LOWER BOUNDS ON CUBICAL DIMENSION OF $C'(1/6)$ GROUPS

KASIA JANKIEWICZ

Abstract. For each $n$ we construct examples of finitely presented $C'(1/6)$ small cancellation groups that do not act properly on any $n$-dimensional CAT(0) cube complex.

1. Introduction

Groups that satisfy the $C'(1/6)$ small cancellation condition were shown to act properly and cocompactly on CAT(0) cube complexes by Wise in [Wis04].

In this note we are interested in the minimal dimension of a CAT(0) cube complex that such groups act properly on.

Definition 1.1. The cubical dimension of $G$ is the infimum of the values $n$ such that $G$ acts properly on an $n$-dimensional CAT(0) cube complex.

Wise’s complex is obtained from Sageev’s construction [Sag95] with walls joining the opposite sides in each relator (after subdividing each edge into two if necessary). However, its dimension is not in general optimal. For example, the dimension of the CAT(0) cube complex associated to the usual presentation for the fundamental group of the surface of genus $g \geq 2$ is $g$, while its cubical dimension equals 2 as it acts on the hyperbolic plane with a CAT(0) square complex structure.

We prove the following:

Theorem 1.2. For each $n \geq 1$ and each $p \geq 6$ there exists a finitely presented $C'(1/p)$ small cancellation group $G$ such that the cubical dimension of $G$ is greater than $n$.

For $n = 1$, the stronger form of Theorem 1.2 was proved by Pride in [Pri83]. He gives an explicit example of an infinite $C'(1/6)$ group with property FA. Pride’s construction has been revisited in [JW17]. We observe that the case $n = 2$ can be deduced from the work of Kar and Sageev who study uniform exponential growth of groups acting freely on CAT(0) square complexes [KS16]. See Remark 1.1. As a consequence, the Kar-Sageev examples have finite cubical dimension that is strictly larger than the geometric dimension.

This note is organized as follows. In Section 2 we recall the classification of isometries of a CAT(0) cube complex with respect to hyperplanes. We
LOWER BOUNDS ON CUBICAL DIMENSION OF $C'(1/6)$ GROUPS

In Section 3 we describe how to build a $C'(1/p)$ presentations where relators are positive products of given words. This technical result is applied in Section 4, which is the heart of the paper and contains the proof of Theorem 1.2. The argument heavily utilizes hyperplanes to create a dichotomy between free subsemigroups and subgroups having polynomial growth. The main ingredient of the proof of Theorem 1.2 is Lemma 4.2 which states that for any two hyperbolic isometries $a, b$ of an $n$-dimensional CAT(0) cube complex one of the following holds: $\langle a^N, b^N \rangle$ is virtually abelian for some $N = N(n)$, or there is a hyperplane stabilized by certain conjugates of some powers of $a$ or $b$, or there is a pair of words in $a, b$ of uniformly bounded length that generates a free semigroup.

Acknowledgements. I would like to thank my supervisors Piotr Przytycki and Daniel Wise. I would also like to thank Carolyn Abbot, Yen Duong, Teddy Einstein, Justin Lanier, Thomas Ng and Radhika Gupta for helpful discussion on [KS16]. The author was partially supported by (Polish) Narodowe Centrum Nauki, grant no. UMO-2015/18/M/ST1/00050.

2. ISOMETRIES AND HYPERPLANES IN CAT(0) CUBE COMPLEXES

In this section we recall relevant facts about isometries of CAT(0) cube complexes and collect some lemmas that will be used in the proof of Theorem 1.2. For general background on CAT(0) cube complexes and groups acting on them we refer the reader to [Sag14].

Throughout the paper $X$ will be a finite dimensional CAT(0) cube complex. The set of all hyperplanes of $X$ is denoted by $\mathcal{H}(X)$ and a cube complex dual to a collection $\mathcal{H}$ of hyperplanes is denoted by $X(\mathcal{H})$. We use letters $h, h^*$ to denote the halfspaces of a hyperplane $h$, and $N(h)$ to denote the closed carrier of $h$, i.e. the convex subcomplex of $X$ that is the union of all the cubes intersecting $h$. We say that a hyperplane $h$ separates subsets $A, B \subset X$, if $A \subset h$ and $B \subset h^*$. The metric $d$ is the $\ell_1$-metric on $X$. All the paths we consider are combinatorial (i.e. concatenations of edges), all the geodesics are with respect to $d$, and all axes of hyperbolic isometries are combinatorial axes. The combinatorial translation length $\delta(x)$ of an isometry $x$ is defined as $\inf_{p \in X^0} d(p, xp)$. If $x$ acts without hyperplane inversions then the infimum is realized and $\delta(x^k) = k\delta(x)$ [Hag07] (see also [Woo16]). In particular, $x$ has an axis and any axis of $x$ is also an axis of $x^k$. The combinatorial minset of $x$ is

$$\text{Min}^0(x) = \{ p \in X^0 : d(p, xp) = \delta(x) \}$$

where $X^0$ is the 0-skeleton of $X$. Every 0-cube $p$ of $\text{Min}^0(x)$ lies on an axis of $x$ (any geodesic joining $\{x^p\}$).

Let $n = \dim X$. Let $x$ be a hyperbolic isometry of $X$ and let $\mathcal{H}$ be a hyperplane. We recall the classification of isometries of a CAT(0) cube complex. More details can be found in [CST11 Sec 2.4 and 4.2].

ref to [LS77] for the background on small cancellation theory. In Section 3 we describe how to build a $C'(1/p)$ presentations where relators are positive products of given words. This technical result is applied in Section 4, which is the heart of the paper and contains the proof of Theorem 1.2. The argument heavily utilizes hyperplanes to create a dichotomy between free subsemigroups and subgroups having polynomial growth. The main ingredient of the proof of Theorem 1.2 is Lemma 4.2 which states that for any two hyperbolic isometries $a, b$ of an $n$-dimensional CAT(0) cube complex one of the following holds: $\langle a^N, b^N \rangle$ is virtually abelian for some $N = N(n)$, or there is a hyperplane stabilized by certain conjugates of some powers of $a$ or $b$, or there is a pair of words in $a, b$ of uniformly bounded length that generates a free semigroup.

Acknowledgements. I would like to thank my supervisors Piotr Przytycki and Daniel Wise. I would also like to thank Carolyn Abbot, Yen Duong, Teddy Einstein, Justin Lanier, Thomas Ng and Radhika Gupta for helpful discussion on [KS16]. The author was partially supported by (Polish) Narodowe Centrum Nauki, grant no. UMO-2015/18/M/ST1/00050.

2. ISOMETRIES AND HYPERPLANES IN CAT(0) CUBE COMPLEXES

In this section we recall relevant facts about isometries of CAT(0) cube complexes and collect some lemmas that will be used in the proof of Theorem 1.2. For general background on CAT(0) cube complexes and groups acting on them we refer the reader to [Sag14].

Throughout the paper $X$ will be a finite dimensional CAT(0) cube complex. The set of all hyperplanes of $X$ is denoted by $\mathcal{H}(X)$ and a cube complex dual to a collection $\mathcal{H}$ of hyperplanes is denoted by $X(\mathcal{H})$. We use letters $h, h^*$ to denote the halfspaces of a hyperplane $h$, and $N(h)$ to denote the closed carrier of $h$, i.e. the convex subcomplex of $X$ that is the union of all the cubes intersecting $h$. We say that a hyperplane $h$ separates subsets $A, B \subset X$, if $A \subset h$ and $B \subset h^*$. The metric $d$ is the $\ell_1$-metric on $X$. All the paths we consider are combinatorial (i.e. concatenations of edges), all the geodesics are with respect to $d$, and all axes of hyperbolic isometries are combinatorial axes. The combinatorial translation length $\delta(x)$ of an isometry $x$ is defined as $\inf_{p \in X^0} d(p, xp)$. If $x$ acts without hyperplane inversions then the infimum is realized and $\delta(x^k) = k\delta(x)$ [Hag07] (see also [Woo16]). In particular, $x$ has an axis and any axis of $x$ is also an axis of $x^k$. The combinatorial minset of $x$ is

$$\text{Min}^0(x) = \{ p \in X^0 : d(p, xp) = \delta(x) \}$$

where $X^0$ is the 0-skeleton of $X$. Every 0-cube $p$ of $\text{Min}^0(x)$ lies on an axis of $x$ (any geodesic joining $\{x^p\}$).

Let $n = \dim X$. Let $x$ be a hyperbolic isometry of $X$ and let $\mathcal{H}$ be a hyperplane. We recall the classification of isometries of a CAT(0) cube complex. More details can be found in [CST11 Sec 2.4 and 4.2].

ref to [LS77] for the background on small cancellation theory. In Section 3 we describe how to build a $C'(1/p)$ presentations where relators are positive products of given words. This technical result is applied in Section 4, which is the heart of the paper and contains the proof of Theorem 1.2. The argument heavily utilizes hyperplanes to create a dichotomy between free subsemigroups and subgroups having polynomial growth. The main ingredient of the proof of Theorem 1.2 is Lemma 4.2 which states that for any two hyperbolic isometries $a, b$ of an $n$-dimensional CAT(0) cube complex one of the following holds: $\langle a^N, b^N \rangle$ is virtually abelian for some $N = N(n)$, or there is a hyperplane stabilized by certain conjugates of some powers of $a$ or $b$, or there is a pair of words in $a, b$ of uniformly bounded length that generates a free semigroup.

Acknowledgements. I would like to thank my supervisors Piotr Przytycki and Daniel Wise. I would also like to thank Carolyn Abbot, Yen Duong, Teddy Einstein, Justin Lanier, Thomas Ng and Radhika Gupta for helpful discussion on [KS16]. The author was partially supported by (Polish) Narodowe Centrum Nauki, grant no. UMO-2015/18/M/ST1/00050.
• $x$ skewers $h$ if $x^k h \subseteq h$ for one of the halfspaces $h$ of $h$ and some $k > 0$. Equivalently, if some (equivalently, any) axis of $x$ intersects $h$ exactly once.
• $x$ is parallel to $h$ if some (equivalently, any) axis of $x$ is in a finite neighbourhood of $h$.
• $x$ is peripheral to $h$ if $x$ does not skewer $h$ and is not parallel to $h$.

Equivalently, $x^k h \subseteq h^*$ for some $k > 0$.

Note that the type of behaviour of $x$ with respect to $h$ is commensurability invariant, i.e. $x^i$ has the same type as $x$ with respect to $h$. The set of all hyperplanes in $X$ skewered by $x$ is denoted by $sk(x)$. The constant $k$ in the above definitions can be chosen to be at most $n$. Indeed, the $n+1$ hyperplanes $\{h, xh, \ldots, x^nh\}$ cannot all intersect in $X$ since $\dim X = n$.

In particular, if $h \in sk(x)$ then $x^n h \subset x^k h \subset \ldots \subset h$ for one of the halfspaces $h \in h$ and for an appropriate $k < n$. Similarly, we have the following:

**Lemma 2.1.** There exists a constant $K_3 = K_3(n)$ such that for each hyperplane $h$ in $X$ and an isometry $x$ there exist $k < k' \leq K_3$ such that the hyperplanes $\{h, x^k h, x^{k'} h\}$ pairwise are disjoint or equal.

**Proof.** Consider the graph $\Gamma$ whose vertices correspond to integers, and two integers $r, q$ are joined by an edge if and only if $x^r h$ and $x^q h$ are distinct and intersect. Cliques in $\Gamma$ correspond to collections of distinct pairwise intersecting hyperplanes. Let $K_3$ be the Ramsey constant for numbers $(n+1)$ and 3. Since $X$ is $n$-dimensional, there are no $(n+1)$-cliques in $\Gamma$. The induced subgraph of $\Gamma$ on vertices $[0, K_3 - 1]$ must contain a 3-anticlique. This corresponds to a triple of hyperplanes $\{x^p h, x^q h, x^r h\}$ where $p < q < r$ that pairwise are disjoint or equal. Hence the hyperplanes $\{h, x^{r-p} h, x^{r-p} h\}$ are pairwise disjoint or equal.

In the above Lemma the hyperplanes $h, x^k h, x^{k'} h$ are pairwise disjoint, or $x^{K_3}$ stabilizes $h$ (and the two cases are not mutually exclusive).

**Lemma 2.2.** [KSI16, Lem 12] Suppose $x$ and $y$ are hyperbolic isometries of $X$ and there exists a hyperplane $h = (h, h^*)$ such that $xh \subset h$, $yh \subset h$ and $xh \subset yh^*$. Then $x, y$ freely generate a free semigroup. See Figure 2.2.

The triple $\{h, xh, yh\}$ as in Lemma 2.2 is called a ping-pong triple. The following Lemma is a higher dimensional version of the All-Or-Nothing Lemma.
Proof. Let $p$ be some 0-cube of $\gamma_x$. Let $h_1, \ldots, h_k$ denote all the hyperplanes separating $p$ and $x^np$ (in particular, $k = n!\delta(x)$). Since $x^nh_i \subset h_i$ for all $i$ and appropriate choice of halfspace $h_i$ of $h_i$, the partition of the set of all hyperplanes skewered by $x$ into $\{x^{ni}h_1\}_{i \in \mathbb{Z}}, \ldots, \{x^{ni}h_k\}_{i \in \mathbb{Z}}$ gives an isometric embedding of Hull($\gamma_x$) into a product of $k$ trees by [CH13]. Since all the hyperplanes are intersected by a single bi-infinite geodesic (an axis of $x$), all the trees are in fact lines, i.e. Hull($\gamma_x$) isometrically embeds in $\mathbb{E}^k$ with the standard cubical structure. The action of $x^nl$ extends to the action to $\mathbb{E}^k$ as a translation by the vector $[1, \ldots, 1]$. Thus every 0-cube of the combinatorial convex hull Hull($\gamma_x$) is translated by $k = n!\delta(x) = \delta(x^nl)$ and therefore the 0-skeleton of Hull($\gamma_x$) is contained in Min$^0(x^{nl})$.

The subcomplex Hull(Min$^0(x)$) is the maximal subcomplex of the $\bigcap\{h : \text{Min}^0(x) \subset h\}$, i.e. Hull(Min$^0(x)$) is dual to $\mathcal{H}_x = \{h : \text{Min}^0(x) \cap h \neq \emptyset\}$. 

\[\text{Lemma 2.3.}\] Let $x$ and $y$ be hyperbolic isometries and let $h \in \text{sk}(x)$. Then one of the following holds

- $y$ skewers all $x^{nl}h$ for $i \in \mathbb{Z}$, or
- $y$ skewers none of $x^{nl}h$ for $i \in \mathbb{Z}$, or
- one of the following pairs of words freely generate a free semigroup for some $1 \leq k \leq n$:

\[
\begin{align*}
(x^{nl}, y^{kn}x^{nl}), \\
(x^{nl}, y^{-kn}x^{nl}), \\
(x^{-nl}, y^{kn-}x^{-nl}), \\
(x^{-nl}, y^{-kn-}x^{-nl}).
\end{align*}
\]

\[\text{(*)}\]

\[\text{Proof.}\] Let $h$ be the halfspace of $h$ such that $x^{nl}h \subset h$. Suppose that $y$ skewers some hyperplane in $P$ but not all of them. Without loss of generality we can assume that $y$ skewers exactly one of $h, x^{nl}h$. First suppose $y$ skewers $h$ but not $x^{nl}h$ i.e. the axis $\gamma_x \subset x^{nl}h^\ast$. Since $\gamma_x$ goes arbitrarily deep in $h^\ast$ we have that $y$ is peripheral to $x^{nl}h$. We either have $y^{nl}h \subset h$ or $y^{-nl}h \subset h$. Let $h$ be such that $y^{kn}x^{nl}h$ and $x^{nl}h$ are disjoint. Either $y^{kn}x^{nl}h \subset y^{kn}h \subset h$ or $y^{-kn}x^{nl}h \subset y^{-kn}h \subset h$ and thus $\{h, x^{nl}h, y^{kn}x^{nl}h\}$ or $\{h, x^{nl}h, y^{-kn}x^{nl}h\}$ is a ping-pong triple. Similarly, if $y$ skewers $x^{nl}h$ but not $h$, then one of $\{x^{nl}h^\ast, h^\ast, y^{kn}h^\ast\}$ or $\{x^{nl}h^\ast, h^\ast, y^{-kn}h^\ast\}$ is a ping-pong triple.

\[\text{□}\]

The \textit{combinatorial convex hull} Hull(A) of a subset $A \subset X$ is the smallest convex cube complex containing $A$.

\[\text{Lemma 2.4.}\]

1. The combinatorial convex hull Hull($\gamma_x$) of an axis $\gamma_x$ of $x$ isometrically embeds in $\mathbb{E}^k$ for some $k \geq 1$.

2. The 0-skeleton of Hull(Min$^0(\gamma_x)$) is contained in Min$^0(x^{nl})$.

\[\text{Proof.}\] Let $p$ be some 0-cube of $\gamma_x$. Let $h_1, \ldots, h_k$ denote all the hyperplanes separating $p$ and $x^{nl}p$ (in particular, $k = n!\delta(x)$). Since $x^{nl}h_i \subset h_i$ for all $i$ and appropriate choice of halfspace $h_i$ of $h_i$, the partition of the set of all hyperplanes skewered by $x$ into $\{x^{ni}h_1\}_{i \in \mathbb{Z}}, \ldots, \{x^{ni}h_k\}_{i \in \mathbb{Z}}$ gives an isometric embedding of Hull($\gamma_x$) into a product of $k$ trees by [CH13]. Since all the hyperplanes are intersected by a single bi-infinite geodesic (an axis of $x$), all the trees are in fact lines, i.e. Hull($\gamma_x$) isometrically embeds in $\mathbb{E}^k$ with the standard cubical structure. The action of $x^nl$ extends to the action to $\mathbb{E}^k$ as a translation by the vector $[1, \ldots, 1]$. Thus every 0-cube of the combinatorial convex hull Hull($\gamma_x$) is translated by $k = n!\delta(x) = \delta(x^{nl})$ and therefore the 0-skeleton of Hull($\gamma_x$) is contained in Min$^0(x^{nl})$.
∅ and Min^0(x) ∩ h^* ≠ ∅}. If h ∉ sk(x), and p, p′ ∈ Min^0(x) are separated by h, i.e. p ∈ h and p′ ∈ h^*, then x is parallel to h. Indeed, x^i p ∈ h and x^i p′ ∈ h^* for all i and since d(x^i p, x^i p′) = d(p, p′) the axis γ_x through p is contained in N_d(h) where d ≤ d(p, p′). Thus the set \( \mathcal{H}_x \) consists of hyperplanes skewed by x or parallel to x. It follows that Hull(Min^0(x)) decomposes as a product Y × Y^⊥ where Y is dual to sk(x) and Y^⊥ is dual to the set of all the hyperplanes of \( \mathcal{H}_x \) that are parallel to x. For each \( p ∈ Y^⊥ \) the complex Y × \{p\} is the combinatorial convex hull of an axis of x. It follows that Hull(Min^0(x)) is the union of the complexes of the form Hull(γ_x) and so the 0-skeleton of Hull(Min^0(x)) is contained in Min^0(x^{n!}). □

**Lemma 2.5.** Let X be a CAT(0) cube complex that is a subcomplex of a CAT(0) cube complex that is quasi-isometric to \( \mathbb{E}^d \). Then any group G acting properly on X does not contain a copy of \( F_2 \). Moreover, if G is torsion-free, then G is virtually abelian.

**Proof.** The growth of \( X^0 \) is a polynomial of degree at most \( d \) and so is the growth of G. Hence G cannot contain a copy of \( F_2 \). The second part follows from the Tits alternative for groups acting properly on CAT(0) cube complexes [SW05] which states that any such group with a bound on the size of finite subgroups either contains a copy of \( F_2 \), or is virtually abelian. □

3. Constructing small cancellation presentations

The main goal of this section is the following.

**Proposition 3.1.** Let \( U = \{(u_i, v_i)\}_{i=1}^m \) be a finite collection of pairs where for each i the elements \( u_i, v_i ∈ F(x, y) \) are not powers of the same element. There exists a \( C'(1/6) \) small cancellation presentation

\[
\langle x, y \mid r_1, \ldots, r_m \rangle
\]

where \( r_i \) is a positive word in \( u_i, v_i \) that is not a proper power for \( i = 1, \ldots, m \).

By \( F(x, y) \) in the above Lemma and throughout the section we denote the free group on generators \( a \) and \( b \). The length of a word \( u \) with respect to \( x, y \) is denoted by \( |u| \). A *spelling* of a nontrivial element \( u ∈ F(x, y) \) is a concatenation \( u_1 \cdots u_m = u \) where each *syllable* \( u_i \) is a nontrivial element of \( F(x, y) \). The *cancellation* in the spelling \( uv \) is the value canc(\( u, v \)) = \( \frac{1}{2}(|u| + |v| - |uv|) \), i.e. the length of the common prefix of the reduced words representing \( u^{-1} \) and \( v \). A spelling is *reduced* if canc(\( u_i, u_{i+1} \)) = 0 for \( i = 1, \ldots, m - 1 \); in other words \( |u| = \sum_i |u_i| \). A spelling is *cyclically reduced* if additionally canc(\( u_m, u_1 \)) = 0. For \( u, v ∈ F(x, y) \) we say \( u, v \) are *virtually conjugate* and write \( u \sim v \) if some powers of \( u \) and \( v \) are conjugate. We denote a free semigroup on \( u, v \) by \( \{u, v\}^+ \). Let \( u^k \) denote an element \( u^k \) for some \( k ≥ 0 \).
Lemma 3.2. Let $H$ be a finitely generated subgroup of $F_k$. There exists a constant $C = C(H < F_k)$ such that the map between the conjugacy classes of maximal $\mathbb{Z}$-subgroups induced by the inclusion $H \hookrightarrow F_k$ is at most $C$-to-1.

Proof. Let $A \hookrightarrow B$ be an immersion of graphs where $B$ is a wedge of $k$ circles, where the induced map on the fundamental groups is the inclusion $H \hookrightarrow F_k$.

For any graph $\Gamma = A, B$, the conjugacy class of a $\mathbb{Z}$-subgroup in $\pi_1 \Gamma$ can be represented by an immersion $L \hookrightarrow \Gamma$ of a line that factors as $L \hookrightarrow S \hookrightarrow \Gamma$ where $S$ is a circle, taken modulo the orientation. Thus different conjugacy classes of $\mathbb{Z}$-subgroups in $H$ that map into the same conjugacy class in $F_k$ are different lifts $A \hookrightarrow B$.

The number of such lifts is bounded by the number of vertices in $A$. \qed

Lemma 3.3. Let $u, v \in F(x, y)$ be such that $u$ and $v$ are not powers of the same element. There are infinitely many pairwise non virtually conjugate elements of the form $u_k v_k$.

Proof. Two elements of $F(x, y)$ are virtually conjugate if and only if they have the following reduced spellings

$$gw^ig^{-1}$$

$$h\bar{w}^jh^{-1}$$

where $g, h, w$ are reduced words in $x, y$ and $\bar{w}$ is a cyclic permutation of $w$. In particular the elements of the set $\{x^k y^k : n \in \mathbb{Z}\}$ are not virtually conjugate, i.e. they are contained in distinct conjugacy classes of maximal $\mathbb{Z}$-subgroups. Since $u, v$ are not powers of the same element, the group $\langle u, v \rangle$ is a rank 2 free group. By Lemma 3.2 there exists a constant $C$ such that the map between the conjugacy classes of maximal $\mathbb{Z}$-subgroups induced by the inclusion $\langle u, v \rangle \hookrightarrow F(x, y)$ is at most $C$-to-1. The lemma follows. \qed

We say that elements $u, v \in F(x, y)$ are non-cancellable, if for any $w_1, w_2 \in \{u, v\}^+$

$$\text{canc}(w_1, w_2) < \frac{1}{2} \min\{|u|, |v|\}.$$  

In particular, we have $|w_1 w_2| \geq \max\{|w_1|, |w_2|\}$. Equivalently, it suffices that $\text{canc}(u, v) < \frac{1}{2} \min\{|u|, |v|\}$ and $\text{canc}(v, u) < \frac{1}{2} \min\{|u|, |v|\}$ for $u, v$ to be non-cancellable. If $u, v$ are non-cancellable then so are any two elements in $\{u, v\}^+$.

Lemma 3.4. Let $u, v \in F(x, y)$ not be powers of the same element. Then there exists elements $u', v' \in \{u, v\}^+$ that are non-cancellable and are not powers of the same element.
Proof. If $\text{canc}(u, v) > \frac{1}{2}\min\{|u|, |v|\}$ replace the pair $(u, v)$ with $(u, uv)$ if $|u| \leq |v|$, and with $(v, uv)$ otherwise. If $\text{canc}(v, u) > \frac{1}{2}\min\{|u|, |v|\}$ replace the pair $(u, v)$ with $(u, vu)$ if $|u| \leq |v|$, and with $(v, vu)$ otherwise. Repeat these steps until $\text{canc}(u, v), \text{canc}(v, u) \leq \frac{1}{2}\min\{|u|, |v|\}$. Since at each step the value $|u| + |v|$ strictly decreases, the procedure terminates in finitely many steps. Note that for any nontrivial element $w \in F(x, y)$ we have $\text{canc}(w, w) < \frac{1}{2}|w|$, i.e. $|w^2| > |w|$. Let $u' = u^2$ and $v' = v^2$. We have $\text{canc}(u', v') = \text{canc}(u, v) \leq \frac{1}{2}\min\{|u|, |v|\} < \frac{1}{2}\min\{|u'|, |v'|\}$ as wanted. Similarly, $\text{canc}(v', u') < \frac{1}{2}\min\{|u'|, |v'|\}$. It follows that $\text{canc}(w_1, w_2) < \frac{1}{2}\min\{|u'|, |v'|\}$ for every $w_1, w_2 \in \{u', v'\}^+$.

**Lemma 3.5.** Let $s, t$ be two cyclically reduced elements in $F(x, y)$ such that $|s| \geq |t| > 0$ such that $s^2$ is a prefix of $t^k$. Then $s, t$ are powers of the same element.

**Proof.** Suppose that $s$ and $t$ are not powers of the same element. In particular, $s$ is not a power of $t$, so there exists a nonempty prefix $w$ of $s$ that is both some prefix of $t$ and some suffix of $t$. See Figure 2. If $|w| \leq \frac{1}{2}|t|$, then $t$ has a reduced spelling $wwu$ for some $u$, and $s$ has a reduced spelling $(wwu)^k wu$ for some $k \geq 1$. Then $s^2$ has a prefix $(wwu)^k wu \cdot wuw = (wwu)^k+1 wuu$ which must coincide with $t^{k+1} = (wwu)^{k+2}$. In particular, $uw = wu$, which means that $w, u$ are powers of the same element. That is a contradiction.

If $|w| > \frac{1}{2}|y|$, then $t$ has reduced spellings $uw$ and $wuw'$ for some $u, u'$ such that $|u| = |u'| < |w|$, and $s$ has a reduced spelling $(uw)^k u$ for some $k \geq 1$. The prefix $(uw)^k uw$ of $s^2$ must coincide with the prefix $(uw)^k u$ of $t^{k+2}$. In particular $uw = wu$, which again is a contradiction.

**Lemma 3.6.** Let $u_i, v_i \in F(x, y)$ for $i = 1, 2$ where for each $i = 1, 2$ the elements $u_i, v_i$ are non-cancellable and are not powers of the same element. Then for each $i = 1, 2$ there exist $s_i, t_i \in \{u_i, v_i\}^+$ such that

- $s_i, t_i$ are non-cancellable and are not virtually conjugate,
- $\text{canc}(s_i, t_i) = \text{canc}(t_i, s_i) = \text{canc}(s_i, s_i) = \text{canc}(t_i, t_i)$, i.e. there exists $g$ such that $s_i = g s_i g^{-1}$ and $t_i = g t_i g^{-1}$ are reduced spellings where $s_i, t_i$ are cyclically reduced and have no cancellation,
- every piece $w$ between a word in $\{s_1, t_1\}^+$ and a word in $\{s_2, t_2\}^+$ we have $|w| < \min\{|s_i|, |t_i|\}$ for $i = 1, 2$.

**Proof.** Since $u_i, v_i$ are non-cancellable, the consecutive cancellations between syllables in any word $r \in \{u_i, v_i\}$ are separated from each other. For $i = 1, 2$ set $s_i' = u_i^{n_{i1}} v_i^{n_{i2}}$ and $t_i' = u_i^{n_{i2}} v_i^{n_{i1}}$ where $n_{i1}, n_{i2}, n_{i1}, n_{i2}$ are chosen.
so that $s_1', t_1', s_2', t_2'$ are pairwise non virtually conjugate. This can be done by Lemma 3.3 Note that for $i = 1, 2$ we have $\text{canc}(s_i', t_i') = \text{canc}(s_i', s_i') = \text{canc}(t_i', t_i') = \text{canc}(v_i, u_i)$. Let $\bar{s}_i'$ denote the cyclically reduced word representing an element conjugate to $s_i'$ such that the spelling $s_i' = g\bar{s}_i'g^{-1}$ is reduced where $|g| = \text{canc}(v_i, u_i) < \frac{1}{2}\min\{|u_i|, |v_i|\}$. We have $t_i' = g\bar{t}_i'g^{-1}$ where $\bar{t}_i'$ is cyclically reduced, and thus any positive word $r(s_i', t_i')$ in $s_i', t_i'$ has the reduced spelling $gr(\bar{s}_i', \bar{t}_i')g^{-1}$.

Let $N = 8 \max\{|s_1'|, |s_2'|, |t_1'|, |t_2'|\}$ and set $s_i' = (s_i')^N$ and $t_i' = (t_i')^N$. Let $w$ be a piece between a word in $\{s_1, t_1\}^+$ and a word in $\{s_2, t_2\}^+$ and suppose that $|w| \geq N$. There exists a subword $w'$ of $w$ of length $\geq \frac{1}{2}N$ that is a subword of $(s_1')^*$ or of $(t_1')^*$. There exists an even shorter subword $w''$ of $w'$ of length $\geq \frac{1}{2}N$ that is also a subword of either $(s_2')^*$ or of $(t_2')^*$. Thus one of $(s_1')^*$, $(t_1')^*$ and one of $(s_2')^*$, $(t_2')^*$ have a common subword of length $\geq 2 \max\{|s_1'|, |s_2'|, |t_1'|, |t_2'|\}$ and by Lemma 3.5 they are virtually conjugate. This is a contradiction. Thus $|w| < N$. We clearly also have $|s_i'| = (|s_i'|)^N \geq N$, and $|t_i'| = (|t_i'|)^N \geq N$ for $i = 1, 2$, and thus we get $|w| < \min\{|s_i|, |t_i|\}$.

Lemma 3.7. Let $s, t$ be cyclically reduced elements that are not proper powers in $F(x, y)$ such that $s, t$ are not virtually conjugate. Let $r = s^{\alpha_1}t^{\beta_1} \cdots s^{\alpha_{2p}}t^{\beta_{2p}}$ for some $p$ and $w$ be a piece in $r$. If $\alpha_j, \beta_j$ are all different and greater than $2 \max\{|s|, |t|\} + 1$, then for every piece $w$ in $r$ we have $|w| \leq (\max\{\alpha_j\} + 2) |s| + (\max\{\beta_j\} + 2) |t|$.

Proof. Let $w$ be a piece in $r$ and consider two subwords of $r$: $\eta_0\eta_1 \cdots \eta_k\eta_{k+1}$ and $\mu_0\mu_1 \cdots \mu_{\ell}\mu_{\ell+1}$ where $\eta_i, \mu_j \in \{s, t\}$ such that $\eta_1 \cdots \eta_k$ and $\mu_1 \cdots \mu_{\ell}$ are maximal words in syllables $s, t$ entirely contained in $w$. We say that two syllables $\eta_i$ and $\mu_j$ are aligned if $\eta_i = \mu_j$ and they entirely overlap in $w$.

Suppose two syllables $\eta_i, \mu_j$ overlap in $w$ and $\eta_i = \mu_j = s$. If they are not aligned, say a proper suffix of $\eta_i$ equals a proper prefix of $\mu_j$ then $\eta_{i+1} = t$ and $\mu_{j-1} = t$ (since $s$ is not equal to any of its conjugates by Lemma 3.5). See Figure 3. Since $s, t$ are not conjugate by Lemma 3.5 we get that $j \geq \ell - 1$ and $i \leq 2$. Thus $|w| < 6 \max\{|s|, |t|\} < 6(|s| + |t|)$. From now on, assume that any two copies of $s$ or $t$ that overlap are aligned.

Suppose $\eta_i = s$ and $\mu_j = s$ are aligned where $1 \leq i \leq k$ and $1 \leq j \leq \ell$. If $\eta_{i+1} = s, \mu_{j+1} = t$, then $i + 2 \geq k + 1$. Indeed, consider three cases:

- $|s| = |t|$: Then necessarily $i = k$ and $j = \ell$.
- $|s| < |t|$: If $\eta_{i+2} = s$, then $i + 2 \geq k + 1$ because otherwise $\eta_{i+1}\eta_{i+2} = s^2$ was a subword of $t^*$ (more specifically a subword of $\mu_{j+1}\mu_{j+2} = t^*$, which is impossible since $s, t$ are not virtually conjugate).
(1/6) groups

If \( \eta_{k+2} = t \) then \( \eta_{k+2} \) and \( \mu_{j+1} \) are two overlapping not aligned copies of \( t \) so \( i + 2 \geq k + 1 \).

- \( |s| > |t| \): If \( \eta_{k+2} = s \), then \( i + 2 \geq k + 1 \) because otherwise \( \mu_{j+1} \mu_{j+2} = t^2 \) was a subword of \( s^* \). If \( \eta_{k+2} = t \), then \( \eta_{k+2} \) and \( \mu_{j+2} \) are two overlapping not aligned copies of \( t \) so \( i + 2 \geq k + 1 \) (\( \mu_{j+2} \) overlaps with \( \eta_{k+2} \) because otherwise \( \mu_{j+1} \mu_{j+2} = t^2 \) was a subword of \( s^* \)).

Similarly, if instead \( \eta_{k-1} = s, \mu_{j-1} = t \), then \( i - 2 \leq 0 \). Similarly we can switch \( s \) and \( t \). We are looking for an upper bound of \( |w| \). If \( w \) contains whole syllable \( s^{\alpha_n} \) as a subword for some \( n \) and \( \eta_{k+1} = \cdots = \eta_{k+\alpha_n} = s \) for \( 0 \leq i \leq k - \alpha_n \). In particular \( \eta_i = t \) and \( \eta_{k+\alpha_n+1} = t \). Since \( \alpha_n \geq 2|y| + 1 \) there must be a syllable \( \mu_j \) contained in the subword spelled by \( \eta_{i+1} \cdots \eta_{i+\alpha_n} \) because otherwise \( t \) and \( s \) were virtually conjugate. By the previous consideration \( \mu_j \) and \( \eta_i \) are aligned for some \( i + 1 \leq i' \leq i + \alpha_n \).

Since \( \alpha_1, \beta_1, \ldots, \alpha_2p, \beta_2p \) are all different, we can find \( i, j \) such that \( \eta_i = s \) and \( \mu_j \) are aligned and either \( \eta_{i+1} \), \( \mu_{j+1} \) or \( \eta_{i-1} \), \( \mu_{j-1} \) are different syllables (i.e. one of them is \( s \) and the other is \( t \)). By the consideration above, the subword \( s^{\alpha_n} \) is contained less than two syllables from the beginning of \( w \) or from the end of \( w \). The same happens with a syllable \( t^{\beta_m} \) contained in \( w \). We conclude that \( |w| \leq (\max\{\alpha_i\} + 2) |s| + (\max\{\beta_i\} + 2) |t| \).

**Proof of Proposition 2.4** First by Lemma 3.4 we can assume that for \( i = 1, \ldots, m \) the elements \( u_i, v_i \) are non-cancellable. Replace the pair \((u_1, v_1)\) and \((u_2, v_2)\) by \((s_1, t_1)\) and \((s_2, t_2)\) respectively as in Lemma 3.6 and continue replacing for each pair of indices \( i < j \leq m \). After \( \binom{m}{2} \) steps we have a collection \( \{(s_i, t_i)\}_{i=1}^{m} \) where for every piece \( w \) between a word in \((s_i, t_i)\) and a word in \((s_j, t_j)\) where \( i \neq j \) we have \( |w| < \max\{|s_i|, |t_i|\} \) and where for any \( i \) the elements \( s_i, t_i \) are not virtually conjugate.

Let \( r_i(s_i, t_i) = s_i^{\alpha_i^{t_i}}t_i^{\beta_i^{t_i}} \cdots s_{i+p}^{\alpha_{i+p}^{t_i}}t_{i+p}^{\beta_{i+p}^{t_i}} \) where \( \alpha_i^{t_i}, \beta_i^{t_i}, \ldots, \alpha_{2p}^{t_i}, \beta_{2p}^{t_i} \) are all distinct. Then for each piece \( w \) between \( r_i \) and \( r_j \) where \( i \neq j \) we clearly have \( |w| < \max\{|s_i|, |t_i|\} < \frac{1}{p}|r_i| \). Moreover, if \( \min\{\alpha_1^{t_1}, \beta_1^{t_1}, \ldots, \alpha_{2p}^{t_1}, \beta_{2p}^{t_1}\} > \frac{1}{2} \max\{\alpha_1^{t_1}, \beta_1^{t_1}, \ldots, \alpha_{2p}^{t_1}, \beta_{2p}^{t_1}\} + 1 \) then also for any piece \( w \) that lies in \( r_i \) in two different ways we also have \( |w| < \frac{1}{p}|r_i| \). Indeed, by Lemma 3.6 \( r_i \) has the reduced form \( gr_i(s_i, t_i)g^{-1} \) where \( g s_i g^{-1}, g t_i g^{-1} \) are reduced spellings of \( s_i, t_i \) respectively with \( s_i, t_i \) cyclically reduced. Let \( s_i^\prime, t_i^\prime \) be the words that are not proper powers such that \( s_i^\prime = (s_i^\prime)^{n_{s_i}} \) and \( t_i^\prime = (t_i^\prime)^{n_{t_i}} \), i.e. neither \( s_i^\prime \) or \( t_i^\prime \) is equal to any of its nontrivial cyclic permutations. Also, by Lemma 3.6 \( s_i^\prime \), \( t_i^\prime \) are not conjugate.

Suppose the piece \( w \) is disjoint from \( g, g^{-1} \). Then \( w \) is a word in \( s_i^\prime, t_i^\prime \) and by Lemma 3.7

\[
|w| \leq \left( \max_j\{n_{s_i} \alpha_j^{t_i} \} + 2 \right) |s_i^\prime| + \left( \max_j\{n_{t_i} \beta_j^{t_i} \} + 2 \right) |t_i^\prime|.
\]
It follows that

\[ |w| \leq \left( \max_j \{ \alpha_j^i, \beta_j^i \} + 2 \right) (|s_i| + |t_i|) < 2 \min_j \{ \alpha_j^i, \beta_j^i \} (|s_i| + |t_i|) = \frac{1}{p} \left( 2p \min_j \{ \alpha_j^i, \beta_j^i \} (|s_i| + |t_i|) \right) < \frac{1}{p} |R_i|. \]

Finally if \( w \) overlaps with the prefix \( g \) or suffix \( g^{-1} \) then \( w \) is a subword of \( gs_i^1 \ell_1^{\beta_1^i} \) or \( s_i^2 \ell_2^{\beta_2^i} g^{-1} \). If we choose \( \alpha_1^i, \beta_1^i, \ldots, \alpha_{2p}^i, \beta_{2p}^i \) sufficiently large so \( \min_j \{ \alpha_j^i, \beta_j^i \} > \frac{1}{2} \left( \max_j \{ \alpha_j^i, \beta_j^i \} + |g| + 2 \right) \) then we have

\[ |w| \leq |g| + \max_j \{ \alpha_j^i, \beta_j^i \} (|s_i| + |t_i|) < 2 \min_j \{ \alpha_j^i, \beta_j^i \} (|s_i| + |t_i|) < \frac{1}{p} |R_i|. \]

\[ \square \]

4. Proof of Theorem 1.2

Remark 4.1. The case \( n = 2 \) of Theorem 1.2 can be deduced from the work of Kar and Sageev who study uniform exponential growth of groups acting freely on CAT(0) square complexes \([KS10]\). They prove that for any two elements \( x, y \) there exists a pair of words of length at most 10 in \( x, y \) that freely generates a free semigroup, unless \( \langle x, y \rangle \) is virtually abelian. One can construct a small cancellation presentation by applying Proposition 3.1 to \( \mathcal{U} = \{ (u, v) \mid |u|, |v| \leq 10 \text{ and } u, v \text{ are not powers of the same element} \} \).

The resulting group cannot act properly on a CAT(0) square complex, since for each pair \( u, v \) there is a relator which is a positive word in \( u, v \).

Let \( R_n(x, y) \) be the union of the following pairs for all \( k < n \) and \( \ell < \ell' \leq K_3 \)

\[
\begin{align*}
(x^{n!}, y^{k!} x^{n!}), & \quad (x^{-n!} y^{-k!} x^{n!} y^{k!}), \\
(x^{n!}, y^{-k!} x^{n!}), & \quad (y^{-k!} x^{-n!} y^{k!} x^{n!}), \\
(x^{-n!} y^{k!} x^{n!}), & \quad (x^{-n!} y^{k!} x^{n!}), \\
(x^{-n!} y^{k!} x^{n!}), & \quad (x^{n!}, y^{k!} x^{n!}).
\end{align*}
\]

Let \( \mathcal{R}_1(x, y) = R_1(x, y) \cup R_1(y, x) \). Let

\[ \mathcal{R}_n(x, y) = R_n(x, y) \cup R_n(y, x) \cup \mathcal{R}_n^{-1}(y^N, x^{-n!} y^N x^{n!}) \cup \mathcal{R}_n^{-1}(x^N, y^{-n!} x^N y^{n!}) \]

where \( N = n!K_3! \).

Lemma 4.2 (The Main Lemma). Suppose \( x \) and \( y \) are hyperbolic isometries of an \( n \)-dimensional CAT(0) cube complex. Then one of the following holds:

- one of the pairs in \( R_n(x, y) \) freely generates a free semigroup, or
- either \( y^N \) and \( x^{-n!} y^N x^{n!} \), or \( x^N \) and \( y^{-n!} x^N y^{n!} \) stabilize a hyperplane, or
- the group \( \langle x^N, y^N \rangle \) is virtually abelian.
Proof. Without loss of generality we may assume that the action of \( \langle x, y \rangle \) is without hyperplane inversions, as we can always subdivide \( X \) to have this property of the action. Let \( \gamma_x, \gamma_y \) be axes of \( x, y \) respectively.

Suppose there exists a hyperplane \( h \in \text{sk}(x) - \text{sk}(y) \). By Lemma 2.3, \( y \) does not skewer \( x^nh \) for any \( n \in \mathbb{Z} \) unless one of the pairs in \( R_n(x, y) \) freely generates a free semigroup. Without loss of generality (by possibly renaming some \( x^nh \) as \( h \)) we can assume that \( \gamma_y \subset h \cap x^nh^* \).

If \( y^N h = h \) and \( y^N x^nh = x^nh \) then the subgroup \( \langle y^N, x^{-n}, y, x^n \rangle \) preserves \( h \). We are now assuming that this is not the case, i.e. at least one of \( h \) and \( x^nh \) is not preserved by \( y^N \).

Suppose that \( y^N \) does not stabilize \( h \). Let \( k \leq n \) be minimal such that \( y^kx^nh \) and \( x^nh \) are disjoint or equal and let \( \ell < \ell' \leq K_3 \) such that \( \{h, y^nh, y^{k\ell}h\} \) are pairwise disjoint (no two can be equal since \( y^N \) does not stabilize \( h \)). If \( y^kx^nh \neq x^nh \), then we have \( y^kx^nh \subset x^nh^* \), and thus also \( x^ny^kx^nh \subset h^* \). Since \( y^k h^* \subset h \) and \( y^{k\ell} h^* \subset h \) there is a ping-pong triple \( \{x^{-n}y^kx^nh^*, y^{k\ell}h^*, y^{k\ell}h^*\} \). See Figure 4. Now suppose \( y^kx^nh = x^nh \). We have \( y^{k\ell}h^* \subset x^nh^* \) because \( h^* \subset x^nh^* \), and thus \( \{x^nh^*, h^*, y^{k\ell}h^*\} \) is a ping-pong triple. Analogously, if \( y^N \) does not stabilize \( x^nh \) then one of \( \{x^ny^kh, y^{k\ell}x^nh, y^{k\ell}x^nh\} \) and \( \{h, x^nh, y^{k\ell}x^nh\} \) is a ping-pong triple for some \( k \leq n \) and \( \ell < \ell' \leq K_3 \).

Similarly, if there exists a hyperplane \( h \in \text{sk}(y) - \text{sk}(x) \), then one of the pairs in \( R_n(x, y) \) freely generates a free semigroup or \( \langle x^N, y^{-n}, x^Ny^N \rangle \) stabilizes a hyperplane. Otherwise \( \text{sk}(x) = \text{sk}(y) \), which we now assume is the case.

Suppose there exists a hyperplane \( h \) separating \( \gamma_x, \gamma_y \) that is not stabilized by either \( x^{K_3} \) or \( y^{K_3} \). Let \( k \leq n \) be minimal such that \( x^k h \subset h^* \) for appropriate choice of halfspace \( h \) of \( h \). Let \( \ell, \ell' \leq K_3 \) such that \( \{h, y^{k\ell}h, y^{k\ell'}h\} \) are pairwise disjoint. The triple \( \{x^kh, y^{k\ell}h, y^{k\ell'}h\} \) is a ping-pong triple.

We can now assume that every hyperplane separating any two axes of \( x \) and \( y \) is stabilized by \( x^{K_3} \) or \( y^{K_3} \). If a hyperplane \( h \) is stabilized by \( x^{K_3} \) then there are axes of \( x^{K_3} \) in both halfspaces \( h, h^* \). In particular, no hyperplane separates \( \text{Min}^0(x^{K_3}) \) and \( \text{Min}^0(y^{K_3}) \), hence \( \text{Hull}(\text{Min}^0(x^{K_3})) \cap
Hull(\(\text{Min}_0(x^{K_3})\)) \(\neq \emptyset\). Let \(p\) be a 0-cube in the intersection Hull(\(\text{Min}_0(x^{K_3})\)) \(\cap\) Hull(\(\text{Min}_0(y^{K_3})\)). By Lemma 2.4 \(p\) lies on axes of both \(x^N\) and \(y^N\). The complex Hull(\(\gamma\)) where \(\gamma\) is an axis of \(x^N\) through \(p\) is a minimal convex subcomplex containing the \(\langle x^N, y^N \rangle\)-orbit of \(p\), Hull(\(\gamma\)) is dual to \(sk(x) = sk(y)\), and \(\langle x^N, y^N \rangle\) acts properly on Hull(\(\gamma\)). By Lemma 2.4 Hull(\(\gamma\)) embeds in \(\mathbb{R}^k\) and by Lemma 2.5 the group \(\langle x^N, y^N \rangle\) is virtually abelian.

In the following proof \(|w|_*\) denotes the minimal number of syllables of the form \(x^{\pm*}, y^{\pm*}\) in a spelling of \(w\).

**Proof of Theorem 7.2.** Let \(G\) be a group given by the \(C'(1/p')\) presentation from Proposition 3.1 with \(U = R_n(x, y)\) where \(p' = \max\{p, 8 \cdot 3^n\}\). In particular, \(G\) is an infinite, torsion-free, non-elementary hyperbolic group.

Since \(p' \geq p\) the group \(G\) is \(C'(1/p)\). Suppose that \(G\) acts properly on an \(n\)-dimensional CAT(0) cube complex.

By definition of \(G\) none of the pairs in \(R_n(x, y)\) can freely generate a free semigroup since there is a relator in the presentation of \(G\) associated to each pair. Also the subgroup \(\langle x^N, y^N \rangle\) is not virtually abelian since the presentation of \(G\) is \(C'(1/6)\), so by Lemma 4.2 one of the pairs \(y^N, x^{-n!} y^N x^{n!}\) or \(x^N, y^{-n!} x^N y^{n!}\) stabilizes a hyperplane and thus these two elements act on an \((n - 1)\)-dimensional CAT(0) cube complex. Since \(R_{n-1}(y^N, x^{-n!} y^N x^{n!}) \subset R_n(x, y)\) and \(R_{n-1}(x^N, y^{-n!} x^N y^{n!}) \subset R_n(x, y)\) we can apply Lemma 4.2 again and we conclude that either one of \(\langle y^N, x^{-n!} y^N x^{n!} \rangle\) and \(\langle x^N, y^{-n!} x^N y^{n!} \rangle\) is virtually abelian, or an appropriate pair of elements stabilizes a hyperplane. We can keep applying Lemma 4.2 As long as the pair of elements \(u, v\) stabilizes a hyperplanes, then by Lemma 4.2 one of the pairs \(v^N, u^{-n!} v^N u^{n!}\) or \(u^N, v^{-n!} u^N v^{n!}\) generates a virtually abelian subgroup or stabilizes a hyperplane. By construction, \(u\) and \(v\) at each step are some conjugates of one of the the original generators \(u\) and \(v\), so \(|u^k|_* = |u|_*\) and \(|v^k|_* = |v|_*\) for any \(k > 0\). Also,

\[
|v^{-n!} u^N v^{n!}|_* \leq |v^{-n!}|_* + |u^N|_* + |v^{n!}|_* = |v|_* + |u|_* + |v|_* \leq 3 \max\{|u|_*, |v|_*\},
\]

and similarly \(|u^{-n!} v^N u^{n!}|_* \leq 3 \max\{|u|_*, |v|_*\}\). By applying Lemma 4.2 up to \(n\) times, we eventually get a pair of elements \(u_0, v_0\) that generates a virtually abelian subgroup and we have \(|u_0|_*, |v_0|_* \leq 3^n\). Since all elements of \(G\) have infinite order and \(G\) contains no abelian groups of rank 2, we conclude that \(\langle u_0, v_0 \rangle\) is (virtually) \(\mathbb{Z}\). In particular, \(u_0^k = v_0^{k'}\) for some \(k, k' \neq 0\) and we have \(|u_0^{-k} v_0^{-k'}|_* \leq 2 \cdot 3^n\). By Greendlinger’s Lemma [LS77] some subword \(w\) of \(u_0^{-k} v_0^{-k'}\) must be also a subword of some relator \(r\) with \(|w| \geq \frac{1}{2} |r|\). On one hand \(|w|_* \leq 2 \cdot 3^n\). On the other hand, the length of each syllable of the form \(x^{\pm*}\) or \(y^{\pm*}\) in \(r\) is at most \(1 + \frac{1}{p} |r| < \frac{2}{p} |r|\) because if \(x^k\) is a subword of \(r\) then \(x^{k-1}\) is a piece in \(r\) and the same for \(y\). Thus for any
LOWER BOUNDS ON CUBICAL DIMENSION OF $C'(1/6)$ GROUPS

subword $w'$ of $r$ of length at most $\frac{|r|}{2}$ we have $|w'| > \frac{2^r}{4}$. Since $\frac{2^r}{4} \geq 2 \cdot 3^n$ we get a contradiction.

□

REFERENCES

[CH13] Victor Chepoi and Mark F. Hagen. On embeddings of CAT(0) cube complexes into products of trees via colouring their hyperplanes. J. Combin. Theory Ser. B, 103(4):428–467, 2013.

[CS11] Pierre-Emmanuel Caprace and Michah Sageev. Rank rigidity for CAT(0) cube complexes. Geometric And Functional Analysis, 21:851–891, 2011. 10.1007/s00039-011-0126-7.

[Hag07] Frédéric Haglund. Isometries of CAT(0) cube complexes are semi-simple. pages 1–17, 2007. Preprint.

[JW17] Kasia Jankiewicz and Daniel T. Wise. Cubulating small cancellation free products. pages 1–11, 2017. Preprint.

[KS16] Aditi Kar and Michah Sageev. Uniform exponential growth for CAT(0) square complexes. arXiv:1607.00052v2, pages 1–15, 2016.

[LS77] Roger C. Lyndon and Paul E. Schupp. Combinatorial group theory. Springer-Verlag, Berlin, 1977. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 89.

[Pri83] Stephen J. Pride. Some finitely presented groups of cohomological dimension two with property (FA). J. Pure Appl. Algebra, 29(2):167–168, 1983.

[Sag95] Michah Sageev. Ends of group pairs and non-positively curved cube complexes. Proc. London Math. Soc. (3), 71(3):585–617, 1995.

[Sag14] Michah Sageev. CAT(0) cube complexes and groups. In Geometric group theory, volume 21 of IAS/Park City Math. Ser., pages 7–54. Amer. Math. Soc., Providence, RI, 2014.

[SW05] Michah Sageev and Daniel T. Wise. The Tits alternative for CAT(0) cubical complexes. Bull. London Math. Soc., 37(5):706–710, 2005.

[Wis04] Daniel T. Wise. Cubulating small cancellation groups. GAFA, Geom. Funct. Anal., 14(1):150–214, 2004.

[Woo16] Daniel Woodhouse. A generalized axis theorem for cube complexes. arXiv:1602.01952, pages 1–14, 2016.

Department of Mathematics, University of Chicago, Chicago, Illinois, 60637

E-mail address: kasia@math.uchicago.edu