A SECOND DERIVATIVE HÖLDER ESTIMATE
FOR WEAK MEAN CURVATURE FLOW

YOSHIHIRO TONEGAWA

Abstract. We give a proof that Brakke's mean curvature flow under the unit density assumption is smooth almost everywhere in space-time. More generally, if the velocity is equal in a weak sense to its mean curvature plus some given $\alpha$-Hölder continuous vector field, then we show $C^{2,\alpha}$ regularity almost everywhere.

1. Introduction

A family $\{M_t\}_{t \geq 0}$ of $k$-dimensional surfaces in $\mathbb{R}^n$ is called the mean curvature flow (hereafter abbreviated MCF) if the velocity of $M_t$ is equal to its mean curvature at each point and time. The MCF has been the subject of intensive research since 1980’s due to its importance in the analytic and geometric context as well as for various applications to physical and information sciences such as image processing and metallurgy. The most pertinent aspect of MCF to the present paper is the fact that the MCF is the natural gradient flow of the $k$-dimensional surface area and hence is equipped with uniquely rich variational structures. In his seminal work [5], Brakke took the advantage to define and study his version of MCF, so called Brakke’s MCF (or we may call ‘weak MCF’ to include more general flows), using the notion of varifold [1] in geometric measure theory. More precisely, given any $k$-dimensional integral varifold $V_0$, which may be considered as a generalized $k$-dimensional surface with possible singularities, Brakke proved the existence of a family of varifolds $\{V_t\}_{t \geq 0}$ each of which satisfies the MCF equation taking the advantage of its variational characterization.

Under the further assumption that the density function is 1 almost everywhere in time and space, Brakke also claimed that the MCF is smooth almost everywhere and that it satisfies the MCF equation in the classical sense. The proof of regularity theorem contains remarkable new insights such as ‘clearing-out’, ‘popping soap film’ and ‘cylindrical growth rates’, to name a few. On the other hand it is technically involved and some part, in particular the graphical approximations of the support of moving varifolds [5 Sec. 6.9, ‘Flattening out’], is particularly difficult to follow. Later a local regularity theorem for special but very useful case was obtained by White [19] which is sufficient for many applications of interest while it does not replace Brakke’s claims in full. Recently Kasai and the author [13] gave a new proof for Brakke’s regularity theorem up to $C^{1,\varsigma}$ for general weak MCF where the velocity can be equal to the mean curvature plus any given ambient vector field in a suitable integrability class. Note that $C^{1,\varsigma}$ here means $C^{1,\varsigma}$ in the space variables and $C^{1,\varsigma}$ in the time variable, which are the usual regularity features of parabolic problems (in the following $C^{2,\alpha}$ should be understood in the similar manner). The additional different aspect of [13] from Brakke’s result is that it is a natural parabolic generalization of Allard’s regularity theorem for varifold [11] since the time-independent case of [13] reduces essentially to Allard’s theorem. The

Key words and phrases. mean curvature flow, local regularity theorem, varifold.

Partially supported by JSPS Grant-in-aid for scientific research (B) #21340033, (S) #21224001 and challenging exploratory research #23654057.
new decisive input to the proof of [13] is Huisken’s monotonicity formula for MCF [10] and its variants which were not known at the time that Brakke obtained his result.

The purpose of the present paper is to extend the regularity result from $C^{1,\kappa}$ to $C^{2,\alpha}$ for Brakke’s MCF and more generally for weak MCF with $C^\alpha$ transport term. In the case of Brakke’s MCF, that is, the case that the transport term is identically equal to 0, $C^{2,\alpha}$ regularity implies $C^\infty$ almost everywhere by the standard linear parabolic regularity theory. This proves Brakke’s original claim of almost everywhere $C^\infty$ regularity for his MCF. We noted in [13] that there is an essential gap in [5] for the step of obtaining $C^2$ regularity from $C^{1,\kappa}$ (see [13, Sec. 10.1]). The present paper thus remedies the situation and proves that Brakke’s claim was correct after all. Just to avoid a possible confusion for the reader, we should point out that $C^{1,\kappa}$ regularity of [13] does not imply $C^{2,\alpha}$ simply by the standard linear parabolic regularity theory. This is because Brakke’s formulation only gives variational inequality even with $C^{1,\kappa}$ estimates, and not equality, thus requiring further nonlinear analysis different from simple applications of linear theory.

We briefly describe the method of proof. We first recall the method in [13] for the close relevance. For obtaining $C^{1,\kappa}$ regularity there, we used the so called blow-up argument. The essence of this argument is that one measures the deviation of moving varifolds from some graph of affine function and proves that the deviation is closely approximated by some graph of solution for the heat equation. If this can be established, then one has a way to take a much better affine function approximation to the moving varifolds in a smaller region. The iteration procedure then gives $C^{1,\kappa}$ estimate of the graph representing the support of moving varifolds. The strategy of the present paper is to measure the deviation of moving varifolds from some graph of polynomial function which is quadratic (respectively, linear) in the space (respectively, time) variables and which satisfies the heat equation, and to prove that the small deviation is closely approximated by some graph of solution for the heat equation. Then one can find a much better approximation by a similar polynomial function in a smaller region, and the iteration argument gives $C^{2,\alpha}$ estimates. The procedure takes advantage of $C^{1,\kappa}$ estimate of [13], another version of $L^2$-$L^\infty$ type estimate different from [13, Sec. 6.2], blow-up argument and it is similar to $C^{1,\kappa}$ estimate in spirit. Since we already know that the support of moving varifolds is a $C^{1,\kappa}$ graph, we need no Lipschitz graph approximation as was done in [13]. Thus the proof is less technical in that respect but more so due to the higher order approximations.

There have been numerous works [2, 6, 8, 9, 16] which show the existence of generalized MCF past singularities and global in time, and we see a significant advance of understandings for the special but important subclass of mean convex hypersurfaces [17, 20, 21]. Numerous works which have even more direct relations to Brakke’s MCF are singular perturbation limit problems such as the Allen-Cahn equation [11, 15] and the parabolic Ginzburg-Landau equation [3, 4, 12, 14]. See [13] for further discussion. We cite [7] as one of the best references for Brakke’s MCF.

The organization of the paper is as follows. Section 2 contains basic definitions and notations. Section 3 describes the assumptions and main results of the paper. Section 4 gives the supremum and Dirichlet energy estimate for the difference of heights between MCF graph and a certain quadratic function in terms of their $L^2$-norm in a larger domain. The estimate is essentially used in the subsequent Section 5, where a blow-up argument shows a decay estimate necessary for $C^{2,\alpha}$ estimate. Section 6 concludes the proof of $C^{2,\alpha}$ estimate and Section 7 describes the application to MCF in submanifold. The last Section 8 contains some technical estimates concerning the change of second derivatives under orthogonal rotations.
2. Preliminaries

Even though the content of this section is more or less identical to [13, Sec. 2], we include this section with a few changes for the reader's convenience.

2.1. Basic notations. Throughout this paper, \( k \) and \( n \) will be positive integers with \( 0 < k < n \). We often identify \( \mathbb{R}^k \) with \( \mathbb{R}^k \times \{ 0 \} \subset \mathbb{R}^n \). Let \( \mathbb{N} \) be the natural number and \( \mathbb{R}^+ := \{ x \geq 0 \} \). For \( 0 < r < \infty \) and \( a \in \mathbb{R}^n \) (or \( \mathbb{R}^k \)) let

\[
B_r(a) := \{ x \in \mathbb{R}^n : |x - a| < r \}, \quad B_r^k(a) := \{ x \in \mathbb{R}^k : |x - a| < r \}
\]

and when \( a = 0 \) let \( B_r := B_r(0) \) and \( B_r^k := B_r^k(0) \). We denote by \( \mathcal{H}^k \) the \( k \)-dimensional Hausdorff measure on \( \mathbb{R}^n \). The restriction of \( \mathcal{H}^k \) to a set \( A \) is denoted by \( \mathcal{H}^k|_A \). Set \( \omega_k := \mathcal{H}^k(B_1^k) \). For an open subset \( U \subset \mathbb{R}^n \) let \( C_c(U) \) be the set of all compactly supported continuous functions on \( U \) and let \( C_c(U; \mathbb{R}^n) \) be the set of all compactly supported, continuous vector fields. The upper subscript of \( C_c(U) \) and \( C_c(U; \mathbb{R}^n) \) indicates continuous \( l \)-th order differentiability. For \( g \in C^1(U; \mathbb{R}^n) \), we regard \( \nabla g(x) \) as an element of \( \text{Hom}(\mathbb{R}^n, \mathbb{R}^n) \). Similarly for \( g \in C^1(U) \), we regard the Hessian matrix \( \nabla^2 g(x) \) as an element of \( \text{Hom}(\mathbb{R}^n, \mathbb{R}^n) \). \( \nabla \) always indicates differentiation with respect to the space variables \( x \), and not with respect to the time variable \( t \).

For any Radon measure \( \mu \) on \( \mathbb{R}^n \) and \( \phi \in C_c(\mathbb{R}^n) \) we often write \( \mu(\phi) \) for \( \int_{\mathbb{R}^n} \phi \, d\mu \). Let \( \text{spt} \mu \) be the support of \( \mu \), i.e., \( x \in \text{spt} \mu \) if \( \mu(B_r(x)) > 0 \) for all \( r > 0 \). Let \( \Theta^k(\mu, x) \) be the \( k \)-dimensional density of \( \mu \) at \( x \), i.e., \( \lim_{r \to 0} \mu(B_r(x))/(\omega_k r^k) \), when the limit exists. For \( \mu \) a.e. defined function \( u \), and \( 1 \leq p \leq \infty \), \( u \in L^p(\mu) \) means \( (\int |u|^p \, d\mu)^{1/p} < \infty \).

For \( -\infty < t < s < \infty \) and \( x, y \in \mathbb{R}^n \), define

\[
(2.1) \quad \rho_{(y,s)}(x,t) := \frac{1}{(4\pi(s-t))^{k/2}} \exp \left( \frac{|x-y|^2}{4(s-t)} \right).
\]

\( \rho_{(y,s)} \) is the \( k \)-dimensional backward heat kernel.

2.2. The Grassmann manifold and varifolds. Let \( G(n, k) \) be the space of \( k \)-dimensional subspaces of \( \mathbb{R}^n \) and let \( A(n, k) \) be the space of \( k \)-dimensional affine planes of \( \mathbb{R}^n \). For \( S \in G(n, k) \), we identify \( S \) with the corresponding orthogonal projection of \( \mathbb{R}^n \) onto \( S \). Let \( S^\perp \in G(n, n-k) \) be the orthogonal complement of \( S \). For two elements \( A \) and \( B \) of \( \text{Hom}(\mathbb{R}^n, \mathbb{R}^n) \), define a scalar product \( A \cdot B := \text{trace} (A^* \circ B) \) where \( A^* \) is the transpose of \( A \) and \( \circ \) indicates the usual composition. The identity of \( \text{Hom}(\mathbb{R}^n, \mathbb{R}^n) \) is denoted by \( I \). Let \( a \otimes b \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^n) \) be the tensor product of \( a, b \in \mathbb{R}^n \). For \( A \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^n) \) define

\[
|A| := \sqrt{A \cdot A}, \quad \| A \| := \sup \{|A(x) : x \in \mathbb{R}^n, |x| = 1\}.
\]

For \( T \in G(n, k) \), \( a \in \mathbb{R}^n \) and \( 0 < r < \infty \) we define the cylinder

\[
C(T, a, r) := \{ x \in \mathbb{R}^n : |T(x - a)| < r \}, \quad C(T, r) := C(T, 0, r).
\]

We recall some notions related to varifold and refer to [1, 13] for more details. For any open set \( U \subset \mathbb{R}^n \), define \( G_k(U) := U \times G(n, k) \). A general \( k \)-varifold in \( U \) is a Radon measure on \( G_k(U) \). Set of all general \( k \)-varifolds in \( U \) is denoted by \( \mathbf{V}_k(U) \). For \( V \in \mathbf{V}_k(U) \), let \( \| V \| \) be the mass measure of \( V \), namely,

\[
\| V \|(\phi) := \int_{G_k(U)} \phi(x) \, dV(x, S), \quad \forall \phi \in C_c(U).
\]
Given any $\mathcal{H}^k$ measurable countably $k$-rectifiable set $M \subset U$ with locally finite $\mathcal{H}^k$ measure, there is a natural $k$-varifold $|M| \in V_k(U)$ defined by

$$|M|(\phi) := \int_M \phi(x, \text{Tan}_x M) d\mathcal{H}^k(x), \quad \forall \phi \in C_c(G_k(U)),$$

where $\text{Tan}_x M \in G(n, k)$ is the approximate tangent space which exists $\mathcal{H}^k$ a.e. on $M$. In this case, $|\cdot|_{\mathcal{H}^k} = |\cdot|_{\mathcal{H}^k \setminus M}$. We say $V \in V_k(U)$ is integral if

$$V(\phi) = \int_M \phi(x, \text{Tan}_x M)\theta(x) d\mathcal{H}^k(x), \quad \forall \phi \in C_c(G_k(U)),$$

with some $\mathcal{H}^k$ measurable countably $k$-rectifiable set $M \subset U$ and $\mathcal{H}^k$ a.e. integer-valued integrable function $\theta$ defined on $M$. Note that for such varifold, $\Theta^k(\|V\|, x) = \theta(x) \in \mathbb{N}$, $\mathcal{H}^k$ a.e. on $M$. Set of all integral $k$-varifolds in $U$ is denoted by $IV_k(U)$. We say $V$ is a unit density $k$-varifold if $V$ is integral and $\theta = 1$ a.e. on $M$, that is, $V = |M|$. When $V$ is integral, we often write $\int_U (g(x))^+ d\|V\|(x)$ for $\int_{G_k(U)} S^+(g(x)) dV(x, S)$, for example, since there should be no ambiguity.

2.3. **First variation and generalized mean curvature.** For $V \in V_k(U)$ let $\delta V$ be the first variation of $V$, namely,

$$\delta V(g) := \int_{G_k(U)} \nabla g(x) \cdot S \, dV(x, S)$$

for $g \in C^1_c(U; \mathbb{R}^n)$. Let $\|\delta V\|$ be the total variation when it exists, and if $\|\delta V\|$ is absolutely continuous with respect to $\|V\|$, we have for some $\|V\|$ measurable vector field $h(V, \cdot)$

$$\delta V(g) = - \int_U g(x) \cdot h(V, x) \, d\|V\|(x).$$

(2.2)

The vector field $h(V, \cdot)$ is called the generalized mean curvature of $V$. We say $V$ is stationary if $h(V, \cdot) = 0$, $\|V\|$ a.e. in $U$, or equivalently, $\delta V(g) = 0$ for all $g \in C^1_c(U; \mathbb{R}^n)$. For any $V \in IV_k(U)$ with integrable $h(V, \cdot)$, Brakke’s perpendicularity theorem of generalized mean curvature [5, Chapter 5] says that we have

$$\int_U (g(x))^+ \cdot h(V, x) \, d\|V\|(x) = \int_U g(x) \cdot h(V, x) \, d\|V\|(x)$$

(2.3)

for all $g \in C_c(U; \mathbb{R}^n)$.

2.4. **The right-hand side of MCF equation.** For any $V \in V_k(U)$, $u \in L^2(\|V\|)$ and $\phi \in C^1_c(U; \mathbb{R}^+)$, define

$$\mathcal{B}(V, u, \phi) := \int_U (-\phi(x)h(V, x) + \nabla \phi(x)) \cdot (h(V, x) + (u(x))^+) \, d\|V\|(x)$$

(2.4)

when $V \in IV_k(U)$, $\|\delta V\|$ is locally finite and absolutely continuous with respect to $\|V\|$, and $h(V, \cdot) \in L^2(\|V\|)$. Otherwise we define $\mathcal{B}(V, u, \phi) = -\infty$. Formally, if a family of smooth $k$-dimensional surfaces $\{M_t\}$ moves by the velocity equal to the mean curvature plus smooth $u$, then, one can check that $V_t = |M_t|$ satisfies

$$\frac{d}{dt}\|V_t\|(\phi) \leq \mathcal{B}(V_t, u(\cdot, t), \phi), \quad \forall \phi \in C^1_c(U; \mathbb{R}^+).$$

(2.5)
In fact, (2.5) holds with equality. Conversely, if (2.5) is satisfied, then one can prove that the velocity is equal to the mean curvature plus $u$. If we allow the time-varying test function $\phi \in C^{1}(U \times (0, \infty); \mathbb{R}^+)$ with $\phi(\cdot, t) \in C^1_c(U)$, one can check that we also have
\begin{equation}
\frac{d}{dt}\|V_t\|(\phi(\cdot, t)) \leq \mathcal{B}(V_t, u(\cdot, t), \phi(\cdot, t)) + \int \frac{\partial \phi}{\partial t}(\cdot, t) \, d\|V_t\|.
\end{equation}
This inequality (2.6) motivates the integral formulation of the motion law (3.3) below.

2.5. Notations related to norms. For $0 < \alpha < 1$, $U \subset \mathbb{R}^n$, $-\infty < t_1 < t_2 < \infty$ and for any function $f : U \times (t_1, t_2) \to \mathbb{R}$ we define the $\alpha$-Hölder semi-norm
\[ [f]_\alpha := \sup_{x, y \in U, t_1 < s_1 < s_2 < t_2} \frac{|f(x, s_1) - f(y, s_2)|}{|x - y|^{\alpha} \max\{|s_2 - s_1|^{\alpha/2}\}}.\]

Though we do not write out the domain of $f$ for the notation, we always implicitly assume that the supremum is taken over the domain. We similarly define $[\cdot]_\alpha$ for vector-valued functions and matrix-valued functions. For $f : U \times (t_1, t_2) \to \mathbb{R}$ (or $\mathbb{R}^n$) we also take the liberty of denoting
\[ \|f\|_0 := \sup_{x \in U, t \in (t_1, t_2)} |f(x, t)|\]
since we use sup norm quite often. Whenever it is important for clarity to specify the domain of definition, we write out the information. We also define the $\alpha$-Hölder norm
\[ \|f\|_\alpha (= \|f\|_{C^0(U \times (t_1, t_2))}) := \|f\|_0 + [f]_\alpha.\]

We note that we have some occasions to define $[f]_\alpha$ differently so that it becomes scale invariant. This will be specified individually.

3. Main results

3.1. Assumptions. For an open set $U \subset \mathbb{R}^n$ and $0 < \Lambda \leq \infty$ suppose that we have a family of $k$-varifolds $\{V_t\}_{0 \leq t < \Lambda}$ and a family of $n$-vector valued functions $\{u(\cdot, t)\}_{0 \leq t < \Lambda}$ both on $U$ satisfying the followings.

(B1) For a.e. $t \in [0, \Lambda)$, $V_t$ is a unit density $k$-varifold.

(B2) For $\bar{U} \subset U$ and $(t_1, t_2) \subset (0, \Lambda)$,
\begin{equation}
\sup_{t_1 \leq t \leq t_2} \|V_t\|(\bar{U}) < \infty.
\end{equation}

(B3) For $0 < \alpha < 1$ assume that $u$ is locally $\alpha$-Hölder continuous, namely for any $\bar{U} \subset U$ and $(t_1, t_2) \subset (0, \Lambda)$,
\begin{equation}
\|u\|_{C^0(\bar{U} \times (t_1, t_2))} < \infty.
\end{equation}

(B4) For all $\phi \in C^1(U \times [0, \Lambda); \mathbb{R}^+)$ with $\phi(\cdot, t) \in C^1_c(U)$ and $0 \leq t_1 < t_2 < \Lambda$, we have
\begin{equation}
\|V_{t_2}\|(\phi(\cdot, t_2)) - \|V_{t_1}\|(\phi(\cdot, t_1)) \leq \int_{t_1}^{t_2} \mathcal{B}(V_t, u(\cdot, t), \phi(\cdot, t)) \, dt + \int_{t_1}^{t_2} \int_U \frac{\partial \phi}{\partial t}(\cdot, t) \, d\|V_t\| \, dt.
\end{equation}

Remark 3.1. As is stated in the previous section, (B4) is a weak integral form of the motion law: velocity = mean curvature + $u$. In particular, if $u = 0$, it is Brakke’s MCF in an integral form. If there exists $\bar{U} \subset U$ such that $\text{spt} \|V_t\| \subset \bar{U}$ for all $t \in [0, \Lambda)$, then we do not need to assume (B2). In this case, (B2) is satisfied automatically. This can be proved easily: choose $\phi \in C^1_c(U; \mathbb{R}^+)$ with $\phi \equiv 1$ on $\bar{U}$ and use (3.3) and the Hölder inequality to
show that $\frac{d}{dt} \|V_t\|(\tilde{U}) \leq \|u\|_{\alpha}^2 \|V_t\|(\tilde{U})$, which gives a uniform bound \( 3.1 \). If we work under periodic boundary conditions (i.e., $U = \mathbb{T}^n$, for example, where $\mathbb{T}^n$ is the $n$-dimensional torus), we do not need (B2) by the same reason.

3.2. Partial regularity.

**Definition 3.2.** A point $x \in U \cap \text{spt} \|V_t\|$ is said to be a $C^{2,\alpha}$ regular point if there exists some open neighborhood $O$ in $\mathbb{R}^n$ containing $x$ such that $O \cap \text{spt} \|V_t\|$ is an embedded $k$-dimensional manifold represented as the graph of $f(\cdot, s) : B^k_R \to O$ for $s \in (t - R^2, t + R^2)$ for some $R > 0$ and with

$$\|f\|_0 + \|\nabla f\|_0 + \|\nabla^2 f\|_\alpha + \|\partial f / \partial s\|_\alpha < \infty.$$ 

**Theorem 3.3.** Under the assumptions (B1)-(B4), for a.e. $t \in (0, \Lambda)$, there exists a (possibly empty) closed set $G_t \subset \text{spt} \|V_t\|$ with $\mathcal{H}^k(G_t) = 0$ such that $\text{spt} \|V_t\| \setminus G_t$ is a set of $C^{2,\alpha}$ regular points. Moreover, we have the motion law in the classical sense, namely, the normal velocity vector is equal to the sum of the mean curvature vector and $u^\perp$ at each $C^{2,\alpha}$ regular point.

**Remark 3.4.** For $u = 0$, Theorem 3.3 combined with the standard linear regularity theory proves that the above $f$ is $C^\infty$ on the set of $C^{2,\alpha}$ regular points. This proves ‘almost everywhere regularity’ of unit density Brakke’s MCF.

3.3. Local regularity theorem. To describe the local regularity theorem, we need the following (cf. [13, Def. 5.1])

**Definition 3.5.** Fix $\phi \in C^\infty([0, \infty))$ such that $0 \leq \phi \leq 1$,

$$\phi(x) = \begin{cases} 1 & \text{for } 0 \leq x \leq (2/3)^{1/k}, \\ > 0 & \text{for } 0 \leq x < (5/6)^{1/k}, \\ = 0 & \text{for } x \geq (5/6)^{1/k}. \\ \end{cases}$$

For $0 < R < \infty$, $x \in \mathbb{R}^n$ and $T \in \mathbf{G}(n, k)$ define

$$(3.4) \quad \phi_{T,R}(x) := \phi(R^{-1}|T(x)|), \quad c := \int_T \phi_{T,1}^2 d\mathcal{H}^k(\cdot) = R^{-k} \int_T \phi_{T,R}^2 d\mathcal{H}^k \quad \text{for all } R > 0.$$ 

With this we have the following

**Theorem 3.6.** Corresponding to $k, n, 1 \leq E_0 < \infty, 0 < \nu < 1, 0 < \alpha < 1$, there exist $0 < \varepsilon_0 < 1, 0 < \sigma_0 \leq 1/2, 2 < \Lambda_0 < \infty$ and $1 < c_0 < \infty$ with the following property. For $T \in \mathbf{G}(n, k), 0 < R < \infty, U = C(T, 3R)$ and $(0, \Lambda)$ replaced by $(-\Lambda_0 R^2, \Lambda_0 R^2)$, suppose that $\{V_t\}_{-\Lambda_0 R^2 \leq t \leq \Lambda_0 R^2}$ and $\{u(\cdot, t)\}_{-\Lambda_0 R^2 \leq t \leq \Lambda_0 R^2}$ satisfy (B1)-(B4). Suppose

$$(3.5) \quad \sup_{-\Lambda_0 R^2 \leq t \leq \Lambda_0 R^2} R^{-k} \|V_t\|(C(T, 3R)) \leq E_0.$$ 

$$(3.6) \quad \mu := \left( R^{-(k+4)} \int_{-\Lambda_0 R^2}^{\Lambda_0 R^2} \int_{C(T, 3R)} |T^\perp(x)|^2 d\|V_t\| dt \right)^\frac{1}{4} < \varepsilon_0,$$

$$(3.7) \quad \|u\|_\alpha := R\|u\|_0 + R^{1+\alpha}[u]_\alpha < \varepsilon_0,$$

$$(3.8) \quad (-\Lambda_0 + 3/2)R^2 \leq \exists t_1 \leq (-\Lambda_0 + 2)R^2 : R^{-k}\|V_{t_1}\|((\phi_{T,R}^2) < (2 - \nu)c,$$

$$(3.9) \quad (\Lambda_0 - 2)R^2 \leq \exists t_2 \leq (\Lambda_0 - 3/2)R^2 : R^{-k}\|V_{t_2}\|((\phi_{T,R}^2) > \nu c.$$
Denote \( D := (T \cap B_{\sigma_0 R}) \times (-R^2/4, R^2/4) \). Then there are \( f : D \to T^1 \) and \( F : D \to \mathbb{R}^n \) such that \( T(F(y, t)) = y \) and \( T^2(F(y, t)) = f(y, t) \) for all \( (y, t) \in D \),

\[
\text{spt } |V_i| \cap C(T, \sigma_0 R) = \text{image } F(\cdot, t) \quad \forall t \in (-R^2/4, R^2/4),
\]

(3.10) \( f \) is twice differentiable w.r.t. \( x \) and differentiable w.r.t. \( (x, t) \) on \( D \),

(3.11) \[
R\left\| R^{-2} f, R^{-1} \nabla f, \nabla^2 f, \frac{\partial f}{\partial t} \right\|_0 + R^{1+\alpha} \left[ \nabla^2 f, \frac{\partial f}{\partial t} \right]_\alpha \leq c_0 \max \{\mu, \|u\|_\alpha\}.
\]

Moreover the motion law (normal velocity = mean curvature vector + \( u^\perp \)) is satisfied on image \( F \).

\[ (3.6) \] requires smallness of deviation from \( k \)-dimensional plane in a weak measure-theoretic sense. \[ (3.8) \] excludes the possibility that there may be two or more almost parallel \( k \)-dimensional planes which may not move for the whole time. Obviously, for such case, we cannot hope to represent the graph as a univalent function. The idea of having possibly large \( \Lambda_0 \) is that, if we have a mass strictly less than that of 2 sheets of \( k \)-dimensional planes near the beginning, we will have a nice univalent representation of graph after sufficiently long time. Asking a certain mass lower bound \[ (3.9) \] is also natural since \( V_i = 0 \) for all time would satisfy (B1)-(B4) as well as \[ (3.3) \] and \[ (3.8) \]. Since one can always set \( V_i = 0 \) after any instance and still obtain a solution satisfying \[ (3.3) \], we need to impose \[ (3.9) \] towards the end of the time interval.

4. \( L^2\)-\( L^\infty \) estimate

In this section we first define function \( Q_g \), which is a (square of) distance function from a graph of solution of the heat equation, roughly speaking. We then prove that the \( L^2 \) norm of \( Q_g \) controls such distance function in sup-norm in Proposition \[ 4.3 \] which is analogous to \( L^2\)-\( L^\infty \) estimate of \[ 1.3 \] Prop. 6.4]. The Dirichlet energy of the distance is also similarly controlled. Throughout this section let \( T \in G(n, k) \) be the projection matrix corresponding to \( \mathbb{R}^k \times \{0\} \).

Suppose that we are given a function \( g = (g_{k+1}, \cdots, g_n) \) defined on \( \mathbb{R}^n \times \mathbb{R} \) with the following conditions. For each \( l = k + 1, \cdots, n \),

\[
g_l(x_1, \cdots, x_n, t) = a_l + b_l t + \sum_{i=1}^{k} a_{li} x_i + \frac{1}{2} \sum_{i,j=1}^{k} a_{lij} x_i x_j
\]

for some \( a_l, b_l, a_{li}, a_{lij} \in \mathbb{R} \) with \( a_{lij} = a_{jil} \) for all \( 1 \leq i, j \leq k \). Note that \( g_l \) depends only on \( t \) and \( x_1, \cdots, x_k \) and we often consider \( g_l \) as a function defined on \( \mathbb{R}^k \times \mathbb{R} \). We additionally assume that

\[
b_l = \sum_{i=1}^{k} a_{li}.
\]

Equivalently, each component function \( g_l \) of \( g \) satisfies the heat equation \( \frac{\partial g_l}{\partial t} = \Delta g_l \). We next define

Definition 4.1. If \( g = (g_{k+1}, \cdots, g_n) \) satisfies \[ (4.1) \] and \[ (4.2) \], then we write \( g \in \mathcal{F} \).
For \( g \in \mathcal{F} \), we define a function \( Q_g \) defined on \( \mathbb{R}^n \times \mathbb{R} \) by

\[
(4.3) \quad Q_g(x, t) := \frac{1}{2} \sum_{l=k+1}^{n} (x_l - g_l(x, t))^2.
\]

Note that \((2Q_g)^{1/2}\) is the vertical distance of the point \( x \) from the graph of \( g \). The expectation is that the MCF should be closely approximated by the solution of the heat equation. We next need the following technical lemma.

**Lemma 4.2.** There exists \( 1 < c_1(n, k) < \infty \) with the following property. Suppose a function \( f = (f_{k+1}, \cdots, f_n) : B^k_1 \times (-1, 1) \to \mathbb{R}^{n-k} \) with continuous \( \nabla f \) is given. Define \( M_t := \text{graph } f(\cdot, t) \subset \mathbb{R}^n \). We assume that

\[
(4.4) \quad \sup_{B^k_1 \times (-1, 1)} |\nabla f| \leq 1.
\]

Suppose \( g = (g_{k+1}, \cdots, g_n) \in \mathcal{F} \) is given and for each \( l = k+1, \cdots, n \) define \( g_l : B^k_1 \times (-1, 1) \to \mathbb{R}^n \) by

\[
(4.5) \quad g_l := (\frac{\partial g_l}{\partial x_1}, \cdots, \frac{\partial g_l}{\partial x_k}, 0, \cdots, -1, \cdots, 0),
\]

where \(-1\) is in the \( l\)-th component of \( g_l \). For each \( (x, t) \in B^k_1 \times (-1, 1) \) let \( S = S(x, t) \in \mathbf{G}(n, k) \) be the tangent space \( T_{(x, f(x, t))} M_t \). Then we have

\[
(4.6) \quad \frac{\partial Q_g}{\partial t} - S \cdot \nabla^2 Q_g \leq c_1 Q_g^{1/2} |\nabla f|^2 |\nabla^2 g| - \frac{1}{4k} |\nabla f - \nabla g|^2 - \frac{1}{2} \sum_{l=k+1}^{n} |S(g_l)|^2
\]

and

\[
(4.7) \quad \frac{\partial g_l}{\partial t} - S \cdot \nabla^2 g_l \leq c_1 |\nabla f|^2 |\nabla^2 g_l|.
\]

Note that \( \frac{\partial Q_g}{\partial t}, \nabla^2 Q_g \) and \( Q_g^{1/2} \) are evaluated at \( (x, f(x, t)) \in M_t \) in \((4.6)\).

**Proof.** One checks that

\[
(4.8) \quad \nabla^2 Q_g = \sum_{l=k+1}^{n} (g_l \otimes g_l - (x_l - g_l) \nabla^2 g_l)
\]

where \( \nabla^2 g_l \) is the \( n \times n \) matrix with non-zero components only in the upper-left \( k \times k \) sub-matrix. Due to \((4.2)\) and \((4.8)\), we have

\[
(4.9) \quad \frac{\partial Q_g}{\partial t} - S \cdot \nabla^2 Q_g = \sum_{l=k+1}^{n} (-S \cdot (g_l \otimes g_l) + (x_l - g_l)(T - S) \cdot \nabla^2 g_l).
\]

We estimate each term of the right-hand side of \((4.9)\). For the first term, since \( S \in \mathbf{G}(n, k) \), we have \( S \cdot (g_l \otimes g_l) = |S(g_l)|^2 \). Fix any \( l = k+1, \cdots, n \) and \( j = 1, \cdots, k \). Since \( S \) is the tangent space of graph \((f_{k+1}, \cdots, f_n)\), note that \( S \) contains \( f_j := (0, \cdots, 1, \cdots, 0, \frac{\partial f_{k+1}}{\partial x_j}, \cdots, \frac{\partial f_n}{\partial x_j}) \), where \( 1 \) is in the \( j \)-th component of \( f_j \). Thus we may conclude that

\[
(4.10) \quad S \cdot (g_l \otimes g_l) = |S(g_l)|^2 \geq |g_l| \cdot |f_j|^2/|f_j|^2 = |\frac{\partial g_l}{\partial x_j} - \frac{\partial f_l}{\partial x_j}|^2 \left(1 + \left|\frac{\partial f_j}{\partial x_j}\right|^2\right).
\]
By (4.4) and summing over \( j \) and \( l \), we obtain from (4.10)

\[
\sum_{l=k+1}^{n} S \cdot (g_l \otimes g_l) \geq \frac{1}{2k} |\nabla f - \nabla g|^2.
\]

In particular, from (4.11), we obtain

\[
\sum_{l=k+1}^{n} S \cdot (g_l \otimes g_l) \geq \frac{1}{4k} |\nabla f - \nabla g|^2 + \frac{1}{2} \sum_{l=k+1}^{n} |S(g_l)|^2.
\]

For the second term of (4.9), we need to know the expression of \( T - S \). The \( k \)-dimensional space corresponding to \( S \) is spanned by \( f_1, \ldots, f_k \). Consider the Gram-Schmidt orthonormalization \( \hat{f}_1, \ldots, \hat{f}_k \) of \( f_1, \ldots, f_k \), namely, \( \hat{f}_1 = f_1/|f_1| \), \( \hat{f}_2 = f_2 - (f_2 \cdot \hat{f}_1)\hat{f}_1 \), \( \hat{f}_2 = \hat{f}_2/|\hat{f}_2| \), \ldots, \( \hat{f}_k = f_k - \sum_{j=1}^{k-1}(f_k \cdot \hat{f}_j)\hat{f}_j \). Then \( S = \sum_{j=1}^{k} \hat{f}_j \otimes \hat{f}_j \). It is not difficult to check that each entry of the upper-left \( k \times k \) sub-matrix of \( T - S \) is bounded by some constant times \( |\nabla f|^2 \), where the constant depends only on \( k \) and \( n \). The reason is as follows. The first \( k \) components of \( \hat{f}_j \) are \( O(|\nabla f|^2), \ldots, O(|\nabla f|^2), 1, 0, \ldots, 0 \), where 1 is in the \( j \)-th component. The last \( n-k \) components of \( \hat{f}_j \) are \( O(|\nabla f|) \). The division by \( 1/\sqrt{1 + O(|\nabla f|^2)} \) for normalization does not change the order of magnitude except that the \( j \)-th component turns \( 1 + O(|\nabla f|^2) \). One sees that the next vector \( \hat{f}_{k+1} \) has the same property. Thus for each \( j = 1, \ldots, k \), \( \hat{f}_j \otimes \hat{f}_j \) has \( O(|\nabla f|^2) \) components for the upper-left \( k \times k \) sub-matrix except for the \( j \)-th component, which is \( 1 + O(|\nabla f|^2) \). Since \( T \) has 1 in the diagonal components for the upper-left \( k \times k \) sub-matrix, we have the above stated property. Note that we only need to consider such entries since \( \nabla^2 g \) has non-zero entries only there. Thus with (4.9) and (4.12), we obtain (4.6). The derivation for (4.7) is similar, which only requires the estimate for \( (T - S) \cdot \nabla^2 g \).

**Proposition 4.3.** There exists \( c_2 = c_2(n, k) \) with the following property. Suppose that \( \{V_t\}_{t < \zeta} \) and \( \{u(t)\}_{t < \zeta} \), where \( V_t = |M_t| \) with \( M_t = \text{graph} f(\cdot, t) \), satisfy (B1) and (B4) on \( C(T, 1) \times (-1, 1) \). Let \( g \in \mathcal{F} \) be given with \( Q_g \) as in (4.3). In addition, assume (4.4),

\[
(4.13) \quad \sup_{B_1^k \times (-1, 1)} |\nabla g| \leq 1
\]

and

\[
(4.14) \quad \|u\|_0 := \sup_{C(T, 1) \times (-1, 1)} |u| \leq 1.
\]

Then we have

\[
(4.15) \quad \sup_{B_1^k \times (-3/4, 1)} |f - g|^2 + \int_{-3/4}^{1} \int_{B_1^k \times (-3/4, 1)} |\nabla f - \nabla g|^2 d\mathcal{H}^k dt \leq c_2 \left( \int_{-1}^{1} \int_{C(T, 1)} Q_g d\|V_t\| dt + \|u\|_0^2 + \|\nabla f\|_0^2 + \|\nabla^2 g\|_0^2 \right).
\]

**Proof.** In the proof let \( \eta \in C^\infty(B_1^k \times (-1, 1)) \) be a non-negative function with \( \eta \equiv 1 \) on \( B_3^k \times (-7/8, 1) \), \( \eta \equiv 0 \) on \( B_3^k \times (-1, 1) \setminus B_1^k \times (-15/16, 1) \), \( 0 \leq \eta \leq 1 \) and \( |\nabla \eta|, |\nabla^2 \eta|, |\partial \eta/\partial t| \leq c(k) \). We then re-define \( \eta(x, t) := \eta(T(x, t)) \) for \( (x, t) \in C(T, 1) \times (-1, 1) \). For \( (y, s) \in C(T, 1/2) \times (-3/4, \infty) \), we use \( \phi(x, t) = Q_y(x, t)\rho(y, s)(x, t)\eta(x, t) \) in (4.3), over the time
interval $t_1 = -1$ and $-1 < t_2 < \min\{s, 1\}$. We then obtain (writing $\rho_{(y,s)}(x, t)$ as $\rho$ and $Q_g(x, t)$ as $Q$)

\[
\int_{C(T, 1)} Q \rho \eta d\|V_i\| \Bigg|_{t=t_2} \leq \int_{-1}^{t_2} \int_{C(T, 1)} \left\{ -h \rho \eta Q + \nabla(\rho \eta Q) \cdot (h + u^\perp) + \frac{\partial}{\partial t}(Q \rho \eta) d\|V_i\| dt \right. 
\]

since $\eta = 0$ for $t = -1$. For a.e. $t \in (-1, t_2)$, we may compute the integrand of the right-hand side of (4.16) as follows. Here we use the perpendicularity of mean curvature (2.3) in deriving $\nabla \rho \cdot h = (\nabla \rho)^\perp \cdot h$.

\[
(4.16)
\]

\[
-\|h\|^2 \rho \eta Q + (\nabla \rho \cdot h) \eta Q + \rho \nabla(\eta Q) \cdot h + (-h \rho \eta Q + \eta Q \nabla \rho) \cdot u^\perp + \rho \nabla(\eta Q) \cdot u^\perp + \frac{\partial}{\partial t}(\rho \eta Q) \\
\leq -\rho \|h\|^2 \eta Q - (\nabla \rho \cdot h) \eta Q + \frac{|(\nabla \rho)^\perp|^2}{\rho} \eta Q + \rho \nabla(\eta Q) \cdot h \\
+ \rho \|h\|^2 \eta Q + \rho \eta Q |u|^2 + \rho \nabla(\eta Q) \cdot u^\perp + \frac{\partial}{\partial t}(\rho \eta Q).
\]

Thus we have

\[
\int_{C(T, 1)} Q \rho \eta d\|V_i\| \Bigg|_{t=t_2} \leq \int_{-1}^{t_2} \int_{C(T, 1)} \left\{ -h \rho \eta Q + \nabla(\eta Q) \cdot h + \frac{|(\nabla \rho)^\perp|^2}{\rho} \eta Q \\
+ \rho \eta Q |u|^2 + \rho \nabla(\eta Q) \cdot u^\perp + \frac{\partial}{\partial t}(\rho \eta Q) d\|V_i\| dt. 
\]

By (2.2), the first two terms of the right-hand side of (4.18) is

\[
\int_{-1}^{t_2} \int_{G_k(C(T, 1))} \nabla(\eta Q \nabla \rho) \cdot S - \nabla\{\rho \nabla(\eta Q)\} \cdot S dV_i(\cdot, S) dt 
\]

\[
= \int_{-1}^{t_2} \int_{G_k(C(T, 1))} (\nabla^2 \rho \cdot S) \eta Q - \rho \nabla^2(\eta Q) \cdot S dV_i(\cdot, S) dt.
\]

Using

\[
\nabla^2 \rho \cdot S + \frac{|(\nabla \rho)^\perp|^2}{\rho} + \frac{\partial \rho}{\partial t} = 0,
\]

we obtain from (4.18) and (4.19)

\[
\int_{C(T, 1)} Q \rho \eta d\|V_i\| \Bigg|_{t=t_2} \leq \int_{-1}^{t_2} \int_{G_k(C(T, 1))} \left\{ -\nabla^2(\eta Q) \cdot S + \frac{\partial}{\partial t}(Q \eta) \\
+ \eta Q |u|^2 + Q \nabla \eta \cdot u^\perp + \eta \nabla Q \cdot u^\perp \right\} \rho dV(\cdot, S) dt =: I_1 + I_2 + I_3 + I_4 + I_5.
\]

{
\textit{Estimate of }I_1 + I_2\textit{. The integrand of }I_1\textit{ is}
\[
\{ -QS \cdot \nabla^2 \eta - 2(\nabla \eta \otimes \nabla Q) \cdot S - \eta \nabla^2 Q \cdot S \}\rho.
\]

Note that, with the notation of (4.3), we have

\[
(4.22)
\]

\[
(\nabla \eta \otimes \nabla Q) \cdot S = \nabla \eta \cdot S(\nabla Q) = -\nabla \eta \cdot \sum_{l=k+1}^n (x_i - g_l)S(g_l).
\]

Thus, we obtain from (4.23) and the Cauchy-Schwarz inequality that

\[
-2(\nabla \eta \otimes \nabla Q) \cdot S \leq 2\sqrt{2}|\nabla \eta| |Q|^2 \left( \sum_{l=k+1}^n |S(g_l)|^2 \right)^{\frac{1}{2}} \leq \frac{1}{2} \sum_{l=k+1}^n |S(g_l)|^2 + 4Q^2 |\nabla \eta|^2.
\]

\[
(4.24)
\]
By (4.22), (4.24) and Lemma 1.2 we obtain
\begin{equation}
I_1 + I_2 \leq \int_{-1}^{t_2} \int_{C(T,1)} Q \rho(\|\nabla^2 \eta\| + 4 \frac{|\nabla \eta|^2}{\eta} + |\frac{\partial \eta}{\partial t}|) + c_1 Q \frac{\rho|\nabla f|^2 |\nabla^2 g| |\eta|}{\eta} - \frac{1}{4k} |\nabla f - \nabla g|^2 \rho |d||V_i||dt.
\end{equation}
(4.25)

For \( \rho = \rho_{(y,s)}(x,t) \) with \((y, s) \in C(T,1/2) \times (-3/4, \infty)\) and \((x,t) \notin \{ \eta = 1 \}\) we have \(|x-y| \geq 1/4 \) or \( s-t \geq 1/8 \). Thus we have
\begin{equation}
\rho(|\nabla^2 \eta| + 4 \frac{|\nabla \eta|^2}{\eta} + |\frac{\partial \eta}{\partial t}|) \leq c(k)
\end{equation}
(4.26)

for a suitable constant depending only on \( k \). Then by the Cauchy-Schwarz inequality, (4.25) and (4.26) give
\begin{equation}
I_1 + I_2 \leq \int_{-1}^{t_2} \int_{C(T,1)} c(k)Q + Q \rho |\nabla f|^2 |\nabla^2 g| |\eta| - \frac{1}{4k} |\nabla f - \nabla g|^2 \rho |d||V_i||dt.
\end{equation}
(4.27)

Since \( \int_{C(T,1)} \rho |d||V_i| \leq c(k) \) by (4.14), we obtain from (4.27)
\begin{equation}
I_1 + I_2 \leq \int_{-1}^{t_2} \int_{C(T,1)} c(k)Q + Q \rho |\nabla f|^2 |\nabla^2 g| |\eta| + 2c^2(k) |\nabla f|_0^2 |\nabla^2 g|_0^2.
\end{equation}
(4.28)

Estimate of \( I_5 \). We have by (4.14)
\begin{equation}
\int_{-1}^{t_2} \int_{C(T,1)} \eta Q |u|^2 \rho |d||V_i||dt \leq \int_{-1}^{t_2} \int_{C(T,1)} \eta Q |d||V_i||dt.
\end{equation}
(4.29)

Estimate of \( I_4 \). By (4.14) and since \( \rho \leq c(k) \) on the support of \( |\nabla \eta| \), we have for any \( t_2 \in (-1, 1) \)
\begin{equation}
\int_{-1}^{t_2} \int_{C(T,1)} Q \nabla \eta \cdot u^t \rho |d||V_i||dt \leq c(k) \int_{-1}^{1} \int_{C(T,1)} Q |d||V_i||dt.
\end{equation}
(4.30)

Estimate of \( I_5 \). By (4.13), one can check that \( |\nabla Q| \leq \sqrt{Q}c(k) \), thus
\begin{equation}
\int_{-1}^{t_2} \int_{C(T,1)} \eta |Q | \cdot u^t \rho |d||V_i||dt \leq c(k) |u|_0^2 + \int_{-1}^{t_2} \int_{C(T,1)} \eta Q |d||V_i||dt.
\end{equation}
(4.31)

Thus, with a suitable \( c = c(k) \), we have from (4.21), (4.28), (4.29)-(4.31)
\begin{equation}
\int_{C(T,1)} Q \rho |d||V_i||_{t=t_2} \leq c \left( \int_{-1}^{1} \int_{C(T,1)} Q |d||V_i||dt + \int_{-1}^{t_2} \int_{C(T,1)} Q \eta |d||V_i||dt \right.
\end{equation}
(4.32)

\[ + \| u |_0^2 + \| \nabla f |_0^2 |\nabla^2 g|_0^2 \right) - \frac{1}{4k} \int_{-1}^{t_2} \int_{C(T,1)} |\nabla f - \nabla g|^2 \eta |d||V_i||dt \]

for all \( t_2 \in (-1, \min(s, 1)) \). Note that the differential inequality \( F' \leq c(F + \tilde{c}) \) with \( F(-1) = 0 \) implies \( F(t) \leq \tilde{c}(e^{c(t+1)} - 1) \) and thus \( F'(t) \leq \tilde{c} e^{c(t+1)} \). Thus by dropping the last term of (4.32), we obtain
\begin{equation}
\int_{C(T,1)} Q \eta |d||V_i||_{t=t_2} \leq c e^{c(t+1)} \left( \int_{-1}^{1} \int_{C(T,1)} Q |d||V_i||dt + \| u |_0^2 + \| \nabla f |_0^2 |\nabla^2 g|_0^2 \right).
\end{equation}
(4.33)

For any \( t_2 \in (-3/4, 1) \) and \( y \in C(T,1/2) \cap \text{spt } |V_i| \), and arbitrarily small \( \varepsilon > 0 \), we use \( \rho = \rho_{(y,t_2+\varepsilon)}(x,t) \) in the above computation. Since \( \rho_{(y,t_2+\varepsilon)}(\cdot, t_2)||V_i|| \rightarrow \delta_y \) as \( \varepsilon \rightarrow 0 \), where \( \delta_y \) is the delta function at \( y \), and since \( \eta = 1 \) on \( C(T,1/2) \times (-3/4, 1) \), we conclude that the
first term of the left-hand side of (4.15) is bounded by the right-hand side of (4.13). For the second term of the right-hand side of (4.15), we use (4.32) with \( \rho \equiv 1 \). Note that the only property we used for the above computation is (1.20). Since

\[
\int_{-3/4}^{1} \int_{B_{1/2}^{k}} |\nabla f - \nabla g|^2 \, d\mathcal{H}^k \, dt \leq \int_{-3/4}^{1} \int_{C(T,1)} |\nabla f - \nabla g|^2 \, d\|V_t\| \, dt,
\]

we prove (4.15).

\[
\square
\]

5. A DECAY ESTIMATE BY BLOW-UP ARGUMENT

The main result of this section is the following Proposition 5.1 which shows that one can find a better approximation in \( \mathcal{F} \) in a smaller scale if the relevant quantities (5.1)-(5.4) are sufficiently small. It is similar to [13, Proposition 8.1], except that the decay we obtain here is \( \theta^{1+\alpha} \) instead of \( \theta^\alpha \).

**Proposition 5.1.** Corresponding to \( n, k \) and \( 0 < \alpha < 1 \) there exist \( 0 < \varepsilon_1 < 1 \), \( 0 < \theta < 1/4 \), \( 1 < c_3 < \infty \) with the following property. For \( 0 < R < \infty \), suppose \( \{V_t\}_{R_2 < t < R_2} \) and \( \{u(\cdot,t)\}_{R_2 < t < R_2} \), where \( V_t = |M_t| \) with \( M_t = \text{graph} f(\cdot,t) \), satisfy (B1) and (B4) on \( C(T,R) \times (-R^2,R^2) \) and \( g \in \mathcal{F} \) is given. Denote \( \| \cdot \|_0 := \sup_{R^2 < t < R^2} | \cdot | \). Assume that

\[
\|\nabla f\|_0 \leq \varepsilon_1,
\]

\[
\|\nabla g\|_0 + R \|\nabla^2 g\|_0 + R \|\frac{\partial g}{\partial t}\|_0 \leq \varepsilon_1,
\]

\[
u(0,0) = 0,
\]

\[
\mu := \left( R^{-(k+4)} \int_{-R^2}^{R^2} \int_{C(T,R)} Q_g \, d\|V_t\| \, dt \right)^{1/2} \leq \varepsilon_1.
\]

Then there exists \( \tilde{g} \in \mathcal{F} \) with

\[
R^{-1} \|g - \tilde{g}\|_0 + \|\nabla g - \nabla \tilde{g}\|_0 + R \|\nabla^2 g - \nabla^2 \tilde{g}\|_0 + R \|\frac{\partial g}{\partial t} - \frac{\partial \tilde{g}}{\partial t}\|_0 \leq c_3 \mu
\]

and

\[
\left( \theta R \right)^{-(k+4)} \int_{-\theta^2 R^2}^{\theta^2 R^2} \int_{C(T,\theta R)} Q_\tilde{g} \, d\|V_t\| \, dt \right)^{1/2} \leq \theta^{1+\alpha} \max \{ \mu, c_3 R^{1+\alpha} \}.
\]

**Proof.** After a change of variables, we may assume \( R = 1 \). Note that the statement is written in a scale invariant manner. If the claim were false, then for each \( m \in \mathbb{N} \) there exist \( \{V_t^{(m)}\}_{-1 < t < 1} \) (represented by \( f^{(m)} \)), \( \{u^{(m)}(\cdot,t)\}_{-1 < t < 1} \) satisfying (5.3), (B1) and (B4) on \( C(T,1) \times (-1,1) \) and \( g^{(m)} \in \mathcal{F} \) such that

\[
\|\nabla f^{(m)}\|_0 \leq \frac{1}{m},
\]

\[
\|\nabla g^{(m)}\|_0 + \|\nabla^2 g^{(m)}\|_0 + \|\frac{\partial g^{(m)}}{\partial t}\|_0 \leq \frac{1}{m},
\]

\[
\mu^{(m)} := \left( \int_{-1}^{1} \int_{C(T,1)} Q_{g^{(m)}} \, d\|V_t^{(m)}\| \, dt \right)^{1/2} \leq \frac{1}{m},
\]

\[
\mu^{(m)} := \left( \int_{-1}^{1} \int_{C(T,1)} Q_{g^{(m)}} \, d\|V_t^{(m)}\| \, dt \right)^{1/2} \leq \frac{1}{m}.
\]
but for any $g \in \mathcal{F}$ with
\begin{equation}
\|g - g^{(m)}\|_0 + \|\nabla g - \nabla g^{(m)}\|_0 + \|\nabla^2 g - \nabla^2 g^{(m)}\|_0 + \|\frac{\partial g}{\partial t} - \frac{\partial g^{(m)}}{\partial t}\|_0 \leq m\mu^{(m)},
\end{equation}
we have
\begin{equation}
\left(\theta^{-(k+4)} \int_{-\theta^2}^{\theta^2} Qg d\|V_t^{(m)}\|dt\right)^{\frac{1}{2}} > \theta^{1+\alpha} \max\{\mu^{(m)}, m[u^{(m)}]_\alpha, m\|\nabla f^{(m)}\|_0^2\}.
\end{equation}
Here, $0 < \theta < 1/4$ will be chosen depending only on $k, n$ and $\alpha$. By using $g = g^{(m)}$ (which satisfies (5.10) trivially), we obtain from (5.11)
\begin{equation}
\max\{[u^{(m)}]_\alpha, \|\nabla f^{(m)}\|_0^2\} \leq \frac{\theta^{- \frac{k+4}{2} - 1 - \alpha}\mu^{(m)}}{m}.
\end{equation}
Thus (5.12) shows
\begin{equation}
\lim_{m \to \infty} (\mu^{(m)})^{-1}[u^{(m)}]_\alpha = \lim_{m \to \infty} (\mu^{(m)})^{-1}\|\nabla f^{(m)}\|_0^2 = 0.
\end{equation}
Next we use Proposition 4.3 for $f = f^{(m)}$, $g = g^{(m)}$, $u = u^{(m)}$. The required conditions (4.4), (4.13) and (4.14) follow from (5.7), (5.8), (5.3) and (5.13). Then we have
\begin{equation}
\sup_{B_t^{1/2} \times (-3/4,1)} |f^{(m)} - g^{(m)}|^2 + \int_{-3/4}^{1/4} \int_{B_t^{1/2}} |\nabla f^{(m)} - \nabla g^{(m)}|^2 dH^k dt \leq 2c_2(\mu^{(m)})^2
\end{equation}
for all sufficiently large $m$ due to (4.15), (5.3), (5.8) and (5.13). We next define a sequence of renormalized functions
\begin{equation}
\tilde{f}^{(m)}(x, t) := (\mu^{(m)})^{-1}(f^{(m)}(x, t) - g^{(m)}(x, t))
\end{equation}
with $\tilde{f}^{(m)} = (\tilde{f}_{k+1}^{(m)}, \ldots, \tilde{f}_n^{(m)})$. From (5.14) and (5.15), we have
\begin{equation}
\sup_{B_t^{1/2} \times (-3/4,1)} |\tilde{f}^{(m)}|^2 + \int_{-3/4}^{1/4} \int_{B_t^{1/2}} |\nabla \tilde{f}^{(m)}|^2 dH^k dt \leq 2c_2 =: (c_4)^2
\end{equation}
for all sufficiently large $m$. By the standard compactness theorem, there exist a convergent subsequence (denoted by the same index) and a limit $\tilde{f} = (\tilde{f}_{k+1}, \ldots, \tilde{f}_n)$ such that
\begin{equation}
\tilde{f}^{(m)} \rightharpoonup \tilde{f} \quad \text{weakly in } L^2, \quad \nabla \tilde{f}^{(m)} \rightharpoonup \nabla \tilde{f} \quad \text{weakly in } L^2
\end{equation}
both on $B_t^{1/2} \times (-3/4,1)$ and
\begin{equation}
\|\tilde{f}\|_{L^\infty(B_t^{1/2} \times (-3/4,1))}^2 + \int_{-3/4}^{1/4} \int_{B_t^{1/2}} |\nabla \tilde{f}|^2 dH^k dt \leq (c_4)^2
\end{equation}
In addition, due to Rellich’s compactness theorem, we may choose such subsequence so that for a countable dense set $\{s_j\}_{j=1}^\infty \subset (-3/4,1)$,
\begin{equation}
\{\tilde{f}^{(m)}(\cdot, s_j)\}_{m=1}^\infty \quad \text{is a Cauchy sequence in } L^2(B_t^{1/2}) \quad \text{for all } j \in \mathbb{N}.
\end{equation}
We next claim

**Lemma 5.2.** Each component function $\tilde{f}_l$ is in $C^\infty(B_t^{1/2} \times (-3/4,1))$ and satisfies the heat equation,
\begin{equation}
\frac{\partial \tilde{f}_l}{\partial t} - \Delta \tilde{f}_l = 0
\end{equation}
on $B_t^{1/2} \times (-3/4,1)$.
Proof of Lemma 5.2

In the following we fix \( l \in \{ k + 1, \ldots, n \} \). Let \( \phi \in C^\infty_c(B^k_{1/2}; (\mathbb{R}^+)) \) be arbitrary and fixed. For \((x, t) \in C(T, 1/2) \times (-3/4, 1) \) and \( m \in \mathbb{N} \) define a function

\[
\phi^{(m)}(x, t) := (x_l - g_l^{(m)}(T(x), t) + 2c_4\mu^{(m)}(\phi(T(x), t)).
\]

Since \( x_l = f_l^{(m)}(x, t) \) for \( x \in \text{spt} \|V_t^{(m)}\| \), note that we have from (5.15) and (5.21)

\[
\phi^{(m)}(x, t) = \mu^{(m)}(f_l^{(m)}(x, t) + 2c_4\phi(T(x), t)
\]

for \( x \in \text{spt} \|V_t^{(m)}\| \). Thus, due to (5.16), \( \phi^{(m)} \) is non-negative on \( \text{spt} \|V_t^{(m)}\| \) for all \( t \in (-3/4, 1) \). Away from \( \bigcup_{t \in (-3/4, 1)}(C(T, 1/2) \cap \text{spt} \|V_t^{(m)}\|) \times \{ t \} \), we may modify \( \phi^{(m)} \) so that the modified \( \tilde{\phi}^{(m)} \) is non-negative smooth function with compact support in \( C(T, 1/2) \times (-3/4, 1) \). This modification justifies the use of \( \tilde{\phi}^{(m)} \) in (5.3) but does not affect the following computations since only the values in some neighborhood of spt \( \|V_t^{(m)}\| \) matter. With this modification, the substitution of \( \tilde{\phi}^{(m)} \) in (5.3) gives (denoting \( h(V_t^{(m)}, \cdot) \) by \( h^{(m)} \))

\[
0 \leq \int_{-3/4}^1 \int_{C(T, 1/2)} (-h^{(m)}\phi^{(m)} + \nabla \phi^{(m)}) \cdot (h^{(m)} + (u^{(m)})^\perp) + \frac{\partial \phi^{(m)}}{\partial t} d\|V_t^{(m)}\| dt.
\]

By the Cauchy-Schwarz inequality and dropping a negative term, (5.23) gives

\[
0 \leq \int_{-3/4}^1 \int_{C(T, 1/2)} |u^{(m)}|^2 \phi^{(m)} + |u^{(m)}| |\nabla \phi^{(m)}| + \mu^{(m)}(f_l^{(m)} + 2c_4) \nabla \phi \cdot h^{(m)}
\]

\[
+ \phi \nabla (x_l - g_l^{(m)}) \cdot h^{(m)} + \frac{\partial \phi^{(m)}}{\partial t} d\|V_t^{(m)}\| dt =: I_1^{(m)} + I_2^{(m)} + I_3^{(m)} + I_4^{(m)} + I_5^{(m)}.
\]

We subsequently identify \( \lim_{m \to \infty} (\mu^{(m)})^{-1} I_j^{(m)} \) for each \( j = 1, \ldots, 5 \).

Estimate of \( I_1^{(m)} \).

By (5.3) and (5.13), we have \( |u^{(m)}| = o(\mu^{(m)}) \). Moreover, by (5.16) and (5.22), we have \( |\phi^{(m)}| \leq 3c_4\mu^{(m)} \sup |\phi| \). Thus we have \( |u^{(m)}|^2 |\phi^{(m)}| = o((\mu^{(m)})^3) \) and since \( \|V_t^{(m)}\|(C(T, 1/2)) \) is uniformly bounded, we have

\[
\lim_{m \to \infty} (\mu^{(m)})^{-1} I_1^{(m)} = 0.
\]

Estimate of \( I_2^{(m)} \).

By (5.8) and (5.21), one observes that \( |\nabla \phi^{(m)}| \leq O(1) \) on \( \text{spt} \|V_t^{(m)}\| \). Since \( |u^{(m)}| = o(\mu^{(m)}) \), we conclude that

\[
\lim_{m \to \infty} (\mu^{(m)})^{-1} I_2^{(m)} = 0.
\]

Estimate of \( I_3^{(m)} \).

Choose \( \tilde{\phi} \in C^\infty_c(B_{1/2}^k; \mathbb{R}^+) \) such that \( \tilde{\phi} = 1 \) on \( \text{spt} \phi(\cdot, t) \) for all \( t \in (-3/4, 1) \). We also re-define \( \tilde{\phi}(x, t) := \tilde{\phi}(T(x)) \) for \( x \in C(T, 1/2) \). Take \(-3/4 < t_1 < t_2 < 1\) so that \( \text{spt} \phi \subset B_{1/2}^k \times (t_1, t_2) \). We first claim that

\[
\lim_{m \to \infty} \int_{t_1}^{t_2} \int_{C(T, 1/2)} \tilde{\phi}|h^{(m)}|^2 d\|V_t^{(m)}\| dt = 0.
\]
For the proof, using $\tilde{\phi}$ in (5.3), we have

$$
(5.28) \quad \int_{C(T,1/2)} \tilde{\phi} \, d\|V_{1}^{(m)}\|_{t} \bigg|_{t=t_{1}}^{t_{2}} \leq \int_{t_{1}}^{t_{2}} \int_{C(T,1/2)} (-h^{(m)} \tilde{\phi} + \nabla \tilde{\phi} \cdot (h^{(m)} + (u^{(m)})^{\perp}) \, d\|V_{1}^{(m)}\|_{t} \, dt.
$$

By (5.7), we have $d\|V_{1}^{(m)}\| \to d\mathcal{H}^{k}|_{T}$ uniformly in $t$ and the left-hand side of (5.28) converges to 0 as $m \to \infty$. The right-hand side of (5.28) is bounded from above by

$$
(5.29) \quad \int_{t_{1}}^{t_{2}} \int_{C(T,1/2)} -h^{(m)}|\tilde{\phi}|^{2} - |u^{(m)}|^{2} \tilde{\phi} + \frac{1}{4} |h^{(m)}|^{2} \tilde{\phi} + |u^{(m)}| \|\nabla \tilde{\phi}\| + |h^{(m)}|(\nabla \tilde{\phi})^{\perp} \, d\|V_{1}^{(m)}\|_{t} \, dt
$$

$$
\leq \int_{t_{1}}^{t_{2}} \int_{C(T,1/2)} -\frac{1}{2} h^{(m)}|\tilde{\phi}|^{2} + |u^{(m)}|^{2} \tilde{\phi} + |u^{(m)}| \|\nabla \tilde{\phi}\| + \frac{1}{\tilde{\phi}} (\nabla \tilde{\phi})^{\perp} \, d\|V_{1}^{(m)}\|_{t} \, dt,
$$

where we used (2.3). Terms involving $u^{(m)}$ converge to 0 since $|u^{(m)}| = o(\mu^{(m)})$. We also have $(\nabla \tilde{\phi})^{\perp} = (T - \text{image} \nabla f^{(m)})/(\nabla \tilde{\phi})$ since $\nabla \tilde{\phi} = T(\nabla \phi)$. By (5.7), we have

$$
(5.30) \quad \frac{|(\nabla \tilde{\phi})^{\perp}|^{2}}{\tilde{\phi}} \leq \|T - \text{image} \nabla f^{(m)}\|^{2} \frac{|\nabla \tilde{\phi}|^{2}}{\tilde{\phi}} \to 0
$$

as $m \to \infty$ uniformly in $(x, t)$. Combining (5.28)-(5.30), we prove (5.27). Since we took $\tilde{\phi}$ so that spt $\tilde{\phi} \subset \{\hat{\phi} = 1\}$, (5.16) and (5.27) show that

$$
(5.31) \quad \lim_{m \to \infty} (\mu^{(m)})^{-1} I_{3}^{(m)} = 0.
$$

Estimate of $I_{4}^{(m)} + I_{5}^{(m)}$.

By arguing via the Gram-Schmidt orthonormalization as in the proof of Lemma 4.2, one can show that there exists a constant $c_{3} = c(n, k)$ such that

$$
(5.32) \quad |\text{image} \nabla f^{(m)} - \sum_{j=1}^{k} f^{(m)}_{j} \otimes f^{(m)}_{j}| \leq c_{3}|\nabla f^{(m)}|^{2},
$$

where $f^{(m)}_{j} = (0, \ldots, 1, \ldots, 0, \frac{\partial f^{(m)}}{\partial x_{j}}, \ldots, \frac{\partial f^{(m)}}{\partial x_{j}})$ with 1 in the $j$-th component. Here we recall that we are identifying image $\nabla f^{(m)}$ with the corresponding $n \times n$ orthogonal projection matrix. Then we use (2.2) to derive

$$
(5.33) \quad I_{4}^{(m)} = \int_{-3/4}^{1/4} \int_{C(T,1/2)} (\nabla \phi \otimes \nabla (x_{l} - g_{l}^{(m)})) + \phi \nabla^{2} g_{l}^{(m)} \cdot (\text{image} \nabla f^{(m)}) \, d\|V_{1}^{(m)}\|_{t} \, dt.
$$

We also have

$$
(5.34) \quad - \nabla \phi \otimes \nabla (x_{l} - g_{l}^{(m)}) \cdot \sum_{j=1}^{k} f^{(m)}_{j} \otimes f^{(m)}_{j} = \nabla (g_{l}^{(m)} - f_{l}^{(m)}) \cdot \nabla \phi
$$

and

$$
(5.35) \quad \phi \nabla^{2} g_{l}^{(m)} \cdot \sum_{j=1}^{k} f^{(m)}_{j} \otimes f^{(m)}_{j} = \phi \Delta g_{l}^{(m)}.
$$
Since $g^{(m)}_t$ satisfies (4.2), we obtain by (5.32)-(5.35) and (5.8) that
\begin{align}
(I_4^{(m)} + I_5^{(m)}) & = \int_{-3/4}^{1} \int_{C(T,1/2)} \nabla(g^{(m)}_t - f^{(m)}_t) \cdot \nabla \phi + \mu^{(m)}(f^{(m)}_t + 2c_4) \frac{\partial \phi}{\partial t} d\|V^{(m)}_t\| dt \\
\leq c_5(\|\phi\|_0 + \|\nabla \phi\|_0 + \|\frac{\partial \phi}{\partial t}\|_0)\|\nabla f^{(m)}\|_0^2.
\end{align}

By (5.13), (5.17) and (5.36), we obtain
\begin{align}
\lim_{m \to \infty} (\mu^{(m)})^{-1}(I_4^{(m)} + I_5^{(m)}) = \int_{-3/4}^{1} \int_{B_{1/2}^k} -\nabla f_t \cdot \nabla \phi + (\hat{f}_t + 2c_4) \frac{\partial \phi}{\partial t} d\mathcal{H}^k dt.
\end{align}

We may repeat the same computation with $\phi^{(m)}$ in (5.21) replaced by $(g^{(m)}_t - x_t + 2c_4 \mu^{(m)})\phi$, which is again nonnegative on $\text{spt} \|V^{(m)}_t\|$. This leads to the same conclusion as in (5.38) with $\hat{f}_t$ replaced by $-\hat{f}_t$. Thus (5.38) holds with equality for all $\phi \in C^\infty_c(B_{1/2}^k \times (-3/4,1); \mathbb{R}^+)$.

We next prove

**Lemma 5.3.**
\begin{align}
\lim_{m \to \infty} \|\hat{f}^{(m)} - \hat{f}\|_{L^2(B_{1/2}^k \times (-3/4,1))} = 0.
\end{align}

**Proof of Lemma 5.3.** Let $l \in \{k + 1, \cdots, n\}$ be fixed. We first claim that for each $s \in (-3/4,1)$,
\begin{align}
\hat{f}^{(m)}_t(\cdot, s) \rightharpoonup \hat{f}_t(\cdot, s) \text{ weakly in } L^2(B_{1/2}^k).
\end{align}

By (5.16), for each fixed $s \in (-3/4,1)$, $\{\hat{f}^{(m)}_t(\cdot, s)\}_{m=1}^\infty$ is bounded in $L^2(B_{1/2}^k)$ in particular. Let $w \in L^2(B_{1/2}^k)$ be any weak limit. Let $\phi^{(m)}$ be as in (5.22) and use (3.3) for $t_1 = -3/4$ and $t_2 = s$. The same computations (5.23)-(5.37) show that we have
\begin{align}
\int_{B_{1/2}^k} (w + 2c_4)\phi(\cdot, s) d\mathcal{H}^k \leq \int_{-3/4}^{s} \int_{B_{1/2}^k} -\nabla \hat{f}_t \cdot \nabla \phi + (\hat{f}_t + 2c_4) \frac{\partial \phi}{\partial t} d\mathcal{H}^k dt.
\end{align}

Since $\hat{f}_t$ is already known to be the solution of the heat equation, we have from (5.41)
\begin{align}
\int_{B_{1/2}^k} (w + 2c_4)\phi(\cdot, s) d\mathcal{H}^k \leq \int_{B_{1/2}^k} (\hat{f}_t(\cdot, s) + 2c_4)\phi(\cdot, s) d\mathcal{H}^k.
\end{align}

Similarly, replacing $\phi^{(m)}$ in (5.21) by $(g^{(m)}_t - x_t + 2c_4 \mu^{(m)})\phi$, we obtain
\begin{align}
\int_{B_{1/2}^k} (2c_4 - w)\phi(\cdot, s) d\mathcal{H}^k \leq \int_{B_{1/2}^k} (2c_4 - \hat{f}_t(\cdot, s))\phi(\cdot, s) d\mathcal{H}^k.
\end{align}
Thus (5.42) and (5.43) show that \( \int_{B_{1/2}^k} (w - \tilde{f}_i(t, s)) \phi(t, s) \, d\mathcal{H}^k = 0 \). Since \( \phi(t, s) \in C_1^1(B_{1/2}^k; \mathbb{R}^+) \) may be chosen arbitrarily, we proved \( w = \tilde{f}_i(t, s) \) a.e. on \( B_{1/2}^k \). Since any weak subsequence converges to \( \tilde{f}_i(t, s) \), the whole sequence converges weakly to \( \tilde{f}_i(t, s) \), proving (5.40). The lower semicontinuity under weak convergence shows that

\[
\| \tilde{f}_i(t, s) \|_{L^2(B_{1/2}^k)} \leq \liminf_{m \to \infty} \| \tilde{f}_i^{(m)}(t, s) \|_{L^2(B_{1/2}^k)}
\]

for all \( s \in (-3/4, 1) \). We next show that for any \(-3/4 < s_j < s < 1\) with \( s_j \) satisfying (5.19) and for any \( \phi \in C_1^\infty(B_{1/2}^k; \mathbb{R}^+) \), we have

\[
\limsup_{m \to \infty} \| \tilde{f}_i^{(m)}(t, s) \|_{L^2(B_{1/2}^k)}^2 \leq \| \tilde{f}_i^{(s)}(t, s_j) \|_{L^2(B_{1/2}^k)}^2 + c(\phi)(s - s_j).
\]

To prove (5.45), we use \((x_l - g_l^{(m)})^2 \phi\) as a test function in (3.3) with time interval \([s_j, s] \). Then by the Cauchy-Schwarz inequality and dropping the positive term, we obtain

\[
\int_{C(T, 1/2)} (x_l - g_l^{(m)})^2 \phi \, d||V_l^{(m)}|| \leq \int_{s_j}^{s} \int_{C(T, 1/2)} (x_l - g_l^{(m)})^2 \phi \, d||V_l^{(m)}||^{2}
+ |u_l^{(m)}||\nabla((x_l - g_l^{(m)})^2 \phi)| + h_l^{(m)} \cdot \nabla((x_l - g_l^{(m)})^2 \phi) + \phi \frac{\partial}{\partial t}(x_l - g_l^{(m)})^2 \, d||V_l^{(m)}||^{2} \, dt.
\]

We divide both sides of (5.46) by \((\mu_l)^2\) and take \( m \to \infty \). By (5.40), (5.19) and (5.7), we have

\[
\int_{B_{1/2}^k} (\tilde{f}_i(t, s))^{2} \, d\mathcal{H}^k \leq \liminf_{m \to \infty} \frac{1}{\mu_l^{(m)}} \int_{C(T, 1/2)} (x_l - g_l^{(m)})^2 \phi \, d||V_l^{(m)}||^{2}
\]

where we emphasize that the strong convergence at \( t = s_j \) is essentially used. By (5.10), (5.3), (5.13) and (5.8), one can check that

\[
\lim_{m \to \infty} \frac{1}{\mu_l^{(m)}} \int_{C(T, 1/2)} (x_l - g_l^{(m)})^2 \phi \, d||V_l^{(m)}||^{2}
= \int_{C(T, 1/2)} (\text{image } \nabla f^{(m)}) \cdot \nabla^2((x_l - g_l^{(m)})^2 \phi) + \phi \frac{\partial}{\partial t}(x_l - g_l^{(m)})^2 \, d||V_l^{(m)}||^{2}.
\]

For any \( S \in G(n, k) \),

\[
-S \cdot \nabla^2((x_l - g_l^{(m)})^2 \phi) \leq -2|S(\nabla(x_l - g_l^{(m)}))|^2 \phi + 2(x_l - g_l^{(m)}) \phi S \cdot \nabla^2 g_l^{(m)}
- (x_l - g_l^{(m)})^2 \phi \, d||V_l^{(m)}||^{2} \phi + 2(x_l - g_l^{(m)})^2 \frac{||\nabla \phi||^2}{\phi}
\]

(5.50)

For \( S = \text{image } \nabla f^{(m)} \), by Lemma 4.2 we have

\[
|S(T) \cdot \nabla^2 g_l^{(m)}| \leq c(n, k)|\nabla f^{(m)}||^2|\nabla^2 g_l^{(m)}|.
\]

(5.51)
Thus \((5.50)\) and \((5.51)\) show that
\[
-\left(\text{image } \nabla f^{(m)} \cdot \nabla^2((x_l - g_l^{(m)})^2)\phi \right) \leq \frac{2}{\theta} (x_l - g_l^{(m)})^2 \phi T \cdot \nabla^2 g_l^{(m)} + c_\theta(n, k, ||\phi||_{C^2}) \{ \|x_l - g_l^{(m)}\| \|\nabla f^{(m)}\| \|\nabla^2 g_l^{(m)}\| \}.
\]
By \((4.2)\), we have \(T \cdot \nabla^2 g_l^{(m)} - \frac{\partial g_l^{(m)}}{\partial t} = 0\), thus \((5.49)\), \((5.52)\), \((5.53)\), \((5.58)\) and \((5.56)\) show that
\[
\limsup_{m \to \infty} (\mu^{(m)})^{-2} \int_s^1 \int_{C(T, t/2)} h^{(m)} \cdot \nabla((x_l - g_l^{(m)})^2) \phi + \frac{\partial}{\partial t} (x_l - g_l^{(m)})^2 d||V_t^{(m)}||_t dt \leq c_\theta(c_\delta)^2(s - s_j).
\]
By combining \((5.40)\), \((5.47)\), \((5.48)\) and \((5.53)\), we obtain \((5.54)\). Using the smoothness of \(\hat{f}_t\), \((5.42)\), \((5.43)\) and the fact that \(\{s_j\}_{j=1}^{\infty}\) is dense, one can prove \(\|\hat{f}_t^{(m)}(\cdot, s)\|_{L^2(B_{t/2}^k)} \to \|\hat{f}_t(\cdot, s)\|_{L^2(B_{t/2}^k)}\) for all \(s \in (-3/4, 1)\), which shows the strong \(L^2(B_{1/3}^k)\) convergence. Since these \(L^2\) norms are all bounded uniformly in \(s\) by \((5.16)\), the dominated convergence theorem proves the desired strong convergence, \((5.39)\). This concludes the proof of Lemma 5.3. \(\square\)

Next define \(p = (p_{k+1}, \cdots, p_n) : \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}^{n-k}\) by
\[
(5.54) \quad p_l(x, t) := \hat{f}_l(0, 0) + \nabla \hat{f}_l(0, 0) \cdot x + \frac{1}{2} \nabla^2 \hat{f}_l(0, 0) \cdot x \otimes x + \frac{\partial \hat{f}_l}{\partial t}(0, 0) t
\]
for \(l = k+1, \cdots, n\). Since \(\hat{f}_t\) satisfies the heat equation by Lemma 5.2, the standard interior estimates with \((5.18)\) gives
\[
(5.55) \quad \|p\|_{C^2(B_{1/2}^k \times (-1, 1))} \leq c_\gamma(n, k).
\]
By the Taylor theorem again with the standard interior estimates, we have for \(0 < \theta < 1/4\)
\[
(5.56) \quad \sup_{B_{\theta^2}^k \times (-\theta^2, \theta^2)} |\hat{f} - p| \leq c_\alpha(n, k, \theta^3).
\]
We define for each \(m \in \mathbb{N}\)
\[
(5.57) \quad \hat{g}^{(m)} := g^{(m)} + \mu^{(m)} p
\]
and define \(Q_{\hat{g}^{(m)}}\) as in \((4.3)\). By \((5.54)\) and Lemma 5.2, we have \(\hat{g}^{(m)} \in \mathcal{F}\). On \(spt \|V_t^{(m)}\|\), by \((5.57)\) and \((5.15)\),
\[
(5.58) \quad Q_{\hat{g}^{(m)}} = \frac{1}{2} |f^{(m)} - g^{(m)} - \mu^{(m)} p|^2 \leq (\mu^{(m)})^2 (|\hat{f}^{(m)} - \hat{f}|^2 + |\hat{f} - p|^2).
\]
Thus Lemma 5.3 \((5.7)\), \((5.56)\) and \((5.58)\) show
\[
(5.59) \quad \limsup_{m \to \infty} (\mu^{(m)})^{-2} \int_{\theta^2}^2 \int_{C(T, \theta)} Q_{\hat{g}^{(m)}} d||V_t^{(m)}||_t dt \leq 2 \omega_k(c_\delta)^2 \theta^{k+8}.
\]
We now choose a small \(0 < \theta < 1/4\) depending only on \(n, k, \alpha\) so that
\[
(5.60) \quad 1 \geq 4 \omega_k(c_\delta)^2 \theta^{2-2\alpha}
\]
holds. By \((5.55)\) and \((5.57)\), we have \(\|\hat{g}^{(m)} - g^{(m)}\|_{C^2(B_{1/2}^k \times (-1, 1))} \leq c_\tau \mu^{(m)}\), thus \((5.11)\) is satisfied for all sufficiently large \(m\). One can check that \((5.11)\) and \((5.59)\) lead to a contradiction due to \((5.60)\) for all sufficiently large \(m\). Thus we complete the proof of Proposition 5.1. \(\square\)
6. $C^{2,\alpha}$ Estimate

Working under the same conditions as in the previous section and iterating the argument, we show a proper decay properties necessary for the proof of $C^{2,\alpha}$ estimates. First we prove

**Proposition 6.1.** Corresponding to $n, k, \alpha$ there exist $0 < \varepsilon_2 < 1$ and $1 < c_9 < \infty$ with the following property. Under the assumptions of Proposition 5.1 with $\varepsilon_1$ replaced by $\varepsilon_2$, with additional assumptions

\begin{align}
R^{1+\alpha}[u]_\alpha &\leq \varepsilon_2, \\
\nabla f(0,0) &= 0, \\
R^{1+\alpha}[\nabla f]^2_{(1+\alpha)} &\leq \varepsilon_2,
\end{align}

we have

1. at $(x,t) = (0,0)$, $f$ is differentiable with respect to $(x,t)$ and $\nabla f$ is differentiable with respect to $x$,
2. there exists $g^{(0)} \in \mathcal{F}$ such that

\begin{align}
f &= g^{(0)}, \quad \nabla g^{(0)} = 0, \quad \nabla^2 f = \nabla^2 g^{(0)}, \quad \frac{\partial f}{\partial t} = \frac{\partial g^{(0)}}{\partial t}
\end{align}

all hold at $(x,t) = (0,0)$ and

\begin{align}
R^{-1}\|g - g^{(0)}\|_0 + \|\nabla(g - g^{(0)})\|_0 + R\|\nabla^2(g - g^{(0)})\|_0 + R\|\frac{\partial}{\partial t}(g - g^{(0)})\|_0 \\
&\leq c_9 \max\{\mu, c_3 R^{1+\alpha}[u]_\alpha, c_3 R^{1+\alpha}[\nabla f]^2_{(1+\alpha)}\}.
\end{align}

Here $\| \cdot \|_0 := \sup_{B_R^+ \times (-2,0)} | \cdot |$.

3. Whenever $0 < r \leq R$, there exists $g^{(r)} \in \mathcal{F}$ such that

\begin{align}
r^{-1}\|g^{(r)} - g^{(0)}\|_0 + \|\nabla(g^{(r)} - g^{(0)})\|_0 + r\|\nabla^2(g^{(r)} - g^{(0)})\|_0 + r\|\frac{\partial}{\partial t}(g^{(r)} - g^{(0)})\|_0 \\
&+ \left(r^{-k-4} \int_{r^{-2}}^{r^2} \int_{C(T,r)} Q_{g^{(r)}} d\|V_i\| dt\right)^{\frac{1}{2}} \\
&\leq c_9 (r/R)^{1+\alpha} \max\{\mu, c_3 R^{1+\alpha}[u]_\alpha, c_3 R^{1+\alpha}[\nabla f]^2_{(1+\alpha)}\}.
\end{align}

Here $\| \cdot \|_0 := \sup_{B^+_{R} \times (-r^2,r^2)} | \cdot |$.

**Proof.** After a change of variables, we may assume that $R = 1$. For any $q \in C^2$ define

\begin{align}
\|q\|_{C^{2,1}(r)} := r^{-1}\|q\|_0 + \|\nabla q\|_0 + r\|\nabla^2 q\|_0 + r\|\frac{\partial q}{\partial t}\|_0
\end{align}

with $\| \cdot \|_0 = \sup_{B^+_{\varepsilon} \times (-r^2,r^2)} | \cdot |$ here. For notational simplicity define

\begin{align}
K := \max\{\mu, c_3 [u]_\alpha, c_3 [\nabla f]^2_{(1+\alpha)}\}.
\end{align}

We choose $0 < \varepsilon_2 < \varepsilon_1$ and $1 < c_9 < \infty$ so that

\begin{align}
c_3 \varepsilon_2 &\leq \varepsilon_1, \\
\varepsilon_2 + (c_3)^2 \varepsilon_2 \sum_{j=1}^{\infty} \theta^{(j-1)(1+\alpha)} &\leq \varepsilon_1,
\end{align}

respectively.
\begin{align}
\tag{6.11}
6c_3\theta^{-2} \sum_{j=0}^{\infty} \theta^{(j-1)\alpha} & \leq c_9, \\
\tag{6.12}
2\theta^{-\left(\frac{1}{2}+3+\alpha\right)} & \leq c_9.
\end{align}

We inductively prove the following claims. We set \( g^{(1)} := g \) and suppose that for \( j = 1, \ldots, m \), there are \( g^{(\theta^j)} \in \mathcal{F} \) such that
\begin{align}
\tag{6.13}
\|g^{(\theta^j)} - g^{(\theta^{j-1})}\|_{C^2,1(\theta^j-1)} & \leq c_3\theta^{(j-1)(1+\alpha)} K, \\
\tag{6.14}
\mu^{(j)} & := \left(\theta^{-j(k+2)} \int_{\theta^{2j}}^{\theta^{2j+1}} \int_{C(T,\theta^j)} Q_{g^{(\theta^j)}} d\|V_i\| dt\right)^\frac{1}{2} \leq \theta^{(1+\alpha)} K.
\end{align}

Consider the case \( j = 1 \). Since \( \varepsilon_2 \leq \varepsilon_1 \), Proposition \( 5.1 \) gives \( \hat{g} \in \mathcal{F} \) which we denote by \( g^{(\theta)} \). Note here that due to (6.2), we have \( \|\nabla f\|_0 \leq [\nabla f]^{(1+\alpha)} \). Assume (6.13) and (6.14) hold up to \( j = m \). We have
\begin{align}
\tag{6.15}
\mu^{(m)} & \leq \theta^{m(1+\alpha)} K \leq \varepsilon_1 \\
\text{by (5.4), (6.1), (6.3) and (6.9).}
\end{align}

With the notation \( \|\cdot\| = \sup_{B_{\theta^m}^{(\alpha)}(-\theta^m, \theta^m)} |\cdot| \) for abbreviation in the next computations, we compute
\begin{align}
\|\nabla g^{(\theta^m)}\| + \theta^m\|\nabla^2 g^{(\theta^m)}\| + \theta^m\|\frac{\partial g^{(\theta^m)}}{\partial t}\| \\
\leq \sum_{j=1}^{m} \left(\|\nabla (g^{(\theta^j)} - g^{(\theta^{j-1})})\| + \theta^m\|\nabla^2 (g^{(\theta^j)} - g^{(\theta^{j-1})})\| + \theta^m\|\frac{\partial (g^{(\theta^j)} - g^{(\theta^{j-1})})}{\partial t}\|\right) \\
\leq \varepsilon_2 + \sum_{j=1}^{m} c_3\theta^{(j-1)(1+\alpha)} K \quad \text{(by (6.13) and (5.2))} \\
\leq \varepsilon_2 + (c_3)^2 \varepsilon_2 \sum_{j=1}^{\infty} \theta^{(j-1)(1+\alpha)} \leq \varepsilon_1 \quad \text{(by (5.4), (6.1), (6.3) and (6.10)).}
\end{align}

Hence for \( R = \theta^m \), (5.11)-(5.24) are all satisfied due to (6.15) and (6.16). By Proposition \( 5.1 \) there exists a new \( \hat{g} \in \mathcal{F} \) denoted by \( g^{(\theta^m+1)} \) with the estimates
\begin{align}
\tag{6.17}
\|g^{(\theta^m+1)} - g^{(\theta^m)}\|_{C^2,1(\theta^m)} & \leq c_3\mu^{(m)} \leq c_3\theta^{m(1+\alpha)} K \\
\text{by (5.3), (6.15) and}
\end{align}

\begin{align}
\mu^{(m+1)} & \leq \theta^{1+\alpha} \max\{\mu^{(m)}, c_3\theta^{m(1+\alpha)}[u]_\alpha, c_3 \sup_{B_{\theta^m}^{(\alpha)}(-\theta^m, \theta^m)} |\nabla f|^2\} \\
\tag{6.18}
\leq g^{(m+1)(1+\alpha)} K \\
\text{by (5.6), (6.2) and (6.15). (6.17) and (6.18) show that (6.13) and (6.14) are satisfied for } \ j = m + 1, \text{ thus they are satisfied for all } j \in \mathbb{N}. \text{ For function } g \text{ satisfying (4.1), we have for } 0 < r \leq 1
\end{align}

\begin{align}
\tag{6.19}
\|g\|_{C^2,1(1)} & \leq |g(0, 0)| + 2|\nabla g(0, 0)| + 3|\nabla^2 g(0, 0)| + 2\left|\frac{\partial g}{\partial t}(0, 0)\right| \leq 3r^{-1}\|g\|_{C^2,1(r)}.
\end{align}

Thus for \( j \in \mathbb{N} \) we have by (6.13) and (6.19)
\begin{align}
\tag{6.20}
\|g^{(\theta^j)} - g^{(\theta^{j-1})}\|_{C^2,1(1)} & \leq 3c_3\theta^{(j-1)\alpha} K.
\end{align}
It is clear from (6.20) that there exists $g^{(0)} = \lim_{r \to 0^+} g^{(0)}$ which belongs to $\mathcal{F}$, and which satisfies (6.5) by (6.11). For $0 < r \leq 1$, choose $m$ such that

$$\theta^{m+1} < r \leq \theta^m$$

and set $g^{(r)} = g^{(\theta^m)}$. Then by the similar computations as in (6.19), we have

$$\|g^{(0)} - g^{(r)}\|_{C^{2,1}(r)} \leq \sum_{j=m+1}^{\infty} \|g^{(\theta^j)} - g^{(\theta^{j-1})}\|_{C^{2,1}(r)}$$

and

$$\leq \sum_{j=m+1}^{\infty} 3\theta^{m-j} \|g^{(\theta^j)} - g^{(\theta^{j-1})}\|_{C^{2,1}(\theta^j-1)} \leq \sum_{j=0}^{\infty} 3\theta^{(m+1)(1+\alpha)-\theta(j-1)\alpha} K$$

by (6.13), (6.11) and (6.21). By (6.21), (6.14) and (6.12), we also have

$$\left( r^{-k-4} \int_{r^2}^{r^2} \int_{C(T,r)} Q_{g^{(r)}} d\|V_t\|dt \right)^{\frac{1}{2}} \leq \theta^{-(k+4)/2} \mu^{(m)}$$

which also gives $\nabla^2 f(0,0) = \nabla^2 g^{(0)}(0,0)$. We note that $\nabla^2 g^{(0)}(0,0) \cdot x = \nabla g^{(0)}(x,0)$. By (6.6), for $|x| < 1/2$, $|\nabla g^{(0)}(x,0) - \nabla g^{(2|x|)}(x,0)| = O(|x|^{1+\alpha})$. Thus to show (6.24), it suffices to prove

$$\lim_{x \to 0} \frac{|\nabla f(x,0) - \nabla^2 g^{(0)}(0,0) \cdot x|}{|x|} = 0,$$

which also gives $\nabla^2 f(0,0) = \nabla^2 g^{(0)}(0,0)$. We note that $\nabla^2 g^{(0)}(0,0) \cdot x = \nabla g^{(0)}(x,0)$. By (6.6), for $|x| < 1/2$, $|\nabla g^{(0)}(x,0) - \nabla g^{(2|x|)}(x,0)| = O(|x|^{1+\alpha})$. Thus to show (6.24), it suffices to prove

$$\lim_{x \to 0} \frac{|\nabla f(x,0) - \nabla^2 g^{(2|x|)}(x,0)|}{|x|} = 0.$$

For any $x$ with $|x| < 1/2$, set $r := |x|$, $\beta := \frac{2\alpha}{k+6}$ and let $A$ be the Affine $k$-dimensional plane which is tangent to the graph $g^{(2r)}$ at $(x, g^{(2r)}(x,0))$. As a graph, $A$ is represented as

$$z \in \mathbb{R}^k \rightarrow g^{(2r)}(x,0) + \nabla g^{(2r)}(x,0) \cdot (z - x).$$

In the following we estimate

$$\mu(x) := \left( r^{-(1+\beta)(k+4)} \int_{r^{2(1+\beta)}} \int_{C(T,x,r^{1+\beta})} \text{dist} \ (y, A)^2 d\|V_t\|d\|y\|dt \right)^{\frac{1}{2}}$$

to apply the gradient estimate of [13, Th. 8.7]. For $y = (z, f(z,t)) \in \text{spt} \ |V_t|$ with $z := T(y)$, $\text{dist} \ (y, A) \leq |f(z,t) - g^{(2r)}(x,0) - \nabla g^{(2r)}(x,0) \cdot (z - x)|$

$$\leq |f(z,t) - g^{(2r)}(z,t)| + |g^{(2r)}(z,t) - g^{(2r)}(x,0) - \nabla g^{(2r)}(x,0) \cdot (z - x)|$$

$$\leq \sqrt{2} Q_{g^{(2r)}}(y,t)\left( |\partial t g^{(2r)}| |t| + |\nabla^2 g^{(2r)}||z - x|^2 \right)$$

$$\leq \sqrt{2} Q_{g^{(2r)}}(y,t) + c_{10}(|t| + |z - x|^2).$$
The existence of $c_{10}$ independent of $r$ follows from (6.20). Substituting (6.28) into (6.27) gives

$$
(6.29) \quad \mu(x) \leq \left( r^{-(1+\beta)\beta+4} \int_{r^{-(1+\beta)}}^{r^{2(1+\beta)}} \int_{-r^{2(1+\beta)}}^{r^{2(1+\beta)}} 2Q_g(2r) \, d\|V_t\| \, dt \right)^{\frac{1}{2}} + 6c_{10}r^{1+\beta}.
$$

Since $C(T, x, r^{1+\beta}) \subset C(T, 2r)$, (6.29) shows

$$
(6.30) \quad \mu(x) \leq 2^{K+5} c_9 r^{1+\alpha-\beta\frac{(K+4)}{2}} + 6c_{10}r^{1+\beta} = (2^{K+5} c_9 K + 6c_{10}) r^{1+\beta}
$$

by the choice of $\beta = \frac{2\alpha}{K+6}$. Note that $u$ is Hölder continuous and $u(0,0) = 0$, thus for any large $p, q$ where we subsequently fix,

$$
(6.31) \quad c_{11}(x) := (r^{1+\beta})^{\frac{1}{2}} \frac{1}{2} |\|u\|_{L^{p,q}(C(T, x, r^{1+\beta}) \times (-r^{2(1+\beta)}, r^{2(1+\beta)}}) \leq \varepsilon_2 c(p, q, k, n) r^{1+\alpha}.
$$

The existence of $t_1$ and $t_2$ there for $\nu = 3/4$, for example, is satisfied since $f$ is a graph with uniformly small spacial gradient. Thus there exists a constant $c_{12}$ depending only on $u, k$ such that

$$
(6.33) \quad |\nabla f(x,0) - \nabla g(2r)(x,0)| \leq c_{12} \max\{\mu(x), c_{11}(x)\} \leq 2c_{12} (2^{K+5} c_9 K + 6c_{10}) r^{1+\beta}
$$

by [13] Th. 8.7], (6.31) and (6.32) for all sufficiently small $r$. Now (6.33) proves (6.25).

Finally, we need to prove

$$
(6.34) \quad \lim_{(x,t) \to 0} \frac{|f(x,t) - g^{(0)}(0,t)|}{\sqrt{|x|^2 + t^2}} = 0
$$

which will prove $f$ is differentiable at $(x,t) = (0,0)$ (recall $\nabla g^{(0)}(0,0) = 0$ and $\frac{\partial f}{\partial t}(0,0) = 2g^{(0)}(0,0)$. Set $r := (|x|^2 + t^2)^{1/4}$. By (6.6), we have for some $\hat{t}$ with $|\hat{t}| \leq r^2$ that

$$
(6.35) \quad |g^{(0)}(0,t) - g^{(2r)}(0,t)| \leq |g^{(0)}(0,0) - g^{(2r)}(0,0)| + |t| \left| \frac{\partial}{\partial t}(g^{(0)} - g^{(2r)})(0,\hat{t}) \right|
$$

$$
\leq c_9 (2r)^{2+\alpha} K + 2^{\alpha} c_9 r^{2+\alpha} K.
$$

Moreover, for some $\hat{x}$ with $|\hat{x}| \leq r^2,$

$$
(6.36) \quad |g^{(2r)}(x,t) - g^{(2r)}(0,t)| \leq |\nabla g^{(2r)}(\hat{x},t)||x| \leq c_9 2^{1+\alpha} r^{3+\alpha} K
$$

by (6.6). Thus (6.35) and (6.36) show that it suffices to prove

$$
(6.37) \quad \lim_{(x,t) \to 0} \frac{|f(x,t) - g^{(2r)}(x,t)|}{\sqrt{|x|^2 + t^2}} = 0
$$

to prove (6.34). We basically repeat the same argument as for $\nabla^2 f$. Set $\beta$ as before and let $A$ be the Affine $k$-dimensional plane which is tangent to the graph $g^{(2r)}$ at $(x, g^{(2r)}(x,t))$. We define

$$
(6.38) \quad \mu(x,t) := \left( r^{-(1+\beta)\beta+4} \int_{r^{-(1+\beta)}}^{r^{2(1+\beta)}} \int_{-r^{2(1+\beta)}}^{r^{2(1+\beta)}} \text{dist}(y,A)^2 \, d\|V_s\| \, ds \right)^{\frac{1}{2}}.
$$
For $y = (z, g(z, s)) \in \text{spt } \|V_s\|$ with $z := T(y)$,

$$\text{dist } (y, A) \leq |f(z, s) - g^{(2r)}(x, t) - \nabla g^{(2r)}(x, t) \cdot (z - x)|$$

$$\leq |f(z, s) - g^{(2r)}(z, s)| + |g^{(2r)}(z, s) - g^{(2r)}(x, t) - \nabla g^{(2r)}(x, t) \cdot (z - x)|$$

(6.39)

$$\leq \sqrt{2Q_{g^{(2r)}}(z, s)} + |s - t|\|\frac{\partial g^{(2r)}}{\partial s}\|_0 + |z - x|^2\|\nabla^2 g^{(2r)}\|_0$$

$$\leq \sqrt{2Q_{g^{(2r)}}(z, s)} + c_{10}(|s - t| + |z - x|^2).$$

Substitute (6.39) into (6.38), and proceed just as before. Note that

(6.40) $$C(T, x, r^{1+\beta}) \times (t - r^{2(1+\beta)}, t + r^{2(1+\beta)}) \subset C(T, 2r) \times (-4r^2, 4r^2)$$

due to $r = (|x|^2 + t^2)^{1/4}$. Then we obtain by (6.38)-(6.40) (with an obvious modification for $c_{11}(x, t)$ and sup estimate instead of gradient estimate of [13, Th. 8.7])

(6.41) $$r^{-(1+\beta)}|f(x, t) - g^{(2r)}(x, t)| \leq c_{12}\max\{\mu(x, t), c_{11}(x, t)\} \leq 2c_{12}(2^{|\kappa|} c_9 K + \epsilon)c_{10}r^{1+\beta}$$

for all sufficiently small $r$. By (6.41), we prove (6.37). This completes the proof of Proposition 6.1.

Next, to apply the estimates of Proposition 6.1 at a given point, we need to make a change of variables so that $\nabla f$ and $u$ are both zero there with respect to the new coordinate system. Suppose that we have $\{V_t\}_{-1 < t < 1}$ and $\{u(\cdot, t)\}_{-1 < t < 1}$ satisfying (B1)-(B4) on $B_1 \times (-1, 1)$. Let $(\tilde{x}, \tilde{t}) \in B_{1/2} \cap \text{spt } \|V_t\|$ with $-1/2 < \tilde{t} < 1/2$ be arbitrary. By suitable rotation and parallel translation, we may choose a coordinate system so that $(\tilde{x}, \tilde{t})$ is translated to the origin $(0, 0)$ and $\text{spt } \|V_0\|$ is tangent to $\mathbb{R}^k \times \{0\}$, so that the graph of $f$ has $\nabla f(0, 0) = 0$. Note that $B_{1/2} \times (-1/2, 1/2)$ (with this new coordinate system) is included in the original domain. To have $u(0, 0) = 0$, we change the variables by $(x, t) \rightarrow (x - tu(0, 0), t)$. Namely, we introduce a new coordinate system so that the frame moves at the constant speed $u(0, 0)$. Define for each $t \in (-1/2, 1/2)$ and $\phi \in C_c(G_k(\mathbb{R}^n))$

(6.42) $$\tilde{V}_t(\phi) := V_t(\phi(\cdot - tu(0, 0), \cdot)), \quad \tilde{u}(x, t) := u(x + tu(0, 0), t) - u(0, 0).$$

If $u(0, 0)$ is assumed to be sufficiently small, $B_{1/4} \times (-1/4, 1/4)$ is included in the original domain under the new coordinate system. It is natural to expect the following.

**Lemma 6.2.** The newly defined $\{\tilde{V}_t\}_{-1/4 < t < 1/4}$ and $\{\tilde{u}(\cdot, t)\}_{-1/4 < t < 1/4}$ satisfy (B4) on $B_{1/4} \times (-1/4, 1/4)$ and $\tilde{u}(0, 0) = 0$.

**Proof.** Obviously $\tilde{u}(0, 0) = 0$ follows from (6.42). Write $a := u(0, 0)$ for simplicity. We need to check that (3.3) holds for $\tilde{V}_t$ and $\tilde{u}$. For any $\phi \in C^1(B_{1/4} \times (-1/4, 1/4); \mathbb{R}^k)$ with $\phi(\cdot, t) \in C^1_c(B_{1/4})$, define $\tilde{\phi}(x, t) := \phi(x - at, t)$. Then for any $-1/4 < t_1 < t_2 < 1/4$, by (6.42) and (3.3),

(6.43) $$\|\tilde{V}_t\|((\phi(\cdot, t)))_{t=t_1}^{t_2} = \|V_t\|((\tilde{\phi}(\cdot, t)))_{t=t_1}^{t_2} \leq \int_{t_1}^{t_2} \int (\nabla \tilde{\phi} - \tilde{\phi} h) \cdot (h + u^+) + \frac{\partial \tilde{\phi}}{\partial t} d\|V_t\|dt.$$
If we denote the mean curvature vector of $\tilde{V}_t$ by $\tilde{h}(\tilde{V}_t, x)$, we have $h(\tilde{V}_t, x - at) = h(V_t, x)$ since the change of variables is simply a translation for each fixed time. Thus

$$\int (\nabla \phi - \tilde{h}) \cdot (h + u^\perp) \, d\|V_t\|$$

(6.44)  

$$= \int \{ \nabla \phi - \tilde{h}(\tilde{V}_t, \cdot - at) \} \cdot (\tilde{h}(\tilde{V}_t, \cdot - at) + \tilde{u}^\perp(\cdot - at) + a^\perp) \, d\|V_t\|$$

$$= \int (\nabla \phi - \tilde{h}) \cdot (h + \tilde{u}^\perp + a^\perp) \, d\|\tilde{V}_t\|.$$  

By (2.2) and (2.3) on the other hand, for a.e. $t$, we have

$$\int (\nabla \phi - \tilde{h}) \cdot a^\perp \, d\|\tilde{V}_t\| = \int \nabla \phi \cdot a^\perp - \dot{\tilde{h}} \cdot a \, d\|\tilde{V}_t\|$$

(6.45)  

$$= \int \nabla \phi \cdot a^\perp \, d\|\tilde{V}_t\| + \int S \cdot (a \otimes \nabla \phi) \, d\tilde{V}_t(\cdot, S) = \int \nabla \phi \cdot a \, d\|\tilde{V}_t\|$$

since $S(a \otimes \nabla \phi) = \nabla \phi \cdot a^\top$, where $\cdot^\top$ is the projection to the tangent space, and $a^\perp + a^\top = a$.

Since $\frac{\partial \phi}{\partial t}(x, t) = -\nabla \phi(\cdot - at, t) \cdot a + \frac{\partial \phi}{\partial t}(\cdot - at, t)$, (6.43)-(6.45) prove

$$\|\tilde{V}_t\|(\phi(\cdot, t)) |_{t=t_1}^{t_2} \leq \int_{t_1}^{t_2} \int (\nabla \phi - \tilde{h}) \cdot (\tilde{h} + \tilde{u}^\perp) + \frac{\partial \phi}{\partial t} \, d\|\tilde{V}_t\|dt.$$  

(6.46) shows the claim of the present lemma. \(\square\)

Finally, assuming that we already have $C^{1, \frac{1}{2}}$ estimate of the graph, we prove the following. Note that $C^{1, \frac{1}{2}}$ estimate has been established in [13] and it will be integrated at the end. Some technical lemma concerning the change of second derivatives under orthogonal rotations is relegated to Section 8.

**Theorem 6.3.** Corresponding to $n, k$, and $0 < \alpha < 1$ there exist $0 < \varepsilon_3 < 1$ and $1 < c_{13} < \infty$ with the following property. For $0 < R < \infty$ suppose $\{V_t\}_{-R^2 < t < R^2}$ and $\{u(\cdot, t)\}_{-R^2 < t < R^2}$, where $V_t = |M_t|$ with $M_t = \text{graph } f(\cdot, t)$, satisfy (B1)-(B4) on $C(T, R) \times (-R^2, R^2)$. Assume

$$\|\nabla f\|_0 + R^{1+\alpha} \|\nabla f\|_{\frac{1}{2+\alpha}} \leq \varepsilon_3,$$

(6.47)

$$\|u\|_{\alpha} := R\|u\|_0 + R^{1+\alpha}[u]_\alpha \leq \varepsilon_3,$$

(6.48)

and assume that for some $g \in \mathcal{F}$ with

$$\|\nabla g\|_0 + R\|\nabla^2 g\|_0 + R\|\frac{\partial g}{\partial t}\|_0 \leq \varepsilon_3,$$

(6.49)

we have

$$\mu := \left( R^{-(k+1)} \int_{-R^2}^{R^2} \int_{C(T,R)} Q_2(x, t) \, d\|V_t\|(x) dt \right)^\frac{1}{2} \leq \varepsilon_3.$$  

(6.50)

Then on $B_{R/2}^k \times (-R^2/4, R^2/4)$, $f$ is differentiable w.r.t. $(x, t)$ and $\nabla f$ is differentiable w.r.t. $x$, and we have

$$R \|\nabla^2(f - g)\|_0 + R^{1+\alpha} \left[ \nabla^2 f, \frac{\partial f}{\partial t} \right]_{\alpha} \leq c_{13} \max\{\mu, \|u\|_{\alpha}, \|\nabla f\|_{\frac{1}{2+\alpha}}\}$$

(6.51)

where the (semi-)norms on the left-hand side of (6.51) are over the domain $B_{R/2}^k \times (-R^2/4, R^2/4)$. Moreover, the normal velocity vector of $M_t$ is equal to $h(|M_t|, x) + u(x, t)^\perp$ at each point $x \in M_t \cap C(T, R/2)$ for $t \in (-R^2/4, R^2/4)$.  


Proof. We may assume that $R = 1$ after a change of variables. For any point $\tilde{x} \in M_t \cap C(T, 1/2)$, $\tilde{t} \in (-1/2, 1/2)$, there is a change of variables by Lemma 6.12 so that the new graph function $\tilde{f}$ has $\nabla \tilde{f}(0, 0) = 0$ and $\tilde{u}(0, 0) = 0$ (where $(0, 0)$ corresponds to $(\tilde{x}, \tilde{t})$ before). Let $\tilde{T}$ be $\mathbb{R}^k \times \{0\}$ in this coordinate system which is also the tangent space to the graph of $\tilde{f}$ at $(0, 0)$. We will apply Proposition 6.1 to $\tilde{f}$. To do so, we need the initial approximation function in $\mathcal{F}$. Consider the graph function $\tilde{g}$ at $(0, 0)$ in the coordinate system which is also the tangent space to the graph of $\tilde{g}$ at $(0, 0)$. The normal velocity is equal to the mean curvature at $(0, 0)$. Thus we have

\begin{equation}
(6.52)
\end{equation}

\[ |\tilde{g} - \tilde{g}| \leq c(|\nabla \tilde{f}(\tilde{x}, \tilde{t})| + |u(\tilde{x}, \tilde{t})|). \]

We then define

\begin{equation}
(6.53)
Q_{\tilde{g}}(x, t) := \frac{1}{2} \sum_{l=k+1}^{n} (x_l - \tilde{g}_l(\tilde{T}(x), t))^2
\end{equation}

where $\tilde{g} = (\tilde{g}_{k+1}, \cdots, \tilde{g}_n)$ and $x \in \mathbb{R}^n$ and similarly for $Q_{\tilde{g}}$. By (6.52), we have for $x \in M_t$ with $t \in (-1, 1)$

\begin{equation}
(6.54)
Q_{\tilde{g}} \leq 2Q_{\tilde{g}} + c(n, k)(|\nabla \tilde{f}(\tilde{x}, \tilde{t})|^2 + |u(\tilde{x}, \tilde{t})|^2).
\end{equation}

The difference between $Q_{\tilde{g}}$ and $Q_g$ on $M_t$ is that the former measure the $|\tilde{f} - \tilde{g}|^2/2$ while the latter measures $|f - g|^2/2$. The translation by $tu(\tilde{x}, \tilde{t})$ does not affect the values of $Q_{\tilde{g}}$. Then a simple computation shows

\begin{equation}
(6.55)
Q_{\tilde{g}} \leq 2Q_g.
\end{equation}

Thus we have

\begin{equation}
(6.56)
\left( \int_{-1/4}^{1/4} \int_{C(T, 1/4)} Q_{\tilde{g}}(x, t) \, d\|V_t\| \, dt \right)^{\frac{1}{2}} \leq 2\mu + c(n, k)(\|\nabla f\|_0 + \|u\|_0),
\end{equation}

where $V_t = |M_t|$ on the left-hand side is understood to be the one after the change of variables. Now we are in the position to apply Proposition 6.1 for sufficiently small $\varepsilon_3$ which is determined by $\varepsilon_2$ and (6.56). This proves that $\nabla \tilde{f}$ is differentiable w.r.t. $x$ and $\tilde{f}$ is differentiable w.r.t. $(x, t)$ at $(0, 0)$. It is geometrically obvious that $f$ is then differentiable at $(\tilde{x}, \tilde{t})$. It requires some calculations to prove that $\nabla f$ is differentiable w.r.t. $x$ via computations as in Lemma 8.1 but we omit the details. Moreover, since $\tilde{g}(0, 0) \in \mathcal{F}$, (6.4) proves that $\partial \tilde{f}/\partial t = \Delta \tilde{f}$ at $(0, 0)$ for each component. Since $\nabla \tilde{f}(0, 0) = 0$, this proves that the normal velocity is equal to the mean curvature at $(0, 0)$. Since the coordinate is ‘moving’ with speed $u(\tilde{x}, \tilde{t})$, we proved that the normal velocity is equal to the sum of the mean curvature and $u^\perp$ in the original coordinate system. The supremum estimates for $\nabla^2 (\tilde{f} - \tilde{g})$
and \( \partial (\hat{f} - \hat{g})/\partial t \) follows from (6.5). This in turns gives

\[
(\text{6.57}) \quad \sup_{B_{1/2}^k \times (-1/2,1/2)} |\nabla^2 (f - g), \frac{\partial (f - g)}{\partial t}| \leq c_{14} \max \\{ \mu, \|u\|_\alpha, \|\nabla f\|_{\alpha+\alpha} \}
\]

via (6.4) and estimates on the difference between \( \nabla^2 \hat{g} \) and \( \nabla^2 \tilde{g} \), which can be bounded by \( c(n,k) \|\nabla f\| \). Finally we need to prove the \( \alpha \)-Hölder norm estimate of (6.51). For \( i = 1, 2 \), let \( \tilde{x}_i \in M_i \cap C(T,1/2), t_i \in (-1/2,1/2) \) be any two points with \( (\tilde{x}_1, \tilde{t}_1) \neq (\tilde{x}_2, \tilde{t}_2) \). Without loss of generality we assume

\[
(\text{6.58}) \quad |\tilde{x}_1 - \tilde{x}_2| < 1/10, \quad 0 < \tilde{t} := \tilde{t}_2 - \tilde{t}_1 < 1/100.
\]

After a change of variables as before, so that \( (0,0) \) and \( (\tilde{x}, \tilde{t}) \) in the new coordinate system correspond to \( (\tilde{x}_1, \tilde{t}_1) \) and \( (\tilde{x}_2, \tilde{t}_2) \), respectively, we may have \( \nabla \hat{f}(0,0) = 0 \) and \( \tilde{u}(0,0) = 0 \). Denote the tangent space to the graph \( \hat{f} \) at the origin by \( \hat{T} \). Restricting \( \varepsilon_3 \) further if necessary, by the first part of the proof and by Proposition 6.1, there exist \( \hat{g}^{(0)}, \hat{g}^{(r)} \in \mathcal{F} \) for \( 0 < r < 1/4 \) with (6.4), (6.5) and (6.6) where \( f, u \) and \( T \) in those statements are replaced by \( \hat{f}, \tilde{u}, \hat{T} \) with \( R = 1/4 \). Corresponding to \( (\tilde{x}, \tilde{t}) \), fix \( \hat{r} := 2 \max \{ |\tilde{x}|, |\tilde{t}|^{1/2} \} \) and consider \( \hat{g}^{(r)} \). For later use, define

\[
(\text{6.59}) \quad \hat{a} := (\hat{a}_{ij})_{1 \leq i,j \leq k} := \left( \frac{\partial^2 \hat{g}^{(r)}}{\partial x_i \partial x_j} \right)_{1 \leq i,j \leq k}, \quad \hat{b} := \frac{\partial \hat{g}^{(r)}}{\partial t}.
\]

Note that (recall \( \nabla g \) is independent of \( t \) for \( g \in \mathcal{F} \))

\[
|\nabla \hat{g}^{(r)}(\hat{T}(\tilde{x}))| \leq |\nabla \hat{g}^{(r)}(0)| + \hat{r}|\hat{a}|
\]

\[
\leq 2c_9(4\hat{r})^{1+\alpha} \max \{ \mu, c_3[\hat{u}]_{\alpha}, c_3[\nabla \hat{f}]^2_{1+\alpha} \} + \hat{r}|\nabla \hat{f}|_{0}
\]

\[
\leq c_{15}\hat{r} \max \{ \mu, \|u\|_\alpha, \|\nabla f\|_{\alpha+\alpha} \}
\]

by (6.6), the triangle inequalities and (6.57). Regarding the graph \( \hat{g}^{(r)} \) as a smooth \( k \)-dimensional manifold in \( \mathbb{R}^n \), let \( \hat{T} \in \mathcal{G}(n,k) \) be the tangent space over \( (\tilde{x}, \tilde{t}) \) and let \( \tilde{g} \) be the graph representation over \( \hat{T} \), that is, \( \text{graph} \tilde{g} = \text{graph} \hat{g}^{(r)} \). We introduce yet another new coordinate system so that \( \hat{T} = \mathbb{R}^k \times \{ 0 \} \) and \( (0,0) \) corresponds to \( (\tilde{x}, \tilde{t}) \). We may take such new coordinate system so that the new one is obtained by a parallel translation and an orthogonal rotation \( A \) with \( |I - A| = O(|\nabla \hat{g}^{(r)}(\tilde{x})|) \). By (8.3) and (6.60), we have

\[
(\text{6.61}) \quad \sup_{(x,t) \in \hat{T} \times (1/4,1/4)} |\nabla^2 \hat{g}(x,t) - \hat{a}| \leq c_1 \hat{r} \max \{ \mu, \|u\|_\alpha, \|\nabla f\|_{\alpha+\alpha} \}.
\]

Similar computations show

\[
(\text{6.62}) \quad \sup_{(x,t) \in \hat{T} \times (1/4,1/4)} \left| \frac{\partial \hat{g}}{\partial t}(x,t) - \hat{b} \right| \leq c_{16}\hat{r} \max \{ \mu, \|u\|_\alpha, \|\nabla f\|_{\alpha+\alpha} \}.
\]

Now we define a function \( \hat{g}^{(r)} \in \mathcal{F} \) which is defined relative to \( \hat{T} \) by

\[
(\text{6.63}) \quad \hat{g}^{(r)}(x,t) := \hat{g}(0,0) + \hat{b}t + \frac{1}{2} \sum_{i,j=1}^k \hat{a}_{ij} x_i x_j.
\]
Since \( g(\bar{r}) \in \mathcal{F} \) and by (6.59), we have \( \tilde{g}(\bar{r}) \in \mathcal{F} \). Moreover by the Taylor expansion and (6.61)-(6.63), we have

\[
\sup_{B_{\tilde{r}}/2 \times (-\tilde{r}^{2}, \tilde{r}^{2})} |\tilde{g} - \tilde{g}(\bar{r})| \leq c(k)\tilde{r}^{2} \sup_{B_{\tilde{r}}/2 \times (-\tilde{r}^{2}, \tilde{r}^{2})} (|\nabla^{2}(\tilde{g} - \tilde{g}(\bar{r}))| + |\frac{\partial}{\partial t}(\tilde{g} - \tilde{g}(\bar{r}))|) \leq 2c(k)c_1\tilde{r}^{2} \max\{\mu, \|u\|_{\alpha}, \|\nabla f\|_{\frac{1+\alpha}{2}}\}.
\]

By \( Q_{\tilde{g}} \leq 2Q_{g(\bar{r})} \) and (6.64), we have

\[
(\tilde{r}^{-k-4} \int -\tilde{r}^{2}/4 \int_{C(T, \tilde{r}/2)} Q_{g(\bar{r})} \, d\|V_i\| \, dt)^{1/2} \leq 2(\tilde{r}^{-k-4} \int -\tilde{r}^{2}/4 \int_{C(T, \tilde{r}/2)} Q_{g(\bar{r})} \, d\|V_i\| \, dt)^{1/2} + c(k)c_1\tilde{r}^{2} \max\{\mu, \|u\|_{\alpha}, \|\nabla f\|_{\frac{1+\alpha}{2}}\}.
\]

By (6.6), the first term on the right-hand side is bounded by \( c_0\tilde{r}^{1+\alpha} \max\{\mu, c_2[u]_{\alpha}, c_3[\nabla f]^{2}_{1+\alpha}\} \).

Now we are in the position to prove our main Theorem 3.6 and Theorem 3.3. Proof of Theorem 3.6. As usual we may assume \( R = 1 \). We apply [13, Theorem 8.7] first. To do so, we need to check the assumptions (A1)-(A4) of [13, Section 3.1] are satisfied. Fix \( \alpha \) proves the desired location of pole, we obtain a uniform estimate (A2) in the interior. Thus, corresponding to the listed relevant constants, we have an interior \( C^{1,\varsigma} \) estimate for \( \text{spt} \|V_i\| \), i.e., we can represent \( \text{spt} \|V_i\| \) as a graph \( f(\cdot, t) \) with the desired estimates for \( \|f\|_{0} + \|\nabla f\|_{\frac{1+\alpha}{2}} \). Then use Theorem 5.3 to obtain the second order derivatives estimates in a smaller region, where we use \( g = 0 \) for the initial approximation. Note that \( \|\nabla f\|_{\frac{1+\alpha}{2}} \) on the right-hand side of (6.51) is already estimated in terms of \( \mu \) and \( \|u\|_{L^{p,\alpha}} \). By choosing sufficiently small \( \varepsilon_0 > 0 \), this proves the desired conclusion. □

Proof of Theorem 3.3. Set \( p, q \) as above. By the same reason as above, we have all the conditions (A1)-(A4) of [13] satisfied. Thus [13, Theorem 3.2] shows a.e. \( C^{1,\varsigma} \) regularity in space-time. Then Theorem 6.3 shows \( C^{2,\alpha} \) regularity there as well. □

7. BRAKKE’S MCF IN SUBMANIFOLD

It may be worthwhile to comment on some consequences of our main theorem in the case that the ambient space \( \mathbb{R}^{n} \) is replaced by a submanifold. Such situation naturally arises when we consider a MCF in general Riemannian manifold via Nash’s isometric imbedding theorem. For \( k \in \mathbb{N} \) with \( 1 \leq k < \tilde{k} \leq n \), suppose we have a \( C^{\infty} \tilde{k} \)-dimensional submanifold \( N \) in an open set \( U \subset \mathbb{R}^{n} \) and a family of \( k \)-varifolds which is Brakke’s MCF in \( N \) in an appropriate weak sense. For the precise definition, we need to have a few preliminaries.
We define the second fundamental form of $N$ at $x \in N$ to be the bilinear form $B_x : \text{Tan}_x N \times \text{Tan}_x N \rightarrow (\text{Tan}_x N)^\perp$ such that

\begin{equation}
B_x(v_1, v_2) := -\sum_{i=1}^{n-k} (v_1 \cdot \nabla v_2 \tau_i) \tau_i |_x, \quad v_1, v_2 \in \text{Tan}_x N.
\end{equation}

Here $\tau_1, \ldots, \tau_{n-k}$ are locally defined vector fields which are orthonormal and which satisfy $\tau_i(y) \in (\text{Tan}_y N)^\perp$ on some neighborhood of $x$. Next, for $x \in N$ and $S \in G(n, k)$ with $S \subset \text{Tan}_x N$, define

\begin{equation}
H_N(x, S) := \sum_{i=1}^k B_x(v_i, v_i) \in (\text{Tan}_x N)^\perp,
\end{equation}

where $v_1, \ldots, v_k$ is an orthonormal basis of $S$. $H_N(x, S)$ is well-defined independent of the choice of the orthonormal basis. Though it is simple, we record the following

**Lemma 7.1.** Suppose $V \in IV_k(U)$ satisfies $\text{spt} \|V\| \subset N$ and has a generalized mean curvature $h(V, \cdot)$ in $U$. Let $M \subset U$ be a countably $k$-rectifiable set such that $V = \theta |M|$ with some integer multiplicity function $\theta$. Then we have

\begin{equation}
h(V, x) - H_N(x, \text{Tan}_x M) \in \text{Tan}_x N
\end{equation}

for $\mathcal{H}^k$ a.e. on $M$. Here $\text{Tan}_x M$ is the approximate tangent space of $M$ at $x$.

**Proof.** It suffices to prove that

\begin{equation}
\int_U (h(V, x) - H_N(x, S)) \cdot f \, dV(x, S) = 0
\end{equation}

for all $f \in C^1_c(U; \mathbb{R}^n)$ with $f(x) \in (\text{Tan}_x N)^\perp$. Let $\tau_1, \ldots, \tau_{n-k}$ be a set of locally defined orthonormal vector fields which form a basis for $(\text{Tan}_y N)^\perp$ on $N$. Since the integration is over $M \subset N$, note that the values of $f$ outside of $N$ do not matter. Thus without loss of generality we may express $f = \sum_{i=1}^{n-k} f_i \tau_i$. Then by (7.2) we have

\begin{equation}
\int_U h(V, \cdot) \cdot f \, dV(\cdot, S) = -\sum_{i=1}^{n-k} \int_U S \cdot \nabla (f_i \tau_i) \, dV(\cdot, S) = -\sum_{i=1}^{n-k} \int f_i S \cdot \nabla \tau_i \, dV(\cdot, S)
\end{equation}

where we used $S \cdot \tau_i = 0$ for $V$ a.e. since $S = \text{Tan}_x M \subset \text{Tan}_x N$. On the other hand, by (7.1) and (7.2), we see that $\int_U H_N(\cdot, S) \cdot f \, dV(\cdot, S)$ is equal to the right-hand side of (7.5). This proves (7.4). \qed

**Remark 7.2.** We should point out that $V$ being integral is not essential, and that it suffices for example to have rectifiable $V$ with its approximate tangent space in $\text{Tan}_x N$ a.e. for Lemma 7.1.

Lemma 7.1 shows that for $V = \theta |M|$, we have a decomposition $h(V, x) = h(V, x)^\perp + h(V, x)^\parallel = h(V, x)^\perp + H_N(x, \text{Tan}_x M) \in \text{Tan}_x N \oplus (\text{Tan}_x N)^\perp$. Furthermore, due to the perpendicularity of the mean curvature vector (2.3), we have $h(V, x)^\perp \in (\text{Tan}_x M)^\perp \cap \text{Tan}_x N$ for $\mathcal{H}^k$ a.e. on $M$. The vector $h(V, \cdot)^\perp$ may be considered as an intrinsic mean curvature vector with respect to $N$ and it is natural to define the mean curvature flow whose velocity is equal to $h(V, \cdot)^\perp$ as follows.

**Definition 7.3.** For $\Lambda \leq \infty$, a family of $k$-varifolds $\{V_t\}_{0 \leq t \leq \Lambda}$ in $U \subset \mathbb{R}^n$ is (unit density) Brakke’s MCF in a smooth $k$-dimensional submanifold $N \subset U$ if the followings are satisfied.

(C0) For all $t \in [0, \Lambda)$, $\text{spt} \|V_t\| \subset N$. 

Further assume that $A = \begin{pmatrix} j \end{pmatrix}$ (8.1)

Suppose that two coordinate systems $C^3$ for all $x \in [0, \Lambda)$, $V^1$ is a unit density $k$-varifolds.

(C2) For $U \subset U$ and $(t_1, t_2) \subset (0, \Lambda)$,

$$\sup_{t_1 \leq t \leq t_2} \|V^1\|(\hat{U}) < \infty.$$ (7.6)

(C3) For all $\phi \in C^1(N \times [0, \Lambda] ; \mathbb{R}^+)$ with $\phi(\cdot, t) \in C^1_c(N)$ and $0 \leq t_1 < t_2 < \Lambda$, we have

$$\|V^1\|((\phi(\cdot, t_2)) - \|C^1\|((\phi(\cdot, t_1))$$ (7.7)

$$\leq \int_{t_1}^{t_2} \int_{G_k(U)} (\nabla_N \phi - \phi h(V^1, \cdot)) \cdot (h(V^1, \cdot) - N_N(\cdot, S)) + \frac{\partial \phi}{\partial t}(\cdot, t) dV^1(\cdot, S) d\tau,$$

where $\nabla_N \phi$ is the tangential derivative of $\phi$ on $N$.

Remark 7.4. We also assume that $h(\phi(\cdot, \cdot))$ exists for a.e. $t$ and locally $L^2$ integrable with respect to $d\|V^1\| dt$. By (C1) and Lemma 7.1, for a.e. $t$, we may replace both the first $h(V^1, \cdot)$ and $h(V^1, \cdot) - N_N(\cdot, S)$ of (7.7) by $h(V^1, \cdot)$ without changing the definition. We may also ask (C3) to hold for $\phi$ defined on $U$ and for $\nabla \phi$ instead of $\nabla_N \phi$ due to Lemma 7.1. In sum, we may equivalently assume the following.

(C3') For all $\phi \in C^1(U \times [0, \Lambda] ; \mathbb{R}^+)$ with $\phi(\cdot, t) \in C^1(U)$ and $0 \leq t_1 < t_2 < \Lambda$, we have

$$\|V^1\|((\phi(\cdot, t_2)) - \|C^1\|((\phi(\cdot, t_1))$$ (7.8)

$$\leq \int_{t_1}^{t_2} \int_{G_k(U)} (\nabla \phi - \phi h(V^1, \cdot)) \cdot \frac{\partial h}{\partial t}(\cdot, t) dV^1(\cdot, S) d\tau.$$

Now let us discuss what can be said under the assumptions (C0)-(C3). Since $H_N(\cdot, S)$ is locally a bounded function with $H_N(x, S) \in (\text{Tan}_x N)^\bot$, we may regard $H_N$ as $u^\bot$ in [13] Theorem 3.2] for any large $p$ and $q$. Thus we may conclude that $M_t := \text{spt} \|V^1\|$ is a $C^{1,\infty}$ graph for a.e. in space-time. This in turn shows that $H_N(x, \text{Tan}_x M_t)$ is $\varsigma$-Hölder continuous since it involves the first derivatives of the graph. This will lead us to the setting of the present paper, which shows partial $C^{2,\alpha}$ regularity with motion law 'velocity $= h(V^1, \cdot)^\top$' being satisfied classically. Then the standard parabolic regularity theory shows partial $C^\infty$ regularity. Thus we proved that any unit density Brakke’s MCF in submanifold is necessarily a.e. smooth, the meaning of a.e. is stated rigorously in Section 3.2. The corresponding statement for general smooth Riemannian manifold setting also follows via Nash’s imbedding theorem.

8. APPENDIX

In this appendix we consider how the second derivatives change under the orthogonal change of variables.

Lemma 8.1. There exist $\beta = \beta(n, k)$ and $c = c(n, k)$ with the following property. Suppose that $A = (a_{ij})_{1 \leq i, j \leq n}$ is an orthogonal matrix with

$$|I - A| \leq \beta.$$ (8.1)

Suppose that two coordinate systems $x = (x_1, \ldots, x_n)$ and $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n)$ are related by $\tilde{x}^i = A x^i$. Suppose that a $k$-dimensional manifold $M$ in $\mathbb{R}^n$ is represented in the $x$ and $\tilde{x}$ coordinate systems as $x_j = f_j(x_1, \ldots, x_n)$ for $j = k + 1, \ldots, n$ and $\tilde{x}_j = \tilde{f}_j(\tilde{x}_1, \ldots, \tilde{x}_k)$ for $j = k + 1, \ldots, n$, respectively. Assume that $f_j$ and $\tilde{f}_j$ are differentiable for $j = k + 1, \ldots, n$. Further assume that

$$|\nabla f|, |\nabla \tilde{f}| \leq 1$$ (8.2)
on the domain of definitions of \( f \) and \( \tilde{f} \). Let two points in \( M \) be expressed in \( x \) coordinate system as \( (x^{(i)}_1, \ldots, x^{(i)}_k, f_{k+1}(x^{(i)}_1, \ldots, x^{(i)}_k), \ldots, f_n(x^{(i)}_1, \ldots, x^{(i)}_k)) \) for \( i = 1, 2 \) and in \( \tilde{x} \) coordinate system as \( (\tilde{x}^{(i)}_1, \ldots, \tilde{x}^{(i)}_k, \tilde{f}_{k+1}(\tilde{x}^{(i)}_1, \ldots, \tilde{x}^{(i)}_k), \ldots, \tilde{f}_n(\tilde{x}^{(i)}_1, \ldots, \tilde{x}^{(i)}_k)) \) for \( i = 1, 2 \), respectively. Then writing \( x^{(i)} = (x^{(i)}_1, \ldots, x^{(i)}_k) \) and \( \tilde{x}^{(i)} = (\tilde{x}^{(i)}_1, \ldots, \tilde{x}^{(i)}_k) \), we have
\[
|\nabla f(x^{(1)}) - \nabla f(x^{(2)})| \leq c(n, k)|\nabla \tilde{f}(\tilde{x}^{(1)}) - \nabla \tilde{f}(\tilde{x}^{(2)})|.
\]
Furthermore, assume that \( f_j \) and \( \tilde{f}_j \) are twice differentiable for \( j = k + 1, \ldots, n \). Then we have
\[
|\nabla^2 f(x^{(i)}) - \nabla^2 \tilde{f}(\tilde{x}^{(i)})| \leq c(n, k)|I - A||\nabla^2 \tilde{f}(\tilde{x}^{(i)})|,
\]
\[
|\nabla^2 f(x^{(1)}) - \nabla^2 f(x^{(2)})| \leq c(n, k)|I - A||\nabla \tilde{f}(\tilde{x}^{(1)}) - \nabla \tilde{f}(\tilde{x}^{(2)})| \max_{i=1,2} |\nabla^2 \tilde{f}(\tilde{x}^{(i)})| + c(n, k)|\nabla^2 \tilde{f}(\tilde{x}^{(1)}) - \nabla^2 \tilde{f}(\tilde{x}^{(2)})|.
\]

**Proof.** For the moment we drop the upper subscript \((i)\) for simplicity. Since two coordinate systems are related by \( \tilde{x}^i = Ax^i \), we have
\[
(\tilde{x}_1, \ldots, \tilde{x}_k, \tilde{f}_{k+1}, \ldots, \tilde{f}_n) = A(x_1, \ldots, x_k, f_{k+1}, \ldots, f_n).
\]
By (6.6), one obtains the following identity for each \( m = k + 1, \ldots, n \),
\[
\sum_{l \leq k} a_{ml}x_l + \sum_{l > k} a_{ml}f_l = \tilde{f}_m \left( \sum_{l \leq k} a_{ml}x_l + \sum_{l \leq k} a_{ml}f_l + \sum_{l > k} a_{ml}x_l + \sum_{l > k} a_{ml}f_l \right).
\]
Differentiating (8.7) with respect to \( x_i \), \( 1 \leq i \leq k \), we have (writing \( f_{i, i} := \frac{\partial f}{\partial x_i} \) and similarly for \( \tilde{f} \))
\[
a_{mi} + \sum_{l > k} a_{ml}f_{l, i} = \sum_{p \leq k} (a_{pi} + \sum_{l > k} a_{pl}f_{l, i}) \tilde{f}_{m, p}.
\]
Assuming that \( f_j \) and \( \tilde{f}_j \) are twice differentiable, and differentiating (8.8) with respect to \( x_j \), \( 1 \leq j \leq k \), we have (writing \( f_{i, ij} := \frac{\partial^2 f}{\partial x_i \partial x_j} \) and similarly for \( \tilde{f} \))
\[
\sum_{l > k} a_{ml}f_{l, ij} = \sum_{p \leq k} \sum_{q \leq k} (a_{pi} + \sum_{l > k} a_{pl}f_{l, i}) (a_{qj} + \sum_{l > k} a_{ql}f_{l, j}) \tilde{f}_{m, pq} + \sum_{p \leq k} \tilde{f}_{m, p} \sum_{l > k} a_{pl}f_{l, ij}.
\]
Moving the last term of (8.9) and (8.10) to the right-hand side, respectively, we obtain
\[
\sum_{l > k} \sum_{p \leq k} (a_{nl} - \sum_{p \leq k} \tilde{f}_{m, p} a_{pl})f_{l, i} = -a_{mi} + \sum_{p \leq k} a_{pi} \tilde{f}_{m, p},
\]
\[
\sum_{l > k} \sum_{p \leq k} (a_{nl} - \tilde{f}_{m, p} a_{pl})f_{l, ij} = \sum_{p \leq k} \sum_{q \leq k} (a_{pi} + \sum_{l > k} a_{pl}f_{l, i}) (a_{qj} + \sum_{l > k} a_{ql}f_{l, j}) \tilde{f}_{m, pq}.
\]
Define \((n - k) \times (n - k)\) matrix-valued function \( \tilde{A} = (\tilde{A}_{ij})_{1 \leq i, j \leq n - k} \) whose \((i, j)\) component is \( a_{i+k+j+k} - \sum_{p \leq k} \tilde{f}_{i+k,p} a_{p, j+k} \). By (8.11) and (8.12), if we restrict \( \varepsilon \) sufficiently small, \( \tilde{A} \) is invertible. Multiplying \( \tilde{A}^{-1} \) from left to (8.10) and (8.11), respectively, we obtain
\[
f_{m, i} = -\sum_{m' > k} (\tilde{A}^{-1})_{mm'} a_{mi} + \sum_{p \leq k, m' > k} (\tilde{A}^{-1})_{mm'} a_{pi} \tilde{f}_{m', p},
\]
\[
f_{m, ij} = \sum_{p \leq k} (a_{pi} + \sum_{l > k} a_{pl}f_{l, i}) (a_{qj} + \sum_{l > k} a_{ql}f_{l, j}) \sum_{m' > k} (\tilde{A}^{-1})_{mm'} \tilde{f}_{m', pq}.
\]
For (8.3), it is not difficult to check (by the definition of inverse matrix) that
\begin{equation}
(8.14) \quad |(\tilde{A}^{-1})_{m,m'}(\tilde{\zeta}^{(1)}) - (\tilde{A}^{-1})_{m,m'}(\tilde{\zeta}^{(2)})| \leq c(n,k)|I - A||\nabla \tilde{f}(\tilde{\zeta}^{(1)}) - \nabla \tilde{f}(\tilde{\zeta}^{(2)})|.
\end{equation}
Then (8.3) follows from (8.12), (8.14) and the triangle inequality. Note that we only need differentiability to obtain (8.3). We next consider the difference between \(f_{m,i,j}\) and \(\tilde{f}_{m,i,j}\). The sum of the right-hand side of (8.13) is separated to \(\sum_{(p,q)=(i,j)} + \sum_{(p,q)\neq (i,j)} =: E_1 + E_2\). Then
\begin{align*}
|E_1 - \tilde{f}_{m,i,j}|&= |(a_{ii} + \sum_{l>k} a_{il}f_{l,i})(a_{jj} + \sum_{l'>k} a_{jl'}f_{l',j}) \sum_{m'>k} (\tilde{A}^{-1})_{m,m'} \tilde{f}_{m',ij} - \tilde{f}_{m,i,j}| \\
&\leq |\tilde{f}_{m,i,j}||1 - (a_{ii} + \sum_{l>k} a_{il}f_{l,i})(a_{jj} + \sum_{l'>k} a_{jl'}f_{l',j})(\tilde{A}^{-1})_{mm'}| \\
&\quad + |(a_{ii} + \sum_{l>k} a_{il}f_{l,i})(a_{jj} + \sum_{l'>k} a_{jl'}f_{l',j}) \sum_{m'\neq m} (\tilde{A}^{-1})_{mm'} \tilde{f}_{m',ij}| \\
&\leq c(n,k)|I - A| \sum_{m'>k} |\tilde{f}_{m',ij}|
\end{align*}
(8.15)
since the off-diagonal elements of \(A\) and \(\tilde{A}^{-1}\) are bounded by \(c(n,k)|A - I|\). For \(E_2\), since \(p \neq i\) or \(q \neq j\), one can check from (8.13) that
\begin{equation}
(8.16) \quad |E_2| \leq c(n,k)|I - A| \sum_{m'>k, p,q \leq k} |\tilde{f}_{m',pq}|.
\end{equation}
Then (8.13), (8.14), (8.16) prove (8.4). For (8.5), we have for \(p \leq k\)
\begin{equation}
(8.17) \quad \sum_{l>k} a_{il}(f_{l,i}(x^{(1)}) - f_{l,i}(x^{(2)})) \leq c(n,k)|I - A||\nabla f(x^{(1)}) - \nabla f(x^{(2)})|.
\end{equation}
Then (8.13), (8.14), (8.17) (with (8.3)) with suitable triangle inequalities prove (8.5). \(\square\)

References

[1] W. Allard, On the first variation of a varifold, Ann. of Math. (2) 95 (1972), 417-491
[2] F.J. Almgren, J.E. Taylor, L.-H. Wang, Curvature-driven flows: a variational approach, SIAM J. Control Optim. 31 (1993), no. 2, 387-438
[3] L. Ambrosio, H. M. Soner, A measure theoretic approach to higher codimension mean curvature flow, Ann. Scuola Norm. Sup Pisa Cl. Sci. 25 (1997), no. 1-2, 27-49
[4] F. Bethuel, G. Orlandi, D. Smets, Convergence of the parabolic Ginzburg-Landau equation to motion by mean curvature, Ann. of Math. (2) 163 (2006), no. 1, 37-163
[5] K. Brakke, The Motion of a Surface by its Mean Curvature, Math. Notes 20, Princeton Univ. Press, Princeton, NJ, 1978
[6] Y.-G. Chen, Y. Giga, S. Goto, Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations, J. Differential Geom. 33 (1991), no. 3, 749-786
[7] K. Ecker, Regularity theory for mean curvature flow, Progress in Nonlinear Differential Equations and their Applications, 57, Birkhäuser Boston, Inc., Boston, MA, 2004
[8] L. C. Evans, J. Spruck, Motion of level sets by mean curvature. I, J. Differential Geom. 33 (1991), no. 3, 635-681
[9] L. C. Evans, J. Spruck, Motion of level sets by mean curvature. IV, J. Geom. Anal. 5 (1995), no. 1, 77-114
[10] G. Huisken, Asymptotic behavior for singularities of the mean curvature flow, J. Differential Geom. 31 (1990), no. 1, 285-299
[11] T. Ilmanen, Convergence of the Allen-Cahn equation to Brakke’s motion by mean curvature, J. Differential Geom. 38 (1993), no. 2, 417-461
[12] R. L. Jerrard, H. M. Soner, Dynamics of Ginzburg-Landau vortices, Arch. Rational Mech. Anal. 142 (1998), no. 2, 99-125
[13] K. Kasai, Y. Tonegawa, A general regularity theory for weak mean curvature flow, preprint.
[14] F. H. Lin, Some dynamical properties of Ginzburg-Landau vortices, Comm. Pure Appl. Math. 49 (1996), no. 4, 323-359
[15] C. Liu, N. Sato, Y. Tonegawa, On the existence of mean curvature flow with transport term. Interfaces Free Bound. 12 (2010), no. 2, 251-277
[16] S. Luckhaus, T. Sturzenhecker, Implicit time discretization for the mean curvature flow equation, Calc. Var. PDE 3 (1995), no. 2, 253-271
[17] J. Metzger, F. Schulze, No mass drop for mean curvature flow of mean convex hypersurfaces, Duke Math. J. 142 (2008), no. 2, 283-312
[18] L. Simon, Lectures on geometric measure theory, Proc. Centre Math. Anal. Austral. Nat. Univ. 3, 1983
[19] B. White, A local regularity theorem for classical mean curvature flows, Ann. of Math. (2) 161 (2005), no. 3, 1487-1519
[20] B. White, The nature of singularities in mean curvature flow of mean-convex surfaces, J. Amer. Math. Soc. 16 (2003), no. 1, 123-138
[21] B. White, The size of the singular set in mean curvature flow of mean-convex surfaces, J. Amer. Math. Soc. 13 (2000), no. 3, 665-695

Department of Mathematics, Hokkaido University, Sapporo 060-0810 Japan.
E-mail address: tonegawa@math.sci.hokudai.ac.jp