ON THE DIRAC OPERATOR FOR A TEST ELECTRON IN A REISSNER–WEYL–NORDSTRÖM BLACK HOLE SPACETIME

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Abstract. The present paper studies the Dirac Hamiltonian of a test electron with a domain of bi-spinor wave functions supported on the static region inside the Cauchy horizon of the subextremal RWN black hole spacetime, respectively inside the event horizon of the extremal RWN black hole spacetime. It is found that this Dirac Hamiltonian is not essentially self-adjoint, yet has infinitely many self-adjoint extensions. Including a sufficiently large anomalous magnetic moment interaction in the Dirac Hamiltonian restores essential self-adjointness; the empirical value of the electron’s anomalous magnetic moment is large enough. The spectrum of the subextremal self-adjoint Dirac operator with anomalous magnetic moment is purely absolutely continuous and consists of the whole real line; in particular, there are no eigenvalues. The same is true for the spectrum of any self-adjoint extension of the Dirac operator without anomalous magnetic moment interaction, in the subextremal black hole context. In the extremal black hole sector the point spectrum, if non-empty, consists of a single eigenvalue, which is identified.
1. Introduction

It is well-known that if $M > 0$ is the ADM mass of the general-relativistic Reissner–Weyl–Nordström (RWN) spacetime and $Q > 0$ its charge, and if $G$ denotes Newton’s constant of universal gravitation, then the RWN spacetime features a naked singularity when $GM^2 < Q^2$ and a black hole when $GM^2 \geq Q^2$; in the borderline case $GM^2 = Q^2$ one speaks of the extremal RWN black hole, while $GM^2 > Q^2$ is called the subextremal black hole parameter sector, cf. [17]. For the original publications, see [26], [34], and [25].

In their paper “The general-relativistic hydrogen atom” [10] Cohen and Powers rigorously studied the general-relativistic Dirac operator $H$ ([35], [28], [6]) for a test electron in the RWN spacetime of a point nucleus for both the naked singularity sector and the subextremal black hole sector. They made the startling discovery that in the naked singularity sector $H$ is not well-defined, while for the black hole sector there is a well-defined $H$ but its essential spectrum is the whole real line, void of any eigenvalues.

The truly startling part of the discoveries of Cohen and Powers [10] concerns the naked singularity sector, for it means that ‘switching on relativistic gravity’ destroys the well-defined (i.e. essentially self-adjoint) special-relativistic purely electrical hydrogenic ion problem for all parameter values which correspond to empirically known nuclei ($1 \leq Z \leq 118$ and $m_p \leq M < 400m_p$; here $m_p$ denotes the proton mass). In more technical language, general-relativistic gravity is not at all a ‘weak perturbation’ (see [22]) of special-relativistic electricity in the atomic realm, notwithstanding the general folklore that ‘the gravitational interaction between an electron and a nucleus is too weak to be significant,’ cf. [12].

\[\text{\textsuperscript{1}One may be tempted to consider this result as a vindication for the widespread opinion that “naked singularities are considered unphysical” (cf. [16], p.562). However, this opinion propagates an unfortunate myth. It is based on a misunderstanding of Penrose’s weak cosmic censorship hypothesis, which surmises that gravitational collapse of cosmic matter does not form a naked singularity. In its strict sense the surmise is wrong, as shown first by Christodoulou [7], [8] for spherically symmetric collapse of scalar matter, and most recently by Rodnianski and Shlapentokh-Rothman [27] for collapsing gravitational waves without symmetry assumption; yet it is expected that these scenarios are not generic (this was confirmed for the spherically symmetric scalar case, also by Christodoulou [9]), and that generically (or: typically) a gravitational collapse of cosmic ‘matter’ will not form a naked singularity. However, the point nuclei used in quantum-mechanical models of hydrogenic ions, whether of the kind created in our laboratories, or the hypothetical ‘hyper-heavy’ type “out there” in space, are not assumed to have formed through gravitational collapse of charged matter in cosmic proportions. In short, the weak cosmic censorship hypothesis, even if generically true, is entirely irrelevant to the problem of general-relativistic hydrogenic ions.}\]
Also the special-relativistic Dirac Hamiltonian for hydrogenic ions with purely electrical Coulomb interactions is not always essentially self-adjoint [on the minimal domain $C_0^\infty(\mathbb{R}^3\setminus\{0\})^4$]. We recall that when $Z \in \mathbb{N}$ counts the number of elementary charges in the nucleus, then the Dirac Hamiltonian is essentially self-adjoint for $Z \leq 118$ [24, 30]. For $119 \leq Z \leq 137$ it has a distinguished self-adjoint extension yet for $Z > 137$ nobody seems to know which one of uncountably many self-adjoint extensions is physically distinguished. The heuristic explanation for the breakdown of analytical self-adjointness in the special-relativistic purely electrical hydrogenic ion problem is that the electrical Coulomb attraction between nucleus and electron becomes too strong for the angular momentum barrier to stabilize, and a collapse of the ground state ensues. Since gravity is generally attractive, one would have expected a worsening of the self-adjointness properties in the general-relativistic problem, but a complete wipeout was presumably not expected by anyone!

The problem with the lack of essential self-adjointness of the special-relativistic hydrogenic Dirac Hamiltonian goes away, however, if one takes the anomalous magnetic moment $\mu_a$ of the electron into account. Indeed, as shown in [2, 15] for the special-relativistic hydrogenic problem, adding an anomalous magnetic moment operator to the Dirac Hamiltonian of a test electron with purely electrostatic interactions produces an essentially self-adjoint Hamiltonian for the electron of any hydrogenic ion, independently of the strength of its non-vanishing anomalous magnetic moment; see [30, 31] for numerically computed eigenvalues as functions of $Z$ beyond $Z = 137$. More recently Belgiorno, Martellini, and Baldicchi [4] showed that the Dirac operator for a test electron with anomalous magnetic moment is essentially self-adjoint in the naked RWN geometry (only) if $|\mu_a| \geq \frac{3}{2}\sqrt{G\hbar c}$. The empirical $|\mu_a| \approx \mu_{\text{class}} := \frac{1}{4\pi} \frac{e^3}{m_ee^2}$, which we call the classical magnetic moment of the electron. Since

$$\sqrt{G\hbar c} \approx 1.3 \cdot 10^{-18} \mu_{\text{class}},$$

the hurdle for essential self-adjointness, $|\mu_a| \geq \frac{3}{2}\sqrt{G\hbar c}$, is easily cleared with the empirical electron data. Here, $m_e$ is the empirical mass of the electron and $-e$ its charge, $c$ is the speed of light in vacuum, and $\hbar$ is Planck’s constant divided by $2\pi$, as usual.

With the Dirac operator for an electron in the naked singularity sector of the RWN spacetime basically understood, in this paper we will revisit the problem of the Dirac operator for an electron in the black hole sector of the RWN spacetime family; we will also include some comparative remarks concerning electrons in the naked singularity sector, though.

$\text{The distinguished self-adjoint extension is defined by allowing } Z \in \mathbb{C} \text{ and demanding analyticity in } Z.$

The real threshold values then become $Z = \sqrt{3}/2\alpha_S$ instead of $Z = 118$, and $Z = 1/\alpha_S$ instead of $Z = 137$. Here, $\alpha_S := e^2/hc \approx 1/137.036$ is Sommerfeld’s fine structure constant.
Our point of departure is the fact that Cohen and Powers [10] considered a Dirac Hamiltonian with the minimal domain of $C^\infty$ bi-spinor functions with compact support outside the event horizon of a subextremal black hole. They proved that this $H$ is essentially self-adjoint, and that it has the whole real line as its essential spectrum. Thus its discrete spectrum is empty and any eigenvalues would have to be embedded in the continuum. Yet in [10] the absence of eigenvalues is shown altogether. Their result means that a test electron outside the event horizon of a subextremal RWN black hole cannot be in a stationary bound state. In concert with a result of Weidmann [32], this now implies that the essential spectrum is purely absolutely continuous, and so in fact is the spectrum of this Dirac Hamiltonian.

Upon reflection, it is not too surprising not to find bound states of an electron whose wave function is supported outside the event horizon of an RWN black hole. After all, one expects the electron to be swallowed by the black hole unless it escapes to spatial $\infty$. The capture of the electron by the black hole is not seen in the treatment by Cohen and Powers, who worked with a coordinate system that near the end of the first quarter of the 20th century gave rise to the “frozen star” scenario. The purpose of this coordinate system was to describe the collapsing evolution of gravitating masses as seen from spatial infinity, and therefore failed to capture the formation of a black hole. Thus, conceivably, a Dirac bound state in the black hole sector of RWN may exist after all, but it would require the domain of the Dirac Hamiltonian to not be restricted to bi-spinor wave functions supported outside the event horizon. Of course, it is often argued on positivistic grounds that physics is not concerned with what goes on inside an event horizon, but positivism is merely a form of philosophy which should not be confused with the foundations of physics. Also Werner Israel and his collaborators have long advocated [11] investigating what’s going on inside an event horizon according to general relativity theory. Finster, Smoller, and Yau [14] in particular have inquired into “time-periodic” Dirac bi-spinor wave functions that are supported both outside and inside the event horizon of a RWN black hole spacetime, and found no nontrivial ones in $L^2$. However, since the region between the Cauchy and the event horizon of a RWN black hole spacetime is not static, insisting on time-periodic bi-spinors also there seems like asking for too much. In this vein, in this paper we will investigate the Dirac Hamiltonian for a test electron in the RWN black hole spacetime with the bi-spinor wave function supported entirely on the static part of the region inside the event horizon of the black hole spacetime, which is a static spherically symmetric spacetime with a naked singularity in its own right — it is not asymptotically flat, though.
Our results, stated informally, are:

**Theorem 1:** The Dirac Hamiltonian $H$ for a test electron in the static interior of a (sub-)extremal RWN black hole, if it interacts with the singularity only electrically and gravitationally, is not essentially self-adjoint, yet has infinitely many self-adjoint extensions. In the subextremal case, each self-adjoint extension has a purely absolutely continuous spectrum that extends over the whole real line.

**Theorem 2:** The Dirac Hamiltonian $H$ for a test electron in the static interior of a (sub-)extremal RWN black hole, if it interacts with the singularity electrically, gravitationally, and through its anomalous magnetic moment, is essentially self-adjoint if and only if $|\mu_a| \geq \frac{3\sqrt{7}}{2}\sqrt{G}\hbar c$. In the subextremal essentially self-adjoint situation, the unique self-adjoint extension has purely absolutely continuous spectrum that covers the whole real line.

Thus, the singularity of the RWN spacetime causes a lack of essential self-adjointness (e.s.a.) if the electron is not shielded from it by the event horizon and is assumed to interact only electrically and gravitationally with the singularity, but e.s.a. is restored if a sufficiently large anomalous magnetic moment of the electron is taken into account. The empirical anomalous magnetic moment of the electron is about $10^{18}$ times larger than the critical value, the same critical value as found in [4] for the naked singularity sector.

However, while in Appendix C of [4] it is shown that the general-relativistic hydrogenic Dirac Hamiltonian of a test electron with anomalous magnetic moment in the naked singularity sector of the RWN spacetime of a nucleus has infinitely many discrete eigenvalues in the gap $(-m_ec^2, m_ec^2)$ of its essential spectrum, the essentially self-adjoint operator of an electron with anomalous magnetic moment inside the Cauchy horizon of a subextremal black hole has no eigenvalues at all. We will also show that in the extremal case there can be at most one eigenvalue, possibly infinitely degenerate, which we identify.

In the remainder of this paper we make all this precise.

In section 2 we explain that normal nuclei are associated with the naked singularity sector of the RWN spacetime, while hypothetical ‘hyper-heavy nuclei’ have to be associated with the RWN black hole sector. We also stipulate our dimensionless notation for discussing both the spacetime and the Dirac operators.

Section 3 is the main technical section. We define the Dirac operators, state our theorems precisely, then present their proofs, using strategies of [33], [21], and [10]. Some of our proofs are overall very similar to proofs in [23] for naked-singularity spacetimes, yet details vary.

We conclude in Section 4 and emphasize open problems.
2. The Reissner–Weyl–Nordström spacetime of a point nucleus

In order to facilitate the comparison of our results with those for hydrogenic ions, including some speculative hyper-heavy ones defined by the inequality \( GMm_e > Ze^2 \) (here, \( m_e \) is the empirical rest mass of the electron, and the inequality means that the gravitational attraction of a positron (!) to the nucleus overcomes their electrical repulsion), from now on we think of the central timelike singularity of the RWN spacetime as a proxy for the worldline of a point nucleus at rest. Thus for the charge parameter \( Q \) of the RWN spacetime we set \( Q = Ze \), where \( e > 0 \) denotes the elementary charge (in Gaussian units), and where \( Z \in \mathbb{N} \) counts the number of elementary charges carried by the nucleus. We let the ADM mass of the RWN spacetime be the nuclear mass, \( M_{\text{ADM}} = M = A(Z, N)m_p \), where \( m_p \) is the proton mass, \( N \in \{0, 1, 2, ...\} \) is the number of neutrons in the nucleus, and \( A(Z, N) \geq 1 \) the nuclear mass number; moreover, \( A(Z, N) \approx Z + N \) to within 1% accuracy.

All empirically known long-lived nuclei are far away from the black hole regime \( GM^2 \geq Z^2e^2 \). This conclusion extends to hypothetical nuclei with arbitrary large \( Z \) if they obey the bounds \( Z \leq A(Z, N) \leq 3Z \) known empirically to hold for all long-lived nuclei with \( Z \leq 92 \) in the current chart of the nuclids. Assuming these empirical bounds, essentially \( N \leq 2Z \) to with 1% accuracy, one finds \( \frac{GM^2}{Ze^2} < 9\frac{Gm_p^2}{e^2} \), and since \( \frac{Gm_p^2}{e^2} \approx 2 \cdot 10^{-37} \ll 1 \) by many powers of 10, also \( \frac{GM^2}{Ze^2} \ll 1 \), thus \( \frac{GMm_p}{Ze^2} \ll 1 \), and so \( \frac{GMm_e}{Ze^2} \ll 1 \) by many powers of 10, too.

On the other hand, hypothetical hyper-heavy nuclei, for which \( GMm_e > Ze^2 \), are associated with the black hole sector of the RWN spacetime. For suppose not. Then both \( GMm_e > Ze^2 \) and \( GM^2 < Z^2e^2 \) (the latter condition means we are in the RWN naked singularity regime). Since \( M = A(N, Z)m_p \) for all nuclei, we have \( M \geq Zm_p \), and since \( m_p = 1836m_e \), we find that \( GM^2 < Z^2e^2 \) implies that \( 1836AGMm_e < Z^2e^2 \), while \( Ze^2 < GMm_e \) implies \( 1836AZe^2 < 1836AGMm_e \). And so, by transitivity, we have \( 1836AZe^2 < Z^2e^2 \), hence \( 1836A < Z \), which is impossible with the empirical \( A(Z, N) \approx Z + N \).

Thus, the assumption \( N \leq 2Z \) cannot be imposed as a condition when inquiring into hyper-heavy hydrogenic ions. Fortunately, neutron stars are in a fair sense examples of gravitationally bound nuclei with \( M \approx (Z + N)m_p \) and \( Z \) very small while \( N \) is very large. Of course, neutron stars are not point-like, yet they are only mentioned as an example of gigantic nuclei in nature not obeying the \( N \leq 2Z \) rule. Hyper-heavy nuclei would not only not obey the \( N \leq 2Z \) rule, they would have to be associated with the black hole sector and therefore de-facto be point singularities covered by an event horizon — as per Einstein’s general relativity theory.
2.1. The naked singularity regime. The electrostatic Reissner–Weyl–Nordström (RWN) spacetime of a naked point nucleus is spherically symmetric, static, asymptotically flat and topologically identical to ‘\( \mathbb{R}^4 \setminus \) a timelike line,’ equivalently \( \mathbb{R} \times (\mathbb{R}^3 \setminus \{0\}) \), covered by a single global chart of ‘spherical coordinates’ \((t, r, \vartheta, \varphi) \in \mathbb{R} \times \mathbb{R}_+ \times [0, \pi] \times [0, 2\pi]\). Here, \( r \) is the so-called area radius: every point in the RWN spacetime is an element of a unique orbit of the Killing vector flow corresponding to its \( SO(3) \) symmetry, and this orbit is a scaled copy of \( S^2 \) with area \( A = 4\pi r^2 \), defining \( r > 0 \). Moreover, the variables \( \vartheta \) and \( \varphi \) are the usual polar and azimuthal angles on \( S^2 \). In dimensionless units where \( r \) is measured in multiples of the electron’s reduced Compton wave length \( \hbar/m_e c \), and \( t \) in multiples of \( \hbar/m_e c^2 \), its metric has the line element
\[
ds^2 = -f^2(r)dt^2 + f^{-2}(r)dr^2 + r^2d\Omega^2,
\]
where \( d\Omega^2 = d\vartheta^2 + \sin^2 \vartheta d\varphi^2 \) is the line element on \( S^2 \), and where
\[
f^2(r) = 1 - 2\frac{GMm_e}{\hbar c} \frac{1}{r} + \frac{Gm^2}{\hbar c} \frac{Z^2 e^2}{r^2} 1.
\]
Here, \( M = A(Z, N)m_p \), and we note that \( \frac{Gm_e^2}{\hbar c} \approx 1.79 \cdot 10^{-45} \) and \( \frac{Gm_m m_e}{\hbar c} \approx 3.3 \cdot 10^{-42} \); incidentally, \( \frac{e^2}{\hbar c} \approx 6.0 \cdot 10^{-39} \). Also, \( \frac{r}{\hbar c} \approx 1/137.036 \) is Sommerfeld’s fine structure constant.

The known long-lived nuclei, for which \( A(Z, N) \approx Z + N \) and \( Z \leq A(Z, N) \leq 3Z \), are associated with the naked singularity sector of the RWN spacetimes, i.e. \( f^2(r) > 0 \forall r > 0 \).

2.2. The black hole regime. The RWN spacetime features a black hole if there is at least one value of \( r > 0 \) for which \( f^2(r) = 0 \). Since \( f^2(r) \) is a quadratic polynomial in \( 1/r \), viz. \( f^2(r) = \frac{1}{r^2}(r-r_+)(r-r_-) \), with the zeros formally given by
\[
r_\pm = \frac{Gm_e m}{\hbar c} \left( 1 \pm \sqrt{1 - \frac{Z^2 e^2}{GM^2}} \right),
\]
and those are real if and only if \( \frac{Z^2 e^2}{GM^2} \leq 1 \). If \( \frac{Z^2 e^2}{GM^2} = 1 \), one says the asymptotically flat spacetime contains an extremal black hole; if \( \frac{Z^2 e^2}{GM^2} < 1 \), the asymptotically flat spacetime contains a subextremal black hole. In the extremal case, \( r_+ = r_- = \frac{Gm_e m}{\hbar c} =: r_0 \), and then
\[
f^2(r) = \left( 1 - \frac{Gm_e m}{\hbar c} \frac{1}{r} \right)^2 = \frac{1}{r^2} (r - r_0)^2.
\]

Continuing an asymptotically flat RWN black-hole spacetime analytically, one finds two static regions: either \( r > r_+ \) or \( r < r_- \); this is true even for the extremal case when \( r_+ = r_- \). The maximal analytically extended spacetime even has infinitely many copies of such regions. We will be concerned with spacetimes given by one copy of the inner static region.
3. The Dirac operators

In this section we formulate the Dirac operator for a test electron with or without anomalous magnetic moment in the RWN spacetime of a naked point nucleus. For the sake of definiteness, we will define the electrons’ anomalous magnetic moment as identical to its highly accurate approximation \( \mu_a = -\mu_{\text{class}} \). However, we multiply \( \mu_a \) by an ‘amplitude’ \( a \): for \( a = 0 \) we obtain the Dirac operator for a point electron without anomalous magnetic moment, whereas \( a = 1 \) if the electron’s anomalous magnetic moment is taken into account. By varying \( a \) continuously we can inquire into the threshold for essential self-adjointness.

Electrons with wave function restricted to the region \( r > r_+ \) on the subextremal RWN black hole spacetime were studied in \cite{10}; no bound states exist, then. We will investigate electrons with wave function restricted to the region \( r < r_- \) on the subextremal RWN black hole spacetime, i.e. to the spacetime given by (2), (3), with \((t,r,\vartheta,\varphi) \in \mathbb{R} \times (0,r_-) \times [0,\pi] \times [0,2\pi)\). We will also study wave functions on the extremal RWN black hole spacetime, supported inside the event horizon at \( r_0 (= r_- = r_+) \). For the purpose of comparison, we will also recall the Dirac operator defined on the naked singularity sector.

Due to the spherical symmetry and static character of the spacetimes, the Dirac operator \( H \) of a test electron in the curved space whose line element \( ds^2 \) is given by (2) separates in the spherical coordinates and their default Cartan frame \cite{10}. More precisely, \( H \) is a direct sum of so-called radial partial-wave Dirac operators \( H_{k}^{\text{rad}} := m_e c^2 K_{k}^{a} \), \( k \in \mathbb{Z} \setminus \{0\} \), with

\[
K_{k}^{a} := \begin{bmatrix}
\frac{f(r) - Z\alpha_s \frac{1}{r}}{\left[\frac{k}{r} - Z\alpha_s \frac{a+1}{4\pi r} \right]} & \left[\frac{k}{r} - Z\alpha_s \frac{a+1}{4\pi r} \right] f(r) - f^2(r) \frac{d}{dr} \\
-f(r) - Z\alpha_s \frac{1}{r} & f(r) + f^2(r) \frac{d}{dr}
\end{bmatrix},
\]

which act on two-dimensional bi-spinor wave function subspaces. The spectrum of \( H \) is the union of the spectra of the \( H_{k}^{\text{rad}} \). This reduces the problem to studying the spectrum of \( K_{k}^{a} \).

3.1. Point nucleus as naked singularity of static spacetime. In this case the bi-spinor wave functions are supported on \( \mathbb{R}^3 \) minus a point. The ‘radial Hilbert space’ consists of pairs \( g(r) := (g_1(r),g_2(r))^T \) equipped with a weighted \( L^2 \) norm given by

\[
\|g\|^2 := \int_0^\infty \frac{1}{f^2(r)} \left( |g_1(r)|^2 + |g_2(r)|^2 \right) dr.
\]

As mentioned in the introduction, Cohen and Powers \cite{10} proved that for \( a = 0 \) the Dirac Hamiltonian is not essentially self-adjoint, but has uncountably many self-adjoint extensions. Belgiorno et al. \cite{4} subsequently showed that \( H \) is essentially self-adjoint on the domain of \( C^\infty \) bi-spinor wave functions which are compactly supported away from the singularity at \( r = 0 \) whenever \( a \) is large enough, viz. if \( a |\mu_{\text{class}}| \geq \frac{3}{2} \sqrt{G\hbar c} \).
3.2. Point nucleus as singularity in static interior of black hole spacetime. In this case the bi-spinor wave functions are supported on $S^2 \times (0, r_-)$. The ‘radial Hilbert space’ consists of pairs $g(r) := (g_1(r), g_2(r))^T$ equipped with a weighted $L^2$ norm given by

$$\|g\|^2 := \int_0^{r_-} \frac{1}{f^2(r)} \left( |g_1(r)|^2 + |g_2(r)|^2 \right) dr.$$  

(8)

In the extremal case, $r_- = r_0$.

We change variables $r \mapsto x$ such that

$$f^2(r) \frac{d}{dr} = \frac{d}{dx},$$

with $x = 0$ when $r = 0$, which for the subextremal sector yields

$$x = r + \frac{r_+^2}{r_+ - r_-} \ln \left( 1 - \frac{r}{r_+} \right) - \frac{r_-^2}{r_+ - r_-} \ln \left( 1 - \frac{r}{r_-} \right); \quad r < r_-;$$

(10)

in the extremal limit $r_- \uparrow r_0$ & $r_+ \searrow r_0$ this becomes

$$x = r_0 \left[ \frac{1}{1 - r/r_0} + 2 \ln \left( 1 - \frac{r}{r_0} \right) - \left( 1 - \frac{r}{r_0} \right) \right], \quad r < r_0.$$  

(11)

Note that $x \to \infty$ when $r \uparrow r_-$, respectively when $r \searrow r_0$. This maps $K_k^2$ into

$$\tilde{K}_k^a = \begin{bmatrix} f(r(x)) - Z\alpha_S \frac{1}{r(x)} \left[ \frac{1}{r(x)} - Z\alpha_S \frac{x^a}{4\pi r^2(x)} \right] f(r(x)) - \frac{d}{dx} & \left[ \frac{b}{r(x)} - Z\alpha_S \frac{x^a}{4\pi r^2(x)} \right] f(r(x)) - \frac{d}{dx} \\ \left[ \frac{k}{r(x)} - Z\alpha_S \frac{x^a}{4\pi r^2(x)} \right] f(r(x)) + \frac{d}{dx} & -f(r(x)) - Z\alpha_S \frac{1}{r(x)} \end{bmatrix},$$  

(12)

$$=: \begin{bmatrix} a(x) - b(x) & k\alpha(x) - ad(x) - \frac{d}{dx} \\ kc(x) - ad(x) + \frac{d}{dx} & -a(x) - b(x) \end{bmatrix}$$

(13)

with the inner product

$$\langle g, h \rangle = \int_0^{\infty} (g_1(r(x))\bar{h}_1(r(x)) + g_2(r(x))\bar{h}_2(r(x))) dx.$$  

(14)

3.2.1. Electron without anomalous magnetic moment.

**Theorem 3.1.** The operator $\tilde{K}_k^0$ given by (12) with $a = 0$ has uncountably many self-adjoint extensions for both the subextremal and the extremal black-hole sector.

**Proof.** We use the strategy of [10], [23] for the naked singularity spacetimes. We start with the subextremal case. Note that, with the change of variable (10) one has

$$z = \frac{r - r_-}{r_+ - r_-} \quad \text{as} \quad x \to \infty \quad \text{and} \quad r \sim x^{1/3} \quad \text{as} \quad x \to 0.$$  

Therefore, in the subextremal case the operator (13) features $a(x) \sim x^{-1/3}$, $b(x) \sim x^{-1/3}$ and $c(x) \sim x^{-2/3}$ as $x \to 0$. Furthermore, as $x \to \infty$, we have $f(r(x)) \sim e^{-\kappa x}$ as well as $a(x), c(x) \sim e^{-\kappa x}$, and $b(x) \to \frac{2\alpha_s}{r_-}$.

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3Here, “$f(x) \sim g(x)$ as $x \to x_*$” means $\exists C > 0$ such that $f(x)/g(x) \to C$ as $x \to x_*$, where $x_* = 0$ or $\infty$. 

Let \( \tilde{K}_k^{0*} \) be the adjoint operator of \( \tilde{K}_k^0 \). The domain \( \mathcal{D}(\tilde{K}_k^0) \) comprises all \( C^\infty \) functions of compact support in \((0, \infty)\), and \( \mathcal{D}(\tilde{K}_k^{0*}) \) includes the functions \( f \) which together with \( f' \) are integrable in any compact subset of \([0, \infty)\). On \( \mathcal{D}(\tilde{K}_k^{0*}) \) we now define the sesquilinear form

\[
[g, h] := \langle \tilde{K}_k^{0*} g, h \rangle - \langle g, \tilde{K}_k^{0*} h \rangle,
\]

with \( \langle \cdot, \cdot \rangle \) defined in [14]. By Theorem 4.1 in [33], \( \tilde{K}_k^{0*} \mid \mathcal{D} \) is a self-adjoint extension of \( \tilde{K}_k^0 \) iff

i) \( \mathcal{D}(\tilde{K}_k^0) \subset \mathcal{D} \subset \mathcal{D}(\tilde{K}_k^{0*}) \)

ii) \( [g, h] = 0 \) for all \( g, h \in \mathcal{D} \)

iii) if \( g \in \mathcal{D}(\tilde{K}_k^{0*}) \) and \( [g, h] = 0 \) holds for every \( h \in \mathcal{D} \) then \( g \in \mathcal{D} \).

Now consider the spaces in which \( [g, h] = 0 \). Take \( g \in \mathcal{D}(\tilde{K}_k^{0*}) \) so that \( \tilde{K}_k^{0*} g = \psi \) for some \( \psi \in L^2([0, \infty)) \). Since \( \mathcal{D}(\tilde{K}_k^0) \subset C^1([0, \infty)) \), \( g \in AC([x_1, x_2]) \) for each \( 0 \leq x_1 < x_2 < \infty \), and so we can integrate to obtain

\[
g_1(x) = e^{-\mu(x)} \left( g_1(0) + \int_0^x e^{\mu(y)} [(a(y)+b(y))g_2(y) + \psi_2(y)]dy \right),
\]

\[
g_2(x) = e^{\mu(x)} \left( g_2(0) + \int_0^x e^{-\mu(y)} [(a(y)-b(y))g_1(y) - \psi_1(y)]dy \right)
\]

for each \( x \leq x_2 < \infty \), where \( \mu(x) = \int_0^x kc(y)dy \sim x^{1/3} \to 0 \) as \( x \to 0 \). We also need \( b(x), a(x) \in L^2([0, x_2]), x_2 < \infty \), for \( g_1, g_2 \) to be defined. Integration by parts yields

\[
[g, h] = \lim_{x_1 \to 0} \lim_{x_2 \to \infty} \int_{x_1}^{x_2} \left( \tilde{K}_k^{0*} g_1 \tilde{h}_2 - (\tilde{K}_k^{0*} g_2) \tilde{h}_1 + g_1 \tilde{K}_k^{0*} \tilde{h}_2 - g_2 \tilde{K}_k^{0*} \tilde{h}_1 \right) dx
\]

\[
= \lim_{x_1 \to 0} \lim_{x_2 \to \infty} \left[ g_1(x_2) \tilde{h}_2(x_2) - g_2(x_2) \tilde{h}_1(x_2) - g_1(x_1) \tilde{h}_2(x_1) + g_2(x_1) \tilde{h}_1(x_1) \right]
\]

\[
= g_2(0) \tilde{h}_1(0) - g_1(0) \tilde{h}_2(0).
\]

To obtain the last equality we used [16], [17], and the fact that \( g, h \in L^2([0, \infty)) \).

Thus any symmetric extension requires \( g_2(0) \tilde{h}_1(0) - g_1(0) \tilde{h}_2(0) = 0 \). By taking \( g = h \) one sees that this is possible iff one of \( g_1(0) \) and \( g_2(0) \) is a real multiple of the other. Therefore,

\[
\tilde{K}_k^{0,0} := \tilde{K}_k^{0*} \mid \mathcal{D}_\theta \text{, where } \mathcal{D}_\theta = \{ g \in \mathcal{D}(\tilde{K}_k^{0*}) : g_1(0) \sin \theta + g_2(0) \cos \theta = 0 \},
\]

gives a symmetric extension for any \( 0 \leq \theta < \pi \), cf. [10]. Note that \( \mathcal{D}_\theta \) satisfies both the conditions i) and ii). To see that condition iii) is also satisfied, let \( h \in \mathcal{D}_\theta \), then

\[
[g, h] = 0 \iff g_2(0) \tilde{h}_1(0) - g_1(0) \tilde{h}_2(0) = 0 \iff \frac{\tilde{h}_2(0)}{\tilde{h}_1(0)} = \frac{g_2(0)}{g_1(0)} = -\tan \theta \in \mathbb{R}.
\]

This completes the proof for the subextremal case (cf. the proof of Thm.3.6 in [23]).
For the extremal case, we consider the change of variable (11) and consider the operator (13). Note that the above argument is valid if \( \mu(x) = \int_{0}^{x} kc(y) dy \to 0 \) as \( x \to 0 \); \( b(x), a(x) \in L^2([0, x_2]), \ x_2 < \infty \).

One can easily see that \( c(x) \sim x^{-2/3} \) as \( x \to 0 \) also in the extremal case. Furthermore, \( b(x), a(x) \sim x^{1/3} \) as \( x \to 0 \), and both are continuous in \([0, b]\). Hence, the proof applies similarly. □

**Remark 3.2.** We remark that the deficiency indices of \( \tilde{K}_k^0 \) are \((1, 1)\), for \([g, h]\) is the difference of two positive rank-one bilinear forms. This already implies that an orbit of self-adjoint extensions must exist. The proof of Thm.3.1 identifies these.

Our next theorem identifies the essential spectrum of any self-adjoint extension of the Dirac operator acting on bi-spinor wave-functions supported inside the inner horizon of either the subextremal and the extremal black-hole spacetime.

**Theorem 3.3.** For each \( \theta \), one has \( \sigma_{\text{ess}}(\tilde{K}_k^0; \theta) = \mathbb{R} \).

To prepare the proof of Theorem 3.3 as in [23] we recall the following lemma from [10].

**Lemma 3.4.** Let

\[
D : = \begin{bmatrix} 0 & -\frac{d}{dx} \\ \frac{d}{dx} & 0 \end{bmatrix}
\]

be defined on the \( C^\infty \) two-component functions of compact support in the positive real half-line. Now take the closure of this operator in \((L^2(\mathbb{R}_+))^2\) with the boundary condition \( f_1(0) \sin \theta + f_2(0) \cos \theta = 0 \) at \( x = 0 \), denoted \( D_\theta \). Let \( A \) be the operator

\[
A = \begin{bmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{bmatrix},
\]

where the \( a_{ij} \) are functions in \( L^2([0, b]) \) for all \( 0 < b < \infty \) and \( a_{ij}(x) \to 0 \) as \( x \to \infty \). Then \( A \) is \( D_\theta \) compact.

**Proof of Theorem 3.3** We prove the theorem explicitly for the subextremal case. Yet note that the extremal case follows verbatim after setting \( r_- \to r_0 \).

We split the operator \( \tilde{K}_k^0 \) in (12) as

\[
\tilde{K}_k^0 = \begin{bmatrix} -\frac{Z\alpha_s}{r_-} & kc(x) - \frac{d}{dx} \\ kc(x) + \frac{d}{dx} & -\frac{Z\alpha_s}{r_-} \end{bmatrix} + \begin{bmatrix} a(x) - \left[b(x) - \frac{Z\alpha_s}{r_-}\right] & 0 \\ 0 & -a(x) - \left[b(x) - \frac{Z\alpha_s}{r_-}\right] \end{bmatrix}
\]

\(=: \tilde{K}_k^{00} + V\). (26)
Note that Theorem 3.1 is valid when \( a(x) = 0 \) and \( b(x) = \frac{2a}{r_-} \). Therefore, \( \widetilde{K}_k^{00} \) has deficiency indices \((1, 1)\) and has multiple self-adjoint extensions similar to \( \widetilde{K}_k^0 \). We define these self-adjoint extensions as \( \widetilde{K}_k^{00} \) similar to \( \widetilde{K}_k^0 \), cf. (21). We define the following Weyl sequence for \( \widetilde{K}_k^{00} \): let \( w = -\frac{2a}{r_-} - \lambda \), with any \( \lambda \in \mathbb{R} \), then

\[
    f_{n, \lambda}(x) = \frac{1}{2n^2} x e^{-\frac{r_-}{2n} + ixw} \begin{bmatrix} 1 \\ -i \end{bmatrix}; \quad n \in \mathbb{N}.
\]

We have that \( \| f_{n, \lambda} \|_{(L^2(\mathbb{R}^+))^2} = 1 \), \( f_{n, \lambda}(x) \to 0 \) weakly, and \( \| (\widetilde{K}_k^{00} - \lambda)f_{n, \lambda}(x) \|_{(L^2(\mathbb{R}^+))^2} \to 0 \) as \( n \to \infty \). Hence, any \( \lambda \in \mathbb{R} \) is in the essential spectrum of \( \widetilde{K}_k^{00} \), and so \( \sigma_{\text{ess}}(\widetilde{K}_k^{00}) = \mathbb{R} \).

Next we will show that \( V \) is \( \widetilde{K}_k^{00} \) compact. This is done essentially verbatim to the pertinent part in the proof of Lemma 3.12 in [23]. We define \( \xi(x) = -\int_0^x kc(y)dy \) for \( 0 \leq x \leq 1 \) and \( \xi(x) = \xi(1) \) for \( x > 1 \). Then the following matrix is bounded,

\[
    S = \begin{bmatrix} e^{-\xi(x)} & 0 \\ 0 & e^{\xi(x)} \end{bmatrix}.
\]

Assume that \( \| g(n) \|_{(L^2(\mathbb{R}^+))^2}, \| \widetilde{K}_k^{00}g(n) \|_{(L^2(\mathbb{R}^+))^2}, n \in \mathbb{N} \), are bounded sequences. Then \( \| Sg(n) \|_{(L^2(\mathbb{R}^+))^2} \) and \( \| D_\theta Sg(n) \|_{(L^2(\mathbb{R}^+))^2} \) are also bounded, the first one is because \( S \) is bounded and the latter one is by the fact that \( D_\theta S = S^{-1}SD_\theta S = S^{-1}(\widetilde{K}_k^{00} + W) \) for some bounded \( W \). Moreover, one can check that \( VS^{-1} \) is \( D_\theta \) compact by Lemma 3.4. Hence,

\[
    VS^{-1}g(n) = Vg(n)
\]

has a convergent subsequence. This proves that \( V \) is \( D_\theta \), and \( \widetilde{K}_k^{00} \) compact. \( \square \)

We next show that the essential spectrum has no singular continuous part. As in [23], for this we will need the following Theorem from [32, 33].

**Theorem 3.5. (Weidmann)** Let

\[
    \tau := \begin{bmatrix} 0 & -\frac{d}{dx} \\ \frac{d}{dx} & 0 \end{bmatrix} + P_1(x) + P_2(x)
\]

be defined on \((a, \infty)\). Further assume that \( |P_1(x)| \in L^1(c, \infty) \) for some \( c \in (a, \infty) \), and \( P_2(x) \) is of bounded variation in \([c, \infty)\) with

\[
    \lim_{x \to \infty} P_2(x) = \begin{bmatrix} \mu_+ & 0 \\ 0 & \mu_- \end{bmatrix} \quad \text{for} \quad \mu_- \leq \mu_+.
\]

Then every self-adjoint realization \( A \) of \( \tau \) has purely absolutely continuous spectrum in \((-\infty, \mu_-) \cup (\mu_+, \infty)\).
Proposition 3.6. In both the subextremal and the extremal case, \( \mathbb{R} \setminus \{-\frac{\alpha}{r_-} Z\} \subset \sigma_{ac}(\tilde{K}^0_{k,\theta}). \)

Proof. Recall that \( \tilde{K}^0_{k,\theta} \) is in the form of \( \tau \), with \( P_1(x) = 0 \), and with \( P_2(x) = P_2^a(x) \) for \( a = 0 \), where

\[
P_2^a(x) = \begin{bmatrix}
f(r(x)) - Z\alpha S \frac{1}{r(x)} & k \frac{1}{r(x)} - Z\alpha S \frac{2 a}{4\pi r^2(x)} f(r(x)) \\
k \frac{1}{r(x)} - Z\alpha S \frac{2 a}{4\pi r^2(x)} f(r(x)) & -k \frac{1}{r(x)} - Z\alpha S \frac{1}{r(x)}
\end{bmatrix};
\]

here, both \( f(r(x)) \) and \( r(x) \) are continuously differentiable and hence of bounded variation. Furthermore,

\[
\lim_{x \to \infty} f(r(x)) = 0, \quad \text{and} \quad \lim_{x \to \infty} r(x) = r_*,
\]

where \( r_* = r_- \) in the subextremal, and \( r_* = r_0 \) in the extremal case. This implies

\[
\lim_{x \to \infty} P_2^a(x) = \begin{bmatrix}
-\frac{\alpha Z}{r} & 0 \\
0 & -\frac{\alpha Z}{r_*}
\end{bmatrix}, \quad \forall \ a \geq 0.
\]

Hence, the spectrum of \( \tilde{K}^0_{k,\theta} \) is purely absolutely continuous on \( \mathbb{R} \setminus \{-\frac{\alpha}{r_-} Z\}. \)

□

Corollary 3.7. The singular continuous spectrum \( \sigma_{sc}(K^0_{k,\theta}) = \emptyset \) in both the subextremal and the extremal case.

Proof. By Proposition 3.6 and Theorem 3.3 the essential spectrum is the closure of \( \sigma_{ac}(K^0_{k,\theta}) \). Since the singular continuous spectrum is a subset of the essential spectrum, and since the interior of the essential spectrum here is purely absolutely continuous, a non-empty \( \sigma_{sc}(K^0_{k,\theta}) \) would have to consist of the single point \(-\alpha S Z/r_*\), which is impossible.

□

The results obtained so far show the absence of a discrete spectrum, but not the absence of point spectrum. Obviously, any point spectrum would have to consist of a single eigenvalue, \(-\alpha S Z/r_*\), which could be infinitely degenerate. We next show that in the subextremal case, \(-\alpha S Z/r_*\) is not an eigenvalue.

Theorem 3.8. In the subextremal case \( \tilde{K}^0_{k,\theta} \) has no eigenvalues.

For the proof of Theorem 3.8 we will utilize the following Lemma of Cohen & Powers [10].

Lemma 3.9. Let \( \mathcal{H} \) be a Hilbert space. Let \( V_t \) for \( t > a \) be a bounded linear operator on \( \mathcal{H} \) so that \( V_t f \) is continuous in \( t \) for each \( f \in \mathcal{H} \). Suppose \( f(t) \) solves the differential equation

\[
\frac{df(t)}{dt} = V_t f(t), \quad \text{for } t > a
\]

where the derivative exists in the strong sense. Suppose \( \int_a^\infty \|V_t\| dt = C < \infty \). Then the \( \lim_{t \to \infty} f(t) \) exists, and if this limit is zero then \( f(t) = 0 \) for all \( t \).
Proof of Theorem 3.8. We recall that by Proposition 3.6, \( \lambda = -\frac{Z\alpha}{r_-} \) is the only possible value which may be an eigenvalue. Hence, it is enough to show that if \((\tilde{K}^0_{k;\theta} + \frac{Z\alpha}{r_*})g = 0 \) and \( g \in L^2 \), then \( g \) is identically zero.

Now note that

\[
\tilde{K}^0_{k;\theta} + \frac{Z\alpha}{r_-} = \begin{bmatrix} -b(x) + \frac{Z\alpha}{r_*} & -\frac{d}{dx} \\ -\frac{d}{dx} & -b(x) + \frac{Z\alpha}{r_*} \end{bmatrix} + \begin{bmatrix} a(x) & kc(x) \\ kc(x) & -a(x) \end{bmatrix}.
\]

(36)

Recall that \( a(x), c(x) \to 0 \) as \( x \to \infty \). More specifically, \( a(x), c(x) \sim e^{-\kappa x} \) in the subextremal case, whereas in the extremal case \( \eta \sim 0 \) and therefore, \( \eta \) is purely absolutely continuous:

\[
\eta \in L^1 \text{ with the help of the method of variation of constants. Thus we make the ansatz}
\]

\[
g(x) = \begin{bmatrix} u(x)e^{i\eta(x)} + v(x)e^{-i\eta(x)} \\ -iuc(x) + iv(x)e^{-i\eta(x)} \end{bmatrix}.
\]

(37)

Inserting (37) into \((\tilde{K}^0_{k;\theta} + \frac{Z\alpha}{r_*})g = 0 \) we obtain

\[
\begin{bmatrix} u'\Re(x) - iv'\Re(x) \\ u'\Im(x) + iv'\Im(x) \end{bmatrix} = \begin{bmatrix} u(x)e^{i\eta(x)}[a(x) - ikc(x)] + v(x)e^{-i\eta(x)}[a(x) + ikc(x)] \\ u(x)e^{i\eta(x)}[ia(x) + kc(x)] + v(x)e^{-i\eta(x)}[-ia(x) + kc(x)] \end{bmatrix} \]

(38)

Written in terms of \( \frac{d}{dx} (u, v)^T \) this yields

\[
\frac{d}{dx} \begin{bmatrix} u(x) \\ v(x) \end{bmatrix} = \begin{bmatrix} 0 & e^{-2i\eta(x)}[ia(x) + kc(x)] \\ e^{2i\eta(x)}[-ia(x) + kc(x)] & 0 \end{bmatrix} \begin{bmatrix} u(x) \\ v(x) \end{bmatrix}.
\]

(39)

Recall that in the subextremal case \( a(x), c(x) \sim e^{-\kappa x} \) and therefore, by Lemma 3.9, we obtain \( u = 0 = v \). This finishes the proof. \( \square \)

Remark 3.10. Lemma 3.9 can be applied only to the subextremal case, where we have \( r - r_- \sim e^{-\kappa x} \) as \( x \to \infty \), and \( e^{-\kappa x} \) is integrable at \( \infty \). On the other hand, in the extremal case, \( r - r_0 \sim x^{-1} \) as \( x \to \infty \) and this does not satisfy the integrability condition in Lemma 3.9.

As an immediate consequence of our Proposition 3.6 and Theorem 3.8, we have

Corollary 3.11. In the subextremal case the essential spectrum, given by the whole real line, is purely absolutely continuous: \( \sigma_{ac}(K^0_{k;\theta}) = \mathbb{R} \).
3.2.2. Electron with anomalous magnetic moment.

We now address the Dirac Hamiltonian for an electron with anomalous magnetic moment in the static interior of a RWN black-hole spacetime.

**Theorem 3.12.** In both the subextremal and the extremal case, the operator $\tilde{K}_k^a$ given by (13) is essentially self-adjoint iff $a \frac{e^3}{4\pi mc^2} \geq \frac{3}{2} \sqrt{G \hbar / c}$.

**Proof.** We will show that the limit point case (LPC) is verified both in the right neighborhood of $x = 0$, and in the left neighborhood of $x = \infty$ iff $a \frac{e^3}{4\pi mc^2} \geq \frac{3}{2} \sqrt{G \hbar / c}$, i.e. then there is at least one non-square integrable solution to $\tilde{K}_k^a g = \lambda g$ for each $\lambda \in \mathbb{C}$, or equivalently for a fixed $\lambda$, see [33, Theorem 5.6].

We start with the left neighborhood of $x = \infty$. As $x \to \infty$, the operator $\tilde{K}_k^a$ approaches $K_* := \begin{bmatrix} -\frac{Z\alpha_s}{r_*} & -\frac{d}{dx} \\ \frac{d}{dx} & -\frac{Z\alpha_s}{r_*} \end{bmatrix}$, (40)

where again $r_* = r_-$ in the subextremal case and $r_* = r_0$ in the extremal case. Clearly, $g_\pm = (e^{\pm i \frac{Z\alpha_s}{r_*} x}, \mp ie^{\pm i \frac{Z\alpha_s}{r_*} x})^T$ are solutions to $K_* g = 0$, and $g_\pm$ is not square integrable at $\infty$. Hence, the LPC is satisfied in the left neighborhood of $x = \infty$.

Next, we address the right neighborhood of $x = 0$ ($r = 0$), and consider the solutions to

$$
\begin{align*}
\left[ f(r) - \frac{Z\alpha_s}{r} \right] g_1 + \left[ k f(r) - \frac{Z\alpha_s^2 a}{4\pi r} f(r) - f^2(r) \frac{d}{dr} \right] g_2 &= 0, \\
\left[ -f(r) - \frac{Z\alpha_s}{r} \right] g_2 + \left[ k f(r) - \frac{Z\alpha_s^2 a}{4\pi r} f(r) + f^2(r) \frac{d}{dr} \right] g_1 &= 0.
\end{align*}
$$

Recall that $g = (g_1, g_2)^T$ is square integrable in the right neighborhood of $r = 0$ with the inner product associated with (8) iff for each $0 < R < r_-$,

$$
\int_0^R \frac{1}{f^2(r)} \left( |g_1(r)|^2 + |g_2(r)|^2 \right) dr < \infty.
$$

Therefore, we aim to find solutions to (41), (42) such that (43) does not hold. Note that as $r \to 0$, $f(r) \sim ar^{-1}$, where $a = (r_- r_+)^\frac{1}{2}$ in the subextremal case and $a = r_0$ in the extremal case, when $r_- = r_+ (= r_0)$.

Hence, around zero equations (41), (42) become

$$
\begin{align*}
g_2' + Z\alpha_s^2 a f_2 - k a g_2 &= O(r), \\
g_1' - Z\alpha_s^2 a f_1 + k a g_1 &= O(r).
\end{align*}
$$

(44)
The above equations imply that in a right neighborhood of $r = 0$ we have $g_1 \sim r^{\frac{2a^2}{4\pi m_e}}$ and $g_2 \sim r^{-\frac{2a^2}{4\pi m_e}}$. Note that (43) implies that local square integrability holds for $g_1$ and $g_2$ if
\[
\int_0^R r^{\pm \frac{2a^2}{4\pi m_e} + 2} dr < \infty.
\] (46)

Recalling the definition of $r_\pm$ from (4), and $r_0$ from (5) we see that $a = \frac{G^{1/2} m_e Z e}{\hbar c}$ in both the subextremal and extremal case, so that $Z$ cancels out in the power of $r$. Therefore, in both the subextremal and extremal case the LPC is satisfied iff $-\frac{1}{2} \pi \alpha^2 S a \hbar c G^1 / 2 m_e e + 2 \geq -1$, which translates into $a e^3 4 / 3 m_e c^2 \geq 3 \sqrt{G \hbar c / c}$. □

Inserting numerical values for the physical and mathematical constants, we conclude that $a \geq 1.3 \cdot 10^{-18}$ implies essential self-adjointness. Therefore we arrive at

**Corollary 3.13.** $\tilde{K}_k^a$ is essentially self-adjoint if the empirical value of the electron’s anomalous magnetic moment is used, in which case $a = 1$ to three significant digits.

In the rest of this section we characterize $\text{spec} \tilde{K}_k^a$ when $a e^3 4 / 3 m_e c^2 > \frac{3}{2} \sqrt{G \hbar c} / c$.

**Theorem 3.14.** The essential spectrum $\sigma_{\text{ess}}(\tilde{K}_k^a) = \mathbb{R}$ for both the subextremal and the extremal case.

**Proof.** We define the operators $\tilde{K}_k^a([0, b])$ and $\tilde{K}_k^a([b, \infty])$ as the restriction of $\tilde{K}_k^a$ to $L^2([0, b])$ and $L^2([b, \infty])$ respectively. Then by Theorem 11.5 in [33], we have
\[
\sigma_{\text{ess}}(\tilde{K}_k^a) = \sigma_{\text{ess}}(\tilde{K}_k^a([0, b])) \cup \sigma_{\text{ess}}(\tilde{K}_k^a([b, \infty])).
\] (47)

Instead of (27) we now use the following Weyl sequence,
\[
f_{n,\lambda}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-b)^2}{2\pi} + iw} \left[ 1 \right] ; \ n \in \mathbb{N}.
\] (48)

Then $\xi(x) = -\int_b^x [kc(y) - ad(y)] dy$ for $b \leq x \leq b + 1$ in (28), and one can show that $\sigma_{\text{ess}}(\tilde{K}_{\mu, k}^b) = \mathbb{R}$ in a similar way as in the proof of Theorem 3.3.

On the other hand, the operator $\tilde{K}_k^a([0, b])$ can only have discrete spectrum. To see that, we use Theorem 2 in [13]. In particular, since the limit point case holds, $\tilde{K}_k^a([0, b])$ has discrete spectrum if also
\[
\int_0^b |kc(x) - ad(x)| dx = \infty.
\] (49)

Notice that $d(x) \sim x^{-1}$ as $x \to 0$, which is not locally integrable around zero. See (13) for the definitions of $c(x)$ and $d(x)$. □
Proposition 3.15. In both the subextremal and the extremal case, \( \mathbb{R} \setminus \{-\frac{aZr}{r^2}\} \subset \sigma_{ac}(\tilde{K}_k^a) \) and \( \sigma_{sc}(\tilde{K}_k^a) = \emptyset \).

Proof. We first note that the fact that \( \sigma_{sc}(\tilde{K}_k^a) = \emptyset \) follows from the claim on the absolutely continuous spectrum, see Corollary 3.7. Therefore, we only prove that \( \mathbb{R} \setminus \{-\frac{aZr}{r^2}\} \subset \sigma_{ac}(\tilde{K}_k^a) \).

Similarly to the proof of Proposition 3.6, we use Theorem 3.5. In particular, we now need to consider the operator \( P_a^2(x) \) defined in (32), with \( a > 0 \). We already proved the limit property (34) for \( P_a^2(x) \). And so, our proof of Proposition 3.6 in concert with Corollary 3.7 also proves Proposition 3.15. \( \square \)

Theorem 3.16. In the subextremal case \( \tilde{K}_k^a \) has no eigenvalues.

Proof. The proof follows similarly to the proof of Theorem 3.8. One needs to consider the operator in (36) with \( kc(x) \) replaced by \( kc(x) - ad(x) \). Note that \( d(x) \sim c(x) \) at \( \infty \), and hence the integrability condition in Lemma 3.9 is satisfied. This concludes that \( g \) has to be identically zero, if \( g \in L^2 \) and \( (\tilde{K}_k^a + \frac{ZaI}{r^2})g = 0 \). \( \square \)

Corollary 3.17. In the subextremal case the continuous spectrum is purely absolutely continuous and given by the whole real line, \( \sigma_{ac}(\tilde{K}_k^a) = \mathbb{R} \).

4. Summary and outlook

Hypothetical ‘hyper-heavy nuclei,’ which by definition obey \( GMm_e > Ze^2 \), also obey \( GM^2 > Ze^2 \) (because \( Zm_e < Zm_p \leq M = A(Z,N)m_p \) with \( A(Z,N) \approx Z + N \)), and thus are associated with the black hole sector of the Reissner–Weyl–Nordström (RWN) spacetime. This means that the number of neutrons \( N \gg Z \), as we have shown in section 2. In this paper we have investigated the Dirac Hamiltonian with and without anomalous magnetic moment of the electron when the electron is assumed to reside in the static subregion of the interior of an RWN black-hole spacetime, i.e.
a = 1 to several significant digits, this Dirac operator is well-defined and generates a unitary dynamics for the electron on the static subregion inside the RWN black hole.

We have characterized the spectrum of any self-adjoint extension in all subextremal cases where we showed they exist, and we found the essential spectrum is the whole real line, consisting of purely absolutely continuous spectrum. So there is no gap in the continuum, and therefore no discrete hyper-heavy hydrogenic ion spectrum in the subextremal black hole sector of RWN, unlike the situation in the naked singularity sector. Worse, the absolute continuity result for the spectrum means the complete absence of point spectrum for an electron in the static interior region of a subextremal RWN black-hole spacetime. An analogous result was proved by Cohen and Powers [10] for an electron outside the event horizon of a subextremal RWN black hole, so therefore we can now conclude that there is no hyper-heavy hydrogenic ion point spectrum at all in the subextremal RWN black-hole sector.

It still remains to discuss the Cohen-Powers setup for the extremal sector, i.e. electron outside of the horizon — but in this paper we were only concerned with electrons in the static part of the interior region. It also remains to settle the issue of the point spectrum in the extremal black hole case, when the electron spinor wave function is supported inside the event horizon. We have shown that the only possible eigenvalue is $-Z\alpha_s/r_0$, where $r_0$ is the area radius at which the horizon is located, but it is not clear whether this value is an eigenvalue, and if so, whether it is simple, finitely degenerate, or even infinitely degenerate. The extremal RWN black hole sector is not generic in the RWN spacetime family, but the open questions are technically challenging, and it is curious to contemplate that this exceptional black hole setting is so far the only one which has not been ruled out of permitting bound states.

It also remains to be seen whether the presence of a horizon generically causes absence of eigenvalues for the Dirac operator, as conjectured in [10] for electrons with wave functions supported outside the event horizon, and which may now be conjectured to be true also for electrons in the static interior of other subextremal black hole spacetimes. To prove such a conjecture in all generality, if indeed true, is a challenging project. Yet there are several feasible generalizations of our study which are worthy of pursuit, and which could cement the conjecture further or, possibly, disprove it.

One further direction of inquiry could be an investigation of the Dirac operator for a test electron in generalizations of the RWN black hole spacetime of a single point nucleus that obey other electrostatic vacuum laws. The naked singularity sector of such spacetimes has been described in [29], and generalized in appendix B of [23]. Such a study has the technical
advantage that the spherical symmetry of the spacetime allows one to work with the partial
wave decomposition of the Dirac Hamiltonian, as done in the present paper.

For the naked singularity sector such a study has recently been carried out in [1] for
singularities with zero bare mass, and in [23] for spacetimes with naked singularities of strictly
negative bare mass, with some surprising results. The perhaps most surprising result of [23]
(to its authors at least) is that the Dirac operator for an electron in the naked singularity
sector of the Hoffmann spacetime [20, 29] (Born [5] or Born–Infeld vacuum law) of a point
nucleus is not essentially self-adjoint, with or without anomalous magnetic moment, unless
the bare mass of the singularity vanishes [1]. A vanishing bare mass is not typical, though,
and so the upshot is that in the naked singularity sector of the Hoffmann spacetime family
the Dirac Hamiltonian of a test electron is typically not well-defined even if the anomalous
magnetic moment is taken into account.

We suspect that the same conclusion will hold for the Dirac operator of a test electron in
the static part of the interior region of a Hoffmann black hole spacetime.

Another generalization of the present work is to study the Dirac equation for a test electron
in the multi-black hole spacetime family of Hartle and Hawking [19], obtained by analytical
completion of the asymptotically flat, static, Majumdar–Papapetrou metrics. These are very
special spacetimes, but there are not many explicit representations of spacetimes with several
black holes in them. Each black hole of the Hartle–Hawking family obeys the RWN extremal
condition $GM^2 = Q^2$. The simplest multi-black hole case is a two-black-holes spacetime,
inevitably having axial symmetry. Separation of variables should again be feasible, even
though perhaps not as explicitly solvable as in the spherically symmetric case. A discrete
reflection symmetry is available in a three-black-holes spacetimes, offering some simplifica-
tion, yet for three or more nuclei (black holes) functional and PDE analysis will have to be
fielded to study the Dirac Hamiltonian, cf. [13] and references therein.
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