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CHIRAL RINGS IN TOPOLOGICAL (W-)GRAVITY

W. LERCHE
CERN, CH 1211 Geneva 23, Switzerland

ABSTRACT

We review the superconformal properties of 2d matter coupled to gravity, and extensions thereof. Focusing on topological strings, we recall how the superconformal structure helps to provide a direct link between Liouville theory coupled to matter, and matrix models. We also construct an infinite class of new theories based on $W$-gravity.

1. Introduction

There has been some recent progress in understanding theories describing 2d matter coupled to 2d gravity. I would like to review here in particular the findings of ref.\(^1\). That paper deals actually with two different and logically independent lines of development: one is to exploit an $N = 2$ superconformal symmetry that all 2d matter-gravity systems have. The other is an extension to $W$-gravity coupled to matter, which adds an infinite sequence of new theories.

Matter-plus-gravity systems are interesting to study because they are, for certain choices of matter theories, supposed to be exactly solvable. More precisely, they are supposed to be equivalent to matrix models\(^2\), which are exactly solvable by themselves as a consequence of an underlying structure of KdV-like integrable hierarchies. To deduce this equivalence directly from Liouville theory appears to be difficult, largely due to technical complications. We will show below that the above-mentioned $N = 2$ superconformal structure provides a manifest and direct relationship of (at least certain of) such models to matrix models.

We like to start by briefly recapitulating ordinary gravity coupled to conformal 2d matter. For simplicity, we will consider mainly minimal matter models, but this is not really important. These matter models, denoted by $M_{p,q}$, where $p, q = 1, 2, \ldots$ are coprime integers, have central charges

\[
c_M = 13 - 6(t + \frac{1}{t}) , \quad t \equiv q/p .
\]

It is well known that such theories can be realized in terms of a single free boson $\phi_M$, with an appropriate amount of background charge $\alpha_0$; the stress tensor has the familiar form: $T_M = -(1/2)(\partial \phi_M)^2 + i(\alpha_0/\sqrt{2})\partial^2 \phi_M$, where $\alpha_0 \equiv (1 - t)/\sqrt{t}$. The primary fields can be represented by vertex operators $\Phi_{r:s} = V_{r:s}^M \equiv e^{i\alpha_{r:s}^M \phi_M}$, with $\alpha_{r:s}^M = \frac{1}{\sqrt{2}}[\alpha_+(r - 1) + \alpha_-(s - 1)]$, where $\alpha_+ \equiv \sqrt{t}$, $\alpha_- \equiv -1/\sqrt{t}$. For minimal
models $\mathcal{M}_{p,q}$, one restricts the labels to the Kac table, that is, to $1 \leq r \leq p - 1$ and $1 \leq s \leq q - 1$.

For $t = 1$, the matter model is not minimal, but becomes the celebrated $c = 1$ theory that received quite some attention due to its relationship with 2d black holes$^2$. For generic $t$, the central charge $c_M$ is not equal to 26. Therefore, when coupling to 2d gravity, one deals with “non-critical” strings in which gravity is governed by the Liouville degree of freedom. Liouville theory is usually described in a way very similar to the above matter theory. One treats the Liouville field $\phi_L$ like a free field, and tries to incorporate the cosmological constant term in a perturbative manner (this is possible only for the “topological” theories; see below). One constructs a stress tensor similar to the one of the matter model:

$$T_L = -\frac{1}{2}(\partial\phi_L)^2 + \frac{\beta_0}{\sqrt{2}}\partial^2\phi_L$$

and chooses the background charge, $\beta_0 = (1+t)/\sqrt{t}$, such that $c_L + c_M = 26$. This precisely offsets the central charge the ghost stress tensor $T_{gh} = -2b(\partial c) - (\partial b)c$, so that the total central charge vanishes. Here, $b$ and $c$ denote the fermionic ghosts with spins equal to 2 and $-1$.

As usual in $BRST$ quantization, the physical states of the combined matter-gravity system are given by the non-trivial cohomology classes of a $BRST$ operator. This operator looks

$$Q_{BRST} = \int \frac{dx}{2\pi t} J_{BRST}, \quad J_{BRST} = c[T_M + T_L + \frac{1}{2} T_{gh}] \quad (1.2)$$

and is nilpotent for $c_L + c_M = 26$. The most prominent physical states correspond to the tachyon operators$^4$:

$$T_{r,s} = c V^L_{r,-s} V^M_{r,s} \quad (1.3)$$

By convention, these operators have $bc$ ghost number equal to one. In addition, there exist$^5$ extra physical states whose number and precise structure depends on the specific value of $t$. For unitary minimal models, where $t = (p+1)/p$, there exist infinitely many of such extra states for each matter primary, whereas for generic $t$, there exists basically only one extra sort of states besides the tachyons: these are the operators with vanishing ghost charge. They form what is called$^6$ the ground ring; we will denote it by $\mathcal{R}^\text{gt}$. It is precisely because these operators have zero ghost charge (and zero dimension like all physical operators), that the set of ground ring operators closes into itself under operator products. In fact, even though this ring is in general infinite, it is finitely generated, i.e., it has two generators by whose action all other ring elements can be generated$^6$:

$$x = [bc - \frac{t}{\sqrt{2t}}(\partial\phi_L - i\partial\phi_M)] V^L_{1,2} V^M_{1,2}$$

$$\gamma^0 = [bc - \frac{1}{\sqrt{2t}}(\partial\phi_L + i\partial\phi_M)] V^L_{2,1} V^M_{2,1} \quad (1.4)$$

The rest of the operators with non-zero ghost numbers fall into modules of this ring$^6,7$. The structure of $\mathcal{R}^\text{gt}$ (and that of its cousins obtained by non-trivial pairings of left- and right moving sectors) characterizes to some extent a given theory.
2. \( N=2 \) superconformal structure

The properties of the ground ring elements remind very much to the typical features of chiral primary fields in \( N=2 \) superconformal theories. The whole point is, of course, that the matter-gravity-ghost system is essentially nothing but a (twisted) \( N=2 \) superconformal theory. More precisely, it is known\(^8\) that one can improve the BRST current (1.2) by an irrelevant total derivative piece,

\[
G^+ = J_{BRST} - \partial \left( \sqrt{\frac{2}{t}} (c \partial \phi_L) + \frac{1}{2} (1 - \frac{2}{t}) \partial c \right),
\]

such that \( G^+ \) together with

\[
G^- = b, \quad T = T_L + T_M + T_{gh}, \quad J = cb + \sqrt{\frac{2}{t}} \partial \phi_L,
\]

indeed generates the (topologically twisted\(^9,10\) \( N=2 \) superconformal algebra,

\[
T(z) \cdot T(w) \sim \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)},
\]

\[
T(z) \cdot G^\pm(w) \sim \frac{\frac{1}{3}(3 \mp 1)G^\pm(w)}{(z-w)^2} + \frac{\partial G^\pm(w)}{(z-w)},
\]

\[
T(z) \cdot J(w) \sim \frac{\frac{1}{3}c^{N=2}}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{(z-w)},
\]

\[
J(z) \cdot J(w) \sim \frac{\frac{1}{3}c^{N=2}}{(z-w)^3}, \quad J(z) \cdot G^\pm(w) \sim \pm \frac{G^\pm(w)}{(z-w)},
\]

\[
G^+(z) \cdot G^-(w) \sim \frac{\frac{1}{3}c^{N=2}}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{T(w) + \partial J(w)}{(z-w)},
\]

\[
G^\pm(z) \cdot G^\pm(w) \sim 0,
\]

with anomaly

\[
c^{N=2} = 3 \left( 1 - \frac{2}{t} \right).
\]

Upon untwisting, \( T \to T - \frac{1}{2} \partial J \), \( c^{N=2} \) becomes the central charge of an ordinary \( N=2 \) algebra. Note that the free-field realization (2.1), (2.2) of the \( N=2 \) algebra is different from the well-known one\(^1\). This is however irrelevant, and one can show\(^1\) that the above realization can be obtained by hamiltonian reduction\(^12\) from a \( SL(2|1) \) WZW model in a way that is analogous and equivalent to the usual way of deriving a free-field realization of the \( N=2 \) algebra.

An immediate question is the one about the significance of the \( N=2 \) superconformal symmetry. For general \( t \), the mere presence of an \( N=2 \) algebra doesn’t really seem to provide any important new insights. On the other hand, for integer \( t \), which we like to write as \( t = 2 + k \), a lot can be learned: namely then the anomaly (2.4) becomes equal to the central charge of the \( N=2 \) minimal models, \( A_{k+2} \):
This is a powerful statement, since minimal models tend to be easily solved purely by representation theory. (For general $t = q/p$, the level $k$ becomes fractional, which corresponds to non-minimal, in general non-unitary $N=2$ superconformal models. From representation theory alone, not much can be said about such theories in general.)

However, this does not yet imply that the minimal models $\mathcal{M}_{1,2+k}$ coupled to gravity are the same as the topologically twisted $N=2$ minimal models, $A_{k+1}^{\text{top}}$. A priori, these theories don’t have, in fact, even the same spectra. The spectrum of topological $N=2$ models is well known\(^{10}\): it is given by the chiral ring\(^{13}\), which is the finite set of primary chiral fields. For $A_{k+1}^{\text{top}}$, this is a nilpotent, polynomial ring generated by one element:

$$\mathcal{R}_{A_{k+1}^{\text{top}}} = \frac{P(x)}{[x^{k+1} \equiv 0]} = \left\{ 1, x, x^2, \ldots, x^k \right\} . \quad (2.5)$$

One can check that powers of the ground ring generator $x$ in (1.4) are indeed primary chiral fields with respect to the $N=2$ currents (2.1) and (2.2), and that $\mathcal{R}_{A_{k+1}^{\text{top}}}$ is identical to the subring of $\mathcal{R}^{\text{gr}}$ that is generated by $x$. (It turns out that the corresponding tachyons (1.3) have the same $N=2$ quantum numbers as the ground ring elements, so that they can be viewed as different representatives\(^{\star}\) of the same set of primary chiral fields. This is not true for general $t$.)

However, the ground ring $\mathcal{R}^{\text{gr}}$ of the matter-gravity system contains infinitely many more operators\(^{14}\):

$$\mathcal{R}^{\text{gr}} = \mathcal{R}_{A_{k+1}^{\text{top}}} \otimes \left\{ (\gamma^0)^n, \ n = 0, 1, 2, \ldots \right\} . \quad (2.6)$$

These extra operators simply do not exist in the topological minimal models; they are not primary with respect to the $N=2$ algebra.

The difference between the spectra (2.5) and (2.6) can be accounted for as follows: it turns out that these extra operators are exact with respect to an additional $BRST$ like operator, $\tilde{Q}$:

$$\gamma^0 = -\left\{ \tilde{Q}, \frac{t+1}{t} \partial c + \frac{1}{\sqrt{2}t} c \partial \phi_L \right\} , \quad \text{where} \quad \tilde{Q} = \oint \frac{dz}{2\pi i} b e^{\frac{1}{\sqrt{2}t} (\phi_L - i \phi_M)} . \quad (2.7)$$

One can show\(^{1}\) that $\tilde{Q}$ is one of the Felder-like screening operators that arise in our particular free field realization of the minimal models. That is, by definition the full $BRST$ operator of the topologically twisted $N=2$ minimal models is the sum of $Q_{BRST}$, $\tilde{Q}$ and other screening operators, and it truncates the infinite free field spectrum precisely to the finite set of physical operators (2.5).

\(^{\star}\) These representatives just happen not to close among themselves under operator products.
We thus see that this full BRST operator of the topological minimal models is not the correct one if we wish to describe the minimal models $M_{1,2+k}$ coupled to gravity. The correct operator obtains if we drop $\widetilde{Q}$ as an extra BRST operator; it can be shown that then the “missing” operators $(\gamma^0)^n$ become physical. There is actually a better way to formulate this in terms of equivariant cohomology\textsuperscript{15}, but for lack of space we refrain from doing so.

It can be shown\textsuperscript{16} that these modified minimal topological models, which contain the operators $(\gamma^0)^n$ and which are equivalent to the minimal models $M_{1,2+k}$ coupled to gravity, are in fact also equivalent to the models $A^\text{top}_{k+1}$ coupled to topological gravity\textsuperscript{15}. A priori, the building blocks of $A^\text{top}_{k+1}$ and of the same models coupled\textsuperscript{17} to topological gravity appear to be quite different. There is, however, the remarkable fact that the total BRST operator of the topological matter plus topological gravity system obeys\textsuperscript{16}

$$Q^\text{tot} \equiv Q^{N=2} + Q_{\text{BRST}} = U^{-1} Q^{N=2} U ,$$

where $U$ is some homotopy operator. The upshot is that the cohomologies of $A^\text{top}_{k+1}$ coupled to topological gravity and of the modified minimal topological models are isomorphic, so that at least at the level of Fock spaces

$$\left[ M_{1,2+k} \otimes \text{Liouville gravity} \right] \cong \left[ A^\text{top}_{k+1} \bigg|_{\text{modified cohomology}} \right] \cong \left[ A^\text{top}_{k+1} \otimes \text{topological gravity} \right].$$

In view of this equivalence, the extra ground ring elements, $(\gamma^0)^n$, can thus be interpreted as topological gravitational descendants, and the full ground ring (2.6) might be called a “gravitationally extended chiral ring” of the topological matter model. Note also that for $k = 0$, which corresponds to the trivial topological matter theory, the LHS of (2.9) turns precisely into Distler’s formulation of topological gravity\textsuperscript{18,19}.

With (2.9) at hand, it is then easy to conclude that

$$\left[ M_{1,2+k} \otimes \text{Liouville gravity} \right] \cong \left[ \text{matrix model of type (1, 2 + k)} \right],$$

as it is supposed to be the case\textsuperscript{20}. This is a direct consequence of the fact\textsuperscript{17} that the recursion relations of $[A^\text{top}_{k+1} \otimes \text{topological gravity}]$ are the same as those of the matrix models\textsuperscript{21}.

One would like to make similar statements for more general models $M_{p,q}$ coupled to gravity; such theories can be considered as deformations of the above, topological ones, and it is an interesting question whether $N = 2$ language would be useful for describing this.
The equivalence (2.10) can be exhibited also in a more direct way. The point is that the Landau-Ginzburg formulation\(^{22}\) of the topological matter models, \(A_{k+1}^{\text{top}}\), can be directly related\(^ {23}\) to the KdV structure of the matrix models. More precisely, the Landau-Ginzburg superpotential, which describes the effect of perturbations \(\exp \int d^2z d^2\theta \sum_{i=0}^{k} t_i x^i\), has the generic form

\[
W(x,t_i) = \frac{1}{k+2} x^{k+2} - \sum_{i=0}^{k} g_i(t_j)x^i ,
\]  

(2.11)

where the coupling constants \(g_i(t_j)\) are certain, in general non-trivial functions of the perturbation parameters. Since the correlation functions can easily be computed\(^ {23}\) from \(W(x,t_i)\), solving the theory amounts to determining these functions. They can be obtained by solving the differential equations

\[
\frac{\partial}{\partial t_i} W(x,t) = \frac{1}{i+1} ((k+2)W)^{i+1}_{i+1} ,
\]  

(2.12)

which just express the fact that the \(t_i\) are “flat” coordinates of the deformation space. The crucial observation\(^ {23}\) is that under the substitutions \(x \rightarrow D \equiv \frac{\partial}{\partial z}\) and \(W(x,t_i) \rightarrow L(D,\frac{t_i}{i+1})\), these equations becomes precisely the dispersionless, quasi-classical limit\(^ {24}\) of the KdV flow equations

\[
\frac{\partial}{\partial t_i} L(D,t) = [ (L^{i+1})^+ , L ] .
\]  

(2.13)

These equations describe\(^ {2,20}\) (or even define) the dynamics of the matrix models of type \((1,k+2)\). This immediately proves the equality of (genus zero) correlation functions of the primary fields of \(A_{k+1}^{\text{top}}\) with the corresponding correlators of the matrix models. These arguments, which involve only \(N=2\) Landau-Ginzburg theory, can in fact be extended to gravitational descendants and to recursion relations they obey\(^ {25,16}\). This is precisely in the spirit of what was said above: the ingredients of the coupling of \(A_{k+1}^{\text{top}}\) to topological gravity are already build in the structure of the models \(A_{k+1}^{\text{top}}\) themselves; all what is necessary to describe the coupling of these models to topological gravity is to modify their cohomological definition.

Of particular interest is the perturbation of these models by the “cosmological constant” term\(^ *\). In our language, it is the perturbation by the top element of \(\mathcal{R}^{A_{k+1}^{\text{top}}}\),

\[
S_{\text{cosm}} = \mu \int d^2z e^{\sqrt{2} \phi_L} = t_k \int d^2z d^2\theta x^k .
\]  

(2.14)

It is known\(^ {26}\) that this perturbation is integrable and leads to the massive quantum \(N=2\) sine-Gordon model; although not invariant under the full (twisted) \(N=2\)

\[\tag{\star}\]

* Note that such perturbations lead away from the conformal point. We restrict here to the “small phase space”, i.e., to perturbations generated by the primary fields.

\[\tag{\star}\]

* We define this here as being the operator with trivial matter piece. For the topological models the dependence on \(\mu\) is analytic and thus perturbation theory around \(\mu = 0\) is perfectly well defined. This is in contrast to general models \(\mathcal{M}_{p,q}\) coupled to gravity.
superconformal symmetry, it is supersymmetric, and the (corrected) supercharge $\mathcal{J}_G$ still serves as a BRST operator. The effective superpotential is given by a Chebyshev polynomial:

$$W(x, t_k = \mu) = \frac{2}{k+2} \mu^{\frac{k+2}{2}} T_{k+2} \left( \frac{1}{2} \mu^{-\frac{1}{2}} x \right) = \frac{1}{k+2} x^{k+2} - \mu x^k + O(\mu^2).$$

(2.15)

At $\mu = 1$, the deformed chiral ring becomes identical\(^\text{27}\) to the fusion ring of the $SU(2)_k$ WZW model, which it is also the same as the operator product algebra of the $SU(2)_k/SU(2)_k$ topological field theory. This observation then allows to finally make contact to the formulation of matter-plus-gravity models in terms of topological $G/G$ theories\(^\text{28}\): it is known\(^\text{28,29}\) that at the level of Fock space cohomology, the (suitably defined) $SU(2)/SU(2)$ model is indeed equivalent to the matter-gravity system. We thus have in addition:

$$\left[ \mathcal{M}_{1,2+k} \otimes \text{Liouville gravity} \right]_{\mu=1} \cong \left[ \frac{SU(2)_k}{SU(2)_k} \right]_{\text{modified cohomology}}$$

(2.16)

### 3. Generalizations

The other line of development taken up in ref.\(^\text{1}\) is the generalization to $W$-gravity\(^\text{30}\) coupled to $W$-matter. Here one considers tensor products

$$W_n^{\text{matter}} \otimes W_n^{\text{Liouville}} \otimes_{j=1}^{n-1} \{b_j, c_j\},$$

(3.1)

which might be called “non-critical $W$-strings”.\(^\text{31}\) Above, $W_n^{\text{matter}}$ denotes conformal field theories that have a $W$-algebra as their maximal chiral algebra, which can be for example $W_n$ minimal models $\mathcal{W}_{p,q}^{(n)}$ with central charges $c_M = (n-1)[1 - n(n+1)(t-1)^2]$, $t = q/p$. Furthermore, $W_n^{\text{Liouville}}$ denotes a $(n-1)$-component generalization of Liouville theory (Toda theory), and $\{b_j, c_j\}$ denotes the Hilbert space of a ghost system with spins $j+1$ and $-j$, respectively. As it turns out, the structure of these theories for arbitrary $n$ is very similar to $n = 2$, which corresponds to ordinary gravity. However, only for $n = 3$ the generalization\(^*\) of the $\text{BRST}$ current is explicitly known\(^\text{32}\):

$$J_{\text{BRST}} = c_2 \left[ \frac{1}{b_L} W_L + \frac{i}{b_M} W_M \right] + c_1 \left[ T_L + T_M + \frac{1}{2} T_{gh}^1 + T_{gh}^2 \right] + \left[ T_L - T_M \right] b_1 c_2 (\partial c_2) + \mu (\partial b_1) c_2 (\partial^2 c_2) + \nu b_1 c_2 (\partial^3 c_2),$$

(3.2)

where $b_{L,M}^2 \equiv \frac{16}{5c_{L,M} + 22}$ and $\mu = \frac{3}{5} \nu = \frac{1}{10b_L^2} (1 - 17b_L^2)$. In this equation, $T_{L,M}$ and $W_{L,M}$ denote the usual stress tensors and $W$-generators of the Liouville and matter sectors, and $T_{gh}^i$ are the stress tensors of the ghosts.

Using this $\text{BRST}$ current, one can study the spectrum of physical operators of $W_3$ matter coupled to $W_3$ gravity, and finds\(^\text{1,33,34}\) that the analogs of ground ring elements and tachyons are states with ghost numbers equal to 0, 1, 2, 3 (the first number corresponds to ground ring elements, and the last one to tachyons). The

* The existence of $\text{BRST}$ currents for arbitrary $n$ can be inferred from indirect arguments\(^\text{1}\).
explicit expressions are however too complicated to be written down here.

The interesting point is that there appears an $N=2$ superconformal symmetry for all $n$. For example, for $W_3$ gravity one finds that

$$G^+ = J_{BRST} + \partial \left[ -c_1 J + 2i \sqrt{\frac{2}{3}} b_1 c_1 c_2 J + i \left( \frac{1}{2} + t \right) \sqrt{\frac{2}{3}} b_1 c_1 (\partial c_2) \right. $$
$$- i \left( \frac{3}{2} + 2t \right) \sqrt{\frac{3}{2t}} b_1 (\partial c_1) c_2 - \left( \frac{7t^2 - 16t - 15}{4t} \right) b_1 (\partial c_2) c_2 + i \left( \frac{t - 9}{3t} \right) b_2 (\partial c_2) c_2 $$
$$- i \left( \frac{3}{2} + 4t \right) \sqrt{\frac{3}{2t}} (\partial b_1) c_1 c_2 - \left( \frac{3(4t^2 - 2t - 3)}{2t} \right) (\partial b_1) (\partial c_2) c_2 + i \left( \frac{t - 3}{t} \right) (\partial c_1) c_2 $$
$$+ i \left( \frac{1 + t}{2} \right) \sqrt{\frac{3}{2}} (\partial c_2) J - i \left( \frac{t^2 - 4t - 1}{2t} \right) \sqrt{\frac{3}{2}} (\partial c_2) + t b_1 (\partial c_2) c_2 J \right] ,$$
\[ (3.3) \]

together with

$$G^- = b_1 , \quad T = T_L + T_M + T_{gh} , $$
$$J = c_1 b_1 + c_2 b_2 + \frac{3}{\sqrt{t}} (\lambda_1 \cdot \partial \phi_L) + i \frac{1}{2} \sqrt{\frac{3}{t}} (t - 1) \partial [b_1 c_2] $$
\[ (3.4) \]
gives a non-standard free field realization of the topological algebra (2.3) with

$$c^{N=2} = 6 \left( 1 - \frac{3}{t} \right) .$$
\[ (3.5) \]

Since we are dealing here with theories with an extended symmetry, coupled to an extended “$W$-geometry”, it is perhaps not too surprising to find that these topological algebras actually extend to topologically twisted $N=2$ $W$-algebras. For $t = n + k$, which corresponds to $W_n$-minimal matter models $W_{1,n+k}$, the anomaly indeed becomes equal to the central charges of the minimal $N=2$ $W_n$ models at level $k$: $c^{N=2} = 3 \frac{1}{n+k}$. These models are just the well-known Kazama-Suzuki models\textsuperscript{35} based on cosets $SU(n)_k/U(n-1)$, which are known to have an $N=2$ $W_n$ chiral algebra\textsuperscript{36}. The models that arise here are of course the topologically twisted versions, which we will denote by $\text{CP}^{top}_{n-1,k}$.

The chiral rings of these topological $W_n$ matter models are well understood\textsuperscript{13,37}. They are generated by primary chiral fields $x_i$, $i = 1, \ldots, n - 1$ (with $U(1)$ charges equal to $i/(n + k)$), and have elements

$$\mathcal{R}^{\text{CP}^{top}_{n-1,k}} = \left\{ \prod_{i=1}^{n-1} (x_i)^{m_i} , \sum_{i=1}^{n-1} n_i \leq k \right\} .$$
\[ (3.6) \]

On the other hand, the full ground rings of the minimal models $W_{1,n+k}$ coupled to $W_n$-gravity contain in addition generators $\gamma^0_i$, $i = 1, \ldots, n - 1$ (with $U(1)$ charges equal to $i$) and appear to be “$W$-gravitationally extended” chiral rings of the Kazama-Suzuki models:

$$\mathcal{R}^{gr} = \mathcal{R}^{\text{CP}^{top}_{n-1,k}} \otimes \left\{ \prod_{i=1}^{n-1} (\gamma^0_i)^{n_i} , n_i = 1, 2, \ldots \right\} .$$
\[ (3.7) \]
Although this has not yet been thoroughly investigated for general \( n \), our considerations seem so far to indicate that the structure is completely analogous to the one of \( n = 2 \). Accordingly, one would have for topological \( W \)-strings:

\[
\left[ W_{1,n+k}^{(n)} \otimes W_n \text{-gravity} \right] \cong \left[ \text{CP}_{n-1,k}^{\text{top}} \otimes \text{topological } W_n \text{-gravity} \right],
\]

as well as the obvious generalization of (2.16). However, the building blocks of topological \( W \)-gravity as constructed in ref.38 look different, and the precise connection of our considerations with those of ref.38 needs to be investigated in more detail.

4. Concluding remarks

One possible motivation for investigating the above kind of generalizations is the wish to step beyond the \( c_M = 1 \) barrier of ordinary gravity. For a given \( W_n \)-gravity, there is a barrier at \( c_M = n - 1 \), below of which there is a finite number of (dressed) primary fields and below of which the theory should be solvable. In analogy to ordinary gravity, one would even expect that such theories should be solvable also at the accumulation points \( c_M = n - 1 \) (where there exists an extra \( SU(n) \) current algebra symmetry). At these points, such models are presumably related to black hole type of objects in spacetimes with signature \( (n-1,n-1) \) and are characterized by topological field theories based on non-compact versions of \( \text{CP}_{n-1,k}^{\text{top}} \).

Another motivation would be to find analogs of the identification (2.10). At the moment it is not clear to us whether any such generalized matrix models really exist, but at least the corresponding generalizations of the KdV hierarchy appear to exist; details will be presented elsewhere.

A further point of view might be that there is in fact nothing special about \( W \)-extensions of minimal models: one is dealing with just particular examples of extended chiral algebras, \( \mathcal{A} \). One is tempted to speculate that the emergence of an \( N = 2 \) structure, of extensions of topological gravity and finally of generalized integrable hierarchies might be a more general phenomenon, and occurs even for arbitrary RCFT (or at least, for a large class thereof). The conjecture that for an arbitrary chiral algebra \( \mathcal{A} \), a theory of the form

\[
\left[ \mathcal{A} \text{-minimal matter } \otimes \text{“A-} \text{gravity”} \right]
\]

might be equivalent to some “topological \( \mathcal{A} \) string theory”, seems to be related to the recent ideas of Gepner. A conjecture put forward in ref.39 is that there is an \( N = 2 \) superconformal field theory associated with \( \text{any} \) RCFT, the chiral ring of the \( N = 2 \) theory being isomorphic to the fusion ring of the RCFT. This association is very similar to what “coupling to \( \mathcal{A} \)-gravity” would achieve: gauging the chiral algebra \( \mathcal{A} \) implies that all \( \mathcal{A} \) descendants become unphysical (presumably by becoming top components of \( N = 2 \) supermultiplets), and the spectrum truncates to the (dressed) primary fields (apart from possible gravitational descendants). The algebra of these fields is then more or less the chiral ring of ref.39.
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