Semiclassical Meson-Baryon Dynamics from Large-\(N_c\) QCD

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Abstract

The large-\(N_c\) limit of the meson-baryon effective Lagrangian is shown to reduce to a semiclassical field theory. A chiral bag structure emerges naturally in the \(N_c \to \infty\) limit. A possible connection between the chiral bag picture and the Skyrme model is discussed. The classical meson-baryon theory is used to reproduce the \(M_\pi^3\) non-analytic correction to the baryon mass obtained previously as a loop correction in chiral perturbation theory.
Quantum chromodynamics has an expansion in \(1/N_c\), where \(N_c\) is the number of colors. This expansion was originally proposed by ’t Hooft \(1\), who used it to solve for the exact meson spectrum in 1+1 dimensions. The baryon sector of QCD in the large-\(N_c\) limit was investigated by Witten \(2\). Recently, progress has been made in calculating baryon properties in a systematic expansion in \(1/N_c\) \(3\)\(4\). The baryon sector of QCD has a contracted \(SU(6)\) spin-flavor symmetry in the \(N_c \to \infty\) limit \(5\)\(3\). The form of the spin-flavor symmetry violation away from \(N_c = \infty\) can be classified in terms of \(1/N_c\) suppressed operators. Results can be obtained for baryons to all orders in \(SU(3)\) breaking \(6\). A complementary approach to the one developed in refs. \(3\)\(4\)\(6\) has been used in refs. \(7\)\(8\)\(9\). The results obtained using the \(1/N_c\) expansion in QCD are closely related to earlier results obtained using the Skyrme model \(10\) \(11\) \(12\).

Meson-baryon interactions at low momentum transfer can be described by an effective Lagrangian. In this paper, we will show how the effective Lagrangian in the \(N_c \to \infty\) limit can be treated as a classical field theory which is similar to the chiral bag picture of hadrons \(13\). This connection was also discussed previously by Gervais and Sakita \(3\). The classical theory will be used to reproduce some known results obtained previously using chiral perturbation theory. In particular, we will show how the classical field theory sums all the non-analytic chiral corrections which are of leading order in \(1/N_c\).

The \(N_c\) counting rules imply that in general, three-meson couplings are of order \(1/\sqrt{N_c}\), four meson couplings are of order \(1/N_c\), etc. Thus the meson self-couplings are described by an effective Lagrangian of the form

\[
\mathcal{L}_M = N_c \mathcal{L}^{(M)} \left( \left\{ \frac{M_i}{\sqrt{N_c}} \right\} \right),
\]

where \(\{M_i\}\) is the set of meson fields. Meson-baryon couplings are of order \(\sqrt{N_c}\), two-meson–baryon couplings are of order one, etc., so the meson-baryon interaction Lagrangian can be written as

\[
\mathcal{L}_B = N_c \mathcal{L}^{(B)} \left( \left\{ \frac{M_i}{\sqrt{N_c}} \right\}, B \right),
\]

where \(B\) is the baryon field. In the large \(N_c\) limit, the baryon is infinitely heavy and can be treated as a static source localized at the origin. The contracted \(SU(4)\) symmetry of large-\(N_c\) QCD implies that there is an infinite degenerate baryon tower of states with \(I = J = 1/2, 3/2, \ldots\) \(5\)\(3\). It is more convenient to represent this infinite tower of baryon states by a collective coordinate \(A\) from which the \(I = J\) baryons can be obtained by projection \(10\) \(11\) \(12\). Since the baryon mass splittings are order \(1/N_c\) \(4\), the collective...
coordinate $A$ does not evolve with time at leading order in $1/N_c$. Thus the large-$N_c$ baryons can be replaced by a classical source with collective coordinate $A$ located at $\vec{x} = \vec{x}_0$. The interaction Lagrangian eq. (2) can be written as

$$\mathcal{L}_A = N_c \mathcal{L}^{(A)} \left( \left\{ \frac{M_i}{\sqrt{N_c}} \right\}, A, \vec{x}_0 \right),$$

where $A$ is a time independent classical field. The total Lagrangian for meson-baryon interactions is

$$\mathcal{L} = N_c \left[ \mathcal{L}^{(M)} (\{M_i\}) + \mathcal{L}^{(A)} (\{M_i\}, A, \vec{x}_0) \right] = N_c \mathcal{L}^{(M)} (\{M_i\}, A, \vec{x}_0),$$

on rescaling the meson fields by $\sqrt{N_c}$. The functional integral is performed over the meson fields $M_i$ (but not over the time-independent collective coordinate $A$),

$$Z = \int \{DM_i\} e^{iN_c \mathcal{S}/\hbar},$$

where

$$\mathcal{S} = \int d^4x \mathcal{L}^{(M)} (\{M_i\}, A, \vec{x}_0).$$

The functional integral in eq. (5) needs to be regulated. The effective meson-baryon Lagrangian is valid for momentum scales small compared to $\Lambda_\chi \sim 1$ GeV. The functional integral eq. (5) can be regulated by using $\Lambda_\chi$ as a cutoff. The details of the cutoff procedure are unimportant; what is important is that the cutoff $\Lambda_\chi$ is $N_c$-independent so that there is no hidden $N_c$ dependence in the functional measure. There is an overall factor of $N_c/\hbar$ in front of the entire action, so the $1/N_c$ expansion is equivalent to the semiclassical expansion in powers of $\hbar$. The leading term is the solution of the classical equations of motion of eq. (4). This produces a baryon source $A$ with a classical meson cloud, and is closely related to the chiral bag model. The quantum corrections are obtained by performing a semiclassical expansion about the classical meson background, and including time-dependence in the baryon collective coordinate $A$. Note that the infinite tower of baryon states is crucial for a classical representation of baryons. A single $I = J = 1/2$ state cannot be treated as a classical object.

We will now study a specific example of the meson-baryon effective Lagrangian in more detail. Consider the interaction of pions with the $I = J = 1/2, 3/2, \ldots$ baryon

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1 The connection between the sum of large-$N_c$ diagrams and the semiclassical approximation was proven by Arnold and Mattis for a certain class of diagrams [14].
tower to lowest order in the chiral expansion for two light flavors, and in the isospin limit \( m_u = m_d = m_q \). The pion chiral Lagrangian is

\[
\mathcal{L}^{(M)} = \frac{f_\pi^2}{4} \text{Tr} \partial_\mu U \partial^\mu U^\dagger + \frac{M_\pi^2 f_\pi^2}{4} \text{Tr} [U + U^\dagger],
\]

(7)

where \( M_\pi \) is the pion mass, \( U \) is the exponential of the pion field

\[
U = e^{2i\pi/f_\pi},
\]

(8)

and \( f_\pi \approx 93 \text{ MeV} \) is the pion decay constant. The lowest order pion-baryon coupling is

\[
\mathcal{L}^{(A)} = 3N_c g \delta(\vec{x}) \ X^{ia} A^{ia}(x),
\]

(9)

where \( X^{ia} \) is the baryon axial current in the large-\( N_c \) limit,

\[
X^{ia} = \text{Tr} A^a A^{-1} \tau^i,
\]

(10)

\( A^{ia} \) is proportional to the pion axial current,

\[
A^{ia} = \frac{i}{4} \text{Tr} T^a \left(U \nabla^i U^\dagger - U^\dagger \nabla^i U\right),
\]

(11)

and the static baryon is located at \( \vec{x} = 0 \). The baryon axial vector current \( g_A \) is of order \( N_c \), so we have written \( g_A \equiv N_c g \). The factor of three in eq. (9) ensures that the axial coupling of the \( I = J = 1/2 \) nucleon is \( g_A \) when we convert from the collective coordinate basis to the \( I, J \) basis \([10]\). Expanding the interaction term eq. (9) to first order in the pion field gives the pion-baryon vertex

\[
\frac{3N_c g}{2f_\pi} \ X^{ia} \nabla^i \pi^a = \frac{3g_A}{2f_\pi} \ X^{ia} \nabla^i \pi^a.
\]

(12)

The leading non-analytic correction to the baryon mass due to pion-baryon interactions is from the graph of fig. 1, and can be computed in chiral perturbation theory using the interaction vertex eq. (12). The mass shift for the spin-1/2 nucleon from fig. 1 is a standard computation using baryon chiral perturbation theory with a static baryon \([15]\) including intermediate nucleon and delta states \([16]\), with \( g_{\pi NN}/g_{\pi N\Delta} \) in the ratio given by \( SU(4) \) spin-flavor symmetry,

\[
\delta m = - \left( \frac{3}{2} + 3 \right) \frac{g_A^2 M_\pi^3}{16\pi f_\pi^2} = - \frac{9g^2 N_c^2 M_\pi^3}{32\pi f_\pi^2}.
\]

(13)
The two coefficients in eq. (13) are the intermediate nucleon and delta contributions, respectively. Note that \( \delta m \) is of order \( N_c \), since \( f_\pi \propto \sqrt{N_c} \), and \( g \) and \( M_\pi \) are of order one. The mass-shift in eq. (13) is non-analytic in the quark mass \( m_q \) (\( \delta m \propto m_q^{3/2} \)), since \( M_\pi \propto m_q^{1/2} \).

The general form of the baryon masses in the \( 1/N_c \) expansion is [4]

\[
m = N_c \, m_0 + m_1 + m_2 \frac{J^2}{N_c} + O \left( \frac{1}{N_c^2} \right),
\]

(14)

where \( m_i \) are arbitrary functions of the quark mass \( m_q \). Eq. (13) is an order \( N_c \) loop correction which does not violate the general form for the baryon masses given in eq. (14). It produces a \( m_q^{3/2} \) non-analytic correction that is the same for the entire baryon tower, and so can be absorbed into \( m_0 \). The numerical magnitude of the \( m_q^{3/2} \) correction due to pion loops is small. However, the corresponding \( m_q^{3/2} \) correction due to kaon loops is naively of order 1 GeV, which is comparable to \( m_0 \) and cannot be treated as a small perturbation. It is therefore useful to have a calculational method that directly sums all the order \( N_c \) loop corrections, so that the remaining corrections are suppressed by \( 1/N_c \), and are small. This can be done using the semiclassical method outlined above.

We now compute the baryon mass correction due to pion interactions using the semiclassical approximation, which sums all the order \( N_c \) terms. The classical field equations are solved for a fixed value of the baryon collective coordinate \( A \). It is convenient to choose \( A = 1 \), so that \( X^{ia} \) in eq. (10) is \( \delta^{ia} \). The solutions for other values of \( A \) can be obtained trivially by an isospin rotation on the solution for \( A = 1 \). The Lagrangian for \( A = 1 \) is

\[
L^{(M)} = \frac{f_\pi^2}{4} \, \text{Tr} \, \partial_\mu U \partial^\mu U^\dagger + \frac{M_\pi^2 f_\pi^2}{4} \, \text{Tr} \left[ U + U^\dagger \right] + 3g_A \, \delta (\vec{x}) \, \delta^{ia} \, A^{ia},
\]

(15)

which is of order \( N_c \) since \( f_\pi \propto \sqrt{N_c} \) and \( g_A \propto N_c \). The classical equations of motion are those for a pion field with a \( \delta \)-function source at the origin. It is clear from the form of the source term that the pion field has the hedgehog form,

\[
U = e^{i \vec{r} \cdot \vec{x} F(r)},
\]

(16)

for some function \( F(r) \). The equations of motion for \( F(r) \) for a static classical pion field are obtained by minimizing the energy functional

\[
E = 2\pi f_\pi^2 \int_0^\infty dr \, r^2 \left[ \left( \frac{\partial F}{\partial r} \right)^2 + 2 \frac{\sin^2 F}{r^2} + 2M_\pi^2 \left( 1 - \cos F \right) \right] - \frac{3}{2}g_A \int d^3 \vec{x} \, J(\vec{x}) \left[ \frac{\partial F}{\partial r} + 2 \frac{\sin F \cos F}{r} \right],
\]

(17)
where we have smeared out the baryon source \( \delta(x) \) into \( J(|\vec{x}|) \) (for reasons which will become clear). The source is normalized to unit baryon number,

\[
\int d^3\vec{x} \ J(|\vec{x}|) = 1. \tag{18}
\]

The equation of motion obtained by varying eq. (17) is

\[
\frac{\partial}{\partial r} \left( r^2 \frac{\partial F}{\partial r} \right) - \sin 2F - M^2_\pi r^2 \sin F = \frac{3 g_A}{2 f_\pi^2} \left[ \frac{\partial}{\partial r} \left( r^2 J(r) \right) - 2 J(r) \ r \cos 2F \right]. \tag{19}
\]

This equation was also considered by Gervais and Sakita in their study of the chiral bag \cite{GervaisSakita}. It is convenient to choose the baryon source \( J(r) \) to be

\[
J(r) = \begin{cases} J_0 & r \leq R_0, \\ 0 & r > R_0. \end{cases} \tag{20}
\]

with

\[
\frac{4}{3} \pi R_0^3 J_0 = 1, \tag{21}
\]

so that the \( \delta \)-function is recovered in the limit \( R_0 \to 0 \). Far away from the baryon source, the pion fields are weak, and eq. (19) can be approximated by keeping the lowest order terms in \( F \),

\[
r^2 \frac{d^2 F}{dr^2} + 2r \frac{dF}{dr} - \left( 2 + M^2_\pi r^2 \right) F = \frac{3 g_A}{2 f_\pi^2} \ r^2 \frac{dJ}{dr}, \tag{22}
\]

and setting \( J = 0 \). The solution that is regular at infinity is

\[
F(r) = a \ k_1 (M_\pi r), \tag{23}
\]

where \( a \) is an overall normalization constant, and \( k_1 \) is a spherical Bessel function. The asymptotic form of eq. (23) is

\[
F(r) \sim \frac{e^{-M_\pi r}}{r},
\]

which produces an exponentially decaying pion tail at infinity.

The chiral Lagrangian is an expansion in derivatives over the chiral symmetry breaking scale \( \Lambda_\chi \sim 1 \) GeV, which is held fixed as \( N_c \to \infty \). The chiral Lagrangian can be used for computing processes in which the momentum transfer \( p \ll \Lambda_\chi \). The \( \delta \)-function baryon source is regarded as “pointlike” on the scale of the typical momentum transfer \( M_\pi \), but slowly varying on the scale \( \Lambda_\chi \). In other words, one considers the baryon source to be smeared over a region \( R_0 \), with \( M_\pi \ll R_0^{-1} \ll \Lambda_\chi \). \( R_0^{-1} \) is a cutoff on the pion-baryon...
interaction vertex. At distances much shorter than $R_0$, the pion-baryon interaction is no longer pointlike. Chiral perturbation theory corresponds to computing in the limit $M_\pi \ll R_0^{-1} \ll \Lambda_X$, and treating $R_0^{-1}$ as a cutoff which is removed at the end of the calculation.

In the limit $M_\pi \ll R_0^{-1} \ll \Lambda_X$, the function $F$ is obtained for all values of $r$ using the linear approximation to eq. (19). The solution of eq. (22) is easy to obtain since the inhomogeneous term on the right hand side is non-zero only at $r = R_0$. The answer is

$$F(r) = \begin{cases} a \, k_1(M_\pi r) & r > R_0, \\ b \, i_1(M_\pi r) & r < R_0, \end{cases} \quad (24)$$

where $a$ and $b$ are determined from the boundary condition at $r = R_0$ to be

$$a = \frac{3g_A}{2f_\pi} J_0 M_\pi R_0^2 i_1(M_\pi R_0),$$

$$b = \frac{3g_A}{2f_\pi} J_0 M_\pi R_0^2 k_1(M_\pi R_0). \quad (25)$$

The mass shift of the baryon due to the pion cloud is obtained by substituting eqs. (24) and (25) into the energy functional, eq. (17). The resulting expression includes all corrections of order $N_c$ to the baryon energy, including any dependence on $M_\pi$ that is of order $N_c$,

$$\delta m = \frac{81 g_A^2}{64\pi f_\pi^2 M_\pi^3 R_0^6} \left[ (1 + MR_0) \left[ (1 - MR_0) - (1 + MR_0) e^{-2MR_0} \right] \right]. \quad (26)$$

Expanding in powers of the cutoff $R_0^{-1}$, one obtains

$$\delta m = -\frac{27 g_A^2}{32\pi f_\pi^2 R_0^3} + \frac{27 g_A^2 M_\pi^2}{80\pi f_\pi^2 R_0^2} - \frac{9 g_A^2 M_\pi^3}{32\pi f_\pi^2} + \ldots. \quad (27)$$

The first term in eq. (27) is a cutoff dependent shift in the baryon mass that is independent of $M_\pi$, and can be absorbed into the chiral invariant baryon mass term in the chiral Lagrangian. The second term in eq. (27) is a cutoff dependent term that is of order $M_\pi^2$, and hence is analytic in the quark masses. It can be reabsorbed into the baryon mass term in the chiral Lagrangian that is linear in the quark mass matrix. There is no term of order $M_\pi/R_0^2$ in eq. (27). Such a term would require a non-analytic counterterm in the chiral Lagrangian, which does not exist. Note that there are also no terms in eq. (27) which are proportional to inverse powers of $M_\pi$. The third term in eq. (27) is of order $M_\pi^3 \propto m_q^{3/2}$ and is non-analytic in the quark masses, so it cannot be absorbed into a local Lagrangian counterterm. The third term is finite, independent of the cutoff $R_0$, and
reproduces correctly the known result eq. (13) obtained using chiral perturbation theory.\footnote{2} The higher order terms in eq. (27) vanish as the cutoff is removed \((R_0^{-1} \to \infty)\). One can sum all the leading (in \(N_c\)) non-analytic corrections to the baryon mass to a given order in the chiral expansion, if one retains the corresponding terms in the derivative expansion in the original Lagrangian, eq. (15).

We have shown above that the large-\(N_c\) limit of QCD in the baryon sector reduces to a classical field theory for mesons coupled to a static source. The pion cloud has a hedgehog form when the baryon is chosen to be in a state of definite collective coordinate \(A\), rather than a state of definite spin and isospin. In general, there will also be a cloud of other mesons (such as the \(\rho\)) around the baryon, if these mesons are included in the Lagrangian. Thus the large-\(N_c\) limit is exactly equivalent to a chiral bag picture of baryons \footnote{3} Note that this identification depends on the \(1/N_c\) expansion, but does not require the chiral limit. We have also seen how the classical equations correctly reproduce the order \(N_c\) parts of the loop diagrams in the standard chiral perturbation theory approach.

In general, the results we have obtained depend on the smearing radius \(R_0\) of the baryon source. This can be eliminated by taking \(R_0 \to 0\) while at the same time including all the higher derivative terms in the chiral lagrangian. For example, form factor effects are included via such higher derivative terms. As \(r \to 0\), the pion field \(F(r)\) becomes stronger, and the higher order and non-linear terms in the chiral lagrangian become important. It is also necessary to renormalize the coupling \(g_A\) in the Lagrangian eq. (9) so that the physical pion-baryon coupling remains finite.\footnote{4} The general solution of the full non-linear theory (including higher derivative terms not included in eq. (15)) is complicated. It would be extremely interesting if one could show that as \(R_0 \to 0\), \(F(r) \to -\pi\) near \(r = 0\). In that case, one could eliminate the central core since \(g_A^{\text{bare}} \to 0\), and since the entire baryon number is carried by the pion cloud \footnote{3}. The remaining object is a baryon whose properties are completely determined by the meson Lagrangian—the Skyrme soliton \footnote{18} \footnote{11}. Whether this possibility holds is being investigated further.

\footnote{2} The first two terms are not present in the chiral perturbation theory calculation \footnote{4} \footnote{16} because they vanish in dimensional regularization.

\footnote{3} The pion cloud couples to the source only at \(r = R_0\) with the special choice of \(J(r)\) in eq. (20). In general, the pions will couple over the entire region where the source is non-zero.

\footnote{4} The physical pion-baryon coupling is determined by the coefficient of \(e^{-M_\pi r}/r\) in the pion tail at \(r = \infty\). \(g_A\) in the Lagrangian must vanish as \(R_0 \to 0\) so that the physical pion-baryon coupling remains finite.
Acknowledgments

A recent paper by Dorey, Hughes and Mattis [19] discusses ideas that are very closely related to those discussed here, and arrives at similar conclusions. I would like to thank M.P. Mattis for discussing their work prior to publication, and for comments on this manuscript. I would also like to thank E. Jenkins for helpful discussions.

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Figure Captions

Fig. 1. One loop correction to the baryon mass.
