SINGULAR VECTORS ON MANIFOLDS OVER NUMBER FIELDS

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Abstract. We extend the notion of singular vectors in the context of Diophantine approximation of real numbers with elements of a totally real number field \(K\). For \(m \geq 1\), we establish a version of Dani correspondence in number fields and prove that under a class of ‘friendly measures’ in \(K^m_S\), the set of singular vectors has measure zero.

Tackling the inheritance question, we show the existence of nondegenerate divergent orbits on manifolds in \(K^m_S\) under the action of a certain one parameter subgroup of \(SL_{m+1}(K_S)\).

1. Introduction

The notion of a singular vector stems from the Dirichlet’s theorem, which is a fundamental result in the theory of Diophantine approximation. Let us denote the standard inner product in \(\mathbb{R}^m\) by \(\mathbf{x} \cdot \mathbf{y}\), where \(\mathbf{x}, \mathbf{y} \in \mathbb{R}^m\). For \(\mathbf{x} \in \mathbb{R}^m\), the Dirichlet’s theorem states that for any large positive integer \(T\), there exists \(p \in \mathbb{Z}, \ 0 \neq q \in \mathbb{Z}^m\) such that \(|q \cdot x + p| < \frac{1}{T^m}\) with \(\|q\| \leq T\). Consequently, \(\mathbf{x} \in \mathbb{R}^m\) is said to be singular if for any \(0 < c\) and for sufficiently large \(T\), we can always find nonzero integral points \(p \in \mathbb{Z}, \ 0 \neq q \in \mathbb{Z}^m\) satisfying

\[|q \cdot x + p| < \frac{c}{T^m}, \|q\| \leq T.\]

Singular vectors were introduced by Khintchine in 1926 ([20]). He showed that on the real line a number is singular if and only if it is rational. He also observed that singular vectors constitute a Lebesgue measure zero set in \(\mathbb{R}^m\) (see [8] Ch V, §7). It is very natural to wonder if similar results hold when real vectors are approximated by vectors from a number field.

In this article we extend the notion of singular vectors in the case of dual approximation, to a totally real number field \(K\) of degree \(d\) over \(\mathbb{Q}\). We prove results about singular vectors on the images of \(K^m_S\) valued maps where, \(S = \infty\) is the set of Archimedean places of \(K\) and \(K_S\) is the adelic ring, which is the product of the completions of \(\sigma(K), \ \sigma \in S\), endowed with the product topology. This paper seems to be the first instance where singular vectors over number fields is studied and it generalizes the results in [29] and [25] to totally real number fields.

Let us explain a rather simple case of our theorem 1.4 for the quadratic extension \(K = \mathbb{Q}(\sqrt{2})\). Suppose one considers the Galois embedding

\[\mathbb{Q}(\sqrt{2}) \rightarrow \mathbb{Q}(\sqrt{2}) \times \mathbb{Q}(\sqrt{2}) \text{ by } (a + b\sqrt{2}) \rightarrow (a + b\sqrt{2}, a - b\sqrt{2}),\]

then it makes sense to approximate vectors in \(\mathbb{R}^2\) by elements \((a + b\sqrt{2}, a - b\sqrt{2}) \in \mathbb{Q}(\sqrt{2}) \times \mathbb{Q}(\sqrt{2})\) since the embedding of \(\mathbb{Q}(\sqrt{2})\) in \(\mathbb{R}^2\) is dense by the weak approximation for \(K\). One can implement the same idea in \(\mathbb{R}^4\) seen as the completion of \(\mathbb{Q}(\sqrt{2})^2 \times \mathbb{Q}(\sqrt{2})^2\).
A simple conclusion from our theorem is that almost every vector \( x \in \mathbb{R}^4 \) is not singular under this approximation. More importantly, our theorems imply that

for almost every \( x \in \mathbb{R}^2, (x, x^2) \in \mathbb{R}^4 \) will not be singular.

On the other hand, we also show that there are infinitely many vectors of the form \( (x, x^2) \in \mathbb{R}^4 \) which are singular, but do not occur ‘trivially’. One can think about ‘trivial’ singular vectors in \( \mathbb{R}^2 \) as the vectors coming from the embedding of \( \mathbb{Q}(\sqrt{2}) \) in \( \mathbb{R}^2 \). It turns out that, in \( \mathbb{R}^2 \) we only have ‘trivial’ singular vectors. Hence, it is natural to wonder if there exist non-‘trivial’ singular vectors in higher powers, for instance, the aforementioned scenario of approximating vectors in \( \mathbb{R}^4 \) by those in \( \mathbb{Q}(\sqrt{2})^2 \). Theorem 1.6 answers this affirmatively.

Structure of the paper is as follows. In section 2 we establish the Dirichlet’s approximation theorem for the case of dual approximation of a vector in \( K_S^m \). In section 3 we describe the embedding of \( K_S^m \) in \( G = SL_{m+1}(K_S) \) and study the orbit of the left regular action of certain one parameter subgroups of \( G \) on \( G/\Gamma \) where, \( O_K \) is the ring of integers in \( K \) and \( \Gamma = SL_{m+1}(O_K) \) is a lattice in \( G \). This study leads to the connection of Diophantine approximation with dynamics on homogeneous spaces. This connection was introduced by S.G.Dani in [11] in the context of approximation by rationals. He related the notion of singular vectors in \( \mathbb{R}^m \) to dynamics in the homogeneous space \( SL_{m+1}(\mathbb{R})/SL_{m+1}(\mathbb{Z}) \), proving that \( x \in \mathbb{R}^m \) is singular if and only if a unimodular lattice in \( \mathbb{R}^{m+1} \) corresponding to \( x \) has a divergent trajectory (leaves every compact subset of \( SL_{m+1}(\mathbb{R})/SL_{m+1}(\mathbb{Z}) \) eventually) under the left action on \( \mathbb{R}^{m+1} \) of a certain diagonalizable semigroup in \( SL_{m+1}(\mathbb{R}) \).

We prove a version of the Dani correspondence in Section 3, relating the singularity property of vectors in \( K_S^m \) with the divergence of the corresponding orbits in \( G/\Gamma \). The main ingredient in the proof is the \( S \)-adic version of Mahler’s criterion for compactness which is also included in this section.

Investigations pertaining to inheritance of Diophantine properties began with a conjecture of K. Mahler’s in 1932 (cf. [5]), settled by V. Sprindžuk in the 1960s (37). Diophantine approximation on manifolds, in particular, the study of singular vectors on manifolds is a well researched area for over many decades, developed with the significant contributions by many mathematicians. Schmidt and Davenport in [14] proved that in \( \mathbb{R}^2 \), almost all points on the parabola \( y = x^2 \) are not singular. This is one of the first results for singular vectors on manifolds. Further development in this direction happened in the late 1970s with the papers [2] and [3] of R. C. Baker, wherein, almost all points of certain submanifolds of \( \mathbb{R}^n \) were seen to be not singular.

Two decades later, more progress came with the works [16] and [15] of M. Dodson et. al., establishing non improvability of the Dirichlet’s theorem on Diophantine approximation for almost all points on a manifold belonging to a general class of smooth manifolds. Y. Bugeaud obtained results similar to Bakers’ for the submanifold \( M = \{ t, t^2, \cdots, t^m \} \subset \mathbb{R}^m \), \( t \in \mathbb{R} \), in the year 2002 (4). For Dirichlet improvability results, readers are referred to [30], [36], and the references therein.

D. Kleinbock and B. Weiss, in [29] and [25], studied singular vectors with weights. We refer the readers to [22] and [21] for details on Diophantine approximation with weights. In [29], it was shown that under the class of friendly measures introduced in [23], the
measure of the set of weighted singular vectors in $\mathbb{R}^n$ is zero. The proof in [29] relies on the quantitative nondivergence results from [24] and [23].

In section 4 we introduce a general class of friendly measures in $K_S^m$ (see subsection S.1) and prove that the friendly measure of the set of singular vectors (ref. Def. S.3) in $K_S^m$ is zero. Our proof relies on a quantitative nondivergence theorem (see theorem S.3) for $O_K$ modules in $K_S^m$.

In section 4 we prove the existence of uncountably many totally irrational singular vectors on nondegenerate submanifolds in $K_S^m$. We request the readers to refer to S.2 for the notion of nondegeneracy and to Definition S.5 for total irrationality in $K_S^m$. With an abuse of terminology, by a submanifold of $K_S^m$ we mean a product of submanifolds of $K_{\sigma}^m$, $\sigma \in S$ (see subsection S.1). The existence of totally irrational singular vectors was first observed by Khintchine, in the year 1926 in [20], in $\mathbb{R}^2$. Such vectors, under the Dani correspondence, display nondegenerate divergent trajectories in $\text{SL}_3(\mathbb{R})/\text{SL}_3(\mathbb{Z})$. Recall that, for a connected linear semisimple group $G$, a one-parameter subgroup $\{g_t\} \subset G$, and an irreducible lattice $\Gamma$ in $G$, we say a trajectory $\{g_t\Gamma\}$ is degenerately divergent if

$$g_tv \to 0 \text{ for some nonzero vector } v \neq 0 \in V,$$

where $V$ is a finite dimensional representation of $G$. In [11], Dani proved that for lattices of $\mathbb{Q}$-rank one every divergent trajectory is degenerately divergent. This means, for instance, that the only singular vectors in $\mathbb{R}$ are the rationals. It is thus natural to expect the same result for $\text{SL}_2(K_S)$, which is of $K$-rank 1. Hence, we show that the only singular vectors in $K_S$ are those in $K$. In the subsequent subsections, we divert our attention to investigate the inheritance of totally irrational singular vectors on manifolds. In [29], it was shown that on nondegenerate (not contained in any affine rational hyperplane) real analytic submanifolds of $\mathbb{R}^m$ of dimension at least 2, there exists totally irrational singular vectors. Note that the results in [29] and also [25], are for the weighted singular vectors. Finally, in this section we prove Theorem S.6 wherein we obtain the inheritance of totally irrational singular vectors on nondegenerate real analytic manifolds in $K_S^m$ (see subsection S.1).

1.1. Diophantine approximation in number fields. Let $K$ be a number field of degree $d$ over $\mathbb{Q}$ and $O_K$ be the ring of integers in $K$. Let $S := \{\sigma_1, \cdots, \sigma_d\}$ be the set of all normalized inequivalent archimedean valuations of $K$. For $\sigma \in S$, $K_\sigma$ will denote the completion of the embedding $\sigma(K)$ in $\mathbb{R}$. We assume $K$ to be a totally real field so that $K_\sigma \cong \mathbb{R}$ for each $\sigma \in S$. Denote by $K_S$ the product space $\prod_{\sigma \in S} K_\sigma (\cong \mathbb{R}^d)$. The diagonal embedding of $K \hookrightarrow K_S$ will be denoted by $\tau$ so that, $\tau(\alpha) = (\sigma_1(\alpha), \cdots, \sigma_d(\alpha))$ for $\alpha \in K$. In the sequel, we will identify $K$ with its image in $K_S$ under this Galois embedding and drop writing $\tau$. For any positive integer $m$, define the inclusion map

$$\tau := (\tau_1, \cdots, \tau_m) : K^m \hookrightarrow K_S^m$$

where, each $\tau_i = \tau : K \longrightarrow K_S$ is the diagonal embedding of $K$ into the $i$-th component of $K_S^m$, $1 \leq i \leq m$. As with $K$ in $K_S$, we identify $K^m$ with its image under $\tau$ and drop writing $\tau$. Along with the above conventions, we collect here a few basic notation used in this article so as to avoid detailing repeatedly in the sequel.

- Throughout this article $K$ will represent a totally real number field.
Elements of $K_S$ and $K_\sigma$ will be denoted by all non bold alphabets, with the corresponding Galois embedding raised on the right side. So, $x^\sigma \in K_\sigma$ and $x = (x^\sigma)_{\sigma \in S} \in K_S$.

When $K_\sigma$ or $K_S$ is raised to the m-th power, the vectors will be denoted by bold alphabets with the corresponding Galois embedding written on top right whenever the vector belongs to $K_\sigma^m$. So, $x^\sigma = (x_1^\sigma, \ldots, x_m^\sigma)$ belongs to $K_\sigma^m$ and $x = (x_1, \ldots, x_m) \in \prod_{i=1}^m K_S = K_S^m$. Under the isomorphism $K_S^m \cong K_{\sigma_1}^m \times \cdots \times K_{\sigma_d}^m$, we will have $x = (x^\sigma)_{\sigma \in S}$. We use either notation depending on the context. Also, an element in a general metric space $X$ will be denoted by a non bold alphabet.

The norm in $K_S$ will be the sup norm given by: For $x = (x^\sigma)_{\sigma \in S} \in K_S$

$$\|x\| := \max_{\sigma \in S} |x^\sigma|.$$

The sup norm in $K_S$ is extended to $K_S^m$ as follows: For $x = (x_1, \ldots, x_m) \in K_S^m$

$$\|x\| := \max_{1 \leq j \leq m} |x_j|.$$

The same notation $\|\|$ is used to represent the sup norm in $K_S$ as well as in $K_S^m$. However, boldness of the alphabets within the norm will help distinguish them in the context.

**Definition 1.1.** An element $x \in K_S$ is said to be singular if for any $c > 0$, there exists a $T_0 > 0$ such that, for all $Q \geq Q_0$ one can find $q_0 \in \mathcal{O}_K$ and $0 \neq q \in \mathcal{O}_K$, having the property $\|qx + q_0\| < c/Q$, with $\|q\| \leq Q$.

We have the following version of Dirichlet’s approximation theorem when approximating $m$-tuples of vectors in $K_S$ in the dual sense.

**Theorem 1.2 (Dirichlet’s theorem).** Suppose $K$ is a number field of degree $d$. Given a vector $x = (x_1, \ldots, x_m) \in K_S^m$, for every $Q >> 0$ there are integral vectors $q_0 \in \mathcal{O}_K$ and $q = (q_1, \ldots, q_m) \in \mathcal{O}_K^m$, not all zero, and a constant $c$ depending only on the number field $K$ such that $\|q \cdot x + q_0\| < \frac{c}{Q^d}$, $\|q\| \leq Q$. 

As mentioned above, one of the first steps in Diophantine approximation is a fundamental result by Dirichlet. A proof of the Dirichlet’s theorem for approximation in $K_S$ can be found in [7, 33, 35, 34] and [18]. We define a singular vector in $K_S$ based on the generalization of Dirichlet’s theorem in [33].
A proof of Theorem 1.2 is given in Section 2. From this theorem we coin the definition of a singular vector in $K^m_S$ likewise.

**Definition 1.3.** A vector $x = (x_1, \cdots, x_m) \in K^m_S$ is said to be singular if for every $c > 0$, for all sufficiently large $Q > 0$ there exists $0 \neq q \in \mathcal{O}^m_K$, $q_0 \in \mathcal{O}_K$ satisfying the following system

$$\|q \cdot x + q_0\| < \frac{c}{Q^m},$$

(1.1)

$$\|q\| \leq Q.$$

The set of singular vectors in $K^m_S$ will be denoted $Sing^m_S$. One of the main theorems in this paper, based on the notions of nondegeneracy in $K^m_S$ and a more general class of friendly measures on $K^m_S$ introduced in Section 4 is the following. This theorem is proved in Section 4

**Theorem 1.4.** Suppose $X = \prod_{\sigma \in S} X_{\sigma}$ is a Besicovitch space and $\mu = \prod_{\sigma \in S} \mu_{\sigma}$ be a Federer measure on $X$ and let $f : X \rightarrow K^m_S \setminus \{0\}$ be a continuous map such that $(f, \mu)$ is nonplanar for $\mu$-almost every point of $X$ and for each $\sigma \in S$, $(f_{\sigma}, \mu_{\sigma})$ is good for $\mu_{\sigma}$-almost every point of $X_{\sigma}$. Then $f, \mu(Sing^m_S) = 0$, where $f, \mu$ is the pushforward measure.

Instead of improving the constant in the Dirichlet’s theorem, one can try to improve the power of $Q$ on the right hand side of the first inequality in the Dirichlet’s theorem. This leads to the definition of the uniform exponent $\hat{\omega}(x)$ of a vector $x \in K^m_S$. This exponent was first introduced in [4] by Y. Bugeaud and M. Laurent, in 2005. It is defined to be the supremum of $v$ such that the following system will have a nonzero solution in $\mathcal{O}^m_K$ for all large enough $Q > 0$,

$$\|q \cdot x + q_0\| < \frac{1}{Q^v}$$

(1.2)

$$\|q\| \leq Q.$$

In Section 5 we will see another interpretation of the uniform exponent. It is immediate from the Dirichlet’s theorem that $\hat{\omega}(x) \geq m$ for every $x \in K^m_S$. Also, it is easy to see that $\hat{\omega}(x) > m$ implies $x \in K^m_S$ is singular.

A real vector $x = (x_1, \cdots, x_m) \in \mathbb{R}^m$, when approximated by rationals, is said to be totally irrational if the coordinates $x_1, \cdots, x_m, 1$ are $\mathbb{Q}$-linearly independent. In $\mathbb{R}$ an element $x$ is singular if and only if it is not totally irrational. In the quest for non-degenerate divergent orbits when approximating a real vector with vectors from a number field, we see that the situation is no different when in $K_S$, i.e., all divergent orbits are degenerately divergent. This result is proved in Section 5. The trivial case of singularity for $x \in K^m_S$ is when there exists $q_0 \in \mathcal{O}_K$, $0 \neq q \in \mathcal{O}^m_K$ such that $\|q \cdot x + q_0\| = 0$, i.e., $\|q^* \cdot x^* + q_0^*\| = 0$ for all $\sigma \in S$. This gives us the following definition of a totally irrational vector in $K^m_S$.

**Definition 1.5.** A vector $x = (x_1, \cdots, x_m) \in K^m_S$ is defined to be totally irrational if for any $0 \neq q \in \mathcal{O}^m_K$, $q_0 \in \mathcal{O}_K$, there exists some $\sigma \in S$ such that $\|q^* \cdot x^* + q_0^*\| \neq 0$. 


The final result in this paper is the following inheritance theorem establishing the existence of nondegenerate divergent orbits on nondegenerate manifolds is $K^m_S$. We refer the reader to Section 5 for a proof of this result.

**Theorem 1.6.** Let $M = \prod_{\sigma \in S} M_{\sigma}$ be a connected real analytic submanifold of $K^m_S$ such that each $M_{\sigma}$ is not contained in any affine $K$-hyperplane in $K^m_{\sigma}$ and $\dim(M_{\sigma}) \geq 2$. Then there exists uncountably many totally irrational singular vectors $x$ in $M$ with $\hat{\omega}(x) = \infty$.

**Remark 1.7.**

1. Although this paper deals only with totally real number fields, the definitions of a singular vector and a totally irrational vector remains the same for a general number field. With the slight modifications necessary in incorporating the complex places, it should be possible to extend the results in this paper to an arbitrary number field $L$, with $S$ being the set of all archimedean places of $L$.

2. In Theorem 1.4, the function $f = (f_{\sigma}) : X \to K^m_S$ is defined componentwise, i.e., if $X = \prod_{\sigma \in S} K^m_{\sigma}$ then, $f(x) := (f_{\sigma}(x^\sigma))$ for $x = (x^\sigma) \in \prod_{\sigma \in S} K^m_{\sigma}$. The functions $f_{\sigma}$ are from $K^m_{\sigma} \to K^m_{\sigma}$. We could consider the more general situation of $f(x) := (f_{\sigma}(x))$ where $f_{\sigma} : \prod_{\sigma \in S} K^m_{\sigma} \to K^m_{\sigma}$ but then, the same proof will work only when $\sum n_{\sigma} \leq m$ and each $f_{\sigma}$ is nondegenerate as a map from $\mathbb{R}^{\sum n_{\sigma}} \to \mathbb{R}^m$.

3. The techniques in this paper should carry forward to yield similar results for the weighted case as well.

4. Finding Hausdorff dimensions of singular vectors in $\mathbb{R}^m$ has been a very active area of research in the recent times. The papers [9], [10], [19], and [12] are a few notable works in this direction. A very recent paper (11) deals with the study of divergent trajectories on a finite product of homogeneous spaces, namely $\prod_i \text{SL}_{m_i+n_i}(\mathbb{R})/\text{SL}_{m_i+n_i}(\mathbb{Z})$. This led them to determine the Hausdorff dimension of a certain jointly singular matrix tuples. It would be interesting to find the Hausdorff dimension of the set of singular vectors in $K^m_S$ introduced in this paper.

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2. **Dirichlet’s theorem for $K^m_S$**

In this section we give the proof of the Dirichlet’s theorem stated in the Introduction, in the context of dual approximation in $K^m_S$.

**Notation:** If $M_K$ is the set of all absolute values on $K$, an $M_K$-divisor $c : M_K \to \mathbb{R}_+$ is a real valued function on the absolute values satisfying

- $c(v) = 1$ for all but finitely many $v \in M_K$.
- For $v$ a non archimedean valuation, there is an element $\alpha \in K$ such that $c(v) = |\alpha|_v$.  


The $K$-size of $\mathfrak{c}$ is defined to be $\|\mathfrak{c}\| = \prod_{v \in M_K} \mathfrak{c}(v)^{N_v}$, where $N_v$ is the multiplicity of $v$. We should interpret $\mathfrak{c}(v)$ as prescribing the lengths of the sides of a box (all but a finite number of which are 1) then, the $K$-size of $\mathfrak{c}$ determines the volume of the box. For our purpose we need consider only the $M_K$-divisors $\mathfrak{c}$ for which $\mathfrak{c}(v) = 1$ for all $v$ non archimedean. Also, $N_v = 1$ for all archimedean $v$.

**Proof of Theorem 1.2.** Let $X$ be the set of points in $K_S^{m + 1}$ given by

$$X = \left\{(q_1, \ldots, q_m, p) \in K_S^{m + 1} \mid \|q_i\| \leq Q(D_K)^{\frac{1}{2d}}, 1 \leq i \leq m; \|\sum_{i=1}^{m} q_i \| \leq \frac{(D_K)^{\frac{1}{2d}}}{Q^d}\right\}$$

where $D_K$ is the discriminant of $K$ and $\sqrt{D_K}$ is the volume of the fundamental domain of the lattice $\mathcal{O}_K$ in any embedding $K \otimes_{\mathbb{Q}} \mathbb{R}$. Clearly, $X$ is a convex subset in $K_S^{m + 1}$.

For each $\sigma_i \in S$, if $v_i$ is the corresponding absolute value, define an $M_K$-divisor $\mathfrak{c}_i$ with $\mathfrak{c}_i(v_i) = 2Q(D_K)^{\frac{1}{2d}}$. Then, for $1 \leq j \leq m$, the contribution to the volume of $X$ from each $K_S$ component equals the $K$-size of $\mathfrak{c}_i$, i.e.,

$$\|\mathfrak{c}_i\| = \prod_{i=1}^{d} 2Q(D_K)^{\frac{1}{2d}} = (2Q)^d \sqrt{D_K}. $$

For $j = m + 1$, we define the $M_K$-divisor $\mathfrak{c}$ to be $\mathfrak{c}(v_i) = \frac{2}{Q^d} Q(D_K)^{\frac{1}{2d}}$. An easy computation gives $\|\mathfrak{c}\| = \prod_{i=1}^{d} \frac{2}{Q^d} Q(D_K)^{\frac{1}{2d}} = \frac{2^d}{Q^{dm}} \sqrt{D_K}$. Now, the volume of $X$

$$\text{Vol}(X) = \|\mathfrak{c}\| \prod_{j=1}^{m} \|\mathfrak{c}_i\| = 2^d(m + 1) \sqrt{D_K} ^{m+1}. $$

Applying Minkowski’s convex body theorem, we get the desired lattice point of $\mathcal{O}_K^{m+1}$. □

3. Mahler’s compactness criterion and Dani Correspondence

For an arbitrary integer $m > 0$, consider, $\mathbf{x} = (\mathbf{x}^{\sigma})_{\sigma \in S} \in \prod_{\sigma \in S} K_{\sigma}^m \simeq K_S^m$. Let $G = \text{SL}_{m+1}(K_S)$ and let $u_{\mathbf{x}} \in G$ be the matrix $u_{\mathbf{x}} = \begin{bmatrix} 1 & \mathbf{x} \\ 0 & I_m \end{bmatrix}$. Since $K$ is assumed to be totally real, each $K_{\sigma}^m \simeq \mathbb{R}^m$ and so $\mathbf{x}^{\sigma} \in K_{\sigma}^m$ embeds in $\text{SL}_{m+1}(K_{\sigma})$ as

$$\mathbf{x}^{\sigma} \mapsto u_{\mathbf{x}^{\sigma}} = \begin{bmatrix} 1 & \mathbf{x}^{\sigma} \\ 0 & I_m \end{bmatrix},$$

giving an embedding of $K_S^m \hookrightarrow G \simeq \prod_{\sigma \in S} \text{SL}_{m+1}(K_{\sigma})$ as $\mathbf{x} \mapsto \prod_{\sigma \in S} u_{\mathbf{x}^{\sigma}}$ and in $\text{SL}_{d(m+1)}(\mathbb{R})$ as

$$\mathbf{x} \mapsto \text{diag}((u_{\mathbf{x}^{\sigma}})_{\sigma \in S}).$$

We consider the one parameter subgroup of $G$ denoted by $\{g_t = (g_t^{\sigma})\}$ and given by the block diagonal matrices comprising blocks of the one parameter subgroup in each
$\text{SL}_{m+1}(K_\sigma)$, that is,

$$g_t^\sigma = \begin{bmatrix} e^{mt} & 0 & \cdots & 0 \\ 0 & e^{-t} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & e^{-t} \end{bmatrix}.$$

The discrete subgroup $\Gamma = \text{SL}_{m+1}(\mathcal{O}_K)$ is a lattice in $G$ and the set $\{g_t\}$ acts on the homogeneous space $G/\Gamma$ on the left. We identify $G/\Gamma$ with the set $\Omega_{S,m+1}$ of discrete rank $(m+1)$ $\mathcal{O}_K$-module $g\mathcal{O}_K^{m+1}$ (corresponding to the elements $g\Gamma$, $g \in G$) in $K_S^{m+1}$, having an $\mathcal{O}_K$-basis $\{p_1, p_2, \cdots, p_{m+1}\}$ for $K_S^{m+1}$ such that, for each $\sigma \in S$, $\{\sigma(p_1), \sigma(p_2), \cdots, \sigma(p_{m+1})\}$ form sides of a parallelopiped of volume 1 in $\mathbb{R}^{m+1}$.

With the above, we have introduced the notation necessary to establish the Dani correspondence. We further require the $S$-adic version of the Mahler’s compactness criterion for the same. A general version of the Mahler’s criterion in the case of a number field, for a finite set of places $S$ containing the archimedean places, is proved in [26] and [27]. We provide below the proof in the case of a totally real number field, for $S = \{\}$. As above, let

$$\Omega_{S,m+1} = \{g\mathcal{O}_K^{m+1} \mid g \in G\}.$$  

**Theorem 3.1** (Mahler’s Criterion). A subset $A \subset \Omega_{S,m+1}$ is relatively compact if and only if there exists an open $\epsilon$-neighbourhood, $B_\epsilon$, of 0 in $K_S^n$ such that, for all $\Lambda \in A$, $\Lambda \cap B_\epsilon = \{0\}$.

**Proof.** Let $K[X] := K[X_1, \cdots, X_{(m+1)(m+1)}]$ for a deg $d$ number field $K \subset \mathbb{C}$. We identify $M_{m+1}(\mathbb{C})$ with $\mathbb{C}^{(m+1)^2}$ in the natural way. Consider the algebraic $K$-group $G$ in $\mathbb{C}^{(m+1)^2}$ determined by the zero set of the polynomial $\det(X) - 1 \in K[X]$. The $K$-points of this group $G(K)$ is $\text{SL}_{m+1}(K)$. Using the Weil restriction of scalars functor $R_K:\mathbb{Q}$, we can construct a $\mathbb{Q}$-algebraic subgroup $G' = R_K/G$ of $\text{SL}_N(\mathbb{Z})$ (suitably large $N$). The construction is such that, the $\mathbb{Q}$-points of $G'$ are the $K$-points of $G$, i.e., $\text{SL}_{m+1}(K) = G'(\mathbb{Q})$. We refer the reader to Section 2.1.2 in [32] and also Section 6.1 in [38] for the construction of $G'$. However, we mention here that if the basis $\{\alpha_1, \cdots, \alpha_d\}$ for $K$ over $\mathbb{Q}$, in the beginning of the construction, is chosen to be an $\mathcal{O}_K$-basis such that $\mathcal{O}_K = \sum_i \mathbb{Z}\alpha_i$, then we also have $\text{SL}_{m+1}(\mathcal{O}_K) = G'(\mathbb{Z})$. Further, as $\sigma_1(K) \subset \mathbb{R}$ for all $i$, we get $\text{SL}_{m+1}(K_S) = G'(\mathbb{R})$. The Mahler’s compactness criterion for $\text{SL}_{m+1}(\mathbb{R})/\text{SL}_{m+1}(\mathbb{Z})$ in [31] now completes the proof. 

**Theorem 3.2** (Dani Correspondence). A vector $x \in K_S^m$ is singular if and only if the corresponding trajectory $\{g_tu_x\mathcal{O}_K^{m+1} \mid t \geq 0\}$ is divergent in $G/\Gamma$.

**Proof.** Suppose $\{g_tu_x\mathcal{O}_K^{m+1}\}$ is not divergent in $G/\Gamma$. By Theorem 3.1 this means that, there exist an $\epsilon > 0$ and a sequence $t_k \to \infty$ as $k \to \infty$ such that, $\|g_{t_k}u_x\mathcal{O}_K^{m+1}\| \geq \epsilon$ for some nonzero $(q_0^{(k)}, \ldots, q_m^{(k)}) \in \mathcal{O}_K^{m+1}$. The inequality above gives

$$\|\left(e^{mt_k}(q_0^{(k)}) + q_1^{(k)}x_1 + \cdots + q_m^{(k)}x_m, e^{-t_k}q_1^{(k)}, \cdots, e^{-t_k}q_m^{(k)}\right)\| \geq \epsilon.$$
Therefore, for $Q_k = \varepsilon e^{t_k}$ the system $[1,1]$ has no nonzero solution in $\mathcal{O}_{K}^{m+1}$ for $c = \varepsilon^{m+1}$. Hence $x$ is not singular. Conversely, suppose that $x$ is not singular. Then, there exists an $\varepsilon > 0$ such that, for $Q_k \to \infty$ the system $[1,1]$ has no nonzero solution in $\mathcal{O}_{K}^{m+1}$. Choosing $t_k > 0$ such that $e^{t_k} = Q_k$ yields $\delta(g_{t_k} u_x \mathcal{O}_{K}^{m+1}) \geq \varepsilon$ as $t_k \to \infty$, i.e., $g_{t_k} u_x \mathcal{O}_{K}^{m+1}$ does not diverge as $t \to \infty$.

\[\square\]

4. Singular vectors and friendly measure

4.1. Friendly measure. A metric space $X$ is called Besicovitch [28] if there exists a constant $N_X$ such that the following holds: for any bounded subset $A$ of $X$ and for any family $\mathcal{B}$ of nonempty open balls in $X$ such that every $x \in A$ is a center of some ball in $\mathcal{B}$, there is a finite or countable subfamily $\{B_i\}$ of $\mathcal{B}$ with

\[1_A \leq \sum_i 1_{B_i} \leq N_X.\]

We now define $D$-Federer measures following [23]. Let $\mu$ be a Radon measure on $X$, and $U$ an open subset of $X$ with $\mu(U) > 0$. We say that $\mu$ is $D$-Federer on $U$ if

\[\sup_{x \in \text{supp} \mu, r > 0} \frac{\mu(B(x, 3r))}{\mu(x, r)} < D.\]

We say that $\mu$ as above is Federer if, for $\mu$-a.e. $x \in X$, there exists a neighbourhood $U$ of $x$ and $D > 0$ such that $\mu$ is $D$-Federer on $U$. We refer the reader to [23] and [28] for examples of Federer measures.

Suppose $\mu = \prod_{\sigma \in S} \mu_{\sigma}$ is a measure on $K_{\sigma}^{m}$, where $\mu_{\sigma}$ is a measure on $K_{\sigma}^{m}$. We call $\mu$ nonplanar if, for all $\sigma \in S$, $\mu_{\sigma}(L_{\sigma}) = 0$ for any affine hyperplane $L_{\sigma}$ of $K_{\sigma}^{m}$. Denote by $d_{L_{\sigma}}(x)$ the (Euclidean) distance from $x$ to an affine subspace $L \subset K_{\sigma}^{m}$, and let

\[L(\varepsilon) := \{x \in K_{\sigma}^{m} : d_{L_{\sigma}}(x) < \varepsilon\}.\]

For $A \subset K_{\sigma}^{m}$ with $\mu_{\sigma}(A) > 0$ and $f$ a $K_{\sigma}$-valued function on $K_{\sigma}^{m}$, denote

\[\|f\|_{\mu_{A}} := \sup_{x \in A \cap \text{supp} \mu_{\sigma}} |f(x)|.\]

Given $C, \alpha > 0$ and an open subset $U_{\sigma}$ in $K_{\sigma}^{m}$, we say that $\mu_{\sigma}$ is $(C, \alpha)$-decaying on $U_{\sigma}$ if, for any non-empty open ball $B \subset U_{\sigma}$ centered in $\text{supp} \mu_{\sigma}$, any affine hyperplane $L \subset K_{\sigma}^{m}$, and any $\varepsilon > 0$ we have

\[\mu_{\sigma}(B \cap L(\varepsilon)) \leq C \left( \frac{\varepsilon}{\|d_{L_{\sigma}}\|_{\mu_{\sigma}, B}} \right)^{\alpha} \mu_{\sigma}(B).\]

Finally, we call the measure $\mu = \prod_{\sigma} \mu_{\sigma}$ as friendly if, $\mu$ is nonplanar and $\forall \sigma \in S$, for $\mu_{\sigma}$-a.e. $x_{\sigma} \in K_{\sigma}^{m}$, there exist a neighborhood $U_{\sigma}$ of $x_{\sigma}$ and positive $C_{\sigma, \alpha_{\sigma}}, D > 0$ such that $\mu_{\sigma}$ is $D$-Federer and $\mu_{\sigma}$ is $(C_{\sigma, \alpha_{\sigma}})$-decaying on $U_{\sigma}$. Let $X = \prod_{\sigma \in S} X_{\sigma}$ be a metric space, $\mu = \prod_{\sigma \in S} \mu_{\sigma}$ a measure on $X$ and $f = (f_{\sigma}) = ((f_{1, \sigma}, \ldots, f_{m, \sigma}))$ a map from $X \to K_{\sigma}^{m}$. The pair $(f, \mu)$ will be called nonplanar at $x_0 = (x_{0, \sigma}) \in X$ if, for any neighborhood $B = \prod_{\sigma \in S} B_{\sigma}$ of $x_0$, the restrictions of $1, f_{1, \sigma}, \ldots, f_{m, \sigma}$ to $B_{\sigma} \cap \text{supp} \mu_{\sigma}$ are linearly independent over $\mathbb{R}$ for all $\sigma \in S$. This is equivalent to saying that for each $\sigma$,
We borrow the following notation from [28]. Let \( \Delta \) be a nonplanar \( K_m^S \) and set \( \text{rank}(\Delta) := \dim_K(K\Delta). \)

4.2. Nondegeneracy in \( K_m^S \). Before we talk about nondegenerate maps in \( K_m^S \), let us recall the notion of \((C, \alpha)\)-good functions which was first introduced in [24]. Let \( X \) be a metric space and \( \mu \) a Borel measure on \( X \). For a subset \( U \) of \( X \) and \( C, \alpha > 0 \), say that a Borel measurable function \( f : U \to \mathbb{R} \) is \((C, \alpha)\)-good on \( U \) with respect to \( \mu \) if for any open ball \( B \subset U \) centered in \( \text{supp} \mu \) and \( \varepsilon > 0 \), we have the condition

\[
\mu \{ x \in B : |f(x)| < \varepsilon \} \leq C \left( \frac{\varepsilon}{\|f\|_{\mu, B}} \right)^\alpha \mu(B).
\]

Here, \( \|f\|_{\mu, B} = \sup \{ c : \mu(\{ x \in B : |f(x)| > c \}) > 0 \} \). Suppose \( f : X \to \mathbb{R}^n \) is a map and \( \mu \) is a Borel measure on \( X \), we say \((f, \mu)\) is good at \( x_0 \in X \) if there exists a neighborhood \( V \) of \( x_0 \) and positive \( C, \alpha \) such that, any linear combination of \( 1, f_1, \ldots, f_n \) is \((C, \alpha)\)-good on \( V \) with respect to \( \mu \).

Let us recall the definition of nondegeneracy as in [24]. Consider a \( d \) dimensional submanifold \( M = \{ f(x) \mid x \in U \} \) of \( \mathbb{R}^n \), where \( U \) is an open subset of \( \mathbb{R}^d \) and \( f = (f_1, \ldots, f_n) \) is a \( C^m \)-imbedding of \( U \) into \( \mathbb{R}^n \). For \( l \leq m \), say that \( y = f(x) \) is an \( l \)-nondegenerate point of \( M \) if, the space \( \mathbb{R}^n \) is spanned by the partial derivatives of \( f \) at \( x \) of order up to \( l \). We will say that \( y \) is nondegenerate if it is \( l \)-nondegenerate for some \( l \). Finally, we call the manifold \( M \) nondegenerate if for \( \lambda \text{-a.e. } x \in U \), \( f(x) \) is nondegenerate, where \( \lambda \) is the Lebesgue measure on \( \mathbb{R}^d \).

We can now define nondegenerate maps in \( K_m^S \). Suppose \( f = (f_\sigma) : \prod_{\sigma \in S} U_\sigma \to K_m^S \) be a continuous map, where \( U_\sigma \subset K_m^d \) be an open set. We say \( f \) is nondegenerate if each \( f_\sigma : U_\sigma \to K_m^S \) is nondegenerate in the above sense. Let us recall the following proposition from [24], which guarantees that for a nondegenerate \( f = (f_\sigma) \) in \( K_m^S \), each \( f_\sigma \) is good with respect to the Lebesgue measure \( \lambda \) and hence, by Lemma 2.2 in [28], \( f \) is \((C, \alpha)\)-good for some positive \( C, \alpha \).

**Proposition 4.1.** Let \( f = (f_1, \ldots, f_n) \) be a \( C^l \) map from an open subset \( U \) of \( \mathbb{R}^d \) to \( \mathbb{R}^n \), and let \( x_0 \in U \) be such that \( \mathbb{R}^n \) is spanned by partial derivatives of \( f \) at \( x_0 \) of order up to \( l \). Then there exists a neighborhood \( V \subset U \) of \( x_0 \) and a positive \( C \) such that any linear combination of \( 1, f_1, \ldots, f_n \) is \((C, 1/dl)\)-good on \( V \).

4.3. Quantitative Nondivergence in \( K_m^S \). In the following discussion we assume,

- \( \mathcal{D} \) is an integral domain, that is, a commutative ring with 1 and without zero divisors;
- \( \mathcal{K} \) is the quotient field of \( \mathcal{D} \);
- \( \mathcal{R} \) is a commutative ring containing \( \mathcal{K} \) as a subring.

If \( \Delta \) is a \( \mathcal{D} \)-submodule of \( \mathcal{R}^m \), we denote by \( \mathcal{K}\Delta \) (resp. \( \mathcal{R}\Delta \)) its \( \mathcal{K} \)- (resp. \( \mathcal{R} \)-) linear span inside \( \mathcal{R}^m \), and define the rank of \( \Delta \) by

\[
\text{rank}(\Delta) := \dim_{\mathcal{K}}(K\Delta).
\]

We borrow the following notation from [28] §6.3. Denote by \( \text{GL}(m, \mathcal{R}) \) the group of \( m \times m \) invertible matrices with entries in \( \mathcal{R} \) and set

\[
\mathcal{M}(\mathcal{R}, \mathcal{D}, m) := \{ g\Delta \mid g \in \text{GL}(m, \mathcal{R}), \Delta \text{ is a submodule of } \mathcal{D}^m \}.
\]
and
\[ \Psi(D, m) := \text{the set of all nonzero primitive submodules of } D^m. \]

A function \( \nu : \mathfrak{M}(R, D, m) \to \mathbb{R}_+ \) is norm-like if the following three properties hold:

(N1) For any \( \Delta, \Delta' \in \mathfrak{M}(R, D, m) \) with \( \Delta' \subset \Delta \) and \( \text{rank} (\Delta') = \text{rank} (\Delta) \) one has \( \nu(\Delta') \geq \nu(\Delta) \);

(N2) there exists \( C_\nu > 0 \) such that for any \( \Delta \in \mathfrak{M}(R, D, m) \) and any \( \gamma \notin R\Delta \) one has \( \nu(\Delta + D\gamma) \leq C_\nu \nu(\Delta) \nu(D\gamma) \);

(N3) for every submodule \( \Delta \) of \( D^m \), the function \( GL(m, R) \to \mathbb{R}_+, \ g \to \nu(g\Delta) \), is continuous.

We will also need the following Theorem from [13], which is an improvement of the Theorem in §6.3 of [28].

\textbf{Theorem 4.2.} Let \( X \) be a metric space, \( \mu \) a uniformly Federer measure on \( X \), and let \( D \subset K \subset R \) be as above, \( R \) being a topological ring. For \( m \in \mathbb{N} \), let a ball \( B = B(x_0, r_0) \subset X \) and a continuous map \( h : \tilde{B} \to GL(m, R) \) be given, where \( \tilde{B} \) stands for \( B(x_0, 3^m r_0) \). Let \( \nu \) be a norm-like function on \( \mathfrak{M}(R, D, m) \). For any \( \Delta \in \Psi(D, m) \) denote by \( \psi_\Delta \) the function \( x \mapsto \nu(h(x)\Delta) \) on \( \tilde{B} \). Now suppose for some \( C, \alpha > 0 \) one has

(i) for every \( \Delta \in \Psi(D, m) \), the function \( \psi_\Delta \) is \((C, \alpha)\)-good on \( \tilde{B} \) with respect to \( \mu \),

(ii) for every \( \Delta \in \Psi(D, m) \), \( \|\psi_\Delta\|_{\mu, B} \geq \rho^{\text{rank}(\Delta)} \),

(iii) \( \forall x \in \tilde{B} \cap \text{supp} \mu, \ #\{\Delta \in \Psi(D, m) \mid \psi_\Delta(x) < \rho\} < \infty. \)

Then for any positive \( \varepsilon \leq \rho \) one has

\[ \mu \left( \left\{ x \in B \mid \nu(h(x)\gamma) < \frac{\varepsilon}{C_\nu} \text{ for some } \gamma \in D^m \setminus \{0\} \right\} \right) \leq mC(N_X D^2 m)^m \left( \frac{\varepsilon}{\rho} \right)^\alpha \mu(B). \]

A few notation are in place before stating the next theorem. Define the content of a point \( y = (y^\sigma) \in K^m_s \) as \( c(y) := \prod_{\sigma \in S} \|y^\sigma\| \). For any \( g \in GL(m, K_S) \), we define \( \delta(gO^m_K) := \min\{c(y) \mid y \in gO^m_K \setminus \{0\}\} \). As a consequence of the aforementioned theorem we have the following.

\textbf{Theorem 4.3.} Let \( X \) be a Besicovitch metric space, \( \mu \) a Federer measure on \( X \), and let \( D \subset K \subset R \) be as above be the set of infinite places of \( K \). For \( m \in \mathbb{N} \), let a ball \( B = B(x_0, r_0) \subset X \) and a continuous map \( h : \tilde{B} \to GL(m, K) \) be given, where \( \tilde{B} \) stands for \( B(x_0, 3^m r_0) \). Now, suppose that for some \( C, \alpha > 0 \) and \( 0 < \rho < 1 \) one has

(i) for every \( \Delta \in \Psi(O_K, m) \), the function \( \text{cov} (h(\cdot)\Delta) \) is \((C, \alpha)\)-good on \( \tilde{B} \) with respect to \( \mu \);

(ii) for every \( \Delta \in \Psi(O_K, m) \), \( \sup_{x \in B \cap \text{supp} \mu} \text{cov} (h(x)\Delta) \geq \rho^{\text{rank}(\Delta)} \).

Then for any positive \( \varepsilon \leq \rho \), one has

\[ \mu \left( \left\{ x \in B \mid \delta(h(x)O_K^m) < \varepsilon \right\} \right) \leq mC(N_X D^2 m)^m \left( \frac{\varepsilon \sqrt{D_K}}{\rho} \right)^\alpha \mu(B). \]

\textbf{Proof.} The proof goes line by line as the Theorem in §6.3 of [28], but using Theorem 4.2 in place of the Theorem in §6.3 of [28]. \( \square \)
4.4. Proof of the main theorem.

**Theorem 4.4.** Suppose \( \mu \) is a friendly measure in \( K^m_S \), then \( \mu(\text{Sing}^m_S) = 0 \).

Note that \( \mu \) is friendly if and only if each \( \mu_\sigma \) is Federer, \( (\text{Id}, \mu) \) is nonplanar and \( \forall \sigma \in S, (\text{Id}, \mu_\sigma) \) is good for \( \mu_\sigma \)-a.e. points. Hence, Theorem 4.4 follows from Theorem 4.4. Also, as a corollary we have the following.

**Corollary 4.5.** Suppose \( f = (f_\sigma) : U = \prod_{\sigma \in S} U_\sigma \rightarrow K^m_S \) be a continuous map, where \( U_\sigma \subset K^d_\sigma \) be an open set. Let \( f \) be a nondegenerate map. Then \( f_*\lambda(U \cap \text{Sing}^m_S) = 0 \), where \( \lambda \) is the Lebesgue measure.

**Proposition 4.6.** Let \( X \) be a Besicovitch metric space and \( \mu \) be a Federer measure on \( X \). Denote \( \tilde{B} := B(x, 3^{m+1}r) \). Suppose we are given a continuous function \( f : X \rightarrow K^m_S \) and \( C, \alpha > 0 \) with the following properties:

(i) \( x \mapsto \text{cov}(g_t u_{f(x)} \Delta) \) is \( (C, \alpha) \)-good with respect to \( \mu \) in \( \tilde{B} \) for all \( \Delta \in \mathcal{P}(O_K, m + 1) \),

(ii) There exists \( c > 0 \) and a sequence \( t_i \rightarrow \infty \) such that, for any \( \Delta \in \mathcal{P}(O_K, m + 1) \) one has

\[
\sup_{x \in B \cap \text{supp} \mu} \text{cov}(g_t u_{f(x)} \Delta) \geq c^{\text{rank}(\Delta)}.
\]

Then, \( \mu\{x \in B \mid f(x) \text{ is singular}\} = 0 \)

**Proof.** We will check that the map \( h = g_t u_f \) satisfies the assumptions of Theorem 4.3 with respect to the measure \( \nu = f_*\mu|_B \) where, condition (i) of this proposition is same as the condition (i) of Theorem 4.3. Then condition (ii) of this proposition gives (ii) of Theorem 4.3 for all integers \( t_i \). Therefore by Theorem 4.3 for any \( 1 > \varepsilon > 0 \) we have that

\[
\mu\left(\left\{x \in B \mid \delta(g_t u_{f(x)}O_{K}^{m+1}) < \varepsilon c\right\}\right) \\
\leq (m + 1)C(N_X D^2_\mu)^{m+1}(\sqrt{D_K})^{\alpha} \varepsilon^\alpha \mu(B) \\
= E \varepsilon^\alpha.
\]

From Theorem 4.2 we have

\[
\mu\{x \in B \mid f(x) \text{ is singular}\} = \mu\{x \in B \mid g_t u_{f(x)} \Gamma, t \geq 0 \text{ is divergent}\}
\]

We want to show that for any \( \varepsilon > 0 \), we have \( \mu\{x \in B \mid f(x) \text{ is singular}\} \leq \varepsilon^\alpha \). Let \( \varepsilon > 0 \) be given. Then for any \( n \in \mathbb{N} \), there exists a set \( B_n \subset \{x \in B \mid f(x) \text{ is singular}\} \) such that \( \mu(\{x \in B \mid f(x) \text{ is singular}\}) \leq \mu(B_n) + \frac{1}{n} \) and there exists a \( t_i \) such that

\[
B_n \subset \left\{x \in B \mid \delta(g_t u_{f(x)}O_{K}^{m+1}) < \varepsilon c\right\}.
\]

Hence, for every \( n \in \mathbb{N} \) and \( \varepsilon > 0 \) we have

\[
\mu(\{x \in B \mid f(x) \text{ is singular}\}) \leq E \varepsilon^\alpha + \frac{1}{n},
\]

and we conclude. \( \square \)
Proof of theorem 1.4. We will let \( R = K_S \) and \( D = \mathcal{O}_K \). Let us denote the set of rank \( j \) submodules of \( D^{m+1} \) as \( S_{m+1,j} \). For any nonzero submodule \( \Delta \in S_{m+1,j} \), there exists an element \( w \) of \( \bigwedge^j(D^{m+1}) \) such that \( \text{cov}(\Delta) \geq (\sqrt{D_K})^j c(w) \) and \( \text{cov}(g_u x \Delta) \geq (\sqrt{D_K})^j c(g_u x w) \), where \( x \in K_S^m \). We take \( e_0, e_1, \ldots, e_m \in R^{m+1} \) as the standard basis of \( R^{m+1} \). It will be convenient for us to use the following notation, \( \{ e_i \} \) forms the standard basis of \( K^m_\sigma \) over \( K_\sigma \). Then the standard basis of \( \bigwedge^j R^{m+1} \) will be \( \{ e_I = e_{i_1} \wedge \cdots \wedge e_{i_j} \mid I \subset \{0, \ldots, m\} \text{ and } i_1 < i_2 < \cdots < i_j \} \). For an element \( a = \sum a_I e_I \), we define \( \|a\| = \max_I \|a_I\| \). We can write any \( w \in D^{m+1} \) as \( w = \sum w_I e_I \), where \( w_I \in D \).

Let us note the action of the unipotent flows on the coordinates of \( w \). Recall, for \( x = (x_1, \cdots, x_m) \) we have \( u_x = \begin{bmatrix} 1 & x \\ 0 & I_m \end{bmatrix} \) where, the \( \sigma \) component is \( u_x^{\sigma} = \begin{bmatrix} 1 & x^\sigma \\ 0 & I_m \end{bmatrix} \). Also, \( g_t \) is taken such that the \( \sigma \) component, \( g^\sigma_t = \begin{bmatrix} e^{mt} & 0 & \cdots & 0 \\ 0 & e^{-t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{-t} \end{bmatrix} \). Now, observe that \( u_x^{\sigma} \) leaves \( e_0^\sigma \) invariant and sends \( e_i^\sigma \) to \( x_i^\sigma e_0^\sigma + e_i^\sigma \) for \( i \geq 1 \). Therefore

\[
 u_x^{\sigma}(e_i^\sigma) = \begin{cases} 
 e_i^\sigma & \text{if } 0 \in I \\
 e_i^\sigma + \sum_{\{i \notin I\}} x_i^\sigma e_I \cup \{0\} & \text{if } 0 \notin I.
\end{cases}
\]

Moreover, under the action of \( g_t^\sigma \), the vectors \( e_i^\sigma \) are eigenvectors with eigenvalue \( e^{-t} \) for \( i \geq 1 \) and \( e_0^\sigma \) is an eigenvector with eigenvalue \( e^{mt} \). Therefore

\[
 g^\sigma_t u_x^{\sigma}(e_i^\sigma) = \begin{cases} 
 e^{(m-j+1)t} e_i^\sigma & \text{if } 0 \in I \\
 e^{-jt} e_i^\sigma \pm e^{(m-j+1)t} \sum_{\{i \notin I\}} x_i^\sigma e_I \cup \{0\} & \text{if } 0 \notin I.
\end{cases}
\]

Thus, for \( w \in \bigwedge^j(D^{m+1}) \), \( w = \sum w_I e_I \) with \( w_I \in D \) and so, we get the \( \sigma \) component of \( g_u x w \) to be

\[
 (g_u x w)^\sigma = e^{-jt} \sum_{\{I \mid 0 \notin I\}} w_I e_I^\sigma + e^{(m-j+1)t} \sum_{\{I \mid 0 \in I\}} \left( w_I + \left( \sum_{i \notin I} \pm w_{I \cup \{0\}} x_i^\sigma \right) \right) e_I^\sigma.
\]

(4.6) \( (g_u x w)^\sigma = e^{-jt} \sum_{\{I \mid 0 \notin I\}} w_I e_I^\sigma + e^{(m-j+1)t} \sum_{\{I \mid 0 \in I\}} \left( w_I + \left( \sum_{i \notin I} \pm w_{I \cup \{0\}} x_i^\sigma \right) \right) e_I^\sigma. \]

It will be convenient for us to use the following notation,

\[
 c(w) = \begin{pmatrix}
 c(w)^0 \\
c(w)^1 \\
 \vdots \\
c(w)^n 
\end{pmatrix},
\]

where, \( c(w)^i = \sum_{J \subseteq \{1, \ldots, m\}} \sum_{\#J = j - 1} w_{J \cup \{i\}} e_J \in \bigwedge^{j-1}(V_0) \) and \( V_0 \) is the \( \mathcal{O}_K \) submodule of \( K_{m+1}^\sigma \) generated by \( e_1, \ldots, e_m \). Moreover, \( V_0^\sigma \) will be the subspace of \( K_{m+1}^\sigma \) generated by \( e_1^\sigma, \cdots, e_m^\sigma \).
so that, $V_0 = \prod_{\sigma \in S} V_0^\sigma$. We may therefore write

\[ (g_t u_x w)^\sigma = e^{-jt} \sum_{\{I|0 \in I\}} w_I e_I^\sigma + e^{(m-j+1)t} \left( e_0^\sigma \wedge \sum_{i=0}^m x_i^\sigma c(w)_i \right) = e^{-jt} \pi_{\sigma}(w) + e^{(m-j+1)t} e_0^\sigma \wedge \tilde{x}^\sigma \cdot c(w), \]

where, $\tilde{x} = (1, x)$ and $\pi_{\sigma}$ is the orthogonal projection from $\wedge^j (K_{\sigma}^{m+1}) \to \wedge^j V_0^\sigma$. Hence, we have

\[
\text{cov}(g_t u_x) \Delta \geq (\sqrt{D_K})^m c(g_t u_x w) = \prod_{\sigma \in S} \max \left( e^{(m-j+1)t} \| \sum_{i=0}^m x_i^\sigma c(w)_i \|, e^{-jt} \| \pi_{\sigma}(w) \| \right) \]

Thus, for $x = f(x)$,

\[
\sup_{x = (x^\sigma) \in B \cap \text{supp} \mu} \text{cov}(g_t u_f(x)) \Delta \\
\geq (\sqrt{D_K})^m \prod_{\sigma \in S} \max \left( e^{(m-j+1)t} \sup_{x^\sigma \in B_\sigma \cap \text{supp} \mu_{\sigma}} \| \tilde{f}_\sigma(x^\sigma) \cdot c(w) \|, e^{-jt} \| \pi_{\sigma}(w) \| \right),
\]

where, $\tilde{f} = (1, f_1, \cdots, f_m)$ and $B = \prod B_\sigma$, $B_\sigma$ is a ball in $X_\sigma$. Since $(f, \mu)$ is nonplanar, for each $\sigma$, the restrictions of $1, f_1, \cdots, f_m$ to $B_\sigma \cap \text{supp} \mu_{\sigma}$ are linearly independent. Hence,

\[
\sup_{x^\sigma \in B_\sigma \cap \text{supp} \mu_{\sigma}} \| \tilde{f}_\sigma(x^\sigma) \cdot c(w) \| \geq \| c(w) \| \ \forall \ \sigma \in S,
\]

which guarantees condition (4.2) of Proposition 4.6. Since for each $\sigma \in S$, $(f_\sigma, \mu_{\sigma})$ is good for $\mu_{\sigma}$-almost every point, we can apply Lemma 2.2 from [28] in order to verify that condition (i) of Proposition 4.6 is satisfied. Therefore, by Proposition 4.6 we can conclude the theorem.

5. Totally irrational singular vectors on manifolds

5.1. Totally irrational vectors. From the discussion about totally irrational vectors in Section 1.1 it is clear that the condition for a vector $x \in K_S^m$ to be not totally irrational is equivalent to saying that $x$ is trivially singular, i.e., $\| q \cdot x + q_0 \| = 0$. This in turn implies that $x$ lies in an affine $K$-hyperplane in $K_S^m$. In $K_S$, this means that all vectors that are not totally irrational are precisely those in $K$. Consequently, as in the case with approximation using rationals, we have

**Theorem 5.1.** $x \in K_S$ is singular if and only if it belongs to $K$.

**Proof.** If a vector belongs to $K$ then it is singular is the easy direction. Let $G' = \text{SL}(2, K_S)$ and $\Gamma' = \text{SL}(2, \mathcal{O}_K)$. To prove the other direction, we begin with recalling the identification of $G'/\Gamma'$ with the set $\Omega_{S,2}$ of discrete rank 2 $\mathcal{O}_K$ modules as described in Section 3. For a singular vector $x \in K_S$, by Dani correspondence, the trajectory of lattices $\{g_t u_x \mathcal{O}_K^2\}$ is divergent. Thus, given $\varepsilon > 0$, there exists a $t_\varepsilon > 0$ such that

\[
(5.1) \text{for all } t \geq t_\varepsilon, \text{ there exists } \left( \frac{q_t}{p_t} \right) \in \mathcal{O}_K^2 \text{ with } \| g_t u_x \left( \frac{q_t}{p_t} \right) \| < \varepsilon.
\]
As each $K^2_\sigma \simeq \mathbb{R}^2$, we can use the following Lemma 5.2 to conclude the nonexistence of two linearly independent vectors $\{v_1, v_2\}$ in a given lattice $g_iu_i \mathcal{O}_K^2$ having simultaneously very small norms. The identification in Section 3 identifies a disjoint countable collection of distinct closed subsets of which falls into the $\varepsilon$-open set $A$ in $K^2_\sigma$. Let $\varepsilon_\sigma > 0$ be the threshold length beyond which any two linearly independent vectors $\{v^\sigma, w^\sigma\}$ of a lattice in $\Omega_{S,2}$ cannot coexist inside the $\varepsilon_\sigma$ ball in $K^2_\sigma \simeq \mathbb{R}^2$. Now, as the number of places is finite, for any $\varepsilon < \min\{\varepsilon_\sigma \mid \sigma \in S\}$, continuity of the map $t \mapsto g_t$ along with the divergence condition in equation (5.1) ensures that for any time $t > t_0 \geq t_\varepsilon$, the nonzero integral vector $v_t = \begin{pmatrix} p_t \\ q_t \end{pmatrix}$ which falls into the $\varepsilon$ ball in $K^2_S$ (centered at 0) for the time $v_{t_0} = \begin{pmatrix} p_{t_0} \\ q_{t_0} \end{pmatrix}$ remains inside the ball, should necessarily be a nonzero integral multiple of $v_{t_0}$ (by Lemma 5.2), i.e., $v_t = \lambda_tv_{t_0}$ for $0 \neq \lambda_t \in \mathcal{O}_K$. Hence, the trajectory diverges degenerately. From the discussions in the beginning of this section, we conclude that $x$ belongs to $K$.

Lemma 5.2. There exists an $\varepsilon > 0$ depending on the choice of the norm on $\mathbb{R}^2$ such that, no unimodular lattice in $\mathbb{R}^2$ contains two linearly independent vectors each of norm less than $\varepsilon$.

A proof for this lemma can be found in [17].

5.2. Inheritance in manifolds over $K_S$. Notation: Let $H : \sum_{i=1}^m h_i x_i = h_0$ denote an affine $K$-hyperplane in $K^m$ where, $(h_0, \cdots, h_m) \in \mathcal{O}_K^{m+1}$ is a primitive lattice point. For the affine $K$-hyperplane $A := \prod_{\sigma \in S} H_\sigma$ in $K^m_S$, we let $|A| := \|(h_1, \cdots, h_m)\|$ be referred to as the norm of $A$. Let the collection of affine $K$-hyperplanes of $K^m_S$ as described in Section [14] be denote $\mathcal{A}$. Note that, this is a countable collection and as a consequence of the Dirichlet’s theorem, it is dense in $K^m_S$.

Let $\Phi := \mathcal{O}_K^m \setminus \{0\} \to \mathbb{R}_+$ be a proper function where, $\mathcal{O}_K^m$ is endowed with the discrete topology. For example, we may consider the norm map. We may extend $\Phi$ to all of $\mathcal{O}_K^m$ by defining $\Phi(0) = 0$. The irrationality measure function is then defined as

$$\eta_{\Phi,n}(t) := \min_{\{q_0,q \in \mathcal{O}_K^{m+1} \setminus \{0\} \mid \Phi(q) \leq t\}} \|q \cdot x + q_0\|.$$

The following abstract result exhibits the existence of totally irrational singular vectors on locally closed subsets of $K^m_S$. This result is a generalization of Theorem 1.1 in [25] to the context of number fields. Call a subset $M \subset K^m_S$ locally closed if $M = \overline{M} \cap W$ for an open set $W$ in $K^m_S$.

Theorem 5.3. Suppose $M = \prod_{\sigma \in S} M_\sigma$ is a locally closed subset of $K^m_S$. Let $\mathcal{L}$ and $\mathcal{L}'$ be a disjoint countable collection of distinct closed subsets of $M$ such that, for each $L_i \in \mathcal{L}$, $A_i$ is an affine $K$-hyperplane in $K^m_S$ containing $L_i$. Assume the following holds:

1) Collection $\mathcal{L}$ is dense in $M$.
2) $\mathcal{L} \cup \mathcal{L}' = \{x \in M \mid x$ is not totally irrational.\}
3) For each \( L \in \mathcal{L} \) and \( T > 0 \), \( L = \bigcup_{i \in \mathcal{L}, |A_i| > T} L \cap L_i \).

4) Transversality condition: For \( L \in \mathcal{L} \), \( L = L \setminus \left( \bigcup_{Y \in \mathcal{F}} Y \cup \bigcup_{Y' \in \mathcal{F}'} Y' \right) \), for any finite subcollection \( \mathcal{F}, \mathcal{F}' \) of \( \mathcal{L}, \mathcal{L}' \) respectively, such that \( L \notin \mathcal{F} \).

Then for any non-increasing function \( \delta : \mathbb{R}_+ \to \mathbb{R}_+ \) and an arbitrary proper function \( \Phi : \mathcal{O}_K^m \setminus \{0\} \to \mathbb{R}_+ \), there exists uncountably many totally irrational singular vectors \( x \in M \) with \( \eta_{\Phi,x}(t) \leq \delta(t) \) for all large \( t \).

**Proof.** Define the set

\[ \mathcal{S} = \{ x \in M \mid \exists t_0 \text{ such that } \forall t \geq t_0, \eta_{\Phi,x}(t) \leq \delta(t) \text{ and } x \text{ is totally irrational} \} \]

We produce elements in \( \mathcal{S} \) and simultaneously prove \( \mathcal{S} \) is uncountable. For this we start with assuming \( \mathcal{S} \) is countably infinite, say \( \mathcal{S} = \{ x_1, x_2, \cdots \} \), and arrive at a contradiction as every \( x \) obtained with the method below essentially belongs to \( \mathcal{S} \) but, by (c) below, is not any of the existing elements \( x_i \) in \( \mathcal{S} \). As \( M \) is locally closed, there exists an open subset \( W \subseteq K_S^m \) such that \( M = M \cap W \). Let \( \mathcal{U}_0 = W \). For the primitive lattice point \( (h^{(i)}_0, h^{(i)}_1, \cdots, h^{(i)}_m) \) determining the affine \( K \)-hyperplane \( A_{r_i} \) containing the closed subset \( L_{r_i} \) in \( M \), we let \( p_i := h^{(i)}_0 \) and \( q_i := (h^{(i)}_1, \cdots, h^{(i)}_m) \). The idea now is to build a nested sequence \( \{ \mathcal{U}_i \}_{i=1}^\infty \) of bounded open subsets of \( W \), and a sequence of strictly increasing indices \( \{ r_i \} \) corresponding to the choice of the closed subsets \( L_{r_i} \) in each stage \( i \). This sequence of sets and indices will satisfy the following conditions:

1. \( \overline{\mathcal{U}_i} \subset \mathcal{U}_{i-1} \) for all \( i \geq 1 \),
2. \( \Phi(q_i) > \Phi(q_{i-1}) \) for all \( i \in \mathbb{N} \),
3. \( \mathcal{U}_i \) is disjoint from \( L_k \cup L'_k \cup \{ x_k \} \) for all \( k < i \),
4. \( \mathcal{U}_i \cap L_{r_i} \neq \emptyset \) for all \( i \in \mathbb{N} \),
5. \( \delta(\Phi(q_i)) \) for all \( i \in \mathbb{N} \) and \( x \in \mathcal{U}_i \), we have the local uniformity condition \( \| q_{i-1} \cdot x + p_{i-1} \| < \delta(\Phi(q_i)) \).

Proving the existence of this sequence of subsets \( \{ \mathcal{U}_i \}_{i=1}^\infty \) suffices as any vector \( x \) belonging to the nonempty set \( M \cap \bigcap \mathcal{U}_i \) is seen to be a totally irrational singular vector. With the above notation in place, the construction of the sets \( \{ \mathcal{U}_i \} \) and the proof of \( \mathcal{S} \) being uncountable follows, almost verbatim, the proof for Theorem 1.1 in [23]. To see \( \eta_{\Phi,x}(t) \leq \delta(t) \), condition (b) along with \( \delta \) being non-increasing yields the irrationality measure function \( \eta_{\Phi,x} \) to be non-increasing. Let \( t_0 = \Phi(q_1) \) then, for any \( t \geq t_0 \) there exist an \( r \in \mathbb{N} \) such that, \( t \in [\Phi(q_r), \Phi(q_{r+1})] \). Using (c), we get

\[ \eta_{\Phi,x}(t) \leq \eta_{\Phi,x}(\Phi(q_r)) \leq \min_{p \in \mathcal{O}_K} \| q_r \cdot x + p \| \leq \| q_r \cdot x + p_r \| < \delta(\Phi(q_{r+1})) \leq \delta(t). \]

In the rest of the article we let \( M = \prod_{\sigma \in S} M_{\sigma} \subset K_S^m \) be a real analytic submanifold of \( K_S^m \) (refer Section [1]). When \( \dim(M_{\sigma}) \geq 2 \), in some cases we can reduce the problem of finding totally irrational singular vectors in higher dimensions to surfaces, making it easier to tackle. We have the following proposition towards this end.

**Proposition 5.4.** Let \( K \) be a totally real number field of degree \( d \) over \( \mathbb{Q} \). Suppose \( M = \prod_{\sigma \in S} M_{\sigma} \) is a connected real analytic submanifold of \( K_S^m \) such that each \( M_{\sigma} \) is not
contained in any affine $K$-hyperplane in $K^m_\sigma$ and $\dim(M_\sigma) \geq 2$. Then $M$ contains a bounded real analytic submanifold $N = \prod_{\sigma \in S} N_\sigma$ such that, each $N_\sigma$ is not contained in any affine $K$-hyperplane of $K^m_\sigma$ and $\dim(N_\sigma) = 2$ for all $\sigma$.

Proof. Let $k_\sigma := \dim(M_\sigma)$ so that $k = \sum_{\sigma \in S} k_\sigma$ is the dimension of $M$. We prove by inducting on $\dim(M) = k$. If $k_\sigma = 2$ for all $\sigma$, the assertion is trivially seen to be true. Assume that the result holds true for all connected real analytic submanifolds $N$ of $K^m_S$ with $2d < \dim(N) < k$. Then, the proof for a connected real analytic manifold $M$ with $\dim(M) = k$ is as follows. For each $\sigma$ we identify $K^m_\sigma$ with its image under the isomorphism $K^m_\sigma \simeq \mathbb{R}^m$. We may then assume that each $M_\sigma$ is the image of the open $k_\sigma$-dimensional cube $I_{k_\sigma} := (0,1)^{k_\sigma}$ under a real analytic immersion $f_\sigma: I_{k_\sigma} \to K^m_\sigma$. Then, $M = \prod_{\sigma \in S} f_\sigma(I_{k_\sigma})$.

Now, consider the connected real analytic submanifold $N$ of $M$ defined as follows. Pick a $\tau \in S$. We then define $N = \prod_{\sigma \in S, \sigma \neq \tau} N_\sigma \times N_\tau := \prod_{\sigma \in S, \sigma \neq \tau} (g_\sigma)_\sigma(I_{k_\sigma}) \times (g_\sigma)_\tau(I_{k_{\tau}-1})$ where, the real analytic immersion $g_\sigma$ is given by

$$N_\sigma = (g_\sigma)_\sigma(I_{k_\sigma}) = f_\sigma(I_{k_\sigma}) = M_\sigma, \text{ for all } \sigma \in S, \sigma \neq \tau,$$

$$N_\tau = (g_\sigma)_\tau(x_1, \ldots, x_{k_{\tau}-1}) = f_\tau(x_1, \ldots, x_{k_{\tau}-1}, \alpha), \text{ for some } \alpha \in (0,1).$$

As $M_\tau$ is not contained in any affine $K$-hyperplane in $K^m_\tau$, it is not contained in $H_\tau$ for any $A = \prod_{\sigma \in S} H_\sigma$ in $A$. Note that as $A$ varies in $A$, $H_\tau$ varies over all affine $K$-hyperplanes in $K^m_\tau$. We have the following

Claim: There exists of an $\alpha \in (0,1)$ such that $N_\tau = (g_\alpha)_\tau(I_{k_{\tau}-1})$ is not contained in $H_\tau$ for any $A = \prod_{\sigma \in S} H_\sigma$. $\text{ in } A$.

If the claim holds true, the submanifold $N$ of $K^m_S$ is connected, real analytic and such that $N_\sigma$ is not contained in any affine $K$-hyperplane in $K^m_\sigma$ for all $\sigma$. As $\dim(N) < k$, the induction hypothesis now completes the proof. We now establish the claim. Every $A \in A$ is of the form $A_h = \prod_{\sigma \in S} H_{h,\sigma}$ for some primitive lattice point $h \in O_K^{m+1}$ (ref. 5.2). Let $P = \{h \in O_K^{m+1} \mid h \text{ is primitive}\}$. If possible let there exist no $\alpha \in (0,1)$ such that $N_\tau = (g_\alpha)_\tau(I_{k_{\tau}-1})$ is not contained in any affine $K$-hyperplane in $K^m_{\tau}$. Thus, for each $\alpha$, there exists an $h \in P$ satisfying $(g_\alpha)_\tau(I_{k_{\tau}-1}) \subset H_{h,\tau}$. As $\bigcup_{\alpha \in (0,1)} (g_\alpha)_\tau(I_{k_{\tau}-1}) = M_\tau$, taking union over all $h \in P$ gives

$$I_{k_\tau} = \bigcup_{h \in P} f_\tau^{-1}(M_\tau \cap H_{h,\tau}).$$

Now, as in Lemma 3.5 in [25], given that the right hand side is a countable union, using Baire category theorem we can conclude the existence of an $h_0 \in P$ with the property that $M_\tau \subset H_{h_0,\tau}$. This gives a contradiction. Hence, the claim. $\square$

Remark 5.5. In Proposition 3.4 of [25], the hypothesis regards $M$ as a connected real analytic submanifold of $\mathbb{R}^m$ not contained in any proper affine rational subspace. The fact that any proper affine rational subspace is contained in an affine rational hyperplane, is observed in the proof therein. Similar fact holds true in the case of affine $K$-subspaces of $K^m_S$. Hence, Proposition 5.3 holds true if affine $K$-hyperplane is replaced by affine $K$-subspace of $K^m_S$. 

\[ \square \]
Proposition 5.6. Let \( M = \prod_{\sigma \in S} M_{\sigma} \) be a bounded connected real analytic submanifold of \( K^m_S \) such that each \( M_{\sigma} \) is not contained in any affine \( K \)-hyperplane in \( K^m_S \). Let \( \dim(M_{\sigma}) = 2 \) for all \( \sigma \in S \). Let \( A \in A \) be given by \( A = \prod_{\sigma \in S} H_{\sigma} \). If \( F \) is the collection of points \( x \in M \cap A \) such that, for some \( \sigma \in S \), there does not exist a neighbourhood \( U_{\sigma} \) of \( x^\sigma \) with the property that \( M_{\sigma} \cap H_{\sigma} \cap U_{\sigma} \) is a real analytic curve then, the connected components of \( M \cap A \cap F^c \) are finitely many with their \( \sigma \)-components being real analytic curves.

Proof. Define for \( \sigma \in S \)

\[
\tilde{F}_\sigma := \left\{ x^\sigma \in M_{\sigma} \cap H_{\sigma} \mid \exists \text{ a neighbourhood } U_{\sigma} \text{ of } x^\sigma \text{ such that } M_{\sigma} \cap H_{\sigma} \cap U_{\sigma} \text{ is a real analytic curve} \right\}.
\]

Then, \( F \) can be written as \( F = \bigcup_{\sigma \in S} \left( \tilde{F}_\sigma \times \prod_{\tau \neq \sigma} M_{\tau} \cap H_{\tau} \right) \). Now,

\[
F^c = \left\{ x \in M \cap A \mid \exists \text{ a neighbourhood } U = \prod_{\sigma \in S} U_{\sigma} \text{ of } x \text{ such that } M_{\sigma} \cap H_{\sigma} \cap U_{\sigma} \text{ is a real analytic curve } \forall \sigma \in S \right\}.
\]

This implies that \( x \in F^c \), if and only if \( x^\sigma \notin \tilde{F}_\sigma \) for all \( \sigma \in S \). We have

\[
M \cap A \cap F^c = \prod_{\sigma \in S} M_{\sigma} \cap H_{\sigma} \cap F^c_{\sigma}.
\]

Arguing along the lines of Proposition 3.2 of [25], we can conclude that for each \( \sigma \in S \), \( \tilde{F}_\sigma \) is a finite set and hence \( M_{\sigma} \cap H_{\sigma} \cap F^c_{\sigma} \) has only finitely many connected components, each of which is a real analytic curve. 

Remark 5.7. Retaining the terminology from [29], a connected component \( \gamma \) of \( M \cap A \cap F^c \), \( A \in A \), in the above Proposition will be called a basic connected component of \( M \cap A \). Note that \( \gamma = \prod_{\sigma \in S} \gamma_{\sigma} \) where, each \( \gamma_{\sigma} \) is a connected component of \( M_{\sigma} \cap H_{\sigma} \cap \tilde{F}^c_{\sigma} \), which is a real analytic curve by the Proposition 3.2 of [25]. As \( \tilde{F}_\sigma \) is finite for each \( \sigma \), the collection \( \{ \gamma_{\sigma} \} \) forms a dense subset of \( M_{\sigma} \cap H_{\sigma} \) and hence, for each \( \sigma \), \( \{ \gamma_{\sigma} \} \supset M_{\sigma} \cap H_{\sigma} \). This implies that the set \( \{ \gamma = \prod_{\sigma \in S} \gamma_{\sigma} \} \) comprising the closures of all the basic connected components of \( M \cap A \), \( A \in A \) will contain all points of \( \{ M \cap A \mid A \in A \} \).

We recall a few notation before the proof of Theorem 1.6. For each \( \sigma \), \( \{ e_{\sigma}^1, \ldots, e_{\sigma}^r \} \) will denote the standard basis of the vector space \( K^m_{\sigma} \) over \( K_{\sigma} \). The bold symbols \( \{ e_i \}_{i=1}^m \) will denote the basis of the \( K_{\sigma} \)-module, \( K^m_{\sigma} \). Thus, \( e_i = (e_i^\sigma)_{\sigma \in S} \). An affine subspace \( T = \prod_{\sigma \in S} T_{\sigma} \) in \( K^m_{\sigma} \) is normal to, say \( e_1 \), will mean that \( T_{\sigma} \) is normal to \( e_1^\sigma \) for all \( \sigma \in S \).

Proof of Theorem 1.6. With Remark 5.5 in place, we prove the theorem considering affine \( K \)-hyperplanes of \( K^m_{\sigma} \) in the place of affine \( K \)-subspaces of \( K^m_{\sigma} \). By Proposition 5.4, we may replace \( M \) with a bounded connected submanifold of \( K^m_{\sigma} \) such that for each \( \sigma \), \( M_{\sigma} \) is a real analytic surface not contained in any affine \( K \)-hyperplane in \( K^m_{\sigma} \). For \( x \in M \), the tangent space \( T_xM \) is the product space \( \prod_{\sigma \in S} T_{x^\sigma} M_{\sigma} \). Let \( \mathcal{A} := \{ A_i = \prod_{\sigma \in S} H_{\sigma}^{(i)} \in A \mid M \cap A_i \neq \emptyset \} \) be the countable subcollection of \( A \) comprising all the affine \( K \)-hyperplanes of \( K^m_{\sigma} \) that are normal to either \( e_1 \) or to \( e_2 \). As each \( M_{\sigma} \) is assumed to be not contained in any affine \( K \)-hyperplane inside \( K^m_{\sigma} \), there exists an \( x \in M \) for which, for all \( \sigma \in S \), \( T_{x_{\sigma}} M_{\sigma} \) is not normal to either \( e_1^\sigma \) or to \( e_2^\sigma \). For each \( \sigma \), we may thus further replace \( M_{\sigma} \) by a smaller connected open subset, possibly an open connected neighbourhood of \( x \) in
\( M \), so as to ensure that for every \( y \in M \), \( T_y M \) is not normal to either \( e_1 \) or \( e_2 \). Now, for each \( \sigma \), \( M_\sigma \) can be seen as the graph of a smooth function over its projection to the vector subspace \( V_\sigma \) of \( K_\sigma^m \) spanned by \( \{ e_1', e_2' \} \). We will choose \( M_\sigma \) small enough so that this projection is a convex set for all \( \sigma \). Modifying \( M \) as above without loss of generality, for any \( A_i = \prod_{\sigma \in S} H^{(i)}_\sigma \in \tilde{A} \) and \( x \in M \cap A \), \( T_x M_\sigma \) intersects \( H^{(i)}_\sigma \) transversally for each \( \sigma \), i.e., \( T_x M_\sigma \cap H^{(i)}_\sigma \) will be an affine one dimensional subspace of \( K_\sigma^m \). Using implicit function theorem, we conclude that \( M_\sigma \cap H^{(i)}_\sigma \) is a real analytic curve.

Our aim is to evoke Theorem 5.3 to show the existence of totally irrational singular vectors on \( M \). For this we first define the collection \( \mathcal{L} := \{ L_i \} \) of closed subsets of \( M \). Let \( L_i = \prod_{\sigma \in S} L_{i,\sigma} \) where, \( L_{i,\sigma} = M_\sigma \cap H^{(i)}_\sigma \) for some \( A_i \in \tilde{A} \). Clearly, each \( L_i \) is a product of real analytic curves \( L_{i,\sigma} \). Further, each \( L_{i,\sigma} \) is connected as the projection of \( M_\sigma \) on \( V_\sigma \) is chosen to be convex for all \( \sigma \). Thus, \( L_i \) is connected for all \( i \).

Now, we define the collection \( \mathcal{L}' := \{ L_j' \} \) where \( L_j' = \prod_{\sigma \in S} L_{j,\sigma}' \). Each \( L_j' \) will be the closure \( \overline{\gamma} \) of a basic connected component of a nonempty intersection \( M \cap A \), \( A \in A \), such that the \( \sigma \)-components \( L_{j,\sigma}' = \overline{\gamma}_\sigma \) are not contained in any \( L_{i,\sigma} \). By Proposition 5.6 for each \( A \in A \), the basic connected components of \( M \cap A \) are finite. Thus, \( \{ \mathcal{L}' \} \) is a countable collection. It follows from the discussion in the remark above, that \( \mathcal{L}' \) will contain all points of \( \{ M \cap A \mid A \in A \setminus \tilde{A} \} \). Hence, \( \mathcal{L} \cup \mathcal{L}' \) will contain all points of \( M \) that are not totally irrational. This establishes (2).

Denote by \( l_{i,\sigma} \) the one dimensional affine \( K \)-subspace in \( V_\sigma \) that is the projection of \( L_{i,\sigma} \). Then, \( \{ l_{i,\sigma} \} \), as \( L_i \) varies in \( \mathcal{L} \), forms a dense collection of vertical and a dense collection of horizontal lines in \( V_\sigma \). Let \( V = \prod_{\sigma \in S} V_\sigma \). Clearly, \( V \) is a \( K_S \)-subspace of \( K_S^m \) of dimension 2 generated by \( e_1 \) and \( e_2 \). Thus, \( \mathcal{L}_1 := \{ l_i \mid A_i \text{ normal to } e_1 \} \) forms a collection of 1 dimensional affine subspaces of \( V \) normal to \( e_1 \) and indexed by elements of \( K \). Similarly, \( \mathcal{L}_2 := \{ l_i \mid A_i \text{ normal to } e_2 \} \) is a dense collection of 1 dimensional subspaces in \( V \) normal to \( e_2 \) and indexed by the elements of \( K \). The collections \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) are dense in \( V \) as \( K \) is dense in \( K_S \) by weak approximation. Given \( x \in K_S \), weak approximation gives a sequence \( p_i / q_i \in K \), such that \( \lim_{i \to \infty} p_i / q_i = x \). Thus, all points \( x \in M \) with \( T_x M \) not normal to \( e_1 \) and \( x_1 \in K_S \) being the \( e_1 \) coordinate in the projection \( V \), can be approximated by a sequence of points \( \{ x_i \} \subset L_i \), where \( x_i \) has \( p_i / q_i \) as the \( e_1 \) coordinate in \( V \), lying on a hyperplane \( l_i \in \mathcal{L}_1 \) and \( \{ p_i / q_i \} \) approximate \( x_1 \in K_S \) via weak approximation. One can argue to approximate a point \( x \in M \) having projection \( x_2 \in K_S \) lying in the span if \( e_2 \) in \( V \). This proves (1), the density of \( \mathcal{L} \) in \( M \).

To prove (3), note that for \( \mathbf{x} = (x_1, x_2) \in l_i \), if \( p_i / q_i \to x_2 \) then, we can get a sequence of points \( \mathbf{x}_i = (x_1, p_i / q_i) \) lying on \( l_i \in \mathcal{L}_2 \). Note that \( |A_j| = \|q_j\| \) tends to \( \infty \) as \( \mathbf{x}_i \) tends to \( \mathbf{x} \). Thus, for any \( T > 0 \), and \( \mathbf{x} \in L_i \) with \( \mathbf{x} \) as its projection in \( V \), we can find affine \( K \)-hyperplanes \( A_j \) with \( |A_j| > T \), giving a sequence of points lying on \( L_i \cap L_j \) converging to \( \mathbf{x} \).

For a given \( L_i \in \mathcal{L} \), each \( L_{i,\sigma} \) is a real analytic curve. Hence, for \( j \neq i \), \( L_{i,\sigma} \cap L_{j,\sigma} \) is either empty or consists of a single point. If \( \mathcal{F} = \{ L_k \}_{k=1}^n \subset \mathcal{L} \), with \( L_i \neq L_k \forall k \) then, the above discussion implies that for each \( \sigma \), \( L_{i,\sigma} \cap (\bigcup_{k} L_{k,\sigma}) \) is dense in \( L_{i,\sigma} \). Thus, \( \prod_{\sigma \in S} \left( L_{i,\sigma} \setminus (\bigcup_{k} L_{k,\sigma}) \right) \) is dense in \( L_i \). Now,
This implies \( L_i \setminus (\bigcup_k L_k) \) is dense in \( L_i \). Following the arguments used to establish condition (c) in the proof of Theorem 1.7 in [25], we can see that each \( L'_{k',\sigma} \cap L_{i,\sigma} \) has no nonempty interior relative to the \( \sigma \)-adic topology on \( L_{i,\sigma} \). Thus, \( L_{i,\sigma} \setminus (\bigcup_k L'_{k',\sigma}) \) is dense in \( L_{i,\sigma} \) for each \( \sigma \). Hence using (5.2) with \( L'_{k',\sigma} \) in place of \( L'_{k,\sigma} \), we may conclude that \( L_i \setminus (\bigcup_k L_k) \) is dense in \( L_i \). This proves condition (4) of Theorem 5.3.

To see \( \hat{\omega}(x) = \infty \), notice that, in the light of the definition of the irrationality measure function \( \eta_{\Phi, x} \), the uniform exponent of \( x \in K_m^S \) defined in the introduction can be interpreted as the following for \( \Phi = \| \cdot \| \):

\[
\hat{\omega}(x) = \sup \left\{ \nu : \lim_{t \to \infty} t^{\nu} \eta_{\| \cdot \|, x}(t) < \infty \right\}.
\]

Considering \( \delta(t) = e^t/t^\nu \) in the inequation \( \eta_{\| \cdot \|, x}(t) \leq \delta(t) \) of Theorem 5.3 now yields the desired conclusion.

\[
\Box
\]

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