Large Deviations of Non-Stochastic Interacting Particles on Random Graphs

James MacLaurin*

November 3, 2020

Abstract

This paper concerns the large deviations of a system of interacting particles on a random graph. There is no time-varying stochasticity, and the only sources of disorder are the random graph connections, and the initial condition. The average number of afferent edges on any particular vertex must diverge to infinity as $N \to \infty$, but can do so at an arbitrarily slow rate. These results are thus accurate for both sparse and dense random graphs. A particular application to sparse Erdos-Renyi graphs is provided. The theorem is proved by pushing forward a Large Deviation Principle for a ‘nested empirical measure’ generated by the initial conditions to the dynamics. The nested empirical measure can be thought of as the density of the density of edge connections: the associated weak topology is more coarse than the topology generated by the graph cut norm, and thus there is a broader range of application.

1 Interacting Particle Systems on Heterogeneous Graphs

Understanding the behavior of interacting particle systems on heterogeneous graphs has been a topic of increasing interest over the last five years. This work has a diverse range of applications, including neuroscience [7, 25], mean-field games [9, 2], Josephson arrays [35] and machine learning [31]. Of the papers on the Large Deviations of interacting particle systems, the majority concern systems with stochasticity [40, 16, 15, 8, 17, 14, 32]. Even if there is disorder in the graph connections (such as in [17, 14, 32, 33, 29]), in the large size limit the noise from the stochasticity dominates any disorder resulting from the connection topology. By contrast, in this paper we prove a Large Deviations result for networks without any stochasticity, so that the only source of randomness is the random graph topology and possibly the initial condition.

The theory of Large Deviations concerns the probability of rare events [20, 21, 19, 23]. In neuroscience, for instance, it is known that networks of neurons can support transient phenomena such as UP / DOWN transitions [34], neural avalanches and waves [36]. It is disputed the extent to which the seeming disorder and chaoticity of much neural activity is due to white noise fluctuations (arising from synaptic transmission failure, for instance), or the underlying disordered structure of networks of neurons [38, 25]. Thus there is a need for a theory that elucidates how the disordered structure of neural networks can support a diversity of different behaviors [36]. This paper provides a step in this direction.

There is a longstanding mathematical literature on the asymptotic behavior of networks of interacting particle systems, dating from the seminal work of Tanaka [40] and Sznitman [39]. In recent years, effort has been directed towards understanding the asymptotic behavior of networks of interacting particles on large random graphs subject to white noise [17, 30, 14, 32, 27, 3, 26, 29, 33, 2, 1]. In our earlier work in [30] we determined the asymptotic behavior (and Large Deviations) of noisy interacting particle systems on large, sparse and recurrent graphs where the total number of connections is of the same order as the system size. Subsequent work by [14, 32] determined the limiting dynamics and large deviations of noisy interacting particle systems on sparse graphs where the total number of connections is asymptotically much larger than the total number of nodes in the network. Most recently, [33] determine the large size limiting behavior of interacting particle systems on Galton-Watson trees (extremely sparse networks,

*New Jersey Institute of Technology. james.n.maclaurin@njit.edu
here the expected total number of edges is a finite multiple of the number of vertices). In the Large Deviations result of [14, 32] the white noise dominates the graph disorder in the large size limit, so that the Large Deviations asymptotic for the distribution of the empirical measure scales as \( O(\exp(-NC)) \) for some constant \( C \) that is the infimum of the rate function over a set. Thus, to leading order, the system still behaves like an all-to-all homogeneous network in the large size limit.

The method of this paper is to demonstrate that the empirical measure containing the path-dynamics can be written as a function of the empirical measure of initial conditions. The Large Deviations rate function for the dynamical system is then the push-forward of the Large Deviations rate function of the initial condition. To the best of this author’s knowledge, this method was first applied to interacting particle systems by Tanaka [40]; see also the recent exposition in [12]. This method has also been used to determine the Large Deviations of extremely sparse interacting particle systems in [30]. For this method to work, the push-forward mapping has to be continuous (or approximately continuous). For this reason, we require a more refined empirical measure that contains the conditional empirical measure of afferent connections on one node. With this richer topology, we are able to adapt classical weak-convergence methods to our context. Another advantage of employing the weak convergence of empirical measures is that we do not require that the typical number of afferent connections to each particle be of the same asymptotic order throughout the network: indeed the number of afferent connections to some of the particles can be \( O(1) \), and the number of afferent connections to other particles can diverge with \( n \).

Many papers concerning the asymptotic convergence of large networks of random graphs [11, 6, 4, 2] utilize the graphon theory of Lovasz [28]. The topology on the space of all graphs is defined using the cut distance metric first developed by Frieze and Kannan [24]. Lucon [29] and Bayraktar et al [2] employ graphon theory to determine the asymptotic behavior of disordered noisy interacting particle systems. Recently [22] use the cut-distance to determine the large deviations of an interacting particle model in the dense regime (the total number of edges scales as \( O(N^2) \)). While graphon theory can be used to study the dynamics of dense graphs, its application to dynamical systems on sparse random graphs is complicated by the lack of a suitable regularity theory [10, 13]. This is a major reason why we instead represent the random structure of the graph in a non-standard ‘nested’ empirical measure \( \hat{\mu}^n \) (this contains the distribution of the connectivities at each node). We can then represent the empirical measure of path-solutions as a push-forward of the empirical measure of initial conditions arising from the graph. In this way we are the first to obtain a large deviations principle for non-stochastic interacting particle systems on sparse random graphs.

We scale the strength of interaction by dividing by the total number of afferent edges. If one were to be merely interested in the large size limit of the network (and not the Large Deviations), then one could equivalently divide by the average number of afferent edges (thanks to the Law of Large Numbers, this is equivalent in the large size limit). However once one is in the Large Deviations regime (which by definition concerns rare events), this equivalence is no longer the case. In particular, the probability of a significant number of vertices being either very sparsely connected (much sparser than the average), or relatively densely connected (must more than the average) is not negligible in the Large Deviations regime. For this reason, in this paper we always scale the net effect of one particle on another by the total number of afferent edges: thus the less edges that are afferent on any particular vertex, the greater effect each connection has (other papers such as [32, 29] also use this scaling). The implication of this is that the large deviations rate function is uniformly upperbounded by a non-infinite constant. This may seem strange on first appearances, because in classical Large Deviations work on the empirical measure for stochastically-interacting particle systems, the rate function is the push-forward of the Relative Entropy, and therefore can be infinite [12]. The reason that the rate function of this paper is bounded is that asymptotically sparse networks can always yield any particular connectivity structure (thanks to the scaling of the effective connection strength by the total connectivity); and the probability of an asymptotically sparse connectivity scales according to equation (114).

The structure of this paper is as follows. In Section 2 we define the interacting particle system and outline our three main results: the first Theorem proves the existence and continuity (with respect to a topology \( \tilde{T} \) that is more refined than the weak topology) of the push-forward map \( \Psi \), the second Theorem proves the Large Deviation Principle in the case that the initial empirical measure satisfies an LDP with respect to the weak topology, and the third Theorem applies these results to interacting particle systems on sparse Erdos-Renyi graphs. In Section 3 we prove Theorem 2.3, and in Section 4 we prove Theorems 2.1 and 2.2.
1.1 Notation:

We write $\mathcal{E} = \{0, 1\}$ (the state space for the connections). For any topological space $\mathcal{X}$, let $B(\mathcal{X})$ denote the set of all Borelian subsets generated from open and closed subsets of $\mathcal{X}$. Let $\mathcal{C}_{b}(\mathcal{X})$ denote the set of bounded continuous functions $\mathcal{X} \to \mathbb{R}$ and let $\mathcal{P}(\mathcal{X})$ denote the set of all Borelian probability measures. Unless otherwise indicated, $\mathcal{P}(\mathcal{X})$ is endowed with the topology of weak convergence $T_{w}$ [9], which is generated by open sets of the form

$$\{\mu \in \mathcal{P}(\mathcal{X}) : |\mathbb{E}^{\mu}[g] - x| < \delta\},$$

for any continuous and bounded $g : \mathcal{X} \to \mathbb{R}$, any $x, \delta \in \mathbb{R}$. If $\mathcal{X}$ possesses a metric $d$, then let $d_{W}$ denote the Wasserstein metric on $\mathcal{P}(\mathcal{X})$, i.e.

$$d_{W}(\mu, \nu) = \inf_{\eta} \mathbb{E}^{\eta}[d(x, y)],$$

the infimum being taken over all $\eta \in \mathcal{P}(\mathcal{X} \times \mathcal{X})$ whose marginal law of the first variable $x$ is $\mu$, and marginal law of the second variable $y$ is $\nu$. Also $\mathcal{C}([0, T], \mathcal{X})$ is endowed with the topology generated by the metric

$$D(\{x_{s}\}_{s \in [0, T]}, \{y_{s}\}_{s \in [0, T]}) = \sup_{s \in [0, T]} d(x_{s}, y_{s}).$$

(1)

$\mathbb{R}^{d}$ and $\mathcal{E}$ are endowed with the standard Euclidean norm $\|\|$.

The index set of the particles is written $I_{n} = \{-n, -n+1, \ldots, n-1, n\}$.

2 Statement of Problem and Main Results

Let $w^{ij} \in \{0, 1\}$ specify the strength of connection between nodes $i$ and $j$. For the moment we make little assumptions on $w^{ij}$, although later on we will take $w^{ij}$ to be sampled randomly from a distribution. Define the total connection strength at node $j$ to be $\kappa_{j}^{n}$, i.e.

$$\kappa_{j}^{n} = \sum_{k=-n}^{n} w^{jk}. \tag{2}$$

The following dynamics becomes singular when $\kappa_{j}^{n} = 0$. Thus we assume that no vertex is disconnected from the rest of the graph, i.e. for all $n > 0$,

$$\inf_{j \in I_{n}} \kappa_{j}^{n} > 0. \tag{3}$$

For $k$ such that $\kappa_{n}^{k} > 0$, the dynamics of the discrete network is assumed to take the form

$$\frac{du^{k}}{dt} = G(u^{k}(t)) + \frac{1}{\kappa_{n}^{k}} \sum_{j \in I_{n}} w^{kj} f(u^{k}(t), u^{j}(t)). \tag{4}$$

where $\Xi_{n}^{k} = \{j \in I_{n} : w^{kj} = 1\}$, and $I_{n} = \{-n, -n+1, \ldots, n-1, n\}$. Notice that the interaction is scaled by the total input, similarly to (for instance) [32, 29].

Here $G, f : \mathbb{R}^{d} \to \mathbb{R}^{d}$ are bounded and uniformly Lipschitz. The initial conditions $(u_{i}^{n})_{i \geq 1}$ may also depend on $n$ (this is omitted from the notation). It is assumed that there exists a uniform bound

$$\sup_{\alpha \geq 0} \sup_{k \in I_{n}} \|u_{i}^{k}\| \leq C_{ini}. \tag{5}$$

It is assumed that

$$u_{i}^{j} = u_{i}^{k} \text{ if and only if } j = k. \tag{6}$$

The empirical measure of afferent inputs on neuron $j$ is written as

$$\hat{\mu}_{j}^{n} = \frac{1}{\kappa_{j}^{n}} \sum_{k \in \Xi_{j}^{k}} \delta_{u_{k}^{j}} \in \mathcal{P}(\mathbb{R}^{d}). \tag{7}$$
The initial empirical measure is
\[ \hat{\mu}_n^0 = (2n + 1)^{-1} \sum_{j=-n}^{n} \delta_{u_j^0} \in \mathcal{P}(\mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d)). \] (8)

We note that this empirical measure is richer (i.e., carries more information) than is usual for interacting particle systems (such as for instance \([40, 39]\)). This is necessary for us to define the push-forward operator \(\Psi\) of Theorem 2.1. The empirical measure, without the connections,
\[ \hat{\mu}_n^\ast = (2n + 1)^{-1} \sum_{j=-n}^{n} \delta_{u_j^\ast}. \] (9)

Let
\[ \mathcal{D} = \{ y \in \mathbb{R}^d : \|y\| \leq C_{ini} \}. \] (10)

Let \(\mathcal{P}(\mathcal{D}) \subset \mathcal{P}(\mathbb{R}^d)\) be the set of all measures that are supported on \(\mathcal{D}\), i.e. \(\mu(y \in \mathcal{D}) = 1\). Observe that \(\hat{\mu}_n^\ast \in \mathcal{P}(\mathcal{D})\).

Define the empirical measure containing the trajectories up to time \(t\),
\[ \hat{\mu}_t^n = \frac{1}{2n + 1} \sum_{j \in I_n} \delta_{u[n,t]} \in \mathcal{P}(\mathcal{C}([0,t], \mathbb{R}^d)), \] (11)
where \(u_{[0,t]}^j := \{u_j^t\}_{0 \leq t \leq t}\).

The significance of the following theorem is that the same mapping \(\Psi\) holds for all \(n\) and every possible empirical measure (subject to the above assumptions). It will allow us to transfer the convergence / Large Deviations of \(\hat{\mu}_n^\ast\) to the convergence / Large Deviations of \(\hat{\mu}_n^\ast\).

**Theorem 2.1.** (i) For any \(T \geq 0\), there exists a measurable map \(\Psi : \mathcal{P}(\mathcal{D} \times \mathcal{P}(\mathcal{D})) \mapsto \mathcal{P}(\mathcal{C}([0,T], \mathcal{D}))\) (defined in Lemma 4.3 below) such that
\[ \hat{\mu}_T^n = \Psi \cdot \hat{\mu}_n^\ast, \] (12)

(ii) There exists a topology \(\tilde{T}\) (this is defined at the start of Section 4) such that \(\Psi : (\mathcal{P}(\mathcal{D} \times \mathcal{P}(\mathcal{D})), \tilde{T}) \mapsto (\mathcal{P}(\mathcal{C}([0,T], \mathcal{D})), \mathcal{T}_w)\) is continuous. The topology \(\tilde{T}\) is Hausdorff and separable, and is a refinement of the weak topology over \(\mathcal{P}(\mathcal{D} \times \mathcal{P}(\mathcal{D}))\). \(\mathcal{P}(\mathcal{D} \times \mathcal{P}(\mathcal{D}))\) is compact with respect to \(\tilde{T}\).

**Theorem 2.2.** Suppose that \(\{u_j^\ast, w_{jk}\}_{j,k \in I_n}\) are (possibly correlated) random variables. Let \(\Pi^n \in \mathcal{P}(\mathcal{P}(\mathcal{D} \times \mathcal{P}(\mathcal{D})))\) be the probability law of \(\hat{\mu}_n^\ast\). Suppose that \(\{\Pi^n\}_{n \geq 2^n}\) satisfy a Large Deviation Principle, where \(\mathcal{P}(\mathcal{D} \times \mathcal{P}(\mathcal{D}))\) is equipped with the weak topology \(\mathcal{T}_w\), with good rate function\(^1\) \(I : \mathcal{P}(\mathcal{D} \times \mathcal{P}(\mathcal{D})) \mapsto \mathbb{R}^+\). This means that for any \(O \in \mathcal{B}(\mathcal{P}(\mathcal{D} \times \mathcal{P}(\mathcal{D})))\) and closed \(F \in \mathcal{B}(\mathcal{P}(\mathcal{D} \times \mathcal{P}(\mathcal{D}))),\)
\[ \lim_{n \to \infty} \alpha_n^{-1} \log \Pi^n(\hat{\mu}_n^\ast \in F) \leq -\inf_{\nu \in F} I(\nu) \] (13)
\[ \lim_{n \to \infty} \alpha_n^{-1} \log \Pi^n(\hat{\mu}_n^\ast \in O) \geq -\inf_{\nu \in O} I(\nu), \] (14)
where \(\alpha_n \to \infty\) as \(n \to \infty\). Then, writing \(\tilde{\Pi}^n \in \mathcal{P}(\mathcal{P}(\mathcal{C}([0,T], \mathbb{R}^d)))\) to be the probability law of \(\hat{\mu}_T^n\), \(\{\tilde{\Pi}^n\}_{n \geq 2^n}\) satisfy a Large Deviation Principle with rate function \(H : \mathcal{P}(\mathcal{C}([0,T], \mathbb{R}^d)) \mapsto \mathbb{R}^+\), where \(H = I \circ \Psi^{-1}\), i.e.
\[ \lim_{n \to \infty} \alpha_n^{-1} \log \tilde{\Pi}^n(\hat{\mu}_T^n \in F_T) \leq -\inf_{\nu \in F_T} H(\nu) \] (15)
\[ \lim_{n \to \infty} \alpha_n^{-1} \log \tilde{\Pi}^n(\hat{\mu}_T^n \in O_T) \geq -\inf_{\nu \in O_T} H(\nu), \] (16)
and \(F_T, O_T \in \mathcal{B}(\mathcal{P}(\mathcal{C}([0,T], \mathbb{R}^d)))\) are sets that are (respectively) closed and open with respect to the weak topology.

\(^1\)A good rate function is lower semicontinuous and has compact level sets \([19]\).
2.1 Large Deviations for Sparse Erdos-Renyi Graphs

We now outline a particular application of the above results to sparse random graphs. In the last ten years there has been an increased emphasis on applying the methods of statistical mechanics to sparse random graphs [17, 27, 3, 18]. The first component of \( w^i(t) \) is taken to be a static position variable \( \theta_n^i \) lying in the ring \((-\pi, \pi]\). The position variables are evenly spaced, i.e. \( \theta_n^i = 2\pi j/(2n + 1) \) mod \( \mathbb{S}^1 \). The other initial conditions \( (u_n^i)_{2 \leq p \leq d} \) are fixed and non-random, and chosen to be a function of the first position variable, i.e.

\[
u_n := U(\theta_n^i). \tag{17}\]

Here \( U : \mathbb{S}^1 \to \mathbb{R}^d \) is bounded and continuous. We follow the convention that self-connections are always present, i.e. \( u^{ij} = 1 \) identically. This ensures that \( (3) \) is satisfied, and therefore the dynamics in \( (4) \) is non-singular.

When \( i \neq j \), we take \( w_{ij} \equiv 1 \) with probability \( \rho_n C(\theta_n^i, \theta_n^j) \), for some prescribed continuous bounded function \( C : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R} \), such that \( C(\theta, \beta) > 0 \). The edges \( \{w_{ij}\}_{i, j \in I_n} \) are sampled independently. It is assumed that the following limits exist

\[
\lim_{n \to \infty} \rho_n = 0, \quad \lim_{n \to \infty} \{n \rho_n\} = \infty. \tag{18}\]

The rest of the assumptions are as outlined in the previous section. Since \( \lim_{n \to \infty} \rho_n = 0 \), the network is sparse, such that the typical number of edges afferent on each node is much less than the system size. One could easily obtain similar results in the case that \( \rho_n = O(1) \).

Define \( \Gamma : \mathbb{S}^1 \times \mathcal{P}(\mathbb{S}^1) \to \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \) to be the function

\[
(\theta, \mu) \mapsto (U(\theta), \mu \circ U^{-1}), \tag{19}\]

and we recall the definition of \( U \) in \( (17) \). The main result of this section is the following.

**Theorem 2.3.** Suppose that \( (17) \) and \( (18) \) hold, and that \( \{w^{jk}\}_{j, k \in I_n} \) are sampled independently.

Let \( F_T, O_T \subset \mathcal{P}(\mathcal{C}([0, T], \mathbb{R}^d)) \), with \( F_T \) closed and \( O_T \) open (with respect to the weak topology \( \mathcal{T}_w \)). Then

\[
\lim_{n \to \infty} \{\rho_n(2n + 1)^2\}^{-1} \log \mathbb{P}(\hat{\mu}^n_T \in F_T) \leq -\inf_{\nu \in F_T} H(\nu) \tag{20}\]

\[
\lim_{n \to \infty} \{\rho_n(2n + 1)^2\}^{-1} \log \mathbb{P}(\hat{\mu}^n_T \in O_T) \geq -\inf_{\nu \in O_T} H(\nu). \tag{21}\]

Here \( H : \mathcal{P}(\mathcal{C}([0, T], \mathbb{R}^d)) \to \mathbb{R}^+ \) has compact level sets, and is such that

\[
H(\nu) = \inf_{\mu \in \mathcal{P}(\mathbb{S}^1 \times \mathcal{P}(\mathbb{S}^1))} \left\{ \tilde{I}(\mu) : \Psi \cdot (\mu \circ \Gamma^{-1}) = \nu \right\} \tag{22}\]

where

\[
\tilde{I} : \mathcal{P}(\mathbb{S}^1 \times \mathcal{P}(\mathbb{S}^1)) \to \mathbb{R}^+ \text{ is such that } \tilde{I}(\mu) = \mathbb{E}^\mu[I_0(\alpha)] \tag{23}\]

in the above expectation \( \theta \in \mathbb{S}^1 \), \( \alpha \in \mathcal{P}(\mathbb{S}^1) \) and \( \Gamma \) is defined in \( (19) \). \( I_0 : \mathcal{P}(\mathbb{S}^1) \to \mathbb{R}^+ \) is defined as follows.

In the case that \( \mu \) does not have a density,

\[
I_0(\mu) = \frac{1}{2\pi} \int_{\mathbb{S}^1} C(\theta, \eta)d\eta. \tag{24}\]

Otherwise, in the case that \( d\mu(\theta) = \frac{C(\theta)}{\zeta(\eta)}d\eta \), define

\[
I_0(\mu) = \frac{1}{2\pi} \int_{\mathbb{S}^1} C(\theta, \eta)d\eta - \exp \left( -\frac{1}{2\pi} \int_{\mathbb{S}^1} \zeta(\eta) \log \left( \frac{C(\eta)}{C(\theta, \eta)} \right)d\eta \right). \tag{25}\]

following the convention that \( 0 \log(0) = 0 \) in the second integral on the right hand side.

This theorem is proved in Section 3.
3 Proof of Theorem 2.3

We assume the model setup of Section 2.1. We need to show that the LDP for the law of the initial condition in the statement of Theorem 2.2 is satisfied: i.e. we need to demonstrate (13) and (14) hold for the sparse model of Section 2.1. Define

$$\hat{\mu}_n^x = \frac{1}{2n+1} \sum_{j \in I_n} \delta_{(\theta_j, \mu_j^x)} \in \mathcal{P}(\mathbb{S}^1 \times \mathcal{P}(\mathbb{S}^1))$$

where the conditional empirical measures are, for \(-n \leq j \leq n,

$$\hat{\mu}_j^x = \frac{1}{|\Xi_j|} \sum_{k \in \Xi_j} \delta_{\theta_k^x} \in \mathcal{P}(\mathbb{S}^1),$$

(26)

where \(\Xi_j = \{k \in I_n : w_{jk} = 1\}\). Since \(\Gamma\) (as defined in (19)) is continuous and \(\hat{\mu}_n^x = \hat{\mu}_n^x \circ \Gamma^{-1}\), thanks to Varadhan’s Contraction Principle [19, Theorem 4.2.1], it suffices to prove the LDP for the sequence of laws \(\{\Pi^n\}_{n \geq 1} \subset \mathcal{P}(\mathcal{P}(\mathbb{S}^1 \times \mathcal{P}(\mathbb{S}^1)))\) of \(\hat{\mu}_n^x\). We first assume that the connection probabilities \(C\) are piecewise constant.

Lemma 3.1. Suppose that there exists a partition of \(\mathbb{S}^1\) into intervals, i.e. \(\bigcup_{i=1}^i R_i\), and such that for each \(\alpha \in \mathbb{S}^1\),

$$\mathcal{C}(\theta_1, \alpha) = \mathcal{C}(\theta_2, \alpha)$$

if \(\theta_1, \theta_2 \in R_i\) for some \(R_i\). Thus \(\mathcal{C}\) is piecewise constant in its first argument, and continuous in its second argument.

For open \(A \subset \mathcal{P}(\mathbb{S}^1 \times \mathcal{P}(\mathbb{S}^1))\) and closed \(F \subset \mathcal{P}(\mathbb{S}^1 \times \mathcal{P}(\mathbb{S}^1))\)

$$\lim_{n \to \infty} \left(2n+1\right) \rho_n \P(\hat{\mu}_n^x \in F) = -\inf_{\nu \in F} \hat{I}(\nu)$$

(28)

$$\lim_{n \to \infty} \left(2n+1\right) \rho_n \P(\hat{\mu}_n^x \in A) \geq -\inf_{\nu \in A} \hat{I}(\nu).$$

(29)

Proof. To prove the LDP we are going to employ the Dawson-Gartner theory of projective limits [16], following the exposition in [19]. Let \(W_m \subset \mathcal{B}(\mathbb{S}^1 \times \mathcal{P}(\mathbb{S}^1))\) be \(W_m = \{O^m\}_{\mu \in \Xi_m}\), where

$$O^m = \{(\theta, \mu) : \theta \in \Omega^i_{m+1} \text{ and } \mu \in \Xi_m\},$$

for some indices \(1 \leq j, k_i \leq m\). Let \(\mathcal{T}_2\) be the topology on \(\mathcal{P}(\mathbb{S}^1 \times \mathcal{P}(\mathbb{S}^1))\) that is generated by open sets of the form

$$\{\mu \in \mathcal{P}(\mathbb{S}^1 \times \mathcal{P}(\mathbb{S}^1)) : | \mu(O^i_{m+1}) - x | < \delta \},$$

for \(x, \delta \in \mathbb{R}\), some \(m \geq 1\) and \(1 \leq i \leq m^2\).

Lemma 3.2. \(\mathcal{P}(\mathbb{S}^1 \times \mathcal{P}(\mathbb{S}^1))\) is separable with respect to \(\mathcal{T}_2\). Also \(\mathcal{T}_2\) is a refinement of the weak topology.

Proof. Suppose that \(V : \mathbb{S}^1 \times \mathcal{P}(\mathbb{S}^1) \to \mathbb{R}\) is continuous, and write \(V_{\max} = \sup_{\theta \in \mathbb{S}^1, \mu \in \mathcal{P}(\mathbb{S}^1)} | V(\theta, \mu) |\). Define the set

$$V = \{\nu \in \mathcal{P}(\mathbb{S}^1 \times \mathcal{P}(\mathbb{S}^1)) : |E^\nu[V] - x| < \delta\},$$

(30)

for some \(x, \delta \in \mathbb{R}\). We assume that \(V\) is nonempty. We must prove that \(V\) is open with respect to \(\mathcal{T}_2\). Fix \(\nu \in V\) such that \(|E^\nu[V] - x| = \delta\) and \(\delta < \delta\). Define \(V^{(m)}, \mathcal{V}^{(m)}\) to be

$$V^{(m)}(\theta, \mu) = \sum_{i=1}^{m^2} \chi_{\{\mu \in O^i_{m+1}\}} \sup_{(\beta, \gamma) \in \Omega^i_{m+1}} V(\beta, \gamma)$$

$$\mathcal{V}^{(m)}(\theta, \mu) = \sum_{i=1}^{m^2} \chi_{\{\mu \in O^i_{m+1}\}} \inf_{(\beta, \gamma) \in \Omega^i_{m+1}} V(\beta, \gamma).$$

Since \(V\) is defined on a compact domain, it must be uniformly continuous, and therefore there must exist \(m\) such that

$$\sup_{\theta \in \mathbb{S}^1, \mu \in \mathcal{P}(\mathbb{S}^1)} | V^{(m)}(\theta, \mu) - V^{(m)}(\theta, \mu) | < \frac{1}{2} (\delta - \delta).$$

(31)
We then see that the set

\[ \mathcal{U} = \{ \mu \in \mathcal{P}(S^1 \times \mathcal{P}(S^1)) : \sup_{1 \leq i \leq m} |\mu(O_i^m) - \nu(O_i^m)| < \frac{1}{2V_{max}}(\delta - \delta) \} \tag{32} \]

is such that \( \mathcal{U} \subseteq \mathcal{V} \). This yields the lemma.

\( \mathcal{T}_2 \) is a projective limit topology (in the notation of [19, Section 4.6]) in the following sense. For \( m \geq 1 \), the projection \( p_m : \mathcal{P}(S^1 \times \mathcal{P}(S^1)) \to [0, 1]^m \) is

\[ p_m \cdot \nu = (\nu(O_1^m), \ldots, \nu(O_m^m)). \]

Clearly \( p_m : \mathcal{P}(S^1 \times \mathcal{P}(S^1)) \to [0, 1]^m \) is continuous (when \( \mathcal{P}(S^1 \times \mathcal{P}(S^1)) \) is endowed with the \( \mathcal{T}_2 \) topology and \([0, 1]^m \) with the Euclidean topology). Let \( p_{qm} : [0, 1]^m \to [0, q]^q \) (for \( q \leq m \)) be the induced projection i.e. for \( 1 \leq j \leq q \),

\[ p_{qm}(x_1, \ldots, x_m)^j = \sum_{i=1}^{m} \chi(O_i^m \subseteq O_j^q) x_i. \]

The above identity means that

\[ p_{qm} \cdot p_m(\nu) = p_q(\nu). \tag{33} \]

It is clear that \( p_{qm} \) is continuous.

To prove the theorem, we will use the Dawson-Gartner Theorem for the Large Deviations of projective limit systems [16],[19, Theorem 4.6.1]. To satisfy the conditions of this theorem, we first prove that for any partition \( P > 0 \) and any set \( A \in \mathcal{B}([0, 1]^P) \) that is either open or closed,

\[ \lim_{n \to \infty} \{ (2n + 1)^2 \rho_n \}^{-1} \mathbb{P}((\hat{\mu}_n^0(O_1^P), \ldots, \hat{\mu}_n^0(O_P^P)) \in A) = -\inf_{x \in A} I_{O_1^P, \ldots, O_P^P}(x) \tag{34} \]

where

\[ I_{O_1^P, \ldots, O_P^P}(x) = \sum_{i=1}^{P} E^\nu \left[ \chi(O_i^P) \inf_{\mu \in Z_i} I_0(\mu) \right], \tag{35} \]

and we have written \( O_i^P = O_i^P \times Z_i \). Let

\[ \mathcal{U}_n = \{ z_n \in A : z_n^j = p_n^i/(2n + 1) \text{ for } p_n^i \in \mathbb{Z}^* \text{ and } |p_n^i| \leq (2n + 1) \text{ and } \sum_{i=1}^{P} z_i = 1 \}. \]

Notice that \( |\mathcal{U}_n| \) is polynomial in \( n \), which means that

\[ \lim_{n \to \infty} (2n + 1)^2 \rho_n \log |\mathcal{U}_n| = 0. \]

We thus find that,

\[ \lim_{n \to \infty} (2n + 1)^2 \rho_n \log \mathbb{P}((\hat{\mu}_n^0(O_1^P), \ldots, \hat{\mu}_n^0(O_P^P)) \in A) \]

\[ = \lim_{n \to \infty} (2n + 1)^2 \rho_n \log \sum_{z_n \in \mathcal{U}_n} \mathbb{P}((\hat{\mu}_n^0(O_1^P), \ldots, \hat{\mu}_n^0(O_P^P)) = z_n) \]

\[ = \lim_{n \to \infty} \sup_{z_n \in \mathcal{U}_n} (2n + 1)^2 \rho_n \log \mathbb{P}((\hat{\mu}_n^0(O_1^P), \ldots, \hat{\mu}_n^0(O_P^P)) = z_n), \]

as long as the last limit exists. Now fix \( z_n \in \mathcal{U}_n \) to be such that the supremum on the right hand side is attained at \( z_n \) (this is possible because \( \mathcal{U}_n \) is of finite size).

Let \( \{ v_j^i \}_{j \in I_n} \subset \mathcal{P}(S^1) \) be any set of measures such that (i) \( \mathbb{P}(\hat{\mu}_n^0 = v_j^i) \neq 0 \), and (ii), defining \( v_n = \frac{1}{(2n + 1)^2} \sum_{j \in I_n} \delta_{\hat{\mu}_n^0, v_j^i} \in \mathcal{P}(S^1 \times \mathcal{P}(S^1)) \),

\[ (v_n^0(O_1^P), \ldots, v_n^0(O_P^P)) = z_n. \]

Employing a multinomial expansion, we find that

\[ \mathbb{P}((\hat{\mu}_n^0(O_1^P), \ldots, \hat{\mu}_n^0(O_P^P)) = z_n) = \prod_{j \in I_n} \mathbb{P}(\hat{\mu}_n^0 = v_j) \tag{36} \]
where $\Upsilon_{n,z,n}$ contains combinatorial factors, and (thanks to Stirling’s Approximation) is such that $\Upsilon_n \leq \exp(Cn)$ for a positive constant $C$. We thus find that (recalling that $\rho_n(2n+1) \to \infty$ as $n \to \infty$), and making use of Lemma 3.6,

$$\lim_{n \to \infty} \{(2n+1)^2 \rho_n\}^{-1} \log P((\hat{\nu}_n^a(\mathcal{O}_T^p), \ldots, \hat{\nu}_n^a(\mathcal{O}_T^p)) = z_n) = - \inf_{z \in A} \sum_{i=1}^p z_i \inf_{\mu \in \mathcal{Z}_i} I_{\delta_i}(\mu) = - \inf_{z \in A} I_{\mathcal{O}_T^p \ldots \mathcal{O}_T^p}(z).$$ (37)

Now, applying the Dawson-Gartner Theorem [19, Theorem 4.6.1], it must be that for any open $A \subset \mathcal{P}(\mathbb{S} \times \mathbb{P}(\mathbb{S}^1))$ and any closed $F \subset \mathcal{P}(\mathbb{S} \times \mathcal{P}(\mathbb{S}^1))$ (these sets are open / closed with respect to the $\mathcal{T}_2$ topology),

$$\lim_{n \to \infty} \{(2n+1)^2 \rho_n\}^{-1} \log P(\hat{\nu}_n^a \in F) \leq - \inf_{\nu \in F} \bar{I}(\nu)$$ (38)

$$\lim_{n \to \infty} \{(2n+1)^2 \rho_n\}^{-1} \log P(\hat{\nu}_n^a \in A) \geq - \inf_{\nu \in A} \bar{I}(\nu)$$

where

$$\bar{I}(\nu) = \sup_{P \to 0} \left\{ \int_{\mathcal{O}_T^p \ldots \mathcal{O}_T^p} (\nu(\mathcal{O}_T^p), \ldots, \nu(\mathcal{O}_T^p)) \right\}$$

In Lemma 3.3 we demonstrate that $\bar{I}(\nu) = \bar{I}(\nu)$. We have thus established (28) and (29). [Q.E.D.]

We now complete the proof of Theorem 2.3.

Proof. We earlier assumed in (64) that the connection probabilities are piecewise constant. We now prove the more general case. Let $\mathcal{C}(\alpha, \theta)$ be piecewise constant in its first argument (satisfying (64)), continuous in its second argument, for some $m \geq 1$, and such that

$$\lim_{m \to \infty} \sup_{\alpha, \theta \in \mathbb{S}^1} \left| \mathcal{C}(\alpha, \theta) - \mathcal{C}(\alpha, \theta) \right| = 0.$$ (41)

It is always possible to make the above approximation because $\mathcal{C}$ is continuous on the compact space $\mathbb{S}^1 \times \mathbb{S}^1$ (and therefore it must be uniformly continuous). Furthermore, since by assumption $\mathcal{C}(\cdot, \cdot) \geq C_{lb} > 0$, it must be that

$$\lim_{m \to \infty} \sup_{\alpha, \theta \in \mathbb{S}^1} \left| \frac{\mathcal{C}(\alpha, \theta)}{\mathcal{C}(\alpha, \theta)} - 1 \right| = 0.$$ (42)

It remains to prove that for open $A \subset \mathcal{P}(\mathbb{S} \times \mathcal{P}(\mathbb{S}^1))$ and closed $F \subset \mathcal{P}(\mathbb{S} \times \mathcal{P}(\mathbb{S}^1))$

$$\lim_{n \to \infty} \{(2n+1)^2 \rho_n\}^{-1} \log P(\hat{\nu}_n^a \in F) \leq - \inf_{\nu \in F} \bar{I}(\nu)$$ (43)

$$\lim_{n \to \infty} \{(2n+1)^2 \rho_n\}^{-1} \log P(\hat{\nu}_n^a \in A) \geq - \inf_{\nu \in A} \bar{I}(\nu).$$ (44)

Let $Q \in \mathcal{P}(\mathcal{E}^{(2n+1)^2})$ be the law of the random connections $\{w^{jk}\}_{j,k \in \mathbb{N}}$ with connection probability $\rho_n \mathcal{C}(\cdot, \cdot)$. Let $Q^{(m)} \in \mathcal{P}(\mathcal{E}^{(2n+1)^2})$ be the law of the random connections $\{w^{jk}\}_{j,k \in \mathbb{N}}$ with connection probability $\rho_n \mathcal{C}^{(m)}(\cdot, \cdot)$. We see that

$$\frac{dQ}{dQ^{(m)}}(\{w^{jk}\}) = \prod_{j,k \in \mathbb{N}} \frac{\chi(w^{jk} = 1) \rho_n \mathcal{C}(\theta^a_n, \theta^a_n) + \chi(w^{jk} = 0) (1 - \rho_n \mathcal{C}(\theta^a_n, \theta^a_n))}{\chi(w^{jk} = 1) \rho_n \mathcal{C}^{(m)}(\theta^a_n, \theta^a_n) + \chi(w^{jk} = 0) (1 - \rho_n \mathcal{C}^{(m)}(\theta^a_n, \theta^a_n))}.$$ (45)

Define, for $a \geq 0$,

$$\Upsilon^a_n = \{\{w^{jk}\} \in \mathcal{E}^{(2n+1)^2} : \sum_{j,k \in \mathbb{N}} w^{jk} \leq \rho_n (2n+1)^2 a\},$$

and

$$\alpha_m = \max \left\{ \sup_{\alpha, \theta \in \mathbb{S}^1} \left| \frac{\mathcal{C}(\alpha, \theta)}{\mathcal{C}(\alpha, \theta)} - 1 \right|, \sup_{\alpha, \theta \in \mathbb{S}^1} \left| \mathcal{C}^{(m)}(\alpha, \theta) - \mathcal{C}(\alpha, \theta) \right| \right\}.$$ (46)

Using the inequality

$$\frac{1 - z}{1 - x} \leq 1 + z + 2xz^2,$$
as long as \( x, z \ll 1 \) are sufficiently small, it must hold that there is a \( K > 0 \) such that

\[
\frac{1 - \rho_n C(\theta_n^j, \theta_n^k)}{1 - \rho_n C(\theta_n^j, \theta_n^k)} \leq 1 + \rho_n \alpha_m + K \rho_n^2.
\] (47)

We thus see that for \( \{w^{jk}\} \in \mathcal{V}_n^m \), it must hold that

\[
\frac{dQ^n}{dQ^n_{(m)}} \left( \{w^{jk}\} \right) \leq (1 + \alpha_m) \rho_n (2n+1)^2 \left( 1 + \rho_n \alpha_m + O(\rho_n^2) \right) (2n+1)^2 \left( 1 - \rho_n^2 \right)
\]

\[
\leq \exp \left( 2\alpha_m \rho_n (2n+1)^2 a + K \rho_n^2 (2n+1)^2 \right),
\] (48)

using the bound \( 1 + y \leq \exp(y) \). Now for any set \( U \in B(\mathcal{P}(S^1 \times \mathcal{P}(S^1))) \),

\[
\{(2n+1)^2 \rho_n\}^{-1} \log Q^n(\tilde{\mu}_n^a \in U) \leq \{(2n+1)^2 \rho_n\}^{-1} \log \left\{ Q^n(\tilde{\mu}_n^a \in U \text{ and } \{w^{jk}\} \in \mathcal{V}_n^m) + Q^n(\{w^{jk}\} \notin \mathcal{V}_n^m) \right\} \leq \{(2n+1)^2 \rho_n\}^{-1} \log 2
\]

\[
+ \max \left\{ \{(2n+1)^2 \rho_n\}^{-1} \log Q^n(\tilde{\mu}_n^a \in U \text{ and } \{w^{jk}\} \in \mathcal{V}_n^m), \{(2n+1)^2 \rho_n\}^{-1} \log Q^n(\{w^{jk}\} \notin \mathcal{V}_n^m) \right\}.
\]

Let \( M > \sup_{\theta, \alpha \in \mathcal{C}} C(\theta, \alpha) \). Thanks to Lemma 3.5, we can find \( a > 0 \) such that

\[
\{(2n+1)^2 \rho_n\}^{-1} \log Q^n(\{w^{jk}\} \notin \mathcal{V}_n^m) \leq -M.
\] (49)

Starting with the upper bound (and taking \( U \) to be closed) we thus find that

\[
\limsup_{n \to \infty} \{(2n+1)^2 \rho_n\}^{-1} \log Q^n(\tilde{\mu}_n^a \in U \text{ and } \{w^{jk}\} \in \mathcal{V}_n^m) \leq - \min \left\{ M, \limsup_{n \to \infty} \{(2n+1)^2 \rho_n\}^{-1} \log Q^n(\tilde{\mu}_n^a \in U \text{ and } \{w^{jk}\} \in \mathcal{V}_n^m) \right\}.
\] (50)

Employing the bound in (48),

\[
\limsup_{n \to \infty} \{(2n+1)^2 \rho_n\}^{-1} \log Q^n(\tilde{\mu}_n^a \in U \text{ and } \{w^{jk}\} \in \mathcal{V}_n^m)
\]

\[
\leq \limsup_{n \to \infty} \{(2n+1)^2 \rho_n\}^{-1} \log Q^n_{(m)}(\tilde{\mu}_n^a \in U \text{ and } \{w^{jk}\} \in \mathcal{V}_n^m) + 2\alpha_m a.
\]

Furthermore

\[
\limsup_{n \to \infty} \{(2n+1)^2 \rho_n\}^{-1} \log Q^n_{(m)}(\tilde{\mu}_n^a \in U \text{ and } \{w^{jk}\} \in \mathcal{V}_n^m) \leq \limsup_{n \to \infty} \{(2n+1)^2 \rho_n\}^{-1} \log Q^n_{(m)}(\tilde{\mu}_n^a \in U)
\]

\[
\leq - \inf_{\nu \in U} \tilde{I}(\nu),
\] (51)

where

\[
\tilde{I}(\nu) = E^\nu(\theta, \mu)[I(\nu)] \text{ and}
\]

\[
I(\nu) = \frac{1}{2\pi} \int_{\mathbb{R}} C^{(m)}(\alpha, \theta) d\theta - \exp \left( -\frac{1}{2\pi} \int_{\mathbb{R}} \zeta(\theta) \log \left( \frac{C^{(m)}(\alpha, \theta)}{C(\alpha)} \right) d\theta \right).
\] (52)

Since \( C^{(m)} \) converges uniformly to \( C \), one easily checks that

\[
\lim_{n \to \infty} \sup_{\alpha \in \mathcal{P}(\mathbb{G})} \sup_{\mu \in \mathcal{P}(\mathbb{G})} \left\{ |I(\nu) - I(\mu)| \right\} = 0.
\]

We now take \( m \to \infty \). Since

\[
\inf_{\nu \in U} \tilde{I}(\nu) \leq \sup_{\theta, \alpha \in \mathcal{P}(\mathbb{G})} C(\theta, \alpha) < M,
\]

it must be that

\[
\lim_{m \to \infty} \min_{\nu \in U} \left\{ M, \limsup_{n \to \infty} \{(2n+1)^2 \rho_n\}^{-1} \log Q^n(\tilde{\mu}_n^a \in U \text{ and } \{w^{jk}\} \in \mathcal{V}_n^m) \right\} \leq - \inf_{\nu \in U} \tilde{I}(\nu)
\] (54)

and therefore (43) must hold.

The lower bound is prove analogously.

\[ \square \]
Lemma 3.3.

\[ \tilde{I}(\nu) = \hat{I}(\nu), \]

where \( \hat{I} \) is defined in (23) and \( I \) is defined in (40).

Proof. Let \( P > 0 \) be such that

\[ \bar{I}_{\mathcal{C}_1^P, \ldots, \mathcal{C}_Q^P}((\nu(\mathcal{C}_1^P), \ldots, \nu(\mathcal{C}_Q^P))) \geq \sup_{Q \geq 2} \{ \bar{I}_{\mathcal{C}_1^Q, \ldots, \mathcal{C}_Q^Q}((\nu(\mathcal{C}_1^Q), \ldots, \nu(\mathcal{C}_Q^Q))) \} - \epsilon, \]

for some \( \epsilon \ll 1 \). For any sequence \( \{r_P\}_{P \geq 1} \) such that

\[ \mathcal{O}_{r_P}^P \subseteq \mathcal{O}_{r_P}, \]

it must be that there exists \( (\theta, \eta) \in \mathbb{S}^1 \times \mathcal{P}(\mathbb{S}^1) \) such that \( \bigcap_{P \geq 1} \mathcal{O}_{r_P}^P = (\theta, \eta) \). For any \( (\theta, \mu) \in \mathbb{S}^1 \times \mathcal{P}(\mathbb{S}^1) \), since \( (\theta, \mu) \in \mathcal{O}_{r_P}^{P+1} \) and \( (\theta, \mu) \in \mathcal{O}_P^P \), it must be that \( \mathcal{Z}_{P+1}^{P+1} \subseteq \mathcal{Z}_P^P \), and therefore

\[ \sum_{i=1}^P \chi((\theta, \mu) \in \mathcal{O}_i^P) \inf_{\gamma \in \mathcal{Z}_i^P} I_\theta(\gamma) \leq \sum_{i=1}^{P+1} \chi((\theta, \mu) \in \mathcal{O}_{i+1}^{P+1}) \inf_{\gamma \in \mathcal{Z}_{i+1}^{P+1}} I_\theta(\gamma). \]

It thus follows from the monotone convergence theorem that

\[ \lim_{P \to \infty} \bar{I}_{\mathcal{C}_1^P, \ldots, \mathcal{C}_Q^P}((\nu(\mathcal{C}_1^P), \ldots, \nu(\mathcal{C}_Q^P))) = \mathbb{E}^\nu \left[ \lim_{P \to \infty} \sum_{i=1}^P \chi((\theta, \mu) \in \mathcal{O}_i^P) \inf_{\gamma \in \mathcal{Z}_i^P} I_\theta(\gamma) \right], \]

and the left-hand-side is non-decreasing in \( P \). It is immediate from the definition that

\[ \mathbb{E}^\nu \left[ \lim_{P \to \infty} \sum_{i=1}^P \chi((\theta, \mu) \in \mathcal{O}_i^P) \inf_{\gamma \in \mathcal{Z}_i^P} I_\theta(\gamma) \right] \leq \mathbb{E}^\nu[I_\theta(\mu)]. \]

Since \( I_\theta(\mu) \) is continuous in \( \theta \), and lower-semi-continuous in \( \mu \), it must be that

\[ \mathbb{E}^\nu \left[ \lim_{P \to \infty} \sum_{i=1}^P \chi((\theta, \mu) \in \mathcal{O}_i^P) \inf_{\gamma \in \mathcal{Z}_i^P} I_\theta(\gamma) \right] \geq \mathbb{E}^\nu[I_\theta(\mu)]. \]

We thus see that

\[ \lim_{P \to \infty} \bar{I}_{\mathcal{C}_1^P, \ldots, \mathcal{C}_Q^P}((\nu(\mathcal{C}_1^P), \ldots, \nu(\mathcal{C}_Q^P))) = \mathbb{E}^\nu[I_\theta(\mu)]. \]

We have thus proved that \( \mathbb{E}^\nu[I_\theta(\mu)] \geq \hat{I}(\nu) - \epsilon \). It is clear from the definition of \( \hat{I} \) that

\[ \mathbb{E}^\nu[I_\theta(\mu)] = \hat{I}(\nu). \]

Taking \( \epsilon \to 0 \), it must be that

\[ \mathbb{E}^\nu[I_\theta(\mu)] = \hat{I}(\nu). \]

\[ \Box \]

3.1 LDP for the conditional empirical measure

Throughout this section, suppose that there exists a partition of \( \mathbb{S}^1 \) into intervals, i.e. \( \bigcup_{i=1}^Q R_i \), and such that for each \( \alpha \in \mathbb{S}^1 \),

\[ C(\theta_1, \alpha) = C(\theta_2, \alpha) \]

if \( \theta_1, \theta_2 \in R_i \) for some \( R_i \). We also assume uniform lower and upper bounds, i.e.

\[ \inf_{\theta, \alpha \in \mathbb{S}^1} C(\alpha, \theta) > 0 \]

\[ \sup_{\theta, \alpha \in \mathbb{S}^1} C(\alpha, \theta) < \infty. \]

Let \( \mathcal{M}^+((\mathbb{S}^1 \times \mathcal{E}) \) be the set of all positive Borel measures on \( \mathbb{S}^1 \times \mathcal{E} \). Define

\[ \hat{\mu}_n = \frac{1}{\rho_n(2n+1)} \sum_{k \in I_n} \delta_{(\theta_{n,k}, \omega_{n,k})} \in \mathcal{M}^+((\mathbb{S}^1 \times \mathcal{E})]. \]
and observe that
\[ \tilde{\mu}_n^j = \pi \cdot \hat{\mu}_n^j \] where \( \pi : \mathcal{M}(\mathbb{S}^1 \times \mathcal{E}) \to \mathcal{P}(\mathbb{S}^1) \) is such that
\[ (\pi \cdot \nu)(B) = \frac{\nu(\theta \in B \text{ and } w = 1)}{\nu(w = 1)} \quad \text{for } B \in \mathcal{B}(\mathbb{S}^1). \] (67)

Since \( w/\pi = 1 \) always, \( \pi \cdot \hat{\mu}_n^j \) is always well-defined. We equip \( \mathcal{M}(\mathbb{S}^1 \times \mathcal{E}) \) with the topology \( \tilde{T} \) generated by open sets of the form
\[ \{ \mu \in \mathcal{M}(\mathbb{S}^1 \times \mathcal{E}) : |E^n[g(\theta)w] - x| < \delta \}, \] for \( x, \delta \in \mathbb{R} \) and continuous \( g : \mathbb{S}^1 \to \mathbb{R} \). Measures with identical expectations for all such functions are identified. Observe that \( \pi \) is uniformly continuous over sets of the form \( \{ \nu : \nu(w = 1) \geq \varepsilon > 0 \} \). Let \( \tilde{\Pi}_n^\alpha \in \mathcal{P}(\mathcal{M}(\mathbb{S}^1 \times \mathcal{E})) \) be the probability law of \( \tilde{\mu}_n^j \).

\textbf{Lemma 3.4.} Let \( \{ j_n \}_{n \geq 1} \) be any sequence such that there exists \( p \geq 1 \) such that \( \theta_n^j \in R_p \) for all \( n \geq 1 \), and \( \theta_n^j \to \alpha \in R_p \). For any open \( O \in \mathcal{B}(\mathcal{P}(\mathbb{S}^1)) \) and closed \( F \in \mathcal{B}(\mathcal{P}(\mathbb{S}^1)) \) (according to the topology generated by open sets of the form (69)),
\[ \lim_{n \to +\infty} (\rho_n(2n + 1))^{-1} \log \mathbb{P}(\tilde{\mu}_n^j \in O) \geq - \inf_{\mu \in \tilde{\Pi}_n^\alpha} I_\alpha(\mu). \] (70)

\[ \lim_{n \to +\infty} (\rho_n(2n + 1))^{-1} \log \mathbb{P}(\tilde{\mu}_n^j \in F) \leq - \inf_{\mu \in \tilde{\Pi}_n^\alpha} I_\alpha(\mu). \] (71)

\textbf{Proof.} Fix \( \varepsilon > 0 \) and define \( U_\varepsilon = \{ \mu \in \mathcal{M}(\mathbb{S}^1 \times \mathcal{E}) : \mu(w = 1) \leq \varepsilon \} \). By a union-of-events bound
\[ \lim_{n \to +\infty} (\rho_n(2n + 1))^{-1} \log \mathbb{P}(\tilde{\mu}_n^j \in O) \geq \max \left\{ \lim_{n \to +\infty} (\rho_n(2n + 1))^{-1} \log \mathbb{P}(\tilde{\mu}_n^j \in O \text{ and } \tilde{\mu}_n^j \not\in U_\varepsilon), \lim_{n \to +\infty} (\rho_n(2n + 1))^{-1} \log \mathbb{P}(\tilde{\mu}_n^j \not\in U_\varepsilon) \right\}. \]

Now
\[ \lim_{n \to +\infty} (\rho_n(2n + 1))^{-1} \log \mathbb{P}(\tilde{\mu}_n^j \in O \text{ and } \tilde{\mu}_n^j \not\in U_\varepsilon) = \lim_{n \to +\infty} (\rho_n(2n + 1))^{-1} \log \mathbb{P}(\tilde{\mu}_n^j \in \pi^{-1}(O) \text{ and } \tilde{\mu}_n^j \not\in U_\varepsilon) \]
\[ \geq - \inf_{\mu \in \pi^{-1}(O) \cap \tilde{\Pi}_n^\alpha} I_\alpha(\mu), \]
using the Large Deviation Principle of Lemma 3.8, since \( \pi^{-1}(O) \cap U_\varepsilon \) is open.

Suppose first that (noting that \( \pi^{-1}(O) \) is always non-empty)
\[ \inf_{\nu \in \pi^{-1}(O)} I_\alpha(\nu) = \infty. \] (72)

In this case, if \( \nu \in \mathcal{M}(\mathbb{S}^1 \times \mathcal{E}) \) is such that \( \pi \cdot \nu = \mu \in O \), then it must be that either \( \nu \) does not have a density, or it has a density \( \zeta(\theta) \) and
\[ \int_{\mathbb{S}^1} \zeta(\theta) \log(\zeta(\theta)) d\theta = \infty. \]
Whichever of these cases holds, it must be that
\[ I_\alpha(\mu) = \frac{1}{2\pi} \int_{\mathbb{S}^1} C(\alpha, \theta) d\theta. \] (73)
Furthermore it follows from Lemma 3.10 that
\[ \lim_{\varepsilon \to 0} \lim_{n \to +\infty} (\rho_n(2n + 1))^{-1} \log \mathbb{P}(\tilde{\mu}_n^j \in U_\varepsilon) = - \frac{1}{2\pi} \int_{\mathbb{S}^1} C(\alpha, \theta) d\theta \]
\[ = - \inf_{\mu \in O} I_\alpha(\mu), \]
using (73). We thus obtain (70) in the case that (72) holds, as required.

We now suppose that \( \inf_{\nu \in \pi^{-1}(O)} I_\alpha(\nu) < \infty \). Then we first claim that
\[ \inf_{\nu \in \pi^{-1}(O) \cap \tilde{\Pi}_n^\alpha} I_\alpha(\nu) \to \inf_{\nu \in \pi^{-1}(O)} I_\alpha(\nu). \] (74)
To see this, let $\hat{\nu} \in \pi^{-1}(O)$ be such that, for some $\delta > 0$,
\[ \hat{I}_n(\hat{\nu}) \leq \inf_{\nu \in \pi^{-1}(O)} \hat{I}_n(\nu) + \delta. \]

The openness of $\pi^{-1}(O)$ implies that there must exist $\epsilon > 0$ such that $\nu \in \pi^{-1}(O) \cap U^c$. Since $\delta$ is arbitrary, we have established (74).

By Lemma 3.9,
\[ \inf_{\nu \in \pi^{-1}(O)} \hat{I}_n(\nu) = \inf_{\mu \in O} I_n(\mu). \]

Now $I_n(\mu) \leq \frac{1}{2\pi} \int_{S^1} C(\alpha, \theta) d\theta$, and we thus find from (114) that
\[ \lim_{\epsilon \to 0} \sup_{n \to \infty} \left\{ \lim_{\epsilon \to 0} \left( \rho_n(2n + 1) \right)^{-1} \log \mathbb{P} \left( \tilde{\mu}_{jn} \in O \text{ and } \tilde{\mu}_{jn} \notin U \right) \right\} \]
\[ = \lim_{\epsilon \to 0} \inf_{\mu \in F} \left\{ \rho_n(2n + 1) \right\}^{-1} \log \mathbb{P} \left( \tilde{\mu}_{jn} \in \pi^{-1}(F) \text{ and } \tilde{\mu}_{jn} \notin U \right) \]
\[ \geq -\inf_{\mu \in F} I_n(\mu). \]

We now turn to the upper bound. Fix $\epsilon > 0$ and define $V_\epsilon = \{ \mu \in \mathcal{M}^+(S^1 \times \mathcal{E}) : \mu(w = 1) \geq \epsilon \}$. By a union-of-events bound
\[ \lim_{n \to \infty} \left( \rho_n(2n + 1) \right)^{-1} \log \mathbb{P} \left( \tilde{\mu}_{jn} \notin F \right) \leq \lim_{n \to \infty} \left( \rho_n(2n + 1) \right)^{-1} \log \mathbb{P} \left( \tilde{\mu}_{jn} \notin F \text{ and } \tilde{\mu}_{jn} \in V_\epsilon \right) \]
\[ = \lim_{n \to \infty} \left( \rho_n(2n + 1) \right)^{-1} \log \mathbb{P} \left( \tilde{\mu}_{jn} \in \pi^{-1}(F) \text{ and } \tilde{\mu}_{jn} \notin V_\epsilon \right) \]
\[ \leq -\inf_{\mu \in F} I_n(\nu). \]

If $\inf_{\nu \in \pi^{-1}(F)} I_n(\nu) = \infty$, then it immediately follows that
\[ \lim_{n \to \infty} \left( \rho_n(2n + 1) \right)^{-1} \log \mathbb{P} \left( \tilde{\mu}_{jn} \notin F \right) \leq -\inf_{\mu \in F} I_n(\nu). \]

Otherwise, suppose that $\inf_{\nu \in \pi^{-1}(F)} I_n(\nu) < \infty$. For any $\mu \in F$ and any $a > 0$, $a\mu \in \pi^{-1}F$. We thus find from Lemma 3.9 that
\[ \inf_{\nu \in \pi^{-1}(F)} \hat{I}_n(\nu) = \inf_{\mu \in F} I_n(\mu). \]

The following Lemma establishes an 'exponential tightness' property.

**Lemma 3.5.** Let $\{j_n\}_{n \geq 1}$ be any sequence such that there exists $p$ such that $\theta_{jn}^k \in R_p$ for all $n \geq 1$. For any $M > 0$, there exists a subset $U_m \subset \mathcal{M}^+(S^1 \times \mathcal{E})$, where for $m \geq 0$,
\[ U_m = \{ \mu \in \mathcal{M}^+(S^1 \times \mathcal{E}) : \mu(w = 1) \leq m \} \quad (75) \]

such that
\[ \lim_{n \to \infty} \left( \rho_n(2n + 1) \right)^{-1} \log \mathbb{P} \left( \tilde{\mu}_{jn} \notin U_m \right) \leq -M \quad (76) \]

**Proof.** By Chernoff's Inequality, for any $a > 0$,
\[ \mathbb{P} \left( \tilde{\mu}_{jn} \notin U_m \right) \leq \exp \left( -\rho_n(2n + 1)am \right) \mathbb{E} \left[ \exp \left( a \sum_{k \in I_n} w^{j_n,k} \right) \right] \]
\[ = \exp \left( -\rho_n(2n + 1)am \right) \prod_{k \in I_n} \left[ 1 + \rho_n C(\theta_{jn}^k, \theta_n^k) (\exp(a) - 1) \right]. \quad (77) \]

Similarly to the proof of Lemma 3.8, we find that
\[ \lim_{n \to \infty} \left( \rho_n(2n + 1) \right)^{-1} \log \prod_{k \in I_n} \left[ 1 + \rho_n C(\theta_{jn}^k, \theta_n^k) (\exp(a) - 1) \right] \to \frac{1}{2\pi} (\exp(a) - 1) \int_{S^1} C(\alpha, \beta) d\beta. \quad (79) \]

Taking $a = 1$ and $m$ to be sufficiently large, the previous two equations imply that
\[ \lim_{n \to \infty} \left( \rho_n(2n + 1) \right)^{-1} \log \mathbb{P} \left( \tilde{\mu}_{jn} \notin U_m \right) \leq -M. \quad (80) \]

\[ \square \]
Lemma 3.6. Let \( \{j_n\}_{n \geq 1} \) be any sequence such that there exists \( p \) such that \( \theta_n^{i} \in R_p \) for all \( n \geq 1 \), and also \( \theta_n^{i} \to \alpha \in R_p \). Let \( \mathcal{O} \subseteq \mathcal{P}(S^1) \) be open and let \( \overline{\mathcal{O}} \) denote its closure. Then
\[
\lim_{n \to \infty} \frac{(2n+1)\rho_n}{(2n+1)\rho_n} \log \mathbb{P}(\hat{\mu}_j^n \in \mathcal{O}) = \lim_{n \to \infty} \frac{(2n+1)\rho_n}{(2n+1)\rho_n} \log \mathbb{P}(\hat{\mu}_j^n \in \overline{\mathcal{O}})
\]
\[
= -\inf_{\mu \in \mathcal{O}} I_\alpha(\mu) = -\inf_{\mu \in \mathcal{O}} I_\alpha(\mu).
\]

Proof. For \( 1 \leq p \leq M \), let \( \mathcal{O}_p \subseteq \mathcal{P}(S^1) \) be any set of the form
\[
\mathcal{O}_p = \{ \mu \in \mathcal{P}(S^1) : \mu(F_i^p) > \lambda_i^p \text{ for all } 1 \leq i \leq m_p \},
\]
for some \( m_p \in \mathbb{Z}^+ \), constants \( \lambda_i^p \in [0,1) \) such that
\[
\sum_{i=1}^{m_p} \lambda_i < 1,
\]
and closed mutually disjoint sets \( F_i^p \subseteq S^1 \), i.e. \( F_i^p \cap F_j^p = \emptyset \) if \( i \neq j \). It follows from the Portmanteau Theorem that \( \mathcal{O}_p \) is open [5].

We now claim that one must be able to find a countably infinite set of such open sets such that
\[
\mathcal{O} = \bigcup_{p \geq 1} \mathcal{O}_p.
\]
This is because \( \mathcal{P}(S^1) \) is separable with respect to the weak topology, and open sets of the type \( \mathcal{O}_p \) generate the weak topology [5]. It follows from (84) that
\[
\mathcal{O} = \bigcup_{p \geq 1} \mathcal{O}_p.
\]

Using the Large Deviations estimate (in the second line) and Lemma 3.7 in the penultimate line,
\[
\lim_{n \to \infty} \frac{(2n+1)\rho_n}{(2n+1)\rho_n} \log \mathbb{P}(\hat{\mu}_j^n \in \mathcal{O}) = \lim_{n \to \infty} \frac{(2n+1)\rho_n}{(2n+1)\rho_n} \log \mathbb{P}(\hat{\mu}_j^n \in \overline{\mathcal{O}})
\]
\[
= -\inf_{\mu \in \mathcal{O}} I_\alpha(\mu) = -\inf_{\mu \in \mathcal{O}} I_\alpha(\mu).
\]

The Large Deviations estimate also implies that
\[
\lim_{n \to \infty} \frac{(2n+1)\rho_n}{(2n+1)\rho_n} \log \mathbb{P}(\hat{\mu}_j^n \in \mathcal{O}) \geq -\inf_{\mu \in \mathcal{O}} I_\alpha(\mu).
\]
We have thus established the lemma. \( \square \)

Lemma 3.7. Let \( \mathcal{O} \subseteq \mathcal{P}(S^1) \) be any set of the form
\[
\mathcal{O} = \{ \mu \in \mathcal{P}(S^1) : \mu(F_i) > \lambda_i \text{ for all } 1 \leq i \leq m \},
\]
for some \( m \in \mathbb{Z}^+ \), constants \( \lambda_i \in [0,1) \) such that
\[
\sum_{i=1}^{m} \lambda_i < 1,
\]
and closed mutually disjoint sets \( F_i \subseteq S^1 \), i.e. \( F_i \cap F_j = \emptyset \) if \( i \neq j \). Then
\[
\inf_{\mu \in \mathcal{O}} I_\alpha(\mu) = \inf_{\mu \in \mathcal{O}} I_\alpha(\mu).
\]

Proof. Since \( I_\alpha \) is lower semi-continuous, there exists \( \nu \in \overline{\mathcal{O}} \) such that
\[
\inf_{\mu \in \mathcal{O}} I_\alpha(\mu) = I_\alpha(\nu).
\]
Since \( I_\alpha(\gamma) \leq \frac{1}{2\pi} \int_{S^1} C(\alpha, \theta) d\theta \) for all \( \gamma \in \mathcal{P}(S^1) \), we may assume that
\[
I_\alpha(\nu) \leq \frac{1}{2\pi} \int_{S^1} C(\alpha, \theta) d\theta.
\]
since otherwise the lemma is obviously true. The definition of \( I_\alpha \) implies that \( \nu \) has a density \( \zeta(\theta)/(2\pi) \).

It follows from the Portmanteau Theorem that

\[
\nu(F_i) \geq \lambda_i \text{ for all } 1 \leq i \leq m
\]

and therefore

\[
\frac{1}{2\pi} \int_{F_i} \zeta(\theta) d\theta \geq \lambda_i.
\]  

(92)

Notice that

\[
I_\alpha(\nu) = \frac{1}{2\pi} \int_{\mathbb{S}^1} \mathcal{C}(\alpha, \theta) d\theta - \exp \left( - R(\nu) \right)
\]

where

\[
R(\nu) = \frac{1}{2\pi} \int_{\mathbb{S}^1} \zeta(\theta) \log \left( \frac{\zeta(\theta)}{\mathcal{C}(\alpha, \theta)} \right) d\theta.
\]  

(94)

Define \( \upsilon \in \mathcal{O} \) to be such that \( \nu \) has a density \( \phi(\theta)/(2\pi) \) and also satisfies the following properties:

\[
\upsilon(F_i) = 1
\]

\[\phi \text{ is constant over each } F_i\]

\[
\nu(F_i) = \lambda_i/(\sum_{i=1}^{m} \lambda_i).
\]  

(97)

Next, for \( q \in \mathbb{Z}^+ \), define the measure \( \nu^q = (1 - q^{-1}) \nu + q^{-1} v \), and observe that \( \nu^q \) has a density \( \zeta^q \) given by \( \zeta^q(\theta) = (1 - q^{-1}) \zeta(\theta) + q^{-1} \phi(\theta) \). It follows from (92) and (97) that \( \nu^q \in \mathcal{O} \). It thus remains for us to demonstrate that \( \lim_{q \to \infty} I_\alpha(\nu^q) = I_\alpha(\nu) \).

Let \( A \subset \mathbb{S}^1 \) be the set

\[
A = \{ \theta : \zeta(\theta) \geq \phi(\theta) \text{ and } \zeta(\theta) \geq \mathcal{C}(\alpha, \theta) \}.
\]  

(98)

We have that

\[
R(\nu^q) = \frac{1}{2\pi} \int_A \zeta^q(\theta) \log \left( \frac{\zeta^q(\theta)}{\mathcal{C}(\alpha, \theta)} \right) d\theta + \frac{1}{2\pi} \int_{\mathbb{S}^1 \setminus A} \zeta^q(\theta) \log \left( \frac{\zeta^q(\theta)}{\mathcal{C}(\alpha, \theta)} \right) d\theta.
\]

Thanks to the dominated convergence theorem, since \( \zeta^q(\theta) \to \zeta(\theta) \) as \( q \to \infty \),

\[
\lim_{q \to \infty} \int_{\mathbb{S}^1 \setminus A} \zeta^q(\theta) \log \left( \frac{\zeta^q(\theta)}{\mathcal{C}(\alpha, \theta)} \right) d\theta = \int_{\mathbb{S}^1 \setminus A} \zeta(\theta) \log \left( \frac{\zeta(\theta)}{\mathcal{C}(\alpha, \theta)} \right) d\theta.
\]

Now one easily checks that for \( \theta \in A \),

\[
\zeta^{q+1}(\theta) \log \left( \frac{\zeta^{q+1}(\theta)}{\mathcal{C}(\alpha, \theta)} \right) \geq \zeta^q(\theta) \log \left( \frac{\zeta^q(\theta)}{\mathcal{C}(\alpha, \theta)} \right),
\]

since the function \( x \mapsto x \log(x) \) is increasing for \( x \geq 1 \). We thus find from the monotone convergence theorem that

\[
\lim_{q \to \infty} \int_A \zeta^q(\theta) \log \left( \frac{\zeta^q(\theta)}{\mathcal{C}(\alpha, \theta)} \right) d\theta = \frac{1}{2\pi} \int_A \zeta(\theta) \log \left( \frac{\zeta(\theta)}{\mathcal{C}(\alpha, \theta)} \right) d\theta.
\]

We can thus conclude that \( \lim_{q \to \infty} R(\nu^q) = R(\nu) \), and therefore \( \lim_{q \to \infty} I_\alpha(\nu^q) = I_\alpha(\nu) \). We have established (90).

\[\blacksquare\]

**Lemma 3.8.** Let \( \{\eta_n\}_{n \geq 1} \) be any sequence such that there exists \( p \) such that \( \theta_{\eta_n}^p \in R_p \) for all \( n \geq 1 \), and also \( \theta_{\eta_n}^p \to \alpha \in R_p \). Then \( \{\tilde{\Pi}_{\eta_n}\}_{n \geq 1} \) satisfy a Large Deviation Principle with rate function

\[
\tilde{I}_\alpha(\nu) = \left\{ \begin{array}{ll}
\frac{1}{2\pi} \int_{\mathbb{S}^1} \left\{ \gamma(\theta) \log \left( \frac{\gamma(\theta)}{\mathcal{C}(\alpha, \theta)} \right) - \gamma(\theta) + \mathcal{C}(\alpha, \theta) \right\} d\theta & \text{if } \nu(\cdot, 1) \text{ does not have a density, otherwise} \\
\frac{1}{2\pi} \int_{\mathbb{S}^1} \left\{ \gamma(\theta) \log \left( \frac{\gamma(\theta)}{\mathcal{C}(\alpha, \theta)} \right) - \gamma(\theta) + \mathcal{C}(\alpha, \theta) \right\} d\theta & \text{if } \nu(\cdot, 1) \text{ does have a density, otherwise}
\end{array} \right.
\]  

(99)

The LDP means that for open sets \( O \subset \mathcal{M}^+ (\mathbb{S}^1 \times \mathcal{E}) \) and closed sets \( F \subset \mathcal{M}^+ (\mathbb{S}^1 \times \mathcal{E}) \),

\[
\lim_{n \to \infty} \frac{1}{\rho_n(2n + 1)^2} \log \tilde{\Pi}_{\eta_n}(F) \leq - \inf_{\nu \in F} \tilde{I}_\alpha(\nu)
\]

(100)

\[
\lim_{n \to \infty} \frac{1}{\rho_n(2n + 1)^2} \log \tilde{\Pi}_{\eta_n}(O) \geq - \inf_{\nu \in O} \tilde{I}_\alpha(\nu)
\]

(101)

\( \tilde{I} \) is lower semicontinuous and has compact level sets.
Proof. We first show that the LDP (relative to the topology $\mathcal{T}$ defined in (69)) holds with the rate function

$$I_\alpha (\nu) = \sup_{h \in \mathcal{C}(\mathbb{S})} \left\{ \mathbb{E}^\nu[h(\theta)w] - \hat{\Lambda}(\alpha, h) \right\},$$

(102)

$$\hat{\Lambda}(\alpha, h) = \frac{1}{2\pi} \int_{-\pi}^\pi C(\alpha, \beta) \left( \exp \{ h(\beta) \} - 1 \right) d\beta$$

(103)

Define the logarithmic moment generating function, for any bounded and continuous $h: \mathbb{S}^1 \to \mathbb{R}$,

$$\hat{\Lambda}(\alpha, h) = \lim_{n \to \infty} \left( \rho_n (2n + 1) \right)^{-1} \log \mathbb{E} \left[ \exp \left( \sum_{k \in I_n} h(\theta_n^k) w_{jn}^k \right) \right],$$

(104)

recalling that $(j_n)_{n=1}^{\infty}$ is any sequence such that $\theta_n^k \to \alpha \in \mathbb{S}^1$. We first establish that for all such sequences,

$$\hat{\Lambda}(\alpha, h) = \hat{\Lambda}(\alpha, h).$$

(105)

Observe that

$$\mathbb{E} \left[ \exp \left( \sum_{k=0}^{n} h(\theta_n^k) w_{jn}^k \right) \right] = \mathbb{E} \left[ \exp \left( h(\theta_n^k) + \sum_{k \in I_n} w_{jn}^k h(\theta_n^k) w_{jn}^k \right) \right]$$

$$= \exp \left( h(\theta_n^k) \right) \prod_{k \in I_n} \left( 1 + \rho_n C(\theta_n^k, \theta_n^k) \left( \exp \left( h(\theta_n^k) \right) - 1 \right) \right)$$

since $w_{jn}^k = 1$ identically. We thus find that

$$\lim_{n \to \infty} \left( \rho_n (2n + 1) \right)^{-1} \log \mathbb{E} \left[ \exp \left( \sum_{k \in I_n} h(\theta_n^k) w_{jn}^k \right) \right] =$$

$$\lim_{n \to \infty} \left( \rho_n (2n + 1) \right)^{-1} \sum_{k \in I_n} \log \left( 1 + \rho_n C(\theta_n^k, \theta_n^k) \left( \exp \left( h(\theta_n^k) \right) - 1 \right) \right)$$

$$= \lim_{n \to \infty} \left( \rho_n (2n + 1) \right)^{-1} \sum_{k \in I_n} \left\{ \rho_n C(\theta_n^k, \theta_n^k) \left( \exp \left( h(\theta_n^k) \right) - 1 \right) + O(\rho_n^2) \right\},$$

(106)

through a second degree Taylor Expansion of the logarithmic function. Since $\rho_n \to 0$, the $O(\rho_n^2)$ term is asymptotically negligible, and we therefore obtain that

$$\lim_{n \to \infty} \left( \rho_n (2n + 1) \right)^{-1} \log \mathbb{E} \left[ \exp \left( \sum_{k=0}^{n} h(\theta_n^k) w_{jn}^k \right) \right] = \frac{1}{2\pi} \int_{-\pi}^\pi C(\alpha, \beta) \left( \exp \left( h(\beta) \right) - 1 \right) d\beta.$$ 

(107)

We have thus proved (105). It follows from [19, Corollary 4.6.14] that the sequence of probability laws $(\Pi_n^\nu)_{n \geq 1}$ satisfy a Large Deviation Principle. This is because the function $h \to \hat{\Lambda}(\alpha, h)$ is Gateaux Differentiable. The exponential tightness property follows from Lemma 3.5. One can adapt the proof of Prokhorov’s Theorem to prove that $\mathcal{U}_m$ is compact for any $m > 0$. We have thus established that the sequence of probability laws $(\Pi_n^\nu)_{n \geq 1}$ satisfy a Large Deviation Principle with good rate function (102).

Next we establish that if $\nu(w) = 0$ and $\nu(\cdot, 1)$ does not have a density, then $I(\nu) = \infty$. To this end, let $B(\theta, \delta) \subset \mathbb{S}^1$ be the open ball centered at $\theta$ of radius $\delta$. Let $\nu \in \mathcal{P}(\mathbb{S}^1)$ be Lebesgue measure on $\mathbb{S}^1$. Suppose that there exists $\theta \in \mathbb{S}^1$ such that

$$\lim_{\delta \to 0} \frac{\nu(B(\theta, \delta))}{2\delta} = \infty.$$ 

(108)

For $\delta > 0$, define the function

$$h_\delta(\alpha) = \begin{cases} 
\delta^{-1} & \text{if } x \in B(\theta, \delta) \\
0 & \text{if } x \in B(\theta, \delta) + \mathcal{S}^1 \setminus B(\theta, \delta)
\end{cases}$$

Notice that $h_\delta(\alpha)$ is continuous and bounded. We easily check that

$$\lim_{\delta \to 0} \mathbb{E}^\nu[h_\delta] = \infty , \quad \lim_{\delta \to 0} \left| \hat{\Lambda}(\alpha, h_\delta) \right| < \infty,$$
which means that $\tilde{I}_\alpha(\nu) = \infty$.

Finally, we suppose that $\nu(\cdot, 1)$ has a density given by $\frac{1}{\gamma} \gamma(\cdot)$. We then see that

$$
\tilde{I}_\alpha(\nu) = \sup_{h \in C_c(S^1)} \frac{1}{2\pi} \int_{S^1} \gamma(\theta) \left\{ h(\theta) \frac{C(\alpha, \theta)}{\gamma(\theta)} \left( \exp(h(\theta)) - 1 \right) \right\} d\theta.
$$

We can find the unique supremum of the integrand by differentiating. This implies that $\tilde{I}_\alpha(\nu)$ takes the form in (99).

**Lemma 3.9.** Let $\mu \in \mathcal{P}(S^1)$ have a density $\zeta(\theta)/(2\pi)$. Assume that

$$
\int_{S^1} \zeta(\theta) \log(\zeta(\theta)) d\theta < \infty.
$$

Then

$$
\inf_{\nu \in \mathcal{M}^*(S^1 \times S^1) : \mu = \nu} \tilde{I}_\alpha(\nu) = I_\alpha(\mu).
$$

**Proof.** It must be that

$$
\int_{S^1} \zeta(\theta) d\theta = 2\pi.
$$

Any $\nu \in \mathcal{M}^*(S^1 \times S^1)$ such that $\pi \cdot \nu = \mu$ must have density $a \zeta(\theta)$, for some $a > 0$. We then find that

$$
\tilde{I}_\alpha(\nu) = \frac{1}{2\pi} \int_{S^1} \left\{ a \zeta(\theta) \log \left( \frac{a \zeta(\theta)}{C(\alpha, \theta)} \right) - a \zeta(\theta) + C(\alpha, \theta) \right\} d\theta =: \Gamma(a).
$$

Differentiating with respect to $a$ to obtain the minimum, we find that

$$
\frac{d\Gamma}{da} = \frac{1}{2\pi} \int_{S^1} \left\{ \zeta(\theta) \log \left( \frac{\zeta(\theta)}{C(\alpha, \theta)} \right) + \zeta(\theta) \log(a) \right\} d\theta.
$$

We thus find that the value of $a$ at the unique critical point is

$$
a_* = \exp \left( \frac{1}{2\pi} \int_{S^1} \zeta(\theta) \log \left( \frac{C(\alpha, \theta)}{\zeta(\theta)} \right) d\theta \right).
$$

$\Gamma$ achieves a local minimum at $a_*$, because $\Gamma \to \infty$ as $a \to \infty$. We thus find that, after substituting $a = a_*$ into (112),

$$
I_\alpha(\mu) = \frac{1}{2\pi} \int_{S^1} C(\alpha, \theta) d\theta - \exp \left( \frac{1}{2\pi} \int_{S^1} \zeta(\theta) \log \left( \frac{C(\alpha, \theta)}{\zeta(\theta)} \right) d\theta \right).
$$

One can double check using Jensen’s Inequality that $I_\alpha(\mu) \geq 0$.

**Lemma 3.10.** Let $\{\eta_n\}_{n \geq 1}$ be any sequence such that there exists $p$ such that $\theta_n^{(i)} \in R_p$ for all $n \geq 1$, and also $\theta_n^{(i)} \to \alpha \in R_p$. Then, recalling the definition of $\mathcal{U}_m$ in (75),

$$
\lim_{m \to 0^+} \lim_{n \to \infty} \left( \rho_n(2n + 1) \right)^{-1} \log \mathbb{P} \left( \tilde{\mu}_n^{(m)} \in \mathcal{U}_m \right) = \frac{1}{2\pi} \int_{S^1} C(\alpha, \theta) d\theta.
$$

**Proof.** Since $\mathcal{U}_m$ is closed, by Lemma 3.8,

$$
\lim_{n \to \infty} \left\{ \rho_n(2n + 1) \right\}^{-1} \log \mathbb{P} \left( \tilde{\mu}_n^{(m)} \in \mathcal{U}_m \right) \leq - \inf_{\nu \in \mathcal{U}_m} \tilde{I}_\alpha(\nu).
$$

Define $\mathcal{V}_m = \left\{ \nu \in \mathcal{M}^*(S^1 \times S^1) : \nu(\nu(1) = 1) < m \right\}$. Since $\mathcal{V}_m$ is open, by Lemma 3.8,

$$
\lim_{n \to \infty} \left\{ \rho_n(2n + 1) \right\}^{-1} \log \mathbb{P} \left( \tilde{\mu}_n^{(m)} \in \mathcal{V}_m \right) \geq - \inf_{\nu \in \mathcal{V}_m} \tilde{I}_\alpha(\nu).
$$

It follows from the previous two equations that it suffices for us to prove that

$$
\lim_{m \to 0^+} \inf_{\nu \in \mathcal{U}_m} \tilde{I}_\alpha(\nu) = \frac{1}{2\pi} \int_{S^1} C(\alpha, \theta) d\theta.
$$
First, define $\nu_m \in \mathcal{U}_m$ to be such that $\nu(\cdot, 1)$ has the uniform density $\frac{\Theta}{2\pi}$. One easily checks that $\tilde{I}_n(\nu_m) \to \frac{1}{2\pi} \int_{\mathbb{S}^1} C(\alpha, \theta) d\theta$ as $m \to 0$. We thus see that

$$\lim_{m \to 0} \inf_{\nu \in \mathcal{U}_m} \tilde{I}_n(\nu) \leq \frac{1}{2\pi} \int_{\mathbb{S}^1} C(\alpha, \theta) d\theta. \quad (118)$$

Now suppose for a contradiction that $\nu^m \in \mathcal{U}_m$, and $\nu^m(\cdot, 1)$ has a density $\zeta^m(\theta)/(2\pi)$, and that there exists $\epsilon, m_0 > 0$ such that for all $m < m_0$,

$$\tilde{I}_n(\nu^m) \leq \frac{1}{2\pi} \int_{\mathbb{S}^1} C(\alpha, \theta) d\theta - \epsilon \quad (119)$$

Thanks to (118) and the definition of $\tilde{I}_n$ in (99), it must be that for all sufficiently large $m$,

$$\frac{1}{2\pi} \int_{\mathbb{S}^1} \{\zeta^m(\theta) \log \left( \frac{\zeta^m(\theta)}{C(\alpha, \theta)} \right) - \zeta^m(\theta) \} d\theta \leq -\epsilon. \quad (120)$$

Our assumption that $C$ is lower bounded implies that there exists $C_{lb} > 0$ such that

$$\inf_{\theta, \alpha} C(\alpha, \theta) \geq C_{lb} > 0.$$  

Thus

$$\frac{1}{2\pi} \int_{\mathbb{S}^1} \{\zeta^m(\theta) \log \left( \frac{\zeta^m(\theta)}{C(\alpha, \theta)} \right) + \zeta^m(\theta) \left( - \log(C_{lb}) - 1 \right) \} d\theta \leq -\epsilon. \quad (121)$$

We thus find that

$$\frac{1}{2\pi} \int_{\mathbb{S}^1} \zeta^m(\theta) \log \left( \zeta^m(\theta) \right) d\theta \leq m \left( \log(C_{lb}) + 1 \right) - \epsilon. \quad (122)$$

However there exists a positive constant $K$ such that

$$x \log x \geq -Kx.$$  

Thus

$$-K \frac{1}{2\pi} \int_{\mathbb{S}^1} \zeta^m(\theta) d\theta \leq m \left( \log(C_{lb}) + 1 \right) - \epsilon, \quad (123)$$

and therefore

$$-Km \leq m \left( \log(C_{lb}) + 1 \right) - \epsilon. \quad (124)$$

Taking $m \to 0$, we obtain a contradiction. We have thus proved the lemma. \qed

## 4 Proof of Theorems 2.1 and 2.2

To push-forward the initial conditions by the dynamics, we wish to find the coarsest topology possible that will ensure that the dynamics depends continuously on the initial condition. The weakest such topology that we could find is $\mathcal{T}$, that we will now specify precisely. We will first define a topology $\mathcal{T}_m$ that ‘knows’ about the distribution of the states $\{u_n^\kappa\}$ that can be reached through no more than $m$ connections. $\mathcal{T}_0$ is the standard weak topology over $\mathcal{P}(\mathcal{D})$. $\mathcal{T}$ will be defined to be the coarsest topology containing $\bigcup_{m \geq 1} \mathcal{T}_m$.

Let $\mathcal{V}_0 = \mathcal{P}(\mathcal{D})$, and for $m \geq 1$, let $\mathcal{V}_m$ be the space

$$\mathcal{V}_m = \mathcal{P}(\mathcal{D} \times \mathcal{V}_{m-1}). \quad (125)$$

Recalling that $\mathcal{D}$ is a compact subset of $\mathbb{R}^d$, define $\Phi_m : \mathcal{P}(\mathcal{D} \times \mathcal{P}(\mathcal{D})) \to \mathcal{V}_{m+1}$ as follows. Define $\Lambda^0 : \mathcal{P}(\mathcal{D} \times \mathcal{P}(\mathcal{D})) \times \mathcal{D} \to \mathcal{P}(\mathcal{P}(\mathcal{D}))$, for $B \in \mathcal{B}(\mathcal{P}(\mathcal{D}))$,

$$\Lambda^0_{\mu, x}(B) = \left\{ \lim_{m \to 0} \frac{g(\mu, x, \beta) \chi(\{y-x|z_n^{-1} \text{ and } \delta B\})}{g(\mu, y, \beta, \chi(\{y-x|z_n^{-1}\}) \delta B\})} \chi(\kappa \in B) \right\} \quad (126)$$

for some fixed $\kappa \in \mathcal{P}(\mathcal{D})$ (it does not matter how we choose $\kappa$). It follows from Levy’s Downwards Theorem that for $\mu$ almost every $x$, the above limit $\delta \to 0$ exists, and $\Lambda^0_{\mu, x}$ is a regular conditional
probability distribution \([37]\). However we emphasize that \(\Lambda_{\mu,x}^0\) is a precisely defined measurable function for every \(x \in \mathcal{D}\), not just for \(\mu\) almost every \(x\).

We next define \(\Phi_m : \mathcal{P}(\mathcal{D} \times \mathcal{P}(\mathcal{D})) \to \mathcal{V}_m\), as follows. First define the map \(\Lambda_m^0 : \mathcal{P}(\mathcal{D}) \to \mathcal{P}(\mathcal{D} \times \mathcal{P}(\mathcal{D}))\) to be

\[
\Lambda_m^0(\beta) := \alpha \quad \text{where} \quad \alpha(A \times B) = \int_A \Lambda_{m,x}^0(B) d\beta(x) \quad \text{for any} \quad A \in \mathcal{B}(\mathcal{D}) \quad \text{and} \quad B \in \mathcal{B}(\mathcal{P}(\mathcal{D})).
\]

Next, for \(m \geq 0\), define \(\Lambda_m^m : \mathcal{V}_m \to \mathcal{V}_{m+1}\),

\[
\Lambda_m^m(\beta) := \alpha \quad \text{where} \quad \alpha(A \times B_m) = \int \chi(x \in A \text{ and } \Lambda_m^{m-1}(\gamma) \in B_m) d\beta(x, \gamma)
\]

for any \(A \in \mathcal{B}(\mathbb{R}^d), B_m \in \mathcal{B}(\mathcal{V}_m)\). Now define \(\Phi_m \cdot \mu\) as follows. \(\Phi_1 \cdot \mu = \mu\). Writing \(\mu\) to be the law of random variables \((x, \beta) \in \mathcal{D} \times \mathcal{P}(\mathcal{D})\), for \(m \geq 2\) define \(\Phi_m \cdot \mu\) to be the law of the random variables \((x, \beta')\), where

\[
\hat{\beta} = \Lambda_m^{m-2}(\Lambda_m^{m-3}(\ldots \Lambda_m^0(\beta))) \in \mathcal{V}_{m-1}.
\]

Define \(\mathcal{T}_m\) to be the topology on \(\mathcal{P}(\mathcal{D} \times \mathcal{P}(\mathcal{D}))\) that is generated by open sets of the form

\[
\{\mu : \Phi_m \cdot \mu \in \mathcal{O}\}
\]

for some \(\mathcal{O} \subset \mathcal{V}_m\) that is open with respect to the weak topology on \(\mathcal{V}_m\). We define \(\mathcal{T}_1\) to be the standard weak topology on \(\mathcal{P}(\mathcal{D} \times \mathcal{P}(\mathcal{D}))\). Let \(\mathcal{T}\) be the coarsest topology on \(\mathcal{P}(\mathcal{D} \times \mathcal{P}(\mathcal{D}))\) containing

\[\bigcup_{i \in \mathbb{Z}} \mathcal{T}_i.\]

Fix \(m \in \mathbb{Z}^+\). We now define a continuous map \(\Psi_m : \mathcal{P}(\mathcal{D} \times \mathcal{P}(\mathcal{D})) \to \mathcal{P}(\mathcal{C}([0,T], \mathbb{R}^d))\) as follows. First, we specify \(\Psi_m \cdot \nu\) in the case that \(\nu\) is an empirical measure of the form

\[
\nu = \frac{1}{2q+1} \sum_{j \in I_q} \delta_{y_j} \cdot \nu_j = \vert \mathcal{E}_j \vert^{-1} \sum_{k \in \mathcal{E}_j} \delta_{y_k},
\]

for arbitrary \(\{y_j\}_{j \in I_q} \subset \mathcal{D}\) and \(\mathcal{E}_j \subset I_q\). Define \(\{y_m^s(s)\}_{s \in \mathbb{Z}} \in \mathcal{C}([0,T], \mathbb{R}^d)\) as follows: for \(0 \leq p < m\) and \(s \in [pTm^{-1}, (p+1)Tm^{-1})\),

\[
y_m^s(s) = y_m^s(pT/m) + (s - pTm^{-1}) \left\{ f(y_m^s(pT/m)) + \mathbb{E}^{\nu_m^{pT/m}}[G(y_m^s(pT/m), \cdot)] \right\}
\]

and

\[
\nu_j^m(s) = \mathcal{E}_j^{-1} \sum_{k \in \mathcal{E}_j} \delta_{y_k}(s).
\]

It can be seen that the variables \(y_m^s(s)\) are well-defined: one obtains the solution at the discretized time steps \((pTm^{-1})\) by Euler-stepping, and obtains the solution at all other times through linear interpolation. We then define

\[
\Psi_m \cdot \nu = \mathbb{P}_{[0,T]} \in \mathcal{P}(\mathcal{C}([0,T], \mathbb{R}^d)) \quad \text{where} \quad \mathbb{P}_{[0,T]}^n = \frac{1}{2q+1} \sum_{j \in I_n} \delta_{y_j^m([0,T])}.
\]

Notice first that \(\Psi_m \cdot \nu\) is consistently defined: a permutation of the indices (remembering also to permute the connection indices) leaves both \(\nu\) and \(\Psi_m \cdot \nu\) unchanged. Noting that empirical measures of the form in (133) are dense in \(\mathcal{P}(\mathcal{D} \times \mathcal{P}(\mathcal{D}))\) (with respect to the topology \(\mathcal{T}\)), we next define, for an arbitrary \(\gamma \in \mathcal{V}_m\),

\[
\Psi_m \cdot \gamma := \lim_{p \to \infty} \Psi_m \cdot \mu_p \quad \text{where} \quad \{\mu_p\}_{p \geq 1}
\]

is any sequence of empirical measures of the general form (133) such that

\[
\Phi_m \cdot \mu_p \to \Phi_m \cdot \gamma \quad \text{with respect to the weak topology on} \ \mathcal{V}_m.
\]
Lemma 4.1. $Ψ_m : \mathcal{P}(\mathcal{D} \times \mathcal{P}(\mathcal{D})) \to \mathcal{P}(C([0,T],\mathbb{R}^d))$ is well-defined. Also $Ψ_m$ is continuous, when $\mathcal{P}(\mathcal{D} \times \mathcal{P}(\mathcal{D}))$ is endowed with the topology $\mathcal{T}_m$, and $\mathcal{P}(C([0,T],\mathbb{R}^d))$ is endowed with the weak topology.

Proof. We wish to define a map $Γ^m : V_m \to \mathcal{P}(C([0,T],\mathbb{R}^d))$ such that

$$Ψ_m = Γ^m \cdot Φ_m.$$  \hfill (138)

$Γ^m$ will be shown to be continuous with respect to the weak topology on $V_m$. Define, for $0 ≤ k ≤ m - 1$, $δt = Tm^{-1}$ and

$$W_0^m = \mathcal{P}(\mathcal{D} \times \mathcal{P}(\mathcal{D} \times \mathcal{P}(\mathcal{D} \times \mathcal{P}(\mathcal{D} \ldots))))$$  \hfill (139)

$$W_k^m = \mathcal{P}((Z_k \times \mathcal{P}(Z_k \times \mathcal{P}(\ldots \mathcal{P}(Z_k \ldots)))) \times \mathcal{P}(Z_k \times \mathcal{P}(\ldots \mathcal{P}(Z_k \ldots))))$$ where

$$Z_k = C([0,kδt],\mathbb{R}^d)$$  \hfill (140)

In the above definition of $W_0^m$, there are $m + 1$ nested $\mathcal{D}$ (i.e. there are $m$ right parentheses on the right side of (139)). In the definition of $W_k^m$, there are $m + 1 - k$ nested $Z_k$. We now define a continuous map $Γ^m_k : W_k^m \to W_{k+1}^m$, as follows. Fix $α \in W_k^m$ and write $β = Γ^m_k \cdot α$.

Write $α$ to be the probability law of the random variables $(z_{0,kδt})^0, (z_{[0,kδt])}^0$, where $(z_{0,kδt})^0 \in Z_k$ and $α^0 \in \mathcal{P}(Z_k \times \mathcal{P}(Z_k \times \mathcal{P}(\ldots \mathcal{P}(Z_k \ldots))))$ (there are $m - k$ nested $Z_k$ in this expression for $α^0$). In turn, write $α^0$ to be the probability law of $(a^{(1)})$, where $a^{(1)} \in \mathcal{P}(Z_k \times \mathcal{P}(Z_k \times \mathcal{P}(\ldots \mathcal{P}(Z_k \ldots))))$ (there are $m - k - 1$ nested $Z_k$). Eventually we obtain $α^{(m-k-1)} \in \mathcal{P}(Z_k \times \mathcal{P}(Z_k))$: write this to be the probability law of the random variables $(z_{0,kδt})^{(m-k)}, \alpha^{(m-k)}$, with $α^{(m-k)} \in \mathcal{P}(Z_k)$.

Now, for $0 ≤ a ≤ m - k$, define new random variables $y_{[a,(k+1)δt]}^{(a)} \in Z_{k+1}$, such that for each $s \in (kδt,(k+1)δt]$,

$$y_{a}^{(a)} = z_{s,kδt}^{(a)} + (s - kδt)\left\{ f(z_{s,kδt}^{(a)}) + \int G(z_{s,kδt}^{(a)},x_{kδt})da^{(a)}(x_{[0,kδt]}) \right\}$$  \hfill (142)

$$y_{a}^{(a)} = z_{s}^{(a)} \text{ for } 0 ≤ s ≤ kδt.$$  \hfill (143)

Define $β^{(m-k-1)} \in \mathcal{P}(Z_{k+1})$ to be the probability law of $(y_{0,(k+1)δt]}^{(m-k-1)})$, and for $0 ≤ a ≤ m - k - 1$, define $β^{(a)}$ to be the probability law of $(y_{0,(k+1)δt]}^{(a-1)}; β^{(a+1)})$. Finally, $β \in W_{k+1}^m$ is defined to be the probability law of $(y^{(0)}, β^{(0)})$.

We observe that the above definition of $Ψ_m$ is consistent with the definition in (136) for empirical measures. Also it can be seen that $Γ^m_k : W_k^m \to W_{k+1}^m$ is continuous when both $W_k^m$ and $W_{k+1}^m$ are endowed with the weak topology. Now define

$$Γ^m = Γ^m_m \cdot Γ^m_{m-1} \cdots Γ^m_1 \cdot Γ^0.$$  \hfill (144)

Evidently $Γ^m$ is continuous as well. By definition of the topology $T_m$, the mapping $Φ_m : (\mathcal{P}(\mathcal{D} \times \mathcal{P}(\mathcal{D})), T_m) \to (V_{m+1}, T_w)$ is continuous. We have thus proved the continuity of $Ψ_m$.

We wish to define $Ψ : \mathcal{P}(\mathcal{D} \times \mathcal{P}(\mathcal{D})) \to \mathcal{P}(C([0,T],\mathbb{R}^d))$ to be such that for all measures in a ‘large’ subset of $\mathcal{P}(\mathcal{D} \times \mathcal{P}(\mathcal{D}))$, $Ψ : μ = \lim_{m \to \infty} Ψ_m \cdot μ$. First we need to better understand the circumstances under which this limit exists. Let $Y \subseteq \mathcal{P}(\mathcal{D} \times \mathcal{P}(\mathcal{D}))$ be the set of all empirical measures, i.e.

$$Y = \left\{ \hat{μ} = \frac{1}{2q+1} \sum_{j \in I_q} δ_{(q_j,q_j)} : \hat{μ}_q = \frac{1}{2q+1} \sum_{k \in Z_q} δ_{q,k} \text{ for some } q \in \mathbb{Z}^+, \{q_j\}_{j \in I_q} \subseteq \mathcal{D}, \{Z_j\}_{j \in I_q} \subseteq I_q \right\}.$$  \hfill (145)

We first study $Ψ_m$ when $ν \in Y$ is an empirical measure of the form

$$ν = \frac{1}{2q+1} \sum_{j \in I_q} δ_{(q_j,v_j)} \; , \; ν_j = \left| \vec{ξ}_j \right|^{-1} \sum_{k \in Z_j} δ_{q,k}.$$  \hfill (146)
for arbitrary \( (y^j)_{j \in I_q} \subset \mathcal{D} \) and \( \Xi_j \subset I_q \). Define
\[
z^j(t) = y^j + \int_0^t \{ f(z^j(s)) + \mathbb{E}^{\nu_j(s)}[G(z^j(s), \cdot)] \} \, ds \tag{147}
\]
and
\[
\nu_j(s) = \frac{1}{|\Xi_j|} \sum_{k \in \Xi_j} \delta_{\xi^j(s)}. \tag{148}
\]
Since the functions \( f \) and \( G \) are Lipschitz and bounded, Picard’s Theorem implies the existence of a unique solution. One easily checks using Gronwall’s Inequality that there exists a constant \( C_m > 0 \) such that \( \lim_{m \to \infty} C_m = 0 \) and for all such empirical measures,
\[
\sup_{j \in I_q} \sup_{s \in [0, T]} \| y^j(s) - z^j(s) \| \leq C_m / 2. \tag{149}
\]
We endow \( \mathcal{P}(C([0, T], \mathbb{R}^d)) \) with the Wasserstein metric, given by
\[
d_W(\mu, \nu) = \inf_{\eta} \mathbb{E}^\eta \left[ \sup_{s \in [0, T]} \| y_s - z_s \| \right]. \tag{150}
\]
the infimum being taken over all measures \( \eta \in \mathcal{P}(C([0, T], \mathbb{R}^d) \times C([0, T], \mathbb{R}^d)) \) with marginal laws \( \mu \) and \( \nu \). We have thus established the following lemma.

**Lemma 4.2.** For all \( n \geq m \),
\[
\sup_{\nu \in \Upsilon} d_W(\Psi_m \cdot \nu, \Psi_n \cdot \nu) \leq C_m \quad \text{and therefore} \tag{151}
\]
\[
\lim_{m \to \infty} \Psi_m \cdot \nu = \frac{1}{2q + 1} \sum_{j \in I_q} \delta_{z^j([0, T])} \quad \text{for all } \nu \in \Upsilon, \tag{152}
\]
of the form in (146), with \( z^j(t) \) defined in (147).

We are now in a position to define \( \Psi \cdot \mu \) for an arbitrary \( \mu \in \mathcal{P}(\mathcal{D} \times \mathcal{D}) \). Let \( \{\nu^p\}_{p \in \mathbb{Z}} \subset \Upsilon \) be a sequence of empirical measures such that \( \nu^p \to \mu \) with respect to the topology \( \mathcal{T} \) (in Lemma 4.5 we prove that \( \Upsilon \) is dense in \( \mathcal{P}(\mathcal{D} \times \mathcal{D}) \) with respect to the topology \( \mathcal{T} \)). Thanks to (151), for any \( r \in \mathbb{Z}^+ \) we can find \( m_r \in \mathbb{Z}^+ \) such that
\[
\sup_{\nu \in \Upsilon} \sup_{n \geq m_r} d_W(\Psi_{m_r} \cdot \nu, \Psi_n \cdot \nu) \leq \frac{1}{2r}. \tag{153}
\]
Furthermore the continuity of \( \Psi_{m_r} \) implies that we can find \( p_r \in \mathbb{Z}^+ \) such that for all \( s \geq p_r \),
\[
d_W(\Psi_{m_r} \cdot \nu^s, \Psi_{m_r} \cdot \mu) \leq \frac{1}{2r}. \tag{154}
\]
We choose \( p_r \) to be the smallest possible integer such that the above identity holds. The previous two equations imply that the sequence \( \{\Psi_{m_r} \cdot \nu^p\}_{r \geq 1} \) is Cauchy, i.e.
\[
\sup_{n \geq m_r} \sup_{s \geq p_r} d_W(\Psi_{m_r} \cdot \mu, \Psi_n \cdot \nu^s) \leq r^{-1}.
\]
Thus, since \( \mathcal{P}(C([0, T], \mathbb{R}^d)) \) is complete, the sequence must have a unique limit in \( \mathcal{P}(C([0, T], \mathbb{R}^d)) \). Furthermore one easily checks that the limit is independent of the choice of the approximating sequence of empirical measures \( \{\nu^p\}_{p \in \mathbb{Z}} \). We define
\[
\Psi \cdot \mu := \lim_{r \to \infty} \Psi_{m_r} \cdot \mu \tag{155}
\]

**Lemma 4.3.** The map \( \Psi : (\mathcal{P}(\mathcal{D} \times \mathcal{D}), \mathcal{T}) \to (\mathcal{P}(C([0, T], \mathbb{R}^d)), \mathcal{T}_w) \) in (155) is continuous.

**Proof.** This follows immediately from the definition of \( \Psi \), and the continuity of \( \Psi_{m_r} \). \qed

Since \( \Psi \cdot \mu^p = \hat{\mu}_p^m \), the above lemma implies Theorem 2.1 as required.

For the rest of this section we prove Theorem 2.2. First we demonstrate that the assumption in the statement of Theorem 2.2 implies that the Large Deviation Principle holds with respect to the topology \( \mathcal{T}_w \).
Lemma 4.4. The sequence of probability laws \( \{\Pi^n\}_{n \in \mathbb{Z}^+} \) satisfy a Large Deviation Principle with respect to the topology \( T \) on \( \mathcal{P}(D \times \mathcal{P}(D)) \). That is, for open \( O, F \in \mathcal{B}(\mathcal{P}(D \times \mathcal{P}(D))) \), where \( O \) is open with respect to \( T \) and \( F \) is closed with respect to \( T \),

\[
\lim_{n \to \infty} \alpha_n^{-1} \log \Pi^n(\hat{\mu}_n \in F) \leq - \inf_{\nu \in F} I(\nu) \quad (156)
\]

\[
\lim_{n \to \infty} \alpha_n^{-1} \log \Pi^n(\hat{\mu}_n \in O) \geq - \inf_{\nu \in O} I(\nu), \quad (157)
\]

where \( \alpha_n \to \infty \) as \( n \to \infty \).

Proof. It is proved in Lemma 4.5 that \( \mathcal{P}(D \times \mathcal{P}(D)) \) is compact with respect to the topology \( T \). This means that the sequence of probability laws \( \{\Pi^n\}_{n \in \mathbb{Z}^+} \) is exponentially tight with respect to the topology \( T \). The Lemma thus follows from the Inverse Contraction Principle [19, Corollary 4.2.6] and the assumption in the statement of Theorem 2.2.

The Large Deviations result of Theorem 2.1 now follows from an application of Varadhan’s contraction principle ([19, Theorem 4.2.1]) to Lemma 4.4. We also make use of the continuity of the map \( \Psi \) in Lemma 4.3.

4.1 The \( \tilde{T} \) topology on \( \mathcal{P}(D \times \mathcal{P}(D)) \)

In this section we study the properties of the topology \( \tilde{T} \) topology on \( \mathcal{P}(D \times \mathcal{P}(D)) \). This topology is more refined than the standard weak topology on \( \mathcal{P}(D \times \mathcal{P}(D)) \).

Lemma 4.5. For all \( m \geq 0 \),

- \( T_m \) is a subtopology of \( T_{m+1} \).
- \( (\mathcal{P}(D \times \mathcal{P}(D)), T_m) \) is Hausdorff.
- \( (\mathcal{P}(D \times \mathcal{P}(D)), T_m) \) is separable.
- \( \mathcal{P}(D \times \mathcal{P}(D)) \) is compact with respect to the topology \( T_m \).
- \( \mathcal{P}(D \times \mathcal{P}(D)) \) is compact with respect to the topology \( \tilde{T} \).

Proof. The first identity is immediate from the definitions. The Hausdorff property follows from the fact that \( T_0 \) is Hausdorff (it’s well known that the standard weak topology for probability measures on a Polish space is Hausdorff and separable).

We now demonstrate the separability of \( T_m \). Define the following set of empirical measures

\[
\tilde{T} = \left\{ \hat{\mu}^q = \frac{1}{2q+1} \sum_{j \in \mathbb{Z}} \delta_{(y_q^j, \hat{\mu}_j)} \right\} \cup \hat{\mu}^q = \frac{1}{2q+1} \sum_{j \in \mathbb{Z}} \delta_{y_q^j}, \text{ for some } q \in \mathbb{Z}^+, \{y_q^j\}_{j \in \mathbb{Z}} \subset D \cap Q^d, \{\xi_j \subset I_q\}. \quad (158)
\]

Clearly \( \tilde{T} \) is countable. Thus to demonstrate the separability of \( (\mathcal{P}(D \times \mathcal{P}(D)), T_m) \), it suffices to show that

\[
\Phi_m \cdot \tilde{T} \text{ is dense in } \mathcal{V}_m, \text{ with respect to the weak topology}. \quad (159)
\]

To demonstrate the compactness of \( \mathcal{P}(D \times \mathcal{P}(D)) \) with respect to \( T_m \), suppose that for some index set \( \mathcal{I} \),

\[
\mathcal{P}(D \times \mathcal{P}(D)) = \bigcup_{i \in \mathcal{I}} \mathcal{O}_i, \quad (160)
\]

where \( \mathcal{O}_i = \Phi_m \uparrow (\mathcal{O}_i) \), and \( \mathcal{O}_i \) is open in \( \mathcal{V}_m \) with respect to the weak topology. It then follows from (159) that

\[
\mathcal{V}_m = \bigcup_{i \in \mathcal{I}} \mathcal{O}_i.
\]

It follows from Prokhorov’s Theorem [37] that \( \mathcal{V}_m \) is compact. Thus there exists a finite subset of \( \mathcal{I} \), written \( \{i_p\}_{p=1}^M \), such that

\[
\mathcal{V}_m = \bigcup_{p=1}^M \mathcal{O}_{i_p}.
\]
This means that
\[ \mathcal{P}(\mathcal{D} \times \mathcal{P}(\mathcal{D})) = \bigcup_{p=1}^{M} O_p, \]
and we have thus demonstrated the compactness of \( \mathcal{P}(\mathcal{D} \times \mathcal{P}(\mathcal{D})) \) with respect to \( \mathcal{T}_m \).

The compactness with respect to \( \tilde{\mathcal{T}} \) follows from Tychonoff’s Theorem. \( \square \)

References

[1] Julien Barré, Paul Dolson, Michela Ottobre, and Ewelina Zatorska. Fast non mean-field networks: uniform in time averaging. (1), 2020.
[2] Erhan Bayraktar, Suman Chakraborty, and Ruoyu Wu. Graphon mean field systems. pages 1–26, 2020.
[3] Shankar Bhamidi, Amarjit Budhiraja, and Ruoyu Wu. Weakly interacting particle systems on inhomogeneous random graphs. *Stochastic Processes and their Applications*, 129(6):2174–2206, 2019.
[4] Bhaswar B. Bhattacharya, Shirshendu Ganguly, Eyal Lubetzky, and Yufei Zhao. Upper tails and independence polynomials in random graphs. *Advances in Mathematics*, 319:313–347, 2017.
[5] Patrick Billingsley. *Convergence of Probability Measures*. 1999.
[6] Charles Bordenave and Pietro Caputo. Large deviations of empirical neighborhood distribution in sparse random graphs. *Probability Theory and Related Fields*, 163(1-2):149–222, 2015.
[7] Paul C. Bressloff and Matthew A. Webber. Front propagation in stochastic neural fields. *SIAM Journal on Applied Dynamical Systems*, 11(2):708–740, 2012.
[8] Amarjit Budhiraja, Paul Dupuis, and Markus Fischer. Large deviation properties of weakly interacting processes via weak convergence methods. *Annals of Probability*, 40(1):74–102, 2012.
[9] René Carmona and François Delarue. *Probabilistic theory of mean field games with applications. I*, volume 83. Springer, 2018.
[10] Sourav Chatterjee. An Introduction to Large Deviations for Random Graphs. 53(4):617–642, 2016.
[11] Sourav Chatterjee and S. R.S. Varadhan. The large deviation principle for the Erdős-Rényi random graph. *European Journal of Combinatorics*, 32(7):1000–1017, 2011.
[12] Michele Coghi, Jean-Dominique Deuschel, Peter Friz, and Mario Maurelli. Pathwise McKean-Vlasov Theory. pages 1–41, 2018.
[13] Nicholas Cook and Amir Dembo. Large Deviations of subgraph counts for Sparse Erdős-Rényi Graphs. *Advances in Mathematics*, 373:1–53, 2020.
[14] Fabio Coppini, Helge Dietert, and Giambattista Giacomin. A law of large numbers and large deviations for interacting diffusions on Erdos-Rényi graphs. *Stochastics and Dynamics*, 2019.
[15] Paolo Dai Pra and Frank den Hollander. McKean-Vlasov limit for interacting random processes in random media. *Journal of statistical physics*, 84(3), 1996.
[16] Donald Dawson and Jurgen Gartner. Large deviations from the mckeavlasov limit for weakly interacting diffusions. *Stochastics*, 20(4), 1987.
[17] Sylvain Delattre, Giambattista Giacomin, and Eric Luçon. A Note on Dynamical Models on Random Graphs and Fokker-Planck Equations. *Journal of Statistical Physics*, 165(4):785–798, 2016.
[18] Amir Dembo and Andrea Montanari. Gibbs measures and phase transitions on sparse random graphs. *Brazilian Journal of Probability and Statistics*, 24(2):137–211, 2010.
[19] Amir Dembo and Ofer Zeitouni. *Large Deviations Techniques and Applications 2nd Edition*. Springer, 1998.
[20] Monroe Donsker and SRS Varadhan. Asymptotic Evaluation of Certain Markov Process Expectations for Large Time I. *Communications on Pure and Applied Mathematics*, 28(1):1–47, 1975.
[21] Paul Dupuis and Richard Ellis. *A Weak Convergence Approach to the Theory of Large Deviations*. Wiley Series in Probability and Mathematical Statistics, 1997.
[22] Paul Dupuis and Georgi Medvedev. The large deviation principle for interacting dynamical systems on random graphs. 2020.

[23] Mark I. Freidlin and Alexander D. Wentzell. Random Perturbations of Dynamical Systems. 3rd Edition. Springer Heidelberg, 2012.

[24] Alan Frieze and Ravi Kannan. Quick approximation to matrices and applications. Combinatorica, 19(2):175–220, 1999.

[25] Chengcheng Huang and Brent Doiron. Once upon a (slow) time in the land of recurrent neuronal networks. . . . Current Opinion in Neurobiology, 46:31–38, 2017.

[26] Christian Kuehn. Network dynamics on graphops. New Journal of Physics, 22(5), 2020.

[27] Daniel Lacker, Kavita Ramanan, and Ruoyu Wu. Locally interacting diffusions as space-time Markov random fields. Arxiv Preprint, pages 1–30, 2019.

[28] Laszlo Lovasz. Large Networks and Graph Limits. 2012.

[29] Eric Luçon. Quenched asymptotics for interacting diffusions on inhomogeneous random graphs. Stochastic Processes and their Applications, pages 1–52, 2020.

[30] James Maclaurin. Large Deviations of a Network of Interacting Particles with Sparse Random Connections. arXiv preprint arXiv:1607.05471, 2018.

[31] Andrea Montanari. Optimization of the Sherrington-Kirkpatrick Hamiltonian. pages 1–27, 2019.

[32] Roberto I. Oliveira and Guilherme H. Reis. Interacting Diffusions on Random Graphs with Diverging Average Degrees: Hydrodynamics and Large Deviations. Journal of Statistical Physics, 176(5):1057–1087, 2019.

[33] Roberto I. Oliveira, Guilherme H. Reis, and Lucas M. Stolerman. Interacting diffusions on sparse graphs: hydrodynamics from local weak limits. Electronic Journal of Probability, 25(0), 2020.

[34] Rodrigo Pena, Michael A Zaks, and Antonio C Roque. Dynamics of spontaneous activity in random networks with multiple neuron subtypes and synaptic noise. Journal of Computational Neuroscience, 45:1–28, 2018.

[35] J. R. Phillips, H. S.J. Van Der Zant, J. White, and T. P. Orlando. Influence of induced magnetic fields on the static properties of Josephson-junction arrays. Physical Review B, 47(9):5219–5229, 1993.

[36] James A. Roberts, Leonardo L. Gollo, Romesh G. Abeysuriya, Gloria Roberts, Philip B. Mitchell, Mark W. Woolrich, and Michael Breakspear. Metastable brain waves. Nature Communications, 10(1):1–17, 2019.

[37] Albert N. Shiryaev. Probability-1. Third Edition. Springer, 2016.

[38] H. Sompolinsky, A. Crisanti, and H. J. Sommers. Chaos in random neural networks. Physical Review Letters, 61(3):259–262, 1988.

[39] Alain-Sol Sznitman. Topics in Propagation of Chaos. In P.L Henneguin, editor, Lecture Notes in Mathematics. Ecole d’Ete de Probabilites de Saint-Flour XIX - 1989. Springer-Verlag, 1989.

[40] H. Tanaka. Limit Theorems for Certain Diffusion Processes with Interaction. In Kiyoshi Ito, editor, Stochastic Analysis. Proceedings of the Taniguchi International Symposium on Stochastic Analysis, Katata and Kyoto, 1982, volume 53, pages 1689–1699. North-Holland, 1982.