A generalization of unit distances, angular properties and convexity

Abhijeet Khopkar
No Institute Given

Abstract. In this paper, we prove that unit distance graphs on convex point sets with \( n \) vertices have \( O(n) \) edges improving the previous known bound of \( O(n \log n) \).

1 Introduction

Unit distance graphs (UDGs) are well studied geometric graphs. In these graphs an edge exists between two points if and only if the Euclidean distance between the points is unity.

Definition 1. A geometric graph \( G = (V, E) \) is called unit distance graph provided that for any two vertices \( v_1, v_2 \in V \), the edge \( (v_1, v_2) \in E \) if and only if the Euclidean distance between \( v_1 \) and \( v_2 \) is exactly unity.

UDGs have been studied extensively for various properties including their edge complexity. The upper bound and the lower bound for the number of the maximum edges in the unit distance graphs (on \( n \) points in \( \mathbb{R}^2 \)) are \( O(n^{3/2}) \) [12] and \( \Omega(n \log \log n) \) (for a suitable constant \( c \)) respectively [7]. Erdős showed an upper bound of \( O(n^{3/2}) \) [7]. The bound was first improved to \( o(n^{3/2}) \) [10], then improved to \( n^{1.44} \ldots \) [2]. Finally, the best known upper bound of \( O(n^{3/2}) \) was obtained by [12]. Alternate proofs for the same bound were given by [3][11]. Bridging the gap in these bounds has been a long time open problem. Unit distance graphs have also been studied for various special point sets most notably the case when all the points lie in convex position. The best known upper bound for the number of edges in a unit distance graph on a convex point set with \( n \) points is \( O(n \log n) \). The first proof for this upper bound was given by Zoltán Füredi [9]. The proof is motivated by characterizing a \( 3 \times 2 \) sub matrix that is forbidden in a 0-1 matrix. The sub matrix is motivated by the definition of UDGs and the convexity of the point set. It was shown that any such \( a \times b \) matrix has at most \( a + (a + b) \lfloor \log_2 b \rfloor \) number of 1s. The argument can be easily extended to show that the adjacency matrix of a UDG on a convex point set of size \( n \) has \( O(n \log n) \) number of 1s that corresponds to the total number of edges. Peter Braß and János Pach provided an alternate and simple proof using a simple divide and conquer technique [4]. Another proof for the same bound using another forbidden pattern supplemented by a divide and conquer technique was

\[ \text{Not to be confused with the unit disk graphs} \]
Obtaining \(\text{PRBGs}\) vertices has \(O\) \(n\). We prove that \(1\). Our Contributions

placed from right to left in the increasing order.

1.1 Our Contributions

Two points \(p_i\) and \(p_j\) in a convex point set \(P\) are called antipodal points if there exist two parallel lines \(\ell_i\) passing through \(p_i\) and \(\ell_j\) through \(p_j\), such that all other points in \(P\) are contained between \(\ell_i\) and \(\ell_j\).

We prove that \(UDGs\) on convex point sets have \(O(n)\) edges.

2 Obtaining \(\text{PRBGs}\) from \(UDGs\)

In this section, we show that a \(UDG\) on convex a point set can be decomposed into two \(\text{PRBGs}\) by removing at most linear number of edges. First, we focus on some fundamental properties of the unit distance graphs on a convex point set. Two points \(p_i\) and \(p_j\) in a convex point set \(P\) are called antipodal points if...
Lemma 1. Let \( G_c = (P_c, E) \) be a unit distance graph on convex point set \( P_c \). If \( p_i \in P_c \) and \( p_j \in P_c \) are two antipodal points, then all but at most \( 2|P_c| \) edges of \( G \) cross the line \( p_ip_j \).

Let \( p_1 \) and \( p_2 \) be two antipodal points in the given convex point set \( P_c \) as shown in Figure 1. Let us divide \( P_c \) into two disjoint subsets \( U \) and \( V \). \( U \) is the set of points above the line \( p_1p_2 \) and \( V \) be the set of the points below this line. Let the vertices in \( U \) and \( V \) be \( u_1, u_2, \ldots, u_n \) and \( v_1, v_2, \ldots, v_m \) respectively (from right to left). Remove all the edges that do not cross the line \( p_ip_j \). Let \( E' \) be the set of the remaining edges. Consider the bipartite graph \( G = (U, V, E') \).

\( E' \) is divided into two disjoint sets \( E_1 \) and \( E_2 \) by the following rule. Consider an edge \((u, v_1)\), let \( v_0 \) and \( v_2 \) be the adjacent vertices to \( v_1 \) in \( V \) on left and right side respectively as shown in Figure 2. By convexity, it can be observed that either \( \angle uv_1v_2 \) or \( \angle uv_1v_0 \) is acute. If \( \angle uv_1v_2 \) is acute then put the edge \((u, v_1)\) in \( E_1 \) else if \( \angle uv_1v_0 \) is acute then put the edge \((u, v_1)\) in \( E_2 \). If both the angles are acute, then the edge can be put arbitrarily in either \( E_1 \) or \( E_2 \). In the graph \( G_1 = (U, V, E_1) \), the vertices are ordered as \( u_1 < u_2 < \ldots < u_n \) in \( U \) and \( v_1 < v_2 < \ldots < v_m \) in \( V \). The ordering is reversed in the graph \( G_2 = (U, V, E_2) \).

Remark 1. In \( G_1 \) and \( G_2 \), no two edges intersect in a forward path.

Remove the extreme left edge incident to every vertex \( v \in V \) from \( G_1 \), the resultant graph is called \( G'_1 \). Similarly, by removing the extreme right edge for every vertex \( v \in V \) in \( G_2 \), the graph \( G'_2 \) is obtained. Let \( G_{UDG} \) denote the class of the ordered bipartite graphs, consisting of the graphs \( G'_1 \) and \( G'_2 \) that are obtained from the unit distance graphs. It can be assumed w.l.o.g. that \( |V| \leq |U| \). Thus, a \( UDG/LGG \) on convex a point set can be decomposed into two \( PRBGs \) by removing at most \( 3n \) edges.

Consider the Locally Gabriel graphs on a convex point set. Observe that the Lemma holds true for Locally Gabriel graphs too. Therefore, a bipartition can be obtained similarly by dividing a convex point set along two antipodal points. Consider the bipartite graph between the two partitions. Similar to \( G_{UDG} \), a new graph class \( G_{LGG} \) can be defined. The procedure to obtain a graph in \( G_{LGG} \)
Abhijeet Khopkar

(from the $UDG$ on a convex point set) can also be applied to an $LGG$ on a convex point set to obtain a graph in $G_{LGG}$.

We show that the graphs in $G_{UDG}$ are path-restricted ordered bipartite graphs.

**Lemma 2.** Any graph $G = (U, V, E)$ in $G_{UDG}$ satisfies the path restricted property. Therefore, $G$ is a $PRBG$.

**Proof.** We show that if $P$ is a forward path in $G = (U, V, E)$ with the range $R_P = \{< u_a, u_b >, < v_c, v_d >\}$, then there does not exist a back edge $(u_i, v_c) \in E$ where $u_i < u_a, u_b >$. The path $P$ and the concerned vertices along with the edges are shown in Figure 3(a). Let $v_{d_0} \in V$ be the vertex preceding $v_d$ in $V$. Note that $(u_b, v_d)$ is an edge in $P$. Now $\angle u_b v_d v_{d_0} < \frac{\pi}{2}$ (by the definition of $G_{LGG}$). By convexity, it can be further inferred that $\angle u_b v_d v_c < \frac{\pi}{2}$. Let $u_{b_0} \in U$ be the vertex in $P$ with an edge incident to $v_c$ (apart from $u_b$) and $v_{c_1} \in V$ be the vertex that immediately succeeds to $v_c$ in $P$. By the definition of $LGGs$, $\angle v_d u_b u_{b_0}, \angle u_a v_c v_{c_1} < \frac{\pi}{2}$. By convexity, $\angle v_d u_b u_{b_0}, \angle u_a v_c v_{c_1} < \frac{\pi}{2}$. Thus, in the quadrilateral $u_a v_c v_{d_0} u_{b_0}$, $\angle u_a v_c v_{d_0} u_{b_0}$ must be greater than $\frac{\pi}{2}$. By convexity, $\angle u_a v_c v_{d_0} u_{b_0} > \frac{\pi}{2}$. Thus, the edges $(u_i, v_c)$ and $(u_{a_0}, v_c)$ conflict with each other. Therefore, the edges $(u_i, v_c)$ cannot exist in $G$ for any $u_i < u_a, u_b >$.

![Fig. 3. $G_{LGG}$ has path restricted properties](image)

Recall that the leftmost edge incident to every vertex $v \in V$ is deleted in the graph $G_1 = (U, V, E_1)$ to obtain a $G_{LGG}$. Similar arguments lead to the following claim. If $P$ is a forward path in $G_{LGG} = (U, V, E)$ with the range $R_P = \{< u_a, u_b >, < v_c, v_d >\}$, then there does not exist a back edge $(u_a, v'') \in E$ where $v'' < v_c, v_d >$ (refer to Figure 3(b)).

Thus, any graph in $G_{LGG}$ satisfies the path restricted property. Therefore, $G_{LGG}$ is a $PRBG$. $\square$
3 Properties of the path restricted ordered bipartite graphs

Let us consider all the forward paths originating from a vertex. These paths could be classified into two sets. The first set consists of all the forward paths visiting to the lower ordered vertices (rightwards) and the second set consists of all the forward paths visiting to the higher ordered vertices (leftwards). Let us consider first the set of the paths visiting rightwards. From the subsequent vertices on these paths, multiple paths can originate visiting to the vertices rightwards. These paths never meet with each other (refer to Lemma 2). Thus, these forward paths originating from a vertex form a tree. Let $T_i(u)$ denotes such a tree originating from $u$. Similarly, $T_i(v)$ denotes a tree that consists of all the forward paths originating from $v$ visiting the higher ordered vertices (leftwards).

Lemma 3. For any vertex $v$ in a PRBG $G = (U, V, E)$, the subgraph induced by the vertices of $T_i(v)$ has $n-1$ edges where $n$ is the number of vertices spanned by $T_i(v).

Proof. We show that for any vertex $v$ (let $v \in V$ w.l.o.g.) in a PRBG, the subgraph induced by the vertices in $T_i(v)$ does not have any edge but the edges in $T_i(v)$. On the contrary, let there exists an edge $(u_i, v_i) \in E$ s.t. this edge is not present in $T_i(v)$ and the vertices ($u_i \in U$ and $v_i \in V$) are spanned by $T_i(v)$. Recall that two forward paths emerging from a vertex in the same direction never meet again (refer to Lemma 2). Therefore, the edge $(u_i, v_i)$ does not belong to any forward path emerging from $v$. Let $u_j \in U$ be the vertex with the highest order incident to $v$. Note that $u_i$ and $u_j$ are not the same vertices and $u_i < u_j$ (refer to Figure 4(a)). $u_i$ cannot have an edge incident to $v$, otherwise the edge $(u_i, v_i)$ belongs to a forward path originating from $v$ as shown in Figure 4(b). But there exists a forward path passing through $v$ and $u_i$. Let $v_f \in V$ be the vertex preceding $u_i$ in the forward path from $v$ to $u_i$. Observe that $v_f < v_i$. Thus, there exists a forward path with the range $\{ < u_i, u_j >, < v_f, v > \}$. Therefore, the back edge $(u_i, v_j)$ is forbidden by the definition of PRBGs. Thus, it leads to a contradiction to the assumption that there exists an edge between $u_i$ and $v_j$. □

Lemma 4. For any vertex $v$ in a PRBG $G = (U, V, E)$, all the forward paths in $T_i(v)$ have disjoint ranges.

Proof. Let us assume w.l.o.g. that $v \in V$. Consider two forward paths in $T_i(v)$ originating from $v$. Consider a path $P_1 = (v, u_1, v_1, ...)$ as shown in Figure 5. Also consider the path $P_2 = (v, u_2, v_2, ...)$ where $v_1 < v_2$ (for $v_1, v_2 \in V$). Observe that there is a restriction that $u_i > u_2$ ($u_1, u_2 \in U$), otherwise the edge $(u_1, v_1)$ is forbidden by the path restricted property. Similarly, let $u_i \in U$ and $v_j \in V$ be the successive vertices in $P_1$ and let $u_j \in U$ and $v_j \in V$ be the successive vertices in $P_2$. By the path restricted property, it can be observed that if $v_1 < v_j$, then $u_j < u_i$. Therefore, the ranges of the paths $P_1$ and $P_2$ are disjoint. □
4 Edge complexity of path restricted ordered bipartite graphs

In this section, we study PRBGs for their edge complexity. We also study the edge complexity of these graphs for a special case when the length of the longest forward path is bounded.

Lemma 5 (Crossing lemma). Consider a PRBG $G = (U, V, E)$ with a separator line $\ell$ partitioning $U$ (resp. $V$) into disjoint subsets $U_1$ and $U_2$ (resp. $V_1$ and $V_2$) s.t. all the vertices in $U_1$ and $V_1$ are placed to the left of $\ell$ and all the vertices in $U_2$ and $V_2$ are placed to the right of $\ell$.

1. If every vertex in $U_1$ has an edge incident to it with the other endpoint in $V_1$, then the number of edges between $U_1$ and $V_2$ (crossing $\ell$) is at most $|U_1| + |V_2|$.  
2. If every vertex in $V_1$ has an edge incident to it with the other endpoint in $U_1$, then the number of edges between $V_1$ and $U_2$ (crossing $\ell$) is at most $|V_1| + |U_2|$.

Proof. An edge crossing the partition line $\ell$ is called the crossing edge. Let us consider only the vertices (in either of $U_1, U_2, V_1$ and $V_2$) that have more than one crossing edges incident to them. We give unit charge to all the vertices initially. A vertex can consume its charge to count for an edge. We show that if every vertex is charged for the leftmost crossing edge incident to it, then all the edges are counted.

Consider the rightmost vertex $u_1 \in U_1$ (the vertex with the least order in $U_1$) that has crossing edges incident to the vertices $v_1, v_2, \ldots, v_k$ as shown in Figure 4(a). We show that any of these vertices except $v_1$ cannot have an edge incident to a vertex in $U_1$ placed to the left of $u_1$. Let us assume on the contrary that $v_2$ has such an edge incident to the vertex $u$. By assumption $u$ has an edge incident to a vertex in $V_1$ (say $v \in V_1$), the edge does not intersect $\ell$ and it is placed to the left of it. Since, $v_1$ is placed to the right of $\ell$, there exists a forward
path with the range \{< u, u_1 >, < v, v_2 >\} and the back edge \((u_1, v_1)\) is forbidden by the path restricted property since \(v_1 \in < v, v_2 >\). Thus, it contradicts to the assumption that \(v_2\) has an edge incident to \(u\). Since \(u_1\) is the rightmost vertex in \(U_1\), the vertices \(v_2, \ldots, v_k\) have only one crossing edge incident to them. These vertices consume their charges to count the corresponding edges. \(u_1\) consumes its charge for the edge \((u_1, v_1)\). Note that all the crossing edges incident to \(u_1\) and its adjacent vertices across \(\ell\) (except \(v_1\)) are counted. Also note that the charge of \(v_1\) is still not consumed. Now, this charging scheme can be applied to the next vertex to the left of \(u_1\). Subsequently, this procedure can be applied to all the vertices in \(U_1\) from right to left and all the edges are counted. Thus, if each vertex in \(U_1\) and \(V_2\) consumes its charge to count the leftmost edge incident to it, all the edges between \(U_1\) and \(V_2\) are counted.

Similarly for the proof of (2), if a vertex \(v_1 \in V_1\) that has crossing edges incident to the vertices \(u_1, u_2, \ldots, u_k\) as shown in Figure 6(b), then the vertices \(u_2, \ldots, u_k\) cannot have an incident to a vertex in \(V_1\) placed to the left of \(v_1\). A similar argument can be made to show that if each vertex in \(V_1\) and \(U_2\) consumes its charge to count the leftmost edge incident to it, then all the edges between \(V_1\) and \(U_2\) are counted.

\(\Box\)

5 Hierarchy of various graph classes

In this section, we study the relationship amongst various graph classes. First we show that Class \(G_{UDG}\) is a strict sub class of the class \(G_{LGG}\). Then, we show that class \(G_{LGG}\) is a strict sub class of the generic path restricted ordered bipartite graphs. We also show that the class of \(UDGs\) on convex point sets is a strict sub class of the \(LGGs\) on convex point sets.

**Lemma 6.** Class \(G_{UDG}\) is a strict sub class of the class \(G_{LGG}\).

**Definition 4.** A PRBG \(G = (U, V, E)\) is called strictly path restricted ordered bipartite graph (SPBG), if two vertices \(v_1, v_2 \in V\) s.t. \(v_1 < v_2\) are spanned...
Fig. 7. A forbidden $G_{UDG}$

Fig. 8. A forbidden $G_{LGG}$

Fig. 9. Hierarchy of various graphs

by some tree $T_r(v), v \in V$ and $u_1$ and $u_2$ be the vertices preceding $v_1$ and $v_2$ respectively in the forward paths from $v$ to $v_1$ and $v_2$ and $u_1 < u_2$, then $u_1$ and $u_2$ cannot have edges incident to the vertices $v_1'$ and $v_2'$ (not spanned by $T_r(v)$) s.t. $v_1' < v_2'$.

Remark 2. In a strictly path restricted ordered bipartite graph $G = (U, V, E)$ if two vertices $u_1 \in U$ and $v_1 \in V$ are spanned by some tree $T_l(u)$, then there does not exist an edge between $u_1$ and $v_1$.

It can be observed that a $UDG$ on a convex point set can be represented as strictly path restricted ordered bipartite graph (refer to Lemma 6).

Lemma 7. Class $G_{LGG}$ is a strict sub class of the generic path restricted ordered bipartite graphs.

Proof. We show a simple example of a graph that is a $PRBG$ and forbidden in the class $G_{LGG}$. Consider the graph shown in Figure 8. The graph does not violate the path restricted property of the $PRBG$s. It can be argued that the graph cannot be represented as $G_{LGG}$. Recall that in an $LGG$ if there exist edges $(u, v_1)$ and $(u, v_2)$, then $\angle uv_1 v_2 < \frac{\pi}{2}$ and $\angle uv_2 v_1 < \frac{\pi}{2}$. Therefore, all the four angles $\angle u_1 v_1 v_2, \angle v_1 u_1 u_2, \angle v_4 u_4 u_3$ and $\angle u_4 v_4 v_3$ need to be acute in an $LGG$. By convexity, $\angle v_1 u_1 u_4, \angle u_1 u_4 v_4, \angle u_4 v_4 v_1$ and $\angle u_4 v_1 u_1$ are acute. That is not
Lemma 8. Two linearly separable modules can have at most two edges incident between them.

Therefore, a strict hierarchy can be established among three families of graphs. \( G_{UDG} \) is a strict sub class of the class of the graphs represented by \( G_{LGG} \). Furthermore, \( G_{LGG} \) is a strict subclass of the ordered bipartite graphs that satisfy path restricted property. The family of strictly path restricted ordered bipartite graphs \( (SPBG) \) is an obvious sub class of the generic \( PRBGs \). The hierarchy is shown pictorially in Figure 8. Though a \( G_{UDG} \) can be represented as a \( SPBG \), it is not known whether there is an equivalence between these two classes of graphs. There exist \( G_{LGG} \) not belonging to the class of \( SPBGs \). It is not clear whether all \( SPBGs \) can be represented as \( G_{LGG} \).

Let \( UDGC \) and \( LGCC \) be the classes of all the unit distance graphs and the locally Gabriel graphs on convex point sets. It can be observed in Figure 7 if the points \( v_3 \) and \( v_4 \) coincide then this graph cannot be embedded as unit distance graphs on a convex point but can be embedded as a locally Gabriel graph on a convex point set. It also establishes that the class \( UDGC \) is a strict subclass of \( LGCC \).

6 Linear number of edges in \( UDGs \) on convex point sets

Here we present an improved bound on edge complexity for \( UDGs \) on convex point sets. It strongly exploits the observations made in Lemma 6. If there exists a vertex \( v \in V_0 \) in a \( G_{UDG}(U, V, E) \) such that apart from all the vertices in \( T_1(v_0), \forall v \in V, v < v_0 \) and \( \forall v \in U, u < u_0 \) where \( u_0 \in U \) is a vertex in \( T_1(v_0) \) with the least order. All the edges in this graph apart from the edges in \( T_1(v_0) \) are crossing the edge \((u_0, v_0)\). Thus, by partition lemma the number of these edges is bounded by \((|U| + |V|)\). Thus, this graph has \( O(|U| + |V|) \) edges. This type of \( G_{UDG} \) is called modular \( G_{UDG} \). The tree part of the module is called the core of module and the remaining edges are called auxiliary edges. The vertices to which the auxiliary edges are incident (not in the core) are called auxiliary vertices.

A high level of our approach is to show that a graph in the class \( G_{UDG} \) can be decomposed into interconnected modular \( G_{UDGs} \) (also called modular units subsequently). A given pair of pairwise disjoint modules can have two kinds of orientation. In the first orientation the modules are linearly separable. In such a pair of modules, there exists a separator line such that all the vertices of both the modules lie on the opposite sides of the line, i.e. two modules \( G_1 = (U_1, V_1, E_1) \) and \( G_2 = (U_2, V_2, E_2) \) are linearly separable if \( \forall u_i \in U_1 \) (resp. \( \forall v_i \in V_1 \)) and \( \forall u_j \in U_2 \) (resp. \( \forall v_j \in V_2 \)) either \( u_i > u_j \) and \( v_i > v_j \) or \( u_i < u_j \) and \( v_i < v_j \).

On the contrary, two modules \( G_1 = (U_1, V_1, E_1) \) and \( G_2 = (U_2, V_2, E_2) \) are cross separable if \( \forall u_i \in U_1 \) (resp. \( \forall v_i \in V_1 \)) and \( \forall u_j \in U_2 \) (resp. \( \forall v_j \in V_2 \)) either \( u_i > u_j \) and \( v_i < v_j \) or \( u_i < u_j \) and \( v_i > v_j \).

Lemma 8. Two linearly separable modules can have at most two edges incident between them.
Proof. Let us consider the edges between two linearly separable modules. Let $G_1$ and $G_2$ be two such modules where all the vertices of $G_2$ have higher order than the vertices in $G_1$. Observe the following.

- No auxiliary vertex in $G_1$ has an edge incident to a vertex in $G_2$.
- Only two core vertices in $G_2$ with the highest order (one in each partition) can have an edge incident to a vertex in $G_2$.

$\square$

Corollary 1. The patterns shown in Figure 11 are forbidden in $G_{UDG}$. Note that the dotted edges indicate any generic forward path.

Proof. The proof follows the same argument as Lemma 6. The distance between $u_3$ and $v_3$ is larger than the distance between $u_3$ and $v_3$. Thus, the distance between $v_1$ and $u_3$ is larger than the distance between $v_1$ and $u_2$. It implies that the configurations shown in Figure 11 are not feasible. $\square$

Let us consider the case when the modular units aren’t linearly separable. Let us consider two modular units $G_1 = (U_1, V_1, E_1)$ and $G_2 = (U_2, V_2, E_2)$ such
that \( \forall u_i \in U_1 > \forall u_j \in U_2 \) and \( \forall v_i \in V_1 < \forall v_j \in V_2 \). Any pair of such modules is called cross separable modules. Let us consider the possible adjacencies between \( V_1 \) and \( U_2 \). We argue that the set of such edges form a matching, i.e. no vertex has more than one edges incident to it.

**Lemma 9.** For cross separable modules \( G_1 = (U_1, V_1, E_1) \) and \( G_2 = (U_2, V_2, E_2) \) such that \( \forall u_i \in U_1 > \forall u_j \in U_2 \) and \( \forall v_i \in V_1 < \forall v_j \in V_2 \), there can be only one-to-one adjacencies between \( V_1 \) and \( U_2 \).

**Proof.** Let us prove it by contradiction. Let us assume that the vertices \( u_1 \) and \( u_2 \) have an edge incident to \( v \) as shown in Figures 12. Let \( u_1 \) be a core vertex and \( u_2 \) be an auxiliary vertex and both of these vertices have an edge incident to \( v \). Let \( v_2 \) be the core vertex with an auxiliary edge incident to \( u_2 \). Since \( v_2 \) is a core vertex, it also has an edge incident to at least one core vertex \( u_2' > u_2 \). By path restriction property, \( u_1 > u_2' > u_2 \). Since \( u_1 \) is a core vertex, it has an edge incident to a core vertex \( v_1 \). Again by path restricted property, \( v_2 > v_1 > v \). Since \( v_1 \) and \( v_2 \) are the core vertices in the same module, there exists another path between them. Note that two core vertices are always connected by a left tree. Thus, this tree provides a path between \( v_1 \) and \( u_2 \). Let \( v_1' \) be the immediate neighbor of \( u_2 \) and \( v_1 = v_1' \) or \( v_1 > v_1' > v \), then both \( u_2 \) and \( u_2' \) cannot have an edge incident to \( v_2 \) (refer to Lemma 6). Similarly, if \( v_1 \) has an edge incident to \( u_2 \) or a vertex between \( u_2' \) and \( u_2 \), then \( v \) and \( v_1 \) both cannot have an edge incident to \( u_1 \) (refer to Lemma 6). Thus, \( u_2 < v_1 < u_1 \) and \( v_1 < v_1' < v_2 \). Therefore, there exists vertices \( v_0 \) and \( v_0' \) such that there exist forward paths with ranges \( \{(v_0, v_1), (v_0, u_1')\} \) and \( \{(v_0, v_1'), (u_0, u_2')\} \) respectively. This configuration is not possible by corollary 1.

Similarly, no two auxiliary vertices can have an edge incident to the same vertex outside a module. Thus, amongst the edges incident between \( G_1 \) and \( G_2 \), any vertex in either module has at most one edge incident to it. \( \square \)

Now we introduce a procedure called **partitioning.** If a module is partitioned along a line \( \ell \), then the module is separated into smaller units such that for any of the resultant module either all the vertices lie on one side of \( \ell \) or the vertices...
Lemma 10. A module can be partitioned along any line.

Proof. Observe the forwards paths in a right tree. Note that all the forwards path in a right tree are linearly inseparable. Let \( \ell \) be the partition line. All the forward paths crossing \( \ell \) can be attributed as other modules. If an edge of a forward path is intersected by \( \ell \), than the path to the right of \( \ell \) can be attributed to a new module while the edge crossing \( \ell \) can be attributed as an auxiliary edge from the corresponding vertex. For an example, refer to the Figure 13(a) for the core of a module and a partition line \( \ell \). The resultant modules after partitioning are shown in Figure 13(b). The dotted edges are the edges between the vertices of different modules.

Consider the case when in a \( G_{UDG} \) there is a pair of overlapping modules, i.e. they are neither linearly separable not cross separable. Such modules can be partitioned down further to ensure that any pair of modules is either linearly separable or cross separable.

Consider two cross separated modules as shown in Figure 10. Note that there exist edges no between \( U_1 \) and \( V_2 \) by the assumption that the modules are cross separable. Edges can exist between \( U_2 \) and \( V_1 \) though. Each vertex in \( U_2 \) or \( V_1 \) can have at most one such edge incident to it (refer to Lemma 9). The union of two cross separable modules with such connecting edges is called a fused module and the abstracting a fused module from two basic modules is called fusing.

Lemma 11. A set of modules fused together don't have an edge incident to a common vertex.

Proof. Let us prove it by contradiction. Let us consider the situation when all the modules are star shaped graphs, i.e. there is only one vertex in one partition connected to one or more vertices in the other partition. It is possible to partition the graph in such way by Lemma 10. The proof for this Lemma closely follows
the arguments in Lemma 6. Let \( u_1 \) and \( u_2 \) have an edge incident to a common vertex \( v_0 \) where \( u_1 \) and \( u_2 \) are the vertices in a fused module. Let \( u_1 < u_2 \). \( u_1 \) and \( u_2 \) cannot be the vertices of the same module by Lemma 6. Thus, these are the vertices of different modules fused together. Note that two such vertices have a zig-zag path between them as shown Figure 14. Let \( v_1 \) and \( v_2 \) respectively be the immediate neighbors of \( u_1 \) and \( u_2 \) in this path. Let \( u'_2 \) be the next neighbor of \( v_1 \) in this path. Note that the distance between \( v_2 \) and \( u'_2 \) is less than the unity. Thus, by applying the argument in Lemma 6, \( u_1 \) and \( u_2 \) cannot have an edge incident to \( v_0 \).

\[ \Box \]

![Fig. 14. Edges between two cross separable modules](image)

A \( G_{UDG} \) can be partitioned either into a set of linearly separable modules or a set of cross separable modules. Thus, by Lemma 11 and Lemma 8, a \( G_{UDG} \) has a linear number of edges. A \( UDG \) on convex point sets can be partitioned into two \( G_{UDG} \)s. Thus, we conclude that a \( UDG \) on convex point sets has a linear number of edges.

**Theorem 1.** A \( UDG \) on convex point set with \( n \) vertices has \( O(n) \) edges.

### 7 Conclusion Remarks

In this note, we defined a family of bipartite graphs known as the path restricted ordered bipartite graphs. We also showed that these graphs can be obtained from various geometric graphs on convex point sets. We studied various structural properties of these graphs and showed that a path restricted ordered bipartite graph on \( n \) vertices has \( O(n \log n) \) edges and this bound is tight. The same upper bound was already known for the unit distance graphs and the locally Gabriel graphs on convex point sets. However, the best known lower bound known to the edge complexity on these graphs for convex point sets is \( \Omega(n) \). We improved the upper bound for unit distance graphs to \( O(n) \). The problem of bridging the gap in the bounds remains an open for the locally Gabriel graphs on a convex
point set.

Acknowledgement: The author is thankful to Subramanya Bharadwaj for useful comments towards the proof of Theorem ??.

References

1. Bernardo M. Ábrego and Silvia Fernández-Merchant, The unit distance problem for centrally symmetric convex polygons, Discrete & Computational Geometry 28 (2002), no. 4, 467–473.
2. József Beck and Joel Spencer, Unit distances, J. Comb. Theory, Ser. A 37 (1984), no. 3, 231–238.
3. Peter Brass, Gyula Károlyi, and Pavel Valtr, A Turán-type extremal theory of convex geometric graphs, Discrete and Computational Geometry. Algorithms and Combinatorics, vol. 25, Springer Berlin Heidelberg, 2003, pp. 275–300 (English).
4. Peter Braß and János Pach, The maximum number of times the same distance can occur among the vertices of a convex n-gon is O(n log n), J. Comb. Theory, Ser. A 94 (2001), no. 1, 178–179.
5. Herbert Edelsbrunner and Pter Hajnal, A lower bound on the number of unit distances between the vertices of a convex polygon., J. Comb. Theory, Ser. A 56 (1991), no. 2, 312–316.
6. P. Erdős, On some metric and combinatorial geometric problems, Discrete Mathematics 60 (1986), no. 0, 147–153.
7. Paul Erdős, On sets of distances of n points, The American Mathematical Monthly 53 (1946), no. 5, pp. 248–250.
8. Peter C. Fishburn and James A. Reeds, Unit distances between vertices of a convex polygon, Comput. Geom. 2 (1992), 81–91.
9. Zoltán Füredi, The maximum number of unit distances in a convex n-gon, J. Comb. Theory, Ser. A 55 (1990), no. 2, 316–320.
10. S. Józsa and E. Szemerédi, The number of unit distances on the plane, Infinite and finite sets, Coll. Math. Soc. J. Bolyai 10 (1973), 939–950.
11. János Pach and Gábor Tardos, Forbidden patterns and unit distances, Proceedings of the twenty-first annual symposium on Computational geometry (New York, NY, USA), SCG '05, ACM, 2005, pp. 1–9.
12. Joel Spencer, Endre Szemerédi, and William T. Trotter, Unit distances in the euclidean plane, pp. 293–308, Academic Press, 1984.
13. László A. Székely, Crossing numbers and hard Erdős problems in discrete geometry, Comb. Probab. Comput. 6 (1997), no. 3, 353–358.