Asymptotic theory of neutral stability curve of the Couette flow of vibrationally excited gas

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Abstract. The asymptotic theory of neutral stability curve of the supersonic plane Couette flow of vibrationally excited gas is constructed. The system of two-temperature viscous gas dynamics equations was used as original mathematical model. Spectral problem for an eighth order linear system of ordinary differential equations was obtained from the system within framework of classical theory of linear stability. Transformations of the spectral problem universal for all shear flows were carried along the classical Dunn — Lin scheme. As a result the problem was reduced to secular algebraic equation with a characteristic division on “inviscid” and “viscous” parts which was solved numerically. The calculated neutral stability curves coincide in limits of 10 % with corresponding results of direct numerical solution of original spectral problem.

1. Introduction

The hydrodynamic stability theory used purely asymptotic methods during its formation. Application of analytical approaches allowed one to study some general mechanisms in hydrodynamic stability and turbulence transition. In particular, estimations of critical parameters of some flows important for practice were obtained with values close to real ones. Currently, asymptotic methods keep its role as independent source of theoretical results in spite of dominance of numerical studies in the flow stability problems. Primarily, it is of interest for investigation of optically active, thermally nonequilibrium and chemically reacting molecular gas flows in tracts of gas lasers, rocket engines and various technological plants. Studies in this domain require generalization of classical results [1] of the hydrodynamic stability theory to mediums and flows which were not considered earlier.

In the paper, constructing an asymptotic theory of neutral stability curve for the supersonic plane Couette flow of vibrationally excited gas is considered. It should be noted that earlier the corresponding spectral problem was not asymptotically investigated even for a perfect gas.

2. Statement of problem and basic equations

A problem of linear stability of a plane viscous Couette flow of a vibrationally excited molecular gas is considered. In the coordinate plane \((x, y)\) the flow is bounded by two infinite parallel planes. It is assumed that the plane \(y = 0\) is in a rest and the boundary \(y = h\) moves uniformly in its own plane with a velocity \(U_0\). The flow is described within the framework of two-temperature
aerodynamics model [2, 3]. The quantities used for scaling are the channel width \( h \), boundary velocity \( U_0 \), density \( \rho_0 \) and temperature \( T_0 \) of the stationary flow on the moving boundary of the channel, time \( t_0 = L/U_0 \), and pressure \( p_0 = \rho_0 U_0^2 \).

All parameters in the steady state flow are assumed depending only on the transverse coordinate \( y \) and vibrational temperature \( T_{v,s}(y) = 0 \). Original system of two-temperature aerodynamics equations is linearized near exact Navier — Stokes stationary solution

\[
U_s(y) = y, \quad T_s(y) = 1 + \frac{(\gamma - 1)PrM^2}{2} (1 - y^2), \quad \rho_s(y) = \frac{1}{T_s(y)}, \quad p_s(x_2) = \frac{1}{\gamma M^2}.
\]

which comply with following boundary conditions

\[
U_s(0) = 0, \quad U_s(1) = 1, \quad \frac{dT_s}{dy} \bigg|_{y=0} = 0, \quad T_s(1) = 1. \tag{2}
\]

Here \( U_s \) is the velocity, \( \rho_s \) is the density, \( T_s \) is the static (translational) temperature of a gas, \( Pr, M \) are the Prandtl and Mach numbers in steady flow.

For linearized small-disturbance system

\[
(\hat{u}_x, \hat{u}_y, \hat{\rho}, \hat{T}, \hat{T}_v, \hat{p})
\]
solutions in the form of travelling plane waves were considered:

\[
q(x, y, t) = q(y) e^{i\alpha (x - ct)}, \quad q(x, y, t) = (\hat{u}_x, \hat{u}_y, \hat{\rho}, \hat{T}, \hat{T}_v, \hat{p}), \quad q(y) = (u, \alpha v, \rho, \theta, \theta_v, p),
\]

where \( \alpha \) is the wave number along periodic variable \( x, c = c_r + ic_i \) is the complex phase velocity, \( i \) is the imaginary unity, \( q(y) \) is the vector of disturbance amplitudes. Substitution of \( q(x, y, t) \) in linearized system gives equations for small-disturbance amplitudes:

\[
D\rho + \alpha \rho_s \alpha v + \rho_s \sigma = 0, \tag{3}
\]

\[
\frac{1}{Re} \Delta u - \rho_s Du - \alpha \rho_s v U_s' - i\alpha \varepsilon = 0, \tag{4}
\]

\[
\frac{\alpha}{Re} \Delta v - \alpha \rho_s Dv - \varepsilon' = 0, \tag{5}
\]

\[
\frac{\gamma}{RePr} \Delta \theta - \rho_s D\theta - \alpha \rho_s v T_s' - (\gamma - 1)\sigma + \frac{2\gamma (\gamma - 1)M^2}{Re} (u'' + i\alpha^2 v) U_s'' + \frac{\gamma \rho_s}{\tau} (\theta_v - \theta) = 0, \tag{6}
\]

\[
\frac{20\gamma \gamma_v}{33RePr} \Delta \theta_v - \gamma_v \rho_s D\theta_v - \alpha \gamma_v \rho_s v T_s' - \frac{\gamma \rho_s}{\tau} (\theta_v - \theta) = 0, \quad \gamma M^2p = \rho_s \theta + \rho T_s. \tag{7}
\]

In the equations (3) — (7), \( \alpha_1 = \eta_b/\eta \) is the ratio of the bulk viscosity to the shear viscosity, \( \gamma \) is the adiabatic exponent, \( \gamma_v \) is a parameter characterizing a degree of nonequilibrium of a vibrational mode, \( \tau \) is the characteristic relaxation time. Flow parameters are defined as follows: \( Re = \rho_0 h U_0/\eta \) and \( M = U_0/\sqrt{\gamma R T_0} \) are the Reynolds and Mach numbers, respectively, \( Pr = \eta c_v/\lambda \) is the Prandtl number; \( R \) is the gas constant, \( \lambda \) is the thermal conductivity coefficient. Hence,

\[
D = i\alpha (U_s - c), \quad \sigma = \alpha (v' + iu), \quad \varepsilon = p - \sigma \left( \frac{\alpha_1 + \frac{1}{3}}{Re} \right), \quad \Delta = \frac{d^2}{dy^2} - \alpha^2;
\]

here and below the primes mean a differentiation with respect to the variable \( y \). The system (3) — (7) is of order eight and with homogeneous boundary conditions

\[
|u|_{y=0} = u|_{y=1} = v|_{y=0} = v|_{y=1} = \theta|_{y=0} = \theta|_{y=1} = \theta_v|_{y=0} = \theta_v|_{y=1} = 0 \tag{8}
\]
it defines a spectral problem, in which the eigenvalues are the complex phase velocities of perturbations \( c = c_r + ic_i \), and the Mach number \( M \) and wave number \( \alpha \) are parameters. The asymptotic solutions of the system (3) — (7) for high Reynolds numbers \( \text{Re} \) are constructed in the form of a perturbation series

\[
q(y) = q_0(y) + \text{Re}^{-1} q_1(y) + \ldots
\]

In the zeroth-order approximation one obtains a system of equations for inviscid perturbations. In the work [3] it has been shown that this system is reduced to the second order linear equation for pressure perturbation. Thus, the zeroth-order approximation allows us to find only two of the linearly independent solutions. One has to determine the remaining six solutions by direct consideration of the complete system.

3. Asymptotics of inviscid and viscous of solutions

Two linearly independent inviscid solutions of equation for pressure perturbation

\[
p'' - \frac{T_s}{(y-c)^2} \left( \frac{y-c)^2}{T_s} \right)' \left( 1 - \frac{M^2 (y-c)^2}{T_s} \right) p = 0
\]

were constructed by the Frobenius method [4] using an asymptotical expansion in neighborhood of regular singular point \( y = c \). Here

\[
M^2 = m_r^2 + m_i^2, \quad m_r^2 = \frac{R_1 (1 + \gamma v) + \Delta^2}{R_1^2 + \Delta^2}, \quad m_i^2 = -\frac{\gamma v (\gamma - 1) \Delta}{\gamma R_1^2 + \Delta^2},
\]

\[
R_1 = 1 + \frac{(\gamma v / \gamma)}{\Delta} = \alpha (y - c).
\]

The solutions have the form

\[
p_1(y) = (y-c)^3 - \frac{3 (y-c)^4 T_{s,c}'}{4 T_{s,c}} + \frac{(y-c)^5}{10} \left( \frac{\alpha^2}{2} - \frac{6 T_{s,c}''}{T_{s,c}} + \frac{6 T_{s,c}^2}{T_{s,c}^2} \right) + \ldots,
\]

\[
p_2(y) = 1 + \frac{\alpha^2 T_{s,c} p_1(y)}{6 T_{s,c}} \ln(y-c) - \frac{\alpha^2 (y-c)^2}{2} \left[ 1 + \frac{\alpha^2 (y-c)^2}{6} \left( \frac{\alpha^2}{2} - \frac{T_{s,c}''}{T_{s,c}} + \frac{T_{s,c}'^2}{12 T_{s,c}} + \frac{m_{s,c}^2 M^2}{T_{s,c}} \right) \right] + \ldots,
\]

where

\[
\ln(y-c) = \begin{cases} 
\ln \eta, & \eta > 0, \\
\ln |y-c| - i\pi, & \eta < 0.
\end{cases}
\]

The index “c” denotes values of variables at point \( y = c \) (in critical layer) where the phase velocity is equal to velocity of undisturbed flow. Other components of inviscid solutions

\[
V_k(y) = \{u_{ki}(y), v_{ki}(y), \theta_{ki}(y), \theta_{v,ki}(y)\}, \quad k = 1, 2,
\]

are expressed through obtained \( p_1, p_2 \).
To find the viscous solutions the system (3) — (7) using some simplifications was reduced to the form which is analo-
gical to the Dunn — Lin “viscous” system [5]:

\[ v' + iu - i(y - c) \frac{\theta}{T_s} = 0, \]  
\[ u''' - i\alpha \text{Re} (y - c) \frac{u'}{T_s} = 0, \]  
\[ \theta'' - i\alpha \text{PrRe} (y - c) \frac{\theta}{T_s} + \frac{\gamma_v \text{PrRe}}{\gamma \tau T_s} (\theta_v - \theta) = 0, \]  
\[ \theta_v'' - \frac{33}{20} i\alpha \text{PrRe} (y - c) \frac{\theta_v}{T_s} - \frac{33 \text{PrRe}}{20 \gamma^2 \tau T_s} (\theta_v - \theta) = 0. \]  

For diatomic gases one can approximately suppose that \( \frac{33}{20\gamma} \approx 1 \). It allows us by summation and subtraction of Eqs (14), (15) to introduce the equations for auxiliary functions

\[ \theta_+ = \theta + \theta_v \frac{\gamma_v}{\gamma}, \quad \theta_- = \theta - \theta_v. \]

Then the temperatures are correspondingly expressed by formulae

\[ \theta = \frac{\gamma \theta_+ - \gamma_v \theta_-}{\gamma + \gamma_v}, \quad \theta_v = \frac{\gamma (\theta_+ + \theta_-)}{\gamma + \gamma_v}. \]  

The momentum equation (13) and equations for functions \( \theta_+, \theta_- \) were reduced to the Airy equations. Their solutions were presented through the generalized Airy functions of the first and second orders \( A_k(z, p) \) [6]. As a result, using (16) the solutions for temperatures were found. This allowed us to splinter the last equation from system (12) — (15) and reduce it in such a way to a sixth order system.

Linearly independent solutions for disturbances of transversal velocity are obtained by integration of Eq. (12) for two alternatives under \( \theta = 0, u \neq 0 \) and \( u = 0, \theta \neq 0 \). They also were expressed through the generalized Airy functions of the first and second orders.

As a result, it was shown that the linearly independent solutions of simplified system have the form:

\[ V_3(y) = \left[ u_{1v}(y), v_{1v}^{(1)}(y), 0 \right], \quad V_4(y) = \left[ 0, v_{1v}^{(2)}(y), \theta_{1v} \right], \]  
\[ V_5(y) = \left[ u_{2v}(y), v_{2v}^{(1)}(y), 0 \right], \quad V_6(y) = \left[ 0, v_{2v}^{(2)}(y), \theta_{2v} \right]. \]

\[
u_{1,2v}(y) = \frac{dY}{d\zeta} \left( \frac{dy}{dY} \right)^{3/2} A_{1,2} (i\zeta, 1), \quad v_{1,2v}^{(1)}(y) = -i \left( \frac{dy}{dY} \right)^{5/2} \left( \frac{dY}{d\zeta} \right)^2 A_{1,2} (i\zeta, 2), \]

\[
u_{1,2v}^{(2)}(y) = \frac{i(y - c)}{(\gamma + \gamma_v) T_s} \left( \frac{dY}{d\zeta} \right) \sqrt{\left( \frac{dy}{dY} \right)^3} \left[ \gamma A_{1,2} (i\zeta, 1) - \frac{\gamma_v}{\sqrt{2}} A_{1,2} \left( i\sqrt{2} \zeta_\theta, 1 \right) \right], \]

\[ \theta_{1,2v} = \frac{1}{\gamma + \gamma_v} \sqrt{\frac{dy}{dY}} \left[ \gamma A_{1,2} (i\zeta_\theta) - \frac{\gamma_v}{\sqrt{2}} A_{1,2} \left( i\sqrt{2} \zeta_\theta \right) \right], \]

where

\[ Y = \sqrt{\frac{3}{2} \int \frac{y}{c} \left( \frac{t - c}{T_s} \right) dt}, \quad \zeta = Y \sqrt{\text{Re}}, \quad Y_\theta = Y \sqrt{\text{Pr}}, \quad \zeta_\theta = \zeta \sqrt{\text{Pr}}. \]
4. Secular equation

For derivation of the secular equation it is necessary to require that a linear combination of independent solutions (11), (17), (18) satisfies the uniform boundary conditions (8)

\[ c_1 V_1(0) + c_2 V_2(0) + c_3 V_3(0) + c_4 V_4(0) + c_5 V_5(0) + c_6 V_6(0) = 0, \]  
\[ c_1 V_1(1) + c_2 V_2(1) + c_3 V_3(1) + c_4 V_4(1) + c_5 V_5(1) + c_6 V_6(1) = 0. \]

Large values of arguments of “viscous” solutions on both boundaries \( \zeta(y) \geq O(10), y = 0, 1 \) allows one to estimate it using the asymptotic formulæ [6] for the generalized Airy functions

\[ A_\pm(x, p) \simeq \frac{(\pm 1)^p}{2\sqrt{\pi}} x^{-(2p+1)} e^{\pm \eta}, \]
\[ \eta = \frac{2}{3} x^{3/2}, \quad A_1(x, p) \sim A_-(x, p), A_2(x, p) \sim A_+(x, p). \]

Normalization of solutions (17), (18) using its maximal values on the one of boundaries gives a possibility to replace exponentially small elements in the determinant of system (19), (20) by zeros. As a result, the determinant obtains a disperse form

\[
\Delta = \begin{vmatrix}
  u_{11}(0) & u_{21}(0) & u_{1i}(0) & 0 & 0 & 0 \\
  v_{11}(0) & v_{21}(0) & v_{1i}(0) & v_{1v}(0) & 0 & 0 \\
  \theta_{1i}(0) & \theta_{2i}(0) & \theta_{1i}(0) & 0 & 0 & 0 \\
  u_{1i}(1) & u_{2i}(1) & 0 & 0 & v_{2v}(1) & 0 \\
  v_{1i}(1) & v_{2i}(1) & 0 & 0 & v_{2v}(1) & v_{2v}(1) \\
  \theta_{1i}(1) & \theta_{2i}(1) & 0 & 0 & 0 & \theta_{2v}(1)
\end{vmatrix} = 0.
\]

This transition unweighted essentially a derivation of a secular equation.

Use of the Airy functions allows us to represent the viscous part of secular equation through the tabulated Tietjens function \( F(Z) \) and its derivative \( F'(Z) \), which are usually applied in the traditional asymptotic theory of stability [1, 6]. The secular equation obtained by such a way has the form:

\[ \Pi(0) = \frac{C_1 p_1(0) + p_2(0)}{\alpha^2 [C_1 p_1(0) + p_2(0)]} = -c I \left[ \frac{F(Z) + KJ_1(Z_0)}{1 - IF(Z)} \right], \]
\[ C_1 = -\frac{v_{2v}(1)}{v_{1i}(1)}, \quad K = \frac{\gamma - 1 \max^2}{T_s(0)} \left[ 1 - \frac{\tau \gamma c}{\gamma + \gamma_c - i \gamma c} \right], \quad I = \frac{3}{2} \sqrt{\frac{T_s}{c^3/2}} \int_0^c \frac{t - c}{T_s(t)} \, dt, \]
\[ J_1 = \frac{\left( \gamma \sqrt{2} - \gamma_c \right) \left( \tilde{G}(Z_0) - \gamma_c \left( \gamma \sqrt{2} - \gamma_c \right)^{-1} \left( \sqrt{2} - 1 \right) Z_0 \right)}{\left( \gamma \sqrt{2} - \gamma_c \right) \left( 1 + \left( \sqrt{2} - 1 \right) \left( \gamma \sqrt{2} - \gamma_c \right)^{-1} Z_0^2 \tilde{G}(Z_0) \left[ 1 - F(Z_0) \right] \right)}, \]
\[ \tilde{G}(Z_0) = -\frac{A_1 (-Z_\theta, 1)}{Z_\theta A_1 (-Z_\theta)}, \quad Z_\theta = -\zeta_\theta. \]

5. Numerical calculations of neutral curves

The secular equation (21) has a characteristic structure, which coincides with analagical equations [5, 7]. The left “inviscid” part of Eq. (21) depends on the phase velocity \( c \) and wave number \( \alpha \). At the same time the right “viscous” part of Eq. (21) is expressed through tabulated functions of variable \( Z \), also depending on the phase velocity \( c \). Therefore, points on neutral stability curves \( \text{Re} (\alpha, \gamma_c, M) \) are calculated on plane \( \text{Re} (\alpha) \) in the next sequence. The
value of the phase velocity \( c \) was set for fixed values of the Mach number \( M \) and the degree of vibrational nonequilibrium \( \gamma_v \) in interval \( c = [0, 1] \) with step \( \Delta c = 10^{-3} \). The integral \( I \) on the right-hand side of Eq. (21) was calculated by the Simpson formula. The real and imaginary parts of the right-hand side of Eq. (21) depend on the variable \( Z \). The arrays of their values were calculated for \( Z = [0, 10] \) using tables [5, 8].

For \( Z = [10, 50] \), we used an asymptotic formula

\[
\Pi(0) = -cqI \left\{ 1 + i + 2iqI + \left[ \frac{5i}{2} + KPr^{-1/2} (1 + i) \right] q \right\},
\]

where \( q = Z^{-3/2}/\sqrt{2} \). In both cases the table step \( \Delta Z = 0.04 \) [5] was picked out.

The arrays on the left-hand side of Eq. (21) for a fixed value of the phase velocity \( c \) were calculated for \( \alpha = [1, 3] \) with a step \( \Delta \alpha = 10^{-3} \). Calculated arrays on the right-hand and left-hand sides of Eq. (21) were compared up to obtaining coincide within the accuracy \( 10^{-8} \) if such a state could be achieved for a given value of \( c \). Then calculation was repeated for the next value of the phase velocity \( c \). As a result, we constructed arrays of values of wave numbers \( \alpha_c \), phase velocities \( \tau_c \), and of variable \( Z_c \), which correspond to points on neutral curve. Using formula

\[
\text{Re}_c(\alpha_c) = \frac{Z_c^3}{\alpha_c} \left[ \frac{3}{2} \int_0^{\tau_c} \sqrt{\frac{\tau_c - y}{T_s(y)}} \, dy \right]^{-2},
\]

the values of the Reynolds number \( \text{Re}_k \) on the neutral curve were calculated. The neutral curve for mode I corresponds to the values of the phase velocity in interval \( \tau_c \in (0.5, 1) \). At the same time the other points for \( \tau_c \in (0, 0.5) \) belong to the neutral stability curve of mode I.

**Figure 1.** Neutral stability curves \( \text{Re}_c(\alpha_c) \) for modes I (a) and II (b) at \( M=3 \), \( \gamma_v = 0 \) (1) and \( \gamma_v = 0.667 \), \( \tau = 1 \) (2). The solid and dashed curves show the results of calculations by spectral problem (3) — (7) and the asymptotic theory, correspondingly.

Figure 1 shows in semi-logarithmic coordinates the results of calculations of neutral stability curves for the perfect and vibrationally excited gases at the Mach number \( M = 3 \). The curves plotted by dashed lines are derived from secular equation (21) of the asymptotic theory. The solid lines are presented for comparison with the corresponding results of numerical calculation of spectral problem (3) — (7). It is seen, that asymptotic curves are in satisfactory agreement with numerical calculation of origin spectral problem (3) — (7). In particular, dissipative effect of the vibrational relaxation is reflected clearly. The transition to the asymptotic theory extends slightly an instability area and reduces the critical Reynolds numbers.
Table 1. Numerical values $\text{Re}_{cr}$ and $\alpha_{cr}$ for modes I and II.

|         | $\text{Re}_{cr}^s$ | $\alpha_{cr}^s$ | $\text{Re}_{cr}^a$ | $\alpha_{cr}^a$ | $\text{Re}_{cr}^s$ | $\alpha_{cr}^a$ |
|---------|--------------------|------------------|---------------------|------------------|---------------------|------------------|
| Mode I  |                    |                  |                     |                  |                     |                  |
| $\gamma_v$ |                   |                  |                     |                  |                     |                  |
| 0       | 123900             | 2.835            | 112000              | 2.842            | 82703               | 2.570            |
| 0.667   | 137303             | 2.901            | 124100              | 2.910            | 91650               | 2.628            |
| Mode II |                    |                  |                     |                  |                     |                  |
| $\gamma_v$ |                   |                  |                     |                  |                     |                  |
| 0       | 50060              | 2.546            | 44700               | 2.552            | 33415               | 2.307            |
| 0.667   | 56061              | 2.603            | 50100               | 2.610            | 37421               | 2.359            |

The table 1 allow one to compare data of numerical calculations of spectral problem (3) — (7) and asymptotic theory with respect to critical values of the Reynolds numbers $\text{Re}_{cr}$ and wave numbers $\alpha_{cr}$. It is seen that values $\text{Re}_{cr}^a$ are approximately $9 \div 10$ percent less than the corresponding values $\text{Re}_{cr}^s$.

6. Conclusion
The asymptotic solutions of spectral problem of linear stability theory for the supersonic plane Couette flow of the vibrationally excited gas are constructed. The found solutions are universal for any two-dimensional perturbations of plane shear flows of molecular gases with one dedicated vibrational mode.

The asymptotic curves of neutral stability satisfactorily approximate results of direct numerical solution of spectral problem (3) — (7). These calculations once again confirm existence in a molecular gas of a dissipative effect, which is caused by relaxation of vibrational modes.

The essential point of the study is derivation and solution of secular equation for the case of compressible gas flow between two impermeable boundaries. When the excited vibrational mode is absent, a continuous transition to the compressible flow of a perfect gas takes place. As known, in such a case an asymptotic theory of neutral stability was not previously constructed.

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