Experimental Realization of an Exact Solution to the Vlasov Equations for an Expanding Plasma

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We study the expansion of ultracold neutral plasmas in the regime in which inelastic collisions are negligible. The plasma expands due to the thermal pressure of the electrons, and for an initial spherically symmetric Gaussian density profile, the expansion is self-similar. Measurements of the plasma size and ion kinetic energy using fluorescence imaging and spectroscopy show that the expansion follows an analytic solution of the Vlasov equations for an adiabatically expanding plasma.

Exactly solvable problems are rare in physics and serve as ideal models that provide a starting point for understanding more complex systems. Here, we report the experimental realization of a laser-produced plasma whose dynamics can be described by an exact analytic solution to the Vlasov equations [1,2], which are central equations in the kinetic theory of plasmas. Expansion into a surrounding vacuum is fundamentally important and typically dominates the dynamics of plasmas created with pulsed lasers [3], such as in experiments pursing inertial confinement fusion [4], x-ray lasers [5], or the production of energetic (>MeV) ions through irradiation of solids [3,4], thin foils [3,5,10,11,12,13,14,15], clusters [16].

We study plasma expansion with ultracold neutral plasmas (UNPs) [17,18], which are created by photoionizing laser-cooled strontium atoms [19] just above the ionization threshold. These systems occupy a new regime of neutral plasma physics, and their well-controlled initial conditions and relatively slow dynamics have enabled precise studies of strongly coupled plasma properties [20,21]. Here we use fluorescence imaging and spectroscopy for the first time on these systems. These powerful diagnostics allows us to apply the advantages of UNPs to a new class of important problems.

The investigation of plasma expansion dates back many decades [22,23]. Recently, exact solutions for spatially finite plasmas expanding into vacuum were identified for a 1-dimensional plasma [1] and later extended to 3-dimensions [2,24]. This work was motivated by plasmas produced with short-pulse lasers.

The Vlasov equations, along with Poisson’s equation, describe the evolution of the electron (α = e) and ion (α = i) distribution functions, \( f_α(r,v) \). The Vlasov equations neglect radiative processes and collisional phenomena such as electron-ion thermalization and three-body recombination [25], but this formalism describes many types plasmas and is part of the foundation of kinetic theory.

Among broad classes of general analytic solutions to the Vlasov equations [24], UNPs realize a particular solution, that even describes collisional systems and is valid for a quasi-neutral plasma with spherical Gaussian distribution functions

\[
\begin{align*}
  f_α \propto \exp \left[ -\frac{r^2}{2\sigma^2} - \frac{m_{\alpha}(v-u)^2}{2k_BT_α} \right].
\end{align*}
\]

Quasi-neutrality is defined by \( n_e \approx n_i \), where the electron and ion densities are \( n_α(r) = \int dv f_α(r,v) \). \( T_α \) are the electron and ion temperatures, and the local average velocity varies in space according to \( u(r,t) = \hat{v}(d) \hat{r} \). The temperatures must scale as \( \sigma^2 T_α = \text{constant} \) [2], which is expected for adiabatic cooling in a spherically symmetric UNP [26,28].

Under these conditions, the plasma dynamics is given by \( T_α(t) = T_α(0) / (1 + t^2/\tau_{\text{exp}}^2) \), where the characteristic expansion time \( \tau_{\text{exp}} \) is given by \( \tau_{\text{exp}} = \sqrt{m_i \sigma(0)^2 / k_B [T_e(0) + T_i(0)]} \). Also,

\[
\begin{align*}
  \sigma(t)^2 &= \sigma(0)^2 (1 + t^2/\tau_{\text{exp}}^2),
\end{align*}
\]

and

\[
\begin{align*}
  v_{i,\text{rms}} &= \sqrt{\frac{k_B \left\{ \frac{t^2}{\tau_{\text{exp}}^2} [T_e(t) + T_i(t)] + T_i(t) \right\}}{m_i \sigma(t)^2}},
\end{align*}
\]

describe the evolution of the characteristic plasma size and ion velocity, which are important experimental observables. We define the rms 1-dimensional ion velocity \( \sqrt{\langle (\mathbf{v} \cdot \hat{y})^2 \rangle} \equiv v_{i,\text{rms}} \), where \( \hat{y} \) is the laser propagation direction, \( \mathbf{v} \) is the total ion velocity including random thermal motion and expansion, and the angled brackets refer to an average over the distribution function.

An intuitive explanation for the self-similar nature of the expansion is that thermal pressure produces an average radial acceleration

\[
\begin{align*}
  \mathbf{u} = -\frac{k_B [T_e(t) + T_i(t)]}{m_i} \nabla n(r,t) &= \frac{k_B [T_e(t) + T_i(t)]}{m_i \sigma(t)^2} \mathbf{r}.
\end{align*}
\]

The simplification implied by the last equality is only valid for a spherical Gaussian plasma, and the linearity in \( \mathbf{r} \) preserves the shape of the distribution functions.

Plasmas produced with solid targets, foils, and clusters are often quasi-neutral and well-described by the...
Vlasov equations, and electrons cool adiabatically during much of their evolution, but experimental conditions studied are typically very complicated and evolve extremely rapidly, which frustrates detailed comparison between experiment and theory. Final ion kinetic energy distributions have been shown to agree with simple models [15, 16], but in general, these systems lack the Gaussian distribution functions necessary to realize the analytically describable self-similar expansion.

For appropriate initial conditions, UNPs fulfill the requirements for the analytic solution. UNPs have ion temperatures of about 1 K determined by disorder-induced heating [20, 27]. Electron temperatures ranging from 1 to 1000 K are set by the detuning, $E_c$, of the ionization laser above threshold. The peak densities are $\sim 10^{15} \text{m}^{-3}$, and the profile follows that of the laser-cooled atom cloud, which we adjust to have a spherically symmetric Gaussian. The photoionization pulse length ($\sim$ 10 ns) is much less than the expansion time scale ($\sim$ 10 µs).

The electron distribution equilibrates locally within 100 ns and globally within 1 µs after photoionization [20, 28]. This ensures a Gaussian electron distribution function at the start of the expansion. Despite these very rapid electron-electron collisions the corresponding collision integral vanishes for the spherically symmetric, Gaussian velocity distribution eq.(1). Hence the highly collisional UNP considered here provides an ideal model for truly collisionless plasmas behavior.

Ions reach local thermal equilibrium within a few 100 ns [20]. They do not equilibrate globally on the time scale of the expansion [21], but the ions are so cold compared to the electrons, that the lack of a global ion temperature does not cause any significant deviation from the exact solution. The low ion temperature also implies that the ions form a strongly coupled fluid [20, 29], which, however, negligibly affects the plasma expansion [26].

The analytic expansion solution has been discussed previously for UNPs [26, 28, 33], and it has been checked against average terminal ion expansion velocities [31] and measured electron temperatures [32] that qualitatively affirm the importance of adiabatic cooling for appropriate initial conditions. The lack of spatial and temporal resolution, available here, however prevented conclusive tests of the analytic predictions. Cummings et al. [33, 34] adapted the formalism of [26, 28, 31] and used light-scattering from a small region of the plasma to study the expansion of ultracold plasmas with an elongated aspect ratio, but they found significant deviations from the predictions of the model that perhaps arose because the condition of spherical symmetry was not fulfilled.

To demonstrate that ultracold neutral plasmas can realize the analytic expansion solution, we will first describe our diagnostic and show that the plasma remains Gaussian during its expansion. Then we will show the size variation and ion velocity are given by Eqs. 2 and 3 respectively.

![Diagram of UNP recording fluorescence.](image)

**FIG. 1:** Recording fluorescence of UNPs. The correlation between position and expansion velocity (red arrows) produces a striped image when the Doppler shift due to expansion exceeds the Doppler shift associated with thermal ion velocity.

Figure 1 shows a schematic of the fluorescence imaging experiment. A laser beam that is near resonance with the $^2S_{1/2} - ^2P_{1/2}$ transition in Sr$^+$ at $\lambda = 422$ nm propagates along $\hat{y}$ and illuminates the plasma. Fluorescence in a perpendicular direction ($\hat{z}$) is imaged with a 1:1 relay telescope onto an image-intensified CCD camera. The 422 nm light is typically applied in a 1 µs pulse to provide temporal resolution, and the intensity is only a few mW/cm$^2$, which is low enough to avoid optical pumping to the metastable $^2D_{3/2}$ state.

A general expression for the fluorescence is

$$F(\nu, x, y) \propto \int ds \frac{1}{1 + \frac{(2(\nu - \nu_0) - 2\gamma_{eff})^2}{2\gamma_{eff}}} \times \left\{ \frac{dz n(r)}{\sqrt{2\pi}\sigma_D[T_{i,\text{therm}}(r)]} \exp \left\{ \frac{s - (\nu_0 + \nu_{\exp}^D(r))^2}{2\sigma_D^2[T_{i,\text{therm}}(r)]} \right\} \right\},$$

where $\nu$ is the laser frequency and $\gamma_{eff} = \gamma_0 + \gamma_{laser}$ is the sum of the natural linewidth of the transition ($2\pi \times 20$ MHz) and the imaging laser linewidth ($2\pi \times 8$ MHz). $T_{i,\text{therm}}(r)$ is the local temperature of the ions describing random thermal motion, which gives rise to the Doppler width $\sigma_D$. Due to the directed expansion velocity, the average resonance frequency of the transition for atoms at $r$ is Doppler-shifted from the value for an ion at rest, $\nu_0$, by $\nu_{\exp}^D(r) = \mathbf{u}(r) \cdot \mathbf{v}_{\exp}/\lambda$. The spatial variation in $T_{i,\text{therm}}$, which varies as $n_i^{1/3}$ [21], is small compared to the directed expansion energy, so $T_{i,\text{therm}}$ can be taken as constant.

Images can be analyzed in several different ways, which each provide access to different plasma properties. Summing a series of images taken at equally spaced frequencies covering the entire ion resonance is equivalent to integrating $F(\nu, x, y)$ over frequency. This yields a signal...
proportional to the areal plasma density, \( \int n(x, y, z)dz \), which for a Gaussian density distribution should take the form \( n_{\text{real}}(x, y) = \sqrt{\frac{2}{\pi}} n_0 \exp \left[ -\left( \frac{x^2 + y^2}{2\sigma^2} \right) \right] \).

Figure 2 shows that Gaussian fits of the areal density are excellent during more than a factor of two change in \( \sigma \) from the earliest times until the signal expands beyond the range of the imaging system. This provides direct confirmation of the self-similar nature of the expansion.

There is no sign of any deviation from the gaussian shape at large radius. This might seem surprising because self-similarity follows from Eq. 4, and this equation must break down at large radius. The areal density (arbitrary units) is found by summing together 50 images taken at equally spaced frequencies that fully cover the ion resonance. The Gaussian fits (solid line) to linear cuts show that the expansion is self-similar. The time indicated is the evolution time since plasma creation. The right-hand axes show that the differences between data and fit are small.

The evolution of \( \sigma \) can be extracted from fits such as in Fig. 2 and \( \sigma(0) \) are fit, while \( \sigma_i(0) \) is taken from the theoretical expression for disorder-induced heating [20, 27]. Uncertainties in \( 2E_e/3k_B \) reflect 1-standard-deviation calibration uncertainty in the wavelength of the photoionizing laser. Quoted uncertainties in \( T_e(0) \) are statistical, but there is an additional systematic uncertainty of a few percent arising from calibration of the imaging-system magnification and overlap of the plasma and fluorescence excitation laser. Statistical uncertainty in the measurement of \( \sigma \) is less than the size of the plotting symbols. Initial peak densities for these samples are \( \sim 10^{16} \text{ m}^{-3} \).

The expansion is sensitive to the electron temperature and initial size, and higher electron temperature and smaller size lead to a faster expansion because this increases the thermal pressure (Eq. 4). Fits of the data using Eq. 2 yield values of \( T_e(0) \) that agree reasonably well with the expected values of \( 2E_e/3k_B \). This confirms that on the time scale of the expansion there are no significant collisional or radiative processes changing the electron temperature such as three-body recombination or electron-ion thermalization, as assumed in the Vlasov equations.

In order to completely characterize the plasma expansion we also measure the light-scattering resonance spectrum, formed from the integrated fluorescence in each of a series of images taken at different frequencies, as shown in Fig. 4. Eq. 5 combined with the expansion velocity \( u(r) \) predicted by the Vlasov equations, implies that the resulting signal \( \int dx dy F(\nu, x, y) \) should take the form of a Voigt profile. The rms width of the Gaussian component of this profile arising from Doppler broadening reflects both thermal ion motion and directed expansion and is given by \( \sigma_D = v_{i, \text{rms}}/\lambda \).

Equation 8 provides an excellent fit to the data, and the extracted values of \( T_e(0) \) are consistent with \( 2E_e/3k_B \) within experimental uncertainty. The small initial offset of \( v_{i, \text{rms}} \) in each plot is due to disorder-induced heating of the ions within the first microsecond [20], which locally produces a thermal ion velocity distribution. But \( v_{i, \text{rms}} \) quickly increases well above this value as the elec-
tron pressure drives the plasma expansion and electron thermal energy is converted into directed radial ion velocity. As $\sigma$ increases and the electrons cool adiabatically, the acceleration decreases (Eq. [1]) and the ions eventually reach terminal velocity when essentially all electron kinetic energy is transferred to the ions.

Our measured density profiles (Fig. 3) confirm the validity of Eq. [8] which shows that the ion acceleration is sensitive at all times to the instantaneous electron temperature and width of the plasma. Agreement between experiment and theory for both the size evolution and the temperature and width of the plasma. Agreement between experiments and theory for both the size evolution and the temperature and width of the plasma. Agreement between experiments and theory for both the size evolution and the temperature and width of the plasma. Agreement between experiments and theory for both the size evolution and the temperature and width of the plasma. Agreement between experiments and theory for both the size evolution and the temperature and width of the plasma.

We have demonstrated a plasma in which the expansion matches an analytic solution to the Vlasov equations [2] proposed as a basic model for the dynamics of quasineutral laser-produced plasmas. To realize this situation experimentally, it is necessary to create UNPs with a spherical Gaussian density distribution. Relatively high initial electron temperature and low density are also required to avoid collisional and radiative processes [28, 31, 38, 39] that would heat electrons or lead to electron-ion equilibration. In future studies, deviations from the model can be used to study these collisional processes when they become important. The expansion dynamics shown here provide a general tool for developing a greater understanding and intuition for the expansion of plasmas.

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