Brownian Motion with Drift on Spaces with Varying Dimension

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Abstract

In this paper, we study Brownian motion with drift on spaces with varying dimension. Such a process can be conveniently defined by a regular Dirichlet form that is not necessarily symmetric. The drift term is in some type of $L^p$ space with $p$ depending on the region of the state space. We show it can also be related to a non-drifted Brownian motion on spaces with varying dimension (BMVD in abbreviation) via Girsanov transform. Through the method of Duhamel’s formula, it is established in this paper that the transition density of BMVD with drift has the same type of sharp two-sided Gaussian bounds as that of BMVD without drift. As a corollary, we obtain Green function estimate for BMVD with drift.

1 Introduction

Brownian motion on spaces with varying dimension has been introduced and studied in details with an emphasis on its transition density estimate in [7]. Such a process can be characterized nicely via Dirichlet form. To introduce the state space of BMVD, with or without darning, we think of the following simplest space that BMVD can live on:

$$\mathbb{R}^2 \times \mathbb{R}_+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = 0 \text{ or } x_1 = x_2 = 0 \text{ and } x_3 > 0\}.$$

As has been pointed out in [7], Brownian motion cannot be defined on such a state space in the usual sense because a two-dimensional Brownian motion does not hit a singleton. Therefore in order to define the desired process, we “short” a closed disc on $\mathbb{R}^2$ to a singleton. In other words, we let the media offer infinite conductance on that closed disc, so that the process travels across it with infinite velocity. The resulting Brownian motion hits the shorted disc in finite time with probability one. Then we install an infinite pole at this “shorted” disc.

To be more precise, the state space of Brownian motion with varying dimension is defined as follows (here and in the sequel, we use “ := ” as a way of definition): For each $\varepsilon > 0$, $B_\varepsilon$ is the closed disc on $\mathbb{R}^2$ centered at $(0, 0)$ with radius $\varepsilon$. Let $D_0 = \mathbb{R}^2 \setminus B_\varepsilon$. By identifying $B_\varepsilon$ with a singleton denoted by $a^\ast$, we can introduce a topological space $E := D_0 \cup \{a^\ast\} \cup \mathbb{R}_+$, with a neighborhood of $a^\ast$ defined as $\{a^\ast\} \cup (D_1 \cap \mathbb{R}_+) \cup (D_2 \cap D_0)$ for some neighborhood $D_1$ of 0 in $\mathbb{R}_+$ and $D_2$ of $B_\varepsilon$ in $\mathbb{R}^2$. For $p > 0$, let $m_p$ be the measure on $E$ whose restriction on $\mathbb{R}$ and $D$ is the Lebesgue measure times $p$ and 1, respectively. In particular, we set $m_p(\{a^\ast\}) = 0$. The following definition for BMVD can be found in [7].

Definition 1.1 (Brownian motion with varying dimension). An $m_p$-symmetric diffusion process satisfying the following properties is called Brownian motion with varying dimension.
(i) its part process in \( \mathbb{R}^+ \) or \( D_0 \) has the same law as standard Brownian motion in \( \mathbb{R}^+ \) or \( D_0 \);
(ii) it admits no killings on \( a^* \).

We denote BMVD without drift by \( X^0 \). It follows from the definition that the process spends zero amount of time under Lebesgue measure (i.e. zero sojourn time) at \( a^* \). The next theorem in \[7\] asserts that BMVD exists and is unique in law. It also gives the Dirichlet form characterization of \( X^0 \).

**Theorem 1.2 (’16 Chen, L.).** For every \( \varepsilon > 0 \) and \( p > 0 \), BMVD \( X^0 \) on \( E \) with parameter \((\varepsilon, p)\) exists and is unique. Its associated Dirichlet form \( (\mathcal{E}^0, \mathcal{D}(\mathcal{E}^0)) \) on \( L^2(E;m_p) \) is given by

\[
\mathcal{D}(\mathcal{E}^0) = \left\{ f : f|_{\mathbb{R}^2} \in W^{1,2}(D_0), \ f|_{\mathbb{R}^+} \in W^{1,2}(\mathbb{R}^+), \ \text{and} \ f(x) = f(0) \ \text{q.e.} \ \text{on} \ \partial D_0 \right\},
\]

\[
\mathcal{E}^0(f,g) = \frac{1}{2} \int_{\mathbb{R}^2 \setminus B_{\varepsilon}} \nabla f(x) \cdot \nabla g(x) dx + \frac{p}{2} \int_{\mathbb{R}^+} f'(x)g'(x) dx.
\]

In this paper, we study Brownian motion with drift on spaces with varying dimension which is a natural extension of BMVD. We seek to establish the stability of the heat kernel asymptotic behavior under the perturbation of the drift. To give a brief overview on the classic results for drifted Brownian motion on Euclidean spaces, we restate the following definition for Kato class functions (see, for instance, \[8\]):

**Definition 1.3** (Kato class functions). For \( d \in \mathbb{N} \), we say a function \( b : \mathbb{R}^n \to \mathbb{R} \) is in Kato class \( K_d \) if

\[
\lim_{r \downarrow 0} \sup_{x \in \mathbb{R}^n} \int_{|x-y|<r} \frac{|b(y)|}{|x-y|^{d-2}} dy = 0, \quad \text{for} \ d \geq 3,
\]

\[
\lim_{r \downarrow 0} \sup_{x \in \mathbb{R}^n} \int_{|x-y|<r} \log \left(|x-y|^{-1}\right) |b(y)| dy = 0, \quad \text{for} \ d = 2,
\]

and

\[
\sup_{x \in \mathbb{R}^n} \int_{|x-y| \leq 1} |b(y)| dy < \infty, \quad \text{for} \ d = 1.
\]

Note that in the definition above, it is not necessary that \( n = d \). On \( \mathbb{R}^d \), Brownian motion with drift can be characterized by its associated generator \( \mathcal{L}^b = \Delta + b \cdot \nabla \) where \( b \in \{ b : b \in K_{d+1} \} \) or \( |b|^2 \in K_d \). It is proved by Chen and Zhao in \[8\] that when \( |b|^2 \in K_d \), the bilinear form associated with \( \mathcal{L}^b = \Delta + b \cdot \nabla \) on \( C^\infty_c(\mathbb{R}^d) \) is lower semibounded, closable, Markovian and satisfies Silverstein’s sector condition, therefore there is a minimal diffusion process associated with this bilinear form. Bass and Chen claim in \[2\] that when \( b \in K_{d+1} \), there is a unique weak solution to the SDE:

\[
dY_t = dB_t + b(Y_t)dt, \quad Y_0 = y_0.
\]

Such a solution is a strong Markov process associated to the generator \( \mathcal{L}^b = \Delta + b \cdot \nabla \). Indeed, on \( \mathbb{R}^d \), \( L^p \subset (K_{d+1} \cap \{ b : |b|^2 \in K_d \}) \) for all \( p > d \).

The sharp two-sided bounds for the transition densities of drifted Brownian motion on Euclidean spaces has been extensively studied in the past few decades. It was first established by Aronson that the transition density \( p(t,x,y) \) of drifted Brownian motion on \( \mathbb{R}^d \) has the following
two-sided Gaussian-type bounds, provided that \( b \in L^p(B(0, R)) \) for some \( p > d \) and \( R > 0 \), and \( b \) is bounded outside \( B(0, R) \).

\[
\frac{C_1}{t^{d/2}} \exp \left( -\frac{C_1|x-y|^2}{t} \right) \leq p(t, x, y) \leq \frac{C_2}{t^{d/2}} \exp \left( -\frac{C_2|x-y|^2}{t} \right), \quad 0 < t \leq 1.
\]

Later it is proved by Zhang in [16] that Aronson-type heat kernel two-sided bounds hold provided that \( b \) satisfying some integral condition. It is also proved in [16] that Kato class \( K_{d+1} \) implies that integral condition. It was later shown in [13, Proposition 2.3] that these integral conditions are indeed equivalent to Kato class \( K_{d+1} \). See also Riahi [15].

To give definition to BMVD with drift, we let \( b : E \to \mathbb{R} \) be a measurable function. We denote a family of such functions by \( L^{p_1,p_2}(E) \) for some \( p_1 \in (1, \infty) \) and \( p_2 \in (2, \infty] \), if \( b \) can be decomposed as \( b = b_1 + b_2 \) such that

1. \( b_1 := b|_{\mathbb{R}^+} \in L^{p_1}(\mathbb{R}^+) \) with \( p_1 \in (1, \infty] \)
2. \( b_2 := b|_{D_0} \in L^{p_2}(D_0) \) with \( p_2 \in (2, \infty] \).

**Definition 1.4** (Drifted Brownian motion with varying dimension). For every pair of positive constants \((\varepsilon, p)\) and every \( b \in L^{p_1,p_2}(E) \),

\[
\mathcal{E}^b(f, g) = \mathcal{E}^0(f, g) - (b \cdot \nabla f, g), \quad \mathcal{D}(\mathcal{E}^b) = \mathcal{D}(\mathcal{E}^0),
\]

is a strongly local regular Dirichlet space which is not necessarily symmetric, therefore there is a unique diffusion process associated with it. We call such a process Brownian motion with drift on space with varying dimension and denote it by \( X \).

Unlike non-drifted BMVD \( X^0 \), drifted BMVD \( X \) in general is non-symmetric. The major goal of this paper is to establish the sharp two-sided bound for the transition density functions of \( X \). Observe that in general \( X \) is not rotationally-invariant on \( D_0 \), we can not employ the method of radial process which has been used to treat \( X^0 \). In this paper, via Duhamel’s formula, we identify the transition densities and the resolvent operators of \( X \) as the transformations of those of non-drifted BMVD, from which we derive the upper bound estimate for its heat kernel.

To state the results of this paper, we need to introduce a few more notations. In this paper, we denote the geodesic metric on the underlying space \( E \) by \( \rho \). Namely, for \( x, y \in E \), \( \rho(x, y) \) is the shortest path distance (induced from the Euclidean space) in \( E \) between \( x \) and \( y \). For notational simplicity, we write \( |x|_\rho \) for \( \rho(x, a^*) \) when \( x \in D_0 \). We use \( |\cdot| \) to denote the usual Euclidean norm. For example, for \( x, y \in D_0 \), \( |x - y| \) is the Euclidean distance between \( x \) and \( y \) in \( \mathbb{R}^2 \). Note that for \( x \in D_0 \), \( |x|_\rho = |x| - \varepsilon \). Apparently,

\[
\rho(x, y) = |x - y| \wedge (|x|_\rho + |y|_\rho) \quad \text{for } x, y \in D_0
\]

and \( \rho(x, y) = |x| + |y| - \varepsilon \) when \( x \in \mathbb{R}^+ \) and \( y \in D_0 \) or vice versa. Here and in the rest of this paper, for \( a, b \in \mathbb{R}, a \wedge b := \min\{a, b\} \).

For any compact set \( K \subset E \), we define \( \sigma_K := \inf\{t > 0 : X_t \in K\} \). For any open domain \( D \subset E \), we define \( \tau_D := \inf\{t > 0 : X_t \notin D\} \). Similar notations will be used for other stochastic processes. Also we set \( \delta_D(x) := \inf\{d(x, y); y \in D^c\} \), where \( d(\cdot, \cdot) \) stands for either Euclidean or geodesic distance according to the context. We will use \( B_\rho(x, r) \) (resp. \( B_c(x, r) \)) to denote the geodesic (resp. Euclidean) open ball centered at \( x \in E \) with radius \( r > 0 \).
For two positive functions $f$ and $g$, $f \asymp g$ means that $f/g$ is bounded between two positive constants. In the following, we will also use notation $f \lesssim g$ (respectively, $f \gtrsim g$) to mean that there is some constant $c > 0$ so that $f \leq cg$ (respectively, $f \geq cg$).

The following two-sided short time heat kernel bounds for non-drifted BMVD $X^0$ are established in [7]. We denote the transition density of $X^0$ by $p(t, x, y)$.

**Theorem 1.5** (‘16 Chen, L.). Let $T > 0$ be fixed. There exist positive constants $C_i$, $1 \leq i \leq 22$, such that the following estimates hold on $t \in (0, T]$:

1. For $x \in \mathbb{R}_+$ and $y \in E$, $$\frac{C_1}{\sqrt{t}} e^{-\frac{c_1 x^2}{t}} \leq p(t, x, y) \leq \frac{C_3}{\sqrt{t}} e^{-\frac{c_3 x^2}{t}}.$$ 

2. For $x, y \in D_0 \cup \{a^*\}$ with $\max\{|x|_\rho, |y|_\rho\} < 1$, $$\frac{C_{13}}{\sqrt{t}} e^{-\frac{c_{13} x^2}{t}} + \frac{C_{13}}{t} \left(1 \wedge \frac{|x|_\rho}{\sqrt{t}}\right) \left(1 \wedge \frac{|y|_\rho}{\sqrt{t}}\right) e^{-\frac{c_{13} y^2}{t}} \leq p(t, x, y) \leq \frac{C_{16}}{\sqrt{t}} e^{-\frac{c_{16} x^2}{t}} + \frac{C_{16}}{t} \left(1 \wedge \frac{|x|_\rho}{\sqrt{t}}\right) \left(1 \wedge \frac{|y|_\rho}{\sqrt{t}}\right) e^{-\frac{c_{16} y^2}{t}};$$

and when $\max\{|x|_\rho, |y|_\rho\} \geq 1$, $$\frac{C_{19}}{t} e^{-\frac{c_{19} x^2}{t}} \leq p(t, x, y) \leq \frac{C_{21}}{t} e^{-\frac{c_{21} x^2}{t}}.$$

The above theorem has covered all the cases for $x, y \in E$ owing to the fact that $p(t, x, y)$ is symmetric in $(x, y)$, since $X^0$ is a symmetric Markov process.

To state the main result, for each $\alpha > 0$, we define the canonical form $p^0_\alpha(t, x, y)$ of the heat kernel on $E$ as follows:

$$p^0_\alpha(t, x, y) := \begin{cases} 
\frac{1}{\sqrt{t}} e^{-\frac{\alpha|x-y|^2}{t}}, & x, y \in \mathbb{R}_+; \\
\frac{1}{\sqrt{t}} e^{-\frac{\alpha|x|^2}{t}}, & x, y \in \mathbb{R}_+; \\
\frac{1}{\sqrt{t}} e^{-\frac{\alpha|y|^2}{t}}, & x, y \in \mathbb{R}_+; \\
\frac{1}{\sqrt{t}} e^{-\frac{\alpha|x|^2}{t}}, & x \in D_0, y \in \mathbb{R}_+; \\
\frac{1}{\sqrt{t}} e^{-\frac{\alpha|y|^2}{t}}, & x \in D_0, y \in \mathbb{R}_+; \\
\frac{1}{\sqrt{t}} e^{-\frac{\alpha|x|^2}{t}} + \frac{1}{t} \left(1 \wedge \frac{|x|_\rho}{\sqrt{t}}\right) \left(1 \wedge \frac{|y|_\rho}{\sqrt{t}}\right) e^{-\frac{\alpha|x-y|^2}{t}}, & \text{otherwise}. 
\end{cases}$$ \hspace{1cm} (1.3)

**Remark 1.6.** Indeed, by comparing Theorem 1.3 with (1.24), it is not too hard to see that there exist $\alpha_1, \alpha_2 > 0$ such that

$$C_{17} p^0_{\alpha_1}(t, x, y) \lesssim p(t, x, y) \lesssim p_{\alpha_2}^0(t, x, y) \quad \text{for all } t \leq T, x, y \in E. \hspace{1cm} (1.4)$$

For example, it is clear that $\rho(x, y)^2 \asymp |x|^2 + |y|^2$ for $x \in \mathbb{R}_+$ and $y \in D_0$; and it follows from elementary geometry that $\rho(x, y) \asymp |x - y|$ for $x, y \in D_0$ when $\max\{|x|_\rho, |y|_\rho\} > 1$. We rewrite the two-sided bounds for the heat kernels of BMVD simply for computation convenience in the remaining of this article.
The main result of this paper is the following:

**Theorem 1.7.** Let \( T > 0 \) be fixed. Let \( b \) be a measurable function on \( E \) in the family of \( L_{p_1,p_2}(E) \), where \( p_1 \in (1,\infty) \) and \( p_2 \in (2,\infty] \). Let \( p^b(t,x,y) \) be the transition density of drifted BMVD \( X \) associated with \( \{1,1\} \). There exists constants \( C_1 > 0, C_2 > 0, 0 < \alpha_1 < \alpha_2 \) such that

\[
C_1 p_{\alpha_2}^0(t,x,y) \leq p^b(t,x,y) \leq C_2 p_{\alpha_1}^0(t,x,y), \quad (t,x,y) \in (0,T) \times E \times E.
\]

Recall that an open set \( D \subset \mathbb{R}^d \) is called to be \( C^{1,1} \) if there exist a localization radius \( R_0 > 0 \) and a constant \( \Lambda > 0 \) such that for every \( z \in \partial U \), there exist a \( C^{1,1} \)-function \( \phi = \phi_z : \mathbb{R}^{d-1} \to \mathbb{R} \) satisfying \( \phi(0) = 0, \nabla \phi(0) = (0,\ldots,0), \| \nabla \phi \|_{\infty} \leq \Lambda_0, |\nabla \phi(x) - \nabla \phi(z)| \leq \Lambda |x-z| \) and an orthonormal coordinate system \( CS_z : y = (y_1,\ldots,y_d) := (\tilde{y},y_d) \) with its origin at \( z \) such that

\[
B(z,R_0) \cap D = \{ y \in B(0,R_0) \text{ in } CS_z : y_d > \phi(\tilde{y}) \}.
\]

For the state space \( E \), an open set \( D \subset E \) will be called \( C^{1,1} \) in \( E \), if \( D \cap \mathbb{R}_+ \) is a \( C^{1,1} \) open set in \( \mathbb{R}_+ \), and \( D \cap (D_0 \cup \{ a^* \}) \) is also a \( C^{1,1} \) open set in \( (D_0 \cup \{ a^* \}) \).

Let \( D \) be a bounded \( C^{1,1} \) domain of \( E \) containing \( a^* \). We denote by \( X^D \) the part process of BMVD with drift killed upon exiting a bounded domain \( D \) of \( E \) and let \( p^b_D(t,x,y) \) be the transition density of \( X^D \). Define the Green function of \( X^D \) to be

\[
G^b_D(x,y) := \int_0^\infty p^b_D(t,x,y)dt
\]

The following theorem established in Section 4 provides a two-sided bound for \( G^b_D \).

**Theorem 1.8.** Let \( D \) be a bounded \( C^{1,1} \) domain of \( E \) containing \( a^* \). Let \( b \) be a measurable function on \( E \) in the family of \( L_{p_1,p_2}(E) \), where \( p_1 \in (1,\infty] \) and \( p_2 \in (2,\infty] \). Let \( G_D(x,y) \) be the Green function of drifted BMVD \( X \) killed upon exiting \( D \). Then for \( x \neq y \) in \( D \), it holds

\[
G^b_D(x,y) \asymp \begin{cases} 
\delta_D(x) \wedge \delta_D(y), & x,y \in D \cap \mathbb{R}_+; \\
(\delta_D(y) \wedge 1)(\delta_D \wedge 1) + \ln \left( 1 + \frac{\delta_D \cap D_0(y)}{|x-y|^2} \right), & x,y \in D \cap D_0; \\
\delta_D(x)\delta_D(y), & x \in D \cap \mathbb{R}_+, y \in D \cap D_0.
\end{cases}
\]

Here \( \delta_D(x) := \text{dist}(x,\partial D) := \text{inf}\{ \rho(x,z) : z \notin D \} \) and \( \delta_D \cap D_0(x) := \text{inf}\{ \rho(x,z) : z \notin D \cap D_0 \} \).

Note that \( G^b(x,y) \) is symmetric in \( x \) and \( y \), so the theorem above has covered all \( (x,y) \in E \times E \).

For notational convenience, in the rest of this paper we set

\[
\overline{p}_D(t,x,y) := p(t,x,y) - p_D(t,x,y),
\]

where \( D \) is a domain of \( E \) and \( p_D(t,x,y) \) is the transition density of the part process killed upon exiting \( D \). In other words, for any non-negative function \( f \geq 0 \) on \( E \),

\[
\int_E \overline{p}_D(t,x,y)f(y)m_y(dy) = \mathbb{E}_x[f(X_t); t \geq \tau_D].
\]
The intuition for $p_D(t,x,y)$ and $\mathcal{P}_D(t,x,y)$ is that for $p_D(t,x,y)$, the trajectory starting from $x$ hits $y$ at time $t$ without exiting $D$, while for $\mathcal{P}_D(t,x,y)$, the trajectory has to exit $D$ before ending at $y$.

The rest of this paper is organized as follows: In Section 2 we claim that the process satisfying Definition 1.4 can also be related to $X^0$ through Girsanov transform as follows:

\[
\frac{dQ}{dP} \bigg|_{\mathcal{F}_t} = M_t := \exp \left( \int_0^t b(X_0^s) dX_0^s - \int_0^t |b(X_0^s)|^2 ds \right),
\]

where $\mathcal{F}_t = \sigma\{X_0^s, s \leq t\}$. Thus $(X,Q)$ and $(X^0,P)$ are different representations of the same process. Also we give in Section 2 the generator of $X$ which is indeed the generator of $X^0$ with gradient perturbation. However, any function in the domain of the generator of $X$ has to satisfy the “zero flux” condition at the darning point $a^*$. The resolvents of $X$ can also be represented via Duhamel’s identity.

In Section 3 we establish two-sided short time heat kernel estimates for $X$. The upper bound is established through the smallness of the gradients of the transition densities of $X^0$, which combined with Duhamel’s identity shows that the heat kernel of $X$ has the same form of upper bound as that of $X^0$. The lower bound, on the other hand, is obtained via the conjunction of near-diagonal lower bound estimate and a chain argument. The two-sided Green function estimate is provided in Section 4.

2 Preliminaries: Girsanov Transform of BMVD and Its Resolvent Kernel

Let $b: \mathcal{E} \to \mathbb{R}$ be a measurable function in the family of $L^\infty(\mathcal{E}) + L^{p_1,p_2}(\mathcal{E})$ with $p_1 \in (1,\infty]$ and $p_2 \in (2,\infty]$. We have defined BMVD with drift in the first section in terms of Dirichlet form. In this section, we characterize such a process by Girsanov transform and identify its resolvent kernel.

In order to rigorously define the following family of probability measures $Q$ in terms of Girsanov transform:

\[
\frac{dQ}{dP} \bigg|_{\mathcal{F}_t} = M_t := \exp \left( \int_0^t b(X_0^s) dX_0^s - \int_0^t |b(X_0^s)|^2 ds \right),
\]

where $(X_0^t)_{t>0}$ is BMVD without drift defined in [7], we first set

$M_t^{(1)} := \int_0^t 1_{\mathcal{R}_+}(X_0^s)dX_0^s$ and $M_t^{(2)} := \int_0^t 1_{D_0}(X_0^s)dX_0^s$.

To see how these stochastic integrals in [22] are rigorously defined, we set $A_\delta := \{x \in \mathbb{R}_+ : |x| > \delta\}$, $B_\delta := \{x \in D_0 : |x|_\rho > \delta\}$. We inductively define a sequence of stopping times as
follows:

\[
S^\delta_1 = \inf\{t > 0 : X^0_t \in A_\delta\}; \\
T^\delta_1 = \sigma_{\{a^*\}}; \\
\ldots \\
S^\delta_{k+1} = \sigma_{A_\delta} \circ \theta_{\hat{T}^\delta_k} + T^\delta_k; \\
T^\delta_{k+1} = \sigma_{\{a^*\}} \circ \theta_{\hat{S}^\delta_{k+1}} + S^\delta_{k+1}; \\
\ldots
\]

For each \( t > 0 \), we define

\[
M^{(1),\delta}_t := \sum_{n \geq 1} \int_{S^\delta_n \wedge t}^{\hat{T}^\delta_n \wedge t} 1 \cdot dX^0_s = \int_0^t \sum_n 1_{[S^\delta_n \wedge t, \hat{T}^\delta_n \wedge t]} dX^0_s.
\]

The above stochastic integral is well-defined because the restriction of \( X_s \) on \( \mathbb{R}^+ \) is 1-dimensional Brownian motion, and the summation is finite. Since we have for each fixed \( t > 0 \),

\[
\sum_n 1_{[S^\delta_{n-1} \wedge t, \hat{T}^\delta_n \wedge t]} \uparrow 1_{\{X^0_s \in \mathbb{R}^+\}} \text{ a.s.}, \quad \text{as } \delta \to 0.
\]

It follows that there is a unique square-integrable martingale

\[
M^{(1)}_t = \int_0^t 1_{\{X^0_s \in \mathbb{R}^+\}} dX^0_s.
\]

Similarly, to define \( M^{(2)}_t \), we define

\[
\hat{S}^\delta_1 = \inf\{t > 0 : X^0_t \in B_\delta\}; \\
\hat{T}^\delta_1 = \sigma_{\{a^*\}}; \\
\ldots \\
\hat{S}^\delta_{k+1} = \sigma_{B_\delta} \circ \theta_{\hat{T}^\delta_k} + \hat{T}^\delta_k; \\
\hat{T}^\delta_{k+1} = \sigma_{\{a^*\}} \circ \theta_{\hat{S}^\delta_{k+1}} + \hat{S}^\delta_{k+1}; \\
\ldots
\]

Then for each \( t > 0 \), we set

\[
M^{(2),\delta}_t := \sum_{n \geq 1} \int_{\hat{S}^\delta_{n-1} \wedge t}^{\hat{T}^\delta_n \wedge t} 1 \cdot dX^0_s = \int_0^t \sum_n 1_{[\hat{S}^\delta_{n-1} \wedge t, \hat{T}^\delta_n \wedge t]} dX^0_s.
\]

It follows that

\[
\sum_n 1_{[\hat{S}^\delta_{n-1} \wedge t, \hat{T}^\delta_n \wedge t]} \uparrow 1_{\{X^0_s \in D_0\}} \text{ a.s.}, \quad \text{as } \delta \to 0.
\]

The discussion above yields that there is a unique square-integrable martingale

\[
M^{(2)}_t = \int_0^t 1_{\{X^0_s \in D_0\}} dX^0_s.
\]
We write $M^{(2)}_t = \left( M^{(2),1}_t, M^{(2),2}_t \right)$. Now we define the probability measure $Q$ as follows:

$$
\frac{dQ}{dP} |_{\mathcal{F}_t} := M_t := \exp \left( \int_0^t b_1(M^{(1)}_s) dM^{1}_s + \int_0^t b_2(M^{(2)}_s) dM^{2}_s - \frac{1}{2} \int_0^t |b(X^0_s)|^2 ds \right),
$$

where $(X^0_t)_{t>0}$ is BMVD without drift, $\mathcal{F}_t := \sigma \{ X^0_s, s \leq t \}$, and $P$ is the distribution law of $X^0$. Let $G^0_\alpha$ be the resolvents of $X^0$. The next theorem identifies the process determined by $Q$ is indeed BMVD with drift $X$.

**Theorem 2.1.** Let $X$ be the process associated with the Dirichlet space $(E, \mathcal{D}(E))$. $(X, Q)$ and $(X^0, P)$ are different descriptions of the same process. Denote the resolvent operators of $X^0$ and $X$ by $(G^0_\alpha)_{\alpha>0}$ and $(G^b_\alpha)_{\alpha>0}$ on $L^2(E, m_p)$, respectively. Then there exists some $\alpha_0 > 0$ such that it holds for any $\alpha > \alpha_0$ and any $f \in L^2(E, m_p)$ that

(i) $G^0_\alpha(b \cdot \nabla G^0_\alpha)^n f$ is in $\mathcal{D}(E^0)$, for every $n = 0, 1, 2, \ldots$.

(ii) $G^b_\alpha f \in \mathcal{D}(E^0)$, and $G^b_\alpha f = \sum_{n=0}^{\infty} G^0_\alpha (b \cdot \nabla G^0_\alpha)^n f$, where the infinite sum converges in norm $\| \cdot \|_{1,2}$.

The norm $\| \cdot \|_{1,2}$ above is understood in the sense of the sum of the $\| \cdot \|_{1,2}$ on $D_0$ and $\mathbb{R}_+$.

**Proof.** The proof follows an argument similar to that in [8] with few minor changes. The details are omitted.

We now describe the infinitesimal generator associated with $X$. For notation simplicity we let

$$
\mathcal{F}^0 = \left\{ f : f|_{D_0} \in W^{1,2}_0(D_0), f|_{\mathbb{R}_+} \in W^{1,2}_0(\mathbb{R}_+) \right\}.
$$

Let $u_0(x) = \mathbb{E}_x[e^{-\sigma_{D_0}}]$ when $x \in D_0$ and $\mathbb{E}_x[e^{-\sigma_{\mathbb{R}_+}}]$ when $x \in \mathbb{R}_+$. It is known that $u_0|_{D_0} \in W^{1,2}(D_0), u_0|_{\mathbb{R}_+} \in W^{1,2}(\mathbb{R}_+)$. For any $u \in \mathcal{D}(E^b)$, we define the its flux at $a^*$ by

$$
\mathcal{N}_p(u)(a^*) := \int_E \nabla u(x) \cdot \nabla u_0(x) m_p(dx) + \int_E \Delta u(x) u_0(x) m_p(dx).
$$

The following theorem characterizes the infinitesimal generator associated with $X$.

**Theorem 2.2.** $X$ has an infinitesimal generator $\mathcal{L}^b := \Delta + b \cdot \nabla$. Its domain $\mathcal{D}(\mathcal{L}^b)$ is a subspace of $\mathcal{D}(E^b)$ such that a function $u \in \mathcal{D}(E^b)$ is in $\mathcal{D}(\mathcal{L}^b)$ if and only if the distributional Laplacian $\Delta u$ exists as an $L^2$-integrable function on $E \setminus \{a^*\}$ and $u$ satisfies zero flux property at $a^*$.

**Proof.** $\mathcal{D}(\mathcal{L}^b) = \mathcal{D}(E^0)$ is the linear span of $\mathcal{F}^0 \cup \{u_0\}$. For $u \in \mathcal{D}(E^b), u \in \mathcal{D}(E^b)$ if and only if there is some $f \in L^2(E; m_p)$ so that

$$
\mathcal{E}^b(u, v) = - \int_E f(x)v(x)m_p(dx) \quad \text{for every } v \in \mathcal{D}(E^b).
$$

If this holds, we set $\mathcal{L}^b u = f$. The above is equivalent to

$$
\frac{1}{2} \int_E \nabla u(x) \cdot \nabla v(x)m_p(dx) - \int_E b(x) \nabla u(x)v(x)m_p(dx) = - \int_E f(x)v(x)m_p(dx)
$$

for every $v \in \mathcal{F}^0$.

(2.3)
Proof. We prove (3.1) by considering different regions of \( x \) and \( u \).

We have
\[
\int_E \nabla u(x) \cdot \nabla u_0(x)m_p(dx) - \int_E b(x)\nabla u(x)u_0(x)m_p(dx) = -\int_E f(x)u_0(x)m_p(dx).
\]

(2.3) is equivalent to that \( \Delta u \in L^2(E) \), and (2.4) is equivalent to \( \mathcal{N}_p(u)(a^*) = 0 \).

\[\square\]

3 Gradient Estimates and Small Time Heat Kernel Estimates

Unless otherwise stated, it is always fixed in this section that \( T > 0 \) and \( b \) is a measurable function in \( L^{p_1,p_2}(E) \) with \( p_1 \in (1, \infty] \) and \( p_2 \in (2, \infty] \). Without loss generality, we assume throughout this section that \( 0 < \varepsilon \leq 1/4 \). To establish two-sided short time heat kernel estimate for BMVD with drift \( X \) via the method of Duhamel’s formula, we first establish the smallness of the gradients of the heat kernel \( p(t, x, y) \) of \( X^0 \). Recall the canonical form \( p^0_\alpha(t, x, y) \) of the heat kernel of \( X^0 \) has been defined in (3.5).

**Proposition 3.1.** There exists a constant \( C_1 > 0 \) such that for some \( \beta_1 > 0 \), it holds that
\[
|\nabla x p(t, x, y)| \leq \frac{C_1}{\sqrt{t}} p^0_{\beta_1}(t, x, y), \quad t \leq T, \ (x, y) \in (E \times E) \setminus (D_0 \times D_0).
\]

**Proof.** We prove (3.1) by considering different regions of \( x \).

**Case 1.** \( x \in \mathbb{R}_+, y \in E \). We denote by \( W \) a one-dimensional Brownian motion and \( p_1(t, x, y) = (2\pi t)^{-1/2} \exp(-|x-y|^2/2t) \) its transition density. Since on \( \mathbb{R}_+ \), \( X^0 \) has the same distribution as \( W \) before exiting \( \mathbb{R}_+ \), we have
\[
\nabla x p(t, x, y) = \nabla x p_{\mathbb{R}_+}(t, x, y) + \nabla x p_{\mathbb{R}_+}(t, x, y)
\]
\[
= \nabla x p_{\mathbb{R}_+}(t, x, y) + \nabla x \int_0^t \mathbb{P}_x (\sigma_{a^*} \in ds) p(t - s, a^*, y).
\]

Owing to the following density of 1-dimensional Brownian motion killed upon exiting \( \mathbb{R}_+ \), which is explicitly known:
\[
p_{1,\mathbb{R}_+}(t, x, y) = \frac{1}{\sqrt{2\pi t}} \left( e^{-(x-y)^2/(2t)} - e^{-(x+y)^2/(2t)} \right),
\]
we have
\[
|\nabla x p_{1,\mathbb{R}_+}(t, x, y)| \leq t^{-1} e^{-\beta|x-y|^2/2t}.
\]

The hitting time distribution for one-dimensional Brownian motion is also explicitly known (for example, see, [13]):
\[
\mathbb{P}_x (\sigma_{(0)} \in ds) = \frac{|x|}{\sqrt{2\pi s^3}} e^{-\frac{x^2}{2s}} ds.
\]
Denote the right hand side above by \( \phi(x, s) := \frac{|x|}{\sqrt{2\pi s^3}} e^{-\frac{x^2}{2s}} \). It yields from direct computation that
\[
\left| \frac{\partial}{\partial x} \phi(x, s) \right| = s^{-3/2} \left( 1 - \frac{x^2}{2s} \right) e^{-x^2/(2s)} \lesssim s^{-3/2} e^{-x^2/(4s)}.
\]

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Since $X^0$ and $W$ have the same distribution on $\mathbb{R}_+$ before hitting 0, in view of Case (i) in Theorem 4.3 we get
\[
\left| \nabla_x \int_0^t \mathbb{P}_x(\sigma_{a^*} \in ds) p(t - s, a^*, y) \right| = \left| \nabla_x \int_0^t \mathbb{P}_x(\sigma_0 \in ds) p(t - s, a^*, y) \right|
\]
\[
= \left| \nabla_x \int_0^t \phi(x, s)p(t - s, a^*, y)ds \right|
\]
\[
= \left| \int_0^t \frac{\partial}{\partial x} \phi(x, s) p(t - s, a^*, y)ds \right|
\]
\[
\leq \int_0^t e^{-\beta_1 y^2/(t-s)} ds
\]
\[
\leq t^{-1/2} e^{-\beta_1 |x-y|^2/t}.
\]
In the above computation we slightly abuse the notation by letting $\mathbb{P}_x$ stands for the distribution of $W$ before hitting zero. Combining (3.3) and (3.5) shows for this case that there exists some $\beta_1 > 0$ such that
\[
\left| \nabla_x p(t, x, y) \right| \leq t^{-1/2} p_{\beta_1}^0(t, x, y).
\]

**Case 2.** $x \in D_0, y \in \mathbb{R}_+$ We consider the radial process of $X^0$. defined as follows:
\[
u(x) = \begin{cases} 
-|x|, & x \in \mathbb{R}_+; \\
|x|, & x \in D_0.
\end{cases}
\] (3.6)

We define $Y_t := u(X^0_t)$ the signed radial process of $X^0$. It has been shown in [4, Proposition 4.3] that $Y$ is characterized by the following SDE:
\[
dY_t = dB_t + \frac{1}{2(Y_t + \varepsilon)} 1_{\{y_{t} > 0\}} dt + \frac{2\pi \varepsilon - p}{2\pi \varepsilon + p} d\hat{L}_t^0(Y),
\] (3.7)

where $\hat{L}_t^0(Y)$ is the symmetric semimartingale local time of $Y$ at 0. Let $\eta := \frac{2\pi \varepsilon - p}{2\pi \varepsilon + p}$ and $Z$ be the skew Brownian motion
\[
dZ_t = dB_t + \eta \hat{L}_t^0(Z),
\]
where $\hat{L}_t^0(Z)$ is the symmetric local time of $Z$ at 0. The diffusion process $Y$ can be obtained from $Z$ through a drift perturbation (i.e. Girsanov transform). The transition density function $p^Z(t, x, y)$ of $Z$ is explicitly known and enjoys the two-sided Aronson-type Gaussian estimates; see, e.g., [4]. One can further verify that for some constants $c_1, c_2 > 0$, it holds
\[
|\nabla_x p^Z(t, x, y)| \leq c_1 t^{-1} \exp(-c_2 |x - y|^2/t).
\]

By setting $f(x) := \frac{1}{x + \varepsilon} 1_{\{x > 0\}}$, it is not hard to see that on $\mathbb{R}$, $f \in K_1$:
\[
\sup_{x \in \mathbb{R}} \int_{|y-x| \leq 1} |f(y)|dy \leq \sup_{x \in \mathbb{R}_+} \int_{0 \wedge (x-1)}^{x+1} \frac{1}{y + \varepsilon} dy < \infty,
\]
By using the same argument as that for Theorem A(b) in Zhang [16 §4], we can derive that

\[ \left| \frac{\partial}{\partial x} p^Y(t, x, y) \right| \lesssim t^{-1} e^{-c_2(x-y)^2/t}, \]

where \( p^Y(t, x, y) \) denotes the transition density of \( Y \). When \( y \in \mathbb{R}_+ \), \( x \in D_0 \), \( p(t, x, y) = |x|^{-1} p^Y(t, |x|_{\rho}, -y) \) which only depends on \( |x| \). For functions on \( \mathbb{R}^d \) that are rotationally invariant, i.e., satisfying \( f(x) = g(r) \) for some function \( g \) on \( \mathbb{R} \) and for all \( |x| = r \), it holds that \( \nabla_x f(x) = g'(r) \cdot \frac{x}{|x|} \) and thus \( |\nabla_x f(x)| = |g'(r)| \). Hence,

\[ |\nabla_x p(t, x, y)| = \left| \frac{d}{dr} \left( \frac{1}{r} p^Y(t, r - \varepsilon, -y) \right) \right|, \quad r = |x| = |x|_{\rho} + \varepsilon. \]

Recall that \( |p^Y(t, x, -y)| \lesssim t^{-1/2} e^{-c_2(x^2+y^2)/t} \) when \( x, y > 0 \). We therefore have

\[ |\nabla_x p(t, x, y)| \lesssim |x|^{-1} t^{-1} e^{-c_2(|x|^2+|y|^2)/t} \lesssim t^{-1} e^{-c_2(|x|^2+|y|^2)/t}. \]

i.e., there exists some \( \beta_1 > 0 \) such that

\[ |\nabla_x p(t, x, y)| \lesssim \frac{1}{\sqrt{t}} p_{\beta_1}^0(t, x, y) \quad \text{when} \quad x \in \mathbb{R}_+, y \in D_0. \]

\[ \Box \]

**Proposition 3.2.** There exist constants \( \beta_2 > 0 \), \( C_2, C_3 > 0 \) such that the following two inequalities hold for all \( t \leq T \):

(i) If \( x, y \in D_0 \) with \( \max\{|x|_{\rho}, |y|_{\rho}\} \geq 1 \), then

\[ |\nabla_x p(t, x, y)| \leq \frac{C_2}{\sqrt{t}} p_{\beta_2}^0(t, x, y); \]

\[ (3.8) \]

(ii) If \( x, y \in D_0 \) with \( \max\{|x|_{\rho}, |y|_{\rho}\} < 1 \), then

\[ |\nabla_x p(t, x, y)| \leq \frac{C_3}{t} e^{-\beta_2(|x|^2+|y|^2)/t} + \frac{C_3}{t^{3/2}} \left( 1 + \frac{|y|_{\rho}}{\sqrt{t}} \right) e^{-\beta_2 |x-y|^2/t}. \]

\[ (3.9) \]

**Proof.** First of all, for both cases we have

\[ \nabla_x p(t, x, y) = \nabla_x p_{D_0}(t, x, y) + \nabla_x \pi_{D_0}(t, x, y). \]

(3.10)

In this proof, we denote by \( W \) two-dimensional Brownian motion and \( p_1(t, x, y) = (2\pi t)^{-1} \exp(-|x-y|^2/2t) \) its transition density. Started from \( y \in D_0 \), \( X^0 \) has the same distribution as \( W \) before hitting \( a^* \). By [5, Theorem 1.1], there exists some \( \beta > 0 \) such that

\[ |\nabla_x p_{D_0}(t, x, y)| = |\nabla_x p_{D_0}(t, x, y)| \lesssim t^{-3/2} \left( 1 + \frac{|y|_{\rho}}{\sqrt{t}} \right) e^{-\beta |x-y|^2/t}. \]

(3.11)
Therefore, for both cases, it holds that

\[ \alpha < \alpha \] such that for all \( \int_0^t P_y (\sigma_{a^*} \in ds) \leq \int_0^t P_y (\sigma_{a^*} < ds) \frac{1}{t-s} e^{-\beta_2 |x|^2/(t-s)}. \]

Thus the desired conclusion for Case (ii) has been proved. To verify the conclusion for Case (i), it suffices to observe that when \( \max \{ |x|, |y| \} > 1 \),

\[ |x| + |y| = |x| + |y| - 2 \varepsilon \geq |x - y| - 2 \varepsilon \geq \frac{1}{2} |x - y|, \] if \( |x - y| \geq 3 \varepsilon \);

while when \( |x - y| < 3 \varepsilon \leq 3/4 \) (since it has been assumed \( \varepsilon \leq 1/4 \),

\[ |x| + |y| = |x| + |y| - 2 \varepsilon \geq 1 - 2 \varepsilon > 1/2 > \frac{2}{3} |x - y|. \]

Therefore, when \( \max \{ |x|, |y| \} > 1 \), \( |x - y|^2/4 \leq |x|^2 + |y|^2 \), the gradient has the following upper bound:

\[ \int_0^t 1 \sum_{z \in E} p_{\alpha}(s, z) j(z)|\nabla_{x} p(t, x, y)| m_p(dz) ds \leq C_0(t)p_{\alpha}(t, x, y), \quad t \leq T, x, y \in \mathbb{R}_+. \] (3.14)

where \( C_4(t) \) is non-decreasing in \( t \), \( C_4(t) \rightarrow 0 \) as \( t \rightarrow 0 \). In particular, there exists some \( \alpha_1 > 0 \) such that for all \( \alpha < \alpha_1 \), it holds

\[ \int_0^t \int_{z \in E} p_{\alpha}(s, z) j(z)|\nabla_{x} p(s, z, y)| m_p(dz) ds \leq C_4(t)p_{\alpha}(t, x, y), \quad 0 < s < t \leq T, x, y \in \mathbb{R}_+. \] (3.15)
Proof. Without loss of generality, we assume \( b \geq 0 \). In view of Proposition 3.1, we use the Chapman-Kolmogorov equation and divide the computation into two parts depending on the position of \( z \). Recall \( b = b_1 + b_2 \) where \( b_1 := b|_{\mathbb{R}^+} \in L^{p_1}(\mathbb{R}^+) \), and \( b_2 := b|_{D_0} \in L^{p_2}(D_0) \). By Hölder’s inequality, for \( p_1, q_1 > 0 \) satisfying \( 1/p_1 + 1/q_1 = 1 \), it holds that

\[
\int_0^t \int_{\mathbb{R}^+} p_0^0(t-s, x, z) b(z) p_0^0(s, z, y) m_p(dz) ds \\
\leq \int_0^t \frac{1}{\sqrt{s}} \int_{\mathbb{R}^+} b(z) e^{-\alpha |x-z|^2/(t-s)} e^{-\beta |z-y|^2/s} m_p(dz) ds
\]

\[
\leq e^{-\alpha |x-y|^2/t} \int_0^t s^{-1+1/(2q_1)(t-s)^{-1/2}} b(z) s^{-1/(2q_1)} e^{-(\beta-\alpha)|z-y|^2/s} m_p(dz) ds
\]

\[
\leq e^{-\alpha |x-y|^2/t} \int_0^t s^{-1+1/(2q_1)(t-s)^{-1/2}} \|b_1\|_{p_1} \left( \int_{\mathbb{R}^+} \left( s^{-1/(2q_1)} e^{-(\beta-\alpha)|z-y|^2/s} \right)^{q_1} m_p(dz) \right)^{1/q_1} ds
\]

\[
\lesssim \|b_1\|_{p_1} e^{-\alpha |x-y|^2/t} \cdot \frac{t^{1/(2q_1)}}{\sqrt{t}}. \tag{3.16}
\]

For \( z \in D_0 \), for \( q_2 \) satisfying \( 1/p_2 + 1/q_2 = 1 \), it holds

\[
\int_0^t \int_{D_0} p_0^0(t-s, x, z) b(z) p_0^0(s, z, y) m_p(dz) ds \\
\leq \int_0^t \frac{1}{\sqrt{s}} \int_{D_0} b(z) e^{-\alpha |x|^2/(t-s)} e^{-\beta |y|^2/s} m_p(dz) ds
\]

\[
\leq e^{-\alpha |x-y|^2/t} \int_0^t s^{-1+1/(2q_2)(t-s)^{-1/2}} b(z) s^{-1/(2q_2)} e^{-\beta |y|^2/s} m_p(dz) ds
\]

\[
\leq e^{-\alpha |x-y|^2/t} \int_0^t s^{-1+1/(2q_2)(t-s)^{-1/2}} \|b_2\|_{p_2} \left( \int_{D_0} \left( s^{-1/(2q_2)} e^{-\beta |y|^2/s} \right)^{q_2} m_p(dz) \right)^{1/q_2} ds
\]

\[
\lesssim \|b_2\|_{p_2} e^{-\alpha |x-y|^2/t} \cdot \frac{t^{1/(2q_2)}}{\sqrt{t}}. \tag{3.17}
\]

The proof is complete by adding up (3.16) and (3.17). \( \square \)

The next proposition is analogous to Proposition 3.3 above but concerning the case \( x \in \mathbb{R}^+ \) and \( y \in D_0 \).

**Proposition 3.4.** There exist \( \alpha_2 > 0 \) and a function \( C_5(t) > 0 \) such that for all \( 0 < \alpha < \alpha_2 \),

\[
\int_0^t \int_{z \in E} p_0^0(t-s, x, z) |b(z)||\nabla z p(s, z, y)| m_p(dz) ds \leq C_5(t) p_0^0(t, x, y), \tag{3.18}
\]

\[0 < s < t \leq T, x \in \mathbb{R}^+, y \in D_0,
\]

where \( C_5(t) \) is non-decreasing in \( t \), \( C_5(t) \to 0 \) as \( t \to 0 \).
Proof. We again assume \( b \geq 0 \) and divide the proof into three cases depending on the position of \( z \).

**Case 1.** \( z \in \mathbb{R}_+ \). In view of Proposition 3.1, it suffices to show the following inequality: For any \( 0 < \alpha < \beta \), it holds

\[
\int_0^t \frac{1}{\sqrt{s}} \int_E p^0_\alpha(t-s, x, z)b(z)p^0_\beta(s, z, y)m_p(dz)ds \leq c_1(t)p^0_\alpha(t, x, y), \quad t, x \in \mathbb{R}_+, y \in D_0,
\]

(3.19)

for some \( c_1(t) \) satisfying the same condition stated in the proposition for \( C_3(t) \). Since \( b_1 \in L^p(\mathbb{R}_+) \) with \( p_1 \in (1, \infty) \), we may pick \( q_1 > 0 \) satisfying \( 1/p_1 + 1/q_1 = 1 \).

\[
\int_0^t \int_{\mathbb{R}_+} p^0_\alpha(t-s, x, z)b(z)p^0_\beta(s, z, y)m_p(dz)ds \\
\lesssim \int_0^t \frac{1}{\sqrt{s}} \int_{\mathbb{R}_+} \frac{1}{\sqrt{t-s}} e^{-\alpha|z-z|^2/(t-s)} \frac{1}{\sqrt{s}} e^{-\beta(|z|^2+|y|^2)/s} b(z)m_p(dz)ds \\
\lesssim \int_0^t \frac{1}{(t-s)^{1/2}} \frac{1}{s^{1/2-q_1}} e^{-\alpha(|z|^2+|y|^2)/t} \left( \int_{\mathbb{R}_+} \left( \frac{1}{s^{1/2-q_1}} e^{-\beta(|z|^2/2)} \right)^{q_1} m_p(dz) \right)^{1/q_1} ds \| b_1 \|_{p_1} \\
\lesssim \| b_1 \|_{p_1} \left( \frac{t^{1/(2q_1)}}{\sqrt{t}} e^{-\alpha(|z|^2+|y|^2)/t} \right),
\]

(3.20)

where in the second inequality it is used the triangle inequality that \( \frac{|x-z|^2}{t-s} + \frac{|z|^2}{s} \geq \frac{|x|^2}{t} \). However, when \( z \in D_0 \), we further need to break this into two subcases depending on \( |z|_\rho \).

**Case 2.** \( z \in D_0 \), and \( |z|_\rho \geq 1 \). To bound the left hand side of (3.19) from above, we again seek to establish the following inequality for \( 0 < \alpha < \beta \):

\[
\int_0^t \frac{1}{\sqrt{s}} \int_{z \in D_0, |z|_\rho \geq 1} p^0_\alpha(t-s, x, z)b(z)p^0_\beta(s, z, y)m_p(dz)ds \leq c_2(t)p^0_\alpha(t, x, y), \quad t, x \in \mathbb{R}_+, y \in D_0,
\]

(3.21)

where \( c_2(t) \to 0 \) as \( t \to 0 \). Again by Hölder’s inequality, we have

\[
\int_0^t \frac{1}{\sqrt{s}} \int_{z \in D_0, |z|_\rho \geq 1} p^0_\alpha(t-s, x, z)b(z)p^0_\beta(s, z, y)m_p(dz)ds \\
\lesssim \int_0^t \frac{1}{(t-s)^{1/2}} \frac{1}{s^{1/2-q_2}} e^{-\alpha(|z|^2+|y|^2)/t} \left( \int_{z \in D_0} \left( \frac{1}{s^{1/2-q_2}} e^{-\beta(|z-y|^2)/2} \right)^{q_2} m_p(dz) \right)^{1/q_2} ds \| b_2 \|_{p_2} \\
\lesssim \frac{t^{1/q_2}}{t} e^{-\alpha(|z|^2+|y|^2)/t} \| b_2 \|_{p_2},
\]

(3.22)

where in the second “ \( \lesssim \) ”, it has been used the inequality \( \frac{|z|^2}{t-s} + \frac{|z-y|^2}{s} \geq \frac{|y|^2}{t} \), which is due to the triangle inequality of distance functions:

\[
(t-s)s|y|^2 \leq (t-s)s(|z|_\rho + |z-y|)^2 \leq st|z|^2 + (t-s)s|z-y|^2.
\]

Note \( 1/q_2 > 1/2 \) as \( 1/p_2 + 1/q_2 = 1 \) and \( p_2 \in (2, \infty) \). The conclusion has been proved.

**Case 3.** \( z \in D_0 \), and \( |z|_\rho \leq 1 \). In this case, if \( |y|_\rho \geq 1 \), following an exact same computation as
that for Case 2, we have

\[
\int_0^t \int_{z \in D_0, |z| < 1} p^0_\alpha (t - s, x, z) b(z) p^0_\beta (s, z, y) m_p (dz) ds \\
\lesssim \int_0^t \frac{1}{(t - s)^{1/2}} e^{-\alpha (|x|^2 + |z|^2)/(t - s)} \int_{D_0} \frac{e^{-\beta |y|^2/s} b(z) m_p (dz)}{s^{3/2}} ds \\
\lesssim e^{-\alpha (|x|^2 + |y|^2)/t} \int_0^t \frac{1}{s^{3/2 - 1/2q}} \left( \int_{z \in D_0} \frac{e^{-\beta (|z| |y|)/s} m_p (dz)}{s^{1/2} (t - s)^{1/2}} \right)^{q_2} ds \| b_2 \|_{L^q_p} \\
\lesssim \frac{t^{1/q_2}}{t} e^{-\alpha (|x|^2 + |y|^2)/t} \| b_2 \|_{L^q_p}.
\]

(3.23)

However, if \(|y|_p < 1\), then by Proposition 3.2, there exists some \(\beta > 0\) such that

\[
|\nabla_z p(s, z, y)| \lesssim \frac{1}{s} e^{-\beta (|x|^2 + |y|^2)/s} + \frac{1}{s^{3/2}} \left( 1 \wedge \frac{|y|_p}{\sqrt{s}} \right) e^{-\beta |y|^2/s} \\
\lesssim \frac{1}{s} e^{-\beta (|x|^2 + |y|^2)/s} + \frac{1}{s^{3/2}} e^{-\beta |y|^2/s}.
\]

(3.24)

Let \(0 < \alpha < \beta\). To take care of the two terms in the last display of (3.24) separately, we first observe that using the exact same computation as that for Case 2, one can show

\[
\int_0^t \frac{1}{\sqrt{t - s}} e^{-\alpha (|x|^2 + |z|^2)/(t - s)} \int_{D_0} \frac{b(z)}{s^{3/2}} e^{-\beta (|z|^2 + |y|^2)/s} m_p (dz) ds \lesssim \frac{t^{1/q_2 - 1/2}}{\sqrt{t}} e^{-\alpha (|x|^2 + |y|^2)/t} \| b_2 \|_{L^q_p}.
\]

(3.25)

On the other hand, to handle the first term in the last display of (3.24), we have the following:

\[
\int_0^t \frac{1}{\sqrt{t - s}} e^{-\alpha (|x|^2 + |z|^2)/(t - s)} \int_{z \in D_0} b(z) e^{-\beta (|z|^2 + |y|^2)/s} m_p (dz) ds \\
\leq e^{-\alpha (|x|^2 + |y|^2)/t} \int_0^t \frac{1}{(t - s)^{1/2}} s^{1 - 1/(2q_2)} \left( \int_{z \in D_0} \frac{b(z)}{s^{1/(2q_2)}} e^{-\beta |z|^2/s} m_p (dz) \right)^{1/q_2} ds \cdot \| b_2 \|_{L^q_p} \\
\leq \| b_2 \|_{L^q_p} \frac{t^{1/(2q_2)}}{\sqrt{t}} e^{-\alpha (|x|^2 + |y|^2)/t}.
\]

(3.26)

This combined with (3.25) has completed the discussion for the situation when \(|y|_p < 1\). Now combining the above two parts: \(|y|_p \geq 1\) and \(|y|_p < 1\), we have shown that there exists \(\beta > 0\) such that for all \(0 < \alpha < \beta\), it holds

\[
\int_0^t \int_{z \in D_0} p^0_\alpha (t - s, x, z) |\nabla_z p(s, z, y)| m_p (dz) ds \\
\lesssim \int_0^t \int_{z \in D_0} p^0_\alpha (t - s, x, z) b(z) \left( \frac{1}{s^{3/2}} e^{-\beta |y|^2/s} + \frac{1}{s} e^{-\beta (|x|^2 + |y|^2)/s} \right) m_p (dz) ds \\
\lesssim \| b_2 \|_{L^q_p} \left( \frac{t^{1/(2q_2)} + t^{1/q_2 - 1/2}}{t^{1/2}} \right) e^{-\alpha (|x|^2 + |y|^2)/t},
\]

where \(1/q_2 > 1/2\). This establishes (3.40) for Case 3. □
The next proposition is for the case when \( x \in D_0 \) and \( y \in \mathbb{R}_+ \). However, for this case, we need to split it further into two propositions depending on whether \( |x|_\rho \leq 1 \) or \( |x|_\rho > 1 \). For \( x \in D_0 \) with \( |x|_\rho \leq 1 \), we rewrite the canonical form of the heat kernel of \( X^0 \) and name it as \( p_{1,\alpha}^0 \) as follows:

\[
p_{1,\alpha}^0(t, x, y) := \begin{cases} 
\frac{1}{\sqrt{t}} e^{-\frac{\alpha |x-y|^2}{t}}, & x \in \mathbb{R}_+, y \in \mathbb{R}_+; \\
\frac{1}{\sqrt{t}} e^{-\frac{\alpha (|x|^2+|y|^2)}{t}}, & x \in \mathbb{R}_+, y \in D_0; \\
\frac{1}{\sqrt{t}} e^{-\frac{\alpha |x|^2+|y|^2}{t}}, & x \in D_0, y \in \mathbb{R}_+; \\
\frac{1}{\sqrt{t}} e^{-\frac{\alpha |x-y|^2}{t}} + \frac{1}{t} \left( 1 \wedge \frac{|x|_\rho}{\sqrt{t}} \right) \left( 1 \wedge \frac{|y|_\rho}{\sqrt{t}} \right) e^{-\frac{2\alpha |x-y|^2}{t}}, & \text{otherwise.}
\end{cases}
\] (3.27)

The only difference between \( p_{\alpha}^0(t, x, y) \) and \( p_{1,\alpha}^0(t, x, y) \) is that in the last display, the coefficient \( \alpha \) on the second exponential term is replaced with \( 2\alpha \). The following theorem is immediate.

**Lemma 3.5.** There exist constants \( C_6 > 0 \) and \( \alpha_3 > 0 \) such that

\[
p(t, x, y) \leq C_6 p_{1,\alpha_3}^0(t, x, y), \quad (t, x, y) \in (0, T] \times E \times E.
\]

**Proposition 3.6.** It holds for any \( 0 < \alpha < \beta/2 \) that

\[
\int_0^t \frac{1}{\sqrt{s}} \int_E p_{1,\alpha}^0(t - s, x, z) b(z) |p_{1,\beta}^0(s, z, y) m_p(dz) ds \leq C_7(t) p_{1,\alpha}^0(t, x, y),
\] (3.28)

\[
t \leq T, x \in D_0, |x|_\rho \leq 1, y \in \mathbb{R}_+, \quad \text{where } C_7(t) \text{ is non-decreasing in } t, \ C_7(t) \to 0 \text{ as } t \to 0. \text{ In particular, there exists some } \alpha_4 > 0 \text{ such that for all } \alpha < \alpha_4,
\]

\[
\int_0^t \int_{z \in E} p_{1,\alpha}^0(t - s, x, z) b(z) |\nabla_z p(s, z, y)| m_p(dz) ds \leq C_7(t) p_{1,\alpha}^0(t, x, y)
\] (3.29)

\[
0 < s < t \leq T, x \in D_0, |x|_\rho \leq 1, y \in \mathbb{R}_+.
\]

**Proof.** Again without loss of generality we assume \( b \geq 0 \). This will be established for \( z \in \mathbb{R}_+ \) and \( z \in D_0 \) separately.

**Case 1.** \( z \in \mathbb{R}_+ \). For \( q_1 > 0 \) such that \( 1/p_1 + 1/q_1 = 1 \), it holds

\[
\int_0^t \frac{1}{\sqrt{s}} \int_{\mathbb{R}_+} p_{1,\alpha}^0(t - s, x, z) b(z) p_{1,\beta}^0(s, z, y) m_p(dz) ds \\
\lesssim \int_0^t \frac{1}{(t-s)^{1/2}} e^{-\alpha (|x|^2+|z|^2)/(t-s)} \int_{\mathbb{R}_+} \frac{1}{s} e^{-\beta |z-y|^2/s} b(z) m_p(dz) ds \\
\lesssim e^{-\alpha (|x|^2+|y|^2)/t} \int_0^t \frac{1}{s^{1-1/2q_1}} \frac{1}{(t-s)^{1/2}} \left( \int_{z \in \mathbb{R}_+} \left( \frac{1}{s^{1/2q_1}} e^{-\beta (z-y)^2/s} \right)^{q_1} m_p(dz) \right)^{1/q_1} ds \|b_1\|_{p_1} \\
\lesssim \frac{t^{1/2q_1}}{\sqrt{t}} e^{-\alpha (|x|^2+|y|^2)/t} \|b_1\|_{p_1},
\] (3.30)
Case 2. $z \in D_0$. We first observe that

$$\beta |z|_p \geq \alpha |z| = \alpha (|z|_p + \varepsilon) \quad \text{when} \quad |z|_p \geq \frac{\alpha \varepsilon}{\beta - \alpha}. \quad (3.31)$$

This implies that when $\alpha < \beta/2$ and $|z|_p \geq \varepsilon$, it holds $\beta |z|_p \geq \alpha |z|$. We write

$$\int_0^t \frac{1}{s} \int_{D_0} p^0_{1, \alpha} (t - s, x, z) b(z) p^0_{1, \beta} (s, z, y) m_p (dz) ds$$

$$= \int_0^t \frac{1}{s} \int_{z \in D_0, |z|_p \geq \varepsilon} p^0_{1, \alpha} (t - s, x, z) b(z) p^0_{1, \beta} (s, z, y) m_p (dz) ds$$

$$+ \int_0^t \frac{1}{s} \int_{z \in D_0, |z|_p < \varepsilon} p^0_{1, \alpha} (t - s, x, z) b(z) p^0_{1, \beta} (s, z, y) m_p (dz) ds. \quad (3.32)$$

For the first integral on the right hand side of (3.32), it holds

$$\int_0^t \frac{1}{s} \int_{z \in D_0, |z|_p \geq \varepsilon} p^0_{1, \alpha} (t - s, x, z) b(z) p^0_{1, \beta} (s, z, y) m_p (dz) ds$$

$$\leq e^{-\beta |y|^2/t} \int_0^t \int_{z \in D_0, |z|_p \geq \varepsilon} \frac{1}{t - s} e^{-\alpha |x - z|^2/(t-s)} \cdot b(z) \frac{1}{s} e^{-\alpha |z|^2/s} m_p (dz) ds$$

$$\leq e^{-\beta |y|^2/t} \int_0^t \frac{1}{(t - s)^{1-1/q_2}} s^{1-1/q_2}$$

$$\times \left( \int_{z \in D_0, |z|_p \geq \varepsilon} e^{-\alpha q_2 |x - z|^2/(t-s)} \frac{1}{s} e^{-\alpha q_2 |z|^2/s} m_p (dz) \right)^{1/q_2} ds \|b_2\|_{p_2}$$

$$\leq \frac{1}{t^{1-1/q_2}} e^{-\alpha (|x|^2 + |y|^2)/t} \|b_2\|_{p_2} \leq \frac{1}{t^{1-1/q_2}} e^{-\alpha (|x|^2 + |y|^2)/t} \|b_2\|_{p_2}. \quad (3.33)$$

The desired conclusion has been proved by taking $C_{10}(t) = t^{1/2 - 1/p_2}$, because $1 - 1/q_2 = 1/p_2 < 1/2$ given that $p_2 \in (2, \infty]$. However, for the second integral on the right hand side of (3.32), we recall that when $\max \{ |x|_p, |z|_p \} < 1$,

$$p^0_{1, \alpha} (t - s, x, z) \lesssim \frac{1}{\sqrt{t - s}} e^{-\alpha (|x|^2 + |z|^2)/(t-s)} + \frac{1}{t - s} \left( 1 \wedge \frac{|x|_p}{\sqrt{t - s}} \right) \left( 1 \wedge \frac{|z|_p}{\sqrt{t - s}} \right) e^{-2\alpha |x - z|^2/(t-s)}$$

$$\leq \frac{1}{\sqrt{t - s}} e^{-\alpha (|x|^2 + |z|^2)/(t-s)} + \frac{1}{t - s} e^{-2\alpha |x - z|^2/s}. \quad (3.34)$$
Therefore, for the first term on the right hand side of (3.34), it holds
\[
\int_0^t \frac{1}{\sqrt{s}} \int_{|z| < \epsilon} \frac{1}{t-s} e^{-\alpha(|z|^2 + |z|^2)/(t-s)} b(z) p_{1,\alpha}^0(s, z, y) m_p(dz) \, ds
\]  
\[\lesssim \int_0^t \frac{1}{(t-s)^{1/2}} e^{-\alpha(|z|^2 + |z|^2)/(t-s)} \int_{|z| < \epsilon} \frac{1}{s} e^{-\beta(|z|^2 + |y|^2)/s} b(z) m_p(dz) \, ds
\]  
\[\lesssim e^{-\alpha(|z|^2 + |y|^2)/t} \int_0^t \frac{1}{s^{1/2} q_2} \frac{1}{(t-s)^{1/2}} \left( \int_{|z| < \epsilon} \left( \frac{1}{s^{1/2} q_2} e^{-\left(\beta - \alpha\right)(|z|^2 + |y|^2)/s} m_p(dz) \right)^{q_2} \right)^{1/q_2} \, ds \|b_2\|_{p_2}
\]  
\[\lesssim \frac{t^{1/2 q_2}}{\sqrt{t}} e^{-\alpha(|z|^2 + |y|^2)/t} \|b_2\|_{p_2}, \quad (3.35)
\]

For the second term on the right hand side of (3.34), we first record the following computation over \( \{z \in D_0, |z| < \epsilon\} \) for any \( c > 0 \) which will be used later:
\[
\int_{z \in D_0, |z| < \epsilon} \frac{1}{\sqrt{t-s}} e^{-\alpha|x-z|^2/(t-s)} \frac{1}{\sqrt{s}} e^{-\alpha|x|^2/s} m_p(dz)
\]  
\[= \int_{z \in D_0, |z| < \epsilon} \frac{1}{\sqrt{t-s}} e^{-\alpha|x-z|^2/(t-s)} \frac{1}{\sqrt{s}} e^{-\alpha|x|^2/s} m_p(dz)
\]  
\[\leq e^{-\alpha|x|^2/t} \int_{z \in D_0, |z| < \epsilon} \frac{1}{\sqrt{t-s}} e^{-\alpha|x-z|^2/(t-s)} \frac{1}{\sqrt{s}} e^{-\alpha|x|^2/s} m_p(dz)
\]  
\[\lesssim e^{-\alpha|x|^2/2} e^{\alpha|x|^2} \int_{|z| < \epsilon} \frac{1}{\sqrt{t-s}} e^{-\alpha(x-r)^2/(t-s)} \frac{1}{\sqrt{s}} e^{-\alpha(x-r)^2/s} p \, dr
\]  
\[\lesssim e^{-\alpha|x|^2/2} e^{\alpha|x|^2} \int_{|z| < \epsilon} \frac{1}{\sqrt{t-s}} e^{-\alpha(x-r)^2/(t-s)} \frac{1}{\sqrt{s}} e^{-\alpha(x-r)^2/s} p \, dr
\]  
\[\lesssim \frac{1}{\sqrt{t}} e^{-\alpha|x|^2/(t-s)^2} = \frac{1}{\sqrt{t}} e^{-\alpha(|z|^2 + |y|^2)/t}. \quad (3.36)
\]

With the computation above, we choose \( q_2', p_2 > 0 \) satisfying \( 1/p_2 + 1/q_2' + 1/r_2 = 1 \). Since \( b \in L^{p_2}(D_0) \) for some \( p_2 \in (2, \infty] \), we may pick \( q_2' \) sufficiently large such that \( 1/(2q_2') + 1/p_2 < 1/q_2' + 1/p_2 < 1/2 \). It follows that
\[
\int_0^t \frac{1}{\sqrt{s}} \int_{z \in D_0, |z| < \epsilon} \frac{1}{t-s} e^{-2\alpha|x-z|^2/(t-s)} b(z) p_{1,\alpha}^0(s, z, y) m_p(dz) \, ds
\]  
\[\lesssim \int_0^t \frac{1}{\sqrt{s}} \int_{|z| < \epsilon} \frac{1}{t-s} e^{\beta(y)^2 + |z|^2} \frac{1}{\sqrt{s}} e^{-\alpha(y)^2/s} m_p(dz) \, ds
\]  
\[\lesssim \int_0^t \frac{1}{s^{1/(2q_2') + 1/p_2 + 1/r_2}} \frac{1}{(t-s)^{1/(2q_2') - 1/r_2}} \left( \int_{|z| < \epsilon} \frac{1}{s^{1/(2q_2')}} e^{-\alpha(y)^2 + |z|^2} \frac{1}{(t-s)^{1/(2q_2')}} e^{-\alpha|x-z|^2} \right)^{1/q_2'} \, ds
\]  
\[\lesssim \int_0^t \frac{1}{s^{1/(2q_2') + 1/p_2 + 1/r_2}} \frac{1}{(t-s)^{1/(2q_2') - 1/r_2}} \, ds \left( \int_{|z| < \epsilon} \frac{1}{\sqrt{s}} e^{-\alpha(y)^2 + |z|^2} \frac{1}{\sqrt{t-s}} e^{-\alpha|x-z|^2} m_p(dz) \right)^{1/q_2'}
\]  
\[\lesssim \int_0^t \frac{1}{s^{1/(2q_2') + 1/p_2 + 1/r_2}} \frac{1}{(t-s)^{1/(2q_2') - 1/r_2}} \, ds \cdot \left( \int_{|z| < \epsilon} \frac{1}{t-s} e^{-\alpha|x-z|^2/t-s} m_p(dz) \right)^{1/r_2} \cdot \|b_2\|_{p_2} \lesssim \|b_2\|_{p_2} \quad (3.37)
\]
Combining (3.35) and (3.53) yields that
\[
\int_{z \in D_0, |z|_\rho < \alpha \varepsilon / (\beta - \alpha)} p_{1,\alpha}^0(t - s, x, z)b(z)|\nabla_z p(s, z, y)| m_p(dz) \leq \frac{C(t)}{\sqrt{t}} e^{-\frac{\alpha(|x|^2 + |y|^2)}{t}} \cdot \|b_2\|_{p_2}.
\]

For the next case which is \( x \in D_0 \) with \( |x|_\rho > 1 \) and \( y \in \mathbb{R}_+ \), we rewrite the canonical form of the heat kernel of \( X^0 \) as follows:
\[
p_{2,\alpha}^0(t, x, y) := \begin{cases} 
1 \frac{1}{\sqrt{t}} e^{-\frac{a|x-y|^2}{t}}, x, y \in \mathbb{R}_+; \\
1 \frac{1}{\sqrt{t}} e^{-\frac{\alpha(|x|^2 + |y|^2)}{t}}, x, y \in D_0 \text{ or } x \in D_0, y \in \mathbb{R}_+; \\
1 \frac{1}{\sqrt{t}} e^{-\frac{2a|x-y|^2}{t}}, x, y \in D_0 \cup \{a^*\} \text{ with } \max\{|x|_\rho, |y|_\rho\} > 1; \\
1 \frac{1}{\sqrt{t}} e^{-\frac{\alpha(|x|^2 + |y|^2)}{t}} + \frac{1}{t} \left( 1 \wedge \frac{|x|_\rho}{\sqrt{t}} \right) \left( 1 \wedge \frac{|y|_\rho}{\sqrt{t}} \right) e^{-\frac{a|x-y|^2}{t}}, \text{ otherwise.}
\end{cases}
\]  

The difference between \( p_{\alpha}^0(t, x, y) \) and \( p_{2,\alpha}^0(t, x, y) \) is that for the case \( x, y \in D_0 \) with \( \max\{|x|_\rho, |y|_\rho\} > 1 \), the coefficient \( \alpha \) on the exponential term is replaced with \( 2\alpha \). As before, the following theorem is immediate.

**Lemma 3.7.** There exist constants \( C_8 > 0 \) and \( \alpha_5 > 0 \) such that it holds for the transition density of \( X^0 \) that
\[
p(t, x, y) \leq C_8 p_{2,\alpha_5}^0(t, x, y), \quad (t, x, y) \in (0, T] \times E \times E.
\]

**Proposition 3.8.** It holds for any \( 0 < \alpha < \beta \) that
\[
\int_0^t \frac{1}{\sqrt{s}} \int_E p_{2,\alpha}^0(t - s, x, z)b(z)p_{2,\beta}(s, z, y)m_p(dz)ds \leq C_{12}(t)p_{2,\alpha}(t, x, y),
\]
\[
t \leq T, \quad x \in D_0, \quad |x|_\rho > 1, \quad y \in \mathbb{R}_+.
\]

In particular, there exists some \( \alpha_6 > 0 \) such that for all \( \alpha < \alpha_6 \),
\[
\int_0^t \int_{z \in E} p_{2,\alpha}^0(t - s, x, z)|b(z)||\nabla_z p(s, z, y)| m_p(dz)ds \leq C_9(t)p_{2,\alpha}^0(t, x, y)
\]
\[
0 < s < t \leq T, \quad x \in D_0, \quad |x|_\rho > 1, \quad y \in \mathbb{R}_+,
\]
where \( C_9(t) \) is non-decreasing in \( t \) and \( C_9(t) \to 0 \) as \( t \to 0 \).

**Proof.** Case 1. \( z \in \mathbb{R}_+ \). Showing (3.38) to this case is identical to the first case of Proposition 3.6, which is omitted here.

Case 2. \( z \in D_0 \). For this case, it holds
\[
\int_0^t \frac{1}{\sqrt{s}} \int_{D_0} p_{2,\alpha}^0(t - s, x, z)b(z)p_{2,\beta}(s, z, y)m_p(dz)ds
\]
\[
= \int_0^t \frac{1}{\sqrt{s}} \int_{z \in D_0} \frac{1}{t - s} e^{-2a|x-z|^2/(t-s)}b(z)\frac{1}{\sqrt{s}} e^{-\beta(|z|^2 + |y|^2)/s}m_p(dz)ds.
\]

\( \int \)
Choose $p_2, q_2, r_2 > 0$ satisfying $1/p_2 + 1/q_2 + 1/r_2 = 1$. Since $b \in L^r(D_0)$ for some $p_2 \in (2, \infty]$, we may pick $q_2$ sufficiently large such that $1/(2q_2) + 1/p_2 < 1/q_2 + 1/p_2 < 1/2$. It follows that

$$
\int_0^t \frac{1}{\sqrt{s}} \int_{D_0} \frac{1}{t-s} b(z) e^{-2\alpha|x-z|^2/(t-s)} \frac{1}{\sqrt{s}} e^{-\beta(|y|^2+|z|^2)/s} m_p(dz) ds
$$

$$
\lesssim \int_0^t \frac{1}{\sqrt{s}} \int_{z \in D_0} \frac{1}{t-s} \frac{1}{s/(2q_2) + 1/p_2 + 1/r_2} \frac{1}{(t-s)^{(2q_2) - 1/r_2}} \int_{z \in D_0} \frac{1}{s/(2q_2)} e^{-\alpha(|y|^2+|z|^2)/s} \frac{1}{(t-s)^{(2q_2) - 1/r_2}} e^{-\alpha|x-z|^2/(t-s)} b(z) \frac{1}{(t-s)^{1/2}} e^{-\alpha|x-z|^2/(t-s)} m_p(dz) ds
$$

$$
\lesssim \int_0^t \frac{1}{\sqrt{s}} \int_{z \in D_0} \frac{1}{t-s} \frac{1}{s/(2q_2) + 1/p_2 + 1/r_2} \frac{1}{(t-s)^{(2q_2) - 1/r_2}} ds \left( \int_{D_0} \frac{1}{\sqrt{s}} e^{\frac{-\alpha |y|^2}{s}} \frac{1}{\sqrt{t-s}} e^{\frac{-\alpha |y|^2}{s}} m_p(dz) \right)^{1/q_2}
$$

$$
\times \left( \int_{z \in D_0} \frac{1}{t-s} e^{-\alpha r_2|x-z|^2/(t-s)} m_p(dz) \right)^{1/r_2} \cdot ||b_2||_{p_2}
$$

$$
\lesssim \frac{1}{t^{1/p_2}} \cdot \frac{1}{t^{1/(2q_2)}} e^{-\alpha(|x|^2 + |y|^2)/t} \cdot ||b_2||_{p_2}.
$$

(3.42)

For the remaining of this paper, we first fix a large constant $M \geq 8$ as follows: Let $\beta_1, \beta_2 > 0$ be the same as in Proposition 3.1 and Proposition 3.2. Let $\beta_3 = \min\{\beta_1, \beta_2\}$. Then we select $M \geq 8$ sufficiently large so that

$$
\beta_3 |z - y|^2 / 4 \geq 8\varepsilon + 4|y|, \quad \text{for all } z \in D_0, |z| < 4, |y| > M.
$$

(3.43)

In particular, observing that $\varepsilon \leq 1/4$, this means that

$$
\beta_3 |y|^2 / 4 \geq 8\varepsilon + 4|y|, \quad \text{for all } |y| > M.
$$

For the next proposition, we let the canonical form be as follows:

$$
p_{3, \alpha}^0(t, x, y) := \begin{cases}
\frac{1}{\sqrt{t}} e^{-\alpha |x|^2 / t}, & x \in \mathbb{R}_+, y \in \mathbb{R}_+; \\
\frac{1}{\sqrt{t}} e^{-4\alpha (|x|^2 + |y|^2) / t}, & x \in \mathbb{R}_+, y \in D_0; \\
\frac{1}{\sqrt{t}} e^{-4\alpha (|x|^2 + |y|^2) / t}, & x \in D_0, y \in \mathbb{R}_+; \\
\frac{1}{\sqrt{t}} e^{-\alpha |x-y|^2 / t}, & x, y \in D_0 \cup \{a^*\} \text{ with } \max\{|x|_\rho, |y|_\rho\} > 1; \\
\frac{1}{\sqrt{t}} e^{-\alpha (|x|_\rho^2 + |y|_\rho^2) / t} \left( 1 + \frac{|x|_\rho}{\sqrt{t}} + \frac{|y|_\rho}{\sqrt{t}} \right) e^{-\alpha |x-y|^2 / t}, & \text{otherwise}.
\end{cases}
$$

(3.44)

Note that the difference between $p_{3, \alpha}^0(t, x, y)$ and $p_\alpha^0(t, x, y)$ is that for the case when one of $x$ and $y$ is in $\mathbb{R}_+$ and the other one is in $D_0$, we let the constant involved in the exponential term be $4\alpha$ instead of $\alpha$. The next lemma follows immediately.
Lemma 3.9. There exists constants $C_{10} > 0$, $\alpha_7 > 0$ such that it holds that
\[ p(t, x, y) \leq C_{10} p_{3, \alpha_7}^0 (t, x, y), \quad t \in (0, T), (x, y) \in E \times E. \]

Proposition 3.10. Let $\beta = \beta_3$ in (3.43) and $M$ be the same as in (3.43). It holds for some function $C_{11}(t) > 0$ and any $0 < \alpha < \beta/4$ that
\[
\int_0^t \frac{1}{\sqrt{s}} \int_{E^+} p_{3, \alpha}^0 (t - s, x, z) b(z) p_{3, \beta}^0 (s, z, y) m_p(dz) ds \leq C_{11}(t) p_{3, \alpha}^0 (t, x, y),
\]
\[ t \leq T, x, y, \in D_0, \text{ and } \| y \|_\rho > M, \] (3.45)
where $C_{11}(t)$ is non-decreasing in $t$, $C_{11}(t) \to 0$ as $t \to 0$. In particular, there exists some $\alpha_8 > 0$ such that for all $\alpha < \alpha_8$,
\[
\int_0^t \int_{z \in E^+} p_{3, \alpha}^0 (t - s, x, z) \| \nabla p(s, z, y) \| dz ds \leq C_{11}(t) p_{3, \alpha}^0 (t, x, y), 0 < s < t \leq T,
\]
\[ x, y, \in D_0, \text{ and } \| y \|_\rho > M. \] (3.46)

Proof. For $z \in \mathbb{R}_+$. Observe that for this case, since $\| y \|_\rho > M \geq 8$ and $\varepsilon < 1/4$,
\[
4(|x|^p + |y|^p) \geq |x|^p + 2|x| |y| + 4\varepsilon (|x| + |y| + \varepsilon) = (|x| + |y| + 2\varepsilon)^2.
\] (3.47)

With the above inequality, we have the following for $z \in \mathbb{R}_+$:
\[
\int_0^t \frac{1}{\sqrt{s}} \int_{E^+} p_{3, \alpha}^0 (t - s, x, z) b(z) p_{3, \beta}^0 (s, z, y) m_p(dz) ds
\leq \int_0^t \frac{1}{\sqrt{s}} \int_{E^+} \frac{1}{\sqrt{t - s}} e^{-4\alpha |x|^p + |y|^p}/(t-s) b(z) \frac{1}{\sqrt{s}} e^{-4\beta |z|^p/s} m_p(dz) ds
\leq e^{-4\alpha |x|^p + |y|^p}/t \int_0^t s^{-1+1/(2q_1)} (t-s)^{-1/2} \int_{E^+} b(z) s^{-1/(2q_1)} e^{-4\beta |z|^p/s} m_p(dz) ds
\leq e^{-4\alpha |x|^p + |y|^p}/t \int_0^t s^{-1+1/(2q_1)} (t-s)^{-1/2} \| b_1 \|_{p_1} \left( \int_{E^+} s^{-1/(2q_1)} e^{-4\beta |z|^p/s} m_p(dz) \right)^{1/q_1} ds
\leq \| b_1 \|_{p_1} e^{-4\alpha |x|^p + |y|^p}/t \cdot \frac{t^{1/(2q_1)}}{\sqrt{t}} \leq \| b_1 \|_{p_1} e^{-\alpha |x|^p + \beta |y|^p}/t \cdot \frac{t^{1/(2q_1)}}{\sqrt{t}}. \] (3.48)
This establishes the desired inequality for $z \in \mathbb{R}_+$ in view of Proposition 3.11.

Case 2. $z \in D_0$, $|z|_\rho \geq 4$. For $q_2 : 1/p_2 + 1/q_2 = 1$, by Hölder’s inequality it holds
\[
\int_0^t \frac{1}{\sqrt{s}} \int_{|z|_\rho \geq 4} p_{3, \alpha}^0 (t - s, x, z) b(z) p_{3, \beta}^0 (s, z, y) m_p(dz) ds
\leq \int_0^t \frac{1}{\sqrt{s}} \int_{|z|_\rho \geq 4} e^{-\alpha |x-z|^p/(t-s)} \frac{1}{s} e^{-\beta |y-z|^p/s} b(z) m_p(dz) ds
\leq \int_0^t \left( \frac{1}{t-s} \right)^{1-1/q_2} \frac{1}{s^{3/2 - 1/q_2}} \left( \int_{|z|_\rho \geq 4} e^{-\alpha q_2 |x-z|^p/(t-s)} \frac{1}{s} e^{-\alpha q_2 |y-z|^p/s} m_p(dz) \right)^{1/q_2} ds \| b_2 \|_{p_2}
\leq e^{-\alpha |x|^p + \beta |y|^p}/t \cdot \frac{\| b_2 \|_{q_2}}{\beta^{3/2 - 1/q_2}}. \] (3.49)
where $\frac{3}{2} - \frac{1}{q_2} < 1$ because $p_2 > 2$ implies $1/q_2 > 1/2$. The conclusion follows for this case in view of Proposition 3.2.

**Case 3.** $z \in D_0$, $|z|_\rho < 4$, $|x|_\rho \geq 1$, we can adapt the same computation as in Case 2 and obtain the following:

\[
\int_0^t \frac{1}{\sqrt{s}} \int_{|z|_\rho < 4} \frac{p_{0, x}(t-s, x, z) b(z) p_{0, y}(s, z, y) m_p(dz) ds}{t-s} \lesssim \int_0^t \frac{1}{\sqrt{s}} \int_{|z|_\rho < 4} \frac{1}{t-s} e^{-\alpha|x-z|^2/(t-s)} \cdot \frac{1}{s} e^{-\beta|y-z|^2/s} b(z) m_p(dz) ds \\
\lesssim \int_0^t \frac{1}{t-s} e^{-\alpha|x-z|^2/(t-s)} \cdot \frac{1}{s} e^{-\alpha_2|x-z|^2/s} m_p(dz) \int_{|z|_\rho < 4} \frac{1}{t-s} e^{-\alpha_2|y-z|^2/s} m_p(dz) ds \\
\lesssim e^{-\alpha|x-y|^2/t} \frac{\|b_2\|_{q_2}}{t^{3/2-1/q_2}},
\]

(3.50)

where again $\frac{3}{2} - \frac{1}{q_2} < 1$ because $p_2 > 2$. This again in view of Proposition 3.2 establishes the desired inequality for the current case.

**Case 4.** $z \in D_0$, $|z|_\rho < 4$, $|x|_\rho < 1$. For this case,

\[
p_{0, x}(t-s, x, z) = \frac{1}{\sqrt{t-s}} e^{-\alpha(|x|^2_\rho + |z|^2_\rho)/(t-s)} + \frac{1}{t-s} e^{-\alpha|x-z|^2/(t-s)}.
\]

(3.51)

We observe that the second term on the right hand side of (3.51) can be handled in exact same way as that for Case 3, i.e.,

\[
\int_0^t \frac{1}{\sqrt{s}} \int_{|z|_\rho < 4} \frac{1}{t-s} e^{-\alpha|x-z|^2/(t-s)} \cdot \frac{1}{s} e^{-\beta|y-z|^2/s} b(z) m_p(dz) ds \lesssim e^{-\alpha|x-y|^2/t} \frac{\|b_2\|_{q_2}}{t^{3/2-1/q_2}}.
\]

(3.52)

where $1/q_2 > 1/2$. For the other term on the right hand side of (3.51), observing that $M$ satisfies (3.38) and that $|x|_\rho < 1$, $|y|_\rho \geq M > 8$ and $\epsilon < 1/4$, we therefore have

\[
(|x|_\rho + |y|_\rho + 2\varepsilon)^2 = |x|^2_\rho + |y|^2_\rho + 4\varepsilon + 4\varepsilon(|x|_\rho + |y|_\rho) \\
\leq |x|^2_\rho + |y|^2_\rho + |y|_\rho + 4\varepsilon + 4 \leq |x|^2_\rho + |y|^2_\rho + 8\varepsilon + 4|y|_\rho.
\]

(3.53)

Thus, for the first term on the right hand side of (3.51), we have

\[
\int_0^t \frac{1}{(t-s)^{1/2}} e^{-\alpha(|x|^2_\rho + |z|^2_\rho)/(t-s)} \int_{|z|_\rho < 4} \frac{1}{s^{3/2}} e^{-\beta|y-z|^2/s} b(z) m_p(dz) ds \\
\lesssim \int_0^t \frac{1}{(t-s)^{1/2}} e^{-\alpha(|x|^2_\rho + |z|^2_\rho)/(t-s)} \cdot e^{-\alpha(8\varepsilon + 4|y|)/t} \int_{|z|_\rho < 4} \frac{1}{s^{3/2}} e^{-(\beta-\alpha)|z|^2/s} b(z) m_p(dz) ds \\
\lesssim e^{-\alpha(|x|_\rho + |y|_\rho + 2\varepsilon)^2/t} \int_{|z|_\rho < 4} \frac{1}{s^{3/2-1/q_2}} \cdot \frac{1}{t-s} e^{-\alpha(8\varepsilon + 4|y|)/t} \int_{|z|_\rho < 4} \frac{1}{s^{3/2}} e^{-(\beta-\alpha)|z|^2/s} m_p(dz) ds \\
\lesssim \frac{t^{1/2-1/2}}{\sqrt{t}} e^{-\alpha(|x|+|y|)^2/t} \frac{\|b_2\|_{p_2}}{t^{1/2-1/2}} \lesssim \frac{t^{1/2-1/2}}{\sqrt{T}} e^{-\alpha|y|^2/t} \frac{\|b_2\|_{p_2}}{T^{1/2-1/2}}.
\]

(3.54)
Lemma 3.11. together with (3.31) yields
\[
\int_0^t \int_{|z| < 4} p^0(t - s, x, z)b(z)p^0(s, z, y)m_p(dz)ds \\
\leq \|b_2\|_{\mathcal{P}_2} \left( \frac{t^{1/q_2}}{t} e^{-\alpha|x-y|^2/t} + \frac{1}{t^{3/2-1/q_2}} e^{-\alpha|x-y|^2/t} \right) \leq \|b_2\|_{\mathcal{P}_2} \frac{1}{t^{3/2-1/q_2}} e^{-\alpha|x-y|^2/t}, \tag{3.55}
\]
where \( p_2 > 2 \) and thus \( 1/q_2 > 1/2 \).
\[\square\]

For the next proposition, we use another version of the canonical forms for \( X^0 \) as follows.
\[
p^0_{4,\alpha}(t, x, y) := \left\{ \begin{array}{ll}
1 & e^{-\alpha|x-y|^2/t}, x, y \in \mathbb{R}_+; \\
\frac{1}{\sqrt{t}} e^{-\alpha x^2/2} & x \in \mathbb{R}_+, y \in D_0; \\
\frac{1}{\sqrt{t}} e^{-\alpha(y^2 + |y|^2)/2} & x \in D_0, y \in \mathbb{R}_+; \\
\frac{1}{t} e^{-\alpha|x-y|^2} & x, y \in D_0 \cup \{a^*\} \text{ with } |y| > 1; \\
\frac{1}{t} e^{-\alpha|x|/2} & x, y \in D_0 \cup \{a^*\} \text{ with } |x| \geq 8, |y| < 1; \\
\frac{1}{\sqrt{t}} \left( 1 + \frac{|x|}{\sqrt{t}} \right) \left( 1 + \frac{|y|}{\sqrt{t}} \right) e^{-\alpha|x-y|^2} & \text{otherwise.}
\end{array} \right. \tag{3.56}
\]

The difference between \( p^0_{3,\alpha}(t, x, y) \) and \( p^0_{4,\alpha}(t, x, y) \) is that for the case \( x, y \in D_0 \) with \( |x| \geq 8 \) and \( |y| < 1 \), we use geodesic distance \( \rho \) instead of the Euclidean distance. It can be seen that in this case, \( |x| - |y| > |x - y| \), however, we use \( |x| - |y| \) for computation convenience. The following lemma follows immediately.

**Lemma 3.11.** There exists a constant \( C_{12} > 0 \), \( \alpha_9 > 0 \) such that it holds
\[
p(t, x, y) \leq C_{12} p^0_{4,\alpha_9}(t, x, y), \quad t \in (0, T], (x, y) \in E \times E.
\]

**Proposition 3.12.** There exists some function \( C_{13}(t) > 0 \) and \( \alpha_{10} > 0 \) such that for all \( \alpha < \alpha_{10} \),
\[
\int_0^t \int_{z \in E} p^0_{4,\alpha}(t - s, x, z)b(z)|\nabla z p(s, z, y)| m_p(dz)ds \leq C_{13}(t)p^0_{4,\alpha}(t, x, y), \quad 0 < s < t \leq T, x, y \in D_0,
\]
and \( |x| > 2M, |y| > M \). \tag{3.57}

where \( C_{13}(t) \) is non-decreasing in \( t \), \( C_{13}(t) \to 0 \) as \( t \to 0 \).

**Proof.** Case 1. \( z \in \mathbb{R}_+ \). For this case, since \( |x| > 8 \), similar (3.37) in the last proposition, again we have
\[
4(|x|^2 + |y|^2) \geq |x|^2 + |y|^2 + 2|x||y| + 4\varepsilon(|x| + |y|) = (|x| + |y| + 2\varepsilon)^2. \tag{3.58}
\]
Therefore, the remaining computation follows the same lines as (3.87) in Proposition 3.10. For the next two cases, we let \( \beta \) be the same as \( \beta_2 \) in Proposition 3.2.

**Case 2.** \( z \in D_0, |z|_\rho \geq 4 \). For \( 1/p_2 + 1/q_2 = 1 \), by Hölder's inequality it holds

\[
\int_0^t \frac{1}{\sqrt{s}} \int_{|z|_\rho \geq 4} \frac{1}{t-s} e^{-\alpha|z|^2/(t-s)} \frac{1}{s} e^{-\beta|y|^2/s} b(z) m_p(dz) ds \\
\begin{align*}
&\lesssim \int_0^t \frac{1}{(t-s)^{1-1/q_2}} \frac{1}{s^{3/2-1/q_2}} ds \left( \int_{|z|_\rho \geq 4} \frac{1}{t-s} e^{-\alpha_2|z|^2/(t-s)} \frac{1}{s} e^{-\alpha_2|y|^2/s} m_p(dz) \right)^{1/q_2} ||b_2||_{p_2} \\
&\lesssim e^{-\alpha|z-y|^2/t} ||b_2||_{q_2} \frac{1}{t^{3/2-1/q_2}}, \tag{3.59}
\end{align*}
\]

**Case 3.** \( z \in D_0, |z|_\rho < 4 \). It has been shown that for some \( \beta = \beta_2 \) in Proposition 3.2

\[
|\nabla z p(s, z, y)| \lesssim \frac{1}{s} e^{-\beta(|z|_\rho^2+|y|_\rho^2)/s} + \frac{1}{s^{3/2}} e^{-\beta|z-y|^2/s}. \tag{3.60}
\]

We first have the following for the first term on the right hand side of (3.60):

\[
\int_0^t \int_{z \in D_0, |z|_\rho < 4} \frac{1}{1-t-s} b(z)e^{-\alpha|z|^2/(t-s)} \frac{1}{s} e^{-\beta|z|^2/s} m_p(dz) ds \\
\begin{align*}
&\lesssim e^{-\beta|y|_\rho^2/t} \int_0^t \frac{1}{(t-s)^{1-1/q_2}} \frac{1}{s^{1-1/q_2}} ds \left( \int_{|z|_\rho < 4} \frac{1}{t-s} e^{-\alpha_2|z|^2/(t-s)} \frac{1}{s} e^{-\alpha_2|y|^2/s} m_p(dz) \right)^{1/q_2} \\
&\lesssim e^{-\beta|y|_\rho^2/t} \frac{1}{t^{1-1/q_2}} \frac{1}{t^{1-1/(2q_2)}} e^{-\alpha(|z|-\epsilon)^2/t} ||b_2||_{q_2} \lesssim \frac{||b_2||_{p_2}}{t^{1-1/(2q_2)}} e^{-\alpha(|z|-|y|)^2/t} \\
&\lesssim \frac{1}{t^{1-1/(2q_2)}} e^{-\alpha(|z|-|y|)^2/t} ||b_2||_{p_2} e^{-\alpha(|z|-|y|)^2/t}, \tag{3.61}
\end{align*}
\]

where \( \frac{1}{3} < \frac{1}{q_2} < 1 \). For the other term on the right hand side of (3.60), for \( q'_2, r_2 \): \( 1/p_2 + 1/q'_2 + 1/r_2 = 1 \), we select \( q'_2 \) sufficiently large so that \( 1/(2q'_2) + 1/r_2 > 1/2 \). This is possible since \( p_2 = 2 \). Therefore,

\[
\int_0^t \frac{1}{\sqrt{s}} \int_{|z|_\rho < 4} \frac{1}{t-s} e^{-\alpha|z|^2/(t-s)} \frac{1}{s} e^{-\beta|y|^2/s} b(z) m_p(dz) ds \\
\begin{align*}
&\lesssim \int_0^t \int_{|z|_\rho < 4} \frac{1}{t-s} e^{-\alpha(|z|-|y|)^2/(t-s)} \frac{1}{s} e^{-\beta|y|^2/s} \frac{1}{s} e^{-\alpha|z|^2/s} m_p(dz) ds \\
&\lesssim \int_0^t \frac{1}{s^{3/2-1/q_2}} \frac{1}{t-s} \frac{1}{r_2} (t-s)^{1-1/q_2} \left( \int_{|z|_\rho < 4} \frac{1}{t-s} e^{-\alpha_2|z|^2/(t-s)} \frac{1}{s} e^{-\alpha_2|y|^2/s} m_p(dz) \right)^{1/q_2} \\
&\quad \times \left( \int_{|z|_\rho < 4} \frac{1}{t-s} e^{-(\beta-\alpha)|z|^2/s} m_p(dz) \right)^{1/r_2} ||b_2||_{p_2} \\
&\lesssim \frac{1}{t^{3/2-1/(2q'_2) - 1/r_2}} \frac{1}{t^{1-1/(2q'_2)}} e^{-\alpha(|z|-|y|)^2/t} ||b_2||_{p_2}, \tag{3.62}
\end{align*}
\]
where $3/2 - 1/(2q_2) - 1/r_2 < 1$ because of our assumptions. In view of (3.60), this together with (3.61) proves the desired inequality for the current case.

For the next and the last case which is $x \in D_0$ with $|x|_\rho > 1$ and $y \in \mathbb{R}_+$, we set the canonical form of the heat kernel of $X^0$ as follows:

$$
p^0_{5, \alpha}(t, x, y) := \begin{cases} 
\frac{1}{\sqrt{t}} e^{-\alpha|x-y|^2/t}, & x \in \mathbb{R}_+, y \in \mathbb{R}_+; \\
\frac{1}{\sqrt{t}} e^{\alpha(|x|^2 + |y|^2)/t}, & x \in \mathbb{R}_+, y \in D_0; \\
\frac{1}{\sqrt{t}} e^{\alpha(|x|^2 + |y|^2)/t}, & x \in D_0, y \in \mathbb{R}_+; \\
\frac{1}{\sqrt{t}} e^{-\alpha|x-y|^2/t} + \frac{1}{t} \left( 1 \wedge \frac{|x|_\rho}{\sqrt{t}} \right) \left( 1 \wedge \frac{|y|_\rho}{\sqrt{t}} \right) e^{-\frac{2\alpha|x-y|^2}{t}}, & \text{otherwise.}
\end{cases}
$$

(3.63)

The difference between $p^0_\alpha(t, x, y)$ and $p^0_{5, \alpha}(t, x, y)$ is that for the case $x, y \in D_0$ with $\max\{|x|_\rho, |y|_\rho\} > 1$, the coefficient $\alpha$ on the exponential term is replaced with $2\alpha$. The following theorem is immediate.

**Lemma 3.13.** There exists a constant $C_{14} > 0$ and $\alpha_{11} > 0$ such that it holds

$$
p(t, x, y) \leq C_{14} p^0_{5, \alpha_{11}}(t, x, y), \quad (t, x, y) \in (0, T] \times E \times E.
$$

**Proposition 3.14.** Let $M \geq 8$ be the same as in (3.43). There exists some $\alpha_{12} > 0$ sufficiently small such that for all $0 < \alpha < \alpha_{12}$,

$$
\int_0^t \int_{z \in E} \left| p^0_{5, \alpha}(t-s, x, z) b(z) \right| |\nabla_z p(s, z, y)| m_p(dz) ds \leq C_{15}(t) p^0_{5, \alpha}(t, x, y), \quad 0 < s < t \leq T, x, y \in D_0,
$$

and $|x|_\rho < 2M, |y|_\rho < M$.

(3.64)

where $C_{15}(t)$ is non-decreasing in $t$, $C_{15}(t) \to 0$ as $t \to 0$.

**Proof.** Case 1. $z \in \mathbb{R}_+$. Letting $\beta$ be the same constant as $\beta_1$ in Proposition 3.11 we have

$$
\int_0^t \frac{1}{\sqrt{s}} \int_{\mathbb{R}_+} p^0_\alpha(t-s, x, z) b(z) p^0_\beta(s, z, y) m_p(dz) ds \\
\lesssim \int_0^t \frac{1}{\sqrt{s}} \int_{\mathbb{R}_+} e^{-\alpha(|x|^2 + |z|^2)/(t-s)} b(z) \frac{1}{\sqrt{s}} e^{-\beta(|z|^2 + |y|^2)/s} m_p(dz) ds \\
\leq e^{-\alpha(|x|^2 + |y|^2)/t} \int_0^t s^{-1/(2q_1)} (t-s)^{-1/2} \int_{\mathbb{R}_+} b(z) s^{-1/(2q_1)} e^{-\beta|z|^2/s} m_p(dz) ds \\
\leq e^{-\alpha(|x|^2 + |y|^2)/t} \int_0^t s^{-1/(2q_1)} (t-s)^{-1/2} \|b_1\|_{p_1} \left( \int_{\mathbb{R}_+} s^{-1/(2q_1)} e^{-\beta|z|^2/s} q_1 m_p(dz) \right)^{1/q_1} ds \\
\lesssim \|b_1\|_{p_1} e^{-\alpha(|x|^2 + |y|^2)/t} \cdot \frac{t^{1/(2q_1)}}{\sqrt{t}}.
$$

(3.65)
In view of Proposition 3.1, this implies the desired conclusion for the case of $z \in \mathbb{R}_+$.

Case 2. $z \in D_0$, $|z|_\rho \geq 4M$. Note that $|x|_\rho < 2M$ and $|y|_\rho < M$, it holds that $|x - z|^2 \geq |x|^2 + |x - z|^2/2$, and that $|y - z|^2 \geq |y|^2 + |y - z|^2/2$. By Proposition 3.2 and letting $\beta$ be the same as $\beta_2$ in Proposition 3.2, it holds for $0 < \alpha < \beta$ that

$$
\int_0^t \int_{z \in D_0, |z|_\rho \geq 4M} p_{0,\alpha}^0(t - s, x, z) b(z) \left| \nabla_p p(s, z, y) \right| m_p(dz) ds
$$

$$
\leq \int_0^t \frac{1}{\sqrt{s}} \int_{|z|_\rho \geq 4M} \frac{1}{t - s} e^{-2\alpha |x - z|^2/(t - s)} \frac{1}{s} e^{-2\beta |y - z|^2/s} b(z) m_p(dz) ds
$$

$$
\leq e^{-2\alpha (|x|^2 + |y|^2)/t} \int_0^t \frac{1}{\sqrt{s}} \int_{|z|_\rho \geq 4M} \frac{1}{t - s} e^{-2\alpha |x - z|^2/(t - s)} \frac{1}{s} e^{-2\beta |y - z|^2/s} m_p(dz) ds
$$

$$
\leq e^{-\alpha (|x|^2 + |y|^2)/t} \int_0^t \frac{1}{(t - s)^{1-1/q_2}} \frac{1}{s^{3/2 - 1/q_2}} ds \left( \int_{|z|_\rho \geq 20} \frac{1}{s} e^{-2\alpha q_2 |y - z|^2/2s} m_p(dz) \right)^{1/q_2}
$$

$$
|x - z|^2 \geq 2M
$$

$$
\leq e^{-2\alpha (|x|^2 + |y|^2)/t} \left( \int_{|z|_\rho \geq 20} \frac{1}{s} e^{-2\alpha q_2 |y - z|^2/2s} m_p(dz) \right)^{1/q_2}
$$

$$
\leq e^{-2\alpha (|x|^2 + |y|^2)/t} \left( \int_{|z|_\rho \geq 20} \frac{1}{s} e^{-2\alpha q_2 |y - z|^2/2s} m_p(dz) \right)^{1/q_2}
$$

where $-\frac{1}{2} < 3 - \frac{2 q_2}{q_2} < \frac{1}{2}$ again because $p_2 > 2$.

Case 3. $z \in D_0$, $|z|_\rho < 4M$. In view of the gradient estimate in Proposition 3.2, we know for $0 < \alpha < \beta$, it holds

$$
\int_0^t \frac{1}{\sqrt{s}} \int_{z \in D_0, |z|_\rho < 4M} p_{0,\alpha}^0(t - s, x, z) b(z) \left| \nabla_p p(s, z, y) \right| m_p(dz) ds
$$

$$
\leq e^{-\alpha (|x|^2 + |y|^2)/t} \int_0^t \frac{1}{\sqrt{s}} \int_{|z|_\rho < 4M} \left[ \frac{1}{t - s} e^{-\alpha |x|^2/(t - s)} + \frac{1}{s} \left( 1 \wedge \frac{|x|_\rho}{\sqrt{t - s}} \right) \left( 1 \wedge \frac{|z|_\rho}{\sqrt{t - s}} \right) e^{-\alpha |x - z|^2/(t - s)} \right]
$$

$$
\cdot b(z) \left[ \frac{1}{\sqrt{s}} e^{-\beta (|x|^2 + |y|^2)/s} + \frac{1}{s} \left( 1 \wedge \frac{|y|_\rho}{\sqrt{s}} \right) e^{-\beta |y - z|^2/s} \right] m_p(dz) ds.
$$

The right hand side of (3.71) can be expanded into the sum of four terms. We now bound the right hand side of (3.71) from above term by term. First of all for $p > 0$ we have by Hölder’s inequality,

$$
\int_0^t \frac{1}{\sqrt{s}} e^{-\alpha (|x|^2 + |y|^2)/t} \int_{z \in D_0, |z|_\rho < 4M} \frac{b(z)}{s} e^{-\beta (|x|^2 + |y|^2)/s} m_p(dz) ds
$$

$$
\leq e^{-\alpha (|x|^2 + |y|^2)/t} \int_0^t (t - s)^{1/2} \frac{1}{s^{1 - 1/(2q_2)}} \int_{z \in D_0} \frac{b(z)}{s^{1/(2q_2)}} e^{-\beta (|x|^2 + |y|^2)/s} m_p(dz) ds
$$

$$
\leq e^{-\alpha (|x|^2 + |y|^2)/t} \int_0^t (t - s)^{1/2} \frac{1}{s^{1 - 1/(2q_2)}} \left( \int_{z \in D_0} \frac{1}{s} e^{-\beta (|x|^2 + |y|^2)/s} m_p(dz) \right)^{1/q_2} ds \cdot \| b_2 \|_{p_2}
$$

$$
\leq \| b_2 \|_{p_2} \frac{1/(2q_2)}{\sqrt{t}} e^{-\alpha (|x|^2 + |y|^2)/t}.
$$

(3.68)
Secondly,\[
\begin{align*}
&\int_0^t \frac{1}{(t-s)^{1/2}} e^{-\alpha(|z|^2 + |\rho|^2)/(t-s)} \int_{D_0} \frac{1}{s^{3/2}} e^{-2\beta|z-y|^2/s} b(z) m_p(dz) ds \\
&\lesssim e^{-\alpha(|z|^2 + |\rho|^2)/t} \int_{s=0}^t \frac{1}{s^{3/2-1/q_2}} (\int_{z\in D_0} \left( \frac{1}{s^{1/q_2}} e^{-2(\beta-\alpha)|z-y|^2/s} \right)^{q_2} m_p(dz) )^{1/q_2} \|b_2\|_{p_2} \\
&\lesssim \frac{t^{1/q_2-1/2}}{\sqrt{t}} e^{-\alpha(|z|^2 + |\rho|^2)/t} \|b_2\|_{p_2},
\end{align*}
\] (3.69)
where \(1/q_2 > 1/2\) because \(p_2 \in (2, \infty]\). Thirdly, for the right hand side of (3.71), we follow the same steps of computation as those for (3.55) and get:
\[
\begin{align*}
&\int_0^t \frac{1}{\sqrt{s}} \int_{z\in D_0, |z|<4M} \frac{1}{t-s} \left( 1 \wedge \frac{|x|}{\sqrt{t-s}} \right) \left( 1 \wedge \frac{|z|}{\sqrt{t-s}} \right) e^{-2\alpha|x-z|^2/(t-s)} \frac{1}{\sqrt{s}} e^{-\beta(|z|^2 + |\rho|^2)/s} m_p(dz) ds \\
&\lesssim \frac{1}{t^{1/p_2}} \cdot \frac{1}{t^{1/(2q_2)} e^{-\alpha(|z|^2 + |\rho|^2)/t}} \|b_2\|_{p_2},
\end{align*}
\] (3.70)
where \(1/(2q_2) + 1/p_2 < 1/2\). Finally, in order to check the last term of (3.71), we need to verify the following:
\[
\begin{align*}
&\int_0^t \frac{1}{\sqrt{s}} \int_{z\in D_0, |z|<4M} \frac{1}{t-s} \left( 1 \wedge \frac{|x|}{\sqrt{t-s}} \right) \left( 1 \wedge \frac{|z|}{\sqrt{t-s}} \right) e^{-2\alpha|x-z|^2/(t-s)} b(z) \cdot \frac{1}{s} \left( 1 \wedge \frac{|y|}{\sqrt{s}} \right) e^{-2\beta|y-z|^2/s} m_p(dz) ds \\
&\lesssim \frac{1}{t} \left( 1 \wedge \frac{|x|}{\sqrt{t}} \right) \left( 1 \wedge \frac{|y|}{\sqrt{t}} \right) \frac{1}{t} e^{-2\alpha|x-y|^2/t} \cdot N(t),
\end{align*}
\] (3.71)
Indeed, taking \(D = D_0\) and \(\nu(dzds) = b_2(z)dzds\) which belongs to parabolic Kato class \(K_2\) in [5, Lemma 3.6] immediately yields that
\[
\begin{align*}
&\int_0^t \frac{1}{\sqrt{s}} \int_{z\in D_0} \frac{1}{t-s} \left( 1 \wedge \frac{|x|}{\sqrt{t-s}} \right) \left( 1 \wedge \frac{|z|}{\sqrt{t-s}} \right) e^{-2\alpha|x-z|^2/(t-s)} \cdot b(z) \\
&\cdot \frac{1}{s} \left( 1 \wedge \frac{|y|}{\sqrt{s}} \right) e^{-2\beta|y-z|^2/s} m_p(dz) ds \\
&\lesssim \frac{1}{t} \left( 1 \wedge \frac{|x|}{\sqrt{t}} \right) \left( 1 \wedge \frac{|y|}{\sqrt{t}} \right) \frac{1}{t} e^{-2\alpha|x-y|^2/t} \cdot N(t),
\end{align*}
\] where, as defined in [5],
\[
N(t) = \sup_{\tau>0, x \in \mathbb{R}^2} \int_{(\tau-t,\tau) \times \mathbb{R}^2} \frac{p \cdot b(y)}{(\tau - s)^{3/2}} e^{-c|x-y|^2/(\tau-s)} ds dy
+ \sup_{s>0, x \in \mathbb{R}^2} \int_{(s,s+t) \times \mathbb{R}^2} \frac{p \cdot b(y)}{(\tau - s)^{3/2}} e^{-c|x-y|^2/(\tau-s)} d\tau dy < \infty, \quad \text{for all } t \leq T.
\]
This verifies (3.71) and completes Case 3. \(\square\)
Now our goal is to establish two-sided heat kernel bounds for drifted BMVD. For this purpose, we set \( k_0(t,x,y) = p(t,x,y) \) and then inductively define
\[
 k_n(t,x,y) := \int_0^t \int_E k_{n-1}(t-s,x,z)b(z) \cdot \nabla_z p(s,z,y) \, dz \, ds, \quad \text{for } n \geq 1. \quad (3.72)
\]

**Lemma 3.15.** There exists \( T_1 > 0 \) such that for all \( x, y \in E \) and \( n \geq 1 \), \( k_n(t,x,y) \) is well-defined on \((0, T_1]\). Furthermore, there exist \( C_i > 0 \), \( 16 \leq i \leq 20 \) such that the following upper bound estimates hold when \( t \in (0, T_1]\):

1. For \( x \in \mathbb{R}_+ \) and \( y \in E \) or \( y \in \mathbb{R}_+ \) and \( x \in E \),
\[
 \sum_{n=0}^{\infty} k_n(t,x,y) \leq \frac{C_{16}}{\sqrt{t}} e^{-\frac{C_{16}(x,y)^2}{t}}.
\]

2. For \( x, y \in D_0 \cup \{a^*\} \) with \( \max \{|x|_\rho, |y|_\rho\} < 1 \),
\[
 \sum_{n=0}^{\infty} k_n(t,x,y) \leq \frac{C_{16}}{\sqrt{t}} e^{-\frac{C_{16}(x,y)^2}{t} + \frac{C_{16}}{t} \left( 1 \wedge \frac{|x|_\rho}{\sqrt{t}} \right) \left( 1 \wedge \frac{|y|_\rho}{\sqrt{t}} \right) e^{-\frac{C_{16}(x,y)^2}{t}}};
\]
and when \( \max \{|x|_\rho, |y|_\rho\} \geq 1 \),
\[
 \sum_{n=0}^{\infty} k_n(t,x,y) \leq \frac{C_{16}}{t} e^{-\frac{C_{16}(x,y)^2}{t}}.
\]

**Proof.** In view of Propositions 3.9, 3.10 by using inductions, we have the following upper bounds on \( k_n(t,x,y) \) for different regions of \( x \) and \( y \).

**Case 1.** \( x \in \mathbb{R}_+, \ y \in E \). By Proposition 3.9 and Proposition 3.11, we can select \( t_1 > 0 \) small enough so that there exist some \( 0 < c_1 < 1/2, \ c_2 > 0 \) such that
\[
 |k_n(t,x,y)| \leq c_1^{n-1} \int_0^t \int_E p_{c_2}^0(t-s,x,z)|b(z)||\nabla_z p(s,z,y)| m_0 dz ds \leq c_1^{n} p_{c_2}^0(t,x,y),
\]
for all \( x \in \mathbb{R}_+, y \in E \), and \( n \geq 1, t \in (0,t_1] \).

Therefore,
\[
 \sum_{n=0}^{\infty} k_n(t,x,y) \leq \sum_{n=0}^{\infty} |k_n(t,x,y)| \leq \frac{1}{1-c_1} p_{c_2}^0(t,x,y), \text{ for } x \in \mathbb{R}_+, y \in E, t \in (0,t_1].
\]

Observing that when \( x \) and \( y \) are both in \( \mathbb{R}_+ \), \( |x-y| = \rho(x,y) \), and when \( x \in \mathbb{R}_+ \) but \( y \in D_0 \), \( |x| \neq |y| \neq \rho(x,y) \), it holds for some \( c_3 > 0 \) that
\[
 \sum_{n=0}^{\infty} k_n(t,x,y) \leq \frac{1}{\sqrt{t}} e^{-c_3 \rho(x,y)^2/t} \text{ on } t \in (0,t_1].
\]

**Case 2.** \( y \in \mathbb{R}_+, \ x \in D_0 \) with \( |x|_\rho \leq 1 \). In view of Proposition 3.6, by taking \( t_2 > 0 \) sufficiently small, there exist some \( 0 < c_4 < 1/2 \) and \( c_5 > 0 \) such that
\[
 |k_n(t,x,y)| \leq c_4^{n-1} \int_0^t \int_E p_{1,c_5}^0(t-s,x,z)|b(z)||\nabla_z p(s,z,y)| m_0 dz ds \leq c_4^{n} p_{1,c_5}^0(t,x,y),
\]
\[ x \in D_0, \ |x|_\rho \leq 1, \ y \in \mathbb{R}_+, \ t \in (0,t_2]. \quad (3.73) \]
By taking the sum of \( k_n(t, x, y) \) over \( n \geq 0 \), we get
\[
\sum_{n=0}^{\infty} k_n(t, x, y) \leq \sum_{n=0}^{\infty} |k_n(t, x, y)| \leq \frac{1}{1 - c_4} p_{1,c_5}^0(t, x, y) \lesssim \frac{1}{\sqrt{t}} e^{-c_5|x|^2/|y|^2/t},
\]
where \( t, x \in D_0, |x|_\rho \leq 1, y \in \mathbb{R}_+, t \in (0, t_2] \).

Again in view of the fact that \( |x|_\rho + |y|_\rho = \rho(x, y) \), it holds for some \( c_6 > 0 \) that
\[
\sum_{n=0}^{\infty} k_n(t, x, y) \lesssim \frac{1}{\sqrt{t}} e^{-c_6\rho(x,y)^2/t} \quad \text{on} \quad t \in (0, t_2].
\]

**Case 3.** \( y \in \mathbb{R}_+, x \in D_0 \) with \( |x|_\rho > 1 \). By Proposition 3.8, there exist some \( t_3 > 0 \) such that for some \( 0 < c_7 < 1/2 \) and \( c_8 > 0 \), it holds
\[
|k_n(t, x, y)| \leq c_7^{n-1} \int_E \int_0^t p_{2,c_8}^0(t-s, z)|b(z)||\nabla_z p(s, z, y)| m_p(dz) ds \leq c_8^0 p_{2,c_8}^0(t, x, y),
\]
where \( x \in D_0, |x|_\rho > 1, y \in \mathbb{R}_+, t \in (0, t_3] \).

Similar to the last case, it holds for some \( c_9 > 0 \) that
\[
\sum_{n=0}^{\infty} k_n(t, x, y) \leq \frac{1}{1 - c_7} p_{2,c_8}^0(t, x, y) \lesssim \frac{1}{\sqrt{t}} e^{-c_8\rho(x,y)^2/t},
\]
where \( x \in D_0, |x|_\rho > 1, y \in \mathbb{R}_+, t \in (0, t_3] \).

**Case 4.** \( x, y \in D_0 \) with \( |y|_\rho > M \) where \( M \) is selected as in (3.43). Proposition 3.10, there exist constants \( t_4 > 0, 0 < c_{10} < 1/2 \) and \( c_{11} > 0 \) such that
\[
|k_n(t, x, y)| \leq c_{10}^{n-1} \int_E \int_0^t p_{3,c_{11}}^0(t-s, z)|b(z)||\nabla_z p(s, z, y)| m_p(dz) ds \leq c_{10} c_{11}^0 p_{3,c_{11}}^0(t, x, y),
\]
where \( x, y \in D_0, |y|_\rho > M, t \in (0, t_4] \).

Therefore, observing that in this case, \( \rho(x, y) \asymp |x - y| \), we get that there exists some \( c_{12} > 0 \) such that
\[
\sum_{n=0}^{\infty} k_n(t, x, y) \leq \frac{1}{1 - c_{10}} p_{3,c_{11}}^0(t, x, y) \lesssim \frac{1}{\sqrt{t}} e^{-c_{12}\rho(x,y)^2/t},
\]
where \( x, y \in D_0, |y|_\rho > M, t \in (0, t_4] \).

**Case 5.** \( x, y \in D_0 \) with \( |x|_\rho \geq 2M \) and \( |y|_\rho < M \). In view of Proposition 3.12, there exist constants \( t_5 > 0, 0 < c_{13} < 1/2 \) and \( c_{14} > 0 \) such that
\[
|k_n(t, x, y)| \leq c_{13}^{n-1} \int_E \int_0^t p_{4,c_{14}}^0(t-s, z)|b(z)||\nabla_z p(s, z, y)| m_p(dz) ds \leq c_{13} c_{14}^0 p_{4,c_{14}}^0(t, x, y),
\]
where \( x, y \in D_0, |y|_\rho < M \) and \( |x|_\rho \geq 2M, t \in (0, T_5] \).
Similar to the Case 4, again we observe that in this case, \( \rho(x, y) \propto |x - y| \propto |x| - |y| \). It hence follows that there exists some \( c_{15} > 0 \) such that

\[
\sum_{n=0}^{\infty} k_n(t, x, y) \leq \frac{1}{1 - c_{13}} p^0_{4, c_{14}}(t, x, y) \lesssim \frac{1}{\sqrt{t}} e^{c_{14}(|x| - |y|)^2/t} \lesssim \frac{1}{\sqrt{t}} e^{c_{15}\rho(x, y)^2/t},
\]

for \( x, y \in D_0, |x| \geq 2M, |y| < M, t \in (0, t_5] \).

**Case 6.** \( x, y \in D_0 \) with \( |x|_\rho < 2M \) and \( |y|_\rho < M \). For this last case, it follows from Proposition 3.14 that there exist constants \( t_6 > 0, 0 < c_{16} < 1/2, c_{17} > 0 \) such that

\[
|k_n(t, x, y)| \leq c_{16}^{n-1} \int_E \int_0^t p^0_{5, c_{17}}(t - s, x, z) |b(z)| |\nabla_z p(s, z, y)| m_p(dz) ds \leq c_{16} \int_0^t p^0_{5, c_{17}}(t, x, y),
\]

for \( x, y \in D_0, |y| < M \) and \( |x|_\rho \geq 2M, t \in (0, t_6] \).

It thus follows that there exists some \( c_{18} > 0 \) such that

\[
\sum_{n=0}^{\infty} k_n(t, x, y) \leq \frac{1}{1 - c_{16}} p^0_{5, c_{17}}(t, x, y) \lesssim \frac{1}{\sqrt{t}} e^{-c_{18}(|x|_\rho^2 + |y|_\rho^2)/t} + \frac{1}{t} \left( 1 + \frac{|x|_\rho}{\sqrt{t}} \right) \left( 1 + \frac{|y|_\rho}{\sqrt{t}} \right) e^{-c_{18}(|x|_\rho^2 - |y|_\rho^2)/t},
\]

for \( x, y \in D_0, |y|_\rho < M \) and \( |x|_\rho \geq 2M, t \in (0, t_6] \).

Combining all the six cases above, the proof to the lemma is thus complete by taking \( T_1 = \min\{t_i : 1 \leq i \leq 6\} \).

We now claim the following lemma:

**Lemma 3.16.** Let

\[
p^b(t, x, y) := \sum_{n=0}^{\infty} k_n(t, x, y), \quad \text{for } t > 0, x, y \in E. \tag{3.74}
\]

Then the family \( \{p^b(t, x, y)\}_{t>0, x,y \in E} \) is indeed the transition density of \( X \). In particular, for any \( T_2 > 0 \), the series \( \sum_{n=0}^{\infty} k_n(t, x, y) \) absolutely converges almost uniformly on \((0, T_2] \times E \times E\) with the same type of upper bound as in Lemma 3.15.

**Proof.** We first claim that \( \sum_{n=0}^{\infty} k_n(t, x, y) \) satisfies Chapman-Kolmogorov equation as follows:

\[
\int_{z \in E} \sum_{i \geq 0} k_i(s, x, z) \sum_{j \geq 0} (t - s, z, y) dz = \sum_{n \geq 0} k_n(t, x, y), \quad \text{for } 0 < s, t - s \leq T_1, \tag{3.75}
\]

where \( T_1 \) is the same as in Lemma 3.15. To show this, we first claim:

\[
\int_{z \in E} \sum_{i+j=n} k_i(s, x, z) k_j(t - s, z, y) dz = k_n(t, x, y), \quad \text{for } 0 < s, t - s \leq T_1. \tag{3.76}
\]

We show (3.76) by induction. First of all, it is apparent that (3.76) holds for \( n = 0 \) as \( k_0(t, x, y) = p(t, x, y) \) by definition. Now assume that (3.76) holds for all \( 0, \ldots, n \), we show it also holds for...
Indeed, by (3.72), the left hand side of (3.76) can be written as
\[
\int_{z \in E} \sum_{i+j=n+1 \atop i \geq 0, j \geq 1} k_i(s, x, z) \int_{\tau=0}^{t-s} \int_{u \in E} k_{j-1}(\tau, z, u) b(u) \nabla u p(t - \tau, u, y) d\tau \, dz
\]
\[+ \int_{z \in E} k_{n+1}(s, x, z) k_0(t - s, z, y) \, dz. \tag{3.77}
\]

For the first term of (3.77), by induction, we have
\[
\int_{z \in E} \sum_{i+j=n \atop i \geq 0, j \geq 0} k_i(s, x, z) \int_{\tau=0}^{t-s} \int_{u \in E} k_{j}(\tau, z, u) b(u) \nabla u p(t - s, \tau, u, y) d\tau \, dz = \int_{u \in E} \int_{\tau=0}^{t-s} \sum_{i+j=n \atop i \geq 0, j \geq 0} k_i(s, x, z) k_j(\tau, z, u) b(u) \nabla u p(t - s, \tau, u, y) dz \, d\tau du
\]
\[
= \int_{u \in E} \int_{\tau=0}^{t-s} k_n(s + \tau, x, u) b(u) \nabla u p(t - s - \tau, u, y) d\tau du = \int_{u \in E} \int_{\xi = s + \tau}^{t} k_n(\xi, x, u) b(u) \nabla u p(t - \xi u, y) d\xi du, \quad \text{for all } 0 < s, t - s \leq T_1. \tag{3.78}
\]

For the second term of (3.77), we have
\[
\int_{z \in E} k_{n+1}(s, x, z) k_0(t - s, z, y) \, dz = \int_{z \in E} \int_{\tau=0}^{s} \int_{u \in E} k_n(\tau, x, u) b(u) \nabla u p(s - \tau, u, z) k_0(t - s, z, y) d\tau \, dz d\tau du
\]
\[= \int_{z \in E} \int_{\tau=0}^{s} \int_{u \in E} k_n(\tau, x, u) b(u) \nabla u p(s - \tau, u, z) p(t - s, z, y) d\tau \, dz d\tau du
\]
\[= \int_{\tau=0}^{s} \int_{u \in E} k_n(\tau, x, u) b(u) \nabla u p(t - \tau, u, y) d\tau. \tag{3.79}
\]

(3.79) and (3.78) together proves (3.76), which yields
\[
\int_{z \in E} \sum_{i+j \leq n} k_i(s, x, z) k_j(t - s, z, y) \, dz = \sum_{l=0}^{n} k_l(t, x, y), \quad \text{for all } 0 < s, t - s \leq T_1. \tag{3.80}
\]

From the proof of Lemma 3.15, we know that there exist constants \(c_1, c_2 > 0, \alpha > 0\) such that
\[
\sum_{n \geq 0} |k_n(t, x, y)| \leq c_1 p_0(t, x, y) \leq c_1 p(c_2 t, x, y), \quad \text{for } t \leq T_1. \tag{3.81}
\]

Therefore, by dominate convergence we know (3.75) holds. Furthermore, taking \(s = T_1\) and \(T_1 < t \leq 2T_1\) in (3.80) implies that \(\sum_{n \geq 0} k_n(t, x, y)\) converges absolutely for \(t \leq 2T_1\) and the
following upper bound holds:

$$\sum_{n \geq 0} |k_n(t, x, y)| \leq c_{14}^2 p(c_2 t, x, y), \quad \text{for } t \leq 2T_1.$$ 

By repeating the same induction, there exist $c_3, c_4, c_5 > 0$ such that

$$\sum_{n \geq 0} |k_n(t, x, y)| \leq c_3 e^{c_4 t} p(c_2 t, x, y) \leq c_3 e^{c_4 t} f_{c_5}^0(t, x, y), \quad \text{for all } t > 0. \quad (3.82)$$

With the above upper bound for $\sum_n k_n(t, x, y)$, we now claim $p^b(t, x, y)$ is indeed the transition density of $X$ by showing that for sufficiently large $\alpha > 0$, the Laplace transform of $p^b(t, x, y)$ is well-defined and indeed the kernel of $G^{b}_{\alpha}$.

For $\alpha > 2c_4$ in $(3.82)$, $n = 0, 1, \ldots$, and $x, y \in E$, we define $u^{(n)}_{\alpha}(x, y)$ as the Laplace transform of $k_n(t, x, y)$ as follows:

$$u^{(n)}_{\alpha}(x, y) := \int_0^\infty e^{-\alpha t} k_n(t, x, y) dt.$$ 

We can therefore define for $\alpha > 2c_4$ that

$$u^b_{\alpha}(x, y) := \sum_{n=0}^{\infty} u^{(n)}_{\alpha}(x, y) = \sum_{n=0}^{\infty} \int_0^\infty e^{-\alpha t} k_n(t, x, y) dt = \int_0^\infty e^{-\alpha t} \left( \sum_{n=0}^{\infty} k_n(t, x, y) \right) dt, \quad (3.83)$$

where the infinite sum converges in view of $(3.82)$. In view of Fubini’s theorem, for $n \geq 1$ and $x, y \in E$,

$$u^{(n)}_{\alpha}(x, y) = \int_E u^{(n-1)}_{\alpha}(x, z) b(z) \cdot \nabla z u_{\alpha}(z, y) dz. \quad (3.84)$$

By induction, when $\alpha > 2c_4$, it holds for every bounded $f \geq 0$ and every $n \geq 1$ that

$$G^0_{\alpha} (b \cdot G^0_{\alpha})^n f(x) = \int_E u^{(n-1)}_{\alpha}(x, y) b(y) G^0_{\alpha} f(y) dy = \int_E u^{(n-1)}_{\alpha}(x, y) \left( \int_E b(y) \cdot \nabla y u_{\alpha}(y, z) f(z) dz \right) dy = \int_E \left( \int_E u^{(n-1)}_{\alpha}(x, y) b(y) \cdot \nabla y u_{\alpha}(y, z) dz \right) f(z) dz = \int_E u^{(n)}_{\alpha}(x, z) f(z) dz.$$ 

Recall that it has been claimed in Theorem 2.1 that $G^b_{\alpha} f = \sum_{n=0}^{\infty} G^0_{\alpha} (b \cdot G^0_{\alpha})^n f$ for all $f \in L^2(E)$, where the sum converges in the sense of $|| \cdot ||_{1,2}$ on $E$. By monotone convergence theorem, it holds for all $\alpha > 2c_4$ and all non-negative functions $f \in L^2(E)$ that

$$G^b_{\alpha} f(x) = \sum_{n=0}^{\infty} \int_E u^{(n)}_{\alpha}(x, y) f(y) dy = \int_E \int_0^\infty e^{-\alpha t} p^b(t, x, y) f(y) dt dy.$$ 

Thus the same relationship as above holds for all bounded measurable functions $f$. This implies that when $\alpha > 2c_4$, for all $x \in E$, $G^b_{\alpha}$ has kernel $\int_0^\infty e^{-\alpha t} p^b(t, x, y) dt$, i.e., the Laplace transform of $p^b(t, x, y)$ defined in $(3.72)$. It follows that the family of $\{p^b(t, x, y)\}$ on $\mathbb{R}^+ \times E \times E$ is indeed the transition density functions of $X$. To prove the last statement of the lemma, we only need
to show $p^b(t, x, y)$ has the same type of upper bounds as those in Lemma 3.15 on $(0, 2T_1]$. For this purpose, observing $p^b(t, x, y)$ satisfies Chapman-Kolmogorov equation globally, we have for some constants $c_1, \alpha_{13}, \alpha_{14} > 0$ that

$$p^b(t, x, y) = \int_{z \in E} p^b(t/2, x, z)p^b(t/2, z, y)m_p(dz)$$

$$\lesssim \int_{E} p^0_{\alpha_{13}}(t/2, x, z)p^0_{\alpha_{13}}(t/2, z, y)m_p(dz)$$

$$\lesssim \int_{E} p(c_1t, x, z)p(c_1t, z, y)m_p(dz)$$

$$= p(2c_1t, x, y) \lesssim p^0_{\alpha_{14}}(t, x, y), \quad t \in (0, 2T_1].$$

Therefore the same type of upper bound holds for $p^b(t, x, y)$ on $t \in (0, T_2]$ for arbitrary $T_2 > 0$ by repeating the same process finitely times.

In view of Lemma 3.15 and (3.74), the following theorem is an immediate consequence which provides the short time heat kernel upper bound for $X$.

**Proposition 3.17.** Fix $T > 0$. There exist positive constants $C_i$, $21 \leq i \leq 25$, so that the transition density $p^b(t, x, y)$ of drifted BMVD satisfies the following estimates when $t \in (0, T]$.

1. For $x \in \mathbb{R}_+$ or $y \in \mathbb{R}_+$,

$$p^b(t, x, y) \leq \frac{C_{21}}{\sqrt{t}} e^{-\frac{C_{22}(x-y)^2}{t}}.$$

2. For $x, y \in D_0 \cup \{a^*\}$ with $\max\{|x|, |y|\} < 1$,

$$p^b(t, x, y) \leq \frac{C_{21}}{\sqrt{t}} e^{-\frac{C_{22}(x-y)^2}{t}} + \frac{C_{21}}{t} \left(1 \wedge \frac{|x|}{\sqrt{t}}\right) \left(1 \wedge \frac{|y|}{\sqrt{t}}\right) e^{-\frac{C_{24}|x-y|^2}{t}};$$

and when $\max\{|x|, |y|\} \geq 1$,

$$p^b(t, x, y) \leq \frac{C_{21}}{t} e^{-\frac{C_{22}(x-y)^2}{t}}.$$

We next establish two-sided estimates for $p^b(t, x, y)$ by showing that it has a lower bound whose form matches the upper bound.

**Proposition 3.18.** Fix $T > 0$. There exist positive constants $C_i$, $26 \leq i \leq 39$, so that the transition density $p^b(t, x, y)$ of drifted BMVD satisfies the following estimates when $t \in (0, T]$.

1. For $x \in \mathbb{R}_+$ or $y \in \mathbb{R}_+$,

$$\frac{C_{26}}{\sqrt{t}} e^{-\frac{C_{27}(x-y)^2}{t}} \leq p^b(t, x, y) \leq \frac{C_{28}}{\sqrt{t}} e^{-\frac{C_{29}(x-y)^2}{t}}.$$

2. For $x, y \in D_0 \cup \{a^*\}$ with $\max\{|x|, |y|\} < 1$,

$$\frac{C_{30}}{\sqrt{t}} e^{-\frac{C_{31}(x-y)^2}{t}} + \frac{C_{30}}{t} \left(1 \wedge \frac{|x|}{\sqrt{t}}\right) \left(1 \wedge \frac{|y|}{\sqrt{t}}\right) e^{-\frac{C_{32}|x-y|^2}{t}} \leq p^b(t, x, y)$$

$$\leq \frac{C_{33}}{\sqrt{t}} e^{-\frac{C_{34}(x-y)^2}{t}} + \frac{C_{33}}{t} \left(1 \wedge \frac{|x|}{\sqrt{t}}\right) \left(1 \wedge \frac{|y|}{\sqrt{t}}\right) e^{-\frac{C_{35}|x-y|^2}{t}};$$

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and when max{\(|x|_\rho, |y|_\rho\)} \geq 1,

\[ \frac{C_{36}}{t} e^{-\frac{C_{37}(x-y)^2}{t}} \leq p^b(t, x, y) \leq \frac{C_{38}}{t} e^{-\frac{C_{39}(x-y)^2}{t}}. \]

Proof. For each of the three cases, we only need to establish the lower bound for sufficiently small \(T > 0\). The conclusion for arbitrary \(T\) then follows from Chapman-Kolmogorov equation. Note that in the proof to Lemma 3.15, the constants \(c_1, c_4, c_7\ldots\) there are all monotonically decreasing in \(t\). Thus in view of the inequality

\[ p^b(t, x, y) = \sum_{n=0}^{\infty} k_n(t, x, y) \geq p(t, x, y) - \sum_{n=1}^{\infty} |k_n(t, x, y)|, \]

we can pick \(T > 0\) sufficiently small in Proposition 3.17 so that for all \(0 < t < T, \rho(x, y) < 2\sqrt{t}\), it holds for some \(c > 0\) that

(i) \(x \in \mathbb{R}_+\) or \(y \in \mathbb{R}_+, p^b(t, x, y) \geq c/\sqrt{t};\)

(ii) \(x, y \in D_0, \min\{|x|_\rho, |y|\} \geq 2, p^b(t, x, y) \geq c/t;\)

(iii) Otherwise, \(p^b(t, x, y) \geq \frac{c}{\sqrt{t}} + \frac{c}{t} (1 + \frac{|x|_\rho}{\sqrt{t}}) (1 + \frac{|y|_\rho}{\sqrt{t}}).\)

With the above “near-diagonal” estimates, in order to get the “off-diagonal” estimates, we again divide the computation into several cases depending on the positions of \(x\) and \(y\). In the following computation, without loss of generality we always assume \(|x - y| \geq \rho(x, y) \geq 2\sqrt{t}\). We fix an sufficiently large constant \(\lambda > 0\) which will be used in all the following cases.

Case 1. \(x, y \in \mathbb{R}_+.\) Let \(m\) be the smallest integer such that \(m \geq \lambda|y - y|_\rho^2/t\). It follows that

\[ p^b(t, x, y) \geq \int_{\mathbb{R}_+ \cap \{|y_k - y_{k-1}| \leq \frac{1}{t} \sqrt{m}, k = 2, \ldots, m\}} p^b(t/m, x, y_1) \cdots p^b(t/m, y_{m-1}, y) dy_1 \cdots dy_{m-1} \]

\[ \geq \left( \frac{c}{\sqrt{t/m}} \right)^m \left( \sqrt{\frac{t}{m}} \right)^{m-1} \geq \frac{c^m}{\sqrt{t}} \geq \frac{1}{\sqrt{t}} e^{-c|x-y|^2/t}, \]  \hspace{1cm} (3.85)

where the last inequality is due to the fact that \(m \geq |x - y|^2/t\).

Case 2. \(x \in \mathbb{R}_+, y \in D_0\) with \(|y|_\rho < \sqrt{t}\). Note that in this case, \(|x| \geq \rho(x, y)\). Using the result for Case 1, we have

\[ p^b(t, x, y) \geq \int_{y_1 \in \mathbb{R}_+, |y_1| < \sqrt{t}/4} p^b(t/4, x, y_1) p^b(3t/4, y_1, y) dy_1 \]

\[ \geq \frac{1}{\sqrt{t}} e^{-c|x-y|^2/t} \cdot \frac{1}{\sqrt{t}} e^{-c|y|^2/t}. \]  \hspace{1cm} (3.86)

Case 3. \(x \in \mathbb{R}_+, y \in D_0\) with \(|y|_\rho > \sqrt{t}\). Again we let \(m\) be the smallest integer such that \(m \geq \lambda|y|_\rho^2/t\). We set a region \(D := \{y_1, \ldots, y_m \in D_0, \sqrt{t}/4 < |y_1|_\rho < \sqrt{t}/2, |y_k - y_{k-1}| < \frac{1}{8} \sqrt{t/m}, k = 1, \ldots, m, \text{ and } |y_m - y| < \frac{1}{8} \sqrt{t/m}\}\). Note that in \(D, |y_k|_\rho > \sqrt{t}/8,\) for all \(1 \leq k \leq m\).
Using the results of both Case 2 and the Dirichlet heat kernel estimate, since all the \(y_k\)'s are at least \(\sqrt{t}/8\) distance away from \(a^*\), we get
\[
p^b(t, x, y) \geq \int_{D} p^b(t/2, x, y_1)p^b(t/2m, y_1, y_2) \cdots p^b(t/2m, y_m, y)dy_1 \cdots dy_m
\]
\[
\geq \frac{1}{\sqrt{t}} e^{-c|x|^2/t} \left( \frac{c}{t/m} \right)^m \cdot \left( \frac{t}{m} \right)^m \geq \frac{1}{\sqrt{t}} e^{-c(|x|^2+|y|^2)/t} \geq \frac{1}{\sqrt{t}} e^{-c|\rho|^2/t},
\]  
(3.87)

where in the last "\(\geq\)" it has been used that \(m \propto |y|^2/\rho\).

**Case 4.** \(x \in D_0, y \in \mathbb{R}^+\). It is easy to see that this case can be handled following the exact same argument as Case 2 and 3.

**Case 5.** \(x, y \in D_0\) with \(\max\{|x|_\rho, |y|_\rho\} > 2\) and \(\min\{|x|_\rho, |y|_\rho\} > \sqrt{t}/2\). For this case, the lower bound follows directly from Dirichlet heat kernel estimates.

**Case 6.** \(x, y \in D_0\) with \(\max\{|x|_\rho, |y|_\rho\} > 2\) and \(\min\{|x|_\rho, |y|_\rho\} \leq \sqrt{t}/2\). Without loss of generality, we assume \(|x|_\rho < \sqrt{t}/2\) and \(|y|_\rho > 2\). With the conclusion for Case 5, we have
\[
p^b(t, x, y) \geq \int_{\{y_1 \in D_0, \sqrt{t}/4 < |y_1|_\rho < \sqrt{t}/2\}} p^b(t/4, x, y_1)p^b(t/4, y_1, y)dy_1
\]
\[
(\rho(x, y_1) \leq 3\sqrt{t}/4) \geq \frac{1}{\sqrt{t}} \cdot \frac{1}{t} e^{-c(|y|_\rho-3\sqrt{t}/4)^2/t} \cdot \sqrt{t}
\]
\[
(|y|_\rho - \sqrt{t}/2 \propto \rho(x, y)) \geq \frac{1}{\sqrt{t}} e^{-c\rho(x,y)^2/t}. \tag{3.88}
\]

**Case 7.** \(x, y \in D_0\) with \(\max\{|x|_\rho, |y|_\rho\} < 2\). Still we assume \(\rho(x, y) \geq 2\sqrt{t}\). Observe that in this case, it must hold that \(\max\{|x|_\rho, |y|_\rho\} \geq \sqrt{t}\) due to the assumption that \(\rho(x, y) \geq 2\sqrt{t}\). By Dirichlet heat kernel estimate, it holds
\[
p^b(t, x, y) = \tilde{p}^b_{D_0}(t, x, y) + \tilde{p}^b_{D_0}(t, x, y)
\]
\[
\geq \frac{1}{t} \left( 1 - \frac{|x|_\rho}{\sqrt{t}} \right) \left( 1 - \frac{|y|_\rho}{\sqrt{t}} \right) e^{-c|x-y|^2/t} + \tilde{p}^b_{D_0}(t, x, y). \tag{3.89}
\]

Therefore, it suffices to provide an lower bound for \(\tilde{p}^b_{D_0}(t, x, y)\). However, using the results for Cases 2-4, we immediately have the following:
\[
\tilde{p}^b_{D_0}(t, x, y) \geq \int_{\{y_1 \in \mathbb{R}^+, |y_1| < \sqrt{t}/16\}} p^b(t/2, x, y_1)p^b(t/2, y_1, y)dy_1
\]
\[
\geq \frac{1}{\sqrt{t}} e^{-c(|x|^2+|y|^2)/t} \cdot \frac{1}{\sqrt{t}} e^{-c(|y_1|^2+|y|^2)/t} \cdot \sqrt{t} \geq \frac{1}{\sqrt{t}} e^{-c(|x|^2+|y|^2)/t}. \tag{3.90}
\]

This combined with (3.89) establishes the desired result for the current case.

\[ \square \]

### 4 Green Function Estimate for Drifted BMVD

In this section, we establish two-sided bounds for the Green function of drifted BMVD \(X\) killed upon exiting a bounded connected \(C^{1,1}\) open set \(D \subset E\). Recall that the Green function \(G_D(x, y)\) is defined as follows:
\[
G_D^b(x, y) = \int_0^\infty p^b_D(t, x, y)dt,
\]
where \( p_D^b(t, x, y) \) is the transition density function of the subprocess \( X^{b, D} \) with respect to \( m_p \). We assume \( a^* \in D \) throughout this section, as otherwise, due to the connectedness of \( D \), either \( D \subset \mathbb{R}^+ \) or \( D \subset D_0 \). Therefore \( G_D^b(x, y) \) is just the standard Green function of a bounded \( C^{1,1} \) domain for Brownian motion with drift in one-dimensional or two-dimensional spaces, whose two-sided estimates are known, see [5]. It is not too hard to see from

\[
p_D^b(t, x, y) = p^b(t, x, y) - \mathbb{E}_x[p^b(t - \tau_D, X_{\tau_D}, y); \tau_D < t],
\]

that \( p_D^b(t, x, y) \) is jointly continuous in \((t, x, y)\).

Recall that for any bounded open set \( D \subset E \), \( \delta_D(\cdot) := \rho(\cdot, \partial D) \) denotes the \( \rho \)-distance to the boundary \( \partial D \). For notational convenience, we set \( D_1 := D \cap (\mathbb{R}^+ \setminus \{a^*\}) \) and \( D_2 := D \cap D_0 \). Note that \( a^* \in \partial D_1 \cap \partial D_2 \).

Before we give the two-sided Green function estimate for \( X \), we first claim that the given any bounded open interval \( I \) on \( \mathbb{R} \), the Green function \( g^b_I(x, y) \) associated with \( A^b : A^b u = u'' + b_1 \cdot u \) is comparable to that of \( A : A u = u'' \) on \( I \), assuming that \( b_1 \in L^{p_1}(I) \) for some \( p_1 > 1 \).

**Proposition 4.1.** Let \( I \) be any bounded open interval on \( \mathbb{R} \). There exist constants \( C_1, C_2 > 0 \) such that

\[
C_1 \cdot g_I(x, y) \leq g^b_I(x, y) \leq C_2 \cdot g_I(x, y), \quad \text{for } x, y \in \bar{I}.
\]

**Proof.** Let \((W_t, \mathbb{P}^{0,x})\) be one-dimensional Brownian motion killed upon exiting \( I \) and let \( p^0(t, x, y) \) be its density. To condition \( W \) on paths of exiting \( I \), define

\[
p^0_2(t, x, y) := \frac{p^0(t, x, y) g_I(y, z)}{g_I(x, z)}, \quad \text{if } z \in I;
\]

and

\[
p^0_1(t, x, y) := \frac{p^0(t, x, y) K_I(y, z)}{K_I(x, z)}, \quad \text{if } z \in \partial I,
\]

where \( K_I(\cdot, \cdot) \) is the Martin kernel with respect to \( W \). Thus for fixed \( z \in \partial I \), \( K_I(\cdot, z) \) is a linear function. Also it is known that \( g_I(y, z) \asymp \delta_I(y) \wedge \delta_I(z) \). Define \( M^b_{t} = \int_0^t b_1(W_s) dW_s \). Then \( \langle M \rangle_t = \int_0^t |b_1(W_s)|^2 ds \). Let \( N_t := \exp(M_t - \frac{1}{2} \langle M \rangle_t) \) and \( H(x, y) := \mathbb{E}^x_{\tau_I} [N_{\tau_I}] \) for \( x, z \in I \). Following the same proof of [11] Theorem 3.1, one can show that \( c_1 \leq H(x, y) \leq c_2 \) when \( x, z \in I \). Then by the same argument as [11] Theorem 3.4, it holds that \( g^b_I(x, y) = g_I(x, y) H(x, y) \asymp g_I(x, y) \) on \( I \times I \).

The following theorem gives two-sided Green function estimates for \( X \) in bounded \( C^{1,1} \) domains, whose proof follows the same idea as that in [12].

**Theorem 4.2.** Let \( b \) be a measurable function on \( E \) in \( L^{p_1, p_2}(E) \) with \( p_1 \in (1, \infty) \) and \( p_2 \in (2, \infty] \). Let \( G_D^b(x, y) \) be the Green function of \( X \) killed upon exiting \( D \), where \( D \) is a connected bounded \( C^{1,1} \) domain of \( E \) containing \( a^* \). We have for \( x \neq y \in D \),

\[
G_D^b(x, y) = \begin{dcases}
\delta_D(x) \wedge \delta_D(y), & x \in D_1 \cup \{a^*\}, y \in D_1 \cup \{a^*\};
\delta_D(x) \delta_D(y) + \ln \left( 1 + \frac{\delta_D(x) \delta_D(y)}{|x-y|^2} \right), & x \in D_2, y \in D_2; \\delta_D(x) \delta_D(y), & x \in D_1 \cup \{a^*\}, y \in D_2 \text{ or vice versa.}
\end{dcases}
\]

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Proof. We first show that $G^b_D(x, a^*)$ is a bounded positive continuous function on $D$. By Theorem 1.5 and Theorem 1.7, there is a constant $c_1 > 0$ so that for every $x \in D$,

$$
P_x(\tau_D < 1) \geq P_x(X_1 \in E \setminus D) = \int_{\mathbb{R}_+ \cap \mathbb{R}^c} p(1, x, z)m_p(dz) + \int_{D_0 \setminus \mathbb{R}^c} p(1, x, z)m_p(dz) \geq c_1.
$$

(4.1)

Thus $P_x(\tau_D \geq 1) \leq 1 - c_1$ for every $x \in D$. By the strong Markov property of $X$, there are constants $c_2, c_3 > 0$ so that $P_x(\tau_D \geq t) \leq c_2e^{-c_3t}$ for every $x \in D$ and $t > 0$. For $t \geq 2$ and $x, y \in D$, we thus have by Theorem 1.7

$$p^b_D(t, x, y) = \int_D p^b_D(t - 1, x, z)p^b_D(1, z, y)m_p(dz) \leq c_4 \int_D p^b_D(t - 1, x, z)m_p(dz) \leq c_5e^{-c_3t}.
$$

By Theorem 1.7 again, we conclude that

$$G^b_D(x, a^*) = \int_0^2 p^b_D(t, x, y)dt + \int_2^\infty p^b_D(t, x, y)dt
$$

converges and is a bounded positive continuous function in $x \in D$. In particular, $G^b_D(a^*, a^*) < \infty$.

We further note that $x \mapsto G^b_D(x, a^*)$ is a harmonic function in the interval $D_1$ with respect to one-dimensional Brownian motion with drift. In other words, it is a harmonic function in $D_1$ with respect to $\Delta + b_1 \cdot \nabla$ for $d = 1$. So one can solve it explicitly and verify that it is comparable to a linear function. As it vanishes at $a_1 := \partial D \cap \mathbb{R}_+$, we have

$$G^b_D(x, a^*) \asymp |a_1 - x| \asymp \delta_D(x) \quad \text{for } x \in D_1.
$$

(4.2)

(i) Assume $x, y \in D_1 \cup \{a^*\}$ and $x \neq y$. If $x = a^*$ or $y = a^*$, the desired estimate holds in view of (4.2). Thus we assume neither $x$ nor $y$ is $a^*$. By the strong Markov property of $X$,

$$G^b_D(x, y) = G^b_{D_1}(x, y) + \mathbb{E}_x[G^b_D(X_{\sigma_{a^*}}, y); \sigma_{a^*} < \tau_D] = G^b_{D_1}(x, y) + G^b_D(a^*, y)P_x(\sigma_{a^*} < \tau_D).
$$

The following Green function estimate for one-dimensional $\Delta + b_1 \cdot \nabla$ has been claimed in Proposition 4.1.

$$G^b_{D_1}(x, y) = g^b_{D_1}(x, y) \asymp \delta_{D_1}(x) \wedge \delta_{D_1}(y), \quad x, y \in D_1 \cup \{a^*\}.
$$

On the other hand, $x \mapsto P_x(\sigma_{a^*} < \tau_D)$ is a harmonic function with respect to $\Delta + b_1 \cdot \nabla$ in $D_1$ that vanishes at $a_1$. Thus by the same reasoning as that for (4.2), we have

$$P_x(\sigma_{a^*} < \tau_D) \asymp c_7|x - a_1| \asymp \delta_D(x) \quad \text{for } x \in D_1.
$$

(4.3)

It follows then

$$G^b_D(x, y) \asymp \delta_{D_1}(x) \wedge \delta_{D_1}(y) + \delta_D(x)\delta_D(y) \asymp \delta_D(x) \wedge \delta_D(y).
$$

(ii) Assume that $x, y \in D_2$. By the strong Markov property of $X$,

$$G^b_D(x, y) = G^b_{D_2}(x, y) + \mathbb{E}_x[G^b_D(X_{\sigma_{a^*}}, y); \sigma_{a^*} < \tau_D] = G^b_{D_2}(x, y) + G^b_D(a^*, y)P_x(\sigma_{a^*} < \tau_D).
$$

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Since both $y \mapsto G^{b}_{D_2}(a^*, y)$ and $x \mapsto \mathbb{P}_x(\sigma_{a^*} < \tau_D)$ are bounded positive harmonic functions with respect to $\Delta + b_2 \cdot \nabla$ on $D \cap D_0$ for $d = 2$ vanishing on $D_0 \cap \partial D$, it follows from the boundary Harnack inequality for Brownian motion with drift in $\mathbb{R}^2$ (see [11, Theorem 3.6, Corollary 3.14]) that
\[
G^{b}_{D_2}(a^*, y) \asymp \delta_D(y) \quad \text{and} \quad \mathbb{P}_x(\sigma_{a^*} < \tau_D) \asymp \delta_D(x)
\] (4.4)
This combined with the Green function estimates of $G^{b}_{D_2}(x, y)$ (see [9]) yields
\[
G^{b}_{D}(x, y) \asymp \ln \left(1 + \frac{\delta_D(x)\delta_D(y)}{|x - y|^2}\right) + \delta_D(x)\delta_D(y).
\] (iii) For the remaining case that $x \in D_1 \cup \{a^*\}$ and $y \in D_2$. When $x = a^*$, the desired estimates follows from (4.4) and so it remains to consider $x \in D_1$ and $y \in D_2$. By the strong Markov property of $X$, (4.3) and (4.4),
\[
G^{b}_{D}(x, y) = \mathbb{E}_x[G^{b}_{D}(X_{\sigma_{a^*}}, y); \sigma_{a^*} < \tau_D] = G^{b}_{D}(a^*, y)\mathbb{P}_x(\sigma_{a^*} < \tau_D) \asymp \delta_D(x)\delta_D(y).
\]
This completes the proof of the theorem.

\[\square\]

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**References**

[1] Bogdan, K., Jakubowski, T., Estimates of heat kernel of fractional Laplacian perturbed by gradient operators, *Commun. Math. Phys.* 271 (2007) 179-198.

[2] Bass, R.F., Chen, Z.-Q., Brownian motion with singular drift, *Ann. Prob.* 31 (2003), 791-817.

[3] Chen, Z.-Q., Topics on Recent Developments in the Theory of Markov Processes, [http://www.math.washington.edu/~zchen/RIMS_lecture.pdf](http://www.math.washington.edu/~zchen/RIMS_lecture.pdf)

[4] Chen, Z.-Q., Fukushima, M., *Symmetric Markov processes, time change and boundary theory*, Princeton University Press, 2011.

[5] Cho S., Kim P., and Park H., Two-sided estimates on Dirichlet heat kernels for time-dependent parabolic operators with singular drifts in $C^{1,\alpha}$-domains. *J. Diff. Equations* 252 (2012), 1101-1145.

[6] Chen, Z.-Q., Kim, P., Song, R. and Vondraček, Z., Sharp Green function estimates for $\Delta + \Delta^{\alpha/2}$ in $C^{1,1}$ open sets and their applications, *Illinois Journal of Mathematics* 54 (2010), 981-1024.

[7] Chen, Z.-Q., Lou, S., Brownian motion on spaces with varying dimension. Preprint.
[8] Chen, Z.-Q., Zhao, Z., Diffusion processes and second order elliptic operators with singular coefficients for lower order terms, *Math. Ann.* 302 (1995) 323-357.

[9] Chung, K. L., Zhao, Z., *Form Brownian Motion to Schrödinger’s Equation*, Springer, 1995.

[10] Cranston, M., Fabes, E., and Zhao Z., Conditional gauge and potential theory for the Schrödinger operator, *Trans. AMS* 307 (1988), 171-194.

[11] Cranston, M., Zhao, Z., Conditional transformation of drifted formula and potential theory for $\Delta + b(\cdot) \cdot \nabla$, *Comm. Math. Phys.* 112 (1987), 613-625.

[12] Karatzas, I., Shreve, S. E., *Brownian Motion and Stochastic Calculus*, Springer, 1988.

[13] Kim, P., Song, R., Two-sided estimates on the density of Brownian Motion with Singular Drift. *Illinois Journal of Mathematics* 50 Vol.3 (2006), 635-688, September 2006

[14] D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion*, Springer, 1991.

[15] Riahi, L., Comparison of Green functions and harmonic measures for parabolic operators. *Potential Anal.* 23 (2005), 381-402.

[16] Q. S. Zhang, Gaussian bounds for the fundamental solutions of $\nabla (A \nabla u) + B \nabla - u_t = 0$. *Manuscripta Math.* 93 (1997), 381-390.

[17] Zhang, Q. S., Global bounds of Schrödinger heat kernels with negative potentials, *Journal of Functional Analysis* 182 (2001), 344-370.

[18] Zhang, Q. S., Some gradient estimates for the heat equation on domains and for an equation by Perelman, *Int. Math. Res. Not.* p.39 (Article ID 92314).