On the calculation of hadron form factors from
Euclidean Dyson-Schwinger equations

M. Burkardt†, M.R. Frank*, and K.L. Mitchell‡

† Department of Physics, New Mexico State University, Las Cruces, NM 88003-0001
∗Institute for Nuclear Theory, University of Washington, Seattle, WA 98195
‡TRIUMF, 4004 Wesbrook Mall, Vancouver, British Columbia, Canada, V6T 2A3

We apply Euclidean time methods to phenomenological Dyson-Schwinger
models of hadrons. By performing a Fourier transform of the momentum
space correlation function to Euclidean time and by taking the large Euclidean
time limit, we project onto the lightest on-mass-shell hadron for given quan-
tum numbers. The procedure, which actually resembles lattice gauge theory
methods, allows the extraction of moments of structure functions, moments
of light-cone wave functions and form factors without ad hoc extrapolations
to the on-mass-shell points. We demonstrate the practicality of the procedure
with the example of the pion form factor.

I. INTRODUCTION

The planned experimental program at the Thomas Jefferson National Accelerator Facil-
ity (TJNAF) will subject the electromagnetic structure of hadrons and nuclei to detailed
scrutiny. The calculation of hadron form factors is therefore of fundamental importance to
the subsequent interpretation of the obtained experimental results. However the nonpertur-
bative description of hadron structure in the Minkowski metric, which has only recently been
pursued in detail [1], is extremely difficult due to the direct confrontation with singularities
and the indefinite norm. Alternatively, Euclidean space is characterized by a positive definite
norm, i.e., \( p^2 \equiv p_1^2 + p_2^2 + p_3^2 + p_4^2 \geq 0 \), and it has long been known that the Euclidean for-
mulation is therefore advantageous in the description of nonperturbative processes through
the Dyson-Schwinger (DSE) and Bethe-Salpeter (BSE) equations.

These components are frequently assembled into diagrams such as that in Fig. 1 for the calculation of form factors [2,3]. The problem with the Euclidean formulation of such processes is that for physical particles the external momenta must satisfy the mass-shell condition $P^2 = -M_1^2$ and $K^2 = -M_2^2$, thus forcing the return to Minkowski space. One is then faced with the problem of complex momenta flowing through the loop in the diagram of Fig. 1. This in turn requires solving the DSEs in the complex plane. Although some progress has been made in this regard [4], it is an extremely difficult problem and most calculations until now have employed entire-function fits on the real axis. Although some success has been obtained with this approach [3], there are uncertainties associated with the extrapolation of these functions into the complex plane.

FIG. 1. The electromagnetic vertex for a composite pion at tree level. The quark Green’s functions, quark-photon vertex, and the pion Bethe-Salpeter amplitudes, are dressed in a consistent, gauge-invariance manner.

Here we offer a fresh approach which avoids these uncertainties by obtaining the mass-
This approach does not circumvent the need to make assumptions about the analytic structure of the components in the diagram of Fig.1, but rather relies on the selection of relevant singularities in much the same way as is accomplished by Euclidean-space lattice calculations.

We consider the pion form factor as a prototype for the purposes of illustration.

The exact calculation of the pion form factor via lattice techniques has been studied previously [6,7], and proceeds as follows: one first considers the Euclidean three point correlation function

$$C_\mu(t_x, t_y) = \int d^3 x d^3 y \langle 0 | T \left[ J^\dagger_\pi(y) J_\mu(0) J_\pi(x) \right] |0\rangle e^{-i\vec{K} \cdot \vec{y}} e^{i\vec{P} \cdot \vec{x}},$$

where $J_\pi(x) = \bar{u}(x)i\gamma_5 d(x)$ is the interpolating field for the pion and $J_\mu(0) = \frac{2}{3}\bar{u}(0)\gamma_{\mu} u(0) - \frac{1}{3}\bar{d}(0)\gamma_\mu d(0)$ is the vector current. $t_x$ and $t_y$ are the time components of $x$ and $y$ respectively. Upon inserting a complete set of states between each pair of operators in Eq.(1) and taking the limit $t_x \to -\infty$ and $t_y \to \infty$, so that only the lightest state contributes significantly, one finds

$$\lim_{t_y, -t_x \to \infty} C_\mu(t_x, t_y) = -\frac{e^{E_\rho t_x} e^{-E_\vec{P} t_y}}{4E_\rho E_\vec{K}} \langle 0 | J^\dagger_\pi(0) | \pi, \vec{K} \rangle \langle \pi, \vec{K} | J_\mu(0) | \pi, \vec{P} \rangle \langle \pi, \vec{P} | J_\pi(0) | 0 \rangle,$$

where $|\pi, \vec{P}\rangle$ represents an on-mass-shell (i.e. $E_{\vec{P}} \equiv \sqrt{M_\pi^2 + \vec{P}^2}$) pion state with three-momentum $\vec{P}$. Note that Eq.(2) is exact. Thus despite the Euclidean formulation (as long as one does not employ any further approximations such as quenching, finite quark masses, finite lattice spacing, finite volume) it allows an exact computation of the on-shell pion electromagnetic form factor

$$(\vec{P}_\mu + \vec{K}_\mu) F_\pi \left( (\vec{P} - \vec{K})^2 \right) = \langle \pi, \vec{K} | J_\mu(0) | \pi, \vec{P} \rangle,$$

1A similar method has been applied in Ref. [5] to 2-point functions, where it has been used as a means to obtain the physical mass. However, in this paper, we would like to demonstrate that Euclidean time projection methods are much more powerful and can be successfully applied to higher point functions as well.
where $\hat{P} \equiv (E, \vec{P})$ and $\hat{K} \equiv (E, \vec{K})$. In practice Eq.(1) is usually evaluated in the quenched approximation, yielding

$$C_{\mu}(t_x, t_y) = \int dU e^{-S(U)} \int d^3 x d^3 y e^{-i \vec{K} \cdot \vec{y}} e^{i \vec{P} \cdot \vec{x}}$$

$$\text{tr} \left[ \gamma_5 M^{-1}(U)(x, 0) \gamma_{\mu} M^{-1}(U)(0, y) \gamma_5 M^{-1}(U)(y, x) \right], \quad (4)$$

where $M^{-1}(U)(y, x)$ is the quark propagator for a given configuration of link fields $U$. Note that the group integration $\int dU$ implies that the quark lines and the vertices are dressed with arbitrary gluon lines.

The above example (plus many other examples in Euclidean lattice gauge theory) shows that, despite the fact that the calculations are performed in a Euclidean metric, it is still possible to extract on-mass-shell matrix elements, without having to make ad hoc assumptions about the analytic continuation back to a Minkowskian metric. The basic idea is that large Euclidean time merely acts as a filter to project out the ground state of the Hamiltonian. In this paper, we present a procedure that uses similar techniques in the context of Euclidean Dyson-Schwinger calculations, which also there allows an unambiguous extraction of the on-mass-shell matrix elements.

II. THE DYSON-SCHWINGER APPROACH TO FORM FACTOR CALCULATIONS

An alternative to performing the exact (or quenched) calculation is to employ two approximations restricting the type of allowed gluon dressing. The first is to consider only dressing by the gluon two-point function, and the second is to consider only ladder/rainbow dressing\(^2\). These approximations are in fact implied by the diagram of Fig.1, and allow the separate calculation of the components, i.e., quark propagators and vertex functions.

\(^2\)Note that ladder/rainbow dressing implicitly leads to a restriction on allowed quark lines that resembles the quenched approximation in lattice QCD.
These approximations further comprise an electromagnetic gauge-invariant description \[2,8\]. Within this approximation the diagram of Fig.1 is given by

\[
\Lambda_5^\mu(K, P) = \int \frac{d^4k}{(2\pi)^4} \text{tr} \left[ \Lambda_5(P, k) G \left( k + \frac{P}{2} \right) \Lambda_\mu \left( K - P, k + \frac{K}{2} \right) \right. \\
\left. \times G \left( k - \frac{P}{2} + K \right) \Lambda_5 \left( K, k - \frac{P}{2} + \frac{K}{2} \right) G \left( k - \frac{P}{2} \right) \right],
\]

where \( \Lambda_5 \) is the solution of the \textit{inhomogeneous} pseudoscalar BSE (quark-pseudoscalar vertex), \( \Lambda_\mu \) is the solution of the \textit{inhomogeneous} vector BSE (quark-photon vertex), and \( G \) is the quark propagator.

We will use the notation \( \Lambda \) to represent the solutions to the \textit{inhomogeneous} BSE, and \( \Gamma \) to represent the solutions to the \textit{homogeneous} BSE. In the vicinity of a pole, they are related by \[8\]

\[
\Lambda(P, k) \approx \frac{M^2 \sqrt{Z(-M^2)}}{P^2 + M^2} \Gamma(P, k),
\]

where \( M \) is the pole mass, \( Z(-M^2) \) is the wave function renormalization, and \( \Gamma(P, k) \) therefore obeys the standard Bethe-Salpeter bound state normalization \[8\].

The vertex function for a photon coupling to an on-shell pion is then given by

\[
\Gamma_\mu^\pi(K, P) = \int \frac{d^4k}{(2\pi)^4} \text{tr} \left[ \Gamma_\pi(P, k) G \left( k + \frac{P}{2} \right) \Lambda_\mu \left( K - P, k + \frac{K}{2} \right) \right. \\
\left. \times G \left( k - \frac{P}{2} + K \right) \Gamma_\pi \left( K, k - \frac{P}{2} + \frac{K}{2} \right) G \left( k - \frac{P}{2} \right) \right],
\]

where in the vicinity of the pion pole

\[
\Lambda_5^\mu(K, P) \approx \frac{M_\pi^4 Z(-M_\pi^2)}{(P^2 + M_\pi^2)(K^2 + M_\pi^2)} \Gamma_\mu^\pi(K, P).
\]

The quantity \( \Gamma_\mu^\pi(K, P) \) has the decomposition \[2\]

\[
\Gamma_\mu^\pi(K, P) = (P_\mu + K_\mu) F_\pi \left( P^2, K^2, (K - P)^2 \right) \\
+ (K_\mu - P_\mu)(P^2 - K^2) H_\pi \left( P^2, K^2, (K - P)^2 \right)
\]

and is commonly used to extract the pion form factor, \( F_\pi \), at the on-shell point \( P^2 = K^2 = -M_\pi^2 \).
Rather than explicitly enforcing the mass-shell condition as has been explored previously, here we instead take an approach that is similar to that taken in the Euclidean-space lattice calculation. The quantity in the DSE approach that is analogous to the three-point correlation function, $C'_\mu(t_x, t_y)$ in (11), is given by

$$C'_\mu(t_x, t_y) \equiv - \int_{-\infty}^{\infty} \frac{dP_4 dK_4}{(2\pi)^2} e^{-iP_4 t_x} e^{iK_4 t_y} \Lambda_5^5(K, P).$$

(10)

The form factor is then obtained implicitly from the expression

$$\lim_{t_y, -t_x \to \infty} C'_\mu(t_x, t_y) = - \frac{e^{E P_t x} e^{-E K_t y}}{4E_e E_K} \left[ M_\pi^2 \sqrt{Z(-M_\pi^2)} \right] \left[ (\tilde{P}_\mu + \tilde{K}_\mu) F_\pi \left( (\tilde{K} - \tilde{P})^2 \right) \right] \left[ M_\pi^2 \sqrt{Z(-M_\pi^2)} \right].$$

(11)

The correctness of Eq.(11) is readily verified by substituting Eqs.(8) and (9) into (10), along with the implicit assumption that there are no poles in the immediate vicinity of the ground-state pion pole of interest. The validity of this assumption is of course model dependent. Eq.(11) can be directly compared with the exact result in Eq.(2).

### III. TOY-MODEL CALCULATION

We now proceed to evaluate Eq.(11) for a simple model which allows direct comparison with the exact result. That is, we make the following choices

$$\Lambda_5(P, k) \equiv \frac{i\gamma_5}{P^2}$$

$$\Lambda_\mu(P, k) \equiv -i\gamma_\mu$$

$$G(k) \equiv \frac{1}{i\gamma \cdot k + m_q}$$

(12)

where $m_q$ is taken to be a constituent quark mass. Eq.(12) implies in particular that $\Gamma_\pi = i\gamma_5$. The exact result can therefore be obtained by directly fixing the mass-shell condition in the evaluation of Eq.(7) and Eq.(9), and is shown by the solid line in Fig.2. The constituent quark mass, $m_q$, is taken to be 300 MeV, and a cutoff of 1 GeV is applied on the relative momentum at the quark-photon vertex to regulate the integration. The external momenta are constrained such that $P^2 = K^2 = 0$.  

6
For comparison, we evaluate Eq.(11) by substituting the toy model (12) into Eq.(5) for the vertex $\Lambda_{\mu}^5$. The results are shown by the series of dashed curves in Fig.2 for time separations $\Delta t \equiv t_y - t_x \sim 2 fm$. The Fourier transform is performed with 4096 equally-spaced points with a grid spacing of 0.02 GeV. Note that the agreement with the exact result diminishes at larger momentum transfers $Q^2$, as is expected. The agreement at large momentum is improved by increasing the number of Fourier transform points to include larger momentum components in the integration.

There is an additional effect due to the presence of poles in the quark propagators. This is revealed in Fig.3 where for example a constituent quark mass of 1 GeV is used with 1024 Fourier transform points at a grid spacing of 0.03 GeV. With this large quark mass the associated singularities are strongly damped and therefore do not contaminate the extraction of the pion pole. The comparison with the exact result is now quite good out to reasonably large momenta. The increasing discrepancy with increasing time separation at large momenta is due to the highly oscillatory nature of the Fourier transform, and is improved with a finer grid spacing. It is expected that the use of quark propagators from DSE studies [9] that reflect the confining nature of QCD will also produce accurate results at higher momenta due to the absence of such poles.
FIG. 2. The exact and extracted “pion” form factors are compared for a constituent quark mass $m_q = 300$ MeV.
FIG. 3. The exact and extracted “pion” form factors are compared for a constituent quark mass $m_q = 1.0$ GeV.
IV. SUMMARY AND OUTLOOK

We have introduced Euclidean time projection methods as a means to project Euclidean metric 3-point functions onto the physical mass shell. The method, which is based on Cauchy’s theorem and the existence of a Lehmann representation, avoids having to make ad hoc assumptions about the analytic behavior of propagators and vertex functions, and thus allows the direct use of Euclidean DSEs for the calculation of form factors. We have demonstrated the feasibility of the method by studying the pion form factor for a toy model where the exact form factor is known. The approach resembles methods that have been used in the context of lattice gauge theory [6,7]. The idea is to first Fourier transform n-point functions from (Euclidean) energy to (Euclidean) time. The Euclidean time difference between hadron interpolating fields and currents that are used to “probe” the hadron structure is then taken to be large. This ensures that the hadron which is being probed is actually the ground state with the quantum numbers of the interpolating field.

Even though we have studied the pion form factor as a showcase example, applications of Euclidean time projection methods should be applicable to a much wider class of physical observables. Starting with 2-point functions, one can obtain masses [3] as well as light-cone moments of hadron wave functions [4] on the mass shell. Examples for 3-point functions that we plan to calculate include the elastic form factor, moments of parton distribution functions (e.g. momentum or spin fraction carried by the quarks in a given hadron), and the $\gamma^*\pi \rightarrow \gamma$ transition form factor. The most simple observables related to 4-point functions that one might consider calculating using Euclidean time projection are hadron polarizabilities and (virtual) Compton scattering processes.

The main limitations of the method are first of all the restriction to the lightest hadrons for given quantum numbers: though theoretically possible, it is in practice rather difficult to use the method to obtain observables that involve excited states, since the whole trick is to use large Euclidean times to suppress excited states (a limitation that is very familiar to the lattice gauge community). Another limitation concerns very large momentum trans-
fers, where rapid oscillations of the integrand require a large number of integrations points. However, even with rather moderate computing power, this still allows access to most of the kinematic range that is available at TJNAF.

ACKNOWLEDGMENTS

MB would like to thank the Department of Energy (contract DE-FG03-96ER40965) as well as TJNAF for support. MRF is supported by the Department of Energy under Grant # DE-FG06-90ER40561. KLM is supported by the Natural Sciences and Engineering Research Council of Canada.

[1] K. Kusaka and A. G. Williams, Phys. Rev. D 51, 7026 (1995).
[2] M. R. Frank and P. C. Tandy, Phys. Rev. C 49, 478 (1994).
[3] C.D. Roberts, Nucl. Phys. A 605, 475 (1996); M.R, Frank, K.L. Mitchell, C.D. Roberts, and P.C. Tandy, Phys. Lett. B 359, 17 (1995); C. J. Burden, C. D. Roberts and M. J. Thomson, Phys. Lett. B 371, 163 (1996); Yu.Kalinovsky, K.L.Mitchell and C.D.Roberts, $K_{l3}$ and $\pi_{e3}$ transition form factors, nucl-th/9610047.
[4] S. J. Stainsby and R. T. Cahill, Int. J. Mod. Phys. A 7, 7541 (1992).
[5] L.C.L. Hollenberg, C.D. Roberts and B.H.J. McKellar, Phys. Rev. C 46, 2057 (1992); F. T. Hawes, C. D. Roberts and A. G. Williams, Phys. Rev. D 49, 4683 (1994); P. Maris, Phys. Rev. D 52, 6087 (1995); A. Bender, D. Blaschke, Y. Kalinovsky and C.D. Roberts, Phys. Rev. Lett. 77, 3724 (1996).
[6] W. Wilcox and R. M. Woloshyn, Phys. Rev. D 32, 3282 (1985); R. M. Woloshyn, Phys. Rev. D 34, 605 (1986).
[7] G. Martinelli and C.T. Sachrajda, Nucl. Phys. B 306, 865 (1988).
[8] M. R. Frank, Phys. Rev. C 51, 987 (1995).

[9] M. R. Frank and C. D. Roberts, Phys. Rev. C 53, 390 (1996).