TRITANGENTS TO SMOOTH Sextic CURVES

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Abstract. We prove that a smooth plane sextic curve can have at most 72 tritangents, whereas a smooth real sextic may have at most 66 real tritangents.

1. Introduction
All algebraic varieties considered in the paper are over $\mathbb{C}$. On a few occasions, we discuss subfields $k \subset \mathbb{C}$ (most notably, $k = \mathbb{R}$: by definition, a real variety is a complex one equipped with an anti-holomorphic involution), but we never consider fields of positive characteristic.

1.1. Principal results. This paper concludes the study of the maximal number of straight lines in a smooth polarized $K3$-surface. The most classical case, viz. that of spatial quartics $X \subset \mathbb{P}^3$, goes as far back as to A. Clebsch [2] (the upper bound of $m(11m - 24)$ lines in a smooth degree $m$ surface $X \subset \mathbb{P}^3$, yielding at most 80 lines in a quartic) and F. Schur [27] (an example of a smooth quartic with 64 lines). The sharp upper bound of 64 lines was established in B. Segre [28]. Two recent papers aroused new interest to this classical problem: first, a minor gap in Segre’s proof was discovered and corrected by S. Rams and M. Schütt [24], where the argument was also extended to all characteristics $p \geq 5$ of the ground field, and second, an alternative, purely arithmetical (or rather lattice theoretic) proof was given in [8]; this approach allowed us to obtain, in addition to mere upper bounds, a complete classification of all large (i.e., more than 52) configurations of lines, prove the uniqueness of the line maximizing quartic, and establish the sharp upper bound of 56 real lines in a real smooth quartic surface (also realized by a unique real quartic). A great deal of other works have appeared almost immediately and, for the moment, the case of spatial quartic surfaces still remains the best studied one: there are sharp upper bounds on the number of lines over algebraically closed fields of positive characteristic (see [4, 23, 24, 25]) and over $\mathbb{R}$ (see [8]), partial bounds over $\mathbb{Q}$ (see [6]), upper bounds for singular quartics, both $K3$ (see [31, 32]) and not (see [11]), explicit equations of quartics with many lines (see [8, 29, 33]), etc.

Lines in smooth polarized $K3$-surfaces $X \to \mathbb{P}^{d+1}$ of all degrees $2d \geq 4$, both birational and hyperelliptic (cf. [26]), were studied in [5], using an arithmetical reduction similar to [8] and an appropriate taxonomy of prospective Fano graphs. (It appears that, so far, the more conventional geometric arguments have failed to produce even reasonable bounds on the number of lines.) Among other results, found in [5] are sharp upper bounds on the number of lines, both over $\mathbb{C}$ and $\mathbb{R}$, and a complete description of all large configurations of lines, especially in the two
most “classical” cases, viz. sextics in \( \mathbb{P}^4 \) and octics (most notably, triquadrics) in \( \mathbb{P}^5 \), which give rise to a number of interesting Fano graphs. An unexpected discovery is the fact that the configurations of lines simplify dramatically when the degree grows: asymptotically, for \( 2d \gg 0 \), all lines are either linearly independent in \( H_2(X) \) or, else, among the fiber components of a certain fixed elliptic pencil; in either case, their number does not exceed 24. (The true sharp bound oscillates between 21, 22, and 24, periodically in the degree \( 2d \gg 0 \), whereas its real counterpart oscillates between 19, 20, and 21, with a larger period.) The other side of the coin is the fact that in the remaining “classical” case, the smallest degree \( 2d = 2 \) (double planes), the dual adjacency graph of lines may be too large: the star of a single vertex is more complicated than the whole Fano graph of a quartic. For this reason, the case \( 2d = 2 \) was left out as not feasible in [5]; it is treated in the present paper by means of considerably different arithmetical techniques (see §1.2).

In [6] it was conjectured that the maximal number of lines in a smooth 2-polarized \( K3 \)-surface is 144, with the maximum realized by the double plane \( X \rightarrow \mathbb{P}^2 \) ramified over the sextic curve

\[
z_0^2 + z_1^2 + z_2^2 = 10(z_0^3 z_1^3 + z_1^3 z_2^3 + z_2^3 z_0^3).
\]

(This equation is borrowed from Sh. Mukai [16], as the surface in question admits a faithful action of the Mukai group \( M_3 \); explicit equations of the predicted 144 lines were found independently by D. Festi and Y. Zaytman, private communication.) The conjecture is motivated by the fact that, like Schur’s quartic [27] and some line maximizing sextics in \( \mathbb{P}^4 \) and octics in \( \mathbb{P}^5 \) (see [5, 6]), this surface minimizes the discriminant of a singular \( K3 \)-surface admitting a smooth model of a given degree.

In the present paper we settle (in the affirmative) and extend the conjecture, see Theorem 1.2, Addendum 1.3, and Theorem 1.4. We state our principal results in terms of tritangents to the ramification locus \( C \subset \mathbb{P}^2 \) (a smooth sextic curve) rather than lines in the surface \( X \rightarrow \mathbb{P}^2 \), dividing the numbers by 2 (see §2.2 below for further details). Certainly, when speaking about tritangents, we allow the collision of some of the tangency points; in other words, a tritangent to a smooth sextic \( C \subset \mathbb{P}^2 \) is merely a line \( L \subset \mathbb{P}^2 \) such that the local intersection index \( (L \circ C)_P \) at each intersection point \( P \in L \cap C \) is even.

**Theorem 1.2** (see §9.1 and §9.2). Let \( t(C) \) denote the number of tritangents to a smooth sextic \( C \subset \mathbb{P}^2 \). Then either

- \( t(C) = 72 \), and then \( C \) is the sextic given by (1.1), or
- \( t(C) = 66 \), and then \( C \) is one of the two sextics described in §9.1(2), (3), or \( t(C) \leq 65 \).

Previously known bounds are \( t(C) \leq 76 \) in N. Elkies [9] (cf. Corollary 2.7 below) and \( t(C) \leq 108 \) given by Plücker’s formulas. Note that, unlike the 28 bitangents to any smooth quartic curve (and like the case of other polarized \( K3 \)-surfaces), counting tritangents to a sextic is not an enumerative problem: tritangents are not stable under deformation and a typical sextic has no tritangents at all.

**Addendum 1.3** (see §9.1). The number \( t(C) \) as in Theorem 1.2 takes all values in the set \{0, 1, \ldots, 65, 66, 72\} except, possibly, 61.

Twelve sextics (six configurations of lines) with \( 62 \leq t(C) \leq 65 \) are described in §9.1(4)–(9), but we do not assert the completeness of this list. In spite of extensive, although not exhaustive, search, we could not find a sextic with 61 tritangents.
There are reasons (e.g., Corollary 2.7 below or the large number of sextics with 60 tritangents) to believe that 61 is a natural threshold in the problem, but taking the classification down to 61 tritangents would require too much computing power.

As a by-product of the partial classification given by Theorem 1.2, we obtain a sharp upper bound on the number of real tritangents to a real sextic.

**Theorem 1.4** (see §9.3). *The number of real tritangents to a real smooth (over \(\mathbb{C}\)) sextic \(C \subset \mathbb{P}^2\) does not exceed 66. Up to real projective transformation, a smooth real sextic with 66 real tritangents is unique, see §9.1(2).*

**Remark 1.5.** At present, I do not know what other values are taken by the number of real tritangents to a real smooth sextic. In the range \(61 \leq t_R \leq 65\), among the known examples, there is but one other configuration, with \(t_R = 63\) tritangents, see §9.1(6) (and §9.3 for the explanation and further remarks).

1.2. **Contents of the paper.** As in [5, 8], the line counting problem has a simple arithmetical reduction (see Theorem 2.1): one can effectively decide whether a given graph \(\Gamma\) can serve as the Fano graph of a polarized \(K3\)-surface. The candidates \(\Gamma\) to be tried were constructed in [5, 8] line by line, starting from a sufficiently large and sufficiently simple graph. Unfortunately, this straightforward approach seems to diverge in the case of degree 2, and we choose another one, viz. we replant the prospective Néron–Severi lattice \(\text{NS} := \mathbb{Z}\Gamma / \ker\) to an appropriate Niemeier lattice. (The idea of embedding \(h^\perp \subset \text{NS}\) to a Niemeier lattice is not new, cf. Kondō [12], Nikulin [19], Nishiyama [20], etc. The novelty is the fact that, as we need to keep track of the polarization \(h\), we have to rebuild the *hyperbolic* lattice \(\text{NS}\) to embed it to a *definite* Niemeier lattice \(N\). As a result, instead of counting roots in \(\text{NS}\), we work with square 4 vectors in a certain root-free sublattice \(S \subset N\); unlike [6], this lattice \(S\) or even its genus is not assumed fixed. This construction is explained in §2.4, see Proposition 2.4.) Then, instead of dealing with abstract graphs of *a priori* unbounded complexity, we merely need to consider subsets \(L\) of several finite sets \(\mathcal{F}(h)\) known in advance. The precise arithmetical conditions on the subsets \(L\) that may serve as Fano graphs are stated in §2.5 and §2.6.

This approach has a number of advantages. First, for most 6-polarized Niemeier lattices \(N \ni h\) we have an immediate bound \(|L| \leq 130\) (often even \(|\mathcal{G}(h)| \leq 130\) obtained as explained in §4. Second, the sets \(\mathcal{G}(h)\) have rich intrinsic structure, splitting into orbits and combinatorial orbits (see §3.1), which can be used in the construction of large geometric subsets: instead of building them line-by-line from scratch, we try to patch together precomputed close to maximal intersections with the combinatorial orbits. These algorithms are described in §3. Finally, since we are working with known sets, all symmetry groups can be expressed in terms of permutations, which makes the computation in GAP [10] extremely effective.

In §5–§8 we treat, one by one, the 23 Niemeier lattices rationally generated by roots, outlining the details of the computation in those few cases where the *a priori* upper bound \(|L| \leq 130\) fails. In §9, we draw a formal punch-line, collecting together our findings for individual Niemeier lattices and completing the proofs of the principal results of the paper.

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2. The reduction

The tritangent problem is reduced to an arithmetical question about the Néron–Severi lattice $\text{NS}(X)$ of a smooth 2-polarized $K3$-surface $X$, see Theorem 2.1. The construction of §2.4, combined with Proposition 2.4, replants $\text{NS}(X)$ to a Niemeier lattice. The invertibility of this construction is discussed in §2.6.

2.1. Lattices (see [18]). The principal goal of this section is fixing the terminology and notation. A lattice is a free abelian group $L$ of finite rank equipped with a symmetric bilinear form $b: L \otimes L \to \mathbb{Z}$. Since $b$ is assumed fixed (and omitted from the notation), we abbreviate $x \cdot y := b(x, y)$ and $x^2 := b(x, x)$. A lattice $L$ is even if $x^2 = 0 \mod 2$ for all $x \in L$; otherwise, $L$ is odd. The determinant $\det L \in \mathbb{Z}$ is the determinant of the Gram matrix of $b$ in any integral basis; $L$ is called nondegenerate (unimodular) if $\det L \neq 0$ (respectively, $\det L = \pm 1$). The inertia indices $\sigma_{\pm} L$ are those of $L \otimes \mathbb{R}$. A nondegenerate lattice $L$ is called hyperbolic if $\sigma_+ L = 1$.

The hyperbolic plane is the only unimodular even lattice of rank 2. Explicitly, $U = \mathbb{Z}a + \mathbb{Z}b$, where $a^2 = b^2 = 0$ and $a \cdot b = 1$. One has $\sigma_+ U = \sigma_- U = 1$.

A nondegenerate lattice $L$ admits a canonical inclusion

$$L \hookrightarrow L^\vee := \{ x \in L \otimes \mathbb{Q} \mid x \cdot y \in \mathbb{Z} \text{ for all } y \in L \}$$

to the dual group $L^\vee$. The finite abelian group $L := \text{discr } L := L^\vee / L (q_L \text{ in [18]}$) is called the discriminant group of $L$. Clearly, $|L| = (-1)^{\sigma_- L} \det L$. This group is equipped with the nondegenerate symmetric bilinear form

$$L \otimes L \to \mathbb{Q} / \mathbb{Z}, \quad (x \mod L) \otimes (y \mod L) \mapsto (x \cdot y) \mod \mathbb{Z},$$

and, if $L$ is even, its quadratic extension

$$\mathcal{L} \to \mathbb{Q} / 2\mathbb{Z}, \quad x \mod L \mapsto x^2 \mod 2\mathbb{Z}.$$

We denote by $\mathcal{L}_p := \text{discr}_p L := \mathcal{L} \otimes \mathbb{Z}_p$ the $p$-primary components of $\text{discr } L$. The 2-primary component $L_2$ is called even if $x^2 \in \mathbb{Z}$ for all order 2 elements $x \in L_2$; otherwise, $L_2$ is odd. The determinant $\det L_p$ is the determinant of the “Gram matrix” of the quadratic form in any minimal set of generators. (This is equivalent to the alternative definition given in [18].) Unless $p = 2$ and $L_2$ is odd (in which case the determinant is not defined or used), we have $\det L_p = u_p / |L_p|$, where $u_p$ is a well-defined element of $\mathbb{Z}_p^* / (\mathbb{Z}_p^*)^2$.

The length $\ell(A)$ of a finite abelian group $A$ is the minimal number of generators of $A$. We abbreviate $\ell_p(A) := \ell(A \otimes \mathbb{Z}_p)$ for a prime $p$.

Given a lattice $L$ and $q \in \mathbb{Q}$, we use the notation $L(q)$ for the same abelian group with the form $x \otimes y \mapsto q(x \cdot y)$, assuming that it is still a lattice. We abbreviate $-L := L(-1)$, and this notation applies to discriminant forms as well. The notation $nL, n \in \mathbb{Z}_+$, is used for the orthogonal direct sum of $n$ copies of $L$.

A root in an even lattice $L$ is a vector of square $\pm 2$. A root system is a positive definite lattice generated by roots. Any root system has a unique decomposition into orthogonal direct sum of irreducible components, which are of types $A_n, n \geq 1$, $D_n, n \geq 4$, $E_6, E_7$, or $E_8$ (see, e.g., [1]), according to their Dynkin diagrams.

A Niemeier lattice is a positive definite unimodular even lattice of rank 24. Up to isomorphism, there are 24 Niemeier lattices (see [17]): the Leech lattice $\Lambda$, which is root free, and 23 lattices rationally generated by roots. In the latter case, the
isomorphism class of a lattice $N := N(D)$ is uniquely determined by that of its maximal root system $D$. For more details, see [3].

2.2. The covering $K3$-surface. Given a smooth sextic curve $C \subset \mathbb{P}^2$, the double covering $\varphi: X \to \mathbb{P}^2$ ramified over $C$ is a $K3$-surface. The “hyperplane section” $\varphi^*\mathcal{O}_{\mathbb{P}^2}(1)$ is a 2-polarization of $X$, i.e., a complete fixed point free degree 2 linear system; it is viewed as an element

$$h \in \text{Pic} X = \text{NS}(X) \subset H_2(X; \mathbb{Z}) \cong -2E_8 \oplus 3U.$$ 

Here, the group $H_2(X; \mathbb{Z}) = H^2(X; \mathbb{Z})$ is regarded as a lattice via the intersection form; it can be characterized as the only unimodular even lattice of rank 22 and signature $\sigma_+ - \sigma_- = -16$. The Néron–Severi lattice $\text{NS}(X) = H^{1,1}(X) \cap H_2(X; \mathbb{Z})$ is a primitive hyperbolic sublattice; in particular, $\rho(X) := \text{rk} \text{NS}(X) \leq 20$. Conversely, any 2-polarization $h$ of a $K3$-surface $X$ gives rise to a degree 2 map $\varphi_h: X \to \mathbb{P}^2$ ramified over a sextic curve $C \subset \mathbb{P}^2$ (see [21, 26]). This curve is smooth if and only if no $(-2)$-curve is contracted by $\varphi_h$, or, equivalently, there is no class $e \in \text{NS}(X)$ such that $e^2 = -2$ and $e \cdot h = 0$. With the ramification locus in mind, a 2-polarized $K3$-surface $(X, h)$ with this extra property is called smooth.

A line in a 2-polarized $K3$-surface $(X, h)$ is a smooth rational curve $L \subset X$ such that $L \cdot h = 1$. Any two distinct lines $L_1, L_2 \subset X$ either are disjoint, $L_1 \cdot L_2 = 0$, or intersect at a single point, $L_1 \cdot L_2 = 1$, or intersect at three points, $L_1 \cdot L_2 = 3$, the latter being the case if and only if $L_1, L_2$ are interchanged by the deck translation of the covering $\varphi_h: X \to \mathbb{P}^2$. Since, on the other hand, $L^2 = -2$, each line is unique in its homology class $[L] \in \text{NS}(X)$. Each 2-polarized $K3$-surface has finitely many lines (typically none). The Fano graph $\text{Fn}(X, h)$ is the set of lines in $X$ in which each pair of lines $L_1, L_2$ (regarded as vertices of the graph) is connected by an edge of multiplicity $L_1 \cdot L_2$ (i.e., no edge, simple edge, or triple edge).

Let $C \subset \mathbb{P}^2$ be a smooth sextic and $\varphi: X \to \mathbb{P}^2$ the covering $K3$-surface. If $L \subset \mathbb{P}^2$ is a tritangent to $C$, its pull-back $\varphi^{-1}(L)$ splits into two lines $L_1, L_2$; they intersect at the three points of tangency of $L$ and $C$ (possibly, infinitely near) and are interchanged by the deck translation $\tau$ of $\varphi$. Conversely, any line in $X$ projects to a tritangent to $C$. Thus, the set of tritangents to $C$ is identified with $\text{Fn}X/\sim$, where the free involution $*: \text{Fn}X \to \text{Fn}X$ induced by $\tau$ is intrinsic to the graph: it sends a vertex $L$ to the only vertex connected to $L$ by a triple edge.

2.3. The arithmetic reduction of the tritangent problem. Throughout this paper, by a 2-polarized lattice we mean a hyperbolic even lattice $\text{NS}$ equipped with a distinguished class $h \in \text{NS}$ of square $h^2 = 2$. The Fano graph of a 2-polarized lattice $\text{NS} \ni h$ is the set

$$\text{Fn}(\text{NS}, h) := \{l \in \text{NS} \mid l^2 = -2, l \cdot h = 1\}$$

with two points (vertices) $l_1, l_2$ connected by an edge of multiplicity $l_1 \cdot l_2$. This graph is equipped with a natural involution

$$l \mapsto l^*: = h - l;$$

the vertex $l^*$, called the dual of $l$, is connected to $l$ by a triple edge. Usually, we assume, in addition, that the orthogonal complement $h^\perp \subset \text{NS}$ is root free. Under this additional assumption,

for $l_1, l_2 \in \text{Fn}(\text{NS}, h)$, one has $l_1 \cdot l_2 = 3$ (iff $l_1 = l_2$), 1, 0, or $-2$ (iff $l_1 \neq l_2$); hence, all edges of $\text{Fn}(\text{NS}, h)$ other than $(l, l^*)$ are simple.
The following statement is well known: it follows from the global Torelli theorem for K3-surfaces [22], surjectivity of the period map [13], and Saint-Donat’s results onprojective K3-surfaces [26] (cf. also [8, Theorem 3.11] or [6, Theorem 7.3]).

**Theorem 2.1.** A graph $\Gamma$ is the Fano graph of a smooth 2-polarized K3-surface if and only if $\Gamma \cong \text{Fn}(\text{NS}, h)$ for some 2-polarized lattice $\text{NS} \ni h$ admitting a primitive embedding $\text{NS} \hookrightarrow -2\mathbf{E}_8 \oplus 3\mathbf{U}$ and such that $h^\perp \subset \text{NS}$ is root free. $\square$

2.4. Embedding to a Niemeier lattice. Let $\text{NS} \ni h$ be a 2-polarized lattice. Consider the orthogonal complement $h^\perp \subset \text{NS}$. Each vector $l \in \text{Fn}(\text{NS}, h)$ projects to $l' := l - \frac{1}{2} h \in (h^\perp)^\vee$, and, assuming $\text{Fn}(\text{NS}, h) \neq \emptyset$, there is a unique index 2 extension

$$-S \ni h^\perp \oplus \mathbb{Z}h, \quad h^2 = -6,$$

containing all vectors $l' + \frac{1}{2} h$, $l \in \text{Fn}(\text{NS}, h)$. The lattice $S := S(\text{NS}, h)$ obtained from $-S$ by reverting the sign of the binary form is positive definite, and there is an obvious canonical bijection between $\text{Fn}(\text{NS}, h)$ and the set

$$\mathcal{L} = \mathcal{L}(S, h) := \{ l \in S \mid l^2 = 4 \text{ and } l \cdot h = 3 \};$$

the elements of $\mathcal{L}$ are called lines in $S$. Furthermore, the sublattice $h^\perp \subset \text{NS}$ is root free if and only if so is $h^\perp \subset S$; in this case, we call $S \ni h$ admissible.

For the images $l_1, l_2 \in \mathcal{L}$ of two lines $l_1', l_2' \in \text{Fn}(\text{NS}, h)$ one has $l_1 \cdot l_2 = 2 - l'_1 \cdot l'_2$. Hence, if $S \ni h$ is admissible, then

$$-S \ni h^\perp \oplus \mathbb{Z}h,$$

for $l_1, l_2 \in \mathcal{L}$, one has $l_1 \cdot l_2 = -1$ (iff $l_1 = l_2^\perp$), 1, 2, or 4 (iff $l_1 = l_2$).

We say that $l_1, l_2$ intersect (are disjoint) if $l_1 \cdot l_2 = 1$ (respectively, $l_1 \cdot l_2 = 2$). Accordingly, we regard $\mathcal{L}$ as a graph, with two distinct vertices $l_1, l_2$ connected by a simple (triple) edge whenever $l_1 \cdot l_2 = 1$ (respectively, $l_1 \cdot l_2 = -1$.)

**Proposition 2.4.** Let $\text{NS} \ni h$ be a primitive 2-polarized sublattice of $-2\mathbf{E}_8 \oplus 3\mathbf{U}$, $\text{Fn}(\text{NS}, h) \neq \emptyset$, and let $S := S(\text{NS}, h)$ be the lattice constructed as in (2.2). Then

1. $S$ admits a primitive embedding to a Niemeier lattice $N$;
2. $S$ admits an embedding $S \hookrightarrow N$ to a Niemeier lattice such that the torsion of $N/S$ is a 3-group and $S$ is orthogonal to a root $\bar{r} \in N$.

**Proof.** Denote $\rho := \text{rk} \text{NS}$ and $\mathcal{N} := \text{discr} \text{NS}$, so that we have $\ell(\mathcal{N}) \leq 22 - \rho$ by Theorem 1.12.2 in [18]. Since $h \notin 2\mathbf{NS}^\perp$ (by the assumption that $\text{Fn}(\text{NS}, h) \neq \emptyset$), we have

$$\text{discr} h^\perp = \left(\frac{1}{2} h\right)^2 \oplus \mathcal{N}, \quad \left(\frac{1}{2} h\right)^2 = \frac{3}{2} \mod 2\mathbb{Z},$$

and the construction changes this to

$$\text{discr} S = \left(\frac{1}{2} h\right)^2 \oplus (-\mathcal{N}), \quad \left(\frac{1}{2} h\right)^2 = \frac{3}{2} \mod 2\mathbb{Z}.$$

In particular, $\ell(\text{discr} S) \leq \ell(\mathcal{N}) + 1 < 24 - \rho$, and Theorem 1.12.2 in [18] implies the existence of a primitive embedding $S \hookrightarrow N$. For the second statement, we compute

$$S := \text{discr}(S \oplus \mathbb{Z}\bar{r}) = \left(\frac{1}{2} h\right)^2 \oplus \left(\frac{1}{2} \bar{r}\right)^2 \oplus (-\mathcal{N}), \quad \left(\frac{1}{2} \bar{r}\right)^2 = \frac{1}{3} \mod 2\mathbb{Z}.$$

This time we have $\ell(S_p) = \ell(N_p) < 23 - \rho = 24 - \text{rk}(S \oplus \mathbb{Z}\bar{r})$ for each prime $p > 3$, whereas $\ell(S_2) = \ell(N_p) + 1 \leq 24 - \text{rk}(S \oplus \mathbb{Z}\bar{r})$ for $p = 2, 3$. Since $S_2$ is odd, the possible equality does not impose any extra restriction at $p = 2$. For $p = 3$, in the case of equality, the “wrong” determinant $\det(-S_3) = -|S| \mod (Z_3^3)^2$ does inhibit the existence of a primitive embedding. However, since $\ell(S_3) \geq 3$ in this case, we may pass to an iterated index 3 extension and reduce the length. $\square$
2.5. Admissible sets. In the rest of the paper, we mainly use statement (2) of Proposition 2.4: it lets us avoid the Leech lattice, although at the expense of the possible imprimitivity (which makes some statements somewhat weaker and more complicated, see, e.g., Proposition 2.9 below). The idea is to construct a lattice \( S \) (or, rather, its set of lines) directly inside a Niemeier lattice. Thus, we fix a Niemeier lattice \( N \), a square 6 vector \( h \in N \), and, optionally, a root \( \bar{r} \in h^\perp \) (which is typically omitted from the notation). Consider the set
\[
\mathfrak{F} := \mathfrak{F}(h) := \{ l \in N \mid l^2 = 4, l \cdot h = 3 \text{ (and } l \cdot \bar{r} = 0) \}.
\]
It is equipped with the involution
\[
*: l \mapsto l^* := \bar{h} - l.
\]
The elements of \( \mathfrak{F}(h) \) are called lines. The span of a subset \( L \subset \mathfrak{F}(h) \) is the lattice
\[
\text{span}_L : = (\mathbb{Z}_3 L + Z_3 h) \cap N \subset N.
\]
If \( L \) is symmetric, \( L^* = L \), the summation with \( Z_3 h \) is redundant as \( \bar{h} \in ZL \). On a few occasions, we also consider the integral and rational span
\[
\text{span}_L : = (\mathbb{Z} L + Zh) \cap N \subset \text{span}_L \subset (\mathbb{Q} L + Qh) \cap N.
\]
(The latter is primitive in \( N \).) Via span, we extend to subsets \( L \subset \mathfrak{F}(h) \) much of the terminology applied to lattices. Thus, the rank of \( L \) is \( \text{rk}_L := \text{rk span}_L \), and we say that \( L \) is generated by a subset \( L' \subset L \) if \( L = \mathfrak{F}(h) \cap \text{span}_L' \).

By definition, the torsion of \( N/\text{span}_L \) is a 3-group and \( h \in 3(\text{span}_L)^\vee \). A finite index extension \( S \supset \text{span}_L \) is called mild if \( S \subset \{ v \in N \mid v \cdot h = 0 \text{ mod 3} \} \) (i.e., \( S \subset N \) and still \( h \in 3S^\vee \)) and \( S \) contains no roots \( r \in h^\perp \subset N \). Note that the latter condition is equivalent to the requirement that \( S \) itself should be root free. Indeed, since \( S \) is positive definite and \( h \in 3S^\vee \), we have \( r \cdot h = 0 \) or \( \pm 3 \) for any root \( r \in S \), and in the latter case \( h + r \) is a root in \( h^\perp \).

Definition 2.5. A subset \( L \subset \mathfrak{F}(h) \) is called admissible if
\[
(1) \ L \text{ is symmetric (or } * \text{-invariant), i.e., } L^* = L, \text{ and}
\]
\[
(2) \text{ the sublattice } h^\perp \cap \text{span}_L \text{ contains no roots.}
\]
A subset \( L \subset \mathfrak{F}(h) \) is complete if \( L = \mathfrak{F}(h) \cap \text{span}_L \). A subset \( L \) is saturated if the identity \( L = \mathfrak{F}(h) \cap S \) holds for any mild extension \( S \supset \text{span}_L \). Finally, we say that \( L \) is \( \mathbb{Q} \)-complete if \( L = \mathfrak{F}(h) \cap \text{span}_\mathbb{Q} L \).

Often, it is easier to check (2.3), which follows from (1), (2) above. Indeed, since \( S \) is definite, we have \(-1 \leq l_1 \cdot l_2 \leq 4 \). Thus, forbidden are \( l_1 \cdot l_2 = 3 \) or 0, as then \( l_1 - l_2 \) or \( l_1 - l_2^2 = l_1 + l_2 - h \), respectively, would be a root in \( h^\perp \).

The following bound is due to N. Elkies.

Theorem 2.6 (N. Elkies, [9]). Let \( V \) be a Euclidean vector space, \( \dim V = n \), and let \( v_1, \ldots, v_N \in V \) be a collection of unit vectors such that the products \( v_i \cdot v_j \), \( i \neq j \), take but two values \( \tau_1, \tau_2 \). Assume that \( \tau_1 + \tau_2 \leq 0 \) and \( 1 + \tau_1 \tau_2 n > 0 \). Then
\[
N \leq \frac{(1 - \tau_1)(1 - \tau_2)n}{1 + \tau_1 \tau_2 n}.
\]

Selecting a single vector from each pair \( l, l^* \in \mathfrak{F} \) and applying Theorem 2.6 to the normalized projections to \( h^\perp \subset \text{span}_L \), we arrive at the following corollary.
Corollary 2.7 (N. Elkies [9]). The size of an admissible set \( \mathcal{L} \) is bounded via

\[
|\mathcal{L}| \leq \frac{48(rk \mathcal{L} - 1)}{26 - rk \mathcal{L}}.
\]

Since \( |\mathcal{L}| \) is even, this gives us \( |\mathcal{L}| \leq 152 \) or 122 for \( rk \mathcal{L} = 20 \) or 19, respectively.

2.6. Geometric sets. According to Theorem 2.1 and Proposition 2.4, the Fano graph of any smooth 2-polarized K3-surface \( X \) can be represented as a complete admissible subset \( \mathcal{L} \subseteq \mathfrak{F}(h) \) for an appropriate pair \( h, \bar{r} \in N \) as in §2.5.

For some lattices (those with few roots), the admissibility condition is not enough to eliminate large sets of lines, and we need to use the full range of restrictions.

Recall that we start with the Néron–Severi lattice \( NS(X) \ni \bar{h} \), and as long as the configurations of lines are concerned, we can assume this lattice rationally generated by lines. Indeed, let \( N := \mathbb{Q} \text{Fn}(X) \cap NS(X) \); clearly, \( \bar{h} \in N \). We can pick a vector \( \omega \in (N^\perp) \otimes \mathbb{C}, \omega^2 = 0 \), in the same component of the positive cone as the period (class of a holomorphic 2-form) \( \omega_X \) of \( X \), and such that \( \omega^\perp \cap H_2(X) = N \). (This condition merely means that \( \omega \) is generic; it can be chosen arbitrary close to \( \omega_X \).) By the surjectivity of the period map [13], there is a K3-surface \( X' \to \mathbb{P}^2 \) such that \( \omega_{X'} = \omega \), so that \( NS(X') = N \). (The fact that \( \bar{h} \in N \) defines a map \( X' \to \mathbb{P}^2 \) with a smooth ramification locus follows from that for \( X \), cf. [26] or §2.2.) Clearly, we have \( \text{Fn}(X') = \text{Fn}(X) \), cf. §2.2, and the lines in \( X' \) generate (over \( \mathbb{Q} \)) its Néron–Severi lattice \( N \) by the construction.

Thus, assuming that \( NS(X) \ni \bar{h} \) is rationally generated by lines, we can pass to the positive definite lattice \( S \ni \bar{h} \) as in (2.2) and embed the latter to a Niemeier lattice \( N \), mapping \( \text{Fn}(X) \) bijectively onto the admissible set \( \mathcal{L} = \mathfrak{F}(h) \cap S \). Most steps of this construction are invertible. However, starting from an admissible set \( \mathcal{L} \subseteq \mathfrak{F}(h) \), we may have to take for \( S \) a mild extension of span \( \mathcal{L} \) rather than span \( \mathcal{L} \) itself and, still, we cannot guarantee that the lattice \( NS \) obtained from \( S \) by the backward construction admits a primitive embedding to \( H_2(X) \cong -2E_8 \oplus 3U \). This discussion motivates the following definition.

Definition 2.8. An admissible set \( \mathcal{L} \subseteq \mathfrak{F}(h) \) is called geometric if \( \mathcal{L} \) is complete in some mild extension \( S \ni \text{span} \mathcal{L} \) such that the lattice \( NS \) obtained from \( S \ni \bar{h} \) by the inverse of construction (2.2) admits a primitive embedding to \( -2E_8 \oplus 3U \).

Using Theorem 1.12.2 in [18], one can recast this property as follows. (For a mild extension \( S \ni \text{span} \mathcal{L} \) there is a splitting \( \text{discr} S = (\frac{1}{2} h) \oplus T \), and we merely restate the restrictions on \( T \cong -\text{discr} NS \) in terms of \( \text{discr} S \).)

Proposition 2.9. Let \( N \ni \bar{h} \) be as above. An admissible set \( \mathcal{L} \subseteq \mathfrak{F}(h) \) is geometric if and only if:

1. \( \text{rk} \mathcal{L} \leq 20; \) we denote \( \delta := 22 - \text{rk} \mathcal{L} \geq 2 \), and there is a mild extension \( S \ni \text{span} \mathcal{L} \) in which \( \mathcal{L} \) is a complete subset and such that the discriminant \( S := \text{discr} S \) has the following properties at each prime \( p \):
   2. if \( p > 3 \), then either \( \ell(S_p) < \delta \) or \( \ell(S_p) = \delta \) and \( \det S_p = 3|S| \mod (Q^x_p)^2 \);
   3. either \( \ell(S_2) < \delta \) or \( \ell(S_2) = \delta \) and \( S_2 \) is odd or \( \det S_2 = \pm 3|S| \mod (Q^x_2)^2 \);
   4. either \( \ell(S_3) \leq \delta \) or \( \ell(S_3) = \delta + 1 \) and \( \det S_3 = |S| \mod (Q^x_3)^2 \).

Remark 2.10. In practice, when eliminating large admissible sets, we use just a few simple consequences of Proposition 2.9. The main rôle is played by condition (1), see §3.2.1 below. Then, conditions (2) and (3) are used, as they apply directly to
the original discriminant \( \text{discr}_p(\text{span} \mathcal{L}) = S_p, \ p \neq 3 \). Condition (4) is typically used when there is an obvious maximal mild extension, and we never insist that \( \mathcal{L} \) should be complete in \( S \), thus eliminating both \( \mathcal{L} \) itself and all its oversets.

3. The Approach

Throughout this section, we consider a Niemeier lattice \( N := N(D) \) generated over \( \mathbb{Q} \) by a fixed root system \( D = \bigoplus_k D_k, \ k \in \Omega \), where \( D_k \) are the irreducible components (aka Dynkin diagrams) and \( \Omega \) is the index set. We construct \( N \) as a subgroup of \( \bigoplus \mathcal{D}_k \); the vectors in
\[
\text{discr} \ D := D^*/D = \bigoplus_k \text{discr} \ D_k
\]
that are declared “integral” are as described in [3, Table 16.1]. (We also use the convention of [3] for the numbering of the discriminant classes of irreducible root systems.) We denote by \( O := O(N) \) the full orthogonal group of \( N \), and by \( R := R(N) \subset O(N) \) its subgroup generated by reflections. Both groups preserve \( D \); the reflection group \( R(N) \) preserves each \( D_k \) and acts identically on \( \text{discr} \ D \).

3.1. Notation. We fix a square 6 vector \( h \in N \) and, sometimes, a root \( \bar{r} \in D \) orthogonal to \( h \). (This root is usually omitted from the notation.) We denote by \( O_h(N) \subset O(N) \) and \( R_h(N) \subset R(N) \) the subgroups stabilizing \( h \) (and \( \bar{r} \)). Let
\[
\mathcal{F} = \mathcal{F}(h) = \mathcal{F}(h, \bar{r}) := \{ l \in N \mid l^2 = 4, l \cdot h = 0 \text{ (and } l \cdot \bar{r} = 0) \}\]
be the set of lines. This set splits into a number of \( O_h(N) \)-orbits \( \bar{o}_n \), which split further into \( R_h(N) \)-orbits \( \bar{o} \subset \bar{o}_n \); the latter are called combinatorial orbits. It is immediate that the duality \( l \mapsto l^* \) preserves orbits and combinatorial orbits; hence, we can speak about the dual orbits \( \bar{o}_n^* \) and \( \bar{o}^* \). The number of combinatorial orbits in an orbit \( \bar{o}_n \) is denoted by \( m(\bar{o}_n) \). The set of all combinatorial orbits is denoted by \( \mathcal{O} := \mathcal{O}(h) \). This set inherits a natural action of the group
\[
\text{stab} \ h := O_h(N)/R_h(N),
\]
which preserves each orbit \( \bar{o}_n \). (By an obvious abuse of notation, occasionally we treat \( \bar{o}_n \) as a subset of \( \mathcal{O} \); likewise, subsets of \( \mathcal{O} \) are sometimes treated as sets of lines.) We denote by \( \text{Orb}_m(\bar{o}_n, k) \) the length \( m \) orbit of the action of \( \text{stab} \ h \) on the set of unordered \( * \)-invariant (if so is \( \bar{o}_n \)) \( k \)-tuples of combinatorial orbits \( \bar{o} \subset \bar{o}_n \). The usage of this notation implies implicitly that such an orbit is unique.

The support of a vector \( v \in N = N(D) \) is the subset
\[
\text{supp} \ v := \{ k \in \mathcal{O} \mid v_k \neq 0 \in D_k^* \} \subset \mathcal{O}.
\]
The support is invariant under reflections; hence, we can speak about the support \( \text{supp} \ \bar{o} \) of a combinatorial orbit \( \bar{o} \).

The count and bound of a combinatorial orbit \( \bar{o} \) are defined via
\[
(3.1) \quad c(\bar{o}) := |\bar{o}|, \quad b(\bar{o}) := \max\{ |\mathcal{L} \cap \bar{o}| \mid \mathcal{L} \subset \mathcal{F} \text{ is geometric} \}.
\]
Clearly, \( c \) and \( b \) are constant within each orbit \( \bar{o}_n \) and invariant under duality. In some cases, we replace \( b(\bar{o}) \) by rough bounds, see §4.4 below for details. We extend these notions to subsets \( \mathcal{C} \subset \mathcal{O} \) by additivity:
\[
c(\mathcal{C}) := \sum_{\bar{o} \in \mathcal{C}} c(\bar{o}), \quad b(\mathcal{C}) := \sum_{\bar{o} \in \mathcal{C}} b(\bar{o}).
\]
Thus, we have a naïve a priori bound

\[ |\mathcal{L}| \leq b(\mathcal{O}) = \sum m(\delta_n)b(\sigma), \quad \sigma \subset \delta_n. \tag{3.2} \]

Clearly, the true count \(|\mathcal{L} \cap \mathcal{C}|\) is genuinely additive, whereas the sharp bound on \(|\mathcal{L} \cap \mathcal{C}|\) is only subadditive; thus, our proof of Theorem 1.2 will essentially consist in reducing (3.2) down to a preset goal. To this end, we will consider the set

\[ B = B(\mathfrak{g}) := \{ \mathcal{L} \subset \mathfrak{g} \mid \mathcal{L} \text{ is geometric} \}/O_h(N) \]

and, for a collection of orbits \(\mathcal{C} = \tilde{\delta}_1 \cup \ldots \) and integer \(d \in \mathbb{N}\), let

\[ B_d(\mathcal{C}) := \{ [\mathcal{L}] \in B \mid \mathcal{L} \text{ is generated by } \mathcal{L} \cap \mathcal{C} \text{ and } |\mathcal{L} \cap \mathcal{C}| \geq b(\mathcal{C}) - d \}. \]

Unless specified otherwise, the sets \(B_d(\mathcal{C})\) (for reasonably small values of \(d\)) are computed by brute force, using patterns (see §3.3 below).

3.2. Idea of the proof. To prove Theorem 1.2, we consider, one by one, all 23 Niemeier lattices generated by roots. For each lattice \(N\), we set a goal

\[ |\mathcal{L}| \geq M := 122 \text{ or } 132 \tag{3.3} \]

and try to find all geometric subsets \(\mathcal{L} \subset N\) satisfying this inequality. First, we list all \(O(N)\)-orbits of square 6 vectors \(h \in N\), compute the naïve bounds \(b(\mathcal{O})\) given by (3.2), and disregard those vectors for which \(b(\mathcal{O}) < M\). In the remaining cases, we list all \(O_h(N)\)-orbits of roots \(\vec{r}\) orthogonal to \(h\) and repeat the procedure. This leaves us with relatively few triples \(h, \vec{r} \in N\), which are treated on a case-by-case basis in §5–§8 below.

A typical argument runs as follows. We choose a self-dual union \(\mathcal{C}\) of orbits \(\tilde{\delta}_n\) and use patterns (see §3.3 below) to compute the set \(B_{M(\mathcal{O}) - M}(\mathcal{C})\). (As a modification, we take \(\mathcal{C}\) disjoint from its dual \(\mathcal{C}^*\) and use the obvious relation \(B_d(\mathcal{C}) = B_2d(\mathcal{C} \cup \mathcal{C}^*)\).) More generally, we can consider several pairwise disjoint self-dual unions of orbits \(\mathcal{C}_1, \ldots, \mathcal{C}_m\) and compute the sets \(B_d(\mathcal{C}_i)\) for appropriately chosen integers \(d_i \geq 0\) such that

\[ d_1 + \ldots + d_m + 2(m - 1) \geq b(\mathcal{O}) - M. \]

As a result of this procedure, we can assert that, apart from a few explicitly listed exceptions \(\mathcal{L}_1, \ldots, \mathcal{L}_s\) contained in the above sets \(B_d(\mathcal{C}_i)\), we have \(|\mathcal{L}| < M\) for any geometric set \(\mathcal{L} \subset \mathfrak{g}\). In each case, we manage to choose the unions \(\mathcal{C}_i\) and goals \(d_i\) so that the exceptional sets \(\mathcal{L}_k\) are sufficiently large, so that they can be analysed further as explained below.

3.2.1. Maximal sets. The best case scenario is that of a maximal (with respect to inclusion, in the class of geometric sets) geometric set \(\mathcal{L}\). Such a set admits no geometric extensions; hence, it can be either discarded, if \(|\mathcal{L}| < M\), or listed as an exception in the respective statement. Besides, maximal sets can be discarded at early stages of the computation, without completing the whole pattern; however, we only use this approach in §8.3, where intermediate lists grow too large.

An obvious sufficient condition of maximality is given by Proposition 2.9.

Lemma 3.4. Any maximal geometric set is saturated. Conversely, any saturated geometric set \(\mathcal{L}\) of the maximal rank \(\text{rk } \mathcal{L} = 20\) is maximal. \(\square\)
3.2.2. Extension by a maximal orbit. If a set $\mathcal{L} \in \mathcal{B}_d(\mathcal{C})$ is not maximal, we try to list its geometric extensions $\mathcal{L}' \supset \mathcal{L}$ satisfying (3.3). Clearly, it suffices to consider $\mathcal{C}$-proper extensions only, i.e., those with the property that $\mathcal{L}' \cap \mathcal{C} = \mathcal{L} \cap \mathcal{C}$. Thus, we merely extend the partial pattern

$$\pi: \mathcal{C} \to N, \quad o \mapsto |\mathcal{L} \cap o|,$$

(see §3.3 below) used to construct $\mathcal{L}$ by a few (usually one or at most two) extra values $\pi(o)$, $o \in \mathcal{O} \setminus \mathcal{C}$.

In many cases, a geometric set $\mathcal{L} \in \mathcal{B}_d(\mathcal{C})$ has the property that

$$\sum (b(o) - b'(o)) \geq b(\mathcal{O}) - M, \quad o \in \mathcal{O}_\delta := \{ o \in \mathcal{O} \mid |\mathcal{L} \cap o| < b(o) \},$$

where $b'(o)$ is the second largest value taken by $|\mathcal{L}' \cap o|$ with $\mathcal{L}'$ admissible. This implies that any admissible extension $\mathcal{L}' \supset \mathcal{L}$ satisfying (3.3) must have maximal intersection, $|\mathcal{L}' \cap o| = b(o)$, for at least one orbit $o \in \mathcal{O}_\delta$. Trying these orbits one by one (i.e., extending the pattern via $\pi(o) = b(o)$), we obtain larger sets, which are usually maximal. (This computation uses patterns, see §3.3 below, and takes into account the symmetry of $\mathcal{L}$.)

3.2.3. Other extensions. In the few remaining cases, we either analyze the lines contained in $\text{span}_Q \mathcal{L}$ (if $\text{rk} \mathcal{L} = 20$) or obtain maximal $\mathcal{C}$-proper extensions $\mathcal{L}' \supset \mathcal{L}$ by adding one or, rarely, two extra lines.

3.3. Patterns. Since we are interested in large geometric sets, we construct them orbit-by-orbit, by piling together maximal or close to maximal intersections $\mathcal{L} \cap o$. This process is guided by patterns, i.e., $*$-invariant functions

$$\pi: \mathcal{O} \to N, \quad \mathcal{O} \mapsto |\mathcal{L} \cap \mathcal{O}|.$$  

Having $h, \bar{r} \in N$ fixed, we start with precomputing all geometric sets $\mathcal{L} \subset \mathcal{O}$ in each combinatorial orbit $\mathcal{O}$. (Certainly, it suffices to consider one representative in each orbit $\mathcal{O}_\delta$; the rest is obtained by translations.) Then, in order to compute one of the sets $\mathcal{B}_d(\mathcal{C})$ in §3.1, we list all (stab $h$)-orbits of restricted patterns $\pi: \mathcal{C} \to N$ satisfying the inequality $\sum \pi(o) \geq b(\mathcal{C}) - d$, $o \in \mathcal{C}$, order the orbits appropriately (typically, by the decreasing of $\pi(o)$), and construct a geometric set $\mathcal{L}$ by adding one orbit at a time, as a sequence $\mathcal{O} = \mathcal{L}_0 \subset \mathcal{L}_1 \subset \mathcal{L}_2 \subset \ldots$. At each step $k$ and for each set $\mathcal{L}_{k-1}$ constructed at the previous step, we proceed as follows:

1. compute the stabilizer $G$ of $\mathcal{L}_{k-1}$ under the action of $R_h(N)$;
2. compute the $G$-orbits of the geometric sets $\mathcal{L}' \subset o_k$ of size $|\mathcal{L}'| = \pi(o_k)$;
3. for a representative $\mathcal{L}'$ of each $G$-orbit, consider the set $\mathcal{L}_k$ generated by the union $\mathcal{L}_{k-1} \cup \mathcal{L}'$; then, select those sets $\mathcal{L}_k$ that are geometric;
4. to reduce the overcounting, select, for the next step, those sets $\mathcal{L}_k$ that satisfy the equality $|\mathcal{L}_k \cap o_i| = \pi(o_i)$ for each $i \leq k$.

If the defect $d$ is not too large, this procedure works reasonably fast and results in a reasonably small collection of sets that are to be analyzed further.

**Remark 3.5.** Although it is not obvious a priori, it turns out that large geometric sets are often determined by their patterns uniquely up to $R_h(N)$. Furthermore, a large set is easily reconstructed from its pattern, as the algorithm above converges very fast. For this reason, we often describe large geometric sets, especially those that are not $\mathbb{Q}$-complete (see Definition 2.5), by their patterns.
A pattern $\pi$ taking a constant value $v_n$ on each orbit $\sigma_n$ is described via

$$\pi = \langle v_1, v_2, \ldots \rangle.$$  

Sometimes, we use a “double value” $v_n = a|b$; this means that a cluster $c_n \subset \sigma_n$ is fixed (and described elsewhere) so that the restriction of $\pi$ to $\sigma_n$ takes two values: $\pi(\sigma) = a$ for $\sigma \subset c_n$, and $\pi(\sigma) = b$ for $\sigma \subset \sigma_n \setminus c_n$.

**Remark 3.6.** In some cases, where $b(\Omega)$ exceeds the goal by just a few units, we use patterns to show directly that $B_{n(\Omega)}(\Omega) = \emptyset$. These cases are marked with a ✓ in the tables, and any further explanation is typically omitted (and so usually is the list of orbits).

### 3.4. Clusters

Sometimes, the number of combinatorial orbits in an orbit $\bar{o}$ is too large, making it difficult to compute all patterns. In these cases, we subdivide $\bar{o}$ into a number of clusters $c_k \subset \bar{o}$, not necessarily disjoint, and compute patterns and, then, geometric sets cluster by cluster. The subdivision is chosen so that stab $h$ acts transitively on the set of clusters. To reduce the overcounting, we assume that the clusters are ordered lexicographically, by the decreasing of the sequence

$$\{ |L \cap c_k|, \delta_0(c_k), \delta_1(c_k), \ldots \}, \quad \delta_i(c_k) := \# \{ \sigma \subset c_k \mid |L \cap \sigma| = b(\sigma) - i \}.$$  

In particular, this convention implies that, when computing the set $B_d(\bar{o})$, for the first cluster $c_1$ one must have $|L \cap c_1| \geq b(c_1) - md/n$, where $n$ is the total number of clusters and the multiplicity $m$ is the number of clusters containing any fixed orbit $\sigma \subset \bar{o}$. More generally, extending a geometric set $L$ from $c_1, \ldots, c_k$ to the next cluster $c_{k+1}$, one must have $|L \cap c_{k+1}| \leq |L \cap c_k|$ and

$$|L \cap c_{k+1}| \geq b(c_{k+1}) - \frac{1}{n-k} \left( md - \sum_{i=1}^{k} (b(c_i) - |L \cap c_i|) \right).$$  

Certainly, if the clusters are not disjoint, we also take into account the intersections $c_{k+1} \cap c_i$, $i = 1, \ldots, k$, when computing the restricted patterns $\pi: c_{k+1} \to \mathbb{N}$.

### 4. Counts and bounds

In this section, we explain the computation of the bounds $b(\sigma)$ on the number of lines in an admissible set within a combinatorial orbit $\sigma$, see (3.1).

### 4.1. Blocks

Consider a combinatorial orbit $\sigma$. In order to estimate the count $c(\sigma)$ and bound $b(\sigma)$, we break the root system $D$ into blocks, $D = B_1 \oplus B_2 \oplus \ldots$, each block $B_k$ consisting of whole components $D_i$. Then, $h$ and $l \in \mathfrak{F}(h) \cap \sigma$ split into $\bigoplus_k h_k$ and $\bigoplus_k l_k$, respectively, with $h_k, l_k \in B^\vee_k$. We denote by $\sigma|_k := \sigma|_{B_k} \subset B^\vee_k$ the restriction of $\sigma$ to $B_k$ (which, in fact, is nothing but the orthogonal projection of $\sigma$ to $B^\vee_k$). This restriction consists of a whole $R_{h_k}(B_k)$-orbit of vectors; in particular, we have a well defined square $l_k^2 \in \mathbb{Q}$, product $l_k \cdot h_k \in \mathbb{Q}$, and discriminant class $l_k \mod B_k \in \text{discr} B_k$. Usually, these data determine an irreducible block up to isomorphism, the reason being the following simple observation (which follows from the fact that all roots in $\mathbb{N}$ are assumed to lie in $D$):  

- each vector $l_k \in \sigma|_k$ is either integral, $l_k \in B_k$ (and then $l_k^2 = 0, 2, or 4$) or shortest vector in its discriminant class;
- each vector $h_k$ is either integral, $h_k \in B_k$ (and then $h_k^2 = 0, 2, 4, or 6$) or shortest or second shortest vector in its discriminant class.
Here, *shortest* are the vectors minimizing the square within a given discriminant class, whereas *second shortest* are those of square (minimum + 2). In fact, $h_k$ can be a second shortest vector in at most one block $B_k$.

The *count* of a block $B$ is defined in the obvious way: $c(B) = |B|_{\mathcal{I}}$. The *bound* is defined via $b(B) = \max|\mathcal{B}|$, where $\mathcal{B} \subset B$ is a $*$-invariant (if $\star = \circ$) subset satisfying the following condition:

\begin{equation}
(4.1) \text{for } l', l'' \in \mathcal{B}, \text{ one has } l'^2 - l' \cdot l'' = 0 \text{ (iff } l' = l'') \text{, } 2, 3, \text{ or } 5 \text{ (iff } l' = (l'')^*) \text{.}
\end{equation}

In other words, we bound the cardinality of subsets $\mathcal{L} \subset \mathcal{O}$ satisfying (2.3) and such that all lines $l \in \mathcal{L}$ have the same fixed restriction to all other blocks $B' \neq B$.

If $D$ is broken into two blocks, $B_1 \oplus B_2$, we obviously have

\begin{equation}
(4.2) \quad c(\mathcal{O}) = c(B_1)c(B_2), \quad b(\mathcal{O}) \leq \min \{c(B_1)b(B_2), b(B_1)c(B_2)\}.
\end{equation}

By induction, for any number of blocks $B_k$, this implies

\begin{equation}
(4.3) \quad c(\mathcal{O}) = \prod_k c(B_k), \quad b(\mathcal{O}) \leq c(\mathcal{O}) \min_k \frac{b(B_k)}{c(B_k)}.
\end{equation}

This bound (with $B_k = D_k$ the irreducible components of $D$) and corresponding bound on $b(\mathcal{O})$ given by (3.2) are always listed first in the tables below. If $b(\mathcal{O}) \geq M$, we try to improve the bounds $b(\mathcal{O})$ using one of the following arguments:

1. *Lemma 4.5* below applied to an appropriate splitting into two blocks;
2. a computation using larger blocks, see §4.2 below;
3. a brute force enumeration of admissible subsets $\mathcal{L} \subset \mathcal{O}$; the bounds whose sharpness is confirmed by this computation are underlined.

In the tables, we refer to this list for the reasons for the improved bounds.

### 4.2. Brute force via blocks

For some large combinatorial orbits $\mathcal{O}$, the exact computation of $b(\mathcal{O})$ by brute force is not feasible, and we improve the original bound given by (4.3) by using larger blocks. Typically, we consider two blocks $B_1$ (one of the irreducible components of $D$) and $B_2$ (the sum of all other components). Then, we compute all *admissible* (rather than just satisfying (4.1)) sets $\mathcal{L}(l_1) \subset \mathcal{O}$ with a fixed restriction $l_1 \in B_1^\circ$, replacing (4.2) with

\[ b(\mathcal{O}) \leq c(B_1) \max|\mathcal{L}(l_1)|. \]

If this bound is still not good enough, we vary $l_1 \in B_1^\circ$ and try to construct an admissible set $\mathcal{L} \subset \mathcal{O}$ by packing together precomputed large (usually maximal or submaximal) sets $\mathcal{L}(l'_1)$, $\mathcal{L}(l''_1)$, etc., obtaining a better bound and, if necessary, a complete list of large admissible sets in $\mathcal{O}$.

### 4.3. Self-dual combinatorial orbits

Let $\mathcal{O}$ be a self-dual combinatorial orbit, $\mathcal{O}^* = \mathcal{O}$, and break $D$ into blocks $B_k$. Each block is also self-dual: $l_k := h_k - l_k \in \mathcal{O}|_k$ whenever $l_k \in \mathcal{O}|_k$. In particular, $l_k = l_k \mod B_k$. Hence, we have

\[ 2l_k \cdot h_k = h_k^2 \quad \text{(since } l_k^2 = l_k^2), \]
\[ l_k \cdot l_k = l_k^2 - \delta_k \quad \text{for some } \delta_k \in \mathbb{Z}. \]

The integer $\delta(B_k) := \delta_k = 2l_k^2 - l_k \cdot h_k$, constant throughout the block, is called the *defect* of the block $B_k$; it takes values in the range $0 \leq \delta_k \leq 5$, and the defects of
where denote ℏ given a vector "subblocks" vectors are pairwise distinct. Let ℏ o

\( \mathcal{L} \) be the lattice, where we abbreviate \( \mathcal{L} \) and \( \mathcal{L}' \) are in \( \mathcal{L} \) or it is not. In the former case, we have

\[ \{ l_3 \oplus l_3, \bar{l}_2 \oplus l_3, l_2 \oplus \bar{l}_3, \bar{l}_2 \oplus \bar{l}_3 \} \subset \mathcal{L} \]

and, by (2.3) and (4.4), no other vector \( l_2 \oplus l_3 \) or \( \bar{l}_2 \oplus l_3 \) with \( l_3 \neq l_3, \bar{l}_3 \) is in \( \mathcal{L} \).

Each 4-element subset of this form consumes two vectors from \( \mathcal{L} \), and all these vectors are pairwise distinct. Let \( U \subset \mathcal{L} \) be the set of vectors \( l_2 \) as above, and denote \( u := |U| \); clearly, \( 0 \leq 2u \leq \min\{c_2, c_3\} \).

Otherwise, in the obvious notation, we have

\[ l_2 \oplus S(l_2) \subset \mathcal{L}, \quad \bar{l}_2 \oplus S(\bar{l}_2) \subset \mathcal{L}, \]

where \( S(l_2) \subset \mathcal{L} \) is a certain subset and \( S(\bar{l}_2) = S(l_2) \). Since \( S(l_2) \cap S(\bar{l}_2) = \emptyset \) by the assumption, all subsets \( S(l_2), l_2 \in \mathcal{L} \), are pairwise disjoint and do not contain any of the \( 2u \) vectors \( l_2 \) coupled with \( l_2 \in U \); hence, their total cardinality does not exceed \( c_3 - 2u \). On the other hand, since \( |S(l_2)| \leq b_3 \) for each \( l_2 \in \mathcal{L} \), this cardinality does not exceed \( (c_2 - 2u)b_3 \). Taking the minimum and maximizing over all values of \( u \), we arrive at the bound in the statement. \( \square \)

4.4. Computing counts and bounds. For "small" blocks \( B_k \cong A_{\leq 7}, D_{\leq 7}, E_6, E_7, E_8 \), the counts \( c(B_k) \) and bounds \( b(B_k) \) used in (4.3) are obtained by a direct computation. For larger blocks, we use the standard combinatorial description of the \( A \) - and \( D \)-type root systems as sublattices of the odd unimodular lattice

\[ \mathbf{H}_n := \bigoplus \mathbb{Z} e_i, \quad e_i^2 = 1, \quad i \in \mathcal{I} := \{1, \ldots, n\}. \]

(When working with this lattice, we let \( 1_{\mathcal{I}} := \sum_{i \in \mathcal{I}} e_i \) for a subset \( \mathcal{I} \subset \mathcal{I} \).) Then, given a vector \( h_k = \sum_i \alpha_i e_i \in \mathbf{H}_n \otimes \mathbb{Q} \), we subdivide the block \( B_k^\circ \subset \mathbf{H}_n \otimes \mathbb{Q} \) into "subblocks"

\[ B_k(\alpha) := \{ \sum_i \beta_i e_i \mid i \in \text{supp}(\alpha) \}, \quad \text{supp}(\alpha) := \{ i \in \mathcal{I} \mid \alpha_i = \alpha \}, \]
on which \( h_k \) is constant. We obtain counts and bounds, in the sense of (4.1), for each subblock and use an obvious analogue of (4.3) to estimate \( b(B_k) \). The technical details are outlined in the next two sections.
4.5. **Root systems** $A_n$. A block $B_k$ of type $A_n$ is $\tilde{I}^2 \subset H_{n+1}$:

$$A_n = \{ \sum_{i} \alpha_i e_i \in H_{n+1} \mid \sum_{i} \alpha_i = 0 \}.$$ 

One has $\text{discr} A_n = \mathbb{Z}/(n+1)$, with a generator of square $n/(n+1)$ mod $2\mathbb{Z}$, and the shortest representatives of the discriminant classes are vectors of the form

$$\vec{e}_o := \frac{1}{n+1} (|o| \vec{1}_o - |o| \vec{1}_o), \quad \vec{e}_o^2 = \frac{|o||\vec{o}|}{n+1},$$

where $o \subset I$ and $\vec{o}$ is the complement. We have $\vec{e}_o = -\vec{e}_o$ and

$$(4.6) \quad \vec{e}_r \cdot \vec{e}_s = |r \cap s| - \frac{|r||s|}{n+1}.$$ 

If $|r| = |s|$, or, equivalently, $e_r$ and $e_s$ are in the same discriminant class, then

$$\vec{e}_r^2 - \vec{e}_r \cdot \vec{e}_s = \frac{1}{2} |r \triangle s|,$$

where $\triangle$ is the symmetric difference. Hence, in the case where $l_k$ is a shortest vector in its (nonzero) discriminant class, the bound $b(B_k(\alpha))$ can be estimated by the following lemma, applied to $S = \text{supp}(\alpha)$.

**Lemma 4.7.** Consider a finite set $S$, $|S| = n$, and let $\mathcal{S}$ be a collection of subsets $s \subset S$ with the following properties:

1. all subsets $s \in \mathcal{S}$ have the same fixed cardinality $m$;
2. if $r, s \in \mathcal{S}$, then $|r \triangle s| \in \{0, 4, 6, 10\}$;
3. in the case $(n, m) = (10, 5)$, if $s \in \mathcal{S}$, then also $\tilde{s} \in \mathcal{S}$.

Then, for small $(n, m)$, the maximal cardinality $|\mathcal{S}|$ is as follows:

$$(n, m) : (n, 1) \quad (n, 2) \quad (6, 3) \quad (7, 3) \quad (8, 3) \quad (9, 3) \quad (10, 3) \quad (11, 3) \quad (8, 4) \quad (9, 4) \quad (10, 5)$$

max$|\mathcal{S}|$ : 1 \quad |n/2| \quad 4 \quad 7 \quad 8 \quad 12 \quad 13 \quad 17 \quad 9 \quad 12 \quad 24

More generally, for $m = 3$ one has

$$|\mathcal{S}| \leq \lfloor n((n-1)/2)/3 \rfloor.$$ 



Note that, if a collection $\mathcal{S}$ is as in the lemma, then so is the collection $\{\tilde{s} \mid s \in \mathcal{S}\}$. Hence, we can always assume that $2m \leq n$.

**Proof of Lemma 4.7.** The first two values are obvious; the others are obtained by listing all admissible collections. The general estimate for $m = 3$ follows from the observation that any two subsets in $\mathcal{S}$ have at most one common point and, hence, each point of $S$ is contained in at most $\lfloor (n-1)/2 \rfloor$ subsets.

There remains to consider a subblock $B_k(\alpha)$ of a block $B_k$ containing vectors of the form $l_k = \tilde{1}_r - \tilde{1}_s$, where $r, s \subset I$, $r \cap s = \emptyset$, and $|r| = |s| = 1$ or 2. In the latter case, one must have $l_k \cdot h_k = 3$, and it follows that $|(r \cup s) \cap \text{supp}(\alpha)| \leq 2$ for each $\alpha \in \mathbb{Q}$. The bounds are as follows:

1. if $|(r \cup s) \cap \text{supp}(\alpha)| = 1$, then, obviously, $b(B_k(\alpha)) = 1$;
2. if $|r \cap \text{supp}(\alpha)| = 2$ (or $|s \cap \text{supp}(\alpha)| = 2$), the distinct sets $r \cap \text{supp}(\alpha)$ must be pairwise disjoint and, hence, $b(B_k(\alpha)) = \lfloor \frac{1}{2} |\text{supp}(\alpha)| \rfloor$;
3. if $|r \cap \text{supp}(\alpha)| = |s \cap \text{supp}(\alpha)| = 1$, then the distinct sets $r \cap \text{supp}(\alpha)$ must also be pairwise disjoint and, hence, $b(B_k(\alpha)) = |\text{supp}(\alpha)|$. 


4.6. Root systems $D_n$. A block $B_k$ of type $D_n$ can be defined as the maximal even sublattice in $H_n$:  

$$D_n = \{ \sum_i \alpha_i e_i \in H_n \mid \sum_i \alpha_i = 0 \text{ mod } 2 \}.$$ 

One has $\text{disc} D_n = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ (if $n$ is even) or $\mathbb{Z}/4$ (if $n$ is odd); the shortest vectors are  

$$e_i, \ i \in I, \text{ and } \bar{e}_o := \frac{1}{2}(\bar{1}_o - \bar{1}_o), \ o \subset I, \quad \bar{e}_o^2 = \frac{n}{4}$$

(the class $\bar{e}_o \mod D_n$ depends on the parity of $|o|$) and we have a literal analogue of (4.6) for any pair $r, s \subset I$. Thus, if $B_k \ni \bar{e}_o$, the bounds $b(B_k(\alpha))$ are estimated by Lemma 4.7 (if $\alpha \neq 0$) or Lemma 4.8 below (if $\alpha = 0$) applied to $S = \text{supp}(\alpha)$.

**Lemma 4.8.** For $n \leq 10$, the maximal cardinality of a collection $\mathcal{S}$ satisfying conditions (2) and (3) (if $n = 10$) of Lemma 4.7 is bounded as follows:

$$n = \begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
|\mathcal{S}| & \leq & 1 & 1 & 1 & 2 & 2 & 4 & 8 & 10 & 16 & 32
\end{array}$$

These bounds are sharp for $n \leq 8$.

**Proof.** If $n \leq 6$, the statement is easily proved by inspection, using Lemma 4.7.

Let $n = 8$. Represent a subset $s \in \mathcal{S}$ as the root $\bar{e}_s \in D_8$. Then, condition (2) implies that all subsets $s \in \mathcal{S}$ have cardinality of the same parity and, hence, all roots are in the same discriminant class; thus, they lie in an extension $E_8 \supset D_8$.

By (2), the roots $\bar{e}_s$ constitute a union $\Gamma$ of (affine) Dynkin diagrams other than $A_1$ admitting an isometry to $E_8$, which gives us a bound $|\mathcal{S}| \leq 12$. Furthermore, the roots $\bar{e}_s$ are distinguished by the property $\bar{e}_s \cdot 2e_1 = 1 \text{ mod } 2$. Thus, each affine component of $\Gamma$ must have even degree. The maximal graph with these properties is $2D_4$, resulting in the bound $|\mathcal{S}| \leq 10$.

If $n = 7$, we extend the ambient set $S$ and each subset by an extra point and argue as above, obtaining roots $\bar{e}_s \in E_8$ with the property $\bar{e}_s \cdot 2e_1 = 1$. This time, the roots are linearly independent and $|\mathcal{S}| \leq \text{rk} E_8 = 8$. This is realized by $2D_4$.

In general, represent $s \in \mathcal{S}$ by the vector $\bar{I}_s \in H_n$. (If $n = 10$, select one subset $s$ from each pair $s, \bar{s}$.) Then $\bar{I}_s^2 = n$ and the products $\bar{I}_r \cdot \bar{I}_s = n - 2|r \Delta s|$, $r \neq s$, take but two values $n - 8$ or $n - 12$. Since $n \leq 10$, Theorem 2.6 applies and bounds the number of vectors by 16. If $n = 10$, this bound is to be doubled. \hfill \Box

The few remaining cases are listed below.

(1) If $B_k(\alpha) \ni \pm 2e_i, i \in \text{supp}(\alpha)$, then $b(B_k(0)) = 1$.

Assume that $\ell_k = \sum_i (\pm e_i), i \in o \subset S, |o| \leq 4$. If $\alpha = 0$, then

(2) $|o \cap \text{supp}(\alpha)| = 0, 1, \text{ or } 2$ and $b(B_k(\alpha)) \leq 1, 2, \text{ or } 4|\text{supp}(\alpha)|$, respectively, similar to §4.5. (Here, the last number is a bound on the size of a union of (affine) Dynkin diagrams other than $A_1$ admitting an isometry to $D_{\text{supp}(\alpha)}$.) If $\alpha \neq 0$, the numbers of signs $\pm$ within $\text{supp}(\alpha)$ are also fixed, and the options are as follows:

(3) $m := |o \cap \text{supp}(\alpha)| \leq 3$ and all signs are the same: by an analogue of (4.6), a bound on $b(B_k(\alpha))$ is given by Lemma 4.7 applied to $S = \text{supp}(\alpha)$;

(4) $|o \cap \text{supp}(\alpha)| = 2$ and the signs differ: $b(B_k(\alpha)) = |\text{supp}(\alpha)|$ as in §4.5(3).

**Remark 4.9.** If $n \geq 5$, the group $O(D_n)$ is an index 2 extension of $R(D_n)$: it is generated by the reflection against the hyperplane orthogonal to any of $e_i$. Hence, up to $O(D_n)$, we can assume that, in the expression $h_k = \sum_i \alpha_i e_i$, all coefficients
Remark 3.6 and §7. In these "trivial" cases marked with a 4.3 describing the orbits and computing counts and bounds.

4.3 systems with few (up to six) irreducible components. We set the goal

|Ω| \geq M := 122

and prove the following theorem.

Theorem 4.1. Fix a root system D with at most six irreducible components and a configuration (h, r) in the Niemeier lattice N(D). Then, with the exception of

\begin{itemize}
  \item |Ω| = 144 and Ω is conjugate to \( \mathfrak{M}_{144}^{1,44} \subset N(4A_5 \oplus D_4) \), see (5.3), or
  \item |Ω| = 130 and Ω is conjugate to \( \mathfrak{M}_{130}^{1,30} \subset N(6A_4) \), see (5.4),
\end{itemize}

one has |Ω| \leq 120 for each geometric set Ω.

Proof. For each configuration (h, r) (or just vector h), we list all \( O_h(N) \)-orbits \( \bar{\sigma}_n \) and indicate the number \( m(\bar{\sigma}_n) \) of combinatorial orbits \( \sigma \subset \bar{\sigma}_n \), the count \( c(\sigma) \), and the naive bound on \( |\Omega \cap \sigma| \) given by (4.3). Sometimes, this bound is improved by one of the arguments (1)–(3) in §4.1; the best bound obtained is denoted by \( b(\sigma) \).

The results are listed in several tables below.

Convention 5.2. In the tables, the number \( m(\bar{\sigma}_n) \) is marked with an * if \( \bar{\sigma}_n \) is self-dual; it is marked with ** if also each combinatorial orbit \( \sigma \subset \bar{\sigma}_n \) is self-dual. If \( \bar{\sigma}_n \) is not self-dual, then its dual \( \bar{\sigma}_n^* = \bar{\sigma}_{n+1} \) is omitted.

For the components \( h_k \) of h we use the notation \( \{h_k^2\}_{d'} \), where \( d' \) is either the discriminant class of \( h_k \) or, if \( h_k \in D_k \), the symbol 0 (if \( h_k = 0 \) or \( h_k^2 = 2 \)), • (if \( h_k^2 = 4 \)), or ♦ (if \( h_k^2 = 6 \)). If these data do not determine \( h_k \), we use a superscript:

\begin{itemize}
  \item + or − to select a second shortest vector (one of the form \( e_\alpha + r \), where \( r = e_i - e_j \in D_k \) is a root and \( i, j \in o \) or \( i, j \in \bar{o} \), respectively, see §5.4 and §6.6) in a discriminant class \( d' \neq 0 \);
  \item ♦, if \( h = h_k \in D_n \subset H_n \) or \( A_n-1 \subset H_n \) is of the form \( 2e_1 - e_2 - e_3 \) rather than \( e_1 + e_2 + e_3 - e_4 - e_5 - e_6 \), see §5.4 and §6.6;
  \item the discriminant class of \( \frac{1}{2}h_k \), if \( h_2 \in D_n \cap 2D_n^\vee \).
\end{itemize}

If \( D_k \) contains the root \( \bar{r} \), this notation is changed to \( \{h_k^2\}_{d'} \), where \( d' \) and an occasional superscript have the same meaning as for \( h \).

Also shown in the tables is the naive a priori estimate \( b(\Omega) \) given by (3.2). For the vast majority of configurations we have \( b(\Omega) \leq M \), and these configurations are omitted. (The complete set of tables is available in [7].) The few cases where \( b(\Omega) \geq M \) are shown in bold, and we treat them separately below, except those marked with a ✓ (see Remark 3.6). In these “trivial” cases marked with a ✓ we usually also omit the list of orbits.

5.1. The lattice \( N(4A_5 \oplus D_4) \). There are 93 configurations to be considered, and the maximal naive bound is \( b(\Omega) = 156 \) (see Table 1).
5.1.1. Configuration 1. There are two sets $\mathcal{L} \in \mathcal{B}_{17}(\delta_2)$, defined by the patterns $\pi$ such that $\pi|_{\delta_3} = \text{const} = 7$ or 9. The former has rank 19, and its only nontrivial geometric $\delta_2$-proper (see §3.2.2) extension has $\pi(\delta) = \pi(\delta^*) = 2$ for a pair of dual orbits $\delta, \delta^* \subset \delta_1$ and $\pi(\delta') = 0$ for all other orbits $\delta' \subset \delta_1$. The other set, which is denoted $\mathfrak{M}_{144}$, is maximal, see §3.2.1. This set of size 144 is determined by the pattern (see Remark 3.5)

\[ \pi = \langle 0, 9, 9 \rangle \]  

(not $\mathbb{Q}$-complete).

5.1.2. Configuration 3. It is not practical to compute the admissible sets for all orbits; thus, we argue as in §4.2 and only compute admissible subsets $\mathcal{L} \subset \delta \subset \delta_1$ of size at least 18. This suffices to show that there is a unique set $\mathcal{L} \in \mathcal{B}_4(\delta_1)$, with the pattern $\pi$ taking values $(22, 22, 18)$ on $\delta_1$ and identical 0 on $\delta_2$. This set is maximal (see §3.2.1).

5.2. The lattice $N(6A_4)$. There are 39 configurations to be considered, and the maximal naïve bound is $b(\mathcal{D}) = 150$ (see Table 2).

5.2.1. Configuration 1. There is a unique set $\mathcal{L}_{130} \in \mathcal{B}_{28}(\mathcal{D})$; it has 130 lines and is described by the pattern (see Remark 3.5)

\[ \pi = \langle 0, 6, 10 \rangle \]  

(not $\mathbb{Q}$-complete).

This case completes the technical details of the proof of Theorem 5.1.  

\( \square \)
Theorem 6.1. Let $\mathcal{L} \subset N(8A_3)$ be a geometric set. Then, unless

1. $|\mathcal{L}| = 132$ and $\mathcal{L}$ is conjugate to $\mathcal{G}_{132}^i$, see (6.2), or $\mathcal{G}_{132}^{ii}$, see (6.3), one has $|\mathcal{L}| \leq 130$.

Proof. We proceed as in §5, listing pairs $h, \tilde{r} \in N(8A_3)$ and respective orbits (and following the notation of §5). There are 28 configurations to be considered, and the maximal naive bound is $b(\mathcal{D}) = 160$ (see Table 3).

6.1. Configuration 1. The only maximal geometric set, denoted $\mathcal{G}_{132}^i$, does have 132 lines. It is characterized by the constant pattern (see Remark 3.5)

(6.2) $\pi = \langle 12, 12 \rangle$. 

Table 3. The lattice $N(8A_3)$

|   |   |   |   |   |   |   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1: | {4}^+ | [2]_0 | [0]_0 | [0]_0 | [0]_0 | [0]_0 | [0]_0 | [0]_0 | 728 | 200 → 132 | ✓ |
| 1: | [2]_2 | [1]_2 | [0]_0 | [0]_0 | [0]_0 | [0]_0 | [0]_0 | [0]_0 | 3'' | 72 | 24 → 12 (3) |
| 2: | [2]_2 | [1]_3 | [0]_0 | [0]_0 | [0]_0 | [0]_0 | 8'' | 64 | 16 → 12 (2) |
| 4: | [4]_3 | [4]_3 | [4]_1 | [1]_2 | [1]_2 | [0]_0 | [4]_1 | [1]_2 | 448 | 160 |
| 5: | [4]_3 | [4]_3 | [4]_1 | [1]_2 | [1]_2 | [0]_0 | [4]_1 | [1]_2 | 420 | 136 |
| 7: | {3}^+ | [1]_2 | [1]_2 | [0]_0 | [0]_0 | [0]_0 | [1]_2 | [0]_0 | 600 | 172 → 136 |
| 10: | [4]_3 | [4]_3 | [1]_2 | [1]_2 | [0]_0 | [0]_0 | [4]_1 | [1]_2 | 472 | 152 → 148 |
| 14: | [4]_3 | [4]_3 | [1]_2 | [1]_2 | [0]_0 | [0]_0 | [4]_1 | [1]_2 | 548 | 152 → 140 |
| 15: | [4]_3 | [4]_3 | [4]_1 | [0]_0 | [0]_0 | [1]_2 | [4]_1 | [0]_0 | 496 | 148 → 138 |
| 18: | [4]_3 | [4]_3 | [4]_1 | [0]_0 | [0]_0 | [3]_2 | [4]_1 | [0]_0 | 600 | 152 → 146 |
| 19: | [4]_3 | [4]_3 | [4]_1 | [0]_0 | [0]_0 | [3]_2 | [4]_1 | [0]_0 | 520 | 144 → 140 |
| 21: | [1]_2 | [1]_2 | [1]_2 | [2]_0 | [0]_0 | [0]_0 | [1]_2 | [0]_0 | 472 | 164 → 140 |
| 27: | [4]_3 | [4]_3 | [4]_3 | [4]_3 | [4]_3 | [4]_3 | [4]_3 | [4]_3 | 420 | 140 |

6. The lattice $N(8A_3)$

Starting from this section, we relax the goal to

$$|\mathcal{L}| \geq M := 132.$$
6.2. Configuration 4. There are 162 sets $\mathcal{L} \in B_{10}(\bar{\delta}_4)$. Most are maximal; one of them, denoted $\mathcal{E}_{132}^4$, has 132 lines and is determined by the pattern

$$(6.3) \quad \pi = \langle 0, 3, 3, 4, 0, 0 \rangle.$$ 

Eleven sets are of rank 19; extending these sets by a maximal orbit (see §3.2.2), we arrive at the bound $|\mathcal{L}| \leq 112$.

On the other hand, there is a unique set $\mathcal{L} \in B_6(\bar{\delta}_1 \cup \bar{\delta}_2 \cup \bar{\delta}_4 \cup \bar{\delta}_6 \cup \bar{\delta}_6^*)$. One has $|\mathcal{L}| = 92$, and this set is maximal, see §3.2.1. \hfill \Box

7. The lattice $N(12A_2)$

The ultimate result of this section is the following theorem.

Theorem 7.1. Let $\mathcal{L} \subset N(12A_2)$ be a geometric set. Then, unless

1. $|\mathcal{L}| = 144$ and $\mathcal{L}$ is conjugate to $\mathcal{M}_{144}$, see (7.5), or
2. $|\mathcal{L}| = 132$ and $\mathcal{L}$ is conjugate to $\mathcal{E}_{132}^4$, see (7.8),

one has $|\mathcal{L}| \leq 130$.

In the course of the proof of this theorem we also discover and describe (by means of their patterns, see Remark 3.5) several geometric sets $\mathcal{L}$ of size $|\mathcal{L}| \geq 124$.

Proof of Theorem 7.1. We proceed as in §5, analyzing pairs $(\bar{h}, \bar{r})$ one by one. (The notation in the table is explained in §5.) There are 9 configurations to be considered, and the maximal na"ive bound is $b(O) = 190$ (see Table 4).

7.1. Configuration 1. We subdivide the orbit $\bar{\delta}_1$ into five pairwise disjoint clusters $c_1, \ldots, c_5$ constituting $\text{Orb}_{5}(\bar{\delta}_1, 5)$. Explicitly,

$$c_k := \{l \in \bar{\delta}_1 \mid l_k \cdot h_k = 1\}, \quad k \in K := \{k \in \Omega \mid h_k = 2\}.$$ 

Then, arguing as in §3.4, we compute $B_{58}(\bar{\delta}_1) = \emptyset$.

7.2. Configuration 2. There are four sets $\mathcal{L} \in B_{30}(\bar{\delta}_1)$. One, denoted $\mathcal{L}_{130}^4$, is maximal and has 130 lines; it is characterized by the pattern

$$(7.2) \quad \pi := \langle 6, 0, 10, 0, 0 \rangle \quad (\text{not } \mathcal{Q}\text{-complete}).$$ 

The three other sets are of rank 19; extending them by an extra orbit (see §3.2.3), we arrive at a number of sets of size $|\mathcal{L}| \leq 118$ and one, up to $O_{124}(N)$, maximal set $\Sigma_{124}^{iii}$ of size 124. The latter is characterized by any of the five patterns

$$(7.3) \quad \pi_c = \langle 5, 4, 2, 8, 0, 0 \rangle, \quad c := c_1 \in \text{Orb}_5(\bar{\delta}_1, 16);$$

explicitly, $c = \{l \in \bar{\delta}_1 \mid l_k \cdot h_k \neq 1\}$ for some $k \in K$ (see §7.1).

On the other hand, there are 13 sets $\mathcal{L} \in B_6(\bar{\delta}_2 \cup \bar{\delta}_3 \cup \bar{\delta}_4 \cup \bar{\delta}_6^*)$, all saturated and with $|\mathcal{L}| \leq 94$. One set is of rank 19; extending it by an extra orbit (see §3.2.3), we obtain a number of sets with at most 92 lines.

7.3. Configuration 3. There is a single set $\mathcal{L} \in B_{14}(\bar{\delta}_3)$; this set is maximal, and one has $|\mathcal{L}| = 120$. 
Table 4. The lattice $N(12A_2)$

| Set  | Description                                                                 | Count |
|------|-----------------------------------------------------------------------------|-------|
| 1:   | $(\frac{4}{3})_2 (\frac{4}{3})_2 (\frac{4}{3})_2 (\frac{4}{3})_2 (0)_{0} (\frac{4}{3})_{0} (\frac{4}{3})_{0} (\frac{4}{3})_{0} (\frac{4}{3})_{0} (\frac{4}{3})_{0}$ | 572   |
| 1:   | $[\frac{4}{3}]_1  [\frac{4}{3}]_2  (\frac{4}{3})_2 (\frac{4}{3})_2 (0)_{0} (\frac{4}{3})_{0} (\frac{4}{3})_{0} (\frac{4}{3})_{0} (\frac{4}{3})_{0} (\frac{4}{3})_{0}$ | 30    |
| 2:   | $[\frac{4}{3}]_1  [\frac{4}{3}]_2  (\frac{4}{3})_2 (\frac{4}{3})_2 (0)_{0} (\frac{4}{3})_{0} (\frac{4}{3})_{0} (\frac{4}{3})_{0} (\frac{4}{3})_{0} (\frac{4}{3})_{0}$ | 1     |
| 2:   | $(\frac{4}{3})_2 (\frac{4}{3})_2 (\frac{4}{3})_2 (\frac{4}{3})_2 (0)_{0} (\frac{4}{3})_{0} (\frac{4}{3})_{0} (\frac{4}{3})_{0} (\frac{4}{3})_{0} (\frac{4}{3})_{0}$ | 492   |
| 2:   | $[\frac{4}{3}]_1  [\frac{4}{3}]_2  (\frac{4}{3})_2 (\frac{4}{3})_2 (0)_{0} (\frac{4}{3})_{0} (\frac{4}{3})_{0} (\frac{4}{3})_{0} (\frac{4}{3})_{0} (\frac{4}{3})_{0}$ | 20    |
| 3:   | $[\frac{4}{3}]_1  [\frac{4}{3}]_2  (\frac{4}{3})_2 (\frac{4}{3})_2 (0)_{0} (\frac{4}{3})_{0} (\frac{4}{3})_{0} (\frac{4}{3})_{0} (\frac{4}{3})_{0} (\frac{4}{3})_{0}$ | 1     |
| 4:   | $(\frac{4}{3})_2 (\frac{4}{3})_2 (\frac{4}{3})_2 (\frac{4}{3})_2 (0)_{0} (\frac{4}{3})_{0} (\frac{4}{3})_{0} (\frac{4}{3})_{0} (\frac{4}{3})_{0} (\frac{4}{3})_{0}$ | 4     |
| 3:   | $[\frac{4}{3}]_1  [\frac{4}{3}]_2  (\frac{4}{3})_2 (\frac{4}{3})_2 (0)_{0} (\frac{4}{3})_{0} (\frac{4}{3})_{0} (\frac{4}{3})_{0} (\frac{4}{3})_{0} (\frac{4}{3})_{0}$ | 464   |
| 4:   | $[\frac{4}{3}]_1  [\frac{4}{3}]_2  (\frac{4}{3})_2 (\frac{4}{3})_2 (0)_{0} (\frac{4}{3})_{0} (\frac{4}{3})_{0} (\frac{4}{3})_{0} (\frac{4}{3})_{0} (\frac{4}{3})_{0}$ | 36    |
| 5:   | $[\frac{4}{3}]_1  [\frac{4}{3}]_2  (\frac{4}{3})_2 (\frac{4}{3})_2 (0)_{0} (\frac{4}{3})_{0} (\frac{4}{3})_{0} (\frac{4}{3})_{0} (\frac{4}{3})_{0} (\frac{4}{3})_{0}$ | 18    |
| 6:   | $[\frac{4}{3}]_1  [\frac{4}{3}]_2  (\frac{4}{3})_2 (\frac{4}{3})_2 (0)_{0} (\frac{4}{3})_{0} (\frac{4}{3})_{0} (\frac{4}{3})_{0} (\frac{4}{3})_{0} (\frac{4}{3})_{0}$ | 44    |
| 7:   | $(\frac{4}{3})_2 (\frac{4}{3})_2 (\frac{4}{3})_2 (\frac{4}{3})_2 (0)_{0} (\frac{4}{3})_{0} (\frac{4}{3})_{0} (\frac{4}{3})_{0} (\frac{4}{3})_{0} (\frac{4}{3})_{0}$ | 468   |
| 8:   | $(\frac{4}{3})_2 (\frac{4}{3})_2 (\frac{4}{3})_2 (\frac{4}{3})_2 (0)_{0} (\frac{4}{3})_{0} (\frac{4}{3})_{0} (\frac{4}{3})_{0} (\frac{4}{3})_{0} (\frac{4}{3})_{0}$ | 440   |

7.4. Configuration 4. There are 733 sets $\Sigma \in B_{14}(\delta_1)$, which are all saturated; we have $|\Sigma| \leq 126$. Extending the 32 sets of rank 19 by a maximal orbit (see §3.2.2), we arrive at $|\Sigma| \leq 112$. The only, up to $O_h(N)$, set $\Sigma_{126}^{4\nu}$ with 126 lines constitutes
\(B_{0}(\mathfrak{b}_{1})\); it is characterized by any of the four patterns

\[(7.4) \quad \pi_{e} = \langle 2, 0, 3 \rangle_{2}, \quad \varepsilon = c_{3} \in \text{Orb}_{4}(\mathfrak{b}_{3}, 6).\]

On the other hand, there are 105 sets \(\mathcal{L} \in B_{14}(\mathfrak{b}_{2} \cup \mathfrak{b}_{3})\), which are all of rank 18 or 19. Extending them by a maximal orbit (see \(\S 3.2.2\)), we arrive at \(|\mathcal{L}| \leq 118\).

7.5. **Configuration 6.** There are 16 sets \(\mathcal{L} \in B_{28}(\mathfrak{b}_{3})\). One of them, denoted \(\mathcal{W}_{144}^{iv}\), is maximal and contains 144 lines. It is determined by the pattern

\[(7.5) \quad \pi = \langle 0, 0, 9 \rangle \quad \text{(not } \mathcal{Q}\text{-complete).}\]

Extending the remaining 15 sets by one or two extra orbits (see \(\S 3.2.3\)), we obtain, among others, a set with 124 lines and one with 128 lines. The latter, denoted by \(\mathcal{W}_{128}^{iv}\), is characterized by the pattern

\[(7.6) \quad \pi = \langle 0, 2, 7 \rangle.\]

The 124-element set \(\mathcal{W}_{124}^{iv}\) is characterized by any of the six patterns

\[(7.7) \quad \pi_{e} = \langle 0, 3, 7 \rangle_{6},\]

where \(c := c_{3} \in \text{Orb}_{6}(\mathfrak{b}_{3}, 8)\) and \(c_{2} \in \text{Orb}_{1}(\mathfrak{b}_{2}, 4, \text{stab } c)\) is determined by \(c\).

7.6. **Configuration 7.** There are 244 sets \(\mathcal{L} \in B_{22}(\mathfrak{b}_{3})\), all saturated. One of these sets, denoted \(\mathcal{W}_{132}^{iv}\), has 132 lines; it is determined by the pattern

\[(7.8) \quad \pi = \langle 3, 0, 4, 0, 0, 0 \rangle.\]

For the other sets, one has \(|\mathcal{L}| \leq 116\). Twenty sets are of rank 19; extending them by a maximal orbit (see \(\S 3.2.2\)), we obtain at most 120 lines.

On the other hand, there are nine sets \(\mathcal{L} \in B_{8}(\mathfrak{b}_{3} \cup \mathfrak{b}_{2} \cup \mathfrak{b}_{4} \cup \mathfrak{b}_{5} \cup \mathfrak{b}_{7})\), which are all saturated and have \(|\mathcal{L}| \leq 104\). Extensions of the two sets of rank 19 by an extra orbit (see \(\S 3.2.3\)) have at most 108 lines.

7.7. **Configuration 8.** There are 94 sets \(\mathcal{L} \in B_{6}(\mathfrak{b}_{5})\). Most are maximal, and one has \(|\mathcal{L}| \leq 116\). The extensions of the two sets of rank 19 by a maximal orbit (see \(\S 3.2.2\)) are maximal sets with at most 110 lines. \(\Box\)

8. **The lattice \(N(24A_{1})\)**

The results of this section are summarized by the following theorem.

**Theorem 8.1.** Let \(\mathcal{L} \subset N(24A_{1})\) be a geometric set. Then, unless

- \(|\mathcal{L}| = 144 \text{ and } \mathcal{L} \text{ is conjugate to } \mathcal{W}_{144}^{iv}, \text{ see (8.10)}, \text{ or}
- \(|\mathcal{L}| = 132 \text{ and } \mathcal{L} \text{ is conjugate to one of the sets } \mathcal{W}_{132}^{iv}, \mathcal{W}_{132}^{vi}, \mathcal{W}_{132}^{vi}, \text{ or } \mathcal{W}_{132}^{vi}, \text{ see (8.2), (8.3), (8.11), or (8.12), respectively},

one has \(|\mathcal{L}| \leq 130\).

**Proof.** We proceed as in the previous sections. Each component \(v_{k} \in D_{k}^{\prime}, k \in \Omega = [1, \ldots, 24]\), of a vector \(v \in N\) is a multiple of the generator \(r_{k} \in D_{k}\). To save space, we use the notation

\[\cdot \text{ (if } v_{k} = 0), \quad - (\text{if } v_{k} = \pm \frac{1}{2}r_{k}), \quad \circ (\text{if } v_{k} = \pm r_{k}), \quad \bullet (\text{the position of } \bar{r}).\]

The signs always agree, so that \(h_{k} \cdot l_{k} \geq 0\) for any line \(l \in \mathfrak{b}(\bar{h})\) and \(k \in \Omega\).

There are three configurations (see Table 5).

Fix a basis \(\{r_{k}\}, k \in \Omega\), for \(24A_{1}\) consisting of roots. The kernel

\[N \mod 24A_{2} \subset \text{discr } 24A_{1} \cong (\mathbb{Z}/2)^{24}\]
of the extension is the Golay code $C_{24}$ (see [3]). The map \text{supp} identifies codewords with subsets of $\Omega$; then, $C_{24}$ is invariant under complement and, in addition to $\emptyset$ and $\Omega$, it consists of 759 octads, 759 complements thereof, and 2576 dodecads.

To simplify the notation, we identify the basis vectors $r_k$ (assumed fixed) with their indices $k \in \Omega$. For a subset $S \subseteq \Omega$, we let $\bar{I}_S := \sum r, r \in S$, and abbreviate $|S| := \frac{1}{2} |\bar{I}_S| \in N$ if $S \subseteq C_{24}$ is a codeword.

8.1. Configuration 1. We have $|\text{stab} \bar{h}| = 5760$ and $\bar{h}$ is the sum of three roots. Using patterns (see §3.3), we compute $B_{98}(\bar{h}_1) = \emptyset$, arriving at $|\mathcal{L}| \leq 120$.

8.2. Configuration 2. We have $|\text{stab} \bar{h}| = 1344$ and $\bar{h} = |O| + r_h$, where $O \subseteq C_{24}$ is an octad and $r_h \notin O$. Let $K := \Omega \setminus (O \cup \{r_h, \bar{r}\})$ and break $\bar{h}_1$ into eight clusters $c_o := \{ o \in \bar{h}_1 \mid (K \cap \text{supp} o) \subset o \}$, $o \in C_{24}$, $|o| = 8$, $o \cap O = \emptyset$, $\bar{r} \notin o$, $r_h \in o$. They constitute the orbit $\text{Orb}_{98}(\bar{h}_1, 14)$. Each combinatorial orbit $o \subseteq \bar{h}_1$ belongs to two clusters, and each pair of clusters intersects in a single pair of dual orbits.

The set $B_{52}(\bar{h}_1)$ is computed cluster by cluster, as explained in §3.4. We arrive at a number of sets $\mathcal{L}$ of size $|\mathcal{L}| \leq 120$ and a few those with $124 \leq |\mathcal{L}| \leq 132$. All sets are maximal. The large sets found can be described as $\mathcal{L} = \bar{h}_1 \cap \text{span}(\bar{r}, \bar{h}, \bar{u}_a, v)^\perp$, where

- $\bar{r}$ and $\bar{h} := |O| - 2r_h = h - 3r_h$ generate the subspace $\bar{h}_1^\perp \subset N$,
- $\bar{u}_a := 1_K - 2s$ for a certain fixed point $s \in O$,

and the extra vector $v$ is specified below, using the \textit{ad hoc} notation

- $\bar{v}_o := [o \setminus O] - [o \cap O]$ for a codeword $o \in C_{24}$.

Then, the large sets are as follows:

\begin{align*}
(8.2) \quad & \mathcal{G}_{132}^{iv} : \quad v = k, \quad k \in K; \\
(8.3) \quad & \mathcal{G}_{132}^{v} : \quad v = \bar{v}_o, \quad \bar{r} \in o, r_h \in o, s \in o, \quad |o \cap O| = 2, |o| = 8; \\
(8.4) \quad & \mathcal{L}_{126}^{vi} : \quad v = \bar{v}_o, \quad \bar{r} \notin o, r_h \in o, s \in o, \quad |o \cap O| = 2, |o| = 8; \\
(8.5) \quad & \mathcal{L}_{126}^{vii} : \quad v = \bar{v}_o + r_h, \quad \bar{r} \in o, r_h \notin o, s \in o, \quad |o \cap O| = 4, |o| = 8; \\
(8.6) \quad & \mathcal{L}_{126}^{viii} : \quad v = [o], \quad \bar{r} \in o, r_h \notin o, s \notin o, \quad |o \cap O| = 0, |o| = 8; \\
(8.7) \quad & \mathcal{L}_{126}^{ix} : \quad v = \bar{v}_o, \quad \bar{r} \notin o, r_h \in o, s \in o, \quad |o \cap O| = 2, |o| = 12; \\
(8.8) \quad & \mathcal{L}_{126}^{x} : \quad v = \bar{v}_o, \quad \bar{r} \in o, r_h \in o, s \in o, \quad |o \cap O| = 2, |o| = 12; \\
(8.9) \quad & \mathcal{L}_{126}^{xi} : \quad v = [o] - r_h, \quad \bar{r} \in o, r_h \in o, s \notin o, \quad |o \cap O| = 2, |o| = 8.
\end{align*}
In each case, it is straightforward that the set of data required for the description is unique up to \(O_{\mathcal{H}}(N)\).

8.3. Configuration 3. We have \(|\text{stab } h| = 7920\) and \(h = [\mathcal{O}]\), where \(\mathcal{O} \in C_{24}\) in a dodecad. Let \(K := \Omega \setminus (\mathcal{O} \cup \bar{r})\). Each support \(o := \text{supp } o\), \(o \in \bar{o}_1\), is an octad, so that \(|o \cap \mathcal{O}| = 6\) and \(|o \cap K| = 2\); conversely, each 2-element set \(s \subset K\) extends to a unique pair of such octads, representing a pair of dual orbits \(o, o^* \subset \bar{o}_1\). We break \(\bar{o}_1\) into eleven clusters

\[c_k := \{o \subset \bar{o}_1 \mid \text{supp } o \ni k\}, \quad k \in K.\]

Each orbit belongs to two clusters, and each pair of clusters intersects in a single pair of dual orbits. We compute the set \(B_{88}(\bar{o}_1)\) cluster by cluster, as explained in §3.4. Note that the first cluster \(c\) has \(|\Sigma \cap c| \geq 24\) (and, hence, \(\delta_0(c) \geq 4\)) and, in case of equality, also \(|\Sigma \cap \mathcal{O}_k| = 24\) for each \(k \in K\). In this latter case, we reduce overcounting by using the following observations:

- if \(\delta_0(c) = 6\), there is exactly one other cluster \(c'\) with \(\delta_0(c') = 6\), so that \(\Sigma \cap \mathcal{O} = \Sigma \cap o^* = \emptyset\) for the two orbits \(o, o^* \subset c \cap c'\);
- if \(\delta_0(c) = 4\), then \(\delta_0(c_k) = 4\) for each \(k \in K\) (thus, no preferred order), and we can choose \(c'\) so that \(|\Sigma \cap o| = |\Sigma \cap o^*| = 1\) for \(o, o^* \subset c \cap c'\).

In these two cases, we start with the pair \(c, c'\) and employ the extra symmetry.

The result is one maximal set \(\mathcal{M}_{744}\) and two submaximal sets \(\mathcal{S}_{132}, \mathcal{S}_{132}^{\text{II}}\). As a by-product, we have found six sets \(\mathcal{L}\) with \(124 \leq |\mathcal{L}| \leq 130\) and a number of sets of size \(|\mathcal{L}| \leq 120\). Most large sets can be described as

\[\mathcal{L} = \bar{o}_1 \cap \text{span}(\bar{r}, \bar{I}_K, r, v)^\perp,\]

where \(r \in K\) is a certain fixed point and the extra vector \(v\) is described below. This description depends on a codeword \(o \in C_{24}\) (we use the shortcut \(\tilde{w}_o := 3\bar{v}_o + [\mathcal{O}]\), where \(\bar{v}_o\) is as in §8.2) and, occasionally, an extra point \(s \in o \cap \mathcal{O}\) or \(t \in K \setminus r\).

Then, the large sets are as follows:

\begin{align*}
(8.10) & \quad \mathcal{M}_{744}^{\text{III}} : \quad v = t; \\
(8.11) & \quad \mathcal{S}_{132}^{\text{II}} : \quad v = [o] - s, \quad \bar{r} \notin o, r \notin o, \quad |o \cap \mathcal{O}| = 2, |o| = 8; \\
(8.12) & \quad \mathcal{S}_{132}^{\text{III}} : \quad v = \tilde{w}_o, \quad \bar{r} \in o, r \in o, \quad |o \cap \mathcal{O}| = 4, |o| = 8; \\
(8.13) & \quad \mathcal{L}_{130}^{\text{II}} : \quad v = [o] - s, \quad \bar{r} \in o, r \notin o, \quad |o \cap \mathcal{O}| = 2, |o| = 8; \\
(8.14) & \quad \mathcal{L}_{128}^{\text{IV}} : \quad v = \tilde{w}_o - 3t, \quad \bar{r} \in o, r \in o, t \in o, \quad |o \cap \mathcal{O}| = 4, |o| = 8; \\
(8.15) & \quad \mathcal{L}_{126}^{\text{V}} : \quad v = \tilde{w}_o, \quad \bar{r} \notin o, r \in o, \quad |o \cap \mathcal{O}| = 4, |o| = 8; \\
(8.16) & \quad \mathcal{L}_{126}^{\text{VI}} : \quad v = \tilde{w}_o, \quad \bar{r} \notin o, r \in o, \quad |o \cap \mathcal{O}| = 4, |o| = 12; \\
(8.17) & \quad \mathcal{L}_{124}^{\text{VII}} : \quad v = \tilde{w}_o, \quad \bar{r} \notin o, r \notin o, \quad |o \cap \mathcal{O}| = 4, |o| = 8.
\end{align*}

In (8.17) we require, in addition, that the 6-element set \((o \cap \mathcal{O}) \cup \{\bar{r}, r\}\) should not be contained in an octad. Under this extra assumption, the set of data needed for the description is unique up to \(O_{\mathcal{H}}(N)\). \hfill \Box

Remark 8.18. In §8.3, there is one more 126-element set \(\mathcal{L}\). However, since \(\mathcal{L}\) is graph isomorphic to \(\mathcal{L}_{126}^{\text{V}} \cong \mathcal{L}_{126}^{\text{VI}}\) and, on the other hand, we do not assert the completeness in this range, we omit its description, which is more complicated.

Remark 8.19. In the course of this computation, we have also observed all even line counts \(38 \leq |\mathcal{L}| \leq 120\) realized by geometric sets of rank 20.
9. Proofs of the main results

In this concluding section, we fill in a few missing links to complete the proofs of the principal results of the paper stated in the introduction.

9.1. Proof of Theorem 1.2 and Addendum 1.3. As explained in §2.2, instead of counting tritangents to smooth sextics one can study (doubling the numbers) the Fano graphs of smooth 2-polarized $K3$-surfaces. By Theorem 2.1, the latter task is equivalent to the study of the Fano graphs of certain 2-polarized lattices $NS \ni h$, and Proposition 2.4 and subsequent definitions reduce it further to the study of geometric subsets $S \subset \mathcal{H}(h)$ in 6-polarized Niemeier lattices $N \ni h$ other than the Leech lattice $\Lambda$ (as we can always assume that there is a root $\check{r} \in h^\perp$). This is done in Theorems 5.1, 6.1, 7.1, and 8.1, and there remains to observe that all sets of size 144 are isomorphic as abstract graphs,

(1) size 144: $\mathcal{M}_{144} \cong \mathcal{M}_{144}^2 \cong \mathcal{M}_{144}^3$, $T = [12, 6, 12]^*$,

and there are two isomorphism classes of sets of size 132:

(2) size 132: $\mathcal{S}^{i,2}_{132} \cong \mathcal{S}_{132} \cong \mathcal{S}_{132}^{i,2} \cong \mathcal{S}_{132}^{i,3} \cong \mathcal{S}_{132}^{i,4} \cong \mathcal{S}_{132}^{i,5}, T = [2, 0, 66]^*$,

(3) size 132: $\mathcal{S}_{132}^{i,6}, T = [4, 0, 32]^*$.

The graphs are compared by means of the GRAPE package [14, 15, 30] in GAP [10]. A posteriori, the large graphs $\mathcal{L}$ found in the paper are distinguished by their size $|2|$, discriminant form $\text{discr}(\text{span}_L \mathcal{L})$, and, in a few cases below, the size $|\text{Aut} \mathcal{L}|$ of the group of abstract graph automorphisms (also computed by GRAPE). Instead of $\text{discr}(\text{span}_L \mathcal{L})$, we give a list of representatives of the genus of the transcendental lattice $T := NS^h$ of the corresponding 2-polarized $K3$-surface, using the inline notation $[2a, b, 2c]$ for the even rank 2 form $T = \mathbb{Z}a + 2\mathbb{Z}v$, $u^2 = 2a$, $u \cdot v = b$, $v^2 = 2c$. The meaning of the superscript * is explained in §9.2(1) below.

This observation establishes the bounds stated in Theorem 1.2; the uniqueness is proved in §9.2 below. For the record, we give a similar classification for the other large geometric sets found in the course of the computation and described elsewhere in the paper:

(4) size 130: $\mathcal{L}_{130}^i \cong \mathcal{L}_{130}^ii \cong \mathcal{L}_{130}^iii, T = [12, 3, 12]^*$;

(5) size 128: $\mathcal{L}_{128}^i \cong \mathcal{L}_{128}^iv \cong \mathcal{L}_{128}^v, T = [12, 2, 12]^*$;

(6) size 126: $\mathcal{L}_{126}^i \cong \mathcal{L}_{126}^ii \cong \mathcal{L}_{126}^iii \cong \mathcal{L}_{126}^iv, T = [2, 1, 72]^*, [6, 1, 24], [8, 1, 18]$;

(7) size 126: $\mathcal{L}_{126}^v \cong \mathcal{L}_{126}^vi, |\text{Aut} \mathcal{L}| = 144, T = [14, 7, 14]^*$;

(8) size 126: $\mathcal{L}_{126}^v, |\text{Aut} \mathcal{L}| = 504, T = [14, 7, 14]^*$;

(9) size 124: $\mathcal{L}_{124}^{i,2} \cong \mathcal{L}_{124}^{i,3} \cong \mathcal{L}_{124}^{i,4} \cong \mathcal{L}_{124}^{i,5} \cong \mathcal{L}_{124}^{i,6}, T = [4, 0, 38]^* \text{ or } [6, 2, 26]$.

Besides, we have found nine isomorphism classes of geometric sets of size 120. Note that, unlike (1)–(3), we do not assert the completeness of these lists.

The statement of Addendum 1.3 is essentially given by Remark 8.19 and the above list, as geometric sets with fewer than 38 lines are easily constructed directly, mostly within an appropriate single combinatorial orbit $\phi$.

9.2. Proof of Theorem 1.2 (uniqueness). In full agreement with Corollary 2.7, all large geometric sets listed in §9.1 are of the maximal rank $\text{rk} \mathcal{L} = 20$. Therefore, the isomorphism classes of the smooth 2-polarized $K3$-surfaces (or, equivalently, the projective equivalence classes of sextics $C \subset \mathbb{P}^2$) with the given Fano graph $\text{Fn}(X, h) \cong \mathcal{L}$ are given by the global Torelli theorem [22] (cf. also [8, Theorem 3.11]) as the classes of primitive embeddings $NS \hookrightarrow L := -2E_8 \oplus 3U$ up to the group $O^+_h(L)$ of auto-isometries of $L$ preserving $h$ and the positive sign structure (i.e.,
Theorem 1.4. Theorem 1.2 applies to all geometric sets listed in Lemma 9.1. Besides, we have the following lemma.

There are obvious identities and inclusions

\[ O_h(\NS) = O_h(S) \subset O_h(\text{span } \mathfrak{L}) \subset O_h(\text{span}_\mathbb{Z} \mathfrak{L}) = \text{Aut } \mathfrak{L}. \]

Besides, we have the following lemma.

**Lemma 9.1.** Let \( \mathfrak{L} \subset N \) be a geometric set, and assume that

\[ \text{rk } \mathfrak{L} = 20, \quad \det(\text{span}_\mathbb{Z} \mathfrak{L}) < 1296. \]

Then the only mild extension \( S \supset \text{span } \mathfrak{L} \) is \( S = \text{span } \mathfrak{L} = \text{span}_\mathbb{Z} \mathfrak{L}. \)

**Proof.** For the Néron–Severi lattice \( \NS(X) \) of a K3-surface \( X \) corresponding to \( S \) we have

\[ |\det \NS(X)| = \det(\text{span}_\mathbb{Z} \mathfrak{L})/3i^2, \quad i := [S : \text{span}_\mathbb{Z} \mathfrak{L}]. \]

On the other hand, since \( X \) is smooth and of the maximal Picard rank 20, we have \( |\det \NS(X)| \geq 108 \) by [6, Theorem 1.5]. This implies \( i < 2 \), hence \( i = 1 \). \( \square \)

Lemma 9.1 applies to all geometric sets listed in \( \S 9.1 \) (and to the nine sets of size 120 mentioned thereafter). Then, a direct computation shows that the natural map \( \text{Aut } \mathfrak{L} \to \text{Aut}(\text{discr } \NS) \) is surjective. This fact renders the other group \( O^+(T) \) redundant and proves that each pair \((\mathfrak{L}, T)\) listed is realized by either

1. a single curve \( C \cong \tilde{C} \) (marked with a * in the list) or
2. a pair \( C, \tilde{C} \) of complex conjugate curves.

(The former is the case whenever \( T \) admits an orientation reversing auto-isometry; note that we do not assert that the curve admits a real structure, although most likely it does.) In particular, each of the three configurations listed in items (1), (2), (3) is realized by a single curve, as stated in **Theorem 1.2**. \( \square \)

9.3. **Proof of Theorem 1.4.** Let \( C \subset \mathbb{P}^2 \) be a real sextic. By definition, this means that \( C \) is invariant under a certain fixed real structure (anti-holomorphic involution) \( c: \mathbb{P}^2 \to \mathbb{P}^2 \). This involution lifts to two commuting anti-holomorphic automorphisms \( c_\pm \) of the covering K3-surface \( \varphi: X \to \mathbb{P}^2, \) so that \( c_+ \circ c_- = \tau \) is the deck translation. *Prima facie*, \( c_\pm \) are either involutions or of order 4, with \( c_+^2 = \tau \).

However, if we assume that \( C \) has a real tritangent \( L \), then at least one of the three tangency points (possibly, infinitely near) must be real; thus, the ramification locus of \( \varphi \) has a real point and both lifts \( c_\pm \) are involutions, *i.e.*, real structures on \( X \). Furthermore, we can select \( c_+ \) so that both pull-backs \( L_1, L_2 \subset \varphi^{-1}(L) \) are real (and then they are complex conjugate with respect to \( c_- \)); then, the pull-backs \( L'_1, L'_2 \subset \varphi^{-1}(L') \) of any other real tritangent \( L' \) are also real, as each of \( L'_i \) intersects exactly one of \( L_1, L_2 \) at a single point, which must be real.

Thus, we reduce the problem to counting real lines in a real 2-polarized K3-surface \((X, h)\). More precisely, this means that we fix a real structure \( c: X \to X, \)
c_*(h) = -h, and count lines \( L \subset X \) satisfying \( c_*[L] = -[L] \). (Recall that each line is unique in its homology class and that anti-holomorphic maps reverse the orientation of complex curves.) Arguing as in [8], we can perturb the period of \( X \) to change \( NS(X) \) to the sublattice rationally generated by the real lines; then, all lines in the new surface \( X \) are real and \( \text{Ker}(1 - c_*) \subset T(X) = NS(X) \perp \). Using the classification of real structures found in [18], we obtain the following statement.

**Lemma 9.2 (cf. [8, Lemma 3.10]).** A smooth 2-polarized \( K_3 \)-surface \( X \) is equilinear deformation equivalent to a real surface \( Y \) in which all lines are real if and only if the orthogonal complement \( \text{Fn}(X, h)^\perp \) contains \( A_1 \) or \( U(2) \) as a sublattice. \( \triangleright \)

If \( \text{rk} \text{Fn}(X, h) = 20 \), the Picard rank \( \text{rk} NS(X) = 20 \) is also maximal, the moduli space is finite, and the statement can be made more precise.

**Lemma 9.3.** Let \( X \) be a 2-polarized \( K_3 \)-surface, \( \text{rk} \text{Fn}(X, h) = 20 \). Then, the real structures \( c: X \rightarrow X \) with respect to which all lines are real are in a one-to-one correspondence with pairs of roots \( \pm r \in T(X) \). Under this correspondence, \(-c_* \) is the reflection against the hyperplane \( r^+ \subset H_2(X; \mathbb{Z}) \). \( \triangleright \)

There remains to examine the list found in §9.1 and observe that the maximum, which is 132 lines, is realized by a unique graph, \textit{viz.} the one in item (2), and the corresponding transcendental lattice \( T \) contains a single pair of roots. The next known examples are item (6) with 126 lines and two of the nine graphs with 120 lines, but we do not assert the completeness of our lists in this range. \( \square \)

**References**

1. Nicolas Bourbaki. *Lie groups and Lie algebras. Chapters 4–6*. Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 2002. Translated from the 1968 French original by Andrew Pressley. MR 1890629 (2003a:17001)

2. A. Clebsch. Zur Theorie der algebraischen Flächen, J. Reine Angew. Math. **58** (1861), 93–108. MR 1579142

3. J. H. Conway and N. J. A. Sloane, *Sphere packings, lattices and groups*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 290, Springer-Verlag, New York, 1988, With contributions by E. Bannai, J. Leech, S. P. Norton, A. M. Odlyzko, R. A. Parker, L. Queen and B. B. Venkov. MR 920369 (89a:11067)

4. Alex Degtyarev. *Lines in supersingular quartics*, To appear, arXiv:1604.05836, 2016.

5. Alex Degtyarev. *Smooth models of singular \( K_3 \)-surfaces*, Rev. Mat. Iberoam. **35** (2019), no. 1, 125–172. MR 3914542

6. Alex Degtyarev. *Lines on Smooth Polarized \( K_3 \)-Surfaces*, Discrete Comput. Geom. **62** (2019), no. 3, 601–648. MR 3996938

7. Alex Degtyarev. *Tritangents to smooth sextic curves*, Preprint MPIM 19-51, 2019.

8. Alex Degtyarev, Ilia Itenberg, and Ali Sinan Sertöz. *Lines on quartic surfaces*, Math. Ann. **368** (2017), no. 1-2, 753–809. MR 3651588

9. Noam D. Elkies. *Upper bounds on the number of lines on a surface*, New Trends in Arithmetic and Geometry of Algebraic Surfaces, BIRS conference 17w5146, 2017.

10. GAP – Groups, Algorithms, and Programming, Version 4.10.1, https://www.gap-system.org, Feb 2019.

11. Vik. S. Kulikov. *Surjectivity of the period mapping for \( K_3 \) surfaces*, Uspehi Mat. Nauk **32** (1977), no. 4(196), 257–258. MR 0480528 (58 #688)

12. Brendan D. McKay. *Nauty user’s guide (version 1.5)*, Tech. Report TR-CS-90-0, Australian National University, Computer Science Department, 1990.
15. Brendan D. McKay and Adolfo Piperno, *Practical graph isomorphism, II*, J. Symbolic Comput. 60 (2014), 94–112. MR 3131381
16. Shigeru Mukai, *Finite groups of automorphisms of K3 surfaces and the Mathieu group*, Invent. Math. 94 (1988), no. 1, 183–221. MR 958597
17. Hans-Volker Niemeier, *Definite quadratische Formen der Dimension 24 und Diskriminante 1*, J. Number Theory 5 (1973), 142–178. MR 0316384
18. V. V. Nikulin, *Integer symmetric bilinear forms and some of their geometric applications*, Izv. Akad. Nauk SSSR Ser. Mat. 43 (1979), no. 1, 111–177, 238, English translation: Math USSR-Izv. 14 (1979), no. 1, 103–167 (1980). MR 525944 (80j:10031)
19. I. I. Pjatecki˘ı-ˇSapiro and I. R. ˇSafareviˇ c, *Torelli’s theorem for algebraic surfaces of type K3*, Izv. Akad. Nauk SSSR Ser. Mat. 35 (1971), 530–572, English translation: Math. USSR-Izv. 5, 547–588. MR 0284440 (44 #1666)
20. I. I. Pjatecki˘ı-ˇSapiro and I. R. ˇSafareviˇ c, *Torelli’s theorem for algebraic surfaces of type K3*, Izv. Akad. Nauk SSSR Ser. Mat. 79 (2015), no. 4, 103–158. MR 3397421
21. Ulf Persson, *Double sextics and singular K3 surfaces*, Algebraic geometry, Sitges (Barcelona), 1983, Lecture Notes in Math., vol. 1124, Springer, Berlin, 1985, pp. 262–328. MR 805337 (87i:14036)
22. I. I. Pjatecki˘ı-ˇSapiro and I. R. ˇSafareviˇ c, *Torelli’s theorem for algebraic surfaces of type K3*, Izv. Akad. Nauk SSSR Ser. Mat. 35 (1971), 530–572, English translation: Math. USSR-Izv. 5, 547–588. MR 0284440 (44 #1666)
23. Slawomir Rams and Matthias Sch¨ utt, *112 lines on smooth quartic surfaces (characteristic 3)*, Q. J. Math. 66 (2015), no. 3, 941–951. MR 3396099
24. I. I. Pjatecki˘ı-ˇSapiro and I. R. ˇSafareviˇ c, *Torelli’s theorem for algebraic surfaces of type K3*, Izv. Akad. Nauk SSSR Ser. Mat. 35 (1971), 530–572, English translation: Math. USSR-Izv. 5, 547–588. MR 0284440 (44 #1666)
25. I. I. Pjatecki˘ı-ˇSapiro and I. R. ˇSafareviˇ c, *Torelli’s theorem for algebraic surfaces of type K3*, Izv. Akad. Nauk SSSR Ser. Mat. 79 (2015), no. 4, 103–158. MR 3397421
26. Ken-ichi Nishiyama, *The Jacobian fibrations on some K3 surfaces and their Mordell-Weil groups*, Japan. J. Math. (N.S.) 22 (1996), no. 2, 293–347. MR 1432379
27. Leonard H. Soicher, *GRAPE, GRaph Algorithms using PErmutation groups*, Version 4.8.1, https://gap-packages.github.io/grape, Oct 2018, Refereed GAP package.
28. Davide Cesare Veniani, *Lines on K3 quartic surfaces in characteristic 2*, Q. J. Math. 68 (2017), no. 2, 551–581. MR 3667213
29. Davide Cesare Veniani, *The maximum number of lines lying on a K3 quartic surface*, Math. Z. 285 (2017), no. 3-4, 1141–1166. MR 3623744
30. Davide Cesare Veniani, *Symmetries and equations of smooth quartic surfaces with many lines*, To appear, arXiv:1708.01219, 2017.