Topologically singular points in the moduli space of Riemann surfaces

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To our friend María Teresa Lozano

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Abstract In 1962 E. H. Rauch established the existence of points in the moduli space of Riemann surfaces not having a neighbourhood homeomorphic to a ball. These points are called here topologically singular. We give a different proof of some of the results of Rauch and also determine the topologically singular and non-singular points in the branch locus of some equisymmetric families of Riemann surfaces.

Keywords Riemann surface · Moduli space · Orbifold · Teichmüller space

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1 Introduction

Let \( M \) be a manifold and \( p : M \to N \) be a regular branched covering; \( N \) has then a structure of (good) orbifold. The set of singular values of \( p \) is called the branch locus and it is the image by \( p \) of the fixed points of automorphisms of the covering \( p \); it consists of both ordinary manifold points and of points we call topologically singular points, meaning that they do not admit a neighbourhood...
homeomorphic to a ball. Note that all the points outside the branch locus are manifold points.

We shall assume, all over this paper, that \( g \) is an integer \( \geq 2 \). The moduli space \( M_g \) of surfaces of genus \( g \) is endowed with the structure of an orbifold given by the Teichmüller space \( T_g \) and the action of the mapping class group that produces a covering \( T_g \to M_g \). In \([6]\) Rauch proves, that for \( g > 3 \), every point in the branch locus \( B_g \) of \( M_g \) is topologically singular, the branch loci \( B_2 \) and \( B_3 \) containing topologically non-singular and singular points. In this article, we present a topological proof of these results. We point out as well an error appearing in \([6]\) in the description of the topologically singular points in the branch locus \( B_2 \) of \( M_2 \).

Finally, we study some equisymmetric families of dimension 4, showing how both topologically singular and non-singular points appear in the branch loci of moduli spaces of these families.

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2 Preliminaries

2.1 Uniformization of Riemann surfaces and automorphisms using Fuchsian groups

A Fuchsian group \( \Delta \) is a discrete subgroup of \( \text{PSL}(2, \mathbb{R}) \), i.e. the group \( \text{Isom}^+(\mathbb{H}^2) \) of direct isometries of \( \mathbb{H}^2 \). If \( \mathbb{H}^2/\Delta \) is compact, the algebraic structure of \( \Delta \) is given by the signature \( s = (h; m_1, ..., m_r) \), where \( h \) is the genus of the quotient surface \( \mathbb{H}^2/\Delta \) and the \( m_i \) are the branched indices of the covering \( \mathbb{H}^2 \to \mathbb{H}^2/\Delta \) (the order of the isotropy groups of the conic points of the orbifold \( \mathbb{H}^2/\Delta \)). The group \( \Delta \) admits a canonical presentation:

\[
\langle a_i, b_i; i = 1, ..., h; x_j; j = 1, ..., r : x_1...x_r \prod^{h}_{i} [a_i b_i] = x_j^{m_j} = 1 \rangle
\]

We shall consider only compact Riemann surfaces. A Riemann surface \( X \) of genus \( g > 1 \), may be uniformized by a surface Fuchsian group, i.e. \( X = \mathbb{H}^2/\Gamma \), where \( \Gamma \) is a Fuchsian group with signature \( (g; \) ), surface group of genus \( g \). The group \( \Gamma \) is isomorphic to the fundamental group of \( X \).

When \( g > 1 \), the group of automorphisms of the Riemann surface \( X \) is a finite group \( \text{Aut}(X) \). If \( G \leq \text{Aut}(X) \) the quotient orbifold \( X/G \) is isomorphic to \( \mathbb{H}^2/\Delta \), where \( \Delta \) is a Fuchsian group containing \( \Gamma \) and such that \( \Delta/\Gamma \cong G \).

If we have an \( n \)-fold covering \( X = \mathbb{H}^2/\Gamma \to \mathbb{H}^2/\Delta \), where \( \Gamma \) is a surface genus \( g \) Fuchsian group and \( \Delta \) has signature \( (h; m_1, ..., m_r) \), the following Riemann-Hurwitz formula holds:

\[
2g - 2 = n(2h - 2 + \sum^r (1 - \frac{1}{m_j})
\]
2.2 Teichmüller and moduli spaces

Let $G$ be an abstract group isomorphic to a Fuchsian group with signature $s$. Two representations $\alpha_1$ and $\alpha_2$ of $G$ in $\text{PSL}(2, \mathbb{R})$ are equivalent if there is $\gamma \in \text{PSL}(2, \mathbb{R})$ such that $\alpha_1(\zeta) = \gamma \alpha_2(\zeta) \gamma^{-1}$, for all $\zeta \in G$. The Teichmüller space $T_s$ is the space of equivalence classes of representations $\rho$ of $G$ in $\text{PSL}(2, \mathbb{R})$ such that $\rho(G)$ is a Fuchsian group with signature $s$. This space with the topology induced by $\text{PSL}(2, \mathbb{R})$ is homeomorphic to a ball of dimension

$$\dim T_{(h; m_1, \ldots, m_r)} = 6h - 6 + 2r$$

Note that the group $G$ is isomorphic to the orbifold fundamental group of $\mathbb{H}^2/\rho(G)$, where $[\rho] \in T_s$. Let $\text{Mod}_g$ be the mapping class group of surfaces of genus $g$. The group $\text{Mod}_g$ acts by composition on $T_g$ and the quotient $T_g/\text{Mod}_g = \mathcal{M}_g$ is the moduli space of Riemann surfaces of genus $g$. Note that $\mathcal{M}_g$ is, by construction, an orbifold and its universal covering is $\Pi : T_g \rightarrow \mathcal{M}_g$. The set of branch values of the covering $\Pi$ is the branch locus $B_g$ of the orbifold $\mathcal{M}_g$. The branch locus $B_g$ is the image by $\Pi$ of the fixed points by finite subgroups of $\text{Mod}_g$ and represents in $\mathcal{M}_g$ the surfaces with non-trivial automorphism group (up to the exception of $\mathcal{M}_2$, since $B_2$ consists of surfaces having non-trivial automorphisms different from the hyperelliptic involution).

Let $\theta : G \rightarrow G$ be an epimorphism from the abstract group $G$ isomorphic to a Fuchsian group with signature $s = (h; m_1, \ldots, m_r)$ and such that $\ker \theta$ is isomorphic to a surface group of genus $g$. There is a natural embedding $i_\theta : T_s \rightarrow T_g$. The image $\Pi(i_\theta(T_s)) \subset \mathcal{M}_g$ consists of the surfaces of genus $g$ having a subgroup of their automorphism groups isomorphic to $G$ with a specific action determined by $\theta$. We say that $\Pi(i_\theta(T_s))$ is the moduli space of an equisymmetric family given by $\theta$. If we consider the subset $S_{s, \theta}$ of $\Pi(i_\theta(T_s))$ consisting of the surfaces whose full automorphism group is $G$, we obtain a stratification of $B_g$ by the sets $S_{s, \theta}$ which are called the equisymmetric strata (see [2]).

3 Topologically singular points in moduli space

**Definition 1** (Topological singular point) A point $X$ in $B_g$ is topologically singular if $X$ has not a neighbourhood in $\mathcal{M}_g$ homeomorphic to a $(6g - 6)$ ball.

In other words:

**Definition 2** (Rauch definition [3]) A point $X$ in $B_g$ is singular if $X$ is not a manifold (or uniformizable) point in $\mathcal{M}_g$.

**Theorem 1** The group $\text{Aut}(X)$, for $X$ in the branch loci of $B_g$, acts as a subgroup of $\text{O}(6g - 6)$ in $S^{6g - 7}$. The point $X$ is topologically non-singular if and only if $S^{6g - 7}/\text{Aut}(X)$ is homeomorphic to $S^{6g - 7}$.
Proof Let $X \in \mathcal{B}_g$ and $Y \in \Pi^{-1}(X) \subset T_g$. Since $\text{Mod}_g$ acts discontinuously on $T_g$, there is a $(6g - 6)$ ball $U \subset T_g$, with center $Y$, such that $h \in \text{Mod}_g$ satisfies the condition $h(U) \cap U \neq \emptyset$ if and only if $h$ fixes $Y$.

An element $h$ of $\text{Mod}_g$ such that $h(U) \cap U \neq \emptyset$ is given by an automorphism of the Riemann surface represented by $Y$ in $T_g$, consequently by $X$ in $\mathcal{M}_g$, and we may identify $h$ with an element (still called $h$) of $\text{Aut}(X)$. Under such identification, $\text{Aut}(X)$ acts on the ball $U$ and it follows, since $h$ is an isometry of $T_g$ (with the Teichmüller metric), that $h$ acts as an isometry on both $U$ and $\partial U \cong S^{6g-7}$; therefore, $\text{Aut}(X)$ acts as a subgroup $G$ of $O(6g - 6)$.

Now $\Pi(U) = U/\text{Aut}(X) = U/G$ is a neighbourhood of $X$, so $X$ is non-singular whenever $U/G$ is homeomorphic to the $(6g - 6)$ ball or, equivalently, whenever $\Pi(\partial U) \cong S^{6g-7}/G$ is the sphere $S^{6g-7}$.

The following theorem will be used to produce our proof of the main Theorem of \cite{4}.

**Theorem 2** Let $X \in \mathcal{B}_g$. If for each equisymmetric stratum $S$ such that $X \in S$, the codimension of $S$ is greater than 2, then $X$ is topologically singular.

**Proof** Let $\Pi : T_g \to \mathcal{M}_g$ be the covering given by the action of $\text{Mod}_g$.

Let $S$ the set of equisymmetric strata $S$ such that $X \in S$. Let $U$ be a ball containing a point of $\Pi^{-1}(X)$ and such that $\Pi(U)$ does not cut $\mathcal{B}_g$ but on the strata in $S$. We shall show that if $\text{codim}(S) > 2$ for all $S \in S$ then $U/\text{Aut}(X)$ is not a ball.

We consider the covering $p = \Pi |_{\Pi^{-1}(\partial U)} : \partial U \to \Pi(\partial U)$ with branch locus $\Pi(\partial U) \cap \mathcal{B}_g = \Pi(\partial U) \cap (\cup S)$. There is a finite triangulation of $U$ in such a way that $\text{Fix}(\text{Aut}(X))$ is a subpolyhedron and the action of $\text{Aut}(X)$ preserves the triangulation (note that $\text{Aut}(X)$ acts as a finite order rotation group of $O(6g - 6)$). Since $\text{codim}(S) > 2$, for all $S \in S$, the codimension of the polyhedron $\Pi(\partial U) \cap \mathcal{B}_g = \Pi(\partial U) \cap (\cup S)$ is greater than 2 in $\Pi(\partial U)$. If $\Pi(\partial U)$ is a manifold then $\pi_1(\Pi(\partial U) - \Pi(\partial U) \cap \mathcal{B}_g) \cong \pi_1(\Pi(\partial U))$ (see \cite{4} Theorem 2.3, page 146). If $g > 2$, the covering $p$ has $|\text{Aut}(X)| \neq 1$ sheets (in case $g = 2$ the covering $p$ has $|\text{Aut}(X)/\langle h \rangle| \neq 1$ sheets, where $h$ is the hyperelliptic involution) and $p$ is branched on $\Pi(\partial U) \cap \mathcal{B}_g$. In both cases $\pi_1(\Pi(\partial U) - \Pi(\partial U) \cap \mathcal{B}_g)$ must be not trivial. Hence either $\pi_1(\Pi(\partial U)) \neq 1$ or $\Pi(\partial U)$ is not a manifold and, in both cases, $\Pi(\partial U)$ cannot be homeomorphic to the sphere $S^{6g-7}$; therefore, $X$ is topologically singular.

**Corollary 1** Let $X \in \mathcal{B}_g$ and suppose that $X$ is isolated in $\mathcal{B}_g$ (see \cite{6}) then $X$ is topologically singular.

**Proof** If $X$ is isolated then $\{X\}$ is the unique equisymmetric stratum $S$ such that $X \in S$ and $\{X\}$ has dimension 0 (codimension greater than 2).

**Remark 1** Note that if $X$ is an isolated point of $\mathcal{B}_g$ and $U$ is a ball in $T_g$ containing a point of $\Pi^{-1}(X)$ such that $\Pi(U) \cap \mathcal{B}_g = \{X\}$, the covering $\partial U \to \Pi(\partial U)$ is a regular unbranched covering with deck transformation group $\text{Aut}(X)$, so $\Pi(\partial U)$ is a manifold and $\pi_1(\Pi(\partial U)) \cong \text{Aut}(X)$.
Theorem 3 For \( g \geq 4 \), every point in \( \mathcal{B}_g \) is topologically singular.

Proof Note that every strata of \( \mathcal{B}_g \) is contained in the closure \( \overline{S}_k \) of some equisymmetric strata defined by the action of a prime order automorphism. We shall then study the dimension of such strata. Note that in some cases the full group of automorphisms of the surfaces in \( \overline{S}_k \) may contain strictly the cyclic group generated by the prime order automorphism.

Assume that \( \overline{S}_k \) is the closure of an equisymmetric stratum of \( \mathcal{B}_g \) defined by a cyclic subgroup \( C_k \) of \( \text{Mod}_g \) of prime order \( k \) such that the corresponding action on surfaces has \( r \) fixed points. Let \( h \) be the genus of all the quotients of the surfaces in \( S_k \) by the action of the group \( C_k \). The Riemann-Hurwitz formula gives:

\[
2g - 2 = k(2h - 2 + \sum_r (1 - \frac{1}{k}))
\]

Since \( k \geq 2 \) then

\[
2g - 2 \geq 2(2h - 2 + r/2)
\]

and \( r \leq 2g - 4h + 2 \). Hence the dimension of \( S_k \) is smaller than \( 4g - 2h - 2 \). Supposing the codimension of \( S_k \) is two, we have \( 6g - 8 = 4g - 2h - 2 \) yielding \( g = 2 \) or \( 3 \), which contradicts our hypothesis. We thus conclude that the codimension of \( S_k \) is greater than 2 and Theorem 2 completes the proof.

Remark 2 The proof in [9] of the Theorem 3 is based on a Theorem of Zariski [8].

Let us now study the topologically singular points of the moduli space of surfaces genus 2 and 3. We need the following corollary of Theorem 2:

**Corollary 2** Let \( X \in \mathcal{B}_g \) and suppose that \( X \in \overline{S} \) where \( S \) is a maximal equisymmetric stratum of codimension greater than 2. Then \( X \) is topologically singular.

Proof Let \( X \in \overline{S} \) where \( S \) is a maximal equisymmetric stratum of codimension greater than 2. By Theorem 2 each point \( Y \in S \) is singular since \( Y \) is the unique stratum containing \( Y \). Since \( X \in \overline{S} \) we have \( X \) is singular.

First we consider the case \( g = 3 \).

**Theorem 4** The points of \( \mathcal{B}_3 \) corresponding to surfaces having an automorphism different from the hyperelliptic involution are topologically singular, while the points of \( \mathcal{B}_3 \) corresponding to surfaces having only the hyperelliptic involution are non-singular.

Proof The orders of prime order automorphisms of surfaces of genus 3 are 2, 3 and 7 (see [1]). Let \( S^{(i)}_k \) be the equisymmetric strata corresponding to genus 3 surfaces where \( C_k \) acts in a determined topological way, \( k = 2, 3, 7 \). Let us denote \( \overline{S}^{(1)}_2 \) the hyperelliptic locus. Every point in \( \mathcal{B}_3 \) is in the closure of some \( S^{(1)}_k \).
Order 2: there are three topological types of automorphisms.

Type A: the hyperelliptic involution with 8 fixed points and quotient of genus 0. The stratum $S_2^{(1)}$ corresponding to this topological type of action has dimension $6 \times 0 - 6 + 2 \times 8 = 10$ (codimension two in $\mathcal{M}_3$). Each point in $S_2^{(1)}$ has a neighbourhood that is homeomorphic to the quotient space of a ball by the action of an order two rotation with fixed point set of codimension 2, then the points in $S_2^{(1)}$ are not topologically singular.

Type B: the involution has 4 fixed points and the quotient has genus 1. In this case the stratum $S_2^{(2)}$ has dimension $6 \times 1 - 6 + 2 \times 4 = 8$ (codimension $> 2$). Since the signature $(1; 2, 2, 2)$ is maximal (see for instance [3]), then this stratum is maximal. By Corollary 2 all the points in $S_2^{(2)}$ are singular.

Type C: for this type of automorphisms there are no fixed points and the quotient surface has genus 2. The dimension of $S_3^{(3)}$ is $6 \times 2 - 6 = 6$ (codimension $> 2$). This stratum is not maximal but is contained in the closure of $S_2^{(3)}$ then all its points are singular.

Order 3: there are two topological types of automorphisms of order three.

Type A: two fixed points and genus of quotient 1. This stratum $S_3^{(1)}$ is not maximal and it is contained in $S_2^{(2)}$, so all the points corresponding to surfaces with this type of action are singular.

Type B: five fixed points and genus of quotient 0. The dimension of this stratum $S_3^{(2)}$ is $6 \times 0 - 6 + 10 = 4$ (codimension $> 2$). The stratum $S_3^{(2)}$ is maximal and all the points in the closure of $S_3^{(2)}$ are singular.

Order 7: A unique point corresponding to the Klein quartic $K$. The surface $K$ admits order three automorphisms, then it is in $S_3^{(i)}$ and $K$ is a singular point.

Finally, we must consider the points in $S_2^{(1)}$ but not in any $S_k^{(i)}$. These points correspond to hyperelliptic surfaces having an automorphism of order 4 that is a square root of the hyperellipticity. Let $S_4 \subset S_2^{(1)}$ be the stratum corresponding to surfaces with full automorphism group $C_4$. The codimension of $S_4$ is greater than two ($\dim S_4 = 6 \times 0 - 6 + 2 \times 5 = 4$). Let $X$ be a point in $S_4$ and $Y \in \Pi^{-1}(X)$. Let $U$ be a neighbourhood of $Y$ in $\mathbb{T}_g$ where $\text{Aut}(X) = C_4 = \langle t : t^4 = 1 \rangle$ acts. We have that $\partial U / \langle t^2 \rangle$ is the sphere but $\partial U / \langle t^2 \rangle \rightarrow \partial U / \langle t \rangle$ is a 2-fold covering branched on a subpolyhedra of codimension $> 2$; then, by a similar argument to the one in the proof of Theorem 2 we prove that $\partial U / \langle t \rangle$ is not simply connected and $X$ is thus topologically singular. Hence all the points in $S_4$ are singular.

Finally we consider the case $g = 2$. In this case, using our approach we cannot give a complete description of the topological singular points in $\mathcal{B}_2$.

**Theorem 5** The points in the stratum $S_2$ of $\mathcal{B}_2$ corresponding to surfaces having full automorphism group $C_2 \times C_2$ are not topologically singular. The isolated point of $\mathcal{B}_2$ corresponding to the Kulkarni surface $y^2 = x^5 - 1$ is singular.
Proof First note that $\dim \mathcal{M}_2 = 6 \times 2 - 6 = 6$.

By Theorem 1 the points of $S_2$ have a neighbourhood $U$ that is homeomorphic to the quotient of a ball $B$ by a rotation of order two having as fixed point set a linear subspace of codimension two (intersection with $B$). Hence, $U$ is a ball and the points in $S_2$ are not singular.

The stratum $S_3$ (surfaces with an order 5 automorphism) has a single point and it is an isolated point of $B_2$ (see [5]). By Corollary 1 this point is singular.

Remark 3 The points in $B_2$ different from Kulkarni isolated surface and the surfaces in $S_2$ are in strata completely included in $S_2$ (then these strata are non-maximal) and our methods do not provide information on the singularity of such points.

Remark 4 In the case 1 of the main Theorem of [6] the author misses the singular point corresponding to Kulkarni surface, different from the Bolza surface $y^2 = x^5 - x$.

4 Singular points of equisymmetric families

In the case of (real) dimension two equisymmetric families, as for all two dimensional orbifolds, the points in the branch locus are not topologically singular. We shall show that in families of greater dimensions the points in the singular locus may be either topologically singular or non-singular. In that direction, let us study the singular points of some equisymmetric families of (real) dimension four.

Example 1 The families $W_{q,w}$ consist of Riemann surfaces that are $q \times w$-fold cyclic coverings of the sphere branched on five points, where $q, w > 5$ are prime integers $q \neq w$. The type of coverings defining the families $W_{q,w}$ will be described now.

Let $T(0; q, q, q, qw, w)$ be the Teichmüller space of groups $\Delta$ with signature $(0; q, q, q, qw, w)$, and

$$\langle x_i, i = 1, ..., 5 : x_i^q = 1, i = 1, 2, 3, x_4^{qw} = x_5^w = 1 \text{ and } x_1 \cdots x_5 = 1 \rangle$$

be a canonical presentation for these Fuchsian groups. The surfaces of the family $W_{q,w}$ are uniformized by the surface groups in the kernel of the epimorphism $\theta : \Delta \to C_{qw} = \langle l : l^{qw} = 1 \rangle$, given by $\theta(x_i) = l^w, i = 1, 2, 3$ and $\theta(x_4) = l^{-3w}l^w, \theta(x_5) = l^{-q}$. The inclusion $\ker \theta \subset \Delta$ induces an embedding $e : T(0; q, q, qw, w) \to T_g$ and the moduli space of $W_{q,s}$ is $H(e(T(0; q, q, qw, w)))$.

The branch locus of the family consists of one point $Y$ with isotropy group $C_3$. The point $X$ corresponds to the case where the group $\Delta$ is inside the triangular group $(0; q, 3qw, 3w)$. The point $Y$ is singular and the boundary of a neighbourhood of $Y$ is homeomorphic to the lens space $L(3, 1)$ (see a survey on lens spaces in [7], there, precisely, the lens spaces are studied as quotient singularities).
Example 2 The family $Q$ consists of Riemann surfaces that are $q$-fold (where $q > 5$ is a prime) cyclic coverings of the sphere branched on five points. The $q$-cyclic coverings defining the family have some special types which we shall describe in terms of Fuchsian groups.

Let $T_{(0;q,\ldots,q)}$ be the Teichmüller space of groups $\Delta$ with signature $(0; q, \ldots, q)$, and
\[
\langle x_i, i = 1, \ldots, 5 : x_i^q = 1, \ i = 1, \ldots, 5 \text{ and } x_1 \cdots x_5 = 1 \rangle
\]
be a canonical presentation for these Fuchsian groups. The family $Q$ has dimension four, the surfaces of the family are uniformized by the surface groups in the kernel of the epimorphism $\theta : \Delta \to C_q = \langle l : l^q = 1 \rangle$, given by $\theta(x_i) = l, \ i = 1, \ldots, 3$ and $\theta(x_4) = \theta(x_5) = l^{2q-2}$. The inclusion $\ker \theta \subset \Delta$ induces an embedding $e : T_{(0;q,\ldots,q)} \to T_q$ and the moduli space of $Q$ is $\Pi(e(T_{(0;q,\ldots,q)}))$.

The branch locus of the family consists of a dimension two subset $L$ corresponding to cone points with isotropy group of order 2 and one point $Y$ with isotropy group $D_3$. The points in $L$ correspond to Fuchsian groups $\Delta$ in $T_{(0;q,\ldots,q)}$ contained in Fuchsian groups $\Lambda$ with signature $(0; 2, q, q, 2q)$ and the point $Y$ corresponds to the case where the group $\Delta$ is inside the triangular group $(0; 2, 2q, 3q)$. The points in $L$ have a neighbourhood $U$ such that the boundary is the tridimensional sphere, since the covering
\[
\Pi(e(T_{(0;q,\ldots,q)})) \cap \Pi^{-1}(\partial U) \to \partial U
\]
is given by the quotient of a rotation around a trivial knot. The singular point $Y$ has also a neighbourhood $V$ whose boundary is homeomorphic to $S^3$, because the intersection $\partial V \cap L$ is the trefoil knot and the covering
\[
\Pi(e(T_{(0;q,\ldots,q)})) \cap \Pi^{-1}(\partial U) \to \partial U
\]
is the universal covering of the orbifold defined on $S^3$ with singular orbifold locus the trefoil knot with isotropy group $C_2$ for the points in the branch locus.

The covering is equivalent to the composition of coverings $S^3 \xrightarrow{3:1} L(3,1) \xrightarrow{2:1} S^3$ (the first covering is unbranched and the second one is the given by the Montesinos involution).

We feel very happy to conclude this article in honour of Professor Maite Lozano with this application of the theory of branched coverings of 3–manifolds.

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