MAXIMAL ABELIAN SUBALGEBRAS OF $\mathcal{O}_n$

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Abstract

We consider maximal abelian subalgebras of $\mathcal{O}_n$ which are invariant to the standard circle action. It turns out that these are all contained in the zero grade of $\mathcal{O}_n$. Then we consider shift invariant maximal abelian subalgebras of the zero grade, which are also invariant to a “second shift” map, and show that these are just infinite tensor products of diagonal matrices in the standard UHF picture of the zero grade.

1 Introduction

In this note, we are concerned with certain abelian subalgebras of Cuntz algebra $\mathcal{O}_n$. Let $\mathcal{O}_n = C^*(s_1, \ldots, s_n)$. As usual, for $\mu = i_1 \ldots i_k$, $i_j \in \{1, \ldots, n\}$, we let $|\mu| = k$ be the length of $\mu$ and denote $s_{i_1} \ldots s_{i_k}$ by $s_{\mu}$. The set of all finite words in $\{1, \ldots, n\}$ is denoted by $W(n)$. Let

\[
\sigma(x) = \sum_{i=1}^{n} s_i x s_i^* \tag{1}
\]

be the canonical endomorphism on $\mathcal{O}_n$,

\[
\omega_t(s_i) = ts_i, \quad i = 1, \ldots, n, \quad t \in \mathbb{T} \tag{2}
\]

the standard circle action, and consider the $C^*$-subalgebra $\mathcal{D}$ of $\mathcal{O}_n$ defined as

\[
\mathcal{D} = C^*\{s_{\mu}s_{\mu}^*; \mu \in W(n)\} \tag{3}
\]

Then $\mathcal{D}$ is an abelian subalgebra (cf. [2]), and below we list some of its properties:

- $\sigma(\mathcal{D}) \subset \mathcal{D}$
- $\omega_t(\mathcal{D}) \subset \mathcal{D}$; in fact, even more is true: $\omega_t(d) = d$, for all $t \in \mathbb{T}$ and $d \in \mathcal{D}$; hence, $\mathcal{D} \subset \mathcal{O}_n^0$, the fixed-point algebra under the circle action
- $\mathcal{D}$ is a maximal abelian subalgebra of $\mathcal{O}_n^0$; furthermore, $\mathcal{D}$ is maximal abelian in $\mathcal{O}_n$ (cf. [3])
The subject of the present work is to give a characterisation of \( D \) in the above terms. More precisely, we prove (cf. 4.4):

**Theorem 1.1** Let \( A \) be a maximal abelian subalgebra of \( \mathcal{O}_n \) such that \( \omega(A) \subset A \), \( \sigma(A) \subset A \) and \( \tilde{\sigma}(A) \subset A \). Then there is an automorphism \( \alpha_U \) of \( \mathcal{O}_n \), determined by a unitary \( U \in \mathbb{M}_n(\mathbb{C}) \), such that \( \alpha_U(A) = D \). Furthermore, \( \alpha_U \) commutes with \( \omega, \sigma \) and \( \tilde{\sigma} \).

We now describe the above notation. Let \( \psi : \mathcal{O}_n \rightarrow \mathbb{M}_n \otimes \mathcal{O}_n \) be given by

\[
\begin{bmatrix}
    s_1 x s_1 & \cdots & s_1 x s_n \\
    \vdots & \ddots & \vdots \\
    s_n x s_1 & \cdots & s_n x s_n
\end{bmatrix}
\]

(4)

Then \( \psi \) is an isomorphism from \( \mathcal{O}_n \) onto \( \mathbb{M}_n \otimes \mathcal{O}_n \) (cf. [1]).

The endomorphism \( \tilde{\sigma} : \mathcal{O}_n \rightarrow \mathcal{O}_n \) is the “second shift”, given by the formula

\[
\tilde{\sigma} = \sum_{i,j,k} s_i s_k s_i^* x s_j s_k^* s_j^*,
\]

(5)

and determined by the following diagram

\[
\begin{array}{ccc}
\mathcal{O}_n & \xrightarrow{\psi} & \mathbb{M}_n \otimes \mathcal{O}_n \\
\tilde{\sigma} \downarrow & & id \downarrow \otimes \sigma \\
\mathcal{O}_n & \xleftarrow{\psi^{-1}} & \mathbb{M}_n \otimes \mathcal{O}_n
\end{array}
\]

(6)

Note that it follows easily that

\[
\tilde{\sigma}(1) = 1, \quad \tilde{\sigma}(s_is_j^*) = s_is_j^*,
\]

and

\[
\tilde{\sigma}(s_is_\mu s_\nu^* s_j^*) = s_i \sigma(s_\mu s_\nu^*) s_j^*,
\]

for all \( \mu, \nu \in W(n) \) such that \( |\mu| = |\nu| \).

Finally, for \( u = [u_{ij}]_{i,j=1}^n \) a unitary in \( M_n(\mathbb{C}) \), let \( U = \sum_{i,j} u_{ij} s_is_j^* \). Then \( U \) is a unitary in \( \mathcal{O}_n \), and the map \( s_i \mapsto Us_i, \ i = 1, \ldots, n \) extends to an isomorphism of \( \mathcal{O}_n \), denoted \( \alpha_U \).

We also prove the same result as 4.4 but starting from slightly different assumptions (cf. 4.5):

**Theorem 1.2** Let \( A \) be a maximal abelian algebra in \( \mathcal{O}_n \) such that \( A \cap \mathcal{O}_0^0 \) is maximal abelian in \( \mathcal{O}_n^0 \), \( \sigma(A) \subset A \) and \( \tilde{\sigma}(A) \subset A \). Then there is an automorphism \( \alpha_U \) of \( \mathcal{O}_n \), determined by a unitary \( U \in M_n(\mathbb{C}) \), such that \( \alpha_U(A) = D \). Furthermore, \( \alpha_U \) commutes with \( \sigma \) and \( \tilde{\sigma} \).

The paper is organised as follows. In Section 2, we show that an algebra which is maximal in the class of abelian algebras that are invariant under the action of the circle is indeed maximal.
abelian. In Section 3, we describe maximal abelian subalgebras that are invariant under two shift maps. Both Section 2 and 3 are done in a slightly more general setting. Finally, in Section 4, we apply these results to the particular case of $O_n$ and easily obtain the stated characterisation of the abelian subalgebra $A$.

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2 Maximal (abelian $T$–invariant $*$–subalgebras)

Take a C∗–algebra $B$, and let $\omega : \mathbb{T} \to Aut(B)$ be a homomorphism that is continuous in the topology of pointwise convergence. This means that for each $b \in B$ the map $t \mapsto \omega_t(b)$ is continuous, and the triple $(B, \mathbb{T}, \omega)$ is called a C∗–dynamical system (cf. [4, 7.4.1]).

Consider the class of $*$–subalgebras $A$ of $B$ which are abelian and $T$–invariant (i.e. $\omega_t(a) \in A$ for all $a \in A$ and all $t \in \mathbb{T}$). We will call a subalgebra which is maximal in this class a maximal (abelian $T$-invariant $*$–subalgebra), using brackets to avoid ambiguity. Our task is to show that we can remove the brackets, that is, we show that a maximal (abelian $T$-invariant $*$–subalgebra) is actually maximal abelian.

The main result of this section is 2.7. For convenience we assume that $B$ is a subalgebra of $B(H)$ for some Hilbert space $H$.

**Definition 2.1** Let $(B, \mathbb{T}, \omega)$ be a C∗–dynamical system, and define $B_n = \{ b \in B : \omega_t(b) = t^*b, \text{ for all } t \in \mathbb{T} \}$. Let $\pi_n : B \to B$ be defined as

$$\pi_n(b) = \int_{\mathbb{T}} t^{-n}\omega_t(b)dt,$$

where $dt$ is the Haar measure on $\mathbb{T}$ (i.e. normalised Lebesgue measure). Then each $B_n$ is a closed linear subspace in $B$, and each $\pi_n$ is a linear contraction with image $B_n$. Also $B_0$ is a subalgebra and $\pi_0$ is a conditional expectation to $B_0$. Furthermore, we have

$$\pi_m(b) = \delta_{n,m}b, \ b \in B_n.$$

**Proposition 2.2** The commutant $A' \cap B$ of $A$ in $B$ is an $T$–invariant $*$–subalgebra of $B$, and $A \subset A'$.

Further the image of $A' \cap B$ under $\pi_n$ is contained in $A' \cap B$.

**Proof:** Suppose that $b \in A' \cap B$. Since $A$ is a $*$–subalgebra, for all $a \in A$, $a^*b = ba^*$, so we see that
Proposition 2.3 The image of $A' \cap B$ under $\pi_0$ is contained in $A$.

Proof: Suppose that $b \in A' \cap B$. By considering $b + b^*$ and $i(b - b^*)$ we may suppose that $b$ is actually Hermitian. Then $\pi_0(b)$ is Hermitian and fixed by the circle action, so the algebra generated by $A$ and $\pi_0(b)$ is an abelian circle invariant *-subalgebra of $B$, and so $\pi_0(b) \in A$ by maximality. □

Proposition 2.4 The image of $A' \cap B$ under $\pi_n$ is contained in $A$ for all $n \in \mathbb{Z}$.

Proof: Suppose that $b \in A' \cap B$. Then $\pi_n(b) \in A'$, and $\pi_n(b)\pi_n(b)^*$ is an Hermitian circle invariant element of $A' \cap B$. By maximality we then have $\pi_n(b)\pi_n(b)^* \in A$, and similarly we have $\pi_n(b)^*\pi_n(b) \in A$. This means that $\pi_n(b)$ commutes with both $\pi_n(b)\pi_n(b)^*$ and $\pi_n(b)^*\pi_n(b)$, and so is normal by the next lemma. Now the algebra generated by $A$, $\pi_n(b)$ and $\pi_n(b)^*$ is an abelian circle invariant *-algebra, so $\pi_n(b) \in A$ by maximality. □

We initially proved the next lemma using polar decomposition. The following much simpler proof is due independently to S. Wassermann and N–C. Wong:

Lemma 2.5 Let $x \in B$ commute with both $xx^*$ and $x^*x$. Then $x$ is normal.

Proof: We need to show that $xx^* - x^*x = 0$. Since $xx^* - x^*x$ is selfadjoint, that is equivalent to $(xx^* - x^*x)^2 = 0$, which follows immediately, since the assumption implies

\[ xx^*x = x(x^*)x^* \text{ and } xx^*x = x^*(xx^*)^x \]

□

Proposition 2.6 $A' \cap B \subset A''$.

Proof: Take $b \in A' \cap B$. For any $\xi, \eta \in H$ we define a continuous function $f : \mathbb{T} \to \mathbb{C}$ by $f(t) = \langle \xi, \omega_t(b)(\eta) \rangle$. By Fourier analysis we get Fourier coefficients $f_n = \langle \xi, \pi_n(b)(\eta) \rangle$, where

\[ \sum_{n=-m}^m t^n f_n \to f(t), \quad t \in \mathbb{T} \] (7)

in the $L^2(\mathbb{T})$ topology as $m \to \infty$. Since $B \subset B(H)$, we can put $\eta = c(\kappa)$ for some $c \in A'$ and $\kappa \in H$. Then as $\pi_n(b) \in A$ we see that $f_n = \langle \xi, c\pi_n(b)(\kappa) \rangle = \langle c^*\xi, \pi_n(b)(\kappa) \rangle$. Now we can write

\[ \sum_{n=-m}^m t^n f_n \to \langle c^*\xi, \omega(b)(\kappa) \rangle = \langle \xi, c\omega(b)(\kappa) \rangle, \quad t \in \mathbb{T} \]

in the $L^2(\mathbb{T})$ topology as $m \to \infty$. The two limits are the same in $L^2(\mathbb{T})$, so $\langle \xi, c\omega_t(b)(\kappa) \rangle = \langle 0, \omega_t(b)c(\kappa) \rangle$ almost everywhere in $\mathbb{T}$. By continuity they are the same at $t = 1$, so $cb = bc$. □
Theorem 2.7 Let \((B, \mathcal{T}, \omega)\) be a \(C^*\)–dynamical system, and suppose that \(A\) is a maximal (abelian \(\mathcal{T}\)–invariant \(*\)–subalgebra) of \(B\). Then \(A\) is a maximal abelian subalgebra of \(B\).

Proof: By the previous proposition we see that \(A' \cap B\) is abelian. Now \(A' \cap B\) is an abelian circle invariant \(*\)–subalgebra of \(B\) which contains \(A\), so \(A = A' \cap B\) by the maximality condition on \(A\).

But the equation \(A = A' \cap B\) means that \(A\) is maximal abelian in \(B\). □

The following example – due to R. Exel – shows that the previous theorem does not hold for an arbitrary dynamical system \((B, G, \omega)\), even with \(G\) compact:

Example 2.8 Consider the adjoint action of \(SU_2\) on \(M_2(\mathbb{C})\). The subalgebra consisting of the complex multiples of the identity is maximal among the class of abelian \(SU_2\)–invariant \(*\)–subalgebras.

However, it is not maximal abelian, as it is properly contained in the diagonal matrices.

3 Maximal abelian \(*\)–subalgebras of \(\mathcal{O}_n\) contained in the zero grade

Let \(B\) be a unital \(C^*\)–algebra with a given isomorphism \(\psi : B \to M_n \otimes B\) with \(\psi(1) = I_n \otimes 1\), \(I_n\) being the identity matrix in \(M_n\). We define isomorphisms \(\psi_m : B \to (M_n) \otimes^m \otimes B\) \((m \geq 0)\) recursively, beginning with \(\psi_0 : B \to B\) the identity, \(\psi_1 = \psi\), and continuing by defining \(\psi_{m+1}\) to be the composition

\[
B \xrightarrow{\psi_m} (M_n) \otimes^m \otimes B \xrightarrow{id \otimes \psi} (M_n) \otimes^{m+1} \otimes B
\]

where \(id : M_n \to M_n\) is the identity map. Now we define an algebra map \(\kappa_m : M_n \otimes^m \otimes B\) by

\[
\kappa_m(x) = \psi^{-1}_m(x \otimes 1).
\]

Since \(\psi(1) = I_n \otimes 1\) we get the commutative diagram

\[
\begin{array}{ccc}
M_n \otimes^m & \xrightarrow{\kappa_m} & B \\
\downarrow id \otimes I_n & & \downarrow id_B \\
M_n \otimes^{m+1} & \xrightarrow{\kappa_{m+1}} & B
\end{array}
\]

Define \(C \subset B\) to be the closure of the union of the subalgebras \(\kappa_m(M_n \otimes^m)\).

We can define shift maps \(\sigma_m : B \to B\) \((m \geq 1)\) by the composition

\[
B \xrightarrow{\psi_m^{-1}} (M_n) \otimes^{m-1} \otimes B \xrightarrow{id \otimes f \otimes \psi} (M_n) \otimes^m \otimes B \xrightarrow{\psi^{-1}} B,
\]

where \(f : B \to M_n \otimes B\) is the algebra map \(f(b) = I_n \otimes b\).

Let \(E_{ij} \in M_n\) be the matrix with entry 1 in row \(i\) column \(j\), and zeros elsewhere. Define a linear map \(e_{ij} : M_n \to \mathbb{C}\) by \(e_{ij}(E_{kl}) = \delta_{ik}\delta_{jl}\). Now we can define a map \(\chi_{mij} : B \to B\) \((m \geq 1)\) by the composition

\[
B \xrightarrow{\psi_m} (M_n) \otimes^m \otimes B \xrightarrow{id \otimes e_{ij} \otimes id} (M_n) \otimes^{m-1} \otimes B \xrightarrow{\psi^{-1}_m} B.
\]
Proposition 3.1 For all \( b \in B \) and \( y \in M_n \otimes B \), \((e_{ij} \otimes \text{id}_B)(f(b).y) = b.((e_{ij} \otimes \text{id}_B)(y))\) and \((e_{ij} \otimes \text{id}_B)(y.f(b)) = ((e_{ij} \otimes \text{id}_B)(y)).b\).

Proof: Take \( y = y_1 \otimes y_2 \in M_n \otimes B \) (linear combinations of terms of this form are dense in \( M_n \otimes B \)). Then
\[ (e_{ij} \otimes \text{id}_B)(f(b).y) = (e_{ij} \otimes \text{id}_B)((I \otimes b)(y_1 \otimes y_2)) = ((e_{ij} \otimes \text{id}_B)(y)).b. \]
The other way round is the same. \( \square \)

Corollary 3.2 For all \( b, c \in B \), \( \chi_{mij}(\sigma_m(b).c) = b.\chi_{mij}(c) \) and \( \chi_{mij}(c.\sigma_m(b)) = \chi_{mij}(c).b \).

Proof: This is essentially the same as the previous proposition. \( \square \)

Corollary 3.3 Suppose that \( A \) is a maximal abelian \( * \)-subalgebra of \( B \), obeying the condition \( \sigma_m(A) \subset A \). Then for all \( 1 \leq i, j \leq n, \chi_{mij}(A) \subset A \).

Proof: Take \( a \in A \). Then for all \( a' \in A \) we have \( \sigma_m(a').a = a.\sigma_m(a') \). Applying \( \chi_{mij} \) to this we get \( a'.\chi_{mij}(a) = \chi_{mij}(a).a', \) so \( \chi_{mij}(a) \in A \) by maximality. \( \square \)

Proposition 3.4 Suppose that \( A \) is a maximal abelian \( * \)-subalgebra of \( B \), obeying the condition \( \sigma_1(A) \subset A \). Then \( \psi_m(A) \subset M_n^m \otimes A \).

Proof: First note that \( \psi(a) = \sum_{ij} E_{ij} \otimes \chi_{1ij}(a) \), so \( \psi(A) \subset M_n \otimes A \) by the last proposition. The rest follows by induction. \( \square \)

Definition 3.5 Take a unital algebra map \( \phi : A \rightarrow \mathbb{C} \), and extend it to a positive contraction \( \phi : B \rightarrow \mathbb{C} \). Then we define a map \( \phi_m : B \rightarrow M_n^{\otimes m} \) by
\[ B \xrightarrow{\psi_m} M_n^{\otimes m} \otimes B \xrightarrow{\text{id}^{\otimes m} \otimes \phi} M_n^{\otimes m}. \]

Since \( \phi_m(1) = 1 \), it follows from [3, 3.1.6] that \( \phi_m \) is a contraction. On the other hand, this is clearly a unital homomorphism when restricted to \( A \). We denote by \( D \) the image of \( \phi_1 : A \rightarrow M_n \).

Proposition 3.6 If \( \sigma_1(A) \subset A \), then \( \phi_{m+1}(A) \subset M_n \otimes \phi_m(A) \).

Proof: We can write \( \phi_{m+1} \) as
so we see that $\phi_{m+1}(A) \subset M_n \otimes \phi_m(A)$.

**Proposition 3.7** If $\sigma_2(A) \subset A$, then $(\text{id} \otimes e_{ij} \otimes \text{id}^{\otimes m-1})\phi_{m+1}(A) \subset \phi_m(A)$ for $m \geq 1$.

**Proof:** The map $(\text{id} \otimes e_{ij} \otimes \text{id}^{\otimes m-1}) \circ \phi_{m+1}$ is

$$B \xrightarrow{\psi_{m+1}} M_n^{\otimes m+1} \otimes B \xrightarrow{\text{id}^{\otimes m+1} \circ \phi} M_n^{\otimes m} \otimes B$$

which can be rewritten as

$$B \xrightarrow{\psi_{m+1}} M_n^{\otimes m+1} \otimes B \xrightarrow{\text{id}^{\otimes m+1} \circ \phi} M_n^{\otimes m} \otimes B \xrightarrow{\text{id} \otimes e_{ij} \otimes \text{id}^{\otimes m-1}} M_n^{\otimes m}.$$

This can be shown to be

$$B \xrightarrow{\psi_2} M_n^{\otimes 2} \otimes B \xrightarrow{\text{id}^{\otimes 2} \circ \text{id} B} M_n \otimes B \xrightarrow{\psi_1^{-1}} B \xrightarrow{\text{id} \otimes e_{ij} \otimes \text{id}^{\otimes m-1}} M_n \otimes B \xrightarrow{\text{id} \otimes e_{ij} \otimes \text{id}^{\otimes m-1}} M_n \otimes B \xrightarrow{\text{id} \otimes e_{ij} \otimes \text{id}^{\otimes m-1}} M_n \otimes B \xrightarrow{\text{id} \otimes e_{ij} \otimes \text{id}^{\otimes m-1}} M_n \otimes B.$$

so we see that $(\text{id} \otimes e_{ij} \otimes \text{id}^{\otimes m-1}) \circ \phi_{m+1} = \phi_m \circ \chi_{2ij} : B \to M_n^{\otimes m}$. Now use $\chi_{2ij}(A) \subset A$.

**Corollary 3.8** If $\sigma_1(A) \subset A$ and $\sigma_2(A) \subset A$, then $\phi_m(A) \subset D^{\otimes m}$.

**Proof:** This is proved by induction. First note that $\phi_1(A) \subset D^{\otimes 1}$ by definition of $D$. Now assume that $\phi_m(A) \subset D^{\otimes m}$, and consider $m + 1$. By the previous proposition we see that $\phi_{m+1}(A) \subset D \otimes M_n \otimes D^{\otimes m-1}$, whereas the proposition before that says that $\phi_{m+1}(A) \subset M_n \otimes D^{\otimes m}$. The result follows by standard linear algebra.

**Proposition 3.9** Given $c \in C$ and $\epsilon > 0$, there is an $m \geq 1$ so that $|\kappa_m(\phi_m(c)) - c| < \epsilon$.

**Proof:** There is an $m \geq 1$ and an $x \in M_n^{\otimes m}$ so that $|c - \kappa_m(x)| < \epsilon/2$. Since $\phi(1) = 1$ we get $\phi_m(\kappa_m(x)) = x$, and since $\phi_m$ is a contraction, $|\phi_m(c) - x| < \epsilon/2$. Finally as $\kappa_m$ is a contraction, $|\kappa_m(\phi_m(c)) - \kappa_m(x)| < \epsilon/2$.

Now, let $D^\infty$ stand for the closure of the union of $\kappa_m(D^{\otimes m})$ for $m \geq 1$. Then we have:

**Theorem 3.10** Suppose that $A \cap C$ is maximal abelian in $C$. Then $A \cap C = D^\infty$ and $D$ is maximal abelian in $M_n(\mathbb{C})$.

**Proof:** 3.8 and 3.9 show that $A \cap C \subset D^\infty$. Since $D^\infty$ is abelian and $A \cap C$ is maximal, it follows that $A \cap C = D^\infty$. Hence, $D$ is maximal abelian in $M_n(\mathbb{C})$. 

7
4 Maximal (abelian $T$–invariant $*$-subalgebras) of $O_n$

In this section, we apply the results from Section 2 and 3 to maximal abelian subalgebras of $O_n$ that are invariant under the standard circle action. The notation is as in the Introduction.

The next lemma is probably well–known, but we couldn’t find a reference:

**Lemma 4.1** Let $x$ be in $O_n^k$ (i.e. $\omega_t(x) = t^k x$), for $k \neq 0$. If $x$ is normal, then $x = 0$.

**Proof:** Suppose $k > 0$. Let $y = x(s_1^k) \in O_n^0$, and let $\tau$ be the faithful normalised trace on $O_n^0 \cong M_{\infty}(\mathbb{C})$. Then $yy^* = xx^*$, $y^*y = s_1^k x^* x(s_1^k)^k$, and $\tau(yy^*) = \tau(y^*y)$ imply

$$\tau(xx^*) = n^{-k} \tau(x^* x)$$

If $xx^* = x^* x$, then

$$\tau(x^* x) = n^{-k} \tau(x^* x),$$

hence $\tau(x^* x) = 0$. \hfill \Box

**Theorem 4.2** Suppose that $A$ is a maximal (abelian $T$–invariant $*$-subalgebra) of $O_n$. Then $A \subset O_n^0$.

**Proof:** By the previous lemma, $\pi_k(a) = 0$ for all $a \in A$ and $k \neq 0$. By Fourier analysis we get

$$\sum_{k=-m}^m t^k \langle \xi, \pi_k(a)(\eta) \rangle \rightarrow \langle \xi, \omega_t(a)(\eta) \rangle, \ t \in \mathbb{T}$$

in the $L^2(\mathbb{T})$ topology as $m \rightarrow \infty$. But then $\langle \xi, \omega_t(a)(\eta) \rangle$ is constant on $\mathbb{T}$, so $\omega_t(a) = a$. \hfill \Box

**Remark 4.3** Note that, if $B$ is a $C^*$–algebra of the form $B = A \rtimes_\alpha \mathbb{N}$, where $A$ has a normalised faithful trace, $\alpha$ is a trace–scaling endomorphism, and the circle action is just the dual action with respect to this crossed–product representation, the above argument shows that any maximal (abelian $T$–invariant $*$-subalgebra) will be contained in the fixed–point algebra for the circle action. This will be the case if $B$ is a simple Cuntz–Krieger algebra, or more generally, for certain Cuntz–Pimsner algebras (cf. [4]).

**Theorem 4.4** Let $A$ be a maximal abelian subalgebra of $O_n$ such that $\omega(A) \subset A$, $\sigma(A) \subset A$ and $\tilde{\sigma}(A) \subset A$. Then there is an automorphism $\alpha_U$ of $O_n$, determined by a unitary $U \in M_n(\mathbb{C})$, such that $\alpha_U(A) = D$. Furthermore, $\alpha_U$ commutes with $\omega$, $\sigma$ and $\tilde{\sigma}$.

**Proof:** By [4.2], $A \subset O_n^0$. The result then follows from [3.10], with $B = O_n^0$, $\sigma = \sigma_1$ and $\tilde{\sigma} = \sigma_2$, while $u$ is any unitary in $M_n(\mathbb{C})$ that diagonalises the maximal abelian subalgebra $D$. \hfill \Box
Theorem 4.5 Let $A$ be a maximal abelian algebra in $\mathcal{O}_n$ such that $A \cap \mathcal{O}_n^0$ is maximal abelian in $\mathcal{O}_n^0$, $\sigma(A) \subset A$ and $\tilde{\sigma}(A) \subset A$. Then there is an automorphism $\alpha_U$ of $\mathcal{O}_n$, determined by a unitary $U \in M_n(\mathbb{C})$, such that $\alpha_U(A) = D$. Furthermore, $\alpha_U$ commutes with $\sigma$ and $\tilde{\sigma}$.

Proof: We apply 3.10, with $B = \mathcal{O}_n$, and $\sigma$ and $\tilde{\sigma}$ as in the previous theorem. That shows that $D^\infty \subset A$. Since $D^\infty$ is maximal abelian in $\mathcal{O}_n$ (cf. [3, 2.18]), $A = D^\infty$. Note that this means that all of $A$ is contained in the zero grade, although no assumptions on the circle action were made. □

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