GOOD MODELS, INFINITE APPROXIMATE SUBGROUPS AND APPROXIMATE LATTICES

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Abstract. We investigate good models of approximate subgroups and use them to study various classes of approximate subgroups. In particular, we show a closed-approximate-subgroup theorem, we extend to all amenable groups a structure theorem for mathematical quasi-crystals due to Meyer, and we prove a generalisation of theorems due to Auslander and Mostow about intersections of lattices and radicals of Lie groups.

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1. Introduction

In a seminal paper [16] Hrushovski showed that a wide range of approximate subgroups are closely related to neighbourhoods of the identity in locally compact...
groups. His ideas were subsequently used in combination with results by Gleason and Yamabe to prove breakthrough structure theorems for finite approximate subgroups (10) and locally compact approximate subgroups (13). Similar ideas were used independently by Meyer to study mathematical models for quasi-crystals in his pioneering work (21). And by Monod and Caprace (12) in order to investigate commensurators of discrete subgroups, and by Tzanev (24) to investigate Hecke pairs, by means of the so-called Schlichting completion.

**Definition 1.** Let $\Lambda$ be an approximate subgroup of a group $\Gamma$. A group homomorphism $f : \Gamma \to H$ with target a locally compact group $H$ is called a *good model* (of $(\Lambda, \Gamma)$) if:

(i) $f(\Lambda)$ is relatively compact;
(ii) there is $U \subset H$ a neighbourhood of the identity such that $f^{-1}(U) \subset \Lambda$.

If $\Gamma$ is generated by $\Lambda$, then we say that $f$ is a good model of $\Lambda$. Any approximate subgroup commensurable to $\Lambda$ will be called a *Meyer subset*.

The term “good models” is inspired by the work of Breuillard, Green and Tao (10) and the “Meyer subsets” refers back to works in aperiodic order following Meyer’s theorem (21).

When $\Lambda$ generates $\Gamma$, Definition 1 is in fact a relaxed version of [10, Definition 3.5] and [16, Theorem 4.2] in a language where all subsets are definable. Furthermore, if $\Lambda$ is a genuine subgroup, then $f$ essentially is the Schlichting completion of $(\Lambda, \Gamma)$ (see [24]).

The link between good models and the works of Meyer is less straightforward. Meyer studied cut-and-project schemes as a paradigm to build certain types of approximate subgroups of Euclidean spaces. A *cut-and-project scheme* $(G, H, \Gamma)$ - as defined in full generality by Björklund and Hartnick in [4] - is the datum of two locally compact groups $G$ and $H$, and a lattice $\Gamma$ in $G \times H$ which projects injectively to $G$ and densely to $H$. Given a symmetric relatively compact neighbourhood of the identity $W_0 \subset H$ we can define the *model set* $P_0(G, H, \Gamma, W_0) := p_G((G \times W_0) \cap \Gamma)$.

**Proposition 1.** Let $\Lambda$ be a strong or uniform approximate lattice in a locally compact second countable group $G$. Then $\Lambda$ is a Meyer subset if and only if it is commensurable to a model set.

Approximate lattices refer to a type of approximate subgroups of locally compact groups introduced in [4] to give a non-commutative generalisation to the mathematical quasi-crystals studied by Meyer. In particular, uniform approximate lattices can be easily defined as those co-compact discrete closed approximate subgroups of locally compact groups: in other words any closed discrete approximate subgroup $\Lambda$ of a locally compact $G$ such that there exists a compact subset $K \subset G$ with $K\Lambda = G$.

The language of good models and the language of cut-and-project schemes both have advantages. On the one hand, cut-and-project schemes relate discrete approximate subgroups of locally compact groups to genuine discrete subgroups. On the other hand, it appears to be easier to prove that a given approximate subgroup has a good model thanks to results such as:

**Theorem 1.** Let $\Lambda$ be an approximate subgroup of a group $G$. There exists a good model $f$ of $\Lambda$ if and only if there is a sequence of approximate subgroups
Let $\Lambda_n = \Lambda$ and for all integers $n \geq 0$ we have $\Lambda_{n+1}^2 \subset \Lambda_n$ and $\Lambda_n$ commensurable to $\Lambda$.

The proof is simple and requires only knowledge of elementary point-set topology. In addition, it provides precise information on the good model $f$ (see the more detailed Theorem 8). Theorem 1 is inspired by [10, §6] and moreover is essentially implicit in works in model theory (see for instance [16, Lemma 3.4]).

Before we move on to applications, we mention a result in stark contrast with Theorem 1.

**Theorem 2.** Let $F_2$ be the free group over two generators $a$ and $b$. For any two reduced words $w, x \in F_2$ define $o(x, w)$ as the number of occurrences of $w$ in $x$. Then for any $w \in F_2 \setminus \{a, b, a^{-1}, b^{-1}, e\}$ of length $l$ the set
$$\{g \in F_2 | o(g, w) - o(g, w^{-1}) \leq 3l\}$$
is an approximate subgroup but not a Meyer subset.

We consider then two interesting classes of approximate subgroups: compact approximate subgroups and amenable approximate subgroups. We will see that these types of approximate subgroups always have good models, and thus are particularly regular types of approximate subgroups. In the case of compact approximate subgroups this leads to a closed-subgroup theorem for approximate subgroups.

**Theorem 3 (Closed-approximate-subgroup theorem).** Let $\Lambda$ be a closed approximate subgroup of a locally compact group $G$. There is an injective locally compact group homomorphism $\phi : H \to G$ and an open approximate subgroup $\Xi$ of $G$ such that $\Lambda \subset \phi(\Xi) \subset \Lambda^2$. Furthermore, if $G$ is a Lie group, then $H$ is a Lie group.

Here, a good model of some compact approximate subgroup contained in $\Lambda^2$ appears implicitly as the inverse of the map $\phi$. Theorem 3 in combination with a theorem due to P.K.Carolino ([13, Theorem 1.25]) yields a structure theorem in the spirit of [10] for all compact approximate subgroups of locally compact groups. More precisely, we are able to remove the openness assumption from [13, Theorem 1.25].

**Theorem 4 (Structure of compact approximate subgroups).** Let $\Lambda$ be a compact approximate subgroup of a locally compact group $G$. Then $\Lambda^\infty$ admits a structure of locally compact group such that the inclusion $\Lambda^\infty$ is continuous and $\Lambda$ is a compact neighbourhood of the identity. Moreover, for every $\varepsilon > 0$ there is an approximate subgroup $\Lambda' \subset \Lambda^{16}$ open in the topology of $\Lambda^\infty$ and a compact subgroup $H \subset \Lambda'$ normalised by $\Lambda'$ such that:

(i) $\Lambda$ can be covered by $O_{K,\varepsilon}(1)$ translates of $\Lambda'$;

(ii) $\langle \Lambda' \rangle / H$ is a Lie group of dimension $O_K(1)$.

If $\mathfrak{l}'$ denotes the Lie algebra of $\langle \Lambda' \rangle / H$ and $\Lambda''$ the image of $\Lambda'$ in $\langle \Lambda' \rangle / H$, then there exists a norm $| \cdot |$ on $\mathfrak{l}'$ such that:

(iii) for $X, Y \in \mathfrak{l}'$ we have $||[X, Y]|| \leq O_K(||X||Y||)$;

(iv) for $g \in \Lambda''$ the operator norm (induced by $| \cdot |$) of $\text{Ad}(g) - \text{Id}$ is $O_K(\varepsilon)$;

(v) there is a convex set $B \subset \mathfrak{l}'$ such that $\mathfrak{l}' / B$ is a finite $O_{K,\varepsilon}(1)$-approximate local group.

Likewise, we show that amenability assumptions force approximate subgroups to have good models.
**Theorem 5.** Let $G$ be a locally compact group, $H \subset G$ be a normal amenable closed subgroup and $K \subset G$ be a compact subset. If $\Lambda$ is a uniformly discrete approximate subgroup contained in $KH$ then $\Lambda^4$ has a good model.

As a corollary, we generalise Meyer’s seminal theorem [21] to all approximate lattices in amenable locally compact groups.

**Theorem 6** (Meyer theorem for amenable groups). If $\Lambda$ is an approximate lattice in an amenable locally compact group $G$, then $\Lambda$ is contained in and commensurable to a model set.

We are thus able to give a positive answer to a question raised by Björklund and Hartnick [4, Prob. 1] extending previous work in the nilpotent and soluble cases ([19, 18]). When $G$ is a Lie group we are able to get more precise information. We then know that $\Lambda$ is always uniform and that we can find a good model of $\Lambda^4$ with target a connected amenable Lie group (see Propositions 9 and 10).

Theorem 5 also allows us to study approximate subgroups in a neighbourhood of the amenable radical of a Lie group. This strategy yields a generalisation of theorems due to Auslander ([1, Theorem 1]) and Mostow ([22, Lemma 3.9]) about intersections of lattices and radicals in Lie groups.

**Theorem 7.** Let $\Lambda$ be a uniform or strong approximate lattice in a connected Lie group $G$. Let $A$ be its amenable radical, $R$ be its soluble radical and $N$ be its nilpotent radical. Then:

(i) the approximate subgroup $\Lambda^2 \cap A$ is a uniform approximate lattice in $A$;
(ii) if no semi-simple factor of $A$ acts trivially on $R$, then $\Lambda^2 \cap N$ is a uniform approximate lattice in $N$;
(iii) if no semi-simple factor of $A$ acts trivially on $R/N$, then $\Lambda^2 \cap R$ is a uniform approximate lattice in $R$.

We mention that in a manuscript in preparation ([?]) Hrushovski has studied a generalisation of good models called good quasi-models. Roughly, good quasi-model is defined similarly to a good model but the map $f$ is only required to be a quasi-homomorphism. A breakthrough result proved by Hrushovski asserts that all approximate subgroups have a good quasi-model (this is in particular coherent with Theorem 2).

1.1. **Proof strategy.** The proof of Proposition 1 is fairly straightforward. However, we mention that the first part of the proof (Lemma 8) consists in showing that the graph of a good model of a discrete approximate subgroup is a discrete subgroup. This observation is key in the rest of the paper. Then Proposition 1 is an elementary application of results from 4 and 5.

Likewise the proof of Theorem 4 is straightforward. We want to use the sequence $(\Lambda_n)_{n \geq 0}$ to define a neighbourhood basis for the identity in $\Lambda^\infty$. However, the sequence $(\Lambda_n)$ is not suitable, so the first step is to consider a ‘normalised’ version of $(\Lambda_n)_{n \geq 0}$ and to define the topology $T$ generated by those subsets. The algebraic properties of $(\Lambda_n)_{n \geq 0}$ then make $\Lambda^\infty$ equipped with $T$ into a topological locally pre-compact group. We are thus able to find a completion of $\Lambda^\infty$ and this completion map is the right candidate as a good model of $\Lambda$.

In particular, since the good model comes from the completion of a topology on $\Lambda^\infty$ we know that a good model and its target will inherit many properties of $\Lambda^\infty$. This observation leads to a proof of Theorem 2. Indeed, we choose an approximate
subgroup that is a ‘quasi-kernel’ of a quasi-morphism, and, thus, has ‘co-dimension’
one. We then manage to show that any good model, if one exists, essentially has
a target of dimension one, and even more precisely that the good model and the
quasi-morphism are within bounded distance. Then \[1\] allows us to conclude.

Theorem 3 and Theorem 4 are also consequences of this observation. A compact
approximate subgroup \(\Lambda\) comes with a topology that provides natural candidates
for the sequence \((\Lambda_n)_{n \geq 0}\), namely any neighbourhood basis for the identity in the
topological space \(\Lambda\). Once we have shown that locally \(\Lambda\) behaves like a group
(Lemma 9) we readily prove that \((\Lambda_n)_{n \geq 0}\) indeed satisfies the conditions of Theorem
1. We then remark that the good model of \(\Lambda\) we obtain has a continuous inverse.
This inverse map turns out to be the one described in the statements of Theorem
3 and Theorem 4.

To prove Theorem 5 and Theorem 6 we define *amenable approximate subgroups.*
This definition is a slight variation of [20, Definition 1] and a particular case of
the near-subgroups from [16]. We know therefore that they have a good model by
[16, \S 4]. For the sake of completeness we provide a short and self-contained proof
based on an argument by Massicot and Wagner ([20, Definition 1]) and Theorem
1. Then the proof of Theorem 5 essentially reduces to showing that a uniformly
discrete approximate subgroup in a neighbourhood of a normal amenable subgroup
is amenable. This part is proved thanks to a combination and adaptation of
techniques from the theory of amenable locally compact groups as found in [10].
Furthermore, the extension of Meyer’s theorem to amenable groups (Theorem 6)
is a direct consequence of Theorem 5 and the equivalence between good models and
cut-and-project schemes for approximate lattices (Proposition 1). Building up on
Theorem 6 we are then able to obtain interesting results on approximate lattices in
connected amenable Lie groups (Propositions 9,10 and 11).

These results are particularly useful in the proof of Theorem 7. To prove The-
orem 7 we roughly follow a proof strategy due to Raghunathan (see [23, \S 8]). We
first prove a Borel density theorem for approximate lattices and a general result
about intersections of approximate lattices and closed subgroups in the spirit of
[23, Theorem 1.13]. We also make a crucial use of Theorem 5 to prove that dis-
crete approximate subgroups around a normal soluble subgroup generate soluble
subgroups (Proposition 13). This part of the proof can be seen as an approximate
version of a classical result due to Auslander (see [23, Theorem 8.24]) as well as
a soluble version of Theorem 5. Finally, we consider an approximate subgroup \(\Xi\)
within bounded distance of the radical of a Lie group that satisfies a minimality
assumption. Combining all of the previous results to study \(\Xi\) we are able to prove
part (i) of Theorem 7. Part (ii) and (iii) are then technical but straightforward con-
sequences. Note that the last step in the proof of part (i) of Theorem 7 is specific
to the case of approximate lattices and does not appear in [23, \S 8].

2. Preliminaries

2.1. Preliminaries on approximate subgroups and commensurability. We
collect here various well-known results about general approximate subgroups.

Lemma 1 (Rusza’s covering lemma). Let \(X,Y\) be subsets of a group \(G\) and \(F \subset X\)
be maximal such that \((fY)_{f \in F}\) is a family of disjoint sets. Then \(X \subset FY^{-1}\).

**Proof.** If \(x \in X\), then there is \(f \in F\) such that \(xX \cap fX\). Thus, \(x \in fXX^{-1} \subset FY^{-1}\). \(\square\)
Lemma 2. Let $X_0, X_1, \ldots, X_r$ be subsets of a group $G$ and $F_1, \ldots, F_r \subset G$ be finite subsets such that $X_0 \subset F_i Y_i$ for all integers $1 \leq i \leq r$. There is $F \subset G$ with $|F| \leq |F_1| \cdots |F_r|$ such that

$$X_0 \subset F \cdot \bigcap_{1 \leq i \leq n} X_i^{-1} X_i.$$  

Proof. Take $f := (f_i) \in F_1 \times \cdots \times F_r$ and if $\bigcap_{1 \leq i \leq r} f_i X_i \neq \emptyset$ choose an element $x_f \in \bigcap_{1 \leq i \leq r} f_i X_i$. If $x$ is any element of $X_0$ then there must be some $f \in F_1 \times \cdots \times F_r$ such that $x \in \bigcap_{1 \leq i \leq r} f_i X_i$. We thus have $x_f^{-1} x \in \bigcap_{1 \leq i \leq n} X_i X_i^{-1} X_i$. So, defining $F := \{ x_f \mid f \in F_1 \times \cdots \times F_n \} X \cap f Y \neq \emptyset \}$, we find

$$X \subset F \cdot \bigcap_{1 \leq i \leq n} X_i^{-1} X_i.$$  

\[\square\]

Lemma 3. Let $k_1, \ldots, k_r$ be positive integers and $\Lambda_1, \ldots, \Lambda_r$ be $K_1, \ldots, K_r$-approximate subgroups of a group $G$. We have:

1. $\bigcap_{1 \leq i \leq r} \Lambda_i^k$ is a $K_1^k \cdots K_r^k$-approximate subgroup;
2. if $(\Xi_i)_{1 \leq i \leq r}$ is a family of approximate subgroups with $\Xi_i$ commensurable to $\Lambda_i$ for all integers $1 \leq i \leq r$, then the subsets $\bigcap_{1 \leq i \leq r} \Lambda_i^k$ and $\bigcap_{1 \leq i \leq r} \Xi_i^k$ are commensurable.

Proof. We know that $\bigcap_{1 \leq i \leq n} \Lambda_i^k$ is covered by $K_1^k$ left-translates of $\Lambda_i$ for all integers $1 \leq i \leq r$. So statement (1) is a consequence of Lemma 2. In a similar fashion, statement (2) is a consequence of Lemma 2 applied to $\bigcap_{1 \leq i \leq r} \Lambda_i^k$ and $\Xi_1, \ldots, \Xi_r$.  

\[\square\]

Remark 1. Part (1) of Lemma 3 is [10, Lemma 10.3].

Corollary 1. Let $\Lambda_1$ and $\Lambda_2$ be two commensurable approximate subgroups of a group $G$. Let $\phi : H \to G$ be a group homomorphism. Then $\phi^{-1}(\Lambda_1)$ and $\phi^{-1}(\Lambda_2)$ are commensurable approximate subgroups of $H$.

Proof. By Lemma 3 the subsets $f(H) \cap \Lambda_1^k$ and $f(H) \cap \Lambda_2^k$ are commensurable approximate subgroups. Take $\{ i, j \} \subset \{ 1, 2 \}$, we can therefore find a finite subset $F_{ij} \subset \Gamma$ such that $(f(H) \cap \Lambda_1^k)^2 \subset f(F_{ij}) (f(H) \cap \Lambda_2^k)$. Thus,

$$f^{-1}(\Lambda_i)^4 \subset F_{ij} f^{-1}(\Lambda_j)^2.$$  

So $f^{-1}(\Lambda_1)^2$ and $f^{-1}(\Lambda_2)^2$ are commensurable approximate subgroups.  

\[\square\]

2.2. Invariant hull, strong approximate lattices, cut-and-project schemes.

Let $G$ be a locally compact second countable group and let $C(G)$ be the set of closed subsets of $G$. We can define a topology on $G$ called the Chabauty-Fell topology. It is generated by the sets

$$U^V = \{ F \in C(G) \mid F \cap V \neq \emptyset \} \text{ and } U_K = \{ F \in C(G) \mid F \cap K = \emptyset \}$$

for all $V \subset G$ open and $K \subset G$ compact. One can check that the map

$$G \times C(G) \to C(G)$$

$$(g, F) \mapsto gF$$

defines a continuous action of the group $G$ on $C(G)$ and that $C(G)$ is a compact metrizable set (see [4, Section 4.1]). Whence the Chabauty-Fell topology is uniquely determined by the notion of convergence. But convergence in the Chabauty-Fell
topology can also be characterised in the following way: A sequence \((F_i)_{i \geq 0}\) converges to \(F \in C(G)\) if and only if (1) for every \(x \in F\) there are \(x_i \in F_i\) for all \(i \in \mathbb{N}\) such that \(x_i \to x\) as \(i \to \infty\); (2) If \(x_i \in F_i\) for all \(i \in \mathbb{N}\) then every accumulation point of \((x_i)_{i \geq 0}\) lies in \(F\) (see [6, Section 2.2]).

**Definition 2.** Let \(F\) be a closed subset of a locally compact second countable group \(G\). The **invariant hull** of \(F\) is the compact \(G\)-space defined as

\[ \Omega_F := G \cdot F \subset C(G) \]

with the induced continuous \(G\)-action.

**Definition 3.** Let \(\Lambda\) be a uniformly discrete approximate subgroup of a locally compact second countable group \(G\). A \(G\)-invariant Borel probability measure \(\nu\) on \(\Omega_\Lambda\) is said **non-trivial** if \(\nu(\{\emptyset\}) = 0\). The approximate subgroup \(\Lambda\) is a **strong approximate lattice** if there is a non-trivial \(G\)-invariant Borel probability measure \(\nu\) on \(\Omega_\Lambda\).

**Proposition 2 (Lemma 2.3, [6]).** If \(H\) is a closed subgroup of a locally compact second countable group \(G\), then \(\Omega_H \setminus \{\emptyset\}\) is isomorphic as a topological \(G\)-space to \(G/H\).

**Corollary 2.** Let \(H\) be a closed subgroup of a locally compact second countable group \(G\), then \(H\) is a lattice if and only if \(H\) is an approximate lattice.

**Lemma 4 (Remark 4.14, (1), [4]).** If \(\Lambda\) is a uniform approximate lattice in an amenable locally compact second countable group, then \(\Lambda\) is a strong approximate lattice.

**Definition 4 (Definitions 2.11 and 2.12, [4]).** A **cut-and-project scheme** \((G, H, \Gamma)\) is the datum of two locally compact groups \(G\) and \(H\), and a lattice \(\Gamma\) in \(G \times H\) which projects injectively to \(G\) and densely to \(H\). Take moreover a symmetric relatively compact neighbourhood of the identity \(W_0 \subset H\). Then the set

\[ P_0(G, H, \Gamma, W_0) := p_G ((G \times W_0) \cap \Gamma) \subset G \]

is called a **model set** where \(p_G : G \times H \to G\) denotes the natural projection.

The interest of model sets and cut-and-project schemes in the study of approximate lattices lies in the following proposition.

**Proposition 3 (Theorem 3.4, [3] and Proposition 2.13, [4]).** Let \((G, H, \Gamma)\) be a cut-and-project scheme and \(W_0\) be a symmetric relatively compact neighbourhood of the identity. If \(G\) is second countable and \(\partial W_0\) is Haar-null, then \(P_0(G, H, \Gamma, W_0)\) is a strong approximate lattice. If \(\Gamma\) is a uniform lattice, then \(P_0(G, H, \Gamma, W_0)\) is a uniform approximate lattice.

### 3. Good models: definition, first properties and examples

#### 3.1. About the definition of good models

Let us recall Definition

**Definition.** Let \(\Lambda\) be an approximate subgroup of a group \(G\). A group homomorphism \(f : \Lambda^\infty \to H\) with target a locally compact group \(H\) is called a **good model** (of \(\Lambda\)) if:

(i) \(f(\Lambda)\) is relatively compact;
(ii) there is \(U \subset H\) a neighbourhood of the identity such that \(f^{-1}(U) \subset \Lambda\).
Any approximate subgroup commensurable to \( \Lambda \) will be called a Meyer subset.

\textbf{Remark 2.} Restricting the range of the good model \( f \) we can always assume that \( f \) has dense image.

\textbf{Lemma 5.} Let \( H \) be a locally compact group, \( \Gamma \) be a discrete group, \( V_1 \) and \( V_2 \) be symmetric relatively compact neighbourhoods of the identity in \( H \) and \( f : \Gamma \to H \) be a group homomorphism. The subsets \( f^{-1}(V_1) \) and \( f^{-1}(V_2) \) are commensurable approximate subgroups.

\textbf{Proof.} Take \( i, j \in \{1, 2\} \). We have \( e \in \hat{V}_j \) so

\[
V_i^2 \subset \bigcup_{h \in V_i^2} h\hat{V}_j.
\]

But \( \hat{V}_j \) is open and \( V_i^2 \) is relatively compact, and, thus, there is a finite subset \( F_{ij} \subset V_i^2 \) such that \( V_i^2 \subset F_{ij}U \). Since \( V_1 \) and \( V_2 \) are moreover symmetric subsets, we have that \( V_1 \) and \( V_2 \) are commensurable approximate subgroups.

Choose now a symmetric open neighbourhood of the identity \( W \) such that \( W^2 \) is contained in \( V_1 \) and \( V_2 \). Then \( f^{-1}(W^2), f^{-1}(V_1^2) \) and \( f^{-1}(V_2^2) \) are commensurable approximate subgroups by Lemma 5 But for \( i = 1, 2 \) we have

\[
f^{-1}(W^2) \subset f^{-1}(V_i) \subset f^{-1}(V_i^2).
\]

Hence, \( f^{-1}(V_1) \) and \( f^{-1}(V_2) \) are commensurable approximate subgroups. \( \square \)

\textbf{Corollary 3.} Let \( \Lambda \) be an approximate subgroup of a group \( G \) and \( f : \Lambda^\infty \to H \) be a good model of \( \Lambda \). If \( U \subset H \) is a symmetric neighbourhood of the identity such that \( f^{-1}(U) \subset \Lambda \), then \( f^{-1}(U) \) is an approximate subgroup commensurable to \( \Lambda \).

\textbf{Proof.} We know that \( f(\Lambda) \) is relatively compact so there is a compact symmetric neighbourhood of the identity \( V \subset H \) such that \( f(\Lambda) \subset V \). We have

\[
f^{-1}(U \cap V) \subset f^{-1}(U) \subset \Lambda \subset f^{-1}(V).
\]

Since \( f^{-1}(U \cap V) \) and \( f^{-1}(V) \) are commensurable approximate subgroups by Lemma 5 and \( f^{-1}(U) \) is symmetric, the subsets \( f^{-1}(U) \) and \( \Lambda \) are commensurable approximate subgroups. \( \square \)

\textbf{Proposition 4.} Let \( \Lambda \) be an approximate subgroup of a group \( \Gamma \). Suppose that \( (\Lambda, \Gamma) \) has a good model. We have:

\begin{enumerate}
\item if \( \phi_1 : \Gamma_1 \to \Gamma \) is a group homomorphism, then \( \phi^{-1}(\Lambda) \) is an approximate subgroup and \( (\phi^{-1}(\Lambda), \Gamma_1) \) has a good model;
\item if \( \phi_2 : \Gamma \to \Gamma_1 \) is a group homomorphism, then \( (\phi_2(\Lambda), \phi_2(\Gamma)) \) has a good model;
\item there is \( \Lambda_3 \subset \Lambda \) commensurable to \( \Lambda \) and a good model of \( \Lambda_3 \) with target a connected Lie group.
\end{enumerate}

Let \( (\Lambda_i)_{i \in I} \) be a directed family of \( K \)-approximate subgroups of the groups \( (\Gamma_i)_{i \in I} \) for some fixed integer \( K \). Suppose that \( \psi_{ij}(\Lambda_i) \subset \Lambda_j \) whenever \( i \leq j \). Then:

\begin{enumerate}
\item the direct limit of \( \lim_{i \to j} \Lambda_i^2 \) is an approximate subgroup;
\item if moreover \( \Lambda_i \) has a good model for all \( i \in I \), then the direct limit \( \lim_{i \to j} \Lambda_i \) has a good model.
\end{enumerate}
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Proof. Let \( f : \Gamma \to H \) be a good model of \((\Lambda, \Gamma)\) and let \( U \subset H \) be open subset as in Definition \([1]\). Set furthermore \( \Lambda_1 := \phi_1^{-1}(\Lambda) \) and \( f_1 := f \circ \phi \). Then \( f_1(\Lambda) = f(\Lambda) \) is relatively compact and \( f_1^{-1}(U) \subset \Lambda_1 \). Hence, \( \Lambda_1 \) is an approximate subgroup by Lemma \([5]\) and \( f_1 \) is a good model for \((\Lambda_1, \Gamma_1)\). Now, that \((\phi_2(\Lambda), \phi_2(\Gamma))\) has a good model is a clear consequence of Theorem \([4]\). Let us now prove (3). By the Gleason–Yamabe theorem there is \( \tilde{H} \) an open subgroup and a normal compact subgroup \( K \) of \( \tilde{H} \) contained in \( U \) such that \( \tilde{H}/K \) is a Lie group. There is moreover a symmetric open subset \( K \subset V \subset U \cap \tilde{H} \) such that the projection of \( f(\Gamma) \cap V \) to \( \tilde{H}/K \) is dense in a connected subgroup. Then \( \Lambda_3 := f^{-1}(V) \) is an approximate subgroup commensurable to \( \Lambda \) according to Lemma \([3]\). But the composite of \( f \) and the natural projection \( \tilde{H} \to \tilde{H}/K \) is a good model of \( \Lambda_3 \).

Let \( \mathcal{U} \) be an ultra-filter on \( I \) that contains the cofinal subsets \( \{ i \in I | j \leq i \} \). Write \( \Lambda \) and \( \Gamma \) for the ultraproducts of \((\Lambda_i)_{i \in I}\) and \((\Gamma_i)_{i \in I}\) over \( \mathcal{U} \) respectively. By the universal property of direct limits, there is a natural map \( \phi : \lim_{\rightarrow \mathcal{U}} \Gamma_i \to \Gamma \) such that \( \phi^{-1}(\Lambda) = \lim_{\rightarrow \mathcal{U}} \Lambda_i \). So \( \lim_{\rightarrow \mathcal{U}} \Lambda_2 \) is an approximate subgroup by Corollary \([1]\). Moreover, the approximate subgroup \( \Lambda \) has a good model by \([13, \text{Theorem 6.6}]\). So \( \lim_{\rightarrow \mathcal{U}} \Lambda_i \) has a good model by Corollary \([1]\). \( \square \)

3.2. Combinatorial characterisation of good models.

Theorem 8. Let \( \Lambda \) be an approximate subgroup of a group \( \Gamma \). The following are equivalent:

1. there is a good model \( f : \Gamma \to H \) for \((\Lambda, \Gamma)\);
2. there exists a sequence \((\Lambda_n)_{n \geq 0}\) of approximate subgroups such that:
   (a) \( \Lambda_0 = \Lambda \);
   (b) for all integers \( n \geq 0 \) and all \( \gamma \in \Gamma \), the approximate subgroups \( \gamma \Lambda_n \gamma^{-1} \) and \( \Lambda \) are commensurable;
   (c) for all integers \( n \geq 0 \), we have \( \Lambda_{n+1} \subset \Lambda_n \);
3. there exists \( B \subset \mathcal{P}(\Gamma) \) such that:
   (a) there is \( \Xi \in B \) with \( \Xi \subset \Lambda \);
   (b) all elements of \( B \) contain \( e \) and are commensurable to \( \Lambda \);
   (c) for all \( \Lambda_1 \in B \), there is \( \Lambda_2 \in B \) such that \( \Lambda_2 \Lambda_1 \subset \Lambda_1 \);
   (d) for all \( \gamma \in \Gamma \) and \( \Lambda_1 \in B \), there is \( \Lambda_2 \in B \) with \( \gamma \Lambda_2 \gamma^{-1} \subset \Lambda_1 \).

Moreover, when any of the three statements above is satisfied:

4. with \( B \) as in (3), we can choose a good model \( f : \Gamma \to H \) such that \( f \) has dense image and \( B \) is a neighbourhood basis for the identity with respect to the initial topology on \( \Gamma \) given by \( f \);
5. when any of there is a good model \( f_0 : \Gamma \to H_0 \) for \((\Lambda, \Gamma)\) such that for any other good model \( f : \Gamma \to H \) for \((\Lambda, \Gamma)\) we have a continuous group homomorphism \( \phi : H_0 \to H \) with \( f = \phi \circ f_0 \).

Proof. (1) \( \Rightarrow \) (2):

Choose a neighbourhood of the identity \( U \subset H \) such that \( f^{-1}(U) \subset \Lambda \). There exists a sequence \((U_n)_{n \geq 0}\) of relatively compact symmetric neighbourhoods of the identity in \( H \) such that \( U_0 = U \) and \( U_{n+1} \subset U_n \) for all integers \( n \geq 0 \). Define now \((\Lambda_n)_{n \geq 0}\) by \( \Lambda_n = \Lambda \) and \( \Lambda_n = f^{-1}(U_n) \). We readily check that for all integers \( n \geq 0 \) we have \( \Lambda_{n+1} \subset \Lambda_n \). Furthermore, \( \gamma \Lambda_n \gamma^{-1} \) is an approximate subgroup commensurable to \( \Lambda \) by Corollary \([3]\). So (1) \( \Rightarrow \) (2) is proved.

(2) \( \Rightarrow \) (3):
Let \((\Lambda_n)_{n \geq 0}\) be as in (2). For any two subsets \(X, Y \subset G\) define

\[ X^Y := \bigcap_{y \in Y} yXy^{-1}. \]

Define the subset \(B\) of \(\Gamma\) by

\[ B := \left\{ (\Lambda_2^1)^F \mid \forall n \in \mathbb{N}, \forall F \subset \Gamma, |F| < \infty \right\}. \]

We know that \(\Lambda_2^1 \subset \Lambda\) and that for all \(\Xi \in B\) we have \(e \in \Xi\) and \(\Xi\) is an approximate subgroup commensurable to \(\Lambda\) (Lemma 3). Now, for all \(n \in \mathbb{N}\), \(F \subset \Gamma\) finite we have

\[ \left( (\Lambda_{n+1}^2)^F \right)^2 \subset (\Lambda_{n+1}^4)^F \subset (\Lambda_n^2)^F \]

and for \(\gamma \in \Gamma\) we find

\[ \gamma (\Lambda_n^2)^F \gamma^{-1} \subset (\Lambda_n^2)^F. \]

So \(B\) satisfies (3).

(3) \(\Rightarrow\) (1):

Choose \((\Lambda_n)_{n \geq 0}\) as in (3). Equip the group \(\Lambda^\infty\) with the topology defined by

\[ T = \{ U \subset \Gamma \mid \forall \gamma \in U, \exists \Xi \in B, \gamma \Xi \subset U \}. \]

By [8, Chapter III, §2, Proposition 1] the topology \(T\) is the unique topology that makes \(G\) into a topological group and such that \(B\) is a neighbourhood basis for the identity in \(\Gamma/\{e\}\).

Now, the closure \([e]\) of the identity is a closed normal subgroup and the group \(\Gamma/\{e\}\) equipped with the quotient topology is the maximal Hausdorff factor of \(\Gamma\). Let \(p : \Gamma \to \Gamma/\{e\}\) be the natural map. Then \(\{p(\Xi)\mid \Xi \in B\}\) is a neighbourhood basis for the identity in \(\Gamma/\{e\}\). But the subsets that belong to \(B\) are pairwise commensurable, so the neighbourhoods \(\{p(\Xi)\mid \Xi \in B\}\) are pre-compact. Hence, the topological group \(\Gamma/\{e\}\) has a completion by [26, Theorem X]. In other words, there is a locally compact group \(H\) and a group homomorphism

\[ i : \Gamma/\{e\} \to H \]

such that \(i\) has dense image and is a homeomorphism onto its image. Define the map \(f := i \circ p\). We will show that \(f\) is a good model. The group \(H\) is a complete space and \(\Lambda\) is pre-compact in the topology \(T\) according to assumption (b). So \(f(\Lambda)\) is a relatively compact subset of \(H\). Also, recall that \(i\) is a homeomorphism onto its image, the map \(p\) is open and \(B\) is a neighbourhood basis for the identity. There is thus a neighbourhood of the identity \(U \subset H\) such that \(f^{-1}(U) \subset \Lambda\) according to assumption (a).

Statement (4) is straightforward from the proof of (3) \(\Rightarrow\) (1). Let us prove (5).

Let \(T_0\) denote the initial topology on \(\Gamma\) with respect to the class of all good models \(f : \Gamma \to H\) for \((\Lambda, \Gamma)\). In other words, the topology \(T_0\) is generated by the subsets \(f^{-1}(U)\) of \(P(\Gamma)\) where \(f : \Gamma \to H\) and \(U\) run through all good models for \((\Lambda, \Gamma)\) and all open subsets \(U \subset H\). Define \(B_0\) as \(\{ U \in T_0 \mid U \subset \Lambda \}\). Since \(\Lambda\) is assumed to have a good model we know that \(B_0\) is not empty. Moreover, take \(\Xi \in B_0\). Then there are good models \((f_i : \Gamma \to H_i)_{1 \leq i \leq r}\) for \((\Lambda, \Gamma)\) and open neighbourhoods of the identity \(U_i \subset H_i\) for all \(1 \leq i \leq r\) such that

\[ \bigcap_{1 \leq i \leq r} f_i^{-1}(U_i) \subset \Xi. \]
Choose symmetric open neighbourhoods of the identity such that $V_i^2 \subset U_i$ for all integers $1 \leq i \leq r$. According to Corollary 3 and Lemma 3 we know that $\Lambda$ is commensurable to $\bigcap_{1 \leq i \leq r} f_i^{-1}(V_i^2)$, hence to $\Xi$. So $\mathcal{B}_0$ satisfies conditions (a) and (b) of (3). But conditions (c) and (d) of (3) are also satisfied since $(G, \mathcal{T}_0)$ is a topological group. Let now $f_0 : \Gamma \to H_0$ be as in the proof of (3). Then every good model $f : \Gamma \to H$ for $(\Lambda, \Gamma)$ is continuous with respect to $\mathcal{T}_0$. According to the universal properties of quotients and completions one can therefore find a continuous group homomorphism $\phi : H_0 \to H$ such that $\phi \circ f_0 = f$. 

\textbf{Remark 3.} We mention two other approaches to Theorem 8. In both, the first step is to embed $\Lambda$ in a “sufficiently saturated elementary extension”. In our situation, this would mean embedding $\Lambda$ in an ultra-power of $\Lambda$ over a sufficiently large ultra-filter. Then one can either consider a language in which the subsets forming the sequence $(\Lambda_n)_{n \geq 0}$ are definable, and define $\mathcal{T}$ as the logic topology ([16, Lemma 3.4], [20]). Or proceed as in [10, Lemma 6.6] constructing a distance via the Birkhoff–Kakutani construction. In both cases this would provide a locally compact group topology on a quotient of the “sufficiently saturated elementary extension”. Notice that since taking the metric completion is essentially taking a quotient of a suitable topology on a quotient of the “sufficiently saturated elementary extension”. Notice that since taking the metric completion is essentially taking a quotient of a suitable ultra-power, the three approaches are essentially the same. Conversely, the extra structure obtained through these two approaches is implicitly obtained in the proof of Theorem 8 since the set $\mathcal{B}$ we construct in step (2) $\Rightarrow$ (3) is contained in any Boolean algebra containing the subsets $(\Lambda_n^2)$.

\textbf{Proof of Theorem 4.} Theorem 1 is the equivalence “(1) $\iff$ (2)” in Theorem 8.

\textbf{Corollary 4.} Let $\Lambda$ be an approximate subgroup of a group $G$. Then $\Lambda$ is a Meyer subset if and only if there is a sequence $(\Lambda_n)_{n \geq 0}$ of approximate subgroups commensurable to $\Lambda$ such that $\Lambda_{n+1}^2 \subset \Lambda_n$ for all $n \geq 0$.

\textbf{Proposition 5.} Let $\Lambda$ be an approximate subgroup of a group $G$. If $\Lambda$ is a Meyer subset then there is a positive integer such that $\Lambda^n$ has a good model.

\textbf{Proof.} By Corollary 4 there is a sequence $(\Lambda_n)_{n \geq 0}$ of approximate subgroups commensurable to $\Lambda$ such that $\Lambda_{n+1}^2 \subset \Lambda_n$ for all integers $n \geq 0$. Define for all $n \geq 1$ the subset $\tilde{\Lambda}_n := \Lambda_\infty \cap \Lambda_n^2$. According to Lemma 3 the subset $\Lambda_n$ is an approximate subgroup. We also have

\[ \Lambda^2 \cap \Lambda_n^2 \subset \tilde{\Lambda}_n \subset \Lambda_n^2, \]

but $\Lambda^2 \cap \Lambda_n^2$ is commensurable to $\Lambda$ by Lemma 3 and $\Lambda_n^2$ is as well. So $\tilde{\Lambda}_n$ is commensurable to $\Lambda$. Since $\tilde{\Lambda}_1 \subset \Lambda_\infty$ and $\tilde{\Lambda}_1$ is commensurable to $\Lambda$, there is an integer $n \geq 0$ such that $\tilde{\Lambda}_1^2 \subset \Lambda^n$. Finally, for all $n \geq 1$ we have

\[ \tilde{\Lambda}_{n+1}^2 \subset \Lambda_\infty \cap \Lambda_{n+1}^2 \subset \Lambda_\infty \cap \Lambda_n^2 = \tilde{\Lambda}_n. \]

Set $\tilde{\Lambda}_0 = \Lambda^n$. Then $(\tilde{\Lambda}_n)_{n \geq 0}$ satisfies the conditions of Theorem 8. So $\Lambda^n$ has a good model.

3.3. An approximate subgroup without a good model.

\textbf{Definition 5.} Let $G$ be a group. A map $f : G \to \mathbb{R}$ is a quasi-morphism if there exists $C \geq 0$ such that for all $g, h \in G$ we have

\[ |f(gh) - f(g) - f(h)| \leq C. \]
We denote the infimum of such $C$’s by $C(f)$. We say that $f$ is **symmetric** if for all $g \in G$ we have $f(g^{-1}) = -f(g)$. Moreover, we say that $f$ is **homogeneous** when for all $n \in \mathbb{Z}$ and $g \in G$ we have $f(g^n) = nf(g)$.

**Lemma 6.** Let $G$ be a group and $f : G \to \mathbb{R}$ be a symmetric quasi-morphism. Then for all $R > C(f)$ the set $f^{-1}([-R; R])$ is an approximate subgroup.

**Proof.** Let $\Lambda$ denote the set $f^{-1}([-R; R])$. First of all, $\Lambda$ is symmetric since $f$ is symmetric. Also, the set $f(\Lambda^2)$ is contained in $[-2R - C(f); 2R + C(f)]$. Set $\delta := R - C(f) > 0$ and choose a finite subset $F \subset \Lambda^2$ such that

$$f(\Lambda^2) \subset \bigcup_{\gamma \in F} f(\gamma) + [-\delta; \delta].$$

We will now prove that $\Lambda^2 \subset FA$. Indeed, let $\lambda \in \Lambda^2$. There is $\gamma \in F$ such that $|f(\lambda) - f(\gamma)| \leq \delta$. But,

$$|f(\gamma^{-1}\lambda) - (f(\lambda) - f(\gamma))| \leq C(f),$$

so

$$|f(\gamma^{-1}\lambda)| \leq C(f) + \delta = R.$$

Hence, $\gamma^{-1}\lambda \in \Lambda$. \hfill $\square$

**Lemma 7.** Let $G$ be a group and $f_1, f_2 : G \to \mathbb{R}$ be two symmetric quasi-morphisms. Suppose that $\sup_{g \in G} |f_1(g) - f_2(g)| < \infty$. Then for all $R_1 > C(f_1)$ and $R_2 > C(f_2)$ the approximate subgroups $f_1^{-1}([-R_1; R_1])$ and $f_2^{-1}([-R_2; R_2])$ are commensurable.

**Proof.** First of all, assume that $f_1 = f_2 = f$. Set $\delta := R_1 - C(f)$ and let $\Lambda_1$ and $\Lambda_2$ denote respectively $f^{-1}([-R_1; R_1])$ and $f^{-1}([-R_2; R_2])$. Notice that $\Lambda_1 \subset \Lambda_2$. Choose now a finite subset $F \subset \Lambda_2$ such that

$$f(\Lambda_2) \subset \bigcup_{\gamma \in F} f(\gamma) + [-\delta; \delta].$$

Then for any $\lambda \in \Lambda_2$ there is $\gamma \in F$ such that $|f(\lambda) - f(\gamma)| \leq \delta$. Thus,

$$|f(\gamma^{-1}\lambda)| \leq C(f) + \delta = R_1.$$

Therefore, $\Lambda_2 \subset F f^{-1}([-R_1; R_1]) = FA_1$.

Now, suppose that $f_1 \neq f_2$. Let $\eta$ denote $\sup_{g \in G} |f_1(g) - f_2(g)|$. Again let $\Lambda_1$ and $\Lambda_2$ denote respectively $f_1^{-1}([-R_1; R_1])$ and $f_2^{-1}([-R_2; R_2])$. Then

$$f_1(\Lambda_2) \subset [-\eta - R_2; R_2 + \eta],$$

so

$$\Lambda_2 \subset f_1^{-1}([-\eta - R_2; R_2 + \eta]).$$

But according to the first part of the proof there is a finite subset $F_1 \subset G$ such that

$$f_1^{-1}([-\eta - R_2; R_2 + \eta]) \subset F_1 A_1.$$

Hence, $\Lambda_2 \subset F_1 A_1$. By a similar argument we conclude that there is a finite subset $F_2 \subset G$ such that $\Lambda_1 \subset F_2 A_2$. Hence, $\Lambda_1$ and $\Lambda_2$ are commensurable. \hfill $\square$

**Proposition 6.** Let $G$ be a finitely generated group and $f : G \to \mathbb{R}$ be a homogeneous quasi-morphism. Choose a real number $R > C(f)$. If the approximate subgroup $f^{-1}([-R; R])$ is a Meyer subset then $f$ is a group homomorphism.
Proof: If \( f \) is bounded, then \( f = 0 \). So assume that \( f \) is unbounded. Let \( \Lambda \) denote \( f^{-1}([-R; R]) \). We know that \( \Lambda \) is an approximate subgroup by Lemma 6. Since \( \Lambda \) is commensurable to \( f^{-1}([-R'; R']) \) for any \( R' > C(f) \) by Lemma 7, the approximate subgroup \( f^{-1}([-R'; R']) \) is a Meyer subset as well. But \( G \) is finitely generated, so choose \( R' > C(f) \) such that \( f^{-1}([-R'; R']) \) generates \( G \). We can replace \( R \) by \( R' \) and assume that \( \Lambda \) generates \( G \). By proposition 5 there is an integer \( n \geq 1 \) such that there is a good model \( \tilde{f} : \Lambda^\infty \to H \) for \( \Lambda^n \). Moreover, we can assume that \( \tilde{f}(\Lambda^\infty) \) is dense in \( H \). Take \( U \subset H \) an open neighbourhood of the identity such that \( f^{-1}(U) \subset \Lambda^n \). Since \( \tilde{f}(\Lambda^\infty) \) is dense in \( H \), we have \( U \subset \tilde{f}(\Lambda^n) \). The subgroup generated by the compact set \( \tilde{f}(\Lambda^n) \) is therefore open, hence closed, and contains \( \Lambda^\infty \). Thus, the compact set \( \tilde{f}(\Lambda^n) \) generates \( H \). We will show that \( f = c\tilde{f} \) for some real number \( c > 0 \). Let us first prove two claims.

Claim 3.3.1. The set of commutators \( \{h_1h_2h_1^{-1}h_2^{-1}|h_1, h_2 \in H\} \) is relatively compact.

By Lemma 7 the approximate subgroup \( \tilde{f}(f^{-1}([-3C(f); 3C(f)])) \) is commensurable to \( \tilde{f}(\Lambda) \subset \tilde{f}(\Lambda^n) \). Hence, the subset \( K := \tilde{f}(f^{-1}([-3C(f); 3C(f)])) \) is compact. Take now \( \gamma_1, \gamma_2 \in \Lambda^\infty \). Then \( \tilde{f}(\gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}) \leq 3C(f) \). So \( \tilde{f}(\gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}) \in K \). But \( \tilde{f}(\Lambda^\infty) \) is dense in \( H \) so \( \{h_1h_2h_1^{-1}h_2^{-1}|h_1, h_2 \in H\} \subset K \).

Claim 3.3.2. Let \( \gamma_0 \in \Lambda^\infty \) be such that \( f(\gamma_0) > 0 \). Then there is a compact subset \( K \subset G \) such that for all \( \gamma \in \Lambda^\infty \) we have

\[
\tilde{f}(\gamma) \subset \{\tilde{f}(\gamma_0^n)\}K.
\]

Set \( R_0 := f(\gamma_0) \). The subset \( f^{-1}([-R_0 - C(f); R_0 + C(f)]) \) is commensurable to \( \Lambda \) by Lemma 7. So, as above, the subset \( K := \tilde{f}(f^{-1}([-R_0 - C(f); R_0 + C(f)])) \) is compact. Now, there is an integer \( l \) such that \( |f(\gamma) - lf(\gamma_0)| \leq R_0 \). Therefore, \( \gamma_0^{-l}\gamma \in K \).

We will now conclude thanks to the following theorem.

Theorem 9 (23). Let \( H \) be a compactly generated locally compact group such that every conjugacy class is relatively compact. Then there is a normal compact subgroup \( K \subset H \) such that \( H/K \simeq \mathbb{R}^k \times \mathbb{Z}^l \) for some non-negative integers \( k, l \).

Let \( h \in H \). The conjugacy class of \( h \) is contained in \( h\{h_1h_2h_1^{-1}h_2^{-1}|h_1, h_2 \in H\} \) so is relatively compact by Claim 3.3.1. As a consequence, we can find a compact normal subgroup \( K \subset H \) and non-negative integers \( k, l \) such that \( H/K \simeq \mathbb{R}^k \times \mathbb{Z}^l \). We will show that \( k + l \leq 1 \) so \( H/K \simeq \mathbb{R}, H/K \simeq \mathbb{Z} \) or \( H/K \simeq \{e\} \). Indeed, let \( p : H \to H/K \) be the natural map. Then the group homomorphism \( p \circ \tilde{f} \) has dense image and \( p \circ \tilde{f}(\Lambda) \) is relatively compact. If \( H/K \) is compact then \( H/K \simeq \{e\} \). Otherwise we can find \( \gamma_0 \in \Lambda^\infty \setminus \Lambda \). We thus know that \( \tilde{f}(\gamma_0) \geq R > 0 \). So by Claim 3.3.2 every \( \gamma \in \Lambda^\infty \) is such that \( \tilde{f}(\gamma) \in \langle p \circ \tilde{f}(\gamma_0) \rangle \) where \( \langle p \circ \tilde{f}(\gamma_0) \rangle \) denotes the subgroup generated by \( p \circ \tilde{f}(\gamma_0) \) and \( L \) is some compact subset of \( H/K \). Therefore \( \langle p \circ \tilde{f}(\gamma_0) \rangle \) is an infinite cyclic co-compact subgroup, and, hence, \( k + l \leq 1 \). Now, choose a neighbourhood \( U \subset H \) of the identity such that \( \tilde{f}^{-1}(U) \subset \Lambda^n \). Since \( K \) is a compact subgroup of \( H \) and the subgroup \( \tilde{f}(\Lambda^\infty) \) is dense we can find an integer \( m \geq 0 \) such that \( K \subset f(\Lambda^m) \). Thus, \( \tilde{f}^{-1}(K) \subset \Lambda^m f^{-1}(U) \subset \Lambda^{m+n} \). Define \( V = p(U) \). Then \( V \subset H/K \) is a neighbourhood of the identity and

\[
\tilde{f}^{-1}(p^{-1}(V)) \subset \tilde{f}^{-1}(UK) \subset \Lambda^{m+2n}.
\]
Therefore \( p \circ \hat{f} \) is a good model of \( \Lambda^{m+2n} \) with image dense in \( \mathbb{R} \), \( \mathbb{Z} \) or \( \{e\} \). So \( \Lambda^{m+2n} \) has a good model with image contained in \( \mathbb{R} \). Since \( f \) is unbounded and \( \hat{f}(\Lambda^{m+2n}) \) is bounded, we can find \( \gamma \in \Lambda^\infty \setminus \Lambda^{m+2n} \). In particular, we know that \( f(\gamma) > 0 \). Then the map

\[
\hat{f} : g \mapsto p \circ \hat{f}(g) - \frac{p \circ \hat{f}(\gamma)}{f(\gamma)} f(g)
\]

is a homogeneous quasi-morphism with \( \hat{f}(\gamma) = 0 \). Moreover, any set commensurable to \( \Lambda \) has bounded image by \( \hat{f} \). But, if \( g \in G \), then there is \( n \in \mathbb{Z} \) such that \( |f(g) - nf(\gamma)| \leq f(\gamma) \). This implies that \( G = \langle \gamma \rangle f^{-1}([-C(f) - f(\gamma); f(\gamma) + C(f)]) \) where \( \langle \gamma \rangle \) is the subgroup generated by \( \gamma \). Since \( f^{-1}([-C(f) - f(\gamma); f(\gamma) + C(f)]) \) is commensurable to \( \Lambda \) according to Lemma 7, we get that it is mapped to a bounded set by \( \hat{f} \). Hence, \( \hat{f} \) has bounded image. But a homogeneous quasi-morphism with bounded image is null, so \( p_f(\gamma) \hat{f} = p \circ \hat{f} \) is a group homomorphism. To conclude we have to prove that \( p \circ \hat{f}(\gamma) \neq 0 \). But \( \gamma \notin \Lambda^{m+2n} \) and \( p \circ \hat{f} \) is a good model of \( \Lambda^{m+2n} \), so \( p \circ \hat{f}(\gamma) \neq 0 \).

**Proof of Theorem 3.** Recall that \( w \) is a non-trivial reduced word of length \( l \) in \( F_2 \) the free group over \( \{a, b\} \). According to \([11, \S 3(a)]\) the map

\[
f_w : F_2 \to \mathbb{R} \quad \quad g \mapsto o(g, w) - o(g, w^{-1})
\]

is a symmetric quasi-morphism with \( C(f_w) \leq 3l - 1 \). Moreover, \( f_w \) is within bounded distance of a unique homogeneous quasi-morphism, \( \hat{f}_w \), say, that is not a group homomorphism as soon as \( w \notin \{a, b, a^{-1}, b^{-1}, e\} \). But according to Lemma 6 the set \{\( g \in F_2 | o(g, w) - o(g, w^{-1}) \leq 3l \} = f_w^{-1}([-3l; 3l]) \) is an approximate subgroup. Moreover, by \([7]\) it is commensurable to \( f_w^{-1}([-C(f_w) - 1; C(f_w) + 1]) \). As a consequence, if \( \{g \in F_2 | o(g, w) - o(g, w^{-1}) \leq 3l \} \) is a Meyer subset, then \( f_w^{-1}([-C(f_w) - 1; C(f_w) + 1]) \) is a Meyer subset, and, hence, \( f_w \) is a group homomorphism according to Proposition 4. If \( w \notin \{a, b, a^{-1}, b^{-1}, e\} \), then this is impossible.

**Remark 4.** Proposition 6 combined with \([11, \S 3(a)]\) in fact proves that any ultrapower of the approximate subgroup \( \{g \in F_2 | o(g, w) - o(g, w^{-1}) \leq 3l \} \) with \( w \notin \{a, b, a^{-1}, b^{-1}, e\} \) is not a Meyer subset. Which means that it is not commensurable to an approximate subgroup that has a good model. In particular, this seems to strongly contradict that “even without the definable amenability assumption a suitable Lie model exists” conjectured in \([21, \text{p. 57}]\). It is interesting to note that another counter-example to this conjecture is given in \([17]\) and that it is also built thanks to Brooks’ quasi-morphisms.

### 3.4. Good models and cut-and-project schemes

We first prove a lemma that is of independent interest.

**Lemma 8.** Let \( \Lambda \) be a discrete approximate subgroup of a locally compact group \( G \). If \( \Lambda \) has a good model \( f \), then the graph of \( f \) defined by \( \Gamma_f := \{(\gamma, f(\gamma)) | \gamma \in \Lambda^\infty \} \) is a discrete subgroup of \( G \times H \).
Proof: Choose a neighbourhood of the identity $U \subset H$ as in Definition 1 and an open set $V \subset G$ such that $V \cap \Lambda = \{e\}$. The subset $V \times U \subset G \times H$ is a neighbourhood of the identity. Take $\gamma \in \Lambda^\infty$. Then $(\gamma, f(\gamma)) \in V \times U$ implies $f(\gamma) \in U$ so $\gamma \in \Lambda$. But $\gamma \in V$, so $\gamma = e$. Thus, $\Gamma_f \cap (V \times U) = \{e\}$. \qed

Proof of Proposition 7 Assume that $\Lambda$ is a model set. Let $H, \Gamma$ and $W_0$ be as in Definition 4 and $p_G : G \times H \to G$ and $p_H : G \times H \to H$ be the natural maps. The map $(p_G)_|\Gamma$ is injective and

$$\Lambda = P_0(G, H, \Gamma, W_0) = p_G \left((G \times W_0) \cap \Gamma\right).$$

But

$$p_G \left((G \times W_0) \cap \Gamma\right) = \left(p_H \circ (p_G)_|\Gamma^{-1}\right)^{-1}(W_0)$$

where we think of $(p_G)_|\Gamma$ as a bijective map from $\Gamma$ to $p_G(\Gamma)$. We know that $\Lambda^\infty \subset p_G(\Gamma)$ so set

$$\tau : \Lambda^\infty \to H$$

$$\gamma \mapsto p_H \circ (p_G)_|\Gamma^{-1}(\gamma).$$

Then $\tau$ is a group homomorphism, $W_0$ is a symmetric relatively compact neighbourhood of the identity and $\tau^{-1}(W_0) = \Lambda$. So $\tau$ is a good model of $\Lambda$.

Conversely, assume that $\Lambda$ has a good model $f$. As in the proof of Proposition 6 we can assume that $f : \Lambda^\infty \to H$ is a group homomorphism with dense image. Now, consider the subgroup of $G \times H$ given by the graph of $f$, namely,

$$\Gamma_f := \{(\gamma, f(\gamma)) | \gamma \in \Lambda^\infty\}.$$

Let $U \subset H$ be symmetric relatively compact neighbourhood of the identity such that $f^{-1}(U) \subset \Lambda$. Then $W_0 := U \cup f(\Lambda)$ is a symmetric relatively compact neighbourhood of the identity in $H$ and

$$\Lambda \ker(f) = f^{-1}(U \cup f(\Lambda)) = p_G \left((G \times W_0) \cap \Gamma_f\right)$$

where $p_G : G \times H \to G$ is the natural map. Moreover, we readily check that $\Gamma_f$ projects injectively to $G$ and densely to $H$. We will now show that $\Gamma_f$ is a lattice in $G \times H$ and this will show that $(G, H, \Gamma_f)$ is a cut-and-project scheme. The subgroup $\Gamma_f$ is discrete by Lemma 8. If $\Lambda$ is a uniform approximate lattice then $\Gamma_f$ is a uniform lattice in $G \times H$ by 4, Proposition 2.3, (iv)]. If $\Lambda$ is a strong approximate lattice then choose a $G$-invariant Borel probability measure $\nu$ on $\Omega_\Lambda$. By 8, Theorem 3.1 and Remark 3.2, (iv)] there is a Borel $G$-equivariant map $\beta : \Omega_\Lambda \to (G \times H) / \Gamma_f$. So the push-forward measure $\beta_* \nu$ is a $G$-invariant probability measure on $Y := (G \times H) / \Gamma_f$. Let $\mu$ denote the semi-invariant measure on $Y$ given by 23, I.1.4]. By 4, Theorem 5.8 we know that there is a continuous group homomorphism $u : H \to \mathbb{R}^+$ such that $(g, h)_* \mu = u(h) \mu$ for all $(g, h) \in G \times H$. Take $p_G \in C_c(G)$ and $p_H \in C_c(H)$ two non-null non-negative functions and let $\mu_G$ and $\mu_H$ be Haar-measures on $G$ and $H$ respectively. Then one checks that the finite measure $(p_1 \mu_G \otimes p_2 \mu_H) * (p_1 \mu_G \otimes p_2 \mu_H) * \beta_* \nu$ is $G$-invariant and that here $\rho \in L^1(Y)$ continuous such that $(p_1 \mu_G \otimes p_2 \mu_H) * (p_1 \mu_G \otimes p_2 \mu_H) * \beta_* \nu = \rho \mu$ (see for instance 8). In particular, we find a non-negative continuous $\rho \in L^1(Y)$ such that for all $g \in G$ and $\mu$-almost every $y \in Y$ we have $\rho(g^{-1}y) = \rho(y)$. But $G$ acts minimally by the duality principle. So the continuous function $\rho$ is constant. Therefore, the measure $\mu$ is finite. So $\Gamma_f$ is a lattice by 23, I.1.5]. \qed
Remark 5. The map $\tau$ introduced in the first part of the proof of Proposition I is well-known in the abelian setting and is called the star-map (see for instance [2, §7.2]). Moreover, a slight adaptation of the above argument would yield a result similar to Proposition I assuming that $\Lambda$ is an approximate lattice and $\Lambda^\infty$ is dense in $G$.

4. A CLOSED-APPROXIMATE-SUBGROUP THEOREM

We will start by proving a general form of Theorem 3.

Theorem 10. Let $\Lambda$ be a compact approximate subgroup of a locally compact group $G$. There is a locally compact group $H$, an injective continuous group homomorphism $i : H \to G$ and a relatively compact symmetric subset of the identity $V \subset H$ such that $i(V) = \Lambda$ and $i(H) = \Lambda^\infty$.

The key observation needed to prove Theorem 10 is the fact that locally a closed approximate subgroup behaves like a group.

Lemma 9. Let $\Lambda$ be a closed approximate subgroup of a locally compact group $G$ and let $\Xi$ be a subset covered by finitely many left-translates of $\Lambda$. There is a neighbourhood of the identity $U \subset \Lambda$ such that:

$$\Xi \cap U = \Lambda^2 \cap U.$$

Proof. Choose a finite subset $F \subset G$ such that $\Xi \subset F \Lambda$. Define the open subset

$$U := G \setminus \left( \bigcup_{f \in F, f \notin \Lambda} f \Lambda \right).$$

Since $e \in \Lambda$ implies $f \in \Lambda^{-1} = \Lambda$, the subset $U$ contains the identity. We thus have

$$U \cap \Xi \subset U \cap F \Lambda \subset \bigcup_{f \in F, f \in \Lambda} f \Lambda \subset \Lambda^2.$$

Proof of Theorem 10. For all $\gamma \in \Gamma$ let $U(\gamma)$ be the neighbourhood of the identity given by Lemma 9 applied to $\Xi := \gamma \Lambda^4 \gamma^{-1}$. Choose neighbourhood basis for the identity $B$ made of closed subsets in $G$ and define $B_\Lambda$ as the subset $\{ \Lambda^2 \cap U^{-1}U \mid U \in B \}$. The set $B_\Lambda$ satisfies (3.a) and (3.b) of Theorem 8 by Lemma 2. Take any $U \in B$ and any $\gamma \in \Gamma$ and choose $V \in B$ symmetric such that $\gamma(V^{-1}V)^2 \gamma^{-1} \subset U(\gamma) \cap U$. Then

$$\gamma \left( V^{-1}V \cap \Lambda^2 \right)^2 \gamma^{-1} \subset \gamma(V^{-1}V)^2 \gamma^{-1} \cap \gamma \Lambda^4 \gamma^{-1} \subset U(\gamma) \cap U \cap \gamma \Lambda^4 \gamma^{-1} \subset U \cap \Lambda^2.$$

So $B_\Lambda$ checks all conditions of (3) of Theorem 8.

Choose a good model $f : \Gamma \to H$ for $(\Lambda^2, \Gamma)$ as in (4) of Theorem 8. In particular, the family $\{ f(\Xi) \mid \Xi \in B_\Lambda \}$ is a neighbourhood basis for the identity in $H$ and
ker(f) = \bigcap_{\Xi \in B_\Lambda} \Xi \subset \bigcap_{U \in B} U^{-1}U = \{e\}. Take \lambda \in \Lambda^2 and take any \( U \in B \) with \( U^{-1}U \subset U(e) \). Then
\[
\Lambda^2 \cap \lambda U^{-1}U = \lambda(\lambda^{-1}\Lambda^2 \cap U^{-1}U) \subset \lambda(\Lambda^4 \cap U^{-1}U) = \lambda(\Lambda^2 \cap U^{-1}U),
\]
so the restriction of \( f \) to \( \Lambda^2 \) is a continuous map. Hence, \( \{ f(\Xi) \mid \Xi \in B_\Lambda \} = \{ f(\Xi) \mid \Xi \in B_\Lambda \} \) is a neighbourhood basis for the identity in \( H \). Since \( f(\Lambda^2) \) has non-empty interior and \( f(\Gamma) \) is dense in \( H \), we find \( f(\Gamma) = f(\Gamma)f(\Lambda^2) = H \). So \( f \) is bijective. But for all \( \Xi \in B_\Lambda \) we have \( f^{-1}(f(\Xi)) = \Xi \). So \( f^{-1} : H \to G \) is a continuous one-to-one group homomorphism and \( \Lambda^2 \) is the image of the compact neighbourhood of the identity \( f(\Lambda^2) \).

Proof of Theorem 3. Apply Theorem 11 to \( (\Lambda^2 \cap V^2, \Lambda^\infty) \) where \( V \) is any symmetric compact neighbourhood of the identity in \( G \). This yields an injective continuous group homomorphism \( i : H \to G \) with image \( \Lambda^{-1} \) and such that \( i^{-1}(\Lambda^2 \cap V^2) \) is a compact neighbourhood of the identity. Now, the approximate subgroup \( i^{-1}(\Lambda) \) is commensurable to the approximate subgroup \( i^{-1}(\Lambda^2) \).

If moreover \( G \) is a Lie group, then that \( H \) is also a Lie group is a consequence of [9, Chapter III, §8, Corollary 1]. \( \square \)

Remark 6. In particular, one can define the Lie algebra associated to a closed approximate subgroup of a Lie group.

**Proposition 7.** Let \( \Lambda \) be a closed approximate subgroup of \( \mathbb{R} \). Then one of the following is true:

- \( \Lambda \) is finite;
- there are real numbers \( 0 < a < b < \infty \) such that \([-a; a] \subset \Lambda^2 \subset [-b; b] \);
- \( \Lambda \) is a uniform approximate lattice;
- there is \( n \in \mathbb{N} \) such that \( \Lambda^n = \mathbb{R} \).

Proof. We know by [13, Theorem 1] that \( \Lambda \) is either bounded or relatively dense (this could also be proved directly). By Theorem 3 there is a Lie group \( H \), an injective lie group homomorphism \( i : H \to \mathbb{R} \) and an open approximate subgroup \( \Xi \) in \( H \) commensurable to \( i^{-1}(\Lambda) \) and contained in \( i^{-1}(\Lambda)^3 \). Let \( H^0 \) denote the connected component of the identity of \( H \). Then \( H^0 \simeq \mathbb{R} \) or \( H^0 \simeq \{e\} \). If \( H^0 \) is \( \{e\} \), then \( \Lambda \) is uniformly discrete so we are in situation (i) or (iii). If \( H^0 = \mathbb{R} \) then \( \Lambda^2 \) is a neighbourhood of the identity in \( \mathbb{R} \). So \( \Lambda \) satisfies either (ii) or (iv). \( \square \)

Proof of Theorem 4. Recall the statement of [13, Theorem 1.25]:

**Theorem.** Let \( \Lambda \) be a relatively compact open \( K \)-approximate subgroup of a locally compact group \( G \). Then for every \( \varepsilon > 0 \) there is an open approximate subgroup \( \Lambda' \subset \Lambda^4 \) and a compact subgroup \( H \subset \Lambda' \) normalised by \( \Lambda' \) such that:

- (i) \( \Lambda \) can be covered by \( O_{K,\varepsilon}(1) \) translates of \( \Lambda' \);
- (ii) \( \langle \Lambda' \rangle/H \) is a Lie group of dimension \( O_K(1) \).

If \( l' \) denotes the Lie algebra of \( \langle \Lambda' \rangle/H \) and \( \Lambda'' \) the image of \( \Lambda' \) in \( \langle \Lambda' \rangle/H \), then there exists a norm \( \| \cdot \| \) on \( l' \) such that:

- (iii) for \( X, Y \in l' \) we have \( \|X,Y\| = O_K(\|X\|\|Y\|) \);
- (iv) for \( g \in \Lambda'' \) the operator norm (induced by \( \| \cdot \| \) of Ad(\( g \)) – Id is \( O_K(\varepsilon) \);
- (v) there is a convex set \( B \subset l'' \) such that \( l''/B \) is a finite \( O_{K,\varepsilon}(1) \)-approximate local group.
Let $\tilde{H}$ and $f : \tilde{H} \to G$ be the locally compact group and the homomorphism given by Theorem 10 and $V \subset \tilde{H}$ be a neighbourhood such that $f(V) = \Lambda^2$. Then $V$ is a compact neighbourhood of the identity, so there is an open symmetric set $\tilde{V}$ such that $V \subset \tilde{V} \subset V^2$. Then $\tilde{V}$ is an open relatively compact $K^6$-approximate subgroup. But $f$ is an injective continuous homomorphism so [13, Theorem 1.25] applied to $\tilde{V}$ yields Theorem 4.

\[ \square \]

5. Amenable approximate subgroups

5.1. Definition and first properties.

**Definition 6.** Let $\Lambda$ be an approximate subgroup of a group $G$. We say that $\Lambda$ is **amenable** if there exists a finitely additive measure $m$ defined on all subsets of $\Lambda^4$ such that:

- (finiteness) $0 < m(\Lambda) < \infty$;
- (local left-invariance) for all $\lambda \in \Lambda^4$, $X \subset \Lambda^4$, $m(\lambda X) = m(X)$.

J.-C. Massicot and F.O. Wagner studied the related, but different, notion of definably amenable approximate subgroups ([20, p.57]). Other objects closely related to Definition 6 are the near-subgroups defined by E. Hrushovski in [16]. In fact every amenable approximate subgroup is a near-subgroup and the study of near-subgroups in [16] yields the following striking and particularly useful following result:

**Proposition 8 ([16]).** Let $\Lambda$ be an approximate subgroup of a group $G$. If $\Lambda$ is amenable, then $\Lambda^4$ has a good model.

Proposition 8 is a consequence of [16, Theorem 4.2] applied to an ultra-power of $\Lambda$. However, [16] deals with subsets satisfying weaker model-theoretic assumptions, and, hence, requires more involved techniques from model theory. We give here an elementary and self-contained proof that relies on Theorem 8 and a slight modification of an argument due to J.-C. Massicot and F.O. Wagner in [20].

**Lemma 10.** Let $\Lambda$ be an amenable approximate subgroup of a group $G$. Let $m$ be a positive integer. Then there is $S \subset \Lambda^4$ an approximate subgroup commensurable to $\Lambda$ such that $S^m \subset \Lambda^4$. Moreover, if $\Lambda$ is a $K$-approximate subgroup for some integer $K$, then we can choose $S$ such that $C_{K,m}$ left-translates of $S$ cover $\Lambda$ where $C_{K,m}$ is some positive number that depends only on $K$ and $m$.

**Proof.** Take $\mu$ a measure as in Definition 6. First of all, let $\Xi \subset \Lambda$ be such that $\mu(\Xi) \geq t \mu(\Lambda)$ for some $t \in (0;1]$. Let

$$X(\Xi) := \{g \in \Lambda^2 | \mu(g \Xi \cap \Xi) \geq st \mu(\Lambda)\}$$

where $s = \frac{1}{K}$. The approximate subgroup $\Lambda$ is covered by at most $N := \lfloor \frac{1}{s} \rfloor$ left translates of $X(\Xi)$. Indeed, suppose not, then we can build inductively a sequence $(g_i)_{0 \leq i \leq N}$ such that $g_i \in \Lambda \setminus \bigcup_{j < i} g_j X(\Xi)$. Thus, we have

$$\mu(g_i \Xi \cap g_j \Xi) < st \mu(\Lambda)$$

for all $0 \leq j < i \leq N$. Then
\[ K\mu(\Lambda) \geq \mu(\Lambda^2) \geq \mu( \bigcup_{0 \leq i \leq N} g_i\Xi) \]
\[ \geq (N + 1)\mu(\Xi) - \sum_{0 \leq i < j \leq N} \mu(g_i\Xi \cap g_j\Xi) \]
\[ > (N + 1)t\mu(\Lambda) - \frac{N(N + 1)}{2}st\mu(\Lambda) \]
\[ \geq (1 - s\frac{N}{2})(N + 1)t\mu(\Lambda) \]
\[ \geq (1 - s\frac{1}{2s})t\mu(\Lambda) = \frac{1 - 2K}{2}t\mu(\Lambda) \]
\[ \geq K\mu(\Lambda). \]

This is a contradiction. Set
\[ f(t) := \inf \left\{ \frac{\mu(\Xi)}{\mu(\Lambda)} | \Xi \subset \Lambda, \mu(\Xi) \geq t\mu(\Lambda) \right\}. \]

There is \( t \geq c_{K,m} \) such that \( f(t^2) \geq (1 - \frac{1}{4m})f(t) \) where \( c_{K,m} \) is a constant that depends only on \( K \) and \( m \). Indeed, define inductively
\[
\begin{cases}
t_0 = 1, \\
t_{n+1} = \frac{t_n}{2K}.
\end{cases}
\]

Let \( n \in \mathbb{N} \) be such that for all \( k \leq n \) we have \( f(t_k) < (1 - \frac{1}{4m})f(t_{k-1}) \). Then
\[ 1 \leq f(t_n) < (1 - \frac{1}{4m})^n f(t_0) \leq (1 - \frac{1}{4m})^n K, \]
so \( n < \frac{\log(K)}{\log((1 - \frac{1}{4m}))} \). Hence, \( c_{K,m} = \frac{1}{(2K)^n} \) with \( n = \left\lceil \frac{\log(K)}{\log((1 - \frac{1}{4m}))} \right\rceil \) works.

Take \( t \geq c_{K,m} \) such that \( f(t^2) \geq (1 - \frac{1}{4m})f(t) \) and \( \Xi \) such that \( \mu(\Xi) \geq t\mu(\Lambda) \) and \( \frac{\mu(\Xi)}{\mu(\Lambda)} \geq (1 + \frac{1}{4m})f(t) \). We prove by induction on \( k \leq m \) that for \( g_1, \ldots, g_k \in X(\Xi) \) we have \( g_1 \cdots g_k \in \Lambda^4 \) and
\[ \mu((g_1 \cdots g_k\Xi\Lambda) \Delta (\Xi\Lambda)) \leq \frac{k}{m} \mu(\Xi\Lambda). \]

If \( k = 1 \), then \( g = g_1 \in X(\Xi) \subset \Lambda^2 \). Furthermore,
\[ \mu(g\Xi\Lambda \cap \Xi\Lambda) \geq \mu((g\Xi \cap \Xi)\Lambda) \]
\[ \geq f(t^2)\mu(\Lambda) \]
\[ \geq (1 - \frac{1}{4m})f(t)\mu(\Lambda) \]
\[ \geq \frac{1 - \frac{1}{4m}}{1 + \frac{4m}{4m}}\mu(\Xi\Lambda). \]

Hence,
\[ \mu((g\Xi\Lambda) \Delta (\Xi\Lambda)) \leq 2 \left( 1 - \frac{1 - \frac{1}{4m}}{1 + \frac{4m}{4m}} \right) \mu(\Xi\Lambda) \leq \frac{1}{m} \mu(\Xi\Lambda). \]
Suppose now that the statement is true for some \( k < m \). Then for \( g_1, \ldots, g_{k+1} \in X(\Xi) \) we have,

\[
\mu((g_1 \cdots g_{k+1} \Xi \Lambda) \Delta (\Xi \Lambda)) \leq \mu((g_1 \cdots g_{k+1} \Xi \Lambda) \Delta (g_1 \cdots g_{k} \Xi \Lambda)) + \mu((g_1 \cdots g_{k} \Xi \Lambda) \Delta (\Xi \Lambda)) + \frac{k}{m} \mu(\Xi \Lambda).
\]

But \( g_1 \cdots g_k \in \Lambda^4 \) and both \( g_{k+1} \Xi \Lambda \) and \( \Xi \Lambda \) are contained in \( \Lambda^4 \). Local left-invariance thus implies,

\[
\mu((g_1 \cdots g_{k+1} \Xi \Lambda) \Delta (\Xi \Lambda)) \leq \mu((g_{k+1} \Xi \Lambda) \Delta (\Xi \Lambda)) + \frac{k}{m} \mu(\Xi \Lambda) < \frac{k+1}{m} \mu(\Xi \Lambda).
\]

In particular,

\[
\mu((g_1 \cdots g_{k+1} \Xi \Lambda) \Delta (\Xi \Lambda)) < \frac{2}{m} \mu(\Xi \Lambda),
\]

so

\[
g_1 \cdots g_{k+1} \Xi \Lambda \cap \Xi \Lambda \neq \emptyset
\]

which implies \( g_1 \cdots g_{k+1} \in \Lambda^4 \). As a consequence, \( X(\Xi)^m \subset \Lambda^4 \) and \( \lfloor \|2K\| \| eK \| \rfloor \leq \frac{2K}{eK \cdot m} + 1 \approx (2K)^{2\log(2K)m} \) translates of \( X(\Xi) \) cover \( \Lambda \).

Before we move on to the proof of Proposition 8 we show one more lemma.

**Lemma 11.** Let \( \Lambda \) and \( \Xi \) be approximate subgroups of a group \( G \). If \( \Lambda \subset \Xi \), the approximate subgroups \( \Lambda \) and \( \Xi \) are commensurable and \( \Xi \) is amenable, then \( \Lambda \) is amenable.

**Proof.** Let \( m \) be as in Definition 6 with respect to \( \Xi \). We have \( \Lambda^8 \subset \Xi^8 \) so we can define the finitely-additive measure \( m_\Lambda \) for all \( X \subset \Lambda^8 \) by \( m_\Lambda(X) = m(X) \). Since \( \Lambda^4 \subset \Xi^4 \) the measure \( m_\Lambda \) is locally left-invariant with respect to \( \Lambda \). But \( \Lambda \) and \( \Xi \) are commensurable so there is a finite subset \( F \subset G \) such that \( \Xi \subset FA \). This implies

\[
\Xi \subset (F \cap \Xi \Lambda^{-1}) \Lambda \subset (F \cap \Xi^2) \Lambda.
\]

By local left-invariance of \( m \) we get \( m(\Xi) \leq |F|m(\Lambda) = |F|m_\Lambda(\Lambda) \). Hence, the measure \( m_\Lambda \) satisfies the finiteness assumption as well. \( \square \)

**Proof of Proposition 8** By Lemma 10 there is an approximate subgroup \( \Lambda_1 \subset \Lambda^4 \) commensurable to \( \Lambda \) and such that \( \Lambda_1^8 \subset \Lambda^4 \). According to Lemma 11 the approximate subgroup \( \Lambda_1 \) is amenable. We can thus build inductively a sequence \( (\Lambda_n)_{n \geq 0} \) such that \( \Lambda_0 = \Lambda \) and \( \Lambda_{n+1}^2 \subset \Lambda_n^4 \) for all integers \( n \geq 0 \). By Theorem 8 applied to the sequence \( (\Lambda_i^4)_{n \geq 0} \) we obtain that \( \Lambda^4 \) has a good model. \( \square \)

### 5.2. Uniformly discrete approximate subgroups, approximate lattices and Amenable locally compact groups.

The proof of Theorem 5 consists in three steps. In the first step we construct a left-invariant mean \( m \) (i.e. a certain non-trivial positive linear functional) on a suitable subspace of the set of continuous functions \( C^0(G) \). Then we extend \( m \) to a mean defined on the space of essentially bounded Borel functions with support contained in certain neighbourhoods of the subgroup \( H \). Finally, we will “restrict” \( m \) to obtain a finitely-additive measure on \( \Lambda^8 \). Most of the proof consists in adapting classical material from the theory of amenable locally compact groups (as found in [12]) to our setting.
Proof of Theorem [3]. Let us first recall some notations and definitions (see [15] for more details). Fix $\mu_G$ a left-Haar measure on $G$. Define the left- and right-translates of a function $f : G \to \mathbb{R}$ by $g \in G$ as the maps $g_f : x \mapsto f(g^{-1}x)$ and $f_g : x \mapsto f(xg)$ respectively. A function $f : G \to \mathbb{R}$ is right-uniformly continuous if for any real number $\epsilon > 0$ there is an open neighbourhood $U(\epsilon) \subset G$ of the identity such that for all $g \in U(\epsilon)$ and $x \in G$ we have $|f(x) - f(gx)| < \epsilon$. The set of right-uniformly continuous bounded functions on $G$ will be denoted by $C^0_{b,ru}(G)$. Likewise the set of continuous bounded functions (resp. continuous functions with compact support) on $G$ will be denoted by $C^0_b(G)$ (resp. $C^0_c(G)$). One readily checks that $G$ acts continuously by left-translations on the normed vector space $C^0_{b,ru}(G)$ equipped with the norm $\| \cdot \|_{\infty}$. The convolution $\phi \ast f$ of $\phi \in L^1(G)$ and $f \in L^\infty(G)$ is defined by

$$\phi \ast f : G \to \mathbb{R}$$

$$x \mapsto \int_G \phi(t)f(t^{-1}x)d\mu_G(t).$$

We have $\|\phi \ast f\|_{\infty} \leq \|\phi\|_1|f|_{\infty}$ and $\phi \ast f \in C^0_{b,ru}(G)$. If moreover $f \in L^1(G)$ then by Fubini $\|\phi \ast f\|_1 \leq \|\phi\|_1|f|_1$. In addition, for any $g \in G$ we have $\phi \ast (gf) = \Delta(g^{-1})\phi \ast f$ (where $\Delta : G \to \mathbb{R}$ is the modular function) and $\phi \ast f = (g\phi) \ast f$. A linear map $F : V \to \mathbb{R}$ is said left-invariant if $V \subset C^0_{b,ru}(G)$ is stable by the $G$-action and if for every $g \in G$ and $f \in V$ we have $m(gf) = m(f)$. It is said positive if for all $f \in V$ with $f \geq 0$ we have $F(f) \geq 0$.

We can now proceed to the first step of the proof. The vector subspace of $C^0(G)$ we want to consider is

$$V := \{ f \in C^0_{b,ru}(G)|p(\text{supp}(f)) \text{ is relatively compact}\}$$

where $p : G \to G/H$ is the natural projection. We will prove the following claim.

Claim 5.2.1. There exists a non-trivial left-invariant positive linear map $m : V \to \mathbb{R}$.

Fix $\mu_{G/H}$ a right-Haar measure on $G/H$. First of all, note that $V$ is stable under the action of $G$. Since $H$ is an amenable locally compact group there is a left-invariant mean $m_H : C^0_b(H) \to \mathbb{R}$ according to [15] Theorem 2.2.1]. This means that $m_H$ is a left-invariant positive linear functional such that for any $f \in C^0_b(H)$ we have $m_H(f) \leq \|f\|_{\infty}$ and $m_H(1) = 1$. Take $f \in V$ and consider the map

$$\tilde{f} : G \to \mathbb{R}$$

$$x \mapsto m_H ((xf)|_H).$$

We will show that $\tilde{f}$ is continuous and invariant under left-translation by elements of $H$. Indeed, if $h, x \in H$ and $g \in G$ then $hgf(x) = (gf)(h^{-1}x)$. But $h^{-1}x \in H$ if and only if $x \in H$. So $hgf|_H = h\cdot (gf|_H)$, and, hence, for $x \in G$ we have

$$h\tilde{f}(x) = \tilde{f}(h^{-1}x) = m_H (h^{-1}xf|_H) = m_H (h^{-1} (xf|_H)) = m_H ((xf)|_H).$$

Moreover, for any $x_1, x_2 \in G$ we have

$$\|\tilde{f}(x_1) - \tilde{f}(x_2)\|_{\infty} = |m_H(x_1f|_H) - m_H(x_2f|_H)| \leq \|x_1f - x_2f\|_{\infty}.$$ 

But $f$ is right-uniformly continuous, so $\tilde{f}$ is continuous. Therefore, there exists a unique continuous function $f_{G/H} : G/H \to \mathbb{R}$ such that $(f_{G/H} \circ p)(x) = m_H(xf|_H)$ where $p : G \to G/H$ is the natural projection. One readily checks that the map
$f \mapsto f_{G/H}$ is linear, sends positive functions to positive functions and $\|f_{G/H}\|_{\infty} \leq \|f\|_{\infty}$ for all $f \in C^0_b(G)$. Furthermore, we have $\text{supp}(f_{G/H}) \subset \text{p}(\text{supp}(f))$, so $f_{G/H}$ is a continuous function with compact support. We are thus able to define

$$m : V \to \mathbb{R}$$

$$f \mapsto \int_{G/H} f_{G/H}(t) d\mu_{G/H}(t).$$

The map $m$ is a positive linear map. Choose a compact neighbourhood of the identity $U \subset G$ and $f \in V$ such that $f(x) = 1$ for all $x \in UH$. Then for all $x \in UH$ we have $xf_{H} = 1$, so $f_{G/H}(p(x)) = 1$. This implies $m(f) \geq \mu_{G/H}(p(U)) > 0$ so $m$ is non-trivial. It only remains to check that $m$ is left-invariant. Take $g \in G$ and $f \in V$. Then

$$(g f)_{G/H} = (f_{G/H})_{p(g)}.$$ But $\mu_{G/H}$ is right-invariant so

$$m(g f) = \int_{G/H} (g f)_{G/H}(t) d\mu_{G/H}(t)$$

$$= \int_{G/H} (f_{G/H})_{p(g)}(t) d\mu_{G/H}(t)$$

$$= \int_{G/H} f_{G/H}(t) d\mu_{G/H}(t)$$

$$= m(f).$$

So Claim 5.2.1 is proved.

Let us move on to the second step. First define

$$P(G) := \{ f \in C^0_c(G) | \forall x \in G, f(x) \geq 0, \|f\|_1 = 1 \}.$$

We will prove the following.

Claim 5.2.2. Set $\mathcal{X} := \{ B \subset G | B \text{ is Borel, } p(B) \text{ is relatively compact} \}$. Then for any $\phi \in P(G)$ the map

$$m_\phi : \mathcal{X} \to \mathbb{R}$$

$$B \mapsto m(\phi * \chi_B),$$

satisfies:

(i) $m_\phi$ takes positive values;
(ii) $m_\phi$ does not depend on $\phi$;
(iii) $\forall B \in \mathcal{X}, g \in G, m_\phi(gB) = m_\phi(B)$;
(iv) $\forall A, B \in \mathcal{X}$ with $A \cap B = \emptyset, m_\phi(A \cup B) = m_\phi(A) + m_\phi(B)$;
(v) for any relatively compact open subset $U \subset G$ we have $m_\phi(UH) > 0$. 

Part (i) is straightforward. Let us then prove (ii). Take $B \in \mathcal{X}$ and consider the linear functional
\[
F : C_0^c(G) \to \mathbb{R}
\]
\[
\phi \mapsto m(\phi \ast \chi_B).
\]
This map is well-defined since $\phi \ast \chi_B \in V$ for every $\phi \in C_0^c(G)$. Since $\chi_B$ takes non-negative values and $m$ is positive, the linear functional $F$ is positive. By the Riesz-Markov-Kakutani representation theorem, $F$ is given by a regular Borel measure. We have moreover that for all $g \in G$
\[
F(g \phi) = m((g \phi) \ast \chi_B) = m(g(\phi \ast \chi_B)) = m(\phi \ast \chi_B) = F(\phi).
\]
So according to the Haar theorem there exists a constant $k(B) \in \mathbb{R}$ such that $F(\phi) = k(B) \int_G \phi(t) d\mu_G(t)$ for all positive functions $\phi \in C_0^c(G)$. In particular, if $\phi_1, \phi_2 \in P(G)$ then $m_{\phi_1}(B) = F(\phi_1) = k(B) = F(\phi_2) = m_{\phi_2}(B)$. This proves (ii).
Now take $g \in G$. Then
\[
m_{\phi}(gB) = m(\phi \ast \chi_{gB}) = m(\phi \ast g \chi_B) = m(\Delta(g^{-1}) \phi_{g^{-1}} \ast \chi_B).
\]
But $\Delta(g^{-1}) \phi_{g^{-1}} \in C_0^c(G)$ is positive and
\[
\int_G \Delta(g^{-1}) \phi_{g^{-1}}(t) d\mu_G(t) = \int_G \phi(t) d\mu_G(t) = 1,
\]
so $\Delta(g^{-1}) \phi_{g^{-1}} \in P(G)$. As a consequence,
\[
m_{\phi}(gB) = m_{\Delta(g^{-1}) \phi_{g^{-1}}}(B) = m_{\phi}(B)
\]
according to (ii). This proves (iii). Now (iv) is easily proved. Indeed, $A \cap B = \emptyset$ implies $\chi_{A \cup B} = \chi_A + \chi_B$. Therefore,
\[
m_{\phi}(A \cup B) = m(\phi \ast \chi_{A \cup B}) = m(\phi \ast \chi_A) + m(\phi \ast \chi_B) = m_{\phi}(A) + m_{\phi}(B).
\]
We will now prove (v). Since $m$ is non-trivial there is $f \in V$ such that $m(f) = 1$. But $f \in V$ so there is a compact neighbourhood of the identity $W \subset G$ such that $f$ is supported in $WH$. In particular, $1 = m(f) \leq m(||f||_\infty \chi_U)$. If $K$ denotes $\text{supp}(\phi)$, then for all $x \in WH$ we have
\[
\phi \ast \chi_{K^{-1}WH}(x) = \int_{K} \phi(t) \chi_{K^{-1}WH}(t^{-1}x) d\mu_G(t) \geq \int_{K} \phi(t) d\mu_G(t) = 1.
\]
Therefore, $m_{\phi}(K^{-1}WH) \geq ||f||_\infty^{-1} > 0$. But $U \subset G$ is open, so there is a finite subset $F \subset G$ such that $K^{-1}WH \subset FUH$. Hence,
\[
0 < m_{\phi}(K^{-1}WH) \leq m_{\phi}(FUH) \leq |F|m_{\phi}(UH)
\]
as a consequence of (i) and (iv). So indeed $m_{\phi}(UH) > 0$ and (v) is proved.

The third step is as follows.

**Claim 5.2.3.** There is $B_\infty \in \mathcal{X}$ such that
\begin{enumerate}[label=(\roman*)]
    
    (i) $\forall g \in G, \forall A \subset \Lambda^8, m_{\phi}(gAB_\infty) = m_{\phi}(AB_\infty)$;
    
    (ii) $\forall A, B \subset \Lambda^8$ with $A \cap B = \emptyset, m_{\phi}((A \cup B)B_\infty) = m_{\phi}(AB_\infty) + m_{\phi}(BB_\infty)$;
    
    (iii) $m_{\phi}(\Lambda B_\infty) \in (0; +\infty)$.
\end{enumerate}

First of all, if $B_\infty \in \mathcal{X}$ and $A$ is any subset of $\Lambda^8$ with $p(A)$ relatively compact, then $AB_\infty \in \mathcal{X}$. So $m_{\phi}(AB_\infty)$ is well defined. Moreover, for any $B_\infty$ part (i) of Claim 5.2.3 is a straightforward consequence of part (iii) of Claim 5.2.2. We will now find an open set $B_\infty \in \mathcal{X}$ that satisfies part (ii) and (iii) of Claim 5.2.3 as well. Since
Λ is a uniformly discrete approximate subgroup, the set Λ⁸ is uniformly discrete as well. Therefore, there exists V ⊂ U an open relatively compact neighbourhood of the identity in G such that \( VV^{-1} \cap Λ^{16} = \{e\} \). Let \((g_n)_{n \geq 0}\) be a sequence of G such that \( UH = \bigcup_{n \geq 0} Vg_n \). Then define inductively \( B_n := Vg_n \setminus \bigcup_{m < n} Λ^{16}B_m \) and \( B_∞ := \bigcup_{n \geq 0} B_n \). We have \( B_∞B_∞^{-1} \cap Λ^{16} = \{e\} \) and \( Λ^{16}B_∞ = G \). Take a finite subset \( F \subset G \) such that \( Λ^{16} \subset F \). Then according to Claim 5.2.2, \( 0 < m_φ(UH) \leq m_φ(Λ^{16}B) \leq m_φ(FA) \).

Therefore,

\[
0 < m_φ(UH) \leq m_φ(Λ^{16}B) \leq m_φ(FA).
\]

As in addition \( AB ∈ X \) we deduce that \( m_φ(AB) ∈ (0; +∞) \). We thus know that \( B_∞ \) satisfies (i) and (iii). Let us check that it satisfies (ii) as well. Indeed, take \( A, B \subset Λ^8 \) with \( A \cap B = \emptyset \). We will first show that \( AB_∞ \cap BB_∞ = \emptyset \). Suppose to the contrary that \( AB_∞ \cap BB_∞ ≠ \emptyset \). Choose \( a ∈ A, b ∈ B \) such that \( aB_∞ \cap bB_∞ ≠ \emptyset \). But \( A \) and \( B \) are subsets of \( Λ^8 \) so this means that \( a = b \) and \( A \cap B ≠ \emptyset \). A contradiction. Thus, \( AB_∞ \cap BB_∞ = \emptyset \). As a consequence, and according to Claim 5.2.2 we obtain

\[
m_φ((A ∪ B)B_∞) = m_φ(AB_∞) + m_φ(BB_∞).
\]

To conclude the proof of Theorem 5 we note that the set function

\[
\mathcal{P}(Λ^8) \rightarrow \mathbb{R}
\]

\[
A \mapsto m_φ(Λ^8A),
\]

is a finitely-additive measure as in Definition 6. □

Corollary 5. Let \( G \) be an amenable locally compact group. If \( Λ ⊂ G \) is a uniformly discrete approximate subgroup, then \( Λ \) is amenable.

Proof. Corollary 5 is a consequence of Theorem 5 applied to \( G \) with \( G = H \). □

Corollary 6. If \( Λ \) is an approximate subgroup of an amenable group \( G \), then \( Λ \) is amenable.

Proof. The group \( G \) equipped with the discrete topology is an amenable locally compact group. Moreover, \( Λ \) is obviously a uniformly discrete approximate subgroup in this topology. So Corollary 5 is a consequence of Corollary 5. □

Proof of Theorem 6. Since \( Λ \) is in particular uniformly discrete the approximate subgroup \( Λ^4 \) has a good model \( f \) by Corollary 5 and Proposition 8. If \( Λ \) is a uniform approximate lattice, then \( Λ^4 \) is a uniform approximate lattice. Similarly, if \( Λ \) is a strong approximate lattice, then \( Λ^4 \) is a strong approximate lattice by [6, 2.11]. Thus, the approximate subgroup \( Λ^4 \ker(f) \) is a model set by Proposition 9. But \( Λ^4 \subset Λ^4 \ker(f) \subset Λ^8 \). □

5.3. Consequences and questions.

Proposition 9. Let \( Λ \) be a strong or uniform approximate lattice in a connected amenable Lie group. Then there is \( Λ' \subset Λ^4 \) commensurable to \( Λ \) that has a good model \( f : Λ^∞ \rightarrow H \) with target a connected amenable Lie group.
Proof. According to Corollary 5 and Proposition 8 there is a good model of \( \Lambda^4 \). In addition, there is an approximate subgroup \( \Lambda' \subset \Lambda^4 \) commensurable to \( \Lambda \) such that \( \Lambda' \) has a good model \( f : (\Lambda')^\infty \to H \) with dense image and target a connected Lie group according to Proposition 4. By Proposition 1 the graph of \( \Gamma_f \), so \( \Gamma_f \) is a lattice in \( G \times H \). Let \( A \) denote the amenable radical (i.e. the maximal normal amenable closed subgroup) of \( H \). Then \( G \times A \) is the amenable radical of \( G \times H \). Let \( p : G \times H \to H/A \) be the natural map. By [14, Corollary 1.4] the image \( p(\Gamma_f) \) is a discrete subgroup of \( H/A \). We can therefore choose \( U \subset H \) a compact neighbourhood of the identity with \( f^{-1}(U) \subset \Lambda' \) and sufficiently small so that \( \Gamma_f \cap (G \times U) \subset G \times A \). But \( f^{-1}(U) \subset U \) since \( f \) has dense image. So \( U \subset A \) which implies that \( A = H \) because \( H \) is connected. Thus, the map \( f : (\Lambda')^\infty \to H \) is a good model of \( \Lambda' \) with dense image and target a connected amenable Lie group. \( \square \)

** Proposition 10.** Let \( \Lambda \) be a strong approximate lattice in a connected amenable Lie group \( G \). Then \( \Lambda \) is a uniform approximate lattice.

Proof. There is \( \Lambda' \subset \Lambda^4 \) an approximate subgroup commensurable to \( \Lambda \) that has a good model \( f : (\Lambda')^\infty \to H \) with target an amenable Lie group and dense image. According to [4, Corollary 2.11] the approximate subgroup \( \Lambda' \) is a strong approximate lattice as well. Thus, by Proposition 4 the strong approximate lattice \( \Lambda' \) is commensurable to \( P_0(G,H,\Gamma_f,W_0) \) where \( \Gamma_f \) is the graph of \( f \) and \( W_0 \subset H \) is some symmetric relatively compact neighbourhood of the identity. In particular, \( \Gamma_f \) is a lattice in the amenable Lie group \( G \times H \), so \( \Gamma_f \) is uniform by [3, Proposition 3.4]. Whence \( P_0(G,H,\Gamma_f,W_0) \) and \( \Lambda \) are uniform approximate lattices by [4 Proposition 2.13]. \( \square \)

** Proposition 11.** Let \( R \) be a connected soluble Lie group. If \( \Lambda \subset R \) is a uniformly discrete approximate subgroup, then \( \Lambda^\infty \) is finitely generated.

Proof. According to Corollary 6 and Proposition 5 the approximate subgroup \( \Lambda \) has a good model. According to Proposition 4 there is an approximate subgroup \( \Lambda' \subset \Lambda \) that has a good model \( f : (\Lambda')^\infty \to H \) with dense image and target a connected Lie group. Since \( (\Lambda')^\infty \) is soluble we obtain that \( H \) is soluble. Now, the graph of \( f \) denoted by \( \Gamma_f \), is a discrete subgroup of the connected soluble Lie group \( R \times H \) and therefore \( \Gamma_f \) is finitely generated by [23 Proposition 3.8]. Let \( F_1 \subset \Lambda' \) be a finite set of generators of \( (\Lambda')^\infty \) and \( F_2 \subset \Lambda^\infty \) be a finite subset such that \( \Lambda \subset F_2\Lambda' \). Then \( F_1 \cup F_2 \) is a finite set that generates \( \Lambda^\infty \). \( \square \)

** Question 1.** Let \( \Lambda \) be an amenable approximate subgroup. Does \( \Lambda^2 \) always have an amenable good model? Can one find a finitely additive probability measure hat is “globally” left-invariant? Are amenable approximate subgroups stable with respect to the usual set operations (e.g. direct limits)?

6. Generalisation of Theorems of Mostow and Auslander

6.1. Intersections of Approximate Lattices and Closed Subgroups.

** Proposition 12.** Let \( \Lambda \) be a uniformly discrete approximate subgroup of a locally compact group \( G \). Assume that \( H \) is a closed subgroup of \( G \) such that \( p(\Lambda) \) is locally finite where \( p : G \to G/H \) is the natural map. We have:
• if $\Lambda$ is a uniform approximate lattice, then $\Lambda^2 \cap H$ is a uniform approximate lattice in $H$;
• if $\Lambda$ is a strong approximate lattice in $G$ second countable and $H$ is normal, then there is a non-trivial $H$-invariant Borel probability measure on $\Omega_{P \cap H}$ for some $P \in \Omega_{\Lambda}$;
• if $\Lambda$ is a strong approximate lattice in $G$ second countable and $H$ is normal and amenable, then $\Lambda^2 \cap H$ is a strong approximate lattice in $H$.

It is interesting to compare Proposition 12 with the lattice version of the statement (see for instance [23, Theorem 1.13]).

Proof. We will first prove (i). We know that $\Lambda^2 \cap H$ is an approximate subgroup according to Lemma 3. Moreover, since $\Lambda$ is uniformly discrete, so is $\Lambda^2 \cap H$. So we only need to prove that $\Lambda^2 \cap H$ is relatively compact in $H$. Let $K \subset G$ be a compact subset such that $K\Lambda = G$. Since $p(\Lambda)$ is locally finite there is $F \subset \Lambda$ finite such that $K^{-1}H \cap \Lambda \subset FH$. Take $h \in H$. There is $\lambda \in \Lambda$ and $k \in K$ such that $k\lambda = h$. Then $\Lambda \in K^{-1}H \cap \Lambda$ so there is $f \in F$ such that $f^{-1}\lambda \in H \cap \Lambda^2$. Therefore, $h \in KF (H \cap \Lambda^2)$.

Let us move on to the proof of (ii). First of all, note that $\Lambda^2 \cap H$ is uniformly discrete and is an approximate subgroup according to Lemma 3. We will prove that $\Omega_{\Lambda^2 \cap H}$ has a non-trivial $H$-invariant Borel probability measure. Fix a non-trivial $G$-invariant Borel probability measure $\nu_0$ on $\Omega_{\Lambda}$. Let $\mathcal{P}(\Omega_{\Lambda}, H)$ denote the set of $H$-invariant Borel probability measures. The set $\mathcal{P}(\Omega_{\Lambda}, H)$ seen as a subset of the topological dual of $\mathcal{C}_0(\Omega_{\Lambda})$ (by the Riesz-Markov-Kakutani representation theorem) is a compact convex subset in the weak-* topology. By the Krein-Milman theorem $\mathcal{P}(\Omega_{\Lambda}, H)$ is thus the closed convex hull of its extreme points. Since $\nu_0 \in \mathcal{P}(\Omega_{\Lambda}, H)$ (for $\nu_0$ is $G$-invariant, hence $H$-invariant) and $\nu_0(\{\emptyset\}) = 0$ there is an extreme point $\nu_1 \in \mathcal{P}(\Omega_{\Lambda}, H)$ such that $\nu_1(\{\emptyset\}) = 0$. Now by [27, Lemma 3.2.21] the measure $\nu_1$ is $H$ ergodic. Then $\nu_1$ is a Borel probability measure on $\Omega_{\Lambda}$ a metrizable compact space ([4, §4.1]), so $\nu_1$ has a well-defined support $K$. The compact subset $K$ is $H$-invariant with $\nu_1(K) = 1$ and for any open subset $U \subset \Omega_{\Lambda}$ we have $\nu_1(U \cap K) = 0$ if and only if $U \cap K = \emptyset$. Thus, according to [27, Proposition 2.1.7], there is $P_1 \in K$ such that $K = H \cdot P_1$. Furthermore, $P_1 \neq \emptyset$ since $H \cdot \emptyset = \{\emptyset\}$ and $\nu_1(\{\emptyset\}) = 0$. Choose now $p_1 \in P_1$. Then $e \in p_1^{-1}P_1 \subset \Lambda^2$ by [4, Lemma 4.6]. Moreover, the subgroup $H$ is normal so $\nu_2 := (p_1^{-1})^{*} \nu_1$ is an $H$-invariant ergodic Borel probability measure with $\nu_2(\emptyset) = 0$ and support $p_1^{-1}K = \overline{H \cdot P_2}$ where $P_2 = p_1^{-1}P_1$. Define the map

$$
\pi : \overline{H \cdot P_2} \to \mathcal{C}(H)
$$

$$
P \mapsto P \cap H.
$$

Then $\pi$ is $H$-equivariant. We claim moreover that $\pi$ is continuous. Indeed, first $P_2 \subset \Lambda^2$. But $p(\Lambda^2)$ is locally finite, so $H\Lambda^2$ is closed. Therefore, any $P \in \overline{H \cdot P_2}$ is a subset of $\Lambda^2 H$ by [4, Lemma 4.1]. In addition, since $p(\Lambda^2)$ is locally finite we know that there is an open subset $U \subset G$ such that $\Lambda^2 H \cap U = H$. So for any open subset $V \subset G$ we have

$$
\pi^{-1}(U^V) = \pi^{-1}(\{P \in \mathcal{C}(H) | P \cap V \neq \emptyset\})
$$

$$
= \{P \in \overline{H \cdot P_2} | P \cap (U \cap W) \neq \emptyset\}
$$

$$
= U^W \cap \overline{H \cdot P_2},
$$
where $W \subset G$ is any open subset such that $H \cap W = V$. Likewise for any compact subset $L \subset H$ we have

$$
\pi^{-1}(U_L) = \pi^{-1}((P \in C(H))|P \cap L = \emptyset)) = \{P \in \overline{H \cdot P_2} | P \cap L = \emptyset\}
= U_L \cap \overline{H \cdot P_2},
$$

where we consider $L \subset H \subset G$. So $\pi$ is indeed a continuous map. Thus, $\pi(H \cdot P_2)$ is a compact subset of $C(H)$ and $\pi(P_2) = P_2 \cap H$ has dense orbit in $\pi(H \cdot P_2)$. So $\pi(H \cdot P_2) = \Omega_{P_2 \cap H}$. Set $\nu_3 := \pi \circ (\nu_2)|_{\overline{H \cdot P_2}}$ where $(\nu_2)|_{\overline{H \cdot P_2}}$ is the restriction of the measure $\nu_2$ to its support $H \cdot P_2$ which is a well defined $H$-invariant ergodic Borel probability measure since $\Omega_\Lambda$ is metric compact. Then $\nu_3$ is a $H$-invariant ergodic Borel probability measure on $\Omega_{P_2 \cap H}$. Suppose now that $\nu_3(\emptyset) > 0$, then $\nu_3(\emptyset) = 1$ by ergodicity. Thus, $\pi^{-1}(\emptyset)$ is an $H$-invariant compact co-null subset of $\overline{H \cdot P_2}$. So $\pi^{-1}(\emptyset) = \overline{H \cdot P_2}$ because $\overline{H \cdot P_2}$ is the support of $\nu_2$. Therefore $\pi(P_2) = \emptyset$. A contradiction. Hence, $\nu_3(\emptyset) = 0$ so $\nu_3$ is a non-trivial $H$-invariant Borel probability measure on $\Omega_{P_2 \cap H}$. Finally, according to (the proof of) [6, Corollary 2.11], the approximate subgroup $\Lambda^2 \cap H \supset P_2 \cap H$ is a strong approximate lattice in $H$. □

Lemma 12. Let $\Lambda$ be a uniformly discrete approximate subgroup of a locally compact group $G$. Assume that $H$ is a closed subgroup of $G$ such that $\Lambda^2 \cap H$ is a uniform approximate lattice in $H$. Then $p(\Lambda)$ is a locally finite subset of $G/H$ where $p: G \to G/H$ is the natural map.

Proof. Let $K \subset G/H$ be a compact subset. Then there is a compact subset $L \subset G$ such that $p(L) = K$. Since $\Lambda^2 \cap H$ is relatively dense in $H$ there is a compact subset $L' \subset G$ such that $LH \subset L'(\Lambda^2 \cap H)$. Take $\lambda \in \Lambda \cap LH$ then

$$
\lambda \in \Lambda \cap (L'(\Lambda^2 \cap H)) \subset \Lambda \cap ((L' \cap \Lambda^3)(\Lambda^2 \cap H)),
$$

so $p(\Lambda) \cap K \subset p(L' \cap \Lambda^3)$ that is indeed finite. □

Corollary 7. Let $\Lambda$ be a uniform approximate lattice in a locally compact group $G$. Then for all $\gamma \in \Lambda^\infty$ the approximate subgroup $\Lambda^2 \cap C(\gamma)$ is a uniform approximate lattice in $C(\gamma)$ the centraliser of $\gamma$. Moreover, if $G$ is a Lie group and $\Lambda^\infty$ is dense in $G$, then $\Lambda^2 \cap Z(G)$ is a uniform approximate lattice in $Z(G)$ the centre of $G$.

Proof. Let $n \geq 0$ be an integer such that $\gamma \in \Lambda^n$ and consider the map

$$
\varphi: G \to G
\begin{align*}
g &\mapsto g\gamma g^{-1}.
\end{align*}
$$

Then $\varphi$ factors as $\varphi = \psi \circ p$ where $\psi: G/C(\gamma) \to G$ is a continuous injective map and $p: G \to G/C(\gamma)$ is the natural map. But $\varphi(\Lambda) \subset \Lambda^{n+2}$ so is locally finite. Hence, $p(\Lambda)$ is locally finite as well. By part (i) of Proposition 12 we deduce that $\Lambda^2 \cap C(\gamma)$ is a uniform approximate lattice in $C(\gamma)$.

Now if $G$ is a Lie group and $\Lambda^\infty$ is dense in $G$, then $Z(G) = \bigcap_{\gamma \in \Lambda^\infty} C(\gamma)$. But $Z(G)$ is a Lie group and so are the $C(\gamma)$’s. Thus, there are $\gamma_1, \ldots, \gamma_n \in \Lambda^\infty$ such that $\dim(Z(G)) = \dim(\bigcap_{1 \leq i \leq n} C(\gamma_i))$. Consider now the map
\[ \varphi : G \rightarrow G^n 
\quad g \mapsto (g\gamma_1g^{-1}, \ldots, g\gamma_ng^{-1}). \]

As above \( \varphi \) factors as \( \varphi = \psi \circ p \) with \( \psi : G/(\bigcap C(\gamma_i)) \rightarrow G^n \) an injective and continuous map and \( p : G \rightarrow G/(\bigcap C(\gamma_i)) \) the natural map. But \( \varphi(\Lambda) \subset \prod_{1 \leq i \leq n} \Lambda^m \) where \( m \) is a positive integer such that \( \{\gamma_1, \ldots, \gamma_n\} \subset \Lambda^m \). Thus, \( \varphi(\Lambda) \) is locally finite and so is \( p(\Lambda) \). By part (i) of Proposition 12 we deduce that \( \Lambda^2 \cap \bigcap_{1 \leq i \leq n} C(\gamma_i) \) is a uniform approximate lattice in \( \bigcap_{1 \leq i \leq n} C(\gamma_i) \). But \( Z(G) \) is an open subgroup of \( \bigcap_{1 \leq i \leq n} C(\gamma_i) \) so \( p'(\Lambda^2 \cap \bigcap_{1 \leq i \leq n} C(\gamma_i)) \) is obviously locally finite where \( p' : \bigcap_{1 \leq i \leq n} C(\gamma_i) \rightarrow \left( \bigcap_{1 \leq i \leq n} C(\gamma_i) \right)/Z(G) \) is the natural map. By part (i) of Proposition 12 once again we have that

\[ \left( \Lambda^2 \cap \bigcap_{1 \leq i \leq n} C(\gamma_i) \right)^2 \cap Z(G) \subset \Lambda^4 \cap Z(G) \]

is a uniform approximate lattice in \( Z(G) \). By Lemma 3 we find that \( \Lambda^2 \cap Z(G) \) is a uniform approximate lattice in \( Z(G) \).

6.2. Borel density for approximate lattices.

**Definition 7** (Definition 1.1, [3]). Let \( G \) be a locally compact group. A closed subset \( X \subset G \) has property \( (S) \) if for all neighbourhoods \( \Omega \subset G \) of the identity and all \( g \in G \) there is \( n \in \mathbb{N} \) such that \( g^n \in \Omega X \Omega \).

**Proposition 13.** Let \( G \) be a locally compact second countable group. We have:

- if \( \Lambda \) is a uniform approximate lattice in \( G \), then \( \Lambda^\infty \) has property \( (S) \);
- if \( X \) is a closed subset and \( \Omega_X \) has a non-trivial \( G \)-invariant Borel probability measure, then \( X^{-1}X \) has property \( (S) \).

**Proof.** Let us start with the case of uniform approximate lattices. Set \( H = \Lambda^\infty \), then the subgroup \( \Lambda^\infty \) has property \( (S) \) if and only if \( H \) has property \( (S) \). The subgroup \( \Lambda^\infty \) is finitely generated by [4, Theorem 2.22] so \( H \) is compactly generated. But \( \Lambda \) is a uniform approximate lattice in the compactly generated Lie group \( H \), so \( H \) is unimodular by [4, Theorem 5.8]. Now \( H \) is a co-compact unimodular subgroup so \( G/H \) has a finite \( G \)-invariant measure by [22, Lemma 1.4] . Hence, according to [23, Lemma 5.4] the subgroup \( H \) has property \( (S) \) and so does \( \Lambda^\infty \).

Consider now the second part of Proposition 13. By assumption there is a non-trivial \( G \)-invariant Borel probability measure \( \nu \) on \( \Omega_X \). If \( \Omega \) is any symmetric neighbourhood of the identity, then the open subset \( U^\Omega \) satisfies \( \nu \left( U^\Omega \right) > 0 \). Indeed, \( U^\Omega \) is open and

\[ \Omega_X \setminus \{\emptyset\} = \bigcup_{g \in G} Ug^{-1} \Omega = \bigcup_{g \in G} gU^\Omega. \]

Since \( \Omega_X \) is metric compact it is in particular second countable so we can find \( D \subset G \) countable such that

\[ \Omega_X \setminus \{\emptyset\} = \bigcup_{d \in D} dU^\Omega. \]
But \( \nu(\Omega_X \setminus \{\emptyset\}) = 1 \) so there is \( d \in D \) such that \( 0 < \nu(dU_\Omega) = \nu(U_\Omega) \). Therefore, for any \( g \in G \) there is \( 1 < n < \left( \nu(U_\Omega) \right)^{-1} \) such that \( \nu(U_\Omega \cap g^nU_\Omega) > 0 \). So we can find \( C \in U_\Omega \cap g^nU_\Omega \). Thus, \( C \cap \Omega \neq \emptyset \) and \( C \cap g^n\Omega \neq \emptyset \). That implies that \( C^{-1}C \cap \Omega g^n\Omega \neq \emptyset \). But \( C^{-1}C \subset X^{-1}X \) so \( g^n \in \Omega X^{-1}X \Omega \). □

**Corollary 8.** Let \( \Lambda \) be a strong or uniform approximate lattice in a connected Lie group \( G \). Let \( A \) be the amenable radical of \( G \) (i.e. the maximal amenable connected normal subgroup) and \( p : G \to G/A \) be the natural projection. If \( H \) is a connected amenable subgroup of \( G/A \) stable under conjugation by elements of \( p(\Lambda) \), then \( H = \{e\} \).

**Proof.** The connected Lie group \( G/A \) is semi-simple and has no compact factor. Moreover, \( \Lambda^\infty \) has property (S) by Proposition 13 and \( p \) is a surjective continuous group homomorphism so \( p(\Lambda^\infty) \) has property (S) as well. Since \( H \) is stable under conjugation by elements of \( p(\Lambda) \) it is also stable under conjugation by elements of \( p(\Lambda^\infty) \). Hence, the subgroup \( H \) is normal according to [23, Corollary 5.16]. So \( H \) is an amenable connected normal subgroup. Which implies that \( H = \{e\} \). □

**Remark 7.** We could also prove Corollary 8 by invoking [6]. However, Proposition 13 has the advantage to yield a self-contained proof that can easily be generalised to cases not covered by [6], namely approximate lattices in \( S \)-adic Lie groups.

6.3. **Proof of Theorem 7.** We start with a technical proposition.

**Proposition 14.** Let \( \Lambda \) be a uniformly discrete approximate subgroup of a connected Lie group \( G \) and \( R \) a normal closed soluble subgroup of \( G \). Let \( p : G \to G/R \) denote the natural map. Assume that \( p(\Lambda) \) is relatively compact. Then there is \( \Lambda' \subset \Lambda^4 \) an approximate subgroup commensurable to \( \Lambda \) that generates a soluble group.

**Proof.** According to Theorem 5 and Theorem 6 the approximate subgroup \( \Lambda^4 \) has a good model. In addition, by Proposition 13 there is \( \Lambda' \subset \Lambda^4 \) an approximate subgroup commensurable to \( \Lambda \) such that \( \Lambda' \) has a good model \( f : (\Lambda')^\infty \to H \) with dense image and target a connected Lie group. Now by Lemma 8 the graph of \( f \) denoted by \( \Gamma_f \) is a discrete subgroup of \( G \times H \). Thus, by [23, Theorem 8.24], the connected component of the identity of \( (p \times id)(\Gamma_f) \subset G/R \times H \) is a soluble Lie group. So we can find symmetric relatively compact neighbourhoods of the identity \( V \subset G \) and \( W \subset H \) such that \( \Gamma_f \cap (VR \times W) \) generates a soluble subgroup of \( G \times H \). Now the map \( p(\Lambda')^\infty \times f : (\Lambda')^\infty \to G/R \times H \) is a good model of \( \Lambda' \). Indeed, \( (p(\Lambda')^\infty \times f)(\Lambda') = p(\Lambda') \times f(\Lambda') \) is relatively compact and for any neighbourhood of the identity \( U \subset G \) such that \( f^{-1}(U) \subset \Lambda' \) we have \( (p(\Lambda')^\infty \times f)^{-1}(G/R \times U) \subset f^{-1}(U) \subset \Lambda' \). Therefore, according to Corollary 8 the approximate subgroup \( \Lambda'' := (p(\Lambda')^\infty \times f)^{-1}(V \times (W \cap U \cap U^{-1})) \subset \Lambda' \) is commensurable to \( \Lambda' \). But \( (\iota \times f)(\Lambda'') \subset \Gamma_f \cap (VR \times W) \) where \( \iota : \Lambda'' \to G \) is the inclusion map. Since \( \Gamma_f \cap (VR \times W) \) generates a soluble subgroup and \( \iota \times f \) is injective, the approximate subgroup \( \Lambda'' \) generates a soluble subgroup. □

**Proof of Theorem 7.** We know that the soluble radical \( R \) is contained in the amenable radical \( A \) and that \( R \) is a co-compact subgroup of \( A \). Let \( U \subset G \) be any compact symmetric neighbourhood of the identity. Then \( U^2R \cap \Lambda^2 \) is an approximate subgroup by Lemma 8. According to Proposition 14 there is an approximate subgroup \( \Xi \) commensurable to \( U^2R \cap \Lambda^2 \) such that \( \Xi^\infty \) is soluble. Choose one such \( \Xi \) with
$H := \Xi_{\infty}R$ of minimal dimension. Choose $\lambda \in \Lambda$, then the approximate subgroup $\lambda \Xi \lambda^{-1}$ is commensurable to $\Lambda((U^2R \cap \Lambda^2)\lambda^{-1}$ and $\Xi$ is commensurable to $(U^2R \cap \Lambda^2)$. So $\Xi$ and $\lambda \Xi \lambda^{-1}$ are commensurable according to Lemma 3. Therefore, according to Lemma 3 the approximate subgroup $C_{\Lambda} := \Xi^2 \cap \lambda \Xi \lambda$ is commensurable to $\Xi$. Since in addition $C_{\Lambda} \subset \Xi^2$ we have by minimality of $\dim(\Xi_{\infty}R)$ that $C_{\Lambda} \subseteq R$ is an open subgroup of $H$. But $C_{\Lambda} \subseteq H \cap \lambda H \lambda^{-1} \subset H$ so $H^0 = \lambda H^0 \lambda^{-1}$ where $H^0$ is the connected component of the identity of $H$. As a consequence, the approximate subgroup $\Lambda$ normalises $H^0$. Let $p : G \to G/A$ be the natural map then $p(H^0)$ is a connected soluble subgroup of $G/A$ normalised by $p(\Lambda)$. By Corollary 3 we have $p(H^0) = \{e\}$ which means $H^0 \subset A$. Since $H^0$ is an open subgroup of $H$ and the inclusion $R \subset H^0$ holds, there is a neighbourhood of the identity $W \subset G$ such that $\Xi^2 \cap W^2R \subset H^0 \subset A$. But $\Xi$ is commensurable to $U^2R \cap \Lambda^2$, so the approximate subgroup $\Xi^2 \cap W^2R$ is commensurable to $U^2R \cap \Lambda^2$ according to Lemma 3. So there is a finite subset $F \subset A$ such that $U^2R \cap \Lambda^2 \subset F(\Xi^2 \cap W^2R)$. As a consequence,

$$p(U^2R \cap \Lambda^2) \subset p(F(\Xi^2 \cap W^2R)) \subset p(FA) \subset p(F).$$

Since $R$ is co-compact in $A$ and the above argument is valid for any relatively compact neighbourhood of the identity $U$, the subset $p(\Lambda)$ is locally finite. Now by Proposition 12 we find that $\Lambda^2 \cap A$ is a uniform or strong approximate lattice in $A$. But $A$ is a connected amenable Lie group so Proposition 11 shows that $\Lambda^2 \cap A$ is a uniform approximate lattice in $A$. This proves part (i).

Now write $\Lambda_A = (\Lambda^2 \cap A)$. Note that $R \subset A$ is also the soluble radical of $A$ and $N$ is the nilpotent radical of $A$. Since moreover $A/R$ is normal connected in $G/R$ this implies that any compact semi-simple factor of $A$ is a compact semi-simple factor of $G$. Now we know that $\Lambda_A$ is a uniform approximate lattice in $A$ so according to Proposition 3 there is $\Lambda_A' \subset \Lambda_A^5$ an approximate subgroup commensurable to $\Lambda_A$ - in particular a uniform approximate lattice - that has a good model $f : (\Lambda_A')^\infty \to H$ with dense image and target a connected amenable Lie group. Let $K \subset H$ be the maximal compact normal subgroup. Then $H/K$ is a connected amenable Lie group as well and for any symmetric relatively compact neighbourhood of the identity $U \subset H$ the subset $\Lambda_A' := f^{-1}(U K)$ is an approximate subgroup commensurable to $\Lambda_A$ by Corollary 3. Moreover, the map $\hat{f} := p_K \circ f$ with $p_K : H \to H/K$ the natural map is a good model of $\Lambda_A'$. By Proposition 1 the graph of $\hat{f}$, denoted by $\Gamma_{\hat{f}}$, is a uniform lattice in $\Lambda_A' \times H/K$. Write $\bar{R}$ for the soluble radical of $H/K$ and $\bar{N}$ for its nilpotent radical. Then $H/K$ has no non-trivial normal compact subgroup so $H/K$ has no semi-simple compact factor that acts trivially on $\bar{R}$. By assumption, $G$ has no semi-simple compact factor that acts trivially on $R$, so $A$ has no semi-simple compact factor that acts trivially on $R$. By [14, Theorem 1.3] this implies that $\Gamma_{\hat{f}} \cap (\bar{N} \times \bar{N})$ is a uniform lattice in $\bar{N} \times \bar{N}$. Choose now $U \subset H/K$ a symmetric relatively compact neighbourhood of the identity such that $\hat{f}^{-1}(U) \subset \Lambda_A'$ then $p_G(\Gamma_{\hat{f}} \cap (\bar{N} \times (\bar{N} \cap U)))$ (where $p_G : G \times H/K \to H/K$ is the natural map) is a uniform approximate lattice in $\bar{N}$ by [4, Proposition 2.13] that is contained in $\Lambda_A' \subset \Lambda_A^5$. Therefore, $\Lambda_A^5 \cap \bar{N}$ is a uniform approximate lattice in $\bar{N}$, and so is $\Lambda^2 \cap \bar{N}$ by Lemma 3. So (ii) is proved.

Finally let us prove (iii). The group $G$ satisfies the conditions of part (i) and (ii). So $\Lambda_A = \Lambda^2 \cap A$ is a uniform approximate lattice in $A$ by (i) and $\Lambda^2 \cap \bar{N}$ is a uniform approximate lattice in $\bar{N}$ by (ii).
approximate lattice in $N$ by (ii). Now according to Lemma 12 the approximate subgroup $p_N(\Lambda_A) \subset A/N$ is locally finite where $p_N : A \to A/N$ is the natural map. Since $\Lambda_A$ is an approximate subgroup this implies that $p_N(\Lambda_A^2)$ is locally finite. As $p_N(\Lambda_A)$ is moreover relatively dense because $\Lambda_A$ is relatively dense, we obtain that $p_N(\Lambda_A)$ is a uniform approximate lattice in $A/N$ by [4, Proposition 2.9]. But $A/N$ is a connected Lie group and the nilpotent and the soluble radical of $A/N$ are both equal $R/N$. Moreover, by assumption no semi-simple factor of $G/N$ acts trivially on $R/N$ so, as above, no semi-simple factor of $A/N$ acts trivially on $R/N$. Thus, according to (ii) the subset $p_N(\Lambda_A^2) \cap R/N$ is a uniform approximate lattice (by Lemma 8) such that $p_N(\Lambda_R)$ is relatively dense in $R/N$ and $\Lambda_R \cap N$ is relatively dense in $N$ as it contains $\Lambda^2 \cap N$. Choose $K_1, K_2 \subset R$ two compact subsets such that $p_N(K_1)p_N(\Lambda_R) = R/N$ and $N \subset K_2 \Lambda_R$. Then

$$R = K_1 \Lambda_R N = K_1 N \Lambda_R = K_1 K_2 \Lambda_R^2,$$

so $\Lambda_R^2$ is relatively dense in $R$, hence a uniform approximate lattice. By Lemma 8 one last time we find that $\Lambda^2 \cap R$ is a uniform approximate lattice in $R$.

\[\square\]

**Question 2.** With the notations of Theorem 7. Let $p : G \to G/A$ denote the natural projection. Suppose that both $p(\Lambda)$ and $\Lambda^2 \cap A$ are contained in model sets. Is $\Lambda$ contained in a model set?

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