WHERE IS THE LARGE RADIUS LIMIT?

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ABSTRACT

By properly accounting for the invariance of a Calabi-Yau sigma-model under shifts of the \( B \)-field by integral amounts (analogous to the \( \theta \)-angle in QCD), we show that the moduli spaces of such sigma-models can often be enlarged to include “large radius limit” points. In the simplest cases, there are holomorphic coordinates on the enlarged moduli space which vanish at the limit point, and which appear as multipliers in front of instanton contributions to Yukawa couplings. (Those instanton contributions are therefore suppressed at the limit point.) In more complicated cases, the instanton contributions are still suppressed but the enlarged space is singular at the limit point. This singularity may have interesting effects on the effective four-dimensional theory, when the Calabi-Yau is used to compactify the heterotic string.

1. Integral Shifts of the \( B \)-Field

To write a Lagrangian for the nonlinear sigma-model on a Calabi-Yau manifold \( X \), we must make a choice of metric \( g_{ij} \) (Ricci-flat in the one-loop approximation) and “\( B \)-field” (a real closed 2-form). This \( B \)-field enters into the action only through integration over the world-sheet, in a term proportional to \( \int_{\Sigma} \phi^* (B) \). (The notation refers to a map \( \phi \) from the worldsheet \( \Sigma \) to the target space \( X \), assumed to satisfy \( h^{2,0}(X) = 0 \).) In fact, this term can naturally be combined with a contribution from the Kähler form \( J \) of the metric to produce a term in the action proportional to \( \int_{\Sigma} \phi^* (B + i J) \). When \( B, J \) and the string tension are normalized properly, the partition function and the correlators involve precisely the quantity \( \exp (2\pi i \int_{\Sigma} \phi^* (B + i J)) \).

Altering \( B \) by adding an exact 2-form to it does not alter any of the quantities \( \int_{\Sigma} \phi^* (B) \), so only the de Rham cohomology class of \( B \) matters for specifying the Lagrangian. Furthermore, if we replace \( B \) by \( B + B_0 \) where \( B_0 \) represents an integral cohomology class, then the crucial quantity \( \exp (2\pi i \int_{\Sigma} \phi^* (B + i J)) \) is left unchanged since each \( \int_{\Sigma} \phi^* (B_0) \) is an integer. (The \( B \)-field therefore plays a rôle somewhat analogous to that of the \( \theta \)-angle in QCD.) The importance of the resulting principle of invariance of physics under integral shifts of the \( B \)-field is not as widely recognized as it ought to be.

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This invariance is manifest in the analysis carried out by Candelas et al. of the mirror map for quintic threefolds. Analyzing the behavior of the metric on the moduli space, these authors found that the usual parameter \( t \) (proportional to \( B + iJ \)) on the Kähler moduli space of the quintic-mirror has an asymptotic relationship to the natural parameter \( z := \psi^{-5} \) on the complex moduli space of the quintic, of the form

\[
t \sim \frac{1}{2\pi i} \log z + \text{constant} + \ldots.
\]

Once one has observed that values of \( t \) which differ by an integer lead to identical physics, one is led to introduce \( e^{2\pi i t} \) as a more natural parameter on the Kähler moduli space. (This effectively modifies the definition of that space by making identifications between points which differ by an integral shift of the \( B \)-field.) The new parameter \( e^{2\pi i t} \) is then a single-valued function of \( z \), consistent with mirror symmetry. This same idea has led to successful mirror map calculations for other one-parameter families of Calabi-Yau threefolds, and more recently for one-parameter families of Calabi-Yau manifolds in higher dimension.

2. The Large Radius Limit

To analyze the large radius limit in general, we choose a basis \( e^1, \ldots, e^r \) of the integral harmonic 2-forms on the target space \( X \), and write \( B + iJ = \sum z_j e^j \). The \( z_j \)'s can be regarded as coordinates on the “complexified Kähler cone,” constrained by some inequalities such as \( \text{Im}(z_j) > 0 \). The identification under integral shifts of the \( B \)-field can be implemented by exponentiating these coordinates, introducing \( w_j := e^{2\pi i z_j} \). Inequalities such as \( 0 < \text{Im}(z_j) < \infty \) on the \( z_j \)'s translate into inequalities such as \( 0 < |w_j| < 1 \) on the \( w_j \)'s. We partially compactify the space by including points for which some \( w_j \) is 0.

To see the large radius limit, we should rescale the metric via \( g_{ij} \mapsto \lambda g_{ij} \), and take \( \lambda \to \infty \). The Kähler form scales as \( J \mapsto \lambda J \), and the exponentiated coordinates transform as \( w_j \mapsto |w_j|^\lambda \cdot \text{arg}(w_j) \). As \( \lambda \to \infty \), all points with \( |w_j| < 1 \) flow towards the origin \((0, \ldots, 0)\), so we can apparently regard the origin in this coordinate system as the “large radius limit point.” This is consistent with the behavior of the instanton expansions of three-point functions, which take the general form

\[
\langle \mathcal{O}_A \mathcal{O}_B \mathcal{O}_C \rangle = A \cdot B \cdot C + \sum_{\Gamma} \frac{w^\Gamma}{1 - w^\Gamma} (A \cdot \Gamma)(B \cdot \Gamma)(C \cdot \Gamma),
\]

where the sum is over rational curves \( \Gamma \) on \( X \), and \( w^\Gamma \) is a monomial in \( w_1, \ldots, w_r \) determined by the homology class of \( \Gamma \). All instanton contributions to this correlation function vanish at the origin in the \( w \)-coordinates.

In order to ensure that \( |w_j| < 1 \) for all points in the Kähler moduli space, we must choose the classes \( e^1, \ldots, e^r \) to lie in the closure of the Kähler cone of \( X \). When \( r = 1 \) (i.e., for the mirrors of one-parameter families), this condition completely specifies the integral basis and determines a “large radius limit point”
unambiguously. However, when \( r > 1 \) the freedom to change the basis causes difficulties.

In all examples with \( r > 1 \) studied in the literature the Kähler cone has a very simple form and the edges of its closure can be used as the desired basis. However, this is not a general feature of Calabi-Yau threefolds: a study from a mathematical perspective reveals some complexities which are not visible in these examples.

3. Blowing Down the Moduli Space

The most concrete way to see the effect of a change of basis is to consider what happens if the moduli space is blown up at the origin in the \( w \)-coordinates. We take \( r = 2 \) for simplicity, and find two new coordinate charts after the blowup, with coordinates \((w_1, \frac{w_2}{w_1})\) and \((\frac{w_1}{w_2}, w_2)\). (The corresponding bases are \(\{e^1 + e^2, e^2\}\) and \(\{e^1, e^1 + e^2\}\).) Rescaling the metric and taking \(\lambda \to \infty\) sends \((w_1, w_2)\) to the origin in the first chart when \(|w_2| < |w_1|\), and to the origin in the second chart when \(|w_1| < |w_2|\). Both “origins” can thus lay claim to being the “large radius limit” associated to at least part of the Kähler moduli space.

Conversely, if we have a partial compactification of the Kähler moduli space which includes more than one large radius limit point (each associated with a different basis \(e^1, \ldots, e^r\), and with a different domain inside the moduli space), we should attempt to blow down this space to produce a partial compactification with a single large radius limit point for the entire moduli space. These blowdowns are similar to those arising in toric geometry and will often lead to singularities in the compactified space. The instanton contributions to correlation functions are still suppressed in such a limit, in spite of the singularities—we must accept the possibility that the “true” large radius limit point is not a smooth point.

(Note that all of the large radius limit points under discussion are associated to a single Kähler cone. It is also possible to consider other large radius limit points associated to the Kähler cones of different birational models of \(X\). This leads to topology-changing transitions and one would not expect to collapse those limit points to a single point by blowing down.)

Even when we expect to be able to blow down and are willing to allow singularities, it may prove to be impossible to perform the desired blowing down, due to the presence of an infinite number of large radius limit points. For example, if the Kähler cone is described as \(\frac{2}{1 - \sqrt{5}} < y < \frac{2}{1 + \sqrt{5}}\), then (as shown in figure 1) attempting to cover the cone using integral bases leads to a sequence of rays with slopes \(-\frac{5}{8}, -\frac{2}{3}, -1, \frac{1}{6}, 2, \frac{5}{3}, \frac{13}{8}, \ldots\) which asymptotically approach the walls of the cone. Each adjacent pair of rays in the sequence gives rise to a distinct large radius limit point.

\(^\dagger\)The figure does not include these walls—the limiting rays with irrational slope \(\frac{1 + \sqrt{5}}{2}\) since they are less than a line-width’s distance from the outer rays as shown (at the level of resolution of the figure).
4. Automorphisms

A Calabi-Yau threefold may have holomorphic automorphisms which act nontrivially on the Kähler cone. If we make the identifications on the Kähler moduli space dictated by those automorphisms, it may become possible to do the blowdowns—an infinite number of large radius limit points may turn into a finite number after these identifications.

In the example above, an automorphism acting on the cone as \((x, y) \mapsto (2x + 3y, 3x + 5y)\) leads from an infinite number of large radius limit points on the original Kähler moduli space to two remaining points on the quotient space. The quotient space can then be blown down explicitly, leading to a singular surface with local equation \(w^2 = (u^3 - v^2)(u^2 - v^3)\). This is illustrated in figure 2.
moduli fields which always remain close to the field theory (large radius) limit, yet which generate global monodromy effects in $M^{3,1}$, somewhat akin to discrete gauge transformations. (Previous examples of such discrete symmetries were only visible upon leaving the sigma-model region of the moduli space, i.e., upon varying the moduli fields to a point far from the field theory limit.) Phenomenological implications of such a scheme are at present unknown.

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