that will imply (3.1); obviously, we assume that:
\[ p(x_t|x_{[-\infty, t-1]}) \geq 0 \] \hspace{1cm} (3.3)

and
\[ \int_0^1 dx_t p(x_t|x_{[-\infty, t-1]}) = 1 \] \hspace{1cm} (3.4)

for all \( x_{[-\infty, t-1]} \). Moreover, we assume that \( p(\cdot | \cdot) \) is invariant under translations of the lattice \( \mathbb{Z} \), in a natural way. The non-trivial assumptions are:
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a) Let, for \( s < t - 2 \),
\[
\delta_{s,t}(x_{s:t}) \equiv p(x_t|x_{s:t-1} \vee 0) - p(x_{t+1}|x_{s:t-1} \vee 0),
\]
where \( x_t \vee 0 \) denotes the configuration equal to \( x_t \) for \( t \in I \) and equal to zero elsewhere. We assume that \( \exists C < 0, m > 0 \) such that \( \forall s, t \in \mathbb{Z} \) as above,
\[
\|\delta_{s,t}\|_{\infty} \leq C \exp(-m|t-s|).
\]
(3.6)

b) Define, for \( N \geq 1 \), the Markov chain on \( \Omega = [0,1]^N \) by the transition probability
\[
P(x_{[1,N]}|x_{[-N+1,0]}) = \prod_{t=1}^{N} p(x_t|x_{t-N:t-1} \vee 0).
\]
(3.7)

We assume that this Markov chain satisfies : \( \exists \delta > 0, \forall B \subset \Omega, \forall x, x' \in \Omega, \)
\[
P(B|x) + P(B^c|x') \geq \delta
\]
(3.8)
where \( \delta \) is independent of \( N \) (see however the Remark following the proof of Proposition 3.1 for a generalization).

**Proposition 3.1.** Under assumptions a) and b) above, (3.1) holds.

**Remark 1.** The techniques used here can also prove the analogue of (3.1) with \( x_T \) replaced by \( x_{[T,T-L]} \), for any finite \( L \), and this, in turn, allows one to associate to the system of conditional probabilities a unique probability distribution on \( [0,1]^\mathbb{Z} \) (which is called, in statistical mechanics, the Gibbs state associated to the system of conditional probabilities), but I will not go into that, because I want to give here only an elementary idea of the techniques used in \( [3] \). Of course, this type of results is not new (see e.g. \( [25] \), Lecture 12, for a similar result, applied to dynamical systems, with a somewhat different proof).

To prove the Proposition, we first use a result of Doob (\( [4] \), p. 197–198):

**Lemma 3.1** For the Markov chain defined in b) above, there exists a probability distribution \( P \) on \( \Omega \) such that \( \forall x \in \Omega, \forall B \subset \Omega, \forall n \geq 1, \)
\[
|P^n(B|x) - P(B)| \leq (1 - \delta)^n.
\]
(3.9)

**Proof.** Let \( \overline{P}(B,n) = \sup_x P^n(B|x) \) and \( \underline{P}(B,n) = \inf_x P^n(B|x) \). It is easy to see that \( \overline{P}(B,n) \) is decreasing in \( n \), while \( \underline{P}(B,n) \) is increasing in \( n \). Thus, it is sufficient to prove the bound (3.9) for the difference \( |\overline{P}(B,n) - \underline{P}(B,n)| \) and, for that, we shall prove:
\[
0 \leq \overline{P}(B,n+1) - \underline{P}(B,n+1) \leq (1 - \delta)(\overline{P}(B,n) - \underline{P}(B,n)).
\]
(3.10)
Since \( \overline{P}(B,n) - \underline{P}(B,n) \leq 1 \), (3.9) follows.
Define a signed measure on subsets of $\Omega$:
\[
\Psi_{x,x'}(E) = P(E|\mathbf{x}) - P(E|\mathbf{x'})
\] (3.11)
and let $S^+$ (resp. $S^-$) denote the set where $\Psi_{x,x'}(E) \geq 0$ for $E \subset S^+$ (resp. $\leq 0$).

We have:
\[
P(B, n+1) - P(B, n+1) = \sup_{x,x'} \int [P(dx''|\mathbf{x}) - P(dx''|\mathbf{x'})]P^n(B|x'')
\]
\[
= \sup_{x,x'} \int \Psi_{x,x'}(dx'')P^n(B|x'')
\]
\[
\leq \sup_{x,x'}(\Psi_{x,x'}(S^+)P(B, n) + \Psi_{x,x'}(S^-)P(B, n)).
\] (3.12)

By definition, $\Psi_{x,x'}(S^-) = -\Psi_{x,x'}(S^+)$, so that
\[
\Psi_{x,x'}(S^+)P(B, n) + \Psi_{x,x'}(S^-)P(B, n) = \Psi_{x,x'}(S^+)(P(B, n) - P(B, n)).
\]

Also, for any set $E \subset \Omega$, (3.8) implies
\[
\Psi_{x,x'}(E) = 1 - (P(E^c|\mathbf{x}) + P(E|\mathbf{x'})) \leq 1 - \delta.
\]

Applying this to $E = S^+$ in (3.12) implies (3.10).

**Remark 2.** We shall use this Lemma under the following form:
\[
\int_{\Omega} dx|P^n(\mathbf{x}|\mathbf{x'}) - P(\mathbf{x})| \leq 2(1 - \delta)^n
\] (3.13)
for all $\mathbf{x'} \in \Omega$; this follows by applying (3.9) separately to the sets where the integrand is positive and negative.

Now let us turn to the

**Proof of Proposition 3.1.**

We write each factor in (3.2) as
\[
p(x_t|x_{[\infty,0]}) = p(x_t|x_{[t-N,t-1]}) + \sum_{|s-t|N} \delta_{s-t}(x_{[s,t]})
\] (3.14)
where $N$ is an integer to be chosen later. Insert this in the product in (3.2), and expand: we get
\[
p(x_T|x_{[\infty,0]}) = \sum_{I \subset \{1,T\}} \sum_s \prod_{t=1}^{T-1} dx_t \prod_{t \in I} \delta_{s,t}(x_{[s,t]}) \prod_{t \notin I} p(x_t|x_{[t-N,t-1]} \cup \mathbf{0}),
\] (3.15)
where the sum over subsets $I$ corresponds to the choice in (3.14) between the first term and the sum, while the sum over $s = (s_t)_{t \in I}$ corresponds to the possible choices of a term in that sum.
Now, let \( T = \bigcup_{t \in I} [s, t] \) and let

\[
[1, T] \setminus T = \bigcup_i J_i \bigcup_{\alpha} I_\alpha, \tag{3.16}
\]

where each \( J_i \) is a union of intervals of length \( N \), containing at least two such intervals, and each \( I_\alpha \) is an interval of length less than \( 2N \) between two connected intervals in \( T \) or an interval of length less than \( N \) between an interval in \( T \) and an interval \( J_i \). The reason for these definitions is that, in the RHS of (3.15), the only functions depending on \( x_s \), with \( s \) in the complement of \( I \), are the factors \( p(x_t | x_{[t-N,t-1]}) \), so that, by integrating over these variables, one can obtain the transition probabilities of the Markov chain defined in condition b) above. For that, we need intervals of length at least \( 2N \), which are the \( J_i \)'s, while the intervals \( I_\alpha \)'s simply cover the leftover sites.

Since the model here is translation invariant, let us fix one interval \( J_i = J \), and write it as a union of disjoint intervals of length \( N \):

\[
J = \bigcup_{l=0}^{n-1} K_l = [t + 1 + lN, t + (l + 1)N].
\]

We have, by definition (3.7) of the transition probability \( P \):

\[
\int \prod_{l=1}^{n-1} d\mathbf{x}_{K_l} \prod_{l=1}^{n} \prod_{t \in K_l} (p(x_t | x_{[t-N,t-1]} \lor 0)) = P^n(\mathbf{x}_{K_n} | \mathbf{x}_{K_0}). \tag{3.17}
\]

Now write this as

\[
P^n(\mathbf{x}_{K_n} | \mathbf{x}_{K_0}) - P(\mathbf{x}_{K_n}) + P(\mathbf{x}_{K_0}), \tag{3.18}
\]

where \( P \) is defined by (3.9). Apply this to each interval \( J_i \) in (3.16), with \( n \) replaced by \( n_i = \frac{1}{N} - 1 \). Insert that identity in (3.15) for each \( J_i \) and expand the corresponding product over \( i \) of \( A_i + B_i \), where \( A = P^n(\mathbf{x}_{K_n} | \mathbf{x}_{K_0}) - P(\mathbf{x}_{K_n}) \) and \( B = P(\mathbf{x}_{K_n}) \).

For \( E \) as in (3.1), integrate over \( E \) each term in the resulting expansion, and write

\[
p(x_T \in E | \mathbf{x}_{[-\infty,0]}) = Q + R, \tag{3.19}
\]

where \( Q \) collects all the terms in the resulting sum where at least one factor \( P(\mathbf{x}_{K_{n_i}}) \) appears and \( R \) all the rest. Now, the presence of one such factor \( P \) 'decouples' \( x_T \) from the initial conditions \( \mathbf{x}_{[-\infty,0]} \), in the sense that, if we consider the difference

\[
p(x_T \in E | \mathbf{x}_{[-\infty,0]}) - p(x_T \in E | \mathbf{x}'_{[-\infty,0]}), \tag{3.20}
\]

for two different past histories, then the \( Q \) sums are equal and only the \( R \) sums contribute to the difference. Indeed, fix a \( K_{n_i} \) and consider all the terms in our expansions where the factor \( P(\mathbf{x}_{K_{n_i}}) \) appears; let \( t_0 \) be the last time before the interval \( K_{n_i} \). By construction, in all the terms under consideration, all the functions
that depend on \( x_t \), for \( t > t_0 \) do not depend on the variables \( x_t \), for \( t \leq t_0 \). So, if we resum, in the expansion, all the terms depending on the variables \( x_t \), for \( t \leq t_0 \), we obtain, for the two terms in (3.20), \( p(x_t_0|x_{-\infty,0}) \), and \( p(x_{t_0}|x_{-\infty,0}) \) (we simply use (3.15) read from right to left, with \( T \) replaced by \( t_0 \)). But performing in (3.2) the integral over \( x_t_0 \) gives 1 in both cases, which shows that the difference between the respective sums cancel.

So, if we show that, \( \exists C < \infty, m > 0 \) such that

\[
|R| \leq Ce^{-mT},
\]

\( \forall x_{-\infty,0} \), we obtain that the absolute value of (3.20) is exponentially small and, from that, (3.1) easily follows.

Using the bound (3.6) on \( \delta_{s,t} \) and (3.13) on

\[
\int \prod_{t \in K_n} dx_t |P^n(x_{K_n}|x_{K_0}) - P(x_{K_n})|,
\]

(3.22)

and the fact that, by (3.4) and \( x_t \in [0,1] \), all the integrals are bounded by 1, we get:

\[
|R| \leq \sum_I \sum_s \prod_{t \in I} (Ce^{-m|t-s_t|}) \prod_i (2(1-\delta)^{n_i}),
\]

(3.23)

where the second product runs over the intervals \( J_i \) in (3.16), and where \( n_i = \frac{|J_i|}{N} - 1 \). Note that the length of each \( I_\alpha \) in (3.11) is less than \( 2N \) and, since such intervals are always adjacent to a connected component of \( \overline{T} \) (unless \( I = \emptyset \), in which case this number is at most 2), the number of intervals \( I_\alpha \) is less than \( 2|I| + 2 \); the same bound holds for the number of intervals \( J_i \) in (3.16) (in fact, a better bound holds here, but we won’t use it). So, we have:

\[
\sum_i n_i \geq \sum_i \frac{|J_i|}{N} - (2|I| + 2) \geq \frac{(T - |\overline{T}|)}{N} - c|I| - 2,
\]

(3.24)

for some number \( c \), where, in the second inequality, we use \( |I_\alpha| \leq 2N \) and (3.11).

Using this, we can, by changing the constant \( C \), bound (3.23) by:

\[
C \sum_I \sum_s \prod_{t \in I} e^{-m|t-s_t|} C^{|I|} (1 - \delta)^{(T - |\overline{T}|)/N}
\]

(3.25)

Since by definition of \( \overline{T} \), \( \sum_{t \in I} |t - s_t| \geq |\overline{T}| \), we can, by considering separately the terms where \( |\overline{T}| \leq \frac{T}{2} \), and those where \( |\overline{T}| > \frac{T}{2} \), bound the sum in (3.25) by

\[
Ce^{-\tilde{m}T} \sum_I \sum_s \prod_{t \in I} e^{-\frac{T}{2}|t-s_t|} C^{|I|}
\]

(3.26)

where

\[
\tilde{m} = \min \left( \frac{m}{4}, \frac{-\ln(1-\delta)}{2N} \right).
\]

(3.27)
Now, choose $N$ so that
\[ \sum_{|t-s| > N} e^{-\frac{m}{2}|t-s|} \leq \eta, \tag{3.28} \]
with
\[ (1 + C\eta) \leq e^{\tilde{m}/2}, \tag{3.29} \]
which is possible since, from (3.28) we see that, for large $N$, $\eta = \exp(-\mathcal{O}(N))$ while, from (3.27), $\tilde{m} = \mathcal{O}(N^{-1})$.

We use (3.28) to control the sum over each $s_t$ in (3.26), and we get
\[ (3.26) \leq Ce^{-\tilde{m}T} \sum_{I \subset [1,T]} (C\eta)^{|I|} \leq C(1 + C\eta)^T e^{-\tilde{m}T} \tag{3.30} \]
and, using (3.28), we get (3.21) with $m = \frac{\tilde{m}}{2}$.

This completes the proof of Proposition 3.1.

**Remark 3.** By considering (3.27, 3.28, 3.29), we see that one can extend the proof to a situation where $\delta$ in (3.8) depends on $N$, as long as $\delta \geq \exp(-cN)$ for a constant $c$ small enough.

Now, let us turn to the real model, and make a list of the difficulties not present in the toy model. The first one is that time is continuous rather than discrete, but that is a minor problem. We can easily introduce a discretization of time. A more serious problem is that one deals with what are called “unbounded spins” in statistical mechanics or what is also known as a “large field problem”, namely the variables $s(\tau)$ in (2.9), which play a role similar to the variables $x_t$, here, take value in $\mathbb{R}^N$ rather than $[0,1]$ (actually, if we consider the variable $s$ over a unit time interval, they take values in a space of functions from that interval into $\mathbb{R}^N$). And, what really causes a problem, is the fact that the bounds (3.6), (3.8) do not hold when the variables $s$ take large values. However, as we shall see in the next section, this is unprobable. Thus, before doing an expansion as in (3.15, 3.18), we must first distinguish between time intervals where the $s$ variables are large and those where they are small. Then, putting aside lots of technicalities, we perform the expansion (3.15) in the latter intervals and use estimates like (4.2) below to control the sum over the intervals where $\omega$ is large.

Finally, there is an additional difficulty coming from the fact that the definition of the probabilities here involve a Girsanov representation. In statistical mechanics, one usually deals with situations where the probabilities (3.3) can be written as:
\[ p(x|_{-\infty,t-1}) = \exp(\sum_{t \in I} \phi_I(x_I)), \tag{3.31} \]
where the $\{\phi_I\}$’s represent “many body interactions’ (suitably normalized so that (3.4) holds) and the sum runs over intervals $I \subset \mathbb{Z}$ whose last point is $t$. Then, a bound of the form
\[ \|\phi_I\|_\infty \leq C \exp(-m|I|), \tag{3.32} \]
with $C < \infty$, $m > 0$, is enough to obtain (3.6) and (3.8). But here the probabilities are not of that form, because of the stochastic integral $\int_0^t f(\tau)\gamma^{-1}d\xi(\tau)$ in (2.9).

4. A priori estimates on the transition probabilities

The memory in the process (2.7) is coming from the dependence of the solution of (2.2) on its initial conditions. By Proposition 2.1, the dependence is weak if $\int_0^t \|\nabla \omega\|^2$ is less than $cRt$ for a suitable $c$. It is convenient to define, for each unit interval $[n-1, n] \equiv n$, a quantity measuring the size of $\omega$ on that interval by:

$$D_n = \frac{1}{2} \sup_{t \in n} \|\omega(t)\|^2 + \int_n \|\nabla \omega(t)\|^2 dt. \quad (4.1)$$

The following Proposition bounds the probability of the unlikely event that we are interested in:

**Proposition 4.1.** There exist constants $c > 0$, $c' < \infty$, $\beta_0 < \infty$, such that for all $t, t', 1 \leq t < t'$ and all $\beta \geq \beta_0$,

$$P\left(\sum_{n=t}^{t' - 1} D_n(\omega) \geq \beta R|t' - t| \omega(0)\right) \leq \exp\left(\frac{1}{R} c' e^{-t} \|\omega(0)\|^2\right) \exp\left(-c\beta|t' - t|\right). \quad (4.2)$$

**Remark 1.** This means that the probability that $\omega$ is large over an interval of time decays exponentially with the length of that interval, provided that $\|\omega(0)\|$ is not too large. And, if $\|\omega(0)\|$ is of order $K$, $D_n(\omega)$ will be, with large probability, of order $R$ after a time of order $\log K$.

The main idea in the proof is a probabilistic analogue of the so-called enstrophy balance: in the deterministic case, using integration by parts and $\nabla \cdot u = 0$, on derives from (1.3) with $db = 0$, the identity:

$$\frac{1}{2} \frac{d}{dt} \|\omega\|^2 = -\|\nabla \omega\|^2,$$

which implies that the enstrophy ($\|\omega\|^2$) decreases in time. This basic property of equation (1.3) makes the proof of the following Lemma rather simple.

**Lemma 4.1.** For all $\omega(0) \in L^2$, and all $t \geq 0$,

$$E\left[e^{\frac{1}{2||\omega(0)||^2} \|\omega(t)\|^2} \bigg| \omega(0)\right] \leq 3e^{\frac{1}{2||\omega(0)||^2}}, \quad (4.3)$$

and

$$P(\|\omega(t)\|^2 \geq D|\omega(0)\|) \leq 3e^{-\frac{D}{3}e^{\frac{1}{2||\omega(0)||^2}}}. \quad (4.4)$$
Proof. Let \( x(\tau) = \lambda(\tau)\|\omega(\tau)\|^2 = \lambda(\tau) \sum_k |\omega_k|^2 \) for \( 0 \leq \tau \leq t \). Then by Ito’s formula (remember that, by (1.6), \( \sum_k \gamma_k = R \) and thus \( \gamma_k \leq R, \forall k \)):

\[
\frac{d}{d\tau} E[e^x] = E[(\dot{\lambda}\lambda^{-1} x - 2\lambda \sum_k k^2 |\omega_k|^2 + \lambda \sum_k \gamma_k + 2\lambda^2 \sum_k \gamma_k |\omega_k|^2) e^x]
\]

\[
\leq E[(\dot{\lambda}\lambda^{-1} - 2 + 2\lambda R) x + \lambda R) e^x],
\]

(4.5)

where \( E \) denotes the conditional expectation, given \( \omega(0) \), and where we used the Navier-Stokes equation (1.3), \( |k| \geq 1 \) for \( \omega_k \neq 0 \), and the fact that the nonlinear term does not contribute (using integration by parts and \( \nabla \cdot u = 0 \)). Take now \( \lambda(\tau) = \frac{1}{4} e^{(\tau-t)} \) so that \( \lambda \leq \frac{1}{4} R, \dot{\lambda}\lambda^{-1} = 1, \dot{\lambda}\lambda^{-1} - 2 + 2\lambda R \leq -\frac{1}{2} \) and \( \lambda R \leq \frac{1}{4} \). So,

\[
\frac{d}{d\tau} E[e^x] \leq E[(1 - \frac{1}{2} x) e^x] \leq \frac{1}{2} - \frac{1}{4} E[e^x],
\]

where the last inequality follows by using \( (1 - 2x)e^x \leq 2 - e^x \). Thus, Gronwall’s inequality implies that:

\[
E[e^{x(\tau)}] \leq e^{-\frac{1}{2} \tau} e^{x(0)} + 2 \leq 3e^{x(0)},
\]

i.e., using the definition of \( \lambda(\tau) \),

\[
E\left[ \exp\left( \frac{\epsilon^t}{4R} \|\omega(\tau)\|^2 \right) \right] \leq 3 \exp\left( \frac{\epsilon^t}{4R} \|\omega(0)\|^2 \right).
\]

This proves (4.3) by putting \( \tau = t \); (1.4) follows from (4.3) by Chebychev’s inequality.

Since the \( D_n \) in (1.2) is the supremum over unit time intervals of

\[
D_t(\omega) = \frac{1}{2} \|\omega(t)\|^2 + \int_{n-1}^{n} \|\nabla \omega\|^2 d\tau \quad n - 1 \leq t \leq n,
\]

(4.6)

which does not involve only \( \|\omega(t)\|^2 \), we need to control also the evolution of \( D_t(\omega) \) over a unit time interval, taken, for now, to be \([0,1]\). From the Navier-Stokes equation (1.3) and Ito’s formula, we obtain

\[
D_t(\omega) = D_0(\omega) + Rt + \int_0^t (\omega, db)
\]

(4.7)

(since the nonlinear term does not contribute, as in (1.5)).

Our basic estimate is:

**Lemma 4.2.** There exist \( C < \infty, c > 0 \) such that, \( \forall A \geq 3D_0(\omega) \)

\[
P( \sup_{t \in [0,1]} D_t(\omega) \geq A|\omega(0)|) \leq C \exp(-\frac{cA}{R}).
\]

(4.8)
Remark 2. While the previous Lemma showed that $\|\omega(t)\|^2$ tends to decrease as long as it is larger than $O(R)$, this Lemma shows that, in a unit interval, $D_t(\omega)$ does not increase too much relative to $D_0(\omega) = \frac{1}{2}\|\omega(0)\|^2$. Thus, by combining these two Lemmas, we see that $D_n(\omega) = \sup_{t \in [n-1, n]} D_t(\omega)$ is, with large probability, less than $\|\omega(0)\|^2$, when the latter is larger than $O(R)$, at least for $n \geq n_0$ not too small. Thus, it is unlikely that $D_n(\omega)$ remains much larger than $R$ over some interval of (integer) times, and this is the basis of the proof of Proposition 4.1.

Without entering into details, here are the main ideas in the proof of (4.8). From (4.7), we see that it is enough to get an upper bound on

$$P \left( \sup_{t \in [0,1]} | \int_0^t (\omega, db) | \geq (A - D_0 - R) | \omega(0) | \right) \tag{4.9}$$

We use (see [3] for more details) Doob’s inequality (see e.g. [27], p.24), to reduce the control over the supremum over $t$ to estimates on $| \int_0^1 (\omega, db) |$. Letting $E$ denote the conditional expectation, given $\omega(0)$, and using Novikov’s bound (see e.g. the proof of Lemma 5.2 below), we get

$$E(e^{+\varepsilon \int_0^1 (\omega, db)}) \leq \left( E(e^{2\varepsilon^2 \int_0^1 d\tau (\omega(\tau), \gamma \omega(\tau))}) \right)^{1/2}$$

$$\leq \left( \int_0^1 d\tau E(e^{2\varepsilon^2 (\omega(\tau), \gamma \omega(\tau))}) \right)^{1/2} \leq \left( \int_0^1 d\tau E(e^{2\varepsilon^2 R ||\omega(\tau)||^2}) \right)^{1/2} \tag{4.10}$$

where the last two inequalities follow from Jensen’s inequality, applied to $e^{2\varepsilon^2 \int_0^1 d\tau (\omega(\tau), \gamma \omega(\tau))}$, and from $\gamma_k \leq R$ (see (1.6)). Now, choosing $\varepsilon$ so that $2\varepsilon^2 R = \frac{1}{4\gamma}$, i.e. $\varepsilon = \frac{1}{\sqrt{8\gamma R}}$, we can use (4.4) to bound the RHS of (4.10). Combining this with Chebychev’s inequality gives bounds on (4.9).

5. Decoupling estimates

In this section, I shall give a very brief sketch of the ideas used to prove the analogue of assumptions (3.3) and (3.8) of section 3 in the present setting, at least in the probable regions where $\omega$ is small. The main point is to understand the analogue of the bound (3.32), which expresses the exponential decay of interactions. What plays the role of the right hand side of (3.31) is, see (2.9):

$$g_t \equiv e^{f_{t-1}^{s-1}(f(\tau), \gamma^{-1}(ds(\tau) - \frac{1}{2}f(\tau)d\tau))} \tag{5.1}$$

where, for simplicity, I consider a unit time interval $[t - 1, t]$. We want to show that this depends weakly on the past; so consider two functions $g_1, g_2$, defined in terms of two functions $f_1, f_2$, themselves defined through different $l_1$ and $l_2$ (see (2.4, 2.7)). And, by analogy with what we did in section 3, we choose $l_1 = l(t, s([0, t]), l(0) = 0), l_2 = l(t, s([1, t]), l(1) = 0)$, i.e. we set the large $k$ modes equal to zero at different times (0 or 1). Using (2.8), we see that $l_1 = l(t, s([1, t]), l_1(1)),$
with \( l_1(1) = l_1(s([0, 1]), l(0) = 0 \), so that we have, at time \( t = 1 \), two initial conditions, \( l_1(1), l_2(1) = 0 \), with \( \|l_1(1) - l_2(1)\|_2 = \|l_1(1)\|_2 \) of order one, if \( \omega \) is small in the interval \([0, 1]\).

Now, if \( g_1 \) depends weakly on the past, it should mean that, for large \( t \), \( g_1 \) and \( g_2 \) are, in some sense, exponentially close. To measure the difference, write:

\[
g_1 - g_2 = (1 - \frac{g_2}{g_1})g_1 \equiv (1 - H)g_1,
\]

(5.2)

which will be convenient, since we deal with unbounded variables for which \( \sup \) norm estimates like in (3.32) are not available. Explicitly:

\[
H = e^{\int_{t=1}^t (\delta f(t), \gamma^{-1}(ds(t) - f_1(t) dt)) - \int_{t=1}^t (\delta f(t), \gamma^{-1} \delta f(t)) dt}
\]

(5.3)

where \( \delta f = f_2 - f_1 \). What we want to show is that \( 1 - H \) is, in a suitable sense, small.

The next Lemma gives a bound on \( \|\delta f\| \) in terms of \( \|\delta l\| \), and \( \|\omega\| \); \( \|\delta l\| \) is controlled by Proposition 2.1, provided that \( \omega \) is small, in the sense discussed in section 4, in which case \( \|\omega\| \) is also controlled, using \( \sup_{t \in \mathbb{N}} \|\omega(t)\| \leq (2D_n)^\frac{1}{2} \).

**Lemma 5.1.** Let \( f(\omega) = PF(\omega) \) and \( \omega = s + l, \omega' = s + l' \). Then,

\[
\|\delta f\| = \|f(\omega) - f(\omega')\| \leq C(R)(2\|\omega\|\|\delta l\| + \|\delta l\|^2)
\]

(5.4)

with \( \delta l = l - l' \) and \( C(R) \) a constant depending on the parameter \( R \) (see (1.4)).

**Proof.** We have

\[
|f_k(\omega) - f_k(\omega')| \leq \sum_p |\omega_{k-p}\omega_p' - \omega'_{k-p}\omega_p| \frac{|k|}{|p|}
\]

which, since \( |k| \leq \sqrt{\kappa R} \) is bounded by

\[
\sqrt{\kappa R} \sum_p |s_{k-p}\delta l_p + s_p\delta l_{k-p} + l_p l_{k-p} - l'_p l'_{k-p}|,
\]

(5.5)

Writing \( l_p l_{k-p} - l'_p l'_{k-p} = l_p \delta l_{k-p} + l_{k-p} \delta l_p - \delta l_p \delta l_{k-p} \) and using Schwarz’ inequality, we get

\[
(5.3) \leq \sqrt{\kappa R}(2\|\omega\|\|\delta l\| + \|\delta l\|^2)
\]

which proves (5.4), since \( f_k \neq 0 \) only for \( k \leq \kappa R \), so that the sum in the \( L^2 \) norm \( \|\delta f\| \) runs over \( C(R) \) terms.

This Lemma would be enough to control \( (1 - H) \) if we had only in (5.3) the factor \( e^{-\frac{1}{2} \int_{t=1}^t (\delta f(t), \gamma^{-1} \delta f(t)) dt} \), which involves only ordinary integrals.

To control the stochastic integral, it is convenient to undo the Girsanov transformation, i.e. to change variables from \( s \) back to \( b \). Let \( E \) denote the expectation...
with respect to the Brownian motion $b$ with covariance $\gamma$ on the time interval $[t-1, t]$. We get, using (2.5):

$$H = e^{\int_{t-1}^t (\delta f(t), \gamma^{-1} db(t)) - \frac{1}{2} \int_{t-1}^t (\delta f(t), \gamma^{-1} \delta f(t)) dt}.$$ (5.6)

Write now $(1 - H)^2 = 1 - 2H + H^2$; to give a flavour of the estimates, let us see how one could show that the expectation with respect to $E$ of $-2H + H^2$ is close to 1, i.e. that the expectation of $(1 - H)^2$ is close to zero. One can rather easily bound from below the expectation of $H$, using Jensen’s inequality; to get an upper bound on the expectation of $H^2$, one uses:

Lemma 5.2. Let $\zeta(t) \in C([0, 1], H_s)$ be progressively measurable. Then

$$Ee^{\int_0^t (\zeta, \gamma^{-1} db(t)) + \lambda \int_0^t (\zeta, \gamma^{-1} \zeta) dt} \leq e^{2(1+\lambda)||\zeta||^2 \rho^{-1}}$$ (5.7)

where $||\zeta|| = \sup_{\tau} ||\zeta(\tau)||$.

Proof. This is just a Novikov bound: we bound the LHS, using Schwarz’ inequality, by

$$\left( Ee^{\int_0^t (2\zeta, \gamma^{-1} db) - 2\int_0^t (\zeta, \gamma^{-1} \zeta) dt} \right)^{\frac{1}{2}} \left( Ee^{2(1+\lambda) \int_0^t (\zeta, \gamma^{-1} \zeta) dt} \right)^{\frac{1}{2}}$$

and note that the expression inside the first square root is the expectation of a martingale and equals one.

We can then apply this Lemma to $\zeta = 2\delta f$, $\lambda = -\frac{1}{4}$, replacing $[0, 1]$ by $[t-1, t]$, and use the estimates coming from Lemma 5.1 and Proposition 2.1 to show that the RHS of (5.7) is exponentially close to 1, for $t$ large. This gives a rough idea of why the “interactions” here are exponentially decaying, but it must be said that the full story is far more complicated and I refer to reader to [3] for more details.

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