CLASSIFICATION OF THREE DIMENSIONAL COMPLEX $\omega$-LIE ALGEBRAS

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ABSTRACT. A complex $\omega$-Lie algebra is a vector space $L$ over the complex field, equipped with a skew symmetric bracket $[-,-]$ and a bilinear form $\omega$ such that

$$[[x,y],z] + [[y,z],x] + [[z,x],y] = \omega(x,y)z + \omega(y,z)x + \omega(z,x)y$$

for all $x,y,z \in L$. The notation of $\omega$-Lie algebras, as a generalization of Lie algebras, originally appeared in Riemannian geometry; they are related to the isoparametric hypersurfaces in spheres ([1, 4]). There are no non-Lie $\omega$-Lie algebras in dimension two. The purpose of this note is to give a classification of three dimensional non-Lie complex $\omega$-Lie algebras.

1. Introduction. Let $k$ be a field of characteristic zero and $L$ be a finite-dimensional vector space over $k$. Let $[-,-] : L \times L \rightarrow L$ be a anti-commutative product on $L$ and $\omega : L \times L \rightarrow k$ be a bilinear form on $L$. The triple $(L, [-,-], \omega)$ is called an $\omega$-Lie algebra if the following condition is satisfied:

$$(1) \quad [[x,y],z] + [[y,z],x] + [[z,x],y] = \omega(x,y)z + \omega(y,z)x + \omega(z,x)y$$

for all $x,y,z \in L$. The equation (1) is called the $\omega$-Jacobi identity. Apparently, the $\omega$ is also skew-symmetric; an $\omega$-Lie algebra is a Lie algebra if and only if the bilinear form $\omega \equiv 0$. So we usually call the Lie algebras trivial $\omega$-Lie algebras.

The notation of $\omega$-Lie algebras, which is related to the study of isoparametric hypersurfaces in Riemannian geometry, was introduced in the recent works of Bobieński and Nurowski ([1, 4]). By the definition, it is easy to see that all $\omega$-Lie algebras are trivial in the case of dimension 1 and 2. The first example of nontrivial 3-dimensional $\omega$-Lie algebra was given by Nurowski [3] in 2007. In that paper, Nurowski finally completed the classification of 3-dimensional $\omega$-Lie algebra over the field of real numbers.

A fundamental development of $\omega$-Lie algebras was by Zusmanovich [5], in which a lot of basic concepts, such as modules, (quasi-) ideals and (generalized) derivations, were introduced; some basic properties of $\omega$-Lie algebras were found. One of Zusmanovich’s results asserts that finite-dimensional nontrivial $\omega$-Lie algebras are either low-dimensional or have an abelian subalgebra of small codimension with some restrictive conditions. In particular, the following useful result is proved.

Lemma 1 ([5], Lemma 8.1). If $L$ is a finite-dimensional $\omega$-Lie algebra with non-degenerate $\omega$, then dim $L = 2$. 

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Recall that a skew-symmetric bilinear form \( \omega \) on \( L \) is degenerate if there exists a nonzero vector \( x \in L \) such that \( \omega(x, y) = 0 \) for all \( y \in L \). Lemma 1 mentioned above means that the bilinear form \( \omega \) on any \( \omega \)-Lie algebra \( L \) must be degenerate if \( \dim L \geq 3 \).

The main purpose of this note is to classify all 3-dimensional nontrivial \( \omega \)-Lie algebras over the field of complex numbers.

In what follows, \( C \) is the field of complex numbers and \( L \) denotes an \( \omega \)-Lie algebra with a basis \( \{x, y, z\} \); we write \( \wedge^2 L \) for the exterior power of \( L \) with the basis \( \{x \wedge y, x \wedge z, y \wedge z\} \) and \( \varphi = [\cdot, -] : \wedge^2 L \rightarrow L \) is the bracket product. We use \( L' = [L, L] \) to denote the commutator subalgebra of \( L \). We call the dimension of \( L' \) the rank of \( \varphi \). In the book [2], Fulton and Harris presented a classification of 3-dimensional complex Lie algebras by considering the rank of \( \varphi \).

In this note, we will follow some ideas in [2] and discuss the rank of \( \varphi \), which may be 0, 1, 2, 3.

The following theorem is our main result.

**Theorem 2.** Let \( L \) be a nontrivial (i.e. non-Lie) 3-dimensional \( \omega \)-Lie algebra, then it must be isomorphic to one of the following algebras:

1. \( L_1 \) : \( [x, z] = 0, [y, z] = z, [x, y] = y \) and \( \omega(y, z) = \omega(x, z) = 0, \omega(x, y) = 1 \).
2. \( L_2 \) : \( [x, y] = 0, [x, z] = y, [y, z] = z \) and \( \omega(x, y) = 0, \omega(x, z) = 1, \omega(y, z) = 0 \).
3. \( A_\alpha \) : \( [x, y] = x, [x, z] = x + y, [y, z] = z + \alpha x \) and \( \omega(x, y) = \omega(x, z) = 0 \),
   \[ \omega(y, z) = -1, \text{ where } \alpha \in C. \]
4. \( B_\alpha \) : \( [x, y] = y, [x, z] = y + z, [y, z] = \alpha x \) and \( \omega(x, y) = \omega(x, z) = 0 \),
   \[ \omega(y, z) = 2\alpha, \text{ where } 0 \neq \alpha \in C. \]
5. \( C_\alpha \) : \( [x, y] = y, [x, z] = \alpha z, [y, z] = x \) and \( \omega(x, y) = \omega(x, z) = 0 \),
   \[ \omega(y, z) = 1 + \alpha, \text{ where } 0 \neq \alpha \in C. \]

2. **Ranks 0 and 1.** We continue to follow the notations in the preceding section.

   If the rank of \( \varphi \) is zero, then \( L \) is abelian. It follows from the \( \omega \)-Jacobi identity (1) that
   \[ \omega(x, y)z + \omega(y, z)x + \omega(z, x)y = 0. \]

   Since \( \{x, y, z\} \) is a basis of \( L \), \( \omega(x, y) = \omega(y, z) = \omega(z, x) = 0 \). Thus in this case, the \( \omega \)-Lie structure on \( L \) is trivial.

   If the rank of \( \varphi \) is 1, then \( \dim L' = 1 \) and the kernel of \( \varphi \) is two dimensional. Suppose that \( \{x, y, z\} \) is a basis of \( L \) such that \( [x, y] = [x, z] = 0 \). We let \( [y, z] = ax + by + cz \) for some \( a, b, c \in C \). By \( \omega \)-Jacobi identity, we have
   \[ \omega(x, y)z + \omega(y, z)x + \omega(z, x)y \]
   \[ = [(x, y), z] + [(y, z), x] + [(z, x), y] \]
   \[ = [ax + by + cz, x] = 0. \]
Since \( x, y, z \) is linearly independent, \( \omega(x, y) = \omega(y, z) = \omega(z, x) = 0 \). Thus \( \omega \) is trivial. The same arguments as in \([2]\) (page 137) will imply that there only exist two Lie algebras:

\[
\begin{align*}
g_1 & : [x, y] = [x, z] = 0, \text{ and } [y, z] = y. \\
g_2 & : [x, y] = [x, z] = 0, \text{ and } [y, z] = x.
\end{align*}
\]

They are trivial \( \omega \)-Lie algebras.

3. **Rank 2.** In this case, we choose \([y, z]\) as a basis of \( L' \) and \( x \not\in L' \). We assume that \([y, z] = ay + bz \). Our arguments will separated into the following two cases: \( a = b = 0 \) or the others.

**Case 1.** If both \( a \) and \( b \) are zero, then \([x, y] \neq 0 \) and \([x, z] \neq 0 \) because the kernel of \( \varphi \) is one dimensional. Thus the linear map \( \text{ad}_x : L' \rightarrow L' \) by \( u \mapsto [x, u] \) is bijective. By linear algebra, we can choose the suitable basis elements \( y, z \) such that \( \text{ad}_x \) is similar to

\[
\begin{pmatrix}
  c & 0 \\
  0 & d
\end{pmatrix}
\]

or

\[
\begin{pmatrix}
  e & 0 \\
  1 & e
\end{pmatrix},
\]

where \( c, d, e \) are nonzero complex numbers. In the first situation, \([x, y] = \text{ad}_x(y) = cy\) and \([x, z] = \text{ad}_x(z) = dz\). Let \( \tilde{x} = c^{-1}x \), then \([\tilde{x}, y] = y \) and \([\tilde{x}, z] = (c^{-1}d)z\). We obtain a family of Lie algebras with one parameter:

\[
b_\alpha : [x, y] = y, [x, z] = az \text{ and } [y, z] = 0, \text{ where } 0 \neq \alpha \in \mathbb{C}.
\]

In the second situation, \([x, y] = ey\) and \([x, z] = y + ez\). Let \( \tilde{\alpha} = e^{-1}x \), then \([\tilde{x}, y] = y \) and \([\tilde{x}, z] = e^{-1}y + z\). Let \( \tilde{y} = e^{-1}y \), then \([\tilde{x}, \tilde{y}] = \tilde{y} \) and \([\tilde{x}, z] = \tilde{y} + z\). Thus we get a Lie algebra:

\[
g_3 : [x, y] = y, [x, z] = y + z \text{ and } [y, z] = 0.
\]

**Case 2.** Assume that one of \( a \) and \( b \) is not zero. We need only to consider the case of \( b \neq 0 \) because if \( a \neq 0 \) then we can transpose \( y \) and \( z \) and will get the same results.

Let \( \tilde{z} = z + b^{-1}ay \), then \([y, \tilde{z}] = [y, z] = ay + bz = b\tilde{z}\). Let \( \tilde{y} = b^{-1}y \) then \([\tilde{y}, \tilde{z}] = \tilde{z}\). So in this case, we can assume that \([y, z] = z\). Since the kernel of \( \varphi \) is one dimensional, one of \([x, z]\) and \([x, y]\) is zero, and the other is not zero.

**Subcase 1.** If \([x, y] = ay + cz \neq 0 \) and \([x, z] = 0\), then

\[
\begin{align*}
\omega(x, y)z + \omega(y, z)x + \omega(z, x)y \\
= & \quad [(x, y), z] + [(y, z), x] + [(z, x), y] \\
= & \quad [ay + cz, z] + [z, x] = az.
\end{align*}
\]

This means that \( \omega(y, z) = \omega(z, x) = 0 \) and \( a = \omega(x, y) \). If \( c = 0 \), then \( a \neq 0 \). We let \( \tilde{x} = a^{-1}x \) and it is easy to check that \([\tilde{x}, z] = 0, [y, z] = z, [\tilde{x}, y] = y, \omega(y, z) = \omega(x, z) = 0 \) and \( \omega(\tilde{x}, y) = 1 \). Thus we get a nontrivial \( \omega \)-Lie algebra:

\[
L_1 : [x, z] = 0, [y, z] = z, [x, y] = y \text{ and } \omega(y, z) = \omega(x, z) = 0, \omega(x, y) = 1.
\]
If \( c \neq 0 \) and \( a = 0 \), then the \( \omega \)-Jacobi identity implies that \( \omega \) is trivial. If \( c \neq 0 \) and \( a \neq 0 \), we assume that \( \tilde{y} = y + a^{-1}cz \), then \( [x, z] = 0, [\tilde{y}, z] = z, [x, \tilde{y}] = a\tilde{y} \) and \( \omega(\tilde{y}, z) = \omega(x, z) = 0, \omega(x, \tilde{y}) = a \), so it is easy to see that the corresponding \( \omega \)-Lie algebra is isomorphic to \( L_1 \).

Subcase 2. If \([z, x] = -[x, z] = ay + cz \neq 0 \) and \([x, y] = 0\), then
\[
\omega(x, y)z + \omega(y, z)x + \omega(z, x)y = \omega([x, y], z) + \omega(y, [z, x]) + \omega([z, x], y) = [z, x] + [cz, y] = ay.
\]

Thus \( \omega(y, z) = 0, \omega(x, y) = 0 \) and \( \omega(z, x) = a \). Now we have
\[
[y, z] = z, [x, y] = 0, [z, x] = ay + cz.
\]

Since the dimension of \( L' \) is two, \( a \neq 0 \). Let \( \tilde{y} = y + a^{-1}cz \), then
\[
[z, x] = a\tilde{y}, [\tilde{y}, z] = z, [x, \tilde{y}] = -c\tilde{y}.
\]

Recall that the kernel of \( \varphi \) is one dimensional, so \( c \) must be zero. Now we can assume that \( \tilde{z} = a^{-1}z \) and \( \tilde{x} = -x \). This yields that
\[
[\tilde{y}, \tilde{z}] = \tilde{z}, [\tilde{x}, \tilde{y}] = 0, [\tilde{x}, \tilde{z}] = \tilde{y}, \omega(\tilde{y}, \tilde{z}) = 0, \omega(\tilde{x}, \tilde{y}) = 0, \omega(\tilde{x}, \tilde{z}) = -1.
\]

Hence there is a nontrivial \( \omega \)-Lie algebra:
\[
L_2 : \quad [x, y] = 0, [x, z] = y, [y, z] = z \text{ and } \omega(x, y) = 0, \omega(x, z) = 1, \omega(y, z) = 0.
\]

4. Rank 3. For the case of \( \omega \equiv 0 \), it follows from the Fulton and Harris’s arguments in \cite{2} (pages 141-142) that there exists only one Lie algebra:
\[
\mathfrak{g}_4 : \quad [x, y] = 2y, [x, z] = -2z \text{ and } [y, z] = x.
\]

Next we consider the nontrivial case. Since the dimension of \( L' \) is 3, the rank of adjoint map \( \text{ad}_x : L \to L \) must be 2 for any nonzero \( x \in L \). Thus the kernel of \( \text{ad}_x \) is equal to \( C \cdot x \).

By Lemma 1, if \( \omega \) is non-degenerate, then \( L \) must have dimension 2. So the bilinear form \( \omega \) we consider here is degenerate. This means that there exists an nonzero element \( x \in L \) such that \( \omega(x, v) = 0 \) for all \( v \in L \). Now we fix \( x \). By the Jordan canonical form, we can choose a suitable basis \( \{u, y, z\} \) of \( L \) such that \( \text{ad}_x \) is similar to
\[
A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 1 & \delta \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix}, \quad \text{or } D = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \tau \end{pmatrix},
\]
where \( \delta, \mu, \nu \) and \( \tau \) are all nonzero and are the eigenvalues of \( \text{ad}_x \). Thus our arguments consist of the following four cases.

Case 1. If \( \text{ad}_x \) is similar to \( A \), then \([x, u] = 0, [x, y] = u \) and \([x, z] = y \). Let \( u = ax \) for some nonzero \( a \in \mathbb{C} \). Then we have \([x, y] = ax \) and \([x, z] = y \). That is, \([a^{-1}x, a^{-1}y] = a^{-1}x \) and \([a^{-1}x, z] = a^{-1}y \). Thus
we can assume that \([x, y, z]\) is a basis of \(L\) such that
\[
[x, y] = x, [x, z] = y.
\]

Next we need to determine the commutator relations of \(y\) and \(z\). Let \([y, z] = bx + cy + dz\), then
\[
\omega(x, y)z + \omega(y, z)x + \omega(z, x)y
\]
\[
= \([x, y], z\] + [[y, z], x] + [[z, x], y]
\]
\[
= [x, z] + [cy + dz, x]
\]
\[
= (1 - d)y - cx.
\]

Thus \(\omega(x, y) = 0, \omega(x, z) = d - 1, \omega(y, z) = -c\). Since \(\omega(x, z) = 0, d = 1\). Notice that \(\omega\) is not trivial, so \(c \neq 0\). In the equation \([y, z] = bx + cy + z\), we can first assume that \(\tilde{z} = z + cy\). Then
\[
[y, \tilde{z}] = \tilde{z} + bx, [x, y] = x, [x, \tilde{z}] = cx + y,
\]
and \(\omega(x, y) = 0, \omega(x, \tilde{z}) = 0\) and \(\omega(y, \tilde{z}) = -c\). Let \(\alpha = c^{-1}b, \beta = c^{-1}, \gamma = c^{-1}\tilde{z}\), then
\[
[y, \gamma'] = \gamma' + \alpha x, [x, y] = x, [x, \gamma'] = x + \beta y
\]
and \(\omega(x, y) = 0, \omega(x, \gamma') = 0\) and \(\omega(y, \gamma') = -1\). We define \(x' = x^{-1}x\) and \(y = \alpha \beta\). Then
\[
[x', y] = x', [x', \gamma'] = x' + y, [y, \gamma'] = \gamma' + y x'
\]
with \(\omega(x', y) = 0, \omega(x', \gamma') = 0, \omega(y, \gamma') = -1\). Hence we get a family of \(\omega\)-Lie algebras with one parameter:

\[
A_\alpha : \begin{array}{l}
x, y, \alpha, x, z = x + y, y, [y, z] = z + \alpha x; \\
\omega(x, y) = 0, \omega(x, z) = 0, \omega(y, z) = -1, \text{ where } \alpha \in \mathbb{C}.
\end{array}
\]

Case 2. If \(\text{ad}_x\) is similar to \(B\), then \([x, u] = 0, [x, y] = \delta y, [x, z] = y + \delta z\). Obviously, \([\delta^{-1}x, \delta^{-1}y] = \delta^{-1}y, [\delta^{-1}x, z] = \delta^{-1}y + z\). Thus we can assume that \([x, y, z]\) is a basis of \(L\) such that
\[
[x, y] = y, [x, z] = y + z.
\]

Let \([y, z] = ax + by + cz\), then
\[
\omega(x, y)z + \omega(y, z)x + \omega(z, x)y
\]
\[
= \([x, y], z\] + [[y, z], x] + [[z, x], y]
\]
\[
= 2[y, z] + [by + cz, x]
\]
\[
= 2ax + (b - c)y + cz.
\]

Thus \(\omega(x, y) = c, \omega(x, z) = c - b\) and \(\omega(y, z) = 2\alpha\). Recall that \(x\) belongs to the kernel of \(\omega\), so \(c = b = 0\). Hence we obtain a family of \(\omega\)-Lie algebras of one parameter:

\[
B_\alpha : \begin{array}{l}
x, y, \alpha, x, z = y + z, y, [y, z] = \alpha x; \\
\omega(x, y) = 0, \omega(x, z) = 0, \omega(y, z) = 2\alpha, \text{ where } 0 \neq \alpha \in \mathbb{C}.
\end{array}
\]
Case 3. If ad$_x$ is similar to $C$, then $[x, u] = 0, [x, y] = \mu y, [x, z] = \nu z$. We can assume that $\{x, y, z\}$ is a basis of $L$. Let $[y, z] = ax + by + cz$. Notice that the dimension of $L'$ is 3, so $\alpha \neq 0$. Thus as before, we can assume that $[x, y] = y, [x, z] = \alpha z$ and $[y, z] = x + by + cz$, where $\alpha \neq 0$. The $\omega$-Jacobi identity implies that $b = c = 0$ and $\omega(x, y) = 0, \omega(x, z) = 0, \omega(y, z) = 1 + \alpha$.

We obtain again a family of $\omega$-Lie algebras of one parameter:

$$C_{\alpha} : \quad [x, y] = y, [x, z] = \alpha z, [y, z] = x;$$

$$\omega(x, y) = 0, \omega(x, z) = 0, \omega(y, z) = 1 + \alpha,$$

where $0 \neq \alpha \in \mathbb{C}$.

Case 4. If ad$_x$ is similar to $D$, then $[x, u] = 0, [x, y] = u, [x, z] = \tau z$. Let $\{x, y, z\}$ is a basis of $L$ and $[y, z] = ax + by + cz$. As before, we can assume that $[x, y] = x, [x, z] = z$ and $[y, z] = ax + by + cz$. By the $\omega$-Jacobi identity we will see that $\omega(x, y) = 1$, which contradicts with the assumption that $\omega(x, v) = 0$ for all $v \in L$. Thus in this situation, there do not exist any $\omega$-Lie algebras.

This completes the proof of Theorem 2.

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