On the speed of convergence of Piterbarg constants

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Abstract
In this paper, we derive an upper bound for the difference between the continuous and discrete Piterbarg constants. Our result allows us to approximate the classical Piterbarg constants by their discrete counterparts using Monte Carlo simulations with an explicit error rate.

Keywords Fractional Brownian motion · Piterbarg constants · Discretization error · Monte Carlo simulation

Mathematics Subject Classification 60G15 · 60G70 · 65C05

1 Introduction
Let $B_t^\alpha, t \in \mathbb{R}$ be a fractional Brownian motion (later on fBM), i.e., a Gaussian process with zero expectation and covariance function given by

$$\text{Cov}(B_t^\alpha, B_s^\alpha) = \frac{|t|^\alpha + |s|^\alpha - |t - s|^\alpha}{2}, \quad t, s \in \mathbb{R}, \alpha \in (0, 2).$$

One of the constants that typically appear in the asymptotics of the ruin probabilities for Gaussian and related processes are the Piterbarg constants defined for $d > 0$ and...
\( \alpha \in (0, 2) \) by

\[
\mathcal{P}_\alpha(d, K) = \mathbb{E}\left\{ \sup_{t \in \mathbb{R} \cap K} e^{\sqrt{2}B_\alpha(t) - (1+d)|t|^\alpha} \right\},
\]

where \( K \) can be both \([0, \infty)\) and \((-\infty, \infty)\). We refer to Theorem 10.1 in [25] for finiteness and positivity of \( \mathcal{P}_\alpha(d, K) \); some other contributions dealing with Piterbarg constants are [2, 9, 16, 22, 24]. We notice that the Piterbarg constants appear also in Gaussian queueing theory results, see [10–15, 18–20].

As for well-known Pickands constants (see, e.g., [25]), the exact value of the classical Piterbarg constant is known only for special case \( \alpha = 1 \), namely \( \mathcal{P}_1(d, (−\infty, \infty)) = 1 + \frac{2}{d} - \frac{1}{2d+1} \) and \( \mathcal{P}_1(d, [0, \infty)) = 1 + \frac{1}{d} \), see [2] and [23], respectively. In other cases naturally arises the question of an approximation of the classical Piterbarg constants.

Since it seems difficult to simulate fBM on a continuous-time scale, one can approximate the Piterbarg constants by their discrete analogous defined for \( \delta > 0 \) by

\[
\mathcal{P}_\alpha^\delta(d, K) = \mathbb{E}\left\{ \sup_{t \in G(\delta) \cap K} e^{\sqrt{2}B_\alpha(t) - (1+d)|t|^\alpha} \right\},
\]

where \( G(\delta) = \delta \mathbb{Z} \) for \( \delta > 0 \). Set in the following \( \mathcal{P}_\alpha^0(d, K) = \mathcal{P}_\alpha(d, K) \) and \( G(0) = \mathbb{R} \).

The question of speed of convergence of the discrete Piterbarg constants to continuous ones is related to the estimation of \( \sup_{t \in [0, 1]} B_\alpha(t) - \sup_{t \in [0, 1] \cap \delta \mathbb{Z}} B_\alpha(t) \) as \( \delta \to 0 \). We refer to [6, 7] for the interesting analysis of the expression above. For Brownian motion case (later on BM), i.e., when \( \alpha = 1 \), we refer to [17] for the survey of the known results for the current moment.

### 2 Main results

Here we present the results needed for approximation of \( \mathcal{P}_\alpha(d, K) \) by simulation methods. The theorem below derives an upper bound for the difference between the continuous and discrete Piterbarg constants.

**Theorem 2.1** For any \( d > 0 \) with some constant \( C > 0 \) that does not depend on \( \delta \), it holds that

\[
\mathcal{P}_\alpha(d, K) - \mathcal{P}_\alpha^\delta(d, K) \leq C\delta^{\alpha/2}(− \ln \delta)^{1/2}, \quad \delta \to 0.
\]

Since it is impossible to simulate a fBM on the infinite-time horizon, we deal with the truncated version of the Piterbarg constants. Namely, for \( d > 0 \) and \( \delta, T \geq 0 \) a truncated Piterbarg constant is defined by

\[
\mathcal{P}_\alpha^\delta(d, K; T) = \mathbb{E}\left\{ \sup_{t \in \mathbb{K} \cap [-T, T]} e^{\sqrt{2}B_\alpha(t) - (1+d)|t|^\alpha} \right\},
\]

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where for real $a, b$ and $\eta > 0$ $[a, b, \eta] := [a, b] \cap \eta \mathbb{Z}$. Set in the following $[a, b]_0 := [a, b]$ for $a < b$.

The following theorem provides us an upper bound for the difference between a Piterbarg constant and the corresponding truncated Piterbarg constant:

**Theorem 2.2** For any $\alpha_0 \in (0, \alpha)$, it holds with some constant $C$ that does not depend on $\delta$ and all sufficiently large $T$, that

$$\mathcal{P}^\delta_a(d, \mathcal{K}) - \mathcal{P}^\delta_a(d, \mathcal{K}; T) \leq e^{-CT^{\alpha_0}}.$$  

The results above imply the following approach for approximation of $\mathcal{P}_a(d, \mathcal{K})$ for $d > 0$:

1) For small $\delta > 0$ take $S_\delta = (-\ln \delta)^{2/\alpha}$ and approximate $\mathcal{P}^\delta_a(d, \mathcal{K}; S_\delta)$ by $\widehat{\mathcal{P}}^\delta_a(d, \mathcal{K}; S_\delta)$ obtained by Monte Carlo simulations.

2) Theorems 2.1 and 2.2 imply that with some constant $C$ that does not depend on $\delta$

$$\mathcal{P}_a(d, \mathcal{K}) - \mathcal{P}^\delta_a(d, \mathcal{K}; S_\delta) \leq C \delta^{\alpha/2} (-\ln \delta)^{1/2}$$

for sufficiently small $\delta > 0$, hence we can approximate $\mathcal{P}_a(d, \mathcal{K})$ by $\widehat{\mathcal{P}}^\delta_a(d, \mathcal{K}; S_\delta)$ with an error term not exceeding $C \delta^{\alpha/2} (-\ln \delta)^{1/2}$ combined with statistical error from estimation of $\widehat{\mathcal{P}}^\delta_a(d, \mathcal{K}; S_\delta)$.

Independence of the increments of Brownian motion allows us to deduce more precise bounds for $\mathcal{P}_1(d, \mathcal{K}) - \mathcal{P}^\delta_1(d, \mathcal{K})$ than in Theorem 2.1. Namely,

**Theorem 2.3** For $d > 0$ as $\delta \to 0$ it holds, that

$$\mathcal{P}_1(d, \mathcal{K}) - \mathcal{P}^\delta_1(d, \mathcal{K}) \sim -\frac{\zeta(1/2)}{\sqrt{\pi}} \sqrt{\delta} \cdot \mathcal{P}^\delta_1(d, \mathcal{K}),$$

where $\zeta$ is the Euler-Riemann zeta function and $-\frac{\zeta(1/2)}{\sqrt{\pi}} > 0$.

It is interesting that the constant $-\frac{\zeta(1/2)}{\sqrt{\pi}}$ appears above, the same constant plays the same role in the difference between the classical Pickands constants, see [5]. This constant appears in many problems concerning the difference between supremum of BM on a continuous and discrete grids, see [17].

### 3 Proofs

Define for $d > 0$ and $\alpha \in (0, 2)$

$$Z_\alpha(t) = \sqrt{2} B_\alpha(t) - (d + 1)|t|^{\alpha}, \quad t \in \mathbb{R}.$$  

Before giving proofs, we present and prove several lemmas needed for the proofs of the main results.

**Lemma 3.1** For all $d > 0$ there exists $C_1, C_2 \in \mathbb{R}$ such that

$$\mathbb{P} \left\{ \sup_{t \in [0, \infty)} e^{Z_\alpha(t)} > x \right\} \sim C_1 (\ln x)^{C_2} x^{-1-d}, \quad x \to \infty.$$
Proof of Lemma 3.1} By the self-similarity of fBM, for \( x > 1 \), we have

\[
\mathbb{P}\left\{ \sup_{t \in [0, \infty)} e^{Z_\alpha(t)} > x \right\} = \mathbb{P}\left\{ \exists t \in [0, \infty) : \sqrt{2} B_\alpha(t) - (d + 1)t^\alpha > \ln x \right\}
\]

\[
= \mathbb{P}\left\{ \exists t \in [0, \infty) : \frac{\sqrt{2} B_\alpha(t)}{(d + 1)t^\alpha + 1} > \sqrt{\ln x} \right\}
\]

\[
= : \mathbb{P}\left\{ \exists t \in [0, \infty) : V(t) > \sqrt{\ln x} \right\}.
\]

The variance of \( V(t) \) for \( t \geq 0 \) achieves its unique maximum at \( t_0 = \frac{1}{\sqrt{d+1}} \) and \( \text{Var}(V(t_0)) = \frac{1}{2(d+1)} \), hence applying [25, Theorem 10.1] we find that there exist some \( C_1 > 0 \) and \( C_2 \in \mathbb{R} \) such that \( \mathbb{P}\left\{ \sup_{t \in [0, \infty)} e^{Z_\alpha(t)} > x \right\} \sim C_1 (\ln x)^{C_2 x^{-1-d}} \), as \( x \to \infty \). \( \square \)

**Lemma 3.2** For any \( p, T > 0 \) and \( \alpha \in (0, 2) \) for sufficiently small \( \delta > 0 \) with some \( C > 0 \) that does not depend on \( \delta \) it holds, that

\[
\left( \mathbb{E}\left\{ \sup_{t, s \in [0, T], |t - s| \leq \delta} |B_\alpha(t) - B_\alpha(s)|^p \right\} \right)^{1/p} \leq C \delta^{\alpha/2} \sqrt{|\ln(\delta/T)|}.
\]

**Proof of Lemma 3.2** We have by Theorem 4.2 in [8], Chapter 'Modulus of Continuity', p.164 with some non-negative random variable \( A \) that does not depend on \( \delta \) and has all finite moments that

\[
\mathbb{E}\left\{ \sup_{t, s \in [0, T], |t - s| \leq \delta} |B_\alpha(t) - B_\alpha(s)|^p \right\} = T^{\alpha p} \mathbb{E}\left\{ \sup_{t, s \in [0, 1], |t - s| \leq \delta T} |B_\alpha(t) - B_\alpha(s)|^p \right\}
\]

\[
\leq T^{\alpha p} \mathbb{E}\left\{ \sup_{t, s \in [0, 1], |t - s| \leq \delta \frac{T}{T}} (|A| t - s)^{p/2} \sqrt{|\ln(|t - s|)|} \right\}^p \mathbb{P}\{ A^p \}
\]

\[
= C \delta^{\alpha p/2} \sqrt{|\ln(\delta/T)|}^p,
\]

and the claim follows. \( \square \)

Now we are ready to prove the main results.

**Proof of Theorem 2.1** We have by Theorem 2.2 with \( T_\delta = (-\ln \delta)^{2/\alpha} \) that

\[
\mathcal{P}_\alpha(d, \mathcal{K}) - \mathcal{P}_\alpha(\delta d, \mathcal{K}) \leq \mathcal{P}_\alpha(d, \mathcal{K}; T_\delta) - \mathcal{P}_\alpha(\delta d, \mathcal{K}; T_\delta) + o(\delta^{\alpha/2}).
\]

\[
= \mathbb{E}\left\{ \sup_{t \in \mathcal{K} \cap [-T_\delta, T_\delta]} e^{Z_\alpha(t)} - \sup_{t \in \mathcal{K} \cap [-T_\delta, T_\delta]} e^{Z_\alpha(t)} \right\} + o(\delta^{\alpha/2})
\]

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Note that for any $y \leq x$ it holds, that $e^x - e^y \leq (x - y)e^x$. Implementing this inequality, we find that for $s, t \in [0, T_\delta]$, 

$$
\left| e^{Z_\alpha(t)} - e^{Z_\alpha(s)} \right| \leq e^{\max(Z_\alpha(t), Z_\alpha(s))} |Z_\alpha(t) - Z_\alpha(s)| 
\leq e^{\sup_{w \in [0, \infty)} Z_\alpha(w)} \left| \sqrt{2}(B_\alpha(t) - B_\alpha(s)) - (t^\alpha - s^\alpha) \right|.
$$

By the lines above, we obtain

$$
\mathbb{E} \left\{ \sup_{t \in [0, T_\delta]} e^{Z_\alpha(t)} - \sup_{t \in [0, T_\delta]} e^{Z_\alpha(t)} \right\} 
\leq \mathbb{E} \left\{ \sup_{t,s \in [0, T_\delta], |t-s| \leq \delta} |e^{Z_\alpha(t)} - e^{Z_\alpha(s)}| \right\} 
\leq \mathbb{E} \left\{ \sup_{t,s \in [0, T_\delta], |t-s| \leq \delta} e^{\max_{w \in [0, \infty)} Z_\alpha(w)} |B_\alpha(t) - B_\alpha(s)| \left| \sqrt{2}(B_\alpha(t) - B_\alpha(s)) - (t^\alpha - s^\alpha) \right| \right\} 
\leq \sqrt{2} \mathbb{E} \left\{ e^{\sup_{t,s \in [0, T_\delta], |t-s| \leq \delta} \max_{w \in [0, \infty)} Z_\alpha(w)} \right\}^{1/q} \left( \mathbb{E} \left\{ \sup_{t,s \in [0, T_\delta], |t-s| \leq \delta} |B_\alpha(t) - B_\alpha(s)|^p \right\} \right)^{1/p} + o(\delta^{\alpha/2}),
$$

where $1/q + 1/p = 1$ and $q$ is chosen such that $\mathbb{E} \left\{ e^{\sup_{w \in [0, \infty)} Z_\alpha(w)} \right\} < \infty$, this is possible to do by Lemma 3.1. Next by Lemma 3.2 and lines above, we have

$$
\mathcal{P}_\alpha(d, K) - \mathcal{P}_{\alpha}^\delta(d, K) \leq C\delta^{\alpha/2} \sqrt{\ln(|\delta/T_\delta|)} + o(\delta^{\alpha/2})
$$

and the claim follows. \qed

**Proof of Theorem 2.2** We have

$$
\mathcal{P}_{\alpha}^\delta(d, K) - \mathcal{P}_{\alpha}(d, K; T) = \mathbb{E} \left\{ \sup_{t \in \mathbb{K} \cap (-\infty, \infty) \delta} e^{Z_\alpha(t)} - \sup_{t \in \mathbb{K} \cap [-T, T] \delta} e^{Z_\alpha(t)} \right\} 
\leq \mathbb{E} \left\{ \sup_{t \in \mathbb{R}} e^{Z_\alpha(t)} \mathbb{E}(M_T) \right\},
$$

where $M_T = \{ \omega \in \Omega : Z_\alpha(\omega, t) > 0 \text{ for some } t \in (-\infty, -T] \cap [T, \infty) \}$ with $\Omega$ being the general probability space. Since trajectories of $\sqrt{2}B_\alpha(t)/t^{\alpha-\alpha_0}/2, t \geq 0$
\(T, \alpha_0 \in (0, \alpha)\) are bounded almost surely, we have by Borell-TIS inequality (see Theorem 2.1.1 in [1]) and the symmetry
\[
P\{M_T\} \leq 2P\left\{ \exists t \geq T : \sqrt{2}B_\alpha(t) - (d + 1)t^\alpha > 0 \right\}
\]
\[
= 2P\left\{ \exists t \geq T : \frac{\sqrt{2}B_\alpha(t)}{(d + 1)t^{\alpha - \alpha_0/2}} > t^{\alpha_0/2} \right\} \leq e^{-CT^{\alpha_0}}.
\]
By Lemma 3.1, \(\sup_{t \in \mathbb{R}} e^{Z_\alpha(t)}\) has the finite moment of order \(1 + d/2\), and hence applying Hölder inequality we have
\[
E\left\{ \sup_{t \in \mathbb{R}} e^{Z_\alpha(t)} I(M_T) \right\} \leq C_1P\{M_T\}^C \leq e^{-CT^{\alpha_0}}
\]
and the claim follows. \(\square\)

**Proof of Theorem 2.3** Let \(Z_1(t) = \sqrt{2}B(t) - (1 + d)|t|\) be defined as in (1). We will show the statement only for \(\mathcal{K} = (-\infty, \infty)\); the proof for \(\mathcal{K} = [0, \infty)\) will be analogous (but slightly simpler). Fix \(d > 0\) for \(J \subseteq \mathbb{R}\), we define
\[
M_J := \sup_{t \in J} Z_1(t), \quad M_\delta := \sup_{t \in J \cap \delta \mathbb{Z}} Z_1(t), \quad \Delta_\delta := M_J - M_\delta / \delta^{1/2}.
\]
For brevity, we write \(M := M_\mathbb{R}, M_\delta := M_\delta^{\mathbb{R}}, \) and \(\Delta_\delta := \Delta_\delta^{\mathbb{R}}.\) Then,
\[
P_1(d, \mathcal{K}) - P_1^\delta(d, \mathcal{K}) = E\left\{ e^M - e^{M_\delta} \right\}.
\]
Using mean value theorem and noticing that \(M_\delta \leq M\) a.s., we obtain the following bounds
\[
E\left\{ (M - M_\delta) \cdot e^{M_\delta} \right\} \leq P_1(d, \mathcal{K}) - P_1^\delta(d, \mathcal{K}) \leq E\left\{ (M - M_\delta) \cdot e^M \right\}.
\]
Till the end, we want to show that
\[
\lim_{\delta \to 0} E\left\{ \Delta_\delta \cdot e^{M_\delta} \right\} = \lim_{\delta \to 0} E\left\{ \Delta_\delta \cdot e^M \right\} = \lim_{\delta \to 0} E\left\{ \Delta_\delta \cdot \mathcal{P}_\alpha(d, \mathcal{K}) \right\} = -\frac{\zeta(1/2)}{\sqrt{\pi}} \mathcal{P}_\alpha(d, \mathcal{K}),
\]
which, in combination with (3), will conclude the proof. We will show (4) in three steps.

**Step 1.** Show that there exists a random variable \(D\) such that \(E\{D\} = -\frac{\zeta(1/2)}{\sqrt{\pi}}\) and such that for any continuity set \(A\) of \(D\) (\(A\) such that \(P\{D \in \partial A\} = 0\)) and an arbitrary event \(B \in \mathcal{F}\) of positive probability, where \(\mathcal{F}\) is the smallest \(\sigma\)-algebra containing all \(\sigma(Z_1(t)) : t \in \mathbb{R}\), for \(J = \mathbb{R}\), as \(\delta \downarrow 0\)
\[
P\left\{ \Delta_\delta^{J} \in A, B \right\} \to P\{D \in A\} P\{B\}.
\]
\(\mathcal{P}\) Springer
The statement (5) for \( J = \mathbb{R} \) implies that \( \Delta_\delta \xrightarrow{d} D \) and, intuitively, that the sequence is ‘asymptotically independent’ of the process \( \{Z_1(t)\}_{t \in \mathbb{R}} \).

Now, [21, Thm. 4] implies that (5) holds for \( J = [0, T) \) as well as \( J = (-T, 0] \) for arbitrary \( T \in (0, \infty) \).

Now, let \( \tau \) and \( \tau_\delta \) be the (almost surely unique) time epochs of the all-time supremum of \( Z \) over \( \mathbb{R} \) and over \( \mathbb{R} \mathbb{Z} \), respectively, i.e., \( Z_1(\tau) = M \) and \( Z_1(\tau_\delta) = M_\delta \). For any \( T > 0 \), let \( C_T^+ \) be the event that \( \{\tau \in [0, T), \tau_\delta \in [0, T)\} \) and similarly, let \( C_T^- := \{\tau \in (-T, 0], \tau_\delta \in (-T, 0]\} \). We have

\[
\mathbb{P} \{\Delta_\delta \in A, B\} = \mathbb{P} \{\Delta_\delta \in A, B, C_T^-\} + \mathbb{P} \{\Delta_\delta \in A, B, C_T^+\}
\]

\[
= \mathbb{P} \{\Delta_\delta^{[0,T)} \in A, B, C_T^+\} + \mathbb{P} \{\Delta_\delta^{(-T,0]} \in A, B, C_T^-\}
\]

\[
+ \mathbb{P} \{\Delta_\delta \in A, B, (C_T^- \cup C_T^+)^c\},
\]

which implies that

\[
\left| \mathbb{P} \{\Delta_\delta \in A, B\} - \left( \mathbb{P} \{\Delta_\delta^{[0,T)} \in A, B, C_T^+\} + \mathbb{P} \{\Delta_\delta^{(-T,0]} \in A, B, C_T^-\} \right) \right| \leq \mathbb{P} \{(C_T^- \cup C_T^+)^c\}.
\]

We now pass \( \delta \to 0 \), use (5) with \( B := B \cap C_T^+ \) and \( B := B \cap C_T^- \), and obtain

\[
\left| \mathbb{P} \{\Delta_\delta \in A, B\} - \mathbb{P} \{D \in A\} \left( \mathbb{P} \{B, C_T^+\} + \mathbb{P} \{B, C_T^-\} \right) \right| \leq \mathbb{P} \{(C_T^- \cup C_T^+)^c\}.
\]

We pass \( T \to \infty \) and notice that \( \mathbb{P} \{C_T^+ \cap C_T^-\} = \mathbb{P} \{\tau = 0\} = 0 \) and \( \mathbb{P} \{C_T^+ \cup C_T^-\} \to 1 \), as \( T \to \infty \), which shows (5) for \( J = \mathbb{R} \). Finally, the fact that \( \mathbb{E} \{D\} = -\frac{C(1/2)}{\sqrt{\pi}} \) was established in [4, Corollary 4].

**Step 2.** Show that the family of random variables \( \Delta_\delta \cdot e^{M_\delta} \), indexed by \( \delta \), is uniformly integrable for \( \delta \) small enough.

According to de la Vallée-Poussin Theorem (see also [3, (25.13)]), it is enough to establish that there exists \( \varepsilon > 0 \) such that the sequence \( \mathbb{E} \{(\Delta_\delta \cdot e^{M_\delta})^{1+\varepsilon}\} \) is bounded for all \( \delta \) small enough. Due to Hölder’s inequality, with \( p, q \in [1, \infty) \) satisfying \( 1/p + 1/q = \frac{1}{2} \) we obtain

\[
\mathbb{E} \{(\Delta_\delta \cdot e^{M_\delta})^{1+\varepsilon}\} \leq \left( \mathbb{E} \left| \Delta_\delta^{p(1+\varepsilon)} \right| \right)^{1/p} \cdot \left( \mathbb{E} \left| e^{q(1+\varepsilon)M_\delta} \right| \right)^{1/q}.
\]

We now establish that for any \( 0 < p < \alpha \), the \( p \)th moment of \( \Delta_\delta \) is bounded for all \( \delta \) small enough. Since \( \Delta_\delta \leq \Delta_\delta^{[0,\infty)} + \Delta_\delta^{(-\infty,0]} \), then for any \( p > 0 \) we have

\[
\mathbb{E} \{\Delta_\delta\}^p \leq \max\{1, 2^{p-1}\} \cdot \left( \mathbb{E} \left| \Delta_\delta^{[0,\infty)} \right| \right)^p + \mathbb{E} \left\{ \Delta_\delta^{(-\infty,0]} \right\}^p,
\]
where we used the fact that \((x + y)^p \leq \max\{1, 2^{p-1}\}(x^p + y^p)\) for all \(x, y > 0\). The boundedness of \(\mathbb{E}\{\Delta_\delta\}^p\) readily follows from the boundedness of \(\mathbb{E}\left\{\Delta_\delta^{[0,\infty]}\right\}^p\) and \(\mathbb{E}\left\{\Delta_\delta^{(-\infty,0]}\right\}^p\), which are established in [4, Theorem 4.2]. Moreover, since \(M\) is exponentially distributed with mean \((1 + a)^{-1}\), then \(\mathbb{E}\{e^{\beta M}\} < \infty\) for any \(\beta < 1 + a\).

So, in (6), if we take \(q > 1\) and \(\varepsilon > 0\) small enough such that \(q(1 + \varepsilon) < 1 + a\), then \(\mathbb{E}\left\{e^{q(1+\varepsilon)M}\right\} \leq \mathbb{E}\left\{e^{qM}\right\}\) will be bounded, as \(\delta \to 0\).

**Step 3.** Since \(e^{M_\delta} \to e^M\) a.s., and the limit is \(\mathcal{F}\)-measurable, then due to the mixing condition in (5), the pair \((\Delta_\delta, e^{M_\delta})\) converges jointly in distribution to \((D, e^M)\), where \(D\) is independent of \(e^M\). Now, from the continuous mapping theorem we find that \(\Delta_\delta \cdot e^{M_\delta} \overset{d}{\to} D \cdot e^M\). Finally, (4) follows from the uniform integrability of the family \(\Delta_\delta \cdot e^{M_\delta}\). \(\square\)

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