Berezinskii–Kosterlitz–Thouless Transition in a Two-Dimensional Random-Bond XY Model on a Square Lattice *

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We perform Monte Carlo simulations to study the two dimensional random-bond XY model on a square lattice. Two kinds of bond randomness with the coupling coefficient obeying the Gaussian or uniform distribution are discussed. It is shown that the two kinds of disorders lead to similar thermodynamic behaviors if their variances take the same value. This result implies that the variance can be chosen as a characteristic parameter to evaluate the strength of the randomness. In addition, the Berezinskii–Kosterlitz–Thouless transition temperature decreases as the variance increases and the transition can even be destroyed as long as the disorder is strong enough.

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The Berezinskii–Kosterlitz–Thouless (BKT) transition in two-dimensional (2D) systems has been extensively studied for decades since the discovery of the exotic quasi-long-range order formed by the binding of vortex-antivortex pairs.[1,2] The simplest model to demonstrate the BKT transition is the so-called 2D XY model and the Hamiltonian takes the form

\[ H = -\sum_{\langle ij \rangle} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j = -\sum_{\langle ij \rangle} J_{ij} \cos \theta_{ij}, \]

where \( \mathbf{S}_i = (S_i^x, S_i^y) = (\cos \theta_i, \sin \theta_i) \) donates spin at site \( i \); \( J_{ij} \) is the coupling coefficient of the two nearest-neighboring sites, \( i \) and \( j \); and \( \theta_{ij} = \theta_i - \theta_j \) is the phase difference between the two sites. Typically, the transition in the 2D XY model is characterized by low temperature power-law decay of a two-point correlation function which gives rise to divergent susceptibility,[3] measurable finite-size-induced magnetization with universal magnetic exponent[4] and a discontinuous jump to zero of the helicity modulus.[5] In experiments, the BKT transition has been confirmed in various real systems, such as \(^4\)He films,[6] Josephson-junction arrays[8] and planar lattice of Bose–Einstein condensates.[8]

Due to the presence of defects and distortions, real systems are always imperfect and are usually subject to certain disorder effects. Therefore it is of interest to study how the imperfection affects the BKT transition. For the 2D XY model, the imperfection can be demonstrated by two parameters, \( J_{ij} \) and \( \theta_{ij} \), and \( J_{ij} \) (or \( \theta_{ij} \)) may be governed by a random distribution \( P(J_{ij}) \) (or \( P(\theta_{ij}) \)). This means that \( J_{ij} \) (or \( \theta_{ij} \)) takes the values subject to a probability distribution, which is called the bond randomness (or phase randomness).

Rubinstein et al.[9] studied 2D XY ferromagnets with random Dzyaloshinskii–Moriya interactions and derived a model with both \( J_{ij} \) and \( \theta_{ij} \) randomness. They showed that the spatial variation in \( J_{ij} \) might be irrelevant at long wavelengths. For this reason, less attention was paid to this type of disorder. However, the case that \( J_{ij} \) obeys a discrete probability distribution has still been intensively studied. A simple choice of \( P(J_{ij}) \) is \( P(J_{ij}) = p \delta(J_{ij} - J_0) + (1 - p) \delta(J_{ij}) \)[10–12] which means that each bond might be vacant with probability \( 1 - p \). This model is often referred to as the bond diluted model.

In fact, the continuous bond randomness case that \( J_{ij} \) obeys a continuous probability distribution should not be neglected. Korshunov[13] argued that the continuous bond randomness of the coupling coefficient can greatly change the critical behavior of the BKT transition, as long as the randomness is strong enough. Recently this point has also been discussed on the basis of the six-state clock model.[14] These works motivate us to study the 2D XY model with continuous bond randomness.

It is plausible that concrete forms of different probability distributions of the coupling may influence the BKT transition differently. In this study we try to make sure whether it is true or not. We consider two kinds of bond randomness, for which the distribution function, \( P(J_{ij}) \), obeys the Gaussian distribution and uniform distribution.

The 2D XY model under present consideration is defined on a square lattice. For the first case, \( P(J_{ij}) \) is given by

\[ P(J_{ij}) = \frac{1}{\sqrt{2\pi}\sigma^2} \exp \left[ -\frac{(J_{ij} - J_0)^2}{2\sigma^2} \right], \]

where \( J_0 = 1 \) is the mean value of the coupling coefficients and \( \sigma^2 \) is the variance. For the second case, \( J_{ij} \)
distributes uniformly in the region \([J_0 - d, J_0 + d]\),

\[ P(J_{ij}) = \frac{1}{2d} = \text{const}, \]  

and variance of the uniform distribution is determined as

\[ \sigma^2 = \langle (J_{ij} - J_0)^2 \rangle = \frac{d^2}{3}. \]  

We will characterize the BKT transition by investigating the thermodynamics of the random-bond 2D \(XY\) model. The thermodynamic quantities calculated in the following include the finite-size magnetization,

\[ m = \langle |M| \rangle, \]  

the susceptibility and the specific heat,

\[ \chi = \frac{N}{k_B T} \langle (M^2) - \langle M \rangle^2 \rangle, \]  
\[ C = \frac{N}{k_B T^2} \langle (H'^2) - \langle H' \rangle^2 \rangle, \]  

where

\[ M = (M_x, M_y) = \left( \frac{1}{N} \sum \sin \theta_i, \frac{1}{N} \sum \cos \theta_i \right), \]  
\[ H' = -\frac{1}{N} \sum_{\langle ij \rangle} J_{ij} S_i \cdot S_j, \]  

\( N \) is the number of sites and \( \langle \cdot \rangle \) denotes thermodynamic average. To calculate \( \chi \) and \( C \), we record \( M \) and \( H' \) after each Monte Carlo step and exploit their fluctuations.\(^{[14]}\)

In addition, the helicity modulus is a useful parameter to study the BKT transition for its characteristic feature of a discontinuous universal jump to zero at critical temperature.\(^{[1]}\) The helicity modulus takes the form\(^{[10]}\)

\[ \langle Y \rangle = \langle e \rangle - \frac{N}{k_B T} \langle s^2 \rangle, \]  

where

\[ e \equiv \frac{1}{N} \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j), \]  
\[ s \equiv \frac{1}{N} \sum_{\langle ij \rangle} \sin(\theta_i - \theta_j). \]  

The notation \( \langle ij \rangle \) means that the sum is taken over all links in one direction only.

All the thermodynamic quantities are calculated numerically by employing the standard Metropolis Monte Carlo method under periodic boundary conditions.\(^{[17]}\) The square lattice includes \( N = 64 \times 64 \) sites and we perform \( 10^6 \) Monte Carlo steps to produce each numerical result.

\[ \text{Fig. 1. Finite-size magnetization (a) and helicity modulus (b) versus temperature for the random-bond } XY \text{ model with Gaussian distribution (marked as g) and uniform distribution (marked as u) of couplings. The dashed lines in (b) and (a) plot Eq. (14) and fitting curves for Gaussian-type magnetization based on Eq. (13), respectively. The error bars are obtained by using the standard deviations of numerical results.} \]

\[ \text{Fig. 2. Susceptibility (a) and specific heat (b) versus temperature for the random-bond } XY \text{ model with Gaussian distribution (marked as g) and uniform distribution (marked as u) of couplings.} \]

In Figs. 1 and 2, we plot the thermodynamic quantities versus temperature for both a Gaussian distribution and uniform distribution. For the two different distributions, all four thermodynamic quantities match pretty well when their variances \( \sigma^2 \) take the same value, especially at high temperatures. Our results suggest that different probability distributions
of couplings bring about similar effect on properties of the random-bond 2D XY model. In addition, it seems that the variance is to some extent a good parameter to evaluate the strength of the disorder of the random system. However, the matching between the two distribution cases becomes worse if the randomness tends to be extremely strong, for example, \( \sigma^2 \geq 0.6 \). A possible reason is that thermodynamic behaviors become dependent on the concrete random distribution of coupling coefficients in the strong random limit. The BKT scenario might be invalid when the disorder is extremely strong.\(^{[14]}\)

To proceed, we attempt to locate the BKT transition temperature \( T_c \). Usually, \( T_c \) can be determined by the finite-size magnetization and the helicity modulus in Fig. 1, while in Fig. 2 the peak of susceptibility is not used to locate \( T_c \) for lack of accuracy.\(^{[14,18]}\) and the peak of specific heat occurs a certain percent above \( T_c \).\(^{[19]}\) The universal critical behavior of finite size induced magnetization,\(^{[4]}\)

\[
m(T \to T^*) \sim (T_c - T)^{0.21},
\]

can be used to estimate the critical temperature by curve fitting,\(^{[17]}\) where \( T^* \) is a temperature near which the power law behavior \( (T_c - T)^{0.21} \) holds best. Figure 1(a) shows the fitting of magnetization. For \( \sigma^2 = 0.0, 0.1, 0.2, 0.4, \) and \( 0.6 \), the estimated value of \( k_B T_c/J_0 \) is 1.03, 0.99, 0.93, 0.81, and 0.68, respectively.

The renormalization theory predicts that the helicity modulus jumps from \((2/\pi)k_B T_c\) to zero in the thermodynamic limit.\(^{[5]}\) It has been proven that this characteristic also exists in a bond diluted model\(^{[18,20]}\) which indicates that it can be applied in our random-bond model. Therefore, \( T_c \) can be estimated from the intersection of \( \Upsilon(T) \) and the straight line,\(^{[18]}\)

\[
\Upsilon = \frac{2}{\pi} k_B T.
\]

Estimating from intersecting points of curves and the straight line in Fig. 1(b), we obtain that \( k_B T_c/J_0 \approx 0.92, 0.87, 0.81, 0.70, \) and 0.52 for \( \sigma^2 = 0.0, 0.1, 0.2, 0.4, \) and 0.6, respectively.

There is a comparable difference between the BKT transition temperatures determined by the two methods. It might owe to the finite size effect, since both methods to locate \( T_c \) are size-dependent while their size dependences are different. In addition, the obtained results of the helicity modulus lose accuracy when the disorder turns stronger, which can be seen in Fig. 1(b) where the error becomes quite large at \( \sigma^2 = 0.6 \) and 0.8. Nevertheless, the tendency that \( T_c \) decreases with the increasing \( \sigma^2 \) can be confirmed, as shown in Figs. 2(a) and 2(b).

For \( \sigma^2 = 0.8 \), curve fitting of the magnetization gives \( k_B T_c/J_0 = 0.53 \), while no appropriate intersection point can be found in the helicity modulus in Fig. 1(b). Thus a question arises: what happens in the strong randomness limit, for example, at \( \sigma^2 \geq 0.8 \)? A possible answer is that the BKT transition is already destroyed by disorder in this case.\(^{[14]}\)

To further verify the existence of the BKT transition we calculate the fourth-order helicity modulus \( \Upsilon_4 \) as suggested in Ref.\(^{[16]}\), which proved that the negative value of \( \Upsilon_4 \) guarantees the discontinuous jump at \( T_c \), and thus guarantees the BKT transition. The fourth-order helicity modulus can be expressed as

\[
\langle \Upsilon_4 \rangle = -\frac{4}{N} \langle Y \rangle + 3 \frac{\langle e^2 \rangle}{N} - \frac{1}{k_B T} \langle (Y - \langle Y \rangle)^2 \rangle
+ \frac{2N^2}{T^2} \langle s^4 \rangle.
\]

The obtained result is shown in Fig. 3. There exists a trough around \( T_c \) in \( \Upsilon_4 \). When \( \sigma^2 \) increases, the depth of the trough may decrease while the negative of \( \Upsilon_4 \) at \( T_c \) is still quite clear for \( \sigma^2 = 0.4 \). For \( \sigma^2 = 0.6 \), we can still identify that \( \Upsilon_4 \) is negative near \( k_B T_c/J_0 \approx 0.52 \), despite the error in the data. However, as \( \sigma^2 \) reaches 0.8, the trough is overshadowed by noise and no discontinuous jump of helicity modulus can be confirmed, which implies that the BKT-type transition disappears.

In summary, thermodynamic properties of 2D random-bond XY model on a square lattice are studied by using Monte Carlo simulations. The randomness may arise from disorder in real systems. Two kinds of random-bond models are considered, with the probability distributions of the coupling coefficient obeying the Gaussian distribution and the uniform distribution. We show that thermodynamic quantities of the two models are in good agreement as long as their variances take the same value. Thus the variance can be taken as a characteristic parameter to evaluate the strength of the randomness. Moreover, it is shown that the temperature of the BKT transition is suppressed as the randomness becomes stronger. Fur-
thermore, the BKT transition could even be destroyed in the strongly disordered cases.

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