On and off-diagonal Sturmian operator: dynamic and spectral dimension

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Abstract

We study two versions of quasicrystal model, both subcases of Jacobi matrices. For off-diagonal model, we show an upper bound of dynamical exponent and the norm of the transfer matrix. We apply this result to the off-diagonal Fibonacci Hamiltonian and obtain a sub-ballistic bound for coupling large enough. In diagonal case, we improve previous lower bounds on the fractal box-counting dimension of the spectrum.

1 Sturmian on and off-diagonal models

On and off-diagonal models are special cases of the Jacobi operators. Given two real-valued sequences $a$ and $b$, a Jacobi operator $H$ acts on $\ell^2(\mathbb{Z})$ in the following way

$$H\psi(n) = a(n)\psi(n+1) + a(n-1)\psi(n-1) + b(n)\psi(n), \quad n \in \mathbb{Z}. \quad (1)$$

$H$ is associated to a self-adjoint tridiagonal matrix with diagonal entries filled with values of $b$ and off-diagonal entries filled with values of $a$.

Let $\beta \in [0, 1]$ be an irrational number, on and off-diagonal Sturmian models will be defined respectively by setting in (1)

$$a(n) = 1, \quad b(n) = \lambda_1\left([\lfloor(n+1)\beta\rfloor] - [n\beta]\right) \quad (2)$$

respectively

$$a(n) = (\lambda_1 - \lambda_2)\left([\lfloor(n+1)\beta\rfloor] - [n\beta]\right) + \lambda_2, \quad b(n) = 0, \quad n \in \mathbb{Z} \quad (3)$$

where $\lambda_1, \lambda_2 \in \mathbb{R}^+$ and for $x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes the largest integer smaller than $|x|$. The term Sturmian refers to quasiperiodicity of sequences $b$ in (2) and $a$ in (3). On-diagonal model is more usually named discrete Schrödinger operator and when $\beta$ is the golden mean $\frac{\sqrt{5}-1}{2}$, the Sturmian model is oftenly called the Fibonacci Hamiltonian.

The study of those models was initiated with the introduction of the Fibonacci operator in the early 1980’s by Kohmoto et al. [KKT] and Ostlund et al. [OPRSS]. At that point in
time, the interest in this model was based mainly on the existence of an exact renormalization group procedure, the appearance of critical eigenstates and zero measure Cantor spectrum. Only shortly thereafter, Shechtman et al. [SBGC] reported their discovery of structures, now called quasicrystals, whose are shown to have aperiodic structure. Sturmian sequences are central models of a quasicrystal in one dimension. Indeed, it is aperiodic moreover, it belongs to virtually all classes of mathematical models of quasicrystals that have since been proposed. We refer to the reader to [BM] for a recent account of the mathematics related to the modelling and study of quasicrystals. Thus, the study of Sturmian operators was further motivated by the interest in electronic spectra and transport properties of one-dimensional quasicrystals. It is shown that those operators give rise to anomalous transport for a large class of irrational number [DT1, M1, M2]. Moreover, it exhibits a number of interesting phenomena, such as Cantor spectrum of Lebesgue measure zero [S1, S3, BIST] and purely singular continuous spectral measure [DL, DKL, BIST]. Consequently, apart from the almost Mathieu operator, the Fibonacci operator has been the most heavily studied quasi-periodic operator in the last three decades; compare the survey articles [D1, D2, S2]. Partly due to the choice of the model in the foundational papers [KKT, OPRSS] the mathematical literature on the Fibonacci operator has so far only considered the diagonal model. Given the connection to quasicrystals and hence aperiodic point sets, and particularly cut-and-project sets, it is however equally (if not more) natural to study the off-diagonal Sturmian model. We do refer the reader to [BM] for background. For further motivation to study also the off-diagonal model, we mention that it has been the object of interest in a number of physics and then mathematics recent papers [EL, EL2, Dah, DG, KST, VP].

In this paper, we are interested in transport properties and in fractal dimension of the spectrum. More precisely, the reader will find in the following:

For off-diagonal Sturmian model,

- We show pseudo spectrum and spectrum are equal (Theorem 1).
- We established a link between outside probabilities and transfer matrices in both time-averaged (Theorem 2) and non time-averaged settings (Theorem 3).
- In the Fibonacci case, we deduce from previous item a dynamical upper bound for the wavepacket spreading (Theorem 4). This bound is sub-ballistic for hopping constants $\lambda_1, \lambda_2$ well choosen.

For on-diagonal Sturmian model,

- We improve previous lower bound for the box-counting dimension of the spectrum (Theorem 5). This bound is valid for a large class of irrational number verifying a Lebesgue measure 1 diophantite condition (Theorem 6).
- Considering a smaller class of irrational number, but still Lebesgue measure 1, we are able to improve our previous lower bound (Theorem 6).
2 The trace map application

Denote the continued fraction expansion of $\beta$ by

$$\beta = [a_1, a_2, \ldots] = \frac{1}{a_1 + \frac{1}{a_2 + \ldots}}$$

and define the best rational approximants,

$$p_{k+1} = a_{k+1}p_k + p_{k-1}, \quad p_0 = 1, \quad p_{-1} = 0,$$

$$q_{k+1} = a_{k+1}q_k + q_{k-1}, \quad q_0 = 0, \quad q_{-1} = 1.$$

A classic tool to investigate one-dimensional model is to write to free equation

$$H\psi(n) = E\psi(n) \quad (4)$$

where $E$ is a real or a complex number. Equation (4) can be rewritten in both on and off-diagonal case respectively as

$$\Psi^b(n) = \begin{pmatrix} \psi(n + 1) \\ \psi(n) \end{pmatrix} = \begin{pmatrix} E - b(n) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi(n) \\ \psi(n - 1) \end{pmatrix}$$

$$\Psi^a(n) = \begin{pmatrix} \psi(n + 1) \\ \psi(n) \end{pmatrix} = \frac{1}{a(n)} \begin{pmatrix} E & -1 \\ a(n)^2 & 0 \end{pmatrix} \begin{pmatrix} \psi(n) \\ a(n - 1)\psi(n - 1) \end{pmatrix}$$

Denote $T^b_n(E) = \begin{pmatrix} E - b(n) & -1 \\ 1 & 0 \end{pmatrix}$ and $T^a_n(E) = \frac{1}{a(n)} \begin{pmatrix} E & -1 \\ a(n)^2 & 0 \end{pmatrix}$. In all the following, objects with exponent $a$ are associated to the off-diagonal model, while those with exponent $b$ are associated to diagonal case. When there is no ambiguity, we will drop it. Then for $i = a, b$ and with $F^i_n(E) = T^i_n(E) \ldots T^i_1(E)$, one constructs recursively a solution $\psi^i$ of the equation (4) with

$$\Psi^i(n) = F^i_n(E)\Psi^i(0).$$

We define the so-called transfer matrices, denoted by $M^i_n(E) = F^i_n(E)$. One important objet is the traces of transfer matrices, denote $x^i_n = \text{Tr}M^i_n$ and $z^i_n = \text{Tr}(M^i_{n-1}M^i_n)$. In the following, we will drop the index $i$ whenever it is possible for sake of simplicity.

The transfer matrix $k$–evolution follows the simple rule (see e.g. [BIST, IR T, R])

$$M_k = M_{k-2}M^a_{k-1}, \quad k \geq 1. \quad (5)$$

We can extend this relation to $k = 0$ by setting $M^a_{-1} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ and $M^b_{-1} = \begin{pmatrix} 1 & -\lambda_1 \\ 0 & 1 \end{pmatrix}$.

The evolution of these sequences is derived from taking the trace in (5) and by Cayley-Hamilton Theorem (see [BIST] for details); for $k \geq 0$

$$x_{k+1} = z_kS_{a_{k+1}-1}(x_k) - x_{k-1}S_{a_{k+1}-2}(x_k), \quad (6)$$

$$z_{k+1} = z_kS_{a_{k+1}}(x_k) - x_{k-1}S_{a_{k+1}-1}(x_k), \quad (7)$$
where the \( S_i \) are Chebyshev second order polynomials
\[
S_{l+1}(x) = xS_l(x) - S_{l-1}(x), \quad \forall l \geq 0, \quad S_0(x) = 1, \quad S_{-1}(x) = 0.
\]

The initial conditions of these two sequences are given by, (see [DG, Dah, BIST])
\[
x_{-1}^a = \frac{\lambda_1^2 + \lambda_2^2}{\lambda_1 \lambda_2}, \quad x_0^a = \frac{E}{\lambda_2}, \quad z_0^a = \frac{E}{\lambda_1},
\]
\[
x_{-1}^b = 2, \quad x_0^b = E, \quad z_0^b = E - \lambda_1.
\]

Sequences \( x^i \) and \( z^i \) verify the Fricke-Vogt invariant, namely for \( k \geq 0 \) and \( E \), one has
\[
(x_k^i)^2 + (x_{k+1}^i)^2 + (z_{k+1}^i)^2 - x_k^i x_{k+1}^i z_{k+1}^i = (c^i)^2 + 4
\]
with \( c^a = \frac{(\lambda_1 - \lambda_2)(\lambda_1 + \lambda_2)}{\lambda_1 \lambda_2} \) and \( c^b = \lambda_1 \).

3 Periodic approximants for off-diagonal Sturmian model

We define the following sequence of sets
\[
\sigma_{n,k} = \{ E \in \mathbb{R} : |\text{Tr}M_{n-1}M_n^k| \leq 2 \}.
\]

In case \( k = 0 \), those sets are called the periodic approximants of \( H \) as \( \sigma_{n,0} \) is the spectrum of a periodic operator defined as in (1) with a \( q_n \)-periodic sequence \( a \), see [SI, BIST].

The pseudo spectrum is defined as \( B_\infty = \{ E \in \mathbb{R} : \{ x_n(E) \}_n \text{ is bounded} \} \) and denote \( N_0(E) \) the first index such that \( |x_{N_0}(E)| \leq 2 \).

**Lemma 1** If \( |x_0| > 2 \) and \( |z_0| > 2 \), then the two sequences \( \{ x_n, z_n \}_n \) are unbounded.

**Proof.** Let us first check the following induction, suppose that at step \( n \), we have \( |z_n| > 2, \quad |x_n| > 2 \) and \( |z_n| > |x_{n-1}| \). Then, (7) implies that for some integer \( l > 0 \),
\[
|z_{n+1}| = |z_n S_l(x_n) - x_{n-1} S_{l-1}(x_n)| \\
\geq |z_n| |S_l(x_n) - S_{l-1}(x_n)| \\
\geq |z_n| (|x_n| - 1)^l.
\]
Since \( |x_n|, |z_n| > 2 \), one has \( |z_{n+1}| > |z_n| \) and \( |z_{n+1}| > |x_n| \). From (6), the same argument gives \( |x_{n+1}| > |z_n| \) completing the induction. Clearly if the induction is fulfilled, it implies that both sequences are increasing and taking logarithme in (11) show they are unbounded.

We check now the initial condition of the induction. Let first assume that \( \lambda_1 < \lambda_2 \). Then \( |x_0 = \frac{E}{\lambda_2}| > 2 \) implies that \( |z_0 = \frac{E}{\lambda_1}| > \frac{2\lambda_1}{\lambda_2} > \frac{\lambda_2}{\lambda_2} + \frac{\lambda_1}{\lambda_2} = x_{-1} \). Now suppose that \( \lambda_2 < \lambda_1 \), we argue that \( |z_1| > |x_0| \) and \( |z_1|, |x_1| > 2 \). First if \( a_1 = 1 \), then
\[
z_1 = z_0 x_0 - x_{-1}
\]
and since \(|x_0| > |x_{-1}|\) and \(|z_0| > 2\), we have that \(|z_1| > |x_0|\). Moreover, in that case \(x_1 = z_0\) completing the proof. Then suppose that \(a_1 = l > 1\). The assumption \(\lambda_2 < \lambda_1\) with \(|x_0|, |z_0| > 2\) implies that \(|E| > 2\lambda_1\). Then there exists a positive constant \(\varepsilon\) such that \(E = \pm (2\lambda_1 + \varepsilon)\),

\[
|z_1| = |(z_0 x_0 - x_{-1}) S_{l-1}(x_0) - z_0 S_{l-2}(x_0)|
\]

\[
= \left| \left( \frac{E^2}{\lambda_1 \lambda_2} - \frac{\lambda_1^2 + \lambda_2^2}{\lambda_1 \lambda_2} \right) S_{l-1}(x_0) - \frac{E}{\lambda_1} S_{l-2}(x_0) \right|
\]

\[
= \left| \left( \frac{3\lambda_1}{\lambda_2} - \frac{\lambda_2}{\lambda_1} + \frac{\varepsilon^2}{\lambda_1 \lambda_2} + \frac{4\varepsilon}{\lambda_2} \right) S_{l-1}(x_0) \mp \left( 2 + \frac{\varepsilon}{\lambda_1} \right) S_{l-2}(x_0) \right|
\]

\[
\geq \left( \frac{2\lambda_1}{\lambda_2} + \frac{\varepsilon^2}{\lambda_1 \lambda_2} + \frac{4\varepsilon}{\lambda_2} \right) |S_{l-1}(x_0) - S_{l-2}(x_0)|
\]

\[
\geq \left( \frac{2\lambda_1}{\lambda_2} + c\varepsilon \right) (|x_0| - 1)^{l-1}
\]

\[
\geq 2|x_0| - 2 > |x_0|.
\]

The same argument upon \(x_1\) implies \(|x_1| > 2\lambda_1 + \varepsilon > 2\) completing the proof. \(\square\)

Lemma 2 Let \(x_n\) and \(z_n\) be the trace map sequence define in (6) and (7), \(\delta > 0\). If there exists an integer \(N\) such that

\[
|x_{N-1}| \leq 2 + \delta, \quad |x_N| > 2 + \delta \quad \text{and} \quad |z_N| > 2 + \delta,
\]

then modulus of the two sequences are superexponentially increasing

\[
|x_{k+1}| \geq |z_k| \geq e^{c(\delta)G_{k-N}} + 1, \quad \forall k > N,
\]

with

\[
G_k = G_{k-1} + a_k G_{k-2}, \quad G_0 = 1, \quad G_{-1} = 1.
\]

Moreover,

\[
B_\infty = \bigcap_{n=N}^{\infty} (\sigma_{n,0} \cup \sigma_{n+1,0}), \text{ any } N \geq N_0.
\]

Proof. First part of the Lemma follows from (6)-(7) (see [M2, M1] for a detailed Proof). The occurrence of \(\delta\) plays no role in the result and will be needed for technical reason in section 5. The condition upon \(z_N\) could also be replaced by \(|x_{N+1}| > 2\), see [BIST] for this proof version.

For the second part, it remains to show is that for \(E \in B_\infty\), \(N_0(E)\) is finite, then the result follows from the first part of the lemma and identical argument that in the on-diagonal case (see [S1, BIST]). Suppose that \(x_n\) is bounded and greater than two in modulus. This implying at any rank \(n\), that \(2 < |x_{n+1}| \leq |x_{n-1}|\). Elsewhere, replacing the hypothesis \(|x_{n-1}| \leq 2 < |x_{n+1}|\) by the weaker one \(|x_{n-1}| < |x_{n+1}|\) in Proposition 4 in [BIST], one can show that the sequence \(x_n\) is growing exponentially. Thus each subsequence on even and odd index has to be decreasing and bounded from below by 2, and thus has a limit \(l_1\). Then the Fricke-Vogt invariant implies the sequence \(z_n\) also has a limit denoted \(l_2\). Therefore the point \((l_2, l_1, l_1)\) is a fixed point of the trace map and the set of equation coming from \(T(l_2, l_1, l_1) = (l_2, l_1, l_1)\) should be verified. This is easily shown to be impossible using the fact that \(|S_\alpha(l_1)| \geq 2\) since \(|l_1| \geq 2\). \(\square\)
Theorem 1 For any $\lambda_1, \lambda_2 > 0$, $\sigma(H)$ the spectrum of $H$ and $B_\infty$ coincide.

Proof. Using Lemma 2 one can argue the same as in [S1], replacing $N \geq 0$ by $N \geq N_0$ anytime it appears. \hfill $\Box$

4 Dynamical upper bound for off-diagonal model

In this section, we are mostly interested in dynamical properties of the wavepacket. It is known for this type of aperiodic model that the unitary group $\psi(t) = e^{-itH} \psi(0)$ will spread with time $t$. Here, $\psi(0)$ is some well localized initial condition, for example, $\delta_1$ the Dirac function on the first site of $\ell^2(\mathbb{Z})$. We want to quantify this spreading, thus we recall one usual way to measure it.

Denote by $a(n, t) = |\langle e^{-itH} \delta_1, \delta_n \rangle|^2$ the probability for the system to be at the $n$th site at time $t$. Here $\{\delta_i\}_{i \in \mathbb{Z}}$ is the canonical basis of $\ell^2(\mathbb{Z})$.

We denote the outside probabilities by $P(N, t) = \sum_{|n| > N} a(n, t)$, and $P_r(N, t) = \sum_{n > N} a(n, t)$, $P_l(N, t) = \sum_{n < -N} a(n, t)$.

Namely, $P(N, t)$ is the probability to be outside the ball of size $N$ at time $t$.

For all $\alpha \in [0, +\infty)$, as it is done in [GKT], define

$$S^-(\alpha) = -\lim_{t \to \infty} \inf \frac{\ln P(t^\alpha -2, t)}{\ln t}, \quad S^+(\alpha) = -\lim_{t \to \infty} \sup \frac{\ln P(t^\alpha -2, t)}{\ln t}.$$

The following critical exponents are particular of interest:

$$\alpha^+_l = \sup\{\alpha \geq 0 : S^+(-\alpha) = 0\}, \quad \alpha^+_u = \sup\{\alpha \geq 0 : S^+(\alpha) < \infty\}.$$

They verify $0 \leq \alpha^+_l \leq \alpha^+_u$. In particular, if $\gamma > \alpha^+_u$ then $P(t^\gamma, t)$ goes to 0 faster than polynomially. $\alpha^+_l$ can be interpreted as the (lower and upper) rates of propagation of the essential part of $\psi$ and $\alpha^+_u$ as the rates of propagation of the fastest part of $\psi$ (see [GKT]).

It is also convenient sometimes to consider these definitions in average in time. We define the time-averaged probability $\langle a(n, T) \rangle = \frac{2}{T} \int_0^T e^{-2t/T} a(n, t)dt$. Replacing then $a(n, t)$ by $\langle a(n, T) \rangle$ we can then define its time-averaged outside probabilities $\langle P(N, T) \rangle$ and all the exponents above, denoted with a tilde.

Now main notations are set up, we extent the link between (time-averaged or not) outside probabilities and transfer matrices to off-diagonal models and show the following:

Theorem 2 Suppose $H$ defined in (1) with $a, b$ defined in (3), and let $K \geq 4$ be such that $\sigma(H) \subset [-K + 1, K - 1]$. Then the time-averaged outside probabilities can be bounded from above in terms of transfer matrix norms as follows:

$$\langle P(N, T) \rangle \lesssim \exp(-cN) + T^3 \int_{-K}^K \left( \max_{1 \leq n \leq N} \|F(n, E + \frac{1}{T})\|^2 \right)^{-1} dE.$$
The proof starting point, as in diagonal case (see [DT1]), is Parseval formula,

\[
\langle a(n, T) \rangle = \frac{1}{T \pi} \int_{-\infty}^{\infty} |\langle (H - E - \frac{i}{T})^{-1} \delta_1, \delta_n \rangle|^2 dE.
\]

Denote \( \varepsilon = \frac{1}{T} \) and \( R(z) = (H - zI)^{-1} \) for \( z \in \mathbb{C} \setminus \mathbb{R} \). Let us assume that \( T > 1 \) and therefore \( 0 < \varepsilon < 1 \). Then, we have

\[
\langle P_r(N, T) \rangle = \frac{\varepsilon}{\pi} \int_{-\infty}^{\infty} M_r(N, E + i\varepsilon) dE,
\]

where

\[
M_r(N, z) = \sum_{n>N} |\langle R(z)\delta_1, \delta_n \rangle|^2 = \|\chi_N R(z)\delta_1\|^2,
\]

and \( \chi_N(n) = 0, n \leq N \), \( \chi_N(n) = 1, n \geq N + 1 \). We need to bound \( M_r(N, z) \) from above. If the energy \( E \) is outside the spectrum of \( H \), one can use Combes-Thomas estimate (see e.g. [CT]). Assume that \( \eta = \text{dist}(E + i\varepsilon, \sigma(H)) \geq 1 \), then we have,

\[
|\langle R(z)\delta_1, \delta_n \rangle| \leq \frac{2}{\eta} \exp(-d \min\{d \eta, 1\}|n - 1|)
\]

with some universal positive constant \( d \). Using this estimate, one can see easily that

\[
\int_{|E| \geq K} M_r(N, E + i\varepsilon) dE \leq C(K) \exp(-dN), d > 0,
\]

and thus the limit goes to 0 for any \( N(T) = T^\alpha, \alpha > 0 \). The remaining problem is to estimate the other part of the integral where \( E + i\varepsilon \) may be \( \varepsilon \)–close to the spectrum of \( H \)

\[
L_r(N, \varepsilon) = \int_{-K}^{K} M_r(N, E + i\varepsilon) dE.
\]

We link \( M_r(N, z) \) to the complex solutions to the stationary equation \( H\psi = z\psi \). First consider the truncated operator \( H_N \), namely

\[
H_N\psi(n) = w(n + 1)\psi(n + 1) + w(n)\psi(n - 1)
\]

(15)

where

\[
w(n) = a(n), n \leq N, \quad w(n) = \lambda_2, n > N.
\]

Denote by \( R_N(z) \) the resolvent of the operator \( H_N \) and define

\[
S(N, z) = \|\chi_N R_N(z)\delta_1\|^2.
\]

**Lemma 3** For any \( E \in [-K, K] \), \( 0 < \varepsilon < 1 \), we have

\[
M_r(N, E + i\varepsilon) \lesssim \varepsilon^{-2} S(N, E + i\varepsilon),
\]

where the implicit constant depends only on \( \lambda_1, \lambda_2 \).
**Proof.** Using the resolvent identity,

\[ R(z)\delta_1 - R_N(z)\delta_1 = R(z)\chi_N(H - \lambda_2\Delta)R_N(z)\delta_1 = R(z)\chi_N(dT^{\pm 1})R_N(z)\delta_1. \]

where \( d \) is a constant, namely \( \lambda_1 - \lambda_2 = 0 \) and \( T \) is the shift operator, \( T\psi(n) = \psi(n+1) \) acting on \( \ell^2(\mathbb{Z}) \).

Since \( \|R(z)\| \leq \varepsilon^{-1} \), \( \varepsilon < 1 \), and the sequence \( a(n) \) is bounded, we get

\[
M_r(N,z) \leq 2S(N,z) + 2\|R(z)(\chi_N H - \lambda_2\Delta)R_N(z)\delta_1\|^2 \\
\leq 2S(N,z) + 2\varepsilon^{-2}\|\chi_N(H - \lambda_2\Delta)R_N(z)\delta_1\|^2 \\
\leq 2S(N,z) + 2\varepsilon^{-2}\|\chi_N(dT^{\pm 1})R_N(z)\delta_1\|^2 \\
\leq C(d)\varepsilon^{-2}S(N,z). \]

The next step is to link the quantity \( S(N,z) \) with the solutions to the stationary equation. For any complex \( z \), define \( u_0(n,z) \) as a solution to \( Hu_0 = zu_0, \ u_0(0,z) = 0, \ u_0(1,z) = 1 \), and define \( u_1(n,z) \) as a solution to \( Hu_1 = zu_1, \ u_1(0,z) = 1, \ u_1(1,z) = 0 \). Since \( H \) and \( H_N \) coincide for all \( n \leq N \), their solutions \( u_0, u_1 \) coincide for all \( n \leq N + 1 \). We will consider \( u_0, u_1 \) only for such \( n \), thus we will use the same notation for \( u_0, u_1 \) for \( H \) and \( H_N \).

Now consider the free equation upon \( H_N \)

\[ \lambda_2u(n-1) + \lambda_2u(n+1) = zu(n), \quad n > N \]

and define

\[ s_{1,2} = \frac{z \pm \sqrt{z^2 - 4\lambda_2^2}}{2\lambda_2} \]

the roots of the characteristic polynomial of the sequence \( u(n) \). Since \( s_1s_2 = 1 \), one has \( |s_1| > 1 \) and \( |s_2| < 1 \). Therefore \( s_1, s_2 \) are the eigenvalues of equation (16) whose general solution is

\[ u(n) = c_1s_1^n + c_2s_2^n. \]

**Lemma 4** Let \( z = E + i\varepsilon \), where \( E \in [-K, K], \ 0 < \varepsilon < 1 \). Then, for \( n \geq N \geq 3 \), we have

\[
|\langle R_N(z)\delta_1, \delta_n \rangle| \leq 2\varepsilon^{-1} \frac{|s_2(z)|^{n-N}}{|s_2(z)u_0(N,z) - u_0(N+1,z)|}, \quad (17)
\]

and

\[
|\langle R_N(z)\delta_1, \delta_n \rangle| \leq \varepsilon^{-1} \frac{|s_2(z)|^{n-N}}{|s_2(z)u_1(N,z) - u_1(N+1,z)|}. \quad (18)
\]

**Proof.** The following formula holds for any off-diagonal model, (see [KKL]):

\[
\langle R(z)\delta_1, \delta_n \rangle = d(z)u_0(n,z) + b(z)u_1(n,z), \ n \geq 1, \quad (19)
\]

\[
\langle R(z)\delta_1, \delta_n \rangle = d(z)u_0(n,z) + c(z)u_1(n,z), \ n < 1, \quad (20)
\]

where

\[
b(z) = \frac{m_-(z)}{b(m_+(z) + m_-)}, \quad c(z) = \frac{-m_+(z)}{b(m_+(z) + m_-)}, \quad d(z) = \frac{-m_+(z)m_-(z)}{b(m_+(z) + m_-)}. \quad (21)
\]
with some complex functions \( m_+(z), m_-(z) \), called the \( m \)-functions which depend on the sequences \( a \) and \( b \). Since \( \|R(z)\delta_1\| \leq \varepsilon^{-1} \) and

\[
d(z) = \langle R(z)\delta_1, \delta_1 \rangle, \quad c(z) = \langle R(z)\delta_1, \delta_0 \rangle,
\]

we get \( |d(z)| \leq \varepsilon^{-1} \), \( |c(z)| \leq \varepsilon^{-1} \). Since \( b(z) = 1 + c(z) \) and \( \varepsilon \leq 1 \), we also get \( |b(z)| \leq 2\varepsilon^{-1} \).

Summarizing,

\[
|d(z)| \leq \varepsilon^{-1}, \quad |c(z)| \leq \varepsilon^{-1}, \quad |b(z)| \leq 2\varepsilon^{-1}.
\]

(22)

One should stress that the bounds (22) hold for any operator and the constants are universal.

Consider the operators \( H_N \) and define \( \phi = R_N(z)\delta_1 \) (the vector \( \phi \) depends on \( N \), of course). Since

\[
(H_N - z)\phi = \delta_1,
\]

the function \( \phi(n) = \langle \phi, \delta_n \rangle \) obeys the equation \( H_N\phi(n) = z\phi(n) \), \( n \geq 2 \). Since \( w(n) = \lambda_1, n \geq N + 1, \phi(n) \) obeys the free equation (16) for \( n \geq N + 1 \). Hence,

\[
(\phi(N + k + 1), \phi(N + k)) = c_1s_1(z)^ke_1 + c_2s_2(z)^ke_2, \quad k \geq 0.
\]

(23)

Here \( e_{1,2} = (s_{1,2}(z), 1)^T \) are two eigenvectors of the matrix corresponding to the equation (16), and the constants \( c_1, c_2 \) are defined by

\[
(\phi(N + 1), \phi(N))^T = c_1e_1 + c_2e_2.
\]

Since \( |s_2(z)| < 1 \), \( |s_1(z)| > 1 \) and \( \phi \in \ell^2(\mathbb{Z}) \), the identity (23) implies that \( c_1 = 0 \), and thus

\[
(\phi(N + 1), \phi(N))^T = D_N(z)(s_2(z), 1)^T, \quad D_N(z) \neq 0.
\]

(24)

On the other hand, (19) implies

\[
\phi^{\pm}(N + 1) = d_N(z)u_0(N + 1, z) + b_N(z)u_1(N + 1, z),
\]

(25)

\[
\phi(N) = d_N(z)u_0(N, z) + b_N(z)u_1(N, z).
\]

(26)

Remark that \( d_N(z) \neq 0 \), since it is a Borel transform of the spectral measure corresponding to the vector \( \delta_1 \) [KKL]. The same is true for \( b_N(z), c_N(z) \) since \( m_+(z), m_-(z) \) are functions with positive imaginary part (it also follows from \( d_N(z) \neq 0 \) and the expressions of \( b_N, c_N \)). Using the fact that \( u_0(n + 1, z)u_1(n, z) - u_0(n, z)u_1(n + 1, z) = 1 \) for any \( n \) (in particular, for \( n = N \)), and (24)–(26), it is easy to calculate \( D_N(z) \):

\[
D_N(z) = \frac{d_N(z)}{s_2(z)u_1(N, z) - u_1(N + 1, z)} = \frac{b_N(z)}{s_2(z)u_0(N, z) - u_0(N + 1, z)}.
\]

(27)

As observed above, the solutions \( u_0, u_1 \) are the same for \( H, H_N \) if \( n \leq N + 1 \). It follows from (23), where \( c_1 = 0 \) and \( c_2 = D \), that

\[
\langle R_N(z)\delta_1, \delta_n \rangle = D_N(z)s_2(z)^n, \quad n \geq N.
\]

The result of the lemma follows now directly from (22) and (27).
Lemma 5 For $z = E + i\varepsilon$ with $E \in [-K, K]$ and $0 < \varepsilon \leq 1$, for $N \geq 3$, we have

$$M_r(N, z) \leq C(K)\varepsilon^{-4} \left( \max_{3 \leq n \leq N} \| \Phi(n, z) \|^2 \right)^{-1}.$$  

Proof. The second bound of Lemma 3 and the bound (17) of Lemma 4 yield

$$M_r(N, z) \leq A(K)\varepsilon^{-4}|s_2(z)u_0(N, z) - u_0(N + 1, z)|^{-2} \sum_{k=0}^{\infty} |s_2(z)|^{2k}$$

with uniform $B(K)$, since $|s_2(z)| < 1$. Then, one has

$$M_r(N, z) \leq C(K)\varepsilon^{-4} (|u_0(N, z)|^2 + |u_0(N + 1, z)|^2)^{-1},$$

using the fact that $u_0$ is exponentially decreasing in $N$, the product $u_0(N, z)u_0(N + 1, z)$ only adds some universal constant to $C(K)$. Using (18), we can prove a similar bound with $u_0$ replaced by $u_1$ and therefore obtain

$$M_r(N, z) \leq C(K)\varepsilon^{-4} \| \Phi(N, z) \|^2.$$  

Since $M_r(n, z)$ is decreasing in $n$, the asserted bound follows.  

Proof of Theorem 2. The assertion is an immediate consequence of (14) and Lemma 5 (and the analogous results on the left half-line). As sequences $a$ and $b$ in (3) are symmetric the same can be done for $\langle P_l(N, T) \rangle$. Note that we can replace $[3, N]$ by $[1, N]$ since this modification only changes the $K$-dependent constant.  

We now show the same result for non time-averaged quantities. As in [DT2], the key is to replace the Parseval formula by a Dunford functional calculus (sometime called Riesz-Dunford).

Lemma 6 For every $n \in \mathbb{Z}$, $t \in \mathbb{R}$, and positively oriented simple closed contour $\gamma$ in $\mathbb{C}$ that is such that the spectrum of $H$ lies inside $\gamma$, we have

$$\langle e^{-itH}\delta_1, \delta_n \rangle = -\frac{1}{2\pi i} \int_{\gamma} e^{-itz} \langle (H - z)^{-1}\delta_1, \delta_n \rangle dz.$$  

Proof This is a direct consequence of Dunford functional calculus, see [Du, DuS, RS].  

Lemma 7 Suppose $H$ defined in (11) and $a$ and $b$ in (3) and $K \geq 4$ is such that $\sigma(H) \subset [-K + 1, K - 1]$. Then,

$$P(N, t) \lesssim \exp(-cN) + \int_{-K}^{K} \sum_{n\geq N} |\langle (H - E - it^{-1})^{-1}\delta_1, \delta_n \rangle|^2 dE.$$  

10
**Proof.** For any $t > 0$, consider the following contour $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$, where

$$
\gamma_1 = \{E + iy : E \in [-K, K], y = t^{-1}\}, \quad \gamma_2 = \{E + iy : E = -K, y \in [-1, t^{-1}]\},
$$

$$
\gamma_3 = \{E + iy : E \in [-K, K], y = -1\}, \quad \gamma_4 = \{E + iy : E = K, y \in [-1, t^{-1}]\}.
$$

Notice that the spectrum of $H$ lies within this contour and that for $z \in \gamma$, $\Im m(z) \leq t^{-1}$ and thus $|e^{-itz}| \leq e$. Lemma 6 then implies,

$$
|\langle e^{-itH}\delta_1, \delta_n \rangle| \lesssim 4 \sum_{j=1}^{4} \int_{\gamma_j} |\langle (H - z)^{-1}\delta_1, \delta_n \rangle||dz|.
$$

If $z \in \gamma_2 \cup \gamma_3 \cup \gamma_4$, then we can again apply Combes-Thomas estimates and bound by $C \exp(-dN)$.

The integral over $\gamma_1$ can be estimated using Cauchy-Schwarz inequality:

$$
\left(\int_{\gamma_1} |\langle (H - z)^{-1}\delta_1, \delta_n \rangle||dz|\right)^2 \leq C(K) \int_{-K}^{K} |\langle (H - E - \frac{i}{t})^{-1}\delta_1, \delta_n \rangle|^2 dE.
$$

**Theorem 3** Suppose $H$ defined in (1) and $a, b$ in (3), and $K \geq 4$ is such that $\sigma(H) \subset [-K + 1, K - 1]$. Then the outside probabilities can be bounded from above in terms of transfer matrix norms as follows:

$$
P(N, t) \lesssim \exp(-dN) + t^4 \int_{-K}^{K} \left( \max_{1 \leq n \leq N} \|\Phi(n, E + \frac{i}{t})\|^2 \right)^{-1} dE.
$$

**Proof.** The proof starts by using Lemma 6 and 7 instead of Parseval formula. All the steps are then the same that in the proof of the Theorem 2.

5 Application to off-diagonal Fibonacci dynamic

We apply Theorem 2 and 3 for Fibonacci off-diagonal model and show a non trivial dynamical upper bound. Namely we show the following Theorem:

**Theorem 4** Consider the off-diagonal Fibonacci Hamiltonian, that is the operator defined in (1) and (3) with $\beta = \frac{\sqrt{5} - 1}{2}$. Assume $c = c^a > 8$, then

$$
\tilde{\alpha}_u^+ \leq \frac{2 \log \frac{\sqrt{5} - 1}{2}}{\log \xi_c},
$$

where $\xi_c = c - 2 + \sqrt{c^2 - 4c + 1}$.

The same holds for non time-averaged dynamical exponent $\alpha_u^+$.

**Remark 1** Picking $c$ large enough, one obtains a non trivial bound, that is better that the so-called ballistic bound 1.
Considering Fibonacci special case simplifies greatly the trace map evolution, since one has to consider only one sequence of trace as $z_n = x_{n+1}$. Indeed evolution of the trace map reduces to the following:

$$x_{n+1} = x_n x_{n-1} - x_{n-2}$$

with

$$x_{-1} = \frac{\lambda_1^2 + \lambda_2^2}{\lambda_1 \lambda_2}, \quad x_0 = \frac{E}{\lambda_2}, \quad x_1 = \frac{E}{\lambda_1}. \quad (30)$$

Denote by $F_k$ the Fibonacci sequence. A direct application of Lemma 1 yields in this case to either $|x_0| \leq 2$ or $|x_1| \leq 2$ and implies $N_0(E) \leq 1$ uniformly in $E$. This allow to extend description in term of periodic band spectrum. We recall some results, classical in on-diagonal case (see Appendix B), and extended to off-diagonal Fibonacci case since proof depend mostly on the trace map application evolution (see [Dah]). For the same technical reasons that in on-diagonal case, we should suppose that $c > 4$.

**Definition 1** Define a band $B_k \subset \sigma_{k,0}$ to be of type A if $B_k \subset \sigma_{k-1,0}$ and a band of type B if $B_k \subset \sigma_{k-2,0}$.

This definition exhausts all possibilities as seen in the following lemma:

**Lemma 8** Let $c > 4$, and $k \geq 2$. Then

(i) Each type A band $B_k \subset \sigma_{k,0}$ contains exactly one type B band $B_{k+2} \subset \sigma_{k+2,0}$ and no other bands from $\sigma_{k+1,0}$ and $\sigma_{k+2,0}$.

(ii) Each type B band $B_k \subset \sigma_{k,0}$ contains exactly one type A band $B_{k+1} \subset \sigma_{k+1,0}$ and two type B bands from $\sigma_{k+2,0}$ positionned around $B_{k+1}$ and no other bands from $\sigma_{k+1,0}$ and $\sigma_{k+2,0}$.

**Proof.** The proof inspired from diagonal case [R] can be found in [Dah].

We recall also some estimates on the band sizes given in [Dah]. Those estimates were there used to compute the fractal dimension of the spectrum.

**Lemma 9** Let $c > 8$, and $k \geq 2$. Then, with $\xi_c = c - 2 + \sqrt{c^2 - 4c + 1}$, we have the following inequalities:

(i) For any type A band $B_{k+1} \subset \sigma_{k+1,0}$, $E \in B_{k+1}$ implies

$$\xi_c \leq \left| \frac{x'_{k+1}(E)}{x_k(E)} \right| \leq 2c + 7. \quad (i)$$

(ii) For any type B band $B_{k+2} \subset \sigma_{k+2,0}$, $E \in B_{k+2}$ implies

$$\xi_c \leq \left| \frac{x'_{k+2}(E)}{x_k(E)} \right| \leq 2(2c + 7). \quad (ii)$$
By now, we consider the periodic approximants spectrum in $\mathbb{C}$.

$$\sigma_{k,0}^\delta = \{ z \in \mathbb{C} : |x_k(z)| \leq 2 + \delta \}.$$ 

All the properties keep true replacing $\sigma_{k,0}$ by $\sigma_{k,0}^\delta$ for some small enough fixed $\delta$ (recall the occurrence of $\delta$ in Lemma 2), in particular statement (13). A condition on $c$ should be added to keep Fricke-Voigt invariant, $c > \lambda(\delta) = [12(1+\delta)^2 + 8(1+\delta)^3 + 4]^{1/2}$.

The following Proposition states, due to classical Koebe distortion theorem, the height of the set $\sigma_{k,0}^\delta$ is almost the same that its length.

**Proposition 1** If $k \geq 3$, $\delta > 0$ and $c > \max(8, \lambda(\delta))$ then there exist constants $c_\delta, d_\delta > 0$ such that

$$\bigcup_{j=1}^{F_{k-1}} B(x_k^{(j)}, r_k) \subseteq \sigma_{k,0}^\delta \subseteq \bigcup_{j=1}^{F_{k-1}} B(x_k^{(j)}, R_k)$$

where $\{x_k^{(j)}\}_{1 \leq j \leq q_{k-1}}$ are the zeros of $x_k$, $r_k$ and $R_k$ are the sizes of respectively the smallest and the largest band in $\sigma_{k,0}^\delta$.

**Proof.** The proof is a direct consequence of properties of the functions $x_k(E)$ which are proper and continuous as polynomials in $E$ and the distortion theorem of Koebe. See [DT1, M2, M1] for details. \hfill \Box

We have now all the required tools to finish the proof of the theorem 4.

**Proof of Theorem 4.** As $x_k^{(j)}$ are real, we have with Proposition 1

$$\sigma_k^\delta \subseteq \{ z \in \mathbb{C} : |\Im z| < R_k \} \subseteq \{ z \in \mathbb{C} : |\Im z| < dF_k^{-\gamma(c)} \}.$$

for a suitable $\gamma(c)$. This implies

$$\sigma_{k,0}^\delta \cup \sigma_{k+1,0}^\delta \subseteq \{ z \in \mathbb{C} : |\Im z| < dF_k^{-\gamma(c)} \}. \quad (31)$$

Let us precise how to choose $\gamma(c)$.

From Lemma 8 and Lemma 9, it is easy to bound $R_k$:

$$R_k \leq \xi^{-k/2}$$

We should have $R_k < dF_k^{-\gamma(c)}$ so a suitable $\gamma$ can be chosen by taking:

$$\gamma(c) \leq \limsup_k \frac{k \log \xi_c}{2 \log F_k}.$$ 

Remarking that $F_k$ behave like $\left(\frac{\sqrt{5} - 1}{2}\right)^k$ leads to $\gamma(c) \leq \frac{\log \xi_c}{2 \log \frac{\sqrt{5} - 1}{2}}$.

For $\varepsilon = |\Im z| > 0$, we get a lower bound for $|x_n(E + i\varepsilon)|$ uniform in $E \in [-K, K] \subset \mathbb{R}$. For a fixed $\varepsilon > 0$, we choose $k$ such that $dF_k^{-\gamma(V)} < \varepsilon$. With (11), this shows $|x_k(E + i\varepsilon)| > 2 + \delta$ and $|x_{k+1}(E + i\varepsilon)| > 2 + \delta$. As $|x_1(E + i\varepsilon)| \leq 2 + \delta$ or $|x_0(E + i\varepsilon)| \leq 2 + \delta$ from Lemma 11 we can apply Lemma 2 and thus

$$|x_j| \geq e^{(\log(1+\delta)F_{j-k}) + 1} \quad \forall j > k.$$ 

All this motivates the following definitions:
**Definition 2** For $\delta > 0, T > 1$, denote by $k(T)$ the unique integer with

$$\frac{F_{k(T)}^{(\varepsilon)}}{d} < T \leq \frac{F_{k(T)}^{(\varepsilon)}}{d}$$

and let

$$N(T) = F_{k(T)}^{(\varepsilon)} + \lceil k(T) \rceil.$$

For every $\nu > 0$, there is a constant $C_\nu > 0$ such that

$$N(T) \leq C_\nu T^{\gamma + \nu}.$$  \hspace{1cm} (32)

Applying Theorem 2 and above estimate, we get

$$P_r(N(T), T) \lesssim \exp(-cN(T)) + T^3 \int_{-K}^{K} \left( \max_{1 \leq F_n \leq N(T)} \|M_n(E + \frac{i}{T})\|^2 \right) dE,$$

$$\lesssim \exp(-cN(T)) + T^3 e^{-2 \log(1+\delta) F_{\lceil \sqrt{k(T)} \rceil}}.$$

From this bound, it is clear that $P_r(N(T), T)$ goes to zero faster than any inverse power of $T$ and thus

$$\tilde{\alpha}_u^+ = \frac{1}{\gamma(c)} + \nu$$

with $\nu$ arbitrary small.

One complete the proof without time averaging, using Theorem 3 instead of Theorem 2. \hspace{1cm} $\square$

### 6 Fractal dimension of the on-diagonal model spectrum

We now consider $H$ defined with (1) and (2). Denotes by $N(\varepsilon)$ the minimal number of balls of diameter at most $\varepsilon$ one need to cover $\sigma$, then the upper (and lower) box-counting dimension are defined respectively by

$$\dim_B^+(\sigma) = \limsup_{\varepsilon \to 0} \frac{\ln N(\varepsilon)}{\ln \varepsilon}, \quad \dim_B^-(\sigma) = \liminf_{\varepsilon \to 0} \frac{\ln N(\varepsilon)}{\ln \varepsilon}.$$

When the limit exists, one denotes it simply by $\dim_B(\sigma)$.

We give a lower bound of minimal number of balls of some explicit decreasing scale $\varepsilon_k$, needed to cover the spectrum. The first idea to cover the spectrum can be to take into account all the bands and take as a scale the smallest band size, but this is a bad idea because this minimal length decreases faster than the number of bands grows. A better idea will be to count how many bands have a length of order $\varepsilon_k \approx \lambda_1^{-k}$. This yields to a better lower bound for the box-counting dimension of the spectrum associated to a large class of irrational number.

**Theorem 5** Set $C_k = 3 \sum_{j=1}^{k} \log(a_j + 2)$. We have for any irrational number $\beta$ verifying $C = \limsup C_k < +\infty$ and $\lambda_1 > 20$:

$$\dim_B^+(\sigma) \geq \frac{\ln \left( \frac{\sqrt{5}+1}{2} \right)}{C + \ln(\lambda_1 + 5)},$$

where $\sigma$ is the spectrum of $H$.  \hspace{1cm} (33)
Remark 2 The diophantine condition $C < +\infty$ is Lebesgue measure 1 (see Appendix A). In particular, it is true for random numbers and degree 2 algebraic numbers.

As in off-diagonal Fibonacci, one can label bands (see appendix B for precise definitions). The following Lemma give precise statement of the counting idea.

Lemma 10 Denote by $n_{k,I}, n_{k,II}$ and $n_{k,III}$ the number of bands of type I, II and III in $\sigma_{k,1}$, $\sigma_{k+1,0}$, $\sigma_{k+1,0}$ with a length greater than $\varepsilon_k = 4\Pi_{j=1}^k(\lambda_1 + 5)^{-1}(a_j + 2)^{-3}$.

For all $k$, we have the following induction:

\begin{align*}
  n_{k+1, I} &= (a_{k+1} + 1)n_{k,II} + a_{k+1}n_{k,III}, \\
  n_{k+1, II} &\geq 1_{\{a_{k+1} \leq 2\}}n_{k, I}, \\
  n_{k+1, III} &= a_{k+1}n_{k,II} + (a_{k+1} - 1)n_{k,III},
\end{align*}

with initial conditions $n_{0,I} = 1$, $n_{0,II} = 0$, $n_{0,III} = 1$.

Moreover,

\begin{align*}
  n_{k,II} \neq 0 \bigvee n_{k,III} \neq 0, \\
  n_{k,I} \neq 0, \\
  n_{k,I} > n_{k,III},
\end{align*}

and

\begin{equation}
  n_{k,II} + n_{k,III} > \left(\frac{\sqrt{5} + 1}{2}\right)^k. \tag{37}
\end{equation}

Proof. The induction relation is obvious with Lemma 12 and Theorem 8 from Appendix B.

The two first properties are made by induction. Initial conditions give level 0. Assume it is true at level $n$, then as $a_{k+1} > 0$, $n_{k,II} \neq 0 \bigvee n_{k,III} \neq 0$, implies $n_{k+1,I} \neq 0$. For the second part, if $a_{k+1} \leq 2$ then $n_{k+1,II} \neq 0$, else $a_{k+1} > 2$ implies $n_{k+1,III} \neq 0$.

To prove $n_{k,I} > n_{k,III}$, it suffices to see that $n_{k,I} \geq n_{k,III} + n_{k-1,II} + n_{k-1,III}$.

Denote by $n_k$ the sum of $n_{k,II}$ and $n_{k,III}$. For the last property, we argue that for $k > 1$,

\begin{equation}
  n_k \geq 2n_{k-2} + n_{k-3}. \tag{38}
\end{equation}

It is easy to verify from (38) that the sequence $n_k$ growth verifies (37).

We now show (38). Using (34)-(36), we get

\begin{align*}
  n_{k,II} &= [(a_{k-1} + 1)n_{k-2,II} + a_{k-1}n_{k-2,III}]1_{\{a_{k-1} \leq 2\}}, \\
  n_{k,III} &= (a_{k-1})(a_{k-1}n_{k-2,II} + (a_{k-1} - 1)n_{k-2,III}) + a_{k-2}n_{k-2,II}1_{\{a_{k-1} \leq 2\}}.
\end{align*}

We distinguish all the cases depending on the values of $a_k$ and $a_{k-1}$ in the following table.

| $n_k \geq$ | $a_{k-1} = 1$ | $a_{k-1} = 2$ | $a_{k-1} = \gamma > 2$ |
|---|---|---|---|
| $a_k = 1$ | $2n_{k-2} + n_{k-3}$ | $3n_{k-2} + n_{k-3}$ | $\gamma n_{k-2}$ |
| $a_k = 2$ | $3n_{k-2} + 2n_{k-3}$ | $5n_{k-2} + 2n_{k-3}$ | $(2\gamma - 1)n_{k-2}$ |
| $a_k = \lambda > 2$ | $(\lambda - 1)n_{k-2} + \lambda n_{k-3}$ | $2(\lambda - 1)n_{k-2} + \lambda n_{k-3}$ | $(\lambda - 1)(\gamma - 1)n_{k-2} + (\lambda - 1)n_{k-3}$ |

Therefore, the slowest growth for the quantity $n_{k,II} + n_{k,III}$ is for $a_k = a_{k-1} = 1$ which ends the proof. \qed
Proof of theorem 5 With Lemma 10 we obtain a bound for \( n_{k,II} + n_{k,III} \), that is the number of bands of length at least \( \varepsilon_k \). Then by definition of box-counting dimension, we have

\[
\dim^+_B(\sigma) \geq \liminf_k \frac{\ln(n_{k,II} + n_{k,III})}{-\ln \varepsilon_k}
\]

and the stated result. \( \square \)

7 Dimension almost sure

In this part, we consider only random numbers, that is number with \( C = 5.04 \ldots \) (see Corollary 1 in Appendix A). This class of course is included in the class of the previous section and is still Lebesgue measure 1. This first Lemma is purely technical and link the ergodicity of a random continued fraction expansion (see Appendix A) with our two step setting.

Lemma 11 Denote \( E(\lambda, \gamma) \) the event \( \{(a_{2k}, a_{2k+1}) = (\lambda, \gamma)\} \) for some \( k > 0 \). For every natural \( \lambda, \gamma \), we have \( P(E(\lambda, \gamma)) = \frac{\ln(1 + \frac{1}{\lambda+2}) \ln(1 + \frac{1}{\gamma+2})}{(\ln(2))^2} \). For all \( \varepsilon > 0 \), there exists \( N_0 \) big enough, such that for every \( N > N_0 \), we have the probability:

\[
P(\text{At least } (P(E(\lambda, \gamma)) - \varepsilon) N \text{ couples verify } E(\lambda, \gamma)) = 1.
\]

Proof. Denote \( A(r) \) the event

\[
A(r) = \{\text{Among the } n \text{ first there is exactly } r \text{ couples verifying } E(\lambda, \gamma)\}.
\]

Since \( a_{2k} \) and \( a_{2k+1} \) are independant random variables, we have \( P(E(\lambda, \gamma)) = \frac{\ln(1 + \frac{1}{\lambda+2}) \ln(1 + \frac{1}{\gamma+2})}{(\ln(2))^2} \) and set for convienience \( \frac{1}{B} = P(E(\lambda, \gamma)) \), we have

\[
P(A(r)) = C_n^r \frac{1}{B^r} \left(1 - \frac{1}{B}\right)^{n-r}.
\]

Then the event \( \{\text{There is less than } r \text{ couples verifying } E(\lambda, \gamma)\} \) is \( \bigcup_{k=0}^r A(k) \) and

\[
P\left(\bigcup_{k=0}^r A(k)\right) = \sum_{k=0}^r C_n^k \frac{1}{B^k} \left(1 - \frac{1}{B}\right)^{n-k} \\
\leq C_n^r \frac{B-1}{B}^n \left(\frac{r+1}{n-r}\right)^r \sum_{k=0}^r \left(\frac{n-r}{(B-1)(r+1)}\right)^k.
\]

Using \( C_{k+1}^k = \frac{n-k}{k+1} C_n^k \geq \frac{n-r}{r+1} C_n^k \) one compare \( C_n^k \) and \( C_n^r \). Then assuming that \( r < \frac{n}{B} \), one shows that

\[
\frac{n-r}{(B-1)(r+1)} \leq \frac{n-n/B}{(B-1)(n/B+1)} = \frac{n}{n+B} < 1.
\]
It follows that the remaining sum in (39) is geometric and is bounded by some constant $K$. Using then Stirling formula, one can estimate $C_r$:

$$C_r \sim \frac{n^n}{(n-r)^{n-r}r^r}.$$ 

This yields to the following

$$P\left(\bigcup_{k=0}^{r} A(k)\right) \leq K \left(\frac{n}{n-r}\right)^n \left(\frac{B-1}{B}\right)^n \left(\frac{r+1}{r}\right)^r.$$ 

If $r \leq \left(\frac{1}{B} - \varepsilon\right)n$, for some $\varepsilon > 0$, then the right term in the above expression tends to 0, which ends the proof.

**Theorem 6** For almost all irrational number $\beta$, an arbitrary small $\nu > 0$ and $\lambda_1 > 20$:

$$\dim^+(\sigma) \geq \frac{D - \nu}{C + \ln(\lambda_1 + 5)}$$

with $D = 1.0382\ldots$

**Remark 3** $D$ is obtained as the limit of an explicit converging series. Value of $D$ is roughly twice bigger than the bound (33) in Theorem 5. It is also worthy to be compare it with optimal bound for golden and silver mean $D = 0.88\ldots$ [DEGT], [LPW].

**Proof.** First, simplify the recursion relation in Lemma 10 by ignoring the smallest term, and thus obtain:

$$n_{2k,II} + n_{2k,III} \geq c_{2k} (n_{2k-2,II} + n_{2k-2,III})$$

where $c_{2k}$ depend only on the values of the couple $(a_{2k}, a_{2k-1})$ according to the table:

| $c_{2k}(a_{2k}, a_{2k-1})$ | $a_{2k} = 1$ | $a_{2k} = 2$ | $a_{2k} = \lambda > 2$ |
|---------------------------|-------------|-------------|---------------------|
| $a_{2k-1} = 1$            | 2           | 3           | $\lambda - 1$      |
| $a_{2k-1} = 2$            | 3           | 5           | $2(\lambda - 1)$   |
| $a_{2k-1} = \gamma > 2$  | $\gamma$    | $2\gamma - 1$ | $(\lambda - 1)(\gamma - 1)$ |

Using now Lemma 11 and probability value for $a_k$ recall in Corollary 1 in Appendix A, one has for all $k > 0$ and almost all $\beta$:

$$n_{2k,II} + n_{2k,III} \geq 2^{k(P(E(1,1))-\varepsilon)} 3^{k(P(E(2,1)) + P(E(1,2))-\varepsilon)} 5^{k(P(E(2,2))-\varepsilon)} \prod_{\lambda=3}^{n} \lambda^{k(P(E(\lambda,1))-\varepsilon)} \times$$

$$\prod_{\lambda=3}^{n} (\lambda - 1)^{k(P(E(\lambda,1))-\varepsilon)} \prod_{\lambda=3}^{n} (2\lambda - 1)^{k(P(E(2,\lambda))-\varepsilon)} \prod_{\lambda=3}^{n} (2\lambda - 2)^{k(P(E(\lambda,2))-\varepsilon)} \times$$

$$\prod_{\lambda,\beta=3}^{n} ((\lambda - 1)(\beta - 1))^k(P(E(\beta,\lambda))-\varepsilon).$$
In the previous equation, we replace all the \( \varepsilon \) constants by only one constant denoted also \( \varepsilon \) and we also take the same \( n \) in all the products. We can do those simplifications since we consider only a finite number of terms. Taking the logarithm on the above expression, one obtains:

\[
\ln \left( \frac{n_{2k,II} + n_{2k,III}}{2k(1 - \varepsilon)} \right) \geq \frac{1}{2(2\ln)^2} \left( \ln(2) \left( \ln \frac{4}{3} \right)^2 + 2 \ln(3) \left( \ln \frac{9}{8} \right) \left( \ln \frac{4}{3} \right) + \ln(5) \left( \ln \frac{9}{8} \right)^2 + \sum_{\lambda=3}^{n} \left( \ln(\lambda) + \ln(\lambda - 1) \right) \left( \ln \frac{4}{3} \right) \ln \left( 1 + \frac{1}{\lambda(\lambda + 2)} \right) + \right.
\]
\[
\sum_{\lambda=3}^{n} \left( \ln(2\lambda - 1) + \ln(2\lambda - 2) \right) \left( \ln \frac{9}{8} \right) \ln \left( 1 + \frac{1}{\lambda(\lambda + 2)} \right) + \sum_{\lambda,\beta=3}^{n} \left( \ln(\lambda - 1) + \ln(\beta - 1) \right) \ln \left( 1 + \frac{1}{\lambda(\lambda + 2)} \right) \ln \left( 1 + \frac{1}{\beta(\beta + 2)} \right) \right) . \quad (40)
\]

We can rewrite the last double sum:

\[
\sum_{\lambda=3}^{n} \sum_{\beta=3}^{n} \left( \ln(\lambda - 1) + \ln(\beta - 1) \right) \ln \left( 1 + \frac{1}{\lambda(\lambda + 2)} \right) \ln \left( 1 + \frac{1}{\beta(\beta + 2)} \right)
\]
\[
= 2 \sum_{\lambda=3}^{n} \sum_{\beta=3}^{n} \ln(\lambda - 1) \ln \left( 1 + \frac{1}{\lambda(\lambda + 2)} \right) \ln \left( 1 + \frac{1}{\beta(\beta + 2)} \right)
\]
\[
= 2 \sum_{\lambda=3}^{n} \ln(\lambda - 1) \ln \left( 1 + \frac{1}{\lambda(\lambda + 2)} \right) \sum_{\beta=3}^{n} \ln \left( 1 + \frac{1}{\beta(\beta + 2)} \right)
\]
\[
= 2 \sum_{\lambda=3}^{n} \ln(\lambda - 1) \ln \left( 1 + \frac{1}{\lambda(\lambda + 2)} \right) \left( \ln \frac{4}{3} \right).
\]

As all the series in \( n \) have positive terms and are convergent, we can bound from below by taking the limit minus some \( \nu > 0 \) arbitrary small.

The limit of r.h.s of (40) can be numerically compute and gives \( D = 1.0382 \ldots \). □

**Appendix A: A theorem by Khintchin**

We recall here some of the ergodic properties of the continued fraction expansion process.

**Theorem 7 [K]** Suppose that \( f(r) \) is a non-negative function of a natural argument \( r \), \((r=1,2,\ldots)\), and suppose that there exist positive constants \( C \) and \( \delta \) such that

\[
f(r) < Cr^{\frac{1}{2} - \delta}.
\]

Then, for all numbers in the interval \((0,1)\), with the exception of a set of measure zero,

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f(a_k) = \sum_{r=1}^{\infty} f(r) \frac{\ln \left( 1 + \frac{1}{r(r+2)} \right)}{\ln 2}.
\]

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Corollary 1  For almost all $\beta$ with respect to Lebesgue measure,

$$C = \limsup_k \frac{3}{k} \sum_{j=1}^{k} \log(a_j + 2) = 5.04\ldots$$

Moreover, the density of the number $i$ in the sequence $\{a_k\}_k$ is almost surely

$$d(i) = \frac{\ln \left(1 + \frac{1}{r(r+2)}\right)}{\ln 2}.$$ 

Proof. Apply Theorem 7 with $f(r) = \ln(r+2)$ and $f(r) = 1_i(r)$ respectively. \qed

Appendix B: Gap labelling and band estimates

We recall precise information on periodic on-diagonal spectrum sets.

Definition 3  For a given $k$, we call

- type I gap: a band of $\sigma_{k,1}$ included in a band of $\sigma_{k,0}$ and therefore in a gap of $\sigma_{k+1,0}$,
- type II band: a band of $\sigma_{k+1,0}$ included in a band of $\sigma_{k,-1}$ and in a gap of $\sigma_{k,0}$,
- type III band: a band of $\sigma_{k+1,0}$ included in a band of $\sigma_{k,0}$ and in a gap of $\sigma_{k,1}$.

These definitions exhaust all the possible configurations:

Lemma 12 [R]  At a given level $k$,

(i) a type I gap contains an unique type II band of $\sigma_{k+2,0}$.

(ii) a type II band contains $(a_{k+1} + 1)$ bands of type I of $\sigma_{k+1,1}$. They are alternated with $(a_{k+1})$ type III bands of $\sigma_{k+2,0}$

(iii) a type III band contains $(a_{k+1})$ bands of type I of $\sigma_{k+1,1}$. They are alternated with $(a_{k+1} - 1)$ type III bands of $\sigma_{k+2,0}$

Let $A = \{I, II, III\}$ be a three letters alphabet. For each band $B$ of spectrum at level $k$, correspond an unique word $i_0i_1\ldots i_k \in A^{n+1}$ such that $B$ is a band of type $i_k$ included in a band of type $i_{k-1}$ at level $k-1,\ldots,\text{included in a band of type } i_0$ at level 0. This word will be called the index of $B$. More than one band can have the same index. Let $T_n = (t_{i,j}(n))_{3^*3}$ be a sequence of matrix and $\tau = i_0i_1\ldots i_k$ an index, we define:

$$L_\tau(T) = t_{i_0,i_1}(1)t_{i_1,i_2}(2)\ldots t_{i_{k-1},i_k}(k).$$

Theorem 8 [LV]  If $\beta = [a_1,a_2\ldots]$ is an irrational number in $(0,1)$ and $H$ defined as in (1) and (2) with $\lambda_1 > 20$ then any band $B$ of index $\tau$ verifies,

$$4L_\tau(Q) \leq |B| \leq 4L_\tau(P)$$

where $P = (P_n)_{n>0}$

$$P_n = \begin{pmatrix}
0 & c_1^{a_{n-1}} & 0 \\
c_1/a_n & 0 & c_1/a_n \\
c_1/a_n & c_1/a_n & 0
\end{pmatrix}$$

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with \( c_1 = \frac{3}{\lambda_1 - 8} \) and \( Q = (Q_n)_{n>0} \)

\[
Q_n = \begin{pmatrix}
0 & c_2^{a_{n-1}} & 0 \\
(c_2(a_n + 2)^{-3} & 0 & c_2(a_n + 2)^{-3} \\
(c_2(a_n + 2)^{-3} & 0 & c_2(a_n + 2)^{-3}
\end{pmatrix}
\]

with \( c_2 = \frac{1}{\lambda_1 + 5} \).

References

[BM] M. Baake, R. Moody (Editors), Directions in Mathematical Quasicrystals, American Mathematical Society, Providence, RI, 2000.

[BIST] J. Bellisard, B. Iochum, E. Scoppola, D. Testard Spectral properties of one dimensional quasicrystals, Commun. Math. Phys. 125 (1989), 527-543

[CT] J. M. Combes, L. Thomas Asymptotic behaviour of eigenfunctions for multiparticle Schrödinger operators. Commun. Math. Phys. 34, 251-270 (1973).

[Dah] J. Dahl, The Spectrum of the Off diagonal Fibonacci Operator, Ph. D. thesis, Rice University, (2011).

[D1] D. Damanik, Gordon-type arguments in the spectral theory of one-dimensional quasicrystals, in [BM], 277–305.

[D2] D. Damanik, Strictly ergodic subshifts and associated operators, in Spectral Theory and Mathematical Physics: a Festschrift in Honor of Barry Simon’s 60th Birthday, 505–538, Proc. Sympos. Pure Math. 76, Part 2, Amer. Math. Soc., Providence, RI, 2007.

[DEGT] D. Damanik, M. Embree, A. Gorodetski, S. Tcheremchantsev, The fractal dimension of the spectrum of the Fibonacci hamiltonian. Commun. Math. Phys. 280 (2008), 499–516.

[DG] D. Damanik, A. Gorodetski, Spectral and Quantum Dynamical Properties of the Weakly Coupled Fibonacci Hamiltonian, Comm. Math. Phys. 305 (2011), 221–277.

[DKL] D. Damanik, R. Killip, D. Lenz, Uniform spectral properties of one-dimensional quasicrystals. III. \( \alpha \)-continuity: Comm. Math. Phys. 212 (2000), no: 1, 191-204

[DL] D. Damanik, D. Lenz, Uniform spectral properties of one-dimensional quasicrystals, I. Absence of eigenvalues, Commun. Math. Phys. 207 (1999), 687–696.

[DT1] D. Damanik, S. Tcheremchantsev, Upper bounds in quantum dynamics, J. Amer. Math. Soc. 20 (2007), no. 3, 799–827.

[DT2] D. Damanik, S. Tcheremchantsev, Quantum dynamics via complex analysis methods: general upper bounds without time averaging and tight lower bound for the strongly coupled Fibonacci Hamiltonian, J. of Funct. Anal. 255, Issue 10, (2008), 2872–2887.
[Du] N. Dunford, Spectral theory. I. Convergence to projections, Trans. Amer. Math. Soc. 54 (1943), 185–217

[DuS] N. Dunford and J. Schwartz, Linear Operators. Part I. General Theory, John Wiley and Sons, Inc., New York, 1988

[EL] S. Even-Dar Mandel, R. Lifshitz, Electronic energy spectra and wave functions on the square Fibonacci tiling, Phil. Mag. 86 (2006), 759–764.

[EL2] S. Even-Dar Mandel, R. Lifshitz, Electronic energy spectra of square and cubic Fibonacci quasicrystals, Philosophical Magazine 88 (2008), 2261-2273.

[GKT] F. Germinet, A. Kiselev, et S. Tcheremchantsev, Transfert matrices and transport for Schrödinger operators, Ann. Inst. Fourier (Grenoble), 54 (2004), 787-830.

[IRT] B. Iochum, L. Raymond, D. Testard, Resistance of one-dimensional quasicrystals Physica A 187 (1992), 353–368.

[K] A. Ya. Khinchin, Continued Fractions (Chicago: University of Chicago Press) 1964.

[KKL] R. Killip, A. Kiselev, and Y. Last, Dynamical upper bounds on wavepacket spreading, Amer. J. Math. 125 (2003), 1165–1198

[KKT] M. Kohmoto, L. Kadanoff, C. Tang, Localization problem in one dimension: mapping and escape, Phys. Rev. Lett. 50 (1983), 1870–1872.

[KST] M. Kohmoto, B. Sutherland, C. Tang, Critical wave functions and a Cantor-set spectrum of a one-dimensional quasicrystal model, Phys. Rev. B 35 (2011), 1020-1033

[LW] Q. Liu, Z. Wen, Hausdorff dimension of spectrum of one-dimensional Schrödinger operator with Sturmian potentials, Potential Anal 20 (2004), 33–59.

[LPW] Q. Liu, J. Peyrière, Z. Wen, Dimension of the spectrum of one-dimensional discrete Schrödinger operators with Sturmian potentials, C. R. Acad. Sci. Paris, Ser. I 345 (2007), 667–672.

[M1] L. Marin Borne Dynamique en Dynamique Quantique, Ph. D. thesis, Université d’Orléans 2009, [http://tel.archives-ouvertes.fr/tel-00482512/fr/](http://tel.archives-ouvertes.fr/tel-00482512/fr/) (2009). For french reader!

[M2] L. Marin, Dynamical bounds for Sturmian Schrödinger operators, Rev. in Math. Ph. 22 (2010).

[OPRSS] S. Ostlund, R. Pandit, D. Rand, H. Schellnhuber, E. Siggia, One-dimensional Schrödinger equation with an almost periodic potential, Phys. Rev. Lett. 50 (1983), 1873–1877.

[R] L. Raymond, A constructive gap labelling for the discrete Schrödinger operator on a quasiperiodic chain, preprint (1997)
[RS] M. Reed and B. Simon, Methods of Modern Mathematical Physics. I. Functional Analysis, Academic Press, Inc., New York, 1980

[Ro] J. A. G. Roberts, Escaping orbits in trace maps, *Physica A: Statistical Mechanics and its Applications* **228** (1996), no. 1-4, 295-325.

[SBGC] D. Shechtman, I. Blech, D. Gratias, J. V. Cahn, Metallic phase with long-range orientational order and no translational symmetry, *Phys. Rev. Lett.* **53** (1984), 1951–1953.

[S1] A. Sütő, The spectrum of a quasiperiodic Schrödinger operator, *Commun. Math. Phys.* **111** (1987), 409–415.

[S2] A. Sütő, Schrödinger difference equation with deterministic ergodic potentials, in Beyond Quasicrystals (Les Houches, 1994), 481–549, Springer, Berlin, 1995.

[S3] A. Sütő, Singular continuous spectrum on a Cantor set of zero Lebesgue measure for the Fibonacci Hamiltonian, *J. Statist. Phys.* **56** (1989), no. 3-4, 525–531.

[VP] M. T. Velhinho, I. R. Pimentel, Lyapunov exponent for pure and random Fibonacci chains, *Phys. Rev. B* **61** (2000), 1043-1050.