New approach to the $k$-independence number of a graph

Yair Caro  
Dept. of Mathematics and Physics  
University of Haifa-Oranim  
Tivon 36006, Israel  
yacaro@kvgeva.org.il

Adriana Hansberg  
Dep. de Matemàtica Aplicada III  
UPC Barcelona  
08034 Barcelona, Spain  
adriana.hansberg@upc.edu

Abstract

Let $G = (V, E)$ be a graph and $k \geq 0$ an integer. A $k$-independent set $S \subseteq V$ is a set of vertices such that the maximum degree in the graph induced by $S$ is at most $k$. With $\alpha_k(G)$ we denote the maximum cardinality of a $k$-independent set of $G$. We prove that, for a graph $G$ on $n$ vertices and average degree $d$, $\alpha_k(G) \geq \frac{k+1}{d+1} n$, improving the hitherto best general lower bound due to Caro and Tuza [Improved lower bounds on $k$-independence, J. Graph Theory 15 (1991), 99-107].

Keywords: $k$-independence, average degree

AMS subject classification: 05C69

1 Introduction

Let $G = (V, E)$ be a graph on $n$ vertices and $k \geq 0$ an integer. A $k$-independent set $S \subseteq V$ is a set of vertices such that the maximum degree in the graph induced by $S$ is at most $k$. With $\alpha_k(G)$ we denote the maximum cardinality of a $k$-independent set of $G$ and it is called the $k$-independence number of $G$. In particular, $\alpha_0(G) = \alpha(G)$ is the usual independence number of $G$. The Caro-Wei bound $\alpha(G) \geq \sum_{v \in V} \frac{1}{\deg(v) + 1}$ [11, 41] is an improvement of the well-known Turán bound for the independent number $\alpha(G) \geq \frac{n}{d(G) + 1}$ [40], where $d(G)$ is the average degree of $G$. Various results concerning possible improvements and generalizations of the Caro-Wei bound are known (see [1–3, 6, 10, 22, 23, 25, 26, 33, 35, 37, 38]). A well known generalization to the $k$-independence number of $r$-uniform hypergraphs was obtained by Caro and Tuza [12] improving earlier results of Favaron [19] and was extended to non-uniform hypergraphs by Thiele [39]. See also the recent papers [15, 17] for updates. An extension of the notion of residue of a graph, notably developed by Fajtlowicz in [18] and Favaron et al. in [20], to the notion of $k$-residue has been developed by Jelen [29]. There has been also much interest in using the Caro-Tuza to algorithmic aspects (see [24, 31, 36]). Yet all these lower bounds give asymptotically $\alpha_k(G) \geq \frac{k+1}{2(d+1)} n$ for $k$ fixed and $d = d(G)$. It is easy to see that in general we cannot hope to get better than $\frac{k+1}{d+1} n$, as can be seen from the graph $G = mK_{d+1}$ for $d \geq k$ with $n = m(d+1)$. So there is still an asymptotic multiplicative gap of a factor of $2^{\frac{k+1}{k+2}}$. It is worth to mention that there is no known modification of the charming probabilistic proof of the lower bound of Caro-Wei theorem to the situation of $k$-independence that gives a better bound than the Caro-Tuza lower bound. Here, for the
sake of being self-contained and to use the same notation, we restate and give the short
proof of the Caro-Tuza theorem for graphs. Then we show how to improve this result using
further ideas and, in particular, we close the multiplicative gap proving, as a corollary of
our main result, that \( \alpha_k(G) \geq \frac{k+1}{d(G)+k+1} n \). Doing so, we solve of a "folklore" conjecture
stated explicitly in [6].

All along this paper, we will use the following notation and definitions. Let \( G \) be a graph.
By \( V(G) \) we denote the set of vertices of \( G \) and \( n(G) = |V(G)| \) is the order of \( G \). \( E(G) \)
stands for the set of edges of \( G \) and \( e(G) \) denotes its cardinality. For a vertex \( v \in V(G) \),
degree \( \deg(v) = \deg_G(v) \) is the degree of \( v \) in \( G \). By \( \Delta(G) \) we denote the maximum degree of \( G \)
and by \( \bar{d}(G) \) the average degree \( \frac{1}{n(G)} \sum_{v \in V(G)} \deg(v) \). For a subset \( S \subseteq V(G) \), we write
\( G[S] \) for the graph induced by \( S \) in \( G \) and \( \deg_S(v) \) stands for the degree \( \deg_{G[S]}(v) \) of \( v \) in
\( G[S] \). Lastly, for a vertex \( v \in V(G) \), \( G - v \) represents the graph \( G \) without vertex \( v \) and all
the edges incident to \( v \).

The paper is divided into five sections. After this introduction section, we deal in section 2
with a first naive approach to obtain a lower bound on \( \alpha_k(G) \) by deleting iteratively vertices
of maximum degree until certain point where an old theorem of Lovász [32] is applied. In
section 3, we proceed the same way, taking however a better control on the number of
vertices that are deleted and we prove that, for a graph \( G \) on \( n \) vertices and average degree
\( d \), \( \alpha_k(G) \geq \frac{k+1}{d(k+1) + k} n \), improving the hitherto best general lower bound due to Caro and
Tuza. For this purpose, we define a parameter \( f(k, d) \) which approaches from below the
best possible ratio \( \frac{\alpha(G)}{n(G)} \) for graphs \( G \) with \( d(G) \leq d \), we calculate the exact value of \( f(1, d) \)
and prove some lower bounds on \( f(k, d) \). In Section 4, we develop some upper bounds on
\( f(k, d) \). Finally, we present in Section 5 some open problems for further research.

2 The naive approach: first improvement

Let \( f : [0, \infty) \to \mathbb{R} \) be the function defined by

\[
f_k(x) = \begin{cases} 
1 - \frac{x}{2(k+1)}, & \text{if } 0 \leq x \leq k + 1 \\
\frac{k+2}{2(x+1)}, & \text{if } x \geq k + 1.
\end{cases}
\]

Observe the following properties of \( f_k(x) \):

(P1) \( f_k(x) \) is a convex function and is strictly monotone decreasing on \([0, \infty)\).

(P2) \( f_k(i) - f_k(i + 1) \geq f_k(j) - f_k(j + 1) \), for \( j \geq i \geq 0 \).

(P3) \( if_k(i - 1) = (i + 1)f_k(i) \), for \( i \geq k + 1 \).

(P4) \( f_k(0) = 1 \) and \( f_k(k + 1) = \frac{1}{2} \).

Theorem 2.1 (Caro-Tuza for Graphs, [12]). Let \( G \) be a graph with degree sequence \( d_1, \ldots, d_n \).
Then \( \alpha_k(G) \geq \sum_{i=1}^{n} f_k(d_i) \).
Proof. For a subset $X \subseteq V(G)$, define $s(X) = \sum_{x \in X} f_k(\deg_X(x))$. Among all subsets of $X \subseteq V(G)$ such that $s(X)$ is maximum, choose $B$ such that $B$ has the smallest cardinality. In particular, $|B| \geq s(B) \geq s(V(G)) = \sum_{x \in V(G)} f_k(\deg(x))$. We will show that $B$ is a $k$-independent set of $G$. Suppose there is a vertex $y \in B$ such that $\deg_B(y) = d \geq k + 1$. Let $y$ be the vertex of maximum degree in $G[B]$. We will show that $s(B \setminus \{y\}) \geq s(B)$, a contradiction to the minimality of $|B|$. For $x \in B \setminus \{y\}$, let $z(x) = 1$ if $xy$ is an edge in $G$ and 0 otherwise. Then

$$s(B \setminus \{y\}) = \sum_{x \in B \setminus \{y\}} f_k(\deg_B(x) \setminus \{y\}) = \sum_{x \in B \setminus \{y\}} f_k(\deg_B(x) - z(x))$$

$$= \left(\sum_{x \in B} f_k(\deg_B(x) - z(x))\right) - f_k(d)$$

$$= s(B) - f_k(d) + \sum_{x \in B} (f_k(\deg_B(x) - z(x)) - f_k(\deg_B(x)))$$

$$= s(B) - f_k(d) + \sum_{x \in B} z(x) (f_k(\deg_B(x) - 1) - f_k(\deg_B(x)))$$

$$= s(B) - f_k(d) + \sum_{x \in B \cap N(y)} (f_k(\deg_B(x) - 1) - f_k(\deg(x))).$$

With (P2) we obtain that the last term is at least $s(B) - f_k(d) + d(f_k(d - 1) - f_k(d)) = s(B) - (d + 1)f_k(d) + df_k(d - 1)$ and, since $df_k(d - 1) = (d + 1)f_k(d)$ by (P3), this is equal to $s(B)$. It follows that $s(B \setminus \{y\}) \geq s(B)$, which is a contradiction to the choice of $B$. Hence, $B$ is a $k$-independent set and thus

$$\alpha_k(G) \geq |B| \geq s(B) \geq s(V) = \sum_{x \in V(G)} f_k(\deg(x)).$$

Note that, for $k = 0$, Theorem 2.1 yields the Caro-Wei bound. By convexity, the above bound yields also the following corollary.

**Corollary 2.2.** For a graph $G$ on $n$ vertices, $\alpha_k(G) \geq f_k(d(G))n$.

Note that, for $k = 0$, Corollary 2.2 yields the Turán bound $\alpha(G) \geq \frac{1}{d(G) + 1} n$. Also, if $d(G) \geq k + 1$, we obtain from this corollary the following one.

**Corollary 2.3.** Let $G$ be a graph on $n$ vertices. If $d(G) \geq k + 1$, then $\alpha_k(G) \geq \frac{k + 2}{2(d(G) + 1)} n$.

For a graph $G$, we will denote with $\chi_k(G)$ the $k$-chromatic number of $G$, i.e. the minimum number $t$ such that there is a partition $V(G) = V_1 \cup V_2 \cup \ldots \cup V_t$ of the vertex set $V(G)$ such that $\Delta(G[V_i]) \leq k$ for all $1 \leq i \leq t$. The following theorem is due to Lovász.

**Theorem 2.4** (Lovász [32], 1966). Let $G$ be a graph with maximum degree $\Delta$. If $k_1, k_2, \ldots, k_t \geq 0$ are integers such that $\Delta + 1 = \sum_{i=1}^t (k_i + 1)$, then there is a partition $V(G) = V_1 \cup V_2 \cup \ldots \cup V_t$ of the vertex set of $G$ such that $\Delta(G[V_i]) \leq k_i$ for $1 \leq i \leq t$. 

3
Several proofs and generalizations of Lovász’s theorem are known. We refer the reader to [8, 9, 13, 14, 34]. An algorithmic analysis of Lovász theorem with running time $O(n^3)$ is given in [24]. An immediate and well known corollary of Lovász’s theorem is Corollary 2.5 which is useful in the study of defective colorings also known as improper colorings (see [4, 16, 21, 27]).

**Corollary 2.5.** If $G$ is a graph of maximum degree $\Delta$, then $\chi_k(G) \leq \left\lceil \frac{\Delta + 1}{k + 1} \right\rceil$.

Since $\alpha_k(G) \geq \frac{n}{\chi_k(G)}$, the following bound proved in 1986 by Hopkins and Staton follows trivially from the above corollary.

**Theorem 2.6** (Hopkins, Staton [28] 1986). Let $G$ be a graph of order $n$ and maximum degree $\Delta$. Then

$$\alpha_k(G) \geq \frac{n}{\left\lceil \frac{\Delta + 1}{k + 1} \right\rceil}.$$ 

The following theorem is a direct consequence of Theorem 2.6 which generalizes and improves several results concerning relations between $\alpha_p(G)$ and $\alpha_q(G)$ (see e.g. [3]).

**Theorem 2.7.** Let $G$ be a graph and $q \geq p \geq 0$ two integers. Then $\alpha_q(G) \leq \left\lceil \frac{q + 1}{p + 1} \right\rceil \alpha_p(G)$.

**Proof.** Let $S$ be a maximum $q$-independent set of $G$. Then $\Delta(G[S]) \leq q$ and, by Theorem 2.6,

$$\alpha_p(G) \geq \alpha_p(G[S]) \geq \frac{|S|}{\left\lceil \frac{\Delta(G[S]) + 1}{p + 1} \right\rceil} \geq \frac{\alpha_q(G)}{\left\lceil \frac{q + 1}{p + 1} \right\rceil},$$

which implies the statement. \(\square\)

Completing $\Delta + 1$ to the next multiple of $k+1$, the following observation is straightforward from Theorem 2.6.

**Observation 2.8.** Let $G$ be a graph of order $n$ and maximum degree $\Delta$ and let $r$ be an integer such that $0 \leq r \leq k$ and $\Delta + 1 + r \equiv 0 \pmod{k + 1}$. Then

$$\alpha_k(G) \geq \frac{k + 1}{\Delta + r + 1} n.$$ 

**Proof.** As clearly $\left\lceil \frac{\Delta + 1}{k + 1} \right\rceil = \frac{\Delta + r + 1}{k + 1}$, Theorem 2.6 implies then $\alpha_k(G) \geq \frac{k + 1}{\Delta + r + 1} n$. \(\square\)

When the graph is $d$-regular, we can set $\Delta = d = d(G)$ in Observation 2.8 and we obtain the following one.

**Observation 2.9.** Let $G$ be a $d$-regular graph on $n$ vertices and let $r$ be an integer such that $0 \leq r \leq k$ and $d + 1 + r \equiv 0 \pmod{k + 1}$. Then $\alpha_k(G) \geq \frac{k + 1}{d + r + 1} n$. 

4
So this observation shows that indeed, for $d$-regular graphs, we can close the multiplicative gap of $2^{k+1}/k+2$ using Lovász's theorem. This serves as an inspiration to try to close the multiplicative gap in general.

Note that, in practice, the Hopkins-Staton bound can be poor if the maximum degree is far from the average degree. So, our first naive strategy will be to delete a vertex with large degree and, if possible, use induction on the number of vertices. Otherwise, if $\Delta(G)$ is near to the average degree $d(G)$, we will apply Theorem 2.6. This is precisely what is done in the next result.

**Theorem 2.10.** Let $G$ be a graph on $n$ vertices. Then $\alpha_k(G) > \frac{k+1}{d(G)+2k+2}n$.

**Proof.** We will proceed by induction on $n$. If $n = 1$, the statement is trivial. If $n = 2$, $G$ is either $K_2$ or $\overline{K}_2$. If $G = K_2$, then $d(G) = 1$ and $\frac{k+1}{d(G)+2k+2}n = \frac{2(k+1)}{3+2k} < 1 \leq \alpha_k(G)$ for any $k \geq 0$. If $G = \overline{K}_2$, then $d(G) = 0$ and thus $\frac{k+1}{d(G)+2k+2}n = 1 < 2 = \alpha_k(G)$ for all $k \geq 0$. Suppose now that $n \geq 3$ and that the statement holds for $n - 1$. Let $G$ be a graph on $n$ vertices and $v \in V(G)$ a vertex of maximum degree $\Delta$. Define $G^* = G - v$. Since any $k$-independent set of $G^*$ is also a $k$-independent set of $G$, $\alpha_k(G) \geq \alpha_k(G^*)$. We distinguish two cases.

**Case 1.** Suppose that $\Delta \leq [d(G)] + k$. Then, by Observation 2.8, we have

$$\alpha_k(G) \geq \frac{k+1}{\Delta + k + 1}n \geq \frac{k+1}{[d(G)] + 2k + 1}n > \frac{k+1}{d(G)+2k+2}n$$

and we are done.

**Case 2.** Suppose that $\Delta \geq [d(G)] + k + 1$. By induction and with $\Delta \geq [d(G)] + k + 1 \geq d(G) + k + 1$, we obtain

$$\alpha_k(G) \geq \alpha_k(G^*) > \frac{(k+1)(n-1)}{d(G^*) + 2k + 2} = \frac{(k+1)(n-1)}{\frac{2e(G^*)}{n-1} + 2k + 2}$$

$$= \frac{(k+1)(n-1)}{\frac{2e(G) - 2\Delta}{n-1} + 2k + 2} = \frac{(k+1)(n-1)}{\frac{d(G)n - 2d(G) + k + 1}{n-1} + 2k + 2}$$

$$\geq \frac{(k+1)(n-1)}{\frac{(d(G))^2 + (d(G) + 2k + 2)}{n-1} + 2k + 2} = \frac{(k+1)n}{(d(G) + 2k + 2)^2}$$

and the statement follows. $\square$

Note that the bound in previous theorem is better than the Caro-Tuza bound for $k = 1$ and $d \geq 8$ and for $k \geq 2$ and $d \geq 2k + 5$. Note also that Theorem 2.10 already closes the multiplicative factor of $2^{k+1}/k+2$ for fixed $k$ as $d(G)$ grows. However, to obtain an even better lower bound, we need to get more control on the number of vertices of large degrees that are deleted and to apply Observation 2.8 in its full accuracy. This will be done in the next section.
We close this section with the following algorithm for obtaining a \( k \)-independent set of cardinality at least \( \frac{k+1}{d(G)+2k+2n} \) for any graph \( G \) on \( n \) vertices that yields us the proof of Theorem 2.10.

**Algorithm 1**

**INPUT:** a graph \( G \) on \( n \) vertices and \( m \) edges.

1. Compute \( \Delta(G) \) and \( d(G) \). GO TO (2).

2. If \( \Delta(G) \leq \lceil d(G) \rceil + k \), perform a Lovász partition into \( k \)-independent sets, choose the largest class \( S \) and END. Otherwise choose a vertex \( v \) be of maximum degree \( \Delta(G) \), set \( G := G - v \) and GO TO (1).

**OUTPUT:** \( S \)

The algorithm terminates as, at some step, \( \Delta(G) \leq \lceil d(G) \rceil + k \) must hold (the latest when \( G \) is the empty graph). As already mentioned, the Lovász partition requires a running time of \( O(n^3) \), while each other step takes at most \( O(n) \) time and the number of iteration steps before performing Lovász partition is at most \( n \). Hence, the algorithm runs in at most \( O(n^3) \) time.

3 Deletions, partitions and a better lower bound on \( \alpha_k(G) \) - second improvement

**Definition 1.** Let \( d, k \geq 0 \) be two integers. We define

\[
 \alpha_k(G) = \inf \left\{ \frac{\alpha_k(G)}{n(G)} : G \text{ is a graph with } d(G) \leq d \right\}.
\]

**Observation 3.1.** Let \( d, k \geq 0 \) be two integers. For every graph \( G \) on \( n \) vertices and average degree \( d(G) \leq d \), \( \alpha_k(G) \geq f(k, d)n \).

The next theorem shows that \( f(k, d) \) is convex as a function of \( d \).

**Theorem 3.2.** Let \( d, k, t \geq 0 \) be integers and \( t \leq d \). Then \( 2f(k, d) \leq f(k, d-t) + f(k, d+t) \).

**Proof.** We will show that for any two graphs \( G_1 \) and \( G_2 \) such that \( d(G_1) \leq d - t \) and \( d(G_2) \leq d + t \), there is a graph \( G \) with \( d(G) \leq d \) such that \( 2\frac{\alpha_k(G)}{n(G)} \leq \frac{\alpha_k(G_1)}{n(G_1)} + \frac{\alpha_k(G_2)}{n(G_2)} \). Let \( G_1 \) and \( G_2 \) be such graphs and let \( n(G_i) = n_i \) and \( V(G_i) = V_i, i = 1, 2 \). Define the graph \( G = n_2G_1 \cup n_1G_2 \). Then

\[
2n_1n_2d(G) = n(G)d(G) = n_2 \sum_{x \in V_1} \deg_{G_1}(x) + n_1 \sum_{y \in V_2} \deg_{G_2}(y)
\]

\[
= n_2n_1d(H_1) + n_1n_2d(G_2)
\]

\[
\leq n_2n_1(d-t) + n_1n_2(d+t) = 2n_1n_2d,
\]
implying that $d(G) \leq d$ and thus $f(k, d) \leq \frac{\alpha_k(G)}{n(G)}$. Moreover,

$$2f(k, d) \leq 2\frac{\alpha_k(G)}{n(G)} = \frac{n_2\alpha_k(G_1) + n_1\alpha_k(G_2)}{n_1n_2} = \frac{\alpha_k(G_1)}{n_1} + \frac{\alpha_k(G_2)}{n_1}.$$ 

As $G_1$ and $G_2$ were arbitrarily chosen, it follows that $2f(k, d) \leq f(k, d - t) + f(k, d + t)$. □

Before coming to the main theorems of this section, we need the following lemmas.

**Lemma 3.3.** Let $d, t \geq 0$ be two integers and let $G$ be a graph on $n$ vertices with average degree $d(G) \leq d$. Then $G$ has a subgraph $H$ such that either $\Delta(H) \leq d + t - 1$ and $n(H) \geq n - \left\lfloor \frac{n}{d+2t+1} \right\rfloor$ or $d(H) \leq d - 1$ and $n(H) = n - \left\lceil \frac{n}{d+2t+1} \right\rceil$.

**Proof.** For an $r \geq 0$, let $\{v_1, v_2, \ldots, v_r\}$ be a set of vertices of maximum cardinality such that $\deg_{G_r+1}(v_i) \geq d + t$, where $G_i = G_1 - v_i$ and $G_0 = G$. Suppose first that $r \leq \left\lfloor \frac{n}{d+2t+1} \right\rfloor$ and let $H = G_{r+1}$. Then $H$ has at least $n - r \geq n - \left\lfloor \frac{n}{d+2t+1} \right\rfloor$ vertices and $\Delta(H) \leq d + t - 1$. Now suppose that $r \geq \left\lceil \frac{n}{d+2t+1} \right\rceil$. Let now $H = G_{q+1}$, where $q = \left\lceil \frac{n}{d+2t+1} \right\rceil$. Then $n(H) = n - \left\lceil \frac{n}{d+2t+1} \right\rceil$. Further,

$$d(H) = \frac{2e(H)}{n(H)} \leq \frac{2(e(G) - (d + t)q)}{n - q} = \frac{dn - 2(d + t)q}{n - q} = \frac{d(n - \frac{2(d+t)q}{d})}{n - q}.$$ 

Since, for any real numbers $a \geq 0$ and $b \geq 1$, the function $\frac{a}{b+x}$ is monotonically decreasing in $[0, \infty)$, setting $a = n$ and $b = \frac{2(d+t)q}{d}$, we obtain with $q = \left\lceil \frac{n}{d+2t+1} \right\rceil \geq \frac{n}{d+2t+1}$

$$d(H) \leq \frac{d(n - \frac{2(d+t)q}{d})}{n - q} \leq \frac{d(n - \frac{2(d+t)q}{d+2t+1})}{n - \frac{n}{d+2t+1}} = \frac{d(d + 2t + 1) - 2(d + t)}{d + 2t} = \frac{d(d + 2t) - (d + 2t)}{d + 2t} = d - 1.$$ 

Hence, we have shown that $G$ has a subgraph $H$ with either $d(H) \leq d - 1$ and $n(H) = n - \left\lceil \frac{n}{d+2t+1} \right\rceil$ or $\Delta(H) \leq d + t - 1$ and $n(H) = n - \left\lfloor \frac{n}{d+2t+1} \right\rfloor$. □

The following corollary is straightforward from this lemma.

**Corollary 3.4.** Let $d, t \geq 0$ be two integers. Let $G$ be a graph on $n$ vertices with average degree $d(G) \leq d$ and such that $d + 2t + 1$ divides $n$. Then $G$ has a subgraph $H$ on $n(H) \geq \frac{d+2t}{d+2t+1}n$ vertices such that either $d(H) \leq d - 1$ or $\Delta(H) \leq d + t - 1$.

**Lemma 3.5.** Let $G$ be a graph on $n$ vertices with average degree $d(G) \leq d$ and such that $d + 2t + 1$ does not divide $n$. Then there is a graph $H$ such that $d + 2t + 1$ divides $m = n(H)$, $d(H) = d(G)$ and $\frac{\alpha_k(H)}{m} = \frac{\alpha_k(G)}{n}$.

**Proof.** Let $H = (d + 2t + 1)G$ be the graph consisting of $d + 2t + 1$ copies of $G$. Then $m = n(H) = (d + 2t + 1)n$ is multiple of $d + 2t + 1$, $d(H) = d(G)$ and $\frac{\alpha_k(H)}{m} = \frac{(d+2t+1)\alpha_k(G)}{(d+2t+1)n} = \frac{\alpha_k(G)}{n}$. □
Let $n$ be an even integer. We denote with $J_n$ the graph consisting of a complete graph on $n$ vertices minus a 1-factor. We are now ready to present the exact value of $f(1, d)$ and the consequences of this result.

**Theorem 3.6.** Let $d \geq 0$ be an integer. Then the following statements hold.

1. $f(1, d) = \frac{2}{d+2}$, if $d \equiv 0 \pmod{2}$

2. The equality $f(1, d) = \frac{\alpha_1(G)}{n(G)}$ is attained by the graph $G = J_{d+2}$, when $d$ is even, and by $G = (d+3)J_{d+1} \cup (d+1)J_{d+3}$, when $d$ is odd.

3. $f(1, d) \geq \frac{2}{d+2}$.

4. For every graph $G$ on $n$ vertices, $\alpha_1(G) \geq \frac{2n}{d(G) + 2}$.

**Proof.** (1) We will prove by induction on $d$ that

$$f(1, d) \geq \frac{2}{d+2}, \quad \text{if } d \equiv 0 \pmod{2}$$

$$f(1, d) \geq \frac{2}{d+2}, \quad \text{if } d \equiv 1 \pmod{2}.$$

For $d = 0$, clearly $f(1, 0) = 1 = \frac{2}{d+2}$, as the only possible graph $G$ with $d(G) \leq 0$ is the empty graph. For $d = 1$, let $G$ be a graph with $d(G) \leq 1$. Setting $t = 1$, we can suppose by Lemma 3.5 that 4 divides $n(G) = n$. Hence, Corollary 3.4 implies that there is a subgraph $H$ of $G$ on $n(H) \geq \frac{3}{4}n$ vertices with $d(H) \leq 0$ or $\Delta(H) \leq 1$. In both cases we have clearly $\alpha_1(G) \geq \alpha_1(H) = n(H) \geq \frac{3}{4}n$ and hence $f(1, 1) = \inf \{ \frac{\alpha_1(G)}{n(G)} : G \text{ graph with } d(G) \leq 1 \} \geq \frac{3}{4} = \frac{2(1+2)}{(1+1)(1+3)}$.

Assume we have proved the statement for $f(1, d-1)$. Now we will prove it for $f(1, d)$, where $d > 1$. Let $G$ be a graph on $n$ vertices such that $d(G) \leq d$. We distinguish two cases.

**Case 1.** Suppose that $d \equiv 0 \pmod{2}$. Setting $t = 0$, we can suppose by Lemma 3.5 that $d+1$ divides $n$. By Corollary 3.4 there is a subgraph $H$ of $G$ on at least $\frac{n}{d+1}n$ vertices with either $d(H) \leq d-1$ or $\Delta(H) \leq d-1$. Hence, in both cases $d(H) \leq d-1$ and thus, by induction, we have

$$\alpha_1(G) \geq \alpha_1(H) \geq f(1, d-1)n(H) \geq \frac{2(d+1)d}{d(d+2)(d+1)}n = \frac{2}{d+2}n.$$

Hence, $f(1, d) = \inf \{ \frac{\alpha_1(G)}{n(G)} : G \text{ graph with } d(G) \leq d \} \geq \frac{d}{d+2}$ and we are done.

**Case 2.** Suppose that $d \equiv 1 \pmod{2}$. Set $t = 1$. By Lemma 3.5 we can suppose that $d+3$ divides $n$. By Corollary 3.4 there is a subgraph $H$ of $G$ on at least $\frac{n}{d+3}n$ vertices with either $d(H) \leq d-1$ or $\Delta(H) \leq d$. If $d(H) \leq d-1$, we can apply the induction hypothesis on $H$ and we obtain

$$\alpha_1(G) \geq \alpha_1(H) \geq f(1, d-1)n(H) \geq \frac{2(d+2)}{(d+1)(d+3)}n.$$
and we are done. Suppose finally that \( \Delta(H) \leq d \). Then, by Theorem 2.6 and as \( d \) is odd, we have
\[
\alpha_1(G) \geq \alpha_1(H) \geq \frac{n(H)}{\frac{d+2}{2}} \geq \frac{d+2}{d+1} n = \frac{2(d+2)}{(d+1)(d+3)} n.
\]
Thus, in both cases, \( f(1, d) = \inf \{ \frac{\alpha_1(G)}{n(G)} : G \text{ graph with } d(G) \leq d \} \geq \frac{2(d+2)}{(d+1)(d+3)} n \). Hence, by induction, the statement holds.

Let \( d \) be even. Clearly \( \alpha_1(J_{d+2}) = 2 \) and hence, \( f(1, d) \leq \frac{\alpha_1(J_{d+2})}{d+2} = \frac{2}{d+2} \). For \( d \) odd, the graph \( G = (d + 3)J_{d+1} \cup (d + 1)J_{d+3} \) has \( \alpha_1(G) = (d + 3)2 + (d + 1)2 = 4(d + 2) \). Hence
\[
f(1, d) \leq \frac{\alpha_1(G)}{n(G)} = \frac{4(d+2)}{2(d+1)(d+3)}. \]
Together with the inequalities proven above, it follows
\[
f(1, d) = \begin{cases} \frac{2}{d+2} & \text{if } d \equiv 0 \pmod{2} \\ \frac{2(d+2)}{(d+1)(d+3)} & \text{if } d \equiv 1 \pmod{2}. \end{cases}
\]

(2) This follows from the discussion in (1).

(3) It is easily seen that \( \frac{2(d+2)}{(d+1)(d+3)} \geq \frac{2}{d+2} \). Hence we have always \( f(1, d) \geq \frac{2}{d+2} \).

(4) From item (2), we obtain \( \alpha_1(G) \geq f(1, \lceil d(G) \rceil) n \geq \frac{2}{\lceil d(G) \rceil + 2} n \). \( \square \)

We can now state and prove our main result generalizing the proof of Theorem 3.6 to arbitrary \( k \) and \( d \).

**Theorem 3.7.** Let \( d, k \geq 0 \) be two integers. Then the following statements hold.

1. \( f(k, d) \geq \frac{(k+1)(d+2)}{(d+k+1)(d+t)} \geq \frac{k+1}{d+k+1} \), where \( t \) is such that \( d \equiv k + 1 - t \pmod{k+1} \) and \( 1 \leq t \leq k + 1 \).

2. For \( k \geq d \), \( f(k, d) \geq \frac{2k+2-d}{2k+2} \). For \( k = 1 \), the bound is realized by the graph \( K_{1, k+1} \cup kK_1 \) and thus \( f(k, 1) = \frac{2k+1}{2k+2} \).

3. For any graph \( G \) on \( n \) vertices, \( \alpha_k(G) \geq \frac{k+1}{\lceil d(G) \rceil + k+1} n \).

**Proof.** (1) We will proceed to prove the inequality \( f(k, d) \geq \frac{(k+1)(d+2)}{(d+k+1)(d+t)} \) by induction on \( d \). If \( d = 0 \), then \( d \equiv (k + 1) - (k + 1) \) and clearly \( f(k, 0) = 1 = \frac{(k+1)(0+2)(k+1)}{(0+k+1)(0+(k+1))} \). Hence the only possible graph \( G \) with \( d(G) \leq 0 \) is the empty graph.

Assume \( f(k, d-1) \geq \frac{(k+1)(d-1+2t)}{(d+k+1)(d-1+1+t')}, \) where \( d-1 \equiv k+1 - t' \pmod{k+1}, 1 \leq t' \leq k+1 \), and \( d \geq 1 \). We will prove the statement for \( d \). Hereafter, we distinguish two cases.

**Case 1.** Suppose that \( d \equiv 0 \pmod{k+1} \). Then \( t = k + 1 \). Let \( G \) be a graph on \( n \) vertices such that \( d(G) \leq d \). By Lemma 3.5 (setting there \( t = 0 \)), we can suppose that \( d+1 \) divides \( n \). Then from Corollary 3.4 it follows that there is a subgraph \( H \) of \( G \) on at least \( \frac{d}{d+1} n \) vertices such that \( d(H) \leq d-1 \) or \( \Delta(H) \leq d-1 \). In both cases we have \( d(H) \leq d-1 \).
Then, as \( d - 1 \equiv (k + 1) - 1 \pmod{k + 1} \), we obtain by induction

\[
\alpha_k(G) \geq \alpha_k(H) \geq \frac{(k+1)(d+1)}{(d+k+1)d}n(H) \geq \frac{k+1}{d+k+1}n
\]

\[
= \frac{(k+1)(d+2t)}{(d+2t)(d+k+1)} = \frac{(k+1)(d+2t)}{(d+k+t+1)(d+t)}n.
\]

Thus, \( f(k, d) = \inf\{\frac{\alpha_k(G)}{n(G)} : \text{G graph with } d(G) \leq d\} \geq \frac{(k+1)(d+2t)}{(d+k+t+1)(d+t)} \) and we are done.

Case 2. Suppose that \( d \equiv k + 1 - t \pmod{k + 1} \) for some \( t \) with \( 1 \leq t \leq k \). By Corollary 3.4, there is a subgraph \( H \) of \( G \) on \( n(H) \geq \frac{d+2t}{d+k+1}n \) vertices with either \( d(H) \leq d - 1 \) or \( \Delta(H) \leq d + t - 1 \). If \( \Delta(H) \leq d + t - 1 \), then Theorem 2.6 yields

\[
\alpha_k(G) \geq \alpha_k(H) \geq \frac{n(H)}{\frac{\Delta(H)+1}{k+1}} \geq \frac{\frac{d+2t}{d+k+1}}{\frac{d+1}{k+1}} = \frac{(k+1)(d+2t)}{(d+2t+1)(d+t)}n
\]

\[
\geq \frac{(k+1)(d+2t)}{(d+k+t+1)(d+t)}n.
\]

Hence, \( f(k, d) = \inf\{\frac{\alpha_k(G)}{n(G)} : \text{G graph with } d(G) \leq d\} \geq \frac{(k+1)(d+2t)}{(d+k+t+1)(d+t)} \) and we are done.

Suppose now that \( d(H) \leq d - 1 \). Since \( d - 1 \equiv (k + 1) - (t + 1) \), we obtain by induction

\[
\alpha_k(G) \geq \alpha_k(H) \geq \frac{(k+1)((d-1) + 2(t+1))}{((d-1) + k + (t+1) + 1)((d-1) + (t+1))}n(H)
\]

\[
\geq \frac{(k+1)(d+2t+1)}{(d+k+t+1)(d+t)} \cdot \frac{d+2t}{d+2t+1}n
\]

\[
= \frac{(k+1)(d+2t)}{(d+k+t+1)(d+t)}n.
\]

Thus, again, \( f(k, d) \geq \frac{(k+1)(d+2t)}{(d+k+t+1)(d+k+1)} \) and Case 2 is done.

Hence, by induction, the statement holds. Finally, the inequality \( \frac{(k+1)(d+2t)}{(d+k+t+1)(d+t)} \geq \frac{k+1}{d+k+1} \) follows easily.

(2) Let \( k \geq d \) and let \( t \) be such that \( d \equiv k + 1 - t \pmod{k + 1} \) and \( 1 \leq t \leq k \). Then \( d = k + 1 - t \). Hence, with (1),

\[
f(d, k) \geq \frac{(k+1)(d+2t)}{(d+k+t+1)(d+t)} = \frac{(k+1)(d+2k+2)}{(d+k+(k+1-d)+1)(d+(k+1-d))} = \frac{2k+2-d}{2k+2}.
\]

Let \( G = K_{1,k+1} \cup K_{1,1} \). Then \( \alpha_k(G) = 2k+1 \), \( n(G) = 2k+2 \) and \( d(G) = \frac{2k+2}{2k+2} = 1 \). Hence, \( \frac{2k+1}{2k+2} \leq f(k, 1) \leq \frac{\alpha_k(G)}{n(G)} = \frac{2k+1}{2k+2} \), obtaining thus equality.

(3) If \( G \) is a graph on \( n \) vertices, then, using (1), we obtain

\[
\frac{\alpha_k(G)}{n} \geq f(k, [d(G)]) \geq \frac{k+1}{[d(G)]+k+1}.
\]
The proofs of Lemma 3.3 and Theorem 3.7 yield us an algorithm for finding, for any graph $G$ on $n$ vertices, a $k$-independent set of cardinality at least $\frac{k+1}{d(G)+k+1}n$. It works the following way. It computes $d = d(G)$ and $\Delta(G)$ and finds the integer $t$ such that $0 \leq t \leq k$ and $d \equiv k+1-t \pmod{k+1}$ (note that the case $t = 0$ corresponds here to the case $t = k+1$ of Theorem 3.7). Then it checks if the graph satisfies the condition $\Delta(G) \leq d + t - 1$. If so, then it performs a Lovász partition into $k$-independent sets, selects the largest set from it and gives this as output. If not, then a vertex of maximum degree is deleted and the condition on the maximum degree is checked again on the remaining graph. This deletion step is repeated up to $\left\lceil \frac{n}{d+2t+1} \right\rceil$ times, as, by Lemma 3.3, if the maximum degree is still larger than $d + t - 1$, then we are left with a graph with smaller average degree, with which the algorithm starts over again, doing here the inductive step of Theorem 3.7.

Algorithm 2

INPUT: a graph $G$ on $n$ vertices and $m$ edges.

1. Compute $\Delta(G)$ and $d(G)$. Set $d = \lceil d(G) \rceil$ and determine $t$ such that $0 \leq t \leq k$ and $d \equiv k+1-t \pmod{k+1}$. Set $r := 0$ and GO TO (2).

2. If $\Delta(G) \leq d + t - 1$, perform a Lovász partition into $k$-independent sets, choose the largest class $S$ and END. Otherwise GO TO (3).

3. Set $r := r + 1$. If $r > \left\lceil \frac{n}{d+2t+1} \right\rceil$, set $n := n - \left\lceil \frac{n}{d+2t+1} \right\rceil$ and GO TO (1). Otherwise choose a vertex $v$ of maximum degree $\Delta(G)$, set $G := G - v$, compute $\Delta(G)$ and GO TO (2).

OUTPUT: $S$

The algorithm terminates as, at some step, $\Delta(G) \leq \lceil d(G) \rceil + t - 1$ must hold (the latest when $G$ is the empty graph). Again, the algorithm has a running time of at most $O(n^3)$.

4 Upper bounds on $f(k, d)$ and determination of $f(k, d)$ for further small values

Observe that after Theorems 3.6(1) and 3.7(2), we know the exact value of $f(k, d)$ in case $\min\{d, k\} \leq 1$. The first pair $(k, d)$ for which an exact value of $f(k, d)$ is not known yet is $(2, 2)$. In this section, we develop several upper bounds on $f(k, d)$ as a starting point to future research to obtain further exact values of $f(k, d)$. We will use the following Theorem.

Theorem 4.1 (see [7], p.108). Let $r, g \geq 3$ be two integers. If $m$ is an integer with $m \geq \frac{(r-1)(g-1)-1}{r-2}$, then there exists an $r$-regular graph of girth at least $g$ and order $2m$.

Define the function $h(r) = \frac{(r-1)^{r+3}-1}{r-2}$. We will use the particular form of this theorem with $m \geq h(r)$, implying that there is an $r$-regular graph of girth at least $r+4$ and order $2m$. 

11
In the proof of the following theorem, we use the following notation. \( \overline{G} \) denotes the complementary graph of \( G \). If \( F \subseteq E(G) \), then \( G - F \) represents the graph \( G \) without the edges contained in \( F \). For \( q \leq n \), \( K_n - E(K_q) \) stands for the complete graph \( K_n \) without the edges of a subgraph \( K_q \). Further given two graphs \( G \) and \( H \), \( G \cup H \) is the graph consisting of one copy of \( H \) and one copy of \( G \) and \( qG \) denotes the union of \( q \) copies of \( G \). Finally, the girth of a graph \( G \) is denoted by \( g(G) \).

**Theorem 4.2.** Let \( d, k \geq 0 \) be two integers. Then the following statements hold.

1. For \( d \geq k \), \( \frac{k+1}{d+k+1} \leq f(k, d) \leq \frac{k+1}{d+1} \).
2. For \( d > k \), \( d \equiv 0 \pmod{2} \) and \( k \equiv 1 \pmod{2} \), \( f(k, d) \leq \frac{k+1}{d+2} \).
3. For \( d > k \), \( f(k, d) \leq \frac{k+2}{d+3} \).
4. For \( k \geq 3 \), \( d \geq 2h(k) - k - 1 \) and \( d + k + 1 \equiv 0 \pmod{2} \), \( f(k, d) \leq \frac{k+2}{d+k+1} \).
5. For \( 2 \leq d \leq 4 + 6q \), where \( q \geq 0 \) is an integer, \( \frac{3}{d+3} \leq f(2, d) \leq \frac{3}{d+1+\frac{q}{d+q}} \).
6. For \( k \geq 2 \), \( \frac{k}{k+1} \leq f(k, 2) \leq \frac{k+1}{k+2+\frac{1}{k+1}} \).
7. For \( k \geq 3 \), there is a constant \( c > 0 \) such that \( f(k, d) < \frac{k+2}{d+c(d+1)} + 1 \).

**Proof.**

1. The lower bound follows from Theorem 3.7. The upper bound follows from \( f(k, d) \leq \alpha_k(K_d+1) = \frac{k+1}{d+1} \).
2. Let \( G \) be the graph \( K_{d+2} \) minus a 1-factor (this is possible, as \( d \) is assumed even). Then \( d(G) = d \). Let \( T \subseteq V(G) \) be any subset of \( k + 2 \) vertices. As \( k + 1 \equiv 0 \pmod{2} \), not every vertex of \( T \) is covered by the edges of the 1-factor in \( \overline{G}[T] \). Hence, at least one vertex from \( T \) is adjacent in \( G \) to all other vertices from \( T \). Hence, no subset of \( k + 2 \) vertices can be a \( k \)-independent set and thus \( \alpha_k(G) \leq k + 1 \). This implies \( f(k, d) \leq \frac{k+1}{d+1} \).
3. Let \( d > k \). Consider the graph \( G = K_{d+3} - E(C_{d+3}) \), where \( C_{d+3} \) is a cycle of length \( d+3 \) in \( K_{d+3} \). Then \( d(G) = d \). Let \( T \subseteq V(G) \) a subset of \( k+3 \) vertices. Since \( k+3 < d+3 = n(G) \), the graph \( \overline{G}[T] \) contains no cycles. Hence it there is at least one vertex in \( v \in V(T) \) which is adjacent in \( G'[T] \) to all but at most one vertex and hence \( \text{deg}_{G'[T]}(v) \geq k + 1 \). This implies that \( \alpha_k(G) \leq k + 2 \) and thus \( f(k, d) \leq \frac{k+2}{d+k+1} \).
4. Let \( k \geq 3 \), \( d \geq 2h(k) - k - 1 \) and \( d + k + 1 \equiv 0 \pmod{2} \). By Theorem 4.1, there is a \( k \)-regular graph \( H \) with \( g(H) \geq k + 4 \) and \( n(H) = d + k + 1 = n \). Consider now the graph \( G = K_n - E(H) \). Then \( d(G) = n - 1 - k = d \). Let \( T \subseteq V(G) \) be a subset of \( k+3 \) vertices. Since \( g(H) \geq k + 4 \), \( \overline{G}[T] \) is a forest. Hence it there is at least one vertex in \( v \in V(T) \) which is adjacent in \( G'[T] \) to all but at most one vertex and hence \( \text{deg}_{G'[T]}(v) \geq k + 1 \). Thus, \( \alpha_k(G) \leq k + 2 \) and we obtain \( f(k, d) \leq \frac{k+2}{d+k+1} \).
5. Consider the graph \( G = (K_{d+2} - E(K_3)) \cup q(K_{d+1} - E(K_3)) \). Then \( n(G) = (q+1)d+q+2 \) and \( d(G) = (d-1)(d+1)+3(d-1)+q((d-2)d+3(d-2)) = (q+1)d^2 + (q+3)d - (4+6q) \). Since \( d \leq 4 + 6q \), it follows that

\[
d(G) = \frac{(q+1)d^2 + (q+3)d - (4+6q)}{(q+1)d + q + 2} \leq \frac{(q+1)d^2 + (q+3)d - d}{(q+1)d + q + 2} = d.
\]
As clearly \( \alpha_2(G) = 3(q + 1) \), we obtain therefore, together with Theorem 3.7 (1),
\[
\frac{3}{d + 3} \leq f(2, d) \leq \frac{3(q + 1)}{(q + 1)d + q + 2} = \frac{3}{d + \frac{2q+2}{q+1}} = \frac{3}{d + 1 + \frac{1}{q+1}}.
\]

(6) Let \( k \geq 2 \) and consider the graph \( G = (K_{k+3} - E(K_{k+1})) \cup kK_{1,k+1} \). Then \( n(G) = k+3+k(k+2) = k^2 + 3k + 3 \) and \( d(G)n(G) = 2(k+2)+(k+1)2+2k(k+1) = 2(k^2 + 3k + 3) = 2n(G) \). Hence, \( d(G) = 2 \). Moreover, it is easy to see that \( \alpha_k(G) = (k + 1)^2 \). Thus this implies that
\[
f(k, 2) \leq \frac{\alpha_k(G)}{n(G)} = \frac{(k + 1)^2}{k^2 + 3k + 3} = \frac{k + 1}{k + 2 + \frac{1}{k+1}}.
\]

Together with the bound from item (2) of Theorem 3.7, we obtain
\[
\frac{k}{k + 1} \leq f(k, 2) \leq \frac{k + 1}{k + 2 + \frac{1}{k+1}}.
\]

(7) By Theorem 4.1 there is an \( r \)-regular graph \( H \) with \( g(H) \geq k + 4 \) and \( n = n(H) \geq \frac{2((r-1)^{k+3}-1)}{r-2} \). Take \( n \) even and let \( G = K_n - E(H) \). Then \( d = d(G) = n - 1 - r \). Let \( T \subseteq V(G) \) be a subset of \( k+3 \) vertices. As \( g(H) \geq k+4 \), \( G[T] \) is a forest and thus there is a vertex in \( T \) which is adjacent in \( G \) to all other vertices from \( T \) with the exception of at most one. Hence, \( T \) cannot be a \( k \)-independent set and thus \( \alpha_k(G) \leq k + 2 \). This implies that
\[
f(k, d) \leq \frac{\alpha_k(G)}{n(G)} \leq \frac{k+2}{d+r+1}. \quad \text{As} \quad d \sim 2r^{k+2}, \quad \text{we have} \quad r \sim (\frac{d}{2})^{\frac{1}{k+2}}, \quad \text{and thus there is a constant} \quad c > 0 \quad \text{such that} \quad r = c(\frac{d}{2})^{\frac{1}{k+2}}, \quad \text{implying that} \quad f(k, d) \leq \frac{k+2}{d+c(\frac{d}{2})^{\frac{1}{k+2}}+1}. \quad \square
\]

5 Open problems

We close this paper with the following open problems.

**Problem 1.** Is \( f(k, d) \) in fact a minimum for every \( k \) and \( d \)? Namely, does
\[
\inf \left\{ \frac{\alpha_k(G)}{n(G)} : G \text{ graph with } d(G) \leq d \right\} = \min \left\{ \frac{\alpha_k(G)}{n(G)} : G \text{ graph with } d(G) \leq d \right\}
\]
hold?

In case the answer to this problem is positive, this may have several consequences in computing \( f(k, d) \).

**Problem 2.** Is the bound \( f(k, d) \geq \frac{2k+2-d}{2k+2} \) of Theorem 3.7 (2) sharp for \( k \geq d \geq 2 \)?

Below are the best possible bounds on \( f(2, d) \) we have for \( k = 0, 1, \ldots, 10 \).
| $d$ | lower bound* | upper bound | graph for upper bound | theorem used for upper bound |
|-----|--------------|-------------|-----------------------|-----------------------------|
| 0   | 1            | 1           | $K_1$                 | -                           |
| 1   | 5/6          | 5/6         | $K_{1,k+1} \cup kK_1$ | 3.7(2)                     |
| 2   | 2/3          | 9/13        | $(K_5 - E(K_3)) \cup 2K_{1,3}$ | 4.2(6)                     |
| 3   | 1/2          | 3/5         | $K_5 - E(K_3)$        | 4.2(5), $q = 0$            |
| 4   | 4/9          | 1/2         | $K_6 - E(K_3)$        | 4.2(5), $q = 0$            |
| 5   | 7/18         | 6/13        | $(K_7 - E(K_3)) \cup (K_6 - E(K_3))$ | 4.2(5), $q = 1$            |
| 6   | 1/3          | 2/5         | $(K_8 - E(K_3)) \cup (K_7 - E(K_3))$ | 4.2(5), $q = 1$            |
| 7   | 11/36        | 6/17        | $(K_9 - E(K_3)) \cup (K_8 - E(K_3))$ | 4.2(5), $q = 1$            |
| 8   | 5/13         | 6/19        | $(K_{10} - E(K_3)) \cup (K_9 - E(K_3))$ | 4.2(5), $q = 1$            |
| 9   | 1/4          | 2/7         | $(K_{11} - E(K_3)) \cup (K_{10} - E(K_3))$ | 4.2(5), $q = 1$            |
| 10  | 7/30         | 6/23        | $(K_{12} - E(K_3)) \cup (K_{11} - E(K_3))$ | 4.2(5), $q = 1$            |

*Lower bounds are from Thm. 3.7(2) in case $d = 1$ and Thm. 3.7(1) else.

**Problem 3.** Improve upon the values given in the table.

In order to better understand $f(k, d)$, we can define

$$f(k, d, \Delta) = \inf \left\{ \frac{\alpha_k(G)}{n(G)} : G \text{ is a graph with } d(G) \leq d \text{ and } \Delta(G) \leq \Delta \right\},$$

where $\Delta \geq d$, and $k$ are all nonnegative integers. Observe that $f(k, d) = \inf \{ f(k, d, \Delta) : \Delta \geq d \}$ and hence a knowledge on $f(k, d, \Delta)$ may help in obtaining better bounds on $f(k, d)$. For instance, let us take $f(2, 2, 3)$. Observe that, from Theorem 3.7(2), $f(2, 2, 3) \geq f(2, 2) \geq \frac{4}{3}$. Further, consider the graph $G = R_8 \cup 4K_{1,3}$ on 24 vertices, where $R_8$ is the graph depicted below (note that $R_8$ is the extremal graph for Reed’s upper bound of $\frac{4}{3}$ on the domination number for graphs on $n$ vertices with minimum degree at least 3), and observe that $\alpha_2(G) = 17$, $n(G) = 24$ and $\Delta(G) = 3$. Then, it follows that $\frac{4}{3} \leq f(2, 2) \leq f(2, 2, 3) \leq \frac{17}{24}$.

The graph $R_8$.

But if we consider for instance the graph $H = (K_5 - E(K_3)) \cup 2K_{1,3}$, then we have there $\alpha_2(H) = 9$, $n(H) = 13$ and $\Delta(H) = 4$ and thus $\frac{2}{3} \leq f(2, 2) \leq f(2, 2, 4) \leq \frac{3}{13}$, which is better than the bound $\frac{17}{24}$ obtained with the graph $G$. Thus, we would like to state the following question.

**Problem 4.** Obtain lower and upper bounds on $f(k, d, \Delta)$.
References

[1] M. Ajtai, J. Komlós, and E. Szemerédi, A note on Ramsey numbers, J. Comb. Theory, Ser. A 29 (1980), 354–360.

[2] N. Alon, J. Kahn, P. D. Seymour, Large induced degenerate subgraphs, Graphs Combin. 3 3 (1987), 203–211.

[3] N. Alon and J.H. Spencer, The probabilistic method (2nd ed.), Wiley, 2000.

[4] J.-C. Bermond, F. Havet, F. Huc, C. Linhares Sales, Improper coloring of weighted grid and hexagonal graphs, Discrete Math. Algorithms Appl. 2 3 (2010), 395–411.

[5] M. Blidia, M. Chellali, O. Favaron, N. Meddah, Maximal $k$-independent sets in graphs, Discuss. Math. Graph Theory 28 1 (2008), 151–163.

[6] A. Bojilov, Y. Caro, A. Hansberg, N. Nenov, Partitions of graphs into small and large sets, submitted.

[7] B. Bollobás, Extremal graph theory, Reprint of the 1978 original, Dover Publications, Inc., Mineola, NY (2004), xx+488 pp.

[8] B. Bollobás, B. Manvel, Optimal vertex partitions, Bull. London Math. Soc. 11 2 (1979), 113–116.

[9] O. V. Borodin, A. V. Kostochka, B. Toft, Variable degeneracy: extensions of Brooks’ and Gallai’s theorems, Discrete Math. 214 1-3 (2000), 101–112.

[10] P. Borowiecki, F. Göring, J. Harant, D. Rautenbach, The potential of greed for independence, J. Graph Theory, DOI: 10.1002/jgt.20644 (2011).

[11] Y. Caro, New results on the independence number, Tech. Report, Tel-Aviv University (1979).

[12] Y. Caro, Z. Tuza, Improved lower bounds on k-independence, J. Graph Theory 15 1 (1991), 99–107.

[13] P. A. Catlin, Another bound on the chromatic number of a graph, Discrete Math. 24 1 (1978), 1–6.

[14] P. A. Catlin, Graph decompositions satisfying extremal degree constraints, J. Graph Theory 2 2 (1978), 165–170.

[15] M. Chellali, O. Favaron, A. Hansberg, Adriana, L. Volkmann, $k$-domination and $k$-independence in graphs: a survey. Graphs Combin. 28 1 (2012), 1–55.

[16] L. J. Cowen, R. H. Cowen, D. R. Woodall, Defective colorings of graphs in surfaces: partitions into subgraphs of bounded valency, J. Graph Theory 10 2 (1986), 187–195.

[17] B. Csaba, T. A. Plick, A. Shokoufandeh, A note on the Caro-Tuza bound on the independence number of uniform hypergraphs, Australas. J. Combin. 52 (2012), 235–242.

[18] S. Fajtlowicz, On conjectures of Graffiti, III. Congr. Numer. 66 (1988), 23–32.

[19] O. Favaron, $k$-domination and $k$-independence in graphs, Ars Combin. 25 C (1988), 159–167.

[20] O. Favaron, M. Maho, J.F. Saclé, On the residue of a graph, J. Graph Theory 15 (1991), 39–64.
[21] M. Frick, M. Henning, Extremal results on defective colorings of graphs, Discrete Math. 126 1-3 (1994), 151–158.

[22] F. Göring, J. Harant, D. Rautenbach and I. Schiermeyer, On F-independence in graphs, Discuss. Math. Graph Theory 29 2 (2009), 377–383.

[23] J. R. Griggs, Lower Bounds on the Independence Number in Terms of the Degrees, Journal of Combinatorial Theory, Series B 34 1 (1983) 22–39.

[24] M. M. Halldorsson and H. Ch. Lau, Low-degree Graph Partitioning via Local Search with Applications to Constraint Satisfaction, Max Cut, and Coloring, Journal of Graph Algorithms and Applications 1 3 (1997),49–61.

[25] J. Harant, D. Rautenbach, Independence in connected graphs, Discrete Appl. Math. 159 1 (2011), 79–86.

[26] J. Harant and I. Schiermeyer, On the independence number of a graph in terms of order and size, Discrete Math. 232 (2001), 131–138.

[27] F. Havet, R. J. Kang, J.-S. Sereni, Improper coloring of unit disk graphs, Networks 54 3 (2009), 150–164.

[28] G. Hopkins and W. Staton, Vertex partition and k-small subsets of graphs, Ars Combin. 22 (1986), 19-24.

[29] F. Jelen, k-Independence and the k-residue of a graph, J. Graph Theory 32 (1999), 241–249.

[30] Y. Li, W. Zang, Differential methods for finding independent sets in hypergraphs, SIAM J. Discrete Math. 20 1 (2006), 96–104.

[31] E. Losievskaja, Approximation algorithms for independent set problems on hypergraphs, Ph.D. thesis, Reykjavik University (2009), http://hdl.handle.net/1946/7410

[32] L. Lovász. On decompositions of graphs, Studia Sci. Math Hungar. 1 (1966) 237–238.

[33] O. Murphy, Lower bounds on the stability number of graphs computed in terms of degrees, Discrete Math. 90 2 (1991), 207–211.

[34] L. Rabern, Partitioning and coloring with degree constraints, arXiv:1202.5855v1.

[35] S.M. Selkow, The independence number of a graph in terms of degrees, Discrete Math. 132 (1994) 363–365.

[36] H. Shachnai, A. Srinivasan, Finding large independent sets in graphs and hypergraphs. SIAM J. Discrete Math. 18 3 (2004/05), 488–500 (electronic).

[37] J.B. Shearer, A note on the independence number of triangle-free graphs, Discrete Math. 46 (1983), 83–87.

[38] J.B. Shearer, A note on the independence number of triangle-free graphs. II, J. Comb. Theory, Ser. B 53 (1991), 300–307.

[39] T. Thiele, A lower bound on the independence number of arbitrary hypergraphs, J. Graph Theory 30 3 (1999), 213–221.

[40] P. Turán, An extremal problem in graph theory (hungarian), Mat Fiz Lapok 48 (1941), 436–452.

[41] V. K. Wei, A lower bound on the stability number of a simple graph, Bell Laboratories Technical Memorandum, 81–11217–9, Murray Hill, NJ (1981).