AN IMPROVED BERRY-ESSÉEN BOUND OF LEAST SQUARES ESTIMATION FOR FRACTIONAL ORNSTEIN-UHLENBECK PROCESSES

YONG CHEN AND XIANGMENG GU

ABSTRACT. The aim of this paper is twofold. First, it offers a novel formula to calculate the inner product of the bounded variation function in the Hilbert space $\mathcal{H}$ associated with the fractional Brownian motion with Hurst parameter $H \in (0, \frac{1}{2})$. This formula is based on a kind of decomposition of the Lebesgue-Stieljes measure of the bounded variation function and the integration by parts formula of the Lebesgue-Stieljes measure. Second, as an application of the formula, we explore that as $T \to \infty$, the asymptotic line for the square of the norm of the bivariate function $f_T(t, s) = e^{-\theta |t-s|} 1_{\{0 \leq s, t \leq T\}}$ in the symmetric tensor space $\mathcal{H}^\otimes 2$ (as a function of $T$), and improve the Berry-Esséen type upper bound for the least squares estimation of the drift coefficient of the fractional Ornstein-Uhlenbeck processes with Hurst parameter $H \in (\frac{1}{4}, \frac{1}{2})$. The asymptotic analysis of the present paper is much more subtle than that of Lemma 17 in Hu, Nualart, Zhou(2019) and the improved Berry-Esséen type upper bound is the best improvement of the result of Theorem 1.1 in Chen, Li (2021). As a by-product, a second application of the above asymptotic analysis is given, i.e., we also show the Berry-Esséen type upper bound for the moment estimation of the drift coefficient of the fractional Ornstein-Uhlenbeck processes where the method is obvious different to that of Proposition 4.1 in Sottinen, Viitasaari(2018).

Keywords: Fractional Brownian motion; Fractional Ornstein-Uhlenbeck process; Berry-Esséen bound.

MSC 2010: 60G15; 60G22; 62M09.

1. INTRODUCTION

Unless otherwise specified, the Hurst parameter in this paper is always assumed to be $H \in (0, \frac{1}{2})$. This article has two main purposes. One is to improve the Berry-Esséen bound of the least squares estimation of the drift coefficients of the fractional Ornstein-Uhlenbeck process based on continuous sample observations. The second is to give an easy to calculate formula for the inner product of Hilbert space...
$H$ connected by fractional Brownian motion when it is restricted to bounded variation function. For the two purposes of this paper, the former can be regarded as a very effective application of the latter. In addition, as an accessory product, we also give the second application of the latter: The method of proving Berry-Esséen bound for moment estimation of drift coefficients of fractional Ornstein-Uhlenbeck process is different from that of Proposition 4.1 of [1] and Theorem 5.4 of [2]. The conclusions of this paper are novel. It is particularly worth emphasizing that, as far as we know, there is no alternative method to obtain the upper bound of the improved Berry-Esséen class for the least squares estimation of drift coefficients.

In addition, we also point out that the binary function in symmetric tensor space $H^{\otimes 2}$ obtained by using this method

$$f_T(t, s) = e^{-\theta|t-s|}1_{\{0 \leq s, t \leq T\}} \quad (1.1)$$

The asymptotic property of norm square (see Proposition 1.9) is much more precise than that of Lemma 17 in [3].

Specifically, we consider the fractional Ornstein-Uhlenbeck process based on continuous time observation

$$dX_t = -\theta X_t dt + \sigma dB^H_t, \quad X_0 = 0, \quad 0 \leq t \leq T, \quad (1.2)$$

Berry-Esséen class upper bounds for two kinds of estimators of drift coefficients, including $\theta > 0$ is the drift coefficient $\sigma > 0$ is the volatility coefficient, $B^H_t$ is the one-dimensional fractional Brownian motion with Hurst parameter $H$, and its covariance function is given by the following formula:

$$R_H(t, s) = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}). \quad (1.3)$$

Without losing generality, the following is constant $\sigma = 1$. Reference [3] minimized the following formula

$$\int_0^T |X_t + \theta X_t|^2 dt, \quad (1.4)$$

and calculating the limit of the second moment of the sample (orbit) of the OU process

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T X_t^2 dt, \quad (1.5)$$
When traversal is constructed (i.e. $\theta > 0$), the least squares estimation and moment estimation of the drift coefficient are respectively:

$$\hat{\theta}_T = -\frac{\int_0^T X_t dX_t}{\int_0^T X_t^2 dt} = \theta - \frac{\int_0^T X_t dB_t^H}{\int_0^T X_t^2 dt},$$  \hspace{1cm} (1.6)

$$\tilde{\theta}_T = \left(\frac{1}{H\Gamma(2H)T} \int_0^T X_t^2 dt\right)^{-1/2},$$  \hspace{1cm} (1.7)

As in reference [3], this paper does not discuss the meaning of the first random integral about the fractional OU process $X_t$ at the right end of (1.6), but only regards it as a formal integral, that is, it is only understood as substituting the direct form of equation (1.2) into the integral. The second random integral about $B_t^H$ at the right end of (1.6) obtained after substitution is understood as a divergent (or skorohold) integral about fractional Brownian motion, but its meaning as a statistic in the sense of standard statistics is not studied. Of course, the statistical meaning of the second statistic moment estimation is completely clear.

Further, by verifying the fourth order moment theorem, reference [3] gives the strong convergence and asymptotic normality of the least squares estimate and the moment estimate. The two asymptotic properties of the norm of the binary function $f_T(t, s)$ and its contraction are the key steps. For the former, they use a formula of the inner product of space $\mathcal{H}$ and tensor space $\mathcal{H} \otimes^2$, see (2.5) for details. This formula is the expression formula for the inner product of bounded variation function in Hilbert space associated with the general second moment process given by the integral by parts formula in combination with [4]: the inner product is equal to the integral of the product of the covariance function of the second moment process with respect to the measure derived from two bounded variation functions. For the norm of the compression of binary function $\frac{1}{\sqrt{T}} f_T(t, s)$, they use Fourier transform to prove that it tends to zero, see (2.6).

Based on the above results in [3], reference [5] gives the convergence rate between the distribution of the least squares estimate and its asymptotic distribution, that is, the upper bound of Berry-Esséen class: when $H \in (0, \frac{1}{2})$ and $T$ are sufficiently large, random variable

$$\sqrt{T}(\hat{\theta}_T - \theta)$$
and the upper bound of Kolmogorov distance of normal random variable is $T^{-\beta}$, we have:

$$\beta = \begin{cases} 
\frac{1}{2}, & H \in [0, \frac{1}{4}], \\
1 - 2H, & H \in (\frac{1}{4}, \frac{1}{2}). 
\end{cases}$$  \hspace{1cm} (1.8)

Here, the method of proving the Berry-Ésséen bound of the least squares estimate is based on the Corollary 1 of [6] and two asymptotic analyses of the binary function $f_T(t, s)$. The method to prove the Berry-Ésséen bound of moment estimation is to transform the fourth order moment into the two asymptotic analyses of the binary function $f_T(t, s)$ through the multiplication formula of multiple Wiener integrals, and to estimate the inner product of $f_T(t, s)$ and $h_T(t, s)$ (see (1.17)), see [7] and [8]. Different from this, Proposition 4.1 of [1] and Theorem 5.4 of [2], the proof of the Berry-Ésséen bound for moment estimation is to transform the fourth-order moment into an asymptotic analysis of the stationary solution of the fractional OU process by the Wick formula, and the latter is known, see [9].

Review (1.8), when $H = \frac{1}{2} - \varepsilon$ and $\varepsilon$ sufficiently small, $\beta$ tends to zero. This is the same as when $H = \frac{1}{2}$, the known Berry-Ésséen bound of $\sqrt{T}(\hat{\theta}_T - \theta)$ is $\frac{1}{\sqrt{T}}$, which is very far away, so a reasonable guess is:

“when $H \in (\frac{1}{4}, \frac{1}{2})$, the upper bound of Berry-Ésséen class is still $\frac{1}{\sqrt{T}}$.”

We will prove this conjecture in this paper. It can be seen from the proof of Theorem 1.1 in [7] that the key problem is a more precise asymptotic analysis of the norm of the bivariate function $f_T(t, s)$. Therefore, obtaining the asymptotic analysis of this binary function norm is the key step of this paper, and we describe the result of this asymptotic analysis as the following theorem:

**Theorem 1.1.** Let $\theta > 0$, $H \in (0, \frac{1}{2})$. For binary function $f_T(t, s)$ in space $\mathcal{H}^{\otimes 2}$, see (1.1), There is a normal number $C_{H, \theta}$ that does not depend on $T$, so that when $T$ is sufficiently large, there is an inequality

$$\left| \| f_T \|_{\mathcal{H}^{\otimes 2}}^2 - 2(H\Gamma(2H))^2\sigma_H^2 T \right| \leq C_{H, \theta}$$  \hspace{1cm} (1.9)

holds, where

$$\sigma_H^2 = (4H - 1) + \frac{2\Gamma(2 - 4H)\Gamma(4H)}{\Gamma(2H)\Gamma(1 - 2H)}.$$  \hspace{1cm} (1.10)

**Remark 1.2.** (1) The upper bound given by formula (1.9) in this paper is a constant $C_{H, \theta}$, which is independent of $T$, that is, the order of the upper bound of $T$ is 0. In contrast, the upper bound corresponding to the result
of Lemma 3.11 in [5] is $T^{2H}$, that is, the order of the upper bound of $T$ is $2H$. Furthermore, the upper bound in this paper is the best upper bound in the sense of the asymptote below.

(2) In fact, the conclusion obtained in this paper is stronger than (1.9). That is, this paper actually obtains the square of the norm of the binary function $f_T(t, s)$, as a function of $T$, the asymptote when $T \to \infty$:

$$\lim_{T \to \infty} \left( \|f_T\|_{2H}^2 - 2(H\Gamma(2H))^2\sigma_H^2 T \right) = C_H,$$

(1.11)

Here $C_H \in \mathbb{R}$ is a constant that depends only on $H$ and is independent of $T$. See the proof of Theorem 1.1 in Section 3 of this paper for details. We emphasize that in this paper, the intercept term $C_H$ of the asymptote is irrelevant, while the existence and slope of the asymptote play a key role.

(3) The standard $o, O$ symbols in asymptotic analysis are used to compare Lemma 17 in [3], Lemma 3.11 in [5], and the formula (1.9) in this paper as follows: when $T \to \infty$, we have:

$$\frac{1}{T} \|f_T\|_{2H}^2 - 2(H\Gamma(2H))^2\sigma_H^2 = o(1),$$

(1.12)

$$\frac{1}{T} \|f_T\|_{2H}^2 - 2(H\Gamma(2H))^2\sigma_H^2 = O(T^{2H-1}),$$

(1.13)

$$\frac{1}{T} \|f_T\|_{2H}^2 - 2(H\Gamma(2H))^2\sigma_H^2 = O(T^{-1}).$$

(1.14)

As a comparison, the method of Lemma 17 in [3] can obtain formula (1.12) succinctly, and the method of Lemma 3.11 in [5] is based on Lemma 17 in [3]. However, this method cannot be further improved, that is, the above formula (1.14) cannot be obtained. In other words, the method of using the new formula for calculating the inner product of fractional Brownian motion given in this paper is, as far as we know, still irreplaceable.

Starting from the asymptotic analysis given by the above theorem, the following theorem shows that when $H \in (\frac{1}{4}, \frac{1}{2})$, the improved Berry-Ésséen bound of the least squares estimate is the $\frac{1}{\sqrt{T}}$ guessed above, and as a by-product, the Berry-Ésséen bound of the moment estimate is also $\frac{1}{\sqrt{T}}$.

**Theorem 1.3.** Let $Z$ be a standard normal random variable and $H \in (0, \frac{1}{2})$. Then there is a normal number $C_{0,H}$, and it does not depend on $T$, so that when $T$ is
large enough, there is Berry–Esséen inequality

\[
\sup_{z \in \mathbb{R}} \left| P\left( \sqrt{\frac{T}{\theta\sigma^2_H}} (\hat{\theta}_T - \theta) \leq z \right) - P(Z \leq z) \right| \leq \frac{C_{\theta,H}}{\sqrt{T}}; \quad (1.15)
\]

\[
\sup_{z \in \mathbb{R}} \left| P\left( \sqrt{\frac{4H^2T}{\theta\sigma^2_H}} (\hat{\theta}_T - \theta) \leq z \right) - P(Z \leq z) \right| \leq \frac{C_{\theta,H}}{\sqrt{T}}; \quad (1.16)
\]

hold, where \( \sigma^2_H \) as (1.10).

Remark 1.4. We point out that in the Berry–Esséen class inequality estimates of two statistics (see (3.24) and (3.25)), part of the source of the upper bound \( \frac{1}{\sqrt{T}} \) is based on the key inequality (3.17) in [3], that is, the upper bound of the function \( f_T(s, t) \) about its own compressed \( f_T \otimes_1 f_T \) norm in space \( \mathcal{H}^{\otimes 2} \) is the key fact of \( \sqrt{T} \). The upper bound estimation is obtained by another formula for calculating the inner product in \( \mathcal{H} \), namely Fourier transform, as shown in formula (2.6). In a word, we finally get the Berry–Esséen class upper bound estimates of the two statistics in this paper using four very different formulas for calculating the inner product of \( \mathcal{H}^{\otimes 2} \): (1.23), (2.3), (2.5) and (2.6). In other words, except for the formula (2.8) for calculating the inner product using the operator \( K^*_H \), all the other four formulas for calculating the inner product of \( \mathcal{H} \) mentioned in Section 2 have been used.

In this paper, the method of proving Berry–Esséen inequality (1.16) of moment estimation is based on the following proposition, which gives the estimation of the inner product of binary functions \( f_T, h_T \) in tensor space \( \mathcal{H}^{\otimes 2} \). Here, binary function

\[
h_T(t, s) = e^{-\theta(T-t) - \theta(T-s)} \mathbb{1}_{\{0 \leq s, t \leq T\}}. \quad (1.17)
\]

Proposition 1.5. Let the binary functions \( f_T, h_T \) be given in (1.1) and (1.17) respectively, then there is a constant \( C_H \) independent of \( T \), which makes the following inequality hold:

\[
|\langle f_T, h_T \rangle_{\mathcal{H}^{\otimes 2}}| \leq C_H. \quad (1.18)
\]

Remark 1.6. Proposition 1.5 and Theorem 1.1 have the same point in that they are both the inner product of two bivariate functions estimated in \( \mathcal{H}^{\otimes 2} \). The difference is that the former actually divides the integral region into nine blocks, and finally
IMPROVED BERRY-ESSÉEN BOUND OF LSE FOR FOU

reduces it to three kinds of integral calculations by symmetry and other methods. The latter takes advantage of the particularity of the function \( h_T(s, t) \) to separate variables, so it regresses to the problem of estimating the inner product of two univariate functions in \( \mathcal{H} \), and conveniently uses the inner product calculation formula in Inference 2.4. Compared with the two methods, the whole process of the former is very complicated and the latter is very simple. However, since the function \( f_T(s, t) \) is not variable separated, the latter method is not applicable to the former. As far as we know, we do not know whether there are other simpler methods to prove the conclusion of Theorem 1.1.

In the second half of this section, we give a new formula for calculating the inner product of \( H \in (0, \frac{1}{2}) \) space-time \( \mathcal{H} \) and symmetric tensor space \( \mathcal{H}^{\otimes 2} \). See Propositions 1.9 and 1.12. The new formula is similar to but also obviously different from the following well-known facts to some extent: When Hurst parameter \( H \in (0, \frac{1}{2}) \), the formula for the inner product of two disjoint functions \( f \) and \( g \) in Hilbert space \( \mathcal{H} \) connected by fractional Brownian motion is the same as that for the inner product when \( H \in (\frac{1}{2}, 1) \), see [9, 12], or see Corollary 2.4. The new formula for calculating the inner product given in Proposition 1.9 of this paper can be explained as follows: The integral region \([0, T]^2\) is divided into the following three parts. For the double integral on region

\[
\kappa_1 := \{(u, v) \in [0, T]^2 : 0 \leq v \leq u - 1 \leq T - 1\}
\]

and

\[
\kappa_2 := \{(u, v) \in [0, T]^2 : 0 \leq u \leq v - 1 \leq T - 1\}
\]

the Partial integral formula on the measure is applied twice, while for the double integral on region

\[
\kappa_3 := \{(u, v) \in [0, T]^2 : 0 \vee (u - 1) \leq v \leq (u + 1) \wedge T\}
\]

the Partial integral formula on the measure is applied only once. The integral domain decomposition is shown in Figure 1.

**Notation 1.7.** Record \( \alpha_H = H(2H - 1) \). Let \( \mathcal{V}_{[0,T]} \) be the whole set of bounded variation functions defined on \([0, T]\). For any \( f \in \mathcal{V}_{[0,T]} \), \( f^0 \) is defined as

\[
f^0(x) = \begin{cases} f(x), & x \in [0, T], \\ 0, & \text{other.} \end{cases}
\]
Record $\nu_f$ is the limit of Lebesgue-Stieljes measure on $([0, T], B([0, T]))$ of $f^0(x)$ connection on $([0, T], B([0, T]))$. In particular, the following more special form is used in this paper, that is, let $0 \leq a < b \leq T$, and $g = f \cdot 1_{[a, b]}$, where $f$ is a differentiable function, then:

$$
\nu_g(dx) = f'(x) \cdot 1_{[a, b]}(x)dx + f(x) \cdot (\delta_a(x) - \delta_b(x))dx,
$$

(1.22)

Here $\delta_a(\cdot)$ is a dirac generalized function whose mass is concentrated at point $a$. For ease of use, we use the notation partial $\frac{\partial g}{\partial x}$ to represent the “density function” in the form of measure (1.22).

Remark 1.8. The details of the above measures can be found in [4], which is the source of the new inner product expression formula in this paper, and is also one of the starting points of this paper. The purpose of introducing this measure is to use the Partial integral formula about this measure. In other words, its convenience is to absorb the values of endpoints $a, b$ into the measure through two Dirac generalized functions (or Dirac single point measure) $\nu_g$, so that it is convenient to use the Partial integral formula of $\nu_g$. See Lemma 2.2 for details.

Proposition 1.9. If $f, g \in V_{[0, T]}$, then

$$
\langle f, g \rangle_H = \alpha_H \left( \int_0^T g(t)dt \int_0^{t-1} f(s)(t-s)^{2H-2}ds + \int_1^T f(s)ds \int_0^{s-1} g(t)(s-t)^{2H-2}dt \right)

- H \int_0^T g(t)dt \int_0^T (t^{2H-1} - \text{sgn}(t-s)|t-s|^{2H-1}) \nu_{\tilde{f}_t}(ds),
$$

(1.23)

including $\tilde{f}_t(s) = f(s) \cdot 1_{[(t-1)\vee 0, (t+1)\wedge T]}(s)$ is a family of functions with $s$ as the independent variable and $t$ as the parameter. The meaning of $\nu_{\tilde{f}_t}(ds)$ is given in
notation 1.7 and (1.22). In addition, take any two positive numbers \( \varepsilon_1, \varepsilon_2 \in (0, T) \), record
\[
\tilde{f}_t(s) = f(s) \cdot 1_{[t-\varepsilon_1, (t+\varepsilon_2)\land T]}(s),
\]
then (1.23) can be generalized as:
\[
\langle f, g \rangle_{\mathcal{H}} = a_H \left( \int_{\varepsilon_1}^{T} g(t) dt \int_{0}^{t-\varepsilon_1} f(s)(t-s)^{2H-2} ds + \int_{\varepsilon_2}^{T} f(s) ds \int_{0}^{s-\varepsilon_2} g(t)(t-s)^{2H-2} dt \right)
- H \int_{0}^{T} g(t) dt \int_{0}^{T} (t^{2H-1} - \text{sgn}(t-s)|t-s|^{2H-1}) \nu_{\tilde{f}_t}(ds).
\] (1.24)

**Remark 1.10.** The significant difference between the inner product calculation formula (Proposition 1.9) obtained by the above division of the integral region and the inner product calculation formula when the supports do not intersect is that the latter requires that the support set of the binary function is \( \{(u, v) : 0 \leq v \leq u \leq T\} \) or \( \{(u, v) : 0 \leq v \leq u \leq T\} \) a rectangular sub region. See Corollary 2.4. This significant difference is reflected in Figure 1 to some extent.

Note that \( \mathcal{H}^{\otimes 2} \) and \( \mathcal{H}^{\otimes 2} \) are quadratic tensor product spaces and quadratic symmetric tensor product spaces of \( \mathcal{H} \). Proposition 1.12 will give the calculation formula of the inner product of binary symmetric functions in \( \mathcal{H}^{\otimes 2} \), which is the direct inference of Proposition 1.9, so the derivation details are omitted below. For the convenience of expression, the following marks are introduced:

**Notation 1.11.** Let \( \mathcal{L}(C_0, \mathbb{R}) \) be the whole set of bounded linear functionals defined on the set of compact supported continuous functions \( C_0 \). Let \( a, b \in [0, T] \), define three linear operators as follows:

1. Operators of \( \mathcal{V}_{[0,T]} \rightarrow \mathcal{L}(C_0, \mathbb{R}) \):
\[
\frac{\partial_a}{\partial s} f(s) = \frac{\partial}{\partial s}(f(s) \cdot 1_{[(a-1)\lor 0,(a+1)\land T]}(s)),
\]
I.e.: \( \frac{\partial_a}{\partial s} f(s) \) is the density function of measure \( \nu_{f_a} \) corresponding to function \( f_a(s) \) in Proposition 1.9, see also (1.22).

2. Operators of \( \mathcal{V}^{\otimes 2}_{[0,T]} \rightarrow \mathcal{L}(C_0, \mathbb{R}) \otimes \mathcal{V}_{[0,T]} \):
\[
\frac{\partial_a}{\partial s} \varphi(s,t) = \frac{\partial}{\partial s}(\varphi(s,t) \cdot 1_{[(a-1)\lor 0,(a+1)\land T]}(s)).
\]

3. Operators of \( \mathcal{V}^{\otimes 2}_{[0,T]} \rightarrow \mathcal{L}(C_0, \mathbb{R})^{\otimes 2} \):
\[
\frac{\partial_a \partial_b}{\partial s \partial t} \varphi(s,t) = \frac{\partial^2}{\partial s \partial t}(\varphi(s,t) \cdot 1_{[(a-1)\lor 0,(a+1)\land T]}(s) \cdot 1_{[(b-1)\lor 0,(b+1)\land T]}(t)).
\]
Proposition 1.12. Let \( \vec{s} = (s_1, s_2) \), \( \vec{t} = (t_1, t_2) \) and \( (\vec{s}, \vec{t}) \in \kappa_i \times \kappa_j \), \( i, j = 1, 2, 3 \), \( \kappa_i \) see (1.19)-(1.21). If \( \phi, \psi \in \mathcal{V}^{\otimes 2}_{[0,T]} \), then

\[
\langle \psi, \phi \rangle_{H^{\otimes 2}} = \alpha_H^2 \sum_{i,j=1}^{2} \int_{\kappa_i \times \kappa_j} \psi(s_1, t_1) \phi(s_2, t_2) |s_1 - s_2|^{2H-2} |t_1 - t_2|^{2H-2} d\vec{s} d\vec{t} \\
- 2\alpha_H \sum_{i=1}^{2} \int_{\kappa_i \times \kappa_i} \psi(s_1, t_1) \frac{\partial_{s_2}}{\partial s_2} \phi(s_2, t_2) \frac{\partial R_H}{\partial s_1}(s_1, s_2) |t_1 - t_2|^{2H-2} d\vec{s} d\vec{t} \\
+ \int_{\kappa_i \times \kappa_i} \psi(s_1, t_1) \frac{\partial R_H}{\partial s_1}(s_1, s_2) \partial_{t_2} \phi(s_2, t_2) \frac{\partial_{s_2}}{\partial s_2} \phi(s_2, t_2) d\vec{s} d\vec{t},
\]

(1.25)

where \( R_H(t_1, t_2) \) is the covariance of fractional Brownian motion (see (1.3)), Operators \( \frac{\partial_{a}}{\partial s}, \frac{\partial_{b}}{\partial s} d\vec{t} \) see mark 1.11.

Remark 1.13. (1) If \( \psi, \phi \) is asymmetric, then

\[
\sum_{i=1}^{2} \int_{\kappa_i \times \kappa_i} \psi(s_1, t_1) \frac{\partial_{s_2}}{\partial s_2} \phi(s_2, t_2) \frac{\partial R_H}{\partial s_1}(s_1, s_2) |t_1 - t_2|^{2H-2} d\vec{s} d\vec{t} \\
\neq \sum_{i=1}^{2} \int_{\kappa_i \times \kappa_i} \psi(s_1, t_1) \frac{\partial_{t_2}}{\partial t_2} \phi(s_2, t_2) \frac{\partial R_H}{\partial t_1}(t_1, t_2) |s_1 - s_2|^{2H-2} d\vec{s} d\vec{t}.
\]

(2) The essence of inner product formula (1.25) is also measure decomposition, that is, for any given \( (s_1, t_1) \in [0,T]^2 \), \( \nu_\phi \), the measure \( \nu_\phi \) on \([0,T]^2\) associated with the binary function \( \phi(s_2, t_2) \in \mathcal{V}^{\otimes 2}_{[0,T]} \) is decomposed into the sum of the measures derived from the limitation of \( \phi(s_2, t_2) \) itself on \( (M_{ij}, \mathcal{B}(M_{ij})) \), \( i, j = 1, 2, 3 \), (see Figure 2 for details), where

\[
M_{11} = \{(s_2, t_2) \in [0,T]^2, s_2 \leq s_1 - 1, t_2 \leq t_1 - 1 \}, \\
M_{33} = \{(s_2, t_2) \in [0,T]^2, s_1 - 1 \leq s_2 \leq s_1 + 1, t_1 - 1 < t_2 \leq t_1 + 1 \},
\]

Others are similar and here \( M_{ij} \) is the closure of \( M_{ij} \).
The rest of this paper is arranged as follows: in Section 2 we briefly review various known calculation formulas of inner product in $\mathcal{H}$ and prove Proposition 1.9. In Section 3 we prove Proposition 1.5, Theorem 1.1 and Theorem 1.3. As an appendix, in Section 4 we give asymptotes of various multiple integrals used in the proof of Theorem 1.1. Finally, we point out that the constants $C_H, C_{H,\theta}$ independent to $T$, and can be different from line to line.

2. Preparation knowledge and proof of new calculation formula of inner product in $\mathcal{H}$

2.1. Preparation knowledge. Record $\mathcal{E}$ as all the real valued ladder functions on $[0,T]$, and assign the inner product on them:

$$\langle 1_{[a,b)}, 1_{[c,d])\rangle_{\mathcal{H}} = \mathbb{E}(\langle B^H_{[b,c]}(B^H_{[a,d]} - B^H_{[c,d]} \rangle).$$  

$\mathcal{H}$ is the Hilbert space of $\mathcal{E}$ after completion. On the premise of preserving the linear structure and norm, the mapping $1_{[0,t]} \mapsto B^H_t$ is extended to $\mathcal{H}$, and the isometric isomorphic mapping is recorded as $\varphi \mapsto B^H(\varphi)$. And called $\{B^H(\varphi), \varphi \in \mathcal{H}\}$ is a Gaussian equidistant process connected with Hilbert space $\mathcal{H}$. The expression formula of $\mathcal{H}$ inner product in Hilbert space is discussed in two cases.

1. When $H > \frac{1}{2}$, The covariance of $B^H_t$ can be written as

$$R_H(t,s) = \alpha_H \int_0^s du \int_0^t |u - v|^{2H-2} dv,$$

where $\alpha_H = H(2H - 1)$. But for any $f, g \in \mathcal{H}$, we have

$$\langle f, g \rangle_{\mathcal{H}} = \alpha_H \int_0^T g(s) ds \int_0^T f(t) |s - t|^{2H-2} dt,$$

It should be noted that the elements in $\mathcal{H}$ are not necessarily ordinary functions.
2. For any given \( s \in [0, T] \), when \( H < \frac{1}{2} \), the defective integral of \( |s - t|^{2H-2} \) on \([0, T]\) does not converge, so the covariance function of \( B_t^H \) cannot be directly expressed in the form of (2.2). At this time, the elements in space \( \mathcal{H} \) are ordinary functions, but the formula (2.3) of inner product in Hilbert space \( \mathcal{H} \) is generally not true. But what is interesting is that the covariance of \( B_t^H \) increment satisfies the formula: If let \( 0 \leq a < b \leq c < d \leq T \), then

\[
\mathbb{E}[(B^H_b - B^H_a)(B^H_d - B^H_c)] = \alpha_H \int_a^b du \int_c^d |u - v|^{2H-2} dv,
\]

(2.4)

This leads to the conclusion that if \( f, g \in \mathcal{H} \) supports are disjoint, the inner product formula (2.3) still holds, see [9, 12]. By the way, we point out that Corollary 2.4 in this paper can also lead to this known conclusion.

In reference [4], a formula is given for limiting the inner product of Hilbert space \( \mathcal{H} \) to the bounded variation function \( \mathcal{V}_{[0,T]} \). If \( f, g \in \mathcal{V}_{[0,T]} \) then

\[
\langle f, g \rangle_{\mathcal{H}} = \int_{[0,T]^2} R_H(s,t)\nu_f(ds)\nu_g(dt) = -\int_{[0,T]^2} g(t)\frac{\partial R_H}{\partial t}(s,t)dt \nu_f(ds).
\]

(2.5)

Refer to [3] [4] [5].

In addition, with the help of Fourier transform, the following formula for the inner product of Hilbert space \( \mathcal{H} \) is sometimes very useful:

\[
\langle f, g \rangle_{\mathcal{H}} = \frac{\Gamma(2H + 1) \sin(\pi H)}{2\pi} \int_{\mathbb{R}} \mathcal{F}f(\xi)\mathcal{F}g(\xi) |\xi|^{1-2H} d\xi,
\]

(2.6)

here \( f, g \) can be taken from some proper subspace of \( \mathcal{H} \), see [13] for details.

Finally, with the help of kernel function

\[
K_H(t,s) = c_H \left[ \left( \frac{t}{s} \right)^{H - \frac{1}{2}} (t - s)^{H - \frac{1}{2}} - (H - \frac{1}{2}) \frac{\sqrt{t}}{\sqrt{s}} - H \int_s^t u^{H - \frac{1}{2}} (u - s)^{H - \frac{1}{2}} du \right]
\]

(2.7)

and operator \( K_H^{*} \)

\[
(K_H^{*}\phi)(t) = K_H(T,t)\phi(T) + \int_t^T \frac{\partial K_H}{\partial s}(s,t)[\phi(s) - \phi(t)]ds,
\]

People transform the inner product of Hilbert space \( \mathcal{H} \) into the inner product of elements in \( L^2([0,T]) \).

\[
\langle \phi, \psi \rangle_{\mathcal{H}} = \langle K_H^{*}\phi(t), K_H^{*}\psi(t) \rangle_{L^2([0,T])}.
\]

(2.8)

This inner product formula establishes the theoretical relationship between \( \mathcal{H} \) and \( L^2([0,T]) \), but people usually do not directly use it to calculate the inner product. For details, see [14].
2.2. A new formula for calculating inner product. Let $f$, $g$ be monotone non decreasing functions on $\mathbb{R}$, then the Lebesgue-Stieljes measure associated with bounded variation functions $(f - g)$ on $\mathbb{R}$ is defined as

$$\bar{\nu}_{f-g} = \bar{\nu}_f - \bar{\nu}_g.$$  

Here, $\bar{\nu}_f$ is the Lebesgue-Stieljes positive measure on $\mathbb{R}$ associated with monotone non decreasing function $f$ on $\mathbb{R}$, and we emphasize that right continuity of $f$ is not required here. In fact, the value of function $f$ at the discontinuous point and its Lebesgue-Stieljes measure $\bar{\nu}_f$ is independent (this point has been implicitly used many times in this paper). For details, see the Theorem 1.7.9 and exercise 1.7.12 in [15]. From the uniqueness theorem of measures, it is easy to deduce the following well-known lemmas:

**Lemma 2.1.** If $F, G$ is are bounded variation functions on $\mathbb{R}$, order $\Psi = F + G$ then

$$\bar{\nu}_\Psi = \bar{\nu}_F + \bar{\nu}_G,$$  \hspace{1cm} (2.9)

In particular, when $F, G \in \mathcal{V}_{[0,T]}$, order $\Psi = F + G$ then

$$\nu_\Psi = \nu_F + \nu_G,$$  \hspace{1cm} (2.10)

Here, $\nu_\Psi$ is the limit of the measure $\bar{\nu}_{\Psi^0}$ associated with the extension $\Psi^0$ of function $\Psi$ on $\mathbb{R}$ on $([0,T], B([0,T]))$, see notation 1.7.

The following Lemma 2.2 on the Partial integral formula of measure is one of the main bases for proving Proposition 1.9, which is taken from Lemma 3.1 of [16]. The key is to regard the value of the function at two endpoints in the general Partial integral formula as a measure about two Dirac points (or called Dirac $\delta$ generalized function). The two point measures are absorbed into the Lebesgue-Stieljes measure associated with the bounded variation function. This processing method first extends the bounded variation function, and then restricts the Lebesgue-Stieljes measure generated by the extended function back to the original support set of the bounded variation function. See [4, 16] for details, and the notation 1.7 and formula (1.22) in this paper.

**Lemma 2.2.** Let $[a,b]$ be a compact interval with positive length, $\phi : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on $(a,b)$. If $\phi'$ is absolutely integrable, then
for any \( f \in V_{[a,b]} \), we have
\[
- \int_{[a,b]} f(t) \phi'(t) dt = \int_{[a,b]} \phi(t) \nu_f(dt).
\]
(2.11)

where \( \nu_f \) is a restriction on \( ([a,b], B([a,b])) \) of the Lebesgue-Stieltjes measure on \( (\mathbb{R}, B(\mathbb{R})) \) associated with
\[
f^0(x) = \begin{cases} f(x), & \text{if } x \in [a,b], \\ 0, & \text{other} . \end{cases}
\]

Remark 2.3. Lemma 2.2 is a rewriting of the Partial integral formula of continuous monotone increasing function (such as [15] exercise 1.7.17). Its proof comes from Proposition 1.6.41 of [4] and [15]. See Lemma 3.1 of [16] for details.

From formulas (2.4), (2.5), and the above lemma, we have the following inference:

**Corollary 2.4.** Let \( 0 \leq a < b \leq c < d \leq T \), the bounded variation functions \( f(s) \) and \( g(t) \) are supported on \( [a,b] \) and \( [c,d] \) respectively. If \( H \in (0, \frac{1}{2}) \), then
\[
(f, g)_H = \alpha_H \int_{a}^{b} f(s) ds \int_{c}^{d} g(t)(t-s)^{2H-2} dt.
\]
(2.12)

Remark 2.5. Since the set of bounded variation functions \( V_{[0,T]} \) is a dense subset of Hilbert space \( \mathcal{H} \), the inner product formula (2.12) based on the continuity of the inner product is still valid for any function supporting disjoint in Hilbert space \( \mathcal{H} \).

In the Partial integration formula of measure, namely Lemma 2.2, take \( \phi \equiv 1 \) to get:

**Corollary 2.6.** Let function \( \varphi \in \mathcal{V}_{[0,T]}^\otimes 2 \), and \( \frac{\partial}{\partial s}, \frac{\partial}{\partial b} \) as shown in mark 1.11. If functions \( f, g \) are bounded Borel measurable functions on set \( [(a-1) \vee 0, (a+1) \wedge T] \) and set \( [(b-1) \vee 0, (b+1) \wedge T] \) respectively, then:
\[
\int_{(a-1)\vee 0}^{(a+1)\wedge T} g(t) dt \int_{(a-1)\vee 0}^{(a+1)\wedge T} \frac{\partial}{\partial s} \varphi(s,t) ds = 0, \\
\int_{(b-1)\vee 0}^{(b+1)\wedge T} f(s) ds \int_{(a-1)\vee 0}^{(a+1)\wedge T} \frac{\partial}{\partial s} \frac{\partial}{\partial b} \varphi(s,t) dt = 0, \\
\int_{(a-1)\vee 0}^{(a+1)\wedge T} f(s) ds \int_{(b-1)\vee 0}^{(b+1)\wedge T} \frac{\partial}{\partial s} \frac{\partial}{\partial b} \varphi(s,t) dt = 0.
\]
In the rest of this section, we give the proof of Proposition 1.9.

**Proof of Proposition 1.9:** The method is to use measure decomposition. First, determine \( t \in [0, T] \), and divide \( s \in [0, T] \) into the following three intervals:
\[
O_1 := [0, (t - 1) \lor 0), \quad O_2 := [(t - 1) \lor 0, (t + 1) \land T], \quad O_3 = ((t + 1) \land T, T];
\]

Then the function \( f(s) \) is decomposed into the restriction of the above three intervals, so that the measure \( \nu_f \) is decomposed into the sum of the Lebesgue-Stieltjes measures associated with the three. The specific steps are as follows:

First, review formula (2.5):
\[
\langle f, g \rangle_H = - \int_0^T g(t)dt \int_0^T \frac{\partial R_H}{\partial t}(s, t) \nu_f(ds).
\] (2.13)

For any given \( t \in [0, T] \), decompose function \( f(s) \in V_{[0, T]} \) as shown in the figure above:
\[
f(s) = f(s)\left(1_{[0,(t-1)\lor 0]}(s) + 1_{[1(t-1)\lor 0),(t+1)\land T]}(s) + 1_{((t+1)\land T,T]}(s)\right)
\]
\[
\quad : = f_1^t(s) + \tilde{f}_t(s) + f_2^t(s).
\]

According to Lemma 2.1 know the measure \( \nu_f \) has the following decomposition:
\[
\nu_f = \nu_{f_1} + \nu_{\tilde{f}_t} + \nu_{f_2}.
\] (2.14)

Here, the four measures are Lebesgue-Stieltjes measures defined on \([0, T], \mathcal{B}([0, T])\).

Substitute formula (2.14) into formula (2.13) to get:
\[
\langle f, g \rangle_H = - \int_0^T g(t)dt \int_0^T \frac{\partial R_H}{\partial t}(s, t) \nu_{f_1}(ds) - \int_0^T g(t)dt \int_0^T \frac{\partial R_H}{\partial t}(s, t) \nu_{\tilde{f}_t}(ds)
\]
\[
- \int_0^T g(t)dt \int_0^T \frac{\partial R_H}{\partial t}(s, t) \nu_{f_2}(ds) := I_1 + I_2 + I_3.
\] (2.15)

Note that the support set of function \( f_1^t(s) \) is \([0, (t - 1) \lor 0]\), then
\[
\int_0^T \frac{\partial R_H}{\partial t}(s, t) \nu_{f_1}(ds) = \begin{cases} 0 & t \in [0, 1], \\ \int_0^T 1_{[0,t-1]}(s) \frac{\partial R_H}{\partial t}(s, t) \nu_{f_1}(ds) & t \in (1, T]. \end{cases}
\]

Note that when \( t \in (1, T] \), integral in the right end of the above equation
\[
\int_0^T 1_{[0,t-1]}(s) \frac{\partial R_H}{\partial t}(s, t) \nu_{f_1}(ds) = \int_0^{t-1} \frac{\partial R_H}{\partial t}(s, t) \nu_{f_1}(ds).
\]
where, in fact, measure \( \nu f^1 \) at the right end can be understood as only defined on \((0, t-1], B([0, t-1])\); however function \( \frac{\partial^2 R_H}{\partial s \partial t}(s, t) \), as a function of argument \( s \), is absolutely integrable on \([0, t-1] \), so it is obtained from Lemma 2.2:

\[
\int_0^{t-1} \frac{\partial R_H}{\partial t}(s, t) \nu f^1(t) ds = - \int_0^{t-1} f(s) \frac{\partial^2 R_H}{\partial s \partial t}(s, t) ds.
\]

Thus:

\[
I_1 = \alpha_H \int_1^T g(t) dt \left( \int_0^{t-1} f(s)(t-s)^{2H-2} ds \right).
\]

(2.16)

Similarly, because the support set of function \( f^2(t) \) is \([(t+1) \land T, T] \), we have

\[
\int_0^T \frac{\partial R_H}{\partial t}(s, t) \nu f^2(ds) = \begin{cases} 
\int_{t+1}^T \frac{\partial R_H}{\partial s}(s, t) \nu f^2(ds) & t \in [0, T-1), \\
0 & t \in [T-1, T],
\end{cases}
\]

and

\[
\int_{t+1}^T \frac{\partial R_H}{\partial t}(s, t) \nu f^2(ds) = - \int_{t+1}^T f(s) \frac{\partial^2 R_H}{\partial s \partial t}(s, t) ds.
\]

and

\[
I_3 = \alpha_H \int_1^T f(s) ds \int_0^{s-1} g(t)(s-t)^{2H-2} dt.
\]

(2.17)

Substitute formula (2.16), (2.17) into formula (2.15) to get formula (1.23). Finally, the proof of formula (1.24) is the same.

\[\square\]

3. PROOF OF MAIN THEOREMS

Without losing generality, this section assumes the parameters \( \theta = 1 \) in definitions (1.1) and (1.17) of binary functions \( f_T(t, s) \) and \( h_T(t, s) \).

**Proof of Proposition 1.5:** Firstly, let \( t \in [0, T] \) be determined, and understand \( f_T(t, \cdot) \) as a function of one variable on \( s \in [0, T] \). Then notice that the binary function \( h_T \) can be expressed as a function of one variable \( \phi_T(t) = e^t T \mathbb{1}_{[0,T]}(t) \) tensor about oneself, namely \( h_T(t, s) = \phi_T(t) \phi_T(s) \). So according to the Fubini theorem, we have:

\[
\langle f_T, h_T \rangle_{\mathcal{H}^2} = \langle f_T(t, \cdot), \phi_T \rangle_{\mathcal{H}}, \langle \phi_T \rangle_{\mathcal{H}}.
\]

(3.1)

Secondly, calculate the inner product \( \langle f_T(t, \cdot), \phi_T \rangle_{\mathcal{H}} \) when \( t \in [0, T] \) is taken. According to the linear property of the inner product, we have:

\[
\langle f_T(t, \cdot), \phi_T \rangle_{\mathcal{H}} = \langle f^1, h^1 \rangle_{\mathcal{H}} + \langle f^1, h^2 \rangle_{\mathcal{H}} + \langle f^2, h^1 \rangle_{\mathcal{H}} + \langle f^2, h^2 \rangle_{\mathcal{H}},
\]

(3.2)
We then assert that there is a constant $C_H$ independent of $T$, so that for any given $t \in [0, T]$, and we have inequality

$$\left| \int_0^T e^{s-T}(1 - \delta_T(s)) \frac{\partial R(t, s)}{\partial t} \, ds \right| \leq C_H \times \left[ e^{-T} t^{2H-1} + e^{t-T} + (T-t)^{2H-1} \mathbb{1}_{(T-1,T]}(t) + (T-t)^{2H-2} \mathbb{1}_{[0,T-1]}(t) \right].$$

(3.4)
Combining inequalities (i, j) hold for any i, j. According to the identities (3.1) and (3.2), we get the inequality (1.18).

Proof of Theorem 1.1: Recall that in Remark 1.1.2, we will draw a conclusion that is stronger than the formula (1.9) required by the theorem. That is, the square of the norm of the binary function \( f_T(t, s) \) is taken as the function of \( T \) and the asymptote (1.11) when \( T \to \infty \). Equation (1.11) is proved in several steps as follows.
Step 1. According to Proposition 1.12, we get the decomposition formula of $\|f_T\|_{H^@2}^2$.

\[
\|f_T\|_{H^@2}^2 = \alpha_H^2 \sum_{i,j=1}^2 \int \int_{\kappa_i \times \kappa_j} e^{-|s_1-t_1|} e^{-|s_2-t_2|} |s_1 - s_2|^{2H-2} |t_1 - t_2|^{2H-2} \, d\bar{s} \, d\bar{t} \\
- 2\alpha_H \sum_{i=1}^2 \int_{\kappa_i} e^{-|s_1-t_1|} \frac{\partial}{\partial s_i} e^{-|s_2-t_2|} \frac{\partial R_H}{\partial s_1} (s_1, s_2) |t_1 - t_2|^{2H-2} \, d\bar{s} \, d\bar{t} \\
+ \int_{\kappa} e^{-|s_1-t_1|} \frac{\partial R_H}{\partial s_1} (s_1, s_2) \frac{\partial R_H}{\partial t_1} (t_1, t_2) \frac{\partial s_1}{\partial t_1} \frac{\partial s_2}{\partial t_2} e^{-|s_2-t_2|} \, d\bar{s} \, d\bar{t} \\
:= \alpha_H^2 \sum_{i,j=1}^2 M_{ij}(T) - 2\alpha_H \sum_{i=1}^2 M_{3i}(T) + M_{33}(T).
\]

Making the change of variables $x = T - s_1$, $y = T - t_1$, $u = T - s_2$, $v = T - t_2$, we have:

\[
M_{11}(T) = M_{22}(T) \quad \text{and} \quad M_{12}(T) = M_{21}(T).
\]

So

\[
\|f_T\|_{H^@2}^2 = M_{33}(T) + 2 \left( \alpha_H^2 (M_{11}(T) + M_{12}(T)) - \alpha_H (M_{31}(T) + M_{32}(T)) \right).
\]

(3.7)

Step 2. Solve the asymptote of function $M_{11}(T) + M_{12}(T)$ when $T \to \infty$. First,

\[
M_{11}(T) = \int_1^T ds_1 \int_1^T dt_1 e^{-|s_1-t_1|} dt_1 \int_0^{s_1-1} (s_1 - s_2)^{2H-2} \, ds_2 \int_0^{t_1-1} (t_1 - t_2)^{2H-2} e^{-|t_2-s_2|} \, dt_2 \\
= 2 \int_1^T e^{-s_1} \, ds_1 \int_1^s \, dt_1 \int_0^{s_1-1} (s_1 - s_2)^{2H-2} \, ds_2 \int_0^{t_1-1} (t_1 - t_2)^{2H-2} e^{-|t_2-s_2|} \, dt_2.
\]

(3.8)

\[
M_{12}(T) = \int_1^T ds_1 \int_0^{T-1} dt_1 \int_0^{s_1-1} (s_1 - s_2)^{2H-2} \, ds_2 \int_{t_1+1}^T (t_2 - t_1)^{2H-2} e^{-|t_2-s_1|} \, dt_2.
\]

(3.9)

Formula (3.8) is the symmetry of integral with respect to two variables $s_1, t_1$. According to Lemma 4.2 and Lemma 4.3, we get the function $M_{11}(T) + M_{12}(T)$,
and the asymptote when $T \to \infty$ is:
\[
T \times \left( 4H - 1 \right) \left( \int_1^\infty e^{-u} u^{2H-2} du \right)^2 + 2 \int_1^\infty (e^{1-u} + e^{-1-u}) u^{2H-2} du \\
+ 2(4H - 1) \int_1^\infty e^{-u} u^{2H-2} du \int_1^u e^v v^{2H-2} dv \right] + C_H. 
\]  
(3.10)

**Step 3.** Solve the asymptote of function $M_{31}(T) + M_{32}(T)$ when $T \to \infty$. First,
\[
M_{31}(T) = \int_{s_1,t_1} e^{-[s_1-t_1]} \frac{\partial s_1}{\partial s_2} e^{-[s_2-t_2]} \frac{\partial R_H}{\partial s_1} (s_1, s_2) (t_1 - t_2)^{2H-2} ds dt \\
= H \int_0^T ds_1 \int_1^T e^{-[s_1-t_1]} dt_1 \int_0^{t_1-1} (t_1 - t_2)^{2H-2} dt_2 \int_{(s_1-1)\vee0}^{(s_1+1)\wedge T} \frac{\partial s_1}{\partial s_2} e^{-[s_2-t_2]} \\
\times \left( s_1^{2H-1} - \text{sgn}(s_1 - s_2)|s_1-s_2|^{2H-1} \right) ds_2. 
\]  
(3.11)

From Inference 2.6, we have:
\[
\int_0^T s_1^{2H-1} ds_1 \int_1^T e^{-[s_1-t_1]} dt_1 \int_0^{t_1-1} (t_1 - t_2)^{2H-2} dt_2 \int_{(s_1-1)\vee0}^{(s_1+1)\wedge T} \frac{\partial s_1}{\partial s_2} e^{-[s_2-t_2]} ds_2 = 0. 
\]

Put it in (3.11), we have
\[
M_{31}(T) = -H \int_0^T ds_1 \int_1^T e^{-[s_1-t_1]} dt_1 \int_0^{t_1-1} (t_1 - t_2)^{2H-2} dt_2 \int_{(s_1-1)\vee0}^{(s_1+1)\wedge T} \frac{\partial s_1}{\partial s_2} e^{-[s_2-t_2]} \\
\times \text{sgn}(s_1 - s_2)|s_1-s_2|^{2H-1} ds_2 := H \times |N(T) - \tilde{N}(T)|, 
\]  
(3.12)

here
\[
N(T) = \int_0^T ds_1 \int_1^T e^{-[s_1-t_1]} dt_1 \int_0^{t_1-1} (t_1 - t_2)^{2H-2} dt_2 \int_{(s_1-1)\vee0}^{(s_1+1)\wedge T} \text{sgn}(s_1 - s_2)|s_1-s_2|^{2H-1} ds_2; 
\]  
(3.13)

\[
\tilde{N}(T) = \int_0^T ds_1 \int_1^T e^{-[s_1-t_1]} dt_1 \int_0^{t_1-1} (t_1 - t_2)^{2H-2} dt_2 \int_{(s_1-1)\vee0}^{(s_1+1)\wedge T} e^{-[t_2-s_2]} \\
\times \text{sgn}(s_1 - s_2)|s_1-s_2|^{2H-1} \left( \delta(s_1-1\vee0)(s_2) - \delta(s_1+1\wedge T)(s_2) \right) ds_2. 
\]  
(3.14)

Similarly, we have:
\[
M_{32}(T) = -H \int_0^T ds_1 \int_1^T dt_2 \int_0^{t_2-1} e^{-[s_1-t_1]} (t_2 - t_1)^{2H-2} dt_1 \int_{(s_1-1)\vee0}^{(s_1+1)\wedge T} \frac{\partial s_1}{\partial s_2} e^{-[s_2-t_2]} \\
\times \text{sgn}(s_1 - s_2)|s_1-s_2|^{2H-1} ds_2 := H \times |U(T) - \tilde{U}(T)|, 
\]  
(3.15)
here

\[
U(T) = \int_0^T ds_1 \int_1^T dt_2 \int_0^{t_2-1} e^{-|t_1-s_1|(t_2-t_1)2H-2} dt_1 \int_{(s_1-1)\cap0}^{(s_1+1)\cap T} \text{sgn}(s_2-t_2)e^{-|t_2-s_2|} \times \text{sgn}(s_1-s_2)|s_1-s_2|^{2H-1} ds_2;
\]

\[
\tilde{U}(T) = \int_0^T ds_1 \int_1^T dt_2 \int_0^{t_2-1} e^{-|t_1-s_1|(t_2-t_1)2H-2} dt_1 \int_{(s_1-1)\cap0}^{(s_1+1)\cap T} e^{-|t_2-s_2|} \times \text{sgn}(s_1-s_2)|s_1-s_2|^{2H-1} ds_2. \tag{3.16}
\]

According to Lemma 4.4, Lemma 4.5 and Lemma 4.6, we get that the asymptote of \(M_{31}(T) + M_{32}(T)\) is:

\[
2HT \times \left[ \int_1^\infty e^{-u}u^{2H-1} du \left[ -2H(e^{-1} + e) + (4H - 1) \int_0^1 (e^x + e^{-x})x^{2H-1} dx \right] + e^{-1} \int_0^1 (e^x - e^{-x})x^{2H-1} dx - (1 + e^{-2}) \right]. \tag{3.18}
\]

**Step 4.** Solve the asymptote of function \(M_{33}(T)\). Similar to the method for dealing with items \(M_{31}(T)\) and \(M_{32}(T)\) in step 3, we expand \(\frac{\partial R}{\partial s_1}(s_1, s_2)\) and \(\frac{\partial R}{\partial t_1}(t_1, t_2)\) in turn, and use Inference 2.6 twice consecutively to obtain:

\[
M_{33}(T) = \int_{\kappa_3 \times \kappa_3} e^{-|s_1-t_1|} \frac{\partial R}{\partial s_1}(s_1, s_2) \frac{\partial R}{\partial t_1}(t_1, t_2) \frac{\partial s_1}{\partial s_2} \frac{\partial t_1}{\partial t_2} e^{-|s_2-t_2|} ds_2 dt_2
\]

\[
= H \int_0^T ds_1 \int_0^T dt_2 e^{-|s_1-t_1|} \int_{(t_1-1)\cap0}^{(t_1+1)\cap T} \frac{\partial R}{\partial t_1}(t_1, t_2) \frac{\partial s_1}{\partial s_2} \frac{\partial t_1}{\partial t_2} e^{-|s_2-t_2|} ds_2
\]

\[
\times \int_{(s_1-1)\cap0}^{(s_1+1)\cap T} \left( s_1^{2H-1} - |s_1-s_2|^{2H-1} \text{sgn}(s_1-s_2) \right) \frac{\partial s_1}{\partial s_2} \frac{\partial t_1}{\partial t_2} e^{-|s_2-t_2|} ds_2
\]

\[
= -H^2 \int_0^T ds_1 \int_0^T dt_2 e^{-|s_1-t_1|} \int_{(t_1-1)\cap0}^{(t_1+1)\cap T} |s_1-s_2|^{2H-1} \text{sgn}(s_1-s_2) ds_2
\]

\[
\times \int_{(t_1-1)\cap0}^{(t_1+1)\cap T} \left( |t_1-t_2|^{2H-1} - |t_1-t_2|^{2H-1} \text{sgn}(t_1-t_2) \right) \frac{\partial s_1}{\partial s_2} \frac{\partial t_1}{\partial t_2} e^{-|s_2-t_2|} dt_2
\]

\[
= H^2 \int_0^T ds_1 \int_0^T dt_2 e^{-|s_1-t_1|} \int_{(t_1-1)\cap0}^{(t_1+1)\cap T} |s_1-s_2|^{2H-1} \text{sgn}(s_1-s_2) ds_2
\]

\[
\times \int_{(t_1-1)\cap0}^{(t_1+1)\cap T} |t_1-t_2|^{2H-1} \text{sgn}(t_1-t_2) \frac{\partial s_1}{\partial s_2} \frac{\partial t_1}{\partial t_2} e^{-|s_2-t_2|} dt_2. \tag{3.19}
\]
Note that the bivariate joint “density function” in the above equation can be expressed as:
\[
\frac{\partial s_1 \partial t_1}{\partial s_2 \partial t_2} e^{-|s_2-t_2|} = e^{-|s_2-t_2|} \times \left( -1 - \text{sgn}(t_2 - s_2)[\delta(s_1 - 1) \wedge 0(s_2) - \delta(s_1 + 1) \wedge T(s_2)] \
- \text{sgn}(s_2 - t_2)[\delta(t_1 - 1) \vee 1(t_2) - \delta(t_1 + 1) \wedge T(t_2)] \
+ \left( \delta(s_1 - 1) \wedge 0(s_2)(s_2)(\delta(t_1 - 1) \vee 0 - \delta(t_1 + 1) \wedge T)(t_2) \right) \right)
\]

Substitute the above joint density function into formula (3.19) to obtain:
\[
M_{33}(T) = H^2 \times [-L(T) + 2P(T) + Q(T)], \quad (3.20)
\]
where
\[
L(T) = \int_{[0,T]^2} e^{-|t_1 - s_1|} ds_1 dt_1 \int_{(s_1 - 1) \wedge 0}^{(s_1 + 1) \wedge T} ds_2 \int_{(t_1 - 1) \vee 0}^{(t_1 + 1) \wedge T} \text{sgn}(s_1 - s_2)|s_1 - s_2|^{2H-1} \
\times \text{sgn}(t_1 - t_2)|t_1 - t_2|^{2H-1} e^{-|t_2 - s_2|} dt_2, \quad (3.21)
\]
\[
P(T) = \int_{[0,T]^2} e^{-|t_1 - s_1|} ds_1 dt_1 \int_{(s_1 - 1) \wedge 0}^{(s_1 + 1) \wedge T} ds_2 \int_{(t_1 - 1) \vee 0}^{(t_1 + 1) \wedge T} e^{-|t_2 - s_2|} \text{sgn}(s_1 - s_2)|s_1 - s_2|^{2H-1} \
\times \text{sgn}(t_1 - t_2)|t_1 - t_2|^{2H-1} \text{sgn}(s_2 - t_2)[\delta(s_1 - 1) \wedge 0(s_2) - \delta(s_1 + 1) \wedge T(s_2)] dt_2, \quad (3.22)
\]
\[
Q(T) = \int_{[0,T]^2} e^{-|t_1 - s_1|} ds_1 dt_1 \int_{(s_1 - 1) \wedge 0}^{(s_1 + 1) \wedge T} ds_2 \int_{(t_1 - 1) \vee 0}^{(t_1 + 1) \wedge T} e^{-|t_2 - s_2|} \text{sgn}(s_1 - s_2)|s_1 - s_2|^{2H-1} \
\times \text{sgn}(t_1 - t_2)|t_1 - t_2|^{2H-1} \left( \delta(s_1 - 1) \wedge 0 - \delta(s_1 + 1) \wedge T \right)(s_2)(\delta(t_1 - 1) \vee 0 - \delta(t_1 + 1) \wedge T)(t_2) dt_2.
\]

For term \(Q(T)\), first integrate Dirac function, then convert it into four double integrals, when \(T \to \infty\), we can directly calculate that its asymptote is
\[
(6e^{-2} + 2)T + C_H.
\]

Lemma 4.7 and Lemma 4.8 respectively give the asymptotes of the terms \(L(T)\) and \(P(T)\) when \(T \to \infty\), and combine the three asymptotes. According to formula (3.20), the asymptote of \(M_{33}(T)\) is:
\[
2H^2T \times \left[ -2(4H + 1) \int_0^1 e^{-u} u^{2H-1} du \int_0^u e^v v^{2H-1} dv + (4H + 1) \left( \int_0^1 e^{-u} u^{2H-1} du \right)^2 \
+ 4H \int_0^1 (e^{-u} - e^{-1+u}) u^{2H-1} du + e^{-2} + 3 \right] + C_H. \quad (3.23)
\]
Finally, the above three steps give the asymptotes of $M_{11}(T) + M_{12}(T)$, $M_{31}(T) + M_{32}(T)$, and $M_{33}(T)$ as functions of $T$ when $T \to \infty$, and obtain (3.10), (3.18) and (3.23) respectively. Then it is known from the decomposition formula (3.7) that the norm of the binary function $f_T(t, s)$ is taken as the function of $T$, and the asymptote exists when $T \to \infty$. Finally, from the uniqueness of (1.12) and function limit, it is concluded that (1.11) holds.

**Remark 3.1.** A by-product of the proof of Theorem 1.1 is the following seemingly tedious analytical identities:

$$2(H\Gamma(2H))^2\left[4H - 1 + \frac{2H(2 - 4H)\Gamma(4H)}{\Gamma(2H)\Gamma(1 - 2H)}\right] = A_3 + 2\alpha_H(\alpha_H A_1 - A_2),$$

Here $A_1, A_2, A_3$ are the slope values of asymptotes (3.10), (3.18) and (3.23) respectively. It should be emphasized that the true meaning of Theorem 1.1 lies in “The norm of the bivariate function $f_T(t, s)$ is taken as a function of $T$. When $T \to \infty$, the existence of asymptote.” As for the specific value of the slope of the asymptote, it is not so important. Therefore, in order to save space, this paper will not verify this analytic identity.

**Proof of Theorem 1.3:** First, we have to prove the Berry–Esséen type inequality (1.15) of the least squares estimator. Therefore, From the proof of [11] Theorem 1.1, we know that

$$\sup_{z \in \mathbb{R}} \left| P\left( \frac{T}{\sqrt{T}}(\hat{\theta}_T - \theta) \leq z \right) - P(Z \leq z) \right| \leq C_{\theta, H} \times \max \left\{ \frac{1}{\sqrt{T}}, T\left\| f_T \right\|_{\mathcal{H}^2}^2 - 2(H\Gamma(2H))^2\sigma_H^2 \right\}. \quad (3.24)$$

Therefore, according to Theorem 1.1, we can know that inequality (1.15) is true.

Then we prove Berry–Esséen inequality of moment estimator (1.16). According to the proof of Theorem 1.1 in [8],

$$\sup_{z \in \mathbb{R}} \left| P\left( \frac{T}{\sqrt{T}}(\hat{\theta}_T - \theta) \leq z \right) - P(Z \leq z) \right| \leq C_{\theta, H} \times \max \left\{ \frac{1}{\sqrt{T}}, T\left\| f_T \right\|_{\mathcal{H}^2}^2 - 2(H\Gamma(2H))^2\sigma_H^2 , \sqrt{\left\langle f_T, h_T \right\rangle_{\mathcal{H}^2}} \right\}. \quad (3.25)$$

Therefore, the inequality (1.16) is established by Theorem 1.1 and Proposition 1.5.\qed
4. Appendix: Asymptotic Analysis

The Lemma 4.1 is a trivial inequality which is used several times in this section.

Lemma 4.1. Let $\alpha < 0$. There is a positive number $C_{\alpha}$ which depends only on $\alpha$ such that for any $x > 1$, we have

$$\int_{1}^{x} e^{u}u^{\alpha}du < C_{\alpha} \times e^{x}. \quad (4.1)$$

Lemma 4.2. Let (3.8) be the expression of the quadruple integral $M_{11}(T)$. When $T \to \infty$, The asymptote of $M_{11}(T)$ is given by

$$2T \times \left[ (4H - 1) \int_{1}^{\infty} e^{-v}v^{2H-2}dv \int_{1}^{v} e^{u}u^{2H-2}du + \int_{1}^{\infty} e^{1-u}u^{2H-2}du \right] + C_{H}. \quad (4.2)$$

Proof. We take the integral variable $(s_{1}, t_{1})$ of the quadruple integral $\frac{1}{2} M_{11}(T)$. We first decompose the region $[0, s_{1} - 1] \times [0, t_{1} - 1]$ of the integral variable $(s_{2}, t_{2})$ into $\{0 \leq t_{2} \leq t_{1} - 1, t_{2} \leq s_{2} \leq s_{1} - 1\} \cup \{0 \leq s_{2} \leq t_{2} \leq t_{1} - 1\}$. The integral $\frac{1}{2} M_{11}(T)$ restricted to the corresponding subregion is called $J_{1}(T)$, $J_{2}(T)$ respectively, where

$$J_{1}(T) = \int_{1}^{T} ds_{1} \int_{1}^{t_{1}} dt_{1} \int_{0}^{t_{1}-1} dt_{2} \int_{t_{2}}^{s_{1}-1} ds_{2}(s_{1} - s_{2})^{2H-2}(t_{1} - t_{2})^{2H-2}e^{(s_{1}-s_{2})-|t_{2}-s_{2}|},$$

$$J_{2}(T) = \int_{1}^{T} ds_{1} \int_{1}^{t_{1}} dt_{1} \int_{0}^{t_{1}-1} dt_{2} \int_{0}^{t_{2}} ds_{2}(s_{1} - s_{2})^{2H-2}(t_{1} - t_{2})^{2H-2}e^{(s_{1}-s_{2})-|t_{2}-s_{2}|}.$$  

Then we try to obtain the asymptote of the quadruple integral $J_{1}(T)$. Making the change of variables $u = s_{1} - s_{2}, v = t_{1} - t_{2}, x = s_{1} - t_{1} + v$. By the symmetry, we have

$$J_{1}(T) = \int_{1}^{T} ds_{1} \int_{1}^{t_{1}} e^{-2x}dx \int_{1}^{x} e^{v}v^{2H-2}dv \int_{1}^{x} e^{u}u^{2H-2}du$$

$$= 2 \int_{1}^{T} ds_{1} \int_{1}^{t_{1}} e^{-2x}dx \int_{1}^{x} e^{v}v^{2H-2}dv \int_{1}^{v} e^{u}u^{2H-2}du.$$

Lemma 4.1 and Partial integration formulate indicate that when $T \to \infty$, the asymptote of $J_{1}(T)$ is

$$T \times \int_{1}^{\infty} e^{-v}v^{2H-2}dv \int_{1}^{v} e^{u}u^{2H-2}du + C_{H}. \quad (4.3)$$
Then we take the asymptote of $J_2(T)$. Making the change of variables $u = s_1 - s_2$, $v = t_1 - t_2$, $x = s_1 - t_1 + v$, we have

$$J_2(T) = \int_1^T ds_1 \int_1^{s_1} e^{-u} u^{2H-2} du \int_1^x dx \int_1^x e^v v^{2H-2} dv$$

Lemma 4.1 and Partial integration formulate imply that when $T \to \infty$, the asymptote of $J_2(T)$ is

$$T \times \int_1^\infty e^{-u} u^{2H-2} du \int_1^u dx \int_1^x e^v v^{2H-2} dv + C_H$$

$$= T \times \int_1^\infty e^{-u} u^{2H-2} du \int_1^u e^v v^{2H-2} (u - v) dv + C_H. \quad (4.4)$$

Finally, the asymptote of $\frac{1}{2} M_{11}(T)$ can be obtained by combining (4.3) and (4.4), and we obtain the asymptote (4.2) of $M_{11}(T)$.

**Lemma 4.3.** Let (3.9) be the expression of the quadruple integral $M_{12}(T)$, then when $T \to \infty$, the asymptote of $M_{12}(T)$ is

$$T \left[ (4H - 1) \left( \int_1^\infty e^{-u} u^{2H-2} du \right)^2 + 2 \int_1^\infty e^{-1-u} u^{2H-2} du \right] + C_H. \quad (4.5)$$

**Proof.** We first decompose the integral region $[1, T] \times [0, T - 1]$ of the integral variable $(s_1, t_1)$ of the quadruple integral $M_{12}(T)$ into $\{0 \leq s_1 - 1 \leq t_1 \leq T - 1\} \cup \{0 \leq t_1 \leq s_1 - 1 \leq T - 1\}$. The integral $M_{12}(T)$ restricted to corresponding sub-region is $\tilde{J}_1(T)$ and $\tilde{J}_2(T)$ respectively, where

$$\tilde{J}_1(T) = \int_1^T ds_1 \int_0^{T-1} dt_1 \int_0^{s_1-1} ds_2 \int_{t_1+1}^T (s_1 - s_2)^{2H-2} (t_2 - t_1)^{2H-2} e^{-|t_1-s_1|-|t_2-s_2|} dt_2,$$

$$\tilde{J}_2(T) = \int_1^T ds_1 \int_0^{s_1-1} dt_1 \int_0^{s_1-1} ds_2 \int_{t_1+1}^T (s_1 - s_2)^{2H-2} (t_2 - t_1)^{2H-2} e^{-|t_1-s_1|-|t_2-s_2|} dt_2.$$

Then we try to obtain the asymptote of $\tilde{J}_1(T)$ when $T \to \infty$. Making the change of variables $u = t_2 - t_1$, $v = u + s_1 - 1 - s_2$ and $x = t_2 - s_2$, we have

$$\tilde{J}_1(T) = \int_1^T dt_2 \int_1^{t_2} e^{-x} dx \int_1^x e^{-|x-v-1|} dv \int_1^u (v - u + 1)^{2H-2} u^{2H-2} du$$
By the Partial integration formulate and Fubini Theorem, when $T \to \infty$, the asymptote is as follow

\[
T \times \int_1^\infty e^{-x} \, dx \int_1^x e^{-|x-v-1|} \, dv \int_1^v (v-u+1)^{2H-2} u^{2H-2} \, du + C_H
\]

\[
= T \times \left[ \int_1^\infty u^{2H-2} \, du \int_1^\infty e^v (v-u+1)^{2H-2} \, dv \int_1^{1+v} e^{1-x} \, dx \right.
\]

\[
+ \int_1^\infty u^{2H-2} \, du \int_1^\infty e^{-v-1} (v-u+1)^{2H-2} \, dv \int_{1+v}^{1+u} \, dx \right] + C_H
\]

\[
= \frac{3}{2} T \times \left( \int_1^\infty e^{-u} u^{2H-2} \, du \right)^2 + C_H. \tag{4.6}
\]

We next obtain the asymptote of the integral $\bar{J}_2(T)$ when $T \to \infty$. Fix the integral variable $(s_1, t_2)$ of $\bar{J}_2(T)$. We decompose the integral region $[0, s_1 - 1]^2$ of integral variable $(t_1, s_2)$ into $\{0 \leq s_2 \leq t_1 \leq s_1 - 1\} \cup \{0 \leq t_1 \leq s_2 \leq s_1 - 1\}$. The integral $\bar{J}_2(T)$ restricted to corresponding subregion is $\bar{J}_{21}(T)$ and $\bar{J}_{22}(T)$ respectively, where

\[
\bar{J}_{21}(T) = \int_1^T ds_1 \int_0^{s_1-1} dt_1 \int_0^{t_1} ds_2 \int_{t_1+1}^T (s_1 - s_2)^{2H-2} (t_2 - t_1)^{2H-2} e^{-|t_1-s_1|-|t_2-s_2|} \, dt_2,
\]

\[
\bar{J}_{22}(T) = \int_1^T ds_1 \int_0^{s_1-1} ds_2 \int_0^{s_2} dt_1 \int_{t_1+1}^T (s_1 - s_2)^{2H-2} (t_2 - t_1)^{2H-2} e^{-|t_1-s_1|-|t_2-s_2|} \, dt_2.
\]

Fix the integral variable $(s_1, t_1, s_2)$ of $\bar{J}_{21}(T)$ again. We decompose the integral region $[t_1+1, T]$ of integral variable $t_2$ into $[t_1 + 1, s_1] \cup [s_1, T]$. The integral $\bar{J}_{21}(T)$ restricted to corresponding subregion is $\bar{J}_{211}(T)$ and $\bar{J}_{212}(T)$ respectively, where

\[
\bar{J}_{211}(T) = \int_1^T ds_1 \int_0^{s_1-1} dt_1 \int_0^{t_1} ds_2 \int_{t_1+1}^{s_1} (s_1 - s_2)^{2H-2} (t_2 - t_1)^{2H-2} e^{-|t_1-s_1|-|t_2-s_2|} \, dt_2,
\]

\[
\bar{J}_{212}(T) = \int_1^T ds_1 \int_0^{s_1-1} dt_1 \int_0^{t_1} ds_2 \int_{s_1}^{T} (s_1 - s_2)^{2H-2} (t_2 - t_1)^{2H-2} e^{-|t_1-s_1|-|t_2-s_2|} \, dt_2.
\]

For integral $\bar{J}_{211}(T)$, making the change of variables $u = t_2 - t_1$, $v = s_1 - t_1$, $x = s_1 - s_2$, we have

\[
\bar{J}_{211}(T) = \int_1^T ds_1 \int_1^{s_1} e^{-u} u^{2H-2} dx \int_1^x e^{-v} u^{2H-2} du.
\]
The Partial integration formulate implies that when $T \to \infty$, the asymptote of $\tilde{J}_{211}(T)$ is

$$T \int_1^\infty e^{-x} x^{2H-2} dx \int_1^x e^{-u} u^{2H-2}(x-u) du + C_H. \quad (4.7)$$

For integral $\tilde{J}_{211}(T)$, making the change of variables $u = t_2 - t_1$, $v = s_1 - t_1$, $x = s_1 - s_2$, we have

$$\tilde{J}_{212}(T) = \int_1^T dt_2 \int_1^{t_2} dy \int_1^{y} u^{2H-2} du \int_1^{u} e^{-v} (y-u+v)^{2H-2} dv.$$  

The Partial integration formulate implies that when $T \to \infty$, the asymptote of $\tilde{J}_{212}(T)$ is

$$T \times \int_1^\infty e^{-y} dy \int_1^{y} u^{2H-2} du \int_1^{u} e^{-v} (y-u+v)^{2H-2} dv + C_H$$

$$= T \times \left[ 2 \int_1^\infty e^{-v} u^{2H-2} du \int_1^{u} e^{-v} v^{2H-1} dv - \left( \int_1^\infty e^{-u} u^{2H-2} du \right)^2 \right] + C_H. \quad (4.8)$$

Fix the integral variable $(s_1, s_2, t_1)$ of $\tilde{J}_{22}(T)$ again. We decompose the integral region $[t_1 + 1, T]$ of integral variable $t_2$ into $[t_1 + 1, s_2 + 1] \cup [s_2 + 1, s_1] \cup [s_1, T]$. The integral $\tilde{J}_{22}(T)$ restricted to corresponding subregion is $\tilde{J}_{221}(T)$, $\tilde{J}_{222}(T)$, $\tilde{J}_{223}(T)$ respectively, where

$$\tilde{J}_{221}(T) = \int_1^T ds_1 \int_0^{s_1-1} ds_2 \int_0^{s_2} dt_1 \int_{t_1+1}^{s_2+1} (s_1 - s_2)^{2H-2} (t_2 - t_1)^{2H-2} e^{-|t_1-s_1|-|t_2-s_2|} dt_2,$$

$$\tilde{J}_{222}(T) = \int_1^T ds_1 \int_0^{s_1-1} ds_2 \int_0^{s_2} dt_1 \int_{s_2+1}^{s_1} (s_1 - s_2)^{2H-2} (t_2 - t_1)^{2H-2} e^{-|t_1-s_1|-|t_2-s_2|} dt_2,$$

$$\tilde{J}_{223}(T) = \int_1^T ds_1 \int_0^{s_1-1} ds_2 \int_0^{s_2} dt_1 \int_{s_1}^{T} (s_1 - s_2)^{2H-2} (t_2 - t_1)^{2H-2} e^{-|t_1-s_1|-|t_2-s_2|} dt_2.$$

For integral $\tilde{J}_{221}(T)$ and integral $\tilde{J}_{222}$, making the change of variables, we have

$$\tilde{J}_{221}(T) = \int_1^T ds_1 \int_1^{s_1} e^{-v} dv \int_1^v (v-z+1)^{2H-2} dz \int_1^{z} x^{2H-2} e^{-|z-x-1|} dx,$$

$$\tilde{J}_{222}(T) = \int_1^T ds_1 \int_1^{s_1} e^{-v} dv \int_1^v e^{-x} x^{2H-2} dx \int_1^{x} e^{z-1}(v-z+1)^{2H-2} dz.$$
The Partial integration formulate and Fubini Theorem imply that when \( T \to \infty \), the asymptotes of \( J_{221}(T) \) and \( J_{222}(T) \) are
\[
T \times \int_1^\infty e^{-v} dv \int_1^{v} (v - z + 1)^{2H-2} dz \int_1^{x} x^{2H-2} e^{-|z-x-1|} dx + C_H \\
= \frac{3}{2} T \times (\int_1^\infty e^{-u} u^{2H-2} du)^2 + C_H \\
= 2T \times \int_1^\infty e^{-x} x^{2H-2} \int_1^{x} e^{-y} (y^{2H-1} - y^{2H-2}) dy + C_H. \tag{4.9}
\]

For integral \( J_{223}(T) \), making the change of variables, we have
\[
J_{223}(T) = \int_1^T dt_2 \int_1^{t_2} e^{-u} u^{2H-2} du \int_1^{u} dy \int_1^{y} e^{-x} x^{2H-2} dx.
\]

By the Partial integration formulate, we obtain when \( T \to \infty \), the asymptotes of \( J_{223}(T) \) is
\[
T \times \int_1^\infty e^{-u} u^{2H-2} du \int_1^{u} e^{-x} x^{2H-2} (u-x) dx + C_H. \tag{4.11}
\]
Combining (4.7)–(4.11), we obtain the asymptote of \( J_2(T) \)
\[
T \left[ (4H - \frac{5}{2}) \left( \int_1^\infty e^{-u} u^{2H-2} du \right)^2 + 2 \int_1^\infty e^{-u} u^{2H-2} du \right]. \tag{4.12}
\]
Finally, we obtain the asymptote (4.5) of \( \tilde{J}(T) \) by combining (4.6) and (4.12) \( \Box \)

**Lemma 4.4.** Let (3.13) be the expression of the quadruple integral \( N(T) \), then when \( T \to \infty \), the asymptote of \( N(T) \) is
\[
T \times \left[ \int_1^\infty e^{-u} u^{2H-2} du \times \left[ e^{-1} - e + (4H - 1) \int_0^1 (e^x - e^{-x}) x^{2H-1} dx \right] \\
+ \int_0^1 (e^{x-1} - e^{-x-1}) x^{2H-1} dx \right] + C_H. \tag{4.13}
\]

**Proof.** We divide the integral region \( \{0 \leq s_1 \leq T, \ (s_1 - 1) \lor 0 \leq s_2 \leq (s_1 + 1) \land T\} \) of integral variable \((s_1, s_2)\) of \( N(T) \) into
\[
\{0 \leq s_1 - 1 \leq s_2 \leq s_1 \leq T\} \cup \{0 \leq s_2 - 1 \leq s_1 \leq s_2 \leq T\} \\
\cup \{0 \leq s_2 \leq s_1 \leq 1\} \cup \{0 \leq s_1 \leq s_2 \leq 1\}.
\]
The integral \( N(T) \) over the corresponding region is \( N_1(T), N_2(T), N_3(T), N_4(T) \), where

\[
N_1(T) = \int_{[1, T]^2} e^{-|t_1 - s_1|} dt_1 ds_1 \int_0^{t_1} (t_1 - t_2)^{2H-2} dt_2 \int_{s_1-1}^{s_1} \times \text{sgn}(s_2 - t_2) e^{-|t_2 - s_2|} (s_1 - s_2)^{2H-1} ds_2,
\]

\[
N_2(T) = - \int_{[1, T]^2} dt_1 ds_2 \int_0^{t_1} (t_1 - t_2)^{2H-2} dt_2 \int_{s_2-1}^{s_2} \times \text{sgn}(s_2 - t_2) e^{-|t_1 - s_1| - |t_2 - s_2|} (s_2 - s_1)^{2H-1} ds_1,
\]

\[
N_3(T) = \int_1^T dt_1 \int_0^{t_1} (t_1 - t_2)^{2H-2} dt_2 \int_0^{1} ds_1 \int_0^{s_1} \times \text{sgn}(s_2 - t_2) e^{-|t_1 - s_1| - |t_2 - s_2|} (s_1 - s_2)^{2H-1} ds_2,
\]

\[
N_4(T) = \int_1^T dt_1 \int_0^{t_1} (t_1 - t_2)^{2H-2} dt_2 \int_0^{1} ds_2 \int_0^{s_2} \times \text{sgn}(s_2 - t_2) e^{-|t_1 - s_1| - |t_2 - s_2|} (s_2 - s_1)^{2H-1} ds_1.
\]

First, by the absolute integrability of the double integral

\[
\int_1^\infty e^{-t_1} dt_1 \int_0^{t_1} (t_1 - t_2)^{2H-2} dt_2 \int_{[0, 1]^2} |s_2 - s_1|^{2H-1} e^{s_1} ds_1 ds_2,
\]

we know the limit of \( N_3(T), N_4(T) \) exists when \( T \to \infty \). Therefore, integral \( N(T) \) and integral \( N_1(T) + N_2(T) \) have asymptotes with the same slope but different intercepts. Next we take the asymptotes of \( N_1(T) \) and \( N_2(T) \) respectively.

We then should decompose the integral region \([1, T]^2\) of integral variable \((s_1, t_1)\) of \( N_1(T) \) into \( 1 \leq t_1 \leq s_1 \leq T \) and \( 1 \leq s_1 \leq t_1 \leq T \) to take the asymptote of \( N_1(T) \). And we have

\[
N_1(T) = \int_1^T ds_1 \int_1^{s_1} dt_1 + \int_1^T dt_1 \int_1^{t_1} ds_1 \int_0^{s_1-1} dt_2 \int_{s_1-1}^{s_1} \times (s_1 - s_2)^{2H-1} (t_1 - t_2)^{2H-2} \text{sgn}(s_2 - t_2) e^{-|t_1 - s_1| - |t_2 - s_2|} ds_2
\]

\[
:= N_{11}(T) + N_{12}(T)
\]

For \( N_{11}(T) \), making the change of variables \( u = t_1 - t_2, v = s_1 - s_2 \), we obtain

\[
N_{11}(T) = \int_0^1 e^v v^{2H-1} dv \int_1^T e^{-2s_1} ds_1 \int_1^{s_1} e^{2t_1} dt_1 \int_1^{t_1} e^{-u} u^{2H-2} du.
\]
Therefore, by the Partial integration formulate, we obtain when $T \to \infty$, the asymptote of $N_{11}(T)$ is

$$\frac{T}{2} \int_0^1 e^v v^{2H-1} dv \int_1^{\infty} e^{-u} u^{2H-2} du + C_H. \quad (4.14)$$

For $N_{12}(T)$, we fix integral variable $t_1$ and decompose the integral region $[1, t_1] \times [0, t_1 - 1]$ of integral variable $(s_1, t_2)$ into $0 \leq t_2 \leq s_1 - 1 \leq t_1 - 1$ and $1 \leq s_1 \leq t_2 + 1 \leq t_1$. Then $N_{12}(T)$ split into the sum of the following two integrals:

$$N_{12}(T) = \int_1^T dt_1 \left( \int_{s_1-1}^{s_1} dt_2 \int_1^{t_1-1} ds_1 + \int_1^{t_2+1} dt_2 \int_1^{s_1-1} ds_1 \right) \times (s_1 - s_2)^{2H-1}(t_1 - t_2)^{2H-2}\text{sgn}(s_2 - t_2) e^{s_1 - t_2 - |t_2 - s_2|} ds_2$$

$$:= O_1(T) + O_2(T)$$

For $O_1(T)$, making the change of variables $u = t_1 - t_2, v = s_1 - t_2, x = s_1 - s_2$, we have

$$O_1(T) = \int_0^1 e^x x^{2H-1} dx \int_1^T dt_1 \int_1^{t_1} e^{-u} u^{2H-2}(u - 1) du.$$ 

Therefore, by the Partial integration formulate, when $T \to \infty$, the asymptotes of $O_1(T)$ is

$$T \times \int_0^1 e^x x^{2H-1} dx \int_1^{\infty} e^{-u} (u^{2H-1} - u^{2H-2}) du + C_H. \quad (4.15)$$

For $O_2(T)$, making the change of variables $u = t_1 - t_2, v = t_1 - s_1 + 1, x = s_1 - s_2$ indicates

$$O_2(T) = \int_1^T dt_1 \int_1^{t_1} e^{1-v} dv \int_1^v u^{2H-2} du \int_0^1 x^{2H-1}$$

$$\times \text{sgn}(u - v - x + 1)e^{-|u-v-x+1|} dx.$$ 

By the Partial integration formulate, when $T \to \infty$, the asymptote of $O_2(T)$ is

$$T \int_1^{\infty} e^{1-v} dv \int_1^v u^{2H-2} du \int_0^1 x^{2H-1} \text{sgn}(u - v - x + 1)e^{-|u-v-x+1|} dx + C_H$$

$$= T \int_1^{\infty} e^{-u} u^{2H-2} du \int_0^1 e^x \left( \frac{1}{2} x^{2H-1} - x^{2H} \right) dx + C_H, \quad (4.16)$$
The last equation is obtained by decomposing the region \( \{1 \leq u \leq v < \infty \} \) into 
\( \{1 \leq u \leq v \leq 1 + u < \infty \} \cup \{1 \leq u \leq v - 1 < \infty \} \), using the Fubini theorem. Combining (4.15) and (4.16), we obtain the asymptote of \( N_1(T) \):

\[
T \left[ \int_0^1 e^{x-1} x^{2H-1} dx - \int_1^\infty e^{1-u} x^{2H-2} du + (4H - \frac{3}{2}) \int_1^\infty e^{-u} x^{2H-2} du \int_0^1 e^{x} x^{2H-1} dx \right] + C_H. 
\]

Combining the equation and the asymptote (4.14), the asymptote of \( N_1(T) \) is

\[
T \left[ \int_0^1 e^{x-1} x^{2H-1} dx - \int_1^\infty e^{1-u} x^{2H-2} du + (4H - 1) \int_1^\infty e^{-u} x^{2H-2} du \int_0^1 e^{x} x^{2H-1} dx \right] + C_H. 
\]

To take the asymptote of \( -N_2(T) \), we decompose the integral region \([1, T]^2\) of integral variable \((t_1, s_2)\) of \( -N_2(T) \) into \( \{1 \leq t_1 \leq s_2 \leq T\} \cup \{1 \leq s_2 \leq t_1 \leq T\} \). Then we obtain

\[
-N_2(T) = (\int_1^T ds_2 \int_1^{s_2} dt_1 + \int_1^T dt_1 \int_1^{t_1} ds_2) \int_1^{t_1-1} dt_2 \int_1^{s_2} ds_1 \times (s_2 - s_1)^{2H-1}(t_1 - t_2)^{2H-2} \text{sgn}(s_2 - t_2)e^{-|t_1-s_1|-|t_2-s_2|}ds_1 
\]

\[
:= N_{21}(T) + N_{22}(T). 
\]

Making the change of variables, we have

\[
N_{21}(T) = \int_1^T ds_2 \int_1^{s_2} e^{-v} dv \int_1^{v} u^{2H-2} du \int_0^1 x^{2H-1} e^{-|x-v+u|} dx. 
\]

The Partial integration formulate and making the change of variable \( z = v - u \) imply that when \( T \to \infty \), the asymptote of \( N_{21}(T) \) is

\[
T \times \int_1^\infty u^{2H-2} du \int_0^\infty e^{-v} dv \int_0^1 x^{2H-1} e^{-|x-v+u|} dx + C_H 
\]

\[
= T \times [(2H + \frac{1}{2}) \int_0^1 e^{-x} x^{2H-1} dx - e^{-1}] \times \int_1^\infty e^{-u} u^{2H-2} du + C_H. 
\]

(4.19)

For \( N_{22}(T) \), we fix integral variable \( t_1 \) and decompose the integral region \([0, t_1-1] \times [1, t_1] \) of integral variable \((t_2, s_2)\) into \( 1 \leq t_2 + 1 \leq s_2 \leq t_1 \) and \( 1 \leq s_2 \leq t_2 + 1 \leq t_1 \).
Then \( N_{22}(T) \) split into the sum of the following two integrals:

\[
N_{22}(T) = \int_1^T dt_1 \left( \int_0^{t_1-1} dt_2 \int_{t_2+1}^{t_1} ds_2 + \int_1^{t_1} ds_2 \int_{s_2-1}^{t_1-1} dt_2 \right) \int_{s_2-1}^{s_2} \times (s_2 - s_1)^{2H-1}(t_1 - t_2)^{2H-2} \text{sgn}(s_2 - t_2)e^{-|t_1-s_1|-|t_2-s_2|} \, ds_1
\]

\[ := O'_1(T) + O'_2(T). \]

Making the change of variables \( u = t_1 - t_2, v = t_1 + s_2, x = s_2 - s_1 \), we have

\[
O'_1(T) = \int_1^T dt_1 \int_1^{t_1} e^{-u} u^{2H-2} \, du \int_1^u e^{-x} x^{2H-1} \, dx,
\]

\[ O'_2(T) = \int_1^T dt_1 \int_1^{t_1} e^{-v} \, dv \int_1^v e^{-|u-v+1|} u^{2H-2} \text{sgn}(u - v + 1) \, du \int_0^1 e^{-x} x^{2H-1} \, dx. \]

The Partial integration formulate implies that when \( T \to \infty \), the asymptotes of \( O'_1(T) \) and \( O'_2(T) \) are

\[
T \times \int_0^1 e^{-x} x^{2H-1} \, dx \int_1^\infty e^{-u} u^{2H-2} (u - 1) \, du + C_H, \quad (4.20)
\]

\[
T \times \int_0^1 e^{-x} x^{2H-1} \, dx \int_1^\infty e^{-v} \, dv \int_1^v e^{-|u-v+1|} u^{2H-2} \text{sgn}(u - v + 1) \, du + C_H
\]

\[ = \frac{T}{2} \times \int_0^1 e^{-x} x^{2H-1} \, dx \times \int_1^\infty e^{-u} u^{2H-2} \, du + C_H. \quad (4.21)\]

Combining (4.20) and (4.21), we obtain the asymptote of \( N_{22}(T) \) is

\[
T \times \int_0^1 e^{-x} x^{2H-1} \, dx \int_1^\infty e^{-u} u^{2H-2} (u - \frac{1}{2}) \, du + C_H. \quad (4.22)
\]

Combining (4.19) and (4.22), the asymptote of \( N_2(T) \) is

\[
T \left[ \int_0^1 e^{-x} x^{2H-1} \, dx - \int_1^\infty e^{-u} u^{2H-2} \, du + (4H - 1) \int_1^\infty e^{-u} u^{2H-2} \, du \int_0^1 e^{-x} x^{2H-1} \, dx \right] + C_H.
\]

The asymptote of \( N(T) \) is (4.13), which is obtained by the asymptote (4.18) of \( N_1(t) \) minus the above equation. \( \square \)

**Lemma 4.5.** Let (3.14) be the expression of the quadruple integral \( \tilde{N}(T) \), then when \( T \to \infty \), the asymptote of \( \tilde{N}(T) \) is

\[
T \times \left[ (1 + e^{-2}) + [(2H + 1)e^{-1} + (2H - 1)e] \int_1^\infty e^{-u} u^{2H-2} \, du \right] + C_H. \quad (4.23)
\]
**Proof.** By integrating Dirac function, we write the quadruple integral \( \tilde{N}(T) \) as the sum of the following two triple integrals:

\[
\tilde{N}(T) = \tilde{N}_1(T) + \tilde{N}_2(T),
\]

(4.24)

where

\[
\tilde{N}_1(T) = \int_0^T ds_1 \int_1^T e^{-|t_1-s_1|} dt_1 \int_0^{t_1-1} (t_1-t_2)^{2H-2} dt_2 \left( (s_1 - 1) \lor 0 \right)^{2H-1} e^{-|t_2-(s_1-1)\lor 0|},
\]

\[
\tilde{N}_2(T) = \int_0^T ds_1 \int_1^T e^{-|t_1-s_1|} dt_1 \int_0^{t_1-1} (t_1-t_2)^{2H-2} dt_2 \left( (s_1 + 1) \land T - s_1 \right)^{2H-1} e^{-|t_2-(s_1+1)\land T|}.
\]

First, solve the asymptote of triple integral \( \tilde{N}_1(T) \). Then divide the integral region \( s_1 \in [0, T] \) into \([0, 1) \cup [1, T] \). Making the change of variable \( u = t_1 - t_2 \), we get that the triple integral of the subinterval \( s_1 \in [0, 1) \) connection is:

\[
\int_0^1 s_1^{2H-1} ds_1 \int_1^T e^{-(t_1-s_1)} dt_1 \int_0^{t_1-1} e^{-t_2(t_1-t_2)^{2H-2}} dt_2
= \int_0^1 e^s s_1^{2H-1} ds_1 \int_1^T e^{-2t_1 dt_1} \int_1^{t_1} e^u u^{2H-2} du.
\]

When \( T \to \infty \), its limit exists. Then the asymptote of \( \tilde{N}_1(T) \) and the triple integral

\[
\tilde{N}_{11}(T) = \int_1^T ds_1 \int_1^T e^{-|t_1-s_1|} dt_1 \int_0^{t_1-1} e^{-|t_2-s_1+1|} (t_1-t_2)^{2H-2} dt_2
\]

connected with \( \tilde{N}_1(T) \) in the integral sub region \( s_1 \in [1, T] \) have the same slope asymptote (different intercept terms). Making the change of variables

\[
w = s_1 \lor t_1, \quad v = |s_1 - t_1|, \quad u = t_1 - t_2
\]

Triple integral \( \tilde{N}_{11}(T) \) is rewritten as

\[
\tilde{N}_{11}(T) = \int_1^T dw \int_0^{w-1} e^{-v} dv \left[ \int_1^w u^{2H-2} e^{-(v+u-1)} du + \int_1^w u^{2H-2} e^{-v+u+1} du \right].
\]

(4.25)

Making the change of variables

\[
u' = u - 1, \quad x = v + u'
\]
We know that the first part of triple integral (4.25) is
\[
\int_1^T \int_0^{w-1} e^{-v} dv \int_1^w u^{2H-2} e^{-(v+u-1)} du \\
= \int_1^T \int_0^{w-1} e^{-2x} dx \int_0^x (1 + u')^{2H-2} e^{u'} du'.
\]

It is easy to know from the Partial integral formula and Fubini theorem that the asymptote of the above formula is
\[
T \int_0^\infty e^{-2x} dx \int_0^x (1 + u')^{2H-2} e^{u'} du' + C_H = \frac{1}{2} T \int_1^\infty e^{1-u} u^{2H-2} du + C_H. \quad (4.26)
\]

Making the change of variable \( u' = u - 1 \) and Fubini theorem, the second part of triple integral (4.25) is
\[
\int_1^T \int_0^{w-1} e^{-v} dv \int_1^w u^{2H-2} e^{-|v-u+1|} du \\
= \int_1^T \int_0^{w-1} e^{-v} dv \left[ \int_1^{1+v} u^{2H-2} e^{v-u-1} du + \int_1^w u^{2H-2} e^{v+u} du \right] \\
= \int_1^T \int_0^{w-1} e^{-v} dv \int_0^v e^u (u' + 1)^{2H-2} du' + \int_1^T \int_0^{w-1} e^{-u} (u' + 1)^{2H-2} u' du'
\]

According to the Partial integral formula, the asymptote of the above formula is
\[
T \left[ \int_0^\infty e^{-2x} dx \int_0^x (1 + u')^{2H-2} e^{u'} du' + \int_0^\infty e^{-u'} (1 + u')^{2H-2} u' du' \right] + C_H \\
= T \left[ \frac{1}{2} \int_1^\infty e^{1-u} u^{2H-2} du + \int_1^\infty e^{1-u} u^{2H-2} (u - 1) du \right] + C_H. \quad (4.27)
\]

Combining (4.26) and (4.27), we get the asymptote of triple integral \( \tilde{N}_1(T) \) as:
\[
T \times \int_1^\infty e^{1-u} u^{2H-1} du + C_H. \quad (4.28)
\]

Next, we solve the asymptote of triple integral \( \tilde{N}_2(T) \). Similarly, we divide the integral region \( s_1 \in [0, T] \) into \([0, T - 1] \cup (T - 1, T] \). Making the change of variable \( u = t_1 - t_2 \), we obtain the limit existence of triple integrals associated with subinterval \( s_1 \in (T - 1, T] \) when \( T \to \infty \). Therefore, the asymptote of \( \tilde{N}_2(T) \) and the following triple integral
\[
\tilde{N}_{21}(T) = \int_0^{T-1} ds_1 \int_1^T e^{-|t_1-s_1|} dt_1 \int_0^{t_1-1} e^{-|t_2-s_1-1|} (t_1 - t_2)^{2H-2} dt_2
\]
have the same slope (different intercept terms). For triple integrals $\tilde{N}_{21}(T)$, first making the change of variable $u = t_1 - t_2$, and then we divide the integral region $[0, T - 1] \times [1, T]$ of the integral variable $(s_1, t_1)$ as follows:

$$\{1 \leq t_1 \leq s_1 \leq T - 1\} \cup \{0 \leq t_1 - 1 \leq s_1 \leq t_1 \land (T - 1) \leq T\} \cup \{0 \leq s_1 \leq t_1 - 1 \leq T - 1\},$$

we have

$$\tilde{N}_{21}(T) = \int_0^{T-1} ds_1 \int_1^T e^{-|t_1 - s_1|} dt_1 \int_1^{t_1} e^{-|t_1 - s_1 - u|} u^{2H-2} du$$

$$= \left[ \int_1^{T-1} ds_1 \int_1^s dt_1 + \int_1^T dt_1 \int_{t_1 - 1}^{t_1 \land (T-1)} ds_1 + \int_1^T dt_1 \int_0^{t_1 - 1} ds_1 \right]$$

$$\times \int_1^{t_1} e^{-|t_1 - s_1| - |t_1 - s_1 - u|} u^{2H-2} du. \quad (4.29)$$

According to the Partial integral formula, the asymptotes of the first, second and third parts of triple integral (4.29) are:

$$\frac{T}{2} \times \int_1^{\infty} e^{-u} u^{2H-2} du + C_H,$$

$$T \times \int_1^{\infty} e^{-u} u^{2H-2} du + C_H,$$

$$T \times \int_1^{\infty} e^{-u} \left[ u^{2H-1} + \frac{1}{2} u^{2H-2} \right] + C_H.$$

Combining the three asymptotes above, we get triple integral the asymptote of $\tilde{N}_2(T)$ is

$$T \times \int_1^{\infty} e^{-u} \left[ u^{2H-1} + 2u^{2H-2} \right] + C_H. \quad (4.30)$$

Finally, we combine the asymptote (4.28) of $\tilde{N}_1(T)$ and the asymptote (4.30) of $\tilde{N}_2(T)$ to obtain the asymptote (4.23) of $\tilde{N}(T)$. □
Lemma 4.6. Record two quadruple integrals $U(T), \tilde{U}(T)$, as given in (3.16) and (3.17). When $T \to \infty$, their asymptotes are:

$$T \times \left[ \int_{1}^{\infty} e^{-u}u^{2H-2}du \times \left[ e^{-1} - e + (4H - 1) \int_{0}^{1} (e^x - e^{-x})x^{2H-1}dx \right] 
+ \int_{0}^{1} (e^{x-1} - e^{-x-1})x^{2H-1}dx \right] + C_H,$$

$$T \times \left[ (1 + e^{-2}) + [(2H + 1)e^{-1} + (2H - 1)e] \int_{1}^{\infty} e^{-u}u^{2H-2}du \right].$$

(4.31)

The proof of Lemma 4.6 is basically consistent with the proof of Lemma 4.4 and Lemma 4.5 above. Considering the length of the article, its details are omitted.

Lemma 4.7. The marked quadruple integral $L(T)$ is given by (3.21). Then the asymptote of $L(T)$ when $T \to \infty$ is:

$$4T \times \left[ (4H + 1) \int_{0}^{1} e^{-u}u^{2H-1}du \int_{0}^{u} e^{v}v^{2H-1}dv - (2H + \frac{1}{2}) \left( \int_{0}^{1} e^{-u}u^{2H-1}du \right)^{2} 
+ \int_{0}^{1} (e^{-u-1} - e^{u-1})u^{2H-1}du \right] + C_H.$$

(4.32)

Proof. The starting point is to remove the following two absolute value symbols from the quadruple integral $L(T)$: $|s_1 - s_2| |s_1 - s_2|$ and $|t_1 - t_2|$. That is, first of all, we divide the integral region of the integral variables $s_2, t_2$ as follows:

$$\int_{(s_1 - 1)\land 0}^{(s_1 + 1)\land T} ds_2 \int_{(t_1 - 1)\land 0}^{(t_1 + 1)\land T} dt_2 = \int_{(s_1 - 1)\land 0}^{s_1} ds_2 \int_{(t_1 - 1)\land 0}^{t_1} dt_2 + \int_{(s_1 - 1)\land 0}^{s_1} ds_2 \int_{(t_1 - 1)\land 0}^{t_1} dt_2.
$$

(4.33)

The integral values of the four integral $L(T)$ in the above four sub regions are recorded as $LL_1(T), L_2(T), L_3(T), L_4(T)$. It is easy to know by symmetry:

$$L_1(T) = L_4(T), \quad L_2(T) = L_3(T).$$

(4.34)

Then remove the two symbols $\lor$ in $L_1(T)$ respectively, and further decompose the integral into the sum of the following four integrals:

$$L_1(T) = \int_{1}^{T} ds_1 \int_{s_1 - 1}^{s_1} ds_2 + \int_{0}^{s_1} ds_1 \int_{s_1}^{s_1} ds_2 \int_{1}^{T} dt_1 \int_{t_1 - 1}^{t_1} dt_2
+ \int_{0}^{1} dt_1 \int_{0}^{1} dt_1 \int_{0}^{1} dt_1 \int_{0}^{1} dt_1
\times (s_1 - s_2)^{2H-1}(t_1 - t_2)^{2H-1} e^{-|t_1 - s_1| - |t_2 - s_2|}dt_2
:= L_{11}(T) + L_{12}(T) + L_{13}(T) + L_{14}(T)$$

(4.35)
First, we notice that the integral $L_{14}(T)$ is independent of $T$, and we get $L_{12}(T) = L_{13}(T)$ through symmetry. Making the change of variables $u = s_1 - s_2, v = t_1 - t_2$, we can deduce:

$$L_{12}(T) = \int_1^T e^{-s_1} ds_1 \int_0^1 u^{2H-1} du \int_0^1 e^{t_1} dt_1 \int_0^{t_1} e^{-|s-t+v-u|} v^{2H-1} dv,$$

Therefore, the $\lim_{T \to \infty} L_{12}(T)$ exists. Again according to symmetry and making the change of variables $u = s_1 - s_2, v = t_1 - t_2, x = s_1 - t_1$, it is obtained that:

$$L_{11}(T) = 2 \int_1^T ds_1 \int_0^{s_1-1} e^{-x} dx \int_0^1 u^{2H-1} du \int_0^1 e^{-|x-u+v|} v^{2H-1} dv$$

It can be seen from the Partial integration formula that the asymptotes of $L_{11}(T)$ and $L_1(T)$ when $T \to \infty$ are both (only the intercept term $C_H$ with different difference):

$$2T \int_0^\infty e^{-x} dx \int_0^1 u^{2H-1} du \int_0^1 e^{-|x-u+v|} v^{2H-1} dv + C_H$$

$$= 2T \left[ (4H + 1) \int_0^1 u^{2H-1} du \int_0^1 e^{-u} v^{2H-1} dv - \int_0^1 e^{-v} v^{2H-1} dv \right] + C_H \quad (4.36)$$

The above equation divides the integral region of $(u,v)$ into $\{0 \leq u \leq v \leq 1\} \cup \{0 \leq v \leq u \leq 1\}$, for the second sub region, we divide the integral region of $x$ into $[0, u-v] \cup (u-v, \infty)$ again, and then get it from Fubini theorem.

Similarly, it is decomposed as follows:

$$L_2(T) = - \left( \int_1^T ds_1 \int_0^{s_1-1} ds_2 \int_0^1 ds_1 \int_0^{s_2} ds_2 \right) \left( \int_1^T dt_2 \int_0^{t_2} dt_1 \right)$$

$$\times (s_1 - s_2)^{2H-1} (t_2 - t_1)^{2H-1} e^{-|t_1-s_1|-|t_2-s_2|}$$

$$:= - (L_{21}(T) + L_{22}(T) + L_{23}(T) + L_{24}(T)),$$

Where the integral $L_{24}(T)$ is independent of $T$, and the existence of $\lim_{T \to \infty} (L_{22}(T) + L_{23}(T))$ is deduced by making the change of variables $v = t_1 - t_2, u = s_1 - s_2$.

Then, it can be seen from making the change of variables

$$w = \max \{s_1, t_2\}, x = |s_1 - t_2|, u = s_1 - s_2, v = t_2 - t_1$$

that the quadruple integral $L_{21}(T)$ is:

$$L_{21}(T) = 2 \int_1^T dw \int_0^w e^{-x} dx \int_0^1 e^{-|x-v-u|} u^{2H-1} v^{2H-1} du dv.$$
It is known from the Partial integration formula that the asymptotes of \(-L_{21}(T)\) and \(L_2(T)\) when \(T \to \infty\) are both (only different intercept terms \(C_H\)):

\[
-2T \int_0^1 e^{-u} u^{2H-1} du \times \int_0^\infty e^{-x} dx \int_0^1 e^{-|x-v|} v^{2H-1} dv + C_H
\]

\[
= -2T \int_0^1 e^{-u} u^{2H-1} du \int_0^1 e^{-v} (v^{2H} + \frac{1}{2} v^{2H-1}) dv + C_H, \tag{4.37}
\]

The above equation decomposes the integral domain of \((x, v)\) into \(\{0 \leq x \leq v \leq 1\} \cup \{0 \leq v \leq 1, x > v\}\), and then obtains it from Fubini theorem.

Finally, from (4.34) and (4.35), combining the asymptotes (4.36) and (4.37) of \(L_1(T)\) and \(L_2(T)\) when \(T \to \infty\), it is obtained that the asymptote of \(L(T)\) is (4.33).

**Lemma 4.8.** The quadruple integral \(P(T)\) is given in equation (3.22). Then the asymptote of \(P(T)\) when \(T \to \infty\) is:

\[
2T \times \left[1 - e^{-2} - (2H + 1) \int_0^1 (e^{v-1} - e^{-u-1}) u^{2H-1} du\right] + C_H. \tag{4.38}
\]

**Proof.** First, by integrating Dirac function, we write \(P(T)\) as the sum of the following two triple integrals:

\[
P(T) = \int_{[0,T]^2} e^{-|t_1-s_1|} ds_1 dt_1 \int_{(t_1+1) \lor 0}^{(t_1+1) \lor T} \sgn(t_1 - t_2) |t_1 - t_2|^{2H-1} dt_2
\times \left[e^{-|t_2-(s_1-1)\lor 0|} \sgn\left((s_1 - 1) \lor 0 - t_2\right)(s_1 - (s_1 - 1) \lor 0)^{2H-1}
\right.
\]

\[
+ e^{-|t_2-(s_1+1)\lor T|} \sgn\left((s_1 + 1) \land T - t_2\right)((s_1 + 1) \land T - s_1)^{2H-1}\right]
:= P_1(T) + P_2(T).
\]

It is required to solve the asymptote of triple integral \(P_1(T)\). First, we divide the region \(\{(s_1, t_1) \in [0, T]^2\}\) into the following parts:

\[
[0, 3] \times [0, 1], [3, T] \times [0, 1], [0, 1] \times [1, T], [1, T-1] \times [T-1, T],
\]

\[
[T-1, T] \times [1, T-1], [T-1, T]^2, [1, T-1]^2.
\]

It is clear that the triple integral \(P_1(T)\) restricted in sub-region \([0, 3] \times [0, 1]\) is independent of \(T\) and when \(T \to \infty\), the limit of triple integral \(P_1(T)\) in sub-region \([3, T] \times [0, 1]\) exists. Making the change of variables \(u = t_2 - t_1\), we have that when \(T \to \infty\), the limit of triple integral \(P_1(T)\) in sub-region \([0, 1] \times [1, T]\) exists; Making the change of variables \(y = T - t_1\), \(u = t_2 - t_1\), when \(T \to \infty\), the
limit of triple integral $P_1(T)$ in sub region $[1, T - 1] \times [T - 1, T]$ exists. It can be seen from making the change of variables

$$x = T - s_1, \; y = T - t_1, \; u = t_2 - t_1$$

that the integral of triple integral $P_1(T)$ in the sub region $[T - 1, T]^2$ is also independent of $T$; When $T \to \infty$, the triple integral $P_1(T)$ is in the sub region $[T - 1, T] \times [1, T - 1]$ the limit of the integral exists.

Making the change of variables

$$w = \max \{ s_1, t_1 \}, \; v = |s_1 - t_1|, \; u = t_2 - t_1$$

we can see that the triple integral $P_1(T)$ in sub region $[1, T - 1]^2$ is:

$$\int_{[1,T-1]^2} e^{-|t_1-s_1|}ds_1dt_1 \int_{t_1-1}^{t_1+1} \sgn(t_1 - t_2)|t_1 - t_2|^{2H-1}e^{-|t_2-(s_1-1)|}\sgn(s_1 - 1 - t_2)dt_2$$

$$= \int_1^{T-1} dw \int_0^{u-1} e^{-v}dv \int_{-1}^{1} [e^{-|v+u+1|} + e^{-|u-v+1|}\sgn(u - v + 1)] |u|^{2H-1} \sgn(u)du.$$

According to the Partial integration formula, When $T \to \infty$, the asymptote of the above formula and the asymptote of triple integral $P_1(T)$ are both (combined with the limit existence of integrals in the first six regions, it can be seen that they are only different from each other in terms of intercept $C_H$):

$$T \times \int_0^{\infty} e^{-v}dv \int_{-1}^{1} [e^{-|v+u+1|} + e^{-|u-v+1|}\sgn(u - v + 1)] |u|^{2H-1} \sgn(u)du + C_H$$

$$= T \times \int_{-1}^{1} e^{-u-1} |u|^{2H-1} \sgn(u)(1 + u)du + C_H.$$

Similarly, the asymptote of triple integral $P_2(T)$ has the same slope as the asymptote of the following integral

$$\int_{[1,T-1]^2} e^{-|t_1-s_1|}ds_1dt_1 \int_{t_1-1}^{t_1+1} \sgn(t_1 - t_2)|t_1 - t_2|^{2H-1}e^{-|t_2-(s_1+1)|}\sgn(s_1 + 1 - t_2)dt_2$$

$$= \int_1^{T-1} dw \int_0^{u-1} e^{-v}dv \int_{-1}^{1} [-e^{-|u-v-1|} + e^{-|u+v-1|}\sgn(u + v - 1)] |u|^{2H-1} \sgn(u)du$$

Thus, we get that the asymptote of the triple integral $P_2(T)$ is:

$$T \times \int_0^{\infty} e^{-v}dv \int_{-1}^{1} [-e^{-|u-v-1|} + e^{-|u+v-1|}\sgn(u + v - 1)] |u|^{2H-1} \sgn(u)du$$

$$= T \times \int_{-1}^{1} e^{u-1} |u|^{2H-1} \sgn(u)(u - 1)du + C_H.$$
Finally, we combine the two asymptotes of $P_1(T)$ and $P_2(T)$ to obtain the asymptote (4.38) of $P(T)$.

\[ \square \]

\textbf{References}

[1] Sottinen T, Viitasaari L. Parameter estimation for the Langevin equation with stationary-increment Gaussian noise. Stat Inference Stoch Process, 2018, 21(3): 569-601

[2] Douissi S, Es-Sebaiy K, Kerchev G, Nourdin I. Berry-Esseen bounds of second moment estimators for Gaussian processes observed at high frequency. Electron J Statist, 2022, 16(1): 636-670

[3] Hu Y, Nualart D, Zhou H. Parameter estimation for fractional Ornstein–Uhlenbeck processes of general Hurst parameter. Statistical Inference for Stochastic Processes, 2019, 22(1): 111-142

[4] Jolis M. On the Wiener integral with respect to the fractional Brownian motion on an interval. Journal of mathematical analysis and applications, 2007, 330(2): 1115-1127

[5] Chen Y, Li Y. Berry-Esseen bound for the parameter estimation of fractional Ornstein-Uhlenbeck processes with the hurst parameter $H \in (0, \frac{1}{2})$. Communications in Statistics-Theory and Methods, 2021, 50(13): 2996-3013

[6] Kim Y T, Park H. S. Optimal Berry-Esséen bound for statistical estimations and its application to SPDE. Journal of Multivariate Analysis, 2017, 155: 284-304.

[7] Chen Y, Zhou H. Parameter estimation for an Ornstein-Uhlenbeck process driven by a general gaussian noise. Acta Mathematica Scientia, 2021, 41B(2): 573-595.

[8] Chen Y, Gu X M, Li Y. Parameter Estimation for an Ornstein-Uhlenbeck Processes driven by a general Gaussian Noise with Hurst Parameter $H \in (0, \frac{1}{2})$. arXiv preprint arXiv:2111.15292, 2021.

[9] Cheridito P, Kawaguchi H, Maejima M. Fractional Ornstein-Uhlenbeck processes. Electron J Probab, 2003, 8(3), 14 pp.

[10] Hu Y, Nualart D. Parameter estimation for fractional Ornstein-Uhlenbeck processes. Statistics & probability letters, 2010, 80(11-12): 1030-1038

[11] Chen Y, Kuang N H, Li Y. Berry-Esseen bound for the parameter estimation of fractional Ornstein-Uhlenbeck processes. Stochastics and Dynamics, 2020, 20(04): 2050023

[12] Mishura Y S. Stochastic calculus for fractional Brownian motion and related processes. Lecture Notes in Mathematics 1929. Springer-Verlag, Berlin, 2008

[13] Pipiras V, Taqqu M. S. Integration questions related to fractional Brownian motion. Probability Theory and Related Fields, 2000, 118(2):251-91

[14] Nualart D. The Malliavin calculus and related topics. Springer, 2006

[15] Tao T. An introduction to measure theory. Providence: American Mathematical Society, 2011

[16] Chen Y, Ding Z, Li Y. Berry-Esseen bounds and almost sure CLT for the quadratic variation of a general Gaussian process. arXiv preprint arXiv:2106.01851, 2021
