Mass formulae and strange quark matter

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Abstract

We have derived the popularly used parametrization formulae for quark masses at low densities and modified them at high densities within the mass-density-dependent model. The results are applied to investigate the lowest density for the possible existence of strange quark matter at zero temperature.

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The possible existence of strange quark matter (SQM) has been a focus of investigations [1] since Witten’s conjecture [2] that quark matter with strangeness per baryon of order unity may be bound. Because of the well-known difficulty of Quantum Chromodynamics in the nonperturbative domain, phenomenological models reflecting quark confinement are widely used in the study of hadron, and many of them have been successfully applied to investigate the stability and properties of SQM. One of the most famous models is the MIT bag model [3] with which Farhi and Jaffe [4] find that SQM is absolutely stable around the normal nuclear density for a wide range of parameters.

Another popularly used model is the mass-density-dependent model in which quark confinement is achieved by requiring [5]

$$\lim_{n_b \to 0} m = \infty,$$  

where $m$ is the quark mass, $n_b$ is the baryon number density. By using this model, Chakrabarty et al. [6] obtain a significantly different result: only at very high densities does SQM have the absolute stability.

Benvenuto and Lugones [7] point out that this is the consequence of an incorrect thermodynamical treatment. They add an extra term to the energy expression, and get similar results to those in the bag model.

S. Chakrabarty et al. [6,8] have already discussed the limitation of the conventional MIT bag model which assumes that the quarks are asymptotically free within the bag. In order to incorporate the strong interactions between quarks, one has to fall back on the perturbative theory, while the mass-density-dependent model mimics not only the quark confinement, but also the interactions between quarks. Therefore, the validity of quark mass dependence on density is of utter importance.

However, the popularly used quark mass formulae [5–7] are pure parameterizations without any real support from underlying field theories up to now. Our motivation in writing this letter is to try to derive the relation between the quark mass and density from more fundamental principles on one hand, and to study the critical density for SQM on the other hand. By “critical density” we mean in this paper that only above the density does SQM have the possibility of existence.

In a recent work [9], we have demonstrated that the quark mass and quark condensates satisfy the following relation

$$\frac{m}{m_c} = \frac{1}{1 - \langle \bar{q}q \rangle_{n_b}/\langle \bar{q}q \rangle_0},$$  

(2)
where we have suppressed the color index, the model parameter is changed to $m_c$ from $m_{q0}$ which is saved for the original mass, $\langle \bar{q}q \rangle_0$ and $\langle \bar{q}q \rangle_{n_b}$ are the quark condensates, respectively, in vacuum and in strange quark matter with baryon number density $n_b$. In this paper, we will see that Eq. (2) has important properties of both confinement and asymptotic freedom.

According to the obvious equality

$$\lim_{n_b \to 0} \frac{\langle \bar{q}q \rangle_{n_b}}{\langle \bar{q}q \rangle_0} = 1,$$

we can expand the relative condensate as

$$\frac{\langle \bar{q}q \rangle_{n_b}}{\langle \bar{q}q \rangle_0} = 1 - \frac{n_b}{\alpha'} + \text{higher orders in } n_b + \cdots,$$  

where

$$\alpha' = -\left( \frac{d}{dn_b} \frac{\langle \bar{q}q \rangle_{n_b}}{\langle \bar{q}q \rangle_0} \right)_{n_b=0}^{-1}.$$  

At low densities, we can naturally ignore all terms in Eq. (4) with orders in $n_b$ higher than 1 and obtain

$$\frac{\langle \bar{q}q \rangle_{n_b}}{\langle \bar{q}q \rangle_0} \approx 1 - \frac{n_b}{\alpha'}.$$  

Substituting Eq. (6) into Eq. (2), we get $m = m_c \alpha'/n_b$.

At high densities, the approximation (6) is no longer valid. We will soon see that the quark mass is inversely proportional to $n_b^{1/3}$, rather than just $n_b$.

The Hellmann-Feynman theorem [10] indicates that

$$\langle \Psi(\lambda) | \frac{d}{d\lambda} H(\lambda) | \Psi(\lambda) \rangle = \frac{d}{d\lambda} \langle \Psi(\lambda) | H(\lambda) | \Psi(\lambda) \rangle,$$

where $H(\lambda)$ is a Hermitian operator, $| \Psi(\lambda) \rangle$ is the normalized eigenvector of $H(\lambda)$, $\lambda$ is an independent real parameter.

We express the effective Hamiltonian density as

$$H_{\text{eff}} = H' + m\bar{q}q,$$
where $H'$ is the kinetic energy term. In the quark mass-density-dependent model, the total (or effective) mass depends on the density. It can be divided into two parts, namely, $m = m_0 + m_I$, where $m_0$ is the original mass independent of density, and $m_I$ is the interacting mass mimicking the strong interaction between quarks. Assuming that the quark condensate depends merely on the total quark number density, or, is blind to the configuration of the system, we, for simplicity, consider only one flavour case in the following derivation.

Substituting $\int d^3x H_{\text{eff}}$ for $H(\lambda)$, and $m_0$ for $\lambda$ in Eq. (7), we have

$$\frac{dm}{dm_0} \langle \Psi | \int d^3x \bar{q}q | \Psi \rangle = \frac{d}{dm_0} \langle \Psi | \int d^3x H_{\text{eff}} | \Psi \rangle. \quad (9)$$

Because the expectation value of $H_{\text{eff}}$ should equal that of $H_{\text{QCD}}$, we write down

$$\frac{dm}{dm_0} \langle \Psi | \int d^3x \bar{q}q | \Psi \rangle = \frac{d}{dm_0} \langle \Psi | \int d^3x H_{\text{QCD}} | \Psi \rangle. \quad (10)$$

Applying this equation respectively to the cases $|\Psi\rangle = |n_b\rangle$ and $|\Psi\rangle = |0\rangle$, where $|n_b\rangle$ denotes the ground state of the quark matter at rest with baryon number density $n_b$, $|0\rangle$ is the vacuum, and then taking the difference, we obtain

$$\frac{\langle \bar{q}q \rangle_{n_b}}{\langle \bar{q}q \rangle_0} = 1 - \frac{1}{\langle \bar{q}q \rangle_0} \frac{d\epsilon}{dm_0} \frac{dm}{dm_0} = 1 - \frac{1}{\langle \bar{q}q \rangle_0} \frac{\partial\epsilon}{\partial m}, \quad (11)$$

where the uniformity of the system has been taken into account, $|\bar{q}q\rangle_0 \equiv -\langle \bar{q}q \rangle_0$, $\epsilon$ is the energy density of the quark matter. Compared with the corresponding formula for nuclear matter [11], they are very similar in form. But the physical contents are very different. The energy density there depends on the nuclear density and the quark current mass which is an independent quantity. Here we have the following chain relation

$$\epsilon \downarrow \quad n_b \quad m_0 \quad m_0 \quad m_I \quad n_b.$$

This is why we should take $m_0$, rather than $m$, as the substitute for $\lambda$.

Combining Eqs. (2) and (11), we obtain

$$m \frac{\partial\epsilon}{\partial m} = m_c |\bar{q}q|_0 \equiv B. \quad (12)$$
At zero temperature, one has already known
\[ \epsilon = \frac{gm^4}{16\pi^2} \left[ x(2x^2 + 1)\sqrt{x^2 + 1} - \text{sh}^{-1}(x) \right], \tag{13} \]
where \( \text{sh}^{-1}(x) = \ln(x + \sqrt{x^2 + 1}), x = p_f/m, \) the Fermi momenta \( p_f = (\frac{18}{g^2}n_b)^{1/3} \), \( g \) is the degeneracy factor (6 for quarks). If replacing the \( \epsilon \) in Eq. (11) by this expression, we can see that the relative condensate indeed has the similar expansion to Eq. (4).

Substituting Eq. (13) into Eq. (12), we have
\[ x\sqrt{x^2 + 1} - \text{sh}^{-1}(x) = \frac{4\pi^2 B}{gm^4}. \tag{14} \]

Generally, this equation can be solved numerically. However, it can be analytically solved both at low and high densities.

At low densities, we expand the left hand side of Eq. (14) to \( x^5 \) term, and get
\[ m = \frac{B}{3n_b} \left[ \frac{1}{2} + \frac{1}{2} \left( 1 + \frac{25/2}{5g^{1/3}B^{2/3}n_b^{1/3}} \right) \right]. \tag{15} \]

If we only expand Eq. (14) to \( x^3 \) term, or equivalently, ignore the second term in the parentheses of Eq. (15), then
\[ m = \frac{B}{3n_b}. \tag{16} \]

This result was obtained from other arguments by the previous authors [5] many years ago. It agrees to the confinement property of quarks, and has been popularly applied to investigate the properties of SQM [6–8].

At high densities, Eq. (14) becomes
\[ p_f^2 - m^2 \ln \left( \frac{2p_f}{m} \right) = \frac{4\pi^2 B}{gm^2}. \tag{17} \]

The first term on the left is much greater than the second one. So we ignore the second term and get
\[ m = \frac{2\pi \sqrt{B/g}}{p_f} = \left( \frac{4\pi/9}{\sqrt{g}} \right)^{1/3} \sqrt{B} \equiv \frac{\beta}{n_b^{1/3}}. \tag{18} \]
A more precise solution can be obtained by substituting this expression for the $m'$s in the second term on the left hand side of Eq. (17):

$$m = \frac{2\pi\sqrt{B/g}}{p_f} \left[ 1 + \frac{\pi^2B/g}{p_f^4} \ln\left( \frac{p_f^4}{\pi^2B/g} \right) \right]. \quad (19)$$

Equation (18) or (19) is in accordance with the asymptotic freedom of QCD.

Table 1
Comparison of the relative errors(%) for different mass formulae. The densities are in unit of the normal nuclear density $n_0 = 0.17$ MeV·fm$^{-3}$. Parameter $\beta$ is taken to be $(137$ MeV)$^2$. “*” means over 100%.

| $n_b$ | 0.1 | 0.2 | 0.5 | 1.0 | 5.0 | 10.0 | 15.0 |
|------|-----|-----|-----|-----|-----|------|------|
| Eq.(15) | 0.00 | 0.10 | 4.57 | 26.30 | * | * | * |
| Eq.(16) | 0.40 | 2.34 | 15.54 | 35.46 | 74.36 | 83.56 | 87.39 |
| Eq.(18) | 72.31 | 56.90 | 31.34 | 16.71 | 3.23 | 1.53 | 0.98 |
| Eq.(19) | 86.74 | 43.09 | 10.75 | 1.66 | 0.44 | 0.23 | 0.15 |

In Table 1, we show the relative errors for different mass formulae. We see that the quark mass is inversely proportional to density at very low densities. When the density exceeds the nuclear density, the logarithmic term occurs, and the quark mass changes asymptotically to be in reverse relation to $n_b^{1/3}$. Therefore, we should apply Eq. (18) to the investigation of SQM, namely,

$$m_{u,d} = \frac{\beta_0}{n_b^{1/3}}, \quad (20)$$

$$m_s = \frac{\beta_s}{n_b^{1/3}}. \quad (21)$$

The parameters $\beta_0$ and $\beta_s$ determine the critical density $n_c$ rather simply

$$n_c = \frac{(\beta_s^2 - \beta_0^2)^{3/4}}{\pi \sqrt{\gamma}}, \quad (22)$$

where $\gamma$ is dimensionless: $1.5 < \gamma < 2$. We use $\gamma = 1.995$ in the calculations.

To prove this expression, we assume the SQM to be a Fermi gas mixture of $u$, $d$, $s$ quarks and electrons with chemical equilibrium maintained by the weak interactions: $d, s \leftrightarrow u + e + \bar{\nu}_e$, $s + u \leftrightarrow u + d$. At zero temperature, the
thermodynamic potential density of the species $i$ is
\[
\Omega_i = -\frac{g_i}{48\pi^2} \left[ \mu_i (2\mu_i^2 - 5m_i^2) \sqrt{\mu_i^2 - m_i^2} + 3m_i^4 \ln \frac{\mu_i + \sqrt{\mu_i^2 - m_i^2}}{m_i} \right],
\tag{23}
\]
where $i = u, d, s, e$ (ignore the contribution from neutrinos); $g_i$ is the degeneracy factor with values 6 and 2 respectively for quarks and for electrons; $m_{u,d}$ and $m_s$ to be replaced by Eqs. (20) and (21).

The corresponding baryon number density is
\[
n_i = -\frac{\partial \Omega_i}{\partial \mu_i} = \frac{g_i}{6\pi^2} \left( \frac{\mu_i^2}{\mu_i^2 - m_i^2} \right)^{3/2}.
\tag{24}
\]

For a given $n_b$, the chemical potentials $\mu_i (i = u, d, s, e)$ are determined by the following equations [4]
\[
\begin{align*}
\mu_d &= \mu_s \equiv \mu, \quad \mu_u + \mu_e = \mu, \quad n_b = \frac{1}{3}(n_u + n_d + n_s),
\tag{25} \\
2n_u - \frac{1}{3}n_d - \frac{1}{3}n_s - n_e &= 0. 
\tag{26}
\end{align*}
\]

The last two equations are equivalent to
\[
\begin{align*}
n_u - n_e &= n_b, 
\tag{29} \\
n_d + n_s + n_e &= 2n_b. 
\tag{30}
\end{align*}
\]

We therefore define a function of $\mu_e$
\[
F(\mu_e) = (\mu^2 - m_{u,d}^2)^{3/2} + (\mu^2 - m_s^2)^{3/2} + \frac{1}{3}(\mu_e^2 - m_e^2)^{3/2} - 2\pi^2n_b, 
\tag{31}
\]
where
\[
\mu = \mu_e + \sqrt{m_{u,d}^2 + [\pi^2n_b + \frac{1}{3}(\mu_e^2 - m_e^2)^{3/2}]^{2/3}}. 
\tag{32}
\]

Because $m_s > m_{u,d}$, the equation $F(\mu_e) = 0$ for $\mu_e$ has solution if and only if
\[ \mu \geq m_s, \quad F(\mu_e) \leq 0. \quad (33) \]

At the critical density \( n_c \), the equality signs in the above should be taken, and so we can easily find

\[ 1.5\pi^2 n_c < (m_s^2 - m_{u,d}^2)^{3/2} < 2\pi^2 n_c. \quad (35) \]

Solving for \( n_c \) from this inequality, we accordingly obtain Eq. (22).

For a more exact \( n_c \), we can numerically solve the following equation

\[
m_s - \sqrt{m_{u,d}^2 + [3\pi^2 n_c - (m_s^2 - m_{u,d}^2)^{3/2}]^{2/3}} = \sqrt{m_s^2 + [3(2\pi^2 n_c - (m_s^2 - m_{u,d}^2)^{3/2})]^{2/3}},
\]

in the range

\[
\left( \frac{\beta_s^2 - \beta_0^2}{{2}^{3/4}} \right) < n_c < \left( \frac{\beta_s^2 - \beta_0^2}{{1.5}^{3/4}} \right).
\]

In fact, the expression (22) is precise enough for practical applications because of the smallness of electron content.

At \( n_c \), the strangeness fraction becomes zero. When the density decreases further, the equation group (25–28) which determines the configuration of the system has no solution. This indicates that \( n_c \) is the lowest density for the possible existence of SQM.

To calculate the critical density, we need to know the concrete values of \( \beta_0 \) and \( \beta_s \).

Generally, there are two constraints on \( \beta_0 [4,6,7] \): firstly, it should be so big that the energy per baryon for light-flavour quark matter is greater than 930 MeV in order not to contradict standard nuclear physics; secondly, because we are interested in the possibility of absolute stability of SQM, it should not exceed an upper limit so that the energy per baryon can be less than 930 MeV for symmetric three flavour quark matter. Therefore, \( \beta_0 \) must be in the range from a fixed lower limit \( \beta_{0\text{min}} \) to an upper limit \( \beta_{0\text{max}} \). By using the method in Ref. [7], we obtain: \( \beta_{0\text{min}} = (135 \text{ MeV})^2 \), \( \beta_{0\text{max}} = (147 \text{ MeV})^2 \).

Because at very low densities the mass formula should be Eq. (16), we apply this formula to estimate the lowest permissible value for \( B \) independently by the same method. The result is that \( B \) should be no less than 69 MeV·fm\(^{-3}\).
This requires $\beta_0$ to be greater than $(137 \text{ MeV})^2$ according to the relation
\[ \beta = \left(\frac{4\pi}{9/\sqrt{g}}\right)^{1/3} \sqrt{B} \] (see Eq. (18)). It is remarkable that the requirement is in accordance with the lower limit of $\beta_0$. With our purpose of studying the lowest density for the possible existence of SQM in mind, we take $\beta_0 = (137 \text{ MeV})^2$.

As for $\beta_s$, there is no reliable method to fix it presently. If regarding the nucleon as a special SQM with zero strangeness, one has
\[ \frac{\beta_s}{\beta_0} = \sqrt{\frac{B_s}{B}} \sim \sqrt{\frac{2\sigma_s}{\sigma_N}} \approx 3.46, \] (38)
where the ratio $\sigma_s/\sigma_N$ has been taken to be 6 [12]. The corresponding $n_c$ is $2.7n_0$, not too far away from the normal nuclear density. Therefore, whether SQM can exist near the nuclear density is still an open question to study.

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