Abstract
We show how modal quantales arise as convolution algebras $Q^X$ of functions from $\ell r$-multisemigroups $X$, that is, multisemigroups with a source map $\ell$ and a target map $r$, into modal quantales $Q$, which can be seen as weight or value algebras. In the tradition of boolean algebras with operators we study modal correspondences between algebraic laws in $X$, $Q$ and $Q^X$. The class of $\ell r$-multisemigroups introduced in this article generalises Schweizer and Sklar’s function systems and object-free categories to a setting isomorphic to algebras of ternary relations as used in boolean algebras with operators and in substructural logics. Our results provide a generic construction recipe for weighted modal quantales from such multisemigroups. This is illustrated by many examples, ranging from modal algebras of weighted relations as used in fuzzy mathematics, category quantales in the tradition of category algebras or group rings, incidence algebras over partial orders, discrete and continuous weighted path algebras, weighted languages of pomsets with interfaces, and weighted languages associated with presimplicial and precubical sets. We also discuss how these results can be combined with previous ones for concurrent quantales and generalised to a setting that supports reasoning with stochastic matrices or probabilistic predicate transformers.

Keywords: multisemigroups, quantales, convolution algebras, (object-free) categories, modal algebras, quantitative program verification.

1 Introduction
Convolution is an ubiquitous operation in mathematics and computing. Let, for instance, $(\Sigma^*, \circ, \varepsilon)$ be the free monoid over the finite alphabet $\Sigma$ and $(S, +, \cdot, 0, 1)$ a semiring. The convolution of two functions $f, g : \Sigma^* \to S$ is then defined as

$$(f * g)(x) = \sum_{x=y \circ z} f(y) \cdot g(z),$$

where $\sum$ represents finitary addition in $S$. The functions $f$ and $g$ associate a value or weight in $S$ with any word in $\Sigma^*$. The weight of the convolution $f * g$ and word $x$ is thus computed by splitting $x$ into all possible words $y$ and $z$ such that $x = y \circ z$, multiplying their weights $f(y)$ and $g(z)$ in $S$, and then adding up the results for all appropriate $y$ and $z$. A finite addition suffices in this example because only finitely many $y$ and $z$ satisfy $x = y \circ z$ for any $x$. If $S = \{0, 1\} = 2$ is the semiring of booleans (with max as + and min as $\cdot$), $f : \Sigma^* \to 2$ becomes a characteristic function for a set; $f(x)$ can be read as $x \in f$ and convolution is language product. The generalisation to arbitrary semirings thus yields weighted languages [DKV09], and it can be shown that convolution algebras $S^{\Sigma^*}$ of weighted languages form again semirings with convolution as multiplication.

This data can be changed in various ways. We are interested in the case where $\Sigma^*$ is generalised to a multisemigroup $(X, \circ)$ with multioperation $\circ : X \times X \to PX$ satisfying a suitable associativity law [KM15], or to a multimonoid that can have many units. Multisemigroups specialise to partial semigroups when $|x \circ y| \leq 1$ for all $x, y \in X$ and to semigroups when $|x \circ y| = 1$. We also replace $S$
by a quantale $Q$, a complete lattice with an additional monoidal composition and unit satisfying certain sup-preservation laws. The existence of arbitrary sups in $Q$ then compensates for the lack of finite decompositions in $X$, which were previously available in $\Sigma^*$. For functions $f, g : X \to Q$, convolution now becomes

$$(f * g)(x) = \bigvee_{x \in y \sqcup z} f(y) \cdot g(z),$$

where $\bigvee$ indicates a supremum and $\cdot$ the multiplication in $Q$. This time, the convolution algebra $Q^X$ can be equipped with a quantale structure \cite{DHS21}. For the quantale $Q = 2$ of booleans it becomes a powerset quantale. Prime examples for this construction are the convolution quantales of binary relations and of quantale-valued binary relations \cite{Gog67}, as well as boolean-valued and quantale-valued matrices. These lift from instances of multimonoids known as pair groupoids.

Yet the relationship between multimonoids, quantales and convolution quantales is not just a lifting. Multimonoids are isomorphic to relational monoids formed by ternary relations with suitable monoidal laws, where $R(x, y, z)$ holds if and only if $x \in y \circ z$. The construction of convolution algebras can thus be seen in light of Jónsson and Tarski's duality between boolean algebras with $n$-ary operators and $n+1$-ary relations \cite{IT51, Go69}, and in particular of the modal correspondences between these algebras and relations. Convolution can be seen as a generalised binary modality on $Q^X$ and $X$ as a ternary frame, so we may ask about correspondences between equations in the algebras $X$ and $Q$ and $Q^X$.

Such correspondences between monoidal properties in $X$ and quantalic properties in $Q$ and $Q^X$ have already been studied and adapted to situations where $Q$ is merely a semiring \cite{CDS20a}, for $Q = 2$, they are well known from substructural logics. They have also been extended to correspondences between concurrent relational monoids and the concurrent semirings and quantales previously introduced by Hoare and coworkers \cite{HMSW11}, where two quantalic compositions are present, which interact via a weak interchange law, as known from higher categories.

Here, our main motivation lies in the study of correspondences between the source and target structures \cite{CDS20b} and the domain and codomain structures that are present in multimonoids and many convolution quantales, including powerset quantales. On the one hand, every element of a multimonoid has a forteriori a unique left and a unique right unit, which can be captured using source and target maps like in a category \cite{CDS20b}, and every small category can of course be modelled in object-free style as a partial semigroup equipped with such maps \cite{ML98}. On the other hand, quantales of binary relations, for instance, have a non-trivial domain and codomain structure, which has led to more abstract definitions of modal semirings and quantales \cite{DMS06, DS11, FJSZ20}. Yet what is the precise correspondence between the source and target structure of the pair groupoid, say, and these relational domain and codomain operations? And how does this correspondence generalise to source and target maps in arbitrary multisemigroups and domain and codomain operations in arbitrary value and convolution quantales?

As a first step towards answers, we introduce $\ell r$-multisemigroups: multisemigroups $(X, \odot)$ equipped with two operations $\ell, r : X \to X$ inspired by similar ones that appear in Schweizer and Sklar’s function systems \cite{SS67} and by the source and target maps of object-free categories \cite{ML98}. We study the basic algebra of $\ell r$-multisemigroups and present a series of examples, including categories and non-categories. Most of these results have been formalised using the Isabelle/HOL proof assistant\footnote{https://github.com/gstruth/lr-multisemigroups}. It turns out, in particular, that the locality laws $\ell(x \odot \ell(y)) = \ell(x \odot y)$ and $r(r(x) \odot y) = r(x \odot y)$ of $\ell r$-multisemigroups, which have previously been studied in the context of modal semigroups, semirings and quantales \cite{DJS09, DMS06, DS11, FJSZ20}, are equivalent to the typical composition pattern of categories, namely that $x \odot y$ is defined (and hence not $\emptyset$) if and only if $r(x) = \ell(y)$ (for categories, the order of composition needs to be swapped). Indeed, partial $\ell r$-semigroups that satisfy locality are (small) categories.

As a second step, we generalise the standard definitions of domain and codomain used in modal semirings and quantales to ones suitable for convolution quantales. When relations are represented at 2-valued matrices, for instance, domain elements in $2^X$ are diagonal matrices in which the value of each diagonal element is 1 if there is a 1 in the respective row of the matrix, and 0 otherwise. More generally, for $Q$-valued relations represented as $Q$-valued matrices, a domain element in $Q^X$ can be seen as a diagonal matrix in which the value of each diagonal element is the domain of the supremum of the respective row values of the matrix, taken in $Q$. 

Figure 1: Triangle of correspondences between identities in families of $\ell r$-multisemigroups $X$, quantales $Q$, and convolution algebras $Q^X$. The box contains a miniature table of content pointing to the sections with the main contributions.

Equipped with $\ell r$-multisemigroups and the generalised domain and codomain operations for $Q^X$, we prove the main results in this article: correspondences between equational properties of source and target operations in $\ell r$-multisemigroups and those of domain and codomain operations in the modal quantales $Q$ and $Q^X$. The resulting triangle, with links to the main theorems that capture them, is shown in Figure 1. We develop these in full generality, but keep an eye on the $Q = 2$ case which has been formalised with Isabelle. From the point of view of boolean algebras with operators, this yields a multimodal setting where the quantalic composition in $Q^X$ is a generalised binary modality associated with a ternary frame in $X$, whereas the domain and codomain operations in $Q^X$ are generalised unary modalities that can be associated with binary frames in $X$ based on $\ell$ and $r$.

These results show how the equational axioms for modal semirings and quantales, as powerset or proper convolution algebras, arise from the much simpler ones of $\ell r$-multisemigroups and how in particular the composition pattern of categories translates into the locality laws of modal convolution algebras. More generally, the lifting from $X$ and $Q$ to $Q^X$ yields a generic construction recipe for modal quantales. It shows that any category, for instance, can be lifted in such a way, but we present other examples such as generalised effect or shuffle algebras, where locality laws are absent or which are proper multisemigroups. Constructing variants of modal quantales is interesting for program verification because these algebras are related to dynamic logics, Hoare logics and algebras of predicate transformers. Conversely, by these correspondences, before constructing a modal quantale one should ask what the underlying $\ell r$-multisemigroup might be. The benefit is that the $\ell r$-multisemigroup laws are much easier to check than those of modal quantales and in particular proof assistants can benefit from our generic construction.

The relevance and potential of the construction is underpinned by many examples. Most of the $\ell r$-multisemigroups presented are categories, for instance categories of paths in quivers or Moore paths in topology, pair groupoids, segments and intervals of posets or pomsets, simplices and cubical sets with interfaces, but some others are neither local nor partial. For all of them we get modal convolution quantales for free, and even modal convolution semirings, if the underlying $\ell r$-multisemigroup is finitely decomposable like $\Sigma^*$ above. This is in particular the case for Rota’s incidence algebras constructed over locally finite posets [Rot64]. Well known constructions from algebra, such as matrix rings, group rings, or category algebras, are closely related, yet typically use rings instead of semirings or quantales as value-algebras.

Beyond the results mentioned, we sketch a combination of the correspondence results for concurrent quantales and those for modal quantales, with obvious relevance to true concurrency semantics in computing, for instance based on pomsets or digraphs with interfaces [Win77, FDC13, FJST20]. Details are left to a successor article. We also discuss the relationship between the domain and codomain operations in modal convolution quantales, and the diamond and box operators that can be defined using
them. This opens the door to convolution algebras of quantitative predicate transformer semantics for programs, including probabilistic or fuzzy ones, just like in the qualitative case, using for instance the Lawvere quantale or related quantales over the unit interval as instances of \( Q \). Yet the definitions of domain and codomain in \( Q^X \) mentioned, despite being natural for correspondence results, turn out to be too strict for dealing with stochastic matrices or Markov chains. We outline how more liberal definitions of domain and codomain in \( Q^X \) can be used instead, yet leave details once again to future work.

The remainder of this article is organised as follows. Section 2 outlines the basics of multisemigroups and multimonomoids with many units, relates them with object-free categories and lists some examples. Section 3 introduces the first contribution of this article: \( \ell r \)-multisemigroups, relates them with Schweizer and Sklar’s function systems \( SS67 \), with domain semigroups \( DJS09 \), with multimonomoids and an alternative axiomatisation of object-free categories using source and target maps. Once again, examples are listed. Section 4 recalls notions of modal quantales \( FJSZ20 \) and discussed their structure. As a warm-up for more general constructions, Section 5 shows how modal powerset quantales can be lifted from \( \ell r \)-multisemigroups; proofs can be found in Appendix C. Section 6 then presents the first main result of this article: the construction of modal convolution quantales \( Q^X \) from \( \ell r \)-semigroups \( X \) and modal value quantales \( Q \), using appropriate definitions of domain and codomain on convolution algebras. It also discusses finitely decomposable \( \ell r \)-multisemigroups, where value quantales can be replaced by value semirings and similar structures. Section 8 contains the remaining main results, which complete the triangle of correspondences shown in Figure 1. A large list of examples is discussed in Section 9. These range from weighted path, interval and pomset languages over \( \ell r \)-semigroups that arise in algebraic topology to the weighted assertion quantales of separation logic. A combination of the results for the algebras in this article with previous ones suitable for concurrency \( CDS20a \) is outlined in Section 10. The final technical section, Section 11 discusses the modal box and diamond operators that arise in convolution quantales and some selected models, before Section 12 concludes the main part of the article. Finally, as many algebraic definitions are scattered across this article, we summarise the most important ones in Appendix A. Appendix B explains the equivalence of multisemigroups and relational semigroups, on which some proofs in this article are based. Finally, Appendix C shows the main proofs of Section 9 as already mentioned.

## 2 Multisemigroups

Categories can be axiomatised in object-free style \( ML98 \), essentially as partial monoids with many units that satisfy a certain locality condition. We have previously studied their correspondence to relational monoids \( CDS20b \), which are sets \( X \) equipped with ternary relations \( R \subseteq X \times X \times X \) with many relational units that satisfy certain relational associativity and unit laws. Here we use an isomorphic representation of multioperations of type \( X \times X \to \mathcal{P}X \) (the isomorphism being that between the category \( \text{Rel} \) and the Kleisli category of the powerset monad). A multioperation on \( X \), like the corresponding ternary relation on \( X \), relates pairs in \( X \times X \) with a set of elements of \( X \), including the empty set. This encompasses partial and total operations, where each such pair is related to at most one element and to precisely one element, respectively. Multioperations have a long history in mathematics, see \( KM15 \) for a discussion, references and additional material.

Dealing with categories that are not small requires classes instead of sets in some of our examples. In the tradition of object-free categories, we then tacitly extend the following definitions to classes, yet do not formally distinguish between sets and classes—the simple constructions in this article do not lead to paradoxes.

So let \( \circ : X \times X \to \mathcal{P}X \) be a multioperation on set \( X \). We write \( D_{xy} \) in place of \( x \circ y \neq \emptyset \) to indicate that the composition \( x \circ y \) is non-empty, which intuitively means defined, and extend \( \circ \) to an operation \( \circ : \mathcal{P}X \times \mathcal{P}X \to \mathcal{P}X \) defined, for all \( A, B \subseteq X \), by

\[
A \circ B = \bigcup \{ x \circ y \mid x \in A \text{ and } y \in B \}.
\]

We write \( x \circ B \) instead of \( \{ x \} \circ B \), \( A \circ x \) instead of \( A \circ \{ x \} \) and \( f(A) \) for \( \{ f(a) \mid a \in A \} \). Note that \( A \circ \emptyset = \emptyset \circ B = \emptyset \).

A multimag magma \(( X, \circ )\) is simply a non-empty set \( X \) with a multioperation \( \circ : X \times X \to \mathcal{P}X \).
The multioperation $\odot$ is associative if $x \odot (y \odot z) = (x \odot y) \odot z$.

It is local if $u \in x \odot y \land D_{yz} \Rightarrow D_{xz}$ for all $u, x, y, z \in X$.

It is a partial operation if $|x \odot y| \leq 1$ and a (total) operation if $|x \odot y| = 1$, for all $x, y \in X$.

An element $e \in X$ is a left unit in $X$ if $\exists x. x \in e \odot x$ and $\forall x, y. x \in e \odot y \Rightarrow x = y$; it is a right unit in $X$ if $\exists x. x \in e \odot y \land \forall x, y. x \in y \odot e \Rightarrow x = y$. We write $E$ for the set of all (left or right) units in $X$.

**Remark 2.1.** Intuitively, locality states that if $x$ and $y$ as well as $z$ can be composed, then every element in the composition of $x$ and $y$ can be composed with $z$. For partial operations this reduces to the composition pattern of categories, binary relations, paths in digraphs and many other examples, as explained below. Locality has previously been called coherence [CDS20]. It will become clear below why we are now adopting a different name.

A multimagma $X$ is unital if for every $x \in X$ there exist $e, e' \in E$ such that $D_{ex}$ and $D_{xe'}$. This definition of units follows that for object-free categories. Equivalently, we may require that there exists a set $E \subseteq X$ such that, for all $x \in X$,

$$E \odot x = \{x\} \quad \text{and} \quad x \odot E = \{x\}.$$ 

A multisemigroup is an associative multimagma; a multimonoid a unital multisemigroup.

These definitions imply that a multisemigroup $(X, \odot)$ is a partial semigroup if $\odot$ is a partial operation and a semigroup if $\odot$ is a total operation—and likewise for partial monoids and monoids.

Spelling out associativity yields

$$x \odot (y \odot z) = \bigcup \{x \odot v \mid v \in y \odot z\} = \bigcup \{u \odot z \mid u \in x \odot y\} = (x \odot y) \odot z.$$ 

Multimagmas and relatives form categories in several ways. A multimagma morphism $f : X \to Y$ satisfies

$$f(x \odot_X y) \subseteq f(x) \odot_Y f(y).$$

The morphism is bounded if, in addition,

$$f(x) \in u \odot_Y v \Rightarrow \exists y, z. x \in y \odot_X z \land u = f(y) \land v = f(z).$$

Obviously, $f$ is a multimagma morphism if and only if $x \in y \odot_X z \Rightarrow f(x) \in f(y) \odot_Y f(z)$. This is a natural generalisation of $x = y \odot_X z \Rightarrow f(x) = f(y) \odot_Y f(z)$, and hence of $f(x \odot_X y) = f(x) \odot_Y f(y)$, for total operations. For partial operations, it implies that the right-hand side of the inclusion must be defined whenever the left-hand side is, and, in the bounded case, that the left-hand side is defined whenever the right-hand side is.

A morphism $f : X \to Y$ of unital multimagmas needs to preserve units as well: $f(e) \in E_Y$ holds for all $e \in E_X$, and $e \in E_Y$ implies that there is an $e' \in E_X$ such that $f(e') = e$ holds if $f$ is bounded. Morphisms of object-free categories are functors.

More generally, bounded morphisms are standard in modal logic as functional bisimulations. The isomorphisms between categories of relational semigroups and monoids and those of multisemigroups and multimonoids are explained in Appendix B.

In every multimagma, every unit $e$ satisfies $e \odot e = \{e\}$ and $D_{ee}$. If $e, e' \in E$, then $D_{ee'}$ if and only if $e = e'$, for if $D_{ee'}$ holds, then $e \odot e' = \{x\}$ for some $x \in X$ and hence $e = x = e'$ by the (implicational) unit axioms. Units are therefore idempotents which are, in a sense, orthogonal.

In every multimonoid, each element has precisely one left and one right unit: if $e, e' \in E$ both satisfy $e \odot x = \{x\} = e' \odot x$ for some $x \in X$, then $\emptyset \neq e \odot x = e \odot (e' \odot x) = (e \odot e') \odot x$, which is only the case when $e = e'$, as explained above (the argument for right units is similar). This functional correspondence allows defining source and target maps $\ell, r : X \to X$ such that $\ell(x)$ denotes the unique left unit and $r(x)$ the unique right unit of $x$. Then $D_{xy}$ implies $r(x) = \ell(y)$ and the converse implication is equivalent to locality. These properties have been proved for relational structures [CDS20], but hold in the corresponding multialgebras via the isomorphism.

**Example 2.2** (Multimonoids).
1. The object-free categories in Chapter I.1 of Mac Lane’s book [ML98] and the local partial monoids are the same class, as already mentioned; the category of such object-free categories and that of local partial monoids (with both types of morphisms mentioned) are isomorphic. Local partial monoids are small categories, and categories if the partial monoid is built on a class.

2. Every monoid \((X, \cdot, 1)\) is a one-element category and therefore local. The diagram

\[
\begin{array}{c}
\bullet & \xrightarrow{a} & \bullet \\
1 & & \end{array}
\]

for instance, corresponds to a partial monoid \(X = \{1, a\}\) with multiplication defined by \(11 = \{1\}\) and \(1a = \{a\} = a1\); in addition we can impose \(aa = \{a\}\). Locality is trivial: composition is total and \(\ell(x) = 1 = r(x)\) for all \(x \in X\).

3. The pair groupoid (or codiscrete groupoid) \((X \times X, \circ, Id_X)\) over any set \(X\) is formed by the set of ordered pairs over \(X\) with

\[
(w, x) \circ (y, z) = \begin{cases} 
\{(w, z)\} & \text{if } x = y, \\
\emptyset & \text{otherwise}
\end{cases}
\]

and \(E = Id_X\), where \(Id_X\) is the identity relation on \(X\). It is a local partial monoid, and hence a category—in fact even a groupoid as the name indicates. Source and target maps are given by \(\ell((x, y)) = (x, x)\) and \(r((x, y)) = (y, y)\). This category is equivalent to a trivial category. \(X \times X\) is nothing but the universal relation on \(X\).

4. The pair groupoid example generalises from universal relations to smaller relations \(R \subseteq X \times X\). If \(R\) is transitive, then \(\circ\) is well-defined and \((R, \circ)\) a local partial semigroup. If \(R\) is also reflexive, and hence a preorder, then \((R, \circ, Id_X)\) is a partial monoid; it is a groupoid if and only if \(R\) is symmetric. Finally, if \(R\) is a partial order, we obtain the poset \((X, R)\) regarded as a poset category.

5. The shuffle multimonom \((\Sigma^*, \|, \{\varepsilon\})\), where \(\Sigma^*\) is the free monoid over the (finite) alphabet \(\Sigma\), is the empty word and the multioperation \(\| : \Sigma^* \times \Sigma^* \rightarrow \mathcal{P}\Sigma^*\) is defined, for \(a, b \in \Sigma, v, w \in \Sigma^*\), by

\[
v \| \varepsilon = \{v\} = \varepsilon \| v \quad \text{and} \quad (av) \| (bw) = a(v \| (bw)) \cup b((av) \| w),
\]

is not a category: \(\|\) is a proper multioperation. Locality is trivial because \(v \| w \neq \emptyset\) and \(\ell(v) = \varepsilon = r(v)\) for all \(v, w \in \Sigma^*\).

6. The monoid in \((2)\) becomes partial and non-local when we break composition and impose \(aa = \emptyset\), because \(\ell(a) = r(a)\) still holds. Instead of a one-element category, it is now a plain digraph.

7. The partial abelian monoid of heaplets \((H, \circ, \varepsilon)\) used in separation logic is formed by the set \(H\) of partial functions between two sets. The partial operation \(\circ\) is defined as

\[
f \circ g = \begin{cases} 
\{w \cup g \mid \text{dom}(f) \cap \text{dom}(g) = \emptyset, \\
\emptyset & \text{otherwise}
\end{cases}
\]

Its unit is the partial function \(\varepsilon\) with empty domain. Locality fails because \(\ell(f) = \varepsilon = r(g)\) for all \(f, g \in H\), but \(f \circ g = \emptyset\) when domains of \(f\) and \(g\) overlap. This algebra of heaplets is a non-local instance of a generalised effect algebra, used for modelling unsharp measurements in the foundations of quantum mechanics: a partially abelian monoid with a single unit (which in addition is cancellative and positive) [HP96].

8. In the multimagma \(\{(x, e, e'), \circ, \{e, e'\}\}\) with composition defined by

\[
\begin{array}{c|ccc}
\circ & e & e' & a \\
\hline
\varepsilon & \{e\} & \emptyset & \{a\} \\
e' & \emptyset & \{e'\} & \{a\} \\
a & \emptyset & \{a\} & \{a\}
\end{array}
\]

the element \(a\) has left units \(e\) and \(e'\) and right unit \(e'\). Associativity fails because \((e \circ e') \circ a \neq e \circ (e' \circ a)\). This shows that units need not be uniquely defined in multimagas.
Moreover, by the retraction laws, $x$.

Finally, every $\ell r$-laws are Axioms (D3) and its opposite (R3) of modal semigroups [DJS09]. Domain axioms for ordered semigroupoids, with a view on allegories, have already been considered by Kahl [Kah08]. We define multisemigroups with such functions. An isomorphic alternative based on relational semigroups has been outlined in [CDS20b].

An $\ell r$-multimagma is a structure $(X, \circ, \ell, r)$ such that $(X, \circ)$ is a multimagma and the operations $\ell, r : X \to X$ satisfy, for all $x, y \in X$,

$$D_{xy} \Rightarrow r(x) = \ell(y), \quad \ell(x) \circ x = \{x\}, \quad x \circ r(x) = \{x\}.$$

Recall that $D_{xy} \Leftrightarrow x \circ y \neq \emptyset$. Henceforth we often write $xy$ instead of $x \circ y$ and $AB$ instead of $A \circ B$.

An $\ell r$-multisemigroup is an associative $\ell r$-multimagma. An $\ell r$-multimagma is $\ell r$-local whenever $r(x) = \ell(y) \Rightarrow D_{xy}$ and therefore $D_{xy} \Leftrightarrow r(x) = \ell(y)$. Locality in the sense of multimonoids and $\ell r$-locality coincide in $\ell r$-multisemigroups; henceforth we simply speak about locality of $\ell r$-multisemigroups.

Duality for $\ell r$-multimagnas arises by interchanging $\ell$ and $r$ as well as the arguments of $\circ$. The classes of $\ell r$-multimagnas and semigroups, hence also $\ell r$-multisemigroups, are closed under this transformation; locality and partiality are self-dual. Hence the dual of any property that holds in any of these classes, obtained by making these replacements, holds as well. This generalises opposition in categories.

**Lemma 3.1.** In any $\ell r$-multimagma,

1. the compatibility laws $\ell \circ r = r, r \circ \ell = \ell$ and retraction laws $\ell \circ \ell = \ell, r \circ r = r$ hold,
2. the idempotency law $\ell(x)\ell(x) = \{\ell(x)\}$ holds,
3. the commutativity law $r(x)\ell(y) = \ell(y)r(x)$ holds,
4. the export laws $\ell(\ell(x)y) = \ell(x)\ell(y)$ and $r(xy) = r(x)r(y)$ hold,
5. the weak twisted laws $\ell(xy)x \subseteq x\ell(y)$ and $xr(y) \subseteq r(y)x$ hold.

All proofs have been checked with Isabelle.

**Remark 3.2.** We compare our axioms and derived laws with the axioms of Schweizer and Sklar’s function systems [SS67] (see Appendix A for a list). The associativity axiom of multisemigroups generalises the associativity axiom of function systems. The compatibility laws are their Axioms (24), the absorption axioms $\ell(x)x = \{x\} = x\ell(x)$ their Axioms (21); the commutativity law is their Axiom (31). The export laws are Axioms (13) and its opposite (R3) of modal semigroups [DJS09]. The relationship between $\ell r$-algebras, function systems and modal semigroups is summarised in Remark 3.3 below.

The compatibility laws imply that $\ell(x) = x \Leftrightarrow r(x) = x$ and further that

$$X_\ell = \{x \mid \ell(x) = x\} = \{x \mid r(x) = x\} = X_r.$$

Moreover, by the retraction laws, $X_\ell = \ell(X)$ and $X_r = r(X)$.

**Lemma 3.3.** In any $\ell r$-multisemigroup,

1. the commutativity laws $\ell(x)\ell(y) = \ell(y)\ell(x)$ and $r(x)r(y) = r(y)r(x)$,
2. the idempotency law $r(x)r(x) = \{r(x)\}$,
3. the orthogonality laws $D_{\ell(x)\ell(y)} \Leftrightarrow \ell(x) = \ell(y)$ and $D_{r(x)r(y)} \Leftrightarrow r(x) = r(y)$.

Finally, every $\ell r$-multimagma is unital with $E = X_\ell = X_r$, and we often write $E$ instead of $X_\ell$ or $X_r$.
Remark 3.4. The locality laws generalise Axioms (3a) for function systems [SS67]; the twisted laws for this article. In sum, (3c) to both classes axiomatises domain and codomain of functions. Law (D3c) is relevant for systems generalise Axiom (3c) and law (D3c), in this order. Function semigroups without (3c) and law (D3c) precodomain, domain and codomain operations for semirings [DMS06]. In this context, predomain \( \ell r \)-multisemigroups, these variants of domain and codomain axioms are at the powerset level. Our results Remark 3.7. Remark 3.8. It seems natural to ask whether the axiom Proof. We have checked with Isabelle that the equational locality laws imply \( \ell r \)-locality in any \( \ell r \)-multimagma. Equality in \( \ell r \)-multisemigroups follows from Lemma 3.5. □

Lemma 3.5. In any local \( \ell r \)-multisemigroup,

1. the equational locality laws \( \ell(x)y \subseteq \ell(x\ell(y)) \) and \( r(xy) = r(xy) \) hold,
2. the twisted laws \( \ell(x)y = \ell(xy) \) and \( r(xy) = r(xy) \) hold.

Once again, all proofs have been checked with Isabelle. Locality is in fact an equational property.

Proposition 3.6. An \( \ell r \)-multisemigroup is \( \ell r \)-local if and only if \( \ell(x\ell(y)) \subseteq \ell(xy) \) and \( r(xy) \subseteq r(xy) \).

Proof. We have checked with Isabelle that the equational locality laws imply \( \ell r \)-locality in any \( \ell r \)-multimagma. Equality in \( \ell r \)-multisemigroups follows from Lemma 3.5.

Remark 3.7. Locality and weak locality have already been studied in the context of predomain, precodomain, domain and codomain operations for semirings [DMS06]. In this context, predomain and precodomain operations satisfy weak locality axioms, but not the strong ones. Relative to \( \ell r \)-multisemigroups, these variants of domain and codomain axioms are at the powerset level. Our results in Sections 6 to 8 explain how they are caused by locality in \( \ell r \)-multisemigroups.

Remark 3.8. It seems natural to ask whether the axiom \( D_{xy} \Rightarrow r(x) = \ell(y) \) in the definition of \( \ell r \)-magnas could be replaced, like locality, by equational axioms. Experiments with Isabelle show that adding the equational properties derived in Lemmas 3.1-3.5 does not suffice. We leave this question open.

The final lemma on \( \ell r \)-multisemigroups yields a more fine-grained view on definedness conditions and \( \ell r \)-locality.

Lemma 3.9. In any \( \ell r \)-multimagma,

1. \( r(x) = \ell(y) \Leftrightarrow D_{r(x)\ell(y)} \) and \( D_{xy} \Rightarrow D_{r(x)\ell(y)} \).
2. \( D_{r(x)\ell(y)} \Rightarrow D_{xy} \) whenever the \( \ell r \)-multimagma is a local \( \ell r \)-multisemigroup.

Once again, the proofs have been obtained by Isabelle. The correspondence between multimonomoids and \( \ell r \)-multisemigroups can now be summarised as follows.

Proposition 3.10. Every multimonomoid \((X, \circ, E)\) is an \( \ell r \)-multisemigroup \((X, \circ, \ell, r)\) in which \( \ell(x) \) and \( r(x) \) indicate the unique left and right unit of any \( x \in X \). Conversely, every \( \ell r \)-multisemigroup \((X, \circ, \ell, r)\) is a multimonomoid \((X, \circ, E)\) with \( E = X_\ell = X_r \).

The result trivially carries over to local structures and extends to isomorphisms between categories of \( \ell r \)-multisemigroups and relational monoids with suitable morphisms. A morphism \( f \) of \( \ell r \)-multimagnas \((X, \circ_X, \ell_X, r_X)\) and \((Y, \circ_Y, \ell_Y, r_Y)\) is of course a multimagma morphism that satisfies \( f \circ \ell_X = \ell_Y \circ f \) and \( f \circ r_X = r_Y \circ f \).
Example 3.11 (ℓr-Multisemigroups). All structures in Example 2.2 are ℓr-multisemigroups by Proposition 3.10. We consider some of them in more detail.

1. Local partial ℓr-semigroups and the object-free categories in Chapter XII.5 of Mac Lane’s book [ML98] are the same class, as already mentioned; the categories of such object-free categories and local partial ℓr-semigroups (with both kinds of morphisms) are isomorphic. Hence local partial ℓr-semigroups are small categories. We briefly recall the relationship to more standard definitions of categories.

A (small) category consists of a set \( O \) of objects and a set \( M \) of morphisms with maps \( s, t : M \to O \) associating a source and target object with each morphism, an operation \( id : O \to M \) associating an identity arrow with each object, and a partial operation of composition; of morphisms such that \( f; g \) is defined whenever \( t(f) = s(g) \) —where \( f; g = g \circ f \). The following axioms hold for all \( X \in O \) and \( f, g, h \in M \) (for Kleene equality, that is, whenever compositions are defined):

\[
\begin{align*}
\sigma(s(id(X))) &= X, \\
\tau(id(X)) &= X, \\
\sigma(f; g) &= \sigma(f), \\
\tau(f; g) &= \tau(g), \\
\sigma(f; (g; h)) &= (f; g); h, \\
\tau(f; (g; h)) &= f; \tau(g), \\
\sigma(id(s(f))) &= f, \\
\tau(id(t(f))) &= f.
\end{align*}
\]

In a category, \( \ell = id \circ s \) and \( r = id \circ t \) leads to ℓr-multisemigroups. Conversely, the elements in \( E \) of a local partial ℓr-semigroup \( X \) serve as objects of a category.

2. The pair groupoid is a local partial ℓr-semigroup.

3. In the broken monoid from Example 2.2(1), \( \ell(aa) = \ell(\emptyset) = \emptyset \subset \{1\} = \ell(a1) = \ell(a\ell(a)) \), hence locality of \( \ell \) fails; that of \( r \) fails by duality.

4. In the partial abelian ℓr-semigroup of heaplets from Example 2.2(7), locality fails, too: if the domains of \( f, g, h \in H \) overlap, then \( \ell(fg) = \ell(\emptyset) = \emptyset \subset \{\varepsilon\} = \ell(f) = \ell(f\varepsilon) = \ell(f\ell(g)) \) and locality of \( r \) fails by duality.

5. A concrete example of a category as a local partial ℓr-semigroup are Elgot’s matrix theories [Elg76].

The set \( MS = \bigcup_{n,m \geq 0} S^{n \times m} \) of matrices over any semiring \( S \) forms a partial monoid with matrix multiplication as composition. \( MS \) is a local partial ℓr-semigroup with \( \ell \) and \( r \) defined, for any \( M \in S^{n \times m} \), by \( \ell(M) = I_n \) and \( r(M) = I_m \); the identity matrices of the appropriate dimensions.

This category is isomorphic to the standard category of matrices, which has natural numbers as objects and \( n \times m \)-matrices as elements of the hom-set \([n,m] \).

4 Convolution Quantales

We have already extended the multioperation \( \circ : X \times X \to PX \) to type \( PX \times PX \to PX \) defining \( A \circ B = \bigcup \{ x \circ y \mid x \in A \text{ and } y \in B \} \) and the maps \( \ell, r : X \to X \) to type \( PX \to PX \) by taking images. We wish to explore the algebraic structure of such powerset liftings over ℓr-multimagnas and their relatives. It is known that powerset liftings of relational monoids yield unital quantales [Ros97, DHS21]. Yet the least law is absent [Ros90].

We write \( \sqcap \) for the sup and \( \sqcup \) for the inf operator, and \( \lor \) and \( \land \) for their binary variants. We also write \( \bot = \bigwedge Q = \sqcup \emptyset \) for the least and \( \top = \bigvee Q = \sqcap \emptyset \) for the greatest element of \( Q \). In the literature, quantales are often defined without \( 1 \) (those with \( 1 \) are then called unital), but we have no use for these. We write \( Q_1 = \{ \alpha \in Q \mid \alpha \leq 1 \} \) for the set of subidentities of quantale \( Q \).

A quantale is boolean if its lattice reduces is a complete boolean algebra—a complete lattice and a boolean algebra.

We write \( \neg \) for boolean complementation. In a boolean quantale, \( Q_1 \) forms a complete boolean subalgebra with complementation \( \lambda x. x' = \lambda x. 1 - x \), and in which composition coincides with meet [FJSZ20].

Our examples in Section 2 require weaker notions of quantale. A prequantale is a quantale in which the associativity law is absent [Ros90].

Proposition 4.1. Let \( (X, \circ, \ell, r) \) be an ℓr-multisemigroup. Then \((PX, \subseteq, \circ, E)\) forms a boolean quantale in which the complete boolean algebra is atomic.
Proof. If \((X, \odot, \ell, r)\) is an \(\ell r\)-multisemigroup, then \((X, \odot, E)\) is a multimonom by Proposition 3.10 hence isomorphic to a relational monoid and its powerset algebra a quantale \([Ros97]\) (see also \([DHS21]\)). The complete lattice on \(P X\) is trivially boolean atomic.

Remark 4.2. If \(X\) is an \(\ell r\)-multimagma instead of a \(\ell r\)-multisemigroup, then \(P X\) forms a prequantale instead of a quantale \([DHS21] , [CDS21a]\). This weaker result is needed in Section 6.

Remark 4.3. Dualities between \(n\)-ary (modal) structures and boolean algebras with \(n\)-ary (modal) operations have been studied by Jónsson and Tarski \([JT51]\): correspondences between relational associativity laws and those at powerset level are well known from substructural logics such as the Lambek calculus \([Lam58]\). Here we study ternary relations—binary multioperations—and the quantalic composition as a binary (modal) operation.

Example 4.4 (Powerset Quantales over \(\ell r\)-Semigroups). While the lifting works for arbitrary \(\ell r\)-multisemigroups, we restrict our attention to categories.

1. Let \(C = (O, M)\) be a category. Then \((PC, \subseteq, \odot, 1)\), with the operations from Section 4 and \(1 = \{id_X \mid X \in O\}\), forms a boolean quantale, in fact an atomic boolean one. This holds by Proposition 4.1 because categories are \(\ell r\)-semigroups (Example 2.2). Note the difference between this construction and that of the powerset functor: we take powersets of morphisms, not of objects.

2. An interesting instance lifts the pair groupoid on set \(X\) to the quantale of binary relations over \(X\).

The quantalic composition is relational composition, the monoidal unit the identity relation, set union is sup and set inclusion the partial order. Relations can be seen as possibly infinite-dimensional boolean-valued square matrices in which the quantalic composition is matrix multiplication (cf. Example 3.11(5)).

The fact that groupoids can be lifted to algebras of binary relations with an additional operation of converse, \(R^c = \{(y, x) \mid (x, y) \in R\}\), was known to Jónsson and Tarski \([JT52]\). This applies in particular to groups as single-object groupoids. Further examples of powerset liftings of categories and other \(\ell r\)-multisemigroups can be found in Section 9.

The powerset lifting to \(P X\) is a lifting to the function space \(2^X\), where \(2\) is the quantale of booleans. It generalises to liftings to function spaces \(Q^X\) for arbitrary quantales \(Q\) \([DHS21]\). We present a multioperational version.

Let \((X, \odot, \ell, r)\) be an \(\ell r\)-multimagma and \((Q, \leq, \cdot, 1)\) a quantale. For functions \(f, g : X \to Q\), we define the convolution \(\ast : Q^X \times Q^X \to Q^X\) as

\[
(f \ast g)(x) = \bigvee_{y \in x \odot z} f(y) \cdot g(z) = \bigvee\{f(y) \cdot g(z) \mid x \in y \odot z\}.
\]

For any predicate \(P\) we define

\[
[P] = \begin{cases} 
1 & \text{if } P, \\
\perp & \text{otherwise,}
\end{cases}
\]

and then the function \(id_E : X \to Q\) as

\[
id_E(x) = [x \in E].
\]

In addition, we extend sups and \(\leq\) pointwise from \(Q\) to \(Q^X\). This leads to the following generalisation of Proposition 4.1.

Theorem 4.5. Let \(X\) be an \(\ell r\)-multisemigroup and \(Q\) a quantale. Then \((Q^X, \leq, \ast, id_E)\) is a quantale.

Proof. If \((X, \odot, \ell, r)\) is an \(\ell r\)-multisemigroup and \(Q\) a quantale, then \((X, \odot, X_E)\) is a multimonom by Proposition 3.10 and a relational monoid up-to isomorphism. Hence \(Q^X\) is a quantale \([DHS21]\) (shown for slightly different, but equivalent relational monoid axioms).

In addition, \(Q^X\) is distributive (more precisely, its underlying complete lattice) if \(Q\) is, and boolean if \(Q\) is \([DHS21]\).

We call \(Q^X\) from Theorem 4.5 the convolution algebra or convolution quantale of \(X\) and \(Q\).
Remark 4.6. If $X$ is an $ℓr$-multimagma and $Q$ a prequantale, then the convolution algebra $Q^X$ is a prequantale. This result is needed in Section 7.

The following laws help calculating with convolutions. First of all, using the function $\delta_x(y) = [x = y]$, we can write $f(x) = \bigvee_{y \in Y} f(y) \cdot \delta_y(x)$ for any $f \in Q^X$ and more generally $f = \bigvee_{x \in X} f(x) \cdot \delta_x$,

viewing $f(x) \cdot \delta_y$ and even $\alpha \cdot f$, for any $\alpha \in Q$ and $f : X \to Q$, as a scalar product in a $Q$-module on $Q^X$, as usual in algebra. This allows us to rewrite convolution as

$$(f \ast g)(x) = \bigvee_{y,z \in X} f(y) \cdot g(z) \cdot [x \in y \circ z]$$

or even as

$$f \ast g = \bigvee_{x,y,z \in X} f(y) \cdot g(z) \cdot [x \in y \circ z] \cdot \delta_x,$$

sup as

$$\bigvee F = \bigvee_{x \in X} \bigvee \{ f(x) \mid f \in F \} \cdot \delta_x$$

and finally the identity function as $id_E = \bigvee_{e \in E} \delta_e$.

Example 4.7 (Convolution Quantales over $ℓr$-Semigroups). Once again we restrict our attention to categories.

1. A category algebra is the convolution algebra of a small category with values in a commutative ring with unity. This generalises the well known group algebras, in particular group rings. Similarly, Theorem 4.3 constructs category quantales, evaluating small categories in quantales, which yields quantales as convolution algebras.

2. An instance are $Q$-fuzzy relations [Gog67], which are binary relations taking values in a quantale $Q$. The associated quantales are convolution quantales over pair groupoids. If $Q$ is the Lawvere quantale described in Example 4.8 below, this yields $t$-norms. $Q$-valued relations correspond to possibly infinite dimensional $Q$-valued square matrices. If the base set is finite, we recover the matrix theories $MQ$ of Example 3.11. Heisenberg’s original formalisation of quantum mechanics [Hei25] used a similar convolution algebra over the pair groupoid $[\text{Con 95}]$, yet with values in the field of complex numbers.

Many additional examples of relational monoids, partial monoids and convolution algebras are discussed in [DHS16, DHS21, CDS20]. Further examples, with a view of lifting categories and non-local $ℓr$-semigroups, can be found in Section 9.

Finally, we list some well known candidates for value-quantales.

Example 4.8 (Quantales).

1. We have already mentioned the quantale of booleans $(2, \leq, \land, 1)$, which has carrier set $\{0,1\}$ and $\land$, in fact min, as composition.

2. The Lawvere quantale $(\mathbb{R}^+, \geq, +, 0)$ has $\bigwedge$ as supremum, + as quantalic composition, extended by $x + \infty = \infty = \infty + x$, and 0 as its unit. It is important for defining generalised metric spaces and $t$-norms.

3. The unit interval $[0,1]$, $\leq, \cdot, 1)$ forms a quantale with $\bigvee$ as supremum. It is isomorphic to the Lawvere quantale via the function $(\lambda x \cdot \ln x)$ as its inverse, and important in probability applications.

4. The structures $(0,1], \leq, \cdot, 1)$ and $[0,1], \geq, \cdot, 0)$ and their variants with $[0,1]$ replaced by $\mathbb{R}_+^\infty$ (and unit $\infty$ for the first) form other quantales over the unit interval, which are at the basis of max-plus algebra [HOvdW06] and similar structures.

All these examples are distributive, but not boolean quantales, and they justify our approach to domain in non-boolean cases below.
5 Modal Quantales

The results of Section 4 do not lift the source and target structure of \(\ell r\)-multisemigroups to \(Q^X\) faithfully: all relational units are mapped to the unit in \(Q\) by \(id_E\) rather bluntly. A more fine-grained approach, in which different elements of \(E\) are mapped to different elements in the powerset quantale or different values or weights in \(Q\), is possible.

Example 5.1 (Relational domain and codomain). In the relation quantale, the standard relational domain and codomain operations are

\[
\text{dom}(R) = \{(a, a) \mid \exists b. (a, b) \in R\} \quad \text{and} \quad \text{cod}(R) = \{(b, b) \mid \exists a. (a, b) \in R\}.
\]

Thus \(\text{dom}(R) = \{\ell(x) \mid x \in R\} = \ell(R)\) and \(\text{cod}(R) = \{r(x) \mid x \in R\} = r(R)\).

We can capture this more abstractly. Formally, a domain quantale [FJSZ20] is a quantale \((Q, \leq, \cdot, 1)\) equipped with a domain operation \(\text{dom} : Q \to Q\) that satisfies, for all \(\alpha, \beta \in Q\),

\[
\alpha \leq \text{dom}(\alpha) \cdot \alpha, \\
\text{dom}(\alpha \cdot \text{dom}(\beta)) = \text{dom}(\alpha) \cdot \text{dom}(\beta), \\
\text{dom}(\bot) = \bot, \\
\text{dom}(\alpha \vee \beta) = \text{dom}(\alpha) \vee \text{dom}(\beta).
\]

We refer to the domain axioms as absorption, locality, subidentity, strictness and (binary) sup preservation, respectively. Absorption can be strengthened to an identity: \(\text{dom}(\alpha) \alpha = \alpha\).

The domain axioms are precisely those of domain semirings [DS11]: domain quantales are thus quantales that are also domain semirings with addition as binary sup. Properties of domain semirings are therefore inherited, for instance

- the export law \(\text{dom}(\text{dom}(\alpha) \beta) = \text{dom}(\alpha) \text{dom}(\beta)\);
- order preservation \(\alpha \leq \beta \Rightarrow \text{dom}(\alpha) \leq \text{dom}(\beta)\);
- the weak twisted law: \(\alpha \text{dom}(\beta) \leq \text{dom}(\alpha \beta) \alpha\);
- least left absorption (lla): \(\text{dom}(\alpha) \leq \rho \Leftrightarrow \alpha \leq \rho \text{dom}(\alpha)\); and
- the adjunction \(\text{dom}(\alpha) \leq \rho \Leftrightarrow \alpha \leq \rho \top\).

In the last two laws, \(\rho \in Q_{\text{dom}} = \{\alpha \mid \text{dom}(\alpha) = \alpha\}\) (they need not hold for all \(\rho \in Q_1 \supseteq Q_{\text{dom}}\)).

Domain axioms for arbitrary sups are unnecessary [FJSZ20]. In every domain quantale, \(\text{dom}\) preserves arbitrary sups, \(\text{dom}(\bigvee A) = \bigvee \text{dom}(A)\), hence in particular \(\text{dom}(\top) = 1\), and distributes weakly with infs, \(\text{dom}(\bigwedge A) \leq \bigwedge \text{dom}(A)\). Domain elements also left-distribute over non-empty infs, \(\text{dom}(\alpha) \cdot \bigwedge A = \bigwedge (\text{dom}(\alpha) A)\) for all \(A \neq \emptyset\).

Much of the structure of the domain algebra induced by \(\text{dom}\) is inherited from domain semirings as well. It holds that \(Q_{\text{dom}} = \text{dom}(Q)\). The domain algebra \((Q_{\text{dom}}, \leq, \cdot, 1)\) is therefore a subquantale of \(Q\) that forms a bounded distributive lattice with \(\cdot\) as binary sup. It contains the largest boolean subalgebra of \(Q\) bounded by \(\bot\) and \(1\) [DS11]. The elements of \(Q_{\text{dom}}\) are called domain elements of \(Q\). Yet, in the quantalic case, the lattice \(Q_{\text{dom}}\) is complete [FJSZ20]: \(\text{dom}(\bigvee \text{dom}(A)) = \bigvee \text{dom}(A)\) follows from sup-preservation of \(\text{dom}\). All sups of domain elements are therefore again domain elements, but sups and infs in \(Q_{\text{dom}}\) need not coincide with those in \(Q\).

Quantales are closed under opposition: interchanging the order of composition in quantale \(Q\) yields a quantale \(Q^{op}\); properties of quantales translate under this duality. The opposite of the domain operation on a domain quantale is of course a codomain operation.

A codomain quantale is the opposite of a domain quantale, just like a codomain semiring is the opposite of a domain semiring. Codomain quantales can be axiomatised using a codomain operation \(\text{cod} : Q \to Q\) that satisfies the dual domain axioms, making \((Q^{op}, \text{cod})\) a domain quantale.

A modal quantale is a domain and codomain quantale \((Q, \leq, \cdot, 1, \text{dom}, \text{cod})\) that satisfies the following compatibility axioms, which make the domain and codomain algebras \(Q_{\text{dom}}\) and \(Q_{\text{cod}}\) coincide:

\[
\text{dom} \circ \text{cod} = \text{cod} \quad \text{and} \quad \text{cod} \circ \text{dom} = \text{dom}.
\]
Remark 5.2. In modal semirings, dom and cod are usually modelled indirectly through their boolean complements in the subalgebras of subidentities, that is, by antidomain and antirange operations. This allows expressing boolean complementation in $Q_{\text{dom}}$, which then becomes the largest complete boolean subalgebra of $Q$ bounded by $\bot$ and $1$ [DST11]. As we intend to lift from $\ell r$-multisemigroups, where complements of source and target operations may not exist, we do not follow this approach.

In a boolean quantale $Q$, the subalgebra $Q_1$ of subidentities forms a complete boolean algebra with quantalic composition as binary inf, hence in particular $Q_{\text{dom}} = Q_1$. One can then axiomatise domain by the adjunction $\text{dom}(\alpha) \leq \rho \Leftrightarrow \alpha \leq \rho^+$, for all $\rho \in Q_1$, and weak locality $\text{dom}(\alpha \beta) \leq \text{dom}(\text{dom}(\alpha \beta))$ [FJSZ20]. Dual results hold for codomain. Using the adjunction alone yields only pre-domain and precodomain operations [DMS06]. Finally, in a boolean quantale, antidomain and antico-domain operations can be defined as $\text{adom} = (\lambda x. x') \circ \text{dom}$ and its dual. The axioms of antidomain and antico-domain semirings [DST11] can then be derived [FJSZ20].

Some of the $\ell r$-magmas, $\ell r$-semigroups and $\ell r$-monoids in the examples in Section 9 fail to yield associativity or locality laws when lifted. This requires a more fine-grained view on modal quantales.

- A modal prequantale is a prequantale in which the locality axioms for dom and cod are replaced by the export axioms

$$\text{dom}(\text{dom}(\alpha \beta)) = \text{dom}(\alpha)\text{dom}(\beta) \quad \text{and} \quad \text{cod}(\alpha \text{cod}(\beta)) = \text{cod}(\alpha)\text{cod}(\beta).$$

The algebra $Q_{\text{dom}} = \text{dom}(Q) = \text{cod}(Q) = Q_{\text{cod}}$ still forms a complete distributive lattice in this case. Yet neither the weak locality laws $\text{dom}(\alpha \beta) \leq \text{dom}(\text{dom}(\alpha \beta))$ or $\text{dom}(\alpha \beta) \geq \text{dom}(\text{dom}(\alpha \beta))$ nor their opposites for cod are derivable in this setting.

- A weakly local modal quantale is a modal quantale in which the locality axioms for dom and cod have once again been replaced by the export axioms above. In the presence of associativity, the weak locality laws

$$\text{dom}(\alpha \beta) \leq \text{dom}(\text{dom}(\alpha \beta)) \quad \text{and} \quad \text{cod}(\alpha \beta) \leq \text{cod}(\text{cod}(\alpha \beta))$$

are now derivable, but not the opposite inequalities; locality therefore does not hold.

As mentioned before, this foliation of definitions is reflected by lifting and modal correspondence properties in Sections 6, 7 and 8, and justified by mathematically meaningful examples in Section 9.

The definitions and properties mentioned have been verified with Isabelle, they also transfer to dioids, but have not yet been developed in greater detail for these.

6 Modal Powerset Quantales

We show how $\ell r$-multisemigroups can be lifted to powerset algebras, using the lifted operations $\ell, r : \mathcal{P}X \to \mathcal{P}X$ defined, for all $A \subseteq X$, by $\ell(A) = \{\ell(x) \mid x \in A\}$ and $r(A) = \{r(x) \mid x \in A\}$.

Remark 6.1. To relate this with the lifting of the ternary relation $R$ to a binary modality $\odot$ on the powerset quantale in the sense of Jönsson and Tarski, consider $\mathcal{R}(\ell) = \{(x, y) \mid y = \ell(x)\}$, the graph of $\ell$, which is a functional binary relation, and similarly the graph $\mathcal{R}(r)$ of $r$. Then

$$\ell(A) = \{y \mid (x, y) \in \mathcal{R}(\ell) \text{ for some } x \in A\} \quad \text{and} \quad r(A) = \{y \mid (x, y) \in \mathcal{R}(r) \text{ for some } x \in A\},$$

which shows that $\ell$ and $r$ are modal diamond operators on $\mathcal{P}X$. The following lemmas show that they indeed preserve binary sups and are strict, and hence are operators on boolean algebras.

Our main aim is to verify the domain quantale axioms and then use properties of domain and codomain to infer that local $\ell r$-multisemigroups can be lifted to boolean modal quantales at powerset level. We develop this theorem step-by-step, starting from $\ell r$-multimagmas, to clarify correspondences.

Lemma 6.2. Let $X$ be an $\ell r$-multimagma. For $A, B \subseteq X$ and $A \subseteq \mathcal{P}X$,

1. the compatibility laws $\ell(r(A)) = r(A)$ and $r(\ell(A)) = \ell(A)$ hold,
2. the absorption laws $\ell(A) \cdot A = A$ and $A \cdot r(A) = A$ hold,

3. the sup-preservation laws $\ell(\bigcup A) = \bigcup \ell(A) | A \in A$ and $r(\bigcup A) = \bigcup \{r(A) | A \in A\}$ hold,

4. the binary sup-preservation laws $\ell(A \cup B) = \ell(A) \cup \ell(B)$, $r(A \cup B) = r(A) \cup r(B)$ and the zero laws $\ell(\emptyset) = \emptyset = r(\emptyset)$ hold,

5. the commutativity laws $f(A)g(B) = g(B)f(A)$ hold for $f, g \in \{\ell, r\}$,

6. the subidentity laws $\ell(A) \subseteq X_\ell$ and $r(A) \subseteq X_r$ hold,

7. the export laws $\ell(\ell(A) \cdot B) = \ell(A)\ell(B)$ and $r(A \cdot r(B)) = r(A)r(B)$ hold.

The proofs have been verified with Isabelle and is subsumed by that of Theorem 7.1. Yet because powerset liftings are important and the proof may be instructive, we present details in Appendix C. Lemma 6.2 shows that the domain axioms, except locality, can already be lifted from $\ell r$-multimagmas. The identities in (4) are subsumed by those in (3) in the powerset quantale, but they are needed for lifting to modal semirings; see Section 7. That is why we are listing them.

Lifting weak variants of locality for $\ell$ and $r$ requires $\ell r$-multisemigroups; lifting locality, in addition, requires locality.

**Lemma 6.3.** Let $X$ be an $\ell r$-multisemigroup and $A, B \subseteq X$. Then

$$\ell(AB) \subseteq \ell(A\ell(B)) \quad \text{and} \quad r(AB) \subseteq r(r(A)B).$$

The converse inclusions hold if $X$ is local.

The proofs have again been checked with Isabelle and can be found in Appendix C. Weak locality holds in any weakly local modal semiring and quantale (which satisfy export axioms, see Section 5).

The results of the previous two lemmas can be summarised as follows.

**Theorem 6.4.** Let $X$ be an $\ell r$-multimagma.

1. Then $(PX, \subseteq, \circ, E, \text{dom}, \text{cod})$ is a boolean modal prequantale in which dom$(A) = \ell(A)$, cod$(A) = r(A)$, for all $A \subseteq X$, and the complete boolean algebra is atomic.

2. It is a weakly local modal quantale if $X$ is an $\ell r$-multisemigroup.

3. It is a modal quantale if $X$ satisfies locality.

**Proof.** We have derived the respective variants of modal prequantale and quantale axioms in Lemmas 6.2 and 6.3 for $\ell r$-magmas, $\ell r$-semigroups and local $\ell r$-multisemigroups. They hold in addition to the boolean prequantale and quantale axioms lifted via Proposition 4.1 and the remark following it.

This shows the particular role of weak locality and locality in the three stages of lifting. The construction shown is one direction of the well known Jónsson-Tarski duality between relational structures and boolean algebras with operators [JT51], which generalises to categories of relational structures and boolean algebras with operators [Go89]. Theorem 6.4 is an instance of this duality. Like in modal logic, there are correspondences between relational structures and boolean algebras with operators. The identities lifted in Lemma 6.2 and 6.3 are one direction of these. They are further investigated in a more general setting in Section 8.

**Example 6.5** (Modal Powerset Quantales over $\ell r$-Semigroups).

1. Any category as a local partial $\ell r$-semigroup can be lifted to a modal powerset quantale. It is boolean and has the arrows of the category as atoms. The domain algebra is the entire boolean subalgebra below the unit of the quantale, the set of all objects of the category (or the identity arrows). In this sense, a modal algebra can be defined over any category.
2. As an instance, in the modal powerset quantale over the groupoid on \( X \), that is, the modal quantale of binary relations, the domain and codomain elements are the relational domains and codomains of relations mentioned in Example \( \text{[5]} \). Domain and codomain elements are precisely the subidentity relations below \( \text{Id}_X \). In the associated matrix algebra, these correspond to (boolean-valued) identity matrices and further to predicates, cf. Example \( \text{[5]} \). The domain and codomain operators allow constructing predicate transformers and algebraic variants of dynamic logics, with applications in program verification; see Section \( \text{[1]} \).

3. In this and the next example, locality at powerset-level fails. Recall that the partial \( \ell r \)-semigroup in the broken monoid (Example \( \text{[5]} \)) is only weakly local. The powerset quantale is only weakly local as well. To check this, we simply replay the non-locality proof for the partial \( \ell r \)-semigroup with \( A = \{ a \} \): \( \text{dom}(AA) = \text{dom}(\emptyset) = \emptyset \subset \{ 1 \} = \text{dom}(A\{ 1 \}) = \text{dom}(\text{Adom}(A)) \). Locality of codomain is ruled out by duality.

4. Locality of domain and codomain of the powerset algebra over the non-local partial abelian monoid of heaplets (Section 9) do not lift to powersets.

Example 6.6.

1. The category \( 1 \xrightarrow{0} 2 \), also known as walking arrow, forms the partial local \( \ell r \)-semigroup with elements \( X = \{ 1, a, 2 \} \), \( \ell \) and \( r \) defined by \( \ell(1) = r(1) = 1 = \ell(a) \) and \( \ell(2) = r(2) = 2 = r(a) \) and composition \( 11 = 1, 1a = a = a2 \) and \( 22 = 2 \). Then, for \( A = \{ 1, a \} \) and \( B = \{ 2 \} \),

\[
A \cdot \text{dom}(B) = A \cdot B = \{ a \} \subset A = \{ 1 \} \cdot A = \text{dom}(A \cdot B) \cdot A
\]

refutes \( \text{[3]} \) in \( \mathcal{P}X \). The opposite law \( \text{[D3c]} \) for \( \text{cod} \) is refuted by a dual example.

2. Axiom \( \text{[D3c]} \) also fails in modal quantales of relations. Encoding the walking arrow on the set \( X = \{ a, b \} \) using the relations

\[
R \xrightarrow{a} a \xrightarrow{R} b \xrightarrow{S}
\]

yields \( R \text{dom}(S) = RS = \{ (a, b) \} \subset R = \{ (a, a) \} R = \text{dom}(RS)R \in \mathcal{P}(X \times X) \). The expression \( \text{dom}(RS) = \text{dom}(R \text{dom}(S)) \) models the relational preimage of \( \text{dom}(S) \) under \( R \). Obviously, executing \( R \) from all those inputs that may lead into \( \text{dom}(S) \) and restricting the outputs of \( R \) to \( \text{dom}(S) \) is only the same when \( R \) is a function.

A dual example for \( \text{cod} \) and \( \text{[D3c]} \) uses the converses of \( R \) and \( S \). From the discussion above it is evident that \( \text{[D3c]} \) does not hold for general functions, yet it does for monos in \( \text{Rel} \).

Hence we remain within the realm of modal quantales as opposed to function systems. Weak variants of Schweizer and Sklar’s axioms \( \text{[3c]} \) and \( \text{[D3c]} \), \( \text{dom}(\beta) \leq \text{dom}(\alpha \beta) \alpha \) and \( \text{cod}(\alpha \beta) \leq \beta \text{cod}(\alpha \beta) \), can be derived in any modal semiring, as already mentioned, yet they need not hold in modal prequantales. The equational \( \ell r \)-multisemigroup variants of Axioms \( \text{[3]} \) and \( \text{[D3c]} \) do not lift to powersets. A well-known theorem by Gautam \[\text{[Gau57]}\] shows that identities lift to the powerset level if and only if all variables occur on each side of the identity (or else the two sides are identical). Neither the generalisations of \( \text{[3c]} \) and \( \text{[D3c]} \) satisfy this condition, nor their equational specialisations.
These definitions imply that DomX. Theorem 7.1. Let quantales and weakly local modal quantales. Yet first we need to generalise the definition of domain and lifting to functions valued in domain quantales.

Remark 6.7. The functor $G : \text{Set} \to \text{Rel}$ that maps functions to their graph associates convolutions of (graphs of) functions with function composition $G(f) \circ G(g) = G(f \circ g)$, and distributes over $\text{dom}$: $G(\text{dom}(f)) = \text{Id}_{\text{dom}(G(f))} = \text{dom}(G(f))$. Axiom $[13c]$ can then be derived in the convolution algebra:

$$G(f) \circ \text{dom}(G(g)) = G(f \circ \text{dom}(g)) = G(\text{dom}(f \circ g); f) = \text{dom}(G(f) \circ G(g)) \circ G(f)$$

Yet this depends on the specific form of convolution for (graphs of) functions and thus goes beyond the general lifting by convolution. The argument for $[13c]$ is analogous.

An asymmetry between $\ell_r$-multisemigroups and modal quantales remains. While the domain axioms for quantales are purely equational, those for $\ell_r$-multisemigroups are based on the implication $D_{xy} \Rightarrow r(x) = \ell(y)$, and we do not know an equational axiomatisation for this class. We leave this as an open question.

7 Modal Convolution Quantales

In this section we prove one of the main theorems in this article, and perhaps its most useful one. It refines Proposition $[4.1]$ as a powerset lifting. Here we aim at a similar refinement of Theorem $[4.5]$ as a lifting to functions valued in domain quantales.

Once again we aim to expose the conditions on the algebras used in the lifting, including modal prequantale and weakly local modal quantale. Yet first we need to generalise the definition of domain and codomain at the powerset level to make it suitable for convolution quantales. For every $\ell_r$-multisemigroup $X$, modal prequantale $Q$ and functions $f, g : X \to Q$ we define the operations $\text{Dom}$ and $\text{Cod}$ by

$$\text{Dom}(f) = \bigvee_{x \in X} \text{dom}(f(x)) \cdot \delta_{\ell(x)} \quad \text{and} \quad \text{Cod}(f) = \bigvee_{x \in X} \text{cod}(f(x)) \cdot \delta_r(x).$$

These definitions imply that $\text{Dom}(f)(x) = \text{Cod}(f)(x) = \perp$ for $x \notin E$. Restricted to functions $E \to Q$, they are equivalent to $\text{Dom}(f) \circ \ell = \text{dom} \circ f$ and $\text{Cod}(f) \circ r = \text{cod} \circ f$.

Theorem 7.1. Let $(X, \odot, \ell, r)$ be an $\ell_r$-multisemigroup and $Q$ a modal prequantale.

1. Then $(Q^X, \leq, *, \text{id}_E, \text{Dom}, \text{Cod})$ is a modal prequantale.

2. It is a weakly local modal quantale if $X$ is an $\ell_r$-multisemigroup and $Q$ a weakly local modal quantale.

3. It is a modal quantale if $X$ is also local and $Q$ a modal quantale.

Proof. Relative to Theorem $[4.1]$ and the remark following it we need to check the domain and codomain axioms as well as the compatibility axioms. We show proofs up-to duality. We point out where an $\ell_r$-multisemigroup $X$ together with a modal prequantale $Q$ or an $\ell_r$-multisemigroup together with a weakly local modal quantale suffices for the lifting; this is the case in (1)-(5) below. Here it is convenient to view $\alpha \cdot \delta_x$ as an element of the convolution algebra or an associated (imaginary) $Q$-module.

1. First we show sup-preservation in $Q^X$, assuming the corresponding law in $Q$:

$$\text{Dom} \left( \bigvee F \right) = \bigvee_{x \in X} \text{dom} \left( \bigvee \{ f(x) \mid f \in F \} \right) \cdot \delta_{\ell(x)}$$

$$= \bigvee \left\{ \bigvee \text{dom}(f(x)) \cdot \delta_{\ell(x)} \mid f \in F \right\}$$

$$= \bigvee \{ \text{Dom}(f) \mid f \in F \}.$$
2. For the first compatibility axiom in $Q^X$, assuming the corresponding laws in $X$ and $Q$,

$$(\text{Dom} \circ \text{Cod})(f) = \bigvee_x \text{dom} \left( \bigvee_y \text{cod}(f(y)) \cdot \delta_{r(y)}(x) \right) \cdot \delta_{\ell(x)}$$

$$= \text{dom} \left( \bigvee_y \text{cod}(f(y)) \right) \cdot \delta_{r(y)}$$

$$= \bigvee_y \text{dom}(\text{cod}(f(y))) \cdot \delta_{r(y)}$$

$$= \bigvee_y \text{cod}(f(y)) \cdot \delta_{r(y)}$$

$$= \text{Cod}(f).$$

3. For the domain subidentity axiom in $Q^X$, assuming the corresponding law in $Q$,

$$\text{Dom}(f) = \bigvee_x \text{dom}(f(x)) \cdot \delta_{\ell(x)} \leq \bigvee_{x \in E} 1 \cdot \delta_{x} = \text{id}_E.$$

4. For domain absorption in $Q^X$, assuming the corresponding laws in $X$ and $Q$,

$$\text{Dom}(f) \ast f = \bigvee_{w,x,y} \left( \bigvee_z \text{dom}(f(z)) \cdot \delta_{\ell(z)}(x) \right) \cdot f(y) \cdot [w \in x \odot y] \cdot \delta_w$$

$$= \bigvee_{w,y,z} \text{dom}(f(z)) \cdot f(y) \cdot [w \in \ell(z) \odot y] \cdot \delta_w$$

$$\geq \bigvee_{w,y} \text{dom}(f(y)) \cdot f(y) \cdot [w \in \ell(y) \odot y] \cdot \delta_w$$

$$= \bigvee_y f(y) \cdot \delta_y$$

$$= f,$$

and $\text{Dom}(f) \ast f \leq f$ follows from (3).

5. For domain export in $Q^X$, assuming the corresponding laws in $X$ and $Q$,

$$\text{Dom}(\text{Dom}(f) \ast g) = \bigvee_x \text{dom} \left( \bigvee_{y,z} \text{dom}(f(w)) \cdot \delta_{\ell(w)}(y) \right) \cdot g(z) \cdot [x \in y \odot z] \cdot \delta_{\ell(x)}$$

$$= \bigvee_{x,z,w} \text{dom}(\text{dom}(f(w)) \cdot g(z)) \cdot [\ell(x) \in \ell(\ell(w) \odot z)]$$

$$= \bigvee_{x,z,w} \text{dom}(f(w)) \cdot \text{dom}(g(z)) \cdot [\ell(x) \in \ell(w) \odot \ell(z)] \cdot \delta_{\ell(x)}$$

$$= \left( \bigvee_w \text{dom}(f(w)) \cdot \delta_{\ell(w)} \right) \ast \left( \bigvee_z \text{dom}(g(z)) \cdot \delta_{\ell(z)} \right)$$

$$= \text{Dom}(f) \ast \text{Dom}(g).$$

This is not a domain quantale axiom, but a domain axiom for prequantales, as already mentioned.
6. For weak domain locality in $Q^X$, assuming the corresponding laws in $X$ and $Q$, 

$$\text{Dom}(f * g) = \bigvee_x \text{dom} \left( \bigvee_{y,z} f(y) \cdot g(z) \cdot [x \in y \circ z] \right) \cdot \delta_{\ell(x)}$$

$$= \bigvee_{x,y,z} \text{dom}(f(y) \cdot g(z)) \cdot [\ell(x) \in \ell(y \circ z)]$$

$$\leq \bigvee_{x,y,z} \text{dom}(f(y) \cdot \text{Dom}(g(z))) \cdot [\ell(x) \in \ell(y \circ \ell(z))]$$

$$= \bigvee_x \text{dom} \left( \bigvee_{y,z} f(y) \cdot \left( \bigvee_w \text{dom}(g(w)) \cdot \delta_{\ell(w)}(z) \right) \cdot [x \in y \circ z] \right) \cdot \delta_{\ell(x)}$$

$$= \text{Dom}(f * \text{Dom}(g)).$$

7. For domain locality, we replay the weak locality proof with equations. This requires locality in $X$ and $Q$.

**Remark 7.2.** Theorem 7.1 comprises the case where $Q$ is merely a (unital) quantale, that is, $\text{dom}$ and $\text{cod}$ map $\bot$ to $\bot$ and all other elements to $1 \neq \bot$ (or another fixed element $\alpha \neq \bot$). A further specialisation leads to the quantale $2$ of booleans, and hence Theorem 6.3.

Full modal correspondence results for $Q$-valued functions are more complicated than for powersets. They are investigated in the next section. Finally, examples for all three cases of Theorem 7.1, beyond mere powerset liftings, are presented in Section 9.

Next we return to our two running examples.

**Example 7.3 (Modal Convolution Quantales over $\ell r$-Semigroups).**

1. In the construction of category quantales, the $\ell r$-structure of the underlying category lifts to the modal structure of the convolution quantale. In this sense, a $Q$-valued modal algebra can be defined over any category.

2. As an instance, for any modal quantale $Q$, the convolution algebra $Q^X$ over the pair groupoid $X$ forms a modal quantale. In this algebra,

$$\text{Dom}(f)(a,b) = \bigvee_c \text{dom}(f(a,c)) \cdot \delta_a(b) = \text{dom} \left( \bigvee_c f(a,c) \right) \cdot \delta_a(b),$$

$$\text{Cod}(f)(a,b) = \bigvee_c \text{cod}(f(c,a)) \cdot \delta_a(b) = \text{cod} \left( \bigvee_c f(c,a) \right) \cdot \delta_a(b).$$

The subalgebra of weighted domain and codomain elements is then the algebra of all weighted elements below the identity relation. These can be identified with $Q$-valued predicates. For $Q = 2$, this reduces to the standard definitions of $\text{dom}$ and $\text{cod}$ in the quantale of binary relations. When $Q$-valued relations are viewed as possibly infinite dimensional matrices, domain and codomain elements correspond to diagonal matrices with values given by domain and codomain elements below 1 in $Q$ along the diagonal according to the formulas above and $\bot$ everywhere else. For instance,

$$\text{Dom} \left( \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right) = \begin{pmatrix} \text{dom}(\alpha) \cup \beta & \bot \\ \bot & \text{dom}(\gamma) \cup \delta \end{pmatrix},$$

$$\text{Cod} \left( \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right) = \begin{pmatrix} \text{cod}(\alpha) \cup \gamma & \bot \\ \bot & \text{cod}(\beta) \cup \delta \end{pmatrix}.$$ 

This can be seen as a refinement of the matrix theories of Example 3.11.5.


The matrix example does not actually require a quantale as value algebra: in a finite weighted relation, the summation used in relational composition, that is, matrix multiplication, is over a finite set and can therefore be represented by a finite supremum. In the absence of domain and codomain in the value algebra, even a semiring can be used.

More generally, we can require that the multioperation \( \odot \) of the \( \ell r \)-multisemigroup satisfies a finite decomposition property. Here are two classical examples beyond matrices.

- Schützenberger and Eilenberg’s approach to weighted formal languages \cite{DKY09} generalises language product to a convolution of functions \( \Sigma^* \to S \) from the free monoid \( \Sigma^* \) over the finite alphabet \( \Sigma \) into a semiring \( S \), as mentioned in the introduction. A semiring suffices in the convolution because the number of prefix/suffix pairs into which any finite word in \( \Sigma^* \) can be split is obviously finite. The resulting convolution algebra is again a semiring.

- Rota’s incidence algebra \cite{Rot64} of functions \( P \to R \) from a poset \( P \) to a commutative ring \( R \) requires \( P \) to be locally finite, that is, every closed segment \( [x,y] = \{ z \mid x \leq z \leq y \} \) must be finite. The incidence algebra is then an associative algebra in which the sup of the convolution is replaced by a summation in \( R \). See also Section \ref{section:rota}. Alternatively, in groups rings, it is usually assumed that functions \( G \to R \) have finite support, yet we focus on the property used for matrices, formal power series and incidence algebras.

A multimagma \((X,\odot)\) is \textit{finitely decomposable} if for each \( x \in X \) the set \( \{(y,z) \mid x \in y \odot z\} \) is finite. The following variant of Theorem \ref{thm:finite-decomposable} is then immediate from corresponding results for relational monoids \cite{CDS20a}.

\textbf{Theorem 7.4.} If \( X \) is a finitely decomposable \( \ell r \)-multisemigroup and \((S,+,\cdot,0,1)\) a semiring, then the convolution algebra \( S^X \) is a semiring.

A direct proof requires verifying that all sups in the proof of Theorem \ref{thm:finite-decomposable} remain finite if \( X \) is finitely decomposable. The operation + of semirings corresponds to \( \lor \) in quantales, 0 corresponds to \( \bot \).

This result easily extends to domain semirings. Formally, a domain semiring \cite{DS11} is a semiring \((S,+,\cdot,0,1)\) equipped with a domain operation \( \text{dom} : S \to S \) that satisfies the same domain axioms as those of domain quantales, replacing \( \bot \) by 0 and \( \lor \) by +. Every domain semiring is automatically additively idempotent, and hence the relation \( \leq \), defined as \( \lambda x,y. x + y = y \), is a partial order.

A modal semiring is a domain semiring and a codomain semiring, defined like for quantales by opposition, which satisfy the compatibility conditions \( \text{cod} \odot \text{dom} = \text{dom} \) and \( \text{dom} \odot \text{cod} = \text{cod} \).

Verifying finiteness of sups in the proof of Theorem \ref{thm:finite-decomposable} then extends Theorem \ref{thm:finite-decomposable} as follows.

\textbf{Proposition 7.5.} If \( X \) is a finitely decomposable local \( \ell r \)-multisemigroup and \( S \) a modal semiring, then the convolution algebra \( S^X \) is a modal semiring.

The small sup-preservation properties of Lemma \ref{lem:sup-pres} are needed in the proof. The result can obviously be adapted to \( \ell r \)-multimagnas and modal presemirings, and to \( \ell r \)-multisemigroups and weakly modal semirings, as in previous sections. These results cover many examples, as we shall see in Section \ref{section:lifting-to-correspondence}.

\section{From Lifttings to Correspondences}

Section \ref{section:lifting} has presented lifting results from \( \ell r \)-multimagnas to modal convolution prequantales. These yield one direction of a triangle of modal correspondences between \( \ell r \)-magnas \( X \), modal prequantales \( Q \) and modal prequantales \( Q^X \), shown in Figure \ref{fig:lifting-triangle}. In the relational setting, the triangular correspondences from \cite{CDS20a} show that certain properties in any two of a relational magma \( X \), a prequantale \( Q \) and a prequantale \( Q^X \) induce corresponding properties in the remaining algebra, for instance associativity. They include the following results for units, translated to multimagnas by isomorphism.

\textbf{Proposition 8.1.} Let \( X \) be a multimagma and \( Q \) a prequantale, not necessarily unital.

1. If the prequantale \( Q^X \) is unital and \( 1 \neq \bot \) in \( Q \), then \( X \) is an \( \ell r \)-magma.

2. If the prequantale \( Q^X \) is unital and \( X \) an \( \ell r \)-magma, then \( Q \) is unital.
Proof. The results are known for relational magmas [CDS20a, Proposition 4.1] and thus hold for multimagmas. Proposition 5.10 translates them to \( \ell r \)-multimagmas.

The restriction \( \perp \neq 1 \) in Proposition 5.1 is very mild as its negation would imply \( Q = \{ \perp \} \).

We now complete the triangle in Figure 1 by refining Proposition 5.1 to variants of modal quantales. Relative to Proposition 5.1 we only consider the \( \ell r \)-axioms and the modal axioms for prequantales and quantales while assuming the multimagma and prequantale axioms. The constructions use a technique from [CDS20a] which in turn has been adapted from representation theory in algebra.

In the following theorems, we tacitly assume that \( \ell, \text{dom} \) and \( \text{Dom} \), and similarly \( r, \text{cod} \) and \( \text{Cod} \), are related as in Theorem 7.1. In particular, therefore

\[
\text{Dom}(\alpha \cdot \delta_x) = \bigvee_y \text{dom}(\alpha \cdot \delta_x(y)) \delta_{\ell(y)} = \text{dom}(\alpha) \cdot \delta_{\ell(x)} \quad \text{and} \quad \text{Dom}(\delta_x) = \delta_{\ell(x)}
\]

with dual laws for \( \text{Cod} \), and

\[
(\alpha \cdot \delta_x \ast \beta \cdot \delta_y)(z) = \bigvee_{u,v} \alpha \cdot \delta_x(u) \cdot \beta \cdot \delta_y(v) \cdot [z \in u \circ v] = \alpha \cdot \beta \cdot [z \in x \circ y]
\]

and thus in particular \( (\delta_x \ast \beta \cdot \delta_y)(z) = [z \in x \circ y] \).

Our first lemma link properties of \( X \) with those of \( Q \) and \( Q^X \) through \( \delta \)-functions.

**Lemma 8.2.** Let \( X \) be a multimagma with functions \( \ell, r: X \to X \), let \( Q \) and \( Q^X \) be prequantales with functions \( \text{dom}, \text{cod}: Q \to Q \) and \( \text{Dom}, \text{Cod}: Q^X \to Q^X \). Then, for all \( \alpha, \beta \in Q \) and \( x, y, z \in X \),

1. \( \text{Dom}(\alpha \cdot \delta_x) \ast (\alpha \cdot \delta_x) = \bigvee_y \text{dom}(\alpha) \cdot \alpha \cdot [y \in \ell(x) \circ x] \cdot \delta_y \)
2. \( \text{Dom}(\text{Dom}(\alpha \cdot \delta_x) \ast (\beta \cdot \delta_y)) = \bigvee_z \text{dom}(\alpha \cdot \beta) \cdot [z \in \ell(x) \circ \ell(y)] \cdot \delta_z \)
3. \( \text{Dom}((\alpha \cdot \delta_x) \ast (\beta \cdot \delta_y)) = \bigvee_z \text{dom}(\alpha \cdot \beta) \cdot [z \in \ell(x) \circ \ell(y)] \cdot \delta_z \)
4. \( \text{Dom}(\text{Dom}(\alpha \cdot \delta_x) \ast (\beta \cdot \delta_y)) = \bigvee_z \text{dom}(\alpha \cdot \beta) \cdot [z \in \ell(x) \circ \ell(y)] \cdot \delta_z \)
5. \( \text{Dom}((\alpha \cdot \delta_x) \ast (\beta \cdot \delta_y)) = \bigvee_z \text{dom}(\alpha \cdot \beta) \cdot [z \in \ell(x) \circ \ell(y)] \cdot \delta_z \)
6. Corresponding properties hold for \( \text{Cod}, \text{cod} \) and \( r \).

**Proof.** We write \( \delta_{\alpha}^x \) instead of \( \alpha \cdot \delta_x \) and drop multiplication symbols wherever convenient.

1. \( \text{Dom}(\delta_{\alpha}^x) \delta_{\alpha}^x = \bigvee_{y, u, v} \delta_{\alpha}^y(u) \delta_{\alpha}^y(v) \cdot [y \in uv] \delta_y = \bigvee_y \text{dom}(\alpha) \alpha \cdot [y \in \ell(x) \circ x] \delta_y \)

2. \( \text{Dom}(\text{Dom}(\delta_{\alpha}^x) \delta_{\beta}^x) = \bigvee_z \text{dom}(\text{Dom}(\alpha \cdot \beta) \cdot [z \in \ell(x) \circ \ell(y)] \cdot \delta_z \)

3. \( \text{Dom}(\delta_{\alpha}^y) \text{Dom}(\delta_{\beta}^y) = \bigvee_{z, u, v} \delta_{\alpha}^z(u) \delta_{\beta}^z(v) \cdot [z \in uv] \delta_z = \bigvee_z \text{dom}(\alpha) \text{dom}(\beta) \cdot [z \in \ell(x) \circ \ell(y)] \delta_z \)

4. \( \text{Dom}((\delta_{\alpha}^x) \text{Dom}(\delta_{\beta}^x)) = \bigvee_z \text{dom}(\alpha \cdot \beta) \cdot [z \in \ell(x) \circ \ell(y)] \cdot \delta_z \)

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Let

\[
\begin{align*}
\text{Dom}(\delta_x \delta_y) &= \bigvee_z \text{dom} \left( \bigvee_{v,w} \delta_x(v) \delta_y(w) \right) \delta_{\ell(z)} \\
&= \bigvee_z \text{dom} \left( \alpha \beta \delta_x \delta_y \right) \delta_{\ell(z)} \\
&= \bigvee_z \text{dom} \left( \alpha \beta \delta_x \delta_y \right) \delta_{\ell(z)} \delta_{\ell(z)} .
\end{align*}
\]

6. Proofs for Cod, cod and \( r \) are dual.

The following statements add structure to Proposition 8.1(1). They expose the laws in \( Q \) and \( Q^X \) needed to derive the \( \ell \)-multimagma and \( \ell r \)-multisemigroup axioms, with and without locality. The \( \ell r \)-multisemigroup structure arises from of [CD82][Corollary 4.7] for relational semigroups together with Proposition 5.10 which translates it to \( \ell r \)-semigroups. Proving them requires very mild assumptions on \( X \) and \( Q \) or \( Q^X \).

**Proposition 8.3.** If \( Q^X \) and \( Q \) are quantales and \( 1 \neq \bot \) in \( Q \), then \( X \) is an \( \ell r \)-multisemigroup.

The \( \ell r \)-multisemigroup structure is therefore completely determined by elements below \( \text{id}_E \), more specifically, functions \( \delta_{\ell(x)} \) and their relations to the elements in \( X \). We calculate the absorption law for \( \ell \) explicitly as an example of the technique used: With Lemma 8.2(1), \( \delta_x = \text{Dom}(\delta_x) \ast \delta_x = \bigvee y \in \ell(x \circ x) \delta_y \).

Hence \( \ell(x \circ x) = \{ x \} \) whenever the corresponding domain absorption law holds in \( Q^X \) and \( 1 \neq \bot \) in \( Q \). The fact that \( \text{Dom} \) appears in the calculation does not go beyond Proposition 8.1 \( \text{Dom}(\delta_x) = \delta_{\ell(x)} \), which is below \( \text{id}_E \) in \( Q^X \).

The next statement adds locality to the picture.

**Theorem 8.4.** Let \( Q^X \) and \( Q \) be modal quantales, with \( 1 \neq \bot \) in \( Q \). Then \( X \) is a local \( \ell r \)-multisemigroup.

**Proof.** It remains to consider locality. Lemma 8.2 yields \( \text{Dom}(\delta_x \ast \text{Dom}(\delta_y)) = \bigvee_z \{ z \in \ell(x \circ y) \} \delta_{\ell(z)} \) and \( \text{Dom}(\delta_x \ast \delta_y) = \bigvee_z \{ z \in \ell(x \circ y) \} \delta_{\ell(z)} \). Therefore \( \text{Dom}(\delta_x \ast \text{Dom}(\delta_y))(z) = \text{Dom}(\delta_x \ast \delta_y)(z) \) implies \( \ell(x \circ y) = \ell(x \circ y) \).

Finally, we turn to the correspondences from \( X \) and \( Q^X \) to \( Q \). We first consider modal axioms in \( Q \) that do not depend on \( \ell r \)-multisemigroups. Similarly to the assumption that \( 1 \neq \bot \) in \( Q \) above, we need to assume the existence of certain elements in \( X \).

**Theorem 8.5.** Let \( X \) be an \( \ell r \)-multimagma in which there exist \( x, y, z, w \in X \), not necessarily distinct, such that \( \ell(x) \circ \ell(y) \neq \emptyset \) and \( z \circ w \neq \emptyset \), let \( Q \) be a prequantale and \( Q^X \) a modal prequantale such that \( \text{id}_E \neq \bot \).

1. Then \( Q \) is a modal prequantale.
2. It is a weakly modal quantale if \( X \) is an \( \ell r \)-multisemigroup and \( Q^X \) a weakly local quantale.
3. It is a modal quantale if \( X \) is also local and \( Q^X \) a modal quantale.

**Proof.** Note that by definition \( \emptyset \neq E \subseteq X \). We verify the modal quantale axioms in \( Q \), using in each case the corresponding axiom in \( X \) and \( Q^X \). Suppose that \( X \) is an \( \ell r \)-magma and \( Q^X \) a modal prequantale with \( \text{id}_E \neq \bot \).

- For domain absorption, using Lemma 8.2(1) with \( \ell(x) \circ x = \{ x \} \),
  
  \[ \text{dom}(\alpha) \ast \alpha = (\text{Dom}(\delta_x) \ast (\delta_y))(x) = \delta_x(x) = \alpha. \]

- For domain export, using Lemma 8.2(2) and (3) with \( z \in \ell(\ell(x) \circ y) = \ell(x) \circ \ell(y) \),
  
  \[ \text{dom}(\text{dom}(\alpha) \ast \beta) = \text{Dom}(\delta_x) \ast \delta_y(z) = (\text{Dom}(\delta_x) \ast \delta_y)(z) = \text{dom}(\alpha) \cdot \text{dom}(\beta). \]

- For the domain subidentity axiom, \( \text{dom}(\alpha) = \text{Dom}(\delta_x)(\ell(x)) \leq \text{id}_E(\ell(x)) = 1. \)
Modal Kleene algebras and higher-dimensional (poly)graphs [CGMS 20].

The powerset quantale over the path category of a digraph has recently been extended to higher-dimensional quantales.

For quantitative analysis of systems or for the design of algorithms, the construction of modal convolution quantales of finite traces are considered, semirings suffice as weight algebras.

Example 9.1

Along Theorem 7.1 the associated modal convolution quantales of Q-algebras of weighted paths are important.

Example 9.1 (Modal Convolution Quantales over Path Categories of Digraphs (Quivers)).

1. A digraph or quiver is a structure K consisting of a set V_K of vertices, a set E_K of edges and source and target functions s, t : E_K \rightarrow V_K. The path category of K [ML98] has elements V_K as objects and sequences (v_1, e_1, v_2, \ldots, e_{n-1}, v_n) : v_1 \rightarrow v_n, in which vertices and edges alternate, as arrows. Composition \pi_1 \cdot \pi_2 of \pi_1 : v_3 \rightarrow v_4 and \pi_2 : v_1 \rightarrow v_2 is defined whenever v_2 = v_3, and it concatenates the two paths while gluing the common end v_2 = v_3. Sequences (v) of length 1 are identities. Path categories become local partial \ell-r-semigroups if we introduce an edge i_v for every vertex v and functions \ell(e) = i_{s(e)} and r(e) = i_{t(e)} for every edge e, and it is common to define path algebras of quivers this way as sequences of arrows. By Theorem [7.3] the convolution algebra or category algebra over the path category of any digraph with values in Q is a modal quantale for any modal quantale Q. All paths are finitely decomposable, hence we can replace Q by a modal semiring.

2. A special path category is generated by the one-point quiver with n arrows. It represents the free monoid with n generators. The \ell-r-structure and hence the modal structure is then trivial. Lifting along Theorem [7.3] yields the quantale or semiring of weighted languages.

3. Forgetting edges represents paths as sequences of vertices. All lifting results transfer.

4. Forgetting the internal structure of paths and keeping only their ends, brings us back to the pair groupoid and weighted binary relations.

The remaining proofs follow by duality.

9 Examples

In this section we list additional examples of modal convolution quantales. We start with those that come from categories.

Example 9.1 (Modal Convolution Quantales over Path Categories of Digraphs (Quivers)).

1. A digraph or quiver is a structure K consisting of a set V_K of vertices, a set E_K of edges and source and target functions s, t : E_K \rightarrow V_K. The path category of K [ML98] has elements V_K as objects and sequences (v_1, e_1, v_2, \ldots, e_{n-1}, v_n) : v_1 \rightarrow v_n, in which vertices and edges alternate, as arrows. Composition \pi_1 \cdot \pi_2 of \pi_1 : v_3 \rightarrow v_4 and \pi_2 : v_1 \rightarrow v_2 is defined whenever v_2 = v_3, and it concatenates the two paths while gluing the common end v_2 = v_3. Sequences (v) of length 1 are identities. Path categories become local partial \ell-r-semigroups if we introduce an edge i_v for every vertex v and functions \ell(e) = i_{s(e)} and r(e) = i_{t(e)} for every edge e, and it is common to define path algebras of quivers this way as sequences of arrows. By Theorem [7.3] the convolution algebra or category algebra over the path category of any digraph with values in Q is a modal quantale for any modal quantale Q. All paths are finitely decomposable, hence we can replace Q by a modal semiring.

2. A special path category is generated by the one-point quiver with n arrows. It represents the free monoid with n generators. The \ell-r-structure and hence the modal structure is then trivial. Lifting along Theorem [7.3] yields the quantale or semiring of weighted languages.

3. Forgetting edges represents paths as sequences of vertices. All lifting results transfer.

4. Forgetting the internal structure of paths and keeping only their ends, brings us back to the pair groupoid and weighted binary relations.

In computing, paths arise as execution sequences of automata or transition systems, and are sometimes called traces. Sets of traces are also models for the behaviours of concurrent or distributed computing systems. Lifting along Theorem [6.3] constructs modal quantales of traces at powerset level; lifting along Theorem [7.3] the associated modal convolution quantales of Q-weighted traces. Once again, as finite traces are considered, semirings suffice as weight algebras. Algebras of weighted paths are important for quantitative analysis of systems or for the design of algorithms. The construction of the modal powerset quantale over the path category of a digraph has recently been extended to higher-dimensional modal Kleene algebras and higher-dimensional (poly)graphs [CGMS20].
Example 9.2 (Modal Incidence Algebras over Categories of Segments and Intervals). The arrows or pairs in poset categories, as in Example 2.2(4), represent (closed) segments of the poset; those of a linear order represent (closed) intervals. By Theorem 6.4 for any modal quantale \( Q \), the convolution algebra or category algebra over a poset category with values in \( Q \) forms a modal quantale.

Without the modal structure, such convolution quantales have already been studied \[DHS21\]. When the underlying partial or total orders are \( \text{locally finite} \), that is, all segments and intervals are finite (all arrows in the poset category are finitely decomposable), values can be taken in (semi)rings and the convolution algebras become the incidence algebras à la Rota \[Rot64\]. The general setting supports algebraic generalisations of duration calculi \[ZH04, DHS21\]; it specialises to interval and interval temporal logics \[HS91, Mos12\] for powerset liftings. The additional modal structure yields richer algebras for reasoning about states as well as intervals and supports mixed modalities over weighted intervals and their endpoints. This, and in particular applications to weighted and probabilistic interval temporal logics and duration calculi, remains to be explored.

Our next example considers \( \ell r \)-multisemigroups that arise from composing digraphs, yet we specialise to finite partial orders. We show how locality can be obtained by introducing interfaces.

Example 9.3 (Weighted Poset Languages). We restrict our attention to finite posets.

1. Finite posets form partial \( \ell r \)-multisemigroups (based on classes) with respect to serial composition, which is the disjoint union of posets with all elements of the first poset preceding that of the second one in the order of the composition. This yields partial monoids with the empty poset as unit, and hence partial \( \ell r \)-semigroups in which \( \ell \) and \( r \) map every poset to the empty poset. The algebra is therefore not local and does not form a category. The convolution algebras are therefore weakly local quantales, but the modal structure is trivial as \( \text{Dom} \) and \( \text{Cod} \) map the empty poset to \( \bot \) and any other element to \( \text{id\{}1\text{\}} \). The powerset lifting yields poset languages.

2. The points of finite posets can be labelled with letters from some alphabet and equivalence classes of such labelled posets can be taken with respect to isomorphism that preserves the order structure and labels, but forgets the names of nodes. This leads to partial words \[Gra81\], which are also known as \( \text{pomsets} \) in concurrency theory. The serial composition passes to a total operation on equivalence classes, which makes the resulting \( \ell r \)-monoid a monoid and hence a category.

3. Pomsets can be equipped with interfaces \[Win77\]. The source interface of a pomset consists of all its minimal elements (with their labels); its target interface of all its maximal elements (again with their labels). Pomsets with interfaces form a partial \( \ell r \)-semigroup with \( \ell \) mapping every pomset to its source interface, \( r \) mapping every poset to its target interface, and composition defined by gluing pomsets on their interfaces whenever they match, and extending the order as in the previous example. The partial \( \ell r \)-semigroup of such pomsets with interfaces is local and hence a category. Winkowski also defines a parallel composition that turns pomsets with interfaces into a weak monoidal category in which parallel composition is a partial tensor. Details are beyond the scope of this article, but see Section \[LU\] for initial steps that extend modal quantales towards concurrency.

Winkowski’s pomsets with interfaces have recently been extended to posets in which interfaces may be arbitrary subsets of the sets of minimal or maximal elements of posets \[FJST20\]. Our lifting results extend to these. Compositions of digraphs with interfaces can also be defined differently, simply by juxtaposing these objects and then making interface nodes disappear in the composition. This yields again categories, yet with different units given by identity relations \[Hot65, FDC13\]. The usual liftings therefore apply. In all these examples one considers an additional operation of parallel composition, which is simply disjoint union of posets or dags. For these, the powerset lifting yields standard models of concurrency.

The following example lifts a local \( \ell r \)-multimagma to a modal prequantale.

Example 9.4 (Weighted Path Languages in Topology).

1. A path in a set \( X \) is a map \( f : [0, 1] \to X \), where it is usually assumed that \( X \) is a topological space and \( f \) continuous. The source of path \( f \) is \( \ell(f) = f(0) \); its target \( r(f) = f(1) \). Two paths \( f \) and \( g \)
in $X$ can be composed whenever $r(f) = f(1) = g(0) = \ell(g)$, and then

$$(f \cdot g)(x) = \begin{cases} f(2x) & \text{if } 0 \leq x \leq \frac{1}{2}, \\ g(2x - 1) & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

The parameterisation destroys associativity of composition, hence $(X^{[0,1]}, \cdot, \ell, r)$ is a local partial $\ell r$-magma. The powerset lifting to $\mathcal{P}(X^{[0,1]})$ satisfies the properties of Lemma 6.2 but even weak locality fails due to the absence of associativity in $X^{[0,1]}$ and, accordingly, $\mathcal{P}(X^{[0,1]})$. The same failure occurs when lifting to a convolution quantale $Q^{X^{[0,1]}}$ along Theorem 7.1: associativity of the modal (pre)quantale $Q$ of weights makes no difference; the convolution algebra forms merely a modal prequantale whenever $Q$ does.

2. Path composition is of course associative up-to homotopy. The associated local partial $\ell r$-semigroup can then be lifted like any other category. Associativity can be enforced in a more fine-grained way when paths are considered up-to reparametrisation equivalence [FR07]: two paths $f, g : [0,1] \rightarrow X$ are reparametrisation equivalent if there is a path $h$ and surjective increasing maps $\varphi, \psi : [0,1] \rightarrow [0,1]$ such that $f = h \circ \varphi$ and $g = h \circ \psi$. Composition of reparametrisation equivalence classes of paths (also called traces) is associative and the convolution algebra a modal quantale.

3. Alternatively, categories of Moore paths can be defined on intervals of arbitrary length [Bro00]. A path is then a (continuous) map $f : [0,n] \rightarrow X$ and, writing $|f|$ instead of $n$ and likewise,

$$(f \cdot g)(x) = \begin{cases} f(x) & \text{if } 0 \leq x \leq |f|, \\ g(x - |f|) & \text{if } |f| \leq x \leq |f| + |g|. \end{cases}$$

Lifting along Theorem 6.4 now yields a modal convolution quantale appropriate, for instance, for interval temporal logics [Mos12] or durational calculi [ZH04] for hybrid or continuous dynamical systems (paths could correspond to trajectories for initial value problems for vector fields or systems of differential equations). Lifting along Theorem 7.1 yields algebras that may be useful for modelling for weighted, probabilistic or stochastic systems with continuous dynamics. □

Example 9.5 ($\Delta$-sets). A presimplicial set $K$ is a sequence of sets $(K_n)_{n \geq 0}$ equipped with face maps $d_i : K_n \rightarrow K_{n-1}$, $i \in \{0, \ldots, n\}$, satisfying the simplicial identities $d_i \circ d_j = d_{i-1} \circ d_j$ for all $i < j$. Elements of $K_n$ are $n$-simplices. Presimplicial sets generalise digraphs: $K_0, K_1$ are the sets of vertices and edges of a digraph, respectively; $d_0, d_1 : K_1 \rightarrow K_0$ are the source and target maps. Higher simplices represent compositions of edges. For edges $x, y \in X_1$, for example, a 2-simplex $z \in K_2$ with $d_2(z) = x$ and $d_0(z) = y$ reflects the fact that the edge $d_1(z)$ is a composition of $x$ and $y$. For given $x$ and $y$ there may be many such 2-simplices $z$, or none at all.

There are at least two ways of constructing an $\ell r$-multisemigroup from a presimplicial set $K$.

First, for $0 \leq i \leq n$ and $x \in K_n$, denote

$$s_i(x) = (d_{i+1} \circ d_{i+2} \circ \cdots \circ d_n)(x) \quad \text{and} \quad t_i(x) = (d_0 \circ d_1 \circ \cdots \circ d_{n-i-1})(x).$$

By definition, $s_i(x), t_i(x) \in K_i$. The set of simplices $K = \bigsqcup_{n \geq 0} K_n$ forms a graded $\ell r$-multisemigroup $(K, \circ, \ell, r)$ with

$$x \in y \circ z \iff \exists i. y = s_i(x) \land z = t_{n-i}(x)$$

and $\ell(x) = s_0(x)$, $r(x) = t_0(x)$. Associativity follows from the relationship $t_j(s_i(x)) = s_j(t_i(x))$ that holds for all $x \in X_n$ and $0 \leq i, j, k \leq n$ such that $i + k = n + j$.

In general, the graded $\ell r$-multisemigroup $(K, \circ)$ is neither local nor partial. Locality and partiality hold if $K$ is the nerve of a category $C$ (we omit degeneracies). In this case, elements of $K_n$ are sequences of morphisms

$$x = (c_0 \xrightarrow{\alpha_1} c_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} c_n)$$

with $\ell(x) = (c_0), r(x) = (c_n)$, while $\circ$ is the concatenation of sequences.

The second construction uses simplices with interfaces in $K$, which are triples $(s_i(x), x, t_j(x))$ for $x \in K_n$, $0 \leq i, j \leq n$. The set $\text{Int}(K)$ of all simplices with interfaces in $K$ forms an $\ell r$-multisemigroup.
with

$$(s_p(x), t_q(x)) \in (s_i(y), t_j(y)) \circ (s_k(z), t_l(z))$$

$$\iff s_p(x) = s_i(y) \land t_j(y) = s_k(z) \land t_l(z) = s_l(y) \land z = t_n(u(x) \land z = t_n(u(x)),$$

for $x \in K_n$, $y \in K_u$, $z \in K_{n-u+j}$, and

$$\ell(s_i(x), t_j(x)) = (s_i(s_i(x)), t_j(s_i(x))) = (s_i(x), s_i(x), s_i(x)),

r(s_i(x), t_j(x)) = (s_j(t_j(x)), t_j(t_j(x)) = (t_j(x), t_j(x), t_j(x)).$$

There is an obvious embedding

$$K \ni x \mapsto (s_0(x), x, t_0(x)) \in \text{Int}(K)$$

of $\ell r$-multisemigroups; hence, $\text{Int}(K)$ is, again, neither partial nor local. As with the first construction, partiality and locality hold for nerves of categories. An element $x$ of $\text{Int}(K)$ is then a sequence of composable morphisms of $C$ with distinguished initial segment $\ell(x)$ and final segment $r(x)$, while $x \circ y$ is a concatenation of $x$ and $y$ with $r(x)$ and $\ell(y)$ identified. \hfill $\square$

**Example 9.6** (Precubical sets). A *precubical set* $X$ \cite{Ser51, Gra09} is a sequence of sets $(X_n)_{n \geq 0}$ equipped with face maps $d^i_n : X_n \to X_{n-1}$, $1 \leq i \leq n$, $\varepsilon \in \{0, 1\}$, satisfying the identities $d^i_n \circ d^j_n = d^j_{n-1} \circ d^i_n$ for $i < j$ and $\varepsilon, \eta \in \{0, 1\}$. Like presimplicial sets, precubical sets generalise digraphs: $X_0$ and $X_1$ may be regarded as sets of vertices and edges, respectively, and $d^i_1, d^i_0 : X_1 \to X_0$ as source and target maps. Higher cells represent equivalences between paths. A square $x \in X_2$, for example, reflects the fact that the paths $(d^i_1(x), d^j_2(x))$ and $(d^i_1(x), d^j_2(x))$ are equivalent.

For a subset $A = \{a_1, \ldots, a_k\} \subseteq [n]$ and $\varepsilon \in \{0, 1\}$ define the iterated face map $d^\varepsilon_A : X_n \to X_{n-|A|}$ by

$$d^\varepsilon_A(x) = d^\varepsilon_{a_1} \circ \cdots \circ d^\varepsilon_{a_k}(x).$$

The precubical set $X$ then forms an $\ell r$-semigroup $(X, \circ, \ell, r)$ where

$$x \in y \circ z \iff \exists A \subseteq [n]. y = d^0_A(x) \land z = d^1_{|A|\setminus A}(x)$$

and $\ell(x) = d^0_{|A|}(x) \in X_0$, $r(x) = d^1_{|A|}(x) \in X_0$ for all $x \in X_n$. Like in the previous example, the $\ell r$-multisemigroup $X$ is neither partial nor local.

A special case of this example is the shuffle multimonoid from Example 2.2. Let $\Sigma$ be a finite alphabet, $X_n$ the set of all words of length $n$, and $d^i_n : X_n \to X_{n-1}$ the map that removes the $i$-th letter. Then $X = (X_n, d^i_n)$ is a precubical set and the associated $\ell r$-multisemigroup $(X, \circ, \ell, r)$ is the shuffle multimonoid on $\Sigma$. \hfill $\square$

Next we present an example of a weakly local modal convolution quantale.

**Example 9.7** (Weighted Assertions in Separation Logic). Revisiting the non-local partial $\ell r$-semigroup of heaps, lifting along Theorem 7.1 yields a weakly local modal quantale as the convolution algebra, yet once again with trivial domain/codomain structure, as the empty heaplet is the only unit \cite{DHS16}. The modal structure is trivial as there are no elements between $\emptyset$ and $\{\varepsilon\}$. This models weighted assertions of separation logic, including fuzzy or probabilistic ones. \hfill $\square$

Our final example is a lifting from a proper $\ell r$-multisemigroup that is not even partial.

**Example 9.8** (Weighted Shuffle languages). We have already seen that words under shuffle form a proper $\ell r$-multisemigroup that is not partial, but local \cite{CDS20b}. There is only one unit—the empty word. Liftings to $Q$-weighted shuffle languages are discussed in \cite{CDS20a}. The domain/codomain structure of the convolution algebra is trivial. \hfill $\square$
10 Modal Concurrent Convolution Quantales

Correspondences for biquantales and relational bimonoids that satisfy certain weak interchange laws have already been studied [CDS20a]. The two operations in this setting are interpreted as a serial or sequential and a concurrent composition, yet the latter need not be commutative. We outline a simple modal extension.

An interchange $\ell r$-multisemigroup is formed by two local $\ell r$-multisemigroups $(X, \odot, \ell, r, i)_{i \in \{0, 1\}}$ that interact via the weak interchange law

$$(w \odot_1 x) \odot_0 (y \odot_1 z) \subseteq (w \odot_0 y) \odot_1 (x \odot_0 z).$$

It is helpful to think of $\odot_0$ as a horizontal multioperation and of $\odot_1$ as a vertical one; weak interchange can then be represented graphically:

\[\begin{array}{ccc}
w & y \\
\downarrow & \uparrow \\
x & z \\
\end{array}\quad \subseteq \quad \begin{array}{ccc}
w & y \\
\downarrow & \uparrow \\
x & z \\
\end{array}\]

Similarly, an interchange quantale is formed by two quantales $(Q, \leq, \cdot, i)_{i \in \{0, 1\}}$ that interact via the weak interchange law

$$(\alpha \cdot \beta) \cdot (\gamma \cdot \delta) \leq (\alpha \cdot \gamma) \cdot (\beta \cdot \delta).$$

In any interchange quantale, substituting $1_0$ and $1_1$ in the interchange law yields $1_0 \leq 1_1$. Moreover, $1_0 = 1_1 = 1$ leads to an Eckmann-Hilton collapse where weak interchange implies the smaller variants

$$\alpha \cdot 0 \beta \leq \alpha \cdot 1 \beta, \quad \alpha \cdot 0 \beta \leq \beta \cdot 1 \alpha,$$

$$\alpha \cdot 0 (\beta \cdot 1 \gamma) \leq (\alpha \cdot 0 \beta) \cdot 1 \gamma, \quad (\alpha \cdot 1 \beta) \cdot 0 \gamma \leq (\beta \cdot 0 \gamma),$$

$$\alpha \cdot 0 (\beta \cdot 1 \gamma) \leq (\alpha \cdot 1 \beta) \cdot (\beta \cdot 0 \gamma), \quad (\alpha \cdot 1 \beta) \cdot 0 \gamma \leq (\beta \cdot 0 \gamma).$$

Correspondence triangles between such interchange laws in double $\ell r$-multimagmas $X$, double prequantales $Q$ and double prequantales $Q^X$ (where no interchange laws are assumed) have already been established [CDS20a].

Here we extend interchange $\ell r$-multisemigroups to modal interchange $\ell r$-multisemigroups by imposing locality for $\ell_i$ and $r_i$. Interchange quantales are extended to modal interchange quantales by adding domain and codomain operations $\text{dom}_i$ and $\text{cod}_i$ for $i \in \{0, 1\}$, with the usual axioms.

Combining the results for interchange algebras [CDS20a] with those in this article then yields the following lifting result.

**Theorem 10.1.** Let $X$ be a local interchange $\ell r$-multisemigroup and $Q$ a modal interchange quantale. Then $Q^X$ is a modal interchange quantale.

In addition, the results in this article combine with those for interchange algebras to structures with only weak locality or even without locality in a modular way and extend to full correspondence triangles. Smaller interchange laws in two of $X$, $Q$ and $Q^X$ also yield a small interchange law in the remaining algebra [CDS20a].

In concrete models such as the pomset and graph categories with interfaces discussed in Section 9 it seems reasonable to impose additional laws. For pomsets with interfaces, the parallel composition is simply disjoint union, and interfaces extend accordingly. The unit of parallel composition is the empty pomset, so that $\ell_1$ and $r_1$ become trivial.

Parallel composition $\odot_1$ can then be made commutative in $X$ [Win77]. This yields commutativity of $*_1$ in $Q^X$ whenever $*_1$ commutes in $Q$, plus the two usual remaining correspondences [CDS20a].

Moreover, the serial source and target maps $\ell_0$ and $r_0$ may be assumed to distribute with parallel composition $\odot_1$ whenever this composition is defined:

$$\ell_0(x \odot_1 y) \subseteq \ell_0(x) \odot_1 \ell_0(y) \quad \text{and} \quad r_0(x \odot_1 y) \subseteq r_0(x) \odot_1 r_0(y).$$
This means that \( \ell_0 \) and \( r_0 \) are multisetmonoid automorphisms with respect to \((X, \odot_1)\). They may be extended to \(\ell r\)-multisetmonoid morphisms on pomsets with interfaces, but we are not interested in this property in this section. Alternatively, equational variants of the two automorphism laws may be assumed. We impose corresponding laws in \(Q\):

\[
dom_0(\alpha \cdot_1 \beta) \leq \dom_0(\alpha) \cdot_1 \dom_0(\beta) \quad \text{and} \quad \cod_0(\alpha \cdot_1 \beta) \leq \cod_0(\alpha) \cdot_1 \cod_0(\beta),
\]

and refer to these laws uniformly as *weak automorphism laws* in the sequel. We use \([P]\)_0 and \([P]\)_1 in the proof that follows, mapping to different units as needed.

**Lemma 10.2.** Let \(X\) be a local interchange \(\ell r\)-semigroup and \(Q\) a modal interchange quantale. If the weak automorphism laws hold in \(X\) and \(Q\), then they hold in hold in \(Q^X\):

\[
\Dom_0(\ell *_1 g) \leq \Dom_0(\ell) *_1 \Dom_0(g) \quad \text{and} \quad \Cod_0(\ell *_1 g) \leq \Cod_0(\ell) *_1 \Cod_0(g).
\]

**Proof.**

\[
\Dom_0(\ell *_1 g)(x) = \bigvee_w \dom_0 \left( \bigvee_{v, w} \ell \cdot_1 g(w) \cdot_1 [u \in v \odot_1 w]_1 \right) \cdot_0 \delta_{\ell_0}(u)(x)
\]

\[
= \dom_0 \left( \bigvee_{v, w} \ell \cdot_1 g(w) \right) \cdot_0 [x \in \ell_0(v \odot_1 w)]_0
\]

\[
\leq \bigvee_{v, w} \dom_0(\ell \cdot_1 g(w)) \cdot_0 [x \in \ell_0(v) \odot_1 \ell_0(w)]_0
\]

\[
= \bigvee_{t, u} \left( \bigvee_{t, u, v} \ell \cdot_1 \delta_{\ell_0}(t)(x) \right) \cdot_1 \left( \bigvee_w \dom_0(g(w)) \cdot_1 \delta_{\ell_0}(w)(u) \cdot_1 [x \in t \odot_1 u]_1 \right)
\]

\[
= (\Dom_0(\ell) *_1 \Dom_0(g))(x).
\]

The proof for \(r\), \(\cod\) and \(\Cod\) is dual. \(\square\)

**Remark 10.3.** For the remaining correspondences, observe that

\[
\Dom_0(\delta^\alpha_x *_1 \delta^\beta_y)(z) = \dom_0(\alpha \cdot_1 \beta) \cdot_0 [z \in \ell(x \odot_1 y)]_0,
\]

\[
(\Dom_0(\delta^\alpha_x) *_1 \Dom_0(\delta^\beta_y))(z) = \dom_0(\alpha) \cdot_1 \dom_0(\beta) \cdot_1 [z \in \ell_0(x) \odot_1 \ell_0(y)]_1.
\]

- For the correspondence from \(X\) and \(Q^X\) to \(Q\), suppose the weak automorphism laws hold in \(X\) and \(Q^X\). Assume that there exist \(x, y, z \in X\) such that \(z \in \ell_0(x) \odot_1 \ell_0(y)\). Then

\[
\dom_0(\alpha \cdot_1 \beta) = \Dom_0(\delta^\alpha_x *_1 \delta^\beta_y)(z) \leq (\Dom_0(\delta^\alpha_x) *_1 \Dom_0(\delta^\beta_y))(z) = \dom_0(\alpha) \cdot_1 \dom_0(\beta).
\]

- For the correspondence from \(Q\) and \(Q^X\) to \(X\), suppose the weak automorphism laws hold in \(Q\) and \(Q^X\). Assume that there exist \(\alpha, \beta \in Q\) such that \(\dom_0(\alpha \cdot_1 \beta) \neq 0\). It then follows from the above observation that \(\Dom_0(\delta^\alpha_x *_1 \delta^\beta_y)(z) \leq (\Dom_0(\delta^\alpha_x) *_1 \Dom_0(\delta^\beta_y))(z)\) implies \(\ell_0(x \odot_1 y) \leq \ell_0(x) \odot_1 \ell_0(y)\).

Further adjustments to concrete semantics for concurrent systems are left for future work.

### 11 Quantitative Modal Algebras

The domain and codomain yield modal diamond and box operators on the convolution algebra, as usual for modal semirings [DS11]. Many properties developed for such semirings transfer automatically to quantales, yet additional properties hold. Forward and backward modal diamond operators can be defined in a modal quantale \(Q\), as on any modal semiring, for \(\alpha, \beta \in Q\), as

\[
|\alpha)\beta = \dom(\alpha\beta) \quad \text{and} \quad (|\alpha)\beta = \cod(\beta\alpha).
\]

The two operators are related by opposition and the following conjugation, as for modal semirings [DS11].
Lemma 11.1. In every modal quantale $Q$, for all $\alpha \in Q$ and $\rho, \sigma \in Q_{\dom}$,

$$\rho \cdot |\alpha\rangle \sigma = \bot \iff (|\alpha\rangle \rho \cdot \sigma = \bot).$$

Proof. $\rho |\alpha\rangle \sigma = \bot \iff \rho \dom(\alpha \sigma) = \bot \iff \rho \sigma = \bot \iff \cod(\rho \alpha) \sigma = \bot \iff (|\alpha\rangle \rho \cdot \sigma = \bot).$

The laws $\cod(\alpha \beta) = \bot \iff \alpha \beta = \bot \iff \alpha \dom(\beta) = \bot$ used in this proof hold in any modal quantale or modal semiring because $\dom(\alpha) = \bot \iff \alpha = \bot \iff \cod(\alpha) = \bot$ and by locality [DS11]. We have seen similar laws for $\ell$-multisemigroups in Lemma [33] in Section [3].

Another consequence of locality is that $|\alpha\rangle \beta = |\alpha\rangle \dom(\beta)$, so that $\beta$ is automatically a domain element. We henceforth indicate this by writing $|\alpha\rangle \rho$ for $\rho \in Q_{\dom}$. Moreover, $|\alpha\beta| = |\alpha\rangle \circ |\beta\rangle$ and $\langle |\alpha\beta\rangle = \langle |\beta\rangle \circ |\alpha\rangle$, as is standard in modal logic. Without locality, only inequalities hold.

Lemma 11.2. In any modal quantale $Q$, the operators $|\alpha\rangle$ and $\langle |\alpha\rangle$ preserve arbitrary sups in $Q_{\dom}$, for all $\alpha \in Q$.

Proof. The $\dom$ and $\cod$ operations preserve all sups in the complete lattice $Q_{\dom}$ (which is equal to $Q_{\cod}$ [FJSZ20]). Therefore, diamonds preserve all sups, too: for all $P \subseteq Q_{\dom}$,

$$\langle |\alpha\rangle \rho = \dom(\alpha \cdot (\bigvee P)) = \dom(\bigvee \{\alpha \rho \cdot \rho \in P\}) = \bigvee \{\dom(\alpha \rho) \cdot \rho \in P\} = \bigvee \{|\alpha\rangle \rho \cdot \rho \in P\}.$$ 

Sup preservation of $\langle |\alpha\rangle$ follows by duality.

Remark 11.3. In modal semirings, diamond operators are strict and preserve all finite sups, that is, $|\bot\rangle \bot = \bot = (|\alpha\rangle \bot \bot = |\alpha\rangle \bot \rho \vee \sigma) = (|\alpha\rangle \rho \vee \langle |\alpha\rangle \sigma$ and $\langle |\alpha\rho \vee \langle |\alpha\rangle \sigma = (|\alpha\rangle \rho \vee \langle |\alpha\rangle \sigma$ (using quantale notation with $\vee$ in place of $+$ and $\bot$ in place of $0$).

An immediate consequence of sup-preservation is that modal quantales admit box operators.

Proposition 11.4. In any modal quantale $Q$, the operators $|\alpha\rangle$ and $\langle |\alpha\rangle$ have right adjoints in $Q_{\dom}$:

$$\langle |\alpha\rangle \rho \leq \sigma \iff \rho \leq |\alpha\rangle \sigma$$

and

$$|\alpha\rangle \rho = \bigvee \{\sigma \mid |\alpha\rangle \sigma \leq \rho\}$$

They satisfy

$$\langle |\alpha\rangle \rho \leq \sigma \iff \rho \leq |\alpha\rangle \sigma$$

and

$$|\alpha\rangle \rho \leq \sigma \iff \rho \leq |\alpha\rangle \sigma.$$ 

Proof. The diamonds $|\alpha\rangle$ and $\langle |\alpha\rangle$ preserve arbitrary sups by Lemma [11.2]. Hence they have right adjoints (by the adjoint functor theorem), defined as above.

It follows that $|\alpha\beta| = |\alpha\rangle \circ |\beta\rangle$, because, for all $\sigma \in Q_{\dom}$,

$$\sigma \leq |\alpha\beta| \rho \iff (|\alpha\rangle \beta \sigma \leq \rho \iff \langle |\alpha\rangle \beta \sigma \leq \rho \iff \langle |\alpha\rangle \sigma \leq \rho \iff \sigma \leq |\alpha\rangle \beta \rho,$$

and dually $|\alpha\beta| = |\beta\rangle \circ |\alpha\rangle$.

The following lemma relates boxes and diamonds with laws that do not mention modalities.

Lemma 11.5. In every modal quantale,

$$|\alpha\rangle \rho \leq \sigma \iff \alpha \rho \leq \sigma \alpha,$$

$$\rho \leq |\alpha\rangle \sigma \iff \rho \alpha \leq \sigma \alpha,$$

Proof. We consider only the first law. The others follow from duality and the adjunctions. First, suppose $|\alpha\rangle \rho \leq \sigma$, that is, $\dom(\alpha \rho) \leq \sigma$. Then $\alpha \rho = \dom(\alpha \rho) \rho \leq \sigma \rho \leq \sigma \alpha$. For the converse direction, suppose $\alpha \rho \leq \sigma \alpha$. Then $\dom(\alpha \rho) \leq \dom(\sigma \alpha) = \sigma \dom(\alpha) \leq \sigma$.

Therefore, $|\alpha\rangle \sigma = \bigvee \{\rho \mid \rho \alpha \leq \sigma \alpha\}$ and $|\alpha\rangle \sigma = \bigvee \{\rho \mid \rho \alpha \leq \sigma \alpha\}$, and the diamond operators, as left adjoints, satisfy $\langle |\alpha\rangle \rho = \bigwedge \{\sigma \mid \rho \alpha \leq \sigma \alpha\}$ and $\langle |\alpha\rangle \rho = \bigwedge \{\sigma \mid \rho \alpha \leq \sigma \alpha\}$.

In any boolean modal semiring and therefore any boolean modal quantale, the following De Morgan duality relates boxes and diamonds [DS11], where $\rho'$ is the complement of $\rho$ in $Q_{\dom}$ as before.

Lemma 11.6. If $Q$ is a boolean modal quantale, then $|\alpha\rangle \rho = (|\alpha\rangle \rho')'$ and $|\alpha\rangle \rho = (\langle |\alpha\rangle \rho')'$. 

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Proof. \( |\alpha\rangle \rho \leq \sigma \Leftrightarrow |\alpha\rangle \rho \cdot \sigma' = \perp \Leftrightarrow \rho \cdot (|\alpha\rangle \sigma' = \perp \Leftrightarrow (|\alpha\rangle \sigma')' = [\alpha] \rho. \) The proof for \( |\alpha\rangle \) follows by opposition.

In boolean modal semirings and quantales, we therefore obtain conjugation laws for boxes as for diamonds.

**Lemma 11.7.** In every boolean modal quantale,

\[
\rho \lor |\alpha\rangle \sigma = 1 \Leftrightarrow |\alpha\rangle \sigma \lor \rho = 1.
\]

**Proof.** Straightforward from De Morgan duality.

The results for boolean quantales are summarised in Figure 2. The green and orange correspondences hold in any modal quantale, the black ones only in the boolean case. While the De Morgan dualities do not depend on locality, the conjugations and Galois connections depend on these laws.

**Remark 11.8.** When the \( \ell r \)-multisemigroup \( X \) is finitely decomposable, one can thus use boolean modal semirings, that is, boolean monoids equipped with the usual domain axioms, in the lifting. Boolean monoids are essentially boolean quantales in which only finite sups and infs are assumed to exist and required to be preserved, including empty ones. It follows that, if \( X \) is a finitely decomposable local \( \ell r \)-multisemigroup and \( Q \) a boolean modal semiring, then \( Q^X \) is a boolean modal semiring.

Next, we consider the modal operators in the convolution quantales \( Q^X \) and relate them with similar properties in \( X \) and \( Q \). For the diamonds,

\[
(f|g)(x) = \bigvee_{x \in \ell(y \circ z)} |f(y)\rangle g(z) \quad \text{and} \quad (|f\rangle g)(x) = \bigvee_{x \in r(z \circ y)} (f(y)| \langle z \rangle g(z),
\]

and as upper adjoints are lower adjoints in the dual lattice,

\[
(|f\rangle g)(x) = \bigwedge_{x \in \ell(y \circ z)} |f(y)\rangle g(z) \quad \text{and} \quad (f|g)(x) = \bigwedge_{x \in r(z \circ y)} (f(y)| \langle z \rangle g(z).
\]

Note that by locality \( \ell(x \circ y) = \ell(x \odot \ell(y)) \) and likewise for \( r \), which makes these identities slightly longer, but more symmetric. Alternatively, we can write

\[
|f\rangle g = \bigvee_{x, y, z \in X} |f(y)\rangle g(z) \cdot [x \in \ell(y \circ z)] \cdot \delta_x,
\]

and the shape of the laws for the remaining boxes and diamonds is then obvious.

Using these definitions we consider the modal operators in relation and path quantales in more detail.

**Example 11.9 (Modal Operators in Powerset Quantales).**

1. In the modal powerset quantale over the path category from Example 9.1(1), elements are sets of paths. Domain elements are formed by the source elements in the set and codomain elements by its target elements. Both kinds of elements are vertices as paths of length one. Therefore, for any set of paths \( A \) and set of vertices \( P \subseteq V \) (as paths of length one),

\[
|A\rangle P = \{ v \in V \mid \exists \pi \in A. v \circ \pi \in P \}
\]
is the set of all vertices in \( V \) from which one may reach a vertex in \( P \) at the end of some path in \( A \). Dually, \( \langle A \rangle P \) models the set of all vertices one may reach from a vertex in \( P \) at the end of some path in \( A \). Moreover,

\[
\langle A \rangle P = \{ v \in V \mid \forall \pi \in A. \ v \circ \pi \in P \}
\]
models the set of vertices from which one must reach a vertex in \( P \) at the end of all paths in \( A \), and \( [A] P \) models the set of vertices one must reach from a vertex in \( P \) at the end of all paths in \( A \). The shape of the modal operators in algebras of continuous paths in Example 9.3 is very similar.

2. The modal operators in the modal powerset quantale of the pair groupoid—the quantale of binary relations—are the standard modal operators with respect to Kripke semantics. Up to isomorphism between predicates and subidentity relations, for any binary relation \( R \) and predicate \( P \), we have

\[
|R| P = \{ a \mid \exists b. (a, b) \in R \land P(b) \}, \quad (R|P = \{ b \mid \exists a. (a, b) \in R \land P(a) \}, \quad |R|P = \{ a \mid \forall b. (a, b) \in R \Rightarrow P(b) \}, \quad (R|P = \{ b \mid \forall a. (a, b) \in R \Rightarrow P(a) \}.
\]

Similar constructions apply to the other powerset algebras over \( \ell r \)-multisemigroups outlined in Section 9. Working out details is routine.

**Example 11.10.**

1. For weighted relations, the forward diamond is spelled out as

\[
\langle |f| \bigvee_{c} ^{\delta_{g^{c}(c)}}(a, b) = \bigvee_{c} |f(a, c)| g(c, c) \cdot \delta_{a}(b).
\]

For finite relations, that is, in the finitely decomposable case, we obtain matrices. In the \( 2 \times 2 \) case, for instance,

\[
\begin{pmatrix}
\alpha_{11} & \alpha_{12} \\
\alpha_{21} & \alpha_{22}
\end{pmatrix}
\begin{pmatrix}
\beta_{11} & \perp \\
\perp & \beta_{22}
\end{pmatrix}
= \text{Dom}
\begin{pmatrix}
\alpha_{11} & \beta_{11} & \alpha_{12} & \beta_{22} \\
\alpha_{21} & \beta_{11} & \alpha_{22} & \beta_{22}
\end{pmatrix}
= \begin{pmatrix}
|\alpha_{11}| & \beta_{11} \lor |\alpha_{12}| & \perp & \perp \\
\perp & \perp & \perp & |\alpha_{22}| \lor |\beta_{22}|
\end{pmatrix}.
\]

The other modal operators satisfy similar laws. The diagonal matrices are isomorphic to vectors and the computation performed in the above example corresponds to a linear transformation of a vector, except for the decoration with diamonds on the diagonal.

2. In the modal quantale of weighted paths,

\[
\langle |f| \bigvee \delta_{v^{\text{dom}(g(v))}}(x) = \bigvee_{v} |f(v)| g(r(v)) \cdot \delta_{v^{\text{cod}(r(v))}}(x).
\]

For finite graphs, the element \( \delta_{v^{\text{dom}(g(v))}} \) can again be seen as a diagonal matrix or vector of weighted vertices, whereas \( f(x) \) can be seen as a matrix labelling pairs of vertices with the set of weights of the (hom-set) of paths between them. The multiplication with \( \delta_{v^{\text{cod}(g(v))}} \) then projects on those paths with targets in \( v \) and taking the diamond computes the supremum of the weight of all paths which end up in a node in \( v \).

From a linear algebra point of view, the use of \( \text{dom} \) or \( \text{cod} \) in the computation of weights is irritating. But the results of Sections 7 and 8 show that the condition that \( Q \) be a modal quantale cannot be weakened. Otherwise, our correspondence results break. Yet the approach is too restricted, for instance, for deriving probabilistic predicate transformer semantics in which states and relations carry more general weights. In the probabilistic quantale from Example 10.3 in particular, it is easy to check that \( Q_{k} = \{ 0, 1 \} \), which precludes any assignment of non-trivial weights to states of a system.

Predicate transformers, however, are more general than modal Kleene algebras: they are simply order-, sup- or inf-preserving functions between (complete) lattices. The absorption laws \( \text{dom}(\alpha) \cdot \alpha = \alpha \) and \( \alpha \cdot \text{cod}(\alpha) = \alpha \) of domain and codomain, in particular, seem irrelevant in this setting. Only an action
on weighted relations is needed beyond properties like sup-preservation. Yet this is given by domain and codomain locality, as discussed at the beginning of this section. Locality, however, lifts from $X$ to $Q^X$ even if $Q$ is only a quantale.

To explain this, we first redefine domain as

$$\text{Dom}(f)(x) = \bigvee_{y \in X} f(y) \cdot \delta_{\ell(y)}(x),$$

using $f(y)$ instead of $\text{dom}(f(y))$, and likewise for $\text{Cod}$. Then, for locality in $Q^X$ and $Q$ merely a quantale,

$$(\text{Dom}(f \ast g))(v) = \bigvee_{x,y,z} f(y) \cdot g(z) \cdot \left[ x \in y \circ z \right] \cdot \delta_{\ell(x)}(v)$$

replaying the proof of Theorem 7.1. The proof for $\text{Cod}$ is dual, as usual. Domain export lifts in the same way:

$$(\text{Dom}(\text{Dom}(f) \ast g))(v) = \bigvee_{x,y,z} \left( \bigvee_{w} f(w) \cdot \delta_{\ell(w)}(y) \right) \cdot g(z) \cdot \left[ x \in y \circ z \right] \cdot \delta_{\ell(x)}(v)$$

and once again the proof for codomain export, with $Q$ being merely a quantale, is dual. Yet it is easy to check that absorption laws cannot be lifted this way.

**Example 11.11.** Defining forward and backward diamonds as before in the quantale of weighted relations, yet using the revised domain definition, yields,

$$\left( \bigvee_{c} \delta_{\ell(c,c)}^{\text{dom}(g(c,c))} \right) \cdot (a,b) = \bigvee_{c} f(a,c) \cdot g(c,c) \cdot \delta_{a}(b).$$

And the shape of the backward diamond is then obvious by duality. In the matrix case, our previous example now reduces to

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \begin{pmatrix} \beta_{11} & \perp \\ \perp & \beta_{22} \end{pmatrix} = \begin{pmatrix} \alpha_{11} \beta_{11} \lor \alpha_{12} \beta_{22} & \perp \\ \perp & \alpha_{21} \beta_{11} \lor \alpha_{22} \beta_{22} \end{pmatrix}.$$

This has the right shape for stochastic matrices and probabilistic predicate transformers, using the probabilistic quantale from Example 11.18.

We have checked with Isabelle that, in the absence of the absorption laws, domain elements still form a subalgebra and that commutativity lifts whenever $Q$ is abelian\footnote{The proofs are routine. We have not added them to the Isabelle repository, as theories from another repository (Archive of Formal Proofs) need to be downloaded by the reader to make them compile.}. Idempotency of domain elements does not lift, which is consistent with the fact that multiplication of diagonal matrices is not in general idempotent. This is appropriate for vector spaces and similar structures. Developing the “modal” algebras of such approaches and exploring applications is left for future work.
12 Conclusion

We have defined $\ell_r$-multisemigroups and used them to obtain a triangle of equational correspondences between modal value quantales and quantale-valued modal convolution quantales, explaining in particular the role of locality in axiomatisations of domain semirings and quantales in light of the typical composition pattern for categories. Our results yield a generic construction recipe for modal quantales, in that one gets such quantales for free as soon as the much simpler underlying $\ell_r$-multisemigroup has been identified. The relevance of this approach is illustrated by many examples from mathematics and computing.

We have sketched two directions for further work as well: First, in combination with previous results for concurrent relational monoids and concurrent quantales [CDS20a], it seems interesting to build models for non-interleaving concurrent systems based on po(m)sets and graphs with interfaces and lift them to modal concurrent semirings and quantales. Second, the relevance of the weakened domain and codomain quantales of Section 11 for the quantitative verification of computing systems with weights or probabilities remains to be explored.

We also aim at a categorification of the approach in terms of Day convolution. While this yields additional generality, we have chosen a simpler algebraic approach in this article with a view on verification applications with proof assistants like Isabelle, where reasoning with monoidal categories, coends or profunctors may become unwieldy. Relational monoids, $\ell_r$-semirings and the construction of convolution quantales have already been formalised with Isabelle [DGHS17]; Isabelle verification components for separation logic have used a somewhat less general approach of partial abelian monoids and generalised effect algebras [DGS15].

Finally, it seems interesting to generalise Jónsson-Tarski duality to our weighted setting and study in particular the role played by the weight algebra $Q$ in this setting.

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A Glossary of Algebraic Structures

Multisemigroups and $\ell r$-Semigroups

- A multimagma $(X, \odot)$ is a set $X$ equipped with a multioperation $\odot : X \times X \to \mathcal{P}X$.
- A multisemigroup is an associative multimagma: for all $x, y, z \in X$,
  \[ \bigcup \{x \odot v \mid v \in y \odot z\} = \bigcup \{v \odot z \mid v \in x \odot y\}. \]
- A multimonoid $(X, \odot, E)$ is a multisemigroup with a set $E \subseteq X$ such that, for all $x \in X$,
  \[ \bigcup \{e \odot x \mid e \in E\} = \{x\} = \bigcup \{x \odot e \mid e \in E\}. \]
- A multimagma $X$ is a partial magma if $|x \odot y| \leq 1$ and a magma if $|x \odot y| = 1$ for all $x, y \in X$. These definitions extend to multisemigroups and multimonoids.
- A multimagma, multisemigroup or multimonoid $X$ is local if $u \in x \odot y \land y \odot z \neq \emptyset \Rightarrow u \odot z \neq \emptyset$.
- An $\ell r$-multimagma $(X, \odot, \ell, r)$ is a multimagma $(X, \odot)$ equipped with two operations $\ell, r : X \to X$ such that, for all $x, y \in X$,
  \[ x \odot y \neq \emptyset \Rightarrow r(x) = \ell(y), \quad \ell(x) \odot x = \{x\}, \quad x \odot r(x) = \{x\}. \]
  The definition extends to $\ell r$-multisemigroups, partial and total algebras.

- An $\ell r$-semigroup is local if
  \[ \ell(x \odot \ell(y)) = \ell(x \odot y) \quad \text{and} \quad r(r(x) \odot y) = r(x \odot y). \]

Function Systems and Modal Semigroups

- A function system [SS67] is a structure $(X, \cdot, L, R)$ such that $(X, \cdot)$ is a semigroup and the following axioms hold (using Schweizer and Sklar’s notation):
  \[
  \begin{align*}
  L(R(x)) &= R(x), & R(L(x)) &= L(x), & (2a) \\
  L(x) \cdot x &= x, & x \cdot R(x) &= x, & (2b) \\
  L(x \cdot L(y)) &= L(x \cdot y), & R(R(x) \cdot y) &= R(x \cdot y), & (3a) \\
  L(x) \cdot R(y) &= R(y) \cdot L(x), & y \cdot R(x \cdot y) &= R(x) \cdot y. & (3b) \\
  y \cdot R(x \cdot y) &= R(x) \cdot y. & (3c)
  \end{align*}
  \]
  In addition, Schweizer and Sklar consider the identity
  \[ L(x \cdot L(y)) = x = x \cdot L(y), \quad (D3c) \]
  which is valid in some function systems, but not in all. They point out that all axioms except [SS67], which has subsequently been called twisted axiom, hold of binary relations. Schweizer and Sklar’s axioms use function composition $\circ$ instead of $\cdot$.

- A domain semigroup [DJS69] is a semigroup $(X, \cdot)$ with a binary operation $\text{dom} : X \to X$ that satisfies
  \[
  \begin{align*}
  \text{dom}(x) &= x, \quad (D1) \\
  \text{dom}(x \cdot y) &= \text{dom}(x \cdot \text{dom}(y)), \quad (D2) \\
  \text{dom}(\text{dom}(x) \cdot y) &= \text{dom}(x) \cdot \text{dom}(y), \quad (D3) \\
  \text{dom}(x) \cdot \text{dom}(y) &= \text{dom}(y) \cdot \text{dom}(x). \quad (D4)
  \end{align*}
  \]

- A modal semigroup is a domain semigroup $X$ equipped with a codomain operation $\text{cod} : X \to X$ that satisfies opposite laws, in which the arguments of composition have been swapped, and the compatibility laws $\text{dom} \circ \text{cod} = \text{cod}$ and $\text{cod} \circ \text{dom} = \text{dom}$.

The axioms [D3] and [D4] as well as the dual codomain axioms are derivable in function systems. Conversely, the function system axiom [SS67] is derivable in modal semigroups. Disregarding the twisted law, function systems are therefore a more compact, but equivalent axiomatisation of domain semigroups.

35
Dioids and Modal Semirings

- A dioid is an additively idempotent semiring \((S, +, \cdot, 0, 1)\), that is, a semiring \(S\) in which \(x + x = x\) for all \(x \in S\). As \((S, +, 0)\) is a semilattice, the relation \(x \leq y \iff x + y = y\) is a partial order with least element 0 and in which \(+\) and \(\cdot\) preserve the order in both arguments.

- A modal semiring is a semiring \(S\) equipped with operations \(\text{dom}, \text{cod} : S \to S\) that satisfy
  \[
  x \leq \text{dom}(x) \cdot x, \quad \text{dom}(x \cdot \text{dom}(y)) = \text{dom}(x \cdot y), \quad \text{dom}(x) \leq 1, \quad \text{dom}(0) = 0,
  \]
  \[
  \text{dom}(x + y) = \text{dom}(x) + \text{dom}(y),
  \]
as well as opposite axioms for \(\text{cod}\) and in which \(\text{dom}\) and \(\text{cod}\) are compatible in the sense that \(\text{dom} \circ \text{cod} = \text{cod} \circ \text{dom} = \text{dom}\). Adding these axioms to any semiring and defining \(x \leq y \iff x + y = y\) forces additively idempotency. Every modal semiring is therefore a dioid.

Quantales and Modal Quantales

- A prequantale is a structure \((Q, \leq, \cdot, 1)\) such that \((Q, \leq)\) is a complete lattice, 1 a unit of composition and \(\cdot\) preserves all sups in both arguments.

- A quantale is a prequantale \(Q\) in which composition is associative (hence \((Q, \cdot, 1)\) is a monoid).

- A modal prequantale is a prequantale \(Q\) with operations \(\text{dom}, \text{cod} : Q \to Q\) such that the modal semiring axioms hold (replacing 0 with \(\bot\) and \(+\) with \(\lor\), yet the second axiom (the locality axiom) for \(\text{dom}\) and its opposite for \(\text{cod}\) are replaced by the export axiom
  \[
  \text{dom}(\text{dom}(x) \cdot y)) = \text{dom}(x) \cdot \text{dom}(y)
  \]
  and its opposite for \(\text{cod}\).

- A weakly local modal quantale is a modal prequantale that is also a quantale.

- A modal quantale is a quantale \(Q\) equipped with operations \(\text{dom}, \text{cod} : Q \to Q\) that satisfy the modal semiring axioms (with the obvious syntactic replacements).

B Multioperations and Ternary Relations

We briefly outline the relationship between multioperations and ternary relations. A relational magma \((X, R)\) is a set \(X\) with a ternary relation \(R\). We write \(R^x_{yz}\) instead of \((x, y, z) \in R\) and \(D_{yz}\) whenever \(R^x_{yz}\) holds for some \(x\). We have previously set up the relationship between object-free categories and such relational structures \([\text{CDS20b}]\). The association with multioperations is obtained via

\[
x \in y \circ z \iff R^x_{yz}.
\]

Accordingly, the ternary relation \(R\) is relationally associative if \(\forall u, x, y, z. R^x_{uy} \land R^y_{xz} \iff \exists v. R^x_{uv} \land R^v_{xz}\). It is local if for all \(u, x, y, z, R^u_{xy}\) and \(D_{yz}\) imply \(D_{uz}\). It is weakly functional if \(|\{ x \mid R^x_{yz}\}| \leq 1\) and functional if \(|\{ x \mid R^x_{yz}\}| = 1\), for all \(x, y\). These properties correspond to partiality and totality of multirelations. Element \(e \in X\) is a relational left unit if \(R^x_{xe}\) for some \(x\) and \(R^e_{xy}\) implies \(x = y\) for all \(x, y\); it is a relational right unit if \(R^x_{ye}\) for some \(x\) and \(R^y_{xe}\) implies \(y = x\) for all \(x, y\). Definitions of relational magmas, unitality, relational semigroups and relational monoids are then analogous to those for multimagmas. Similar notions have been studied in \([\text{DHS21}]\).

A relational magma morphism \(f : (X, R) \to (Y, S)\) satisfies \(R^x_{yz} \Rightarrow S^{f(x)}_{f(y)f(z)}\) for all \(x, y, z \in X\). It is bounded if \(S^u_{v}\) implies that there are \(x, y \in X\) such that \(R^x_{yz}, u = f(y)\) and \(v = f(z),\) for all \(x \in X\) and \(u, v \in Y\).

Lemma B.1. Categories of relational magmas and those of multimagmas are isomorphic (both for morphisms and for bounded ones).
Proof. Relations of type $X \times Y \times Z$ are in bijective correspondence with functions of type $Y \times Z \rightarrow \mathcal{P}X$ via the maps

$$F(R) = \lambda y, z. \{ x \mid R^y_z \} \quad \text{and} \quad \mathcal{R}(F) = \{(x, y, z) \mid x \in F(y, z)\}.$$ 

We extend $F$ and $\mathcal{R}$ to functors between categories of relational magmas and multimagas by stipulating $F(f) = f = \mathcal{R}(f)$. Suppose $f$ is a relational magma morphism. Then $F(f)$ is a multimagma morphism because

$$f(F(R)(y, z)) = \{ f(x) | R^y_z \} \subseteq \{ f(x) | R^f(y, z) \} = F(S)(f(y), f(z)).$$

Suppose $f$ is a multimagma morphism. Then $\mathcal{R}(f)$ is a relational magma morphism because

$$(\mathcal{R}(\circ_Y))^y_z \iff x \in y \circ_X y \Rightarrow f(x) \in f(y) \circ_Y f(z) \iff (\mathcal{R}(\circ_Y))^{f(x)}_{f(y), f(z)}.$$ 

Moreover, if the relational magma morphism $f$ is bounded, then so is $F(f)$, because

$$f(x) \in F(S)(u, v) \iff S_{uv}^{f(x)}$$

$$\Rightarrow R^u_v, u = f(y) \text{ and } v = f(z) \text{ for some } y, z$$

$$\iff x \in F(R)(y, z), u = f(y) \text{ and } v = f(z) \text{ for some } y, z.$$

Finally, if the multimagma morphism $f$ is bounded, then so is $\mathcal{R}(f)$, because

$$(\mathcal{R}(\circ_Y))^{f(x)}_{u, v} \iff f(x) \in u \circ_Y v$$

$$\Rightarrow x \in y \circ_X z, u = f(y) \text{ and } v = f(z) \text{ for some } y, z$$

$$\iff (\mathcal{R}(\circ_X))^y_z \wedge u = f(y) \wedge v = f(z) \text{ for some } y, z.$$ 

It is then easy to check that $F$ and $\mathcal{R}$ are mutually inverse functors. This shows that the categories of relational magmas with (bounded) morphisms and those of multimagas with (strong) morphisms are isomorphic. 

The isomorphisms extend to relational and multisemigroups, relational monoids and multimonomoids, and their local, partial and total variants. They transfer theorems between relational structures \cite{CDS20b} and the corresponding multialgebras.

C Proofs of Lemmas 6.2 and 6.3

Proof of Lemma 6.2. We show proofs up-to opposition.

1. $\ell(r(A)) = \{ \ell(r(x)) \mid x \in A \} = \{ r(x) \mid x \in A \} = r(A)$.

2. $\ell(A)A = \bigcup \{ \ell(x)y \mid x, y \in A \text{ and } D_{\ell(x)y} \}$

$$= \bigcup \{ \ell(x)y \mid x, y \in A, D_{\ell(x)y} \text{ and } r(\ell(x)) = \ell(y) \}$$

$$= \bigcup \{ \ell(x)y \mid x, y \in A, D_{\ell(x)y} \text{ and } \ell(x) = \ell(y) \}$$

$$= \bigcup \{ \ell(y)y \mid y \in A \}$$

$$= \bigcup \{ y \mid y \in A \}$$

$$= A.$$ 

3. $\ell(\bigcup A) = \{ \ell(x) \mid x \in \bigcup A \} = \{ \ell(x) \mid x \in A \text{ for some } A \in A \} = \bigcup \{ \ell(A) \mid A \in A \}.$$

4. Immediate from (3).
5. We only prove the identity for \( \ell(A)r(B) \); the remaining ones then follow from (1).

\[
\ell(A)r(B) = \bigcup\{\ell(x)r(y) \mid x \in A \text{ and } y \in B\} = \bigcup\{r(y)\ell(x) \mid x \in A \text{ and } y \in B\} = r(B)\ell(A).
\]

6. \( \ell(A) = \{\ell(x) \mid x \in A\} \subseteq \{\ell(x) \mid x \in X\} = \{x \mid \ell(x) = x\} = E. \)

7. 

\[
\ell(\ell(A)B) = \bigcup\{\ell(\ell(x)y) \mid x \in A, y \in B \text{ and } D_{\ell(x)y}\}
= \bigcup\{\ell(x)\ell(y) \mid x \in A, y \in B, D_{\ell(x)y} \text{ and } r(\ell(x)) = \ell(y)\}
= \bigcup\{\ell(x)\ell(y) \mid x \in A, y \in B, D_{\ell(y)x} \text{ and } \ell(x) = \ell(y)\}
= \bigcup\{\ell(x)\ell(y) \mid x \in A, y \in B \text{ and } D_{\ell(y)x}\}
= \bigcup\{\ell(x)\ell(y) \mid x \in A \text{ and } y \in B\}
= \ell(A)\ell(B).
\]

Proof of Lemma 6.3. Suppose \( X \) is an \( \ell r \)-multisemigroup. For the first inclusion, it is routine to derive (lla) from Lemma 6.2. The claim then follows from Proposition 4.1 and the adjunction for domain. The second inclusion holds by opposition.

Finally, suppose that \( X \) is local. Then, writing \( r(x) = \ell(y) \) in place of \( D_{xy} \) owing to locality,

\[
\ell(A\ell(B)) = \bigcup\{\ell(x\ell(y)) \mid x \in A, y \in B \text{ and } r(x) = \ell(\ell(y))\}
= \bigcup\{\ell(xy) \mid x \in A, y \in B \text{ and } r(x) = \ell(y)\}
= \ell(AB)
\]
and the opposite result is obvious.