FINITE ORBIT DECOMPOSITION OF REAL FLAG MANIFOLDS

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Abstract. Let $G$ be a connected real semi-simple Lie group and $H$ a closed connected subgroup. Let $P$ be a minimal parabolic subgroup of $G$. It is shown that $H$ has an open orbit on the flag manifold $G/P$ if and only if it has finitely many orbits on $G/P$. This confirms a conjecture by T. Matsuki.

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Date: July 9, 2013.

2000 Mathematics Subject Classification. 22F30, 14M17, 22E46.

Key words and phrases. Flag manifold, orbit decomposition, spherical subgroup.
1. Introduction

Let $G$ be a connected real semi-simple Lie group and $P$ a minimal parabolic subgroup. Let $H < G$ be a closed and connected subgroup. The following theorem was conjectured by T. Matsuki in [12].

**Theorem 1.1.** If there exists an open $H$-orbit on the real flag variety $G/P$ then the double coset space $H \backslash G/P$ is finite.

The purpose of this paper is to give a proof of Matsuki’s conjecture. Note that the converse statement is easy: if $H \backslash G/P$ is finite, then at least one double coset must be open as a consequence of the Baire category theorem. Further we remark that the theorem becomes false if the parabolic subgroup $P$ is not minimal. A standard counterexample is $G = \text{SL}(3, \mathbb{R})$ with $P$ a maximal parabolic and $H$ the unipotent part of $P$.

In case $G$ is a complex algebraic reductive group, the minimal parabolic $P$ equals a Borel subgroup $B$ of $G$. A complex algebraic subgroup which has an open orbit on $G/B$ is called spherical. In this case the finiteness of $H \backslash G/B$ for a spherical subgroup $H$ is a result of Brion [3] and Vinberg [13] with a simplified proof by Knop [9]. The spherical subgroups of a complex algebraic group have been classified by Krämer [10] and Brion [4], but to our knowledge there exists no such classification for $G$ real.

For $G$ real and $H$ a symmetric subgroup (that is, it is the identity component of the set of fixed points for an involution), it was shown by Wolf [14] that the conclusion (and hence the assumption) of Theorem 1.1 is always fulfilled.

Our proof of Theorem 1.1 proceeds in two steps. In the first step we reduce the assertion to the case where the real rank of $G$ is one. For rank one groups we then treat the cases where $H$ is reductive or non-reductive in $G$ separately. In case $H$ is non-reductive, one shows that $H$ is contained in a conjugate of $P$ and that there are 2, 3 or 4 $H$-orbits on $G/P$. For reductive $H$ we prove a refined statement:

**Theorem 1.2.** Suppose that $G$ is of real rank one and that $H$ is a connected reductive subgroup with an open orbit on $G/P$. Then there is a symmetric subgroup $H' \supset H$ such that the $H'$-orbit decomposition of $G/P$ equals the $H$-orbit decomposition.

This concludes the proof of Theorem 1.1 since as mentioned above, $H \backslash G/P$ is finite for all symmetric subgroups of $G$.

Finally we remark that the conclusion of Theorem 1.2 is false in higher real rank. For example, $H = \text{SL}(2, \mathbb{R})$ diagonally embedded in the triple product $G = \text{SL}(2, \mathbb{R})^3$ admits an open orbit in $G/P$ (see [7]).
Let $P = P_1^3$, where $P_1$ is a parabolic subgroup of $\text{SL}(2, \mathbb{R})$, then the $H$-orbit through the origin of $G/P$ is one-dimensional. On the other hand, the proper symmetric subgroups containing $H$ have the form $H' = \text{SL}(2, \mathbb{R})^2$, embedded by $(x, y) \mapsto (x, x, y)$ up to permutation, and for these groups the orbit through the origin is two-dimensional.

2. Reduction to the rank one case

Let us call the pair $(G, H)$ real spherical provided there are open $H$-orbits on $G/P$. This means that the corresponding infinitesimal pair $(\mathfrak{g}, \mathfrak{h})$ of Lie algebras satisfies $\mathfrak{g} = \mathfrak{h} + \text{Ad}(x)\mathfrak{p}$ for some $x \in G$. Since all our groups will be real, we will just say spherical. As customary we denote Lie subgroups of $G$ by upper case roman letters and their Lie algebras by the corresponding lower case German letters.

Let $(G, H)$ be a spherical pair. Matsuki remarked in [12], p. 813, that Theorem 1.1 holds true provided it is valid for all spherical pairs $(G, H)$ where $G$ is a semisimple Lie group of real rank one. The purpose of this section is to provide a proof of this remark. We follow closely the proof of Theorem 4 in [12].

We assume that the assertion of Theorem 1.1 holds for all rank one groups. Let $(G, H)$ be a spherical pair, then there exists an open orbit in $G/P$, say $Hx_0P$. We fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$, with corresponding involution $\theta$, and a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{s}$. We assume that $P \supset A$ and denote by $\Pi \subset \mathfrak{a}^*$ the set of simple roots attached to $P$. For $\alpha \in \Pi$ we define the parabolic subgroup $P_\alpha := Ps_\alpha P \cup P$ where $s_\alpha \in G$ is a Weyl group representative of the reflection associated to $\alpha$, and write $P_\alpha = L_\alpha U_\alpha$ for its Levi decomposition relative to $A$. Then $L_\alpha$ has real rank one. Write $G = P_{\alpha_1} \cdots P_{\alpha_n}$ as a product of such parabolics. Set

$$P^i := P_{\alpha_1} \cdots P_{\alpha_i} \quad (0 \leq i \leq n)$$

with the convention that $P^0 = P$. We will prove by induction on $i$ that $H \setminus Hx_0P^i/P$ is finite. The theorem is reached after $n$ steps.

The case of $i = 0$ is clear. Assume that

$$(2.1) \quad Hx_0P^i = Hg_1P \cup \ldots \cup Hg_kP$$

for elements $g_1, \ldots, g_k \in G$ and let $\alpha = \alpha_{i+1} \in \Pi$. Then it is sufficient to show for any $g \in Hx_0P^i$ that $HgP_\alpha$ breaks into finitely many $H \times P$-orbits.

We shall first prove that there exists a relatively open $H \times P$-orbit in $HgP_\alpha$. More precisely, we will show that for some $r = 1, \ldots, k$ we have $Hg_rP \subset HgP_\alpha$ open.
Note that $p_\alpha = p + g^{-\alpha} + g^{-2\alpha}$ and set $V := \exp(g^{-\alpha} + g^{-2\alpha})$. Then $VP$ is an open neighborhood of $1$ in $P_\alpha$. As $Hx_0P^i$ is open and contains $g$, we obtain an open subset $O \subset V$ with $1 \in O$ such that $gx \in Hx_0P^i$ for all $x \in O$. Moreover, because of (2.1) we have $O = \bigcup_{r=1}^k O_r$ with

$$O_r = \{x \in O \mid gx \in Hg_rP\},$$

and at least one of these sets has non-empty interior by Baire’s theorem. Fix such an $r$, then $HgxP = Hg_rP$ for every $x \in O_r$, and as $O_rP$ has non-empty interior in $VP$ it follows that $Hg_rP$ has non-empty interior in $HgP_\alpha$, hence is an open subset by transitivity of $H \times P$.

We can now show that $HgP_\alpha$ decomposes finitely. Notice that we have $HgP_\alpha = Hg_rP_\alpha$. For simplicity we replace $H$ by $g_rHg_r^{-1}$, and claim that if $HP$ is open in $HP_\alpha$ then the latter set is a finite union of $H \times P$ orbits.

We write $p_\alpha = l_\alpha + u_\alpha$ for the Levi decomposition of the Lie algebra of $P_\alpha$. Further we denote by

$$\pi_\alpha : p_\alpha \to l_\alpha$$

the projection along $u_\alpha$ and remark that the map is a Lie algebra homomorphism. Set $h_\alpha := \pi_\alpha(p_\alpha \cap h)$. As $HP$ is open we find

$$h + p = h + p_\alpha$$

and hence $p_\alpha = (p_\alpha \cap h) + p$. In turn this implies that

$$l_\alpha = h_\alpha + (l_\alpha \cap p).$$

In other words, $(l_\alpha, h_\alpha)$ is a rank-one spherical pair. We thus get that $H_\alpha\backslash L_\alpha/(L_\alpha \cap P)$ is finite (the fact that $L_\alpha$ can be non-connected does not matter, because all its components intersect non-trivially with $P$). We write

$$L_\alpha = \bigcup_{j=1}^m H_\alpha x_j(L_\alpha \cap P)$$

and claim that

$$HP_\alpha = \bigcup_{j=1}^m Hx_jP.$$  

As $P_\alpha = L_\alpha P$, it suffices to show that $hx_j \in Hx_jP$ for all $h \in H_\alpha$ and all $j$. Note that $h_\alpha$ is contained in the subalgebra $(p_\alpha \cap h) + u_\alpha$ of $p_\alpha$, and hence $H_\alpha$ is contained in the subgroup $(P_\alpha \cap H)U_\alpha$ of $P_\alpha$. It follows that

$$hx_j \in (P_\alpha \cap H)U_\alpha x_j = (P_\alpha \cap H)x_jU_\alpha \subset Hx_jP$$

as claimed.

Hence the proof of Theorem 1.1 is reduced to the following result.
Proposition 2.1. Let $G$ be a semisimple Lie group of real rank one and $H$ a connected spherical subgroup. Then the number of $H$-orbits on $G/P$ is finite.

Example: $G = \text{SL}(2, \mathbb{R})$. Every one-dimensional subalgebra is conjugate to $\mathfrak{k}$, $\mathfrak{a}$ or $\mathfrak{n}$. The first two are symmetric, and in the third case finiteness of $H \backslash G/P$ follows from the Bruhat decomposition. Hence $H \backslash G/P$ is finite for every non-trivial connected subgroup $H$.

2.1. Simple groups. Once the real rank is one we can easily reduce to the case that $G$ is simple. Otherwise $G$ is locally isomorphic to $G_1 \times K_2$ where $G_1$ is simple of real rank one and $K_2$ is compact. Then $P = P_1 \times K_2$ where $P_1 \subset G_1$ is minimal parabolic, and hence $G/P = G_1/P_1$. Moreover, if $H_1$ denotes the projection of $H$ on $G_1$, then $H$-orbits on $G/P$ are the same as $H_1$-orbits on $G_1/P_1$.

3. Non-reductive spherical subgroups

In this section we prove Proposition 2.1 for spherical subgroups which are not reductive. The following lemma can be deduced from [6] Theorem 4.4.1 for $G$ classical and from [5] Section 8 for $G$ exceptional. The proof given below does not refer to classification.

Lemma 3.1. Let $\mathfrak{g}$ be a simple real rank one Lie algebra and $\mathfrak{h}$ a subalgebra, which is not reductive in $\mathfrak{g}$. Then, up to conjugation, $\mathfrak{h}$ is contained in $\mathfrak{p} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}$.

Proof. Recall that a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is said to be reductive in $\mathfrak{g}$ if the adjoint representation of $\mathfrak{h}$ on $\mathfrak{g}$ is completely reducible. Every subalgebra $\mathfrak{h}$ allows a decomposition $\mathfrak{h} = \mathfrak{l} \ltimes \mathfrak{n}_1$ such that $\mathfrak{l}$ is reductive in $\mathfrak{g}$ and $\mathfrak{n}_1$ an ideal which acts on $\mathfrak{g}$ nilpotently. Our assumption on $\mathfrak{h}$ is then that the ideal $\mathfrak{n}_1$ does not vanish.

Since $\text{ad}_{\mathfrak{g}}(\mathfrak{n}_1)$ is nilpotent, it follows that a $G$-conjugate of $\mathfrak{n}_1$ is contained in $\mathfrak{n}$. We may thus assume that $\mathfrak{n}_1 \subset \mathfrak{n}$. Notice that $[\mathfrak{h}, \mathfrak{n}_1] \subset \mathfrak{n}_1$. Thus it suffices to show that $\mathfrak{p}$ contains the normalizer in $\mathfrak{g}$ of any non-zero subspace $\mathfrak{n}_1 \subset \mathfrak{n}$.

Consider an element $X \in \mathfrak{g}$ which normalizes $\mathfrak{n}_1$. It has the form

$$X = X_{-2} + X_{-1} + X_0 + X_1 + X_2$$

with $X_j \in \mathfrak{g}^{j\alpha}$ for $j = -2, \ldots, 2$. Here $\alpha$ is the indivisible positive root of $\mathfrak{a}$. We need to prove that $X_j = 0$ for $j < 0$.

Let $Y = Y_1 + Y_2 \in \mathfrak{n}_1$, where $Y_1 \in \mathfrak{g}^\alpha$ and $Y_2 \in \mathfrak{g}^{2\alpha}$. From

$$[X, Y] \in \mathfrak{n}_1 \subset \mathfrak{g}^\alpha + \mathfrak{g}^{2\alpha}$$

(3.1)
we infer that

\[(3.2) \quad [X_{-2}, Y_1] = [X_{-1}, Y_1] + [X_{-2}, Y_2] = 0\]

Assume first that \(X_{-2} \neq 0\). Then \(a = \mathbb{R}[X_{-2}, \theta(X_{-2})]\) and the elements \((X_{-2}, \theta(X_{-2}), [X_{-2}, \theta(X_{-2})])\) form an \(\mathfrak{sl}(2)\)-triple. Since \(g^a\) and \(g^{2a}\) carry positive \(a\)-weights, it follows from finite dimensional \(\mathfrak{sl}(2)\)-theory that \(\text{ad}(X_{-2})\) is injective on both of these spaces. Hence \(3.2\) implies \(Y_1 = Y_2 = 0\), which contradicts that \(n_1 \neq 0\). Hence \(X_{-2} = 0\).

Assume next that \(X_{-1} \neq 0\). Then \((X_{-1}, \theta(X_{-1}), [X_{-1}, \theta(X_{-1})])\) forms an \(\mathfrak{sl}(2)\)-triple, and as before \(\text{ad}(X_{-1})\) is injective on \(g^a\) and \(g^{2a}\). With \(X_{-2} = 0\) we obtain from \((3.2)\) that \([X_{-1}, Y_1] = 0\), and hence \(Y_1 = 0\). We conclude that \(n_1 \subset g^{2a}\). Then \((3.1)\) reads \([X, Y] \in g^{2a}\), and in addition to \((3.2)\) we obtain \([X_{-1}, Y_2] = 0\). Hence \(Y_2 = 0\), again contradicting that \(n_1 \neq 0\). Hence \(X_{-1} = 0\) as claimed. \(\square\)

For a rank one group \(G\) we let \(s\) be a Weyl group representative and recall the Bruhat decomposition \(G = P_sP \cup P\).

**Lemma 3.2.** Let \(g\) be a simple real rank one Lie algebra and \(h\) a spherical subalgebra. Suppose that \(h \subset p = m + a + n\). Then the Bruhat decomposition \(G/P = P \cup NsP\) is \(H\)-stable and there are at most four \(H\)-orbits on \(G/P\).

**Proof.** The cells of the Bruhat decomposition are \(P\)-stable, hence also \(H\)-stable. In particular, the closed cell \(P \subset G/P\) is an \(H\)-orbit. Hence by assumption the open cell \(O := NsP\) admits at least one open \(H\)-orbit, and the assertion is that then it decomposes into at most three \(H\)-orbits.

As in the proof of Lemma 3.1 we decompose \(h = l \ltimes n_1\) with \(l\) reductive in \(g\) and \(n_1 \subset n\). As \(h \subset p\) it is no loss of generality to assume that \(l \subset m + a\). Then with \(m_1 = l \cap m\) we have

\[h = m_1 + \mathbb{R}X + n_1\]

for some \(X = Y + Z\) with \(Y \in m\) and \(Z \in a\). Since \([l, l] \subset m_1\), the element \(X\) belongs to the center of \(l\) and commutes with \(m_1\). Let \(s_1 = \mathbb{R}X\), then both \(s_1\) and \(m_1\) normalize \(n_1\) and we have

\[H = M_1S_1N_1\]

for the corresponding subgroups of \(G\). Since \(M_1S_1s \subset sP\) we see that the \(H\)-orbits in \(NsP\) are the sets \(N_1c_{M_1S_1}(x)sP\) for \(x \in N\), where \(c_g(x) = gxg^{-1}\). In particular, \(N_1sP \subset NsP\) is an \(H\)-orbit. If \(N_1 = N\) we are done, hence we may assume \(n_1 \not\subset n\).
Note that $Z \neq 0$. Otherwise $X = Y$, hence $X \in \mathfrak{m}_1$ and $\mathfrak{h} = \mathfrak{m}_1 + \mathfrak{n}_1$. Then $H = M_1 N_1$ and $H$-orbits have the form $N_1 c_{M_1}(x)sP$. Since $N_1 c_{M_1}(x)$ cannot be open in $N$ for any $x$, a contradiction is reached.

The element $X$ acts semisimply on $\mathfrak{n}_C$ and preserves the subspace $\mathfrak{n}_1C$. Note that $\mathfrak{n} = \mathfrak{g}^\alpha \oplus \mathfrak{g}^{2\alpha}$ and that $[\mathfrak{n}, \mathfrak{n}] \subset \mathfrak{g}^{2\alpha}$. Since $Z \neq 0$ the spaces $\mathfrak{g}^\alpha$ and $\mathfrak{g}^{2\alpha}$ have no eigenvalues in common for $\text{ad}(X)$. It follows that

$$\tag{3.3} \mathfrak{n}_1 = (\mathfrak{n}_1 \cap \mathfrak{g}^\alpha) \oplus (\mathfrak{n}_1 \cap \mathfrak{g}^{2\alpha}).$$

Let $\mathfrak{n}_0$ be the orthogonal complement of $\mathfrak{n}_1$ in $\mathfrak{n}$, then $\mathfrak{n}_0 \neq 0$ and (3.3) implies

$$\tag{3.4} \mathfrak{n}_0 = (\mathfrak{n}_0 \cap \mathfrak{g}^\alpha) \oplus (\mathfrak{n}_0 \cap \mathfrak{g}^{2\alpha}).$$

It follows from (3.3) and (3.4) together with [8] Ch. IV, Lemma 6.8, that the exponential map induces a diffeomorphism of $\mathfrak{n}_0$ with the left coset space $N_1 \backslash N$. Note that $\mathfrak{n}_0$ is $M_1 S_1$-invariant. We conclude that the $H$-orbits in $NsP$ correspond to the orbits of $\text{Ad}(M_1 S_1)$ on $\mathfrak{n}_0$. In particular, the $H$-orbit $N_1 sP$ corresponds to $\{0\} \subset \mathfrak{n}_0$.

Since $M$ acts isometrically on $\mathfrak{n}$ it follows that $M_1$ acts isometrically on $\mathfrak{n}_0$. Furthermore, let $X = Y + Z$ be normalized such that $\alpha(Z) = 1$ and put $s_t = \exp(tX) \in S_1$, then for $j = 1, 2$

$$\tag{3.5} \| \text{Ad}(s_t)x \| = e^{jt}\|x\|, \quad x \in \mathfrak{n}_0 \cap \mathfrak{g}^{j\alpha}, t \in \mathbb{R}.$$ 

It follows that if $O_1 \neq \{0\}$ is an $\text{Ad}(M_1 S_1)$-orbit in $\mathfrak{n}_0$, then the intersection of $O_1$ with every sphere in $\mathfrak{n}_0$ is non-empty and is an $\text{Ad}(M_1)$-orbit.

Assume first that $\dim \mathfrak{n}_0 > 1$. Then spheres in $\mathfrak{n}_0$ are connected, and by compactness an open $\text{Ad}(M_1)$-orbit is the entire sphere. Hence we conclude that the open $\text{Ad}(M_1 S_1)$-orbit in $\mathfrak{n}_0$ is $\mathfrak{n}_0 \setminus \{0\}$. In this case $NsP$ decomposes in two orbits, $N_1 sP$ and its complement.

Assume finally that $\dim \mathfrak{n}_0 = 1$. In this case it follows from (3.5) that $\text{Ad}(s_t)x = e^{jt}x$ for all $x \in \mathfrak{n}_0$ and all $t \in \mathbb{R}$ (where $j = 1$ or 2). Hence in this case there are three orbits in $\mathfrak{n}_0$, corresponding to $\{0\}$ and the two components of its complement. \hfill $\square$

**Remark 3.3.** The proof shows a bit more. The open $N$-orbit $NsP$ breaks into at most three $H$-orbits. If we identify $N$ with $\mathbb{R}^n$ and $N_1$ with $\mathbb{R}^k$, then these $H$-orbits are of the following type:

1. $\mathbb{R}^n$ (one orbit, the case where $H \supset N$) when $k = n$.
2. $\mathbb{R}^k$ and $\mathbb{R}^{n-k} \setminus \{0\} \times \mathbb{R}^k$ when $0 \leq k < n - 1$.
3. $\mathbb{R}^{n-1}$, $\mathbb{R}^+ \times \mathbb{R}^{n-1}$, $\mathbb{R}^- \times \mathbb{R}^{n-1}$ when $k = n - 1$. 
4. Some results in real rank one

Lemma 4.1. Let \( g \) be a real reductive Lie algebra with Cartan involution \( \theta \), and let \( g_n \subset g \) be its maximal non-compact ideal. Let \( h \subset g \) be a \( \theta \)-stable subalgebra such that \( g = h + \mathfrak{k} \). Then \( g_n \subset h \).

Proof. It follows from the assumption that \( s \subset h \) and this implies the conclusion as \( g_n \) is generated by \( s \).

Recall that a subalgebra \( h' \subset g \) is called symmetric if it is the fixed points of an involution of \( g \). Recall also that for every involution there exists a commuting Cartan involution. Given an involution \( \sigma \), we write \( g = h' + q \) for the corresponding decomposition of \( g \).

Lemma 4.2. Let \( g \) be simple of real rank one and let \( h' \) be a proper symmetric subalgebra defined by an involution \( \sigma \) commuting with \( \theta \). Let \( p = m + a + n \) be a minimal parabolic subalgebra for which \( a \subset s \cap q \).

Then

\[ g = h' + p \quad \text{and} \quad h' \cap p \subset m. \]

Proof. This follows from [11], Theorem 3, since \( \sigma(n) = \theta(n) \).

Lemma 4.3. Let \( G \) be a simple Lie group of real rank one and let \( h \subset h' \subset g \) be reductive subalgebras such that

1. \( h \) is spherical
2. \( h' \) is symmetric and defined by an involution commuting with \( \theta \).

Then:

1. There exists a minimal parabolic subalgebra \( p \) with \( a \subset s \cap q \) such that \( h' = h + (h' \cap m) \) and

\[ H' = H(H' \cap M). \]

2. Let \( h'_n \) be the maximal non-compact ideal in \( h' \), then \( h'_n \subset h \).

Proof. Let \( p_1 \) be a minimal parabolic subalgebra for which \( h + p_1 = g \). Then \( h' + p_1 = g \) and it follows from [11], Theorems 1 and 3, that \( p_1 \) is \( H' \)-conjugate to a minimal parabolic subalgebra \( p \) with \( a \subset s \cap q \). Thus \( g = h + \text{Ad}(x)p \) for some \( x \in H' \). Since \( h \subset h' \) this implies that

\[ h' = h + h' \cap \text{Ad}(x)p = h + \text{Ad}(x)(h' \cap p). \]

From Lemma 4.2 we find \( h' \cap p = h' \cap m \), which is compact. It follows that \( H \times (H' \cap M) \) is open and closed in \( H' \), hence equal to \( H' \). Hence \( H \times (H' \cap M) \) is left\( \times \)right transitive on \( H' \), and (3) follows.

Since \( h \) is reductive in \( g \), it is reductive in \( h' \). Hence some \( H' \)-conjugate \( h_1 \) of it is \( \theta \)-stable. The conclusion in (3) is valid for \( h_1 \) and hence \( h' = h_1 + h' \cap \mathfrak{k} \). It now follows from Lemma 4.1 that \( h'_n \subset h_1 \).

Since \( h'_n \) is an ideal this implies \( h'_n \subset h \) as well.
Proposition 4.4. Let $G$ be a connected simple Lie group of real rank one and let $H \subset H'$ be connected reductive subgroups such that $H$ is spherical and $H'$ is symmetric and proper in $G$. Then $H$ is transitive on each $H'$-orbit in $G/P$.

Proof. Choose a Cartan involution which commutes with the involution which defines $H'$. Let $p$ be as in Lemma 4.3. Since the real rank of $G$ is one, it follows from Matsuki’s orbit description in [11] that $H'$ has only open and closed orbits in $G/P$. The open orbits are of the form $H'xP$ for $x \in N_K(a)$, the normalizer in $K$ of $a$, and the closed orbits are of the form $H'yP$ with $y \in K$ such that $\text{Ad}(y)(a) \subset h'$.

It follows from Lemma 4.3 that

$$H'xP = H(H' \cap M)xP = HxP$$

for $x \in N_K(a)$.

Let $h_c'$ denote the ideal in $h'$ which is complementary to $h'_n$. Then $h' = h'_n \oplus h'_c$ and $H' = H'_nH'_c$. If $\text{Ad}(y)(a) \subset h'$ then $\text{Ad}(y)(a) \subset h'_n$, and hence $\text{Ad}(y)(a)$ is centralized by $H'_c$. It follows that $H'_c \subset yMy^{-1}$ and hence

$$H'yP = H'_nyP = HyP$$

since $H'_n \subset H$ by Lemma 4.3. □

5. Example: The Lorentzian groups

Before we treat the general case, it is instructive to see the proof of Theorem 1.2 for the case of $\text{SO}_0(1, n)$ for $n \geq 2$.

Proof. We observe that $G = \text{SO}_0(1, n)$ acts on $\mathbb{R}^{n+1}$. In the sequel we write the elements of $x \in \mathbb{R}^{n+1}$ as $x = (x_0, x')$ with $x' \in \mathbb{R}^n$. The stabilizer $P \subset G$ of the line $\mathbb{R}(1, 1, 0, \ldots, 0) \in \mathbb{P}(\mathbb{R}^{n+1})$ is a minimal parabolic subgroup. Note that $G/P = S^{n-1}$ is an $n - 1$-dimensional sphere which we shall identify with the projective quadric:

$$(5.1) \quad S^{n-1} = \{ [x] \in \mathbb{P}(\mathbb{R}^{n+1}) \mid x_0^2 = ||x'||^2 = x_1^2 + \ldots + x_n^2 \}.$$

Let $h$ be a reductive spherical subalgebra, and let $h = h_n \oplus h_c$ be the decomposition of $h$ in ideals, such that $h_n$ is non-compact and $h_c$ is compact. Since $\text{so}(1, n)$ has rank one and root multiplicity $m_{2\alpha} = 0$, the same must be true for $h_n$. Hence $h_n = \text{so}(1, p)$ for some $0 \leq p \leq n$. Furthermore, by conjugation of $h$ we can arrange that $h_n = \text{so}(1, p)$ is realized in the left upper corner of $g = \text{so}(1, n)$, and accordingly:

$$(5.2) \quad h = h_n \oplus h_c, \quad H = \text{SO}_0(1, p) \times H_c$$

with $h_c \subset \text{so}(n-p)$ and $\text{so}(n-p)$ embedded in the lower right corner. It now follows from Proposition 4.4 that orbits on $G/P$ for $H$ are the
same as for the symmetric subgroup \( H' = \text{SO}_0(1, p) \times \text{SO}(n - p) \) of \( G \). Thus the proof of Theorem 1.2 is complete for this case. □

Remark 5.1. It follows from the above that every spherical subgroup in \( \text{SO}_0(1, n) \) is conjugate to a subgroup of \( H' = \text{SO}_0(1, p) \times \text{SO}(n - p) \) of the form (5.2) for some \( 0 \leq p \leq n \). Furthermore since \( H' \cap M = \text{SO}(n - p - 1) \) in this case, it follows that such a subgroup is spherical if and only if \( H_c \) is transitive on \( S^{n-p-1} = \text{SO}(n - p)/\text{SO}(n - p - 1) \). Besides \( H = H' \) this can be attained in case \( p \) satisfies certain parity conditions. A typical example is \( p = n - 2k \) and

\[
H = \text{SO}_0(1, n - 2k) \times \text{SU}(k),
\]

since \( H_c = \text{SU}(k) \) acts transitively on the spheres in \( \mathbb{R}^{2k} \). For \( p = n - 4k \) we can also take \( H_c = \text{Sp}(k) \) which again acts transitively on spheres. Besides these two series there are three exceptional cases (see [2] for the classification of transitive actions of compact Lie groups on spheres).

6. Classifications

In this section we prepare for the proof of Theorem 1.2 by recalling and applying some results which are seen from classifications of simple Lie groups and their subgroups.

We first recall the classification of the simple real rank one Lie algebras

\[
\begin{align*}
\mathfrak{so}(1, n), & \quad \mathfrak{su}(1, n), \quad \mathfrak{sp}(1, n), \quad \mathfrak{f}_4,
\end{align*}
\]

where \( \mathfrak{f}_4 = \mathfrak{f}_4(-20) \), the real form of \( \mathfrak{f}_4 \mathbb{C} \) with maximal compact \( \mathfrak{so}(9) \). In the first series \( n \) is limited to \( n \geq 2 \). In the second and third series \( n \geq 1 \) is allowed, but as \( \mathfrak{so}(1, 2) \simeq \mathfrak{su}(1, 1) \) and \( \mathfrak{so}(1, 4) \simeq \mathfrak{sp}(1, 1) \) an exhaustive list is obtained by taking \( n \geq 2 \) in all cases.

6.1. Symmetric subalgebras.

Lemma 6.1. The symmetric pairs (excluding \( \mathfrak{h} = \mathfrak{g} \) and \( \mathfrak{h} = \mathfrak{k} \)) for the simple real rank one Lie algebras are

\[
\begin{align*}
\mathfrak{g} = \mathfrak{so}(1, n), & \quad \mathfrak{h}_m = \mathfrak{so}(1, m) \times \mathfrak{so}(n - m), & 0 < m < n, \\
\mathfrak{g} = \mathfrak{su}(1, n), & \quad \mathfrak{h}_m = \mathfrak{s}(\mathfrak{u}(1, m) \times \mathfrak{u}(n - m)), & 0 < m < n, \\
& \quad \mathfrak{h} = \mathfrak{so}(1, n) \\
\mathfrak{g} = \mathfrak{sp}(1, n), & \quad \mathfrak{h}_m = \mathfrak{sp}(1, m) \times \mathfrak{sp}(n - m), & 0 < m < n, \\
& \quad \mathfrak{h} = \mathfrak{u}(1, n) \\
\mathfrak{g} = \mathfrak{f}_4, & \quad \mathfrak{h}_1 = \mathfrak{so}(1, 8), \quad \mathfrak{h}_2 = \mathfrak{sp}(1, 2) \times \mathfrak{sp}(1),
\end{align*}
\]

Proof. This is seen from Berger’s table ([1] pages 157–161). □
6.2. Maximal reductive subalgebras. We are particularly interested in reductive subalgebras which are maximal. The following lemma provides the key to the reduction to symmetric pairs.

**Lemma 6.2.** Let \( g \) be a simple Lie algebra of real rank one and let \( h \subset g \) be a maximal proper reductive subalgebra. Then either \( h \) is a symmetric subalgebra, or

1. \( g = \text{sp}(1, n) \) and \( h \) is conjugate to \( \text{so}(1, n) \times \text{sp}(1) \), where \( n > 1 \).
2. \( g = f_4 \) and \( h \) is conjugate to \( \text{su}(1, 2) \times \text{su}(3) \).
3. \( g = f_4 \) and \( h \) is conjugate to \( \text{so}(1, 2) \times g_2 \) where \( g_2 \) denotes the compact real form of \( g \).

None of the pairs in (1)-(3) are spherical.

**Proof.** It is well known that the symmetric subalgebras of a simple Lie algebra are maximal proper reductive subalgebras. Reductive subalgebras are listed in [5], pages 276 and 284, and it is easily seen from these lists together with the list in Lemma 6.1 that only the subalgebras in (1)-(3) are maximal and non-symmetric. The fact that these pairs are not spherical will be proved in the following subsections.

6.2.1. \((g, h) = (\text{sp}(1, n), \text{so}(1, n) \times \text{sp}(1))\) is not spherical when \( n > 1 \).
If \( n = 2 \) then \( \dim h = 6 \) and \( \dim(g/p) = 7 \), so we may assume \( n \geq 3 \).

Like in the Lorentzian cases we identify for \( G = \text{Sp}(1, n) \) the flag variety \( G/P = S^{4n-1} \) with a quadric in the quaternion projective space \( \mathbb{P}(V) \) where \( V = \mathbb{H}^{n+1} \).

\[ S^{4n-1} = \{ [z] \in \mathbb{P}(V) \mid |z_0|^2 = |z_1|^2 + \ldots + |z_n|^2 \} . \]

Here the action of \( G \) on \( V \) is from the left, and \[ [z] = \{ zh \mid h \in \mathbb{H}, h \neq 0 \} \in \mathbb{P}(V) . \]

As a representation for \( \text{SO}_0(1, n) \), the space \( V \) decomposes in four copies of \( V_0 = \mathbb{R}^{n+1} \) with standard action. Hence the stabilizer in \( H_n = \text{SO}_0(1, n) \) of an element \( v \in V \) is the stabilizer of four elements in \( V_0 \), hence the centralizer of an at most four-dimensional subspace in \( V_0 \). The centralizer in \( \text{SO}_0(1, n) \) of a four dimensional subspace of \( \mathbb{R}^{n+1} \) is conjugate to the centralizer in \( \text{SO}_0(1, n-3) \) of a one dimensional subspace of \( \mathbb{R}^{n-2} \). Since all non-trivial orbits of \( \text{SO}_0(1, n-3) \) in \( \mathbb{R}^{n-2} \) have codimension one, a simple computation shows that the codimension in \( \text{SO}_0(1, n) \) of such a subgroup is \( 4n - 6 \). Hence orbits of \( H_n \) in \( V \) are at most of this dimension and orbits of \( H \) in \( S^{4n-1} \) are at most of dimension \( 4n - 3 \).
6.2.2. \((f_4, \mathfrak{so}(1,2) \times \mathfrak{g}_2)\) is not spherical. Since \(\dim G/P = 15\), it suffices to show that the subgroup \(G_2 \subset K\) with Lie algebra \(\mathfrak{g}_2\) has orbits in \(K/M = G/P\) of dimension at most 11. Recall that \(K = \text{Spin}(9)\) and that we can realize \(K/M\) as the unit sphere in the 16-dimensional real spin representation \(V_{16}\) of \(K\). This representation decomposes for the standard inclusions \(\text{Spin}(7) \subset \text{Spin}(8) \subset \text{Spin}(9)\) into a direct sum of two copies of the spin representation \(V_8\) of \(\text{Spin}(7)\). Now \(G_2\) is the isotropy subgroup of a spinor in \(V_8\), and hence as a \(G_2\)-representation

\[
V_{16} = V_7 \oplus V_7 \oplus \mathbb{R} \oplus \mathbb{R},
\]

with a 7-dimensional representation of \(G_2\). It follows that every orbit of \(G_2\) lies in a product \(K_1S^6 \times K_2S^6 \subset V_7 \oplus V_7\) of spheres of radii \(K_1, K_2 \geq 0\). Furthermore, the action of \(G_2\) on \(S^6 \times S^6\) is not transitive as the diagonal is invariant. Since \(G_2\) is compact, we conclude that there are no open orbits on \(S^6 \times S^6\). This proves the claim.

6.2.3. \((f_4, \mathfrak{su}(2,1) \times \mathfrak{su}(3))\) is not spherical. Note that \(\dim H = 16\) and \(\dim G/P = 15\). Let us first collect a few facts about \(f_4\). We refer to [5] for more details. Consider the Jordan algebra

\[
(6.2) \quad \text{Herm}(3, \mathbb{O})_{2,1} = \left\{ x = \begin{pmatrix} \alpha_1 & c_3 & -\bar{c}_2 \\ \bar{c}_3 & \alpha_2 & c_1 \\ c_2 & -\bar{c}_1 & \alpha_3 \end{pmatrix} \mid \alpha_i \in \mathbb{R}, c_i \in \mathbb{O} \right\}.
\]

The group \(G\) of automorphisms of \(W := \text{Herm}(3, \mathbb{O})_{2,1}\) is a real Lie group with Lie algebra \(f_4\). Moreover the trace free elements \(V := W_{\text{tr}=0}\) is an irreducible real representation for \(G\) with a non-zero \(K\)-fixed vector. Let \(v_0 \in V\) be a highest weight vector, then \(P \cdot v_0 = \mathbb{R}^+ v_0\) and we can realize the flag manifold as the image of \(G \cdot v_0\) in \(\mathbb{P}(V)\). According to [5], p. 275, \(\mathbb{R}^+ G \cdot v_0 = \mathcal{C}\), where

\[
\mathcal{C} := \{ x \in V \mid x^2 = 0, x \neq 0 \},
\]

and thus \(G/P = \mathbb{P}(\mathcal{C})\).

Note that \(H = \text{SU}(2,1) \times \text{SU}(3)\) acts naturally on \(V\). The factor \(H_n = \text{SU}(2,1)\) acts by matrix conjugation. Further, the automorphism group of \(\mathbb{O}\) is \(G_2\) and \(H_c = \text{SU}(3)\) is the subgroup which commutes with complex multiplication on \(\mathbb{O}\). We will show that every element \([x] \in \mathbb{P}(\mathcal{C})\) has an at least 2-dimensional stabilizer in \(H\).

A straightforward matrix computation shows that if \(x\) in (6.2) satisfies \(x^2 = 0\) and has trace zero, then up to multiplication by a real number

\[
(6.3) \quad x = \begin{pmatrix} |c_2|^2 & -\bar{c}_2 \bar{c}_1 & -\bar{c}_2 \\ c_1 c_2 & |c_1|^2 & c_1 \\ c_2 & -\bar{c}_1 & -1 \end{pmatrix}
\]
with $|c_1|^2 + |c_2|^2 = 1$.

In the sequel we decompose $\mathcal{O} = \mathbb{C} + \mathbb{C}^\perp$ and regard $\mathcal{O}_I := \mathbb{C}^\perp$ as a complex vector space for the left action of $\mathbb{C}$. Then as a module for $\text{SU}(3)$ it is equivalent with the standard complex representation on $\mathbb{C}^3$.

Having said that we write the elements $x \in V$ as

$$x = x_C + x_I$$

where $x_C \in i\text{su}(2,1) \subset V$ and $x_I$ is of the form

$$(6.4) \quad x_I = \begin{pmatrix} 0 & c_3 & -\bar{c}_2 \\ \bar{c}_3 & 0 & c_1 \\ c_2 & -\bar{c}_1 & 0 \end{pmatrix}$$

with $c_1, c_2, c_3 \in \mathcal{O}_I$. Note that this gives us a decomposition of $H$-modules.

We see from (6.3) that $x_C \neq 0$ for all $x \in \mathcal{C}$. Hence the map

$$\mathbb{P}(\mathcal{C}) \to \mathbb{P}(i\text{su}(2,1)), \ [x] \mapsto [x_C]$$

is defined. As this is an open map, the image of an open $H$-orbit will be a non-empty open set. Since the semisimple elements in $i\text{su}(2,1)$ are dense, it suffices to consider $x$ in (6.3) with $x_C$ semisimple. If $x_C \in i\text{su}(2,1)$ is semisimple it is $\text{SU}(2,1)$-conjugate to one of the following

$$\begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & -\alpha_1 - \alpha_2 \end{pmatrix}, \quad \begin{pmatrix} 2\alpha & 0 & 0 \\ 0 & -\alpha & \gamma i \\ 0 & \gamma i & -\alpha \end{pmatrix},$$

where $\alpha_1, \alpha_2, \alpha, \gamma \in \mathbb{R}$. However, the second case does not conform with (6.3). Hence we may assume that $x = x_C + x_I$ with $x_C$ diagonal and with $x_I$ as in (6.4). It follows from (6.3) that $c_1c_2 = \bar{c}_3$.

The fact that $c_1c_2 \in \mathcal{O}_I$ implies that $c_1$ and $c_2$ are orthogonal elements. Let $\mathcal{O}_I = \mathbb{C}j \oplus \mathbb{C}l \oplus \mathbb{C}n$ in the standard notation. After application of $\text{SU}(3)$ to $x_I$, it is no loss of generality to assume that $c_1 = aj$ and $c_2 = bl$ for some $a, b \in \mathbb{R}$. Then $c_3 = -abn$ and

$$x_I = \begin{pmatrix} 0 & -abn & bl \\ abn & 0 & aj \\ bl & aj & 0 \end{pmatrix}.$$ 

The diagonal torus $T < \text{SU}(2,1)$ commutes with the diagonal matrix $x_C$, and embedded into $H$ via

$$t = \text{diag}(t_1, t_2, t_3) \mapsto (t, t)$$

it also stabilizes $x_I$ – note that for $z \in \mathbb{C}$ and $x \in \mathcal{O}_I$ one has $xz = \bar{z}x$. Hence the stabilizer of $x$ in $H$ has dimension at least 2.

This concludes the proof of Lemma 6.2. \qed
7. Proofs

All ingredients for the proofs have already been prepared.

7.1. Proof of Theorem 1.2. Let \( G \) be semisimple of real rank one and \( H \subset G \) a connected reductive spherical subgroup. As seen in Section 2.1 we may assume \( G \) is simple.

Let \( \mathfrak{h}' \) be a maximal proper reductive subalgebra which contains \( \mathfrak{h} \). It follows from Lemma 6.2 that \( \mathfrak{h}' \) is symmetric, and then it follows from Proposition 4.4 that \( H \)-orbits and \( H' \)-orbits agree on \( G/P \).

7.2. Proof of Proposition 2.1 and Theorem 1.1. Theorem 1.2 implies the statement of Proposition 2.1 for reductive subgroups by the results of [14] or [11]. By combining with Lemmas 3.1 and 3.2 we obtain the proposition for general subgroups. This also concludes the proof of Theorem 1.1.

References

[1] M. Berger, Les espaces symétriques noncompacts. Ann. Sci. École Norm. Sup. (3) 74 (1957), 85–177.
[2] A. Borel, Le plan projectif des octaves et les sphères comme espaces homogènes, C. R. Acad. Sci. Paris 230 (1950), 1378–1380.
[3] M. Brion, Quelques propriétés des espaces homogènes sphériques, Manuscripta Math. 55 (1986), 191–198.
[4] M. Brion, Classification des espaces homogènes sphériques, Compositio Math. 63 (1987), 189–208.
[5] S. Chen, On subgroups of the noncompact real exceptional Lie group \( F_4^* \), Math. Ann. 204, 271–284 (1973).
[6] S. Chen and L. Greenberg, Hyperbolic spaces. Contributions to analysis (a collection of papers dedicated to Lipman Bers), pp. 49–87. Academic Press, New York, 1974.
[7] T. Danielsen, B. Krötz and H. Schlichtkrull, Decomposition theorems for triple spaces, arXiv:1301.0489
[8] S. Helgason, Groups and Geometric Analysis, Academic Press 1984.
[9] F. Knop, On the set of orbits for a Borel subgroup, Comment. Math. Helv. 70 (1995), 285–309.
[10] M. Krämer, Sphärische Untergruppen in kompakten zusammenhängenden Gruppen, Compositio Math. 38 (2) (1979), 129–153.
[11] T. Matsuki, The orbits of affine symmetric spaces under the action of minimal parabolic subgroups, J. Math. Soc. Japan 31 (1979), no. 2, 331–357.
[12] , Orbits on flag manifolds, Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990), 807–813, Math. Soc. Japan, Tokyo, 1991.
[13] E.B. Vinberg, Complexity of actions of reductive groups, Funktsional. Anal. i Prilozhen. 20 (1986), 1–13, 96.
[14] J. Wolf, Finiteness of orbit structure for real flag manifolds, Geom. Dedicata 3 (1974), 377–384.