I. INTRODUCTION

The fluctuation theorem (FT) deals with the relative probability that some extensive quantities of a system which is currently away from thermodynamic equilibrium will increase or decrease over a given amount of time. For non-equilibrium statistical mechanics, the ergodic hypothesis usually does not apply. Starting from the deterministic equation such as Liouville’s equation or the von Neumann equation is a safer way. However, the exact solutions of these equations are very difficult to obtain. The FT is a suitable method for non-equilibrium statistical mechanics. The proof of FT is based on the deterministic equation. With some simple assumptions, it can describe some universal properties of nonequilibrium fluctuations such as irreversible work fluctuations and entropy fluctuations. It also gives a generalization of the second law of thermodynamics. The FT bridges the microscopic dynamics and the macroscopic observations. Hence, FT is of fundamental importance to non-equilibrium statistical mechanics.

Research on fluctuation theorems (FTs) for closed quantum systems has been fruitful in recent years [1, 2]. FTs reflect the symmetry between forward evolution and backward evolution processes. In a closed quantum system, the forward evolution process is denoted as $U$, which is unitary. The backward evolution process is the time-reverse evolution process $U^\dagger$, which is also unitary. Under backward evolution, all the initial states can be recovered. In open quantum systems, these properties are not available. For example, when a system and its environment are initially uncorrelated, the dynamical map of the open system is completely positive and trace preserving (CPTP). The volume of the quantum state space of the system can shrink under the dynamical map [3]. The backward map cannot simultaneously guarantee exact recovery and the CPTP properties. A recent work [4] chose the Petz recovery map as a backward map and proved a general FT for quantum channels. Although the evolutions of open systems are derived from closed system environments, their FTs look very different. The average entropy production of a closed system environment is equal to the quantum relative entropy between the final density matrix of the forward evolution process and the initial density matrix of the backward evolution process. The average entropy production of the open system is equal to the decrease in the quantum relative entropy between the initial state and reference state. These seemingly different results are actually closely related. It has been proven that they are equal when the global unitary operation $U$ satisfies the strict energy conservation condition [2].

When memory effects are present, the associated dynamical map cannot be CPTP [3]. The FTs for quantum channels do not seem to apply to such circumstances. Moreover, in the FTs for quantum channels, irrecoverability is the key to the fluctuations. The distinguishability of the states should not increase in the FTs for quantum channels. This is in conflict with the fact that the states can be recovered in non-Markovian processes [6]. Furthermore, the average entropy production is non-negative in the FTs for quantum channels. This means that the corresponding physical observation monotoni...
cally increases during evolution procedure, and this is also in conflict with a non-Markovian process. These contradictions suggest that the common FTs derived from two-point measurements (TPMs) are not applicable to non-Markovian processes. To resolve this issue and obtain the FTs for non-Markovian processes, one approach is to consider the entire system of interest, including its environment.

Here, we propose a different approach inspired by the process tensor [1]. In the framework of the process tensor, the quantum system is undergoing a process that one can split into arbitrary discrete time steps. There multi-time processes fully characterize the interaction between system and environment. The process tensor is separated from the Choi representation for multi-time processes. This tensor is the Choi state of a many-body channel, which contains all the available evolutionary information. The many-body channel itself is a CPTP map regardless of whether the evolution is Markovian. One can trace a multi-time process back by executing the process tensor on identity operations. For more general CPTP operators, we obtain other multi-time channels. By applying the FTs developed for quantum channels, it is easy to obtain the FTs for many-body channels and derived channels.

In this paper, we study the FTs for many-body channels and derived channels. In section II, we first rewrite the proof of the FTs for quantum channels with superoperators and then with Choi representation. This helps to simplify the proof. We also find an equivalence relation between the FTs for closed systems and the FTs for open systems without strict energy conservation conditions. After these preparations, we study the FTs for multi-time processes in section III. We first present a many-body channel and several derived channels that are closely related to multipoint measurements. We show that the Petz recovery map of the many-body channel cannot derive channels as we do for the many-body channel. Only for a Markovian process can the Petz recovery map be time-ordered and linkable. Consistently, one can trace the backward evolution process from the Petz recovery map of the many-body channel for a Markovian process. In such a form, the intermediate state of the system is completely measurable [3]. For non-Markovian processes, we can only trace the backward evolution process from the Petz recovery map of the derived channel. We mainly focus on a derived channel that is closely related to a general measurement [11] of the intermediate state. We prove the corresponding FTs and show how the memory effects reduce the system fluctuations.

II. PRELIMINARIES

A. Proving FTs with a superoperator

We reformulate the FTs for quantum channels [4] with a superoperator [11]. We regard an operator $O$ as a state $|O\rangle$. The inner product of the operators $(O_A|O_B) := \text{Tr}(O_A^\dagger O_B) = (O_B|O_A)^*$. The operator vector space is orthonormalized as follows:

$$(\Pi_{ij}|\Pi_{ij}) = \delta_{ik}\delta_{jl},$$

where $\Pi_{ij} = |i\rangle \langle j|$. The completeness relation is

$$\hat{I} = \sum_{ij} |\Pi_{ij}\rangle \langle \Pi_{ij}|.$$  

The quantum channel $\mathcal{N}$ is a superoperator that maps a density matrix $|\rho\rangle$ to another density matrix $|\rho'\rangle = \mathcal{N}|\rho\rangle$.

The forward transition matrices are defined as

$$T_{ij\rightarrow kj} = (\Pi_{ij}|\mathcal{N}|\Pi_{kj}),$$

where quantum channel $\mathcal{N}(\cdot) = \sum M_i(\cdot)M_i^\dagger$ is a CPTP map. The forward transition matrices contain all evolution information. It is easy to obtain the transition probabilities from this information. By using the definition of the inner product of operators, we have

$$T_{ij\rightarrow kj}^\dagger = (\Pi_{ij}|\mathcal{N}^\dagger|\Pi_{kj}^\dagger)^*.$$  

The superoperator $\mathcal{N}^\dagger(\cdot) = \sum M_i^\dagger(\cdot)M_i$ is not a CPTP map, so it cannot be used as a backward map. The Petz recovery map [12]

$$\mathcal{R}_\gamma = \mathcal{J}_\gamma^{1/2} \circ \mathcal{N}\circ \mathcal{J}_\gamma^{-1/2}$$

is commonly used as a backward map. $\gamma$ is called the reference state, and $\mathcal{J}_\gamma^\alpha(\cdot) := O^\alpha(\cdot)O^\dagger$ is defined as a rescaling map. The Petz recovery map is CPTP. It can fully recover the reference state $\mathcal{R}_\gamma \circ \mathcal{N}(\gamma) = \gamma$ but generally cannot recover other states. The trace preserving property of the Petz recovery map is easily obtained from

$$(I|\mathcal{J}_\gamma^{1/2} \circ \mathcal{N}\circ \mathcal{J}_\gamma^{-1/2}|O) = (O|\mathcal{J}_\gamma^{-1/2} \circ \mathcal{N}(\gamma)^* = \text{Tr}(O).$$

(6)

With the Petz recovery map, Eq. (1) becomes

$$T_{ij\rightarrow kj} = (\mathcal{J}_\gamma^{-1/2}\Pi_{ij}|\mathcal{R}_\gamma|\mathcal{J}_\gamma^{1/2}\Pi_{kj}^\dagger)^*.$$  

(7)

The operators obtained from the rescaling map are no longer normalized. After normalization, we have so-called reference-rescaled operators [13]:

$$|\Pi_{kj}^\gamma\rangle = \frac{1}{Z_{kj}^\gamma} |\mathcal{J}_\gamma^{1/2}\Pi_{kj}^\dagger\rangle,$$

(8)

where the factor

$$Z_{kj}^\gamma = \sqrt{\langle \mathcal{J}_\gamma^{1/2}\Pi_{kj}^\dagger|\mathcal{J}_\gamma^{1/2}\Pi_{kj}^\dagger \rangle} = \|\mathcal{J}_\gamma^{1/2}\Pi_{kj}^\dagger\|_2 = \sqrt{\langle (\Pi_{kj}^\gamma\Pi_{kj}^\gamma)^* \rangle}.$$  

(9)

Similarly, we have $|\Pi_{ij}\rangle = |\mathcal{J}_\gamma^{-1/2}\Pi_{ij}\rangle/Z_{ij}^\gamma$ and

$$Z_{ij} = \sqrt{\langle \mathcal{J}_\gamma^{-1/2}\Pi_{ij}^\dagger|\mathcal{J}_\gamma^{-1/2}\Pi_{ij}^\dagger \rangle} = \|\mathcal{J}_\gamma^{-1/2}\Pi_{ij}\|_2 = \sqrt{\langle (\Pi_{ij}^\gamma\Pi_{ij}^\gamma)^* \rangle}.$$  

(10)
With reference-rescaled operations, Eq. (7) can be rewritten as

\[ T_{ij \rightarrow k'l'} = \tilde{T}_{ij \rightarrow k'l'} Z_{ij} Z_{k'l'}, \quad (11) \]

where

\[ \tilde{T}_{ij \rightarrow k'l'} = (\Pi_{ij} | \mathcal{R}_\gamma | \Pi_{k'l'}^\prime) \quad (12) \]
denotes the backward transition matrices. Eq. (11) shows a clear relationship between the forward transition matrices and the backward transition matrices.

Suppose that the system evolves from \( \rho = \sum_u p_u |\psi_u \rangle \langle \psi_u| \) to \( \rho' = \mathcal{N}(\rho) = \sum_{u'} p'_{u'} |\phi'_{u'} \rangle \langle \phi'_{u'}| \). Then, the two-point measurement (TPM) quasiprobability distribution for the forward process is

\[ P_{ij,k'l'}^{u,u'} = p_u (\Pi_{\phi'_{u'}} | \Pi_{k'l'}^\prime) (\Pi_{k'l'}^\prime | \mathcal{R}_\gamma | \Pi_{ij}) (\Pi_{ij} | \Pi_{\psi_u}), \quad (13) \]

where \( \Pi_{\psi_u} = |\psi_u \rangle \langle \psi_u| \) and \( \Pi_{\phi'_{u'}} = |\phi'_{u'} \rangle \langle \phi'_{u'}| \). It is easy to show that

\[ (P_{ij,k'l'}^{u,u'})^* = p_u (\Pi_{\phi'_{u'}} | \Pi_{k'l'}^\prime) (\Pi_{k'l'}^\prime | \mathcal{R}_\gamma | \Pi_{ij}) (\Pi_{ij} | \Pi_{\psi_u}) = P_{ji,k'l'}^{u,u'}. \quad (14) \]

The TPM quasiprobability distribution for the backward process is

\[ P_{ij,k'l'}^{u,u'} = p_{u'} (\Pi_{\psi_u} | \Pi_{ij}^\dagger) (\Pi_{ij}^\dagger | \mathcal{R}_\gamma | \Pi_{k'l'}^\prime) (\Pi_{k'l'}^\prime | \Pi_{\phi'_{u'}}). \quad (15) \]

The entropy production can be defined as

\[ \sigma_{ij \rightarrow k'l'} = \delta s_{ij \rightarrow k'l'} = -\delta q_{ij \rightarrow k'l'} \quad (16) \]

where \( \delta s_{ij \rightarrow k'l'} = -\log(p_{k'l'}) + \log(p_u) \) and \( \delta q_{ij \rightarrow k'l'} = -\log(Z_{ij} Z_{k'l'}^*) \).

According to its definition, the entropy production satisfies

\[ \sigma_{ij \rightarrow k'l'} = \sigma_{ji \rightarrow k'l'} \quad (17) \]

The entropy production distribution can be obtained directly from the TPM quasiprobability as follows:

\[ P_{\sigma}(\sigma) = \sum_{u,i,j} \sum_{u',v',k,l} P_{ij,k'l'}^{u,u'} \delta(\sigma - \sigma_{ij \rightarrow k'l'}), \quad (18) \]

The entropy production distribution under the recovery map can be defined as

\[ P_{\sigma}(\sigma) = \sum_{u,i,j} \sum_{u',v',k,l} \tilde{P}_{ij,k'l'}^{u,u'} \delta(\sigma + \sigma_{ij \rightarrow k'l'}^*). \quad (19) \]

The basis \( \{|i\} \) is chosen such that it diagonalizes the reference state \( \gamma \), and \( \{|k'\} \) is chosen as the eigenbasis of \( \mathcal{N}(\gamma) \). These facts lead to \( |\Pi_{ij}\rangle = |\Pi_{ij}\rangle \) and \( |\Pi_{k'l'}^\prime\rangle = |\Pi_{k'l'}^\prime\rangle \). Combining this with Eqs. (11), (14) and (17) to (19), we derive the following relation:

\[ \frac{P_{\sigma}(\sigma)}{P_{\sigma}(-\sigma)} = e^\sigma. \quad (20) \]

The FT \( \langle e^\sigma \rangle = 1 \) can be easily obtained from this relation, and the generalized second law yields

\[ \langle \sigma \rangle = S(\rho || \gamma) - S(\mathcal{N}(\rho) || \mathcal{N}(\gamma)) \geq 0, \quad (21) \]

which is consistent with the conclusion that the distinguishability of quantum states does not increase under a CPTP map.

**B. Relationship between two types of FTs**

Here, we show the relationship between the FTs for quantum channels and the FTs for closed systems. The premise is that their evolution processes must be consistent; that is, the dynamical map of an open system is

\[ \mathcal{N}(\cdot) = \text{Tr}_E[U_{SE} \cdot \otimes \rho_E^0 U_{SE}^\dagger]. \quad (22) \]

Under a strict energy conservation condition and by choosing \( \rho_{SE} = \rho_S^0 \otimes \rho_E^0 \), the entropy production of a system environment is \[ \Sigma = S(\rho_{SE} || \rho_{SE}^0) \geq S(\rho_S^0 || \rho_S^0). \quad (23) \]

\( \Sigma \) is equal to the entropy production \( \langle \sigma \rangle \) yielded when setting the global fixed points \( \rho_{SE}^0 = \rho_E^0 \) as the reference state.

The strict energy conservation condition is not necessary to bridge two FTs. The reference state does not have to be the global fixed points. For an arbitrary reference state, the entropy production of Eq. (21) becomes

\[ \langle \sigma \rangle = S(\rho_S \otimes \rho_E^0 || \gamma_S \otimes \rho_E^0) - \text{Tr}_E(\rho_{SE} (\ln \rho_{SE} - \ln \gamma_{SE}^0)) = \text{Tr}_E(\rho_{SE} (\ln \rho_{SE}^0 - (\ln \rho_S^0 + \ln \gamma_{SE} - \ln \gamma_{SE}^0))) = S(\rho_{SE} || \tilde{\rho}_{SE}) \quad (24) \]

where \( \tilde{\rho}_{SE} = \exp(\ln \rho_S^0 + \ln \gamma_{SE} - \ln \gamma_{SE}^0) \). We use Eq. (22) in the equality.

Here, we briefly discuss the meaning of \( \tilde{\rho}_{SE} \). If the dynamical map does not change the relative entropy between \( \rho \) and \( \gamma \), then we have

\[ \langle \sigma \rangle = 0 = S(\rho_{SE} || \gamma_{SE}) - S(\rho_S^0 || \gamma_{SE}^0). \quad (25) \]

In the sandwiched Rényi divergence, the equivalent condition of Eq. (25) is \[ \gamma_S^{\alpha} (\gamma_S \otimes \rho_S^{\alpha})^{-1} \gamma_S^{\alpha} \otimes I_E \]

\[ = \gamma_S^{\alpha} (\gamma_S \otimes \rho_S^{\alpha})^{-1} \gamma_S^{\alpha} \quad (26) \]

where \( \beta = (1 - \alpha)/2 \alpha \). Condition (25) can be rewritten as

\[ \rho_{SE} = \mathcal{R}_{S \rightarrow SE}^{\alpha} (\rho_S^0) := J_{SE}^{\beta} (J_{SE}^{\beta} \otimes J_{SE}^{\beta} (J_{SE}^{\beta} \rho_S^{\alpha - 1})^{1/\beta}) \quad (27) \]

It is easy to show that

\[ \lim_{\alpha \rightarrow 1} \mathcal{R}_{S \rightarrow SE}^{\alpha} (\rho_S^0) = \tilde{\rho}_{SE}. \quad (28) \]
\( \hat{\rho}_{SE} \) is given by the map \( R_{S \rightarrow SE}^{\alpha, \gamma, N} \) and is very different from the normal chosen \( \hat{\rho}_{SE} = R_{S \rightarrow SE}(\rho_S') = \rho_S' \otimes \rho_E \). The map is also different from the maps listed in [2]. System-environment correlations are allowed in \( R_{S \rightarrow SE}^{\alpha, \gamma, N}(\rho_S') \). To our limited knowledge, the map \( R_{S \rightarrow SE}^{\alpha, \gamma, N} \) is new and needs further research.

When we choose \( \hat{\rho}_{SE} = R_{S \rightarrow SE}(\rho_S) \) as the initial state of the backward process, the given entropy production of the entire system, including the environment, is

\[
\Sigma = S(\rho'_{SE}||\hat{\rho}_{SE})
\]

(29)

according to the FTs for closed quantum systems [1]. It is equal to the entropy production given in Eq. [21]. Hence, the two FTs are consistent with each other.

One advantage of the FTs for quantum channels is that the entropy production \( \langle \sigma \rangle \) is also closely related to the TP. If the reference state can be described with intensive properties and extensive properties, such as

\[
\gamma(t) = \exp(-\sum \beta_i O^i(t)/Z(t)),
\]

(30)

then we have that \( \ln \gamma = -\sum \beta_i O^i(0) + \beta F(0) \) and \( \ln N(\gamma) = -\sum \beta_i O^i(t) + \beta F(0) \). Under these circumstances, the second law given by Eq. [21] is related to the observables

\[
\beta \langle w \rangle - \Delta S \leq \beta \Delta F,
\]

(31)

where \( \Delta S = S(\rho') - S(\rho) \) and

\[
\beta \langle w \rangle = \sum \beta_i [\text{Tr} O^i(t) \rho_S' - \text{Tr} O^i(0) \rho_S].
\]

(32)

Any map involving a thermal environment necessarily has its thermal state as a fixed point when strict energy conservation holds [2, 13], so the above assumption [20] is fairly general.

Another advantage of the FTs for quantum channels is that the initial state of the system is arbitrary. We do not need to suppose that the initial state of the system is in equilibrium, while this assumption is often used in the FTs for closed systems.

C. Rewriting FTs with the Choi-Jamiolkowski isomorphism

We rewrite the FT proof with the Choi-Jamiolkowski isomorphism. According to the proof in section II A, the TPM and quantum channel are the keys to the FTs. The Choi-Jamiolkowski isomorphism can convert the quantum channel to a Choi state [16] and realize the TPM with one operator. This helps to generalize the FTs for quantum channels to multitime processes.

In the Choi-Jamiolkowski isomorphism, one must introduce an auxiliary system \( A \) with the same dimensionality as the original system. The density matrix of the maximally entangled state between the ancilla and Choi-Jamiolkowski system is as follows:

\[
\Phi^{AS} = 1/N \sum_{ij} \Pi^A_{ij} \otimes \Pi^S_{ij},
\]

(33)

where \( N \) is the Hilbert space dimensionality of the system. The density operator \( I^A \otimes N^S(\Phi^{AS}) \) is called the Choi state. The maximally entangled state has the following property:

\[
N(\rho_A|\Phi^{AS}) = \rho_S = |\rho_S\rangle,
\]

(34)

with which we can rewrite the forward transition matrices in Eq. [8] as

\[
T_{ij \rightarrow k'l'} = N(\Pi^A_{ij} \otimes \Pi^S_{k'l'}|N_{AS}|\Phi^{AS}),
\]

(35)

where \( N^{AS} = I^A \otimes N^S \) is still a CPTP map. In this form, the TPM is realized with the operator \( N(\Pi^A_{ij} \otimes \Pi^S_{k'l'}). \) The forward transition matrices can also be treated as the inner product of the measurement operator and the Choi state. The relation in Eq. [21] becomes

\[
T_{ij \rightarrow k'l'} = N((\mathcal{J}^{AS}_{\gamma, N(\gamma)})^{1/2} \Pi^A_{ij} \otimes \Pi^S_{k'l'}|*).
\]

(36)

For arbitrary operators \( O^1 \) and \( O^2 \) and a superoperator \( \mathcal{M} \), the maximally entangled state \( \Phi^{AS} \) can switch the remapping:

\[
(O^A_{\alpha} \otimes O^2_{\beta} |\mathcal{M}_S |\mathcal{J}_{\alpha S} \Phi^{AS}) = (O^2_{\beta} |\mathcal{M}_S |\mathcal{J}_{\alpha S} \Phi^{AS})
\]

(33)

This allows us to move the full remapping in Eq. [36] to the same side and obtain

\[
T_{ij \rightarrow k'l'} = N(\Phi^{AS})(\mathcal{J}^{AS}_{\gamma, N})^{1/2} \Pi^A_{ij} \otimes \Pi^S_{k'l'}|*),
\]

(38)

where \( \mathcal{J}^{AS}_{\gamma, N} = (\mathcal{J}^{AS}_{\gamma})^{-1} \otimes \mathcal{J}^{AS}_{\gamma} \). The backward transition matrices can be defined as

\[
T_{ij \rightarrow k'l'} = N(\Phi^{AS} |\mathcal{J}^{AS}_{\gamma, N})^{1/2} \Pi^A_{ij} \otimes \Pi^S_{k'l'}|*).
\]

(39)

Eq. [11] still holds, and the TPM quasiprobability distribution for the forward process is now written as

\[
P_{ij, k'l'}^{u, v} = p_u N(\Pi^A_{ij} \otimes \Pi^S_{k'l'}|N_{ij} \otimes N_{k'l'}) N(\Phi^{AS}).
\]

(41)

The TPM quasiprobability distribution for the backward process becomes

\[
P_{ij, k'l'}^{u, v} = p'_u N(\Phi^{AS} |\mathcal{J}^{AS}_{\gamma, N})^{1/2} \Pi^A_{ij} \otimes \Pi^S_{k'l'}|N_{ij} \otimes N_{k'l'}) N(\Phi^{AS}).
\]

(42)

The other proofs and results are not different from those in section II A.
III. MULTITIME PROCESSES

As in the procedure of the process tensor $\mathcal{A}$, we assume that the system and environment are initially uncorrelated. After the first step, the unrestricted unitary evolution of the system environment enables the creation of correlations. Such multitime processes can map the initial system $|\rho_S\rangle$ to

$$|\rho_S\rangle = (I_E \rho_{SE}^0 \otimes \cdots \otimes \rho_{SE}^0), \quad \text{(43)}$$

where $\rho_{SE}^i$ is the initial density matrix of the environment. The corresponding process tensor is

$$|T_n\rangle = \text{Tr}_E \rho_{S^n E}^0 \otimes \cdots \otimes \rho_{S^n E}^0 \otimes A (\prod_{i=1}^n \Phi^{A_i S^i}), \quad \text{(44)}$$

where $A(\cdot) = (\cdot) \otimes \rho_E^0$. This process tensor is the Choi state of the following many-body channel:

$$\mathcal{N}^{S^n}_{S^n} = \text{Tr}_E \rho_{S^n E}^0 \otimes \cdots \otimes \rho_{S^n E}^0 \otimes A, \quad \text{(45)}$$

which can also be treated as the process by which multiple systems interact with a single environment in turn. If the evolution is a Markovian process, the many-body channel can be decomposed into several CPTP maps $\mathcal{F}$:

$$\mathcal{N}^{S^n}_{S^n} = \mathcal{N}^{S^n}_0 \cdots \mathcal{N}^{S^n}_1. \quad \text{(46)}$$

Such a decomposition does not hold when the evolution process is non-Markovian. The output state of the Markovian many-body channel in Eq. (40) is a tensor product state when the input state is a tensor product state:

$$|\rho_{S^n}^{1 \cdots n}\rangle = N^n (\rho_1^A \otimes \cdots \otimes \rho_n^A |T_n\rangle = |\rho_{S^n}^{1 \cdots n}\rangle. \quad \text{(47)}$$

The decomposition also does not hold for non-Markovian evolution.

The many-body channel in Eq. (41) is time-ordered, which means that the future process will not affect the current state:

$$(I^{S^{i+1} \cdots S^n} |N^{S^{i+1} \cdots S^n} |\rho_1 \otimes \cdots \otimes \rho_n) = N^{S^{i+1} \cdots S^n} |\rho_1 \otimes \cdots \otimes \rho_n). \quad \text{(48)}$$

However, the current state can affect the future evolution:

$$(I^{S^1 \cdots S^i} |N^{S^1 \cdots S^i} |\rho_1 \otimes \cdots \otimes \rho_n) = N^{S^{1+1} \cdots S^n} |\rho_{i+1} \otimes \cdots \otimes \rho_n), \quad \text{(49)}$$

where the maps $N^{S^{1+1} \cdots S^n}$ vary with the historical states $\rho_1 \otimes \cdots \otimes \rho_i$. Only when the evolution process is Markovian can the future evolution be history-independent. The many-body channel in Eq. (41) is also linkable; that is, a new quantum channel can be derived by linking the output state of the previous step with the input state of the next step:

$$\mathcal{N}^{S^{1} \cdots S^{i+2} \cdots S^n} = \text{Tr}_E \rho_{S^n E}^0 \otimes \cdots \otimes \rho_{S^{i+1} E}^0 \otimes \rho_{S^i E}^0 \otimes A. \quad \text{(50)}$$

The multitime evolution can be obtained by linking all steps:

$$\mathcal{N}^S = \text{Tr}_E \rho_{S^n E}^0 \otimes \cdots \otimes \rho_{S^1 E}^0 \otimes A. \quad \text{(51)}$$

The Choi state of $\mathcal{N}^S$ can be obtained by linking the process tensor $|T_n\rangle$ as follows:

$$|T^S\rangle = N^{2n-2} (\prod_{i=1}^{n-1} \Phi^{A_i+1 S^i} |T_n\rangle. \quad \text{(52)}$$

The evolution $\mathcal{N}^S$ does not contain any measurements or control operations. If needed, one can apply CPTP operations $A_i$ between the steps for manipulation purposes at intermediary time steps. This results in the following quantum channel:

$$\mathcal{N}^S_{n-1} = \text{Tr}_E \rho_{S^n E}^0 \otimes (A_{n-1} \rho_{S^{n+1}}) \otimes \cdots \otimes (A_1 \rho_{S^2}) \otimes A. \quad \text{(53)}$$

We graphically illustrate the many-body quantum channel and its derived channel in fig. 1.

One can derive channels by linking the indices of the many-body quantum channel:

$$\mathcal{N}^S (\Phi^{A_i S^{i+1}}) = N^2 (\Phi^{A^{i+1} S^i} |N^S \Phi^{A_i S^i} \otimes \Phi^{A^{i+1} S^{i+1}}). \quad \text{(54)}$$

Therefore, it is natural to wonder whether the corresponding FTs can be obtained from the FTs for a many-body quantum channel. Unfortunately, this is not possible. A many-body quantum channel is time-ordered and linkable, and this does not necessarily mean that the corresponding Petz recovery is time-ordered and linkable. Therefore, the Petz recovery map of a derived channel cannot be obtained by linking the indices of the Petz recovery map of the many-body quantum channel.

$$\mathcal{N}^S (\Phi^{A^{i+1} S^{i+1}} |R') \neq N^2 (\Phi^{A^{i+1} S^{i+1}} \otimes \Phi^{A^{i+1} S^{i+1}} |R |\Phi^{A^{i+1} S^{i+1}}). \quad \text{(55)}$$

Consequently, one cannot obtain all those FTs in one step. These relations are depicted in fig. 2.

As we cannot obtain all the FTs at one time, we analyze these channels separately. In this section, we first give the FTs for two ordinary channels. After that, we study the FTs for Markovian evolution, which allows for the complete measurement of intermediate states. However, these procedures are not applicable to non-Markovian evolution. We analyze the underlying causes of this fact. For non-Markovian evolution, we use a general measurement for the intermediate state and obtain the FTs that contain the effects of intermediate measurements.
FIG. 1: The many-body channel (A) can be extracted from multitime processes. It is time-ordered and linkable. The many-body channel itself can be treated as multitime processes that freshly prepare the system state at each step. When linking the steps without measurements, channel (B) emerges. For a Markovian process, the measurements over the intermediate state do not lead to any conflict. Therefore, we can link the steps with the measurements and obtain (C). For a non-Markovian process, we can only insert operations and discuss the FTs with derived channel (D).

FIG. 2: A many-body channel is linkable, which does not mean that its Petz recovery map is also linkable. The Petz recovery map of the derived channel cannot be obtained with the Petz recovery map of the corresponding many-body channel.

A. Two ordinary channels

Here, we present the FTs for many-body channels and channels that evolve without intermediate measurements. Both many-body channels $N^{S_1 \ldots S_n}$ and $N^S$ in Eq. (51) still follow single-step evolution. Therefore, the approaches and results shown in section II A are applicable.

The many-body channels $N^{S_1 \ldots S_n}$ map the initial tensor product states $\rho_I = \rho_{S_1}^1 \otimes \cdots \otimes \rho_{S_n}^n = \sum_{u_1 \cdots u_n} \prod_{i=1}^n T_{u_i} \Pi_{\psi_{u_i}}^{S_i}$ to their final states $\rho_F = \rho_{S_1 \ldots S_n}^{k' \cdots l'} = \sum_{u'} \prod_{i=1}^n T_{u_i} \Pi_{\psi_{u_i}'}^{S_i \ldots S_n}$, which are not of tensor product form unless the evolution process is Markovian. The forward transition matrices can be defined as

$$T_{i_1 j_1 \cdots i_n j_n \rightarrow k' l'} = N^n(\prod_{i=1}^n T_{i_1 j_1} \otimes \cdots \otimes T_{i_n j_n} \otimes \Pi_{k' l'}) |T_n).$$

The TPM quasiprobability distribution for the forward process can be defined as

$$P_{i_1 j_1 \cdots i_n j_n, k' l'} = T_{i_1 j_1 \cdots i_n j_n \rightarrow k' l'} \times$$

$$(\prod_{i=1}^n T_{i_1 j_1} \otimes \cdots \otimes T_{i_n j_n} \otimes \Pi_{k' l'}).$$
The entropy production can be defined as
\[\sigma_{i_1,j_1 \cdots i_n,j_n \to k' l'} = -\log(p_{i'}) + \log(p_{i_1}^1) + \cdots + \log(p_{i_n}^n) - (-\log(Z_{i_1,j_1} \cdots Z_{i_n,j_n} Z'_{k,l'})). \tag{58}\]
The quantities for the backward process can be similarly defined. The relation in Eq. (20) still holds for the quantities defined here, and the generalized second law becomes
\[\langle \sigma \rangle = S(p_i^S|\gamma_S^S) - S(p_{i'}^S|\gamma_S^S). \tag{60}\]

Suppose that \(N_2 \otimes N_1\) is the Markov process that is closest to process \(N_2, N_1\). We can rewrite Eq. (60) as
\[\langle \sigma \rangle = S(p_i^S|\gamma_S^S) - (N_2(p_i^S)|N_1(\gamma_S^S)) + S(p_i^S|\gamma_S^S) - S(N_2(p_i^S)|N_2(\gamma_S^S)) - \sigma_{NM}, \tag{61}\]

where \(\rho_i^S = N_2(p_i^S)\). The nonnegative quantity
\[\sigma_{NM} = S(p_i^S|\gamma_S^S) - S(N_2(p_i^S)|N_2(\gamma_S^S)) \tag{62}\]
is related to the increase in distinguishability, which implies the memory effects. The quantity \(\sigma_{NM}\) is similar to the quantity used in the Breuer-Laine-Piilo (BLP) measure \(\sigma\). Eq. (61) tells us that memory effects can reduce the system fluctuations.

### B. Markovian process

The Petz recovery map of a Markovian process is time-ordered and linkable. The measurement of the intermediate state does not affect the future evolution. These properties make Markovian processes special.

In a Markovian process, the system state evolves from the initial state \(\rho_1 = \sum u p_u \Pi_{uv}^S\) to the intermediate state \(\rho_{M} = N_1 \cdots N_n(\rho_1) = \sum \rho_i \Pi_{uv}^S\), and finally to \(\rho_F = N_n \cdots N_1(\rho_1) = \sum \rho_i \Pi_{uv}^S\). We can completely measure the intermediate state. The linking operation \(\Phi^{A^{i+1}S^i}\) in Eq. (54) is the Choi state of the identity mapping \(\Phi^{A^{i+1}S^i} = I^S \otimes \Phi^{A^{i+1}S^i}\). If we replace this operation with projection measurements \(M_\mu\), where \(M_\mu(\rho) = \Pi_\mu(\rho) \Pi_\mu^\dagger\), we obtain the operation
\[M_\mu \rho^{A^{i+1}S^i} = \sum \Pi_{\mu \nu}^{A^{i+1}} \otimes \Pi_{\mu \nu}^{S^i}/N. \tag{63}\]

If such an operation is applied to the full intermediate state, we obtain a tensor similar to that in Eq. (52):
\[\sum_{v_1 \cdots v_{n-1}} ((\prod_{i=1}^{n-1} \Pi_{\mu \nu_i}^{A_{i+1}} \otimes \Pi_{\mu \nu_i}^{S^i})|N^{S^1 \cdots S^n}|(\prod_{i=1}^{n} \Phi^{A_i S^i})). \tag{64}\]

When we ignore the intermediate measurements, this tensor returns the unmeasured tensor:
\[\sum_{\mu_1 \cdots \mu_{n-1}} |T_{\mu_1 \cdots \mu_{n-1}}\rangle = |T^S\rangle. \tag{65}\]

Now, we prove the FT. For simplicity, we only consider the two-step evolution process. The system state evolves from \(\rho_1 = \sum u p_u \Pi_{uv}^S\) to \(\rho_M = \sum \rho_i \Pi_{uv}^S\) and finally to \(\rho_F = \sum \rho_i \Pi_{uv}^S\). Since the many-body channel of Markovian evolution gives tensor product states, we can define the forward transition matrices as

\[T_{ij,kl \to k' l', m'n'} = N^2(\Pi_{ij}^A \otimes \Pi_{k'l'}^S \otimes \Pi_{kl}^A \otimes \Pi_{m'n'}^S|N_2 \circ N_1|\Phi^{AS} \otimes \Phi^{A'S'}). \tag{66}\]

As in the procedure of Eq. (38), the forward transition matrices can be expressed with a Petz recovery map:

\[T_{ij,kl \to k' l', m'n'} = N^2(\Phi^{AS} \otimes \Phi^{A'S'}|R_{\gamma, \gamma'}((J^{AS \gamma S'}_{\gamma, \gamma'}N_2^i \circ N_1^S)|\Phi^{AS} \otimes \Phi^{A'S'}). \tag{67}\]

where the Petz recovery map \(R_{\gamma, \gamma'} = (J^{S}_{\gamma'})^{-1/2} \circ (J^{S}_{\gamma'})^{1/2} \circ N_1^S \circ N_2^S \circ (J^{S}_{\gamma S'} \otimes \gamma S')^{-1/2}\) is a CPTP map and \(J^{AS \gamma S'} = (J^{S}_{\gamma'})^{-1} \otimes (J^{S}_{\gamma'})^{-1} \otimes J^{S}_{\gamma S'} \otimes N_1^S \otimes \gamma S').\) The final reference state here is the tensor product state \(N(\gamma S \otimes \gamma S') = N_1^S(\gamma S) \otimes N_2^S(\gamma S')\), which makes the Petz recovery map divisible: \(R_{\gamma, \gamma'} = R_1^S \circ \circ \circ \otimes R_2^S\). The factor of the rescaled
We are free to choose the basis of $\Pi^\gamma$ according to Eq. (64), we define the quasiprobability distribution of the three-point measurement for the forward process

$$Z_{ij}^{-1} Z_{kl}^{-1} Z_{k'l'}^{-1} Z_{m'n'}^{-1} = (\hat{J}_{ij}^{A^S, A^I} | J_{ij}^{A^S, A^I} = 1/2 | \Pi^A_{ij} \otimes \Pi^S_{k'l'} \otimes \Pi^A_{k'l'} \otimes \Pi^S_{m'n'} | 2).$$

The backward transition matrices can be defined as

$$T_{ij,kl\to k'l',m'n'} = N^2(\Phi^{AS} \otimes \Phi^{A'S'}) | \mathcal{R}_{ij,kl} | \Pi^A_{ij} \otimes \Pi^S_{k'l'} \otimes \Pi^A_{k'l'} \otimes \Pi^S_{m'n'} | 2).$$

where $|\Pi^A_{ij} \otimes \Pi^S_{k'l'} \otimes \Pi^A_{k'l'} \otimes \Pi^S_{m'n'} | = (\hat{J}_{ij}^{A^S, A^I} | J_{ij}^{A^S, A^I} = 1/2 | \Pi^A_{ij} \otimes \Pi^S_{k'l'} \otimes \Pi^A_{k'l'} \otimes \Pi^S_{m'n'} | 2)$. The relation between the forward transition matrices and the backward transition matrices is

$$T_{ij,kl\to k'l',m'n'} = T_{ij,kl\to k'l',m'n'} = Z_{ij}^{-1} Z_{kl}^{-1} Z_{k'l'}^{-1} Z_{m'n'}^{-1}.$$ (69)

Since we need to measure the intermediate state, the TPM should be turned into a three-point measurement. According to Eq. (64), we define the quasiprobability distribution of the three-point measurement for the forward process as

$$P^{u',w',u''}_{i,j,k',l',m',n'} = p_u \sum_{\nu} N^2(\Pi^A_{\psi_u} \otimes \Pi^S_{\nu} \otimes \Pi^A_{\nu} \otimes \Pi^S_{\psi_{w'}} | \Pi^A_{ij} \otimes \Pi^S_{k'l'} \otimes \Pi^A_{k'l'} \otimes \Pi^S_{m'n'} | 2) \times (\Pi^A_{ij} \otimes \Pi^S_{k'l'} \otimes \Pi^A_{k'l'} \otimes \Pi^S_{m'n'} | \Delta^{S'}_2 \otimes \Delta^{S'}_1 | \Phi^{AS} \otimes \Phi^{A'S'} |).$$

We are free to choose the basis of $\Pi^S_{\nu}$ and $\Pi^A_{\nu}$. Here, we set it equal to the basis of $\{\psi_{w'}, \psi_u\}$. It is easy to prove that $P^{u',w',u''}_{i,j,k',l',m',n'}$ satisfies the marginality condition:

$$\sum_{u',w',u''} \sum_{i,j,k',l',m',n'} P^{u',w',u''}_{i,j,k',l',m',n'} = p_u N^2(\Pi^A_{\psi_u} \otimes \Pi^S_{\nu} \otimes \Pi^A_{\nu} \otimes \Pi^S_{\psi_{w'}} | \Pi^A_{ij} \otimes \Pi^S_{k'l'} \otimes \Pi^A_{k'l'} \otimes \Pi^S_{m'n'} | 2) = p_u.$$ (70)

$$\sum_{u',w',u''} \sum_{i,j,k',l',m',n'} P^{u',w',u''}_{i,j,k',l',m',n'} = \sum_{\nu} N^2(\rho^A_{\psi_u} \otimes \Pi^S_{\nu} \otimes \Pi^A_{\nu} \otimes \Pi^S_{\psi_{w'}} | \Pi^A_{ij} \otimes \Pi^S_{k'l'} \otimes \Pi^A_{k'l'} \otimes \Pi^S_{m'n'} | 2) = \sum_{\nu} (\Pi^S_{\psi_{w'}} | \Delta^S_2 | \Pi^S_1 | \rho^A_1) = p'_{\nu}. (71)

$$\sum_{u',w',u''} \sum_{i,j,k',l',m',n'} P^{u',w',u''}_{i,j,k',l',m',n'} = \sum_{\nu} (\Pi^S_{\psi_{u''}} | \Delta^S_2 | \Pi^S_{\nu} | \Pi^S_1 | \rho^S_1) = (\Pi^S_{\psi_{w'}} | \Delta^S_2 | \Pi^S_{\nu} | \Pi^S_1 | \rho^S_1) = p''_{\nu', w'}. (72)

The entropy production can be defined as

$$\delta_{ijkl\to k'l'm'n'} = (\delta_{sij\to k'l'} - \delta_{qij\to k'l'}) + (\delta_{sij\to k'l'} - \delta_{qkl\to m'n'}).$$

(73)
where $\delta s_{\mu \rightarrow \mu} = -\log(p'_{\mu}) + \log(p_{\mu})$, $\delta s_{\mu \rightarrow \mu''} = -\log(p''_{\mu}) + \log(p'_{\mu})$, $\delta q_{ij \rightarrow k'k'} = -\log(Z_{ij}^{-1}Z_{m'k'}^{\nu}(\gamma'))$ and $\delta q_{kl \rightarrow m'n'} = -\log(Z_{kl}^{-1}Z_{m'k'}^{\nu}(\gamma'))$. The entropy production distribution $\sigma$ is

$$P_{\rightarrow}(\sigma) = \sum_{\mu, ij, k, k'} \sum_{\mu', \nu} p_{\mu', \nu, \mu''}^{ij, k, k', m', n'} \delta(\sigma - \sigma_{ij, k \rightarrow k', m', n'})$$

(74)

The quasiprobability distribution of the three-point measurement for the backward process can be defined as

$$P_{\mu', \nu, \mu''}^{ij, k, k', m', n'} = p_{\mu'', \nu}^{ij, k, k', m', n'} N^{2}(\Phi^{AS} \otimes \Phi^{A'S'}|R_{1}^{S} \otimes R_{2}^{S'}[\Pi_{ij}^{A} \otimes \Pi_{k'k'}^{A'} \otimes \Pi_{m'n'}^{S'}])$$

$$\times (\Pi_{ij}^{A} \otimes \Pi_{k'k'}^{A'} \otimes \Pi_{m'n'}^{S'} | \Pi_{\psi_{\nu}}^{A} \otimes \Pi_{\mu'}^{S'} \otimes \Pi_{\mu''}^{A'} \otimes \Pi_{\mu'''}^{S'}).$$

(75)

The relation in Eq. (20) still holds with the quantities defined here. Combining Eqs. (72) to (74), we obtain

$$\langle \sigma \rangle = S(\rho_{I}||\gamma) - S(\rho_{M}||N_{I}^{S}(\gamma)) + S(\rho_{M}||\gamma') - S(\rho_{M}||N_{I}^{S}(\gamma'))$$

(77)

Different from the previous result (30), the average entropy production in Eq. (77) contains the intermediate state $\rho_{M}$. Therefore, a change in the intermediate state will affect the entropy production defined here. The FTs described here are extensions of the previous procedure. If we choose $\gamma' = N_{I}^{S}(\gamma)$, then the average entropy production returns to the previous result:

$$\langle \sigma \rangle = S(\rho_{I}||\gamma) - S(\rho_{M}||N_{I}^{S}(\gamma)) = S(\rho_{I}||\gamma) - S(N_{I}^{S} \otimes N_{I}^{S}(\rho_{I})||N_{I}^{S} \otimes N_{I}^{S}(\gamma)).$$

(78)

The freely chosen intermediate reference state $\gamma'$ can bring some convenience. For example, the reference state is often selected from the global fixed points of the quantum channel. In multitime processes, the problem is that one cannot ensure that the evolved reference state $N_{I}(\gamma)$ is always the global fixed point. In contrast, the extra reference state $\gamma'$ can always be selected from the global fixed points of $N_{I}^{S}$. Therefore, the method proposed in this section is more suitable for multitime processes.

Even for the cases in which $\gamma' = N_{I}^{S}(\gamma)$, Eq. (77) has deeper meaning. The quantity in (73) is the composite of two parts, both of which satisfy the fluctuation relation. $\sigma_{1} := \delta s_{\mu \rightarrow \mu} - \delta q_{ij \rightarrow k'k'}$ is the entropy production of quantum channel $N_{I}$ according to the time-ordered property. Hence, its fluctuation relation is obvious from section II A. For $\sigma_{2} := \delta s_{\mu \rightarrow \mu''} - \delta q_{kl \rightarrow m'n'}$, the distribution can be derived from Eq. (73):

$$P_{\rightarrow}(\sigma_{2}) = \sum_{\mu', \nu, \mu''} \sum_{ij, k, k', l} p_{\mu', \nu, \mu''}^{ij, k, k', l} \delta(\sigma - \sigma_{2})$$

(79)

Combining this with Eq. (72), we obtain

$$P_{\mu', \nu, \mu''}^{kl, m', n'} = \sum_{\nu} N^{2}(\rho_{A}^{S} \otimes \Pi_{\mu'}^{S} \otimes \Pi_{\mu''}^{A'} \otimes \Pi_{\mu'''}^{S'}|\Pi_{ij}^{A'} \otimes \Pi_{k'k'}^{S'}|\Pi_{m'n'}^{S'}|\Phi^{AS} \otimes \Phi^{A'S'})$$

$$\times (\Pi_{ij}^{A'} \otimes \Pi_{k'k'}^{S'} \otimes \Pi_{m'n'}^{S'} | \Pi_{\psi_{\nu}}^{A} \otimes \Pi_{\mu'}^{S'} \otimes \Pi_{\mu''}^{A'} \otimes \Pi_{\mu'''}^{S'}).$$

(80)
C. The conflict between intermediate measurements and non-Markovian processes

In the above section, we extend the FTs for non-Markovian processes. We realize intermediate measurements with the operation in [83] and prove that the FTs contain the intermediate state of the system. It is natural to ask whether this procedure is applicable to non-Markovian processes. The answer is no, and we discuss this from different perspectives.

We first discuss the conflict through concrete examples. We still consider the two-step evolution process here. Since the considered evolution process is non-Markovian, the final state of the two-body channel is not of tensor product form. The forward transition matrices should be

\[
T_{ij,kl \rightarrow k'l',m'n'} = N^2 (\Pi_{ij}^A \otimes \Pi_{kl}^{A'} \otimes \Pi_{m'n'}^{S''} | \Phi^{AS} \otimes \Phi^{A'S'} ).
\]  

(81)

If we still define the quasiprobability distribution of the three-point measurement according to Eq. (84), we obtain

\[
P_{ij,k'l',kl,m'n'}^{u,u',w''} = p_u \sum_\nu N^2 (\Phi^{AS} \otimes \Phi^{A'S'} | \mathcal{R}_{\gamma \otimes \gamma'}^{S''} | \Pi_{ij}^A \otimes \Pi_{k'l'}^{A'} \otimes \Pi_{kl}^{S''} \otimes \Pi_{m'n'}^{S''} | \Phi^{AS} \otimes \Phi^{A'S'} ).
\]

(82)

\[
\text{The main problem comes from the fact that the Petz recovery map}
\]

\[
\mathcal{R}_{\gamma \otimes \gamma'}^{S''} = J^{1/2}_{\gamma \otimes \gamma'} \circ \mathcal{N}^{S''1} \circ J^{-1/2}_{\gamma \otimes \gamma'}
\]

(83)

is neither time-ordered nor linkable. These issues cause the quasiprobability distribution for the backward process

\[
P_{ij,k'l',kl,m'n'}^{u,u',w''} = p_u \sum_\nu N^2 (\Phi^{AS} \otimes \Phi^{A'S'} | \mathcal{R}_{\gamma \otimes \gamma'}^{S''} | \Pi_{ij}^A \otimes \Pi_{k'l'}^{A'} \otimes \Pi_{kl}^{S''} \otimes \Pi_{m'n'}^{S''} | \Phi^{AS} \otimes \Phi^{A'S'} ).
\]

(84)

\[
\text{to not satisfy the marginality condition:}
\]

\[
\sum_{u,u',w''} P_{ij,k'l',kl,m'n'}^{u,u',w''} = \sum_\nu N^2 (\Phi^{AS} \otimes \Phi^{A'S'} | \mathcal{R}_{\gamma \otimes \gamma'}^{S''} | I^A \otimes \Pi_{\mu\nu}^{S''} \otimes \Pi_{\mu\nu}^{S''} | \rho_F^{S''} )
\]

\[
= \sum_\nu (I^S \otimes \Pi_{\mu\nu}^{S''} | \mathcal{R}_{\gamma \otimes \gamma'}^{S''} | \Pi_{\mu\nu}^{S''} | \rho_F^{S''} ) \neq p_u. \tag{85}
\]

Hence, the quasiprobability distribution in Eq. (84) is ill-defined. The procedure of section III B is not suitable for non-Markovian processes.

On the other hand, TPMs need to know the complete information about the initial and final states, so they are complete measurements [9]. Complete measurements mean that we can copy or broadcast these states, which requires that these states can be maximally entangled with an auxiliary state [10]. However, the final state \(\rho_{ij}^{S_1 \ldots S_n}\) of a non-Markovian process shows that correlations are present between the intermediate state and the other states. According to the exclusivity of entanglement, the complete measurements of the intermediate state conflict with a non-Markovian process.

These conclusions are consistent with the analysis in section III A. In a non-Markovian process, the measurements taken over the intermediate state influence the later evolution process and conflict with the FTs. To obtain the FTs for non-Markovian processes, one approach is to avoid measuring the intermediate states, as in section III A. Another approach is to include a measurement component in the evolution process and use derived channels to deduce the FTs. In the following section, we show how to construct a suitable channel and prove the corresponding FTs.
D. The FTs for non-Markovian processes

As shown in Eq. (53), we can derive a quantum channel by inserting operations between the process steps. A simple and direct approach is to insert a projective measurement \( \mathcal{A} = \sum_k \Pi_k(\cdot)\Pi_k \). However, such a measurement itself causes entropy production, which makes it difficult to separate the contributions of various components. In addition, we cannot obtain information about the intermediate states because the measured results are sent to the next step and not retained.

Here, we use the following unitary evolution

\[
\mathcal{A}' = \mathcal{A}'_{SS'}(\cdot) \otimes |0\rangle_S' \langle 0| = \sum_{kl} \Pi_k^{S'}(\cdot)\Pi_k^S \otimes \Pi_k^{S'} \otimes \Pi_k^S
\]

(86)
to realize general quantum measurements \( \Pi_k \) for intermediate states. The operation \( \mathcal{A}' \) is a unitary transformation for \( SS' \). This operation does not lead to entropy production. The ancillary \( S' \) also records the probability distributions of the intermediate state with respect to the basis \( \{\Pi_k\} \), i.e., \( \text{Tr}_{SS'}\mathcal{A}_{SS'}(\rho_{NM}) = \sum_k \Pi_k \rho_{NM} \Pi_k^S \Pi_k^{S'} \). The measurement here cannot obtain the complete information about the intermediate states. The correlations between the system and the environment or the process itself produce natural limitations regarding the available information. That is why the measurement procedure utilized here does not conflict with a non-Markovian process.

Here, we still consider the two-step evolution process. The derived channel

\[
\mathcal{N}_{SS'} = \text{Tr}_{E} U_{SS'} \circ \mathcal{A}_{SS'} \circ U_{SS'} \circ \mathcal{A}
\]

(87)
is a CPTP map for \( SS' \), which can also be expressed as \( \mathcal{N}_{SS'} = \sum_i M_{SS'}^S(\cdot)M_{SS'}^{S'} \). It maps the initial state \( \rho_I = (\sum_u p_u \Pi_u^S \otimes \Pi_u^{S'}) \otimes \rho_{PF} = \sum_v p_v' \Pi_v^{S'} \otimes \rho_{PF} \). The forward transition matrices can be defined as

\[
T_{ij\rightarrow k'l'} = N(\Pi_i^A \otimes \Pi_{k'l'}^S | \mathcal{N}_{SS'}[\Phi_{AS} \otimes \Pi_0^S]).
\]

(88)
As in the procedure of Eq. (86), we have

\[
T_{ij\rightarrow k'l'}^* = N(\Phi_{AS} \otimes \Pi_0^S | \mathcal{R}_{SS'}^\gamma[\mathcal{J}_{SS'}^\gamma(\cdot)]^{1/2} \Pi_i^A \otimes \Pi_{k'l'}^S),
\]

(89)
where the Petz recovery map

\[
\mathcal{R}_{SS'}^\gamma = (\mathcal{J}_{\Pi_0^S})^{1/2} \otimes (\gamma_{SS'})^{1/2} \Delta_{SS'} \circ (\mathcal{J}_{SS'}^{\gamma(\gamma_{SS'})})^{-1/2}.
\]

(90)
The rescaling map \( \mathcal{J}_{SS'}^{\gamma(\gamma_{SS'})} = (\mathcal{J}_{\Pi_0^S})^{1/2} \otimes \gamma_{SS'} \), where \( \gamma_{SS'} = N_{SS'}(\gamma_{SS'}) \) is the final reference state and the reference state \( \gamma_{SS'} = \gamma_{S} \otimes \Pi_{S'} \). The trace preserving property of this Petz recovery map is obvious from

\[
(\gamma_{SS'})^{1/2} \otimes (\gamma_{SS'})^{-1/2} \Delta_{SS'}(\gamma_{SS'})^{-1/2} \gamma_{SS'} = (\mathcal{R}_{SS'}^\gamma(\cdot))^{1/2} \mathcal{R}_{SS'}^\gamma(\cdot).
\]

(91)

The rescaling map \( (\mathcal{J}_{\Pi_0^S})^{1/2} \) allows the final states of \( \mathcal{R}_{SS'}^\gamma \), to maintain the form \( \rho_F^S \otimes \Pi_0^S \). The factor of the rescaled operators becomes

\[
Z_{ij}^{\gamma_0} Z_{k'l'}^{\gamma_F} := ||(\mathcal{J}_{\gamma_{SS'}}^\gamma)^{1/2} \Pi_i^A \otimes \Pi_{k'l'}^{S'}||_2 = ||J_{ij}^{\gamma_0} \Pi_i^A||_2 \times ||J_{k'l'}^{\gamma_0} \Pi_{k'l'}^{S'}||_2.
\]

(92)
The backward transition matrices can be defined as

\[
\tilde{T}_{ij\rightarrow k'l'} = N(\Phi_{AS} \otimes \Pi_0^S | \mathcal{R}_{SS'}^{\gamma_0}[\Pi_i^A \otimes \Pi_{k'l'}^{S'}]),
\]

(93)
where the reference-rescaled operators

\[
\Pi_i^A \otimes \Pi_{k'l'}^{S'} = ((\mathcal{J}_{\gamma_{SS'}}^{\gamma_0})^{1/2} \Pi_i^A \otimes \Pi_{k'l'}^{S'})(Z_{ij}^{\gamma_0} Z_{k'l'}^{\gamma_F}).
\]

(94)
The relation between the forward transition matrices and the backward transition matrices is

\[
T_{ij\rightarrow k'l'} = \tilde{T}_{ij\rightarrow k'l'} \times (Z_{ij}^{\gamma_0} Z_{k'l'}^{\gamma_F}).
\]

(95)
Since a general measurement of the intermediate state is realized with ancilla measurements, the TPM of the system should be turned into a TPM for the system ancilla. The quasiprobability distribution of the TPM for the forward process can be defined as

\[
P_{u,v'}_{ij,k'l'} = p_u N(\Pi_{u,v'}^A \otimes \Pi_{u,v'}^{S'} | \mathcal{N}_{SS'}[\Phi_{AS} \otimes \Pi_0^S]),
\]

(96)
It is easy to prove that \( P_{u,v'}_{ij,k'l'} \) satisfies the marginality condition:

\[
\sum_{v',ij,k'l'} P_{u,v'}_{ij,k'l'} = p_u N(\Pi_{u,v'}^A \otimes \Pi_{u,v'}^{S'} | \mathcal{N}_{SS'}[\Phi_{AS} \otimes \Pi_0^S]) = p_u,
\]

\[
\sum_{u,ij,k'l'} P_{u,ij,k'l'} = N(\rho_i^A \otimes \Pi_{ij}^{S'} | \mathcal{N}_{SS'}[\rho_F \otimes \Pi_0^S]) = (\Pi_{ij}^{S'} \otimes \rho_F^{S'}) = p_F,
\]

\[
\sum_{u,v',ij,k'l'} P_{u,v',ij,k'l'} = N(\rho_i^A \otimes \Pi_{ij}^{S'} | \mathcal{N}_{SS'}[\Pi_0^S]),
\]

\[
\times (\Pi_{ij}^{S'} \otimes \rho_F^{S'}) = \delta_{ij} (\Pi_{ij}^A \otimes \rho_F^S),
\]

\[
\sum_{u,v',ij,k'l'} P_{u,v',ij,k'l'} = N(\rho_i^A \otimes \Pi_{ij}^{S'} | \mathcal{N}_{SS'}[\Pi_0^S]),
\]

(97)
\[
\times (\Pi_{ij}^{S'} \otimes \rho_F^{S'}) = \delta_{k'l'} (\Pi_{ij}^{S'} \otimes \rho_F^{S'}).\]

The entropy production can be defined as

\[
\sigma_{ij\rightarrow k'l'} = \delta_{ij\rightarrow k'l'} - \delta_{ij\rightarrow k'l'}
\]

(98)
where \( \delta_{ij\rightarrow k'l'} = -\log(Z_{ij}^{\gamma_0} Z_{k'l'}^{\gamma_F}) \). The entropy production distribution \( \sigma \) is the same as Eq. (18). The
The quasiprobability distribution for the backward process can be defined as

\[ P_{ij,k'}^{a,v'} = p_{ij}^{v'} (\Phi^{AS} \otimes \Pi_0^S | R_{SS'}^{\gamma} | \Pi_{S}^{A} \otimes \Pi_{S'}^{S'}) \times (\Pi_{ij}^S \otimes \Pi_{k'}^{S'} | \Pi_{ij'}^S \otimes \Pi_{k'}^{S'}) \]  

(99)

The quasiprobability distribution for the backward process also satisfies the marginality condition:

\[ \sum_{u,w'} P_{ij,k'}^{a,v'} = p_{ij}^{v'} (\Pi_{ij}^S \otimes \Pi_0^S | R_{SS'}^{\gamma} | \Pi_{ij}^S \otimes \Pi_{ij'}^S) = p_{ij}^{v'}, \]

(100)

where we use the property that the final states of \( R_{SS'}^\gamma \) are always in the form of \( \rho_S^S \otimes \Pi_0^S \). The relation in Eq. (20) still holds. Combining Eqs. (97) and (98), we obtain

\[ \langle \sigma \rangle = S(\rho_1^S | \gamma^S) - S(\rho_F^{SS'} | \gamma_F^{SS'}). \]

(101)

Comparing this result with Eq. (61), we find that the average entropy production in Eq. (101) contains partial information about the intermediate state \( \text{Tr}_F \rho_F^{SS'} = \sum_k \Pi_k \rho_M \Pi_k^\dagger \). Compared with that provided by Eq. (77), the information here is not complete. Similar to Eq. (61), the entropy production here can also be related to the degree of non-Markovianity. Unlike the Markov process \( N_{\text{Markov}}^\gamma = N_0^\gamma \circ N_1^\gamma \) used in Eq. (62), the process \( N_{\text{Markov}}^{SS'} = N_S^\gamma \circ A_{SS'} \circ N_M^\gamma \) that is closest to \( N_{SS'}^\gamma \). Eq. (10) is used here as a sufficient condition for \( N_{SS'}^\gamma = N_{\text{Markov}}^{SS'} \). This means that the quasidistance between \( N_{SS'} \) and \( N_{\text{Markov}}^{SS'} \) can measure the degree of non-Markovianity. To the best of our knowledge, such a non-Markovianity measure has not yet been discussed. We briefly discuss it in appendix A. Further research will be needed to better understand this non-Markovianity measure.

The mapping \( A^t \) does not change the quantum relation entropy, so we can rewrite Eq. (101) as

\[ \langle \sigma \rangle = S(\rho_1^S | \gamma^S) - S(\rho_M^{SS'} | \gamma_M^{SS'}) + S(\rho_M^{SS'} | \gamma_M^{SS'}) - S(\rho_F^{SS'} | \gamma_F^{SS'}), \]

(102)

where \( \rho_M^{SS'} = A_{SS'}(\rho_S^S \otimes \Pi_0^S) \) and \( \rho_M^\gamma = N_0^\gamma(\rho_M^\gamma) \). The quantity \( S(\rho_1^S | \gamma^S) - S(\rho_M^{SS'} | \gamma_M^{SS'}) \) is nonnegative. When setting \( N_1^\gamma = \text{Tr}_F U_F^\dagger F \circ A \), this value matches the entropy production of the first step. The quantity \( S(\rho_M^{SS'} | \gamma_M^{SS'}) - S(\rho_F^{SS'} | \gamma_F^{SS'}) \) is nonnegative when no memory effect is present. For a non-Markovian process, we can separate the contributions as follows:

\[ \langle \sigma \rangle = S(\rho_1^S | \gamma^S) - S(\rho_M^{SS'} | \gamma_M^{SS'}) + S(\rho_M^{SS'} | \gamma_M^{SS'}) - S(\rho_F^{SS'} | \gamma_F^{SS'}) - \sigma_{NM}, \]

(103)

where the nonnegative quantity

\[ \sigma_{NM} = S(\rho_F^{SS'} | \gamma_F^{SS'}) - S(\mathcal{N}_2^\gamma(\rho_M^{SS'}) | \mathcal{N}_2^\gamma(\gamma_M^{SS'})) \]

is similar to Eq. (62). This quantity also reflects memory effects and can reduce system fluctuations.

**IV. CONCLUSION AND OUTLOOK**

In this paper, we first discuss the relation between the FTs for closed quantum systems and the FTs for quantum channels. We find that the FTs are equivalent when utilizing a special map to determine the initial state of the backward process. After that, we extend the FTs for quantum channels to multitime processes. We use a many-body channel and its derived channel to provide a general framework for multimode processes. For Markovian processes, we show that the two-point measurements can be extended to a multipoint measurement. We prove the corresponding FTs and find that the total fluctuation is the aggregation of the fluctuations of each step. For non-Markovian processes, we find that the multipoint measurement yields an ill-defined quasiprobability distribution. The complete measurements of the intermediate states lead to conflicts. Then, we insert operations between the steps and use the derived channel to obtain the corresponding FTs. The inserted operations convey partial information about the intermediate states. The given FTs show that memory effects can reduce the system fluctuations.

The transition matrices are formed by the inner product of the measurement operator and the Choi state, and it would be interesting to research the FTs based on the Choi state. According to the Choi-Jamiołkowski isomorphism, a trace preserving map is completely positive if and only if its Choi state is nonnegative \( \mathcal{N}^\gamma(\Phi^{AS}) \geq 0 \). The positivity of the Choi state may be the key to this approach.

The non-Markovianity measure in appendix A is related to the general measurements of the intermediate system. It is crucial to understand how memory effects influence fluctuations. It would be interesting to find the deeper physical meaning of \( \sigma_{NM} \). Comparing it with other non-Markovianity measures might help to find the answer to this question.

The average entropy production depends on the initial system state. In appendix B, we find that this quantity is not a linear function of the density matrix. The induced change in Holevo information must be accounted for. It may be useful to understand how the initial state of a system impacts its entropy production.

**Acknowledgments**

ZH is supported by the National Natural Science Foundation of China under grant nos. 12047556, 11725524.
and the Hubei Provincial Natural Science Foundation of China under Grant No. 2019CFA003.

Appendix A: Non-Markovianity measure

For a general many-body channel $\mathcal{N}^{S_1 S_2}$, the operation $A^i$ in section III yields

$$\sum_{kl} (\Pi_{kl}^{S_1} |\mathcal{N}^{S_1 S_2} [\Pi_{kl}^{S_1} \otimes \Pi_{kl}^{S_1'} ]),$$

(A1)

which maps the initial state $\rho_{F_1}^{S_1}$ to the final state $\rho_{F_2}^{S_1 S_2'}$ ($\rho_{F_1}^{S_1}$ to $\rho_{F_2}^{S_1 S_2'}$ in the main text). For a Markovian process, $\mathcal{N}^{S_1 S_2} = \mathcal{N}^{S_2} \circ \mathcal{N}^{S_1}$. And Eq. (A1) can be expressed as

$$\sum_{kl} \mathcal{N}^{S_2} [\Pi_{kl}^{S_1} \otimes \Pi_{kl}^{S_1'} ] (\Pi_{kl}^{S_1} |\mathcal{N}^{S_1'}),$$

(A2)

which corresponds to $\mathcal{N}^{S_2}_M \circ A_{SS'} \circ \mathcal{N}^{S_2}_M$ in the main text. Hence, Eq. (A1) is a sufficient condition for obtaining $\mathcal{N}_{SS'} = \mathcal{N}_{SS'}^M$.

When $\mathcal{N}_{SS'} = \mathcal{N}_{SS'}^M$, the $\sigma_{NM}$ in Eq. (101) is equal to zero. Only memory effects can lead to an increase in distinguishability and allow $S(\rho_{F_2}^{S_1 S_2'} || \tau_{F_2}^{S_1 S_2'}) > S(\rho_{F_2}^{S_1 S_2'} || \tau_{F_2}^{S_1 S_2'})$.

Recall that in Eq. (14), the quasidistance between the generalized Choi state of a non-Markovian process and the closest Choi state of a Markov process measures the degree of non-Markovianity:

$$D_{NM} := \min_{\mathcal{N}^M} D(\mathcal{Y} || \mathcal{N}^M),$$

(A3)

where $\mathcal{Y}$ is the Choi state of a many-body channel $\mathcal{N}^{S_1 \ldots S_n}$ and $\mathcal{N}^M$ is the Choi state of a Markov process $\mathcal{N}^{S_1}_n \circ \ldots \circ \mathcal{N}^{S_1}$.

This non-Markovianity measure is also suitable for the derived channels in Eq. (53). The Markovian process yields

$$\mathcal{N}^M_{A_{n-1} : A_1 - 1} = \mathcal{N}_n \circ (A_{n-1} \circ A_n) \circ \ldots \circ (A_1 \circ A_1).$$

Any CP-inducing quasidistance between the Choi state of $\mathcal{N}^M_{A_{n-1} : A_1 - 1}$, and the closest Choi state of $\mathcal{N}^M_{A_{n-1} : A_1 - 1}$ also measures the degree of non-Markovianity.

Appendix B: Holevo information

The entropy production in Eq. (21) is state-dependent. Suppose that the initial state $\rho = \sum_a p_a \rho_a$ yields $\langle \sigma \rangle$ and the initial states $\rho_a$ produce $\langle \sigma_a \rangle$; then, it is easy to show that

$$\langle \sigma \rangle = \sum_a p_a \langle \sigma_a \rangle + \delta \chi,$$

(B1)

where $\delta \chi = \chi_I - \chi_F$. $\chi_I$ is the Holevo information of $\rho$, and $\chi_F$ is the Holevo information of $\mathcal{N}(\rho)$.

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