Characterisations of algebraic properties of groups in terms of harmonic functions

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Abstract

We prove various results connecting structural or algebraic properties of graphs and groups to conditions on their spaces of harmonic functions. In particular: we show that a group with a finitely supported symmetric measure has a finite-dimensional space of harmonic functions if and only if it is virtually cyclic; we present a new proof of a result of V. Trofimov that an infinite vertex-transitive graph admits a non-constant harmonic function; we give a new proof of a result of T. Ceccherini-Silberstein, M. Coornaert and J. Dodziuk that the Laplacian on an infinite, connected, locally finite graph is surjective; we give lower bounds on the dimensions of the spaces of polynomially growing harmonic functions on a virtually nilpotent group, asymptotically optimal in the virtually abelian case; and we show that the positive harmonic functions on a non-virtually nilpotent linear group span an infinite-dimensional space.

An important tool is a version of the Garden of Eden theorem for linear cellular automata, due to Ceccherini-Silberstein and Coornaert. Our version is valid in non-amenable groups and even locally finite graphs (with linear cellular automata replaced by a notion appropriate to that setting).

We present some other applications of our Garden of Eden theorem. In particular: we generalise Ceccherini-Silberstein and Coornaert’s original Garden of Eden theorem to non-amenable groups in the case of a symmetric or self-adjoint linear cellular automaton (such as a symmetric Laplacian), and with a simpler proof; we combine our result with some of the arguments of Ceccherini-Silberstein and Coornaert to recover their result in the general case; and we explain how our result leads to a reformulation of a conjecture of I. Kaplansky.

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One can often obtain algebraic information about a group by considering it as a geometric object. For example, if $G$ is a group and $S \subset G$ is a finite, symmetric set then one can construct the Cayley graph $(G,S)$ of $G$ with respect to $S$ by declaring the elements of $G$ to be vertices and saying that $x$ and $y$ are joined by an edge if and only if there is some non-identity element $s \in S$ such that $xs = y$.

One way of studying the geometry of a Cayley graph, or indeed any graph, is to consider the behaviour of probabilistic processes on it. In this paper we are particularly concerned with linking the algebra and geometry of groups and graphs to spaces of harmonic functions on them. Harmonic functions are in fact naturally defined on slightly more general objects called weighted graphs, also sometimes called electric networks. A weighted graph is a graph in which to each edge $xy$ we associate a real number $\omega_{xy} = \omega_{yx} > 0$ called a weight; the degree of a vertex $x$ is then given by

$$\deg x = \sum_{y \sim x} \omega_{xy}.$$ 

An graph in the ordinary sense can be thought of as a weighted graph in which all weights are equal to 1. A harmonic function on a weighted graph $\Gamma$ without isolated vertices is then a function $h : \Gamma \to \mathbb{R}$ satisfying

$$h(x) = \frac{1}{\deg x} \sum_{y \sim x} \omega_{xy} h(y).$$

Perhaps the most famous example of a result linking the algebraic structure of a group to the geometry of a Cayley graph is M. Gromov’s celebrated theorem on groups of polynomial growth, which states that a certain geometric condition on a Cayley graph $(G,S)$ of $G$ is characteristic of a certain algebraic condition on the subgroup of $G$ generated by $S$ (virtual nilpotence) [13]. A recent proof of Gromov’s theorem due to B. Kleiner [21] provides an example of how harmonic functions are related to the algebra and geometry of groups, since a key step in Kleiner’s proof is to show that if a group has polynomial growth then the vector space of harmonic functions on the Cayley graph that grow at most linearly in the Cayley-graph distance from the identity is finite dimensional.

While Kleiner’s proof of Gromov’s theorem essentially uses the space of linearly growing harmonic functions as a tool to characterise an algebraic condition on a group in terms of a geometric condition, in principle it should be possible to characterise certain algebraic or geometric conditions purely in terms of spaces of harmonic functions. Indeed, in a very recent preprint, T. Meyerovitch and A. Yadin [24, Theorem 1.4] have shown that in the case of a finitely generated linear group over $\mathbb{C}$, being virtually nilpotent is equivalent to having a finite-dimensional space of linearly growing harmonic functions. This equivalence is, moreover, conjectured to hold in arbitrary finitely generated groups [24].

The first result of this paper shows that finite-dimensionality of the space of all harmonic functions on a group is also equivalent to a simple algebraic condition.

**Theorem 1.1.** Let $G$ be an infinite group with a finite, symmetric generating set $S$. Then the space of harmonic functions on the Cayley graph $(G,S)$ is finite dimensional if and only if $G$ contains a finite-index subgroup isomorphic to $(\mathbb{Z}, +)$. 

In fact, our result is slightly more general than Theorem 1.1. Define a probability measure $\mu$ on $G$ to be a generating probability measure if the semigroup generated by its support, $\text{supp} \mu$, is $G$, and say that $\mu$ is symmetric if $\mu(g) = \mu(g^{-1})$ for every $g \in G$. If $\mu$ is a (not necessarily symmetric) finitely supported generating probability measure on $G$, we say that a function $h : G \to \mathbb{R}$ is harmonic with respect to $\mu$ if we have

$$h(x) = \sum_{s \in \text{supp} \mu} \mu(s)h(xs)$$

for every $x \in G$. More generally, then, we prove the following.

**Theorem 1.2.** Let $G$ be an infinite group, and let $\mu$ be a symmetric, finitely supported generating probability measure on $G$. Then the space of functions on $G$ that are harmonic with respect to $\mu$ is finite dimensional if and only if $G$ contains a finite-index subgroup isomorphic to $\langle \mathbb{Z}, + \rangle$.

The ‘direct’ statement of Theorem 1.2, that a group with a finite-index subgroup isomorphic to $\mathbb{Z}$ has a finite-dimensional space of harmonic functions, is straightforward, and we prove it as Proposition 9.1 below. The real content is therefore in the ‘inverse’ statement, which we separate as follows.

**Theorem 1.3 (Inverse result).** Let $G$ be an infinite group, and let $\mu$ be a symmetric, finitely supported generating probability measure on $G$. Suppose that $G$ has a finite-dimensional space of harmonic functions with respect to $\mu$. Then $G$ contains a finite-index subgroup isomorphic to $\langle \mathbb{Z}, + \rangle$.

*Remark 1.4.* It follows from Gromov’s theorem that, for infinite groups, being virtually cyclic (and hence, by Theorem 1.2, having a finite-dimensional space of harmonic functions) is also equivalent to having linear volume growth, and to having subquadratic volume growth. Van den Dries and Wilkie [30] give an elementary proof of this fact without appealing to Gromov’s theorem.

It follows immediately from the well-known maximum principle (Lemma 2.2 below), that the only harmonic functions on a finite group are the constants. Moreover, it is not hard to show (see, for example, [27], or Lemma 9.6 below) that an infinite virtually cyclic group admits a non-constant harmonic function.

As a corollary of Theorem 1.3, therefore, one can obtain another characterisation of an algebraic condition in terms of a condition on a space of harmonic functions: the space of harmonic functions on a group with a symmetric, finitely supported generating probability measure is 1-dimensional if and only if the group is finite.

In fact, this was already known, and with a far simpler proof than via Theorem 1.3, by a result of V. Trofimov [28]. Indeed, Trofimov shows that this characterisation holds, more generally, for vertex-transitive graphs. The second result of this paper is a new proof of Trofimov’s result, valid in the even more general setting of vertex-transitive weighted graph (Trofimov’s proof could conceivably also work in this more general setting).

**Proposition 1.5.** Let $\Gamma$ be an infinite, locally finite, vertex-transitive weighted graph. Then $\Gamma$ admits a non-constant harmonic function.

Proposition 1.5 is trivial if $\Gamma$ is not connected. In Section 7, we prove it in the case that $\Gamma$ is connected and the random walk on $\Gamma$ is transient; for the case in which $\Gamma$ is connected and the random walk is recurrent we refer the reader to [28] (where the transient and recurrent cases are also treated separately). See Section 2 for definitions of transient, recurrent and random walk.

*Remarks 1.6.* Trofimov’s result is in fact stronger than Proposition 1.5 as it proves the existence of a function whose growth rate is bounded in terms of the rate of volume growth of metric balls in $\Gamma$. Nonetheless, it seems to be of interest to have an alternative proof of the qualitative statement, and in any case recording this proof requires very little extra effort, as we deduce Proposition 1.5 fairly immediately from a slightly more general result (Proposition 7.1 below) that is an important ingredient in our proof of Theorem 1.3.

Proposition 1.5 does not necessarily hold if $\Gamma$ is not vertex transitive, as can be seen by considering the graph in Figure 1. This example was presented explicitly in a talk of Coornaert [1], having been observed by Trofimov [28, Remark 2].

1. http://www-irma.u-strasbg.fr/~coornaer/florence-laplacian-2012.pdf
Harmonic functions are often thought of in terms of operators called *discrete Laplacians*, traditionally denoted by $\Delta$. Given a weighted graph $\Gamma$, we define the *Laplacian* $\Delta = \Delta_\Gamma$ on $\Gamma$ by setting

$$\Delta f(x) = f(x) - \frac{1}{\deg x} \sum_{y \sim x} \omega_{xy} f(y)$$

for every function $f : \Gamma \to \mathbb{R}$. For a group $G$ with a finitely supported generating probability measure $\mu$ we write $\Delta = \Delta_\mu$ for the Laplacian defined by $\mu$, given by setting

$$\Delta f(x) = f(x) - \sum_{s \in \text{supp } \mu} \mu(s) f(xs)$$

for every function $f : G \to \mathbb{R}$. Note that if $\mu$ is symmetric then $\Delta_\mu$ is equal to the Laplacian on the Cayley graph of $G$ with respect to $S$, weighted such that $\omega_{xy} = \mu(x^{-1}y)$.

In each of the above cases, a function $h$ is harmonic if and only if $\Delta h = 0$. The results we have stated so far can therefore be thought of as characterising algebraic conditions on a group in terms of the kernel of the Laplacian. Our next result characterises a structural condition on a graph in terms of the *image* of the Laplacian.

A simple rank-nullity argument shows that the Laplacian on a finite graph is not surjective, since its kernel contains the constant functions. T. Ceccherini-Silberstein, M. Coornaert and J. Dodziuk [10, Theorem 1.1] show that the converse is also true for connected graphs. In Section 4 we give a new proof of this result, valid in weighted graphs (Ceccherini-Silberstein, Coornaert and Dodziuk’s proof could conceivably also work in this more general setting).

**Proposition 1.7.** The Laplacian on an infinite, connected, locally finite weighted graph $\Gamma$ is surjective onto $\mathbb{R}^\Gamma$.

**Corollary 1.8.** The Laplacian on a locally finite weighted graph $\Gamma$ is surjective onto $\mathbb{R}^\Gamma$ if and only if every connected component of $\Gamma$ is infinite.

Our proof of Proposition 1.7 is inspired by an earlier, less general, result of Ceccherini-Silberstein and Coornaert, which states that the Laplacian on an infinite Cayley graph is surjective [7, Theorem 1.1]. We also show that this holds for more general Laplacians on groups.

**Proposition 1.9.** Let $G$ be an infinite group, and let $\mu$ be a finitely supported generating probability measure on $G$. Then $\Delta_\mu$ is surjective.

It is quite likely that the argument of Ceccherini-Silberstein and Coornaert [7] could also give a proof of Proposition 1.9. However, our proof of Proposition 1.9 is simpler than the argument of [7] (see Remarks 4.3, below), and since Proposition 1.9 cannot be concluded directly from either Proposition 1.7 or [7, Theorem 1.1], it seems, in any case, worth recording a proof here.

An important tool in this paper is the so-called *Garden of Eden theorem* for linear cellular automata, originally due to Ceccherini-Silberstein and Coornaert [6]. Given its importance to our arguments, we introduce it briefly here.
If $G$ is a group and $A$ is a set, called the alphabet, then $G$ acts on the set $A^G$ of maps $f : G \to A$ via
\[ g \cdot f(x) = f(g^{-1}x). \]

If $f : G \to A$ and $M \subseteq G$ then we denote by $f|_M$ the restriction of $f$ to $M$. A cellular automaton over $G$ on the alphabet $A$ is a map $\tau : A^G \to A^G$ with the property that there is some finite set $M \subseteq G$ and a map $\lambda : A^M \to A$ such that
\[ \tau(f)(g) = \lambda((g \cdot f)|_M). \]

The set $M$ is called a memory set for $\tau$, and $\lambda$ is called a local defining map.

Given an initial state $f_0 \in A^G$, one can consider $\tau$ as defining a dynamical process on $A^G$ by setting $f_{i+1} = \tau(f_i)$ to obtain a sequence $f_0, f_1, f_2, \ldots$ of configurations in $A^G$. A configuration $f \in A^G$ is then said to be a Garden of Eden configuration if it is not in the image of $\tau$, and hence can appear only as an initial configuration in this dynamical process.

The term Garden of Eden theorem for a class of cellular automata is often used to describe a result giving a necessary and sufficient condition for the existence of Garden of Eden configurations, or, to put it another way, a necessary and sufficient condition for a cellular automaton in the class to be surjective. There are various results depending on the alphabet and the group; we refer the reader to [9, Theorem 8.9.6] for more detailed background to this area.

The class of interest to us is the class of linear cellular automata, in which $A$ is a finite-dimensional vector space $V = K^r$ over a field $K$ and a linear cellular automaton is a cellular automaton that is also a linear map $V^G \to V^G$. In order to state the Garden of Eden theorem in this context, we need one more definition. A linear map on $V^G$ is pre-injective if its restriction to the subspace $V_0^G$ of finitely supported functions in $V^G$ is injective.

**Theorem 1.10** (Garden of Eden theorem for linear cellular automata; Ceccherini-Silberstein–Coornaert [9, Theorem 8.9.6]). Let $V$ be a finite-dimensional vector space and let $G$ be an amenable group. Then a linear cellular automaton $\tau : V^G \to V^G$ is surjective if and only if it is pre-injective.

The Laplacian on a group with a finitely supported generating probability measure is an example of a linear cellular automaton, and so Theorem 1.10 can readily be applied to such a Laplacian, provided that the group is amenable. However, in this paper we are concerned with Laplacians on arbitrary groups, and even graphs, and so we seek a version of Theorem 1.10 that holds in this greater generality.

Before we can state such a result, we must replace the notion of cellular automaton over a group with one that is valid in the more general setting of a graph. To that end, given a locally finite graph $\Gamma$ and an alphabet $A$, we say that a map $\tau : A^\Gamma \to A^\Gamma$ is locally specifiable if $\tau(f)(x)$ depends only on $f(x)$ and $f(y)$ for $y \sim x$. Note, in particular, that if $\tau$ is a cellular automaton on a group $G$ with memory set $M$ then $\tau$ is a locally specifiable map on the Cayley graph $(G, M \cup M^{-1})$. The Laplacian on a locally finite graph is also locally specifiable.

The result underpinning much of this paper is the following.

**Theorem 1.11.** Let $V$ be a finite-dimensional vector space and let $\Gamma$ be a locally finite graph. Then a locally specifiable linear map $\tau : V^\Gamma \to V^\Gamma$ is surjective if and only if its transpose $\tau^t$ is pre-injective.

Here the transpose of $\tau$ is defined in terms of the natural (possibly infinite) matrix representation of $\tau$, which we define precisely in Section 2. The transpose of $\tau$ is then simply the locally specifiable linear map whose corresponding matrix is the transpose of the matrix corresponding to $\tau$.

Let us emphasise here that Theorem 1.11 applies, in particular, to linear cellular automata on non-amenable groups, and, in that sense, considerably more general than Theorem 1.10. In the specific case of a symmetric linear cellular automaton, such as the Laplacian defined by a symmetric, finitely supported generating measure on a group, this leads to the following generalisation of Theorem 1.11.

**Corollary 1.12** (Garden of Eden theorem for symmetric linear cellular automata over non-amenable groups). Let $V$ be a finite-dimensional vector space and let $G$ be a group. Then a symmetric linear cellular automaton $\tau : V^G \to V^G$ is surjective if and only if it is pre-injective.
Remarks 1.13. Corollary 1.12 is valid only for symmetric linear cellular automata, whereas Theorem 1.10 is valid for all linear cellular automata. The reader may refer to [6, §5] or [9, §8.10-8.11] for examples of (non-symmetric) linear cellular automata on non-amenable groups for which Theorem 1.10 fails; thus, generalisations to non-amenable groups in the spirit of Corollary 1.12 must necessarily have some additional hypothesis on the map $\tau$.

With a bit more work, one can adapt some of the techniques from [6] to recover Theorem 1.10 in full from Theorem 1.11; see Appendix B.

Outline of the paper

In Section 2 we give more detailed definitions of the concepts we use in this paper, and record some standard facts about harmonic functions on groups and graphs.

In Section 3 we further motivate, and more importantly prove, Theorem 1.11. In Section 4 we apply Theorem 1.11 to prove Propositions 1.7 and 1.9. In Section 5 we use it to develop a tool for proving the existence of harmonic functions on a graph or group, and then in Section 7 we use this tool to prove a slight technical generalisation of Proposition 1.5.

In Section 6 we prove Theorem 1.2, and in particular a quantitative version of Theorem 1.3, in the case of a virtually nilpotent group. In Section 10 we prove Theorem 1.3, and hence Theorem 1.2, in the case of a non-virtually nilpotent linear group. In Sections 11-13 we reduce Theorem 1.3 to the linear case, and hence complete the proof of Theorem 1.2.

In Sections 6 and 8 we make two brief technical reductions.

In Appendix A we prove a technical result about coordinates in nilpotent Lie groups that we need in Section 9. In Appendix B we present two additional applications of Theorem 1.11. In the first, we reformulate a conjecture of I. Kaplansky, the so-called ‘stable-finiteness’ conjecture.

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Theorem 1.3 answers a question of Ben Green (private communication). The author previously made public a claimed proof of that result, by a method completely different to that contained in this paper, that turned out to contain a serious error. Thanks are due to an anonymous referee for noticing this.

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2 Background and notation

In this section we set much of our notation and present general background material from the literature. Most of this is standard; see, for example, [25].

Throughout this paper, by a graph we mean an undirected weighted graph with no loops and no multiple edges. We denote by $e$ some distinguished vertex, and write $x \sim y$ to indicate that $x$ and $y$ are neighbours. An isomorphism of weighted graphs is an isomorphism of graphs that preserves weights. A weighted graph is called regular if

$$\deg x = \sum_{y \sim x} \omega_{xy}$$

is independent of the vertex $x$. 

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We noted in the introduction that if $\mu$ is a symmetric, finitely supported generating probability measure on a group $G$ then $\Delta_\mu$ is equal to the Laplacian on the Cayley graph of $G$ with respect to $S$, weighted such that $\omega_{xy} = \mu(x^{-1}y)$; thus, in particular, any result about weighted graphs also applies immediately to groups with symmetric, finitely supported generating probability measures. We denote this weighted Cayley graph by $(G, \mu)$. In this case, the distinguished vertex $e$ of $(G, \mu)$ is always the identity element of $G$. If $G$ is abelian then we usually switch to additive notation and write 0, rather than $e$, for the identity element.

We denote by $d = d_G$ the graph metric on a graph $\Gamma$; thus, for vertices $x \neq y \in \Gamma$ the quantity $d(x, y)$ is equal to length of a path of minimum length joining $x$ to $y$. If $G$ is a group with a finite generating set $S$ then we denote by $d = d_S$ the word metric on $G$ with respect to $S$; if $S$ is symmetric then this agrees with the graph metric on the Cayley graph $(G, \mu)$. In that instance, we occasionally abbreviate $|g| = |g|_S = d(e, g)$. More generally, if $\mu$ is a finitely supported generating probability measure on $G$ then we define

$$S := \text{supp } \mu,$$

and set $d = d_\mu = d_S$ and $|g| = |g|_\mu = |g|_S$.

If $\Gamma$ is a graph or group and $V = K^n$ is a vector space then for each vertex or element $x \in \Gamma$ and each $i = 1, \ldots, n$ we denote by $\delta^i_x : \Gamma \to V$ the map defined by $\delta^i_x(x) = e_i$, and $\delta^i_x(y) = 0$ for every $y \neq x$. In the event that $n = 1$ we drop the superscript and define $\delta_x : \Gamma \to K$ by $\delta_x(x) = 1$ and $\delta_x(y) = 0$ for every $y \neq x$. The $\delta^i_x$ form a basis for the space $V^\Gamma_0$ of finitely supported $V$-valued functions on $\Gamma$ (and, for the purposes of this paper, should ‘morally’ be thought of as a basis for $V^\Gamma$).

This space $V^\Gamma_0$ is invariant under any locally specifiable linear map $\tau : V^\Gamma \to V^\Gamma$, and so we may consider the (possibly infinite) matrix of the restriction $\tau|_{V^\Gamma_0}$ with respect to this basis. In fact, $\tau$ is entirely determined by its restriction to $V^\Gamma_0$, and so the matrix of $\tau|_{V^\Gamma_0}$ with respect to this basis completely determines $\tau$. Moreover, the composition of such matrices respects the composition of the corresponding linear maps. Throughout this paper, when we refer to the matrix of a locally specifiable linear map $\tau : V^\Gamma \to V^\Gamma$, we mean the matrix of its restriction $\tau|_{V^\Gamma_0}$ with respect to the basis $\{\delta^i_x\}$.

Given a finite set $Y$ and a function $f : Y \to \mathbb{R}$, we generally denote by $\mathbb{E}_{y \in Y} f(y)$ the average

$$\mathbb{E}_{y \in Y} f(y) = \frac{1}{|Y|} \sum_{y \in Y} f(y). \quad (2.1)$$

However, in the specific case that $\mu$ is a generating probability measure on a group $G$, and $S$ is the support of $\mu$, given a function $f : S \to \mathbb{R}$ the notation $\mathbb{E}_{s \in S} f(s)$ means the average

$$\mathbb{E}_{s \in S} f(s) = \sum_{s \in S} \mu(s) f(s).$$

Note that this agrees with $(2.1)$ only if $\mu$ is the uniform probability measure on $S$.

If $G_1, G_2$ are groups and $\phi : G_1 \to G_2$ a surjective homomorphism, then given a finitely supported generating probability measure $\mu$ on $G_1$ we define a finitely supported generating probability measure $\phi(\mu)$ on $G_2$ by setting

$$\phi(\mu)(g) = \sum_{\gamma \in \phi^{-1}(g)} \mu(\gamma).$$

Note that if $\mu$ is symmetric then so is $\phi(\mu)$.

**Lemma 2.1.** Let $G_1$ be a group with a finitely supported generating probability measure $\mu$. Suppose that $\phi : G_1 \to G_2$ is a surjective homomorphism, and that $f : G_2 \to \mathbb{R}$. Then $f \circ \phi$ is harmonic with respect to $\mu$ if and only if $f$ is harmonic with respect to $\phi(\mu)$.

**Proof.** Given an arbitrary $g_2 \in G_2$, the surjectivity of $\phi$ implies that there exists $g_1 \in G_1$ such that $\phi(g_1) = g_2$. On the other hand, given an arbitrary $g_1 \in G_1$, we may simply define $g_2 \in G_2$ by $g_2 = \phi(g_1)$. In either case,

$$f(g_2) = f \circ \phi(g_1)$$
and

$$E_{s \in \phi(S)} f(g_2 s) = E_{s \in S} f(g_2 \phi(s)) = E_{s \in S} f \circ \phi(g_1 s),$$

from which the lemma follows easily. □

Given a subset \( A \) of a graph \( \Gamma \), or of a group \( G \) with a finitely supported generating probability measure \( \mu \), we define the \textit{neighbourhood} \( A^+ \) of \( A \) to be the set

\[ A^+ = \{ x \in \Gamma : d(x, A) \leq 1 \}; \]

the \textit{interior} \( A^\circ \) of \( A \) to be the set

\[ A^\circ = \{ x \in A : \{ x \}^+ \subset A \}; \]

the \textit{inner boundary} \( \partial^- A \) of \( A \) to be the set

\[ \partial^- A = A \setminus A^\circ; \]

and the \textit{outer boundary} \( \partial^+ A \) of \( A \) to be the set

\[ \partial^+ A = A^+ \setminus A. \]

Let \( \Gamma \) be a locally finite weighted graph, or a group with a finitely supported generating probability measure \( \mu \). Let \( A \) be a subset of \( \Gamma \), and let \( A^+ \) be a subset of \( \Gamma \) containing \( A^+ \). Then we say that a function \( h : D \to \mathbb{R} \) is \textit{harmonic on} \( A \) if for each \( x \in A \) we have

\[ h(x) = \frac{1}{\deg x} \sum_{y \sim x} \omega_{xy} h(y) \]

(in the graph case), or

\[ h(x) = E_{s \in S} h(x s) \]

(in the group case).

The following is an immediate consequence of the definition of harmonicity.

\textbf{Lemma 2.2 (Maximum principle).} Let \( \Gamma \) be a locally finite graph, or a group with a finitely supported generating probability measure \( \mu \), and let \( A \) be a connected subset of \( \Gamma \). Suppose that \( f : A^+ \to \mathbb{R} \) is harmonic on \( A \) and achieves a maximum on \( A \). Then \( f \) is constant.

Harmonic functions on graphs and groups are intimately connected to random walks. Given a graph \( \Gamma \) and a vertex \( x \in \Gamma \), the \textit{random walk} starting at \( x \) is a sequence of \( \Gamma \)-valued random variables \( X_0, X_1, X_2, \ldots \), with \( X_0 = x \) with probability 1 and each subsequent \( X_n \) chosen from among the neighbours of \( X_{n-1} \) such that \( X_n = y \) with probability \( \omega_{X_{n-1}y} / \deg X_{n-1} \). Given a group \( G \) with a finitely supported generating probability measure \( \mu \), the random walk on the pair \((G, \mu)\) starting at \( x \in G \) is a sequence of \( G \)-valued random variables \( X_0, X_1, X_2, \ldots \), with \( X_0 = x \) with probability 1 and each subsequent \( X_n \) taking the value \( X_{n-1} \) with probability \( \mu(s) \). We say that the random walk on \((G, \mu)\) is \textit{symmetric} if \( \mu \) is symmetric.

Given an event \( B \), we denote by \( \mathbb{P}_x[B] \) the conditional probability \( \mathbb{P}[B \mid X_0 = x] \). Given another event \( C \), we denote by \( \mathbb{P}_x[B \mid C] \) the conditional probability \( \mathbb{P}[B \mid C \text{ and } \{X_0 = x\}] \). We use the conditional expectation notion \( E_x \) similarly.

If \( A \) is a subset of \( \Gamma \), we write

\[ T_A := \inf \{ t : X_t \in A \}, \]

with \( T_A = \infty \) if \( X_t \notin A \) for all \( t \). The random variable \( T_A \) is often called a \textit{stopping time} for the random walk. If \( A \) is the singleton \( \{x\} \) then we abbreviate

\[ T_x := T_{\{x\}}. \]
Lemma 2.3 (Harmonic functions are determined by their boundary values). Let $\Gamma$ be a graph, or a group with a finitely supported generating probability measure, and let $A$ be a finite subset of $\Gamma$ with non-empty outer boundary. Let $f_0 : \partial^+ A \to \mathbb{R}$. Then the function $f : A^+ \to \mathbb{R}$ defined by

$$ f(x) = E_x [f_0 (X_{T_x + A})] $$

is harmonic on $A$ and agrees with $f_0$ on $\partial^+ A$, and is unique with respect to these two properties.

Proof. It is easy to see that $f$ is harmonic on $A$ and agrees with $f_0$ on $\partial^+ A$, so we just need to show that $f$ is unique. To see this, note that if $f'$ is another such function then at least one of $f - f'$ or $f' - f$ must achieve a maximum on $A$, and so the maximum principle (Lemma 2.2) implies that $f - f'$ is constant; this constant must be 0 since $f = f'$ on $\partial^+ A$.

Corollary 2.4. Let $G$ be a group and let $A$ be a finite subset of $G$. Suppose that $f_1, f_2 : A^+ \to \mathbb{R}$ are harmonic on $A$, and that $f_1 \geq f_2$ on $\partial^+ A$. Then $f_1 \geq f_2$ on the whole of $A^+$.

We define the kernel of a random walk on a graph or group $\Gamma$ to be the matrix with rows and columns indexed by the vertices or elements of $\Gamma$ with $xy$ entry defined by

$$ K_{xy} = \mathbb{P}_x [X_1 = y]. $$

Observe then that

$$ \mathbb{P}_x [X_n = y] = (K^n)_{xy} = \langle \delta_x, K^n \delta_y \rangle, $$

where $\delta : \Gamma \to \mathbb{R}$ is the function taking the value 1 at $x$ and 0 elsewhere, and $\langle \cdot , \cdot \rangle$ is the standard inner product on $\mathbb{R}^\Gamma$.

Lemma 2.5. Let $x, y$ be vertices in a vertex-transitive weighted graph $\Gamma$. Then

$$ \mathbb{P}_x [X_{2n} = y] \leq \mathbb{P}_e [X_{2n} = e]. $$

Remark 2.6. Lemma 2.5 does not necessarily hold if $2n$ is replaced by $n$. For example, if $n$ is odd then in the Cayley graph $(\mathbb{Z}, \pm 1)$ we have $\mathbb{P}_0 [X_n = 0] = 0$.

Proof of Lemma 2.5. Let $K$ be the kernel of the random walk, and note that regularity of $\Gamma$ implies that $K$ is symmetric. Then

$$ \mathbb{P}_x [X_{2n} = y] = \langle \delta_x, K^{2n} \delta_y \rangle = \langle K^n \delta_x, K^n \delta_y \rangle $$

(by symmetry of $K$)

$$ \leq \langle K^n \delta_x, K^n \delta_y \rangle^{1/2} \langle K^n \delta_y, K^n \delta_y \rangle^{1/2} $$

(by Cauchy–Schwarz)

$$ = \langle \delta_x, K^{2n} \delta_y \rangle^{1/2} \langle \delta_y, K^{2n} \delta_y \rangle^{1/2} $$

(by symmetry of $K$)

$$ = \mathbb{P}_x [X_{2n} = x]^{1/2} \mathbb{P}_y [X_{2n} = y]^{1/2} $$

$$ = \mathbb{P}_e [X_{2n} = e] $$

(by vertex transitivity),

and so the lemma is proved. 

A vertex $x$ of a graph, or a group with a finitely supported generating probability measure, is called recurrent for the random walk on the graph or group if $\mathbb{P}_x [T_x < \infty] = 1$, and transient for the random walk otherwise. In the case of a connected graph or a group this is independent of the choice of vertex, and so it makes sense to define the random walk on a connected graph, or on a group with a finitely supported generating probability measure, to be recurrent if $\mathbb{P}_e [T_e < \infty] = 1$, and transient otherwise.

Write $R_e$ for the number of times the random walk visits the vertex $x$.

Lemma 2.7. The random walk on a connected graph, or group with a finitely supported generating probability measure, is transient if and only if $\mathbb{E}_e [R_e] < \infty$. 

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Proof. If the random walk is transient then $\mathbb{P}_e[T_e < \infty] = \alpha$, say, with $\alpha < 1$, and so

$$\mathbb{P}_e[R_e = k] = \alpha^{k-1}(1-\alpha).$$

This means that $R_e$ is a geometric random variable with parameter $1-\alpha$, and so

$$\mathbb{E}_e[R_e] = \frac{1}{1-\alpha} < \infty,$$

as desired.

Conversely, if the random walk is recurrent then $\mathbb{P}_e[R_e = \infty] = 1$, and so $\mathbb{E}_e[R_e] = \infty$. 

In the case of a group, if we require probability measures to be symmetric then recurrence or transience of the random walk is even independent of the choice of finitely supported generating probability measure $\text{Varopoulos}[29,31]$. Proposition 4.2. It therefore makes sense simply to define a finitely generated group to be recurrent if some symmetric random walk on it is recurrent, and transient otherwise.

N. Varopoulos has characterised those groups that are recurrent.

**Proposition 2.8 (Varopoulos [29,31]).** Let $G$ be a group with a symmetric, finitely supported generating probability measure $\mu$. Then the random walk on $(G,\mu)$ is recurrent if and only if $G$ is finite or has a finite-index subgroup isomorphic to $\mathbb{Z}$ or $\mathbb{Z}^2$.

## 3 A Garden of Eden theorem

In this section we prove Theorem 1.11. However, before we do so, let us discuss why it is a natural variant of Theorem 1.10.

The equivalence between surjectivity and pre-injectivity established in Theorem 1.10 is a very natural generalisation of the finite-dimensional phenomenon that a linear map is surjective if and only if it is injective, injectivity and pre-injectivity of course being the same thing in a finite-dimensional setting. The philosophy driving this section, however, is that it is, in a sense, even more natural to compare surjectivity of a linear map $\tau : U \to W$ to injectivity of its dual map $\tau^* : W^* \to U^*$.

In the finite-dimensional setting it is easy to establish that injectivity and surjectivity of a map, and injectivity of its transpose, and hence its dual, are all equivalent, and so the distinction is somewhat obscured. Nonetheless, even in that setting the equivalence of surjectivity of a map and injectivity of its transpose is extremely elementary, whereas the equivalence of surjectivity of a map and injectivity of the same map relies on a dimension argument that does not carry over to the infinite-dimensional setting.

In the case of a locally specifiable linear map $\tau : V^T \to V^T$ on an infinite, connected, locally finite graph $\Gamma$, the dual map $\tau^*$ is very easy to write down. First let us note that the dual of the vector space $K^\Gamma$ of $K$-valued functions on $\Gamma$ is isomorphic to the subspace $K^\Gamma_0$ of finitely supported functions. Since $\Gamma$ is connected and locally finite it is also countable, and so since $V$ is finite dimensional, $V^T$ is isomorphic to $K^\mathbb{Z}$. The dual of $V^T$ can therefore be viewed as the subspace $V^T_0$ of finitely supported $V$-valued functions on $\Gamma$.

An convenient feature of a locally specifiable linear map $\tau$ on the space $V^T$ is that if we view $\tau$ as an infinite matrix then its transpose $\tau'$ also determines a well-defined locally specifiable linear map on $V^T$. However, it is also straightforward to check that, in the case that $\Gamma$ is connected, if we identify $(V^T)^*$ with $V^T_0$, as described in the previous paragraph, then upon restriction to $V^T_0$ the transpose $\tau'$ acts precisely as $\tau^*$. In particular, injectivity of the dual map $\tau^*$ on $(V^T)^* = V^T_0$ is equivalent to pre-injectivity of the transpose $\tau'$ on $V^T$. In light of the preceding paragraphs, this is very suggestive of pre-injectivity as a dual property to surjectivity.

The technical tool that allows us to relate the ‘local’ property of pre-injectivity to the ‘global’ property of surjectivity is the following result, which is essentially [6, Lemma 3.1] and was initially used by Ceccherini-Silberstein and Coornaert to prove Theorem 1.10 in the case of a countable amenable group [6]. Here, and throughout this section, we write $e$ for an arbitrary distinguished vertex of the graph $\Gamma$, and write $B(n) = B_e(n)$ for the ball of radius $n$ about $e$. 

\[ \text{Proof.} \] If the random walk is transient then $\mathbb{P}_e[T_e < \infty] = \alpha$, say, with $\alpha < 1$, and so

$$\mathbb{P}_e[R_e = k] = \alpha^{k-1}(1-\alpha).$$

This means that $R_e$ is a geometric random variable with parameter $1-\alpha$, and so $\mathbb{E}_e[R_e] = \frac{1}{1-\alpha} < \infty$, as desired.

Conversely, if the random walk is recurrent then $\mathbb{P}_e[R_e = \infty] = 1$, and so $\mathbb{E}_e[R_e] = \infty$. 

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\[ \text{Proof.} \] If the random walk is transient then $\mathbb{P}_e[T_e < \infty] = \alpha$, say, with $\alpha < 1$, and so

$$\mathbb{P}_e[R_e = k] = \alpha^{k-1}(1-\alpha).$$

This means that $R_e$ is a geometric random variable with parameter $1-\alpha$, and so $\mathbb{E}_e[R_e] = \frac{1}{1-\alpha} < \infty$, as desired.

Conversely, if the random walk is recurrent then $\mathbb{P}_e[R_e = \infty] = 1$, and so $\mathbb{E}_e[R_e] = \infty$. 

In the case of a group, if we require probability measures to be symmetric then recurrence or transience of the random walk is even independent of the choice of finitely supported generating probability measure $\text{Varopoulos}[29,31]$. Proposition 4.2. It therefore makes sense simply to define a finitely generated group to be recurrent if some symmetric random walk on it is recurrent, and transient otherwise.

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Lemma 3.1 (Ceccherini-Silberstein–Coornaert). Let $\Gamma$ be a connected, locally finite graph and let $\tau : V^T \to V^T$ be a locally specifiable linear map. Suppose that $f : \Gamma \to V$ is such that for every $n$ there is a function $v_n : \Gamma \to V$ such that $\tau(v_n)$ and $f$ agree on the ball $B(n)$. Then there is a function $w : \Gamma \to V$ such that $f = \tau(w)$.

Proof. [Lemma 3.1]. For each $n \geq 2$, denote by $\tau_n$ the linear map $V^{B(n)} \to V^{B(n-1)}$ induced by $\tau$, and define $L_n$ to be the affine subspace of $V^{B(n)}$ given by $L_n = \tau_n^{-1}(f|_{B(n-1)})$. Note in particular that $v_{n-1}|_{B(n)} \in L_n$, so that $L_n$ is non-empty.

For $n \leq m$, the restriction map $V^{B(m)} \to V^{B(n)}$ induces an affine map $\pi_{n,m} : L_m \to L_n$, and so we may define an affine subspace $K_{n,m} \subseteq L_n$ by $K_{n,m} = \pi_{n,m}(L_m)$. Since

$$\pi_{n_1,n_3} = \pi_{n_1,n_2} \circ \pi_{n_2,n_3}$$

whenever $n_1 \leq n_2 \leq n_3$, for any fixed $n$ we have

$$K_{n,n} \supset K_{n,n+1} \supset K_{n,n+2} \supset \ldots,$$

and so the sequence $K_{n,n}, K_{n,n+1}, K_{n,n+2}, \ldots$ is a decreasing sequence of non-empty finite-dimensional affine subspaces. This sequence therefore stabilises at some non-empty affine subspace $J_n$ of $L_n$.

Now (3.1) also implies that whenever $n \leq n' \leq m$ we have

$$\pi_{n,n'}(K_{n',m}) \subseteq K_{n,m},$$

and so by taking $m$ sufficiently large we see in particular that $\pi_{n,n'}(J_{n'}) \subseteq J_n$. We claim that in fact

$$\pi_{n,n'}(J_{n'}) = J_n.$$ (3.2)

Indeed, given $u \in J_n$, let $m$ be sufficiently large that $J_n = K_{n,m}$ and $J_{n'} = K_{n',m}$. By definition of $K_{n,m}$, there is some $v \in L_m$ such that $u = \pi_{n,m}(v)$, and then yet another application of (3.1) then shows that

$$u = \pi_{n,n'}(\pi_{n',m}(v)).$$ (3.3)

However, $\pi_{n',m}(v) \in K_{n',m} = J_{n'}$ by definition of $K_{n',m}$, and so (3.3) implies that $u \in \pi_{n,n'}(J_{n'})$. Since, $u \in J_n$ was arbitrary, this proves (3.2), as claimed.

We now construct recursively a sequence of functions $w_n \in J_n$, $n \in \mathbb{N}$, as follows. Initially, choose an arbitrary function $w_1 \in J_1$. Then, given $w_n \in J_n$, choose $w_{n+1}$ arbitrarily from the set $\pi_{n+1,n}(w_n) \subseteq J_{n+1}$, which is non-empty by (3.2). Since $w_{n+1}$ and $w_n$ agree on $B(n)$, there exists $w \in V^T$ such that $w|_{B(n)} = w_n$ for every $n$. However, $\tau(w)|_{B(n-1)} = \tau_n(w_n) = f|_{B(n-1)}$ for every $n$ by construction, and so $\tau(w) = f$.

Proof of Theorem 1.1. A locally specifiable map is pre-injective on $\Gamma$ if and only if it is pre-injective on every connected component of $\Gamma$, and surjective on $\Gamma$ if and only if it is surjective on every connected component of $\Gamma$, and so we may assume that $\Gamma$ is connected. This is essentially the same as a reduction to the countable case made by Ceccherini-Silberstein and Coornaert in their original proof of Theorem 1.1.8.

We first prove that surjectivity of $\tau$ implies pre-injectivity of $\tau'$. Given $v, w \in V = \mathbb{K}^r$, write $v = \sum_{i=1}^{r} v_i w_i$, and given $f_1, f_2 \in V^T$ write $f_1 \cdot f_2 = \sum_{x \in \Gamma}(f_1(x) \cdot f_2(x))$. Then if $\tau$ is surjective and $\varphi \in V_0^T$, we have

$$\tau'(\varphi) = 0 \Rightarrow \tau'(\varphi) \cdot f = 0 \text{ for every } f \in V^T \Rightarrow \varphi \cdot \tau(f) = 0 \text{ for every } f \in V^T \Rightarrow \varphi = 0$$

by surjectivity of $\tau$, and so $\tau'$ is pre-injective.

We now prove the harder direction, namely that pre-injectivity of $\tau'$ implies surjectivity of $\tau$. Lemma 3.1 means that in order to prove that $\tau$ is surjective it suffices to show that the linear map $\tau_n : V^{B(n)} \to V^{B(n-1)}$ induced by $\tau$ is surjective. Since $\tau_n$ is a map between finite-dimensional spaces, it therefore suffices to show that its dual $\tau_n^* : V^{B(n-1)} \to V^{B(n)}$ is injective. However, the matrix of $\tau_n^*$ is precisely $\tau'$ restricted to $V^{B(n-1)}$ in domain and $V^{B(n)}$ in range, and so pre-injectivity of $\tau'$ implies injectivity of $\tau_n^*$, which in turn implies surjectivity of $\tau_n$, as required.
4 Transpose-harmonic functions and surjectivity of Laplacians

In this section we prove Propositions 1.7 and 1.9. The proofs essentially consist of a fairly direct applications of Theorem 1.11. In that respect they are strongly reminiscent of an argument of Ceccherini-Silberstein and Coornaert proving Proposition 1.9 in the amenable case [7, §3, Case 1], the main difference being that where we apply Theorem 1.11 they applied Theorem 1.10.

The main difficulty in applying Theorem 1.11, as opposed to Theorem 1.10, is that it forces us to consider the transpose of the Laplacian. In a regular graph, or on a group with a symmetric generating probability measure, the Laplacian is symmetric, and so this difficulty disappears. In the general setting, however, overcoming this matter requires a little more work.

Definition 4.1 (Transpose-harmonic function). Given a Laplacian $\Delta$ on a graph or a group $\Gamma$, we denote by $\Delta'$ the transpose of $\Delta$, and say that a function $h : \Gamma \to \mathbb{R}$ is transpose harmonic if $\Delta' h = 0$.

If $\Delta = \Delta_\mu$ is the Laplacian on a group defined by a finitely supported generating probability measure $\mu$ then, writing $\mu' : \Gamma \to \mathbb{R}$ defined by $\mu'(g) = \mu(g^{-1})$ we have

$$(\Delta_\mu)' = \Delta_{\mu'}.$$ (4.1)

In the case of the Laplacian on a weighted graph, on the other hand, we have the following.

Lemma 4.2. Let $\Delta$ be the Laplacian on a locally finite weighted graph $\Gamma$, and let $f : \Gamma \to \mathbb{R}$ be a function. Then for each $x \in \Gamma$ we have

$$\Delta' f(x) = f(x) - \sum_{y \sim x} \frac{\omega_{xy} f(y)}{\deg y}.$$

In particular, $f$ is transpose harmonic at $x$ if and only if the function $\hat{f} : \Gamma \to \mathbb{R}$ defined by

$$\hat{f}(y) = \frac{f(y)}{\deg y}$$

is harmonic at $x$.

Proof. The matrix of $\Delta$ is not hard to describe. In the row corresponding to the point $x$, the matrix has 1 in the column corresponding to $x$; it has $-\omega_{xy}/\deg x$ in each column corresponding to a neighbour $y$ of $x$; and every other entry is zero. The $x$ row in the matrix of $\Delta'$ therefore has 1 in the column corresponding to $x$; for each neighbour $y$ of $x$ it has $-\omega_{xy}/\deg y$ in the column corresponding to $y$; and every other entry is zero. The desired result follows immediately.

Proof of Propositions 1.7 and 1.9. In each case, Theorem 1.11 shows that it is sufficient to prove that a finitely supported transpose-harmonic function is identically zero.

In the case of the Laplacian on an infinite, connected, locally finite weighted graph (as in Proposition 1.7), Lemma 4.2 implies that the required statement is equivalent to showing that a finitely supported harmonic function is identically zero, since $\hat{f}(x) = 0$ if and only if $f(x) = 0$.

In the case of the Laplacian defined by a finitely supported generating probability measure $\mu$ (as in Proposition 1.9), (4.1) implies that the required statement is equivalent to showing that a finitely supported $\mu'$-harmonic function is identically zero.

In each case, the required statement follows from the maximum principle (Lemma 2.2), and so the propositions are both proved.

Remarks 4.3. The proof just presented is modelled on the amenable case of the proof of [7, Theorem 1.1], which is Proposition 1.9 in the special case of the Laplacian on a Cayley graph. The proof of [7, Theorem 1.1] in the amenable case uses Theorem 1.10 in place of Theorem 1.11. The fact that Theorem 1.10 does not necessarily hold in non-amenable groups forces the authors to use a different argument in
that case, in particular relying on a spectral criterion for amenability of finitely generated groups due to Kesten and Day. Our use of Theorem 1.1 allows us to avoid this complication.

Our arguments would also prove Proposition 1.7 for an asymmetrically weighted graph, which is to say if we were to drop the assumption that $\omega_{xy} = \omega_{yx}$, provided it satisfied

$$\sum_{y \sim x} \omega_{xy} = \sum_{y \sim x} \omega_{yx}$$

for every $x$.

5 A duality result for harmonic functions

The aim of this section is to prove the following result.

**Proposition 5.1** (Duality result for harmonic functions). Let $\Gamma$ be an infinite, connected, locally finite weighted graph, and let $X$ be a finite subset of $\Gamma$. Then the following statements are equivalent.

1. Every function $f : X \to \mathbb{R}$ extends to a harmonic function on all of $\Gamma$.
2. There is no non-zero finitely supported function on $\Gamma$ that is harmonic on $\Gamma \setminus X$.

**Remarks 5.2.** Proposition 5.1 fails in a finite graph, or a graph with a finite connected component, since statement (2) never holds in a finite graph, but statement (1) holds in an arbitrary graph when $X$ is a singleton. See Remark 5.5 for details on where the proof breaks down.

In many cases in which the truth or otherwise of one of the two statements of Proposition 5.1 is very easily established, the truth or otherwise of the other statement is also straightforward. The reader may find it instructive to consider the following examples of this phenomenon.

(i) When $X$ is a singleton then both statements of Proposition 5.1 are easily seen to be true (using the maximum principle (Lemma 2.2) in the case of statement (2)).

(ii) When $X$ consists of a vertex or group element and all of its neighbours then both statements are easily seen to be false.

(iii) If $\Gamma$ is the Cayley graph $(\mathbb{Z}, \pm 1)$ and $X$ is a subset of size 2 then both statements of Proposition 5.1 hold. If $X$ has size 3, on the other hand, then neither statement holds.

(iv) If $\Gamma$ is the subgraph of $(\mathbb{Z}, \pm 1)$ induced by the positive integers, and $X$ is a subset of size 2, then neither statement of Proposition 5.1 holds.

Given a subset $Y$ of $\Gamma$, we denote by $\mathbb{R}^Y_\Gamma$ the subspace of $\mathbb{R}^\Gamma$ consisting of those functions supported on $Y$. Proposition 5.1 then follows from combining the following two lemmas with Proposition 1.7, which implies that $\Delta(\mathbb{R}^\Gamma) = \mathbb{R}^\Gamma$.

**Lemma 5.3.** Let $\Gamma$ be a locally finite weighted graph, and let $X \subset \Gamma$ be a finite set. Then the following statements are equivalent.

1. We have $\Delta(\mathbb{R}^\Gamma_{\Gamma \setminus X}) = \mathbb{R}^\Gamma$.
2. There is no non-zero finitely supported function on $\Gamma$ that is harmonic on $\Gamma \setminus X$.

**Lemma 5.4.** Let $\Gamma$ be a locally finite weighted graph, and let $X \subset \Gamma$ be a finite set. Then the following statements are equivalent.

1. We have $\Delta(\mathbb{R}^\Gamma_{\Gamma \setminus X}) = \Delta(\mathbb{R}^\Gamma)$.
2. Every function $f : X \to \mathbb{R}$ extends to a harmonic function on all of $\Gamma$.

**Proof of Lemma 5.3.** First note that by Lemma 4.2 and the fact that for every function $f : \Gamma \to \mathbb{R}$ we have $\hat{f}(x) = 0$ if and only if $f(x) = 0$, statement (2) of Lemma 5.3 is equivalent to the following statement.
(2') There is no non-zero finitely supported function on $\Gamma$ that is transpose harmonic on $\Gamma\backslash X$.

Abusing notation slightly, we identify the operator $\Delta$ with its (possibly infinite) matrix. Statement (1) of the lemma is then equivalent to saying that the matrix $\Delta_{\Gamma\backslash X}$ obtained by replacing the columns of $\Delta$ corresponding to the elements of $X$ with columns of zeros is surjective.

Statement (2'), on the other hand, means that if $f \in \mathbb{R}_{\Gamma}^0$ is non-zero then $\Delta'(f)$ cannot be zero on $\Gamma\backslash X$. Put another way, this says that even if we replace the rows of $\Delta'$ corresponding to the elements of $X$ with columns of zeros then $\Delta'$ will be pre-injective.

However, $\Delta'$ with the rows corresponding to $X$ replaced by zeros is equal to the transpose of $\Delta_{\Gamma\backslash X}$. Replacing some entries of $\Delta$ by zeros does not change the fact that it is a locally specifiable map, and so the equivalence of (1) and (2') therefore follows from Theorem 1.11.

**Proof of Lemma 5.4.** We first prove that (1) implies (2). Let $f : X \to \mathbb{R}$ be arbitrary, and define $\overline{f}$ to be the function on $\Gamma$ that agrees with $f$ on $X$ and takes the value 0 elsewhere. By (1) we can find a function $h$ supported on $\Gamma\backslash X$ such that $\Delta(h) = \Delta(-\overline{f})$. The function $h + \overline{f}$ is then a harmonic extension of $f$, and so (2) is proved.

Conversely, note that in order to prove (1) it suffices to prove that for every $x \in X$ the function $\Delta(\delta_x)$ lies in the space $\Delta(\mathbb{R}_{\Gamma\backslash X}^0)$. However, if we assume (2) then in particular we have a harmonic extension $h$ of the function $f : X \to \mathbb{R}$ taking the value 1 at $x$ and 0 on $X\backslash\{x\}$, and it immediately follows that $\Delta(\delta_x) = \Delta(-h|_{\Gamma\backslash X})$.

**Remark 5.5.** In the case that $\Gamma$ has a finite connected component, Proposition 1.7 no longer holds, and so Lemmas 5.3 and 5.4 no longer combine to prove Proposition 5.1.

6 A standing assumption on $\mu$

Let $G$ be a group, and let $\mu$ be a finitely supported generating probability measure. Note that a function $f : G \to \mathbb{R}$ is harmonic with respect to the measure $\mu$ if and only if it is harmonic with respect to the measure $\overline{\mu}$ defined by

$$\overline{\mu}(g) = \begin{cases} 0 & \text{if } g = e \\ \frac{\mu(g)}{1-\mu(e)} & \text{otherwise}, \end{cases}$$

and so we may, and do throughout the rest of this paper, assume that $\mu(e) = 0$.

7 Existence of non-constant harmonic functions on graphs

Proposition 5.1 could, in principle, be used to bound from below the space of harmonic functions on an infinite connected weighted graph $\Gamma$. Specifically, if one could show that there were some set $X$ of vertices of $\Gamma$ such that $\Gamma$ admitted no non-zero finitely supported function harmonic except on $X$, then one could then use Proposition 5.1 to conclude that the space of harmonic functions on $\Gamma$ was of dimension at least $|X|$.

It is trivial that on every graph the harmonic functions are at least one dimensional, since the constants are always harmonic. It is certainly not trivial, however, to show that an infinite graph must in general admit a non-constant harmonic function. The simplest meaningful application one could hope for from Proposition 5.1 would therefore be to give a lower bound of two for the dimension of the space of harmonic functions on an infinite graph.

In this section we show that Proposition 5.1 can indeed be used to prove such a result. Specifically, we prove the following result, which generalises Proposition 1.5 in the transient case.

**Proposition 7.1.** Let $\Gamma$ be a locally finite vertex-transitive weighted graph, and suppose that the random walk on $\Gamma$ is transient. Suppose that $K$ is finitely generated subgroup of $\text{Aut} \Gamma$ such that the orbit $Ke$ is infinite. Then there exists a harmonic function on $\Gamma$ that is not constant on $Ke$. 

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Proof. Let $G$ be a connected, locally finite, vertex-transitive weighted graph. Then there is a finitely generated subgroup $G < \text{Aut} \Gamma$ that is transitive.

Proof. Let $e \in \Gamma$. By the transitivity of $\text{Aut} \Gamma$, for each neighbour $y$ of $e$ there is an automorphism $g_y$ of $\Gamma$ such that $g_y e = y$. We claim that $$G = \langle g_y : y \sim e \rangle$$ is transitive; since $\Gamma$ is locally finite, this is sufficient to prove the lemma.

Since $\Gamma$ is connected, it suffices to show that if $z \in Ge$ and $x \sim z$ then $x \in Ge$. To see this, note that for $z \in Ge$ there exists $h \in G$ such that $e = hz$. However, this means that we have $hx \sim e$, and so $x = h^{-1}g_x e \in Ge$, as desired, and the lemma is proved.

We also recover from Proposition 7.1 the following well-known fact.

Corollary 7.4. Let $G$ be an infinite group with a symmetric, finitely supported generating probability measure $\mu$. Then $(G, \mu)$ admits a non-constant harmonic function.

Proof. If the random walk on $(G, \mu)$ is transient then the corollary follows immediately from Proposition 7.1 if we let $G$ act on its own Cayley graph by left multiplication and take $K = G$. If the random walk is recurrent then Proposition 2.8 implies that $G$ has either $\mathbb{Z}$ or $\mathbb{Z}^2$ as a finite-index subgroup, in which case the corollary follows from [27] or from Lemma 9.6 below.

For the remainder of this section we are concerned with proving Proposition 7.1. Throughout, $\Gamma$ is a locally finite vertex-transitive weighted graph with distinguished vertex $e$.

By Proposition 5.1 in order to prove Proposition 7.1 for $\Gamma$ in the connected case it suffices to find two points $x, y \in Ke$ with the property that there is no non-zero finitely supported function on $\Gamma$ that is harmonic except at $x, y$. The following result gives a necessary condition for the existence of such a function.

Lemma 7.5. Let $x, y \in \Gamma$ and suppose that there exists a finitely-supported non-zero function $f : \Gamma \to \mathbb{R}$ that is harmonic except at $x$ and $y$. Then there exists some $N > 0$ such that the conditional probability $P_{g}[T_x < T_y \mid \min\{T_x, T_y\} < \infty]$ is independent of $g$ for $d(e, g) \geq N$.

Proof. Since $f$ is finitely supported, there is some $N > d(e, x), d(e, y)$ such that $f(g) = 0$ whenever $d(e, g) \geq N$. We prove that the lemma holds with this $N$.

For $M \in \mathbb{N}$ we denote by $B(M) = B_e(M)$ the ball of radius $M$ about the vertex $e$, and by $\tau_M$ the quantity $$\tau_M = \min\{t : X_t \in (G \setminus B(M)) \cup \{x, y\}\},$$
where \(X_0, X_1, \ldots\) is, as usual, the random walk on \(\Gamma\). By countable additivity of \(\mathbb{P}\), for \(g \in B(M)\) we have

\[
\begin{align*}
P_g[X_{\tau_M} = x | X_{\tau_M} \in \{x, y\}] &\to P_g[T_x < T_y | \min\{T_x, T_y\} < \infty], \quad (7.1) \\
P_g[X_{\tau_M} = y | X_{\tau_M} \in \{x, y\}] &\to P_g[T_y < T_x | \min\{T_x, T_y\} < \infty] \quad (7.2)
\end{align*}
\]
as \(M \to \infty\).

Let \(M \geq N\). By Lemma 2.3 there is a unique function \(f_M : B(M + 1) \to \mathbb{R}\) that is harmonic on \(B(M) \setminus \{x, y\}\) and satisfies the following conditions:

\[
\begin{align*}
f_M(x) &= f(x); \quad (7.3) \\
f_M(y) &= f(y); \quad (7.4) \\
f_M(z) &= 0 \text{ for } z \notin B(M); \quad (7.5)
\end{align*}
\]

indeed, Lemma 2.3 implies that

\[
f_M(g) = f(x) \cdot \mathbb{P}_g[X_{\tau_M} = x] + f(y) \cdot \mathbb{P}_g[X_{\tau_M} = y] \tag{7.6}
\]

for \(g \in B(M)\).

The restriction \(f|_{B(M+1)}\) is of course harmonic on \(B(M) \setminus \{x, y\}\), and trivially satisfies (7.3) and (7.4); by the definitions of \(N\) and \(M\) it also satisfies condition (7.5), and so by the uniqueness of \(f_M\) it follows that

\[
f_M = f|_{B(M+1)}. \tag{7.7}
\]

By the maximum principle (Lemma 2.2), and since \(f\) is not identically zero, \(f\) must take a positive value on one of \(x\) and \(y\), and a negative value at the other; this implies in particular that

\[
f(x) \neq 0; \quad f(y) \neq 0. \tag{7.8}
\]

If \(N \leq |g| \leq M\) then (7.4) and the definition of \(N\) together imply that \(f_M(g) = 0\), and so (7.6) and (7.8) imply that

\[
\frac{\mathbb{P}_g[X_{\tau_M} = x]}{\mathbb{P}_g[X_{\tau_M} = y]} = \frac{f(y)}{f(x)}.
\]

and hence that

\[
\frac{\mathbb{P}_g[X_{\tau_M} = x | X_{\tau_M} \in \{x, y\}]}{\mathbb{P}_g[X_{\tau_M} = y | X_{\tau_M} \in \{x, y\}]} = \frac{f(y)}{f(x)}.
\]

Letting \(M \to \infty\), we therefore see from (7.1) and (7.2) that

\[
\frac{\mathbb{P}_g[T_x < T_y | \min\{T_x, T_y\} < \infty]}{\mathbb{P}_g[T_y < T_x | \min\{T_x, T_y\} < \infty]} = \frac{f(y)}{f(x)}.
\]

Since the numerator and denominator of the left-hand side of (7.9) always sum to 1, this determines \(\mathbb{P}_g[T_x < T_y | \min\{T_x, T_y\} < \infty]\) uniquely and independently of \(g\), and so the lemma is proved.

An important tool in converting Lemma 7.5 into a proof of Proposition 7.1 is the following result of D. Aldous [11, Proposition 2], which shows that vertex transitivity of a graph imposes some non-trivial symmetry on the random walk.

**Proposition 7.6** (Aldous). Let \(x, y \in \Gamma\). Then for each \(n\) we have

\[
\mathbb{P}_x[T_y = n] = \mathbb{P}_y[T_x = n].
\]

**Remarks.** Proposition 7.6 does not necessarily hold in a regular graph that is not vertex transitive; see Figure 2. Aldous proved Proposition 7.6 via an analytic argument in the case of a finite vertex-transitive graph. The same argument could well work for infinite graphs as well, but here we present a combinatorial proof of the infinite case.
A key step in the proof of Proposition 7.6 is the following lemma.

**Lemma 7.8.** Let \( n \in \mathbb{N} \). Then for every \( x, y \in \Gamma \) we have

\[
P_x[X_n = x, \text{ and } X_i \neq y \text{ for all } i = 1, \ldots, n - 1] = P_y[X_n = y, \text{ and } X_i \neq x \text{ for all } i = 1, \ldots, n - 1]
\]

**Proof.** If \( n = 0 \) then the lemma is trivial, so by induction we may fix \( n > 0 \) and assume that

\[
P_x[X_r = x, \text{ and } X_i \neq y \text{ for all } i = 1, \ldots, r - 1] = P_y[X_r = y, \text{ and } X_i \neq x \text{ for all } i = 1, \ldots, r - 1] = u_r,
\]

say, for every \( r < n \). Moreover, since \( \Gamma \) is regular, if \( z_0, \ldots, z_r \) is a path from \( x \) to \( y \) then

\[
P_x[X_0 = z_0, \ldots, X_r = z_r] = P_y[X_0 = z_r, \ldots, X_r = z_0].
\]

This means, in particular, that if \( v_r(x, y) \) is the probability of moving from \( x \) to \( y \) in \( r \) steps, without visiting either \( x \) or \( y \) in between, then

\[
v_r(x, y) = v_r(y, x) = v_r,
\]

say, for every \( r \).

It is immediate from the vertex transitivity of \( \Gamma \) that we have

\[
P_x[X_n = x] = P_y[X_n = y],
\]

and so it suffices to show that we have

\[
P_x[X_n = x, \text{ and } X_i = y \text{ for some } i = 1, \ldots, n - 1] = P_y[X_n = y, \text{ and } X_i = x \text{ for some } i = 1, \ldots, n - 1].
\]

Given \( k \geq 1 \) and a sequence \( 0 \leq a_1 < b_1 \leq a_2 < b_2 \leq \ldots \leq a_k < b_k \leq n \) of integers, define the event

\[
L_{x,y}(n; k; a_1, \ldots, a_k; b_1, \ldots, b_k)
\]

to be the event that \( X_0 = X_n = x \) and, if \( 0 = t_1 < \ldots < t_l = n \) are all the times \( t \) at which \( X_t \in \{x, y\} \) and we set

\[
A = \{ t_i : X_t \neq X_{t_{i+1}} \}
\]

and

\[
B = \{ t_{i+1} : X_{t_i} \neq X_{t_{i+1}} \},
\]

then we have \( A = \{a_1, \ldots, a_k\} \) and \( B = \{b_1, \ldots, b_k\} \). Setting \( a_{k+1} = n \) and \( b_0 = 0 \) for notational convenience, we have

\[
P_x[L_{x,y}(n; k; a_1, \ldots, a_k; b_1, \ldots, b_k)] = \prod_{i=0}^{k} u_{a_{i+1} - b_i} \prod_{j=1}^{k} v_{b_j - a_j}
\]

\[
= P_y[L_{y,x}(n; k; a_1, \ldots, a_k; b_1, \ldots, b_k)].
\]
However, the event \( \{ X_0 = X_n = x, \text{ and } X_i = y \text{ for some } i = 1, \ldots, n-1 \} \) is precisely the disjoint union of all events \( L_{x,y}(n; k; a_1, \ldots, a_k; b_1, \ldots, b_k) \) with \( k \geq 1 \), and so (7.12) follows immediately from (7.13). The lemma is then immediate from (7.11) and (7.12).

**Proof of Proposition 7.9.** We prove the more precise statement that

\[
P_x[T_y = n \text{ and } \max\{t < n : X_t = x\} = r] = P_y[T_x = n \text{ and } \max\{t < n : X_t = x\} = r]
\]

for every \( r \geq 0 \). Indeed, this follows readily from Lemma 7.8 and (7.10), and the observation that

\[
P_x[T_y = n \text{ and } \max\{t < n : X_t = x\} = r] = v_{n-r}(x,y)P_x[X_r = x, \text{ and } X_i \neq y \text{ for all } i = 1, \ldots, r-1].
\]

**Remarks 7.9.** The only properties of \( \Gamma \) that we used in the proof of the graph case of Proposition 7.6 were its regularity and (7.11). These properties are satisfied, more generally, by walk-regular (unweighted) graphs, which are defined to be graphs in which the number of distinct loops based at a vertex \( x \) is independent of the choice of path. Thus, Proposition 7.6 also holds in walk-regular unweighted graphs.

Walk-regular unweighted graphs are genuinely more general than vertex-transitive unweighted graphs; C. Godsil and B. McKay [13, Fig. 2] have constructed a walk-regular graph that is not vertex transitive.

The following lemma proves the intuitively reasonable result that if the random walk is more likely to hit \( x \) than \( y \) eventually, then it is also more likely to hit \( x \) first.

**Lemma 7.10.** If \( x, y \in \Gamma \) satisfy

\[
P_x[T_x < \infty] > P_x[T_y < \infty]
\]

then they also satisfy

\[
P_x[T_x < T_y | \min\{T_x, T_y\} < \infty] > 1/2.
\]

If the random walk on \( \Gamma \) is transient then (7.14) and (7.15) are equivalent.

**Remark 7.11.** The conditions (7.14) and (7.15) are not necessarily equivalent in a vertex-transitive graph with a recurrent random walk, as can be seen by setting \( e = 0, x = 1 \) and \( y = 2 \) in the Cayley graph \((Z, \{ \pm 1 \})\).

**Proof of Lemma 7.11.** Write \( p(x, y) = P_x[T_y < \infty] \), the probability that the random walk starting at \( x \) hits \( y \) eventually. If \( P_x[T_x < \infty] > P_x[T_y < \infty] \) then this implies in particular that \( P_x[T_z < \infty] \) is not constant in \( z \), which implies that the random walk is transient. We may therefore assume that the random walk is transient and prove that (7.14) and (7.15) are equivalent.

Note that by Proposition 7.6 we have

\[
p(x, y) = p(y, x).
\]

Moreover, since the random walk is transient we have

\[
p(x, y) < 1.
\]

Write

\[
p(x) = P_x[T_x < \infty | \min\{T_x, T_y\} < \infty],
\]

and

\[
p(y) = P_x[T_y < \infty | \min\{T_x, T_y\} < \infty],
\]

and note that condition (7.14) is equivalent to

\[
p(x) > p(y).
\]
Write
\[ f(x) = \mathbb{P}_e[T_x < T_y | \min\{T_x, T_y\} < \infty] \]
and
\[ f(y) = \mathbb{P}_e[T_y < T_x | \min\{T_x, T_y\} < \infty]. \]
Condition (7.15) is that \( f(x) > 1/2 \), or equivalently that
\[ f(x) > f(y), \]
since \( f(x) + f(y) = 1 \). However, we have
\[ p(y) = f(y) + f(x)p(x, y), \]
and by (7.16) we also have
\[ p(x) = f(x) + f(y)p(x, y). \]
The equivalence of (7.14) and (7.15) therefore follows from (7.17).

**Proposition 7.12.** If the random walk on \( \Gamma \) is transient then \( \mathbb{P}_x[T_y < \infty] \to 0 \) as \( d(x, y) \to \infty \).

**Proof.** It is clear that
\[ \mathbb{P}_x[T_y < \infty] \leq \sum_{n=0}^{\infty} \mathbb{P}_x[X_n = y]. \]
However, since \( \mathbb{P}_x[X_n = y] = 0 \) for \( n < d(x, y) \), we in fact have the stronger bound
\[ \mathbb{P}_x[T_y < \infty] \leq \sum_{n=d(x, y)}^{\infty} \mathbb{P}_x[X_n = y]. \] (7.18)

If \( n \) is even then we have \( \mathbb{P}_x[X_n = y] \leq \mathbb{P}_e[X_n = e] \) by Lemma 2.5. If \( n \) is odd, on the other hand, then we have
\[ \mathbb{P}_x[X_n = y] = \mathbb{P}_e[X_{n-1} = y] \leq \mathbb{P}_e[X_{n-1} = e], \]
again by Lemma 2.5. Combining these two inequalities with (7.18) shows that
\[ \mathbb{P}_x[T_y < \infty] \leq 2 \sum_{n \geq d(x, y)-1} \mathbb{P}_e[X_n = e] \] (7.19)

Recall that \( R_e \) is the number of times the random walk hits the vertex \( e \). In particular,
\[ R_e = \sum_{n=0}^{\infty} 1_{\{X_n = e\}}, \]
and so by linearity of expectation we have
\[ \mathbb{E}_e[R_e] = \sum_{n=0}^{\infty} \mathbb{P}_e[X_n = e]. \]
Lemma 2.7 therefore implies that
\[ \sum_{n=0}^{\infty} \mathbb{P}_e[X_n = e] \leq \infty, \]
which, combined with (7.19), shows that \( \mathbb{P}_x[T_y < \infty] \to 0 \) as \( d(x, y) \to \infty \), as desired.

**Proof of Proposition 7.1.** If the orbit \( K_e \) has non-trivial intersection with two connected components of \( \Gamma \) then the result follows by taking a function that takes the value 1 on one of these components and 0 elsewhere on \( \Gamma \). We may therefore assume that \( \Gamma \) is connected, and so by Proposition 5.1 it suffices to find two points \( x, y \in K_e \) with the property that there is no non-zero finitely supported function on \( \Gamma \) that is harmonic except at \( x, y \).

We consider two cases.
(1) The subgroup $K$ contains an element $v$ such that the vertices $v^n e$ are all distinct for $n \in \mathbb{N}$.

(2) For every element $u$ of the subgroup $K$ there is some $m$ such that $u^m e = e$.

In case (1), Proposition 7.12 implies that $\mathbb{P}_e[T_{v^n e} < \infty] \to 0$ and $\mathbb{P}_e[T_{v^{-n} e} < \infty] \to 0$ as $n \to \infty$. This implies that there are infinite increasing sequences $n_1^+, n_2^+, n_3^+ \ldots$ and $n_1^-, n_2^-, n_3^- \ldots$ such that

$$\mathbb{P}_e[T_{v^n e} < \infty] > \mathbb{P}_e[T_{v^n e} < \infty]$$

and

$$\mathbb{P}_e[T_{v^{-n} e} < \infty] < \mathbb{P}_e[T_{v^{-n} e} < \infty],$$

which by Lemma 7.10 means that

$$\mathbb{P}_{v^n e}[T_e > T_{v e} \mid \min \{T_e, T_{v e}\} < \infty] > 1/2$$

and

$$\mathbb{P}_{v^{-n} e}[T_e > T_{v e} \mid \min \{T_e, T_{v e}\} < \infty] < 1/2.$$

Since $v^{-n_1} e \to \infty$ and $v^{n_1} e \to \infty$, Lemma 7.5 therefore implies that there exists no non-zero finitely supported function on $\Gamma$ that is harmonic except at $e, e$.

In case (2), let $R$ be a finite symmetric generating set for $K$. We claim that there are elements $x_1, x_2, \ldots \in K$ with $d(e, x_n e) \to \infty$ such that, for each $n$, there is some $r_n \in R$ such that $\mathbb{P}_e[T_{x_n e} < \infty] < \mathbb{P}_e[T_{x_n e} < \infty]$. Indeed, for each $n = 1, 2, \ldots$, let $x_n$ be a point of minimal distance from the identity in the Cayley graph $(K, R)$ such that $\mathbb{P}_e[T_{x_n e} < \infty] < 1/n$. Such a point always exists by Proposition 7.12 and the assumption that the orbit $K e$ is infinite, and by the regularity and local finiteness of $\Gamma$ we have

$$d(e, x_n e) \to \infty$$

as $n \to \infty$. By definition of $x_n$, and using (7.20), for sufficiently large $n$ there is some $r_n \in R$ such that $\mathbb{P}_e[T_{x_n e} < \infty] < 1/n > \mathbb{P}_e[T_{x_n e} < \infty]$, as claimed.

By the finiteness of $R$, upon passing to a subsequence if necessary we may in fact assume that there is some $u \in R$ such that for each $n$ we have

$$\mathbb{P}_e[T_{x_n e} < \infty] < \mathbb{P}_e[T_{x_n e} < \infty].$$

We claim that there is no non-zero finitely supported function on $\Gamma$ that is harmonic except at $e, u e$.

As in case (1), condition (7.21) and Lemma 7.10 imply that

$$\mathbb{P}_{x_n^{-1} e}[T_e > T_{u e} \mid \min \{T_e, T_{u e}\} < \infty] < 1/2;$$

indeed, applying the automorphism $u^m$, we see that

$$\mathbb{P}_{u^m x_n^{-1} e}[T_{u^m e} > T_{u^{m+1} e} \mid \min \{T_{u^m e}, T_{u^{m+1} e}\} < \infty] < 1/2$$

for every $m \in \mathbb{N}$. Moreover, (7.20) implies that for each $m \in \mathbb{N}$ we have $d(u^m e, x_n^{-1} e) \to \infty$ as $n \to \infty$, and so

$$d(e, u^m x_n^{-1} e) \to \infty$$

as $n \to \infty$. If the claim is false, and there does exist some non-zero finitely supported function on $\Gamma$ that is harmonic except at $e, u e$, then translating this function by $u^m$ we see that there is also a function on $\Gamma$ that is harmonic except at $u^m e, u^{m+1} e$. Combining (7.20) and (7.22) with Lemma 7.5 therefore implies that for each $m$ there is some $N_m > 0$ such that

$$\mathbb{P}_x[T_{u^m e} > T_{u^{m+1} e} \mid \min \{T_{u^m e}, T_{u^{m+1} e}\} < \infty] < 1/2$$

for every $x \in \Gamma$ such that $d(e, x) \geq N_m$; since the orbit of $e$ under $u$ is finite we may assume that the $N_m$ are all equal to some $N > 0$. Fixing some $x$ with $d(e, x) \geq N$ and applying Lemma 7.10 once more, this means that

$$\mathbb{P}_x[T_{u^m e} < \infty] < \mathbb{P}_x[T_{u^{m+1} e} < \infty]$$
for every \( m \in \mathbb{N} \), which implies by induction that
\[
P_x[T_e < \infty] < P_x[T_{wu^m} e < \infty]
\]
for every \( m \in \mathbb{N} \). This is impossible, however, since there is some \( m \in \mathbb{N} \) such that \( u^m e = e \), and so it must have been the case that there was no non-zero finitely supported function on \( \Gamma \) harmonic except at \( e, ue \). This proves the claim, and hence the proposition in case (2).  

\[\square\]

8 Passing to a normal subgroup of finite index

We make repeated use, throughout the rest of this paper, of the following standard reduction.

**Lemma 8.1.** Let \( G \) be a group and let \( H \) be a finite-index subgroup of \( G \). Then there exists a finite-index subgroup \( H' < H \) that is normal in \( G \).

**Proof.** It is easy to verify that the subgroup
\[
H' = \bigcap_{gH \in G/H} gHg^{-1}
\]
is well defined, normal and of finite index in \( G \).  

\[\square\]

9 Harmonic functions on virtually nilpotent groups

In the case of virtually nilpotent groups, we prove Theorem \[1.2\] in a purely algebraic fashion. The main purpose of this section is to present this argument.

Throughout this section, \( G \) is a group with a finite-index \( m \)-step nilpotent subgroup \( N \), and \( \mu \) is a symmetric, finitely supported generating probability measure on \( G \). The nilpotent subgroup \( N \) possesses a torsion-free subgroup \( N' \) of finite index \[26\]; as is noted in \[2, §1.3\], upon replacing \( N \) by \( N' \) if necessary we may therefore assume that \( N \) itself is torsion free. Moreover, by Lemma \[8.1\] we may also assume that \( N \) is normal in \( G \).

The first result of this section is the direct statement of Theorem \[1.2\].

**Proposition 9.1** (Direct result). If \( N = \mathbb{Z} \) then \( G \) has a finite-dimensional space of harmonic functions.

In the second result, we prove the inverse statement, Theorem \[1.3\] for virtually nilpotent groups. In fact, we give quantitative lower bounds for certain spaces of harmonic functions on a virtually nilpotent group. Write \( H^k(G, \mu) \) for the space of harmonic functions on \( (G, \mu) \) of polynomial growth of exponent at most \( k \); thus
\[
H^k(G, \mu) = \{ h : G \to \mathbb{R} : \Delta_\mu h = 0 \text{ and } |h(x)| \ll x^k \text{ for all } x \neq e \}.
\]

Our aim is to give lower bounds on the dimensions of these spaces.

It follows from a result of Alexopoulos \[2, Theorem 1.12\] that the space \( H^k(G, \mu) \) consists of functions that restrict to certain so-called polynomials on \( N \). This result leads directly to an upper bound on the dimension of \( H^k(G, \mu) \), but it is also of use in obtaining the lower bound that we seek, since it gives a fairly explicit form that elements of \( H^k(G, \mu) \) must take. We therefore describe these polynomial functions now.

An embedding theorem of Mal’cev \[26\] states that \( N \) embeds as a discrete, cocompact subgroup of a simply connected nilpotent Lie group \( \mathfrak{N} \) of the same nilpotency class, \( m \), as \( N \). Let \( \mathfrak{n} \) be the Lie algebra of \( \mathfrak{N} \), and write
\[
\mathfrak{n} = \mathfrak{n}_0 \supset \mathfrak{n}_1 \supset \cdots \supset \mathfrak{n}_m \supset \mathfrak{n}_{m+1} = \{0\}
\]
for the lower central series of \( \mathfrak{n} \). Denote \( \dim \mathfrak{n} \) by \( d \), and for each \( i \) write
\[
d_i = \dim \mathfrak{n}_i - \dim \mathfrak{n}_{i+1}
\]
and

\[ n_i = \dim n - \dim n_i = d_1 + \cdots + d_{i-1}. \]

Another result of Mal’cev \[22\] then states that there is a basis \(X_1, \ldots, X_d\) for \(n\) that satisfies the following properties.

(i) For each \(j = 1, \ldots, m\) we have \(n_j = \text{span}\{X_{n_j+1}, \ldots, X_d\}\).

(ii) For every \(x \in N\) there is a unique \(d\)-tuple \((x_1, \ldots, x_d) \in \mathbb{R}^d\) such that

\[ x = \exp(x_1X_1) \cdots \exp(x_dX_d). \quad (9.1) \]

(iii) The subgroup \(N\) of \(N\) consists precisely of those \(x \in N\) for which \((x_1, \ldots, x_d) \in \mathbb{Z}^d\).

The basis \(X_1, \ldots, X_d\) is called a Mal’cev basis, and the coordinates \((x_1, \ldots, x_d)\) for \(x \in \mathbb{N}\) given by \(9.1\) are called Mal’cev coordinates, or coordinates of the second kind. By identifying each \(x \in N\) with its \(d\)-tuple \((x_1, \ldots, x_d)\) of Mal’cev coordinates, we may identify \(N\) with \(\mathbb{R}^d\). Moreover, by property (iii) above we may similarly identify \(N\) with \(\mathbb{Z}^d\).

Following \[2\], once we have made these identifications we may define a monomial on \(N\) to be a monomial, in the usual sense, on \(\mathbb{R}^d\), and a monomial on \(N\) to be the restriction of a monomial on \(N\) (or, equivalently by (iii), a monomial on \(\mathbb{Z}^d\)). A polynomial on \(N\) is then a finite sum of monomials. For each \(i = 1, \ldots, d\) we define \(\sigma(i) \in \mathbb{N}\) by requiring that

\[ X_i \in n_{\sigma(i)} \setminus n_{\sigma(i)+1}. \quad (9.2) \]

We then define the homogeneous degree of the monomial \(q(x) = x_1^{\alpha_1} \cdots x_d^{\alpha_d}\) by

\[ \deg q = \sigma(1)\alpha_1 + \cdots + \sigma(d)\alpha_d, \]

and if \(q_1, \ldots, q_r\) are monomials then we define the homogeneous degree of the polynomial \(p = q_1 + \cdots + q_r\) by

\[ \deg p = \max_i \deg q_i. \]

It follows that

\[ |p(x)| \ll_p |x|^{\deg p} \quad (9.3) \]

for \(x \neq e\); see, for example, \[2\] §1.7 & §6.1. Note also that the homogeneous degree behaves similarly under multiplication to the degree on \(\mathbb{R}^d\), in that if \(p, p' : N \to \mathbb{R}\) are polynomials then their pointwise product \(pp'\) is also a polynomial on \(N\), and satisfies

\[ \deg pp' = \deg p + \deg p'. \quad (9.4) \]

Finally, if \(q_1, \ldots, q_r\) are monomials, each of homogeneous degree exactly \(k\), then we say that the polynomial \(p = q_1 + \cdots + q_r\) is a homogeneous polynomial of homogeneous degree \(k\). Writing \(P^k(N)\) for the space of homogeneous polynomials of degree \(k\) on \(N\), we are now in a position to state the main result of this section.

**Theorem 9.2.** Let \(G\) be a group with a finite-index subgroup \(N\) isomorphic to a discrete, cocompact subgroup of some simply connected nilpotent Lie group \(\mathbb{N}\), equipped with some Mal’cev basis. Let \(\mu\) be a symmetric, finitely supported generating probability measure on \(G\). Then

\[ \dim H^k(G, \mu) \geq \dim P^k(N). \]

**Remark 9.3.** The definition of a polynomial on \(N\) depends on the embedding \(N \hookrightarrow \mathbb{N}\) and the Mal’cev basis, which is why in the hypotheses Theorem 9.2 we assume that \(N\) comes with a fixed embedding and a fixed Mal’cev basis. In the virtually abelian case, this is nothing more than assuming from the outset that \(N = \mathbb{Z}^d\) (with an implicit fixed set of standard generators).
Corollary 9.4 (Theorem 1.2 for virtually nilpotent groups). Let $G$ be a virtually nilpotent group and let $\mu$ be a symmetric, finitely supported generating probability measure on $G$. Then the space of harmonic functions on $(G,\mu)$ is finite dimensional if and only if $G$ is virtually cyclic. In the event that the space of harmonic functions is infinite dimensional, the subspace of polynomially growing harmonic functions is also infinite dimensional.

Proof. In the event that $G$ is virtually cyclic the result is just Proposition 9.1. We may therefore assume that $G$ is not virtually cyclic, and apply Theorem 9.2 with $N \neq \mathbb{Z}$, which implies, in particular, that
\[ d_1 \geq 2. \]  
(9.5)

Considering only the polynomials in the variables $x_1, \ldots, x_{d_1}$, we certainly have
\[ \dim P^k(N) \geq \dim P^k(\mathbb{Z}^{d_1}). \]  
(9.6)

However, it is well known and easy to check that
\[ \dim P^k(\mathbb{Z}^{d_1}) = \binom{d_1 + k - 1}{k}, \]  
(9.7)

and so Theorem 9.2 and (9.6) imply that
\[ \dim H^k(G,\mu) \geq \binom{d_1 + k - 1}{k}. \]  
(9.8)

By (9.5), this means that $\dim H^k(G,\mu) \to \infty$ as $k \to \infty$, and the result follows. \[ \square \]

Remarks 9.5. In order to prove a bound of the form (9.8), and hence to prove Corollary 9.4, it would be sufficient to prove Theorem 9.2 only in the special case that $N$ is abelian. Indeed, let $N$ be a nilpotent, non-virtually cyclic subgroup of finite index in $G$. By Lemma 8.1 we may assume that $N$ is normal, which in turn implies that $[N,N]$ is normal in $G$ (since $[N,N]$ is characteristic in $N$). Write $\pi : G \to G/[N,N]$ for the canonical projection. Since $N$ is not virtually cyclic, neither is its abelianisation $N/[N,N]$, and so $N/[N,N]$ is an abelian subgroup of finite index in $G/[N,N]$ with torsion-free rank
\[ r \geq 2. \]  
(9.9)

It follows from (9.7) and the abelian case of Theorem 9.2 that
\[ \dim H^k(G/[N,N],\pi(\mu)) \geq \binom{r + k - 1}{k}, \]  
and so by Lemma 2.4 we have
\[ \dim H^k(G,\mu) \geq \binom{r + k - 1}{k}. \]  

Corollary 9.4 then follows from (9.9).

We note in Remark 9.13 below, that the proof of Theorem 9.2 is far more elementary in the abelian case than in the general case; in particular, in the abelian case one does not need the Mal’cev embedding theorem or the existence of Mal’cev coordinates (or, at least, one needs only the trivial abelian incarnations of these results). The preceding paragraph therefore gives a more elementary proof of Corollary 9.4 than the one presented here. Indeed, the proof of Corollary 9.4 is the only place in this paper in which we use Theorem 9.2 — in particular, in proving Theorem 1.3 we apply Corollary 9.4 but do not apply Theorem 9.2 directly — and so this also yields a more elementary proof of Theorem 1.3.

The best upper bound currently available for $\dim H^k(G,\mu)$ is due to B. Hua and J. Jost [18, Theorem 1.1]. Their upper bound is given in terms of the homogeneous dimension $D$ of $N$, defined by
\[ D = \sum_{i=1}^{m} id_i \]
virtually nilpotent case, combining Theorem 9.2 with the Hua–Jost bound gives, for example, \[ H \]
then (9.8) implies that \( \dim H \approx k \) matching the upper bound of Hua and Jost. In the general virtually nilpotent case, combining Theorem 9.2 with the the Hua–Jost bound gives, for example,

\[ k^\frac{d}{d-1} \leq \dim H_k(G, \mu) \leq k^{D-1}. \]

It would be interesting to know whether either asymptotic bound could be improved in this case.

In the case that \( G \) is abelian and \( \mu \) is the uniform probability measure on a finite, symmetric generating set \( S \) for \( G \), the dimension of \( H^k(G, \mu) \) is known precisely. The first result was due to Heilbronn [17], who calculated \( \dim H^k(G, \mu) \) for \( S \) the standard generating set for \( \mathbb{Z}^d \). Hua, Jost and X. Li-Jost [19] generalised this to an arbitrary abelian group \( \mathbb{Z}^d \oplus \mathbb{Z} \), with \( \mathbb{Z} \) finite and abelian, and an arbitrary generating set \( S \). These results state that for any \( S \) the dimension is given by \( \dim H^k(\mathbb{Z}^d \oplus \mathbb{Z}, \mu) = \sum_{k} (\frac{d+k-1}{k}) + (\frac{d+k-2}{k-1}) \).

In the final result of this section, we note an explicit description of the space \( H^1(G, \mu) \). We say that a function \( \varphi : G \to \mathbb{R} \) factors through \( G/N \) if \( \varphi(x) = \varphi(t) \) for every \( x \in N \) and every \( t \in T \).

**Lemma 9.6** [27] Let \( G \) be a group with a finite-index normal subgroup isomorphic to \( \mathbb{Z}^d \), and let \( \mu \) be a symmetric, finitely supported generating probability measure on \( G \). Then for each \( i = 1, \ldots, d \) there is a function \( \varphi_i : G \to \mathbb{R} \) that factors through \( G/\mathbb{Z}^d \) such that the function \( f_i \in B^1 \) given by

\[ f_i(xt) = x_i + \varphi_i(t) \]

(9.10)
is harmonic on \((G, \mu)\). Moreover, \( H^1(G, \mu) \) is spanned by the set \( \{1, f_1, \ldots, f_d\} \).

**Proof.** The existence of harmonic functions of the form (9.10) follows directly from [27] Theorem 3.6]. The fact that \( \{1, f_1, \ldots, f_d\} \) spans \( H^1(G, \mu) \) is then precisely the linear-growth case of [2 Theorem 1.12].

**Remark 9.7.** An essentially identical argument shows that if \( G \) has a finite-index torsion-free nilpotent subgroup \( N \) then there are functions \( f_1, \ldots, f_d \) of the form (9.10) such that

\[ H^1(G, \mu) = \text{span}\{1, f_1, \ldots, f_d\}. \]

Since we do not need this fact we leave the details to the reader.

Before we embark on the outstanding proofs, let us describe the structure of \( G \) a little more precisely. Recall that we may assume that \( N \) is torsion-free and normal in \( G \). We may also define a right-transversal \( T \) of \( N \) containing the identity, which is to say a finite set \( T \) such that

\[ N \triangleleft NT = G, \]

and such that each \( g \in G \) can be expressed uniquely as

\[ g = \zeta(g) \tau(g) \]

with \( \zeta(g) \in N \) and \( \tau(g) \in T \).

As in Section 2 we write

\[ S := \text{supp } \mu, \]

and for a function \( f : S \to \mathbb{R} \) we denote by \( \mathbb{E}_{s \in S} \) the average

\[ \mathbb{E}_{s \in S} f(s) := \sum_{s \in S} \mu(s) f(s). \]
Lemma 9.8. Let $d_\mathcal{N}$ be the Cayley-graph distance on $N$ with respect to the generating set 
\{\exp X_1, \ldots, \exp X_d\}.

Then there exists $M \in \mathbb{N}$ such that for every $g \in G$ and every $s \in S$ we have $d_\mathcal{N}(\zeta(gs), \zeta(g)) \leq M$.

Proof. Given $g \in G$ and $s \in S$, write $t = \tau(g)$, so that $gs = \zeta(g)ts = \zeta(g)\zeta(ts)\tau(ts)$. This implies, in particular, that $\zeta(gs) = \zeta(g)\zeta(ts)$, and so we may take $M$ to be the maximum over the (finite) set \{\zeta(ts)\}_{s \in S, t \in T}.

Proof of Proposition 9.1. Lemma 9.8 implies that for each $n \in \mathbb{Z}$ we have $[-n, n]TS \subset [-n-M, n+M]T$. It follows that $([-n, n]T)^+ \subset [-n-M, n+M]T$, and so $\partial^+([-n, n]T)$ has cardinality at most $2M|T|$. Lemma 9.3 therefore implies that the space of functions on $\partial^+([-n, n]T)$ that are harmonic on $[-n, n]T$ is of dimension at most $2M|T|$. However, $G = \bigcup_{n=1}^{\infty} [-n, n]T$, and so the space of harmonic functions on $G$ is also of dimension at most $2M|T|$.

Remark 9.9. Taking $G = \mathbb{Z}$ and setting $\mu$ to be the uniform probability measure on $[-M, M]$ shows that the bound $2M|T|$ on the dimension of the space of harmonic functions in the proof of Proposition 9.1 can be tight. In particular, the precise dimension depends on the measure $\mu$ as well as on the group $G$.

Let us now begin the task of proving Theorem 9.2. From now on, we assume that $G$ and $N$ are fixed. Our approach is very much modelled on that used by Heilbronn 17 in the case $G = \mathbb{Z}^d$. He exploited the fact that the linear operator $\Delta$ maps the space of polynomials of degree at most $k$ onto the space of polynomials of degree at most $k-2$ in order to apply a rank-nullity argument. Below we define a certain space $B^k$ of polynomials on $G$ and show that $\Delta$ maps it into a space $A^{k-1}$ of lower dimension, thus giving a lower bound on the dimension of the subspace of $B^k$ consisting of harmonic functions. Let $p_0 = 1, p_1, p_2, \ldots$ be the sequence of monomials on $N$, ordered such that $p_{0k-1}, \ldots, p_{0k}$ are the monomials of homogeneous degree $k$ and arbitrarily otherwise. Define a space $A^k$ of functions $G \to \mathbb{R}$ of the form
\[f(x) = \sum_{i=1}^{\nu_k} \psi_i(t)p_i(x), \tag{9.11}\]
with each $\psi_i : G \to \mathbb{R}$ a function that factors through $G/N$. Thus $A^k$ consists of functions of the form
\[f(x) = P_t(x), \tag{9.12}\]
where for each $t \in T$ the function $P_t : N \to \mathbb{R}$ is a polynomial on $N$ of homogeneous degree at most $k$.

Define the subspace $B^k$ of $A^k$ to consist of those functions in the form (9.11) with the additional constraint that the functions $\psi_{0k-1}, \ldots, \psi_{0k}$ are constant. Thus, when a function in $B^k$ is written in the form (9.12), the coefficients of the monomials of homogeneous degree $k$ in $P_t$ are independent of $t$.

It follows from the work of Alexopoulos 2 that $H^k(G, \mu) \subset B^k$, and so by (9.3) we have
\[\dim H^k(G, \mu) = \dim B^k - \dim \Delta(B^k).\]

The basic strategy of our proof of Theorem 9.2 is to bound from above the quantity $\dim \Delta(B^k)$.

Given a function $p : N \to \mathbb{R}$ and an element $u \in N$ we define the discrete derivative $\partial_u p : N \to \mathbb{R}$ by $\partial_u p(x) = p(xu) - p(x)$. The following property is familiar from polynomials on $\mathbb{Z}^d$.

Lemma 9.10. If $p : N \to \mathbb{R}$ is a polynomial of homogeneous degree $k$ then $\partial_u p$ is a polynomial of homogeneous degree at most $k-1$, for any $u \neq e$.

The proof of Lemma 9.10 is based on the following result, the proof of which we defer until Appendix A since it essentially proceeds by routine nilpotent algebra.

Lemma 9.11. There are polynomials $q_1, \ldots, q_d : N \times N \to \mathbb{R}$ such that $\deg q_i = \sigma(i)$ for each $i$, and such that if $x, u \in N$ then the $i$th coordinate of $xu$ is given by $q_i(x, u)$.
Proof of Lemma 9.10. Lemma 9.11 and (9.3) imply that \( p(xu) \) is a polynomial in \( x_1, \ldots, x_d, u_1, \ldots, u_d \) of homogeneous degree \( k \). Setting \( u = e \), and hence \( u_i = 0 \) for every \( i \), we see that the terms of \( p(x) \) are precisely the terms of \( p(xu) \) that do not have any \( u_i \) factors, and so \( \partial_u p(x) \) is a polynomial in \( x_1, \ldots, x_d, u_1, \ldots, u_d \) of homogeneous degree \( k \), all of whose terms have a non-trivial \( u_i \)-factor. In particular, if \( u \) is fixed then \( \partial_u p(x) \) is a polynomial in \( x_1, \ldots, x_d \) of homogeneous degree strictly less then \( k \).

\[ \square \]

Lemma 9.12. For every \( k \in \mathbb{N} \) we have \( \Delta(B^k) \subset A^{k-1} \).

Proof. Let \( f \) be of the form (9.11) with the functions \( \psi_{v_{k-1}+1}, \ldots, \psi_{v_k} \) constant. Then we have

\[
\Delta f(xt) = E_{s \in S}[f(xt) - f(xts)] \\
= E_{s \in S}[f(xt) - f(x\zeta(ts)\tau(ts))] \\
= E_{s \in S} \left[ \sum_{i=1}^{v_k} (\psi_i(t)p_i(x) - \psi_i(ts)p_i(x\zeta(ts))) \right] \\
= E_{s \in S} \left[ \sum_{i=1}^{v_{k-1}} (\psi_i(t) - \psi_i(ts))p_i(x) - \sum_{i=1}^{v_k} \psi_i(ts)\partial_{\zeta(ts)} p_i(x) \right] \\
= \sum_{i=1}^{v_{k-1}} \Delta \psi_i(t)p_i(x) - \sum_{i=1}^{v_k} E_{s \in S}[\psi_i(ts)\partial_{\zeta(ts)} p_i(x)],
\]

and so the lemma follows from Lemma 9.10.

\[ \square \]

Proof of Theorem 9.2. Theorem 9.2 follows immediately from (9.3), Lemma 9.12 and the observation that

\[
\dim B^k = \dim A^{k-1} + \dim P^k(N).
\]

Remark 9.13. The proof of Theorem 9.2 is far more elementary in the virtually abelian case, in which the classification of finitely generated abelian groups, rather than the Mal’cev embedding theorem, allows us to identify \( N = \mathbb{Z}^d \), and then Lemma 9.11 is trivial.

10 Positive harmonic functions on linear groups

If \( G \) is a group and \( \mu \) is a finitely supported generating probability measure then a positive harmonic function on \( (G, \mu) \) is a harmonic function \( h : G \to \mathbb{R} \) that takes only positive values. G. Margulis [23] showed that a nilpotent group admits no non-constant positive harmonic functions. More generally, we have the following result of W. Hebisch and L. Saloff Coste.

Proposition 10.1 (Hebisch–Saloff Coste [15]). Let \( G \) be a virtually nilpotent group with a symmetric, finitely supported generating probability measure \( \mu \). Then \( (G, \mu) \) admits no non-constant positive harmonic functions.

P. Bougerol and L. Elie show that for linear groups the converse is also true.

Proposition 10.2 (Bougerol–Elie [5]). Let \( G \) be a subgroup of \( GL_d(\mathbb{R}) \) that is not virtually nilpotent, and let \( \mu \) be a symmetric, finitely supported generating probability measure on \( G \). Then \( (G, \mu) \) admits a non-constant positive harmonic function.

The purpose of this section is to show that, in that case, there are in fact many positive harmonic functions.

Proposition 10.3. Let \( G \) be a group with a symmetric, finitely supported generating probability measure \( \mu \), and suppose that \( (G, \mu) \) admits at least one non-constant positive harmonic function. Then the set of positive harmonic functions on \( (G, \mu) \) spans an infinite-dimensional space.
The following is then immediate.

**Corollary 10.4.** Let $G$ be a subgroup of $GL_d(\mathbb{R})$ that is not virtually nilpotent, and let $\mu$ be a symmetric, finitely supported generating probability measure on $G$. Then the positive harmonic functions on $(G, \mu)$ span an infinite-dimensional space.

**Question 10.5.** Does an arbitrary non-virtually nilpotent group with a symmetric, finitely supported generating probability measure admit a non-constant positive harmonic function?

In proving Proposition 10.3 we make use of the Martin boundary of $(G, \mu)$.

**Definition 10.6 (Minimal harmonic function).** Given a group $G$ with a finitely supported generating probability measure $\mu$, a **minimal harmonic function** on $(G, \mu)$ is a positive harmonic function $f : G \to \mathbb{R}$ with the property that every other positive harmonic function $f' : G \to \mathbb{R}$ satisfying $f' \leq f$ is a constant multiple of $f$. A **normed minimal harmonic function** $f : G \to \mathbb{R}$ is a minimal harmonic function satisfying $f(e) = 1$.

**Definition 10.7 (Martin boundary).** The **Martin boundary** $\Delta(G, \mu)$ of the pair $(G, \mu)$ is the compact closure, in the topology of pointwise convergence, of the set of normed minimal harmonic functions on $(G, \mu)$.

Each positive harmonic function $f : G \to \mathbb{R}$ has a unique representing measure $\nu_f$ on the Martin boundary, which is to say a measure $\nu_f$ such that

$$f(x) = \int_{\Delta} h(x) d\nu_f(h)$$

for every $x \in G$ (see [20 §0.3] or [31 §7, p. 32]).

**Lemma 10.8.** The set of normed minimal harmonic functions on a group $G$ with respect to a finitely supported generating probability measure $\mu$ is linearly independent.

**Proof.** Suppose that $h_1, \ldots, h_r$ are distinct minimal harmonic functions and let $\alpha_1, \ldots, \alpha_r$ be such that

$$\sum_{i=1}^{m} \alpha_i h_i = 0.$$ 

Without loss of generality we may assume that $\alpha_i \leq 0$ for $i \leq k$, and that $\alpha_i \geq 0$ for $i > k$, and so in fact we have

$$\sum_{i=1}^{k} (-\alpha_i) h_i = \sum_{i=k}^{m} \alpha_i h_i. \quad (10.2)$$

However, both the left-hand side and the right-hand side of $\alpha_i$ are non-negative harmonic functions, and so it follows from the uniqueness of the representation $\alpha_i$ that the $\alpha_i$ are all zero. \qed

**Proof of Proposition 10.3.** We prove the contrapositive. Suppose that the set of positive harmonic functions on $G$ does not span an infinite-dimensional space. By Lemma 10.8 this implies in particular that the set of normed minimal harmonic functions is finite, so we may enumerate them as $h_1, \ldots, h_m$.

The group $G$ acts on the space of all harmonic functions via

$$g \cdot f(x) = f(g^{-1} x).$$

The image of a minimal harmonic function under this action is another minimal harmonic function, and so in particular for each $i = 1, \ldots, m$ and each $g \in G$ we have some $\alpha_{g, i} \in \mathbb{R}$ and some $g \cdot i \in [m]$ such that

$$g \cdot h_i = \alpha_{g, i} h_{g \cdot i}.$$
As the notation $g \cdot i$ implicitly suggests, this defines an action of $G$ on the set $[m]$. By the orbit-stabiliser theorem, for each $i$ the stabiliser $H_i$ of $i$ is of finite index in $G$; by Lemma 8.1 we may set $H$ to be a normal subgroup of $G$ that has finite index in $\bigcap_{i=1}^{m} H_i$, and hence in $G$. For every $g \in H$ we have $g \cdot i = \alpha_{g,i} h_i$, which is to say that

$$h_i(g^{-1}x) = \alpha_{g,i} h_i(x).$$

(10.3)

for every $x \in G$ and every $i$. Taking $x = e$, and noting that $h_i(e) = 1$, we see that $\alpha_{g,i} = h_i(g^{-1})$, and so (10.3) implies that

$$h_i(g^{-1}x) = h_i(g^{-1}) h_i(x)$$

(10.4)

for every $g \in H$ and every $x \in G$.

This implies in particular that the restriction of $h_i$ to $H$ is a homomorphism into $\mathbb{R}^\times$. Combined with (10.4), this means that $h_i(cx) = h_i(x)$ for every $c \in [H, H]$ and every $x \in G$, and so we conclude that each $h_i$ factors through $G/[H, H]$ (noting that $[H, H]$ is characteristic in $H$, and hence normal in $G$).

Let $p : G \to \mathbb{R}$ be a positive harmonic function. Since $p$ can be expressed in the form (10.4), $p$ must also factor through $G/[H, H]$, and so writing $\phi : G \to G/[H, H]$ we have $p = \tilde{p} \circ \phi$, with $\tilde{p} : G/[H, H] \to \mathbb{R}$ harmonic by Lemma 2.1. However, the abelian group $H/[H, H]$ is of finite index in $G/[H, H]$, and so Proposition 10.1 therefore implies that $\tilde{p}$, and hence $p$, is constant.

## 11 Random walks on virtually cyclic groups

In this section we consider an infinite group $G$ with a finite-index normal cyclic subgroup $\mathbb{Z}$ and a symmetric, finitely supported generating probability measure $\mu$. In a similar fashion to Section 9, we consider a finite set $T$ such that each $g \in G$ can be expressed uniquely as

$$g = \zeta(g) \tau(g)$$

with $\zeta(g) \in \mathbb{Z}$ and $\tau(g) \in T$.

In general we continue to denote the identity of $G$ by $e$, the inverse of an element $g$ by $g^{-1}$, and the composition of two group elements $g, h$ by $gh$. However, when composing elements of $\mathbb{Z}$ with one another we often switch to additive notation to emphasise the integer structure. Thus, for example, we sometimes denote the identity by 0, the inverse of $m$ by $-m$ and the composition of $m$ and $n$ by $m + n$, provided $m, n \in \mathbb{Z}$. This should not cause confusion since, whilst the notation for a given group element is not unique, neither is it ambiguous (in particular, we never multiply together two elements of $\mathbb{Z}$). For the avoidance of doubt, the notation 1 always represents a generating element of the subgroup $\mathbb{Z}$, and never the identity element of $G$.

For each $n \in \mathbb{N}$ write

$$T^+_n = \min\{t \geq 0 : \zeta(X_t) \geq n\}, \quad T^-_n = \min\{t \geq 0 : \zeta(X_t) \leq n\},$$

noting that these quantities are almost surely finite. The purpose of this section is then to prove the following result.

**Lemma 11.1.** Let $m \in \mathbb{Z}$, and suppose that $g \in G$ with $m < \zeta(g) < m + R$. Let $M$ be as in Lemma 9.3. Then

$$\mathbb{P}_x\left[T_{m+R}^+ < T^-_m\right] = \frac{\zeta(g) - m}{R + M} + O\left(\frac{1}{R}\right)$$

**Proof.** By Lemma 9.4 there exists a function $\varphi : T \to \mathbb{R}$ such that the function $f : G \to \mathbb{R}$ given by

$$f(nt) = n + \varphi(t)$$

is harmonic on $G$. Let $t_{\min} \in T$ be the point at which $\varphi$ takes its minimum value, and let $t_{\max} \in T$ be the point at which $\varphi$ takes its maximum value, and define two further harmonic functions $f^+, f^- : G \to \mathbb{R}$ by

$$f^+ = \frac{1}{R + M} \left(f - f((m - M)t_{\min})\right); \quad f^- = \frac{1}{R + M} \left(f - f((m + R + M)t_{\max})\right) + 1.$$

Note the following properties of $f^+, f^-$. 28
(i) We have $f^-(nt) \leq 0 \leq f^+(nt)$ whenever $n \in [m-M,m]$.

(ii) We have $f^-(nt) \leq 1 \leq f^+(nt)$ whenever $n \in [m+R,m+R+M]$.

Moreover, $f^+ - f^-$ is constant and given by
\begin{equation}
    f^+ - f^- = \frac{\varphi(t_{\max}) - \varphi(t_{\min}) + M}{R + M},
\end{equation}
and we have
\begin{equation}
    f^+(nt) = f^-(nt) = \frac{1}{R + M},
\end{equation}
for every $n \in \mathbb{Z}$ and every $t \in T$.

Now define $h : [m-M,m+R+M]T \to \mathbb{R}$ by setting
\begin{equation*}
    h(nt) = \begin{cases} 
        0 & \text{when } n \in [m-M,m] \\
        1 & \text{when } n \in [m+R,m+R+M],
    \end{cases}
\end{equation*}
and requiring that $h$ be harmonic elsewhere. Lemma 2.3 and the definition of $M$ imply that $h$ is well defined by these stipulations, and moreover that
\begin{equation}
    h(g) = \mathbb{P}_g \left[ T^+_{m+R} < T^-_m \right].
\end{equation}

Now Corollary 2.4, properties (i) and (ii) of $f^+, f^-$ and the definition of $h$ imply that
\begin{equation*}
    f^{-} \leq h \leq f^+,
\end{equation*}
and hence (i), (ii), (11.1) and (11.2) imply that
\begin{equation*}
    h(g) = \frac{\zeta(g) - m}{R + M} + O \left( \frac{1}{R} \right).
\end{equation*}

The desired result then follows from (11.3). \hfill \Box

12 Harmonic functions on groups with virtually cyclic quotients

In this section we consider groups with virtually cyclic quotients. A well-known example of a group with a genuinely cyclic quotient is the lamplighter group. If $L$ is the $\mathbb{Z}/2\mathbb{Z}$-vector space of finitely supported functions $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$, viewed as an additive group, then the lamplighter group $G$ is the semidirect product $G = \mathbb{Z} \ltimes L$ defined by the action of $m \in \mathbb{Z}$ on $L$ given by $m \cdot f(x) = f(x - m)$. Explicitly, the group operation is defined by $(m,f) \cdot (m',f') = (m + m', f + m \cdot f')$.

I. Benjamini, G. Kozma and Yadin [4] give an explicit construction of a positive harmonic function on the lamplighter group.

Proposition 12.1 (Benjamini–Kozma–Yadin, unpublished). Let $G$ be the lamplighter group, and let $\mu$ be a symmetric, finitely supported generating probability measure on $G$. Denote the random walk on the lamplighter group by
\begin{equation*}
    (M_0, F_0), (M_1, F_1), (M_2, F_2), \ldots.
\end{equation*}

Let $\tau_r = \min \{ t \geq 0 : |M_t| \geq r \}$, and define $h_r : G \to \mathbb{R}$ by
\begin{equation*}
    h_r(g) = \mathbb{P}_g \left[ F_{\tau_r}(n) = 0 \text{ for all } n < 0 \right].
\end{equation*}

Then $r h_r$ converges pointwise to a positive harmonic function on $G$.

\footnote{At the time of writing, the work [4] in which this result will appear is still in preparation. There are some notes on the construction, based on a lecture by Kozma, available at http://metric2011.wordpress.com/2014/01/27/notes-of-gady-kozmas-lecture/}
In order to prove Theorem 1.3 we need a slightly more general result. The purpose of this section is to show that the construction of Benjamini, Kozma and Yadin can be adapted fairly easily to obtain harmonic functions on a more general family of finitely generated groups with virtually cyclic quotients.

**Proposition 12.2.** Let $G$ be a group with a symmetric, finitely supported generating probability measure $\mu$, and suppose that there is a homomorphism $\psi$ from $G$ onto an infinite virtually cyclic group such that $K = \ker \psi$ is not finitely generated. Then $(G, \mu)$ admits a harmonic function that is not constant on $K$.

**Remark 12.3.** The function we construct in proving Proposition 12.2 is positive, and so Proposition 10.3 implies that $G$ has an infinite-dimensional space of harmonic functions, although we do not use this fact in proving Theorem 1.3. The reader may find it an instructive exercise to modify the construction described below to obtain an infinite-dimensional space of harmonic functions directly.

We start our proof of Proposition 12.2 by expressing $G$ in a particularly convenient form.

**Lemma 12.4.** The group $G$ possesses an infinite cyclic subgroup $\mathbb{Z}$ such that $K\mathbb{Z}$ is normal in $G$, and a finite set $T$ containing the identity such that each $g \in G$ can be expressed uniquely as

$$g = \kappa(g) \zeta(g) \tau(g)$$

with $\kappa(g) \in K$, $\zeta(g) \in \mathbb{Z}$ and $\tau(g) \in T$. Moreover, $\psi$ is an isomorphism on $\mathbb{Z}$ and injective on $T$, and each $\bar{g} \in \psi(G)$ can be expressed uniquely as

$$\bar{g} = \bar{\zeta}(\bar{g}) \bar{\tau}(\bar{g})$$

with $\bar{\zeta}(\bar{g}) \in \psi(\mathbb{Z})$ and $\bar{\tau}(\bar{g}) \in \psi(T)$.

**Proof.** The image $\psi(G)$ possesses an infinite cyclic subgroup $(z)$ of finite index, and by Lemma 8.1 we may assume that $(z)$ is normal in $\psi(G)$. Let $\bar{\tau} \in \psi^{-1}(z)$. The element $\bar{\tau}$ is of infinite order, and we denote by $\mathbb{Z}$ the infinite cyclic subgroup that it generates. Note that $\psi$ is injective on $\mathbb{Z}$, and hence an isomorphism on $\mathbb{Z}$, as required.

Since $\psi(\mathbb{Z}) = \langle z \rangle$ is of finite index in $\psi(G)$, we may choose a finite set $\bar{T}$ containing $e$ such that each $\bar{g} \in \psi(G)$ can be expressed uniquely in the form (12.2), with $\bar{\tau}(\bar{g}) \in \bar{T}$. For each $t \in \bar{T}$, pick an arbitrary $t \in \psi^{-1}(T)$, and define $T = \{ t \in \bar{T} \}$. It immediately follows that the element $\bar{\tau}(\bar{g})$ in (12.2) belongs to $\psi(T)$, and that $\psi$ is injective on $T$, as required. The injectivity of $\psi$ on $\mathbb{Z}$ additionally implies that each $g \in G$ can be expressed uniquely in the form (12.1).

The fact that $K\mathbb{Z}$ is normal in $G$ follows immediately from the fact that $\psi(\mathbb{Z}) = \langle z \rangle$ is normal in $\psi(G)$. □

From now on in this section $\psi$ and $K$ are as in Proposition 12.2 and $\mathbb{Z}$ and $T$ are fixed as in Lemma 12.4. Note that we have $\bar{\tau}(\bar{\psi}(g)) = \psi(\bar{\tau}(g))$, and that if we abuse notation slightly and identify $\mathbb{Z}$ with its isomorphic image $\psi(\mathbb{Z})$ we have $\bar{\zeta}(\bar{\psi}(g)) = \zeta(g)$.

As in Section 11 when composing elements of $\mathbb{Z}$ or $\psi(\mathbb{Z})$ with one another we often switch to additive notation to emphasise the integer structure.

Since $K$ is normal, the group $\mathbb{Z}$ acts on $K$ by conjugation. We may therefore define an automorphism $\varphi : K \to K$ by $\varphi(k) = 1k1^{-1}$. More generally, this means that

$$\varphi^n(k) = nkn^{-1}.$$

As in Section 9 we denote

$$S := \operatorname{supp} \mu.$$

If $g = knt$ is a group element with $k \in K$, $n \in \mathbb{Z}$ and $t \in T$, then the elements adjacent to $g$ in the Cayley graph $(G, S)$ are the elements $gs = knst$ with $s \in S$. 

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Lemma 12.5. Let $k \in K$, $n \in \mathbb{Z}$, $t \in T$ and $s \in G$. Then
\[
\kappa(knts) = k\varphi^n(\kappa(ts));
\]
\[
\zeta(knts) = n + \zeta(ts);
\]
\[
\tau(knts) = \tau(ts).
\]

Proof. Expressing $ts$ in the form (12.1), we have $knts = kn\kappa(ts)n\zeta(ts)n\tau(ts)$, and hence $knts = k\varphi^n(\kappa(ts))n\zeta(ts)n\tau(ts)$, as claimed.

For each set $A \subset \mathbb{Z}$ define a subgroup $U_A$ of $K$ by $U_A = \langle \varphi^n(\kappa(ts)) : s \in S, t \in T, n \in A \rangle$, and for each $n \in \mathbb{Z}$ abbreviate by $U_n$ the subgroup $U_n = U_{[n,\infty)}$.

Note that $K = U_{\mathbb{Z}}$.

Lemma 12.6. If $K$ is not finitely generated then, possibly after relabelling each $n \in \mathbb{Z}$ as $-n$, we have
\[
\cdots \supseteq U_{-2} \supseteq U_{-1} \supseteq U_0 \supseteq U_1 \supseteq U_2 \supseteq \cdots .
\]

Proof. The containments of (12.4) are immediate by definition, so we just need to prove that they are strict. We start by showing that either $U_{\{0\}} \not\subset U_N$ or $U_{\{0\}} \not\subset U_{-N}$. Indeed, suppose that $U_{\{0\}} \subset U_N$ and $U_{\{0\}} \subset U_{-N}$, which, since $U_{\{0\}}$ is finitely generated, implies in particular that there is some $M \in \mathbb{N}$ such that $U_{\{0\}} \subset U_{[M]}$ and $U_{\{0\}} \subset U_{[-M]}$. Since $\varphi$ is an automorphism, (12.5) also implies that $U_{\{-1\}} \subset U_{\{0\} \cup \{M-1\}}$, and hence by (12.6) that $U_{\{-1\}} \subset U_{[M]}$. Repeating this argument, we conclude that $U_{\{-n\}} \subset U_{[M]}$ for every $n \in \mathbb{N}$. Similarly, (12.6) implies that $U_{\{n\}} \subset U_{[-M]}$ for every $n \in \mathbb{N}$, and so in fact we have $U_{\mathbb{Z}} = U_{[-M,M]}$.

By (12.3), this contradicts the assumption that $K$ is not finitely generated, and so either $U_{\{0\}} \not\subset U_N$ or $U_{\{0\}} \not\subset U_{-N}$, as claimed. Upon relabelling each $n \in \mathbb{Z}$ by $-n$ if necessary, we may assume the former, which implies in particular that $U_0 \not\subset U_1$.

Repeatedly using the fact that $\varphi$ is an automorphism then yields the lemma.

We assume from now on that $\mathbb{Z}$ is labelled in such a way that (12.3) holds.

As usual, we denote by $X_0, X_1, X_2, \ldots$ the random walk on $G$ defined by $\mu$. In this section, we additionally denote by $X_0, X_1, X_2, \ldots$ the random walk on $\psi(G)$ defined by $\psi(\mu)$. Note that the projected walk $(\psi(X_t))$ is isomorphic to the random walk $(X_t)$.

For each $n \in \mathbb{N}$, write
\[
T^+_n = \min \{ t \geq 0 : \zeta(X_t) \geq n \}, \quad T^-_n = \min \{ t \geq 0 : \zeta(X_t) \leq n \}.
\]
We claim that Lemma 12.7.

There exist some \( n < R \) and \( \gamma \), such that if \( u \in [t, T_u^+ \cap T_u^-] \) then \( T_u^+ = T_u^- \).

Define \( B_R = \min\{t \geq 0 : \zeta(X_u) \geq 0 \text{ for all } u \in [t, T_u^+]\} \).

More generally, for each \( n < R \) set

\[ B_R^n = \min\{t \geq 0 : \zeta(X_u) \geq n \text{ for all } u \in [t, T_u^+]\}. \]

**Lemma 12.7.** There exist some \( l > \max \zeta(TS) \) and some \( \alpha \in (0,1) \) such that if \( R > l \), and if \( g \) is such that \( -l \leq \zeta(g) \leq 0 \), then either

\[ \mathbb{P}_g[\zeta(X_t) \geq -l \text{ for all } t \leq T_U^+] = 0 \]

or

\[ \mathbb{P}_g[\kappa(X_{B_R}) U_0 \mid \zeta(X_t) \geq -l \text{ for all } t \leq T_U^+] \leq \alpha \]

**Proof.** Fix an element \( u \in K \setminus U_0 \), and for each \( t \in T \) and each \( j \) satisfying \( 0 \leq j < \max \zeta(TS) \) fix a path

\[ x_j^t = e, x_1^j, t, \ldots, x_j^t = t^{-1} \varphi^{-j}(u)t \]

from \( e \) to \( t^{-1} \varphi^{-j}(u)t \) in the Cayley graph \((G, S)\), chosen so that

\[ \zeta(tx_j^t) < - \max \zeta(TS) \]

for at least one \( i \).

Let

\[ l = (1 + \max_{j,t} r_{j,t}) \max \zeta(TS). \]

Write \( \gamma = \min_{s \in S} \mu(s) \), and set

\[ \beta = \gamma^{\max_{j,t} r_{j,t}}. \]

Note that for each \( j, t \) there is a probability of at least \( \beta \) that the random walk starting at \( e \) has \( x_0^t, x_1^j, \ldots, x_j^t \) as an initial segment.

Write \( A \) for the set of (finite) paths \( p \) from \( g \) whose images \( \zeta(p) \) in \( Z \) finish at \( R \) or above, but stay in the range \([-l, R-1]\) until then. If \( A = \emptyset \) then \( \mathbb{P}_g[\zeta(X_t) \geq -l \text{ for all } t \leq T_u^+] = 0 \) and the lemma holds, and so we may assume that \( A \neq \emptyset \). For each \( p \in A \), write \( f_p, m_p, t_p \) for the final position of \( p \), with \( f_p \in K, m_p \in Z \) and \( t_p \in T \); thus \( m_p \geq R \), but all earlier positions of \( \zeta(p) \) are below \( R \). Also, let \( \sigma_p \) be the largest final segment of \( p \) whose image in \( Z \) lies entirely in the non-negative integers, and let \( k_p m_p t_p \) be the first position of this final segment, with \( k_p \in K, m_p \in Z \) and \( t_p \in T \). Note that

\[ 0 \leq m_p < \max \zeta(TS). \]

**Lemma 12.5** implies that \( \{p \in A : k_p \in U_0\} = \{p \in A : f_p \in U_0\} \), and so we may define

\[ A_\varepsilon = \{p \in A : k_p U_0\} = \{p \in A : f_p U_0\}, \]

\[ A_\varepsilon = A \setminus A \varepsilon. \]

We claim that

\[ \mathbb{P}_g(A_\varepsilon) \gg \mathbb{P}_g(A_\varepsilon). \]

(12.10)
This is sufficient to prove the lemma, since the conditional probability we are aiming to estimate is equal to
\[
\frac{P_g(A_e)}{P_g(A_e) + P_g(A_c)}.
\]

We define a map \( c \) from \( A_e \) to the set of finite paths starting at \( g \) as follows. Given \( p \in A_e \), let \( c(p) \) be the path that agrees with \( p \) up until \( k_pm_plp \), then has positions
\[
k_pm_plp x_1^{mp,tlp}, \ldots, k_pm_plp x_{mp,tlp}^{mp,tlp},
\]
and then continues with the same increments as the original path \( p \) had after position \( k_pm_plp \). This is well defined by (12.9).

We claim that \( c(p) \in A_{\delta} \) for every \( p \in A_e \). To see that \( c(p) \in A \), note that
\[
k_pm_plp x_{mp,tlp}^{mp,tlp} = k_pum_plp.
\]
(12.11)

This implies in particular that
\[
\psi(k_pm_plp x_{mp,tlp}^{mp,tlp}) = mp\psi(tlp) = \psi(k_pm_plp).
\]
(12.12)

By definition of \( l \), at no point between \( k_pm_plp \) and \( k_pm_plp x_{mp,tlp}^{mp,tlp} \) does \( \zeta(p) \) drop below \(-l\), and so it follows that \( c(p) \in A \). To see, more specifically, that \( c(p) \in A_{\delta} \), note that the definition of \( k_pm_plp \) combines with (12.12) to imply that \( \zeta(c(p)) \) doesn’t drop below zero after \( k_pm_plp x_{mp,tlp}^{mp,tlp} \). Lemma 12.5 and (12.12) therefore imply that \( F_c(p) \) is in the same left coset of \( U_0 \) as \( k_p u \). In particular, since \( k_p \in U_0 \) and \( u \notin U_0 \) we have \( F_c(p) \notin U_0 \), and so \( c(p) \in A_{\delta} \), as claimed.

The fact that \( c(A_e) \subset A_{\delta} \) of course implies that
\[
P_g(A_{\delta}) \geq P_g(c(A_e)).
\]
(12.13)

We claim, moreover, that \( c \) is \( O(1) \)-to-one. Write \( a(p) \) for the segment that was added to \( c \) to obtain \( c(p) \), and note that one can, in principal at least, recover \( p \) from \( c(p) \) simply by deleting the segment \( a(p) \). Note that (12.8) and (12.9) combine with Lemma 12.5 and the fact (noted in the preceding paragraph) that \( \zeta(c(p)) \) doesn’t drop below zero after \( k_pm_plp x_{mp,tlp}^{mp,tlp} \) to imply that \( \zeta(p) \) drops below zero for the last time at some point during \( a(p) \). This means that knowledge of \( c(p) \) only is sufficient to identify, to within \( \max_1 \tau_{r_j,t} \) positions, where in \( c(p) \) the segment \( a(p) \) begins. Furthermore, the increments of \( a(p) \) coincide with those of one of the finitely many paths \( (x_j^{-1}) \). There are therefore at most \( O(1) \) possibilities for \( a(p) \), given \( c(p) \), and so \( c \) is \( O(1) \)-to-one, as claimed.

This implies, in particular, that
\[
P_g(c(A_e)) \gg \sum_{p \in A_e} P_g(c(p)).
\]
(12.14)

However, it follows from the definition of \( \beta \) that for every \( p \in A_e \) we have
\[
P_g(c(p)) \geq \beta P_g(p).
\]

In combination with (12.13) and (12.14), this implies that
\[
P_g(A_{\delta}) \geq P_g(c(A_e)) \gg \sum_{p \in A_e} P_g(c(p)) \geq \beta \sum_{p \in A_e} P_g(p) = \beta P_g(A_e),
\]
and so (12.10) holds as claimed and the lemma is proved.

**Lemma 12.8.** Let \( l \) and \( \alpha \) be as given by Lemma 12.7. Let \( n \leq 0 \). Then if \( R > l \), and if \( g \) is such that \( n - l \leq \zeta(g) \leq n \), then either
\[
P_g[\zeta(X_t) \geq n - l \text{ for all } t \leq T_R^+] = 0
\]
or
\[
P_g[\kappa(X_{t_{R,\alpha}^n}) \in U_n | \zeta(X_t) \geq n - l \text{ for all } t \leq T_R^+] \leq \alpha.
\]

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Applying Lemma 12.6, we see that
implies in particular that
or
Proof. Everthing in this lemma is conditional on the event $\{ \zeta(X_t) \geq k \}$ for all $t \leq T_R^+$, hence that it is sufficient to show that
$C_q := \{ \zeta(X_t) \geq k \}$ for all $t \leq T_R^+$.

Applying Lemma 12.9, we see that $\kappa(X_{T_R^+}) \notin U_0$ precisely when $\kappa(X_{B_R^+}) \notin U_0$ for each $n < 0$. This implies in particular that
$$\{ \zeta(X_{T_R^+}) \in U_0 \} \subset \{ \kappa(X_{B_R^+}) \in U_0 \} \text{ for each } n = k + l, 2l, \ldots, k + ml,$$
and hence that it is sufficient to show that
$$\mathbb{P}_g [ \kappa(X_{B_R^+}) \in U_0 \text{ for each } n = k + l, 2l, \ldots, k + ml \mid C_k ] \leq \alpha^m \quad (12.15)$$
whenever $\mathbb{P}_g [C_k] \neq 0$. We show this by induction on $m$.

If
$$\mathbb{P}_g [ \kappa(X_{B_R^+}) \in U_0 \mid C_k ] = 0$$
then either $\mathbb{P}_g [C_k] = 0$ or the left-hand side of (12.15) is 0; in either case the lemma holds, so we may assume that
$$\mathbb{P}_g [ \kappa(X_{B_R^+}) \in U_0 \mid C_k ] \neq 0.$$\nThis implies that the left-hand side of (12.15) is at most
$$\mathbb{P}_g [ \kappa(X_{B_R^+}) \in U_{k+l} \mid C_k ] \cdot \mathbb{P}_g [ \kappa(X_{B_R^+}) \in U_n \text{ for } n = k + 2l, \ldots, k + ml \mid C_k \cap \{ \kappa(X_{B_R^+}) \in U_{k+l} \} ].$$\nHowever, it follows immediately from Lemma 12.3 that
$$\mathbb{P}_g [ \kappa(X_{B_R^+}) \in U_{k+l} \mid C_k ] \leq \alpha,$$
and so in fact the left-hand side of (12.15) is at most
$$\alpha \cdot \mathbb{P}_g [ \kappa(X_{B_R^+}) \in U_n \text{ for } n = k + 2l, \ldots, k + ml \mid C_k \cap \{ \kappa(X_{B_R^+}) \in U_{k+l} \} ].$$\nConditioning on the position of the random walk on $G$ immediately after the projected walk on $\mathbb{Z}$ has left the set $[k, k + l - 1]$ for the last time before reaching $R$, this is at most
$$\alpha \cdot \sum_{y: \kappa(y) \in U_{k+l}} \mathbb{P}_y [ \kappa(X_{B_R^+}) \in U_n \text{ for } n = k + 2l, \ldots, k + ml \mid C_{k+l} ] \cdot \mathbb{P}_g [ X_{B_R^+}^y = y \mid C_{k+l} ] \quad (12.16)$$
Note that if $\mathbb{P}_y [C_{k+l}] = 0$ then $\mathbb{P}_y [X_{B_R^+}^y = y] = 0$, so elements $y$ for which
$$\mathbb{P}_g [ \kappa(X_{B_R^+}) \in U_n \text{ for } n = k + 2l, \ldots, k + ml \mid C_{k+l} ]$$
is not defined do not appear in the sum in the numerator of (12.16), and so that sum is well defined. This means, moreover, that given $X_0 = y$, for every possible value $y$ of $X_{B_R^+}^y$ the first factor of the summand of (12.16) is at most $\alpha^{m-1}$ by induction, and so (12.16), and hence the left-hand side of (12.15), is at most $\alpha^m$, as required.
Define

\[ M_R = \min \{ \zeta(X_t) : t \leq T_R^+ \}, \]

so that \( M_R \) is the minimum point hit by \( \zeta(X_t) \) before it first exceeds \( R \).

**Lemma 12.10.** Let \( n \in \mathbb{N}; \) let \( l \) and \( \alpha \) be as given by Lemma 12.7; let \( m \) be such that \((-m+1)l < -n \leq -ml; \) and let \( R > l \). Then either

\[ P_g[M_R = -n] = 0 \]

or

\[ P_g[\kappa(X_{T_R^n}) \in U_0 | M_R = -n] \leq \alpha^m. \]

**Proof.** We may assume that \( P_g[M_R = -n] \neq 0 \), and hence in particular that \( P_g[C_{-n}] \neq 0 \), and so \( P_g[\kappa(X_{T_R^n}) \in U_0 | M_R = -n] \) is well defined and equal to

\[
\sum_{y \in G} P_g \left[ \left\{ T_{b \in G; \zeta(b) = -n} < T_R^+ \text{ and } X_{T_{b \in G; \zeta(b) = -n}} = y \right\} \mid C_{-n} \right] \cdot P_g[\kappa(X_{T_R^n}) \in U_0 | C_{-n}],
\]

which is at most

\[
\sum_{y \in G; \zeta(y) = -n} P_g \left[ X_{T_{b \in G; \zeta(b) = -n}} = y \mid \left\{ T_{b \in G; \zeta(b) = -n} < T_R^+ \text{ and } C_{-n} \right\} \right] \cdot P_g[\kappa(X_{T_R^n}) \in U_0 | C_{-n}].
\]

If \( P_g[C_{-n}] = 0 \) then

\[ P_g \left[ X_{T_{b \in G; \zeta(b) = -n}} = y \mid \left\{ T_{b \in G; \zeta(b) = -n} < T_R^+ \text{ and } C_{-n} \right\} \right] = 0,
\]

and so elements \( y \) for which

\[ P_g[\kappa(X_{T_R^n}) \in U_0 | C_{-n}] \]

is not defined do not appear in the sum (12.17) and that sum is well defined. The sum (12.17) is, moreover, the expectation of the quantity (12.18) with respect to some probability measure on the set \( \{ y \in G : \zeta(y) = -n \} \). However, for each \( y \) in that set for which the quantity (12.18) is defined, the quantity (12.18) is at most \( \alpha^m \) by Lemma 12.9 and so (12.17) is at most \( \alpha^m \) and the lemma is proved. \( \square \)

Define a real-valued function \( h_R \) on the subset \( K[-R, R]T \) of \( G \) by

\[ h_R(g) = P_g[T_R^+ < T_R^- \text{ and } \kappa(X_{T_R^n}) \in U_0]. \]

**Lemma 12.11.** The function \( h_R \) satisfies the following properties.

(i) The function \( h_R \) is harmonic on the interior of \( K[-R, R]T \).

(ii) For every \( g \in (K[-R, R]T)^\circ \) we have \( h_R(g) \ll_g 1/R \).

(iii) If \( \zeta(g) \geq 0 \) and \( \kappa(g) \notin U_0 \) then \( h_R(g) \ll 1/R \).

(iv) If \( \zeta(g) \geq 0 \) and \( \kappa(g) \in U_0 \) then \( h_R(g) \gg \zeta(g)/R \).

**Proof.** The harmonicity of \( h_R \) is clear, so we prove properties (ii), (iii) and (iv). We may rewrite \( h_R(g) \) by conditioning on \( M_R \) as follows:

\[ h_R(g) = \left( \sum_{n=0}^{R} P_g[M_R = -n] \cdot P_g[\kappa(X_{T_R^n}) \in U_0 | M_R = -n] \right) + P_g[M_R > 0] \cdot P_g[\kappa(X_{T_R^n}) \in U_0 | M_R > 0]. \]

Let us examine these probabilities in turn, starting with \( P_g[M_R = -n] \). This corresponds to the event that \( \zeta(X_t) \) hits \(-n\) before reaching or exceeding \( R \), but then reaches or exceeds \( R \) before dropping below \(-n\). In particular,

\[ P_g[M_R = -n] \leq P_{\psi(g)} \left[ T_{-n}^- < T_R^+ \right] \cdot \max_{t \in T} P_{(-n)\psi(t)} \left[ T_R^+ < T_{-(n+1)}^- \right]. \]
Applying Lemma 12.11 for each \( n \geq 0 \) we have
\[
P_{\psi(g)} \left[ T_{-n} \leq T_R^+ \right] = \frac{R - \zeta(g)}{R + n + O(1)} + O \left( \frac{1}{R} \right) \tag{12.21}
\]
and
\[
\max_{t \in \mathbb{T}} P_{(-n)\psi(t)} \left[ T_R^+ < T_{-(n+1)}^- \right] = \frac{1}{R + n + O(1)} + O \left( \frac{1}{R} \right). \tag{12.22}
\]
If \( \zeta(g) \leq 0 \) then of course \( P_g[M_R > 0] = 0 \); another application of Lemma 12.11 implies that more generally we have
\[
P_g[M_R > 0] = \begin{cases} 
\frac{\zeta(g)}{n + O(1)} + O \left( \frac{1}{n} \right) & \text{if } \zeta(g) > 0; \\
0 & \text{if } \zeta(g) \leq 0. 
\end{cases} \tag{12.23}
\]
We now consider \( P_g[\kappa(X_{T_R}) \in U_0 \mid M_R = -n] \) when \( n \in \mathbb{N} \) and \( P_g[M_R = -n] \neq 0 \). Let \( l \) and \( \alpha \) be as given by Lemma 12.7, noting in particular that \( \alpha < 1 \), and let \( m \) be such that \(-(m + 1)l < -n \leq -ml\). Lemma 12.10 then implies that
\[
P_g[\kappa(X_{T_R}) \in U_0 \mid M_R = -n] \leq \alpha^m. \tag{12.24}
\]
Finally, the condition that \( M_R > 0 \) implies that \( \zeta(X_t) \) does not drop below zero until after time \( T_R^+ \), which by Lemma 12.5 means that \( \kappa(X_t) \) is in the same left coset of \( U_0 \) as \( \kappa(g) \) for every \( t \leq T_R^+ \). We therefore have
\[
P_g[\kappa(X_{T_R}) \in U_0 \mid M_R > 0] = \begin{cases} 
1 & \text{if } \kappa(g) \in U_0 \\
0 & \text{otherwise.} 
\end{cases} \tag{12.25}
\]
Properties (ii), (iii) and (iv) then follow from (12.21), (12.22), (12.23), (12.24) and (12.25) and the fact that \( \alpha < 1 \).

Proof of Proposition 12.12 Property (ii) of Lemma 12.11 implies that \( R \cdot h_R(g) = O_g(1) \), so for each \( g \) there is a convergent subsequence of \( R \cdot h_R(g) \) as \( R \to \infty \). Since \( G \) is countable, a simple diagonal argument therefore gives a subsequence of \( R \cdot h_R \) that converges pointwise to a function \( h : G \to \mathbb{R} \). The limit function \( h \) is harmonic by property (i) of Lemma 12.11 and not constant on \( K \) by properties (iii) and (iv).

13 Groups with finite-dimensional spaces of harmonic functions

In this section we complete the proof of Theorem 1.3. The results of Sections 9 and 10 prove it for linear groups; in this section we perform a very natural reduction to that case using the results of the subsequent sections.

The group \( G \) in Theorem 1.3 acts on the space \( H \) of harmonic functions on \((G, \mu)\) via the linear transformations \( g \cdot f(x) = f(g^{-1}x) \). This action defines a homomorphism \( G \to GL(H) \), which we denote by \( \psi : G \to GL(H) \) throughout this section. If \( H \) is finite dimensional then \( \psi \) is in fact a homomorphism into \( GL_n(\mathbb{R}) \) for some \( n \). Moreover, harmonic functions on \((G, \mu)\) are in direct correspondence with those on \((\psi(G), \psi(\mu))\), as follows.

Lemma 13.1. A function \( h : G \to \mathbb{R} \) is harmonic with respect to \( \mu \) if, and only if, there is some function \( \overline{h} : \psi(G) \to \mathbb{R} \), harmonic with respect to \( \psi(\mu) \), such that \( h = \overline{h} \circ \psi \).

Proof. If \( h : G \to \mathbb{R} \) is harmonic and \( k \in \ker \psi \) then \( h(kg) = h(g) \) for every \( g \), so there exists \( \overline{h} : \psi(G) \to \mathbb{R} \) such that \( h = \overline{h} \circ \psi \). The desired result then follows from Lemma 2.7.

Proof of Theorem 1.3 If \((G, \mu)\) has a recurrent random walk then the result follows from Proposition 2.8 and Corollary 9.4. We may therefore assume that \((G, \mu)\) has a transient random walk.

If the space of harmonic functions is finite dimensional then \( \psi \) gives a homomorphism \( \psi : G \to GL_n(\mathbb{R}) \). By Lemma 13.1 \((\psi(G), \psi(\mu))\) has a finite-dimensional space of harmonic functions, and so by Corollaries 9.4 and 10.3 the linear group \( \psi(G) \) is virtually cyclic.
If $\psi(G)$ is finite then, by the maximum principle (Lemma 2.2) and Lemma 13.1 every harmonic function on $(G, \mu)$ is constant, contradicting Corollary 13.3. We may therefore assume that $\psi(G)$ is infinite. If $\ker \psi$ is not finitely generated, Proposition 12.2 therefore gives a harmonic function on $(G, \mu)$ that is not constant on $\ker \psi$. On the other hand, since we are assuming that the random walk on $(G, \mu)$ is transient, if $\ker \psi$ is finitely generated and infinite then Proposition 7.1 gives a harmonic function that is not constant on $\ker \psi$. In either case this contradicts Lemma 13.1 and so $\ker \psi$ must in fact be finite. Since $\psi(G)$ is virtually cyclic, it follows that $G$ is itself virtually cyclic, as desired.

Remarks 13.2. Meyerovitch and Yadin [24, Theorem 1.4] have very recently shown that if $G$ is a non-virtually nilpotent subgroup of $GL_d(\mathbb{C})$ with a symmetric, finitely supported generating probability measure $\mu$ then $\dim H^1(G, \mu) = \infty$. This could be used in place of Corollary 10.4 in the above proof of Theorem 1.3.

It is conjectured that if $G$ is any group with a symmetric, finitely supported generating probability measure $\mu$ then $\dim H^1(G, \mu) = \infty$ [24]. A verification of this conjecture would immediately reduce the proof of proof of Theorem 1.3 to the virtually nilpotent case, and hence to the material in Section 9 of this paper.

A Coordinates and polynomials on nilpotent Lie groups

In this appendix we prove Lemma 9.11, the statement of which is as follows.

**Lemma 9.11.** There are polynomials $q_1, \ldots, q_d : N \times N \to \mathbb{R}$ such that $\deg q_i = \sigma(i)$ for each $i$, and such that if $x, u \in N$ then the $i$th coordinate of $xu$ is given by $q_i(x, u)$.

Here, and throughout this appendix, all notation is as in Section 9.

The main tool in this appendix is the Baker–Campbell–Hausdorff formula, which states that if $n$ is the Lie algebra of some Lie group then for any elements $X, Y \in n$ we have

$$\exp(X) \exp(Y) = \exp(X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] + \cdots),$$

(A.1)

where the series on the right is a sum of distinct nested Lie brackets in $X$ and $Y$ with (fixed) rational coefficients. Note that in the case with which we are concerned the Lie algebra $n$ is nilpotent, and so this series is in fact a finite sum.

For the remainder of this appendix, $H$ is a discrete, cocompact subgroup of some simply connected nilpotent Lie group $\mathbb{H}$, equipped with some Mal’cev basis. For the avoidance of doubt, for $p : H \to \mathbb{R}$ a polynomial on $H$, the notation $\deg p$ refers to the homogeneous degree of $p$ on $H$, whereas $\sigma(t)$ is still defined in terms of lower central series of the nilpotent group $N$, as in (10.2).

**Lemma A.1.** Let $1 \leq i, j \leq d$, and let $p_1, \ldots, p_d, p'_1, \ldots, p'_d$ be polynomials on $H$ such that $\deg p_t \leq \sigma(t)$ for every $t \geq i$ and $\deg p'_t \leq \sigma(t')$ for every $t' \geq j$. Then there are polynomials $q_{\sigma(i) + \sigma(j) + 1}, \ldots, q_d$ on $H$ satisfying $\deg q_t \leq \sigma(t)$ for every $t$ such that

$$[(p_1 X_1 + \cdots + p_d X_d) \cdot (p'_1 X_j + \cdots + p'_d X_d)] = q_{\sigma(i) + \sigma(j) + 1} X_{\sigma(i) + \sigma(j) + 1} + \cdots + q_d X_d.$$

(A.2)

**Proof.** The bilinearity of the Lie bracket implies that the left-hand side of (A.2) can be expressed as the sum of terms of the form $p_t p'_t [X_i, X_j]$, with $t \geq i$ and $t' \geq j$. The element $[X_i, X_j]$ lies in $\mathfrak{n}_{\sigma(t) + \sigma(t')} = \text{span} \{X_{\sigma(t) + \sigma(t')} : t \geq i, t' \geq j \}$, and so the result follows from (A.1).

**Lemma A.2.** Let $p_1, \ldots, p_d, p'_1, \ldots, p'_d$ be polynomials on $H$ such that $\deg p_t \leq \sigma(t)$ and $\deg p'_t \leq \sigma(t')$ for every $t$. Then there are polynomials $q_1, \ldots, q_d$ on $H$ satisfying $\deg q_t \leq \sigma(t)$ for every $t$ such that

$$\exp(p_1 X_1 + \cdots + p_d X_d) \exp(p'_1 X_1 + \cdots + p'_d X_d) = \exp(q_1 X_1 + \cdots + q_d X_d).$$

**Proof.** This follows from the Baker–Campbell–Hausdorff formula (A.1) and Lemma A.1.
Lemma A.3. Let $1 \le i \le d$, and let $p_1, \ldots, p_d$ be polynomials on $H$ such that $\deg p_i \le \sigma(t)$ for every $t$. Then there are polynomials $q_1, \ldots, q_d$ on $H$ satisfying $\deg q_t \le \sigma(t)$ for every $t$ such that
\[
\exp(p_i X_i + \cdots + p_d X_d) = \exp(q_i X_i) \cdots \exp(q_d X_d).
\]

Proof. The Baker–Campbell–Hausdorff formula (A.1) and Lemma A.1 imply that there are polynomials $q_1+1, \ldots, q_d$ on $H$ satisfying $\deg q_t \le \sigma(t)$ for every $t$ such that
\[
\exp(-p_i X_i) \exp(p_i X_i + \cdots + p_d X_d) = \exp(q_{i+1} X_{i+1} + \cdots + q_d X_d),
\]
and hence that
\[
\exp(p_i X_i + \cdots + p_d X_d) = \exp(p_i X_i) \exp(q_{i+1} X_{i+1} + \cdots + q_d X_d).
\]
The result then follows by induction. \hfill \Box

Proof of Lemma A.3. Set $H = \mathbb{N} \times \mathbb{N}$ and $\mathbb{P} = \mathbb{N} \times \mathbb{N}$, and note that both $x_i$ and $u_i$ are polynomials of degree $\sigma(i)$ on $H$. It therefore follows from repeated application of Lemma A.2 that there are polynomials $p_i, \ldots, p_d : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ such that $\deg p_i \le \sigma(t)$ for every $t$ and such that $xu = \exp(p_1(x,u)X_1 + \cdots + p_d(x,u)X_d)$. The desired result then follows from Lemma A.3. \hfill \Box

B Further applications of our Garden of Eden theorem

In this appendix we describe how our Garden of Eden theorem, Theorem 1.11, relates to other work in the literature. First, we explain how Theorem 1.11 can be used to recover Theorem 1.10. After that we explain how it can be used to reformulate a conjecture of I. Kaplansky.

The Ceccherini-Silberstein–Coornaert Garden of Eden theorem

Theorem 1.10 follows immediately from Theorem 1.11 and the following result.

Proposition B.1. Let $V$ be a finite-dimensional vector space and let $\tau : V^G \to V^G$ be a linear cellular automaton with memory set $M$ on an amenable group $G$. Then $\tau$ is pre-injective if and only if $\tau'$ is pre-injective.

Remark B.2. As we noted at the start of the proof of Theorem 1.11 a locally specifiable map on a locally finite graph is pre-injective if and only if it is pre-injective on every connected component; in proving Proposition B.1 we may therefore assume that $G$ is generated by $M$, and hence that $G$ is countable.

From now on in this appendix, $G$ is a fixed countable amenable group and $V$ is a fixed finite-dimensional vector space.

In proving Theorem 1.10 Ceccherini-Silberstein and Coornaert make use of the notion of mean dimension, the use of which in connection to Theorem 1.10 appears to have been first suggested by Gromov [15] [8.4].

Let $X$ be a subspace of $V^G$. Given a subset $\Omega$ of $G$ and an element $f$ of $V^G$, denote by $f_\Omega$ the function that agrees with $f$ on the subset $\Omega$ and takes the value 0 elsewhere, and denote by $X_\Omega$ the subspace of $V^G$ defined by
\[
X_\Omega = \{f_\Omega : f \in X\}.
\]
Since $G$ is countable and amenable, it admits a Følner sequence, which is to say a sequence $(\Omega_n)_{n \in \mathbb{N}}$ of subsets of $G$ with the property that for every $g \in G$ we have
\[
\frac{|\Omega_n \triangle \Omega_n g|}{|\Omega_n|} \to 0 \tag{B.1}
\]
as $n \to \infty$ [11]. The mean dimension of $X$ with respect to $(\Omega_n)_{n \in \mathbb{N}}$ is then denoted $\operatorname{mdim} X$, and defined by
\[
\operatorname{mdim} X = \lim_{n \to \infty} \inf_{n \in \mathbb{N}} \frac{\dim X_{\Omega_n}}{|\Omega_n|}.
\]
For the remainder of this appendix, \( \{\Omega_n\}_{n \in \mathbb{N}} \) is a fixed Følner sequence in \( G \), and the mean dimension of a subspace \( X \) of \( V^G \) is always computed with respect to \( \{\Omega_n\}_{n \in \mathbb{N}} \). We define the neighbourhood \( \Omega^+ \) of a subset \( \Omega \subset G \) to be its neighbourhood in the Cayley graph \( (G, M) \). Note in particular from \( \text{(B.1)} \) that
\[
\frac{|\partial^+ \Omega_n|}{|\Omega_n|} \to 0. \tag{B.2}
\]

Ceccherini-Silberstein and Coornaert \[6\] originally obtained Theorem \[1.10\] in the case of a countable amenable group as an immediate consequence of the following more precise statement.

**Proposition B.3** (Ceccherini-Silberstein–Coornaert \[5\] Theorem 4.10]). Let \( \tau : V^G \to V^G \) be a linear cellular automaton. Then the following statements are equivalent:

1. \( \tau \) is surjective;
2. \( \tau \) is pre-injective;
3. \( \mathrm{mdim} \tau(V^G) = \dim V \).

The key observation that allows us to prove Proposition \[B.1\] is that the mean dimension of \( \tau \) is equal to that of its transpose \( \tau' \).

**Proposition B.4.** Let \( \tau : V^G \to V^G \) be a locally specifiable linear map, with local specifiability defined in terms of the Cayley graph \( (G, M) \). Then \( \mathrm{mdim} \tau(V^G) = \mathrm{mdim} \tau'(V^G) \).

In proving Proposition \[B.4\] we make use of the following straightforward lemma.

**Lemma B.5.** Let \( X \) be a subspace of \( V^G \). Then \( \dim X_{\Omega_n^+} = \dim X_{\Omega_n} + o(|\Omega_n|) \).

**Proof.** We have \( X_{\Omega_n^+} \subset X_{\Omega_n} \oplus V^G_{\partial^+ \Omega_n} \), and so
\[
\dim X_{\Omega_n^+} \leq \dim X_{\Omega_n} + \dim V^G_{\partial^+ \Omega_n} = \dim X_{\Omega_n} + |\partial^+ \Omega_n| \dim V.
\]

The desired result then follows from \( \text{(B.2)} \). \qed

Before we prove Proposition \[B.4\] we introduce some notation. Given a locally specifiable linear map \( \tau : V^G \to V^G \) and finite subsets \( A, B \subset G \), we denote by \( \tau_{AB} \) the \( |B| \times |A| \) matrix formed by taking the rows of \( \tau \) corresponding to elements of \( B \) and the columns of \( \tau \) corresponding to elements of \( A \).

**Proof of Proposition B.4.** Note that \( \dim \tau(V^G)_{\Omega_n} = \dim \tau(V^G_{\Omega_n^+})_{\Omega_n} \) and \( \dim \tau'(V^G)_{\Omega_n} = \dim \tau'(V^G_{\Omega_n^+})_{\Omega_n} \), which, by Lemma \[B.5\] implies that
\[
\dim \tau(V^G)_{\Omega_n} - \dim \tau'(V^G)_{\Omega_n} = \dim \tau(V^G_{\Omega_n^+})_{\Omega_n} - \dim \tau'(V^G_{\Omega_n^+})_{\Omega_n} + o(|\Omega_n|). \tag{B.3}
\]

However, \( \tau(V^G_{\Omega_n^+})_{\Omega_n^+} \) is isomorphic to the image of \( \tau_{\Omega_n^+}^{\Omega_n^+} \), and \( \tau'(V^G_{\Omega_n^+})_{\Omega_n^+} \) is isomorphic to the image of \( \tau_{\Omega_n^+}^{\Omega_n^+} \). Since \( \tau_{\Omega_n^+}^{\Omega_n^+} \) and \( \tau_{\Omega_n^+}^{\Omega_n^+} \) are finite and transposes of one another, this implies that \( \dim \tau(V^G_{\Omega_n^+})_{\Omega_n^+} = \dim \tau'(V^G_{\Omega_n^+})_{\Omega_n^+} \), and so \( \text{(B.3)} \) implies that
\[
\dim \tau(V^G)_{\Omega_n} - \dim \tau'(V^G)_{\Omega_n} = o(|\Omega_n|).
\]

The desired result then follows immediately from the definition of mean dimension. \qed

**Proof of Proposition B.4.** By Remark \[B.2\] we may assume that \( G \) is generated by \( M \) and, in particular, that \( G \) is countable. Proposition \[B.1\] then follows directly from Lemma \[B.4\] and the equivalence \( (2) \Leftrightarrow (3) \) of Proposition \[B.3\]. The equivalence \( (2) \Leftrightarrow (3) \) of Proposition \[B.3\] follows from \[6\] Lemmas 4.8 & 4.9. \qed
Remark B.6. We remarked at the start of Section 3 that, in the case of a finite-dimensional linear transformation $\tau$, one can see very elementarily that surjectivity of $\tau$ is equivalent to injectivity of the dual $\tau^*$, and then use a slightly less elementary dimension argument to show that injectivity of $\tau^*$ is equivalent to injectivity of $\tau$. The scheme of the above argument for proving Theorem 1.10 is closely analogous to this, with dimension replaced by the asymptotic notion of mean dimension.

It would be stretching reality somewhat to claim that this represented a new proof of Theorem 1.10, since there is considerable overlap between our proof of Proposition B.1 and Ceccherini-Silberstein and Coornaert’s original proof of Theorem 1.10. However, arranging the proof in this way probably shortens the proof slightly, and perhaps makes clearer the role of amenability; note, in particular, that it is only in using the mean-dimension to convert a statement about $\tau'$ to a statement about $\tau$ that we use the amenability of $G$.

Kaplansky’s stable-finiteness conjecture

A group $G$ is called linear surjunctive if every injective linear cellular automaton is surjective. Since injectivity is stronger than pre-injectivity, Theorem 1.10 immediately implies that an amenable group is linear surjunctive. It turns out, however, that linear surjunctivity is a property satisfied in far greater generality.

Proposition B.7 (Ceccherini-Silberstein–Coornaert [9, Theorem 8.14.4]). Every sofic group is linear surjunctive.

Remark B.8. Gromov and B. Weiss [9, Theorem 7.8.1] showed that every sofic group $G$ is also surjunctive, meaning that every injective cellular automaton over $G$ on a finite alphabet is surjective.

Ceccherini-Silberstein and Coornaert [9, §8.15] note that surjunctivity of a group $G$ is related to a certain condition on group algebras, called stable finiteness. Before we state their result, let us record the prerequisite definitions. A ring $R$ is said to be directly finite if whenever two elements $a, b \in R$ satisfy $ab = 1$ they also satisfy $ba = 1$. The ring $R$ is said to be stably finite if the matrix ring $\text{Mat}_d(R)$ is directly finite for every $d \geq 1$. See [9] §8.15 for more detailed background and examples. Given a field $K$, the group algebra $K[G]$ is defined by taking the vector space $K^G_0$ of finitely supported $K$-valued functions on $G$ with the convolution product. See [9, §8.4] for further details.

Proposition B.9 ([9, Corollary 8.15.6]). Let $G$ be a group and let $K$ be a field. Then the following conditions are equivalent.

1. For every finite-dimensional vector space $V$ over $K$, every injective linear cellular automaton $\tau : V^G \to V^G$ is surjective.

2. The group algebra $K[G]$ is stably finite.

In light of Proposition B.7, we therefore have the following corollary.

Corollary B.10 ([9, Corollary 8.15.8]). Let $G$ be a sofic group and let $K$ be a field. Then the group algebra $K[G]$ is stably finite.

It is natural to ask whether Proposition B.7 and Corollary B.10 hold in more general groups; Ceccherini-Silberstein and Coornaert [9, p. 418, (OP-15)] attribute this question to Kaplansky.

Question B.11 (Kaplansky). Does either, and hence both, of the following equivalent statements hold?

1. Given any group $G$ and any field $K$, the group algebra $K[G]$ is stably finite.

2. Every group is linear surjunctive.

By Theorem 1.11 this question can be reformulated as follows.

Corollary B.12. Statements (1) and (2) of Question B.11 are equivalent to the following statement.

3. If $\tau$ is an injective linear cellular automaton over an arbitrary group then its transpose $\tau'$ is pre-injective.
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