FACTORIZATIONS OF POLYNOMIALS OVER NONCOMMUTATIVE ALGEBRAS AND SUFFICIENT SETS OF EDGES IN DIRECTED GRAPHS

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To the memory of Felix Alexandrovich Berezin

Abstract. To directed graphs with unique sink and source we associate a noncommutative associative algebra and a polynomial over this algebra. Edges of the graph correspond to pseudo-roots of the polynomial. We give a sufficient condition when coefficients of the polynomial can be rationally expressed via elements of a given set of pseudo-roots (edges). Our results are based on a new theorem for directed graphs also proved in this paper.

0. Introduction

Let $R$ be an associative ring with unit and $P(t) = a_0 t^n + a_1 t^{n-1} + \cdots + a_n$ be polynomial in $R[t]$. Here $t$ is an independent central variable. Recall [GGRSW] that an element $x \in R$ is a pseudo-root of $P(t)$ if there exist polynomials $Q_1(t), Q_2(t) \in R[t]$ such that

$$P(t) = Q_1(t)(t - x)Q_2(t).$$

The element $x$ is a right root of $P(t)$ if $Q_2(t) = 1$ and is a left root of $P(t)$ if $Q_1(t) = 1$. It is easy to check that $x$ is a right root of $P(t)$ if and only if $a_0 x^n + a_1 x^{n-1} + \cdots + a_n = 0$. Similarly, $x$ is a left root if and only if $x^n a_0 + x^{n-1} a_1 + \cdots + a_n = 0$.

For noncommutative $R$, a theory of polynomials over $R$ should be based not only on properties of right (left) roots but on pseudo-roots as well.

Suppose now that $P(t)$ is a monic polynomial (i. e., $a_0 = 1$). We say that a subset $Y$ of pseudo-roots of $P(t)$ is a defining set if $P(t)$ can be factored as $P(t) = (t - y_n)(t - y_{n-1})\cdots(t - y_1)$, where $y_k \in Y$ for $k = 1, 2, \ldots, n$.

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When $R$ is a commutative integral domain, the set of pseudo-roots of $P(t)$ coincides with the set of roots $X$. In this case if $\text{card}(X) = n$ then $X$ is a defining set and the coefficients of $P(t)$ can be written as polynomial expressions in $x_1, x_2, \ldots, x_n \in X$ (the Viète theorem).

When $R$ is not commutative one faces new phenomena. To describe these we introduce the following notation. For a subset $Y \subseteq R$, denote by $R(Y)$ the subring generated by $Y$ and by $\tilde{R}(Y)$ the subring of all $r \in R$ such that $r$ can be written as a rational expression in $y \in Y$. Then

1. The coefficients of $P(t)$ can be written only as rational expressions in left or right roots of $P(t)$ (see [GR1] and other papers [GR2, GGRW]). More precisely, $a_1, a_2, \ldots, a_n$ can be written as polynomial expressions in $z \in Z$ where $Z$ is a defining set of pseudo-roots of $P(t)$, but elements $z \in Z$ can only be expressed rationally in terms of left or right roots of $P(t)$;
2. There might be a large number of pseudo-roots: if a polynomial $P(t)$ of degree $n$ has $n$ right roots in a generic position (see section 1.2) then $P(t)$ has at least $n^{2n-1}$ pseudo-roots;
3. Not any subset $Y$ of pseudo-roots of $P(t)$ of cardinality greater or equal $n$ is a defining set. Moreover, the subring $\tilde{R}(Y)$ (and, therefore, $R(Y)$) may not even contain a defining set.

In this paper we study the following problem: describe subsets $Y$ of pseudo-roots of a monic polynomial $P(t)$ such that the subring $\tilde{R}(Y)$ contains a defining subset of pseudo-roots of $P(t)$. In fact, we replace $\tilde{R}(Y)$ by a smaller subring containing $R(Y)$. However, item (1) above indicates that a consideration of subrings $R(Y)$ for this problem is too restrictive.

Surprisingly enough, for a large class of noncommutative algebras the answer may be given in terms of properties of directed graphs.

The content of the paper is the following. In Section 1 we formulate our results for the universal algebra of pseudo-roots of polynomials over noncommutative algebras. In Section 2 we introduce a class of noncommutative algebras associated with directed graphs and show their relations with factorizations of noncommutative polynomials. In Section 3 we introduce sufficient sets of edges in directed graphs, study properties of the sufficient sets of edges, and apply those properties for a description of sufficient sets of pseudo-roots of noncommutative polynomials. We believe that the results obtained in this section are of interest for a “pure theory” of directed graphs.

1. Main example: Universal algebra of pseudo-roots

1.1. Algebra $Q_n$. The universal algebra of pseudo-roots $Q_n$ over a field $k$ was introduced in [GRW] and studied in [GGR, GGRSW, SW, P]. Algebra $Q_n$ was used to construct a natural family of defining sets of pseudo-roots for noncommutative
Recall that generators of $Q_n$ are elements $x_{A,i}$ where $A \subseteq \{1, 2, \ldots, n\}$, $i = 1, 2, \ldots, n$ and $i \notin A$. The defining relations in $Q_n$ are as follows:

\begin{align}
(1.1) \quad x_{A \cup \{i\}, j} + x_{A,i} &= x_{A \cup \{j\}, i} + x_{A,j}, \\
(1.2) \quad x_{A \cup \{i\}, j} \cdot x_{A,i} &= x_{A \cup \{j\}, i} \cdot x_{A,j}
\end{align}

for each $A \subseteq \{1, 2, \ldots, n\}$, $i, j \notin A$, $i \neq j$. We will call the generators $x_{A,i}$ the pseudo-roots.

There is a canonical polynomial $P(t) \in Q_n[t]$ having $x_{A,i}$’s as its pseudo-roots. Let $\{i_1, i_2, \ldots, i_n\}$ be any ordering of $\{1, 2, \ldots, n\}$. Set $A_1 = \emptyset$, $A_2 = \{i_1\}$, $A_3 = \{i_1, i_2\}$, $\ldots$, $A_n = \{i_1, i_2, \ldots, i_{n-1}\}$. Set

$$P(t) = (t - x_{A_n,i_n})(t - x_{A_{n-1},i_{n-1}})\cdots(t - x_{A_1,i_1}).$$

**Theorem 1.1.1.** ([GRW]) Polynomial $P(t)$ does not depend on an ordering of $\{1, 2, \ldots, n\}$.

**Theorem 1.1.2.** ([GRW]) There exists the canonical epimorphism

$$c : Q_n \to F[t_1, t_2, \ldots, t_n]$$

given by the formulas $c(x_{A,i}) = t_i$.

Here $F$ is the ground field, $t_1, t_2, \ldots, t_n$ are independent commuting variables.

### 1.2. Universality of algebra $Q_n$.

Let $R$ be an associative algebra and $P(t) \in R[t]$ be a monic polynomial of degree $n$. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a set of right roots of $P(t)$. We call the set $X$ **generic** if the Vandermonde matrix

$$V(i_1, i_2, \ldots, i_{k+1}) = \begin{pmatrix}
  x_{i_1}^k & x_{i_2}^k & \cdots & x_{i_{k+1}}^k \\
  x_{i_1} & x_{i_2} & \cdots & x_{i_{k+1}} \\
  1 & 1 & \cdots & 1
\end{pmatrix}$$

is invertible in $R$ for each $k = 1, 2, \ldots, n - 1$ and distinct $1 \leq i_1, i_2, \ldots, i_{k+1} \leq n$.

In this case one can define the Vandermonde quasideterminants $v(i_1, i_2, \ldots, i_{k+1})$ ([GR1, GR2, GGRW] as follows. Set $r$ to be the row matrix $(x_{i_1}^{k-1}, x_{i_2}^{k-2}, \ldots, x_{i_k}^1)^T$ and $c$ to be the column matrix $(x_{i_{k+1}}^{k-1}, x_{i_{k+1}}^{k-2}, \ldots, 1)^T$. Then

$$v(i_1, i_2, \ldots, i_{k+1}) = x_{i_{k+1}}^k - rV(i_1, i_2, \ldots, i_k)^{-1}c.$$
Example. \(v(1,2) = x_2 - x_1\).

The following proposition follows from the general properties of quasideterminants ([GR1, GR2, GRRW]).

**Proposition 1.2.1.** If \(X\) is a generic set then all quasideterminants \(v(i_1, i_2, \ldots, i_{k+1})\) are defined and invertible in \(R\).

**Theorem 1.2.2.** [GRW] If \(X = \{x_1, x_2, \ldots, x_n\}\) is a generic set of right roots of a monic polynomial \(P(t) \in R[t]\) of degree \(n\) then there exists a canonical homomorphism \(\phi : Q_n \to R\) such that \(\phi(x_{\emptyset,k}) = x_k\) for \(k = 1, 2, \ldots, n\).

Moreover,

\[
\phi(x_{\{i_1, i_2, \ldots, i_k\}, i_{k+1}}) = v(i_1, i_2, \ldots, i_{k+1})x_{i_{k+1}}v(i_1, i_2, \ldots, i_{k+1})^{-1}
\]

and the canonical extension of \(\phi\) to the homomorphism of \(Q_n[t]\) to \(R[t]\) maps \(P(t)\) to \(P(t)\).

By abusing notation we denote elements \(\phi(x_{\{i_1, i_2, \ldots, i_k\}, i}) \in R\) via \(x_{i_1, i_2, \ldots, i_k, i}\).

**Corollary 1.2.3.** For a generic set \(X\) and an ordering \(\{i_1, i_2, \ldots, i_n\}\) of \(1, 2, \ldots, n\) set \(y_1 = x_{i_1}\), \(y_2 = x_{i_1, i_2}\), \ldots, \(y_n = x_{i_1, i_2, \ldots, i_{n-1}, i_n}\). Then \(P(t) = (t - y_n)(t - y_{n-1})\ldots(t - y_1)\).

Corollary 1.2.3 implies that \(Y = \{y_1, y_2, \ldots, y_n\}\) is a defining set of pseudo-roots of \(P(t)\).

**Example.** If \(n = 2\) then \(P(t) = (t - x_{i,j})(t - x_i)\), where \(x_{i,j} = (x_j - x_i)x_j(x_j - x_i)^{-1}\), \(i, j = 1, 2\) and \(i \neq j\).

If \(n = 3\) then \(P(t) = (t - x_{i,j,k})(t - x_{i,j})(t - x_k)\), where

\[
x_{i,j,k} = (x_{i,k} - x_{i,j})x_{i,k}(x_{i,k} - x_{i,j})^{-1} = (x_{j,k} - x_{j,i})x_{j,k}(x_{j,k} - x_{j,i})^{-1}.
\]

Here \(i, j, k = 1, 2, 3\) and \(i, j, k\) are all distinct.

Note that elements \(y_k\), \(k = 1, 2, \ldots, n\) are rational expressions in \(x_{i_1}, x_{i_2}, \ldots, x_{i_k}\) and do not depend on ordering of \(i_1, i_2, \ldots, i_{k-1}\).

### 1.3. Algebras \(Q_n\) and graphs.

For each natural \(n\) we define the directed graph \(\Gamma_n\) as follows. The vertices of \(\Gamma_n\) are the subsets of the set \(\{1, 2, \ldots, n\}\) (including the empty subset). The edges of \(\Gamma_n\) are all directed pairs of subsets \((A \cup \{i\}, A)\), where \(A \subseteq \{1, 2, \ldots, n\}\) and \(i \notin A\); we denote such an edge by \((A, i)\). The subset \(A \cup \{i\}\) is the tail of the edge \((A, i)\) and the subset \(A\) is the head of \((A, i)\).

There is a one-to-one correspondence between the generators \(x_{A,i}\) of algebra \(Q_n\) and the edges \((A, i)\) of the graph \(\Gamma_n\). Note that two edges \((A, i)\) and \((B, j)\) have a common head if \(A = B\) and a common tail if \(A \cup \{i\} = B \cup \{j\}\). Also, the generators occurring in relations (1.1) and (1.2) for the algebra \(Q_n\) correspond to the edges of the diamond \(D\) in \(\Gamma_n\) with vertices \(A, A \cup \{i\}, A \cup \{j\}, A \cup \{i, j\}\).
Definition 1.3.1. We sat that the edge \((A, i)\) in \(D\) is obtained by the \(D\)-operation from the ordered pair of edges \((A \cup \{i\}, j)\) and \((A \cup \{j\}, i)\) and the edge \(A \cup \{i\}, j)\) is obtained by the \(U\)-operation from the ordered pair of edges \((A, i), (A, j)\).

Note that \((A, j)\) is obtained by the \(D\)-operation from the pair of edges \((A \cup \{j\}, i)\) and \((A \cup \{i\}, j)\), and the edge \(A \cup \{i\}, j)\) is obtained by the \(U\)-operation from the ordered pair of edges \((A, j), (A, i)\).

Definition 1.3.2. We say that a pseudo-root \(\xi \in Q_n\) is obtained from an ordered pair of pseudo-roots \(x_{A,i}, x_{B,j}\) by the \(u\)-operation if the edges \((A, i), (B, j)\) have a common head and \((x_{A,i} - x_{B,j})x_{A,i} = \xi(x_{A,i} - x_{B,j})\).

A pseudo-root \(\eta \in Q_n\) is obtained from an ordered pair of pseudo-roots \(x_{A,i}, x_{B,j}\) by the \(d\)-operation if the edges \((A, i), (B, j)\) have a common tail and \((x_{A,i} - x_{B,j})\eta = x_{A,i}(x_{A,i} - x_{B,j})\).

A connection between \(d\)- and \(u\)-operations in \(Q_n\) and \(D\)- and \(U\)-operations in \(\Gamma_n\) is given by the following proposition.

Proposition 1.3.3. The element \(x_{A \cup \{i\}, j}\) is obtained by the \(u\)-operation from the pair \(x_{A,i}, x_{A,j}\).

The element \(x_{A,i}\) is obtained by the \(d\)-operation from the pair \(x_{A \cup \{j\}, i}, x_{A \cup \{i\}, j}\).

Proof. Look at the diamond \(D\) and use formulas (1.1) and (1.2).

1.4. Sufficient sets of pseudo-roots in \(Q_n\). Let \(Z\) be a subset of pseudo-roots in \(Q_n\).

Definition 1.4.1. The set of elements in \(Q_n\) that can be obtained from elements of \(Z\) by a successive applications of \(d\)- and \(u\)-operations is called the \(du\)-envelope of \(Z\).

The following proposition is obvious.

Proposition 1.4.2. Let \(f : Q_n \rightarrow D\) be a homomorphism of \(Q_n\) into a division ring \(D\). If \(r \in Q_n\) belongs to a \(du\)-envelope of \(Z\) then \(f(r)\) can be written as a rational expression in elements belonging to \(f(Z)\).

Remark. In fact, \(f(r)\) can be obtained from elements \(f(z), z \in Z\) by operations of addition, subtractions, multiplication, left conjugation \((a, b) \mapsto (a - b)a(a - b)^{-1}\) and right conjugation \((a, b) \mapsto (a - b)^{-1}a(a - b)\).

Recall that a subset \(Y\) of pseudo-roots of \(P(t)\) is a defining set is \(P(t)\) can be factorized as \(P(t) = (t - y_n)(t - y_{n-1}) \ldots (t - y_1)\), where \(y_k \in Y\) for \(k = 1, 2, \ldots, n\).

Definition 1.4.3. A set \(Z \subseteq Q_n\) is called sufficient if the \(du\)-envelope of \(Z\) contains a defining set of pseudo-roots of \(P(t)\).
By Proposition 1.4.2, instead of asking when the rational envelope of a set \( W \) of pseudo-roots of a monic polynomial \( P(t) \in D[t] \) contains a defining set of pseudo-roots of \( P(t) \) we ask a more restrictive question: when the set \( W \) is an image of a sufficient set of pseudo-roots in \( Q_n \)?

Any defining set of elements is a sufficient set. Other examples of sufficient sets in \( Q_n \) are given by the following statement.

**Proposition 1.4.4.** The sets \( \{x_{\emptyset,k}, \, k = 1, 2, \ldots, n\} \) and \( \{x_{12\ldots k\ldots n,k}, \, k = 1, 2, \ldots, n\} \) are sufficient in \( Q_n \) for \( \mathcal{P} \).

**Proof.** Use successively Proposition 1.3.3.

A necessary condition of a subset in \( Q_n \) to be sufficient for \( \mathcal{P}(t) \) is given by the following theorem.

**Theorem 1.4.5.** If \( Z = \{x_{A_1,i_1}, x_{A_2,i_2}, \ldots, x_{A_n,i_n}\} \) is a sufficient subset of \( Q_n \) then all \( i_1, i_2, \ldots, i_n \) are distinct.

**Proof.** Let \( R \) be a commutative algebra without zero divisors, \( P(t) \) be a monic polynomial over \( R \) and \( X = x_1, x_2, \ldots, x_n \) a generic set of right roots of \( P(t) \). According to Theorem 1.2.2 there exists a homomorphism \( \phi : Q_n \to R \) such that \( \phi(x_{A_k,i_k}) = x_{i_k} \) for all \( k = 1, 2, \ldots, n \). Let \( \hat{Z} \) be the du-envelope of \( Z \). Clearly, \( \phi(Z) = \phi(\hat{Z}) \). Therefore, \( \phi(Z) \) is a defining set in \( R \). It implies that \( \phi(Z) = X \) and so all \( i_1, i_2, \ldots, i_n \) are distinct.

A set of edges in a directed graph is connected if it is connected in the associated non-directed graph (see Section 3 below for details).

**Theorem 1.4.6.** Let \( Z = \{x_{A_1,i_1}, x_{A_2,i_2}, \ldots, x_{A_n,i_n}\} \) be a subset of \( Q_n \) such that all \( i_1, i_2, \ldots, i_n \) are distinct. If the set of edges \( \{(A_1,i_1),(A_2,i_2),\ldots,(A_n,i_n)\} \) in \( \Gamma_n \) is connected then the set \( Z \) is sufficient.

Let \( f : Q_n \to D \) be a homomorphism of \( Q_n \) into a division ring \( D \), \( \hat{f} : Q_n[t] \to D[t] \) be the induced homomorphism of the polynomial rings and \( P(t) = \hat{f}(\mathcal{P}(t)) \).

**Corollary 1.4.7.** Let \( Z = \{x_{A_1,i_1}, x_{A_2,i_2}, \ldots, x_{A_n,i_n}\} \) be a subset of \( Q_n \) such that all \( i_1, i_2, \ldots, i_n \) are distinct and the set of edges \( \{(A_1,i_1),(A_2,i_2),\ldots,(A_n,i_n)\} \) in \( \Gamma_n \) is connected.

Then all coefficients of \( P(t) \in D[t] \) can be obtained from elements \( f(z), \, z \in Z \) by operations of addition, subtractions, multiplication, and left and right conjugation.

**Examples.** 1. The set \( X = \{x_{\emptyset,1}, x_{\emptyset,2}, \ldots, x_{\emptyset,n}\} \) is connected (the corresponding edges have a common head \( \emptyset \)). Therefore, \( X \) is a sufficient set.

2. For \( n = 2 \) the sufficient sets are \( \{x_{\{i\},j}, x_{\emptyset,i}\}, \{x_{\emptyset,j}, x_{\emptyset,i}\}, \{x_{\{i\},j}, x_{\{j\},i}\} \). The sets \( \{x_{\{i\},j}, x_{\emptyset,j}\} \) are not sufficient. Here \( i, j = 1, 2, \, i \neq j \).
3. Let $n = 3$. The set $\{x_{\{1\},2}, x_{\{2\},1}, x_{\{2\},3}\}$ is sufficient because

$$(x_{\{2\},1} - x_{\{1\},2})x_{\emptyset,1} = x_{\{1\},2}(x_{\{2\},1} - x_{\{1\},2}),$$

$$x_{\{1\},3}(x_{\{1\},3} - x_{\{1\},2}) = (x_{\{1\},3} - x_{\{1\},2})x_{\{1\},3}$$

and $\{x_{\{1\},2}, x_{\{1\},2}, x_{\emptyset,1}\}$ is a defining set of pseudo-roots.

The sets $\{x_{\{1\},2}, x_{\{2\},1}, x_{\emptyset,3}\}$ and $\{x_{\{1\},3}, x_{\{1\},2}, x_{\emptyset,1}\}$ also are sufficient but not defining sets in $Q_3$.

4. The set $W = \{x_{\{1\},2}, x_{\{3\},2}, x_{\emptyset,1}\}$ is not sufficient because $d$- and $u$-operations are not defined on elements of $W$. We believe that coefficients of $P(t)$ cannot be written as rational expressions in elements of $W$, but we do not have a proof of this statement.

A proof of Theorem 1.4.6 follows from a more general theorem for algebras associated with directed graphs (quivers). We will describe such algebras in the next section.

2. Algebras associated with directed graphs

The class of algebras defined in this section contains the universal algebra of pseudo-roots $Q_n$.

2.1. Directed graphs. In this subsection we recall some well-known definitions and results about directed graphs.

A directed graph $\Gamma$ is a pair of sets $\Gamma = (V, E)$ and a map $\phi : E \to V \times V$. Elements of $V$ are called vertices of $\Gamma$ and elements $e \in E$ are called edges of $\Gamma$.

Let $\phi(e) = (t(e), h(e))$. The vertex $t(e)$ is called the tail of $e$ and the vertex $h(e)$ is called the head of $e$.

A directed path $P$ of length $k$ in $\Gamma$ from a vertex $u$ to a vertex $v$ is a sequence of edges $e_1, e_2, \ldots, e_k$ such that $t(e_{i+1}) = h(e_i)$ for $i = 1, 2, \ldots, k-1$ and $t(e_1) = u$, $h(e_k) = v$. The vertex $u$ is denoted by $t(P)$ and called the tail or origin of $P$. The vertex $v$ is denoted by $h(P)$ and called the head or terminus of $P$. The length of $P$ is denoted by $l(P)$.

An edge $e \in E$ is called essential if there is no path $P$ such that $l(P) \geq 2$ and $t(e) = t(P)$, $h(e) = h(P)$. A directed path $P$ is called maximal if all its edges are essential.

A vertex $u \in V$ is called a source if there is no edge $e \in E$ such that $h(e) = u$. A vertex $v \in V$ is called a sink if there is no edge $f \in E$ such that $t(f) = v$.

A directed graph $\Gamma = (V, E)$ is called a layered graph if there is a function $r$ from $V$ to the set of non-negative integers $\mathbb{Z}_+$ such that $r(t(e)) - 1 = r(h(e))$ for any edge $e \in E$. 
2.3. The graph of right divisors. Let $P(t)$ be a monic polynomial over an associative algebra $R$ and $S$ be a set of pseudo-roots of $P(t)$. Denote by $R_S$ the subalgebra in $R$ generated by pseudo-roots $x \in S$.

Construct a layered graph $\Gamma(P,S) = (V,E)$,

$$V = V(n) \sqcup V(n-1) \sqcup \ldots V(1) \sqcup V(0)$$
as follows. The vertices of $V(k) = \{v \in V : r(v) = k\}$ are monic polynomials $B(t) \in R[t]$ such that $\deg B(t) = k$ and

$$P(t) = Q(t)B(t)$$

in $R[t]$.

We say that there is an edge from vertex $B_1(t)$ to $B_2(t)$ in $\Gamma$ if

$$B_1(t) = (t - x)B_2(t)$$

for some $x \in S$.

Note that $V(n)$ consists of one vertex $v = P(t)$ and $V(0)$ consists of one vertex $w = 1$.

2.4. From graphs to algebras and polynomials. Let $\Gamma = (V, E)$ be a directed graph. Fix a field $k$. Let $T(E)$ be the free associative algebra over $k$ with generators $e \in E$. To any directed path $\pi = (e_1, e_2, \ldots, e_k)$ in $\Gamma$ there corresponds a polynomial $U_{\pi}(t) \in T(E)[t]$

$$U_{\pi}(t) = (t - e_1)(t - e_2)\ldots(t - e_k).$$

The following definition was introduced in [GRSW].

Definition 2.4.1. The algebra $A(\Gamma)$ is the quotient algebra of $T(E)$ modulo the following relations:

(2.4.1)

$$U_{\pi_1}(t) = U_{\pi_2}(t)$$

in $T(E)$ for any two paths $\pi_1$ and $\pi_2$ that common beginning and common end.

For two vertices $v, w$ in $\Gamma$ define the following polynomial $P_{v, w}(t)$ over $A(\Gamma)$:

If there exists a path $\pi = (f_1, f_2, \ldots, f_m)$ from $v$ to $w$, set

$$P_{v, w}(t) = (t - f_1)(t - f_2)\ldots(t - f_m)$$

in $A(\Gamma)[t]$. If there are no paths from $v$ to $w$, we set $P_{v, w} = 1$.

The construction of algebra $A(\Gamma)$ implies that the polynomial $P_{v, w}(t)$ does not depend of the choice of a path from $v$ to $w$.

If the set of vertices of $\Gamma$ contains one source $u$ and one sink $z$, then the polynomial $P(t) = P_{u, z}(t)$ is called the polynomial associated to graph $\Gamma$.

The following theorem implies that in many important cases algebras $A(\Gamma)$ are defined by linear and quadratic relations. These relations correspond to diamonds in $\Gamma$.

Theorem 2.4.2. If $\Gamma$ is a modular layered graph then defining relations for algebra $A(\Gamma)$ can be given by formulas (2.4.1) when $l(P_1) = l(P_2)$. 

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**Example.** If $\Gamma$ is the Hasse graph for the set of all subsets of $\{1, 2, \ldots, n\}$ then $A(\Gamma)$ coincides with the universal algebra of pseudo-roots $Q_n$.

**Remark.** It is natural to study quotient algebras $A_0(\Gamma)$ of algebras $A(\Gamma)$ given by relations $ef = 0$ for all pairs of edges $e, f$ in $\Gamma$ such that $h(e) \neq t(f)$.

2.5. Universality of $A(\Gamma)$.

Let $R$ be an algebra, $P(t)$ a monic polynomial of degree $n$ over $R$, and $S$ a set of pseudo-roots of $P(t)$. Let $\Gamma(P, S)$ be the layered graph constructed in Section 2.3. It contains two marked vertices: the source $v = P(t)$ and the sink $w = 1$.

Assume that the set $S$ contains pseudo-roots $a_1, a_2, \ldots, a_n$ such that

$$P(t) = (t - a_1)(t - a_2) \ldots (t - a_n).$$

Then the graph $\Gamma(P, S)$ contains a directed path from $v = P(t)$ to $w = 1$.

Following Section 2.4, we construct algebra $A(\Gamma(P, S))$ and polynomial $\mathcal{P}(t)$ over this algebra.

**Theorem 2.5.1.** There is a canonical homomorphism

\begin{equation}
\alpha : A(\Gamma(P, S)) \to R
\end{equation}

such that the induced homomorphism of polynomial algebras

$$\hat{\alpha} : A(\Gamma(P, S))[t] \to R[t]$$

maps $\mathcal{P}(t)$ to $P(t)$.

**Proof.** To an edge $e$ in $\Gamma$ there corresponds a pair of polynomials $B_1(t), B_2(t)$ in $R[t]$ such that $B_1(t), B_2(t)$ divide $P(t)$ from the right and $B_1(t) = (t - a)B_2(t)$.

Set

$$\alpha(e) = a.$$

One can see that this map can be uniquely extended to the homomorphism (2.5.1) and that $\hat{\alpha}(\mathcal{P}(t)) = P(t)$.

2.6. Iterated constructions. Let $\Gamma = (V, E)$ be a layered graph, $V = \bigcup_{k=0}^{n} V(k)$. Assume that $V(n)$ contains exactly one vertex $u$, $V(0)$ contains exactly one vertex $z$ and there is a directed path from $u$ to $z$. Denote by $E_0$ the subset of all edges $e \in E$ such that there is a directed path from $u$ to $z$ containing $e$. Denote by $\Gamma_0$ the subgraph of $\Gamma$ generated by $E_0$.

Following Section 2.4, construct the algebra $A(\Gamma)$ and the polynomial $\mathcal{P}(t)$. It is clear that $E_0$ can be canonically identified with a set of pseudo-roots of $\mathcal{P}(t)$. This identification establishes an isomorphism between the graphs $\Gamma_0$ and $\Gamma(\mathcal{P}(t))$. 

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3. Sufficient sets of edges for directed graphs

In this section we will define and study sufficient set of edges in directed graphs $\Gamma = (V, E)$. These sets will provide us with a construction of sufficient sets of pseudo-roots of polynomials $P(t)$ over algebras $A(\Gamma)$. All graphs considered in this sections are simple (i.e., if $t(e) = t(f)$ and $h(e) = h(f)$ then $e = f$) and acyclic (i.e., there are no directed paths $P$ such that $t(P) = h(P)$).

3.1. $D$- and $U$-operation for directed graphs. Let $\Gamma = (V, E)$ be a directed graph.

**Definition 3.1.1.**

1. A pair of edges $f_1, f_2$ with a common head is obtained from the pair $e_1, e_2$ with a common tail by $D$-operation if $h(e_i) = t(f_i)$ for $i = 1, 2$;
2. A pair of edges $e'_1, e'_2$ with a common tail is obtained from the pair $f'_1, f'_2$ with a common head by $U$-operation if $h(e'_i) = t(f'_i)$ for $i = 1, 2$.

**Remark.** We do not require the uniqueness of $D$- and $U$-operations.

**Definition 3.1.2.** A subset $E_0 \subseteq E$ is called $DU$-complete (or simply complete) if the results of any $D$-operation or any $U$-operation applied to edges from $E_0$ belong to $E_0$.

**Proposition 3.1.3.** For any subset $F \subseteq E$ there exists a minimal complete set $\hat{F} \subseteq E$ containing $F$.

We call $\hat{F}$ the completion of $F$.

To any directed graph $\Gamma = (V, E)$ there corresponds the double directed graph $\hat{\Gamma} = (V, E \sqcup E_-)$ with the same set of vertices and the doubled set of edges $E \sqcup E_-$. Here $E_-$ is a copy of $E$ and to any edge $e \in E$ there corresponds the opposite edge $-e \in E_-$ such that $h(-e) = t(e)$ and $t(-e) = h(e)$.

Edges $e \in E$ are called positive edges and edges $-e \in E_-$ are called negative edges.

Recall that a path from $v \in V$ to $w \in V$ is a set of edges $(f_1, f_2, \ldots, f_k)$ in the double directed graph $\hat{\Gamma}$ such that $t(f_1) = v$, $h(f_k) = w$ and $t(f_{i+1}) = h(f_i)$ for $i = 1, 2, \ldots, k - 1$. A path is called positive if all its edges are positive.

Any subset of edges $G$ generates a subgraph of $\Gamma(G) = (V(G), G)$ where $V(G)$ is the union of all heads and tails of edges $e \in G$.

A set of vertices $W \subseteq V$ is connected if for any vertices $v, w \in W$ there exists a path from $v$ to $w$. A set of edges $G$ is connected if $V(G)$ is connected.

3.2. Ample sets and sufficient sets. Let $\Gamma = (V, E)$ be a directed graph.

**Definition 3.2.1.** A set of edges $G$ in $\Gamma$ is called sufficient if its completion $\hat{G}$ contains a positive path from a source to a sink.
**Definition 3.2.2.** A set of vertices $W \subseteq V$ is called ample if

(1) For any non-sink vertex $v \in V$ there exists a vertex $u \in W$ such that there is no positive path in $\Gamma$ from $u$ to $v$;

(2) For any non-source vertex $v \in V$ there exists a vertex $w \in W$ such that there is no positive path in $\Gamma$ from $v$ to $w$.

A set of edges is called ample if the set of its tails and heads is ample.

As an example, consider the graph $\Gamma_n$ of all subsets of $\{1, \ldots, n\}$. It has one source $\{1, \ldots, n\}$ and one sink $\emptyset$. It is clear that a family of vertices $F = \{B_1, \ldots, B_K\}$ is ample if and only if for each $l \in \{1, \ldots, n\}$ there is $B_k \in F$ such that $l \in B_k$, and $B_{k'} \in F$ such that $l \notin B_{k'}$. This immediately provides examples of ample sets of edges in $\Gamma_n$.

**Proposition 3.2.3.** A set of edges $(A_1, i_1), (A_2, i_2), \ldots, (A_n, i_n)$ in $\Gamma_n$ is ample if all $i_1, i_2, \ldots, i_n$ are distinct.

**Proof.** Since all $i_1, \ldots, i_n$ are distinct, we can renumber the edges in our set so that this set becomes $\{(A_1, 1), \ldots, (A_n, n)\}$ with $i \notin A_i$. Then the corresponding set of vertices is

$$\{A_1, A_1 \cup \{1\}, A_2, A_2 \cup \{2\}, \ldots, A_n, A_n \cup \{n\}\}$$

(possibly with repetitions), and it is clear that for each $l \in \{1, \ldots, n\}$ we have

$$l \in A_l \cup \{l\}, \quad l \notin A_l.$$

**Theorem 3.2.4.** Any ample connected set of edges of a finite modular directed graph is a sufficient set.

Our proof of the theorem is based on the following lemma. Let $F$ be a complete connected set of edges in a modular directed graph $\Gamma = (V, E)$.

**Lemma 3.2.5.** Let $v, u \in V(F)$ and there is no positive paths in $\Gamma(F)$ from $u$ to $v$. Then

(1) There exists an edge $f \in F$ such that $t(f) = v$, and

(2) There exists an edge $e \in F$ such that $h(e) = u$.

**Proof.** We will prove the first part of the lemma. The second part can be proved in a similar way.

Let $\mathcal{P}$ be the set of all shortest paths in $\Gamma(F)$ from $u$ to $v$. By assumption, $\mathcal{P}$ does not contain a positive path. Then any path $P \in \mathcal{P}$ is defined by the sequence of edges $(\ldots, -f_{n-k(P)}, f_{n-k(P)}+1, \ldots, f_n)$ from $F$.

Take $P \in \mathcal{P}$ such that $k(P)$ is minimal. We claim that $k(P) = 0$. In fact, if $k(P) \neq 0$, then one can take

$$D(f_{n-k(P)+1}, f_{n-k(P)}) = (g_{n-k(P)+1}, g_{n-k(P)}).$$
Note that \( g_{n-k(P)+1}, g_{n-k(P)} \in F \) because \( F \) is complete. Also, \( g_{n-k(P)+1} \neq f_{n-k(P)+2} \) otherwise \( P \) is not the shortest path.

Construct a new path \( P' \in P \) by replacing the edges \(-f_{n-k(P)}, f_{n-k(P)}+1\) with the edges \( g_{n-k(P)}, g_{n-k(P)} \). Then \( k(P') = k(P) - 1 \) and we got a contradiction.

Therefore, \( k(P) = 0 \) and we may set \( f = f_n \). This proves the lemma.

**Proof of Theorem 3.2.4.** Let \( H \) be an ample set of edges in a finite direct modular graph \( \Gamma = (V, E) \) and let \( \hat{H} \) be the completion of \( H \). Let \( P \) be a path of maximal length in \( \Gamma(\hat{H}) \). Denote by \( u \) the beginning of this path and by \( v \) the end of this path.

We claim that \( u \) is a source and \( v \) is a sink in \( \Gamma \).

Indeed, suppose \( v \) is not a sink. Since \( G \) is an ample set, there exist a vertex \( w \in V(H) \) such that there is no positive path in \( \Gamma(\hat{H}) \) from \( w \) to \( v \). Lemma 3.2.5 implies that there is an edge \( f \in \hat{H} \) such that \( t(f) = v \). By adding \( f \) to \( P \) one gets a direct path in \( \Gamma(\hat{H}) \) from \( u \) to \( h(v) \). Then \( P \) is not a path of maximal length in \( \Gamma(\hat{H}) \) and we got a contradiction. It proves that \( v \) is a sink.

Similarly, one can prove that \( u \) is a source. The theorem is proved.

### 3.3. Sufficient sets of edges and sufficient sets of pseudo-roots.

Now let \( \Gamma = (V, E) \) be a directed graph such that

1. \( \Gamma \) contains a unique source \( M \) and a unique sink \( m \);
2. For each vertex \( v \in V \) there exist a positive path from \( M \) to \( v \) and a positive path from \( v \) to \( M \).

Recall that we associate to \( \Gamma \) an algebra \( A(\Gamma) \) and a polynomial \( \mathcal{P}(t) \in A(\Gamma)[t] \). The polynomial \( \mathcal{P}(t) \) is constructed using a positive path from \( M \) to \( m \) in \( \Gamma \), but it does not depend on the path. To any edge \( e \in E \) there corresponds to a pseudo-root \( e \in A(\Gamma) \) of \( \mathcal{P}(t) \), and to any positive path \( (e_1, e_2, \ldots, e_n) \) from \( M \) to \( m \) in \( \Gamma \) there corresponds the factorization

\[
\mathcal{P}(t) = (t - e_1)(t - e_2) \cdots (t - e_n)
\]

of \( \mathcal{P}(t) \) over \( A(\Gamma) \).

Let \( e_1, e_2 \in E \) be edges with the common tail and \( f_1, f_2 \in E \) be edges with the common head such that \( h(e_i) = t(f_i) \) for \( i = 1, 2 \). From the definition of algebra \( A(\Gamma) \) it follows that \( e_1 + f_1 = e_2 + f_2 \) and \( e_1 f_1 = e_2 f_2 \). These formulas imply the following proposition.

**Proposition 3.3.1.** For \( i, j = 1, 2 \), \( i \neq j \) we have

\[
e_i(f_i - f_j) = (f_i - f_j)f_i,
\]

\[
e_i(e_i - e_j) = (e_i - e_j)f_i.
\]
Corollary 3.3.2. If the pair $f_1, f_2$ is obtained from the pair by $e_1, e_2$ by $D$-operation then the elements $f_1, f_2$ in $A(\Gamma)$ are obtained from the elements $e_1, e_2$ by $d$-operations. If the pair $g_1, g_2$ is obtained from the pair by $h_1, h_2$ by $U$-operation then the elements $g_1, g_2$ in $A(\Gamma)$ are obtained from the elements $h_1, h_2$ by $d$-operations.

Corollary 3.3.3. If $S \subseteq E$ is an ample set then there exists a factorization (3.1) of $P(t)$ such that $du$-completion of $S$ contains elements $e_1, e_2, \ldots, e_n$ and, therefore, coefficients of $P(t)$.

Corollary 3.3.3 and Theorem 3.2.4 implies the following theorem.

Theorem 3.3.4. Let $S \subseteq E$ be an ample connected set of edges in a modular directed graph $\Gamma = (V, E)$. Then there exists a factorization (3.1) of $P(t)$ such that $du$-completion of $S$ contains elements $e_1, e_2, \ldots, e_n$ and, therefore, coefficients of $P(t)$.

Let $\Gamma$ be the graph of all subsets of $\{1, 2, \ldots, n\}$. Theorem 1.4.6 follows from Theorem 3.3.4 and Proposition 3.2.3 applied to the graph $\Gamma$.

We will also construct two “perpendicular” examples of connected ample sets of edges in $\Gamma$.

Example 1. Take $S = \{x_{\emptyset, 1}, x_{\emptyset, 2}, \ldots, x_{\emptyset, n}\}$. The set $S$ is connected and ample. It is easy to see that any edge in $\Gamma$ belongs to the $DU$-completion of $S$. Therefore, the $du$-completion of $S$ contains elements $e_1, e_2, \ldots, e_n$ for any factorization of $P(t)$.

Example 2. Take $T = \{x_{12\ldots n-1, n}, x_{12\ldots n-2, n-1}, \ldots, x_{\emptyset, 1}\}$. The set $T$ is also connected and ample. Its $DU$-completion coincides with $T$ and its $du$-completion defines a factorization of $P(t)$.

Other examples of sufficient sets of pseudo-roots were considered in Section 1.

References

[GR1] I. Gelfand, V. Retakh, Gelfand Mathematical Seminars 1993-95, Birkhauser Boston, 1996, pp. 93-100.

[GR2] I. Gelfand, V. Retakh, Quasdeterminants I, Selecta Math. (N.S.) 3 (1997), 517-546.

[GGRSW] I. Gelfand, S. Gelfand, V. Retakh, S. Serconek, and R. Wilson, Hilbert series of quadratic algebras associated with decompositions of noncommutative polynomials, J. Algebra 254 (2002), 279–299.

[GGRW] I. Gelfand, S. Gelfand, V. Retakh, R. Wilson, Quasdeterminants, Advances in Math. 193 (2005), 56-141.

[GRW] I. Gelfand, V. Retakh, and R. Wilson, Quadratic-linear algebras associated with decompositions of noncommutative polynomials and differential polynomials, Selecta Math. (N.S.) 7 (2001), 493–523.
