EQUIVARIANT STRONGLY PROJECTIVELY FLAT MAPS OF
COMPACT HOMOGENEOUS KÄHLER MANIFOLDS

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Abstract. In [4], the author defines projectively flat maps of a compact Kähler manifold into complex Grassmannian manifold. In this article, by focusing on the essence of the result in [4] we define strongly projectively flat maps and study such maps. Finally we prove a rigidity of equivariant strongly projectively flat maps of simply connected homogeneous Kähler manifold.

1. Introduction

Holomorphic maps into the complex projective space have been studied for a long time. In [1] E. Calabi have proved that holomorphic isometric immersions of Kähler manifolds into the complex projective space are rigid and equivariant with respect to the group of automorphisms of the domain. In [7] M. Takeuchi has constructed all holomorphic isometric immersions of homogeneous Kähler manifolds into the complex projective space and has classified the holomorphic isometric immersions of Hermitian symmetric spaces.

In this article, we study holomorphic maps of a compact homogeneous Kähler manifold into the complex Grassmannian manifold.

Let $\mathbb{C}^n$ be an $n$-dimensional complex vector space with a Hermitian inner product $(\cdot, \cdot)_n$ and $Gr_p(\mathbb{C}^n)$ the complex Grassmannian manifold of complex $p$-planes in $\mathbb{C}^n$ with the Hermitian metric of Fubini-Study type induced by $(\cdot, \cdot)_n$. Let $M$ be a compact Kähler manifold and $f : M \to Gr_p(\mathbb{C}^n)$ be a holomorphic map. Then we have a holomorphic vector bundle $f^*Q \to M$ over $M$ which is the pull-back bundle of the universal quotient bundle $Q \to Gr_p(\mathbb{C}^n)$ over $M$ by $f$.

Definition 1 (cf. [4]).

1. A holomorphic map $f : M \to Gr_p(\mathbb{C}^n)$ is called projectively flat if the pull-back bundle $f^*Q \to M$ is projectively flat.

2. A holomorphic map $f : M \to Gr_p(\mathbb{C}^n)$ is called strongly projectively flat if there exists a holomorphic Hermitian line bundle $L \to M$ such that $f^*Q \to M$ is isomorphic to $\tilde{L} \to M$ with holomorphic structures and fiber metrics, where $\tilde{L} \to M$ is orthogonal direct sum of $q$-copies of $L \to M$.

Strongly projectively flat condition is a kind of simple extension of a map into the complex projective space with Fubini-Study metric. (For a detail, see the section 3).

Projectively flat condition is defined by the author in [2]. A strongly projectively flat map is projectively flat since the orthogonal direct sum of $q$-copies of a holomorphic line bundle is projectively flat. In general, the inverse of this assertion is not true. However, we can show the following assertion.
Proposition 1. Assume that a holomorphic map $f : M \to Gr_p(\mathbb{C}^n)$ is an isometric. Then $f$ is strongly projectively flat if and only if $f$ is projectively flat.

Proof. For a detail, see [3]. Let $M$ be a compact Kähler manifold and $f : M \to Gr_p(\mathbb{C}^n)$ be a holomorphic isometric projectively flat immersion. It follows from the result in [3] that the curvature $R^*Q$ of pull-back bundle $f^*Q \to M$ is expressed as

$$R^*Q = \frac{-1}{q} \omega_M \text{Id}_{Q_*},$$

where $q$ is the rank of $Q \to M$, $\omega_M$ is the Kähler form on $M$ and $\text{Id}_{Q_*}$ is the identity map of the fiber of $f^*Q \to M$ at $x \in M$. Since $\omega_M$ is parallel, it follows from the holonomy theorem that there exists a Hermitian line bundle $L \to M$ such that $f^*Q \to M$ is isomorphic to the orthogonal direct sum of $q$-copies of $L \to M$ as a Hermitian vector bundle.

Therefore $f : M \to Gr_p(\mathbb{C}^n)$ is strongly projectively flat.

In the present paper, our goal is to show the following theorem.

Theorem 1. Let $M = G/K$ be a compact simply connected Kähler manifold such that $G$ is the isometry group of $M$ and $K$ an isotropy subgroup of $G$. Let $f : M \to Gr_p(\mathbb{C}^n)$ be a full holomorphic strongly projectively flat map into the complex Grassmannian manifold. If $f$ is $G$-equivariant, then there exists an $N$-dimensional complex vector space $W$ and a holomorphic map $f_0 : M \to Gr_{N-1}(W)$ such that $\mathbb{C}^n$ is regarded as the orthogonal direct sum of $q$-copies of $W$ and $f$ is congruent to the following composed map:

$$f : M \to Gr_{N-1}(W) \times \cdots \times Gr_{N-1}(W) \to Gr_{q(N-1)}(\mathbb{C}^n),$$

$$x \mapsto (f_0(x), \cdots, f_0(x)) \mapsto f_0(x) \oplus \cdots \oplus f_0(x).$$

$G$-equivariance of a holomorphic map $f : M \to Gr_p(\mathbb{C}^n)$ means that there exists a Lie group homomorphism $\rho : G \to SU(n)$ which satisfies the following equation:

$$f(gx) = \rho(g)f(x), \quad \text{for } x \in M, g \in G.$$

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2. Preliminaries

For a detail of the argument of this section, see [3]. Let $\mathbb{C}^n$ be an $n$-dimensional complex vector space with a Hermitian inner product $(\cdot, \cdot)_n$ and $Gr_p(\mathbb{C}^n)$ be the complex Grassmannian manifold of complex $p$-planes in $\mathbb{C}^n$. We denote by $S \to Gr$ the tautological bundle and by $\underline{\mathbb{C}}^n := Gr_p(\mathbb{C}^n) \times \mathbb{C}^n \to Gr$ the trivial bundle over $Gr_p(\mathbb{C}^n)$. They are holomorphic vector bundles. The trivial bundle $\underline{\mathbb{C}}^n \to Gr$ has a Hermitian fiber metric induced by $(\cdot, \cdot)_n$, which is denoted by the same notation.

Since $S \to Gr$ is a subbundle of $\underline{\mathbb{C}}^n \to Gr$, The bundle $S \to Gr$ has a Hermitian fiber metric $h_S$ induced by $(\cdot, \cdot)_n$ and we obtain a holomorphic vector bundle $Q \to Gr$ satisfying the following short exact sequence:

$$0 \to S \to \underline{\mathbb{C}}^n \to Q \to 0.$$  

This is called the universal quotient bundle over $Gr_p(\mathbb{C}^n)$. When we denote by $S^\perp \to Gr$ the orthogonal complement bundle of $S \to Gr$ in $\underline{\mathbb{C}}^n \to Gr$, $Q \to Gr$
is isomorphic to $S^\perp \to Gr$ as a $C^\infty$-complex vector bundle. Thus $Q \to Gr$ has a Hermitian fiber metric $h_Q$ induced by the Hermitian fiber metric of $S^\perp \to Gr$.

These vector bundles are all homogeneous vector bundles. We set $\tilde{G} := SU(n)$ and $\tilde{K} := S(U(p) \times U(q))$. Then $Gr_p(\mathbb{C}^n) \cong \tilde{G}/\tilde{K}$. Let $\mathbb{C}^p$ be a $p$-dimensional complex subspace of $\mathbb{C}^n$ such that $\mathbb{C}^p$ is an irreducible representation space of $\tilde{K}$. We denote by $\mathbb{C}^q$ the orthogonal complement space of $\mathbb{C}^p$ in $\mathbb{C}^n$, which is also an irreducible representation space of $\tilde{K}$. Then $S \to Gr$ and $S^\perp \to Gr$ are expressed as the following:

$$S = \tilde{G} \times_{\tilde{K}} \mathbb{C}^p, \quad S^\perp = \tilde{G} \times_{\tilde{K}} \mathbb{C}^q.$$ 

For the exact sequence (3), the inclusion $S \to \mathbb{C}^n$ is expressed as

$$S = \tilde{G} \times_{\tilde{K}} \mathbb{C}^p \ni [g, v] \mapsto ([g], gv) \in \tilde{G}/\tilde{K} \times \mathbb{C}^q = \underline{\mathbb{C}}^n,$$

for $g \in \tilde{G}$ and $v \in \mathbb{C}^p$. Similarly $Q \to Gr$ is regarded as a subbundle of $\underline{\mathbb{C}}^n \to M$:

$$Q \cong S^\perp \ni [g, v] \mapsto ([g], gv) \in \underline{\mathbb{C}}^q,$$

for $g \in \tilde{G}$ and $v \in \mathbb{C}^q$. When we regard $Q \to Gr$ a subbundle of $\underline{\mathbb{C}}^n \to Gr$ as above, the action of $G$ to $Q \to Gr$ is expressed as the following:

$$g \cdot ([\tilde{g}], \tilde{g}v) = ([g], gv), \quad \text{for } g, \tilde{g} \in \tilde{G}, v \in \mathbb{C}^q.$$

Since the holomorphic tangent bundle $T_{1,0}Gr \to Gr$ over $Gr_p(\mathbb{C}^n)$ is identified with $S^* \otimes Q \to Gr$, where $S^* \to Gr$ is the dual bundle of $S \to Gr$, complex manifold $Gr_p(\mathbb{C}^n)$ has a homogeneous Hermitian metric $h_{Gr} := h_S \otimes h_Q$. This is called the Hermitian metric of Fubini-Study type of $Gr_p(\mathbb{C}^n)$ induced by $(\cdot, \cdot)_n$.

**Remark 1.** When we consider the case that $p = n - 1$, $(Gr_{n-1}(\mathbb{C}^n), h_{Gr})$ is the complex projective space with Fubini-Study metric of constant holomorphic sectional curvature 2. (See [3].)

Let $M$ be a compact complex manifold, $V \to M$ a holomorphic vector bundle with a Hermitian fiber metric $h_V$ and $W$ the space of holomorphic sections of $V \to M$. We denote by $(\cdot, \cdot)_W$ the $L_2$-Hermitian inner product of $W$. Let $\tilde{W}$ be a subspace of $W$ and we denote by $ev$ a bundle homomorphism:

$$ev : \tilde{W} := M \times \tilde{W} \to V : (x, t) \mapsto t(x).$$

This is called an *evaluation map*. We assume that the evaluation map $ev$ is surjective. In this case, $V \to M$ is called *globally generated* by $\tilde{W}$. For each $x \in M$ we set the linear map $ev_x : \tilde{W} \to V_x : t \mapsto t(x)$. Since $V \to M$ is globally generated by $\tilde{W}$, $ev_x$ is surjective for each $x \in M$. Thus the dimension of the kernel $\text{Ker} ev_x$ of $ev_x$ are independent of $x$, which is denoted by $p$. Therefore we obtain a holomorphic map

$$f : M \to Gr_p(\tilde{W}) : x \mapsto \text{Ker} ev_x.$$ 

This is called an *induced map* by $(L \to M, \tilde{W})$. When $\tilde{W} = W$, the induced map is called *standard map* by $L \to M$.

On the other hand, let $f : M \to Gr_p(\mathbb{C}^n)$ be a holomorphic map and $f^*Q \to M$ the pull-back vector bundle of $Q \to Gr$ by $f$ with pull-back metric $h_Q$ and connection $\nabla^Q$. We assume that $f^*Q \to M$ is isomorphic to $V \to M$ as a holomorphic

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1In this paper, the complex projective space means the complex Grassmannian manifold $Gr_{n-1}(\mathbb{C}^n)$, not $Gr_1(\mathbb{C}^n)$. 
Hermitian vector bundle. For [6] there exists a semi-positive Hermitian endomorphism $T$ of $W$ such that $f$ is expressed as the following map:

$$f : M \rightarrow Gr_p(W) : x \mapsto T^{-1} \left( f_0(x) \cap (Ker T)^\perp \right),$$

where $T^{-1}$ is the inverse of $T : Ker T^\perp \rightarrow Ker T^\perp$. The semi-positive Hermitian endomorphism $T : W \rightarrow W$ is obtained by the following construction: by Borel-Weil Theory the complex vector space $\mathbb{C}^n$ is regarded as the space of holomorphic sections of $Q \rightarrow Gr_p(\mathbb{C}^n)$. We have a linear map $\iota : \mathbb{C}^n \rightarrow W$ by restricting sections to $M$.

**Definition 2** ([6]). A holomorphic map $f : M \rightarrow Gr_p(\mathbb{C}^n)$ is called full if $\iota : \mathbb{C}^n \rightarrow W$ is injective.

We assume that $f : M \rightarrow Gr_p(\mathbb{C}^n)$ is full. Then $\mathbb{C}^n$ can be considered as a subspace of $W$ by $\iota$. Let $ev_C : \mathbb{C}^n \rightarrow V$ and $ev : W \rightarrow V$ be evaluation maps. Then for any $x \in M$, $Ker ev_C = Ker ev_x \cap \mathbb{C}^n$. Therefore $f : M \rightarrow (Gr_p(\mathbb{C}^n), (,\cdot)_n)$ is expressed as

$$f(x) = Ker ev_x \cap \mathbb{C}^n.$$

The Hermitian inner product $(,\cdot)_n$ is not always coincide with $(,\cdot)_W$. Let $T$ be the positive Hermitian endomorphism of $\mathbb{C}^n$ which satisfies that

$$\langle Tu, Tv \rangle_n = (u, v)_W$$

for any $u, v \in \mathbb{C}^n$. We have an isometry

$$T^{-1} : (Gr_p(\mathbb{C}^n), (,\cdot)_n) \rightarrow (Gr_p(\mathbb{C}^n), (,\cdot)_W) : U \mapsto T^{-1}U.$$

Let $\pi : W \rightarrow \mathbb{C}^n$ be the orthogonal projection onto $\mathbb{C}^n$ with respect to $(,\cdot)_W$. We denote by $T := T \circ \pi$ an endomorphism of $W$, which is semi-positive Hermitian. Consequently $f : M \rightarrow Gr_p(\mathbb{C}^n)$ is expressed as

$$f : M \rightarrow (Gr_p(W), (,\cdot)_W) : x \mapsto T^{-1}(f_0(x) \cap Ker T^\perp).$$

This means that a holomorphic map which has the pull-back bundle $f^*Q \rightarrow M$ isomorphic to $V \rightarrow M$ is expressed a deformation of the standard map induced by $V \rightarrow M$.

Let $M = G/K$ be a compact homogeneous Kähler manifold, $V_0$ an irreducible $K$-representation space and $V = G \times_K V_0 \rightarrow M$ a holomorphic homogeneous vector bundle. We denote by $W$ the space of holomorphic sections of $V$ with $L_2$-Hermitian inner product $(,\cdot)_W$. It follows from Bott-Borel-Weil theory that $W$ is an irreducible $G$-representation space. For $g \in G$ and $t \in W$, the action of $G$ to $W$ is expressed as

$$(g \cdot t)([g_0]) := g(t(g^{-1}[g_0])), \quad g_0 \in G.$$

We assume that the evaluation map $ev : M \times W \rightarrow V$ is surjective.

**Proposition 2** ([6]). $V_0$ is considered as a subspace of $W$.

We set $\pi_0 := ev_{|e} : W \rightarrow V_0$. For $g \in G$ and $t \in W$, we can calculate

$$ev([g], t) = t([g]) = g \cdot g^{-1}t(g \cdot [e])$$

$$= g \cdot ev([e], g^{-1}t) = g(e, \pi_0(g^{-1}t)) = [g, \pi_0(g^{-1}t)].$$

The adjoint map $ev^*$ of $ev$ is expressed as $ev^*([g, v]) = ([g], gv)$.
Let $U_0 := \text{Ker} v_0$. Then $U_0$ is also $K$-representation space and $W$ is orthogonal direct sum of $U_0$ and $V_0$. It follows from (5) that $\text{Ker} v_0 = gU_0$. Therefore the standard map $f_0 : M \to \text{Gr}(W)$ induced by $V \to M$ is that

$$f_0([g]) = gU_0.$$ 

Thus the standard map induced by a holomorphic homogeneous vector bundle is $G$-equivariant.

Let $f : M \to \text{Gr}_p(\mathbb{C}^n)$ be a full holomorphic map such that $f^*Q \to M$ is holomorphic isomorphic to $V \to M$. Then there exists a semi-positive Hermitian endomorphism $T : W \to W$ such that the map $f$ is expressed as the following:

$$f : M \to (\text{Gr}_p(\mathbb{C}^n), \langle \cdot, \cdot \rangle_W) : [g] \mapsto T^{-1}(gU_0 \cap (\text{Ker} T)^\perp).$$

Pulling the sequence (3) back, we have a short exact sequence:

$$0 \to f^*S \to \mathbb{C}^n := M \times \mathbb{C}^n \to f^*Q \to 0. \tag{10}$$

The vector bundle $f^*Q \to M$ is regarded as a subbundle of $\mathbb{C}^n \to M$, which is orthogonal complement bundle of $f^*S \to M$:

$$f^*Q = \{(|[g], v) \in \mathbb{C}^n | f([g]) \perp v\}.$$ 

It follows from (5) that we have

$$0 = (v, T^{-1}(gU_0 \cap \mathbb{C}^n))_W = (g^{-1}T^{-1}v, U_0 \cap \mathbb{C}^n)_W.$$ 

Then we have $g^{-1}T^{-1}v \in V_0 \iff v \in TgV_0$. Therefore the isomorphism $\phi : V \to f^*Q \subset \mathbb{C}^n$ is expressed as the following:

$$[g, v] \mapsto ([g], Tgv), \quad \text{for } g \in G, \ v \in V_0. \tag{11}$$

3. Main Theorem and its Proof

Let $M$ be a compact simply connected homogeneous Kähler manifold, $G$ the isometry group of $M$ and $K$ an isotropy subgroup of $G$. Let $f : M \to \text{Gr}_p(\mathbb{C}^n)$ be a full holomorphic strongly projectively flat map. By definition of strongly projectively flatness there exists a Hermitian line bundle $L \to M$ such that $f^*Q \to M$ is isomorphic to $\tilde{L} \to M$ as a Hermitian vector bundle, where $\tilde{L} \to M$ is orthogonal direct sum of $q$-copies of $L \to M$. Since $M$ is a compact simply connected homogeneous Kähler, $L \to M$ is homogeneous. We set $L_0$ the $1$-dimensional $K$-representation space such that $L = G \times_K L_0$. Then we have

$$\tilde{L} = L \oplus \cdots \oplus L = G \times_K (L_0 \oplus \cdots \oplus L_0) = G \times_K \tilde{L}_0,$$

where $\tilde{L}_0$ is $q$-orthogonal direct sum of $L_0$. We denote by $W$ and $\tilde{W}$ the spaces of holomorphic sections of $L \to M$ and $\tilde{L} \to M$ respectively and we set $N$ the dimension of $W$. By definition of $\tilde{L} \to M$, $\tilde{W}$ is regarded as $q$-orthogonal direct sum of $W$. Let $\pi_j : \tilde{W} \to W$ be the orthogonal projection onto the $j$-th component of $W$. It follows from Proposition (2) that $L_0$ is a subspace of $W$ and $\tilde{L}_0$ is a subspace of $\tilde{W}$ as a $K$-representation space. When we restrict $\pi_j$ to $\tilde{L}_0$, $\pi_j|_{\tilde{L}_0}$ is orthogonal projection of $\tilde{L}_0$ onto the $j$-th component of $\tilde{L}_0$. We denote by

$$ev : M \times W \to L, \quad \tilde{ev} : M \times \tilde{W} \to \tilde{L}$$
the evaluation maps respectively and by \( f_0 : M \rightarrow Gr_{N-1}(W) \) and \( \tilde{f}_0 : M \rightarrow Gr_{q(N-1)}(\tilde{W}) \) the standard maps induced by \( L \rightarrow M \) and \( \tilde{L} \rightarrow M \) respectively. Since \( \tilde{W} \) is orthogonal direct sum of \( q \)-copies of \( W \), we have

\[
\begin{align*}
\tilde{e}v([g], t) &= \tilde{e}v([g], t_1 + \cdots + t_q) \\
&= ev([g], t_1) + \cdots + ev([g], t_q) \in L + \cdots + L,
\end{align*}
\]

where \( t = t_1 + \cdots + t_q \) is orthogonal decomposition with respect to \( \tilde{W} = W + \cdots + W \). We set \( U_0 := \text{Ker}ev_{[g]} \). Then it follows from (11) that \( f_0([g]) = gU_0 \). It follows from (12) that the map \( \tilde{f}_0 \) is expressed as

\[
\tilde{f}_0 : M \rightarrow Gr_{N-1}(W) \times \cdots \times Gr_{N-1}(W) \rightarrow Gr_{q(N-1)}(\tilde{W}),
\]

\[
[g] \rightarrow (gU_0, \cdots, gU_0) \rightarrow gU_0 \oplus \cdots \oplus gU_0 = g \cdot (U_0 + \cdots + U_0).
\]

Since \( f : M \rightarrow Gr_p(\mathbb{C}^n) \) is full, \( \mathbb{C}^n \) is a subspace of \( \tilde{W} \). It follows from the previous section that there exists a semi-positive Hermitian endomorphism \( T : \tilde{W} \rightarrow \tilde{W} \) and a bundle isomorphism \( \phi : \tilde{L} \rightarrow f^*Q \) such that maps \( f : M \rightarrow Gr_p(\mathbb{C}^n) \) and \( \phi : \tilde{L} \rightarrow f^*Q \) is expressed as the following:

\[
\begin{align*}
f([g]) &= T^{-1} \left( \tilde{f}_0([g]) \cap (\text{Ker}T)^\perp \right), \\
\phi([g, v]) &= ([g], Tgv),
\end{align*}
\]

for \( g \in G \) and \( v \in \tilde{L}_0 \).

By using the above notations, we rewrite Main theorem as followings:

**Theorem 2.** Let \( M \) be a compact simply connected homogeneous Kähler manifold and \( G \) the isometry group of \( K \). Let \( f : M \rightarrow Gr_p(\mathbb{C}^n) \) be a full holomorphic strongly projectively flat map into the complex Grassmannian manifold. Then \( f \) is \( G \)-equivariant if and only if \( f \) is the standard map.

In order to prove this theorem, it is sufficient to show that the Hermitian endomorphism \( T : \tilde{W} \rightarrow \tilde{W} \) is the identity map of \( \tilde{W} \).

From now on, we assume that \( f : M \rightarrow Gr_p(\mathbb{C}^n) \) is \( G \)-equivariant. Then there exists a Lie group homomorphism \( \rho : G \rightarrow SU(n) \) which satisfy the following equation:

\[
f(g\tilde{g}) = \rho(g)f(\tilde{g}), \quad g, \tilde{g} \in G.
\]

By definition \( \mathbb{C}^n \) is \( G \)-representation space and a vector subspace of \( \tilde{W} \).

**Lemma 1.** \( f^*Q \rightarrow M \) is homogeneous.

**Proof.** The definition of the pull-back bundle \( f^*Q \rightarrow M \) is that

\[
f^*Q = \{ ([g], v) \in M \times Q | f([g]) = \pi(v) \},
\]

where \( \pi : Q \rightarrow Gr_p(\mathbb{C}^n) \) is the natural projection. For any \( ([\tilde{g}], v) \in f^*Q \) and \( g \in G \), we have an action of \( G \) to \( f^*Q \rightarrow M \) by

\[
g \cdot ([\tilde{g}], v) = (g[\tilde{g}], \rho(g)v).
\]

Since \( G \) acts to \( M \) transitively, \( f^*Q \rightarrow M \) is homogeneous. \( \square \)

Since \( f^*Q \rightarrow M \) is homogeneous, the space of holomorphic sections of \( f^*Q \rightarrow M \) is \( G \)-representation space. Let \( t \) be a holomorphic section of \( f^*Q \rightarrow M \). For \( g \in G \) and \( x \in M \), we have

\[
(g \cdot t)(x) = g \left( t(g^{-1}x) \right).
\]
For \( t \in \mathbb{C}^n \), we obtain a holomorphic section of \( f^*Q \to M \) which is expressed as
\[
    t(x) = (x, t(f(x))), \quad \text{for } x \in M.
\]
Thus for \( g, \tilde{g} \in G \) we obtain
\[
(g \cdot t)(x) = g(t(g^{-1}x)) = g(g^{-1}x, t(f(g^{-1}x))),
\]
where
\[
= (x, \rho(g)t(\rho(g^{-1})f(x))) = (x, (\rho(g)t)(f(x))),
\]
\[
= (\rho(g)t)(x).
\]
Therefore \( \mathbb{C}^n \) is a \( G \)-representation subspace of the space of holomorphic sections of \( f^*Q \to M \).

**Lemma 2.** The holomorphic isomorphism \( \phi : \tilde{L} \to f^*Q \) is \( G \)-equivariant.

**Proof.** At first we show that \( f^*Q \to M \) is isomorphic to \( \tilde{L} \) as a homogeneous vector bundle. Since \( \phi \) preserves Hermitian connections, bundles \( \tilde{L} \to M \) and \( f^*Q \to M \) have same holonomy groups and \( \phi \) is holonomy equivariant. Since the action of \( K \) to \( f^*Q \to M \) and \( L = M \to M \) at \([e]\), where \( e \) is the unit element in \( G \), is expressed as an action of the holonomy group, \( \phi \) is \( K \)-equivariant. Thus \( f^*Q_\chi \) is isomorphic to \( L_0 \oplus \cdots \oplus L_0 \) as a \( K \)-representation space. Therefore \( f^*Q \to M \) is isomorphic to \( \tilde{L} \to M \) as a homogeneous vector bundle.

Finally we show that a holomorphic isomorphism \( \phi : L \to \tilde{L} \) is \( G \)-equivariant. We denote by \( \tilde{L} \overset{\cong}{=} L_1 \oplus \cdots \oplus L_q \) and \( L_j = G \times_K L_{(j)}, \) where \( L_j \to M \) is the \( j \)-th component and \( L_{(j)} \) is isomorphic to \( L_0 \) as a \( K \)-representation space for \( j = 1, \cdots, q \). Then we have
\[
   \tilde{L} = G \times_K (L_{(1)} \oplus \cdots \oplus L_{(q)}).
\]

Let \( \phi_j : L_j \to \tilde{L} \) be the restriction of \( \phi : \tilde{L} \to \tilde{L} \) to \( L_j \to M \). Then \( \phi_j \) is expressed as the following:
\[
\phi_j([g, v]) = [g, \varphi_1(g)(v) \oplus \cdots \oplus \varphi_q(g)(v)], \quad \text{for } g \in G, v \in L_{(j)},
\]
where \( \varphi_i : L_{(j)} \to L_{(i)} \) is a linear map for \( i = 1, \cdots, q \). Since \( L_{(i)} \) and \( L_{(j)} \) are isomorphic 1-dimensional \( K \)-representation spaces, there exists a complex number \( \alpha_i(g) \) such that \( \varphi_i(g)(v) = \alpha_i(g)v \). Thus we have
\[
\phi_j([g, v]) = [g, \alpha_1(g)v \oplus \cdots \oplus \alpha_q(g)v], \quad \text{for } g \in G, v \in L_{(j)}.
\]

Since \( \phi_j \) is a bundle homomorphism, we obtain
\[
\phi_j([gk, kv]) = [gk, \alpha_1(gk)v \oplus \cdots \oplus \alpha_q(gk)v] = [g, \alpha_1(g)kv \oplus \cdots \oplus \alpha_q(g)kv],
\]
\[
\phi_j([g, kv]) = [g, \alpha_1(g)kv \oplus \cdots \oplus \alpha_q(g)kv],
\]
for \( g \in G, k \in K \) and \( v \in L_{(j)} \). It follows that \( \alpha_i(gk) = \alpha_i(g) \) for \( i = 1, \cdots, q, \ g \in G \) and \( k \in K \). Therefore \( \alpha_i \) is a complex valued function on \( G/K \). Since \( \phi_j \) is holomorphic, so is \( \alpha_i \) for \( i = 1, \cdots, q \), which implies that \( \alpha_i \) is a constant function for each \( i \) because \( G/K \) is compact. We regard \( \alpha_i \) as a complex number. Then we have
\[
\phi_j([g, v]) = [g, \alpha_1v \oplus \cdots \oplus \alpha_qv], \quad \text{for } g \in G, v \in L_{(j)}.
\]
This is \( G \)-equivariant for each \( j = 1, \cdots, q \). Consequently \( \phi : \tilde{L} \to \tilde{L} \) is \( G \)-equivariant.

It follows from Lemma 1 and Lemma 2 that \( \mathbb{C}^n \) is a \( G \)-representation subspace of \( \mathbb{C}^n \).
Lemma 3. The semi-positive Hermitian endomorphism $T : \tilde{W} \rightarrow \tilde{W}$ is $G$-equivariant.

Proof. Since $\phi : \tilde{L} \rightarrow f^*Q$ is $G$-equivariant, it follows from [1] and [15] that we can calculate

\[
\phi(g_1 \cdot [g_2, v]) = \phi([g_1 g_2, v]) = ([g_1 g_2], Tg_1 g_2 v),
\]
\[
\phi(g_1 \cdot [g_2, v]) = g_1 \cdot \phi([g_2, v]) = g_1 \cdot ([g_2], Tg_2 v) = ([g_1 g_2], g_1 Tg_2 v).
\]

Therefore we have

\[Tgv = gTv, \quad \text{for } g \in G, v \in \tilde{L}_0.\]

We denote by $GL_0$ a subspace of $W$ spanned by $gv$ for any $g \in G$ and $v \in L_0$ and similarly we denote by $GL_0$. Then $GL_0$ is regarded as $q$-orthogonal direct sum of $GL_0$. $GL_0$ is a $G$-representation subspace of $W$. Since $W$ is irreducible and $GL_0$ is not empty, we obtain $W = GL_0$. Consequently, we obtain $W = GL_0$. It follows that for any $w \in \tilde{W}$ there exists $\alpha_i \in \mathbb{C}$, $g_i \in G$ and $v_i \in \tilde{L}_0$ such that $w = \sum \alpha_i g_i v_i$, where the right hand side of this equation is a finite sum. For any $g \in G$, we have

\[Tgw = Tg \sum \alpha_i g_i v_i = \sum Tg\alpha_i g_i v_i = \sum gT\alpha_i g_i v_i = gT \sum \alpha_i g_i v_i = gT w.\]

Therefore $T$ is $G$-equivariant. \[\square\]

Since $T : \tilde{W} \rightarrow \tilde{W}$ is $G$-equivariant, $T$ is also $K$-equivariant.

Lemma 4.

\[T(\tilde{L}_0) \subset \tilde{L}_0.\]

Proof. Since the orthogonal projection $\pi_j : \tilde{W} \rightarrow W$ is $K$-equivariant for each $j = 1, \cdots, q$, $\pi_j \circ T : \tilde{W} \rightarrow W$ is a $K$-equivariant endomorphism. Thus $\pi_j \circ T(\tilde{L}_0) \subset \tilde{L}_0$ is a $K$-representation subspace of $W$. It follows from Schur’s lemma and Borel-Weil theory that $\pi_j \circ T(L_0) \subset \tilde{L}_0$. Consequently $T(L_0) \subset (L_0)$. \[\square\]

We denote by the same notation $T : \tilde{L}_0 \rightarrow \tilde{L}_0$ the restriction of $T : \tilde{W} \rightarrow \tilde{W}$ to $\tilde{L}_0$.

Theorem 3. The endomorphism $T : \tilde{W} \rightarrow \tilde{W}$ is the identity map.

Proof. Since the bundle isomorphism $\phi : \tilde{L} \rightarrow f^*Q$ preserves fiber metrics and $T$ is Hermitian, we have

\[(v_1, v_2)_{\tilde{L}_0} = ([e, v_1], [e, v_2])_{\tilde{L}} = ([e, Tv_1], [e, Tv_2])_{\tilde{L}} = (Tv_1, Tv_2)_{\tilde{L}_0} = (T^2 v_1, v_2)_{\tilde{L}_0},\]

for any $v_1, v_2 \in \tilde{L}_0$. Therefore $T^2 : \tilde{L}_0 \rightarrow \tilde{L}_0$ is the identity map. Since $W$ is $G$-irreducible and $T$ is $G$-equivariant, $T^2 : \tilde{W} \rightarrow \tilde{W}$ is the identity map and so is $T$ because $T$ is semi-positive Hermitian. \[\square\]

Consequently, a holomorphic strongly projectively flat $G$-equivariant map $f : M \rightarrow Gr_p(\mathbb{C}^n)$ is the standard map induced by a $q$-orthogonal direct sum bundle of Hermitian line bundle $L \rightarrow M$. 

References

[1] E. Calabi, Isometric imbeddings of complex manifolds, Ann. of Math., 58 (1953), 1–23.
[2] S. Kobayashi, Differential geometry of Complex Vector Bundles, Iwanami Shoten and Princeton University, Tokyo (1987).
[3] ————, On compact Kähler manifolds with positive definite Ricci tensor, Ann. of Math. (2), 74 (1961), 570–574.
[4] I. Koga, Projectively flat immersions of Hermitian symmetric spaces of compact type, accepted.
[5] I. Koga, Y. Nagatomo, A Study of Submanifolds of the Complex Grassmannian Manifold with Parallel Second Fundamental Form, accepted.
[6] Y. Nagatomo, Harmonic maps into Grassmannian manifolds, a preprint.
[7] M. Takeuchi Homogeneous Kähler submanifolds in complex projective spaces, Japan. J. Math. 4 (1977), 171–219.

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