Abstract

In this essay we propose a realization of Lurie’s claim that inner fibrations \( p : X \to C \) are classified by \( C \)-indexed diagrams in a "higher category" whose objects are \( \infty \)-categories, morphisms are correspondences between them and higher morphisms are higher correspondences. We will obtain this as a corollary of a more general result which classifies all simplicial maps.

Correspondences between \( \infty \)-categories, and simplicial sets in general, are a generalization of the concept of profunctor (or bimodule) for categories. While categories, functors and profunctors are organized in a double category, we will exhibit simplicial sets, simplicial maps, and correspondences as part of a simplicial category. This allows us to make precise statements and proofs. Our main tool is the theory of double colimits.
Notation and terminology

We will mostly observe standard notation and terminology. Generic categories are denoted by calligraphic capital letters $\mathcal{C}, \mathcal{D}$ etc., while particular categories are denoted by the name of their objects in bold characters. For example $\textbf{Set}$, $\textbf{Cat}$, $\textbf{sSet}$ are the categories of sets, categories and simplicial sets.

$\Delta$ denotes the category of finite ordinals and order preserving maps as usual, while $\Delta^n$ denotes the standard $n$-simplex. For a simplicial set $X$, $X_n$ refers to its set of $n$-simplices.

Given the nature of our discussion the term "simplicial category" means simplicial object in the category of categories. We refer to simplicially enriched categories as $\textbf{sSet}$-categories. Also, given our references, the term $\infty$-category means quasi-category, that is a simplicial set with the inner horn filling property.

1 Introduction and summary

1.1 Correspondences

Besides functors, another interesting notion of morphism between categories is that of a profunctor. A profunctor $u$ between two categories $\mathcal{C}$ and $\mathcal{D}$, or $(\mathcal{C}, \mathcal{D})$-profunctor, is a functor

$$u : \mathcal{C}^{\text{op}} \times \mathcal{D} \to \textbf{Set}$$

where $\textbf{Set}$ is the category of sets and functions.

A profunctor records a right action of $\mathcal{C}$ and a left action of $\mathcal{D}$ simultaneously. If $\mathcal{C}$ and $\mathcal{D}$ are groups profunctors are also known as bisets. If we enrich the above definition over abelian groups and let $\mathcal{C}$ and $\mathcal{D}$ be rings then profunctors are simply bimodules. If we enrich over the category $\mathbf{2} = \{0 \to 1\}$, in which case $\mathcal{C}$ and $\mathcal{D}$ are posets, when $\mathcal{C}$ and $\mathcal{D}$ are discrete sets profunctors are simply relations. This is why they are referred to as relators sometimes.

The perspective of interest from our point of view is one of a more combinatorial flavour. Given a profunctor $u$ we may record all its information in a category called the collage of $u$. This category is simply constructed by first starting with a copy of $\mathcal{C}$ and $\mathcal{D}$ and then considering the elements of the sets $u(c,d)$ for $c \in \mathcal{C}, d \in \mathcal{D}$ as actual arrows $c \to d$. Composition in the collage is given by the functoriality of $u$.

It is easy to see that
defining a \((C, D)\)-profunctor is the same as constructing a new category \(U\) from \(C\) and \(D\) by adding new morphisms from objects of \(C\) to those of \(D\) (but not in the reverse direction).

Hence, profunctors are simply \textit{collages}.

To make things even more interesting, observe that a collage \(U\) is naturally equipped with a map
\[ p : U \to \Delta^1 \]
where \(\Delta^1 = \{0 \to 1\}\) is the usual categorical 1-simplex, with \(p^{-1}(0) \cong C\) and \(p^{-1}(0) \cong D\). It is easy to see that \(U\) is the collage of a profunctor if and only if it comes equipped with a map \(p\) to \(\Delta^1\). So we see that profunctors are simply maps to \(\Delta^1\)

We would like to reserve the term \textit{correspondence} when we have the latter perspective in mind. Nonetheless all the above terms are synonymous (at least for categories) and these labels are purely the author’s own preference.

The reader can work out a lot of examples by picking one of the above perspectives. For example given a functor \(F : C \to D\) there is a profunctor \(F^*\) given by
\[ F^*(c, d) = D(Fc, d) \]

We may interpret \(F^*\) the \textit{induced} bimodule, but also as the correspondence representing the mapping cylinder of \(F\). In a formal double categorical setting \(F^*\) is obtained as a certain Kan extension and it is also called the \textit{companion} of \(F\).

The author still marvels at the fact that all of the above are equivalent and considers the true value of category theory to be precisely the unveiling of such patterns in mathematics. This allows us not only to reinterpret known constructions more conceptually but to prove new theorems as well. For example in this essay we will consider correspondences between simplicial sets and prove a classification result about simplicial maps (and inner fibrations).

We want to think of profunctors as morphisms between categories so we need to specify a composition. Given that profunctors are bimodules, their composition will be a tensor product. Let \(u\) be a \((C, D)\)-profunctor and \(v\) be a \((D, E)\)-profunctor. We define their composition to be “the” \((C, E)\)-profunctor \(v \otimes_D u\) whose evaluation at a pair of objects \(c \in C, e \in E\) is given by the coend formula
\[ (v \otimes_D u)(c, e) = \int_{d \in D} v(d, e) \times u(c, d) \]

While this coend formula has the virtue of applying in any enrichment it might not be very illuminating. If we view profunctors as collages we may interpret their composition as follows. Consider \(\text{col}(u)\) and \(\text{col}(v)\) and juxtapose them along \(D\). What results is not a category because we cannot compose \(x \in u(c, d)\) with \(y \in v(d, e)\). Resolve this issue by generating a free category out of the data by declaring a new morphism \(y \otimes x : c \to e\) serving as a composite (subject to the obvious relations). Finally we remove the objects of \(D\) to obtain the collage of \(v \otimes_D u\).
If we think of profunctors as maps to $\Delta^1$ we obtain yet another description of composition. Let $p : U \to \Delta^1$ and $q : V \to \Delta^1$ be composable correspondences, i.e. $p^{-1}(1) \cong D \cong q^{-1}(0)$ for some $D$. By taking the pushout along $D$ we obtain a map $p \coprod_D q : U \coprod_D V \to \Delta^2 \cong \Delta^1 \coprod_\Delta \Delta^1$ (this is precisely the above picture). Then obtain the tensor product by taking the pullback

$$V \otimes_D U \longrightarrow U \coprod_D V$$

$$\Delta^1 \overset{d_1}{\longrightarrow} \Delta^2$$

It is easy to see that the above three descriptions are equivalent. This composition operation does not produce a category because it is unital and associative only up to canonical isomorphism (we defined it using universal properties after all). Given $C$, the identity profunctor is defined to be $\text{hom}_C : C^{op} \times C \to \text{Set}$ which assigns to a pair of objects $c, c'$ the set of morphisms in $C$ between them. With this data we obtain a weak 2-category $\text{Prof}$ with categories as objects, profunctors as 1-morphisms and their natural transformations as 2-morphisms.

1.2 The meaning of inner fibrations

Let $p : X \to C$ be an inner fibration between simplicial sets. For an $n$-simplex $\sigma \in C_n$ consider the fiber $p^{-1}(\sigma)$ obtained by the pullback square

$$
\begin{array}{ccc}
\Delta^n & \overset{\sigma}{\longrightarrow} & C \\
p \downarrow & & \downarrow p \\
p_{\sigma} & \longrightarrow & X
\end{array}
$$

Fibrations are stable under pullback, so $p_{\sigma}$ is an inner fibration as well. Moreover, if the target of an inner fibration is a category then the source is an $\infty$-category. In our case we conclude that fibers $p^{-1}(\sigma)$ over each simplex $\sigma$
have to be ∞-categories. It is not difficult to see that the converse is also true: if a map \( p : X \to C \) of simplicial sets is such that fibers over each simplex are ∞-categories then it is an inner fibration.

Referring to the above observations we quote the authors of [BS18]

"So in a strong sense, we’ll understand the ”meaning” of inner fibrations once we understand the ”meaning” of functors from ∞-categories to \( \Delta^n \)

Given that for \( n = 1 \) these maps are understood as correspondences between ∞-categories, we will refer to functors to \( \Delta^n \) as higher correspondences.

A higher correspondence \( p : X \to \Delta^n \) may be thought of as consisting of \( n + 1 \) ∞-categories \( X_i = p^{-1}(i), i = 0, 1 \ldots n \), and a big collage between them. By the latter we mean that \( X \) is formed from the \( X_i \)'s by ”adding” 1-simplicies which join vertices of \( X_i \) and \( X_j \) only if \( i < j \), and higher simplicies after that.

Given an inner fibration \( p : X \to C \) for each simplex \( \sigma \in C_n \) we obtain a correspondence \( p^{-1}(\sigma) \). We would like to see the association

\[
\sigma \mapsto p^{-1}(\sigma)
\]

as being functorial. This means there should be a ”higher category” in which objects are ∞-categories, morphisms are correspondences and higher morphisms are higher correspondences. Then the above would yield a map from \( C \) to this higher category. As Lurie points out (Lur09), this higher category cannot be realized as an ∞-category because higher morphisms do not have to be invertible.

1.3 Our strategy and results

We will propose a (1-categorical) realization of the above. Our treatment has three characteristics which distinguish it from the rest of the literature (to the best of our knowledge):

- We will treat the relationship between inner fibrations and higher correspondences as part of a larger pattern.
- We will define a simplicial structure which binds together correspondences of various dimension.
- We will use double categorical methods.

The pattern we observe can be put as follows: in certain categories all the information about a morphism \( f : X \to A \) is contained in its fibers. The basic example is the category of sets. Fibrationally, a function \( f \) whose target is a set \( A \) is the same (up to isomorphism) as an \( A \)-indexed family of sets.

Another instance occurs in the category of categories. Given a functor \( F : \mathcal{X} \to \mathcal{A} \), its fibers over objects of \( \mathcal{A} \) are categories and its fibers over morphisms of \( \mathcal{A} \) are profunctors. These fibers vary functorially over \( \mathcal{A} \) and
completely characterize $F$, i.e. a functor $F$ whose target is $\mathcal{A}$ is the same (up to isomorphism) as an $\mathcal{A}$-indexed diagram in $\textbf{Prof}$. We briefly discuss this in \[2.1\]

We will revisit the latter result from the double categorical point of view. Observe that in the case of sets the correspondence mentioned above is established by the disjoint union operation. We would like a colimiting process to witness the analogous result for categories. Once we see categories and profunctors as being the horizontal part of a double category (in which functors serve as vertical morphisms) we find the concept of double colimit (as developed in [GP99]) plays exactly the role we want. This is the content of \[2.2\]

If $f : X \to A$ is a map of simplicial sets then its fibers over simplices of $A$ are higher correspondences, and we will show that indeed they do capture all the information about $f$. In order to make our assertion precise we need a double category theory for simplicial sets. Such a theory was studied in our previous work [Had19], from which we extract definitions and some results.

We incarnate the double category theory of simplicial sets in the form of a simplicial category $\text{sSet}^\sharp : \Delta^{op} \to \text{Cat}$ whose category of $n$-simplices is the corresponding category of higher correspondences, i.e. the slice category over $\Delta^n$

$$\text{sSet}^\sharp_n = \text{sSet}/\Delta^n$$

The details are discussed in \[3.1\]

It turns out a lot of double category theory can be carried out in a simplicial category, in particular double colimits \[3.2\]. We will prove \[2\] that taking double colimits produces an equivalence of categories

$$\text{dcolim} : [A, \text{sSet}^\sharp] \to \text{sSet}/A$$

between the category of $A$-indexed diagrams of higher correspondences and the slice category over $A$. The corresponding result for $\infty$-categories follows as a corollary \[1\].

## 2 The fibrational perspective

### 2.1 The pattern

We would like to study the relationship between higher correspondences and inner fibrations as part of a larger pattern. This pattern may be seen, on a fundamental level, to begin with sets and functions. This is so because all the information about a function $f : X \to A$ lies in its fibers.

The pair $(X, f)$ gives us a map

$$A \to \text{Set}$$

by assigning $a \mapsto f^{-1}(a)$ for $a \in A$. Then we may recover $X \cong \bigsqcup_{a \in A} f^{-1}(a)$. In other words
giving a function with codomain $A$ is the same (up to isomorphism) as giving an $A$-indexed family of sets

More precisely, the disjoint union operation establishes an equivalence of categories

\[ \prod : [A, \text{Set}] \to \text{Set}/A \]

between the category of $A$-indexed sets and the slice category over $A$.

The situation becomes more complex when we study categories and functors. Let $F : X \to A$ be a functor, and denote the fibers $F^{-1}(a) = X_a$ for simplicity. To understand $F$ fibrationally we must also take into considerations morphisms in $A$.

Let $f : a \to b$ be a morphism in $A$. According to our previous conclusions, the fiber $X_f = F^{-1}(f)$ will be an $(X_a, X_b)$-correspondence. For a pair of composable arrows $(f, g)$ in $A$ we obtain a morphism $X_g \otimes X_f \Rightarrow X_{gf}$. Hence the assignment $a \mapsto X_a$ produces a lax 2-functor

\[ A \to \text{Prof} \]

into the weak 2-category of categories, profunctors and their transformations.

The above map has enough information for us to reconstruct $X$ and $F$. Hence we conclude

functors with codomain $A$ are classified by lax $A$-indexed diagrams of correspondences

### 2.2 Double colimits

As illuminating as this conclusion may be it is certainly not very satisfactory. The reason being there must be a colimiting process which allows us to retrieve $X$ out of the induced diagram of correspondences, and we have not provided one. This gap can be filled with the theory of double colimits developed in [GP99].

Recall that a double category is a category object in the category of categories. If we unpack the contents of a double category then it consists of

- objects
- two types of morphisms between objects, called vertical and horizontal, which compose within their type but not with each other
- square-shaped 2-cells whose boundary consists of vertical and horizontal morphisms in the obvious way

```
\[
\begin{array}{c}
\bullet \\
\downarrow \\
\bullet
\end{array}
\]
```
• horizontal and vertical composition of 2-cells subject to various unital and associative laws analogous to those of 2-category theory

Double categories are the perfect organizing principle for categories, functors and profunctors because they accommodate two types of morphisms. There is a double category \textbf{Prof} in which

- objects are categories
- vertical morphisms are functors
- horizontal morphisms are profunctors
- 2-cells

\[\begin{array}{ccc}
\mathcal{C}_0 & \xrightarrow{u} & \mathcal{C}_1 \\
F & \downarrow & G \\
\mathcal{D}_0 & \xrightarrow{v} & \mathcal{D}_1
\end{array}\]

are functors \(\alpha : \text{col}(u) \to \text{col}(v)\) such that \(\alpha|_{\mathcal{C}_0} = F\) and \(\alpha|_{\mathcal{C}_1} = G\)

A double category has incorporated in it a vertical 2-category and a horizontal one whose 2-morphisms are cells of the form

\[\begin{array}{ccc}
\bullet & \xrightarrow{id} & \bullet \\
\downarrow & \quad & \downarrow \\
\bullet & \xrightarrow{id} & \bullet
\end{array}\]

\textbf{Prof} contains the 2-category of categories, functors and natural transformations in the vertical direction and the 2-category of categories, profunctors and their transformations in the horizontal direction.

Let \(\mathbb{D}\) be a double category, \(\mathbb{D}_h\) be its horizontal 2-category and

\[F : \mathcal{J} \to \mathbb{D}_h\]

be a small \(\mathcal{J}\)-indexed horizontal diagram. The \textbf{double colimit} of \(F\) is defined to be an object dcolim\(F \in \mathbb{D}\) equipped with

- a vertical morphism \(\alpha_i : F(i) \to \text{dcolim}F\) for all \(i \in \mathcal{J}\)
- a cell \(\alpha_f\)
for each morphism \( f : i \rightarrow j \) in \( J \)
such that everything commutes and it is universal with respect to the vertical direction. The definition allows for \( F \) to be strict, weak or lax by simply factoring the relevant extra structure in "everything commutes".

The following example (for which we are indebted to the authors of [GP99]) is paradigmatic. Let \( u \) be a correspondence between two categories. When we regard \( u \) as a diagram \( \Delta^1 \rightarrow \text{Prof} \) we obtain

\[
d\text{colim} u \cong \text{colim} u
\]

So, constructing the collage of a profunctor is a colimiting process! This is one of the many advantages of organizing profunctors in a double category.

Back to our problem, we already observed that a functor \( F : \mathcal{X} \rightarrow \mathcal{C} \) allows us to regard \( \mathcal{X} \) as a big collage formed by its fibers. Now we can make this precise. Let \( \phi_F : \mathcal{A} \rightarrow \text{Prof} \) be the diagram corresponding to \( F \). Then we have

\[
d\text{colim} \phi_F \cong \mathcal{X}
\]

It is easy to exhibit \( \mathcal{X} \) as a double colimit and verify the universal property so we leave this proof to the reader who would like to try his hand on double colimits.

We may put the above in a more concise manner as follows. Given two horizontal diagrams \( F, G : \mathcal{A} \rightarrow \mathcal{D}_h \) in a double category \( \mathcal{D} \) we may define a \textit{vertical transformation} \( \alpha : F \Rightarrow v G \) between them to consist of:

- a vertical morphism \( \alpha_a : Fa \rightarrow Ga \) for all \( a \in \mathcal{A} \)
- a 2-cell \( \alpha_f \) for all \( f : a \rightarrow b \) in \( \mathcal{A} \)

\[
\begin{array}{c}
Fa \xrightarrow{Ff} Fb \\
\alpha_a \downarrow \alpha_f \downarrow \alpha_b \\
Ga \xrightarrow{Gf} Gb
\end{array}
\]

such that everything commutes (in the most obvious sense). As always, when \( F \) is weak or lax we factor the extra data of \( F \) in the commutativity conditions.
If we assume $D$ to be strict in the vertical direction then we obtain a category $[A, D]_{v}$ of diagrams and vertical transformations.

We can also construct the slice category $D/F$ whose objects are pairs $(x, \alpha)$ of objects $x \in D$ and vertical transformations $\alpha : F \Rightarrow x$ to the constant diagram at $x$, and morphisms $(x, \alpha) \to (y, \beta)$ are vertical morphisms $x \to y$ which respect $\alpha$ and $\beta$. Then the double colimit of $F$ is the initial object in this category.

Now we have enough language to state all of our observations in the form of a theorem.

**Theorem 1.** Taking double colimits produces an equivalence of categories

\[ dcolim : [A, Prof]_{lax} \cong \text{Cat/}A \]

between the category of lax $A$-indexed horizontal diagrams in $Prof$ (with vertical transformations as morphisms) and the slice category over $A$.

### 2.3 Corollary: reinterpreting the Grothendieck construction

Theorem 1 is a more general version of a result from our previous work [Had19] which states that the Grothendieck construction for categories is a double colimit in $Prof$. Now we may obtain this result as a corollary.

Recall that given a small diagram of categories $F : A \to \text{Cat}$ the Grothendieck construction (aka homotopy colimit) of $F$ is the category $\text{Gro}(F)$ with:

- objects those of $F(a)$ for $a \in A$
- a morphism between $x \in F(a)$ and $y \in F(b)$ being a pair $(f, \alpha)$ where $f : a \to b$ is a morphism in $A$ and $\alpha : Ff(x) \to y$ is a morphism in $F(b)$

\[ x \xrightarrow{f} Ff(x) \]

\[ \downarrow_{\alpha} \]

\[ y \]

- Composition given by the formula

\[ (g, \beta) \circ (f, \alpha) = (gf, \beta \circ Fg(\alpha)) \]

whenever it makes sense.
There is a natural map
\[ p_F : \text{Gro}(F) \to A \]
defined by \( p_F(x) = a \) if \( x \in F(a) \) on objects and \( p_F(f, \alpha) = f \) on morphisms. This functor is an example of a Grothendieck fibration in the sense of the following definition.

**Definition 1.** A functor
\[ p : \mathcal{X} \to A \]
is a **Grothendieck fibration** if for any fixed \( f : a \to b \) in \( A \) and \( x \in p^{-1}(a) \) there is a lift \( \bar{f} : x \to \bar{x} \) in \( p^{-1}(f) \) such that any other lift \( x \to y \) of \( f \) factors uniquely through \( \bar{f} \).

These fibrations are also called **cartesian fibrations.** In this case we refer to \( \bar{f} \) as the cartesian lift of \( f \). (This should remind topologists of the unique path lifting property of covering maps). It is clear by construction that any functor \( p_F \) associated to a diagram of categories as above is a Grothendieck fibration by simply putting \( \bar{f} = (f, \text{id}_x) \).

It is not hard to see the converse: any cartesian fibration comes from a diagram of categories. This way we say that the Grothendieck construction gives an equivalence of categories
\[ \text{Gro} : [\mathcal{A}, \text{Cat}] \rightleftharpoons (\text{Cat}/A)_{\text{cart}} \]
between the category of \( \mathcal{A} \)-indexed categories and the full subcategory of the slice category over \( A \) consisting of cartesian fibrations.

On the other hand Theorem 1 tells us that the slice category over \( A \) is equivalent to the category of lax \( \mathcal{A} \)-indexed diagrams of profunctors. It turns out that Grothendieck fibrations correspond precisely to diagramas of profunctors induced by functors.

More precisely, every functor \( F : \mathcal{C} \to \mathcal{D} \) gives us a \((\mathcal{C}, \mathcal{D})\)-profunctor \( F^* \) defined by \( F^*(c, d) = \mathcal{D}(c, Fr_d) \) for all \( c \in \mathcal{C}, d \in \mathcal{D} \). Creating the induced profunctor \( F \mapsto F^* \) produces a weak 2-functor
\[ (\cdot)^* : \text{Cat} \to \text{Prof} \]
as we have isomorphisms $G^* \otimes F^* \cong (G \circ F)^*$. By postcomposition we have a map, which we still denote $(\cdot)^*$, between the diagram categories

$$(\cdot)^* : [A, \text{Cat}] \to [A, \text{Prof}]$$

Then we have a commutative square:

$$
\begin{array}{ccc}
[A, \text{Cat}] & \xrightarrow{\text{Gro}} & (\text{Cat}/A)_{\text{cart}} \\
(\cdot)^* \downarrow & & \downarrow \text{inclusion} \\
[A, \text{Prof}]_v & \xrightarrow{\text{dcolim}} & \text{Cat}/A \\
\end{array}
$$

which is simply telling us that the Grothendieck construction is a double colimit, i.e. given a diagram of categories $F$ we have

$$\text{Gro}(F) \cong \text{dcolim}(F^*)$$

More general homotopy colimits are shown to be double colimits in [Had19].

3 Higher correspondences and simplicial maps

Our goal is to develop results similar to those we introduced above for simplicial sets. Making Lurie’s claim precise falls within this scope as well. In order to achieve this we have to develop double category theory for simplicial sets.

Of course, given that correspondences between simplicial sets cannot be tensored we cannot have an actual double category whose objects are simplicial sets, vertical morphisms are simplicial maps and horizontal morphisms are correspondences. Nonetheless we claim that such a double category theory is achievable.

The key idea we propose is to use simplicial categories instead of double categories. This will turn out to make sense from many angles:

- Higher correspondences themselves look and feel like simplices. Once we define this simplicial structure it is possible to study functors valued in higher correspondences.

- Categories and functors are part of a 2-category, which we recorded as the vertical 2-category of $\text{Prof}$. Simplicial sets and simplicial maps are part of a simplicially enriched category. In general, we will see that a simplicial category, looked at from the double categorical perspective, has a vertical $\text{sSet}$-category as part of it. Thus the latter become good candidates for our purposes.

- The theory of double colimits is available in simplicial categories.

- Some of the expected results can be proved, especially a classification of simplicial maps.
3.1 A simplicial category for higher correspondences

**Definition 2.** An \( n \)-correspondence of simplicial sets is a simplicial set \( X \) equipped with a map \( p : X \to \Delta^n \)

As mentioned in the introduction, the structure map \( p \) allows us to think of \( X \) as a collage of simplicial sets. First, we can think of the fibers \( X_i = p^{-1}(i) \) as the vertices of \( X \). Then we can describe \( X \) to be formed from the \( X_i \)'s by adding new simplices in the increasing direction. The only rule is that we cannot add simplices with all vertices inside one of the \( X_i \)'s.

This picture indicates that an \( n \)-correspondence itself should be regarded as some sort of \( n \)-simplex. For example we can easily define the \( i \)-th face of \( X \) to be obtained by deleting the vertex \( X_i \). More precisely, define \( d_i X \) to be the fiber of the \( i \)-th face of \( \Delta^n \) via the pullback square

\[
\begin{array}{ccc}
\Delta^{n-1} & \xrightarrow{d_i} & \Delta^n \\
\downarrow & & \downarrow p \\
\end{array}
\]

It is easy to see that

\[ d_i X \cong X - X_i \]

The more intricate part is defining degeneracies. The example of profunctors indicates that if \( X \) is a 0-correspondence its degeneracy ought to be the cylinder \( X \times \Delta^1 \to \Delta^1 \). If we keep taking further degeneracies of \( X \) we should produce the higher cylinders \( X \times \Delta^n \), which are higher correspondences via the projection map.

This way, we will define the \( i \)-th degeneracy of an \( n \)-correspondence \( X \) to be obtained by ”extending \( X \) with a cylinder at \( X_i \)”. This construction is given by the pullback

\[
\begin{array}{ccc}
s_i X & \longrightarrow & X \times \Delta^1 \\
\downarrow & & \downarrow p \times \Delta^1 \\
\Delta^{n+1} & \xrightarrow{\iota_i} & \Delta^n \times \Delta^1 \\
\end{array}
\]

where the map \( \iota_i \) is given by components \( s^i : \Delta^{n+1} \to \Delta^n \) and \( \chi_{>i} : \Delta^{n+1} \to \Delta^1 \), the latter being the characteristic function of the subset \( \{i+1, \ldots, n\} \subseteq [n] \)

\[ \chi_{>i}(j) = \begin{cases} 0 & \text{if } j \leq i \\ 1 & \text{if } j > i \end{cases} \]

The following ”formula” is easy to see and perhaps more illuminating:

\[ s_i X \cong X \times \Delta^1 - (X_0, 1) - \cdots - (X_{i-1}, 1) - (X_{i+1}, 0) - \cdots - (X_n, 0) \]
For clarity we will work out an example. Let $X_0 = (a)$ be a copy of $\Delta^0$ and $X_1 = (b \to c)$ be a copy of $\Delta^1$. Consider $X = \Delta^2$ as a 1-correspondence between them.

We first form the cylinder $X \times \Delta^1$

Then we obtain $s_0X$ and $s_1X$ by the prescribed deletions

The simplicial identities are clear from the above formulas. It is also clear from the example that they do not hold strictly but up to isomorphism. After all, we defined faces and degeneracies as pullbacks. Hence we have defined a (weak) simplicial category

$$\text{sSet}^\triangledown : \Delta^{op} \to \text{Cat}$$

whose category of $n$-simplices is precisely the category of $n$-correspondences

$$\text{sSet}^\triangledown_n = \text{sSet}/\Delta^n$$

It is only fair that when we study simplicial objects in a 2-category we are faced with weak functoriality. Nonetheless the coherence theorem tells us it is "safe" to work with them as if they were strict (in some sense).
3.2 The analogy with double categories

We will briefly discuss the double categorical aspects of simplicial categories. Let

\[ \mathbb{E} : \Delta^{op} \rightarrow \textbf{Cat} \]

be a simplicial category. Composing \( \mathbb{E} \) with the nerve functor \( N : \textbf{Cat} \rightarrow \text{sSet} \) allows us to regard it as a bisimplicial set. Bisimplicial sets have a vertical and horizontal direction, both consisting of simplices, tied together by bisimplices.

More explicitly \( \mathbb{E} \) may be thought of as a double categorical structure with:

- objects those of \( \mathbb{E}_0 \)
- vertical morphisms those of \( \mathbb{E}_0 \)
- the objects of \( \mathbb{E}_n \) for various \( n \) as horizontal simplices
- morphisms of \( \mathbb{E}_n \) as prism-shaped cells of various dimensions. For example a morphism \( f : x \rightarrow y \) in \( \mathbb{E}_1 \) looks like a square

\[
\begin{array}{ccc}
  x_0 & \xrightarrow{x} & x_1 \\
  f_0 & \downarrow & f_1 \\
  y_0 & \xrightarrow{y} & y_1
\end{array}
\]

with \( f_0 \) and \( f_1 \) being the faces of \( f \). If \( f \in \mathbb{E}_2 \) it looks like a prism

\[
\begin{array}{ccc}
  x_0 & \xrightarrow{x_1} & x_2 \\
  f_0 & \downarrow & f_1 \\
  y_0 & \xrightarrow{y_1} & y_2
\end{array}
\]

And so on in higher dimensions.

If \( \mathbb{D} \) is a double category we extract its vertical 2-category \( \mathbb{D}_v \) by taking cells whose top and button horizontal morphisms are identities. In a simplicial context the role of identities is played by degeneracies. If \( \mathbb{E} \) is a simplicial category, when we consider only cells in which the top an bottom are totally degenerate we are left with a simplicially enriched category \( \mathbb{E}_v \).

More precisely, let \( s^n \) denote the iterated degeneracy functor

\[ s^n : \mathbb{E}_0 \xrightarrow{s^0} \mathbb{E}_1 \xrightarrow{s^0} \cdots \xrightarrow{s^0} \mathbb{E}_n \]
Then \( \mathbb{E}_v \) is the \( s\text{Set} \)-category whose objects are those of \( \mathbb{E} \), and whose \( n \)-simplices of the mapping space \( \mathbb{E}_v(x, y) \), for \( x, y \in \mathbb{E} \), are given by

\[
\mathbb{E}_v(x, y)_n = \mathbb{E}_n(s^n x, s^n y)
\]

Faces and degeneracies are defined in the obvious way using the simplicial structure of \( \mathbb{E} \).

What the double categorical lens unveiled for us is that

"2-categories are to double categories what \( s\text{Set} \)-categories are to simplicial categories"

Categories, functors and natural transformation are part of a 2-category which serves as the vertical part of \( \text{Prof} \). Simplicial sets, simplicial maps, homotopies and higher homotopies are part of a \( s\text{Set} \)-category \( s\text{Set} \), and we have

\[
s\text{Set}_v^2 = s\text{Set}
\]

We see that under our observations \( s\text{Set}_v^2 \) is indeed analogous to \( \text{Prof} \).

With degeneracies playing the role of identities and simplices the role of composites we can easily extend the theory of double colimits to simplicial categories.

Let \( J \) be a small category (or just a simplicial set). Regarding \( J \) as a discrete simplicial category, a horizontal \( J \)-indexed diagram in \( \mathbb{E} \) is simply a transformation

\[
F : J \to \mathbb{E}
\]

In fact, recall that since \( \text{Cat} \) is a 2-category then the collection of simplicial categories \( s\text{Cat} \) attains a 2-category structure as well. In this 2-category the functor \( F \) as above is a 1-morphism and given diagrams \( F, G \) a 2-morphism between them is precisely what we would call vertical transformation.

For an object \( x \in \mathbb{E} \) we will denote by \( x : J \to \mathbb{E} \) the constant diagram which assigns to each \( n \)-simplex of \( J \) the totally degenerate simplex \( s^n x \in \mathbb{E}_n \).

Then we define the \textbf{double colimit} of \( F \) to be an object \( \text{dcolim} F \) equipped with a vertical transformation \( \Rightarrow \text{dcolim} F \) which is initial with respect to vertical morphisms

\[
F \xrightarrow{x} \text{dcolim} F
\]

It is easy to verify the existence of double colimits and to compute them as well. It turns out that double colimits in \( \mathbb{E} \) exist if and only if \( \mathbb{E}_0 \) is cocomplete and the double colimit of each horizontal simplex exist. Following [GP99] we call the latter cotabulators.
More precisely, for $x \in E_n$ we define its **cotabulator** $\perp_x$ to be the double colimit of the corresponding map $x : \Delta^n \to E$. Let $\text{spx}\mathcal{J}$ be the category of simplices of $\mathcal{J}$. Then we can prove

$$\text{dcolim} F \cong \text{colim}_{\sigma \in \text{spx}\mathcal{J}} F(\sigma)$$

(see Theorem 2 in [Had19]).

### 3.3 Classifying simplicial maps and inner fibrations

Now that we have enough language at our disposal we can state and prove our results. As it happens a lot in category theory, once the correct setup is in place the proofs are really easy.

Let $f : X \to A$ be a map of simplicial sets. For a simplex $\sigma \in A_n$ let $X_\sigma = f^{-1}(\sigma)$ be the preimage, given by the pullback square

\[
\begin{array}{ccc}
X_\sigma & \xrightarrow{f} & X \\
\downarrow & & \downarrow \\
\Delta^n & \xrightarrow{\sigma} & A
\end{array}
\]

By definition $X_\sigma$ is an $n$-correspondence. The assignment $\sigma \mapsto X_\sigma$ produces the classifying diagram of correspondences

$$\phi_f : A \to \text{sSet}^\sharp$$

This way we have

$$X \cong \text{dcolim}(\phi_f)$$

**Proof.** Since any simplicial set is a colimit of its simplices, in virtue of [1] we may simply consider the case $A = \Delta^n$. Then we may simply exhibit $X$ as the desired double colimit (in this case, the cotabulator of itself).

The structure map $X \to s^n X$, given that $s^n X$ is the correspondence given by the projection $X \times \Delta^n \to \Delta^n$, is the morphism $(id_X, f) : X \to X \times \Delta^n$ in $\text{sSet}/\Delta^n$. For a simplicial set $Y$ we have:

$$\text{sSet}_n^\sharp(X, s^n Y) \cong \text{sSet}/\Delta^n(X, Y \times \Delta^n)$$

$$\cong \text{sSet}(X, Y) \times \text{sSet}/\Delta^n(X, \Delta^n)$$

$$\cong \text{sSet}(X, Y)$$

What we have proved is a theorem analogous to Theorem [1].
Theorem 2. For a simplicial set $A$ taking double colimits gives an equivalence of categories

$$dcolim : [A, sSet^\#] \xrightarrow{\sim} sSet/A$$

between the category of $A$-indexed horizontal diagrams in $sSet^\#$ and vertical transformations and the slice category over $A$.

We also obtain the corresponding result for $\infty$-categories as a corollary. As mentioned in the introduction, a simplicial map $p : X \rightarrow A$ is an inner-fibration if and only if each fiber $X_\sigma$, $\sigma \in A_n$, is an $\infty$-category. Given that inner fibrations are stable under pullback and products we may define the simplicial category $\infty Cat^\#$ with

$$\infty Cat^\#_n = \infty Cat/\Delta^n$$

exactly in the same manner we defined $sSet^\#$. Then we have the desired result.

Corollary 1. Taking double colimits in $sSet^\#$ produces an equivalence of categories

$$dcolim : [A, \infty Cat^\#] \xrightarrow{\sim} (\infty Cat/A)_{\text{inner}}$$

between the category of $A$-indexed diagrams of correspondences of $\infty$-categories and the slice category over $A$ consisting of inner fibrations.

Remark. This result is not fully satisfactory given that the above should be seen as an equivalence of $\infty$-categories. Stating such a result seems to require a coherent version of the double category theory we used here. We leave this for future work.
References

[BS18] Clark Barwick and Jay Shah. Fibrations in $\infty$-category theory. In *2016 MATRIX Annals*, pages 17–42. Springer, 2018.

[GP99] Marco Grandis and Robert Paré. Limits in double categories. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 40(3):162–220, 1999.

[Had19] Redi Haderi. Higher equipments, double colimits and homotopy colimits. *arXiv preprint arXiv:1908.06201* 2019.

[Lur09] Jacob Lurie. *Higher Topos Theory (AM-170)*, volume 189. Princeton University Press, 2009.