ON DISTRIBUTION OF ZEROS OF RANDOM POLYNOMIALS IN COMPLEX PLANE

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Abstract. Let $G_n(z) = \xi_0 + \xi_1 z + \cdots + \xi_n z^n$ be a random polynomial with i.i.d. coefficients (real or complex). We show that the arguments of the roots of $G_n(z)$ are uniformly distributed in $[0, 2\pi]$ asymptotically as $n \to \infty$. We also prove that the condition $E \ln(1 + |\xi_0|) < \infty$ is necessary and sufficient for the roots to asymptotically concentrate near the unit circumference.

1. Introduction: problem and results

Let $\{\xi_k\}_{k=0}^\infty$ be a sequence of independent identically distributed real- or complex-valued random variables. It is always supposed that $P(\xi_0 = 0) < 1$.

Consider the sequence of random polynomials

$$G_n(z) = \xi_0 + \xi_1 z + \cdots + \xi_n z^n.$$ 

By $z_1, \ldots, z_n$ denote the zeros of $G_n$. It is not hard to show (see [1]) that there exist an indexing of the zeros such that for each $k = 1, \ldots, n$ the $k$-th zero $z_{k,n}$ is a one-valued random variable. For any measurable subset $A$ of $\mathbb{C}$ put $N_n(A) = \# \{z_{k,n} : z_{k,n} \in A \}$. Then $N_n(A)/n$ is a probability measure on the plane (the empirical distribution of the zeros of $G_n$). For any $a, b$ such that $0 \leq a < b \leq \infty$ put $R_n(a, b) = N_n(\{z : a \leq |z| \leq b \})$ and for any $\alpha, \beta$ such that $0 \leq \alpha \leq \beta \leq 2\pi$ put $S_n(\alpha, \beta) = N_n(\{z : \alpha \leq \arg z \leq \beta \})$. Thus $R_n/n$ and $S_n/n$ define the empirical distributions of $|z_{k,n}|$ and $\arg z_{k,n}$.

In this paper we study the limit distributions of $N_n, R_n, S_n$ as $n \to \infty$.

The equation $G_n(z) = 0$ is always supposed to have $n$ distinct zeros. The asymptotic study of $R_n, S_n$ has been initiated by Shparo and Shur in [16]. To describe their results let us introduce the function

$$f(t) = \left[ \log^+ \log^+ \cdots \log^+ t \right]^{1+\varepsilon} \prod_{i=1}^m \log^+ \log^+ \cdots \log^+ t,$$

where $\log^+ s = \max(1, \log s)$. We assume that $\varepsilon > 0, m \in \mathbb{Z}^+$ and $f(t) = (\log^+ t)^{1+\varepsilon}$ for $m = 0$.

Shparo and Shur have proved in [16] that if

$$E f(|\xi_0|) < \infty$$

for some $\varepsilon > 0, m \in \mathbb{Z}^+$, then for any $\delta \in (0, 1)$ and $\alpha, \beta$ such that $0 \leq \alpha < \beta \leq 2\pi$

$$\frac{1}{n} R_n(1 - \delta, 1 + \delta) \xrightarrow{P} 1, \quad n \to \infty,$$

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1
The first relation means that under quite weak constraints imposed on the coefficients of a random polynomial, almost all its roots “concentrate uniformly” near the unit circumference with high probability; the second relation means that the arguments of the roots are asymptotically uniformly distributed.

Later Shepp and Vanderbei \[15\] and Ibragimov and Zeitouni \[5\] under additional conditions imposed on the coefficients of \(G_n\) got more precise asymptotic formulas for \(R_n\).

What kind of further results could be expected? First let us note that if, e.g., 
\(E|\xi_0| < \infty\), then for \(|z| < 1\)
\[G_n(z) \to G(z) = \sum_{k=0}^{\infty} \xi_k z^k\]
as \(n \to \infty\) a.s. The function \(G(z)\) is analytical inside the unit disk \(|z| < 1\). Therefore for any \(\delta > 0\) it has only a finite number of zeros in the disk \(|z| > 1/(1-\delta)\).

At the other hand, the average number of zeros in the domain \(|z| > 1/(1-\delta)\) is the same (it could be shown if we consider the random polynomial \(G(1/z)\)). Thus one could expect that under sufficiently weak constraints imposed on the coefficients of a random polynomial the zeros concentrate near the unit circle \(\Gamma = \{z : |z| = 1\}\) and a measure \(R_n/n\) converges to the delta measure at the point one. We may expect also from the consideration of symmetry that the arguments \(\arg z_{kn}\) are asymptotically uniformly distributed. Below we give the conditions for these hypotheses to hold. We shall prove the following three theorems about the behavior of \(N_n/n, R_n/n, S_n/n\).

For the sake of simplicity, we assume that \(P\{\xi_0 = 0\} = 0\). To treat the general case it is enough to study in the same way the behavior of the roots on the sets \(\{\theta'_n = k, \theta''_n = l\}\), where
\[\theta'_n = \max\{i = 0, \ldots, n \mid \xi_i \neq 0\}, \quad \theta''_n = \min\{j = 0, \ldots, n \mid \xi_j \neq 0\}\.

**Theorem 1.** The sequence of the empirical distributions \(R_n/n\) converges to the delta measure at the point one almost surely if and only if
\[(1) \quad E \log(1 + |\xi_0|) < \infty.\]

In other words, \((1)\) is necessary and sufficient condition for
\[(2) \quad P\left\{\frac{1}{n}R_n(1-\delta, 1+\delta) \xrightarrow{n \to \infty} 1\right\} = 1\]
hold for any \(\delta > 0\).

We shall also prove that if \((1)\) does not hold then no limit distribution for \(\{z_{nk}\}\) exist.

**Theorem 2.** Suppose the condition \((1)\) holds. Then the empirical distribution \(N_n/n\) almost surely converges to the probability measure \(N(\cdot) = \mu(\cdot \cap \Gamma)/(2\pi)\), where \(\Gamma = \{z : |z| = 1\}\) and \(\mu\) is the Lebesgue measure.

**Theorem 3.** The empirical distribution \(S_n/n\) almost surely converges to the uniform distribution, i.e.,
\[P\left\{\frac{1}{n}S_n(\alpha, \beta) \xrightarrow{n \to \infty} \frac{\beta - \alpha}{2\pi}\right\} = 1\]
for any $\alpha, \beta$ such that $0 \leq \alpha < \beta \leq 2\pi$.

Let us remark here that Theorem 3 does not require any additional conditions on the sequence $\{\xi_k\}$.

The next result is of crucial importance in the proof of Theorem 3.

**Theorem 4.** Let $\{\eta_k\}_{k=0}^{\infty}$ be a sequence of independent identically distributed real-valued random variables. Put $g_n(x) = \sum_{k=0}^{n} \eta_k x^k$ and by $M_n$ denote the number of real roots of the polynomial $g_n(x)$. Then

\[
P \left\{ \frac{M_n}{n} \to 0 \right\} = 1, \quad E M_n = o(n), \quad n \to \infty.
\]

Theorem 4 is also of independent interest. In a number of papers it was shown that under weak conditions on the distribution of $\eta_0$ one has $E M_n \sim c \times \log n, n \to \infty$ (see [2], [3], [4], [6], [9], [10]). L. Shepp proposed the following conjecture: for any distribution of $\eta_0$ there exist positive numbers $c_1, c_2$ such that $E M_n \geq c_1 \times \log n$ and $E M_n \leq c_2 \times \log n$ for all $n$. The first statement was disproved in [17], [18]. There was constructed a random polynomial $g_n(x)$ with $E M_n < 1 + \varepsilon$. It is still unknown if the second statement is true. However, Theorem 4 shows that an arbitrary random polynomial can not have too much real roots (see also [14]).

In fact, in the proof of Theorem 3 we shall use a slightly generalized version of Theorem 4:

**Theorem 5.** For some integer $r$ consider a set of $r$ non-degenerate probability distributions. Let $\{\eta_k\}_{k=0}^{\infty}$ be a sequence of independent real-valued random variables with distributions from this set. As above, put $g_n(x) = \sum_{k=0}^{n} \eta_k x^k$ and by $M_n$ denote the number of real roots of the polynomial $g_n(x)$. Then

\[
P \left\{ \frac{M_n}{n} \to 0 \right\} = 1, \quad E M_n = o(n), \quad n \to \infty.
\]

2. **Proof of Theorem 4**

Let us establish the sufficiency of (1). Let it hold and fix $\delta \in (0, 1)$. Prove that the radius of convergence of the series

\[
G(z) = \sum_{k=0}^{\infty} \xi_k z^k
\]

is equal to one with probability one.

Consider $\rho > 0$ such that $P\{|\xi_0| > \rho\} > 0$. Using the Borel-Cantelli lemma we obtain that with probability one the sequence $\{\xi_k\}$ contains infinitely many $\xi_k$ such that $|\xi_k| > \rho$. Therefore the radius of convergence of the series (4) does not exceed 1 almost surely.

On the other hand, for any non-negative random variable $\zeta$

\[
\sum_{k=1}^{\infty} P(\eta \geq k) \leq E \zeta \leq 1 + \sum_{k=1}^{\infty} P(\zeta \geq k).
\]

Therefore, it follows from (1) that

\[
\sum_{k=1}^{\infty} P(|\xi_k| \geq e^{\gamma k}) < \infty
\]
for any positive constant $\gamma$. It follows from the Borel-Cantelli lemma that with probability one $|\xi_k| < e^{\gamma k}$ for all sufficiently large $k$. Thus, according to the Cauchy-Hadamard formula (see, e.g., [11]), the radius of convergence of the series $A$ is at least 1 almost surely.

Hence with probability one $G(z)$ is an analytical function inside the unit ball $\{|z| < 1\}$. Therefore if $0 \leq a < b < 1$, then $R(a, b) < \infty$, where $R(a, b)$ denotes the number of the zeros of $G$ inside the domain $\{z : a \leq |z| \leq b\}$. It follows from the Hurwitz theorem (see, e.g., [11]) that with probability one $R_n(0, 1 - \delta) \leq R(0, 1 - \delta/2)$ for all sufficiently large $n$. This implies

$$\mathbf{P}\left\{\frac{1}{n} R_n(0, 1 - \delta) \xrightarrow{n \to \infty} 0\right\} = 1.$$  

In order to conclude the proof of (2) it remains to show that

$$\mathbf{P}\left\{\frac{1}{n} R_n(1 + \delta, \infty) \xrightarrow{n \to \infty} 0\right\} = 1.$$  

In other words, we need to prove that $\mathbf{P}\{A\} = 0$, where $A$ denotes the event that there exists $\varepsilon > 0$ such that

$$R_n(1 + \delta, \infty) \geq \varepsilon n$$

holds for infinitely many values $n$.

By $B$ denote the event that $G(z)$ is an analytical function inside the unit disk $\{|z| < 1\}$. For $m \in \mathbb{N}$ put

$$\zeta_m = \sup_{k \in \mathbb{Z}^+} |\xi_k e^{-k/m}|.$$  

By $C_m$ denote the event that $\zeta_m < \infty$. It was shown above that $\mathbf{P}\{B\} = \mathbf{P}\{C_m\} = 1$ for $m \in \mathbb{N}$. Therefore, to get $\mathbf{P}\{A\} = 0$, it is sufficient to show that $\mathbf{P}\{ABC_m\} = 0$ for some $m$.

Let us fix $m$. The exact value of it will be chosen later. Suppose the event $ABC_m$ occurred. Index the roots of the polynomial $G_n(z)$ according to the order of magnitude of their absolute values:

$$|z_1| \leq |z_2| \leq \cdots \leq |z_n|.$$  

Fix an arbitrary number $C > 1$ (an exact value will be chosen later). Consider indices $i, j$ such that

$$|z_i| < 1 - \delta/C, \quad |z_{i+1}| \geq 1 - \delta/C, \quad |z_j| \leq 1 + \delta, \quad |z_{j+1}| > 1 + \delta.$$  

If $|z_1| \geq 1 - \delta/C$, then $i = 0$; if $|z_n| \leq 1 + \delta$ then $j = n$.

It is easily shown that if

$$|z| < \min \left(1, \frac{|\xi_0|}{n \max_{k=1, \ldots, n} |\xi_k|}\right),$$  

then

$$|\xi_0| > |\xi_1 z| + |\xi_2 z^2| + \cdots + |\xi_n z^n|.$$  

Therefore such $z$ can not be a zero of the polynomial $G_n$. Taking into account that the event $C_m$ occurred, we obtain a lower bound for the absolute values of the zeros for all sufficiently large $n$:

$$|z_1| \geq \min \left(1, \frac{|\xi_0|}{n \max_{k=1, \ldots, n} |\xi_k|}\right) \geq \frac{|\xi_0|}{n^{\zeta_m e^{\gamma/2}}} \geq |\xi_0| \zeta_m^{e^{-2n/m}}.$$  

Therefore for any integer $l$ satisfying $j + 1 \leq l \leq n$ and all sufficiently large $n$

$$|z_1 \ldots z_l| = |z_1 \ldots z_i||z_{i+1} \ldots z_j||z_{j+1} \ldots z_l|$$

$$\geq |\xi_0|^l \frac{\zeta}{\varsigma_m} e^{-2n/\zeta} \left( 1 - \frac{\delta}{C} \right)^{j-i} (1 + \delta)^{l-j}.$$ 

Since $A$ occurred, $n - j \geq n\varepsilon$ for infinitely many values of $n$. Therefore if $l$ satisfies $n - \sqrt{n} \leq l \leq n$, then the inequalities $j + 1 \leq l \leq n$ and $l - j \geq n\varepsilon/2$ hold for infinitely many values of $n$. According to the Hurwitz theorem, $i \leq R_n(0, 1 - \delta/(2C)) \leq R(0, 1 - \delta/(2C))$ for all sufficiently large $n$. Therefore for infinitely many values of $n$

$$|z_1 \ldots z_l| \geq \left( \frac{|\xi_0|}{\zeta_m} \right)^{R(0, 1 - \delta/(2C))} e^{-2n} R(0, 1 - \delta/(2C))/m \left( 1 - \frac{\delta}{C} \right)^{n} (1 + \delta)^{n\varepsilon/2}.$$ 

Choose now $C$ large enough to yield

$$(1 - \frac{\delta}{C}) (1 + \delta)^{\varepsilon/2} > 1.$$ 

Furthermore, holding $C$ constant choose $m$ such that

$$b = e^{-2R(0, 1 - \delta/(2C))/m} \left( 1 - \frac{\delta}{C} \right)^{(1 + \delta)^{\varepsilon/2}} > 1.$$ 

Since

$$\left( \frac{|\xi_0|}{\zeta_m} \right)^{R(0,/(2C))/n} \xrightarrow{n \to \infty} 1,$$

there exists a random variable $a > 1$ such that for infinitely many values of $n$

$$|z_1 \ldots z_l| \geq \left( \frac{|\xi_0|}{\zeta_m} \right)^{R(0, 1 - \delta/(2C))} b^n = \left( \frac{|\xi_0|}{\zeta_m} \right)^{R(0, 1 - \delta/(2C))} a^n.$$ 

On the other hand, it follows from $n - \sqrt{n} \leq l$ and Viète’s formula that

$$|z_{i+1} \ldots z_n| \geq \left( \frac{n}{n - \sqrt{n}} \right)^{-1} \sum_{i_1 < \cdots < i_{n-1}}|z_{i_1} \ldots z_{i_{n-1}}| = \left( \frac{n}{n - \sqrt{n}} \right)^{-1} |\xi_0| |\xi_1| |\xi_2| \ldots |\xi_n|.$$ 

We combine these two inequalities to obtain for infinitely many values of $n$

$$\frac{|\xi_0|}{|\xi_n|} = \frac{|z_1 \ldots z_n|}{|\xi_0|^n} \geq a^n \left( \frac{n}{n - \sqrt{n}} \right)^{-1} \frac{|\xi_0|}{|\xi_n|} \geq c_1 a^n \left( \frac{|\xi_0|}{|\xi_n|} \right)^{n} \frac{\sqrt{n}^{n+\frac{1}{2}} (n - \sqrt{n})^{n-\sqrt{n}+\frac{1}{2}} |\xi_0|}{|\xi_n|} \geq c_2 a^n \left( \frac{\sqrt{n}^n}{\sqrt{n}} \right)^{n} \left( 1 - \frac{1}{\sqrt{n}} \right)^n \frac{|\xi_0|}{|\xi_n|} \geq c_3 \exp \left( n \log a - \frac{\sqrt{n} \log n}{2} - \sqrt{n} \right) \frac{|\xi_0|}{|\xi_n|} \geq e^{\alpha n} \frac{|\xi_0|}{|\xi_n|},$$

where $\alpha$ is a positive random variable. Multiplying left and right parts by $|\xi_0|$, we get

$$ABC_m \subset \bigcup_{i=1}^{\infty} D_i,$$

where $D_i$ denotes the event that $|\xi_0| > e^{\alpha/4} \max_{n - \sqrt{n} \leq l \leq n} |\xi_1|$ for infinitely many values of $n$. 

To complete the proof it is sufficient to show that $P\{D_i\} = 0$ for all $i \in \mathbb{N}$. Having in mind to apply the Borel-Cantelli lemma, let us introduce the following events:

$$H_{in} = \left\{ |\xi_0| > e^{n/i} \max_{n - \sqrt{n} \leq i \leq n} |\xi_i| \right\}.$$

Considering $\theta > 0$ such that $P\{|\xi_0| \leq \theta\} = F(\theta) < 1$, we have

$$H_{in} \subset \left\{ |\xi_0| > e^{|\xi_0|^i} \max_{n - \sqrt{n} \leq i \leq n} |\xi_i| \leq \theta \right\},$$

consequently,

$$\sum_{n=1}^{\infty} P\{H_{in}\} \leq \sum_{n=1}^{\infty} P\{|\xi_0| > e^{|\xi_0|^i}\} + \sum_{n=1}^{\infty} (F(\theta))^{\sqrt{n}} < \infty$$

and, according to the Borel-Cantelli lemma, $P\{D_i\} = 0$.

We prove the implication (2) $\Rightarrow$ (1) arguing by contradiction. Suppose (1) does not hold, i.e.,

$$E \log(1 + |\xi_0|) = \infty.$$

It follows from (5) that

$$\sum_{n=1}^{\infty} P\{|\xi_n| \geq e^{\gamma n}\} = \infty$$

for an arbitrary positive $\gamma$. For $k \in \mathbb{N}$ introduce an event $F_k$ that $|\xi_n| \geq e^{kn}$ holds for infinitely many values of $n$. It follows from (6) and the Borel-Cantelli lemma that $P\{F_k\} = 1$ and, consequently, $P\{\bigcap_{k=1}^{\infty} F_k\} = 1$. This yields

$$P\left\{\limsup_{n \to \infty} |\xi_n|^{1/n} = \infty\right\} = 1.$$

Therefore with probability one for infinitely many values of $n$

$$|\xi_n|^{1/n} > \max_{i=0, \ldots, n-1} |\xi_i|^{1/i}, \quad |\xi_n|^{1/n} > \frac{3}{\varepsilon}, \quad |\xi_0| < 2^{n-1},$$

where $\varepsilon > 0$ is an arbitrary fixed value. Let us hold one of those $n$. Suppose $|z| \geq \varepsilon$. Then

$$|\xi_0 + \xi_1 z + \cdots + \xi_{n-1} z^{n-1}|$$

$$\leq 2^{n-1} + |\xi_n z^{n-1/n}| + |\xi_n z^{n/2}| + \cdots + |\xi_n z^n|^{(n-1)/n}$$

$$= \frac{2n}{2} - 1 + \frac{|\xi_n z^n| - 1}{|\xi_n z^n| - 1} \leq \frac{|\xi_n z^n| - 1}{2} - 1 + \frac{|\xi_n z^n| - 1}{(3/\varepsilon) \times \varepsilon - 1} < |\xi_n z^n|.$$

Thus with probability one for infinite number of values of $n$ all the roots of the polynomial $G_n$ are located inside the circle $\{z : |z| = \varepsilon\}$, where $\varepsilon$ is an arbitrary positive constant. This means that (2) does not hold for any $\delta \in (0, 1)$. 
3. Proof of Theorem 2

The proof of Theorem 2 follows immediately from Theorem 1 and Theorem 3. However, the additional assumption (1) significantly simplifies the proof.

Consider a set of sequences of reals

\[ \{a_{11}, a_{12}, a_{22}, \ldots, a_{nn}, a_{2n}, \ldots, a_{nn}\}, \ldots, \]

where all \( a_{jn} \in [0, 1] \). We say that \( \{a_{jn}\} \) are uniformly distributed in \([0, 1]\) if for any \( 0 \leq a < b \leq 1 \)

\[
\lim_{n \to \infty} \frac{\#\{j \in \{1, 2, \ldots, n\} : a_{jn} \in [a, b]\}}{n} = b - a.
\]

The definition is an insignificant generalization of the notion of uniformly distributed sequences (see, e.g., [7]). It is easy to see that the Weyl criterion (see Ibid.) continues to be valid in this case:

The set of sequences \( \{a_{jn}, j = 1, \ldots, n\}, n = 1, 2, \ldots, \) is uniformly distributed if and only if for all \( l = 1, 2, \ldots \)

\[
\frac{1}{n} \sum_{j=1}^{n} e^{2\pi il a_{jn}} \to 0, \quad n \to \infty.
\]

Let \( z_{jn} = r_{jn} e^{i\theta_{jn}} \) be a zero of \( G_n(z) \), \( r_{jn} = |z_{jn}|, \theta_{jn} = \arg z_{jn}, 0 \leq \theta_{jn} < 2\pi \).

The asymptotic uniform distribution of the arguments is equivalent to the statement that the set of sequences \( \{\theta_{jn}/(2\pi)\} \) is uniformly distributed. Thus, according to Weyl’s criterion, it is enough to show that for any \( l = 1, 2, \ldots \)

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} e^{il\theta_{jn}} = 0
\]

with probability 1.

For the simplicity we assume that \( \xi_0 \neq 0 \). Consider the random polynomial

\[ \tilde{G}_n(z) = \xi_1 z_1 + \xi_2 z_2 + \cdots + \xi_n z^n + \xi_0 z^n. \]

Its roots are \( z_{kn}^{-1} \). According to Newton’s formulas (see, e.g., [8]),

\[
\sum_{j=1}^{n} \frac{1}{z_{jn}} = \varphi_l \left( \frac{\xi_1}{\xi_0}, \ldots, \frac{\xi_l}{\xi_0} \right),
\]

where \( \varphi_l(x_1, \ldots, x_l) \) are polynomials which do not depend on \( n \). (For example, \( \varphi_1(x) = -x \).) It follows that

\[
\frac{1}{n} \sum_{j=1}^{n} e^{-il\theta_{jn}} = \frac{1}{n} \sum_{j=1}^{n} e^{-il\theta_{jn}} \left( 1 - \frac{1}{r_{jn}} \right) + \frac{\varphi_l}{n}.
\]

As was shown in the proof of Theorem 1, for \( |z| < 1 \) the polynomials \( G_n(z) \) converge to the analytical function \( G(z) = \sum_{k=0}^{\infty} \xi_k z^k \) with probability 1. Since \( \xi_0 \neq 0 \), the function \( G(z) \) has no zeros inside a circle \( \{z : |z| \leq \rho\} \), \( \mathbb{P}\{\rho > 0\} = 1 \).

Hence for \( n \geq N \), \( \mathbb{P}\{N < \infty\}, \) the polynomials \( G_n(z) \) have no zeros inside \( \{z : |z| \leq \rho\} \). Let \( \gamma > 0 \) be a positive number. It follows from (7) that

\[
\left| \frac{1}{n} \sum_{j=1}^{n} e^{il\theta_{jn}} \right| \leq (l+1) \frac{\gamma}{(1-\gamma)^2} + \frac{1}{n} \left( 1 + \frac{1}{\rho} \right) \sum_{j=1}^{N} \frac{\varphi_l}{n}.
\]
Theorem[1] implies that the second member on the right-hand side goes to zero as $n \to \infty$ with probability 1. Hence

$$\frac{1}{n} \sum_{j=1}^{n} e^{i\theta j} \to 0, \quad n \to \infty,$$

with probability 1 and the theorem follows.

4. Proof of Theorem 3

Consider integer numbers $p, q_1, q_2$ such that $0 \leq q_1 < q_2 < p - 1$. Put $\varphi_j = q_j/p, \ j = 1, 2$, and try to estimate $S_n = S_n(2\pi \varphi_1, 2\pi \varphi_2)$. Evidently $S_n = \lim_{R \to \infty} S_{nR}$, where $S_{nR}$ is the number of zeros of $G_n(z)$ inside the domain $A_R = \{ z : |z| \leq R, 2\pi \varphi_1 \leq \arg z \leq 2\pi \varphi_2 \}$. It follows from the argument principle (see, e.g., [11]) that $S_{nR}$ is equal to the change of the argument of $G_n(z)$ divided by $2\pi$ as $z$ traverses the boundary of $A_R$. The boundary consists of the arc $\Gamma_R = \{ z : |z| = R, 2\pi \varphi_1 \leq \arg z \leq 2\pi \varphi_2 \}$ and two intervals $L_j = \{ z : 0 \leq |z| \leq R, \arg z = \varphi_j \}$, $j = 1, 2$. It can easily be checked that if $R$ is sufficiently large, then the change of the argument as $z$ traverses $\Gamma_R$ is equal to $n(\varphi_2 - \varphi_1) + o(1)$ as $n \to \infty$. If $z$ traverses a subinterval of $L_j$ and the change of the argument of $G_n(z)$ is at least $\pi$, then the function $|G_n(z)| \cos(\arg G_n(z))$ has at least one root in this interval. It follows from Theorem 3 that with probability one the number of real roots of the polynomial

$$g_{n,j}(x) = \sum_{k=0}^{n} x^k \Re(\xi_k e^{2\pi i k \varphi_j}) = \sum_{k=0}^{n} x^k \eta_{k,j}$$

is $o(n)$ as $n \to \infty$. Thus the change of the argument of $G_n(z)$ as $z$ traverses $L_j$ is $o(n)$ as $n \to \infty$ and

$$P \left\{ \frac{1}{n} S_n(2\pi \varphi_1, 2\pi \varphi_2) = (\varphi_2 - \varphi_1) + o(1), \quad n \to \infty \right\} = 1 .$$

The set of points of the form $\exp\{2\pi i q/p\}$ is dense in the unit circle $\{ z : |z| = 1 \}$. Therefore

$$P \left\{ \frac{1}{n} S_n(\alpha, \beta) \to_{n \to \infty} \frac{\beta - \alpha}{2\pi} \right\} = 1$$

for any $\alpha, \beta$ such that $0 \leq \alpha < \beta \leq 2\pi$.

5. Proof of Theorem 5

First we convert the problem of counting of real zeros of $g_n(x)$ to the problem of counting of sign changes in the sequence of the derivatives $\{g_n^{(j)}(1)\}_{j=0}^{n}$.

Let $\{a_j\}_{j=0}^{n}$ be a sequence of real numbers. By $Z(\{a_j\})$ denote the number of sign changes in the sequence $\{a_j\}$, which is defined as follows. First we exclude all zero members from the sequence. Then we count the number of the neighboring members of different signs.

For any polynomial $p(x)$ of degree $n$ put $Z_p(x) = Z(\{p^{(j)}(x)\})$, i.e., the number of sign changes in the sequence $p(x), p'(x), \ldots, p^{(n)}(x)$.

**Lemma 1** (Budan-Fourier Theorem). Suppose $p(x)$ is a polynomial such that $p(a), p(b) \neq 0$ for some $a < b$. Then the number of the roots of $p(x)$ inside $(a, b)$ does not exceed $Z_p(a) - Z_p(b)$. Moreover, the difference between $Z_p(a) - Z_p(b)$ and the number of the roots is an even number.
Proof. See, e.g., [8]. □

Corollary 1. The number of the roots of \( p(x) \) inside \([1, \infty)\) does not exceed \( Z_p(1) \).

Proof. For all sufficiently large \( x \) the sign of \( p^{(j)}(x) \) coincides with the sign of the leading coefficient. □

Corollary 2. The function \( Z_p(x) \) does not increase.

Let us turn back to the random polynomial \( g_n(x) \). Here and elsewhere we shall omit the index \( n \) when it can be done without ambiguity. By \( M_n(a, b) \) denote the number of zeros of \( g(x) \) inside the interval \([a, b]\).

First let us prove that

\[
E Z_g(1) = o(n), \quad n \to \infty .
\]

Fix some \( \varepsilon > 0 \) and \( \lambda \in (0, 1/2) \). Since the distributions of \( \{\eta_j\} \) belong to a finite set, there exists \( K = K(\varepsilon) \) such that

\[
\sup_{j \in \mathbb{Z}} P \{|\eta_j| \geq K\} \leq \varepsilon .
\]

Let \( I \) be a subset of \( \{0, 1, \ldots, n\} \) consisting of indices \( j \) such that \( \lfloor \lambda n \rfloor \leq j \leq \lfloor (1 - \lambda)n \rfloor \) and \( |\eta_j| < K \). Put

\[
g_1(x) = \sum_{j \in I} \eta_j x^j, \quad g_2(x) = g(x) - g(x) .
\]

Let \( \tau_j \) be the indicator of \( \{|g_1^{(k)}(1)| \geq |g_2^{(k)}(1)|\} \) and \( \chi_j \) be the indicator of \( \{|\eta_j| \geq K\} \).

Lemma 2. Let \( a_1, a_2, b_1, b_2 \) be real numbers. If \( (a_1 + a_2)(b_1 + b_2) < 0 \) and \( a_2 b_2 \geq 0 \), then either \( |a_1| \geq |a_2| \) or \( |b_1| \geq |b_2| \).

Proof. The proof is trivial. □

It follows from Lemma 2 that

\[
Z_g(1) = Z_{g_1 + g_2}(1) \leq Z_{g_2}(1) + 2 \sum_{j=0}^{n} \tau_j \leq Z_{g_2}(1) + 2 \lambda n + 2 + 2 \sum_{j=\lfloor \lambda n \rfloor}^{\lfloor (1-\lambda)n \rfloor} \tau_j .
\]

Owing to the monotonicity of the function \( Z_{g_2}(x) \), one has

\[
Z_{g_2}(1) \leq Z_{g_2}(0) \leq \sum_{j=0}^{n} \chi_j .
\]

Hence,

\[
Z_g(1) \leq 2 \lambda n + 2 + \sum_{j=0}^{n} \chi_j + 2 \sum_{j=\lfloor \lambda n \rfloor}^{\lfloor (1-\lambda)n \rfloor} \tau_j .
\]

Using 9 we have \( E \chi_j = P \{|\eta_j| \geq K\} \leq \varepsilon \), therefore,

\[
E Z_g(1) \leq 2 \lambda n + 2 + \varepsilon(n + 1) + 2E \sum_{j=\lfloor \lambda n \rfloor}^{\lfloor (1-\lambda)n \rfloor} \tau_j .
\]
Let us now estimate $E \tau_k$. Note that $g^{(k)}(x) = \sum_{l=k}^{n} \eta_l A_{k,l} x^{l-k}$, where $A_{k,l} = l(l-1) \cdots (l-k+1)$. Fix some integer $k$ such that $\lambda n \leq k \leq (1-\lambda)n$. If $n-1 \geq j \geq k$, then

$$A_{k,j} \leq (1-\lambda)A_{k,j+1},$$

which implies

$$A_{k,j} \leq A_{k,(1-\lambda)n} (1-\lambda)^{(1-\lambda)n-j}$$

for $\lambda n \leq k \leq j \leq (1-\lambda)n$. Consequently,

$$|g^{(k)}_1(1)| = \left| \sum_{j \in J,j \geq k} \eta_j A_{k,j} \right| \leq K A_{k,(1-\lambda)n} \sum_{j=0}^{[(1-\lambda)n]} (1-\lambda)^j \leq \frac{K}{\lambda} A_{k,(1-\lambda)n}.$$

This yields that

$$E \tau_k = P \left\{ |g^{(k)}_1(1)| \geq |g^{(k)}_2(1)| \right\}$$

$$\leq P \left\{ |g^{(k)}_1(1)| \geq |g^{(k)}_1(1) + g^{(k)}_2(1) - g^{(k)}_1(1)| \right\}$$

$$= P \left\{ |g^{(k)}(1)| \leq 2 |g^{(k)}_1(1)| \right\} \leq P \left\{ |g^{(k)}(1)| \leq \frac{2K}{\lambda} A_{k,(1-\lambda)n} \right\}.$$

For an arbitrary random variable $X$ define the concentration function $Q(h; X)$ as follows:

$$Q(h; X) = \sup_{a \in \mathbb{R}} P \left\{ a \leq X \leq a + h \right\}.$$

If $X, Y$ are independent random variables, then (see, e.g., [12])

$$Q(h; X + Y) \leq \min \left( Q(h; X), Q(h; Y) \right).$$

Therefore,

$$E \tau_k \leq P \left\{ |g^{(k)}(1)| \leq \frac{2K}{\lambda} A_{k,(1-\lambda)n} \right\}$$

$$\leq P \left\{ \frac{g^{(k)}(1)}{A_{k,(1-\lambda)n}} \leq \frac{2K}{\lambda} \right\} \leq Q \left( \frac{2K}{\lambda}; \frac{g^{(k)}(1)}{A_{k,(1-\lambda)n}} \right)$$

$$= Q \left( \frac{2K}{\lambda}; \sum_{j=k}^{n} \frac{A_{k,j}}{A_{k,(1-\lambda)n}} \eta_j \right) \leq Q \left( \frac{2K}{\lambda}; \sum_{j=(1-\lambda)n}^{n} \frac{A_{k,j}}{A_{k,(1-\lambda)n}} \eta_j \right).$$

To estimate the right-hand side of (12) we use the following result.

**Lemma 3** (the Kolmogorov-Rogozin inequality). Let $X_1, X_2, \ldots, X_n$ be independent random variables. Then for any $0 < h_j \leq h$, $j = 1, \ldots, n$,

$$Q(h; X_1 + \cdots + X_n) \leq \frac{Ch}{\sqrt{\sum_{j=1}^{n} h_j^2 (1 - Q(h_j; X_j))}},$$

where $C$ is an absolute constant.

**Proof.** See [13].
Since the distributions of \( \{ \eta_j \} \) belong to a finite set, we get
\[
\delta = \delta(\varepsilon, \lambda) = \inf_{j \in \mathbb{Z}} \left\{ 1 - Q \left( \frac{2K}{\lambda} ; \eta_j \right) \right\} > 0.
\]

Putting \( h = h_j = 2K/\lambda \) in (13) and using (12), we obtain
\[
E \tau_k \leq C \left[ \sum_{j=\lfloor (1-\lambda)n \rfloor}^{n} \left\{ 1 - Q \left( \frac{2K}{\lambda} ; A_{k,j} A_{k,\lfloor (1-\lambda)n \rfloor} \eta_j \right) \right\} \right]^{-1/2} \leq \frac{C}{\sqrt{\delta \lambda n}}.
\]

Combining this with (11), we have
\[
E Z_g(1) \leq 2\lambda n + 2 + \varepsilon(n + 1) + \frac{2C}{\sqrt{\delta \lambda \varepsilon}} n^{1/2}.
\]

Since \( \lambda, \varepsilon \) are arbitrary positive numbers, we obtain (3), which together with the corollary from Lemma 1 implies
\[
E M_n(1, \infty) = o(n) , \quad n \to \infty .
\]

Considering the random polynomials \( g(1/x) \) and \( g(-x) \), it is possible to obtain similar estimates for \( M_n(0,1) \) and \( M_n(-\infty,0) \). Thus the second part of (3) holds. To prove the first one, we estimate the probabilities of large deviations for the sums \( \sum x_j \) and \( \sum \tau_j \). The elementary considerations or the application of Bernstein inequalities (see, e.g., [12]) leads to
\[
P \left\{ \left| \sum_{j=0}^{n} x_j \right| > 2(n + 1) \varepsilon \right\} \leq 2e^{-n\varepsilon/8} .
\]

The analysis of the behavior of \( \sum \tau_j \) is slightly more difficult.

Henceforth we shall use the following notation: for any positive functions \( f_1, f_2 \) we write \( f_1 \ll f_2 \), if there exists an absolute constant \( C \) such that \( f_1 \leq C f_2 \) in the domain of these functions.

**Lemma 4.** There exists a constant \( c \) depending only on \( \lambda, \varepsilon \) and the distributions of \( \{ \eta_j \} \) such that
\[
E \tau_k \leq c n^{-2}
\]
for \( \lambda n \leq k \leq (1 - \lambda)n \).

**Proof.** As was shown in (12),
\[
E \tau_k \leq Q \left( \frac{2K}{\lambda} ; \sum_{j=\lfloor (1-\lambda)n \rfloor}^{n} \frac{A_{k,j}}{A_{k,\lfloor (1-\lambda)n \rfloor}} \eta_j \right) .
\]

\( \Box \)
To estimate the concentration function in the right-hand side we use the result of Esseen (see, e.g., [12]). Let \( X \) be a random variable with a characteristic function \( f(t) \). Then

\[
Q(h; X) \ll \max\left(h, \frac{1}{T}\right) \int_{-T}^{T} |f(t)| \, dt
\]

uniformly for all \( T > 0 \).

Putting \( T = \lambda/(KA_{k,(1-\lambda)n}) \) and applying (15), we obtain

\[
E \tau_k \ll \frac{1}{T} \int_{-T}^{T} \prod_{j=1(1-\lambda)n}^{n} |f_j(A_kj t)| \, dt,
\]

where \( f_j(t) \) is a characteristic function of \( \eta_j \). Further,

\[
E \tau_k \ll \frac{1}{T} \int_{-T}^{T} \left[ \prod_{j=1(1-\lambda)n}^{n} |f_j(A_kj t)|^2 \right]^{rac{1}{2}} \, dt
\]

\[
\ll \frac{1}{T} \int_{-T}^{T} \exp \left\{ -\frac{1}{2} \sum_{j=1(1-\lambda)n}^{n} (1 - |f(A_kj t)|^2) \right\} \, dt
\]

\[
= \frac{1}{T} \int_{-T}^{T} \exp \left\{ -\frac{1}{2} \sum_{j=1(1-\lambda)n}^{n} \int_{-\infty}^{\infty} [1 - \cos(A_kj tx)] \, P_j(dx) \right\} \, dt,
\]

where \( P_j \) is a distribution of the symmetrized \( \eta_j \), i.e., a distribution of \( \eta_j - \eta'_j \), where \( \eta'_j \) is an independent copy of \( \eta_j \).

There are at most \( r \) different distributions among \( \{P_j\}_{1 = (1-\lambda)n, j \leq n} \). Therefore there exist a distribution \( \mathcal{P} \) and a subset \( J \subset \{ j : (1-\lambda)n \leq j \leq n \} \) such that \( |J| \geq n \lambda / r \) and \( P_j = \mathcal{P} \) for all \( j \in J \). By \( \sum' \) denote the summation taking over all indices such that \( j \in J \). Thus,

\[
E \tau_k \ll \frac{1}{T} \int_{-T}^{T} \exp \left\{ \frac{1}{2} \sum_{j=1(1-\lambda)n}^{n'} \int_{-\infty}^{\infty} [1 - \cos(A_kj tx)] \, \mathcal{P}(dx) \right\} \, dt.
\]

Choose \( \delta > 0 \) such that \( \gamma = \mathcal{P}\{x : |x| > \delta\} > 0 \). Since the integrands are non-negative, we get

\[
E \tau_k \ll \frac{1}{T} \int_{-T}^{T} \exp \left\{ -\frac{1}{2} \sum_{j=1(1-\lambda)n}^{n'} \int_{|x| > \delta} [1 - \cos(A_kj tx)] \, \mathcal{P}(dx) \right\} \, dt
\]

\[
= \frac{1}{T} \int_{-T}^{T} e^{-\beta n + s(t)} \, dt,
\]

where \( \lambda_r = \lambda(2r - 1)/(2r) \), \( \beta = |J \cap \{ j : (1-\lambda_r)n \leq j \leq n \}|/(2n) \) and

\[
s(t) = \frac{1}{2} \int_{|x| > \delta} \sum_{j=1(1-\lambda_n)n}^{n'} \cos(A_kj tx) \, \mathcal{P}(dx).
\]
Put $\alpha = \lambda \gamma/(4r)$ and consider $\Lambda_1 = \{ t \in [-T, T] : |s(t)| < \alpha n/2 \}$ and $\Lambda_2 = [-T, T] \setminus \Lambda_1$. Since $|J| \geq n \lambda / r$ and by the definition of $\beta$, we have $\beta \geq \alpha$. Therefore,

$$
E T_k \ll e^{-\alpha n/2} + \frac{\mu(\Lambda_2)}{T},
$$

where $\mu$ denotes the Lebesgue measure.

Let us estimate $\mu(\Lambda_2)$. It follows from Chebyshev’s and Hølder’s inequalities that

$$
\mu(\Lambda_2) \leq \frac{16}{\alpha^4 n^4} \int_{-T}^{T} |s(t)|^4 dt \leq \frac{1}{\alpha^4 n^4} \int_{|x|>\delta} dP \int_{-T}^{T} \left| \sum_{j=[(1-\lambda r)n]}^{n} \cos(A_{kj} tx) \right|^4 dt.
$$

Put

$$
S(x) = \int_{-T}^{T} \left| \sum_{j=[(1-\lambda r)n]}^{n} \cos(A_{kj} tx) \right|^4 dt
$$

and assume, for simplicity, that $r = 1$, i.e., $\lambda_r = \lambda/2$, $\sum = \sum'$ and the summation is taken over all $j$. The general case is considered in a similar way.

We have

$$
S(x) = \int_{-T}^{T} \left( \sum_{j_1} \cos^4(A_{k_{j_1}} tx) + \sum_{j_1 \neq j_2} \cos^3(A_{k_{j_1}} tx) \cos(A_{k_{j_2}} tx) \right. \\
+ \sum_{j_1 \neq j_2} \cos^2(A_{k_{j_1}} tx) \cos^2(A_{k_{j_2}} tx) \right. \\
+ \sum_{j_1 \neq j_2 \neq j_3} \cos(A_{k_{j_1}} tx) \cos(A_{k_{j_2}} tx) \cos(A_{k_{j_3}} tx) \\
+ \sum_{j_1 \neq j_2 \neq j_3 \neq j_4} \cos(A_{k_{j_1}} tx) \cos(A_{k_{j_2}} tx) \cos(A_{k_{j_3}} tx) \cos(A_{k_{j_4}} tx) \left. \right) dt.
$$

The first three summands in (18) are easily estimated as follows:

$$
\left| \int_{-T}^{T} \left( \sum_{j_1} \cos^4(A_{k_{j_1}} tx) + \sum_{j_1 \neq j_2} \cos^3(A_{k_{j_1}} tx) \cos(A_{k_{j_2}} tx) \\
+ \sum_{j_1 \neq j_2} \cos^2(A_{k_{j_1}} tx) \cos^2(A_{k_{j_2}} tx) \right) dt \right| \ll Tn^2.
$$

The next two summands have a common method of estimation. We consider only the last one. From the formula $\cos y = (e^{iy} + e^{-iy})/2$ it is easily shown that

$$
\left| \int_{-T}^{T} \sum_{j_1 \neq j_2 \neq j_3 \neq j_4} \cos(A_{k_{j_1}} tx) \cos(A_{k_{j_2}} tx) \cos(A_{k_{j_3}} tx) \cos(A_{k_{j_4}} tx) dt \right| \\
\ll \sum_{j_1 \neq j_2 \neq j_3 \neq j_4} \min \left( T, |x|^{-1} \pm A_{k_{j_1}} \pm A_{k_{j_2}} \pm A_{k_{j_3}} \pm A_{k_{j_4}}^{-1} \right) \right)
$$

The summation in the middle term is taken over all possible combinations of signs.
Consider the partition of the index set \( \{ j = (j_1, j_2, j_3, j_4) : j_1 > j_2 > j_3 > j_4 \} = K_1 \cup K_2 \), where

\[
K_1 = \left\{ j : j_1 - j_2 \leq \frac{10}{\lambda}, j_1 - j_3 \leq \frac{10}{\lambda} \ln \lambda \right\}
\]

and \( K_2 \) is the complement of \( K_1 \). Clearly, \(|K_1| \ll n^2 |\ln \lambda|/\lambda^2\). Therefore,

\[
\sum_{j \in K_1} \min \left( T, |x|^{-1} A_{k_1}^{-1} \right) \leq T n^2 |\ln \lambda|/\lambda^2.
\]

Consider now

\[
\sum_{j \in K_2} A_{k_1}^{-1} \left| 1 - \frac{A_{k_j}}{A_{k_1}} - \frac{A_{k_{j_3}}}{A_{k_1}} - \frac{A_{k_{j_4}}}{A_{k_1}} \right|^{-1}.
\]

Putting \( p = j_1 - j_2 \), we have

\[
\frac{A_{k_{j_2}}}{A_{k_1}} = \frac{(j_1 - p) \cdots (j_1 - p - k + 1)}{j_1 \cdots (j_1 - k + 1)} = \left( 1 - \frac{p}{j_1} \right) \cdots \left( 1 - \frac{p}{j_1 - k + 1} \right) \leq \exp \left\{ -p \sum_{t=j_1-k+1}^{j_1} 1/t \right\}.
\]

Since for any natural \( l \)

\[
\frac{1}{l} > \ln \left( 1 + \frac{1}{l} \right) = \ln(l + 1) - \ln l,
\]

we get

\[
\sum_{t=j_1-k+1}^{j_1} \frac{1}{t} > \ln(j_1 + 1) - \ln(j_1 - k + 1) = -\ln \left( 1 - \frac{k}{j_1 + 1} \right).
\]

Taking into account \( \lambda n \leq k \leq (1 - \lambda)n \) and \((1 - \lambda/2)n \leq j_1 \leq n\) and using the inequality

\[-\ln(1 - t) \geq t, \quad t \in [0,1],
\]

we get

\[
\sum_{t=j_1-k+1}^{j_1} \frac{1}{t} \geq \frac{\lambda n}{n + 1} \geq \frac{1}{2} \lambda.
\]

Therefore,

\[
\frac{A_{k_{j_2}}}{A_{k_1}} \leq \exp \left\{ -\frac{\lambda}{2} p \right\} = \exp \left\{ -\frac{\lambda}{2} (j_1 - j_2) \right\}.
\]

If \( j \in K_2 \) and \( j_1 - j_2 > 10/\lambda \), then

\[
\frac{A_{k_{j_2}}}{A_{k_1}} \leq \frac{A_{k_{j_3}}}{A_{k_1}} \leq \frac{A_{k_{j_4}}}{A_{k_1}} \leq e^{-5} < \frac{1}{4},
\]

which implies

\[
1 - \frac{A_{k_{j_2}}}{A_{k_1}} - \frac{A_{k_{j_3}}}{A_{k_1}} - \frac{A_{k_{j_4}}}{A_{k_1}} \geq \frac{1}{4}.
\]
Suppose now \( j \in K_2 \) and \( j_1 - j_3 > 10 |\ln \lambda|/\lambda \). Using (22) and \( \lambda \in (0, 1/2) \), we get
\[
1 - \frac{A_{kj}}{\lambda_{kj}} \geq 1 - e^{-\lambda/2} \geq \frac{\lambda}{2} \left( 1 - \frac{\lambda}{4} \right) \geq \frac{7}{16} \lambda.
\]
Further, (22) also holds for \( j_3 \). Therefore,
\[
\frac{A_{kj}}{\lambda_{kj}} \leq \frac{A_{kj}}{\lambda_{kj}} \leq \exp \left\{ -\frac{\lambda}{2} (j_1 - j_3) \right\} \leq \exp \left\{ -\frac{10}{2} \ln \lambda \right\} \leq \lambda^5 \leq \frac{1}{16} \lambda.
\]
Thus,
\[
(24) \quad 1 - \frac{A_{kj}}{\lambda_{kj}} - \frac{A_{kj}}{\lambda_{kj}} - \frac{A_{kj}}{\lambda_{kj}} \geq \frac{5}{16} \lambda.
\]
It follows from (23) and (24) that
\[
\sum_{j \in K_2} A_{kj}^{-1} \left| 1 - \frac{A_{kj}}{\lambda_{kj}} - \frac{A_{kj}}{\lambda_{kj}} - \frac{A_{kj}}{\lambda_{kj}} \right|^{-1} \ll \frac{(\lambda n)^4}{A_{k,[(1-\lambda/2)n]}}.
\]
Taking into account the structure of the index set \( \{ j \} \), we have
\[
\sum_{j \in K_2} A_{kj}^{-1} \ll \frac{(\lambda n)^4}{A_{k,[(1-\lambda/2)n]}}.
\]
consequently,
\[
(25) \quad \sum_{j \in K_2} A_{kj}^{-1} \left| 1 - \frac{A_{kj}}{\lambda_{kj}} - \frac{A_{kj}}{\lambda_{kj}} - \frac{A_{kj}}{\lambda_{kj}} \right|^{-1} \ll \frac{\lambda^4 n}{A_{k,[(1-\lambda/2)n]}}.
\]
Combining (18), (19), (20), (21) and (25), we obtain
\[
S(x) \ll T n^2 + T n^2 |\ln \lambda| + \frac{\lambda^3 n^4}{|x| A_{k,[(1-\lambda/2)n]}}.
\]
Applying this to (17), we get
\[
\mu(\lambda_2) \ll \frac{T}{\alpha^4 n^2} + \frac{T |\ln \lambda|}{\lambda^2 \alpha^4 n^2} + \frac{\lambda^3}{\alpha^4 \delta A_{k,[(1-\lambda/2)n]}}.
\]
By (16),
\[
E \tau_k \ll e^{-\alpha n/2} + \frac{1}{\alpha^4 n^2} + \frac{|\ln \lambda|}{\lambda^2 \alpha^4 n^2} + \frac{\lambda^3}{T \alpha^4 \delta A_{k,[(1-\lambda/2)n]}}.
\]
Recalling that \( T = \lambda/(KA_{k,[(1-\lambda)n]}) \), we obtain
\[
E \tau_k \ll e^{-\alpha n/2} + \frac{1}{\alpha^4 n^2} + \frac{|\ln \lambda|}{\lambda^2 \alpha^4 n^2} + \frac{\lambda^2 K A_{k,[(1-\lambda)n]}}{\alpha^4 \delta A_{k,[(1-\lambda/2)n]}}.
\]
It follows from (22) that
\[
\frac{A_{k,[(1-\lambda)n]}}{A_{k,[(1-\lambda/2)n]}} \leq e^{-\lambda^2 n/4}.
\]
Thus,
\[
E \tau_k \ll e^{-\alpha n/2} + \frac{1}{\alpha^4 n^2} + \frac{|\ln \lambda|}{\lambda^2 \alpha^4 n^2} + \frac{\lambda^2 K A_{k,[(1-\lambda)n]}}{\alpha^4 \delta e^{-\lambda^2 n/4}}.
\]
Recalling that \( \alpha = \gamma \lambda/4 \), we obtain
\[
E \tau_k \ll e^{-\gamma \lambda n/8} + \frac{1}{\gamma^4 \lambda^4 n^2} + \frac{|\ln \lambda|}{\gamma^4 \lambda^4 n^2} + \frac{K}{\gamma^4 \lambda^2 \delta} e^{-\lambda^2 n/4}.
\]
Since $K$ is defined by $\varepsilon$ and $\gamma, \delta$ are defined by the distributions of $\{\eta_j\}$, Lemma 3 is proved.

Now we are ready to complete the proof of Theorem 5. It follows from (10) that

$$M_n(1, \infty) \leq 2 \lambda n + 2 + \sum_{j=0}^{n} \chi_j + 2 \sum_{j=\lfloor \lambda n \rfloor}^{\lfloor (1-\lambda)n \rfloor} \tau_j.$$  

By Lemma 4 and Chebyshev’s inequality,

$$P \left\{ \sum_{k=\lfloor \lambda n \rfloor}^{\lfloor (1-\lambda)n \rfloor} \tau_k > n^{3/4} \right\} \leq \frac{\sum_{j=\lfloor \lambda n \rfloor}^{\lfloor (1-\lambda)n \rfloor} E \tau_k}{n^{3/4}} \leq c_1 n^{-5/4}.$$  

Further, it follows from (14) that there exists a constant $c_2 > 0$ depending only on $\varepsilon$ such that

$$P \left\{ \sum_{j=0}^{n} \chi_j > 2\varepsilon n \right\} \leq c_2 n^{-2}.$$  

Combining (26), (27) and (28), we get

$$P \left\{ M_n(1, \infty) > 2 \lambda n + 2 + 2n^{3/4} + 2 \varepsilon n \right\} \leq c_1 n^{-5/4} + c_2 n^{-2}.$$  

Considering the random polynomials $g(1/x)$ and $g(-x)$, it is possible to obtain similar estimates for $M_n(0, 1)$ and $M_n(-\infty, 0)$. Thus there exist positive constants $c'_1, c'_2$ such that

$$P \left\{ M_n > 2 \lambda n + 2 + 2n^{3/4} + 2 \varepsilon n \right\} \leq c'_1 n^{-5/4} + c'_2 n^{-2}.$$  

According to the Borel-Cantelli lemma, with probability one there exists only a finite number of $n$ such that $M_n > 2 \lambda n + 2 + 2n^{3/4} + 2 \varepsilon n$. Since $\lambda, \varepsilon$ are arbitrary small,

$$P \left\{ \frac{M_n}{n} \to 0 \right\} = 1.$$  

Theorem 5 is proved.

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