EXAMPLES OF ABELIAN SURFACES FAILING THE LOCAL-GLOBAL PRINCIPLE FOR ISOGENIES

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Abstract. We provide examples of abelian surfaces over number fields $K$ whose reductions at almost all good primes possess an isogeny of prime degree $\ell$ rational over the residue field, but which themselves do not admit a $K$-rational $\ell$-isogeny. This builds on work of Cullinan and Sutherland. When $K = \mathbb{Q}$, we identify certain weight-2 newforms $f$ with quadratic Fourier coefficients whose associated modular abelian surfaces $A_f$ exhibit such a failure of a local-global principle for isogenies.

1. Introduction

Let $A$ be an abelian variety over a number field $K$, and $\ell$ a prime number. If $A$ admits a $K$-rational $\ell$-isogeny, then necessarily, at every prime $p$ of good reduction not dividing $\ell$, the reduction $\tilde{A}_p$ over $\mathbb{F}_p$ also admits an $\ell$-isogeny, rational over $\mathbb{F}_p$. One may ask the converse question:

If $A$ admits a rational $\ell$-isogeny locally at every prime of good reduction away from $\ell$, must $A$ admit a $K$-rational $\ell$-isogeny?

If the answer to this question for a given pair $(A/K, \ell)$ is ‘No’, we refer to $\ell$ as an exceptional prime for $A$, and refer to $A$ as a Hasse at $\ell$ variety over $K$. We think of Hasse at $\ell$ varieties as being counterexamples to a local-global principle for $\ell$-isogenies.

This problem has been studied extensively in the case where $A$ is an elliptic curve, starting with the work of Sutherland [Sut12] who provided a characterisation of Hasse curves in terms of the projective mod-$\ell$ Galois image (whose definition we recall in Section 2), and found all such counterexamples in the case when $K = \mathbb{Q}$ (of which there is only one up to isomorphism over $\mathbb{Q}$).

Cullinan [Cul12] initiated the study of this question in the case of $\text{dim}(A) = 2$, by identifying the subgroups of $\text{GSp}_4(\mathbb{F}_\ell)$ that the mod-$\ell$ Galois image of a Hasse at $\ell$ variety must be isomorphic to, and remarked that, while his classification could be used to generate Hasse surfaces over arbitrary base fields, it “would be interesting to create “natural” examples of such surfaces”.

In this paper we provide the first examples of Hasse at $\ell$ surfaces that are simple over $\mathbb{Q}$, by studying the abelian varieties $A_f$ associated by Shimura to weight 2 newforms $f$:

Example 1.1. Consider the weight two newform of level $\Gamma_1(189)$, Nebentypus the non-primitive Dirichlet character modulo 189 of conductor 21, sending the two generators 29 and 136 of the group $(\mathbb{Z}/189\mathbb{Z})^\times$ to $-1$ and $\zeta_6^5$, where $\zeta_6 := e^{2\pi i/6}$ respectively, whose first few Fourier coefficients are as follows:

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f(z) = q + (-2 + 2ζ_6)q^4 + (-1 + 3ζ_6)q^7 + O(q^{10}).

Then $A_f$ is a Hasse at 7 abelian surface over $\mathbb{Q}$. This $f$ has label $189,2.p.a$ in the LMFDB.

This $f$ is a CM newform, having complex multiplication by the field $\mathbb{Q}(\sqrt{-3})$ (see [Rib77] for the basic theory of newforms with CM). Newforms with complex multiplication are known to have projective dihedral image at all primes (see e.g. Proposition 4.4 in [Rib77]), and it is tempting to believe that if $A_f$ is Hasse at some prime, then $f$ must be a CM newform. While we have not been able to prove this, we establish the existence of a congruence between $f$ and a CM newform:

**Theorem 1.2.** Let $f$ be a weight 2 newform such that the corresponding modular abelian variety $A_f$ is Hasse at some prime $\ell$ which splits completely in $O_f$. Then $f$ is congruent modulo $\ell$ to a newform with complex multiplication.

The structure of the paper is as follows. In Section 2 we survey previous and related work on this question, including Sutherland’s group-theoretic reformulation of Hasse at $\ell$ varieties. Section 3 studies the modular abelian varieties $A_f$ indicated above, yielding sufficient conditions on $f$ to ensure that $A_f$ is Hasse. Section 4 explains the algorithmic ingredients required to find examples of newforms satisfying the sufficient conditions. In Section 5 we prove Theorem 1.2, and in Section 6 we make some remarks on the prospect of finding examples of absolutely simple Hasse surfaces.

2. **Background and Preliminaries**

For an abelian variety $A$ over a number field $K$, the absolute Galois group $G_K := \text{Gal}(\overline{K}/K)$ acts on the $\ell$-torsion subgroup $A(\overline{K})[\ell]$, yielding the mod-$\ell$ representation

$$\overline{\rho}_{A,\ell} : G_K \to \text{GL}_{2d}(\overline{\mathbb{F}}_{\ell}),$$

whose image $G_{A,\ell} := \text{Im} \overline{\rho}_{A,\ell}$ is well-defined up to conjugacy; we refer to $G_{A,\ell}$ as the mod-$\ell$ image of $A$. We let $H_{A,\ell} := G_{A,\ell}$ modulo scalars, which we refer to as the projective mod-$\ell$ image of $E$, viewed as a subgroup of $\text{PGL}_{2d}(\overline{\mathbb{F}}_{\ell})$. If $A$ admits a polarisation of degree coprime to $\ell$, then the symplectic property of the Weil pairing on $A[\ell]$ ensures that $G_{A,\ell}$ is contained in $\text{GSp}_{2d}(\overline{\mathbb{F}}_{\ell})$, and consequently that $H_{A,\ell} \subseteq \text{PGSp}_{2d}(\overline{\mathbb{F}}_{\ell})$. Henceforth we will assume that $A$ is principally polarised.

By an $\ell$-isogeny $\phi : A \to A'$ of principally polarised abelian varieties of dimension $d$ defined over a field $k$ with char$(k) \neq \ell$ we mean a surjective morphism with kernel isomorphic to $\mathbb{Z}/\ell\mathbb{Z}$. We note that these isogenies are not compatible with the principal polarisations of $A$ and $A'$, since this kernel is not a maximal isotropic subgroup of $A[\ell]$ with respect to the $\ell$-Weil pairing. To consider isogenies that are compatible with the polarisations, one would need to consider certain isogenies with kernel isomorphic to $(\mathbb{Z}/\ell\mathbb{Z})^d$, often denoted as $(\ell, \cdots, \ell)$-isogenies (see e.g. [CS20]). One may well formulate a local-global question for such isotropic isogenies, and the results in [Orr17] are likely to be relevant here; but we do not address this problem in the present paper.

Sutherland’s characterisation of Hasse curves mentioned in the Introduction is expressed in terms of the canonical faithful action of $H_{A,\ell}$ on the projective space
hasse_surf_aces_3

P2d−1(Fl). Following our previous paper [BC14], given a subgroup H of PGSp2d(Fl), we say that H is Hasse if its action on P2d−1(Fl) satisfies the following two properties:

• every element h ∈ H fixes a point in P2d−1(Fl);
• there is no point in P2d−1(Fl) fixed by the whole of H.

The following result is then used by Sutherland in the case of dim A = 1: the details of the general case are entirely analogous, and may be found spelled out in [Ban13], Section 2.2:

Proposition 2.1 (Sutherland). An abelian variety A/K is Hasse at ℓ if and only if H_A,ℓ is Hasse.

In the case dim A = 1, it is easy to show that no subgroup of PGL2(Fl) is Hasse, so for elliptic curves the prime 2 is never an exceptional prime. For an odd prime ℓ, define ℓ* := +ℓ if ℓ ≡ 1 (mod 4), and ℓ* := −ℓ otherwise.

Sutherland provides necessary conditions for an elliptic curve E over a number field K to be Hasse at an odd prime ℓ, under the assumption that \sqrt{ℓ*} ∉ K, which is equivalent to the determinant of the projective representation \overline{P}_{E,ℓ} being surjective (see Lemma 2.1 in [BC14]). These conditions were shown to be sufficient in Section 7 of [BC14]. In the following Proposition, by D2n we mean the dihedral group of order 2n.

Proposition 2.2 ([Sut12], [BC14]). Let ℓ be an odd prime, and assume that \sqrt{ℓ*} ∉ K. Then E/K is Hasse at ℓ if and only if the following hold:

1. the projective mod-ℓ image of E is isomorphic to D2n, where n > 1 is an odd divisor of (ℓ − 1)/2;
2. ℓ ≡ 3 (mod 4);
3. the mod-ℓ image of E is contained in the normaliser of a split Cartan subgroup of GL2(Fl);
4. E obtains a rational ℓ-isogeny over K(\sqrt{ℓ*}).

Remark 2.3. The case of \sqrt{ℓ*} ∈ K was dealt with independently by [Ban13] and [Ann13] (see also [Ann14]).

The property of an elliptic curve E being Hasse at some prime ℓ depends only on j(E), provided j(E) ≠ \{0, 1728\}. Sutherland therefore defines an exceptional pair to be a pair (ℓ, j0) of a prime ℓ and an element j0 ≠ 0, 1728 of a number field K such that there exists a Hasse at ℓ curve over K of j-invariant j0.

Sutherland moreover shows, in the proof of Theorem 2 in [Sut12], that a Hasse curve cannot have CM if ℓ > 7; therefore, specialising now to K = Q, elliptic curves with level structure given by (3) above arise as non-trivial points on the modular curve X_s(ℓ) (the trivial points being the cusps and CM points). That such points exist only for ℓ ∈ \{2, 3, 5, 7, 13\} follows from the work of Bilu, Parent and Rebolledo [BPR13], although Sutherland was able to deduce the following remarkable result using the earlier work of Parent [Par05], as well as an explicit study of the modular curve X_{D0}(7) and its rational points.

Theorem 2.4 (Sutherland). The only exceptional pair for Q is

\[ (7, \frac{2268945}{128}) \].
The analogue of Proposition 2.2 providing precisely which subgroups of $\text{PGSp}_4(\mathbb{F}_\ell)$ are Hasse was given by Cullinan [Cul12]. Given a subgroup $H \subseteq \text{PGSp}_4(\mathbb{F}_\ell)$, let $\pi^{-1}(H)$ denote the pullback of $H$ to $\text{GSp}_4(\mathbb{F}_\ell)$.

**Theorem 2.5** (Cullinan). A subgroup $H \subseteq \text{PGSp}_4(\mathbb{F}_\ell)$ is Hasse if and only if $\pi^{-1}(H) \cap \text{Sp}_4(\mathbb{F}_\ell)$ is isomorphic to one of the groups in Table 2.1.

| Type | Group | Condition |
|------|-------|-----------|
| $C_2$ | $D_{l-1}/2 \wr S_2$ | None |
|      | $C_2^+$ | $l \equiv 1(4)$ |
|      | $(l-1)/2.\text{SL}_2(\mathbb{F}_3).2$ | $l \equiv 1(24)$ |
|      | $(l-1)/2.\text{GL}_2(\mathbb{F}_3).2$ | $l \equiv 1(24)$ |
|      | $(l-1)/2.\hat{S}_4.2$ | $l \equiv 1(24)$ |
|      | $(l-1)/2.\text{SL}_2(\mathbb{F}_5).2$ | $l \equiv 1(60)$ |
|      | $\text{SL}_2(\mathbb{F}_3) \wr S_2$ | $l \equiv 1(48)$ |
|      | $\hat{S}_4 \wr S_2$ | $l \equiv 1(48)$ |
|      | $\text{SL}_2(\mathbb{F}_5) \wr S_2$ | $l \equiv 1(120)$ |
| $C_6$ | $2^{l+1}.O_4^+(2)$ | $l \equiv 1(120)$ |
|      | $2^{1+4}.3$ | $l \equiv 5(24)$ |
|      | $2^{1+4}.5$ | $l \equiv 5(40)$ |
|      | $2^{1+4}.S_3$ | $l \equiv 5(24)$ |
| $S$ | $2.S_6$ | $l \equiv 1(120)$ |
|      | $\text{SL}_2(\mathbb{F}_5)$ | $l \equiv 1(30)$ |
|      | $\text{SL}_2(\mathbb{F}_3)$ | $l \equiv 1(24)$ |

Table 2.1. Hasse subgroups of $\text{PGSp}_4(\mathbb{F}_\ell)$. See [Cul12] for the group-theoretic notation used in this table.

At this point we may readily engineer Hasse surfaces over arbitrary number fields. For example, suppose we would like to construct an abelian surface $A$ whose mod-$\ell$ image satisfies $G_{A,\ell} \cap \text{Sp}_4(\mathbb{F}_\ell) \cong \text{SL}_2(\mathbb{F}_5)$ for some prime $\ell \equiv 1 \pmod{30}$; by Table 2.1 this would give a Hasse surface. We would first take an abelian surface over $\mathbb{Q}$ with absolute endomorphism ring isomorphic to $\mathbb{Z}$; a quick search in the LMFDB [LMF20] yields the genus 2 curve 249.a.249.1:

$$C : y^2 + (x^3 + 1)y = x^2 + x,$$

whose Jacobian variety $A$ has conductor 249 and $\text{End}_{\mathbb{Q}}(A) \cong \mathbb{Z}$. Serre’s Open Image Theorem, which also holds for abelian surfaces with absolute endomorphism ring $\mathbb{Z}$ [Hal11] ensures that, for all sufficiently large primes $\ell$, we have $G_{A,\ell} \cong \text{GSp}_4(\mathbb{F}_\ell)$. Moreover, Dieulefait [Die02] provides an algorithm to determine a bound on the primes of non-maximal image. This algorithm has recently been implemented [BBK+20] in Sage [The20] at an ICERM workshop funded by the Simons collaboration, and for this $A$ we find that any prime $\ell \geq 11$ ensures maximal image. Choose such an $\ell$ which is congruent to 1 (mod 30), e.g. $\ell = 31$. We finally base-change $A$ to force $G_{A,\ell} \cap \text{Sp}_4(\mathbb{F}_\ell) \cong \text{SL}_2(\mathbb{F}_5)$, using the Galois correspondence.

**Example 2.6.** The Jacobian variety of the curve $C$ above is a Hasse at 31 surface over the number field $K$ such that $\text{Gal}(\mathbb{Q}(A[31])/K) \cong \text{SL}_2(\mathbb{F}_5)$. 
Remark 2.7. We indicate here other work on this subject. These local-global type questions for abelian varieties go back to Katz in 1980 [Kat80], who studied the analogous local-global question for rational torsion points; for elliptic curves this goes even further back to the exercises in I-1.1 and IV-1.3 in Serre’s seminal book [Ser68]. Etropolski [Etr15] considers a local-global question for arbitrary subgroups of $\text{GL}_2(\mathbb{F}_\ell)$, and Vogt [Vog20] generalises the prime-degree-isogeny problem to composite degree isogenies. Very recently Mayle [May20] bounds by $\frac{3}{4}$ the density of prime ideals for elliptic curves $E/K$ which do not satisfy either of the “everywhere-local” conditions for torsion or isogenies, and Cullinan, Kenney and Voight study a probabilistic version of the torsion local-global principle for elliptic curves [CKV20].

3. Split modular abelian surfaces which are Hasse

The example constructed in the last section raises the question of whether there are Hasse surfaces over $\mathbb{Q}$, pre-empting this somewhat contrived base-change method. In approaching this question, we establish the following lemma.

Lemma 3.1. Let $A$ be an abelian surface over a number field $K$ whose projective mod-$\ell$ Galois image $H_{A,\ell}$ is contained in the direct sum of two subgroups $H, H'$ of $\text{PGL}_2(\mathbb{F}_\ell)$:

$$H_{A,\ell} \subseteq \begin{pmatrix} H & 0 \\ 0 & H' \end{pmatrix}.$$ 

If one of $\{H, H'\}$ is Hasse, and the other is not contained in a Borel subgroup, then $A$ is a Hasse at $\ell$ surface over $K$.

Proof. By Proposition 2.1 we need to establish that $H_{A,\ell}$ is a Hasse subgroup. Since the mod-$\ell$ Galois representation in this case decomposes as a direct sum of two subrepresentations, we write $V, V'$ such that $A[\ell] = V \oplus V'$.

We first show that $H_{A,\ell}$ does not fix a point in $\mathbb{P}(A[\ell])$. If it did, then that point lifts to a point $w \in A[\ell]$. We may write $w = v \oplus v'$, with $v \in V$, $v' \in V'$. Since at least one of $v, v'$ must be non-zero, we suppose that $v$ is non-zero. Then $H$ must fix the image of $v$ in $\mathbb{P}(V)$, which is not allowed under the hypotheses on $\{H, H'\}$.

Without loss of generality we suppose that $H$ is Hasse. Each element of $H_{A,\ell}$ may be written as $y = \begin{pmatrix} h & 0 \\ 0 & h' \end{pmatrix}$ for $h \in H, h' \in H'$. Since $h$ fixes a point in $V$, $y$ fixes the same point; thus every element of $H_{A,\ell}$ fixes a point. □

An immediate corollary provides an example of a Hasse surface over $\mathbb{Q}$, using Sutherland’s $j$-invariant defined above:

Corollary 3.2. Let $E/\mathbb{Q}$ be any elliptic curve with $j$-invariant $2268945128$. Then the abelian surface $E^2$ is Hasse at 7 over $\mathbb{Q}$. □

This prompts the question of whether there exist simple Hasse surfaces over $\mathbb{Q}$. We provide an affirmative answer to this question by restricting to the class of modular abelian surfaces over $\mathbb{Q}$, whose definition we now recall.

Let $f$ be a weight 2 cuspidal newform of level $\Gamma_1(N)$ for some $N > 1$, with Fourier coefficient field $K_f$, a number field whose ring of integers we will denote as $\mathcal{O}_f$. In the course of constructing the $\ell$-adic Galois representations of $f$, Shimura (Theorem 7.14 in [Shi71]) defined the abelian variety $A_f$ associated to $f$, whose dimension is
It is a theorem of Ribet (Corollary 4.2 in [Rib80]) that these abelian varieties are simple over $\mathbb{Q}$, and that $K_f$ is the full algebra of endomorphisms of $A_f$ which are defined over $\mathbb{Q}$. In this paper we refer to these varieties $A_f$ as modular abelian varieties, and in the case where $[K_f : \mathbb{Q}] = 2$, we call them modular abelian surfaces. (The reader is warned however that the adjective modular is used by different authors throughout the literature to mean different things.)

Furthermore, these varieties are of $\text{GL}_2$-type: the $\ell$-adic Tate module, for each $\ell$, splits as a direct sum $T_{\ell}A_f = \bigoplus_{\lambda | \ell} T_{f,\lambda}$, where each $T_{f,\lambda}$ is a free module of rank 2 over the $\lambda$-adic completion $O_{f,\lambda}$ of $O_f$.

(See Exercise 9.5.2 in [DS05]; to obtain the integrality one may need to replace $T_{f,\lambda}$ with a similar representation, as explained in the discussion immediately preceding Definition 9.6.10 in loc. cit. This decomposition is also explained in Section 2 of [Rib77]). This formula allows us to consider the $\ell$-adic representation $T_{\ell}A_f$ as a direct sum of the 2-dimensional $\lambda$-adic representations associated to $f$.

Consider the case in which $K_f$ is a quadratic field, and $\left(\ell, \lambda\right)$ splits in $O_f$. By taking the reduction mod $\ell$ of the above formula, we obtain a splitting $A_f[\ell] = T_{f,\lambda} \oplus T_{f,\lambda'}$ of the 4-dimensional $G_{\mathbb{Q}}$-representation $A_f[\ell]$ as a sum of two 2-dimensiona representations, all considered as representations over $\mathbb{F}_\ell$. Thus $G_{A_f,\ell}$ is contained in the block sum of two subgroups $G, G'$ of $\text{GL}_2(\mathbb{F}_\ell)$:

$$G_{A_f,\ell} \subseteq \left( \begin{array}{cc} G & 0 \\ 0 & G' \end{array} \right).$$

We choose $G$ and $G'$ minimally; i.e., $G$ is the image of $G_{\mathbb{Q}}$ acting on $T_{f,\lambda}$, and $G'$ the image of $G_{\mathbb{Q}}$ acting on $T_{f,\lambda'}$. We denote by $H$ and $H'$ the corresponding projective images, as subgroups of $\text{PGL}_2(\mathbb{F}_\ell)$.

We may therefore state sufficient conditions for a modular abelian surface $A_f$ to be Hasse, as a corollary of Proposition 2.2 and Lemma 3.1 above:

**Corollary 3.3.** Let $f$ be a weight 2 newform of level $\Gamma_1(N)$ with Fourier coefficient field $K_f$. Suppose:

- $K_f$ is a quadratic field;
- $\ell \geq 7$ is a prime congruent to 3 (mod 4) which splits in $O_f$ as $(\ell) = \lambda\lambda'$;
- among the projective mod-$\lambda$ and mod-$\lambda'$ images, one is isomorphic to $D_{2n}$, where $n > 1$ is an odd divisor of $\frac{\ell-1}{2}$, and the other is not contained in a Borel subgroup.

Then $A_f$ is Hasse at $\ell$ over $\mathbb{Q}$.

**Remark 3.4.** We do not deal with the case of $\ell$ remaining inert or ramifying in $O_f$ in this paper. This would likely involve a group-theoretic investigation of the Hasse subgroups of $\text{PGL}_2(\mathbb{F}_\ell)$.

In the next section we apply an algorithm of Anni [Ann13] which determines when a weight $k$ newform has projective dihedral image, in order to find an $f$ satisfying the assumptions in the above corollary.
4. Constructing examples using Anni’s thesis

Section 10.1 of [Ann13] describes an algorithm (Algorithm 10.1.3 in loc. cit.) to determine whether or not a weight $k$ newform has projective dihedral image modulo a prime ideal $\lambda$ of the ring of integers $\mathcal{O}_f$ of $K_f$. The main idea can be encapsulated in the following:

**Proposition 4.1** (Anni, Ribet, Serre). Let $f$ be a weight $k$ newform of level $N$, and let $\rho$ be the mod-$\lambda$ Galois representation associated to $f$. Assume that $\rho$ is irreducible. Then the following are equivalent:

1. $\rho$ has projective dihedral image;
2. there exists a quadratic character $\alpha$ of modulus $q$ such that $\alpha \otimes \rho \cong \rho$, where $q$ is the product of all primes dividing $N$ such that their square divides $N$;
3. there exists a quadratic field $K$, and characters $\chi, \chi'$ on $G_K$, such that the restriction of $\rho$ to $G_K$ is reducible:
   \[ \rho|_{G_K} = \chi \oplus \chi'. \]

Moreover, if these hold, then the order of the dihedral group is $2n$, where $n$ is the order of $\chi^{-1} \chi'$.

We refer to the relevant results in the literature for more details: in chronological order, Proposition 4.4 and Theorem 4.5 in [Rib77], Section 7 of [Ser77], and Section 10.1 of [Ann13].

Anni’s algorithm then consists in checking whether one of the finitely many Dirichlet characters as described in (2) above satisfies $\alpha \otimes \rho \cong \rho$, noting that only the primes up to the Sturm bound need to be checked.

At this point, if Anni’s algorithm yields a quadratic character for such a newform $f$, then either it has projective dihedral image, or the representation is reducible, and would mean it has cyclic image. This reducible case is equivalent to $f$ being congruent mod-$\ell$ to an Eisenstein series of the same weight and level, which is a finite check.

If the representation is indeed dihedral, then we compute the characteristic polynomials of Frobenius at several rational primes to determine its order.

We implemented this algorithm in Sage, and ran it on all two-dimensional weight-two newforms $f$ (those with $[K_f : \mathbb{Q}] = 2$) of level $\leq 189$, for which the prime 7 splits in $\mathcal{O}_f$. For each such level $N$, the implementation first constructs the entire space of newforms $S_2(\Gamma_1(N))^{\text{new}}$, and thereafter takes only those of dimension 2; as such, it is very inefficient. A faster approach to finding two-dimensional newforms of projective dihedral image would be to run the algorithm on the already-computed newforms in the LMFDB, which would obviate the need to recompute them on the fly.

The results obtained are summarised in Table 4.1. We found that the projective images in all of these cases were isomorphic for each of the prime ideals above 7, and that all of the forms had CM. To save space in the table, we note here that the Fourier coefficient field of all of these newforms is the quadratic field $\mathbb{Q}(\sqrt{-3})$, and remind the reader that with $D_n$ we mean the dihedral group of order $n$ (and not $2n$).

For the newforms in the table whose level is prime to 7, we verified the irreducibility of the mod-$\lambda$ Galois representation with Corollary 2.2 of [Die01]: if it was reducible, then there would exist a Dirichlet character $\chi$ of conductor dividing the
level and valued in \( \mathbb{P}^2 \) such that, for all primes \( p \) away from the level, we would have

\[
a_p \equiv \chi(p) + p \frac{\epsilon(p)}{\lambda(p)} \pmod{\lambda},
\]

where \( \epsilon \) is the Nebentypus of \( f \). Since there are only finitely many such \( \chi \), we can test all possible candidates, and find that none of them satisfy all of these congruences, whence the representation must be irreducible.

The implementation of the algorithms used in this section may be found here: [Ban20]. The last example in the above table is given in Example 1.1.

5. **Modular Hasse surfaces are congruent to CM newforms**

In this section we prove Theorem 1.2.

Let \( f \in S_2(\Gamma_1(N)) \) be a newform, and \( \ell \) a prime which splits completely in the ring of integers \( \mathcal{O}_f \) of the Fourier coefficient field \( K_f \). By assumption that \( A_f \) is Hasse, there exists a prime ideal \( \lambda|\ell \) such that the projective image of \( \overline{\rho}_{f,\lambda} \) is a Hasse subgroup of \( \text{PGL}_2(\mathbb{F}_\ell) \). Therefore, by Proposition 2.2, we have that \( \text{Im} \overline{\rho}_{f,\lambda} \) is a dihedral group.

Henceforth, for ease of notation, write \( \overline{\rho} \) for \( \overline{\rho}_{f,\lambda} \). Since \( \det \overline{\rho} \) is surjective in \( \mathbb{P}^2 \), we have that \( \det \overline{\rho} \) is surjective in \( \{ \pm 1 \} \). The kernel of \( \det \) is an index-2 subgroup of a dihedral group of order \( 2n \) with \( n \) odd, and therefore is cyclic of order \( n \). We thus obtain that the kernel of the composition

\[
G_Q \xrightarrow{\overline{\rho}} D_{2n} \rightarrow D_{2n}/C_n \rightarrow \{ \pm 1 \}
\]

corresponds to the imaginary quadratic field \( \mathbb{Q}(\sqrt{-\ell}) \).

We may now apply Théorème 1.1 of [BNMMdC18] to obtain the existence of a CM newform \( g \) such that \( \overline{\rho} \) is isomorphic to the mod-\( \lambda' \) reduction of the \( \lambda' \)-adic \( G_Q \)-representation \( \rho_{g,\lambda'} \), for some prime ideal \( \lambda' \) lying over \( \ell \) in the Fourier coefficient field of \( g \) (which need not be the same as that of \( f \)). Moreover, from the proof of Corollaire 1.3 in loc. cit., we have that the weight of \( g \) is 2. This yields the desired congruence. \( \square \)

**Remark 5.1.** Theorem A in [OS18] tells us that there are only finitely many \( \overline{\rho} \)-isomorphism classes of abelian surfaces over \( \mathbb{Q} \) with complex multiplication. There

| LMFDB Label | CM field | \( q \)-expansion | \( \mathbb{P} \rho(G_Q) \) |
|--------------|----------|------------------|------------------|
| 49.2.c.a     | \( \mathbb{Q}(\sqrt{-7}) \) | \( q - \zeta_6 q^2 + (1 - \zeta_6) q^3 - 3 q^6 + 3 \zeta_6 q^7 + O(q^{10}) \) | \( C_5 \) |
| 63.2.e.a     | \( \mathbb{Q}(\sqrt{-3}) \) | \( q + 2 \zeta_6 q^2 + (1 - 3 \zeta_6) q^3 + O(q^{10}) \) | \( D_4 \) |
| 81.2.c.a     | \( \mathbb{Q}(\sqrt{-3}) \) | \( q + 2 \zeta_6 q^2 + (1 - \zeta_6) q^3 + O(q^{10}) \) | \( D_{12} \) |
| 117.2.g.a    | \( \mathbb{Q}(\sqrt{-3}) \) | \( q + 2 \zeta_6 q^2 + \zeta_6 q^3 + O(q^{10}) \) | \( D_{12} \) |
| 117.2.q.b    | \( \mathbb{Q}(\sqrt{-3}) \) | \( q - 2 \zeta_6 q^2 + (6 - 3 \zeta_6) q^3 + O(q^{10}) \) | \( D_{12} \) |
| 189.2.c.a    | \( \mathbb{Q}(\sqrt{-3}) \) | \( q + 2 q^2 + (-1 + 3 \zeta_6) q^3 + O(q^{10}) \) | \( D_6 \) |
| 189.2.e.b    | \( \mathbb{Q}(\sqrt{-3}) \) | \( q + (2 - 2 \zeta_6) q^2 + (1 - 3 \zeta_6) q^3 + O(q^{10}) \) | \( D_{12} \) |
| 189.2.p.a    | \( \mathbb{Q}(\sqrt{-3}) \) | \( q + (-2 + 2 \zeta_6) q^2 + (-1 + 3 \zeta_6) q^3 + O(q^{10}) \) | \( D_6 \) |
are therefore only finitely many \( \overline{\mathbb{Q}} \)-isomorphism classes of Hasse modular abelian surfaces with CM. Since the field of complex multiplication in this case must be an imaginary quadratic field of class number 1 or 2, there are only finitely many such. Note that González (Theorem 3.2 in [Gon11]) has enumerated the possible pairs \((\text{End}^0_{\mathbb{Q}}(A_f), \text{End}^0_{\mathbb{Q}}(A_f))\), for \(A_f\) a two-dimensional modular abelian surface with complex multiplication; there are 83 such pairs.

6. ON ABSOLUTE SIMPLICITY OF HASE MODULAR ABELIAN SURFACES

Modular abelian varieties with CM are known to be isogenous over \( \mathbb{Q} \) to a power of a CM elliptic curve, so the \(A_f\) in Example 1.1 is not absolutely simple. One may then ask the following:

**Question 6.1.** Does there exist an absolutely simple Hasse modular abelian surface?

We have been unable to find an example of this. We instead collect here some facts about absolutely simple modular abelian varieties from the literature that would help in the determination of such an example.

**Proposition 6.2** (Cremona, Jordan, Ribet). Let \(f\) be a weight 2 newform of level \(\Gamma_1(N)\) such that the corresponding modular abelian surface \(A_f\) is Hasse at some prime \(\ell\) which splits completely in \(O_f\). Assume that \(f\) is not a CM newform.

- If \(f\) does not have inner twists, then \(A_f\) is absolutely simple.
- If \(f\) does have inner twists, then \(A_f\) is absolutely simple if and only if \(\text{End}^0_{\mathbb{Q}}(A_f)\) is an indefinite quaternion division algebra with centre \(\mathbb{Q}\) of degree 4 over \(\mathbb{Q}\). Moreover, if this holds, then this algebra is realised over a totally complex field, \(A_f\) has potential good reduction everywhere, and for every prime \(p\) dividing \(N\), we have \(\text{ord}_p(N) \geq 2\).

**Proof.** Write \(\mathcal{X} = \text{End}^0_{\mathbb{Q}}(A_f)\). We have the following facts:

1. The centre of \(\mathcal{X}\) is a subfield \(F\) of \(K_f\), and \(\mathcal{X} \cong M_n(\cdot)\), where \(\cdot\) is either \(F\), or else an indefinite quaternion division algebra over \(F\) of dimension \(t^2\) over \(F\), where \(t\) is the Schur index of \(\mathcal{X}\) (Proposition 5.2 in [Rib04]);
2. The degree of \(\mathcal{X}\) over \(\mathbb{Q}\) is \(2[K_f:F]\) (Theorem 5.1 in [Rib80]);
3. \(F\) is a totally real number field, and \(\text{Gal}(K_f/F)\) is the group of inner twists of \(f\) (Corollary 5.4 in [Rib04]);
4. \([K_f:F] = nt\) (Proposition 5.2 in [Rib04]).

If \(f\) does not have inner twists, then \(\text{Gal}(K_f/F)\) is trivial, and so \(n = 1\); i.e., \(A_f\) is absolutely simple.

If \(f\) does have inner twists, then we have \(2 = nt\), so \(n = 1 \iff t = 2\); i.e., \(A_f\) is absolutely simple if and only if \(\mathcal{X}\) is an indefinite quaternion division algebra over \(F\) of degree 4 over \(\mathbb{Q}\).

The statements about the endomorphisms being realised over a totally complex field, and \(A_f\) having potential good reduction everywhere, follow from the observation that \(A_f/K\), when base-changed to the field \(K\) over which all endomorphisms are defined, satisfies the definition of a fake elliptic curve, and thus follow from the known properties of these objects; see e.g. Section 4 of [SS18], who attribute this to Jordan (Section 3 in [Jor86]).
The statement about the valuations of primes dividing \( N \) follows from Theorem 3 in [Cre92].

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