Linear Coding Schemes for the Distributed Computation of Subspaces

V. Lalitha, N. Prakash, K. Vinodh, P. Vijay Kumar and S. Sandeep Pradhan

Abstract

Let $X_1, \ldots, X_m$ be a set of $m$ statistically dependent sources over the common alphabet $F_2$, that are linearly independent when considered as functions over the sample space. We consider a distributed function computation setting in which the receiver is interested in the lossless computation of the elements of an $s$-dimensional subspace $W$ spanned by the elements of the row vector $[X_1, \ldots, X_m] \Gamma$ in which the $(m \times s)$ matrix $\Gamma$ has rank $s$. A sequence of three increasingly refined approaches is presented, all based on linear encoders.

The first approach uses a common matrix to encode all the sources and a Korner-Marton like receiver to directly compute $W$. The second improves upon the first by showing that it is often more efficient to compute a carefully chosen superspace $U$ of $W$. The superspace is identified by showing that the joint distribution of the $\{X_i\}$ induces a unique decomposition of the set of all linear combinations of the $\{X_i\}$, into a chain of subspaces identified by a normalized measure of entropy. This subspace chain also suggests a third approach, one that employs nested codes. For any joint distribution of the $\{X_i\}$ and any $W$, the sum-rate of the nested code approach is no larger than that under the Slepian-Wolf (SW) approach. Under the SW approach, $W$ is computed by first recovering each of the $\{X_i\}$. For a large class of joint distributions and subspaces $W$, the nested code approach is shown to improve upon SW. Additionally, a class of source distributions and subspaces are identified, for which the nested-code approach is sum-rate optimal.

I. INTRODUCTION

In [2], Korner and Marton consider a distributed source coding problem with two discrete memoryless binary sources $X_1$ and $X_2$ and a receiver interested in recovering their modulo-two sum $Z = X_1 + X_2 \mod 2$. An obvious approach to this problem would be to first recover both $X_1$ and $X_2$ using a Slepian-Wolf encoder [3] and then compute their modulo-two sum thus yielding a sum-rate of $H(X_1, X_2)$. Korner and Marton present an interesting, alternative approach in which they first select a $(k \times n)$ binary matrix $A$ that is capable of efficiently compressing $Z = \sum_{i=1}^{k} A_i X_i \mod 2$, where $X_1, X_2$ and $Z$ correspond to i.i.d. $n$-tuple realizations of $X_1, X_2$ and $Z$ respectively. The two sources then transmit $AX_1$ and $AX_2$ respectively. The receiver first computes $AX_1 + AX_2 = AZ \mod 2$ and then recovers $Z$ from $AZ$. Since optimal linear compression of a finite field discrete memoryless source is possible [4], the compression rate $\frac{k}{n}$ associated with $A$ can be chosen to be as close as desired to $H(Z)$, thereby implying the achievability of the sum rate $2H(Z)$ for this problem. For a class of symmetric distributions, it is shown that this rate not only improves upon the sum rate, $H(X_1, X_2)$, incurred under the Slepian-Wolf approach, but is also optimum.

In this paper, we consider a natural generalization of the above problem when there are more than two statistically dependent sources and a receiver interested in recovering multiple linear combinations of the sources. Our interest is in finding achievable sum rates for the problem and we restrict ourselves to linear encoding in all our schemes.

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A. System Model

Consider a distributed source coding problem involving \( m \) sources \( X_1, \ldots, X_m \) and a receiver that is interested in the lossless computation (i.e., computation with arbitrarily small probability of error) of a function of these sources. All sources are assumed to take values from a common alphabet, the finite field \( \mathbb{F}_q \) of size \( q \). The sources are assumed to be memoryless and possessing a time-invariant joint distribution given by \( P_{X_1 \ldots X_m} \). We will assume this joint distribution \( P_{X_1 \ldots X_m} \) to be “linearly non-degenerate”, by which we mean that when the random variables \( \{X_i, 1 \leq i \leq m\} \) are regarded as functions over the sample space \( \Omega \), they are linearly independent, i.e.,

\[
\sum_{i=1}^{m} a_iX_i(\omega) = 0, \quad \text{all } \omega \in \Omega, \quad a_i \in \mathbb{F}_q,
\]

iff \( a_i = 0 \), all \( i \). For simplicity in notation, we will henceforth drop \( \omega \) in the notation. By identifying the linear combination \( \sum_{i=1}^{m} a_iX_i \) with the vector \([a_1 \ a_2 \ \cdots \ a_m]^T\), we see that the vector space \( V \) of all possible linear combinations of the \( \{X_i\} \) can be identified with \( \mathbb{F}_q^m \).

The function of interest at the receiver is assumed to be the set of \( s \) linear combinations \( \{Z_i \mid i = 1, 2, \ldots, s\} \) of the \( m \) sources given by:

\[
[Z_1, \ldots, Z_s] = [X_1, \ldots, X_m]\Gamma,
\]

in which \( \Gamma \) is an \((m \times s)\) matrix over \( \mathbb{F}_q \) of full rank \( s \) and where matrix multiplication is over \( \mathbb{F}_q \). Note that a receiver which can losslessly compute \( \{Z_i, \ i = 1, \ldots, s\} \) can also compute any linear combination \( \sum_{i=1}^{s} \beta_iZ_i, \ \beta_i \in \mathbb{F}_q \), of these random variables. The set of all linear combinations of the \( \{Z_i, \ i = 1, \ldots, s\} \) forms a subspace \( W \) of the vector space \( V \), which can be identified with the column space of the matrix \( \Gamma \). This explains the phrase ‘computation of subspaces ’ appearing in the title. Throughout this paper, we will interchangeably refer to the \( \{X_i, i = 1, \ldots, m\} \) as random variables (when they refer to sources) and as vectors (when they are considered as functions on the sample space). We will also write \( V = \langle X_1, \ldots, X_m \rangle \) to mean that \( V \) is generated by the vectors \( \{X_i, 1 \leq i \leq m\} \). Similarly with other random variables and their vector interpretations.

![Diagram](image)

**Fig. 1.** The common structure for all three approaches to subspace computation.

**Encoder:** All the encoders will be linear and will operate on \( n \)-length i.i.d. realizations, of the corresponding sources \( X_i, 1 \leq i \leq m \). Thus the \( i \)th encoder will map the \( n \)-length vector \( X_i \) to \( A_i^{(n)}X_i \) for some \((k_i \times n)\) matrix \( A_i^{(n)} \) over \( \mathbb{F}_q \). The rate of the \( i \)th encoder, in bits per symbol, is thus given by

\[
P_{i}^{(n)} = \frac{k_i}{n} \log q.
\]

**Receiver:** The receiver is presented with the problem of losslessly recovering the \( \{Z_i, 1 \leq i \leq s\} \), which are \( n \)-length extensions of the random variables \( \{Z_i, 1 \leq i \leq s\} \) defined in (1), from the \( \{A_iX_i, 1 \leq i \leq m\} \). Let \( W \) be the space spanned by the \( \{Z_i, 1 \leq i \leq s\} \). Then lossless recovery of the \( \{Z_i\} \) amounts to lossless computation of the subspace \( W \). Thus in the present notation, \( W \) is to \( W \) as \( Z_i \) is to \( Z_i \).

For \( 1 \leq i \leq s \), let \( \hat{Z}_i \) denote the receiver’s estimates of \( Z_i \). We will use \( P_{e}^{(n)} \) to denote the probability of error in decoding, i.e.,

\[
P_{e}^{(n)} = P \left( (Z_1 \ldots Z_s) \neq (\hat{Z}_1 \ldots \hat{Z}_s) \right).
\]
Achievability: A rate tuple \((R_1, \ldots, R_m)\) is said to be achievable, if for any \(\delta > 0\), there exists a sequence of matrix encoders \(\left\{ (A_1^{(n)}, \ldots, A_m^{(n)}) \right\}_{n=1}^{\infty}\) (and receivers) such that \(R_i^{(n)} \leq R_i + \delta, 1 \leq i \leq m\), for sufficiently large \(n\), and \(\lim_{n \to \infty} P_{e}^{(n)} = 0\). A sum rate \(R\) will be declared as being achievable, whenever \(R = \sum_{i=1}^{m} R_i\) for some achievable rate tuple \((R_1, \ldots, R_m)\). By rate region we will mean the closure of set of achievable rate \(m\)-tuples.

In situations where all encoders employ a common matrix encoder, we will then use the term minimum symmetric rate to simply mean the minimum of all values \(R\) such that the symmetric point \((R_1 = R, \ldots, R_m = R)\) lies in the rate region.

B. Our Work

In this paper, we present three successive approaches to the subspace computation problem along with explicit characterization of the corresponding achievable sum rates. As illustrated (Fig. 1), all three approaches will use linear encoders, but will differ in their selection of encoding matrices \(A_i\). We provide an overview here, of the three approaches along with a brief description of their achievable sum rates, and some explanation for the relative performance of the three approaches. Details and proofs appear in subsequent sections.

Common Code (CC) approach: Under this approach, the encoding matrices of all the sources are assumed to be identical, i.e., \(A_i = A, 1 \leq i \leq m\). It is also assumed, that the receiver decodes \([Z_1 \ldots Z_s]\) by first computing (Fig. 2)

\[
[AZ_1 \cdots AZ_s] = [AX_1 \cdots AX_m] \Gamma,
\]

and thereafter processing the \(\{AZ_i\}\). Thus the CC approach could be regarded as the analogue of the Korner-Marton approach for the modulo-two sum of two sources described earlier. The minimum symmetric rate under this approach is characterized in Theorem 1.

Selected Subspace Approach: It turns out interestingly, that compression rates can often be improved by using the CC approach to compute a larger subspace \(U \subseteq V\) that contains the desired subspace \(W\), i.e., \(W \subseteq U\). We will refer to this variation of the common code approach under which we compute the superspace \(U\) of \(W\), as the Selected Subspace (SS) approach. Thus when we speak of the SS approach, we will mean the selected-subspace variation of the common-code approach. We will present in the sequel (Theorem 5), an analytical means of determining for a given subspace \(W\), the best subspace \(U \supseteq W\) upon which to apply the CC approach. This is accomplished by showing (Theorem 3) that the joint distribution of the \(\{X_i\}\) induces a unique decomposition of the \(m\)-dimensional space \(V\) into a chain of subspaces identified by a normalized measure of entropy. Given this subspace chain, it is a simple matter to determine the optimum subspace \(U\) containing the desired subspace \(W\).

Example 1: Consider a setting where there are 4 sources, \(X_1, \ldots, X_4\) having a common alphabet \(\mathbb{F}_2\), whose

Fig. 2. The Common Code approach.
The joint distribution is described as follows:

\[
\begin{bmatrix}
  X_1 \\
  X_2 \\
  X_3 \\
  X_4 \\
\end{bmatrix} = \begin{bmatrix}
  1 & 1 & 1 & 1 \\
  0 & 1 & 1 & 1 \\
  0 & 0 & 1 & 1 \\
  0 & 0 & 0 & 1 \\
\end{bmatrix} \begin{bmatrix}
  Y_1 \\
  Y_2 \\
  Y_3 \\
  Y_4 \\
\end{bmatrix},
\]  

(5)

where \(\{Y_i\}_{i=1}^4\) are independent random variables such that \(Y_1, Y_2 \sim \text{Bernoulli}(p_1), Y_3 \sim \text{Bernoulli}(p_2), Y_4 \sim \text{Bernoulli}(\frac{1}{2})\), \(0 < p_1 < p_2 < \frac{1}{2}\). Assume that the receiver is interested in decoding the single linear combination \(Z = X_1 + X_2 + X_3 + X_4 \mod 2\). In the subspace notation, this is equivalent to decoding the one dimensional space \(W = \langle Z \rangle\). The CC approach would choose the common encoding matrix \(A\) so as to compress \(Z\) to its entropy, \(H(Z)\), thus yielding a sum rate

\[
R_{CC}^{(\text{sum})}(W) = 4H(Z) = 4H(Y_1 + Y_3).
\]

(6)

It will be shown in Theorem 3 that there is a unique subspace chain decomposition of \(V = \langle X_1, X_2, X_3, X_4 \rangle\) given by \(\emptyset \subsetneq W^{(1)} \subsetneq W^{(2)} \subsetneq W^{(3)} = V\), where

\[
\begin{align*}
  W^{(1)} &= \langle X_1 + X_2, X_2 + X_3 \rangle \\
  W^{(2)} &= \langle X_1 + X_2, X_2 + X_3, X_3 + X_4 \rangle.
\end{align*}
\]

(7)

With respect to this chain of subspaces, the best superspace \(U \supseteq W\) to consider under the SS approach is the smallest subspace in the chain that contains \(W\), which in this case, is \(U = W^{(2)}\) (Fig. 3). The sum rate in this case, identified by Theorem 3, turns out to be given by

\[
R_{SS}^{(\text{sum})}(W) = 4H(Y_3) < R_{CC}^{(\text{sum})}(W),
\]

(8)

where the second inequality in (8) follows as the \(\{Y_i\}\) are independent.

![Fig. 3. The subspace chain decomposition of \(V\) in Example 1 and its application in determining the optimal subspace \(U\) to compute \(W\), under the SS approach.](image)

**Nested Codes (NC) approach** : This approach is motivated by the subspace-chain decomposition and may be viewed as uniting under a common framework, both the CC approach of using a common linear encoder as well as the SW approach of employing different encoders at each source. To illustrate the approach, we continue to work with Example 1. Under the NC approach, the decoding happens in two stages. In the first stage the receiver, using the CC approach decodes the subspace \(W^{(1)}\). In the next stage, using \(W^{(1)}\) as side information, \(W^{(2)}\) is decoded (using a modified CC approach which incorporates side information). The encoding matrices of the various sources are as shown in Fig. 4. The matrix \(B_1\) appearing in the figure is the common encoding matrix that would be used if it was desired to compute subspace \(W^{(1)}\) alone. The block matrix \([B_1^T \quad B_2^T]^T\) is the common encoder that would have been used if one were only interested in computing the complement of \(W^{(1)}\) in \(W^{(2)}\) with \(W^{(1)}\) as side information. It can be shown that there is a rearrangement of the \(\{X_i\}\) under which the complement of \(W^{(1)}\) in \(W^{(2)}\) can be made to be a function only of two of the random variables which, we have assumed without loss of generality here to be \(X_3, X_4\). This explains why the submatrix \(B_2\) appears only in the encoding of sources \(X_3, X_4\).
The sum rate in this case turns out to be given by
\[ R^{(\text{sum})}_{\text{NC}}(W) = 2H(Y_1) + 2H(Y_3), \]
which can be shown to be less than \( R^{(\text{sum})}_{\text{SS}}(W) \) as well as the sum rate, \( R^{(\text{sum})}_{\text{SW}}(W) = H(X_1, X_2, X_3, X_4) \), of the Slepian-Wolf (SW) approach under which the subspace \( W \) is computed by first recovering each of the four random variables \( \{X_i\} \).

A graphical depiction of the sum rates achieved by the various schemes is provided in Fig. 5 with the sum rates appearing on the vertical axis on the far right. It turns out that in general, we have
\[
R^{(\text{sum})}_{\text{CC}}(W) \geq R^{(\text{sum})}_{\text{SS}}(W) \geq R^{(\text{sum})}_{\text{NC}}(W),
\]
\[
R^{(\text{sum})}_{\text{SW}}(W) \geq R^{(\text{sum})}_{\text{NC}}(W).
\]

In the particular case of the example, we have that there exist choices of probabilities \( p_1, p_2 \) such that
\[
R^{(\text{sum})}_{\text{SW}}(W) > R^{(\text{sum})}_{\text{CC}}(W) > R^{(\text{sum})}_{\text{SS}}(W) > R^{(\text{sum})}_{\text{NC}}(W).
\]

C. Other Related Work

Some early work on distributed function computation can be found in [4], [5], [6]. In [4], the general problem of distributed function computation involving two sources is considered and the authors identify conditions on the
function such that the SW approach itself is optimal. In [5], the authors address the Korner-Marton problem of computing the modulo-two sum of two binary sources and produce an achievable-rate pair which is strictly outside the time shared region of the SW and Korner-Marton rate regions. \[^{[4]}\] For the same problem it is shown in [6], that if \( H(X_1 + X_2) \geq \min(H(X_1), H(X_2)) \), then the SW scheme is sum-rate optimal. In [7], the authors showed how linear encoders are suitable for recovery of functions that are representable as addition operations within an Abelian group.

The problem of compressing a source \( X_1 \), when \( X_2 \) is available as side information to the receiver, and where the receiver is interested in decoding a function \( f(X_1, X_2) \) under a distortion constraint, is studied in [8]. For the case of zero distortion, the minimum rate of compression is shown to be related to the conditional graph entropy of the corresponding characteristic graph in [9]. The extension of the nonzero distortion problem for the case of noisy source and side information measurements is investigated in [10].

In [11], Doshi et al. consider the lossless computation of a function of two correlated, but distributed sources. They present a two-stage architecture, wherein in the first stage, the input sequence at each source is divided into blocks of length \( n \) and each block is coloured based on the corresponding characteristic graph at the source. In the second stage, SW coding is used to compress the coloured data obtained at the output of first stage. The achievable rate region using this scheme is given in terms of a multi-letter characterization and the optimality of the scheme is shown for a certain class of distributions. In [12], the authors derive inner and outer bounds for lossless compression of two distributed sources \( X, Y \) to recover a function \( f(X, Y, Z) \), when \( Z \) is available as side information to the receiver. The bound is shown to be tight for partially invertible functions, i.e., for functions \( f \) such that \( X \) is a function of \( f(X, Y, Z) \) and \( Z \).

For the case of two distributed Gaussian sources, computation of a linear combination of the sources is studied in [13], [14], wherein lattice-based schemes are shown to provide a rate advantage. Zero-error function computation in a network setting, has been investigated in [15], [16], [17].

A notion of normalized entropy is introduced in Section II. Section III discusses the rate regions under the CC and SS approaches. The unique decomposition of the \( m \)-dimensional space \( V \) into a chain of subspaces identified by a normalized measure of entropy, is presented in Section IV. It is shown how this simplifies determination of the minimum symmetric rate under the SS approach. An example subspace computation along with an attendant class of distributions for which the SS approach is optimal, are also presented here. The nested-code approach is presented in Section V along with conditions under which this approach improves upon the SW approach as well as examples for which the NC approach is sum-rate optimal. Most proofs are relegated to appendix.

### II. Normalized Entropy

We will use \( \rho_U \) to denote the dimension of a subspace \( U \).

**Entropy of a subspace:** To every subspace \( U \) of \( V \), we will associate an entropy, which is the entropy of any set of random variables that generate \( U \). We will denote this quantity by \( \mathcal{H}(U) \) and refer to this quantity loosely as the entropy of the subspace \( U \). Thus, if \( U = \{ Y_1, \ldots, Y_{\rho_U} \} \),

\[
\mathcal{H}(U) = H(Y_1, \ldots, Y_{\rho_U}).
\]  

\( \mathcal{H}(U) \) can also be viewed as the joint entropy of the collection of all random variables contained in the subspace \( U \), i.e., \( \mathcal{H}(U) := H(\{ U \}) \). Next, given any two subspaces \( U_1 \) and \( U_2 \), we define the conditional entropy of the subspace \( U_2 \) conditioned on \( U_1 \) as

\[
\mathcal{H}(U_2|U_1) \triangleq H(\{ U_2 \}|\{ U_1 \}).
\]  

Let \( U_1 + U_2 \) denote the sum space of \( U_1, U_2 \). Clearly, \( \mathcal{H}(U_1 + U_2) = H(\{ U_1 \}, \{ U_2 \}) \). Hence we can rewrite the above equation as

\[
\mathcal{H}(U_2|U_1) = \mathcal{H}(U_1 + U_2) - \mathcal{H}(U_1).
\]  

\(^1\)The sum rate of this achievable rate-pair is however, still larger than the minimum of the SW and Korner-Marton sum rates. 

\(^2\)We have used \( \mathcal{H}(U) \) in place of \( H(U) \) so as to avoid confusion with the entropy of a random variable whose every realization is a subspace.
Normalized entropy: We define the normalized entropy $\mathcal{H}_N(U)$ of a non-zero subspace $U$ of $V$ as the entropy of $U$ normalized by its dimension i.e.,

$$\mathcal{H}_N(U) \triangleq \frac{\mathcal{H}(U)}{\rho_U}. \quad (12)$$

For any pair of subspaces $U_1, U_2$, $U_2 \nsubseteq U_1$, we define the normalized, conditional entropy of $U_2$ conditioned on $U_1$, to be given by

$$\mathcal{H}_N(U_2|U_1) \triangleq \frac{\mathcal{H}(U_2|U_1)}{\rho_{U_2} - \rho_{U_1 \cap U_2}}. \quad (13)$$

Note that since $\rho_{U_1+U_2} = \rho_{U_1} + \rho_{U_2} - \rho_{U_1 \cap U_2}$, we can equivalently write

$$\mathcal{H}_N(U_2|U_1) \triangleq \frac{\mathcal{H}(U_2 + U_1) - \mathcal{H}(U_1)}{\rho_{U_2} + \rho_{U_1} - \rho_{U_1}}. \quad (14)$$

![Diagram of normalized entropies](image)

**Fig. 6.** An illustration of normalized entropies

The above definitions are illustrated in Fig. 6 where the $x$-axis corresponds to the dimension of subspaces and $y$-axis corresponds to the entropy of subspaces. The slope of the line $L_1$ is the normalized entropy, $\mathcal{H}_N(U_1)$, of $U_1$ and the slope of the line $L_2$ is the normalized conditional entropy, $\mathcal{H}_N(U_2|U_1)$.

### III. COMMON CODE AND SELECTED SUBSPACE APPROACHES

The minimum symmetric rate of the CC and the SS approaches to the distributed subspace computation problem described in Section II-B are presented here.

#### A. Rate Region Under the CC Approach

**Theorem 1:** Consider the distributed source coding setting shown in Fig. 2 where there are $m$ correlated sources $X_1, \ldots, X_m$ and receiver that is interested in decoding the $s$ dimensional subspace $W$ corresponding to the space spanned by the set $\{Z_i\}$ of random variables defined in (1). Then minimum symmetric rate under the CC approach is given by

$$R_{CC}(W) = \max_{W_1 \subseteq W} \mathcal{H}_N(W|W_1). \quad (15)$$

**Proof:** See Appendix A.

The best sum rate $R_{CC}^{(sum)}(W)$ under the CC approach is given by $R_{CC}^{(sum)}(W) = mR_{CC}(W)$. Note that in the special case when the receiver is interested in just a single linear combination, $Z$, of all the sources, the sum rate is simply $mH(Z)$. The following example illustrates the minimum symmetric rate for the case, when the receiver is interested in decoding a two dimensional subspace.
Example 2: Let \( m = 3 \) and the source alphabet be \( \mathbb{F}_2 \). Consider a receiver interested in computing \( Z_1 = X_1, Z_2 = X_2 + X_3 \) i.e, \( W = \langle X_1, X_2 + X_3 \rangle \). Then the minimum symmetric rate under the CC approach is given by

\[
R_{CC}(W) = \max \left\{ \frac{H(Z_1, Z_2)}{2}, H(Z_1, Z_2|Z_1), H(Z_1, Z_2|Z_2), H(Z_1, Z_2|Z_1 + Z_2) \right\}.
\]

(16)

Remark 1: The CC approach is sum-rate optimal for the case when \( W = V \) iff

\[
\mathcal{H}_N(V) \geq \mathcal{H}_N(V|V_1), \quad \forall V_1 \subseteq V.
\]

This follows directly from Theorem 1 by setting \( W = V \) and noting that the optimal sum-rate in this case is simply \( H(X_1, \cdots, X_m) \).

B. Rate Region Under the SS Approach

This approach recognizes that it is often more efficient to compute a superspace of \( W \) rather than \( W \) itself. The identification of the particular superspace that offers the greatest savings in compression rate is taken up in Section IV.

Theorem 2: Under the same setting as in Theorem 1, the minimum symmetric rate under the SS approach is given by

\[
R_{SS}(W) = \min_{U \supseteq W} \max_{U_1 \subseteq U} \mathcal{H}_N(U|U_1).
\]

(18)

Proof: Follows directly from Theorem 1.

The (best) sum rate \( R_{SS}^{(sum)}(W) \) under the SS approach is given by \( R_{SS}^{(sum)}(W) = mR_{SS}(W) \). Any subspace \( U \supseteq W \) which minimizes \( \max_{U_1 \subseteq U} \mathcal{H}_N(U|U_1) \) will be referred to as an optimal subspace for computing \( W \) under the SS approach. There can be more than one optimal subspace associated with a given \( W \).

IV. A DECOMPOSITION THEOREM FOR THE VECTOR SPACE \( V \) BASED ON NORMALIZED ENTROPY

While the results of this section are used to identify the superspace \( U \) that minimizes the quantity \( \max_{U_1 \subseteq U} \mathcal{H}_N(U|U_1) \) appearing in Theorem 2 they are also of independent interest as they exhibit an interesting interplay between linear algebra and probability theory. Also included in this section, are example subspace-computation problems and a class of distributions for which the SS approach is sum-rate optimal, while the CC and the SW approaches are not.

Theorem 3 (Normalized-Entropy Subspace Chain): In the vector space \( V \), there exists for some \( r \leq m \), a unique, strictly increasing sequence of subspaces \( \{0\} = W^{(0)} \subset W^{(1)} \subset \cdots \subset W^{(r)} = V \), such that, \( \forall j \in \{1, \ldots, r\} \),

1) amongst all the subspaces of \( V \) that strictly contain \( W^{(j-1)} \), \( W^{(j)} \) has the least possible value of normalized conditional entropy conditioned on \( W^{(j-1)} \) and

2) if any other subspace that strictly contains \( W^{(j-1)} \) also has the least value of normalized conditional entropy conditioned on \( W^{(j-1)} \), then that subspace is strictly contained in \( W^{(j)} \).

Furthermore,

\[
\mathcal{H}_N(W^{(1)}|W^{(0)}) < \mathcal{H}_N(W^{(2)}|W^{(1)}) < \cdots < \mathcal{H}_N(W^{(r)}|W^{(r-1)}).
\]

(19)

Proof: See Appendix B.

We illustrate Theorem 3 below, by identifying the chain of subspaces \( \{W^{(j)}\} \) for the case when the random variables \( X_1, \ldots, X_m \) are derived via an invertible linear transformation of a set of \( m \) statistically independent random variables \( Y_1, \ldots, Y_m \).
Lemma 4: Let \([X_1, \ldots, X_m] = [Y_1, \ldots, Y_m]G\), where \(G\) is an \((m \times m)\) invertible matrix over \(\mathbb{F}_q\) and \(\{Y_i, i = 1, \ldots, m\}\) are \(m\) independent random variables, each of which takes values in the finite field \(\mathbb{F}_q\). Without loss of generality, let the entropies of \(\{Y_i, i = 1, \ldots, m\}\) be ordered according to
\[
0 < H(Y_1) = \ldots = H(Y_{\ell_1}) < H(Y_{\ell_1+1}) = \ldots = H(Y_{\ell_r+1}) < \ldots < H(Y_{\sum_{i=1}^r \ell_i+1}) = \ldots = H(Y_{\sum_{i=1}^r \ell_i}),
\]
where \(1 \leq \ell_i \leq m, i = 1, \ldots, r\) and \(\sum_{i=1}^r \ell_i = m\). Then, the unique subspace chain identified by Theorem 3 is given by
\[
\{0\} \subset < Y_1, \ldots, Y_{\ell_1} > \subset < Y_1, \ldots, Y_{\ell_1+\ell_2} > \subset \ldots \subset < Y_1, \ldots, Y_m >.
\]

Proof: See Appendix C.  

Remark 2: While Theorem 3 guarantees the existence of \(r\), the above lemma shows that \(r\) can take any value between 1 and \(m\) depending on the joint distribution of the \(\{X_i, 1 \leq i \leq m\}\).

A. Identifying the Optimal Subspace Under the SS Approach

Theorem 5 (Optimal Rate under SS approach): Consider the distributed source coding problem shown in Fig. 2 having \(m\) sources \(X_1, \ldots, X_m\) and a receiver that is interested in decoding the \(s\) dimensional subspace \(W\). Let \(W^{(0)} \subset W^{(1)} \subset \ldots \subset W^{(r)}\) be the unique subspace-chain decomposition of the vector space \(V = < X_1, \ldots, X_m >\), identified in Theorem 3. Then an optimal subspace for decoding \(W\) under the SS approach is given by \(U = W^{(j_0)}\), where \(j_0\) is the unique integer, \(1 \leq j_0 \leq r\), satisfying
\[
W \subset W^{(j_0)}, \quad W \not\subset W^{(j_0-1)}.
\]
Furthermore,
\[
R_{SS}(W) = \mathcal{H}_N(W^{(j_0)}|W^{(j_0-1)}).
\]

Proof: See Appendix D.  

Corollary 6: With the \(W^{(j)}\), \(1 \leq j \leq r\), as above,
\[
R_{SS}(W^{(j)}) = \mathcal{H}_N(W^{(j)}|W^{(j-1)}),
\]
\[
R_{SS}(W^{(1)}) < R_{SS}(W^{(2)}) < \ldots < R_{SS}(W^{(r)}).
\]

B. A Subspace Computation Problem for which SS Approach is Sum-Rate Optimal

Consider the setting where there are \(m\) sources \(X_1, \ldots, X_m\) having a common alphabet \(\mathbb{F}_2\), with \(m\) even, and a receiver interested in computing the sum \(Z = (X_1 + \ldots + X_m) \mod 2\). Let the joint distribution of the \(\{X_i, 1 \leq i \leq m\}\) be specified as follows:
\[
\begin{bmatrix}
X_1 \\
X_2 \\
\vdots \\
X_{m-1} \\
X_m
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & \ldots & 0 \\
1 & 1 & 0 & \ldots & 0 \\
\vdots & & & & \\
1 & 1 & \ldots & 1 & 0 \\
1 & 1 & \ldots & 1 & 1
\end{bmatrix}
\begin{bmatrix}
Y_1 \\
Y_2 \\
\vdots \\
Y_{m-1} \\
Y_m
\end{bmatrix},
\]
where the \(\{Y_i\}_{i=1}^m\) are statistically independent random variables such that \(Y_i \sim \text{Bernoulli}\left(\frac{1}{2}\right)\), for \(i\) odd and \(Y_i \sim \text{Bernoulli}(p), 0 < p < \frac{1}{2}\), for \(i\) even. When \(m = 2\), \(X_1\) and \(X_2\) can be verified to possess a doubly-symmetric joint distribution (see [2]), and this is precisely the class of distribution for which Korner and Marton showed that a common linear encoder is sum-rate optimal for the computation of the modulo-2 sum, \(Z = X_1 + X_2\). We now assume \(m > 2\) in the above setting and show that the SS approach yields optimal sum rate while the CC or SW approach do not. Note that by optimal sum rate we mean that this is best sum rate that can be achieved for the subspace computation problem, even if encoders other than linear encoders were permitted in Fig. 1.
From Lemma 4 we know that the unique subspace chain for \( V = < X_1, \ldots, X_m > \) is given by \( \{0\} \subseteq W^{(1)} \subseteq W^{(2)} \), where

\[
W^{(1)} = < Y_2, Y_4, \ldots, Y_m > = < X_1 + X_2, X_3 + X_4, \ldots, X_{m-1} + X_m >
\]

and \( W^{(2)} = V \). Clearly, the subspace of interest \( W = < X_1 + X_2 + \ldots + X_m > \subseteq W^{(1)} \) and hence by Theorem 5 \( W^{(1)} \) is an optimal subspace to decode \( W \), under the SS approach. The minimum symmetric rate is given by

\[
R_{SS}(W) = H_N(W^{(1)}) = \frac{H(Y_2, Y_4, \ldots, Y_m)}{\binom{m}{2}} = h(p),
\]

yielding a sum rate \( R_{SS}^{(sum)} = mh(p) \).

Now, under the CC approach, since we directly decode the single linear combination \( Z \), the sum rate is given by (Theorem 1)

\[
R_{CC}^{(sum)} = mH(X_1 + \ldots + X_m) \\
\text{subject to } (a) \quad mH(Y_2 + Y_4 + \ldots + X_m) \geq R_{SS}^{(sum)},
\]

where (a) follows from (23) and (b) follows since \( \{Y_i\} \) are independent and \( p < \frac{1}{2} \). Also, under the SW approach in which the whole space \( V \) is first decoded before computing \( W \), the sum rate is given by

\[
R_{SW}^{(sum)} = H(Y_1, \ldots, Y_m) = \frac{m}{2}(1 + h(p)) > R_{SS}^{(sum)}.
\]

We now show that the SS approach is sum-rate optimal. If \( (R_1, \ldots, R_m) \) is any achievable rate tuple, then \( \forall i = 1, \ldots, m \), it must be true that

\[
R_i \geq H(X_1 + \ldots + X_m | X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_m) \\
= H(X_i | X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_m) \geq H(Y_1 + \ldots + Y_i | Y_1, \ldots, Y_{i-1}, X_{i+1}, \ldots, X_m) \geq H(Y_i | Y_1, \ldots, Y_{i-1}, Y_i + Y_{i+1}, Y_{i+2}, \ldots, Y_m) \geq \begin{cases} H(Y_i Y_{i+1}), & i < m \\ H(Y_i), & i = m \end{cases} \geq h(p),
\]

where (a) follows by considering a system in which we give \( \{X_1, \ldots, X_m\} \setminus \{X_i\} \) as side information at the receiver, (b), (c) follow from (23) and (d) follows from the independence of the \( \{Y_i, i = 1, \ldots, m\} \). The bound in (24) holds true for all sources and hence \( \sum_{i=1}^{m} R_i \geq mh(p) \). Since \( R_{SS}^{(sum)} = mh(p) \), it follows that the SS approach is sum-rate optimal.

V. NESTED CODES APPROACH

The NC approach to the subspace computation problem is a natural outgrowth of our decomposition theorem for the vector space \( V \). Under this approach, a sequential decoding procedure is adopted in which \( W^{(j)} \) is decoded using \( W^{(j-1)} \) as side information.

A. CC APPROACH WITH SIDE INFORMATION

As before, we have a receiver that is interested in computing a subspace \( W \) of \( V \) with the difference this time, that the receiver possesses knowledge of a subspace \( S \) of \( W \), \( S = < Y_1, \ldots, Y_{\rho S} > \) as side information. Let \( T \) be a subspace of \( W \) complementary to \( S \) in \( W \), i.e., \( W \) is a direct sum of \( S \) and \( T \) which is denoted by \( W = S \oplus T \). Then clearly, it suffices to compute \( T \) given \( S \) as side information.

We claim that there exists a complement \( T \) of \( S \) in \( W \) which is a function of at most \( (m - \rho_S) \) of the sources. This follows from noting that a basis for \( S \) can be extended to a basis for \( V \) by adding \( (m - \rho_S) \) of the \( \{X_i\} \). Without loss of generality, we may assume that these are the random variables \( X_{\rho_S+1}, \ldots, X_m \). Also, the intersection of
Fig. 7. CC approach for decoding with side information, when side information is linearly independent of the sources.

$W$ with any complement of $S$ in $V$ is clearly a complement of $S$ in $W$. It follows that there is a complement $T$ of $S$ in $W$ which is only a function of the $(m - \rho_S)$ sources $X_{\rho_S+1}, \ldots, X_m$ and thus it is enough to encode the sources $X_{\rho_S+1}, \ldots, X_m$.

We adopt the CC approach here and hence, $n$-length realizations of all the $(m - \rho_S)$ sources $X_{\rho_S+1}, \ldots, X_m$ are encoded by a common matrix encoder $A$. Now, if $T = \langle [X_{\rho_S+1}, \ldots, X_m] \Gamma_T \rangle$ for some $(m - \rho_S) \times \rho_T$ matrix $\Gamma_T$ of rank $\rho_T$, then the receiver, as a first step, multiplies the received matrix $[AX_{\rho_S+1} \ldots AX_m]$ by $\Gamma_T$ on the right. These are then decoded using the side information $Y_1, \ldots, Y_{\rho_S}$ to obtain estimates of $T$ (see Fig. 7). The minimum symmetric rate for this approach is presented below.

**Theorem 7:** Consider a distributed source coding problem, where the receiver is interested in computing the subspace $W$, given that $S \subsetneq W$ is available as side information to the receiver. The minimum symmetric rate under the CC based approach presented above, is given by

$$R_{CC}(W|S) = \max_{T_1 \subseteq T} \{H_N(T|T_1 \oplus S)\}$$

$$= \max_{W_i \subseteq W \text{ s.t. } W_i \supseteq S} \{H_N(W|W_i)\}. \quad (25)$$

**Proof:** Similar to the proof of Theorem 1. \[ \Box \]

Note from (26) that irrespective of the particular complementary subspace $T$ that we choose to compute, the symmetric rate remains the same. The specific choice of $T$ determines however, the number of $\{X_i, i = 1 \ldots m\}$ that are actually encoded. Since $T$ has been selected such that only $(m - \rho_S)$ sources are encoded, the achievable sum rate in this case, is given by $R_{CC}^{(sum)}(W|S) = (m - \rho_S)R_{CC}(W|S)$.

**Corollary 8:** Consider the case where $W = W^{(j)}$ and $S = W^{(j-1)}$. Then

$$R_{CC}(W^{(j)}|W^{(j-1)}) = \max_{U_i \subseteq W^{(j)}} \{H_N(W^{(j)}|U_1)\}$$

$$= H_N(W^{(j)}|W^{(j-1)}), \quad (27)$$

where the last equality follows since

$$H_N(W^{(j)}|W^{(j-1)}) \leq \max_{U_i \subseteq W^{(j)}} \{H_N(W^{(j)}|U_1)\}$$

$$\leq \max_{U_i \subseteq W^{(j)}} \{H_N(W^{(j)}|U_1)\}$$

$$\equiv \max_{U_i \subseteq W^{(j)}} \{H_N(W^{(j)}|U_1)\} \quad \text{(a)}$$

$$H_N(W^{(j)}|W^{(j-1)}), \quad (28)$$

where (a) follows from Corollary 6. Thus, $R_{CC}(W^{(j)}|W^{(j-1)}) = R_{CC}(W^{(j)})$, i.e., the rates per encoder are the same in this instance with and without side information. The difference between the two cases is that in the presence of side information, we need encode only $(m - \rho_{W^{(j-1)}})$ sources as opposed to $m$ leading to a reduced sum rate by the fraction $\frac{(m - \rho_{W^{(j-1)}})}{m}$. 
B. NC Approach for Subspace Computation

Consider the chain of subspaces \( W^{(1)} \subseteq \ldots \subseteq W^{(r)} \) as obtained from Theorem 3. Assume that we are interested in decoding the subspace \( W^{(j)}, j \leq r \). We will now describe a scheme for decoding \( W^{(j)} \), that operates in \( j \) stages. At stage \( \ell, \ell \leq j \), we decode \( W^{(\ell)} \) using \( W^{(\ell-1)} \) as side information, using the CC based approach described above in Section V-A. Using the same argument as in Section V-A, it follows that at stage \( \ell, 1 \leq \ell \leq j \), without loss of generality, it is enough to encode the sources \( X_{m-\rho_{W^{(\ell-1)}}+1}, \ldots, X_m \).

From Corollary 3, the rate of each of the sources that are encoded in the \( \ell \)th stage is given by

\[
R_\ell \equiv R_{CC}(W^{(\ell)}|W^{(\ell-1)}) = H_N(W^{(\ell)}|W^{(\ell-1)}). \tag{29}
\]

Also, let \( A^{(\ell)} \) denote the common encoding matrix used in the \( \ell \)th stage. Since \( R_1 < R_2 < \ldots < R_j \) (see Theorem 3), it can be shown, via a random coding argument and by invoking a union-bound argument on the probability of error calculation, that it is possible to choose the encoding matrices \( A^{(1)}, \ldots, A^{(j)} \) having the following nested structure:

\[
A^{(1)} = [B_1], A^{(2)} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \ldots, A^{(j)} = \begin{bmatrix} B_1 \\ \vdots \\ B_j \end{bmatrix}. \tag{30}
\]

Please see Appendix E for a proof of this statement.

Thus, the sum rate for decoding the subspace \( W^{(j)} \), under the NC approach, is given by

\[
R_{NC}^{(\text{sum})}(W^{(j)}) = \sum_{\ell=1}^{j-1} (\rho_{W^{(\ell)}} - \rho_{W^{(\ell-1)}}) H_N(W^{(\ell)}|W^{(\ell-1)}) + (m - \rho_{W^{(j-1)}}) H_N(W^{(j)}|W^{(j-1)}) = H(W^{(j)}) + (m - \rho_{W^{(j-1)}}) H_N(W^{(j)}|W^{(j-1)}). \tag{31}
\]

As in the case of the SS approach, a scheme for decoding an arbitrary subspace \( W \) under the NC approach would be to decode the subspace \( W^{(j_0)} \), where \( j_0 \) is the unique integer such that \( W \subseteq W^{(j_0)} \) and \( W \subseteq W^{(j_0-1)} \).

Note that whereas the one-stage CC approach for decoding \( W^{(j)} \) would have used the highest-rate matrix \( A^{(j)} \) for all the sources, the NC approach uses it only for sources \( X_{\rho_{W^{(j-1)}}+1}, \ldots, X_m \) and uses lower-rate matrices for the remaining sources. Thus the NC approach clearly outperforms the SS approach for all subspaces with the exception of \( W^{(1)} \). Even beyond this, the NC approach sum rate improves upon the SW sum rate for all subspaces \( W \subseteq W^{(r-1)} \), while in all other cases it equals the SW sum rate. These comparisons are made explicit in the two theorems below.

**Theorem 9:** The sum rate \( R_{NC}^{(\text{sum})}(W^{(j)}) \) incurred in using the nested code approach for decoding the subspace \( W^{(j)}, 1 \leq j \leq r \) satisfies \( R_{NC}^{(\text{sum})}(W^{(j)}) \leq mR_{SS}(W^{(j)}) \), the sum rate for decoding \( W^{(j)} \) using the SS approach. Equality holds iff \( j = 1 \).

**Proof:**

\[
R_{NC}^{(\text{sum})}(W^{(j)}) = \sum_{\ell=1}^{j-1} (\rho_{W^{(\ell)}} - \rho_{W^{(\ell-1)}}) H_N(W^{(\ell)}|W^{(\ell-1)}) + (m - \rho_{W^{(j-1)}}) H_N(W^{(j)}|W^{(j-1)}) \\ \leq \sum_{\ell=1}^{j-1} (\rho_{W^{(\ell)}} - \rho_{W^{(\ell-1)}}) H_N(W^{(j)}|W^{(j-1)}) + (m - \rho_{W^{(j-1)}}) H_N(W^{(j)}|W^{(j-1)}) \\ = mH_N(W^{(j)}|W^{(j-1)}) = mR_{SS}(W^{(j)}), \tag{32}
\]

\(^3\)Similar proofs regarding existence of nested linear codes have been shown in the past, for example see [7].
where \( (a) \) follows by Theorem 5. Since the rates given in Theorem 5 are strictly increasing, \( (a) \) is an equality iff \( j = 1 \).

**Theorem 10:** The sum rate \( R_{\text{NC}}^{(\text{sum})}(W^{(j)}) \) incurred in using the nested code approach for decoding the subspace \( W^{(j)} \), \( 1 \leq j \leq r \) satisfies \( R_{\text{NC}}^{(\text{sum})}(W^{(j)}) \leq H(V) \), the sum rate for decoding \( W^{(j)} \) using the SW approach. Equality occurs iff \( j = r \).

**Proof:**

\[
H(V) = H(W^{(j-1)}) + \sum_{\ell=j}^{r} H(W^{(\ell)}|W^{(\ell-1)}) \\
= H(W^{(j-1)}) + \sum_{\ell=j}^{r} (\rho_{W^{(\ell)}} - \rho_{W^{(\ell-1)}}) H_N(W^{(\ell)}|W^{(\ell-1)}) \\
\overset{(a)}{\succeq} H(W^{(j-1)}) + \sum_{\ell=j}^{r} (\rho_{W^{(\ell)}} - \rho_{W^{(\ell-1)}}) H_N(W^{(j)}|W^{(j-1)}) \\
= H(W^{(j-1)}) + (m - \rho_{W^{(j-1)}}) H_N(W^{(j)}|W^{(j-1)}) \\
= R_{\text{NC}}^{(\text{sum})}(W^{(j)}),
\]

where \( (a) \) follows from Theorem 5. Since the rates given in Theorem 5 are strictly increasing, \( (a) \) holds with equality iff \( j = r \).

**C. An example subspace computation problem for which NC approach is optimal**

We now revisit Example 1 introduced in Section I and show that the NC approach is sum-rate optimal if the subspace of interest is \( W = W^{(2)} \). It is not hard to show that the subspace chain decomposition for the joint distribution in Example 1 is indeed as given in (7). Thus, from (31), the sum rate achievable using the NC scheme is given by

\[
R_{\text{NC}}^{(\text{sum})}(W^{(2)}) = H(W^{(1)}) + (m - \rho_{W^{(1)}}) H_N(W^{(2)}|W^{(1)}) \\
= 2h(p_1) + 2h(p_2).
\]

To show sum-rate optimality, note that if \( (R_1, R_2, R_3, R_4) \) is any achievable rate tuple, then we must have

\[
R_4 \overset{(a)}{\geq} H(X_1 + X_2, X_2 + X_3, X_3 + X_4|X_1, X_2, X_3) \\
= H(X_4|X_1, X_2, X_3) \overset{(b)}{=} H(Y_4|Y_1, Y_2, Y_3 + Y_4) \overset{(c)}{=} H(Y_4|Y_3 + Y_4) = h(p_2),
\]

where \( (a) \) follows by a considering a system in which \( X_1, X_2, X_3 \) is given as side information, \( (b) \) follows from (5) and \( (c) \) follows from the independence of \( \{Y_i, i = 1, \ldots, 4\} \). Next, if we consider a second system in which \( X_4 \) alone is given as side information, then it must be that

\[
R_1 + R_2 + R_3 \geq H(X_1 + X_2, X_2 + X_3, X_3 + X_4|X_4) \\
= H(Y_1, Y_2, Y_3|Y_4) \overset{(c)}{=} h(p_2) + 2h(p_1).
\]

Combining (34) and (35), we get the lower bound on the sum rate given by \( R_1 + R_2 + R_3 + R_4 \geq 2h(p_1) + 2h(p_2) \). This, along with (33) implies sum-rate optimality of the NC approach. Note from Theorems 9 and 10 that the subspace and the SW approaches are both strictly suboptimal in this case.
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APPENDIX A

PROOF OF THEOREM 1

Before proceeding to prove the theorem, for notational simplicity we shall denote $[Z_1, \ldots, Z_s]$ by $Z^{[1:s]}$ and assume that $W_1$ is generated by $Z^{[1:s]}G$, where $G$ is a full rank $s \times \nu$ matrix, where $\nu = \rho_{W_1}$. Thus, the set of achievable rates per encoder under the CC approach given by Theorem 1 can be rewritten as follows.

$$R_{CC}(W) = \left\{ R \mid R \geq \frac{1}{(s-\nu)} H(Z^{[1:s]} \mid Z^{[1:s]}G) \right\} \quad (36)$$

for every choice of $G$, whose column space corresponds to a $\nu$-dimensional subspace of $\mathbb{F}_q^s$, $0 \leq \nu \leq s - 1$.

In order to prove the theorem, we shall work with the system model shown in Fig. 8 which is equivalent to that of CC approach. This system is same as SW system except that all the sources are encoded by a common matrix $A$. A rate $R$ per encoder is achievable in this equivalent system iff it is achievable in the original system of interest (see Fig. 2).

**Achievability:** The notion of typical sets will be required to prove the achievability. The following definition of $\varepsilon$—typical set of a random variable $X$ will be used:

$$A^{(n)}_\varepsilon(X) = \left\{ x : \left| \frac{N(a|x)}{n} - P_X(a) \right| \leq \varepsilon P_X(a), \forall a \in \mathcal{X} \right\}, \quad (37)$$

where $\mathcal{X}$ is the alphabet of the random variable $X$, $N(a|x)$ denotes the number of occurrences of the symbol $a$ in the realization $x$. We refer the reader to [18] for properties of this typical set as well as the related notions of conditional and joint typical sets. We shall make use of the following lemma in the proof of the achievability.
Lemma 11: Let $X$ and $Y$ be two discrete random variables taking on values over a finite alphabet $\mathcal{X}$ and $\mathcal{Y}$ respectively. Let $Y = f(X)$ be a deterministic function of $X$. Let $x$ be an $n-$length realization of the i.i.d. random variable $X$ and $y = f^n(x)$, where $f^n(y) = (f(y_1), \ldots, f(y_n))$. Then we have

$$\{x \in A_e^{(n)}(X) \mid f^n(x) = y\} = A_e^{(n)}(X|y).$$

(38)

Proof: The proof can be shown by using the definition of the typical set as given by (37).

Achievability will be shown using a random coding argument by averaging over the set of all matrix encoders of the form $A : \mathbb{F}_q^n \rightarrow \mathbb{F}_q^k$, where the $k \times n$ matrix $A$ is assumed to be a realization of the random matrix $A$, distributed uniformly on the ensemble $\mathcal{M}_{k \times n}$. We will apply the joint typical set decoder and calculate the probability of error $P_e^{(n)}$ averaged over all the source symbols and also over all realizations of $A$.

Let the source sequences to be $z^{[1:s]}$. Then the decoder will declare $\hat{z}^{[1:s]}$ to be the transmitted sequence if it is the unique sequence that belongs to $A_e^{(n)}(Z^{[1:s]})$ and $A\hat{z}^{[1:s]} = Az^{[1:s]}$. Thus, the decoder will make an error if any one of the following events happen:

$$E_1 : z^{[1:s]} \notin A_e^{(n)}(Z^{[1:s]})$$

$$E_2 : \exists v^{[1:s]} \in A_e^{(n)}(Z^{[1:s]}) \text{ such that } v^{[1:s]} \neq z^{[1:s]} \text{ and } Av^{[1:s]} = Az^{[1:s]}.$$  

(39)

(40)

Let us denote $\Delta^{[1:s]} = v^{[1:s]} - z^{[1:s]}$. Then, the probability of error is upper bounded as

$$P_e^{(n)} \leq P(E_1) + P(E_2) \leq \delta_n + \sum_{z^{[1:s]} \in A_e^{(n)}(Z^{[1:s]})} \sum_{\substack{v^{[1:s]} \in A_e^{(n)}(Z^{[1:s])}: \\ v^{[1:s]} \neq z^{[1:s]}}} P(A \Delta^{[1:s]} = 0),$$

(41)

where $\delta_n \rightarrow 0$ as $n \rightarrow \infty$. We will now compute $P(A \Delta^{[1:s]} = 0)$ as follows. Let $N(\Delta^{[1:s]})$ be the nullspace of $\Delta^{[1:s]}$ and $\nu$ be its rank. Since $\Delta^{[1:s]} \neq 0$, we have $0 \leq \nu \leq s - 1$. The rank of $\Delta^{[1:s]}$ is $(s - \nu)$ and hence the rank of the left nullspace of $\Delta^{[1:s]}$ is $n - (s - \nu)$. Thus, the number of matrices which satisfy $A \Delta^{[1:s]} = 0$ is $(q^{n-(s-\nu)})^k$. Since there are $q^{kn}$ choices for the matrix $A$, we get

$$P(A \Delta^{[1:s]} = 0) = \frac{(q^{n-(s-\nu)})^k}{q^{kn}} = q^{-k(s-\nu)}.$$  

(42)
Thus partitioning the set of all $\Delta^{[1:s]}$ based on the rank of $\mathcal{N}(\Delta^{[1:s]})$, we can rewrite $P_1$ in (41) as

$$P_1 \leq \sum_{\nu=0}^{s-1} \sum_{W_i: \dim(W_i)=\nu} \sum_{\mathcal{N}(\Delta^{[1:s]})} q^{-k(s-\nu)},$$

(43)

where $W_1$ is a $\nu$ dimensional subspace of $\mathbb{F}_q^s$. We shall now provide an alternative expression for the set $\{v^{[1:s]} \in A^{(n)}(Z^{[1:s]}) \mid \mathcal{N}(\Delta^{[1:s]}) = W_1\}$ as follows. Let $\{g_1, g_2, \ldots, g_s\}$ denote a basis for $W_1$ and let $G_{W_i} = [g_1 \ldots g_s]$. Then

$$\begin{align*}
\{v^{[1:s]} \in A^{(n)}(Z^{[1:s]}) \mid \mathcal{N}(\Delta^{[1:s]}) = W_1\} &= \{v^{[1:s]} \in A^{(n)}(Z^{[1:s]}) \mid \Delta^{[1:s]} G_{W_i} = 0\} = \{v^{[1:s]} \in A^{(n)}(Z^{[1:s]}) \mid v^{[1:s]} G_{W_i} = z^{[1:s]} G_{W_i}\}. 
\end{align*}$$

(44)

Now applying Lemma 11 to the above equation wherein we set $X = Z^{[1:s]}$, $f(X) = Z^{[1:s]} G_{W_i}$ and by noting that $v^{[1:s]}$ is an $n$-length realization of $Z^{[1:s]}$, we get

$$\begin{align*}
\left\{v^{[1:s]} \in A^{(n)}(Z^{[1:s]}) \mid v^{[1:s]} G_{W_i} = z^{[1:s]} G_{W_i}\right\} &= A^{(n)}(Z^{[1:s]} \mid z^{[1:s]} G_{W_i}). 
\end{align*}$$

(45)

We substitute the above equation in (43) and use the resulting expression in (41) to get

$$P^{(n)}_e \leq \delta_n + \sum_{z^{[1:s]} \in A^{(n)}(Z^{[1:s]})} P(z^{[1:s]}) \sum_{\nu=0}^{s-1} \sum_{W_i: \dim(W_i)=\nu} 2^{n[H(Z^{[1:s]} G_{W_i}|z^{[1:s]} G_{W_i})(1+\epsilon)-(s-\nu)\frac{k}{n} \log(q)].}

(46)

where we used the fact that the size of the conditional typical set is bounded as

$$\left| A^{(n)}(Z^{[1:s]} \mid z^{[1:s]} G_{W_i}) \right| \leq 2^{n[H(Z^{[1:s]} G_{W_i}|z^{[1:s]} G_{W_i})(1+\epsilon)]}.\n
(47)

Thus, a sufficient condition for $P^{(n)}_e \rightarrow 0$ is that

$$\frac{k}{n} \log(q) \geq \frac{1}{(s-\nu)} H(Z^{[1:s]} \mid Z^{[1:s]} G_{W_i})(1+\epsilon)\n
(48)

for every choice of $\nu$-dimensional subspace $W_1$ of $\mathbb{F}_q^s, 0 \leq \nu \leq s-1$.

We will now show the necessity of the inequalities in (46) if reliable decoding of the sources $Z^{[1:s]}$ is desired thereby proving that $R_{CC}(W)$ is the minimum symmetric rate achievable under CC approach. Let $(\Omega, \Omega^c)$ denote a partition of the sources $Z^{[1:s]}$, such that $|\Omega| = (s-\nu), 0 \leq \nu \leq (s-1)$ and $Z^{\Omega} = \{Z_j, j \in \Omega\}$. It follows from SW lower bound [19] that

$$R \geq \frac{1}{(s-\nu)} H(Z^{\Omega} \mid Z^{\Omega^c})\n
(49)

is a necessary condition. Note that the above inequalities are exactly those in (46) obtained by choosing the columns of $G$ from the set of standard basis vectors for $\mathbb{F}_q^s$.

We will now show that the necessity of remaining inequalities in (46), corresponding to other choices of $G$, is due to our restriction to a common encoding matrix. Consider a new system (which is also reliable) constructed as shown in Fig. 3 with the same encoder and decoder as that of the system in Fig. 8 where $Y^{[1:s]} = Z^{[1:s]} P$, $P$ being an $s \times s$ invertible matrix. Applying the SW bounds [19] to this new system we get,

$$R \geq \frac{1}{(s-\nu)} H(Y^{\Omega} \mid Y^{\Omega^c}),\n
(50)$$
where $\Omega$ is some subset of the sources $Y^{[1:s]}$, such that $|\Omega| = (s - \nu), 0 \leq \nu \leq (s - 1)$. Since, $Y^{[1:s]} = Z^{[1:s]}P$ and $P$ is an invertible matrix, the above equation can be written as

$$R \geq \frac{1}{(s - \nu)} H(Z^{[1:s]} | Z^{[1:s]}G),$$

(51)

where $G$ is an $s \times \nu$ sub matrix of $P$ containing the $\nu$ columns corresponding to $Y_\Omega$. Since the above bound is true for every invertible matrix $P$, we run through all subspaces of $F^s_q$ of dimension less than or equal to $s - 1$ thereby establishing the necessity of the inequalities in (36).

### Appendix B

**Proof of Theorem 3**

We will first present a few properties of normalized and conditional normalized entropies, which will subsequently be used to prove the theorem.

**Lemma 12:** Consider $U, W \subseteq V$ such that $U \not\subseteq W$. Then

$$\mathcal{H}_N(U + W | W) = \mathcal{H}_N(U | W), \quad \forall W \subseteq V.$$

(52)

**Proof:** Follows directly by invoking the equivalent definition of conditional normalized entropy in (14).

**Lemma 13:** Consider $U, W \subseteq V$ such that $U \not\subseteq W$. Then

$$\mathcal{H}_N(U | W) \leq \mathcal{H}_N(U | U \cap W).$$

(53)

**Proof:** Follows from the definition of conditional normalized entropy in (13) and by using that fact that $\mathcal{H}(U | W) \leq \mathcal{H}(U | U \cap W)$.

**Lemma 14:** Consider $W, U_1, U \subseteq V$ such that $W \subseteq U_1 \subseteq U$. Then, one of the following three conditions is true.

a) $\mathcal{H}_N(U_1 | W) < \mathcal{H}_N(U | W) < \mathcal{H}_N(U | U_1)$.

b) $\mathcal{H}_N(U_1 | W) = \mathcal{H}_N(U | W) = \mathcal{H}_N(U | U_1)$.

c) $\mathcal{H}_N(U_1 | W) > \mathcal{H}_N(U | W) > \mathcal{H}_N(U | U_1)$.

**Proof:** $\mathcal{H}_N(U | W)$ can be written as a convex combination of $\mathcal{H}_N(U_1 | W)$ and $\mathcal{H}_N(U | U_1)$ as

$$\mathcal{H}_N(U | W) = \alpha \mathcal{H}_N(U_1 | W) + (1 - \alpha) \mathcal{H}_N(U | U_1),$$

(54)

where $\alpha = \frac{\rho_{U_1} - \rho_W}{\rho_U - \rho_W}$. The lemma now follows. □

**Proof of Theorem 3** Consider the set

$$S_{W_0} = \{U | W_0 \subseteq U \text{ and } \mathcal{H}_N(U | W_0) \leq \mathcal{H}_N(W | W_0) \forall W \subseteq V, W_0 \not\subseteq W\},$$

(55)

i.e., $S_{W_0}$ is the set of all subspaces of $V$ which contain $W_0$ and have the least normalized conditional entropy conditioned on $W_0$. We claim that $S_{W_0}$ is closed under subspace addition, i.e., if $U_1, U_2 \in S_{W_0}$, then $U_1 + U_2 \in S_{W_0}$.
The claim will be proved shortly. Since $S_{W_0}$ is a finite set, this will imply that $Q(W_0) \triangleq \sum_{U \in S_{W_0}} U$ is the unique maximal element of $S_{W_0}$. Now, consider the chain obtained sequentially as follows:

$$W^{(j)} \triangleq Q \left( W^{(j-1)} \right), \quad \forall j > 1,$$

where $W^{(0)} = \{ \emptyset \}$. The construction proceeds until the $r$th stage, where $W^{(r)} = V$. It is clear that the chain obtained from (56) satisfies conditions 1) and 2) in Theorem 3 and is also unique. To prove that this chain also satisfies (19), apply Lemma 14 to the three element subspace chain $W^{(j-1)} \subseteq W^{(j)} \subseteq W^{(j+1)}$, $1 \leq j \leq r - 1$. Since $W^{(j)} = Q(W^{(j-1)})$, we have that $\mathcal{H}_N(W^{(j)}|W^{(j-1)}) < \mathcal{H}_N(W^{(j+1)}|W^{(j)})$. Hence condition a) of Lemma 14 is true in this case and thus

$$\mathcal{H}_N(W^{(j)}|W^{(j-1)}) < \mathcal{H}_N(W^{(j+1)}|W^{(j)}).$$

Now, we will prove our claim that $S_{W_0}$ is closed under subspace addition. Let $U_1, U_2 \in S_{W_0}$. If $U_2 \subseteq U_1$ or $U_1 \subseteq U_2$, the claim is trivially true. Thus, assume that $U_2 \nsubseteq U_1$, $U_1 \nsubseteq U_2$ and consider the following chain of inequalities.

$$\mathcal{H}_N(U_1|W_0) \overset{(a)}{\leq} \mathcal{H}_N(U_1 + U_2|U_1) \overset{(b)}{=} \mathcal{H}_N(U_2|U_1) \overset{(c)}{=} \mathcal{H}_N(U_2|U_1 \cap U_2) \overset{(d)}{=} \mathcal{H}_N(U_2|W_0),$$

where (a) follows by applying Lemma 14 to $W_0 \subseteq U_1 \subseteq U_1 + U_2$ and using the fact that $\mathcal{H}_N(U_1|W_0) \leq \mathcal{H}_N(U_1 + U_2|W_0)$ (since $U_1 \in S_{W_0}$), (b) follows from Lemma 12 (c) follows from Lemma 13 and finally, (d) follows trivially, if $W_0 = U_1 \cap U_2$; else if $W_0 \nsubseteq U_1 \cap U_2$, by applying Lemma 14 to $W_0 \subseteq U_1 \cap U_2 \subseteq U_2$ and noting that $U_2 \in S_{W_0}$.

But, $\mathcal{H}_N(U_1|W_0) = \mathcal{H}_N(U_2|W_0)$ and thus all inequalities in (60) are equalities. Especially, from (a), we get that $\mathcal{H}_N(U_1 + U_2|U_1) = \mathcal{H}_N(U_1|W_0)$. Lemma 14 now implies that $\mathcal{H}_N(U_1 + U_2|W_0) = \mathcal{H}_N(U_1|W_0)$ and thus $U_1 + U_2 \in S_{W_0}$.

**APPENDIX C**

**PROOF OF LEMMA 4**

Since $\{X_i\}$ and $\{Y_i\}$ are related via an invertible matrix, $< X_1, \ldots, X_m > = < Y_1, \ldots, Y_m >$. Thus, we will just find the subspace chain for the $\{Y_i\}$. Set $U_0 = \{ \emptyset \}$ and $U_j = < Y_1, Y_2, \ldots, Y_{\sum_{i=1}^{j-1} \epsilon_i} >$, $1 \leq j \leq r$. Let $U \subseteq V$ be such that $U_{j-1} \subseteq U$. We will now show that $\mathcal{H}_N(U|U_{j-1}) \geq \mathcal{H}_N(U_j|U_{j-1})$ with equality only if $U \subseteq U_j$. This will imply that the chain $U_0 \subseteq U_1 \subseteq \ldots \subseteq U_r$ satisfies the conditions of Theorem 4 and hence, is the required chain.

Let $U \cap U_j = U_{j-1} \oplus A$ and $U = (U \cap U_j) \oplus B$, for some subspaces $A, B$. Then, $\mathcal{H}_N(U|U_{j-1})$ can be expanded as

$$\mathcal{H}_N(U|U_{j-1}) \overset{(a)}{=} \frac{\mathcal{H}(A \oplus B|U_{j-1})}{\rho_A + \rho_B} \overset{(b)}{=} \frac{\mathcal{H}(A|U_{j-1}) + \mathcal{H}(B|A \oplus U_{j-1})}{\rho_A + \rho_B} \overset{(c)}{=} \frac{\mathcal{H}(A|U_{j-1}) + \mathcal{H}(B|U_j)}{\rho_A + \rho_B} \overset{(d)}{=} \frac{\rho_A H(Y_{\sum_{i=1}^{j-1} \epsilon_i+1}) + \rho_B H(Y_{\sum_{i=1}^{j-1} \epsilon_i+1})}{\rho_A + \rho_B} \overset{(e)}{=} H(Y_{\sum_{i=1}^{j-1} \epsilon_i+1}) = \mathcal{H}_N(U_j|U_{j-1}),$$

(61)
where (a) follows from Lemma 12, (b), (c) both follow directly from the definition of conditional entropy in (11), (d) follows from the fact that if \( U' \) is a subspace such that \( U' \cap U_j = \emptyset \), for any \( j \), then

\[
\mathcal{H}_N(U'|U_j) \geq H(Y_{\sum_{i=1}^{t_i+1}}).
\]  

This will be proved shortly. Finally, (e) follows from the assumption on the ordering of the entropies of \( \{Y_i\} \) (see (20)). Note that equality holds in (e) only if \( B = \emptyset \).

We will now prove (62). Let \( \Gamma_U' = \langle Y_1, \ldots, Y_m | \Gamma_{U'} \rangle \), for some \((m \times \rho_U')\) full rank matrix \( \Gamma_{U'} \). Column reduce \( \Gamma_{U'} \) by selecting, for any column, the last row which has a non zero entry and using that entry to make all the other entries in that row as zeros. Let \( S = \{t_1, \ldots, t_{\rho_U'}\} \) denote the the row indices corresponding to the identity sub matrix which occur after the column reduction. Since \( U' \cap U_j = \emptyset \), it must be true that

\[
t_i' \geq \sum_{i=1}^{j} \ell_i + 1, \quad 1 \leq i' \leq \rho_U'.
\]  

Now, if we let \( S^c = \{1, \ldots, m\} \setminus S \), we have

\[
\mathcal{H}_N(U'|U_j) = \frac{\mathcal{H}(U'|U_j)}{\rho_U'} \geq \frac{\mathcal{H}(U'|Y_i, i \in S^c)}{\rho_{U'}} \geq H(Y_{\sum_{i=1}^{t_i+1}}) \geq H(Y_{\sum_{i=1}^{t_i+1}}),
\]  

where (a) follows since \( U_j \subseteq \langle Y_i, i \in S^c \rangle \), (b) follows from (63) and (c) follows from the assumption on the ordering of the entropies of \( \{Y_i\} \) (see (20)).

**APPENDIX D**

**PROOF OF THEOREM 5**

The proof involves two steps, which are outlined next. Each step will be proved subsequently.

**Step 1**: Consider the chain of subspaces \( W^{(0)} \subseteq W^{(1)} \subseteq \ldots \subseteq W^{(r)} \) as obtained from Theorem 3. We will show that that the infimum of the achievable rates for decoding the subspace \( W^{(j)} \) under the CC approach (see Section III-A) is given by

\[
R_{CC}(W^{(j)}) = \mathcal{H}_N \left( W^{(j)} \mid W^{(j-1)} \right), \quad \forall 1 \leq j \leq r.
\]

**Step 2**: Next, consider any subspace \( W \subseteq V \), such that \( W \not\subseteq W^{(j-1)} \) and \( W \subseteq W^{(j)} \). We show that an optimal subspace to decode \( W \) under the SS approach is \( W^{(j)} \), by showing that for any other subspace \( W' \supseteq W \), we have

\[
R_{CC}(W') \geq \mathcal{H}_N \left( W^{(j)} \mid W^{(j-1)} \right) = R_{CC}(W^{(j)}).
\]

**A. Proof of Step 1**

Proof by induction on \( j \). Statement follows for \( j = 1 \), since from (15), we have

\[
R_{CC}(W^{(1)}) = \max_{U_1 \subseteq W^{(1)}_U} \mathcal{H}_N(W^{(1)}|U_1) \equiv \mathcal{H}_N(W^{(1)}),
\]

where (a) follows by applying Lemma 14 to \( \{1\} \subseteq U_1 \subseteq W^{(1)} \) and noting that from Theorem 3 that \( W^{(1)} \) has the least normalized entropy among all subspaces of \( V \). Now, assume that the statement is true for \( j - 1 \), i.e.,

\[
\max_{U_1 \subseteq W^{(j-1)}} \left\{ \mathcal{H}_N(W^{(j-1)}|U_1) \right\} = \mathcal{H}_N(W^{(j-1)}|W^{(j-2)}).
\]

(66)
Then, we need to prove that

\[
\max_{U_1 \subseteq W^{(j)}} \left\{ \mathcal{H}_N(W^{(j)}|U_1) \right\} = \mathcal{H}_N(W^{(j)}|W^{(j-1)}),
\]

which will imply \( R_{CC}(W^{(j)}) = \mathcal{H}_N(W^{(j)}|W^{(j-1)}) \). For any \( U_1 \subseteq W^{(j)} \), let \( A \triangleq U_1 \cap W^{(j-1)} \) and let \( A^c \) be its complement in \( U_1 \). Then

\[
\mathcal{H}(W^{(j)}|U_1) \\
= \mathcal{H}(W^{(j)}) - \mathcal{H}(A \oplus A^c) \\
= \mathcal{H}(W^{(j-1)}) + \mathcal{H}(W^{(j)}|W^{(j-1)}) - \mathcal{H}(A) - \mathcal{H}(A^c|A) \\
\overset{(a)}{=} \mathcal{H}(W^{(j-1)}|A) + \mathcal{H}(W^{(j)}|W^{(j-1)}) - \mathcal{H}(A^c|A) \\
\overset{(b)}{=} \mathcal{H}(W^{(j-1)}|A) + \mathcal{H}(W^{(j)}|W^{(j-1)}) - \mathcal{H}(A^c|W^{(j-1)}) \\
\overset{(c)}{=} \mathcal{H}(W^{(j-1)}|A) + \mathcal{H}(W^{(j)}|W^{(j-1)}) - \rho_A \mathcal{H}_N(W^{(j)}|W^{(j-1)}) \\
\overset{(d)}{=} (\rho_{W^{(j-1)}} - \rho_A) \mathcal{H}_N(W^{(j-1)}|W^{(j-2)}) + \mathcal{H}(W^{(j)}|W^{(j-1)}) - \rho_A \mathcal{H}_N(W^{(j)}|W^{(j-1)}) \\
\overset{(e)}{=} (\rho_{W^{(j-1)}} - \rho_A) \mathcal{H}_N(W^{(j)}|W^{(j-1)}) + \mathcal{H}(W^{(j)}|W^{(j-1)}) - \rho_A \mathcal{H}_N(W^{(j)}|W^{(j-1)}) \\
= (\rho_{W^{(j-1)}} - \rho_{U_1}) \mathcal{H}_N(W^{(j)}|W^{(j-1)}),
\]

which implies that \( \mathcal{H}_N(W^{(j)}|U_1) \leq \mathcal{H}_N(W^{(j)}|W^{(j-1)}) \). Here, (a) and (b) follow since \( A \subseteq W^{(j-1)} \), (c) follows trivially if \( A^c = \{0\} \); else from Theorem 3 \( \mathcal{H}_N(W^{(j)}|W^{(j-1)}) < \mathcal{H}_N(A^c + W^{(j-1)}|W^{(j-1)}) = \mathcal{H}_N(A^c|W^{(j-1)}) \). (d) follows trivially if \( A = W^{(j-1)} \); else by induction hypothesis on \( j - 1 \) (put \( U_1 = A \) in \( 66 \)) and finally, (e) follows since by Theorem 3 \( \mathcal{H}_N(W^{(j-1)}|W^{(j-2)}) < \mathcal{H}_N(W^{(j)}|W^{(j-1)}) \).

**B. Proof of Step 2**

\[
R_{CC}(W^j) = \max_{U_1 \subseteq W^j} \left\{ \mathcal{H}_N(W^j|U_1) \right\}
\]

\[
\overset{(a)}{=} \mathcal{H}_N(W^j|W^j \cap W^{(j-1)})
\]

\[
\overset{(b)}{=} \mathcal{H}_N(W^j|W^{(j-1)})
\]

\[
\overset{(c)}{=} \mathcal{H}_N(W^j + W^{(j-1)}|W^{(j-1)})
\]

\[
\overset{(d)}{=} \mathcal{H}_N(W^{(j)}|W^{(j-1)})
\]

\[
\overset{(e)}{=} R_{CC}(W^{(j)}),
\]

where (a) follows by substituting \( U_1 = W^j \cap W^{(j-1)} \), (b) follows from Lemma 13 (c) follows from Lemma 12 (d) follows since by Theorem 3 \( W^{(j)} \) is least normalized entropy subspace conditioned on \( W^{(j-1)} \) and finally, (e) follows from Step 1.

**APPENDIX E**

**Existence of Nested Codes**

For any \( k_1 \leq k_2 \leq \ldots \leq k_j \), let \( B_{\ell}, \ell = 1, \ldots, j \), denote a random matrix uniformly picked from the set of all \( (k_\ell - k_{\ell-1}) \times n \) matrices over \( \mathbb{F}_q \) (note here \( k_0 = 0 \)). The encoding matrix for the \( \ell^{th} \) stage, \( A^{(\ell)} \), is assumed to have the nested form \( A^{(\ell)} = [B_1^{(\ell)}, \ldots, B_j^{(\ell)}] \), \( 1 \leq \ell \leq j \). For any \( \ell \leq j \), let \( W^{(\ell)} = \{Y_1, \ldots, Y_{\rho_{W^{(\ell)}}}\} \). As discussed in Section 5-B the \( \ell^{th} \) stage computes the complement of \( W^{(\ell-1)} \) in \( W^{(\ell)} \) using \( \hat{W}^{(\ell-1)} \) (all the output up till the
(ℓ − 1)th stage) as side information. Now, for any fixed set of encoding matrices, let \( \mathcal{E}_\ell \) denote the error event up till the \( \ell \)th stage, i.e.,

\[
\mathcal{E}_\ell : (\hat{Y}_1, ..., \hat{Y}_{\rho_W(\ell)}) \neq (Y_1, ..., Y_{\rho_W(\ell)}).
\] (71)

Also let \( P^{(n)}_{e,\ell} \) denote the probability of error in the \( \ell \)th stage assuming that all the previous stages were decoded correctly (i.e., when the \( \ell \)th stage receives \( W^{(\ell−1)} \) as side information). Thus, \( P^{(n)}_{e,\ell} = P(\mathcal{E}_\ell | \mathcal{E}^c_{\ell−1}) \). Thus the overall source averaged probability of error, \( P^{(n)}_e \), in computing \( W^{(j)} \) can be upper bounded as

\[
P^{(n)}_e = P(\mathcal{E}_j) \leq P(\mathcal{E}_{j−1}) + P(\mathcal{E}_j | \mathcal{E}^c_{j−1}) \leq \sum_{\ell=1}^j P^{(n)}_{e,\ell},
\] (72)

\[
P^{(n)}_e \leq P^{(n)}_{e,j} \leq \sum_{\ell=1}^j \sum_{A^{(\ell)}} P^{(n)}_{A^{(\ell)}} P^{(n)}_{e,\ell},
\] (73)

where the last equation follows by repeating steps from (72)-(74). Averaging \( P^{(n)}_e \) further over the ensemble of encoding matrices, we get

\[
\langle P^{(n)}_e \rangle = \sum_{A^{(j)}} \sum_{\ell=1}^j P^{(n)}_{A^{(j)}} P^{(n)}_{e,\ell} = \sum_{\ell=1}^j \sum_{A^{(\ell)}} P^{(n)}_{A^{(\ell)}} P^{(n)}_{e,\ell},
\] (76)

where, in the last equation we have interchanged the order of the two summations and also used the fact that stage \( \ell \) depends only on \( B_1, ..., B_\ell \) and hence \( P^{(n)}_{A^{(j)}} (A^{(j)} \mid A^{(\ell)}) \) can be marginalized over the incremental matrices of the remaining stages to get \( P^{(n)}_{A^{(\ell)}} (A^{(\ell)} \mid A^{(\ell)}) \). But, now from the achievability proof of Theorem 7 we know that the inside term \( \sum_{A^{(\ell)}} P^{(n)}_{A^{(\ell)}} (A^{(\ell)} \mid A^{(\ell)}) \xrightarrow{n \to \infty} 0 \) if

\[
\frac{k_f}{n} \log q > \mathcal{H}_N \left( W^{(\ell)} \mid W^{(\ell−1)} \right), \quad \forall \, 1 \leq \ell \leq j.
\] (77)

This proves the existence of the nested codes as claimed.