∂-HARMONIC FORMS ON 4-DIMENSIONAL ALMOST-HERMITIAN MANIFOLDS

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Abstract. Let \((X, J)\) be a 4-dimensional compact almost-complex manifold and let \(g\) be a Hermitian metric on \((X, J)\). Denote by \(\Delta_\partial := \partial\bar{\partial} + \partial^*\bar{\partial}\) the \(\partial\)-Laplacian. If \(g\) is \emph{globally conformally Kähler}, respectively \emph{(strictly) locally conformally Kähler}, we prove that the dimension of the space of \(\partial\)-harmonic \((1, 1)\)-forms on \(X\), denoted as \(h^{1,1}_\partial\), is a topological invariant given by \(b_+ + 1\), respectively \(b_-\). As an application, we provide a one-parameter family of almost-Hermitian structures on the Kodaira-Thurston manifold for which such a dimension is \(b_-\). This gives a positive answer to a question raised by T. Holt and W. Zhang. Furthermore, the previous example shows that \(h^{1,1}_\partial\) depends on the metric, answering to a Kodaira and Spencer’s problem. Notice that such almost-complex manifolds admit both almost-Kähler and \((\text{strictly})\) locally conformally Kähler metrics and this fact cannot occur on compact complex manifolds.

1. Introduction

Let \((X, J)\) be an almost-complex manifold, then if \(J\) is not integrable one has that \(\bar{\partial} \neq 0\) and so the Dolbeault cohomology of \(X\)

\[ H^{\bullet, \bullet}_\partial(X) := \frac{\text{Ker } \partial}{\text{Im } \partial} \]

is not well defined. However, if \(g\) is a Hermitian metric on \((X, J)\) and \(*\) denotes the associated Hodge-\(*\)-operator, then

\[ \Delta_\partial := \bar{\partial}\partial + \partial^*\bar{\partial} \]

is a well-defined second order, elliptic, differential operator, without assuming the integrability of \(J\). In particular, if \(X\) is compact, then \(\text{Ker } \Delta_\partial\) is a finite-dimensional vector space and we will denote as usual with \(h^{\bullet, \bullet}_\partial\) its dimension. If \(J\) is integrable, then the \((p, q)\)-Dolbeault cohomology groups \(H^{p,q}_\partial(X)\) of the compact complex manifold \((X, J)\) are isomorphic to the Kernel of \(\Delta_\partial\), that is

\[ H^{p,q}_\partial(X) \simeq \text{Ker } \Delta_\partial|_{A^{p,q}(X)} \]

where \(A^{p,q}(X)\) denotes the space of smooth \((p, q)\)-forms on \((X, J)\) and, in particular, the dimension of the space of \(\partial\)-harmonic forms is a holomorphic invariant, not depending on the choice of the Hermitian metric. In [6, Problem 20] Kodaira and

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Spencer asked whether this is the case also when $J$ is not integrable. More precisely,

**Question I** Let $(M, J)$ be an almost-complex manifold. Choose a Hermitian metric on $(M, J)$ and consider the numbers $h^{p,q}_{\partial}$. Is $h^{p,q}_{\partial}$ independent of the choice of the Hermitian metric?

In [7] Holt and Zhang answered negatively to this question, showing that there exist almost-complex structures on the Kodaira-Thurston manifold such that the Hodge number $h^{1,1}_{\partial}$ varies with different choices of Hermitian metrics. Furthermore, in [7, Proposition 6.1] the authors showed that for a compact 4-dimensional almost-Kähler manifold $h^{1,1}_{\partial}$ is independent of the metric, and more precisely $h^{1,1}_{\partial} = b_- + 1$, where $b_-$ denotes the dimension of the space of the anti-self-dual harmonic 2-forms. In [7, Question 6.2] the authors asked the following

**Question II** Let $(M, J)$ be a compact almost-complex 4-dimensional manifold which admit an almost-Kähler structure. Does it have a non almost-Kähler Hermitian metric such that $h^{1,1}_{\partial} \neq b_- + 1$?

In this paper we study this problem. In fact, first we show that on compact almost-complex 4-dimensional manifolds $h^{1,1}_{\partial}$ is a conformal invariant of Hermitian metrics (see Lemma 3.2). In particular, this means that [7, Proposition 6.1] can be extended to Hermitian metrics that are globally conformal to an almost-Kähler metric (for simplicity we will call these metrics globally conformally Kähler, even though the almost-complex structure may not be integrable).

Next we show, using the existence (and uniqueness up to omotheties) of the Gauduchon representative in every conformal class, the following, see Theorem 3.6.

**Theorem 1** Let $(X^4, J)$ be a compact almost-complex manifold of dimension 4, then, with respect to a (strictly) locally conformally Kähler metric, $h^{1,1}_{\partial} = b_-$.  

Here, by (strictly) locally conformally Kähler metric we mean a Hermitian metric $\omega$, such that 

$$d\omega = \theta \wedge \omega$$

with $\theta$ a $d$-closed, non $d$-exact, differential 1-form.

Then we show the following, see Theorem 3.7.

**Theorem 2** Let $(X^4, J)$ be a compact almost-complex manifold of dimension 4 and let $\omega$ be a Hermitian metric, then if $\omega$ is globally conformally Kähler (in particular if it is almost-Kähler), it holds

$$h^{1,1}_{\partial} = b_- + 1.$$  

If $\omega$ is (strictly) locally conformally Kähler, 

$$h^{1,1}_{\partial} = b_-.$$  

In particular, for locally conformally Kähler and globally conformally Kähler metrics on compact 4-dimensional almost-complex manifolds, $h^{1,1}_{\partial}$ is a topological invariant. Notice that this was already known in the integrable case, by [5, Proposition II.6].

We also show in Proposition 3.8 that on every compact 4-dimensional almost-Hermitian manifold, $b_-$ is a lower bound for $h^{1,1}_{\partial}$, that is optimal in view of Theorem 3.6.
Finally, we discuss these results on explicit examples on the Kodaira-Thurston manifold, denoted with $X$. This is a compact 2-step nilmanifold of dimension 4 that can be endowed with both complex and symplectic structures but cannot admit any Kähler metrics (see [9], [11]) and it has the structure of a principal $S^1$-bundle over a 3-torus. This manifold turns out to be a very valuable source of examples in non-Kähler geometry.

First, in example 4.1, we construct a family of almost-complex structures $J_a$, with $a \in \mathbb{R} \setminus \{0\}$, $a^2 < 1$, on $X$ that admit both almost-Kähler and (strictly) locally conformally Kähler metrics (notice that in view of [13] this could not happen in the integrable case). Namely, $(X, J_a)$ is a compact almost-complex 4-dimensional manifold which admit an almost-Kähler metric $\tilde{\omega}_a$ and a non almost-Kähler Hermitian metric $\omega_a$ such that

$$h^{1,1}_a = b_+ \neq b_- + 1.$$  

Hence, this example answers affirmatively to [7, Question 6.2] in the case of the Kodaira-Thurston manifold endowed with the 1-parameter family of almost-complex structures $J_a$.

Moreover, this answers to Kodaira and Spencer’s question, showing that also the Hodge number $h^{1,1}_a$ depends on the Hermitian metric and not just on the almost-complex structure.

Then, in example 4.3 we construct on $X$ a left-invariant almost-complex structure compatible with a family of non left-invariant globally conformally Kähler metrics. In particular, in this case we will have

$$h^{1,1}_a = b_+ + 1.$$  

In both examples we write down explicitly the $\overline{\partial}$-harmonic representatives in $\mathcal{H}^{1,1}_a$.

2. Preliminaries

In this Section we recall some basic facts about almost-complex and almost-Hermitian manifolds and fix some notations. Let $X$ be a smooth manifold of dimension $2n$ and let $J$ be an almost-complex structure on $X$, namely a $(1,1)$-tensor on $X$ such that $J^2 = -\text{Id}$. Then, $J$ induces on the space of forms $A^\ast(X)$ a natural bigrading, namely

$$A^\ast(X) = \bigoplus_{p+q=\bullet} A^{p,q}(X).$$

Accordingly, the exterior derivative $d$ splits into four operators

$$d : A^{p,q}(X) \to A^{p+2,q-1}(X) \oplus A^{p+1,q}(X) \oplus A^{p,q+1}(X) \oplus A^{p-1,q+2}(X)$$

$$d = \mu + \partial + \overline{\partial} + \bar{\mu},$$

where $\mu$ and $\bar{\mu}$ are differential operators that are linear over functions. In particular, they are related to the Nijenhuis tensor $N_J$ by

$$(\mu \alpha + \bar{\mu} \alpha)(u,v) = \frac{1}{4} \alpha (N_J(u,v))$$

where $\alpha \in A^1(X)$. Hence, $J$ is integrable, that is $J$ induces a complex structure on $X$, if and only if $\mu = \bar{\mu} = 0$. 

In general, since $d^2 = 0$ one has
\[
\begin{aligned}
\mu^2 &= 0 \\
\mu \partial + \partial \mu &= 0 \\
\partial \bar{\partial} + \mu \partial + \partial \bar{\partial} &= 0 \\
\bar{\partial} \partial + \mu \bar{\partial} + \bar{\partial} \mu &= 0 \\
\bar{\partial}^2 + \mu \bar{\partial} + \partial \bar{\partial} &= 0 \\
\mu \bar{\partial} + \partial \bar{\partial} &= 0 \\
\bar{\partial} \partial^2 &= 0
\end{aligned}
\]
In particular, $\bar{\partial}^2 \neq 0$ and so the Dolbeault cohomology of $X$
\[
H^{\bullet \bullet}_{\bar{\partial}}(X) := \frac{\text{Ker} \bar{\partial}}{\text{Im} \bar{\partial}}
\]
is well defined if and only if $J$ is integrable.

If $g$ is a Hermitian metric on $(X, J)$ with fundamental form $\omega$ and $*$ is the associated Hodge-$*$-operator, one can consider the following differential operator
\[
\Delta_{\bar{\partial}} := \bar{\partial} \partial^2 + \bar{\partial} \partial + \partial \bar{\partial}.
\]
This is a second order, elliptic, differential operator and we will denote its kernel by
\[
H^{p,q}_{\bar{\partial}}(X) := \text{Ker} \Delta_{\bar{\partial}}|_{\mathcal{A}^p,q(X)}.
\]
In particular, if $X$ is compact this space is finite-dimensional and its dimension will be denoted by $h^{p,q}_{\bar{\partial}}$. By [7] we know that these numbers are not holomorphic invariants, more precisely they depend on the choice of the Hermitian metric. When needed we will use the notations $H^{p,q}_{\bar{\partial}, \omega}$, $h^{p,q}_{\bar{\partial}, \omega}$ in order to stress on the dependence on the Hermitian metric $\omega$.

However, if the Hermitian metric is almost-Kähler and $2n = 4$, in [7, Proposition 6.1] it was shown that $h^{1,1}_{\bar{\partial}} = b_+ - 1$, depends only on the topology of $X$.

We recall that one can consider also other elliptic differential operators on almost-Hermitian manifolds, as done in [11], as generalizations of the classical Dolbeault, Bott-Chern and Aeppli Laplacians defined on complex manifolds. Moreover, a generalization of the Dolbeault cohomology of the non integrable setting was introduced and studied in [2] and [3].

Let us fix now an almost-Hermitian metric $g$ on a compact $2n$-dimensional almost-complex manifold $(X, J)$, which we will identify with its associated $(1,1)$-form $\omega$. Then $J$ acts as an isomorphism on $\wedge^{p,q}X$ by $J \alpha = \sqrt{-1}^{q-p} \alpha$, $\alpha \in \wedge^{p,q}X$.

Via this extension, it follows that $J^2 = (-1)^{k} \text{id}$, so that $J^{-1} = (-1)^{k} J = J^*$ on $\wedge^k X$, where $J^*$ is the pointwise adjoint of $J$ with respect to some (and so any) Hermitian metric. We denote by $d^c$ the differential operator $d^c := -J^{-1} d J$.

Then, we can consider the linear operator $L := \omega \wedge -$ and its adjoint $\Lambda := L^*$. We recall that $L^{n-1} : \wedge^1 X \to \wedge^{2n-1} X$ is an isomorphism, therefore one can define the Lee form of $\omega$, as:
\[
\theta := \Lambda d \omega = J d^c \omega \in \wedge^1 X
\]
such that
\[
d \omega^{n-1} = \theta \wedge \omega^{n-1}.
\]
We will say that $\omega$ is (strictly) locally conformally Kähler if
\[
d \omega = \alpha \wedge \omega
\]
where $\alpha$ is a $d$-closed, non $d$-exact, 1-form. In particular, in this case, the Lee form of $\omega$ is
\[
\theta = \frac{1}{n-1} \alpha.
The metric $\omega$ will be called *globally conformally Kähler* if
\[
d\omega = \alpha \wedge \omega
\]
with $\alpha$ $d$-exact 1-form. Indeed, if $\alpha = df$ then the metric $e^{-f} \omega$ is almost-Kähler.

Notice that, if $\tilde{\omega} = \Phi \omega$, with $\Phi \in C^\infty(X, \mathbb{R})$, $\Phi > 0$, are two conformal Hermitian metrics, then the associated Lee forms are related by
\[
\theta_{\tilde{\omega}} = \theta_\omega + (n-1)d \log \Phi,
\]
in particular, $d\theta_{\tilde{\omega}} = d\theta_\omega$. Hence, all the Hermitian metrics conformal to a (strictly) locally conformally Kähler are still (strictly) locally conformally Kähler.

Another important class of Hermitian metrics is given by the *Gauduchon metrics*, which are defined by $dd^c \omega = 0$ or equivalently as having co-closed Lee form.

These metrics are a very useful tool in conformal and almost-Hermitian geometry, in view of the celebrated result by Gauduchon, [4, Théorème 1], which states that if $(M,J)$ is an $n$-dimensional compact almost-complex manifold with $n > 1$, then any conformal class of any given almost-Hermitian metric contains a Gauduchon metric, unique up to multiplication with positive constants.

3. $h^{1,1}_\partial$ on compact almost-Hermitian 4-dimensional manifolds

In this section we study the Hodge number $h^{1,1}_\partial$ on compact almost-Hermitian 4-dimensional manifolds.

We first show in arbitrary dimension the following Lemma, that ensures that in suitable degrees the Hodge numbers are conformal invariants.

**Lemma 3.1.** Let $(X^{2n}, J)$ be a compact almost-complex manifold of dimension $2n$, then, for $p+q = n$, $h^{p,q}_\partial$ is a conformal invariant of Hermitian metrics.

**Proof.** Let $\tilde{\omega} = \Phi \omega$ be two conformal Hermitian metrics, with $\Phi$ smooth positive function on $X$. Then, for $(p,q)$-forms on a $2n$-dimensional manifold we have that the associated Hodge-$*$-operators are related by,
\[
*_{\tilde{\omega}} = \Phi^{n-p-q} *_\omega.
\]
In general we would have that $\psi \in A^{1,1}(X)$ is $\partial$-harmonic with respect to $\tilde{\omega}$ if and only if
\[
\overline{\partial} \psi = 0, \quad \partial *_{\tilde{\omega}} \psi = 0
\]
if and only if
\[
\overline{\partial} \psi = 0, \quad \partial (\Phi^{n-p-q} *_{\omega} \psi) = 0
\]
Now we compute
\[
\partial (\Phi^{n-p-q} *_{\omega} \psi) = \partial \Phi^{n-p-q} \wedge *_{\omega} \psi + \Phi^{n-p-q} \partial *_{\omega} \psi = (n-p-q) \Phi^{n-p-q-1} \partial \Phi \wedge *_{\omega} \psi + \Phi^{n-p-q} \partial *_{\omega} \psi.
\]
Clearly, if $p+q = n$ we have that
\[
*_{\tilde{\omega}} = *_{\omega},
\]
and
\[
\partial *_{\tilde{\omega}} \psi = 0 \iff \partial *_{\omega} \psi = 0.
\]
Hence, for $p+q = n$
\[
\mathcal{H}^{p,q}_{\partial, \Phi \omega} = \mathcal{H}^{p,q}_{\partial, \omega},
\]
In particular, their dimensions coincide,
\[
h^{p,q}_{\partial, \Phi \omega} = h^{p,q}_{\partial, \omega}.
\]

As a corollary we have the following
Lemma 3.2. Let \((X^4, J)\) be a compact almost-complex manifold of dimension 4, then \(h^{1,1}_{\overline{\partial}}\) is a conformal invariant of Hermitian metrics.

As an immediate application we have the following

Proposition 3.3. Let \((X^4, J)\) be a compact almost-complex manifold of dimension 4, then, with respect to a globally conformally Kähler metric, \(h^{1,1}_{\overline{\partial}} = b_− + 1\).

Proof. Since \(h^{1,1}_{\overline{\partial}}\) is a conformal invariant, the result follows by [7]. Indeed, for almost-Kähler metrics \(h^{1,1}_{\overline{\partial}} = b_− + 1\). □

We first prove the following (cf. [5, Proposition II.6] for the integrable case)

Proposition 3.4. Let \((X^4, J)\) be a compact almost-complex manifold of dimension 4 and let \(\omega\) be a Gauduchon metric, then the trace of a \(\overline{\partial}\)-harmonic \((1,1)\)-form is constant. Namely, if \(\psi \in \mathcal{H}^1_{\overline{\partial}}\) is written as

\[
\psi = f\omega + \gamma \quad \text{with } *\gamma = -\gamma,
\]

then \(f\) is constant.

Proof. Let \(\psi \in A^{1,1}(X)\) be a \(\overline{\partial}\)-harmonic \((1,1)\)-form. Then,

\[
\psi = f\omega + \gamma
\]

with \(\gamma\) anti-self dual \((1,1)\)-form, namely \(*\gamma = -\gamma\), and \(f = \frac{1}{2} \text{tr} \psi = \frac{1}{2} \Lambda \psi = \frac{1}{2} \langle \psi, \omega \rangle\).

Hence,

\[
*\psi = f\omega - \gamma.
\]

Since \(\psi\) is \(\overline{\partial}\)-harmonic, then \(\overline{\partial}\psi = 0\) and \(\partial *\psi = 0\), namely

\[
\overline{\partial}(f\omega) = -\overline{\partial}\gamma
\]

and

\[
\partial(f\omega) = \partial\gamma.
\]

Recalling that for \((1,1)\)-forms on a 4-dimensional manifolds we have that

\[
d^c = i(\overline{\partial} - \partial)
\]

and

\[
d\overline{d}^c + d^c d = 0;
\]

summing up the previous two equations we get

\[
d(f\omega) = id^c \gamma,
\]

hence

\[
d\overline{d}^c(f\omega) = 0.
\]

Since \(\omega\) is Gauduchon this implies that \(f\) is constant. For completeness we recall here the proof (cf. for instance also the proof in [11, Theorem 10]). Since, on a 4-dimensional manifold \(d\omega = \theta \wedge \omega\), where \(\theta\) is the Lee form of \(\omega\), we have

\[
\begin{align*}
\overline{d}^c(f\omega) &= d(\overline{d}^c f \wedge \omega + f J\theta \wedge \omega) \\
&= \overline{d}^c f \wedge \omega - d^c f \wedge \theta \wedge \omega + df \wedge J\theta \wedge \omega \\
&\quad + dJ\theta \wedge \omega + f \theta \wedge J\theta \wedge \omega \\
&= (\overline{d}^c f - d^c f \wedge \theta + df \wedge J\theta + f dJ\theta + f \theta \wedge J\theta) \wedge \omega \\
&\quad + \Lambda (\overline{d}^c f - d^c f \wedge \theta + df \wedge J\theta + f dJ\theta + f \theta \wedge J\theta) \frac{\omega^2}{2}.
\end{align*}
\]

Therefore, \(d\overline{d}^c(f\omega) = 0\) is equivalent to:

\[
\Lambda (\overline{d}^c f - d^c f \wedge \theta + df \wedge J\theta + f dJ\theta + f \theta \wedge J\theta) = 0
\]
Recall now that from [1, Lemma 7] on an almost-Hermitian manifold we have, for every 1-form $\alpha$,
\[ \Lambda(dJ\alpha) = -d^*\alpha - \langle \alpha, \theta \rangle. \]
Therefore,
\[ \Lambda(dd^c f - d^c f \wedge \theta + df \wedge J\theta + f dJ\theta + f \theta \wedge J\theta) = 0 \]
if and only if
\[ -\Delta f - \langle df, \theta \rangle - \Lambda(df \wedge \theta) + \Lambda(df \wedge J\theta) - fd^*\theta - f|\theta|^2 + f\Lambda(\theta \wedge J\theta) = 0. \]
This holds if and only if
\[ -\Delta f + 2\langle df, \theta \rangle - f|\theta|^2 + f|\theta|^2 = 0. \]
Therefore, we have obtained that $dd^c(f\omega) = 0$ if and only if
\[ -\Delta f + \langle df, \theta \rangle = 0. \]
Namely, $f \in \text{Ker} L^*$ where, for Gauduchon metrics,
\[ L^*(h) = \Delta h - \langle dh, \theta \rangle \]
is the adjoint of the operator $L$, with
\[ L(h) = \Delta h + \langle dh, \theta \rangle. \]
Now, by [5] (cf. also [4]) $f$ is either positive or negative (unless $f = 0$). Suppose that $f > 0$ (otherwise one can argue with $-f$), then by $dd^c(f\omega) = 0$ we have that $f\omega$ is a Gauduchon metric conformal to $\omega$, and so $f$ is constant. 

**Remark 3.5.** In the proof of the previous proposition we had that
\[ d(f\omega) = id^c\gamma. \]
Since, $f$ is constant and $d\omega = \theta \wedge \omega$ we have that applying the Hodge-$*$-operator,
\[ f*(\theta \wedge \omega) = i*d^c\gamma. \]
Since $\theta$ is a primitive form, one has that $*(\theta \wedge \omega) = *L\theta = -J\theta$. Moreover, using that $J\gamma = \gamma$ and $*\gamma = -\gamma$ one obtain that
\[ f\theta = -id^c\gamma. \]
In particular, if $\theta \neq 0$ (i.e., $\omega$ is not almost-Kähler), we have that
\[ f = 0 \iff d^c\gamma = 0 \iff \gamma \text{ is harmonic}. \]
As a consequence, we prove that with respect to (strictly) locally conformally Kähler structures, $h^{1,1}_{\mathcal{T}_\gamma}$ is a topological invariant.

**Theorem 3.6.** Let $(X^4, J)$ be a compact almost-complex manifold of dimension 4 and suppose that there exists a (strictly) locally conformally Kähler metric, then
\[ h^{1,1}_{\mathcal{T}_\gamma} = b_{-}. \]

**Proof.** Let $\tilde{\omega}$ be a (strictly) locally conformally Kähler metric on $(X^4, J)$. Since, by Lemma 3.2 $h^{1,1}_{\mathcal{T}_\gamma}$ is a conformal invariant we fix in the conformal class of $\tilde{\omega}$ the Gauduchon representative $\omega$ of volume 1. Clearly, $\omega$ is still (strictly) locally conformally-Kähler.
Let $\psi \in A^{1,1}(X)$ be a $\mathcal{T}_\gamma$-harmonic $(1,1)$-form. Then,
\[ \psi = f\omega + \gamma \]
with \( *\gamma = -\gamma \) and \( f \) constant by Proposition 6.4. By Remark 6.5
\[
f \theta = d^*(-i\gamma) \in \text{Im } d^*.
\]
Now, we want to show that \( f = 0 \). Suppose by contradiction that \( f \neq 0 \), hence
\[
\theta = d^* \left( -\frac{i}{f} \gamma \right) \in \text{Im } d^*.
\]
but, since \( \omega \) is still (strictly) locally conformally-Kähler, \( d\theta = 0 \). So \( \theta \) is \( d \)-closed and \( d^* \)-exact and so \( \theta = 0 \), but this is absurd since \( d\omega \neq 0 \). Therefore, \( f = 0 \) and
\[
\psi = \gamma \quad \text{with} \quad *\gamma = -\gamma
\]
with \( d^*\gamma = 0 \), that is \( \gamma \) harmonic, concluding the proof. \( \square \)

In particular, as a consequence of Lemma 3.2, Theorem 3.6 and [7, Proposition 6.1], for locally conformally Kähler and globally conformally Kähler metrics on compact 4-dimensional almost-complex manifolds, \( h_{1,1}^{1,1} \) is a topological invariant. Namely, we have proven the following

**Theorem 3.7.** Let \((X^4, J)\) be a compact almost-complex manifold of dimension 4 and let \( \omega \) be a Hermitian metric, then if \( \omega \) is globally conformally Kähler (in particular if it is almost-Kähler), it holds
\[
h_{1,1}^{1,1} = b_- + 1.
\]
If \( \omega \) is (strictly) locally conformally Kähler,
\[
h_{1,1}^{1,1} = b_-.
\]
Notice that in the integrable case, \( h_{1,1}^{1,1} \) only depends on the complex structure and for compact complex surfaces this result is known (cf. [5, Proposition II.6]). Indeed, recall that a compact complex surface is Kähler if and only if \( b_1 \) is even and in this case \( h_{1,1}^{1,1} = b_- + 1 \). On the other side, a compact complex surface is non-Kähler if and only \( b_1 \) is odd and in this case \( h_{1,1}^{1,1} = b_- \). This is coherent with our result since on compact Kähler surfaces there exist no (strictly) locally conformally Kähler metrics by [13]. In fact, this last statement holds more generally, indeed by ([13, Theorem 2.1, Remark (1)]), on compact complex manifolds satisfying the \( \partial\bar{\partial} \)-lemma every locally conformally Kähler structure is also globally conformally Kähler. We want to point out that the first non-integrable examples of almost-Kähler manifolds admitting (strictly) locally conformally Kähler manifolds appeared in [12]. In Section 4 we will construct a new family of examples on the Kodaira-Thurston manifold.

In view of these results, we ask whether there exist examples of Hermitian metrics on compact almost-complex 4-dimensional manifolds with \( h_{1,1}^{1,1} \) different from \( b_- \) and \( b_- + 1 \).

The following result gives a general estimate for \( h_{1,1}^{1,1} \).

**Proposition 3.8.** Let \((X^4, J)\) be a compact almost-complex manifold of dimension 4 and let \( \omega \) be a Hermitian metric, then
\[
H^-_g \subseteq H_{1,1}^{1,1},
\]
where \( H^-_g \) denotes the space of anti-self-dual harmonic \((1,1)\)-forms.

In particular,
\[
h_{1,1}^{1,1} \geq b_-.
\]
Proof. Let $\gamma$ be an anti-self-dual $(1,1)$-form, namely $*\gamma = -\gamma$. Suppose that $\gamma$ is harmonic, that is equivalent to $d\gamma = 0$. Hence, for degree reasons
\[
\overline{\partial}\gamma = 0 \quad \text{and} \quad \partial * \gamma = -\partial \gamma = 0.
\]
Therefore, $\gamma$ is $\overline{\partial}$-harmonic. \hfill \square

In particular, Theorem 3.7 shows that the minimum $b^{1,1}_{\overline{\partial}}$ is reached by strictly locally conformally Kähler metrics.

4. Explicit constructions on the Kodaira-Thurston manifold

In this section we apply the results obtained in Section 3 to construct explicit examples. First, we recall the definition of the Kodaira-Thurston manifold $X$. Let $H_3(\mathbb{R}) := \left\{ \begin{bmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{bmatrix} \mid x_1, x_2, x_3 \in \mathbb{R} \right\}$ be the 3-dimensional Heisenberg group and let $\Gamma$ be the subgroup of $H_3(\mathbb{R})$ of the matrices with integral entries. Then, $X := \frac{H_3(\mathbb{R})}{\Gamma} \times S^1$ is a compact 4-dimensional manifold admitting both complex and symplectic structures but no Kähler structures. Denoting with $x_4$ the coordinate on $S^1$, a global frame on $X$ is given by
\[
e_1 := \partial x_1, \quad e_2 := \partial x_2 + x_1 \partial x_3, \quad e_3 := \partial x_3, \quad e_4 := \partial x_4,
\]
and its dual coframe is
\[
e^1 := dx_1, \quad e^2 := dx_2, \quad e^3 := dx_3 - x_1 dx_2, \quad e^4 := dx_4.
\]
In particular, the only non trivial bracket is $[e_1, e_2] = e_3$ and so the structure equations become
\[
de^1 = de^2 = de^4 = 0, \quad de^3 = -e^1 \wedge e^2.
\]
In the sequel we denote by $e^{ij} = e^i \wedge e^j$ and similarly.

Then
\[
H^2_{d\overline{\partial}}(X; \mathbb{R}) \cong \text{Span}_{\mathbb{R}}([e^{13} - e^{24}], [e^{14} + e^{23}], [e^{13} + e^{24}], [e^{14} - e^{23}])
\]
where all the representatives are harmonic, with respect to the Riemannian metric $g = \sum_{j=1}^{4} e^j \otimes e^j$. Furthermore, the space of $d$-harmonic anti-self dual forms is isomorphic to
\[
\text{Span}_{\mathbb{R}}([e^{13} + e^{24}], [e^{14} - e^{23}]),
\]
so that $b_{-}(X) = 2$.

Example 4.1. Now we construct the following family of almost-complex structures $J_a$ on $X$, with $a \in \mathbb{R}$, setting as coframe of $(1,0)$-forms
\[
\Phi^1_a := (e^1 + ae^4) + ie^3, \quad \Phi^2_a := e^2 + ie^4.
\]
We will use the notation $\Phi^1 = \Phi^1_a$, $\Phi^2 = \Phi^2_a$. The dual $(1,0)$-frame of vector fields is given by
\[
V_1 := \frac{1}{2}(e_1 - ie_3), \quad V_2 := \frac{1}{2}(e_2 - i(e_4 - ae_1)).
\]
One can show directly that the complex structure equations become
\[ \partial \Phi^1 = -\frac{i}{4} \Phi^{12} - \frac{i}{4} \Phi^{1\bar{2}} + \frac{i}{4} \Phi^{21} + \frac{a}{2} \Phi^{22} - \frac{i}{4} \Phi^{1\bar{2}}, \]
\[ \partial \Phi^2 = 0, \]
and in particular, \( J_a \) is a non integrable almost-complex structure. For every \( a \in \mathbb{R} \), we fix the following Hermitian metric
\[ \omega_a := \frac{i}{2} (\Phi^1 \wedge \bar{\Phi}^1 + \Phi^2 \wedge \bar{\Phi}^2). \]
A direct computation gives that
\[ d\omega_a = i\frac{a}{4} \Phi^{12\bar{2}} - i\frac{a}{4} \Phi^{21\bar{2}} = \theta_a \wedge \omega_a. \]
with \( \theta_a = \frac{a}{2} (\Phi^1 + \bar{\Phi}^1) \). In particular,
- \( d\theta_a = 0 \),
- \( \omega_a \) is an almost-Kähler metric if and only if \( a = 0 \).
Therefore, for \( a \neq 0 \), the Lee form \( \theta_a \) of the almost-Hermitian metric \( \omega_a \) is closed and not \( d \)-exact, hence \( \omega_a \) is a strictly locally conformally Kähler metric.

Hence, by Theorem 3.7 for the Hodge numbers we have on \((X, J_a)\), with \( a^2 < 1 \),
- \( h^{1,1}_{\omega_a} = b_- = 2 \) for \( a \neq 0 \)
- \( h^{1,1}_{\omega_{a=0}} = b_- + 1 = 3 \)
- \( h^{1,1}_{\tilde{\omega}_a} = b_- + 1 = 3 \)

This, in particular, partially answers to Question 6.2 in [7], giving explicit examples of compact almost-complex 4-dimensional manifolds which admit an almost-Kähler metric and also admit a non almost-Kähler Hermitian metric with \( h^{1,1}_{\omega_a} \neq b_- + 1 \).
Moreover, this answers to Kodaira and Spencer’s question, showing that also the Hodge number \( h^{1,1}_{\omega_a} \) depend on the Hermitian metric and not just on the almost-complex structure.

For the sake of completeness we write down the PDE’s system that one should solve in order to find a basis for \( H^{1,1}_{\omega_a} \).
Let \( \psi \in A^{1,1}(X) \) be an arbitrary \((1,1)-form on X\), then \( \psi \) can be written as
\[ \psi = A \Phi^{11} + B \Phi^{12} + L \Phi^{21} + M \Phi^{22} \]
where \( A, B, L, M \) are smooth functions on \( X \).

By the complex structure equations, we get
\[ \overline{\partial} \psi = V_2(A) \Phi^{112} - V_1(B) \Phi^{112} + V_2(L) \Phi^{212} - V_1(M) \Phi^{212} - \frac{a}{2} A \Phi^{212} + \frac{i}{4} B \Phi^{212} - \frac{i}{4} L \Phi^{212}, \]
hence \( \overline{\partial} \psi = 0 \) if and only if
\[ V_2(A) - V_1(B) = 0, \]
\[ V_2(L) - V_1(M) - \frac{a}{2} A + \frac{i}{4} B - \frac{i}{4} L = 0. \]
Now, if we denote with $\ast$ the Hodge-$\ast$-operator with respect to the metric $\omega$, we have

$$\ast \psi = M \Phi^{11} - B \Phi^{12} - L \Phi^{21} + A \Phi^{22}$$

and

$$\partial \ast \psi = -V_2(M) \Phi^{121} + V_2(B) \Phi^{122} - V_1(L) \Phi^{121} + V_1(A) \Phi^{122} + \frac{a}{2} M \Phi^{122} + \frac{i}{4} B \Phi^{122} - \frac{i}{4} L \Phi^{122},$$

hence $\partial \ast \psi = 0$ if and only if

$$V_2(M) + V_1(L) = 0,$$

$$V_2(B) + V_1(A) + \frac{a}{2} M + \frac{i}{4} B - \frac{i}{4} L = 0.$$ 

Therefore, $\psi$ is harmonic if and only if

$$\begin{cases}
V_2(A) - V_1(B) = 0, \\
V_2(L) - V_1(M) - \frac{a}{2} A + \frac{i}{4} B - \frac{i}{4} L = 0, \\
V_2(M) + V_1(L) = 0,
\end{cases}$$

$$V_2(B) + V_1(A) + \frac{a}{2} M + \frac{i}{4} B - \frac{i}{4} L = 0.$$ 

Since we know that $h^{1,1}_{\omega_a} = b_- = 2$, the solution of this system is given by $A, B, L, M$ constants and

$$A = -M = \frac{i}{2a} B - \frac{i}{2a} L.$$ 

Hence,

$$\mathcal{H}^{1,1}_{\omega_a} = C \left\langle \frac{i}{2a} \Phi^{11} + \Phi^{12} - \frac{i}{2a} \Phi^{22}, - \frac{i}{2a} \Phi^{11} + \Phi^{21} + \frac{i}{2a} \Phi^{22} \right\rangle.$$ 

Since in the integrable case, on Kähler manifolds every locally conformally Kähler metric is globally conformally Kähler we want to put in evidence the following

**Proposition 4.2.** The Kodaira-Thurston manifold with the almost-complex structure $J_a$ constructed above admits both almost-Kähler metrics and (strictly) locally conformally Kähler metrics.

For other examples we refer to [12].

**Example 4.3.** Now we define a different, non left-invariant, Hermitian structure on the Kodaira-Thurston manifold $X$.

First we define a non-integrable left-invariant complex structure $J$ on $X$ setting as global co-frame of $(1,0)$-forms

$$\varphi^1 := e^1 + ie^3, \quad \varphi^2 := e^2 + ie^4$$

and the corresponding structure equations become

$$d\varphi^1 = -\frac{i}{4} \varphi^{12} - \frac{i}{4} \varphi^{12} + \frac{i}{4} \varphi^{21} - \frac{i}{4} \varphi^{12}, \quad d\varphi^2 = 0.$$ 

We will denote with $\{W_1, W_2\}$ its dual frame, more precisely

$$W_1 := \frac{1}{2} (e_1 - ie_3), \quad W_2 := \frac{1}{2} (e_2 - ie_4),$$

namely,

$$W_1 = \frac{1}{2} (\partial_x - i \partial_x), \quad W_2 = \frac{1}{2} (\partial_x + x_1 \partial_x - i \partial_x).$$
We consider now on \((X, J)\) the following 1-parameter family of almost-Hermitian metrics
\[ \omega_{tf} = e^{2tf(x_2)}e^1 \wedge e^3 + e^2 \wedge e^4 \]
where \(f = f(x_2)\) is a \(\mathbb{Z}\)-periodic smooth function with \(e_2(f) \neq 0\). In particular, \(f\) induces a smooth function on \(X\). Then it is immediate to check that the almost-complex structure \(J\) is compatible with \(\omega_{tf}\) so that it defines a positive definite Hermitian metric on \(X\). In fact, the metric \(\omega_{tf}\) is almost-Kähler if and only if \(t = 0\). Indeed, by the structure equations and the fact that \(e_2(f) = f'(x_2) \neq 0\) by assumption, we have
\[ d\omega_{tf} = -2te^{2tf(x_2)}e_2(f)e^{123}. \]
Hence \(\omega_{tf}\) is a Hermitian deformation of an almost-Kähler metric on \(X\).

Moreover, notice that
\[ \omega_{tf} = 2tf(x_2)e_1 \wedge e^3 = 2tf(x_2)e_2 \wedge e^4 = 2tf(x_2)e_3 \wedge e^1, \]
with \(\omega_{tf} = 2tf(x_2)e_1^2 = 2tf(x_2)e_2^2 = 2tf(x_2)e_3^2\), namely the Lee form of \(\omega_{tf}\) is \(d\)-exact, which means that \(\omega_{tf}\) is globally conformally Kähler. A direct computation shows that, in fact
\[ e^{-2tf(x_2)}\omega_{tf} \]
is an almost-Kähler metric. Hence, by Theorem 3.7 for the Hodge numbers we have on \((X, J)\),
\[ h^{1,1}_{\omega_{tf}} = b_+ + 1 = 3. \]

For completeness we write down the system that one should solve in order to find a basis for \(H^{1,1}_{\omega_{tf}}\).

Let \(\psi \in A^{1,1}(X)\) be an arbitrary \((1, 1)\)-form on \(X\), then \(\psi\) can be written as
\[ \psi = A\varphi^{11} + B\varphi^{12} + L\varphi^{21} + M\varphi^{22} \]
where \(A, B, L, M\) are smooth functions on \(X\).

By the complex structure equations, we get
\[ \overline{\partial}\psi = W_2(A)\varphi^{112} - \bar{W}_1(B)\varphi^{112} + W_2(L)\varphi^{212} - \bar{W}_1(M)\varphi^{212} + \frac{i}{4}B\varphi^{212} - \frac{i}{4}L\varphi^{212}, \]

hence \(\overline{\partial}\psi = 0\) if and only if
\[ W_2(A) - \bar{W}_1(B) = 0, \]
\[ W_2(L) - \bar{W}_1(M) + \frac{i}{4}B - \frac{i}{4}L = 0. \]

Now, if we denote with \(*\) the Hodge-* operator with respect to the metric \(\omega_{tf}\) we have that a unitary frame is given by
\[ \psi_1 = \frac{1}{\sqrt{2}}e^{tf(x_2)}\varphi^1, \quad \psi_2 = \frac{1}{\sqrt{2}}\varphi^2 \]
and hence we have
\[ *\psi = Ae^{-2tf(x_2)}\varphi^{22} - B\varphi^{12} - L\varphi^{21} + M\varphi^{22} \]
and
\[ \partial *\psi = -W_2(Me^{2tf(x_2)})\varphi^{121} + W_2(B)\varphi^{122} - W_1(L)\varphi^{121} + W_1(Ae^{-2tf(x_2)})\varphi^{122} + \frac{i}{4}B\varphi^{22} - \frac{i}{4}L\varphi^{22}, \]

hence \(\partial *\psi = 0\) if and only if
\[ W_2(Me^{2tf}) + W_1(L) = 0, \]
\[ W_2(B) + W_1(Ae^{-2tf(x_2)}) + \frac{i}{4}B - \frac{i}{4}L = 0. \]
Therefore, $\psi$ is harmonic if and only if
\[
\begin{align*}
\vec{W}_2(A) - \vec{W}_1(B) &= 0, \\
\vec{W}_2(L) - \vec{W}_1(M) + \frac{i}{4}B - \frac{i}{4}L &= 0, \\
\vec{W}_2(Me^{2tf(z_2)}) + \vec{W}_1(L) &= 0, \\
\vec{W}_2(B) + \vec{W}_1(Ae^{-2tf(z_2)}) + \frac{i}{4}B - \frac{i}{4}L &= 0.
\end{align*}
\]
Since we know that
\[b\frac{1,1}{\omega_{1,1}} = b_+ + 1 = 3,
\]
the solution of this system is given by $A$ complex constant, $B = L$ complex constants and $M = M_0 e^{-2tf(z_2)}$ where $M_0$ is a complex constant. Hence,
\[
\mathcal{H}_{\text{Harm}}^{1,1} = \mathbb{C}\left\langle \varphi^{11}, \varphi^{12}, \varphi^{21}, e^{-2tf(z_2)}\varphi^{22} \right\rangle.
\]

**Remark 4.4.** It has to be remarked that solving these kind of PDE's systems is not an easy task. Indeed, as a general method, one could use Fourier analysis to expand the unknown complex valued functions $A, B, L, M$, obtaining a first order ODE's system on the Fourier coefficients of $A, B, L, M$, which, as far as we know, is very challenging to solve (cf. also [7, 8] for further comments).

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