The analytical value of the electron ($g$-2) at order $\alpha^3$ in QED.

S.Laporta §
E.Remiddi ◦

Dipartimento di Fisica, Università di Bologna,
and INFN, Sezione di Bologna,
Via Irnerio 46, I-40126 Bologna, Italy

Abstract

We have evaluated in closed analytical form the contribution of the three-loop non-planar ‘triple-cross’ diagrams contributing to the electron ($g$-2) in QED; its value, omitting the already known infrared divergent part, is

$$a_e(3\text{- cross}) = \frac{1}{2}\pi^2\zeta(3) - \frac{55}{12}\zeta(5) - \frac{16}{135}\pi^4 + \frac{32}{3}\left(a_4 + \frac{1}{24}\ln^4 2\right)$$

$$+ \frac{14}{9}\pi^2\ln^2 2 - \frac{1}{3}\zeta(3) + \frac{23}{3}\pi^2\ln 2 - \frac{47}{9}\pi^2 - \frac{113}{48}.$$  

This completes the analytical evaluation of the ($g$-2) at order $\alpha^3$, giving

$$a_e(3\text{- loop}) = \left(\frac{\alpha}{\pi}\right)^3 \left\{ \frac{83}{72}\pi^2\zeta(3) - \frac{215}{24}\zeta(5) + \frac{100}{3}\left[a_4 + \frac{1}{24}\ln^4 2\right] - \frac{1}{24}\pi^2\ln^2 2 \right\}$$

$$- \frac{239}{2160}\pi^4 + \frac{139}{18}\zeta(3) - \frac{298}{9}\pi^2\ln 2 + \frac{17101}{810}\pi^2 + \frac{28259}{5184} \right\}$$

$$= \left(\frac{\alpha}{\pi}\right)^3 (1.181241456...).$$

PACS: 12.20Ds; 13.40.Em; 06.20.Jr; 12.20Fv

Keywords: Quantum ElectroDynamics; Anomalous magnetic moment of the electron; Analytical evaluation of 3-loop radiative corrections.

§ E-mail: laporta@bo.infn.it
◦ E-mail: remiddi@bo.infn.it
Following the work of Ref. [1] we have completed the evaluation in close analytical form of the contribution to the electron anomaly in three-loop QED due to the triple-cross graphs depicted in Fig. 1).

The results are

\[
a_e(3 - \text{cross}; 1a) = \frac{215}{24} \zeta(5) - \frac{1}{3} \pi^2 \zeta(3) - \frac{53}{2160} \pi^4 + 4 \left[ \left( a_4 + \frac{1}{24} \ln^4 2 \right) - \frac{1}{24} \pi^2 \ln^2 2 \right] \\
- \frac{1229}{144} \zeta(3) - \frac{7}{6} \pi^2 \ln 2 + \frac{4165}{2592} \pi^2 - \frac{515}{864} + \frac{1}{2} \ln \lambda \\
= 1.285068495... + \frac{1}{2} \ln \lambda ,
\]

(1)

\[
a_e(3 - \text{cross}; 1b) = -\frac{275}{24} \zeta(5) + \frac{29}{36} \pi^2 \zeta(3) + \frac{43}{1080} \pi^4 - \frac{5}{6} \pi^2 \ln^2 2 + \frac{623}{144} \zeta(3) + \frac{35}{9} \pi^2 \ln 2 \\
- \frac{1951}{648} \pi^2 - \frac{493}{864} \\
= -0.878968171... ,
\]

(2)

\[
a_e(3 - \text{cross}; 1c) = \frac{5}{12} \zeta(5) - \frac{4}{9} \pi^2 \zeta(3) - \frac{161}{1080} \pi^4 + \frac{8}{3} \left( a_4 + \frac{1}{24} \ln^4 2 \right) + \frac{32}{9} \pi^2 \ln^2 2 \\
+ \frac{97}{12} \zeta(3) + \frac{20}{9} \pi^2 \ln 2 - \frac{1043}{432} \pi^2 - \frac{1}{48} \\
= -0.026799490... ,
\]

(3)
from which

\[ a_e(3 - \text{cross}) = 2a_e(3 - \text{cross}; 1a) + 2a_e(3 - \text{cross}; 1b) + a_e(3 - \text{cross}; 1c) \]

\[ = \frac{1}{2} \pi^2 \zeta(3) - \frac{55}{12} \zeta(5) - \frac{16}{135} \pi^4 + \frac{32}{3} \left( a_4 + \frac{1}{24} \ln^4 2 \right) \]

\[ + \frac{14}{9} \pi^2 \ln^2 2 - \frac{1}{3} \zeta(3) + \frac{23}{3} \pi^2 \ln 2 - \frac{47}{9} \pi^2 - \frac{113}{48} + \ln \lambda \]

\[ = 0.785401156... + \ln \lambda . \]

As it is customary in this kind of calculations, \( \lambda \) is the regularizing photon mass (in electron mass units), \( \zeta(p) = \sum_{n=1}^{\infty} 1/n^p, \zeta(2) = \pi^2/6, \ a_4 = \sum_{n=1}^{\infty} \frac{1}{2^n n^4}. \)

The above results are in excellent agreement with the numerical results

\[ a_e(3 - \text{cross}; 1a; \text{Ref.2}) = 1.291 (7) , \]
\[ a_e(3 - \text{cross}; 1b; \text{Ref.2}) = -0.882 (10) , \]
\[ a_e(3 - \text{cross}; 1c; \text{Ref.2}) = -0.021 (100) , \]
\[ a_e(3 - \text{cross}; \text{Ref.3}) = 0.785419 (40) , \]

where the \( \ln \lambda \)'s are understood. As the previous graphs were the last graphs for which the analytical value of the anomaly was still missing, on account the previously known results [4,5,6,7] the complete analytical expression of the anomaly in three-loop QED can now be written as

\[ a_e(3 - \text{loop}) = \left( \frac{\alpha}{\pi} \right)^3 \left\{ \frac{83}{72} \pi^2 \zeta(3) - \frac{215}{24} \zeta(5) + \frac{100}{3} \left[ \left( a_4 + \frac{1}{24} \ln^4 2 \right) - \frac{1}{24} \pi^2 \ln^2 2 \right] \right. \]

\[ - \frac{239}{2160} \pi^4 + \frac{139}{18} \zeta(3) - \frac{298}{9} \pi^2 \ln 2 + \frac{17101}{810} \pi^2 + \frac{28259}{5184} \left\} \right. \]

\[ = \left( \frac{\alpha}{\pi} \right)^3 (1.181241456...) . \]

By using the best numerical value of \( a_e(4 - \text{loop}) = -1.557 (70) , \text{Ref.[8]}, \) and

\[ 1/\alpha = 137.0359979 (32) , \]

Ref.[9], one finds

\[ a_e(\text{th}) = 1159652201.2 (2.1) (27.1) \times 10^{-12} , \]

(6)

to be compared with the experimental value, Ref.[10],

\[ a_e(\text{exp}) = 1159652188.4 (4.3) \times 10^{-12} ; \]

3
conversely, by using Ref.[10] as an input, one obtains

\[ \frac{1}{\alpha(a_e)} = 137.03599941(56) . \]  

If \( p \) is the electron momentum and \( \Delta \) the momentum transfer of some vertex amplitude, the corresponding \((g-2)\) contribution is extracted by keeping \( p \) on the mass shell, expanding the vertex up to first order in \( \Delta \), multiplying it by a suitable spinor projection operator and then performing the appropriate traces [11]. The \((g-2)\) is then expressed as the sum a few hundred terms all of the kind

\[ \int d^4k_1d^4k_2d^4k_3 \frac{N}{D} , \]  

each term having its own specific numerator \( N \) and denominator \( D \). The numerators \( N \) are in general simple monomials in the scalar products of \( p \) and the loop momenta \( k_i \), such as for instance \((p \cdot k_i)\), \((k_i \cdot k_j)\) etc, up to \((p \cdot k_2)^4\), while the denominators \( D \) have in general the form

\[ D = D_1^{n_1}D_2^{n_2}D_3^{n_3}D_4^{n_4}D_5^{n_5}D_6^{n_6}D_7^{n_7}D_8^{n_8} , \]  

the exponents \( n_i \) ranging from 0 to 2 (0 means that the denominator is absent; the powers 2 arise from the expansion in \( \Delta \); in each term there are at most two denominators with \( n_i = 2 \)); in \( m_e = 1 \) units the denominators, see Fig.2), are defined as

\[ D_1 = (p - k_1)^2 + 1 - i\epsilon , \quad D_2 = (p - k_1 - k_2)^2 + 1 - i\epsilon , \]
\[ D_3 = (p - k_1 - k_2 - k_3)^2 + 1 - i\epsilon , \quad D_4 = (p - k_2 - k_3)^2 + 1 - i\epsilon , \]
\[ D_5 = (p - k_3)^2 + 1 - i\epsilon , \quad D_6 = k_1^2 - i\epsilon , \quad D_7 = k_2^2 - i\epsilon , \quad D_8 = k_3^2 - i\epsilon . \]  

Besides the 8 denominators, there are in the problem 9 linearly independent scalar products \( p \cdot k_i, k_i \cdot k_j \), etc; 8 independent identities between the scalar products and the denominators are easily written, such as for instance

\[ p \cdot k_1 = \frac{1}{2} (D_6 - D_1) , \quad p \cdot k_2 . \]  

The identities can be used to express the scalar products in the numerator as a combination of the denominators, so obtaining a combination of integrands with a smaller number of denominators and therefore simpler from the point of view of the analytical
integration. It is to observed however that, as there are 9 scalar products and only 8 denominators, one scalar product (which we choose to be \( p \cdot k_2 \)) must remain anyhow in the numerator; furthermore, the simpler terms obtained in that way can be in general non convergent when taken separately.

![Diagram of loop integrations]

**Fig.2.** The notation for the 3-loop integrations.

In Ref. [1] we gave the analytical value of the simplest term

\[
J_0 \equiv \frac{(-i)^3}{\pi^6} \int d^4k_1d^4k_2d^4k_3 \frac{1}{D_1D_2D_3D_4D_5D_6D_7D_8} = 4\pi^2 \ln^2 2 - \frac{1}{6}\pi^4 .
\]

With a straightforward extension of the results of Ref. [1] one can obtain, for instance

\[
J_1 \equiv \frac{(-i)^3}{\pi^6} \int d^4k_1d^4k_2d^4k_3 \frac{p \cdot k_2}{D_1D_2D_3D_4D_5D_6D_7D_8} = 5\zeta(5) - \frac{1}{2}\pi^2\zeta(3) .
\]

The extension to all the other terms appearing in the expression giving the \((g-2)\) is in principle straightforward but in fact rather wearisome (it implies, among other things, to build, for each tensor in the loop momenta, the scalar amplitudes free from kinematical singularities for which unsubtracted dispersion relations can be written). It was found convenient to work out another approach, based on the integration by part identity method first developed in Ref.[12]. In our case the method consists in writing

\[
\int \left( \prod_{i=1}^3 d^nk_i \right) \frac{\partial}{\partial k_{j,\mu}} \left( v_{l,\mu} \frac{N}{D} \right) = 0 ,
\]

where \( v_l \) is either the external momentum \( p \) or one of the three loop momenta \( k_i \), \( N/D \) stands for the ratio of all the pairs of numerators and denominators of the kind described above; note however that while in the original integrals the loop momenta have dimension 4, the identities are written for momenta having continuous \( n \) dimension, in which case their validity is obvious. The required \( n = 4 \) values are then recovered by means of a suitable limiting procedure. Needless to say, if the \((g-2)\) from some graph is obtained as the \( n \to 4 \) limit of an \( n \)-dependent expression, this expression must be
evaluated for consistency in \( n \) dimensions from the very beginning; in particular, the traces of the \( \gamma \)-matrix must be evaluated in \( n \)-dimensions too.

By explicitly performing the derivatives, using the identities (10) and rearranging the obtained terms in an identity like Eq.(13), one obtains a relation between various integrals which amounts to express one of them as a linear combination of the others. The number of all the possible identities is large (a few thousands, as there are twelve identities for each of the several hundreds \( N/D \) possibilities); they form a huge linear system, whose coefficients are in turn polynomials in \( n \), from which the overwhelming majority of the occurring integrals can be expressed as a linear combination of a few basic integrals. As we are interested in the \( n \to 4 \) limit, we put \( n = 4 - 2\omega \) and then deal with the limit by expanding everything in powers of \( \omega \) for \( \omega \to 0 \). Fortunately, the expansion and the extremely lengthy processing of the various identities can be carried out in an almost mechanical way.

A thorough investigation shows that all the three loop integrals (and therefore also the complete \((g-2)\) expression in which we are interested) can be expressed in terms of the following 18 basic integrals:

\[
I_1 = \left( \frac{-i}{\pi^{n-2}} \right)^3 \int d^n k_1 d^n k_2 d^n k_3 \frac{p \cdot k_2}{D_1 D_2 D_3 D_4 D_5 D_6 D_7 D_8},
\]

\[
I_2 = \left( \frac{-i}{\pi^{n-2}} \right)^3 \int d^n k_1 d^n k_2 d^n k_3 \frac{1}{D_1 D_2 D_3 D_4 D_5 D_7 D_8},
\]

\[
I_3 = \left( \frac{-i}{\pi^{n-2}} \right)^3 \int d^n k_1 d^n k_2 d^n k_3 \frac{1}{D_1 D_2 D_4 D_5 D_6 D_8},
\]

\[
I_4 = \left( \frac{-i}{\pi^{n-2}} \right)^3 \int d^n k_1 d^n k_2 d^n k_3 \frac{1}{D_2 D_3 D_4 D_6 D_7 D_8},
\]

\[
I_5 = \left( \frac{-i}{\pi^{n-2}} \right)^3 \int d^n k_1 d^n k_2 d^n k_3 \frac{1}{D_1 D_3 D_4 D_5 D_7 D_8},
\]

\[
I_6 = \left( \frac{-i}{\pi^{n-2}} \right)^3 \int d^n k_1 d^n k_2 d^n k_3 \frac{1}{D_1 D_3 D_5 D_6 D_7 D_8},
\]

\[
I_7 = \left( \frac{-i}{\pi^{n-2}} \right)^3 \int d^n k_1 d^n k_2 d^n k_3 \frac{1}{D_2 D_4 D_5 D_6 D_7 D_8},
\]

\[
I_8 = \left( \frac{-i}{\pi^{n-2}} \right)^3 \int d^n k_1 d^n k_2 d^n k_3 \frac{1}{D_1 D_2 D_3 D_4 D_5},
\]

\[
I_9 = \left( \frac{-i}{\pi^{n-2}} \right)^3 \int d^n k_1 d^n k_2 d^n k_3 \frac{1}{D_2 D_3 D_5 D_6 D_7}.
\]
\[
I_{10} = \left( \frac{-i}{\pi^{n-2}} \right)^3 \int d^n k_1 d^n k_2 d^n k_3 \frac{1}{D_2 D_4 D_6 D_7 D_8},
\]
\[
I_{11} = \left( \frac{-i}{\pi^{n-2}} \right)^3 \int d^n k_1 d^n k_2 d^n k_3 \frac{p \cdot k_2}{D_1 D_3 D_5 D_7},
\]
\[
I_{12} = \left( \frac{-i}{\pi^{n-2}} \right)^3 \int d^n k_1 d^n k_2 d^n k_3 \frac{1}{D_1 D_3 D_5 D_7},
\]
\[
I_{13} = \left( \frac{-i}{\pi^{n-2}} \right)^3 \int d^n k_1 d^n k_2 d^n k_3 \frac{1}{D_1 D_2 D_4 D_5},
\]
\[
I_{14} = \left( \frac{-i}{\pi^{n-2}} \right)^3 \int d^n k_1 d^n k_2 d^n k_3 \frac{1}{D_3 D_5 D_6 D_7},
\]
\[
I_{15} = \left( \frac{-i}{\pi^{n-2}} \right)^3 \int d^n k_1 d^n k_2 d^n k_3 \frac{1}{D_2 D_3 D_4 D_5},
\]
\[
I_{16} = \left( \frac{-i}{\pi^{n-2}} \right)^3 \int d^n k_1 d^n k_2 d^n k_3 \frac{1}{D_3 D_4 D_7 D_8},
\]
\[
I_{17} = \left( \frac{-i}{\pi^{n-2}} \right)^3 \int d^n k_1 d^n k_2 d^n k_3 \frac{1}{D_3 D_6 D_7 D_8},
\]
\[
I_{18} = \left( \frac{-i}{\pi^{n-2}} \right)^3 \int d^n k_1 d^n k_2 d^n k_3 \frac{1}{D_1 D_4 D_5}.
\]

Note that in the above list there is only one integral with all the 8 denominators ($I_1$, the limit of which is Eq.(12) above), while there are no integrals with 7 denominators. With the exception of $I_1$, all the basic integral are divergent in the $n \to 4$ limit; having put $n = 4 - 2\omega$, the divergences show up as $1/\omega$ singularities, so that evaluating them amounts to evaluate the coefficients of their expansions in $\omega$. As an example, the value of $I_{18}$, which perhaps the easiest to obtain as it factorizes in the product of three 1-loop integrals, is

\[
I_{18} = C(\omega) \left( -\frac{1}{\omega^3} - \frac{3}{\omega^2} - \frac{6}{\omega} - 10 - 15\omega - 21\omega^2 - 28\omega^3 + O(\omega^4) \right),
\]

where $C(\omega)$, defined as

\[
C(\omega) = (\pi^{\omega} \Gamma(1 + \omega))^3,
\]

is an overall normalization factor, whose limiting value at $\omega = 0$ is 1.
As an example of the kind of formulae obtained by expressing a non-trivial integral in terms of the basic integrals, let us give the relation

\[
\frac{(-i)^3}{\pi^6} \int d^4 k_1 d^4 k_2 d^4 k_3 \frac{(p \cdot k_2)^2}{D_1 D_2 D_3 D_4 D_5 D_6 D_7 D_8} = \lim_{\omega \to 0} \left[ -I_1 \\
+ I_2 (-3\omega) + I_3 \left( \frac{1}{4} - \frac{3}{4} \omega - 2\omega^2 - 6\omega^3 \right) \\
+ I_4 \left( \frac{3}{2} \omega \right) + I_5 (-\omega - 2\omega^2 - 4\omega^3) + I_6 \left( \frac{1}{2} - \frac{1}{2} \omega - 3\omega^2 - 16\omega^3 \right) \\
+ I_7 (3\omega + 14\omega^2 + 64\omega^3) + I_8 \left( -\frac{1}{6\omega} \right) + I_9 \left( -\frac{1}{2\omega} - \frac{1}{2} + \omega + 4\omega^2 + 176\omega^3 \right) \\
+ I_{10} \left( -\frac{1}{2} - \frac{5}{2} \omega - \frac{27}{2} \omega^2 - \frac{137}{2} \omega^3 \right) \\
+ I_{11} \left( \frac{1}{6\omega} \right) - \frac{139}{36} - \frac{1249}{108} \omega - \frac{19225}{324} \omega^2 - \frac{238333}{972} \omega^3 \\
+ I_{12} \left( \frac{3}{64\omega^2} + \frac{13}{128\omega} + \frac{383}{128} + 2\omega - \frac{353}{16} \omega^2 + \frac{3041}{8} \omega^3 \right) \\
+ I_{13} \left( -\frac{1}{12\omega} - \frac{3}{8} - \frac{1}{6} \omega + \frac{29}{24} \omega^2 + \frac{37}{4} \omega^3 \right) \\
+ I_{14} \left( -\frac{1}{4\omega} + \frac{13}{8} + \frac{87}{8} \omega + \frac{439}{8} \omega^2 + \frac{623}{2} \omega^3 \right) \\
+ I_{15} \left( -\frac{7}{12\omega} - \frac{11}{8} - \frac{19}{24} \omega + \frac{55}{12} \omega^2 + \frac{73}{2} \omega^3 \right) \\
+ I_{16} \left( -\frac{1}{2\omega} - \frac{7}{4} - \frac{7}{4} \omega + \frac{9}{4} \omega^2 + 196\omega^3 \right) \\
+ I_{17} \left( \frac{3}{2\omega} + \frac{3}{4} + \frac{29}{6} \omega + \frac{50}{3} \omega^2 \right) \\
+ I_{18} \left( \frac{3}{64\omega^2} - \frac{2}{3\omega} - \frac{55}{36} - \frac{1391}{864} \omega - \frac{94423}{5184} \omega^2 + \frac{353507}{7776} \omega^3 \right) \right].
\]

(15)

Similar expressions hold for all the other integrals contributing to the anomaly. In all those expressions, the coefficient of \( I_1 \) is always finite (i.e. not singular for \( \omega \to 0 \)), so that only its \( \omega = 0 \) limit, given by \( J_1 \) of Eq.(12), is required.

Besides \( I_1 \), we have already given the explicit analytical value of \( I_{18} \), Eq.(14). In order to complete the calculation of all the basic integrals, we have evaluated directly only a subset of them (namely \( I_8, I_9, I_{10} \) and from \( I_{13} \) to \( I_{18} \)). For the others, we found it more convenient to write a large number of identities of the form of Eq.(15), in which however the analytical value of the integral on the l.h.s. is already known from previous work; the relation is then expressing such a known result in terms of the still unknown
basic integrals. In the case of $J_0$, Eq.(11), one finds for instance

\[
J_0 = \lim_{\omega \to 0} \left[ I_2 \left( \frac{5}{2} - \frac{15}{2} \omega \right) + I_3 \left( \frac{3}{4} - \frac{9}{4} \omega \right) + I_4 \left( -\frac{5}{4} + \frac{5}{4} \omega \right) + I_5 \left( \frac{1}{2} - \frac{3}{2} \omega \right) + I_6 \left( \frac{3}{4} - \frac{3}{4} \omega + \frac{3}{2} \omega^2 + 6 \omega^3 \right) + I_7 \left( -\frac{5}{2} + \frac{5}{2} \omega - 5 \omega^2 - 20 \omega^3 \right) + I_8 \left( -\frac{2}{3 \omega} + \frac{4}{3} - \frac{16}{3} \omega - \frac{64}{9} \omega^2 - 128 \omega^3 \right) + I_9 \left( \frac{1}{8 \omega^2} - \frac{1}{\omega} + 1 + \omega \right) + I_{10} \left( \frac{1}{4 \omega} + 1 + \frac{7}{2 \omega} + \frac{39}{2} \omega^2 + 201 \omega^3 \right) + I_{11} \left( \frac{7}{6 \omega} - \frac{49}{36} + \frac{11}{108} \omega - \frac{1819}{324} \omega^2 - \frac{34249}{972} \omega^3 \right) + I_{12} \left( -\frac{3}{256 \omega^3} + \frac{59}{256 \omega^2} - \frac{203}{256 \omega} \right) + I_{13} \left( \frac{1}{3 \omega} - \frac{17}{24} \right) + \frac{211 \omega}{48} - \frac{423 \omega^2}{16} - \frac{1263 \omega^3}{8} \right) + I_{14} \left( \frac{1}{16 \omega^2} - \frac{49}{32 \omega} + \frac{53}{32} \right) + \frac{3 \omega}{8} - \frac{119 \omega}{16} \omega^2 + \frac{427 \omega^3}{8} \right) + I_{15} \left( \frac{1}{3 \omega} - \frac{17}{6} \right) + \frac{211 \omega}{12} - \frac{423 \omega^2}{4} - \frac{1263 \omega^3}{2} \right) + I_{16} \left( \frac{1}{8 \omega^2} - \frac{13}{16 \omega} + \frac{7}{16} \right) + \frac{5 \omega}{4} - \frac{9 \omega^2}{2} - \frac{67 \omega^3}{4} \right) + I_{17} \left( \frac{3}{4 \omega^2} + \frac{23}{8 \omega} - \frac{69}{8} \right) + \frac{40 \omega}{3} - \frac{587 \omega^2}{6} \right) + I_{18} \left( -\frac{3}{128 \omega^3} + \frac{7}{32 \omega^2} - \frac{29}{12 \omega} - \frac{3799}{576} \right) - \frac{140051}{3456} \omega - \frac{1193035}{5184} \omega^2 - \frac{10388627}{7776} \omega^3 \right].
\]

(16)

Similarly, in Ref.[7] the following 7-denominator integral was evaluated

\[
\frac{(-i)^3}{\pi^6} \int d^4 k_1 d^4 k_2 d^4 k_3 \frac{1}{D_1 D_3 D_4 D_5 D_6 D_7 D_8} = \frac{21}{12} \pi^2 \zeta(3) - \frac{45}{4} \zeta(5);
\]

(17)
in terms of the basic integrals one has

\[
\frac{(-i)^3}{\pi^6} \int d^4k_1 d^4k_2 d^4k_3 \frac{1}{D_1D_3 D_4D_5D_6D_7D_8} = \lim_{\omega \to 0} \left[ I_5 \left( \frac{1}{2\omega} - \frac{3}{2} \right) + I_6 \left( -\frac{1}{2\omega} + \frac{1}{2} - \omega - 4\omega^2 - 16\omega^3 \right) + I_9 \left( -\frac{1}{4\omega} + \frac{1}{2} \right) + I_{11} \left( \frac{2}{\omega} + \frac{14}{3} + \frac{445}{18} \omega + \frac{3104}{27} \omega^2 + \frac{85177}{162} \omega^3 \right) + I_{12} \left( -\frac{3}{16} - \frac{19}{32} \omega - \frac{39}{16} \omega^2 - \frac{79}{8} \omega^3 \right) + I_{14} \left( -\frac{11}{8\omega} - \frac{53}{16} - \frac{299}{16} \omega - \frac{715}{8} \omega^2 - \frac{1667}{4} \omega^3 \right) + I_{15} \left( -\frac{1}{4\omega} + \frac{1}{8} - \frac{1}{8} \omega - \frac{1}{4} \omega^2 - \frac{1}{2} \omega^3 \right) + I_{16} \left( \frac{1}{4\omega} - \frac{1}{8} + \frac{1}{8} \omega + \frac{1}{4} \omega^2 + \frac{1}{2} \omega^3 \right) + I_{17} \left( -\frac{1}{2\omega} - \frac{7}{12} - \frac{10}{3} \omega - \frac{40}{3} \omega^2 \right) + I_{18} \left( \frac{3}{8\omega} + \frac{29}{48} + \frac{521}{144} \omega + \frac{7691}{432} \omega^2 + \frac{111629}{1296} \omega^3 \right) \right].
\]

(18)

It is to be stressed here that such relations have quite a broad generality, which extends
to cover scalar amplitudes occurring in graphs with different topology; the l.h.s. integral
of Eq.s(17),(18) was indeed evaluated in Ref.[7], dealing with “corner-ladder” graphs.

By exploiting our previous work, we can write a redundant number of such rela-
tions expressing unknown basic integrals in terms of already known integrals (they
are not written here for the sake of brevity); in so doing we obtain a redundant set
of linear equations, which are easily solved in terms of the unknown basic integrals
(the redundancy provides with additional consistency checks). As a side remark, let
us observe that for each basic integral one must evaluate the first few terms (up to 7)
of its expansion in \( \omega \), see for instance \( I_{18} \), so that the number of unknown quantities
increases, but on the other hand each relation, being an identity in \( \omega \) provides with
several independent equations for the unknown.

As a result the following integral table is established

\[
I_1 = C(\omega) \left[ 5\zeta(5) - \frac{1}{2} \pi^2 \zeta(3) + O(\omega) \right],
\]

\[
I_2 = C(\omega) \left[ \frac{2}{\omega} \frac{\zeta(3)}{90} - \frac{13}{180} \pi^4 - \frac{1}{3} \pi^2 + 10\zeta(3) + \omega \left( \frac{385}{2} \zeta(5) - \frac{85}{6} \pi^2 \zeta(3) - \frac{7}{15} \pi^4 - 82\zeta(3) - 4\pi^2 \ln 2 + 16\pi^2 \right) + O(\omega^2) \right],
\]

As a result the following integral table is established

\[
I_1 = C(\omega) \left[ 5\zeta(5) - \frac{1}{2} \pi^2 \zeta(3) + O(\omega) \right],
\]

\[
I_2 = C(\omega) \left[ \frac{2}{\omega} \frac{\zeta(3)}{90} - \frac{13}{180} \pi^4 - \frac{1}{3} \pi^2 + 10\zeta(3) + \omega \left( \frac{385}{2} \zeta(5) - \frac{85}{6} \pi^2 \zeta(3) - \frac{7}{15} \pi^4 - 82\zeta(3) - 4\pi^2 \ln 2 + 16\pi^2 \right) + O(\omega^2) \right],
\]
\[ I_3 = C(\omega) \left[ \frac{1}{3\omega^3} + \frac{7}{3\omega^2} + \frac{31}{3\omega} - \frac{2}{15}\pi^4 - \frac{4}{3}\zeta(3) + \frac{103}{3} + \omega \left( 95\zeta(5) - \frac{25}{3}\pi^2\zeta(3) ight) \
- \frac{1}{15}\pi^4 - \frac{184}{3}\zeta(3) - 8\pi^2\ln 2 + \frac{44}{3}\pi^2 + \frac{235}{3} + O(\omega^2) \right], \]

\[ I_4 = C(\omega) \left[ 2\zeta(3)\omega - \frac{7}{90}\pi^4 + 2\zeta(3) + \frac{1}{3}\pi^2 \
+ \omega \left( \frac{385}{2}\zeta(5) - \frac{85}{6}\pi^2\zeta(3) - \frac{7}{15}\pi^4 - 82\zeta(3) - 4\pi^2\ln 2 + 16\pi^2 \right) + O(\omega^2) \right], \]

\[ I_5 = C(\omega) \left[ \frac{1}{6\omega^3} + \frac{3}{2\omega^2} + \frac{1}{\omega} \left( -\frac{1}{3}\pi^2 + \frac{55}{6} \right) - \frac{1}{45}\pi^4 - \frac{14}{3}\zeta(3) - \frac{7}{3}\pi^2 + \frac{95}{2} \
+ \omega \left( -\frac{2}{9}\pi^4 - 44\zeta(3) - \frac{29}{3}\pi^2 + \frac{1351}{6} \right) + O(\omega^2) \right], \]

\[ I_6 = C(\omega) \left[ \frac{1}{6\omega^3} + \frac{7}{3\omega^2} + \frac{31}{3\omega} - \frac{4}{45}\pi^4 + \frac{2}{3}\zeta(3) + \frac{1}{3}\pi^2 + \frac{103}{3} + \omega \left( \frac{45}{2}\zeta(5) - \frac{7}{2}\pi^2\zeta(3) ight) \
+ \frac{11}{45}\pi^4 + \frac{14}{3}\zeta(3) - 4\pi^2\ln 2 + \frac{14}{3}\pi^2 + \frac{235}{3} + O(\omega^2) \right], \]

\[ I_7 = C(\omega) \left[ \frac{1}{6\omega^3} + \frac{3}{2\omega^2} + \frac{1}{\omega} \left( -\frac{1}{3}\pi^2 + \frac{55}{6} \right) - \frac{1}{15}\pi^4 - \frac{8}{3}\zeta(3) - 2\pi^2 + \frac{95}{2} + \omega \left( \frac{45}{2}\zeta(5) ight) \
- \frac{17}{6}\pi^2\zeta(3) - \frac{7}{9}\pi^4 - 50\zeta(3) - 4\pi^2\ln 2 + \frac{1}{3}\pi^2 + \frac{1351}{6} + O(\omega^2) \right], \]

\[ I_8 = C(\omega) \left[ -\frac{1}{\omega^3} - \frac{16}{3\omega^2} - \frac{16}{\omega} + 2\zeta(3) - \frac{8}{3}\pi^2 - 20 \
+ \omega \left( -\frac{3}{10}\pi^4 - \frac{200}{3}\zeta(3) + 16\pi^2\ln 2 - 28\pi^2 + \frac{364}{3} \right) \
+ \omega^2 \left( -126\zeta(5) + 21\pi^2\zeta(3) + \frac{46}{15}\pi^4 - 512\zeta_4 - \frac{64}{3}\ln^4 2 \
- \frac{80}{3}\pi^2\ln^2 2 - \frac{776}{3}\zeta(3) + 168\pi^2\ln 2 - 188\pi^2 + 1244 \right) + O(\omega^3) \right], \]

\[ I_9 = C(\omega) \left[ -\frac{2}{3\omega^3} - \frac{10}{3\omega^2} + \frac{1}{\omega} \left( -\frac{1}{3}\pi^2 - \frac{26}{3} \right) - \frac{16}{3}\zeta(3) - \frac{11}{3}\pi^2 - 2 \
+ \omega \left( -\frac{13}{45}\pi^4 - \frac{248}{3}\zeta(3) + 16\pi^2\ln 2 - \frac{73}{3}\pi^2 + \frac{398}{3} \right) \
+ \omega^2 \left( 96\zeta(5) - \frac{8}{3}\pi^2\zeta(3) + \frac{3}{5}\pi^4 - 512\zeta_4 - \frac{64}{3}\ln^4 2 \
- \frac{128}{3}\pi^2\ln^2 2 - \frac{1888}{3}\zeta(3) + 160\pi^2\ln 2 - 129\pi^2 + 1038 \right) + O(\omega^3) \right], \]
\[ I_{10} = C(\omega) \left[ -\frac{1}{3\omega^3} - \frac{5}{3\omega^2} + \frac{1}{\omega} \left( -\frac{2}{3}\pi^2 - 4 \right) - \frac{26}{3}\zeta(3) - \frac{7}{3}\pi^2 + \frac{10}{3} \right. \\
\left. + \omega \left( -\frac{35}{18}\pi^4 - \frac{94}{3}\zeta(3) - \pi^2 + \frac{302}{3} \right) + O(\omega^2) \right], \]

\[ I_{11} = C(\omega) \left[ \frac{1}{2\omega^3} + \frac{37}{24\omega^2} + \frac{43}{16\omega} + 2\zeta(3) + \frac{139}{96} + \omega \left( -\frac{1}{10}\pi^4 + \frac{19}{3}\zeta(3) - \frac{773}{96} \right) \right. \\
\left. + \omega^2 \left( -\frac{447}{2}\zeta(5) + \frac{53}{2}\pi^2\zeta(3) - \frac{67}{60}\pi^4 + \frac{235}{2}\zeta(3) \right) \right. \\
\left. + 12\pi^2\ln 2 - 18\pi^2 - \frac{27869}{384} \right) + O(\omega^2) \right], \]

\[ I_{12} = C(\omega) \left[ \frac{1}{\omega^3} + \frac{7}{2\omega^2} + \frac{253}{36\omega} + \frac{2501}{216} + \omega \left( -\frac{64}{9}\pi^2 + \frac{59437}{1296} \right) \right. \\
\left. + \omega^2 \left( -\frac{1792}{9}\zeta(3) + \frac{256}{3}\pi^2\ln 2 - \frac{2272}{27}\pi^2 + \frac{2831281}{7776} \right) \right. \\
\left. + \omega^3 \left( \frac{2752}{135}\pi^4 - \frac{8192}{3}a_4 - \frac{1024}{9}\ln^4 2 - \frac{3584}{9}\pi^2\ln^2 2 \right. \\
\left. - \frac{663616}{27}\zeta(3) + \frac{9088}{9}\pi^2\ln 2 - \frac{49840}{81}\pi^2 + \frac{117529021}{46656} \right) + O(\omega^4) \right], \]

\[ I_{13} = C(\omega) \left[ \frac{2}{3\omega^3} + \frac{23}{9\omega^2} + \frac{35}{2\omega} + \frac{127}{12} + \omega \left( \frac{112}{3}\zeta(3) - \frac{189}{8} \right) + \omega^2 \left( -\frac{136}{15}\pi^4 \right. \\
\left. + 256a_4 + \frac{32}{3}\ln^4 2 - \frac{32}{3}\pi^2\ln^2 2 + 280\zeta(3) - \frac{14917}{48} \right) + O(\omega^3) \right], \]

\[ I_{14} = C(\omega) \left[ \frac{1}{3\omega^3} + \frac{7}{6\omega^2} + \frac{25}{12\omega} + \frac{1}{3}\zeta(3) - \frac{5}{24} \right. \\
\left. + \omega \left( -\frac{2}{15}\pi^4 + \frac{28}{3}\zeta(3) - \frac{959}{48} \right) \right. \\
\left. + \omega^2 \left( 48\zeta(5) - \frac{7}{15}\pi^4 + \frac{50}{3}\zeta(3) - \frac{10493}{96} \right) + O(\omega^3) \right], \]

\[ I_{15} = C(\omega) \left[ \frac{3}{2\omega^3} + \frac{23}{4\omega^2} + \frac{105}{16\omega} + \frac{4}{3}\pi^2 + \frac{275}{16} + \omega \left( 28\zeta(3) - 8\pi^2\ln 2 + 10\pi^2 - \frac{567}{32} \right) \right. \\
\left. + \omega^2 \left( -\frac{62}{15}\pi^4 + 192a_4 + 8\ln^4 2 + 16\pi^2\ln^2 2 + 210\zeta(3) - 60\pi^2\ln 2 \right. \\
\left. + \frac{145}{3}\pi^2 - \frac{14917}{64} \right) + O(\omega^3) \right], \]
\[ I_{16} = C(\omega) \left[ \frac{1}{2\omega^3} + \frac{7}{4\omega^2} + \frac{1}{\omega} \left( \frac{1}{3} \pi^2 + \frac{25}{8} \right) + 4\zeta(3) + \frac{7}{6} \pi^2 - \frac{5}{16} \right. \\
+ \omega \left( \frac{16}{45} \pi^4 + 14\zeta(3) + \frac{25}{12} \pi^2 - \frac{959}{32} \right) \\
+ \omega^2 \left( 72\zeta(5) + \frac{8}{3} \pi^2 \zeta(3) + \frac{56}{45} \pi^4 + 25\zeta(3) - \frac{5}{24} \pi^2 - \frac{10493}{64} \right) + O(\omega^3) \right], \]

\[ I_{17} = C(\omega) \left[ -\frac{1}{6\omega^2} - \frac{35}{36\omega} - \frac{1}{3} \pi^2 - \frac{559}{216} + \omega \left( -\frac{16}{3} \zeta(3) - \frac{35}{18} \pi^2 + \frac{2737}{1296} \right) \\
+ \omega^2 \left( -\frac{37}{45} \pi^4 - \frac{280}{9} \zeta(3) - \frac{559}{108} \pi^2 + \frac{552041}{7776} \right) + O(\omega^3) \right]. \]

By using the above table in combination with the integration by parts identities, we obtain the analytical value of the required integrals; from Eq.(15) we have for instance

\[ \frac{(-i)^3}{\pi^6} \int d^4k_1 d^4k_2 d^4k_3 \frac{(p \cdot k_2)^2}{D_1 D_2 D_3 D_4 D_5 D_6 D_7 D_8} = \]

\[ = \frac{1}{2} \zeta(3) \pi^2 - 5\zeta(5) - \frac{2}{45} \pi^4 + \zeta(3) - 2\pi^2 \ln 2 + \frac{3}{2} \pi^2. \]
In terms of the basic integrals, the \((g-2)\) of the graph of Fig.1c, for instance, reads

\[
(g - 2) (\text{Fig.1c}) = \lim_{\omega \to 0} \left[ I_1 \left( \frac{7}{6} \right) + I_2 \left( -\frac{1}{2\omega} + \frac{19}{12} - \frac{1637}{72} \omega \right) + I_3 \left( -\frac{13}{48\omega} + \frac{253}{144} - \frac{823}{108} \omega + \frac{102797}{2592} \omega^2 - \frac{979525}{3888} \omega^3 \right) + I_4 \left( \frac{3}{8\omega} - \frac{5}{24} + \frac{229}{18} \omega \right) \\
+ I_5 \left( -\frac{1}{12\omega} - \frac{7}{18} - \frac{587}{216} \omega + \frac{9133}{324} \omega^2 - \frac{340685}{1944} \omega^3 \right) + I_6 \left( -\frac{1}{24\omega} + \frac{55}{72} - \frac{827}{108} \omega + \frac{19075}{648} \omega^2 - \frac{874721}{3888} \omega^3 \right) + I_7 \left( \frac{19}{24\omega} - \frac{25}{18} + \frac{632}{27} \omega - \frac{56983}{648} \omega^2 + \frac{1488295}{1944} \omega^3 \right) + I_8 \left( \frac{1}{8\omega^2} - \frac{5}{9\omega} + \frac{1585}{216} - \frac{36581}{1296} \omega + \frac{1051253}{3888} \omega^2 - \frac{743606}{729} \omega^3 \right) + I_9 \left( \frac{11}{96\omega^2} - \frac{565}{288\omega} + \frac{467}{36} - \frac{11225}{162} \omega + \frac{1629641}{3888} \omega^2 - \frac{1068433}{432} \omega^3 \right) + I_{10} \left( -\frac{11}{72\omega} - \frac{613}{216} - \frac{187}{324} \omega + \frac{234293}{8064} \omega^2 + \frac{1054147}{23328} \omega^3 \right) + I_{11} \left( -\frac{1}{6\omega^2} - \frac{1}{18\omega} - \frac{2143}{216} + \frac{5767}{144} \omega - \frac{2472821}{7776} \omega^2 + \frac{20840363}{11664} \omega^3 \right) + I_{12} \left( -\frac{11}{512\omega^3} + \frac{525}{1024\omega^2} - \frac{2601}{2304\omega} + \frac{577597}{27648} - \frac{4981223}{41472} \omega + \frac{46026487}{62208} \omega^2 \\
- \frac{1632171647}{373248} \omega^3 \right) + I_{13} \left( \frac{7}{96\omega^2} + \frac{13}{192\omega^2} + \frac{2375}{576} - \frac{95}{432} \omega + \frac{339883}{2592} \omega^2 + \frac{60055}{3888} \omega^3 \right) + I_{14} \left( \frac{35}{192\omega^2} - \frac{1051}{1152\omega^2} + \frac{29419}{3456} - \frac{220349}{486} \omega + \frac{264997}{11664} \omega^2 - \frac{31472155}{11664} \omega^3 \right) + I_{15} \left( \frac{1}{4\omega^2} + \frac{7}{36\omega^2} + \frac{6229}{432} + \frac{187}{27} \omega + \frac{1934333}{3888} \omega^2 + \frac{1580447}{11664} \omega^3 \right) + I_{16} \left( \frac{11}{96\omega^2} - \frac{1075}{576} + \frac{8179}{864} - \frac{10925}{192} \omega + \frac{2919977}{7776} \omega^2 - \frac{49397413}{23328} \omega^3 \right) + I_{17} \left( -\frac{17}{24\omega^2} + \frac{1429}{144\omega} - \frac{12223}{216} + \frac{146749}{432} \omega - \frac{15117413}{7776} \omega^2 \right) + I_{18} \left( -\frac{11}{512\omega^3} + \frac{1315}{1536\omega^2} - \frac{3107}{2304\omega} + \frac{292849}{6912} - \frac{206467}{4608} \omega + \frac{181740889}{124416} \omega^2 \\
- \frac{177573299}{93312} \omega^3 \right) \right].
\]

(19)
Similar expressions hold for the other graphs.

By using the above table Eq.s(2,3) are immediately obtained.

Graph 1a is ultraviolet divergent and requires renormalization. From the unrenormalized amplitude we obtain

$$a_e(1a; \text{not ren.}) = -\frac{1}{16\omega} + \frac{215}{24} \zeta(5) - \frac{1}{3} \pi^2 \zeta(3)$$

$$- \frac{53}{2160} \pi^4 + 4 \left[ a_4 + \frac{1}{24} \left( \ln^4 2 - \pi^2 \ln^2 2 \right) \right]$$

$$- \frac{1229}{144} \zeta(3) - \frac{7}{6} \pi^2 \ln 2 + \frac{3571}{2592} \pi^2 + \frac{133}{864} .$$  (20)

To obtain Eq.(1), one must account for the charge renormalization of the inserted 4th order vertex amplitude, to be carried out by means of a suitable subtraction constant, which will be indicated here as $Z$. For consistency with the previous analytic ($g$-2) calculations, $Z$ must correspond to on mass-shell renormalization. It turns out therefore that $Z$ is infrared divergent (this is the only part of the triple-cross ($g$-2) calculation in which infrared divergences appear); again for consistency with previous work, the infrared divergence is parametrized by giving to the photon a fictitious infinitesimal mass $\lambda$. The ultraviolet divergences of the counterterm are still parametrized by means of the $n$-dimensional regularization; the result for the renormalization counterterm of the inserted 4th order graph reads

$$Z = -\frac{1}{8\omega} - \frac{11}{24} \pi^2 + 2 - \ln \lambda .$$

In order to renormalize the contribution of graph 1a, one must subtract from Eq.(20) $F_2^{(2)}(0)Z$ where $F_2^{(2)}(0)$ is the second order (1 loop) magnetic form factor, to be evaluated in $n$-dimensions as well; as $Z$ has a $1/\omega$ singularity, use must be done of the value of $F_2^{(2)}(0)$ evaluated up to first order in $\omega$,

$$F_2^{(2)}(0) = \frac{1}{2} + 2\omega .$$

By subtracting $F_2^{(2)}(0)Z$ from Eq.(20), Eq.(1) is recovered.
Acknowledgements.
As the results presented in this note have been obtained by intensive use of the algebra manipulating programs FORM and ASHMEDAII, we want to express our gratitude to their authors, J. Vermaseren and M.J. Levine, for their help and advice in the early stages of the work.

References

[1] S. Laporta and E. Remiddi, Phys.Lett. B356, 390 (1995).
[2] M. J. Levine and J. Wright, Phys.Rev. D8, 3171 (1973).
[3] T. Kinoshita, Phys.Rev.Lett. 75, 4728 (1995).
[4] M. J. Levine, E. Remiddi and R. Roskies, in Quantum Electrodynamics, edited by T. Kinoshita, Advanced series on Directions in High Energy Physics, Vol. 7, (World Scientific, Singapore, 1990), 162, see p.214-216. Note that the result quoted there in p.215 for the graph $L_2$, which corresponds to Eq.(3) of R. Barbieri, M. Caffo and E. Remiddi, Nuovo Cimento Lett.5, 769 (1972), is incorrect in that context and should be replaced by Eq.(5) of that same reference, which reads $-\frac{293}{72} + \frac{19}{27}\zeta(2) + \frac{335}{144}\zeta(3)$.
We thank B. N. Taylor and P. Mohr for pointing out the oversight.
[5] S. Laporta and E. Remiddi, Phys.Lett. B265, 181 (1991).
[6] S. Laporta, Phys.Rev. D47, 4793 (1993).
[7] S. Laporta, Phys.Lett. B343, 421 (1995).
[8] T. Kinoshita, IEEE Trans. Instrum. Meas. 44, 498 (1995).
[9] M. E. Cage et al., IEEE Trans. Instrum. Meas. 38, 284 (1989).
[10] R. S. Van Dick, Jr., P. B. Schwinberg and H. G. Dehmelt, Phys.Rev.Lett. 59, 26 (1987).
[11] see Ref.[4], Sec. 2.1, p. 167.
[12] K. G. Chetyrkin and F. V. Tkachov, Nucl. Phys. B192, 159 (1981);
F. V. Tkachov, Phys.Lett. B100, 65 (1981).
We acknowledge also a discussion with D. Broadhurst on this point.