Some results on the Bazilevic functions $B_1(\alpha)$ related to the Lemniscate Bernoulli (LB)

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Abstract. This research is concerned with Bazilevic $B_1(\alpha)$ on the unit disc $D = \{z : |z| < 1\}$, related to the Lemniscate Bernoulli (LB), defined by kind of subordination for some positive alpha. There will be determined the Hankel determinant, especially the third Hankel determinant which $B_1(\alpha)$ subordinates to LB, and we start with the case for starlike and convex functions, subset of $B_1(\alpha)$. In this article we improve the result of Kumar and Ravichandran.

1. Introduction and Preliminaries

Let $S$ be the class of analytic normalized univalent functions $f$ defined in $z \in D = \{z : |z| < 1\}$ and given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Then for $\alpha \geq 0$, $f \in B_1(\alpha) \subset S$, if and only if, for $z \in D$

$$\text{Re} \left( \frac{z^{1-\alpha} f'(z)}{f(z)^{1-\alpha}} \right) > 0. \tag{2}$$

The Bazilević functions (2) with logarithmic growth, $B_1(\alpha) \subset S$ have been extensively studied (see e.g. [8], [5], [6], [2]). Some results have been obtained for the class $B_1(\alpha)$. Amongst other results, Singh [7], found sharp estimates for the modulus of the first four coefficients, $|a_2|$, $|a_3|$ and $|a_4|$.

Next, the coefficient bounds yield information regarding the geometric properties of some subclass of univalent functions. In 1916, Bieberbach [1] computed an estimate for the second coefficient of normalized univalent analytic function (1) and this bound provides the growth, distortion, and covering theorems.

The $q$th Hankel determinant of a function $f$ given by (1) is defined for $q > 1$ and $n > 0$ as follows,
\[ H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ a_{n+q-1} & \cdots & a_{n+2(q-1)} & \end{vmatrix} \]

In recent years a great deal of attention has been devoted to finding estimates of Hankel determinants whose elements are the coefficients of univalent (and $p$-valent) functions. For $f \in S$, growth results have been established for the general Hankel determinant $H_q(n)$. The second Hankel determinant $H_2(2) = |a_2a_4 - a_3^2|$ has received more attention of Janteng [3] and Marjono and Thomas [5], with significant results being obtained for $f \in S$.

In this paper we will see this for subclass Bazilevic functions $B_1(\alpha)$ and try to learn about third Hankel determinant which is defined as follows:

\[ H_{3,1}(f) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_5 - a_3^2). \]

We will also want to see about starlike and convex functions related to the third Hankel determinant as a subclass of Bazilevic functions $B_1(\alpha)$.

2. Known Result

Denote by $P$, the class of functions $p$ satisfying $\text{Re} \; (p(z)) > 0$ for $z \in D$, with Taylor series

\[ p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \]

There are some examples $p \in P$ i.e. $p(z) = \frac{1+z}{1-z}$, $p(z) = \frac{1+z^2}{1-z^2}$ and also we can define Lemniscate Bernoulli $(1+z)^{1/2}$.

We will use the following Lemmas:

**Lemma 2.1.** If $p \in P$ with coefficient $p_n$ as above then for some complex valued $x$ with $|x| \leq 1$ and some complex valued $\zeta$ with $|\zeta| \leq 1$,

\[ 2p_2 = p_1^2 + x(4 - p_1^2) \]
\[ 4p_3 = p_1^3 + 2(4 - p_1^2)p_1x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)\zeta \]

**Lemma 2.2.** If $p \in P$, then $|p_n| \leq 2$ for $n \geq 1$ and

\[ |p_2 - \frac{\mu}{2}p_1| \leq \max \{2,2||\mu - 1||\} \]

**Lemma 2.3.** Let $p \in P$, then for all $n, m \in N$,

\[ |\mu p_n p_m - p_{m+n}| \leq \begin{cases} 2, & 0 \leq \mu \leq 1; \\ 2|2\mu - 1|, & \text{elsewhere}, \end{cases} \]

The following theorem of Kumar and Ravichandran [4] gives an improvement to the existing estimate on the third Hankel determinant related to the starlike and convex functions with respect to the symmetric points.

**Theorem 2.1** The third Hankel determinant for the functions in the classes $S*{s}$ and $K{S}$ are $5/4$ and $91/1728$, respectively.
Proof: In proving this theorem \cite{4} start by equating
\[ \frac{zf(z)}{f(z) - f(-z)} = p(z), \]
then comparing coefficients on both sides of the above equation, such that we have
\[ a_2 = \frac{p_1}{2}, \quad a_3 = \frac{p_2}{2}, \quad a_4 = \frac{1}{8} (p_1 p_2 + 2 p_3) \quad \text{and} \quad a_5 = \frac{1}{8} (p_2^2 + 2 p_4). \]
Using the above we can write
\[ H_{3,1}(f) = a_5 (a_3 - a_2^2) + a_3 (a_2 a_4 - a_2^2 a_3) - a_4 (a_4 - a_2 a_3) \]
\[ = \frac{1}{64} \left( p_2^2 (p_2^2 - 4 p_4) + 4 p_1 p_2 p_3 - 4 (p_3^2 - 2 p_2 p_4 + p_4^2) \right). \]
Further, by suitable arranging the terms, we have
\[ |64 H_{3,1}(f)| = |8 p_2 p_4 - 4 p_2^2 p_4 + p_2^2 p_2^2 - 4 p_3^2 + 4 p_1 p_2 p_3 - 4 p_3^2| \]
\[ \leq 4 p_4 (2 p_2 - p_2^2) + |p_2^2 (p_2^2 - 4 p_2)| + |4 p_3 (p_1 p_2 - 4 p_2)|. \] (3)
By using Lemma 2.3, we see that
\[ |4 p_4 (2 p_2 - p_2^2)| \leq 32, \quad |p_2^2 (p_2^2 - 4 p_2)| \leq 32. \quad \text{and} \quad |4 p_3 (p_1 p_2 - 4 p_2)| \leq 16 \] (4)
Thus using (3) and (4), we have
\[ |H_{3,1}(f)| \leq \frac{80}{64} = \frac{5}{4}. \] (5)
Now, let us take the class \( K_s \), by still considering class of positive real part \( p(z) \). Again we can write
\[ \frac{2(z f'(z))'}{(f(z) - f(-z))'} = p(z) \]
On comparing on both sides of the above equation, we have
\[ a_2 = \frac{p_1}{4}, \quad a_3 = \frac{p_2}{6}, \quad a_4 = \frac{1}{32} (p_1 p_2 + 3 p_3) \quad \text{and} \quad a_5 = \frac{1}{40} (p_2^2 + 2 p_4). \]
Using the above, we have
\[ H_{3,1}(f) = \frac{1}{138240} \left( 9 p_2^2 (p_2^2 - 48 p_4) + 180 p_2 p_3 - 4 (16 p_3^2 - 288 p_2 p_4 + 135 p_3^2) \right) \]
By suitably arranging terms, we obtain
\[ H_{3,1}(f) = 1152 p_4 \left( p_2 - \frac{432}{1152} p_2^2 \right) + 64 p_2^2 \left( \frac{9}{64} p_2^2 - p_2 \right) + 540 p_3 \left( \frac{180}{540} p_1 p_2 - p_3 \right) \]
Next by using the fact that \( |p_4| \leq 2 \) and also using Lemma 2.3, we have
\[ 1152 \left| p_4 \left( p_2 - \frac{432}{1152} p_2^2 \right) \right| \leq 4608, \]
\[ 64 \left| p_2^2 \left( \frac{9}{64} p_2^2 - p_2 \right) \right| \leq 512, \]
\[ \text{and} \quad 540 \left| 4 p_3 \left( \frac{180}{540} p_1 p_2 - p_3 \right) \right| \leq 2160 \] (6)
Now, by using (6), we get
\[ |H_{3,1}(f)| \leq \frac{4608 + 512 + 2160}{138240} = \frac{91}{1728} \approx 0.052662 \approx 0.052662 < \frac{19}{135} \approx 0.140741 \]

By using the same method we obtain for \( f \in K_s \)
\[ |H_{3,1}(f)| \leq \frac{91}{1728} \]

This completes the proof.

3. Results

Relate to the above result, we can remind two important classes \( M(\lambda) \) and \( N(\lambda) \) respectively subclasses of \( S \), which is defined as the following.
\[ \Re\left(\frac{zf'(z)}{f(z)}\right) < \lambda \]
and
\[ \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < \lambda \]
for \( z \in D \).

Kumar and Ravichandran had the following theorem.

**Theorem 3.1.** The third Hankel determinant for the functions in the classes \( M \) and \( N \) are bounded by 
\( \frac{579 + 8\sqrt{3}}{1728} \) dan \( \frac{144431 + 96\sqrt{141}}{6497280} \), respectively.

The proof is similar to the above proof, first start by equating the class which is functions with positive real part.

(a) Let \( f \in M \). Then, we can associate a function \( p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + \cdots \in P \) such that
\[ \frac{zf'(z)}{f(z)} = \frac{1}{2}(3 - p(z)). \]

Next continued by comparing its coefficients on both sides related to class \( M \) first. Then we compute the value of \( H_{3,1}(f) \) and arrange this into better representation. Finally we simplify and obtained the result for subclass \( M \), i.e.
\[ |H_{3,1}f(z)| \leq \frac{579 + 8\sqrt{3}}{1728} \approx 0.343088 < \frac{81 + 16\sqrt{3}}{216} \approx 0.5033 \]

(a) Let \( f \in N \). Then, we can associate a function \( p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + \cdots \in P \) such that
\[ 1 + \frac{zf''(z)}{f'(z)} = \frac{1}{2}(3 - p(z)). \]

Next continued by comparing its coefficients on both sides related to class \( N \) first. Then we compute the value of \( H_{3,1}(f) \) and arrange this into better representation. Finally we simplify and obtained the result for subclass \( N \), i.e.
\[ |H_{3,1}f(z)| \leq \frac{579144431 + 96\sqrt{141}}{6497280} \approx 0.0224049 < \frac{139}{5760} \approx 0.0241319 \]

We have the following theorem for analytic functions especially convex and starlike functions.
**Theorem 3.2.** The sharp bound on the third Hankel determinant for the classes of starlike and convex functions with respect to symmetric point are 1/4 and 4/135, respectively.

Proof. In this case we will substitute $p(z)$ of Kumar and Ravichandran [4] i.e. $\frac{zf'(z)}{f(z)}$ by $\frac{zf'(z)}{f(z)}$, so we have

\[ \frac{zf'(z)}{f(z)} = p(z). \] (7)

It means that $zf'(z) = f(z)p(z)$ such that by comparing its coefficients we have

\begin{align*}
2a_2 &= a_2 + p_1 \\
3a_3 &= a_3 + a_2p_1 + p_2 \\
4a_4 &= a_4 + a_3p_1 + a_2p_2 + p_3 \\
5a_5 &= a_5 + a_3p_2 + a_2p_3 + p_4
\end{align*}

i.e. we have

\begin{align*}
a_2 &= p_1 \\
a_3 &= \left(p_1^2 + p_2\right)/2 \\
a_4 &= \frac{1}{6}(p_1^3 + p_2p_1) + \frac{1}{3}(p_1p_2) + \frac{p_3}{3} \\
4a_5 &= \frac{1}{2}\left(p_1^2p_2 + p_2^2\right) + p_1p_3 + p_4
\end{align*}

Next by the similar method we obtain

\[ H_{3,1}(f) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2). \]

It means that

\[ H_{3,1}(f) = A - B + C \]

where

\begin{align*}
A &= a_3(a_2a_4 - a_3^2), \\
B &= a_4(a_4 - a_2a_3),
\end{align*}

and

\[ C = a_5(a_3 - a_2^2). \] (8)

So we can write (8) as the following

\begin{align*}
A &= \frac{p_1^2 + p_2}{12}\left[p_1^4 - 6p_1^3 + p_2p_1^2 - 2p_1p_2 + 2p_3\right] \\
B &= p_1^3 + \frac{1}{6}(5p_1p_2 + 2p_1 - 2p_3) \\
C &= \frac{1}{16}\left(p_1^2p_2 + p_2^2 + 2p_1p_3 + 2p_4\right)(p_2 - p_1^2) \tag{9}
\end{align*}

By considering (9) and that $|p_i| \leq 2$, finally we obtain

\[ |H_{3,1}(f)| \leq \frac{1}{4} \]

and the equation will be hold when we choose function $f$ equal to $f_0$ such that

\[ \frac{2zf_0'(z)}{f_0(z) - f_0(-z)} = 1 + z^3 \]

Next, for the convex functions we can present similar (7) a follows
1 + \frac{zf''(z)}{f'(z)} = p(z)

It means that

\[f'(z) + zf'''(z) = f'(z)p(z)\]  \hspace{1cm} (10)

where

\[f(z) = z + \sum_{n=2}^{\infty} a_n z^n\]

and

\[p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n\]

Comparing coefficients in both sides of (10), we have 4 expressions of \(a_2, a_3, a_4\) and \(a_5\) in terms of combination of \(p_i\) and \(p_j\). Then we can calculate \(H_{3,1}(f)\) such that finally we obtain the following

\[|H_{3,1}(f)| \leq \frac{4}{135}\]

for convex function \(f\) and the equation will be hold when we choose \(f = f_1\) such that

\[
\frac{2zf'_1(z)}{f_1(z) - f_1(-z)} = \frac{1 + z^2}{1 - z^2}
\]

The proof is complete. \hfill \Box

4. Conclusion

In this paper, we are working on the univalent functions based on the result of Marjono and Thomas [5]. We give the boundary for the third Hankle determinant for starlike and convex functions as subset of Bazilevic \(B_1(\alpha)\) as an improvement of the work of Kumar and Ravichandran. To prove this we use lemma for the functions with positive real part \(p(z)\).

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