Koszulity for skew PBW extensions over fields

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Abstract

Koszul and homogeneous Koszul algebras were defined by Priddy in [18]. There exist some relations between these algebras and the skew PBW extensions introduced in [8]. In this paper we give conditions to guarantee that skew PBW extensions over fields are Koszul or homogeneous Koszul. We also show that a constant skew PBW extension of a field is a PBW deformation of its homogeneous version.

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1 Introduction

Koszul and homogeneous Koszul algebras were introduced by Priddy in [18]; despite of that these type of algebras have not been enough studied, they have important applications in algebraic geometry, Lie theory, quantum groups, algebraic topology and combinatorics. The structure and history of Koszul homogeneous algebras were detailed in [17]. There exist numerous equivalent definitions of homogeneous Koszul algebras (see for example [3]); in addition, Koszul algebras have been defined in a more general way by some authors and they are commonly called “Generalized Koszul algebras” (see for example [4], [7], [14], [27]). In this paper we work with the definition given by Priddy, which is commonly called “Koszul algebras in the classical sense”.

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Skew PBW extensions or $\sigma$-PBW extensions were defined in [8]. Several properties of these extensions have been recently studied (see for example [1], [2], [9], [10], [11], [12], [19], [20], [21], [22], [25], [26]). There exist some relations between Koszul and homogeneous Koszul algebras with the skew PBW extensions. Our interest in this article is to study the Koszul property for the skew PBW extensions over fields. For this purpose we classify the skew PBW extensions in five sub-classes: constant, bijective, pre-commutative, quasi-commutative and semi-commutative, and we show that a skew PBW extension $A$ of a field is Koszul (homogeneous Koszul) when $A$ is pre-commutative and constant (semi-commutative). Finally, following the ideas presented in [6], we show that a constant skew PBW extension of a field is a PBW deformation of its homogeneous version.

2 Skew PBW extensions

In this section we recall some elementary properties of skew PBW extensions; in addition, we will introduce some sub-classes of them: constant, pre-commutative and semi-commutative. Examples of these sub-classes are presented.

2.1 Definitions and properties

**Definition 2.1** ([8], Definition 1). Let $R$ and $A$ be rings. We say that $A$ is a *skew PBW extension of $R$* (also called a $\sigma$-PBW extension of $R$) if the following conditions hold:

(i) $R \subseteq A$;

(ii) there exist finitely many elements $x_1, \ldots, x_n \in A$ such $A$ is a left $R$-free module with basis

$$\text{Mon}(A) := \{x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n\}, \text{ with } \mathbb{N} := \{0, 1, 2, \ldots \}.$$

The set $\text{Mon}(A)$ is called the set of standard monomials of $A$.

(iii) For each $1 \leq i \leq n$ and any $r \in R \setminus \{0\}$, there exists an element $c_{i,r} \in R \setminus \{0\}$ such that

$$x_ir - c_{i,r} x_i \in R. \quad (2.1)$$

(iv) For any elements $1 \leq i, j \leq n$ there exists $c_{i,j} \in R \setminus \{0\}$ such that

$$x_jx_i - c_{i,j}x_i x_j \in R + Rx_1 + \cdots + Rx_n. \quad (2.2)$$

Under these conditions we will write $A := \sigma(R)\langle x_1, \ldots, x_n \rangle$.

The notation $\sigma(R)\langle x_1, \ldots, x_n \rangle$ and the name of the skew PBW extensions are due to the following proposition.
Proposition 2.2 ([8], Proposition 3). Let $A$ be a skew PBW extension of $R$. For each $1 \leq i \leq n$, there exists an injective endomorphism $\sigma_i : R \to R$ and a $\sigma_i$-derivation $\delta_i : R \to R$ such that

$$x_i r = \sigma_i(r)x_i + \delta_i(r), \quad r \in R.$$  

(2.3)

Definition 2.3. Let $A$ be a skew PBW extension of $R$, $\Sigma := \{\sigma_1, \ldots, \sigma_n\}$ and $\Delta := \{\delta_1, \ldots, \delta_n\}$, where $\sigma_i$ and $\delta_i$ ($1 \leq i \leq n$) are as in the Proposition 2.2.

(a) $A$ is called \textit{pre-commutative} if the condition (iv) in Definition 2.1 is replaced by:

For any $1 \leq i, j \leq n$ there exists $c_{i,j} \in R \setminus \{0\}$ such that

$$x_j x_i - c_{i,j} x_i x_j \in Rx_1 + \cdots + Rx_n.$$  

(2.4)

(b) $A$ is called \textit{quasi-commutative} if the conditions (iii) and (iv) in Definition 2.1 are replaced by

(iii') for each $1 \leq i \leq n$ and all $r \in R \setminus \{0\}$ there exists $c_{i,r} \in R \setminus \{0\}$ such that

$$x_i r = c_{i,r} x_i;$$  

(2.5)

(iv') for any $1 \leq i, j \leq n$ there exists $c_{i,j} \in R \setminus \{0\}$ such that

$$x_j x_i = c_{i,j} x_i x_j.$$  

(2.6)

(c) $A$ is called \textit{bijective} if $\sigma_i$ is bijective for each $\sigma_i \in \Sigma$, and $c_{i,j}$ is invertible for any $1 \leq i < j \leq n$.

(d) Any element $r$ of $R$ such that $\sigma_i(r) = r$ and $\delta_i(r) = 0$ for all $1 \leq i \leq n$ will be called a \textit{constant}. $A$ is called \textit{constant} if every element of $R$ is constant.

(e) $A$ is called \textit{semi-commutative} if $A$ is quasi-commutative and constant.

Recall that a \textit{filtered ring} is a ring $B$ with a family $F(B) = \{F_n(B) \mid n \in \mathbb{Z}\}$ of subgroups of the additive group of $B$ where we have the ascending chain $\cdots \subset F_{n-1}(B) \subset F_n(B) \subset \cdots$ such that $1 \in F_0(B)$ and $F_n(B)F_m(B) \subseteq F_{n+m}(B)$ for all $n, m \in \mathbb{Z}$. From a filtered ring $B$ it is possible to construct its associated graded ring $Gr(B)$ taking $Gr(B)_n := F_n(B)/F_{n-1}(B)$. The following proposition establishes that one can construct a quasi-commutative skew PBW extension from a given skew PBW extension of a ring $R$.

Proposition 2.4 ([12], Proposition 2.1). Let $A$ be a skew PBW extension of $R$. Then, there exists a quasicommutative skew PBW extension $A^\sigma$ of $R$ in $n$ variables $z_1, \ldots, z_n$ defined by the relations $z_i r = c_{i,r} z_i$, $z_j z_i = c_{i,j} z_i z_j$, for $1 \leq i \leq n$, where $c_{i,r}, c_{i,j}$ are the same constants that define $A$. Moreover, if $A$ is bijective then $A^\sigma$ is also bijective.
The next proposition computes the graduation of a skew PBW extension.

**Theorem 2.5** ([12], Theorem 2.2). Let $A$ be an arbitrary skew PBW extension of $R$. Then, $A$ is a filtered ring with increasing filtration given by

$$F_m(A) := \begin{cases} R & \text{if } m = 0 \\ \{ f \in A \mid \deg(f) \leq m \} & \text{if } m \geq 1 \end{cases}$$

(2.7)

and the corresponding graded ring $Gr(A)$ is isomorphic to $A^\sigma$.

### 2.2 Examples and classification

**Examples 2.6.** In [8] and [12] was presented a list of examples of quasi-commutative or bijective skew PBW extensions. We also classify these examples according to Definition 2.3. Through this paper, $\mathbb{K}$ will denote a field and $K$ a commutative ring.

1. Classical polynomial ring; Ore extensions of bijective type and Weyl algebras; Universal enveloping algebra of a Lie algebra; Tensor product; crossed product; Algebra of $q$-differential operators; Algebra of shift operators; Mixed algebras; Algebra of discrete linear systems; Linear partial differential operators; Linear partial shift operators; Algebra of linear partial difference operators; Algebra of linear partial $q$-dilation operators; Algebra of linear partial $q$-differential operators; Diffusion algebra 1 ([21]); Diffusion algebra 2 ([12]); Additive analogue of the Weyl algebra; Multiplicative analogue of the Weyl algebras; Quantum algebras; Dispin algebras; Woronowicz algebras; Complex algebras; Algebra $U$; Manin algebras; Algebra of quantum matrices; $q$-Heisenberg algebras; Quantum enveloping algebras of $\mathfrak{sl}(2, \mathbb{K})$; The algebra of differential operators on a quantum space; Witten’s deformation of $U(\mathfrak{sl}(2, \mathbb{K}))$; Quantum Weyl algebra of Maltsiniotis; Quantum Weyl algebras; Multiparameter quantized Weyl algebras; Quantum symplectic space and Quadratic algebras in 3 variables.

2. Jordan plane. The Jordan plane $A$ is the free $\mathbb{K}$-algebra generated by $x, y$ with relation $yx = xy + x^2$, so $A = \mathbb{K}\langle x, y \rangle/\langle yx - xy - x^2 \rangle \cong \sigma(\mathbb{K}[x])\langle y \rangle$.

3. Particular Sklyanin algebra. The Sklyanin algebra (Example 1.14, [23]) is the $\mathbb{K}$-algebra $S = \mathbb{K}\langle x, y, z \rangle/\langle axy + bxy + cz^2, axz + bzx + cy^2, ayz + byz + cx^2 \rangle$, where $a, b, c \in \mathbb{K}$. If $c \neq 0$ then $S$ is not a skew PBW extension. If $c = 0$ and $a, b \neq 0$ then in $S$: $yx = -\frac{b}{a}xy; zx = -\frac{a}{b}xz$ and $zy = -\frac{a}{b}yz$, therefore $S \cong \sigma(\mathbb{K})\langle x, y, z \rangle$ is a skew PBW extension of $\mathbb{K}$, and we call this algebra a particular Sklyanin algebra.

4. Multi-parameter quantum affine $n$-spaces. Let $n \geq 1$ and $q$ be a matrix $(q_{ij})_{n \times n}$ with entries in a field $\mathbb{K}$ where $q_{ii} = 1, q_{ij}q_{ji} = 1$ for all $1 \leq i, j \leq n$. Then multi-parameter quantum affine $n$-space $O_q(\mathbb{K}^n)$ is defined to be $\mathbb{K}$-algebra generated by $x_1, \ldots, x_n$ with the relations $x_jx_i = q_{ij}x_ix_j$ for all $1 \leq i, j \leq n$. 

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5. Homogenized enveloping algebra ([24], Chapter 12). Let $G$ a finite dimensional Lie algebra over $K$ with basis $\{x_1, \ldots, x_n\}$ and $\mathcal{U}(G)$ its enveloping algebra. The homogenized enveloping algebra of $G$ is $A(G) := T(G \oplus Kz)/\langle R \rangle$, where $T(G \oplus Kz)$ is the tensor algebra, $z$ is a new variable, and $R$ is spanned by $\{z \otimes x - x \otimes z \mid x \in G\} \cup \{x \otimes y - y \otimes x - [x, y] \otimes z \mid x, y \in G\}$. From the PBW Theorem for $G \otimes K(z)$, considered as a Lie algebra over $K(z)$, we get that $A(G)$ is a skew PBW extension of $K[z]$.

**Classification 2.7.** We classify the above examples of skew PBW extensions as constant (C), bijective (B), pre-commutative (P), quasi-commutative (QC) and semi-commutative (SC); the classification is presented in the next table, where the symbols $\star$ and $\checkmark$ denote negation and affirmation, respectively.

| Skew PBW extension | C | B | P | QC | SC |
|---------------------|---|---|---|----|----|
| Classical polynomial ring | $\checkmark$ | $\checkmark$ | $\checkmark$ | $\checkmark$ | $\checkmark$ |
| Ore extensions of bijective type | $\star$ | $\checkmark$ | $\checkmark$ | $\star$ | $\star$ |
| Weyl algebra | $\star$ | $\checkmark$ | $\checkmark$ | $\star$ | $\star$ |
| Jordan plane | $\star$ | $\checkmark$ | $\checkmark$ | $\star$ | $\star$ |
| Particular Sklyanin algebra | $\checkmark$ | $\checkmark$ | $\checkmark$ | $\checkmark$ |
| Universal enveloping algebra of a Lie algebra | $\checkmark$ | $\checkmark$ | $\checkmark$ | $\star$ | $\star$ |
| Homogenized enveloping algebra $A(G)$ | $\checkmark$ | $\checkmark$ | $\star$ | $\star$ |
| Tensor product | $\checkmark$ | $\checkmark$ | $\checkmark$ | $\star$ | $\star$ |
| Crossed product | $\star$ | $\checkmark$ | $\star$ | $\star$ |
| Algebra of $q$-differential operators | $\star$ | $\checkmark$ | $\checkmark$ | $\star$ | $\star$ |
| Algebra of shift operators | $\star$ | $\checkmark$ | $\checkmark$ | $\star$ | $\star$ |
| Mixed algebra | $\star$ | $\checkmark$ | $\star$ | $\star$ |
| Algebra of discrete linear systems | $\star$ | $\checkmark$ | $\star$ | $\checkmark$ |
| Linear partial differential operators | $\star$ | $\checkmark$ | $\star$ | $\star$ |
| Linear partial shift operators | $\star$ | $\checkmark$ | $\star$ | $\star$ |
| Algebra of linear partial difference operators | $\star$ | $\checkmark$ | $\star$ | $\star$ |
| Algebra of linear partial $q$-dilation operators | $\star$ | $\checkmark$ | $\star$ | $\star$ |
| Algebra of linear partial $q$-differential operators | $\star$ | $\checkmark$ | $\star$ | $\star$ |
| Diffusion algebra 1 | $\checkmark$ | $\checkmark$ | $\checkmark$ | $\star$ | $\star$ |
| Diffusion algebra 2 | $\checkmark$ | $\checkmark$ | $\checkmark$ | $\star$ | $\star$ |
| Additive analogue of the Weyl algebra | $\checkmark$ | $\checkmark$ | $\star$ | $\star$ |
| Multiplicative analogue of the Weyl algebra | $\checkmark$ | $\checkmark$ | $\checkmark$ | $\star$ |
| Quantum algebra | $\checkmark$ | $\checkmark$ | $\star$ | $\star$ |
| Dispin algebra | $\checkmark$ | $\checkmark$ | $\checkmark$ | $\star$ |
| Woronowicz algebra | $\checkmark$ | $\checkmark$ | $\star$ | $\star$ |
| Complex algebra | $\star$ | $\checkmark$ | $\checkmark$ | $\star$ |
| Algebra $\mathcal{U}$ | $\star$ | $\checkmark$ | $\star$ | $\star$ |
| Mann algebra | $\star$ | $\checkmark$ | $\star$ | $\star$ |
| $q$-Heisenberg algebra | $\checkmark$ | $\checkmark$ | $\star$ | $\star$ |
| Quantum enveloping algebra of $\mathfrak{sl}(2, K)$ | $\star$ | $\checkmark$ | $\star$ | $\star$ |
| Hayashi’s algebra | $\star$ | $\checkmark$ | $\star$ | $\star$ |
| Multi-parameter quantum affine $n$-space | $\checkmark$ | $\checkmark$ | $\checkmark$ | $\star$ |
| The algebra of differential operators on a quantum space $\mathcal{S}_q$ | $\star$ | $\checkmark$ | $\star$ | $\star$ |
| Witten’s deformation of $U(\mathfrak{sl}(2, K))$ | $\star$ | $\checkmark$ | $\star$ | $\star$ |
| Quantum Weyl algebra of Maltsiniotis | $\star$ | $\checkmark$ | $\star$ | $\star$ |
| Quantum Weyl algebra | $\star$ | $\checkmark$ | $\star$ | $\star$ |
| Multiparameter quantized Weyl algebra | $\star$ | $\checkmark$ | $\star$ | $\star$ |
| Quantum symplectic space | $\star$ | $\checkmark$ | $\star$ | $\star$ |
| Quadratic algebras in 3 variable | $\star$ | $\checkmark$ | $\star$ | $\star$ |
Example 2.8. Sridharan enveloping algebra of 3-dimensional Lie algebra \( \mathcal{G} \). Let \( \mathcal{G} \) be a finite dimensional Lie algebra, and let \( f \in Z^2(\mathcal{G}, \mathbb{K}) \) be an arbitrary 2-cocycle, that is, \( f : \mathcal{G} \times \mathcal{G} \to \mathbb{K} \) such that \( f(x, x) = 0 \) and

\[
  f(x, [y, z]) + f(z, [x, y]) + f(y, [z, x]) = 0
\]

for all \( x, y, z \in \mathcal{G} \). The Sridharan enveloping algebra of \( \mathcal{G} \) is defined to be the associative algebra \( \mathcal{U}_f(\mathcal{G}) = T(\mathcal{G})/I \), where \( T(\mathcal{G}) \) is the tensor algebra of \( \mathcal{G} \) and \( I \) is the two-sided ideal of \( T(\mathcal{G}) \) generated by the elements

\[
  (x \otimes y) - (y \otimes x) - [x, y] - f(x, y), \quad \text{for all } x, y \in \mathcal{G}.
\]

Note that if \( f = 0 \) then \( \mathcal{U}_f(\mathcal{G}) = \mathcal{U}_0(\mathcal{G}) = \mathcal{U}(\mathcal{G}) \). For \( x \in \mathcal{G} \), we still denote by \( x \) its image in \( \mathcal{U}_f(\mathcal{G}) \). \( \mathcal{U}_f(\mathcal{G}) \) is a filtered algebra with the associated graded algebra \( \text{gr}(\mathcal{U}_f(\mathcal{G})) \) being a polynomial algebra. Let \( \mathbb{K} \) be an algebraically closed field of characteristic zero. If \( \mathcal{G} \) is a Lie \( \mathbb{K} \)-algebra of dimension three, then the Sridharan enveloping algebra \( \mathcal{U}_f(\mathcal{G}) \) for \( f \in Z^2(\mathcal{G}, \mathbb{K}) \) is isomorphic to one of ten associative \( \mathbb{K} \)-algebras (see [16], Theorem 1.3), which is defined by three generators \( x, y, z \) and the commutation relations as the following table shows. Therefore, the Sridharan enveloping algebra \( \mathcal{U}_f(\mathcal{G}) \) is a skew PBW extension of \( \mathbb{K} \), i.e. \( \mathcal{U}_f(\mathcal{G}) \cong \sigma(\mathbb{K}\langle x, y, z \rangle) \), and it is classified as follows:

| Type | \([x, y]\) | \([y, z]\) | \([z, x]\) | C | B | P | QC | SC |
|------|------------|------------|------------|---|---|---|----|----|
| 1    | 0          | 0          | 0          | ✓ | ✓ | ✓ | ✓  | ✓  |
| 2    | 0          | \(x\)      | \(0\)      | ✓ | ✓ | ✓ | ✓  | ✓  |
| 3    | \(x\)      | 0          | \(0\)      | ✓ | ✓ | ✓ | ✓  | ✓  |
| 4    | \(x\)      | \(2y\)     | \(-x\)     | ✓ | ✓ | ✓ | ✓  | ✓  |
| 5    | \(y\)      | \(-2y\)    | \(-x\)     | ✓ | ✓ | ✓ | ✓  | ✓  |
| 6    | \(z\)      | \(0\)      | \(-y\)     | ✓ | ✓ | ✓ | ✓  | ✓  |
| 7    | 1          | \(x\)      | \(0\)      | ✓ | ✓ | ✓ | ✓  | ✓  |
| 8    | \(x\)      | 0          | \(0\)      | ✓ | ✓ | ✓ | ✓  | ✓  |
| 9    | \(x\)      | \(0\)      | \(0\)      | ✓ | ✓ | ✓ | ✓  | ✓  |
| 10   | \(y\)      | \(x\)      | \(0\)      | ✓ | ✓ | ✓ | ✓  | ✓  |

where \( \alpha \in \mathbb{K}\setminus\{0\} \).

## 3 Koszulity

Some authors have defined Koszul algebras in a more general sense than [18] (see for example [4], [14], [27]). Our focus is to study the Koszul property for skew PBW extensions taking into account the definition given in [18]. In this section we give sufficient conditions to guarantee that skew PBW extensions are Koszul or homogeneous Koszul. For this purpose, we show the relationship between \( \mathbb{K} \)-algebras that are skew PBW extensions and certain classes of algebras defined in [18] containing the Koszul and homogeneous Koszul algebras. Let \( L := \mathbb{K}\langle x_1, \ldots, x_n \rangle \) the free associative algebra (tensor algebra) in \( n \) generators \( x_1, \ldots, x_n \). Note that \( L \) is positively graded with graduation given by \( L := \bigoplus_{j \geq 0} L_j \), where \( L_0 = \mathbb{K} \) and \( L_j \) spanned by all words of length \( j \) in the alphabet \( \{x_1, \ldots, x_n\} \), for \( j > 0 \).
3.1 Pre-Koszul algebras

We present a definition of pre-Koszul and homogeneous pre-Koszul algebras, analogous to the definition given by Priddy in [18].

**Definition 3.1.** Let $L = \mathbb{K}\langle x_1, \ldots, x_n \rangle$ and let $B := L/I$.

(i) $B$ is said to be a pre-Koszul algebra if $I$ is a two sided ideal generated by elements of the form

$$\sum_{i=1}^{n} c_i x_i + \sum_{1 \leq j, k \leq n} c_{j,k} x_j x_k, \text{ where } c_i \text{ and } c_{j,k} \text{ are in } \mathbb{K}, \quad (3.1)$$

(ii) A pre-Koszul algebra is said to be pre-Koszul homogeneous if $c_i = 0$, for $1 \leq i \leq n$ in (3.1).

Presentations of special types of skew PBW extensions are given in the following remark.

**Remark 3.2.** Let $A = \sigma(\mathbb{K})\langle x_1, \ldots, x_n \rangle$ be a skew PBW extension of a field $\mathbb{K}$.

1. We note that $A = \mathbb{K}\langle x_1, \ldots, x_n \rangle/I$, where $I$ is the two sided ideal generated by elements as in (iii) and (iv) of the Definition 2.1, i.e., elements of the form

$$c_r + x_i r - c_{i,r} x_i, \quad r_0 + r_1 x_1 + \cdots + r_n x_n + x_j x_i - c_{i,j} x_i x_j, \quad (3.2)$$

where $r \neq 0$, $c_r, c_{i,r} \neq 0$, $r_0, r_1, \ldots, r_n, c_{i,j} \neq 0$ are elements in $\mathbb{K}$, with $1 \leq i, j \leq n$.

2. If $A$ is pre-commutative, then $A = \mathbb{K}\langle x_1, \ldots, x_n \rangle/I$, where $I$ is the two sided ideal generated by elements of the form

$$c_r + x_i r - c_{i,r} x_i, \quad r_1 x_1 + \cdots + r_n x_n + x_j x_i - c_{i,j} x_i x_j, \quad (3.3)$$

where $r \neq 0$, $c_r, c_{i,r} \neq 0$, $r_1, \ldots, r_n, c_{i,j} \neq 0$ are elements in $\mathbb{K}$, with $1 \leq i, j \leq n$.

3. If $A$ is constant, then $A = \mathbb{K}\langle x_1, \ldots, x_n \rangle/I$, where $I$ is the two sided ideal generated by elements of the form

$$r_0 + r_1 x_1 + \cdots + r_n x_n + x_j x_i - c_{i,j} x_i x_j, \quad (3.4)$$

where $r_0, r_1, \ldots, r_n, c_{i,j} \neq 0$ are elements in $\mathbb{K}$, with $1 \leq i, j \leq n$.

4. If $A$ is quasi-commutative then $A = \mathbb{K}\langle x_1, \ldots, x_n \rangle/I$, where $I$ is the two sided ideal generated by elements as in (iii') and (iv') of the Definition 2.3, i.e., elements of the form

$$x_i r - c_{i,r} x_i, \quad x_j x_i - c_{i,j} x_i x_j, \quad (3.5)$$

where $r \neq 0$, $c_{i,r} \neq 0$, $c_{i,j} \neq 0$ are elements in $\mathbb{K}$, with $1 \leq i, j \leq n$. 

5. If $A$ is semi-commutative then $A = \mathbb{K}\langle x_1, \ldots, x_n \rangle / I$, where $I$ is the two sided ideal generated by elements of the form

$$x_jx_i - c_{i,j}x_ix_j$$

(3.6)

where $c_{i,j} \neq 0$ are elements in $\mathbb{K}$, with $1 \leq i, j \leq n$.

If otherwise is not assumed, in this paper all skew PBW extensions are $\mathbb{K}$-algebras and extensions of the field $\mathbb{K}$ (i.e., $R = \mathbb{K}$ in Definition 2.1), so $A = \sigma(\mathbb{K})\langle x_1 \ldots, x_n \rangle$ is necessarily a constant skew PBW extension.

**Proposition 3.3.** Let $A = \sigma(\mathbb{K})\langle x_1, \ldots, x_n \rangle$ be a skew PBW extension. If $A$ is pre-commutative then $A$ is pre-Koszul.

**Proof.** From (3.3) and (3.4) we have that $A = \mathbb{K}\langle x_1, \ldots, x_n \rangle / I$, where $I$ is the two sided ideal generated by elements of the form

$$r_1x_1 + \cdots + r_nx_n + x_jx_i - c_{i,j}x_ix_j.$$  

(3.7)

Then we conclude that $A$ is pre-Koszul.

**Example 3.4.** According to the classifications presented in the tables of Section 2, the following skew PBW extensions are pre-Koszul algebras: classical polynomial ring over a field; particular Sklyanin algebra; universal enveloping algebra of a Lie algebra; algebra of linear partial q-dilation operators; additive analogue of the Weyl algebra; multiplicative analogue of the Weyl algebra; quantum algebra $\mathcal{U}(\mathfrak{so}(3, K))$; dispin algebra; Woronowicz algebra; $q$-Heisenberg algebra; multi-parameter quantum affine n-space; types 1, 2, 3, 4, 5 and 6 of Sridharan enveloping algebra of 3-dimensional Lie algebras.

**Proposition 3.5.** Let $A$ be a skew PBW extension. If $A$ is semi-commutative then $A$ is pre-Koszul homogeneous.

**Proof.** If $A$ is a semi-commutative skew PBW extension then $A$ from (3.6) and Proposition 3.3 we get that $A$ is pre-Koszul homogeneous.

**Example 3.6.** From Example 3.4 we obtain the following examples of pre-Koszul homogeneous skew PBW extensions: classical polynomial ring over a field; particular Sklyanin algebras; multiplicative analogue of the Weyl algebra; multi-parameter quantum affine n-space; the Sridharan enveloping algebra of 3-dimensional Lie algebra with $[x, y] = [y, z] = [z, x] = 0$.

Let $B$ be a pre-Koszul algebra. One can truncate the relations in (3.1) leaving only their homogeneous quadratic parts. Let $B^{(0)}$ be the obtained algebra. Then $B^{(0)}$ is called the associated homogeneous pre-Koszul algebra of $B$. Note that $B$ is homogeneous if and only if $B^{(0)} \cong B$ as algebras.
**Proposition 3.7.** Let $A$ be a pre-Koszul skew PBW extension, then $A'$ is the associated homogeneous pre-Koszul algebra of $A$. 

*Proof.* Let $A$ be a pre-Koszul skew PBW extension. By Proposition 2.4 there exists a quasi-commutative skew PBW extension $A'$ of $K$ in $n$ variables $z_1, \ldots, z_n$ defined by the relations $z_i r = c_{i,r} z_i$, $z_j z_i = c_{i,j} z_i z_j$, for $1 \leq i \leq n$, where $c_{i,r}, c_{i,j}$ are the same constants that define $A$. Since $A$ is pre-Koszul then by Proposition 3.3 $A'$ is constant and therefore $A'$ is defined by the relations $z_j z_i = c_{i,j} z_i z_j$. Then $A(0) \cong A'$. 

### 3.2 Koszul algebras and skew PBW extensions

Let $B$ be a finitely graded algebra generated in degree 1; consider the *Yoneda algebra* of $B$ defined by

$$E(B) := \bigoplus_{i \geq 0} Ext_B^i(K, K);$$

the $Ext$ groups here are computed in the category of graded $B$-modules with graded $Hom$ spaces; the product in $E(B)$ is defined in the following way: Let $\{P_i \xrightarrow{d_i} P_{i-1}\}_{i \geq 0}$ be a graded projective resolution of $K$ that defines the groups $Ext_B^i(K, K)$, with $P_{-1} := K$; moreover, let $\overline{f} \in Ext_B^i(K, K) = \ker(d_{i+1}^*)/Im f_i^*$ with $f \in \ker(d_{i+1}^*) \subseteq Hom_B(P_i, K)$ and $\overline{g} \in Ext_B^j(K, K) = \ker(d_{j+1}^*)/Im f_j^*$ with $g \in \ker(d_{j+1}^*) \subseteq Hom_B(P_j, K)$, then we define

$$Ext^i_B(K, K) \times Ext^j_B(K, K) \to Ext^{i+j}_B(K, K)$$

$$(\overline{f}, \overline{g}) \mapsto \overline{fg'} := \overline{f} \overline{g},$$

where $g' : P_{i+j} \to P_i$ is defined inductively by the following commutative diagrams:

![Diagram](https://via.placeholder.com/150)

Can be proved that this product is well defined, i.e., it does not depend of the projective resolution of $K$ and the choosing of $g_0, g_1, \ldots, g_{i-1}, g_i$; moreover, $fg' \in \ker(d_{i+j+1}^*)$. In fact, from the step $i + 1$ in the previous inductive procedure we have that $d_{i+1} g_{i+1} = g_i d_{i+j+1}$, so $fd_{i+1} g_{i+1} = fg_i d_{i+j+1}$, i.e., $0 = d_{i+1}^*(f)(g_{i+1}) = d_{i+j+1}^*(fg_i)$. Thus, $E(B)$ is a graded algebra; note that the $K$-vector space $Ext^i_B(K, K)$ is graded

$$Ext_B^i(K, K) = \bigoplus_{j \geq 0} Ext^{i,j}_B(K, K),$$

with
\[
\text{Ext}^i_j(B, \mathbb{K}) := (\text{Ext}^i_B(\mathbb{K}, \mathbb{K}))_{-j} := \text{Ext}^i_B(\mathbb{K}, \mathbb{K}(-j)),
\]
so setting \( E^{i,j}(B) := \text{Ext}^i_j(B, \mathbb{K}) \) we get that
\[
E(B) = \bigoplus_{i,j \geq 0} E^{i,j}(B)
\]
is a bigraded algebra. For \( i \geq 0 \), we write
\[
E^i(B) := \bigoplus_{j \geq 0} E^{i,j}(B);
\]
in particular,
\[
E^0(B) = \bigoplus_{j \geq 0} \text{Hom}^j_B(\mathbb{K}, \mathbb{K}) = \bigoplus_{j \geq 0} (\text{Hom}_B(\mathbb{K}, \mathbb{K}))_{-j} = \bigoplus_{j \geq 0} \text{Hom}_B(\mathbb{K}, \mathbb{K}(-j)),
\]
with \( \text{Hom}_B(\mathbb{K}, \mathbb{K}(-j)) := \{ f \in \text{Hom}_B(\mathbb{K}, \mathbb{K}) | f(\mathbb{K}_l) \subseteq \mathbb{K}_{l-j}, l \in \mathbb{Z} \} \).

**Definition 3.8.** Let \( B \) be a homogeneous pre-Koszul algebra, \( B \) is called **homogeneous Koszul** if the following equivalent conditions hold:

(i) \( \text{Ext}^i_j(B, \mathbb{K}) = 0 \) for \( i \neq j \);

(ii) \( E(B) \) is generated by \( E^{1,1}(B) \);

(iii) The module \( \mathbb{K} \) admits a linear free resolution, i.e., a resolution by free \( B \)-modules

\[
\cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow \mathbb{K} \rightarrow 0
\]
such that \( P_i \) is generated in degree \( i \).

**Definition 3.9** ([18], Page 43). We say that a pre-Koszul algebra \( B \) is a **Koszul algebra** if \( B^{(0)} \) is a homogeneous Koszul algebra.

**Remark 3.10.** Notice that if \( B \) is homogeneous Koszul algebra then \( B \) is Koszul. In fact, as \( B \) is homogeneous then \( B^{(0)} \cong B \) as algebras and so \( B^{(0)} \) is homogeneous Koszul, therefore \( B \) is Koszul.

In the current literature, homogeneous Koszul algebras are called simply Koszul algebras. Some authors have studied Koszul algebras defined by Priddy in [18]. For example Koszul algebras are defined in [17], analogous to the Definition 3.9. Let \( P \subseteq \mathbb{K} \oplus L_1 \oplus L_2 \) a subspace of \( F_2(L) \) and \( A = L/(P) \). Let \( A^{(0)} = L/(R) \), where \( R \) is obtained by taking homogeneous part of \( P \). \( A \) is said to be (nonhomogeneous) Koszul if \( A^{(0)} \) is homogeneous Koszul (see [17], page 140). In [15] Koszul algebras are defined as follows. Let \( V \) a graded vector space and a degree homogeneous subspace \( P \subseteq V \oplus V \otimes 2 \), the algebra \( A = T(V)/(P) \) is called (nonhomogeneous quadratic) Koszul if \( P \cap V = \{0\} \), \( \{P \otimes V + V \otimes P\} \cap V \otimes 2 \subseteq P \cap V \otimes 2 \) and \( T(V)/(\pi(P)) \) is homogeneous Koszul, where \( \pi : T(V) \rightarrow V \otimes 2 \) is the projection onto the quadratic part of the tensor algebra. R. Berger in [5] defined the notion of \( N \)-Koszul algebra, if \( N = 2 \), the notion of homogeneous Koszul
algebra is obtained. To avoid confusion, we still use the names given in the Definition 3.8 (homogeneous Koszul) and the Definition 3.9 (Koszul).

Let $L = \mathbb{K}\langle x_1, \ldots, x_n \rangle$ the free associative algebra in $n$ generators $x_1, \ldots, x_n$. Let $R$ a subspace of $F_2(L) = \mathbb{K} \bigoplus L_1 \bigoplus L_2$, the algebra $L/\langle R \rangle$ is called (nonhomogeneous) quadratic algebra. $L/\langle R \rangle$ is called homogeneous quadratic algebra if $R$ is a subspace of $L_2$, for $\langle R \rangle$ the two-sided ideal of $L$ generated by $R$. Let $A = \mathbb{K}\langle x_1, \ldots, x_n \rangle/\langle R \rangle$ be a quadratic algebra with a fixed generators $\{x_1, \ldots, x_n\}$. For a multiindex $\alpha := (i_1, \ldots, i_m)$, where $1 \leq i_k \leq n$, we denote the monomials in $\mathbb{K}\langle x_1, \ldots, x_n \rangle$ by $x^\alpha := x_{i_1}x_{i_2}\cdots x_{i_m}$. For $\alpha = \emptyset$ we set $x^\emptyset := 1$. Now let us equip the subspace $L_2$ with the basis consisting of the monomials $x_{i_1}x_{i_2}$. Let $S^{(1)} := \{1, 2, \ldots, n\},\ S^{(1)} = S^{(1)} \times S^{(1)}$ the cartesian product, then for $R \subseteq L_2$ we obtain the set $S \subseteq S_1 \times S_1$ of pairs of indices $(l, m)$ for which the class of $x_lx_m$ in $L_2/R$ is not in the span of the classes of $x_rx_s$ with $(r, s) < (l, m)$, where $<$ denotes the lexicographical order ([17], 4.1-Lemma 1.1). Hence, the relations in $A$ can be written in the following form:

$$x_{i_1}x_{i_2} = \sum_{(r, s)<(i,j)} c_{i_1}^{r,s}x_r x_s, \quad (i, j) \in S^{(1)} \times S^{(1)} \setminus S.$$ 

Define further $S^{(0)} := \{\emptyset\}$, and for $m \geq 2$,

$$S^{(m)} := \{(i_1, \ldots, i_m) \mid (i_k, i_{k+1}) \in S, \ k = 1, \ldots, m-1\}$$

and consider the monomials $\{x_{i_1}\cdots x_{i_m} \in A_m \mid (i_1, \ldots, i_m) \in S^{(m)}\}$. Note that these monomials always span $A_m$ as a vector space and and the monomials

$$(A, S) := \{x_{i_1}\cdots x_{i_m} \mid (i_1, \ldots, i_m) \in \bigcup_{m>0} S^{(m)}\}$$

linearly span the entire $A$. We call $(A, S)$ in (3.8) a PBW-basis of $A$ if they are linearly independent and hence form a $\mathbb{K}$-linear basis. The elements $x_1, \ldots, x_n$ are called PBW-generators of $A$. A PBW-algebra is a homogeneous quadratic algebra admitting a PBW-basis, i.e., there exists a permutation of $x_1, \ldots, x_n$ such that the standard monomials in $x_1, \ldots, x_n$ form a $\mathbb{K}$-basis of $A$.

**Proposition 3.11.** Let $A$ be a semi-commutative skew PBW extension. Then $A$ is a PBW algebra.

**Proof.** If $A = A(\mathbb{K}\langle x_1, \ldots, x_n \rangle)$ is a semi-commutative skew PBW extension, then $A = \mathbb{K}\langle x_1, \ldots, x_n \rangle/\langle x_jx_i - c_{i,j}x_ix_j \rangle$ (as in (3.6)) is a homogeneous quadratic algebra with generators $x_1, \ldots, x_n$ and relations $x_jx_i - c_{i,j}x_ix_j$. Using the above notation we have that for $1 \leq i \leq j \leq n$, the class of $x_ix_j$ is not in the span of the classes of $x_rx_s$ with $(r, s) < (i, j)$, but, the class of $x_jx_i$ is in the span of the class.
of $x_i x_j$ with $(i, j) < (j, i)$. Therefore $S = \{(i, j) \mid 1 \leq i \leq j \leq n\} = S^{(2)}$ and $S^{(m)} = \{(i_1, \ldots, i_m) \mid 1 \leq i_1 \leq \cdots \leq i_m, 1 \leq i_k \leq n\}$ for $m \geq 3$. Then
\[(A, S) = \{x_1^{m_1} \cdots x_n^{m_n} \mid m_1, \ldots, m_n \geq 0\} = \text{Mon}(A) := \{x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n\}.
\]

By Definition 2.1 (ii), \text{Mon}(A) is a $\mathbb{K}$-basis for $A$ and therefore $A$ is a PBW algebra.

**Theorem 3.12** ([18], Theorem 5.3). *If $B$ is a PBW algebra then $B$ is a homogeneous Koszul algebra.*

The Theorem 3.12 and this proof can also be found in [17], Theorem 3.1, page 84; there they also present an example of a homogeneous Koszul algebra which is not PBW algebra.

**Corollary 3.13.** *Every semi-commutative skew PBW extension is homogeneous Koszul algebra.*

**Proof.** If following from Proposition 3.11 and Theorem 3.12.

**Theorem 3.14.** *Every pre-commutative skew PBW extension is Koszul.*

**Proof.** If $A$ is a pre-commutative skew PBW extension then by Remark 3.2, $A = \mathbb{K}\langle x_1, \ldots, x_n \rangle / I$, where $I$ is the two-sided ideal generated by relations of the form $x_j x_i - c_{i,j} x_i x_j + \sum_{t=1}^n k_t x_t$, $0 \neq c_{i,j}, k_t \in \mathbb{K}$, $1 \leq i, j, t \leq n$. By Proposition 3.3 $A$ is pre-Koszul, therefore from Proposition 3.7, $A^{(0)} = A^\sigma = \mathbb{K}\langle x_1, \ldots, x_n \rangle / \langle x_j x_i - c_{i,j} x_i x_j \rangle$ is the associated homogeneous pre-Koszul algebra of $A$. Note that $A^\sigma$ is semi-commutative, so by Corollary 3.13, $A^{(0)}$ is a homogeneous Koszul algebra, i.e., $A$ is Koszul.

**Corollary 3.15.** *If $A$ is a pre-commutative skew PBW extension then $\text{Gr}(A)$ is homogeneous Koszul.*

**Examples 3.16.** Next we present some examples of homogeneous Koszul skew PBW extensions, many of which had already been presented by other authors with the name of Koszul algebras. For this purpose we use the classification given in Subsection 2.2 and Corollary 3.13: classical polynomial ring; particular Sklyanin algebra; Algebra of linear partial q-dilation operators; multiplicative analogue of the Weyl algebra; multi-parameter quantum affine $n$-spaces; the Sridharan enveloping algebra of 3-dimensional Lie algebra with $[x, y] = [y, z] = [z, x] = 0$.

**Examples 3.17.** Recall that every homogeneous Koszul algebra is Koszul (Remark 3.10), so Examples 3.16 are Koszul skew PBW extensions. According to classification given in Subsection 2.2 and Theorem 3.14 the next skew PBW extensions are Koszul: universal enveloping algebra of a Lie algebra, with $\mathbb{K}$ a field; diffusion algebra 1; quantum algebra; Dispın algebra; Woronowicz algebra; $q$-Heisenberg algebra; types 1, 2, 3, 4, 5 and 6 of Sridharan enveloping algebra of 3-dimensional Lie algebra (Example 2.8).

Note that some particular classes of skew PBW extensions in Examples 3.16 and 3.17 represent the same algebra. For example, Sridharan enveloping algebra of 3-dimensional Lie algebra of type 1 and the classical polynomial ring $\mathbb{K}[x, y, z]$ are the same algebra.
4 PBW deformations

Let $V$ be a vector space over a field $K$ and let $T(V) = \bigoplus T^i(V)$ be its tensor algebra over $K$. Consider the natural filtration $F_i(T) = \bigoplus T^i(V) \mid j \leq i$ of $T(V)$. Fix a subspace $P \subseteq F_2(T) = K \bigoplus V \bigoplus (V \otimes V)$, and let us consider the two-sided ideal $\langle P \rangle$ in $T(V)$ generated by $P$. Let $A = T(V)/\langle P \rangle$ be a nonhomogeneous quadratic algebra. It inherits a filtration $A_0 \subseteq A_1 \subseteq \cdots A_n \subseteq \cdots$ from $T(V)$, let $Gr(A)$ the associated graded algebra. Consider the natural projection $\pi: F_2(T) = K \bigoplus V \bigoplus (V \otimes V) \rightarrow V \otimes V$ on the homogeneous component, set $R = \pi(P)$ and consider the homogeneous quadratic algebra $T(V)/(R)$. $T(V)/(R)$ is called the homogeneous version (or the induced homogeneous quadratic) algebra of $A$ determined by $P$. We have the natural epimorphism $p: T(V)/(R) \rightarrow Gr(A)$ (induced by the projection $T(V) \rightarrow A$).

**Definition 4.1** ([6], Page 316). With the above notation, a nonhomogeneous quadratic algebra $A := T(V)/(P)$ is a Poincaré-Birkhoff-Witt (PBW) deformation (or satisfies the PBW property with respect to the subspace $P$ of $F_2(T)$) of $B := T(V)/(R)$ if the natural projection $p: T(V)/(R) \rightarrow Gr(A)$ is an isomorphism.

**Proposition 4.2** ([15], Theorem 3.6.4). Let $A = T(V)/(P)$ with $P \subseteq V \bigoplus V^{\otimes 2}$, $P \cap V = \{0\}$ and $\{P \otimes V + V \otimes P\} \cap V^{\otimes 2} \subseteq P \cap V^{\otimes 2}$. If $A$ is Koszul, then the epimorphism $p: B = T(V)/(R) \rightarrow Gr(A)$ is an isomorphism of graded algebras, i.e., $A$ is a PBW deformation of $B$.

**Corollary 4.3** ([15], Corollary 3.6.5). Let $A = T(V)/(P)$, with $P \subseteq V \bigoplus V^{\otimes 2}$. If $B = T(V)/(R)$ is homogeneous Koszul, then:

1. $P \cap V = \{0\} \iff \langle P \rangle \cap V = \{0\}$
2. $\{P \otimes V + V \otimes P\} \cap V^{\otimes 2} \subseteq P \cap V^{\otimes 2} \iff \langle P \rangle \cap \{V \bigoplus V^{\otimes 2}\} = P$.

**Remark 4.4.** [6], Lemma 0.4 establishes that if the algebra $T(V)/(P)$ is a PBW deformation of $T(V)/(R)$ then it satisfies the following conditions:

(I) $P \cap F_1(T) = 0$;

(J) $(F_1(T) \cdot P \cdot F_1(T)) \cap F_2(T) = P$.

If a nonhomogeneous quadratic algebras satisfy (I) then the subspace $P \subseteq F_2(T)$ can be described in terms of two maps $\alpha: R \rightarrow V$ and $\beta: R \rightarrow K$ as $P = \{x - \alpha(x) - \beta(x) \mid x \in R\}$. If $A = T(V)/(P)$ is a PBW deformations of its homogeneous version then $P$ can not have relations in $F_1(T)$, so:

(i) If $A$ is a PBW deformation of some skew PBW extension $B$, then $A$ is constant.
(ii) The homogeneous version of a skew PBW extension $A$ is the skew PBW extension $B$ such that the conditions (i) and (ii) of Definition 2.1 for $A$ are satisfy for $B$, and the conditions (iii) and (iv) are replaced by $x_jx_i - c_{i,j}x_ix_j = 0$, where $c_{i,j}$ are the same that for $A$.

(iii) The homogeneous version of a skew PBW extension is homogeneous Koszul.

For example the homogeneous version for the universal enveloping algebra of a Lie algebra $\mathcal{G}$, $\mathcal{U}(\mathcal{G})$ is the symmetric algebra $S(\mathcal{G})$.

**Proposition 4.5.** Let $A$ be a constant skew PBW extension of a field $K$. Then $A$ is a PBW deformation of its homogeneous version $B$.

**Proof.** Let $A$ be a constant skew PBW extension of a field $K$ then, $x_jx_i - c_{i,j}x_ix_j + r_0 + r_1x_1 + \cdots + r_nx_n$ (as in Definition 2.1) are the generated relations of the subspace $P$, that is, $A = K\langle x_1, \ldots, x_n \rangle / \langle P \rangle$. Then the subspace $\pi(P) = R$ is generated by the relations $x_jx_i - c_{i,j}x_ix_j$, i.e, $K\langle x_1, \ldots, x_n \rangle / \langle R \rangle = B$ is the homogeneous version of $A$. Now for Theorem 2.5, $Gr(A) \cong A^\sigma$ where $A^\sigma$ is a skew PBW extension of $K$ in $n$ variables $z_1, \ldots, z_n$ defined by the relations $z_jz_i = c_{i,j}z_iz_j$, for $1 \leq i \leq n$. So by Remark 4.4, $A^\sigma \cong B$ and therefore $Gr(A) \cong B$, i.e., $A$ is a PBW deformation of $B$. \qed

Note that if a skew PBW extension $A$ is not constant then the Proposition 4.5 fails, indeed: the homogeneous version of $A$ is the skew PBW extension $B$ with relations $x_jx_i - c_{i,j}x_ix_j = 0$, where $c_{i,j}$ are the same that for $A$, but $Gr(A)$ is defined defined by the relations $z_ir = c_{i,r}z_i$, $z_jz_i = c_{i,j}z_iz_j$ (see Theorem 2.5 and Proposition 2.4), so $Gr(A) \not\cong B$.

Let $T(V) / \langle P \rangle$ be a nonhomogeneous quadratic algebra. Take $R = p(P) \subseteq T^2(V)$ and consider the corresponding homogeneous quadratic algebra $A = T(V) / \langle R \rangle$. The main theorem of [6] establishes that if $A$ is a homogeneous Koszul algebra then conditions (I) and (J) in Remark 4.4 imply that the algebra $T(V) / \langle P \rangle$ is a PBW deformation of $A$.

**Proposition 4.6.** If $A$ is a PBW deformation of a skew PBW extension $B$, then $B$ is homogeneous Koszul.

**Proof.** Let $A = \sigma(K)\langle x_1, \ldots, x_n \rangle$ be a PBW deformation of $B$, then

$$A = K\langle x_1, \ldots, x_n \rangle / \langle x_jx_i - c_{i,j}x_ix_j + k_0 + k_1x_1 + \cdots + k_nx_n \rangle,$$

with $c_{i,j} \in K \setminus \{0\}$, $k_l \in K$, $1 \leq i, j \leq n$, $0 \leq l \leq n$ and $B \cong K\langle x_1, \ldots, x_n \rangle / \langle x_jx_i - c_{i,j}x_ix_j \rangle$. Then $B$ is a semi-commutative skew PBW extension of $K$, and by Corollary 3.13, we conclude that $B$ is homogeneous Koszul. \qed

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