LAMPS IN SLIM RECTANGULAR PLANAR SEMIMODULAR LATTICES

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Dedicated to George Grätzer on his forthcoming eighty-fifth birthday

Abstract. A planar (upper) semimodular lattice $L$ is slim if the five-element nondistributive modular lattice $M_3$ does not occur among its sublattices. (Planar lattices are finite by definition.) Slim rectangular lattices are particular slim planar semimodular lattices defined by G. Grätzer and Knapp in 2007. In 2009, they also proved that the congruence lattices of slim planar semimodular lattices with at least three elements are the same as those of slim rectangular lattices. In order to give an effective toolkit for studying these congruence lattices, we introduce the concept of lamps of slim rectangular lattices. This toolkit allows us to prove in a new and easy way that the congruence lattices of slim planar semimodular lattices satisfy all previously known properties. Also, we use lamps to prove that these congruence lattices satisfy two new properties called the two-pendant four-crown property and the forbidden marriage property.

1. Introduction

The theory of planar semimodular lattices has been an intensively studied part of lattice theory since Grätzer and Knapp’s pioneering [20]. By definition, planar lattices are finite. A slim planar semimodular lattice is a planar (upper) semimodular lattice $L$ such that one of the following three conditions holds:

(i) $M_3$, the five-element nondistributive modular lattice, is not a sublattice of $L$,
(ii) $M_3$ is not a cover-preserving sublattice of $L$,
(iii) $J(L)$, the set of nonzero join-irreducible elements of $L$, is the union of two chains;

see Grätzer and Knapp [20] and Czédli and Schmidt [11], or the book chapter Czédli and Grätzer [6] for the equivalence of these three conditions for planar semimodular lattices (but not for other lattices.) The importance of slim planar semimodular lattices is surveyed, for example, in Czédli and Kurusa [9], Czédli and Grätzer [6], Czédli and Schmidt [11], and Grätzer and Nation [22]. The study of their congruence lattices goes back to Grätzer and Knapp [21]. These congruence lattices are finite distributive lattices, of course, but it appears from Czédli [4] and Grätzer [15],

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that they have special further properties. In addition to develop an effective toolkit to derive these properties in a new and easy way, our target is also to present two new properties. It remains a problem whether the properties recalled or proved here and in Czédli and Grätzer [7] characterize the congruence lattices of slim planar semimodular lattices.

Next, let $L$ be a finite lattice. Since $\text{Con} L$ is distributive, it is determined by the poset (partially ordered set) $\langle J(\text{Con} L); \leq \rangle$ of its nonzero join-irreducible elements. There are three known ways to describe this poset.

First, one can use the join dependency relation defined on $J(L)$; see Lemma 2.36 of the monograph Freese, Ježek, and Nation [13], where this relation is attributed to Day [12].

Second, Grätzer [15] takes (and well describes) the prime-perspectivity relation on the set of prime intervals of $L$. His description becomes more powerful if $L$ happens to be a slim planar semimodular lattice: then Grätzer’s Swing Lemma applies, see [17] and see also Czédli, Grätzer, and Lakser [8] and Czédli and Makay [10].

Third, but only for a slim rectangular lattice $L$, Czédli [2] defined a relation on the set of trajectories of $L$ while Grätzer [17] defined an analogous relation on the set of prime intervals. Although it happened in a different way, Theorem 7.3(ii) of Czédli [2] indicates that the dual of the approach based on join dependency relation could also have been used to derive a more or less similar description of $\langle J(\text{Con} L); \leq \rangle$.

No matter if the underlying set $X$ consists of join-irreducible elements, prime intervals, or trajectories, the common feature of the above-mentioned approaches is that only a quasiorder (also called preorder), that is, a reflexive and transitive relation $\rho$ is defined on $X$, and we have to form the quotient set $X/(\rho \cap \rho^{-1})$ and equip it with the quotient relation $\rho/(\rho \cap \rho^{-1})$ to obtain $\langle J(\text{Con} L); \leq \rangle$ up to isomorphism. For slim rectangular lattices, we are going to improve the situation in this aspect.

**Target.** One of our goals is to prove that the congruence lattices of slim planar semimodular lattices satisfy two new properties, the two-pendant four-crown property and the forbidden marriage property. To accomplish another goal in connection with the just-mentioned one, we are going to define the so-called lamps of a slim rectangular lattice so that the set of lamps becomes a poset isomorphic to $J(L)$. The progress is that while lamps are visual and easy to recognize and understand, we do not have to form any quotient set. In fact, as opposed to the previously used join-irreducible elements, prime intervals, and trajectories, lamps are in bijective correspondence with join-irreducible congruences. Also, it is easy to see in the diagram of $L$ when a lamp is smaller than another lamp. The use of lamps is particularly effective if we use diagrams belonging to class $C_1$ of lattice diagrams introduced in Czédli [5]. In case of slim rectangular lattices, we believe that diagrams outside $C_1$ are not effective enough to deal with complicated problems. Except for distributivity, all previously known properties of the congruence lattices of slim planar semimodular lattices follow easily from our approach with lamps.

**Outline.** The rest of the paper is structured as follows. In Section 2 we recall the concept of slim rectangular lattices and that of their $C_1$-diagrams, introduce the concept of lamps of these lattices, and prove our (Main) Lemma 2.11. This lemma provides the main tool for “enlightening” and understanding the congruence lattices of slim planar semimodular lattices. In Section 3 further tools are given
and several consequences of (the Main) Lemma 2.11 are proved. In particular, the section proves that the congruence lattices of slim planar semimodular lattices satisfy all previously known properties. Section 4 defines the two-pendant four-crown property and the forbidden marriage property, and proves that congruence lattices of slim planar semimodular lattices satisfy these two properties.

2. From diagrams to lamps

Let \( L \) be a slim planar semimodular lattice; we always assume that a planar diagram of \( L \) is fixed. The left boundary chain and the right boundary chain of \( L \) are denoted by \( C_{\text{left}}(L) \) and \( C_{\text{right}}(L) \), respectively. Here and at several other concepts occurring later, we heavily rely on the fact that the diagram of \( L \) is fixed; indeed, \( C_{\text{left}}(L) \) and \( C_{\text{right}}(L) \) depend on the diagram, not only on \( L \).

Following Grätzer and Knapp [21], a slim planar semimodular lattice is called a slim rectangular lattice if \( |L| \geq 4 \), \( C_{\text{left}}(L) \) has exactly one doubly irreducible element, \( lc(L) \), \( C_{\text{right}}(L) \) has exactly one doubly irreducible element, \( rc(L) \), and these two doubly irreducible elements are complementary, that is, \( lc(L) \lor rc(L) = 1 \) and \( lc(L) \land rc(L) = 0 \). Here \( lc(L) \) and \( rc(L) \) are called the left corner (element) and the right corner (element) of the rectangular lattice \( L \). Note that \( |L| \geq 4 \) above can be replaced by \( |L| \geq 3 \). Note also that the definition of rectangularity does not depend on how the diagram is fixed since \( lc(L) \) and \( rc(L) \) are the only doubly irreducible elements of \( L \). Finally, let us emphasize that a slim rectangular lattice is planar and semimodular by definition, whereby the title of the paper is redundant; the purpose of this redundancy is to give more information about the content of the paper. The (principal) ideals \( ↓lc(L) \) and \( ↓rc(L) \) are chains and they are called the bottom left boundary chain and the bottom right boundary chain, respectively, while the filters \( ↑lc(L) \) and \( ↑rc(L) \) are also chains under the names top left boundary chain and top right boundary chain, respectively. The lower boundary and the upper boundary of \( L \) are \( ↓lc(L) \cup ↓rc(L) \) and \( ↑lc(L) \cup ↑rc(L) \), respectively. Note also that \( J(L) \cup \{0\} \) equals the lower boundary \( ↓lc(L) \cup ↓rc(L) \), but for the set \( Mi(L) \) of non-unit meet-irreducible elements, we only have that \( ↑lc(L) \cup ↑rc(L) \subseteq Mi(L) \cup \{1\} \).

For example, the lattices \( S_7^{(n)} \) for \( n \in N^+ := \{1, 2, 3, \ldots\} \), defined in Czédli [2], and presented here in Figure 1 for \( n \leq 4 \) are slim rectangular lattices.

![Figure 1. S_7^{(1)}, S_7^{(2)}, S_7^{(3)}, and S_7^{(4)}](image)

If \( p \) and \( q \) are elements of a lattice such that \( p \prec q \), then the prime interval, that is, two-element interval \([p, q]\) is an edge of the diagram; this edge is also denoted by \([p \prec q]\). Following Czédli [5], we need the following concepts.

**Definition 2.1 (Types of diagrams).** The slope of the line \( \langle x, x \rangle : x \in \mathbb{R} \) and that of the line \( \langle x, -x \rangle : x \in \mathbb{R} \) are called normal slopes. This allows us to speak
of lines, line segments, and edges of normal slopes. For example, an edge \([p \prec q]\) of a lattice diagram is of normal slope iff the angle that this edge makes with a horizontal line is \(\pi/4\) (45°) or \(3\pi/4\) (135°). If this angle is strictly between \(\pi/4\) and \(3\pi/4\), then the edge is precipitous. For examples, vertical edges are precipitous. We say that a diagram of a slim rectangular lattice \(L\) belongs to \(C_1\) or, in other words, it is a \(C_1\)-diagram if every edge \([p \prec q]\) such that \(p \in \text{Mi}(L) \setminus (\text{C}_{\text{left}}(L) \cup \text{C}_{\text{right}}(L))\) is precipitous and all the other edges are of normal slopes. A \(C_1\)-diagram of \(L\) belongs to \(C_2\) or, in other words, it is a \(C_2\)-diagram if any two edge on the lower boundary are of the same geometric length.

The diagrams in Figures 1, 3, 7, and 11, the diagrams in Figures 1, 3, 7, and 11 are the \(C_1\)-diagrams; this harmonizes with our belief that only \(C_1\)-diagrams can give satisfactory insight into the congruence lattices of slim rectangular lattices.

Conception 2.2. In the rest of the paper, all diagrams are assumed to be \(C_1\)-diagrams. Furthermore, \(L\) will always denote a slim rectangular lattice with a fixed \(C_1\) diagram.

Definition 2.3 (Lamps). Let \(L\) be a slim rectangular lattice with a fixed \(C_1\)-diagram.

(i) An edge \(n = [p \prec q]\) of \(L\) is a neon tube if \(p \in \text{Mi}(L)\). The elements \(p\) and \(q\) are the foot, denoted by \(\text{Foot}(n)\), and the top of this neon tube. Clearly, a neon tube is determined by its foot.

(ii) If \([p \prec q]\) is neon tube such that \(p\) belongs to the boundary of \(L\) (equivalently, if \(\downarrow p\) contains a doubly irreducible element), then the singleton set consisting of \([p \prec q]\) is a boundary lamp. If \(I\) is a boundary lamp with neon tube \([p \prec q]\), then \(p\) is called the foot of \(I\) and is denoted by \(\text{Foot}(I)\) while \(\text{Peak}(I) := q\) is the peak of \(I\).

(iii) Assume that \(q \in L\) is the top of a neon tube whose foot is not on the boundary \(\text{C}_{\text{left}}(L) \cup \text{C}_{\text{right}}(L)\) of \(L\), and let

\[\beta_q := \bigwedge\{p_i : [p_i \prec q]\text{ is a neon tube and }p_i \notin \text{C}_{\text{left}}(L) \cup \text{C}_{\text{right}}(L)\}.\]  \(2.1\)

Then the interval \(I := [\beta_q, q]\) is an internal lamp of \(L\). The prime intervals \([p_i \prec q]\) such that \(p_i \in \text{Mi}(L)\) but \(p_i \notin \text{C}_{\text{left}}(L) \cup \text{C}_{\text{right}}(L)\) are the neon tubes of this lamp. An internal lamp has either a unique neon tube, or it has more than one neon tubes. The element \(q\) is the peak of the lamp \(I\) and it is denoted by \(\text{Peak}(I)\) while \(\text{Foot}(I) := \beta_q\) is the foot of \(I\).

(iv) The lamps of \(L\) are its boundary lamps and its internal lamps. Clearly, for every lamp \(I\) of \(L\),

\[\text{Foot}(I) = \bigwedge\{\text{Foot}(n) : n\text{ is a neon tube of }I\}.\]  \(2.2\)

Since a slim rectangular lattice \(L\) has only two doubly irreducible elements, \(\text{lc}(L)\) and \(\text{rc}(L)\), the non-containment in \(2.1\) is equivalent to the condition that “\(\downarrow p_i\) does not contain a doubly irreducible element”. Therefore, the concept of lamps does not depend on the diagram of \(L\). Note that in a reasonable sense, the \(C_1\) diagram
of \( L \) is unique, and so is its \( C_2 \) diagram; see Czédli [3]. To help the reader in finding the lamps in our diagrams, let us agree to the following.

**Convention 2.4.** In our diagrams of slim rectangular lattices, the foots of lamps are exactly the black-filled elements. (Except possibly for Figure 10 which can but need not be the whole lattice in question.) The thick edges are always neon tubes (but there can be neon tubes that are not thick edges).

Note that in (the slim rectangular lattices of) Figures 1, 2, 4, and 11, the neon tubes are exactly the thick edges. In addition to the fact that neon tubes are easy to recognize as edges with bottom elements in \( \text{Mi}(L) \), there is another way to recognize them even more easily; the following remark follows from definitions.

**Remark 2.5.** Neon tubes in a \( C_1 \)-diagram of a slim rectangular lattice (and so in our figures) are exactly the precipitous edges and the edges on the upper boundary \( \uparrow \text{lc}(L) \cup \uparrow \text{rc}(L) \).

**Regions** of a slim rectangular lattice \( L \) are defined as closed planar polygons surrounded by some edges of (the fixed diagram) of \( L \); see Kelly and Rival [24] for an elaborate treatise of these regions, or see Czédli and Grätzer [6]. Note that every interval of a planar lattice determines a region; possibly of area 0 if the interval is a chain. The affine plane on which diagrams are drawn is often identified with \( \mathbb{R}^2 \) via the classical coordinatization. By the full geometric rectangle of a slim rectangular lattice \( L \) with a fixed diagram we mean the closed geometric rectangle whose boundary is the union of all edges belonging to \( C_{\text{left}}(L) \cup C_{\text{right}}(L) \). It is a rectangle indeed since we allow \( C_1 \) diagrams only. Smaller geometric rectangles are also relevant; this is why we have the second part of the following definition.

**Definition 2.6** (geometric shapes associated with lamps). Keeping Convention 2.2 in mind, let \( I = [p, q] = [\text{Foot}(I), \text{Peak}(I)] \) be a lamp of \( L \).

(i) The body of \( I \), denoted by \( \text{Body}(I) = \text{Body}([p, q]) \) is the region determined by \([p, q] \). Note that \( \text{Body}(I) \) is a line segment if \( I \) has only one neon tube, and (by Remark 2.5) it is a quadrangle of positive area having two precipitous upper edges and two lower edges of normal slopes otherwise.

(ii) Assume that \( I \) is an internal lamp, and define \( r \) as the meet of all lower covers of \( q \). Then the interval \([r, q] \) is a region; this region is denoted by \( \text{CircR}(I) = \text{CircR}([p, q]) \) and it is called the circumscribed rectangle of \( I \).

For example, if \( I \) is the only internal lamp of \( S_7^{(n)} \), then \( \text{CircR}(I) \) is the full geometric rectangle of \( S_7^{(n)} \) for all \( n \in \mathbb{N}^+ \) and \( \text{Body}(I) \) is the dark-grey area for \( n \in \{2, 3, 4\} \) in Figure 1. To see another example, if \( E \) is the lamp with two neon tubes labelled by \( e \) on the left of Figure 2, then \( \text{Body}(I) \) is the dark-grey area and the vertices of \( \text{CircR}(I) \) are \( x, y, z, \) and \( t \). Note that by the dual of Czédli [3, Proposition 3.13],

\[
\text{r in Definition 2.6(ii)} \text{ can also be defined as the meet of the leftmost lower cover and the rightmost lower cover of q.} \tag{2.3}
\]

**Definition 2.7** (line segments associated with lamps). Let \( I := [p, q] \) be a lamp of a slim rectangular lattice \( L \) with a fixed \( C_1 \)-diagram, and let \( F \) stand for the full geometric rectangle of \( L \). Let \( (p_x, p_y) \in \mathbb{R}^2 \) and \( (q_x, q_y) \in \mathbb{R}^2 \) be the geometric points corresponding to \( p = \text{Foot}(I) \) and \( q = \text{Peak}(I) \). As usual, \( \mathbb{R}^+ \) will stand for
the set of non-negative real numbers. We define the following four (geometric) line segments of normal slopes; see Figure 3 where these line segments are dashed edges of normal slopes.

\[
LRoof(I) := \{ (\xi, \eta) \in F : (\exists t \in \mathbb{R}^+) (\xi = qx - t \text{ and } \eta = qy - t) \},
\]

\[
RRoof(I) := \{ (\xi, \eta) \in F : (\exists t \in \mathbb{R}^+) (\xi = qx + t \text{ and } \eta = qy - t) \},
\]

\[
LFloor(I) := \{ (\xi, \eta) \in F : (\exists t \in \mathbb{R}^+) (\xi = px - t \text{ and } \eta = py - t) \},
\]

\[
RFloor(I) := \{ (\xi, \eta) \in F : (\exists t \in \mathbb{R}^+) (\xi = px + t \text{ and } \eta = py - t) \}.
\]

These line segments are called the left roof, the right roof, the left floor, and the right floor of \(I\), respectively. We defined the roof of \(I\) and the floor of \(I\) as follows:

\[
\text{Roof}(I) := LRoof(I) \cup RRoof(I), \quad \text{and} \quad \text{Floor}(I) := LFloor(I) \cup RFloor(I).
\]

If \(I\) is an internal lamp then roof of \(I\) and floor of \(I\) are \(\Lambda\)-shaped broken lines (that is, chevrons pointing upwards), but they are line segments if \(I\) is a boundary lamp.

In real life, neon tubes and lamps are for emitting light beams. Our lamps do this only downwards with normal slopes; the photons they emit can only go to the directions \((1, -1)\) and \((-1, -1)\). To be more precise, we have the following definition; Convention 2.2 is still in effect.

**Definition 2.8** (Enlightened sets). Let \(I := [\text{Foot}(I), q] = [\text{Foot}(I), \text{Peak}(I)]\) be a lamp of \(L\). A geometric point \((x, y)\) of the full geometric rectangle of \(L\) is **enlightened by \(I\) from the left** if the lamp has a neon tube \([p_i, q]\) such that the edge \([p_i, q]\) as a geometric line segment has a nonempty intersection with the half-line \(\{ (x-t, y+t) : 0 \leq t \in \mathbb{R} \}\). Similarly, a point \((x, y)\) of the full geometric rectangle of \(L\) is **enlightened by \(I\) from the right** if the half-line \(\{ (x+t, y+t) : 0 \leq t \in \mathbb{R} \}\) has a nonempty intersection with at least one of the neon tubes of \(I\). If \((x, y)\) is enlightened from the left or from the right, then we simply say that this point is **enlightened by the lamp \(I\)**. The set of points enlightened by the lamp \(I\), that of points enlightened by
Figure 3. Four line segments associated with $I$

$I$ from the right, and that from the left are denoted by

$$\begin{align*}
\text{Enl}(I) &= \text{Enl}([\text{Foot}(I), \text{Peak}(I)]), \\
\text{LeftEnl}(I) &= \text{LeftEnl}([\text{Foot}(I), \text{Peak}(I)]), \text{ and} \\
\text{RightEnl}(I) &= \text{RightEnl}([\text{Foot}(I), \text{Peak}(I)])
\end{align*}$$

(respectively. Let us emphasize that, say, $\text{LeftEnl}(I)$ consist of points enlightened from the right; the notation is explained by the fact that the geometric points of $\text{LeftEnl}(I)$ are on the left of (and down from) $I$. Finally, we also define

$$\text{Enl}^+(I) := \text{Enl}(I) \setminus \text{Floor}(I).$$

(2.5)

Note that $\text{Enl}(I)$ is $\text{LeftEnl}(I) \cup \text{RightEnl}(I)$. By Definition 2.7 and 2.8, $\text{Enl}(I)$ is (topologically) bordered by $\text{Roof}(I)$, $\text{Floor}(I)$, and appropriate line segments of $C_{\text{left}}(L)$ and $C_{\text{right}}(L)$, and so it is bordered by line segments of normal slopes.

Note also that the intersection $\text{LeftEnl}(I) \cap \text{RightEnl}(I)$ can be of positive (geometric) area in the plane and that both $\text{LeftEnl}(I)$ and $\text{RightEnl}(I)$ are of positive area if and only if $I$ is an internal lamp.

For example, each of $S_7^{(1)}$, $S_7^{(2)}$, $S_7^{(3)}$, and $S_7^{(4)}$ of Figure 1 has a unique internal lamp, namely, the interval spanned by the black-filled enlarged element in the middle and 1. The enlightened set of this lamp is the “$\mathbf{A}$-shaped” grey-filled hexagon (containing light-grey and dark-grey points). Also, $S_7^{(n)}$ has exactly two boundary lamps and the enlightened set of each of these two lamps is the whole geometric rectangle of $S_7^{(n)}$, for every $n \in \mathbb{N}^+$. If $E$ denotes the lamp with two $e$-labelled neon tubes on the right of Figure 2 then $\text{Enl}(I)$ is the “$\mathbf{A}$-shaped” grey-filled hexagon. In Figure 3 let $E$ and $G$ denote the lamps consisting of the $e$-labelled edges and the $g$-labelled edges, respectively. On the left of this figure, $\text{LeftEnl}(E)$...
and $\text{LeftEnl}(G)$ are the grey-filled trapezoids while the grey-filled trapezoids on the right are $\text{RightEnl}(E)$ and $\text{RightEnl}(G)$. Let us note that, for any slim rectangular lattice $L$,

two distinct internal lamps can never have the same peak, \hspace{1cm} (2.7)

Next, we introduce some relations on the set of lamps. Even though not all of these relations are applied in the subsequent sections, they and Lemma 2.11 will hopefully be useful in future investigation of the congruence lattices of slim planar semimodular lattices; for example, in Czédli and Grätzer [7].

**Definition 2.9 (Relations defined for lamps).** Let $L$ be a slim rectangular lattice with a fixed $C_1$-diagram. The set of lamps of $L$ will be denoted by $\text{Lamps}(L)$. On this set, we define eight irreflexive binary relations; seven in geometric ways based on Definitions 2.6–2.8 and one in a purely algebraic way; these relations will soon be shown to be the same. For $I, J \in \text{Lamps}(L)$,

(i) let $(I, J) \in \rho_{\text{Body}}$ mean that $I \neq J$, $\text{Body}(I) \subseteq \text{Enl}(J)$, and $I$ is an internal lamp;

(ii) let $(I, J) \in \rho_{\text{CircR}}$ mean that $I$ is an internal lamp, $\text{CircR}(I) \subseteq \text{Enl}(J)$, and $I \neq J$;

(iii) let $(I, J) \in \rho_{\text{alg}}$ mean that $\text{Peak}(I) \leq \text{Peak}(J)$ but $\text{Foot}(I) \nleq \text{Foot}(J)$;

(iv) let $(I, J) \in \rho_{\text{LRBody}}$ mean that $I$ is an internal lamp, $I \neq J$, and $\text{Body}(I) \subseteq \text{LeftEnl}(J)$ or $\text{Body}(I) \subseteq \text{RightEnl}(J)$;

(v) let $(I, J) \in \rho_{\text{LRCircR}}$ mean that $I$ is an internal lamp, $I \neq J$, and $\text{CircR}(I) \subseteq \text{LeftEnl}(J)$ or $\text{CircR}(I) \subseteq \text{RightEnl}(J)$;

(vi) let $(I, J) \in \rho_{\text{foot}}$ mean that $I \neq J$, $\text{Foot}(I) \in \text{Enl}(J)$, and $I$ is an internal lamp;

(vii) let $(I, J) \in \rho_{\text{infoot}}$ mean that $I \neq J$ and $\text{Foot}(I)$ is in the geometric (or, in other words, topological) interior of $\text{Enl}(J)$; and, finally,

(viii) let $(I, J) \in \rho_{\text{in+foot}}$ mean that $\text{Foot}(I) \in \text{Enl}^+(J)$.

**Remark 2.10.** In each of (i), (ii), . . . , (viii) of Definition 2.9, $I$ is an internal lamp and $I \neq J$; this follows easily from other stipulations even where this is not explicitly mentioned.
Now we are in the position to formulate the main achievement of this section. The congruence generated by a pair $(x, y)$ of elements will be denoted by $\text{con}(x, y)$. If $p = [x, y]$ is an interval, then we can write $\text{con}(p)$ instead of $\text{con}(x, y)$.

Lemma 2.11 (Main Lemma). If $L$ is a slim rectangular lattice with a fixed $C_1$-diagram, then the following four assertions hold.

(i) The relations $\rho_{\text{Body}}, \rho_{\text{CircR}}, \rho_{\text{Alg}}, \rho_{\text{LRBody}}, \rho_{\text{LRCircR}}, \rho_{\text{Foot}}, \rho_{\text{InfFoot}}$, and $\rho_{\text{InfFoot}}$ are all the same.

(ii) The reflexive transitive closure of $\rho_{\text{Alg}}$ is a partial ordering of the set $\text{Lamps}(L)$ of all lamps of $L$; we denote this reflexive transitive closure by $\leq$.

(iii) The poset $\langle \text{Lamps}(L); \leq \rangle$ is isomorphic to the poset $\langle \text{Con}(L); \leq \rangle$ of nonzero join-irreducible congruences of $L$ with respect to the ordering inherited from $\text{Con} L$, and the map $\varphi : \text{Lamps}(L) \to \text{Con}(L)$, defined by $[p, q] \mapsto \text{con}(p, q)$, is an order isomorphism.

(iv) If $I \prec J$ (that is, $J$ covers $I$) in $\text{Lamps}(L)$, then $\langle I, J \rangle \in \rho_{\text{Alg}}$.

The proof of this lemma heavily relies upon Czédi [2] and [5]. Before presenting this proof, we need some preparations. First, we need to recall the multifork structure theory from Czédi [2]. Minimal regions of a planar lattice are called cells. Every slim planar semimodular lattice, in particular, every slim rectangular lattice $L$ is a 4-cell lattice, that is, its cells are formed by four edges; see Grätzer and Knapp [20]. So the cells of $L$ are of the form $[a \land b, a \lor b]$ such that $a \parallel b$ and $[a \land b, a \lor b] = [a \land b, a, b, a \lor b]$. This cell is said to be a distributive cell if the principal ideal $\downarrow (a \lor b)$ is a distributive lattice. Let $n \in \mathbb{N}^+$. To obtain the multifork extension (of rank $n$) of $L$ at a distributive 4-cell $J$ means that we change $J$ to a copy of $F_7^{(n)}$ and keep adding new elements while going to the southeast and the southwest to preserve semimodularity. This is visualized by Figure 6, where $L$ is drawn on the left, $J$ is the grey-filled 4-cell, $n = 3$, and the slim rectangular lattice $L'$ we obtain by the multifork extension of $L$ at $J$ of rank 3 is $L'$, drawn on the right. The new elements, that is, the elements of $L' \setminus L$, are the pentagon-shaped ones. (For a more detailed definition of a multifork extension, which we do not need here since Figures 5 and 6 are sufficient for our purposes, the reader can resort to Czédi [2]. Note that [2] uses the term “$n$-fold” rather than “of rank $n$”.)

By a grid we mean the direct product of two finite nonsingleton chains or a $C_1$-diagram of such a direct product. Note that a slim rectangular lattice is distributive if and only if it is a grid. Note also that, in a slim rectangular lattice with a fixed $C_1$-diagram, a 4-cell $I = [p, q]$ is distributive if and only if every edge in the ideal $\downarrow q$ is of a normal slope:

$$\text{(2.8)}$$

the “only if” part of (2.8) is Corollary 6.5 of Czédi [5] while the “if” part follows easily by using that if all edges of $\downarrow q$ are of normal slopes then $\downarrow q$ is a (sublattice of a) grid.

Lemma 2.12 (Theorem 3.7 of Czédi [2]). Each slim rectangular lattice is obtained from a grid by a sequence of multifork extensions at distributive 4-cells, and every lattice obtained in this way is a slim rectangular lattice.

Figure 6 illustrates how we can obtain the lattice $L$ defined by Figure 2 from the initial grid $L_0$ in six steps. For $i = 1, 2, \ldots, 6$, $L_i$ is obtained from $L_{i-1}$ by a multifork extension at the grey-filled 4-cell of $L_{i-1}$. We still need one important
Figure 5. \( L' \) is the multifork extension of rank 3 at the 4-cell \( J \)

concept, which we recall from Czédli [2]; it was originally introduced in Czédli and Schmidt [11].

Figure 6. Illustrating Lemma 2.12: a sequence of multifork extensions

**Definition 2.13 (Trajectories).** Let \( L \) be a slim rectangular lattice with a fixed \( C_1 \)-diagram. The set of its edges, that is, the set of its prime intervals is denoted by Edges(\( L \)). We say that \( p, q \in \text{Edges}(L) \) are **consecutive edges** if they are opposite sides of a 4-cell. Maximal sequences of consecutive edges are called **trajectories**. In other words, the blocks of the least equivalence relation on Edges(\( L \)) that includes the consecutiveness relation are called trajectories. If a trajectory has an edge on the upper boundary (equivalently, if it has an edge that is the unique neon tube of a boundary lamp), then this trajectory a **straight trajectory**. Otherwise, it is a **hat trajectory**. Each trajectory has a unique edge that is a neon tube; this edge is called the **top edge** of the trajectory. The top edge of a trajectory \( u \) will be denoted by TopE(\( u \)) while \( \text{Trajs}(L) \) will stand for the set of trajectories of \( L \).

For example, if \( L \) is the lattice on the left of Figure 5 then it has eight trajectories. One of the eight trajectories is a hat trajectory and consists of the \( h \)-labelled
edges. There are seven straight trajectories and one of these seven consists of the $s$-labelled edges. Note that all neon tubes of $L$ are drawn by thick lines. On the right of the same figure, only the neon tubes of the new lamp are thick. Also, $L'$ has exactly four hat trajectories and seven straight trajectories. One of the hat trajectories consists of the $h$-labelled edges while the $s$-labelled edges form a straight trajectory.

Proof of Lemma 2.11. Let $L$ be a slim rectangular lattice with a fixed $C_1$-diagram. To prove part (i), we need some preparations. We know from Lemma 2.12, visualized by Figure 6, that $L$ is obtained by a sequence $L_0, L_1, \ldots, L_k = L$ such that $L_0$ is a grid and $L_i$ is a multifork extension of $L_{i-1}$ at a distributive 4-cell $H_i$ of $L_{i-1}$ for $i \in \{1, \ldots, k\}$. (2.9)

In Lamps($L_0$), there are only boundary lamps. It is clear by definitions that each internal lamp arises from the replacement of the distributive 4-cell $H_i$ of $L_{i-1}$, grey-filled in Figures 5 and 6, by a copy of $S^{(n_i)}_4$, for some $i \in \{1, \ldots, k\}$ and $n_i \in \mathbb{N}^+$. Shortly saying, every internal lamp comes to existence from a multifork extension. Furthermore, if a lamp $K$ comes by a multifork extension at a 4-cell $H_i$, then CircR($I$) is the geometric region determined by $H_i$; the second half of (2.10) follows from (2.8) and Convention 2.2. For example, on the right of Figure 5, the new lamp what the multifork extension has just brought is the lamp with pentagon-shaped black-filled foot and the thick edges are its neon tubes. Similarly, the new lamp is the one with thick neon tube(s) in each of $L_1, \ldots, L_6 = L$ in Figure 6. Keeping Convention 2.2 in mind, it is clear that if $[p, q]$ is the new lamp that the multifork extension of $L_{i-1}$ brings into $L_i$, then $[\text{lcf}(L_i) \wedge p, p]$ and $[\text{rcf}(L_i) \wedge p, p]$ are chains with all of their edges being of normal slopes. Observe that no geometric line segment that consists of some edges of $L_i$ disappears at further multifork extension steps, (2.11) but there can appear more vertices on it. This fact, Convention 2.2, (2.8), (2.10), and the fact that $L_{i-1}$ is a sublattice of its multifork extension $L_i$ for all $i \in \{1, \ldots, k\}$ yield that if $I = [p, q] \in \text{Lamps}(L)$, then the lowest point of $\text{LRoof}(I)$ and that of $\text{LFloor}(I)$ are $\text{lcf}(L) \wedge q$ and $\text{lcf}(L) \wedge p$, respectively, and the intervals $[\text{lc}(L) \wedge q, q]$ and $[\text{lc}(L) \wedge p, p]$ are chains. That is, $\text{LRoof}(I)$ and $\text{LFloor}(I)$ correspond to intervals that are chains. The edges of these chains are of the same normal slope. Similar statements hold for “right” instead of “left”.

For later reference, note another easy consequence of (2.9) and (2.11), or (2.12):

With reference to (2.9), assume that $i < j \leq k$ and a lamp $I$ is present in $L_i$. Then Enl($I$) is the same in $L_i$ as in $L_j$ and $L$. (2.13)

For an internal lamp $I$, the leftmost neon tube and the (not necessarily distinct) rightmost neon tube are the right upper edge and the left upper edge of a 4-cell,
respectively. Combining this fact, (2.10), and Observation 6.8(iii) of Czédli [5], we conclude that

\[
\text{the two upper (geometric) sides of the circumscribed rectangle } \text{CircR}(I) \text{ of an internal lamp } I \text{ are edges (that is, prime intervals) of } L.
\]

(2.14)

Since \(\text{LeftEnl}(J) \subseteq \text{Enl}(J)\) and \(\text{RightEnl}(J) \subseteq \text{Enl}(J)\) for every \(J \in \text{Lamps}(L)\),

\[
\rho_{\text{LRCircR}} \subseteq \rho_{\text{CircR}} \quad \text{and} \quad \rho_{\text{LRBody}} \subseteq \rho_{\text{Body}}.
\]

(2.15)

Next, we are going to prove that \(\rho_{\text{CircR}} \subseteq \rho_{\text{Body}}\) and \(\rho_{\text{LRCircR}} \subseteq \rho_{\text{LRBody}}\).

(2.16)

To do so, assume that \(\langle I, J \rangle \in \rho_{\text{CircR}}\) and \(\langle I', J' \rangle \in \rho_{\text{LRCircR}}\). This means that \(\text{CircR}(I) \subseteq \text{Enl}(J)\) and \(\text{CircR}(I') \subseteq \text{LeftEnl}(J')\) or \(\text{CircR}(I') \subseteq \text{RightEnl}(J')\), respectively. The definition of \(S_7^n\) and multifork extensions, (2.10), and (2.11) yield that \(\text{Body}(I) \subseteq \text{CircR}(I)\) and \(\text{Body}(I') \subseteq \text{CircR}(I')\). Combining these inclusions with the earlier ones, we have that \(\text{Body}(I) \subseteq \text{Enl}(J)\) and \(\text{Body}(I') \subseteq \text{LeftEnl}(J')\) or \(\text{Body}(I') \subseteq \text{RightEnl}(J')\). Hence, \(\langle I, J \rangle \in \rho_{\text{Body}}\) and \(\langle I', J' \rangle \in \rho_{\text{LRBody}}\), proving the validity of (2.16).

We claim that

\[
\rho_{\text{Body}} \subseteq \rho_{\text{alg}}
\]

(2.17)

To show this, assume that \(\langle I, J \rangle \in \rho_{\text{Body}}\), that is, \(\text{Body}(I) \subseteq \text{Enl}(J)\), \(I \neq J\), and \(I\) is an internal lamp. We know from (2.6) that \(\text{Enl}(J)\) is bordered by geometric line segments of normal slopes. Hence, Corollary 6.1 of Czédli [5] and \(\text{Body}(I) \subseteq \text{Enl}(J)\) yield that \(\text{Peak}(I) \leq \text{Peak}(J)\). Figure 1 and Convention 2.2 imply that \(\text{Foot}(I)\) is not on \(\text{Floor}(J)\), the lower geometric boundary of \(\text{Enl}(J)\). It is trivial by \(\text{Body}(I) \subseteq \text{Enl}(J)\) that \(\text{Foot}(I)\) is not (geometrically and strictly) below \(\text{Floor}(J)\). Hence, the just mentioned Corollary 6.1 of [5] shows that \(\text{Foot}(I) \not\leq \text{Foot}(J)\). We have obtained that \(\langle I, J \rangle \in \rho_{\text{alg}}\). Thus, \(\rho_{\text{Body}} \subseteq \rho_{\text{alg}}\), proving (2.17).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure7.png}
\caption{Proving that \(\rho_{\text{foot}} \subseteq \rho_{\text{LRCircR}}\)}
\end{figure}

By construction, see Figure 1, \(\text{CircR}(I)\) contains \(\text{Foot}(I)\), as a geometric point, in its topological interior for every internal lamp \(I \in \text{Lamps}(L)\). This yields the first inclusion in (2.18) below clear:

\[
\rho_{\text{CircR}} \subseteq \rho_{\text{infoot}} \subseteq \rho_{\text{in+foot}} \subseteq \rho_{\text{foot}} \subseteq \rho_{\text{LRCircR}}.
\]

(2.18)
The second and the third inclusions above are trivial. In order to show the fourth inclusion, $\rho_{\text{foot}} \subseteq \rho_{\text{LRCircR}}$, assume that $\langle I, J \rangle \in \rho_{\text{foot}}$. We know that $I \neq J$, $I$ is an internal lamp, and Foot$(I) \in \text{Enl}(J)$. Since Enl$(J) = \text{LeftEnl}(J) \cup \text{RightEnl}(J)$, left-right symmetry allows us to assume that the geometric point Foot$(I)$ belongs to LeftEnl$(J)$. But, as it is clear from Figure 1 and (2.10), Foot$(I)$ is in the (topological) interior of CircR$(I)$. Hence,

there is an open set $U \subseteq \mathbb{R}^2$ such that $U \subseteq \text{CircR}(I) \cap \text{LeftEnl}(J)$. (2.19)

We claim that CircR$(I) \subseteq \text{LeftEnl}(J)$; suppose the contrary. We know from (2.6), (2.10), (2.11), (2.13), and Remark 2.5 that LeftEnl$(J)$ is surrounded by edges of normal slopes except its top right precipitous side. Also, we obtain from (2.8), (2.10), and (2.14) that CircR$(I)$ is a rectangle whose sides are of normal slopes, and the upper two sides are edges (that is, prime intervals). These facts, (2.19), and CircR$(I) \not\subseteq \text{LeftEnl}(J)$ yield that a side of CircR$(I)$ crosses a side of LeftEnl$(J)$. But edges do not cross in a planar diagram, whence no side of LeftEnl$(J)$ crosses an upper edge of CircR$(I)$. So if we visualize LeftEnl$(J)$ by light-grey color and CircR$(I)$ is dark-grey, then none of the six little dark-grey rectangles on the left of Figure 7 can be CircR$(I)$. Note that, in both part of this Figure, $J$ is formed by the three thick neon tubes. Since the above-mentioned six dark-grey rectangles represent generally, the only possibility is that the top vertex Peak$(I)$ of CircR$(I)$ is positioned like the top of the dark-grey rectangle on the right of Figure 7. (There can be lattice elements not indicated in the diagram, and we took into consideration that the top sides of CircR$(J)$ are also edges and cannot be crossed and that (2.7) holds). But then Corollary 6.1 of Czédli [5] yields that Peak$(J) \leq$ Peak$(I)$. With reference to (2.6) and (2.10), let $i$ and $j$ denote the subscripts such that $I$ and $J$ came to existence in $L_i$ and $L_j$, respectively. Since none of the lattices $S^{(n)}_j$, $n \in \mathbb{N}^+$, is distributive, $j > i$. So the elements of $L_i$ are old when $L_j$ is constructed as a mult fork extension of $L_{j-1}$. It is clear by the concept of mult fork extensions, that of LeftEnl$(J)$, (2.8), and (2.10), that no old element, that is, no $x \in L_{j-1}$, belongs to the topological interior of Enl$(J)$. But Foot$(I) \in L_i \subseteq L_{j-1}$ does belong to this interior, which is a contradiction showing that CircR$(I) \not\subseteq \text{LeftEnl}(J)$. Thus, $\langle I, J \rangle \in \rho_{\text{LRCircR}}$, and we obtain that $\rho_{\text{foot}} \subseteq \rho_{\text{LRCircR}}$. This completes the proof of (2.18).

Next, we are going show that

$\rho_{\text{alg}} \subseteq \rho_{\text{foot}}$. (2.20)

To verify this, assume that $\langle I, J \rangle \in \rho_{\text{alg}}$. We know that Peak$(I) \leq$ Peak$(J)$ but Foot$(I) \not\leq$ Foot$(J)$. Observe that Foot$(I) \subseteq$ Peak$(I) \leq$ Peak$(J)$. If $J$ is an internal lamp, then (2.6) gives that Peak$(J)$ and Foot$(J)$ are the vertices of the A-shaped Roof$(J)$ and Floor$(J)$, respectively, and Roof$(J)$ and Floor$(J)$ are of normal slopes. If $J$ is a boundary lamp, then Roof$(J)$ and Floor$(J)$ are line segments of the same normal slope with (geometrically) highest points Peak$(J)$ and Foot$(J)$, respectively. Hence, no matter if $J$ is internal or not, it follows from Corollary 6.1 of Czédli [5] and Foot$(I) \leq$ Peak$(J)$ that Foot$(I)$ is geometrically below or on Roof$(J)$. On the other hand, Corollary 6.1 of [5] and Foot$(I) \not\leq$ Foot$(J)$ imply that Foot$(I)$ is neither geometrically below, nor on Floor$(J)$. So Foot$(I)$ is above Floor$(J)$. Therefore, Foot$(I)$ is geometrically between Floor$(J)$ and Roof$(J)$. Thus, (2.6) gives that Foot$(I) \in \text{Enl}(J)$. This shows that $\langle I, J \rangle \in \rho_{\text{foot}}$, and we have shown the validity of (2.20).
Figure 8. An overview of what has already been proved

The directed graph in Figure 8 visualizes what we have already shown. Each (directed) edge $\rho_1 \rightarrow \rho_2$ of the graph means that $\rho_1 \subseteq \rho_2$ has been formulated in the displayed equation given by the label of the edge. The directed graph is strongly connected, implying part (i).

Next, we turn our attention to parts (ii) and (iii). For a trajectory $u$ of $L$, $\text{TopE}(u)$ is a neon tube, so it belongs to a unique lamp; we denote this lamp by $\text{Lmp}(u)$. Lamps are special intervals. Hence, in agreement with how the notation $\text{con}()$ was introduced right before Lemma 2.11, $\text{con}(\text{Lmp}(u))$ will denote the congruence generated by the interval $\text{Lmp}(u)$. We claim that, for any trajectory $u$ of $L$,

$$\text{con}(\text{TopE}(u)) = \text{con}(\text{Lmp}(u)).$$  \hfill (2.21)

To show (2.21), observe that this assertion is trivial if $\text{TopE}(u)$ is the only neon tube of $\text{Lmp}(u)$ since then the same prime interval generates the congruence on both sides of (2.21).

Hence, we can assume that $\text{Lmp}(u)$ has more than one neon tubes; clearly, then $\text{Lmp}(u)$ is an internal lamp. Let $p := \text{Foot}(\text{Lmp}(u))$, $q := \text{Peak}(\text{Lmp}(u))$, and let $[p_1, q] = \text{TopE}(u)$, $[p_2, q]$, $\ldots$, $[p_m, q]$ be a list of all neon tubes of $\text{Lmp}(u)$. With this notation, (2.21) asserts that $\text{con}(p_1, q) = \text{con}(p, q)$. Since $p \leq p_1 \leq q$ and congruence blocks of a finite lattice are intervals, the inequality $\text{con}(p_1, q) \leq \text{con}(p, q)$ is clear. It follows in a straightforward way by inspecting the lattice $S^m_2$, see Figure 1, or it follows trivially by the Swing Lemma, see Grätzer [17] or Czédli, Grätzer and Lakser [8], that $\text{con}(p_1, q) = \text{con}(p_2, q) = \ldots = \text{con}(p_m, q)$. Hence, $(p_1, q) \in \text{con}(p_1, q)$ for $i \in \{1, \ldots, m\}$, and we obtain that

$$\text{(p, q)} \overset{2.21}{\rightarrow} (p_1 \land \cdots \land p_m, q \land \cdots \land q) \in \text{con}(p_1, q).$$  \hfill (2.22)

This yields the converse inequality $\text{con}(p_1, q) \geq \text{con}(p, q)$ and proves (2.21).

As usual, the smallest element and largest element of an interval $I$ will often be denoted by $0_I$ and $1_I$, respectively. Following Czédli [2, Definition 4.3], we define two relations on $\text{Trajs}(L)$. For trajectories $u, v \in \text{Trajs}(L)$, we let

$$\langle u, v \rangle \in \sigma \overset{\text{def}}{\leftrightarrow} \left\{ \begin{array}{ll} 1_{\text{TopE}(u)} \leq 1_{\text{TopE}(v)}, & 0_{\text{TopE}(u)} \nleq 0_{\text{TopE}(v)}, \\
\text{and } u \text{ is a hat trajectory.} & \end{array} \right.$$  \hfill (2.23)

The second relation is $\tau$, the reflexive transitive closure of $\sigma$. We also need a third relation on $\text{Trajs}(L)$; it is $\Theta := \tau \cap \tau^{-1}$. The $\Theta$-block of a trajectory $u \in \text{Trajs}(L)$ will be denoted by $u/\Theta$. Since $\tau$ is a quasordering, that is, a reflexive and transitive relation, it is well known that the quotient set $\text{Trajs}(L)/\Theta$ turns into a poset...
\[ \langle \text{Trajs}(L)/\Theta, \tau/\Theta \rangle \text{ by defining} \]
\[
\langle u/\Theta, v/\Theta \rangle \in \tau/\Theta \iff \langle u, v \rangle \in \tau 
\]
for \( u, v \in \text{Trajs}(L) \); see, for example, (4.1) in Czédi [2]. We claim that, for any \( u, v \in \text{Trajs}(L) \),
\[
\langle u/\Theta, v/\Theta \rangle \in \tau/\Theta \iff \text{Lmp}(u) \leq \text{Lmp}(v). 
\]
(2.25)
As the first step towards (2.25), we show that, for any \( u, v \in \text{Lamps}(L) \),
\[
\langle u/\Theta, v/\Theta \rangle \in \tau/\Theta \iff \text{Lmp}(u) = \text{Lmp}(v). 
\]
(2.26)
The \( \leq \) part of (2.26) is quite easy. Assume that \( \text{Lmp}(u) = \text{Lmp}(v) \). Then it is clear by Figure 1 and (2.23) that \( u/v \) are in \( \tau \). Hence, both \( \langle u, v \rangle \) and \( \langle v, u \rangle \) are in \( \tau \), whence \( u/\Theta = v/\Theta \).

To show the converse implication, assume that \( u/\Theta = v/\Theta \) and, to exclude a trivial case, \( u \neq v \). This assumption gives that \( \langle u, v \rangle \in \tau \), and thus there is a \( k \in \mathbb{N}^+ \) and there are elements \( w_0 = u, w_1, \ldots, w_k = v \) such that \( \langle w_{i-1}, w_i \rangle \in \sigma \) for all \( i \in \{1, \ldots, k\} \).

By (2.23), \( u = u_0 \) is a hat trajectory and \( 1_{\text{TopE}(u)} = 1_{\text{TopE}(w_0)} \leq 1_{\text{TopE}(w_1)} \leq \cdots \leq 1_{\text{TopE}(w_k)} = 1_{\text{TopE}(v)} \). That is, \( 1_{\text{TopE}(u)} \leq 1_{\text{TopE}(v)} \). Since \( \langle v, u \rangle \) is also in \( \tau \), we also have that \( v \) is a hat trajectory and \( 1_{\text{TopE}(v)} \leq 1_{\text{TopE}(u)} \). Hence, \( 1_{\text{TopE}(v)} = 1_{\text{TopE}(u)} \) and so \( \text{Peak}(\text{Lmp}(u)) = \text{Peak}(\text{Lmp}(v)) \). Thus, \( \text{Lmp}(u) = \text{Lmp}(v) \) by (2.27), proving (2.26).

In the next step towards (2.25), the just-proved (2.26) allows us to assume that \( u/\Theta \neq v/\Theta \), which is equivalent to \( \text{Lmp}(u) \neq \text{Lmp}(v) \). First, we show that
\[
\text{if } \langle u, v \rangle \in \sigma, \text{ then } \text{Lmp}(u) \leq \text{Lmp}(v). 
\]
(2.27)
Let \( u = u_1, \ldots, u_k \) and \( v = v_1, \ldots, v_l \) be the neon tubes of \( \text{Lmp}(u) \) and \( \text{Lmp}(v) \), respectively. With reference to (2.9) and (2.10), let \( i \) and \( j \) denote the subscripts such that \( \text{Lmp}(u) \) and \( \text{Lmp}(v) \) came to existence in \( L_i \) and \( L_j \), respectively. We obtain from \( \langle u, v \rangle \in \sigma \) that \( \text{Peak}(\text{Lmp}(u)) = 1_{\text{TopE}(u)} \leq 1_{\text{TopE}(v)} = \text{Peak}(\text{Lmp}(v)) \).

By (2.23), \( u \) is a hat trajectory, whence \( \text{Lmp}(u) \) is an internal lamp and so \( i \geq 1 \). Since none of the lattices \( S_n^{(n)} \), \( n \in \mathbb{N}^+ \), is distributive, \( i > j \). Since the sequence (2.9) is increasing, we obtain that \( 0_{\text{TopE}(v)} \in L_{i-1} \). It is clear from (2.10) and the description of multifork extensions (see Figures 5 and 6 and see also Figure 1) that
\[
\text{if, according to (2.9), a lamp } K \text{ comes to existence in } L_l, x \in L_{i-1}, \text{ and } \text{Foot}(K) \leq x, \text{ then } \text{Peak}(K) \leq x. 
\]
(2.28)
Suppose for a contradiction that \( \text{Foot}(\text{Lmp}(u)) \leq 0_{\text{TopE}(v)} \). Applying (2.28) with \( \ell = i \) and \( K = I \), the already-mentioned \( 0_{\text{TopE}(v)} \in L_{i-1} \) leads to \( \text{Peak}(\text{Lmp}(u)) \leq 0_{\text{TopE}(v)} \). But then \( 0_{\text{TopE}(u)} < 1_{\text{TopE}(u)} = \text{Peak}(\text{Lmp}(u)) \leq 0_{\text{TopE}(v)} \), which is a contradiction since \( 0_{\text{TopE}(u)} \not\leq 0_{\text{TopE}(v)} \) by \( \langle u, v \rangle \in \sigma \). Hence, \( \text{Foot}(\text{Lmp}(u)) \not\leq 0_{\text{TopE}(v)} \). Combining this with \( \text{Foot}(\text{Lmp}(v)) \leq 0_{\text{TopE}(v)} \), see (2.2), we obtain that \( \text{Foot}(\text{Lmp}(u)) \not= \text{Foot}(\text{Lmp}(v)) \). We have already seen that \( \text{Peak}(\text{Lmp}(u)) \leq \text{Peak}(\text{Lmp}(v)) \), whereby \( (\text{Lmp}(u), \text{Lmp}(v)) \in \rho_{\text{alg}} \). This yields that \( \text{Lmp}(u) \leq \text{Lmp}(v) \) since “\( \leq \)” is the reflexive transitive closure of \( \rho_{\text{alg}} \). Thus, we have shown (2.27).

Next, we assert that, for any \( u, v \in \text{Trajs}(L) \),
\[
\text{if } (\text{Lmp}(u), \text{Lmp}(v)) \in \rho_{\text{alg}}, \text{ then } \langle u, v \rangle \in \tau. 
\]
(2.29)
To show this, let \( v_1 = v, v_2, \ldots, v_k \) be the neon tubes of \( Lmp(v) \). Assume that the pair \( \langle Lmp(u), Lmp(v) \rangle \) belongs to \( \rho_{\text{alg}} \). Then

\[
1_{\text{TopE}(u)} = \text{Peak}(Lmp(u)) \leq \text{Peak}(Lmp(v)) = 1_{\text{TopE}(v_j)} \quad \text{for all } j \in \{1, \ldots, k\},
\]

and we know from Remark 2.10 that \( Lmp(u) \) is an internal lamp. Hence, \( u \) is a hat trajectory. On the other hand, \( 0_{\text{TopE}(u)} \not\leq \text{Foot}(Lmp(v)) \) since otherwise \( \text{Foot}(Lmp(u)) \leq 0_{\text{TopE}(u)} \leq \text{Foot}(Lmp(v)) \) would contradict \( \langle Lmp(u), Lmp(v) \rangle \in \rho_{\text{alg}} \). If we had that \( 0_{\text{TopE}(u)} \leq 0_{\text{TopE}(v_j)} \) for all \( j \in \{1, \ldots, k\} \), then we would obtain that

\[
0_{\text{TopE}(u)} \leq \bigwedge_{j \in \{1, \ldots, k\}} 0_{\text{TopE}(v_j)} \text{Foot}(Lmp(v)),
\]

contradicting \( 0_{\text{TopE}(u)} \not\leq \text{Foot}(Lmp(v)) \). Hence, there exists a \( j \in \{1, \ldots, k\} \) such that \( 0_{\text{TopE}(u)} \not\leq 0_{\text{TopE}(v_j)} \). Thus, using (2.30), (2.23), and the fact that \( u \) is a hat trajectory, we obtain that \( \langle u, v_j \rangle \in \sigma \). Hence, \( \langle u, v_j \rangle \in \tau \). Also, \( Lmp(v_j) = Lmp(v) \) and (2.26) yield that \( \langle v_j, v \rangle \in \tau \). By transitivity, \( \langle u, v \rangle \in \tau \), proving (2.29).

By (2.24), what (2.25) asserts is equivalent to the statement that \( \langle u, v \rangle \in \tau \iff Lmp(u) \leq Lmp(v) \). Since \( \tau \) on \( \text{Trajs}(L) \) and “\( \leq \)” on \( \text{Lamps}(L) \) are the reflexive transitive closures of \( \sigma \) and \( \rho_{\text{alg}} \), respectively, the desired (2.25) follows from (2.27) and (2.29).

Next, we are going to call a certain map a quasi-coloring: for more about this concept, the reader can but need not look into Czédli [1] and [2], where the concept was introduced. Following Definition 4.3(iv) of [2], we define a map \( \psi \) from the set \( \text{Edges}(L) \) of edges of \( L \) to \( \text{Trajs}(L) \) by the rule \( p \in \xi(p) \) for every \( p \in \text{Edges}(L) \). That is, \( \xi \) maps an edge to the unique trajectory containing it. By Theorem 4.4 of [2], \( \xi \) is a quasi-coloring. It follows from Lemma 4.1 of [2] that the posets \( \langle J(L); \leq \rangle \) and \( \langle \text{Trajs}(L)/\Theta; \tau/\Theta \rangle \) are isomorphic. (2.31)

By (2.25) and its particular case, (2.26), the structures

\[
\langle \text{Trajs}(L)/\Theta, \tau/\Theta \rangle \text{ and } \langle \text{Lamps}(L); \leq \rangle \text{ are also isomorphic.} \tag{2.32}
\]

We have already mentioned around (2.24) and we also know from (2.31) that \( \langle \text{Trajs}(L)/\Theta, \tau/\Theta \rangle \) is a poset. Thus, \( \langle \text{Lamps}(L); \leq \rangle \) is also a poset by (2.32), proving part (iii) of Lemma 2.11.

Combining (2.31) and (2.32), it follows that \( \langle J(L); \leq \rangle \cong \langle \text{Lamps}(L); \leq \rangle \), which is the first half of part (iii) of Lemma 2.11. It is straightforward to extract from (2.6)–(2.8) of Czédli [1], or from the proof of (2.31) or that of Theorem 7.3(i) of [2] that

\[
\text{the map } \psi_1 : \langle \text{Trajs}(L)/\Theta; \tau/\Theta \rangle \to \langle J(\text{Con } L); \leq \rangle \text{ defined by } u/\Theta \mapsto \text{con}(\text{TopE}(u)) \text{ is a poset isomorphism.} \tag{2.33}
\]

Observe that every lamp \( I \) is of the form \( Lmp(u) \) for some \( u \in \text{Trajs}(L) \); indeed, we can choose \( u \) as \( \xi(p) \) for some (in fact, any) neon tube \( p \) of \( I \). By (2.25) and (2.26),

\[
\text{the map } \psi_2 : \langle \text{Lamps}(L); \leq \rangle \to \langle \text{Trajs}(L)/\Theta; \tau/\Theta \rangle, \text{ defined by } Lmp(u) \mapsto u/\Theta, \text{ is also a poset isomorphism.} \tag{2.34}
\]

Combining (2.21), (2.33), and (2.34), we obtain that \( \varphi = \psi_2 \cdot \psi_1 \). Hence, \( \varphi \) is also a poset isomorphism, proving part (iii) of Lemma 2.11.
Finally, if a partial ordering \( \leq \) is the reflexive transitive closure of a relation \( \rho \), then the covering relation \( \prec \) with respect to \( \leq \) is obviously a subset of \( \rho \). This yields part (iv) and completes the proof of Lemma 2.11.

3. Easy consequences of Lemma 2.11 and some other easy statements

Some statements of this section are explicitly devoted to congruence lattices of slim planar semimodular lattices; they are called corollaries since they are derived from Lemma 2.11. The rest of the statements of the section deal with lamps.

Convention 2.4 raises the (easy) question whether lamps are determined by their feet and how. For an element \( u \in L \setminus \{1\} \), let \( u^+ \) denote the join of all covers of \( u \), that is,

\[
u^+ := \bigvee \{ y \in L : u \prec y \}, \quad \text{provided } u \neq 1. \tag{3.1}\]

Note that each \( u \in L \setminus \{1\} \) has either a single cover, or it has exactly two covers; see Grätzer and Knapp [20, Lemma 8]. Let \( x \in L \) and define the element \( \text{lifted}(x) \) by induction on the number of elements of the principal filter \( \uparrow x \) as follows.

\[
\text{lifted}(x) = \begin{cases} 
1, & \text{if } x = 1; \\
x^+, & \text{if } \exists y \in \text{Mi}(L) \text{ such that } y \prec x^+; \\
\text{lifted}(x^+), & \text{otherwise}.
\end{cases} \tag{3.2}
\]

**Lemma 3.1.** If \( L \) is a slim rectangular lattice, then \( \text{Peak}(I) = \text{lifted}(\text{Foot}(I)) \) holds for every \( I \in \text{Lamps}(L) \).

**Proof.** The proof is trivial by Figure 1 and (2.10). \( \Box \)

**Lemma 3.2** (Maximal lamps are boundary lamps). If \( L \) is a slim rectangular lattice, then the maximal elements of \( \text{Lamps}(L) \) are exactly the boundary lamps.

**Proof.** We know from Grätzer and Knapp [21] that each slim planar semimodular lattice \( L \) with at least three elements has a so-called congruence-preserving rectangular extension \( L' \); see also Czédi [5] and Grätzer and Schmidt [23] for stronger versions of this result. Among other properties of this \( L' \), we have that \( \text{Con } L' \cong \text{Con } L \).

Hence, to simplify the notation,

we can assume in the proof that \( L \) is a slim rectangular lattice with a fixed \( C_1 \)-diagram. \( \tag{3.3} \)

Note that the same assumption will be made in many other proofs when we know that \( |L| \geq 3 \). Armed with (3.3), Lemma 3.2 follows trivially from (2.10) and Lemma 2.11. \( \Box \)

The set of maximal elements of a poset \( P \) will be denoted by \( \text{Max}(P) \).

**Corollary 3.3** (P2 property from Grätzer [19]). If \( L \) is a slim planar semimodular lattice with at least three elements, then \( \text{Con } L \) has at least two coatoms or, equivalently, \( J(\text{Con } L) \) has at least two maximal elements.

**Proof.** Assume (3.3). The well-known representation theorem of finite distributive lattices (see, for example, Grätzer [14, Theorem 107]) easily implies that \( \text{Con } L \) has at least two coatoms if an only if \( J(\text{Con } L) \) has at least two maximal elements. Hence, Lemma 3.2 applies. \( \Box \)
The covering relation in a poset $P$ is denoted by $\prec$, or if confusion threatens, by $\preccurlyeq$. For sets $A_1$, $A_2$, and $A_3$, the notation $A_3 = A_1 \cup A_2$ will stand for the conjunction of $A_1 \cap A_2 = \emptyset$ and $A_3 = A_1 \cup A_2$.

**Corollary 3.4** (Bipartite maximal elements property). If $L$ is a slim planar semi-modular lattice with at least three elements and $D := \text{Con} L$, then there exist nonempty sets $\text{LeftMax}(J(D))$ and $\text{RightMax}(J(D))$ such that

$$\text{Max}(J(D)) = \text{LeftMax}(J(D)) \cup \text{RightMax}(J(D))$$

and for each $x \in J(D)$ and $y, z \in \text{Max}(J(D))$, if $x \prec_{J(D)} y$, $x \prec_{J(D)} z$, and $y \neq z$, then neither $\{y, z\} \subseteq \text{LeftMax}(J(D))$, nor $\{y, z\} \subseteq \text{RightMax}(J(D))$. Furthermore, when $J(D) = J(\text{Con} L)$ is represented in the form $\text{Lamps}(L)$ according to Lemma 2.11[iii], then the members of $\text{LeftMax}(J(D))$ correspond to the boundary lamps on the top left boundary chain of $L$ while those of $\text{RightMax}(J(D))$ to the boundary lamps on the top right boundary chain.

**Proof.** Armed with (3.3) again, we know from Lemma 3.2 that the maximal lamps are on the upper boundary. Let $\text{LeftMax}(\text{Lamps}(L))$ and $\text{RightMax}(\text{Lamps}(L))$ denote the set of boundary lamps on the top left boundary chain $\uparrow \text{lc}(L)$ and those on the top right boundary chain $\uparrow \text{rc}(L)$, respectively. Since $L$ is rectangular, none of these two sets is empty. So these two sets form a partition of $\text{max Lamps}(L)$.

Let us say that, for $I', J' \in \text{Lamps}(L)$,

$$\text{Enl}(I') \text{ and } \text{Enl}(J') \text{ are sufficiently disjoint} \text{ if for every line segment } S \text{ of positive length in the plane},$$

(3.4)

if $S \subseteq \text{Enl}(I') \cap \text{Enl}(J')$, then $S$ is of a normal slope.

Clearly, if $\text{Enl}(I')$ and $\text{Enl}(J')$ are sufficiently disjoint, then no nonempty open set of $\mathbb{R}^2$ is a subset of $\text{Enl}(I') \cap \text{Enl}(J')$.

Let $I, J \in \text{LeftMax}(\text{Lamps}(L))$ such that $I \neq J$. There are two easy ways to see that $\text{Enl}(I)$ and $\text{Enl}(J)$ are sufficiently disjoint: either we apply (2.12), or we use (2.13) with $(i, j) = (0, k)$. Suppose, for a contradiction, that $K \prec_{\text{Lamps}(L)} I$ and $K \prec_{\text{Lamps}(L)} J$. Then, by parts [i] and [iv] of Lemma 2.11 $(K, I) \in p_{\text{Body}}$. Similarly, $(K, J) \in p_{\text{Body}}$, and so we have that $\text{Body}(K) \subseteq \text{Enl}(I) \cap \text{Enl}(J)$. Since $K$ is an internal lamp, it contains a precipitous neon tube $S$, which contradicts the sufficient disjointness of $\text{Enl}(I)$ and $\text{Enl}(J)$. By Lemma 2.11 and left-right symmetry, we conclude Corollary 3.4.

**Corollary 3.5** (Dioecious maximal elements property). If $L$ is a slim planar semi-modular lattice, $D := \text{Con} L$, $x \in J(D)$, $y \in \text{Max}(J(D))$, and $x \prec_{J(D)} y$, then there exists an element $z \in J(D)$ such that $z \neq y$ and $x \prec_{J(D)} z$.

The adjective “dioecious” above is explained by the idea of interpreting $x \prec y$ as “$x$ is a child of $y$”.

**Proof of Corollary 3.5** If $|L| < 3$, then $|J(L)| \leq 1$ an the statement is trivial. Hence, we can assume that $|L| \geq 3$ and that $L$ is rectangular; see (3.3). Assume that $I \in \text{Lamps}(L)$, $J \in \text{Max}(\text{Lamps}(L))$, and $I \prec J$ in $\text{Lamps}(L)$. We know from Lemma 3.2 that $I$ is an internal lamp. For the sake of contradiction, suppose that $J$ is the only cover of $I$ in $\text{Lamps}(L)$. By left-right symmetry and Lemma 3.2 we can assume that (the only neon tube of) $J$ is on the top left boundary chain of $L$. With reference to (2.9), the enlightened sets of the lamps of $L_0$, which are boundary lamps, divide the full geometric rectangle of $L_0$ into pairwise sufficiently
disjoint (topologically closed) rectangles $T_1, \ldots, T_m$. That is, $T_1, \ldots, T_m$ are the squares (that is, the 4-cells) of the initial grid $L_0$. By \eqref{2.13}, the enlightened sets of the boundary lamps of $L$ (rather than $L_0$) divide the full geometric rectangle of $L$ into the same rectangles, and the same holds for all $L_i$, $i \in \{0, 1, \ldots, k\}$. We know from \eqref{2.10} that, for some $i \in \{1, \ldots, k\}$, each of the four sides of the rectangle $\text{CircR}(I)$ is an edge in $L_i$. Since no two edges of $L_i$ cross each other by planarity (see also Kelly and Rival \cite[Lemma 1.2]{21}), it follows from \eqref{2.11} that the sides (in fact, edges) of $\text{CircR}(I)$ in $L_i$ do not cross the sides of $T_1, \ldots, T_m$. Hence, still in $L_i$, $\text{CircR}(I)$ is fully included in one of the $T_1, \ldots, T_m$. This also holds in $L$ since $\text{CircR}(I)$ is the same in $L$ as in $L_i$ by \eqref{2.10}. Hence, there is lamp $K$ on the top right boundary chain of $L$ such that $\text{CircR}(I) \subseteq \text{Enl}(K)$. Hence, $(I, K) \in \rho_{\text{CircR}}$, whence parts [i] and [iv] of Lemma 2.11 give that $I < K$ in Lamps($L$). By finiteness, we can pick a lamp $K'$ such that $I < K' < K$ in Lamps($L$). We have assumed that $J$ is the only cover of $I$, whereby $K' = J$ and so $J = K' \leq K$. The inequality here cannot be strict since both $J$ and $K$ belong to $\text{Max}(\text{Lamps}(L))$. Hence, $J = K$, but this is a contradiction since $J$ is on the top left boundary chain of $L$ while $K$ is on the upper right boundary chain. Since $J(D) \cong \text{Lamps}(L)$ by Lemma 2.11 we have proved Corollary 3.5. \hfill \square

**Corollary 3.6** (Two-cover Theorem from Grätzer \cite{21}). If $L$ is a slim planar semimodular lattice and $D := \text{Con} L$, then for every $x \in J(D)$, the set $\{y \in J(D) : x \prec_{J(D)} y\}$ of covers of $x$ with respect to $\prec_{J(D)}$ consists of at most two elements.

**Proof.** Since the case $|L| < 3$ is trivial, we assume $|L| \geq 3$. In virtue of Lemma 2.11, we can work in Lamps($L$) rather than in $J(D)$. For lamps $I$ and $J$ of our slim rectangular lattice $L$, we define

$$I \prec_{\text{left}} J \overset{\text{def}}{=} \text{CircR}(I) \subseteq \text{LeftEnl}(L)$$

and, similarly,

$$I \prec_{\text{right}} J \overset{\text{def}}{=} \text{CircR}(I) \subseteq \text{RightEnl}(L).$$

By parts [i] and [iv] of Lemma 2.11 for any $I, J \in \text{Lamps}(L)$,

$$I \prec_{\text{left}} J \text{ and } I \prec_{\text{right}} J \text{ can simultaneously hold. Based on \eqref{2.10}, a straightforward induction on } i \text{ occurring in \eqref{2.9}} \text{ yields that} \quad \forall I \in \text{Lamps}(L), \text{ there is at most one } J \in \text{Lamps}(L) \text{ such that } I \prec_{\text{left}} J.$$  

Note that $I \prec_{\text{left}} J$ and $I \prec_{\text{right}} J$ can simultaneously hold. Based on \eqref{2.10}, a straightforward induction on $i$ occurring in \eqref{2.9} yields that

$$I \prec_{\text{right}} K \text{ holds for at most one } K \in \text{Lamps}(L).$$

Finally, \eqref{3.6} and \eqref{3.7} imply Corollary 3.6. \hfill \square

**Definition 3.7.** For non-horizontal parallel geometric lines $T_1$ and $T_2$, we say that $T_1$ is to the left of $T_2$ if $T_1$ is of the form $\{(a_i, 0) + t \cdot \langle v_x, v_y \rangle : t \in \mathbb{R}\}$ for $i \in \{1, 2\}$ such that $a_1 < a_2$. Here the vector $\langle v_x, v_y \rangle$ is the common direction of $T_1$ and $T_2$ while $(a_i, 0)$ is the intersection point of $T_i$ and the $x$-axis. We denote by $T_1 \setminus T_2$ that $T_1$ is left to $T_2$. For parallel line segments $S_1$ and $S_2$ of positive lengths, we say that $S_1$ is to the left of $S_2$, in notation, $S_1 \setminus S_2$, if the line containing $S_1$ is to the left of the line containing $S_2$. Let us emphasize that if $S_1$ or $S_2$ is of zero length, then $S_1 \setminus S_2$ fails! Next, let $L$ be a slim rectangular lattice, and let $J_0$ and
be distinct lamps of $L$. With reference to Definition 2.7, we say that $J_0$ and $J_1$ are left separatory lamps if there is a (unique) $i \in \{0, 1\}$ such that
\[
\text{LRoof}(J_i) \lambda \text{LRoof}(J_{1-i}) \lambda \text{LFloor}(J_i) \lambda \text{LFloor}(J_{1-i}).
\] (3.8)

Replacing the left roofs and left floors in (3.8) by right roofs and right floors, respectively, we obtain the concept of right separatory lamps. We say that $J_0$ and $J_1$ are separatory lamps if $J_0$ and $J_1$ are left separatory or right separatory. Finally, if the line segments $\text{LFloor}(J_0)$ and $\text{LFloor}(J_1)$ lie on the same line, then $J_0$ and $J_1$ are left floor-aligned. If $\text{RFloor}(J_0)$ and $\text{RFloor}(J_1)$ are segments of the same line, then $J_0$ and $J_1$ are right floor-aligned. They are floor-aligned if they are left floor-aligned or right floor-aligned.

Lemma 3.8. If $I$ and $J$ are distinct lamps of a slim rectangular lattice, then these two lamps are neither separatory, nor floor-aligned.

Proof. Suppose the contrary and let $J_0 \triangleq I$ and $J_1 \triangleq J$. Using (2.11), (2.14), and that Foot($J_j$) is in the (topological) interior of CircR($J_j$) for $j \in \{0, 1\}$, we can find an $i \in \{0, 1\}$ such that LRoof($J_i$) or LFloor($J_i$) crosses the upper right edge of CircR($J_{1-i}$) or left-right symmetrically. This contradicts planarity and completes the proof. Alternatively, we can use an induction on $i$ occurring in (2.9). □

Since this section is intended to be a toolkit, we formulate the following lemma here; not only its proof but also its complete formulation are left to the next section.

Lemma 3.9. If $L$ is a slim rectangular lattice, then $\text{Lamps}(L) = \text{Lamps}(L_k)$ satisfies (4.3).

Figure 9. The two-pendant four-crown

4. Two new properties of congruence lattices of slim planar semimodular lattices

For posets $X$ and $Y$, we say that $X$ is a cover-preserving subposet of $Y$ if $X \subseteq Y$ and, for all $u, v \in X$, $u \leq_X v \iff u \leq_Y v$ and $u \prec_X v \iff u \prec_Y v$. The poset $R$ given in Figure 9 will be called the two-pendant four-crown; it is a four-crown decorated with two “pendants”, $z$ and $w$. This section is devoted to the following two properties.

Definition 4.1 (Two-pendant four-crown property). We say that a finite distributive lattice $D$ satisfies the two-pendant four-crown property if $R$ given in Figure 9 is not a cover-preserving subposet of $J(D)$ such that the maximal elements of $R$ are maximal in $J(D)$.

Definition 4.2 (Forbidden marriage property). We say that a finite distributive lattice $D$ satisfies the forbidden marriage property if for every $x, y \in J(D)$ and $z \in \text{Max}(J(D))$, if $x \neq y$, $x \prec_{J(D)} z$, and $y \prec_{J(D)} z$, then there is no $p \in J(D)$ such that $p \prec_{J(D)} x$ and $p \prec_{J(D)} y$. 
Now we are in the position to formulate the main theorem of the paper.

**Theorem 4.3 (Main Theorem).** If $L$ is a slim planar semimodular lattice, then

1. $\text{Con} L$ satisfies the the forbidden marriage property, and
2. $\text{Con} L$ satisfies the two-pendant four-crown property;

see Definitions 4.1 and 4.2.

**Remark 4.4.** The smallest distributive lattice $D$ that fails to satisfy the forbidden marriage property is the eight-element $D_8$ given in Czéldi [4]. Now the result of [4], stating that $D_8$ cannot be represented as the congruence lattice of a slim planar semimodular lattice, becomes an immediate consequence of Theorem 4.3. In fact, part (i) of Theorem 4.3 is a generalization of [4].

**Remark 4.5.** By the well-known representation theorem of finite distributive lattices, see, for example, Grätzer [14, Theorem 107], there is a unique finite distributive lattice $D_R$ such that $J(D_R) \cong R$. By Theorem 4.3(ii), there is no slim planar semimodular lattice $L$ such that $\text{Con} L \cong D_R$. Since $D_R$ satisfies the properties mentioned in Corollaries 3.3–3.6 so all the previously known properties, and even the forbidden marriage property, it follows that part (ii) of Theorem 4.3 is really a new result.

**Remark 4.6.** A straightforward calculation shows that $D_R$ mentioned in Remark 4.5 consists of 56 elements. Furthermore, we are going to prove that every finite distributive lattice with less than 56 elements satisfies the two-pendant four-crown property.

![Figure 10. Illustration for the proof of (4.3)](image)

**Proof of Theorem 4.3.** As usual, the case $|L| < 3$ is trivial. Hence, (3.3) is assumed. Also, we use the notation $D := \text{Con} L$. By Lemma 2.11 we can also assume that
\[ J(D) = \text{Lamps}(L). \] For \( J_0, J_1 \in \text{Lamps}(L) \), we say that

the lamps \( J_0 \) and \( J_1 \) are independent if there is a

(unique) \( i \in \{0, 1\} \) such that \( \text{Peak}(J_i) \leq \text{Foot}(J_{1-i}) \). \hfill (4.1)

First, we deal with part \((i)\), that is, with the forbidden marriage property. In

Figure 10, which is either \( L \) or only a part (in fact, an interval) of \( L \), there are four

internal lamps, \( S, U, V, \) and \( W \); their feet are \( s, u, v, \) and \( w \), respectively. In

this figure, for example, \( U \) and \( V \) are independent but \( S \) and \( W \) are not. Actually,

\( \{S, W\} \) is the only two-element subset of \( \{S, U, V, W\} \) whose two members are not

independent. It follows from (2.6) and Corollary 6.1 of Czédli \[5\] that, using the

terminology of (3.4),

\[ \text{Foot}(\rho) \geq \text{Peak}(\rho) \geq \text{Foot}(\rho) \text{ if } \rho \text{ is abundant.} \hfill (4.2) \]

If \( J_0 \) and \( J_1 \) are independent lamps, then

\[ \text{Enl}(J_0) \text{ and } \text{Enl}(J_1) \text{ are sufficiently disjoint.} \hfill (4.3) \]

Now assume that \( Z \) is a boundary lamp. By left-right symmetry, we can assume that

it is on the top left boundary chain; see Figure 10 where \( Z = \text{Foot}(Z) \). The

intersection of \( \text{Enl}(Z) = \text{RightEnl}(Z) \) with the right boundary chain \( C_{\text{right}}(L) \) will

be denoted by \( E(Z) \); it is the (topologically closed) line segment with endpoints \( a \) and \( b \) in the Figure. Also, we define the following set of geometric points

\[ F(Z) := \{ q \in E(Z) : (\exists U \in \text{Lamps}(L)) ((U, Z) \in \rho_{\text{CircR}} \text{ and } q \in \text{Enl}(U)) \}. \]

For example, \( F(Z) \) in Figure 10 is the line segment with endpoints \( b \) and \( c \). If

\( F(Z) = \emptyset \) or \( F(Z) \) is a line segment with its upper endpoint being the same as that

of \( E(Z) \), then we say that there is no gap in \( F(Z) \). For example, there is no gap

in \( F(Z) \) in Figure 10. With reference to (2.9) and (2.10), we claim that

\[ \text{for } i = 0, 1, \ldots, k, \text{ the lower covers of } Z \]

\[ \text{in } \text{Lamps}(L_i) \text{ are pairwise independent} \hfill (4.3) \]

and, still in \( L_i \), there is no gap in \( F(Z) \):

note that our boundary lamp \( Z \) is in \( L_0 \) and so \( 4.3 \) makes sense. Note also that

\[ (4.3) \text{ implies Lemma } 3.9 \text{ because } L = L_k. \]

We prove \((4.3)\) by induction on \( i \). The case \( i = 0 \) is trivial since \( Z \) has no lower

cover in \( \text{Lamps}(L_0) \). In Figure 10, \( Z \) has three lower covers: \( U, V \) and \( W \). (Since

\( S < W < Z \), \( S \) is not a lower cover.) Assume that this figure is the relevant

discriminant (that is, \( \text{Enl}(Z) \)) of \( L_i \) for some \( i \in \{0, 1, \ldots, k - 1\} \). Assume also that in

\( L_{i+1} \), \( Z \) obtains a new lower cover, \( T \). We know from \((ii)\) and \((iv)\) of Lemma 2.11

that \( \langle T, Z \rangle \in \rho_{\text{CircR}} \). Due to (2.9) and (2.10), \text{CircR}(T) \) is a distributive 4-cell

in the figure. For every lamp \( G \in \text{Lamps}(L_i) \) such that \( G < Z \) (in particular, if

\( \langle G, Z \rangle \in \rho_{\text{CircR}} \) ), \text{CircR}(T) cannot be a 4-cell of \( \text{Enl}(G) \) since otherwise \( \langle T, G \rangle \in \rho_{\text{CircR}} \) would lead to \( T < G < Z \), contradicting \( T \prec Z \). Also, there can be no \( G \in \text{Lamps}(L_i) \) such that \( \langle G, Z \rangle \in \rho_{\text{CircR}} \) and \( G \) is the “south-east” of \( T \),
because otherwise \( \text{CircR}(T) \) would not be distributive. It follows that \( T \) is one

of the dark-grey cells in the figure, whereby even in \( L_{i+1} \), the lower covers of \( Z \) remain pairwise independent and there is no gap in \( F(Z) \). Since the figure clearly

represents generality, we are done with the induction step from \( i \) to \( i + 1 \). This

proves \((4.3)\).

Finally, Lemmas 2.11 and 3.2 translate part \((i)\) of Theorem 4.3 to the following

statement on \( \text{Lamps}(L) \):

\[ \text{if } Z \text{ is a boundary lamp, } X \prec Z, Y \prec Z, \text{ and } X \neq Y, \text{ then} \]

\[ \text{there exists no } P \in \text{Lamps}(L) \text{ such that } P \prec X \text{ and } P \prec Y. \hfill (4.4) \]
To see this, assume the premise. For the sake of contradiction, suppose that there does exists a P described in (4.4). By \( [\text{iv}] \) and \( \{\text{iv}\} \) of Lemma 2.11, \((P, X) \in \rho_{\text{CircR}} \) and \((P, Y) \in \rho_{\text{CircR}} \). Hence, \( \text{CircR}(P) \subseteq \text{Enl}(X) \cap \text{Enl}(Y) \), whereby \( \text{Enl}(X) \) and \( \text{Enl}(Y) \) are not sufficiently disjoint. On the other hand, we obtain from \( L = L_k \) and \( \{\text{iv}\} \) that \( \text{Enl}(X) \) and \( \text{Enl}(Y) \) are independent, whence they are sufficiently disjoint by \( \{\text{iv}\} \). This is a contradiction proving (4.4) and part \( \{\text{i}\} \) of Theorem 4.3.

Next, we deal with part \( \{\text{ii}\} \). For the sake of contradiction, suppose that \( D = \text{Con} L \) fails to satisfy the two-pendant four-crown property. This assumption and Lemma 2.11 yield that \( R \) is a cover-preserving subposet of \( \text{Lamps}(L) = J(D) \) such that the four maximal elements of \( R \) are also maximal in \( \text{Lamps}(L) \); see Definition 4.1. For \( a, b, \cdots \in R \), the corresponding lamp will be denoted by \( A, B, \ldots \), that is, by the capitalized version of the notation used in Figure 9. By Lemma 3.2, \( A, B, C, D \) are boundary lamps. Each of these four lamps is on the top left boundary chain or on the top right boundary chain.

By Corollary 3.4, any two consecutive members of the sequence \( A, B, C, D, A \) belong to different top boundary chains since they have a common lower cover. By left-right symmetry, we can assume that \( A \) and \( C \) are on the top left boundary chain while \( B \) and \( D \) on the top right one. We can assume that \( C \) is above \( A \) in the sense that \( \text{Foot}(A) \not< \text{Foot}(C) \) since otherwise we can relabel \( R \) according to the “rotational” automorphism that restricts to \( \{a, b, c, d\} \) as

\[
\begin{pmatrix}
a & b & c & d \\
c & d & a & b
\end{pmatrix}.
\]

Also, we can assume that \( D \) is above \( B \) since otherwise we extend

\[
\begin{pmatrix}
a & b & c & d \\
a & d & c & b
\end{pmatrix}.
\]

to a “reflection” automorphism of \( R \) and relabel \( R \) accordingly. Note that \( A \) and \( C \) are not necessarily neighboring boundary lamps, that is, we have \( \text{Peak}(A) \not\subseteq \text{Foot}(C) \) but \( \text{Peak}(A) \not< \text{Foot}(C) \) need not hold. Similarly, we only have that \( \text{Peak}(B) \not\subseteq \text{Foot}(D) \). The situation is outlined in Figure 11. The foots of lamps in the figure are black-filled and any of the two distances marked by curly brackets can be zero. The enlightened sets \( \text{Enl}(X) \) and \( \text{Enl}(Y) \) are dark-grey while \( \text{Enl}(A) \) and \( \text{Enl}(C) \) are (dark and light) grey. Since \( X \not< A \) and \( X \not< B \), we know from \( \{\text{i}\} \) and \( \{\text{iv}\} \) of Lemma 2.11 that \( \langle X, A \rangle \in \rho_{\text{Body}} \) and \( \langle X, B \rangle \in \rho_{\text{Body}} \). Hence, \( \text{Body}(X) \subseteq \text{Enl}(A) \cap \text{Enl}(B) \), in accordance with the figure. Similarly, \( Y \not< C \) and \( Y \not< D \) lead to \( \text{Body}(Y) \subseteq \text{Enl}(C) \cap \text{Enl}(D) \), as it is indicated in Figure 11.  

Note that due to (2.6), the figure is satisfactorily correct in this aspect. Hence, the \( A \)-shaped \( \text{Enl}(Y) \) is above the \( A \)-shaped \( \text{Enl}(X) \). Thus, using (2.6), we obtain that

\[
\text{Enl}(X) \text{ and } \text{Enl}(Y) \text{ are sufficiently disjoint; }
\]

see \( \{\text{iv}\} \) for this concept. On the other hand, \( Z \not< X \) and \( Z \not< Y \) together with \( \{\text{i}\} \) and \( \{\text{iv}\} \) of Lemma 2.11 gives that \( \langle Z, X \rangle \in \rho_{\text{Body}} \) and \( \langle Z, Y \rangle \in \rho_{\text{Body}} \). Hence, \( \text{Body}(Z) \subseteq \text{Enl}(X) \cap \text{Enl}(Y) \). Thus, since \( Z \) is an internal lamp by Lemma 3.2, \( \text{Enl}(X) \cap \text{Enl}(Y) \) contains a precipitous neon tube. This contradicts (4.5). Note that there is another way to get a contradiction: since \( \langle Z, X \rangle \in \rho_{\text{CircR}} \) and \( \langle Z, Y \rangle \in \rho_{\text{CircR}} \), we have that \( \text{CircR}(Z) \subseteq \text{Enl}(X) \cap \text{Enl}(Y) \), which contradicts the fact that \( \text{CircR}(Z) \) is of positive area (two-dimensional measure) while \( \text{Enl}(X) \cap \text{Enl}(Y) \) is
of area 0. Any of the two contradictions in itself implies part (iii) and completes the proof of Theorem 4.3. □

**Figure 11.** Illustration for the proof of Theorem 4.3 (ii)

**Proof of Remark 4.6.** Suppose the contrary. Then there is a distributive lattice $D$ such that $D$ fails to satisfy the two-pendant four-crown property but $|D| < 56$. Let $Q := J(D)$. By our assumption, $R$ is a subposet of the poset $Q$. Denote by $\text{DnSt}(R)$ and $\text{DnSt}(Q)$ the lattice of down sets (that is, order ideals and the emptyset) of $R$ and $Q$, respectively. For $X \subseteq Q$, let $\downarrow_Q X := \{ y \in Q : y \leq x \text{ for some } x \in X \}$. The map $\varphi: \text{DnSt}(Q) \rightarrow \text{DnSt}(R)$, defined by $X \mapsto X \cap R$ is surjective since, for each $Y \in \text{DnSt}(R)$, we have that $\downarrow_Q Y \in \text{DnSt}(Q)$ and $\varphi(\downarrow_Q Y) = Y$. Hence, using the structure theorem mentioned in Remark 4.5, $|D| = |\text{DnSt}(Q)| \geq |\text{DnSt}(R)| = |D_R| = 56$, which is a contradiction proving Remark 4.6. □

We conclude the paper with a last remark.

**Remark 4.7.** Lamps have several geometric properties. Many of these properties have already been mentioned, and there are some technical other ones, too. These properties would allow us to represent the congruence lattices of slim planar semi-modular lattices in a purely geometric (but quite technical) way. However, this does not seem to be more useful than our technique based on (2.9)–(2.10) and the toolkit presented in the paper.

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