ELLiptic equations in Weighted Besov Spaces on Asymptotically Flat Riemannian Manifolds

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Abstract. This paper deals with the applications of weighted Besov spaces to elliptic equations on asymptotically flat Riemannian manifolds, and in particular to the solutions of Einstein’s constraints equations. We establish existence theorems for the Hamiltonian and the momentum constraints with constant mean curvature and with a background metric that satisfies very low regularity assumptions.

These results extend the regularity results of Holst, Nagy and Tsoptgerel about the constraint equations on compact manifolds in the Besov space $B^s_{p,p}$ [23], to asymptotically flat manifolds. We also consider the Brill–Cantor criterion in the weighted Besov spaces. Our results improve the regularity assumptions on asymptotically flat manifolds [15, 29], as well as they enable us to construct the initial data for the Einstein–Euler system.

1. Introduction

A special feature of the Einstein equations is that initial data cannot be prescribed freely. They must satisfy constraint equations. To prove the existence of a solution of the Einstein equations, it is first necessary to prove the existence of a solution of the constraints. The usual method of solving the constraints relies on the theory of elliptic equations. Since asymptotically flat metric falls off only as $r^{-1}$ as $r \to \infty$ on a spacelike slice and the positive mass theorem [34] implies that any attempt to impose faster fall-off excludes all but the trivial solution the metric does not belong to a Sobolev space. The usual way to get around this is to replace the ordinary Sobolev space by a weighted one.

Therefore much attention has been devoted to solutions of the Einstein constraint equations in asymptotically flat space–times by means of weighted Sobolev spaces as an essential tool. These spaces are defined as the completion of $C^\infty_0(\mathbb{R}^n)$ with respect to the norm

\[ \|u\|_{m,p,\delta} = \left( \sum_{|\alpha| \leq m} \int \left| (1 + |x|)^{\delta + |\alpha|} \partial^\alpha u \right|^p dx \right)^{1/p}, \]

and denoted by $W^{p}_{m,\delta}$.

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Elliptic equations on $W^p_{m,\delta}$ spaces were first considered by Nirenberg and Walker in [31]. This paper led to numerous publications dealing with its applications to the solutions of Einstein constraint equations in asymptotically flat space–times. Some significant contributions include the papers of Bartnik [3], Cantor [8, 9], Choquet–Bruhat and Christodoulou [14] and Christodoulou and O’Murchadha [17]. Afterward the regularity assumptions were improved, by Choquet–Bruhat, Isenberg, Pollack and York for the Einstein–scalar field gravitational constraint equations [15, 16], and by Maxwell in the vacuum case and with boundary conditions [29]. In both papers the authors assumed that the background metric is locally in $W^p_2$ when $p > \frac{n}{2}$, and they obtained a solution to the constraint equations with a conformal metric in the same space. In the case $p = 2$, Maxwell constructed low regularity solutions of the vacuum Einstein constraint equations with a metric in the weighted Bessel potential spaces $H^s_{\text{loc}}$ for $s > \frac{n}{2}$ [30].

On compact manifolds one seeks solutions to the Einstein constraint equations in the unweighted Sobolev spaces. Choquet–Bruhat obtained solutions with a metric in $W^p_2$ and for $p > \frac{n}{2}$ [12]. Later Maxwell improved the regularity in the Bessel potential spaces $H^s$ for $s > \frac{n}{2}$ [28]. Holst, Nagy and Tsogtgerel [23] study solutions of the Einstein constraints in the Sobolev–Sobolevskij spaces $W^p_s$, and obtained solutions to the Hamiltonian and momentum constraints in these spaces when $s \in (\frac{n}{p}, \infty) \cap [1, \infty)$ and $p \in (1, \infty)$. Thus their results cover [12] in the case $s = 2$ and [28] in the case $p = 2$.

One of our main results is the extension of the regularity result of [23] to asymptotically flat manifolds. In doing so we use Triebel’s extension of the $W^p_{m,\delta}$–spaces to the fractional order spaces $W^p_{s,\delta}$–spaces, where $s$ and $\delta$ are real numbers, [38] (see definition 2.2). The $W^p_{s,\delta}$–spaces are constructed by means of the Besov space $B^s_{p,p}$ (see (2.1)). The Besov spaces coincide with the Sobolev–Sobolevskij spaces whenever $s$ is positive (see e.g. [4, Ch.6]), and they are suitable for interpolation both for negative and positive $s$. This is a vital property of the Besov spaces that enable us to prove certain properties by means of interpolation.

In the present paper we prove existence and uniqueness results for the Hamiltonian and momentum constraints with constant mean curvature (CMC) in the $W^p_{s,\delta}$–spaces, and under the conditions $s \in (\frac{n}{p}, \infty) \cap [1, \infty)$, $\delta \in (-\frac{n}{p}, n - 2 - \frac{n}{p})$ and for all $p \in (1, \infty)$.

Thus we improve the regularity [13, 15, 16, 29] and extend the range of $p$ to $(1, \infty)$. [15, 29]. Holst et al. achieved solutions to the constrain equations on compact manifolds with low regularity, without the CMC–condition. We believe that the present paper is a step toward developing a theory of rough solutions in asymptotically flat manifolds for the non–CMC case.

The Brill–Cantor condition (5.1) suggests a criterion under which a given metric in an asymptotically flat manifold can be rescaled to yield a conformal metric with zero scalar curvature (see §5). This criterion is related to the Yamabe conformal invariant classes, but has a different interpretation on asymptotically flat manifolds. For an enlightening discussion about this criterion see [20]. Cantor and Brill suggested this condition and
showed its equivalent to the existence of a flat metric for an integer $m$ greater than $\frac{n}{p} + 2$ and $1 < p < \frac{2n}{n-2}$ [11]. Since then the regularity assumptions were improved by several authors [13, 15, 16, 29], however, they dealt only with Sobolev spaces of integer order, and under the restriction that $p > \frac{n}{2}$. For $p = 2$ it was proved for all $s > \frac{n}{2}$ in [30].

We treat the Bill–Cantor condition in the weighted Besov spaces $W^p_{s,\delta}$ and establish its equivalent to the existence of a metric with zero scalar curvature for $s \in \left(\frac{n}{p}, \infty\right) \cap [1, \infty)$, $\delta \in (-\frac{n}{p}, -2 - \frac{n}{p})$ and all $p \in (1, \infty)$. To conclude, our results generalize [30] to $p \in (1, \infty)$, and improve the regularity of [13, 15, 16, 29].

One of the essential difficulties is to prove the continuity of the Yamabe functional (5.1) in terms of the norm of $W^p_{s,\delta}$. In previous publications the restriction of $p > \frac{n}{2}$ was caused by two reasons: the Sobolev embedding theorem; and a certain application of the generalized Hölder inequality that requires the condition $p > \frac{n}{2}$. We, however, overcome this obstacle by improving the multiplication property of functions in $W^p_{s,\delta}$–spaces (see Proposition 2.15).

The outline of the paper is as follows. In the first part of Section 2 we sketch Triebel’s construction of the weighted Besov spaces and state their main properties. We also pay attention to the bilinear form on these spaces since the dual representation of the norm plays a role in the proof of existence of solutions of nonlinear equations. In the second part we establish tools which are needed for PDE in these spaces, including embeddings, pointwise multiplication and Moser type estimates.

Section 3 is devoted to elliptic linear systems on asymptotically flat Riemannian manifolds. In the first subsection we establish a priori estimates for second order elliptic operators with coefficients in the $W^p_{s,\delta}$ spaces in $\mathbb{R}^n$ and show that these systems are semi–Fredholm operators. This property plays an essential role in the study of the non–linear equations. The definition of asymptotically flat manifolds of the class $W^p_{s,\delta}$ is done in subsection 3.2. We also define there the norms and the bilinear forms on a Riemannian manifold. In subsection 3.3 we study weak solutions that meet very low regularity requirements. This demands special attention to the extension of a $L^2$-bilinear form to the bilinear form acting on $W^p_{s,\delta}$ and its dual. We then define weak solutions on the manifolds and derive a weak maximum principle for all $p \in (1, \infty)$.

In Section 4 we prove the existence and uniqueness theorem of a semi–linear equation, where the linear part is the Laplace–Beltrami operator of an asymptotically flat Riemannian manifold. The method of sub and super solution is the common method for these types of non–linearity, however, we shall implement Cantor’s homotopy argument [9] in the weighted Besov spaces.

In Section 5 we discuss the Brill–Cantor criterion. We show that for $s \in \left(\frac{n}{p}, \infty\right) \cap [1, \infty)$, $\delta \in (-\frac{n}{p}, -2 - n + \frac{n}{p})$ and all $p \in (1, \infty)$ condition (5.1) is necessary and sufficient for the existence of a metric that belongs to $W^p_{s,\delta}$ and has zero scalar curvature. In Section 6 we consider the construction of initial data for the Einstein–Euler system. In this system the equations for the gravitational fields are coupled to a perfect fluid and certain relations
between the source terms of the constraint equations and the fluid variables must be fulfilled. In [6] the authors discussed this problem in detail using the weighted Hilbert spaces $W^p_{s,\delta}$. Here we extend these results to the $W^p_{s,\delta}$ spaces.

Finally, in the Appendix we discuss the extension of several properties $W^p_{s,\delta}$ spaces in $\mathbb{R}^n$ to the $W^p_{s,\delta}$ spaces on a Riemannian manifold. The norm on a Riemannian manifold is defined by means of a collection of charts and partition of unity. Though the extension to a Riemannian manifold seems to be an obvious matter, sometimes it requires a certain attention and watchfulness.

Some notations: For $p \in (1, \infty)$, $p'$ will stand for the dual index to $p$, that is $\frac{1}{p} + \frac{1}{p'} = 1$. The scaling of a distribution $u$ with a positive number $\varepsilon$ is denoted by $u_\varepsilon$. A Riemannian manifold is denoted by $\mathcal{M}$ and $g = g_{ab}$ is a metric on $\mathcal{M}$, $\nabla u$ is the covariant derivative and $|\nabla u|^2 = g^{ab}\partial_a u\partial_b u$, where $g^{ab}$ is the inverse matrix of $g_{ab}$. Latin indexes $a, b$ take the values $1, \ldots, n$ and the dimension $n$ is greater or equal to two throughout this paper. We will use the notation $A \lesssim B$ to denote an inequality $A \leq CB$ where the positive constant $C$ does not depend on the parameters in question.

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2. The weighted Besov spaces

2.1. The construction of the Spaces $W^p_{s,\delta}$. In this subsection we sketch the construction of the weighted Besov spaces. We start by fixing the notations and recalling the definition of the Besov spaces $B^s_{p,p}$ [4, 40]. Let $\mathcal{S}$ denote the Schwartz class of rapidly decreasing functions in $\mathbb{R}^n$ and $\mathcal{S}'$ its dual. Let $\{\phi_j\}_{j=0}^\infty \subset C^\infty_0(\mathbb{R}^n)$ be a dyadic partition of unity of $\mathbb{R}^n$ such that supp$(\phi_0) \subset \{|\xi| \leq 2\}$, supp$(\phi_j) \subset \{2^{j-1} \leq |\xi| \leq 2^{j+1}\}$ for $j \in \mathbb{N}$, all the function $\phi_j$ are nonnegative and $\sum_{j=0}^\infty \phi_j(\xi) = 1$. Let $\mathcal{F}(u)$ be the Fourier transform of a distribution $u$, $s \in \mathbb{R}$ and $1 \leq p < \infty$, then

$$W^p_s := B^s_{p,p} = \left\{ u \in \mathcal{S}' : \|u\|_{W^p_s} := \left( \sum_{j=0}^\infty 2^{jsp} \|\mathcal{F}^{-1}(\phi_j \mathcal{F}(u))\|^p_{L^p} \right)^{1/p} < \infty \right\}. \tag{2.1}$$

For $1 < p < \infty$, the dual space $(W^p_s)'$ is isomorphic to $W^p_{-s}$, where $1/p + 1/p' = 1$ (see e.g. [4, Corollary 6.2.8]). Let $u \in W^p_s$ and $\varphi \in W^p_{-s}$, then

$$\langle u, \varphi \rangle = \sum_{j=0}^\infty \sum_{k=j+2}^{\infty} \int \left[ \mathcal{F}^{-1}(\phi_j \mathcal{F}(u))(x) \right] \left[ \mathcal{F}^{-1}(\phi_k \mathcal{F}(\varphi))(x) \right] dx \tag{2.2}$$

$$:= \sum_{j=0}^\infty \sum_{k=j}^{\infty} \left( \mathcal{F}^{-1}(\phi_j \mathcal{F}(u)), \mathcal{F}^{-1}(\phi_k \mathcal{F}(\varphi)) \right)_{L^2(\mathbb{R}^n)},$$
is a bilinear form on $W^p_s \times W^p_{-s}$.

Here and throughout the paper $(\cdot, \cdot)_{L^2}$ denote the $L^2$ bilinear form, also whenever $k < 0$, sums as in (2.2) start from zero. For the proof of this formula see [2, Proposition 2.76].

**Proposition 2.1.** If $s > 0$ $u \in W^p_s$ and $\varphi \in \mathcal{S}$, then

\begin{equation}
(u, \varphi) = (u, \varphi)_{L^2(\mathbb{R}^n)}.
\end{equation}

**Proof.** Assuming that $u$ is smooth, then we have by Parseval’s formula that

\[
(u, \varphi)_{L^2(\mathbb{R}^n)} = c_n \left( \mathcal{F}(u), \overline{\mathcal{F}(\varphi)} \right)_{L^2(\mathbb{R}^n)} = c_n \left( \sum_{j=0}^{\infty} \phi_j \mathcal{F}(u), \sum_{k=0}^{\infty} \phi_k \mathcal{F}(\varphi) \right)_{L^2(\mathbb{R}^n)}
\]

\[
= c_n \sum_{j=0}^{\infty} \sum_{k=j-2}^{j+2} \left( \phi_j \mathcal{F}(u), \phi_k \mathcal{F}(\varphi) \right)_{L^2(\mathbb{R}^n)} = \sum_{j=0}^{\infty} \sum_{k=j-2}^{j+2} \left( \mathcal{F}^{-1}(\phi_j \mathcal{F}(u)), \mathcal{F}^{-1}(\phi_k \mathcal{F}(\varphi)) \right)_{L^2(\mathbb{R}^n)}
\]

\[
= (u, \varphi).
\]

Now if $u \in W^p_s$, then $u \in L^p$ since $s > 0$. So take a sequence $\{u_k\} \subset C_0^\infty(\mathbb{R}^n)$ such that $u_k$ tends to $u$ in $W^p_s$. Then $|(u_k, \varphi)_{L^2(\mathbb{R}^n)}| \leq \|u_k\|_{L^p} \|\varphi\|_{L^{p'}} \leq \|u_k\|_{W^p_s} \|\varphi\|_{L^{p'}}$. Thus (2.3) follows.

We turn now to the construction of the weighted-$W^p_s$ space. We also use a dyadic partition of unity, which is denoted by $\{\psi_j\}_{j=0}^\infty$, and is such that the support of $\psi_j$ is contained in the dyadic shell $\{x : 2j-2 \leq |x| \leq 2j+1\}$, $\psi_j(x) = 1$ on $\{x : 2j-1 \leq |x| \leq 2j\}$ for $j = 1, 2, \ldots$, while $\psi_0$ has a support in the ball $\{x : |x| \leq 2\}$ and $\psi_0(x) = 1$ on $\{x : |x| \leq 1\}$. In addition we require that $\{\psi_j\}_{j=0}^\infty \subset C_0^\infty(\mathbb{R}^n)$ and satisfies the inequalities

\begin{equation}
|\partial^\alpha \psi_j(x)| \leq C_\alpha 2^{-|\alpha|j},
\end{equation}

where the constant $C_\alpha$ does not depend on $j$. For a positive number $\varepsilon$ we denote the scaling $u(\varepsilon x)$ by $u_\varepsilon(x)$.

**Definition 2.2** (Weighted Besov spaces $W^p_{s,\delta}$). Let $s, \delta \in \mathbb{R}$ and $p \in [1, \infty)$, the $W^p_{s,\delta}(\mathbb{R}^n)$-space is the set of all tempered distributions $u$ such that the norm

\begin{equation}
\|u\|_{W^p_{s,\delta}(\mathbb{R}^n)} := \sum_{j=0}^{\infty} 2^{(\delta + \frac{\alpha}{p})j} \left\| (\psi_j u)(2^j) \right\|_{W^p_s}.
\end{equation}

is finite.

The $W^p_{s,\delta}$-norm of distributions in an open set $\Omega \subset \mathbb{R}^n$ is given by

\[
\|u\|_{W^p_{s,\delta}(\Omega)} = \inf_{f \mid_{\Omega} = u} \|f\|_{W^p_{s,\delta}(\mathbb{R}^n)}.
\]

The following basic properties were established in Triebel [38, 39].

**Theorem 2.3** (Triebel, Basic properties). Let $s, \delta \in \mathbb{R}$ and $p \in (1, \infty)$. 

(a) The space $W^p_{s,\delta}(\mathbb{R}^n)$ is a Banach space and different choices of the dyadic resolution $\{\psi_j\}$ which satisfies (2.4) result in equivalent norms.

(b) $C_0^{\infty}(\mathbb{R}^n)$ is a dense subset in $W^p_{s,\delta}(\mathbb{R}^n)$.

(c) The dual space of $W^p_{s,\delta}(\mathbb{R}^n)$ is $W^{p'}_{-s,-\delta}(\mathbb{R}^n)$, where $p' = \frac{p}{(p-1)}$.

(d) Interpolation (real): Let $0 < \theta < 1$, $s = \theta s_0 + (1-\theta)s_1$, $\delta = \theta \delta_0 + (1-\theta)\delta_1$ and $1/p = \theta/p_0 + (1-\theta)/p_1$, then

\[ (W^{p_1}_{s_1,\delta_1}(\mathbb{R}^n), W^{p_2}_{s_2,\delta_2}(\mathbb{R}^n))_{\theta,p} = W^p_{s,\delta}(\mathbb{R}^n). \]

For the definition of the bilinear form on $W^p_{s,\delta}(\mathbb{R}^n) \times W^{p'}_{-s,-\delta}(\mathbb{R}^n)$ we choose a dyadic resolution $\{\psi_j\}$ such that $\sum_{j=0}^{\infty} \psi_j(x) = 1$ and set

\[ \langle u, \varphi \rangle_W = \sum_{j=0}^{\infty} \sum_{k=j-2}^{j+2} 2^n \langle \psi_j u_{(2^j)}, \psi_k \varphi_{(2^j)} \rangle, \]

\[ \langle \cdot, \cdot \rangle \] denotes the form appearing on the left hand side and is of course given by (2.2). It satisfies the inequality

\[ |\langle u, \varphi \rangle_W| \leq C \|u\|_{W^p_{s,\delta}(\mathbb{R}^n)} \|\varphi\|_{W^{p'}_{-s,-\delta}(\mathbb{R}^n)} \]

(see the proof of Theorem 2 in [39]). In a similar manner to Proposition 2.1, we have that:

**Proposition 2.4.** If $s > 0$, $u \in W^p_{s,\delta}(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}$, then

\[ (u, \varphi)_{L^2(\mathbb{R}^n)} = \langle u, \varphi \rangle_W. \]

**Proof.** We use Proposition 2.1 and also the same idea as in its proof, but here the dyadic resolution $\{\psi_j\}$ replaces $\{\phi_j\}$. \hfill \Box

**Remark 2.5.** Let $f \in W^p_{s,\delta}(\mathbb{R}^n)$, then it follows from (c) above that

\[ \|f\|_{W^p_{s,\delta}(\mathbb{R}^n)} = \sup\{ \langle f, \varphi \rangle_W : \|\varphi\|_{W^{p'}_{-s,-\delta}(\mathbb{R}^n)} \leq 1, \varphi \in C_0^{\infty}(\mathbb{R}^n) \}, \]

and in particular, by (c) of Theorem 2.3 and Proposition 2.4, if $f \geq 0$, then

\[ \|f\|_{W^p_{s,\delta}(\mathbb{R}^n)} = \sup\{ \langle f, \varphi \rangle_W : \varphi \geq 0, \|\varphi\|_{W^{p'}_{-s,-\delta}(\mathbb{R}^n)} \leq 1, \varphi \in C_0^{\infty}(\mathbb{R}^n) \}. \]

For $s > 0$ the Besov norm (2.1) is equivalent to the norm of the Sobolev–Sobolevskij spaces (see e.g. [4, Ch. 6], [37, §35] or [40]). Their norm is defined as follows. Let $s = m + \lambda$, where $m$ is a nonnegative integer and $0 < \lambda < 1$, then

\[ \|u\|_{s,p}^p = \begin{cases} \sum_{|\alpha| \leq m} \|\partial^\alpha u\|^p_{L^p}, & s = m \\ \sum_{|\alpha| \leq m} \|\partial^\alpha u\|^p_{L^p} + \sum_{|\alpha| = m} \int \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^p}{|x - y|^{n+\lambda p}} \, dx \, dy, & s = m + \lambda. \end{cases} \]
Moreover, he showed that the constants in the above equivalence depend only on the dimension and the constants \(c\) (2.12)

Then the norms \((2.11)\) and \((2.12)\) we introduce the homogeneous norm, that is,

\[
\|u\|_{s,p,\text{hom}}^p = \begin{cases} 
\sum_{|\alpha|=m} \|\partial^\alpha u\|_{L^p}^p, & s = m \\
\sum_{|\alpha|=m} \left( \sum_{|\alpha|=m} \int \int \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^p}{|x-y|^{n+\lambda p}} \right)^{1/p} dx dy, & s = m + \lambda 
\end{cases}
\]

In order to show the equivalence between the norms \((2.5)\) and \((2.11)\) we introduce the homogeneous norm, that is,

\[
\|u\|_{s,p,\text{hom}}^p = \begin{cases} 
\sum_{|\alpha|=m} \|\partial^\alpha u\|_{L^p}^p, & s = m \\
\sum_{|\alpha|=m} \left( \sum_{|\alpha|=m} \int \int \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^p}{|x-y|^{n+\lambda p}} \right)^{1/p} dx dy, & s = m + \lambda 
\end{cases}
\]

Taibleson [36, Theorem 10], proved that the norm \(\|u\|_{s,p}\) is equivalent to \((\|u\|_{L^p}^p + \|u\|_{s,p,\text{hom}}^p)^{1/p}\) (see also [35, Ch. V, §4-5]), and by using this equivalence Triebel [38] proved that

\[
c_0\|u\|_{s,p,\delta}^p \leq \sum_{j=0}^{\infty} 2^{j\delta} \|\psi_j u\|_{L^p}^p + 2^{(\delta+s)pj} \|\psi_j u\|_{s,p,\text{hom}}^p \leq c_1 \|u\|_{s,p,\delta}^p.
\]

Moreover, he showed that the constants in the above equivalence depend only on \(s, \delta, p, \) the dimension and the constants \(C_\alpha\) of inequalities (2.4).

Taking into account the homogeneous properties, that is, \(\|\psi_j u\|_{L^p}^p = 2^{-jn}\|\psi_j u\|_{L^p}^p\) and \(\|\psi_j u\|_{s,p,\text{hom}}^p = 2^{-j(n-sp)}\|\psi_j u\|_{s,p,\text{hom}}^p\), and combining them with the equivalence (2.12), we obtain

\[
\|u\|_{s,p,\delta}^p \sim \sum_{j=0}^{\infty} 2^{j(\delta+p)\delta} \left( \|\psi_j u\|_{L^p}^p + \|\psi_j u\|_{s,p,\text{hom}}^p \right)
\]

\[
\sim \sum_{j=0}^{\infty} 2^{j(\delta+p)\delta} \|\psi_j u\|_{s,p}^p \sim \sum_{j=0}^{\infty} 2^{j(\delta+p)\delta} \|\psi_j u\|_{W^p_{s,p}}^p = \|u\|_{W^p_{s,p}}^p.
\]

This proves the following theorem of Triebel [38].

**Theorem 2.6** (Triebel, Equivalence of norms). Let \(s > 0, 1 \leq p < \infty\) and \(-\infty < \delta < \infty\). Then the norms \((2.5)\) and \((2.11)\) are equivalent. In particular, when \(s\) is a non–negative integer, then the norm \((2.5)\) is equivalent to the norm \((1.1)\).

**Remark 2.7.** Note that the Besov space \(B^0_{p,p}\) is continuously included in \(L^p\) for \(p \in [1, 2]\), and \(L^p\) is continuously included in \(B^0_{p,p}\) for \(p \in [2, \infty)\) (see [2, Theorem 2.41]). This phenomenon occurs also in the weighted spaces. Let \(L^p_\delta\) denote the Lebesgue space with
the weight \((1 + |x|)^\delta\), then it follows from the dyadic representation of the norm that \(W_{0,\delta}^p\) is continuously included in \(L_\delta^p\) for \(p \in [1, 2]\), and \(L_\delta^p\) is continuously included in \(W_{0,\delta}^p\) for \(p \in [2, \infty)\).

### 2.2. Some Properties of \(W_{s,\delta}^p(\mathbb{R}^n)\)-spaces

In this subsection we establish several useful tools for PDEs in these spaces, including embeddings, pointwise multiplications, fractional powers and Moser type estimates.

**Proposition 2.8.** If \(u \in W_{s,\delta}^p(\mathbb{R}^n)\), then

\[
\|\partial_i u\|_{W_{s-1,\delta+1}^p(\mathbb{R}^n)} \leq C\|u\|_{W_{s,\delta}^p(\mathbb{R}^n)},
\]

where the constant \(C\) depends on the constant of the equivalence of Theorem 2.6.

**Proof.** If \(s \geq 1\), then (2.11) implies \(\|\partial_i u\|_{s-1,p,\delta+1} \leq \|u\|_{s,p,\delta}\), so (2.13) follows from Theorem 2.6 in that case. For \(s \leq 0\), we have by the previous step

\[
|\langle \partial_i u, \varphi \rangle_W| = |\langle u, \partial_i \varphi \rangle_W| \leq \|u\|_{W_{s,\delta}^p(\mathbb{R}^n)} \|\partial_i \varphi\|_{W_{s-1,\delta}^p(\mathbb{R}^n)} \leq C\|u\|_{W_{s,\delta}^p(\mathbb{R}^n)} \|\varphi\|_{W_{s+1,\delta-1}^p(\mathbb{R}^n)}
\]

for all \(\varphi \in C_0^\infty(\mathbb{R}^n)\). Hence by (2.9), \(\|\partial_i u\|_{W_{s,\delta}^p(\mathbb{R}^n)} \leq C\|u\|_{W_{s+1,\delta-1}^p(\mathbb{R}^n)}\). For the remaining value of \(s\) we use interpolation in order to obtain (2.13). \(\square\)

**Proposition 2.9.** Let \(N\) be a nonnegative integer and assume \(\zeta\) is a smooth function such that

\[
|\partial^\alpha \zeta(x)| \leq C_N \text{ for all } |\alpha| \leq N \text{ and } x \in \mathbb{R}^n.
\]

If \(u \in W_{s,\delta}^p(\mathbb{R}^n)\), \(|s| < N\) and \(1 < p < \infty\), then

\[
\|\zeta u\|_{W_{s,\delta}^p(\mathbb{R}^n)} \leq C_N\|u\|_{W_{s,\delta}^p(\mathbb{R}^n)}.
\]

**Proof.** For such smooth function \(\|\zeta u\|_{W_{s}^p} \leq C_N\|u\|_{W_{s}^p}\) holds. This inequality can be proven by interpolation, for details see the Lemma in [39]. By (2.14), \(|\langle \partial^\alpha \zeta \rangle(2^j x)| \leq C_N\), and hence

\[
\|\zeta u\|_{W_{s,\delta}^{p,\infty}(\mathbb{R}^n)} = \sum_{j=0}^{\infty} 2^{(\delta + \frac{\alpha}{p})p_j} \left\|\left\langle \psi_{j} \zeta u \right\rangle_{(2^j)}\right\|_{W_{s}^{p}} \leq C_N^p \sum_{j=0}^{\infty} 2^{(\delta + \frac{\alpha}{p})p_j} \left\|\left\langle \psi_{j} u \right\rangle_{(2^j)}\right\|_{W_{s}^{p}} = C_N^p \|u\|_{W_{s,\delta}^{p,\infty}(\mathbb{R}^n)}
\]

\(\square\)

**Proposition 2.10.** Let \(\chi_R \in C^\infty(\mathbb{R}^n)\) be a cut-off function such that \(\chi_R(x) = 1\) for \(|x| \leq R\), \(\chi_R(x) = 0\) for \(|x| \geq 2R\) and \(|\partial^\alpha \chi_R| \leq c_\alpha R^{-|\alpha|}\). Then for \(\delta' < \delta\)

\[
\|(1 - \chi_R) u\|_{W_{s,\delta}^p(\mathbb{R}^n)} \lesssim R^{-(\delta - \delta')}\|u\|_{W_{s,\delta}^p(\mathbb{R}^n)}
\]

holds.
Proof. Let \( J_0 \) be the smallest integer such that \( R \leq 2^{J_0+1} \). Then \( (1 - \chi_R)\psi_j = 0 \) for \( j = 0, 1, ..., J_0 - 1 \), and hence

\[
\| (1 - \chi_R)u \|_{W^p_{s,\delta}(\mathbb{R}^n)}^p = \sum_{j=J_0}^{\infty} 2^{(\delta + \frac{n}{p})j} \| (\psi_j (1 - \chi_R)u)_{2j} \|_{W^p_{s,\delta}}^p \\
\leq \sum_{j=J_0}^{\infty} 2^{-(\delta - \delta')j} 2^{(\delta + \frac{n}{p})j} \| (\psi_j u)_{2j} \|_{W^p_{s,\delta}}^p \lesssim 2^{-(\delta - \delta')mJ_0} \sum_{j=J_0}^{\infty} 2^{(\delta + \frac{n}{p})j} \| (\psi_j u)_{2j} \|_{W^p_{s,\delta}}^p \\
\lesssim \left( R^{-(\delta - \delta')} \| u \|_{W^p_{s,\delta}(\mathbb{R}^n)} \right)^p.
\]

\( \square \)

The next proposition deals with embeddings. It concerns also the embedding into the weighted space of continuously differentiable functions, \( C^m_{\beta}(\mathbb{R}^n) \) where \( m \) is a nonnegative integer, \( \beta \in \mathbb{R} \) and which posses the following norm

\[
(2.15) \quad \| u \|_{C^m_{\beta}(\mathbb{R}^n)} = \sum_{|\alpha| \leq m} \sup_{\mathbb{R}^n} ((1 + |x|)^{\beta + |\alpha|} |\partial^\alpha u(x)|).
\]

**Proposition 2.11 (Embedding).**

(a) Let \( s_1 \leq s_2 \) and \( \delta_1 \leq \delta_2 \), then the inclusion \( i : W^{p}_{s_2,\delta_2}(\mathbb{R}^n) \to W^{p}_{s_1,\delta_1}(\mathbb{R}^n) \) is continuous.

(b) Let \( s_1 < s_2 \) and \( \delta_1 < \delta_2 \), then the embedding \( i : W^{p}_{s_2,\delta_2}(\mathbb{R}^n) \to W^{p}_{s_1,\delta_1}(\mathbb{R}^n) \) is compact.

(c) Let \( s > \frac{n}{p} + m \) and \( \delta + \frac{n}{p} \geq \beta \), then the embedding \( i : W^{p}_{s,\delta}(\mathbb{R}^n) \to C^m_{\beta}(\mathbb{R}^n) \) is continuous.

Proof. From the definitions of the norms (2.1) and (2.5), we see that they are increasing functions of both \( s \) and \( \delta \). Hence \( \| u \|_{W^{p}_{s,\delta}(\mathbb{R}^n)} \leq \| u \|_{W^{p}_{s,\delta}(\mathbb{R}^n)} \) and this proves (a). To prove (b), let \( N \) be a positive integer and set \( i_N(u) = \sum_{j=0}^{N} (\psi_j u)_{2j} \). Since \( i_N(u) \) has support in \( \{|x| \leq 2^{N+2}\} \) and \( s_1 < s_2 \), \( i_N : W^{p}_{s_2} \to W^{p}_{s_1} \) is a compact operator (see e.g. [19, §3.3.2]). In addition, by Proposition 2.10 we have

\[
\| i_N(u) - i(u) \|_{W^{p}_{s_1,\delta_1}(\mathbb{R}^n)} \lesssim 2^{-N(\delta_2 - \delta_1)} \| u \|_{W^{p}_{s_2,\delta_2}(\mathbb{R}^n)}.
\]

Thus the embedding \( i \) is a norm limit of compact operators, hence it is itself compact (see e.g. [33, Theorem 4.11]).

\[\text{In this Theorem the authors estimate the entropy numbers of the embedding and show that it tends to zero. This, however, implies the compactness.}\]
We turn now to (c). Assume first that $m = 0$ and $s > \frac{2}{p}$, then $\sup_{\mathbb{R}^n} |u(x)| \lesssim \|u\|_{W^p_s}$ (see e. g. [37, §32]). Applying it term-wise to the norm (2.5), we have

$$
\sup_{\mathbb{R}^n} (1 + |x|)^\beta |u(x)| \leq 2^\beta \sup_{j \geq 0} \left( 2^{\beta j} \sup_{\mathbb{R}^n} |\psi_j(x)u(x)| \right)
$$

(2.16)

$$
\leq 2^\beta \sup_{j \geq 0} \left( 2^{\beta j} \sup_{\mathbb{R}^n} |\psi_{2^j}(x)u(2^j x)| \right) \lesssim 2^\beta \sup_{j \geq 0} \left( 2^{\beta j} \|\psi_j u\|_{W^p_s} \right)
$$

$$
\leq 2^\beta \sup_{j \geq 0} \left( 2^{(\delta + s/2)j} \|\psi_j u\|_{W^p_s} \right) \lesssim 2^\beta \|u\|_{W^p_{s,\delta}(\mathbb{R}^n)}.
$$

If $m \geq 1$ and $|\alpha| \leq m$, then $\partial^\alpha u \in W^p_{s-|\alpha|,\delta+|\alpha|}(\mathbb{R}^n)$ by Proposition 2.8. So applying (2.16) to $\partial^\alpha u$ with $\delta' = \delta + |\alpha|$ and $\beta' = \beta + |\alpha|$, we obtain $\|\partial^\alpha u\|_{C^0_{\beta+|\alpha|}(\mathbb{R}^n)} \leq C \|\partial^\alpha u\|_{W^p_{s-|\alpha|,\delta+|\alpha|}(\mathbb{R}^n)}$.

For further applications we discuss the construction of the sequence $\{\psi_j\}$ that appears in Definition 2.2. Let $h$ be a $C^\infty(\mathbb{R})$ function such that $h(t) = -1$ for $t \leq \frac{1}{4}$, $h(t) = 0$ for $1/2 \leq t \leq 1$ and $h(t) = 1$ for $2 \leq t$. Let

$$
g(t) = \begin{cases} 
\frac{e^{-(1-t)^2}}{\sqrt{\pi}}, & |t| < 1 \\
0, & |t| \geq 1
\end{cases}
$$

(2.17)

Then the functions $\psi_j(x) = g(h(2^{-j}|x|))$ satisfy the requirements of the dyadic resolution above, Definition 2.2. Moreover, for any positive $\gamma$, $\psi_j^\gamma(x) = g^\gamma(h(2^{-j}|x|))$ and from (2.17) we see that there are two constants $C_1(\gamma, \alpha)$ and $C_2(\gamma, \alpha)$ such that

$$
C_1(\gamma, \alpha) |\partial^\alpha \psi_j(x)| \leq |\partial^\alpha \psi_j^\gamma(x)| \leq C_2(\gamma, \alpha) |\partial^\alpha \psi_j(x)|
$$

for any multi–index $\alpha$, and these inequalities are independent of $j$. Therefore the family $\{\psi_j^\gamma\}$ satisfies condition (2.4) and hence by Theorem 2.3 (a) we obtain:

**Proposition 2.12.** Let $\gamma$ be positive number, then

$$
\|u\|_{W^p_{s,\delta}(\mathbb{R}^n)} \simeq \sum_{j=0}^{\infty} 2^{(\delta + \frac{s}{p})j} \|\psi_j^\gamma u\|_{W^p_s}.
$$

Using Proposition 2.12 we establish multiplication and fractional power properties of the weighted Besov spaces.

**Proposition 2.13.** Assume $s \leq \min\{s_1, s_2\}$, $s_1 + s_2 > s + \frac{n}{p}$, $s_1 + s_2 \geq n \cdot \max\{0, (\frac{2}{p} - 1)\}$ and $\delta \leq \delta_1 + \delta_2 + \frac{n}{p}$, then the multiplication

$$
W^p_{s_1,\delta_1}(\mathbb{R}^n) \times W^p_{s_2,\delta_2}(\mathbb{R}^n) \to W^p_{s,\delta}(\mathbb{R}^n)
$$

is continuous.
and the Cauchy–Schwarz inequality we have

\[ \| (\psi^2 u v)_{2j} \|_{W_p^s} \lesssim \| (\psi u)_{2j} \|_{W_p^{s_1}} \| (\psi v)_{2j} \|_{W_p^{s_2}}. \]

For the proof of these types of results see [32, §4.6.1]. Set \( a_j = \| (\psi u)_{2j} \|_{W_p^{s_1}} \) and \( b_j = \| (\psi v)_{2j} \|_{W_p^{s_2}} \), then by Proposition 2.12 and the Cauchy–Schwarz inequality we have

\[ \| uv \|_{W_p^{s_1, \delta_1}(\mathbb{R}^n)} \lesssim \sum_{j=0}^{\infty} 2^{(\delta_1 + \frac{n}{p})p j} \left( (\psi^2 u v)_{2j} \right)_{j}^\frac{1}{p} \lesssim \sum_{j=0}^{\infty} 2^{(\delta_1 + \delta_2 + \frac{n}{p})p j} a_j b_j. \]

\[ \lesssim \left( \sum_{j=0}^{\infty} 2^{(\delta_1 + \frac{n}{p})p j} a_j \right)^{\frac{1}{p}} \left( \sum_{j=0}^{\infty} 2^{(\delta_2 + \frac{n}{p})p j} b_j \right)^{\frac{1}{p}} \lesssim \| u \|_{W_p^{s_1, \delta_1}(\mathbb{R}^n)} \| v \|_{W_p^{s_2, \delta_2}(\mathbb{R}^n)}. \]

\[ \square \]

**Corollary 2.14.** Let \( s > \frac{n}{p} \) and \( \delta \geq -\frac{n}{p} \), then the space \( W_p^{s, \delta} \) is an algebra.

Proposition 2.13 can be extended to a multiplication of three functions and with relaxed conditions on the \( \delta \)'s.

**Proposition 2.15.** Assume \( s \leq \min\{s_1, s_2\} \), \( s_1 + s_2 > s + \frac{n}{p} \), \( s_1 + s_2 \geq n \cdot \max\{0, (\frac{2}{p} - 1)\} \) and \( \delta \leq \delta_1 + \delta_2 + \delta_3 + \frac{2n}{p} \), then the multiplication

\[ W_p^{s_1, \delta_1} \times W_p^{s_2, \delta_2} \times W_p^{s_3, \delta_3} \to W_p^{s, \delta} \]

is continuous.

**Proof.** Similar to the above proof, by the multiplication properties in the Besov spaces, we have

\[ \| (\psi^3 u v w)_{2j} \|_{W_p^s} \lesssim \| (\psi u)_{2j} \|_{W_p^{s_1}} \| (\psi v)_{2j} \|_{W_p^{s_2}} \| (\psi w)_{2j} \|_{W_p^{s_3}}. \]

Let \( a_j \) and \( b_j \) be as in the previous proof and let \( c_j = \| (\psi w)_{2j} \|_{W_p^{s_3}} \). Replacing the Cauchy–Schwarz inequality by the Hölder inequality, we get that

\[ \| uvw \|_{W_p^{s_1, \delta_1}(\mathbb{R}^n)}^p \lesssim \left( \sum_{j=0}^{\infty} 2^{(\delta_1 + \frac{n}{p})p j} a_j \right)^{\frac{1}{p}} \left( \sum_{j=0}^{\infty} 2^{(\delta_2 + \frac{n}{p})p j} b_j \right)^{\frac{1}{p}} \left( \sum_{j=0}^{\infty} 2^{(\delta_3 + \frac{n}{p})p j} c_j \right)^{\frac{1}{p}} \lesssim \| u \|_{W_p^{s_1, \delta_1}(\mathbb{R}^n)} \| v \|_{W_p^{s_2, \delta_2}(\mathbb{R}^n)} \| w \|_{W_p^{s_3, \delta_3}(\mathbb{R}^n)}. \]

\[ \square \]
Proposition 2.16. Let \( u \in W^p_{s,\delta} \cap L^\infty, 1 \leq \beta, 0 < s < \beta + \frac{1}{p} \) and \( \delta \in \mathbb{R} \), then

\[
\| |u|^\beta \|_{W^p_{s,\delta}(\mathbb{R}^n)} \leq C(\| u \|_{L^\infty}) \| u \|_{W^p_{s,\delta}(\mathbb{R}^n)}.
\]

Proof. The unweighted inequality,

\[
(2.19) \quad \| |u|^\beta \|_{W^p_{s,\delta}(\mathbb{R}^n)} \leq C(\| u \|_{L^\infty}) \| u \|_{W^p_{s,\delta}(\mathbb{R}^n)}.
\]

was proven by Bourdaud and Meyer [5] for \( \beta = 1 \) and by Kateb [24] for \( 1 < \beta \). Applying (2.19) term-wise to the equivalent norm (2.18), we get

\[
\| |u|^\beta \|_{W^p_{s,\delta}(\mathbb{R}^n)} \cap L^\infty(\mathbb{R}^n) = 2(\delta + \frac{1}{p})\| (|\psi_j|^\beta |u|^\beta) \|_{W^p_{s,\delta}(\mathbb{R}^n)}^2 \leq (C(\| u \|_{L^\infty}))^p \| u \|_{W^p_{s,\delta}(\mathbb{R}^n)}^p.
\]

Proposition 2.17. Let \( F : \mathbb{R}^m \to \mathbb{R}^l \) be a \( C^{N+1} \) function such that \( F(0) = 0 \) and \( 0 < s \leq N \). Then there exists a positive constant \( C \) such that

\[
(2.20) \quad \| F(u) \|_{W^p_{s,\delta}(\mathbb{R}^n)} \leq C \| F \|_{C^{N+1}} (1 + \| u \|_{L^\infty}) \| u \|_{W^p_{s,\delta}(\mathbb{R}^n)}
\]

for any \( u \in W^p_{s,\delta}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \). In particular, if \( s > \frac{n}{p} \) and \( \delta \geq -\frac{n}{p} \), then

\[
(2.21) \quad \| F(u) \|_{W^p_{s,\delta}(\mathbb{R}^n)} \leq C \| u \|_{W^p_{s,\delta}(\mathbb{R}^n)}.
\]

Proof. The Moser type estimate

\[
(2.22) \quad \| F(u) \|_{W^p_{s,\delta}(\mathbb{R}^n)} \leq C \| F \|_{C^{N+1}} (1 + \| u \|_{L^\infty}) \| u \|_{W^p_{s,\delta}(\mathbb{R}^n)}
\]

in the Besov spaces was proven in [32, §5.3.4]. Let \( \{\psi_j\} \) be the dyadic resolution of unity used in the definition of the norm (2.5) and set \( \Psi_j(x) = (\varphi(x))^{-1}\psi_j(x) \), where \( \varphi(x) = \sum_{j=0}^\infty \psi_j(x) \). Then the sequence \( \{\Psi_j\} \subset C_0^\infty(\mathbb{R}^n) \), satisfies (2.4) and \( \sum_{j=0}^\infty \Psi_j(x) = 1 \). Since \( F(0) = 0 \), we obtain

\[
(2.23) \quad (\psi_j F(u))_{2j} = \left( \psi_j F \left( \sum_{k=0}^{\infty} \Psi_k u \right) \right)_{2j} = \left( \psi_j F \left( \sum_{k=j-2}^{j+1} \Psi_k u \right) \right)_{2j},
\]

for each \( j \). Here we use the convention that a summation starts from zero whenever \( k < 0 \). By the Moser type estimate (2.22), we have

\[
(2.24) \quad \left\| \psi_j F \left( \sum_{k=j-2}^{j+1} \Psi_k u \right) \right\|_{W^p_{s,\delta}(\mathbb{R}^n)} \leq C \left\| \left( \sum_{k=j-2}^{j+1} \Psi_k u \right) \right\|_{W^p_{s,\delta}(\mathbb{R}^n)} \leq C \sum_{k=j-2}^{j+1} \| (\Psi_k u)_{2j} \|_{W^p_{s,\delta}(\mathbb{R}^n)}.
\]
where $C = C(∥F∥_{C^{N+1}}, ∥u∥_{L^∞})$. Taking into account the known scaling properties of the Besov’s norm and the form of $Ψ_k$, we get that
\begin{equation}
(2.25) \quad ∥(Ψ_k u)_{2j}∥_{W^p_s} = ∥((Ψ_k u)_{2^{j-k}})_{2j-k}∥_{W^p_s} \lesssim 2^{(k-j)n/p2^s} ∥(Ψ_k u)_{2^k}∥_{W^p_s}.
\end{equation}
Combining (2.23), (2.24) with inequality $∥(Ψ_k u)_{2^k}∥_{W^p_s} \leq C ∥(Ψ_k u)_{2^k}∥_{W^p_s}$, we obtain that
\begin{align*}
∥F(u)∥_{W^p_s(R^n)}^p &= \sum_{j=0}^{∞} 2^{(δ+\frac{n}{p})pj} ∥((ψ_j F(u))_{2j})∥_{W^p_s}^p \\
&\leq \left(C(∥F∥_{C^{N+1}}, ∥u∥_{L^∞}^N))^p \sum_{j=0}^{∞} 2^{(δ+\frac{n}{p})pj} \sum_{k=j-2}^{j+1} 2^{(k-j)\frac{n}{p}} ∥(ψ_k u)_{2^k}∥_{W^p_s}^p \\
&\leq 4 \left(C(∥F∥_{C^{N+1}}, ∥u∥_{L^∞}^N))^p \sum_{k=0}^{∞} 2^{(δ+\frac{n}{p})pj} ∥(ψ_k u)_{2^k}∥_{W^p_s}^p \\
&= 4 \left(C(∥F∥_{C^{N+1}}, ∥u∥_{L^∞}^N))^p ∥u∥_{W^p_{s,δ}(R^n)}^p.
\end{align*}
When $s > \frac{n}{p}$ and $δ ≥ \frac{n}{p}$, then (2.21) follows from Proposition 2.11(c).

\[\square\]

3. Linear Elliptic Systems on Asymptotically Flat Riemannian Manifolds

In this section we study second order linear elliptic systems whose coefficients are in the weighted Besov spaces. We emphasize the study of operators with the Laplace Beltrami operator of an asymptotically flat Riemannian manifold as the principal part. The range of $δ$ is restricted to the interval $(-\frac{n}{p}, -2 + \frac{n}{p})$, since for these values of $δ$ the Laplace operator is an isomorphism between $W^p_{s,δ}(R^n)$ and $W^p_{s-2,δ+2}(R^n)$.

3.1. Linear elliptic operators in $R^n$. We consider second order linear elliptic systems of the form
\begin{equation}
(3.1) \quad (Lu)^i = (a_2)^{ab}_{ij} \partial_a \partial_b u^j + (a_1)^{a}_{ij} \partial_a u^j + (a_0)_ij u^j,
\end{equation}
where $a_k$ are $N × N$ block matrices. The operator $L$ is elliptic in $R^n$ if
\begin{equation}
(3.2) \quad \det \left( (a_2)^{ab}_{ij}(x)ξ_a ξ_b \right) \neq 0 \quad \text{for all } ξ ∈ R^n \setminus \{0\} \quad \text{and } x ∈ R^n.
\end{equation}
Let $A_∞$ be a matrix with constant coefficients, the symbol $A_∞$ stands also for a second order differential operator of the form
\begin{equation}
(3.3) \quad (A_∞ u)^i = (A_∞)^{ab}_{ij} \partial_a \partial_b u^j.
\end{equation}
We assume $\det \left( (A_∞)^{ab}_{ij} ξ_a ξ_b \right) \neq 0$, hence $A_∞$ is an elliptic operator.

Definition 3.1. We say that operator $L$ belongs $\text{Asy}(A_∞, s, δ, p)$ if
\begin{equation}
(3.4) \quad a_2 - A_∞ ∈ W^p_{s,δ}(R^n), \quad a_1 ∈ W^p_{s-1,δ+1}(R^n) \quad \text{and} \quad a_0 ∈ W^p_{s-2,δ+2}(R^n).
\end{equation}
The following Corollary is a consequence of Propositions 2.8 and 2.13.

**Corollary 3.2.** Let \( L \in (\text{Asy}A_\infty, s, \delta, p) \), \( s \in (\frac{n}{p}, \infty) \cap [1, \infty) \), \( \delta \in [-\frac{n}{p}, \infty) \) and \( p \in (1, \infty) \), then

\[
L : W^{p}_{s, \delta}(\mathbb{R}^n) \to W^{p}_{s-2, \delta+2}(\mathbb{R}^n)
\]

is a bounded operator.

**Lemma 3.3.** Let \( \delta \in (-\frac{n}{p}, -2 + \frac{n}{p'}) \) and \( p \in (1, \infty) \), \( s \in \mathbb{R} \), then the operator

\[
A^{\infty} : W^{p}_{s, \delta}(\mathbb{R}^n) \to W^{p}_{s-2, \delta+2}(\mathbb{R}^n)
\]

is an isomorphism.

**Proof.** (of Lemma 3.3) For an integer \( s \) that is greater or equal two the isomorphism of system (3.5) was proven by Lockhart and McOwen [26, Theorem 3]². Hence, by interpolation, Theorem 2.3(d), it is an isomorphism for all \( s \geq 2 \).

For \( s \leq 2 \) we shall consider the adjoint operator. Let \( u \) and \( \varphi \) be two smooth functions, then \((A_\infty)^{ib}_j\partial_{a} u, \varphi\)\(_{L^2(\mathbb{R}^n)}\) = \( \langle u, (A_\infty)^{ib}_j\partial_{a} \varphi\rangle_{L^2(\mathbb{R}^n)} \), and by Proposition 2.4 and approximation, \( \langle (A_\infty)^{ib}_j\partial_{a} u, \varphi\rangle_{W} = \langle u, (A_\infty)^{ib}_j\partial_{a} \varphi\rangle_{W} \). Thus letting \( A^{\infty}_s \) denote the adjoint matrix of \( A_\infty \), we conclude from Theorem 2.3(c) that

\[
A^{\infty}_s : W^{p}_{s'+2, -\delta-2}(\mathbb{R}^n) \to W^{p}_{s', -\delta}(\mathbb{R}^n)
\]

is the adjoint operator of (3.5). Note that \( -\frac{n}{p} < -\delta - 2 < -2 + \frac{n}{p} \), so the previous part implies that (3.6) is an isomorphism for \( s \leq 0 \). Since the adjoint of an isomorphism is also an isomorphism (see e.g. [33, Theorem 5.15]), we conclude that (3.5) is an isomorphism for all negative integers, and by interpolation for all \( s \).

In order to prove a priori estimates for \( L \in (A_\infty, s, \delta, p) \) we need the corresponding result in the unweighted Besov spaces. The following Lemma was proven in [23, Lemma 32].

**Lemma 3.4** (Holst, Nagy and Tsogtgerel). Assume that the coefficients of \( L \) satisfy the conditions: \( a_i \in W^{p}_{s-i} \) for \( i = 0, 1, 2 \), \( s \in (\frac{n}{p}, \infty) \cap [1, \infty) \) and \( p \in (1, \infty) \), and (3.2). Then for all \( u \in W^{p}_{s} \) with support in a compact set \( K \), there exists a constant \( C \) that depends on \( K \) and the \( W^{p}_{s-i} \)-norms of the coefficients \( a_i \) such that

\[
\|u\|_{W^{p}_{s}} \leq C \left\{ \|Lu\|_{W^{p}_{s-2}} + \|u\|_{W^{p}_{s-1}} \right\}.
\]

**Lemma 3.5.** Let \( L \in \text{Asy}(A_\infty, s, \delta, p) \), \( s \in (\frac{n}{p}, \infty) \cap [1, \infty) \), \( \delta \in (-\frac{n}{p}, -2 + \frac{n}{p'}) \), \( p \in (1, \infty) \) and \( \delta' < \delta \). Then for any \( u \in W^{p}_{s, \delta}(\mathbb{R}^n) \),

\[
\|u\|_{W^{p}_{s, \delta}(\mathbb{R}^n)} \leq C \left\{ \|Lu\|_{W^{p}_{s-2, \delta+2}(\mathbb{R}^n)} + \|u\|_{W^{p}_{s-1, \delta'}(\mathbb{R}^n)} \right\},
\]

² Lockhart and McOwen considered the Douglas Nirenberg elliptic system, which is an extension of the common elliptic systems (3.1). The ellipticity of this system is expressed by means of two \( N \) tuples \( (s_1, \ldots, s_N) \) and \( (t_1, \ldots, t_N) \) of nonnegative integers. Setting \( s_i = m \) and \( t_i = m + 2 \) for \( i = 1, \ldots, N \), reduces the constant coefficients Douglas Nirenberg system of [26] to (3.3).
where the constant $C$ depends on $W^p_{s,\delta}$-norms of the coefficients of $L$, $s, \delta, p$ and $\delta'$.

A continuous linear operator $L : E \to F$, where $E$ and $F$ are Banach spaces, is called Semi-Fredholm if its kernel is finite dimensional and it has a closed range. This is equivalent to the inequality
\[ \|u\|_E \leq C \{\|Lu\|_F + |u|\}, \]
where the norm $|\cdot|$ is compact relative to the norm $\|\cdot\|_E$ (see e.g. [33, Theorem 6.2]).

Thus as a consequence of the estimates (3.8) and the compact embedding Proposition 2.11(b), we obtain:

**Corollary 3.6** (Semi-Fredholm). *Let us assume that the conditions of Lemma 3.5 hold, then $L : W^p_{s,\delta}(\mathbb{R}^n) \to W^p_{s-2,\delta+2}(\mathbb{R}^n)$ is a semi-Fredholm operator.*

**Proof of Lemma 3.5.** Let $\chi_\rho$ be a cut-off function such that $\text{supp}(\chi_\rho) \subset B_{2\rho}$, $\chi_\rho(x) = 1$ on $B_\rho$, and $|\partial^\alpha \chi_\rho| \leq C_\rho \rho^{-|\alpha|}$. Here $B_\rho$ denotes a ball of radius $\rho$. We decompose $u = (1 - \chi_\rho)u + \chi_\rho u$ and estimate each term separately. Lemma 3.3 implies that $A_\infty$ is an isomorphism, hence there is a constant $C$ such that
\[ \| (1 - \chi_\rho)u \|_{W^p_{s,\delta}(\mathbb{R}^n)} \leq C \|A_\infty ((1 - \chi_\rho)u)\|_{W^p_{s-2,\delta+2}(\mathbb{R}^n)}. \]
Let $[A_\infty, (1 - \chi_\rho)]$ denote a commutation, that is,
\[ (A_\infty, (1 - \chi_\rho))u = -(A_\infty)_{ij}^a \partial_i \partial_j \chi_\rho u^a + 2\partial_i \partial_j \chi_\rho \partial^a u. \]
Then
\[ A_\infty ((1 - \chi_\rho)u) = [A_\infty, (1 - \chi_\rho)]u + (1 - \chi_\rho)L(u) - (1 - \chi_\rho)(L - A_\infty)u. \]
The coefficients of the commutator (3.10) have compact support, hence we may replace $\delta + 2$ in its $W^p_{s-2,\delta+2}(\mathbb{R}^n)$-norm by any other $\delta'$, and that will result in an equivalent norm. So by Propositions 2.8 and 2.9 we obtain that
\[ \|[A_\infty, (1 - \chi_\rho)]u\|_{W^p_{s-2,\delta+2}(\mathbb{R}^n)} \leq C_1(\rho) \left\{\|u\|_{W^p_{s-2,\delta'}(\mathbb{R}^n)} + \|Du\|_{W^p_{s-2,\delta'+1}(\mathbb{R}^n)}\right\} \]
\[ \leq C_1(\rho) \|u\|_{W^p_{s-1,\delta'}(\mathbb{R}^n)}. \]
Letting $\delta_1 = -\frac{2}{p}$ and $\delta_2 = \delta + 2$ allow us to apply Proposition 2.13 and with a combination of Proposition 2.8, we get that
\[ \|(1 - \chi_\rho)(A_\infty - a_2)D^2 u\|_{W^p_{s-2,\delta+2}(\mathbb{R}^n)} \lesssim \|(1 - \chi_\rho)(A_\infty - a_2)\|_{W^p_{s,\delta_1}(\mathbb{R}^n)} \|D^2 u\|_{W^p_{s-2,\delta+2}(\mathbb{R}^n)}, \]
\[ \lesssim \|(1 - \chi_\rho)(A_\infty - a_2)\|_{W^p_{s,\delta_1}(\mathbb{R}^n)} \|u\|_{W^p_{s,\delta}(\mathbb{R}^n)}. \]
Since $\delta > \delta_1 = -\frac{2}{p}$, we can apply Proposition 2.10 and obtain that
\[ \|(1 - \chi_\rho)(A_\infty - a_2)\|_{W^p_{s,\delta_1}(\mathbb{R}^n)} \lesssim \rho^{-(\delta - \frac{2}{p})} \|(A_\infty - a_2)\|_{W^p_{s,\delta}(\mathbb{R}^n)}. \]
Repeating similar arguments with the other terms, we conclude that
\[ \|(1 - \chi_\rho)(L - A_\infty)u\|_{W^p_{s-2,\delta+2}(\mathbb{R}^n)} \leq \rho^{-(\delta - \frac{2}{p})} \Lambda \|u\|_{W^p_{s,\delta}(\mathbb{R}^n)}, \]
where
\[ \Lambda \simeq \| (a_2 - A_\infty) \|_{W^p_{s,\delta} (\mathbb{R}^n)} + \| a_1 \|_{W^p_{s-1,\delta+1} (\mathbb{R}^n)} + \| a_0 \|_{W^p_{s-2,\delta+2} (\mathbb{R}^n)} . \]
Thus from inequalities (3.9), (3.12) and (3.13), and the identity (3.11), we obtain that
\[ \begin{equation}
(1 - \chi_\rho) u \|_{W^p_{s,\delta} (\mathbb{R}^n)} \leq C \| Lu \|_{W^p_{s-2,\delta+2} (\mathbb{R}^n)} + C_1 (\rho) \| u \|_{W^p_{s-1,\delta} (\mathbb{R}^n)} \\
+ \rho^{-(\delta - \frac{n}{p})} \| u \|_{W^p_{s,\delta} (\mathbb{R}^n)} .
\end{equation} \tag{3.14} \]

We turn now to the second term. Since \( \chi_\rho u \) has compact support, \( \| \chi_\rho u \|_{W^p_{s,\delta} (\mathbb{R}^n)} \simeq \| \chi_\rho u \|_{W^p_{s,\delta} (\mathbb{R}^n)} \), so by Lemma 3.4,
\[ \| \chi_\rho u \|_{W^p_{s,\delta} (\mathbb{R}^n)} \simeq \| \chi_\rho u \|_{W^p_{s,\delta} (\mathbb{R}^n)} \leq C \left\{ \| L(\chi_\rho u) \|_{W^p_{s-2}} + \| \chi_\rho u \|_{W^p_{s-1}} \right\}. \tag{3.15} \]

Now \( L(\chi_\rho u) = \chi_\rho Lu + [L, \chi_\rho] u \), where the commutator \([L, \chi_\rho] \) is an operator of order one and with coefficients with compact support in \( B_{2\rho} \). Thus as in the estimate of the commutator (3.10), similar arguments provide that
\[ \| L(\chi_\rho u) \|_{W^p_{s-2}} \leq \| \chi_\rho (Lu) \|_{W^p_{s-2}} + \| [L, \chi_\rho] u \|_{W^p_{s-2}} \leq C_2 (\rho) \left\{ \| Lu \|_{W^p_{s-2,\delta+2} (\mathbb{R}^n)} + \| u \|_{W^p_{s-1,\delta} (\mathbb{R}^n)} \right\}. \tag{3.16} \]

Combining inequalities (3.14), (3.15) and (3.16) yields
\[ \| u \|_{W^p_{s,\delta} (\mathbb{R}^n)} \leq \left\{ C + C_2 (\rho) \right\} \| Lu \|_{W^p_{s-2,\delta+2} (\mathbb{R}^n)} + \left\{ C_1 (\rho) + C_2 (\rho) \right\} \| u \|_{W^p_{s-1,\delta} (\mathbb{R}^n)} \\
+ \rho^{-(\delta - \frac{n}{p})} \| u \|_{W^p_{s,\delta} (\mathbb{R}^n)}. \]

Thus choosing \( \rho \) sufficiently large such that \( \rho^{-(\delta - \frac{n}{p})} \Lambda \leq \frac{1}{2} \) completes the proof. \( \square \)

The next proposition asserts that solutions to the homogeneous equation have a lower growth at infinity.

**Proposition 3.7.** Assume \( L \in \text{Asy}(A_\infty, s, \delta, p) \) \( s \in (\frac{n}{p}, \infty) \cap [1, \infty) \) and \( \delta \in (-\frac{n}{p}, -2 + \frac{n}{p}) \). If \( Lu = 0 \), then \( u \in W^p_{s,\delta} (\mathbb{R}^n) \) for any \( \delta' \in (-\frac{n}{p}, -2 + \frac{n}{p}) \).

**Proof.** We follow the idea of Christodoulou and O’Murchadha [17]. By Proposition 2.11 (a) it suffices to prove the statement for \( \delta' > \delta \). Let
\[ f = (L - A_\infty) u. \]
At the first stage we chose \( \delta' > \delta \) so that \( \frac{n}{p} + \delta + (\delta + 2) \geq \delta' + 2 \). Then by Proposition 2.13 we obtain that
\[ \| f \|_{W^p_{s-2,\delta'+2} (\mathbb{R}^n)} \lesssim \left( \| a_2 - A_\infty \|_{W^p_{s,\delta} (\mathbb{R}^n)} + \| a_1 \|_{W^p_{s-1,\delta+1} (\mathbb{R}^n)} + \| a_0 \|_{W^p_{s-2,\delta+2} (\mathbb{R}^n)} \right) \| u \|_{W^p_{s,\delta} (\mathbb{R}^n)}. \]
Since \( Lu = 0 \), \( A_\infty u = f \), so by Lemma 3.3 we obtain that \( \| u \|_{W^p_{s,\delta} (\mathbb{R}^n)} \lesssim \| f \|_{W^p_{s-2,\delta'+2} (\mathbb{R}^n)}. \)
We now may repeat this procedure with \( \delta' \) replacing \( \delta \) and \( \delta'' \) replacing \( \delta' \), which can be done iteratively until \( \delta'' = -2 + \frac{n}{p} \). \( \square \)
3.2. Asymptotically flat manifold. A Riemannian manifold \((M, g)\) is asymptotically flat if the Riemannian metric tends to the Euclidean metric at infinity. In case the Riemannian metric \(g\) is smooth, then is often required that the convergence to the Euclidean metric has a certain decay rate at infinity, as for example in [25]. Our definition follows essentially the one of Bartnik [3].

**Definition 3.8.** Let \(M\) be \(n\) dimensional smooth connected manifold and let \(g\) be a metric on \(M\) such that \((M, g)\) is complete. We say that \((M, g)\) is **asymptotically flat** of the class \(W^{p}_{s,δ}\), if \(g \in W^{p}_{s,loc}(M)\) and there is a compact set \(K \subset M\) such that:

1. There is a finite collection of charts \(\{(U_{i}, \phi_{i})\}_{i=1}^{N}\) which covers \(M \setminus K\);
2. For each \(i\), \(\phi_{i}^{-1}(U_{i}) = E_{r_{i}} := \{x \in \mathbb{R}^{n} : |x| > r_{i}\}\) for some positive \(r_{i}\);
3. The pull-back \((\phi_{i}^{*}g)\) is uniformly equivalent to the Euclidean metric \(e\) on \(E_{r_{i}}\);
4. For each \(i\), \((\phi_{i}^{*}g)_{ab} - \delta_{ab} \in W^{p}_{s,δ}(E_{r_{i}})\), where \(\delta_{ab}\) the is the Kronecker symbol.

The weighted Sobolev space on \(M\) is denoted by \(W^{p}_{s,δ}(M)\) and its norm is defined as follows. Let \((V_{j}, \Theta_{j})\) be collections of charts which cover \(K\) and where \(\Theta_{j}\) is a diffeomorphism between a ball \(B_{j}\) in \(\mathbb{R}^{n}\) and \(V_{j} \subset K\). Let \(\{\chi_{i}, \alpha_{j}\}\) be a partition of unity subordinate to \(\{U_{i}, V_{j}\}\), then

\[
\|u\|_{W^{p}_{s,δ}(M)} := \sum_{i=1}^{N} \|\phi_{i}^{*}(\chi_{i}u)\|_{W^{p}_{s,δ}(\mathbb{R}^{n})} + \sum_{j=1}^{N_{0}} \|\Theta_{j}^{*}(\alpha_{j}u)\|_{W^{p}_{s,δ}(\mathbb{R}^{n})}
\]

is a norm of the weighted Besov space \(W^{p}_{s,δ}(M)\). Note that the norm (3.17) depends on the partition of unity, but different partitions result in equivalent norms. A bilinear form on \(W^{p}_{s,δ}(M) \otimes W^{p'}_{-s,-δ}(M)\) is defined in a similar way:

\[
\langle u, \varphi \rangle_{M} = \sum_{i=1}^{N} \langle \phi_{i}^{*}(\chi_{i}u), \phi_{i}^{*}(\chi_{i}\varphi) \rangle_{W} + \sum_{j=1}^{N_{0}} \langle \Theta_{j}^{*}(\alpha_{j}u), \Theta_{j}^{*}(\alpha_{j}\varphi) \rangle_{W},
\]

where \(\langle \cdot, \cdot \rangle\) is a bilinear form (2.2) and \(\langle \cdot, \cdot \rangle_{W}\) is the bilinear form (2.6). Using (2.2) and inequality (2.7), and combining these with an elementary inequality and the norm 3.17, we see that

\[
|\langle u, \varphi \rangle_{M}| \leq C \|u\|_{W^{p}_{s,δ}(M)} \|\varphi\|_{W^{p'}_{-s,-δ}(M)}.
\]

The form (3.18) depends on the partition of unity, but each one induces a topological dual \((W^{p}_{s,δ}(M))^{*}\) isomorphic to \(W^{p'}_{-s,-δ}(M)\) (see Appendix §7).

The norm of the spaces \(C^{m}_{β}(M)\) is defined in an analogous manner, (3.17), that is,

\[
\|u\|_{C^{m}_{β}(M)} := \sum_{i=1}^{N} \|\phi_{i}^{*}(\chi_{i}u)\|_{C^{m}_{β}(\mathbb{R}^{n})} + \sum_{j=1}^{N_{0}} \|\Theta_{j}^{*}(\alpha_{j}u)\|_{C^{m}(\mathbb{R}^{n})}.
\]
It follows from Condition 3 of Definition 3.8 that both types of the norms do not depend on the metric. This fact is a known property of the ordinary Sobolev spaces on compact Riemannian manifolds (see [22, Proposition 2.3]).

Let $E$ and $F$ be two smooth vector bundles over a Riemannian manifold $(\mathcal{M}, g)$. A second order linear differential operator $L$ on $\mathcal{M}$ is a linear map from $C^\infty(E)$ to $C^\infty(F)$ that can be written in local coordinates in the form

$$
(Lu)^i = (a_2)_{ij}^a \nabla_a \nabla_b u^j + (a_1)_{ij}^a \nabla_a u^j + (a_0)_{ij} u^j,
$$

where $\nabla$ is the covariant derivative and the coefficients $a_k$ are tensors. The operator is elliptic at $x \in \mathcal{M}$, if $\det ((a_2(x))_{ij}^a) \neq 0$ for all covectors $\xi \neq 0$.

The operator $L$ belongs to $\text{Asy}(A_\infty, s, \delta, p)$ on $(\mathcal{M}, g)$ asymptotically flat manifold if $(a_2 - A_\infty) \in W^{p}_{s,\delta}(\mathcal{M})$, $a_1 \in W^{p}_{s-1,\delta+1}(\mathcal{M})$ and $a_0 \in W^{p}_{s-2,\delta+2}(\mathcal{M})$, where $A_\infty$ is a constant tensor.

The properties of the $W^{p}_{s,\delta}(\mathbb{R}^n)$ spaces proven in Sections 2.2 and 3.1 are also valid for $W^{p}_{s,\delta}(\mathcal{M})$. These can be proven by using a finite covering of the manifold and a partition of unity subordinate to the covering. We will discuss some of these in the Appendix (§7).

Let $L = \Delta_g$ be the Laplace Beltrami operator on the Riemannian manifold $\mathcal{M}$, then in local coordinates it has the form

$$
\Delta_g u = g^{ab} \nabla_a \nabla_b u = g^{ab} \partial_a \partial_b u + \partial_b (g^{ab}) \partial_a u + \frac{1}{2} \left( g^{ab} \partial_b g_{ab} \right) g^{ab} \partial_a u,
$$

where $g^{ab}$ denote the inverse matrix of $g_{ab}$. The principle symbol is $g^{ab} \xi_a \xi_b = |\xi|^2$, so it is obviously an elliptic operator on the manifold $\mathcal{M}$. For each of the charts $(U_i, \varphi_i)$ we get from Propositions 2.8, 2.13 and 2.17, and condition 4. of Definition 3.8, that $(g^{ab} - \delta^{ab}) \in W^{p}_{s,\delta}(E_{r_i})$. Also the first order terms belong to $W^{p}_{s-1,\delta+1}(E_{r_i})$. Hence it follows from Definition 3.8 and the norm (3.17) that for any $a_0 \in W^{p}_{s-2,\delta+2}(\mathcal{M})$, the operator

$$
L = -\Delta_g + a_0
$$

belongs to $\text{Asy}(-\Delta, s, \delta, p)$ on $(\mathcal{M}, g)$. Here $\Delta u = \partial^a \partial_a u$ in local coordinates.

The coefficients of the Laplacian (3.22) in the exterior of the ball $E_{r_i}$ can be extended to the entire space $\mathbb{R}^n$ so that the extension remains an elliptic operator. Hence, if $s \in (\frac{n}{p}, \infty) \cap [1, \infty)$ and $\delta \geq -\frac{n}{p}$, then as a consequence of the above definition of the norm (3.17) and Lemma 3.5 on manifolds (see Proposition 7.7 in Appendix §7) we obtain:

**Corollary 3.9.** Let $s \in (\frac{n}{p}, \infty) \cap [1, \infty)$, $\delta \in (-\frac{n}{p}, -2 + \frac{n}{p})$, $a_0 \in W^{p}_{s-2,\delta+2}(\mathcal{M})$ and assume $(\mathcal{M}, g)$ is an asymptotically flat manifold of the class $W^{p}_{s,\delta}$. Then

$$
-\Delta_g + a_0 : W^{p}_{s,\delta}(\mathcal{M}) \rightarrow W^{p}_{s-2,\delta+2}(\mathcal{M})
$$

is a semi–Fredholm operator.
3.3. Weak solutions of linear systems on manifolds. In this subsection we consider weak solution of the equation \(-\Delta_g u + a_0 u = f\), where \(\Delta_g\) is the Laplace Beltrami operator \((3.22)\) and \(a_0, f \in W^{p}_{s-2,\delta+2}(\mathcal{M})\).

But prior to the definition of weak solutions, we have to extend the \(L^2\)-form

\[
(u, v)_{(L^2, g)} := \int_{\mathcal{M}} uv d\mu_g, \quad u, v \in C^\infty_0(\mathcal{M}),
\]

to a continuous bilinear form on \(W^{p}_{s,\delta}(\mathcal{M}) \otimes W^{p'}_{-s,-\delta}(\mathcal{M})\). Here \(\mu_g\) is the volume element with respect to the metric \(g\).

So assume \((\mathcal{M}, g)\) is an asymptotically flat manifold of the class \(W^{p}_{s,\delta}\), \(s > \frac{n}{p}\) and \(\delta \geq -\frac{n}{p}\); let \(|g|\) denote the determinant of \(g\) and \(\{\chi_i, \alpha_j\}\) be the partition of unity as in the definition of the norm \((3.17)\), then \(\{\chi_i^2, \alpha_j^2\} := \left(\sum_{i=1}^{N} \chi_i^2 + \sum_{j=1}^{N_0} \alpha_j^2\right)^{-1} \{\chi_i^2, \alpha_j^2\}\) is also a partition of unity. Using Propositions 2.1 and 2.4 and the bilinear form \((3.18)\), we obtain that

\[
(u, v)_{(L^2, g)} = \sum_{i=1}^{N} \int_{E_{\chi_i}} \phi_i^* \left(\chi_i^2 u \sqrt{|g|}\right) dx + \sum_{j=1}^{N_0} \int_{B_j} \Theta_j^* \left(\alpha_j^2 u \sqrt{|g|}\right) dx
\]

\[
= \sum_{i=1}^{N} \langle \phi_i^* \chi_i^2 u \sqrt{|g|}, \phi_i^* \chi_i^2 v \rangle \left(\mathcal{M}\right) + \sum_{j=1}^{N_0} \langle \Theta_j^* \alpha_j^2 u \sqrt{|g|}, \Theta_j^* \alpha_j^2 v \rangle \left(\mathcal{M}\right)
\]

Thus we have proven:

\[
|\langle u, v \rangle_{(\mathcal{M}, g)}| \leq C \left\| \sqrt{|g|} u \right\|_{W^{p}_{s,\delta}(\mathcal{M})} \left\| \varphi \right\|_{W^{p'}_{-s',-\delta'}(\mathcal{M})}
\]

\[
\leq C \left(\sqrt{|g|} - 1\right) \left\| u \right\|_{W^{p}_{s,\delta}(\mathcal{M})} \left\| \varphi \right\|_{W^{p'}_{s',s}(\mathcal{M})}
\]

\[
\leq C_g \left\| u \right\|_{W^{p}_{s,\delta}(\mathcal{M})} \left\| \varphi \right\|_{W^{p'}_{s',s}(\mathcal{M})}.
\]

This also implies that the topological dual with respect to \(\langle \cdot, \cdot \rangle_{(\mathcal{M}, g)}\) is isomorphic to \(W^{p'}_{-s',-\delta'}(\mathcal{M})\) (see Proposition 7.5). Thus we have proven:

**Proposition 3.10.** Let \(s \in \left(\frac{n}{p}, \infty\right) \cap [1, \infty)\), \(\delta \geq -\frac{n}{p}\), \(s - 2 \leq s \leq s\), \(\delta' \in \mathbb{R}\) and \((\mathcal{M}, g)\) be an asymptotically flat manifold of the class \(W^{p}_{s,\delta}\). Then the \(L^2\) inner–product \((3.23)\) extends to a continuous bilinear form \(\langle \cdot, \cdot \rangle_{(\mathcal{M}, g)} : W^{p}_{s,\delta}(\mathcal{M}) \otimes W^{p'}_{-s',-\delta'}(\mathcal{M}) \rightarrow \mathbb{R}\) that satisfies the inequality

\[
|\langle u, v \rangle_{(\mathcal{M}, g)}| \leq C_g \left\| u \right\|_{W^{p}_{s,\delta}(\mathcal{M})} \left\| v \right\|_{W^{p'}_{-s',-\delta'}(\mathcal{M})},
\]
where the constant $C_g$ depends on the metric $g$. Furthermore, if $s' > 0$, $u \in W^p_{s',\delta} (\mathcal{M})$ and $v \in \mathcal{S}$, then $\langle u, v \rangle_{(M,g)} = (u,v)_{(L^2,g)}$.

In a similar manner we can treat the $L^2$–bilinear form between two smooth vector fields $X$ and $Y$, that is,

\[
( X, Y )_{(L^2,g)} := \int_{\mathcal{M}} \langle X, Y \rangle_g d\mu_g
\]

(3.27) is true.

Applying again Propositions 3.11 and 2.17, we conclude:

**Proposition 3.11.** Let $(\mathcal{M},g)$ be an asymptotically flat manifold of the class $W^p_{s,\delta}$, $s \in (\frac{n}{p}, \infty) \cap [\frac{1}{2}, \infty)$ and $\delta \geq -\frac{n}{p}$. Then the $L^2$–bilinear form (3.27) has a continuous extension to a form $\langle X, Y \rangle_{(\mathcal{M},g)}$ on $W^p_{s-1,\delta+1}(\mathcal{M}) \otimes W^p_{s-1,\delta+1}(\mathcal{M})$ that satisfies the inequality

\[
|\langle X, Y \rangle_{(\mathcal{M},g)}| \leq C_g \| X \|_{W^p_{s-1,\delta+1}(\mathcal{M})} \| Y \|_{W^p_{s-1,\delta+1}(\mathcal{M})}.
\]

(3.28) Furthermore, if $s > 1$, $X \in W^p_{s-1,\delta+1}(\mathcal{M})$ and $Y \in \mathcal{S}$, then $\langle X, Y \rangle_{(\mathcal{M},g)} = (X,Y)_{(L^2,g)}$.

If $\varphi$ has compact support in a certain chart, then by integration by parts, we obtain

\[
\langle \nabla u, \nabla \varphi \rangle_{(L^2,g)} = \int \sqrt{\det g} g^{ab} \partial_a u \partial_b \varphi dx = -\int \Delta_g u \varphi d\mu_g.
\]

Note that $W^p_{0,\delta} \neq L^p_{\delta}$ (see Remark 2.7), however, if $u \in W^p_{s,\delta}(\mathcal{M})$ and $s \geq 1$, then by Theorem 2.6 \( \nabla u \in L^p_{s+1} \). Therefore it follows from Propositions 2.1 and 2.4 that $\langle \nabla u, \nabla \varphi \rangle_{(L^2,g)} = \langle \nabla u, \nabla \varphi \rangle_{(\mathcal{M},g)}$, whenever $\varphi$ is smooth. This justifies the following definition.

**Definition 3.12** (Weak solutions). Let $a_0, f \in W^p_{s-2,\delta+2}(\mathcal{M})$ and $s \geq 1$. A distribution $u \in W^p_{s,\delta}(\mathcal{M})$ is a weak solution of the equation

\[
-\Delta_g u + a_0 u = f,
\]

(3.29) if

\[
\langle \nabla u, \nabla \varphi \rangle_{(L^2,g)} + \langle a_0 u, \varphi \rangle_{(\mathcal{M},g)} = \langle f, \varphi \rangle_{(\mathcal{M},g)} \quad \text{for all } \varphi \in C^\infty_0(\mathcal{M}).
\]

(3.30) In the case that (3.29) were an inequality, then the equality in (3.30) would be replaced by the corresponding inequality and the test functions would be non–negative.

Next we prove the weak maximum principle for the operator $-\Delta_g + a_0$ when $a_0 \geq 0$. For $p = 2$ it was proven by Maxwell [30], and on compact manifolds in the $W^p_2$–spaces by Holst, Nagy and Tsogtgerel [23]. We recall that the distribution $a_0 \geq 0$ if and only if $\langle a_0, \varphi \rangle_{(\mathcal{M},g)} \geq 0$ for all non–negative $\varphi \in C^\infty_0(\mathcal{M})$. 


**Lemma 3.13.** Assume \((\mathcal{M}, g)\) is an asymptotically flat manifold of the class \(W^p_{s,\delta}, a_0 \geq 0, a_0 \in W^{p}_{s-2,\delta+2}, s \in \left(\frac{n}{p}, \infty\right) \cap [1, \infty)\) and \(\delta > -\frac{n}{p}\). If \(u \in W^p_{s,\delta}(\mathcal{M})\) satisfies
\[
(3.31) \quad -\Delta_g u + a_0 u \leq 0,
\]
then \(u \leq 0\) in \(\mathcal{M}\).

In order to prove it we need a pointwise multiplication in \(W^p_s\) with different values of \(p\). Such properties were established in \([32, \S 4.4]\), but for our needs it suffices to use Zolesio’s formulation and his result \([41]\).

**Proposition 3.14** (Zolesio). Let \(0 \leq s \leq \min\{s_1, s_2\}\), and \(1 \leq p_i, p < \infty\) be real numbers satisfying
\[
s_1 - s \geq n \left(\frac{1}{p_i} - \frac{1}{p}\right) \quad \text{and} \quad s_1 + s_2 - s > n \left(\frac{1}{p_i} + \frac{1}{p_2} - \frac{1}{p}\right).
\]
Then the pointwise multiplication \(W^p_{s_1}(\mathbb{R}^n) \times W^p_{s_2}(\mathbb{R}^n) \to W^p_s(\mathbb{R}^n)\) is continuous.

We shall also need the following known embedding (see e.g. \([4, \text{Theorem 6.5.1}]\)).

**Proposition 3.15.** If \(s - \frac{n}{p} \geq s_0 - \frac{n}{p_0}\) and \(p \leq p_0\), then the embedding \(W^p_s(\mathbb{R}^n) \to W^{p_0}_{s_0}(\mathbb{R}^n)\) is continuous.

**Remark 3.16.** We shall use Propositions 3.14 and 3.15 on an open bounded set \(\Omega_0 \subset \mathcal{M}\). The extension of them to bounded subsets of manifolds can be carried out by standard methods. To see this let \(\{(V_j, \Theta_j)\}\) be a finite collection of charts such that \(\Omega_0 \subset \cup_j V_j\) and let \(\{\alpha_j\}\) be partition of unity subordinate to \(\{V_j\}\). Then
\[
\|u\|_{W^p_s(\Omega_0)} = \sum_j \|\Theta^*_j \alpha_j u\|_{W^p_s(\mathbb{R}^n)}
\]
is a norm, and applying Propositions 3.14 and 3.15 to each of the norms in \(\mathbb{R}^n\) yields the results on subsets of a manifold.

**Proof of Lemma 3.13.** We will show that \(u \leq \epsilon\) for an arbitrary positive \(\epsilon\). Since \(\delta > -\frac{n}{p}\), \(u\) tends to zero at each end of \(\mathcal{M}\) by Proposition 2.11(c). Hence \(\{u > \epsilon\}\) is a bounded set in \(\mathcal{M}\). Let \(w := \max\{u - \epsilon, 0\}\) and \(\Omega_0 \subset \mathcal{M}\) be an open set such that \(\text{supp}(w) \in \Omega_0\). We recall that if a certain function, say \(v\), has support in the closure of \(\Omega_0\), then \(\|v\|_{W^p_s(\mathcal{M})} \simeq \|v\|_{W^p_s(\Omega_0)}\). Because of the limitations of the embeddings of Proposition 3.15, we split the proof into two cases, \(p \geq 2\) and \(p \leq 2\).

Starting with \(p \geq 2\), we have that \(W^1_p(\Omega_0) \subset W^2_p(\Omega_0)\). Hence \(w := \max\{u - \epsilon, 0\}\) also belongs to \(W^2_p(\Omega_0)\). We now claim that \(uw \in W^{p'}_{2-s}(\Omega_0)\), but since \(2 - s \leq 1\), it suffices to show that \(uw \in W^1_p(\Omega_0)\). Applying Proposition 3.14, we have that
\[
\|uw\|_{W^{p'}_{1}(\Omega_0)} \lesssim \|u\|_{W^p_{1}(\Omega_0)} \|w\|_{W^{p'}_{1}(\Omega_0)},
\]
and since $p' \leq 2$, we have by the Hölder inequality that
\[ \|w\|_{W^{p'}(\Omega_0)} \lesssim (\text{Vol}(\Omega_0, g))^{\frac{n}{p(p'-2)}} \|w\|_{W^p(\Omega_0)}^p. \]
Hence $uw$ belongs $W^{p'}_{2-s}(\Omega_0)$, and since $a_0|_{\Omega_0} \in W^{p}_{s-2}(\Omega_0)$ the bilinear form $\langle a_0, uw \rangle_{(M,g)}$ is finite. Moreover $uw \geq 0$, so by the density property of Besov spaces $\langle a_0, uw \rangle_{(M,g)} \geq 0$. Combining these with (3.31) and (3.30), we obtain that
\[ 0 \leq \langle a_0, uw \rangle_{(M,g)} = \langle a_0u, w \rangle_{(M,g)} \leq -(\nabla u, \nabla w)_{(L^2, g)} = -(\nabla w, \nabla w)_{(L^2, g)} \leq -\|\nabla w\|_{L^2(\Omega_0)}^2. \]
Thus $w$ is a constant in $\Omega_0$ and since it vanishes on the boundary, it is identically zero in $\Omega_0$. Consequently $u \leq \epsilon$, and that completes the proof when $p \geq 2$.

In the case of $p \leq 2$, we first claim that $a_0 \in W^n_{n-1}(\Omega_0)$. To see this we apply Proposition 3.15 with $s_0 = -1$ and $p_0 = n$, then the inequality $s - 2 - \frac{n}{p} \geq -1 - \frac{n}{n}$ holds, since $s \geq \frac{n}{p}$. The requirement $p \leq p_0 = n$ holds since throughout the paper $n \geq 2$. So we conclude $a_0 \in W^n_{n-1}(\Omega_0)$.

Applying again Proposition 3.15 we have that $u \in W^n_{n}(\Omega_0)$, hence $w = \max\{u - \epsilon, 0\}$ belongs to $w \in W^n_{n}(\Omega_0)$, and Proposition 3.14 yields that
\[ \|uw\|_{W^{n'}(\Omega_0)} \lesssim \|u\|_{W^n(\Omega_0)} \|w\|_{W^n(\Omega_0)}. \]
Therefore by (2.2) the bilinear form $\langle a_0, uw \rangle_{(M,g)}$ is well defined. We now complete the proof as in the case $p \geq 2$. \hfill \Box

Let $e$ be the metric such that in any local coordinates $e_{ab} = \delta_{ab}$. In case $M$ has one end, then $e$ is the Euclidean metric. For $t \in [0, 1]$ we consider a family of metrics $tg + (1-t)e$. Then $M$ equipped with this metric is asymptotically flat of class $W^{p}_{s,\delta}$, since $(tg + (1-t)e)_{ab} - \delta_{ab} = t(g_{ab} - \delta_{ab}) \in W^{p}_{s,\delta}(E_r)$. So by the weak maximum principle, the operator
\[ \Delta_{\{tg+(1-t)e\}} + ta_0 : W^{p}_{s,\delta}(M) \to W^{p}_{s-2,\delta+2}(M) \tag{3.32} \]
is injective for all $t \in [0, 1]$. Lemma 3.3 implies the for $t = 0$ the operator (3.32) is an isomorphism, and by Corollary 3.9 it has a closed range. In this situation we can apply standard homotopy arguments (see e.g. [10, Lemma 2]) and obtain the following lemma:

**Lemma 3.17.** Assume $(M, g)$ is an asymptotically flat manifold of the class $W^{p}_{s,\delta}$, $a_0 \geq 0$, $a_0 \in W^{p}_{s-2,\delta+2}$, $s \in \left(\frac{n}{p}, \infty\right) \cap [1, \infty)$ and $\delta \in \left(-\frac{n}{p}, -2 + \frac{n}{p}\right)$. Then for any $f \in W^{p}_{s-2,\delta+2}(M)$ equation
\[ -\Delta_g u + a_0 u = f \]
has a unique solution $u$ satisfying
\[ \|u\|_{W^{p}_{s,\delta}(M)} \leq C \|f\|_{W^{p}_{s-2,\delta+2}(M)}, \tag{3.33} \]
where the constant $C$ is independent on $f$. \hfill \Box
4. Semi–linear elliptic equations

In this section we establish an existence and uniqueness theorem for a semi–linear equation whose principal part is the Laplace–Beltrami operator on an asymptotically flat Riemannian manifold. The method of sub and super solutions is used frequently in such types of problems, however, we will employ a homotopy argument similar to the one presented by Cantor [9]. The authors applied this method in [6] for $p = 2$ and $s \geq 2$, and here, beside extending it to the $W^{p}_{s,\delta}$-spaces, we simplify some of the steps of the proof by computing the norm by means of the bilinear form (2.10). The conditions of Theorem 4.1 below could be relaxed to some extensions, but we refrain dealing with it here.

Let

$$F(u, x) := h_1(u)m_1(x) + \cdots + h_N(u)m_N(x),$$

be a function, where $h_i : (-1, \infty) \to [0, \infty)$ is $C^1$ non–increasing function, $m_i \geq 0$ and $m_i \in W^{p}_{s-2,\delta+2}(\mathcal{M})$. The typical example of $h_i(t)$ is $(1 + t)^{-\alpha_i}$ with $\alpha_i > 0$.

**Theorem 4.1.** Assume $(\mathcal{M}, g)$ is an asymptotically flat manifold of the class $W^{p}_{s,\delta}$, $a_0 \in W^{p}_{s-2,\delta+2}$, $a_0 \geq 0$, $s \in (\frac{2}{p}, \infty) \cap [1, \infty)$ and $\delta \in (-\frac{2}{p}, -2 + \frac{n}{p})$. Then the equation

$$-\Delta_g u + a_0u = F(u, \cdot)$$

has a unique non–negative solution $u \in W^{p}_{s,\delta}(\mathcal{M})$.

**Proof.** We define a map $\Phi : (W^{p}_{s,\delta}(\mathcal{M}) \cap \{u > -1\}) \times [0, 1] \to W^{p}_{s-2,\delta+2}(\mathcal{M})$ by

$$\Phi(u, \tau) = -\Delta_g u + a_0u - \tau F(u, \cdot)$$

and set $J = \{\tau \in [0, 1] : \Phi(u, \tau) = 0\}$. Lemma 3.17 implies that $0 \in J$ and therefore it suffices to show that $J$ is an open and closed set. Since the functions $h_i$ are non–increasing, $\frac{\partial F}{\partial u}(u, \cdot) \leq 0$, and therefore the operator

$$Lw := \left(\frac{\partial \Phi}{\partial u}(u, \tau)\right) w = -\Delta_g w + a_0w - \tau \frac{\partial F}{\partial u}(u, \cdot)w.$$ 

satisfies the assumptions of Lemma 3.17. Hence $\frac{\partial \Phi}{\partial u}$ is an isomorphism and this implies that $J$ is an open set (see e.g. [21, §17.2]). The essential difficulty is to show that $J$ is a closed set. So let $u(\tau)$ be a solution to $\Phi(u, \tau) = 0$. We first claim that there is a positive constant $C_0$ independent of $\tau$ such that

$$\|u(\tau)\|_{W^{p}_{s,\delta}(\mathcal{M})} \leq C_0.$$ 

Now the weak maximum principle, Lemma 3.13, implies that $u(\tau) \geq 0$, and hence, $h_i(u(\tau)) \leq h_i(0)$. Therefore by Proposition 3.10, for any nonnegative $\varphi \in C^\infty_0(\mathcal{M})$ we have that

$$0 \leq \langle h_i(u(\tau))m_i, \varphi\rangle_{(\mathcal{M}, g)} \leq \langle h_i(0)m_i, \varphi\rangle_{(\mathcal{M}, g)} \leq C_g h_i(0) \|m_i\|_{W^{p}_{s-2,\delta+2}(\mathcal{M})} \|\varphi\|_{W^{-p'}_{2-s, -\delta-2}(\mathcal{M})}.$$ 


Hence by formulas (3.18), (3.24) and inequality (2.10)
\[(4.4)\]
\[\|h_i(u(\tau))m_i\|_{W^p_{s,2,\delta+2}(\mathcal{M})} \leq C_g h_i(0)\|m_i\|_{W^p_{s,2,\delta+2}(\mathcal{M})}\]
for \(i = 1, \ldots, N\). By Lemma 3.17, and inequalities (3.33) and (4.4) we obtain that
\[\|u(\tau)\|_{W^p_{s,\delta}(\mathcal{M})} \leq C \|F(u(\tau), \cdot)\|_{W^p_{s,2,\delta+2}(\mathcal{M})} \leq CC_g \sum_{i=1}^{N} h_i(0)\|m_i\|_{W^p_{s,2,\delta+2}(\mathcal{M})} =: C_0,\]
and that proves (4.2).

Differentiating (4.1) with respect to \(\tau\) gives
\[(4.5)\]
\[-\Delta_g u_\tau + a_0 u_\tau - \tau \frac{\partial F}{\partial u}(u(\tau), \cdot)u_\tau = F(u(\tau), \cdot),\]
where \(u_\tau\) denotes the derivative of \(u(\tau)\) with respect to \(\tau\). By Propositions 2.13 and 2.17, both \(\|F(u(\tau), \cdot)\|_{W^p_{s,2,\delta+2}(\mathcal{M})}\) and \(\|\frac{\partial F}{\partial u}(u(\tau), \cdot)\|_{W^p_{s,2,\delta+2}(\mathcal{M})}\) are bounded by \(\|u(\tau)\|_{W^p_{s,\delta}(\mathcal{M})}\).

In addition, \(\frac{\partial F}{\partial u} \leq 0\), thus the operator (4.5) satisfies the conditions of Lemma 3.17, and hence it possesses a solution \(u_\tau\) in \(W^p_{s,\delta}(\mathcal{M})\).

We now show that \(\|u_\tau\|_{W^p_{s,\delta}(\mathcal{M})}\) is bounded by a constant independent of \(\tau\). By Lemma 3.17, equation
\[-\Delta_g w + a_0 w = F(u(\tau), \cdot),\]
has a solution \(w\) that satisfies the inequality \(\|w\|_{W^p_{s,\delta}(\mathcal{M})} \leq C \|F(u(\tau), \cdot)\|_{W^p_{s,2,\delta+2}(\mathcal{M})}\).

Since the bound of \(\|F(u(\tau), \cdot)\|_{W^p_{s,2,\delta+2}(\mathcal{M})}\) is independent of \(\tau\) by (4.2), we conclude that
\[\|w\|_{W^p_{s,\delta}(\mathcal{M})} \leq K\]
and the constant \(K\) is independent of \(\tau\). From the weak maximum principle, Lemma 3.13, we get that \(u_\tau(\cdot) \geq 0\) and hence \((-\Delta_g + a_0)(w - u_\tau) = -\tau \frac{\partial F}{\partial u}(u(\tau), \cdot)u_\tau \geq 0\). Thus \((w - u_\tau) \geq 0,\) again by the maximum principle, and by Proposition 3.10 we have that
\[0 \leq \langle u_\tau, \varphi \rangle_{(\mathcal{M}, g)} \leq \langle w, \varphi \rangle_{(\mathcal{M}, g)} \leq C_g \|w\|_{W^p_{s,\delta}(\mathcal{M})} \|\varphi\|_{W^{p', s, -\delta}(\mathcal{M})} \leq C_g K \|\varphi\|_{W^{p', s, -\delta}(\mathcal{M})}\]
for any non-negative \(\varphi \in C^0_{c, s}(\mathcal{M})\). Thus using again the dual estimate of the norm (2.10), we have obtained that
\[\|u_\tau\|_{W^p_{s,\delta}(\mathcal{M})} \leq C_g K,\]
This implies that the norm \(\|u(\tau)\|_{W^p_{s,\delta}(\mathcal{M})}\) is a Lipschitz function of \(\tau\), that is,
\[|\|u(\tau_1)\|_{W^p_{s,\delta}(\mathcal{M})} - \|u(\tau_2)\|_{W^p_{s,\delta}(\mathcal{M})}| \leq C_g K|\tau_1 - \tau_2|\].
Therefore if \(\{\tau_k\} \subset J\) and \(\tau_k \to \tau_0\), then \(\{u(\tau_k)\}\) is a Cauchy sequence in \(W^p_{s,\delta}(\mathcal{M})\) and hence \(J\) is a closed set. That completes the proof of the existence.

As for the uniqueness, assume there are two different solutions \(u_1\) and \(u_2\). Then at least one of the sets \(\Omega_+ := \{x \in \mathcal{M} : u_1(x) - u_2(x) > 0\}\) or \(\Omega_- := \{x \in \mathcal{M} : u_1(x) - u_2(x) > 0\}\) is non-empty. Suppose that \(\Omega_+ \neq \emptyset\), then \(w = u_1 - u_2\) satisfies the equation 
\[-\Delta_g w + a_0 w = F(u_1) - F(u_2) \leq 0\] in \(\Omega_+\), since \(F(u, x)\) is non-increasing as a function of \(u\). Now the weak
maximum principle, Lemma 3.13, implies that \( w \leq 0 \) in \( \Omega_+ \) and hence it must be an empty set. A similar contradiction occurs if \( \Omega_- \neq \emptyset \).

\[ \square \]

5. The Brill–Cantor criterion

Let \( (\mathcal{M}, g) \) be an asymptotically flat manifold of class \( W^p_{s,\delta} \) and \( R(g) \) be the scalar curvature. Throughout this section \( n \geq 3 \). We set \( 2^* = \frac{2n}{n-2} \) and \( s_n = \frac{n-2}{4(n-1)} \). Following Choquet–Bruhat, Isenberg, and York [16] and Maxwell [29], we define.

**Definition 5.1.** An asymptotically flat manifold \( (\mathcal{M}, g) \) is in the positive Yamabe class if

\[ (5.1) \quad \inf_{\varphi \in C_0^\infty(\mathcal{M})} \frac{\langle \nabla \varphi, \nabla \varphi \rangle_{(L^2, g)} + s_n \langle R(g), \varphi^2 \rangle_{(\mathcal{M}, g)}}{\|\varphi\|^2_{L^{2^*}}} > 0. \]

This condition is conformal invariant under the scaling \( g' = \phi^{\frac{4}{n-2}} g \) [16]. For \( s \geq 2 \) the metric \( g \in W^p_{2,\delta}(\mathcal{M}) \) and by Theorem 2.6, \( R(g) \in L^p(\mathcal{M}) \). So in this case formula (3.24) implies that \( \langle R(g), \varphi^2 \rangle_{(\mathcal{M}, g)} = \langle R(g), \varphi^2 \rangle_{(L^2, g)} \) and condition (5.1) takes the common form

\[ \inf_{\varphi \in C_0^\infty(\mathcal{M})} \int_\mathcal{M} ((\nabla \varphi, \nabla \varphi)_g + s_n R(g) \varphi^2) \, d\mu_g > 0. \]

Though condition (5.1) is similar to the Yamabe functional on compact manifolds ([1, Ch. 5], [13, Ch. 7]), it has a different interpretation on asymptotically flat manifolds, namely, in that case being in the positive Yamabe class is equivalent to the existence of a conformal flat metric.

**Theorem 5.2.** Let \( (\mathcal{M}, g) \) be an asymptotically flat manifold of the class \( W^p_{s,\delta} \), and assume that \( s \in \left(\frac{n}{p}, \infty\right) \cap [1, \infty) \) and \( \delta \in (-\frac{n}{p}, -2 + \frac{n}{p}) \). Then \( (\mathcal{M}, g) \) is in the positive Yamabe class if and only if there is a conformally equivalent metric \( g' \) such that \( R(g') = 0 \).

This type of result was first proven in [11] for \( s > \frac{n}{p} + 2 \) and \( 1 < p < \frac{2n}{n-2} \). Since then the regularity assumptions were improved by several authors [13, 15, 16, 29], however, they dealt only with Sobolev spaces of integer order, and when \( s = 2 \) they are restricted to \( p > \frac{n}{2} \). For \( p = 2 \) it was proven for all \( s > \frac{n}{2} \) in [30]. Thus Theorem 5.2 improves regularity and extends the range of \( p \) to \((1, \infty)\).

**Proof of Theorem 5.2.** We prove only that condition (5.1) implies the existence of a flat metric. The converse assertion requires no special attention of the weighted Besov spaces and we refer to [15, 16] for this part.

We consider the following conformal transformation \( g' = \phi^{\frac{4}{n-2}} g \). It is known that the metric \( g' \) has scalar curvature zero if and only if equation (see e.g. [1])

\[ (5.2) \quad -\Delta_g \phi + s_n R(g) \phi = 0, \]
possesses a positive solution $\phi$ such that $\phi - 1 \in W^p_{s,\delta}(\mathcal{M})$. Setting $u = \phi - 1$, then (5.2) becomes
\begin{equation}
-\Delta_g u + s_n R(g) u = -s_n R(g).
\end{equation}
In order to assure that equation (5.3) has a solution, it suffices to show that the operator $-\Delta_g + \tau s_n R(g)$ has a trivial kernel for each $\tau \in [0, 1]$. The crucial point is to estimate the numerator of (5.1) in terms of the $W^p_{s,\delta}(\mathcal{M})$-norm. Starting with the second term, we have by Proposition 3.10 that
\begin{equation}
\|\langle R(g), \varphi^2 \rangle_{(M, g)} \| = \|\langle R(g)\varphi^2, 1 \rangle_{(M, g)} \| \lesssim \| R(g)\varphi^2 \|_{W^p_{2-2,\delta''}(\mathcal{M})} \| 1 \|_{W^{p'}_{1-3,\delta''}(\mathcal{M})}.
\end{equation}
Obviously, $\| 1 \|_{W^{p'}_{1-3,\delta''}(\mathcal{M})} \leq \| 1 \|_{W^{p'}_{1-3,\delta''}(\mathcal{M})}$ and the last term is finite if $\delta'' > \frac{n}{p}$. Take now $\delta'$ satisfying the condition
\begin{equation}
\frac{n}{p'} < \delta'' \leq \delta + 2 + 2\delta' + \frac{2n}{p},
\end{equation}
and Proposition 2.15, with $\delta_1 = \delta + 2$ and $\delta_2 = \delta_3 = \delta'$, then we obtain that
\begin{equation}
\| R(g)\varphi^2 \|_{W^p_{2-2,\delta''}(\mathcal{M})} \lesssim \| R(g)\varphi^2 \|_{W^p_{2-2,\delta''+2}(\mathcal{M})} \left( \| \varphi \|_{W^p_{s,\delta'}(\mathcal{M})} \right)^2.
\end{equation}
For the first term of the numerator of (5.1), we have by the identity (3.24) that
\begin{equation}
(\nabla \varphi, \nabla \varphi)_{(L^2, g)} = (|\nabla \varphi|^2)_{(L^2, g)} = \langle \sqrt{|g|} g^{ab} \partial_a \varphi \partial_b \varphi, 1 \rangle_{\mathcal{M}}.
\end{equation}
So by inequality (3.19),
\begin{equation}
\| (\nabla \varphi, \nabla \varphi)_{(L^2, g)} \| \lesssim \| \sqrt{|g|} g^{ab} \partial_a \varphi \partial_b \varphi \|_{W^p_{1-4,\delta''}(\mathcal{M})} \| 1 \|_{W^{p'}_{1-3,\delta''}(\mathcal{M})}.
\end{equation}
As in the previous term, $\| 1 \|_{W^{p'}_{1-3,\delta''}(\mathcal{M})}$ is finite if $\delta'' > \frac{n}{p}$. Writing
\begin{equation}
\sqrt{|g|} g^{ab} \partial_a \varphi \partial_b \varphi = \left( \sqrt{|g|} g^{ab} - \delta^{ab} \right) \partial_a \varphi \partial_b \varphi + \partial^a \varphi \partial_a \varphi
\end{equation}
and assuming that $\delta'$ satisfies (5.5), we then can apply again Propositions 2.15, with $\delta_1 = \delta$ and $\delta_2 = \delta_3 = \delta' + 1$, Proposition 2.17, and get that
\begin{equation}
\| \left( \sqrt{|g|} g^{ab} - \delta^{ab} \right) \partial_a \varphi \partial_b \varphi \|_{W^p_{s-1,\delta''}(\mathcal{M})} \lesssim \| g - 1 \|_{W^p_{s,\delta}(\mathcal{M})} \left( \| \nabla \varphi \|_{W^p_{s-1,\delta''}(\mathcal{M})} \right)^2.
\end{equation}
By Proposition 2.13,
\begin{equation}
\| |\nabla \varphi|^2 \|_{W^p_{s-1,\delta''}(\mathcal{M})} \lesssim \left( \| |\nabla \varphi| \|_{W^p_{s-1,\delta''}(\mathcal{M})} \right)^2
\end{equation}
whenever $\delta'$ also satisfies the condition
\begin{equation}
\frac{n}{p'} < \delta'' \leq 2(\delta' + 1) + \frac{n}{p}.
\end{equation}
We are now in a position to show that if $(\mathcal{M}, g)$ is in the positive Yamabe class, then $-\Delta_g + \tau s_n R(g)$ is an injective operator. For $\tau = 0$ it is injective by the weak maximum
and the above estimate and its version on manifolds Lemma 5.3, we conclude by Corollary 3.9 and the homotopy argument as in Lemma 3.17 that equation (5.3) has a unique solution. Let \( u \) be the solution and set \( \phi = 1 + u \), then it remains to show that \( \phi > 0 \). We follow here [11, 29]. Let \( u_\lambda \) be a solution to \( -\Delta_g u_\lambda + \lambda s_n R(g) u_\lambda = -\lambda s_n R(g) \) and set \( J = \{ \lambda \in [0, 1] : \phi_\lambda(x) = 1 + u_\lambda(x) > 0 \} \). By Lemma 3.5 and its version on manifolds Lemma 7.7, there is a constant \( C \) independent of \( \lambda \) and \( \delta' < \delta \) such that

\[
\|u_\lambda\|_{W^{p}_{s, \delta}(\mathcal{M})} \leq C \left\{ \|\lambda s_n R(g)\|_{W^{p}_{s-2, \delta+2}(\mathcal{M})} + \|u_\lambda\|_{W^{p}_{s-1, \delta'}(\mathcal{M})} \right\}.
\]

Now it follows from the compact embedding, Proposition 2.11 (b) and the above estimate that \( \|u_\lambda\|_{W^{p}_{s, \delta}(\mathcal{M})} \) is a continuous function of \( \lambda \). Hence, by the embedding into the continuous, Proposition 2.11(c), \( \phi_\lambda - 1 \) is continuous in \( C^0_\beta \) for some \( \beta > 0 \) with respect to \( \lambda \). Thus \( J \) is a open and non–empty set, since \( 0 \in J \). So if \( J \neq [0, 1] \), then there exists a \( 0 < \lambda_0 < 1 \) such that \( \phi_{\lambda_0} \geq 0 \). Then by the Harnack inequality \( \phi_{\lambda_0} > 0 \) and consequently \( \phi_1 = \phi > 0 \). For details how to apply the Harnack inequality under the present regularity assumption see [23, Lemma 35] and [30, lemma 5.3].

6. Applications to the Constraint Equations of the Einstein–Euler Systems

In this section we describe briefly the initial data for the Einstein–Euler system, for more details we refer to [6, 7]. In [6] we constructed the initial data in the Hilbert space \( W^{2}_{s, \delta}(\mathcal{M}) \) and here we apply the results of the previous sections in order to construct the initial data in the weighted Besov spaces \( W^{p}_{s, \delta}(\mathcal{M}) \) for \( 1 < p < \infty \).

The Einstein–Euler system describes a relativistic self–gravitating perfect fluid. The fluid quantities are the energy density \( \rho \), the pressure \( p \) and a unite time–like velocity vector \( u^\alpha \).
In this section Greek indexes take the values 0, 1, 2, 3. The evolution of the gravitational fields is described by the Einstein equations

\[ R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 8\pi T_{\alpha\beta}, \]

where \( g_{\alpha\beta} \) is a semi Riemannian metric with signature \((-, +, +, +)\), \( R_{\alpha\beta} \) is the Ricci curvature tensor and \( T_{\alpha\beta} \) is the energy–momentum tensor of the matter, which in the case of a perfect fluid takes the form

\[ T^{\alpha\beta} = (\rho + p)u^\alpha u^\beta + p g^{\alpha\beta}. \]

The evolution of the fluid is described by the Euler equations \( \nabla_\alpha T^{\alpha\beta} = 0 \). This system contains more unknowns than equations and therefore an additional relation is indispensable. The usual strategy is to introduce an equation of state, which connects \( p \) and \( \rho \). Here we consider the analogue of the non–relativistic polytropic equation of state and that is given by

\[ p = p(\rho) = \kappa \rho^\gamma, \quad 1 < \gamma, \quad \kappa \in \mathbb{R}^+. \]

In the context of astrophysics, isolated systems cannot have a density that is bounded below by a positive constant. It either falls off at infinity, or has compact support. That causes the corresponding symmetric hyperbolic system to degenerate (see [7] for details). Following Makino [27], we regularize the symmetric hyperbolic system by the variable change

\[ w = \rho^{\frac{\gamma - 1}{2}}. \]

The initial data of the Einstein–Euler system are a proper Riemannian metric \( g \) and a symmetric \((2, 0)\)–tensor \( K_{ab} \), given on a three dimensional manifold \( \mathcal{M} \). The matter variables are \((z, j^a)\), where \( z \) denotes the energy density and \( j^a \) the momentum density, and in addition, there are initial data for the fluid. These are the Makino variable \( w \) and the velocity vector \( u^a \). The data must satisfy the constraint equations

\[ \begin{cases} R(g) - K_{ab}K^{ab} + (g_{ab}K_{ab})^2 = 16\pi z & \text{Hamiltonian constraint} \\ (\nabla_b K^{ab} - \nabla^b (g_{bc}K_{bc})) = -8\pi j^a & \text{Momentum constraint} \end{cases} \]

Let \( \tilde{u}^a \) denote the projection of the velocity vector \( u^a \) on the initial manifold \( \mathcal{M} \). The projections of the energy–momentum tensor \( T_{\alpha\beta} \) twice on the unit normal \( n^\alpha \), once on \( n^\alpha \) and once on \( \mathcal{M} \), lead to the following relations

\[ \begin{cases} z = \rho + (\rho + p)g_{ab}\tilde{u}^a\tilde{u}^b \\ j^\alpha = (\rho + p)\tilde{u}^a\sqrt{1 + g_{ab}\tilde{u}^a\tilde{u}^b} \end{cases} \]

We use the well–known conformal method for solving the constraint equations (6.4). This method starts by giving some free quantities and the solutions of the constraints are obtained in the end by rescaling these with appropriate powers of a scalar function \( \phi \). This function is the solution of the Lichnerowicz equation (6.9). In the case of the fluid the quantities which can be rescaled in a way which is consistent with the general scheme are
z and \( j^a \), and not the quantities \( w \) and \( \tilde{u}^a \). Therefore, in order to provide initial data for the fluid variables \((w, \tilde{u}^a)\), equations (6.5) must be inverted.

Taking into account the variable change (6.3) and the equation of state (6.2), then (6.5) is equivalent to the inversion of the map (see [6, §4] for details)

\[
\Phi (w, \tilde{u}^a) := \left( w \left\{ 1 + (1 + \kappa w^2) \left( g_{ab} \tilde{u}^a \tilde{u}^b \right) \right\}^{\frac{n-1}{2}}, \frac{1 + \kappa w^2}{1 + (1 + \kappa w^2) \left( g_{ab} \tilde{u}^a \tilde{u}^b \right)} \right) \tag{6.6}
\]

The inversion of this map under certain condition was established in [6].

**Theorem 6.1** (Reconstruction theorem for the initial data). Let \( g \) be a Riemannian metric, then there is a continuous function \( S : [0, 1] \to \mathbb{R} \) such that if

\[
0 \leq z^{\frac{n-1}{2}} \leq S \left( z^{-1} \sqrt{g_{ab} j^a j^b} \right), \tag{6.7}
\]

then system (6.6) has a unique inverse.

Since Condition (6.7) is not invariant under scaling, the unscaled initial data for the energy and momentum densities must satisfy it.

Therefore there are two types of free data, the geometric data \((\bar{g}, \bar{A}^{ab})\) where \( \bar{g} \) is a Riemannian metric, \( \bar{A}^{ab} \) is divergence and trace free form, and the matter data \((\dot{z}^{\frac{n-1}{2}}, \dot{j}^a)\), which are constructed using (6.5) but with the flat metric \( \hat{g} \).

We also assume that \((M, \bar{g})\) belongs in the positive Yamabe class (see Definition 5.1) and has no Killing vector fields in \( W^p_{s,\delta}(M) \) (for \( p = 2 \) and \( s > \frac{3}{2} \) this assumption was verified in [30]).

**Theorem 6.2** (Solution of the constraint equations). Let \( M \) be a Riemannian manifold and \((\bar{g}, \bar{A}^{ab}, \dot{z}^{\frac{n-1}{2}}, \dot{j}^a)\) be free data such that \((M, \bar{g})\) is asymptotically flat of the class \( W^p_{s,\delta} \) and belongs to the positive Yamabe class, \( \bar{A}^{ab} \in W^p_{s-1,\delta+1}(M) \), \( (\dot{z}^{\frac{n-1}{2}}, \dot{j}^a) \in W^p_{s,\delta+2}(M) \), \( s \in \left( \frac{2}{p}, \frac{2}{\gamma-1} + \frac{1}{p} \right) \cap [1, \infty) \) and \( \delta \in \left( -\frac{2}{p}, n - 2 - \frac{2}{p} \right) \).

1. Assume \((\dot{z}, \dot{j}^a)\) satisfy (6.7) with respect to a flat metric \( \hat{g} \), then \((w, \tilde{u}^a) = \Phi^{-1}(\dot{z}^{\frac{n-1}{2}}, \dot{j}^a)\) are initial data for the fluid and satisfy the compatibility condition (6.5) in the term of the metric \( g = \phi^4 \hat{g} \), where \( z = \phi^{-8} \dot{z} \) and \( j^a = \phi^{-10} \dot{j}^a \) and \( \phi \) is the solution to the Lichnerowicz equation (6.9). Moreover, \((w, \tilde{u}^a - 1, \tilde{a}^a) \in W^p_{s,\delta+2}(M) \).
2. There exists a conformal metric \( g \), \((2, 0)\)-symmetric form \( K_{ab} \) which satisfy the constraint equation (6.4) with the right hand side \((z, j^a)\). The pair \((M, g)\) is asymptotically flat of the class \( W^p_{s,\delta} \) and \( K_{ab} \in W^p_{s-1,\delta+1}(M) \).

**Remark 6.3.** The upper bound \( \frac{2}{\gamma-1} + \frac{1}{p} \) for the regularity index \( s \) is caused by the equation of state (6.3), and it is not needed whenever \( \frac{2}{\gamma-1} \) is an integer.
Proof Theorem 6.2. We first replace the metric $\hat{g}$ by a conformal flat metric $\tilde{g}$. The metric $\tilde{g}$ is given by the conformal transformation $\tilde{g} = \varphi^4 \hat{g}$, where $\varphi - 1 \in W^p_{s,\delta}(\mathcal{M})$. The existence and the uniqueness of such a $\varphi$ is assured by Theorem 5.2.

In the second stage we set $\hat{A}^{ab} = \varphi^{-10} A^{ab}$ and

$$\hat{K}^{ab} = \hat{A}^{ab} + \left( \hat{\nabla}_j W^j \right)^{ab},$$

where $\hat{\nabla}$ is the Killing fields operator with respect to the metric $\hat{g}$, that is,

$$\left( \hat{\nabla}_j W^j \right)^{ab} = \hat{\nabla}_a W_b + \hat{\nabla}_b W_a - \frac{1}{3} g^{ab} \left( \hat{\nabla}_i W^i \right).$$

Then $\hat{K}$ satisfies the momentum constraint (6.4), if the vector $W$ is a solution to the Lichnerowicz Laplacian

$$\left( \Delta_{L_0} W \right)^b = \hat{\nabla}_a \left( \hat{\nabla}_j W^j \right)^{ab} = \Delta_{\hat{g}} W^b + \frac{1}{3} \hat{\nabla}^b \left( \hat{\nabla}_a W^a \right) + \hat{R}_a^b W^a = -8\pi \hat{j}^b.$$

Here $\hat{R}_a^b$ is the Ricci curvature tensor with respect to the metric $\hat{g}$. The Lichnerowicz Laplacian (6.8) is a strongly elliptic operator (see e.g. [16]) and belongs to $\text{Asy}(\Delta, s, \delta, p)$, since $(\mathcal{M}, \hat{g})$ is asymptotically flat of the class $W^p_{s,\delta}$. Its kernel consists of Killing vector fields in $W^p_{s,\delta}(\mathcal{M})$, since we assume there are no such fields, then Corollary 3.6 implies that $\Delta_{L_0}$ is an isomorphism, and consequently equation (6.8) possesses a solution.

The solution to the Hamiltonian constraint is constructed by an additional conformal transformation $\tilde{g} = \phi^4 \hat{g}$. Setting $K^{ab} = \phi^{-10} \hat{K}^{ab}$ and $j^b = \phi^{-10} \hat{j}^b$ preserves the momentum constraint of (6.4) with respect to the metric $g$. Under this transformation, the scalar curvature $R(\tilde{g})$ satisfies the equation

$$\phi^5 R(\tilde{g}) = R(\hat{g}) - 8\Delta_{\hat{g}} \phi,$$

(see e.g. [1, Ch. 5]), and since $R(\hat{g}) = 0$, the Hamiltonian constraint in (6.5) is satisfied provided that $\phi$ is a solution to the Lichnerowicz equation

$$-\Delta_{\hat{g}} \phi = 2\pi \hat{\phi}^{-3} + \frac{1}{8} \hat{K}_a^b \hat{K}_b \phi^{-7}.$$

Setting $u = \phi - 1$, then the Lichnerowicz equation (6.9) takes the form

$$-\Delta_{\hat{g}} u = 2\pi \hat{\phi} (u + 1)^{-3} + \frac{1}{8} \hat{K}_a^b \hat{K}_b (u + 1)^{-7},$$

which is then in a form suitable for the application of Theorem 4.1. This theorem provides a non–negative solution $u \in W^p_{s,\delta}(\mathcal{M})$. Hence $\phi \geq 1$.

It remains to construct the initial data for the fluid variables $(w, \tilde{u}^a)$ in terms of the metric $g = \phi^4 \hat{g}$. Setting $z = \phi^{-8} \hat{z}$, preserves the quantity $\hat{z}^{-2} \hat{g}_{ab} \hat{\phi}^a \hat{\phi}^b$, while $z^{\frac{1}{z}} = \phi^{-4(\gamma - 1)} \hat{z}^{\frac{1}{\hat{z}}}$. Since the adiabatic constant $\gamma > 1$ and $\phi \geq 1$, $\phi^{-4(\gamma - 1)} \leq 1$ and consequently $z^{\frac{1}{z}} \leq \hat{z}^{\frac{1}{\hat{z}}}$. Therefore, if $(\hat{z}^{\frac{1}{\hat{z}}}, \hat{z}^{\frac{1}{\hat{z}}})$ satisfies (6.7), then the pair $(\hat{z}^{\frac{1}{\hat{z}}}, \hat{z}^{\frac{1}{\hat{z}}})$ does it too.
Hence, by Theorem 6.1 we can let \((w, \tilde{u}^a) = \Phi^{-1}(z^{\frac{2s}{2s+1}}, \frac{z}{z^{\frac{2s}{2s+1}}})\), and then obviously the compatibility conditions (6.5) are satisfied in terms of the metric \(g\). Notice that \(z^{\frac{2s}{2s+1}} \in W_{s,\delta+2}^p(M)\), so now we can apply the estimate of the fractional power, Proposition 2.16, with \(\beta = \frac{2}{\gamma - 1}\) and obtain that \(z \in W_{s,\delta+2}^p(M)\). At this stage appears the upper bound of the regularity index \(s\). From Propositions 2.13 and 2.17 we get that \((w, \tilde{u}^a) = \Phi^{-1}(z^{\frac{2s}{2s+1}}, \frac{z}{z^{\frac{2s}{2s+1}}})\) are also in \(W_{s,\delta+2}^p(M)\). Finally, since the velocity vector is a time–like unit vector, we set \(\tilde{u}^0 = 1 + g_{ab}\tilde{u}^a\tilde{u}^b\).

7. Appendix

The tools and the elliptic theory which we developed in \(\mathbb{R}^n\) hold also on an asymptotically flat manifolds of the class \(W_{s,\delta}^p\). Here we will discuss the extension of several properties to Riemannian manifolds. The unproven properties are to be demonstrated by similar techniques and methods.

We recall some of the notations of §3.2: From the Definition 3.8 there is a compact set \(K \subset M\), and a collection of charts \(\{(U_i, \phi_i)\}_{i=1}^{N}\) such that \(M \setminus K \subset \bigcup_{i=1}^{N} U_i\), where \(\phi_i : E_{r_i} \rightarrow U_i\) is a homeomorphism and \(E_{r_i} = \{x \in \mathbb{R}^n : |x| > r_i\}\). Let \(\{(V_j, \Theta_j)\}_{j=1}^{N_0}\) be a collection of charts that cover \(K\), where each \(\Theta_j\) is a homeomorphism between a ball \(B_j\) in \(\mathbb{R}^n\) and \(V_j \subset K\). Let \(\chi_i, \alpha_j\) be a partition of unity subordinate to \(\{U_i, V_j\}\), then

\[
\|u\|_{W_{s,\delta}^p(M)} := \sum_{i=1}^{N} \|\phi_i^*(\chi_i u)\|_{W_{s,\delta}^p(\mathbb{R}^n)} + \sum_{j=1}^{N_0} \|\Theta_j^*(\alpha_j u)\|_{W_{s,\delta}^p(\mathbb{R}^n)}
\]

is a norm in \(W_{s,\delta}^p(M)\), and

\[
\langle u, \varphi \rangle_M = \sum_{i=1}^{N} \langle \phi_i^*(\chi_i u), \phi_i^*(\chi_i \varphi) \rangle_W + \sum_{j=1}^{N_0} \langle \Theta_j^*(\alpha_j u), \Theta_j^*(\alpha_j \varphi) \rangle_W
\]

is a bilinear form in \(W_{s,\delta}^p(M)\), where \(\langle \cdot, \cdot \rangle_W\) are the bilinear forms (2.2) and (2.6) respectively.

We shall need the following elementary two Propositions.

**Proposition 7.1.** Let \(F : O_1 \rightarrow O_2\) be a \(C^\infty\)-diffeomorphism between two open set of \(\mathbb{R}^n\) such that \(\det(DF) \geq \epsilon_0 > 0\). Then

\[
\|F^*(u)\|_{W_{s,\delta}^p(O_1)} = \|u \circ F\|_{W_{s,\delta}^p(O_2)} \leq C\|u\|_{W_{s,\delta}^p(O_2)}.
\]

The constant \(C\) depends on \(s, \delta\) and \(\epsilon_0\).

**Proof.** Let \(s = k\) be a positive integer and consider the norm (2.11) restricted to the open sets \(O_1\) and \(O_2\). Then standard calculations give that

\[
\sum_{\alpha \leq k} \int_{O_1} (1 + |x|)^{(\delta+|\alpha|)p} |\partial^\alpha (u \circ F)|^p dx \leq \left( \frac{C}{\epsilon_0} \right)^p \sum_{\alpha \leq k} \int_{O_2} (1 + |x|)^{(\delta+|\alpha|)p} |\partial^\alpha u|^p dx.
\]
Hence, by the equivalence of the norms (1.1) and (2.5) (see Theorem 2.6) we have that
\[ \|F^*(u)\|_{W^p_{k,\delta}(O_1)} \leq \frac{C}{\epsilon_0} \|u\|_{W^p_{k,\delta}(O_2)}. \]

For a negative integer \( k \) we compute the norm in the dual form (2.9). Note that
\( (u \circ F, \varphi)_{L^2(O_1)} = (u, ((\det(DF)^{-1}) \circ F^{-1})_{L^2(O_2)} \), hence by Propositions 2.1 and 2.4, and approximation, we have that
\[ \|u \circ F\|_{W^p_{k,\delta}(O_1)} = \sup \{|\langle u \circ F, \varphi \rangle_{W^p_{k,\delta}(O_1)} : \varphi \in C^\infty_0(O_1)\} \]
\[ = \sup \{|\langle u, (\det(DF)^{-1}) \circ F^{-1}\rangle_{W^p_{k,\delta}(O_1)} : \varphi \in C^\infty_0(O_1)\}. \]

Replacing \( F \) by \( F^{-1} \), \( p \) by \( p' \) and \( \delta \) by \( -\delta \) in the previous estimate, we obtain that
\[ |\langle u, \varphi(\det(DF)^{-1}) \circ F^{-1}\rangle_{W^p_{k,\delta}(O_1)}| \leq C \|u\|_{W^p_{k,\delta}(O_2)} \|\varphi\|_{W^p_{k,\delta}(O_1)}. \]

Thus the linear operator (7.1) is bounded whenever \( s \) is an integer. We now complete the proof by interpolation, Theorem 2.3 (d).

\[ \Box \]

**Remark 7.2.** Obviously the proposition holds also in the unweighted spaces \( W^p_s \).

The following proposition can be proven by Proposition 7.1 and by standard techniques of finite covering of manifolds (see e.g. [18]).

**Proposition 7.3.** Suppose \( u \in W^p_{s,\delta}(\mathcal{M}) \) and \( \text{supp}(u) \subset U_i \) or \( \text{supp}(u) \subset V_j \), then
\[ \|u\|_{W^p_{s,\delta}(\mathcal{M})} \leq C \|\phi^*_i(u)\|_{W^p_{s,\delta}(E_{r_i})} \quad \text{or} \quad \|u\|_{W^p_{s,\delta}(\mathcal{M})} \leq C \|\Theta^*_j(u)\|_{W^p_{s,\delta}(B_j)}. \]

We are now in a position to extend several properties to the \( W^p_{s,\delta}(\mathcal{M}) \)-spaces.

**Proposition 7.4.** The class \( C^\infty_0(\mathcal{M}) \) is dense in \( W^p_{s,\delta}(\mathcal{M}) \).

**Proof.** Let \( u \in W^p_{s,\delta}(\mathcal{M}) \), \( \epsilon \) be a positive arbitrary number. Since \( \phi^*_i(\chi_i u) \) has support in \( E_{r_i} \), there is, by Theorem 2.3 (b), \( h_i \in C^\infty(C^\infty(\mathbb{R}^n)) \) such that \( \|\phi^*_i(\chi_i u) - h_i\|_{W^p_{s,\delta}(E_{r_i})} < \epsilon \). Similarly, \( \Theta^*_j(\alpha_j u) \in W^p(B_j) \) has compact support in the ball \( B_j \), so by the approximation in the Besov spaces, there is \( h_j \in C^\infty(B_j) \) such that \( \|\Theta^*_j(\alpha_j u) - h_j\|_{W^p(B_j)} < \epsilon \). Setting
\[ h = \sum_{i=1}^{N} h_i \circ \phi^{-1}_i + \sum_{j=1}^{N_0} h_j \circ \Theta^{-1}_j, \]
then \( h \in C^\infty_0(\mathcal{M}) \) and from Proposition 7.3 we obtain
that
\[ \|u - h\|_{W^p_{s,\delta}(\mathcal{M})} = \left\| \sum_{i=1}^{N} (\chi_i u - h_i \circ \phi_i^{-1}) + \sum_{j=1}^{N_0} (\alpha_j u - h_j \circ \Theta_j^{-1}) \right\|_{W^p_{s,\delta}(\mathcal{M})} \]
\[ \leq \sum_{i=1}^{N} \| (\chi_i u - h_i \circ \phi_i^{-1}) \|_{W^p_{s,\delta}(\mathcal{M})} + \sum_{j=1}^{N_0} \| (\alpha_j u - h_j \circ \Theta_j^{-1}) \|_{W^p_{s,\delta}(\mathcal{M})} \]
\[ \leq C \sum_{i=1}^{N} \| \phi_i^*(\chi_i u) - h_i \|_{W^p_{s,\delta}(E_r)} + C \sum_{j=1}^{N_0} \| \Theta_j^*(\alpha_j u) - h_j \|_{W^p_{s,\delta}(B_j)} \]
\[ \leq \epsilon C(N + N_0). \]

The next proposition characterizes the topological dual of \( W^p_{s,\delta}(\mathcal{M}) \).

**Proposition 7.5.** Let \((\mathcal{M}, g)\) be an asymptotically flat manifold of the class \( W^p_{s,\delta} \), then \( W^p_{-s,-\delta}(\mathcal{M}) = (W^p_{s,\delta}(\mathcal{M}))^* \).

**Proof.** From the bilinear (3.18) form and the inequality (3.19) we see that \( W^p_{-s,-\delta}(\mathcal{M}) \subset (W^p_{s,\delta}(\mathcal{M}))^* \). Thus it suffices to show the reverse inclusion.

So let \( T \in (W^p_{s,\delta}(\mathcal{M}))^* \) and set \( \|T\| = \sup\{|T(\varphi)| : \|\varphi\|_{W^p_{s,\delta}(\mathcal{M})} \leq 1\} \). For \( i = 1, \ldots, N \), we define \( \phi_i^*(\chi_i T)(u) = T(\phi_i^*(\chi_i)u \circ \phi_i^{-1}) \), where \( u \in W^p_{s,\delta}(\mathbb{R}^n) \), and for \( j = 1, \ldots, N_0 \), we define \( \Theta_j^*(\alpha_j T)(u) = T(\Theta_j^*(\alpha_j)u \circ \Theta_j^{-1}) \), where \( u \in W^p_{\delta}(\mathbb{R}^n) \). Then by Proposition 7.3,

\[ |\phi_i^*(\chi_i T)(u)| = |T(\phi_i^*(\chi_i)u \circ \phi_i^{-1})| \leq \|T\|\|\phi_i^*(\chi_i)u \circ \phi_i^{-1}\|_{W^p_{s,\delta}(\mathcal{M})} \]
\[ \leq C\|T\|\|\phi_i^*(\chi_i)u\|_{W^p_{s,\delta}(\mathbb{R}^n)}, \]

and

\[ |\Theta_j^*(\alpha_j T)(u)| = |T(\Theta_j^*(\alpha_j)u \circ \Theta_j^{-1})| \leq \|T\|\|\Theta_j^*(\alpha_j)u \circ \Theta_j^{-1}\|_{W^p_{s,\delta}(\mathcal{M})} \]
\[ \leq C\|T\|\|\Theta_j^*(\alpha_j)u\|_{W^p_{\delta}(\mathbb{R}^n)}. \]

Thus \( \phi_i^*(\chi_i T) \in (W^p_{s,\delta}(\mathbb{R}^n))^* \), and hence Theorem 2.3 (c) implies that \( \phi_i^*(\chi_i T) \in W^p_{-s,-\delta}(\mathbb{R}^n) \). Similarly \( \Theta_j^*(\alpha_j T) \in W^p_{-\delta}(\mathbb{R}^n) \). Computing the norm of \( T \) according to
Hence from Proposition 7.6, we shall now extend the multiplicity property to an asymptotically flat manifold of the class $W^p_{s,\delta}$.

**Proposition 7.6 (Proposition 2.13).** Let $\mathcal{M}$ be an asymptotically flat manifold of the class $W^p_{s,\delta}$, and assume $s \leq \min\{s_1, s_2\}$, $s_1 + s_2 > s + \frac{n}{p}$, $s_1 + s_2 \geq n \cdot \max\{0, \left(\frac{2}{p} - 1\right)\}$ and $\delta \leq \delta_1 + \delta_2 + \frac{2}{p}$, then the multiplication

$$W^p_{s_1,\delta_1}(\mathcal{M}) \times W^p_{s_2,\delta_2}(\mathcal{M}) \to W^p_{s,\delta}(\mathcal{M})$$

is continuous.

**Proof.** Note that $\{\chi^2_j, \alpha^2_j\} := \left(\sum_{i=1}^N \chi_i^2 + \sum_{j=1}^{N_0} \alpha_j^2\right)^{-1}$ $\{\chi^2_j, \alpha^2_j\}$ is also a partition of unity. Since different partitions of unity result in equivalent norms, we have that

$$\|uv\|_{W^p_{s,\delta}(\mathcal{M})} \simeq \sum_{j=1}^{N_0} \|\Theta_j^*(\alpha^2_juv)\|_{W^p_{s_2,\delta_2}(\mathcal{M})} + \sum_{i=0}^N \|\phi_i^*(\chi^2_iuv)\|_{W^p_{s,\delta}(\mathcal{M})}.$$

From Proposition 2.13 we obtain that

$$\|\phi_i^*(\chi^2_iuv)\|_{W^p_{s,\delta}(\mathcal{M})} \lesssim \|\phi_i^*(\chi_iu)\|_{W^p_{s_1,\delta_1}(\mathcal{M})} \|\phi_i^*(\chi_iu)\|_{W^p_{s_2,\delta_2}(\mathcal{M})},$$

and by the corresponding estimate in the unweighted Besov spaces (e.g. [32, §4.6.1]), we also have that

$$\|\Theta_j^*(\alpha^2_juv)\|_{W^p_{s_2,\delta_2}(\mathcal{M})} \lesssim \|\Theta_j^*(\alpha_ju)\|_{W^p_{s_1,\delta_1}(\mathcal{M})} \|\Theta_j^*(\alpha_ju)\|_{W^p_{s_2,\delta_2}(\mathcal{M})}.$$

Now we set $a_i = \|\phi_i^*(\chi_iu)\|_{W^p_{s_1,\delta_1}(\mathcal{M})}$, $a_j = \|\Theta_j^*(\alpha_ju)\|_{W^p_{s_1,\delta_1}(\mathcal{M})}$, $b_i = \|\phi_i^*(\chi_iu)\|_{W^p_{s_2,\delta_2}(\mathcal{M})}$ and $b_j = \|\Theta_j^*(\alpha_ju)\|_{W^p_{s_2,\delta_2}(\mathcal{M})}$. Using the above two estimates and an elementary inequality, we have that

$$\|uv\|_{W^p_{s,\delta}(\mathcal{M})} \lesssim \sum_{i=1}^N a_i b_i + \sum_{j=1}^{N_0} a_j b_j \leq (N_0 + N) \left(\sum_{i=1}^N a_i + \sum_{j=1}^{N_0} a_j\right) \left(\sum_{i=1}^N b_i + \sum_{j=1}^{N_0} b_j\right) \lesssim \|u\|_{W^p_{s_1,\delta_1}(\mathcal{M})} \|v\|_{W^p_{s_2,\delta_2}(\mathcal{M})}.$$

\[\square\]
Finally we show Lemma 3.5 on asymptotically flat manifolds.

**Lemma 7.7** (Lemma 3.5). Let $L$ be an elliptic operator on asymptotically flat manifold $(\mathcal{M}, g)$ of class $W_{s,\delta}^p$ and assume $L \in \text{Asy}(A_{\infty}, s, \delta, p)$, $s \in (\frac{p}{p}, \infty) \cap [1, \infty)$, $\delta \in (-\frac{n}{p}, -2 + \frac{n}{p})$ $p \in (1, \infty)$ and $\delta' < \delta$. Then for any $u \in W_{s,\delta}^p(\mathcal{M}),$

$$\|u\|_{W_{s,\delta}^p(\mathcal{M})} \leq C \left\{ \|Lu\|_{W_{s-2,\delta+2}^p(\mathcal{M})} + \|u\|_{W_{s-1,\delta'}^p(\mathcal{M})} \right\},$$

and the constant $C$ depends on $W_{s,\delta}^p$-norms of the coefficients of $L$, $s, \delta, p$ and $\delta'$.

**Proof.** Let $u \in W_{s,\delta}^p(\mathcal{M})$ and set $\tilde{u} = \phi_i^*(u)$, $\tilde{\chi} = \phi_i^*(\chi_i)$ and $\tilde{L} = \phi_i^*(L)$, then in local coordinates

$$(\tilde{L}\tilde{u})^i = (\tilde{\alpha}_2)_{ij} \left( \partial_a \partial_b \tilde{u} - \Gamma^c_{ab} \partial_c \tilde{u} \right) + (\tilde{\alpha}_1)_{ij} \partial_a \tilde{u} + (\tilde{\alpha}_0)_{ij} \tilde{u},$$

where $\Gamma^c_{ab}$ denote the Christoffel symbols. Then $\tilde{L}\tilde{u}$ is an elliptic operator in the set $E_{\epsilon_1}$, and therefore it can be extend to $\mathbb{R}^n$ such that it will remain elliptic in $\mathbb{R}^n$. Using Definition 3.8, the multiplication property, Proposition 2.13, as well as Propositions 2.8 and 2.17, we obtain that $(\tilde{\alpha}_2 - A_{\infty}) \in W_{s,\delta}^p(\mathbb{R}^n)$ and $(-\tilde{\alpha}_2 \Gamma^c_{ab} + \tilde{\alpha}_1) \in W_{s-1,\delta+1}^p(\mathbb{R}^n)$. Hence we can apply Lemma 3.5 and obtain that

$$(7.6) \quad \|\tilde{\chi}^2 \tilde{u}\|_{W_{s,\delta}^p(\mathbb{R}^n)} \leq C \left\{ \|\tilde{L}(\tilde{\chi}^2 \tilde{u})\|_{W_{s-2,\delta+2}^p(\mathbb{R}^n)} + \|\tilde{\chi}^2 \tilde{u}\|_{W_{s-1,\delta'}^p(\mathbb{R}^n)} \right\}.$$ 

We can now write that $\tilde{L}(\tilde{\chi}^2 \tilde{u}) = \tilde{\chi}^2 \tilde{L}(\tilde{u}) + \tilde{\chi} R\tilde{u}$, where $R$ is an operator of first order and its coefficients contain derivatives of $\tilde{\chi}$. Since $\tilde{\chi}(x) = 1$ for large $|x|$, the coefficients of $R$ have compact support, and therefore the norm $\|\tilde{\chi} R\tilde{u}\|_{W_{s-2,\delta+2}^p(\mathbb{R}^n)}$ is equivalent to $\|\tilde{\chi} R\tilde{u}\|_{W_{s-2,\delta''}^p(\mathbb{R}^n)}$ for any choice of $\delta''$. Applying the multiplication properties, Proposition 2.13, we have that

$$\|\tilde{\chi} R\tilde{u}\|_{W_{s-2,\delta+2}^p(\mathbb{R}^n)} \leq C \left\{ \|\tilde{\chi} \partial \tilde{u}\|_{W_{s-2,\delta+1}^p(\mathbb{R}^n)} + \|\tilde{\chi} \tilde{u}\|_{W_{s-2,\delta}^p(\mathbb{R}^n)} \right\}$$

$$\leq C \left\{ \|\tilde{\chi} \tilde{u}\|_{W_{s-1,\delta'}^p(\mathbb{R}^n)} + \|\tilde{\chi} \tilde{u}\|_{W_{s-1,\delta}^p(\mathbb{R}^n)} \right\} \leq C \|\tilde{\chi} \tilde{u}\|_{W_{s-1,\delta}^p(\mathbb{R}^n)}.$$

Since multiplication with $\tilde{\chi}$ or $\tilde{\chi}^2$ results in equivalent norms, we conclude from the above inequality and (7.6) that

$$\|\tilde{\chi} \tilde{u}\|_{W_{s,\delta}^p(\mathbb{R}^n)} \leq C \left\{ \|\tilde{\chi} \tilde{L}(\tilde{u})\|_{W_{s-2,\delta+2}^p(\mathbb{R}^n)} + \|\tilde{\chi} \tilde{u}\|_{W_{s-1,\delta}^p(\mathbb{R}^n)} \right\}.\quad (7.7)$$

For the covering of the compact part we apply Lemma 32 of [23] and obtain that

$$\|\Theta_j^*(\alpha_j u)\|_{W_{p}^p(\mathbb{R}^n)} \leq C \left\{ \|\Theta_j^*(\alpha_j L u)\|_{W_{p-2}^p(\mathbb{R}^n)} + \|\Theta_j^*(\alpha_j u)\|_{W_{p-1}^p(\mathbb{R}^n)} \right\}.\quad (7.8)$$

Recalling the form of the norm (3.17), we see that inequalities (7.7) and (7.8) complete the proof. $\square$
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