The Penrose inequality for asymptotically locally hyperbolic spaces with nonpositive mass

Dan A. Lee  André Neves

October 14, 2013

Abstract

In the asymptotically locally hyperbolic setting it is possible to have metrics with scalar curvature $\geq -6$ and negative mass when the genus of the conformal boundary at infinity is positive. Using inverse mean curvature flow, we prove a Penrose inequality for these negative mass metrics. The motivation comes from a previous result of P. Chruściel and W. Simon, which states that the Penrose inequality we prove implies a static uniqueness theorem for negative mass Kottler metrics.

1 Introduction

The Penrose inequality for asymptotically flat 3-manifolds $M$ with mass $m$ and nonnegative scalar curvature states that if $\partial M$ is an outermost minimal surface (i.e., there are no compact minimal surfaces separating $\partial M$ from infinity), then

$$m \geq \sqrt{\frac{A}{16\pi}},$$

where $A$ is the area of $\partial M$.

G. Huisken and T. Ilmanen [12] first proved this inequality for $A$ equal to the largest area of a connected component of $\partial M$. H. Bray [3] later proved the more general inequality described above using a different method, and this result was later extended to dimensions less than 8 by Bray and the first author [4].

We are interested in an analog of this theorem for a class of asymptotically locally hyperbolic manifolds which we now define.

Definition. We say that a $C^i$ Riemannian metric $g$ on a smooth manifold $M^3$ is $C^i$ asymptotically locally hyperbolic if there exists a compact set $K \subset$
M and a constant curvature surface $(\hat{\Sigma}, \hat{g})$, called the **conformal infinity** of $(M, g)$, such that $M \setminus K$ is diffeomorphic to $(1, \infty) \times \hat{\Sigma}$ with the metric satisfying

$$g = (\hat{k} + \rho^2)^{-1} d\rho^2 + \rho^2 \hat{g} + \rho^{-1} h + Q,$$

where

- $\hat{k}$ is the constant curvature of $(\hat{\Sigma}, \hat{g})$;
- $\rho$ is the coordinate on $(1, \infty)$;
- $Q$ is a $C^1$ symmetric two-tensor on $M$ so that

$$|Q|_b + \rho |\hat{\nabla}Q|_b + \cdots + \rho^i |\hat{\nabla}^i Q|_b = o(\rho^{-3}),$$

where $b$ is the hyperbolic metric $(\hat{k} + \rho^2)^{-1} d\rho^2 + \rho^2 \hat{g}$ and $\hat{\nabla}$ are derivatives taken with respect to $b$. We will use the notation $Q = o_i(\rho^{-3})$ as a convenient abbreviation;

- $h$ is a $C^1$ symmetric two-tensor on $\hat{\Sigma}$ depending on $\rho$ in such a way that there exists a function $\mu$ on $\hat{\Sigma}$, called the **mass aspect function**, such that $\lim_{\rho \to \infty} \frac{2}{3} \tr_{\hat{g}} h = \mu$, where the convergence is in $C^1$.

For the sake of convenience, we assume that $\hat{k}$ is 1, 0, or $-1$, and in the case $\hat{k} = 0$, we further assume that $|\hat{\Sigma}|_{\hat{g}} = 4\pi$. (These assumptions simply serve the purpose of normalization.)

Finally, we define the **mass** to be

$$m = \int_{\hat{\Sigma}} \mu \, d\hat{g} = \frac{1}{|\hat{\Sigma}|_{\hat{g}}} \int_{\hat{\Sigma}} \mu \, d\hat{g},$$

and we also define

$$\bar{m} = \sup_{\Sigma} \mu.$$

An important class of asymptotically locally hyperbolic manifolds is given by the Kottler metrics (see [8] for instance), which are static metrics with cosmological constant $\Lambda = -3$.

**Definition.** Let $(\hat{\Sigma}, \hat{g})$ be a surface with constant curvature $\hat{k}$ equal to 1, 0, or $-1$, with area equal to $4\pi$ in the $\hat{k} = 0$ case. Let $m \in \mathbb{R}$ be large enough so that the function

$$V(r) = \sqrt{r^2 + \hat{k} - \frac{2m}{r}}$$

\[2\]
has a nonnegative zero. Let \( r_m \) be the largest zero of \( V \), and define the metric
\[
g = V^{-2}dr^2 + r^2\hat{g}
\]
on \((r_m, \infty) \times \hat{\Sigma}\). Define \((M, g)\) to be the metric completion of this Riemannian manifold. We say that \((M, g)\) is a Kottler space with conformal infinity \((\hat{\Sigma}, \hat{g})\) and mass \( m \).

**Remark.**

- The Kottler metrics have scalar curvature \( R = -6 \).
- The most familiar situation is when \( \hat{\Sigma} \) is a sphere \( (\hat{k} = 1) \), in which case the metric is also called an anti-de Sitter–Schwarzschild metric in the literature.
- As long as \( V(r) \) has a positive largest zero \( r_m \), we obtain \( M = [r_m, \infty) \times \hat{\Sigma} \) with \( \partial M = \{r_m\} \times \hat{\Sigma} \) as an outermost minimal surface boundary. One can see that these metrics are asymptotically locally hyperbolic by performing a substitution, in which case one has \( h = \frac{2}{3}m\hat{g} \).
- In order for \( V(r) \) to have a positive zero, we must have \( m > 0 \) when \( \hat{k} = 0 \) or \( \hat{k} = -1 \). However, when \( \hat{k} = -1 \), the parameter \( m \) need not be positive but only greater than a critical mass \( m_{\text{crit}} = -\frac{1}{3\sqrt{3}} \).
- When the largest zero of \( V(r) \) is exactly 0, we say that \((M, g)\) is a critical Kottler space. When \( \hat{k} = -1 \) and \( m = m_{\text{crit}} \), the metric \( g \) on \((0, \infty) \times \hat{\Sigma}\) is a two-ended complete Riemannian manifold, with one end asymptotically locally hyperbolic and the other end asymptotic to the cylindrical metric \( dt^2 + \frac{1}{3}\hat{g} \) on \( \mathbb{R} \times \hat{\Sigma} \). When \( \hat{k} = 0 \) and \( m = 0 \), the metric \( g \) can be written as \( dt^2 + e^{2t}\hat{g} \) on \( \mathbb{R} \times \hat{\Sigma} \) after a coordinate change, and of course, when \( \hat{k} = 1 \) and \( m = 0 \), the completed metric \((M, g)\) is just hyperbolic space.

We can now state our main theorem.

**Theorem 1.1** (Penrose Inequality for nonpositive mass). Let \((M^3, g)\) be a \( C^2 \) asymptotically locally hyperbolic manifold with \( \bar{m} \leq 0 \) and conformal infinity \((\hat{\Sigma}, \hat{g})\), whose genus is \( g \).

Assume that \( R \geq -6 \), \( \partial M \) is an outermost minimal surface, and there is a boundary component \( \partial_1 M \) of genus \( g \). Then
\[
\bar{m} \geq \frac{1}{7} \sqrt{\frac{A}{16\pi}} \left( 1 - g + \frac{A}{4\pi} \right),
\] (1)
where $A$ is the area of $\partial_1 M$, and $\gamma = (\max\{1, g - 1\})^{3/2}$ is a topological constant.

Furthermore, equality occurs if and only if $(M, g)$ is isometric to the Kottler space with infinity $(\hat{\Sigma}, \hat{g})$ and mass $\hat{m}$.

If one replaces $\hat{m}$ by $m$ and removes the $\hat{m} \leq 0$ condition in our theorem, one obtains the natural analog of Huisken and Ilmanen’s Penrose inequality \cite{12} in the asymptotically locally hyperbolic setting. Versions of this statement have been conjectured by P. Chruściel and W. Simon \cite{8}, Section VI], and in the $g = 0$ case, by X. Wang \cite{19} Section 1. The graph case of the conjecture has recently been established by L. de Líma and F. Girão when $m \geq 0$ \cite{9}. See also a related volume comparison result by S. Brendle and O. Chodosh \cite{5}.

**Corollary 1.2.** Let $(M^3, g)$ be a $C^2$ asymptotically locally hyperbolic manifold with conformal infinity of genus $g$.

Assume that $R \geq -6$, $\partial M$ is an outermost minimal surface, and there is a boundary component $\partial_1 M$ of genus $g$. Then

$$\bar{m} > 0 \quad \text{for } g = 0 \text{ or } 1,$$

$$\bar{m} > -\frac{1}{3\sqrt{3}} \quad \text{for } g > 1.$$

**Proof.** If $g = 0$ or $g = 1$ and $\bar{m} \leq 0$ we obtain at once from Theorem 1.1 that $|\partial_1 M| = 0$, which is a contradiction.

If $g > 1$, then it is an easily verifiable fact that

$$\sqrt{\frac{x}{16\pi}} \left(1 - g + \frac{x}{4\pi}\right) \geq -\frac{(g - 1)^{3/2}}{3\sqrt{3}} \quad \text{for all } x \geq 0.$$

This inequality and Theorem 1.1 imply at once that $\bar{m} \geq -\frac{1}{3\sqrt{3}}$. If equality holds then we are in the equality case of Theorem 1.1 and so $M$ must be isometric to a critical Kottler space of mass $-\frac{1}{3\sqrt{3}}$. But this is impossible because a critical Kottler space has no compact minimal surfaces. (They are foliated by strictly mean convex surfaces.)

Of course, the $g = 0$ case of Corollary 1.2 is just the positive mass theorem for asymptotically hyperbolic manifolds, proved by Chruściel and M. Herzlich \cite{7} and by Wang \cite{19}, for the case of a manifold with minimal

\footnote{Although we are unable to replace $\bar{m}$ by $m$ in general, we do obtain a slightly stronger inequality than \cite{1}. See Lemma 3.13.}
boundary, proved by V. Bonini and J. Qing [1]. While the assumption on
the boundary is not desirable, we note that there exist examples (the AdS
solitons due to G. Horowitz and R. Myers [11]) of asymptotically locally
hyperbolic manifolds with $g = 1$, no boundary, and negative mass.

We note that the inverse mean curvature flow technique allows us to a
give a new proof of a weakened version of the positive mass theorem for
asymptotically hyperbolic manifolds that was mentioned above.

**Theorem 1.3** (Positive $\bar{m}$ theorem for asymptotically hyperbolic mani-
folds). Let $(M^3, g)$ be a complete $C^2$ asymptotically hyperbolic
(with or without a minimal boundary) manifold with scalar curvature $R \geq -6$. Then $\bar{m} \geq 0$. Moreover, if $\bar{m} = 0$, then $(M, g)$ must be hyperbolic space.

We prove Theorem 1.1 following the inverse mean curvature flow theory
developed by Huisken and Ilmanen in [12]. The general idea is to flow $\partial_1 M$
outward with speed inversely proportional to the mean curvature and obtain
a (weak) flow of surfaces $(\Sigma_t)_{t \geq 0}$ (where $\Sigma_0 = \partial_1 M$). To each compact
surface one considers its Hawking mass to be

$$m_H(\Sigma) := \sqrt{\frac{|\Sigma|}{16\pi}} \left(1 - g - \frac{1}{16\pi} \int_\Sigma (H^2 - 4) \right),$$

(2)

where $|\Sigma|$ denotes the area of $\Sigma$, and $H$ is its mean curvature. Observe
that our notation $m_H(\Sigma)$ leaves out the dependence on $g$. The key property
of inverse mean curvature flow is that $m_H(\Sigma_t)$ is non-decreasing in time.
Therefore, since $m_H(\Sigma_0)$ coincides with the right-hand side of the inequality
in Theorem 1.1, the desired result follows if

$$\lim_{t \to \infty} m_H(\Sigma_t) \leq \bar{m} \cdot (\max\{1, g - 1\})^{3/2}.$$

In the asymptotically locally hyperbolic setting this inequality is subtle,
and in fact the second author constructed well-behaved examples (with $g = 0$)
where the above inequality does not hold [15]. The central observation
of this paper is that this inequality holds if $\bar{m} \leq 0$. The difference between
our result here and the one in [15] can be traced to the fact that if the mass
aspect is positive, then a desired inequality goes in the wrong direction,
whereas for nonpositive mass aspect, it goes in the right direction. See the
end of the proof of Lemma 3.13 to see the exact place where the condition
$\bar{m} \leq 0$ is used.

\footnote{Here we define this to mean asymptotically locally hyperbolic with infinity equal to
the round sphere.}
Acknowledgements: The authors would like to thank Piotr Chruściel and Walter Simon for bringing this problem to our attention and for their interest in this work. We also thank Richard Schoen for some helpful conversations.

2 An application to static uniqueness

Chruściel and Simon proved in [8] that Theorem 1.1 implies a uniqueness theorem for static metrics with negative mass that we now explain.

Definition. We say that $(M^3, g, V)$ is a complete vacuum static data set with cosmological constant $\Lambda = -3$ if and only if $M$ is a smooth manifold (possibly with boundary) equipped with a complete $C^2$ Riemannian metric $g$ and a nonnegative $C^2$ function $V$ such that $\partial M = \{ V = 0 \}$ and

$$\Delta_g V = -\Lambda V$$

$$\text{Ric}(g) = \frac{1}{V} \text{Hess}_g V + \Lambda g.$$ (3)

If $(M^3, g, V)$ is a vacuum static data set as above, then the Lorentzian metric $h = -V^2 dt^2 + g$ on $\mathbb{R} \times M$ is a solution to Einstein's equations with cosmological constant $\Lambda$. In the $\Lambda = 0$ case a 1987 result of G. Bunting and A. Masood-ul-Alam [6] shows that Schwarzschild spaces are the only asymptotically flat vacuum static data sets (with the case of connected boundary originally proved by H. Müller zum Hagen, D. Robinson, and H. Seifert in 1973 [14]).

We are interested in a similar characterization of the Kottler metrics defined in the previous section, which are known to be vacuum static data sets with cosmological constant $\Lambda = -3$. A static uniqueness theorem for the hyperbolic space was proved in work of Boucher-Gibbons-Horowitz [2], Qing [18], and Wang [20].

The following definition is equivalent to the one given in [8, Section III.A].

Definition. Let $i \geq 2$. We say that a complete vacuum static data set $(M^3, g, V)$ with cosmological constant $\Lambda = -3$ is $C^i$ conformally compactifiable if $g$ and $V$ are $C^i$ and there exists a smooth compact manifold $M'$ with boundary and a $C^{i+1}$ embedding of $M$ into $M'$ such that $M' \cong M \cup \partial_{\infty} M$,

- the function $V^{-1}$ extends to a $C^i$ function on $M'$ with $d(V^{-1}) \neq 0$ at $\partial_{\infty} M$, and
- the formula $\hat{g} = V^{-2} g$ near $\partial_{\infty} M$ defines a Riemannian metric.
If \((M, g, V)\) is a complete vacuum static data set, the fact that \(\text{Hess}_g V = 0\) on \(\partial M\) implies that \(|\nabla V|\) is constant on each component of \(\partial M\), and we call this constant the surface gravity \(\kappa\) of that component.

The surface gravity is strictly positive for the following reason: Given \(p \in \partial M\), let \(\gamma\) be the unit speed geodesic that starts at \(p\) perpendicular to \(\partial M\) and set \(f(t) = V(\gamma(t))\). From the static equation we see the existence of \(c > 0\) and \(t_0 > 0\) so that \(f'' \leq cf\) for all \(0 \leq t \leq t_0\). Standard o.d.e. comparison shows that if \(f'(0) = f(0) = 0\) then \(f(t) \leq 0\) for all \(0 \leq t \leq t_0\), which is impossible because \(V\) is strictly positive on the interior of \(M\).

The (non-critical) Kottler metrics have exactly one component of \(\partial M\) and, for fixed \(\hat{k}\), there is a bijection between possible surface gravities in \((0, \infty)\) and possible masses in \((-\frac{1}{3\sqrt{3}}, \infty)\) when \(\hat{k} = -1\), or in \((0, \infty)\) when \(\hat{k}\) is 0 or 1. Therefore, for fixed \(\hat{k}\) we can define a bijection \(m(\kappa)\) according to the fixed relationship between mass and surface gravity for Kottler metrics whose infinities have curvature equal to \(\hat{k}\). For this bijection, when \(\hat{k} = -1\) one has \(\lim_{\kappa \to 0} m(\kappa) = -\frac{1}{3\sqrt{3}}\) and \(m(1) = 0\) (see \([8\), Section II\]) for details).

We can now state a static uniqueness theorem for Kottler metrics of negative mass, which will follow from Theorem 1.1 combined with some of the results of \([8\) that we will describe later.

**Theorem 2.1** (Static uniqueness with nonpositive mass). Let \((M^3, g, V)\) be a complete vacuum static data set with cosmological constant \(\Lambda = -3\), and assume that it is \(C^5\) conformally compactifiable with conformal infinity \(\partial\infty M\) of constant curvature \(\hat{k} = -1\).

Assume that there is a component \(\partial_1 M\) of \(\partial M\) such that \(\partial_1 M\) is homeomorphic to \(\partial\infty M\) and \(\partial_1 M\) has the largest surface gravity \(\kappa\) of any component.

If \(m(\kappa) \leq 0\) (or equivalently \(\kappa \leq 1\)), then \((M, g)\) must be isometric to the Kottler metric with infinity \(\partial\infty M\) and mass \(m(\kappa)\), while \(V\) is equal to the usual static potential of the Kottler metric, up to a constant multiple.

Before we present the proof some comments are in order.

- It follows from Lemma 2.4 and the proof of Lemma 3.3 that \(\partial_1 M\) can never have larger genus than \(\partial\infty M\). It would be nice to also rule out the possibility that \(\partial_1 M\) has strictly smaller genus.

- The Horowitz-Myers AdS solitons \([11\) described in the Introduction do not only have negative mass, no boundary, and \(\hat{k} = 0\), but they are also static. A static uniqueness theorem for these examples was proved by G. Galloway, S. Surya, and E. Woolgar \([10\).
• It would be interesting to remove the condition on $m(\kappa)$.

2.1 Proof of Theorem 2.1

The following proposition is a consequence of Theorem I.1 and Proposition III.7 of [8], together with a coordinate change.

**Proposition 2.2.** Let $i \geq 3$. Let $(M^3, g, V)$ be a $C^i$ conformally compactifiable complete vacuum static data set with cosmological constant $\Lambda = -3$. Further assume that the induced metric $\hat{g}$ on $\partial_\infty M$ (as defined in the previous section) has locally constant Gauss curvature $\hat{k}$ equal to 1, 0, or $-1$.

Then $\partial_\infty M$ is connected, $(M, g)$ is asymptotically locally hyperbolic (as defined in the Introduction) with conformal infinity $(\partial_\infty M, \hat{g})$, and

$$V^2 = \rho^2 + \hat{k} - \frac{4\mu}{\rho} + o_1(\rho^{-1}),$$

where $\rho$ is the coordinate used in the definition of asymptotically locally hyperbolic, and $\mu$ is the mass aspect.

In particular, static data sets with $\Lambda = -3$ have a well-defined mass $m$ and $\bar{m}$.

The key theorem of [8] for the purposes of this article is Theorem I.5:

**Theorem 2.3 (Chruściel-Simon).** Let $(M^3, g, V)$ be a $C^3$ conformally compactifiable complete vacuum static data set with cosmological constant $\Lambda = -3$, conformal infinity $(\partial_\infty M, \hat{g})$ of constant curvature $\hat{k} = -1$, and $\partial M \neq \emptyset$.

Let $\partial_1 M$ denote the boundary component with the largest surface gravity $\kappa$ and suppose $m_0 := m(\kappa) \leq 0$ (i.e., $0 < \kappa \leq 1$).

If $(M_0, g_0, V_0)$ denotes the Kottler space with infinity $(\partial_\infty M, \hat{g})$ and mass $m_0$ (i.e., the one with surface gravity $\kappa$), then

$$\frac{\chi(\partial_1 M)}{|\partial_1 M|} \geq \frac{\chi(\partial M_0)}{|\partial M_0|_{g_0}} \quad \text{and} \quad \bar{m} \leq m_0,$$

where $|\partial M_0|_{g_0}$ is the area with respect to $g_0$.

In the case where $\partial_1 M$ has the same genus as $\partial_\infty M$, this theorem provides a simple comparison between the masses and boundary areas of a vacuum static data set and its so-called reference solution $(M_0, g_0, V_0)$. For the sake of completeness we provide the proof of this theorem in Section 4.
Lemma 2.4. Let $\mathcal{M}$ be $\mathcal{C}^3$ asymptotically locally hyperbolic, complete vacuum static data set with cosmological constant $\Lambda = -3$. If $\partial \mathcal{M} \neq \emptyset$, then $\partial \mathcal{M}$ is an outermost minimal surface. In fact, there are no compact minimal surfaces in the interior of $\mathcal{M}$.

Proof. First note that the static equations imply that $\partial \mathcal{M}$ is totally geodesic, so we need only show that there are no other compact minimal surfaces. Consider the trapped region $K$ of $\mathcal{M}$, which is the union of all compact minimal surfaces in $\mathcal{M}$, together with all regions of $\mathcal{M}$ that are bounded by these minimal surfaces. The boundary of the trapped region, $\partial K$, must itself be a smooth compact minimal surface. (See the proof of Lemma 4.1(i) of [12].) Following [12], we define the exterior region $\mathcal{M}'$ of $\mathcal{M}$ to be the metric completion of $\mathcal{M} \setminus K$. Thus $(\mathcal{M}', g, V)$ is a vacuum static data set, except for the requirement that $\{x \in \mathcal{M}' | V(x) = 0\} = \partial \mathcal{M}'$. The exterior region $\mathcal{M}'$ has a strictly outward minimizing minimal boundary and no interior compact minimal surfaces (see [12]), where strictly outward minimizing means that $|\partial \mathcal{M}'|$ is strictly less than the area of any other surface that encloses it.

Suppose that $\mathcal{M}$ has a compact minimal surface other than $\partial \mathcal{M}$. Since $V > 0$ away from $\partial \mathcal{M}$, it follows from the definition of $\mathcal{M}'$ that $V$ does not vanish identically on $\partial \mathcal{M}'$. We consider the outward normal flow of surfaces $(\Sigma_t)_{t \geq 0}$ with initial condition $\Sigma_0 = \partial \mathcal{M}'$ that flows with speed $V$. Since $V \geq 0$ does not vanish on $\partial \mathcal{M}'$, this flow is nontrivial. According to the formula for the variation of mean curvature, we see that the mean curvature of $\Sigma_t$ evolves according to

$$
\frac{\partial H}{\partial t} = -\Delta_{\Sigma_t} V - \langle \text{Ric}(\nu, \nu) + |A_{\Sigma_t}|^2 V \rangle V
$$

$$
= -(\Delta g V - \nabla_\nu \nabla_\nu V + \langle H, \nabla V \rangle) - (\nabla_\nu \nabla_\nu V - 3V + |A_{\Sigma_t}|^2 V)
$$

$$
= -\langle H, \nabla V \rangle - |A_{\Sigma_t}|^2 V,
$$

where $\nu$ is the outward unit normal, and $A_{\Sigma_t}$ is the second fundamental form. Since $V \geq 0$, it follows that $\Sigma_t$ must have $H \leq 0$ for all small $t$. By the first variation of area formula, $\Sigma_t$ must have area less than or equal to that of $\Sigma_0 = \partial \mathcal{M}'$. But this contradicts the strictly outward minimizing property of $\partial \mathcal{M}'$.

We can now prove Theorem 2.1 following the description in [8].

Let $(\mathcal{M}, g, V)$ be as in the statement of the theorem. In particular, all of the hypotheses of Chruściel-Simon’s Theorem (Theorem 2.3) are satisfied
and so $\bar{m} \leq 0$. Hence Proposition 2.2 and Lemma 2.4 imply that all of the hypotheses of our Penrose inequality (Theorem 1.1) are also satisfied.

Since we are assuming that $\partial_1 M$ is homeomorphic to $\partial_\infty M$, Theorem 2.3 tells us that $A \geq A_0$ and $\bar{m} \leq m_0$, where $A$ is the area of $\partial_1 M$ and $A_0$ is the area of $\partial M_0$ in the reference solution. Moreover, since the curvature $\hat{k}$ of $\hat{g}$ is equal to $-1$ our Penrose inequality (Theorem 1.1) tells us that

$$\bar{m}^2 \geq \frac{1}{16\pi} \left(1 - g + \frac{A}{4\pi}\right). \quad (5)$$

It is convenient to define constants

$$\tau := \sqrt{\frac{A}{4\pi(g - 1)}} \quad \text{and} \quad r_0 := \sqrt{\frac{A_0}{4\pi(g - 1)}},$$

so that we have $\tau \geq r_0$.

Inequality (5) then becomes

$$\bar{m} \geq \frac{1}{2} \tau(1 + \tau^2) \implies 2\bar{m} + \tau - \tau^3 \geq 0.$$

Meanwhile, on the reference space we know that $r_0$ is the largest root of

$$2m_0 + r_0 - r_0^3 = 0.$$

and so $r_0 \geq \sqrt[3]{\frac{1}{3}}$ using elementary reasoning. Thus

$$0 \leq 2\bar{m} + \tau - \tau^3 = (2\bar{m} + \tau - \tau^3) - (2m_0 + r_0 - r_0^3)$$

$$= 2(\bar{m} - m_0) + (\tau - r_0) - (\tau^3 - r_0^3)$$

$$= 2(\bar{m} - m_0) + (\tau - r_0)[1 - (\tau^2 + \tau r_0 + r_0^2)]$$

$$\leq 2(\bar{m} - m_0) + (\tau - r_0)(1 - 3r_0^2) \leq 0.$$

Therefore all of the inequalities must be equalities and so it follows from the rigidity part of Theorem 1.1 that $(M, g)$ is isometric to the Kottler space with infinity $\partial_\infty M$ and mass $m_0$.

It is simple to check that any two static potentials on the Kottler space $(M, g)$ must be proportional and so $V$ is the usual static potential, up to a constant multiple.
3 Proof of Theorem 1.1

Assume that \((M^3, g)\) be a \(C^2\) asymptotically locally hyperbolic manifold with \(R \geq -6\) such that \(\partial M\) is an outermost minimal surface. In order to establish existence of the weak inverse mean curvature flow, we first need to find a weak subsolution.

**Lemma 3.1.** Let \((M^3, g)\) be a \(C^2\) asymptotically locally hyperbolic metric with radial coordinate \(\rho\) as in the definition of asymptotically locally hyperbolic. There exists \(r_0\) so that for all \(r \geq r_0\),

\[
\Sigma^-_t = \{\rho = (r + 1)e^{t/2} - 1\} \quad \text{and} \quad \Sigma^+_t = \{\rho = (r - 1)e^{t/2} + 1\}, \quad t \geq 0
\]

are, respectively, subsolutions and supersolutions for inverse mean curvature flow with initial condition \(\{\rho = r\}\).

**Proof.** We will prove that \(\Sigma^+_t\) is a supersolution. (The proof for \(\Sigma^-_t\) is similar.) Using the asymptotics of \(g\), one can see that the inverse mean curvature of the constant \(\rho\) sphere in \((M, g)\) is

\[
H^{-1} = \frac{\rho}{2\sqrt{k + \rho^2}} + O(\rho^{-2}).
\]

Since that \(\Sigma^+_t\) is just the constant \(\rho\) sphere with \(\rho = (r - 1)e^{t/2} + 1\), we see that the speed of the flow is just

\[
\left. \frac{d\rho}{dt} \right|_{\rho} = \frac{1}{2}(r-1)e^{t/2}[(\hat{k} + \rho^2)^{-1/2} + O(\rho^{-3})] = \frac{\rho}{2\sqrt{k + \rho^2}} - \frac{1}{2\sqrt{k + \rho^2}} + O(\rho^{-2}).
\]

Clearly, for sufficiently large \(\rho\), this speed is less than \(H^{-1}_{\Sigma^+_t}\), showing that \(\Sigma^+_t\) is a supersolution. \(\square\)

Given Lemma 3.1, we may now apply Huisken and Ilmanen’s Weak Existence Theorem 3.1 of [12] to find a weak solution \(u\) for inverse mean curvature flow with initial condition \(\partial M\). More precisely, \(u\) is a proper, locally Lipschitz nonnegative function \(u\) defined on \(M\) with \(u = 0\) on \(\partial M\) that satisfies a certain variational property (defined on page 365 of [12]). The surfaces \(\Sigma_t := \partial \{u < t\}\) are \(C^{1,\alpha}\) and strictly outward minimizing\(^3\) as defined in the proof of Lemma 2.4. In particular, each \(\Sigma_t\) is mean convex. There are only

\(^3\)Huisken and Ilmanen instead describe the region enclosed by \(\Sigma_t\) as a strictly minimizing hull [12].
countably many “jump times,” that is, values of $t$ for which $\Sigma_t := \partial \{ u < t \}$ does not equal $\Sigma_t^+ := \partial(\text{int} \{ u \leq t \})$. In a nonrigorous sense, $\Sigma_t$ may be regarded as flowing by smooth inverse mean curvature flow, except when it ceases to be strictly outward minimizing, at which time it “jumps” to a strictly outward minimizing surface $\Sigma_t^+$ of equal area.

In case $\partial M$ is not connected, Huisken and Ilmanen explained how one can single out a component $\partial_1 M$ of $\partial M$ as the initial surface while treating the other components of $\partial M$ as “obstacles.” See Section 6 of [12] for details. Essentially, we arbitrarily “fill in” all other components $\partial_2 M, \ldots, \partial_n M$ of $\partial M$ to obtain a new space $\tilde{M}$ and then run the weak inverse mean curvature flow in $\tilde{M}$ with initial condition $\partial_1 M$, except that whenever the surface $\Sigma_t$ is about to enter the filled-in region, we jump to a connected strictly outward minimizing surface $F$ enclosing both $\Sigma_t$ and one or more of the filled-in regions. We then restart the flow with initial condition $F$.

There is another important alteration introduced by Huisken and Ilmanen [12, Section 4]. We consider the exterior region $M'$ of $M$, as defined in the proof of Lemma 2.4. Since $\partial M$ was the outermost minimal surface of $M$, it follows that $\partial M$ is still part of the boundary of $M'$, but now there might be more minimal boundary components. The exterior region $M'$ is an improvement over $M$ because it is completely free of compact minimal surfaces in its interior. We will actually run the weak inverse mean curvature flow in the exterior region of $M$ rather than in $M$ itself. So for our proof of Theorem 1.1, we may assume without loss of generality that $M$ is an exterior region.

### 3.1 Monotonicity of inverse mean curvature flow

Fix an integer $g$ and recall our definition of the Hawking mass of a surface $\Sigma$ in $(M, g)$ to be

$$m_H(\Sigma) := \sqrt{\frac{|\Sigma|}{16\pi}} \left( 1 - g - \frac{1}{16\pi} \int_\Sigma (H^2 - 4) \right).$$

The proof of the Geroch Monotonicity Formula 5.8 in [12] adapts straightforwardly to the locally hyperbolic setting to show the following:

**Theorem 3.2** (Huisken-Ilmanen). Let $(M^3, g)$ be a complete, one-ended, $C^2$ asymptotically locally hyperbolic manifold with outermost minimal boundary, and let $\partial_1 M$ be one of its boundary components. Let $\Sigma_t$ be a weak solution to inverse mean curvature flow (possibly with obstacles, as described above),
with initial surface \( \partial_1 M \). Then for \( 0 \leq \xi < \eta \), if there are no obstacles between \( \Sigma_{\xi} \) and \( \Sigma_{\eta} \), then

\[
m_H(\Sigma_{\eta}) - m_H(\Sigma_{\xi}) \geq \frac{1}{2} (16\pi)^{-3/2} \int_{\xi}^{\eta} |\Sigma_t|^{1/2} \left[ 8\pi (\hat{\chi} - \chi(\Sigma_t)) + \int_{\Sigma_t} (2(R + 6) + |\hat{A}|^2 + 4H^{-2}|\nabla H|^2) \right] dt,
\]

where \( \hat{\chi} := 2 - 2g \), and \( \hat{A} \) is the trace-free part of the second fundamental form.

To make use of this theorem we need the following lemma.

**Lemma 3.3.** Let \((M^3, g)\) be a complete, one-ended, \( C^2 \) asymptotically locally hyperbolic manifold, which is an exterior region. Let \( \partial_1 M \) be a component of \( \partial M \) with genus \( g \), and let \( \Sigma_t \) be a weak solution to inverse mean curvature flow (possibly with obstacles), with initial surface \( \partial_1 M \). For all \( t \), the surface \( \Sigma_t \) is connected and has genus at least \( g \). In particular, \( \hat{\chi} \geq \chi(\Sigma_t) \).

**Proof.** Let \( \tilde{M} \) be the manifold \( M \) with the obstacles filled in. Let \( u \) be the function defining the weak flow (possibly with obstacles). For each \( t > 0 \), let \( \tilde{M}_t \) be the closure of \( \{ x \in \tilde{M} | u(x) < t \} \), so that \( \partial \tilde{M}_t = \Sigma_t \cup \partial_1 M \). We claim that \( \tilde{M}_t \) is connected. If it were not connected, one of the components \( \Omega \) of \( \tilde{M}_t \) would be disjoint from \( \partial_1 M \). By the variational property that characterizes \( u \), one can deduce that \( u \) must be constant over \( \Omega \). (See the proof of [12, Connectedness Lemma 4.2(i)].) Since \( \tilde{M} \) is connected, \( \Omega \) must meet \( \Sigma_t \), and thus \( u = t \) on \( \Omega \), which is a contradiction to the definition of \( \tilde{M}_t \). Since \( \tilde{M}_t \) is connected, it follows that \( M_t := \tilde{M}_t \cap M \) is connected. Note that \( \partial M_t = \Sigma_t \cup \partial_1 M \cup \cdots \cup \partial_k M \), where \( \partial_2 M, \ldots, \partial_k M \) is some labeling of the other components of \( \partial M \) that touch \( M_t \).

The rest of the proof does not use inverse mean curvature flow. It is essentially a topological argument that relies only on the following facts about \( \Sigma_t \): There exists a connected manifold \( M_t \) whose boundary is \( \Sigma_t \cup \partial_1 M \cup \cdots \cup \partial_k M \), \( \Sigma_t \) is mean convex, each \( \partial_i M \) is minimal, and the \( \partial_i M \)’s are the only compact minimal surfaces in \( M_t \). This last part is where we use the assumption that \( M \) is an exterior region.

Let \( \Sigma^1, \ldots, \Sigma^k \) be the connected components of \( \Sigma_t \). We minimize area in the isotopy class of \( \Sigma^1 \) in \( M_t \). Note that Theorem 1 of Meeks, Simon, and Yau applies because \( M_t \) has mean convex boundary, as explained in Section 6 of [13]. According to Theorem 1 and Remark 3.27 of [13], there exists some
surface $\tilde{\Sigma}^1$ obtained from $\Sigma^1$ via isotopy and a series of $\gamma$-reductions such that each component of $\tilde{\Sigma}^1$ is a parallel surface of a connected minimal surface, except for one component that may be taken to have arbitrarily small area. Recall that a $\gamma$-reduction is a surgery procedure that deletes an annulus and replaces it with two disks in such a way that the annulus and two disks bound a ball in $M_t$. (See [13, Section 3] for details.)

Note that $\gamma$-reduction preserves homology class. Since the only compact minimal surfaces in $M_t$ are the $\partial_i M$’s, and because a surface of small enough area must be homologically trivial, it follows that

$$[\Sigma^1] = [\tilde{\Sigma}^1] = \sum_{i=1}^{k} n_i [\partial_i M] \text{ in } H_2(M_t, \mathbb{Z}),$$

for some integers $n_i$. Using the long exact sequence for the pair $(M_t, \partial M_t)$, we have exactness of

$$H_3(M_t, \partial M_t) \xrightarrow{\partial} H_2(\partial M_t, \mathbb{Z}) \xrightarrow{i_*} H_2(M_t, \mathbb{Z}).$$

Since $M_t$ is connected, $\ker i_*$ must be generated by

$$\partial [M_t] = \sum_{i=1}^{\ell} [\Sigma^i] - \sum_{i=1}^{k} [\partial_i M],$$

where $\Sigma^i$ and $\partial M$ are oriented using the outward normal in $M$ as usual. Since equation (7) says that $[\Sigma^1] - \sum_{i=1}^{k} n_i [\partial_i M] \in \ker i_*$, it follows that $\Sigma_t$ must be connected and hence equal to $\tilde{\Sigma}^1$. In particular,

$$[\Sigma^1] = \sum_{i=1}^{k} [\partial_i M] \text{ in } H^2(M_t, \mathbb{Z}).$$

Since each component of $\tilde{\Sigma}^1$ is either isotopic to one of the $\partial_i M$’s (with some orientation) or is null homologous, and since there are no relations among $[\partial_i M]$ in $H^2(M_t, \mathbb{Z})$, the previous equation implies that at least one component of $\tilde{\Sigma}^1$ is isotopic to $\partial_1 M$. Finally, since $\gamma$-reduction can only reduce the total genus of all components of a surface, we know that $\Sigma_t$ has genus at least as large as that of $\partial_1 M$.

Note that if we apply the reasoning in the proof of Lemma 3.3 above to the “conformal infinity” $\hat{\Sigma}$ of $M$, we see that the genus of $\hat{\Sigma}$ is at least as large as the genus of $\partial_1 M$.  

14
Corollary 3.4 (Geroch monotonicity). Let \((M^3, g)\) be a complete, one-ended, \(C^2\) asymptotically locally hyperbolic manifold, which is an exterior region. Let \(\partial_1 M\) be a component of \(\partial M\) with genus \(g\), and let \(\Sigma_t\) be a weak solution to inverse mean curvature flow (possibly with obstacles), with initial surface \(\partial_1 M\). Then the Hawking mass of \(\Sigma_t\) is nondecreasing in \(t\).

Proof. The result follows immediately from Theorem 3.2 and Lemma 3.3 in the absence of obstacles. Since there are only finitely many obstacles, all that is left to show is that the mass cannot drop when we jump over an obstacle. Let \(t\) be the first time that we have to jump over an obstacle, and let \(F\) denote the strictly outward minimizing surface that \(\Sigma_t\) jumps to. Then we know from [12, Equation 6.1] that

\[
|\Sigma_t| \leq |F| \quad \text{and} \quad \int_{\Sigma_t} H^2_{\Sigma_t} \geq \int_F H^2_F, 
\]

where the second inequality essentially follows from the fact that the strictly minimizing hull of a surface should be minimal away from where it agrees with the original surface. In the case \(g < 2\), these inequalities combine with nonnegativity of \(m_H(\Sigma_t)\) (from monotonicity in the absence of obstacles) to immediately show that \(m_H(\Sigma_t) \leq m_H(F)\), just as in the asymptotically flat case [12, Section 6].

To handle the case \(g \geq 2\) we proceed as follows.

\[
m_H(F) - m_H(\Sigma_t) = |F|^{1/2} \frac{m_H(F)}{|F|^{1/2} - |\Sigma_t|^{1/2}} m_H(\Sigma_t) - |\Sigma_t|^{1/2} \frac{m_H(\Sigma_t)}{|\Sigma_t|^{1/2}} \\
= (|F|^{1/2} - |\Sigma_t|^{1/2}) \frac{m_H(F)}{|F|^{1/2} - |\Sigma_t|^{1/2}} + |F|^{1/2} \left( \frac{m_H(F)}{|F|^{1/2} - |\Sigma_t|^{1/2}} - \frac{m_H(\Sigma_t)}{|\Sigma_t|^{1/2}} \right) \\
= (|F|^{1/2} - |\Sigma_t|^{1/2}) \frac{m_H(\Sigma_t)}{|\Sigma_t|^{1/2}} \left[ 1 + |F|^{1/2} (16\pi)^{-3/2} \left( - \int_F (H^2_F - 4) + \int_{\Sigma_t} (H^2_{\Sigma_t} - 4) \right) \right] \\
\geq (|F|^{1/2} - |\Sigma_t|^{1/2}) \frac{m_H(\Sigma_t)}{|\Sigma_t|^{1/2}} + |F|^{1/2} (16\pi)^{-3/2} (4|F| - 4|\Sigma_t|) \\
\geq (|F|^{1/2} - |\Sigma_t|^{1/2}) \frac{m_H(\Sigma_t)}{|\Sigma_t|^{1/2}} + |\Sigma_t|^{1/2} (16\pi)^{-3/2} (4|F| - 4|\Sigma_t|) \\
= (|F|^{1/2} - |\Sigma_t|^{1/2}) \left( \frac{m_H(\Sigma_t)}{|\Sigma_t|^{1/2}} + 4(16\pi)^{-3/2} |\Sigma_t|^{1/2} (|F|^{1/2} + |\Sigma_t|^{1/2}) \right) \\
\geq (|F|^{1/2} - |\Sigma_t|^{1/2}) \left( \frac{m_H(\Sigma_t)}{|\Sigma_t|^{1/2}} + 8(16\pi)^{-3/2} |\Sigma_t| \right),
\]

15
Therefore it only remains to show that \( m_H(\Sigma_t) \geq -8(16\pi)^{-3/2}|\Sigma_t|^{3/2} \). Note that for any \( A \geq 0 \),
\[
\sqrt{\frac{A}{16\pi}} \left( 1 - \frac{1}{4\pi} A \right) \geq - \left( \frac{g - 1}{3} \right)^{3/2}.
\]
Taking \( A = |\partial_1 M| \) and using monotonicity in the absence of obstacles, we then have
\[
m_H(\Sigma_t) \geq m_H(\partial_1 M) \geq - \left( \frac{g - 1}{3} \right)^{3/2}.
\]
On the other hand, observe that a stable compact minimal surface of genus \( g \) in a 3-manifold with \( R \geq -6 \) must have area at least \( \frac{4\pi}{3}(g-1) \). (This follows from a standard computation using the second variation of area, see [17, Section 2]). Thus
\[
8(16\pi)^{-3/2}|\Sigma_t|^{3/2} \geq 8(16\pi)^{-3/2}|\partial_1 M|^{3/2}
\]
\[
\geq 8(16\pi)^{-3/2} \left[ \frac{4\pi}{3}(g-1) \right]^{3/2}
\]
\[
\geq \left( \frac{g - 1}{3} \right)^{3/2},
\]
which completes the proof.

3.2 The long-time limit of inverse mean curvature flow

First we compute the asymptotics of Ricci curvature for asymptotically locally hyperbolic manifolds. Proceeding as in Lemma 3.1 of [16], we deduce the following:

**Lemma 3.5.** Let \((M^3, g)\) be a asymptotically locally hyperbolic, with radial coordinate \( \rho \). If \( \frac{\nabla \rho}{|\nabla \rho|}, e_1, e_2 \) is an orthonormal frame at a point in \( M \), then
\[
\text{Ric} \left( \frac{\nabla \rho}{|\nabla \rho|}, \frac{\nabla \rho}{|\nabla \rho|} \right) = -2 - 2\mu \rho^{-3} + o(\rho^{-3})
\]
\[
\text{Ric}(e_i, e_j) = -2\delta_{ij} + O(\rho^{-3})
\]
\[
\text{Ric} \left( \frac{\nabla \rho}{|\nabla \rho|}, e_i \right) = o(\rho^{-3})
\]
\[
R = -6 + o(\rho^{-3}).
\]
In order to make certain computations easier, we consider a conformal compactification \( \tilde{g} = \rho^{-2}g \) of the exterior region of \((M, g)\). Then if we set \( s = \rho^{-1} \), we have
\[
\tilde{g} = ds^2 + \hat{g} + \tilde{Q},
\]
on the space \((0, s_1) \times \hat{\Sigma}\) for small enough \( s_1 \), where
\[
|\tilde{Q}| + s|\nabla \tilde{Q}| + s^2|\nabla^2 \tilde{Q}| = O_2(s^2).
\]

**Lemma 3.6.** There is a constant \( C \) such that for sufficiently large \( t \), the \( \rho \) coordinate on \( \Sigma_t \) lies in \((\frac{1}{C}e^{t/2}, Ce^{t/2})\) and the \( s \) coordinate on \( \Sigma_t \) must lie in \((\frac{1}{C}e^{-t/2}, Ce^{-t/2})\).

**Proof.** This follows immediately from the subsolutions and supersolutions of inverse mean curvature flow described in Lemma 3.1. \( \square \)

We will use the following area bound repeatedly.

**Lemma 3.7.** Let \( \hat{\Sigma}_t \) denote the surface \( \Sigma_t \) endowed with the metric induced from \( \tilde{g} \). The area of \( \hat{\Sigma}_t \) is uniformly bounded in time.

**Proof.** This follows immediate from the fact that \(|\Sigma_t| = |\Sigma_0| e^t\), the definition of \( \tilde{g} \), and the previous Lemma. \( \square \)

**Lemma 3.8.** \( \int_{\hat{\Sigma}_t} (H^2 - 4) \) is uniformly bounded above and below.

**Proof.** The upper bound follows easily from Corollary 3.4. Since
\[
m_H(\Sigma_0) \leq m_H(\Sigma_t) = \sqrt{\frac{|\Sigma_t|}{16\pi}} \left( 1 - g + \frac{1}{16\pi} \int_{\Sigma_t} (H^2 - 4) \right),
\]
we have
\[
\int_{\Sigma_t} (H^2 - 4) \leq 16\pi \left( 1 - g + m_H(\Sigma_0) \sqrt{\frac{16\pi}{|\Sigma_t|}} \right) = 16\pi(1 - g) + O(e^{-t/2}).
\]

To prove the lower bound, we consider the Gauss-Codazzi equations in \((M, g)\):
\[
2|\hat{A}|^2 + 4K = H^2 + 2R - 4\text{Ric}(\nu, \nu) \tag{8}
\]
Performing the same computation in the compactified metric \( \tilde{g} \) and using the fact that integral of the left-hand side is conformally invariant, we see that
\[
\int_{\Sigma_t} (H^2 + 2R - 4\text{Ric}(\nu, \nu)) = \int_{\hat{\Sigma}_t} (\tilde{H}^2 + 2\tilde{R} - 4\tilde{\text{Ric}}(\tilde{\nu}, \tilde{\nu})).
\]
By Lemma 3.5 we see that
\[
\int_{\Sigma_t} (H^2 - 4) = \int_{\Sigma_t} (\tilde{H}^2 + 2\tilde{R} - 4\tilde{\text{Ric}}(\tilde{\nu}, \tilde{\nu})) + O(e^{-t/2}) \\
\geq -C + \int_{\Sigma_t} \tilde{H}^2 \geq -C,
\]
for some constant $C$ independent of $t$, where we used the fact that the curvature of $\tilde{g}$ is bounded. 

\begin{equation}
\label{eq:inequality}
\end{equation}

Lemma 3.9. We have
\[
\int_{\Sigma_t} |\nabla T_s|^2 = O(e^{-t/2}).
\]

Proof.
\[
\int_{\Sigma_t} |\nabla T_s|^2 = -\int_{\Sigma_t} s \Delta \hat{\Sigma}_s = \int_{\Sigma_t} s (\Delta_{\tilde{g}} s - \nabla_{\tilde{\nu}} \nabla_{\tilde{\nu}} s + \langle \tilde{H}, \partial_s \rangle) \\
\leq \int_{\Sigma_t} s |\Delta_{\tilde{g}} s - \nabla_{\tilde{\nu}} \nabla_{\tilde{\nu}} s| + \int_{\Sigma_t} s |\tilde{H}| \cdot |\nabla s| \\
= \int_{\Sigma_t} O(s^2) + \int_{\Sigma_t} |\tilde{H}| \cdot O(s) \\
\leq O(e^{-t}) + O(e^{-t/2}) \left( \int_{\Sigma_t} \tilde{H}^2 \right)^{1/2},
\]
where we used the Hölder inequality in the last line. The result now follows because inequality \eqref{eq:inequality} states that
\[
\int_{\Sigma_t} \tilde{H}^2 \leq C + \int_{\Sigma_t} (H^2 - 4),
\]
while Lemma 3.8 implies that the right-hand side is bounded. 

Lemma 3.10. $|\hat{\Sigma}_t| = |\hat{\Sigma}| + O(e^{-t/2})$.

Proof. Choose $\tilde{\nu}$ to be the inward pointing normal of $\hat{\Sigma}_t$. We will first show that $|\hat{\Sigma}_t| \leq |\hat{\Sigma}| + O(e^{-t/2})$.

Using the fact that $s$ is approximately a distance function with respect to $\tilde{g}$,
\[
|\hat{\Sigma}_t| = \int_{\Sigma_t} (|\nabla s|^2 + O(s^2)) = \int_{\Sigma_t} (|\nabla T_s|^2 + |\nabla N s|^2) + O(e^{-t})
\]

18
= \int_{\tilde{\Sigma}_t} |\tilde{\nu}(s)|^2 + O(e^{-t/2}),

where the last line follows from Lemma 3.9. So it suffices to estimate \( \int_{\tilde{\Sigma}_t} |\tilde{\nu}(s)|^2 \) in terms of \( |\tilde{\Sigma}| \).

We now divide \( \tilde{\Sigma}_t \) into three parts:

\[
A_t = \{ x \in \tilde{\Sigma}_t \mid \tilde{\nu}(s) \leq -e^{-t/4} \}, \quad B_t = \{ x \in \tilde{\Sigma}_t \mid -e^{-t/4} < \tilde{\nu}(s) \leq 0 \}
\]

and

\[
C_t = \{ x \in \tilde{\Sigma}_t \mid 0 < \tilde{\nu}(s) \}. \tag{10}
\]

Then

\[
\int_{\tilde{\Sigma}_t} |\tilde{\nu}(s)|^2 = \int_{A_t} |\tilde{\nu}(s)|^2 + \int_{B_t} |\tilde{\nu}(s)|^2 + \int_{C_t} |\tilde{\nu}(s)|^2 \\
\leq (1 + O(e^{-t}))|A_t| + e^{-t/2}|B_t| + (1 + O(e^{-t}))\int_{C_t} \tilde{\nu}(s) \\
\leq |A_t| + \int_{\tilde{\Sigma}_t} \tilde{\nu}(s) + O(e^{-t/2})
\]

If we take \( \tilde{M}_t \) to be the region of \((0, s_1) \times \tilde{\Sigma}\) so that \( \partial \tilde{M}_t = \{0\} \times \tilde{\Sigma} \cup \Sigma_t \), then the divergence theorem tells us that

\[
\int_{\tilde{\Sigma}_t} \tilde{\nu}(s) = \int_{\{0\} \times \Sigma} \tilde{\nu}(s) + \int_{\tilde{M}_t} \Delta s = |\tilde{\Sigma}| + O(e^{-t}).
\]

Thus

\[
\int_{\tilde{\Sigma}_t} |\tilde{\nu}(s)|^2 \leq |A_t| + |\tilde{\Sigma}| + O(e^{-t/2}). \tag{10}
\]

It remains to estimate \( |A_t| \). The mean curvature changes under under the conformal change \( g = s^{-2}\tilde{g} \) according to the formula

\[
H = s\tilde{H} + 2\tilde{\nu}(s).
\]

Since \( \Sigma_t \) evolves by inverse mean curvature flow, we know that \( H \geq 0 \). Therefore

\[
\tilde{\nu}(s) \geq -\frac{s\tilde{H}}{2}.
\]

So the definition of \( A_t \) tells us that

\[
-e^{-t/4} \geq \tilde{\nu}(s) \geq -\frac{s\tilde{H}}{2}
\]

on \( A_t \). Squaring this and integrating over \( A_t \) gives

\[
\int_{A_t} e^{-t/2} \geq \int_{A_t} \frac{s^2\tilde{H}^2}{4} \quad \Rightarrow \quad |A_t| \leq O(e^{-t/2}) \int_{\Sigma_t} \tilde{H}^2.
\]
As mentioned in the proof of Lemma 3.9, $\hat{\tilde{H}}^2$ is bounded, and therefore this inequality combined with inequality (10) gives us

$$\int_{\Sigma_t} |\tilde{\nu}(s)|^2 \leq |\hat{\Sigma}| + O(e^{-t/2}),$$

completing the proof of one side of the inequality.

The reverse inequality follows from the fact that $|\tilde{\Sigma}|$ is within error $O(e^{-t})$ from the area of $\hat{\Sigma}_t$ as measured in the product metric $ds^2 + \hat{g}$ on $(0, s_1) \times \hat{\Sigma}$, and the projection map of the product metric onto $\hat{\Sigma}$ is area-nonincreasing.

Lemma 3.11. There exists a sequence of times $t_i$ such that

$$\lim_{i \to \infty} \int_{\Sigma_{t_i}} |A|^2 = 0.$$

Proof. We use formula (5.22) from [12]. For $\xi < \eta$, we have

$$\int_{\Sigma_\xi} H^2 \geq \int_{\Sigma_\eta} H^2 + \int_{\Sigma_\eta} \int_{\Sigma_t} \left(2\frac{|DH|^2}{H^2} + 2|A|^2 + 2\text{Ric}(\nu, \nu) - H^2\right) dt$$

$$\geq \int_{\Sigma_\eta} H^2 + \int_{\Sigma_\eta} \int_{\Sigma_t} (2|A|^2 + 2\text{Ric}(\nu, \nu))$$

Since $\frac{d}{dt}|\Sigma_t| = |\Sigma_t|$, it follows that

$$\int_{\Sigma_\xi} (H^2 - 4) \geq \int_{\Sigma_\eta} (H^2 - 4) + \int_{\Sigma_\eta} \int_{\Sigma_t} (2|A|^2 + 2\text{Ric}(\nu, \nu) + 2)) dt.$$

The last term is easily bounded:

$$\int_{\xi}^{\eta} \int_{\Sigma_t} 2(\text{Ric}(\nu, \nu) + 2) dt = \int_{\xi}^{\eta} \int_{\Sigma_t} O(p^{-3}) dt = \int_{\xi}^{\eta} O(e^{-t/2}) dt = O(e^{-\xi/2}).$$

Hence,

$$\int_{\xi}^{\eta} \int_{\Sigma_t} 2|A|^2 dt \leq \int_{\Sigma_\xi} (H^2 - 4) - \int_{\Sigma_\eta} (H^2 - 4) + O(e^{-\xi/2}).$$

Since $\int_{\Sigma_\xi} (H^2 - 4)$ is bounded above and below by Lemma 3.8, the integral on the left-hand side is bounded as $\eta \to \infty$, completing the proof.
Lemma 3.12. Using the same sequence as in Lemma 3.11, we have $\chi(\Sigma_t) = \chi(\tilde{\Sigma})$ for sufficiently large $i$.

Proof. Note that by Lemma 3.3, we already know that $\chi(\Sigma_t) \leq \chi(\tilde{\Sigma})$, so we need to show that $\chi(\Sigma_t) \geq \chi(\tilde{\Sigma})$ for all $i$ large.

The Gauss-Codazzi equations in the compactified metric tell us that

$$2|\tilde{\partial}^2| + 4\tilde{K} = \tilde{H}^2 + 2\tilde{R} - 4\tilde{Ric}(\tilde{\nu}, \tilde{\nu}),$$

where $\tilde{\nu}$ is the normal vector to the surface. If we apply this to the surfaces $\{s\} \times \tilde{\Sigma}$ we obtain that

$$\lim_{s \to 0} (2\tilde{R} - 4\tilde{Ric}(\tilde{\nabla}_s, \tilde{\nabla}_s)) = 4\hat{k}, \quad (11)$$

where $\hat{k}$ is the constant Gaussian curvature of $(\tilde{\Sigma}, \tilde{g})$.

Decompose

$$\tilde{\nu} = a \tilde{\nabla}_s + v,$$

so that $v$ is orthogonal to $\tilde{\nu}$, we have $|v| = |\tilde{\nabla}_s| = |\tilde{\nabla}^T_s| + O(s^2)$.

Integrating Gauss-Codazzi equations on $\tilde{\Sigma}_t$ and recalling Lemma 3.9 we obtain

$$\left( \int_{\tilde{\Sigma}_t} 2|\tilde{\partial}^2| \right) + 8\pi \chi(\tilde{\Sigma}_t) = \int_{\tilde{\Sigma}_t} \tilde{H}^2 + 2\tilde{R} - 4\tilde{Ric}(\tilde{\nu}, \tilde{\nu}) \geq \int_{\tilde{\Sigma}_t} 2\tilde{R} - 4\tilde{Ric}(\tilde{\nu}, \tilde{\nu})$$

$$= \int_{\tilde{\Sigma}_t} (2\tilde{R} - 4a^2\tilde{Ric}(\tilde{\nabla}_s, \tilde{\nabla}_s) - 8\tilde{Ric}(\tilde{\nabla}_s, v) - \tilde{Ric}(v, v)) + O(s^2)$$

$$\geq \int_{\tilde{\Sigma}_t} (2\tilde{R} - 4\tilde{Ric}(\tilde{\nabla}_s, \tilde{\nabla}_s) - C(|\tilde{\nabla}^T_s| + |\tilde{\nabla}^T_s|^2)) + O(s^2)$$

$$= \int_{\tilde{\Sigma}_t} (2\tilde{R} - 4\tilde{Ric}(\tilde{\nabla}_s, \tilde{\nabla}_s)) + O(e^{-t/4}).$$

Thus, using (11), Lemma 3.10 and Lemma 3.11 we obtain

$$\lim_{i \to \infty} 8\pi \chi(\tilde{\Sigma}_t) \geq 4\hat{k} |\tilde{\Sigma}| = 8\pi \chi(\tilde{\Sigma}).$$

This implies $\chi(\tilde{\Sigma}_t) \geq \chi(\tilde{\Sigma})$ for all $i$ sufficiently large. \qed

Lemma 3.13. If $\bar{m} \leq 0$, then

$$\lim_{t \to \infty} m_H(\Sigma_t) \leq -\left( \int_{\Sigma} \mu^{2/3} \right)^{3/2} \left( \frac{\mid\tilde{\Sigma}\mid}{4\pi} \right)^{3/2}.$$
Proof. Using Gauss-Codazzi equations, Gauss-Bonnet Theorem, and Lemma 3.5, we have

\[ m_H(\Sigma_t) = \sqrt{\frac{\Sigma_t}{16\pi}} \left( 1 - g - \frac{1}{16\pi} \int_{\Sigma_t} (H^2 - 4) \right) \]

\[ = \sqrt{\frac{\Sigma_t}{(16\pi)^3}} \left( 16\pi (1 - g) - \int_{\Sigma_t} (-2R + 4K_{\Sigma_t} + 4\text{Ric}(\nu, \nu) + |\hat{A}|^2 - 4) \right) \]

\[ = \sqrt{\frac{\Sigma_t}{(16\pi)^3}} \left( 8\pi \chi(\hat{\Sigma}) - \int_{\Sigma_t} (4K_{\Sigma_t} + 4(\text{Ric}(\nu, \nu) + 2) + |\hat{A}|^2 + o(\rho^{-3})) \right) \]

\[ = \sqrt{\frac{\Sigma_t}{(16\pi)^3}} \left( 8\pi \chi(\hat{\Sigma}) - 8\pi \chi(\Sigma_t) - \int_{\Sigma_t} |\hat{A}|^2 - 4 \int_{\Sigma_t} (\text{Ric}(\nu, \nu) + 2) + o(e^{-\frac{t}{2}}) \right) \]

If we choose \( t \) to be one of the times from the sequence described in Lemma 3.12, we have

\[ m_H(\Sigma_t) \leq -\frac{1}{2}(4\pi)^{-3/2} \sqrt{\Sigma_t} \int_{\Sigma_t} (\text{Ric}(\nu, \nu) + 2) + o(1). \quad (12) \]

Decompose

\[ \nu = a \frac{\nabla \rho}{|\nabla \rho|} + v \]

so that \( v \) is orthogonal to \( \nabla \rho \). We have \( |v| = \frac{\nabla^T \rho}{|\nabla \rho|} \) while \( a^2 = 1 - |v|^2 \). It follows from Lemma 3.5 that

\[ \text{Ric}(\nu, \nu) = a^2(-2 - 2\mu \rho^{-3}) + |v|^2(-2 + O(\rho^{-3})) + o(\rho^{-3}) \]

\[ = -2 - 2\mu \rho^{-3} + |v|^2O(\rho^{-3}) + o(\rho^{-3}). \quad (13) \]

Note that

\[ \int_{\Sigma_t} |v|^2 \rho^{-3} = \int_{\Sigma_t} \left( \frac{\nabla^T \rho}{|\nabla \rho|} \right)^2 \rho^{-3} = \int_{\Sigma_t} \left( \frac{\nabla^T s}{|\nabla s|} \right)^2 s^{-3} \]

\[ = \int_{\Sigma_t} \left( \frac{\nabla^T s}{|\nabla s|} \right)^2 s = O(e^{-t}), \]

where we used Lemma 3.9 in the last line. Putting this together with formulas (12) and (13) we obtain

\[ m_H(\Sigma_t) \leq (4\pi)^{-3/2} \sqrt{\Sigma_t} \int_{\Sigma_t} \mu \rho^{-3} + o(1) \]

22
\[ (4\pi)^{-3/2} \left( \int_{\Sigma_t} s^{-2} \right)^{1/2} \left( \int_{\Sigma_t} \mu s \right) + o(1). \]  

(14)

We use the Hölder inequality with \( p = 3 \) and \( q = 3/2 \) to see that

\[ \int_{\Sigma_t} \mu^{2/3} = \int_{\Sigma_t} s^{-2/3} (\mu s)^{2/3} \]

\[ \leq \left( \int_{\Sigma_t} s^{-2} \right)^{1/3} \left( \int_{\Sigma_t} |\mu| s \right)^{2/3}. \]

Taking the \( 3/2 \) power of both sides, and using the fact that \( \mu \leq 0 \), we have

\[ -\left( \int_{\Sigma_t} \mu^{2/3} \right)^{3/2} \geq \left( \int_{\Sigma_t} s^{-2} \right)^{1/2} \left( \int_{\Sigma_t} \mu s \right). \]

Combining this with inequality (14) to conclude that

\[ m_H(\Sigma_t) \leq -\left( \frac{1}{4\pi} \int_{\Sigma_t} \mu^{2/3} \right)^{3/2} + O(e^{-t/2}), \]

for any sequence of \( t \)'s as in the previous lemma. Now the result follows from taking the limit as this sequence approaches infinity and applying monotonicity of \( \dot{\Sigma}_t \) and Lemma 3.10.

\[ \square \]

**Proof of Theorem 1.1** Recalling that \( \hat{g} \) has constant curvature \(-1, 0, \) or \(1\) on \( \hat{\Sigma} \) and our normalization of area in the \( k = 0 \) case, we have

\[ |\hat{\Sigma}| = 4\pi \max(1, g - 1)^{3/2}. \]

Using Geroch Monotonicity (Corollary 3.4) and the previous lemma, we can now conclude

\[ m_H(\partial_1 M) \leq \lim_{t \to \infty} m_H(\Sigma_t) \leq -\left( \int_{\Sigma_t} \mu^{2/3} \right)^{3/2} \left( \frac{|\Sigma|}{4\pi} \right)^{3/2} \]

\[ \leq \bar{m} \max(1, g - 1)^{3/2}, \]

where we used the definition \( \bar{m} = \sup \mu \).

All that remains is to prove the rigidity. We assume all of the hypotheses of Theorem 1.1, as well as equality in the Penrose inequality. By Corollary 3.4, the Hawking mass \( m_H(\Sigma_t) \) must be constant, and moreover, by equation (6),

\[ \int_{\Sigma_t} (2(R + 6) + |A|^2 + 4H^{-2} |\nabla H|^2) = 0. \]  

(15)
As argued on page 422 of [12], it follows that $H$ is a positive constant on each $\Sigma_t$, each $\Sigma_t$ is smooth, and there are no jump times. By the Smooth Start Lemma 2.4 of [12], $\Sigma_t$ is a classical solution of inverse mean curvature flow foliating the manifold $M$. This allows us to think of inverse mean curvature flow as defining a diffeomorphism from $[0, \infty) \times \partial_1 M$ to $M$. (In particular, $\partial M$ must be connected.) We can write the metric as

$$g = H^{-2} dt^2 + g_{\Sigma_t}.$$  

Note that equation (15) implies that $R = -6$ everywhere, and $\dot{A} = 0$ on each $\Sigma_t$. In particular, $A = \frac{H}{2} g_{\Sigma_t}$ and thus $\partial_t g_{\Sigma_t} = \frac{H}{2} A = g_{\Sigma_t}$. Hence

$$g = H^{-2} dt^2 + e^t g_{\Sigma_0}. \quad (16)$$

Recall that

$$\frac{dH}{dt} = \Delta H^{-1} - (|A|^2 + \text{Ric}(\nu, \nu))H^{-1},$$

which implies $\text{Ric}(\nu, \nu)$ is constant on each $\Sigma_t$ and so, by the Gauss-Codazzi equations, so is the Gauss curvature $K$. In particular, $\Sigma_0 = \partial M$ has constant curvature. From here it is clear that $(M, g)$ must be the Kottler space, since Kottler spaces are the unique asymptotically locally hyperbolic manifolds with $R = -6$ of the form $g = f(r)^2 dr^2 + r^2 g_{\Sigma_0}$, where $g_{\Sigma_0}$ is a constant curvature metric on $\Sigma_0$.

**Proof of Theorem 1.3.** Assume the hypotheses of Theorem 1.3. If there exists at least one compact minimal surface in $M$, then the result follows from Corollary 1.2 applied to the exterior region of $M$.

So let us assume that $M$ contains no compact minimal surfaces, and suppose that $\bar{m} < 0$. In particular, the mass aspect $\mu$ is everywhere negative. We can consider a weak inverse mean curvature flow whose initial condition is effectively a point, just as Huisken and Ilmanen did in [12, Section 8]. Then the same arguments used to prove Theorem 1.1 show that

$$0 = \lim_{t \rightarrow 0} m_H(\Sigma_t) \leq \lim_{t \rightarrow \infty} m_H(\Sigma_t) \leq \bar{m},$$

where the Hawking mass here has $g = 0$. Rigidity follows according to an argument similar to the one used in Theorem 1.1 above.

**4 The results of Chruściel-Simon: Proof of Theorem 2.3**

Assume the hypotheses of Theorem 2.3. In particular, we have a static data set $(M, g, V)$ and a reference Kottler space $(M_0, g_0, V_0)$ that has the same
surface gravity. We also have the important assumption that $m_0 \leq 0$. We define

$$W = |\nabla V|^2,$$

and also a reference function $W_0$ on $M$ as follows: On the reference space $M_0$, $|\nabla V_0|^2$ is constant on each level set of $V_0$ and, because of this, one may regard it as a function composed with $V_0$. That is, there exists a single-variable function $\omega$ with the property that $|\nabla V_0|^2 = \omega(V_0)$ as functions on $M_0$. We define the function $W_0$ on $M$ to be the function

$$W_0 = \omega(V).$$

Recall the formula $V_0 = \sqrt{r^2 + \frac{k - 2m}{r}}$ for an appropriate coordinate $r$ on $M_0$. Inverting this formula, we may think of $r$ as some function of $V_0$. In a manner similar to the way we defined $W_0$, we can define a reference function $r$ on $M$ by composing this function with $V$. (It might be logical to call this function $r_0$, but that is unnecessary because there is no ambiguity here.)

**Lemma 4.1.** Under the hypotheses of Theorem 2.3 and using the notation introduced above, if we consider an open set in $M$ on which $W$ does not vanish, then on that open set $W - W_0$ satisfies the elliptic inequality

$$\Delta(W - W_0) + \langle \xi, \nabla(W - W_0) \rangle + \alpha(W - W_0) \geq 0,$$

where $\xi$ is a smooth vector field and $\alpha$ is a smooth function whose sign is the opposite of $m_0$.

This is the critical ingredient of the proof of Theorem 2.3 and it corresponds to equation (VII.15) in [8].

**Proof.** Since $\nabla V \neq 0$, the level sets of $V$ are smooth surfaces which we denote $\Sigma_V$. We introduce the notation

$$\partial_V = \frac{\nabla V}{|\nabla V|^2} = \frac{\nabla V}{W}.$$

Note that $\partial_V$ has the nice property that if $f$ is a single-variable function, then

$$\partial_V(f(V)) = f'(V).$$

Recall the static equations

$$\Delta_g V = 3V \quad \text{and} \quad \text{Ric}(g) = \frac{1}{V} \text{Hess}(V) - 3g.$$
Also note that $\nabla W = 2 \text{Hess} V(\nabla V, \cdot)$ and thus

$$\partial_t W = 2 \text{Hess} V \left( \frac{\nabla V}{|\nabla V|}, \frac{\nabla V}{|\nabla V|} \right) = \frac{2}{W} \text{Hess} V(\nabla V, \nabla V).$$

Choose an orthonormal frame $e_1, e_2, e_3$ such that $e_3 = \frac{\nabla V}{|\nabla V|}$. Starting with the Böchner formula and the static equations above, we have

$$\Delta W = 2|\text{Hess} V|^2 + 2\text{Ric}(\nabla V, \nabla V) + 2\langle \nabla(\Delta V), \nabla V \rangle$$

$$= 2 \sum_{i,j = 1}^3 |\text{Hess} V(e_i, e_j)|^2 + \frac{2}{W} \text{Hess} V(\nabla V, \nabla V)$$

$$= 2|\text{Hess} V(e_3, e_3)|^2 + 4 \sum_{i = 1}^2 |\text{Hess} V(e_3, e_i)|^2$$

$$+ 2 \sum_{i,j = 1}^2 |\text{Hess} V(e_i, e_j)|^2 + \frac{2}{W} \text{Hess} V(\nabla V, \nabla V)$$

$$= 2 \left| \text{Hess} V \left( \frac{\nabla V}{|\nabla V|}, \frac{\nabla V}{|\nabla V|} \right) \right|^2 + \frac{4}{|\nabla V|^2} \sum_{i = 1}^2 |\text{Hess} V(\nabla V, e_i)|^2$$

$$+ 2|\nabla V|^2 |A_{\Sigma V}|^2 + \frac{2|\nabla V|^2}{W} \text{Hess} V \left( \frac{\nabla V}{|\nabla V|}, \frac{\nabla V}{|\nabla V|} \right)$$

$$= \frac{1}{2} (\partial_t W)^2 + \frac{1}{W^2} |\nabla^T W|^2 + 2W (|\hat{A}_{\Sigma V}|^2 + \frac{1}{2} H_{\Sigma V}^2) + \frac{W}{\sqrt{W}} \partial_t W$$

$$= \frac{1}{2} (\partial_t W)^2 + WH_{\Sigma V}^2 + \frac{W}{\sqrt{W}} \partial_t W + \frac{1}{W^2} |\nabla^T W|^2 + 2W |\hat{A}_{\Sigma V}|^2$$

$$= \frac{1}{2} (\partial_t W)^2 + [\text{Hess} V(e_1, e_1) + \text{Hess} V(e_2, e_2)]^2 + \frac{W}{\sqrt{W}} \partial_t W + \frac{1}{W^2} |\nabla^T W|^2 + 2W |\hat{A}_{\Sigma V}|^2$$

$$= \frac{1}{2} (\partial_t W)^2 + [\Delta V - \text{Hess} V(e_3, e_3)]^2 + \frac{W}{\sqrt{W}} \partial_t W + \frac{1}{W^2} |\nabla^T W|^2 + 2W |\hat{A}_{\Sigma V}|^2$$

$$= \frac{1}{2} (\partial_t W)^2 + [3V - \text{Hess} V(e_3, e_3)]^2 + \frac{W}{\sqrt{W}} \partial_t W + \frac{1}{W^2} |\nabla^T W|^2 + 2W |\hat{A}_{\Sigma V}|^2$$

$$= \frac{1}{2} (\partial_t W)^2 + \left[ 3V - \frac{1}{2} \partial_t W \right]^2 + \frac{W}{\sqrt{W}} \partial_t W + \frac{1}{W^2} |\nabla^T W|^2 + 2W |\hat{A}_{\Sigma V}|^2$$

$$= \frac{3}{2} (\partial_t W)^2 - 3V \partial_t W + 9V^2 + \frac{W}{\sqrt{W}} \partial_t W + \frac{1}{W^2} |\nabla^T W|^2 + 2W |\hat{A}_{\Sigma V}|^2$$
Meanwhile,
\[
\Delta W_0 = \text{div}(\nabla(\omega(V))) = \text{div}(\omega'(V)\nabla V) \\
= \omega''(V)W + \omega'(V)\Delta V = \omega''(V)W + 3\omega'(V)V.
\]

Notice that both of the computations above provide a way to compute \(\Delta_0(\omega(V_0))\) in the reference space \(M_0\). Since \(\omega(V_0)\) is constant on level surfaces of \(V_0\) and since those level surfaces are umbilic, the first computation yields
\[
\Delta_0(\omega(V_0)) = \frac{3}{4}|\partial_0(\omega(V_0))|^2 - 3V\partial_0(\omega(V_0)) + 9V_0^2 + \frac{\omega(V_0)}{V_0} - \partial_0(\omega(V_0)) \\
= \frac{3}{4}|\omega'(V_0)|^2 - 3V\omega'(V_0) + 9V_0^2 + \frac{\omega(V_0)}{V_0} - \omega'(V_0).
\]

Meanwhile, if we do the second computation in the reference space, we find
\[
\Delta_0(\omega(V_0)) = \omega''(V_0)\omega(V_0) + 3\omega'(V_0)V_0.
\]

Equating the right-hand sides of the two equations above, we actually obtain a differential equation for the single-variable function \(\omega\), which means that we can replace the \(V_0\) by \(V\) to obtain the following equation on the original space \(M\):
\[
\omega''(V)\omega(V) + 3\omega'(V)V = \frac{3}{4}|\omega'(V)|^2 - 3V\omega'(V) + 9V^2 + \frac{\omega(V)}{V} - \omega'(V) \\
\omega''(V)V_0 + 3\omega'(V)V = \frac{3}{4}|\partial_0 W_0|^2 - 3V\partial_0 W_0 + 9V^2 + \frac{\omega(V)}{V} - \omega'(V).
\]

Picking up from our expression for \(\Delta W_0\) above, we now have
\[
\Delta W_0 = \omega''(V)W + 3\omega'(V)V \\
= \omega''(V)W_0 + 3\omega'(V)V + \omega''(V)(W - W_0) \\
= \frac{3}{4}|\partial_0 W_0|^2 - 3V\partial_0 W_0 + 9V^2 + \frac{\omega(V)}{V} - \omega'(V)(W - W_0).
\]

Now we can subtract this expression from our expression for \(\Delta W\) (except for the last two nonnegative terms which we ignore) to find
\[
\Delta(W - W_0) \geq \frac{3}{4}|\partial_0 W|^2 - 3V\partial_0 W + 9V^2 + \frac{\omega(V)}{V} - \omega'(V) \\
- \left[\frac{3}{4}|\partial_0 W_0|^2 - 3V\partial_0 W_0 + 9V^2 + \frac{\omega(V)}{V} - \omega'(V)(W - W_0)\right] \\
= \frac{3}{4}\partial_0(W - W_0)\partial_0(W - W_0) - 3V\partial_0(W - W_0).
\]
\[
\begin{align*}
&+ \frac{W}{V} \partial_V W - \frac{W}{V} \partial_V W_0 + \frac{W}{V} \partial_V W_0 - \frac{W}{V} \partial_V W_0 - \omega''(V)(W - W_0) \\
&= \left[ \frac{2}{3} \partial_V (W + W_0) - 3V + \frac{W}{V} \right] \partial_V (W - W_0) + (W - W_0) \frac{\omega'(V)}{V} - \omega''(V)(W - W_0) \\
&= \left[ \frac{2}{3} \partial_V (W + W_0) - 3V + \frac{W}{V} \right] \partial_V (W - W_0) + \frac{\omega'(V)}{V} - \omega''(V)(W - W_0).
\end{align*}
\]

It only remains to compute the sign of the zero order coefficient. Using the formula

\[V_0 = \left( r^2 + k - \frac{2m_0}{r} \right)^{1/2}\]

in \(M_0\), we can perform simple computations using the chain rule to show that

\[\omega(V) = \left( r + \frac{m_0}{r^2} \right)^2.\]

and

\[\frac{dV}{dr} = \frac{\sqrt{\omega(V)}}{V}.\]

Therefore

\[\omega'(V) = 2V \left( 1 - \frac{2m_0}{r^3} \right),\]

and

\[\omega''(V) = 2V \left( 1 - \frac{2m_0}{r^3} \right) + 2V \frac{6m_0}{r^3} \frac{dr}{dV} = \frac{\omega'(V)}{V} + \frac{12V^2}{\sqrt{\omega(V)}} \frac{m_0}{r^4}.\]

Thus

\[\frac{\omega'(V)}{V} - \omega''(V) = -\frac{12V^2 m_0}{\sqrt{W_0} r^4}.\]

So we see that as long as \(m_0 \leq 0\), the coefficient of \(W - W_0\) is nonnegative.

\[\boxed{}\]

**Corollary 4.2.** Under the hypotheses of Theorem 2.3 and using the notation introduced above, we have \(W \leq W_0\) on \(M\).

**Proof.** By the definition of \(W_0\), we know that \(W = W_0\) on \(\partial_1 M\), and \(W \leq W_0\) on all of \(\partial M\) since \(\partial_1 M\) was assumed to have the largest surface gravity of any boundary component. We also know from Proposition 2.2 that \(W - W_0 = 0\) at the conformal infinity. (We will see a more detailed calculation below.) Suppose that \(W > W_0\) somewhere in \(M\). Since \(W \leq W_0\) at the boundary and also “at infinity,” the function \(W - W_0\) must achieve
its positive maximum value at some point \( p \) in the interior of \( M \). Since \( W(p) > W_0(p) > 0 \), we see that the previous lemma applies to some open set \( U \) containing \( p \). By the maximum principle for the elliptic inequality given by the Lemma, \( W - W_0 \) cannot have a local maximum in \( U \), which is a contradiction. \( \square \)

**Proof of Theorem 2.3.** We claim that for any point \( p \in \partial_1 M \),

\[
K_{\partial_1 M}(p) \geq K_{\partial M_0},
\]

where \( K_{\partial_1 M}(p) \) is the Gaussian curvature of \( \partial_1 M \) at \( p \), while \( K_{\partial M_0} \) is the constant Gaussian curvature of the \( \partial M_0 \) in the reference space. Integrating this inequality and using the Gauss-Bonnet Theorem, the claim clearly implies that

\[
\frac{\chi(\partial_1 M)}{|\partial_1 M|} \geq \frac{\chi(\partial M_0)}{|\partial_1 M_0|},
\]

so we now focus on proving the claim.

For the following, we use the same notation as in the proof of Lemma 4.1. Since \( W \neq 0 \) at \( \partial M \), the vector field \( \partial_V \) is well-defined near \( \partial_1 M \), and we can consider the flow \( \varphi \) generated by \( \partial_V \) near \( \partial_1 M \). Choose a point \( p \in \partial_1 M \). Applying Taylor’s Theorem to the function \( W \) restricted to the flow line starting at \( p \), we have

\[
W(\varphi(p)) = W(p) + [(\partial_V W)(p)]v + [(\partial_V^2 W)(p)]v^2 + O(v^3).
\]

We know that \( W(p) = \kappa^2 \), where \( \kappa \) is the surface gravity of \( \partial_1 M \). Next,

\[
\partial_V W = 2 \text{Hess } V(\nabla V, \partial_V) = 2 \text{Hess } V(e_3, e_3) = V(2\text{Ric}(e_3, e_3) + 6) = 0,
\]

where the last identity follows because \( V \) vanishes on \( \partial M \). Following same reason and using the Gauss-Codazzi equations plus the fact that \( \partial M \) is totally geodesic, we have

\[
\partial_V^2 W = (2\text{Ric}(e_3, e_3) + 6) + V \partial_V(2\text{Ric}(e_3, e_3) + 6) = 2\text{Ric}_p(e_3, e_3) + 6 = R - 2K_{\partial_1 M}(p) + 6 = -2K_{\partial_1 M}.
\]

Thus

\[
W(\varphi(p)) = \kappa^2 - 2[K_{\partial_1 M}(p)]v^2 + O(v^3).
\]

Performing the same computation in the reference solution, we obtain

\[
W_0(\varphi(p)) = \kappa^2 - 2K_{\partial M_0}v^2 + O(v^3).
\]
The claim now follows from Corollary 4.2.

We now focus on proving the mass inequality $\mu \leq m_0$. Recall from Proposition 2.2 that we have

\[
V^2 = \rho^2 + \hat{k} - \frac{4}{3}\mu\rho^{-1} + o_1(\rho^{-1}),
\]

where $\rho$ is a coordinate as in the definition of asymptotically locally hyperbolic. Differentiating this, we find

\[
2V\nabla V = (2\rho + \frac{4}{3}\mu\rho^{-2})\nabla \rho + o(\rho^{-1}).
\]

Taking the norm-square of both sides, and using the asymptotically locally hyperbolic property,

\[
V^2 W = (\rho^2 + \frac{4}{3}\mu\rho^{-3})^2|\nabla \rho|^2 + o(\rho^{-1}),
\]

Then

\[
W = \frac{\rho^2 + \hat{k}\rho^2 + \frac{4}{3}\mu\rho + o(\rho)}{\rho^2 + \hat{k} - \frac{4}{3}\mu\rho^{-1} + o(\rho^{-1})}
\]

Recall that we also have

\[
r^2 + \hat{k} - \frac{2m_0}{r} = V^2 = \rho^2 + \hat{k} - \frac{4}{3}\mu\rho^{-1} + o_1(\rho^{-1}),
\]

by definition of the function $r$. In particular,

\[
r^{-1} = \rho^{-1} + o(\rho^{-1}).
\]

By changing variables from $r$ to $\rho$, we have

\[
W_0 = \left(r + \frac{m_0}{r^2}\right)^2 = r^2 + 2m_0r^{-1} + O(\rho^{-4})
\]

Comparing these asymptotic expansions for $W$ and $W_0$ and using Corollary 4.2 we see that as $\rho \to \infty$, we have

\[
\frac{8}{7}\mu \leq -\frac{4}{3}\mu + 4m_0,
\]

from which the result follows. \qed
References

[1] Vincent Bonini and Jie Qing. A positive mass theorem on asymptotically hyperbolic manifolds with corners along a hypersurface. *Ann. Henri Poincaré*, 9(2):347–372, 2008.

[2] W. Boucher, G. W. Gibbons, and Gary T. Horowitz. Uniqueness theorem for anti-de Sitter spacetime. *Phys. Rev. D (3)*, 30(12):2447–2451, 1984.

[3] Hubert L. Bray. Proof of the Riemannian Penrose inequality using the positive mass theorem. *J. Differential Geom.*, 59(2):177–267, 2001.

[4] Hubert L. Bray and Dan A. Lee. On the Riemannian Penrose inequality in dimensions less than eight. *Duke Math. J.*, 148(1):81–106, 2009.

[5] Simon Brendle and Otis Chodosh. A volume comparison theorem for asymptotically hyperbolic manifolds, 2013, arXiv:1305.6628.

[6] Gary L. Bunting and A. K. M. Masood-ul Alam. Nonexistence of multiple black holes in asymptotically Euclidean static vacuum space-time. *Gen. Relativity Gravitation*, 19(2):147–154, 1987.

[7] Piotr T. Chruściel and Marc Herzlich. The mass of asymptotically hyperbolic Riemannian manifolds. *Pacific J. Math.*, 212(2):231–264, 2003.

[8] Piotr T. Chruściel and Walter Simon. Towards the classification of static vacuum spacetimes with negative cosmological constant. *J. Math. Phys.*, 42(4):1779–1817, 2001.

[9] Levi Lopes de Lima and Frederico Girão. A Penrose inequality for asymptotically locally hyperbolic graphs, 2013, arXiv:1304.7887.

[10] G. J. Galloway, S. Surya, and E. Woolgar. On the geometry and mass of static, asymptotically AdS spacetimes, and the uniqueness of the AdS soliton. *Comm. Math. Phys.*, 241(1):1–25, 2003.

[11] Gary T. Horowitz and Robert C. Myers. AdS-CFT correspondence and a new positive energy conjecture for general relativity. *Phys. Rev. D (3)*, 59(2):026005, 12, 1999.

[12] Gerhard Huisken and Tom Ilmanen. The inverse mean curvature flow and the Riemannian Penrose inequality. *J. Differential Geom.*, 59(3):353–437, 2001.
[13] William Meeks, III, Leon Simon, and Shing Tung Yau. Embedded minimal surfaces, exotic spheres, and manifolds with positive Ricci curvature. *Ann. of Math. (2)*, 116(3):621–659, 1982.

[14] H. Müller zum Hagen, David C. Robinson, and H. J. Seifert. Black holes in static vacuum space-times. *General Relativity and Gravitation*, 4:53–78, 1973.

[15] André Neves. Insufficient convergence of inverse mean curvature flow on asymptotically hyperbolic manifolds. *J. Differential Geom.*, 84(1):191–229, 2010.

[16] André Neves and Gang Tian. Existence and uniqueness of constant mean curvature foliation of asymptotically hyperbolic 3-manifolds. II. *J. Reine Angew. Math.*, 641:69–93, 2010.

[17] Ivaldo Nunes. Rigidity of area-minimizing hyperbolic surfaces in three-manifolds, 2011, arXiv:1103.4805.

[18] Jie Qing. On the rigidity for conformally compact Einstein manifolds. *Int. Math. Res. Not.*, (21):1141–1153, 2003.

[19] Xiaodong Wang. The mass of asymptotically hyperbolic manifolds. *J. Differential Geom.*, 57(2):273–299, 2001.

[20] Xiaodong Wang. On the uniqueness of the AdS spacetime. *Acta Math. Sin. (Engl. Ser.)*, 21(4):917–922, 2005.