A note on the relationship between action accessible and weakly action representable categories

James Richard Andrew Gray

July 14, 2022

Abstract

The main purpose of this paper is to show that the converse of the known implication weakly action representable implies action accessible is false. In particular we show that both action accessibility, as well as the (at least formally stronger) condition requiring the existence of all normalizers do not imply weakly-action-representability even for varieties. In addition we show that in contrast to both action accessibility and the condition requiring the existence of all normalizers, weakly-action representability is not necessarily inherited by Birkoff subcategories.

1 Introduction

Recall that for a pointed category \( C \), a split extension (of \( B \) with kernel \( X \)) is a diagram in \( C \)

\[
\begin{array}{c}
X \xrightarrow{\kappa} A \xrightarrow{\alpha} B \\
\end{array}
\]

where \( \kappa \) is the kernel of \( \alpha \), and \( \alpha \beta = 1_B \). A morphism of split extensions is a diagram in \( C \)

\[
\begin{array}{c}
X \xrightarrow{\kappa} A \xrightarrow{\alpha} B \\
\downarrow{u} \downarrow{v} \downarrow{w} \\
X' \xrightarrow{\kappa'} A' \xrightarrow{\alpha'} B'
\end{array}
\]

where the top and bottom rows are split extensions (the domain and codomain respectively), and \( v\kappa = \kappa'u \), \( v\beta = \beta'w \) and \( w\alpha = \alpha'v \). Let us denote by \( \text{SplExt}(C) \) the category of split extensions in \( C \), and by \( K \) and \( P \) the functors sending a split extension to its kernel and codomain, respectively. These data together form a span

\[
\begin{array}{c}
\text{SplExt}(C) \xrightarrow{P} C \\
\downarrow{K} \downarrow{C} \\
C
\end{array}
\]
Recall also that when $C$ is pointed protomodular \cite{2}, for each object $X$ in $C$, the assignment of each object $B$ to the isomorphism class of split extensions $E$ with $K(E) = X$ and $P(E) = B$, determines a functor $\text{SplExt}(-, X) : C^{\text{op}} \to \text{Set}$ which assigns to each morphism $p : E \to B$ the morphism $\text{SplExt}(p, X)$ defined by pulling back along $p$. The category $C$ is action representable in the sense of \cite{1} when each of these functors is representable and is weakly action representable \cite{7} when for each $X$ in $C$ there exists a weak representation, that is there is a pair $(M, \mu)$ where $M$ is an object in $C$ and $\mu : \text{SplExt}(-, X) \to \text{hom}(-, M)$ is a monomorphism. Note that when the functor $\text{SplExt}(-, X)$ is representable the representing object will be written $[X]$.

Action representability can also be rephrased as requiring that for each $X$ in $C$ the fiber $K^{-1}(X)$ has a terminal object, and action accessibility \cite{4} can be phrased, by the weakening of this, to instead require that for each $X$ in $C$ the fiber $K^{-1}(X)$ has enough sub-terminal objects, that is, each object admits a morphism into a sub-terminal object (= an object admitting at most one morphism into it). The sub-terminal objects in $K^{-1}(X)$ are called faithful extensions.

In \cite{7}, G. Janelidze proved for a semi-abelian category (in the sense of Janelidze, Marki, Tholen \cite{8}) weakly-action-representability implies action accessibility (Theorem 4.6 of \cite{7}). The main purpose of this paper is to show that the converse does not hold. We show that a Birkhoff subcategory of a (weakly) action representable category is not necessarily weakly action representable. This should be contrasted with the fact that a Birkhoff sub-category of action accessible category is necessarily action accessible \cite{4}, and the immediate Proposition 2.1 below which shows if $C$ is a category admitting all normalizers (in the sense of \cite{5} or in the sense of \cite{3}), then every full subcategory of $C$ closed under sub-objects and finite limits, admits all normalizers. Combining these two facts we show that the each category of $n$-solvable groups ($n \geq 3$) is action accessible and has normalizers, but is not weakly action representable.

\section{The results}

In this section we prove our main results.

Recall that the normalizer of a monomorphism $f : W \to X$ in \cite{5} was defined to be the universal factorization of $f$ as normal monomorphism followed by a monomorphism

$$W \xrightarrow{n} N \xrightarrow{m} X.$$ 

A different definition was given in \cite{3}, which in pointed, finitely complete con-
texted can be formulated as a commutative diagram

\[
\begin{array}{c}
W \xrightarrow{\kappa} R \\
\downarrow f \quad \downarrow (r_1, r_2) \\
N \xrightarrow{(0,1)} N \times N \\
\downarrow m \quad \downarrow m' \\
X
\end{array}
\]

where the upper square is a pullback and the morphisms \( r_1, r_2 : R \to N \) are the projections of an equivalence relation, which is universal amongst such commutative diagrams. The two definitions coincide in the pointed exact protomodular context, where \( r_1, r_2 : R \to N \) is necessarily the kernel pair of its coequalizer, which in turn is necessarily a normal epimorphism with kernel \( n \).

**Proposition 2.1.** Let \( C \) be a pointed category admitting normalizers (in either sense). If \( X \) is a full sub-category of \( C \) closed under subobjects and finite limits, then \( X \) admits normalizers (in the same sense).

*Proof.* It is easy to check that under the conditions above the normalizer in \( C \) of a monomorphism \( f \) in \( X \) is also the normalizer of \( f \) in \( X \). \( \square \)

Recall that a span of monomorphisms \( m : S \to B \) and \( m' : S \to B' \) (sometimes called an amalgum) in a category \( C \) can be amalgamated in \( C \) if there exist monomorphisms \( u : B \to D \) and \( u' : B' \to D \) in \( C \) such that \( mu = m'u' \).

**Proposition 2.2.** Let \( C \) be a action representable category, and let \( X \) be a Birkoff subcategory of \( C \). The category \( X \) is not weakly action representable (and hence not action representable), if there exist monomorphisms \( m : S \to B \) and \( m' : S \to B' \) in \( X \), monomorphisms \( u : B \to D \) and \( u' : B' \to D \) in \( C \), and \( X \) in \( X \) with a monomorphism \( v : D \to [X] \) in \( C \) such that

(i) \( um = u'm' \);

(ii) \( m \) and \( m' \) cannot be amalgamated in \( X \);

(iii) the split extensions corresponding to \( vu \) and \( vu' \) in \( C \) are in \( X \).

*Proof.* The monomorphisms \( vu, vu' \), and \( vmu = vu'm' \) produce the span of faithful extensions in \( X \).
If $X$ were weakly action representable, then $X$ would have weak representation $M$ and there would (by Corollary 4.3 of [7]) be monomorphisms $i : B \to M$ and $i' : B' \to M$ in $X$ such that $im = i'm'$. This is impossible since $m$ and $m'$ can’t be amalgamated in $X$.

Example 2.3. Let $C$ be the category of groups. Recall that: $C$ is action representable [1] with $[X] = \text{Aut}(X)$ (the automorphism group of $X$), $C$ admits normalizers (in the sense of [2] or equivalently – in this context – in the sense of [3]), and amalgamation holds in $C$ (which according to [9] was first proved in [12]). It is well-known that every group can be embedded in the automorphism group of an abelian group (to prove this one can recall that every group can be embedded in the symmetric group on its underlying set, and the symmetric group on a set $W$ can be embedded in $\text{Aut}(\mathbb{Z}_W^2)$). Now let $X$ be the sub-variety of $n$-solvable groups ($n \geq 3$). In [11] B. H. Neumann has shown that there exists an abelian group $S$, a 2-nilpotent group $B$ and two monomorphisms $m : S \to B$ and $m' : S \to B$ which can’t be amalgamated as a solvable group. Since $n$-nilpotent implies $n$-solvable, and split extensions with kernel abelian and codomain 2-solvable are at most 3-solvable it follows by the previous proposition that $X$ is not weakly action representable (take $D$ any group with $u : B \to D$ and $u' : B' \to D$ monomorphisms such that $um = u'm'$ and then take $X = \mathbb{Z}_2^D$). It seems worth pointing out that a finite such $D$ does exist (see e.g. Corollary 15.2 [10]). Action accessibility of $X$ follows from Proposition 2.3 of [4] together with action accessibility of $C$ (which in turn follows immediately from $C$ being action representable). Note that $X$ also has normalizers by Proposition 2.1. Action accessibility of $X$ can then also be obtained from Proposition 4.5 of [3] (see also [6] where it is proved that action accessibility is equivalent to the existence of certain normalizers).

References

[1] F. Borceux, G. Janelidze, and G. M. Kelly, Internal object actions, Commentationes Mathematicae Universitatis Carolinae 46(2), 235–255, 2005.

[2] D. Bourn, Normal subobjects and abelian objects in protomodular categories, Journal of Algebra 228(1), 143–164, 2000.

[3] D. Bourn and J. R. A. Gray, Normalizers and split extensions, Applied Categorical Structures 23(6), 753–776, 2015.

[4] D. Bourn and G. Janelidze, Centralizers in action accessible categories, Cahiers de Topologie et Géométrie Différentielles Catégoriques 50(3), 211–232, 2009.

[5] J. R. A. Gray, Normalizers, centralizers and action representability in semi-abelian categories, Applied Categorical Structures 22(5-6), 981–1007, 2014.

[6] J. R. A. Gray, Normalizers, centralizers and action accessibility, Theory and Applications of Categories 30(12), 410–432, 2015.
[7] G. Janelidze, *Central extensions of associative algebras and weakly action representable categories*, arXiv:2206.02744v3 [math.CT].

[8] G. Janelidze, L. Márki, and W. Tholen, *Semi-abelian categories*, Journal of Pure and Applied Algebra 168, 367–386, 2002.

[9] E. W. Kiss, L. Márki, P. Pröhle, and W. Tholen, *Categorical algebraic properties. A compendium on amalgamation, congruence extension, epimorphisms, residual smallness, and injectivity*, Studia Scientiarum Mathematicarum Hungarica. Combinatorics, Geometry and Topology (CoGeTo) 18(1), 79–140, 1982.

[10] B. H. Neumann, *An essay on free products of groups with amalgamations*, Phil. Trans. Roy. Soc. London (A) 246, 503–554, 1954.

[11] B. H. Neumann, *Permutational products of groups*, Australian Mathematical Society. Journal. Series A. Pure Mathematics and Statistics 1, 299–310, 1959/1960.

[12] O. Schreier, *Die Untergruppen der freien Gruppen*, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg 5(1), 161–183, 1927.