ABSTRACT

In this work, a diffusive eco-epidemiological model where the predator’s population consume their species to survive. The proposed model is studied from both points of view, theoretical and numerical. Firstly, we deal with the behaviour of the constant positive steady state then nonconstant positive steady state. Sufficient conditions for local asymptotic stability and global asymptotic stability for a constant positive steady state are derived by linearization and Lyapunov function technique. Prior estimates of the positive steady state given, conditions are obtained for the non-existence of non-constant positive solution, by Cauchy and Poincaré inequality. The existence of non-constant positive steady states is studied by Leray-Schauder degree theory. These results indicate the importance of the large diffusivity which is responsible for the appearance and non-appearance of stationary patterns. We have discussed Turing instability, which ensures the existence of Turing patterns. Further, the effect of the cannibalistic attack rate and disease transmission rate observed on the dynamics of the proposed model system. Even it followed that in the absence of cannibalism, the spatial distribution is not possible and increment in the cannibalistic attack rate and disease transmission rate promote the Turing patterns. The similar effect observed for the disease transmission rate. Further, we have calculated Lyapunov exponents and saw the chaotic and stable fixed dynamics for various parameter settings. In the last, we have performed extensive numerical simulation and obtained Turing and spatiotemporal patterns.

Keywords cannibalism · local stability · global stability · nonconstant positive steady state · priori estimates · Turing instability and Turing patterns · spatiotemporal patterns · Lyapunov exponents.

1 Introduction

Cannibalism is a widespread phenomenon which takes place when resources are scarce, and population densities are high. It is a process of killing and consumption of full or a part of an individual of same species (conspecifics). The classical work of Polis [21], cited more than 1300 species amongst which it occurs. Bacteria, protozoans, invertebrates, and vertebrates, including human, practised cannibalism. Ecological, social and psychological circumstances greatly influence the dynamic cannibalistic behaviour of interacting species. Santosh et al. [2] observed cannibalism as a self-regulatory mechanism to control disease transmission among the predator species. In 2009, Sun et al. [25] studied the spatial patterns in the distribution of organism where they consider cannibalism in the predator population and found spatial pattern formation greatly influenced by cannibalism. This work reveals a connection between pattern formation and cannibalism.

Diffusion gives rise to a pattern formation phenomenon. Diffusion plays an important role in population biology. Starting from the Turing seminal work [27], Brusselator model [3], Gierer-Meinhardt model [10, 30]. Sel’kov model [5, 29], Lotka-Volterra predator-prey model [17, 6] and the references therein, self diffusion and cross-diffusion are
observed as a causes of stationary patterns. These patterns arise due to the diffusion driven stability which is also called as Turing instability studied by many authors [11, 23, 16, 12]. The constant positive solution of the temporal model system (ODE) indicates the homogeneous distribution of the species. While, in the case of spatially inhomogeneous, the existence of non-constant positive solution also called stationary pattern indicates the rich dynamic of a spatiotemporal model system (PDE). Existence and non-existence results of non-constant positive steady state demonstrate stationary patterns for certain regions of diffusion coefficients. Pang and Wang [18] studied two species reaction-diffusion system, through the analysis they observed that the resulted patterns are caused by diffusion. Chen and Wang [4] studied a two species diffusive model with the Beddington-DeAngelis functional response and using topological degree theory, they have established the existence of the non-constant steady state of the model system. This non-constant steady-state phenomenon has been studied even on cross diffusive models [32], [19], [26], [31], [13], food chain and food web models [23], [20], [7], epidemic model [8] and eco-epidemiological models [33], [11].

Here we have studied a diffusive eco-epidemiological model to understand the influence of cannibalism and disease on pattern formation as well as the effect on dynamics we have observed. Also we have calculated the Lyapunov exponents to observe the chaotic dynamics.

The organisation of the manuscript is as follows. In section 2 we have proposed our spatiotemporal model system. In this section we have studied even on cross diffusive models [32], [19], [26], [31], [13], food chain and food web models [23], [20], [7], epidemic model [8] and eco-epidemiological models [33], [11].

In the last section 9, a detailed discussion about various results obtained throughout this article presented.

### 2 Model formulation

In this work, we have considered a temporal model system for studying the concept of non-constant steady states of a spatiotemporal model system. The temporal model proposed by Santosh et al. [2] with the assumption that disease spreads into predator species only. Therefore the predator species divided into susceptible predator and infected predator. Since the predators have cannibalistic behaviour, thus, the disease spreads among them. Santosh et al. [2] proposed the following model system:

\[
\begin{align*}
\frac{du}{dt} &= ru \left(1 - \frac{u}{k}\right) - \frac{(\alpha_1 v + \alpha_2 w)u}{\gamma + u} \equiv G_1(u, v, w), \\
\frac{dv}{dt} &= \frac{\alpha_1 (\alpha_1 v + \alpha_2 w)u}{\gamma + u} + c_1 \sigma (\beta v + w) v + c_2 \sigma (\beta v + w) w - \sigma (\beta v + w) v - \sigma l f v w - \lambda v w - d v \equiv G_2(u, v, w), \\
\frac{dw}{dt} &= \lambda v w + \sigma l f v w - \sigma (\beta v + w) w - (d + c) w \equiv G_3(u, v, w),
\end{align*}
\]

where \(u(t)\) is prey population density, \(v(t)\) is susceptible predator population density and \(w(t)\) is infected predator population density. Predator population follows Holling type II functional response with a half-saturation constant \(\gamma\). The biological meaning of the other parameters given in table [1].

Taking into account the inhomogeneous distribution of the species at different spatial locations within a fixed bounded domain \(\Omega \in \mathbb{R}^2\) with a smooth boundary at any given time, and the dynamics of the population under such environment, we are led to the following spatiotemporal (reaction-diffusion) system:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= ru \left(1 - \frac{u}{k}\right) - \frac{(\alpha_1 v + \alpha_2 w)u}{\gamma + u} + d_1 \Delta u, \\
\frac{\partial v}{\partial t} &= \frac{\alpha_1 (\alpha_1 v + \alpha_2 w)u}{\gamma + u} + c_1 \sigma (\beta v + w) v + c_2 \sigma (\beta v + w) w - \sigma (\beta v + w) v - \sigma l f v w - \lambda v w - d v + d_2 \Delta v, \\
\frac{\partial w}{\partial t} &= \lambda v w + \sigma l f v w - \sigma (\beta v + w) w - (d + c) w + d_3 \Delta w.
\end{align*}
\]
The boundary and initial conditions are given by,
\[
\begin{cases}
\frac{\partial u}{\partial v} = \frac{\partial v}{\partial u} = \frac{\partial w}{\partial y} = 0, \\
u(x, 0) > 0, v(x, 0) > 0, w(x, 0) > 0,
\end{cases}
\]
where, \(x \in \partial \Omega,\)
\[\] \(x \in \Omega.\)

In the above, system defined in a bounded domain \(\Omega \in \mathbb{R}^N\) with positive initials and zero-flux boundary conditions. Where \(d_1, d_2\) and \(d_3\) are diffusion coefficients, \(v\) is the outward unit normal vector of the boundary \(\partial \Omega\). \(\Delta \equiv (\partial^2 / \partial x^2 + \partial^2 / \partial y^2)\) is the Laplacian operator in two dimensional space.

From [2], it is observed that the temporal system (1) has a constant positive solution, \(u^* = (u^*, v^*, w^*)^T\), where \(w^* = -\frac{1}{2}((d + e) - (\lambda + \beta f - \sigma) v^*)\), \(u^*\) and \(v^*\) are obtained by solving this system of equations
\[
\begin{align*}
b_1 u^2 + b_2 u + b_3 v + b_4 &= 0, \\
d_1 v^2 + \frac{u v}{\gamma + u} + d_3 v + d_4 \frac{u}{\gamma + u} + d_5 &= 0
\end{align*}
\]
where, \(b_1 = r (\frac{r}{\lambda} - 1)\), \(b_3 = \alpha + \alpha_2 (lf - \beta + \frac{e}{\gamma})\), \(b_4 = -\{r\gamma + \frac{\alpha_2}{\gamma} (d + e)\}\), \(d_1 = c_2 (\lambda + \beta f - \sigma) \frac{e}{\gamma} + (c_1 \alpha - \sigma - \alpha) f - \lambda (\lambda + \beta f - \sigma) + (c_1 \alpha - \sigma - \alpha) f \lambda\), \(d_2 = \alpha (c_1 \alpha - \sigma - \alpha) f - \lambda (\lambda + \beta f - \sigma)\), \(d_3 = -\{c_1 \alpha - \sigma - \alpha) f - \lambda (d + e) + 2c_2 (d + e) (\lambda + \beta f - \sigma) + df\}, \(d_4 = -\alpha (\alpha) (d + e)\), and \(d_5 = c_2 (d + e)^2\).

The constant positive solution or steady state \(u^*\) exists if and only if
\[
(\lambda + \beta f - \sigma) v^* > (d + e).
\]
For the simplicity, taking \(u = (u, v, w)^T\) and \(G(u) = (G_1(u), G_2(u), G_3(u))^T\), then problem (2) can be written in this form
\[
\frac{\partial u}{\partial t} = D \Delta u + G(u),
\]
where \(D = \text{Diag}(d_1, d_2, d_3)\) is a matrix of diffusion coefficients.

### Table 1: List of parameters

| Parameter | Biological meaning |
|-----------|--------------------|
| \(r\)    | Intrinsic growth rate of prey |
| \(k\)    | Carrying capacity of prey |
| \(\lambda\) | Disease transmission rate |
| \(\alpha_1\) | Predation rate of susceptible predator |
| \(\gamma\) | Half-saturation constant |
| \(\alpha\) | Conversion efficiency of predator |
| \(\alpha_2\) | Predation rate of infected predator |
| \(d\) | Natural death rate of predator |
| \(e\) | Additional disease-related mortality rate |
| \(\sigma\) | Attack rate due to cannibalism |
| \(c_1\) | Conversion rate of predator for cannibalism of susceptible predator |
| \(\beta\) | Dimensionless quantity |
| \(l\) | Probability of transmission for cannibalistic interaction |
| \(c_2\) | Conversion rate of predator for cannibalism of infected predator |
| \(f\) | Number of predator sharing one conspecific predator |

### 3 Stability of the constant positive steady state

In this section, we have discussed the local stability of constant positive solution \(u^* = (u^*, v^*, w^*)^T\) of [2]. But before that we will set up these useful notations.
Let $0 = \mu_0 < \mu_1 < \mu_2 < \mu_3 < \cdots$ be the eigenvalues of the operator $-\Delta$ on $\Omega$ with the homogeneous Neumann boundary condition, and $E(\mu_i)$ be the eigenspace corresponding to $\mu_i$ in $C^1(\bar{\Omega})$. Let

$$X = \{ u \in [C^1(\bar{\Omega})]^3 ] \mid \partial \nu u = 0 \text{ on } \partial \Omega \},$$

$\{ \phi_{ij}, j = 1, \ldots, \dim E(\mu_i) \}$ be an orthonormal basis of $E(\mu_i)$, and $X_{ij} = \{ c \phi_{ij} \mid c \in \mathbb{R}^3 \}$. Then,

$$X = \bigoplus_{i=1}^{\infty} X_i \quad \text{and} \quad X = \bigoplus_{j=1}^{\dim E(\mu_i)} X_{ij}. $$

**Theorem 3.1.** If the conditions

$$M_1 > D_1, M_2 > E_1, \frac{r}{k} > \frac{4}{\sigma^2},$$

$$u^* > \max \left\{ \frac{1}{w^*(\mu_i - D_1 + \sigma w^*)} \frac{-(M_1 - D_1) + \sqrt{(M_1 - D_1)^2 + 4 \left( \frac{r}{k} - \frac{A}{B^2} \right) \sigma}}{2 \left( \frac{r}{k} - \frac{A}{B^2} \right)} \right\} $$

(7)

where, $A, B, C, M_1, M_2, E_1, D_1$ and $P_i$ are given in (15), hold, then the constant positive solution $u^*$ of the spatiotemporal system (2) is locally asymptotically stable.

**Proof.** Linearization of (2) at constant positive solution $u^*$ is

$$U_t = L U,$$

(8)

where $U = (u_1, v_1, w_1)^T$, and $L = D \Delta + G_u(u^*)$. The jacobian matrix $G_u(u^*)$ is given by

$$G_u(u^*) = \begin{pmatrix}
    a_{11} & a_{12} & a_{13} \\
    a_{21} & a_{22} & a_{23} \\
    a_{31} & a_{32} & a_{33}
\end{pmatrix}$$

(9)

where,

$$a_{11} = -\frac{ru^*}{k} + \frac{(\alpha_1 v^* + \alpha_2 w^*)u^*}{(\gamma + u^*)^2}, \quad a_{12} = -\frac{\alpha_1 u^*}{\gamma + u^*}, \quad a_{13} = -\frac{\alpha_2 u^*}{\gamma + u^*},$$

$$a_{21} = \frac{\alpha(\alpha_1 v^* + \alpha_2 w^*)}{\gamma + u^*} - \frac{\alpha(\alpha_1 v^* + \alpha_2 w^*)u^*}{(\gamma + u^*)^2},$$

$$a_{22} = \frac{\alpha_1 u^*}{\gamma + u^*} + c_1 \sigma(2\beta v^* + w^*) + c_2 \sigma \beta w^* - \sigma(2\beta v^* + w^*) - \sigma(l f w^* - \lambda w^* - d),$$

$$a_{23} = \frac{\alpha_2 u^*}{\gamma + u^*} + c_1 \sigma v^* + c_2 \sigma(\beta v^* + 2w^*) - \sigma v^* - \sigma(l f v^* - \lambda v^*),$$

$$a_{31} = 0, \quad a_{32} = \lambda w^* + \sigma(l f w^* - \beta w^*), \quad a_{33} = -\sigma w^*. $$

For each $i \geq 0$, $X_i$ is invariant under the operator $L$, and $\eta$ is an eigenvalue of $L$ on $X_i$, iff it is an eigenvalue of the matrix $-\mu_i D + G_u(u^*)$.

The characteristic polynomial of the matrix $G_u(u^*)$ can be written as

$$\psi(\xi) = \xi^3 + A_1 \xi^2 + A_2 \xi + A_3,$$

(10)

where,

$$A_1 = (M_1 - D_1) + \left( \frac{r}{k} - \frac{A}{B^2} \right) u^*, $$

$$A_2 = u^* \left( \frac{r}{k} - \frac{A}{B^2} \right) (M_1 - D_1 + \sigma w^* + \sigma w^*(M_1 - D_1)) + P_i w^*(M_2 - E_1) + \frac{\alpha_1 u^* C}{B},$$

$$A_3 = \left( \frac{r}{k} - \frac{A}{B^2} \right) \{ \sigma(M_1 - D_1) + P_i (M_2 - E_1) \} + \frac{\sigma \alpha_1 C}{B},$$

(11)  (12)  (13)
\[ A_1A_2 - A_3 = \left\{ (M_1 - D_1) + \left( \frac{r}{k} - \frac{A}{B^2} \right) u^* \right\} \times \left\{ u^* \left( \frac{r}{k} - \frac{A}{B^2} \right) (M_1 - D_1 + \sigma w^*) \right\} + \sigma w^*(M_1 - D_1) + P_1 w^*(M_2 - E_1) + \frac{\alpha_1 u^* C}{B} \right\} - \left\{ \left( \frac{r}{k} - \frac{A}{B^2} \right) \right\} \times \{ \sigma(M_1 - D_1) + P_1(M_2 - E_1) \} + \frac{\sigma \alpha_1 C}{B} \right\}. \]

Where,

\[ A = \frac{\alpha_1 v^* + \alpha_2 w^*}{\gamma + u^*}, \quad B = \frac{\alpha (\alpha_1 v^* + \alpha_2 w^*) - \alpha (\alpha_1 v^* + \alpha_2 w^*) u^*}{(\gamma + u^*)^2}, \]

\[ D_1 = \frac{\alpha_1 v^*}{\gamma + u^*} + c_1 \sigma (2\beta v^* + w^*) + c_2 \sigma \beta w^*, \quad M_1 = \sigma (2\beta v^* + w^*) \]

\[ + \sigma f w^* + \lambda w^* + d, \quad E_1 = \frac{\alpha_0 u^*}{\gamma + u^*} + c_1 \sigma v^* + c_2 \sigma \beta v^* + 2w^*), \]

\[ M_2 = \sigma v^* + \sigma f v^* + \lambda v^* \quad \text{and} \quad P_1 = \lambda + \sigma f - \sigma \beta. \]

By using the conditions stated in the theorem\([5,1]\) it is easy to verify that \( A_1 > 0, A_2 > 0, A_3 > 0 \) and \( A_1 A_2 - A_3 > 0 \).

The characteristic polynomial of \(-\mu_i D + G_u(u^*)\) is

\[ \phi_i(\eta) = \eta^3 + B_{1i} \eta^2 + B_{2i} \eta + B_{3i}, \]

with

\[ B_{1i} = \mu_i(d_1 + d_2 + d_3) + A_1, \]

\[ B_{2i} = \mu_i^2(d_1 d_2 + d_2 d_3 + d_1 d_3) - \mu_i \{ d_1(a_{22} + a_{33}) + d_2(a_{11} + a_{33}) + d_3(a_{11} + a_{22}) \} + A_2, \]

\[ B_{3i} = \mu_i^3d_1d_2d_3 - \mu_i^2 \{ d_1d_2a_{33} + d_2d_3a_{11} + d_1d_3a_{22} \} + \mu_i \{ d_1(a_{22}a_{33} - a_{23}a_{32}) + d_2(a_{11}a_{33} - a_{13}a_{31}) + d_3(a_{11}a_{22} - a_{12}a_{21}) \} + A_3, \]

where \(a_{ij}\) and \(A_i\) are given in \([9]\) and \([11, 13]\) respectively. From \([16]\) and \([18]\), it follows that \( B_{1i}, B_{2i}, B_{3i} > 0 \). A series of calculations yields

\[ B_{1i}B_{2i} - B_{3i} = M_1 \mu_i^3 + M_2 \mu_i^2 + M_3 \mu_i + A_1 A_2 - A_3, \]

in which:

\[ M_1 = (d_1 d_2 + d_2 d_3 + d_1 d_3)(d_1 + d_2 + d_3) - d_1 d_2 d_3, \]

\[ M_2 = - \{ (a_{11} + a_{22}) \{ d_3(d_1 + d_2 + d_3) + d_1 d_2 \} \} \}

\[ + \{ a_{11} + a_{33} \} \{ d_3(d_1 + d_2 + d_3) + d_1 d_3 \}, \]

\[ M_3 = d_1 [ A_2 - A_1(a_{22} + a_{33}) - (a_{22}a_{33} - a_{23}a_{32}) ] + d_2 [ A_2 - A_1(a_{11} + a_{33}) - a_{11}a_{33} ]

\[ + d_3 [ A_2 - A_1(a_{11} + d_2) - a_{11}a_{22} - a_{12}a_{21}]. \]

From the above we can conclude that \( B_{1i}B_{2i} - B_{3i} > 0 \), for all \( i \geq 0 \). Therefore by using Routh-Hurwitz criterion, for each \( i \geq 0 \), the three roots \( \eta_{i,1}, \eta_{i,2}, \eta_{i,3} \) of \( \phi_i(\eta) = 0 \) all have negative real parts.

Finally, Theorem 5.1.1 of Dan Henry\([9]\) concludes the results. \(\square\)

In the next theorem, we proof the result of global stability of \(u^*\) for \([2]\), which means that the species will be spatially homogeneously distributed as time goes to infinity.

**Theorem 3.2.** If the conditions

\[ \begin{cases} \frac{\alpha_1 v^* + \alpha_2 w^*}{\gamma + u^*} + \frac{\alpha \gamma (\alpha_1 + \alpha_2 w^*)}{2(\gamma + u^*)} \leq \frac{r}{k} \\ \frac{\alpha \gamma (\alpha_1 + \alpha_2 w^*)}{2(\gamma + u^*)} + c_2 \sigma (w' + w^* + \beta) + \alpha \alpha_2 k \leq \sigma \beta + \sigma (1 - c_1) \end{cases} \]

satisfy then the constant positive solution \(u^*\) of the system \([2]\) is globally asymptotically stable.
Proof. Define the Lyapunov function

$$E(t) = \int_{\Omega} \left( u - u^* - u^* \log \frac{u}{u^*} \right) + \int_{\Omega} \left( v - v^* - v^* \log \frac{v}{v^*} \right) + \int_{\Omega} \left( w - w^* - w^* \log \frac{w}{w^*} \right)$$

We note that $E(t) \leq 0$ for all $t > 0$. Furthermore, referring to (2), we compute

$$E'(t) = \int_{\Omega} \left\{ \frac{u - u^*}{u} G_1(u) + \frac{v - v^*}{v} G_2(u) + \frac{w - w^*}{w} G_3(u) \right\} dx$$

$$+ \int_{\Omega} \left\{ \frac{u - u^*}{u} d_1 \Delta u + \frac{v - v^*}{v} d_2 \Delta v + \frac{w - w^*}{w} d_3 \Delta w \right\} dx$$

$$= \int_{\Omega} \left\{ (u - u^*) \left( r \left( \frac{1}{k} - \frac{\alpha_1 v + \alpha_2 w}{\gamma + u} \right) + (v - v^*) \left( \frac{\alpha (\alpha_1 v + \alpha_2 w) u}{(\gamma + u) v} \right) \right) \right\} dx$$

$$+ c_1 \sigma (\beta v + w) + c_2 \sigma \left( \frac{w}{v} \right) (w - \sigma (\beta v + w) - \sigma^f w - \lambda w - d)$$

$$(w - w^*) \left( \lambda v + \sigma^f v - \sigma (\beta v + w) - (d + e) \right) \right\} dx$$

$$- \int_{\Omega} \left\{ \frac{d_1 u^*}{u^2} | \nabla u |^2 + \frac{d_2 v^*}{v^2} | \nabla v |^2 + \frac{d_3 w^*}{w^2} | \nabla w |^2 \right\} dx \triangleq I_1(t) + I_2(t)$$

Because of Neumann boundary condition, it is obvious that:

$$I_2(t) = - \int_{\Omega} \left\{ \frac{d_1 u^*}{u^2} | \nabla u |^2 + \frac{d_2 v^*}{v^2} | \nabla v |^2 + \frac{d_3 w^*}{w^2} | \nabla w |^2 \right\} dx \leq 0$$

Now

$$I_1(t) = \int_{\Omega} \left\{ - \frac{r}{k} (u - u^*) 2 - \frac{\alpha_1}{(\gamma + u)} (u - u^*) (v - v^*) - \frac{\alpha_2}{(\gamma + u)} (u - u^*) (w - w^*) \right\} dx$$

$$+ \left( \frac{\alpha_1 v + \alpha_2 w}{\gamma + u} \right) (u - u^*) 2 + \left( \frac{\alpha_2 w}{\gamma + u} \right) (u - u^*) (v - v^*)$$

$$+ c_2 \sigma \left( \frac{w}{v} \right) - \sigma (1 - c_1) - c_2 \sigma \beta + c_2 \sigma (v - v^*)^2 + \left( \lambda + \sigma^f v - \sigma \beta (v - v^*) (w - w^*) \right)$$

$$- \sigma (w - w^*) \right\} dx$$

Using the inequality $a^2 + b^2 \geq 2ab$ and by Theorem 4.3, we derive for $t > T$ that:

$$I_1(t) \leq \int_{\Omega} \left\{ \left( \frac{\alpha_1 v^* + \alpha_2 w^*}{\gamma + u^*} + \frac{\alpha_1 \gamma}{(\gamma + u^*)} + \frac{\alpha_2 \gamma w^*}{(\gamma + u^*)} \right) (u - u^*) 2 + \left( \frac{\alpha_1 \gamma}{(\gamma + u^*)} + \frac{\alpha_2 \gamma w^*}{(\gamma + u^*)} \right) (u - u^*) (v - v^*) \right\} dx$$

$$+ \left( c_2 \sigma \left( \frac{w}{v} \right) - \sigma (1 - c_1) - c_2 \sigma \beta + c_2 \sigma (v - v^*) ^2 + \left( \lambda + \sigma^f v - \sigma \beta (v - v^*) (w - w^*) \right) \right\} dx$$

$$- \sigma (w - w^*) \right\} dx$$

Here, $w' = \max w$, under the assumptions (19), we obtain that $I_1(t) \leq 0$, and thus

$$E'(t) = I_1(t) + I_2(t) \leq 0,$$

and the equality holds if $(u, v, w) = (u^*, v^*, w^*)$. The proof is completed. \qed
### 4 Non-constant positive steady states

In this section, we have obtained the conditions for the non-existence and existence of the non-constant positive solutions of the following elliptic system:

\[
\begin{cases}
-d_1 \Delta u = ru \left(1 - \frac{u}{k}\right) - \frac{(\alpha_1 v + \alpha_2 w)u}{\gamma + u}, \\
-d_2 \Delta v = \frac{\alpha(\alpha_1 v + \alpha_2 w)u}{\gamma + u} + c_1 \sigma (\beta v + w)v \\
+ c_2 \sigma (\beta v + w)v - (\beta v + w)v - \sigma (\beta v + w)v - \sigma (\beta v + w)v - (d + e)w, \\
-d_3 \Delta w = \lambda vw + \sigma (\beta v + w)v - \sigma (\beta v + w)v - (d + e)w, \\
\frac{\partial u}{\partial v} = \frac{\partial v}{\partial v} = \frac{\partial w}{\partial v} = 0. \quad x \in \partial \Omega.
\end{cases}
\]

(20)

For the purpose, we are going to establish a priori upper and lower bounds for positive solutions of the problem (20).

#### 4.1 A priori estimates

The main purpose of this subsection is to give a priori upper and lower bounds for the positive solutions of (20). For that, we have used two important results: Harnack Inequality and Maximum Principle. Which are due to Lin et al. [14], and Lou and Ni [15]. For our convenience, let us denote the constants collectively by \( \Lambda \).

**Lemma 4.1.** Assume that \( c(x) \in C(\bar{\Omega}) \) and let \( \omega(x) \in C^2(\Omega) \cap C^1(\bar{\Omega}) \) be a positive solution to: \( \Delta \omega + c(x)\omega = 0, \ x \in \Omega \) where \( c(x) \), satisfying the homogeneous Neumann boundary condition, then there exists a positive constant \( C_* = C_*(N,\Omega, ||c||_\infty) \) such that

\[
\max_\Omega \omega \leq C_* \min_\Omega \omega.
\]

**Lemma 4.2.** Suppose that \( g \in C(\Omega \times \mathbb{R}^1) \) and \( b_j \in C(\bar{\Omega}), \ j = 1, 2, ..., N \).

(i) If \( \omega(x) \in C^2(\Omega) \cap C^1(\bar{\Omega}) \) satisfies \( \Delta \omega + \sum_{j=1}^N b_j(x) \omega_{x_j} + g(x, \omega(x)) \geq 0, \ x \in \Omega, \ \frac{\partial \omega}{\partial v} \leq 0, \ x \in \partial \Omega, \) and \( \omega(x_0) = \max_\Omega \omega(x) \) then \( g(x_0, \omega(x_0)) \geq 0 \).

(ii) If \( \omega(x) \in C^2(\Omega) \cap C^1(\bar{\Omega}) \) satisfies \( \Delta \omega + \sum_{j=1}^N b_j(x) \omega_{x_j} + g(x, \omega(x)) \leq 0, \ x \in \Omega, \ \frac{\partial \omega}{\partial v} \geq 0, \ x \in \partial \Omega, \) and \( \omega(x_0) = \min_\Omega \omega(x) \) then \( g(x_0, \omega(x_0)) \leq 0 \).

By using above lemmas the result of upper bounds can be stated as follows:

**Theorem 4.3.** For any positive solution \( u = (u, v, w)^\top \) of (20), it can be drawn that

\[
\max_\Omega u \leq k,
\]

\[
\max_\Omega v \leq \frac{ak}{d_2} \left( d_1 + \frac{d_2 r}{d} + \frac{d_3 r}{d + e} \right), \tag{21}
\]

\[
\max_\Omega w \leq \frac{ak(\lambda + \sigma f)}{\sigma d_2} \left( d_1 + \frac{d_2 r}{d} + \frac{d_3 r}{d + e} \right).
\]

**Proof.** Applying lemma 4.2 in first equation of (2) yields

\[
ru(x_0) \left(1 - \frac{u(x_0)}{k}\right) \geq 0
\]

\[
\Rightarrow \max_\Omega u(x) \leq k.
\]

Let \( s = \alpha d_1 u + d_2 v + d_3 w \), we can obtain

\[
\begin{cases}
-\Delta s = \alpha ru \left(1 - \frac{u}{k}\right) - \sigma (\beta v + w)[1 - c_1(v + 1 - c_2)w] \\
- dv - (d + e)w, \quad x \in \Omega, \\
\frac{\partial s}{\partial v} = 0, \quad x \in \partial \Omega.
\end{cases}
\]
Let \( s(x_1) = \max_{\Omega} s(x) \). Again, from Lemma 4.2, we have
\[
v(x_1) \leq \frac{\alpha r k}{d} \quad \text{and} \quad w_1(x) \leq \frac{\alpha r k}{d + c} \quad \text{(since \( c_1, c_2 < 1 \)).}
\]
Consequently,
\[
\max_{\Omega} v(x) \leq \frac{1}{d_2} \max_{\Omega} s(x) = \frac{s(x_1)}{d_2} = \frac{\alpha d_1 u(x_1) + d_2 v(x_1) + d_3 w(x_1)}{d_2} = \frac{\alpha k}{d_2} \left( d_1 + \frac{d_2 r}{d} + \frac{d_3 r}{d + c} \right).
\]
Now, applying Maximum principle in the third equation of (2)
\[
\lambda v(x_2) w(x_2) + \sigma f v(x_2) w(x_2) - \sigma (\beta v(x_2) + w(x_2)) w(x_2) - (d + e) w(x_2) \geq 0,
\]
\[
\Rightarrow \max_{\Omega} w(x) = \frac{\alpha k(\lambda + \sigma f)}{\sigma d_2} \left( d_1 + \frac{d_2 r}{d} + \frac{d_3 r}{d + c} \right).
\]
The proof is completed.

**Theorem 4.4.** Let \( \Lambda \) and \( d_1, d_2, d_3 \) be fixed positive constants. Assume that
\[
(d_1, d_2, d_3) \in [d_1, \infty) \times [d_2, \infty) \times [d_3, \infty),
\]
and
\[
(\lambda + \sigma f - \beta v') < \sigma w' + (d + e), \quad \text{(22)}
\]
where, \( v' = \max_{\Omega} v(x) \) and \( w' = \max_{\Omega} w(x) \), given in theorem 4.3. Then there exist a positive constant \( \bar{C} = \bar{C}(\Lambda, d_1, d_2, d_3) \), such that any positive solution \((u, v, w)\) of (20) satisfies
\[
\min_{\Omega} u > \bar{C}, \quad \min_{\Omega} v > \bar{C}, \quad \min_{\Omega} w > \bar{C}. \quad \text{(23)}
\]

**Proof.** Let
\[
c_1(x) \triangleq \frac{1}{d_1} \left( ru \left( 1 - \frac{u}{k} \right) - \frac{(\alpha_1 v + \alpha_2 w)}{\gamma + u} \right),
\]
\[
c_2(x) \triangleq \frac{1}{d_2} \left( \frac{\alpha (\alpha_1 v + \alpha_2 w)}{\gamma + u} + c_1 \sigma (\beta v + w) v + c_2 \sigma (\beta v + w) w - \sigma (\beta v + w) v - \sigma f v w - \lambda v w - d v \right),
\]
\[
c_3(x) \triangleq \frac{1}{d_3} \left( \lambda v w + \sigma f v w - \sigma (\beta v + w) w - (d + e) w \right).
\]
Then, in view of (21), there exists a positive constant \( \bar{C} = \bar{C}(\Lambda, \bar{d}) \) such that
\[
||c_1(x)||_{\infty}, \quad ||c_2(x)||_{\infty}, \quad ||c_3(x)||_{\infty} \leq \bar{C}, \quad \text{if} \quad d_1, d_2, d_3 \geq \bar{d}.
\]
Lemma 4.1 shows that there exists a positive constant \( C^* = C^*(\Lambda, \bar{d}) \) such that
\[
\max_{\Omega} u \leq C^* \min_{\Omega} u, \quad \max_{\Omega} v \leq C^* \min_{\Omega} v, \quad \max_{\Omega} w \leq C^* \min_{\Omega} w.
\]
Presently, assume, on the contrary, that (23) does not hold. Then there will be a sequence \( \{ (d_{1n}, d_{2n}, d_{3n}) \}_{n=1}^{\infty} \) with \( (d_{1n}, d_{2n}, d_{3n}) \in [d_1, \infty) \times [d_2, \infty) \times [d_3, \infty) \) such that the corresponding positive solutions \((u_n, v_n, w_n)\) of (20) satisfy
\[
\max_{\Omega} u_n \to 0 \quad \text{or} \quad \max_{\Omega} v_n \to 0 \quad \text{or} \quad \max_{\Omega} w_n \to 0.
\]
The standard regularity theorem for the elliptic equations and \( d_1n \geq d_1, d_{2n} \geq d_2, d_{3n} \geq d_3 \), yields that there exists a subsequence of \( \{ (u_n, v_n, w_n) \}_{n=1}^{\infty} \) which is till denoted by \( \{ (u_n, v_n, w_n) \}_{n=1}^{\infty} \) and non-negative functions
Let \( u, v, w \in C^2(\Omega) \), such that \((u_n, v_n, w_n) \to (u, v, w)\) as \( n \to \infty \). By (23), it’s noted that \( u \equiv 0 \) or \( v \equiv 0 \) or \( w \equiv 0 \).

For all \( n \geq 1 \), integrating by parts, it is obtained that

\[
\int_{\Omega} \left( ru_n \left(1 - \frac{u_n}{k}\right) - \frac{(\alpha_1 v_n + \alpha_2 w_n)u_n}{\gamma + u_n}\right) dx = 0,
\]

\[
\int_{\Omega} \left( \frac{\alpha (\alpha_1 v_n + \alpha_2 w_n)u_n}{\gamma + u_n} + c_1 \sigma (\beta v_n + w_n) v_n + c_2 \sigma (\beta v_n + w_n) w_n \right. \\
- \sigma (\beta v_n + w_n) v_n - \sigma f v_n w_n - \lambda v_n w_n - dv_n \left. \right) = 0,
\]

\[
\int_{\Omega} \left( \lambda v_n w_n + \sigma f v_n w_n - \sigma (\beta v_n + w_n) w_n - (d + e) w_n \right) dx = 0.
\]

Letting \( n \to \infty \) in (24), it can be obtained that

\[
\int_{\Omega} \left( ru \left(1 - \frac{u}{k}\right) - \frac{(\alpha_1 v + \alpha_2 w)u}{\gamma + u}\right) dx = 0,
\]

\[
\int_{\Omega} \left( \frac{\alpha (\alpha_1 v + \alpha_2 w)u}{\gamma + u} + c_1 \sigma (\beta v + w) v + c_2 \sigma (\beta v + w) w \right. \\
- \sigma (\beta v + w) v - \sigma f v w - \lambda v w - dv \left. \right) = 0,
\]

\[
\int_{\Omega} \left( \lambda v w + \sigma f v w - \sigma (\beta v + w) w - (d + e) w \right) dx = 0.
\]

Here, we have mentioned all the three cases as follows:

**Case 1.** \( u \equiv 0, \ v \neq 0, \ w \neq 0 \) on \( \bar{\Omega} \).

From the Hopf boundary lemma \( v > 0, \ w > 0 \) on \( \bar{\Omega} \), then we have

\[
\begin{cases}
-\bar{d}_2 \Delta v = c_1 \sigma (\beta v + w) v + c_2 \sigma (\beta v + w) w - \sigma (\beta v + w) \vspace{1.4pt} \\
\quad - \sigma f v w - \lambda v w - dv, \\
\quad x \in \Omega, \\
\quad \partial v / \partial v = 0, \\
\quad x \in \partial \Omega.
\end{cases}
\]

Let \( v(y_0) = \max_{\Omega} v(x) > 0 \). Applying Maximum principle and the third inequality of (21)

\[
c_1 \sigma (\beta v(y_0) + w(y_0)) v(y_0) + c_2 \sigma (\beta v(y_0) + w(y_0)) w(y_0) - \sigma (\beta v(y_0) + w(y_0)) v(y_0) \vspace{1.4pt} \\
- \sigma f v(y_0) w(y_0) - \lambda v(y_0) w(y_0) - dv(y_0) \geq 0,
\]

which implies,

\[
v(y_0) \leq \frac{c_2 \sigma w'}{(\lambda + \sigma f)} \leq \frac{\sigma w'}{(\lambda + \sigma f)},
\]

provided that \( c_1 < 1, c_2 < 1 \) and \( c_1 + c_2 \beta < 1 \).

Based on assumption (22) stated above, it is easy to see that

\[
\lambda v_n w_n + \sigma f v_n w_n - \sigma (\beta v_n + w_n) w_n - (d + e) w_n < 0.
\]

Integrating the differential equation for \( w_n \) over \( \Omega \) by parts, the result is that

\[
0 = d_{3n} \int_{\Omega} \partial_n w_n dS = -d_{3n} \int_{\Omega} \Delta w_n dx
\]

\[
= \int_{\Omega} (\lambda v_n w_n + \sigma f v_n w_n - \sigma (\beta v_n + w_n) w_n - (d + e) w_n) dx < 0,
\]

which is a contradiction.
Case 2. \( v \equiv 0, u \not\equiv 0 \) on \( \bar{\Omega} \).

Since \( v_n \to v \equiv 0 \) as \( n \to \infty \), then
\[
\frac{\alpha(\alpha_1 v_n + \alpha_2 w_n)u_n}{\gamma + u_n} + c_1 \sigma (\beta v_n + w_n)v_n + c_2 \sigma (\beta v + w_n)w_n
- \sigma (\beta v + w)v - \sigma f v w - \lambda v w - d v > 0,
\]
as \( n \gg 1 \). Next, integrating by parts, the differential equation for \( v_n \) over \( \Omega \), then we have
\[
0 = d_{2n} \int_{\bar{\Omega}} \partial_n v_n dS = -d_{2n} \int_{\partial \Omega} \Delta v_n dS
= \int_{\partial \Omega} \left( \frac{\alpha(\alpha_1 v_n + \alpha_2 w_n)u_n}{\gamma + u_n} + c_1 \sigma (\beta v_n + w_n)v_n
+ c_2 \sigma (\beta v + w_n)w_n - \sigma (\beta v + w_n)v_n - \sigma f v_n w_n - \lambda v_n w_n - d v_n \right) dS > 0,
\]
which is a contradiction.

Case 3. \( w \equiv 0, u \not\equiv 0, v \not\equiv 0 \) on \( \bar{\Omega} \).

Then by Hopf boundary lemma, \( u > 0, v > 0 \) on \( \bar{\Omega} \). And \( u \) and \( v \) satisfy
\[
\begin{aligned}
&-d_3 \Delta v = \frac{\alpha \alpha_1 v}{\gamma + u} + c_1 \sigma \beta v^2 - \sigma \beta v^2 - d v, \\
&\left( \frac{\partial v}{\partial n} = 0, \right. \quad \left. x \in \partial \Omega. \right)
\end{aligned}
\]
Let \( v(y_1) = \max_{\Omega} v(x) > 0 \). Applying Lemma 4.2 and the first inequality of (21),
\[
\frac{\alpha \alpha_1 u(y_1) v(y_1)}{\gamma + u(y_1)} + c_1 \sigma \beta v^2(y_1) - \sigma \beta v^2(y_1) - d v(y_1) \geq 0,
\]
which implies,
\[
\frac{\alpha \alpha_1 k}{\sigma \beta \gamma (1 - c_1)} \leq \frac{\alpha k}{d_2} \left( \frac{d_1 + d_2 r}{d} + \frac{d_3 r}{d + e} \right).
\]
Based on assumption (22) stated above, it is easy to see that
\[
\lambda v_n w_n + \sigma f v_n w_n - \sigma (\beta v_n + w_n)w_n - (d + e) w_n < 0.
\]
Now, integrating the differential equation by parts, for \( w_n \) over \( \Omega \), then we get
\[
0 = d_{3n} \int_{\partial \Omega} \partial_n w_n dS = -d_{3n} \int_{\partial \Omega} \Delta w_n dS
= \int_{\partial \Omega} \left( \lambda v_n w_n + \sigma f v_n w_n - \sigma (\beta v_n + w_n)w_n - (d + e) w_n \right) dS < 0,
\]
which is a contradiction. Hence, proof of the theorem is completed.

\[\square\]

### 4.2 Non-existence of non-constant positive steady states

This section deals with the non-constant positive solution of (20), here, for the results of non-existence we vary the diffusion coefficient \( d \) and fix the other parameters \( d_1, d_2, \) and \( \Lambda \).

**Theorem 4.5.** Let \( d_1^* \) and \( d_2^* \) are fixed positive constants such that
\[
d_1^* \geq \frac{\mu_1}{\mu_2}, \quad d_2^* \geq \frac{\sigma \alpha_1 v}{\mu_1}.
\]

Where \( \mu_1 = \max_{\Omega} w(x) \) and \( \mu_1 \) is the smallest positive eigenvalue of the operator \( -\Delta \) on \( \Omega \) with homogeneous Neumann boundary condition. Then there exists a positive constant \( D_3 = D_3(\Lambda, d_1^*, d_2^*) \), such that, when \( d_3 > D_3, d_1 \geq d_1^* \) and \( d_2 \geq d_2^* \), problem (20) has no non-constant positive solution.
Proof. Let \((\bar{u}, \bar{v}, \bar{w})\) be a positive solution of (20), where

\[
\bar{u} \geq \frac{1}{|\Omega|} \int_{\Omega} u \, dx, \quad \bar{v} \geq \frac{1}{|\Omega|} \int_{\Omega} v \, dx, \quad \bar{w} \geq \frac{1}{|\Omega|} \int_{\Omega} w \, dx.
\]

Now we multiply first equation of (20) by \((u - \bar{u})\) second equation by \((v - \bar{v})\) and third equation by \((w - \bar{w})\) then,

\[
\begin{align*}
-d_1(u - \bar{u}) \nabla^2 u &= (u - \bar{u}) G_1(u, v, w), \\
-d_2(v - \bar{v}) \nabla^2 v &= (v - \bar{v}) G_2(u, v, w), \\
-d_3(w - \bar{w}) \nabla^2 w &= (w - \bar{w}) G_3(u, v, w).
\end{align*}
\]

(27)

Now we integrate (27) over \(\Omega\) and apply Green’s first identity, then we have

\[
\begin{align*}
d_1 \int_{\Omega} |\nabla u|^2 \, dx + d_2 \int_{\Omega} |\nabla v|^2 \, dx + d_3 \int_{\Omega} |\nabla w|^2 \, dx \\
= \int_{\Omega} (u - \bar{u})(G_1(u, v, w) - G_1(\bar{u}, \bar{v}, \bar{w})) \, dx + \int_{\Omega} (v - \bar{v})(G_2(u, v, w) - G_2(\bar{u}, \bar{v}, \bar{w})) \, dx \\
+ \int_{\Omega} (w - \bar{w})(G_3(u, v, w) - G_3(\bar{u}, \bar{v}, \bar{w})) \, dx
\end{align*}
\]

\[
\begin{align*}
= \int_{\Omega} (u - \bar{u})^2 \left( r \left( 1 - \frac{u + \bar{u}}{k} \right) - \frac{\gamma (1 - \alpha)(\alpha_1 v + \alpha_2 w)}{(\gamma + u)(\gamma + u)} \right) \, dx \\
+ \int_{\Omega} (v - \bar{v})^2 \left( \sigma (c_1 + c_2 \beta) \bar{w} - (1 - c_1) \sigma \beta (v + \bar{v}) - (\sigma + \lambda + \sigma \lambda) f \right) \, dx \\
+ \int_{\Omega} (w - \bar{w})^2 \left( \lambda \sigma f - \sigma \beta \right) \bar{v} - \sigma (w + \bar{w}) - (d + e) \, dx \\
- \int_{\Omega} (u - \bar{u})(v - \bar{v}) \left( \frac{(1 - \alpha) \alpha_1 u}{\gamma + u} \right) \, dx - \int_{\Omega} (u - \bar{u})(w - \bar{w}) \left( \frac{(1 - \alpha) \alpha_2 u}{\gamma + u} \right) \, dx \\
+ \int_{\Omega} (v - \bar{v})(w - \bar{w}) \left( \lambda \sigma f - \sigma \beta \right) \bar{v} - \sigma (w + \bar{w}) - (d + e) \, dx.
\end{align*}
\]

By Cauchy inequality, we have

\[
\begin{align*}
\int_{\Omega} \left\{ d_1 |\nabla u|^2 + d_2 |\nabla v|^2 + d_3 |\nabla w|^2 \right\} \, dx \\
\leq \int_{\Omega} (u - \bar{u})^2 \left( r \left( 1 - \frac{u + \bar{u}}{k} \right) - \frac{\gamma (1 - \alpha)(\alpha_1 v + \alpha_2 w)}{(\gamma + u)(\gamma + u)} \right) \, dx \\
+ \int_{\Omega} (v - \bar{v})^2 \left( \sigma w' (c_1 + c_2 \beta) - (1 - c_1) \sigma \beta (v + \bar{v}) - (\sigma + \lambda + \sigma \lambda) f \right) \, dx \\
+ \int_{\Omega} (w - \bar{w})^2 \left( \lambda \sigma f - \sigma \beta \right) \bar{v} - \sigma (w + \bar{w}) - (d + e) \, dx \\
+ \frac{1}{4e} \int_{\Omega} (w - \bar{w})^2 \left( (v - \bar{v})^2 \left( \lambda \sigma f - \sigma \beta \right) \bar{v} - \sigma (w + \bar{w}) - (d + e) \right) \, dx,
\end{align*}
\]

where, \( w' = \max_{\bar{\Omega}} w(x) \) and \( \epsilon \) is an arbitrary positive constant.

Using Poincaré Inequality, we have

\[
\begin{align*}
\int_{\Omega} \left\{ d_1 |\nabla u|^2 + d_2 |\nabla v|^2 + d_3 |\nabla w|^2 \right\} \, dx \\
\geq \int_{\Omega} d_1 \mu_1 (u - \bar{u})^2 \, dx + \int_{\Omega} d_2 \mu_1 (v - \bar{v})^2 \, dx + \int_{\Omega} d_3 \mu_1 (w - \bar{w})^2 \, dx.
\end{align*}
\]

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Since the upper bound \([21]\), a sufficiently small \(\epsilon_0\) can be chosen such that
\[
d_1\mu_1 > r \left(1 - \frac{u + \bar{u}}{k}\right) - \frac{\gamma(1 - \alpha)(\alpha_1 v + \alpha_2 w)}{(\gamma + \bar{u})(\gamma + u)},
\]
\[
d_2\mu_1 > \sigma w'(c_1 + c_2\beta) - (1 - c_1)\sigma\beta(v + \bar{v}) - (\sigma + \lambda + \sigma f)w - d + \epsilon_0.
\]
Lastly, by taking \(D_3 > \frac{1}{\mu_1}\left[\left(\lambda + \sigma f - \sigma\beta\bar{v} - \sigma(w + \bar{w}) - (d + e)\right) + \frac{1}{4\epsilon_0}(v - \bar{v})^2(\lambda + \sigma f - \sigma\beta\bar{v} - \sigma(w + \bar{w}) - (d + e))^2\right]\),

then we can conclude that \(u = \bar{u}, v = \bar{v}, w = \bar{w}\), which establishes proof of the theorem. \(\square\)

### 4.3 Existence of non-constant positive steady states

In this section, we have derived the conditions for the existence of non-constant positive solutions of \(20\), that means how the parameters can properly be chosen that exhibits beautiful Turing patterns. For the purposes, we vary the diffusion coefficient \(d_2\) and fix the other parameters \(\Lambda, d_1\), and \(d_3\).

To obtain the results, first we study the linearization of \(20\) at constant positive solution \(u^*\). Let’s denote
\[
\mathbf{X} = \{\mathbf{u} \in [C^2(\Omega)]^3 \mid \partial_\nu \mathbf{u} = 0 \text{ on } \partial \Omega\}, \quad \mathbf{X}^+ = \{\mathbf{u} \in \mathbf{X} \mid u > 0, v > 0, w > 0, x \in \bar{\Omega}\},
\]
and
\[
\mathbb{B}(\Theta) = \{\mathbf{u} \in \mathbf{X} \mid \Theta^{-1} < u, v, w < \Theta, x \in \bar{\Omega}\}.
\]

Where \(\Theta\) is a positive constant and guaranteed to obtained from Theorems [4.3 and 4.4]. The elliptic problem \(20\) can be written in this form:
\[
\begin{align*}
-\mathcal{D}\Delta \mathbf{u} &= \mathbf{G}(\mathbf{u}), & x & \in \Omega, \\
\partial_\nu \mathbf{u} &= 0, & x & \in \partial \Omega.
\end{align*}
\]

(28)

Here, \(\mathbf{u}\) be a positive solution of \(28\) if and only if
\[
\mathbf{F}(\mathbf{u}) \triangleq \mathbf{u} - (\mathbf{I} - \Delta)^{-1}\{\mathcal{D}^{-1}\mathbf{G}(\mathbf{u}) + \mathbf{u}\} = 0 \text{ in } \mathbf{X}^+,
\]
where \((\mathbf{I} - \Delta)^{-1}\) is the inverse of \(\mathbf{I} - \Delta\) in \(\mathbf{X}\). As \(\mathbf{F}(\cdot)\) is a compact perturbation of the identity operator, for any \(\mathbb{B} = \mathbb{B}(\Theta)\). The Leray-Schauder degree \(\deg(\mathcal{F}(\cdot), 0, \mathbb{B})\) is well-defined if \(\mathbf{F}(\mathbf{u}) \neq 0\) on \(\partial \mathbb{B}\). Further, we notice that
\[
\mathcal{D}_\mathbf{u}\mathbf{F}(\mathbf{u}^*) = \mathbf{I} - (\mathbf{I} - \Delta)^{-1}\{\mathcal{D}^{-1}\mathbf{G}(\mathbf{u}^*) + \mathbf{I}\},
\]
and if \(\mathcal{D}_\mathbf{u}\mathbf{F}(\mathbf{u}^*)\) is invertible, then the index of \(\mathbf{F}\) at \(\mathbf{u}^*\) is denoted as \(\text{index}(\mathcal{F}(\cdot), \mathbf{u}^*) = (-1)^\rho\), where \(\rho\) is counting multiplicities of eigenvalues with negative real parts of \(\mathcal{D}_\mathbf{u}\mathbf{F}(\mathbf{u}^*)\).

We notice that, \(\mathbf{X}_{ij}\) is invariant under \(\mathcal{D}_\mathbf{u}\mathbf{F}(\mathbf{u}^*)\) for each integer \(i \geq 1\) and each integer \(1 \leq j \leq \dim E(\mu_i)\). Thus, \(\lambda\) is an eigenvalue of the matrix
\[
\mathbf{I} - \frac{1}{1 + \mu_i}[\mathcal{D}^{-1}\mathbf{G}(\mathbf{u}^*) + \mathbf{I}] = \frac{1}{1 + \mu_i}[\mu_i\mathbf{I} - \mathcal{D}^{-1}\mathbf{G}(\mathbf{u}^*)].
\]

Thus the matrix \(\mathcal{D}_\mathbf{u}\mathbf{F}(\mathbf{u}^*)\) is invertible if and only if, for all \(i \geq 1\), the matrix \(\mathbf{I} - \frac{1}{1 + \mu_i}[\mathcal{D}^{-1}\mathbf{G}(\mathbf{u}^*) + \mathbf{I}]\) is non-singular. Write
\[
\mathbb{H}(\mu) = \mathbb{H}(\mu; \mu) \triangleq \det\{\mu\mathbf{I} - \mathcal{D}^{-1}\mathbf{G}(\mathbf{u}^*)\} = \frac{1}{d_1d_2d_3} \det\{\mu\mathcal{D} - \mathbf{G}(\mathbf{u}^*)\}
\]
(29)

Furthermore, we note that if \(\mathbb{H}(\mu_i) \neq 0\) then for each \(1 \leq j \leq \dim E(\mu_i)\), the number of negative eigenvalues of \(\mathcal{D}_\mathbf{u}\mathbf{F}(\mathbf{u}^*)\) on \(\mathbf{X}_{ij}\) is odd if \(\mathbb{H}(\mu_i) < 0\). From this, we can conclude the following result.

**Proposition 4.6.** Suppose that for all \(i \geq 0\), the matrix \(\mu_i\mathbf{I} - \mathcal{D}^{-1}\mathbf{G}(\mathbf{u}^*)\) is non-singular. Then
\[
\text{index}(\mathcal{F}(\cdot), \mathbf{u}^*) = (-1)^\rho \quad \text{where} \quad \rho = \sum_{i \geq 0, \mathbb{H}(\mu_i) < 0} \dim E(\mu_i)
\]

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According to this proposition, in order to calculate index($F(\cdot), u^*$), we should consider carefully the sign of $A(t)$. Therefore the direct calculation gives

$$A(\mu) \triangleq \det \{ \mu D - G_w(u^*) \} = A_3(d_2)\mu^3 + A_2(d_2)\mu^2 + A_1(d_2)\mu - \det(G_w(u^*))$$

with

$$A_3(d_2) = d_1d_2d_3, \quad A_2(d_2) = -(d_1d_2a_{33} + d_2d_3a_{11} + d_1d_3a_{22}),$$

$$A_1(d_2) = d_1(a_{22}a_{33} - a_{23}a_{32}) + d_2a_{11}a_{33} + d_3(a_{11}a_{22} - a_{12}a_{21}),$$

where $a_{ij}$ are given in (9).

Here, we have considered the dependence of $A$ on $d_2$. If $\tilde{\mu}_1(d_2), \tilde{\mu}_2(d_2), \tilde{\mu}_3(d_2)$ be the roots of $A(d_2; \mu) = 0$ with $\text{Re}\{\tilde{\mu}_1\} \leq \text{Re}\{\tilde{\mu}_2\} \leq \text{Re}\{\tilde{\mu}_3\}$, then

$$\tilde{\mu}_1(d_2)\tilde{\mu}_2(d_2)\tilde{\mu}_3(d_2) = \det(G_w(u^*)).$$

Note that $\det(G_w(u^*)) < 0$ and $A_3(d_2) > 0$. Thus, one of $\tilde{\mu}_1(d_2), \tilde{\mu}_2(d_2), \tilde{\mu}_3(d_2)$ is real and negative, and the product of two is positive.

Now, we will perform the following limits:

$$\lim_{d_2 \to \infty} \frac{A(\mu)}{d_2} = \lim_{d_2 \to \infty} \frac{A_3\mu^3 + A_2\mu^2 + A_1\mu}{d_2} = \mu[d_1d_3\mu^2 - (d_1a_{33} + d_3a_{11})\mu + a_{11}a_{33}].$$

If the parameters $\Lambda, d_1, d_3$ satisfy $a_{11}d_3 + a_{33}d_1 > 0$, the we can establish the following result:

**Proposition 4.7.** Assume the condition (3) holds, and $a_{11} > 0$. Then there exists a positive constant $D_2$, such that when $d_2 \geq D_2$, then the three roots $\tilde{\mu}_1(d_2), \tilde{\mu}_2(d_2), \tilde{\mu}_3(d_2)$ of $A(d_2; \mu) = 0$, all are real and satisfy

$$\begin{cases}
\lim_{d_2 \to \infty} \tilde{\mu}_1(d_2) = \frac{d_3a_{11} + d_1a_{33} - \sqrt{(d_3a_{11} + d_1a_{33})^2 - 4d_1d_3a_{11}a_{33}}}{2d_1d_3} = a_{33}d_3 < 0, \\
\lim_{d_2 \to \infty} \tilde{\mu}_2(d_2) = 0, \\
\lim_{d_2 \to \infty} \tilde{\mu}_3(d_2) = \frac{d_3a_{11} + d_1a_{33} + \sqrt{(d_3a_{11} + d_1a_{33})^2 - 4d_1d_3a_{11}a_{33}}}{2d_1d_3} = a_{11}d_1 > 0.
\end{cases}$$

Moreover,

$$\begin{cases}
-\infty < \tilde{\mu}_1(d_2) < 0 < \tilde{\mu}_2(d_2) < \tilde{\mu}_3(d_2), \\
A(d_2; \mu) < 0 \text{ if } \mu \in (-\infty, \tilde{\mu}_1(d_2)) \cup (\tilde{\mu}_2(d_2), \tilde{\mu}_3(d_2)), \\
A(d_2; \mu) > 0 \text{ if } \mu \in (\tilde{\mu}_1(d_2), \tilde{\mu}_2(d_2)) \cup (\tilde{\mu}_3(d_2), \infty).
\end{cases}$$

Next, we will prove the existence of non-constant positive solutions of (20), when $d_2$ is sufficiently large and the other parameters are fixed.

**Theorem 4.8.** Assume the parameters $d_1$ and $d_3$ are fixed, $a_{11} > 0$ and $a_{11}a_{33} < 0$ holds. If $\tilde{\mu} \in (\mu_n, \mu_{n+1})$ for some $n \geq 1$, and the sum $\sigma_n = \sum_{i=1}^{n} \text{dim} E(\mu_i)$ is odd, then there exists a positive constant $D_2$ such that, if $d_2 \geq D_2$, then the problem (20) has at least one non-constant positive solution to generate Turing patterns.

**Proof.** If $a_{11}a_{33} < 0$, by proposition 4.7 here, we have a positive constant $D_2$, such that when $d_2 \geq D_2$, (32) holds and

$$0 = \mu_0 < \tilde{\mu}_2(d_2) < \tilde{\mu}_1(d_2) < \tilde{\mu}_3(d_2) \in (\mu_n, \mu_{n+1}).$$

The motivation of the theorem is to prove the existence of non-constant positive solution of (20) for any $d_2 \geq D_2$. This proof is based on the homotopy invariance of the topological degree. Assume that the attestation on the contrary isn’t valid for some $d_2 = d_2 \geq D_2$. To the continuation of proof, we fixed $d_2 = d_2$, $d_1 = \frac{r}{\mu_1}$, $d_2 = \frac{\alpha k(c_1 + c_2\beta)(\lambda + \sigma f)}{\mu_1d_2} \cdot \left(d_1 + \frac{d_2r}{d} + \frac{d_3r}{d_e}\right)$.

By Theorem 4.5, we obtain $D_3 = D_3(\Lambda, d_1^*, d_2^*)$. Fix $d_2 \geq d_2^*, d_1^* \geq \max\{d_1^*, d_1\}, d_3 \geq \max\{D_3, d_3\}$. For $t \in [0, 1]$, define $D(t) = \text{diag}(d_1(t), d_2(t), d_3(t))$ with $d_i(t) = td_i + (1 - t)d_i, i = 1, 2, 3$. Now, consider the problem
where we have
\begin{equation}
\begin{cases}
-D(t)u = G(u), & x \in \Omega, \\
\partial_\nu u = 0, & x \in \partial\Omega.
\end{cases}
\end{equation}

Then \(u\) is a non-constant positive solution of \((20)\) iff it is a positive solution of \((34)\) for \(t = 1\). It is obvious that \(u^*\) is the unique constant positive solution of \((34)\) for any \(0 \leq t \leq 1\). For any \(0 \leq t \leq 1\), \(u\) is a positive solution of \((34)\) iff
\[
F(t; u) \triangleq u - (I - \Delta)^{-1}\{D^{-1}(t)G(u) + u\} = 0 \text{ in } X^+.
\]

Clearly, \(F(1; u) = F(u)\), Theorem 4.5 shows that \(F(0; u) = 0\) has only the positive solution \(u^*\) in \(X^+\). The direct calculation gives,
\[
D_\mu F_u(t; u^*) \triangleq I - (I - \Delta)^{-1}\{D^{-1}(t)G_u(u^*) + I\},
\]
and in particular,
\[
D_\mu F_u(0; u^*) \triangleq I - (I - \Delta)^{-1}\{D^{-1}G_u(u^*) + I\},
\]
\[
D_\mu F_u(1; u^*) \triangleq I - (I - \Delta)^{-1}\{D^{-1}G_u(u^*) + I\} = D_\mu F(u^*),
\]
where \(\hat{D} = \text{diag}(\hat{d}_1, \hat{d}_2, \hat{d}_3)\). From \((29)\) and \((30)\)
\[
\mathbb{H}(\mu) = \frac{1}{d_1d_2d_3}A(d_2; \mu) \quad (35)
\]
In view of \((32)\) and \((33)\), it follows from \((35)\) that
\[
\begin{cases}
\mathbb{H}(\mu_0) = \mathbb{H}(0) > 0, \\
\mathbb{H}(\mu_i) < 0, & 1 \leq i \leq n, \\
\mathbb{H}(\mu_{i+1}) > 0, & i \geq n + 1
\end{cases} \quad (36)
\]
Therefore, \(0\) is not an eigenvalue of the matrix \(\mu_1I - \hat{D}^{-1}G_u(u^*)\) for all \(i \geq 0\), and
\[
\dim E(\mu_i) = \sum_{i=1}^{n} \dim E(\mu_i) = \sigma_n, \quad \text{which is odd.}
\]

Thanks to Proposition 4.6 we have
\[
\text{index}(F(1; \cdot), u^*) = (-1)^\rho = (-1)^n = -1
\]
Similarly, it is possible to prove
\[
\text{index}(F(0; \cdot), u^*) = (-1)^0 = 1 \quad (37)
\]
By Theorems 4.3 and 4.4 there exists a positive constant \(\Theta\) such that, for all \(0 \leq t < 1\), the positive solutions of \((34)\) satisfy \(\Theta^{-1} < u, v, w < \Theta\). Therefore \(F(t; u)\) on \(\partial \bar{\Theta}(\Theta)\) for all \(\Theta^{-1} < u, v, w < \Theta\). By homotopy invariance of the topological degree,
\[
\deg(F(1; \cdot), 0, \bar{\Theta}(\Theta)) = \deg(F(0; \cdot), 0, \bar{\Theta}(\Theta)) \quad (39)
\]
On the other hand, by our assumption, both equations \(F(1; u) = 0\) and \(F(0; u) = 0\) have only the positive solution \(u^*\) in \(\bar{\Theta}(\Theta)\), and hence, by \((37)\) and \((38)\),
\[
\deg(F(0; \cdot), 0, \bar{\Theta}(\Theta)) = \text{index}(F(0; \cdot), u^*) = 1, \\
\deg(F(1; \cdot), 0, \bar{\Theta}(\Theta)) = \text{index}(F(1; \cdot), u^*) = -1.
\]
This contradicts \((39)\) and the proof is completed.
4.4 Bifurcation

In this section, we have discussed the bifurcation of nonconstant positive solutions of (20), and for that we fix the parameters $\Lambda, d_1, d_3$. Here, we have considered the bifurcation with respect to the diffusive rate parameter $d_2$. Let $(\tilde{d}_2; u^*) \in \mathbb{R}^+ \times X$ is a bifurcation point of (20) if for any $\delta \in (0, d_2)$, there exists $d_2 \in [\tilde{d}_2 - \delta, \tilde{d}_2 + \delta]$ such that (20) has a non-constant positive solution. Otherwise, $(\tilde{d}_2, u^*)$ is called as regular point. Let $S_{\mu} = \{\mu_1, \mu_2, \ldots\}$ be positive spectrum of $-\Delta$ on $\Omega$ with the homogeneous Neumann boundary condition and also define $\mathcal{N}(d_2) = \{\mu > 0 \mid \mathbb{H}(d_2; \mu) = 0\}$, for $d_2 > 0$, where $\mathbb{H}(d_2; \mu)$ is introduced by (29). Then $\mathcal{N}(d_2)$ contains at most two elements.

**Theorem 4.9.** Assume that the parameter $\Lambda$ satisfies (5) and $\tilde{d}_2 > 0$.

(i) If $S_{\mu} \cap \mathcal{N}(\tilde{d}_2) = \emptyset$, then $(\tilde{d}_2; u^*)$ is a regular point of (20).

(ii) Suppose $S_{\mu} \cap \mathcal{N}(\tilde{d}_2) \neq \emptyset$, and the positive roots of $\mathbb{H}(d_2; \mu) = 0$ are simple. If the sum $\sum_{\mu_i \in \mathcal{N}(\tilde{d}_2)} \dim E(\mu_i)$ is odd, then $(\tilde{d}_2; u^*)$ is a bifurcation point of (20).

**Proof.** Let $\kappa(x) = u(x) - u^*$. Then the problem (20) is equivalent to
\[
\begin{cases}
-\Delta \kappa = D^{-1}G(u^* + \kappa), & x \in \Omega, \\
\partial_\nu \kappa = 0, & x \in \partial \Omega.
\end{cases}
\]
which, in turn, is equivalent to
\[
f(d_2; \kappa) \triangleq \kappa - (I - \Delta)^{-1}\{D^{-1}G(u^* + \kappa) + \kappa\} = 0 \quad \text{on } X.
\]
By direct computation, we have
\[
D_{\kappa}f(d_2; 0) = I - (I - \Delta)^{-1}\{D^{-1}G(u^*) + I\},
\]
and as in subsection 4.3 for each $i, \xi$ is an eigenvalue of $D_{\kappa}f(d_2; 0)$ on $X_i$ if and only if $\xi(1 + \mu_i)$ is an eigenvalue of the matrix $\mathbb{H}(d_2; \mu_i)$.

(i) If $S_{\mu} \cap \mathcal{N}(\tilde{d}_2) = \emptyset$, then $\det \mathbb{H}(d_2; \mu_i) \neq 0$ for all $i$, i.e., 0 is not the eigenvalue of $D_{\kappa}f(d_2; 0)$. This implies that $D_{\kappa}f(d_2; 0)$ is a homeomorphism from $X$ to itself. The implicit function theorem shows that for all $d_2$ close to $\tilde{d}_2$, $\kappa = 0$ is the only solution to $f(d_2; \kappa) = 0$ in a small neighborhood of the origin, i.e., $(\tilde{d}_2; u^*)$ is a regular point of (20).

(ii) If $S_{\mu} \cap \mathcal{N}(\tilde{d}_2) \neq \emptyset$, it is easy to show that 0 is a simple eigenvalue of $\mathbb{H}(d_2; \mu_i)$ for any $i$ satisfying $\mu_i \in S_{\mu} \cap \mathcal{N}(\tilde{d}_2)$. Now, suppose on the contrary that the assertion of the theorem is false. Then there exists $\tilde{d}_2 > 0$ such that the following are true:

(a) $S_{\mu} \cap \mathcal{N}(\tilde{d}_2) = \emptyset$, and $\sum_{\mu_i \in \mathcal{N}(\tilde{d}_2)} \dim E(\mu_i)$ is odd.

(b) There exists $\delta \in (0, \tilde{d}_2)$ such that for every $d_2 \in [\tilde{d}_2 - \delta, \tilde{d}_2 + \delta], \kappa = 0$ is the only solution to $f(d_2; \kappa) = 0$ in a neighbourhood $B_\delta$ of the origin.

Since $f(d_2; \cdot)$ is a compact perturbation of an identity function, in view of (b), the Leray-Schauder degree $\text{deg}(F(\cdot), B_\delta, 0)$ is well defined and doesn’t depend on $d_2 \in [\tilde{d}_2 - \delta, \tilde{d}_2 + \delta]$. In addition, for those $d_2 \in [\tilde{d}_2 - \delta, \tilde{d}_2 + \delta], D_{\kappa}f(d_2; 0)$ is invertible. Since the positive roots of $\mathbb{H}(d_2; \mu) = 0$ are simple, by Proposition 4.6 we have $\text{deg}(f(d_2; 0), B_\delta, 0) = (-1)^{\sigma(d_2)}$. Let
\[
\tilde{\mathbb{H}}(d_2; \mu) = d_1d_2d_3\mathbb{H}(d_2; \mu).
\]
For $\mu_i \in S_{\mu} \cap \mathcal{N}(\tilde{d}_2)$, as $\tilde{\mathbb{H}}(d_2; \mu_i) = 0$, a direct computations yields,
\[
\frac{\partial}{\partial d_2} \tilde{\mathbb{H}}(d_2; \mu_i) = \tilde{d}_2^{-1}[d_1d_3a_{22}\mu_i^2 - \mu_i(d_1a_{22}a_{33} - a_{23}a_{32}) - d_3(a_{11}a_{22} - a_{12}a_{21}) - A_3] < 0.
\]
Since $S_{\mu} \cap \mathcal{N}(\tilde{d}_2)$ contains at most two elements, there exists $\delta < 1$ such that
\[
\frac{\partial}{\partial d_2} \tilde{\mathbb{H}}(d_2; \mu_i) < 0,
\]
for all \( d_2 \in [\bar{d}_2 - \delta, \bar{d}_2 + \delta] \) and \( \mu_i \in S_p \cap \mathcal{N}(\bar{d}_2) \). Therefore
\[
\mathbb{H}(\bar{d}_2 - \delta; \mu_i) \mathbb{H}(\bar{d}_2 + \delta; \mu_i) < 0,
\]
and in turn,
\[
\mathbb{H}(\bar{d}_2 - \delta; \mu_i) \mathbb{H}(\bar{d}_2 + \delta; \mu_i) < 0, \quad \forall \mu_i \in S_p \cap \mathcal{N}(\bar{d}_2).
\] (42)

Since \( S_p \) does not have any accumulation point, by taking \( \delta \) sufficiently small, we may assume that \( S_p \cap \mathcal{N}(\bar{d}_2) = \emptyset \) for all \( d_2 \in [\bar{d}_2 - \delta, \bar{d}_2 + \delta] \). Therefore, \( D \kappa f(d_2; 0) \) is invertible for all \( d_2 \in [\bar{d}_2 - \delta, \bar{d}_2 + \delta] \). Now, for each \( i \) and \( d_2 \in [\bar{d}_2 - \delta, \bar{d}_2 + \delta] \), \( \mathcal{X} \) is invariant under \( D \kappa f(d_2; 0) \), and the number of eigenvalues with negative real parts of \( D \kappa f(d_2; 0) \) on \( \mathcal{X} \) is the same as that of the matrix \( \mathbb{H}(\bar{d}_2; \mu_i) \). If \( \mu_i, \mathcal{N}(\bar{d}_2) \), then the number of eigenvalues with negative real parts of \( D \kappa f(d_2; 0) \) on \( \mathcal{X} \) is independent of \( d_2 \in [\bar{d}_2 - \delta, \bar{d}_2 + \delta] \); whereas if \( \mu_i \in \mathcal{N}(\bar{d}_2) \) then the difference between the number of eigenvalues with negative real parts of \( D \kappa f(d_2; 0) \) on \( \mathcal{X} \), for \( d_2 = d_2 - \delta \) and \( d_2 = d_2 + \delta \) is 1 by (42). Thus, \( \sigma(\bar{d}_2 + \delta) - \sigma(\bar{d}_2 - \delta) \) is equal to the sum \( \sum_{\mu_i \in \mathcal{N}(\bar{d}_2)} \dim \mathcal{E}(\mu_i) \), which is odd. Therefore
\[
\deg(f(d_2 - \delta, \cdot), B_\delta, 0) \neq \deg(f(d_2 + \delta, \cdot), B_\delta, 0).
\]
and we have a contradiction. This shows that \( (\bar{d}_2; u^*) \) is a bifurcation point of (20). \( \square \)

**Theorem 4.10.** Assume that the parameter \( \Lambda \) satisfies (5), \( S_p \cap \mathcal{N}(\bar{d}_2) = \emptyset \), and the positive roots of \( \mathbb{H}(\bar{d}_2; \mu) = 0 \) are simple. If the sum \( \sum_{\mu_i \in \mathcal{N}(\bar{d}_2)} \dim \mathcal{E}(\mu_i) \), is odd, then there exists an interval \( (\alpha, \beta) \subset \mathbb{R}^+ \) such that for every \( d_2 \in (\alpha, \beta) \), the problem (20) admits a non-constant positive solution \( u = u(d_2) \). Moreover, one of the following holds:

\( (i) \) \( \bar{d}_2 = \alpha < \beta < \infty \) and \( S_p \cap \mathcal{N}(\beta) \neq \emptyset \);

\( (ii) \) \( 0 < \alpha < \beta < \bar{d}_2 \) and \( S_p \cap \mathcal{N}(\alpha) \neq \emptyset \);

\( (iii) \) \( u(\alpha) = u^* \) or \( u(\beta) = u^* \).

\( (iv) \) \( (\alpha, \beta) = (\bar{d}_2, \infty) \).

\( (v) \) \( (\alpha, \beta) = (0, \bar{d}_2) \).

**Proof.** Let \( \Gamma = \{ d_2 > 0 \mid S_p \cap \mathcal{N}(d_2) \neq \emptyset \} \), \( S = \text{closure} \{ (d_2, u) \in \mathbb{R}^+ \times \mathcal{X} \mid u > 0, u \neq u^*, u \) solves (20) \}. In view of the estimates (21) and (23), following the arguments of (22) or (24), and incorporating the calculation of the degree \( \deg(f(d_2; \cdot), B_\delta, 0) \) that we presented in the proof of Theorem 4.9, we can conclude that \( S \) contains a maximal connected subset \( \mathcal{C} \) which emanates from \( (\bar{d}_2; u^*) \) and

1. \( \mathcal{C} \) meets \( \Gamma \times \{ u^* \} \) at a point \( (\bar{d}_2; u^*) \); or
2. \( \mathcal{C} \) meets \( \{ d_2 > 0 \} \times \{ u^* \} \) at a point \( (d_2; u^*) \) with \( d_2 \neq \bar{d}_2 \); or
3. \( \mathcal{C} \) is non-compact in \( (0, \infty) \times \mathcal{X} \).

Corresponding to the case (1), either the assertion (i) or the statement (ii) of the theorem holds. If (2) happens, then (iii) holds. Finally, if (3) holds, then, applying the estimates (21) and (23), we can quickly show that either (iv) or (v) holds. This completes the proof. \( \square \)

Above discussed theoretical results are only sufficient conditions. The theorem associated with the existence of non-constant solution shows that we can obtain Turing patterns for certain values of diffusion coefficients. Similarly, in a theoretical way, we have proved the result for the non-existence of non-constant steady states, i.e. when the Turing patterns would not occur. Which based upon the definition of \( \mu_1 \), and it clearly understood that the determination of \( \mu_1 \) is not an easy task. Here \( \mu_1 \) is the smallest eigenvalue of \( -\Delta \). Hence the condition for non-existence cannot be verified for the suitable choice of parameter values.

The next section, we will discuss the Turing instability, in which if the conditions demonstrated are satisfied for some parameter values then the Turing patterns arise.
5 Turing Instability

We perform a linear stability analysis of the spatial model system (2) and for this purpose we linearize the spatio-temporal model (2) about the interior equilibrium point $E^*(u^*, v^*, w^*)$, perturbed with the following two-dimensional spatio-temporal perturbation of the form

$$
\begin{align*}
    u &= u^* + \epsilon_1 \exp(\lambda_k t + i(k_xx + k_yy)) = u^* + u_1 \\
    v &= v^* + \epsilon_2 \exp(\lambda_k t + i(k_xx + k_yy)) = v^* + v_1 \\
    w &= w^* + \epsilon_3 \exp(\lambda_k t + i(k_xx + k_yy)) = w^* + w_1
\end{align*}
$$

(43, 44, 45)

where $\epsilon_i$, $i = 1, 2, 3$, are positive and sufficiently small constants, $k_x$ and $k_y$ are the components of wave number $k$ along $x$ and $y$ directions respectively and $\lambda_k$ is the wavelength. The system is linearized about the non-trivial interior equilibrium point $E^*(u^*, v^*, w^*)$. The characteristic equation of the linearized version of the spatial model system (2) is given by

$$
(J_s - \lambda_k I_3) \begin{pmatrix} u_1 \\ v_1 \\ w_1 \end{pmatrix} = 0
$$

(46)

with

$$
J_{xyz} = \begin{pmatrix}
    a_{11} - d_1 k^2 & a_{12} & a_{13} \\
    a_{21} & a_{22} - d_2 k^2 & a_{23} \\
    a_{31} & a_{32} & a_{33} - d_3 k^2
\end{pmatrix}
$$

(47)

where $k$ is the wave number given by $k^2 = k_x^2 + k_y^2$ and $I_3$ is a $3 \times 3$ identity matrix and $a_{ij}$ are given in (9). Our interest is the stability properties of the attracting interior equilibrium point $E^*$, which will lead to the conditions for Turing instability. From (46) to (47), we get the characteristic equation of the form

$$
\det(J_{xyz} - \lambda_k I_3) = \lambda_k^3 + \rho_1 \lambda_k^2 + \rho_2 \lambda_k + \rho_3 = 0
$$

(48)

where,

$$
\rho_1 = -\text{tr}(J_{xyz}) = -(a_{11} + a_{22} + a_{33}) + (d_1 + d_2 + d_3)k^2
$$

$$
\rho_2 = k^4(d_1 d_2 + d_2 d_3 + d_3 d_1) - (a_{11} d_2 + d_3 + a_{22}(d_3 + d_1)
+ a_{33}(d_1 + d_2))k^2 + (a_{11} a_{33} + a_{22} a_{33} + a_{11} a_{22}) - a_{12} a_{21} - a_{23} a_{32}
$$

$$
\rho_3 = -\det(J_{xyz}) = d_1 d_2 d_3 k^6 - k^4(a_{11} d_2 d_3 + d_3 a_{22} d_1 + a_{33} d_1 d_2)
+ k^2(a_{11} a_{33} d_2 + a_{22} a_{33} d_1 + a_{11} a_{22} d_3 - d_3 a_{12} a_{21} - d_1 a_{23} a_{32})
+ (a_{12} a_{21} + a_{33} a_{12} a_{21} - a_{21} a_{32} a_{33} - a_{11} a_{22} a_{33}).
$$

**Theorem 5.1.** Diffusion-driven instability occurs if one of the following conditions is satisfy:

(i) If $p_2 < 0$ and $p_2^2 - 4p_1p_3 > 0$ then $k_c^2$ is positive and real, where

$$
k_c^2 = \frac{p_2}{2p_1} \text{ and } \rho_2(k_c^2) = p_3 - \frac{p_2^2}{4p_1} < 0
$$

(ii) If $q_2 < 0$ and $q_2^2 - 3q_1q_3 > 0$ then $k_d^2$ is positive and real, where

$$
k_d^2 = -q_2 + \sqrt{q_2^2 - 3q_1q_3} \quad \frac{3q_1}{3q_1}
$$

$$
\rho_3(k_d^2) = \frac{2q_3^2 - 9q_1q_2q_3 + 27q_1^2q_4 - 2(q_2^2 - 3q_1q_3)^{\frac{3}{2}}}{27q_1^{\frac{3}{2}}} < 0
$$

(iii) If $r_2 < 0$ and $r_2^2 - 3r_1r_3 > 0$ then $k_f^2$ is positive and real, where

$$
k_f^2 = -r_2 + \sqrt{r_2^2 - 3r_1r_3} \quad \frac{3r_1}{3r_1}
$$

$$
\Phi(k_f^2) = \frac{2r_3^2 - 9r_1r_2r_3 + 27r_1^2r_4 - 2(r_2^2 - 3r_1r_3)^{\frac{3}{2}}}{27r_1^{\frac{3}{2}}} < 0
$$
Proof. The spatially homogeneous state will be unstable provided that at least one eigenvalue of the characteristic equation (48) is positive. It is clear that the homogeneous steady state $E^*$ is asymptotically stable if and only if $\rho_1(0) > 0$, $\rho_3(0) > 0$ and $\rho_1(0)\rho_2(0) - \rho_3(0) > 0$. But it will be driven to an unstable state by diffusion if any of the conditions $\rho_1(k^2) > 0$, $\rho_3(k^2) > 0$ and $\rho_1(k^2)\rho_2(k^2) - \rho_3(k^2) > 0$ fail to hold. However, it can be easily seen that diffusion-driven instability cannot occur by contradicting $\rho(k^2) > 0$ because $d_1, d_2, d_3$ and $k^2$ are positive, the inequality $\rho_1(k^2) > 0$ always hold due to the the stability condition of the interior equilibrium point in homogeneous state of non-spatial model (2).

So for the diffusion driven instability, it is sufficient to show that one of the following satisfy:

$$\rho_2(k^2) < 0, \rho_3(k^2) < 0 \text{ and } \rho_1(k^2)\rho_2(k^2) - \rho_3(k^2) < 0$$

For $\rho_2(k^2) < 0$, we can write it from the equation (48) and let us suppose $z = k^2$, so we get

$$\rho_2(z) = p_1 z^2 + p_2 z + p_3, \quad (49)$$

where,

$$p_1 = d_1 d_2 + d_2 d_3 + d_3 d_1$$
$$p_2 = -(a_{11} d_2 + d_3 + a_{22} d_3 + d_1 + a_{33} d_1 + d_2)$$
$$p_3 = a_{11} a_{33} + a_{22} a_{33} + a_{11} a_{22} - a_{12} a_{21} - a_{23} a_{32}$$

The equation (49) has two roots $z_{1,2} = \frac{-p_2 \pm \sqrt{p_2^2 - 4p_1p_3}}{2p_1}$ and for the instability we need to show that $p_2 < 0$ and $p_2^2 - 4p_1p_3 > 0$. For the minimum of $\rho_2(z)$ we take $\frac{d\rho_2}{dz} = 0$ gives $z = \frac{p_2}{2p_1}$ and $\frac{d^2\rho}{dz^2} = 2p_1 > 0$. Therefore $\rho_2(k^2) < 0$ at $z = k^2_\epsilon = \frac{p_2}{2p_1}$ and the minimum value is $\rho_2(k^2_\epsilon) = p_3 - \frac{p_2}{4p_1}$. Hence Turing instability occurs for the range $z_1 < k^2_\epsilon < z_2$.

Now similarly from the equation (48), for $\rho_3(k^2)$ we have

$$\rho_3(z) = q_1 z^3 + q_2 z^2 + q_3 z + q_4 \quad (50)$$

where,

$$q_1 = d_1 d_2 d_3$$
$$q_2 = -(a_{11} d_2 d_3 + a_{22} d_1 d_3 + a_{33} d_1 d_2)$$
$$q_3 = a_{11} a_{33} d_2 + a_{22} a_{33} d_1 + a_{11} a_{22} d_3 - d_3 a_{12} a_{21} - d_1 a_{23} a_{32}$$
$$q_4 = a_{32} a_{11} a_{23} + a_{33} a_{12} a_{21} - a_{21} a_{32} a_{13} - a_{11} a_{22} a_{33}$$

For the minimum of $\rho_3(z)$ we have $\frac{d\rho_3}{dz} = 0$ which gives $3q_1 z^2 + 2q_2 z + q_3 = 0$ has two roots $z_{1,2} = \frac{-q_2 \pm \sqrt{q_2^2 - 3q_1q_3}}{3q_1}$. We find $\frac{d^2\rho}{dz^2} = \pm 2\sqrt{q_2^2 - 3q_1q_3}$ so minimum at $k^2_\delta = \frac{-q_2 + \sqrt{q_2^2 - 3q_1q_3}}{3q_1}$ and the minimum value is

$$\rho_3(k^2_\delta) = \frac{2q_3^2 - 9q_1 q_2 q_3 + 27q_1^2 q_4 - 2(q_2^2 - 3q_1 q_3)^2}{27q_1^2} < 0.$$ 

So for the instability we need to show that $q_2 < 0$ and $q_2^2 - 3q_1 q_3 > 0$. Now for the third condition $\rho_1(k^2)\rho_2(k^2) - \rho_3(k^2) < 0$ we take $\Phi(k^2) = \rho_1(k^2)\rho_2(k^2) - \rho_3(k^2)$, for some wave number $k \geq 0$. From the equation (48) we get,

$$\Phi(z) = r_1 z^3 + r_2 z^2 + r_3 z + r_4 \quad (51)$$

where,

$$r_1 = b p_1 - q_1$$
$$r_2 = b p_2 - a p_1 - q_2$$
$$r_3 = b p_3 - a p_2 - q_3$$
$$r_4 = -(a p_3 + q_4)$$

for the suitability in calculation we choose $a$ and $b$ such that $a = a_{11} + a_{22} + a_{33}$ and $b = d_1 + d_2 + d_3$ for the minimum of $\Phi(z)$ we have $\frac{d\Phi}{dz} = 0$ which gives $3r_1 z^2 + 2r_2 z + r_3 = 0$ has two roots $z_{1,2} = \frac{-r_2 \pm \sqrt{r_2^2 - 3r_1 r_3}}{3r_1}$. 

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We find \( \frac{d^2 \Phi}{dz^2} = \pm 2\sqrt{r_2^2 - 3r_1 r_3} \) so minimum at \( k_2^2 = -r_2 + \sqrt{r_2^2 - 3r_1 r_3} \) and the minimum value is \( \Phi(k_2^2) = 2r_2^2 - 9r_1 r_2 r_3 + 27r_2^2 r_3^2 - 2(r_2^2 - 3r_1 r_3)^2 \frac{27r_1^2}{27r_1^2} < 0 \). So for the instability we need to show that \( r_2 < 0 \) and \( r_2^2 - 3r_1 r_3 > 0 \).

The existence of Turing pattern can be checked by planar stability, in which at wave number \( k = 0 \), \( \rho_n(k = 0) > 0 \), \( n = 1, 2, 3 \), and \( \rho_1(0) \rho_2(0) > \rho_3(0) \); however there exist at least one non-zero mode such that one or more of the conditions \( \rho_n(k) > 0 \), \( \rho_1(k) \rho_2(k) > \rho_3(k) \), is violated. And it is clear from table 2 that the patterns shown in Fig. 2 are Turing patterns because at \( k = 0 \) all \( \rho_n > 0 \) and at non-zero mode \( k = 15 \), \( \rho_3 \) is negative for both (A) and (B). Here in two dimensional case, \( k^2 = k_x^2 + k_y^2 \), we assume \( k_y = 0 \) and obtained the results in table 2.

### 6 Effect of disease transmission rate

In this manuscript, we have chosen a different parameter set to observe the spatial interaction between the species. We have observed both Turing patterns and spatiotemporal patterns. If we take disease transmission rate \( \lambda = 0 \), then there is no positive equilibrium, and with any positive initial we didn’t get any spatiotemporal and Turing patterns, shown in Fig 7(a). Since we want to deal with positive equilibrium which is most biological feasible, we keep \( \lambda = 0.007 \) all the species settle down to a stable position and Turing Patterns obtained (patterns not shown but the same as given in Fig 9). Further, for \( \lambda = 0.008 \), at the initial value (22.9113,19.0387,30.1736), the system shows a chaotic oscillation in the species same as given in Fig 5 and also settle down to a stable position for other initials. Further, we calculate Lyapunov exponents corresponding to Fig. 1(a) and obtain two positive and one negative Lyapunov exponents, shown in Fig. 1(d). The system (1) has at least one positive Lyapunov exponent, and the sum of the exponents are negative that means the system is dissipative and has a chaotic dynamic.

### 7 Cannibalistic attack rate

In the absence of cannibalism, i.e., when \( \sigma = 0 \) and other parameters fixed as used for spatiotemporal patterns, all the species coexist in oscillatory dynamic while neither Turing nor spatiotemporal patterns arise, as shown in Fig 7(b). When we increase the value of \( \sigma \) from 0.005 to 0.025, then the amplitude of oscillation increases and at 0.025 an attracting disc obtained at the initial (7.4427,18.5188,6.5779), as shown in Fig 7. Further at \( \sigma = 0.026 \), stable focus observed at initial (15.1342,20.5234,6.3140) shown in Fig 8 and two different Turing patterns can be seen in Fig 9(A) and 9(B). We have also plotted the space-series corresponding to Fig 9 with diffusive rates \( 10^{-5}, 10^{-3} \) and \( 10^{-10} \) (similar space-series obtained for the diffusive rates \( 10^{-6}, 10^{-3} \) and \( 10^{-10} \) but result not shown) and observe that due to the effect of diffusion, the species coexist in oscillatory dynamics. Next, when we increase \( \sigma \) to 0.30, at the initial value (14.7832,23.1122,8.2449), the species coexist into a chaotic oscillation (shown in Fig 8 and also settle down to a stable position for other initials. Next, we have calculated the Lyapunov exponents in Fig. 2(d), where two of them are positive, and one is negative, that means the system (1) is chaotic. Since the sum of the exponents is negative, that means the system (1) is dissipative as well. We have also calculated Lyapunov exponents in Fig. 8(d) and 8(d), and observed that all three Lyapunov exponents in both figures are negative, that means the system (1) has stable fixed points.

### 8 Numerical simulation

In this section, we are going to discuss the spatial interaction between the species. Which can be seen through pattern formation for specific choices of parameter values. To obtain these patterns, we have taken a square domain
Figure 1: Dynamics of model system (1) at parameter used for spatiotemporal patterns. (a) three-dimensional phase plot. (b) two-dimensional phase plot (c) time series plot and (d) dynamics of the Lyapunov exponents.

Figure 2: Dynamics of model system (1) at $\sigma = 0.025$ and other parameters are fixed. (a) three-dimensional phase plot. (b) two-dimensional phase plot (c) time series plot and (d) dynamics of the Lyapunov exponents.

with a zero-flux boundary condition and applied the FTCS numerical scheme to solve the spatial model system (2). Spatial interaction in one-dimensional is possible, but in this article, our assumption is to interacting the species in two-dimensional space. For the reaction part, we have used the forward difference scheme while standard five-point explicit finite difference scheme used for the diffusion part. The initial distribution is considered with small perturbation of the form $0.1 \cos^2(10x) \cos^2(10y)$ about the steady state. Here, we have simulated both Turing and spatiotemporal patterns for different parameter values as given in table (3). Turing patterns demonstrated in Fig (9) and Spatiotemporal patterns demonstrated in Figs. [5]–[11].
Figure 3: Dynamics of model system (1) at parameters used for Turing patterns. (a) three-dimensional phase plot. (b) two dimensional phase plot (c) time series plot and (d) dynamics of the Lyapunov exponents.

Figure 4: Space-series plot corresponding to Fig 3. (a) space-series for Prey (b) space-series for susceptible predator and (c) space-series for infected predator.

All the simulations are performed in Matlab for two different sets of parameters. In each snapshot, the yellow colour represents the high population density, while the blue colour represents the low population density. The Turing patterns are obtained when we increase the value of cannibalistic attack rate $\sigma$ from 0.005 to 0.026. Earlier, we have proved and validate the existence of Turing patterns through Turing instability. The perturbations are considered the interior equilibrium point $(u^*, v^*, w^*)$ obtained using these parameters is $(8.1844, 19.0716, 6.7682)$. Here, we perform simulation over $[0, \pi] \times [0, \pi]$ with spatial resolution $\Delta x = \Delta y = 0.01$ and time step $\Delta t = 0.01$. It has been observe that in the absence of cannibalism i.e., when $\sigma = 0$ no Turing patterns are arise, shown in Fig. 7 even the higher values of cannibalistic attack rate promote Turing patterns. The snapshots of Fig. 9(A), shows distribution of the species (Prey, susceptible predator and infected predator) with diffusive rates $10^{-5}$, $10^{-3}$ and $10^{-10}$ and Fig. 9(B) represents at diffusive rates $10^{-5}$, $10^{-3}$ and $10^{-10}$ in two dimensional space ($x$ and $y$).

| Fig. | Patterns       | $d_1$  | $d_2$  | $d_3$  | $\sigma$ | Rest of the parameters |
|------|----------------|-------|-------|-------|---------|------------------------|
| 9(A) | Turing         | $10^{-9}$ | $10^{-3}$ | $10^{-10}$ | 0.026   | $r = 0.4, K = 68, \alpha_1 = 0.3,$ |
|      |                |        |       |       |         | $\alpha_2 = 0.1, \gamma = 10, \alpha = 1,$ |
|      |                |        |       |       |         | $c_2 = 0.2, l = 0.08, f = 10,$ |
|      | spatiotemporal | $10^{-6}$ | $10^{-4}$ | $10^{-10}$ | 0.005   | $\lambda = 0.003, d = 0.02, e = 0.01,$ |
|      | spatiotemporal | $10^{-10}$ | $10^{-10}$ | $10^{-10}$ | 0.005   | $\beta = 0.5, c_1 = 1,$ |
Figure 5: Chaotic oscillations in the species at $\sigma = 0.030$. (a) three dimension phase plot (b) two dimension phase plot (c) time-series plot and (d) dynamics of the Lyapunov exponents.

Figure 6: Time-series plot at $\sigma = 0.030$, for the (a) prey species (b) susceptible predator and (c) infected predator.

Figure 7: Contour plot of the population densities, when (a) disease transmission rate $\lambda = 0$ and (b) cannibalistic attack rate $\sigma = 0$.

We have observed both (A) and (B) for $t = 15000$ but no changes are observed therefore we have presented the snapshots at $t = 1000$. The resulted patterns are like labyrinth patterns.

Similarly, we have obtained spatiotemporal patterns as parameters are given in table 3. Only here the cannibalistic attack rate $\sigma$ changed to 0.005 and the diffusive rates are changed the snapshots of Fig. 8 shows the distribution of the species (Prey, susceptible predator and infected predator) with diffusive rates $10^{-6}$, $10^{-6}$ and $10^{-10}$ in two-dimensional space. Each individual species exhibits different spatial patterns at initial stages, i.e. $t = 500, 1000, 1500, 2000$ but after $t = 2000$ the distribution of the species at each stage was almost same. At
at \( t = 500, 1000 \) most of the patterns are like circular, rectangular or mixed spot patterns while at \( t = 1500 \) prey and susceptible predator species exhibit star and spot like patterns. At \( t = 2000 \) system exhibits grid mixed spot patterns. At \( t = 5000 \) the species exhibits patchy patterns while the interesting part occurs at \( t = 10000 \) when these patchy patterns change into spirals. These spirals are not regular. At each stage, pattern changes, which shows instability in the system behaviour. Actually, we have done so many observations on diffusive rates to generate patterns, all are not shown in this manuscript and from there we conclude that for the other combination of diffusive rates like \( 10^{-10}, 10^{-4}, 10^{-10} \) and \( 10^{-10}, 10^{-6}, 10^{-10} \), after certain stages all the species exhibits almost same patterns and stop to change after some time. While the spatial patterns we have shown in Fig.8 corresponding to the diffusive rates \( 10^{-6}, 10^{-6}, 10^{-10} \) are changing continuously with time. The spatial patterns in Fig.10 shows distribution of the species with same initial distribution and with diffusive rates \( 10^{-10}, 10^{-4} \) and \( 10^{-10} \). Same parameter values are taken to generate Fig.10. At \( t = 2000 \) patterns are appeared and change until \( t = 5000 \). The spatial distribution of the species was almost same at each stage. Next, in Fig.11 we obtain spatial pattern for diffusive rates \( d_1 = 10^{-10}, d_2 = 10^{-6} \) and \( d_3 = 10^{-10} \) and it is observed that each individual species exhibits different spatial patterns at initial stages, i.e. \( t = 500, 1000, 1500 \) but after \( t = 1500 \), the distribution was almost same.

![Figure 8](image_url)

Figure 8: Spatiotemporal patterns of prey, susceptible predator and infected predator densities of the model system (4), at (A) \( t = 500 \), (B) \( t = 1000 \), (C) \( t = 1500 \), (D) \( t = 2000 \), (E) \( t = 4000 \), and (F) \( t = 10000 \).
Figure 9: Turing patterns of prey, susceptible predator and infected predator densities of the model system (4). (A) at $t = 1000$, for the diffusive rates $10^{-5}$, $10^{-3}$ and $10^{-10}$. (B) at $t = 1000$, for the diffusive rates $10^{-6}$, $10^{-3}$ and $10^{-10}$. 
Figure 10: Spatiotemporal patterns of prey, susceptible predator and infected predator densities of the model system (4), at (A) $t = 2000$, (B) $t = 3000$, (C) $t = 4000$, and (D) $t = 5000$. 
Figure 11: spatiotemporal patterns of prey, susceptible predator and infected predator densities of the model system (4), at (A) $t = 500$, (B) $t = 1000$, and (C) $t = 1500$.

9 Conclusion

We have obtained the conditions on diffusive rates that are responsible for the appearance of stationary patterns. We have established sufficient conditions for the local asymptotic stability as well as for global asymptotic stability. For the purpose, we have used the linearization and Lyapunov method technique. In the case of large diffusivity, we observed that population evolution influenced by diffusion. From theorem 4.5, under certain conditions, there is no non-constant positive solution provided that diffusion coefficient $d_1$ and $d_3$, are fixed and $d_2$ is sufficiently large that means the variation in the diffusive rate of infected predator prevent the emergence of patterns. From theorem 4.8, under the certain condition there exist at least one constant positive solution provided the diffusion coefficient $d_1$ and $d_3$, are fixed and $d_2$ is sufficiently large that means a diffusive rate of the susceptible predator is much responsible for the appearance of Turing patterns. Thus we can conclude that large diffusivity may be one of the reasons for the appearance and non-appearance of stationary patterns. Further, we have discussed bifurcation of non-constant positive solutions along with the theorems 4.9 and 4.10. We have discussed Turing instability in theorem 5.1, which ensures the existence of Turing patterns. Further, we have chosen two sensitive parameters of the model system (1), disease transmission rate $\lambda$ and cannibalistic attack rate $\sigma$, which exhibit various dynamics such as limit cycle, stable focus, and chaotic oscillation. We have calculated Lyapunov exponents at different parameter set and observe that the system shows chaotic as well as stable fixed dynamics. Even it seen that in the absence of cannibalism and disease, the distribution of the species in space is not possible means patterns are not possible. The increment in the cannibalistic attack rate and disease transmission rate promote the Turing patterns.

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