Quantum spirals

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Abstract
Quantum systems often exhibit fundamental incapability to entertain vortex. The Meissner effect, a complete expulsion of the magnetic field (the electromagnetic vorticity), for instance, is taken to be the defining attribute of the superconducting state. Superfluidity is another, close-parallel example; fluid vorticity can reside only on topological defects with a limited (quantized) amount. Recent developments in the Bose–Einstein condensates produced by particle traps further emphasize this characteristic. We show that the challenge of imparting vorticity to a quantum fluid can be met through a nonlinear mechanism operating in a hot fluid corresponding to a thermally modified Pauli–Schrödinger spinor field. The thermal baroclinic effect is represented by a nonlinear, non-Hermitian Hamiltonian, which, in conjunction with spin vorticity, leads to new interesting quantum states; a spiral solution is explicitly worked out in a simple field-free model.

Keywords: quantum fluid, vortex, baroclinic effect, spin

(Some figures may appear in colour only in the online journal)

1. Introduction

The magic of quantum correlations has been often invoked to explain the exotic and extraordinary phenomena like the superfluidity, superconductivity, and Bose–Einstein condensation (BEC), displayed by, what may be called, the quantum fluids [1–9]. What links all these diverse systems is the absence of vorticity. The Meissner effect, a complete expulsion of the magnetic field (the electromagnetic vorticity), for instance, is taken to be the defining attribute of the superconducting state [1, 3] (see also [10] for a characterization of superconductivity as vanishing of the total vorticity, a sum of the fluid and electromagnetic vorticities).
Of course, these highly correlated quantum states are accessible only under very special conditions, in particular at very low temperatures. It is, perhaps, legitimate to infer that these quantum fluids, when they are not in their super phase, may, in fact, entertain some sort of vorticity. In this article, we explore how such a vortical state may emerge in a quantum system, whose standard dynamical equations (like the Schrödinger equation) are not fundamentally suitable for hosting a vortex.

The investigation of the quantum vortex states constitutes a fundamental enquiry, because such states, like their classical counterparts, aught to be ubiquitous. And again as to their classical counterparts, the vorticity will lend enormous diversity and complexity to the behavior and properties of quantum fluids. Unveiling the mechanisms responsible for creating/sustaining vorticity will, surely, advance our understanding of the dynamics of the phase transition—from zero vorticity to finite vorticity and vice versa. We will carry out this deeper enquiry, aimed at bridging the vorticity gap, by exploring a quantum system equivalent to a hot fluid/plasma [11]. We will show that the thermodynamic forces induce two distinct fundamental changes in the dynamics: the Hamiltonian becomes (1) nonlinear by the thermal energy, and (2) non-Hermitian by the entropy. By these new effects, a finite vorticity becomes accessible to the quantum fluid. Such a vorticity-carrying hot quantum fluid could define a new and interesting state of matter. Within the framework of a hot Pauli–Schrödinger quantum fluid, we will demonstrate the existence of one such state—the quantum spiral. We believe that it marks the beginning of a new line of research.

To highlight the new aspect of our construction, we present a short overview of papers on the classical-quantum interplay. The first set of investigations [12–26] is devoted to deriving (and studying) the fluid-like systems from the standard equations of quantum mechanics, while the second set [27–33] constructs a quantum mechanics equivalent to a given classical system. Building from the energy momentum tensor for a perfect isotropic hot fluid, Mahajan & Asenjo [11] have recently demonstrated that the emergent quantum mechanics of an elementary constituent of the Pauli–Schrödinger hot fluid (called a fluidon) is nonlinear as distinct from the standard linear quantum mechanics; the thermal interactions manifest as the fluidon self-interaction. Through a reexamination of this thermal interaction, we begin our quest for the mechanism of creating quantum vorticity. In addition to being a source of nonlinearity, the thermal interaction for the quantum fluid also endows it with an entropy. In the present work, the dynamics is isentropic. The formulated model equation will, thus, have a proper PT symmetry, satisfying the particle and energy conservation laws—the entropy production, by some irreversible processes, is a different subject; some dissipative models have been introduced [34–39], which do not conserve the particle number. An inhomogeneity of the entropy, leading to the baroclinic effect, is the key element of our study. In the next section, we elucidate the fundamental role of inhomogeneous entropy in the creation mechanism of vortexes.

2. Circulation law—the vortex and heat

There exists a deep relationship between the vortex and a heat cycle, which is mediated by an entropy. We may consider a vortex in general space; by vortex we mean finiteness of a circulation (or, non-exact differential). A heat cycle $\oint TdS = 0$ ($T$ is the temperature and $S$ is the entropy) epitomizes such a circulation.

Upon the realization of the thermodynamic law on a fluid, the heat cycle is related to the circulation of momentum, i.e., the mechanical vortex. For a fluid with a density $\rho$, pressure $P$, and enthalpy $H$, obeying the thermodynamic relation $TdS = dH - \rho^{-1}dP$, a finite heat cycle
\( \oint TdS \equiv 0 \) is equivalent to the baroclinic effect \( \oint \rho^{-1}dP \equiv 0 \) (the exact differential \( dH \) does not contribute a circulation; \( \oint dH \equiv 0 \)). Notice that the entropy, a deep independent attribute of thermodynamics, is the source of the baroclinic effect; such an effect will be encountered whenever a field has a similar internal degree of freedom represented by some scalar like an entropy.

Kelvin’s circulation theorem says that, as far as the specific pressure force \( \rho^{-1}dP \) (or, the heat \( TdS \)) is an exact differential, the fluid (so-called barotropic fluid) conserves the circulation \( \oint \nabla \times \mathbf{v} \) of the momentum \( \mathbf{v} \) along an arbitrary co-moving loop \( \Gamma \). Therefore, a vorticity-free flow remains so forever. As the antithesis, non-exact thermodynamic force \( \rho^{-1}dP \) (or, equivalently, a heat cycle \( \oint TdS \equiv 0 \)) violates the conservation of circulation of momentum, leading to the vortex creation.

3. Vortex in spinor fields

To formulate a quantum-mechanical baroclinic effect by quantizing a classical fluid model, the correspondence principle is best described by the Madelung representation of wave functions [12, 13] (see the appendix). We must, however, remember that a scalar (zero-spin) Schrödinger field falls short of describing a vortex, because the momentum field is the gradient of the eikonal of the Schrödinger field (or, in the language of classical mechanics, the momentum is the gradient of the action), which is evidently curl-free. This simple fact is, indeed, what prevents a conventional Schrödinger quantum regime from hosting vorticity.

In a set of classic papers [14–17], Takabayasi showed that the differences in the phases of spinor components could generate a spin vorticity in the ‘Madelung fluid’ equivalent of the Pauli–Schrödinger quantum field. In this paper, we investigate an additional source of vorticity provided by a baroclinic mechanism in a thermally modified nonlinear Pauli–Schrödinger system, obtained here, by adding a thermal energy \( U \) to the Pauli–Schrödinger Hamiltonian.

In order to delineate the new effect in the simplest way, we consider a minimum, field-free Hamiltonian

\[
\mathcal{H} = \int \left[ \frac{1}{2m} \left( \frac{\hbar}{i} \nabla \Psi \right)^{*} \cdot \left( \frac{\hbar}{i} \nabla \Psi \right) + U\Psi^{*} \cdot \Psi \right] dx,
\]

where \( \Psi = (\psi_{1}, \psi_{2}) \) is a two-component spinor field, and \( U \) is a thermal energy. Notice that the conventional potential energy is replaced by the thermal energy. The formulation of \( U \) as a function of \( \Psi \) is the most essential element of our construction. In classical thermo/fluid dynamics, the thermal energy is generally expressed as \( U = U(\rho, S) \) with the density \( \rho \) and the entropy \( S \). Although \( \rho \) is readily expressed as \( \Psi^{*}\Psi \), finding an expression for \( S \) is more challenging. It is the right juncture to inform the reader that for \( U = U(\rho) \), the sought after baroclinic effect is absent; see [11].

The Madelung representation of the wave function

\[
\psi_{j}(x, t) = \sqrt{\rho_{j}(x, t)} e^{iS_{j}(x, t)/\hbar} \quad (j = 1, 2)
\]

converts the two complex field variables \( (\psi_{1}, \psi_{2}) \) into four real variables \( (\rho_{1}, S_{1}, \rho_{2}, S_{2}) \). It will be, however, more convenient to work with an equivalent set:
The four-momentum becomes

\[ p^\nu = \frac{1}{\rho} \mathcal{R}(\Psi^* \cdot i\hbar \partial^\nu \Psi) = -\left( \partial^\nu \varphi + \frac{\mu}{\rho} \partial^\nu \sigma \right), \]

where \((x_0, x_1, x_2, x_3) = (t, -x, -y, -z)\), and \((\partial^0, \partial^1, \partial^2, \partial^3) = (\partial_t, -\nabla)\). The spatial part of (4),

\[ p = \nabla \varphi + \frac{\mu}{\rho} \nabla \sigma \]

reads as the Clebsch-parameterized momentum field [40–44]. The second term of the right-hand side of (5) yields a vorticity:

\[ \nabla \times p = \nabla \left( \frac{\mu}{\rho} \right) \times \nabla \sigma. \]

In (6), we have assumed that \( \varphi \) does not have a phase singularity. If \( \varphi \) is an angular (multi-valued) field, as in the example of quantum spirals given later, a circulation, representing a point vortex, will be created by the singularity of \( \nabla \times (\nabla \varphi) \) (mathematically, a cohomology).

We may regard \( \sigma \) as a Lagrangian label of scalar fields co-moving with the fluid (see the appendix). Hence, for an isentropic process, we parameterize entropy as \( S = S(\sigma) \) to put \( U = U(\rho, \sigma) \), completing the process of identification of the thermal variables with the wave function. The enthalpy and temperature are, respectively, given by

\[ H = \frac{\partial (\rho U)}{\partial \rho}, \quad T = \frac{\partial U}{\partial S}. \]

Denoting \( S'(\sigma) = dS(\sigma)/d\sigma \), we may evaluate \( T = (\partial U/\partial \sigma)/S'(\sigma) \).

4. Thermally modified nonlinear Pauli–Schrödinger equation

We are ready to derive the determining equation. In terms of \( \Psi \), the canonical 1-form reads \( \Theta = \int p^0 \rho \, dx \). The variation of the action \( \int (\Theta - \mathcal{H}) \, dr \) by \( \Psi \) yields a thermally modified nonlinear Pauli–Schrödinger equation:

\[ i\hbar \partial_t \psi_j = -\frac{\hbar^2}{2m} \nabla^2 \psi_j + (H - iG_j) \psi_j \quad (j = 1, 2), \]

where

\[ G_j = (-1)^j \hbar \frac{S'(\sigma) T \rho}{4\rho_j} \quad (j = 1, 2). \]

The following results can be readily derived by the new equation (8):

(i) The terms \(-iG_j \psi_j\) \((j = 1, 2)\) on the right-hand side of (8) represent the baroclinic effect, by which the generator of the system
\[ A = \begin{pmatrix} \frac{\hbar^2}{2m} \nabla^2 + H - iG_1 & 0 \\ 0 & -\frac{\hbar^2}{2m} \nabla^2 + H - iG_2 \end{pmatrix} \] (10)

is non-Hermitian (see remark 1). However, the particle number \((\int \rho \, dx)\) and the energy \((\mathcal{H})\) are preserved as constants of motion (in fact, the determining equation is derived from a canonical action principle). This is in marked contrast to other non-Hermitian systems that model dissipative processes \([39]\). Our system is isentropic, and is time-reversal invariant with respect to the transformation

\[ T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^*, \] (11)

where * denotes the map to the complex conjugate.

(ii) When \(S'(\sigma) = 0\) (i.e., the fluid is homentropic), the baroclinic terms are zero. Then, (8) consists of two coupled nonlinear Schrödinger equations (the nonlinear coupling comes through \(H\) being a function of \(\rho\)). Of course, we may put \(\psi_2 \equiv 0\), and then, the system reduces into the standard scalar-field nonlinear Schrödinger equation governing \(\psi_1\). It is well known that, in a one-dimensional space, we obtain solitons when \(H = a \rho\) (bright solitons with \(a < 0\), and dark solitons with \(a > 0\)). The nonlinear coupling of the two components \(\psi_1\) and \(\psi_2\) induces chaotic behavior interestingly, however, \(\rho = |\psi_1|^2 + |\psi_2|^2\) remains ordered. These features are displayed in figure 1, where a representative solution in the barotropic \((G_i = 0)\) limit is plotted. The reader is referred, for example, to \([45]\) for a more complex coupling of differently polarized solitons (sometimes called ‘vector’ solitons) propagating on a birefringent medium, where we assume a Hamiltonian with nonlinearity of the form \(H(\rho_1, \rho_2)\).

(iii) When the baroclinic terms are finite, there is no one-dimensional (plane wave) solution.

In fact, upon substitution of \(\psi_j = e^{i(k_j x + k_j z - \omega t)}/\sqrt{2}\phi_j(x)\) \((j = 1, 2)\) into (8), we obtain an eigenvalue problem

\[ \frac{\hbar^2}{2m} \frac{d^2}{dx^2} \phi_j = (\lambda + H - iG_i) \phi_j \quad (j = 1, 2), \] (12)

where \(\lambda = (k_1^2 + k_2^2)/2m - \omega\). Half of the eigenvalues of this operator (evaluated locally at each \(\lambda\)), \(\pm \sqrt{\lambda + H - iG_i}\), have positive real parts whenever \(G_i \neq 0\), and thus (12), cannot have a bounded solution for any \(\lambda\).

The non-existence of a one-dimensional (plane-wave) solution in a baroclinic system emphasizes the fact that the baroclinic effect is absent in a one-dimensional system. However, we do find interesting solutions in two-dimensional space.

Remark 1. An operator \(A\) is Hermitian, if \((Au, v) = (u, Av)\) for all \(u\) and \(v\) belonging to the domain of \(A\). When \(A\) is a densely defined linear operator, it can be resolved with real spectra (von Neumann’s theorem); hence the group \(\{e^{itH}; t \in \mathbb{R}\}\) is a unitary group, implying the particle conservation (note that a non-Hermitian operator does not necessarily have complex spectra). When the generator \(A\) of (10) is linearized (i.e., evaluated for a fixed \(\Psi\)), it has complex spectra. Hence, the general solution of the linearized system does not conserve the

The time-reversal operator \(T\) may be written as \(i\sigma_2 \ast (\sigma_2)\) (the second Pauli matrix), by which the spin flips the sign. Among the Clebsch parameters of (3), the total density \(\rho\) and the entropy label \(\sigma\) are unchanged, while \(\mu\) and \(\varphi\) are changed.
particle number (i.e., the group generated by the linearized operator is not unitary). The conservation of the particle number $x_\text{d} \Omega$ in the exact system (8) is due to its nonlinearity that matches the baroclinic terms $G_1$ and $G_2$ to cancel the changes in $\int \rho_1 \, dx$ and $\int \rho_2 \, dx$.

5. Quantum spirals

Let us assume a solution of a spiral form:

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} e^{i(n\theta + \beta_j(r) - \omega t)} \phi_1(r) \\ e^{i(n\theta + \beta_j(r) - \omega t)} \phi_2(r) \end{pmatrix}. $$

The azimuthal mode number $n$ gives the number of arms. The phase factor $\beta_j(r)$ determines their shape; for example, when $\beta_j(r)$ is a linear function of $r$, we obtain a Archimedean spiral. The factor $\phi_j(r)$ yields the radial modulation of amplitudes. We find that the nonlinear terms

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Figure 1. A typical nonlinear Pauli–Schrödinger field in one-dimensional space. Here, $\hat{\sigma}_y = 0$ (stationary field), and $H = a \rho$ with $a = -2 \times (\hbar^2/2m)$. The horizontal axis is the coordinate $x$ (in the unit of an arbitrary system size). (a) The two spinor components (here, real-valued functions) $\psi_1(x)$ (blue solid) and $\psi_2(x)$ (red dashed) are coupled though the nonlinear enthalpy coefficient $H = a (|\psi_1|^2 + |\psi_2|^2)$, exhibiting chaotic oscillations. (b) The densities $\rho_1 = |\psi_1|^2$ (purple solid) and $\rho_2 = |\psi_2|^2$ (red dashed) oscillates irregularly, while the total density $\rho = \rho_1 + \rho_2$ (blue bold) remains ordered.
are functions only of \( r \); the azimuthal mode number \( n \), therefore, is a good quantum number. Inserting (13) into (8), we obtain (14) into (8),

\[
\phi''_j + \left( \frac{1}{r} + i \beta'_j \right) \phi'_j = \left( -\omega + \frac{n^2}{r^2} + (\beta'_j)^2 + H \right)
- i \left( \beta''_j + \frac{\beta'_j}{r} - G_j \right) \phi_j. \tag{16}
\]

Bounded solutions are obtained when the non-Hermitian term vanishes; one must, then, solve the system of simultaneous equations

\[
\phi''_j + \left( \frac{1}{r} + i \beta'_j \right) \phi'_j = \left( -\omega + \frac{n^2}{r^2} + (\beta'_j)^2 + H \right) \phi_j, \tag{17}
\]

\[
\beta''_j + \frac{\beta'_j}{r} - G_j = 0. \tag{18}
\]

By (18), the baroclinic term \( G_j \) generates \( \beta_j(r) \) that determines the shape of spiral. Evidently, in a barotropic field (\( G_j = 0 \)), spirals do not appear (\( \beta_j(r) = 0 \)).

To construct explicit examples, let us consider an ideal gas that has an internal energy such that

\[
U = c_v (\rho \exp^{S - \alpha})^{1/c_v}, \tag{19}
\]

where \( c_v \) (specific heat at constant volume per particle) and \( \alpha_0 \) are constants. We normalize the Boltzmann constant \( k = 1 \) by measuring the temperature \( T \) in energy units. For simplicity, we assume \( S = \sigma \) (thus, the constant \( S(\sigma) = 1 \) absorbs the dimension of \( h \)) to evaluate

\[
T = (\rho \exp^{\sigma - \alpha})^{1/c_v}, \quad H = (c_v + 1) T. \tag{20}
\]

Substituting (14) and (15), we may write the coefficients \( H \) and \( G_j \) in terms of \( \phi_j \) and \( \beta_j \) (\( j = 1, 2 \)).

As an example of the numerical solutions of this system (with \( c_v = 1 \) and \( \alpha_0 = 0 \)), we display in figure 2, a typical \( n = 2 \) solution exhibiting twin spirals (opposite sense) of the two

Figure 2. Dual spirals created in a baroclinic Pauli–Schrödinger field; the density plots of \( \Re \psi_1 \) (left) and \( \Re \psi_2 \) (right) show opposite-sense spirals. Here \( \omega = 4.5 \).
components of the spinor $\Psi$. Figure 3 picks up the phase factors $\beta_j(r)$, the amplitude factors $r_j^2$, and the temperature $T$ from the spiral spinor fields of figure 2.

We note that this quantum spiral is a kind of 'point vortex' that has zero vorticity excepting the central phase singularity. Both $\rho$ and $\sigma$ (thus $S(\sigma)$) are functions of only $r$, hence the classical baroclinic term $\nabla S \times \nabla T$ is zero for $r > 0$. However, the quantum baroclinic effect, represented by the non-Hermitian terms $G_j$, is in function to create the spiral arms. Figure 4(a) shows that when the baroclinic term is zero, (then, the generator is Hermitian), no spiral structures, even with a finite $n$, are created. This is because the phase factors $\beta_j(r)$ become zero when $G_j = 0$; see (18). On the contrary, the baroclinic Pauli–Schrödinger equation has axisymmetric ($n = 0$) solutions also (see figure 4(b)).

6. Conclusion

We have shown that the quantum fluid with internal thermal energy is capable of supporting entirely new quantum states like the quantum spirals we have derived in the present simplified model. The harnessing of the profound effect of entropy, in a thermal spin quantum system, has led us to a new mechanism (whose classical counterpart is the famous baroclinic effect
generating, for example, hurricanes) that substantially extends the range of vortical states accessible to quantum systems.

We note that although the conventional spin forces (either spin-magnetic field interactions or spin–spin interaction; see the appendix) may amplify or sustain vorticities, they cannot generate it from zero. The baroclinic effect is a creation mechanism that works without a seed. A close analogy is the respective roles of dynamo amplification of magnetic field [46] and the Biermann battery mechanism [47] in a classical plasma; the former needs a seed magnetic field.

We also note that the baroclinic effect we have studied is an isentropic process, which differs from dissipation mechanisms like friction. For example, a ‘phenomenological Hamiltonian’ such that \((1 + i\gamma)\hat{H}\) (\(\gamma\) is a damping coefficient, and \(\hat{H}\) is a Hermitian operator) was proposed to describe damping of vortexes in BEC [34, 35]. Or, a finite-temperature Bose gas is modeled by a modified Gross–Pitaevskii equation including an effective friction term, which is coupled with a quantum Boltzmann equation describing a thermal cloud [36–38] (see also tutorial [39]). In contrast to the presently formulated isentropic model, dissipation destroys the time-reversal symmetry, resulting in non-conservation of the particle number and the energy of the condensate component. The reader is referred to [26] for a dissipative process in nonlinear optical systems (quantum fluids of light). Such dissipative systems cannot be formulated by canonical action principles; the reader is referred to other variational approaches, for example, applying Rayleigh’s dissipation function [48] or the stochastic variational method [49].

One expects to find a variety of new/interesting states when this work is extended for more encompassing quantum systems: the thermal Pauli–Schrödinger system and the relativistic thermal Dirac (Feynman–Gellmann) equation coupled to the electromagnetic field. For both of these systems, baroclinic terms can be readily incorporated in known formalisms [11, 24, 32]. Coriolis force, a close cousin of the magnetic force, could also be included as a gauge field [33]. When we consider higher-order spinors (for example, spin-1 representation of SU(2) fields may be applied for Bose gas in a trap [50, 51]), the number of Clebsch parameters increases, and the field starts to have a helicity [44].
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Appendix. Correspondence principle

We explain the correspondence principle relating the classical and quantized fields. In terms of the real variables (3), the Hamiltonian (1) reads \( \mathcal{H} = \mathcal{H}_c + \mathcal{H}_q \) with

\[
\mathcal{H}_c = \int \left[ \frac{\nabla \varphi + \frac{\mu}{2m} \nabla \sigma}{\rho} + U(\rho, \sigma) \right] \rho \, dx, \tag{21}
\]

\[
\mathcal{H}_q = \int \frac{\hbar^2}{8m} \left( \frac{\nabla \rho^2}{\rho^2} + \sum_{\ell} \nabla S_\ell^2 \right) \rho \, dx, \tag{22}
\]

where \( S_\ell = \rho^{-1} \psi^* \sigma \psi \) (\( \sigma \) are the Pauli matrices). Here \( \mathcal{H}_c \) is a classical Hamiltonian generating fluid dynamics [41]. With the canonical 1-form \( \Theta = \int \rho^0 \, dx = -\int (\rho \partial_\varphi + \mu \partial_\sigma), \) the variation of the classical action \( \int (\Theta - \mathcal{H}_c) \, dt \) with respect to the canonical variables \( (\rho, \varphi, \mu, \sigma) \) yields Hamilton’s equation:

\[
\partial_t \rho = \partial_\rho \mathcal{H}_c = -\nabla \cdot (\rho \mathbf{v}), \tag{23}
\]

\[
\partial_t \varphi = -\partial_\varphi \mathcal{H}_c = -\mathbf{v} \cdot \nabla \varphi + m |\mathbf{v}|^2 / 2 - H, \tag{24}
\]

\[
\partial_t \mu = \partial_\mu \mathcal{H}_c = -\nabla \cdot (\mathbf{v} \mu) + \rho T, \tag{25}
\]

\[
\partial_t \sigma = -\partial_\sigma \mathcal{H}_c = -\mathbf{v} \cdot \nabla \sigma, \tag{26}
\]

where \( \mathbf{v} = \mathbf{p} / m \) is the fluid velocity, (23) is the particle conservation law, (26) is the entropy conservation law (justifying the parameterization \( U(\rho, \sigma) \)), and the combination of all equations with the thermodynamic relation \( \nabla H - T \nabla \sigma = \rho^{-1} \nabla P \) (\( P \) is the pressure) yields the momentum equation

\[
\partial_t \mathbf{p} + (\mathbf{v} \cdot \nabla) \mathbf{p} = -\rho^{-1} \nabla P. \tag{27}
\]

The system (23)–(26) is an infinite-dimensional Hamiltonian system endowed with a canonical Poisson bracket such that

\[
\{ \rho (x), \varphi (y) \} = \delta (x - y), \quad \{ \mu (x), \sigma (y) \} = \delta (x - y); \tag{28}
\]

all other brackets are zero. By (2) and (3), the Poisson algebra (28) is equivalent to the Lie algebra of the second-quantized Pauli–Schrödinger field acting on the Fock space of either Bosons or Fermions [42]:

\[
[\psi_j (x), \psi_k^* (y)]_\pm = (i \hbar)^{-1} \delta_{jk} \delta (x - y),
\]

\[
[\psi_j (x), \psi_j (y)]_\pm = 0, \quad [\psi_k^* (x), \psi_j^* (y)]_\pm = 0. \tag{29}
\]

Based on this correspondence principle, we can quantize the classical field by adding \( \mathcal{H}_q \) to \( \mathcal{H}_c \); the action principle with respect to \( \Psi \) yields (8). The same action principle with respect to the real variables (3) yields the fluid representation. The particle conservation law (23) is the
same. On the right-hand side of (27), \( \mathcal{H}_q \) adds quantum forces [14, 15]

\[
F_q = \nabla \left( \frac{\hbar^2 \nabla^2 \sqrt{\rho}}{2m \sqrt{\rho}} \right) - \sum_{\ell} \frac{\hbar^2}{4m \rho} \nabla \cdot \left[ \rho \nabla S_d \otimes \nabla S_d \right],
\]

(30)

by which the fluid picture becomes considerably complicated. The first term is the so-called Bohm pressure, which vanishes in the limit of \( \hbar \to 0 \). We note that \( \nabla S_d \) produces a term of order \( \hbar^{-1} \); hence, the spin force (second term) includes an order-unity contribution. The transport of the entropy label \( \sigma \) is also complicated in a spinor fluid.

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