THE DIRICHLET PROBLEM FOR PRESCRIBED CURVATURE EQUATIONS OF $p$-CONVEX HYPERSURFACES

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Abstract. In this paper, we study the Dirichlet problem for $p$-convex hypersurfaces with prescribed curvature. We prove that there exists a graphic hypersurface satisfying the prescribed curvature equation with homogeneous boundary condition. An interior curvature estimate is also obtained.

Keywords: Dirichlet problem; $C^2$ estimates; interior curvature estimates, $p$-convex hypersurfaces.

1. Introduction

In this paper, we are interested in finding a graphic hypersurface of prescribed curvature on a bounded domain $\Omega \subset \mathbb{R}^n$. Suppose that the hypersurface $M_u := \{(x, u(x)) : x \in \Omega \} \subset \mathbb{R}^{n+1}$ is given by the graph of a smooth function $u : \Omega \to \mathbb{R}$. Denote by $\kappa[M_u] = (\kappa_1, \cdots, \kappa_n)$ the principal curvatures of $M_u$ with respect to the downward unit normal of $M_u$. Given an integer $p$, where $1 \leq p \leq n$, it is interesting in geometry analysis to find a graphic hypersurface $M_u$ with its principal curvatures satisfying the following equation

$$(1.1) \quad \Pi_{1 \leq i_1 < \cdots < i_p \leq n}(\kappa_{i_1} + \cdots + \kappa_{i_p}) = f(x, u, \nu) \text{ in } \Omega,$$

and with the prescribed homogeneous boundary condition as below

$$(1.2) \quad u = 0 \text{ on } \partial \Omega,$$

where $\nu$ is the upward unit normal vector to the graphic hypersurface at $X := (x, u(x))$ and $f(x, z, \nu) > 0$ is a smooth function defined on $\Omega \times \mathbb{R} \times \mathbb{R}^n$. For convenience, we denote that $M^p_u := \Pi_{1 \leq i_1 < \cdots < i_p \leq n}(\kappa_{i_1} + \cdots + \kappa_{i_p})$. We also simply write $M_u$ when there is no ambiguity.

The equation (1.1) is a fully nonlinear partial differential equation for $p < n$. It is natural to consider the problem in the class of $p$-convex hypersurfaces so that the equation is elliptic. Recall that a $C^2$ regular hypersurface $M_u$ is called (strictly) $p$-convex if $\kappa[M_u](x)$ satisfies, at each point $x \in \Omega$, that

$$\kappa_{i_1} + \cdots + \kappa_{i_p} \geq (> ) 0, \forall 1 \leq i_1 < \cdots < i_p \leq n.$$

Accordingly, we say that the $C^2$ function $u$ is admissible. The notion of $p$-convexity can be dated back to Wu [51]. Sha in [41, 42] studied Riemannian manifolds with $p$-convex boundaries and proved a complete characterization for such Riemannian manifolds. Recently, there were some new discoveries in $p$-convex geometry made by Harvey and Lawson. We refer the reader to [23, 24] for these results and also to a nice survey [25].

There is a vast literature on the existence of closed Weingarten hypersurfaces of codimension one in the Euclidean space. The Gaussian curvature case corresponding to $p = 1$ in (1.1) was studied by Oliker [36]. The mean curvature equation
corresponding to \( p = n \) in \[ (1.1) \] was studied by Bakelman-Kantor \[ 1 \] and Treibergs-Wei \[ 47 \]. Caffarelli-Nirenberg-Spruck in \[ 5 \] was concerned with prescribed curvature equations of very general type, and it was then generalized to Riemannian manifolds by Gerhardt in \[ 13 \]. The existence of a starshaped \((n-1)\)-convex hypersurface of prescribed curvature \( (1.1) \) was proved by Chu-Jiao \[ 5 \]. The case \( p \geq \frac{n}{2} \) was explored in a subsequent work by the author in \[ 10 \]. In complex settings, when \( p = n-1 \), the operator appears in the Gauduchon conjecture which was solved by Székelyhidi-Tosatti-Weinkove \[ 44 \]. Some previous work on this topic can be found in Tosatti-Weinkove \[ 45, 46 \] and Fu-Wang-Wu \[ 11, 12 \]. A counterpart to \( (1.1) \) that was studied extensively is the higher order mean curvature equation

\[
(1.3) \quad \sigma_k(\kappa) = f(X, \nu(X)), \quad \forall X \in M,
\]

where \( 1 \leq k \leq n \) and \( \sigma_k \) is the \( k \)-th elementary symmetric function. In warped product manifolds, Chen-Li-Wang \[ 7 \] established the curvature estimate for convex hypersurfaces satisfying \( (1.3) \). Jin-Li \[ 30 \] and Li-Oliker \[ 33 \] investigated the star-shaped hypersurfaces in hyperbolic and elliptic space respectively. For the study of many authors, due to its role in solving a Plateau type problem for locally convex Weingarten hypersurfaces. For its own sake, there still are many works, such as Caffarelli-Nirenberg-Spruck \[ 6 \], Lions \[ 35 \], Ivochkina \[ 26, 27 \], Ivochkina-Lin-Trudinger \[ 28 \], Lin-Trudinger \[ 34 \] and Trudinger \[ 48 \]. We remark that in \[ 28 \] the Dirichlet problem with non-homogeneous boundary data was considered for smooth solutions and in \[ 48 \] it was investigated in the context of viscosity solutions. In this paper, we are focused on the Dirichlet problem of homogeneous boundary data. We shall return to the non-homogeneous boundary case in future work. Now, we recall the \( p \)-convex cones introduced by Harvey and Lawson \[ 25 \].

**Definition 1.1.** Let \( p \in \{1, \cdots, n\} \). The \( \mathcal{P}_p \) cone is a subset in \( \mathbb{R}^n \) with element \((\lambda_1, \cdots, \lambda_n)\) such that \( \lambda_1 + \cdots + \lambda_p > 0 \) for all \( 1 \leq i_1 < \cdots < i_p \leq n \). The \( \mathcal{P}_p \) cone of symmetric \( n \times n \) matrices is defined as: \( A \in \mathcal{P}_p \) if the \( n \)-tuple of the eigenvalues of matrix \( A \) is in \( \mathcal{P}_p \). We call \( A \) \( p \)-positive if \( A \in \mathcal{P}_p \).

Denote by \((\kappa_1^b, \cdots, \kappa_{n-1}^b)\) the principal curvatures of the boundary \( \partial \Omega \) taken with respect to the exterior unit normal to \( \partial \Omega \) (see Section 14.6 in [14]). We call \( \Omega \) is strictly convex if that \((\kappa_1^b(x), \cdots, \kappa_{n-1}^b(x)) \in \mathcal{P}_1 \subset \mathbb{R}^{n-1} \) for every \( x \in \partial \Omega \). Now, we state our main result.

**Theorem 1.2.** Let \( \Omega \) be a strictly convex bounded domain in \( \mathbb{R}^n \) with smooth boundary \( \partial \Omega \) and \( f = f(x, z, \nu) \in C^\infty(\Omega \times \mathbb{R} \times S^n) \) be a positive function with \( f_z \geq 0 \). Suppose that \( p > \frac{n}{2} \). Assume that there is an admissible subsolution \( u \in C^2(\Omega) \) satisfying

\[
(1.4) \quad M_p^{\nu} \geq f(x, u, \nu) \text{ in } \Omega \text{ and } u = 0 \text{ on } \partial \Omega,
\]

where \( \nu \) denotes the upward unit normal vector to the graphic hypersurface \( M_\nu \) at \( X := (x, u(x)) \). Then there exists a unique admissible solution \( u \in C^\infty(\Omega) \) to \( (1.1) \) and \( (1.2) \).
Remark 1.3. The endpoint case $p = n$ is excluded from our discussion, since this case belongs to the realm of quasilinear elliptic equations. The case $p = 1$ is also well known. Jiao-Sun [29] proved the theorem for $p = n - 1$ recently. It remains an interesting question to prove the theorem for $1 < p < \frac{n}{2}$.

The approach to solve the fully nonlinear elliptic equation (1.1) is the well-known continuity method and degree theory. So the main task of this paper is to establish a priori estimates for admissible solutions up to second order derivatives as higher order estimates follow from the Evans-Krylov theorem and Schauder theory. By the assumption of the existence of a subsolution, $C^0$ and $C^1$ estimates are easy to derive. The most difficult part is to establish a priori $C^2$ estimates, including the global $C^2$ estimate and the boundary $C^2$ estimate. Moreover, since $f$ depends on $\nu$, the bad term $-Cv_{11}$ appears in the inequality (3.9) when applying the maximum principle to the test function $G$ in Section 3, which arise from differentiating the equation (1.1) twice (see (3.8)). This also cause trouble for us to deal with third order terms. To overcome these difficulties, one can impose various assumptions on $f$ as in [26, 27, 16, 17]. Due to the special nature of the operator of equation (1.1), we use idea from [8, 10] to control $-Cv_{11}$ by good terms without imposing any extra conditions on $f$. Then the bad third order terms can be eliminated by the good term $-F_{1i}$. See Lemma 2.4 and Lemma 2.5. As to the boundary $C^2$ estimate, the issue is to estimate the tangential-normal derivatives of solutions. Fortunately, we can use idea from Ivochkina [26] to tackle this. Actually, we have enough good third order terms to derive an interior curvature estimate.

**Theorem 1.4.** Let $\Omega$ be a bounded domain in $\mathbb{R}^n$. Suppose that $u \in C^4(\Omega) \cap C^2(\bar{\Omega})$ is an admissible solution to (1.1) and (1.2). Then there is a positive constant $C$ and $\beta$ such that the second fundamental form $h$ of $M_u$ satisfies

$$\sup_{\Omega} (-u)^\beta |h| \leq C,$$

where $C$ and $\beta$ depend on $n$, $p$, $|u|_{C^1}$, $\inf \hat{f}$ and $|f|_{C^2}$.

Some applications of the interior curvature estimate can be found in Sheng-Urbas-Wang [43]. The above theorem holds for $p \geq \frac{n}{2}$ if $f \equiv f(x,u,\nu)$ and for $1 \leq p \leq n$ if $f \equiv f(x,u)$ as explained below. The assumption $p \geq \frac{n}{2}$ in Theorem 1.2 is only used to prove Lemma 2.4 to control $-Cv_{11}$ in the inequality (3.10) (and in (3.21) for the interior curvature estimate). If the right hand side function $\tilde{f}$ in the equation (1.1) does not depend on $\nu$, the term $-Cv_{11}$ does not appear in (3.10) (also not appear in (3.21)), for which we refer the reader to examine the inequalities (3.8) and (3.9). Therefore, we have the following result.

**Theorem 1.5.** Let $\Omega$ be a strictly convex bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial \Omega$ and $f = f(x,z) \in C^\infty(\bar{\Omega} \times \mathbb{R})$ be a positive function with $f_z \geq 0$. Suppose $1 \leq p \leq n$. Assume that there is an admissible subsolution $\underline{u} \in C^2(\bar{\Omega})$ satisfying

$$M^p \geq f(x,\underline{u}) \text{ in } \Omega \text{ and } \underline{u} = 0 \text{ on } \partial \Omega,$$

where $\nu$ denotes the upward unit normal vector to the graphic hypersurface $M_\underline{u}$ at $X := (x, \underline{u}(x))$. Then there exists a unique admissible solution $u \in C^\infty(\Omega)$ to (1.1) and (1.2).
Before we end the introduction, let’s review a conjecture on curvature estimates for hypersurfaces with prescribed higher order mean curvature as it is closely related to our topic. The Gaussian curvature and the mean curvature correspond to $k = n$ and $k = 1$ in (1.3) respectively. Ren-Wang conjectured that the curvature estimate holds for $k > \frac{n}{2}$ in [38, 39]. While the case $k = n$ proved a long time ago by Caffarelli-Nirenberg-Spruck [3], it was only recently that Ren-Wang [37, 38] proved the conjecture for $k \geq n - 2$. Guan-Ren-Wang in [21] established the curvature estimate when $k = 2$, and in the same paper they also proved the curvature estimate for convex solutions of equation (1.3) when $3 \leq k \leq n$.

The paper is organized as follows. In Section 2, we recall some properties of the operator from [9] and prove some key lemmas which are crucial to the global and boundary $C^2$ estimate. Then, we prove the global and boundary $C^2$ estimate in Section 3 and Section 4 respectively. In Section 5, we provide a proof of the gradient estimate.

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2. Preliminaries

Let $M$ be a hypersurface in $\mathbb{R}^{n+1}$ given by the graph of $u : \Omega \to \mathbb{R}$. Then the induced metric $g$ and second fundamental form $h$ of $M$ are given by

$$g_{ij} = \delta_{ij} + u_i u_j$$

and

$$h_{ij} = \frac{u_{ij}}{\sqrt{1 + |Du|^2}}.$$

The upward unit normal vector is

$$\nu = \frac{(-Du, 1)}{\sqrt{1 + |Du|^2}}.$$

The principal curvatures $\kappa = (\kappa_1, \cdots, \kappa_n)$ of the hypersurface $M$ are the eigenvalues of $h_{ij}$ with respect to $g_{ij}$, i.e. the eigenvalues of the following matrix

$$\frac{1}{w} \left( I - \frac{Du \otimes Du}{w^2} \right) D^2 u,$$

where $w = \sqrt{1 + |Du|^2}$ and $g^{ij} = \delta_{ij} - \frac{u_i u_j}{w^2}$ is the inverse of $g_{ij}$. Note that the above matrix generally is not symmetric. As in [6], we consider the following matrix

$$a_{ij} = \frac{1}{w} \gamma_{ik} u_{kl} \gamma_{lj},$$

where $\gamma^{ik} = \delta_{ik} - \frac{u_i u_k}{w(1+w)}$ is the inverse matrix of $\gamma_{ij} = \delta_{ij} + \frac{u_i u_j}{1+w}$ and $\gamma_{ij}$ is the square root of $g_{ij}$.

For convenience, we introduce the following notations

$$F(a_{ij}) := F(\kappa) = \Pi_{1 \leq i_1 < \cdots < i_p \leq n} (\kappa_{i_1} + \cdots + \kappa_{i_p})$$

and $\tilde{F} = F \frac{1}{C_p}$. Equation (1.1) then can be written as

$$\tilde{F}(a_{ij}) := \tilde{F}(\kappa) = \tilde{f}(x, u, \nu),$$

where $\kappa = (\kappa_1, \cdots, \kappa_n)$ and $\tilde{f} = f \frac{1}{C_p}$. It is also convenient to rewrite the equation as below

$$G(D^2 u, Du) := \tilde{F}(a_{ij}) = \tilde{f},$$
where \( a_{ij} \) is defined in (2.2). Denote
\[
F^{ij} = \frac{\partial F}{\partial a_{ij}}, \quad F^{ij,kl} = \frac{\partial^2 F}{\partial a_{ij} \partial a_{kl}}, \quad G^{ij} = \frac{\partial G}{\partial a_{ij}}, \quad \sigma^i = \frac{\partial \sigma}{\partial a_i}.
\]

We remark that \( G \) is concave and homogeneous one with respect to \( D^2w \) since \( \bar{F} \) is concave and homogeneous one with respect to \( a_{ij} \). See [9]. The equation is elliptic as the matrix \( \{ \frac{\partial \bar{F}}{\partial a_{ij}} \} \) is positive definite for \( \{a_{ij}\} \in P_p \). It is easy to verify that
\[
\frac{1}{w} \sum_i \bar{F}^{ii} \geq \sum_i G^{ii} \geq \frac{1}{w^3} \sum_i \bar{F}^{ii}.
\]

By a straightforward calculation, one get the following lemma. See formula (2.21) in Lemma 2.3 of [18].

**Lemma 2.1 ([18]).** We have
\[
G^s = -\frac{u_d}{w^2} \sum_i \frac{\partial \bar{F}}{\partial \kappa_i} - \frac{2}{w(1+w)} \sum_{ij} \frac{\partial \bar{F}}{\partial a_{ij}} a_{ij} (wu_i \gamma_{sj} + u_j \gamma_{is}). \tag{2.5}
\]

Next, we calculate the derivatives of \( F \). The calculations are carried out at a point \( X_0 \) on the hypersurface \( M_n \) where \( a_{ij} \) is diagonal. Note that \( F^{ij} \) is also diagonal at \( X_0 \) and we have the following formulas
\[
F^{kk} = \frac{\partial F}{\partial \kappa_k} = \sum_{k \in \{i_1, \ldots, i_p\}} \frac{F(\kappa)}{\kappa_{i_1} + \cdots + \kappa_{i_p}},
\]
for which we refer to Lemma 1.10 in [9]. We have formulas for the second order derivatives of \( F \) at \( X_0 \) as below
\[
F^{kk,ii} = \frac{\partial^2 F}{\partial \kappa_k \partial \kappa_i} = \sum_{k \in \{i_1, \ldots, i_p\}, l \in \{j_1, \ldots, j_p\}} \frac{F(\kappa)}{(\kappa_{i_1} + \cdots + \kappa_{i_p})(\kappa_{j_1} + \cdots + \kappa_{j_p})},
\]
and, for \( k \neq r \),
\[
F^{kr, rk} = \frac{F^{kk} - F^{rr}}{\kappa_k - \kappa_r} = -\sum_{k \notin \{i_1, \ldots, i_p\}, r \notin \{j_1, \ldots, j_p\}} \frac{F(\kappa)}{(\kappa_{i_1} + \cdots + \kappa_{i_p})(\kappa_{j_1} + \cdots + \kappa_{j_p})}.
\]

Otherwise, we have \( F^{ij,kl} = 0 \). See Lemma 1.12 in [9] for the above formulas, or see [10]. These formulas can also be easily obtained from Theorem 5.5 in [2]. The following properties of the function \( F \) which are very similar to the properties of \( \sigma_k \) were proved by Dinew [9].

**Lemma 2.2 ([9]).** Suppose that the diagonal matrix \( A = \text{diag}(\lambda_1, \ldots, \lambda_n) \) belongs to \( P_p \) and that \( \lambda_1 \geq \cdots \geq \lambda_n \). Then,
\begin{enumerate}
\item \( \bar{F}^{ii}(A) \lambda_1 \geq \frac{1}{w} \bar{F}(A) \);
\item \( \sum_{k=1}^n \bar{F}^{kk}(A) \geq p \);
\item \( \sum_{k=1}^n F^{kk}(A) \lambda_k = C_p F(A) \);
\item there is a constant \( \theta = \theta(n, p) \) such that \( F^{jj}(A) \geq \theta \sum F^{ii} \) for all \( j \geq n - p + 1 \).
\end{enumerate}
The proof of the above Lemma can also be found in the Appendix of [10]. Next, we prove a few lemmas which were used to derive second order estimates.

**Lemma 2.3.** For every $C > 0$ and every compact set $K \subset \mathcal{P}_p$, there exists a number $R = R(C, K)$ such that the inequality

$$F(\lambda_1, \cdots, \lambda_n + R) \geq C$$

holds for all $\lambda = (\lambda_1, \cdots, \lambda_n) \in K$.

**Proof.** Since $K$ is compact, there is a positive constant $\epsilon = \epsilon(K)$ such that, for all $\lambda \in K$ and $\{i_1, \cdots, i_p\} \subset \{1, \cdots, n-1\}$,

$$\lambda_{i_1} + \cdots + \lambda_{i_p} \geq \epsilon.$$

Therefore, for all $\lambda \in K$, we have

$$F(\lambda_1, \cdots, \lambda_n + R) = \Pi_{1 \leq i_1 \cdots \leq i_p \leq n-1}(\lambda_{i_1} + \cdots + \lambda_{i_p} + \lambda_n + R)$$

$$\geq \frac{C^{n-1}}{C^{n-1}} \times \Pi_{1 \leq i_1 \cdots \leq i_p \leq n-1}(\lambda_{i_1} + \cdots + \lambda_{i_p})$$

which goes to infinity when $R$ goes to infinity. Thus, the lemma is proved. \qed

The following two lemmas were essentially proved in [10] (see Lemma 3.2 and Lemma 3.4), which were used to deal with the bad third order terms. Here, we restate them in the following form.

**Lemma 2.4.** Suppose $p \geq \frac{n}{2}$ and $A = \text{diag}(\lambda_1, \cdots, \lambda_n) \in \mathcal{P}_p$ with $\lambda_1 \geq \cdots \geq \lambda_n$. If $\lambda_n \geq -\delta \lambda_1$ where $0 < \delta \leq \frac{1}{2(p-1)}$, we have

$$C \sum F^{ii}(A) \geq \lambda_1$$

for $C$ sufficiently large.

**Proof.** We divide the proof into two cases.

Case 1. Suppose $\lambda_{n-1} + \lambda_{n-2} + \cdots + \lambda_n < \frac{1}{\lambda_1}$. Since

$$F^{nn} \geq \frac{F(k)}{\lambda_{n-1} + \lambda_{n-2} + \cdots + \lambda_n},$$

we see that

$$F^{nn} \geq c_0 \lambda_1$$

for some $c_0 > 0$ depending on $\inf f$. The lemma follows for sufficiently large $C$ depending on $\inf f$.

Case 2. Suppose $\lambda_{n-1} + \lambda_{n-2} + \cdots + \lambda_n \geq \frac{1}{\lambda_1}$. For a fixed $(p-1)$-tuple $2 \leq i_1 < \cdots < i_{p-1} \leq n$, we have

$$\lambda_1 + \lambda_{i_1} + \cdots + \lambda_{i_{p-1}} \geq \frac{\lambda_1}{2}.$$
by our assumption. Hence, we have
\[
F^{n_n} \geq \prod_{2 \leq i_1 < \cdots < i_{p-1} \leq n} (\lambda_{i_1} + \lambda_{i_2} + \cdots + \lambda_{i_{p-1}}) 
\times \prod_{2 \leq i_1 < \cdots < i_p \leq n} (\lambda_{i_1} + \lambda_{i_2} + \cdots + \lambda_{i_p}) 
\geq \frac{\lambda_{1}}{2} C_{n-1}^{p-1} \frac{1}{\lambda_{1}} C_{n-1}^{p}.
\]

For \( p \geq \frac{n}{2} \), a direct calculation shows that
\[
P_{n-1} - C_{n-1}^{p} = \frac{(n - 1) \cdots (n - p + 1)}{(p - 1)!} \left(1 - \frac{n - p}{p}\right) \geq 0.
\]
Therefore, we obtain
\[
2 C_{n-1}^{p} F^{n_n} \geq \lambda_{1}.
\]

Lemma 2.5. Suppose \( A = \text{diag}(\lambda_1, \cdots, \lambda_n) \in P_p \) with \( \lambda_1 \geq \cdots \geq \lambda_n \). For any given \( 1 > \epsilon > 0 \), there is a \( \delta = \delta(\epsilon) > 0 \) such that, if \( \lambda_n \geq -\delta \lambda_1 \), the inequality
\[
-2F^{1i,1i}(A) + (1 + \epsilon) \frac{F^{11}(A)}{\lambda_1} \geq (1 + \epsilon) \frac{F^{ii}(A)}{\lambda_1}
\]
holds for \( i = 2, 3, \cdots, n \).

Proof. Observe that \( \frac{F^{ii}}{\lambda_1} = \frac{\lambda_1 - \lambda_i}{\lambda_1} (-F^{1i,1i}) + \frac{F^{11}}{\lambda_1} \), from which we obtain that
\[
\frac{F^{ii}}{\lambda_1} \leq -F^{1i,1i} + \frac{F^{11}}{\lambda_1} \text{ for } \lambda_i \geq 0.
\]
Since \(-\lambda_n \leq \delta \lambda_1 \), we have
\[
\frac{F^{ii}}{\lambda_1} \leq -(1 + \delta) F^{1i,1i} + \frac{F^{11}}{\lambda_1} \text{ for } \lambda_i \leq 0.
\]
By the above two inequalities, we get the desired inequality for sufficiently small \( \delta \).

3. Global and interior \( C^2 \) estimates

Let \( \langle \cdot, \cdot \rangle \) be the inner product in \( \mathbb{R}^{n+1} \) and \( \{e_1, \cdots, e_n, e_{n+1}\} \) be the standard basis of \( \mathbb{R}^{n+1} \). Then \( X = (x, u(x)) = x_1 e_1 + \cdots + x_n e_n + u(x) e_{n+1} \) and \( |X|^2 = \langle X, X \rangle = \sum_i x_i^2 + u(x)^2 \).

3.1. Global \( C^2 \) estimates. Now we prove the global \( C^2 \) estimate. The main idea which is from [3] is to control the bad third order terms.

Theorem 3.1. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \). Suppose that \( u \in C^4(\Omega) \cap C^2(\bar{\Omega}) \) is an admissible solution to (1.1). Then there is a positive constant \( C \) such that
\[
(3.1) \sup_{\Omega} |D^2 u| \leq C(1 + \sup_{\partial \Omega} |D^2 u|),
\]
where \( C \) depends on \( n, p, |u|_{C^1}, \inf f \) and \( |f|_{C^2} \).
Proof. By the relation between $h_{ij}$ and $u_{ij}$, to prove the global $C^2$ estimate it is equivalent to prove the curvature estimate. Let $a$ be a positive constant depending on $|Du|_{C^0}$ such that $\frac{1}{w} \geq 2a$. Consider the following test function

$$G(x, \xi) = \left(\frac{1}{w} - a\right)^{-1} e^{\frac{3}{2} |x|^2} h_{\xi \xi},$$

where $X \in M_u$, $\xi \in T_X M_u$ is a unit vector and $A$ is a large constant to be determined later. Suppose the maximum of $G$ is achieved at a point $X_0 = (x_0, u(x_0)) \in M_u$ and $\xi_0 \in T_{X_0} M_u$. We choose new orthonormal vectors $\{\epsilon_1, \cdots, \epsilon_n, \epsilon_{n+1}\}$ at $X_0$ such that $\xi_0 = \epsilon_1$ and $\nu(X_0) = \epsilon_{n+1}$. Denote $x_0 = (x_0^1, \cdots, x_0^n)$. As in [4], we can represent the hypersurface near $X_0$ by the tangential coordinates $y_1, \cdots, y_n$ and $v(y)$:

$$X = \sum_{j=1}^n x_j^0 e_j + u(x_0) e_{n+1} + \sum_{j=1}^n y_j \epsilon_j + v(y) \epsilon_{n+1}.$$ 

Thus, we have $v(0) = 0$ and $\nabla v(0) = 0$. Using the same notations as in [6], set $\omega = (1 + |\nabla v|^2)^{1/2}$. Then the principal curvature in the $\epsilon_1$ direction is

$$\kappa_1 = \frac{v_{11}}{(1 + v_1^2)\omega}.$$

Under the new coordinates, we denote the point $X \in M_u$ near $X_0$ by $Y = \sum_{j=1}^n y_j \epsilon_j + v(y) \epsilon_{n+1}$. The normal in the new coordinates still denoted by $\nu$ is given as

$$\nu(Y) = -\frac{1}{\omega} \sum_{j=1}^n v_j \epsilon_j + \frac{1}{\omega} \nu_{n+1}.$$

Recall that $\epsilon_{n+1} = \frac{1}{w(x_0)} (-u_1(x_0), \cdots, -u_n(x_0), 1)$. Then,

$$\frac{1}{w} = \nu(Y) \cdot \epsilon_{n+1} = \frac{1}{\omega w(x_0)} - \frac{1}{\omega} \sum_{j=1}^n a_j v_j,$$

where $a_j = \epsilon_j \cdot \epsilon_{n+1}$. It is easy to see that $\sum_j a_j^2 \leq 1$.

We compute that $|Y|^2 = \sum y_j^2 + v(y)^2$ and $|X|^2 = |X_0|^2 + |Y|^2 + 2 \langle X_0, Y \rangle$. At the point $y = 0$ the function

$$(3.3) \quad G(y) = (1/w - a)^{-1} e^{\frac{3}{2} |x|^2} \frac{v_{11}}{(1 + v_1^2)\omega}$$

attains its maximum that is equal to $G(x_0, \xi_0)$. At this point, we also have $v_{1j} = 0$ for $j = 2, \cdots, n$ since the $y_1$ axe lines along the principal direction $\epsilon_1$. By a rotation of coordinates $y_2, \cdots, y_n$, we may assume that $v_{ij}(0)$ is diagonal. By (2.2) and $\nabla v(0) = 0$, we find that $a_{il} = v_{il}$, $\delta_{il} \kappa_i$ at the origin. This implies that $\tilde{F}^{ij}$ is diagonal. Without loss of generality, we assume that

$$\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_n.$$

At the origin, by direct calculations (see (2.12) in [3]), we see that

$$\frac{\partial^2 a_{ij}}{\partial y_1^2} = a_{i11} = v_{i11} - v_{11}^2 (v_{il} + \delta_{i1} v_{1l} + \delta_{i1} v_{1l}).$$
Recall that \( v(0) = 0 \) and \( \nabla v(0) = 0 \). Differentiating \( \log G(y) \) at \( y = 0 \) twice yields that, for \( 1 \leq i \leq n \),
\[
0 = -\frac{(1/w)_i}{1/w - a} + A(y_i + (X_0, c_i)) + \frac{v_{1i1}}{v_{11}} - \frac{2v_{1i}v_{11}}{1 + v_i} - \frac{\omega_i}{\omega}
\]
and
\[
0 \geq -\left(\frac{(1/w)_i}{1/w - a}\right)_i + A + \frac{v_{1i1}}{v_{11}} - \left(\frac{v_{1i1}}{v_{11}}\right)^2 - 2v_{1i}^2 - v_i^2.
\]
Contracting (3.5) with \( \tilde{\omega} \geq 0 \) and \( v_f \) for sufficiently large \( v_{11} \), we further derive that
\[
\text{Using (3.4), we derive that}
\]
\[
\text{By (3.2), differentiate } 1/w \text{ with respect to } y_i \text{ to obtain}
\]
\[
(1/w)_i = -a_i v_{ii} \text{ and } (1/w)_i = - \sum_j a_{ij} v_{jii} - \frac{v_{ii}^2}{w(x_0)}
\]
for all \( 1 \leq i \leq n \). Combining the above formulas with the derivatives of \( a_{il} \), we therefore have from (3.6) that
\[
0 \geq \tilde{\nabla}_{ii} a_{ii1} v_{11} - \tilde{\nabla}_{ii} v_{ii} - \tilde{\nabla}_{ii} \left(\frac{v_{1i1}}{v_{11}}\right)^2 + a_j \tilde{\nabla}_{ii} a_{ij1} + \frac{\omega_i}{\omega}
\]
\[
+ \tilde{\nabla}_{ii} v_{ii}^2 \left(\frac{1/w}_i\right)_i - \tilde{\nabla}_{ii} v_{ii}^2 + A \sum \tilde{\nabla}_{ii}.
\]
By differentiating the equation \( \tilde{F}(a_{ij}) = \tilde{f}(X, \nu(X)) = \tilde{f}(Y, \nu(Y)) \) twice, we obtain
\[
\tilde{\nabla}_{ii} a_{ii1} = (\tilde{f})_j \geq -C - C v_{11}
\]
and
\[
\tilde{\nabla}_{ii} a_{ii1} = -\tilde{\nabla}_{ij,kl} a_{ij1} a_{kl1} + (\tilde{f})_{11}
\]
\[
\geq -\tilde{\nabla}_{ij,kl} a_{ij1} a_{kl1} - C v_{11}^2 - \sum_j \tilde{f}_{ij} \frac{v_{1i1}}{\omega}
\]
for sufficiently large \( v_{11} \), where \( C \) depends on \( |f|_{C^2} \) and \( |u|_{C^1} \). Substituting the above inequalities into (3.7), we arrive at
\[
0 \geq - \frac{2}{v_{11}} \sum_{i \neq 2} \tilde{\nabla}_{ii,il} a_{i1i} a_{i1i1} - C v_{11} - \sum_j \tilde{f}_{ij} \frac{v_{1i1}}{v_{11}} - C
\]
\[
- \tilde{\nabla}_{ii} \left(\frac{v_{1i1}}{v_{11}}\right)^2 + \tilde{\nabla}_{ii} \left(\frac{1/w}_i\right)^2 + \frac{aw}{1-aw} \tilde{\nabla}_{ii} v_{ii}^2 + A \sum \tilde{\nabla}_{ii},
\]
provided that \( v_{11} \) is large enough, where we used by Lemma 2.2 (3) that \( \tilde{\nabla}_{ii} a_{ii1} = \tilde{f} \). Using (3.4), we further derive that
\[
0 \geq \frac{aw}{1-aw} \tilde{\nabla}_{ii} v_{ii}^2 - \frac{2}{v_{11}} \sum_{i \neq 2} \tilde{\nabla}_{ii,il} a_{i1i} a_{i11} - CA
\]
\[
- \tilde{\nabla}_{ii} \left(\frac{v_{1i1}}{v_{11}}\right)^2 + \tilde{\nabla}_{ii} \left(\frac{1/w}_i\right)^2 + A \sum \tilde{\nabla}_{ii} - C v_{11}
\]
for large $A$.

We now begin to deal with the third order terms. The following lemma ensures us that in what follows we can assume that $\kappa_n \geq -\delta \kappa_1$, where $\delta \leq \frac{1}{2(\rho - 1)}$ is a small positive constant to be determined.

**Lemma 3.2.** Suppose that $\kappa_n \leq -\delta \kappa_1$. Then, we can derive an upper bound for $\kappa_1$ from (3.10).

**Proof.** By the critical equation (3.4) and the Cauchy-Schwarz inequality, we have
\[
\tilde{F}^{ii} \left( \frac{v_{11i}}{v_{11}} \right)^2 \leq (1 + \varepsilon) \tilde{F}^{ii} \left( \frac{1}{1/w} \right)^2 + \left( 1 + \frac{1}{\varepsilon} \right) A^2 \tilde{F}^{ii} (y_i + \langle X_0, \epsilon_i \rangle)^2
\]
for any $\varepsilon > 0$. From (3.10) and $-\tilde{F}^{1i,1i} \geq 0$, we see that
\[
0 \geq \frac{aw}{1 - aw} \tilde{F}^{ii} v_{11i}^2 - \varepsilon \tilde{F}^{ii} \left( \frac{1}{1/w} \right)^2 - \frac{CA^2}{\varepsilon} \sum \tilde{F}^{ii} - C v_{11}
\]
for $v_{11}$ sufficiently large. Using $\frac{1}{w} = -a_i v_{11i}$ and choosing $\varepsilon$ sufficiently small, we obtain from (3.11) that
\[
0 \geq \frac{a}{2} \tilde{F}^{ii} v_{11i}^2 - \frac{CA^2}{\varepsilon} \sum \tilde{F}^{ii} - C v_{11}
\]
\[
\geq \frac{a}{2} \delta^2 \kappa_1^2 - \frac{CA^2}{\varepsilon} \sum \tilde{F}^{ii} - C v_{11},
\]
where we used Lemma 2.2 (4) in the second inequality. This implies an upper bound for $\kappa_1$ since $\sum \tilde{F}^{ii} \geq p$. □

For an $\varepsilon > 0$ sufficiently small, we choose $\delta = \delta(\varepsilon)$ such that Lemma 2.5 holds. Now we assume $\kappa_n \geq -\delta \kappa_1$. By Lemma 2.4, the inequality (3.10) becomes
\[
0 \geq \frac{aw}{1 - aw} \tilde{F}^{ii} v_{11i}^2 - \frac{2}{v_{11}} \sum_{i \geq 2} \tilde{F}^{1i,1i} v_{11i} v_{1i11} - CA
\]
\[
- \tilde{F}^{ii} \left( \frac{v_{11i}}{v_{11}} \right)^2 + \tilde{F}^{ii} \left( \frac{1}{1/w} \right)^2 + \frac{A}{2} \sum \tilde{F}^{ii}
\]
as long as $A$ is sufficiently large. For any given $\varepsilon > 0$, by Lemma 2.5, (3.12) becomes
\[
0 \geq a \tilde{F}^{ii} v_{11i}^2 + \sum_{i \geq 2} \tilde{F}^{ii} \left( \frac{v_{11i}}{v_{11}} \right)^2 - \tilde{F}^{ii} \left( \frac{v_{11i}}{v_{11}} \right)^2 - CA
\]
\[
- (1 + \varepsilon) \sum_{i \geq 2} \tilde{F}^{1i} \left( \frac{v_{11i}}{v_{11}} \right)^2 + \tilde{F}^{ii} \left( \frac{1}{1/w} \right)^2 + \frac{A}{2} \sum \tilde{F}^{ii}.
\]
By the critical equation (3.4) and the Cauchy-Schwarz inequality, we see that
\[
\left( \frac{v_{11i}}{v_{11}} \right)^2 \leq (1 + \varepsilon) \left( \frac{1}{1/w} \right)^2 + \frac{CA^2}{\varepsilon}
\]
and, for $i \geq 2$,
\[
(1 + \varepsilon) \left( \frac{v_{11i}}{v_{11}} \right)^2 \leq (1 + \varepsilon)^2 \left( \frac{1}{1/w} \right)^2 + \frac{CA^2}{\varepsilon}.
\]
Substituting (3.14) and (3.15) into (3.13), we have

\[ 0 \geq a \tilde{F}^{ii} v_i^2 - (1 + \epsilon) \tilde{F}^{11} \left( \frac{(1/w)_i}{1/w - a} \right)^2 \frac{CA^2}{\epsilon} \tilde{F}^{11} - CA \]

\[ - (1 + 3\epsilon) \tilde{F}^{ii} \sum_{i,j \geq 2} \left( \frac{(1/w)_i}{1/w - a} \right)^2 \tilde{F}^{ii} \left( \frac{(1/w)_j}{1/w - a} \right)^2 + A \sum \tilde{F}^{ii}. \]

(3.16)

Observe that \( |v_i| \leq n v_{11} \) and \( \tilde{F}^{11} \leq \cdots \leq \tilde{F}^{nn} \). We further derive that

\[ 0 \geq a \tilde{F}^{ii} v_i^2 - \frac{3n^2 \epsilon}{\tilde{f}} \tilde{F}^{11} v_{11}^2 - \frac{CA^2}{\epsilon} \tilde{F}^{11} - CA. \]

(3.17)

Choose \( \epsilon \leq \frac{1}{6n^2} \). We then derive an upper bound for \( v_{11} \) by Lemma 2.2 (1).

\[ \square \]

3.2. Interior \( C^2 \) estimates. As in Sheng-Urbas-Wang [13], we actually can derive an interior curvature estimate for equation (1.1) with \( p \geq \frac{3}{2} \) by the same argument as that of Theorem 1.3 in [10]. Furthermore, if \( f \) does not depend on \( \nu \), the interior estimate holds for all \( p \).

**Theorem 3.3.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \). Suppose that \( u \in C^4(\Omega) \cap C^2(\overline{\Omega}) \) is an admissible solution to (1.1) and (1.2). Then there is a positive constant \( C \) and \( \beta \) such that

\[ \sup_{\Omega} (-u)^{\beta} |h| \leq C, \]

where \( C \) and \( \beta \) depend on \( n, p, |u|_{C^1}, \inf \tilde{f} \) and \( |f|_{C^2} \).

**Proof.** We consider the test function \( G(x, \xi) = (-u)^{\beta} \left( \frac{1}{w} - a \right)^{-1} e^{\frac{1}{2} |x|^2} h \xi \). Assume its maximum is attained at a point \( X_0 = (x_0, u(x_0)) \in M_u \) and \( \xi_0 \in T_{x_0} M_u \). As before, after choosing new coordinates at \( X_0 \), the maximum of

\[ G(y) = (-u)^{\beta} \left( \frac{1}{w} - a \right)^{-1} e^{\frac{1}{2} |x|^2} \frac{v_{11}}{v_{i1}} \]

is achieved at \( y = 0 \) and \( a_{ii} = v_{ii} = \kappa \delta_{ii} \). Without loss of generality, we also assume that \( \kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_n \); then \( \tilde{F}^{11} \leq \tilde{F}^{22} \leq \cdots \leq \tilde{F}^{nn} \). Note that \( u = X \cdot \kappa_{n+1} = X_0 \cdot \kappa_{n+1} + \sum_{j=1}^n a_j y_j + \frac{v(y)}{w(x_0)} \), where \( a_j \) are defined as before. Differentiate \( u \) with respect to \( y_i \) to obtain

\[ u_i = a_i + \frac{v_i}{w(x_0)} \text{ and } u_{ii} = \frac{v_{ii}}{w(x_0)}. \]

Differentiating \( \log G(y) \) at \( y = 0 \) twice yields that, for \( 1 \leq i \leq n \),

\[ 0 = \beta \frac{u_{ii}}{u} \left( \frac{1}{w} - a \right)_i + A(y_i + \langle X_0, e_i \rangle) + \frac{v_{11i}}{v_{11}}, \]

and

\[ 0 \geq \beta \frac{u_{ii}}{u} - \beta \left( \frac{u_i}{u} \right)^2 \left( \frac{1}{w} - a \right)_i + A \frac{v_{11i}}{v_{11}} - \frac{(v_{11i})^2}{v_{11}} - 2 v_{11i} - v_{ii}. \]

(3.19)

(3.20)
Contracting (3.20) with \( \tilde{F}^{ii} \) and using \( \tilde{F}^{ii}v_{ii} = \tilde{F}^{ii}a_{ii} = \tilde{f} \) and (3.19), similar to (3.10), we will arrive at

\[
0 \geq \frac{C\beta}{u} - \beta \tilde{F}^{ii} \left( \frac{u_i}{u} \right)^2 - \frac{2}{v_{11}} \sum_{i \geq 2} \tilde{F}^{ii} a_{ii} a_{i11} - CA - Cv_{11}
\]

(3.21)

\[
- \tilde{F}^{ii} \left( \frac{v_{11i}}{v_{11}} \right)^2 + \tilde{F}^{ii} \left( \frac{(1/w)_i}{1/w - a} \right)^2 + \frac{aw}{1 - aw} \tilde{F}^{ii} v_{i1}^2 + A \sum \tilde{F}^{ii}
\]

as long as \( v_{11} \) large enough.

If \( \kappa_n \leq -\delta \kappa_1 \) for some \( \delta > 0 \), similar to Lemma 3.2, by the critical equation (3.19) and the Cauchy-Schwarz inequality, we have

\[
(3.22)
\]

\[
\left( \frac{v_{11i}}{v_{11}} \right)^2 \leq (1 + \epsilon) \left( \frac{(1/w)_i}{1/w - a} \right)^2 + \frac{CA^2}{\epsilon} + \frac{C\beta^2}{\epsilon} \frac{1}{u^2}
\]

for any \( \epsilon > 0 \), where \( C \) depends on \( |Du|_{C^0} \). Substituting the above inequality into (3.21), we then have

\[
0 \geq \frac{a}{2} \tilde{F}^{ii} v_{i1}^2 + \frac{C\beta}{u} - \frac{C\beta^2}{\epsilon} \sum \tilde{F}^{ii} - \frac{CA^2}{\epsilon} \sum \tilde{F}^{ii} - Cv_{11}
\]

(3.23)

if \( \epsilon > 0 \) is sufficiently small and \( v_{11} > A \). Utilizing \( \kappa_n \leq -\delta \kappa_1 \) and Lemma 2.2 (4), we then obtain

\[
0 \geq \frac{a\delta^2 \theta}{2} \sum \tilde{F}^{ii} + \frac{C\beta}{u} \sum \tilde{F}^{ii} - \frac{C\beta^2}{\epsilon} u^2 - \frac{CA^2}{\epsilon} \sum \tilde{F}^{ii} - Cv_{11},
\]

(3.24)

By Lemma 2.2 (2), we derive, for some \( C \) depending on \( n, p, |u|_{C^2} \), and \( |f|_{C^2} \), that

\[ u^2 v_{11}^2 \leq C, \]

which implies (3.13).

For any \( \epsilon > 0 \) sufficiently small, choose \( \delta = \delta(\epsilon) \) such that Lemma 2.5 holds. In the following, we will assume \( \kappa_n \geq -\delta \kappa_1 \). By Lemma 2.4 and 2.5, the inequality (3.21) becomes

\[
0 \geq \frac{C\beta}{u} - \beta \tilde{F}^{ii} \left( \frac{u_i}{u} \right)^2 + (1 + \epsilon) \sum_{i \geq 2} \tilde{F}^{ii} \left( \frac{v_{11i}}{v_{11}} \right)^2 - \tilde{F}^{ii} \left( \frac{v_{11i}}{v_{11}} \right)^2 - CA
\]

(3.25)

\[- (1 + \epsilon) \sum_{i \geq 2} \tilde{F}^{ii} \left( \frac{v_{11i}}{v_{11}} \right)^2 + \tilde{F}^{ii} \left( \frac{(1/w)_i}{1/w - a} \right)^2 + a \tilde{F}^{ii} v_{i1}^2 + A \frac{1}{2} \sum \tilde{F}^{ii} \]

as long as \( A \) is sufficiently large. By (3.19) and the Cauchy-Schwarz inequality, we have, for \( i \geq 2 \),

\[- \beta \tilde{F}^{ii} \left( \frac{u_i}{u} \right)^2 \geq - \frac{3}{\beta} \tilde{F}^{ii} \left( \frac{v_{11i}}{v_{11}} \right)^2 - \frac{3}{\beta} \tilde{F}^{ii} \left( \frac{(1/w)_i}{1/w - a} \right)^2 - \frac{CA^2}{\beta} \tilde{F}^{ii}. \]
Taking \( i = 1 \) and \( \varepsilon = \epsilon \) in (3.22) and substituting it together with the above inequality into (3.25), we obtain

\[
0 \geq a \tilde{F}_{ii}^1 v_{1i}^2 - \beta \tilde{F}_{11}^1 \left( \frac{u_{11}}{u} \right)^2 + \left( \epsilon - \frac{3}{\beta} \right) \sum_{i \geq 2} \tilde{F}_{ii}^i \left( \frac{u_{11}}{v_{11}} \right)^2 - \frac{CA^2}{\beta} \sum \tilde{F}_{ii}^i
\]

\[
- \frac{\varepsilon \tilde{F}_{11}^1 \left( \frac{1}{w} \right)^2}{w} - \frac{CA^2}{\epsilon} \tilde{F}_{11}^1 - \frac{C\beta^2}{\epsilon} \tilde{F}_{11}^i + \frac{A}{2} \sum \tilde{F}_{ii}^i - CA
\]

(3.26)

Recall that \( \tilde{F}_{11}^1 \leq \tilde{F}_{22}^2 \leq \cdots \leq \tilde{F}_{nn}^n \). Choosing \( \varepsilon \) sufficiently small and \( \beta \) sufficiently large and using (3.22) for \( i \geq 2 \), we arrive at

\[
0 \geq a \tilde{F}_{ii}^1 v_{1i}^2 - (3\varepsilon + \frac{3}{\beta}) \sum_{i \geq 2} \tilde{F}_{ii}^i \left( \frac{1}{w} \right)^2
\]

\[
+ \frac{C\beta}{u} - \frac{C\beta^2}{\epsilon} \tilde{F}_{11}^1 + \frac{A}{4} \sum \tilde{F}_{ii}^i - CA
\]

(3.27)

\[
\geq a \tilde{F}_{ii}^1 v_{1i}^2 + \frac{C\beta}{u} - \frac{C\beta^2}{\epsilon} \tilde{F}_{11}^1 - \frac{CA^2}{\epsilon} \tilde{F}_{11}^i - CA.
\]

By Lemma 2.2 (1), we conclude

\[
u_{11}^2 \leq C
\]

from the last inequality of (3.27), where \( C \) depends on \( n, p, |u|_{C^1}, \inf \tilde{f} \) and \( |f|_{C^2} \).

This completes the proof of Theorem 3.3.

\[\square\]

4. Boundary \( C^2 \) estimates

In this section, we establish the boundary \( C^2 \) estimate. The main theorem in this section is stated as below.

**Theorem 4.1.** Let \( \Omega \) be a strictly convex bounded domain in \( \mathbb{R}^n \) with smooth boundary \( \partial \Omega \). Suppose \( u \in C^3(\bar{\Omega}) \) is an admissible solution to (1.1) and (1.2). Then, there exists a positive constant \( C \) such that

\[
\max_{\partial \Omega} |D^2 u| \leq C,
\]

where \( C \) depends on \( n, p, |u|_{C^1}, |f|_{C^1}, \inf f \) and \( \partial \Omega \).

Without loss of generality, we assume \( 0 \in \partial \Omega \). Suppose that the boundary \( \partial \Omega \) around the origin is given by

\[
x_n = \rho(x') = \frac{1}{2} \sum_{\alpha < n} \kappa^b_{\alpha} x_n^\alpha + O(|x'|^3),
\]

where \( x' = (x_1, \ldots, x_{n-1}) \) and \( \kappa^b_1, \ldots, \kappa^b_{n-1} \) are the principal curvatures of \( \partial \Omega \) at the origin. Differentiating the boundary condition \( u = 0 \) at the origin twice, we obtain that

\[
|u_{\alpha\beta}(0)| \leq C, \text{ for } 1 \leq \alpha, \beta \leq n - 1.
\]
For the tangential-normal estimate, we need some idea from [26]. Let \( d(x) = \text{dist}(x, \partial \Omega) \) be the distance function to the boundary and \( \Omega_\delta := \{ x \in \Omega : |x| < \delta \} \). We consider
\[
W := u_\alpha + \rho_\alpha u_n - \frac{1}{2} \sum_{1 \leq \beta \leq n-1} u_\beta^2,
\]
where \( 1 \leq \alpha \leq n-1 \). Similar to Lemma 5.1 in [26] and Lemma 5.3 in [29], we can prove the following lemma.

**Lemma 4.2.** For sufficiently small \( \delta > 0 \), we have,
\[
G^{ij} W_{ij} \leq C(1 + |DW| + \sum_i G^{ii} + G^{ij} W_i W_j) \text{ in } \Omega_\delta,
\]
where \( C \) depends on \( n, p, |u|_{C^1}, |\tilde{f}|_{C^1}, \inf \tilde{f} \) and \( \partial \Omega \).

**Proof.** Note that
\[
G^{ij} W_{ij} + G^s W_s = \tilde{f}_\alpha + \rho_\alpha \tilde{f}_n - \sum_{\beta \leq n-1} u_\beta \tilde{f}_\beta + 2G^{ij} u_n \rho_{\alpha j} - \sum_{\beta \leq n-1} G^{ij} u_{\beta i} \rho_{\alpha j} + u_n G^s \rho_{\alpha s}.
\]
It is easy to see that
\[
G^{ij} = \frac{1}{w} \sum_{s,t} \tilde{F}^{st} \gamma^{si} \gamma^{jt} \text{ and } u_{ij} = w \sum_{s,t} \gamma_{is} a_{st} \gamma_{tj}.
\]
It follows that
\[
\sum_{\beta \leq n-1} G^{ij} u_{\beta i} \rho_{\alpha j} = \sum_{\beta \leq n-1} \sum_{s,t} \frac{\partial \tilde{F}}{\partial \alpha_{ij}} \gamma_{\beta s} \gamma_{\beta t} a_{st} a_{ij}.
\]
Suppose \( b_{ij} \) is the orthogonal matrix that diagonalize \( a_{ij} \) and \( \tilde{F}^{ij} \) simultaneously, i.e.
\[
\tilde{F}^{ij} = \sum_s b_{is} \frac{\partial \tilde{F}}{\partial \kappa_s} b_{js} \text{ and } a_{ij} = \sum_s b_{is} \kappa_s b_{js}.
\]
Therefore, we get
\[
\sum_{\beta \leq n-1} G^{ij} u_{\beta i} u_{\beta j} = \sum_{\beta \leq n-1} \sum_i \left( \sum_s \gamma_{\beta s} b_{si} \right)^2 \frac{\partial \tilde{F}}{\partial \kappa_i} \kappa_i^2.
\]
Define the matrix \( \eta = (\eta_{ij}) \) by
\[
\eta_{ij} = \sum_s \gamma_{is} b_{sj}.
\]
It is easy to verify that \( \eta \cdot \eta^T = g \) and \( |\det(\eta)| = \sqrt{1 + |Du|^2} \). Hence,
\[
\sum_{\beta \leq n-1} G^{ij} u_{\beta i} \rho_{\alpha j} = \sum_{\beta \leq n-1} \sum_i (\eta_{\beta i})^2 \frac{\partial \tilde{F}}{\partial \kappa_i} \kappa_i^2.
\]
By (4.6) and (4.7), we obtain
\[
|G^{ij} u_{ni} \rho_{\alpha j}| \leq C \sum_i \frac{\partial \tilde{F}}{\partial \kappa_i} |\kappa_i| \text{ and } \tilde{F}^{ij} a_{ij} = \sum_i \frac{\partial \tilde{F}}{\partial \kappa_i} \kappa_i b_{ji} b_{ti}.
\]
By (2.5) and the above equality, we derive that

\[ G^s \rho_{as} \leq C \sum_i \frac{\partial F}{\partial \kappa_i} |\kappa_i|. \]

A direct calculation shows that

\[ |\tilde{f}_\alpha + \tilde{f}_n \rho_{\alpha} - \sum_{\beta \leq n-1} u_\beta \tilde{f}_\beta| \leq C(1 + |DW|). \]

We therefore arrive at

\[ G^{ij} W_{ij} \leq C \left(1 + |DW| + \sum G^{ii}\right) - G^s W_s \]

(4.10)

\[ + C \sum_i \frac{\partial F}{\partial \kappa_i} |\kappa_i| - \sum_{\beta \leq n-1} G^{ij} u_{\beta i} u_{\beta j}. \]

Next we estimate \( G^s W_s \). By Lemma 2.1 and (4.9), we see that

\[ (4.11) \]

\[-G^s W_s = \frac{u_s W_s}{w^2} \tilde{f} + \frac{2}{w} \sum_{i,t} \frac{\partial F}{\partial \kappa_i} \kappa_i (b_{ti} u_t)(b_{ji} \gamma^s) W_s \]

\[ \leq C|DW| + \frac{2}{w} \sum_{i,j} \frac{\partial F}{\partial \kappa_i} \kappa_i (b_{ti} u_t)(b_{ji} \gamma^s) W_s. \]

Now we divide the proof into two cases: (a) \( \sum_{\beta \leq n-1} \eta^2_{\beta i} \geq \epsilon \) for all index \( i \); (b) \( \sum_{\beta \leq n-1} \eta^2_{\beta r} < \epsilon \) for some index \( 1 \leq r \leq n \), where \( \epsilon > 0 \) is a small constant to be determined later.

**Case (a).** By our assumption and (4.8), we have

\[ (4.12) \]

\[ \sum_{\beta \leq n-1} G^{ij} u_{\beta i} u_{\beta j} \geq \epsilon \sum_i \frac{\partial F}{\partial \kappa_i} \kappa_i^2. \]

The second term in the right hand side of the inequality (4.11) can be estimated as

\[ (4.13) \]

\[ \frac{\partial F}{\partial \kappa_i} \kappa_i (b_{ti} u_t)(b_{ji} \gamma^s) W_s \leq \frac{\epsilon}{4} \frac{\partial F}{\partial \kappa_i} \kappa_i^2 + C \frac{\partial F}{\partial \kappa_i} (b_{ji} \gamma^s W_s)^2 \]

by the Cauchy-Schwarz inequality. Note that

\[ (4.14) \]

\[ \frac{\partial F}{\partial \kappa_i} (b_{ji} \gamma^s W_s)^2 = G^{ij} W_i W_j. \]

Combining (4.14), (4.13) with (4.11), we obtain

\[ (4.15) \]

\[ -G^s W_s \leq C|DW| + \frac{\epsilon}{2} \sum_i \frac{\partial F}{\partial \kappa_i} \kappa_i^2 + \frac{C}{\epsilon} G^{ij} W_i W_j. \]

Substituting (4.15) and (4.12) into (4.10) and applying the Cauchy-Schwarz inequality to \( \sum \frac{\partial F}{\partial \kappa_i} |\kappa_i| \), we get the desired inequality (4.4).

**Case (b).** We may assume that the principal curvatures are ordered as \( \kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_n \). Then, we have \( \frac{\partial F}{\partial \kappa_1} \leq \frac{\partial F}{\partial \kappa_2} \leq \cdots \leq \frac{\partial F}{\partial \kappa_n} \). Taking into account the fact that \( |\eta_{nr}| \leq \sqrt{1 + |Du|^2_{L^2}|} \), we see that

\[ 1 \leq |\det(\eta)| \leq \sqrt{1 + |Du|^2_{L^2} \det(\eta')} + C\epsilon, \]
where \( \eta' = \{ \eta_{pq} \}_{p \neq n, q \neq r} \), which implies
\[
|\det(\eta')| \geq \frac{1}{2\sqrt{1 + |Du|_{C^0}^2}}
\]
for sufficiently small \( \epsilon \). On the other hand, for any fixed index \( i \neq r \), it holds that
\[
|\det(\eta')| \leq C \sum_{\beta \neq n} |\eta_{\beta i}|^2
\]
for some positive constant \( C \) depending on \( n \) and \( |Du|_{C^0} \). By the above two inequalities, we derive that, for any \( i \neq r \),
\[
\sum_{\beta \leq n-1} |\eta_{\beta i}|^2 \geq c_1
\]
for some positive constant \( c_1 \) depending on \( n \) and \( |Du|_{C^0} \). By the above two inequalities, we obtain
\[
\sum_{\beta \leq n-1} G^{ij} u_{\beta i} u_{\beta j} \geq c_1 \sum_{i \neq r} \frac{\partial \tilde{F}}{\partial \kappa_i} \kappa_i^2.
\]
If \( \kappa_r \leq 0 \), by the same argument as that of Lemma 2.20 in [15] and \( \sum \frac{\partial \tilde{F}}{\partial \kappa_i} \kappa_i \geq 0 \), we obtain
\[
\sum_{i \neq r} \frac{\partial \tilde{F}}{\partial \kappa_i} \kappa_i^2 \geq \frac{1}{n} \sum_i \frac{\partial \tilde{F}}{\partial \kappa_i} \kappa_i^2.
\]
We then get the desired inequality (4.13) by the same argument as Case (a) with \( \epsilon \) in (4.12) replaced by \( \frac{\epsilon}{n} \). Therefore, we may assume \( \kappa_r \geq 0 \). Note that \( \kappa_r \geq \kappa_n \). It is easy to see that
\[
(b_{jr} \gamma^s) W_s = w \left( \eta_{nr} + \rho_\alpha \eta_{nr} - \sum_{\beta \leq n-1} u_{\beta \eta_{\beta r}} \right) \kappa_r + b_{jr} \gamma^s \rho_\alpha u_n,
\]
which implies that
\[
(b_{jr} \gamma^s) W_s \leq C w (\epsilon + |\rho_\alpha|) \kappa_r + C.
\]
First, suppose that \( \kappa_n \geq -\frac{\kappa_r}{2(n-1)} \). We see \( \kappa_r + \kappa_i_1 + \cdots + \kappa_{i_{p-1}} \geq \frac{\kappa_r}{2} \). It follows that
\[
\frac{\partial F}{\partial \kappa_r} = \sum_{r \notin \{i_1, \ldots, i_{p-1}\}} \frac{F(\kappa)}{\kappa_r + \kappa_i_1 + \cdots + \kappa_{i_{p-1}}} \leq \frac{C}{\kappa_r}.
\]
By (4.18) and the Cauchy-Schwarz inequality, we have
\[
\sum_{l,i} \frac{\partial \tilde{F}}{\partial \kappa_i} \kappa_i (b_{li} u_i) (b_{jr} \gamma^s) W_s \leq C |DW| + \frac{\epsilon}{4} \sum_{i \neq r} \frac{\partial \tilde{F}}{\partial \kappa_i} \kappa_i^2 + \frac{C}{\epsilon} G^{ij} W_i W_j.
\]
Substituting the above inequality into (4.11), we obtain
\[
-G^s W_s \leq C |DW| + \frac{\epsilon}{4} \sum_{i \neq r} \frac{\partial \tilde{F}}{\partial \kappa_i} \kappa_i^2 + \frac{C}{\epsilon} G^{ij} W_i W_j.
\]
By (4.19) and (4.18), and applying the Cauchy-Schwarz inequality to \( \sum_{i \neq r} \frac{\partial \tilde{F}}{\partial \kappa_i} |\kappa_i| \), we conclude (4.4) from (4.10).
Now suppose that \( \kappa_n \leq -\frac{\partial ^2 \overline{\Phi}}{\partial x^2} \). Note that \( r \neq n \) in this case. Since \( \frac{\partial \overline{\Phi}}{\partial x_r} \kappa_r = \delta - \sum_{i \neq r} \frac{\partial \overline{\Phi}}{\partial x_i} \kappa_i \), we have, by (4.11) and the Cauchy-Schwarz inequality,

\[
\sum_{i, j} \frac{\partial \overline{\Phi}}{\partial x_i} \kappa_i (b_{ij} u_i) (b_{ij} \gamma^{ij}) W_i \\
\leq C \frac{\partial \overline{\Phi}}{\partial x_r} \kappa_r + C(\epsilon + |\rho_\alpha|) \frac{\partial \overline{\Phi}}{\partial x_r} \kappa_n + \epsilon \sum_{i \neq r} \frac{\partial \overline{\Phi}}{\partial x_i} \kappa_i^2 + \frac{C}{\epsilon} \sum_{i \neq r} \frac{\partial \overline{\Phi}}{\partial x_i} \kappa_i^2 + C \frac{\partial \overline{\Phi}}{\partial x_n} \kappa_n W_i W_j.
\]

Combining (4.10), (4.11) and (4.16) with the above inequality and choosing \( \epsilon, \delta \) sufficiently small such that \( C(\epsilon + \delta) < c_1 \), we can get (4.14). Thus, the proof of Lemma 4.2 is finished.

With the above lemma in hand, we can construct a suitable barrier function to prove the following estimate

\[
(4.20) \quad |u_{\alpha n}(0)| \leq C, \text{ for } 1 \leq \alpha \leq n - 1.
\]

Since the boundary \( \partial \Omega \) is smooth and strictly convex, we can find a smooth strictly convex function \( v \) satisfying \( v = 0 \) on \( \partial \Omega \) and \( v \leq 0 \) in \( \Omega \). Therefore, we have \( D^2 v(x) \in P_1 \) for \( x \in \overline{\Omega} \). Moreover, we can find a small positive constant \( \theta \) such that \( \overline{v} = v - \theta |x|^2 \) satisfies \( D^2 \overline{v}(x) \in P_1 \) for \( x \in \overline{\Omega} \). Now we can finish the proof of (4.20). Consider the following barrier function

\[
(4.21) \quad \Psi = \overline{v} + \frac{\theta}{2} |x|^2 - td + \frac{N}{2} d^2,
\]

where \( t \) and \( N \) are two positive constants to be chosen later. A direct calculation shows that

\[
(4.22) \quad G^{ij} \Psi_{ij} = G^{ij} (D^2 \overline{v} + N Dd \otimes Dd)_{ij} + G^{ij} (N d - t) d_{ij} + \theta \sum G_{ii}.
\]

Concerning \( D^2 \overline{v} \in P_1 \), we see that \( D^2 \overline{v} + N Dd \otimes Dd \in P_1 \). By the concavity and homogeneity of \( G \) with respect to \( D^2 u \), we see that

\[
(4.23) \quad G^{ij} (D^2 \overline{v} + N Dd \otimes Dd)_{ij} \geq G(D^2 \overline{v} + N Dd \otimes Dd, Du) = \tilde{F}(V_u),
\]

where \( V_u := \frac{1}{\lambda} (\gamma^{ik} V_{kl} \gamma^{ij}) \) and \( V := \{ v_{kl} + N d_k d_l \} \). We adopt the convention that the eigenvalues are arranged in algebraically nondecreasing order:

\[
\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n.
\]

Then, by the Weyl theorem, it is easy to see that \( \lambda_\alpha (V) \geq \lambda_\alpha (D^2 \overline{v}) \) for \( \alpha = 1, 2, \cdots, n - 1 \) and \( \lambda_n (V) \geq \lambda_1 (D^2 \overline{v}) + N. \) Denote \( \theta_0 = \inf_{x \in \Omega} \frac{\lambda_1 (D^2 \overline{v})}{\lambda_n (D^2 \overline{v})} \). We further derive that \( \lambda_\alpha (V) \geq \theta_0 \lambda_\alpha (D^2 \overline{v}) \) for \( \alpha = 1, 2, \cdots, n - 1 \) and \( \lambda_n (V) \geq \theta_0 (\lambda_n (D^2 \overline{v}) + N). \) On the other hand, by the Ostrowski theorem, there exist \( \theta_k > 0 \) such that \( \lambda_k (V_u) = \theta_k \lambda_k (V) \) for \( k = 1, 2, \cdots, n \), where \( \theta_k \) is uniformly positive only depending on \( |Du|_{C^0} \). Substituting (4.23) into (4.22) and by Lemma 4.2 we have, for sufficiently large \( N \) and sufficiently small \( \delta \) and \( t, \)

\[
G^{ij} \Psi_{ij} \geq \tilde{\theta} N\delta + \frac{\theta}{2} \sum G_{ii} \geq N \alpha + \frac{\theta}{2} \sum G_{ii},
\]
where $\bar{\theta}$ depends on $\theta_0, \theta_1, \ldots, \theta_n$ and $\lambda(D^2v)$. Also, note that $|D\Psi| = |D\bar{v} + \theta x - tDd + Ndd| \leq C + N\delta$ in $\Omega_\delta$. Choosing $\delta$ sufficiently small, we obtain

$$G^{ij} \Psi_{ij} \geq N^{1/2} |D\Psi| + \frac{\theta}{2} \left( \sum G^{ii} + 1 \right).$$

Let $\tilde{W} = 1 - e^{-BW}$, where constant $B > 0$ is sufficiently large. It follows that

$$G^{ij}(R\Psi - \tilde{W})_{ij} \geq \sqrt{N}(|DR\Psi| - |D\tilde{W}|)$$

if we choose $R \gg B \gg 1$. Since $\Psi|_{\partial\Omega} \leq -\frac{\theta}{2} |x'|^2$, $\Psi|_{\partial\Omega} \leq -\frac{\theta}{2} \delta^2$ and $-W|_{\partial\Omega} = \frac{1}{2} \sum \rho^2 u^2 \leq C|x'|^2$ for some $C$ depending on $|\rho|_{C_2}$ and $|u|_{C_1}$, it is easy to verify that on $\partial\Omega_\delta$ we have $R\Psi - \tilde{W} \leq 0$ if $R$ is large enough. By the maximum principle we get

$$R\Psi - \tilde{W} \leq 0 \text{ in } \Omega_\delta.$$

As $(R\Psi - \tilde{W})(0) = 0$ we deduce that $u_{nn}(0) \leq -C$. Consider $W' = -u_n - \rho_n u_n - \frac{1}{2} \sum_{1 \leq i \leq n - 1} u_i^2$. By the same argument we deduce $u_{nn}(0) \leq C$. The proof of (4.20) is completed.

Finally, we prove the double normal estimate

$$|u_{nn}(0)| \leq C.$$  

(4.25)

We only need to derive an upper bound for $u_{nn}(0)$ by $\sum \kappa_i > 0$ since $P_p \subset P_n$. Actually, by the inequality of arithmetic and geometric means, we have $\sum \kappa_i \geq \bar{F}(\kappa) \geq \inf f > 0$. By (1.8) in [3], i.e. $ad(x) \leq -u(x) \leq \frac{1}{a} d(x)$ in $\Omega$ for some constant $a > 0$ depending on $\inf f$, we obtain

$$u_n(0) \leq -a.$$  

We have, with respect to a principal coordinate system at the origin, $u_{\alpha\beta}(0) = -u_n \kappa^b_{\alpha} \delta_{\alpha\beta}$ for $1 \leq \alpha, \beta \leq n - 1$ and $g^{ij} = \delta_{ij} - \frac{|D\alpha|^2}{w^2} \delta_{in} \delta_{jn}$. Therefore, the matrix in (2.2) has the following form

$$
\begin{pmatrix}
-u_n \kappa^b_1 & 0 & \cdots & 0 & \frac{u_n}{w} \\
0 & -u_n \kappa^b_2 & \cdots & 0 & \frac{u_n}{w} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & -u_n \kappa^b_{n - 1} & \frac{u_n}{w} \\
\frac{u_n}{w} & \frac{u_n}{w} & \cdots & \frac{u_n - 1}{w} & \frac{u_n}{w}
\end{pmatrix}.
$$

By Lemma 1.2 of [4], when $|u_{nn}|$ goes to infinity, the eigenvalues $w(\kappa_1, \cdots, \kappa_n)$ of the above matrix behave like

$$w_{\kappa_n} = -u_n \kappa^b_\alpha + o(1), \quad 1 \leq \alpha \leq n - 1, \quad w_{\kappa_n} = \frac{u_n}{w^2} + O(1),$$

where $o(1)$ and $O(1)$ are uniform only depending on $-u_n \kappa^b_\alpha$ and $|u_{nn}(0)|$. Since $-u_n(0) \geq a$ and $(\kappa^b_1, \cdots, \kappa^b_{n - 1}) \in P_p$, there exists a uniform positive constant $\varsigma$ such that

$$\Pi_{1 \leq i_1 < \cdots < i_p \leq n - 1}(\kappa_{i_1} + \cdots + \kappa_{i_p}) \geq \varsigma,$$

as long as $u_{nn}$ is large enough. Returning to the equation (1.1), we have

$$\Pi_{1 \leq i_1 < \cdots < i_{p - 1} \leq n - 1}(\kappa_n + \kappa_{i_1} + \cdots + \kappa_{i_{p - 1}}) \leq \frac{C'}{\varsigma},$$

from which we can derive an upper bound for $\kappa_n$. Therefore, $u_{nn}(0)$ is bounded from above. Combining (1.3), (4.20) with (4.25), we obtain (4.1).
5. Gradient Estimates

In this section, we establish the gradient estimate. First, by the existence of a subsolution \( u \) and the maximum principle, it is easy to derive

\[
\sup_\Omega |u| + \sup_{\partial \Omega} |Du| \leq C,
\]

where \( C \) depends on \( |u|_{C^1} \). Next, we prove the global gradient estimate following the argument in [6]. We have

**Theorem 5.1.** Let \( u \in C^3(\Omega) \cap C^1(\bar{\Omega}) \) be an admissible solution to the problem \((1.1)\) and \((1.2)\). Suppose \( f(x, z, \nu) \in C^\infty(\bar{\Omega} \times \mathbb{R} \times S^n) > 0 \) and \( f_z \geq 0 \). Then, the estimate

\[
\sup_\Omega |Du| \leq C(1 + \sup_{\partial \Omega} |Du|)
\]

holds for a positive constant \( C \) depending on \( n, p, |u|_{C^0}, \inf f \) and \( |f|_{C^1} \).

**Proof.** As in [6], we consider the following quantity

\[
Q = |Du| e^{Au},
\]

where \( A \) is a large constant to be determined later. Suppose that \( Q \) achieves its maximum at \( x_0 \in \Omega \). At this point we may rotate the coordinates such that \( u_1 = |Du| > 0 \) and \( u_\alpha = 0 \) for \( \alpha \geq 2 \).

Differentiating \( \log Q \) at \( x_0 \) once, we obtain

\[
\frac{u_{1i}}{u_1} + Au_i = 0 \quad \text{for } i = 1, \ldots, n.
\]

By the above equality, we may further rotate the coordinates so that \( u_{ij} \) is diagonal at \( x_0 \). From \((2.2)\), we find that

\[
a_{11} = \frac{u_{11}}{w_3}, \quad a_{ii} = \frac{u_{ii}}{w} \quad \text{for } i \geq 2 \quad \text{and} \quad a_{ij} = 0 \quad \text{for } i \neq j.
\]

Denote by \( \alpha_{ij} \) the positive semidefinite matrix defined as

\[
\alpha_{11} = \frac{1}{w_3}, \quad \alpha_{ii} = \frac{1}{w_i}, \quad i \geq 2, \quad \alpha_{ij} = \frac{1}{w}, \quad j \geq i \geq 2.
\]

As in [49], denote by \( \tilde{F}^{ij} = F^{ij} \alpha_{ij} \) the Hadamard product of the matrix \( F^{ij} \) and \( \alpha_{ij} \), which is positive definite since \( F^{ij} \) is diagonal and positive definite. Differentiating \( \log Q \) at \( x_0 \) a second time and contracting with \( \tilde{F}^{ij} \), we get

\[
0 \geq \frac{\tilde{F}^{ij} u_{ij}}{u_1} - \frac{\tilde{F}^{ij} u_{1i} u_{1j}}{u_1^2} + A \tilde{F}^{ij} u_{ij}.
\]

Differentiating \( a_{ij} \) and the equation \((1.1)\), we obtain

\[
F^{ij} a_{ij} = \tilde{F}^{ij} u_{ij} - \frac{2 F^{11} u_1^2 u_{ij}}{u_1^5} - \frac{u_{11} u_1}{u_1^2} F^{ij} a_{ij} = (f)_1.
\]

Note that \((f)_1 = f_{x_1} + f_u u_1 + f_{\nu_1} (\nu_1)_1 \). By \((5.3)\) and \( f_u \geq 0 \), we have

\[
\frac{(f)_1}{u_1} \geq - \frac{C}{u_1} + \frac{u_{11} f_{x_1}}{u_1} \left( \frac{u_1^2}{u_1^2 - 1} \right) - f_{\nu_{u_1}} \frac{u_{11}}{u_1^2} \geq - \frac{C A}{u_1},
\]
where $C$ depends on $|Df|_{C^0} \equiv \sup_{\Omega \times [-C_0,C_0] \times \mathbb{S}^n} |Df|$. Substituting (5.5) and (5.6) into (5.4), we have

\begin{equation}
0 \geq \frac{2F^{ij}u_{ij}^2}{w^5} + \frac{u_{11}F^{ij}u_{ij}}{w^2} - \frac{CA}{u_1} - \frac{\hat{F}^{ij}u_{ij}}{u_1^2} + A\hat{F}^{ij}u_{ij}
\end{equation}

(5.7)

\begin{equation}
\geq \frac{2u_1^2}{w^2} - 1) \frac{F^{ij}u_{ij}^2}{w^3u_1^2} + (1 - \frac{u_2^2}{w^2})AC^n f - \frac{CA}{u_1}
\end{equation}

where in the last inequality we used $F^{ij}u_{ij} = \hat{F}^{ij}u_{ij} = C_p f$. From (5.3), we see $u_{11} = -Au_1^2 < 0$. By Lemma 2.2 (4) and (2), there exists a positive constant $c_2$ depending on $n$, $p$ and $\inf f$ such that

\begin{equation}
0 \geq c_2A^2u_{11}^4 - \frac{CA}{u_1},
\end{equation}

(5.8)

which is a contradiction to large $A$ and $|Du|$. Therefore, $Q$ attains its maximum on $\partial \Omega$ and hence (5.2) is proved.

\[ \square \]

**Remark 5.2.** The interior gradient estimate was proved by Li [32] for Weingarten curvature equations of general type by the normal perturbation method.

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