TATE COHOMOLOGY OF WHITTAKER LATTICES AND BASE CHANGE OF GENERIC REPRESENTATIONS OF GL\(_n\)

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**Abstract.** Let \(p\) and \(l\) be two distinct odd primes, and let \(n \geq 2\) be a positive integer. Let \(E\) be a finite Galois extension of degree \(l\) of a \(p\)-adic field \(F\). Let \(q\) be the cardinality of the residue field of \(F\). Let \(\pi_F\) be an integral \(l\)-adic generic representation of \(\text{GL}_n(F)\), and let \(\pi_E\) be the base change of \(\pi_F\). Let \(J_l(\pi_E)\) (resp. \(J_l(\pi_F)\)) be the unique generic component of the mod-\(l\) reduction \(r_l(\pi_F)\) (resp. \(r_l(\pi_E)\)). Assuming that \(l\) does not divide \(|\text{GL}_{n-1}(\mathbb{F}_q)|\), we prove that the Frobenius twist of \(J_l(\pi_F)\) is the unique generic subquotient of the Tate cohomology group \(\hat{H}^0(\text{Gal}(E/F), J_l(\pi_E))\) considered as a representation of \(\text{GL}_n(F)\).

**1. Introduction**

Let \(l\) be a prime number, and let \(F\) be a number field. Let \(G\) be a reductive algebraic group defined over \(F\), and let \(\sigma\) be an automorphism of order \(l\) of \(G\). D.Treumann and A.Venkatesh have constructed a functorial lift of a mod-\(l\) automorphic form for \(G^G\) to a mod-\(l\) automorphic form for \(G\) (see [TV16]). They conjectured that the mod-\(l\) local functoriality at ramified places must be realised in Tate cohomology, and they defined the notion of linkage (see [TV16, Section 6.3] for more details). Among many applications of this set up, we focus on local base change lifting from \(G^G = \text{GL}_n/F\) to \(G = \text{Res}_{E/F} \text{GL}_n/E\), where \(E/F\) is a Galois extension of \(p\)-adic fields with \([E : F] = l\). Truemund and Venkatesh’s conjecture on linkage in Tate cohomology is verified for local base change of depth-zero cuspidal representations by N.Ronchetti, and a precise conjecture in the context of local base change of \(l\)-adic higher depth cuspidal representations was formulated in [Ron16, Conjecture 2]. In this article, using Whittaker models and Rankin-Selberg zeta functions, we prove this conjecture for \(\text{GL}_n\) under the assumption that \(l\) does not divide the pro-order of \(\text{GL}_{n-1}(F)\) whenever \(n > 2\). In fact, when the prime \(l\) does not divide the pro-order of \(\text{GL}_{n-1}(F)\), we prove a much stronger theorem that the Frobenius twist of a mod-\(l\) generic representation of \(\text{GL}_n(F)\) occurs as a sub-quotient of the zeroth Tate cohomology of its base change lifting to \(\text{GL}_n(E)\) (see Theorem 6.7 for the precise result).

Let us introduce some notations to state the results of this article. From now, we assume that \(F\) is a finite extension of \(\mathbb{Q}_p\) with residue field \(\mathbb{F}_q\). Let \(E\) be a finite Galois extension of \(F\) with \([E : F] = l\), where \(l\) and \(p\) are distinct odd primes. Let \(\pi_F\) be an integral \(l\)-adic generic representation of \(\text{GL}_n(F)\). The mod-\(l\)-reduction of \(\pi_F\) has a unique generic component and it is denoted by \(J_l(\pi_F)\) (see Vig01, Section 1.8.4). The base change lift of \(\pi_F\) to \(\text{GL}_n(E)\) is denoted by \(\pi_E\) (for the definition, see subsection (4.2)). Note that \(\pi_E\) is also an integral \(l\)-adic generic representation of \(\text{GL}_n(E)\). Moreover, there is an isomorphism \(T : \pi_E \sim \pi_E^\gamma\), where \(\pi_E^\gamma\) is the twist of \(\pi_E\) by a generator \(\gamma\) of \(\text{Gal}(E/F)\). Then the unique generic component \(J_l(\pi_E)\) of the mod-\(l\) reduction \(r_l(\pi_E)\) is also stable under the action of \(\text{Gal}(E/F)\)–induced by \(T\). In this article, Tate cohomology groups are always with respect to the action of \(\text{Gal}(E/F)\). We prove the following theorem:

**Theorem 1.1.** Let \(F\) be a finite extension of \(\mathbb{Q}_p\), and let \(E\) be a finite Galois extension of \(F\) with \([E : F] = l\), where \(p\) and \(l\) are distinct odd primes such that \(l\) does not divide \(|\text{GL}_{n-1}(\mathbb{F}_q)|\). Let \(\pi_F\) be an integral \(l\)-adic generic representation of \(\text{GL}_n(F)\), and let \(\pi_E\) be the base change lifting of \(\pi_F\) to \(\text{GL}_n(E)\). Then, the Frobenius twist of \(J_l(\pi_F)\) occurs a subquotient of the zeroth Tate cohomology group \(\hat{H}^0(J_l(\pi_E))\), considered as a representation of \(\text{GL}_n(F)\).

We note some immediate remarks on the hypothesis in Theorem 1.1. As a consequence of Proposition 6.3 in Section 6, the Frobenius twist of \(J_l(\pi_F)\), defined as \(J_l(\pi_F) \otimes_{\text{Frob}} \mathbb{F}_l\), where \(\text{Frob}\) is the Frobenius automorphism of \(\mathbb{F}_l\), is in fact the unique generic sub-quotient of the Tate cohomology group \(\hat{H}^0(J_l(\pi_E))\). We use Kirillov and Whittaker models of generic representations to prove our main result. The hypothesis
that $l$ does not divide the pro-order of $GL_{n-1}(F)$ is required in the proof of a vanishing result of Rankin–Selberg integrals on $GL_{n-1}(F)$ (the analogue of [JPSS81, Lemma 3.5] or [BH03, 6.2.1]). This condition on $l$ may be removed using $\gamma$-factors defined over local Artinian $F$-algebras as defined in the work of G.Moss and N.Matringe in [MM22]. However, the right notion of base change over local Artinian $F$-algebras is not clear to the authors and hence, we use the mild hypothesis that $l$ does not divide $[GL_{n-1}(\mathbb{F}_q)]$. If $\pi_F$ and $\pi_E$ are both cuspidal, then using the Kirillov model for cuspidal representations, one observes that the Tate cohomology group $\hat{H}^0(r_1(\pi_E))$ is an irreducible $GL_n(F)$ representation, and the above theorem says that this Tate cohomology space is isomorphic to the Frobenius twist of mod-$l$ reduction of $\pi_F$ (Corollary 6.9) when $l$ does not divide the pro-order of $GL_{n-1}(F)$. This is conjectured by Ronchetti in [Ron16, Conjecture 2].

Let $\mathcal{K}$ be the maximal unramified extension of $Q_l$ in an algebraic closure $\overline{Q}_l$ of $Q_l$. Let $\Lambda$ be the ring of integers of $\mathcal{K}$. We also prove an integral version of Theorem 1.1. To be precise, say $\pi_E$ is an integral generic $\mathcal{K}$-representation of $GL_n(E)$—which is absolutely irreducible (i.e., $\pi_E \otimes_K \overline{Q}_l$ is irreducible), such that $\pi_E \otimes_K Q_l$ is the base change lift of an $l$-adic integral generic representation $\pi_F$ of $GL_n(F)$. We show that the Frobenius twist of $J_l(\pi_F)$ occurs as the unique generic subquotient of the zeroth Tate cohomology group $\hat{H}^0(\mathcal{W}_L(\pi_E, \psi_E))$ (Corollary 6.8), where $\mathcal{W}_L(\pi_E, \psi_E)$ is the space of all $\Lambda$-valued functions in the Whittaker model of $\pi_E$, with respect to a $Gal(E/F)$-equivariant character $\psi_E$. A priori, Vigneras showed that $\mathcal{W}_L(\pi_E, \psi_E)$ is a $GL_n(E)$-invariant $\Lambda$-lattice in $\mathbb{W}(\pi_E, \psi_E)$. When $l$ does not divide the pro-order of $GL_n(F)$, we obtain a much precise version of Theorem 1.1. We can show that the first Tate cohomology group of any $Gal(E/F)$ invariant lattice $\mathcal{L}$ in a generic representation $\pi_E$ as in Theorem 1.1 is trivial. Moreover, we show that the zeroth Tate cohomology group of the mod-$l$ reduction $r_l(\pi_E)$ is an irreducible representation of $GL_n(F)$ (see Theorem 8.3 and Corollary 8.4). Our method can also be extended to some non-generic representations as well. Especially for those irreducible representations of $GL_n(E)$ which remains irreducible when restricted to the mirabolic subgroup, denoted by $P_n(E)$. This class of representations are exactly the Zelevinsky sub-representations. Assume that $\sigma_E$ is an $l$-adic cuspidal representation obtained as a base change lifting of $\pi_E$ to $GL_n(E)$. Let $\Delta$ be a segment (see Section 2.7.2) on the cuspidal line of $\pi_E$ (defined in Theorem 1.1). We apply Theorem 1.1 to compute the Tate cohomology of mod-$l$ Zelevinsky sub-representations $\mathcal{Z}(\Delta)$ (see Theorem 7.3), where $\Delta$ is the segment on the cuspidal line of $r_l(\sigma_E)$.

When $F$ is a local function field, the above theorem follows from the work of T.Feng [Fen24]. T.Feng uses the constructions of V. Lafforgue and A. Genestier-V.Lafforgue [GL17]. Assuming that $l$ and $p$ do not divide $n$, Ronchetti proved the above result for depth-zero cuspidal representations using the compact induction model. Our methods are very different from the work of N.Ronchetti and the work of T.Feng. We rely on Rankin–Selberg integrals and Whittaker models. We do not require the explicit construction of cuspidal representations. We use various properties of local $\epsilon$ and $\gamma$-factors both in $l$-adic and mod-$l$ situations associated with the representations of the $p$-adic group and the Weil group. The machinery of local $\epsilon$ and $\gamma$-factors of both $l$-adic and mod-$l$ representations of $GL_n(F)$ is made available by the seminal works of D.Helm, G.Moss, N.Matringe and R.Kurinczuk (see [HM18], [Mos16], [KM21], [KM17]).

The case where $\pi_E$ is a cuspidal representation of $GL_2(E)$ is considered in Theorem 6.5, the general case is proved in Theorem 6.7 using induction on $n$. The reader might quickly follow the proof of Theorem 6.5 before going to the general case. We sketch the proof of Theorem 1.1. The theorem is proved, inductively on $n$, using the Kirillov model. Let $\psi_F : F \rightarrow \overline{Q}_l^K$ be a non-trivial additive character and let $\psi_E$ be the character $\psi_F \circ Tr_{E/F}$, where $Tr_{E/F} : E \rightarrow F$ is the trace function. Let $(\pi_F, V)$ be an integral generic $l$-adic representation of $GL_n(F)$. In particular, $V$ is a $Q_l^K$-vector space. Let $N_n(F)$ be the group of unipotent upper triangular matrices in $GL_n(F)$. Let $\Theta_F : N_n(F) \rightarrow Q_l^K$ be a non-degenerate character corresponding to $\psi_F$. We denote by $\mathbb{W}(J_l(\pi_F), \overline{\psi}_F)$ the Whittaker model of the unique generic sub-quotient, denoted by $J_l(\pi_F)$, of the mod-$l$ reduction of $\pi_F$. Here, $\overline{\psi}_F$ is the mod-$l$ reduction of $\psi_F$. Let $\pi_E$ be the base change lift of $\pi_F$ to $GL_n(E)$. Similar notations for $\pi_E$ are followed where $\overline{\psi}_F$ is replaced with $\overline{\psi}_E$. It is easy to note that (Lemma 2.4) $\mathbb{W}(J_l(\pi_F), \overline{\psi}_F)$ is stable under the action of $Gal(E/F)$ on the space $Ind_{N_n(E)}^{GL_n(E)}(\overline{\psi}_E)$. Let $\mathcal{K}(J_l(\pi_F), \overline{\psi}_F)$ be the Kirillov model of $J_l(\pi_F)$. Using the result [MM22, Theorem 4.2], we get that the restriction to $P_n(F)$ map from $\mathbb{W}(J_l(\pi_F), \overline{\psi}_F)$ to $\mathcal{K}(J_l(\pi_F), \overline{\psi}_F)$ is a bijection.

The Kirillov model $\mathcal{K}(J_l(\pi_F), \overline{\psi}_E)$ contains the space of all smooth and compactly supported functions $Ind_{N_n(E)}^{GL_n(E)}(\overline{\psi}_E)$. Recall that the Tate cohomology group $\hat{H}^0(\mathcal{K}(J_l(\pi_F), \overline{\psi}_E))$ (For definition, see Section 5) is an
$\mathbb{F}_l$-representation of $P_n(F)$. Let $\Phi$ be the following map obtained by restriction of functions to $P_n(F)$:

$$\Phi : \tilde{H}^0(\mathbb{K}(J_{l}(\mathbb{F}), \mathbb{F})) \longrightarrow \text{Ind}_{N_n(F)}^{G_n(F)}(\mathbb{F}).$$

Using compactly supported functions, one can show that the inverse image of $\mathbb{K}(J_{l}(\mathbb{F}), \mathbb{F})$ under the map $\Phi$ is non-zero, and it is denoted by $M(\mathbb{F}, \mathbb{F})$. Here, $J_{l}(\mathbb{F})$ is the Frobenius twist of $J_{l}(\mathbb{F})$. To prove the main theorem, we show that the space $M(\mathbb{F}, \mathbb{F})$ is stable under $GL_n(F)$ and the restriction of $\Phi$ to the space $M(\mathbb{F}, \mathbb{F})$ is $GL_n(F)$ equivariant. This is just equivalent to showing that

$$I(X, \Phi(J_{l}(\mathbb{F})(w_n)W), \sigma(w_{n-1})W') = I(X, J_{l}(\mathbb{F})(w_n)\Phi(W), \sigma(w_{n-1})W'),$$

for all $W \in M(\mathbb{F}, \mathbb{F})$ and $W' \in \mathcal{W}(\sigma, \mathbb{F})^{-1}$, where $w_{n-1}, w_n$ are defined in subsection (2.2); and $\sigma$ is an $l$-modular generic representation of $GL_{n-1}(F)$. Here, $I(X, W, W')$ is a mod-$l$ Rankin–Selberg zeta functions written as a formal power series in the variable $X$ instead of the traditional $q^{-s}$ ([KM17, Section 3]). We transfer the local Rankin–Selberg zeta functions $I(X, \Phi(J_{l}(\mathbb{F})(w_n)W), \sigma(w_{n-1})W')$ made from integrals on $GL_{n-1}(F)/N_{n-1}(F)$ to Rankin–Selberg zeta functions defined by integrals on $GL_{n-1}(E)/N_{n-1}(E)$. Then, using local Rankin–Selberg functional equation, we show that the equality in (1.1) is equivalent to certain identities of mod-$l$ local $\gamma$-factors, such as (6.13).

We briefly explain the contents of this article. In Section 2, we recall various notations, conventions on integral representations, Whittaker models and Kirillov models. In Section 3, we collect various results on local constants both in mod-$l$ and $l$-adic settings. In Section 4, we put some well known results from $l$-adic local Langlands correspondence. In Section 5, we recall and set up some initial results on Tate cohomology of smooth integral representations as well as mod-$l$ representations. In Section 6, we begin with a few observations on compatibility of Jacquet and twisted Jacquet functors with Tate cohomology. Then we prove our main result Theorem 6.7. In Sections 7 and 8, in the banal case, we completely compute the Tate cohomology of the representations $Z(\Delta)$ and $L(\Delta)$ using Theorem 6.7.

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2. Preliminaries

2.1. Let $K$ be a non-Archimedean local field and let $\mathfrak{o}_K$ be the ring of integers of $K$. Let $p_K$ be the maximal ideal of $\mathfrak{o}_K$ and let $\varpi_K$ be a uniformizer of $K$. Let $q_K$ be the cardinality of the residue field $k_K = \mathfrak{o}_K/p_K$. Let $v_K : K^\times \rightarrow \mathbb{Z}$ be the normalized valuation. We denote by $\nu_K$ the normalized absolute value of $K$ corresponding to $v_K$. Let $l$ and $p$ be two distinct odd primes. Let $F$ be a finite extension of $\mathbb{Q}_p$, and let $E$ be a finite Galois extension of $F$ with $[E : F] = l$. We denote the group $\text{Gal}(E/F)$ by $\Gamma$.

2.2. For any ring $A$, let $M_{r \times s}(A)$ be the $A$-algebra of all $r \times s$ matrices with entries from $A$. Let $GL_n(K) \subseteq M_{n \times n}(K)$ be the group of all invertible $n \times n$ matrices. We denote by $G_n(K)$ the group $GL_n(K)$ and $G_n(K)$ is equipped with locally compact topology induced from the local field $K$. For $r \in \mathbb{Z}$, let

$$G_r^n(K) = \{ g \in G_n(K) : v_K(\det(g)) = r \}.$$

We set $P_n(K)$, the mirabolic subgroup, defined as the group:

$$\left\{ \begin{pmatrix} A & M \\ 0 & 1 \end{pmatrix} : A \in G_{n-1}(K), M \in M_{(n-1) \times 1}(K) \right\}.$$

Let $B_n(K)$ be the group of all invertible upper triangular matrices in $M_{n \times n}(K)$, and let $N_n(K)$ be its unipotent radical. We denote by $w_n$ the following matrix of $G_n(K)$:

$$w_n = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let $X_K^n$ denote the coset space $N_{n-1}(K) \setminus G_{n-1}(K)$. For $r \in \mathbb{Z}$, we denote the coset space $\{ N_{n-1}(K)g : g \in G_{n-1}^r(K) \}$ by $X^n_{K_r}$. 
2.3. Fix an algebraic closure $\overline{\mathbb{Q}}_l$ of the field $\mathbb{Q}_l$. Let $\mathbb{Z}_l$ be the integral closure of $\mathbb{Z}$ in $\overline{\mathbb{Q}}_l$ and let $\mathfrak{p}_l$ be the unique maximal ideal of $\mathbb{Z}_l$. We have $\mathbb{Z}_l/\mathfrak{p}_l \simeq \mathbb{F}_l$. We fix a square root of $q_F$ in $\overline{\mathbb{Q}}_l$, and it is denoted by $q_F^{1/2}$. The choice of $q_F^{1/2}$ is required for transferring the complex local Langlands correspondence to a local $l$-adic Langlands correspondence (see [BH06, Chapter 8]). Let $K$ denote the maximal unramified extension of $\mathbb{Q}_l$ in $\overline{\mathbb{Q}}_l$, and let $\Lambda$ be the ring of integers of $K$. Let $W(\mathbb{F}_l)$ be the ring of Witt vectors of $\mathbb{F}_l$. The prime number $l$ is called banal for $G_n(K)$ if $l$ does not divide $|\text{GL}_n(k_K)|$.

2.4. Smooth representations and Integral representations. Let $G$ be a locally compact and totally disconnected group. A representation $(\pi, V)$ is said to be smooth if, for every vector $v \in V$, the $G$-stabilizer of $v$ is an open subgroup of $G$. All the representations are assumed to be smooth and the representation spaces are vector spaces over $R$, where $R = \overline{\mathbb{Q}}_l$ or $\mathbb{F}_l$. A representation $(\pi, V)$ is called $l$-adic when $R = \overline{\mathbb{Q}}_l$ and $(\pi, V)$ is called $l$-modular when $R = \mathbb{F}_l$. We denote by $\text{Irr}(G, R)$, the set of all irreducible smooth $R$-representations of $G$. Let $C_c^\infty(G, R)$ denote the set of all locally constant and compactly supported functions on $G$ taking values in a ring $R$.

Let $(\pi, V)$ be an $l$-adic representation of $G$. A lattice in $V$ is a free $\mathbb{Z}_l$-module $\mathcal{L}$ such that $\mathcal{L} \otimes_{\mathbb{Z}_l} \overline{\mathbb{Q}}_l = V$. The representation $(\pi, V)$ is said to be integral if it has finite length as a representation of $G$ and there exists a $G$-invariant lattice $\mathcal{L}$ in $V$. A character is a smooth one-dimensional representation $\chi : G \rightarrow R^\times$. For $G = G_n(K)$, a character $\chi : K^\times \rightarrow R^\times$ induces a character $\chi \circ \det : G_n(K) \rightarrow R^\times$. By abuse of notation, we denote the character $\chi \circ \det$ by $\chi$. In particular, the normalized absolute value of $K$ gives a character $\nu_K$ of $G_n(K)$. We say that a character $\chi : G \rightarrow \overline{\mathbb{Q}}_l^*$ is integral if it takes values in $\mathbb{Z}_l$.

Let $(\pi, V)$ be an integral $l$-adic representation of $G$. Choose a $G$-invariant lattice $\mathcal{L}$ in $V$. Then the group $G$ acts on $\mathcal{L} \otimes_{\mathbb{Z}_l} \mathbb{F}_l$, which is a vector space over $\mathbb{F}_l$. This gives an $l$-modular representation, which depends on the choice of the $G$-invariant lattice $\mathcal{L}$. By [Vig96, II. 5.11.a and 5.11.b], the representation $(\pi, \mathcal{L} \otimes_{\mathbb{Z}_l} \mathbb{F}_l)$ is of finite length and its semisimplification is independent of the choice of the $G$-invariant lattice $\mathcal{L}$ in $V$. We denote by $r_l(\pi)$ the semisimplification of $(\pi, \mathcal{L} \otimes_{\mathbb{Z}_l} \mathbb{F}_l)$. The representation $r_l(\pi)$ is called the mod-$l$ reduction of the $l$-adic representation $\pi$. We say that an $l$-modular representation $\sigma$ lifts to an integral $l$-adic representation $\pi$ if there exists a $G$-invariant lattice $\mathcal{L} \subseteq \pi$ such that $\mathcal{L} \otimes_{\mathbb{Z}_l} \mathbb{F}_l \simeq \sigma$.

2.5. Parabolic induction. Let $H$ be a closed subgroup of $G$. Let $\text{Ind}^G_H$ and $\text{ind}^G_H$ be the smooth induction functor and compact induction functor respectively. We follow [BZ77] for the definitions.

Set $G = G_n(K)$, $P = P_n(K)$ and $N = N_n(K)$, where $G_n(K)$, $P_n(K)$ and $N_n(K)$ are defined in subsection (2.2). Let $\lambda = (n_1, n_2, ..., n_t)$ be an ordered partition of $n$. Let $Q_\lambda \subseteq G_n(K)$ be the group of matrices of the form

$$
\begin{pmatrix}
A_1 & * & * & * \\
A_2 & * & * & * \\
& * & * & * \\
& & & & A_t
\end{pmatrix},
$$

where $A_i \in G_{n_i}(K)$, for all $1 \leq i \leq t$. Then $Q_\lambda = M_\lambda \ltimes U_\lambda$, where $M_\lambda$ is the group of block diagonal matrices of the form

$$
\begin{pmatrix}
A_1 \\
A_2 \\
& \ddots \\
& & & & A_t
\end{pmatrix},
\quad A_i \in G_{n_i}(K),
$$

for all $1 \leq i \leq t$ and $U_\lambda$ is the unipotent radical of $Q_\lambda$ consisting of matrices of the form

$$
U_\lambda = \begin{pmatrix}
I_{n_1} & * & * & * \\
* & I_{n_2} & * & * \\
& * & \ddots & * \\
& & & I_{n_t}
\end{pmatrix},
$$
where $I_{n_i}$ is the $n_i \times n_i$ identity matrix.

Let $\sigma$ be an $R$-representation of $M_\lambda$. Then the representation $\sigma$ is considered as a representation of $Q_\lambda$ by inflation via the map $Q_\lambda \to Q_\lambda/U_\lambda \simeq M_\lambda$. The induced representation $\text{Ind}^G_{Q_\lambda}(\sigma)$ is called the parabolic induction of $\sigma$. We denote the normalized parabolic induction of $\sigma$ corresponding to the partition $\lambda$ by $i^G_{Q_\lambda}(\sigma)$. For details, see [BZ77]. Let $\lambda = (n_1, \ldots, n_s)$ be a partition of $n$ and let $\sigma_i$ be $R$-representation of $G_{n_i}$ for each $i$. We denote the parabolic induction $i^G_{Q_\lambda}(\sigma_1 \otimes \cdots \otimes \sigma_s)$ by the product symbol $\sigma_1 \times \cdots \times \sigma_s$.

2.5. Cuspidal and Supercuspidal representation. Keeping the notation as in (2.5). Let $\pi$ be an irreducible $R$-representation of $G$. Then $\pi$ is called a cuspidal representation if for all proper subgroups $Q_\lambda = M_\lambda \rtimes U_\lambda$ of $G$ and for all irreducible $R$-representations $\sigma$ of $M_\lambda$, we have

$$\text{Hom}_{\lambda}(\pi, i^G_{Q_\lambda}(\sigma)) = 0.$$ 

The representation $\pi$ is called supercuspidal if for all proper subgroups $Q_\lambda = M_\lambda \rtimes U_\lambda$ of $G$ and for all irreducible $R$-representations $\sigma$ of $M_\lambda$, the representation $(\pi, V)$ is not a subquotient of $i^G_{Q_\lambda}(\sigma)$.

Remark 2.1. Let $k$ be an algebraically closed field, and let $\pi$ be a smooth $k$-representation of $G$. If the characteristic of $k$ is 0, then $\pi$ is cuspidal if and only if $\pi$ is supercuspidal. But when characteristic of $k$ is $l > 0$, there are cuspidal representations of $G$ which are not supercuspidal. For details, see [Vig96, Section 2.5, Chapter 2].

2.7. Generic representation. Let $\psi_K : K \to R^\times$ be a non-trivial additive character of $K$. Let $\Theta_K$ be the character of $N_n(K)$, defined by

$$\Theta_K(x_{ij}) := \psi_K(\sum_{i=1}^{n-1} x_{i,i+1}).$$

Let $(\pi, V)$ be an irreducible $R$-representation of $G_n(K)$. Then recall that

$$\dim_R(\text{Hom}_{N_n(K)}(\pi, \Theta_K)) \leq 1.$$ 

For the proof, see [BZ76] when $R = \mathbb{Q}_l$ and see [Vig96] when $R = \mathbb{F}_l$. An irreducible $R$-representation $(\pi, V)$ of $G_n(K)$ is called generic if

$$\dim_R(\text{Hom}_{N_n(K)}(\pi, \Theta_K)) = 1.$$ 

2.7.1. Whittaker Model. Let $(\pi, V)$ be a generic $R$-representation of $G_n(K)$. By Frobenius reciprocity, the representation $\pi$ is embedded in the space $\text{Ind}^{G_n(K)}_{N_n(K)}(\Theta_K)$. Let $W_\pi$ be a non-zero linear functional in the space $\text{Hom}_{N_n(K)}(\pi, \Theta_K)$. Let $\mathcal{W}(\pi, \psi_K) \subset \text{Ind}^{G_n(K)}_{N_n(K)}(\Theta_K)$ be the space consisting of functions $W_v$, $v \in V$, where

$$W_v(g) := W_\pi(\pi(g)v),$$ 

for all $g \in G_n(K)$. Then the map $v \mapsto W_v$ induces an isomorphism from $(\pi, V)$ to $\mathcal{W}(\pi, \psi_K)$. 

2.7.2. Segments. In this subsection, we recall the notion of segments and its associated representations. For details, see [Zel80] for $R = \mathbb{Q}_l$ and [KM17], [MS14] for $R = \mathbb{F}_l$.

Let $r, t \in \mathbb{Z}$ with $r \leq t$. A segment is a sequence $\Delta = (\nu^r_K \sigma, \nu^{r+1}_K \sigma, \ldots, \nu^t_K \sigma)$, with $\sigma$ a cuspidal $R$-representation of $G_n(K)$. The length of $\Delta$ is defined to be $t - r + 1$. In [MS14, Definition 7.5] the authors, using Bushnell-Kutzko’s simple types and the Hecke algebras associated with them, defined a certain quotient of the parabolically induced representation

$$\tau = \nu^r_K \sigma \times \nu^{r+1}_K \sigma \times \cdots \times \nu^t_K \sigma,$$

denoted by $\mathcal{L}(\Delta)$. The normalised Jacquet module of $\mathcal{L}(\Delta)$ with respect to the opposite of the parabolic subgroup $P_{(n,\ldots,n)}$ is equal to

$$\nu^r_K \sigma \otimes \nu^{r+1}_K \sigma \otimes \cdots \otimes \nu^t_K \sigma.$$

Moreover, there is a unique generic sub-quotient of $\tau$, denoted by $\text{St}(\sigma, [r, t])$ and it is called the generalised Steinberg representation associated to $\Delta$. We denote by $\text{St}(\sigma, k)$ the representation $\text{St}(\sigma, [0, k-1])$, for $k \geq 1$.

2.7.3. Let $\sigma$ be a cuspidal $R$-representation of $G_n(K)$. The set $\{\nu^r_K \sigma : r \in \mathbb{Z}\}$ is called the cuspidal line of $\sigma$. The following theorem is proved by [MS14, Section 5.2] defines a positive integer $e(\sigma)$ as follows:

$$e(\sigma) = \begin{cases} +\infty & \text{if } R = \mathbb{Q}_l; \\ o(\sigma) & \text{if } R = \mathbb{F}_l \text{ and } o(\sigma) > 1; \\ t & \text{if } R = \mathbb{F}_l \text{ and } o(\sigma) = 1. \end{cases}$$

Then for a segment $\Delta = (\nu^r_K \sigma, \ldots, \nu^t_K \sigma)$, with $r \leq t$, the representation $\mathcal{L}(\Delta)$ is equal to $\text{St}(\sigma, [r, t])$ if and only if the length of the segment $\Delta$ is less than $e(\sigma)$. Recall [MS14, Remarque 8.14]). In this case, the segment $\Delta$ is called a generic segment. Note that every segment is generic for $R = \mathbb{Q}_l$.

2.7.4. Two segments $\Delta_1$ and $\Delta_2$ are said to be linked if $\Delta_1 \not\subseteq \Delta_2$, and $\Delta_2 \not\subseteq \Delta_1$ and $\Delta_1 \cup \Delta_2$ is a segment. The theorem is proved by [MS14, Theorem 9.10] for $R = \mathbb{F}_l$ and [Zel80, Theorem 9.7] for $R = \mathbb{Q}_l$.

**Theorem 2.2.** Let $\pi = \mathcal{L}(\Delta_1) \times \cdots \times \mathcal{L}(\Delta_t)$ be an $R$-representation of $G_n(K)$, where each $\Delta_j$ is a generic segment. Then $\pi$ is irreducible if and only if the segments $\Delta_i$ and $\Delta_j$ are not linked for all $i, j$ with $i \neq j$.

An $R$-representation of the form $\mathcal{L}(\Delta_1) \times \cdots \times \mathcal{L}(\Delta_t)$, where each $\Delta_i$ is generic, is called a representation of Whittaker type. In [Zel80, Theorem 9.7] and [Vig98, Proposition V.3], it is shown that

**Theorem 2.3.** An $R$-representation $\pi$ of $G_n(K)$ is generic if and only if $\pi$ is an irreducible $R$-representation of Whittaker type.

2.7.5. In this subsection, we fix a standard lift of an $l$-modular generic representation of $G_n(K)$. First, recall that any $l$-modular cuspidal representation of $G_m(K)$ can be lifted to an $l$-adic cuspidal representation of $G_m(K)$ (see [Vig96, Chapter 3, 4.25]). Let $\rho$ be an $l$-modular cuspidal representation of $G_m(K)$ and let $\Delta = (\rho, \widetilde{\sigma}_K \rho, \ldots, \widetilde{\sigma}_K^{l-1} \rho)$ be a segment associated with $\rho$, where $\widetilde{\sigma}_K$ is the mod-$l$ reduction of $\nu_K$. Let $\sigma$ be a cuspidal lift of $\sigma$. Then the segment $D = (\sigma, \nu_K \sigma, \ldots, \nu_K^{l-1} \sigma)$ is called a standard lift of $\Delta$. When $\mathcal{L}(\Delta) = \text{St}(\rho, r)$, then the mod-$l$ representation $\mathcal{L}(\Delta)$ lifts to $\mathcal{L}(\bar{D}) = \text{St}(\sigma, r)$ ([KM17, Proposition 2.16]).

Let $\pi$ be a generic $l$-modular representation of $G_n(K)$. Then $\pi$ is of the form $\mathcal{L}(\Delta_1) \times \cdots \times \mathcal{L}(\Delta_t)$, where each $\Delta_i$ is a generic segment. For each $1 \leq i \leq t$, let $D_i$ be a standard lift of $\Delta_i$. Then the $l$-adic representation $\tau = \mathcal{L}(D_1) \times \cdots \times \mathcal{L}(D_t)$ is generic, and $\pi$ lifts to $\tau$ ([KM17, Remark 2.31]). The representation $\tau$ is called a standard lift of $\pi$.

2.7.6. Let $(\pi, V)$ be an integral, $l$-adic, generic representation of $G_n(K)$. We fix a non-trivial additive character $\psi_K : K \to \Lambda^\times$. By abuse of notation, the composition $K \ni \psi_K \to \Lambda^\times \to \mathbb{Q}_l^\times$ is also denoted by $\psi_K$. Consider the space $\mathcal{W}_l(\pi, \psi_K)$ consisting of $W \in \mathcal{W}(\pi, \psi_K)$, taking values in $\mathbb{Q}_l$. It follows from [Vig04, Theorem 2] that the $\mathbb{Z}_l$-module $\mathcal{W}_l(\pi, \psi_K)$ is a $G_n(K)$-invariant lattice in $\mathcal{W}(\pi, \psi_K)$. The lattice $\mathcal{W}_l(\pi, \psi_K)$ is also called the integral Whittaker model or Whittaker lattice. Let $\tau$ be an $l$-modular generic representation of $G_n(K)$, and let $\pi$ be an $l$-adic generic representation of $G_n(K)$. Then, the representation $\pi$ is called a Whittaker lift of $\tau$ if there exists a lattice $\mathcal{L} \subseteq \mathcal{W}_l(\pi, \psi_K)$ such that

$$\mathcal{L} \otimes_{\mathbb{Z}_l} \mathbb{F}_l \simeq \mathcal{W}(\tau, \psi_K),$$
where $\overline{\psi}_K$ is the reduction mod-$l$ of $\psi_K$. Note that any standard lift of an $l$-modular generic representation $\pi$ is a Whittaker lift (see [KM17, Theorem 2.26]). Let $(\pi, V)$ be an $l$-adic integral generic representation defined over the field $\mathcal{K}$. Let $V'$ be a $\mathcal{K}$-structure in $V$. There exists a $\Lambda[G_n(K)]$-lattice in $V'$ (see [EH14, Corollary 4.4.4]). Thus, we get that the set of $\Lambda$-valued functions, denoted by $\mathcal{W}_\Lambda(\pi, \psi_K)$, in $\mathcal{W}^0(\pi, \psi_K)$ is a $G_n(K)$-stable lattice.

2.7.7. Now we follow the notations as in (2.1). Choose a generator $\gamma$ of $\Gamma$. Let $\pi$ be an $R$-representation of $G_n(E)$. The group $\Gamma = \text{Gal}(E/F)$ acts on $G_n(E)$ component-wise i.e., for $\gamma \in \Gamma, g = (a_{ij})_{i,j=1}^n \in G_n(E)$, we set
\[
\gamma.g := (\gamma(a_{ij}))_{i,j=1}^n.
\]
Let $\pi^\gamma$ be the representation of $G_n(E)$ on $\gamma$, defined by
\[
\pi^\gamma(g) := \pi(\gamma.g), \text{ for all } g \in G_n(E).
\]
We say that the representation $\pi$ of $G_n(E)$ is $\Gamma$-equivariant if the representations $\pi$ and $\pi^\gamma$ are isomorphic.

We now prove a lemma concerning the $\Gamma$ invariance of the Whittaker model of a $\Gamma$-equivariant representation $\pi$ of $G_n(E)$. Let $\psi_F$ and $\psi_E$ be the non-trivial additive characters of $F$ and $E$ respectively such that $\psi_E = \psi_F \circ \text{Tr}_{E/F}$ where, $\text{Tr}_{E/F}$ is the trace map of the extension $E/F$. Let $\Theta_F$ and $\Theta_E$ be the characters of $N_n(F)$ and $N_n(E)$ respectively, as defined in (2.7). Now consider the action of $\Gamma$ on the space $\text{Ind}_{G_n(E)}^{G_n(K)}(\Theta_E)$, given by
\[
(\gamma.f)(g) := f(\gamma^{-1}g),
\]
for all $\gamma \in \Gamma, g \in G_n(E)$ and $f \in \text{Ind}_{G_n(E)}^{G_n(K)}(\Theta_E)$.

**Lemma 2.4.** Let $(\pi, V)$ be a generic $R$-representation of $G_n(E)$ such that $(\pi, V)$ is $\Gamma$-equivariant. Then the Whittaker model $\mathcal{W}(\pi, \psi_E)$ of $\pi$ is invariant under the action of $\Gamma$.

**Proof.** Let $W_v$ be a Whittaker functional on the representation $\pi$. For $v \in V$, we have
\[
W_v(\pi^\gamma(x)v) = \Theta_E(\gamma(x))W_v(v) = (\psi_F \circ \text{Tr}_{E/F})\left(\sum_{i=1}^{n-1} \gamma(x_{i,i+1})\right)W_v(v) = \Theta_E(x)W_v(v),
\]
for all $x \in N_n(E)$. Thus, $W_v$ is also a Whittaker functional for the representation $(\pi^\gamma, V)$. Let $W_v \in \mathcal{W}(\pi, \psi_E)$. Then
\[
(\gamma^{-1}.W_v)(g) = W_v(\pi^\gamma(g)v).
\]
From the uniqueness of the Whittaker model, we have $\gamma^{-1}.W_v \in \mathcal{W}(\pi, \psi_E)$. Hence the lemma. \hfill $\square$

2.8. **Kirillov Model.** Let $\pi$ be a generic $R$-representation of $G_n(K)$. Following the notations as in the subsections (2.5) and (2.7), consider the space $\mathcal{K}(\pi, \psi_K)$ of all elements $W$ restricted to $P_n(K)$, where $W$ varies over $\mathcal{W}(\pi, \psi_K)$. Then the space $\mathcal{K}(\pi, \psi_K)$ is $P_n(K)$-invariant. By Frobenius reciprocity, there is a non-zero (unique up to a scalar) linear map $A_{\pi} : V \rightarrow \text{Ind}_{N_n(K)}^{P_n(K)}(\Theta_K)$, which is injective and compatible with the action of $P_n(K)$. In fact,
\[
A_{\pi}(V) = \mathcal{K}(\pi, \psi_K) \cong \mathcal{W}(\pi, \psi_K) \cong \mathcal{K}(\pi, \psi_K).
\]
Moreover, $\mathcal{K}(\psi_K) = \text{ind}_{N_n(K)}^{P_n(K)}(\Theta_K) \subseteq \mathcal{K}(\pi, \psi_K)$ and the equality holds if $\pi$ is cuspidal. The space of all $\mathbb{Z}_l$-valued functions in $\mathcal{K}(\pi, \psi_K)$ (resp. $\mathcal{K}(\psi_K)$) is denoted by $\mathcal{K}^0(\pi, \psi_K)$ (resp. $\mathcal{K}^0(\psi_K)$).

2.8.1. We now recall the Kirillov model for $n = 2$ and some of its properties. For details, see [BH06]. Up to isomorphism, any irreducible representation of $P_2(K)$, which is not a character, is isomorphic to
\[
J_{\psi} := \text{ind}_{N_2(K)}^{P_2(K)}(\psi), \quad (2.2)
\]
for some non-trivial smooth additive character $\psi$ of $K$, viewed as character of $N_2(K)$ via standard isomorphism $N_2(K) \cong K$. Two different non-trivial characters of $N_2(K)$ induce isomorphic representations of $P_2(K)$. The space (2.2) is identified with the space of locally constant compactly supported functions on $K^\times$, to be denoted by $C_c(K^\times, \mathbb{Q}_l)$. The action of $P_2(K)$ on the space $C_c^\infty(K^\times, \mathbb{Q}_l)$ is given by
3.1. Keeping the notation as in Section 2, we briefly discuss about the Weil group and its Weil-Deligne representations. For a reference, see [BH06, Chapter 7] and [Del73, Chapter 4].

We choose a separable algebraic closure $\overline{K}$ of $K$. Let $\Omega_K$ be the absolute Galois group $\text{Gal}(\overline{K}/K)$ and Let $\mathcal{I}_K$ be the inertia subgroup of $\Omega_K$. Let $W_K$ denote the Weil group of $K$. Fix a geometric Frobenius element $\text{Frob}$ in $W_K$. Then we have

$$W_K = \mathcal{I}_K \rtimes \text{Frob}^\mathbb{Z}.$$ 

There is a natural Krull topology on the absolute Galois group $\Omega_K$ and the inertia group $\mathcal{I}_K$, as a subgroup of $\Omega_K$, is equipped with the subspace topology. Let the fundamental system of neighbourhoods of the Weil group $W_K$ be such that each neighbourhood of the identity $W_K$ contains an open subgroup of $\mathcal{I}_K$. Then under this topology, the Weil group $W_K$ becomes a locally compact and totally disconnected group. If $K_1/K$ is a finite extension with $K_1 \subseteq \overline{K}$, then the Weil group $W_{K_1}$ is considered as a subgroup of $W_K$.

An $R$-representation $\rho$ of $W_K$ is called unramified if $\rho$ is trivial on $I_K$. Let $\nu$ be the unramified character of $W_K$ which satisfies $\nu(\text{Frob}) = q_K^{-1}$. We now define semisimple Weil-Deligne representations of $W_K$.

3.2. Semisimple Weil-Deligne representation. A Weil-Deligne representation of $W_K$ is a pair $(\rho, U)$, where $\rho$ is a finite dimensional $R$-representation of $W_K$ and $U$ is a nilpotent endomorphism of the vector space underlying $\rho$ and intertwining the actions of $\nu \rho$ and $\rho$. A Weil-Deligne representation $(\rho, U)$ of $W_K$ is called semisimple if $\rho$ is semisimple as a representation of $W_K$. Note that any semisimple representation $\rho$ of $W_K$ is considered as a semisimple Weil-Deligne representation of the form $(\rho, 0)$. For two Weil-Deligne representations $(\rho, U)$ and $(\rho', U')$ of $W_K$, let

$$\text{Hom}_D((\rho, U), (\rho', U')) = \{ f \in \text{Hom}_{W_K}(\rho, \rho') : f \circ U = U' \circ f \},$$

We say that $(\rho, U)$ and $(\rho', U')$ are isomorphic if there exists a map $f \in \text{Hom}_D((\rho, U), (\rho', U'))$ such that $f$ is bijective. Let $G^s_n(K)$ be the set of all $n$-dimensional semisimple Weil-Deligne representations of the Weil group $W_K$.

3.3. Local Constants of Weil-Deligne representation. Keep the notations as in sections (3.1) and (3.2). In this subsection, we consider the local constants for $l$-adic Weil-Deligne representations of $W_K$.

3.3.1. $L$-factors. Let $(\rho, U)$ be an $l$-adic semisimple Weil-Deligne representation of $W_K$. Then the $L$-factor corresponding to $(\rho, U)$ is the following rational function in $X$:

$$L(X, (\rho, U)) = \det((\text{id} - X \rho(\text{Frob}))|_{\ker(U)^{2g}})^{-1}.$$
3.3.2. Local $\epsilon$-factors and $\gamma$-factors. Let $\psi_K : K \rightarrow \overline{\mathbb{Q}}_{\ell}^X$ be a non-trivial additive character and choose a self-dual additive Haar measure on $K$ with respect to $\psi_K$. Let $\rho$ be an $l$-adic representation of $\mathcal{W}_K$. The epsilon factor $\epsilon(X, \rho, \psi_K)$ of $\rho$, relative to $\psi_K$ is defined in [Del73]. Let $K'/K$ be a finite extension inside $\overline{K}$. Let $\psi_{K'}$ denote the character of $K'$, where $\psi_{K'} = \psi_K \circ \text{Tr}_{K'/K}$. Then the epsilon factor satisfies the following properties:

1. If $\rho_1$ and $\rho_2$ are two $l$-adic representations of $\mathcal{W}_K$, then
   \[
   \epsilon(X, \rho_1 \oplus \rho_2, \psi_K) = \epsilon(X, \rho_1, \psi_K) \epsilon(X, \rho_2, \psi_K).
   \]
2. $\rho$ is an $l$-adic representation of $\mathcal{W}_{K'}$, then
   \[
   \epsilon(X, \text{ind}_{\mathcal{W}_{K'}}^{\mathcal{W}_K} (\rho), \psi_K) \epsilon(X, \rho, \psi_{K'}) = \left\{ \frac{\epsilon(X, \text{ind}_{\mathcal{W}_{K'}}^{\mathcal{W}_K} (1_{K'}), \psi_K)}{\epsilon(X, 1_{K'}, \psi_{K'})} \right\}^{\dim(\rho)},
   \]
   where $1_{K'}$ denotes the trivial character of $\mathcal{W}_{K'}$.
3. If $\rho$ is an $l$-adic representation of $\mathcal{W}_K$, then
   \[
   \epsilon(X, \rho, \psi_K) \epsilon(q_{K'}^{-1}X^{-1}, \rho^\vee, \psi_K) = \det(\rho(-1)),
   \]
   where $\rho^\vee$ denotes the dual of the representation $\rho$.
4. For an $l$-adic representation $\rho$ of $\mathcal{W}_K$, there exists an integer $n(\rho, \psi_K)$ for which
   \[
   \epsilon(X, \rho, \psi_K) = (q_{K'}^X)^{n(\rho, \psi_K)} \epsilon(\rho, \psi_K).
   \]

Now for an $l$-adic semisimple Weil-Deligne representation $(\rho, U)$, the $\epsilon$-factor is defined as
\[
\epsilon(X, (\rho, U), \psi_K) = \epsilon(X, \rho, \psi_K) \frac{L(q_{K'}^{-1}X^{-1}, \rho^\vee)}{L(X, \rho)} \frac{L(\rho, (\rho, U)^\vee)}{L(q_{K'}^{-1}X^{-1}, (\rho, U)^\vee)},
\]
where $((\rho, U)^\vee = (\rho^\vee, -U^\vee)$. Set
\[
\gamma(X, (\rho, U), \psi_K) = \epsilon(X, (\rho, U), \psi_K) \frac{L(\rho, (\rho, U))}{L(q_{K'}^{-1}X^{-1}, (\rho, U)^\vee)}.
\]
The element $\gamma(X, (\rho, U), \psi_K)$ is called the $\gamma$-factor of the Weil-Deligne representation $(\rho, U)$.

Now we state a result [KM21, Proposition 5.11] which concerns the fact that the $\gamma$-factors are compatible with reduction modulo $l$. For $P \in \mathbb{Z}[X]$, we denote by $r_l(P) \in \mathbb{F}_l[X]$ the polynomial obtained by reduction mod-$l$ to the coefficients of $P$. For $Q \in \mathbb{Z}[X]$ such that $r_l(Q) \neq 0$, we set $r_l(P/Q) = r_l(P)/r_l(Q)$.

**Proposition 3.1.** Let $\rho$ be an integral $l$-adic semisimple representation of $\mathcal{W}_K$. Then
\[
r_l(\gamma(X, \rho, \psi_K)) = \gamma(X, r_l(\rho), \psi_K),
\]
where $\psi_K$ is the reduction mod-$l$ of $\psi_K$.

We end this subsection with a lemma which will be needed later in the proof of Theorem (1.1).

**Lemma 3.2.** Let $E/F$ be a cyclic Galois extension of prime degree $l$ and assume $l \neq 2$. Let $\rho$ be an $l$-adic representation of $\mathcal{W}_E$ of even dimension. Then
\[
\epsilon(X, \rho, \psi_E) = \epsilon(X, \text{ind}_{\mathcal{W}_E}^{\mathcal{W}_{\mathbb{Q}}}(\rho), \psi_F).
\]

**Proof.** Let $C_{E/F}(\psi_F) = \epsilon(X, \text{ind}_{\mathcal{W}_E}^{\mathcal{W}_{\mathbb{Q}}}(1_E), \psi_F) / \epsilon(X, 1_{E}, \psi_E)$, where $1_E$ denotes the trivial character of $\mathcal{W}_E$. Then $C_{E/F}(\psi_F)$ is independent of $X$ (see [BH06, Corollary 30.4, Chapter 7]). Using the equality (3.1), we get
\[
\frac{\epsilon(X, \text{ind}_{\mathcal{W}_E}^{\mathcal{W}_{\mathbb{Q}}}(\rho), \psi_F)}{\epsilon(X, \rho, \psi_E)} = (C_{E/F}(\psi_F))^{\dim(\rho)}.
\]
In view of the functional equation (3.2), we have
\[
C_{E/F}(\psi_F)^2 = \xi_{E/F}(-1),
\]
where $\xi_{E/F} = \det(\text{ind}_{W_E}^W(1_E))$, a quadratic character of $W_F$. Note that $\xi_{E/F}$ is a character of the quotient group $W_F/W_E$ which is a cyclic group of order $l$, and this implies that $\xi_{E/F}^l = 1$. Since $l$ is odd, we get that $\xi_{E/F} = 1_F$, the trivial character of $W_F$. Hence the lemma.

\[ \square \]

3.4. **Local constants of $p$-adic representations.** Following the notations as in Section (2.7), we now define the $L$-factors and $\gamma$-factors for irreducible $R$-representations of $G_n(K)$. For details, see [KM17]. Let $\pi$ be an $R$-representation of Whittaker type of $G_n(K)$ and let $\pi'$ be an $R$-representation of Whittaker type of $G_{n-1}(K)$. Let $W \in \mathcal{W}(\pi, \psi_K)$ and $W' \in \mathcal{W}(\pi', \psi_K^{-1})$. The function $W \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} W'(g)$ is compactly supported on $Y_K^r$ [KM17, Proposition 3.3]. Then the following integral

$$c^K_r(W, W') = \int_{Y_K^r} W \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} W'(g) dg,$$

is well defined for all $r \in \mathbb{Z}$, and vanishes for $r << 0$. In this paper, we deal with base change where two different $p$-adic fields are involved. So to avoid confusion, we use the notation $c^K_r(W, W')$ instead of the notation $c_r(W, W')$ used in [KM17, Proposition 3.3] for these integrals on $Y_K^r$. Now consider the functions $\tilde{W}$ and $\tilde{W}'$, defined as

$$\tilde{W}(x) = W(w_n(x^r)^{-1})$$

and

$$\tilde{W}'(g) = W'(w_{n-1}(g^r)^{-1}),$$

for all $x \in G_n(K)$, $g \in G_{n-1}(K)$. Then making a change of variables, we have the following relation:

$$c^K_r(\tilde{W}, \tilde{W}') = c^K_r(\pi(w_n)W, \pi'(w_{n-1})W').$$

Let $I(X, W, W')$ be the following power series:

$$I(X, W, W') = \sum_{r \in \mathbb{Z}} c^K_r(W, W')q^{r/2}X^r \in R((X)).$$

Note that $I(X, W, W')$ is a rational function in $X$ (see [KM17, Theorem 3.5]).

3.4.1. **$L$-factors.** Let $\pi$ and $\pi'$ be two $R$-representations of Whittaker type of $G_n(K)$ and $G_{n-1}(K)$ respectively. Then the $R$-submodule spanned by $I(X, W, W')$ as $W$ varies in $\mathcal{W}(\pi, \psi_K)$ and $W'$ varies in $\mathcal{W}(\pi', \psi_K^{-1})$, is a fractional ideal of $R[X, X^{-1}]$ and it has a unique generator which is an Euler factor denoted by $L(X, \pi, \pi')$. The generator $L(X, \pi, \pi')$ called the $L$-factor associated to $\pi$, $\pi'$ and $\psi_K$.

**Remark 3.3.** If $\pi$ and $\pi'$ are $l$-adic representations of Whittaker type of $G_n(K)$ and $G_{n-1}(K)$ respectively, then $1/L(X, \pi, \pi') \in \mathcal{O}_l[X]$.

We conclude this section with a theorem [KM17, Theorem 4.3,] which describes $L$-factors of cuspidal representations.

**Theorem 3.4.** Let $\pi_1$ and $\pi_2$ be two cuspidal $R$-representations of $G_n(K)$ and $G_m(K)$ respectively. Then $L(X, \pi_1, \pi_2)$ is equal to 1, except in the following case : $\pi_1$ is banal in the sense of [MS14] and $\pi_2 \simeq \chi \pi_1^r$ for some unramified character $\chi$ of $K^\times$.

In the proof of Theorem 1.1, we only consider the case when $m = n - 1$, and by the above theorem the $L$-factor $L(X, \pi_1, \pi_2)$ associated with the cuspidal $R$-representations $\pi_1$ and $\pi_2$ is equal to 1.

3.4.2. **Functional Equations and Local $\gamma$-factors.** Let $\pi$ and $\pi'$ be two $R$-representations of Whittaker type of $G_n(K)$ and $G_{n-1}(K)$ respectively. Then there is an invertible element $\epsilon(X, \pi, \pi', \psi_K)$ in $R[X, X^{-1}]$ such that for all $W \in \mathcal{W}(\pi, \psi_K)$, $W' \in \mathcal{W}(\pi', \psi_K^{-1})$, we have the following functional equation:

$$\frac{I(q_K^{-1}X^{-1}, \tilde{W}, \tilde{W}')}{L(q_K^{-1}X^{-1}, \pi, \pi')} = \omega_{\pi'}(-1)^{n-2} \epsilon(X, \pi, \pi', \psi_K) \frac{I(X, W, W')}{L(X, \pi, \pi')},$$

where $\tilde{W}$ is defined as in (3.4) and $\omega_{\pi'}$ denotes the central character of the representation $\pi'$. We call $\epsilon(X, \pi, \pi', \psi_K)$ the local $\epsilon$-factor associated to $\pi$, $\pi'$ and $\psi_K$. Moreover, if $\pi$ and $\pi'$ be $l$-adic representations
of Whittaker type of \( G_n(K) \) and \( G_{n-1}(K) \) respectively, then the factor \( \epsilon(X, \pi, \pi', \psi_K) \) is of the form \( cX^k \) for a unit \( c \in \mathbb{Z}_l \). In particular, there exists an integer \( n(\pi, \pi', \psi_K) \) such that

\[
\epsilon(X, \pi, \pi', \psi_K) = (\tilde{q}_K^X)^{n(\pi, \pi', \psi_K)} \epsilon(\pi, \pi', \psi_K).
\] (3.5)

Now the local \( \gamma \)-factor associated with \( \pi, \pi' \) and \( \psi \) is defined as:

\[
\gamma(X, \pi, \pi', \psi_K) = \epsilon(X, \pi, \pi', \psi_K) \frac{L(q_K^{1-1}X^{-1}, \tilde{\pi}, \tilde{\pi}')}{L(X, \pi, \pi')},
\]

3.4.3. Compatibility with reduction modulo \( l \). Let \( \tau \) and \( \tau' \) be two \( l \)-modular representations of Whittaker type of \( G_n(K) \) and \( G_{n-1}(K) \) respectively. Let \( \pi \) and \( \pi' \) be the respective Whittaker lifts of \( \tau \) and \( \tau' \). Then

\[
L(X, \tau, \tau')|_{l_1} (L(X, \pi, \pi'))
\]

and

\[
B_l((\gamma(X, \pi, \pi', \psi_K)) = \gamma(X, \tau, \tau', \tilde{\psi}_K).
\]

For details, see [KM17, Section 3.3].

3.4.4. Generic part of mod-\( l \) reduction. Let \( \pi \) be an integral \( l \)-adic generic representation of \( G_n(K) \). The mod-\( l \)-reduction of \( \pi \), denoted by \( r_l(\pi) \), has a unique generic component and it is denoted by \( J_l(\pi) \) (see [Vig01, Section 1.8.4]). Let \( \sigma \) be an \( l \)-adic generic representation of \( G_{n-1}(K) \). Now, the functional equation for the pair \((J_l(\pi), J_l(\sigma))\) gives

\[
I(q_K^{1-1}X^{-1}, \tilde{W}, \tilde{W}') = \omega_\sigma(-1)^{n-2} \gamma(X, J_l(\pi), J_l(\sigma), \tilde{\psi}_K)I(X, W, W'),
\]

for all \( W \in \mathbb{W}(J_l(\pi), \tilde{\psi}_K) \) and \( W' \in \mathbb{W}(J_l(\sigma), \tilde{\psi}_K^{-1}) \). Let us consider the following commutative diagram:

\[
\begin{array}{ccc}
\mathbb{W}^0(\pi, \psi_K) & \xrightarrow{\Lambda_\pi} & \Ind_{G_n(K)}^{G_{n-1}(K)} \mathcal{O}_K \\
\Res_{P_n(K)} & \downarrow & \Res_{P_n(K)} \\
K^0(\pi, \psi_K) & \xrightarrow{\lambda_\pi} & \Ind_{G_{n-1}(K)}^{G_{n-1}(K)} \mathcal{O}_K
\end{array}
\]

Note that the restriction to \( P_n(K) \) map on \( \mathbb{W}^0(\pi, \psi_K) \) is an isomorphism. Here \( \Lambda_\pi \) and \( \lambda_\pi \) are the pointwise mod-\( l \) reduction maps. Since \( K^0(\psi_K) \) is contained in \( K^0(\pi, \psi_K) \) and \( \lambda_\pi \) maps \( K^0(\psi_K) \) onto \( K(\tilde{\psi}_K) \), the \( P_n(K) \)-equivariant map \( \lambda_\pi \) is non-zero. It then follows from the commutativity of the above diagram that \( \Lambda_\pi \) is non-zero. Since \( J_l(\pi) \) is the unique generic subquotient of \( r_l(\pi) \), the image of \( \Lambda_\pi \) contains the Whittaker space \( \mathbb{W}(J_l(\pi), \tilde{\psi}_K) \). Similarly, the image of \( \Lambda_\sigma \) contains \( \mathbb{W}(J_l(\sigma), \tilde{\psi}_K) \). Let \( U \) (resp. \( U' \)) be an element of \( \mathbb{W}^0(\pi, \psi_K) \) (resp. \( \mathbb{W}^0(\sigma, \psi_K) \)) such that \( \Lambda_\pi(U) = W \) (resp. \( \Lambda_\sigma(U') = W' \)). From the functional equation for the pair \((\pi, \sigma)\), we get the following relation

\[
I(q_K^{1-1}X^{-1}, \tilde{U}, \tilde{U}') = \omega_\sigma(-1)^{n-2} \gamma(X, \pi, \sigma, \psi_K)I(X, U, U').
\]

After reducing the above equality modulo-\( l \), we have

\[
I(q_K^{1-1}X^{-1}, \tilde{W}, \tilde{W}') = \omega_\sigma(-1)^{n-2} r_l(\gamma(X, \pi, \sigma, \psi_K))I(X, W, W'),
\]

Thus, we get that

\[
r_l(\gamma(X, \pi, \sigma, \psi_K)) = \gamma(X, J_l(\pi), J_l(\sigma), \tilde{\psi}_K).
\] (3.6)
4. Local Langlands Correspondence

4.1. The $l$-adic local Langlands correspondence. In this subsection, we recall the $l$-adic local Langlands correspondence. Keep the notation as in section (2). Let $\psi_K$ be a non-trivial additive character of $K$. Recall that local Langlands correspondence over $\mathbb{Q}_l$ is the bijection

$$\Pi_K : \text{Irr}(GL_n(K), \mathbb{Q}_l) \rightarrow G^\text{ss}_n(K)$$

such that

$$\gamma(X, \sigma \times \sigma', \psi_K) = \gamma(X, \Pi_K(\sigma) \otimes \Pi_K(\sigma'), \psi_K)$$

and

$$L(X, \sigma \times \sigma') = L(X, \Pi_K(\sigma) \otimes \Pi_K(\sigma'))$$

for all $\sigma \in \text{Irr}(G_n(K), \mathbb{Q}_l)$, $\sigma' \in \text{Irr}(G_n(K), \mathbb{Q}_l)$. Moreover, the set of all cuspidal $l$-adic representations of $GL_n(K)$ is mapped onto the set $n$-dimensional irreducible $l$-adic representations of the Weyl group $W_K$ via the bijection $\Pi_K$ (see [HT01], [Hen00] or [Sch13]). Note that the classical local Langlands correspondence is a bijection between $\text{Irr}(GL_n(K), \mathbb{C})$ and the isomorphism classes of $n$-dimensional, complex semisimple Weil–Deligne representations. To get a correspondence over $\mathbb{Q}_l$, one twists the original correspondence by the character $\nu^{(1-n)/2}$. For details see [Clo90, Conjecture 4.4, Section 4.2], [Hen01, Section 7] and for $n = 2$ see [BH06, Theorem 35.1].

4.2. Local base Change for the extension $E/F$. Now we recall local base change for a cyclic extension of a $p$-adic field. The base change operation on irreducible smooth representations of $GL_n(F)$ over complex vector spaces is characterised by certain character identities (see [AC89, Chapter 3]). Let us recall the relation between $l$-adic local Langlands correspondence and local base change for $GL_n$. Let $\pi'_F$ be an $l$-adic irreducible smooth representation of $GL_n(F)$. Let $(\rho_F, U)$ be a semisimple Weil-Deligne representation such that $\Pi_F(\pi'_F) = \rho_F$, where $\Pi_F$ is the $l$-adic local Langlands correspondence as described in the previous section. Let $\pi_E$ be the $l$-adic irreducible representation of $GL_n(E)$ such that

$$\text{res}_{W_E}(\Pi_F(\pi'_F)) \simeq \Pi_E(\pi_E).$$

The representation $\pi_E$ is the base change of $\pi'_F$. Note that in this case $\pi_E \simeq \pi'_E$, for all $\gamma \in \Gamma$.

4.2.1. Base change for $L(\Delta)$. Let $k$ be a positive integer. Let $\Delta = \{\tau_F, \tau_F\nu_F, ..., \tau_F\nu_F^{k-1}\}$ be a segment, where $\tau_F$ is an $l$-adic cuspidal representation of $G_m(F)$. Consider the generic representation $L(\Delta)$ of $G_m(F)$. Then

$$\Pi_F(L(\Delta)) = \Pi_F(\tau_F) \otimes \text{Sp}_F(k),$$

where $\text{Sp}_F(k)$ is the semisimple Weil-Deligne representation of $W_F$, defined as in [BH06, Section 31, Example 31.1]. If $l$ does not divide $m$, then there exists a cuspidal representation $\tau_E$ of $G_m(E)$ such that $\tau_E$ is a base change of $\tau_F$ that is,

$$\text{res}_{W_E}(\Pi_F(\tau_F)) = \Pi_E(\tau_E).$$

Then we have

$$\text{res}_{W_E}(\Pi_F(L(\Delta))) = \Pi_E(\tau_E) \otimes \text{Sp}_E(k) = \Pi_E(L(D)),$$

where $D$ is the segment $\{\tau_E, \tau_E\nu_E, ..., \tau_E\nu_E^{k-1}\}$. Hence, it follows that the generic representation $L(D)$ of $G_m(E)$ is a base change of $L(\Delta)$. Next, we prove a lemma about base change lifting and integrality.

Lemma 4.1. Let $\pi_F$ be an irreducible $l$-adic representation of $G_n(F)$, and let $\pi_E$ be the base change lifting of $\pi_F$ to $G_n(E)$. Then $\pi_F$ is integral if and only if the base change lifting $\pi_E$ is integral.

Proof. Let $\pi$ be an irreducible $l$-adic smooth representation of $G_n(K)$. Let $\text{scs}(\pi)$ be the supercuspidal support of $\pi$ (see [Vig98, III.3] for the definition). The representation $\pi$ is integral if and only if $\text{scs}(\pi)$ is integral (see [Vig01, Section 1.4] and for general reductive groups see [DHKM24, Corollary 1.6] for a reference). The representation $\text{scs}(\pi)$ is integral if and only if the central character of $\text{scs}(\pi)$ is integral.

Assume that $\pi_F$ is integral. Let $(\rho_F, U)$ be the $l$-adic, semisimple Weil-Deligne representation of $W_F$ associated with $\pi_F$ under the local Langlands correspondence (LLC) $\Pi_F$. Under LLC, the representation $\text{scs}(\pi_F)$ corresponds to the $W_F$-representation $\rho_F$. Since the determinant character of each irreducible component of $\rho_F$ is integral, we get that $\rho_F$ is integral. This implies that the restriction $\text{res}_{W_E}(\rho_F)$ is integral. Under LLC, the supercuspidal support of $\pi_E$ corresponds to the restriction $\text{res}_{W_E}(\rho_F)$. Thus, the supercuspidal support of $\pi_E$ is integral and we get that $\pi_E$ is integral.
Conversely assume that \( \pi_E \) is integral. Let \((\rho_E, U_E)\) be the \( l \)-adic, semisimple Weil-Deligne representation of \( \mathcal{W}_E \) associated with \( \pi_E \) under LLC. Since the supercuspidal support of \( \pi_E \) is integral, the \( \mathcal{W}_E \)-representation \( \rho_E \) (which is \( \text{res}_{\mathcal{W}_E}(\rho_F) \)) is integral. Let \( \mathcal{L}_E \) be a \( \mathcal{W}_E \)-stable lattice in \( \rho_E \). Then,

\[
\sum_{x \in \mathcal{W}_E/\mathcal{W}_K} \rho_F(x) \mathcal{L}_E.
\]

is a \( \mathcal{W}_F \)-stable lattice in \( \rho_F \). Thus, the representation \( \rho_F \) is also integral, which implies that the supercuspidal support \( \text{scs}(\pi_F) \) is integral. Hence, \( \pi_F \) is integral. \( \square \)

5. Tate Cohomology

In this section, we recall Tate cohomology and some useful results on \( \Gamma \)-equivariant \( l \)-sheaves of \( \Lambda \)-modules on an \( l \)-space \( X \) equipped with an action of \( \Gamma \). We refer to [TV16, Section 3] for details.

5.1. Fix a generator \( \gamma \) of \( \Gamma \). Let \( M \) be a \( \Lambda[\Gamma] \)-module, and let \( T_\gamma \) be the automorphism of \( M \) defined by

\[
T_\gamma(m) = \gamma \cdot m, \quad \text{for} \quad \gamma \in \Gamma, \ m \in M.
\]

Let \( N_\gamma = \text{id} + T_\gamma + T_\gamma^2 + \ldots + T_\gamma^{l-1} \) be the norm operator. The Tate cohomology groups \( \hat{H}^0(M) \) and \( \hat{H}^1(M) \) are defined as:

\[
\hat{H}^0(M) = \frac{\ker(\text{id} - T_\gamma)}{\text{Im}(N_\gamma)}, \quad \hat{H}^1(M) = \frac{\ker(N_\gamma)}{\text{Im}(\text{id} - T_\gamma)}.
\]

5.2. Tate Cohomology of sheaves on \( l \)-spaces. Let \( X \) be an \( l \)-space equipped with an action of a finite group \( \langle \gamma \rangle \) of order \( l \). Let \( \mathcal{F} \) be an \( l \)-sheaf of \( \Lambda \) modules on \( X \). Write \( \Gamma_c(X, \mathcal{F}) \) for the space of compactly supported sections of \( \mathcal{F} \). In particular, if \( \mathcal{F} \) is the constant sheaf with stalk \( \Lambda \), then \( \Gamma_c(X, \mathcal{F}) = C_\infty(X, \Lambda) \). The assignment \( \mathcal{F} \mapsto \Gamma_c(X, \mathcal{F}) \) is a covariant exact functor. If \( \mathcal{F} \) is \( \gamma \)-equivariant, then \( \gamma \) can be regarded as a map of sheaves \( \mathcal{F}|_{X^\gamma} \to \mathcal{F}|_{X^\gamma} \) and the Tate cohomology is defined as

\[
\hat{H}^0(\mathcal{F}|_{X^\gamma}) := \ker(1 - \gamma)/\text{Im}(N), \quad \hat{H}^1(\mathcal{F}|_{X^\gamma}) := \ker(N)/\text{Im}(1 - \gamma).
\]

A compactly supported section of \( \mathcal{F} \) can be restricted to a compactly supported section of \( \mathcal{F}|_{X^\gamma} \). The following result is often useful in calculating Tate cohomology groups.

**Proposition 5.1** (Treumann-Venkatesh, [TV16]). The restriction map induces an isomorphism of the following spaces:

\[
\hat{H}^i(\Gamma_c(X; \mathcal{F})) \to \Gamma_c(X^\gamma; \hat{H}^i(\mathcal{F})) \quad \text{for} \quad i = 0, 1.
\]

5.2.1. The above proposition is very useful to compute the Tate cohomology groups of compactly induced representations. For instance, the following argument is used at many places in the paper. Let \( E/F \) be a Galois extension of degree \( l \) with Galois group \( \Gamma \). Let \( \psi_E : F \to \bar{\mathbb{Z}}^\times_l \) be an additive character and let \( \psi_E \) be the character \( \psi_F \circ \text{Tr}_{E/F} \), where \( \text{Tr}_{E/F} \) is the trace function. Let \( \Theta_E \) and \( \Theta_F \) be the non-degenerate characters of \( N_n(E) \) and \( N_n(F) \) associated with \( \psi_F \) and \( \psi_E \) respectively (see Subsection 2.7). We will use the notations \( \bar{\psi}_E, \bar{\psi}_F, \bar{\Theta}_E, \bar{\Theta}_F \) for the respective mod-\( l \) reductions. Recall the notation \( \mathcal{K}(\psi_K) \) for the compact induction \( \text{ind}_{N_n(K)}^{P_n(K)} \psi_K \). Note that the Galois group \( \Gamma \) acts on the representation \( \mathcal{K}(\bar{\psi}_E) \), by setting

\[
(\gamma f)(x) = f(\gamma^{-1}(x)), \quad \gamma \in \Gamma, \ x \in P_n(E), \ f \in \mathcal{K}(\bar{\psi}_E).
\]

The restriction to \( P_n(F) \) map:

\[
\mathcal{K}(\bar{\psi}_E) \to \mathcal{K}(\bar{\psi}_F), \quad f \mapsto \text{res}_{P_n(F)} f
\]

factorizes through

\[
\hat{H}^0(\mathcal{K}(\bar{\psi}_E)) \to \mathcal{K}(\bar{\psi}_F).
\]

We set \( Y_K = P_n(K)/N_n(K) \). Note that the pointed set \( H^1(\Gamma, N_n(E)) \) is trivial and hence \( Y^+_E = Y_F \). Applying Proposition 5.1 for the case where \( X = Y_E \) and \( \mathcal{F} \) equals the sheaf associated with the induced representation \( \text{ind}_{N_n(E)}^{P_n(E)} \bar{\Theta}_E \), we get that the map (5.1) is an isomorphism. Proposition 5.1 also shows that \( \hat{H}^1(\mathcal{K}(\psi_E)) \) is trivial. Here, \( \mathcal{K}(\psi_E) \) is the space of \( \bar{\mathbb{Z}}_l \)-valued functions in the \( l \)-adic representation \( \text{ind}_{N_n(E)}^{P_n(E)} \psi_E \).
5.3. Comparison of integrals of smooth functions. The group $\Gamma = \langle \gamma \rangle$ acts on the space $X_E = G_{n-1}(E)/N_{n-1}(E)$ and hence its action on the space $C_c^\infty(X_E, \mathbb{F})$ is given by the following equality:

$$(\gamma \phi)(x) := \phi(\gamma^{-1}x), \text{ for all } x \in X_E, \text{ and } \phi \in C_c^\infty(X_E, \mathbb{F}).$$

Let $C_c^\infty(X_E, \mathbb{F})^\Gamma$ be the space of all $\Gamma$-invariant functions in $C_c^\infty(X_E, \mathbb{F})$. We end this section with a proposition comparing the integrals on the spaces $X_E$ and $X_F$.

**Proposition 5.2.** Let $d\mu_E$ and $d\mu_F$ be Haar measures on $X_E$ and $X_F$ respectively. Then, there exists a non-zero scalar $c \in \mathbb{F}$, such that

$$\int_{X_E} \phi \, d\mu_E = c \int_{X_F} \phi \, d\mu_F,$$

for all $\phi \in C_c^\infty(X_E, \mathbb{F})^\Gamma$.

**Proof.** Since $N_{n-1}(E)$ is stable under the action of $\Gamma$ on $G_{n-1}(E)$, we have the following long exact sequence of non-abelian cohomology [Ser, Chapter VII, Appendix]:

$$0 \longrightarrow N_{n-1}(E) \longrightarrow G_{n-1}(E) \longrightarrow X_E^E \longrightarrow H^1(\Gamma; N_{n-1}(E)) \longrightarrow H^1(\Gamma; G_{n-1}(E)).$$

Since $H^1(\Gamma; N_{n-1}(E)) = 0$, we get from the above exact sequence that

$$X_E^E \simeq X_F.$$

Since $X_F$ is closed in $X_E$, we have the following exact sequence of $\Gamma$-modules:

$$0 \longrightarrow C_c^\infty(X_E \setminus X_F, \mathbb{F}) \longrightarrow C_c^\infty(X_E, \mathbb{F}) \longrightarrow C_c^\infty(X_F, \mathbb{F}) \longrightarrow 0. \quad (5.2)$$

Now, the action of $\Gamma$ on $X_E \setminus X_F$ is free, and it follows from Proposition 5.1 that

$$H^1(\Gamma, C_c^\infty(X_E \setminus X_F, \mathbb{F})) = 0. \quad (5.3)$$

Using (5.2) and (5.3), we get the following exact sequence:

$$0 \longrightarrow C_c^\infty(X_E \setminus X_F, \mathbb{F})^\Gamma \longrightarrow C_c^\infty(X_E, \mathbb{F})^\Gamma \longrightarrow C_c^\infty(X_F, \mathbb{F})^\Gamma \longrightarrow 0.$$

Again the free action of $\Gamma$ on $X_E \setminus X_F$ gives a fundamental domain $U$ such that $X_E \setminus X_F = \bigsqcup_{i=0}^{l-1} \gamma^i U$, and we have

$$\int_{X_E \setminus X_F} \phi \, d\mu_E = l \sum_{i=0}^{l-1} \int_U \phi \, d\mu_E = 0,$$

for all $\phi \in C_c^\infty(X_E \setminus X_F, \mathbb{F})^\Gamma$. Therefore the linear functional $d\mu_E$ induces a $G_{n-1}(F)$-invariant linear functional on $C_c^\infty(X_F, \mathbb{F})$, and we have

$$\int_{X_E} \phi \, d\mu_E = c \int_{X_F} \phi \, d\mu_F,$$

for some scalar $c$. Now we will show that $c \neq 0$. By [Vig96, Chapter 1, Section 2.8], we have a surjective map $\Psi : C_c^\infty(G_n(E), \mathbb{F}) \longrightarrow C_c^\infty(X_E, \mathbb{F})$, defined by

$$\Psi(f)(g) := \int_{N_n(E)} f(ng) \, dn,$$

for all $f \in C_c^\infty(G_n(E), \mathbb{F})$, where $dn$ is a Haar measure on $N_n(E)$. Then there exists a $\Gamma$-invariant compact open subgroup $I \subseteq G_n(E)$ such that $\Psi(1_I) \neq 0$, where $1_I$ denotes the characteristic function on $I$. So the Haar measure $d\mu_E$ is non-zero on the space $C_c^\infty(X_E, \mathbb{F})^\Gamma$, and this implies that $c \neq 0$. Hence the proposition follows. \qed

**Remark 5.3.** Keep the notations and hypothesis in Proposition 5.2. From now, the Haar measures $d\mu_E$ and $d\mu_F$ on $X_E$ and $X_F$ respectively, are chosen so as to make $c = 1$. Then we have

$$\int_{X_E} \phi \, d\mu_E = \int_{X_F} \phi \, d\mu_F.$$
Moreover, if \( e \) is the ramification index of the extension \( E \) over \( F \), then for all \( r \notin \{ te : t \in \mathbb{Z} \} \), we have
\[
\int_{(X^1_p)^F} \phi d\mu_F = 0
\]
and for all \( r \in \{ te : t \in \mathbb{Z} \} \), we have
\[
\int_{(X^1_p)^F} \phi d\mu_F = \int_{X^1_p} \phi d\mu_F.
\]

5.4. Finiteness of Tate cohomology. In this part, we prove some results on finiteness of the Tate cohomology of finite length representations of \( \text{GL}_n \). First, we introduce some notations. Let \( U_n(K) \) be the subgroups of \( G_n(K) \), given by
\[
U_n(K) = \left\{ \begin{pmatrix} 1 & C \\ 0 & 1 \end{pmatrix} : C \in K^{n-1} \right\}
\]
respectively. Note that \( U_n(K) \) is contained in the mirabolic subgroup \( P_n(K) \). We use the short notation \( Z_{K,n} \) to denote the coset space \( P_n(K)/P_{n-1}(K)U_n(K) \). Let \( \text{Rep}_R(G) \) denote the category of smooth \( R \)-representations of a locally profinite group \( G \), where \( R \) denotes either \( \mathbb{Q}_l \) or \( \mathbb{F}_l \). Then we have four fundamental functors:
\[
\Psi^- : \text{Rep}_R(P_n) \rightarrow \text{Rep}_R(G_{n-1}), \Psi^+ : \text{Rep}_R(G_{n-1}) \rightarrow \text{Rep}_R(P_n)
\]
\[
\Phi^- : \text{Rep}_R(P_n) \rightarrow \text{Rep}_R(P_{n-1}), \Phi^+ : \text{Rep}_R(P_{n-1}) \rightarrow \text{Rep}_R(P_n).
\]
For the definitions of the functors \( \Phi^\pm \) and \( \Psi^\pm \), see [BZ77, Section 3] for \( R = \mathbb{Q}_l \), and [Vig96, Chapter III, Section 1] for \( R = \mathbb{F}_l \).

5.4.1. Let \( \tau \) be a smooth \( R \)-representation of \( P_n(K) \). The \( m \)-th derivative of \( \tau \), denoted by \( \tau^{(m)} \), is defined as the representation \( \Psi^-(\Phi^-)^{m-1}(\tau) \) of \( G_{n-m}(K) \). There is a functorial filtration on \( \tau \), given by
\[
0 \subseteq \tau_n \subseteq \tau_{n-1} \subseteq \cdots \subseteq \tau_2 \subseteq \tau_1 = \tau,
\]
where \( \tau_m = (\Phi^+)^{m-1}(\Psi^-)^{m-1}(\tau) \) and \( \tau_m/\tau_{m+1} = (\Phi^+)^{m-1}(\Psi^-)^{m}(\tau) \). We have the following easy lemma.

Lemma 5.4. Let \( \rho \) be a finite length representation of \( G_t(K) \), where \( 1 \leq t < n \). Then
\[
(\Phi^+)^{n-t-1}(\Psi^-)(\rho)
\]
is also of finite length as a representation of \( P_n(K) \).

Proof. This is an immediate consequence of [Vig96, Chapter III, Subsection 1.5] and the exactness of the functor \( (\Phi^+)^{n-t-1}(\Psi^-) \).

5.4.2. Let \( E \) be a finite Galois extension of a \( p \)-adic field \( F \) with \([ E : F ] = l \), where \( l \) and \( p \) are distinct primes. Fix a non-trivial additive character \( \psi_E : F \rightarrow \mathbb{A}^\times \). By abuse of notation, the composition
\[
F \xrightarrow{\psi_E} \mathbb{A}^\times \rightarrow \mathbb{Q}_l^\times
\]
is also denoted by \( \psi_F \). Let \( \psi_E \) be the character of \( E \), defined by the composition \( \psi_F \circ \text{Tr}_{E/F} \), where \( \text{Tr}_{E/F} \) denotes the trace map of the extension \( E/F \). The mod-\( l \) reductions of \( \psi_F \) and \( \psi_E \) are denoted by \( \overline{\psi}_F \) and \( \overline{\psi}_E \), respectively. Then, we have the following finiteness result of the Tate cohomology groups.

Proposition 5.5. Let \( \Pi \) be a finite length \( l \)-modular representation of \( G_{\alpha}(E) \) with an isomorphism \( T : \Pi \rightarrow \Pi^\gamma \) and \( T^l = \text{id} \). Then, the Tate cohomology \( \hat{H}^t(\Pi) \), with respect to the operator \( T \), is a finite length representation of \( G_n(F) \).

Proof. We prove the proposition using induction on the integer \( n \). The case \( n = 1 \) is clear. So, we assume that the proposition is true for all finite length \( l \)-modular representations of \( G_{\alpha}(E) \times \Gamma \) and for all \( t < n \). Now, we consider \( \Pi \) as a representation of the mirabolic subgroup \( P_n(E) \). Since \( \psi_E(\gamma(x)) = \overline{\psi}_E(x) \), for \( x \in E \), we get the isomorphism
\[
\Phi^-(\Pi^\gamma) \simeq \Phi^-(\Pi)^\gamma,
\]
as representation of \( P_{n-1}(E) \). Similarly, for any smooth \( l \)-modular representation \( \tau \) of \( P_{n-1}(E) \), we have \( P_n(E) \)-equivariant isomorphism
\[
\Phi^+(\tau^\gamma) \simeq \Phi^+(\tau)^\gamma.
\]
Using (5.4) and (5.5), and the isomorphism $T$, we get an isomorphism between the representations $\Pi_m$ and $\Pi'_m$, and also between the representations $\Pi^{(m)}$ and $\Pi^{(m)}$, for all $m \leq n$.

Recall that $Z_{E,m}$ denote the coset space $P_m(E)/P_{m-1}(E)U_m(E)$. Since $P_{m-1}(E)U_m(E)$ is a $\Gamma$-stable subgroup of $P_m(E)$, we have the following long exact sequence of non-abelian cohomology ([Ser, Appendix, Proposition 1]):

$$0 \to P_{m-1}(E)U_m(F) \to P_m(F) \to Z_{E,m}^\Gamma \to H^1(\Gamma, P_{m-1}(E)U_m(E)) \to H^1(\Gamma, P_m(E)). \quad (5.6)$$

Consider the short exact sequence of non-abelian $\Gamma$-modules

$$0 \to U_m(E) \to P_{m-1}(E)U_m(E) \to P_{m-1}(E) \to 0. \quad (5.7)$$

From Hilbert’s theorem 90, we get that $H^1(\Gamma, U_m(E))$ and $\tilde{H}^1(\Gamma, P_{m-1}(E))$ are trivial. Then, from the long exact sequence of non-abelian cohomology corresponding to (5.7), we have $H^1(\Gamma, P_{m-1}(E)U_m(E)) = 0$. Hence, the long exact sequence (5.6) gives the equality $Z_{E,m}^\Gamma = Z_{F,m}^\Gamma$. Now, using Proposition 5.1 repeatedly $(m-1)$-times, we get the $P_n(F)$-equivariant isomorphism

$$\tilde{H}^i(\Pi_m/\Pi_{m+1}) \cong (\Phi^+)^{m-1}(\Psi^+)(\tilde{H}^i(\Pi^{(m)})).$$

Using Leibniz formula for derivatives ([Vig96, Lemma 1.10, Chapter 3]), we get that $\Pi^{(m)}$ is a finite length representation of $G_{n-m}(E)$. By induction hypothesis, the $G_{n-m}(F)$-representation $\tilde{H}^i(\Pi^{(m)})$ is of finite length, for all $m < n$. In view of Lemma 5.4, it follows from the above isomorphism that the $P_n(F)$-representation $\tilde{H}^i(\Pi_m/\Pi_{m+1})$ is of finite length for all $m < n$.

Now, for each $m \in \{1, 2, \ldots, n-1\}$, we consider the short exact sequence of $P_n(E)$-representations

$$0 \to \Pi_{m+1} \to \Pi_m \to \Pi_m/\Pi_{m+1} \to 0.$$ 

Since $\Gamma$ is cyclic, the corresponding long exact sequence of Tate cohomology gives the following diagram:

$$\xymatrix{ \tilde{H}^0(\Pi_{m+1}) \ar[d] \ar[r] & \tilde{H}^0(\Pi_m) \ar[d] \ar[r] & \tilde{H}^0(\Pi_m/\Pi_{m+1}) \ar[d] \ar[l] \ar[r] & \tilde{H}^1(\Pi_m/\Pi_{m+1}) \ar[d] \ar[l] \ar[r] & \tilde{H}^1(\Pi_m) \ar[l] \ar[r] & \tilde{H}^1(\Pi_{m+1}) }$$

We denote the above exact sequence by $S(m)$. Now, consider the largest integer $r$ for which $\Pi_r$ is non-zero. Using induction hypothesis, the Tate cohomology groups $\tilde{H}^i(\Pi_r)$ is of finite length as a representation of $P_n(F)$. Now, using the exact sequence $S(r-1)$ and the finiteness of $\tilde{H}^i(\Pi_{r-1}/\Pi_r)$, we get that $\tilde{H}^i(\Pi_{r-1})$ is a finite length representation of $P_n(F)$. Again, using the finiteness of both the representations $\tilde{H}^i(\Pi_{r-1})$ and $\tilde{H}^i(\Pi_{r-2}/\Pi_{r-1})$, it follows from the exact sequence $S(r-2)$ that $\tilde{H}^i(\Pi_{r-2})$ is of finite length. Thus, inductively, we get that $\tilde{H}^i(\Pi)$ is of finite length as a representation of $P_n(F)$ and hence of $G_n(F)$. This completes the proof.

As a corollary, we have

**Corollary 5.6.** Let $(\Pi, V)$ be an integral $\mathcal{K}$-representation of $G_n(F)$ with an isomorphism $T : (\Pi, V) \to (\Pi', V)$ and $T^l = \text{id}$. Then, for any $G_n(E) \rtimes \Gamma$-invariant $\Lambda$-lattice $\mathcal{L}$ in $V$ (here, $\Gamma$ acts on $V$ via $T$), the Tate cohomology groups $\tilde{H}^i(\mathcal{L})$, $i \in \{0, 1\}$, are of finite length as representations of $G_n(F)$.

**Proof.** Recall that $\mathcal{L}$ is a free $\Lambda$-module and $\mathcal{L}/l\mathcal{L}$ is a finite length $G_n(E)$-representation (see [Vig96, II.5.11.a] for finiteness). Consider the short exact sequence of $G_n(E) \rtimes \Gamma$-modules

$$0 \to \mathcal{L} \to \mathcal{L} \to \mathcal{L}/l\mathcal{L} \to 0.$$
By Proposition 5.5, the Tate cohomology group $\hat{H}^1(\mathcal{L}/\mathcal{L})$ has finite length as representations of $G_n(F)$. From the long exact sequence of Tate cohomology corresponding to the above short exact sequence, we have

$$0 \to \hat{H}^0(\mathcal{L}) \to \hat{H}^0(\mathcal{L}/\mathcal{L}) \to \hat{H}^1(\mathcal{L}) \to 0.$$ 

and

$$0 \to \hat{H}^1(\mathcal{L}) \to \hat{H}^1(\mathcal{L}/\mathcal{L}) \to \hat{H}^0(\mathcal{L}) \to 0.$$ 

Thus, we get that each $\hat{H}^i(\mathcal{L})$ is of finite length. \qed

5.5. Frobenius Twist. Let $G$ be a locally profinite group (i.e., locally compact and totally disconnected). Let $(\sigma, V)$ be an $l$-modular representation of $G$. Consider the vector space $V^{(l)}$, where the underlying additive group structure of $V^{(l)}$ is same as that of $V$ but the scalar action $*$ on $V^{(l)}$ is given by

$$c * v = c^* v, \text{ for all } c \in \mathbb{F}_l, v \in V.$$ 

Then the action of $G$ on $V$ induces a representation $\sigma^{(l)}$ of $G$ on $V^{(l)}$. The representation $(\sigma^{(l)}, V^{(l)})$ is called the Frobenius twist of the representation $(\sigma, V)$.

We end this subsection with a lemma which will be used in the main result.

Lemma 5.7. Let $\psi$ be a non-trivial $l$-modular additive character of $F$ and let $\Theta$ be the non-degenerate character of $N_n(F)$ corresponding to $\psi$. If $(\pi, V)$ and $(\sigma, V)$ are two $l$-modular generic representations of $G_n(F)$ and $G_{n-1}(F)$ respectively, then

$$\gamma(X, \pi, \sigma, \psi)^l = \gamma(X^{(l)}, \pi^{(l)}, \sigma^{(l)}, \psi^{(l)}).$$

Proof. Let $W_\pi$ be a Whittaker functional on the representation $\pi$. Then the composite map

$$V_\pi \xrightarrow{W_\pi} \mathbb{F}_l \xrightarrow{x \mapsto x^l} \mathbb{F}_l,$$

denoted by $W_{\pi^{(l)}}$, is a Whittaker functional (with respect to $\psi^{(l)} : N_n(F) \to \mathbb{F}_l^\times$) on the representation $\pi^{(l)}$, as we have:

$$W_{\pi^{(l)}}(c.v) = W_\pi((c^* v)^l) = c W_{\pi^{(l)}}(v)$$

and

$$W_{\pi^{(l)}}(\pi^{(l)}(n)v) = (\Theta(n) W_\pi(v))^l = \Theta^l(n) W_{\pi^{(l)}}(v),$$

for all $v \in V_\pi$, $c \in \mathbb{F}_l$ and all $n \in N_n(F)$.

So the Whittaker model $\mathcal{W}(\pi^{(l)}, \psi^{(l)})$ consists of the functions $W^{\psi^{(l)}}_v$, where $W_v$ varies in $\mathcal{W}(\pi, \psi)$. Similarly the Whittaker model $\mathcal{W}(\sigma^{(l)}, \psi^{(l)})$ of $\sigma^{(l)}$ consists of the functions $U^{\psi^{(l)}}_v$, where $U_v$ varies in $\mathcal{W}(\sigma, \psi)$. Then by the Rankin-Selberg functional equation in the subsection (3.4.2), we have

$$\sum_{r \in \mathbb{Z}} c_r^{(l)} (W_v, \tilde{U}_v) q_F^{-r/2} X^{-r} = \omega_\sigma (-1)^{-n-2} \gamma(X, \pi, \sigma, \psi)^l \sum_{r \in \mathbb{Z}} c_r^{(l)} (W_v, U_v) q_F^{r/2} X^r$$

and

$$\sum_{r \in \mathbb{Z}} c_r^{(l)} (\tilde{W}_v, \tilde{U}_v) q_F^{-r/2} X^{-r} = \omega_{\sigma^{(l)}} (-1)^{-n-2} \gamma(X^{(l)}, \pi^{(l)}, \sigma^{(l)}, \psi^{(l)}) \sum_{r \in \mathbb{Z}} c_r^{(l)} (W_v, U_v) q_F^{r/2} X^r.\quad (5.8)$$

Replace $X$ by $X^{(l)}$ to the equation (5.9), we have

$$\sum_{r \in \mathbb{Z}} c_r^{(l)} (\tilde{W}_v, \tilde{U}_v) q_F^{-r/2} X^{-r} = \omega_{\sigma^{(l)}} (-1)^{-n-2} \gamma(X^{(l)}, \pi^{(l)}, \sigma^{(l)}, \psi^{(l)}) \sum_{r \in \mathbb{Z}} c_r^{(l)} (W_v, U_v) q_F^{r/2} X^r.\quad (5.10)$$

Then from the equations (5.8) and (5.10), we get

$$\gamma(X, \pi, \sigma, \psi)^l = \gamma(X^{(l)}, \pi^{(l)}, \sigma^{(l)}, \psi^{(l)}).$$

\qed

6. TATE COHOMOLOGY OF WHITTAKER LATTICES

As before, we fix a non-trivial additive character $\psi_F : F \to \Lambda^\times$. Let $\psi_E$ be the composition $\psi_F \circ \text{Tr}_{E/F}$, where $\text{Tr}_{E/F}$ is the trace map of the extension $E/F$. The mod-$l$ reductions of $\psi_F$ and $\psi_E$ are denoted by $\tilde{\psi}_F$ and $\tilde{\psi}_E$ respectively. Let $\Theta_F$ and $\Theta_E$ be the characters of $N_n(F)$ and $N_n(E)$ respectively, as defined in (2.7). Similarly, we denote by $\overline{\Theta}_F$ and $\overline{\Theta}_E$ the mod-$l$ reductions of $\Theta_F$ and $\Theta_E$ respectively.
6.1. Let \((\pi, V)\) be a generic \(R\)-representation of \(G_n(E)\), where \(R = \mathbb{Q}_l\) or \(\mathbb{F}_l\), such that \(\pi\) is isomorphic to \(\pi^\gamma\), for all \(\gamma \in \Gamma\). Let \(\mathcal{W}(\pi, \psi_E)\) be the Whittaker model of \(\pi\). For \(W \in \mathcal{W}(\pi, \psi_E)\), we recall that \(\gamma.W\) is a function given by

\[(\gamma.W)(g) = W(\gamma^{-1}(g)),\]

for all \(g \in G_n(E)\). Note that \(\gamma.W \in \mathcal{W}(\pi, \psi_E)\) (see Lemma 2.4). Thus, we define

\[T_\gamma : \mathcal{W}(\pi, \psi_E) \to \mathcal{W}(\pi, \psi_E)\]

by setting \(T_\gamma(W) = \gamma.W\), for all \(W \in \mathcal{W}(\pi, \psi_E)\). The map \(T_\gamma\) gives an isomorphism between \((\pi^\gamma, V)\) and \((\pi, V)\) as we have

\[T_\gamma(\pi(g)W)(h) = \pi(g)W(\gamma^{-1}(h)) = W(\gamma^{-1}(h)g)\]

and

\[[\pi^\gamma(g)T_\gamma(W))(h) = T_\gamma(W)(h\gamma(g)) = W(\gamma^{-1}(h)g),\]

for all \(g, h \in G_n(E)\).

6.2. Jacquet-functors and Tate cohomology. We begin with a few elementary results on the compatibility of Jacquet (twisted Jacquet) functors with Tate cohomology. Let \((\pi, \mathcal{L})\) be a smooth \(\Lambda[G_n(E) \rtimes \Gamma]\)-module. Let \(\lambda = (n_1, n_2, \ldots, n_r)\) be a partition of \(n\) and let \(P_\lambda = M_\lambda N_\lambda\) be a parabolic subgroup of \(G_n\) with \(N_\lambda\) its unipotent radical and \(M_\lambda\) is a standard Levi subgroup. Let \(\mathcal{L}(N_\lambda(E))\) be the space spanned by the set of vectors

\[\{v - \pi(n)v : v \in \mathcal{L}, n \in N_\lambda(E)\}.\]

Note that the space \(\mathcal{L}(N_\lambda(E))\) is stable under the action of \(\Gamma\).

**Lemma 6.1.** The image of the natural map \(\hat{H}^0(\mathcal{L}(N_\lambda(E))) \to \hat{H}^0(\mathcal{L})\) is equal to \(\hat{H}^0(\mathcal{L})(N_\lambda(F))\).

**Proof.** Let \(\phi\) be the natural map \(\hat{H}^0(\mathcal{L}(N_\lambda(E))) \to \hat{H}^0(\mathcal{L})\). Let \(v \in \text{img}(\phi)\) and let \(\tilde{v}\) be a lift of \(v\) in \(\mathcal{L}(N_\lambda(E))^G\). Then there exists a compact open subgroup \(N\) of \(N_\lambda(E)\) such that

\[
\int_N \pi(n)\tilde{v}\, dn = 0.
\]

(6.1)

Since \(\pi\) is smooth, there exists a compact open subgroup \(N'\) of \(N\) of finite index such that

\[
\int_N \pi(n)\tilde{v}\, dn = \sum_{n \in N/N'} \pi(n)\tilde{v}\, dn.
\]

Since \(N_\lambda(E)\) has a filtration of \(\Gamma\)-stable compact open subgroups, we may assume that \(N\) is \(\Gamma\)-stable. If \(X\) denotes the coset space \(N'/N\), then we have

\[
\sum_{x \in X} \pi(x)\tilde{v} = \sum_{y \in X^G} \pi(y)\tilde{v} + \sum_{z \in X \setminus X^G} \pi(z)\tilde{v}.
\]

(6.2)

Since the \(\Gamma\)-action on \(X \setminus X^G\) is free, there exists a subset \(U\) such that \(X \setminus X^G\) is the disjoint union of \(\gamma^iU\), \(1 \leq i \leq l\). As \(\tilde{v}\) is \(\Gamma\)-invariant, we have

\[
\sum_{z \in X \setminus X^G} \pi(z)\tilde{v} = \sum_{i=1}^l \sum_{u \in U} \pi(\gamma^i(u))\tilde{v} = N\left(\sum_{u \in U} \pi(u)\tilde{v}\right),
\]

where \(N = 1 + \gamma + \cdots + \gamma^{l-1}\). This shows that

\[
\sum_{z \in X \setminus X^G} \pi(z)v = 0
\]

in \(\hat{H}^0(\mathcal{L})\). Therefore, it follows from (6.1) and (6.2) that

\[
\sum_{y \in X^G} \pi(y)v = 0.
\]
Using Lemma 6.1. The long exact sequence of Tate cohomology groups associated with the exact sequence

\[ 0 \rightarrow V(N_\Lambda(E)) \rightarrow V \rightarrow V_{N_\Lambda(E)} \rightarrow 0, \]

is equal to:

\[ 0 \rightarrow \widehat{H}^0(V(N_\Lambda(E))) \xrightarrow{\phi} \widehat{H}^0(V) \rightarrow \widehat{H}^0(V_{N_\Lambda(E)}) \rightarrow \widehat{H}^1(V(N_\Lambda(E))) \rightarrow \widehat{H}^1(V) \rightarrow 0. \]

Using Lemma 6.1, we get that \( \phi(\widehat{H}^0(V(N_\Lambda(E)))) \) is equal to \( \widehat{H}^0(V)(N_\Lambda(F)) \), and therefore the \( N_\Lambda(F) \)-coinvariants \( \widehat{H}^0(V)_{N_\Lambda(F)} \) is isomorphic to a subrepresentation of \( \widehat{H}^0(V_{N_\Lambda(E)}) \). Since \( \widehat{H}^0(V_{N_\Lambda(E)}) \) is non-zero, the lemma follows from the irreducibility of \( \widehat{H}^0(V_{N_\Lambda(E)}) \).

Using similar ideas, we can prove that zeroth Tate cohomology of a generic representation has a unique generic subquotient. For any integral \( l \)-adic generic representation \( (\pi, V) \) of \( G_n(K) \), defined over \( \mathcal{K} \), we recall that \( \mathbb{W}_\Lambda(\pi, \psi_K) \) denotes the set of functions in \( \mathbb{W}(\pi_E, \psi_E) \) with values in \( \Lambda \). The \( \Lambda \)-module \( \mathbb{W}_\Lambda(\pi, \psi_K) \) gives a \( \Lambda \) structure of \( \pi \) (see 2.7.6).

**Proposition 6.3.** Let \( \pi_E \) be an \( l \)-modular generic representation (or an integral \( l \)-adic generic representation defined over \( \mathcal{K} \)) of \( G_n(E) \). Assume that \( \pi_E \) is stable under the action of \( \Gamma \). Then there exists a unique generic subquotient of the \( G_n(F) \) representation \( \widehat{H}^0(\mathbb{W}(\pi_E, \psi_E)) \) (resp. \( \widehat{H}^0(\mathbb{W}_\Lambda(\pi_E, \psi_E)) \)).

**Proof.** Let \( W \) be the Whittaker space \( \mathbb{W}(\pi_E, \psi_E) \) (resp. \( \mathbb{W}_\Lambda(\pi_E, \psi_E) \)). Let \( W(N_n(E), \Theta_E) \) be the \( \overline{\mathbb{F}_l} \) (resp. \( \Lambda \))-span of the vectors of the form \( \pi_E(n)v - \Theta_E(n)v \) (resp. \( \pi_E(n)v - \Theta_E(n)v \)), for all \( v \in W \) and \( n \in N_n(E) \). We have the following exact sequence:

\[ 0 \rightarrow W(N_n(E), \Theta_E) \rightarrow W \rightarrow W_{N_n(E), \Theta_E} \rightarrow 0. \]

The space \( W_{N_n(E), \Theta_E} \) is a one dimensional vector space over \( \overline{\mathbb{F}_l} \) (resp. a free \( \Lambda \)-module of rank one). The long exact sequence in the Tate cohomology gives us

\[ \widehat{H}^0(W(N_n(E), \Theta_E)) \xrightarrow{f} \widehat{H}^0(W) \xrightarrow{\delta} \widehat{H}^0(W_{N_n(E), \Theta_E}) \rightarrow \widehat{H}^1(W(N_n(E), \Theta_E)) \rightarrow \widehat{H}^1(W). \]

Using arguments of Lemma 6.1, the image of the morphism \( f \) is equal to \( \widehat{H}^0(W)(N_n(F), \Theta'_F) \). The Tate cohomology of the Kirillov model \( \mathbb{K}(\pi_E, \psi_E) \) contains \( \mathbb{K}(\psi'_F) \) as \( P_n(F) \) subrepresentation (see 6.5.1). Since
Γ acts trivially on \( W_{N_n(E)} \), the Tate cohomology space \( \hat{H}^0(W_{N_n(E)}, \mathcal{O}_E) \) is a one dimensional vector space over \( \mathbb{F}_l \). Hence, the map \( g \) induces the isomorphism:

\[
\hat{H}^0(W)_{N_n(F), \mathcal{O}_E} \cong \hat{H}^0(W_{N_n(E)}, \mathcal{O}_E).
\]

Now, it follows from the exactness of the Jacquet functor and Proposition 5.5, that \( \hat{H}^0(W) \) admits a unique generic subquotient. \( \square \)

**Remark 6.4.** The above lemmas will be used to compute the Tate cohomology of the base change of mod-\( l \) the Zelevinsky sub-representation \( Z(\Delta) \). The Jacquet functor of \( Z(\Delta) \) with respect to the parabolic subgroup of type \( (n/k, n/k, \ldots, n/k) \), where \( k \) is the length of the segment \( \Delta \), is an \( l \)-modular cuspidal representation and the hypothesis in Lemma 6.2 are applicable. The precise definitions will be recalled in the next section.

### 6.3. The \( \text{GL}_2 \) case.

**Theorem 6.5.** Let \( F \) be a finite extension of \( \mathbb{Q}_p \), and let \( E \) be a finite Galois extension of \( F \) with \( [E : F] = l \). Assume that \( l \) and \( p \) are distinct odd primes. Let \( \pi_F \) be an integral \( l \)-adic cuspidal representations of \( G_2(F) \) and let \( \pi_E \) be the representation of \( G_2(E) \) such that \( \pi_E \) is the base change of \( \pi_F \). Then

\[
\hat{H}^0(r_l(\pi_E)) \cong r_l(\pi_F)(l).
\]

**Proof.** First note that the base change lift \( \pi_E \) is cuspidal, and hence the mod-\( l \) reduction \( r_l(\pi_E) \) is also cuspidal. Let \( \psi_E \) and \( \psi_F \) be defined as in subsection (2.7.7). Let \( (\mathbb{K}^{\infty}_{\psi_E}, C^{\infty}_c (E^\times, \mathbb{F}_l)) \) be a Kirillov model of \( r_l(\pi_E) \). Recall that the group \( \Gamma \) acts on \( C^{\infty}_c (E^\times, \mathbb{F}_l) \). We denote by \( \hat{H}^0(r_l(\pi_E)) \) the cohomology group \( \hat{H}^0(C^{\infty}_c (E^\times, \mathbb{F}_l)) \). Then, using Proposition 5.1, we have

\[
\hat{H}^0(r_l(\pi_E)) \cong C^{\infty}_c (F^\times, \mathbb{F}_l).
\]

The space \( \hat{H}^0(r_l(\pi_E)) \) is isomorphic to \( \text{ind}^{P_2(F)}_{N_2(F)}(\overline{\psi}_F \psi) \) as a representation of \( P_2(F) \), where \( \overline{\psi}_F \psi \) is the mod-\( l \) reduction of \( \psi_F \); and the induced action of the operator \( \mathbb{K}^{\infty}_{\psi_E}(w) \) on \( \hat{H}^0(r_l(\pi_E)) \) is denoted by \( \mathbb{K}^{\infty}_{\psi_E}(w) \). The theorem now follows from the following claim.

**Claim 1.** \( \mathbb{K}^{\infty}_{\psi_E}(w)(f) = \mathbb{K}^{\infty}_{\psi_F}(w)(f) \), for all \( f \in C^{\infty}_c (F^\times, \mathbb{F}_l) \).

Now, for a function \( f \in C^{\infty}_c (F^\times, \mathbb{F}_l) \), any covering of \( \text{supp}(f) \) by open subsets of \( F^\times \) has a finite refinement of pairwise disjoint open compact subgroups of \( F^\times \). So we may assume that \( \text{supp}(f) \subseteq \mathfrak{w}^r \times U^1_F \), where \( r \in \mathbb{Z} \), \( \mathfrak{w}_F \) is an uniformizer of \( F \) and \( x \) is a unit in \( (\mathfrak{o}_F/\mathfrak{p}_F)^\times \) embedded in \( F^\times \). Therefore it is sufficient to prove the claim for functions \( f \in C^{\infty}_c (F^\times, \mathbb{F}_l) \) with \( \text{supp}(f) \subseteq U^1_F \), and we have

\[
f = c_{\chi_F} \sum_{\chi_F \in U^1_F} \xi_{\{\chi_F, 0\}},
\]

where \( c_{\chi_F} \in \mathbb{F}_l \) and \( U^1_F \) is the set of smooth characters of

\[
F^\times = (\mathfrak{w}_F)^r \times k_F^r \times U^1_F
\]

which are trivial on \( k_F^r \) and \( \mathfrak{w}_F \). We now prove the claim for the function \( \xi_{\{\chi_F, 0\}} \) for \( \chi_F \in U^1_F \). There exists a character \( \chi_0 \in U^1_F \) such that \( \chi_0^r = \chi_F \). Let \( \chi_0 \) be the \( l \)-adic lift of the character \( \chi_0 \). Define the characters \( \chi_E \) and \( \tilde{\chi}_E \) of \( E^\times \) as follows

\[
\chi_E(x) = \chi_0(\text{Nr}_{E/F}(x))
\]

and

\[
\tilde{\chi}_E(x) = \tilde{\chi}_0(\text{Nr}_{E/F}(x)),
\]

for \( x \in E^\times \). Here, \( \text{Nr}_{E/F} : E^\times \to F^\times \) denotes the norm map. Note that \( \tilde{\chi}_E \) and \( \chi_E \) extends the character \( \chi_F \). We have the following relations:

\[
\mathbb{K}^{\infty}_{\psi_E}(w)(\xi_{\{\chi_F, 0\}}) = \epsilon(\chi_E r_l(\pi_E), \overline{\psi}_E) \xi_{\{\chi_F, -n(\chi_E^{-1} r_l(\pi_E), \overline{\psi}_E)\}} \quad (6.3)
\]
and
\[ K_{\psi_F}^{(\pi_F)^{(i)}}(w)(\xi\{\chi_F,0\}) = \epsilon(\chi_F^{-1} r_1(\pi_F)^{(i)}, -n(\chi_F^{-1} r_1(\pi_F)^{(i)})) \],
(6.4)
where \( e \) denotes the ramification index of the extension \( E/F \). Next, we aim to prove the following identity:
\[ \epsilon(X, \chi_F^{-1} r_1(\pi_F), -n(\chi_F^{-1} r_1(\pi_F))) = \epsilon(X, \chi_F^{-1} r_1(\pi_F)^{(i)})) \]\
(6.5)
It follows from Theorem 3.4 that the \( \epsilon \)-factor is same as the \( \gamma \)-factor in both \( l \)-adic and mod-\( l \) cases. Now, using the identity in [AC89, Proposition 6.9], we get
\[ \epsilon(X, \chi_F^{-1} \pi_E, -n(\chi_F^{-1} \pi_E)) \]
where \( \eta \) runs over all the characters of the group \( F^* \) which is isomorphic to \( \text{Gal}(E/F) \) via local class field theory. Using the identity (3.6), we have
\[ r_1(\epsilon(X, \chi_F^{-1} \pi_E, -n(\chi_F^{-1} \pi_E))) = \epsilon(X, \chi_F^{-1} r_1(\pi_E)) \]
and
\[ r_1(\epsilon(X, \chi_F^{-1} \pi_E, -n(\chi_F^{-1} \pi_E))) = \epsilon(X, \chi_F^{-1} r_1(\pi_E)) \]
for each character \( \eta \). Using the above identities and taking mod-\( l \) reduction to (6.5), we get
\[ \epsilon(X, \chi_F^{-1} r_1(\pi_E), -n(\chi_F^{-1} r_1(\pi_E))) = \epsilon(X, \chi_F^{-1} r_1(\pi_E), -n(\chi_F^{-1} r_1(\pi_E))) \]
Using Lemma 5.7, we obtain the following identity
\[ \epsilon(X, \chi_F^{-1} r_1(\pi_E), -n(\chi_F^{-1} r_1(\pi_E))) = \epsilon(X, \chi_F^{-1} r_1(\pi_E), -n(\chi_F^{-1} r_1(\pi_E))) \]
Now, using the identity (3.5) and comparing the degree of \( X \) from above relation, we get
\[ n(\chi_F^{-1} r_1(\pi_E)) = n(\chi_F^{-1} r_1(\pi_E)) \]
and
\[ \epsilon(\chi_F^{-1} r_1(\pi_E), -n(\chi_F^{-1} r_1(\pi_E))) = \epsilon(\chi_F^{-1} r_1(\pi_E), -n(\chi_F^{-1} r_1(\pi_E))) \]
Thus it follows from (6.3) and (6.4) that
\[ K_{\psi_F}^{(\pi_F)^{(i)}}(w)(\xi\{\chi_F,0\}) = K_{\psi_F}^{(\pi_F)^{(i)}}(w)(\xi\{\chi_F,0\}) \]
Hence we prove the claim, and the theorem follows.

6.4. Our main result uses the following lemma which is the analogue of completeness of Whittaker models in the complex case.

**Lemma 6.6.** Assume that \( l \) does not divide \( |G_n(K)| \) and let \( \overline{\gamma}_K \) be the mod-\( l \) reduction of \( \psi_K \). Let \( \overline{\gamma}_K \) be the non-degenerate character of \( N_n(K) \) associated with \( \overline{\gamma}_K \) (see Section 2.7). Let \( \phi \in \text{ind}_{N_n(K)}(\overline{\gamma}_K) \). If
\[ \int_{N_n(K)/G_n(K)} \phi(t)\overline{W}(t) W(t) dt = 0, \]
for all \( W \in \mathbb{W}(\sigma, \overline{\psi}_K^{-1}) \) and for all generic representations \( \sigma \) of \( G_n(K) \), then \( \phi = 0 \).

**Proof.** Suppose \( \phi \) is non-zero. Let \( \text{Rep}_{\overline{\mathbb{F}}_l}(G_n(K)) \) be the category of smooth \( W(\overline{\mathbb{F}}_l)[G_n(K)]\)-modules, and let \( Z_n \) be its center. Let \( W_n \) be the smooth \( \text{Rep}_{\overline{\mathbb{F}}_l}[G_n(K)]\)-module \( \text{ind}_{N_n(K)}^{G_n(K)}(\Theta_K) \). Recall that for any primitive idempotent \( \epsilon \) in \( Z_n \), the space \( \epsilon W_n \) is a smooth \( \text{co-Whittaker} \) \( \epsilon Z_n[G_n(K)]\)-module (see [Hel16b, Theorem 6.3]). According to [Mos21, Corollary 4.3], there exists a primitive idempotent \( \epsilon' \) of \( Z_n \) and an element \( U \in \mathbb{W}(\epsilon' W_n \otimes_{\text{Rep}_{\overline{\mathbb{F}}_l}} \overline{\mathbb{F}}_l, \overline{\psi}_K^{-1}) \) such that the integral
\[ \langle \phi, U \rangle := \int_{N_n(K)/G_n(K)} \phi(t) \otimes U(t) dt \]
is non-zero in \( \mathbb{W}(\epsilon' W_n \otimes_{\text{Rep}_{\overline{\mathbb{F}}_l}} \overline{\mathbb{F}}_l, \overline{\psi}_K^{-1}) \). As described in [Hel16a], the primitive idempotent \( \epsilon' \) corresponds to an inertial equivalence class of pairs \((M, \pi)\), where \( M \) is a Levi subgroup of \( G_n(K) \) and \( \pi \) is a supercuspidal \( \overline{\mathbb{F}}_l\)-representation of \( M \). Let \( R' \) denote the ring \( \overline{\mathbb{F}}_l \otimes_{\text{Rep}_{\overline{\mathbb{F}}_l}} \epsilon' Z_n \).
For the inertial equivalence class \([M, \pi]\), consider the subcategory \(\text{Rep}_{W(\pi)}(G_n(K))_{[M, \pi]}\), consisting of objects \(I\) in \(\text{Rep}_{W(\pi)}(G_n(K))\) whose irreducible sub-quotients have mod-\(l\) inertial supercuspidal support \([M, \pi]\). Let \(A_{[M, \pi]}\) denote the center of the subcategory \(\text{Rep}_{W(\pi)}(G_n(K))_{[M, \pi]}\). Since \(l\) does not divide \(|G_n(k\mathbf{F})|\), it follows from [Hel16a, Example 13.9] that

\[ A_{[M, \pi]} = C_{[M, \pi]}, \]

where \(C_{[M, \pi]}\) is a \(W(\mathbf{F})\)-subalgebra of \(A_{[M, \pi]}\), as defined in [Hel16a, Theorem 12.5]. Then, there is an isomorphism of \(C_{[M, \pi]} \otimes_{W(\pi)} \mathbf{F}_l\) with the reduced quotient of \(A_{[M, \pi]} \otimes_{\mathbf{F}_l} \mathbf{F}_l\) (see [Hel16a, Corollary 12.13]), and hence we get that the \(W(\mathbf{F})\)-algebra \(R^t\) is reduced. In particular, the element \(\langle \phi, U \rangle\) is not nilpotent.

Therefore, the basic open set \(D(\langle \phi, U \rangle)\) is non-empty, and hence intersects the dense set of closed points of the affine \(W(\mathbf{F})\)-scheme associated with \(R^t\). This implies that there exists a map \(f : R^t \to \mathbf{F}_l\) such that the image of \(\langle \phi, U \rangle\) under \(f\), which is equal to \(\int_{N_{2n}(K)\backslash G_n(K)} \phi(t)W_0(t) dt\) for some \(W_0 \in \mathbb{W}(\epsilon W_n \otimes R^t, \mathbf{F}_l, \psi_{K^n})\), is non-zero in \(\mathbf{F}_l\). Note that \(\epsilon^t W_n \otimes R^t, \mathbf{F}_l, \psi_{K^n}\), as \(\mathbf{F}_l\)-representation, admits a generic quotient with same Whittaker space. Hence, the lemma.

\[ \square \]

6.5. The general case. Let \(\pi_F\) be an integral generic \(l\)-adic representation of \(G_n(F)\), and let \(\pi_E\) be the base change lifting of \(\pi_F\) to \(G_n(E)\). We observe that the unique generic component \(J_1(\pi_E)\) of the mod-\(l\) reduction of \(\pi_E\) is stable under the action of \(\Gamma\). We will now prove the main theorem of our article.

**Theorem 6.7.** Let \(F\) be a finite extension of \(\mathbb{Q}_p\), and let \(E\) be a finite Galois extension of \(F\) with \([E : F] = l\), where \(p\) and \(l\) are distinct primes such that \(l\) does not divide \(|G_{n-1}(k_F)|\). Let \(\pi_F\) be an integral \(l\)-adic generic representation of \(G_n(F)\) with \(J_1(\pi_F)\), the unique generic component of the mod-\(l\) reduction of \(\pi_F\). Let \(\pi_E\) be the base change lift of \(\pi_F\). Then, the representation \(J_1(\pi_F)^{(l)}\) is the unique generic sub-quotient of \(\hat{H}^0(J_1(\pi_E))\).

**Proof.** We begin with a summary of the proof. We prove the above theorem using induction on the integer \(n\). The proof is divided into four parts. In the first part, we isolate a subspace \(\mathcal{M}(\pi_F, \psi_F)\) of the Tate cohomology of the Kirillov model of \(J_1(\pi_E)\) which will eventually give \(J_1(\pi_F)^{(l)}\) as a quotient. In the second part, we will set up comparison of Zeta integrals on homogeneous spaces of \(F\) with those on homogeneous spaces of \(E\). In the third part we reduce the theorem to an identity of local \(\gamma\)-factors. In the fourth part we deal with these local \(\gamma\)-factor identities and we show that \(\mathcal{M}(\pi_F, \psi_F)\) is stable under the action of \(G_n(F)\). At the end of the fourth part, we get a natural onto map from \(\mathcal{M}(\pi_F, \psi_F)\) to the mod-\(l\) Kirillov model \(K(J_1(\pi_F)^{(l)}, \psi_F)\) as \(G_n(F)\) representations.

6.5.1. Notations on Whittaker and Kirillov models are defined in the subsections (2.7.1) and (2.8). Consider the Whittaker model \(\mathcal{W}(J_1(\pi_E), \psi_E)\) of \(J_1(\pi_E)\). The restriction map \(W \to \text{res}_{P_n(E)}(W)\) is an isomorphism from \(\mathcal{W}(J_1(\pi_E), \psi_E)\) onto \(K(J_1(\pi_E), \psi_E)\) (see [MM22, Theorem 4.2]). Recall that \(K(\psi_E)\) denotes the compactly induced representation \(\text{Ind}_{P_n(E)}^{G_n(E)}\mathcal{O}_{\psi_E}\). Note that \(K(\psi_E)\) is contained in \(K(J_1(\pi_E), \psi_E)\). Let \(I_n\) be the following natural map:

\[ I_n : \hat{H}^0(K(\psi_E)) \to \hat{H}^0(K(J_1(\pi_E), \psi_E)). \]

Let \(\Phi_n : K(J_1(\pi_E), \psi_E) \to \text{Ind}_{P_n(E)}^{G_n(E)}\mathcal{O}_F\) be the restriction to \(P_n(F)\) map. Note that the map \(\Phi_n\) factorizes through

\[ \Phi_n : \hat{H}^0(K(J_1(\pi_E), \psi_E)) \to \text{Ind}_{P_n(E)}^{G_n(E)}\mathcal{O}_{\psi_F}. \]

The composition \(\Phi_n \circ I_n\) is induced by the restriction to \(P_n(F)\) map from \(K(\psi_F)^{(l)}\) to \(K(\psi_F)\), and hence, \(\Phi_n \circ I_n\) is an isomorphism onto the space \(K(\psi_F)\) by Proposition 5.1 (see Subsection 5.2.1). This implies that the image of \(\Phi_n\) contains \(K(\psi_F)\). Let \(\mathcal{M}(\pi_F, \psi_F)\) be the space \(\Phi_n^{-1}(K(J_1(\pi_F)^{(l)}, \psi_F))\). The space \(\mathcal{M}(\pi_F, \psi_F)\) is a non-zero \(P_n(F)\) sub-representation of \(\hat{H}^0(K(J_1(\pi_E), \psi_E))\), and the map

\[ \Phi_n : \mathcal{M}(\pi_F, \psi_F) \to K(J_1(\pi_F)^{(l)}, \psi_F) \]

is non-zero. Then, using induction on \(n\), we will show that the space \(\mathcal{M}(\pi_F, \psi_F)\) is stable under the action of \(G_n(F)\) and the map \(\Phi_n\) is \(G_n(F)\)-equivariant.
6.5.2. Let \( \overline{J_l(\pi_E)}(w_n) \) be the induced action of \( J_l(\pi_E)(w_n) \) on the space \( \widehat{H}^0(\mathbb{K}(\pi_l), \overline{\psi}_E) \). Let \( V \) be an element in \( \mathcal{M}(\pi_F, \psi_F) \). Then there exists \( W \in \mathbb{W}(J_l(\pi_E), \overline{\psi}_E)^\Gamma \) such that \( W \) is mapped to \( V \) under the map
\[
\mathbb{W}(J_l(\pi_E), \overline{\psi}_E)^\Gamma \rightarrow \mathbb{K}(\pi_l), \overline{\psi}_E)^\Gamma \rightarrow \widehat{H}^0(\mathbb{K}(\pi_l), \overline{\psi}_E).
\]
Let \( \sigma_F \) be an arbitrary \( l \)-modular generic representation of \( G_{n-1}(F) \), and let \( \sigma_F \) be its \( l \)-adic lift. In this case, the generic mod-\( l \) representation \( J_l(\sigma_F) \) is equal to \( \sigma_F \). Let \( \sigma_E \) be an \( l \)-adic generic representation of \( G_{n-1}(E) \) obtained as a base change of \( \sigma_F \). Note that the map
\[
\hat{\Phi}_{n-1} : \widehat{H}^0(\mathbb{W}(J_l(\sigma_E), \overline{\psi}_E^{-1})) \rightarrow \text{Ind}_{N_{n-1}(F)}^{G_{n-1}(F)}(\sigma_F)^{-1}
\]
is non-zero. Here, \( \hat{\Phi}_{n-1} \) is the restriction to \( G_{n-1}(F) \) map on the space \( \text{Ind}_{N_{n-1}(E)}^{G_{n-1}(E)}(\sigma_F) \). Assuming the induction hypothesis for \( n-1 \) and using the fact that the representation \( \widehat{H}^0(\mathbb{W}(J_l(\sigma_E), \overline{\psi}_E^{-1})) \) has a unique generic subquotient (Proposition 6.3), the image of \( \hat{\Phi}_{n-1} \) contains \( \mathbb{W}(\sigma_F, \overline{\psi}_F) \). Thus, for any \( W' \in \mathbb{W}(\sigma_F, \overline{\psi}_F) \), there exists an element \( S \in \mathbb{W}(J_l(\sigma_E), \overline{\psi}_E)^\Gamma \) such that \( \hat{\Phi}_{n-1}(S) = W' \) and
\[
\hat{\Phi}_{n-1}(J_l(\sigma_E)(w_n)S) = \sigma_F^{(l)}(w_n)W'.
\]
Now the functional equation in (3.4.2) gives the following relation:
\[
\sum_{r \in \mathbb{Z}} c_r^E(W, \tilde{S})q_F^{-rf}X^{-fr} = \omega_{J_l(\sigma_E)}(-1)^{n-2}\gamma(X, J_l(\pi_E), J_l(\sigma_E), \psi_E) \sum_{r \in \mathbb{Z}} c_r^E(W, S)q_F^{-r}X^{fr},
\]
where \( f \) denotes the residue degree of the extension \( E/F \). Note that \( \omega_{\sigma_E}(-1) = \omega_{\sigma_F}(-1) \) as \( l \) is an odd prime. Applying Proposition 5.2, we get
\[
\int_{(X_E)^r} W \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} S(g)dg = \int_{X_E^r} W \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} S(g)dg,
\]
for each \( r \in \mathbb{Z} \). Using the above equalities and Remark 5.3, the functional equation (6.7) becomes
\[
\sum_{r \in \mathbb{Z}} c_r^E(W, \tilde{S})q_F^{-rf}X^{-fr} = \omega_{J_l(\sigma_E)}(-1)^{n-2}\gamma(X, J_l(\pi_E), J_l(\sigma_E), \overline{\psi}_E) \sum_{r \in \mathbb{Z}} c_r^E(W, S)q_F^{-r}X^{fr}.
\]
Using the modification as in (3.3), the above equality becomes
\[
\sum_{r \in \mathbb{Z}} c_r^E(J_l(\pi_E)(w_n)W, \sigma_F^{(l)}(w_n)W')q_F^{-rf}X^{-fr} = \omega_{J_l(\sigma_E)}(-1)^{n-2}\gamma(X, J_l(\pi_E), J_l(\sigma_E), \overline{\psi}_E) \sum_{r \in \mathbb{Z}} c_r^E(W, W')q_F^{-r}X^{fr}.
\]
6.5.3. For any \( V \in \mathcal{M}(\pi_F, \psi_F) \), we show that
\[
\Phi_n(J_l(\pi_E)(w_n)V) = J_l(\pi_F)^{(l)}(w_n)\Phi_n(V).
\]
Let \( U \) be an element of \( \mathbb{W}(J_l(\pi_F)^{(l)}, \overline{\psi}_F) \) such that \( \text{res}_{p_n(F)}(U) \) is equal to \( \Phi_n(V) \). By Lemma 6.6, the assertion (6.10) is equivalent to the following equality:
\[
\sum_{r \in \mathbb{Z}} c_r^E(J_l(\pi_E)(w_n)W, \sigma_F^{(l)}(w_n)W')q_F^{-r/2}X^{-r} = \sum_{r \in \mathbb{Z}} c_r^E(J_l(\pi_F)^{(l)}(w_n)U, \sigma_F^{(l)}(w_n)W')q_F^{-r/2}X^{-r},
\]
for all \( W' \in \mathbb{W}(\sigma_F, \overline{\psi}_F^{-1}) \) and for all \( l \)-modular generic representations \( \sigma_F \) of \( G_{n-1}(F) \). Now, consider an \( l \)-modular generic representation \( \sigma_F \) of \( G_{n-1}(F) \) and take an \( l \)-adic lift of \( \sigma_F \), say \( \sigma_F \) (see subsection 2.7.5).
Note that $J_l(\sigma_F) = \sigma_F$. Let $\sigma_E$ be the $l$-adic generic representation of $G_{n-1}(E)$ obtained as a base change lift of $\sigma_F$. From the functional equation with its modifications as in (3.3), we have
\[
\sum_{r \in \mathbb{Z}} c_{-r}(J_l(\pi_F)^{(l)}(w_n)U, \sigma_F^{(l)}(w_{n-1})W')q_F^{-r}X^{-lr} = \\
\omega_{J_l(\sigma)}(-1)^{n-2}\gamma(X^l, J_l(\pi_F)^{(l)}, \sigma_F^{(l)}, \overline{\psi}_F) \sum_{r \in \mathbb{Z}} c_{-r}(U, W')q_F^{-r}X^{-lr},
\]
where we replace the variable $X$ by $X^l$. Note that $\text{res}_{p_F}(W)$ is equal to $\text{res}_{p_F}(U)$. Thus, comparing the above functional equation with (6.9), the relation (6.10) is now equivalent to the following equality:
\[
\gamma(X, J_l(\pi_E), J_l(\sigma_E), \overline{\psi}_E) = \gamma(X^l, J_l(\pi_F)^{(l)}, \sigma_F^{(l)}, \overline{\psi}_F).
\]

6.5.4. Recall that
\[
\gamma(X, \pi_E, \sigma_E, \psi_E) = \epsilon(X, \pi_E, \sigma_E, \psi_E) \frac{L(q_F^{-1}X^{-1}, \sigma_E^\sigma, \overline{\psi}_E)}{L(X, \pi_E, \sigma_E)}.
\]
Now using the identity in [AC89, Proposition 6.9], we have
\[
L(X, \pi_E, \sigma_E) = \prod_\eta L(X, \pi_F, \sigma_F \otimes \eta)
\]
and
\[
\epsilon(X, \pi_E, \sigma_E, \psi_E) = C_{E/F}(\psi_F)^{n(n-1)} \prod_\eta \epsilon(X, \pi_F, \sigma_F \otimes \eta, \psi_F),
\]
where $\eta$ runs over all the characters of the group $F^x/N_{E/F}(E^x)$, which is isomorphic to $\text{Gal}(E/F)$ via local class field theory. Here, $C_{E/F}(\psi_F)$ is the Langlands constant, defined as in the proof of Lemma 3.2, and $C_{E/F}(\psi_F)^2 = 1$. Then the above relations implies that
\[
\gamma(X, \pi_E, \sigma_E, \psi_E) = \prod_\eta \gamma(X, \pi_F, \sigma_F \otimes \eta, \psi_F). \tag{6.12}
\]
Now, using the identity (3.6), we have
\[
r_l(\gamma(X, \pi_E, \sigma_E, \psi_E)) = \gamma(X, J_l(\pi_E), J_l(\sigma_E), \overline{\psi}_E)
\]
and
\[
r_l(\gamma(X, \pi_F, \sigma_F \otimes \eta, \psi_E)) = \gamma(X, J_l(\pi_F), \sigma_F, \overline{\psi}_F),
\]
for each character $\eta$. Taking mod-$l$ reduction to the identity (6.12) and using these relations, we get
\[
\gamma(X, J_l(\pi_E), J_l(\sigma_E), \overline{\psi}_E) = \gamma(X, J_l(\pi_F), \sigma_F, \overline{\psi}_F)^l. \tag{6.13}
\]
Finally, it follows from Lemma 5.7 that
\[
\gamma(X, J_l(\pi_E), J_l(\sigma_E), \overline{\psi}_E) = \gamma(X^l, J_l(\pi_F)^{(l)}, \sigma_F^{(l)}, \overline{\psi}_F).
\]
The identity (6.10) shows that space $\mathcal{M}(\pi_F, \psi_F)$ is stable under the action of $G_n(F)$ and the map
\[
\Phi_n : \mathcal{M}(\pi_F, \psi_F) \to \mathbb{F}(J_l(\pi_F)^{(l)}, \overline{\psi}_F)
\]
is surjective. Using Proposition 6.3, the $G_n(F)$ representation $\tilde{\mathcal{H}}^0(\mathcal{W}(J_l(\pi_E), \overline{\psi}_E))$ has a unique generic subquotient which is necessarily equal to $J_l(\pi_F)^{(l)}$. This completes the proof. \hfill \square

Now we deduce some corollaries of Theorem 6.7. We keep the same assumptions that $E/F$ is a finite Galois extension $p$-adic fields with $[E : F] = l$, where $l$ and $p$ are distinct primes, and $l$ does not divide $|G_{n-1}(k_F)|$.

**Corollary 6.8.** Let $\pi_E$ be an integral generic $K$-representation of $G_n(E)$ which is absolutely irreducible. Assume that $\pi_E^{\sigma} \simeq \pi_E$, for all $\gamma \in \Gamma$. Let $\mathcal{W}(\pi_E, \psi_E)$ be the space of all $\Lambda$-valued functions in the Whittaker model of $\pi_E$. Let $\pi_F$ be the integral generic $\mathbb{Q}_l$-representation of $G_n(F)$ such that $\pi_E \otimes_K \mathbb{Q}_l$ is the base change lift of $\pi_F$. Then the Frobenius twist of $J_l(\pi_F)$ occurs as a unique generic subquotient of the zeroth Tate cohomology group $\tilde{\mathcal{H}}^0(\mathcal{W}_\Lambda(\pi_E, \psi_E))$. 

Proof. The outline of the proof is same as Theorem 6.7. For the sake of completeness, we discuss some crucial steps. As one can observe, the previous and the present theorems are similar in spirit to local converse theorem for \((n, n - 1)\), we precisely use Theorem 6.7 at the \((n - 1)\) step.

First, note that the \(\Lambda\)-lattice \(\mathbb{W}_\Lambda(\pi_E, \psi_E)\) is stable under the action of \(G_n(E) \times \Gamma\) (see [Vig04, Theorem 2] and Lemma 2.4). Consider the integral Kirillov model \(\mathbb{K}_\Lambda(\pi_E, \psi_E)\). The restriction map \(W \mapsto \text{res}_{P_n(E)}(W)\) is then a bijection from \(\mathbb{W}_\Lambda(\pi_E, \psi_E)\) onto \(\mathbb{K}_\Lambda(\pi_E, \psi_E)\). Let \(\Phi_n\) be the following \(P_n(F)\)-equivariant map, defined as the composition of restriction to \(P_n(F)\) map and (pointwise) mod-\(l\) reduction map

\[
\Phi_n : \mathbb{K}_\Lambda(\pi_E, \psi_E)^F \rightarrow \mathbb{K}(J_1(\pi_F)^{(l)}, \overline{\psi}_F).
\]

Then \(\Phi_n\) is non-zero and it factorizes through the Tate cohomology space \(\widehat{H}^0(\mathbb{K}_\Lambda(\pi_E, \psi_E))\). As before, we consider the non-zero space \(\Phi_n^{-1}(\mathbb{K}(J_1(\pi_F)^{(l)}, \overline{\psi}_F))\) and denote it by \(\mathcal{M}(\pi_F, \psi_F)\). To prove the above corollary, it is sufficient prove that

\[
\Phi_n(\pi_E(w_n)V) = J_1(\pi_F)^{(l)}(w_n)\Phi_n(V),
\]

for all \(V \in \mathcal{M}(\pi_F, \psi_F)\). It is enough to prove the following identity of Laurent series

\[
\sum_{r \in \mathbb{Z}} c_r^F(\Phi_n(\pi_E(w_n)V), W')q_r^F X^r = \sum_{r \in \mathbb{Z}} c_r^F(J_1(\pi_F)^{(l)}(w_n)\Phi_n(V), W')q_r^F X^r,
\]

for all \(W' \in \mathbb{W}(\sigma_F^{(l)}, \overline{\psi}_F^{(l)})\) and for all \(l\)-modular generic representations \(\sigma_F\) of \(G_{n-1}(F)\). Take such mod-\(l\) generic representation \(\sigma_F\) of \(G_{n-1}(F)\). Let \(\sigma_F\) be an \(l\)-adic lift of \(\overline{\sigma}_F\) and let \(\sigma_E\) be the base change lift of \(\sigma_F\) to \(G_{n-1}(E)\). Theorem 6.7 gives a \(G_{n-1}(F)\)-stable subspace \(N(\sigma_F, \psi_F)\) of the Tate cohomology group \(\widehat{H}^0(\mathbb{W}(J_1(\sigma_E), \psi_E))\) with the following \(G_{n-1}(F)\)-equivariant surjection

\[
\Phi_n^{-1} : \mathcal{N}(\sigma_F, \psi_F) \rightarrow \mathbb{W}(\sigma_F^{(l)}, \overline{\psi}_F^{(l)}).
\]

Now, lifting the function \(W'\) via \(\Phi_n^{-1}\) and the function \(V\) to the respective \(\Gamma\)-invariant Kirillov models and using the identities (6.8), the above relation (6.14) is then equivalent to the following identity of gamma factors:

\[
r_t(\gamma(X, \pi_E, \sigma_E, \psi_E)) = \gamma(X^l, J_1(\pi_F)^{(l)}, \sigma_F^{(l)}, \overline{\psi}_F).
\]

This follows from the arguments of the subsection (6.5.4) in the proof of Theorem 6.7.

**Corollary 6.9.** Let \(n \geq 3\), and let \(\pi_F\) be an integral \(l\)-adic cuspidal representation of \(G_n(F)\) and let \(\pi_E\) be the base change lift of \(\pi_F\). We further assume that \(l\) does not divide \(n\). Then we have

\[
\widehat{H}^0(\mathbb{W}(r_1(\pi_E))) \simeq r_1(\pi_F)^{(l)}.
\]

Proof. Since \(l\) does not divide \(n\), the representation \(\pi_E\) is cuspidal. As the Kirillov model \(\mathbb{K}(r_1(\pi_E), \overline{\psi}_E)\) is equal to \(\mathcal{K}(\overline{\psi}_E)\), we get that \(\widehat{H}^0(\mathbb{K}(r_1(\pi_E), \overline{\psi}_E))\) is equal to \(\mathcal{K}(\overline{\psi}_F)\). Thus, the action of \(G_n(F)\) on \(\widehat{H}^0(\mathbb{K}(r_1(\pi_E), \overline{\psi}_E))\) is irreducible, and the corollary follows from Theorem 6.7.

**7. Base change for \(Z(\Delta)\)**

In this section, we study the Tate cohomology of the base change of the Zelevinsky subrepresentations of the form \(Z(\Delta)\). In [Zel80], Zelevinsky uses the notation \(\{\Delta\}\) for \(Z(\Delta)\). In this section, we continue with the assumptions in Corollary 6.9, i.e., \(l \neq p\) and \(l\) does not divide \(|G_{n-1}(\mathbb{F}_q)|\) and the integer \(n\). Recall that \(q\) is the cardinality of the residue field of \(F\). We will crucially use the fact that \(Z(\Delta)\) remains irreducible under the restriction to \(P_n\) and it is characterised by this property.

7.1. Keeping the notations as in subsection (2.7.2), let \(\Delta = \{\sigma, \sigma v_K, \ldots, \sigma v_K^{r-1}\}\) be a segment, where \(K\) is a \(p\)-adic field and \(\sigma\) is a cuspidal \(l\)-adic representation of \(G_m(K)\). We denote by \(\ell(\Delta)\) the length of \(\Delta\), i.e., the integer \(r\). The parabolic induction

\[
\sigma \times \sigma v_K \times \cdots \sigma v_K^{r-1}
\]

admits a unique irreducible subrepresentation, denoted by \(Z(\Delta)\). Moreover, \(Z(\Delta)\) can be characterised as those irreducible representation of \(G_m(K)\) that remain irreducible after restricting to \(P_m(K)\), and the restriction is isomorphic to \((\Phi^+)^{m-1} \circ \Psi^+(Z(\Delta^-))\), where \(\Delta^- = \Delta \setminus \{\sigma v_K^{r-1}\}\). We refer to [BZ77, Section 3], [Vig96, Section 1.2, Chapter III] for the definitions of the functors \(\Phi^+\) and \(\Psi^+\), and for the definition of \(Z(\Delta)\) and its restriction to \(P_n(K)\), we refer to [Zel80, Section 3].
7.2. Let $F$ be a finite extension of $\mathbb{Q}_p$, and let $E$ be a finite Galois extension of $F$ of prime degree $l$ with $l \neq p$. Let $\Gamma$ denote the Galois group $\text{Gal}(E/F)$ with generator, say $\gamma$. Let $\sigma_F$ and $\sigma_E$ be the integral cuspidal $l$-adic representations of $G_m(F)$ and $G_m(E)$ respectively, such that $\sigma_E$ is a base change lift of $\sigma_F$.

Consider the segments
\[
\Delta_F = \{ \sigma_F, \sigma_F^{l}, \ldots, \sigma_F^{k-1} \}, \\
\Delta_E = \{ \sigma_E, \sigma_E^{l}, \ldots, \sigma_E^{k-1} \}.
\]

Then we have the irreducible $l$-adic representations $Z(\Delta_F)$ and $Z(\Delta_E)$ of $G_n(F)$ and $G_n(E)$ respectively, where $n = km$. If we let $\sigma_F'$ (resp. $\sigma_E'$) be the representation $\sigma_F^{k-1}$ (resp. $\sigma_E^{k-1}$), then we have
\[
\Pi_F(Z(\Delta_F)) = \Pi_F(\sigma_F') \oplus \Pi_F(\sigma_F^{k-1}) \oplus \cdots \oplus \Pi_F(\sigma_F) \\
\Pi_E(Z(\Delta_E)) = \Pi_E(\sigma_E') \oplus \Pi_E(\sigma_E^{k-1}) \oplus \cdots \oplus \Pi_E(\sigma_E),
\]

where $\Pi_F$ and $\Pi_E$ are the local Langlands correspondences defined as in subsection (4.1). This shows that
\[
\text{Res}_{\mathbb{Q}_p}(\Pi_F(Z(\Delta_F))) \simeq \Pi_E(Z(\Delta_E)).
\]

Thus the representation $Z(\Delta_E)$ is the base change of $Z(\Delta_F)$.

7.3. Let $L_0$ be a $G_m(E)$-invariant lattice in $\sigma_E$, and let $S_\gamma : \sigma_E \to \sigma_E^\gamma$ be an isomorphism with $S_\gamma^l = \text{id}$ and $S_\gamma(L_0) = L_0$. Recall that the representation $\pi_E = \sigma_E \times \sigma_E \times \cdots \times \sigma_E$ admits a $G_n(E)$-invariant lattice, say $L'$, which is induced via $L_0$. Then $L = L' \cap Z(\Delta_E)$ is a $G_n(E)$-invariant lattice in $Z(\Delta_E)$. Now, the map $S_\gamma$ induces an isomorphism $T_\gamma : Z(\Delta_E) \to Z(\Delta_E)^\gamma$ such that $T_\gamma^l = \text{id}$ and $T_\gamma$ stabilizes $L$. Moreover, choosing a $G_n(E)$-invariant, $S_\gamma$-stable lattice in $\sigma_E$ is equivalent to choosing a $G_n(E)$-invariant, $T_\gamma$-stable lattice in $Z(\Delta_E)$.

Remark 7.1. Since $\sigma_E$ is cuspidal, the restriction $\sigma_E|_{P_n(E)}$ is isomorphic to the compact induction $K(\psi_E)$ as $\mathbb{Q}_l$-representations. So, the restriction of $L_0$ to the subgroup $P_n(E)$ is isomorphic to the space of $\mathbb{Z}_l$ valued functions in $K(\psi_E)$. This implies that $\hat{H}^1(L_0) = 0$. For details, see [Ron16, Theorem 6].

From this, we now deduce the following result.

Proposition 7.2. Let $L$ be a lattice in $Z(\Delta_E)$ that is stable under the action of both $G_n(E)$ and $T_\gamma$. Then we have $\hat{H}^1(L) = 0$.

Proof. We prove the above claim using induction on $l(\Delta_E)$. If the length of $\Delta_E$ is 1, then the proposition clearly follows from Remark 7.1. Recall that
\[
Z(\Delta_E)|_{P_n(E)} \simeq (\Phi^+)^{m-1} \circ \Psi^+(Z(\Delta_E)).
\]

Here, $Z(\Delta_E)^\gamma$ is identified with the $m$-th derivative of $Z(\Delta_E)$. The $m$-th derivative is the composition of the following maps:
\[
Z(\Delta_E) \xrightarrow{r_{N_{m-1}(E)}} Z(\Delta_E) \otimes \sigma_E \xrightarrow{id \otimes r_{N_{m}(E)}, \Theta_E} Z(\Delta_E),
\]

where $r_{N_{m-1}(E)}$ and $r_{N_{m}(E), \Theta_E}$ are the natural quotient maps of corresponding Jacquet module and twisted Jacquet module. Note that the maps $r_{N_{m-1}(E)}$ and $r_{N_{m}(E), \Theta_E}$ preserve integral structures (see [Dat05, Proposition 1.4(i)] for $r_{N_{m-1}(E)}$, and [Vig04, Theorem III.2] for $r_{N_{m}(E), \Theta_E}$). Thus, we get that $L^{-}$ defined as
\[
L^{-} = \Psi^{-} \circ (\Phi^{-})^{m-1}(L)
\]
is a lattice in $Z(\Delta_E^-)$. We have the following isomorphism of $P_n(E) \rtimes \Gamma$-modules
\[
L|_{P_n(E)} \simeq (\Phi^+)^{m-1} \circ \Psi^+(L^-).
\]

Applying Proposition 5.1 repeatedly $(m - 1)$-times, we get that
\[
\hat{H}^1(L) \simeq (\Phi^+)^{m-1} \circ \Psi^+(\hat{H}^1(L^-)).
\] (7.1)

When $k$ is 2, we have $\Delta_E = \{ \sigma_E, \sigma_{E/E}\}$ and in this case, the representation $Z(\Delta_E^-)$ is equal to $\sigma_E$ and $L^{-}$ is a $G_n(E) \rtimes \Gamma$-stable lattice in $\sigma_E$. Then, it follows from Remark 7.1 and the isomorphism (7.1) that $\hat{H}^1(L) = 0$. Suppose the result is true for all $Z(\Delta)$’s where the length of $\Delta$ is strictly less than $k$. Recall
that the length of $\Delta_E^-$ is $k - 1$. By induction hypothesis, we have $\hat{H}^1(L^-) = 0$. Then, using (7.1), we get that $\hat{H}^1(L) = 0$.

7.4. We now recall the mod-$l$ reduction of the representation $Z(\Delta_F)$. Let us introduce the following notations:

$$r_l(\Delta_F) = \{r_l(\sigma_F, \tau_l(\sigma_F))\}$$

and

$$r_l(\Delta_F)^{(l)} = \{r_l(\sigma_F)^{(l)}, (\tau_l(\sigma_F))^{(l)}, \ldots, (\tau_l(\sigma_F))^{k-1(l)}\},$$

where $r_l(\sigma_F)$ is the mod-$l$ reduction of $\sigma_F$ and $r_l(\sigma_F)^{(l)}$ is the Frobenius twist of $r_l(\sigma_F)$. Then the mod-$l$ reduction of $Z(\Delta_F)$ ([MS14, Theorem 9.39]) is given by

$$r_l(Z(\Delta_F)) = Z(r_l(\Delta_F)).$$

This shows, in particular, that $r_l(Z(\Delta_F))$ is irreducible. Moreover, the Frobenius twist of $r_l(Z(\Delta_F))$ equals $Z(r_l(\Delta_F)^{(l)})$. We conclude this section with the following theorem.

**Theorem 7.3.** Let $E/F$ be a finite Galois extension with $[E:F] = l$, where $l$ and $p$ are distinct primes such that $l$ does not divide $n$ and $|G_{n-1}(\mathbb{F}_q)|$. Let $\sigma_F$ be an integral cuspidal $l$-adic representation of $G_m(F)$, and let $\sigma_E$ be an integral $l$-adic representation of $G_m(E)$ obtained as a base change of $\sigma_F$ (Note that $\sigma_E$ is also cuspidal). Let $\Delta_F = \{\sigma_F, \tau_F, \nu_F, \ldots, \nu_F^{k-1}\}$ and $\Delta_E = \{\sigma_E, \tau_E, \nu_E, \ldots, \nu_E^{k-1}\}$ be two segments (Here $n = km$). Then we have

$$\hat{H}^0(r_l(Z(\Delta_E))) \simeq r_l(Z(\Delta_F)^{(l)}).$$

**Proof.** We use induction on $l(\Delta_F)$. For $k = 1$, we have $Z(\Delta_E) = \sigma_E$ and $Z(\Delta_F) = \sigma_F$, and the theorem follows from Corollary 6.9. Suppose the result is true for all segments $\Delta_F'$ and $\Delta_E'$ with $l(\Delta_F') = l(\Delta_E') < k$. Let $\tau_F$ and $\tau_E$ be the mod-$l$ Zelevinsky representations $Z(r_l(\Delta_F))$ and $Z(r_l(\Delta_E))$, respectively. We denote by $\tau_F^+$ and $\tau_E^+$ the mod-$l$ representations $Z(r_l(\Delta_F))$ and $Z(r_l(\Delta_E))$ respectively. Since the restriction $\tau_E|_{P_n(E)}$ is isomorphic to $(\Phi^+)^{m-1} \circ \Psi^+(\tau_E^+)$, it follows from Proposition 5.1 that

$$\hat{H}^0(\tau_E|_{P_n(E)}) \simeq (\Phi^+)^{m-1} \circ \Psi^+(\hat{H}^0(\tau_E^+)). \quad (7.2)$$

By induction hypothesis, we have

$$\hat{H}^0(\tau_E^+) \simeq (\tau_E^+)^{(l)}. \quad (7.3)$$

Thus it follows from (7.2) and [Vig96, Chapter 3, 1.5] that $\hat{H}^0(\tau_E)$ is an irreducible representation of $P_n(F)$ and hence irreducible as a representation of $G_n(F)$. Let $\lambda = (m, m, \ldots, m)$ be the partition of $n$ and let $P_\lambda = M_\lambda N_\lambda$ be the parabolic subgroup of $G_n$. The isomorphism implies that $\hat{H}^0(\tau_E)|_{N\lambda(F)}$ is non-zero. Then, using Lemma 6.2 and Corollary 6.9, we get the following isomorphism of $M_\lambda(F)$-representations

$$\hat{H}^0(\tau_E)|_{N_\lambda(F)} \simeq ((\tau_E^+)^{(l)})|_{N_\lambda(F)}. \quad (7.4)$$

The irreducibility of $\hat{H}^0(\tau_E)$ and the isomorphism (7.3) implies that

$$\hat{H}^0(\tau_E) \simeq (\tau_E^+)^{(l)}$$

as a representation of $G_n(F)$ (see [Vig98, Proposition V.9.1]).

8. Irreducibility of Tate Cohomology of Generic Representations

In this section, we discuss the Tate cohomology groups of representations of the form $L(\Delta)$, where $L(\Delta)$ is defined in subsection (2.7.2). We assume that $l$ does not divide the pro-order of $G_n(F)$. We continue with the notation that $\sigma_F$ is an $l$-adic cuspidal representation of $G_n(F)$ and $\sigma_E$ is the base change lift of $\pi_F$ to $G_n(E)$. \hfill \Box
8.1. Keep the notations as in subsection (7.2). Recall that \( L(\Delta_E) \) is the unique generic quotient of the parabolically induced representation \( \sigma_E \times \sigma_E \nu^E \times \cdots \times \sigma_E \nu^{E,k-1}_E \). Now fix a \( G_n(E) \)-invariant lattice \( \mathcal{L}_0 \) in \( \sigma_E \). Then we have the \( G_n(E) \)-invariant lattice \( \mathcal{L}_0 \times \cdots \times \mathcal{L}_0 \) in \( \sigma_E \times \cdots \times \sigma_E \nu^{E,k-1}_E \), and the image of \( \mathcal{L}_0 \times \cdots \times \mathcal{L}_0 \) under the surjection

\[
\sigma_E \times \sigma_E \nu^E \times \cdots \times \sigma_E \nu^{E,k-1}_E \rightarrow L(\Delta_E),
\]

say \( \mathcal{L} \), is again a \( G_n(E) \)-invariant lattice in \( L(\Delta_E) \). As in subsection (7.3), an isomorphism between \( \sigma_E \) and \( \sigma_E^\gamma \) induces an isomorphism \( T_\gamma : L(\Delta_E) \rightarrow L(\Delta_E)^\gamma \) with \( T_\gamma = \text{id} \) and \( T_\gamma(\mathcal{L}) = \mathcal{L} \). Here, the group \( \Gamma \) acts on the lattice \( \mathcal{L} \) by \( T_\gamma \).

**Proposition 8.1.** Let \( \mathcal{L} \) be a lattice in \( L(\Delta_E) \) that is stable under the action of \( G_n(E) \) and \( T_\gamma \). Then \( \hat{H}^1(\mathcal{L}) = 0 \).

**Proof.** We proceed by induction on \( \ell(\Delta_E) \), which equals \( k \). When \( \ell(\Delta_E) = 1 \), then \( L(\Delta_E) = \sigma_E \). In this case, the proposition follows from [Ron16, Theorem 6]. Suppose the result is true for all representations \( L(\Delta) \), where \( \ell(\Delta) \) is strictly less than \( k \). Let \( \tau \) be the restriction \( \text{res}_{P_n(E)}(L(\Delta_E)) \). Consider the filtration of \( P_n(E) \)-representations:

\[
(0) \subseteq \tau_n \subseteq \cdots \subseteq \tau_2 \subseteq \tau_1 = \tau,
\]

where \( \tau_i/\tau_{i+1} = (\Phi^+)^{i-1} \circ \psi^+(\tau^{(i)}) \) and \( \tau^{(i)} = \psi^-(\Phi^-)^{i-1} \). The map \( T_\gamma \) induces an isomorphism between \( \tau^{(i)} \) and \( (\tau^{(i)})^\gamma \), and also between the representations \( \tau_i \) and \( \tau_i^\gamma \). Hence, there is an action of \( \Gamma \) on both \( \tau^{(i)} \) and \( \tau_i \). From [Zel80, Proposition 9.6], we get that

\[
\tau^{(j)} = 0, \text{ if } j \text{ is not divisible by } m, \quad \text{and} \quad \tau^{(rm)} = L(\{ \sigma_E \nu^{E}_1, \ldots, \sigma_E \nu^{E,k-1}_E \}), \text{ for } r = 0, 1, \ldots, k - 1.
\]

For each \( r \in \{1, 2, \ldots, k - 1\} \), let \( \Delta'_E \) and \( \Delta''_E \) be the segments \( \{ \sigma_E \nu^{E}_1, \ldots, \sigma_E \nu^{E,k-1}_E \} \) and \( \{ \sigma_E, \ldots, \sigma_E \nu^{E-1}_E \} \) respectively. The \( rm \)-th derivative of \( L(\Delta_E) \) is the composition of the following maps:

\[
\tau \xrightarrow{r_{N_n-\text{rm}}(E)} L(\Delta'_E) \otimes L(\Delta''_E) \xrightarrow{\text{id} \otimes r_{N_n-\text{rm}}(E) \otimes E} \tau^{(rm)},
\]

where \( r_{N_n-\text{rm}}(E) \) and \( \text{id} \otimes r_{N_n-\text{rm}}(E) \otimes E \) are the quotient maps of the corresponding Jacquet module and twisted Jacquet module. Note that the maps \( r_{N_n-\text{rm}}(E) \) and \( r_{N_n-\text{rm}}(E) \otimes E \) preserve integral structures (see [Dat05, Proposition 1.4(i)] and [Vig04, Theorem III.2]). Thus, we get that \( \mathcal{L}^{(rm)} \), defined as

\[
\mathcal{L}^{(rm)} = \psi^- \circ (\Phi^-)^{rm-1}(\mathcal{L}),
\]

is a \( G_{n-\text{rm}}(E) \times \Gamma \)-invariant lattice in \( L(\Delta_E)^{\text{rm}} \). Let \( \mathcal{L}_i \subset \tau_i \) be the \( P_n(E) \times \Gamma \)-invariant \( \mathbb{Z}_l \)-lattice

\[
\mathcal{L}_i = (\Phi^+)^{i-1} \circ (\Phi^-)^{i-1}(\mathcal{L}),
\]

for all \( 1 \leq i \leq n \). For \( 1 \leq r \leq k - 1 \), we have the short exact sequence of \( P_n(E) \times \Gamma \)-modules

\[
0 \rightarrow \mathcal{L}_{(r+1)m} \rightarrow \mathcal{L}_{rm} \rightarrow (\Phi^+)^{rm-1} \circ \psi^+(\mathcal{L}^{(rm)}) \rightarrow 0 \quad (8.1)
\]

By induction hypothesis, we have \( \hat{H}^1(\mathcal{L}^{(rm)}) = 0 \). Then the long exact sequence of Tate cohomology corresponding to (8.1) gives

\[
\cdots \rightarrow \hat{H}^1(\mathcal{L}_{(r+1)m}) \rightarrow \hat{H}^1(\mathcal{L}_{rm}) \rightarrow 0 \rightarrow \hat{H}^0(\mathcal{L}_{(r+1)m}) \rightarrow \cdots
\]

For \( r = k - 1 \), the representation \( \tau_k \), is equal to \( \text{ind}_{N_{n-k}}^{P_n(E)} \theta_E \). In this case, \( \hat{H}^1(\mathcal{L}_n) = 0 \) by Proposition 5.1. Then, from the above long exact sequence, we get that \( \hat{H}^1(\mathcal{L}_{(k-1)m}) = 0 \). Again using the above long exact sequence for \( r = k - 2 \), we get that \( \hat{H}^1(\mathcal{L}_{(k-2)m}) = 0 \). Thus, an inductive process gives

\[
\hat{H}^1(\mathcal{L}) = \hat{H}^1(\mathcal{L}_m) = 0.
\]

\( \square \)
8.2. Let \( \pi_E \) be a generic, integral \( l \)-adic representation of \( G_n(E) \). Then \( \pi_E \) is of the form

\[
\mathcal{L}(\Delta_1) \times \mathcal{L}(\Delta_2) \times \cdots \times \mathcal{L}(\Delta_t),
\]

where for each \( j \in \{1, 2, \ldots, t\} \), the representation \( \mathcal{L}(\Delta_j) \) is integral. Let \( \mathcal{L}_j \) be a lattice in \( L(\Delta_j) \), defined as in subsection (8.1). Let \( T_{\gamma,j} \) be the isomorphism between \( L(\Delta_j) \) and \( L(\Delta_j)^{\gamma} \) such that \( T_{\gamma,j}(\mathcal{L}_j) = \mathcal{L}_j \). Now consider the \( \mathbb{Z}_l \)-module \( \mathcal{L} = \mathcal{L}_1 \times \cdots \times \mathcal{L}_t \). Then \( \mathcal{L} \) is a lattice in \( \pi_E \) that is stable under the action of \( G_n(E) \). Moreover, we have an isomorphism \( T_\gamma : \pi_E \to \pi_E^{\gamma} \), induced by \( \{ T_{\gamma,j} \}_{j=1}^t \), such that \( T_\gamma(\mathcal{L}) = \mathcal{L} \). Note that \( l \) is banal for \( G_n(E) \). Since the mod-\( l \) reduction of \( \pi_E \) is irreducible, any lattice in \( \pi_E \) is homothetic to \( \mathcal{L} \).

**Corollary 8.2.** Assume that \( l \) does not divide \( |G_n(\mathbb{F}_q)| \). Let \( \pi_E \) be a generic, integral \( l \)-adic representation of \( G_n(E) \) as above. Let \( \mathcal{L} \) be a lattice in \( \pi_E \) that is stable under the action of \( G_n(E) \) and \( T_\gamma \). Then \( \hat{H}^1(\mathcal{L}) = 0 \).

**Proof.** Using Proposition 5.1, we have

\[
\hat{H}^1(\mathcal{L}) = \hat{H}^1(\mathcal{L}_1) \times \cdots \times \hat{H}^1(\mathcal{L}_t).
\]

Now applying Proposition 8.1, we get that \( \hat{H}^1(\mathcal{L}_i) = 0 \), for each \( i \). Hence, the theorem. \( \square \)

**Theorem 8.3.** Let \( E/F \) be a finite Galois extension with \( [E:F] = l \), where \( l \) and \( p \) are distinct primes such that \( l \) does not divide \( |G_n(\mathbb{F}_q)| \). Let \( \sigma_F \) be an integral cuspidal \( l \)-adic representation of \( G_m(F) \), and let \( \sigma_E \) be an integral cuspidal \( l \)-adic representation of \( G_m(E) \) obtained as a base change of \( \sigma_F \). Let \( \Delta_F = \{ \sigma_F, \sigma_{F^\ell}, \ldots, \sigma_{F^{\ell^k-1}} \} \) and \( \Delta_E = \{ \sigma_E, \sigma_{E^\ell}, \ldots, \sigma_{E^{\ell^k-1}} \} \) be two segments (Here \( n = km \)). Then

\[
\hat{H}^0(r_l(L(\Delta_E))) \simeq r_l(L(\Delta_F))^{(l)}.
\]

**Proof.** We prove the theorem using induction on \( l(\Delta_F) \). Since \( l \) does not divide \( |G_n(\mathbb{F}_q)| \), the mod-\( l \) reduction of the irreducible integral representations \( L(\Delta_F) \) and \( L(\Delta_E) \) are also irreducible, and we have

\[
\hat{r}_l(L(\Delta_F)) = \hat{r}_l(L(\Delta_E))
\]

and

\[
\hat{r}_l(L(\Delta_E)) = \hat{r}_l(L(\Delta_F))
\]

where \( \hat{r}_l(\Delta) \)'s are defined as in subsection (7.3). Using the long exact sequence in Tate cohomology for the exact sequence (8.1) we get a filtration

\[
\text{res}_{P_n(F)} \hat{H}^0(\hat{r}_l(L(\Delta_E))) = \eta_1 \supseteq \eta_2 \supseteq \cdots \supseteq \eta_n,
\]

such that \( \eta_i/\eta_{i+1} \neq 0 \) if and only if \( i \) is a multiple of \( m \). By induction hypothesis, we get that \( \eta_ms/\eta_{m(s+1)} \) is an irreducible representation of \( P_n(F) \). Theorem 6.7 says that the Frobenius twist \( \hat{r}_l(L(\Delta_F))^{(l)} \) is the unique generic sub-quotient of \( \hat{H}^0(\hat{r}_l(L(\Delta_E))) \). Since the lengths of \( P_n(F) \) representations \( \hat{H}^0(\hat{r}_l(L(\Delta_E))) \) and \( \hat{r}_l(L(\Delta_F)) \) are the same, we get that \( \hat{r}_l(L(\Delta_F))^{(l)} \) is isomorphic to \( \hat{H}^0(\hat{r}_l(L(\Delta_E))) \). \( \square \)

Let us continue with the hypothesis as in Theorem 8.3. Let \( \pi \) be an integral \( l \)-adic generic smooth representation of \( G_n(E) \). Since \( \pi \) does not divide \( |G_n(\mathbb{F}_q)| \), the mod-\( l \)-reduction \( \hat{r}_l(\pi) \) is irreducible, and hence generic. Then we have

**Corollary 8.4.** Let \( E/F \) be a finite Galois extension with \( [E:F] = l \), where \( l \) and \( p \) are distinct primes such that \( l \) does not divide \( |G_n(\mathbb{F}_q)| \). Let \( \pi_F \) be an integral \( l \)-adic generic representation of \( G_n(E) \), and let \( \pi_E \) be a base change lift of \( \pi_F \) (Note that \( \pi_E \simeq \pi_F^\ell \)). Then

\[
\hat{H}^0(\hat{r}_l(\pi_E)) \simeq \hat{r}_l(\pi_F)^{(l)}.
\]

**Proof.** This follows from Proposition 5.1 and Theorem 8.3. \( \square \)
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