HOMOGENEOUS COSMOLOGIES FROM THE QUASI-MAXWELL FORMALISM

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Abstract. We show how to use the quasi-Maxwell formalism to obtain solutions of Einstein’s field equations corresponding to homogeneous cosmologies – namely Einstein’s universe, Gödel’s universe and the Ozsvath-Farnsworth-Kerr class I solutions – written in frames for which the associated observers are stationary.

INTRODUCTION

A particularly intuitive framework for obtaining and interpreting stationary solutions of Einstein’s field equations is the so-called quasi-Maxwell formalism ([2], [5]). Although such solutions have been extensively treated in the past ([1], [7]), this approach has been successfully used in recent times ([3], [4]). In this paper we apply the quasi-Maxwell formalism in the case when the space manifold is a Lie group with left-invariant metric and fields, and rediscover Einstein’s universe, Gödel’s universe and the Ozsvath-Farnsworth-Kerr class I solutions, sometimes written in unconventional frames.

The organization of the paper is as follows: In the first section we briefly review the quasi-Maxwell formalism for stationary spacetimes. In the second section we analyze the form taken by the quasi-Maxwell equations when the space manifold is a Lie group. In the third section we further specialize to Lie groups with class A Lie algebras. Finally, solutions of the quasi-Maxwell equations for these space manifold are obtained and identified in the last section.

We use Einstein summation convention, irrespective of the position of the indices (which will often be irrelevant as we will be leading with orthonormal frames on Riemannian manifolds). We will take Latin indices $i, j, \ldots$ to run from 1 to 3.

1. Quasi-Maxwell formalism

In this section we briefly review the quasi-Maxwell formalism for stationary spacetimes. For more details, see [5].

Recall that a stationary spacetime $(M, g)$ is a Lorentzian 4-manifold with a global timelike Killing vector field $T$. We assume that there exists a global time function $t : M \to \mathbb{R}$ such that $T = \frac{\partial}{\partial t}$. The quotient of $M$ by the integral curves of $T$ is a 3-dimensional manifold $\Sigma$ to which we refer as the space manifold. If $\{x^i\}$ are local coordinates in $\Sigma$, we can write the line element of $(M, g)$ as

$$ds^2 = -e^{2\phi} \left(dt + A_i dx^i\right)^2 + \gamma_{ij} dx^i dx^j$$

where $\phi$, $A_i$ and $\gamma_{ij}$ do not depend on $t$. This allows us to interpret $\phi$, $A_i = A_i dx^i$ and $\gamma = \gamma_{ij} dx^i \otimes dx^j$ as tensor fields on the space manifold. It turns out that $\gamma$ is a Riemannian metric in $\Sigma$, independent of the choice of the global time function $t$. The differential forms $G = -d\phi$ and $H = -e^\phi dA$ are also independent of this choice, and play a central role in the so-called quasi-Maxwell formalism. We define the gravitational and gravitomagnetic (vector) fields $G$ and $H$ through

(1) \hspace{2cm} G = \gamma(G, \cdot)

(2) \hspace{2cm} H = \epsilon(H, \cdot, \cdot)

where $\epsilon$ is a Riemannian volume form in $(\Sigma, \gamma)$ (which we assume to be orientable).
We identify a vector \( v \in T_p\Sigma \) with the unique vector field \( v \) along the integral curve of \( T \) through \( p \) which is orthogonal to \( T \) and satisfies \( \pi_* v = v \) (\( \pi : M \to \Sigma \) being the quotient map). Let \( \{X_0, X_i\} \) be a local orthonormal frame on \( M \), where \( X_0 = (-g(T,T))^{-\frac{1}{2}} T \). If
\[
  u = u^0 X_0 + u^i X_i = u^0 X_0 + u
\]
represents the unit tangent vector to a timelike geodesic, the motion equation
\[
  \nabla_u u = 0
\]
is equivalent to
\[
  \nabla_u u = u^0 (u^0 G + u \times H)
\]
with \( u^0 = (1 + u^2)^{\frac{1}{2}} \) (where \( \nabla \) is the Levi-Civita connection of \( (M, g) \), \( \nabla \) is the Levi-Civita connection of \( (\Sigma, \gamma) \) and \( u^2 = \gamma(u, u) \)).

If we let \( R_{ij} \) and \( \nabla_i G_j \) represent the components of the Ricci tensor of \( \nabla \) and of the covariant derivative of \( G \), Einstein’s equations for a perfect fluid with density \( \rho \), pressure \( p \) and 4-velocity \( u \) reduce to the quasi-Maxwell equations \( (QM) \)
\[
  (QM.1) \quad \text{div } G = G^2 + \frac{1}{2} H^2 - 8\pi (\rho + p) u^2 - 4\pi (\rho + 3p)
\]
\[
  (QM.2) \quad \text{curl } H = 2G \times H - 16\pi (\rho + p) u^0 u
\]
\[
  (QM.3.ij) \quad R_{ij} + \nabla_i G_j = G_i G_j + \frac{1}{2} H_i H_j - \frac{1}{2} H^2 \gamma_{ij} + 8\pi \left( (\rho + p) u_i u_j + \frac{1}{2} (\rho - p) \gamma_{ij} \right).
\]

We can use \( QM \) to solve Einstein’s equations by writing down a Riemannian metric for the space manifold (eventually depending on unknown functions), and solving for the fields (see [2], [3], [4], [5]). For instance, the Schwarzschild solution is the static solution (i.e., \( H = 0 \)) obtained when we consider a spherically symmetric space manifold with radial \( G \).

A word of caution must be issued here: the quasi-Maxwell decomposition does depend on the choice of the timelike Killing vector field \( T \). Therefore when one solves the \( QM \) equations, one is really solving for \( (M, g, T) \). If a given spacetime has a large enough isometry group, it can yield many different solutions of \( QM \).

The goal of this paper is the classification of solutions whose space manifolds are Lie groups with left-invariant metrics, and whose vector fields \( G \) and \( H \) are left-invariant.

2. QUASI-MAXWELL EQUATIONS FOR A LIE GROUP

Let the space manifold \( \Sigma \) be a 3-dimensional Lie group. To choose a left-invariant metric we fix a frame \( \{X_i\} \) of left-invariant vector fields and declare it to be orthonormal. All the information about the geometry of the space manifold will then be encoded in the structure constants, defined by
\[
  [X_i, X_j] = C^k_{ij} X_k = C_{kij} X_k.
\]
The last equality emphasizes that there is no need to worry about the vertical position of the indices, as we’re working with an orthonormal frame. The Christoffel symbols of the Levi-Civita connection are then given by
\[
  \Gamma^j_{ik} = \frac{1}{2} (C_{ijk} + C_{kij} - C_{jki}).
\]
Letting \( G = G_i \omega^i \), where \( \{\omega^i\} \) is the dual basis of \( \{X_i\} \), we have
\[
  \nabla_i G_j = -\Gamma^k_{ij} G_k.
\]
Consequently,
\[
  \text{div } G = \nabla_i G_i = -\Gamma^k_{ii} G_k.
\]
The Maurer-Cartan formula
\[
  d\omega^i = \frac{1}{2} C^i_{jk} \omega^j \wedge \omega^k
\]
assures us that the exterior derivative is a linear transformation between the spaces $\Omega^1_L(\Sigma)$ and $\Omega^2_L(\Sigma)$ of the left-invariant 1 and 2-forms, whose matrix for the bases $\{\omega^1, \omega^2, \omega^3\}$ of $\Omega^1_L(\Sigma)$ and $\{\omega^2 \wedge \omega^3, \omega^1 \wedge \omega^3, \omega^1 \wedge \omega^2\}$ of $\Omega^2_L(\Sigma)$ is

$$D = \begin{pmatrix}
C_{132} & C_{232} & C_{332} \\
C_{113} & C_{213} & C_{313} \\
C_{121} & C_{221} & C_{321}
\end{pmatrix}. $$

By definition, $\text{curl} H$ is the only vector field satisfying

$$\epsilon(\text{curl} H, \cdot, \cdot) = d(\gamma(H, \cdot)).$$

Since vectors, 1 and 2-forms are related by the isomorphisms given by the metric and the volume element of $\Sigma$, we obtain

$$\text{curl} H = (X_1 X_2 X_3) \cdot D \cdot \begin{pmatrix}
H_1 \\
H_2 \\
H_3
\end{pmatrix}. $$

The fact that $\gamma, G$ and $H$ are left-invariant imposes restrictions on the fluid generating the gravitational field:

**Proposition 2.1.** The density and pressure are constant and $\mathbf{u}$ is left-invariant.

**Proof.** QM.3.ii gives us

$$(\rho + p) u^2_i = -\frac{1}{2}(\rho - p) + \text{constant}. $$

Adding these three equations we obtain

$$(\rho + p) \mathbf{u}^2 = -\frac{3}{2}(\rho - p) + \text{constant}, $$

which substituted in QM.1 yields

$$-3(\rho - p) + \rho + 3p = \text{constant} \iff 3p - \rho = \text{constant}. $$

From QM.2 we see that

$$(\rho + p) u^0 u_i = \text{constant} \Rightarrow (\rho + p)^2 (u^0)^2 u^2_i = \text{constant}$$

$$\iff (\rho + p) (\mathbf{u}^2 + 1) (\rho + p) u^2_i = \text{constant}$$

$$\iff \left[ -\frac{3}{2}(\rho - p) + \text{const.} + (\rho + p) \right] \cdot \left[ -\frac{1}{2}(\rho - p) + \text{const.} \right] = \text{const}. $$

$$\iff \frac{1}{4} \rho^2 + \frac{5}{4} p^2 - \frac{3}{2} \rho p + \text{first order terms} = \text{constant}. $$

But since $\rho = 3p + \text{constant}$, we get

$$-p^2 + \text{first order terms} = \text{constant}. $$

We conclude that $\rho$ and $p$ can take at most two distinct values in $\Sigma$, and, being so, the result follows from their continuity.

It is now clear that for $\rho + p \neq 0$ the components of $\mathbf{u}$ are constant, which suffices to insure that it is left-invariant. For $\rho + p = 0$, $\mathbf{u}$ becomes undefined and we can take it to be left-invariant (e.g. zero) without loss of generality.

**Corollary 2.2.** The vector field $\mathbf{u}$ has the following properties:

$$\tilde{\text{div}} \mathbf{u} = 0$$

and

$$\tilde{\nabla}_u \mathbf{u} = 0.$$
Proof. We have seen that we only have to consider the case $\rho + p \neq 0$. Euler’s equation for a perfect fluid is

$$\tilde{\text{div}} T = 0 \iff \begin{cases} \tilde{\text{div}} (\rho u) + p \tilde{\text{div}} u = 0 \\ (\rho + p) \nabla_u u = - (\tilde{\text{grad}} p) \perp \end{cases}$$

where $(\tilde{\text{grad}} p) \perp$ designates the component of $\tilde{\text{grad}} p$ orthogonal to $u$. Since $\rho$ and $p$ are constant with $\rho + p \neq 0$, it follows that

$$\begin{cases} (\rho + p) \tilde{\text{div}} u = 0 \\ (\rho + p) \nabla_u u = 0 \end{cases} \iff \begin{cases} \tilde{\text{div}} u = 0 \\ \nabla_u u = 0 \end{cases}.$$ 

\[\Box\]

Corollary 2.3. The vector fields $u$ and $G$ are orthogonal.

Proof. The motion equation yields

$$\tilde{\nabla}_u u = 0 \Rightarrow \nabla_u u = u^0 (u^0 G + u \times H).$$

Since $u$ is left-invariant and $u^0$ is a nonzero constant,

$$0 = \frac{d}{d\tau} \gamma(u, u) = 2\gamma(\nabla_u u, u) = 2(u^0)^2 \gamma(G, u) \Rightarrow \gamma(G, u) = 0.$$ 

\[\Box\]

The following result relates solutions corresponding to conformally related left-invariant metrics:

**Proposition 2.4. (Rescaling Lemma)** From a solution $(G_i, H_j, u_k, \rho, p)$ of $QM_i$, where the left-invariant metric is associated to the frame $\{X_i\}$, we can construct a solution $(\hat{G}_i, \hat{H}_j, \hat{u}_k, \hat{\rho}, \hat{p})$ for the left-invariant metric associated to the frame $\{\hat{X}_i = \lambda X_i\}$, by setting

$$\hat{G}_i = \lambda G_i, \quad \hat{H}_j = \lambda H_j, \quad \hat{u}_k = u_k, \quad \hat{\rho} = \lambda^2 \rho, \quad \hat{p} = \lambda^2 p.$$ 

Proof. Since

$$[\hat{X}_i, \hat{X}_j] = \lambda^2 [X_i, X_j] = \lambda^2 C_{kij} X_k = \lambda C_{kij} \hat{X}_k,$$

we obtain

$$\hat{C}_{kij} = \lambda C_{kij},$$

from which follows

$$\hat{\Gamma}^i_{jk} = \lambda \Gamma^i_{jk} \quad \text{and} \quad \hat{D} = \lambda D$$

and consequently

$$\hat{R}_{ij} = \lambda^2 R_{ij}, \quad \hat{\text{div}} \hat{G} = \lambda^2 \text{div} G \quad \text{and} \quad \hat{\text{curl}} \hat{H} = \lambda^2 \text{curl} H.$$ 

Since that, by construction, $\hat{\gamma}_{ij} = \gamma_{ij} = \delta_{ij}$, the result follows. 

\[\Box\]

It is easy to see that this rescaling corresponds to rescaling the full spacetime metric by $\frac{1}{\lambda^2}$. 

\[\Box\]
3. Class A Lie Algebras

If we take $\Sigma$ to be connected and simply connected, Lie's theorem [9] guarantees that the space manifold will be uniquely determined, up to isomorphism, by its Lie algebra. Therefore, the consideration of all possible space manifolds becomes the classification of three-dimensional Lie algebras - a much simpler task!

Following [8], we learn that this classification may be realized by means of a $\binom{n}{2}$ symmetric tensor $M$ and a covector $\nu \in \ker M$, whose components in a given basis for the Lie algebra are $\nu_i = C^k_{ki}$. It becomes natural to divide the classification in two classes: class $A$ for Lie algebras with $\nu = 0$, and class $B$ for Lie algebras with $\nu \neq 0$.

We shall restrict ourselves to class $A$ algebras. These are classified by the rank and signature of the symmetric tensor $M$, and are six in total: the abelian algebra (corresponding to rank $M = 0$), the Heisenberg algebra (corresponding to rank $M = 1$), the semidirect products $\mathfrak{so}(1, 1) \ltimes \mathbb{R}^2$ (corresponding to the two possible signatures for rank $M = 2$) and the simple algebras $\mathfrak{sl}(2)$ and $\mathfrak{so}(3)$ (corresponding to the two possible signatures for rank $M = 3$). In terms of the more usual Bianchi classification, these are Bianchi types I, II, V I with parameter $k = 1$, VII with parameter $h = 0$, VIII and IX, respectively (see [6], [7]).

Since $C^k_{ki} = 0$, $M$ can be identified, using the left-invariant metric on which the Lie algebra basis is orthonormal, with minus the linear operator yielding the exterior derivative restricted to $\Omega^1$. Therefore, class $A$ Lie algebras are classified by the rank and signature of the matrix $D$ of the previous section. This matrix is also useful for computing the Ricci tensor:

**Proposition 3.1.** In a Lie group with class $A$ Lie algebra and left-invariant metric, the matrix of components of the Ricci tensor in the basis $\{\omega^i \otimes \omega^j\}$, where $\{\omega^i\}$ is an orthonormal left-invariant co-frame, is given by

$$(R_{ij}) = D^2 - \frac{1}{2} \text{tr}(D^2) I + \text{cof}(D).$$

The proof of this result is straightforward but lengthy and will be omitted.

Since $D$ is symmetric, we are guaranteed the existence of a left-invariant orthonormal co-frame $\{\omega_i\}$ for which

$$D = \text{diag}(C_{132}, C_{213}, C_{321}).$$

Consequently, we can eliminate two unknowns in $QM$:

**Proposition 3.2.** There exists a left-invariant orthonormal frame $\{\hat{X}_i\}$ for which the exterior derivative matrix in the basis $\{\gamma(\hat{X}_k, \cdot)\}$ and $\{\epsilon(\hat{X}_k, \cdot)\}$ is diagonal and $G = G\hat{X}_i$.

**Proof.** Choose $\{X_i\}$ such that $D = \text{diag}(a, b, c)$ and let $G = G_iX_i$. Since $G$ is a closed 1-form, we get

$$dG = d(\gamma(G, \cdot)) = 0 \iff aG_1X_1 + bG_2X_2 + cG_3X_3 = 0.$$

Rearranging the indices if necessary, the last equation tells us that:

1. $\text{rank}(D) = 3 \Rightarrow a, b, c \neq 0 \Rightarrow G = 0$;
2. $\text{rank}(D) = 2 \Rightarrow a = 0, b, c \neq 0 \Rightarrow G_2 = G_3 = 0$;
3. $\text{rank}(D) = 1 \Rightarrow a, b = 0, c \neq 0 \Rightarrow G_3 = 0 \Rightarrow G \perp X_3$. For the nontrivial case (i.e., $G \neq 0$) it suffices to choose $\hat{X}_1 = \frac{1}{\sqrt{c}}G$, $\hat{X}_3 = X_3$ and $\hat{X}_2$ in such a way as to complete the basis as an orthonormal basis;
4. $\text{rank}(D) = 0$: identical to (3).

We end this section with three useful results easily proved from the diagonalization of the exterior derivative matrix.

**Proposition 3.3.** Left-invariant vector fields have vanishing divergence.

**Proof.** If we choose a basis for which $D$ is diagonal, we conclude that the only structure constants not necessarily zero are those with no repeated indices, and consequently

$$\Gamma_{jk}^i \neq 0 \Rightarrow (i, j, k) \text{ is a permutation of (1, 2, 3)}.$$
The result then follows from the equation
\[ \text{div } \mathbf{G} = -\Gamma^k_i G_k. \]
\[ \square \]
Equivalently, we have

**Proposition 3.4.** \( d(\Omega^2_L) = 0. \)

**Corollary 3.5.** \( \mathbf{G} \) and \( \mathbf{H} \) are orthogonal.

**Proof.** Since \( \mathbf{H} \) is a left-invariant 2-form, the last result tells us that
\[ d \mathbf{H} = 0 \Leftrightarrow d (-e^\phi d\mathbf{A}) = 0 \]
\[ \Leftrightarrow -e^\phi d\phi \wedge d\mathbf{A} - e^\phi d(d\mathbf{A}) = 0 \]
\[ \Leftrightarrow \mathbf{G} \wedge \mathbf{H} = 0. \]

Using proposition 3.2, we get
\[ \mathbf{G}_1 \omega^1 \wedge (\mathbf{H}_1 \omega^2 \wedge \omega^3 + \mathbf{H}_2 \omega^3 \wedge \omega^1 + \mathbf{H}_3 \omega^1 \wedge \omega^2) = 0 \]
\[ \Leftrightarrow \mathbf{G}_1 \mathbf{H}_1 \omega^1 \wedge \omega^2 \wedge \omega^3 = 0 \Leftrightarrow \mathbf{G}_1 \mathbf{H}_1 = 0 \Leftrightarrow \gamma(\mathbf{G}, \mathbf{H}) = 0. \]
\[ \square \]

4. **Classification**

For now on we will consider only orthonormal bases \( \{X_i\} \) of left-invariant vector fields for the class \( A \) Lie algebras of the space manifold such that \( D = \text{diag}(a, b, c) \). From Proposition 3.1 we have
\[ (R_{ij}) = \text{diag} \left( \frac{1}{2} a^2 - \frac{1}{2} b^2 - \frac{1}{2} c^2 + bc, -\frac{1}{2} a^2 + \frac{1}{2} b^2 - \frac{1}{2} c^2 + ac, -\frac{1}{2} a^2 - \frac{1}{2} b^2 + \frac{1}{2} c^2 + ab \right). \]

4.1. **Vacuum solutions with cosmological constant.** For convenience, we begin with the computation of \( QM \) solutions such that \( \rho + p = 0 \). These correspond to vacuum solutions with cosmological constant.

**Proposition 4.1.** The only \( QM \) vacuum solution with cosmological constant \( (\rho + p = 0) \) is Minkowski spacetime, i.e., \( \mathbf{G} = \mathbf{H} = 0 \) and \( p = 0 \). The space manifold is then Ricci-flat (Ricci = 0), and hence we necessarily have \( D = \text{diag}(0, b, b) \) for some \( b \in \mathbb{R} \) in an appropriate basis of the space manifold’s Lie algebra.

**Proof.** Let \( \rho + p = 0 \). The indefiniteness of \( \mathbf{u} \) allows us to assume, without loss of generality (wlg), that \( \mathbf{u} = 0 \). From the motion equation we get
\[ 0 = (u^0)^2 \mathbf{G} = 0 \Leftrightarrow \mathbf{G} = 0. \]

Therefore,
\[ QM.1 \Leftrightarrow 0 = \frac{1}{2} \mathbf{H}^2 - 4\pi(\rho + 3p) \Leftrightarrow \mathbf{H}^2 = 16\pi p. \]

Since \( (R_{ij}) \) is diagonal,
\[ QM.3.ij \ (i \neq j) \Leftrightarrow 0 = H_i H_j. \]

Therefore, two of the components of \( \mathbf{H} \) must vanish. Taking, wlg, \( \mathbf{H} = H X_1 \) and writing \( D = \text{diag}(a, b, c) \), we get
\[ QM.2 \Leftrightarrow D \cdot \mathbf{H} = 0 \Leftrightarrow a H = 0. \]

If \( H = 0 \), we obtain \( p = 0 \Rightarrow \rho = 0 \).
If \( a = 0, \)
\[ QM.3.ii \ (i \neq 1) \Leftrightarrow R_{22} = R_{33} = -\frac{1}{2} \mathbf{H}^2 + 4\pi(\rho - p) = 4\pi(\rho - 3p). \]
But
\[ R_{22} = R_{33} \Leftrightarrow \frac{1}{2} b^2 - \frac{1}{2} c^2 = -\frac{1}{2} b^2 + \frac{1}{2} c^2 \Leftrightarrow b^2 = c^2 \Rightarrow R_{22} = R_{33} = 0, \]
yielding $\rho - 3p = 0$, and therefore $\rho = p = 0$ (hence $H = 0$).

Thus the only solution with $\rho + p = 0$ is Minkowski spacetime, and verifies $Ricci = 0$. From the diagonalization of $D$, it is easily seen that a space manifold is Ricci-flat if and only if there is a basis for its Lie algebra such that $D = \text{diag}(0, b, b)$, $b \in \mathbb{R}$.

For the remaining computations we will therefore assume that $\rho + p \neq 0$.

4.2. Solutions with a flat space manifold. In this section we will compute all solutions of QM with flat space manifold $(\Sigma, \gamma)$. Since this is a 3-dimensional manifold, the curvature tensor is completely determined by the Ricci tensor, and therefore flatness is equivalent to Ricci-flatness.

**Theorem 4.2.** The QM solutions with flat space manifold (i.e., with Ricci = 0) correspond to Lie algebras with a basis for which $D = \text{diag}(0, b, b)$, $b \in \mathbb{R}$, and such that:

1. (Gödel’s universe) $b = 0$, $G = \sqrt{16\pi p}X_1$, $H = \sqrt{32\pi p}X_2$, $u = X_3$ and $\rho = p \in \mathbb{R}^+$;
2. (Minkowski spacetime) $G = H = 0$, $p = \rho = 0$ is a solution, for all $b \in \mathbb{R}$ (cf. Proposition 4.1).

**Proof.** We already saw that Ricci-flatness implies that we can choose $D = \text{diag}(0, b, b)$, $b \in \mathbb{R}$. Arguing as in the demonstration of proposition 3.2, we can take $G = GX_1$ and $H = H_1 X_1 + H_2 X_2$.

Suppose first that $G = 0$. In this case,

$QM.2 \Leftrightarrow \begin{cases}
0 = 16\pi (\rho + p)u^0 u_1 \\
0 = 16\pi (\rho + p)u^0 u_3 \\
b H_2 = -16\pi (\rho + p)u^0 u_2 \\
H_2 = -16\pi (\rho + p)u^0 u_2 \\
\Rightarrow u_1 = u_3 = 0 \Leftrightarrow u = u X_2.
\end{cases}$

We then have as the only non trivial equation $QM.3.ij (i \neq j)$

$QM.3.12 \Leftrightarrow H_1 H_2 = 0$.

If $H_1 = 0$, then $H = H X_2$. Therefore

$QM.3.ii \Leftrightarrow \begin{cases}
0 = -\frac{1}{2} H^2 + 4\pi (\rho - p) \\
0 = \frac{1}{2} H^2 - \frac{1}{2} H^2 + 8\pi (\rho + p)u^2 + 4\pi (\rho - p) \\
0 = -\frac{1}{2} H^2 + 4\pi (\rho - p)
\end{cases}$

$\Rightarrow \begin{cases}
H^2 = 8\pi (\rho - p) \\
8\pi (\rho + p)u^2 = -4\pi (\rho - p) = -\frac{1}{2} H^2
\end{cases}$

and

$QM.1 \Leftrightarrow 0 = \frac{1}{2} H^2 - 8\pi (\rho + p)u^2 - 4\pi (\rho + 3p) \Leftrightarrow H^2 = 4\pi (\rho + 3p)$.

Consequently,

$4\pi (\rho + 3p) = 8\pi (\rho - p) \Leftrightarrow \rho = 5p,$

and therefore

$QM.22 \Leftrightarrow 8\pi (\rho + p)u^2 = -4\pi (\rho - p) \Leftrightarrow 12pu^2 = -4p \Rightarrow p = 0 \Rightarrow \rho = 0$.

If $H_2 = 0 (\Rightarrow H = H X_1)$, we get

$QM.3.ii \Leftrightarrow \begin{cases}
0 = 4\pi (\rho - p) \\
0 = -\frac{1}{2} H^2 + 8\pi (\rho + p)u^2 + 4\pi (\rho - p) \\
0 = -\frac{1}{2} H^2 + 4\pi (\rho - p)
\end{cases}$

$\Rightarrow \begin{cases}
\rho = p \\
u = H = 0
\end{cases}$

and

$QM.1 \Leftrightarrow 0 = 4\pi (\rho + 3p).$

But since $\rho = p$, we obtain $\rho = p = 0$.

Let us now consider the case $G \neq 0$. From corollary 3.5 we have

$\gamma(G, H) = 0 \Leftrightarrow H_1 = 0 \Leftrightarrow H = H X_2.$
If \( b = 0 \),
\[
\text{QM.2} \iff \begin{cases} 
0 = u_1 \\
0 = u_2 \\
0 = 2\Gamma u - 16\pi(\rho + p)u_0 u_3
\end{cases} \iff \begin{cases} 
\mathbf{u} = uX_3 \\
\Gamma = 8\pi(\rho + p)u^0 u
\end{cases}
\]
and since
\[
\nabla \Gamma = \nabla_i \Gamma \omega^i \otimes \omega^j \\
= -\Gamma^k_{ij} \Gamma_k \omega^i \otimes \omega^j \\
= -\Gamma^1_{23} \Gamma G_1 \omega^2 \otimes \omega^3 - \Gamma^1_{32} \Gamma G_1 \omega^3 \otimes \omega^2 \\
= -\frac{1}{2}(C_{123} + C_{312} - C_{231})G_2 \omega^2 \otimes \omega^3 - \frac{1}{2}(C_{132} + C_{213} - C_{321})G_3 \omega^3 \otimes \omega^2 \\
= -\frac{1}{2}(0 - b)G \omega^2 \otimes \omega^3 - \frac{1}{2}(0 + b)G \omega^3 \otimes \omega^2 = 0,
\]
equations QM.3.ij (\( i \neq j \)) are trivial.
On the other hand,
\[
\text{QM.3.ii} \iff \begin{cases} 
0 = G^2 - \frac{1}{2}\Gamma^2 + 4\pi(\rho - p) \\
0 = 4\pi(\rho - p) \\
0 = -\frac{1}{2}\Gamma^2 + 8\pi(\rho + p)u^2 + 4\pi(\rho - p) \\
G^2 = \frac{1}{2}\Gamma^2 \Rightarrow H \neq 0 \\
\rho = p \\
16\pi p u^2 = \frac{1}{2}H^2
\end{cases}
\]
from which
\[
\text{QM.1} \iff 0 = G^2 + \frac{1}{2}\Gamma^2 - 8\pi(\rho + p)u^2 - 4\pi(\rho + 3p) \\
\iff G^2 = 4\pi(\rho + 3p) = 16\pi p \Rightarrow H^2 = 32\pi p \quad \text{and} \quad p > 0.
\]
Consequently,
\[
16\pi p u^2 = \frac{1}{2}H^2 = 16\pi p \Leftrightarrow u^2 = 1.
\]
Equation QM.2.3 is immediately satisfied if we respect its only imposition: \( \Gamma H u > 0 \). It can be shown that this solution is in fact Gödel’s universe (see section 4.6).
We are now left with the case \( \mathbf{G} \neq \mathbf{0}, \ b \neq 0 \). We have
\[
\text{QM.2} \iff \begin{cases} 
0 = u_1 \\
bH = -16\pi(\rho + p)u^0 u_2 \\
0 = 2\Gamma - 16\pi(\rho + p)u^0 u_3
\end{cases}
\]
Since \( \nabla \Gamma = 0 \), \( \mathbf{H} = \mathbf{H}X_2 \) and \( u_1 = 0 \), all of the QM.3.ij (\( i \neq j \)) are trivial with the exception of
\[
\text{QM.3.23} \iff \text{QM.3.32} \iff 0 = 8\pi(\rho + p)u_3 \Rightarrow 0 = u_3.
\]
But since the components of \( \mathbf{u} \) are constant,
\[
\nabla \mathbf{u} \mathbf{u} = u_i u_j \nabla_j X_i = u_2 u_3 (\nabla X_2 X_3 + \nabla X_3 X_2) = 0.
\]
If \( u_3 = 0 \) we obtain \( \mathbf{u} \) parallel to \( \mathbf{H} \) and hence
\[
\text{Motion Equation} \Leftrightarrow 0 = (u^0)^2 \mathbf{G} \Leftrightarrow \mathbf{G} = 0,
\]
yielding a contradiction.
If \( u_2 = 0 \), QM.2.2 \( \Rightarrow \mathbf{H} = 0 \) and again the motion equation will lead us to \( \mathbf{G} = 0 \). Therefore we must have \( b = 0 \) whenever \( \mathbf{G} \neq \mathbf{0} \). \( \square \)
4.3. Solutions for Lie algebras with rank $D = 3$. It is easily seen that a change of basis from \{X_1, X_2, X_3\} to \{-X_1, X_2, X_3\} changes the exterior derivative matrix from $D = \text{diag}(a, b, c)$ to $D = \text{diag}(-a, -b, -c)$. Therefore we can assume wlg that $a > 0$.

**Theorem 4.3.** The QM solutions with rank $D = 3$ correspond to Lie algebras with a basis such that $a > 0$ and:

1. (Einstein’s universe) $D = \text{diag}(a, b, b)$, with $b > 0$, $a \geq b$, $G = 0$, $H = \sqrt{a(a-b)}X_1$, $u = -\sqrt{\frac{a}{2}}X_1$ and $\rho = -3p = \frac{3b}{2}$;
2. (Gödel’s universe) $D = \text{diag}(a, b, b)$, with $b < 0$, $G = 0$, $H = \sqrt{a(a-2b)}X_1$, $u = -\sqrt{-\frac{a}{2}}X_1$ and $\rho = p = -\frac{ab}{16\pi}$;
3. (Ozsvath-Farnsworth-Kerr class I) $D = \text{diag}(a, a-b)$, with $16b(a-b) > 3a^2 \Leftrightarrow \frac{1}{4}a < b < \frac{4}{3}a$, $G = 0$, $H = \sqrt{4b(a-b) - \frac{3}{4}a^2}X_1$, $u = -\frac{a}{\sqrt{16b(a-b)-3a^2}}X_1$, $p = -\frac{a^2}{64\pi}$ and $\rho = \frac{32b(a-b) - 5a^2}{64\pi}$.

**Proof.** Let $D = \text{diag}(d_1, d_2, d_3)$, with $\Pi_i d_i \neq 0$. Then $G = 0$, and consequently

$$QM.2 \Leftrightarrow d_i H_i = -16\pi(\rho + p)u^0 u_i \Leftrightarrow H_i = -\frac{16\pi(\rho + p)}{d_i}u^0 u_i$$

(the Einstein summation convention will not apply for the duration of this proof). Therefore,

$$QM.3.ij (i \neq j) \Leftrightarrow 0 = H_i H_j + 16\pi(\rho + p)u_i u_j$$

$$\Leftrightarrow 0 = \left[\frac{16\pi(\rho + p)}{d_i d_j}u^0\right]^2 u_i u_j + 16\pi(\rho + p)u_i u_j$$

$$\Leftrightarrow 0 = 16\pi(\rho + p)u_i u_j \left(\frac{16\pi(\rho + p)}{d_i d_j}u^0 + 1\right)$$

$$\Leftrightarrow 0 = u_i u_j \text{ or } (u^0)^2 = -\frac{d_i d_j}{16\pi(\rho + p)}.$$

We have to consider the following cases:

1. $u_i = 0$ and:
   a. $u_{i2} = 0$;
   b. $(u^0)^2 = -\frac{d_i d_j}{16\pi(\rho + p)}$ (where $i_1, i_2, i_3$ is an arbitrary permutation of $(1, 2, 3)$);
2. $(u^0)^2 = -\frac{d_i d_j}{16\pi(\rho + p)}$, for all $i, j \in \{1, 2, 3\}$ with $i \neq j$.

Let us do so:

1. (a) Suppose, wlg, that $u_2 = u_3 = 0 \Rightarrow u = u X_1$. Then $QM.2 \Rightarrow H = H X_1$, and therefore

$$QM.3.ii (i \neq 1) \Leftrightarrow R_{22} = R_{33} = -\frac{1}{2}H^2 + 4\pi(\rho - p).$$

However,

$$R_{22} = R_{33} \Leftrightarrow -\frac{1}{2}a^2 + \frac{1}{2}b^2 - \frac{1}{2}c^2 + ac = -\frac{1}{2}a^2 - \frac{1}{2}b^2 + \frac{1}{2}c^2 + ab$$

$$\Leftrightarrow b^2 - c^2 + ac - ab = 0$$

$$\Leftrightarrow (b - c)(b + c) - a(b - c) = 0$$

$$\Leftrightarrow (b - c)(b + c - a) = 0$$

$$\Leftrightarrow c = b \text{ or } c = a - b,$$

which leads us to the consideration of two sub-cases:

(i) $c = b$;
(ii) $c = a - b$. 

Let us do so:

(i) We have $D = \text{diag}(a, b, b)$. The Rescaling Lemma (Proposition 2.4) allows us to choose $a = 1$. Let $\Omega = 8\pi(\rho + p) \neq 0$. The QM equations are:

$QM.1 \Leftrightarrow \frac{1}{2} H^2 = \Omega u^2 + \frac{1}{2} \Omega + 8\pi p$;

$QM.2 \Leftrightarrow H = -2\Omega u^0 u$;

$QM.3.ij (i \neq j)$ are already satisfied;

$QM.3.ii \Leftrightarrow \begin{cases} R_{11} = \frac{1}{2} = \Omega u^2 + 4\pi(\rho - p) \\
R_{22} = R_{33} = b - \frac{1}{2} = -\frac{1}{2} H^2 + 4\pi(\rho - p) \end{cases}$

We then have

$QM.3.11 + QM.3.22 \Leftrightarrow b = -\frac{1}{2} H^2 + \Omega u^2 + 8\pi(\rho - p)$.

Inserting $QM.1$ in the last equation yields

$b = -\Omega u^2 - \frac{1}{2} \Omega - 8\pi p + \Omega u^2 + 8\pi(\rho - p)$

$\Leftrightarrow b = -\frac{1}{2} \Omega - 8\pi p + 8\pi(\rho + p) - 16\pi p = \frac{1}{2} \Omega - 24\pi p$

$\Leftrightarrow p = \frac{1}{24\pi} \left( \frac{1}{2} \Omega - b \right)$.

On the other hand,

$QM.3.11 \Leftrightarrow \Omega u^2 = \frac{1}{2} - 4\pi(\rho - p) = \frac{1}{2} - 4\pi(\rho + p) + 8\pi p$

$= \frac{1}{2} - \frac{1}{3} \Omega - \frac{1}{3} b$.

Therefore,

$u^2 = \frac{3 - 2b}{6\Omega} - \frac{1}{3}$

Similarly,

$QM.1 \Leftrightarrow H^2 = \frac{2}{3} (\Omega - 2b) + 1$.

Now

$QM.2 \Leftrightarrow H = -2\Omega u^0 u$

$\Rightarrow H^2 = 4\Omega^2 (u^0)^2 u^2 \Leftrightarrow \frac{1}{4\Omega^2} H^2 = u^4 + u^2$

$\Leftrightarrow \frac{1}{4\Omega^2} \left( \frac{2}{3} (\Omega - 2b) + 1 \right) - \left( \frac{3 - 2b}{6\Omega} - \frac{1}{3} \right)^2 - \left( \frac{3 - 2b}{6\Omega} - \frac{1}{3} \right) = 0$

$\Leftrightarrow \left( \frac{b}{3} + \frac{1}{4} - \frac{(3 - 2b)^2}{36} \right) \frac{1}{\Omega^2} + \left( \frac{1}{6} + \frac{6 - 4b}{18} + \frac{2b - 3}{6} \right) \frac{1}{\Omega} + \frac{2}{9} = 0$

$\Leftrightarrow -b^2 \frac{1}{\Omega^2} + b \frac{1}{\Omega} + 2 = 0 \Leftrightarrow \frac{1}{\Omega} = \frac{-b \pm 3b}{-2b^2}$

$\Leftrightarrow \Omega = \frac{b}{2}$ or $\Omega = -b$. 
Let $\Omega = \frac{b}{2}$. We easily obtain
\[
H^2 = 1 - b;
\]
\[
a^2 = \frac{1 - b}{b};
\]
\[
p = -\frac{b}{32\pi};
\]
\[
\rho = -3p.
\]
To obtain the general solution, i.e., for $D = \text{diag}(a, b, b)$, we have to use the Rescaling Lemma. We have
\[
H^2(a, b, b) = a^2 H^2 \left(1, \frac{b}{a}, \frac{b}{a}\right) = a^2 \left(1 - \frac{b}{a}\right) = a(a - b)
\]
yielding the condition $a \geq b$.

Similarly,
\[
u^2(a, b, b) = H^2 \left(1, \frac{b}{a}, \frac{b}{a}\right) = \frac{1 - b}{b} a - b \frac{b}{a}
\]
yielding the condition $b > 0$. QM.2 requires only that $H$ and $u$ satisfy $Hu \leq 0$.

Finally,
\[
p(a, b, b) = a^2 \left(-\frac{b}{a} \frac{b}{32\pi}\right) = -\frac{ab}{32\pi}.
\]

It can be shown that all these solutions of QM are in fact Einstein’s universe in different frames (see section 4.6).

If $\Omega = -b$, the procedure above yields the second family of solutions, corresponding to G"odel’s universe.

(ii) We have $D = \text{diag}(a, b, a - b)$. Let us set $a = 1$. The only changes with respect to the previous case occur in
\[
QM.3.ii \iff \begin{cases} R_{11} = 2b(1 - b) = 8\pi(\rho + p)u^2 + 4\pi(\rho - p) \\ R_{22} = R_{33} = 0 = -\frac{1}{2} H^2 + 4\pi(\rho - p) \iff H^2 = 8\pi(\rho - p) \end{cases}
\]
From the last equation we obtain
\[
QM.1 \iff 4\pi(\rho - p) = 8\pi(\rho + p)u^2 + 4\pi(\rho + 3p) \implies 8\pi(\rho + p)u^2 = -16\pi p \implies u^2 = -\frac{2p}{\rho + p} \implies (p \leq 0 \text{ and } \rho + p > 0) \text{ or } (p \geq 0 \text{ and } \rho + p < 0).
\]
The second condition implies $\rho - p < -2p \leq 0$, contradicting $H^2 = 8\pi(\rho - p)$. Since $\rho + p > 0$, $H$ and $u$ must have opposite signs and
\[
QM.2 \iff \sqrt{8\pi(\rho - p)} = 16\pi(\rho + p) \sqrt{1 - \frac{2p}{\rho + p}} \sqrt{\frac{-2p}{\rho + p}} \iff \rho - p = 64\pi p(p - \rho).
\]
But $\rho - p = 0 \Rightarrow H = 0 \nRightarrow u = 0 \nRightarrow R_{11} = 0 \iff bc = 0$, which is absurd.
We then have
\[
p = -\frac{1}{64\pi}
\]
and
\[
QM.3.11 \iff \rho = \frac{32b(1 - b) - 5}{64\pi}.
\]
From the equations above we can then obtain the expression for $H^2$ and $u^2$, in the special case $a = 1$. To obtain the general solution and the restrictions over $a$ and $b$, we proceed as in the previous case. This third family of solutions can be shown to be the Ozsváth-Farnsworth-Kerr class I family of solutions. 

(b) Let $u_1 = 0$ ($\Rightarrow H_1 = 0$) and $(u^0)^2 = -\frac{bc}{16\pi(\rho + p)}$. We will prove that there are no solutions satisfying these hypotheses. We start by checking that

$$u^2 + 1 = -\frac{bc}{16\pi(\rho + p)} \Rightarrow 8\pi(\rho + p)u^2 = -\frac{1}{2}bc - 8\pi(\rho + p),$$

and hence

$$QM.1 \Rightarrow H^2 = -bc - 8\pi(\rho - p).$$

On the other hand,

$$QM.3.11 \Rightarrow 8\pi(\rho - p) = R_{11} - \frac{1}{2}bc$$

and so

$$QM.3.22 + QM.3.33 \Rightarrow R_{22} + R_{33} = 4\pi(\rho - p) - 16\pi p \Rightarrow p = \frac{2R_{11} - 4R_{22} - 4R_{33} - bc}{64\pi}.$$ 

It is now immediate that

$$\rho = \frac{10R_{11} - 4R_{22} - 4R_{33} - 5bc}{64\pi}.$$ 

On the other hand,

$$QM.2 \Rightarrow \begin{cases} bH_2 = -16\pi(\rho + p)u^0u_2 \\ cH_3 = -16\pi(\rho + p)u^0u_3 \end{cases} \Rightarrow \begin{cases} b^2(H_2)^2 = [16\pi(\rho + p)]^2(u^0)^2(u_2)^2 \\ c^2(H_3)^2 = [16\pi(\rho + p)]^2(u^0)^2(u_3)^2 \end{cases} \Rightarrow \begin{cases} (H_2)^2 = -16\pi(\rho + p)\frac{b}{c}(u_2)^2 \\ (H_3)^2 = -16\pi(\rho + p)\frac{b}{c}(u_3)^2 \end{cases}$$ 

Using $(QM.2.2)^2$, we get

$$QM.3.22 \Rightarrow R_{22} = \frac{1}{2}(H_2)^2 - \frac{1}{2}H^2 + 8\pi(\rho + p)(u_2)^2 + 4\pi(\rho - p) \Rightarrow \left(1 - \frac{c}{b}\right)8\pi(\rho + p)(u_2)^2 = R_{22} - R_{11}.$$ 

It is easily checked that there are no solutions with $b = c$, and hence

$$8\pi(\rho + p)(u_2)^2 = \frac{b}{b - c}(R_{22} - R_{11}).$$ 

As a consequence of $(QM.2.2)^2$, we have

$$(H_2)^2 = \frac{2c}{c - b}(R_{22} - R_{11}).$$ 

A similar procedure will give us

$$8\pi(\rho + p)(u_3)^2 = \frac{c}{c - b}(R_{33} - R_{11});$$ 

$$(H_3)^2 = \frac{2b}{b - c}(R_{33} - R_{11}).$$ 

From equation

$$(H_2)^2 + (H_3)^2 = -bc - 8\pi(\rho - p)$$

we obtain the restriction

$$-3a^2b + 3a^2c + 4ab^2 - 4ac^2 - b^3 - 4b^2c + 4bc^2 + c^3 = 0.$$
To simplify this last expression we use the Rescaling lemma to set $a = 1$ and divide the resulting polynomial equation by $b - c$, thus obtaining
\[ b^2 + c^2 + 5bc - 4b - 4c + 3 = 0. \]
We have
\[ (u^0)^2 = -\frac{bc}{16\pi(\rho + p)} \Rightarrow bc(\rho + p) < 0. \]
The expression for $(H_3)^2$ implies that $\frac{b}{c}(R_{33} - R_{11}) \geq 0$, and since
\[ 8\pi(\rho + p)(u_2)^2 = \frac{b}{b-c}(R_{22} - R_{11}) \Leftrightarrow 8\pi bc(\rho + p)(u_2)^2 = \frac{b^2c}{b-c}(R_{22} - R_{11}), \]
the restriction implied by the expression for $(u^0)^2$ gives us
\[ \frac{c}{c-b}(R_{22} - R_{11}) \geq 0. \]
If we proceed in a similar fashion using the expressions for $(H_2)^2$ and $8\pi(\rho + p)(u_3)^2$, and then compute the components of the Ricci tensor in terms of $a$ and $b$, we will obtain the following restrictions:
(i) $b^2 + c^2 + 5bc - 4b - 4c + 3 = 0$;
(ii) $c(c-b)(b-1)(b-c+1) \geq 0$;
(iii) $b(b-c)(c-1)(c-b+1) \geq 0$;
(iv) $bc(\rho + p) < 0 \Rightarrow 64\pi bc(\rho + p) < 0 \Rightarrow be[3(b^2 + c^2 - 3bc + 4(b + c) - 7] > 0$.
From (i) we obtain $b^2 + c^2 = -5bc + 4b + 4c - 3$, which when used in (iv) yields
\[ (iv') \ bc(-9bc + 8b + 8c - 8) > 0. \]
It is now easy to use a geometrical argument to determine the incompatibility of restrictions (ii), (iii) and (iv'): we just have to check that the regions determined in the $bc$-plane by these restrictions do not intersect.

(2) It is obvious that
\[ ab = bc = ac \Leftrightarrow a = b = c. \]
Symmetry allows us to consider $u_2 = u_3 = 0$, and thus we are back to the very first case we analyzed.

\[ \square \]

4.4. Solutions with $G = 0$. The next two results complete the classification of $QM$ class $A$ solutions with zero gravitational field.

**Proposition 4.4.** There are no class $A$ $QM$ solutions with zero gravitational field corresponding to Lie algebras with rank $D = 1$.

**Proof.** Let $D = \text{diag}(0, 0, 1)$. Symmetry allows us to take $H = H_2X_2 + H_3X_3 \equiv u = uX_3$. Therefore, $QM.3.23 \Leftrightarrow 0 = H_2H_3$.

If $H_3 = 0 \iff u = 0$, we have
\[ QM.3.11 \Leftrightarrow H^2 = -1. \]
If $H_2 = 0$, we have
\[ QM.3.11 \Leftrightarrow \begin{cases} H^2 = 8\pi(\rho - p) + 1 \\ 8\pi(\rho + p)u^2 = \frac{1}{2} - 4\pi(\rho - p) \end{cases} \]
and
\[ QM.1 \Leftrightarrow \rho = 5p \Rightarrow p \neq 0. \]
Using all this in $QM.2.2$ leads to $p = 0$.

\[ \square \]

**Proposition 4.5.** The only class $A$ solution of $QM$ with zero gravitational field corresponding to a Lie algebra with rank $D = 2$ is Minkowski spacetime.
Proof. We can assume $a = 0$. Thus $QM.2 \Rightarrow u_1 = 0$, and therefore

$$\nabla u = u_2 u_3 (\Gamma_{23}^1 + \Gamma_{32}^1) X_1$$

$$= \frac{1}{2} u_2 u_3 (C_{123} + C_{312} - C_{231} + C_{132} + C_{213} - C_{321}) X_1$$

$$= \frac{1}{2} u_2 u_3 (0 - c + b + 0 + b - c) X_1$$

$$= (b - c) u_2 u_3 X_1.$$

We then have

$$\text{Motion equation} \iff \begin{cases} (b - c) u_2 u_3 = u^0 (u_2 H_3 - u_3 H_2) \\ u_3 H_1 = 0 \\ u_2 H_1 = 0 \end{cases} \Rightarrow H_1 = 0 \text{ or } u = 0.$$ 

It can be easily seen that no solutions exist for $u = 0$, and that solutions featuring $H_1 = 0$ and $u_2 u_3 \neq 0$ must verify $b = c$, and hence are Minkowski spacetime (cf. Theorem 4.2).

We are left with the case $H_1 = 0$ and $u_2 u_3 \neq 0$. Using $QM.2$ we obtain

$$\text{Motion equation} \iff QM.3.ij \ (i \neq j) \iff (u^0)^2 = -\frac{bc}{16\pi(\rho + p)}.$$ 

The situation is now quite similar to the one in the demonstration of case 1b of Theorem 4.3. Using the same procedure we obtain

$$u_2 = \pm \frac{2\sqrt{3}}{3} \sqrt{-\frac{b^2}{-b^2 - c^2 + bc}} \Rightarrow -b^2 - c^2 + bc > 0$$

and

$$H_2 = \pm \sqrt{-2bc} \Rightarrow bc < 0.$$ 

But

$$1 + (u_2)^2 + (u_3)^2 = -\frac{bc}{16\pi(\rho + p)} \iff b^2 + c^2 + 5bc = 0$$

and, therefore,

$$-b^2 - c^2 + bc > 0 \iff b^2 + c^2 + 5bc - 6bc < 0 \iff bc > 0,$$

yielding a contradiction. \hfill \Box

4.5. Solutions with $G \neq 0$. For solutions with $G \neq 0$ we can assume $G = GX_1$ with $G \neq 0$, which implies $H_1 = u_1 = 0$ and $D = \text{diag}(0, b, c)$. It is then easy to see that $QM1 + QM3.22 + QM3.33$ yields

$$G^2 + 4\pi(\rho - 5p) = 0.$$ 

These solutions must of course include the two-parameter family given by

$$H_2 = \sqrt{2}G \cos \theta;$$

$$H_3 = \sqrt{2}G \sin \theta;$$

$$u_2 = -\sin \theta;$$

$$u_3 = \cos \theta;$$

$$p = \rho = \frac{G^2}{16\pi};$$

$$b = c = 0,$$

corresponding to the Gödel universe. Apart from these, one can show that there exist further solutions, belonging to the category 2 of Ozsvath classification (see section 4.6). Unfortunately, it is not possible to obtain simple expressions for these solutions.
4.6. **Identifying the solutions.** Recall that a solution of Einstein’s field equations is said to be spacetime homogeneous if it admits a transitive action by isometries. This will happen if, for instance, the solution is a left-invariant metric on a (four-dimensional) Lie group.

The solutions we have been considering have in fact a Lie group structure, as \( M = \mathbb{R} \times \Sigma \) and \( \Sigma \) is a three-dimensional Lie group.

**Proposition 4.6.** A stationary spacetime \((M, g)\) corresponding to a solution of \(QM\) for which the space manifold \((\Sigma, \gamma)\) is a Lie group with a left-invariant Riemannian metric and whose fields \( G \) and \( H \) are left-invariant is a Lie group with a left-invariant Lorentzian metric.

*Proof.* One just has to check that \( \{X_0, X_i\} \) is a left-invariant orthonormal frame, where \( \{X_i\} \) are the vector fields in \( M \) associated to a left-invariant orthonormal frame on the space manifold. \( \Box \)

Since all spacetime homogeneous perfect fluid solutions which are left-invariant Lorentzian metrics on a Lie group have been classified (see \([6],[7]\)), we can use this classification to identify the solutions we have obtained. One must be careful to use frame-independent quantities when comparing solutions; in most cases it suffices to compare the equations of state.

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