Instanton representation of Plebanski gravity for Type D spacetimes

Eyo Eyo Ita III

July 13, 2010

Department of Applied Mathematics and Theoretical Physics
Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road
Cambridge CB3 0WA, United Kingdom
eei20@cam.ac.uk

Abstract

In this paper we demonstrate the construction of some GR solutions for spacetimes of Petrov Type D with vanishing SO(3,C) angles. The solutions are obtained purely by application of the initial value constraints in the instanton representation.
1 Introduction

In [1] a new formulation of nonmetric general relativity is presented, where the basic phase space variables are a self-dual connection and the self-dual part of the Weyl curvature.\textsuperscript{1} The action for this formulation is dual to the Ashtekar theory and arises from the starting Plebanski action. It is shown in [1] that the instanton representation reproduces the Einstein equations up to terms which are spatial gradients of the initial value constraints. This implies that a necessary condition for the new formulation to yield the Einstein equations is that the initial value constraints must be satisfied. We would like to investigate the notion that this should as well be a sufficient condition. This would imply that by satisfying the initial value constraints, one should be able to construct solutions to the Einstein equations in the form of the spacetime metric \( g_{\mu\nu} \). In the instanton representation the metric is a derived quantity, which should in principle be expressible directly in terms of the physical degrees of freedom. The 3+1 decomposition of the metric clearly delineates the separation of the physical from the unphysical degrees of freedom of GR. It happens that the inputs for these degrees of freedom come directly from the instanton representation phase space, via implementation of the constraints. In this paper we demonstrate this feature for some simple nontrivial cases.

The organization of this paper is as follows. In section 2 we present the initial value constraints of the instanton representation. The constraints are written entirely in the language of the 3-dimensional complex orthogonal group, and do not contain any reference to a metric or to coordinates. Section 3 performs a transformation of the constraints from the instanton representation into the Ashtekar variables, which implies an inherent choice of coordinates. We then construct solutions to the initial value constraints in the instanton representation for spacetimes of Petrov Type D, and reconstruct the spacetime metric from these solutions. The only inputs into this construction are the eigenvalues of the self-dual Weyl curvature (CDJ matrix) and the connection, and for simplicity We have limited ourselves in this paper to spacetimes where the \( SO(3,C) \) angles vanish. We produce some known GR solutions including the Schwarzschild and DeSitter blackhole solutions using this scheme, which is shown in sections 5 and 6. In section 7 we apply the scheme to different permutations of the eigenvalues, which leads to some new metrics which we display. Section 8 is a brief discussion

\textsuperscript{1}The nonmetric formulation in [2] due to Jaboson, Caovpilla and Dell writes GR almost completely in terms of the connection. Our new formulation can be seen as the analogous action at the level prior to elimination of the Weyl curvature, which therefore contains additional fields.
section. Section 9 is an appendix where we derive the polar representation of
the Gauss’ law constraint as differential equations for the $SO(3,C)$ angles.

2 Initial value constraints

In the instanton representation of Plebanski gravity the basic phase space
variables are $(A^a_i, \Psi_{ae})$, where $A^a_i$ is a $SO(3,C)$ gauge connection and $\Psi_{ae}$ is
the CDJ matrix, which takes its values in $SO(3,C) \otimes SO(3,C)$. The initial
value constraints in this representation are given by

$$w_e\{\Psi_{ae}\} = 0; \quad \epsilon_{dae} \Psi_{ae} = 0; \quad \Lambda + \text{tr}\Psi^{-1} = 0.$$  \hspace{1cm} (1)

The middle equation of (1) implies $\Psi_{ae} = \Psi_{ea}$ is symmetric. A complex sym-
metric matrix can be diagonalized by a special complex orthogonal transform-
ation, when there exist three linearly independent eigenvectors. Hence
for nondegenerate $\Psi_{ae}$ we can write the following polar decomposition

$$\Psi_{ae} = (e^{\theta \cdot T})_{af} \lambda_f (e^{-\theta \cdot T})_{fe};$$  \hspace{1cm} (2)

where $\vec{\theta} = (\theta^1, \theta^2, \theta^3)$ are three complex rotation parameters and $T \equiv (T^A)_{fg}$
are the $SO(3,C)$ generators. Then the third equation of (1) reduces to

$$\Lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} = 0$$  \hspace{1cm} (3)

on account of the cyclic property of the trace, where $\lambda_1$, $\lambda_2$ and $\lambda_3$ are the
eigenvalues of (the now symmetric) $\Psi_{ae}$. The first equation of (1) is

$$w_e\{\Psi_{ae}\} = v_e\{\Psi_{ae}\} + C^{fg}_a \Psi_{fg} = 0,$$  \hspace{1cm} (4)

where we have defined

$$C^{fg}_a = (f_{abf} \delta_{ge} + f_{ebg} \delta_{af}) C_{be}$$  \hspace{1cm} (5)

with $SO(3,C)$ structure constants $f_{abc}$. Also, using the $SO(3,C)$ magnetic
field we have defined the vector fields $v_a$ and the magnetic helicity density
matrix $C_{ae}$ by

$$v_a = B^i_a \partial_i; \quad C_{ae} = A^a_i B^i_e;$$  \hspace{1cm} (6)
where \( B^i_a \) is the magnetic field for \( A^a_i \), given by

\[
B^i_a = \epsilon^{ijk} \partial_j A^a_k + \frac{1}{2} \epsilon^{ijk} f_{abc} A^b_j A^c_k. \tag{7}
\]

Substituting (2) into (4), we obtain the following matrix representation (See Appendix A for derivation)

\[
\begin{pmatrix}
0 & (\lambda_3 - \lambda_1) v'_3 & (\lambda_1 - \lambda_2) v'_2 \\
(\lambda_2 - \lambda_3) v'_2 & 0 & (\lambda_1 - \lambda_2) v'_1 \\
(\lambda_2 - \lambda_3) v'_2 & (\lambda_3 - \lambda_1) v'_1 & 0
\end{pmatrix}
\begin{pmatrix}
\theta^1 \\
\theta^2 \\
\theta^3
\end{pmatrix}
\]

\[
= \begin{pmatrix}
v'_1 - C'_{[23]} & -C'_{32} & C'_{23} \\
C'_{31} & v'_2 - C'_{[31]} & -C'_{13} \\
-C'_{21} & C'_{12} & v'_3 - C'_{[12]}
\end{pmatrix}
\begin{pmatrix}
\lambda_1 \\
\lambda_2 \\
\lambda_3
\end{pmatrix}
\]

where we have defined

\[
C'_{eh} = (e^{-\theta T})_{eb} (e^{-\theta T})_{hg} C_{bg}; \quad v'_a = (e^{-\theta T})_{de} v_e. \tag{8}
\]

Note that (8) are the \( SO(3,C) \) rotated versions of their unprimed counterparts (6). We have obtained a system of three nonlinear differential equations in the unknown angles \( \vec{\theta} \). Each triple of eigenvalues satisfying (3), combined with a connection \( A^a_i \) determines the angles \( \vec{\theta} \). We will argue that each well-defined \( \vec{\theta} \) thus determined corresponds to a GR solution, by showing that (1) are actually the initial value constraints of GR.

### 3 Ashtekar variables

The eigenvalues \( \lambda_1, \lambda_2 \) and \( \lambda_3 \) are coordinate-invariant and invariant under \( SO(3,C) \) rotations. The CDJ matrix \( \Psi_{ae} \) is also coordinate-invariant and can be seen, from the polar representation in (2), as the rotation of these eigenvalues into a new \( SO(3,C) \) frame \( \vec{\theta} \). The existence of a magnetic field \( B^i_a \), due to its spatial index \( i \), amounts to the specification of coordinates \( x^i \). If one regards the vector fields \( v_a \) as fundamental, then \( B^i_a \) becomes a derived quantity via

\[
B^i_a = v_a \{ x^i \}. \tag{9}
\]

For each set of coordinate functions \( x^i \), the CDJ matrix can be written as
$$\Psi_{ae}^{-1} = B^i_a (B^{-1})^i_f \Psi_f^{-1}. \quad (10)$$

Let us perform a re-definition of variables \((\bar{\sigma}^{-1})^i_e = (B^{-1})^i_f \Psi_f^{-1}\). Then one has the relation

$$\bar{\sigma}^i_a = \Psi_{ae} B^i_e, \quad (11)$$

which holds when all quantities are nondegenerate. Hence, \(\bar{\sigma}^i_a\) and \(B^i_a\) both imply the choice of coordinates and the existence of a \(SO(3, C)\) frame due to having both internal and spatial indices.

We will now write the constraints (1) in terms of \(\bar{\sigma}^i_a\). For the first equation let us first define the \(SO(3, C)\) covariant derivative of \(\Psi_{ae}\)

$$D_i \Psi_{ae} = \partial_i \Psi_{ae} + A^b_i (f_{abf} \Psi_{fe} + f_{ebg} \Psi_{ag}), \quad (12)$$

for some \(SO(3, C)\) connection \(A^a_i\). Multiplication of (12) by \(B^i_e\) yields

$$w_e \{ \Psi_{ae} \} = B^i_e D_i \Psi_{ae} = 0, \quad (13)$$

which can be written as

$$w_e \{ \Psi_{ae} \} = D_i (\Psi_{ae} B^i_e) - \Psi_{ae} D_i B^i_e = 0. \quad (14)$$

We will now declare \(B^i_a\), as defined by (9), to be the magnetic field for connection \(A^a_i\) appearing in (12). Then \(B^i_a\) satisfies the Bianchi identity

$$D_i B^i_a = 0 \quad (15)$$

Multiplication of the second equation of (1) by \((\text{det} B)(B^{-1})^d_i \) yields

$$(\text{det} B)(B^{-1})^d_i \epsilon_{dae} \Psi_{ae} = \epsilon_{ijk} B^j_d B^k_e \Psi_{ae}, \quad (16)$$

where we have used the property of determinants for nondegenerate 3 by 3 matrices. Then upon use of (11), equation (16) becomes

$$H_i = \epsilon_{ijk} \bar{\sigma}^j_a B^k_a = 0. \quad (17)$$

Finally, upon use of (11) the third equation of (1) becomes
\[
\Lambda + \text{tr}\Psi^{-1} = \Lambda + B^i_a (\tilde{\sigma}^{-1})^a_i \\
= (\det\tilde{\sigma})^{-1} (\Lambda (\det\tilde{\sigma}) + \frac{1}{2} \epsilon_{ijk} \epsilon^{abc} B^i_a \tilde{\sigma}^j_b \tilde{\sigma}^k_c ).
\]  
(18)

Hence for \((\text{det}\tilde{\sigma}) \neq 0\), equation (18) is equivalent to

\[
H = \epsilon_{ijk} \epsilon^{abc} \tilde{\sigma}^i_a \tilde{\sigma}^j_b \left( B^k_c + \frac{\Lambda}{3} \tilde{\sigma}^k_c \right) = 0.
\]
(19)

Equations (15), (17) and (19) are none other than the Gauss’ law, diffeomorphism and the Hamiltonian constraints in Ashtekar variables [3],[4],[5]. Note that the analogous constraints in (1) contain only \(SO(3,C)\) indices. It will be more convenient to solve the constraints on the instanton representation phase space \(\Omega_{Inst}\), since the physical degrees of freedom are explicit.

In the instanton representation of Plebanski gravity the spacetime metric \(g_{\mu\nu}\) is a derived quantity. The prescription for constructing \(g_{\mu\nu}\) solving the Einstein equations starts first by constructing the spatial 3-metric \(h_{ij}\) in terms of the instanton representation variables. This is given by

\[
h_{ij} = (\det\Psi)(\Psi^{-1})^{ae}(B^{-1})^a_i (B^{-1})^e_j (\det B),
\]
(20)

which implies the relation \(hh^{ij} = \tilde{\sigma}^j_a \tilde{\sigma}^i_a\) upon use of the relation \(\tilde{\sigma}^i_a = \Psi^{ae} B^i_e\). \(^2\)

Then one must supplement (20) with the choice of a lapse-shift combination \(N^\mu = (N, N^i)\), which in loose terms corresponds to a choice of gauge. Then the line element is given by

\[
ds^2 = -N^2 dt^2 + h_{ij} \omega^i \otimes \omega^j,
\]
(21)

where we have defined the one form \(\omega^i = dx^i + N^i dt\).

If one adopts the view that the CDJ matrix is fundamental, then the information about coordinate systems is encoded in \(B^i_a\). Equation (20) depicts a clean separation of these degrees of freedom. We would like to go one step further to the level of the eigenvalues of \(\Psi^{ae}\), which determine its algebraic classification. Since we have assumed nondegenerate \(\Psi^{ae}\), then we are restricted to Petrov Types I, D and O where \(\Psi^{ae}\) have respectively three, two and one distinct eigenvalues.

\(^2\)This is the relation of the induced contravariant 3-metric to the Ashtekar densitized triad \(\tilde{\sigma}^i_a\).
4 Spacetimes of Petrov Type D

For the purposes of this paper we will demonstrate solution of the initial value constraints for spacetimes of Petrov Type D, where $\Psi_{ae}$ has two equal eigenvalues. Denote these eigenvalues by $\lambda_f = (\varphi_1, \varphi, \varphi)$ and all permutations thereof. Then the Hamiltonian constraint reduces to

$$\frac{1}{\varphi_1} + \frac{2}{\varphi} + \Lambda = 0. \tag{22}$$

Equation (22) yields the following relations which we will use later

$$\varphi_1 = -\left(\frac{\varphi}{\Lambda \varphi + 2}\right); \quad \varphi_1 - \varphi = -\varphi \left(\frac{\Lambda \varphi + 3}{\Lambda \varphi + 2}\right). \tag{23}$$

The self-dual Weyl curvature for a spacetime of Type D is of the form $\psi_{ae} = \Psi(-2, 1, 1)$ for an arbitrary function $\Psi$, which is traceless. The diagonalized form of a Type D CDJ matrix is given by adding to this a cosmological contribution, which in matrix form is given by

$$\Psi_{ae}^{-1} = -\frac{\Lambda}{3} \delta_{ae} + \psi_{ae} = \begin{pmatrix} \frac{-\Lambda}{3} - 2\Psi & 0 & 0 \\ 0 & -\frac{\Lambda}{3} + \Psi & 0 \\ 0 & 0 & -\frac{\Lambda}{3} + \Psi \end{pmatrix}.$$

One can then read off the value of $\varphi$ in (22) as

$$\varphi = \frac{1}{-\frac{\Lambda}{3} + \Psi}; \quad \Lambda \varphi + 2 = \left(\frac{\Lambda}{3} + 2\Psi\right); \quad \Lambda \varphi + 3 = \frac{3\Psi}{-\frac{\Lambda}{3} + \Psi}. \tag{24}$$

From (24) the following quantities $\Phi$ and $\psi$ can be constructed

$$\Phi = \frac{\varphi(\Lambda \varphi + 3)^2}{(\Lambda \varphi + 2)^3} = 9 \left(\frac{1}{2\Psi^{1/3} + \frac{\Lambda}{3}\Psi^{-2/3}}\right)^3;$$

$$\psi = \varphi^2(\Lambda \varphi + 3) = 3 \left(\frac{1}{-\frac{\Lambda}{3} \Psi^{-1/3} + \Psi^{2/3}}\right)^3, \tag{25}$$

which will become useful later. We will now choose a connection, for illustrative purposes, given by

$$A^a_i = \begin{pmatrix} A^1_r & A^1_\theta & A^1_\phi \\ A^2_r & A^2_\theta & A^2_\phi \\ A^3_r & A^3_\theta & A^3_\phi \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cos \theta \\ 0 & 0 & -\frac{\sin \theta}{g} \\ 0 & \frac{1}{g} & 0 \end{pmatrix}.$$
where \( g = g(r) \) is an arbitrary function only of radial distance \( r \) from the origin. By this choice we have also made the choice of a coordinate system \((r, \theta, \phi)\) to whose axes various quantities will be referred. The Ashtekar magnetic field derived from \( A^a_i \) is given by

\[
B^a_i = \varepsilon^{ijk} \partial_j A^a_k + \frac{1}{2} \varepsilon^{ijk} f_{abc} A^b_j A^c_k = \left( \begin{array}{ccc} -\left(1 - \frac{1}{g^2}\right) \sin \theta & 0 & 0 \\ 0 & \sin \theta \frac{d}{dr} g^{-1} & 0 \\ 0 & 0 & \frac{d}{dr} g^{-1} \end{array} \right),
\]

and the magnetic helicity density matrix \( C_{ae} \) is given by

\[
C_{ae} = A^a_i B^i_e = \frac{\partial}{\partial r} \left( \begin{array}{ccc} 0 & 0 & \frac{\cos \theta}{g} \\ 0 & 0 & \frac{\sin \theta}{2} (1 - \frac{1}{g^2}) \\ 0 & \frac{\sin \theta}{2} (1 - \frac{1}{g^2}) & 0 \end{array} \right).
\]

The vector fields \( v_a = B^i_a \partial_i \) can be read off from the magnetic field matrix

\[
v_1 = -\sin \theta \left(1 - \frac{1}{g^2}\right) \frac{\partial}{\partial r}; \quad v_2 = \frac{d}{dr} \left(\frac{1}{g}\right) \sin \theta \frac{\partial}{\partial \theta}; \quad v_3 = \frac{d}{dr} \left(\frac{1}{g}\right) \frac{\partial}{\partial \phi}.
\]

These will constitute the differential operators in the Gauss’ law constraint.

### 5 First permutation of eigenvalues

The simplest nontrivial solution should correspond to configurations where the \( SO(3, C) \) angles \( \vec{\theta} \) are zero. Then all primed quantities (72) are the same as their unprimed counterparts (6). There are three distinct permutations of eigenvalues to consider, and the first permutation is given by \( \vec{\lambda} = (\varphi_1, \varphi, \varphi) \). Then (4) for this case reduces to

\[
\left[ \begin{array}{ccc} v_1 - C_{[23]} & -C_{32} & C_{23} \\ C_{31} & v_2 - C_{[31]} & -C_{13} \\ -C_{21} & C_{12} & v_3 - C_{[12]} \end{array} \right] \left( \begin{array}{c} \varphi_1 \\ \varphi \\ \varphi \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right),
\]

where \( C_{[ae]} = C_{ae} - C_{ea} \). This leads to the following equations

\[
v_1 \{ \varphi_1 \} = C_{[23]} (\varphi_1 - \varphi); \quad v_2 \{ \varphi \} = C_{31} (\varphi - \varphi_1); \quad v_3 \{ \varphi \} = C_{21} (\varphi_1 - \varphi).
\]

Using the results from (23), the first equation of (27) implies that

\[
-v_1 \left( \frac{\varphi}{\Lambda \varphi + 2} \right) = -C_{[23]} \varphi \left( \frac{\Lambda \varphi + 3}{\Lambda \varphi + 2} \right).
\]

(28)
Using (23), equation (28) can be written as

\[ v_1\{-\left(\frac{\varphi}{\Lambda \varphi + 2}\right)\} = -C_{[23]}\varphi \left(\frac{\Lambda \varphi + 3}{\Lambda \varphi + 2}\right); \]

\[ \rightarrow \frac{1}{\varphi} \left(\frac{\Lambda \varphi + 2}{\Lambda \varphi + 3}\right) v_1\{\left(\frac{\varphi}{\Lambda \varphi + 2}\right)\} = C_{[23]} \]

\[ = \frac{1}{\varphi} \left(\frac{\Lambda \varphi + 2}{\Lambda \varphi + 3}\right) \left[\frac{(\Lambda \varphi + 2) v_1\{\varphi\} - \varphi v_1\{\Lambda \varphi + 2\}}{(\Lambda \varphi + 2)^2}\right] \] \hspace{1cm} (29)

where we have used the Liebniz rule. Equation (29) then simplifies to

\[ \frac{2v_1\{\varphi\}}{\varphi(\Lambda \varphi + 2)(\Lambda \varphi + 3)} = \frac{1}{3} v_1\{\Phi\} = C_{[23]}, \] \hspace{1cm} (30)

which gives

\[ v_1\{\ln \Phi\} = 3C_{[23]} \] \hspace{1cm} (31)

with \( \Phi \) given by (25).

The second equation of (27) implies that

\[ v_2\{\varphi\} = C_{31} \varphi \left(\frac{\Lambda \varphi + 3}{\Lambda \varphi + 2}\right). \] \hspace{1cm} (32)

Using (23), equation (32) simplifies to

\[ \frac{1}{\varphi} \left(\frac{\Lambda \varphi + 2}{\Lambda \varphi + 3}\right) v_2\{\varphi\} = \frac{1}{3} v_2\{\varphi^2(\Lambda \varphi + 3)\} = C_{31}, \] \hspace{1cm} (33)

which gives

\[ v_2\{\ln \psi\} = 3C_{31}. \] \hspace{1cm} (34)

The manipulations of the third equation of (27) are directly analogous to (33) and (34), which implies that

\[ v_3\{\varphi\} = -C_{21} \varphi \left(\frac{\Lambda \varphi + 3}{\Lambda \varphi + 2}\right) \rightarrow v_3\{\ln \psi\} = -3C_{21}. \] \hspace{1cm} (35)

Hence the three equations can be written as

\[ v_1\{\ln \Phi\} = 3C_{[23]}; \hspace{0.5cm} v_2\{\ln \psi\} = 3C_{31}; \hspace{0.5cm} v_3\{\ln \psi\} = -3C_{21}, \] \hspace{1cm} (36)
where $\Phi$ and $\psi$ are given by (25). When the vector fields $\mathbf{v}_a$ are invertible, then (36) implies the relations

$$
\Phi = e^{3\mathbf{v}_1^{-1}\{C_{23}\}}; \quad \psi = e^{3\mathbf{v}_2^{-1}\{C_{31}\}} = e^{-3\mathbf{v}_3^{-1}\{C_{21}\}}.
$$

(37)

Hence, the imposition of vanishing $SO(3,C)$ angles in conjunction with the equality of two eigenvalues from (37) results in severe constraints on the allowable configurations $A_a^\alpha$.

The first equation of (36) for the chosen configuration reduces to

$$
\mathbf{v}_1\{\ln \Phi\} = 3C_{23} \longrightarrow -\sin \theta \left(1 - \frac{1}{g^2}\right) \frac{\partial}{\partial r} \ln \Phi = 3\sin \theta \frac{\partial}{\partial r} \left(1 - \frac{1}{g^2}\right),
$$

(38)

which integrates to

$$
\Phi = c(\theta, \phi)\left(1 - \frac{1}{g^2}\right)^{-3}
$$

(39)

for some arbitrary function of two variables $c$. The second equation of (36) is given by

$$
\mathbf{v}_2\{\ln \psi\} = 3C_{31} \longrightarrow \left(\frac{d}{dr} g^{-1}\right) \sin \theta \frac{\partial \ln \psi}{\partial \theta} = 0,
$$

(40)

which implies that $\psi = \psi(r, \phi)$. The third equation of (36) is given by

$$
\mathbf{v}_3\{\ln \psi\} = -3C_{21} \longrightarrow \left(\frac{d}{dr} g^{-1}\right) \frac{\partial \ln \psi}{\partial \phi} = 0.
$$

(41)

In conjunction with the results from (40), one has that $\psi = \psi(r)$ must be a function only of $r$. Note that this is consistent with $\Phi$ only being a function of $r$ as in (39), which requires that $c(\theta, \phi) = c$ be a numerical constant.

Continuing from (39), we have

$$
\left(\frac{1}{2\Psi^{1/3} + \frac{1}{3}\Psi^{-2/3}}\right)^3 = c\left(1 - \frac{1}{g^2}\right)^{-3},
$$

(42)

which upon redefining the parameter $c$ yields the solution

$$
g^2 = \left(1 - \frac{2}{c} \Psi^{1/3} - \frac{\Lambda}{3c} \Psi^{-2/3}\right)^{-1}.
$$

(43)

So knowing $\Psi$, which comes directly from the CDJ matrix for Type D and the fact that $\vec{\theta}$ have been chosen to be zero, enables us to determine the connection $A_a^\alpha$ explicitly in this case.
Knowing the CDJ matrix eigenvalues and the magnetic field, we can now proceed to compute the 3-metric $h_{ij}$ for the chosen configuration. The CDJ matrix constituent of (20) is given by

$$\text{(det}\Psi)(\Psi^{-1}\Psi^{-1})^{ae} = -\left( \begin{array}{ccc} \frac{\Lambda}{3} + 2\Psi & 0 & 0 \\ 0 & \frac{1}{\frac{\Lambda}{3} + 2\Psi} & 0 \\ 0 & 0 & \frac{1}{\frac{\Lambda}{3} + 2\Psi} \end{array} \right),$$

and the magnetic field constituent is given by\(^3\)

$$\left( B^{-1}\right)^{a}_i\left( B^{-1}\right)^e_j (\text{det}B) \rightarrow -\left( \begin{array}{ccc} \frac{(\Psi^{-1})^2}{1 - \Psi^2} & 0 & 0 \\ 0 & 1 - \frac{1}{\Psi} & 0 \\ 0 & 0 & (1 - \frac{1}{\Psi^2}) \sin^2 \theta \end{array} \right).$$

We would rather like to express the metric directly in terms of $\Psi$, which is the fundamental degree of freedom from the given Petrov Type. Hence from (42) we have

$$1 - \frac{1}{g^2} = \frac{1}{c}\Psi^{-2/3}\left(2\Psi + \frac{\Lambda}{3}\right), \quad (44)$$

and

$$\frac{d}{dr}g^{-1} = -\left(\frac{1}{3c}\right)\Psi^{-5/3}\left(1 - \frac{2}{c}\Psi^{1/3} - \frac{\Lambda}{3c}\Psi^{-2/3}\right)^{-1/2}\left(\frac{\Lambda}{3} + \Psi\right)\Psi', \quad (45)$$

where $\Psi' = \frac{d\Psi}{dr}$. Then in terms of $\Psi$ we have

$$\left( B^{-1}\right)^{a}_i\left( B^{-1}\right)^e_j (\text{det}B) \rightarrow \frac{1}{c}\left( \begin{array}{ccc} \frac{1}{9} \frac{\Psi^{-8/3}(\Psi')^2}{1 - \Psi^{1/4} - \frac{\Lambda}{3}\Psi^{-2/3}} & \frac{\Lambda + \Psi}{2\Psi + \frac{\Lambda}{3}} & 0 \\ 0 & \Psi^{-2/3}(2\Psi + \frac{\Lambda}{3}) & 0 \\ 0 & 0 & \Psi^{-2/3}(2\Psi + \frac{\Lambda}{3}) \sin^2 \theta \end{array} \right).$$

The fact that $B^a_i$ is fixed by $\Psi$ is a consequence of the imposition of vanishing $SO(3,C)$ angles. This is a strong constraint, without which one would expect $B^a_i$ and $\lambda_f$ to be independent quantities. Multiplying these matrices together, we obtain the 3-metric

\(^3\)We have, in an abuse of notation, anticipated the result of multiplying all of the matrices needed for (20). This is allowed since the matrices are all diagonal.
This is a general solution for the first permutation sequence of the eigenvalues. To doublecheck this result, let us see whether this can lead to any known solutions. First let us eliminate the constant of integration $c$ via the rescaling $\Psi \rightarrow \Psi c^{-3/2}$. For the $A^i_a$ we have chosen, using the zero shift $N^i = 0$ gauge, this yields a spacetime metric of

$$ds^2 = -N^2 dt^2 + \frac{1}{9} \left( \frac{\Psi^{-8/3}(\Psi')^2}{1 - 2\Psi^{1/3}c^{-2/3} - \frac{1}{3}\Psi^{-2/3}} \right) dr^2 + \Psi^{-2/3}(d\theta^2 + \sin^2\theta d\phi^2),$$

(46)

Already, it can be seen in this special gauge that (46) leads to some known GR solutions. (i) Taking $\Psi = \frac{1}{r^3}$, $c = (GM)^{-2/3}$, $N^i = 0$ and $N^2 = 1 - \frac{2GM}{r} - \frac{\Lambda}{3} r^2$, we obtain

$$g_{\mu\nu} = \begin{pmatrix}
1 - \frac{2GM}{r} - \frac{\Lambda}{3} r^2 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & r^2 & 0 \\
0 & 0 & 0 & r^2 \sin^2\theta
\end{pmatrix},$$

which is the solution for a Euclidean DeSitter blackhole. This can be changed to Lorentzian signature by Wick rotation $N \rightarrow iN$. Choosing $\Lambda = 0$ gives the Schwarzchild blackhole and choosing $G = 0$ gives the DeSitter metric.\footnote{Setting $M = 0$ corresponds to a transition from Type D to Type O spacetime, where $\Psi = 0$.}

7 Remaining permutations of eigenvalues

Using a specific permutation of the eigenvalues of $\Psi_{ae}$ we have obtained some known GR solutions. Let us now examine whether any of the other permutations yield new solutions. For the second permutation of eigenvalues we have $\bar{\varphi} = (\varphi, \varphi_1, \varphi)$, which leads to the Gauss’ law constraint

$$\begin{pmatrix}
v_1 - C_{23} & -C_{32} & C_{23} \\
C_{31} & v_2 - C_{31} & -C_{13} \\
-C_{21} & C_{12} & v_3 - C_{12}
\end{pmatrix} \begin{pmatrix}
\varphi \\
\varphi_1 \\
\varphi
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix},$$

which yields the equations
\( \nu_2\{\varphi_1\} = C_{31}(\varphi_1 - \varphi); \quad \nu_3\{\varphi\} = C_{12}(\varphi - \varphi_1); \quad \nu_1\{\varphi\} = C_{32}(\varphi_1 - \varphi). \) (47)

The first equation of (47) yields

\[
\nu_2\{\ln \Phi\} = 3C_{31} \rightarrow \left( \frac{d}{dr} g^{-1} \right) \sin \theta \frac{\partial \ln \Phi}{\partial \theta} = 3 \frac{\partial}{\partial r} \left( -\frac{\cos \theta}{g} \right)
\]

which integrates to

\[
\Phi = c(r, \phi) \sin^{-3} \theta
\]

for some arbitrary function \( c \). The second equation of (47) yields

\[
\nu_3\{\ln \psi\} = 3C_{12} = 0 \rightarrow \left( \frac{d}{dr} g^{-1} \right) \frac{\partial \ln \psi}{\partial \phi} = 0,
\]

which implies that \( \psi = \psi(r, \theta) \). The third equation of (47) yields

\[
\nu_1\{\ln \psi\} = -3C_{32} \rightarrow -\sin \theta \left( 1 - \frac{1}{g^2} \right) \frac{\partial \ln \psi}{\partial r} = \frac{3}{2} \sin \theta \frac{\partial}{\partial r} \left( 1 - \frac{1}{g^2} \right),
\]

which integrates to

\[
\psi = k(\theta, \phi) \left( 1 - \frac{1}{g^2} \right)^{-3/2}.
\]

For consistency with the results of (49) and (50), we must have that

\[
\psi = 3 \left( -\frac{\Lambda}{3} \Psi^{-1/3} + \Psi^{2/3} \right)^{-3} = k(\theta) \left( 1 - \frac{1}{g^2} \right)^{-3/2};
\]

\[
\Phi = 9 \left( \frac{\Lambda}{3} \Psi^{-2/3} + 2 \Psi^{1/3} \right)^{-3} = c(r) \sin^{-3} \theta.
\]

Equations (53) yield

\[
-\frac{\Lambda}{3} \Psi^{-1/3} + \Psi^{2/3} = k(\theta) \sqrt{1 - \frac{1}{g^2}}; \quad \frac{\Lambda}{3} \Psi^{-2/3} + 2 \Psi^{1/3} = c(r) \sin \theta.
\]

But (54) itself seems a rather stringent condition on \( \Psi \), so we will be content for the purposes of this paper to examine the \( \Lambda = 0 \) case for simplicity. Setting \( \Lambda = 0 \) in (54) implies the following consistency condition that
\((c(r)\sin\theta)^2 = k(\theta)\sqrt{1 - \frac{1}{g^2}};\)

\[
\rightarrow c(r) = \left(1 - \frac{1}{g^2}\right)^{1/4}; \quad k(\theta) = \sin\theta. \quad (55)
\]

From this we obtain

\[
\Psi = \Psi(r, \theta) = \left(1 - \frac{1}{g^2}\right)^{3/4} \sin^3 \theta. \quad (56)
\]

For illustrative purposes we will now compute the 3-metric that this implies. For \(\Lambda = 0\) we have that

\[
(\det \Psi)(\Psi^{-1} \Psi^{-1})^{ae} = -\frac{1}{\Psi}\left(\begin{array}{ccc}
2 & 0 & 0 \\
0 & \frac{1}{g} & 0 \\
0 & 0 & \frac{1}{2}
\end{array}\right).
\]

Using the magnetic field for the configuration chosen, which is the same as for the previous permutation, then (20) yields a 3-metric

\[
h_{ij} = \frac{1}{2}\left(1 - \frac{1}{g^2}\right)^{-3/4} \sin^{-3} \theta \left(\begin{array}{ccc}
\frac{4(\frac{d}{dr}g^{-1})^2}{1-g^2} & 0 & 0 \\
0 & \frac{1}{2}\left(1 - \frac{1}{g^2}\right) & 0 \\
0 & 0 & \frac{1}{2}(1 - \frac{1}{g^2})\sin^2 \theta
\end{array}\right).
\]

Let us choose a specific function for \(g\), for example

\[
g = (1 - r^2)^{-1/2} \quad \frac{dg^{-1}}{dr} = -r(1 - r^2)^{-1/2}, \quad (57)
\]

where we have taken \(r\) to be dimensionless. Then the 3-metric is given by

\[
h_{ij} = \frac{1}{2}r^{-3/2}\sin^{-3} \theta \left(\begin{array}{ccc}
\frac{4}{1-r^2} & 0 & 0 \\
0 & r^2 & 0 \\
0 & 0 & r^2\sin^2 \theta
\end{array}\right),
\]

which corresponds to a new solution. The 3-metric blows up for \(\theta = 0\) and \(\theta = 0\), as well as at \(r = 0\) and \(r = 1\).
7.1 Third permutation of eigenvalues

For the third permutation of eigenvalues we have \( \vec{\varphi} = (\varphi, \varphi, \varphi_1) \), which leads to the Gauss’ law constraint

\[
\begin{pmatrix}
\begin{pmatrix}
v_1 - C_{[23]} & -C_{32} & C_{23} \\
C_{31} & v_2 - C_{[31]} & -C_{13} \\
-C_{21} & C_{12} & v_3 - C_{[12]}
\end{pmatrix}
\end{pmatrix}
\begin{pmatrix}
\varphi \\
\varphi \\
\varphi_1
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\]

This leads to the equations

\[
v_3\{\varphi_1\} = C_{[12]}(\varphi_1 - \varphi); \quad v_1\{\varphi\} = C_{23}(\varphi - \varphi_1); \quad v_2\{\varphi\} = C_{13}(\varphi_1 - \varphi).
\]

The first equation from (58) is given by

\[
v_3\{\ln \Phi\} = 3C_{[12]} \rightarrow \left( \frac{d}{dr} g^{-1} \right) \frac{\partial \ln \Phi}{\partial \varphi} = 0,
\]

which implies that \( \Phi = \Phi(r, \theta) \). The second equation of (58) is given by

\[
v_1\{\ln \psi\} = 3C_{23} \rightarrow -\sin \theta \left( 1 - \frac{1}{g^2} \right) \frac{\partial \ln \psi}{\partial r} = \frac{3}{2} \sin \theta \frac{\partial}{\partial r} \left( 1 - \frac{1}{g^2} \right),
\]

which integrates to

\[
\psi = k(\theta) \left( 1 - \frac{1}{g^2} \right)^{-3/2}.
\]

This is consistent with the results from (59), since there must be no \( \phi \) dependence. The third equation of (58) is given by

\[
v_2\{\ln \psi\} = -3C_{13} \rightarrow \frac{\partial}{\partial r} \left( \frac{\sin \theta}{g} \right) \frac{\partial \ln \psi}{\partial \theta} = -3 \frac{\partial}{\partial r} \left( \frac{\cos \theta}{g} \right),
\]

which integrates to

\[
\psi = c(r) \sin^{-3} \theta.
\]

From (59) \( \Phi = \Phi(r, \theta) \) can be an arbitrary function of \( r \) and \( \theta \), and hence we are free to determine this dependence entirely from \( \psi \). Consistency of (61) with (63) implies that
\[
\psi = -\frac{\Lambda}{3} \psi^{-1/3} + \psi^{2/3} = \sin^{-3}\theta \left( 1 - \frac{1}{g^2} \right)^{-3/2}.
\] (64)

Unlike the previous permutation of eigenvalues, there is no restriction in (64) in choosing a nonzero \( \Lambda \), since the functional dependence of \( \Phi \) is not constrained. Hence we are free to solve the cubic polynomial (64) for \( \Psi \), which determines the function \( \Phi \). Making the definition \( z = \Psi^{1/3} \), equation (64) reduces to the cubic equation

\[
z^3 - \psi z = \frac{\Lambda}{3},
\] (65)

with solution

\[
\Psi = \left[ 2\sqrt{-\psi/3}\sinh \left( \frac{1}{3} \sinh^{-1} \left( \frac{\sqrt{3}\Lambda}{2} (-\psi)^{-3/2} \right) \right) \right]^3.
\] (66)

For the purposes of constructing a 3-metric we will be content with the \( \Lambda = 0 \) case, which yields

\[
\Psi = \Psi(r, \theta) = \frac{\sin^{-9/2}\theta}{1 - \frac{1}{g^2}}.
\] (67)

Using the previous configuration, equation (67) yields a 3-metric

\[
h_{ij} = \frac{1}{2} \left( 1 - \frac{1}{g^2} \right) \sin^{9/2}\theta \begin{pmatrix}
4 \left( \frac{d}{g^2} \right)^{1} & 0 & 0 \\
0 & \frac{1}{2} \left( 1 - \frac{1}{g^2} \right) & 0 \\
0 & 0 & \frac{1}{2} (1 - \frac{1}{g^2}) \sin^2\theta
\end{pmatrix}.
\]

Choosing the same function as before \( 1 - \frac{1}{g^2} = r^2 \), then we obtain a 3-metric given by

\[
h_{ij} = \frac{1}{2} r^2 \sin^{9/3}\theta \begin{pmatrix}
\frac{4}{1-r^2} & 0 & 0 \\
0 & r^2 & 0 \\
0 & 0 & r^2 \sin^2\theta
\end{pmatrix},
\]

which exhibits behavior complementary to the solution for the previous permutation.
8 Conclusion and discussion

In this paper we have constructed some solutions to the Einstein equations, using the physical degrees of freedom of the instanton representation provided by the initial value constraints. We have applied this scheme to spacetimes of Petrov Type D, producing some known solutions. This construction is possible due to the clear separation of physical from unphysical degrees of freedom made possible in this formulation of gravity. In the algebraically general case one should expect the connection $A^a_i$ and the eigenvalues of $\Psi_{ae}$ to be independent quantities, which form the inputs needed to construct the $SO(3, C)$ frame solving the Gauss' law constraint.\(^5\) In the present paper we imposed the condition that the $SO(3, C)$ angles vanish, which implies that the intrinsic $SO(3, C)$ frame is the same as the frame solving the Gauss' law constraint. The result was that the eigenvalues and the connection became interrelated, a feature which is apparent in the solutions constructed. We have constructed spherically symmetric blackhole and other solutions via this method. For future directions are at least two generalizations of this procedure to be investigated: (i) One generalization is to attempt to find algebraically general solutions with vanishing $SO(3, C)$ angles, and then to lift this restriction by substituting different connections $A^a_i$. (ii) Another generalization is to construct type D solutions using different connections to obtain corresponding non-vanishing $SO(3, C)$ angles. (iii) It is also of interest to classify the different solutions constructable into equivalence classes, to ascertain which sectors of GR are encapsulated and which sectors are not via this approach.\(^6\)

As regards the physical interpretation of the instanton representation phase space variables, a choice of connection appears to be tantamount to the choice of a coordinate system, while the choice of eigenvalues is independent of coordinates and also independent of the $SO(3, C)$ frame. With this interpretation, the range of metric solutions constructible for different connections is simply a given Petrov type (which is the fundamental entity) expressed in different coordinate systems.

---

\(^5\)There also seems to be no restriction to doing this for Petrov Type D, by simply choosing a connection different from the one used in this paper to construct solutions. This will be a future direction of research.

\(^6\)It is already known that this method applies only to spacetimes of Petrov Type I, D and O, where the CDJ matrix has three linearly independent eigenvectors.
9 Appendix A: Polar representation of the Gauss’ law constraint

Let us now compute (4), using (2). The part of (4) involving vector fields is given by

\[ \mathbf{v}_e \{ (e^{\theta T})_{af} + (e^{\theta T})_{ef} \} = (e^{\theta T})_{af} (e^{\theta T})_{ef} \mathbf{v}_e \{ \lambda_f \} \]

\[ + \mathbf{v}_e \{ \theta^A \} \left[ (e^{\theta T})_{ag} (T_A) g_f \lambda_f (e^{-\theta T})_{fe} - (e^{\theta T})_{ag} \lambda_f (T_A) g_f (e^{-\theta T})_{fe} \right] \]

\[ = (e^{\theta T})_{af} (e^{\theta T})_{ef} \mathbf{v}_e \{ \lambda_f \} + (e^{\theta T})_{ag} (e^{\theta T})_{ef} (T_A) g_f (\lambda_f - \lambda_g) \mathbf{v}_e \{ \theta^A \}. \] (68)

In (68) we have used the Liebniz rule in conjunction with matrix multiplication. Combining the results of (68) and (5) in (4), we have

\[ \mathbf{w}_e \{ \Psi_{ae} \} = (e^{\theta T})_{af} (e^{-\theta T})_{fe} \mathbf{v}_e \{ \lambda_f \} \]

\[ + (e^{\theta T})_{ag} (T_A) g_f (\lambda_f - \lambda_g) (e^{-\theta T})_{fe} \mathbf{v}_e \{ \theta^A \} + C^g_a (e^{\theta T})_{fh} (e^{\theta T})_{gh} \lambda_h = 0. \] (69)

Multiplying (69) by \((e^{-\theta T})_{da}\) and using the relation

\[ (e^{-\theta T})_{da} C^f_a (e^{\theta T})_{fh} (e^{\theta T})_{gh} = (e^{-\theta T})_{da} (f_{abf} \delta_{ge} + f_{ebg} \theta_{af}) (e^{\theta T})_{fh} (e^{\theta T})_{gh} \]

\[ = (e^{-\theta T})_{da} (e^{\theta T})_{hf} \mathbf{f}_{abf} (e^{-\theta T})_{gh} C_{bg} + f_{ebg} (e^{-\theta T})_{dh} C_{be}, \] (70)

we will now explicitly compute the required terms. The first term on the right hand side of (70) can be rewritten as

\[ (e^{-\theta T})_{da} (e^{\theta T})_{hf} \mathbf{f}_{abf} (e^{-\theta T})_{gh} C_{bg} \]

\[ = (e^{-\theta T})_{da} (e^{\theta T})_{be} \mathbf{c}_{ec} (e^{\theta T})_{hf} \mathbf{f}_{acf} (e^{-\theta T})_{gh} C_{bg} \]

\[ = (e^{-\theta T})_{da} \mathbf{c}_{ec} (e^{\theta T})_{hf} \mathbf{f}_{acf} (e^{-\theta T})_{eh} C_{bg} \]

\[ = (e^{-\theta T})_{da} (e^{\theta T})_{eh} C_{bg} \] (71)

where we have used \(e^{\theta T} e^{-\theta T} = 1\), and the second term is given by \(- (e^{-\theta T})_{dh} C_{g}\)

where \(C_g = f_{gbe} C_{be}\). Making the definitions

\[ C^e_{eh} = (e^{-\theta T})_{eb} (e^{-\theta T})_{gh} C_{bg}; \quad \mathbf{v}^e_{a} = (e^{-\theta T})_{da} \mathbf{v}_e, \]

(72)

which identifies the primed quantities as the \(SO(3,C)\) rotation of their unprimed counterparts in (6) on all free indices, and using the special orthogonal property

\[ (e^{-\theta T})_{da} (e^{-\theta T})_{ec} (e^{-\theta T})_{hf} \mathbf{f}_{acf} = (\det e^{-\theta T}) f_{deh} = f_{deh}, \] (73)
then the contraction of (70) with $\lambda_h$ is given by

$$f_{deh}C_{eh}^\prime \lambda_h - \delta_{dh} C_d^\prime \lambda_h.$$  \hspace{1cm} (74)

Then (4) can be written as

$$v_d^\prime \{\lambda_d\} - C_d^\prime \lambda_d + f_{deh} C_{eh}^\prime \lambda_h + (T_A)d_j (\lambda_f - \lambda_d) v_f^\prime \{\theta^A\} = 0,$$  \hspace{1cm} (75)

which in matrix form is given by

$$\begin{pmatrix}
0 & (\lambda_3 - \lambda_1) v_3^\prime & (\lambda_1 - \lambda_2) v_2^\prime \\
(\lambda_2 - \lambda_3) v_3^\prime & 0 & (\lambda_1 - \lambda_2) v_1^\prime \\
(\lambda_2 - \lambda_3) v_2^\prime & (\lambda_3 - \lambda_1) v_1^\prime & 0
\end{pmatrix} \begin{pmatrix}
\theta^1 \\
\theta^2 \\
\theta^3
\end{pmatrix}$$

$$= \begin{pmatrix}
v_1^\prime - C_{[23]}^\prime & -C_{32}^\prime & C_{23}^\prime \\
C_{31}^\prime & v_2^\prime - C_{[31]}^\prime & -C_{13}^\prime \\
-C_{21}^\prime & C_{12}^\prime & v_3^\prime - C_{[12]}^\prime
\end{pmatrix} \begin{pmatrix}
\lambda_1 \\
\lambda_2 \\
\lambda_3
\end{pmatrix}$$

Our goal, as a first check on our results, will be to obtain the spherically symmetric vacuum blackhole solutions.

**References**

[1] Eyo Eyo Ita III ‘Instanton representation of Plebanski gravity: A brief summary of the classical theory arXiv:gr-qc/0911.0604

[2] Riccardo Capovilla and Ted Jacobson ‘General Relativity without the Metric’ Phys. Rev. Lett. 20 (1989) 2325-2328

[3] Ahbay Ashtekar. ‘New perspectives in canonical gravity’, (Bibliopolis, Napoli, 1988).

[4] Ahbay Ashtekar ‘New Hamiltonian formulation of general relativity’ Phys. Rev. D36(1987)1587

[5] Ahbay Ashtekar ‘New variables for classical and quantum gravity’ Phys. Rev. Lett. Volume 57, number 18 (1986)