Stability conditions for spatially modulated phases

Sophia K. Domokos\(^1\), Carlos Hoyos\(^2\), Jacob Sonnenschein\(^2\),

\(^1\) Weizmann Institute of Science, Rehovot 76100, Israel

\(^2\) Raymond and Beverly Sackler Faculty of Exact Sciences
School of Physics and Astronomy
Tel-Aviv University, Ramat-Aviv 69978, Israel.
E-mail: sophia.domokos@weizmann.ac.il, choyos@post.tau.ac.il, cobi@post.tau.ac.il

Abstract: We introduce a novel set of stability conditions for vacua with broken Lorentz symmetry. The first class of conditions require that the energy be minimized under small geometric deformations, which translates into requiring the positivity of a “stiffness” four-tensor. The second class of conditions requires that stress forces be restoring under small deformations. We then apply these conditions to examples of recently-discovered spatially modulated (or “striped”) phases in holographic models of superconductors and high-density QCD. For backreacted solutions we find that the pressure condition is equivalent to thermodynamic stability. For probe solutions, however, these conditions are in conflict with the minimization of the free energy. This suggests that either the solutions are unstable or the definition of the free energy in the probe approximation must be revised for these solutions.

Keywords: AdS/CFT, Spatial Modulation, Inhomogeneous Solutions, Stability.
1. Introduction

In systems at finite density, the energetically-preferred vacuum may break translational and rotational invariance. For instance, in large-$N_c$ QCD a chiral density wave is the ground state at low temperatures $[1,2]^1$, and in the presence of a magnetic field a chiral magnetic wave $[4]$ or a chiral magnetic spiral $[5]$ can form. Inhomogeneous phases are also important in condensed matter systems, where they may prove relevant for the description of high-$T_c$ superconductors $[6–8]$.

Instabilities of the spatially homogeneous vacuum abound in holographic duals of strongly coupled systems. First identified in $[9]$ for zero-temperature, finite-density QCD,

---

$^1$Although for $N = 3$ in four dimensions the ground state is expected to be a color superconductor $[3]$. 

---

---
the authors in [10] provided a more general treatment, noting that Chern-Simons-like terms may generically induce spatially modulated instabilities for sufficiently large values of the density or of the Chern-Simons coupling. These instabilities have since been found in a variety of models, from the probe-limit $AdS_5$ Reissner-Nordstrom black hole [10] and the Sakai-Sugimoto model at zero temperature and large axial chemical potential [11], to duals of Fermi liquids in the probe limit [12, 13] and beyond [13], as well as in broad classes of gravity duals for superconducting materials ([14–18]). [19] showed, furthermore, that duals of charge density waves can arise even without the parity- and time-translation-invariance-violating Chern-Simons interaction. Meanwhile, factors which can stabilize the homogeneous vacuum even in the presence of the Chern-Simons coupling include $R^2$ gravitational corrections [20], magnetic fields [21, 22], and coarse-grain holographic models of quarkonium [23].

The actual striped vacuum state has been explicitly constructed in holographic QCD at high temperature and density [25, 26], at zero temperature and large axial chemical potential [11], in supergravity duals of superconductors [15, 27–32]. Though examples of striped phases are quite common, these constitute just one of many types of inhomogeneous vacua. Examples of solitons and more complicated inhomogeneous phases include [33–41]. Though models with explicitly broken translation invariance are also of great interest (e.g. for modeling lattice effects in condensed matter systems [42–47]), in this note we focus exclusively on systems with spontaneously broken translation invariance.

We can gain some intuition about inhomogeneous solutions from solid state physics. A crystalline solid is an example of an inhomogeneous state: the atoms are arranged in a periodic structure that at large distances looks like a continuous medium. The translational symmetries are spontaneously broken, with the phonons acting as Goldstone bosons. Deformations of the solid displace atoms from their equilibrium positions, costing a finite amount of energy. When slightly deformed, such a solid may experience internal stress forces that tend to bring it back to its original shape: such deformation are elastic. Large deformations which change the shape of the solid permanently are termed plastic.

With this framework in mind, we study the stability of inhomogeneous solutions under small deformations. Our strategy differs from the usual linear stability analysis, in which one studies time-dependent harmonic perturbations close to the static solution in configuration space. If the solution remains close to the original solution for all times (i.e. it represents oscillations around a minimum of the energy), it is declared stable. This strategy often involves solving highly non-trivial second order partial differential equations (PDEs). We, however, deform the static solution to another static configuration, still close to the original solution. In general the deformed configuration is not a solution of the equations of motion, but one can interpret it as the result of applying small external forces. If the energy of the deformed configuration is lower than the original solution, it is clear that when the forces are turned off, either suddenly or gradually, the evolution of the system

\[^2\text{It is important to note that the spatially-modulated instability can occur for solutions to the full } D = 11 \text{ supergravity equations of motion of string- or M-theory (see e.g. [24]): though it can be disrupted by varying additional parameters, it does not seem to be an artifact of bottom-up or probe-limit models.}\]

\[^3\text{A partial classification of possible phases was given in [31].}\]
will not take it back to the original solution, but rather to some different vacuum.

We develop two different types of stability conditions. The first uses the changes in the system’s energy due to small geometrical deformations, while the second uses changes in the momentum. In the first case, we expand the energy to second order in the deformation and determine whether the inhomogeneous solution is truly a minimum of the energy, or whether there are unstable directions. The second set of conditions evaluates the stress forces on deformed solutions using the energy-momentum tensor. The latter approach was used for instance by Gibbons to study multiple BIon solutions [48]. We will apply these checks of stability to the spatially modulated solutions found by Ooguri and Park in a Maxwell-Chern-Simons model [25] and in the Sakai-Sugimoto model [26]. We do not observe any sign of instability from the condition of energy minimization, but we find the stress force condition to be in tension with the condition of thermodynamic stability obtained from minimizing the free energy. However, we find that a similar analysis of backreacted solutions (using recent results of Donos and Gauntlett [50]), reveals that in such cases the two conditions are equivalent. This suggests either that the definition of the free energy in the probe limit is inconsistent, or that the backreaction is crucial to stabilizing the solutions.

Though we apply our minimal energy condition specifically to solutions periodic along a spatial direction, most of our results apply generally to static solutions. The minimum energy condition, in particular, applies also to soliton (finite energy) solutions, an extension of our previous work [49], where we derived conditions on the existence of solitons from first-order deformations.

The paper is organized as follows: In section § 2 we derive the minimum energy condition for generic theories with scalar or gauge fields, and compare it to stability conditions in elasticity. In section § 3 we find a condition on the variation of the pressure with respect to the period of the spatially modulated solutions using a stress force analysis. In section § 4 we apply the stability conditions from the previous sections on striped solutions. In section § 5 we conclude and suggest several directions for future work. We have gathered some calculational details and additional material in two appendices.

2. Minimum energy condition

Consider a static solution to the equations of motion in a given field theory. Now perform a geometrical transformation – such as a shear or dilation – on the solution. A solution is energetically stable if its energy increases (or stays the same) under any such deformation. This requirement leads to a series of constraints on static stable solutions.

In what follows we will consider for simplicity only scalar fields or Abelian vector fields. The extension to more generic cases is straightforward.

2.1 Deformed configurations to first order

In this section we review some of the results of [49] in order to introduce our methodology and establish some notation. Consider the simple case of a theory with one or more
scalars $\phi^a$, that possesses a static solution, $\phi_0^a(x)$ with finite energy density. How do small deformations affect the energy of this solution?

Generically, the energy of the solution is a function of the fields and its derivatives,

$$E[\phi_0^a] = \int d^d x \mathcal{E}(\phi_0^a, \partial_i \phi_0^a),$$

(2.1)

where $\mathcal{E}$ is the energy density. Let’s say we deform the configuration by a geometric deformation in space, $\Lambda x$. The deformed solution takes the form

$$\phi_\Lambda^a(x) = \phi_0^a(\Lambda x).$$

(2.2)

For small deformations we can do an expansion around the undeformed solution:

$$\Lambda x^i \simeq x^i + \xi^i(x).$$

(2.3)

We will use Latin indices for spatial components $i, j, \cdots = 1, \ldots, d$ and Greek indices for spacetime components $\mu, \nu, \cdots = 0, 1, \ldots, d$. The energy of the deformed configuration is, to leading order,

$$E[\phi_\Lambda^a] = \int d^d x \mathcal{E}(\phi_\Lambda^a(x), \partial_i \phi_\Lambda^a(x))$$

$$= \int d^d \tilde{x} \left| \frac{\delta x^i}{\delta \tilde{x}^j} \right| \mathcal{E} \left( \phi_\Lambda^a(\tilde{x}), \frac{\partial \tilde{x}^j}{\partial x^i} \partial_j \phi_0^a(\tilde{x}) \right)$$

$$\simeq \int d^d \tilde{x} \mathcal{E} \left( \phi_\Lambda^a(\tilde{x}), \partial_i \phi_0^a(\tilde{x}) \right) + \int d^d \tilde{x} \partial_i \xi^j \left[ \frac{\delta^j}{\delta \partial_i \phi^a} \delta \mathcal{E} - \frac{\delta \mathcal{E}}{\delta \partial_i \phi^a} \partial_j \phi_0^a \right]$$

$$= E[\phi_0^a] - \int d^d \tilde{x} \partial_i \xi^j \Pi^i_j(\phi_0^a) + \ldots. $$

(2.4)

In the second line we have made the change of variables $\tilde{x}^i = (\Lambda x)^i$ and in the next lines we have expanded for $x^i \simeq \tilde{x}^i - \xi^i$. Here $\Pi^i_j$ is a stress tensor for static configurations based on the energy density $\mathcal{E}$:

$$\Pi^i_j = \frac{\delta \mathcal{E}}{\delta \partial_i \phi^a} \partial_j \phi^a - \delta^i_j \mathcal{E}. $$

(2.5)

The difference in the energy of the deformed solution compared to the original one is given by the stress tensor $(\Pi^i_j)$ evaluated at the soliton solution:

$$E[\phi_\Lambda^a] - E[\phi_0^a] = \int d^d x \delta \mathcal{E} = - \int d^d x \partial_i \xi^j \Pi^i_j. $$

(2.6)

For a static solution to the equations of motion this term must be a total derivative.$^4$

For deformations that vanish sufficiently fast at infinity, or at least leave the boundary conditions unaffected, the first order variation just vanishes. This is simply the statement that the static configuration is an extremum of the potential energy (including spatial derivatives).

$^4$For scalars this is obvious, as $\mathcal{E} = -\mathcal{L}$. For theories with gauge fields the situation is a bit more complicated, as the energy density and stress tensor include improvement terms; however, the variation is still a total derivative, as described in [49].
2.2 Deformed configurations to second order

The first-order variation condition guarantees that the solution lies at an extremum of the energy. We can, however, expand the energy to second order in the small deformation parameter $\xi^i$. The deformed energy takes the generic form

$$E[\phi_\Lambda^\alpha] \approx E[\phi_0^\alpha] + \int d^d \tilde{x} \tilde{\partial}_i \tilde{\partial}_j \Pi^i_j + \int d^d \tilde{x} \frac{1}{2} \tilde{\partial}_i \xi^j \tilde{\partial}_j \xi^m C_{jm}^{il} + \ldots .$$

(2.7)

Stability thus requires that

$$\int d^d \tilde{x} \tilde{\partial}_i \xi^j \tilde{\partial}_m \xi^m C_{jm}^{il} \geq 0.$$

(2.8)

Note that this condition must be satisfied for arbitrary deformations. This is guaranteed if

$$C_{jm}^{il} \hat{a}_i \hat{a}_l \hat{b}_j \hat{b}_m \geq 0$$

(2.9)

for arbitrary unit vectors $\hat{a}$, and $\hat{b}$. Flat directions are those along which the inequality is saturated.

Note that the second order condition (2.9) applies locally: unless a solution obeys (2.9), one could engineer some arbitrarily complicated $\xi^i(x)$ in such a way as to make the integrand negative. Furthermore, the condition holds even for field configurations of infinite energy and infinite extent (as long as one can regularize them locally).

This analysis has strong parallels with the theory of elasticity of solids (see e.g. [51]). One can describe the deformation of a solid in terms of a vector $u^i(x)$, which parametrizes the displacement of an element of the solid from the equilibrium configuration at any given point in spacetime. The energy changes as

$$d\varepsilon = \sigma^{ij} u_{ij},$$

(2.10)

where $\sigma^{ij}$ is the “stress” tensor and $u_{ij}$ is the “strain” tensor,

$$u_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i).$$

(2.11)

For instance, the stress tensor of an ideal isotropic fluid is simply proportional to the pressure, $\sigma^{ij} = -p \delta^{ij}$, and measures how the energy changes with changes in volume. The stress tensor can be defined as the variation of the energy with respect to the strain for adiabatic processes

$$\sigma^{ij} = \left( \frac{\partial \varepsilon}{\partial u_{ij}} \right)_s.$$

(2.12)

If the deformations are small ($u_{ij} << 1$), the stress and the strain are proportional to each other:

$$\sigma^{ij} = S^{ijkl} u_{kl},$$

(2.13)

where $S^{ijkl}$ is the “stiffness” tensor. The system is stable if the energy is minimized for all possible deformations. This imposes conditions on the stiffness tensor, that can be summarized as the strong ellipticity condition: for any two unit vectors $\hat{a}$ and $\hat{b}$

$$S^{ilm} \hat{a}_i \hat{a}_l \hat{b}_j \hat{b}_m > 0.$$

(2.14)

analogous to our equation 2.9, if the deformation $\xi^i$ plays the same role as the displacement vector, and the tensor $C$ is the stiffness.
2.2.1 Stiffness in scalar field theories

Let us now compute the stiffness for a general scalar field theory. We transform the coordinates as \( \tilde{x}^i = x^i - \xi^i(x) \), so the energy to second order in fluctuations becomes

\[
E[\phi^n_\lambda] = \int d^d\tilde{x} E(\phi^n_\lambda(x), \partial_i \phi^n_\lambda(x))
\]

\[
= \int d^d\tilde{x} \left| \frac{\partial \tilde{x}^j}{\partial x^i} \right| E\left(\phi^n_\lambda(\tilde{x}), \frac{\partial \tilde{x}^j}{\partial x^i} \tilde{\partial}_j \phi^n_\lambda(\tilde{x})\right)
\]

\[
= E[\phi^n_0] + \int d^d\tilde{x} \tilde{\partial}_i E (\delta^i_j E - \tilde{\partial}_j \phi^n_0 \frac{\delta E}{\delta \partial_i \phi^n_0})
\]

\[
+ \int d^d\tilde{x} \frac{1}{2} \tilde{\partial}_i \xi^i \tilde{\partial}_j \xi^j E + \tilde{\partial}_j \phi^n_0 \tilde{\partial}_m \phi^n_0 \frac{\delta^2 E}{\delta \partial_j \phi^n_0 \delta \partial_m \phi^n_0}
\]

(2.15) where we have used the expansions

\[
\frac{\partial \tilde{x}^j}{\partial x^i} = \delta^j_i - \tilde{\partial}_i \xi^j
\]

(2.16)

\[\left| \frac{\partial \tilde{x}^j}{\partial x^i} \right| = 1 + \partial_i \xi^i + \frac{1}{2} \partial_i \xi^i \partial_j \xi^j + \frac{1}{2} (\partial_i \xi^i)^2 + O(\xi^3).
\]

(2.17)

Rewriting all factors of \( \partial/\partial x^i \) in terms of \( \partial/\partial \tilde{x}^i \), we find the variation of the energy:

\[
E[\phi^n_\lambda] - E[\phi^n_0] = \int d^d\tilde{x} \tilde{\partial}_i \xi^i E (\delta^i_j E - \tilde{\partial}_j \phi^n_0 \frac{\delta E}{\delta \partial_i \phi^n_0})
\]

\[
+ \int d^d\tilde{x} \frac{1}{2} \tilde{\partial}_i \xi^i \tilde{\partial}_j \xi^j E + \tilde{\partial}_j \phi^n_0 \tilde{\partial}_m \phi^n_0 \frac{\delta^2 E}{\delta \partial_j \phi^n_0 \delta \partial_m \phi^n_0}
\]

(2.18)

As noted above, the first order term is a total derivative, so to leading order the variation gives

\[
E[\phi^n_\lambda] - E[\phi^n_0] = \frac{1}{2} \int d^d\tilde{x} C^{il}_{jm} \tilde{\partial}_i \xi^j \tilde{\partial}_l \xi^m,
\]

(2.19) where the tensor \( C^{il}_{jm} \) is

\[
C^{il}_{jm} = \left(\delta^i_j \delta^l_m - \delta^i_j \delta^l_m\right) E + \tilde{\partial}_j \phi^n_0 \tilde{\partial}_m \phi^n_0 \frac{\delta^2 E}{\delta \partial_l \phi^n_0 \delta \partial_l \phi^n_0}
\]

\[
+ \delta^i_j \delta^l_m \phi^n_0 \frac{\delta E}{\delta \partial_l \phi^n_0} + \delta^i_j \tilde{\partial}_j \phi^n_0 \frac{\delta E}{\delta \partial_l \phi^n_0} - \delta^i_j \tilde{\partial}_j \phi^n_0 \frac{\delta E}{\delta \partial_l \phi^n_0} - \delta^i_j \tilde{\partial}_j \phi^n_0 \frac{\delta E}{\delta \partial_l \phi^n_0}.
\]

(2.20)

2.2.2 Stiffness in gauge field theories

For theories with vector fields, the definition of \( C^{il}_{jm} \) is slightly more complicated. In some cases it may be more straightforward to compute the tensor \( \tilde{C}^{\mu\nu}_{\rho\sigma} \), whose spatial part is \( C^{il}_{jm} \).

We present the computation of \( \tilde{C}^{\mu\nu}_{\rho\sigma} \) for gauge-invariant energy densities in Appendix A. This tensor also arises as the quadratic part of an effective Hamiltonian for time-dependent fluctuations around the spatially modulated vacuum, an investigation which we leave for future work.
We now turn to calculating $C_{lm}^{ij}$ for gauge theories (including those with Chern-Simons terms). After the deformation $x_i \rightarrow \tilde{x}_i$, the energy density depends on the

$$
A_0(\tilde{x}), \quad \frac{\partial \tilde{x}_k}{\partial x^i} A_k(\tilde{x}), \quad \frac{\partial \tilde{x}_k}{\partial x^i} F_{lk}(\tilde{x}), \quad \frac{\partial \tilde{x}_k}{\partial x^i} \frac{\partial \tilde{x}_l}{\partial x^j} F_{kl}(\tilde{x}),
$$

(2.21)

The expansion of the energy density to second order is now

$$
E_\Lambda = E - \partial_i \xi^i \pi_l^i + \frac{1}{2} \partial_i \xi^i \partial_l \xi^m \left[ \delta^i_j \pi^i_m - \delta^i_m \pi^i_j - \delta^i_j \pi^i_m + 2 F_{jm} \frac{\delta E}{\delta F_{il}} + \phi^{il} \right]
$$

where we define

$$
\pi^i_l = \frac{\delta E}{\delta A_i} A_l + 2 \frac{\delta E}{\delta F_{il}} F_{l^\mu}
$$

(2.23)

$$
\phi^{il} = \frac{\delta^2 E}{\delta A_i \delta A_l} A_m A_j + 2 \left( \frac{\delta^2 E}{\delta F_{il} \delta A_i} A_m F_{j^\mu} + \frac{\delta^2 E}{\delta F_{il} \delta A_j} A_m F_{m^\mu} \right) + 4 \frac{\delta^2 E}{\delta F_{il} \delta F_{j^\mu}} F_{j^\mu} F_{m^\nu}.
$$

(2.24)

As for the case with scalar fields, the change in the energy to second order is

$$
E_\Lambda - E = \frac{1}{2} \int d^d x \ C_{jm}^{il} \tilde{\partial}_i \xi^j \tilde{\partial}_l \xi^m.
$$

(2.25)

Where the tensor $C$ is

$$
C_{jm}^{il} = \left( \delta^i_j \delta^l_m - \delta^l_j \delta^i_m \right) E + \delta^i_j \pi^i_m + \delta^i_m \pi^i_j - \delta^i_j \pi^i_m - \delta^i_m \pi^i_j + 2 F_{jm} \frac{\delta E}{\delta F_{il}} + \phi^{il}.
$$

(2.26)

We can now use the result of this and the previous subsection to analyze the stability of static solutions.

### 2.2.3 Global deformations to second order

In systems with global (or internal) symmetries, a larger set of constraints must be satisfied to guarantee that the energy is minimized. Since this analysis is not directly related to the spatially modulated configurations we study in the next section, a reader interested mainly in the latter can proceed directly to section §3.

As described in [49], in addition to geometrical deformations of static solutions, one can also deform static solutions using global symmetries. This corresponds to elevating the constant parameters of the global symmetry transformations to space-dependent ones.

$\delta_\theta \phi^a(x) = \theta^A(T_A)_a^b \phi_b(x) \Rightarrow \delta_\theta \phi^a(x) = \theta^A(x) T_A^b \phi_b(x).$  

(2.27)

Requiring that the energy be extremized under such deformations yields the constraint

$$
\delta_\theta E[\phi^a] = \int d^d x [\partial_i (A_i^A J_A^a + \partial_i (\delta_\theta \Psi_0^a))] \Rightarrow \int d^d x J_A^a = 0,
$$

(2.28)

where $J_A^a$ is a space component of the current associated with some global symmetry that is broken by the static solution. $\theta^A$ is the parameter of the transformation, and in the
last step we assumed that the surface term vanishes and we took \( \theta^A = \lambda_A^i x^i \) for constant \( \lambda_A^i \). As above, we can derive an additional condition by requiring that the extremum be a minimum. Let us expand the energy to second order in the global deformation. For a constant parameter, the transformation is a symmetry of the energy. Overall,

\[
0 = E[\phi_0^a] - E[\phi_0^a] \\
= \frac{1}{2} \int \left[ d^d x \left[ \frac{\delta^2 E}{\delta \phi^a_0 \delta \phi^{b_0}} (\theta^A)^a_0 (\theta^B)^b_0 + 2 \frac{\delta^2 E}{\delta \phi^a_0 \delta \phi^b} (\theta^A)^a_0 \partial_i (\theta^B)^b_0 + \frac{\delta^2 E}{\delta \partial_i \phi^a_0 \delta \partial_j \phi^b} \partial_i (\theta^A)^a_0 \partial_j (\theta^B)^b_0 \right] \right] \\
= \frac{1}{2} \int \left[ d^d x \left( \theta^A \theta^B (T_A^c) (T_B^d)_c^d \right) + \frac{\delta^2 E}{\delta \phi^a_0 \delta \partial_i \phi^b} (\theta^A)^a_0 (\theta^B)^b_0 \delta \partial_i (\theta^A)^a_0 \delta \partial_j (\theta^B)^b_0 \right] \partial_i (\theta^A \theta^B) \right] \\
= \frac{1}{2} \int \left[ d^d x \left( (T_A^c) (T_B^d)_c^d \right) + \frac{\delta^2 E}{\delta \phi^a_0 \delta \partial_i \phi^b} (\theta^A)^a_0 (\theta^B)^b_0 \delta \partial_i (\theta^A)^a_0 \delta \partial_j (\theta^B)^b_0 \right] \partial_i (\theta^A \theta^B) \right] \\
(2.29)
\]

Now we allow the parameters of transformation \( \theta^A \) to be space-dependent. Using the symmetry transformation (2.29) the variation of the energy is

\[
E[\phi_0^a] - E[\phi_0^a] = \frac{1}{2} \int \left[ d^d x \left( (T_A^c) (T_B^d)_c^d \right) + \frac{\delta^2 E}{\delta \phi^a_0 \delta \partial_i \phi^b} (\theta^A)^a_0 (\theta^B)^b_0 \delta \partial_i (\theta^A)^a_0 \delta \partial_j (\theta^B)^b_0 \right] \partial_i (\theta^A \theta^B) \right] \\
+ \frac{1}{2} \int \left[ d^d x \left( (T_A^c) (T_B^d)_c^d \right) + \frac{\delta^2 E}{\delta \phi^a_0 \delta \partial_i \phi^b} (\theta^A)^a_0 (\theta^B)^b_0 \delta \partial_i (\theta^A)^a_0 \delta \partial_j (\theta^B)^b_0 \right] \partial_i (\theta^A \theta^B) \right] \\
(2.30)
\]

Thus the generalization of the stiffness tensor for global deformations (we may call it a ‘susceptibility’ tensor) takes the form

\[
E[\phi_0^a] - E[\phi_0^a] = \int d^d x C_{AB}^{ij} \partial_i \theta^A \partial_j \theta^B + C_{AB}^{ij} (\partial_i \theta^A \theta^B + \theta^A \partial_i \theta^B) > 0 \quad (2.31)
\]

where

\[
C_{AB}^{ij} = \frac{1}{2} \frac{\delta^2 E}{\delta (\theta^A)^a_0 \delta (\theta^B)^b_0} (T_A^c) (T_B^d)_c^d \partial^i (\theta^A)^a_0 \partial^j (\theta^B)^b_0 \\
C_{AB}^{ij} = \frac{1}{2} \frac{\delta^2 E}{\delta (\theta^A)^a_0 \delta (\theta^B)^b_0} \partial^i (\theta^A)^a_0 \partial^j (\theta^B)^b_0 \\
(2.32)
\]

The inequality at the end of (2.31) is of course the requirement that the extremum is indeed a minimum. Obviously the energy is invariant under global transformations, namely, constant \( \theta^A \).

For scalar theories with a flavor symmetry and terms at most quadratic in derivatives the susceptibility tensor takes the form

\[
\mathcal{L} = G_{ab}(\phi) \frac{1}{2} \partial_i \phi^a \partial^i \phi^b - V(\phi^a) \quad , \quad (2.33)
\]

we find

\[
C_{AB}^{ij} G_{ab}(\phi)(T_A^c) (T_B^d)_c^d \partial^i (\theta^A)^a_0 \partial^j (\theta^B)^b_0 \\
C_{AB}^{ij} = \frac{1}{2} \left[ \eta_{ij} G_{ab}(\phi) \partial^i (\theta^A)^a_0 \partial^j (\theta^B)^b_0 + \partial_k G_{ac}(\phi) \partial_l (\theta^A)^a_0 \partial^i (\theta^B)^b_0 \right] (T_A^c) (T_B^d)_c^d \quad , \quad (2.34)
\]
Note that there are terms that are not only proportional to the derivative of the parameter $\theta^A$ but to the parameter itself. These appear when the symmetry is non-Abelian. If we perform a transformation $\theta^A$ followed by a transformation $\theta^B$, this should be equivalent to a transformation $\theta^C$, but the relation between $\theta^C$, $\theta^A$ and $\theta^B$ is in general non-linear.

We can further extend the analysis by considering both global and geometric deformations together, with possible mixed terms between the two. We will not study this possibility here.

3. Stress forces

If a body is in mechanical equilibrium, the sum of all the forces inside a volume element vanish. The forces acting on the volume should be precisely equal to the force of the volume acting on the surrounding medium.

The force acting on a volume $V$ is the time variation of the momentum

$$F_i = \partial_0 P_i = \int_V d^d x \partial_0 T^0_i,$$

where $d$ is the number of spatial dimensions and $T^0_i$ is the momentum density. In a field theory this is part of the energy-momentum tensor $T^\mu_\nu$. Using energy-momentum conservation we can write the force in terms of the stress tensor as

$$F_i = -\int_V d^d x \partial_k T^k_i = -\oint d^{d-1} \sigma \hat{n}_k T^k_i.$$

We define $\hat{n}_k$ as the unit vector orthogonal to the surface, pointing out from the volume. Therefore, in the absence of external sources, the force acting on a volume element is determined by the stresses at the boundary. Consider for example an isotropic medium with a pressure\(^5\) that depends on one of the spatial coordinates $z$:

$$T^k_i = p(z) \delta^k_i.$$  \hspace{1cm} (3.3)

Focus on a cylindrical block (length $L$, area $A$) within the material, with its axis along the $z$ direction of length $L$ and caps of area $A$. The $z$-direction force acting on the block

$$F_z = - A(p(L/2) - p(-L/2)).$$  \hspace{1cm} (3.4)

If $p(L/2) > p(-L/2)$, the cylinder is pushed towards negative values of $z$ by the higher pressure at $z = L/2$.

We can now apply this type of analysis to solutions periodic along one spatial direction. We can perform arbitrary local deformations on the solution — each such deformation will change the balance of internal forces in the material. For instance, we can decrease the size of a unit cell, while at the same time deform the neighboring cells so the periodic solution remains unchanged farther away. The forces on the faces of the unit cell no longer cancel: the net force on the surface after the transformation could be pointing either in of

---

\(^5\)The pressure $p$ could also be negative in which case we call it a tension.
Figure 1: When a cell is compressed relative to its neighboring cells there are two possible situations. In the first case (upper arrows in blue), the force on the surfaces of the compressed cell point outward, so the force is restoring and the cell will go back to its original size when external forces are turned off. In the second case (lower arrows in red), the surface forces point inward, so the cell will continue compressing even further instead of returning to its original size.

or out of the unit cell. The first case is clearly unstable, since the deformed cell will now continue decreasing in size. In the second case the force tends to push the configuration back toward the original equilibrium (see figure 1). If the period of the unit cell is \( L = 2\pi/k \), the condition of stability becomes

\[
\partial_k p > 0. \tag{3.5}
\]

Simply put, when the cell is compressed its pressure should increase, and viceversa.

We can realize the situation above by slightly modifying a solution characterized by a given wavenumber \( k \): we make one strip shorter by a small amount and the neighboring strips larger so that the solution is not modified further in the material outside the strip. We can approximate this change by gluing a solution with \( k + \delta k \) between \( z = \pm \pi/(k + \delta k) \) and the solution with \( k' = k \frac{k+\delta k}{k+2\delta k} \) in the intervals \([ -\frac{3\pi}{k}, -\frac{\pi}{k+\delta k} ] \), \([ \frac{\pi}{k+\delta k}, \frac{3\pi}{k} ] \). The solution for \( |z| > 3\pi/k \) is the same as before.

We can also perform other deformations, such as changing the shape of the cell. In all cases the system will be stable if forces are restoring and unstable otherwise.

We apply this argument to spatially modulated solutions in holographic models of QCD in section §4.

4. Application to spatially modulated solutions

A spatially inhomogeneous vacuum (in holographic models with Chern-Simons term) was identified in [25, 26]. [25] studies Maxwell-Chern-Simons (MCS) theory in an \( AdS_3 \) black hole, while [26] treats gauge fields living on D8 branes in the compactified D4 black hole geometry. The latter is the deconfined phase of the Sakai-Sugimoto model [52, 53]. Both
examples were studied in the probe approximation, with no backreaction from the metric. Later works (e.g. [27]) include backreaction in the MCS theory, with qualitatively similar results. For simplicity we will study the stability conditions in the probe MCS in detail, and later on comment on the backreacted solutions.

4.1 Spatially modulated phases in $AdS_5$ black holes

A simple example of spatially modulated structure exists in in Maxwell-Chern-Simons theory [25],

$$\mathcal{L} = \frac{\sqrt{-g}}{\alpha^2} \left[ -\frac{1}{4} \tilde{F}_{IJJ} \tilde{F}^{IJJ} + \frac{1}{3! \sqrt{-g}} \epsilon^{IJKLM} \tilde{A}_I \tilde{F}_{JK} \tilde{F}_{LM} \right], \quad (4.1)$$

in an $AdS_5$ black hole background. The $\tilde{A}_I = \alpha A_I$ are gauge fields rescaled with respect to the Chern-Simons coupling such that entire action becomes proportional to $\alpha^{-2}$.

The holographic dual of this model is a conformal field theory at finite temperature with a probe sector having a global symmetry. The associated global current is dual to the gauge field in the bulk, and the Chern-Simons term implies that the global symmetry is anomalous in the field theory. This model is interesting as a simplified version of QCD at finite temperature, taking into account effects of the chiral anomaly.

The authors of [25] explore this model in the limit $\alpha \rightarrow \infty$ with a finite chemical potential. ($A_0(r=\infty) = \mu \rightarrow 0$ with $\alpha \mu$ finite.) The background metric is given by

$$ds^2 = -H(r)dt^2 + H(r)^{-1}dr^2 + r^2d\vec{x}^2 \quad \text{with} \quad \vec{x} = (x_2, x_3, x_4) \quad (4.2)$$

and

$$H(r) = r^2 \left[ 1 - \left( \frac{T+}{r} \right)^4 \right]. \quad (4.3)$$

The background electric field corresponds to the asymptotic value of the field strength,

$$\lim_{r \rightarrow \infty} \tilde{F}_{0r} = \frac{\bar{E}}{r^3} = -\frac{2r^3}{\pi T r^3} \quad (4.4)$$

where $\tau = T/\mu \alpha$ is the rescaled temperature of the black hole. A spatially modulated solution of the equations of motion is found in [25], of the form

$$\tilde{A}_0 = f(r), \quad \tilde{A}_3 + i\tilde{A}_4 = h(r)e^{ikx_2} \quad (4.5)$$

with all other gauge field components vanishing.\(^6\) The equations of motion under this ansatz are

$$\partial_r \left( r^3 f' + 2kh^2 \right) = 0 \quad (4.6)$$

$$\partial_r \left( rHh' \right) - \frac{k^2}{r} h + 4f'kh = 0 \quad (4.7)$$

\(^6\)We have absorbed a negative sign into $k$ with respect to the convention of [25] for later convenience.
Figure 2: Amplitude $h_0$ of the spatially modulated solution as a function of wave number, $k$ in units of $1/r_+$. We take $\tau = 0.35$.

Integrating the first line allows us to express $f'$ with an integral of motion $\tilde{E}$, using

$$r^3f' + 2kh^2 = -\tilde{E}.$$  \hfill (4.8)

This leaves a single equation

$$r^3\partial_r (rHh') - r^2k^2h - 4kh \left( \tilde{E} + 2kh^2 \right) = 0.$$ \hfill (4.9)

As explained in [25], this equation has non-trivial solutions – that is, with amplitude $h(r_+) = h_0$ nonzero – only for a limited set of values of $k$. Solving the above equation under the boundary condition and assuming no sources for the currents (i.e. no non-normalizable modes in $A_3, A_4$) one finds a relation between $h_0$ and the wavenumber of the modulation, $k$ (see Figure 2).

In holographic models with D-branes the components of the energy-momentum tensor of the brane integrated along the radial direction are identified with the energy-momentum tensor of the dual field theory [54],

$$\langle T^\mu_{\nu} \rangle = \int_{Dp} dr \sqrt{g_{Dp}} T^\mu_{Dp} \nu,$$ \hfill (4.10)

as a result of the map between symmetries of the gravity and gauge theories. It is natural to extend this map to the probe gauge fields, so that the energy density of the field theory can be defined as the integral of the bulk $T_{00}$ along the radial direction

$$\langle \mathcal{E} \rangle = \int_{r_+}^{\infty} dr \left[ \frac{1}{2r^3} \left( \tilde{E} + 2kh^2 \right)^2 + \frac{k^2h^2}{2r} + \frac{rHh'^2}{2} \right].$$ \hfill (4.11)

This is also identified with the free energy,\textsuperscript{7} plotted in Fig. 3. Its lies at roughly $kr_+ = 2.38$.\textsuperscript{8} The corresponding solution is thus thermodynamically stable within this family of spatially modulated solutions.

\textsuperscript{7}There is, in addition, a boundary contribution that we have not included. However, when we Legendre transform to an ensemble with fixed density, this term cancels out. The details can be found in the Appendix of [11].

\textsuperscript{8}The energy density, equation 3.11 in [25] seems to be missing a $r^3$ factor in the integrand.
Figure 3: Free energy $F$ of the spatially modulated solution as a function of wave number, $k$, in units of $1/r_+$. We set $\tau = 0.35$. The function is minimized for $kr_+ \simeq 2.4$.

4.1.1 Stiffness tensor

Using the results of section § 2 we can compute the stiffness tensor to check the stability of these solutions under small geometrical deformations. The energy density is

$$
E = \int_{r_+}^{\infty} dr \sqrt{-g} \left[ \frac{1}{2} g^{00} |g^{ij} F_{0i} F_{0j} + \frac{1}{2} g^{rr} g^{ij} F_{0r} F_{0r} + \frac{1}{2} g^{rr} g^{ij} F_{rj} F_{rj} + \frac{1}{4} g^{ik} g^{jl} F_{ij} F_{kl} \right].
$$

The expression for the stiffness tensor is quite complicated, but one can show that the condition (2.9) is satisfied. We illustrate only a part of the result. Let us take the unit vectors

$$\hat{a} = (\cos \theta_1, \sin \theta_1 \cos \varphi_1, \sin \theta_1 \sin \varphi_1), \quad \hat{b} = (\cos \theta_2, \sin \theta_2 \cos \varphi_2, \sin \theta_2 \sin \varphi_2).$$

Then,

$$C_{jm}^{il} \hat{a}_i \hat{a}_j \hat{b}^m = \frac{1}{32} \int_{r_+}^{\infty} dr \frac{k^2 h^2}{r} \left[ 18 - \cos (2kx_2 + 4\theta_1 - 2\phi_1) - \cos (2kx_2 - 2(2\theta_1 + \phi_1)) + 64 \sin (\theta_1) \sin (\theta_2) \cos^2 (\theta_1) \sin (kx_2 - \phi_1) \sin (kx_2 - \phi_2) - 8 \sin^2 (\theta_1) \cos (2kx_2 - 2\phi_2) + \cos (2\theta_2) (8 \sin^2 (\theta_1) \cos (2kx_2 - 2\phi_2) - 4) + 2 \cos (2kx_2 - 2\phi_1) + 12 \cos (2\theta_1) + 2 \cos (4\theta_1) + 2 \cos (2(\theta_1 - \theta_2)) + 2 \cos (2(\theta_1 + \theta_2)) \right].$$

Note that there is a shift symmetry

$$x_2 \to x_2 + \delta x_2, \quad \phi_1 \to \phi_1 + k\delta x_2, \quad \phi_2 \to \phi_2 + k\delta x_2,$$

which allows us to evaluate the tensor at a fixed value of $x_2$, for instance $x_2 = 0$, this gives an expression of the form

$$C_{jm}^{il} \hat{a}_i \hat{a}_j \hat{b}^m = \frac{1}{32} \int_{r_+}^{\infty} dr \frac{k^2 h^2}{r} C(\theta_1, \theta_2, \phi_1, \phi_2).$$
One can easily check that there are some flat directions where $C = 0$, for instance for $\theta_1 = \pi/2$, $\theta_2 = 0$. We have also checked that (4.16) is never negative, by first finding the values of $(\theta_1, \theta_2, \phi_1, \phi_2)$ where $C$ is extremized

$$\partial_A C = 0, \quad A = \theta_1, \theta_2, \phi_1, \phi_2,$$

(4.17)

and evaluating $C$ at those points. For all extremal points $C \geq 0$, so (2.9) is satisfied and there are no unstable directions. Note that this result is valid for any spatially modulated solution on the curve in Figure 2, not only for the one which minimizes the free energy.

4.1.2 Adiabatic deformations

In addition to obtaining information about the stability of the solutions under geometric deformations, we can compute the work done by adiabatic changes. We introduce time-dependent deformations and eventually neglect contributions from time derivatives.

We take $\xi^0 = \xi^r = 0$ for the deformations. Note that the deformed configurations are still normalizable, since to second order

$$A_M \rightarrow A_M - \partial_M \xi^N A_N + \partial_M \xi^L \partial_L \xi^N A_N.$$  

(4.18)

The only possible non-normalizable contribution would appear from a term proportional to $A_0$, but since $\xi^0 = 0$, no such term exists.

The first order term in the variation of the energy is a total derivative, which we take to vanish, plus an additional term proportional to

$$-2 \partial_0 \xi^i \frac{\delta \mathcal{E}}{\delta F_{0\alpha}} F_{i\alpha}.$$  

(4.19)

This term appears because there is a net flow of momentum in the spatially modulated solution. $\partial_0 \xi^i$ is the velocity of the ‘volume element’ and the energy increases or decreases depending on whether the volume element moves in the same direction as the momentum flow. In the spatially modulated background the only non-zero terms appear for $\alpha = r$ and $i = 3, 4$, and are proportional to

$$F_{3r} = -\partial_r A_3 = -h' \cos(kx_2), \quad F_{4r} = -\partial_r A_4 = -h' \sin(kx_2).$$  

(4.20)

We see from this expression that momentum flows in the $x_3, x_4$ plane and that its direction rotates as we move along the $x_2$ direction. The first order term will vanish for deformations transverse to the momentum flow, namely

$$\xi^2 = \xi^L(t, x, r), \quad \xi^3 = \xi^T(t, x, r) \sin(kx_2), \quad \xi^4 = -\xi^T(t, x, r) \cos(kx_2).$$  

(4.21)

We compute the quadratic order contributions to the energy for these deformations, using the results of Appendix A. From equation (4.21) we can see that $\tilde{C}^{0r}, C^{\alpha\gamma}_{0\delta}, C^{\alpha\gamma}_{xy}$ and $C^{\alpha\gamma}_{x\delta}$ are irrelevant to us (we assume $\xi^0 = \xi^r = 0$).

The final result for the change in the energy density is

$$\mathcal{E}_2 = \mathcal{E}_T + \mathcal{E}_L + \mathcal{E}_{\text{int}},$$  

(4.22)
where \((a, b = 3, 4)\)

\[
\mathcal{E}_L = rk^2h^2 \left( \frac{1}{H}(\partial_t \xi^L)^2 + H(\partial_r \xi^L)^2 + \frac{1}{r^2}(\partial_\phi \xi^L)^2 + \frac{1}{r^2}L^{ab} \partial_a \xi^L \partial_b \xi^L \right),
\]

\[
\mathcal{E}_T = rk^2h^2 \left( \frac{1}{H}(\partial_t \xi^T)^2 + H(\partial_r \xi^T)^2 + H \partial_r (hh' (\xi^T)^2) + \frac{1}{r^2} \delta^{ab} \partial_a \xi^T \partial_b \xi^T \right),
\]

\[
\mathcal{E}_{\text{int}} = 2k f' h' \xi^T \partial_t \xi^L + \partial_\phi \xi^L M_{LT}^a \partial_a \xi^T + \partial_\phi \xi^T M_{TL}^a \partial_a \xi^L + \xi^T N^a \partial_a \xi^L.
\]

We have defined

\[
L^{ab} = \hat{z}^a \hat{z}^b,
\]

\[
M_{LT}^a = \left( \frac{k^2h^2}{r} - r^2(f')^2 - rH(h')^2 \right) \hat{y}^a,
\]

\[
M_{TL}^a = \left( -\frac{3k^2h^2}{r} + r^2(f')^2 + rH(h')^2 \right) \hat{y}^a,
\]

\[
N^a = \left( \frac{k^2h^2}{r} - r^2(f')^2 + rH(h')^2 \right) \hat{z}^a,
\]

where the unit vectors \(\hat{y}\) and \(\hat{z}\) are

\[
\hat{y} = (- \sin(kx_2), \cos(kx_2)), \quad \hat{z} = (\cos(kx_2), \sin(kx_2)).
\]

Note that \(\hat{z}\) is the direction of the momentum flow in the spatially modulated solution and \(\hat{y}\) the direction orthogonal to it in the plane transverse to \(x_2\). Let us now define the coordinates

\[
y = \hat{y}^a x_a, \quad z = \hat{z}^a x_a.
\]

Note that for \(x_2 = 0, y = x_4\) and \(z = x_3\). For \(x_2 \neq 0\), the \(y, z\) coordinates are related to the \(x_3, x_4\) coordinates by a rotation.

We now assume that \(\xi^T = \xi^T(t, r, y, z), \xi^L = \xi^L(t, r, y, z)\). For a function \(F(y, z)\) the spatial derivatives take the form \((A, B = y, z)\)

\[
\partial_a F = \hat{y}_a \partial_y F + \hat{z}_a \partial_z F,
\]

\[
\partial_2 F = y \partial_y F - z \partial_z F \equiv \epsilon^{AB} x_A \partial_B F.
\]

This allows us to simplify the energy density in such a way that the dependence on the coordinates is polynomial:

\[
\mathcal{E}_L = rk^2h^2 \left( \frac{1}{H}(\partial_t \xi^L)^2 + H(\partial_r \xi^L)^2 + \frac{1}{r^2} \epsilon^{AB} x_A \partial_B \xi^L \partial_\phi \xi^L \right),
\]

\[
\mathcal{E}_T = rk^2h^2 \left( \frac{1}{H}(\partial_t \xi^T)^2 + H(\partial_r \xi^T)^2 + H \partial_r (hh' (\xi^T)^2) + \frac{1}{r^2} \delta^{ab} \partial_a \xi^T \partial_b \xi^T \right),
\]

\[
\mathcal{E}_{\text{int}} = 2k f' h' \xi^T \partial_t \xi^L + (M_{LT}^a \hat{y}_a) \epsilon^{AB} x_A \partial_B \xi^L \partial_\phi \xi^T + (M_{TL}^a \hat{y}_a) \epsilon^{AB} x_A \partial_B \xi^T \partial_\phi \xi^L
\]

\[
+ (N^a \hat{z}_a) \xi^T \partial_z \xi^L.
\]
We can further simplify these expressions if we take \( \xi^T = c \xi^L = c \xi \) for some constant \( c \). Then,

\[
\mathcal{E}_2 = (1 + c^2) r k^2 h^2 \left( \frac{1}{H} (\partial_t \xi)^2 + H (\partial_r \xi)^2 + \frac{1}{r^2} \frac{(\epsilon^{AB} x_A \partial_B \xi - c \partial_y \xi)^2}{1 + c^2} + \frac{1}{r^2} (\partial_z \xi)^2 \right)
+ c f' h' \partial_t (\xi^2) + \frac{c}{2} (N^a \tilde{z}_a) \partial_z (\xi^2).
\]

(4.37)

Note that

\[
\epsilon^{AB} x_A \partial_B \xi - c \partial_y \xi = y \partial_z \xi - (z + c) \partial_y \xi,
\]

(4.38)

so if we define a new coordinate \( u = z + c \), the energy density is simply

\[
\mathcal{E}_2 = (1 + c^2) r k^2 h^2 \left( \frac{1}{H} (\partial_t \xi)^2 + H (\partial_r \xi)^2 + \frac{1}{r^2} \frac{(y \partial_u - u \partial_y \xi)^2}{1 + c^2} + \frac{1}{r^2} (\partial_u \xi)^2 \right)
+ c f' h' \partial_t (\xi^2) + \frac{c}{2} (N^a \tilde{z}_a) \partial_z (\xi^2).
\]

(4.39)

We can read some interesting physics from this expression. First, when we integrate the term \( \propto \partial_t (\xi^2) \) over time, it gives a change in the energy proportional to the square of the total displacement. It is analogous to a spring with a spring constant

\[
K_{\text{spring}} = c \int_{r^+}^{\infty} dr \ k' h'.
\]

(4.40)

If we set \( c = 0 \) – that is, if the deformation is only a displacement along the \( x_2 \) direction – then \( K_{\text{spring}} = 0 \) and the variation of the energy will depend only on derivatives of the displacement field if we make \( \partial_r \xi = 0 \). Note that beyond the adiabatic regime this is an issue, since the term proportional to \( (\partial_t \xi)^2 \) diverges at the horizon. Neglecting the time derivatives we are left with the contribution from the spatial components of the stiffness tensor

\[
\mathcal{E} = \frac{k^2 h^2}{r^2} \left( ((y \partial_z - z \partial_y) \xi)^2 + (\partial_z \xi)^2 \right).
\]

The first term is the square of the angular momentum in the \( y, z \) plane, while the second term is the square of the linear momentum in the \( z \) direction. Recall that \( z \) is the direction of the flow of momentum in the spatially modulated solution and that the \( y, z \) plane rotates relative to the \( x_3, x_4 \) axes along the \( x_2 \) direction. This result suggests the possibility of having modes with unusual dispersion relations, which depend on the angular momentum in the transverse plane rather than on the usual linear momentum.

### 4.1.3 Stress forces

Let us now check the stability condition (3.5) derived from the condition on the stress forces when the periodicity of the solution is changed locally. The pressure along the direction of spatial modulation is

\[
p_2 = \sqrt{-g} T^2_2 = -\frac{1}{2r^3} \left( \tilde{E} + 2kh^2 \right)^2 - \frac{k^2 h^2}{2r} + \frac{rHh^2}{2}.
\]

(4.41)
Using the relation between the boundary and bulk stress tensor, we identify the pressure in the field theory dual as
\[
\langle p_2 \rangle = \int_{r_+}^{\infty} dr \sqrt{-g} T_{2}^{2}.
\]
(4.42)

The pressure is negative (it is actually a tension) for all \( k \) and has a maximum at around \( kr_+ \approx 2.25 \) (see Fig. 4). Note that in the region where \( \partial_k \langle p_2 \rangle < 0 \) a deformation that changes the value of \( k \) in a finite region will not go back to the original equilibrium. The minimum of the free energy is inside this region, at \( kr_+ \approx 2.4 \). This suggests that either the endpoint of the instability of the homogeneous solution is a different kind of inhomogeneous solution, or that the probe approximation does not suffice to describe the energetics of the configuration. We will comment more on this in the next section.

Our arguments so far were based on deformations of the solutions into configurations that do not satisfy the equations of motion and therefore require the presence of external forces. Based on the analysis of stresses we have argued that after the external forces are turned off the evolution of the system will take it away from the original solution if the stress forces are not restoring. In order to show that such initial states are indeed possible we give an explicit example of such an initial condition in Appendix B.

### 4.2 Stress forces and thermodynamic stability in backreacted solutions

Spatially modulated solutions of Maxwell-Chern-Simons, including backreaction, were constructed by Donos and Gauntlett in [27]. Equilibrium configurations were found numerically by computing the solutions as a function of the temperature and \( k \) and minimizing the free energy at each temperature. In a more recent paper [50], the authors derive an analytic condition that determines the minimum of the free energy with respect to \( k \).

---

\(^9\)In principle one can derive the same relations using the results of [55]. We thank Ioannis Papadimitriou for pointing this out to us.
Their derivation is not limited to MCS, but extends to other spatially modulated solutions as well. In the spontaneously broken case,

\[ 0 = k \frac{\delta w}{\delta k} \bigg|_{eq} = w + \langle p_2 \rangle \bigg|_{eq}. \]  \hspace{1cm} (4.43)

This implies that

\[ k \frac{\delta^2 w}{\delta k^2} \bigg|_{eq} = \partial_k \langle p_2 \rangle \bigg|_{eq}. \]  \hspace{1cm} (4.44)

Therefore, the condition on the pressure we have found using the stress force analysis (3.5) is equivalent to the condition that the free energy has a minimum. This suggests that there may be an issue with the definition of the free energy in the probe approximation. If the free energy were properly defined, the equilibrium configuration would be located in the stable region as determined by the stress force analysis.

### 4.3 Spatially modulated solutions in the Sakai-Sugimoto model

The Sakai-Sugimoto model consists of a D8 brane embedded in the D4 soliton geometry, localized in the 5th compact direction and wrapping an \( S^4 \). At low energies it is the holographic dual to a confining QCD-like theory. As temperature is increased, the system undergoes a transition to a deconfined phase, described by a black hole geometry. In this phase (for large enough temperatures) the D8 splits in a \( D8 \) and \( \overline{D8} \) falling straight into the horizon. This is seen as the dual version of chiral symmetry restoration. The global chiral symmetries map to gauge fields on the branes worldvolumes, and the chiral anomalies to five-dimensional Chern-Simons terms.

A D8 brane embedded in the D4 black hole geometry has an effective action of the form

\[ S_{D8} = -T_8 \int dt d^3x du u^{1/4} \sqrt{\det(g_{\alpha\beta} + \tilde{F}_{\alpha\beta})} + \frac{\alpha}{6} T_8 \int dt d^3x du \epsilon^{\mu_1\mu_2\mu_3\mu_4\mu_5} \tilde{A}_{\mu_1} \tilde{F}_{\mu_2\mu_3} \tilde{F}_{\mu_4\mu_5}. \]  \hspace{1cm} (4.45)

We have integrated already over the \( S^4 \) directions, so the effective metric is five-dimensional

\[ ds^2 = u^{3/2}(-f(u)dt^2 + d\vec{x}^2) + \frac{du^2}{u^{3/2} f(u)}, \quad f(u) = 1 - \frac{u^3}{u^3}. \]  \hspace{1cm} (4.46)

The value of the Chern-Simons coupling is \( \alpha = \frac{3}{4} \). The authors of [26] demonstrate that a homogeneous D8 embedding with an electric field \( F_{0u} \) is unstable for fluctuations of the gauge field with momentum below some threshold. For a density \( \rho = 5u_T^{5/2} \) this threshold lies at \( k = 2.39u_T^{1/2} \). In the field theory dual, this implies that a finite density homogeneous state is unstable towards the appearance of inhomogeneous currents.

[26] propose as the endpoint of this instability a spatially modulated phase that in the holographic dual is described by a configuration of the D8 gauge fields:

\[ \tilde{A}_t = a(u), \quad \tilde{A}_x + i\tilde{A}_y = h(u)e^{-ikz}. \]  \hspace{1cm} (4.47)
The solution proceeds as for the AdS black hole case in the previous subsection. Under the ansatz for the solution, the equations of motion become

\[
0 = \partial_u \left( \frac{ua'(u)\sqrt{k^2h(u) + u^3}}{\sqrt{-a'(u)^2 + f(u)h'(u)^2 + 1}} \right) + 4k\alpha h(u)h'(u), \quad (4.48)
\]

\[
0 = \partial_u \left( \frac{uf(u)\sqrt{k^2h(u) + u^3h'(u)}}{\sqrt{-a'(u)^2 + f(u)h'(u)^2 + 1}} \right) + 4k\alpha h(u)a'(u) - \frac{k^2uh(u)\sqrt{-a'(u)^2 + f(u)h'(u)^2 + 1}}{\sqrt{k^2h(u) + u^3}}. \quad (4.49)
\]

The first equation can be integrated, and solved for \(a'(u)\):

\[
a'(u) = \sqrt{f(u)h'(u)^2 + 1} \left( \rho - 2k\alpha h(u)^2 \right) \quad (4.50)
\]

\(\rho\) is an integration constant proportional to the charge density.

The equation of motion for \(h(u)\) becomes

\[
K(u)\partial_u \left( K(u)f(u)h'(u) \right) + 4k\alpha h(u) \left( \rho - 2k\alpha h(u)^2 \right) - k^2u^2h(u) = 0, \quad (4.51)
\]

where

\[
K(u) = \sqrt{kh(u)^2 \left( 4k\alpha^2h(u)^2 + ku^2 - 4\alpha\rho \right) + u^5 + \rho^2}. \quad (4.52)
\]

In order to solve the equation of motion one needs to fix the initial conditions at the horizon \(h_0 = h(u_T)\) and

\[
h'(u_T) = \frac{h_0k\alpha h(u_T) \left( 8h_0^2k\alpha^2 + ku_T^2 - 4\alpha\rho \right)}{3 \left( h_0k \left( 4h_0^2k\alpha^2 - 4h_0\alpha\rho + ku_T^2 \right) + u_T^5 + \rho^2 \right)}. \quad (4.53)
\]

One can determine the initial condition for the derivative by evaluating the equations of motion at the horizon and eliminating the term \(a''(u_T)\).

For the numerical calculation we rescale coordinates, density and momentum in such a way that the horizon is effectively at \(u_T = 1\). Following [26], we then use the shooting method to solve for solutions with \(\rho = 5\), \(\alpha = 3/4\), demanding that the solution be normalizable at the boundary. We find solutions in the interval \(k \in [1.485, 4.3494]\) that resemble closely those found by Ooguri and Park. The value of the initial condition \(h_0\) for different values of \(k\) is plotted in figure 5.

We can now check whether this solution is stable under the deformation conditions. The longitudinal pressure density in the \(z\) direction is

\[
p_z = T_z = 2 \frac{\partial \mathcal{L}}{\partial F_{zx}} F_{zx} + 2 \frac{\partial \mathcal{L}}{\partial F_{zy}} F_{zy} - \mathcal{L}. \quad (4.54)
\]

This gives

\[
p_z = T_8 u^4 \sqrt{\frac{-a'(u)^2 + f(u)h'(u)^2 + 1}{k^2h(u)^2 + u^3}}. \quad (4.55)
\]
The pressure density is divergent when $u \to \infty$

$$p_z \simeq u^{5/2} + O\left(u^{-5/2}\right), \quad (4.56)$$

but the divergence is independent of the solution, so we can simply subtract it in order to compute the finite pressure in the dual field theory

$$\langle p_z \rangle = \int_{uT}^{\infty} du \left(p_z - u^{5/2}\right). \quad (4.57)$$

The result is plotted in figure 6. We observe that the condition $\partial_k \langle p_z \rangle > 0$ is satisfied only in a region of low $k$, $k \lesssim 1.89 u^{1/2}_T$. As for the case of Maxwell-Chern-Simons in the probe approximation, the stress force analysis seems to be in disagreement with the quoted values of the free energy minimum. We have seen that in the backreacted MCS solution the stress force condition is consistent with the thermodynamic analysis, so for the D8 branes it may also be related to the probe approximation and/or the definition of the free energy. The MCS example suggests that the issue may be solved by taking into account the backreaction and the contribution of closed string fields to the free energy.

5. Conclusions and future directions

We have presented new tests of stability for inhomogeneous phases based on small geometric deformations. We used these tests to verify the stability of spatially modulated solutions, which are holographic duals of finite density states. First, we expanded the energy to second order in the deformations and established under what condition the solution is a local minimum. We then presented a version of these conditions generalized to global symmetries. The second test of stability amounts to demanding that local deformations induce a restoring force in the material. For the solutions constructed in [25] we found that the energy minimization condition is satisfied, but the force condition is not satisfied. The force condition is also not satisfied in general in [26], and furthermore the minimum of the free energy lies outside the stable region. However, analyzing the backreacted solution
Figure 6: Pressure of the spatially modulated solutions in the Sakai-Sugimoto model. The momentum is normalized in units of $u^1_T/2$ and the pressure in units of the effective D8 brane tension. When the slope is positive the solutions are stable according to the stress force analysis.

for the system of [25] (using the results of [50]), we find that the restoring force condition and thermodynamic stability are equivalent. This suggests that the free energy of the probe solution has not been properly defined. In other words, that simply minimizing the energy of the brane while neglecting the background is not a valid approximation. Another possibility is that the probe solution is indeed unstable.

Although we have worked out the details of only a few examples, the methods we present here can be applied to a variety of inhomogeneous solutions. An obvious advantage of the stability checks we have presented is that they do not require solving the linearized fluctuation equations for time-dependent configurations. Of course this renders the analysis less comprehensive, since there may be unstable directions in field space not captured by the kind of deformations we have presented.

The stability analysis can be improved by including additional types of deformations, such as global (or gauge) symmetry deformations. This extension is relevant to myriad holographic applications, such as solutions involving non-Abelian gauge fields dual to p-wave superfluids [56,57].

A further extension of this work would exploit the connection between Goldstone bosons and symmetry transformations. As the solutions we study spontaneously break translation and/or rotation invariance, we expect that at low energies there should be gapless modes, the Goldstone bosons. The precise number and dispersion relations of these modes is a more complicated issue when Lorentz invariance is broken (see for instance [58,59], and [60] for an example in holography). In principle we expect the Goldstone modes to take a form similar to the geometric deformations we use to test the stability of the background configuration, so the stiffness tensor' will determine at least in part the effective action of Goldstone modes to quadratic order. The fact that we observe flat directions implies that such modes are possible. The dependence of the energy on the transverse angular momentum suggests that indeed the dispersion relation of the Goldstone modes is not simply linear in momentum. We hope to explore these questions and more in the future.
Acknowledgements

We would like to thank Hirosi Ooguri and Chang-Soon Park for their help in clarifying some aspects of [25, 26]. The work of C.H. and J.S. is partially supported by the Israel Science Foundation (grant 1665/10). The work of J.S. on this project was partially supported by the Einstein Center for Theoretical Physics at the Weizmann Institute.

A. General deformation for a gauge-invariant energy density

We can formally generalize the analysis of static deformations to allow deformations that are time-dependent and involve the time directions (such as boosts). One can use these methods to derive an effective action for deformations. With this in mind, we will now consider a general deformation of a theory with Abelian gauge fields.

The change of coordinates is

$$\tilde{x}^\mu = x^\mu - \xi^\mu(x). \quad (A.1)$$

From

$$\partial_\nu \xi^\mu = \frac{\partial \tilde{x}^\nu}{\partial x^\mu} \partial_\alpha \xi^\mu = (\delta^\alpha_\nu - \partial_\nu \xi^\alpha) \partial_\alpha \xi^\mu, \quad (A.2)$$

we get

$$\partial_\nu \xi^\alpha = \tilde{\partial}_\nu \xi^\sigma (\delta^\alpha_\sigma - \tilde{\partial}_\sigma \xi^\alpha). \quad (A.3)$$

The field strength is, to second order

$$\frac{\partial \tilde{x}^\alpha}{\partial x^\mu} \frac{\partial \tilde{x}^\beta}{\partial x^\nu} F_{\alpha\beta} = F_{\mu\nu} + \delta F^{(1)}_{\mu\nu} + \delta F^{(2)}_{\mu\nu}, \quad (A.4)$$

where

$$\delta F^{(1)}_{\mu\nu} = -\tilde{\partial}_\sigma \xi^\rho (\delta^\sigma_\mu \delta^\rho_\nu F_{\rho\nu} + \delta^\sigma_\nu \delta^\rho_\mu F_{\rho\mu}), \quad (A.5)$$

and

$$\delta F^{(2)}_{\mu\nu} = \frac{1}{2} \tilde{\partial}_\sigma \xi^\rho \tilde{\partial}_\lambda \xi^\tau (\delta^\sigma_\mu \delta^\lambda_\rho \delta^\tau_\nu F_{\rho\nu} + \delta^\sigma_\nu \delta^\lambda_\rho \delta^\tau_\mu F_{\rho\mu} + (\sigma, \rho) \leftrightarrow (\lambda, \tau)). \quad (A.6)$$

The Jacobian to second order is

$$\left| \frac{\partial x^\mu}{\partial \tilde{x}^\nu} \right| = 1 + J^{(1)} + J^{(2)}, \quad (A.7)$$

where

$$J^{(1)} = \tilde{\partial}_\sigma \xi^\sigma, \quad (A.8)$$

and

$$J^{(2)} = \frac{1}{2} \tilde{\partial}_\sigma \xi^\rho \tilde{\partial}_\lambda \xi^\tau (\delta^\sigma_\rho \delta^\lambda_\tau - \delta^\sigma_\tau \delta^\lambda_\rho). \quad (A.9)$$

The energy to second order is then

$$E_\Lambda = E + \int d^d x \left( J^{(1)} \mathcal{E} + \frac{\delta \mathcal{E}}{\delta F_{\mu\nu}} \delta F^{(1)}_{\mu\nu} \right) + \int d^d x \left( J^{(2)} \mathcal{E} + J^{(1)} \frac{\delta \mathcal{E}}{\delta F_{\mu\nu}} \delta F^{(2)}_{\mu\nu} + J^{(1)} \frac{\delta \mathcal{E}}{\delta F_{\mu\nu}} \delta F^{(1)}_{\mu\nu} + \frac{1}{2} \frac{\delta^2 \mathcal{E}}{\delta F_{\mu\nu} \delta F_{\alpha\beta}} \delta F^{(1)}_{\mu\nu} \delta F^{(1)}_{\alpha\beta} \right). \quad (A.10)$$
We can now compute terms which are second order in the deformations,

$$\delta E^{(2)} = \frac{1}{2} \int d^4x \tilde{C}^{\alpha\gamma}_{\beta\delta} \partial_\alpha \xi^\beta \partial_\gamma \xi^\delta,$$  \hspace{1cm} (A.11)

where $\tilde{C}^{ij}_{kl} = C^{ij}_{kl}$ is the same tensor as in (2.26) (for $\delta E/\delta A_{\mu} = 0$), but $\tilde{C}$ has also temporal indices:

$$\tilde{C}^{\alpha\gamma}_{\beta\delta} = E \left( \delta^\alpha_\beta \delta^\gamma_\delta - \delta^\alpha_\delta \delta^\gamma_\beta \right) - \frac{\delta E}{\delta F_{\mu\nu}} \left( F_{\mu\beta} \delta^\alpha_\delta \delta^\gamma_\delta + F_{\beta\nu} \delta^\alpha_\mu \delta^\gamma_\delta + \alpha, \beta \leftrightarrow \gamma, \delta \right)$$

$$+ \frac{\delta E}{\delta F_{\mu\nu}} \left( \delta^\alpha_\mu \delta^\gamma_\beta F_{\nu\delta} + \delta^\alpha_\nu \delta^\gamma_\beta F_{\mu\delta} + \alpha, \beta \leftrightarrow \gamma, \delta \right)$$

$$+ \frac{\delta^2 E}{\delta F_{\mu\nu} \delta F_{\rho\sigma}} \left( F_{\mu\beta} \delta^\alpha_\rho \delta^\gamma_\sigma + F_{\beta\nu} \delta^\alpha_\rho \delta^\gamma_\sigma + \alpha, \beta \leftrightarrow \gamma, \delta \right).$$ \hspace{1cm} (A.12)

### B. Deformed configuration as force-free initial condition

The generic form of the background solution is

$$A_3 = h_k \cos(kx_2 + \varphi),$$ \hspace{1cm} (B.1)

$$A_4 = h_k \sin(kx_2 + \varphi),$$ \hspace{1cm} (B.2)

$$A_0 = f_k.$$ \hspace{1cm} (B.3)

We will modify this solution by a shift of the momentum and the phase,

$$k \rightarrow k + \delta k, \hspace{0.5cm} \varphi \rightarrow \varphi + \delta \varphi.$$ \hspace{1cm} (B.4)

To leading order

$$\delta A_3 = a_3 = \delta k \partial_k h_k \cos(kx_2 + \varphi) - (x_2 \delta k + \delta \varphi) \sin(kx_2 + \varphi),$$ \hspace{1cm} (B.5)

$$\delta A_4 = a_4 = \delta k \partial_k h_k \sin(kx_2 + \varphi) + (x_2 \delta k + \delta \varphi) \cos(kx_2 + \varphi),$$ \hspace{1cm} (B.6)

$$\delta A_0 = a_0 = \delta k \partial_k f_k.$$ \hspace{1cm} (B.7)

We will restrict the change of the solution of wavelength $k$ to an interval $[-\pi/k,\pi/k]$. A change of the interval by $\delta k$ will modify the solution only by $\delta k^2$ terms, so we can neglect it.

$$\delta k = \delta k_0 \Delta(x_2), \hspace{0.5cm} \delta \varphi = \delta \varphi(x_2, r),$$ \hspace{1cm} (B.8)

where

$$\Delta(x) \simeq \left( \Theta \left(x + \frac{\pi}{k}\right) - \Theta \left(x - \frac{\pi}{k}\right) \right),$$ \hspace{1cm} (B.9)

Here $\Theta(x)$ is a step function. We can regularize the solution by changing the step functions to

$$\left( \Theta \left(x + \frac{\pi}{k}\right) - \Theta \left(x - \frac{\pi}{k}\right) \right) \rightarrow \frac{1}{2} \left( \tanh \left[ M \left(x + \frac{\pi}{k}\right) \right] - \tanh \left[ M \left(x - \frac{\pi}{k}\right) \right] \right),$$ \hspace{1cm} (B.10)
with \( M \gg 1 \). In this case
\[
\Delta(x) = \frac{1}{2} \left( \tanh \left[ M \left( x + \frac{\pi}{k} \right) \right] - \tanh \left[ M \left( x - \frac{\pi}{k} \right) \right] \right).
\] (B.11)

Expanding the gauge fields in background plus fluctuations \( F_{MN} + f_{MN} \), the linearized equations of motion are
\[
\partial_M (\sqrt{-g} g^{MA} g^{NB} f_{AB}) + \alpha e^{NABC} F_{AB} f_{CD} = 0.
\] (B.12)

For the spatially modulated solution where \( F_{r0} F_{23}, F_{r3}, F_{24} \) and \( F_{r4} \) are different from zero this leads to
\[
0 = \partial_M (\sqrt{-g} g^{MA} g^{32} f_{A2}) + 4\alpha h'(\cos(kx_2) f_{04} - \sin(kx_2) f_{03}) - 4\alpha A'_0 f_{34},
\] (B.13)
\[
0 = \partial_M (\sqrt{-g} g^{MA} g^{rr} f_{Ar}) + 4\alpha h(\sin(kx_2) f_{04} + \cos(kx_2) f_{03}),
\] (B.14)
\[
0 = \partial_M (\sqrt{-g} g^{MA} g^{33} f_{A3}) + 4\alpha A'_0 f_{24} + 4\alpha h \cos(kx_2) f_{r0} + 4\alpha h' \sin(kx_2) f_{02},
\] (B.15)
\[
0 = \partial_M (\sqrt{-g} g^{MA} g^{14} f_{A4}) - 4\alpha A'_0 f_{23} + 4\alpha h \sin(kx_2) f_{r0} - 4\alpha h' \cos(kx_2) f_{02},
\] (B.16)
\[
0 = \partial_M (\sqrt{-g} g^{MA} g^{00} f_{A0}) - 4\alpha h(\sin(kx_2) f_{r4} + \cos(kx_2) f_{r3}) - 4\alpha h'(\cos(kx_2) f_{24} - \sin(kx_2) f_{23}).
\] (B.17)

For an ansatz where \( a_2 = 0 \) and there is no dependence on the \( x^3 \) and \( x^4 \) coordinates, the equations (B.13), (B.14) and (B.17) are constraints depending only on first or zero time derivatives of the gauge potential. The other two equations, (B.15) and (B.16) are dynamical, they contain two time derivatives of \( a_3 \) and \( a_4 \). We want to study the time evolution of the system starting with an initial configuration similar to (B.8). The constraint equations (B.13) and (B.14) are satisfied automatically. When we evaluate explicitly the third constraint (B.17)) there is a term proportional to \( \delta k \)
\[
\delta k \left[ (r^2 \partial_k f')' + 4kh \partial_k h' + 4kh' \partial_k h + 4hh' \right].
\] (B.18)

This is the variation with respect to \( k \) of the equation of motion for the background solution \( f_k \), so it will vanish. The remaining contributions are
\[
\frac{r}{H} \partial_k f \partial_k^2 \delta k + 4h' h(2\partial_2 \delta k + \partial_2 \delta \varphi) = 0.
\] (B.19)

This is satisfied if
\[
\delta \varphi = \delta \varphi_0 - x_2 \delta k + \int dx_2 \delta k - \frac{r}{H} \frac{\partial_k f_k}{4hh'} \partial_2 \delta k.
\] (B.20)

For (B.9),
\[
\delta \varphi = \delta \varphi_0 - \delta k_0 x_2 \Delta(x_2) + \frac{\delta k_0}{2M} \log \left[ \frac{\cosh \left[ M \left( x_2 + \frac{\pi}{k} \right) \right]}{\cosh \left[ M \left( x_2 - \frac{\pi}{k} \right) \right]} \right] - \frac{r}{H} \frac{\partial_k f_k}{8hh'} M \delta k_0 \left( \sech^2 \left[ M \left( x_2 + \frac{\pi}{k} \right) \right] - \sech^2 \left[ M \left( x_2 - \frac{\pi}{k} \right) \right] \right) .
\] (B.21)

The change in the phase of the solution is, up to terms that are exponentially localized around \( x_2 = \pm \pi/k \),
\[
\delta \theta = \delta k x_2 + \delta \varphi \simeq \delta \varphi_0 + \frac{\delta k_0}{2M} \log \left[ \frac{\cosh \left[ M \left( x_2 + \frac{\pi}{k} \right) \right]}{\cosh \left[ M \left( x_2 - \frac{\pi}{k} \right) \right]} \right] .
\] (B.22)
The asymptotic value is a constant

$$\delta \theta_{\pm \infty} = \lim_{x \to \pm \infty} \delta \theta = \delta \phi_0 \pm \frac{\pi \delta k_0}{k}.$$  \hspace{1cm} (B.23)

One can see that $\delta \theta \sim \delta k_0 x_2 + \delta \phi_0$ in the interval $[-\pi/k, \pi/k]$ and $\delta \theta \simeq \delta \theta_{\pm \infty}$ outside the interval. Therefore the initial configuration has the desired form, where the shift in $k$ is restricted to the interval, but the phase of the solution outside the interval is shifted in such a way that it is continuous in the limit $M \to \infty$. In the linearized approximation we can add three of these solutions in such a way that the asymptotic phase of the solution at positive and negative infinity is the same. We simply add the solution that shifts $k$ by $\delta k_0$ in the interval $[-\pi/k, \pi/k]$ with the solutions that shift $k$ by $-\delta k_0/2$ in the intervals $[-3\pi/k, -\pi/k]$ and $[\pi/k, 3\pi/k]$.

Up to exponentially localized terms in the boundaries of the interval, the initial configuration with $\delta k_0 < 0$ describes precisely the situation where a strip is stretched while the neighbouring strips are compressed. The force analysis tells us that in the regions where $\partial_k p < 0$, the forces would tend to increase the deformation even more.

References

[1] D. V. Deryagin, D. Y. Grigoriev and V. A. Rubakov, “Standing wave ground state in high density, zero temperature QCD at large N(c),” Int. J. Mod. Phys. A 7, 659 (1992).
[2] B. Bringoltz, “Solving two-dimensional large-N QCD with a nonzero density of baryons and arbitrary quark mass,” Phys. Rev. D 79, 125006 (2009) [arXiv:0901.4035 [hep-lat]].
[3] E. Shuster and D. T. Son, “On finite density QCD at large N(c),” Nucl. Phys. B 573, 434 (2000) [hep-ph/9905448].
[4] D. E. Kharzeev and H. -U. Yee, “Chiral Magnetic Wave,” Phys. Rev. D 83, 085007 (2011) [arXiv:1012.6026 [hep-th]].
[5] G. Basar, G. V. Dunne and D. E. Kharzeev, “Chiral Magnetic Spiral,” Phys. Rev. Lett. 104, 232301 (2010) [arXiv:1003.3464 [hep-ph]].
[6] A. I. larkin and Y. N. Ovchinnikov, “Nonuniform state of superconductors,” Zh. Eksp. Teor. Fiz. 47, 1136 (1964) [Sov. Phys. JETP 20, 762 (1965)].
[7] P. Fulde R. A. Ferrell, “Superconductivity in a Strong Spin-Exchange Field,” Phys. Rev. A 135, 550 (1964)
[8] M. Vojta, “Lattice symmetry breaking in cuprate superconductors: stripes, nematics, and superconductivity,” Advances in Physics bf 58, 699 (2009) [arXiv:0901.3145 [cond-mat.supr-con]]
[9] S. K. Domokos and J. A. Harvey, “Baryon number-induced Chern-Simons couplings of vector and axial-vector mesons in holographic QCD,” Phys. Rev. Lett. 99 (2007) 141602, arXiv:0704.1604 [hep-ph].
[10] S. Nakamura, H. Ooguri, and C.-S. Park, “Gravity Dual of Spatially Modulated Phase,” Phys. Rev. D 81 (2010) 044018, arXiv:0911.0679 [hep-th].
[11] C. A. B. Bayona, K. Peeters, and M. Zamaklar, “A non-homogeneous ground state of the low-temperature Sakai-Sugimoto model,” JHEP 06 (2011) 092, arXiv:1104.2291 [hep-th].
[12] O. Bergman, N. Jokela, G. Lifschytz, and M. Lippert, “Striped instability of a holographic Fermi-like liquid,” *JHEP* **10** (2011) 034, arXiv:1106.3883 [hep-th].

[13] N. Iizuka and K. Maeda, “Stripe Instabilities of Geometries with Hyperscaling Violation,” arXiv:1301.5677 [hep-th].

[14] A. Donos and J. P. Gauntlett, “Holographic helical superconductors,” *JHEP* **12** (2011) 091, arXiv:1109.3866 [hep-th].

[15] A. Donos and J. P. Gauntlett, “Helical superconducting black holes,” arXiv:1203.0533 [hep-th].

[16] A. Donos, J. P. Gauntlett, and C. Pantelidou, “Spatially modulated instabilities of magnetic black branes,” *JHEP* **01** (2012) 061, arXiv:1109.0471 [hep-th].

[17] A. Donos, J. P. Gauntlett, and C. Pantelidou, “Magnetic and electric AdS solutions in string- and M-theory,” arXiv:1112.4195 [hep-th].

[18] A. Donos, J. P. Gauntlett, J. Sonner and B. Withers, “Competing orders in M-theory: superfluids, stripes and metamagnetism,” *JHEP* **1303**, 108 (2013) [arXiv:1212.0871 [hep-th]].

[19] A. Donos and J. P. Gauntlett, “Holographic charge density waves,” arXiv:1303.4398 [hep-th].

[20] S. Takeuchi, “Modulated Instability in Five-Dimensional U(1) Charged AdS Black Hole with R**2-term,” *JHEP* **01** (2012) 160, arXiv:1108.2064 [hep-th].

[21] A. Ballon-Bayona, K. Peeters and M. Zamaklar, “A chiral magnetic spiral in the holographic Sakai-Sugimoto model,” *JHEP* **1211**, 164 (2012) [arXiv:1209.1953 [hep-th]].

[22] N. Jokela, G. Lifschytz and M. Lippert, “Magnetic effects in a holographic Fermi-like liquid,” *JHEP* **1205**, 105 (2012) [arXiv:1204.3914 [hep-th]].

[23] J. de Boer, B. D. Chowdhury, M. P. Heller and J. Jankowski, “Towards a holographic realization of the Quarkyonic phase,” *Phys. Rev. D* **87**, 066009 (2013) [arXiv:1209.5915 [hep-th]].

[24] A. Donos and J. P. Gauntlett, “Holographic striped phases,” *JHEP* **08** (2011) 140, arXiv:1106.2004 [hep-th].

[25] H. Ooguri and C.-S. Park, “Holographic End-Point of Spatially Modulated Phase Transition,” *Phys. Rev. D* **82** (2010) 126001, arXiv:1007.3737 [hep-th].

[26] H. Ooguri and C.-S. Park, “Spatially Modulated Phase in Holographic Quark-Gluon Plasma,” *Phys. Rev. Lett.* **106** (2011) 061601, arXiv:1011.4144 [hep-th].

[27] A. Donos and J. P. Gauntlett, “Black holes dual to helical current phases,” *Phys. Rev. D* **86**, 064010 (2012) [arXiv:1204.1734 [hep-th]].

[28] M. Rozali, D. Smyth, E. Sorkin and J. B. Stang, “Striped Order in AdS/CFT,” arXiv:1304.3130 [hep-th].

[29] B. Withers, “The moduli space of striped black branes,” arXiv:1304.2011 [hep-th].

[30] A. Donos, “Striped phases from holography,” *JHEP* **1305**, 059 (2013) [arXiv:1303.7211 [hep-th]].

[31] N. Iizuka, S. Kachru, N. Kundu, P. Narayan, N. Sircar, S. P. Trivedi and H. Wang, “Extremal Horizons with Reduced Symmetry: Hyperscaling Violation, Stripes, and a Classification for the Homogeneous Case,” *JHEP* **1303**, 126 (2013) [arXiv:1212.1948 [hep-th]].
[32] M. Rozali, D. Smyth, E. Sorkin and J. B. Stang, “Holographic Stripes,” arXiv:1211.5600 [hep-th].
[33] T. Albash and C. V. Johnson, “A Holographic Superconductor in an External Magnetic Field,” JHEP 0809, 121 (2008) [arXiv:0804.3466 [hep-th]].
[34] T. Albash and C. V. Johnson, “Phases of Holographic Superconductors in an External Magnetic Field,” arXiv:0906.0519 [hep-th].
[35] T. Albash and C. V. Johnson, “Vortex and Droplet Engineering in Holographic Superconductors,” Phys. Rev. D 80, 126009 (2009) [arXiv:0906.1795 [hep-th]].
[36] M. Montull, A. Pomarol and P. J. Silva, “The Holographic Superconductor Vortex,” Phys. Rev. Lett. 103, 091601 (2009) [arXiv:0906.2396 [hep-th]].
[37] V. Keranen, E. Keski-Vakkuri, S. Nowling and K. P. Yogendran, “Dark Solitons in Holographic Superfluids,” Phys. Rev. D 80, 121901 (2009) [arXiv:0906.5217 [hep-th]].
[38] K. Maeda, M. Natsuume and T. Okamura, “Vortex lattice for a holographic superconductor,” Phys. Rev. D 81, 026002 (2010) [arXiv:0910.4475 [hep-th]].
[39] Y. -Y. Bu, J. Erdmenger, J. P. Shock and M. Strydom, “Magnetic field induced lattice ground states from holography,” JHEP 1303, 165 (2013) [arXiv:1210.6669 [hep-th]].
[40] V. Kaplunovsky, D. Melnikov and J. Sonnenschein, “Baryonic Popcorn,” JHEP 1211, 047 (2012) [arXiv:1201.1331 [hep-th]].
[41] V. Kaplunovsky and J. Sonnenschein, “Dimension Changing Phase Transitions in Instanton Crystals,” arXiv:1304.7540 [hep-th].
[42] D. Vegh, “Holography without translational symmetry,” arXiv:1301.0537 [hep-th].
[43] S. A. Hartnoll and D. M. Hofman, “Locally Critical Resistivities from Umklapp Scattering,” Phys. Rev. Lett. 108, 241601 (2012) [arXiv:1201.3917 [hep-th]].
[44] G. T. Horowitz, J. E. Santos and D. Tong, “Optical Conductivity with Holographic Lattices,” JHEP 1207, 168 (2012) [arXiv:1204.0519 [hep-th]].
[45] G. T. Horowitz, J. E. Santos and D. Tong, “Further Evidence for Lattice-Induced Scaling,” JHEP 1211, 102 (2012) [arXiv:1209.1098 [hep-th]].
[46] A. Donos and S. A. Hartnoll, “Metal-insulator transition in holography,” arXiv:1212.2998 [hep-th].
[47] Y. Liu, K. Schalm, Y. -W. Sun and J. Zaanen, “Lattice Potentials and Fermions in Holographic non Fermi-Liquids: Hybridizing Local Quantum Criticality,” JHEP 1210, 036 (2012) [arXiv:1205.5227 [hep-th]].
[48] G. W. Gibbons, “Born-Infeld particles and Dirichlet p-branes,” Nucl. Phys. B 514, 603 (1998) [hep-th/9709027].
[49] S. K. Domokos, C. Hoyos and J. Sonnenschein, “Deformation Constraints on Solitons and D-branes,” arXiv:1306.0789 [hep-th].
[50] A. Donos and J. P. Gauntlett, “On the thermodynamics of periodic AdS black branes,” arXiv:1306.4937 [hep-th].
[51] L. D. Landau and E. M. Lifshitz, “Theory of Elasticity,” Course of Theoretical Physics, Volume 7 Pergamon Press, 1975.
[52] T. Sakai and S. Sugimoto, “Low energy hadron physics in holographic QCD,” Prog. Theor. Phys. 113, 843 (2005) [hep-th/0412141].

[53] O. Aharony, J. Sonnenschein and S. Yankielowicz, “A Holographic model of deconfinement and chiral symmetry restoration,” Annals Phys. 322, 1420 (2007) [hep-th/0604161].

[54] A. Karch, A. O’Bannon and E. Thompson, “The Stress-Energy Tensor of Flavor Fields from AdS/CFT,” JHEP 0904, 021 (2009) [arXiv:0812.3629 [hep-th]].

[55] I. Papadimitriou and K. Skenderis, “Thermodynamics of asymptotically locally AdS spacetimes,” JHEP 0508, 004 (2005) [hep-th/0505190].

[56] S. S. Gubser and S. S. Pufu, “The Gravity dual of a p-wave superconductor,” JHEP 0811, 033 (2008) [arXiv:0805.2960 [hep-th]].

[57] M. Ammon, J. Erdmenger, M. Kaminski and P. Kerner, “Superconductivity from gauge/gravity duality with flavor,” Phys. Lett. B 680, 516 (2009) [arXiv:0810.2316 [hep-th]].

[58] H. B. Nielsen and S. Chadha, “On How to Count Goldstone Bosons,” Nucl. Phys. B 105, 445 (1976).

[59] H. Watanabe and T. Brauner, “On the number of Nambu-Goldstone bosons and its relation to charge densities,” Phys. Rev. D 84, 125013 (2011) [arXiv:1109.6327 [hep-ph]].

[60] I. Amado, D. Arean, A. Jimenez-Alba, K. Landsteiner, L. Melgar and I. S. Landea, “Holographic Type II Goldstone bosons,” arXiv:1302.5641 [hep-th].