GENERAL DECAY OF THE SOLUTION TO A NONLINEAR VISCOELASTIC MODIFIED VON-KÁRMÁN SYSTEM WITH DELAY

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Abstract. In this paper we consider a viscoelastic modified nonlinear Von-Kármán system with a linear delay term. The well posedness of solutions is proved using the Faedo-Galerkin method. We use minimal and general conditions on the relaxation function and establish a general decay results, from which the usual exponential and polynomial decay rates are only special cases.

1. Introduction. In this paper, we are concerned with the following nonlinear modified Von-Kármán system with time delay and a memory term,

\[
\begin{align*}
\rho h D \psi_{tt} + \psi_{xxxx} - \left[ \psi_x (\eta_x + \frac{1}{2} \psi_x^2) \right]_x - \psi_{xxt} - g \ast \psi_{xxxx} &= 0, \\
\rho h \eta_{tt} - [\eta_x + \frac{1}{2} (\psi_x)^2]_x + \alpha_1 \eta_t + \alpha_2 \eta(t - \tau) &= 0,
\end{align*}
\]

where \( D = (I - \frac{h^2}{12} \Delta^2) \) and the interval \( I = (0, L) \) is the segment occupied by the beam. The unknowns \( \psi = \psi(x, t) \), and \( \eta = \eta(x, t) \) represent, respectively, the vertical displacement, and the longitudinal displacement at time \( t \) of the cross section located \( x \) units from the end-point \( x = 0 \).

In (1), subscripts mean partial derivatives and \( h > 0 \) is a parameter related to the rotational inertia of the beam.

When \( \alpha_1 = \alpha_2 = 0 \), this system describe approximately the planar motion of a uniform prismatic beam of length \( L \) with memory term. Here, \( h \) and \( \rho \) are two strictly positive constants represent respectively the thickness and the mass density per unit volume of the beam. In the system (1), \( \alpha_1 \eta_t \) represents a frictional damping. The time delay is given by \( \alpha_2 \eta(t - \tau) \), where \( \alpha_1, \alpha_2, \tau \) are positive constants.

In (1), \( (g \ast f)(t) \) is defined by

\[
(g \ast f)(t) = \int_0^t g(t - s)f(x, s)ds.
\]

This integral term or the viscoelastic damping term that appears in the equations describes the relationship between the stress and the history of the strain in the
beam, according to Boltzmann theory. The function \( g \) represents the kernel of the memory term or the relaxation function.

To the system (1) we add the boundary conditions
\[
\begin{align*}
\psi(0, .) &= \psi(L, .) = 0, \\
\psi_x(0, .) &= \psi_x(L, .) = 0, \\
\eta_x(0, .) &= \eta_x(L, .) = 0 \text{ in } \mathbb{R}^*.
\end{align*}
\] (2)

and the initial initial conditions
\[
(\psi(., t), \psi_t(., 0), \eta(., 0), \eta_t(., 0)) = (\psi_0(., t), \psi_1, \eta_0, \eta_1) \text{ in } \mathbb{R}. \] (3)

The main purpose about problems (1) − (3) is to deal with the well posedness and asymptotic behavior of solutions. Before stating and proving our results, let us recall some other results related to our work.

Several authors have studied the Mindlin-Timoshenko system of equations. This Model is a widely used and fairly complete mathematical model for describing the transverse vibrations of beams. It is a more accurate model than the Euler-Bernoulli one, since it also takes into account transverse shear effects. The Mindlin-Timoshenko system is used, for example, to model aircraft wings (see, e.g., [23]).

For a beam of length \( L > 0 \), this one-dimensional nonlinear system reads as
\[
\begin{align*}
\frac{\hbar}{12} \phi_{tt} - \phi_{xx} + k [\phi + \psi_x] &= 0, \text{ in } Q, \\
\rho h \psi_{tt} - k [\phi + \psi_x]_x + \left[ \psi_x \left( \eta_x + \frac{1}{2} \psi_x^2 \right) \right]_x &= 0, \text{ in } Q, \\
\rho h \eta_{tt} - \left[ \eta_x + \frac{1}{2} (\psi_x)^2 \right]_x &= 0, \text{ in } Q,
\end{align*}
\] (4)

where \( Q = (0, L) \times (0, T) \) and \( T \) is a given positive time. Here, the unknown \( \phi = \phi(x, t) \) represent the angle of rotation. The parameter \( k \) is the so called modulus of elasticity in shear. It is given by the expression \( k = \hat{k} E h / 2 (1 + \epsilon) \), where \( \hat{k} \) is a shear correction coefficient, \( E \) is the Young’s modulus and \( \epsilon \) is the Poisson’s ratio, \( 0 < \epsilon < 1/2 \).

For Mindlin-Timoshenko system, there is a large literature, addressing problems of existence, uniqueness and asymptotic behavior in time when some damping effects are considered, as well as some other important properties (see [18, 30, 32] and references therein).

When one assumes the linear filament of the beam to remain orthogonal to the deformed middle surface, the transverse shear effects are neglected, and one obtains, from the Mindlin-Timoshenko system of equations, the following von Kármán system (see [32]).
\[
\begin{align*}
\rho D \psi_{tt} + \psi_{xxxx} - \left[ \psi_x (\eta_x + \frac{1}{2} \psi_x^2) \right]_x &= 0, \text{ in } [0, L] \times \mathbb{R}^*_+ , \\
\rho h \eta_{tt} - \left[ \eta_x + \frac{1}{2} (\psi_x)^2 \right]_x &= 0, \text{ in } [0, L] \times \mathbb{R}^*_+.
\end{align*}
\] (5)

There is also an extensive literature about system (5) (see [18, 30, 25, 31, 43, 54, 56, 57, 58, 59] and references therein).

Lagnese and Leugering [31] considered a one-dimensional version of the von Kármán system describing the planar motion of a uniform prismatic beam of length \( L \). More precisely, in [31] the following system was considered:
\[
\begin{align*}
\psi_{tt} + \psi_{xxxx} - h \psi_{xxxx} - \left[ \psi_x (\eta_x + \frac{1}{2} \psi_x^2) \right]_x &= 0 \text{ in } [0, L] \times \mathbb{R}^*_+, \\
\eta_{tt} - \left[ \eta_x + \frac{1}{2} (\psi_x)^2 \right]_x &= 0 \text{ in } [0, L] \times \mathbb{R}^*_+. 
\end{align*}
\] (6)
In [31], suitable dissipative boundary conditions at \( x = 0, x = L \) and initial conditions at \( t = 0 \) were given and the stabilization problem was studied.

In [4], Araruna et al. have showed how the so called von Kármán model (6) can be obtained as a singular limit of a modified Mindlin-Timoshenko system (4) when the modulus of elasticity in shear \( k \) tends to infinity, provided a regularizing term through a fourth order dispersive operator is added. Introducing damping mechanisms, the authors also show that the energy of solutions for this modified Mindlin-Timoshenko system decays exponentially, uniformly with respect to the parameter \( k \). As \( k \to \infty \), the authors obtain the damped von Kármán model with associated energy exponentially decaying to zero as well.

**Remark 1.** Since \( k \) is inversely proportional to the shear angle, we note that neglecting the shear effects of the beam is formally equivalent to considering the modulus \( k \) tending to infinity in the Mindlin-Timoshenko system.

The subject of stability of von Kármán system has received a lot of attention in the last years. It is important to mention that the authors in [17, 25, 33] proved uniform decay rates for the von Kármán system with frictional dissipative effects in the boundary. The stability for a von Kármán system with memory and boundary memory conditions was treated in [25, 44, 53, 38]. They proved the exponential or polynomial decay rate when the relaxation function decay is at the same rate. As for the works about general decay for viscoelastic system, we refer the reader to [14, 55] and references therein.

Delay effects are very important because most natural phenomena are in many cases very complicated and do not depend only on the current state but also on the past history of the system. The presence of delay can be a source of instability. In recent years, the stabilization of PDEs with delay effects has draw attention for many author and become an active area of research (see [28, 14, 22, 51, 52, 60, 62, 64, 63]).

In [9], Benaissa et al. in studied a system of viscoelastic wave equations with a linear frictional damping term and a delay

\[
\begin{align*}
\sigma_{tt}(x,t) - \Delta u(x,t) + \int_0^t g(t-s) \Delta u(x,s) ds \\
+ \alpha_1 h_1(u_t(x,t)) + \alpha_2 h_2(u_t(x,t - \tau)) &= 0, \quad \text{in } \Omega \times \mathbb{R}_+,
\end{align*}
\]

\[
\begin{align*}
u(x,t) &= 0, \quad \text{in } \Gamma \times \mathbb{R}_+,
\end{align*}
\]

\[
\begin{align*}
u(x,0) = u_0(x) \quad \nu_t(x,0) = \nu_t(x), \quad \text{in } \Omega,
\end{align*}
\]

\[
\begin{align*}
u(x,t - \tau) = f_0(x,t - \tau), \quad \text{in } \Omega \times [0, \tau[,
\end{align*}
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \), \( n \in \mathbb{N} \), with a smooth boundary \( \partial \Omega = \Gamma \), \( g \) is a positive non-increasing function defined on \( \mathbb{R}^n \), \( h_1 \) and \( h_2 \) are two functions, \( \tau > 0 \) is a time delay, \( \alpha_1 \) and \( \alpha_2 \) are positive real numbers and the initial data \((u_0, u_1, f_0)\) belong to a suitable function space.

In the case \( g \equiv 0 \), problem (7) has been studied by many authors (see [51, 64, 8]).

In the case \( g \neq 0 \), Cavalcanti et al. [13] studied (7) for \( h_2 \equiv 0 \) and with a linear localised frictional damping \( a(x) u_t \). This work was later improved by Berrimi and Messaoudi [10] by introducing a different functional which allowed them to weaken the conditions on \( g \).
For a wider class of relaxation functions, Messaoudi \cite{41, 42} considered

\[ u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds = b | u |^\gamma u, \]  

(8)

for \( \gamma > 0 \) and \( b = 0 \) or \( b = 1 \), and the relaxation function satisfies

\[ g'(t) \leq -\zeta(t)g(t), \]  

(9)

where \( \zeta \) is a differentiable nonincreasing positive function. He established a more general decay result, from which the usual exponential and polynomial decay results are only special cases. Such a condition was then employed in a series of papers, see for instance \cite{26, 46, 47, 55}.

Recently, Mustafa and Messaoudi \cite{50} studied the problem (8) with \( b = 0 \) for the relaxation functions satisfying

\[ g'(t) \leq -H(g(t)), \]  

(10)

where \( H \) is a nonnegative function, with \( H(0) = H'(0) = 0 \) and \( H \) is strictly increasing and strictly convex on \( [0,k] \) for some \( k_0 > 0 \). The authors showed a general relation between the decay rate for the energy and that of the relaxation function \( g \) without imposing restrictive assumptions on the behavior of \( g \) at infinity.

On the other hand, a condition of the form (10) where \( H \) is a convex function satisfying some smoothness properties, was introduced by Alabau-Boussouira and Cannarsa \cite{3} and used then by several authors with different approaches. We refer to \cite{37} where not only general but also optimal result was established by Lasiecka and Wang.

Our purpose in this paper is to give a global solvability in Sobolev spaces and energy decay estimates of the solutions to the problem (1) for linear damping, time delay terms and finite memory. We would like to see the influence of frictional and viscoelastic dampings on the rate of decay of solutions in the presence of linear degenerate delay term.

Our aim is to investigate (1) for relaxation functions \( g \) of more general type than the ones in (9) and (10). We consider the condition

\[ g'(t) \leq -\zeta(t)H(g(t)), \]  

(11)

where \( H \) is increasing and convex without any additional constraints on \( H \) and the coefficients, and establish energy decay results that address both the optimality and generality. The energy decay rates are optimal in the sense that they decay qualitatively the same as the viscoelastic kernels \( g \) do.

To obtain global solutions of problem (1)–(3), we use the Galerkin approximation scheme (see Lions \cite{39}) together with the energy estimate method. The technique based on the theory of non linear semigroups used in Nicaise and Pignotti \cite{51} does not seem to be applicable in our case.

To prove decay estimates, we use a perturbed energy method and some properties of convex functions. In order to accomplish this goal, we shall pursue a strategy based on an adaptation of non linear differential inequalities technique developed in \cite{45, 48, 49}. Arguments of convexity were introduced and developed by many authors \cite{12, 20, 35, 36, 40, 24, 2}.

We observe that our problem is set in a context where:

**a:** The memory damping is defined only on the equation for the vertical displacement.
b: The presence of a frictional damping and a time delay on the equation for the longitudinal displacement.

c: Energy decay estimates under a nonlinear tension.

Our work is organized as follows. In the next section, we prepare some material needed in the proof of our result, like some lemmas (Poincaré’s and Young’s inequalities) and some useful notations. We introduce the different functionals by which we modify the classical energy to get an equivalent useful one. In Section 3, we state and prove the well-posedness of the problem. Finally, in Section 4, we will prove our main results concerning the exponential decay of the energy associated to the solutions of the problem.

2. Statement of results. As in [51], we introduce the new variable

\[ z(x, t, p) = \eta(t - pt, x), \]

which satisfies

\[
\begin{align*}
\tau z_t(x, t, p) + z_p(x, t, p) &= 0 \quad \text{in } Q \times (0, 1), \\
z_x(x, t, p) &= 0 \quad \text{in } \Sigma \times (0, 1), \\
z_0(x, p) &= z(x, p, 0) = f_0(x, -pt) \quad \text{in } I \times (0, 1),
\end{align*}
\]

where \( \Sigma = \{0\} \times \mathbb{R}_+ \cup \{L\} \times \mathbb{R}_+ \).

Therefore, problem (1) – (3) is equivalent to

\[
\begin{align*}
\rho h D\psi_{tt} + \psi_{xxxx} - \left[ \psi_x(\eta_x + \frac{1}{2} \psi_x^2) \right]_x - \psi_{xt} - g \ast \psi_{xxxx} &= 0 \quad \text{in } Q \\
\rho \eta_{tt} - (\eta_x + \frac{1}{2} \psi_x^2)_x + \alpha_1 \eta_t + \alpha_2 \psi(1) &= 0 \quad \text{in } Q \\
\tau z_t(p) + z_p(p) &= 0, \quad \text{in } Q \times (0, 1)
\end{align*}
\]

with Boundary conditions

\[
\begin{align*}
\psi(x, t) &= \psi_x(x, t) = \eta_x(x, t) = 0 \quad \forall (x, t) \in \Sigma, \\
z_x(x, t, p) &= 0 \quad \forall (x, t, p) \in \Sigma \times (0, 1), \\
\end{align*}
\]

and initial conditions

\[
\begin{align*}
(\psi(\cdot, 0), \psi_t(\cdot, 0), \eta(\cdot, 0), \eta_t(\cdot, 0)) &= (\psi_0, \psi_1, \eta_0, \eta_1, f_0(\cdot, -pt)) \text{ on } I, \\
z(\cdot, 0, p) &= z_0 = f_0(\cdot, -pt) \text{ on } I \times (0, 1).
\end{align*}
\]

2.1. Assumptions. To state and prove our result, we use the following assumptions:

\( A_1 \) To preserve the hyperbolicity of our system, we assume that the kernel is such that \( g \in L^1(\mathbb{R}^+) \cap C^2(\mathbb{R}^+) \) with \( g(0) > 0 \) and

\[
l = 1 - \int_0^\infty g(s)ds > 0. \tag{15}\]

\( A_2 \) There exists a \( C^1 \) function \( H : (0, \infty) \rightarrow (0, \infty) \) which is linear or strictly increasing and strictly convex \( C^2 \)-function on \([0, r) \ (r \leq g(0)) \) with \( H(0) = H'(0) = 0 \), such that

\[
g'(t) \leq -\zeta(t)H(g(t)), \quad \forall t \geq 0, \tag{16}\]

where \( \zeta \) is a positive nonincreasing differentiable function.
Now, we prepare some notations and hypotheses which will be needed in the proof of our result. Let $L^2(0,L)$ be the usual Hilbert space with the inner product $(\cdot,\cdot)$ and the inner product induced norm $\|\cdot\|$. Throughout this paper, we define

$$H^2_0(\Omega) = \{ v \in H^2(\Omega), v(0) = v(L) = v_x(0) = v_x(L) = 0 \}$$

equipped with the norm $\|w\|_{H^2_0(\Omega)} = \|w_{xx}\|$,.

$$H^1_1(\Omega) = \{ v \in H^1(\Omega), \int_0^L v(x)dx = 0 \}$$

equipped with the norm $\|w\|_{H^1_1(\Omega)} = \|w_x\|$, and

$$W = \{ v \in H^2_0(\Omega) \cap H^4(0,L)/v_{xx}(L) = 0 \}$$
equipped with the norm $\|v\|_{H^1_1(\Omega)}$.

$C$ and $c$ denote some general positive constants, which may be different in different estimates.

The following inequality will be used repeatedly in the sequel.

**Lemma 2.1.** We have

$$ab \leq \delta a^2 + \frac{1}{4\delta} b^2, \quad a, b \in \mathbb{R}, \quad \delta > 0. \quad (17)$$

We shall use the following inequalities.

**Lemma 2.2.** (See Hardy et al. [27]) Under our boundary conditions (13), we have

$$v^2(x,t) \leq L \|v_x(t)\|^2,$$

$$v^2(x,t) \leq L^3 \|v_{xx}(t)\|^2,$$

$$v_{xx}^2(x,t) \leq L \|v_{xx}(t)\|^2, \quad \forall x \in [0, L], \quad (18)$$

and

$$\|v(t)\|^2 \leq L^2 \|v_x(t)\|^2 \leq L^4 \|v_{xx}(t)\|^2, \quad \forall t \geq 0,$$  \quad (19)

where $\|\cdot\|$ is the norm in $L^2(0,L)$.

Also, we define the Hilbert space $L^2(0,L;L^2(0,1))$ is endowed with the inner product

$$\langle\langle z, \bar{z}\rangle\rangle = \int_0^L \int_0^1 z(x,p,t)\bar{z}(x,p,t)dpdx.$$  

Note that the norms

$$\|z\|^2 = \int_0^L \int_0^1 z^2(x,p,t)dpdx$$

and

$$\|\|z\|\|^2 = \int_0^L \int_0^1 e^{-2\tau p}z^2(x,p,t)dpdx$$

are equivalent in $L^2(0,L;L^2(0,1))$. 
Lemma 2.3. Assume that \((\psi, \psi_t, \eta, \eta_t, z)\) is a strong solution of the problem (12) – (14). Then we have
\[
\frac{\xi}{\tau} \frac{d}{dt} \int_0^L \int_0^1 z^2(x, p, t) dp dx = -\frac{\xi}{2\tau} \int_0^L \int_0^1 \frac{\partial}{\partial p} z^2(x, p, t) dp dx
\]
\[
= \frac{\xi}{2\tau} \int_0^L \{z^2(x, 0, t) - z^2(x, 1, t)\} dx.
\]  
(20)

Proof. We multiply the third equation in (12) by \(\xi z\) and integrate the result over \((0, L) \times (0, 1)\) with respect to \(p\) and \(x\), respectively, to get
\[
\frac{\xi}{\tau} \frac{d}{dt} \int_0^L \int_0^1 z^2(x, p, t) dp dx = -\frac{\xi}{2\tau} \int_0^L \int_0^1 \frac{\partial}{\partial p} z^2(x, p, t) dp dx
\]
\[
= \frac{\xi}{2\tau} \int_0^L \{z^2(x, 0, t) - z^2(x, 1, t)\} dx
\]
which gives (20).

Throughout this paper, we denote by \(\circ\) and \(\diamond\) the binary operators defined by
\[
(g \circ u)(t) = \int_0^t g(t - s) \| u(t) - u(s) \|^2 ds
\]  
(21)
and
\[
(g \diamond u)(t) = \int_0^t g(t - s) (u(s) - u(t)) ds,
\]  
(22)
where \(u \in C([0, T]; L^2(0, L))\).

We define the energy associated with the solution of system (12) – (14) by
\[
E(t) = \frac{1}{2} \left\{ \rho h \| \psi_t \|^2 + \frac{\rho h^3}{12} \| \psi_{xx} \|^2 + \frac{1}{2} (1 - \int_0^t g(s) ds) \| \psi_{xx} \|^2
\]
\[
+ \rho h \| \eta_t \|^2 + \| \eta_x \| + \frac{1}{2} (\psi_x)^2 \| (g \circ \psi_{xx})(t) + \xi \| z \|^2 \right\}
\]  
(23)
where \(\xi\) is a positive constant such that
\[
\alpha_2 \tau < \xi < (2\alpha_1 - \alpha_2) \tau
\]  
(24)
with \(\alpha_1\) and \(\alpha_1\) satisfying
\[
\alpha_2 < \alpha_1.
\]  
(25)

The next lemma gives an identity for the convolution product.

Lemma 2.4. (See Lemma 2.2, [15]) For real functions \(g, \varphi \in C^1(\mathbb{R}^+)\), we have
\[
\int_0^L (g \ast \varphi) \varphi_t dx = -\frac{g(t)}{2} \| \varphi(t) \|^2 + \frac{1}{2} (g' \circ \varphi)(t)
\]
\[
- \frac{1}{2} \frac{d}{dt} \left( g \circ \varphi - \left( \int_0^t g(s) ds \right) \| \varphi(t) \|^2 \right).
\]  
(26)
Lemma 2.5. Assume that \((\psi, \psi_t, \eta, \eta_t, z)\) is a strong solution of the problem (12)–(14). Then the derivative of \(E(t)\) satisfies
\[
\frac{dE(t)}{dt} = -\alpha_1 \| \eta_t \|^2 - \| \psi_{xx} \|^2 - \alpha_2 \int_0^L \eta_t(x, t) z(x, 1, t) dx
\]
\[-\frac{1}{2} g(t) \| \psi_{xx} \|^2 + \frac{1}{2} (g' \circ \psi_{xx})(t). \]
Moreover, for all \(t \geq 0\), we have
\[
\frac{dE(t)}{dt} \leq \left( \frac{\xi}{2\tau} + \frac{\alpha_2}{2} - \alpha_1 \right) \| \eta_t \|^2 + \frac{\alpha_2}{2} - \frac{\xi}{2\tau} \| z(1) \|^2
\]
\[-\frac{1}{2} g(t) \| \psi_{xx} \|^2 + \frac{1}{2} (g' \circ \psi_{xx})(t)
\]
\[\leq 0. \]

Proof. Multiplying the first equation in (12) by \(\psi_t\), the second and the fourth equations by \(\eta_t\) and \(\xi z\), respectively, taking into account (20), (24) and the boundary conditions (13), we obtain the identity (27) after integration over \((0, L)\). Making use Young’s inequality, then (28) follows from (24).

Lemma 2.6 (Jensen inequality). Let \(F\) be a convex function on \([a, b]\), \(f : \Omega \to [a, b]\) and \(h\) are integrable functions on \(\Omega\), \(h(x) \geq 0\), and \(\int_{\Omega} h(x) dx = k > 0\), then Jensen’s inequality states that
\[
F \left[ \frac{1}{k} \int_{\Omega} f(x) h(x) dx \right] \leq \frac{1}{k} \int_{\Omega} F[f(x)] h(x) dx.
\]

3. Global well-posedness. In this section we show the existence and regularity of solutions of the the one dimensional viscoelastic Von-Karman system (12)–(14).

The existence and uniqueness result of problem (12)–(14) is stated as follows.

Theorem 3.1 (Well-posedness). Assume that the initial datum satisfy
\[
(\psi_0, \psi_1) \in \left[H_0^2(I) \cap H^3(I) \right] \times H_0^1(I),
\]
\[
(\eta_0, \eta_1) \in \left[H^2(I) \cap H^1(I) \right] \times H^1(I),
\]
\[
f_0 \in H^1(I; H^1(0, 1)),
\]
with the compatibility condition \(f(., 0) = \eta_1\).

Moreover, assume that the Hypotheses \((A_1), (A_2)\) hold. Then problem (12)–(14) admits a unique weak solution
\[
(\psi, \psi_t, \psi_{tt}) \in L^\infty_{loc}\left(0, \infty; \left[H^2_0(I) \right]^2 \times H_0^1(I) \right),
\]
\[
(\eta, \eta_t, \eta_{tt}) \in L^\infty_{loc}\left(-\tau, \infty; \left[H^1(I) \right]^2 \times L^2(I) \right).
\]

Proof. The proof will be organized as follows. First, using the Galerkin method, we prove the existence of weak solutions, then using elliptic regularity and second order estimates we show the regularity of the solution.

Let \(T > 0\) be fixed and and denote by \(\mathcal{V}^m\) and \(\mathcal{W}^m\) the spaces generated by \((e_i)_{1 \leq i \leq m}\) and \((\sigma_i)_{1 \leq i \leq m}\), and where the sets \(e_k, k \in \mathbb{N}\) and \(w_k, k \in \mathbb{N}\) are basis of spaces \(H_0^2(I)\) and \(H_1^1(I)\) respectively, that is
\[
\mathcal{V}^m = \text{span} \{e_1, e_2, ..., e_m\}, \quad \mathcal{W}^m = \text{span} \{\sigma_1, \sigma_2, ..., \sigma_m\}.
\]
Obviously, with initial conditions
\begin{align*}
\psi^m(0) &= \sum_{j=1}^m \psi^{m,j}(0) e_j, \quad \psi^m_1(0) = \sum_{j=1}^m \psi^{m,j}_1(0) e_j, \\
\eta^m(0) &= \sum_{j=1}^m \eta^{m,j}(0) \sigma_j, \quad \eta^m_1(0) = \sum_{j=1}^m \eta^{m,j}_1(0) \sigma_j, \\
z^m(0, p) &= \sum_{j=1}^m z^{m,j}(0, p) \phi_j, \quad z^m_1(0) = \sum_{j=1}^m z^{m,j}_1(0) \phi_j,
\end{align*}

where
\begin{align*}
\psi^{m,j}(0) &= \langle \psi_0, e_j \rangle, \quad \psi^{m,j}_1(0) = \langle \psi_1, e_j \rangle, \\
\eta^{m,j}(0) &= \langle \eta_0, \sigma_j \rangle, \quad \eta^{m,j}_1(0) = \langle \eta_1, \sigma_j \rangle, \\
z^{m,j}(0, p) &= \langle z_0(p), \phi_j \rangle, \quad z^{m,j}_1(0, p) = \langle z_1, \phi_j \rangle,
\end{align*}

\( j = 1, \ldots, m. \)

Obviously,
\begin{align*}
\psi^m(0) &\rightarrow \psi_0 \text{ strongly in } H^2_0(I), \\
\psi^m_1(0) &\rightarrow \psi_1 \text{ strongly in } H^2_0(I), \\
\eta^m(0) &\rightarrow \eta_0 \text{ strongly in } H^2_1(I), \\
\eta^m_1(0) &\rightarrow \eta_1 \text{ strongly in } H^2_1(I), \\
z^m(0, p) &\rightarrow z_0 \text{ strongly in } H^1(I; H^1(0, 1)).
\end{align*}

By virtue of the theory of ordinary differential equations, the system (33) – (34) has a unique local solution which is extended to a maximal interval \([0, T_m]\) (with \(0 < T_m \leq \infty\)), by Zorn lemma since the nonlinear terms in (33) are locally Lipschitz continuous. Note that \(u^m_t(t), v^m_t(t)\) are from the class \(C^2\).

In the next step, we obtain a priori estimates for the solution such that it can be extended beyond \([0, T_m]\) to obtain a single solution defined for all \(t > 0\).

In order to use a standard compactness argument for the limiting procedure, it suffices to derive some a priori estimates for \(u^m_t(t), v^m_t(t)\). **The first estimate.** Since the sequences \((\psi^m), (\eta^m), \text{ and } (z^m)\) converge, standard calculations, using (33) – (34), similar to those used to derive (28), yield
\begin{align*}
\frac{d\|\psi^m\|}{dt}(t) + \left(\frac{\alpha_2}{2} + \alpha_1 - \frac{\xi}{2\tau}\right) \|\eta^m\|^2 + \left(\frac{\xi}{2\tau} - \frac{\alpha_2}{2}\right) \|z^m(1)\|^2 \\
+ \frac{1}{2} g(t) \|\psi^m_{xx}(t)\|^2 - \frac{1}{2} \left(g' \circ \psi^m_{xx}(t)\right) \leq 0.
\end{align*}

\( \square \)
where
\[
E_m(t) = \frac{1}{2} \left\{ \rho h \| \psi_t^m \|^2 + \frac{\rho h^3}{12} \| \psi^m \|^2 + \left( 1 - \int_0^t g(s) ds \right) \| \psi^m \|^2 + \rho h \| \eta^m \|^2 \\
+ \left\| \eta^m_x + \frac{1}{2} (\psi^m)^2 \right\|^2 + g \circ \psi^m_{xx} + \xi \| z^m \|^2 \right\}.
\]
(37)

Integrating (36) over \((0, t)\) yields
\[
E_m(t) \leq E_m(0) \leq C, \forall t > 0.
\]
(38)
for some positive constant \(C\) independent of \(m \in \mathbb{N}\).

Also, to get an apriori estimate for \(\eta\), we use the Poincaré’s-Wirtinger inequality and the boundedness of \(E\) to obtain
\[
\| \eta^m_x \|^2 \leq L^2 \left( \left\| \eta^m_x + \frac{1}{2} (\psi^m)^2 \right\|_{L^1(I)}^2 + \frac{L^2}{4} \| \psi^m \|^4 \right),
\]
\[
\leq L^3 \left( \left\| \eta^m_x + \frac{1}{2} (\psi^m)^2 \right\|^2 + \frac{L^6}{4} E_m(0) \| \psi^m \|^2 \right),
\]
\[
\leq c E_m(0) \leq C.
\]
(39)
where \(C\) is a positive constant independent of \(m \in \mathbb{N}\).

These estimates imply that the solution \((\psi^m, \eta^m, z^m)\) of the system (33) – (34) exists globally in \([0, +\infty[\).

Estimate (38) yields
\[
\psi^m \text{ is bounded in } L^\infty(0, T, H_0^2(I)),
\psi^m \text{ is bounded in } L^\infty(0, T, H_1^1(I)),
\eta^m \text{ is bounded in } L^\infty(0, T, H_0^1(I)),
\eta^m \text{ is bounded in } L^\infty(0, T, L^2(I)),
\eta^m_x + \frac{1}{2} (\psi^m)^2 \text{ is bounded in } L^\infty(0, T, L^2(I)),
z^m \text{ is bounded in } L^\infty(0, T, L^2(I \times (0, 1))).
\]
(40)
for any \(T > 0\).

The second estimate. We have to estimate \(\psi_{tt}^m(0), \psi_{ttx}^m(0)\) and \(\eta_{tt}^m(0)\) in \(L^2\) norm.

Considering \(t = 0\) in the first equation of (33), then multiplying it by \((u^m_{11})^\prime(0)\) and summing up over \(i\) from 1 to \(m\), it follows that
\[
\rho h \| \psi_{tt}^m(0) \|^2 + \frac{\rho h^3}{12} \| \psi_{ttx}^m(0) \|^2 + \langle \psi_{0xx}^m, \psi_{ttx}^m(0) \rangle
+ \left\langle \psi_{0x}^m \left( \eta_{0x}^m + \frac{1}{2} (\psi_{0x}^m)^2 \right), \psi_{ttx}^m(0) \right\rangle
+ \langle \psi_{xt}^m, \psi_{ttx}^m(0) \rangle = 0.
\]
(41)
Integrating by parts and using Young’s inequality, we get
\[
\langle \psi_{0xx}^m, \psi_{ttx}^m(0) \rangle = - \langle \psi_{0xx}^m, \psi_{ttx}^m(0) \rangle,
\]
\[
\leq C \delta \| \psi_{0xx}^m \|^2 + \delta \| \psi_{ttx}^m(0) \|^2.
\]
(42)
Similarly
\[
\left\langle \psi_{0x}^m \left( \eta_{0x}^m + \frac{1}{2} (\psi_{0x}^m)^2 \right), \psi_{ttx}^m(0) \right\rangle \leq \delta \| \psi_{ttx}^m(0) \|^2 + C \delta \| \psi_{0x}^m \left( \eta_{0x}^m + \frac{1}{2} (\psi_{0x}^m)^2 \right) \|^2.
\]
(43)
Using the embedding $H^1(I) \hookrightarrow L^\infty(I)$, we estimate the second term of the right hand side of (43) as follows

$$\left\| \psi_m^{ox} \left( \eta_{ox} + \frac{1}{2} (\psi_m^{ox})^2 \right) \right\|^2 \leq \left\| \psi_m^{ox} \right\|_{L^\infty(I)}^2 \left\| \eta_{ox} + \frac{1}{2} (\psi_m^{ox})^2 \right\|^2 \leq c \left\| \psi_m^{ox} \right\|^2 \left\| \eta_{ox} + \frac{1}{2} (\psi_m^{ox})^2 \right\|^2 \leq C. \quad (44)$$

From the embedding $H^1(I) \hookrightarrow L^\infty(I)$ and Young’s inequality, we deduce

$$\langle \psi_m^{ox}, \psi_m^{ox}(0) \rangle \leq \left\| \psi_m^{ox} \right\|_{L^\infty(I)} \left\| \psi_m^{ox}(0) \right\| \leq C + \delta \left\| \psi_m^{ox}(0) \right\|^2. \quad (45)$$

After choosing a suitable $\delta$, we infer from (34)-(35) and (41)-(45) that there exists a positive constant $C$ independent of $m$ such that

$$\left\| \psi_m^{ox}(0) \right\|^2 + \left\| \psi_m^{ox}(0) \right\|^2 \leq C. \quad (46)$$

Next, multiplying the second equation of (33) by $(v_m''(0))$, choosing $t = 0$ and summing up over $i$ from 1 to $m$, we get

$$\rho h \left\| \eta_{it}^m(0) \right\|^2 + \left\langle \eta_{ox}^m + \frac{1}{2} (\psi_{ox}^m)^2, \eta_{it}^m(0) \right\rangle + \alpha_1 \left\langle \eta_{it}^m, \eta_{it}^m(0) \right\rangle + \alpha_2 \left\langle \zeta_m^m(1), \eta_{it}^m(0) \right\rangle = 0 \quad (47)$$

Applying Young’s and Poincaré’s inequalities, using the embedding $H^1(I) \hookrightarrow L^\infty(I)$ and the fact that $\mathcal{E}$ is nonincreasing, we conclude that

$$\left\langle \eta_{ox}^m + \frac{1}{2} (\psi_{ox}^m)^2, \eta_{it}^m(0) \right\rangle = -\left\langle \eta_{ox}^m + \psi_{ox}^m \psi_{ox}^m, \eta_{it}^m(0) \right\rangle \leq \delta \left\| \eta_{it}^m(0) \right\|^2 + C_\delta \left\| \eta_{ox}^m + \psi_{ox}^m \psi_{ox}^m \right\|^2 \leq \delta \left\| \eta_{ot}^m(0) \right\|^2 + C_\delta \left\| \eta_{ox}^m \right\|^2 + C_\delta \left\| \psi_{ox}^m \right\|_{L^\infty(I)} \left\| \psi_{ox}^m \right\|^2 \leq \delta \left\| \eta_{ot}^m(0) \right\|^2 + C. \quad (48)$$

Then we use Young’s inequality to obtain, for any $\delta > 0$,

$$\langle \eta_{it}^m, \eta_{it}^m(0) \rangle \leq \delta \left\| \eta_{ot}^m(0) \right\|^2 + C_\delta \left\| \eta_{it}^m \right\|^2 \quad (49)$$

and

$$\langle \zeta_m^m(1), \eta_{it}^m(0) \rangle \leq \delta \left\| \eta_{ot}^m(0) \right\|^2 + C_\delta \left\| \zeta_m^m(1) \right\|^2. \quad (50)$$

Hence, from (34)-(35) and with a suitable choice of $\delta$, there exists a positive constant $C$ independent of $m$ such that

$$\left\| \eta_{ot}^m(0) \right\| \leq C. \quad (51)$$

Next, differentiating the first equation of (33) with respect to $t$, using Lemma 2.3 and multiplying the result by $(u_m'')$, adding from $i = 1$ to $m$, we obtain

$$\frac{1}{2} \frac{d}{dt} \left\{ \rho h \left\| \psi_{it}^m \right\|^2 + \frac{\rho h^3}{12} \left\| \psi_{it}^m \right\|^2 + \left\| \psi_{it}^m \right\|^2 \right\} + g(0) \left\| \psi_{it}^m \right\|^2
We have
\[
\frac{d}{dt} \left( \psi_x^m (\eta_x^m + \frac{1}{2} (\psi_x^m)^2) \right) = \frac{d}{dt} \left( g' \ast \psi_{xx}^m, \psi_{txx}^m \right) - (g'' \ast \psi_{xx}^m, \psi_{txx}^m) - g'(0) \langle \psi_{xx}^m, \psi_{txx}^m \rangle \\
+ g(0) \frac{d}{dt} \langle \psi_{xx}^m, \psi_{txx}^m \rangle.
\]
(52)

Making use of Young’s inequality, the embedding \( H^1(I) \hookrightarrow L^\infty(I) \) and (39), we conclude that
\[
\left\langle \psi_{tx}^m (\eta_{tx}^m + \frac{1}{2} (\psi_{tx}^m)^2), \psi_{txx}^m \right\rangle \leq C \delta \left\| \psi_{tx}^m (\eta_{tx}^m + \frac{1}{2} (\psi_{tx}^m)^2) \right\|^2 + \delta \left\| \psi_{txx}^m \right\|^2 ,
\]
\[
\leq C + \delta \left\| \psi_{txx}^m \right\|^2.
\]
(54)

Making use of the embedding \( H^1(I) \hookrightarrow L^\infty(I) \), we get
\[
\langle \psi_x^m (\eta_{tx}^m + \psi_{xt}^m \psi_x^m), \psi_{txx}^m \rangle = \langle \psi_x^m \eta_{tx}^m, \psi_{txx}^m \rangle + \langle \psi_{xt} (\psi_x^m)^2, \psi_{txx}^m \rangle \\
\leq \delta \left\| \psi_x^m \right\|^2_{L^\infty(I)} \left\| \eta_{tx}^m \right\|^2 + C \delta \left\| \psi_{txx}^m \right\|^2 \\
+ \delta \left\| \psi_{xt}^m \right\|^2_{L^\infty(I)} \left\| (\psi_x^m)^2 \right\|^2 \\
\leq C \delta \left\| \psi_x^m \right\|^2 \left\| \eta_{tx}^m \right\|^2 + C \delta \left\| \psi_{txx}^m \right\|^2 \\
+ \delta C \left( \left\| \psi_{txx}^m \right\|^2 + C \left\| \psi_{txx}^m \right\|^2 \right)
\]
(55)

Since \( g' \) and \( g'' \) are continuous functions on \([0, T]\) then \( m_1 = \sup_{t \in [0, T]} |g'| \) and \( m_2 = \sup_{t \in [0, T]} |g''| \) exist for all \( T < \infty \).

Using Cauchy-Shwarz and Young’s inequalities produce the estimates
\[
\langle g' \ast \psi_{xx}^m, \psi_{txx}^m \rangle \leq m_1^2 C \delta \int_0^t \left\| \psi_{xx}^m (s) \right\|^2 ds + \delta \left\| \psi_{txx}^m \right\|^2 ,
\]
(56)

and
\[
\langle g'' \ast \psi_{xx}^m, \psi_{txx}^m \rangle \leq m_2^2 C \delta \int_0^t \left\| \psi_{xx}^m (s) \right\|^2 ds + \delta \left\| \psi_{txx}^m \right\|^2.
\]
(57)

Employing Young’s inequality, combining (53) – (57), then integrating (52) over \((0, t)\), we obtain
\[
\rho h \left\| \psi_{tx}^m \right\|^2 + \frac{\rho h^3}{12} \left\| \psi_{txx}^m \right\|^2 + (1 - \delta g(0)) \left\| \psi_{txx}^m \right\| \\
\leq C \delta \int_0^t \left\| \psi_{txx}^m (s) \right\|^2 ds + C \delta \int_0^t \left\| \psi_{txx}^m (s) \right\|^2 ds \\
+ C \delta \int_0^t \left\| \eta_{tx}^m (s) \right\|^2 ds + C \delta \int_0^t \int_0^s \left\| \psi_{xx}^m (s) \right\|^2 dz ds + \rho h \left\| \psi_{tx}^m (0) \right\|^2 + \frac{\rho h^3}{12} \left\| \psi_{txx}^m (0) \right\|^2 + C,
\]
(58)
where $C$ is a positive constant independent of $m$ but depends on $T$ and the initial data.

The term $\int_0^t \int_0^T \| \psi_{xx}^m \|^2 \, dz \, ds$ can be estimated as follows
\[
\int_0^t \int_0^T \| \psi_{xx}^m(z) \|^2 \, dz \, ds \leq C \mathcal{E}(0) \int_0^t \int_0^T \, dz \, ds \leq \frac{CT^2}{2}.
\]

Next, multiplying the second equation of (33) by $(\psi_{xx}^m)^\prime$ and summing up over $i$ from 1 to $m$, we arrive at
\[
\frac{1}{2} \frac{d}{dt} \left\{ \rho h \| \eta_t^m \|^2 + \| \eta_{tt}^m \|^2 \right\} + \langle \psi_{xx}^m \psi_{x}^m, \eta_{tt}^m \rangle + \alpha_1 \| \psi_{tt}^m \|^2 + \alpha_2 \langle z_t^i(1), \eta_{tt}^m \rangle = 0. \tag{60}
\]

Now, applying Young’s and Poincaré’s inequalities, we get
\[
\langle \psi_{xx}^m \psi_{x}^m, \eta_{tt}^m \rangle \leq 2\delta \| \eta_{tt}^m \|^2 + C\delta \| \psi_{xx}^m \|_{L^\infty(t)} \| \psi_{xxxx}^m \|^2 \leq 2\delta \| \eta_{tt}^m \|^2 + C\delta \| \psi_{xx}^m \|^2 \leq 2\delta \| \eta_{tt}^m \|^2 + C\delta \| \psi_{xx}^m \|^2.
\]

A differentiation with respect to $t$ of the third equation of (33) implies
\[
\langle z_t^m, \phi_i \rangle - \frac{1}{\tau} \langle z^m_{tt}, \phi_i \rangle = 0. \tag{62}
\]

Multiplying (62) by $(w_i^m)^\prime(t)$, integrating by parts and adding from $i = 1$ to $m$, we obtain
\[
\tau \frac{1}{2} \frac{d}{dt} \| z_t^m \|^2 + \frac{1}{2} \frac{d}{dp} \| z_t^m \|^2 = 0. \tag{63}
\]

Integrating (63) over $(0,1)$ with respect to $p$, taking the sum of (60) and (63) and integrating over $(0,t)$, we obtain
\[
\frac{\rho h}{2} \| \eta_t^m \|^2 + \frac{1}{2} \| \eta_{tt}^m \|^2 + \frac{\tau}{2} \| z_t^m \|^2 + \alpha_1 \int_0^t \| \eta_{tt}^m(s) \|^2 \, ds + \frac{1}{2} \int_0^t \| z_t^m(s,1) \|^2 \, ds \leq \frac{1}{2} \| \eta_t^m(0) \|^2 + \frac{1}{2} \| \eta_{tt}^m \|^2 + \frac{\tau}{2} \| z_t^m \|^2 + C\delta \int_0^t \| \eta_{tt}^m(s) \|^2 \, ds + C\delta \int_0^t \| \psi_{xx}^m(s) \|^2 \, ds + c\delta \int_0^t \| z_t(s,1) \|^2 \, ds.
\]

Combining (58) and (64) with a suitable choice of $\delta$, then using Gronwall’s lemma, we arrive at
\[
\| \psi_t^m \|^2 + \| \psi_{tt}^m \|^2 + \| \psi_{xxxx}^m \|^2 + \| \eta_{tt}^m \|^2 + \| \eta_{tt}^m \|^2 + \| z_t^m \|^2 \leq C.
\]

where $C$ is independent of $m \in \mathbb{N}$.

Replacing $\phi_i$ by $-\phi_{i,xx}$ in the third equation of (33), multiplying the resulting equation by $w_i^m(t)$, summing over $i$ from 1 to $m$, leads to
\[
\tau \frac{d}{dt} \| z_t^m \|^2 + \frac{d}{dp} \| z_t^m \|^2 = 0,
\]

A integration over $(0,t) \times (0,1)$ yields
\[
\tau \| z_t^m \|^2 + \int_0^t \| z_t^m(s,1) \|^2 \, ds = \| \eta_{tt}^m \|^2 + \| z_t^m \|^2 \leq C \tag{66}
\]

where $C$ independent of $m$. 

From (38), (39), (65) and (66), we infer that
\[ \psi_m \] is bounded in \( L^\infty(0, T, H^2_0(I)) \),
\[ \psi^*_m \] is bounded in \( L^\infty(0, T, H^0_0(I)) \),
\[ \eta^m \] is bounded in \( L^\infty(0, T, H^1(I)) \),
\[ \eta^*_m \] is bounded in \( L^\infty(0, T, H^1_0(I)) \),
\[ z^m \] is bounded in \( L^\infty(0, T, L^2(I, H^1(I))) \). (67)

Now, using (40) and (67), we deduce that
\[ \psi^*_m \to \psi \text{ weak-star in } W^{1,\infty}(0, T; H^2(I)) \cap W^{2,\infty}(0, T; H^1_0(I)) \text{,} (68) \]
\[ \eta^*_m \to \eta \text{ weak-star in } W^{1,\infty}(0, T; H^1(I)) \cap W^{2,\infty}(0, T; L^2(I)) \text{,} (69) \]
\[ \eta^*_m + \frac{1}{2} (\psi^*_m)^2 \to f \text{ weak-star in } L^\infty(0, T; L^2(I)), (70) \]
\[ z^m \to z \text{ weak-star in } L^\infty(0, T; H^1(I)) \cap W^{1,\infty}(0, T; L^2(I)) \cap W^{1,\infty}(0, T; L^2(I)) \times (0, 1)) \text{,} (71) \]

as \( m \to \infty \), for a suitable function \( f(x, t) \in L^\infty(0, T; L^2(I)) \).

According to (67), \( \psi^*_m \) is uniformly bounded in \( W^{1,\infty}(0, T; H^2_0(I)) \). In this way, due to the Aubin-Lions compactness theorem (see [61], Corollary 4), we can extract a subsequence \( \psi^*_m \) such that
\[ \psi^*_m \to \psi \text{ strongly in } L^\infty(0, T; H^{2-\delta}(I)) \text{ as } m \to \infty, (72) \]
for any \( \delta > 0 \) which, by Sobolev’s embeddings, implies the strong convergence in \( L^\infty(0, T; L^p(I)) \) for all \( 1 \leq p < \infty \). Therefore,
\[ (\psi^*_m)^2 \to \psi^2 \text{ almost everywhere in } Q \text{ as } m \to \infty, (73) \]

Similarly, by using the embedding \( H^1(I) \hookrightarrow L^2(I) \), we infer that
\[ \eta^*_m \to \eta \text{ strongly in } L^2(Q), (74) \]

which implies
\[ \eta^*_m \to \eta \text{ almost everywhere in } Q \text{ as } m \to \infty. \]

Combining (68) – (71) and (73), it follows that \( f = \eta_x + \frac{1}{2} \psi^2_x \), and
\[ \psi^*_x (\eta^*_m + \frac{1}{2} (\psi^*_m)^2) \to \psi_x (\eta_x + \frac{1}{2} \psi^2_x) \text{ weakly in } L^2(Q). \]

As a consequence of (74) we have
\[ z^m(1) \to z(1) \text{ strongly } L^2(Q) \text{, as } m \to \infty (75) \]
which implies
\[ z^m(1) \to z(1) \text{ almost everywhere in } Q \text{ as } m \to \infty. \]

The convergence (72) – (75) allows us to pass to the limit in (33) – (34).

Thus, the problem (12) – (14) admits a global weak solution \((\psi, \eta, z)\).

Uniqueness can be proved by the straightforward methods and Gronwall’s inequality.
4. General decay. In this section we consider a wider class of kernel functions, and we establish a general decay result, which contains the usual exponential and polynomial decay rates as special cases. The main result of general decay is the following.

**Theorem 4.1.** Assume that \((A_1), (A_2)\) holds. Then for any solution of \((12)-(14)\), there exists two positive constants \(k_1 \leq 1\) and \(k_2\) such that the energy functional satisfies

\[
E(t) \leq k_2 H_1^{-1}(t) \left( k_1 \int_0^t \xi(s) ds \right),
\]

where

\[
H_1(t) = \int_t^r \frac{1}{sH'(s)} ds
\]

and, \(H_1\) is strictly decreasing and convex function on \((0, r)\), with \(\lim_{t \to 0} H_1(t) = +\infty\).

To prove Theorem 4.1, we first proceed to prepare a series of useful lemmas.

**Lemma 4.2.** The following inequalities hold

\[
(g \circ \psi)^2(t) \leq \int_0^t g(t-s) (\psi(t) - \psi(s))^2 ds, \quad (77)
\]

\[
(g' \circ \psi)^2(t) \leq -c \int_0^t g'(t-s) (\psi(t) - \psi(s))^2 ds, \quad (78)
\]

where \(g \circ \psi\) is given by \((22)\).

**Proof.** For inequality \((77)\), we have

\[
(g \circ \psi)^2(t) = \left( \int_0^t g(t-s) (\psi(t) - \psi(s)) ds \right)^2 = \left( \int_0^t \sqrt{g(t-s)} \sqrt{g(t-s)} (\psi(t) - \psi(s)) ds \right)^2.
\]

Cauchy-Schwarz inequality leads to

\[
(g \circ \psi)^2(t) \leq \int_0^t g(s) ds \int_0^t g(t-s) (\psi(t) - \psi(s))^2 ds \\
\leq \int_0^t g(t-s) (\psi(t) - \psi(s))^2 ds.
\]

Similarly, we prove \((78)\) by replacing \(\sqrt{g(t-s)}\) by \(\sqrt{-g'(t-s)}\).

Let \(\mathcal{F}\) be the functional defined by

\[
\mathcal{F}(t) = \frac{N_1}{2} I(t) + N_1 J(t) + N_2 K(t) + L(t), \quad (79)
\]

where

\[
I(t) = \langle ph \psi_{tt}, \psi \rangle, \quad (80)
\]

\[
J(t) = \langle ph \eta_t, \eta \rangle, \quad (81)
\]

\[
K(t) = -\langle ph \psi_t, g \circ \psi \rangle, \quad (82)
\]

\[
L(t) = \int_0^L \int_0^1 e^{-2\tau} \varphi(p) dp dx, \quad (83)
\]
$N_1$ and $N_2$ are positive constants that will be chosen later.

Let $\lambda > 0$ and define $Y(t)$ by

$$Y(t) = \lambda E(t) + F(t)$$

(84)

where $E(t)$ is defined in (23).

The following proposition gives the equivalence between $E(t)$ and the functional $Y(t)$.

**Proposition 1.** Assume that $(A_1)$ hold, then there exists two positive constants $\delta_1, \delta_2$ such that

$$\delta_1 E(t) \leq Y(t) \leq \delta_2 E(t).$$

(85)

**Proof.** To compare $Y(t)$ with $E(t)$, we have to estimate the terms of the right hand side of (79) and show that.

$$|F(t)| \leq c^* E(t), \quad c^* > 0.$$  

From (80), (81), (82) and (83), we obtain

• **Estimate for $I(t)$**

$$I(t) = \rho h \langle D\psi_t, \psi \rangle = \rho h \langle \psi_t - \frac{h^2}{12} \psi_{txx}, \psi \rangle,$$

A integration by parts yields

$$I(t) = \frac{\rho h}{2} \|\psi_t\|^2 + \frac{\rho h^3}{24} \|\psi_{tx}\|^2 \leq c_1 E(t).$$

(86)

• **Estimate for $J(t)$**

Using Young’s inequality, we deduce

$$J(t) = \rho h \langle \eta_t, \eta \rangle \leq \frac{\rho h}{2} \|\eta_t\|^2 + \frac{\rho h}{2} \|\eta\|^2.$$  

Applying Poincaré-Wirtinger inequality, one gets

$$\|\eta\|^2 \leq L^3 \|\eta_x\|^2 + \frac{\rho h^3}{4} \|\psi_{xx}\|^4.$$

Applying Poincaré’s inequality on the term $\|\psi_{xx}\|^4$, and then using the fact that $E(t)$ is decreasing, yields

$$\|\eta\|^2 \leq L^3 \|\eta_x + \frac{1}{2} \psi_x\|^2 + \frac{L^6}{4} \|\psi_{xx}\|^4$$

(87)

• **Estimate for $K(t)$**

Using (77), applying Young’s and then Poincaré inequalities, one gets

$$K(t) = - \langle \rho h D\psi_t, g \circ \psi \rangle = - \rho h \langle \psi_t, g \circ \psi \rangle + \frac{\rho h^3}{12} \langle \psi_{tx}, g \circ \psi \rangle$$

$$\leq \frac{\rho h}{2} \|\psi_t\|^2 + \frac{\rho h^3}{24} \|\psi_{tx}\|^2 + \left( \frac{\rho h L^4}{2} + \frac{\rho h^3 L^2}{24} \right) g \circ \psi_{xx}$$

$$\leq c_3 E(t).$$

(88)
Estimate for $L(t) := \int_0^L \int_0^1 e^{-2\tau p} z^2 dp dx$

Since $L(t)$ defines a norm in $L^2(0, L; L^2(0, 1))$ which is equivalent to the one induced by $L^2(0, L; L^2(0, 1))$, then we have

$$L(t) \leq \int_0^L \int_0^1 e^{-2\tau p} z^2 dp dx \leq \int_0^L \int_0^1 z^2 dp dx \leq E(t). \quad (89)$$

According to (86), (87), (88) and (89), we have

$$|F(t)| \leq c^* E(t), \quad c^* > 0.$$ for

$$c^* = \max\{c_1, c_2, c_3, 1\}.$$ Therefore, we obtain

$$|Y(t) - \lambda E(t)| \leq c^* E(t),$$

that is

$$(\lambda - c^*) E(t) \leq Y(t) \leq (\lambda + c^*) E(t).$$ So, we can choose $\lambda$ large enough such that $\delta_1 = \lambda - c^* > 0$, $\delta_1 = \lambda + c^* > 0$. Then (85) holds true.

This completes the proof. \hfill \Box

In order to proof the main theorem, we need some additional lemmas.

**Lemma 4.3.** Suppose that $(\psi, \psi_t, \eta, \eta_t, z)$ is the solution of (12) – (14). Then the derivative of the functional $I(t)$ satisfies

$$\frac{d}{dt} I(t) \leq -(l - \varepsilon) \|\psi_{xx}\|^2 + \rho h \|\psi_t\|^2 + \frac{\rho h^3}{12} \|\psi_{tx}\|^2$$

$$- \left\langle \psi_x(\eta_x + \frac{1}{2} \psi_x^2), \psi_x \right\rangle + C_g \circ \psi_{xx}, \quad (90)$$

where $\varepsilon$ is an arbitrary positive constant.

**Proof.** Using (12) and (77), we have

$$\frac{d}{dt} I(t) = \langle \rho D\psi_t, \psi \rangle + \langle \rho D\psi_t, \psi_t \rangle$$

$$= I_1 + I_2 + I_3 + \left\langle \rho h \left(I - \frac{h^2}{12} \frac{d^2}{dx^2}\right) \psi_t, \psi_t \right\rangle,$$

where

$$I_1 = - \int_0^L \psi_{xxxx}(t) \psi(t) dx,$$

$$I_2 = \int_0^L \psi(t) \int_0^t g(t-s) \psi_{xxxx}(s) ds dx,$$

$$I_3 = \int_0^L \left[ \psi_x \left( \eta_x + \frac{1}{2} \psi_x^2 \right) \right] \psi(t) dx.$$
Integrating $I_1$ and $I_2$ by parts twice, and using the boundary conditions, we obtain
\[ I_1 = -\|\psi_{xx}\|^2, \]  
\[ I_2 = \int_0^L \psi_{xx}(t) \int_0^t g(t-s)\psi_{xx}(s)ds dx. \]  
A integration by parts in $I_3$, leads to
\[ I_3 = -\int_0^L \left[ \psi_x \left( \eta_x + \frac{1}{2} \psi_x^2 \right) \right] \psi_x(t) dx. \]  
Substituting (92)–(93) in (91), we get
\[ \frac{d}{dt} I(t) = -\|\psi_{xx}\|^2 + \int_0^L \psi_{xx}(t) \int_0^t g(t-s)\psi_{xx}(s)ds dx - \int_0^L \left[ \psi_x \left( \eta_x + \frac{1}{2} \psi_x^2 \right) \right] \psi_x(t) dx \]
\[ + \rho h \|\psi_t\|^2 + \frac{\rho h^3}{12} \|\psi_{tx}\|^2. \]
But
\[ \int_0^L \psi_{xx}(t) \int_0^t g(t-s)\psi_{xx}(s)ds dx = \int_0^L \psi_{xx}(t) \int_0^t g(t-s)(\psi_{xx}(s) - \psi_{xx}(t)) ds dx \]
\[ + \int_0^t g(s)ds \|\omega_{xx}\|^2. \]
Substituting (94) in (94), we obtain
\[ \frac{d}{dt} I(t) = -(1 - \frac{l}{t}) \|\psi_{xx}\|^2 + \int_0^L \psi_{xx}(t) \int_0^t g(t-s)(\psi_{xx}(s) - \psi_{xx}(t)) ds dx \]
\[ - \int_0^L \left[ \psi_x \left( \eta_x + \frac{1}{2} \psi_x^2 \right) \right] \psi_x(t) dx \]
\[ + \rho h \|\psi_t\|^2 + \frac{\rho h^3}{12} \|\psi_{tx}\|^2. \]
Making use of Young’s and Cauchy-Schwarz’s inequalities for the issecond term in the right-hand side of (94), we get, for any $\varepsilon > 0$,
\[ \int_0^L \psi_{xx}(t) \int_0^t g(t-s)(\psi_{xx}(s) - \psi_{xx}(t)) ds dx \leq \varepsilon \|\psi_{xx}\|^2 + C_\varepsilon g \circ \psi_{xx}. \]
From (94)-(94) and (15), we infer that
\[ \frac{d}{dt} I(t) \leq -\left( \frac{l}{t} - \varepsilon \right) \|\psi_{xx}\|^2 - \left\langle \psi_x \left( \eta_x + \frac{1}{2} \psi_x^2 \right), \psi_x \right\rangle + \rho h \|\psi_t\|^2 \]
\[ + \frac{\rho h^3}{12} \|\psi_{tx}\|^2 + C_\varepsilon g \circ \psi_{xx}, \]
which proves the lemma 4.3.

Lemma 4.4. Assume that \((\psi, \psi_t, \eta, \eta_t, z)\) is the solution of (12) – (14). Then the derivative of the functional \(\mathcal{J}(t)\) satisfies

\[
\frac{d}{dt} \mathcal{J}(t) \leq \rho h \|\eta_t\|^2 - \left\langle \eta_x + \frac{1}{2}\psi^2_x, \eta_t \right\rangle + \varepsilon c \left( \|\psi_x + \frac{1}{2}\psi^2_x\|^2 + \|\psi_{xx}\|^2 \right) + C_\varepsilon \|\eta_t\|^2 + C_\varepsilon \|z (1)\|^2.
\]

where \(\varepsilon\) is an arbitrary positive constant.

Proof. A differentiation of \(\mathcal{J}(t)\) yields

\[
\frac{d}{dt} \mathcal{J}(t) \leq \rho h \|\eta_t\|^2 - \left\langle \eta_x + \frac{1}{2}\psi^2_x, \eta_t \right\rangle + (\alpha_1 + \alpha_2) \varepsilon \|\eta\|^2 + C_\varepsilon \|\eta_t\|^2 + C_\varepsilon \|z (1)\|^2.
\]

Applying Young’s inequality, we obtain

\[
\frac{d}{dt} \mathcal{J}(t) \leq \rho h \|\eta_t\|^2 - \left\langle \eta_x + \frac{1}{2}\psi^2_x, \eta_t \right\rangle + (\alpha_1 + \alpha_2) \varepsilon \|\eta\|^2 + C_\varepsilon \|\eta_t\|^2 + C_\varepsilon \|z (1)\|^2.
\]

Lemma 4.5. Suppose that \((\psi, \psi_t, \eta, \eta_t, z)\) is the solution of (12) – (14). Then the derivative of the functional \(\mathcal{K}(t)\) satisfies

\[
\frac{d}{dt} \mathcal{K}(t) \leq -\rho h \left( g_0 - \frac{h^2}{12} \varepsilon \right) \|\psi_t\|^2 - \left( g_0 - \varepsilon \right) \frac{\rho h^3}{12} \|\psi_{xx}\|^2 + (\varepsilon c^2 + 2\varepsilon + \rho h\varepsilon) \|\psi_{xx}\|^2 + \varepsilon c \left( \|\psi_{xx}\|^2 \right) + C_\varepsilon \psi_t \cdot g' \circ \psi_x + C_\varepsilon g \circ \psi_{xx},
\]

where \(\varepsilon\) is an arbitrary positive constant and

\[
g_0 := \int_0^{t_0} g(s) ds \leq \int_0^t g(s) ds, \forall t \geq t_0.
\]

Proof. Differentiating \(\mathcal{K}\) and using (12), we obtain

\[
\frac{d}{dt} \mathcal{K}(t) = -\left\langle \rho h \psi_t, \int_0^t g(t - s)\psi(s)ds \right\rangle - \left\langle \rho h \psi_t, g' \circ \psi + \int_0^t g(s)ds \psi_t \right\rangle
\]

\[
= -\int_0^t g(s)ds \left( \rho h \|\psi_t\|^2 + \frac{\rho h^3}{12} \|\psi_{xx}\|^2 \right) + \left\langle \psi_{xx}, g \circ \psi_{xx} \right\rangle
\]

\[
- \left\langle \psi_t (\eta_x + \frac{1}{2}\psi^2_x), g \circ \psi_x \right\rangle
\]

\[
+ \left\langle \int_0^t g(t - s)\psi_{xx}(s)ds, g \circ \psi_{xx} \right\rangle
\]

\[
- \rho h \left\langle \psi_t, g' \circ \psi \right\rangle + \frac{\rho h^3}{12} \left\langle \psi_t, g' \circ \psi_{xx} \right\rangle
\]
\[ : = -g_0 \left( \rho h \| \psi_t \|^2 + \frac{\rho h^3}{12} \| \psi_{tx} \|^2 \right) \]
\[ + J_1(t) + J_2(t) + J_3(t) + J_4(t) + J_5(t). \tag{97} \]

Next, we shall analyze the terms such as \( J_1 - J_5 \) of the right hand side of (97).

**Estimate for \( J_1 \)**

Applying Young’s inequality and (77), we get
\[ J_1(t) = \langle \psi_{xx}, g \circ \psi_{xx} \rangle \leq \varepsilon \| \psi_{xx} \|^2 + C \varepsilon \| g \circ \psi_{xx} \|. \tag{98} \]
where \( \varepsilon \) is an arbitrary positive constant.

**Estimate for \( J_2 \)**

We follow the previous steps, with applying Poincaré’s inequality, we get
\[ J_2(t) = -\langle \psi_x \left( \eta_x + \frac{1}{2} \psi_x^2 \right), g \circ \psi_x \rangle \leq \varepsilon^2 \| \psi_x \|^2 L_\infty \eta_x + \frac{1}{2} \psi_x^2 \|^2 \]
\[ \leq c \| \psi_{xx} \|^2 \| \eta_x + \frac{1}{2} \psi_x^2 \|^2 \]
\[ \leq \varepsilon c \| \psi_{xx} \|^2 + \frac{c}{\varepsilon} \| \eta_x + \frac{1}{2} \psi_x^2 \|^2. \]

This implies that
\[ J_2(t) \leq c \varepsilon^3 \| \psi_{xx} \|^2 + \frac{c \varepsilon^3}{4} \| \eta_x + \frac{1}{2} \psi_x^2 \|^2 + C \varepsilon \| g \circ \psi_{xx} \|. \]

**Estimate for \( J_3 \)**

We invoke (77) to get
\[ J_3(t) = \left\langle \int_0^t g(t-s) \psi_{xx}(s) ds, g \circ \psi_{xx} \right\rangle \tag{100} \]
\[ = -\left| \int_0^t g(t-s) (\psi_{xx}(t) - \psi_{xx}(s)) ds \right|^2 \]
\[ + \int_0^t g(s) ds (\psi_{xx}(t), g \circ \psi_{xx}) \]
\[ \leq \varepsilon \| \psi_{xx} \|^2 + C \varepsilon \| g \circ \psi_{xx} \|. \]

**Estimate for \( J_4 \)**

Applying Young’s, Poincaré’s inequalities and using (78), we conclude that
\[ J_4(t) = -\rho h \langle \psi_t, g' \circ \psi \rangle \leq \rho \varepsilon \| \psi_{xx} \|^2 - C \varepsilon \| g' \circ \psi_{xx} \|. \tag{101} \]

**Estimate for \( J_5 \)**

Finally, for \( J_5 \), invoking (78), we obtain:
\[ J_5(t) = \frac{\rho h^3}{12} \langle \psi_t, g' \circ \psi_{xx} \rangle \leq \frac{\rho h^3}{12} \varepsilon \| \psi_t \|^2 - C \varepsilon \| g' \circ \psi_{xx} \|. \tag{102} \]
Combining (98) – (102), we arrive at the proof of (4.5).

**Lemma 4.6.** Suppose that \((\psi, \psi_t, \eta, \eta_t, z)\) is the solution of (12) – (14). Then the time derivative of the functional \(L(t)\) satisfies

\[
\frac{d}{dt}L(t) \leq -2L(t) + \frac{\alpha_1 e^{-2\tau}}{\tau} \|z(1)\|^2 + \frac{\alpha_2}{\tau} \|\eta_t\|^2. \tag{103}
\]

**Proof.** Keeping in mind that \(z_t(p) = -\frac{1}{\tau}z_p(p)\), we infer

\[
\frac{d}{dt}I_4(t) = -\frac{2}{\tau} \int_0^1 e^{-2\tau p} z_p^2 dp dx = -\frac{1}{\tau} \int_0^1 e^{-2\tau p} (\tau^2)^2 dp dx \\
= -2L(t) + \frac{\alpha_1 e^{-2\tau}}{\tau} \|z(1)\|^2 + \frac{\alpha_2}{\tau} \|\eta_t\|^2.
\]

**Proposition 2.** Assume that \((A_1)\) and \((A_1)\) hold, then there exists two positive constants \(\beta_1, \beta_2\) such that

\[
\frac{d}{dt}Y(t) \leq -\beta_1 E(t) + \beta_2 g \circ \psi_{xx}. \tag{104}
\]

**Proof.** By using (79), (84) and combining (90) – (103), we get

\[
\frac{d}{dt}Y(t) \leq \left\{-\frac{N_1}{2} (l - \varepsilon) + N_1 c\varepsilon + N_2 (c\varepsilon^3 + 2\varepsilon + \rho \varepsilon)\right\} \|\psi_{xx}\|^2 \\
+ \rho h \left\{N_2 \left[g_0 - \frac{\hat{h}^2}{12} \varepsilon\right] + \frac{N_1}{2}\right\} \|\psi_t\|^2 + \left\{N_1 C_{\varepsilon} + \lambda \left(\frac{\varepsilon}{2\tau} + \frac{\alpha_2}{2} - \alpha_1\right)\right\} \|\eta_t\|^2 \\
+ \frac{\rho h^3}{12} \left\{\frac{N_1}{2} - N_2 (g_0 - \varepsilon)\right\} \|\psi_{xx}\|^2 + \left\{N_1 (c\varepsilon - 1) + cN_2 \varepsilon\right\} \|\eta_x + \frac{1}{2} \psi_x\|^2 \\
- \left(N_2 C_{\varepsilon} - \frac{\lambda}{2}\right) g \circ \psi_{xx} + C_{\varepsilon} (N_1 + N_2) g \circ \psi_{xx} - 2M(t) \\
+ \left\{N_2 C_{\varepsilon} - \frac{e^{-2\tau} \alpha_1}{\tau} + \lambda \left(\frac{\alpha_2}{2} - \frac{\xi}{2\tau}\right)\right\} \|z(1)\|^2. \tag{105}
\]

We want to impose suitable conditions on the different parameters so that the coefficients on the right hand side of (105) are all strictly negative. That is to obtain the following inequalities

\[
N_1 c\varepsilon + N_2 (c\varepsilon^3 + 2\varepsilon + \rho \varepsilon) < \frac{N_1}{2} (l - \varepsilon), \tag{106}
\]

\[
N_1 < 2N_2 \left(\frac{\hat{h}^2}{12} \varepsilon - g_0\right), \tag{107}
\]

\[
N_1 < 2N_2 (\varepsilon - g_0), \tag{108}
\]

\[
cN_2 \varepsilon < N_1 (1 - c\varepsilon), \tag{109}
\]

\[
N_2 C_{\varepsilon} < \frac{\lambda}{2}, \tag{110}
\]

\[
N_2 C_{\varepsilon} < \frac{e^{-2\tau} \alpha_1}{\tau} - \lambda \left(\frac{\alpha_2}{2} - \frac{\xi}{2\tau}\right), \tag{111}
\]

\[
N_1 C_{\varepsilon} < -\lambda \left(\frac{\xi}{2\tau} + \frac{\alpha_2}{2} - \alpha_1\right). \tag{112}
\]
We observe that (106) and (107) will be satisfied if we choose \( \varepsilon > 0 \) small enough and such that
\[
\varepsilon < \max \left\{ 1, \frac{12g_0}{h^2}, g_0, \frac{1}{c} \right\}.
\]
To make (108) and (109) hold we can choose
\[
N_1 < \min \left\{ 2N_2 \left( \frac{h}{12} \varepsilon - g_0 \right), 2N_2 (\varepsilon - g_0) \right\}.
\]
Concerning (110), (111) and (112), we pick
\[
\lambda = \max \left\{ \frac{\alpha_2}{a_1 r}, -\frac{N_2C_\varepsilon}{(\frac{\alpha_2}{T} - \frac{\xi}{2T})}, -\frac{N_1C_\varepsilon}{(\frac{\xi}{2T} + \frac{\alpha_2}{T} - \alpha_1)}, c^* \right\},
\]
This completes the proof.

We consider the following two cases.

**Case I.** \( H(t) \) is linear:

By Multiplying (104) by \( \xi(t) \) and using (28), we get
\[
\frac{d}{dt} \mathcal{Y}(t)\xi(t) \leq -\beta_1 \xi(t) \mathcal{E}(t) + \beta_2 \xi(t) g \circ \psi_{xx}
\]
\[
\leq -\beta_1 \xi(t) \mathcal{E}(t) + \beta_2 g \circ \psi_{xx}
\]
\[
\leq -\beta_1 \xi(t) \mathcal{E}(t) - \beta_2 g' \circ \psi_{xx}
\]
\[
\leq -\beta_1 \xi(t) \mathcal{E}(t) - c\mathcal{E}'(t)
\]
which gives, as \( \xi \) is nonincreasing,
\[
\frac{d}{dt} \left( \mathcal{Y}(t)\xi(t) + c\mathcal{E}(t) \right) \leq -\beta_1 \xi(t) \mathcal{E}(t), \quad \forall t \geq t_1.
\]
Hence, using the fact that \( \mathcal{Y}(t)\xi(t) + c\mathcal{E}(t) \) is equivalent to \( \mathcal{E}(t) \), it is easy to see that
\[
\frac{d}{dt} \left( \mathcal{Y}(t)\xi(t) + c\mathcal{E}(t) \right) \leq -\beta_1 \xi(t) \mathcal{Y}(t)\xi(t) + c\mathcal{E}(t)), \quad \forall t \geq t_1.
\]
for some \( \beta_1 > 0 \). Then
\[
\mathcal{Y}(t)\xi(t) + c\mathcal{E}(t) \leq \gamma_2 e^{-\beta_1 \int_{t_1}^{t} \xi(s)ds}, \quad \forall t \geq t_1
\]
from which we deduce
\[
\mathcal{E}(t) \leq \gamma_2 e^{-\beta_1 \int_{t_0}^{t} \xi(s)ds}, \quad \forall t \geq t_1
\]
for some \( \gamma_2 > 0 \).

Furthermore, using the continuity and boundedness of \( \mathcal{E}(t) \) in \([0, t_1]\), we get
\[
\mathcal{E}(t) \leq \gamma_2 e^{-\beta_1 \int_{0}^{t} \xi(s)ds}, \quad \forall t \geq 0.
\]

**Case II.** \( H(t) \) is nonlinear:

Next, with \( f(t) = \int_{t}^{\infty} g(s)ds \), we use the functional
\[
\mathcal{K}(t) = \int_{0}^{t} f(t - s) \| \psi_{xx} \|^2 ds,
\]
**Lemma 4.7.** Assume that (A) and (B) hold. The functional \( \mathcal{K} \) satisfies, for any \( \varepsilon > 0 \), the estimate
\[
\frac{d}{dt} \mathcal{K}(t) \leq (2\varepsilon - 1) g \circ \psi_{xx} + (f(t) + C_\varepsilon) \| \psi_{xx} \|^2.
\]
Proof. By Young’s inequality and the fact $f'(t) = -g(t)$, we see that
\[
\frac{d}{dt} K(t) = f(0) \|\psi_{xx}\|^2 - \int_0^t g(t-s) \|\psi_{xx}(s)\|^2 ds
\]
\[
= -g \circ \psi_{xx} - 2 \langle \psi_{xx}, g \circ \psi_{xx} \rangle + f(t) \|\psi_{xx}\|^2. \tag{114}
\]
But
\[
-2 \langle \psi_{xx}, g \circ \psi_{xx} \rangle \leq C \varepsilon \|\psi_{xx}\|^2 + 2 \varepsilon g \circ \psi_{xx}, \tag{115}
\]
Combining (114) and (115), we obtain (113).

Let us introduce the functional
\[
\tilde{Y}(t) = Y(t) + \kappa K(t),
\]
where $\kappa$ is a positive constant. Then we have
\[
\tilde{Y}(t) \sim E(t).
\]
Therefore, it is always possible to pick $N_1$ (in 105) and $\kappa$ large enough to get
\[
\frac{d}{dt} \tilde{Y}(t) \leq -C E(t).
\]
Integrating over $(t_0, \infty)$, we get
\[
C \int_{t_0}^\infty E(s) ds \leq \tilde{Y}(t_0) < \infty. \tag{116}
\]
Next, let us define the functional $\mathcal{L}(t)$
\[
\mathcal{L}(t) = q \int_{t_0}^t \|\psi_{xx}(s) - \psi_{xx}(t-s)\|^2 ds, \quad \forall t \geq t_0.
\]
where $q > 0$. Thanks to (116), we can always choose $q$ such that
\[
\mathcal{L}(t) < 1, \quad \forall t \geq t_0. \tag{117}
\]
Next we define
\[
\mathcal{L}_g(t) = - \int_{t_0}^t g'(s) \|\psi_{xx}(t) - \psi_{xx}(t-s)\|^2 ds, \quad \forall t \geq t_0.
\]
Observe that
\[
\mathcal{L}_g(t) \leq -C E'(t),
\]
for some positive constant $C$.

Since $H$ is strictly convex on $(0, r]$ and $H(0) = 0$ we have
\[
H(\theta x) \leq \theta H(x), \quad (\theta, x) \in [0, 1] \times (0, r]. \tag{118}
\]
Using (A2), we get:
\[
\mathcal{L}_g(t) = \frac{1}{q \mathcal{L}(t)} \int_{t_0}^t \mathcal{L}(t) (-g'(s)) q \|\psi_{xx}(t) - \psi_{xx}(t-s)\|^2 ds
\]
\[
\geq \frac{1}{q \mathcal{L}(t)} \int_{t_0}^t \mathcal{L}(t) \xi(s) H(g(s)) q \|\psi_{xx}(t) - \psi_{xx}(t-s)\|^2 ds
\]
\[
\geq \frac{\xi(t)}{q \mathcal{L}(t)} \int_{t_0}^t H(\mathcal{L}(t)g(s)) q \|\psi_{xx}(t) - \psi_{xx}(t-s)\|^2 ds,
\]
Keeping in mind (117) and applying inequality (118) for \( \theta := L(t) \) and \( x = g(s) \), yields
\[
\mathcal{L}_g(t) \geq \frac{\xi(t)}{qL(t)} \int_{t_0}^{t} H(L(t)g(s)) q \| \psi_{xx}(t) - \psi_{xx}(t-s) \|^2 \, ds. \tag{119}
\]
Applying Jensen’s inequality in (29) for \( f(t)g(s) = L(t)g(s) \) and \( h(s) = q \| \psi_{xx}(t) - \psi_{xx}(t-s) \|^2 \), we obtain
\[
\mathcal{L}_g(t) \geq \frac{\xi(t)}{q} H \left( \frac{1}{L(t)} \int_{t_0}^{t} L(t)g(s) q \| \psi_{xx}(t) - \psi_{xx}(t-s) \|^2 \, ds \right)
\]
\[
= \frac{\xi(t)}{qL(t)} \left[ \int_{t_0}^{t} g(s) q \| \psi_{xx}(t) - \psi_{xx}(t-s) \|^2 \, ds \right].
\]
where \( \mathcal{H} \) is an extension of \( H \) such that \( \mathcal{H} \) is strictly increasing and strictly convex \( C^2 \) function on \((0, \infty)\) and this leads to
\[
\int_{t_0}^{t} g(s) q \| \psi_{xx}(t) - \psi_{xx}(t-s) \|^2 \, ds \leq \frac{1}{qH^{-1}} \left( \frac{qL_g(t)}{\xi(t)} \right),
\]
So (104) becomes
\[
\tilde{\mathcal{Y}}(t) \leq -\beta_1 \mathcal{E}(t) + \beta_2 \frac{1}{qH^{-1}} \left( \frac{qL_g(t)}{\xi(t)} \right), \quad \forall t > t_0. \tag{120}
\]
Let \( \epsilon_0 < r \), using the fact that \( \mathcal{E}' \leq 0, \mathcal{H} > 0, \mathcal{H}'' > 0 \), we observe that the functional \( \mathcal{N} \) defined by
\[
\mathcal{N}(t) := \mathcal{H}' \left( \epsilon_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)} \right) \mathcal{Y}(t) + \mathcal{E}(t),
\]
is equivalent to \( \mathcal{E} \).

Using (120), we find that \( \mathcal{N} \) satisfies
\[
\frac{d}{dt} \mathcal{N}(t) = \frac{\mathcal{E}'(t)}{\mathcal{E}(0)} \mathcal{H}'' \left( \epsilon_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)} \right) \mathcal{Y}(t) + \mathcal{H}' \left( \epsilon_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)} \right) \mathcal{Y}'(t) + \mathcal{E}'(t)
\]
\[
\leq \epsilon_0 \frac{\mathcal{E}'(t)}{\mathcal{E}(0)} \mathcal{H}'' \left( \epsilon_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)} \right) \mathcal{Y}(t)
\]
\[
+ \mathcal{H}' \left( \epsilon_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)} \right) \left[ -\beta_1 \mathcal{E}(t) + \beta_2 \frac{1}{qH^{-1}} \left( \frac{qL_g(t)}{\xi(t)} \right) \right] + \mathcal{E}'(t)
\]
\[
\leq -\beta_1 \mathcal{E}(t) \mathcal{H}' \left( \epsilon_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)} \right)
\]
\[
+ \frac{\beta_2}{q} \mathcal{H}^{-1} \left( \frac{qL_g(t)}{\xi(t)} \right) \mathcal{H}' \left( \epsilon_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)} \right) + \mathcal{E}'(t). \tag{121}
\]
Let us denote by \( \mathcal{L}^* \) the conjugate function of the convex function \( \mathcal{L} \) defined by
\[
\mathcal{L}^*(s) = \text{Sup}_{t \in \mathbb{R}^+} (st - \mathcal{L}(t)), \quad \text{then}
\]
\[
st \leq \mathcal{L}^*(s) + \mathcal{L}(t), \tag{122}
\]

and, thanks to the arguments given in [5, 12, 20, 34, 35]
\[ \mathcal{G}^+(s) = s(\mathcal{G}^{-1})^{-1}(s) - \mathcal{G}[(\mathcal{H}^{-1})^{-1}(s)], \quad \forall s \geq 0. \]
This and the definition of \( H \) give
\[ \mathcal{H}^+(s) = s(\mathcal{H}^{-1})^{-1}(s) - \mathcal{H}[(\mathcal{H}^{-1})^{-1}(s)], \quad \forall s \geq 0. \] (123)
Taking \( s := \frac{c_2}{q} \mathcal{H}' \left( \frac{\epsilon_0 \xi(t)}{E(0)} \right) \) and \( t := \mathcal{H}^{-1} \left( \frac{q \xi(t)}{\xi(t)} \right) \) in (122), then making use of (121), (122) and (123), we arrive at
\[
\frac{d}{dt} \mathcal{N}(t) \\
\leq -\beta_1 \mathcal{E}(t) \mathcal{H}' \left( \frac{\epsilon_0 \xi(t)}{E(0)} \right) + \mathcal{H}^{-1} \left( \frac{q \xi(t)}{\xi(t)} \right) + \mathcal{H}' \left( \frac{\epsilon_0 \xi(t)}{E(0)} \right) + \mathcal{E}'(t) \\
\leq -\beta_1 \mathcal{E}(t) \mathcal{H}' \left( \frac{\epsilon_0 \xi(t)}{E(0)} \right) + \mathcal{H}^{-1} \left( \frac{q \xi(t)}{\xi(t)} \right) + \beta_2 \mathcal{H}' \left( \frac{\epsilon_0 \xi(t)}{E(0)} \right) + \mathcal{E}'(t) \\
\leq -\beta_1 \mathcal{E}(t) \mathcal{H}' \left( \frac{\epsilon_0 \xi(t)}{E(0)} \right) + \mathcal{H}^{-1} \left( \frac{q \xi(t)}{\xi(t)} \right) + \beta_2 \epsilon_0 \mathcal{E}(t) \mathcal{H}' \left( \frac{\epsilon_0 \xi(t)}{E(0)} \right) + \mathcal{E}'(t). \] (124)
Next, multiplying (124) by \( \xi(t) \) and using the fact that \( \epsilon_0 \frac{\xi(t)}{E(0)} < r, \mathcal{H}' \left( \frac{\epsilon_0 \xi(t)}{E(0)} \right) = H' \left( \epsilon_0 \frac{\xi(t)}{E(0)} \right) \), we get
\[
\xi(t) \frac{d}{dt} \mathcal{N}(t) \\
\leq -\beta_1 \mathcal{E}(t) \mathcal{H}' \left( \frac{\epsilon_0 \xi(t)}{E(0)} \right) + q \mathcal{E}_g(t) \\
+ \beta_2 \mathcal{E}_0 \mathcal{E}(t) \mathcal{H}' \left( \frac{\epsilon_0 \xi(t)}{E(0)} \right) + \mathcal{E}'(t) \\
\leq -\beta_1 \mathcal{E}(t) \mathcal{H}' \left( \frac{\epsilon_0 \xi(t)}{E(0)} \right) + \beta_2 \mathcal{E}_0 \mathcal{E}(t) \mathcal{H}' \left( \frac{\epsilon_0 \xi(t)}{E(0)} \right) - \epsilon \mathcal{E}'(t),
\]
where \( \epsilon \) is a positive constant.

Now, let us define the functional \( \tilde{\mathcal{N}} \)
\[ \tilde{\mathcal{N}}(t) = \mathcal{N}(t) \xi(t) + \mathcal{E}(t). \]
It is not difficult to see that there exist positive constants \( \rho_1 \) and \( \rho_2 \) for which we have
\[ \rho_1 \mathcal{N}(t) \leq \mathcal{E}(t) \leq \rho_2 \mathcal{N}(t). \] (125)
Consequently, with an appropriate choice of \( \epsilon_0 \), then there exists a positive constant \( k \) such that
\[
\frac{d}{dt} \tilde{\mathcal{N}}(t) \leq -k \xi(t) \mathcal{H}' \left( \frac{\epsilon_0 \xi(t)}{E(0)} \right) = -k \xi(t) H_2 \left( \frac{\epsilon_0 \xi(t)}{E(0)} \right), \quad \forall t \geq t_0 \] (126)
where \( H_2(s) = sH'(\epsilon_0s) \).
Since \( H_2(s) = H'(\epsilon_0s) + \epsilon_0 s H''(\epsilon_0s) \), we use the strict convexity of \( H \) on \([0, r]\),
we observe that \( H_2 > 0, H_2' > 0 \) on \((0, r]\).
Defining now
\[ \mathcal{R}(t) = \frac{\delta \mathcal{N}(t)}{E(0)}, \]

thanks to (125) and (126) we have $E \sim \mathcal{R}$ and for a positive constant $\tilde{k}$

$$\frac{d}{dt} \mathcal{R}(t) \leq -\tilde{k} \xi(t) H_2(\mathcal{R}(t)), \quad \forall t \geq t_0.$$  

Then, integrating over $(t_0, t)$ yields

$$\int_{t_0}^{t} \frac{\mathcal{R}'(s)}{H_2(\mathcal{R}(s))} \leq -\int_{t_0}^{t} \tilde{k} \xi(s) ds,$$

and this leads to

$$\int_{t_0}^{\mathcal{R}(t)} \frac{\mathcal{R}'(s)}{H'(\mathcal{R}(s))} \geq \tilde{k} \int_{t_0}^{t} \xi(s) ds,$$

which gives us

$$\mathcal{R}(t) \leq \frac{1}{\tilde{k} t_0} H_1^{-1}\left(\tilde{k} \int_{t_0}^{t} \xi(s) ds\right),$$

where $H_1(t) = \int_{t_0}^{t} \frac{ds}{H'(s)}$.

This completes the proof.

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**REFERENCES**

[1] C. Abdallah, P. Dorato, J. Benitez-Read and R. Byrne, Delayed positive feedback can stabilize oscillatory systems, *Proceedings of the American Control Conference*, 2-4 June, San Francisco, CA, (1993), 3106–3107.

[2] F. Alabau-Boussouira, On convexity and weighted integral inequalities for energy decay rates of nonlinear dissipative hyperbolic systems, *Applied Mathematics and Optimization*, 51 (2005), 61–105.

[3] F. Alabau-Boussouira and P. Cannarsa, A general method for proving sharp energy decay rates for memory dissipative evolution equations, *C. R. Acad. Sci. Paris*, Ser. I, 347 (2009), 867–872.

[4] F. D. Araruna, P. B. E Silva and E. Zuazua, Asymptotic limits and stabilization for the 1-D Nonlinear Mindlin-Timoshenko system, *J. Syst. Sci. Complex.*, 23 (2010), 1–17.

[5] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer-Verlag, New York, 1989.

[6] V. Barbu, *Nonlinear Differential Equations of Monotone Types in Banach Spaces*, Springer, New York, 2010.

[7] A. Benaissa and A. Guesmia, Energy decay for wave equations of \( \varphi \)-Laplacian type with weakly nonlinear dissipation, *Electronic Journal of Differential Equations*, 2008, 1–22.

[8] A. Benaissa and N. Louhibi, Global existence and Energy decay of solutions to a nonlinear wave equation with a delay term, *Georgian Mathematical Journal*, 20 (2013), 1–24.

[9] A. Benaissa, A. Benguessoum and S. A. Messaoudi, Global existence and energy decay of solutions to a viscoelastic wave equation with a delay term in the nonlinear internal feedback, *Int. J. Dynamical Systems and Differential Equations*, 5 (2014), 1–26.

[10] S. Berrimi and S. A. Messaoudi, Existence and decay of solutions of a viscoelastic equation with a nonlinear source, *Nonlinear Analysis: Theory, Methods and Applications*, 64 (2006), 2314–2331.

[11] H. Brezis, *Analyse Fonctionnelle: Théorie et Applications*, Dunod, Paris, 1983.

[12] M. M. Cavalcanti, V. D. Cavalcanti and I. Lasiecka, Well-posedness and optimal decay rates for the wave equation with nonlinear boundary damping-source interaction, *Journal of Differential Equations*, 236 (2007), 407–459.

[13] M. M. Cavalcanti, V. N. Domingos Cavalcanti and J. A. Soriano, Exponential decay for the solution of semilinear viscoelastic wave equations with localized damping, *Electronic Journal of Differential Equations*, 2002, 1–14.
[14] M. M. Cavalcanti, V. N. Domingos Cavalcanti and P. Martinez, General Decay Rate Estimates for Viscoelastic Dissipative Systems, *Nonlinear Analysis*, 68 (2008), 177–193.

[15] M. M. Cavalcanti, V. N. Domingos Cavalcanti and M. L. Santos, Existence and uniform decay rates of solutions to a degenerate system with memory conditions at the boundary, *Appl. Math. Comput.*, 150 (2004), 439–465.

[16] I. Chueshov, M. Eller and I. Lasiecka, On the attractor for a semilinear wave equation with critical exponent and nonlinear boundary dissipation, *Comm. Partial Differ. Equ.*, 27 (2002), 1901–1951.

[17] I. Chueshov and I. Lasiecka, Global Attractors for von Kármán Evolutions with a Nonlinear Boundary Dissipation, *Journal of Differential Equations*, 198 (2004), 196–231.

[18] I. Chueshov and I. Lasiecka, Global attractors for Mindlin-Timoshenko plates and for their Kirchhoff limits, *Milan J. Math.*, 74 (2006), 117–138.

[19] C. M. Dafermos, Asymptotic stability in viscoelasticity, *Arch. Rational Mech. Anal.*, 37 (1970), 297–308.

[20] D. Toundykov, Uniform energy decay for a wave equation with partially supported nonlinear boundary dissipation without growth restrictions, *Discrete and Continuous Dynamical Systems*, 2 (2009), 67–94.

[21] R. Datko, J. E. Lagnese and M. P. Polis, An example on the effect of time delays in boundary feedback stabilization of wave equations, *SIAM Journal on Control and Optimization*, 24 (1986), 152–156.

[22] R. Datko, Not All Feedback Stabilized Hyperbolic Systems Are Robust with Respect to Small Time Delays in Their Feedbacks, *SIAM Journal on Control and Optimization*, 26 (1988), 697–713.

[23] J. F. Doyle, *Wave Propagation in Structures*, Springer-Verlag, New York, 1997.

[24] M. Eller, J. E. Lagnese and S. Nicaise, Decay rates for solutions of a Maxwell system with nonlinear boundary damping, *Computational and Applied Mathematics*, 21 (2002), 135–165.

[25] A. Favini, M. A. Horn, I. Lasiecka and D. Tartaru, Global existence, uniqueness and regularity of solutions to a von Kármán system with nonlinear boundary dissipation, *Diff. Integ. Eqns*, 9 (1996), 267–294.

[26] J. E. Lagnese, *Boundary Stabilization of Thin Plates*, SIAM, 1989.

[27] J. E. Lagnese and G. Leucering, Uniform stabilization of a nonlinear beam by nonlinear boundary feedback, *Journal of differential equation*, 91 (1991), 355–388.

[28] J. E. Lagnese and J. L. Lions, *Modelling Analysis and Control of Thin Plates*, RMA 6, Masson, Paris, 1988.

[29] I. Lasiecka, *Mathematical Control Theory of Coupled PDE’s (CBMS-NSF Regional Conference Series in Applied Mathematics)*, SIAM, Philadelphia, PA, 2002.

[30] I. Lasiecka and D. Toundykov, Energy decay rates for the semilinear wave equation with nonlinear localized damping and a nonlinear source, *Nonlinear Analysis*, 64 (2006), 1757–1797.

[31] I. Lasiecka and D. Tataru, Uniform boundary stabilization of semilinear wave equations with nonlinear boundary damping, *Differential and Integral Equations*, 6 (1993), 507–533.

[32] I. Lasiecka and X. Wang, Intrinsic decay rate estimates for semilinear abstract second order equations with memory, in: *New Prospects in Direct, Inverse and Control Problems for Evolution Equations*, in: *Springer INdAM*, Springer, Cham, 10 (2014), 271–303.

[33] J. L. Lions, *Quelques Methodes de Resolution des Problemes Aux Limites non Lineaires*, Dunod, Paris, (in French) 1969.
[40] W. J. Liu and E. Zuazua, Decay rates for dissipative wave equations, *Ricerche Mat.*, **48** (1999), 61–75.

[41] S. A. Messaoudi, General decay of the solution energy in a viscoelastic equation with a nonlinear source, *Nonlinear Analysis*, **69** (2008), 2589–2598.

[42] S. A. Messaoudi, General decay of solutions of a viscoelastic equation, *J. Math. Anal. Appl.*, **341** (2008), 1457–1467.

[43] J. E. Munoz Rivera and G. P. Menzala, Decay rates of solutions of a von Kármán system for viscoelastic plates with memory, *Quarterly of Applied Mathematics*, **57** (1999), 181–200.

[44] J. E. Munoz Rivera, H. Portillo Oquendo and M. L. Santos, Asymptotic behavior to a von Kármán plate with boundary memory conditions, *Nonlinear Analysis*, **62** (2005), 1183–1205.

[45] J. E. Munoz Rivera, H. Portillo Oquendo and M. L. Santos, Asymptotic behavior to a von Kármán plate with boundary memory conditions, *Nonlinear Analysis*, **62** (2005), 1183–1205.

[46] J. E. Munoz Rivera, H. Portillo Oquendo and M. L. Santos, Asymptotic behavior to a von Kármán plate with boundary memory conditions, *Nonlinear Analysis*, **62** (2005), 1183–1205.

[47] M. I. Mustapha, General decay result for nonlinear viscoelastic equation, *J. Math. Anal. Appl.*, **457** (2018), 134–152.

[48] M. I. Mustapha, Laminated Timoshenko beams with viscoelastic damping, *J. Math. Anal. Appl.*, **466** (2018), 619–641.

[49] M. I. Mustapha and S. A. Messaoudi, General stability result for viscoelastic wave equations, *J. Math. Phys.*, **53** (2012), 053702, 14pp.

[50] M. I. Mustapha and S. A. Messaoudi, General stability result for viscoelastic wave equations, *J. Math. Phys.*, **53** (2012), 053702, 14pp.

[51] S. Nicaise and C. Pignotti, Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks, *SIAM Journal on Control and Optimization*, **45** (2006), 1561–1585.

[52] S. Nicaise and C. Pignotti, Stabilization of the wave equation with boundary or internal distributed delay, *Differ. Integral Equ.*, **21** (2008), 935–958.

[53] J. Y. Park and S. H. Park, Uniform decay for a von Kármán Plate Equation with a Boundary Memory Condition, *Mathematical Methods in the Applied Sciences*, **28** (2005), 2225–2240.

[54] J. Y. Park and S. H. Park, Uniform decay for a von Kármán Plate Equation with a Boundary Memory Condition, *Mathematical Methods in the Applied Sciences*, **28** (2005), 2225–2240.

[55] J. Y. Park and S. H. Park, Uniform decay for a von Kármán Plate Equation with a Boundary Memory Condition, *Mathematical Methods in the Applied Sciences*, **28** (2005), 2225–2240.