A Riccati equation based approach to isotropic scalar field cosmologies

Tiberiu Harko,1 Francisco S. N. Lobo,2 and M. K. Mak1

1Department of Mathematics, University College London, Gower Street, London WC1E 6BT, United Kingdom
2Centro de Astronomia e Astrofísica da Universidade de Lisboa, Campo Grande, Edifício C8, 1749-016 Lisboa, Portugal
3Department of Computing and Information Management, Hong Kong Institute of Vocational Education, Chai Wan, Hong Kong, P. R. China
(Dated: May 10, 2019)

Gravitationally coupled scalar fields $\phi$, distinguished by the choice of an effective self-interaction potential $V(\phi)$, simulating a temporarily non-vanishing cosmological term, can generate both inflation and late time acceleration. In scalar field cosmological models the evolution of the Hubble function is determined, in terms of the interaction potential, by a Riccati type equation. In the present work we investigate scalar field cosmological models that can be obtained as solutions of the Riccati evolution equation for the Hubble function. Four exact integrability cases of the field equations are presented, representing classes of general solutions of the Riccati evolution equation. The solutions correspond to cosmological models in which the Hubble function is proportional to the scalar field potential plus a linearly decreasing function of time, models with the time variation of the scalar field potential proportional to the potential minus its square, models in which the potential is the sum of an arbitrary function and the square of the function integral, and models in which the potential is the sum of an arbitrary function and the derivative of its square root, respectively. The cosmological properties of all models are investigated in detail, and it is shown that they can describe the inflationary or the late accelerating phase in the evolution of the Universe.

keywords: Riccati equation; isotropic scalar field; cosmology

PACS numbers: 04.50.+h, 04.20.Jb, 04.20.Cv, 95.35.+d

I. INTRODUCTION

The standard model of cosmology is remarkably successful in accounting for the observed features of the Universe. However, there remain a number of fundamental open questions at the foundation of the standard model. In particular, we lack a fundamental understanding of the acceleration of the late universe. In fact, the standard model of cosmology has favoured dark energy models, involving time-dependent scalar fields, as fundamental candidates responsible for the cosmic expansion. Indeed, scalar fields naturally arise in particle physics, including string theory, and in addition to this, the underlying dynamics in inflationary models depend essentially on a single scalar field, with the inflaton rolling in some underlying potential. A plethora of candidates exist for the scalar field potential proportional to the potential minus its square, models in which the Hubble function is proportional to the scalar field potential plus a linearly decreasing function of time, models with the time variation of the scalar field potential proportional to the potential minus its square, models in which the potential is the sum of an arbitrary function and the square of the function integral, and models in which the potential is the sum of an arbitrary function and the derivative of its square root, respectively. The cosmological properties of all models are investigated in detail, and it is shown that they can describe the inflationary or the late accelerating phase in the evolution of the Universe.

$H$ considered as a function of the scalar field $\phi$. In fact, a number of integrable one-scalar spatially flat cosmologies, which play a natural role in the inflationary scenarios, were studied in \cite{9}. Recently, a general method for the study of scalar field cosmologies, based on the reduction of the Klein-Gordon equation to a first order non-linear differential equation, was proposed in \cite{8}.

It is interesting to note that the possibility of describing the cosmological dynamics for a barotropic fluid in terms of a Riccati equation was discussed in \cite{3}. For a cosmological fluid satisfying an equation of state of the form $p = (\gamma - 1)\rho$, the Friedmann equations give for the scale factor the evolution equation in the conformal time $\eta$ written as $a''/a + (c - 1)(a'/a)^2 + ck = 0$, where $c = 3\gamma/2 - 1$, and $k = \pm 1$, and a prime denotes the derivative with respect to $\eta$. By introducing the transformation $u = a'/a$, we obtain for $u$ the Riccati type equation $u' + cu^2 + ck = 0$, which is easily integrable. The Riccati equation based study of the different properties of the isotropic FRW type cosmological models was performed in \cite{10,11}. In fact, the integrability conditions of the Riccati equation obtained in \cite{10,11} allow the integration of the structure equations of isotropic general relativistic compact objects \cite{12} in the context of general relativity. Furthermore, the applications of the Riccati equation to stellar and cosmological models have been extensively discussed in the literature, and we refer the reader to \cite{13–22}. Very recently, using the Chiellini type integrability condition for the generalized first kind Abel differential equation, consequently, the new class of exact
analytical solution of the Riccati type equation has been obtained in [23].

Therefore, the theoretical investigation of scalar field models is an essential task in cosmology. It is the purpose of the present paper to explore alternative approach in solving the cosmological gravitational field equations in the presence of self-interacting scalar fields, based on the mathematical analysis of the Riccati type equation that gives the Hubble function in terms of the scalar field potential. We present several cases of integrability of this equation, corresponding to specific forms of the scalar field potential, or of the Hubble function. The physical properties of the obtained solutions are analyzed in detail.

The present paper is organized as follows. The basic Riccati evolution equation for scalar field cosmologies with an arbitrary self-interaction potential is derived in Section II and four classes of exact scalar field solutions are obtained in Section III. We discuss and conclude our results in Section IV. Throughout this paper, we use natural units $c = 8\pi G = \hbar = 1$, and adopt as our signature for the metric $(+1, -1, -1, -1)$.

## II. THE RICCATI EVOLUTION EQUATION FOR THE HUBBLE FUNCTION: GENERAL FORMALISM

Consider in the Einstein frame the following Lagrangian density, which represents a general class of scalar field models, minimally coupled to the gravitational field,

$$L = \frac{1}{2} \sqrt{|g|} \left( R + [g^\mu\nu (\partial_\mu \phi) (\partial_\nu \phi) - 2V(\phi)] \right),$$

where $R$ is the curvature scalar, $\phi$ is the scalar field, and $V(\phi)$ is the self-interaction potential.

Assume a flat FRW scalar field dominated Universe given by the following the line element

$$ds^2 = dt^2 - a^2(t) (dx^2 + dy^2 + dz^2),$$

where $a$ is the scale factor. Thus, the evolution of a cosmological model is governed by the system of the field equations

$$3H^2 = \rho_\phi = \frac{\dot{\phi}^2}{2} + V(\phi),$$

$$2\dot{H} + 3H^2 = -p_\phi = -\frac{\dot{\phi}^2}{2} + V(\phi),$$

where $\rho_\phi$ is the energy density, and $p_\phi$ is the pressure due to the scalar field $\phi$, respectively, and the evolution equation for the scalar field

$$\ddot{\phi} + 3H \dot{\phi} + V'(\phi) = 0,$$

where $H = \dot{a}/a > 0$ is the Hubble expansion rate function. In the following the overdot denotes the derivative with respect to the time-coordinate $t$, and the prime denotes the derivative with respect to the scalar field $\phi$, respectively. By adding Eqs. (3) and (4), we obtain the Riccati type equation satisfied by $H$, of the form

$$\dot{H} = V(t) - 3H^2(t).$$

The deceleration parameter $q$, indicating the accelerating/decelerating nature of the cosmological expansion, is defined as

$$q(t) = \frac{d}{dt} \left( \frac{1}{H} \right) - 1 = -\frac{\dot{H}}{H^2} - 1 = 2 - \frac{V(t)}{H^2(t)}.\quad (7)$$

In the following Section, we will obtain several classes of exact solutions of the Riccati Eq. (6).

## III. EXACT SOLUTIONS OF THE RICCATI EVOLUTION EQUATION

### A. The case $H(t) = \alpha V(t) + g(t)/3\beta$

We assume that $H$ has the general form

$$H(t) = \alpha V(t) + \frac{1}{3\beta} g(t),$$

where $H(t)$, $V(t)$ and $g(t) \in C^\infty(I)$ are arbitrary functions defined on a real time interval $I \subseteq \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}$ are arbitrary constants. Hence Eq. (6) can be written as

$$\alpha \dot{V} + \frac{1}{3\beta} \ddot{g} = V - 3\alpha^2 V^2 - 2\frac{\alpha}{\beta^2} V - \frac{1}{3\beta^2} g^2.\quad (9)$$

1. The first class of solutions: $g(t) = \beta/t$

As a first example of an exact scalar field cosmological model we consider that the arbitrary function $g(t)$ satisfies the differential equation

$$\ddot{g} = -\frac{1}{\beta} g^2,$$

which gives for $g(t)$ the expression $g(t) = \beta/t$, where without loss of generality, an arbitrary constant of integration has been taken as zero. Then the scalar field potential satisfies the Bernoulli differential equation

$$V = \left( \frac{1}{\alpha} - \frac{2}{\beta} \right) V - 3\alpha V^2.\quad (11)$$

Therefore we have obtained the following theorem:

**Theorem 1.** If the self-interaction potential of a cosmological scalar field has the time dependence given by

$$V(t) = \frac{e^t/\alpha}{t \left[ V_0 t + 3t E_1 \left( \frac{1}{\alpha} \right) - 3ae^{t/\alpha} \right]},$$


where \( Ei(z) = -\int_{-z}^{\infty} e^{-t}dt/t \) is the exponential integral function \[29\], and \( V_0 \) is an arbitrary constant of integration, then the time variation of the Hubble function is obtained as

\[
H(t) = \left[ 3t - \frac{9\alpha e^{t/\alpha}}{V_0 + 3Ei\left(\frac{t}{\alpha}\right)} \right]^{-1}.
\]

Thus, the time variation of the scale factor is obtained as

\[
a(t) = a_0 \left\{ 3e^{t/\alpha} - t \frac{V_0 + 3Ei\left(\frac{t}{\alpha}\right)}{\alpha} \right\}^{1/3},
\]

where \( a_0 \) are arbitrary constants of integration. The time variations of the scalar field potential, of the scale factor, of the deceleration parameter, and the potential–scalar field dependence \( V = V(\phi) \) are presented, for different values of \( \alpha \), and for a fixed value of \( V_0 \), in Figs. 1-2.

The Universe starts its evolution from an accelerating phase, with \( q \approx -1.5 \), but enters, after a short time interval, into a decelerating expansionary phase, with \( q > 0 \). Hence the present model can describe the early inflationary phase in the evolution of the Universe. However, the accelerated expansion is not of de Sitter type \( (q = -1) \).

\[ \phi_{\pm}(t) - \phi_{0\pm} = \pm \sqrt{\frac{2}{3}} \int_{t}^{\infty} \sqrt{\frac{9\xi \text{Ei}\left(\frac{\xi}{\alpha}\right)^2 - 9\alpha e^{\xi/\alpha} \text{Ei}\left(\frac{\xi}{\alpha}\right) + 6V_0 \xi \text{Ei}\left(\frac{\xi}{\alpha}\right) + 9\alpha e^{\xi/\alpha} + V_0^2 \xi - 3V_0 \xi e^{\xi/\alpha}} \left[ 3\xi \text{Ei}\left(\frac{\xi}{\alpha}\right) - 3\alpha e^{\xi/\alpha} + V_0 \xi \right]^2 d\xi, \]

where \( \phi_{0\pm} \) are arbitrary constants of integration. The time variations of the scalar field potential, of the scale factor, of the deceleration parameter, and the potential–scalar field dependence \( V = V(\phi) \) are presented, for different values of \( \alpha \), and for a fixed value of \( V_0 \), in Figs. 1-2.

The second class of solutions

As a second example of an exact solution of the Riccati evolution equation for the Hubble function we consider the case in which the potential satisfies the Bernoulli differential equation

\[
\dot{V} = \frac{1}{\alpha} V - 3\alpha V^2.
\]

Hence the time dependence of the scalar field potential can be obtained as

\[
V(t) = \frac{e^{t/\alpha} V_0}{V_0 + 3\alpha^2 e^{t/\alpha}},
\]

where \( V_0 \) is an arbitrary constant of integration. By assuming that at \( t = 0 \), \( V(0) = V_{in} \), it follows that \( V_0 = 1/V_{in} - 3\alpha^2 \). Then the function \( g(t) \) must satisfy the following Bernoulli differential equation

\[
\frac{1}{3} \dot{g} + 2\alpha \frac{e^{t/\alpha}}{V_0 + 3\alpha^2 e^{t/\alpha}} g + \frac{1}{3\beta} g^2 = 0.
\]
FIG. 1: Variation of the scalar field potential (left plot), and of the scale factor (right plot), as a function of $t$ for the first solution of the Riccati equation, for different values of $\alpha$: $\alpha = -0.1$ (solid curve), $\alpha = -0.15$ (dotted curve), $\alpha = -0.20$ (short dashed curve), and $\alpha = -0.35$ (dashed curve), respectively. In all cases $V_0 = 12$, and $a_0 = 1$.

FIG. 2: Time variation of the deceleration parameter $q(t)$ (left plot), and of $V(\phi)$ (right plot), for the first solution of the Riccati equation, for different values of $\alpha$: $\alpha = -0.1$ (solid curve), $\alpha = -0.15$ (dotted curve), $\alpha = -0.20$ (short dashed curve), and $\alpha = -0.35$ (dashed curve), respectively. In all cases $V_0 = 12$.

**Theorem 2.** If the time dependence of the scalar field potential is given by Eq. (19), then the Hubble function $H(t)$ is obtained as

$$H(t) = \frac{1}{3} \left\{ \frac{1}{\beta V_0^2 g_0 - \alpha \ln |V_0 + 3\alpha^2 e^{t/\alpha}| + t}^{-1} + 3 (\alpha/V_0) e^{t/\alpha} \right\}^{-1},$$

(21)

where $g_0$ is an arbitrary constant of integration. The scale factor is given by

$$a(t) = a_0 \left[ 3\alpha^2 e^{t/\alpha} (\beta V_0^2 g_0 + t) - \alpha \left( V_0 + 3\alpha^2 e^{t/\alpha} \right) \times \ln |V_0 + 3\alpha^2 e^{t/\alpha}| + V_0 (\alpha + \beta V_0^2 g_0 + t) \right]^{1/3},$$

(22)

where we have denoted $a_0 = 1/V_0$. It is interesting to note that the scale factor $a(t)$ is non-singular at $t = 0$, and

$$\lim_{t \to 0} a(t) = a_0 \left[ V_0 (\alpha + 3\alpha^2 \beta V_0 g_0 + \beta V_0^2 g_0) - \alpha (3\alpha^2 + V_0) \ln |3\alpha^2 + V_0| \right]^{1/3}.$$
The deceleration parameter in this model is given by

\[ q(t) = -1 - 3 \left[ \frac{3a^{2/\alpha}}{V_0} - \frac{1}{(3a^2e^{2/\alpha} + V_0)}/(\beta g_0 V_0^2 - \alpha \ln[3a^2e^{2/\alpha} + V_0] + 2)^2 \right] \]

(24)

The time variation of the scalar field can be obtained from the equation \( \dot{\phi}_\pm(t) - \phi_0 \pm = \pm \sqrt{2} \int f \sqrt{3H_0^2}(\xi) - V(\xi)d\xi \), and the dependence of the potential on the scalar field is obtained in a parametric form.

In the limit of small times, from Eq. (19), it follows that \( V(t) \approx 1/(V_0 + 3a^2) = V_\text{ini} = \text{constant} \), while \( \lim_{t \to 0} H(t) \approx \left\{ \alpha + \left[ \beta V_0^2 g_0 - \alpha \ln[V_0 + 3a^2] \right]^{-1} + 3(\alpha/V_0) \right\}^{-1}/3 = H_0 = \text{constant} \). The deceleration parameter has an initial value given by

\[ \lim_{t \to 0} q(t) = -1 - \frac{9}{(3a^2 + V_0)}/(\beta g_0 V_0^2 - \alpha \ln[3a^2e^{2/\alpha} + V_0])^2 + \frac{1}{V_0} \]

(25)

and, depending on the numerical values of the free parameters of the model, a large number of initial states can be constructed. During the initial stages of the expansion, the scalar field is obtained as \( \phi_\pm(t) - \phi_0 \pm = \pm \sqrt{3H_0^2} - V_\text{ini}t \), and shows a linear increase in time during the early phases of the evolution of the Universe.

In the limit \( t \to \infty \), the scalar field potential tends to a constant, \( \lim_{t \to \infty} V(t) = 1/3a^2 \), while \( \lim_{t \to \infty} H(t) = 1/3a^2 \), and \( \lim_{t \to \infty} q(t) = -1 \), respectively. Therefore for this scalar field potential the Universe ends in a de Sitter-type exponential expanding phase. During its entire evolution, the Universe remains in an accelerating phase, with the deceleration parameter having negative values \( q \leq -1 \). Such a scalar field model may be used for the description of the late evolutionary stages of the expansion of the Universe, and could represent an effective dark energy model. Alternatively, from a cosmological point of view this model can be interpreted as describing an eternally inflating Universe.

**B. Exact solution of the field equations for**

\[ V(t) = f_1(t) + 3 \left[ \int f_1(\xi) d\xi + V_1 \right]^2 \]

We assume that the potential \( V(t) \) satisfies the integral condition

\[ V(t) = f_1(t) + 3 \left[ \int f_1(\xi) d\xi + V_1 \right]^2 - 3H^2. \]

(27)

Therefore we obtain the following:

**Theorem 3.** If the scalar field potential \( V(t) \) satisfies the integral condition (26), then the general solution of the Riccati equation (6) is given by

\[ H(t) = \frac{e^{-6V(t) - f^t f^t f_1(\xi) d\xi d\psi}}{C_1 + 3 \int e^{-6V(t) - f^t f^t f_1(\xi) d\xi d\psi} d\eta} \]

\[ + \int f_1(\xi) d\xi + V_1, \]

(28)

where \( C_1 \) is an arbitrary constant of integration.

Equation (28) can be immediately integrated to give the scale factor in the form

\[ a(t) = a_0 e^{V(t) + f^t f^t f_1(\xi) d\xi} \times \left[ C_1 + 3 \int e^{-6V(t) - f^t f^t f_1(\xi) d\xi d\psi} d\eta \right]^{1/3} \]

(29)

where \( a_0 \) is an arbitrary constant of integration. The deceleration parameter \( q \) takes the form

\[ q(t) = 2 - \left[ f_1(t) + 3 \left( \int f_1(\xi) d\xi + V_1 \right) \right] \times \left[ e^{-6V(t) - f^t f^t f_1(\xi) d\xi d\psi} \right] \]

\[ 1/C_1 + 3 \int e^{-6V(t) - f^t f^t f_1(\xi) d\xi d\psi} d\eta \]

\[ + \int f_1(\xi) d\xi + V_1 \]

(30)
The scalar field \( \phi(t) \) can be written as
\[
\phi(t) = \phi_{0\pm} \pm \sqrt{2} \int_{t_0}^{t} \left\{ -f_1(\zeta) - 3 \left[ f_1(\xi) d\xi + V_1 \right]^2 
\right. 
\left. + 3 \left[ e^{-6\eta_1(\zeta)} \int e^{-6\eta} f_1(\xi) d\xi d\eta \n\right. 
\left. + \int_{\zeta}^{\infty} f_1(\xi) d\xi + V_1 \right]^2 \right\}^{1/2} d\zeta,
\]
where \( \phi_{0\pm} \) are arbitrary constants of integration.

Note that the physical behavior of the Universe is determined by the choice of the arbitrary function \( f_1(t) \).
The potential \( V(\phi) \) can be uniquely determined in a parametric form from Eqs. (26) and (31), with the time \( t \) taken as a parameter.

As a simple application of the integrability condition of the gravitational field equations given by Theorem 3 we consider the particular case for which \( f_1(t) = f_0 = \text{constant} > 0 \). Moreover, for simplicity, we also assume \( V_1 = 0 \). Therefore, in this model, the time dependence of the scalar field potential is given by
\[
V(t) = 3f_0^2t^2 + f_0.
\]

For the Hubble function we obtain the expression
\[
H(t) = \frac{2\sqrt{f_0 e^{-3f_0t^2}}}{2H_1\sqrt{f_0} + \sqrt{3\pi} \text{erf}(\sqrt{3f_0t})} + f_0t,
\]
where \( H_1 \) is an arbitrary constant of integration, and \( \text{erf}(z) = (2/\sqrt{\pi}) \int_{0}^{z} \exp(-t^2) dt \) is the error function, giving the integral of the Gaussian distribution [20].
The scale factor \( a(t) \) can be obtained as
\[
a(t) = a_0 e^{\frac{f_0t}{f_0}} \left[ H_1 + \frac{1}{2} \sqrt{\frac{3\pi}{f_0}} \text{erf}(\sqrt{3f_0t}) \right]^{1/3}.
\]
The deceleration parameter is given by
\[
q(t) = -1 + \frac{1}{f_0t^2} \times
\left\{ \frac{4}{\sqrt{f_0 e^{-3f_0t^2}}} \left[ 2H_1\sqrt{f_0} + \sqrt{3\pi} \text{erf}(\sqrt{3f_0t}) \right] + 2 - \frac{4}{\sqrt{f_0 e^{-3f_0t^2}}} \left[ 2H_1\sqrt{f_0} + \sqrt{3\pi} \text{erf}(\sqrt{3f_0t}) \right] + 2 \right\}^{1/2}.
\]
The time dependence of the scalar field can be obtained in an integral form as
\[
\phi(t) = \phi_0 \pm \sqrt{2} \int_{t_0}^{t} \left[ -3f_0^2\zeta^2 - 3f_0 \times
\left( \frac{2\sqrt{f_0 e^{-3f_0t^2}}}{2H_1\sqrt{f_0} + \sqrt{3\pi} \text{erf}(\sqrt{3f_0t})} + f_0\zeta \right) \right]^{1/2} d\zeta.
\]

Eqs. (32) and (33) give the functional dependence of the scalar field potential \( V \) of the scalar field \( \phi \) in a parametric form, with \( t \) taken as parameter.

In the small time limit the Hubble parameter can be approximated as
\[
H(t) \approx \left( f_0 - \frac{3}{H_1^2} \right) t + \frac{1}{H_1},
\]
while in the same order of approximation the expression \( \sqrt{3H^2 - V} \) can be obtained as
\[
\sqrt{3H^2 - V} \approx \frac{\sqrt{3 - f_0H_1^2(H_1 - 3t)}}{H_1}.
\]

Therefore in the small time limit for the scalar field evolution we find
\[
\phi(t) \approx \frac{\sqrt{3 - f_0H_1^2(H_1 - 3t)}}{H_1^2}.
\]

In the first order we obtain the time-scalar field dependence as
\[
t(\phi) \approx \frac{H_1 \phi}{\sqrt{3 - f_0H_1^2}}.
\]

which gives the scalar field potential as a function of \( \phi \) in the form
\[
V(\phi) \approx \frac{3f_0^2H_1^2}{3 - f_0H_1^2}\phi^2 + f_0.
\]

Quadratic potentials have been extensively investigated in the recent literature [31–33], and they allow to recover the connection with particle physics. The effective mass of the scalar field is given by \( m_\phi^2 = 3f_0^2H_1^2/2(3 - f_0H_1^2) \). Moreover, the constant term in the potential naturally generates a cosmological constant. The early time evolution of the deceleration parameter can be approximated as
\[
q(t) \approx -3f_0\left( f_0^2H_1^4 - 3f_0H_1^2 + 3 \right)t^2 + 2f_0H_1(3f_0H_1^2 - 3)t - f_0H_1^2 + 2.
\]

If \( f_0H_1^2 > 2 \), the Universe starts its evolution from an accelerating phase. Hence the present model can describe a generalized effective power law type scalar field potential, which in the small time limit reduces to the quadratic potential.

C. Exact solution of the field equations for
\[
V_\pm(t) = 3f_2(t) \pm \frac{d}{dt} \sqrt{f_2(t)}
\]

We assume that the potential \( V_\pm(t) \) satisfy the differential condition
\[
V_\pm(t) = 3f_2(t) \pm \frac{d}{dt} \sqrt{f_2(t)}.
\]
where we have introduced a new arbitrary function $f_2(t) \in C^\infty(I)$ defined on a real interval $I \subseteq \mathbb{R}$. By inserting Eq. (43) into Eq. (6), the latter takes the form

$$\frac{dH_\pm}{dt} = 3f_2(t) \pm \frac{d}{dt}\sqrt{f_2(t)} - 3H_\pm^2.$$  

(44)

Therefore we have obtained the following:

**Theorem 4.** If the scalar field potential $V(t)$ satisfies the differential condition (43), then the general solutions where we have introduced a new arbitrary function $f_2(t) \in C^\infty(I)$ defined on a real interval $I \subseteq \mathbb{R}$. By inserting Eq. (43) into Eq. (6), the latter takes the form

$$\frac{dH_\pm}{dt} = 3f_2(t) \pm \frac{d}{dt}\sqrt{f_2(t)} - 3H_\pm^2.$$  

(44)

Therefore we have obtained the following:

**Theorem 4.** If the scalar field potential $V(t)$ satisfies the differential condition (43), then the general solutions are given by

$$H_\pm(t) = \frac{e^{\pm f_t \sqrt{f_2(\phi)}}}{C_1 \pm 3 \int_c^t e^{\pm f_\phi \sqrt{f_2(\phi)}}d\eta} \pm \sqrt{f_2(t)},$$  

(45)

where $C_1 \pm$ are arbitrary constants of integration. Equation (45) can be integrated to give the form

$$a_\pm(t) = a_{0\pm} e^{\pm f_t \sqrt{f_2(\phi)}} \times \left[ C_1 \pm + 3 \int_c^t e^{\pm f_\phi \sqrt{f_2(\phi)}}d\eta \right]^{1/3},$$  

(46)

where $a_{0\pm}$ are arbitrary constants of integration.

With the help of Eqs. (6), (15), and (16), respectively, the deceleration parameter $q$ can be written as

$$q_\pm(t) = 2 - \frac{3f_2(t) \pm \frac{d}{dt}\sqrt{f_2(t)}}{\frac{e^{\pm f_t \sqrt{f_2(\phi)}}}{C_1 \pm 3 \int_c^t e^{\pm f_\phi \sqrt{f_2(\phi)}}d\eta} \pm \sqrt{f_2(t)}}^2.$$  

(47)

The scalar field $\phi(t)$ can be written as

$$\phi_\pm(t) = \phi_{0\pm} + \sqrt{2} \int_c^t \left\{ -3f_2(\phi) + \frac{d}{d\phi}\sqrt{f_2(\phi)} + \frac{3}{2} \left[ \frac{e^{\pm f_\phi \sqrt{f_2(\phi)}}}{C_1 \pm 3 \int_c^t e^{\pm f_\phi \sqrt{f_2(\phi)}}d\eta} \pm \sqrt{f_2(\phi)} \right] \right\} \frac{1}{2} d\phi.$$  

(48)

where $\phi_{0\pm}$ are arbitrary constants of integration.

As an example of the application of Theorem 4 we consider the case in which the function $f_2(t)$ has the form $f_2(t) = f_{02}/t^2$, with $f_{02} = \text{constant} > 0$. In this case the scalar field potential takes the form

$$V_\pm(t) = \frac{V_{0\pm}}{t^2},$$  

(49)

where for simplicity, we have introduced the arbitrary constants $V_{0\pm}$ defined as $V_{0\pm} = 3f_{02} \mp \sqrt{f_{02}}$. The Hubble function can then be obtained immediately either from Eqs. (16), (17), (49) or from Eq. (45) as

$$H_\pm(t) = \frac{1}{6t} \left[ 1 + V_{0\pm} \left( 1 - \frac{2H_{1\pm}}{H_{1\pm} + V_{1\pm}} \right) \right],$$  

(50)

where $H_{1\pm}$ are arbitrary constants of integration, and $V_{1\pm} = \sqrt{2V_0 \mp 1}$. For the scale factor we obtain

$$a_\pm(t) = a_{0\pm} t^{(1-V_{1\pm})/6} \left( H_{1\pm} + V_{1\pm} \right)^{1/3},$$  

(51)

where $a_{0\pm}$ are arbitrary constants of integration. The deceleration parameter is given by

$$q_\pm(t) = \frac{4 \left[-30H_{1\pm}V_{1\pm}tV_{1\pm} - H_{1\pm}^2q_{1\pm} + q_{2\pm}tV_{1\pm} \right]}{\left[H_{1\pm} (1 - V_{1\pm}) + (V_{1\pm} + 1) tV_{1\pm} \right]^2},$$  

(52)

where $q_{1\pm} = 3V_{0\pm} + V_{1\pm} - 1$, and $q_{2\pm} = -3V_{0\pm} + V_{1\pm} + 1$. The scalar field $\phi(t)$ can be written as

$$\phi_\pm(t) = \phi_{0\pm} \pm \frac{1}{\sqrt{6}} \times \int_c^t \sqrt{\left[ V_{1\pm} \left( \frac{2H_{1\pm}}{H_{1\pm} + \sqrt{V_{1\pm}}} - 1 \right) \right]^2 - 12V_{0\pm} \frac{d\phi}{\sqrt{V_{1\pm}}}.$$  

(53)

Note that the integral on the right hand side of Eq. (53) can be evaluated exactly with a very complicated expression. However in order to have a concise representation we keep the integral form of the scalar field.

In the following we restrict our analysis to the case $V_{0\pm} = -1/12$, corresponding to the value $f_{02} = 1/36$. In this case we obtain a complete particular solution of the gravitational field equations describing the time evolution of the flat FRW Universe with the self interaction potential $V_+(\phi)$, given by

$$H_+(t) = \frac{1}{6t}, a_+(t) = a_{0+} t^{1/6}, q_+ = 5,$$

$$\phi_+(t) = \phi_{0+} + \ln |t| \sqrt{3}, V_+(\phi) = V_{2+} e^{-2\sqrt{3}\phi},$$  

(54)

where for simplicity, we have denoted $V_{2+} = -e^{2\sqrt{3}\phi}/12$. Thus we have regained the simple power law solution for the cosmological model with the potential expressed as the exponential function of the scalar field (34). This solution represents a decelerating cosmology, with $q > 0$, and it may be useful for the description of the post-inflationary decelerating phase of the early Universe, or during the reheating period.

**IV. DISCUSSIONS AND FINAL REMARKS**

In the present paper, we have shown that the time evolution and dynamics of the Hubble function in scalar field cosmologies can be formulated in terms of solutions of the first order Riccati equation, with the cosmological dynamics entirely determined by the time variation of the scalar field potential. This equation immediately leads to the identification of some classes of scalar field potentials which the field equations can be solved exactly, and it allows the formulation of very general integrability conditions. We have obtained the complete solution of the
gravitational field equations describing the time evolution of the flat FRW Universe in the presence of the scalar field $\phi(t)$ for four functional forms of the self-interaction potential $V$. The first two solutions are obtained for fixed forms of the scalar field potential, while in the last two solutions the form of the potential is arbitrary, and determined by a general integrability condition. The integrability conditions determine the allowed form of the scalar field self-interaction potential in terms of some arbitrary time dependent functions $f_1(t)$ and $f_2(t)$, respectively, thus leading to the possibility of constructing very general solutions of the field equations, and to reconstruct easily the Hubble function, once the evolution of the potential is given.

In conclusion, we have obtained several exact solutions of the gravitational field equations in the presence of a scalar field. In order to obtain a deeper physical understanding of the solutions comparisons with the observational data are necessary. Work under these lines is presently underway, and the results will be presented in a future publication.

Acknowledgments

We would like to thank the anonymous referee for comments and suggestions that helped us to significantly improve our manuscript. FSNL is supported by a Fundação para a Ciência e Tecnologia Investigador FCT Research contract, with reference IF/00859/2012, funded by FCT/MCTES (Portugal). FSNL also acknowledges financial support of the Fundação para a Ciência e Tecnologia through the grants CERN/FP/123615/2011 and CERN/FP/123618/2011. MKM acknowledges financial support from the Vocational Training Council, Hong Kong.

[1] D. H. Weinberg, M. J. Mortonson, D. J. Eisenstein, C. Hirata, A. G. Riess, and E. Rozo, Physics Reports 530, 87 (2013).
[2] B. A. Bassett, S. Tsujikawa, and D. Wands, Rev. Mod. Phys. 78, 537 (2006).
[3] E. J. Copeland, M. Sami and S. Tsujikawa, Int. J. Mod. Phys. D 15, 1753 (2006).
[4] A. G. Muslimov, Class. Quantum Grav. 7, 231 (1990).
[5] D. S. Salopek and J. R. Bond, Phys. Rev. D 42, 3936 (1990).
[6] P. Fre, A. Sagnotti, and A. S. Sorin, to appear in Nucl. Phys. B, [arXiv:1307.1910] (2013).
[7] P. Fre, A.S. Sorin, and M. Trigiante, [arXiv:1310.5340] (2013).
[8] T. Harko, F. S. N. Lobo, and M. K. Mak, Eur. Phys. J. C 74, 2784 (2014).
[9] V. Faraoni, Am. J. Phys. 67, 732 (1999).
[10] H. C. Rosu, Mod. Phys. Lett. A 13, 227 (1998).
[11] H. C. Rosu, Mod. Phys. Lett. A 17, 667 (2002).
[12] H. C. Rosu and P. Ojeda-May, Int. J. Theor. Phys. 45, 1191 (2006).
[13] H. C. Rosu and K. V. Khmelnitskaya, SIGMA 7, 013 (2011).
[14] H. C. Rosu and K. V. Khmelnitskaya, Mod. Phys. Lett. A 28, 1340017 (2013).
[15] M. K. Mak and T. Harko, Applied Mathematics and Computation 218, 10974 (2012).
[16] M. K. Mak and T. Harko, Applied Mathematics and Computation 219, 7465 (2013).
[17] M. K. Mak and T. Harko, The European Physical Journal C 73, 2585 (2013).
[18] T. Harko and M. K. Mak, J. Math. Phys. 41, 4752 (2000).
[19] M. K. Mak, J. A. Belinchon and T. Harko, Int. J. Mod. Phys. D 11, 1265 (2002).
[20] Chiang-Mei Chen, T. Harko and M. K. Mak, Phys. Rev. D64, 124017 (2001).
[21] M. K. Mak and T. Harko, Europhysics Letter. 56, 762 (2001).
[22] T. Harko, F. S. N. Lobo, and M. K. Mak, Universal Journal of Applied Mathematics, 2, 109 (2014).
[23] T. Harko, F. S. N. Lobo, and M. K. Mak, Universal Journal of Applied Mathematics, 1, 101 (2013).
[24] J.D. Barrow, Phys. Lett. B 187, 12 (1987).
[25] A. B. Burd and J. D. Barrow, Nucl. Phys. B 308, 929 (1988).
[26] L. P. Chimento, Class. Quant. Grav. 15, 965 (1998).
[27] J. G. Russo, Phys. Lett. B 600, 185 (2004).
[28] C. Rubano, P. Scudellaro, E. Piedipalumbo, S. Capozziello, and M. Capone, Phys. Rev. D 69, 103510 (2004).
[29] A. D. Polyanin and V. F. Zaitsev, Handbook of exact solutions for ordinary differential equations, CRC Press, Boca Raton, New York, London, Tokyo (1995).
[30] J. Ellis, M. A. G. Garcia, D. V. Nanopoulos, and K. A. Olive, [arXiv:1403.7518] (2014).
[31] L. N. Granda, JCAP 1104, 016 (2011).
[32] L. Arturo Urena-Lopez and M. J. Reyes-Ibarra, Int. J. Mod. Phys. D 18, 621 (2009).
[33] M. H. Dehghani, J. Pakravan, and S. H. Hendi, Phys. Rev. D 74, 104014 (2006).
[34] V. Gorini, A. Y. Kamenshchik, U. Moschella and V. Pasquier, Phys. Rev. D 69, 123512 (2004).