Unitarity in periodic potentials: a renormalization group analysis

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We explore the universal properties of interacting fermionic lattice systems, mostly focusing on the development of pairing correlations from attractive interactions. Using renormalization group we identify a large number of fixed points and show that they correspond to resonant scattering in multiple channels. Pairing resonances in finite-density band insulators occur between quasiparticles and quasiholes living at different symmetry-related wavevectors in the Brillouin zone. This allows a BCS-BEC crossover interpretation of both Cooper and particle-hole pairing. We show that in two dimensions the run-away flows of relevant attractive interactions lead to charged-boson-dominated low energy dynamics in the insulating states, and superfluid transitions in bosonic mean-field or XY universality classes. Analogous phenomena in higher dimensions are restricted to the strong coupling limit, while at weak couplings the transition is in the pair-breaking BCS class. The models discussed here can be realized with ultra-cold gases of alkali atoms tuned to a broad Feshbach resonance in an optical lattice, enabling experimental studies of pairing correlations in insulators, especially in their universal regimes. In turn, these simple and tractable models capture the emergence of fluctuation-driven superconducting transitions in fermionic systems, which is of interest in the context of high temperature superconductors.

I. INTRODUCTION

Fermionic ultra-cold atoms with nearly resonant scattering in the unitarity regime realize the strongest possible form of Cooper pairing, revealed by critical velocity. It is natural to expect that zero-temperature normal states near unitarity would be very strongly correlated with a host of unconventional properties, possibly bearing some resemblance to those found in cuprates. The unitarity limit is therefore an excellent starting point for studies of correlated fermionic superconductors and insulators, which has not been exploited enough in literature. The benefits are both theoretical and experimental. Systematic perturbative and renormalization group calculations are feasible mainly because the unperturbed ground state (fixed point) is a simple state, the vacuum or a band insulator. Experimentally, the unitarity limit is routinely accessed in cold gases of alkali atoms tuned near a broad Feshbach resonance.

Our ultimate goal is to address the long-standing questions about the nature of unconventional normal states proximate to strongly paired fermionic superfluids or superconductors. We design here a simple and tractable model in which the fermionic excitation gap is opened by an external periodic potential, rather than strong interactions. Attractive interactions between quasiparticles whose energy scale exceeds this gap can still give rise to pairing and superfluidity. The need for strong interactions justifies asking if the normal state proximate to the superfluid might have some unconventional properties reflecting strong correlations, especially in universal regimes shaped by resonant scattering.

The inquiry into unconventional superfluidity from the resonant scattering point of view began a long time ago. The present interest in this subject is driven in parallel by a variety of unconventional superconductors in condensed matter physics, and ultra-cold atoms with nearly resonant scattering. The recent theoretical studies of scattering resonances in lattice potentials often rely on two-channel tight-binding models, featuring fermionic atoms resonantly coupled to closed channel bosonic particles. It has been argued that models of this kind provide a good effective description of the microscopic lattice systems of interest. The findings of these studies include lattice Feshbach resonances shifted from their empty-space values.

Most of the mentioned theoretical works approach resonant scattering from a somewhat microscopic angle, exemplified by perturbation theory with the vacuum unperturbed ground-state. Indeed, the usual universal behavior of particles tuned to a broad Feshbach resonance is established in the low density limit. However, universality is a many-body phenomenon and its complete description requires field-theoretical tools such as renormalization group. This issue becomes pressing in the present problem of interest, a band insulator whose non-interacting ground-state contains a macroscopic number of fermions in fully populated bands. These fermions may not be dynamically inert despite the Pauli exclusion principle, due to the strong interactions which bring the system to its unitarity regime.

In this paper we take a field-theoretical approach to nearly resonant pairing between gapped fermions. Abandoning all microscopic details in a renormalization group (RG) calculation allows us to gain a perspective on the generic and universal behavior of a large class of fermionic lattice systems. The effective theories we subject to RG are constructed to preserve the universality class of the microscopic system. This ensures that the universal phase diagram and other macroscopic properties of the microscopic system are correctly captured despite the neglect of microscopic details. The price to pay is
the necessity to deal with multiple flavors of low energy quasiparticles, such as particle and hole excitations which may exist at multiple wavevectors in the first Brillouin zone. Note that the two-channel models with dynamical bosonic fields used in many previous studies are not guaranteed to be in the same universality class as the microscopic system, and may in some cases describe different physics than this paper. We characterize the universality stemming from resonant scattering of quasiparticles in band insulators, and discover generalized unitarity regimes in which quasiparticles of different flavors scatter resonantly. The manifestations of unitarity which we discuss include universal ratios of measurable quantities such as critical temperature, pressure and density. We also analyze the types and conditions for pairing instabilities, conventional versus unconventional superfluid transitions, and emphasize the existence of correlated bosonic Mott insulating states in the phase diagram.

The RG analysis reveals why pairing fluctuations indeed play the crucial role in systems of gapped fermions with short-range attractive interactions. Unlike previous studies of unitarity in continuum lattice potentials, which focused on the zero-density limit, we point out that the unitarity regime in the same universality class can be found at finite densities near any zero temperature band insulator to superfluid transition. The structure of fixed points depends on whether both particles and holes participate equally in the dynamics, or just one of the two quasiparticle types. In the latter case, the exact RG equations can be derived, which allows one to track the run-away flows of attractive interaction couplings (as in some studies of Iron-pnictides). It is these run-away flows that can lead to boson-dominated dynamics at low energies. We find that instabilities in the particle-hole channel are discouraged by attractive interactions.

The run-away flows imply the ultimate RG breakdown when the diverging couplings reach cut-off scales. However, the resulting low energy bosonic dynamics is known to introduce additional fixed points associated with superfluid transitions, which appear as strong-coupling fixed points in the present RG. The superfluid transition in this regime can be either in the bosonic mean-field, or XY universality class. The mean-field universality with dynamical exponent $z = 2$ emerges as the result of run-away flows from the unitarity dominated by either particles or holes, while the $z = 1$ XY universality is related in the same fashion to the unitarity shaped by both particles and holes. Therefore, the analysis here provides a glimpse of the more complete structure of fixed points in theories of fermionic particles with attractive interactions, sketched in Fig.1.

The mentioned finite-density fixed points describe unitarity in zero-density effective theories of particle and hole excitations. Therefore, one can relate a nearly critical interaction strength to scattering lengths in collisions among particles and holes. Any attractive interaction in two dimensions effectively puts the low energy quasiparticles into their Bose-Einstein condensate (BEC) limit, so quasiparticles injected in the insulating state immediately combine into bound-state pairs (whose size can be very large at weak couplings). The effective Bardeen-Cooper-Schrieffer (BCS) regime exists only above two dimensions, at least in the weak coupling limit $\epsilon = 0$ (XY universality). In the latter case, one expects an additional fixed point (XY) in $d \leq 3$. The Gaussian fixed point of the bosonic effective theory appears at the transition line in the limit $U \rightarrow -\infty$, $E_g \rightarrow \infty$. The shaded area is the superfluid or superconducting phase, and the red thick line is the second order superfluid-insulator transition. The dashed green line encloses the region in which the fermionic RG is valid.

FIG. 1: (color online) A hypothetical renormalization group (RG) flow diagram at zero temperature of a fermionic lattice theory with attractive interactions in $d \geq 2$ dimensions. The parameters are fermion density-density interaction $U$ and bandgap (or negative chemical potential) $E_g$. This paper explores in detail the vicinity of two weak-coupling fixed points which govern the pair-breaking superfluid to insulator transition: Gaussian (G) and unitarity (U), separated in proportion to $\epsilon = d - 2$. The negative couplings $U$ below unitarity experience run-away flows under RG and lead to the boson-dominated dynamics. A superfluid-insulator transition in this regime is captured by a bosonic effective theory with dynamical exponent $z = 2$ (mean-field universality) or $z = 1$ (XY universality). In the latter case, one expects an additional fixed point (XY) in $d \leq 3$. The Gaussian fixed point of the bosonic effective theory appears at the transition line in the limit $U \rightarrow -\infty$, $E_g \rightarrow \infty$. The shaded area is the superfluid or superconducting phase, and the red thick line is the second order superfluid-insulator transition. The dashed green line encloses the region in which the fermionic RG is valid.
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A. Model

As a generic model of a band insulator we consider the imaginary-time action of neutral fermionic particles with interactions \( U \), in a lattice potential \( V(\mathbf{r}) \):

\[
S = \int d\tau \left[ \frac{d^d r}{2m} \left( \frac{\partial}{\partial \tau} - \nabla^2 + V(\mathbf{r}) - \mu \right) \psi_\alpha \right] + \int d^d r_1 d^d r_2 U(|\mathbf{r}_1 - \mathbf{r}_2|) \psi_\alpha^\dagger (\mathbf{r}_1) \psi_\alpha (\mathbf{r}_1) \psi_\beta^\dagger (\mathbf{r}_2) \psi_\beta (\mathbf{r}_2) ,
\]

where the summation over spins \( \alpha \in \{\uparrow, \downarrow\} \) is implicit. This is a microscopic multi-band model defined in continuum space and not a priori tied to the vicinity of a critical point. The density of particles is tuned to any number of completely populated bands at zero temperature by placing the chemical potential \( \mu \) in a bandgap. The dynamics of the resulting band insulator can be described by an effective theory of low-energy quasiparticles belonging to the valence and conduction bands. Sufficiently strong attractive interactions \( U \) can drive the system into a superfluid state, and a similar instability can be created by bringing the chemical potential sufficiently close to a band edge, even at weak couplings. A qualitative example of the superfluid-insulator transitions is shown in Fig. 2.

In order to derive the effective theory we formally integrate out the fermion fields from high-energy bands in the path integral. At best, this can be done perturbatively, for example using the Feynman diagram technique. Unless the perturbative integration breaks down, the effective theory takes form

\[
S_{\text{eff}} = \sum_n \int \frac{d\omega}{2\pi} \frac{d^d k}{(2\pi)^d} f_{n,k,\alpha}^\dagger (-i\omega + E_n(\mathbf{k})) f_{n,k,\alpha} + \sum_{n_1,n_2} \sum_{m_1,m_2} U_{n_1,n_2}^{m_1,m_2} \int \frac{d\omega_1}{2\pi} \frac{d^d k_1}{(2\pi)^d} \frac{d\omega_2}{2\pi} \frac{d^d k_2}{(2\pi)^d} \frac{d\Omega}{2\pi} \frac{d^d q}{(2\pi)^d} \times f_{m_1,k_1+q,\alpha}^\dagger f_{n_1,k_1,\alpha} f_{m_2,k_2-q,\beta}^\dagger f_{n_2,k_2,\beta}
\]

in terms of the quasiparticle fermion fields \( f \). In this paper we will consider cases in which the band indices \( n_i, m_i \) denote one or two bands. The most generic transitions are driven by the chemical potential \( \mu \), so that only one band is important (see Fig. 2 (p) and (h) trajectories). The transitions involving particles and holes (the (ph) trajectory in Fig. 2) require at least two bands for a complete description.

A potential problem recognized in a number of studies is that in the vicinity of resonant scattering the microscopic interactions may correspond to energy scales (much) larger than the bandgap. Then, many high energy bands may be significantly hybridized with the conduction and valence bands. In the present formulation of the problem, this can lead to a strong renormalization of

![FIG. 2: (color online) Superfluid transitions (thick red lines) out of a two-dimensional band insulator at zero temperature. The three shown transitions correspond to arbitrarily chosen different strengths of contact attractive interactions, becoming stronger going from top to bottom. The lattice potential is given by \( V(\mathbf{r}) = 2V[\cos(2\pi x/a_L) + \cos(2\pi y/a_L)] \), where \( V \) is the lattice amplitude and \( a_L \) the lattice spacing. Both \( V \) and \( \mu \) are measured in the units of “recoil energy” \( E_0 = \hbar^2/2ma_L^2 \). The thin red line outlines the band edge of the corresponding non-interacting model. The dashed blue lines are trajectories in the parameter space along which the transitions dominated by particles (p), holes (h), or both (ph) can occur.](image-url)
we seek to describe corresponds to the BCS regime in empty space for the same microscopic interactions, where $U/E_{hc}$ is not too large.

II. RENORMALIZATION GROUP ANALYSIS

Here we apply renormalization group (RG) to a band insulator in the unitarity regime. We will identify various fixed points associated with unitarity (resonant two-body scattering) which emerge in the presence of an external periodic potential $V(r)$, but otherwise are analogous to the unitarity fixed point of a uniform system at $V(r) = 0$. The main difference is that the fixed points we shall discuss occur at finite densities of microscopic particles, corresponding to fully occupied bands, whereas universality in the uniform system stems from a zero density fixed point.

There are two characteristic situations which will be considered separately. First, one fermion species (either particles or holes) generally dominates dynamics in the unitarity regime, so the renormalization group equations can be derived exactly to all orders of perturbation theory. This is extremely useful because run-away flows of interaction couplings can be traced more reliably. The second situation is more special and occurs when both particles and holes participate equally in dynamics. Then, the fixed point structure becomes intricate, but can be accessed only in an ($\epsilon$ or large-$N$) expansion. At the end we briefly discuss extensions to more realistic cases with multiple relevant fermion species, and measurable manifestations of the universality class.

A. Transitions involving one fermion species

A transition dominated by either particles or holes, but not both, is generally caused by chemical potential changes as illustrated in Fig.2 with (p) and (h) dashed lines. As a natural starting point one can imagine a band insulator either in the deep BCS limit, or with a very deep lattice potential, where the chemical potential is brought over in that limit, associated with superfluid to Mott-insulator transition with dynamical exponent $z = 2$ at the intersection of the thick red and dashed-blue lines (p) or (h) in Fig.2.

A characteristic weak-coupling interacting fixed point contained in a theory of interacting fermions is unitarity, and it is found at zero temperature when the chemical potential lies exactly at the boundary between a fermion band and a bandgap (or vacuum). The effective action contains a single species of interacting spinful fermions, not much different from the theory of the system without a lattice. If these fermions live in a valence band, we can immediately reformulate the theory in terms of holes. Therefore, we can always write the critical effective theory in the limit of zero quasiparticle density. The universality class at unitarity will be the same as discussed in Ref. 36-38.

The critical theory of interest in $d$ dimensions, augmented by the relevant chemical potential is:

\[
S_1 = \int \mathcal{D}k \ f_{k,\alpha}^\dagger (-i\omega + E(k)) f_{k,\alpha} + U \int \mathcal{D}k_1 \mathcal{D}k_2 \mathcal{D}q \ f_{k_1,\alpha}^\dagger f_{k_1+q,\alpha} f_{k_2,\beta}^\dagger f_{k_2-q,\beta}
\]

where $k = (\omega, \mathbf{k}), \mathcal{D}k = d\omega d^d\mathbf{k}/(2\pi)^{d+1}$, and

\[
E(k) = E_0 + \frac{k^2}{2m}.
\]

All fermion loop Feynman diagrams vanish (the poles of the Green’s functions on the loop lie in the same complex half-plane), and only the interaction coupling is renormalized by a summable geometric progression of ladder diagrams. The exact RG equations are found to be:

\[
\frac{dE_g}{dl} = 2E_g, \quad \frac{dU}{dl} = (2 - d)U - \Pi U^2,
\]

where $l$ is the scale parameter, $E_g = E_0 + U$ is the effective bandgap, and $\Pi$ is a positive cutoff-dependent constant. A fixed point is always found at $E_g = 0, U = 0$. An additional non-trivial fixed point is found in $d \neq 2$ at $E_g = 0, U = U^* = (2 - d)\Pi^{-1}$, which describes attractive interactions if $d > 2$. The schematic flow of interaction couplings is shown in Fig.3.

Any attractive interaction $U < 0$ in $d = 2$ has a runaway flow to $U \to -\infty$, while in $d > 2$ it needs to be large enough to flow toward $U \to -\infty$. Repulsive interactions $U > 0$, on the other hand, flow to the Gaussian fixed point in $d \geq 2$. Therefore, only attractive interactions can produce strongly correlated states. Since the RG equations are exact, we can precisely characterize the run-away flow, assuming that the effect of multi-body collisions remains negligible at least in the insulating state.

FIG. 3: The RG flow of the interaction coupling $U$ in the theory 39.
Solving for $U(l)$ in two and three dimensions we obtain:

$$U(l) = \begin{cases} \frac{U(0)}{1+|U(0)|^{-2}} & , d = 2 \\ \frac{U(0)}{1+|U(0)|^{-3}} & , d = 3 \end{cases} \tag{6}$$

In both cases, the run-away flows have vertical asymptotes so that $U(l)$ diverges at a finite value of $l$ ($l = |U(0)|^{-1}$ in $d = 2$). This indicates that Cooper pairs become stable at a finite length scale and have a finite coherence length despite the fermion bandgap. However, the interpretation of the run-away flow breaks-down at the cut-off scale because so large $U$ will pair-up the high-energy fermions, which were assumed to be unpaired in this RG procedure. A boson-dominated dynamics takes over the shortest length-scales under consideration.

Note that $U(l) \to -\infty$ at a finite $l$ cannot be immediately interpreted as a signal of superfluidity. This is due to the fact that at finite $l$ we do not yet have a theory which transparently describes dynamics at macroscopic scales, while superfluidity is verified only by macroscopic long-range correlations. Since RG is based on integrating out high energy modes, it does not provide a precise answer to the question of what phase the system lives in, but only gives an indication.

The fermion gap in an insulating state grows exponentially under RG, $E_g(l) = E_g(0) e^{2l}$. If $E_g(l)$ is the first to reach the cutoff scale, the further RG flow is halted (RG breaks down) in a state apparently devoid of particles. This is a band insulator. It is obtained in $2 + \epsilon$ dimensions for any sufficiently weak interaction $|U(0)| < |U^{*}| \sim \epsilon$, or even for $|U(0)| > |U^{*}|$ provided that the gap $E_g(0)$ is large enough. If $U(l)$ is the first to reach its cut-off instead, then boson-dominated dynamics at shortest length-scales requires switching to a purely bosonic effective theory in order to determine what happens at large length-scales. Both insulating and superfluid phases are possible in this limit, despite a finite fermion gap, but the transition between them is in a different universality class than the BCS pair-breaking transition.

B. Transitions involving particles and holes

A special case is obtained in the vicinity of vanishing gaps for both particle and hole excitations. A pairing transition influenced by a corresponding fixed point cannot be obtained by changing the chemical potential alone, but can be accessed by tuning interaction strength or lattice depth, at a fixed particle density (see Fig 2 the (ph) dashed line). The values of $\mu$ and $V$ should lie at the intersection of effective conduction and valence bands where the effective bandgap closes. We shall again discover a run-away flow of interaction couplings, but this time it quickly invalidates the perturbative RG. The run-away flow is expected to eventually lead to a strong-coupling fixed point in the XY universality class, associated with the superconductor to Mott-insulator transition at an integer number of bosons per lattice site.

The critical theory for valence ($v$) and conduction ($c$) electrons is:

$$S_2 = \sum_n \int Dk f_{n,k,\alpha}^{\dagger} (-i\omega + E_n(k)) f_{n,k,\alpha} + \sum_{n_1 n_2} \sum_{m_1 m_2} U_{n_1 n_2}^{m_1 m_2} \int Dk_1 Dk_2 Dq \times f_{m_1, k_1 + q, \alpha}^{\dagger} f_{n_1, k_1, \alpha} f_{m_2, k_2 - q, \beta} f_{n_2, k_2, \beta} \tag{7}$$

where $n \in \{v, c\}$ and

$$E_v(k) = -E_v0 - \frac{k^2}{2m_v}, \quad E_c(k) = E_c0 + \frac{k^2}{2m_c}.$$  

Here we made the simplest assumption that the bandgap $E_0 = E_{v0} + E_{c0} \geq 0$ is direct and small (or vanishing) at only one wavevector in the Brillouin zone. Modifications of this assumption are straight-forward and the appropriate more realistic circumstances will be discussed in the following section. Performing a particle-hole transformation for the valence band cannot help us construct an exact RG procedure. Instead, it is convenient to work directly with the native particle degrees of freedom.

The interaction couplings $U_{n_1 n_2}^{m_1 m_2}$ in the band representation are derived from microscopic short-range interactions in real-space. For example, a pure contact interaction $U \psi_{i}^{\dagger}(r) \psi_{\alpha}(r) \psi_{\beta}^{\dagger}(r) \psi_{\beta}(r)$ in $3$ gives:

$$U_{n_1 n_2}^{m_1 m_2}(k_1, k_2, q) = \int_{UC} d^3r u_{m_1, k_1 + q}(r) u_{n_1, k_1}(r) u_{m_2, k_2 - q}(r) u_{n_2, k_2}(r) \tag{8}$$

where UC indicates integration over the lattice unit-cell, and $\psi_{n,k}(r) = u_{n,k}(r)e^{ikr}$ are Bloch wave-functions. This expression illustrates an important property of interactions in the band representation which follows from the overlap features of the Bloch wavefunctions. As a rule of thumb, the couplings $U_{n_1 n_2}^{m_1 m_2}$ are largest by magnitude if $n_i = m_i$ for both $i = 1, 2$ and smallest if $n_i \neq m_i$ for both $i = 1, 2$. The strongest interaction channels involve a single band, while the interband couplings are weaker. This is a natural situation for generic band structures and short-range interactions, but it could be reversed in principle.

It is important to note that the spatial dependence of the microscopic interaction potential $U(r)$ on the distance $r$ between the interacting particles is not automatically irrelevant (in the RG sense) in the presence of the lattice. Short-range variations of $U(r)$, at or below the lattice spacing length-scale, affect the relative strength of the couplings $U_{n_1 n_2}^{m_1 m_2}$, which may lead to non-trivial interacting fixed points as discussed below. Only the long-range variations and the related crystal momentum dependence of the interaction couplings are not relevant in the vicinity of the fixed points of interest.

After normal ordering the interactions take the form:
\[ \sum_{n_1, m_1, n_2, m_2} U_{n_1 n_2}^{m_1 m_2} \int Dk_1 Dk_2 Dq \; f_{m_1, k_1 + q, \alpha} f_{n_1, k_1, \alpha} f_{m_2, k_2 - q, \beta} f_{n_2, k_2, \beta} = \]

\[ \sum_{n_1, m_1, n_2, m_2} U_{n_1 n_2}^{m_1 m_2} \int Dk_1 Dk_2 Dq \; f_{m_1, k_1 + q, \alpha} f_{n_1, k_1, \alpha} f_{m_2, k_2 - q, \beta} f_{n_2, k_2, \beta} + \sum_{n m} U_n^m \int Dk \; f_{m, k, \alpha} f_{n, k, \alpha} \]  

The generated quadratic terms \( U_n^m = n_0/2 \times \sum_{m'} U_{n m'}^m \), where \( n_0 \) is the particle density in the ground-state, are similar to a “charging energy” since they effectively shift the chemical potential as a result of interactions. However, they also couple the particles in the two bands, so we must redefine the bare Green’s function. One way to accomplish this is to treat \( \text{two bands} \), so we must redefine the bare Green’s function \( [i\omega - E_n(k)]^{-1} \delta_{nm} \):

\[
G_n^m(k, i\omega) = \left( i\omega - E_c(k) - U_n^m \right) \left[ i\omega - E_v(k) - U_v^m \right]^{-1} = \frac{g_n^m(k, i\omega)}{(i\omega - \xi_1(k))(i\omega - \xi_2(k))}.
\]

It is convenient to define \( \xi_n(k) = E_n(k) + U_n^m \) and \( \xi = \sqrt{U_c^m U_v^m} \) (note that \( U_c^m = (U_v^m)^* \)). Then:

\[
\xi_1/2(k) = \frac{\xi_c + \xi_v}{2} \pm \left[ \left( \frac{\xi_c - \xi_v}{2} \right)^2 + \xi^2 \right]^{1/2}
\]

are the new poles of the bare fermion excitations, and

\[ g_n^m(k, i\omega) = (i\omega - \xi_c - \xi_v + \xi_n)\delta_{nm} + \xi(1 - \delta_{nm}). \]

It is easy to show that both poles are always real, one being positive (particle-like) and the other negative (hole-like) as long as \( \xi^2 > \xi_c^2 \xi_v \). We will assume that the latter condition is satisfied, so that the system is a band insulator despite the “charging energy”. Consequently, we can expand the poles up to \( \mathcal{O}(k^2) \):

\[
z_1(k) = \xi_1 + \frac{k^2}{2M_1}, \quad z_2(k) = -\xi_2 - \frac{k^2}{2M_2},
\]

where \( \xi_i \) are the bare quasiparticle gaps, and \( M_i \) are the quasiparticle masses given by:

\[
M_1/2 = \alpha \times \frac{m_c^{-1} + m_v^{-1}}{2} \pm \frac{m_c^{-1} - m_v^{-1}}{2}.
\]

The parameter

\[
\alpha = \frac{E_g}{\sqrt{E_g^2 + 4\xi^2}}
\]

captures the amount of mixing between the two bands \((0 \leq \alpha \leq 1); \ E_g = (\xi_c - \xi_v)|_{k=0} \) is the effective fermion bandgap. For \( \alpha = 1 \) there is no band mixing since \( M_i \in \{m_c, m_v\} \). In general, \( \alpha > |\beta| = |m_c - m_v|/(m_c + m_v) \) is required in order for both \( M_i \) to remain positive. Otherwise, band inversion is caused by large interband couplings and it must be taken into account by redefining the low-energy quasiparticles, which then live at some different momenta in the first Brillouin zone. We shall come back to this situation at the end.

Now we set up the RG. As usual, we keep the masses \( m_c \) and \( m_v \) in the absence of interactions fixed under RG. While this does not imply that \( M_i \) will be fixed, it sets the scaling dimension for the field operators to \( d/2 \). The scaling of coordinates and couplings

\[
r' = re^{-l}, \quad r' = re^{-2l}, \quad \epsilon_i' = \epsilon_r e^{2l}, \quad U' = U e^{-(2-d)|l|},
\]

is followed by the diagrammatic integration of high-energy fields living at all Matsubara frequencies and momenta within a shell \( |k| \leq \Lambda e^{-\delta l} \). The resulting one-loop renormalization of the quadratic and quartic couplings is summarized in Table 1. The relevant cutoff-dependent renormalization scales are:

\[
K_{1k} = S_d \Lambda^d \alpha - (-1)^k \left( \frac{2\pi}{d} \right)^{d/2} 2^n, \]

\[
K_{2kk} = S_d \Lambda^d \left( m_c m_v (1 + \alpha^2 - 2\delta_{kk}) \right) \left( \frac{2\pi}{d} \right)^{d/2} 2^n (m_c + m_v), \]

\[
K_{2kk}'' = S_d \Lambda^d \left( (m_c + m_v)(\alpha^2 - 1) + 4m_c \delta_{kk} \right) \left( \frac{2\pi}{d} \right)^{d/2} 2^n (m_c + m_v)^2 - (m_c - m_v)^2).
\]

The RG equations involving all four \( U_n^m \) and all sixteen \( U_{n_1 n_2}^{m_1 m_2} \) couplings (not all of which are independent) are too complicated to be fully solved. A part of the problem is that the parameter \( \alpha \) can also flow under RG, as a result of the renormalization of the couplings \( U_n^m \). In order to simplify notation let us absorb the bare fermion gaps into the quadratic “charging” couplings since they flow the same way under RG: \( U_n^m = U_n^m + E_{n0} \). We begin by noting that the RG equation for the quadratic couplings is:

\[
\frac{dU_n^m}{dl} = 2U_n^m - \sum_k K_{1k} \left( U_{kn}^{mk} + U_{nk}^{km} - 2U_{nk}^{mk} - 2U_{kn}^{km} \right).
\]

In \( d = 2 + \epsilon \) dimensions the interacting fixed points will be at \( U_{n_1 n_2}^{m_1 m_2} \propto \epsilon \), implying \( U_n^m \propto \epsilon \). Finite values for
the Bloch wavefunction properties, the intraband behavior, a repulsive one keeps growing, while an attractive one stays constant. However, analytical solutions for a subset of fixed points can be found if the couplings are small enough. In two dimensions, there is only one weak-coupling fixed point, at $U_n = 0$, $U_{mn} = 0$. The flow of $U_{cc}$, $U_{cv}$ and $U_{vv}$ is determined by the same equations as in two dimensions, and their complex conjugates are fixed at $U_{cc} = U_{vv}$ and $U_{cv} = U_{vc}^*$. The attractive intraband interaction $U_{cc}$ and $U_{vv}$ are of the same type as in two dimensions, and their complex conjugates are fixed at $U_{cc} = U_{vv}$ and $U_{cv} = U_{vc}^*$. The attractive intraband interaction $U_{cc}$ and $U_{vv}$ have the opposite behavior, a repulsive one stays constant, while an attractive one grows. In normal circumstances, due to the Bloch wavefunction properties, the intraband couplings $U_{cc}$ and $U_{vv}$ are larger than the interband $U_{cv}$ and $U_{vc}$, so the attractive intraband channels dominate at macroscopic scales and lead to Cooper pairing even if the interband channels are repulsive. Instabilities in the particle-hole channel are possible only if all interactions are repulsive, or if for some reason $U_{cv}$ is repulsive and stronger than the attractive intraband interactions.

In $d = 2 + \epsilon$ dimensions with $\epsilon > 0$ it is convenient to define:

$$m_c = m(1 + \beta), \quad m_v = m(1 - \beta), \quad \beta = \frac{m_c - m_v}{m_c + m_v}$$

and the rescaled independent dimensionless couplings $(u_c, u_v, u_{cv}, u_m, u_e, e_g)$:

$$U_{cc} - U_v^\gamma = \frac{\Lambda^2 \epsilon}{m} e_g$$

$$U_{cc} = K \epsilon \frac{u_c}{1 + \beta}, \quad U_{vv} = K \epsilon \frac{u_v}{1 - \beta}, \quad U_{cv} + U_{vc} = K \epsilon \frac{u_m}{1 - \beta^2}, \quad U_{cc} = K \epsilon \frac{u_{cv} e^{i\theta}}{\sqrt{1 - \beta^2}}, \quad U_{vv} = K \epsilon \frac{u_{cv} e^{-i\theta}}{\sqrt{1 - \beta^2}}$$

where $K = (2\pi)^d/(S_d \Lambda^m)$. These interactions are diagrammatically represented in the Table II. The RG equations for $\alpha = 1$ are:

$$\frac{du_c}{dl} = \epsilon \left( -u_c - 4u_c^2 - 4u_v^2 \right)$$

$$\frac{du_v}{dl} = \epsilon \left( -u_v - 4u_c^2 - 4u_v^2 \right)$$

$$\frac{du_{cv}}{dl} = \epsilon \left( -u_{cv} + 2u_{cv}^2 + 8(1 - \beta^2) u_c^2 \right)$$

$$\frac{du_m}{dl} = \epsilon \left( -u_m - 4u_m^2 + 4u_{cv} u_m \right)$$

$$\frac{du_e}{dl} = \epsilon \left( -u_e + u_e \left( -4u_c - 4u_v + 8u_{cv} - 4u_m \right) \right)$$

$$\frac{d e_g}{dl} = 2 e_g - \frac{2u_v}{1 - \beta} + \frac{2u_{cv}}{1 - \beta^2} - \frac{u_m}{1 - \beta^2}$$

Above two dimensions, there are seventeen fixed points with $\alpha = 1$. Sixteen of these fixed points $F_1 - F_{16}$ are
given by all possible combinations of:

\[ u_c \in \{0, -\frac{1}{4}\}, \quad u_v \in \{0, -\frac{1}{4}\}, \quad u_e = 0 \]

\[ (u_{cv}, u_m) \in \left\{ \left( \frac{1}{2}, 0 \right), \left( \frac{1}{2}, \frac{1}{4} \right), \left( 0, -\frac{1}{4} \right), \left( 0, 0 \right) \right\} \]

Note that here \( u_e \) is always zero. The RG eigenvalues in the subspace of \( (u_c, u_v, u_{cv}, u_m) \) are \( \pm \epsilon \) at all of these fixed points, and allow enumerating \( F_1 - F_{16} \) simply by the variations of relevant/irrelevant flows of \( (u_c, u_v, u_{cv}, u_m) \). This is illustrated in Fig.3 for the first fifteen fixed points at which at least one of the \( u_c, u_v, u_{cv}, u_m \) couplings is zero. Only the Gaussian fixed point is fully stable, while \( F_{16} \) with all \( u_c, u_v, u_{cv}, u_m \) non-zero is fully unstable. The coupling \( u_e \) is irrelevant only at the Gaussian fixed point, while it is found to be marginal at \( (u_c, u_v, u_{cv}, u_m) \in \{(-1/4, 0, 0, 0), (0, -1/4, 0, 0), (0, 0, 0, -1/4)\} \) and relevant otherwise.

The remaining fixed point \( F_{17} \) is the only one with \( u_e > 0 \):

\[ u_e^2 = \frac{15}{64 \left( 11 - 4 \beta^2 + 8 \sqrt{4 \beta^4 - 7 \beta^2 + 4} \right)}, \quad u_m = 0 \]

\[ u_c = u_v = u_{cv} = -\frac{1}{8} \left( \frac{3}{5 - 4 \beta^2 + 2 \sqrt{4 \beta^4 - 7 \beta^2 + 4}} \right) \]

It has only one relevant direction with RG eigenvalue \( \epsilon \), the \( u_c \) component being the largest in the corresponding eigenvector. The RG flow in the vicinity of this fixed point is illustrated in Fig.4. Note that formally there are other solutions stemming from (20), but they have \( u_e^2 < 0 \) corresponding to time-reversal symmetry violations.

Whenever \( u_c, u_v, u_{cv}, u_m \) are relevant, their RG eigenvalue is \( \epsilon \), the same as the RG eigenvalue at the unitarity fixed point of a uniform zero-density system which corresponds to vacuum resonant scattering. While this is not surprising when it comes to pairing of two quasiparticles \( u_c \) or two quasiholes \( u_v \), it is interesting to note that the same resonant scattering interpretation can be applied to the couplings \( u_{cv} \) and \( u_m \). We can identify the resonantly scattering quasiparticles by tuning to a fixed point with only one finite coupling and then taking a closer look at the operator corresponding to that coupling. In the case of \( u_m \), the operator is

\[ f_1^{a} f_\alpha^{d} f_\beta^{d} f_\gamma^{d} f_{\Phi} = -f_1^{a} f_\alpha^{d} f_\beta^{d} f_{\Phi} f_{\alpha} \]

\[ = \left| \Phi_{\gamma} \right|^2 - \left| \Phi_{\alpha} \right|^2 - \left| \Phi_{\beta} \right|^2 - \left| \Phi_{\gamma} \right|^2 \]

where the operator \( \Phi_{\gamma} = (f_{\beta} f_{\gamma} - f_{\gamma} f_{\beta})/\sqrt{2} \) annihilates an interband singlet and the operators \( \Phi_{\beta} = f_{\gamma} f_{\beta}, \Phi_{\gamma} = f_{\beta} f_{\gamma} \) and \( \Phi_{\beta} = (f_{\gamma} f_{\beta} + f_{\beta} f_{\gamma})/\sqrt{2} \) annihilate triplet pairs. The fixed point(s) at \( u_m < 0 \) can now be associated with the resonant scattering in the interband singlet Cooper channel. Note that the absence of fixed points at \( u_m > 0 \) rules out resonant scattering in the attractive triplet channel (the fixed point at \( u_{cv} = 1/2, u_m = 1/4 \) is fully repulsive in the particle-particle channel).

The interaction \( u_{cv} > 0 \) at its resonant-scattering fixed point is repulsive in the particle-particle channel and cannot lead to a Cooper pair resonance. However, it becomes attractive in the particle-hole channel. Keeping only \( u_{cv} \) finite allows performing a particle-hole transformation in the valence band, after which the theory contains two similarly dispersing fermion fields (particles and holes) in their vacuum states, interacting attractively. Denoting the particle and hole annihilation operators as \( f_{\alpha} \) and \( f_{\alpha}^{\dagger} \) respectively, where \( \alpha \) is the opposite spin

![FIG. 4: The fixed points and RG flow of interaction couplings](image)

![FIG. 5: The fixed points and RG flow involving \( u_m \neq 0 \) for \( u_m = 0, u_v = u_c \). The shaded semi-infinite surface encloses the basin of attraction of the Gaussian fixed point.](image)
of $\alpha$, the $u_{cv}$ operator can be written as
\[
f_{\alpha a}^f f_{\gamma b}^f f_{\delta c}^f = -f_{\alpha a}^f f_{\gamma b}^f f_{\delta c}^f
\]
\[
= -|B_a|^2 - |B_{\gamma a}|^2 - |B_{\delta a}|^2.
\]
Now the operators $B_s = (f_{\alpha} f_{\gamma} - f_{\delta} f_{\gamma})/\sqrt{2}$, $B_{\gamma \delta} = f_{\gamma} f_{\delta}$, $B_{\gamma \beta} = f_{\gamma} f_{\beta}$ and $B_{\alpha 0} = (f_{\alpha} f_{\beta} + f_{\delta} f_{\gamma})/\sqrt{2}$ annihilate singlet and triplet particle-hole pairs. This interaction does not make any distinction between different spins, so that a scattering resonance appears simultaneously in the singlet and all triplet channels. The bound-state resulting from this resonance is an exciton, and symmetry breaking at finite particle and hole density can be either a singlet exciton condensate, or a ferromagnetic state, depending on the other couplings as well as higher order terms in the action. In both cases, the present effective theory would favor ordering at zero wavevector, but the circumstances discussed in the following section could lead to antiferromagnetic and other kinds of ordering at finite wavevectors.

The behavior of $u_c$ does not fit this generic resonant scattering picture. Only at the $F_{17}$ fixed point we find the flow of $u_c$ reminiscent of resonant scattering. The absence of other similar fixed points with $u_c \neq 0$, and the fact that the relevant direction at $F_{17}$ is an almost even-amplitude linear combination of multiple couplings, indicate different physics: an “assisted scattering resonance” in the Cooper channel between a pair of fermions dynamically resonating between the conduction and valence bands. In fact, assuming $\theta = 0$ in \[19\], a sufficiently strong interaction of this type would give rise to an extended “sign-changing” $s$-wave superfluidity in which the pairing gap on the conduction and valence bands has opposite signs. Such an $s^\pm$ pairing is proposed to occur in iron pnictides\[20\]. Other kinds of pairings with different relative phases between the conduction and valence band pairing gaps could be obtained for other values of $\theta$.

The run-away flows in the vicinity of these fixed points are also very important. They indicate the kinds of instabilities of interacting fermions in lattice potentials and circumstances in which they can develop. This information has greater practical use than the detailed properties of the fixed points, because realistic systems can hardly be tuned very close to these fixed points (except cold atom systems which are tunable to the $u_c = -1/4$ and/or $u_{cv} = -1/4$ fixed points). In generic lattice systems with attractive interactions we find that the favored phases are featureless insulators and superconductors. A singlet superconductor is indicated by the flow of interaction couplings $u_\alpha$, $u_{\gamma}$, and $u_m$ toward $-\infty$, although as emphasized in the previous section such run-away flows can also produce bosonic Mott insulators in certain cases. Instabilities in the particle-hole channel are discouraged in normal circumstances with attractive microscopic interactions. Even if the interband couplings end up having repulsive character, the generic lattice and microscopic interaction potentials produce relatively small $u_{cv}$ in comparison to $u_c$ and $u_{cv}$, so that a typical system with attractive interactions in $d + \epsilon$ dimensions flows either to a charge-dynamics influenced insulator state, or toward particle-particle instabilities. With repulsive interactions, however, the same kind of flows near the fixed points featuring $u_{cv}$ take the system either to spin-dynamics influenced insulators, or toward the particle-hole instabilities.

Finding the full structure of fixed points for any $\alpha > |\beta|$ requires allowing all mixing interband couplings to be finite. Preliminary numerical calculations indeed reveal the existence of additional fixed points with finite mixing interactions and $\alpha < 1$. However, a systematic search for these fixed points is very difficult due to the large parameter space and the highly non-linear nature of the RG equations that allow $\alpha$ to flow. The details of these fixed points are not crucial for the present discussion and will not be pursued further.

Now we return to the possibility of band inversion which occurs for $\alpha < |\beta|$. First, we note that in normal microscopic circumstances $\alpha$ is close to unity because the intraband couplings $U_n^m$ are larger than the interband ones $U_n^m$, $n \neq m$. The RG flow further accentuates this situation as the flow of all $U_n^m$ is exponential. However, if the interband couplings are large enough in comparison to the intraband ones, we must cure the resulting effective band inversion by identifying the true low energy quasiparticles, which must live at some new crystal wavevectors. The appropriate RG equations need to deal with more than two fermion flavors. Attractive interactions in such strong interband channels would naturally lead to paired states which spontaneously break translational symmetry, while repulsive interactions would give rise to patterned exciton condensates.

C. Transitions involving multiple fermion species, and universality classes

The lowest energy quasiparticles in band insulators are often concentrated around multiple symmetry-related wavevectors in the Brillouin zone. For example, the simple cubic periodic potential in three dimensions
\[
V(r) = 2V \left[ \cos \left( \frac{2\pi x}{aL} \right) + \cos \left( \frac{2\pi y}{aL} \right) + \cos \left( \frac{2\pi z}{aL} \right) \right]
\]
produces a band insulator with two fermions per site (for not too small $V$) whose lowest hole excitations live at $k_\alpha = (\pi, \pi, \pi)/aL$ in the valence band and lowest particle excitations live at $k_{c1} = (\pi, 0, 0)/aL$ and two other symmetry-related wavevectors $k_{c2}, k_{c3}$ in the conduction band. An effective fermionic theory of this band insulator requires either one hole or three particle fields for generic pairing transitions of the type discussed in section II A. The discussion in section \[11\] has to be extended to one hole and three particle fields in this case.

An effective theory will generally include couplings among all of its fermion fields, and some of the couplings will have the same value by symmetries. As a prototype theory we can take the action \[2\] allowing the la-
bels $n, m\ldots$ to identify any relevant fermion flavor. Like before, the RG analysis would reveal fixed points and run-away flows corresponding to same-flavor pairing and flavor-mixing instabilities. The latter kind could lead to supersolid phases in the particle-particle channel, or exciton condensates in the particle-hole channel, both bringing translational symmetry breaking and new universality classes. On the other hand, the same-flavor pairing instabilities are the most likely outcome of generic attractive interactions in lattice potentials due to the typically dominant couplings in the same-flavor channel.

All superfluid transitions in the unitarity limit which involve only the same-flavor pairing always belong to the same universality class. This universality class can be characterized by critical ratios between pressure, temperature, energy per particle, and chemical potential (relative to the band edge) at small but finite quasiparticle densities. A useful way of calculating the critical ratios involves applying a Hubbard-Stratonovich transformation to the model \( \Phi \) to decouple the short-range interaction, and then promoting the obtained two-channel model to an \( \text{Sp}(2N) \) symmetry group by introducing \( N \) copies of the spinful fermion fields which couple to the same Hubbard-Stratonovich field. Fluctuation corrections to the mean-field thermodynamic functions take the form of 1\( /N \) expansions, so at least in the limit of large \( N \) one can obtain systematic perturbative expressions in the absence of a natural small parameter near unitarity. Taking the physical value \( N = 1 \) and including only the lowest order correction (“Gaussian fluctuations” of the order parameter) already produces very good estimates in the uniform system\(^{37,40}\).

Provided that the inter-flavor scattering vanishes at the fixed point, trivial adjustments are needed to accommodate multiple fermion flavors in the presence of a lattice, most notably in the ratios derived from extensive quantities, such as those containing pressure and energy density. For example, the critical pressure \( P \) at the finite-temperature \( T = T_c \) superfluid transition (in \( d = 3 \))

\[
\frac{(P - P_0)/N}{(2m)^{d/2}T^{d/2}N_f} \bigg|_{T = T_c} = 0.13188 + \frac{0.4046}{N} + \mathcal{O}(1/N^2)
\]

acquires a factor of \( n_f \) in the denominator on the left-hand side, the total number of low-energy particle and hole flavors in the Brillouin zone (\( P_0 \) is the zero-temperature degeneracy pressure of the band insulator). The value of \( n_f \) depends on the bandgap \( E_g \); in the limit \( T_c \gg E_g \) both particle and hole flavors should be counted in \( n_f \), otherwise only particles or holes are important based on the chemical potential.

Another small adjustment of the uniform system \( 1/N \) expansions in Ref\(^{37,40}\) is needed in the critical ratios involving the chemical potential. We need to express the chemical potential \( \mu \) relative to the nearest band edge. If the conduction band is nearest, then the finite quasiparticle density at zero temperature is obtained when \( \mu > 0 \), while finite density requires \( \mu < 0 \) if the valence band is nearest. The critical temperatures at both \( \mu > 0 \) and \( \mu < 0 \) are universal functions of \( |\mu| \):

\[
\frac{|\mu|}{T} \bigg|_{T = T_c} = 1.50448 + \frac{2.785}{N} + \mathcal{O}(1/N^2) .
\]

This expression applies even in the limit \( T_c \gg E_g \) when both particles and holes are important, because this limit can be interpreted as \( |\mu| \gg E_g \). If the lattice depth is so small that the bandgap closes (\( E_g = 0 \)), we must take the larger of the two values of \( |\mu| \) obtained by measuring the chemical potential with respect to the overlapping “conduction” and “valence” band edges. Additional phase transitions below \( T_c \) are possible for \( T_c \sim |\mu| \gg E_g \), involving the onset of pairing in different channels: particle, hole and interband, each characterized by its own order parameter (see previous section). Re-entrant behavior can be anticipated in this regime when only one fermion species is paired at \( T = 0 \) and another one is separated from the chemical potential by a gap smaller than \( T_c \). Then, the thermal population of the fermions across the gap can lead to pairing in additional channels at \( 0 < T_c' < T < T_c \).

Anisotropy associated with low-energy quasiparticles at symmetry-transforming wavevectors in the Brillouin zone is equally easily treated. For any quasiparticle flavor with dispersion

\[
E(k) = E_0 + \sum_{i=1}^{d} \frac{k_i^2}{2m_i} .
\]

we redefine momentum so that \( k_i^2/2m = k_i^2/2m_i \), where \( m \) is a mass to be determined. The measure in path integrals acquires a factor of \( \sqrt{\prod_i m_i}/m^d \) from this change of variables, which can be absorbed into the redefinition of matter fields. This also leads to a renormalization of all interaction couplings. The choice

\[
m = \left( \prod_i m_i \right)^{\frac{1}{d}}
\]

converts the quasiparticle dispersion into an isotropic one without renormalizing any fields or couplings. It is therefore this geometric mean which should replace the mass in all \( 1/N \) expansions.

III. DISCUSSION AND CONCLUSIONS

We considered a band insulator subjected to pairing in the unitarity regime as a model system. The simplest realization of such a system is found in trapped neutral ultracold gases of alkali atoms placed in an optical lattice. The density of atoms can be chosen to correspond to two atoms per lattice site in the central portion of the trap, while the strength of attractive interactions among them is routinely controlled by the Feshbach resonance. A superfluid transition from a thermally excited
band insulator has been already experimentally studied in this kind of a system in the vicinity of the BCS-BEC crossover.2

The focus of our analysis was the characterization of the universal phase diagram featuring $T = 0$ transitions between band insulators and superfluid states. In $d > 2$ dimensions we identified a BCS limit in which this transition is pair-breaking, meaning that its universal properties are transparently captured by a BCS-like theory. A special limiting case of the pair-breaking transition is found at unitarity, where all interaction effects become independent of microscopic scales, leading to the universal dependence of critical temperature and other thermodynamic functions on the particle density in the superfluid state.

The BEC limit, found at any interaction strength in $d = 2$ or at sufficiently strong interactions in $d > 2$, brings a different universality class to superfluid transitions. Fermionic excitations belong to high energies, so the effective theory capturing the transition has only bosonic fields. The transition occurs between the superfluid and a bosonic Mott insulator. The universality class is characterized either by the dynamical exponent $z = 2$ (generic bosonic mean-field transitions driven by the chemical potential), or $z = 1$ (XY transitions driven at fixed density of two fermions per lattice site).

The BCS and BEC limits considered here are relative to a particular band insulator with a fixed lattice potential and particle density at zero temperature. While the particle density is finite in the ground-state, the unitarity regime between these BCS and BEC limits is found at zero quasiparticle density in the effective low-energy theory describing the band insulator. The full microscopic model includes short-range interactions and multiple fermion bands, with the chemical potential residing in a bandgap. Integrating out high-energy fermions leaves behind the effective theory featuring at most two bands immediately adjacent to the chemical potential (the conduction and valence bands). The remaining low energy fermions experience renormalized interactions, and may exist in multiple flavors as quasiparticles and quasiholes concentrated around different symmetry-related wavevectors in the first Brillouin zone (individually having anisotropic dynamics). All of this complexity reduces to a few relevant interaction couplings in the vicinity of renormalization group (RG) fixed points that signify universal behavior, the most naturally occurring ones corresponding to unitarity in the same universality class as if the system were microscopically uniform.

The physical meaning of these fixed points, revealed by RG, is the resonant scattering of quasiparticles. Multiple flavors of quasiparticles give rise to multiple possibilities for resonant scattering. An interesting discovered possibility is the resonant scattering between particles and holes in the presence of repulsive interactions, the unitarity limit in the particle-hole channel separating the regimes with non-existing and existing exciton bound states (excitonic “BCS” and “BEC” regimes respectively). Other possibilities not elaborated here also exist in generic circumstances with multiple fermion flavors, leading to translational symmetry breaking in ordered states. However, most of these universal regimes may be inaccessible in realistic systems because they involve tuning either the details of lattice potentials, or short-range spatial features (at the lattice spacing scales) of the microscopic interaction potential.

The notable exception are unitarity regimes in the uniform particle-particle and hole-hole channels, which can be reached in cold atom systems using Feshbach resonances. The simplest to obtain is the unitarity at a transition driven by the chemical potential, which is naturally found in a trapped gas of cold atoms at an interface between the superfluid and insulating atom clouds. In this case the RG identifies only one relevant interaction parameter, which is tuned by the Feshbach resonance. The transitions driven at fixed density by changing the interaction strength or lattice depth are harder to push to the full unitarity because there are two RG relevant operators (particle-particle and hole-hole scattering lengths) which need to be tuned to their fixed point values. Nevertheless, manifestations of this kind of unitarity can be observed at finite temperatures if the critical temperature is larger than the bandgap.

The RG also provides an indication of the macroscopic properties of states away from the fixed points. If the strength of attractive interactions $U$ is smaller by magnitude than its fixed-point value $|U^*| \propto 1/d$ in $d + \epsilon$ dimensions, then a gapped fermion system is macroscopically a band insulator. Otherwise, the coupling $U$ flows toward $-\infty$ under RG at finite length scales, implying the formation of Cooper pairs at short length scales before the onset of superfluidity at large length scales. It is in this manner that the fermionic RG predicts the existence of bosonic Mott insulators, but a bosonic effective theory is then required to access the superfluid transition at macroscopic scales.

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transitions among fermions and the corresponding universality classes can be traced down to the well-known physics of BCS-BEC crossovers. Even if interactions are not strong enough to bring the system close to its unitarity limit in empty space, the presence of a lattice frustrates the motion of particles and promotes interaction effects, effectively pushing the system toward its lattice unitarity\textsuperscript{18,27}. Furthermore, in two dimensions there is no BCS limit strictly speaking. Two quasiparticles injected into the conduction band will form a bound state regardless of how weak the attractive interactions are. Of course, the size of this “vacuum” bound state might be much larger than the spacing between particles, but this does not preclude the bosonic universality of the superfluid transition.

One potentially important aspect of this is that a conceptually similar situation is found in cuprate high temperature superconductors. Cuprates are quasi two-dimensional systems in which the underdoped normal state (pseudogap) exhibits gapped fermionic quasiparticles, albeit with a specific d-wave pairing symmetry and a gap of completely different origin than in this paper. A number of unconventional properties of cuprates can be qualitatively understood as being related to a fluctuation-driven transition.

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