Closed Abrikosov Vortices in a Superconducting Cylinder

V. A. Kozlov and A. V. Samokhvalov

Institute of Applied Physics, Russian Academy of Sciences

46 Uljanov Street, 603600 Nizhny Novgorod, Russia

Abstract

The new type of solutions of the London equation for type-II superconductors is obtained to describe the ring-shaped (toroidal) Abrikosov vortices. The specific feature of these solutions is the self-consistent localization of both the supercurrent and the magnetic field, enabling one to construct compact magnetic structures inside a superconductor. The torus vortex contraction caused by the vortex instability leads to the destruction of the Cooper pairing and the formation of a normal electron stream in the vicinity of the torus axis. The thermodynamic condition for the excitation of a small closed vortex by a bunch of charged particles contains the fine-structure constant as a determining parameter.

I. INTRODUCTION

A magnetic flux penetrates into type-II superconductors (SC) as lines of Abrikosov vortices having one magnetic flux quantum \( \Phi \). These vortices and their interaction with inhomogeneities and defects of the material (pinning) determine both the magnetic properties of type-II SC and the ability of SC to carry the superconducting current \( \mathcal{I} \). In the vortex core, where the magnetic field achieves its maximum, superconductivity is destroyed, and the modulus of the complex order parameter \( \psi \), describing the superconducting properties of the material, goes to zero. Central points of the vortex core where \( |\psi| = 0 \) define the
location of vortex line, (VL), while the phase of $\psi$ changes by a multiple of $2\pi$ in going round the loop encircling this line. The latter circumstance leads to the flux quantization in the vortex, so that a solitary vortex with a rectilinear VL contains a single flux quantum, $\Phi_0 = \pi \hbar c/e$.

To describe the behavior of Abrikosov vortices in type-II SC with small coherence length compared to the magnetic field penetration depth $\lambda$, it is convenient to apply the London model involving the principle of superposition for currents and fields [3,4]. In the London approximation, the vortex is determined by the VL location as an external parameter. The vortex lines in type-II superconductors reproduce the structure of the magnetic field lines. Thus, the homogeneous magnetic field, $H_0$, generates a two-dimensional lattice of rectilinear VLs [3,4]. More complex vortex structures are possible when the magnetic field is produced by an external current flowing through the SC. In the case of a SC cylinder with a bias current, the magnetic field lines represent a set of concentric circles and any VL is a circle with its center lying at the cylinder axis [3,4]. The current-induced magnetic structures in cylindrical SC wires have been discussed in [5,9]. The continuously collapsing magnetic structures in cylindrical wires have been considered in [10–12] to explain the SC resistive state.

The present paper deals with the new type of solutions of the London equation. These solutions describe the ring-shaped (or toroidal) Abrikosov vortices in a type-II SC. The VL of such a vortex is a circle of a finite radius. The toroidal vortex arising in SC cylinders exhibit some peculiarities compared to the Abrikosov vortex with a rectilinear VL. In Sec. I, the magnetic field, the magnetic flux and the free energy of a solitary closed toroidal vortex in a unbounded SC are calculated. In Sec. II, the toroidal solution of the London equation for a superconducting cylinder for arbitrary relations between the London penetration depth, the cylinder radius and the toroidal vortex size is derived, and the vortex stability is discussed. In Sec. III, the excitation of toroidal vortices by a bunch of charged particles is considered.
II. A CLOSED ABRIKOSOV VORTEX IN AN UNBOUNDED TYPE-II SUPERCONDUCTOR

The London approach to superconductors is appropriate for the case \( \lambda \gg \xi \), where \( \lambda \) is the London penetration depth and \( \xi \) is the coherence length. This approximation is expected to be justified for new high-T superconductors. To find the structure of a solitary closed Abrikosov vortex, we consider the solution of the London equation with a closed VL consisting of a circle of radius, \( R_s \), inside an unbounded SC. In the London model, the magnetic field distribution in the vortex, \( H(\mathbf{r}) \), at distances \( r \) from the center of the normal core, is yielded by the London equation \( [3] \):

\[
H + \lambda^2 \nabla \times \nabla H = \Phi_0 \mathbf{e}_v \delta(\mathbf{r} - \mathbf{r}_s).
\]

Here \( \mathbf{r}_s \) is the radius-vector defining the VL location in space and \( \mathbf{e}_v \) is the unit vector directed along the VL (Fig. 1). In the cylindrical coordinate system \((r, \varphi, z)\), whose \( z = 0 \) plane coincides with the VL plane, the magnetic field, \( H(\mathbf{r}) \), has only an azimuthal component, \( H(r, z) \), and the basic equation (1) can be written in the form:

\[
\frac{\partial^2 h_v}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial h_v}{\partial \rho} + \frac{\partial^2 h_v}{\partial \zeta^2} - \left(1 + \frac{1}{\rho^2}\right) h_v = -\delta(\rho - \rho_s) \delta(\zeta).
\]

Here \( \rho = r/\lambda \), \( \zeta = z/\lambda \), \( \rho_s = R_s/\lambda \), and \( h_v(\rho, \zeta) = H(r, z) \lambda^2 / \Phi_0 \) is the dimensionless magnetic field. To solve Eq.(2) the Fourier-Bessel transform has been used and the magnetic field, \( h_v(\rho, \zeta) \), has been represented as follows:

\[
h_v(\rho, \zeta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dp \exp(ip\zeta) \int_{0}^{+\infty} dq f_v(p, q) q J_1(\rho q),
\]

\[
f_v(p, q) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\zeta \exp(-ip\zeta) \int_{0}^{+\infty} d\rho h_v(\rho, \zeta) \rho J_1(\rho q),
\]

where \( J_1 \) is the Bessel function of the first kind. The differential equation (2) for the Fourier-Bessel components, \( f_v \), expressed in the spectral variables \((p, q)\), is reduced to the simple algebraic relation:
\[ f_v(p, q)(1 + p^2 + q^2) = \rho_s J_1(\rho_s q)/\sqrt{2\pi}. \] (4)

Using Eq. (3.a) one can return to the initial coordinates \((\rho, \zeta)\) and obtain the magnetic field distribution, \(h_v\), in a solitary closed Abrikosov vortex:

\[ h_v(\rho, \zeta) = \frac{\rho_s}{2} \int_0^{+\infty} dq J_1(\rho q)J_1(\rho_s q) \exp\left(-|\zeta|\sqrt{1 + q^2}\right). \] (5)

Expression (5) has a logarithmic divergence at the VL center \((\rho = \rho_s, \zeta = 0)\), because the London equation (1) is not valid in the region of the normal vortex core, where \(|\mathbf{r} - \mathbf{r}_s| \leq \xi\). The magnetic field was thus truncated in the usual way at distances \(\xi\) from the VL center. The toroidal-shaped vortex with a large radius, \(R_s \gg \lambda\), is similar to the linear vortex, and the magnetic field in this vortex varies significantly over distances of the order of \(\lambda\). In the opposite limit \(\xi \ll R_s \ll \lambda\), the spatial region occupied by the toroidal vortex is determined primarily by the radius \(R_s\).

The London equation (1) implies the following expression for the Gibbs free energy, \(G\):

\[ G = \frac{\lambda^2}{8\pi} \int_S dS [\mathbf{H} \cdot \nabla \mathbf{H}] + \frac{1}{8\pi} \int_V dV \mathbf{H} \cdot \left(\mathbf{H} + \nabla \times \mathbf{H}\right). \] (6)

For the considered solitary toroidal vortex, the surface integral is zero and the free energy, \(G_v\), is determined by:

\[ G_v(\rho_s) = \frac{\Phi_0^2}{8\pi \lambda^2} L_s h_s(\rho_s), \] (7)

where \(h_s(\rho_s) \equiv h_v(\rho = \rho_s, \zeta = 0)\) is the magnetic field value at the VL center and \(L_s\) is the VL length. In case \(R_s \gg \lambda\), the free energy of the toroidal vortex coincides with the energy \(G_L\), of the rectilinear Abrikosov vortex of the length \(L_s\):

\[ G_v \approx G_L = \left(\frac{\Phi_0}{4\pi \lambda}\right)^2 L_s K_0(\xi/\lambda), \xi \ll \lambda \ll R_s, \] (8)

where \(K_0\) is the McDonald function. Figure 2 shows the field, \(h_s\), and the energy, \(G_v\), versus the radius, \(R_s\). Since the energy, \(G_v\), grows monotonically with \(R_s\), the toroidal vortex is unstable, tending to contract toward the \(z\) axis and ultimately collapsing.
By integrating the distribution (5) over the \((\rho, \zeta)\) half-plane, we can calculate the magnetic flux, \(\Phi_v\), in the annular vortex:

\[
\Phi_v(\rho_s) = \Phi_0 \left(1 - \rho_s K_1(\rho_s)\right),
\]

where \(K_1\) is the modified Bessel function of the second kind. Therefore, the flux, \(\Phi_v\), depends on the VL radius, \(R_v\), and tends to the asymptotic value \(\Phi_0\) at \(R_s \gg \lambda\) (Fig. 2).

Utilizing the expression

\[
j = \left(\frac{\hbar e}{m}\right)|\psi|^2 \left(\nabla \chi - \frac{2e}{\hbar c} A\right)
\]

for the superconducting current density, \(j\), one can write down the condition for the fluxoid of the closed vortex:

\[
\Phi_v + \frac{2\pi \lambda^3}{c} \int_{-\infty}^{+\infty} d\zeta j_z(\rho = 0, \zeta) = \Phi_0.
\]

Here \(\psi = |\psi| \exp(i\chi)\) is the complex order parameter in the Ginzburg-Landau theory, \(A\) is the vector potential of the magnetic field, and \(j_z(\rho = 0, \zeta)\) is the \(z\) component of the current density, \(j\), yielded by expression (10). Using Eqs. (9,11) and assuming that the relation

\[
\int_{-\infty}^{+\infty} d\zeta j_z(\rho = 0, \zeta) \simeq 2R_s j_z/\lambda
\]

is valid for a small VL circle of radius, \(R_s \ll \lambda\), one can estimate the average current density, \(\overline{j_z}\), at the \(z\) axis:

\[
\overline{j_z} \simeq \frac{c\Phi_0}{4\pi \lambda^2 R_s}.
\]

A decrease of a closed vortex of a radius, \(R_s\), is thus accompanied by an increase in the superconducting current density at the \(z\) axis. One can easily obtain that for the VL radii, \(R_s \leq R^d_s\), where

\[
R^d_s = 3\sqrt{3} \pi \xi,
\]

the average current density, \(\overline{j_z}\), exceeds the current density of the superconductivity destruction, \(j_c\).
Note that $R_s^d \gg \xi$, so the London approximation is still valid. If the average current density $\overline{J_z}$ exceeds the depairing current density $j_c$, then the Cooper pairing is destroyed and a normal electron stream forms in the vicinity of the $z$ axis. The closed vortex contraction finishes in confluence of the additional normal region occurring at the $z$ axis with a normal region existing in the vortex core. The phase difference of the order parameter, present in going round VL, disappears due to collapse and induces thereby the voltage $V$, yielded by the relation $\partial \varphi / \partial t = 2eV/\hbar$. This voltage leads to the vortex energy dissipation. The processes taking place near the torus axis during the closed vortex collapse, resemble the formation of the phase-slip centers in narrow superconducting channels (whiskers), when the current density in these channels exceeds the critical value [13]. Since the topology of the magnetic field in the toroidal vortex is determined by the external current flowing through the SC, one can assume that the periodic occurrence and collapse of these vortices create the resistive state in the superconducting channels with a bias current. This is valid when the transverse dimensions of the channel are larger than $\xi$ and the intrinsic magnetic field of the current affects essentially the transverse structure of the solution [13].

### III. A CLOSED ABRIKOSOV VORTEX IN A SUPERCONDUCTING CYLINDER OF AN ARBITRARY RADIUS

Let us consider the influence of the SC boundary on the structure and the properties of a closed vortex. In a superconducting cylinder of an arbitrary radius $R_c$ (Fig. 1), the dimensionless field distribution $h(\rho, \zeta) = H(r, z)\lambda^2/\Phi_0$, in a toroidal vortex with a VL of radius $R_s \leq R_c$, is described by the following equation:

$$\frac{\partial^2 h}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial h}{\partial \rho} + \frac{\partial^2 h}{\partial \zeta^2} - (1 + \frac{1}{\rho^2})h = -\delta(\rho - \rho_s)\delta(\zeta).$$

(16)

which coincides with Eq.(2). The corresponding boundary condition for the magnetic field produced by the vortex on the surface of the superconducting cylinder, is:
\[ h(\rho_c = R_c/\lambda, \zeta) = 0. \] (17)

Therefore, to determine the structure of the closed toroidal Abrikosov vortex, one should solve Eq.(16) with the boundary condition (17) in the region \( \rho \leq \rho_c \). Due to linearity of (16,17) we represent their solution as a superposition of the solutions for the solitary closed vortex (5) and for the homogeneous boundary problem inside the cylinder:

\[
\frac{\partial^2 h_c}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial h_c}{\partial \rho} + \frac{\partial^2 h_c}{\partial \zeta^2} - \left(1 + \frac{1}{\rho^2}\right)h_c = 0,
\]

(18)

\[ h_c(\rho = \rho_c, \zeta) = -h_v(\rho = \rho_c, \zeta), \rho \leq \rho_c. \] (19)

By representing the solution in the form

\[ h(\rho, \zeta) = h_v(\rho, \zeta) + h_c(\rho, \zeta), \]

(20)

we generalize the well-known procedure of determining the structure of the Abrikosov vortex with a rectilinear VL parallel to the plane surface of the SC, by means of supplementing the vortex with its mirror image, \( h_c(\rho, \zeta) \). The field and the current produced by the image are directed opposite to the corresponding values of the primary vortex, \( h_v(\rho, \zeta) \)[3]. In this case the image is distorted by the curvilinear surface of the cylinder. The solution of the homogeneous equation (18) for the cylinder \( \rho \leq \rho_c \) can be written as:

\[ h_c(\rho, \zeta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dp \exp (ip\zeta)C(p)I_1(\rho\sqrt{1 + p^2}), \]

(21)

where \( I_1 \) is the modified Bessel function of the first kind. The integration constant, \( C(p) \), is given by the boundary condition (19):

\[ C(p) = -\frac{\rho_s}{\sqrt{2\pi}} \frac{I_1(\rho_s\sqrt{1 + p^2})K_1(\rho_c\sqrt{1 + p^2})}{I_1(\rho_c\sqrt{1 + p^2})}. \] (22)

The solution of the homogeneous boundary problem (18,19), valid for \( |\rho_c - \rho_s| \geq \xi/\lambda \), is:

\[ h_c(\rho, \zeta) = -\frac{\rho_s}{\pi} \int_0^{+\infty} dp \cos(p\zeta) \frac{I_1(\rho_\sqrt{1 + p^2})I_1(\rho_s\sqrt{1 + p^2})K_1(\rho_c\sqrt{1 + p^2})}{I_1(\rho_c\sqrt{1 + p^2})}. \] (23)

Thus, the magnetic field structure, \( h(\rho, \zeta) \), of the solitary closed Abrikosov vortex inside a superconducting cylinder of an arbitrary radius, \( R_c \), is fully determined by the relations
An example of the magnetic field structure in the closed vortex, computed by (5,20,23), is given in Figure 3. It is essential that the cylinder surface affects the vortex structure only in the surface layer of a thickness $\lambda$. It is convenient to represent the magnetic flux, $\Phi$, in this vortex as:

$$\Phi = \Phi_v(\rho_s) - \Phi_v(\rho_c) \frac{\rho_s I_1(\rho_s)}{\rho_c I_1(\rho_c)}.$$  \hspace{1cm} (24)

Here the magnetic flux, $\Phi_v$, in a solitary toroidal vortex, determined by (9), is expressed explicitly. According to the boundary condition (17), the magnetic field produced by the vortex, vanishes on the surface and exists only inside the superconductor cylinder. The Gibbs free energy in this case is given by the formula

$$G(\rho_s) = \frac{\Phi_0^2}{8\pi \lambda^2} L_s h_s(\rho_s),$$  \hspace{1cm} (25)

where $h_s(\rho_s)$ is the magnetic field in the core, $h_s(\rho_s) = h_s(\rho = \rho_s, \zeta = 0)$, with allowance for the surface influence:

$$h_s(\rho_s) = h_v(\rho_s, 0) + h_c(\rho_s, 0).$$  \hspace{1cm} (26)

The magnetic flux, $\Phi(\rho_s)$, and the free energy, $G(\rho_s)$, versus the position of the toroidal vortex inside the cylinder are shown in Fig. 4. It can be readily seen that the closed vortex in the cylinder remains unstable and the interaction with the surface does not stabilize the vortex. Depending on the radius of the created VL, the vortex either moves towards the cylinder axis or disappears on the surface.

**IV. THE EXCITATION OF A CLOSED TOROIDAL VORTEX BY A CHARGED PARTICLE BUNCH**

Now we discuss the possibility for exciting a closed toroidal vortex by an external bunch of charged particles. The azimuthal structure which is required for forming a toroidal vortex is offered by the self-magnetic field, $H_q$, of a charge, $q$, moving at a velocity, $V$, parallel to the $z$ axis [14]:

---
\[ H_q = \frac{q \beta \gamma}{\lambda^2} \frac{\rho}{(\rho^2 + \gamma^2 \zeta^2)^{3/2}}. \]  

(27)

where \( \beta = V/c \), \( c \) is the velocity of light and \( \gamma = (1-\beta^2)^{-1/2} \) is the relativistic factor. Instead of dealing with the rigorous nonstationary problem, we use the following thermodynamic condition:

\[ G_q = G_v - \frac{1}{4\pi} \int d^3r(H_qH) \leq 0, \]  

(28)

which estimates the vortex formation energy in the external magnetic field, \( H_q \). Inequality (28) imposes an upper limit, \( R^m_s \left( R_s \leq R^m_s \right) \), on the radius of the vortex line which can be excited in this fashion. For a charge in relativistic motion (\( \beta \approx 1 \)), the dimensionless maximum radius, \( \rho^m_s = R^m_s / \lambda \), satisfies the equation:

\[ \frac{2}{\pi h_s(\rho^m_s)} \frac{1 - \exp(-\rho^m_s)}{\rho^m_s} = \frac{\alpha^{-1}}{Z}. \]  

(29)

Here \( \alpha = e^2/\hbar c \) is the fine-structure constant (spin-orbit constant), \( Z = q/e \) and \( e \) is the charge of an electron. Therefore, the only physical constant which determines the excitation of a toroidal vortex, as a macroscopic entity, is the fine-structure constant, \( \alpha \). Figure 3 plots the maximum vortex radius, \( \rho^m_s \), yielded by Eq.(29) versus the magnitude of the charge, \( q \). It should be mentioned, however, that the London approximation used here, is valid only when \( R^m_s \gg \xi \). Hence a toroidal vortex can be excited only by a rather large magnitude of the moving charge, i.e. \( Z = q/e \approx \alpha^{-1} \approx 137 \). This may be realized, for example, by a bunch of charged particles. Note, that the closed vortex excitation by a bunch of charged particles is, in a sense, the inverse process of the vortex collapse described in Sec. II of the present paper.

V. CONCLUSION

The magnetic field structure, the magnetic flux and the free energy of a closed Abrikosov vortex with a toroidal structure of a VL in an unbounded SC and inside a superconducting cylinder of an arbitrary radius, have been calculated in the London approximation. The
magnetic flux and the free energy of the toroidal vortex strongly depend on the VL radius. The individual toroidal vortex is always unstable: it either contracts towards the torus axis or emerges on the cylinder surface. The VL contraction (decrease of its radius) is accompanied by breaking of the Cooper pairs and generates a normal electron stream in the vicinity of the torus axis. The possibility of exciting a closed toroidal vortex by an external bunch of charged particles has been analyzed. It has been shown that the fine-structure constant is the determining parameter of the thermodynamic condition for the excitation of a vortex by a moving bunch of charged particles.
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FIGURES

FIG. 1. Toroidal Abrikosov vortex inside a superconducting cylinder of radius \( R_c \). VL (circumference of radius \( R_s \)) is shown by a dash-dotted line.

FIG. 2. 1 - Magnetic field at the center of the vortex line, \( h_s \); 2 - the magnetic flux, \( \Phi_v \); inset - the free energy, \( G_v \), all as functions of the radius of the vortex line, \( R_s \). Shown for comparison by the dashed line is the free energy of a linear Abrikosov vortex with a length \( L_s = 2\pi R_s \) \( (\kappa = \lambda/\xi = 100) \).

FIG. 3. Magnetic field structure of a closed toroidal Abrikosov vortex inside a superconducting cylinder \( (R_s = 3\lambda; R_c = 5\lambda; \kappa = \lambda/\xi = 100) \).

FIG. 4. 1 - Magnetic flux, \( \Phi \); 2 - free energy, \( G \), of a closed toroidal Abrikosov vortex inside a superconducting cylinder versus the vortex line radius, \( R_s \) \( (R_c = 5\lambda; \kappa = \lambda/\xi = 100) \).

FIG. 5. Maximum permissible radius of a vortex line, \( R_{m}^{m} = \lambda \rho_{m}^{m} \), versus the magnitude of the charge of moving particles, \( q \) \( (\kappa = \lambda/\xi = 100) \).