Multi-Server Verifiable Delegation of Computations:
Unconditional Security and Practical Efficiency

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Abstract

Outsourcing computation has gained significant popularity in recent years due to the prevalence of cloud computing. There are two main security concerns in outsourcing computation: how to guarantee the cloud server performs the computation correctly and how to keep the client’s data secret. The single-server verifiable computation (SSVC) of Gennaro, Gentry and Parno (Crypto’10) enables a client to delegate the computation of a function \( f \) on any input \( x \) with both concerns highly relieved, but only results in computationally secure schemes that lack practical efficiency.

While the SSVC schemes use a single server, in this paper we develop a multi-server verifiable computation (MSVC) model where the client shares both \( f \) and \( x \) among multiple servers, each server performs a set of computations on its shares, and finally the client reconstructs \( f(x) \) from all servers’ results. In this MSVC model we propose a generic construction for outsourcing computations of the form \( Fx \), where \( F \) is a matrix and \( x \) is a vector. Our generic construction achieves information-theoretic security, input privacy and function privacy. By optimizing the parameters, we obtain both a 3-server scheme, which uses the least number of servers, and a 4-server scheme, which incurs the least workload. By decomposing many polynomial computations as a two-stage computation, where the first-stage has the form \( Fx \) and the second-stage is fast, and delegating
1. Introduction

Outsourcing of computation and data has gained significant popularity in recent years due to the prevalence of cloud computing. The computationally weak devices such as smartphones can offload the storage of large-scale data and expensive computations on the data to powerful cloud services in a pay-per-use manner, which is both scalable and economical. There are two fundamental security concerns in outsourcing: how to ensure that the cloud server performs computations correctly; and how to keep the client’s data secret.

There are many emerging solutions [9, 15, 23, 24, 28] for verifying the cloud servers’ work. Among them is the single-server verifiable computation (SSVC) of [23], which enables the client to outsource the computation of a function \( f \) on any input \( x \) to a cloud server, receive both the result \( y \) and a cryptographic proof from the server, and then verify if \( y = f(x) \). The proof is designed such that no malicious server is able to persuade the client to both accept its result and output a wrong value. Such solutions should also satisfy certain efficiency requirements: the client-side computation in outsourcing should be substantially faster than the naive computation of \( f(x) \). Minimizing the client’s computational cost has been one of the main objectives in this field. The function \( f \) should be encoded and then given to the server. Preparing the encoding may be a heavy task and the SSVC of [23] adopts an amortized model where the one-time cost of encoding \( f \) is amortized over the computations of \( f \) on many different inputs \( x \).

The problem of keeping both \( f \) and \( x \) secret has also been studied as well. The scheme of [23] attains both input privacy and function privacy by using the
very expensive machineries such as fully homomorphic encryption (FHE) and garbled circuits (GCs). As these machineries incur significant latencies in both the client-side computation and the server-side computation, the scheme of [23] is rather inefficient. Achieving input privacy in SSVC schemes is highly non-trivial. In particular, when \( f \) requires high-degree computations on \( x \), the input privacy somehow requires the client to encrypt \( x \) with a semantically secure encryption, which must allow the server to compute \( f \) on the ciphertext of \( x \). As a result, the encryption must be homomorphic. However, the theoretical achievements in homomorphic encryption [12, 25] hasn’t changed the situation that few SSVC schemes based on them are practical.

### 1.1. Our Work

In this paper we try to resolve the conflicts between security/privacy and efficiency with a multi-server verifiable computation (MSVC) model. In this model, we propose a generic construction for outsourcing matrix-vector multiplications of the form \( Fx \), where \( F \) is a matrix and \( x \) is a column vector. Our construction is information-theoretically secure such that the servers are not able to persuade the client to both accept the servers’ results and output a wrong value. Our construction is information-theoretically input private and function private such that no single server is able learn any information about \( F \) or \( x \). Our construction is free of FHE, GCs, and any other public-key operations and thus achieves practical efficiency. By optimizing the parameters, we get two instantiations of the generic construction: one uses 3 servers and the other uses 4 servers. The 3-server scheme is optimal in terms of the total number of needed servers and the 4-server scheme is optimal in terms of the total workload. We also decompose various polynomial computations into two-stage computations. By delegating the heavier first-stage computations, which are matrix-vector multiplications, we obtain polynomial outsourcing schemes that have the same properties of security, privacy and efficiency. We also show applications of our schemes in analysis of sensitive data, polynomial outsourcing, and outsourced private information retrieval (PIR) [29, 36].
Multi-server verifiable computation. In our MSVC model, the client shares its function $f$ and input $x$ among multiple servers, each server performs a set of computations on the function and input shares, and finally the client reconstructs $f(x)$ from all servers’ results. An MSVC scheme is secure if the servers cannot persuade the client to both accept their computation results and reconstruct a wrong function value $\hat{y} \neq f(x)$. In order to achieve the similar security properties, the existing multi-server schemes [1, 14] for outsourcing computations have required that the servers should not collude with each other. In contrast, our schemes will be secure even if all servers are malicious and colluding with each other. However, when it comes to function/input privacy, we require that the servers should not collude with each other; otherwise, the servers could be able to easily recover $f$ or $x$ from their joint shares. The requirement of non-colluding servers for input/function privacy can be met when the servers belong to different cloud services or have conflicts of interest. An MSVC scheme is input private if no individual server is able to distinguish between its shares of two different inputs. An MSVC scheme is function private if no individual server is able to distinguish between its shares of two different functions. In the formal definitions of input (resp. function) privacy, it suffices to require that each server’s input (resp. function) shares are statistically independent of the input (resp. function). Our security and privacy will be information-theoretic and require no number-theoretic assumptions.

Matrix outsourcing. Let $\mathbb{Z}_q$ be a finite field of $q$ elements and let $m, d$ be integers. We interpret any matrix $F \in \mathbb{Z}_q^{m \times d}$ as a function that takes any vector $x \in \mathbb{F}_q^d$ as input and outputs $Fx$. Our first contribution is a generic construction of MSVC schemes for outsourcing the computation of $Fx$. In order to protect $F$ and $x$ from the servers in a $k$-server scheme, we choose $a$ matrices $F_1, \ldots, F_a$ uniformly at random and subject to $F_1 + \cdots + F_a = F$; choose $b$ vectors $x_1, \ldots, x_b$ uniformly at random and subject to $x_1 + \cdots + x_b = x$; and distribute the chosen matrices and vectors among all servers such that: (1) each server only learns a proper subset of the matrices; (2) each server only learns a proper subset of the
vectors; (3) the servers can perform computations on their respective shares in order to generate a set of results, from which the client is able to extract $F\mathbf{x}$. While (1) and (2) allow us to attain the function privacy and input privacy easily, the method of distributing function/input shares should be properly designed such that (3) is also satisfied. Suppose that for every $\ell \in [k]$, the $\ell$-th server is given

$$\rho_\ell = \{F_u : u \in A_\ell\} \text{ and } \sigma_\ell = \{\mathbf{x}_v : v \in B_\ell\},$$

where $A_\ell \subseteq [a]$ and $B_\ell \subseteq [b]$. Then (3) will be satisfied if there exist subsets $C_1 \subseteq A_1 \times B_1, \ldots, C_k \subseteq A_k \times B_k$ such that $\bigcup_{\ell=1}^k C_\ell = [a] \times [b]$. In fact, when the equality holds, the $\ell$-th server only needs to compute and return $y_\ell = \{F_u \mathbf{x}_v : (u,v) \in C_\ell\}$ and the client will be able to reconstruct $F\mathbf{x}$ as

$$F\mathbf{x} = \sum_{\ell=1}^k \sum_{(u,v) \in C_\ell} F_u \mathbf{x}_v.$$ 

In order to make the computation of each $F_u \mathbf{x}_v$ verifiable, we need a critical observation on matrix-vector multiplication [10, 11]: if we choose a vector $\mathbf{r} \leftarrow \mathbb{Z}_q^n$ uniformly at random and set $s_u = \mathbf{r} F_u$, then the output $y_{u,v} = F_u \mathbf{x}_v$ will satisfy the equality $s_u \cdot \mathbf{x}_v = \mathbf{r} \cdot y_{u,v}$, ($s_u \cdot \mathbf{x}_v$ stands for the inner product of $s_u$ and $\mathbf{x}_v$). Without knowing $(\mathbf{r}, s_u)$, an adversary will not be able to choose a vector $\hat{y}_{u,v} \neq y_{u,v}$ such that $s_u \cdot \mathbf{x}_v = \mathbf{r} \cdot \hat{y}_{u,v}$, except with probability $1/q$. Furthermore, the verification takes $O(m + d)$ arithmetic operations modulo $q$, which are significantly faster than the $O(md)$ arithmetic operations required by the naive computation of $F_u \mathbf{x}_v$. By properly applying this verification technique to all computations, we obtain the generic construction with the expected security, input privacy, function privacy, and practical efficiency.

We optimize the generic construction in two directions: (i) minimizing the number $k$ of required servers; (ii) minimizing the workload of the client and the server. Minimizing $k$ is meaningful as that will make the assumption of non-colluding servers most practical. In order to achieve all expected properties of security and input/function privacy, we show that the smallest $k$ is 3 and then give $\Pi_s$, an instantiation of the generic construction with 3 servers. In our generic construction, the client’s workload is dominated by $O(ab(m + d))$ arithmetic operations modulo $q$; the servers’ total workload is dominated by $O(abmd)$ arithmetic operations modulo $q$. By appropriately choosing the parameters we show that when $k = 4$, $ab$ can be minimized. We then give $\Pi_w$, an
instantiation of the generic construction with 4 servers.

**Polynomial outsourcing.** In the literature of SSVC, achieving input privacy has been highly non-trivial, especially when the function $f$ requires high-degree computations on the input $x$. On one hand, in order to keep $x$ private, the client somehow has to encrypt $x$ as $\text{Enc}(x)$ with a semantically secure encryption $\text{Enc}$. On the other hand, the server has to compute $f$ on $\text{Enc}(x)$, but without decrypting $\text{Enc}(x)$. The latter situation requires that $\text{Enc}$ should be homomorphic. As such an $\text{Enc}$ would require heavy public-key operations, the SSVC schemes are hardly practical. In this paper, we try to tackle this problem with our generic construction. We observe that the evaluation of many polynomials, including univariate polynomials, bivariate polynomials, quadratic multivariate polynomials, and multivariate polynomials that have bounded degree in each variable, can be decomposed into a two-stage computation, where the first stage is a heavy matrix-vector multiplication and the second stage is a light inner product computation. By delegating the first-stage computation with our MSVC schemes (say $\Pi_s$ and $\Pi_w$) for matrices and performing the second-stage computation on its own, the client is able to offload most workload to the cloud and achieve all of the expected properties of information-theoretic security and input/function privacy, and practical efficiency.

**Performance.** We implemented both $\Pi_s$ and $\Pi_w$ on a DELL Precision Tower T7810 that runs with an Intel Xeon E5-2650 Processor (2.30 GHz) and a RAM of 128GB. We set $q$ to be a 256-bit prime and consider the multiplication of a random matrix $F$ with a random vector $x$. For $m = d = 3000$, our experiments show that the client-side computations in $\Pi_s$ and $\Pi_w$ require around 177ms and 90ms, respectively. On the other hand, the naive computation of $Fx$ require around 2600ms. The client-side computations in our MSVC schemes are significantly faster than the naive computation. Our MSVC schemes outperform many existing SSVC schemes, which have been considered as practical. For example, for $m = d = 100$, the client in Pinocchio [40] requires 10ms, which is worse than the 5.73ms and 3.31ms used by $\Pi_s$ and $\Pi_w$, respectively. These
experiments show that our schemes are among the most practical schemes for outsourcing computation.

1.2. Related Work

The study of efficient verification of arbitrary computations dates back to the work on interactive proofs [3, 27], probabilistically checkable proofs (PCPs) [31, 32], computationally sound (CS) proofs [37] and the muggle proofs [26, 42]. These constructions yield single-server schemes for securely outsourcing computations, which are either interactive or require random oracles, and achieve no input/function privacy.

The single-server schemes that are both non-interactive and free of random oracles have been extensively studied in the past decade. Among them is the single-server verifiable computation (SSVC) of [23, 2, 4, 19], which enables a client to delegate the computation of any boolean circuit \( f \). These schemes keep the input \( x \) semantically secure from an untrusted cloud server but heavily depend on FHE and GCs. They are rather impractical. There is a long list of SSVC schemes [8, 16, 18, 20, 21, 26, 39, 41] that focus on the delegation of specific functions such as polynomials and matrices. These schemes are free of FHE and GCs but still require the server/client to perform heavy public-key operations such as modular exponentiations and paring computations, which are not as practical as the arithmetic operations in our schemes. Furthermore, they don’t achieve any input privacy or function privacy. In the SSVC schemes of [22, 30, 34], the input \( x \) is encrypted and stored on cloud servers and the function \( f \) is provided by the client. These schemes only achieve one-side privacy, such as the input privacy, or function privacy, but not both. The client and the server in these schemes must perform a large number of expensive public-key operations as well.

Verifiable computation schemes in different multi-server models such as [1, 14] have been studied as well. In the model of Ananth et. al. [1], the client and all servers form a directed cycle and each player has to receive a message from its predecessor and send a message to its successor, i.e., the communications
between servers are required. The computationally secure schemes of [1] are input private and FHE-free. However, they still heavily depend on GCs and so are not as practical as the schemes of this paper. In the model of Canetti et al. [14], the servers do not need to communicate with each other and the client is a referee that determines which server is honest. The schemes of [14] are neither input private nor function private. The security of [1, 14] requires that all servers do not collude with each other and at least one of the servers is honest. In contrast, when considering security, we neither need non-colluding servers nor require at least one honest server. “Non-colluding” is not needed in our MSVC unless the input/function privacy is considered. Ben-Or et al. proposed an MIP model [7], where the security is guaranteed only as long as no two servers collude and the malicious servers can potentially prevent honest ones from convincing the client. In that model, and assuming collisions resistant hash functions, Canetti et al. [13] constructed a computationally secure delegation protocol whose number of rounds is logarithmic in the time needed to compute \( f(x) \).

1.3. Organization

Our MSVC model is defined in Section 2; in Section 3, we propose the generic construction of MSVC schemes for matrix outsourcing; we also obtain a scheme with the smallest number of servers and a scheme with least workload; we implement both schemes and give performance analysis; in Section 4 we obtain the new schemes for polynomial outsourcing and outsourced multi-server PIR schemes where the servers’ work is verifiable. Section 6 contains our concluding remarks.

2. Model and Definitions

For a set \( S \), we denote with “\( x \leftarrow S \)” the process of choosing \( x \) uniformly at random from \( S \). For a probabilistic algorithm \( A \), we denote with “\( x \leftarrow A(\cdot) \)” the process of running \( A \) on some appropriate input and assigning its output
to $x$. For an integer $n > 0$, we denote with $\{n\}$ the set $\{1, \ldots, n\}$. We use uppercase letters for matrices (e.g. $F$), and lowercase bold letters for vectors (e.g. $x$). We denote with $F^\top$ the transpose of a matrix $F$. We denote with $F[i,j]$ the $(i,j)$-entry of the matrix $F$ and denote with $x[i]$ the $i$-th entry of $x$.

Given two vectors $u = (u_1, \ldots, u_m), v = (v_1, \ldots, v_m)$, we denote with $u \cdot v$ the dot product of $u, v$, i.e., $u \cdot v = \sum_{i=1}^{m} u_i v_i$.

In this paper, we work in a new multi-server verifiable computation (MSVC) model. Informally, a $k$-server verifiable computation scheme is a protocol between a client and $k$ servers. The client provides both $k$ shares of the function $F$ and $k$ shares of the input $x$ to all servers. Each server is expected to perform a set of computations on its shares, and produce an output, such that the $k$ outputs together enable the client to reconstruct $F(x)$. Furthermore, the servers’ results can be verified in order to guarantee correct reconstruction. The goal of MSVC is to make the client’s work as efficient as possible, and in particular much faster than the naive computation of $F(x)$. Let $F$ be a function family. A $k$-server verifiable computation scheme $\Pi = (\text{KeyGen}, \text{ProbGen}, \text{Compute}, \text{Verify})$ for $F$ consists of the following algorithms:

- **KeyGen($\lambda, F$):** This is a randomized key generation algorithm. It takes a security parameter $\lambda$ and a function $F$ as input and produces a value $PK_F$, which will be used by a client to prepare its input, $k$ function shares $\rho_1, \ldots, \rho_k$, which will be used by the servers to perform their respective computations, a value $VK_F$, which will be used by the client to perform verification and reconstruct the function’s output.

- **ProbGen($PK_F, x$):** This is a problem generation algorithm. It takes $PK_F$ and $x \in \text{Domain}(F)$ as input and produces $k$ input shares $\sigma_1, \ldots, \sigma_k$, which will be given to the servers to compute with, and a value $VK_x$, which will be used for verification.

- **Compute($\ell, \rho_\ell, \sigma_\ell$):** This is the server-side algorithm. For every $\ell \in [k]$, it performs a set of computations on the function share $\rho_\ell$ and the input share $x_\ell$, and produces an output $y_\ell$. 
Verify($VK_F, VK_x, \{y_\ell\}_{\ell=1}^k$): This is the verification algorithm. It uses the keys $VK_F, VK_x$ to determine if $\{y_\ell\}_{\ell=1}^k$ form a valid encoding of $F(x)$. If $\{y_\ell\}_{\ell=1}^k$ is valid, this algorithm converts the servers’ results to $y = F(x)$ and outputs $y$; otherwise, this algorithm outputs $\bot$ (indicating that some servers are cheating).

Our MSVC model will consist of two phases: a preprocessing phase and a computing phase. In the preprocessing phase, the client takes a function $F$ as input, runs the key generation algorithm to produce the values $PK_F, \{\rho_\ell\}_{\ell=1}^k, VK_F$, and then sends $\rho_\ell$ to the $\ell$-th server. In the computing phase, the client takes $PK_F$ and a value $x \in \text{Domain}(F)$ as input, runs the problem generation algorithm to produce the values $\{\sigma_\ell\}_{\ell=1}^k, VK_x$, and then sends $\sigma_\ell$ to the $\ell$-th server. The $\ell$-th server runs the server-side algorithm to produce an output $y_\ell$. Then the client uses $VK_F, VK_x$ to verify if $\{y_\ell\}_{\ell=1}^k$ is a valid encoding of $F(x)$, and depending on the verification result either outputs $F(x)$ (when valid) or $\bot$ (when invalid).

The scheme $\Pi$ is said to be publicly delegatable if $PK_F$ is public such that any client, without executing KeyGen, is able to run ProbGen to prepare its input. Otherwise, the scheme is said to be privately delegatable. The scheme $\Pi$ is said to be publicly verifiable if $VK_F$ and $VK_x$ are public such that any client, without executing KeyGen or ProbGen, is able to run Verify to determine whether $\{y_\ell\}_{\ell=1}^k$ is a valid encoding of $F(x)$. Otherwise, the scheme is said to be privately verifiable. In this paper, we shall construct publicly delegatable and privately verifiable schemes.

The scheme $\Pi$ is said to be correct if KeyGen and ProbGen produce values that always enable the honest servers to compute values that will verify successfully and be converted into the correct value of $F(x)$.

**Definition 1. (Correctness)** The scheme $\Pi$ is correct if for any $F \in \mathcal{F}$, any $(PK_F, \{\rho_\ell\}_{\ell=1}^k, VK_F) \leftarrow \text{KeyGen}(\lambda, F)$, any $x \in \text{Domain}(F)$, any $(\{\sigma_\ell\}_{\ell=1}^k, VK_x) \leftarrow \text{ProbGen}(PK_F, x)$ and any $\{y_\ell \leftarrow \text{Compute}(\ell, \rho_\ell, \sigma_\ell)\}_{\ell=1}^k$, it holds that Verify($VK_F, VK_x, \{y_\ell\}_{\ell=1}^k$) = $F(x)$. 

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\( (PK_F, \{\rho_t\}_{t=1}^k, VK_F) \leftarrow \text{KeyGen}(\lambda, F); \)

for \( h = 1 \) to \( p \) do (remark: \( p \) is the number of attempts that \( A \) can make)

\[
\begin{align*}
  x^{(h)} &\leftarrow A(PK_F, \{\rho_t\}_{t=1}^k, \{\sigma_t^{(t)}\}_{t=1}^k, \{\hat{y}_t^{(t)}\}_{t=1}^k, b_t)_{t=1}^{h-1}; \\
  (\{\sigma_t^{(h)}\}_{t=1}^k, VK_{x^{(h)}}) &\leftarrow \text{ProbGen}(PK_F, x^{(h)}); \\
  \{y_t^{(h)}\} &\leftarrow A(PK_F, \{\rho_t\}_{t=1}^k, \{\sigma_t^{(t)}\}_{t=1}^k, \{\hat{y}_t^{(t)}\}_{t=1}^k, b_t, \{\sigma_t^{(h)}\}_{t=1}^k)_{t=1}^{h-1}; \\
  \hat{y}^{(h)} &\leftarrow \text{Verify}(VK_F, VK_{x^{(h)}}, \{y_t^{(h)}\}_{t=1}^k) \\
  \text{if } \hat{y}^{(h)} = \perp, \text{ set } b_h = 0; \text{ otherwise, set } b_h = 1;
\end{align*}
\]

if there is an \( h \in [p] \) such that \( \hat{y}^{(h)} \notin \{F(x^{(h)}), \perp\} \), then output 1; otherwise, output 0.

Fig. 1: Experiment \( \text{Exp}_{A,\Pi}^{\text{verif}}(F, p) \)

In our MSVC model, the scheme \( \Pi \) is considered as secure if no malicious servers can persuade the verification algorithm to output a result \( \hat{y} \notin \{F(x), \perp\} \).

This intuition can be formalized with an experiment \( \text{Exp}_{A,\Pi}^{\text{verif}}(F, p) \). In this experiment (Fig. 1), the challenger firstly runs the key generation algorithm to produce the necessary keys \( PK_F, \{\rho_t\}_{t=1}^k, VK_F \) to initialize the scheme. The adversary \( A \) then makes \( p \) attempts to choose a function input, learn the encoding of that input, craft the servers’ results for that input, and then see if these crafted results will be able to cause the verification algorithm to output a wrong function value. The adversary succeeds if its crafted results ever cause \( \text{Verify} \) to output a wrong value. For the scheme \( \Pi \) to be secure, \( A \) is allowed to succeed only with a very small probability \( \epsilon \).

**Definition 2. (Security)** The scheme \( \Pi \) is \((p, \epsilon)\)-secure if for all \( F \in \mathcal{F} \), for any adversary \( A \), it holds that \( \Pr[\text{Exp}_{A,\Pi}^{\text{verif}}(F, p) = 1] \leq \epsilon \), where the probability is taken over the randomness used by \( A \) and the experiment.

In our definition of security we do not limit the computational power of the adversary \( A \). Consequently, our security will be information-theoretic. However, we do upper bound the number of attempts that can be made by \( A \). In
particular, the success probability $\epsilon$ will be bounded by a function of $p$. In order to compare with the computationally secure SSVC schemes, we usually require that $\epsilon$ should be negligible in the statistical security parameter $\lambda$, as long as $p$ is a polynomial function of $\lambda$.

Intuitively, the scheme $\Pi$ is said to be input private if each individual server learns absolutely no information about the client’s input. This property will be captured by the requirement that each individual server should receive an input share that is statistically independent of the client’s input.

**Definition 3. (Input privacy)** The scheme $\Pi$ is input private if for any $F \in \mathcal{F}$, any $x^{(0)}, x^{(1)} \in \text{Domain}(F)$, any $\ell^* \in [k]$, any $\{(\sigma^{(b)}_{\ell})_{\ell=1}^k, VK_{x^{(b)}}\} \leftarrow \text{ProbGen}(PK_F, x^{(b)}), \sigma^{(0)}_{\ell^*}$, and $\sigma^{(1)}_{\ell^*}$ are identically distributed.

Intuitively, the scheme $\Pi$ is said to be function private if each individual server learns absolutely no information about the client’s function. This property will be captured by the requirement that each individual server should receive a function share that is statistically independent of the client’s function.

**Definition 4. (Function privacy)** The scheme $\Pi$ is function private if for any $F^0, F^1 \in \mathcal{F}$, for any $\ell^* \in [k]$, and any $\{(PK_{F^{(b)}}, \rho^{(b)}_{\ell})_{\ell=1}^k, VK_{F^{(b)}}\} \leftarrow \text{KeyGen}(\lambda, F^{(b)}), (PK_{F^{(0)}}, \rho^{(0)}_{\ell^*})$ and $(PK_{F^{(1)}}, \rho^{(1)}_{\ell^*})$ are identically distributed.

The scheme $\Pi$ is said to be outsourceable if the client’s work in the computing phase is substantially faster than the naive computation of the function.

**Definition 5. (Outsourceable)** The scheme $\Pi$ is outsourceable if it permits efficient problem generation and result verification. That is, for any $F \in \mathcal{F}$ and any $x \in \text{Domain}(F)$, the total time $T_e$ required for $\text{ProbGen}(PK_F, x)$ and $\text{Verify}(VK_F, VK_x, \{ye\}_{\ell=1}^k)$ is $o(T_n)$, where $T_n$ is the time required by the naive computation of $F(x)$.

As in the existing SSVC protocols [8, 23], the client’s work in the preprocessing phase may be as expensive as the naive computation of the function. However, executing $\text{KeyGen}$ is a one-time computation that can be amortized
over the computation of $F$ on many different inputs, and thus acceptable. In other words, we will also work in the *amortized* model of [8, 23].

3. Multi-Server Schemes for Matrix Outsourcing

In this section, we propose an MSVC scheme for outsourcing the matrix-vector multiplications of the form $Fx$. The scheme provides public delegation and private verification; it is information-theoretically secure, input private and function private. The function family supported by our scheme is $\mathcal{F} = \mathbb{Z}_q^{m \times d}$, the set of all $m \times d$ matrices over a finite field $\mathbb{Z}_q$. Each matrix $F \in \mathcal{F}$ is interpreted as a function that takes a vector $x \in \mathbb{Z}_q^d$ as input and outputs $y = Fx \in \mathbb{Z}_q^m$. In a nutshell, the efficient verification in our scheme is based on the following technical lemma:

**Lemma 1.** Let $\hat{y}, y \in \mathbb{Z}_q^m$ be distinct vectors. Then $\Pr_{r \leftarrow \mathbb{Z}_q^m} [r \cdot \hat{y} = r \cdot y] \leq \frac{1}{q}$.

To the best of our knowledge, Lemma 1 dates back to [10, 11] and has been used to construct SSVC schemes [38]. Here are the critical observations:

- If one chooses $r \leftarrow \mathbb{Z}_q^m$ and keeps both $r$ and $s = rF$ secret, then for any $x \in \mathbb{Z}_q^d$, the function value $y = Fx$ will satisfy the following equation due to the associative law of matrix multiplications:
  \[ s \cdot x = r \cdot y. \]  

- In a protocol where the server computes $y = Fx$, the client can simply confirm the correctness of $y$ after seeing that it satisfies (1); and a server responding with $\hat{y} \neq Fx$ will pass the verification of (1) with probability at most $1/q$, because $\hat{y}$ verifies if and only if $r \cdot \hat{y} = r \cdot y$.

- The verification of (1) can be done with $O(m + d)$ arithmetic operations; it is significantly faster than the naive computation of $Fx$, which requires $O(md)$ arithmetic operations.
These observations give us an inspiring SSVC scheme: in the preprocessing phase, the client sends the matrix $F$ to the server and keeps a private key $(r, s)$ for verification; in the computing phase, client simply sends $x$ to the server; the server returns $y = Fx$; and finally the client checks (1). However, this scheme is neither input private nor function private.

### 3.1. Generic Construction

We try to achieve both the input privacy and function privacy in the MSVC model. The basic idea is secret-sharing both the function $F$ and the input $x$ among multiple servers such that each server learns absolutely no information about $F$ or $x$, but the servers are still able to perform certain computations on their shares and the computation results together enable the reconstruction of $Fx$. We shall propose a generic construction and then instantiate it with various parameters to attain the best efficiency.

In our generic construction, it suffices to use an additive secret sharing [5] where the secret $\alpha$ is an element of a ring $R$, the shares $\alpha_1, \ldots, \alpha_n \leftarrow R$ are uniformly chosen subject to $\alpha_1 + \cdots + \alpha_n = \alpha$, and each server is given a proper subset of the shares $\{\alpha_1, \ldots, \alpha_n\}$ such that any single server learns absolutely no information about $\alpha$ and a subset of the servers together are able to reconstruct $\alpha$ if and only if their shares cover all of $\alpha_1, \ldots, \alpha_n$.

Let $a, b, k > 1$ be integers and let $A = [a]$ and $B = [b]$. In a $k$-server scheme, we shall decompose the function as $F = F_1 + \cdots + F_a$, decompose the function input as $x = x_1 + \cdots + x_b$, and distribute the additive shares $\{F_u : u \in A\}$ and $\{x_v : v \in B\}$ among $k$ servers $S_1, \ldots, S_k$. For every $\ell \in [k]$, let

$$A_\ell = \{u \in A : F_u \text{ is given to } S_\ell\};$$

$$B_\ell = \{v \in B : x_v \text{ is given to } S_\ell\}. \quad (2)$$

Then each server $S_\ell$ is able to compute $F_u x_v$ for all $(u, v) \in A_\ell \times B_\ell$. Because

$$Fx = \sum_{u=1}^{a} F_u \cdot \sum_{v=1}^{b} x_v = \sum_{(u,v) \in A \times B} F_u x_v, \quad (3)$$
the $k$ servers’ results, i.e., $\{F_{ux} : (u,v) \in \bigcup_{\ell=1}^k (A_\ell \times B_\ell)\}$, suffice to reconstruct $Fx$ if and only if

\[ (A \times B) \subseteq \bigcup_{\ell=1}^k (A_\ell \times B_\ell), \] (4)

i.e., $\{A_\ell \times B_\ell\}_{\ell=1}^k$ form a cover of $A \times B$. For every $\ell \in [k]$, the function $F$ is secret from $S_\ell$ if and only if

\[ A_\ell \neq A. \] (5)

For every $\ell \in [k]$, the input $x$ is secret from $S_\ell$ if and only if

\[ B_\ell \neq B. \] (6)

When $\{A_\ell \times B_\ell\}_{\ell=1}^k$ form a cover of $A \times B$, there must exist $k$ sets $C_1, \ldots, C_k$ such that

\[ C_\ell \subseteq A_\ell \times B_\ell \text{ for every } \ell \in [k]; \]
\[ C_\ell \cap C_{\ell'} = \emptyset \text{ for all } \ell \neq \ell'; \text{ and} \]
\[ C_1 \cup \cdots \cup C_k = A \times B, \] (7)

i.e., $\{C_1, \ldots, C_k\}$ form a partition of $A \times B$. Based on (4), (5), (6) and (7), our generic construction works as follows: in the preprocessing phase, the client sends $\{F_{u} : u \in A_\ell\}$ to $S_\ell$ for every $\ell \in [k]$; in the computing phase, the client sends $\{x_\ell : \ell \in B_\ell\}$ to $S_\ell$ for every $\ell \in [k]$; the server $S_\ell$ returns $\{F_{ux} : (u,v) \in C_\ell\}$. The verification will be done with Lemma 1.

Let $k, a, b, A, B, \{A_\ell\}_{\ell=1}^k, \{B_\ell\}_{\ell=1}^k$ and $\{C_\ell\}_{\ell=1}^k$ be the parameters and sets that satisfy (4), (5), (6) and (7). Our generic construction $\Pi = (\text{KeyGen}, \text{ProbGen}, \text{Compute}, \text{Verify})$ of a $k$-server verifiable computation scheme for matrix outsourcing can be specified as below.

- **KeyGen($\lambda, F$):** This algorithm takes the security parameter $\lambda$ and a matrix $F \in \mathbb{Z}_q^{m \times d}$ as input. It chooses $F_1, \ldots, F_a \leftarrow \mathbb{Z}_q^{m \times d}$ uniformly subject to $F_1 + \cdots + F_a = F$; chooses $r \leftarrow \mathbb{Z}_q^m$, computes $s_u = rF_u$ for every $u \in A$, defines $\rho_\ell = \{F_u : u \in A_\ell\}$ for every $\ell \in [k]$, and finally outputs

  $PK_F = \bot, \{\rho_\ell\}_{\ell=1}^k$, and $VK_F = (r, \{s_u\}_{u=1}^a)$. 

\[ 15 \]
• ProbGen($PK_F, x$): This algorithm takes $PK_F = \perp$ and a vector $x \in \mathbb{Z}_q^d$ as input. It chooses $x_1, \ldots, x_b \leftarrow \mathbb{Z}_q^d$ uniformly subject to $x_1 + \cdots + x_b = x$, defines $\sigma_\ell = \{x_v : v \in B_\ell\}$ for every $\ell \in [k]$, and finally outputs $\{\sigma_\ell\}_{\ell=1}^k$, and $VK_x = (x_1, \ldots, x_b)$.

• Compute($\ell, \rho_\ell, \sigma_\ell$): For every $\ell \in [k]$, this algorithm takes a set $\rho_\ell = \{F_u : u \in A_\ell\}$ of function shares and a set $\sigma_\ell = \{x_v : v \in B_\ell\}$ of input shares as input. It computes $y_{u,v} = F_u x_v$ for all $(u, v) \in C_\ell$ and outputs $y_\ell = \{y_{u,v} : (u, v) \in C_\ell\}$.

• Verify($VK_F, VK_x, \{y_\ell\}_{\ell=1}^k$): This algorithm takes $VK_F = (r, \{s_u\}_{u=1}^a)$, $VK_x = (x_1, \ldots, x_b)$ and the $k$ servers’ computation results $\{y_\ell\}_{\ell=1}^k$ as input. For every $\ell \in [k]$ and $(u, v) \in C_\ell$, it checks the equality

$$r \cdot y_{u,v} = s_u \cdot x_v. \tag{8}$$

If (8) always holds, the algorithm outputs $y = \sum_{\ell=1}^k \sum_{(u,v) \in C_\ell} y_{u,v}$; otherwise, it outputs $\perp$.

**Correctness.** The correctness of $\Pi$ requires that for any function $F \in \mathbb{Z}_q^{m \times d}$, any input $x \in \mathbb{Z}_q^d$, any $(PK_F, \{y_\ell\}_{\ell=1}^k) \leftarrow \text{KeyGen}(\lambda, F)$, any $(\{\sigma_\ell\}_{\ell=1}^k, VK_x) \leftarrow \text{ProbGen}(PK_F, x)$, if $y_\ell$ is output by Compute($\ell, \rho_\ell, \sigma_\ell$) for all $\ell \in [k]$, then Verify($VK_F, VK_x, \{y_\ell\}_{\ell=1}^k$) will always output $FX$. When the server-side algorithm Compute($\ell, \rho_\ell, \sigma_\ell$) is honestly executed for all $\ell \in [k]$, for every $(u, v) \in C_\ell$ and $i \in [m]$, we must have that $y_{u,v}[i] = \sum_{j=1}^d F_u[i,j] \cdot x_v[j]$. It follows that

$$r \cdot y_{u,v} = \sum_{i=1}^m r[i] \cdot y_{u,v}[i] = \sum_{i=1}^m r[i] \sum_{j=1}^d F_u[i,j] \cdot x_v[j]$$

$$= \sum_{j=1}^d x_v[j] \sum_{i=1}^m r[i] \cdot F_u[i,j] = x_v \cdot s_u, \tag{9}$$

which is exactly the equality (8). It follows that the verification algorithm will output

$$\sum_{\ell=1}^k \sum_{(u,v) \in C_\ell} y_{u,v} = \sum_{(u,v) \in A \times B} F_u \cdot x_v = \sum_{u=1}^a F_u \cdot \sum_{v=1}^b x_v = Fx.$$
Input privacy. In the generic construction, each server $S_\ell$ is given a set $\sigma_\ell = \{x_v : v \in B_\ell\}$ of input shares. As $B_\ell$ is a proper subset of $B = [b]$ and all shares of $x$ are chosen uniformly subject to $x_1 + \cdots + x_b = x$, $\sigma_\ell$ must be truly random and independent of $x$. Any single server will learn absolutely no information about $x$, even if it has unlimited computing power. Hence, $\Pi$ achieves information-theoretic input privacy (as defined in Definition 3).

Function privacy. In the generic construction, each server $S_\ell$ is given a set $\rho_\ell = \{F_u : u \in A_\ell\}$ of function shares. As $A_\ell$ is a proper subset of $A = [a]$ and all shares of $F$ are chosen uniformly subject to $F_1 + \cdots + F_a = F$, $\rho_\ell$ must be truly random and independent of $F$. Any single server will learn absolutely no information about $F$, even if it has unlimited computing power. Hence, $\Pi$ achieves information-theoretic function privacy (as defined in Definition 4).

Security. The security of an MSVC scheme requires that no adversary is able to persuade the verification algorithm to both accept the dishonest servers’ results and output a wrong value, except with a very small probability. In our generic construction the client requires each server to compute a set of matrix-vector multiplications and the verification of each matrix-vector multiplication is done with the inspiring SSVC scheme. As a result, the security will follow from Lemma 1.

\textbf{Theorem 1.} The generic construction $\Pi$ is $(p, pab/(q - pab))$-secure. That is,

$$\Pr[\text{Exp}_{\text{verif}}(\mathcal{A},\Pi)(F, p) = 1] \leq \frac{pab}{q - pab}.$$ 

\textbf{Proof:} Let $F \in \mathbb{Z}_q^{m \times d}$ be any admissible function. Let $\mathcal{A}$ be any adversary that makes at most $p$ attempts in the security experiment. We show that $\Pr[\text{Exp}_{\text{verif}}^{\Pi}(F, p) = 1] \leq \frac{pab}{q - pab}$. By Definition 2, the experiment $\text{Exp}_{\text{verif}}^{\Pi}(F, p)$ will be done between $\mathcal{A}$ and the challenger as follows:

- Given $F$, the challenger runs the key generation algorithm $\text{KeyGen}(\lambda, F)$ and then invokes $\mathcal{A}$ as below:
Choose $F_1, \ldots, F_a \leftarrow \mathbb{Z}_q^{m \times d}$ uniformly at random subject to $F_1 + \cdots + F_a = F$; choose $r \leftarrow \mathbb{Z}_q^m$ and set $s_u = rF_u$ for every $u \in A$; set $\rho = \{F_u : u \in A_k\}$ for every $\ell \in [k]$;

- Invoke $A$ with $PK_F = \perp$ and $\{\rho\}_{\ell=1}^k$; keep $VK_F = (r, \{s_u\}_{u=1}^n)$ secret.

- for $h = 1$ to $p$ do (remark: $p$ is the total number of attempts that will be made by the adversary $A$)

  - Based on the current view $(PK_F, \{\rho\}_{\ell=1}^k, \{\{\sigma^{(i)}_{\ell}\}_{\ell=1}^h, \{\hat{y}^{(i)}_{\ell}\}_{\ell=1}^h, \{b_t\}_{t=1}^{h-1}\})$, $A$ chooses an input $x^{(h)} \in \mathbb{Z}_q^d$ and gives it to the challenger;

  - The challenger runs $\text{ProbGen}(PK_F, x^{(h)})$ as follows: choose $x_1^{(h)}, \ldots, x_b^{(h)} \leftarrow \mathbb{Z}_p^d$ uniformly at random and subject to $x_1^{(h)} + \cdots + x_b^{(h)} = x^{(h)}$;

    define $\sigma^{(i)}_{\ell} = \{x_v^{(h)} : v \in B_t\}$ for every $\ell \in [k]$; define $VK_{x^{(h)}} = (x_1^{(h)}, \ldots, x_b^{(h)})$. Finally, it gives $\{\sigma^{(i)}_{\ell}\}_{\ell=1}^k$ to $A$.

  - Based on the current view $(PK_F, \{\rho\}_{\ell=1}^k, \{\{\sigma^{(i)}_{\ell}\}_{\ell=1}^h, \{\hat{y}^{(i)}_{\ell}\}_{\ell=1}^h, \{b_t\}_{t=1}^{h-1}\}, \{\sigma^{(i)}_{\ell}\}_{\ell=1}^k)$, $A$ crafts a set $\{\hat{y}^{(i)}_{\ell}\}_{\ell=1}^k$ of server results and gives them to the challenger, where $\hat{y}^{(i)}_{\ell} = \hat{y}^{(i)}_{u,v} : (u, v) \in C_t$ for every $\ell \in [k]$;

  - The challenger runs $\text{Verify}(VK_F, VK_{x^{(h)}}, \{\hat{y}^{(i)}_{\ell}\}_{\ell=1}^k)$ to compute a value $\hat{y}^{(i)}$; if $\hat{y}^{(i)} = \perp$, it sets $b_h = 0$; otherwise, it sets $b_h = 1$.

- If there is an $h \in [p]$ such that $\hat{y}^{(i)} \notin \{Fx^{(h)}, \perp\}$, then output 1; otherwise, output 0.

For every $h \in [p]$, let $E_h$ be the event that $\hat{y}^{(i)} \notin \{Fx^{(h)}, \perp\}$. Then (i) $E_h$ occurs if and only if $b_h = 1$ and $\hat{y}^{(i)} \neq Fx^{(h)}$; and (ii) The event $\text{Exp}_{A,\Pi}^\text{verif}(F, p) = 1$ occurs if and only if $\bigcup_{h=1}^p E_h$ occurs. It suffices to show that $\Pr\left[\bigcup_{h=1}^p E_h\right] \leq pab/(q - pab)$.

For every $h \in [p]$, the event $b_h = 1$ occurs if and only if for all $\ell \in [k]$ and $(u, v) \in C_\ell$, the equality

$$r \cdot \hat{y}^{(i)}_{u,v} = s_u \cdot x^{(h)}_v$$

(10)
is true. On the other hand, for every \( h \in [p] \) and \( \ell \in [k] \), let \( y^{(h)}_\ell = \{ y^{(h)}_{u,v} : (u,v) \in C_\ell \} \) be the servers’ results generated by executing \textbf{Compute} faithfully.

The correctness of \( \Pi \) implies that for all \( \ell \in [k] \) and \( (u,v) \in C_\ell \),

\[
\mathbf{r} \cdot \mathbf{y}^{(h)} = \mathbf{s}_{u,v} \cdot \mathbf{x}^{(h)}_{u,v}
\]  

(11)

must be true. Due to (10) and (11), the event \( b_h = 1 \) occurs if and only if \( \mathbf{r} \cdot \mathbf{y}^{(h)}_{u,v} = \mathbf{r} \cdot \mathbf{y}^{(h)}_{u,v} \) is true for all \( \ell \in [k] \) and \( (u,v) \in C_\ell \). Equivalently, the event \( b_h = 1 \) occurs if and only if \( S_h := \{ \mathbf{y}^{(h)}_{u,v} - \mathbf{y}^{(h)}_{u,v} : \ell \in [k], (u,v) \in C_\ell \} \) is a set of solution vectors of the following linear equation system

\[
\mathbf{r} \cdot \mathbf{y} = 0,
\]  

(12)

where \( \mathbf{r} \) is the coefficient matrix and \( \mathbf{y} \) is the vector of unknowns.

Based on the specifications and the correctness of \( \Pi \), for every \( h \in [p] \), we have that

\[
\mathbf{y}^{(h)} = \sum_{\ell=1}^{k} \sum_{(u,v) \in C_\ell} \mathbf{y}^{(h)}_{u,v}, \quad \text{and} \quad \sum_{\ell=1}^{k} \sum_{(u,v) \in C_\ell} \mathbf{y}^{(h)}_{u,v} = F \mathbf{x}^{(h)}.
\]

Then for every \( h \in [p] \), the event \( \mathbf{y}^{(h)} \neq F \mathbf{x}^{(h)} \) occurs only if there is at least one \( \ell \in [k] \) and at least one \( (u,v) \in C_\ell \) such that \( \mathbf{y}^{(h)}_{u,v} \neq \mathbf{y}^{(h)}_{u,v} \).

In order to understand the events \( b_h = 1 \) and \( \mathbf{y}^{(h)} \neq F \mathbf{x}^{(h)} \), for every \( h \in [p] \) we define three subsets

\[
X_h = \{ (u,v) \in A \times B : \mathbf{y}^{(h)}_{u,v} - \mathbf{y}^{(h)}_{u,v} \text{ is not a solution of (12)} \};
\]

\[
Y_h = \{ (u,v) \in A \times B : \mathbf{y}^{(h)}_{u,v} - \mathbf{y}^{(h)}_{u,v} \text{ is a nonzero solution of (12)} \};
\]

\[
Z_h = \{ (u,v) \in A \times B : \mathbf{y}^{(h)}_{u,v} - \mathbf{y}^{(h)}_{u,v} \text{ is a zero solution of (12)} \},
\]

which form a partition of \( A \times B \). For every \( h \in [p] \), the above analysis shows that:

(i) The event \( b_h = 1 \) occurs if and only if \( X_h = \emptyset \); (ii) The event \( \mathbf{y}^{(h)} \neq F \mathbf{x}^{(h)} \) occurs only if \( Y_h \neq \emptyset \). It follows that for every \( h \in [p] \), the event \( E_h \) occurs only if \( X_h = \emptyset \) and \( Y_h \neq \emptyset \). For every \( h \in [p] \), let \( F_h \) be the event that \( h = \min\{ t \in [p] : Y_t \neq \emptyset \} \). It is easy to see that \( (\cup_{h=1}^{p} E_h) \subseteq (\cup_{h=1}^{p} F_h) \) and thus

\[
\Pr[\cup_{h=1}^{p} E_h] \leq \Pr[\cup_{h=1}^{p} F_h] = \sum_{h=1}^{p} \Pr[F_h].
\]  

(13)
Note that $F_h$ occurs only if none of the events $F_1, \ldots, F_{h-1}$ occurs, i.e., $Y_1 = \cdots = Y_{h-1} = \emptyset$. Hence, when $F_h$ occurs, we must have that $X_t \cup Z_t = A \times B$ for every $t \in [h-1]$. For every $(u,v) \in Z_t$, $\hat{y}^{(h)}_{u,v} - y^{(h)}_{u,v}$ is a zero solution of (12) and gives $A$ absolutely no information about $r$; for every $(u,v) \in X_t$, $\hat{y}^{(h)}_{u,v} - y^{(h)}_{u,v}$ is not a solution of (12) and gives $A$ exactly the information that $r \cdot (\hat{y}^{(h)}_{u,v} - y^{(h)}_{u,v}) \neq 0$, (14) allows $A$ to rule out $\leq q^{m-1}$ possibilities of $r$. When $F_h$ occurs, $X_1 \cup \cdots \cup X_{h-1}$ contains $\leq (h-1)ab$ elements and helps $A$ to rule out at most $(h-1)abq^{m-1}$ possibilities of $r$. As a result, in the $h$-th attempt, the $r$ is still uniformly distributed over the set of all remaining vectors that have not been ruled out. By providing a set $\{\hat{y}^{(h)}_{u,v} : \ell \in [k], (u,v) \in C_\ell\}$, $A$ would define $\leq ab$ new equation systems of the form (14) that have a nonzero coefficient matrix. The solution spaces of these nontrivial equation systems together cover at most $abq^{m-1}$ vectors out of the $q^m - (h-1)abq^{m-1}$ remaining vectors. Given that none of the events $F_1, \ldots, F_{h-1}$ occurs, the secret vector $r$ is still uniformly distributed over the set of all remaining vectors. Therefore, we must have that

$$\Pr[F_h] \leq \frac{abq^{m-1}}{q^m - (h-1)abq^{m-1}} \leq \frac{ab}{q - pab},$$

(15)
i.e., the $r$ will fall into the union of these solution spaces with probability at most $ab/(q - pab)$. Due to (13) and (15), we have that $\Pr[\operatorname{Exp}_{A,\Pi}^\text{verif}(F, p) = 1] \leq \frac{pab}{q - pab}$.

In Theorem 1, any adversary making $\leq p$ attempts succeeds in breaking the security of $\Pi$ with probability $\leq pab/(q - pab)$. The upper bound can be made negligible in the statistical security parameter $\lambda$ as long as $q \approx 2^\lambda$ and $a, b, p$ are all polynomial functions in $\lambda$. In our generic construction, the key $PK_F$ is empty such that anyone, even without executing $\text{KeyGen}$, is able to execute $\text{ProbGen}(PK_F, x)$ to prepare its input vector $x$. Hence, the scheme $\Pi$ allows public delegation. One the other hand, the verification keys $VK_F, VK_x$ must be kept private. Otherwise, an adversary will be able to easily to persuade the
client to both accept a set of wrong server results and output a wrong function value. Hence, Π is privately verifiable.

For any $F \in \mathbb{Z}_q^{m \times d}$, the computational cost of running KeyGen($\lambda, F$) is dominated by $2 amd$ additions modulo $q$ and $amd$ multiplications modulo $q$. For any $x \in \mathbb{Z}_q^d$, the computational cost of running ProbGen($PK_F, x$) is dominated by $bd$ additions modulo $q$. For every $\ell \in [k]$, the cost of running Compute($\ell, \rho_\ell, \sigma_\ell$) at the $\ell$-th server is dominated by $|C_\ell|md$ additions modulo $q$ and $|C_\ell|md$ multiplications modulo $q$. The total cost of running all $k$ server-side algorithms is dominated by $\sum_{\ell=1}^{k} |C_\ell|md = abmd$ additions modulo $q$ and $\sum_{\ell=1}^{k} |C_\ell|md = abmd$ multiplications modulo $q$. In verification, the client needs to compute $ab$ inner products of dimension-$m$ vectors, $ab$ inner product of dimension-$d$ vectors, and possibly $ab$ additions of dimension-$m$ vectors. The total computational cost of executing Verify is dominated by $ab(2m + d)$ additions modulo $q$ and $ab(m + d)$ multiplications modulo $q$. If we denote with Add$_q$ additions modulo $q$ and denote Mul$_q$ multiplications modulo $q$, then the computational cost of all algorithms in Π can be summarized in the following figure:

| Algorithms | Add$_q$   | Mul$_q$   |
|------------|-----------|-----------|
| KeyGen     | $2amd$    | $amd$     |
| ProbGen    | $bd$      | 0         |
| Compute    | $abmd$    | $abmd$    |
| Verify     | $ab(2m + d)$ | $ab(m + d)$ |

Fig. 2: Computational Cost (Π)

The client’s total computational cost of executing ProbGen and Verify in the computing phase is dominated by $2abm + abd + bd$ additions modulo $q$ and $ab(m+d)$ multiplications modulo $q$. On the other hand, in the naive computation of $Fx$, the client has to do around $md$ additions modulo $q$ and $md$ multiplications modulo $q$. As the parameters $a, b$ are typically constants (see Section 3.2 for parameter selection, and Sections 3.3 and 3.4 for instantiations), we have that $2abm + abd + bd = o(md)$ and $ab(m+d) = o(md)$ as long as $m, d$ are large enough.
That is, the client’s total computational cost in the computing phase of \( \Pi \) will be substantially less than its cost in a naive computation of \( Fx \). Therefore, our generic construction yields MSVC schemes that are outsourceable.

### 3.2. Parameter Selection and Optimization

When our generic construction \( \Pi \) is instantiated, the parameters \( k, a, b, \{ A_\ell \}_{\ell=1}^k, \{ B_\ell \}_{\ell=1}^k \) and \( \{ C_\ell \}_{\ell=1}^k \) should be chosen to both meet the requirements on input/function privacy and also optimize the computational cost. In our MSVC model, the servers do not communicate with each other; otherwise, the input/function privacy may be compromised. In an ideal instantiation, we prefer to choose a smallest \( k \) such that the number \( \binom{k}{2} \) of all possible 2-server collisions is minimized, in order to guarantee the highest level of input/function privacy. On the other hand, the client’s total computational cost in \( \Pi \)’s computing phase is roughly equal to \( 2abm + abd + bd \) additions modulo \( q \) and \( ab(m + d) \) multiplications modulo \( q \); and the servers’ total computational cost is roughly equal to \( abmd \) additions modulo \( q \) and \( abmd \) multiplications modulo \( q \). For the fixed \( m, d \), both workloads are minimized as long as \( ab \) is minimized. Therefore, in an ideal instantiation, we prefer to choose \( a, b \) such that \( ab \) is minimized, in order to give the best efficiency. Minimizing both \( k \) and \( ab \) simultaneously is infeasible, which will be seen soon. In this section, we minimize these parameters separately and obtain two independent schemes: one attains the highest level of privacy, and the other attains the best efficiency.

**Definition 6.** Let \( k, a, b \geq 2 \) be integers and let \( A = [a], B = [b] \). The nonempty sets \( A_1 \times B_1, \ldots, A_k \times B_k \) form a k-covering of \( A \times B \) if \( A_1, \ldots, A_k \subset A, B_1, \ldots, B_k \subset B, \) and \( (A \times B) \subseteq \bigcup_{\ell=1}^k (A_\ell \times B_\ell) \). For all \( a, b \geq 2 \), we define \( k(a, b) \) to be the smallest integer \( k \geq 2 \) such that there is a \( k \)-covering of \( [a] \times [b] \). For every integer \( k \geq 2 \), we define \( w(k) \) to be the least values of \( ab \) such that \( [a] \times [b] \) has a \( k \)-covering, where \( a, b \geq 2 \). We agree that \( w(k) = \infty \) if for all \( a, b \geq 2 \) there is no \( k \)-covering of \( [a] \times [b] \).

**Theorem 2.** We have that \( k(2, b) = k(a, 2) = 4 \) and \( k(a, b) = 3 \) for all \( a, b \geq 3 \).
Proof: For all \( a, b \geq 2 \), the sets \( \{1\} \times \{1\}, \{1\} \times \{2, \ldots, b\}, \{2, \ldots, a\} \times \{1\}, \) and \( \{2, \ldots, a\} \times \{2, \ldots, b\} \) form a 4-covering of \([a] \times [b]\). Therefore, \( k(a, b) \leq 4 \). Due to Definition 6, it is easy to see that \( k(a, b) \geq 2 \) for all \( a, b \geq 2 \). Hence, \( k(a, b) \in \{2, 3, 4\} \) for all \( a, b \geq 2 \).

Below we show that \( k(a, b) > 2 \) for all \( a, b \geq 2 \). Assume for contradiction that there are integers \( a, b \geq 2 \) such that \( k(a, b) = 2 \). Then there is a 2-covering \( \{A_1 \times B_1, A_2 \times B_2\} \) of \([a] \times [b]\), where the definition of covering shows that \( \emptyset \neq A_1, A_2 \subseteq [a], \emptyset \neq B_1, B_2 \subseteq [b] \) and \([a] \times [b] \subseteq (A_1 \times B_1) \cup (A_2 \times B_2) \). Let \( x \in [a] \setminus A_1 \) and \( y \in [b] \setminus B_2 \). Then it is easy to see that \((x, y) \notin A_1 \times B_1, (x, y) \notin A_2 \times B_2\), which contradicts the requirement that \([a] \times [b] \subseteq (A_1 \times B_1) \cup (A_2 \times B_2)\).

As a result, we have that \( k(a, b) \in \{3, 4\} \) for all \( a, b \geq 2 \). Whenever \( a, b \geq 3 \), the sets \( \{1, 2\} \times \{1, 2\}, \{1, 3, \ldots, a\} \times \{1, 3, \ldots, b\}, \{2, 3, \ldots, a\} \times \{2, 3, \ldots, b\} \) would be a 3-covering of \([a] \times [b]\). Hence, we have that \( k(a, b) = 3 \) for all \( a, b \geq 3 \). At last, we show that \( k(a, 2) = 4 \). A similar proof for \( k(2, b) = 4 \) exists and will be omitted from here. Assume for contradiction that \( \{A_1 \times B_1, A_2 \times B_2, A_3 \times B_3\} \) is a 3-covering of \([a] \times [2]\). Due to the definition of covering, we have that \( B_1, B_2, B_3 \in \{\{1\}, \{2\}\} \). Then at least one of \( \{1\} \) and \( \{2\} \) appears at most once in \( B_1, B_2, B_3 \). Without loss of generality, suppose that \( \{1\} \) appears at most once in \( B_1, B_2, B_3 \). We distinguish between two cases: (i) \( \{1\} \) does not appear in \( \{B_1, B_2, B_3\} \); (ii) \( \{1\} \) appears once in \( \{B_1, B_2, B_3\} \). In the first case, \( \{A_1 \times B_1, A_2 \times B_2, A_3 \times B_3\} \) cannot form a covering of \([a] \times [2]\) because for every \( x \in [a], (x, 1) \in [a] \times [2] \) but \((x, 1) \notin (A_1 \times B_1) \cup (A_2 \times B_2) \cup (A_3 \times B_3) \). In the second case, we suppose that \( B_1 = \{1\} \) and \( B_2 = B_3 = \{2\} \). As \( A_1 \) is a proper subset of \([a]\), we can choose \( x \in [a] \setminus A_1 \). Then we would have \((x, 1) \in [a] \times [2]\), but \((x, 1) \notin (A_1 \times B_1) \cup (A_2 \times B_2) \cup (A_3 \times B_3) \), which shows a contradiction. Hence, \([a] \times [2]\) cannot have a 3-covering and \( k(a, 2) \) must be equal to 4. □

Theorem 3. We have that \( w(2) = \infty, w(3) = 9 \) and \( w(k) = 4 \) for all \( k \geq 4 \).

Proof: The proof of Theorem 2 shows that there is no 2-covering of \([a] \times [b]\)
for all \(a, b \geq 2\). Hence, \(w(2) = \infty\). There is a 3-covering of \([a] \times [b]\) if and only if \(a, b \geq 3\). Among the choices of \(a, b\), the product \(ab\) is minimized when \(a = b = 3\). Therefore, we have that \(w(3) = 9\). When \(k \geq 4\), there is a \(k\)-covering of \([a] \times [b]\) for all \(a, b \geq 2\). Among the choices of \(a, b\), the product \(ab\) is minimized when \(a = b = 2\). Therefore, we have that \(w(k) = 4\) for all \(k \geq 4\). □

3.3. Instantiation with the Least Number of Servers

Theorem 2 shows that the smallest number of required servers is 3 when the generic construction \(\Pi\) is instantiated. Let \(k = 3, a = b = 3, A = [3], B = [3], A_1 = \{1, 2\}, B_1 = \{1, 2\}, A_2 = \{1, 3\}, B_2 = \{1, 3\}, A_3 = \{2, 3\}\) and \(B_3 = \{2, 3\}\). Then it is easy to verify that \(A_1 \times B_1\) and \(A_2 \times B_2\) form a cover of \(A \times B\) and \(C_1 = \{(1, 1), (1, 2), (2, 1), (2, 2)\} \subseteq A_1 \times B_1, C_2 = \{(1, 3), (3, 1), (3, 3)\}\) and \(C_3 = \{(2, 3), (3, 2)\}\) form a partition of \(A \times B\). By instantiating the generic construction \(\Pi\) with the parameters \((k, a, b, A, B, \{A_\ell\}_{\ell=1}^3, \{B_\ell\}_{\ell=1}^3, \{C_\ell\}_{\ell=1}^3)\), we will get 3-server verifiable computation scheme, denoted as \(\Pi_s\), which requires the smallest number of servers among all possible instantiations of \(\Pi\). Based on Figure 2, the computational cost of the algorithms in \(\Pi_s\) can be summarized with Fig. 3.

| Algorithms | Add_q | Mul_q |
|------------|-------|-------|
| KeyGen     | 6md   | 3md   |
| ProbGen    | 3d    | 0     |
| Compute    | 9md   | 9md   |
| Verify     | 9(2m + d) | 9(m + d) |

Fig. 3: Computational Cost (\(\Pi_s\))

3.4. Instantiation with the Least Workload

Theorem 3 shows that the smallest amount of client/server computation will be done when the generic construction \(\Pi\) is instantiated to as a 4-server scheme. Let \(k = 4, a = b = 2, A = [2], B = [2], A_1 = \{1\}, B_1 = \{1\}, A_2 = \{1\}, B_2 = \{2\}, A_3 = \{2\}, B_3 = \{1\}, A_4 = \{2\}\) and \(B_4 = \{2\}\). Then it is
easy to verify that $A_1 \times B_1, A_2 \times B_2, A_3 \times B_3$ and $A_4 \times B_4$ form a cover of $A \times B$ and $C_1 = \{(1, 1)\} \subseteq A_1 \times B_1$, $C_2 = \{(1, 2)\}$, $C_3 = \{(2, 1)\}$ and $C_4 = \{(2, 2)\} \subseteq A_4 \times B_4$ form a partition of $A \times B$. By instantiating the generic construction $\Pi$ with the parameters $(k, a, b, A, B, \{A_\ell\}_{\ell=1}^3, \{B_\ell\}_{\ell=1}^3, \{C_\ell\}_{\ell=1}^3)$, we get a 4-server verifiable computation scheme, denoted as $\Pi_w$, which has the fastest client/server computation among all possible instantiations of $\Pi$. Based on Figure 2, the computational cost of the algorithms in $\Pi_w$ can be summarized with Fig. 4.

| Algorithms | Add$_q$ | Mul$_q$ |
|------------|---------|---------|
| KeyGen     | 4$m+d$  | 2$m$    |
| ProbGen    | 2$d$    | 0       |
| Compute    | 4$m+d$  | 4$m$    |
| Verify     | 4$(2m + d)$ | 4$(m + d)$ |

Fig. 4: Computational Cost ($\Pi_w$)

3.5. Implementation

Our generic construction is practical and suitable for implementation. In order to show its practicality, we have implemented $\Pi_s$ and $\Pi_w$ on a DELL Precision Tower T7810 workstation that runs with the Intel Xeon E5-2650 (2.30 GHz) Processor. In both cases, we implemented all algorithm on the same platform, in order to compare between the computational costs. In our implementations, we set $q = 8243401665430070934609707337531854135999471015108634126889281238621513052057$ to be a 256-bit prime such that the success probability of any adversary that makes $p$ attempts in breaking $\Pi_s$ and $\Pi_w$ is bounded by $\frac{9p}{2^{256}} - \frac{9p}{2^{256}}$ and $\frac{4p}{2^{256}} - \frac{4p}{2^{256}}$ respectively.
In our implementations, we choose $m = d$ and consider both a random function $F \in \mathbb{Z}_q^{m \times d}$ and a random input $x \in \mathbb{Z}_q^d$. We record both the time $T_n$ required by the naive computation of $Fx$ and the total time $T_c$ required by the client’s execution of $\text{ProbGen}$ and $\text{Verify}$ in the computing phase. In both experiments we get the results for $m \in \{100, 200, \ldots, 3000\}$, which are shown in Fig. 5 and Fig. 6, respectively. Our experiments show that $T_c$ is much smaller than $T_n$, i.e., the client computation in $\Pi_s$ and $\Pi_w$ are substantially faster than the naive computation of $Fx$. For example, when $m = d = 3000$, we have that $(T_n, T_c) = (2609.08\,\text{ms}, 176.74\,\text{ms})$ in $\Pi_s$ and $(T_n, T_c) = (2801.35, 89.77\,\text{ms})$ in $\Pi_w$. In the literature of SSVC, few schemes allow us to observe that $T_c < T_n$, due to the use of expensive operations such as FHE, GCs, or heavy public-key operations.
4. Applications

4.1. Analysis of Sensitive Data

In some applications the function $F$ may be a trading secret that can be expressed as a matrix function or a polynomial function. The developer of this algorithm wants to provide paid services by applying the algorithm to the user’s sensitive data. However, the developer may not be willing to maintain his own computing infrastructures, which will incur significant financial cost. In this scenario, the developer can outsource the computation of $F$ to multiple clouds with our MSVC schemes such that: (1) both the algorithm $F$ and the user’s data are kept secret from the servers; (2) the servers’ results are verifiable in order to guarantee the correct reconstruction of output; (3) the result verification and reconstruction are substantially faster than the naive computation of $F$.

For example, in molecular anthropology the single nucleotide polymorphisms (SNPs) are usually analyzed in order to help diagnosis, give medication and conduct research. The human genome may be a sequence of \{A, C, G, T\} of length $\approx 3 \times 10^9$. Genomes of two individuals are 99.6% similar and part of the differences is contributed by SNPs. Many SNPs explain why some people have higher chance to have diabetes, cancers, and other inherited diseases than others. It is well-known that how likely a patient will have a disease can be computed from his SNPs. Let \{SNP$_1$, SNP$_2$, \ldots, SNP$_d$\} be a set of related SNPs when a set \{D$_1$, D$_2$, \ldots, D$_m$\} of diseases are considered. A vector $\mathbf{x} = (x[1], x[2], \ldots, x[d]) \in \{0, 1, 2\}^d$ can be defined such that for every $j \in [d]$, $x[j]$ stands for the number of times that SNP$_j$ occurs in a person’s genome. An $m \times d$ matrix $F = (F[i, j])_{m \times d}$ can be defined such that for every $i \in [m]$, the $i$-th row of $F$ is a set of important factors for the disease D$_i$ and the risk of a person to have disease D$_i$ can be computed as $\sum_{j=1}^{m} F[i, j] x[j]$. Then the risk of the person to have the diseases \{D$_1$, D$_2$, \ldots, D$_m$\} can be computed as $F \mathbf{x}$. In this example, the developer (e.g., a doctor) of the algorithm may have spent significant research efforts in developing $F$ and thus consider $F$ as a trading secret; any user (e.g., a patient) of the algorithm $F$ must keep his data $\mathbf{x}$ secret as well. Our MSVC schemes would...
allow the doctor to outsource the computation of $F \mathbf{x}$ to multiple clouds, such that (1)-(3) are all satisfied.

4.2. Polynomial Outsourcing

Achieving input privacy in SSVC has been highly non-trivial, especially when the function $F$ requires high degree computations on the input $x$. On one hand, in order to keep $x$ private from the server, the client has to encrypt $x$ as $\text{Enc}(x)$ with a semantically secure encryption $\text{Enc}$. On the other hand, the server has to compute $F(x)$ with $F$ and $\text{Enc}(x)$, which requires $\text{Enc}$ to be homomorphic. However, the homomorphic encryptions are still far from practical [35] today. As a result, the resulting SSVC schemes are usually impractical, in terms of the client-side computation and the server-side computation.

Our MSVC schemes in Section 3 enable the computation of $F \mathbf{x}$ for any $F \in \mathbb{Z}_{q}^{m \times d}$ and $\mathbf{x} \in \mathbb{Z}_{q}^{d}$; allow public delegation and private verification; have information-theoretic security and privacy (of input and function); and are practically efficient. They successfully resolved the conflict between input privacy and practicality. In this section, we shall show how to use these schemes to resolve the same conflict in the delegation of various polynomial functions, which may have very high degrees. Our main observation is that the special algebraic structure of a polynomial $f$ may allow us to decompose the computation of $f(x)$ as a two-stage computation, where the first-stage computation is a matrix-vector multiplication of the form $\mathbf{u} = F \mathbf{x}$ and the second-stage computation is an inner product computation $\mathbf{y} \cdot \mathbf{u}$. In particular, $F$ is a matrix determined by $f$ and $\mathbf{x}, \mathbf{y}$ are vectors determined by $x$. Usually the computation of $F \mathbf{x}$ is heavy and that of $\mathbf{y} \cdot \mathbf{u}$ is fast. By delegating the heavy computation of $F \mathbf{x}$ with our MSVC scheme and leaving the light computation of $\mathbf{y} \cdot \mathbf{u}$ to the client, the client would be able to offload the main workload to the servers. Furthermore, the resulting delegation scheme for $f$ would inherit all nice properties of MSVC, such as correctness, security, input/function privacy, and practical efficiency. The method of decomposing polynomial evaluations into two-stage computations will be detailed as below.
• **Univariate polynomials.** For any univariate polynomial \( f(x) = \sum_{i=0}^{d} f_i x^i \), define \( m = \lceil \sqrt{d + 1} \rceil \):

\[
F = \begin{pmatrix}
    f_0 & f_1 & \cdots & f_{m-1} \\
    f_m & f_{m+1} & \cdots & f_{2m-1} \\
    \vdots & \vdots & \ddots & \vdots \\
    f_{m^2-m} & f_{m^2-m+1} & \cdots & f_{m^2-1}
\end{pmatrix},
\]

where \( f_i = 0 \) for all \( i > d \); \( x = (1, x, \ldots, x^{m-1})^\top \); and \( y = (1, x^m, \ldots, x^{m^2-m}) \).

Then we will have that \( f(x) = y(Fx) \). The first-stage computation requires \( O(m^2) = O(d) \) arithmetic operations modulo \( q \), and the second-stage computation requires \( O(m) = O(\sqrt{d}) \) arithmetic operations modulo \( q \).

• **Bivariate polynomials.** For any bivariate polynomial \( f(x, y) = \sum_{i,j=0}^{d} f_{i,j} x^i y^j \), define \( F = (f_{i,j}) \), \( x = (1, x, \ldots, x^d)^\top \) and \( y = (1, y, \ldots, y^d) \). Then we will have that \( f(x, y) = y(Fx) \). The first-stage computation requires \( O(d^2) \) arithmetic operations modulo \( q \), and the second-stage computation requires \( O(d) \) arithmetic operations modulo \( q \).

• **Quadratic multivariate polynomials.** For a polynomial \( f(x_1, \ldots, x_d) = \sum_{i,j=1}^{d} f_{i,j} \cdot x_i x_j \), define \( F = (f_{i,j}) \), \( x = (x_1, \ldots, x_d)^\top \) and \( y = x^\top \). Then we will have that \( f(x_1, \ldots, x_d) = y(Fx) \). The first-stage computation requires \( O(d^2) \) arithmetic operations modulo \( q \), and the second-stage computation requires \( O(d) \) arithmetic operations modulo \( q \).

• **Multivariate polynomials of bounded degree in each variable.** For a polynomial \( f(x_1, \ldots, x_m) = \sum_{i_1,\ldots,i_m=1}^{d} f_{i_1,\ldots,i_m} \cdot x_1^{i_1} \cdots x_m^{i_m} \), define \( \ell = \lfloor m/2 \rfloor \), \( y = (x_1^{\ell+1} \cdots x_m^{\ell}) \in \mathbb{Z}_q^d \), \( x = (x_{\ell+1}^{\ell+1} \cdots x_m^{m-1})^\top \in \mathbb{Z}_q^{m-1} \); and let \( F = (F_{(i_1,\ldots,i_\ell),(i_{\ell+1},\ldots,i_m)}) \) be a matrix of size \( d^\ell \times d^{m-\ell} \) such that \( F_{(i_1,\ldots,i_\ell),(i_{\ell+1},\ldots,i_m)} = f_{i_1,\ldots,i_m} \) for all \( (i_1,\ldots,i_\ell) \in [d]^\ell \) and \( (i_{\ell+1},\ldots,i_m) \in [d]^{m-\ell} \). Then we will have that \( f(x_1, \ldots, x_m) = y(Fx) \). The first-stage computation requires \( O(d^m) \) arithmetic operations modulo \( q \), and the
second-stage computation requires $O(d^{\lceil m/2 \rceil})$ arithmetic operations modulo $q$.

In all the above decompositions, the first-stage computation is as heavy as the naive computation of $f(x)$ and the second-stage computation is substantially faster. By delegating the first-stage computations with our MSVC schemes, we would obtain the expected MSVC schemes for polynomials.

4.3. Protocol Design

Our MSVC schemes have interesting applications in the design of cryptographic protocols such as outsourced private information retrieval. A $t$-private $k$-server information-theoretic private information retrieval (PIR) [17] is a protocol between a client and $k$ servers, where each server has a database $f = (f_1, f_2, \ldots, f_N)$ and the client is interested in $f_i$ for some $i \in [N]$. Such a protocol allows the client to retrieve $f_i$ from the servers such that $i$ is still hidden from any $\leq t$ servers. Its efficiency is mainly measured with two parameters: (1) communication complexity, which is the total number of bits communicated for retrieving a single entry of $f$; (2) server computation complexity: which is the total number of database entries accessed by the servers in each retrieval.

While the communication complexity can be sublinear in $N$, Beimel et al. [6] has shown that the servers’ computation complexity is $\Omega(N)$ for any PIR. When $N$ is large, it would be nice to outsource PIR servers’ computation to cloud services [29, 36], which have numerous computing resources. However, outsourcing requires a stronger adversarial model as the cloud may be untrusted. Most of the existing PIR protocols [33] assume that the PIR servers should be honest-but-curious. In the cloud scenario, this weak model should be strengthened to resist malicious servers. A natural way is by making the PIR server’s computation verifiable.

In the literature, the specific computations studied in this paper, such as matrix-vector multiplications and polynomial evaluations, have been widely used in PIR construction. For example, one can consider $f$ as the entries of
a square matrix

\[ F = (F_{i,j}) = \begin{pmatrix} f_1 & f_2 & \cdots & f_d \\ f_{d+1} & f_{d+2} & \cdots & f_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ f_{d^2-d+1} & f_{d^2-d+2} & \cdots & f_{d^2} \end{pmatrix}, \tag{16} \]

where \( d = \lceil \sqrt{N} \rceil \) and \( f_j = 0 \) for all \( j > N \). When the client is interested in \( f_i \) and \( f_i \) is the \((r,c)\)-entry of \( F \) for some \( r, c \in [d] \), it suffices for the client to privately retrieve the \( c \)-th column of \( F \), i.e., \((F_{1,c}, \ldots, F_{d,c})\). Then the server-side computation can be captured by \( Fx \) for \( x = (0, \ldots, 1, \ldots, 0)^\top \), where the \( c \)-th component of \( x \) is equal to 1 and all other components are equal to 0. Our MSVC schemes would enable the client to delegate the computation of \( Fx \) to the PIR cloud servers such that: (1) \( x \) is kept secret from each server; (2) the server’s computation results become verifiable.

In this simple solution, the client and the server only need to communicate \( O(\sqrt{N}) \) bits, which gives a nontrivial outsourced PIR. As an additional property, the database \( f \) is information-theoretically hidden from each server.

5. Conclusion

In this paper, we defined an MSVC model where each server performs a partial computation on the function shares and input shares. We give a generic construction that achieves public delegation and private verification, information-theoretic security, input and function privacy. We also apply these schemes to construct MSVC schemes for various polynomial functions. Our schemes are free of public-key operations and practically efficient. Our schemes also yield information-theoretic PIR schemes that are secure against malicious servers. We leave it as a future work to design MSVC schemes with public verification.

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