ADDENDUM TO: A SURVEY OF MUČNIK AND MEDVEDEV DEGREES

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Abstract. We include here some material that did not make its way into the published version [2], in particular a proof of Theorem K to the effect that there is an initial segment of the strong degrees with dual theory IPC, the intuitionistic propositional calculus.

Contents

17. Implicative lattices (2) 1
18. Open algebras and completeness 6
19. Proof of Theorem K 14
References 18

§17. Implicative lattices (2). This section presents some additional facts about (dual-)implicative lattices in general and $\mathbb{D}_s$ in particular.

Lemma 15.1 showed that when $L \hookrightarrow K$ is a lattice embedding that respects $\to$, then $\text{Th}(K) \subseteq \text{Th}(L)$. We write $K \twoheadrightarrow L$ iff there exists a surjection of $K$ onto $L$ that respects the lattice operations and $\to$ and have similarly

Lemma 17.1. For any implicative lattices $K$ and $L$, if $K \twoheadrightarrow L$, then $\text{Th}(K) \subseteq \text{Th}(L)$.

Proof. If $\eta$ is a surjection of $K$ onto $L$ as described, then for any $L$-valuation $w$, there exists a function $v_0$ such that for each atomic sentence $p$, $w(p) = \eta(v_0(p))$ and hence a $K$-valuation $v$ such that $w = \eta \circ v$. Then for any $\phi \in \text{Th}(K)$ and any $L$-valuation $w$,

$$w(\phi) = \eta(v(\phi)) = \eta(1_K) = 1_L,$$

so $\phi \in \text{Th}(L)$.

We shall need a particular family of instances of this (see Definition 5.9):

Lemma 17.2. For any implicative lattice $L$ and $c,d,e \in L$, $L[c,d] \twoheadrightarrow L[c \land e, d \land e]$. 

PROOF. Given $\mathcal{L}$ and $c, d,$ and $e$, set $\eta(a) := a \land e$. $\eta$ is surjective because for any $f$ with $c \land e \leq f \leq d \land e$, $f = \eta(c \lor f)$. $\eta$ respects $0, 1, \land$ and $\lor$ by definition and distributivity. For $\eta$ to respect $\to$ (see Lemma 5.10), means that for any $a, b \in L[c, d]$

$$(a \to_d b) \land e = (a \land e) \to_d (b \land e),$$

that is,

$$(a \to b) \land d \land e = (a \land e \to b \land e) \land d \land e.$$  

This follows because by the definition of $\to_d$, $(a \to b) \leq (a \land e \to b \land e)$, since

$$(a \land e) \land (a \to b) = a \land (a \to b) \land e \leq b \land e,$$

and $(a \land e \to b \land e) \land e \leq a \to b$ because

$$a \land (a \land e \to b \land e) \land e = (a \land e) \land (a \land e \to b \land e)$$

$$\leq b \land e \leq b.$$

COROLLARY 17.3. For any lattice $\mathcal{L}$ and $c, d, e \in L$,

(i) if $\mathcal{L}$ is implicative, then $Th(\mathcal{L}[c, d]) \subseteq Th(\mathcal{L}[c \land e, d \land e])$;

(ii) if $\mathcal{L}$ is dual-implicative, then $Th^o(\mathcal{L}[c, d]) \subseteq Th^o(\mathcal{L}[c \lor e, d \lor e])$.

PROOF. Part (i) is immediate from the preceding two lemmas and (ii) is its dual—explicitly,

$$Th^o(\mathcal{L}[c, d]) = Th(\mathcal{L}^o[d, c]) \subseteq Th(\mathcal{L}^o[d \lor e, c \lor e])$$

$$= Th(\mathcal{L}[c \lor e, d \lor e]^o) = Th^o(\mathcal{L}[c \lor e, d \lor e]).$$

Next we consider some questions of distributivity; we will need only a few of the clauses of the next lemma but include them all for reference.

LEMMA 17.4. For any lattice $\mathcal{L}$ and $a, b, c, \text{and } d \in L$,

(i) if $\mathcal{L}$ is implicative, then

1. $a \to (c \land d) = (a \to c) \land (a \to d)$
2. $a \to (c \lor d) \geq (a \to c) \lor (a \to d)$
3. $(a \lor b) \to c = (a \to c) \land (b \to c)$
4. $(a \land b) \to c \geq (a \to c) \lor (b \to c)$;

(ii) if $\mathcal{L}$ is dual-implicative, then

1. $a \to_d (c \land d) \leq (a \to_d c) \land (a \to_d d)$
2. $a \to_d (c \lor d) = (a \to_d c) \lor (a \to_d d)$
3. $(a \lor b) \to_d c \leq (a \to_d c) \land (b \to_d c)$
4. $(a \land b) \to_d c = (a \to_d c) \lor (b \to_d c)$.

PROOF. These are all straightforward calculations.
Of interest in the next section will be an extension of (1)° in a certain special case. Recall that before Lemma 2.2 we defined for $P \subseteq \omega \omega$ the **upward Turing closure**

$$P^{\geq_T} := \{ g : (\exists f \in P) f \leq_T g \};$$

we call $P$ ($p \in D_s$) **upward Turing closed** iff $P = P^{\geq_T}$ (for some $P \in p$, $P = P^{\geq_T}$).

By the proofs of Lemma 2.2 and Proposition 2.5, **Lemma 17.5.** For any upward Turing closed sets $P, Q \subseteq \omega \omega$,

(i) $P \leq_s Q$ $\iff$ $P \supseteq Q$;

(ii) $P \uparrow Q = P \cap Q$.

**Proposition 17.6.** For all upward Turing closed $P$ and all $Q, Q_0, Q_1 \subseteq \omega \omega$,

(i) $Q \rightarrow P \equiv_s \{ h : (\forall g \in Q) g \oplus h \in P \}$;

(ii) $d g_s (P)$ is meet-irreducible;

(iii) $P \rightarrow (Q_0 \wedge Q_1) \equiv_s (P \rightarrow Q_0) \wedge (P \rightarrow Q_1)$.

**Proof.** For (i), set $I$ to the right-hand set. Then

$$Q \rightarrow P := \{ (a)^{\sim} h : (\forall g \in Q) \{ a \}^{g \oplus h} \in P \} \leq_s I$$

witnessed by the mapping $h \mapsto (\bar{a})^{\sim} h$ where $\{ \bar{a} \}^{g \oplus h} = g \oplus h$. On the other hand, $(a)^{\sim} h \in (Q \rightarrow P) \implies h \in I$, since $\{ a \}^{g \oplus h} \leq_T g \oplus h$, so $\{ a \}^{g \oplus h} \in P \implies g \oplus h \in P$. Hence $I \leq_s (Q \rightarrow P)$ via the mapping $(a)^{\sim} h \mapsto h$.

For (ii), suppose that $Q_0 \wedge Q_1 \leq_s P$, so some recursive $\Phi : P \rightarrow Q_0 \wedge Q_1$. Then there exist $\sigma$ and $i < 2$ such that $\Phi(\sigma)(0) = i$. Set $\Psi(f)(n) := \Phi(\sigma^{-f})(n + 1)$. Then $\Psi : P \rightarrow Q_i$ because

$$f \in P \implies \sigma^{-f} \in P \implies \Phi(\sigma^{-f}) = (i)^{-f} \Psi(f) \in Q_0 \wedge Q_1.$$

Finally, for (iii), by (1)° above, we need only construct a recursive

$$\Psi : (P \rightarrow (Q_0 \wedge Q_1)) \rightarrow ((P \rightarrow Q_0) \wedge (P \rightarrow Q_1)).$$

From $(a)^{\sim} h \in P \rightarrow (Q_0 \wedge Q_1)$ we can compute a least pair $(\sigma_{a,h}, i_{a,h})$ such that $\{ a \}^{\sigma_{a,h} \oplus h}(0) = i_{a,h}$ and hence an index $b_{a,h}$ such that for any $f$,

$$\{ b_{a,h} \}^{f \oplus h}(n) = \{ a \}^{(\sigma_{a,h}^{-f})}(n + 1).$$

Set $\Psi((a)^{\sim} h) := (i_{a,h}, b_{a,h})^{\sim} h$. Then for all $f \in P$, also $\sigma_{a,h}^{-f} \in P$, so $\{ b_{a,h} \}^{f \oplus h} \in Q_{i_{a,h}}$. Hence $(b_{a,h})^{\sim} h \in P \rightarrow Q_{i_{a,h}}$ and thus $\Psi((a)^{\sim} h) \in (P \rightarrow Q_0) \wedge (P \rightarrow Q_1)$.

Finally, we review some standard algebraic notions as they apply to implicative lattices.
Definition 17.7. For any lattice $\mathcal{L}$ and $F \subseteq L$, $F$ is a filter on $\mathcal{L}$ iff
\[ \emptyset \neq F \neq L, \text{ and for all } a, b \in L, \]
(i) $a \in F$ and $a \leq b$ $\implies$ $b \in F$;
(ii) $a, b \in F$ $\implies$ $a \wedge b \in F$.

$F$ is a prime filter on $\mathcal{L}$ iff additionally
(iii) $a \vee b \in F$ $\implies$ $a \in F$ or $b \in F$.

The easiest examples of filters are the principal filters: for $e \neq 0$,
\[ H_e := \{ a \in L : e \leq a \}. \]

Easily, if $\mathcal{L}$ is distributive, then $H_e$ is prime iff $e$ is join-irreducible. More generally, for any $A \subseteq L$, if $A$ has the finite intersection property (FIP):
\[ \text{for all finite } B \subseteq A, \ \bigwedge B \neq 0, \]
then
\[ H_A := \left\{ a \in L : (\exists \text{ finite } B \subseteq A) \ \bigwedge B \leq a \right\} \]
is called the filter generated by $A$.

Lemma 17.8. For any distributive lattice $\mathcal{L}$ bounded below, any filter $F$ on $\mathcal{L}$ and any $c, d \in L$ such that $c \vee d \in F$ but both $c, d \notin F$,
(i) both $F \cup \{c\}$ and $F \cup \{d\}$ have the FIP;
(ii) $H_{F \cup \{c\}} \cap H_{F \cup \{d\}} = F$.

Proof. For (i), suppose that $F \cup \{c\}$ does not have the FIP; since $F$ is closed under meet, there exists $a \in F$ such that $a \wedge c = 0$. But then
\[ d \geq a \wedge d = (a \wedge c) \vee (a \wedge d) = a \wedge (c \vee d) \in F, \]
so $d \in F$ contrary to hypothesis. For (ii), for any $e \in H_{F \cup \{c\}} \cap H_{F \cup \{d\}}$ there exist $a, b \in F$ such that
\[ a \wedge c \leq e \ \text{ and } \ b \wedge d \leq e. \]
Then also $a \wedge b \wedge c \leq e$ and $a \wedge b \wedge d \leq e$, so
\[ e \geq (a \wedge b \wedge c) \vee (a \wedge b \wedge d) = (a \wedge b) \wedge (c \vee d) \in F, \]
and thus $e \in F$.

A filter is called maximal in a class $\mathcal{G}$ of filters iff $F \in \mathcal{G}$ but $\mathcal{G}$ contains no proper extension of $F$ — that is there is no filter $G$ such that $F \subsetneq G \in \mathcal{G}$. If $\mathcal{G} \neq \emptyset$ is closed under unions of chains, then by Zorn’s Lemma (for the infinite case) $\mathcal{G}$ has a maximal element.

Corollary 17.9. For any distributive lattice $\mathcal{L}$ and any filter $F$ on $\mathcal{L}$,
(i) $F$ is maximal in the class of all filters on $\mathcal{L}$ $\implies$ $F$ is prime;
(ii) for any $a, b \in L$, if $(\forall e \in F) e \wedge a \not\leq b$, then there exists a prime filter $G \supseteq F$ such that $a \in G$ but $b \notin G$.

**Proof.** For (i), if $F$ is maximal but not prime, then for some $c, d \notin F$, $c \lor d \in F$. But then by (i) of the lemma, $H_{F \cup \{c\}}$ would be a proper extension of $F$, a contradiction.

For (ii), given $a$ and $b$ as described, set

$$G := \{ G : \text{ } G \supseteq F \text{ is a filter such that } a \in G \text{ and } b \notin G \}.$$

$G \neq \emptyset$ since $H_{F \cup \{a\}} \in G$. Easily $G$ is closed under unions of chains, so has a maximal element $\overline{G}$. Suppose towards a contradiction that $\overline{G}$ is not prime, so for some $c, d \notin \overline{G}$, $c \lor d \in \overline{G}$. Then by (ii) of the lemma, $b$ does not belong to both $H_{\overline{G} \cup \{c\}}$ and $H_{\overline{G} \cup \{d\}}$, say $b \notin H_{\overline{G} \cup \{c\}}$. Then $H_{\overline{G} \cup \{c\}}$ is a proper extension of $\overline{G}$ in $G$, contrary to the choice of $\overline{G}$. $\dashv$

**Corollary 17.10.** For any implicative lattice $\mathcal{L}$, any filter $F$ on $\mathcal{L}$ and any $a, b \in L$,

$$(a \rightarrow b) \in F \iff (\forall \text{ prime filters } G \supseteq F) a \in G \implies b \in G.$$

**Proof.** Since $a \wedge (a \rightarrow b) \leq b$, if both $a$ and $(a \rightarrow b)$ are in $G$, then so is $b$. Conversely, if $(a \rightarrow b) \notin F$, then for each $e \in F$, $e \not\leq (a \rightarrow b)$ so $e \wedge a \not\leq b$. Thus the result follows by (ii) of the preceding corollary. $\dashv$

**Remark 17.11.** All of these phenomena may also be described in terms of ideals: $I \subseteq L$ is an **ideal on** $\mathcal{L}$ iff $\emptyset \neq I \neq L$ and for all $a, b \in L$,

(i) $b \in I$ and $a \leq b \implies a \in I$;
(ii) $a, b \in I \implies a \lor b \in I$;

and a **prime ideal** iff additionally

(iii) $a \wedge b \in I \implies a \in I$ or $b \in I$.

A filter (ideal) on $\mathcal{L}^\circ$ is an ideal (filter) on $\mathcal{L}$ and $F$ is a prime filter (ideal) on $\mathcal{L}$ iff $L \setminus F$ is a prime ideal (filter) on $\mathcal{L}$.

**Remark 17.12.** As in other branches of algebra, ideals and filters on lattices lead to factor structures. If $F$ is a filter on a lattice $\mathcal{L}$, then

$$a \leq_F b :\iff (\exists e \in F) a \wedge e \leq b$$

is a reflexive and transitive relation on $L$ so

$$a \sim_F b :\iff a \leq_F b \text{ and } b \leq_F a$$

is an equivalence relation and $\leq_F$ induces a partial ordering on the set $L/F$ of equivalence classes. Defining $\wedge_F$ and $\lor_F$ in the obvious way leads to a lattice $\mathcal{L}/F$. In particular, $\mathcal{L}/H_e$ is isomorphic to the initial segment $\mathcal{L}[0, e]$. Note that if $\mathcal{L}$ is implicative, then $a \leq_F b \iff (a \rightarrow b) \in F$. 
§18. Open algebras and completeness. To prepare for the proof of Theorem K, but also for independent interest, we develop in this section some new examples of (dual-)implicative lattices and refinements of the IPC- and WEM-completeness theorems of Section 14. Recall that we call a partial ordering bounded (above) (below) iff it has a greatest or least element or both.

**Definition 18.1.** For any partial ordering $\mathfrak{P} = (P, \leq)$,

(i) $O(\mathfrak{P}) := \{ A \subseteq P : (\forall a, b \in P) \ a \in A \text{ and } a \leq b \implies b \in A \}$;

(ii) $\mathfrak{D}(\mathfrak{P}) := (O(\mathfrak{P}), \emptyset, P, \cap, \cup, \subseteq)$.

Members of $O(\mathfrak{P})$ are described as the open sets in the order topology. Hence it is just a variation on the example preceding Proposition 4.3 to observe that

**Lemma 18.2.** For every partial ordering $\mathfrak{P}$,

(i) $\mathfrak{D}(\mathfrak{P})$ is an implicative lattice;

(ii) if $\mathfrak{P}$ is bounded below, then $\mathfrak{D}(\mathfrak{P})$ is 1-irreducible;

(iii) if $\mathfrak{P}$ is bounded above, then $\mathfrak{D}(\mathfrak{P})$ is 0-irreducible.

**Proof.** Part (i) is straightforward using the implication operator

$$A \to B := \bigcup \{ C \in O(\mathfrak{P}) : A \cap C \subseteq B \}.$$ If $\mathfrak{P}$ has least element $0$, then for $A \in O(\mathfrak{P})$, $A = 1_{O(\mathfrak{P})} = P$ iff $0 \in A$, so if $A = B \cup C$, then one of $B$ or $C$ contains $0$. If $\mathfrak{P}$ has greatest element $1$, then $A = 0_{O(\mathfrak{P})} = \emptyset$ iff $1 \notin A$, so if $A = B \cap C$, then one of $B$ or $C$ fails to contains $1$.

Our first refinement of the completeness theorems is:

**Proposition 18.3.**

IPC $= \bigcap \{ \text{Th}(\mathfrak{D}(\mathfrak{P})) : \mathfrak{P} \text{ is a finite partial ordering bounded below} \}$;

WEM $= \bigcap \{ \text{Th}(\mathfrak{D}(\mathfrak{P})) : \mathfrak{P} \text{ is a finite bounded partial ordering} \}$.

**Proof.** We shall show that for any finite 1-irreducible implicative lattice $\mathfrak{L}$, there exists a finite partial ordering $\mathfrak{P}$ bounded below such that $\mathfrak{L} \rightarrow \mathfrak{D}(\mathfrak{P})$, and if $\mathfrak{L}$ is also 0-irreducible, then we may choose $\mathfrak{P}$ bounded. Then by Lemma 15.1, $\text{Th}(\mathfrak{D}(\mathfrak{P})) \subseteq \text{Th}(\mathfrak{L})$, so by the completeness theorems,

$$\bigcap \{ \text{Th}(\mathfrak{D}(\mathfrak{P})) : \mathfrak{P} \text{ is a finite partial ordering bounded below} \}$$

$$\subseteq \bigcap \{ \text{Th}(\mathfrak{L}) : \mathfrak{L} \text{ is a finite 1-irreducible implicative lattice} \}$$

$$= \text{IPC},$$
and
\[ \bigcap \{ \text{Th}(\mathcal{D}(\mathcal{P})) : \mathcal{P} \text{ is a finite bounded partial ordering} \} \]
\[ \subseteq \bigcap \{ \text{Th}(\mathcal{L}) : \mathcal{L} \text{ is a finite 0- and 1-irreducible implicative lattice} \} \]
\[ = \text{WEM}. \]

The converse inclusions are immediate from the lemma.

Fix such an \( \mathcal{L} \) and set
\[ P := \{ F : F \text{ is a prime filter on } \mathcal{L} \}; \]
\[ \mathcal{P} := (P, \subseteq). \]

\( \mathcal{P} \) is bounded below by the unit filter \( \{ 1 \} \), which is prime because \( 1 \)
is join-irreducible, and if \( \mathcal{L} \) is 0-irreducible, then also \( \mathcal{P} \) has a greatest
element \( L \setminus \{ 0 \} \). Define \( \eta : \mathcal{L} \to \mathcal{O}(\mathcal{P}) \) by
\[ \eta(a) := \{ F \in P : a \in F \}. \]

\( \eta \) is injective and respects \( \leq \) by Corollary 17.9 (ii). It is straightforward
to check respect of the lattice operations:

\[ \eta(a \land b) = \{ F \in P : a \land b \in F \} = \{ F \in P : a \in F \text{ and } b \in F \} = \eta(a) \cap \eta(b), \]
\[ \eta(a \lor b) = \{ F \in P : a \lor b \in F \} = \{ F \in P : a \in F \text{ or } b \in F \} = \eta(a) \cup \eta(b), \]

and that \( \eta(0) = \emptyset \) and \( \eta(1) = P \). Finally, by Corollary 17.10, for any
\( F \in P, \)
\[ F \in \eta(a \to b) \iff (\forall \text{ prime } G \supseteq F) \ G \in \eta(a) \Rightarrow G \in \eta(b) \]
\[ \iff \eta(a) \cap \{ G \in P : F \subseteq G \} \subseteq \eta(b) \]
\[ \iff F \in (\eta(a) \to \mathcal{O}(\mathcal{P}) \eta(b)). \]

Remark 18.4. The existence of embeddings \( \mathcal{L} \hookrightarrow \mathcal{O}(\mathcal{P}) \) in the pre-
ceding proof can be viewed as a representation theorem for implicative
lattices. It is quite parallel to the well-known Stone Representation The-
orem for Boolean algebras, both in statement and proof.

The proposition can also be viewed as an alternative formulation of the
Kripke semantics for intuitionistic propositional logic. Given a partial
ordering \( \mathcal{P} = (P, \subseteq) \) and a valuation \( v : \mathcal{P}S \to \mathcal{O}(\mathcal{P}) \), define
\[ (*) \quad a \vdash \phi \iff a \in v(\phi). \]
This forcing relation on \( \mathcal{P} \) easily satisfies the conditions \( a \not\models \bot \),

\[
\begin{align*}
    a \models \phi \text{ and } a \leq b & \implies b \models \phi \\
    a \models \phi \land \psi & \iff a \models \phi \text{ and } a \models \psi \\
    a \models \phi \lor \psi & \iff a \models \phi \text{ or } a \models \psi \\
    a \models \phi \rightarrow \psi & \iff (\forall b \geq a) b \models \phi \implies b \models \psi.
\end{align*}
\]

Any \( \mathfrak{M} = (P, \leq, \models) \) with these properties is called a Kripke model. Conversely, given a Kripke model, the function \( v \) defined by (*) is a valuation.

A sentence \( \phi \) is true in \( \mathfrak{M} \) — in symbols, \( \models_{\mathfrak{M}} \phi \) — iff \( (\forall a \in P) a \models \phi \) — that is, \( v(\phi) = P = 1_{\mathfrak{M}(\mathcal{P})} \). Hence

\[
\text{Th}(\mathfrak{M}(\mathcal{P})) = \left\{ \phi : \text{for all forcing relations } \models \text{ on } \mathcal{P}, \vDash_{(P, \leq, \models)} \phi \right\}
\]

so by the proposition,

\[
\text{IPC} = \left\{ \phi : \text{for all Kripke models } \mathfrak{M}, \vDash_{\mathfrak{M}} \phi \right\},
\]

which is one version of the Kripke Completeness Theorem. Of course, we have also the stronger version which restricts to Kripke models based on finite partial orders bounded below.

Looking at these algebras in a slightly different way, leads to another useful algebra.

**Definition 18.5.** For any partial ordering \( \mathcal{P} = (P, \leq) \),

1. For any \( A \subseteq P \), \( A^* := \{ b \in P : (\exists a \in A) a \leq b \} \);
2. \( O^\omega(\mathcal{P}) := \{ A^* : A \subseteq P \text{ and } A \text{ is finite} \} \).

Obviously \( O(\mathcal{P}) = \{ A^* : A \subseteq P \} \), so \( O^\omega(\mathcal{P}) \subseteq O(\mathcal{P}) \) with equality for finite \( \mathcal{P} \). In general, \( O^\omega(\mathcal{P}) \) is not naturally the domain of a lattice because it may fail to be closed under intersection. But this problem vanishes under simple natural conditions. Note below that although the notion of dual-implicativity was formally defined only for lattices it applies also to upper semi-lattices.

**Proposition 18.6.** For any upper semi-lattice \( \mathcal{P} = (P, \leq, 0, \lor) \) that is bounded below,

1. \( O^\omega(\mathcal{P}) := (O^\omega(\mathcal{P}), 0, P, \cap, \lor, \subseteq) \) is a bounded lattice;
2. if \( \mathcal{P} \) is dual-implicative, then \( O^\omega(\mathcal{P}) \) is implicative.

**Proof.** Fix \( \mathcal{P} \) as described. Clearly \( O^\omega(\mathcal{P}) \) is closed under union and contains the least element \( 0 \). The greatest element \( P = \{0\}^* \in O^\omega(\mathcal{P}) \), and for finite \( A, B \subseteq P \),

\[
A^* \cap B^* = \{ a \lor b : a \in A \text{ and } b \in B \}^* \in O^\omega(\mathcal{P}).
\]
Thus $\mathfrak{O}^\omega(\mathfrak{P})$ is a lattice. Suppose that $\mathfrak{P}$ is dual-implicative via $\rightarrow \rightarrow$. Then

$$A^* \rightarrow B^* := \bigcap_{a \in A} \{ a \rightarrow b : b \in B \}^*$$

is an implication operator for $\mathfrak{O}^\omega(\mathfrak{P})$ because for all finite $A, B, C \subseteq P$,

$$C^* \subseteq A^* \rightarrow B^* \iff (\forall c \in C)(\forall a \in A)(\exists b \in B) (a \rightarrow b) \leq c$$

$$\iff (\forall c \in C)(\forall a \in A) b \leq a \lor c$$

$$\iff (\forall c \in C)(\forall a \in A) a \lor c \in B^*$$

$$\iff A^* \cap C^* \subseteq B^*.$$

Next we look at some particular choices for $\mathfrak{P}$; here, as usual, $n = \{0, 1, \ldots, n - 1\}$. Much of the material of the rest of this section is taken from Maksimova et. al. [4].

**Definition 18.7.** For all $n > 0$,

(i) $\mathfrak{P}_n := (\wp(n), \supseteq)$; $\mathfrak{P}_n^- := (\wp(n) \setminus \{\emptyset\}, \supseteq)$;

(ii) $\mathfrak{O}_n := \mathfrak{O}(\mathfrak{P}_n)$; $\mathfrak{O}_n^- := \mathfrak{O}(\mathfrak{P}_n^-)$;

(iii) $\mathfrak{P}_\omega := (\{ a \subseteq \omega : a \text{ is finite or cofinite} \}, \supseteq)$;

(iv) $\mathfrak{O}_\omega := \mathfrak{O}^\omega(\mathfrak{P}_\omega)$.

Of course, in (iv), the hypotheses of Proposition 18.6 are satisfied because $\mathfrak{P}_\omega$ is a Boolean algebra. Immediately from Proposition 18.2,

**Corollary 18.8.** For all $n > 0$,

(i) $\mathfrak{O}_n^-$ is a $1$-irreducible implicative lattice;

(ii) $\mathfrak{O}_n$ and $\mathfrak{O}_\omega$ are $0$- and $1$-irreducible implicative lattices.

By Propositions 5.5 and 5.7, we have

$$\text{IPC} \subseteq \bigcap \{ \text{Th}(\mathfrak{O}_n^-) : n > 0 \} \quad \text{and} \quad \text{WEM} \subseteq \bigcap \{ \text{Th}(\mathfrak{O}_n) : n > 0 \}.$$  

We shall see that the second inclusion is in fact an equality, but the first is not and leads to a new logic, which is denoted $\text{LM}$ for the Logic of Medvedev, who first considered it (with a different definition – see [5] and Section 6 of Gabbay [1]).

**Definition 18.9.** $\text{LM} := \bigcap \{ \text{Th}(\mathfrak{O}_n^-) : n > 0 \}$.

**Definition 18.10.** For any partial orderings $\mathfrak{P} = (P, \leq_P)$ and $\mathfrak{Q} = (Q, \leq_Q)$, $\mathfrak{P}$ *cone-covers* $\mathfrak{Q}$ — in symbols, $\mathfrak{P} \triangleright \mathfrak{Q}$ — iff there exists a surjective function $f : P \to Q$ such that for all $a \in P$, $f$ maps $\{ b \in P : a \leq_P b \}$ onto $\{ d \in Q : f(a) \leq_Q d \}$. We write $f : \mathfrak{P} \triangleright \mathfrak{Q}$. 

Proposition 18.11. For any partial orderings \( \mathfrak{P} = (P, \leq_P) \) and \( \mathfrak{Q} = (Q, \leq_Q) \)

\[
\mathfrak{P} \stackrel{\Delta}{\rightarrow} \mathfrak{Q} \quad \Rightarrow \quad \text{Th}(\mathcal{O}(\mathfrak{P})) \subseteq \text{Th}(\mathcal{O}(\mathfrak{Q})).
\]

Proof. Fix a function \( f \) witnessing \( \mathfrak{P} \stackrel{\Delta}{\rightarrow} \mathfrak{Q} \) and for any \( \mathcal{O}(\mathfrak{Q}) \) valuation \( v \), set

\[
v^f(\phi) := f^{-1}(v(\phi)).
\]

The properties of \( f \) guarantee that \( f \) is order-preserving, so easily \( v^f : \mathcal{P} \to \mathcal{O}(\mathfrak{P}) \). We show first that in fact \( v^f \) is an \( \mathcal{O}(\mathfrak{P}) \)-valuation. The conditions

\[
v^f(\phi \land \psi) = v^f(\phi) \land v^f(\psi) \quad \text{and} \quad v^f(\phi \lor \psi) = v^f(\phi) \lor v^f(\psi)
\]

follow from the elementary properties of inverse images: for \( C, D \in \mathcal{O}(\mathfrak{Q}) \),

\[
f^{-1}(C \land D) = f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D) = f^{-1}(C) \land f^{-1}(D)
\]

and similarly for \( \lor \). Since also obviously \( f^{-1}(0_{\mathcal{O}(\mathfrak{Q})}) = 0_{\mathcal{O}(\mathfrak{P})} \), the remaining two conditions follow once we establish that for all \( C, D \in \mathcal{O}(\mathfrak{Q}) \),

\[
f^{-1}(C \rightarrow D) = f^{-1}(C) \rightarrow f^{-1}(D),
\]

where the lattice implications are respectively those of \( \mathfrak{Q} \) and \( \mathfrak{P} \). For this, we need to verify that for any \( X \in \mathcal{O}(\mathfrak{P}) \),

\[
f^{-1}(C) \cap X \subseteq f^{-1}(D) \quad \iff \quad X \subseteq f^{-1}(C \rightarrow D).
\]

For (\( \Leftarrow \)) we have by the properties of \( \rightarrow \) in \( \mathcal{O}(\mathfrak{Q}) \),

\[
f^{-1}(C) \cap f^{-1}(D) = f^{-1}(C \cap (C \rightarrow D)) \subseteq f^{-1}(D).
\]

Towards (\( \Rightarrow \)), suppose that \( f^{-1}(C) \cap X \subseteq f^{-1}(D) \) and \( a \in X \). Then for all \( b \in P, a \leq_P b \implies b \in X \), so

\[
a \leq_P b \quad \text{and} \quad f(b) \in C \quad \implies \quad f(b) \in D.
\]

By cone-covering, for each \( d \geq_Q f(a) \) there exists \( b \geq_P a \) with \( d = f(b) \), so we have

\[
f(a) \leq_Q d \quad \text{and} \quad d \in C \quad \implies \quad d \in D.
\]

Hence

\[
f(a) \in Y := \{ d \in Q : f(a) \leq_Q d \} \in \mathcal{O}(\mathfrak{Q}) \quad \text{and} \quad C \cap Y \subseteq D,
\]

so \( Y \subseteq C \rightarrow D \) and in particular \( f(a) \in C \rightarrow D \) and thus \( a \in f^{-1}(C \rightarrow D) \).
Finally, we have the desired conclusion: for any propositional sentence $\phi$,

\[
\models_{\mathcal{D}(\Omega)} \phi \quad \implies \quad \text{for all } \mathcal{D}(\Omega)\text{-valuations } v, v^f(\phi) = 1_{\mathcal{D}(\Omega)}
\]

\[
\implies \quad \text{for all } \mathcal{D}(\Omega)\text{-valuations } v, v(\phi) = 1_{\mathcal{D}(\Omega)}
\]

\[
\implies \quad \models_{\mathcal{D}(\Omega)} \phi.
\]

**Proposition 18.12.** For any finite bounded partial ordering $\Omega$, there exists $m > 0$ such that $\mathcal{P}_m \triangleleft \Omega$.

**Proof.** We proceed by induction on the size of $\Omega$. If $\Omega$ has only one element, the conclusion is clear. Let $0_\Omega$ and $1_\Omega$ denote respectively the least and greatest element of $\Omega$ and $e_0, \ldots, e_k$ be the immediate successors of $0_\Omega$. Suppose first that $k = 0$ and set $R := Q \setminus \{0_\Omega\}, \leq_R$ is $\leq_Q$ restricted to $R$ and $\mathcal{R} := (R, \leq_R)$. By the induction hypothesis, for some $m > 0$ and $g, g : \mathcal{P}_m \triangleleft \mathcal{R}$. Then we show that $f : \mathcal{P}_{m+1} \triangleleft \Omega$ with, for $A \subseteq m + 1$,

\[
f(A) := \begin{cases} 0_\Omega, & \text{if } m \in A \text{ and } g(A \cap m) = e_0; \\ g(A \cap m), & \text{otherwise}. \end{cases}
\]

Clearly $f$ is surjective. The cone-covering condition is, for all $d \in Q$,

\[
(*) \quad f(A) \leq_Q d \iff (\exists B \subseteq A) f(B) = d.
\]

The implication ($\iff$) follows because $B \subseteq A \implies B \cap m \subseteq A \cap m$ and $g$ is order-preserving. For ($\implies$), if $d = 0_\Omega$, take $B = A$; otherwise $d \in R$. If $f(A) = 0_\Omega$, then $g(A \cap m) = e_0 \leq_R d$ and there exists $B \subseteq A \cap m$ with $g(B) = d$, so also $f(A) = d$. If $f(A) = g(A \cap m)$, then again the desired $B$ exists by the cone-covering property of $g$.

Now, consider the case $k > 0$. For $i < k$, set $R_i := \{c \in Q : e_i \leq_Q c\} \cup \{0_\Omega\}, \leq_{R_i}$ is $\leq_Q$ restricted to $R_i$ and $\mathcal{R}_i := (R_i, \leq_{R_i})$. By the induction hypothesis, for all $i < k$ there exist $m_i$ and $g_i$ with $g_i : \mathcal{P}_{m_i} \triangleleft \mathcal{R}_i$. Let $m := m_0 + \cdots + m_k$. We shall show that $f : \mathcal{P}_m \triangleleft \Omega$ with $f$ defined as follows. For $A \subseteq m$ and $i < k$, set

\[
A^{(i)} := \{j < m_i : m_0 + \cdots + m_{i-1} + j \in A\}.
\]

\[
f(A) := \begin{cases} g_i(A^{(i)}), & \text{if } (\forall j \neq i) g_j(A^{(j)}) = 0_\Omega; \\ 1_\Omega, & \text{if for at least two } i < k, g_i(A^{(i)}) \neq 0_\Omega. \end{cases}
\]

To establish $(*)$($\iff$) suppose that $B \subseteq A$. If for at least two $i < k$, $g_i(B^{(i)}) \neq 0_\Omega$, then $f(B) = 1_\Omega$ so $f(A) \leq_Q f(B)$. Otherwise, for some
We show that \( f(A) = g_i(A^{(i)}) \leq g_i(B^{(i)}) = f(B) \).

Now, towards \((*) \implies \cdot\), suppose that \( f(A) \leq d \). If for at least two \( i < k \)
\( g_i(A^{(i)}) \neq 0_\mathfrak{Q} \), then \( f(A) = 1_\mathfrak{Q} \) so also \( d = 1_\mathfrak{Q} \). Suppose that for at most one \( i < k \), \( g_i(A^{(i)}) \neq 0_\mathfrak{Q} \). Then either \( d = 0_\mathfrak{Q} \), so \( f(A) = d \), or for some \( i \), \( e_i \leq d \), \( f(A) = g_i(A^{(i)}) \leq d \) and \((\forall j \neq i) g_j(A^{(j)}) = 0_\mathfrak{Q} \). Then by the cone-covering property of \( g_i \), there exists \( C \subseteq A^{(i)} \) such that \( g_i(C) = d \) and \( f(B) = d \) for \( B \) such that \( B^{(i)} = C \) and for \( j \neq i \), \( B^{(j)} = A^{(j)} \).

**Corollary 18.13.** \( WEM = \cap \{ \text{Th}(\mathfrak{Q}_n) : n > 0 \} \).

**Proof.** Immediate from the preceding proposition and Propositions 18.3 and 18.11.

**Lemma 18.14.** For each \( n > 0 \), \( \Psi_{n+1} \xrightarrow{\triangle} \Psi_n \).

**Proof.** For nonempty \( X \subseteq n + 1 \), set \( \mu(X) := \) the smallest element of \( X \) and

\[
f(X) := \{ i : i + 1 \in X \setminus \{ \mu(X) \} \}.
\]

We show that \( f : \Psi_{n+1} \xrightarrow{\triangle} \Psi_n \). \( f \) is surjective because, for any \( Z \subseteq n \),
\( f(Y) = Z \) for \( Y = \{ i + 1 : i \in Z \} \cup \{ 0 \} \). To see that \( f \) is cone-covering,
for any \( W \subseteq f(X) \), set \( Y = \{ i + 1 : i \in W \} \cup \{ \mu(X) \} \). Then \( Y \subseteq X \),

\[
i + 1 \in Y \setminus \{ \mu(X) \} \implies i \in W \implies i \in f(X) \implies i + 1 \in X,
\]

so \( \mu(Y) = \mu(X) \) and thus \( f(Y) = W \).

**Corollary 18.15.** \( LM \subseteq WEM \).

**Proof.** By the lemma and Proposition 18.11, \( \text{Th}(\mathfrak{Q}_n) \subseteq \text{Th}(\mathfrak{Q}_n) \).

Hence the conclusion follows from the definition of \( LM \) and Proposition 18.13.

We defined the notion of positive propositional sentence just before Lemma 14.2; let \( \text{Pos} \) denote the set of these.

**Proposition 18.16.** \( WEM \cap \text{Pos} = \text{IPC} \cap \text{Pos} \); hence also \( LM \cap \text{Pos} = \text{IPC} \cap \text{Pos} \).

**Proof.** The inclusion \((\supseteq)\) is clear. Fix \( \phi \in WEM \cap \text{Pos} \) and \( \mathfrak{Q} \) a finite \( 1 \)-irreducible implicative lattice. By Lemma 14.11, \( \mathfrak{Q}_0 \) is also \( 0 \)-irreducible, so by the WEM-Completeness Theorem, \( \models \phi \). But then, since \( \phi \) is positive, it follows from Lemma 14.12 that also \( \models \phi \). Hence by the IPC-Completeness Theorem, \( \phi \in \text{IPC} \). The second clause now follows by the preceding corollary.
Next we adapt these to yield characterizations of $\text{IPC}$ in terms of intervals $\mathcal{L}[d,1]$.

**Definition 18.17.** To each propositional sentence $\phi$ we associate a positive sentence $\phi^+$ as follows. Let $n_\phi$ be the smallest $n$ such that all atomic sentences occurring in $\phi$ are among $p_0, \ldots, p_n$. Define $\psi^\phi$ recursively on subsentences of $\phi$ by:

$$
\begin{align*}
p_i^\phi &:= p_i \text{ for each atomic sentence } p_i; \\
(\psi \land \theta)^\phi &:= \psi^\phi \land \theta^\phi; \\
(\psi \lor \theta)^\phi &:= \psi^\phi \lor \theta^\phi; \\
(\psi \rightarrow \theta)^\phi &:= \psi^\phi \rightarrow \theta^\phi; \\
(\neg \psi)^\phi &:= \psi^\phi \rightarrow p_0 \land \cdots \land p_{n_\phi} \land p_{n_\phi+1}.
\end{align*}
$$

Then $\phi^+ := \phi^\phi$.

**Lemma 18.18.** For any propositional sentence $\phi$ and any implicative lattice $\mathcal{L}$,

(i) $\models_{\mathcal{L}} \phi^+$ $\implies$ $\models_{\mathcal{L}} \phi$;

(ii) $\not\models_{\mathcal{L}} \phi^+$ $\implies$ ($\exists d \in L$) $\not\models_{\mathcal{L}[d,1]} \phi$.

**Proof.** Fix $\mathcal{L}$ and $\phi$. For each $\mathcal{L}$-valuation $v$, let $v_\phi$ be the $\mathcal{L}$-valuation that agrees with $v$ except that $v_\phi(p_{n_\phi+1}) = 0$. By an easy induction on subsentences $\psi$ of $\phi$, $v_\phi(\psi^\phi) = v(\psi)$, so in particular $v_\phi(\phi^+) = v(\phi)$. Thus, if $\models_{\mathcal{L}} \phi^+$, then for each $\mathcal{L}$-valuation $v$, $v_\phi(\phi^+) = 1$ so also $v(\phi) = 1$. Hence $\models_{\mathcal{L}} \phi$.

Towards (ii), suppose that for some $\mathcal{L}$-valuation $v$, $v(\phi^+) \neq 1$. Set

$$d := v(p_0) \land \cdots \land v(p_{n_\phi}) \land v(p_{n_\phi+1}).$$

There is a unique $\mathcal{L}[d,1]$-valuation $w$ such that for $i \leq n_\phi + 1$, $w(p_i) = v(p_i)$ and for $i > n_\phi + 1$, $w(p_i) = 1$. Then for each subsentence $\psi$ of $\phi$, $w(\psi) = v(\psi^\phi)$, since inductively

$$w(\neg \psi) = w(\psi) \rightarrow 0_{\mathcal{L}[d,1]} = v(\psi^\phi) \rightarrow d = v((\neg \psi)^\phi).$$

In particular, $w(\phi) = v(\phi^+) \neq 1$, so $\not\models_{\mathcal{L}[d,1]} \phi$. $\square$

**Proposition 18.19.** For any collection $\mathcal{X}$ of implicative lattices, if

$$\text{IPC} \cap \text{Pos} = \bigcap_{\mathcal{L} \in \mathcal{X}} \text{Th}(\mathcal{L}) \cap \text{Pos},$$

then

$$\text{IPC} = \bigcap_{\mathcal{L} \in \mathcal{X}} \bigcap_{d \in L} \text{Th}(\mathcal{L}[d,1]).$$
Proof. Assume the hypothesis. The inclusion (⊆) of the conclusion is immediate. Suppose that a sentence \( \phi \) belongs to the right-hand side. Then for each \( \mathfrak{L} \in \mathcal{X} \), by (ii) of the preceding lemma, \( \models_{\mathfrak{L}} \phi^+ \), so since \( \phi^+ \) is positive, \( \phi^+ \in \text{IPC} \). Hence, for every finite implicative lattice \( \mathfrak{L} \), \( \models_{\mathfrak{L}} \phi \), so by (i) of the lemma, \( \models_{\mathfrak{L}} \phi \). Thus by the IPC-Completeness Theorem, \( \phi \in \text{IPC} \). ⊣

Corollary 18.20.

\[
\bigcap_{n>0} \bigcap_{D \in \mathcal{O}(\mathfrak{P}_n)} \text{Th}(\mathcal{D}_n[D, 1]) = \text{IPC} = \bigcap_{n>0} \bigcap_{D \in \mathcal{O}(\mathfrak{P}_n)} \text{Th}(\mathcal{D}^-_n[D, 1]).
\]

Proof. Immediate from Corollary 18.13 and Propositions 18.16 and 18.19. ⊣

Corollary 18.21.

\[
\bigcap_{r \in \mathcal{D}_s} \text{Th}^\circ(\mathcal{D}_s[r, 1]) = \text{IPC}
\]

\[
\bigcap_{r \in \mathcal{D}_w} \text{Th}^\circ(\mathcal{D}_w[r, 1]) = \text{IPC}
\]

\[
\bigcap_{r \in \mathcal{D}_w} \text{Th}(\mathcal{D}_w[r, 1]) = \text{IPC}.
\]

Proof. This follows by Theorem J and Propositions 18.16 and 18.19. For example, since

\[
\text{Th}(\mathcal{D}_s^\circ) \cap \text{Pos} = \text{WEM} \cap \text{Pos} = \text{IPC} \cap \text{Pos},
\]

we have

\[
\text{IPC} = \bigcap_{r \in \mathcal{D}_s} \text{Th}(\mathcal{D}_s[r, 1])
\]

\[
= \bigcap_{r \in \mathcal{D}_s} \text{Th}(\mathcal{D}_s[0, r]^\circ)
\]

\[
= \bigcap_{r \in \mathcal{D}_s} \text{Th}^\circ(\mathcal{D}_s[0, r]).
\]

§19. Proof of Theorem K. Improving on the preceding corollary, we shall construct a single strong degree \( r \) such that \( \text{Th}^\circ(\mathcal{D}_s[0, r]) = \text{IPC} \). We follow generally the presentation of Skvortsova [6]. Fix an enumeration \( \langle (n_k, D_k) : k \in \omega \rangle \) of all pairs \( (n, D) \) with \( D \in \mathcal{O}_n \) so that by Corollary 18.20,

\[
\bigcap_{k \in \omega} \text{Th}(\mathcal{D}_{n_k}[D_k, 1]) = \text{IPC}. \tag{1}
\]
We shall construct strong degrees $p_k, q_k$ and $r$ ($k \in \omega$) such that

(2) \[ \mathcal{O}_{nk}[D_k, 1] \hookrightarrow \mathbb{D}_s[p_k, q_k], \]

and

(3) \[ p_k \lor r = q_k. \]

Then

\[ \mathcal{O}_{nk}[D_k, 1] \ hookslash \mathbb{D}_s[p_k, q_k], \]

so by Corollary 17.3(ii) and Lemma 15.1,

\[ \text{Th}^\circ(\mathbb{D}_s[0, r]) \subseteq \bigcap_{k \in \omega} \text{Th}^\circ(\mathbb{D}_s[p_k, q_k]) \subseteq \bigcap_{k \in \omega} \text{Th}(\mathcal{O}_{nk}[D_k, 1]) = \text{IPC}. \]

Towards (2), note first that for each $n$ and each $D \in \mathcal{O}_n$,

\[ \mathcal{O}_n[D, 1] = \mathcal{O}_\omega[D, \varphi(n)]. \]

Hence, if we show that $\mathcal{O}_\omega \hookrightarrow \mathbb{D}_s$, then there are $p, q \in \mathbb{D}_s$ such that

\[ \mathcal{O}_n[D, 1] \subseteq \mathcal{O}_\omega[D, \varphi(n)]. \]

We then achieve (3) by careful choice of $p$ and $q$ for different pairs $(n, D)$.

The basis of the construction is the following classical result.

**Proposition 19.1** (Lachlan and Lebeuf [3]). For any countable upper semi-lattice $\mathcal{P} = (P, \leq, 0, \lor, \triangleright)$ that is bounded below, there exists an embedding of $\mathcal{P}$ into the Turing degrees $\mathbb{D}_T$ as an initial segment — that is, a function $\xi : P \to \mathbb{D}_T$ such that for $a, b \in P$,

(i) $a \leq b \iff \xi(a) \leq_T \xi(b);$ 
(ii) $\xi(0) = 0_T;$ 
(iii) $\xi(a \lor b) = \xi(a) \lor \xi(b);$ 
(iv) $\text{Im}(\xi)$ is closed downward in $\mathbb{D}_T.$

**Proposition 19.2.** For any countable dual-implicative upper semi-lattice $\mathcal{P} = (P, \leq, 0, \lor, \triangleright)$, there exists a dual embedding $\eta : \mathcal{O}^\circ(\mathcal{P}) \hookrightarrow \mathbb{D}_s$ such that for each $D \in \mathcal{O}^\circ(\mathcal{P})$, $\eta(D)$ is the meet of finitely many upward Turing closed degrees.

**Proof.** By Proposition 18.6, $\mathcal{O}^\circ(\mathcal{P})^\circ$ is dual-implicative. Fix an upper semi-lattice embedding $\xi : \mathcal{P} \to \mathbb{D}_T$ as in the preceding proposition. We first transform this into an upper semi-lattice embedding $\eta : \mathcal{P} \to \mathbb{D}_s$:

\[ \eta(a) := \text{dg}_s(S_a) \quad \text{where} \]

\[ S_a := \{ f \in {}^{\omega} \omega : \xi(a) \leq_T \text{dg}_T(f) \text{ or } \text{dg}_T(f) \notin \text{Im}(\xi) \}. \]
Each \( S_a \) is upward Turing closed. It follows by Lemma 17.5 that \( \eta \) respects \( \leq \):

\[
a \leq b \iff \xi(a) \leq_T \xi(b) \iff S_b \subseteq S_a \iff S_a \leq_S S_b.
\]

\( \eta(0) = dg_s(S_0) = dg_s(\omega) = 0_s \), and again by Lemma 17.5, \( \eta \) respects \( \lor \):

\[
S_a \lor b = \{ f : \xi(a) \lor \xi(b) \leq_T \text{dg}_T(f) \text{ or } \text{dg}_T(f) \notin \text{Im}(\xi) \}
= S_a \cap S_b \equiv_S S_a \lor S_b.
\]

Next we show that \( \eta \) respects \( \circ \). For \( \text{dg}_T(f) \notin \text{Im}(\xi) \), \( f \in S_a \circ b \) by definition, and by Proposition 17.6, also

\[
f \in S_a \circ b = \{ h : (\forall g \in S_a) g \oplus h \in S_b \},
\]

since

\[
\text{dg}_T(f) \notin \text{Im}(\xi) \implies \text{dg}_T(g \oplus f) \notin \text{Im}(\xi) \implies g \oplus f \in S_b.
\]

For \( \text{dg}_T(f) \in \text{Im}(\xi) \), fix \( c \in P \) such that \( \xi(c) = \text{dg}_T(f) \). Then

\[
f \in S_a \circ b \iff \xi(a) \circ b \leq_T \text{dg}_T(f)
\iff (a \circ b) \leq c
\iff b \leq a \lor c
\iff \xi(b) \leq_T \xi(a) \lor \text{dg}_T(f)
\iff (\forall g[\xi(a) \leq_T \text{dg}_T(g) \implies \xi(b) \leq_T \text{dg}_T(f \oplus g)]
\iff (\forall g \in S_a \cap \text{Im}(\xi)) \xi(b) \leq_T \text{dg}_T(f \oplus g)
\iff f \in S_a \circ b,
\]

with the last step by Proposition 17.6 (i).

Now, extend \( \eta \) to \( \overline{\eta} : \mathcal{O}^\omega(\mathcal{P})^\circ \to \mathbb{D}_s \) by setting for finite \( A \subseteq P \),

\[
\overline{\eta}(A^*) := \bigwedge_{a \in A} \eta(a).
\]

It remains to show that \( \overline{\eta} \) is well-defined and is a dual-implicative lattice embedding — that is

(i) \( A^* \supseteq B^* \iff \overline{\eta}(A^*) \leq_s \overline{\eta}(B^*) \);

(ii) \( \overline{\eta}(\emptyset) = \infty_s; \quad \overline{\eta}(P) = 0_s \);

(iii) \( \overline{\eta}(A^* \cap B^*) = \overline{\eta}(A^*) \lor \overline{\eta}(B^*) \);

(iv) \( \overline{\eta}(A^* \cup B^*) = \overline{\eta}(A^*) \land \overline{\eta}(B^*) \);

(v) \( \overline{\eta}(A^* \rightarrow B^*) = \overline{\eta}(A^*) \rightarrow \overline{\eta}(B^*) \).
For (i), which also implies that $\eta$ is well-defined, we have
\[ A^* \supseteq B^* \iff (\forall b \in B) (\exists a \in A) a \leq b \]
\[ \iff (\forall b \in B) (\exists a \in A) \eta(a) \leq_T \eta(b) \]
\[ \iff (\forall b \in B) \eta(A^*) \leq_s \eta(b) \]
\[ \iff \eta(A^*) \leq_s \eta(B^*), \]
where the third equivalence uses the meet irreducibility of $\eta(b)$ from Proposition 17.6(ii). Part (ii) is immediate. For (iii), we have
\[ \eta(A^* \cap B^*) = \eta(\{ a \lor b : a \in A \text{ and } b \in B \}^*) \]
\[ = \bigwedge_{a \in A} \bigwedge_{b \in B} \eta(a \lor b) \]
\[ = \bigwedge_{a \in A} \bigwedge_{b \in B} \eta(a) \lor \eta(b) \]
\[ = \eta(A^*) \lor \eta(B^*), \]
by distributivity. Part (iv) is straightforward:
\[ \eta(A^* \cup B^*) = \eta((A \cup B)^*) = \bigwedge_{a \in A \cup B} \eta(a) = \eta(A^*) \land \eta(B^*). \]
Finally for (v) we have
\[ \eta(A^* \rightarrow B^*) = \eta\left( \bigcap_{a \in A} \{ a \rightarrow b : b \in B \}^* \right) \]
\[ = \eta\left( \bigvee_{a \in A} (a \rightarrow F(a)) : F \in B^A \right)^* \]
\[ = \bigwedge_{F \in B^A} \eta\left( \bigvee_{a \in A} (a \rightarrow F(a)) \right) \]
\[ = \bigwedge_{F \in B^A} \bigvee_{a \in A} \eta(a) \rightarrow \eta(b) \]
\[ = \bigvee_{a \in A} \left( \eta(a) \rightarrow \eta(b) \right) = \eta(A^*) \rightarrow \eta(B^*) \] by Proposition 17.6(iii)
\[ = \eta(A^*) \rightarrow \eta(B^*) \] by Lemma 17.4 (4).
\[ \dashv \]

We proceed now to establishing (2) and (3) to complete the proof of Theorem K. By the preceding proposition we may fix $\eta : \mathcal{O}^\omega :\rightarrow \mathcal{D}_\omega$. For each $k \in \omega$ choose $a_k \subseteq \omega$ such that $a_k$ has $n_k$-many elements and $k \neq l \Rightarrow a_k \cap a_l = \emptyset$. Clearly, for each $k$ there exists
\( E_k \in \mathcal{O}(\varphi(a_k)) \) such that
\[
\mathcal{O}_{n_k}[D_k, 1] \cong \mathcal{O}_\omega[E_k, \varphi(a_k)]
\]
and
\[
\mathcal{O}_{n_k}[D_k, 1] = \mathcal{O}_{n_k}^\circ[1, D_k] \cong \mathcal{O}_\omega^\circ[\varphi(a_k), E_k].
\]
Set
\[
p_k := \eta(\varphi(a_k)) \quad \text{and} \quad q_k := \eta(E_k).
\]
Then immediately,
\[
\mathcal{O}_{n_k}[D_k, 1] \overset{\circ}{\to} \mathcal{D}_k[p_k, q_k]
\]
as required by (2). Towards (3), note that obviously
\[
p_k \leq s q_k \quad \text{so} \quad p_k \lor q_k = q_k.
\]
For \( k \neq \ell \), we have, since \( E_\ell \subseteq a_\ell \),
\[
p_k \lor q_\ell = \eta(\varphi(a_k)) \lor \eta(E_\ell) = \eta(\varphi(a_k) \cap E_\ell) = \eta(\emptyset) \geq s \eta(E_k) = q_k.
\]
Hence, if \( \mathcal{D}_s \) were a complete and completely distributive lattice, we could set
\[
r := \bigwedge_{\ell \in \omega} q_\ell
\]
and compute
\[
p_k \lor r = \bigwedge_{\ell \in \omega} (p_k \lor q_\ell) = q_k
\]
as required by (3). Since it isn’t, we need a slightly more cumbersome construction using the “pseudo meet” of Remark 2.6. Choose \( P_k \in p_k \).

By Proposition 19.2 for each \( \ell \in \omega \) we may choose \( Q_\ell \in q_\ell, m_\ell \in \omega \) and upward Turing closed sets \( Q_{\ell,i} \) such that \( Q_\ell = \bigwedge_{i < m_\ell} Q_{\ell,i} \). Set
\[
R := \bigwedge_{\ell \in \omega} Q_\ell := \{ (\ell)^g : \ell \in \omega \text{ and } g \in Q_\ell \} \quad \text{and} \quad r := \text{dg}_s(R).
\]
Clearly \( p_k \lor r \leq s q_k \). For \( k \neq \ell \), we have as above
\[
\text{dg}_s(p_k \lor Q_\ell) = \eta(\emptyset) \geq s \text{dg}_s(Q_{k,0}),
\]
so since \( Q_{k,0} \) is upward Turing closed, by Proposition 17.5, \( p_k \lor Q_\ell \subseteq Q_{k,0} \). Hence, if we set
\[
\Phi_k(f \oplus (\ell)^g) := \begin{cases} g, & \text{if } k = \ell; \\ (0)^{(f \oplus g)}, & \text{otherwise}; \end{cases}
\]
we have \( \Phi_k : P_k \lor R \to Q_k \) — that is, \( q_k \leq s p_k \lor r \) as required.

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