Cyclic bases
of zero-curvature representations:
five illustrations to one concept

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Abstract

The paper contains five examples of using cyclic bases of zero-curvature representations in studies of weak and strong Lax pairs, hierarchies of evolution systems, and recursion operators.

1 Introduction

Many remarkable equations of nonlinear mathematical physics can be represented as the compatibility condition

$$Z \equiv D_t X - D_x T + [X, T] = 0$$ (1)

of the overdetermined linear system

$$\Psi_x = X \Psi, \quad \Psi_t = T \Psi,$$ (2)

where $D_x$ and $D_t$ stand for the total derivatives; $X$ and $T$ are $n \times n$ matrix functions of the independent variables $x$ and $t$, dependent variables $u^i(x, t)$ and derivatives of $u^i; \Psi(x, t)$ is an $n$-component column; the square brackets denote the matrix commutator. The condition (1), usually referred to as the zero-curvature representation (ZCR), is said to represent a given equation if any solution $u^i$ of the equation satisfies (1). The linear transformation

$$\Psi'(x, t) = G \Psi(x, t)$$ (3)
of the auxiliary vector function $\Psi$ with any nondegenerate $n \times n$ matrix function $G(x, t, u^i, u_x^i, \ldots)$, $\det G \neq 0$, generates the gauge transformation of the matrices $X$ and $T$:

$$
X' = GXG^{-1} + (D_x G) G^{-1}, \quad T' = GTG^{-1} + (D_t G) G^{-1}.
$$

(4)

Since the matrix $Z$ in (1) is transformed under (4) as

$$
Z' = GZG^{-1},
$$

(5)

any two ZCRs related by a gauge transformation (4) are considered as equivalent (see, e.g., [1]). The existence of such an equivalence between ZCRs suggests that ZCRs should be studied by gauge-invariant methods.

In the present paper, we give five examples illustrating the use of the cyclic bases of ZCRs—one of the central concepts of our work [2], where a gauge-invariant description of ZCRs of evolution equations was proposed[1]. The paper is organized as follows. In Section 2, we study a continual class of second-order evolution equations admitting weak Lax pairs, and construct an extension of this class to any order. In Section 3, we consider removability of a parameter from a weak Lax pair of the Burgers equation, and then, in Section 4, we derive the Burgers equation’s recursion operator from the cyclic basis of a strong Lax pair. In Section 5, we apply this method of deriving recursion operators from ZCRs to a system of coupled KdV equations, in order to show how the method works with systems, and then, in Section 6, we use the same method again, in order to explain a strange structure of a recursion operator of one KdV–mKdV system. Section 7 contains concluding remarks.

2 Second-order evolution equations

Recently, Marvan [3] proved that every second-order scalar evolution equation possessing an irreducible sl2-valued ZCR can be brought by a contact transformation into the form

$$
u_t = \beta_x u^2 u_{xx} + 2\beta_{xx} u^2 u_x + 4\beta u_x + (\beta_{xxx} - 4\beta_x) u^3 - 4\beta_x u,$$

(6)

A much more general (but very abstract, indeed) gauge-invariant description, applicable to ZCRs of any non-overdetermined systems of PDEs, was introduced by Marvan in [4]; later, in [5], Marvan gave explicit expressions for the objects involved.
where $\beta(x,t)$ is any function with $\beta_x \neq 0$, and then the ZCR of (6) is given by (1) with

$$X = \begin{pmatrix} 1 & 1 \\ u & -1/u \end{pmatrix},$$

(7)

$$T = \begin{pmatrix} -\beta_x u_x + 4\beta/u - \beta_x x u & 4\beta + 2\beta_x u \\ 4\beta - 2\beta_x u & \beta_x u_x - 4\beta/u + \beta_x x u \end{pmatrix}.$$  

(8)

One may wonder why the class of evolution equations (6) contains the arbitrary function $\beta(x,t)$. The origin of $\beta(x,t)$ can be revealed by the technique of cyclic bases of ZCRs [2].

Let us solve the problem of finding all the (1+1)-dimensional scalar evolution equations

$$u_t = f(x,t,u,u_x,\ldots,u_x\ldots x)$$

(9)

which admit ZCRs (1) with the predetermined matrix $X$ (7) and any $2 \times 2$ matrices $T(x,t,u,u_x,\ldots,u_x\ldots x)$ (traceless, without loss of generality), the highest orders of derivatives $u_x\ldots x$ in $f$ and $T$ being not fixed. Since $X$ (7) contains no parameter, this problem is equivalent to finding the complete class of evolution equations possessing weak Lax pairs with the predetermined spatial part.

First, we rewrite the ZCR of (9) in its equivalent (characteristic) form

$$fC = \nabla_x T,$$

(10)

where $C$ is the characteristic matrix of the ZCR (in the present case, $C$ is simply $C = \partial X/\partial u$, because $X$ (7) contains no derivatives of $u$ [4]), and the operator $\nabla_x$ is defined by $\nabla_x Y = D_x Y - [X, Y]$ for any (here, $2 \times 2$) matrix $Y$.

Second, we compute $C$, $\nabla_x C$, $\nabla_x^2 C$ and $\nabla_x^3 C$ for the matrix $X$ (7), and find that the cyclic basis (i.e. the maximal sequence of linearly independent matrices $\nabla_x^i C$, $i = 0,1,2,\ldots$) is three-dimensional in this case, $\{C, \nabla_x C, \nabla_x^2 C\}$, and that the closure equation for the cyclic basis is

$$\nabla_x^3 C = a_1 C + a_2 \nabla_x C + a_3 \nabla_x^2 C$$

(11)
with
\[
\begin{align*}
a_1 &= \frac{8u_x - 2u_x^3}{u^3} - \frac{8u_x u_{xx}}{u^2} + \frac{12u_x - 2u_{xxx}}{u}, \\
a_2 &= \frac{4 - 10u_x^2}{u^2} - \frac{6u_{xx}}{u} + 4, \quad a_3 = -\frac{7u_x}{u}.
\end{align*}
\] (12)

Third, we decompose the unknown matrix \(T\) over the cyclic basis as
\[
T = b_1 C + b_2 \nabla_x C + b_3 \nabla_x^2 C,
\] (13)
where \(b_1, b_2\) and \(b_3\) are unknown scalar functions of \(x, t, u, u_x, \ldots, u_{xxx}\). Substituting (13) into (10) and using the closure equation (11) and the fact of linear independence of \(C, \nabla_x C\) and \(\nabla_x^2 C\), we obtain the following relations:
\[
\begin{align*}
b_2 &= -D_x b_3 - a_3 b_3, \\
b_1 &= -D_x b_2 - a_2 b_3, \\
f &= D_x b_1 + a_1 b_3,
\end{align*}
\] (14)
where the function \(b_3\) remains arbitrary.

Fourth, we combine (12), (13) and (14), use the explicit form of \(C, \nabla_x C\) and \(\nabla_x^2 C\), and thus obtain the following expressions for \(T\) and \(f\) in terms of one arbitrary function \(b_3 = p(x, t, u, u_x, \ldots, u_{xxx})\), where the order of \(u_{xxx}\) is also arbitrary:
\[
T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & -T_{11} \end{pmatrix}
\] (15)
with
\[
\begin{align*}
T_{11} &= \left( -\frac{1}{u^2} D_x^2 + \frac{5u_x}{u^3} D_x + \frac{4 - 9u_x^2 + 3uu_{xx}}{u^4} \right) p, \\
T_{12} &= \left( \frac{2}{u^2} D_x + \frac{4 - 6u_x}{u^3} \right) p, \\
T_{21} &= \left( -\frac{2}{u^2} D_x + \frac{4 + 6u_x}{u^3} \right) p,
\end{align*}
\] (16)
and
\[
\begin{align*}
f &= \left( D_x^3 - \frac{7u_x}{u} D_x^2 - \frac{4 + 4u^2 - 24u_x^2 + 8uu_{xx}}{u^2} D_x \\
&\quad + \frac{16u_x + 12u^2 u_x - 36u_x^2 + 27uu_x u_{xx} - 3u^2 u_{xxx}}{u^3} \right) p.
\end{align*}
\] (17)

The problem has been solved. Every evolution equation (9), which belongs to the continual class determined by (17) with any function \(p(x, t, u,\)
... of any order in \( u_{x...x} \), admits the ZCR (1) with \( X \) given by (7) and \( T \) determined by (15)–(16). As it was shown in [2], such a continual class of evolution equations, which all possess ZCRs with a predetermined matrix \( X \), exists for every matrix \( X \) containing no parameter. Of course, the studied case of \( X \) (7) is not an exception.

Now we can see that the origin of the arbitrary function \( \beta(x, t) \) in (6) is the arbitrary function \( p(x, t, u, u_x, \ldots, u_{x...x}) \) in (17). Indeed, if we require that \( f \) (17) is a function of \( x, t, u, u_x \) only, i.e. that the order of the represented evolution equations is two, then \( p = p(x, t, u) \) (otherwise \( f \) contains \( u_{x...x} \) of order four or higher) and \( \partial p / \partial u = 3 p / u \) (this follows from \( \partial f / \partial u_{xxx} = 0 \)), that is

\[
p = \beta(x, t) u^3
\]  

with any function \( \beta \) \( (\beta_x \neq 0 \) if we need \( \partial f / \partial u_{xx} \neq 0 \)). Finally, the equation (18) relates (17) with (3), as well as (15)–(16) with (3).

3 Burgers equation’s weak Lax pair

Our next example is the ZCR (1) with

\[
X = \begin{pmatrix}
\frac{1}{2} \eta & \frac{1}{4} u + \frac{1}{2} \eta \\
\frac{1}{4} u - \frac{1}{2} \eta & -\frac{1}{2} \eta
\end{pmatrix},
\]  

(19)

\[
T = \begin{pmatrix}
\frac{1}{4} \eta u & \frac{1}{4} u_x + \frac{1}{8} u^2 + \frac{1}{4} \eta u \\
\frac{1}{4} u_x + \frac{1}{8} u^2 - \frac{1}{4} \eta u & -\frac{1}{4} \eta u
\end{pmatrix},
\]  

(20)

where \( \eta \) is a parameter. This ZCR of the Burgers equation \( u_t = u_{xx} + uu_x \) was introduced by Cavalcante and Tenenblat [3]. The parameter \( \eta \), however, is not an essential (‘spectral’) parameter: it can be removed (‘gauged out’) from (19) and (20) by a gauge transformation (4) (therefore this ZCR is a weak Lax pair of the Burgers equation). More precisely, it was recently shown by Marvan [4] that any nonzero value of \( \eta \) in these \( X \) and \( T \) can be changed into any other nonzero value (e.g., into \( \eta = 1 \)) by an appropriate gauge transformation.

Let us see how the technique of cyclic bases allows to discover, in a short computational way, that the case of \( \eta = 0 \) is essentially different from any

\[\text{The fact of removability of } \eta \text{ from } X \text{ [13] was also noted in [2], in slightly different notations, in relation with ZCRs of third-order evolution equations [8].}\]
other case of nonzero $\eta$, and that the parameter $\eta$ should be expected to be removable from this ZCR. The necessary information can be obtained from the dimension of the cyclic basis and from the coefficients of the closure equation.

Following the scheme of Section 3, we find the characteristic matrix and its lower-order covariant derivatives to be

$$C = \frac{1}{4} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\nabla_x C = \frac{\eta}{4} \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}, \quad \nabla_x^2 C = \frac{\eta u}{8} \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}.$$ (21)

We see from (21) that, when $\eta \neq 0$, the cyclic basis is two-dimensional, $\{C, \nabla_x C\}$, with the closure equation

$$\nabla_x^2 C = \frac{u}{2} \nabla_x C,$$ (22)

whereas in the case of $\eta = 0$ the cyclic basis is one-dimensional, consisting of the characteristic matrix $C$ itself, with the closure equation

$$\nabla_x C = 0.$$ (23)

Since the matrices $\nabla_x^i C$ ($i = 0, 1, 2, \ldots$) are transformed under (4) as $\nabla_x^i C' = G (\nabla_x^i C) G^{-1}$ [B, F], the dimension of the cyclic basis and the coefficients of the closure equation are invariants with respect to gauge transformations. Therefore no gauge transformation can change the dimension of the cyclic basis and relate the case of $\eta \neq 0$ with the case of $\eta = 0$. Moreover, the coefficients of the closure equation (22) do not contain $\eta$, and this is highly indicative of the parameter removability. On the contrary, if there is a parameter in a coefficient of a closure equation of a ZCR, then the parameter is essential, not removable from the ZCR, for the reason of the gauge invariance of coefficients of closure equations, and Sections 4–6 provide examples illustrating this phenomenon.

Using the method described in Section 2, we can construct continual classes of evolution equations admitting ZCRs with $X$ given by (19), separately for the two cases of the cyclic basis structure:

$$u_t = D_x \rho \quad \text{for} \quad \eta = 0,$$ (24)

$^3$An easy way to prove this removability of $\eta$ is to solve (4) with respect to $G$ directly, taking $X' = X|_{\eta=1}$ and $T' = T|_{\eta=1}$ and starting from the evident necessary conditions $G = G(x, t)$ and $[C, G] = 0$. A different approach was used in [B].
\[ u_t = D_x \left( D_x + \frac{u}{2} \right) p \quad \text{for} \quad \eta \neq 0, \tag{25} \]

where \( p \) is any function of \( x, t, u, u_x, \ldots, u_{x...x} \), of any order in \( u_{x...x} \). Note that the orders of the linear differential operators in the right-hand sides of (24) and (25) are equal to the dimensions of the corresponding cyclic bases, in accordance with the general theorem on continual classes [2].

The class (24) consists of evolution equations admitting the conserved density \( u \). This is a natural consequence of the cyclic basis structure in the case of \( \eta = 0 \). Indeed, if we consider a \((1 \times 1)\)-dimensional ZCR with \( X = u \) and any function \( T \), then this ZCR is simply a conservation law, \( D_t X - D_x T = 0 \), due to \([X, T] = 0\), and the closure equation for \( X = u \) is exactly (23).

The class (25) is more narrow than (24) but much wider than habitual discrete hierarchies of integrable equations. See [2] for the results of the singularity analysis which suggest that most of equations in such continual classes are non-integrable.

The fact that the Burgers equation \( u_t = u_{xx} + uu_x \) admits a ZCR with \( X \) (19) is equivalent to the fact that the equation belongs to the classes (24) and (25). It tells nothing about the Burgers equation’s integrability.

4 Burgers equation’s recursion operator

Now, let us study a strong Lax pair of the Burgers equation

\[ u_t = u_{xx} + 2uu_x, \tag{26} \]

namely, the ZCR (1) with

\[ X = \begin{pmatrix} \frac{1}{2} (u + \lambda) & 0 \\ 1 & -\frac{1}{2} (u + \lambda) \end{pmatrix}, \tag{27} \]

\[ T = \begin{pmatrix} \frac{1}{2} (u_x + u^2 - \lambda^2) & 0 \\ u - \lambda & -\frac{1}{2} (u_x + u^2 - \lambda^2) \end{pmatrix}, \tag{28} \]

where \( \lambda \) is any constant.\footnote{We are unable to indicate the first appearance of this ZCR in the literature. In implicit form, this ZCR can be found in [3], among non-Abelian pseudopotentials of the Burgers equation.} Note that these \( X \) and \( T \) are lower triangular matrices. However, contrary to the approach adopted in [3], we do not exclude such lower triangular (or reducible) ZCRs from our consideration.
Computing $C$, $\nabla_x C$ and $\nabla_x^2 C$ for the matrix $X$ \((27)\), we find that the cyclic basis is two-dimensional, \(\{C, \nabla_x C\}\), with the closure equation

$$\nabla_x^2 C = (u + \lambda) \nabla_x C.$$  \((29)\)

The presence of $\lambda$ in the gauge-invariant coefficient $u + \lambda$ of \((29)\) proves that the parameter $\lambda$ is essential, not removable by gauge transformations. The matrix $T$ \((28)\) can be decomposed over the cyclic basis as

$$T = (u_x + u^2 - \lambda^2) C + (\lambda - u) \nabla_x C.$$  \((30)\)

Let us find all the evolution equations \((9)\) which admit ZCRs \((1)\) with $X$ determined by \((27)\) and $T$ decomposable over the cyclic basis as

$$T = b_1 C + b_2 \nabla_x C,$$  \((31)\)

where $b_1$ and $b_2$ are any scalar functions of $x, t, u, u_x, \ldots, u_{x...x}$ and $\lambda$. Using \((10)\), \((31)\), \((29)\) and the fact of linear independence of $C$ and $\nabla_x C$, we find that

$$b_1 = (D_x + u + \lambda) p,$$  \((32)\)

$$f = (D_x^2 + (u + \lambda) D_x + u_x) p,$$  \((33)\)

where $p(\lambda, x, t, u, u_x, \ldots, u_{x...x}) = -b_2$ is any function, of any order in $u_{x...x}$. Note that the linear differential operator in the right-hand side of \((33)\) contains the parameter $\lambda$. If $\lambda$ is any fixed constant, then our problem is completely solved by the expression \((33)\) for the right-hand side of every represented equation \((1)\), and we obtain a typical continual class. However, $\lambda$ is a free parameter, and we have to determine the set of functions $p$ satisfying the condition $\partial f / \partial \lambda = 0$ imposed on \((33)\). Here we give a detailed derivation of those admissible $p$, in order to omit details of the same way of reasoning in Sections 5 and 6.

First, we introduce the operators

$$M = D_x^2 + u D_x + u_x, \quad N = D_x,$$  \((34)\)

expand the function $p$ as

$$p = p_0 + \lambda p_1 + \lambda^2 p_2 + \lambda^3 p_3 + \cdots,$$  \((35)\)
where \( p_i (i = 0, 1, 2, \ldots) \) are unknown functions of \( x, t, u, u_x, \ldots, u_{xxx} \), and thus rewrite (33) in the form

\[
f = (M + \lambda N) \left( p_0 + \lambda p_1 + \lambda^2 p_2 + \cdots \right). \tag{36}
\]

Second, we obtain from (36) the expression for \( f \) satisfying the condition \( \partial f / \partial \lambda = 0 \),

\[
f = M p_0, \tag{37}
\]

as well as the recursion relation determining the functions \( p_i \),

\[
M p_{i+1} + N p_i = 0, \quad i = 0, 1, 2, \ldots. \tag{38}
\]

Third, taking into account that \( M \) and \( N \) (34) are differential operators of orders two and one, respectively, we conclude from (38) that there exists a sufficiently large number \( m \), such that \( p_m \) does not contain \( u \) and derivatives of \( u \). Then the recursion relation (38) for \( i \geq m \) and the explicit expressions for \( M \) and \( N \) (34) lead to \( p_m = \phi (t) \), with any \( \phi \), and \( p_i = 0 \) for \( i > m \).

The problem has been solved: the right-hand side of every evolution equation sought is determined by (37), where \( p_0 \) is given by the recursion (38) starting from \( p_m = \phi (t) \), with any function \( \phi (t) \) and any integer \( m \geq 0 \), the operators \( M \) and \( N \) being defined by (34).

However, fourth, the result can be represented in a more compact form. Using the operator \( N^{-1} \), formally inverse to \( N \), we express \( p_0 \) from (38) as \( p_0 = (-N^{-1} M)^m p_m \), rewrite (37) as \( f = M (-N^{-1} M)^m N^{-1} N p_m \), take into account that \( N p_m = 0 \), and thus obtain

\[
f = R^{m+1} 0, \tag{39}
\]

where

\[
R = M N^{-1} = D_x + u + u_x D_x^{-1}. \tag{40}
\]

Consequently, the matrix \( X \) (27) is related not to a continual class of evolution equations, as it would be if there was no essential parameter in \( X \), but to the discrete hierarchy determined by (39), which consists of the evolution equations

\[
\begin{align*}
  u_t &= \phi_1 (t) u_x + \phi_2 (t) (u_{xx} + 2 u u_x) \\
  &\quad + \phi_3 (t) (u_{xxx} + 3 u u_{xx} + 3 u_x^2 + 3 u^2 u_x) + \cdots
\end{align*} \tag{41}
\]

5If a singularity is suspected in \( p \) at \( \lambda = 0 \), one may use (36) after an infinitesimal shift of \( \lambda \), \( \lambda \rightarrow \lambda + \epsilon, M \rightarrow M - \epsilon N, N \rightarrow N, \epsilon \rightarrow 0 \), with no effect on the final result (40).

6In the sense that \( N^{-1} a \) denotes any function \( b (x, t, u, u_x, \ldots, u_{xxx}) \) such that \( a = N b \) if \( b \) exists.
with any functions $\phi_1, \phi_2$, etc. All the equations (41) can be linearized by the Cole–Hopf transformation. The obtained recursion operator $R$ (40), which generates the hierarchy of equations (41), is well known as a recursion operator of the Burgers equation in the sense of higher symmetries (or generalized symmetries, or Lie–Bäcklund algebras) (see, e.g., [10]).

5 Coupled KdV equations

The method, used in Section 4 to derive the Burgers equation’s hierarchy and recursion operator, can be applied to systems of evolution equations as well. Therefore our next example is

$$u_t = u_{xxx} - 12uu_x, \quad v_t = 4v_{xxx} - 6vu_x - 12uv_x. \quad (42)$$

These coupled KdV equations were found by Gürses and Karasu [11] as a new system possessing a second-order recursion operator (in the sense of higher symmetries). Later, the system (42) appeared in [12], in a list of coupled KdV equations that passed the Painlevé test for integrability well, and its ZCR (1) with $3 \times 3$ matrices $X$ and $T$ containing an essential parameter was also found there. For what follows, we need to know the explicit form of the matrix $X$ from [12],

$$X = \begin{pmatrix} 0 & u + \sigma & 0 \\ 2 & 0 & 0 \\ 0 & v & 0 \end{pmatrix}, \quad (43)$$

where $\sigma$ is a parameter. No explicit expression for the matrix $T$ is necessary: it is sufficient to know that $T$ can be decomposed over the cyclic basis.

In this case of a system of two equations, we have two characteristic matrices, $C_u$ and $C_v$, which are simply $C_u = \partial X/\partial u$ and $C_v = \partial X/\partial v$ because $X$ (13) contains no derivatives of $u$ and $v$ [2, 4]. Computing $\nabla_x C_u$, $\nabla_x C_v$, $\nabla_x^2 C_u$, $\nabla_x^2 C_v$ and $\nabla_x^3 C_u$, we find that the cyclic basis is five-dimensional, $\{C_u, C_v, \nabla_x C_u, \nabla_x C_v, \nabla_x^2 C_u\}$, with the two closure equations:

$$\nabla_x^2 C_v = 2(u + \sigma) C_v,$$

$$\nabla_x^3 C_u = 4u_x C_u + 2v_x C_v + 8(u + \sigma) \nabla_x C_u + 6v \nabla_x C_v. \quad (44)$$

The presence of $\sigma$ in the gauge-invariant coefficients of (44) proves that the parameter $\sigma$ is essential, not removable from the studied ZCR by gauge transformations.
Next we pose the problem of finding the hierarchy of all systems

\[
\begin{align*}
    u_t &= f(x, t, u, v, \ldots, u_{x...x}, v_{x...x}), \\
    v_t &= g(x, t, u, v, \ldots, u_{x...x}, v_{x...x})
\end{align*}
\]  

(45)

which admit ZCRs (43) with \(X\) determined by (43) and \(T\) decomposable over the cyclic basis as

\[
T = c_1 C_u + c_2 C_v + c_3 \nabla_x C_u + c_4 \nabla_x C_v + c_5 \nabla^2_x C_u,
\]

(46)

where \(c_1, \ldots, c_5\) are any scalar functions of \(x, t, u, v, \ldots, u_{x...x}, v_{x...x}\) and \(\sigma\). Substituting \(T\) (46) into the characteristic form of ZCRs

\[
f C_u + g C_v = \nabla_x T
\]

(47)

and using the closure equations (44), we obtain

\[
\begin{align*}
    c_1 &= -D_x c_3 - 8(u + \sigma)c_5, \\
    c_2 &= -D_x c_4 - 6vc_5, \\
    c_3 &= -D_x c_5, \\
    f &= D_x c_1 + 4u_x c_5, \\
    g &= D_x c_2 + 2(u + \sigma)c_4 + 2v_x c_5,
\end{align*}
\]

(48)

where the functions \(c_4\) and \(c_5\) remain arbitrary. The relations (48) lead to the following expression for the right-hand sides of the represented equations (45):

\[
\begin{pmatrix} f \\ g \end{pmatrix} = (M + \lambda N)p,
\]

(49)

where \(\lambda = -8\sigma\), \(p = (c_5, -c_4)^T\) (T denotes transposing),

\[
M = \begin{pmatrix} D^3_x - 8u D_x - 4u_x & 0 \\
                    -6v D_x - 4v_x & D^2_x - 2u \end{pmatrix},
\]

(50)

\[
N = \begin{pmatrix} D_x & 0 \\
                    0 & \frac{1}{4} \end{pmatrix}.
\]

(51)

Finally, we impose the conditions \(\partial f/\partial \lambda = 0\) and \(\partial g/\partial \lambda = 0\) on (49), in order to find the set of admissible two-component functions \(p\). Following the same way of reasoning as used in Section 4, we conclude that the systems (45) represented by (1) with \(X\) (43) constitute a discrete hierarchy generated by the recursion operator \(R = MN^{-1}\) with \(M\) (50) and \(N\) (51). Inverting the matrix differential operator \(N\),

\[
N^{-1} = \begin{pmatrix} D_x^{-1} & 0 \\
                    0 & 4 \end{pmatrix},
\]

(52)
we obtain the explicit expression for the recursion operator \( R \),

\[
R = \begin{pmatrix}
D_x^2 - 8u - 4u_x D_x^{-1} & 0 \\
-6v - 4v_x D_x^{-1} & 4D_x^2 - 8u
\end{pmatrix},
\]

(53)

which coincides with the one found in [11].

6 Coupled KdV–mKdV equations

Recently, Kersten and Krasil’chik [13] introduced the new system of coupled KdV–mKdV equations

\[
\begin{align*}
    u_t &= -u_{xxx} + 6uu_x - 3ww_{xxx} - 3w_xw_{xx} + 3u_xw^2 + 6uw w_x, \\
    w_t &= -w_{xxx} + 3w^2 w_x + 3uw_x + 3u_x w
\end{align*}
\]

(54)

and found its recursion operator (in the sense of higher symmetries). The recursion operator of (54) has an unusual structure: its nonlocal part consists of terms of the form \( e_1 D_x^{-1}, e_2 D_x^{-1} e_3 \) and \( e_4 D_x^{-1} e_5 D_x^{-1} \), and those expressions \( e_i \) are polynomials of \( u, u_x, w, w_x, w_{xx}, w_{xxx}, y, \sin(2z) \) and \( \cos(2z) \), where \( y \) and \( z \) are nonlocal variables,

\[
y : \ y_x = u, \quad z : \ z_x = w.
\]

(55)

The appearance of trigonometric functions of a nonlocal variable in the recursion operator derived in [13] looks strange and, of course, may be explained in various ways. One possible explanation can be obtained by the method described in Sections 4 and 5, which allows us to re-derive the recursion operator of (54) in the quotient form \( R = MN^{-1} \), where \( M \) and \( N \) are \( 2 \times 2 \) matrix linear differential operators of orders four and two, respectively, whose coefficients contain only local variables. This approach makes use of the matrix \( X \) of the ZCR of (54), found very recently in [14],

\[
X = \begin{pmatrix}
\alpha & u - w^2 + 9\alpha^2 & w \\
1 & \alpha & 0 \\
0 & 6\alpha w & -2\alpha
\end{pmatrix},
\]

(56)

where \( \alpha \) is a parameter.

Starting from the characteristic matrices \( C_u = \partial X / \partial u \) and \( C_v = \partial X / \partial v \) and repeatedly applying the operator \( \nabla_x = D_x - [X, \ ] \), we find the cyclic basis to be eight-dimensional,

\[
\{ C_u, C_v, \nabla_x C_u, \nabla_x C_v, \nabla_x^2 C_u, \nabla_x^2 C_v, \nabla_x^3 C_u, \nabla_x^3 C_v \},
\]

(57)
for all nonzero values of $\alpha$. Omitting here the cumbersome closure equations, i.e. decompositions of $\nabla^4_x C_u$ and $\nabla^4_x C_v$ over (57), we only note that their gauge-invariant coefficients contain the parameter $\alpha$, which is an essential parameter therefore. Then, the problem of finding the hierarchy of coupled evolution equations $\{u_t = f, w_t = h\}$ represented by (1) with $X$ (56) leads us to

$$
\begin{pmatrix} f \\ h \end{pmatrix} = (M + \lambda N) \begin{pmatrix} c_7 \\ c_8 \end{pmatrix},
$$

(58)

where $\lambda = 36\alpha^2$; the unknown functions $c_7$ and $c_8$ of $\lambda, x, t, u, w, \ldots, u_{x...x}, w_{x...x}$ are the coefficients at $\nabla^3_x C_u$ and $\nabla^3_x C_v$, respectively, in the decomposition of $T$ over (57); and the cumbersome differential operators $M$ and $N$ are given below, in a simplified form. Finally, the conditions $\partial f / \partial \lambda = \partial h / \partial \lambda = 0$ lead us to the conclusion that the represented hierarchy is generated by the recursion operator $R = MN^{-1}$.

The second-order differential operator $N$ can be factorized as

$$N = DP : \quad D = \begin{pmatrix} D_x & 0 \\ 0 & D_x \end{pmatrix}, \quad P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix},$$

(59)

where

$$P_{11} = D_x + \frac{w_x}{w} + \frac{uw + 2w^3}{w_x},$$

(60)

$$P_{12} = -2wD_x - 12w_x - \frac{u_x}{w}$$

$$- \frac{u^2 + 6uw^2 + 8w^4 - 2ww_{xx}}{w_x} + \frac{ww_{xx}}{ww_x},$$

(61)

$$P_{21} = \frac{w^2}{w_x},$$

(62)

$$P_{22} = D_x - \frac{ww + 4w^3 - w_{xx}}{w_x}.$$ 

(63)

The fourth-order differential operator $M$ can be expressed as

$$M = R_{\text{loc}}N + Q,$$

(64)
where $R_{\text{loc}}$ turns out to be exactly the local part of the recursion operator $R$ found in [13] (so we are on the right way),

$$R_{\text{loc}} = \begin{pmatrix} -D_x^2 + 4u + w^2 & -2wD_x^2 - w_xD_x + 3uw - 2w_{xx} \\ 2w & -D_x^2 + 2u + w^2 \end{pmatrix},$$

(65)

and the components of the first-order differential operator $Q$,

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix},$$

(66)

are determined by

$$Q_{11} = (2u_x + uw_x)D_x + uw^2 + w_x^2 + w_{xx}w_x + \frac{2u_xw_x}{w} + \frac{(2uw + 5w^3)u_x}{w} - \frac{w^2w_{xxx}}{w},$$

(67)

$$Q_{12} = (-3wu_x + (2u - 4w^2)w_x - w_{xxx})D_x - u^2w - 4uw^3 - 23uw_xw_x - 8ww_x^2 + (2u - 4w^2)w_{xx} - \frac{2u_x^2}{w} - \frac{(2u^2 + 13uw^2 + 20w^4 - 5ww_{xx})u_x}{w} - \frac{(uw + 4w^3 - w_{xx})w_{xxx}}{w},$$

(68)

$$Q_{21} = w_xD_x + 2uw + 5w^3 + \frac{w_x^2}{w} + \frac{w^2u_x}{w},$$

(69)

$$Q_{22} = (u_x + uw_x)D_x - 2u^2 - 13uw^2 - 20w^4 - 14w_x^2 + 5ww_{xx} - \frac{u_xw_x - uw_{xx}}{w} - \frac{(uw + 4w^3 - w_{xx})u_x}{w}.$$

(70)

Now, let us proceed to the nonlocal part $QN^{-1}$ of the recursion operator $R = MN^{-1}$. Since $N^{-1} = P^{-1}D^{-1}$ due to (59), we have to invert the operator $P$ determined by (60)–(63). Considering

$$S = P^{-1} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} : \quad SP = PS = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

(71)
we obtain
\begin{align*}
S_{11} &= -P_{21}^{-1}P_{22}S_{21}, \\
S_{12} &= -P_{21}^{-1}P_{22}S_{22} + P_{21}^{-1}, \\
S_{21} &= -(P_{11}P_{21}^{-1}P_{22} - P_{12})^{-1}, \\
S_{22} &= (P_{11}P_{21}^{-1}P_{22} - P_{12})^{-1}P_{11}P_{21}^{-1}.
\end{align*}
(72)

Then the relations (72), (60)–(63) and (67)–(70) lead us to a cumbersome explicit expression for the nonlocal part \(QP^{-1}D^{-1}\) of the recursion operator \(R\) in terms of local variables and operators \(D_x, D_x^{-1}\) and \(K^{-1}\), where the operator \(K^{-1}\) is formally inverse to the second-order differential operator
\begin{equation}
K = P_{11}P_{21}^{-1}P_{22} - P_{12} = \frac{w_x}{w^2}D_x^2 + \left(\frac{2w_{xx}}{w^2} - \frac{w_x^2}{w^3}\right)D_x + \left(4w_x + \frac{w_{xxx}}{w^2} - \frac{w_xw_{xx}}{w^3}\right).
\end{equation}
(73)

Note, however, that no nonlocal variables have appeared as yet. The nonlocal variable \(z\) (55) appears only when we make the factorization of the operator \(K\) (73),
\begin{equation}
K = \frac{e^{-2iz}}{w}D_x\frac{e^{4iz}}{w}D_xe^{-2iz}w_x,
\end{equation}
(74)
in order to express the inverse operator \(K^{-1}\) in a more conventional form, i.e. in terms of \(D_x^{-1}\):
\begin{equation}
K^{-1} = \frac{e^{2iz}}{w_x}D_x^{-1}e^{-4iz}wD_x^{-1}e^{2iz}w.
\end{equation}
(75)

Eventually, when we rewrite the nonlocal part of \(R\) in the form given in (13), these exponentials \(e^{2iz}\) and \(e^{-4iz}\) from \(K^{-1}\) (73) give rise to \(\sin(2z)\) and \(\cos(2z)\), while the nonlocal variable \(y\) (55) appears simply as a result of the identity \(u + yD_x - D_y = 0\) used.

7 Conclusion

Throughout the paper, we only used the cyclic bases of ZCRs as a convenient tool, well adapted to the gauge-covariant nature of ZCRs. We believe, however, that the cyclic bases of ZCRs can admit some interesting and useful geometric interpretation.

All the examples, studied in this paper, concerned evolution equations and systems of evolution equations. Of course, this is not a sign of a serious
limitation for the described technique. It is well known that a very wide class
of systems of PDEs, including at least all normal systems \[15\], can be brought
into the evolutionary form (see \[16\] for a nice explanation of this). There
are many advantages of using the evolutionary form of a studied system.
One of them, related to ZCRs, consists in that the cyclic basis of a ZCR
is constructed of the characteristic matrices and their successive covariant
\( x \)-derivatives only, because the covariant \( t \)-derivatives of the characteristic
matrices (i.e. \( \nabla_t C_u = D_tC_u - [T, C_u] \), etc.) can be expressed through the
covariant \( x \)-derivatives in the evolutionary case \[4\].

Further examples of using cyclic bases of ZCRs can be seen in \[17, 18, 19\].
In \[17\], where a ZCR with a parameter was found for the Bakirov system
that admits only one non-Lie local generalized symmetry, a gauge-invariant
coefficient of a closure equation was used as an indicator of non-removability
of the parameter, like we did it in Sections \[4-6\] of the present paper. Note,
however, that other types of gauge invariants can also be helpful in problems
of gauge non-equivalence of ZCRs and non-removability of parameters \[20\].

In \[18\], a recursion operator for a new integrable system of coupled KdV
equations was derived from the eight-dimensional cyclic basis of the sys-
tem’s ZCR. The recursion operator, obtained in \[13\], has a strange non-
local part, expressed in terms of local variables and operators \( D_x^{-1} \) and
\( (3D_x^3 - 4vD_x - 2v_x)^{-1} \). Note, however, that the nonlocal variable \( w : v = 3w_{xx}/w \)
and the factorization \( 3D_x^3 - 4vD_x - 2v_x = 3w^{-2}D_xw^2D_xw^2D_xw^{-2} \)
make it possible to express this unusual operator \( (3D_x^3 - 4vD_x - 2v_x)^{-1} \) in a
more conventional form, i.e. in terms of \( D_x^{-1} \), like we did it by \( (75) \) for the
Kersten–Krasil’shchik recursion operator.

In \[19\], the technique of cyclic bases was applied to the problem of how
to distinguish the fake Lax pairs (introduced by Calogero and Nucci) from
the true ones. A clear difference was found in the structure of cyclic bases,
namely, the closure equations in the ‘fake’ case turned out to be first-order
and typical for conservation laws, like \( (23) \). Though the fake Lax pairs are
simply gauge-transformed matrix conservation laws, it still remains not so
clear what are those true Lax pairs which are generally expected to represent
only integrable (in some reasonable sense) equations. See, in this connection,
one irreducible \( \mathfrak{sl}_2 \)-valued ZCR with an essential parameter, quoted in \[2\] as
Example 6.

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