Invariance Properties of Generalized Polarization Tensors and Design of Shape Descriptors in Three Dimensions

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Abstract
We derive transformation formulas for the generalized polarization tensors under rigid motions and scaling in three dimensions, and use them to construct an infinite number of invariants under those transformations. These invariants can be used as shape descriptors for dictionary matching.

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Key words. Generalized polarization tensors, multi-static response matrix, rigid and scaling transformations, invariant, target identification, dictionary matching, shape descriptor.

1 Introduction
Shape of a domain can be represented in terms of various physical and geometric quantities such as eigenvalues, capacity and moments. The generalized polarization tensor (GPT) is one of them. GPTs are an (infinite) sequence of tensors associated with inclusions (domains) and they appear naturally in the far field expansion of the perturbation of the electrical field in the presence of the inclusion. They are geometric quantities in the sense that the full set of GPTs completely determines the shape of the inclusion as proved in [5]. This suggests that GPTs has richer information on the shape than eigenvalues since the full set of eigenvalues does not determine the shape uniquely [10]. Moreover, recent studies [4, 8] show that we can use a first few terms of GPTs to recover a good approximation of actual shape of the inclusion. Even the topology (the number of components) can be recovered. GPTs have been used for imaging diametrically small inclusions and computation of effective properties of dilute composites. We refer to [6] and references therein for these applications. It is worth mentioning that the notion of GPTs has been used not only for imaging but also for invisibility cloaking [7].

When the domain is transformed by a rigid motion or a dilation, the corresponding GPTs change according certain rules and it is possible to construct as combinations of GPTs invariants under these transformations. This property makes GPTs suitable for the dictionary matching problem. The dictionary matching problem is to identify the object in the dictionary when the target object is identical to one of the objects in the dictionary up to shifting, rotation, and scaling. The standard method of dictionary matching is to construct invariants, called shape descriptors, under rigid motions and scaling, and to compare those invariants, and a common way to construct such invariants uses the moments [11]. In the recent paper [4], new invariants are constructed using GPTs in two dimensions: it is shown that a first few terms of GPTs and hence invariants of the target can be computed using the measurement of the multi-static response matrix, and then

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the dictionary matching technique is applied for target identification. Viability of the method is demonstrated by numerical experiments.

It is the purpose of this paper to extend results of [4] to construct invariants using GPTs in three dimensions. In fact, using contracted GPTs (CGPT), which are harmonic combinations of GPTs, we are able to construct an infinite number of shape descriptors which are invariant under rigid motions and scaling. Since the method of above mentioned paper uses the complex structure of the two dimensional space, it can not be applied to three dimensions. Using transformation formulas of spherical harmonics under rigid motions, which can be found in [13] for example, we are able to derive transformation formulas obeyed by CGPTs. We then use these formulas and method of registration (see [14]) to construct invariants which can be used for target shape description and position and orientation tracking [11, 12].

2 Neumann-Poincaré operator and CGPT

For a given bounded domain \( D \) in \( \mathbb{R}^3 \) with the Lipschitz boundary, the Neumann-Poincaré operator \( \mathcal{K}_D \) for a density function \( \phi \in L^2(\partial D) \) and its \( L^2 \)-adjoint \( \mathcal{K}_D^* \) are defined in the principal value by

\[
\mathcal{K}_D[\phi](x) = \frac{1}{4\pi} \int_{\partial D} \frac{(y-x, \nu(y))}{|x-y|^3} \phi(y) d\sigma(y), \quad x \in \partial D,
\]

\[
\mathcal{K}_D^*[\phi](x) = \frac{1}{4\pi} \int_{\partial D} \frac{(x-y, \nu(x))}{|x-y|^3} \phi(y) d\sigma(y), \quad x \in \partial D.
\]

Here \((\cdot, \cdot)\) denotes the scalar product and \( \nu(x) \) the unit outward normal vector along the boundary at \( x \). Let \( \lambda \) be a real number such that \(|\lambda| > 1/2\). The generalized polarization tensor (GPT) \( M_{\alpha\beta} \) for multi-indices \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \) and \( \beta = (\beta_1, \beta_2, \beta_3) \) associated with \( \lambda \) and \( D \) is defined by

\[
M_{\alpha\beta}(\lambda, D) = \int_{\partial D} y^\beta (\lambda I - \mathcal{K}_D^*)^{-1} [\nu \cdot \nabla y^\alpha] d\sigma.
\] (2.1)

Here \( y^\alpha = y_1^{\alpha_1}y_2^{\alpha_2}y_3^{\alpha_3} \). Throughout this paper \( \lambda \) is fixed, so we use \( M_{\alpha\beta}(D) \) for \( M_{\alpha\beta}(\lambda, D) \).

The contracted GPTs (CGPT) are harmonic combinations of GPTs. To be more precise, let \( Y_n^m, -n \leq m \leq n \), be the (complex) spherical harmonic of homogeneous degree \( n \) and order \( m \), i.e.,

\[
Y_n^m(\theta, \varphi) = (-1)^m \left[ \frac{2n+1}{4\pi} \frac{(n-m)!}{(n+m)!} \right]^{1/2} e^{im\varphi} P_n^m(\cos \theta), \quad -n \leq m \leq n,
\]

where \( P_n^m \) are the associated Legendre polynomials of degree \( n \) and order \( m \). If

\[
y^m Y_n^m(\theta, \varphi) = \sum_{|\alpha|=n} a_{\alpha}^{mn} x^\alpha,
\]

then CGPT \( M_{mnkl} \) is defined by

\[
M_{mnkl} = \sum_{|\alpha|=n, |\beta|=l} a_{\alpha}^{mn} a_{\beta}^{kl} M_{\alpha\beta}, \quad m, n, k, l = 1, 2, \ldots
\] (2.2)

In other words, we have

\[
M_{mnkl} = \int_{\partial D} y^k Y_l^m(\theta_y, \varphi_y)(\lambda I - \mathcal{K}_D^*)^{-1} \left[ \frac{\partial}{\partial n} y^m Y_n^m(\theta_y, \varphi_y) \right]_{\partial D} (y) d\sigma(y),
\] (2.3)

where \( y = r_y (\cos \varphi_y \sin \theta_y, \sin \varphi_y \sin \theta_y, \cos \theta_y) \).

We now show that the operator \( \mathcal{K}_D^* \) is invariant under the shift, scaling, and rotation of the domain \( D \), which will be a crucial fact in study of invariance properties of CGPTs in following sections. Let \( z \) be a point in \( \mathbb{R}^3 \), \( s \) a positive number, and \( R \) a \( 3 \times 3 \) orthogonal matrix, and define
It is known that the solution $u$ to the electric potentials with and without the conductivity inclusions, and it's by GPTs. We will investigate invariance of the block matrix consisting of CGPTs rather than individual MSR matrix and the CGPT block matrix. Then one can see easily that the following invariance holds:

$$
\phi^r(y) = \phi(y), \quad \phi^s(y) = \phi(y), \quad \phi^R(y_R) = \phi(y).
$$

Then one can see easily that the following invariance holds:

$$
K^s_{D^2}[\phi^s](x) = K^s_D[\phi](x) \quad (2.4)
$$

$$
K^*_{sD}[\phi^s](sx) = K^*_D[\phi](x) \quad (2.5)
$$

$$
K^*_{sD^*}[\phi^R](x_R) = K^*_D[\phi](x). \quad (2.6)
$$

In fact, for example, we have by the simple change of variables $y = \tilde{y}/s$

$$
K^*_{sD}[\phi^s](sx) = \frac{1}{4\pi} \int_{\partial(sD)} \frac{(sx - \tilde{y}, v(sx))}{|sx - \tilde{y}|^3} \phi^s(y) d\sigma(\tilde{y})
$$

$$
= \frac{1}{4\pi} \int_{\partial D} \frac{s(x - y, v(sx))}{s^3|x - y|^3} \phi(y) s^2 d\sigma(y) = K^*_D[\phi](x).
$$

The other two relation can be seen similarly.

### 3 The MSR matrix and the CGPT block matrix

We will investigate invariance of the block matrix consisting of CGPTs rather than individual GPTs $M_oq$. The way to construct the block matrix can be seen most clearly using the multi-static response (MSR) matrix.

Let $\{x_r\}_{r=1}^N$ and $\{x_s\}_{s=1}^N$ be a set of electric potential point detectors and electric point sources. Let $u_s(x)$ be the solution to the transmission problem in the presence of inclusion $D$

$$
\begin{align*}
\nabla \cdot (1 + (\kappa - 1)\chi_D)\nabla u_s(x) &= \delta_{x_s}(x), \quad x \in \mathbb{R}^2 \setminus \partial D, \\
u(x) \cdot (\nabla u_s)|_{+} &= \kappa \nu(x) \cdot (\nabla u_s)|_{-}, \quad x \in \partial D, \\
u(x) \cdot (\nabla u_s)|_{+} &= \kappa \nu(x) - \Gamma_s(x) = O(|x|^2) \quad |x - x_s| \to \infty,
\end{align*}
$$

(3.1)

where the notation $\phi|_{\pm}(x)$ means the limit $\lim_{t \to 0} \phi(x \pm t\nu(x))$, $\kappa = (2\lambda + 1)/(2\lambda - 1)$, and

$$
\Gamma_s(x) = \Gamma(x - x_s) = \frac{1}{4\pi} \frac{1}{|x - x_s|}.
$$

Without the inclusion $D$ the solution is $\Gamma_s(x)$. The MSR matrix $V$ is the matrix of differences of electric potentials with and without the conductivity inclusions, and its $rs$-components are defined by

$$
V_{rs} = u_s(x_r) - \Gamma_s(x_r), \quad 1 \leq r, s \leq N.
$$

Let $S_D$ be the single layer potential associated with $D$:

$$
S_D[\phi](x) := \int_{\partial D} \Gamma(x - y)\phi(y) d\sigma(y), \quad x \in \mathbb{R}^3.
$$

It is known that the solution $u_s$ to $\Gamma_s$ can be represented as

$$
u
u_s(x) = \Gamma_s(x) + S_D[\phi_s](x) \quad (3.2)
$$
where \( \phi_s \in L^2(\partial D) \) solves

\[
(\lambda I - K_D^c)[\phi_s] = \frac{\partial \Gamma_s}{\partial \nu} |_{\partial D}.
\]

So, the MSR matrix is given by

\[
V_{sr} = \int_{\partial D} \Gamma(x_r - y)(\lambda I - K_D^c)^{-1} \left[ \frac{\partial \Gamma_s}{\partial \nu} \right] (y) d\sigma(y).
\]

Let \( x = r_x (\cos \varphi_x \sin \theta_x, \sin \varphi_x \sin \theta_x, \cos \theta_x) \) and \( y = r_y (\cos \varphi_y \sin \theta_y, \sin \varphi_y \sin \theta_y, \cos \theta_y) \) in spherical coordinates, and suppose that \( r_y < r_x \). Then it is well known (see [13, 12, 9] for example) that

\[
\Gamma(x - y) = -\sum_{l=0}^{\infty} \sum_{k=0}^{l} \frac{1}{2l+1} Y^l_k(\theta_x, \varphi_x) Y^l_k(\theta_y, \varphi_y) \frac{r^l_{l+1}}{r_x^{l+1}}. \tag{3.3}
\]

So, by assuming the inclusion \( D \) is away from the sources, we have from \( \ref{eq:3.3} \)

\[
V_{rs} = \int_{\partial D} \left( \sum_{l=0}^{\infty} \sum_{k=0}^{l} \frac{1}{2l+1} Y^l_k(\theta_x, \varphi_x) Y^l_k(\theta_y, \varphi_y) \frac{r^l_{l+1}}{r_x^{l+1}} \right)
\]

\[
\times (\lambda I - K_D^c)^{-1} \left[ \frac{\partial}{\partial \nu} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{1}{2n+1} Y^m_n(\theta_y, \varphi_y) Y^{-m}_n(\theta_x, \varphi_x) \frac{r^n_{n+1}}{r_x^{n+1}} \right] (y) d\sigma(y).
\]

\[
= \sum_{l=1}^{\infty} \sum_{l=1}^{\infty} \sum_{n=0}^{\infty} \frac{1}{(2l+1)(2n+1)} \int_{\partial D} Y^l_k(\theta_x, \varphi_x) \left( \lambda I - K_D^c \right) Y^m_n(\theta_y, \varphi_y) \frac{r^{l+1}_{l+1}}{r_x^{l+1}} \frac{r^n_{n+1}}{r_x^{n+1}} M_{nlk} \frac{1}{r_x^{n+1}} Y^m_n(\theta_x, \varphi_x). \tag{3.4}
\]

Here we have used the fact that \( M_{nlk} = 0 \) for \( n = 0 \) or \( l = 0 \).

We now introduce matrix \( M_{ln} \) by

\[
(M_{ln})_{km} := M_{nlk}, \quad -l \leq k \leq l, \quad -n \leq m \leq n. \tag{3.5}
\]

We emphasize that the dimension of \( M_{ln} \) is \((2l+1) \times (2n+1)\). We also define \( 1 \times (2l+1) \) and \( 1 \times (2n+1) \) matrices (vectors) \( \mathbf{Y}_{rl} \) and \( \mathbf{Y}_{sn} \) by

\[
(\mathbf{Y}_{rl})_k := Y^l_k(\theta_x, \varphi_x) \frac{1}{(2l+1)r^l_{l+1}}, \quad -l \leq k \leq l, \tag{3.6}
\]

\[
(\mathbf{Y}_{sn})_m := Y^m_n(\theta_x, \varphi_x) \frac{1}{(2n+1)r^n_{n+1}}, \quad -n \leq m \leq n. \tag{3.7}
\]

Then \( \ref{eq:3.4} \) yields, after truncating terms corresponding to \( l > K \) and \( n > K \) for some integer \( K \),

\[
V_{sr} = \sum_{l,n=1}^{K} \mathbf{Y}_{rl} M_{ln} \mathbf{Y}_{sn}^*, \tag{3.8}
\]

where \( * \) denotes the Hermitian transpose \((A^T = (\mathbf{A}^*)^t, t\) for transpose).
Let $N$ be the number of (coincident) source transmitters and receivers and $K$ be the truncation order as before. We further define

$$
M := \begin{bmatrix}
M_{11} & M_{12} & \cdots & M_{1K} \\
M_{21} & M_{22} & \cdots & M_{2K} \\
\vdots & \vdots & \ddots & \vdots \\
M_{K1} & M_{K2} & \cdots & M_{KK}
\end{bmatrix}, \quad Y := \begin{bmatrix}
Y_{11} & Y_{12} & \cdots & Y_{1K} \\
Y_{21} & Y_{22} & \cdots & Y_{2K} \\
\vdots & \vdots & \ddots & \vdots \\
Y_{N1} & Y_{N2} & \cdots & Y_{NK}
\end{bmatrix}.
$$

(3.9)

Then $M$ and $Y$ are $(K^2 + 2K) \times (K^2 + 2K)$ and $N \times (K^2 + 2K)$ matrices, respectively, and the MSR matrix $V$ can be written as

$$
V = YMY^*.
$$

(3.10)

after truncation. We call $M_{\text{In}}$ and $M$ a CGPT matrix and the CGPT block matrix of order $K$, respectively.

**Proposition 3.1** The CGPTs matrix $M$ is Hermitian, i.e., $M = \overline{M^T}$. Furthermore, the matrices $M_{\alpha n}$ are invertible for all $n \geq 1$.

**Proof.** Define the coefficients $a^{kl}_{\alpha}$ and $a^{mn}_{\alpha}$ so that

$$
r^n Y^n_\alpha (\theta, \varphi) = \sum_{|\beta|=l} a^{kl}_{\alpha} y^\beta \quad \text{and} \quad r^n Y^n_{\alpha \beta} (\theta, \varphi) = \sum_{|\alpha|=n} a^{mn}_{\alpha} y^n_\alpha
$$

Then from (2.2) and the symmetry property of $M_{\alpha, \beta}$ on the coefficients of harmonic polynomials [6, Theorem 4.11], we have

$$
M_{\alpha n \beta \delta} = \sum_{|\alpha|=n, |\beta|=l} a^{kl}_{\beta} a^{mn}_{\alpha} M_{\alpha \beta} = \sum_{|\alpha|=n, |\beta|=l} a^{kl}_{\beta} a^{mn}_{\alpha} M_{\alpha \beta} = M_{\lambda \gamma}.
$$

which yields $M_{\text{In}} = M_{\text{In}}$. So $M$ is Hermitian.

To prove the invertibility of $M_{\alpha n}$, it suffices to show $v^* M_{\alpha n} v \neq 0$ for any $v \in \mathbb{C}^{2n+1}$, $v \neq 0$. By definition of the CGPT matrix, we have

$$
v^* M_{\alpha n} v = \sum_{k=1}^{2n+1} \sum_{m=1}^{2n+1} v_k^* M_{nmnk} v_m
$$

$$
= \int_{\partial D} \sum_{k=1}^{2n+1} v_k r^n_\alpha Y^n_\alpha (\theta_y, \varphi_y) (\lambda I - K_D^*)^{-1} \left[ \frac{\partial}{\partial \nu} \sum_{m=1}^{2n+1} v_m r^n_\beta Y^n_{\alpha \beta} (\theta_y, \varphi_y) \right] (y) d\sigma(y).
$$

Note that $\sum_{k=1}^{2n+1} v_k r^n_\alpha Y^n_\alpha (\theta_y, \varphi_y)$ is a harmonic polynomial, and we introduce the coefficients $\gamma_\alpha$ so that

$$
\gamma_\alpha = \sum_{|\alpha|=n} \sum_{k=1}^{2n+1} v_k r^n_\alpha Y^n_\alpha (\theta_y, \varphi_y).
$$

And write $\gamma_\alpha = \gamma^1_\alpha + i \gamma^2_\alpha$, with $\gamma^1_\alpha$ and $\gamma^2_\alpha$ the real and imaginary part, respectively. Remark that both $\gamma^1_\alpha$ and $\gamma^2_\alpha$ are coefficients of some harmonic polynomials. Then using the fact that $v^* M_{\alpha n} v$ is a real number, we get

$$
v^* M_{\alpha n} v = \sum_{|\alpha|=n, |\beta|=n} \gamma^1_\alpha M_{\alpha \beta} \gamma^1_\beta = \sum_{|\alpha|=n, |\beta|=n} \gamma^2_\alpha M_{\alpha \beta} \gamma^2_\beta,
$$

while this quantity is strictly positive if $\lambda > 1/2$, and strictly negative if $\lambda \leq 1/2$ by the positivity of $M_{\alpha \beta}$ [6, Theorem 4.11]. This completes the proof. \(\square\)
4 Transformation formulas for the CGPT matrix

In this section, we derive transformation formulas of the CGPT matrix $M_{in}$ defined in and the CGPT block matrix under rigid motions and dilation. These formulas play an essential role in finding the invariant under these transformations in the sequel.

4.1 Scaling

We consider first the scaling of $M_{in}$. Let us denote

$$\phi_{D, nm}(y) = (\lambda - K^*_D)^{-1} \left[ \frac{\partial}{\partial y} r^n Y^n_n(\theta, \varphi) \right] (y).$$

Using the change of variables $y_s = sy$ ($r_{ys} = sr_y, \theta_{ys} = \theta_y, \varphi_{ys} = \varphi_y$), we obtain

$$M_{nmik}(D) = \int_{\partial D} r^{l}_{ys} Y^l_k(\theta_y, \varphi_y) \phi_{D, nm}(y) d\sigma(y)
= \int_{\partial(sD)} s^{-l} r^{l}_{ys} Y^l_k(\theta_y, \varphi_y) \phi_{D, nm}(y) s^{-2} d\sigma(y_s)
= s^{-(l+2)} \int_{\partial(sD)} r^{l}_{ys} Y^l_k(\theta_y, \varphi_y) \phi_{D, nm}(y) d\sigma(y_s).$$

Since

$$\langle \nu(y), \nabla(r^n Y^n_n(\theta_y, \varphi_y)) \rangle = s^{-n+1} \langle \nu(y_s), \nabla(r^n Y^n_n(\theta_y, \varphi_y)) \rangle,$$

it follows from (2.5) that

$$(\lambda I - K^*_D) [\phi_{D, nm}](y_s) = (\lambda I - K^*_D) [\phi_{D, nm}](y)
= \langle \nu(y), \nabla(r^n Y^n_n(\theta_y, \varphi_y)) \rangle
= s^{-n+1} \langle \nu(y_s), \nabla(r^n Y^n_n(\theta_y, \varphi_y)) \rangle,$$

and hence

$$\phi_{D, nm}^s(y_s) = s^{-n+1} \phi_{D, nm}(y) = s^{-n+1} \phi_{D, nm}(y_s).$$

Thus, we have

$$M_{nmik}(D) = s^{-(l+n+1)} \int_{\partial(sD)} r^{l}_{ys} Y^l_k(\theta_y, \varphi_y) \phi_{D, nm}(y) d\sigma(y) = s^{-(l+n+1)} M_{nmik}(sD).$$

Lemma 4.1 (Scaling) For any positive integers $l, n$ and scaling parameter $s > 0$, the following holds:

$$M_{in}(sD) = s^{l+n+1} M_{in}(D).$$

(4.1)

4.2 Shifting

To deal with a shifting of $M_{in}$, we need the translation of the regular spherical harmonics $r^n Y^n_n(\theta, \varphi)$. For $y = (r, \theta, \varphi), z = (r_z, \theta_z, \varphi_z), y_z = y + z = (r', \theta', \varphi'),$ we have the following expression of the translation of the regular spherical harmonic,

$$r'^n Y'^n_n(\theta', \varphi') = \sum_{(\nu, \mu)} C_{\nu\mu mn} r^{n-\nu} Y^{n-\mu}_n(\theta_z, \varphi_z) r^n Y^n_n(\theta, \varphi),$$

(4.2)

where

$$C_{\nu\mu mn} = \frac{4\pi (2n+1)(n-m)!(n+m)!}{(2n-2\nu+1)(2\nu+1)(n-\nu-m+\mu)!(n-\nu+m-\mu)!(\nu-\mu)!(\nu+\mu)!}^{1/2}.$$
Here we use special summation notation:

$$
\sum_{(\nu, \mu)}^{(n, m) = \min(n, -\nu + n + m)} {\mu = \min(\nu, -\nu + n + m)}
$$

We refer to [13] for a proof of formula (4.2). Using (2.4) and (4.2), we obtain

$$
G_i(l) = nmlk
\sum_{i=0}^{k} \sum_{i=0}^{\mu = \max(-n, -\nu + n + m)}

\text{We compute}

\sum_{i=0}^{k} \sum_{i=0}^{\mu = \max(-n, -\nu + n + m)}

\text{into account, we immediately have the following property.}

\text{For any positive integer } l, n \text{ and the shifting parameter } z, \text{ the following holds:}

\text{Lemma 4.2 (Shifting) For any positive integer } l, n \text{ and the shifting parameter } z, \text{ the following holds:}

$$
M_{ln}(D_z) = \sum_{i=1}^{l} \sum_{\nu=1}^{n} G_{liz}(z)M_{i\nu}(D)G_{n\nu}(z)
$$

\text{To gain better understanding of the matrix } G_{li}, \text{ we compute } G_{21}. \text{ Indeed it will play a role in}

\text{Section 5} \text{ } G_{21} \text{ is given by}

\text{For } k = -2, -1, 0, 1, 2 \text{ (} l = 2 \text{ and } i = 1), \text{ the conditions } \max(-i, -i - l + k) \leq j \leq \min(i, -i + l + k) \text{ can be written as follows:}

$$
k = -2: \max(-1, -1 + k) \leq j \leq \min(1, 1 + k) \Rightarrow j = -1
$$

$$
k = -1: \max(-1, -1 + k) \leq j \leq \min(1, 1 + k) \Rightarrow j = -1, 0
$$

$$
k = 0: \max(-1, -1 + k) \leq j \leq \min(1, 1 + k) \Rightarrow j = -1, 0, 1
$$

$$
k = 1: \max(-1, -1 + k) \leq j \leq \min(1, 1 + k) \Rightarrow j = 0, 1
$$

$$
k = 2: \max(-1, -1 + k) \leq j \leq \min(1, 1 + k) \Rightarrow j = 1
$$

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Then using (4.4) and the definition of the spherical harmonics, we have

\[
G_{21} = \begin{bmatrix}
G_{1,1,2,2} & 0 & 0 \\
G_{1,1,2,1} & G_{1,0,2,1} & 0 \\
G_{1,1,2,0} & G_{1,0,2,0} & G_{1,1,2,0} \\
0 & G_{1,0,2,1} & G_{1,1,2,1} \\
0 & 0 & G_{1,1,2,2}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
C_{1,1,2,2}Y_{\theta,\varphi}^{-1}(\theta, \varphi) & 0 & 0 \\
C_{1,1,2,1}Y_{\theta,\varphi}^{0}(\theta, \varphi) & C_{1,0,2,1}Y_{\theta,\varphi}^{0}(\theta, \varphi) & 0 \\
C_{1,1,2,0}Y_{\theta,\varphi}^{1}(\theta, \varphi) & C_{1,0,2,0}Y_{\theta,\varphi}^{1}(\theta, \varphi) & C_{1,1,2,0}Y_{\theta,\varphi}^{1}(\theta, \varphi) \\
0 & C_{1,0,2,1}Y_{\theta,\varphi}^{1}(\theta, \varphi) & C_{1,1,2,1}Y_{\theta,\varphi}^{1}(\theta, \varphi) \\
0 & 0 & C_{1,1,2,2}Y_{\theta,\varphi}^{1}(\theta, \varphi)
\end{bmatrix}
\]

\[
= \begin{bmatrix}
-\sqrt{r_z}\sin\theta_ze^{-i\varphi_z} & 0 & 0 \\
\sqrt{r_z}\cos\theta_z & -\sqrt{2r_z}\sin\theta_ze^{-i\varphi_z} & 0 \\
\sqrt{2r_z}\cos\theta_z & \sqrt{2r_z}\sin\theta_ze^{-i\varphi_z} & \sqrt{r_z}\cos\theta_z \\
0 & \sqrt{2r_z}\sin\theta_ze^{i\varphi_z} & \sqrt{r_z}\sin\theta_ze^{i\varphi_z}
\end{bmatrix}.
\] (4.6)

On the other hand, from the equation (4.3), one can easily see that the \(G_{njk}\) is non-zero only when \(k = j\), and

\[G_{njk} = C_{njk}Y_0^0(\theta_z, \varphi_z) = \sqrt{4\pi}Y_0^0(\theta_z, \varphi_z) = 1 .\]

Thus, \(G_{nn}\) is the identity \((2n + 1) \times (2n + 1)\) matrix. So, (4.5) yields, for instance,

\[M_{21}(D_z) = G_{21}(z)M_{11}(D) + M_{21}(D) .\] (4.7)

### 4.3 Rotation

Rotations in three dimensions may be described in many different ways. Among them we use the Euler angles which can be conveniently used to represent the rotation formula for spherical harmonics. The rotation \(R\) is given by

\[
R = \begin{bmatrix}
\cos\gamma & -\sin\gamma & 0 \\
\sin\gamma & \cos\gamma & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
\cos\beta & 0 & -\sin\beta \\
0 & 1 & 0 \\
\sin\beta & 0 & \cos\beta
\end{bmatrix} \begin{bmatrix}
\cos\alpha & -\sin\alpha & 0 \\
\sin\alpha & \cos\alpha & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

That is, we rotate by an angle \(\alpha\) about the \(y\)-axis, and by an angle \(\beta\) about the new \(y\)-axis, and finally by an angle \(\gamma\) about the new \(z\)-axis. Let \(R\) be a rotation matrix. Since the homogeneous polynomials and Laplace operator are invariant under rotation, \(Y_n^m(\mathbf{R}\xi)\) is also spherical harmonic of degree \(n\), moreover \(Y_n^m(\mathbf{R}\xi)\) can be written as follows:

\[Y_n^m(\mathbf{R}\xi) = \sum_{m'=-n}^{n} \rho_n^{m',m}Y_n^{m'}(\xi) ,\] (4.8)

where

\[\rho_n^{m',m} = e^{im'\gamma}d_n^{m',m}(\beta)e^{im\alpha} ,\] (4.9)

Here,

\[d_n^{m',m}(\beta) = \frac{[n + m']!(n - m')!(n + m)!(n - m)!]^1/2}{(n + m - s)!(m' + s)!(m + s)!(n - m' - s)!} \times \sum_s \left(\frac{-1}{2}\right)^s \left(\frac{\cos \frac{\beta}{2}}{2}\right)^{2(m' + s) + m - m'} \left(\frac{\sin \frac{\beta}{2}}{2}\right)^{2s + m + m'} .\]

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where the sum is over values \(s\) such that the factorials are nonnegative:

\[
\max(0, m - m') \leq s \leq \min(n - m', n + m).
\]

A proof of (13) can be found in [13].

We want to find the transformation formula \(M_{nmk}, \text{i.e.,} M_{nmk}(D_R)\). We have

\[
M_{nmk}(D_R) = \int_{\partial D_R} \rho^k Y^k_t(R\xi)(\lambda I - K^s_{D_R})^{-1} \left[ \frac{\partial}{\partial \nu} r^m Y^m_n(R\xi) \right]_{\partial D_R} (Ry) d\sigma(y)
\]

\[
= \int_{\partial D} \rho^k \sum_{l=0}^{l} \rho^l \frac{\partial}{\partial \nu} r^m Y^m_n(R\xi) \left( \lambda I - K^s_{D_R} \right)^{-1} \left[ \frac{\partial}{\partial \nu} r^m Y^m_n(R\xi) \right]_{\partial D} (y) d\sigma(y)
\]

\[
= q^m_k M_{ln}(a^m_n), \tag{4.10}
\]

where

\[
a^m_n = (\rho_{n-m}, \ldots, \rho_{n+m}).
\]

Let for each positive integer \(n\)

\[
Q_n = Q_n(R) := \begin{bmatrix}
\rho_{n-n, n} & \rho_{n-n, n+1} & \cdots & \rho_{n-n, n+1} \\
\rho_{n-n, n+1} & \rho_{n-n+1, n} & \cdots & \rho_{n-n+1, n+1} \\
\cdots & \cdots & \cdots & \cdots \\
\rho_{n-n, n+1} & \rho_{n-n+1, n} & \cdots & \rho_{n-n, n+1}
\end{bmatrix}, \tag{4.11}
\]

The rotation matrix \(Q_n\) is called Wigner \(D\)-matrix and known to be is unitary (see [13]).

**Lemma 4.3 (Rotation)** For a unitary matrix \(R\) the following relation holds:

\[
M_{ln}(D_R) = Q(R)M_{ln}(D)Q^t(R). \tag{4.12}
\]

### 4.4 CGPT block matrices and its properties

In addition to the CGPT block matrix, we define two block matrices: Let \(K\) be the truncation order, \(s\) be a scaling parameter, \(z\) shifting factor, and \(R\) rotation, and define

\[
G(R) := \begin{bmatrix}
G_{11} & 0 & \cdots & 0 \\
G_{21} & G_{22} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
G_{K1} & G_{K2} & \cdots & G_{KK}
\end{bmatrix}, \quad Q(s, R) := \begin{bmatrix}
sQ_1 & 0 & \cdots & 0 \\
0 & s^2Q_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & s^KQ_K
\end{bmatrix}, \tag{4.13}
\]

where \(G_{ln} = G_{ln}(z)\) and \(Q_n = Q_n(R)\) are defined by (4.3) and (4.11), respectively. Then we have the following theorem

**Theorem 4.4** Let \(T_z\) be the shift by \(z\), \(T^s\) scaling by \(s\), and \(R\) a unitary matrix. Then the CGPT block matrix \(M\) satisfies

\[
M(T_zT^sR(D)) = sG(z)Q(s, R)M(D)Q(s, R)^tG(z)^t. \tag{4.14}
\]

**Proof.** We have from (4.3) that

\[
M(T_zT^sR(D)) = \overline{G}M(T^sR(D))G^t.
\]
We also have from \([4.11]\) and \([4.12]\) that
\[
M(T^sR(D)) = s \begin{bmatrix}
  s^2M_{11}(R(D)) & \cdots & s^{1+K}M_{1K}(R(D)) \\
  \vdots & \ddots & \vdots \\
  s^{K+1}M_{K1}(R(D)) & \cdots & s^{2K}M_{KK}(R(D)) \\
\end{bmatrix} = s \begin{bmatrix}
  s^2Q_1M_{11}(D)Q_1^t & \cdots & s^{1+K}Q_1M_{1K}(D)Q_1^t \\
  \vdots & \ddots & \vdots \\
  s^{K+1}Q_KM_{K1}(D)Q_1^t & \cdots & s^{2K}Q_KM_{KK}(D)Q_1^t \\
\end{bmatrix}.
\]
So we obtain \([5.1]\).
\[\square\]

5 Transform invariant shape descriptors

In this section, we construct the invariants using CGPT matrices under shifting, scaling, and rotation.

Let \(B\) be a reference domain and \(D\) be the one obtained by rotating \(B\) by \(R\), scaling by \(s\), and shifting by \(z\), i.e.,
\[
D = T_zT^sR(B).
\]

Since \(Q_n\) is unitary, we have \(Q_n^{-1} = Q_1^t\). One can easily see that \(M_{11}\) and \(Q_1\) commute, i.e.,
\[
M_{11}Q_1 = Q_1M_{11}.
\]

Since \(G_{nn} = I_{2(n+1)}\) (\((2n + 1) \times (2n + 1)\) identity matrix), we have
\[
M_{11}(D) = s^3Q_1M_{11}(B)Q_1^t = s^3M_{11}(B),
\]
\[
M_{21}(D) = s^3G_{21}Q_1M_{11}(B)Q_1^t + s^4Q_2M_{21}(B)Q_1^t = G_{21}M_{11}(D) + s^4Q_2M_{21}(B)Q_1^t.
\]

Define
\[
U_D = M_{21}(D)M_{11}(D)^{-1} \quad \text{and} \quad U_B = M_{21}(B)M_{11}(B)^{-1}.
\]

Then we have
\[
U_D = G_{21} + s^4G_{21}U_B Q_1^t.
\]

5.1 Invariance by registration

We first recall that the matrices \(G_{1n}\) and \(Q_n\) are determined by the shift factor \(z\) and rotation \(R\), respectively. Moreover, in view of \([5.4]\), we may write \(G_{21}\) in terms of rectangular coordinates as
\[
\overline{G}_{21}(z) = \sqrt{5} \begin{bmatrix}
  -(z_1 + z_2i) & 0 & 0 \\
  z_3 & -\sqrt{\frac{5}{2}}(z_1 + z_2i) & 0 \\
  \sqrt{\frac{1}{10}}(z_1 - z_2i) & \sqrt{\frac{5}{4}}z_3 & -\sqrt{\frac{5}{10}}(z_1 + z_2i) \\
  0 & \sqrt{\frac{1}{2}}(z_1 - z_2i) & z_3 \\
  0 & 0 & z_1 - z_2i
\end{bmatrix}.
\]

We present here a method of registration to construct invariants. This method is based on a linear mapping \(u : M_{5x3}(\mathbb{C}) \rightarrow \mathbb{C}^5\) (\(M_{5x3}(\mathbb{C})\) is the collection of 5 \(\times\) 3 complex matrices) which satisfies
\[
u(\overline{G}_{21}(z)) = z
\]
\[\text{(5.5)}\]
Lemma 5.2 Let \( \tilde{D} = T_{-u_D} D \). For any indices \( l, n \) and scaling parameter \( s > 0 \) and rotation \( R \), we have

\[
\mathcal{J}_l n (sR(D)) = \mathcal{J}_l n (sR(\tilde{D})) = M_{ln}(sR(\tilde{D})) = s^{l+n+1} M_{ln}(R(\tilde{D})).
\]

In particular, we have

\[
\mathcal{J}_l n (sD) = s^{l+n+1} \mathcal{J}_l n (D),
\]

and

\[
\mathcal{J}_l n (R(D)) = Q_{l n}(R) \mathcal{J}_l n (D) Q_{l n}(R)^T.
\]

Proof. Since \( \mathcal{J}_l n \) is translation invariant, we have for any \( z \in \mathbb{R}^3 \):

\[
\mathcal{J}_l n (sR(D)) = \mathcal{J}_l n (T_z sR(D)) = \mathcal{J}_l n (sR(T_z D)).
\]

Then by taking \( z = -u_D \), we obtain the first identity in (5.11). Moreover, we have from (5.8)

\[
u_{\tilde{D}} = -u_D + u_D = 0.
\]
So we have \( u(sR\tilde{D}) = sRu\tilde{D} = 0 \), which implies the second identity in (5.11). The third one is (4.1). By taking \( R = I \), we have (5.12). (5.13) follows from the definition of \( J_n \) and (4.12). □

In particular, it can be seen from (5.11) that all \( J_{nn} \) are invertible. So we define the second invariant:

\[
S_{ln}(D) := J_{nn}(D)^{-1}J_{nl}(D)J_{ll}(D)^{-1}J_{ln}(D).
\]

(5.14)

It is worth emphasizing that \( S_{ln}(D) \) is a square matrix of dimension \((2n+1)\).

**Proposition 5.3** For any indices \( l, n \), the quantity \( S_{ln}(D) \) is translation and scaling invariant:

\[
S_{ln}(TzsD) = S_{ln}(D) \quad \text{for any } z \in \mathbb{R}^3, \ s > 0.
\]

(5.15)

Moreover, for any rotation \( R \):

\[
S_{ln}(R(D)) = Q_n(R)S_{ln}(D)Q_n(R)^t.
\]

(5.16)

**Proof.** Since \( J_{nn} \)'s are translation invariant, so are \( S_{ln} \)'s. Scaling invariance of \( S_{ln} \) follows from (5.12). Formula (5.16) can be seen using (5.13) as follows:

\[
S_{ln}(R(D)) = Q_n(R)^t J_{nn}(D)^{-1}J_{nl}(D)J_{ll}(D)^{-1}J_{ln}(D)Q_n(R).
\]

This completes the proof. □

Since the Frobenius norm of a matrix remains unchanged after the multiplication by a unitary matrix, so finally we define the shift descriptor

\[
\mathcal{I}_{ln}(D) = \|S_{ln}(D)\|_F
\]

(5.17)

which is clearly invariant by any rotation, scaling and translation.

### 6 Concluding remarks

In this paper we have constructed new shape descriptors in three dimensions which are invariant under translation, rotation, and scaling. These shape descriptors can be used to efficiently identify a target using a dictionary of precomputed CGPTs data. They can be also used for tracking the position and orientation of a mobile three-dimensional target from multistatic measurements.

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