ON GENERALIZED HILBERT-KUNZ FUNCTION AND MULTIPLICITY

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Abstract. Let \( (R, \mathfrak{m}) \) be a local ring of characteristic \( p > 0 \) and \( M \) a finitely generated \( R \)-module. In this note we consider the limit: 
\[
\lim_{n \to \infty} \frac{\ell(H^0_m(F^n(M)))}{p^{nd}}
\]
where \( F(-) \) is the Peskine-Szpiro functor. A consequence of our main results shows that the limit always exists when \( R \) is excellent and has an isolated singularity. Furthermore, if \( R \) is a complete intersection, then the limit is 0 if and only if the projective dimension of \( M \) is less than the Krull dimension of \( R \). Our results work quite generally for other homological functors and can be used to prove that certain limits recently studied by Brenner exist over projective varieties.

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1. Introduction

Let \( R \) be a local ring of characteristic \( p \) and \( M \) a finitely generated \( R \)-module. Let \( F^n_R(M) = M \otimes_R F^n R \) denote the \( n \)-fold iteration of the Frobenius functor given by base change along the Frobenius endomorphism. Let \( \dim R = d \) and \( q = p^n \). In this paper we study the following, first introduced under different notations by Epstein and Yao (15):

\[
f_{gHK}^M(n) := \ell(H^0_m(F^n(M)))
\]
and

\[
e_{gHK}(M) := \lim_{n \to \infty} \frac{f_{gHK}^M(n)}{p^{nd}},
\]

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which we call the generalized Hilbert-Kunz function and generalized Hilbert-Kunz multiplicity of \( M \), respectively. Obviously, when \( M = R/I \) is Artinian, we have the usual Hilbert-Kunz function and multiplicity. These notions, introduced by Kunz and Monsky, have been intensely studied due to their connections to tight closure, \( F \)-signature and their inherent very interesting and mysterious behavior. See the recent survey [17].

The first place where \( f_{gHK}^M \) was studied that we are aware of is a paper by Ian Aberbach [1] where it was shown that when \( R \) is a domain essentially of finite type over a field and \( M = R/I \) with \( \dim R/I = 1 \), then \( f_{gHK}^M(n) \) is bounded from above. Recently, they have been studied in much more general form by Epstein-Yao in [15] (see Definition 2.3).

However, the main inspiration for our work is a recent astonishing result by Cutkosky ([8, Corollary 11.3]), who showed that under very mild conditions, the limit:

\[
\lim_{n \to \infty} \frac{\ell(H^0_m(R/I^n))}{n^d}
\]

exists. This limit was first defined as a \( \lim sup \) by Katz and Validashti in [18].

Our main results show that under some assumptions, the generalized versions of Hilbert-Kunz multiplicity behave in a remarkably similar manner to the classical ones. Let \( \text{IPD}(M) \) denote the set of prime ideals such that \( \text{pd}_{R_p} M_p = \infty \). Let \( m \) be the maximal ideal. The category of modules \( M \) with \( \text{IPD}(M) \subseteq \{ m \} \) contains all modules of finite lengths, modules of finite projective dimensions and is closed under extension, direct summand, kernel of epimorphism and cokernel of monomorphism. In particular, when \( R \) has an isolated singularity, all finitely generated modules satisfy this property.

Some of our main results are summarized below, see Corollaries 3.4 and 4.5 for the proofs.

**Theorem 1.1.** Suppose that \( \text{IPD}(M) \subseteq \{ m \} \). Then

1. If \( R \) is excellent and Cohen-Macaulay on the punctured spectrum, \( e_{gHK}(M) \) exists.
2. If \( R \) is Cohen-Macaulay, \( e_{gHK}(M) \) exists.
3. If \( R \) is a complete intersection, then \( e_{gHK}(M) = 0 \) if and only if \( \text{pd} M < \dim R \).

Statement (3) is related to two different facts about complete intersections: Dutta and Miller have proved in [12, 23] that if \( \ell(M) < \infty \) then \( e_{gHK}(M) = \ell(M) \) if and only if \( \text{pd} M < \infty \); and it was shown in [10] that under the assumptions of the Theorem, \( f_{gHK}^M(n) = 0 \) for some \( n \) iff \( \text{pd} M < \dim R \). There are examples to show that (3) is not true if we only assume \( R \) is Gorenstein or \( \text{IPD}(M) \) is not in \( \{ m \} \).

In fact, the methods we employ, which are homological, give more general results on the asymptotic behavior of various functors on iterations of the Frobenius endomorphism, see [4.7, 4.9, 4.11, 4.12]. Results in the same spirit have been obtained by other authors ([2, 5, 13, 22, 24]).

For instance, we are able to prove that limits exist for higher local cohomology modules (see 3.3).

**Corollary 1.2.** Let \( R \) be a local ring of dimension \( d \). Suppose that either \( R \) is Cohen-Macaulay or excellent and Cohen-Macaulay on the punctured spectrum. Let \( M \) a finitely
generated \( R \)-module such that \( \text{pd}_{R_p} M_p < d - k \) on the punctured spectrum, then

\[
\lim_{n \to \infty} \frac{\ell_R(H^k_{m}(F^n(M)))}{p^{nd}}
\]

exists.

In particular, if \( M \) is locally free on the punctured spectrum, the limit above exists for all \( k < d \).

These limits have been studied recently by Brenner, who denote them as \( \text{HK}^i(M) \). In fact our results, Theorems 4.15 and 5.2 can be combined with results of Brenner to show that there exists a finite length module over a local hypersurface with irrational Hilbert-Kunz multiplicity ([6]).

Several other applications follow. For example, one can prove that certain limits, studied by Brenner ([6]) over projective varieties, exist. In the following \( F^n \) denote the \( n \)th-iteration Frobenius pull-back.

**Corollary 1.3.** Let \( X \) be a polarized projective variety over a field \( k \) of characteristic \( p \) of dimension \( d \), with a fixed very ample invertible sheaf \( \mathcal{O}_X(1) \). Let \( F \) be a vector bundle on \( X \). Then for each \( 0 < k < \dim X \) the limit

\[
\lim_{n \to \infty} \frac{\sum_{m \in \mathbb{Z}} h^k((F^n F)(m))}{p^{n(d+1)}}
\]

exists if \( X \) is \((S_{k+2})\).

As another application we can prove an effective version of the fact that over a complete intersection, the Picard group of punctured spectrum has no torsion elements (see Theorem 4.15).

**Theorem 1.4.** Let \( R \) be a local ring satisfying Serre’s condition \((S_2)\) and \( \dim R \geq 3 \). Let \( I \) be a reflexive ideal that is locally free on \( \text{Spec} \ R - \{m\} \). Then

1. \((1)\) \( \lim_{n \to \infty} \frac{\ell_R(H^q_{m}(I^{[q]}))}{q^d} = \lim_{n \to \infty} \frac{\ell_R(H^q_{m}(I^{[q^n]}))}{q^{nd}} \) exists.

2. When \( R \) is a complete intersection, the limit in part (1) is 0 if and only if \( I \) is principal. In particular, the Picard group of \( \text{Spec} \ R - \{m\} \) has no torsion elements.

2. Preliminaries

Let \( R \) be a commutative ring of characteristic \( p \) and \( f : R \to R \) be the Frobenius endomorphism. The Frobenius functor, introduced by Peskine and Szpiro in [20], is given by base change along the Frobenius endomorphism: \( F_R(M) = M \otimes_R f^R \) for any \( R \)-module \( M \). Its compositions are given by \( F^n_R(M) = M \otimes_R f^nR \), namely, the base change along the compositions \( f^n \) of \( f \). We omit the subscript \( R \) if there is no ambiguity about \( R \). Note particularly that the module structure on \( F^n(M) \) is via usual multiplication in \( R \) on the right hand factor of the tensor product. The values of the derived functors \( \text{Tor}^R_i(M, f^nR) \) are similarly viewed as \( R \)-modules via the target of the base change map \( f^n \).

For the rest of the paper we will denote \( f^m M \), i.e. the restriction of scalars from the iterated Frobenius map, by \( ^m M \).
It is easy to verify that $F^n(R) \cong R$ and that for cyclic modules $F^n(R/I) \cong R/I^{[q]}$, where $q = p^n$ and $I^{[q]}$ denotes the ideal generated by the $q$th powers of the generators of $I$. For convenience, we frequently use $q$ to denote the power $p^n$, which may vary.

In the sequel, we use $\ell(M)$ and $\text{pd}_M$ to denote the length and projective dimension, respectively, of the module $M$. By codimension of $M$, we mean $\dim R - \dim M$. We use the notation $x$ for a sequence of elements of $R$ and often write simply $R/x$ for $R/(x)$ to save space. We say that a ring $R$ is $F$-finite, if $IR$ is a finitely generated $R$-module. Let $\text{mod}(R)$ denote the category of finitely generated $R$-modules.

If $f, g : \mathbb{N} \to \mathbb{R}$ are sequences, we say that $f = O(g)$ if there exists a constant $C$ such that $f(n) \leq Cg(n)$ for all $n$. We say that $f = o(g)$ if $\lim_{n \to \infty} f(n)/g(n) = 0$.

**Definition 2.1.** Let $M$ be an $R$-module. One defines the infinite projective dimension locus of $M$ as

$$\text{IPD}(M) = \{ p \in \text{Spec } R \mid \text{pd}_{R_p} M_p = \infty \}.$$

**Definition 2.2.** We say that a function $g : \text{mod}(R) \to \mathbb{N}$ is subadditive if for any short exact sequence $0 \to A \to B \to C \to 0$ we have $g(B) \leq g(A) + g(C)$.

**Definition 2.3.** Let $M$ be a finitely generated module over $R$. One defines

$$e^+_{g\text{HK}}(M) = \limsup_{n \to \infty} \frac{\ell(H^0_m(F^n(M)))}{p^n \dim R}$$

and

$$e^-_{g\text{HK}}(M) = \liminf_{n \to \infty} \frac{\ell(H^0_m(F^n(M)))}{p^n \dim R}.$$ 

If $e^+_{g\text{HK}}(M) = e^-_{g\text{HK}}(M)$ we denote the same value by $e_{g\text{HK}}(M)$. We call $f^M_{g\text{HK}}(n) := \ell(H^0_m(F^n(M)))$ the generalized Hilbert-Kunz function of $M$.

This can be seen as a special case of relative multiplicity defined by Epstein-Yao in [15]. In their notation $e^+_{g\text{HK}}(M) = u^+_{M}(0, M)$ and $e^-_{g\text{HK}}(M) = u^-_{M}(0, M)$.

The following Theorem is due to Seibert([27]).

**Theorem 2.4.** Let $R$ be local $F$-finite ring of characteristic $p$ with perfect residue field and $\mathcal{C}$ be a family of finitely generated $R$-modules such that for any short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

$M \in \mathcal{C}$ if and only if $M', M'' \in \mathcal{C}$. Let $g$ be a subadditive function $\mathcal{C} \to \mathbb{R}$ such that $g$ is additive on direct sums of modules. Suppose that all modules in $\mathcal{C}$ have dimension at most $j$, then for any $M \in \mathcal{C}$ there is a constant $c(M)$ such that

$$g^{(n)}(M) = c(M)q^j + O(q^{j-1}).$$

If not stated otherwise, we will apply this theorem for the family of all finite generated modules $\text{mod}(R)$. However, we will need a full version to get better estimates.

**Example 2.5.** Let $i \geq 0$ be an integer and $N$ be a module such that $\text{Tor}_i(N, X)$ (respectively $\text{Ext}^i(N, X)$) has finite length for all $X \in \text{mod}(R)$. Then $g(M) = \ell(\text{Tor}_i(N, M))$ (respectively $g(M) = \ell(\text{Ext}^i(N, M))$) satisfies the conditions of Theorem [2.4].
Remark 2.6. If we are interested in the sequence $f_{gHK}^n(M)$, we can usually assume that $R$ is complete with perfect residue field and, thus, $F$-finite by Corollary 2.6 of [20]. Namely, we can use the following well-known construction

$$R \rightarrow \hat{R} \rightarrow \hat{R} \otimes_{k[[T_1, \ldots, T_d]]} k^\infty[[T_1, \ldots, T_d]] =: S,$$

where $k^\infty$ is the perfect closure of a coefficient field $k$ of $\hat{R}$.

Then $S$ is faithfully flat over $R$ with the maximal ideal $mS$ and is a complete and $F$-finite ring with a perfect residue field. By usual properties of local cohomology, $H^i_{mS}(M \otimes_R S) \cong H^i_m(M \otimes_R S)$ for any $R$-module $M$. Note that

$$F^n_R(M) \otimes_R S = M \otimes_R^nR \otimes_R S \cong M \otimes_R^nS = (M \otimes_R S) \otimes_S^nS = F^n_S(M \otimes_R S).$$

Therefore $H^i_m(F^n_R(M)) \otimes_R S = H^i_m(F^n_S(M \otimes_R S))$ and, since the extension is faithfully flat, $\ell_R(H^i_m(F^n_R(M))) = \ell_S(H^i_m(F^n_S(M \otimes_R S)))$.

Also, if IPD$(M) \subseteq \{m\}$, then $M \otimes_R S$ has the same property as an $S$-module. Namely, let $q$ be a non-maximal prime ideal of $S$, then $p = q \cap R$ is a non-maximal prime of $R$, and $R_p \rightarrow S_q$ is faithfully flat. So, if $F$ is a finite free resolution of $M$ over $R$, then $F \otimes_R S$ is a finite free resolution of $M \otimes_R S$.

Moreover, if $R$ is excellent and Cohen-Macaulay on the punctured spectrum, then $S$ will be also Cohen-Macaulay on the punctured spectrum.

3. Existence of limits

Let $D_R$ denote the dualizing complex of a local ring $R$. Local duality says that for a finitely generated module $M$, $H^d_m(M) \cong (H^{d-i}(\text{Hom}(M, D_R)))^\vee$. By convention, the depth of the zero module is $\infty$.

Theorem 3.1. Let $R$ be an $F$-finite ring of dimension $d > 0$ and $M$ a finitely generated $R$-module, such that IPD$(M) \subseteq \{m\}$. Let $d > k \geq 0$ be an integer, and assume that for all prime ideals $p \neq m$, one of the following holds: $R_p$ is Cohen-Macaulay, $M_p$ is free, or depth $M_p \geq \dim R_p - d + k + 2$. Then, if $R$ satisfies $(S_{k+1})$ on the punctured spectrum,

$$\lim_{n \to \infty} \frac{1}{q^n} \left( \ell \left( H^k_m(F^n(M)) \right) - \ell(\text{Ext}^{-d-k}_R(M, F^n(D_R))) \right) = 0.$$

Proof. Since $R$ is $F$-finite, it is excellent by the work of Kunz([20, Theorem 2.5]), so we can assume that $R$ is complete and has perfect residue field by Remark 2.6.

For clarity we shall denote $^nR$ by $S$.

First, we want to use the spectral sequence

$$E_2^{pq} = H^p \text{Hom}_R(H^{-q}(M \otimes_R S), D_R) \Rightarrow H^{p+q} \text{RHom}(M \otimes_R S, D_R)$$

to compute $H^d \text{RHom}(M \otimes_R S, D_R)$. Note that $H^{-i}(M \otimes_R S) = \text{Tor}^R_i(M, S)$, so it has finite length for any $i > 0$ as $M$ has finite projective dimension on the punctured spectrum, by a theorem of Peskine-Szpiro([26]). Therefore, for any $i > 0$

$$H^{d-i} \text{Hom}_R(H^i(M \otimes_R S), D_R) = H^i_m(\text{Tor}^R_i(M, S)) = 0.$$
Hence the spectral sequence collapses and shows the equality
\[ H^k_m(M \otimes_R S) = \text{H}^{d-k} \text{Hom}_R(M \otimes_R S, D_R) = \text{H}^{d-k} \text{RHom}(M \otimes_R S, D_R). \]
Furthermore, since
\[ \text{RHom}(M \otimes_R S, D_R) = \text{RHom}(M, \text{RHom}(S, D_R)) = \text{RHom}(M, D_S), \]
we obtain that
\[ H^k_m(M \otimes_R S) = H^{d-k}(\text{RHom}(M, D_S)) = H^{d-k}(\text{Hom}_R(F, D_S)) = H^{d-k}(F^* \otimes_R D_S), \]
where \( F \) is a free resolution of \( M \). We will use a spectral sequence
\[ E_2^{pq} = H^p(F^* \otimes_R H^q(D_S)) \implies H^{p+q}(F^* \otimes_R D_S) \]
to compute its homology.

Note that \( E_2^{p,q} = \text{Ext}_R^p(M, H^q(D_S)) \). Since \( H^q(D_S) = \text{Ann} H^q(D_R) \), we obtain that \( E_2^{p,q} = \text{Ext}_R^p(M, \text{Ann} H^q(D_R)) \).

Let \( C_i \) be the family of all finitely generated modules \( X \) such that \( \text{Supp} X \subseteq \text{Supp} H^i(D_R) \). Then the hypothesis of the theorem guarantees that \( \text{Ext}_R^j(M, X) \) has finite length for all \( i, j > 0 \) such that \( i + j \geq d - k - 1 \) and all \( X \in C_i \). Namely, for any prime \( p \) in the punctured spectrum, we want either \( H^i(D_{R_p}) = 0 \) or \( \text{pd} M_p < j \). Note, that is is enough to force these conditions for \( i + j = d - k - 1 \).

Since \( H^i(D_{R_p}) \) is the dual of \( H^p_{\text{dim} \text{R}_{d-i}}(R_p) \), \( H^i(D_{R_p}) = 0 \) whenever depth \( R_p > \text{dim} \text{R}_{d-i} \).
So, if \( R_p \) is Cohen-Macaulay, \( H^i(D_{R_p}) = 0 \) for all \( i > 0 \). Also, if \( M_p \) is free, \( \text{Ext}_R^j(M_p, X_p) = 0 \) for all \( j > 0 \). Moreover, note that if depth \( M_p \geq \text{dim} R_p - d + k + 2 \), then for all \( 0 < j < d - k - 1 \), either \( j \geq \text{pd} M_p \) or \( i = d - k - 1 - j > \text{dim} R_p - \text{depth} R_p \).

So, we can apply Theorem 2.4 to the family \( C_k \) and the function \( g(X) = \ell(\text{Ext}_R^j(M, X)) \).
Since \( \text{dim} H^i(D_R) \leq d - i \), Seibert’s theorem shows that
\[ \lim_{q \to \infty} \frac{\ell(\text{Ext}_R^j(M, nH^i(D_R)))}{q^d} = 0 \]
for any \( i, j > 0 \). Furthermore, since \( R \) is \( S_{k+1} \) on the punctured spectrum, \( H^j_p(R_p) = 0 \) for all \( j \leq \min(k + 1, d - 1) \), so \( H^n(D_R) \) has finite length for \( n \geq \max(d - k - 1, 1) \). Hence,
\[ \lim_{q \to \infty} \frac{\ell(\text{Hom}_R(M, nH^{d-k}(D_R)))}{q^d} = 0, \]
and, if \( k < d - 1 \),
\[ \lim_{q \to \infty} \frac{\ell(\text{Hom}_R(M, nH^{d-k-1}(D_R)))}{q^d} = 0. \]

We claim that for any \( i > 0 \) the contribution of \( E^{d-k-i,i}_{\infty} \) to \( H^{d-k}(F^* \otimes_R D_S) \) is \( o(q^d) \). To see this, note that the entry \( E^{d-k-i,i}_{s+1} \) is a quotient of a submodule of \( E^{d-k-i,i}_s \), so \( \ell(E^{d-k-i,i}_{s+1}) \leq \ell(E^{d-k-i,i}_s) \leq \ell(E^{d-k-i,i}_2) \). But
\[ \lim_{q \to \infty} \frac{\ell(E^{d-k-i,i}_2)}{q^d} = \lim_{q \to \infty} \frac{\ell(\text{Ext}_R^{d-k-i}(M, nH^i(D_R)))}{q^d} = 0 \]
as we have shown above.
At last, we want to estimate the contribution of $E_{d-k,0}$. Note that for any $s \geq 2$ the map on the $s$th sheet from $E_{s}^{d-k,0}$ is zero. Hence
\[
\ell(E_{s}^{d-k,0}) \geq \ell(E_{s+1}^{d-k,0}) \geq \ell(E_{s}^{d-k,s}) \geq \ell(E_{s}^{d-k,s-1}) - \ell(E_{2}^{d-k,s-1}).
\]
Therefore,
\[
\ell(E_{2}^{d-k,0}) \geq \ell(E_{\infty}^{d-k,0}) \geq \ell(E_{2}^{d-k,0}) - \sum_{s=2}^{d-k} \ell(E_{2}^{d-k,s-1}).
\]
Since $E_{2}^{d-k,s-1} = \text{Ext}^{d-k-s}(M, H^{s-1}(D_{s}))$, we have that
\[
\sum_{i=1}^{d-k} \ell(\text{Ext}^{d-k-i}(M, nH^{i}(D_{R}))) \geq \ell(\text{Ext}^{d-k}(M, nH^{0}(D_{R}))) - \ell(E_{\infty}^{d-k,0}) \geq 0.
\]
But, since the sum in the left-hand side is $o(q^d)$ by the first part of the proof, the statement follows.

This theorem allows us to prove the existence of $e_{g\text{HK}}(M)$ (and a bit more), as long as we are able to get the limit for the Ext from Seibert’s theorem. The following corollaries are all instances of this strategy.

**Corollary 3.2.** In the assumptions of the theorem, if $\text{pd}_{R_{p}} M_{p} < d - k$ on the punctured spectrum, then
\[
\lim_{n \to \infty} \frac{\ell(R(H^{k}_{n}(F^{n}(M))))}{p^{nd}}
\]
exists.

**Proof.** Since $\text{pd}_{R_{p}} M_{p} < d - k$ on the punctured spectrum, $\text{Ext}^{d-k}(M, X)$ has finite length for any finite module $X$. Hence
\[
\lim_{n \to \infty} \frac{\ell(\text{Ext}^{d-k}(M, nH^{0}(D_{R})))}{p^{nd}}
\]
exists by Seibert’s theorem, and the assertion follows from Theorem 3.1.

**Corollary 3.3.** Let $R$ be a local ring of dimension $d$. Suppose that either $R$ is Cohen-Macaulay or excellent and Cohen-Macaulay on the punctured spectrum. Let $M$ a finitely generated $R$-module such that $\text{pd}_{R_{p}} M_{p} < d - k$ on the punctured spectrum, then
\[
\lim_{n \to \infty} \frac{\ell(R(H^{k}_{n}(F^{n}(M))))}{p^{nd}}
\]
exists.

In particular, if $M$ is locally free on the punctured spectrum, the limit above exists for all $k < d$.

**Proof.** The Artinian case follows from Seibert’s theorem, so assume $d > 0$. By Remark 2.6 we can assume that $R$ is $F$-finite without affecting relevant issues, so we can apply the previous corollary.

As particular cases we obtain the following corollaries.
Corollary 3.4. Let $R$ be a Cohen-Macaulay local ring, or an excellent local ring that is Cohen-Macaulay on the punctured spectrum. Suppose $M$ is a finite $R$-module such that:

$$\text{IPD}(M) \subseteq \{m\}.$$ 

Then $e_{gHK}(M)$ exists.

Corollary 3.5. Let $R$ be an excellent local ring and $M$ be a finite module. Assume that for any prime $p \neq m$, $\text{pd} M_p < \infty$ and either $R_p$ is Cohen-Macaulay or $M_p$ is free. Then $e_{gHK}(M)$ exists.

In particular, $e_{gHK}(M)$ exists for any finitely generated module $M$, such that $\dim M \leq 1$ and $\text{IPD}(M) \subseteq \{m\}$.

Proof. The first part follows from Corollary 3.2. If $\dim M = 1$, then for any prime ideal $p$ of dimension one, $M_p$ is a finite length module of finite projective dimension, so the New Intersection Theorem implies that $R_p$ is Cohen-Macaulay.

Corollary 3.6. Let $R$ be an excellent local ring which is $(S_1)$ on the punctured spectrum and $M$ be a finite module, that is free on the punctured spectrum. Then $e_{gHK}(M)$ exists.

Corollary 3.7. Let $R$ be an excellent isolated singularity, then $e_{gHK}(M)$ exists for any finitely generated module $M$.

It follows from the proof of Theorem 3.1, that an upper bound exists in more general situation, since we only need to control the $(d - k)$th diagonal of the spectral sequence.

Theorem 3.8. Let $R$ be an $F$-finite ring of dimension $d > 0$ and $M$ a finitely generated $R$-module, such that $\text{IPD}(M) \subseteq \{m\}$. Let $k \geq 0$ be an integer, and assume that for all prime ideals $p \neq m$, one of the following holds: $R_p$ is Cohen-Macaulay, $M_p$ is free, or $\text{depth} M_p \geq \dim R_p - d + k + 1$. Then, if $R$ satisfies $S_k$ and $\text{pd} M_p < d - k$ on the punctured spectrum,

$$\ell \left( H^k_m(F^n(M)) \right) = O(q^d).$$

Proof. Following the proof and the notations of the theorem, we are going to use that $\ell(E^d_{s+1},i) \leq \ell(E^d_s,k-i,i) \leq \ell(E^d_{s+1},i)$. Then to be able to use Seibert’s theorem, we need that $\text{Ext}^j_R(M,X)$ has finite length for all $i,j > 0$ such that $i + j \geq d - k$ and all $X \in C_j$. As in Theorem 3.1, our assumptions guarantee that. Moreover, $S_k$ implies that $\text{Hom}(M,X)$ has finite length for $X \in C_{d-k}$, and, since $\text{pd} M_p < d - k$ on the punctured spectrum, $\text{Ext}^d_{R_p}(M_p,-) = 0$.

It follows that an upper bound for the generalized Hilbert-Kunz function exists quite generally.

Corollary 3.9. Let $R$ be a local ring. Then for any finite $R$-module $M$ such that $\text{IPD}(M) \subseteq \{m\}$, $e_{gHK}^+(M)$ is finite.

Proof. We just note that $\text{depth} M_p \geq \dim R_p - d + 1$ is trivially true on the punctured spectrum, so there is no need in extra assumptions, in particular, since we do not need to preserve the Cohen-Macaulay locus.
The following simple lemma shows that existence of the limit of $H^0_m(Tor^R_1(M, {}^nR))$ follows from existence of $e_{gHK}(syz M)$.

**Lemma 3.10.** Let $R$ be a local Noetherian ring of positive depth, then

$$H^0_m(Tor^R_1(M, {}^nR)) \cong H^0_m(F^n(syz M)).$$

**Proof.** From a long exact sequence for tensor product

$$0 \to Tor^R_1(M, {}^nR) \to F^n(syz M) \xrightarrow{f} F^n(F) \to F^n(M) \to 0$$

we obtain an exact sequences of local cohomology:

$$0 \to H^0_m(Tor^R_1(M, {}^nR)) \to H^0_m(F^n(syz M)) \to H^0_m(K)$$

and

$$0 \to H^0_m(K) \to H^0_m(F^n(F)),$$

here $K = \text{im } f$. Since depth $F > 0$, $H^0_m(K) = 0$, so

$$Tor^R_1(M, {}^nR) \cong H^0_m(F^n(syz M)).$$

$\square$

**Corollary 3.11.** Let $R$ be a local ring and $M$ is a finite $R$-module such that $\text{IPD}(M) \subseteq \{m\}$. Then $\ell(Tor^R_i(M, {}^nR)) = O(q^d)$. Moreover, if $R$ is Cohen-Macaulay, then

$$\lim_{q \to \infty} \frac{\ell(Tor^R_i(M, {}^nR))}{q^d}$$

exists for any $i$ and is equal to $e_{gHK}(syz_i M)$.

**Proof.** If $\text{dim } R = 0$, $M$ has finite length, so the result follows from Seibert’s theorem.

The general case follows from the lemma above. We only want to comment that $Tor^R_i(M, {}^nR)$ has finite length, since, by a theorem of Peskine-Szpiro\cite{26}, $\text{Tor}^R_i(P, {}^nR) = 0$ for any $i > 0$ and any $P$ with $\text{pd}_R P < \infty$. $\square$

We are not able to prove the existence of the limit in the full generality, as our applications of Seibert’s theorem require finite projective dimension on the punctured spectrum. However, we are able to get an upper bound assuming a well-known conjecture.

**Definition 3.12.** A module satisfies condition (LC) if there exists integer $l$ such that

$$m^lq H^0_m(F^n(M)) = 0$$

for all $q$.

This condition was studied for its connection to the localization for tight closure (for example, see \cite{19} Theorem 6). In fact, if all cyclic modules $R/I$ satisfy (LC) then being weakly $F$-regular implies $F$-regular.

**Proposition 3.13.** Let $R$ be a local ring that satisfies countable prime avoidance. If all finitely generated modules over $R$ satisfy (LC) then $e_{gHK}^+(M)$ is finite for any finitely generated module $M$. 
Proof. We proceed by induction on \(\dim M\), the base case \(\dim M = 0\) follows from Theorem 2.4. Let \(S\) be the (countable) set of associated primes of \(F^n(M)\) except possibly \(m\). By assumption one can find \(x \in m^l\) such that \(x\) is not in any prime of \(S\). We know that \(x^q H_m^0(F^n(M)) = 0\) for all \(q\).

Consider the exact sequence

\[
0 \to H_m^0(F^n(M)) \to F^n(M) \to C \to 0.
\]

We claim that \(x\) is a nonzero-divisor on \(C\). Clearly, \(\text{depth } C > 0\), so it suffices to show that \(\text{Ass}(C) \subseteq S\). Let \(p \in \text{Ass}(C)\) and consider the localization of the sequence above at \(p\). Since \(p\) is not \(m\), it follows that \(F^n(M)_p \cong C_p\), so \(p \in S\). The claim follows.

Tensoring the exact sequence with \(R/(x^q)\), and using that \(\text{Tor}_R^1(C, R/(x^q)) = 0\) and \(x^q H_m^0(F^n(M)) = 0\) we get:

\[
0 \to H_m^0(F^n(M)) \to F^n(M)/x^q F^n(M) \to C/x^q C \to 0.
\]

After taking local cohomology of this sequence, we get the embedding

\[
0 \to H_m^0(F^n(M)) \to H_m^0(F^n(M)/x^q F^n(M)).
\]

Moreover, applying the right-exact functor \(F^n(\,\cdot\,)\) to the exact sequence

\[
0 \to M \xrightarrow{x} M \to M/xM \to 0,
\]

we see that \(F^n(M)/x^q F^n(M) = F^n(M/xM)\). Thus, \(H_m^0(F^n(M))\) can be embedded in \(H_m^0(F^n(M/xM))\), and, since \(\dim M/xM < \dim M\), the result follows.

It is well-known that \(R\) has countable prime avoidance if it is complete ([7]), or if the residue field is uncountable.

Next we want to discuss applications to projective varieties over a field of positive characteristic. In this situation our results can be used to prove that certain limits, recently studied by Brenner in [6], exist.

**Corollary 3.14.** Let \(X\) be a polarized projective variety over a field \(k\) of characteristic \(p\) of dimension \(d\), with a fixed very ample invertible sheaf \(O_X(1)\). Let \(F\) be a vector bundle on \(X\). Then for each \(0 < k < \dim X\) the limit

\[
\lim_n \sum_{m \in \mathbb{Z}} h^k((F^n F)(m)) \overline{p^{(d+1)}}
\]

exists if \(X\) is \((S_{k+2})\).

**Proof.** We embed \(X\) into a projective space using \(O_X(1)\) and let \(R\) be the local ring at the vertex of the coordinate ring of \(X\) with respect to said embedding. One can find, up to shifts, a (non-unique) finitely generated module \(R\)-module \(M\) such that \(\tilde{M} \cong F\). There are well-known isomorphisms

\[
H^{i+1}_m(M) \cong \oplus_{i \in \mathbb{Z}} H^i(X, F(i))
\]

which allow us to use Theorem 3.1.

\(\square\)
4. Positivity of $e_{gHK}(M)$

It is well-known that the Hilbert-Kunz multiplicity of a finite length module is positive. So, it is natural to study whether the same holds for the generalized version. Surprisingly, it is indeed positive over a complete intersection unless if the module has non-maximal projective dimension.

First we establish a special case which will be used in the key result, Corollary 4.5. A part of the proof easily follows from the general result in the previous section, but we need it to get the full statement.

**Lemma 4.1.** Let $R$ be local Gorenstein ring and $M$ be a module of finite projective dimension. Suppose $\text{pd} M \leq k$ and $\text{pd}_{R_p} M_p < k$ on the punctured spectrum, then

$$\lim_{n \to \infty} \frac{\ell(H^n(M))}{q^d}$$

exists. Moreover, the limit is positive if and only if $\text{pd} M = k$.

**Proof.** If $\text{pd}_R < k$ then depth $M > d - k$ and also depth$(F^n(M)) > d - k$ for all $n$, so $H^{d-k}_m(F^n(M)) = 0$ for all $n$ and there is nothing to prove. Thus we assume $\text{pd}_R M = k$. By Local duality $\ell(H^{d-k}_m(F^n(M))) = \ell(\text{Ext}^k_R(F^n(M), R))$.

Let

$$0 \to F_k \delta_k \to F_{k-1} \delta_{k-1} \to \cdots \to F_0 \to M \to 0$$

be a minimal resolution of $M$. A resolution of $F^n(M)$ would look like:

$$0 \to F_k \delta^{[q]}_k \to F_{k-1} \delta^{[q]}_{k-1} \to \cdots \to F_0 \to M \to 0$$

It follows that $\text{Ext}^k_R(F^n(M), R)$ is the cokernel of the map $F_{k-1} \delta_{k-1} \to F_k$ where $\delta^*$ represents the transpose matrix of $\delta$. Thus we obtain that $\text{Ext}^k_R(F^n(M), R) \cong \text{Ext}^k_R(M, R \otimes_R X)$. By what we need follows from Theorem 2.4 applied to $g(X) = \ell(\text{Ext}^k_R(M, R \otimes_R X))$. Note that $\ell(\text{Ext}^k_R(M, R)) < \infty$, since $\text{pd} M_p < k$ on the punctured spectrum.

Since $\text{Ext}^k_R(M, R) \neq 0$, $g(X) \geq \ell(k \otimes_R X)$ for any $X$. In particular, $g^{(n)R} \geq \ell(R/m^{[q]})$. Since the Hilbert-Kunz multiplicity is positive, the generalized multiplicity will be positive too.  

**Corollary 4.2.** Let $R$ be a local Gorenstein ring and $M$ be a module of finite projective dimension. Then $e_{gHK}(M)$ exists, and is positive if and only if $\text{pd} M = 0$.

**Proposition 4.3.** Suppose $R$ is a Gorenstein local ring. The following are equivalent:

1. $e_{gHK}(M) > 0$ for all $M$ such that $\text{IPD}(M) = \{m\}$.
2. $e_{gHK}(M) > 0$ for all $M$ such that $\text{IPD}(M) = \{m\}$ and $\dim M = \dim R$.
3. $e_{gHK}(M) > 0$ for all maximal Cohen-Macaulay $M$ such that $\text{IPD}(M) = \{m\}$.

Moreover, if $R$ is a complete intersection, then the following condition is also equivalent to the first three.

4. $e_{gHK}(M) > 0$ for all $M$ such that $\text{IPD}(M) = \{m\}$ and depth $M = 0$. 

Proof. Clearly (1) implies (2) and (3), so we need to prove the other directions.

For the other direction, we claim that \( e_{gHK}(M \oplus F) = e_{gHK}(M) \) for a free module \( F \). Since \( F \) is free, we have an exact sequence

\[
0 \to F^n(M) \to F^n(M \oplus F) \to F^n(F) \to 0,
\]

thus

\[
0 \to H^0_m(F^n(M)) \to H^0_m(F^n(M \oplus F)) \to H^0_m(F^n(F))
\]
is also exact. But \( H^0_m(F^n(F)) = 0 \), so we get that \( H^*_m(F^n(M)) \cong H^*_m(F^n(M \oplus F)) \).

Now, we prove that (3) \( \Rightarrow \) (2). Since \( R \) is Gorenstein, we can use the Auslander-Bridger approximation to get

\[
0 \to N \to M \oplus F \to H \to 0,
\]

where \( N \) is maximal Cohen-Macaulay, \( F \) is free, and \( H \) is a module of finite projective dimension such that \( \text{depth } H = \text{depth } M \). From this exact sequence it is easy to see that

\[
\ell(H^0_m(F^n(M)))) = \ell(H^0_m(F^n(M \oplus F)))) \geq \ell(H^0_m(F^n(N)))).
\]

Now, we prove (3) \( \Rightarrow \) (1) and the other direction is trivial. Let \( M \) be any module such that IPD\((M) = \{m\} \). Then, by Theorem 2.5 of [10], \( m \in \text{Ass } F^n(M) \) any \( n > 0 \), in particular depth \( F(M) = 0 \). Therefore

\[
e_{gHK}(M) = \lim_{n \to \infty} \frac{\ell(H^0_m(F^n(M))))}{p^{nd}} = \frac{1}{p^d} \lim_{n \to \infty} \frac{\ell(H^0_m(F^n(F(M)))))}{p^{nd}} > 0.
\]

Note, that \( F(M) \) can have finite projective dimension, but the limit is still positive by the previous corollary.

\[\square\]

**Theorem 4.4.** Let \( R \) be a Cohen-Macaulay local ring and \( x \) be a regular element. Suppose \( M \) is a finitely generated \( R/\mathfrak{x} \)-module such that IPD\((M) \subseteq \{m\} \). Then \( e_{gHK,R}(M) \leq e_{gHK,R/\mathfrak{x}}(M) \).

Proof. Assume \( R \) is complete, see Remark 2.6 so \( R \) has a canonical module \( \omega_R \). Let \( d \) be the dimension of \( R \). Since, \( \omega_R/\mathfrak{x}R \cong \omega_R/\mathfrak{x}R \), by Theorem 3.1 we need to compare limits

\[
e_{gHK,R/\mathfrak{x}}(M) = \lim_{n \to \infty} \frac{\ell(\text{Ext}^{d-1}_R(M, n(\omega_R/\mathfrak{x}R)))}{q^{d-1}}
\]

and

\[
e_{gHK,R}(M) = \lim_{n \to \infty} \frac{\ell(\text{Ext}^d_R(M, n\omega_R))}{q^d}.
\]

Since \( M \) is an \( R/\mathfrak{x} \)-module and \( x \) is regular, \( \text{Ext}^d_R(M, n\omega_R) \cong \text{Ext}^{d-1}_R(M, n\omega_R/\mathfrak{x}^n\omega_R) \). Moreover, applying \( n\mathfrak{R} \otimes_R - \) to the exact sequence

\[
0 \to R \xrightarrow{x^q} R \to R/\mathfrak{x}^q \to 0,
\]

one can see that \( n\omega_R/\mathfrak{x}^n\omega_R \cong n(\omega_R/\mathfrak{x}^q\omega_R) \). Therefore,

\[
e_{gHK,R}(M) = \lim_{n \to \infty} \frac{\ell(\text{Ext}^{d-1}_R(M, n(\omega_R/\mathfrak{x}^q\omega_R)))}{q^d}.
\]
Note, since $x$ is regular on $\omega$, the following sequences are exact for all $m$

$$0 \rightarrow \omega_R/x^{m-1}\omega_R \rightarrow \omega_R/x^m\omega_R \rightarrow \omega_R/x\omega_R \rightarrow 0.$$  

Since $^nX$ is an exact functor, this gives a filtration of $^n(\omega_R/x^n\omega_R)$ by copies $^n(\omega_R/x^m\omega_R)$.

Applying Hom$_R(M, -)$ to the filtration, we get the exact sequences

$$\text{Ext}^{d-1}_R(M, ^n(\omega_R/x^{m-1}\omega_R)) \rightarrow \text{Ext}^{d-1}_R(M, ^n(\omega_R/x^m\omega_R)) \rightarrow \text{Ext}^{d-1}_R(M, ^n(\omega_R/x\omega_R)) \rightarrow 0.$$  

Therefore, there is a sequence of inequalities

$$\ell(\text{Ext}^{d-1}_R(M, ^n(\omega_R/x^n\omega_R))) \leq \ell(\text{Ext}^{d-1}_R(M, ^n(\omega_R/x))) + \ell(\text{Ext}^{d-1}_R(M, ^n(\omega_R/x^{m-1}\omega_R))).$$  

Thus $\ell(\text{Ext}^{d-1}_R(M, ^n(\omega_R/x^n\omega_R))) \leq q\ell(\text{Ext}^{d-1}_R(M, ^n(\omega_R/x\omega_R)))$ and the assertion follows. \hfill \Box

**Corollary 4.5.** Suppose $R$ is a local complete intersection and $M$ such that IPD($M$) $\subseteq \{m\}$. Then $e_{gHK}(M) > 0$ unless depth $M > 0$ and pd$_R M < \infty$ (i.e., pd$_R M < \text{dim } R$).

**Proof.** Assume $R$ is complete, see Remark 2.6. Hence $R = S/x$ for a regular local ring $R$ and a regular sequence $x = x_1, \ldots, x_d$. By Proposition 4.3, we can assume that depth $M = 0$, then, by Corollary 4.2, we know that $e_{gHK_S}(M) > 0$. Now, we can apply Theorem 4.4 to get

$$0 < e_{gHK_S}(M) \leq e_{gHK_S/x_1}(M) \leq \ldots \leq e_{gHK_R}(M).$$  

\hfill \Box

The statement is not true for Gorenstein rings ([10, Example 3.2]) or without the condition on IPD($M$) (see Example below).

**Example 4.6.** Let $R = k[[x, y, z]]/(x^2y - z^2)$, with char $k = 2$. Then $(x, z)^q = (x^q)$, so $e_{gHK}(R/(x, z)) = 0$, but pd$_R R/(x, z) = \infty$.

**Corollary 4.7.** Let $R$ be a local complete intersection with isolated singularity. The following are equivalent:

1. $e_{gHK}(M) = 0$.
2. $f^M_{gHK}(n) = 0$ for all $n$.
3. pd$_R M < \text{dim } R$.

The next corollary gives an asymptotic version of rigidity of $^nR$ over complete intersection([5]) for a particular class of modules.

**Corollary 4.8.** Suppose $R$ is a local complete intersection and $M$ is such that IPD($M$) $\subseteq \{m\}$. Then for any $i > 0$

$$\lim_{n \rightarrow \infty} \frac{\ell(\text{Tor}^R_i(M, ^nR))}{q^d}$$  

exists and is 0 if and only if pd$_R M < \infty$.

**Proof.** If dim $R = 0$, then $M$ has finite length and we are done by the rigidity of $^nR$. Hence we may assume that dim $R > 0$.

As it follows from Lemma 3.10 and Corollary 3.11

$$\lim_{n \rightarrow \infty} \frac{\ell(\text{Tor}^R_i(M, ^nR))}{q^d} = e_{gHK}(\text{syz}^i M).$$
Since depth(syz\(^i\)M) > 0, it follows from Corollary 4.5 that pd\(_R\)M < ∞ if and only if e\(_{\text{HK}}\)(syz\(^i\)M) = 0. □

**Theorem 4.9.** Let R be a complete intersection. C\(_\bullet\) is a complex of finite free modules. Let i be an integer, and denote \(G(X) = H_i(C\_\bullet \otimes X)\). Moreover, assume that \(H_i(C\_\bullet)\) has finite length and \(\text{IPD}(\text{coker}(\delta_i)) \subseteq \{m\}\). Then

\[
\lim_{n \to \infty} \frac{\ell(G^nR)}{q^d} = 0 \Rightarrow H_i(C\_\bullet) = 0.
\]

Moreover, if dim \(R > 0\) then the following are equivalent:

1. \(\lim_{n \to \infty} \frac{\ell(G^nR)}{q^d} = 0\).
2. \(H_i(C\_\bullet) = 0\) and \(\text{pd}_R\text{coker}(\delta_i) < \infty\).

**Proof.** Assume the first condition holds. Let \(C = \text{coker}(\delta_i)\). By a theorem of Auslander (see [16, Proposition 3.6]) there exists an exact sequence of functors

\[
\text{Tor}_2^R(C, X) \to G(R) \otimes X \to G(X) \to \text{Tor}_1^R(C, X) \to 0.
\]

Hence,

\[
\text{if } \lim_{n \to \infty} \frac{\ell(G^nR)}{q^d} = 0 \text{ then } \lim_{n \to \infty} \frac{\text{Tor}_1^R(C, nR)}{q^d} = 0.
\]

Therefore, pd\(_R\)C < ∞ and both Tor-modules vanish by Corollary 4.8 so we have an isomorphism \(G(R) \otimes nR \cong G(nR)\). But

\[
\lim_{n \to \infty} \frac{\ell(G(R) \otimes nR)}{q^d} = 0
\]

if and only if \(G(R) = 0\), i.e. \(H_i(C\_\bullet) = 0\).

The other direction easily follows from the sequence above. □

**Corollary 4.10.** Let R be complete intersection and M is a finitely generated module such that \(\text{IPD}(M) \subseteq \{m\}\). Then

\[
\lim_{n \to \infty} \frac{\ell(\text{Ext}_R^i(M, nR))}{q^d} = 0 \Rightarrow \text{Ext}_R^i(M, R) = 0.
\]

**Proof.** Let \(C\_\bullet = \text{Hom}_R(P\_\bullet, R)\), where \(P\_\bullet\) is a free resolution of \(M\). Then we can compute Ext by taking homology of \(C\_\bullet \otimes nR\). Assumptions on \(M\) will force \(C\_\bullet\) to satisfy the conditions of the last theorem. □

The converse to this corollary is not true. Indeed, for \(M, N\) maximal CM modules over a hypersurface \(R\) we have:

\[
\text{Ext}_R^1(M, N) \cong \text{Tor}_1^R(\Omega M\_\bullet, N)
\]

(see sequence 4.4). So let \(N = nR\), we know that \(\text{Ext}_R^1(M, R) = 0\) but the limit is positive.

**Corollary 4.11.** Suppose \(R\) is a local complete intersection of dimension \(d > 0\) and \(M\) is a finitely generated module such that \(\text{IPD}(M) \subseteq \{m\}\). Let \(i > d - \text{depth} \ M\). Then

\[
\lim_{n \to \infty} \frac{\ell(\text{Ext}_R^i(M, nR))}{q^d} = 0 \text{ if and only if } \text{pd}_R M < d.
\]
Proof. One direction is trivial.

For the other direction, by taking syzygy we will obtain that $\text{Ext}_R^i(M,^nR) = \text{Ext}_R^1(N,^nR)$ for a maximal Cohen-Macaulay module $N$. Then $\text{Hom}_R(-, R)$ will preserve exactness of a projective resolution of $N$. By Theorem 4.9, the first cosyzygy of $N^*$ has finite projective dimension, thus $N$ is free. Therefore, $\text{pd} M < d$. □

Theorem 4.12. Suppose $R$ is a local complete intersection of dimension $d$ and $M$ is a finitely generated module such that $\text{IPD}(M) \subseteq \{m\}$. Assume that $\text{depth} M \geq k$, then

$$\lim_{n \to \infty} \frac{\ell(H_m^k(F^n(M)))}{q^d} = 0 \text{ if and only if } \text{pd}_R M < d - k.$$  

Proof. If $\text{pd}_R M < \dim R - k$, the limit is zero by Lemma 4.1.

For the other direction, use induction on $k$. The base is Corollary 4.5 so we assume that $k > 0$.

We can use Cohen-Macaulay approximation ([4, 1.8]) to obtain an exact sequence

$$0 \to M \to Q \to N \to 0,$$

where $\text{pd} Q \leq d - k$ and $N$ is maximal Cohen-Macaulay. Tensoring with $^nR$ we obtain

$$0 \to \text{Tor}_1^R(N, ^nR) \to F^n(M) \xrightarrow{g} F^n(Q) \to F^n(N) \to 0.$$ 

Let $C = \text{im} g$, then, since $\text{Tor}_1^R(N, ^nR)$ has finite length, $H_m^k(F^n(M)) = H_m^k(C)$. Moreover, since $\text{depth} Q \geq k$, the long exact sequence for local cohomology gives

$$0 \to H_m^{k-1}(F^n(N)) \to H_m^k(C).$$

Therefore,

$$\lim_{n \to \infty} \frac{\ell(H_m^{k-1}(F^n(N)))}{q^d} \leq \lim_{n \to \infty} \frac{\ell(H_m^k(F^n(M)))}{q^d} = 0.$$ 

Hence, by induction hypotheses, $\text{pd} N < \infty$, and, so, $N$ is free. Therefore, $\text{pd} M \leq d - k$ and we can use Lemma 4.1 to finish the proof. □

Remark 4.13. The assumption on depth $M$ is necessary. For example, let $R$ be a regular local ring of dimension $d > 1$, and $M$ be locally free on the punctured spectrum such that depth $M = 0$. By Theorem 3.1, $\ell(H_1^k(F^n(M))) = \ell(\text{Ext}_{^dR}^{d-1}(M, ^nR))$, so, by Theorem 4.9,

$$\lim_{n \to \infty} \frac{\ell(H_1^k(F^n(M)))}{q^d} = 0$$

if and only if $\text{Ext}_{^dR}^{d-1}(M, R) = 0$. Thus, if we take $M = R \oplus k$, then

$$\lim_{n \to \infty} \frac{\ell(H_1^k(F^n(M)))}{q^d} = 0.$$ 

However $\text{pd} M = d$.

Next we discuss an application on local cohomology of symbolic powers of reflexive ideals.

Proposition 4.14. Let $R$ be a local ring which satisfies Serre’s condition $(S_2)$ and $M$ a finitely generated $R$-module which is locally free on $\text{Spec} R - \{m\}$. Then $H_m^2(M) \cong H_m^2(M^{**})$ and they have finite length.
Proof. First, there is a natural map $M \to M^{**}$ which has finite length kernel and cokernel. Then the exact sequence of local cohomology establishes the isomorphism $H^2_m(M) \cong H^2_m(M^{**})$.

To show that these modules have finite length, we can complete and assume that $R$ has a dualizing complex $D_R$. By Local duality, we have to show that $H^{d-2}(\text{Hom}(M, D_R))$ has finite length. Moreover, localizing at any prime $p \neq m$, we only need to concern with the case $\ell p \geq d - 2$. But then $M_p \cong R^n_p$, so $H^{d-2}(\text{Hom}(M, D_R))_p$ is dual to $H^2_{pR_p}(R^n_p)$. However, since $\ell \ p - d + 2 \leq 1$ and $R$ is $(S_2), H^{ht_p-d+2}(R^n_p) = 0$. \hfill\Box

For an ideal $I$, recall that $I^{(n)}$ denote the $n$th symbolic power of $I$. The second part of the following Theorem is an effective version of the the main result in [10].

**Theorem 4.15.** Let $R$ be a local ring satisfying Serre’s condition $(S_2)$ and $d = \dim R \geq 3$. Let $I$ be a reflexive ideal that is locally free on $\text{Spec } R - \{m\}$. Then

1. There exist elements $a, b \in R$ such that $|\ell(H^2_m(I^{(q)})) - \ell(H^2_m(R/(a, b)^{[q]}))|$ is bounded by a constant. In particular, $\lim_{n \to \infty} \frac{\ell(H^2_m(I^{(q)}))}{q^n} = \lim_{n \to \infty} \frac{\ell(H^2_m(I^{(q)}))}{q^n} = e_{gHK}(R/(a, b))$.

2. When $R$ is a complete intersection, the limit in part (1) is 0 if and only if $I$ is principal. In particular, the Picard group of $\text{Spec } R - \{m\}$ has no non-trivial torsion elements.

**Proof.** First we note that $\ell(H^2_m(I^{(q)})) = \ell(I^{(q)})$ follows from the previous Proposition and the fact that $I^{(n)} \cong (I^{\oplus n})^{**}$.

As in the proof of [10] Theorem 2.9] one can write $I = (a) : (b)$ for $a, b \in R$, furthermore $I^{(n)} = (a^n) : b^n$.

Taking the long exact sequence of local cohomology of

$$0 \to R/(a^n : b^q) \to R/(a^q) \to R/(a^n, b^q) \to 0$$

we get an exact sequence

$$0 \to H^0_m(R/(a, b)^{[q]}) \to H^1_m(R/I^{(q)}) \to H^1_m(R/(a^q)).$$

The long exact sequence of local cohomology induced by $0 \to I^{(q)} \to R \to R/I^{(q)} \to 0$ tells us that $0 \leq \ell(H^2_m(I^{(q)})) - \ell(H^2_m(R/I^{(q)})) \leq \ell(H^2_m(R)).$

On the other hand, the long exact sequence of local cohomology applied to $0 \to R \to R \to R/(a^q) \to 0$ implies that $H^1_m(R/(a^q))$ is a submodule of $H^2_m(R)$, so its length is bounded.

Thus, (1) follows (for the limit statement we use Corollary [3.6]). The first assertion of (2) follows from [4.8]. If $I$ represents a torsion elements in the Picard group of $\text{Spec } R - \{m\}$, then $H^2_m(I^{(q)})$ must be periodic, hence the limit is 0. \hfill\Box

5. More precise behavior of $f^M_{gHK}(n)$

Numerical evidences suggest (see Section 3) that when $R$ has an isolated singularity, the behavior of $f^M_{gHK}(n)$ follows the case of classical HK functions (the condition of isolated singularity may be necessary, see Example [6.3]). We discuss this phenomenon and establish some special cases.
Let \( M \) be a finite module over a hypersurface. If \( \text{IPD}(M) \subseteq \{m\} \), then we can define Hochster’s theta function to be
\[
\theta^R(M, X) = \ell(\text{Tor}^R_{2d}(M, X)) - \ell(\text{Tor}^R_{2d+1}(M, X))
\]
for any module \( X \). The following lemma is implicitly contained in [9]. We place a proof here for convenience, but refer to the paper for more information.

**Lemma 5.1.** Let \( R \) be a hypersurface and \( M \) a finite \( R \)-module with \( \text{IPD}(M) \subseteq \{m\} \). Then \( \theta^R(M, \mathfrak{m}R) = 0 \).

**Proof.** We consider \( \theta^R(M, -) \) to be a function on the Grothendieck group of \( R \). It is known (see [21, Remark 2.8]), that in our assumptions \( [\mathfrak{m}R] = q^d[R] \) in the Grothendieck group. Therefore, \( \theta^R(M, \mathfrak{m}R) = q^d\theta^R(M, R) = 0 \). \( \square \)

**Theorem 5.2.** Let \( R \) be a hypersurface with perfect residue field and \( M \) a finite \( R \)-module with depth \( M > 0 \) and \( \text{IPD}(M) \subseteq \{m\} \). Then there are modules \( M_1, M_2 \) of finite length such that:
\[
f^M_{gHK}(n) = \frac{f^{M_1}_{gHK}(n) - f^{M_2}_{gHK}(n)}{2}.
\]

**Proof.** First, consider the Auslander-Buchweitz approximation ([4, 1.8]) of \( M \). Namely, there exists an exact sequence
\[
0 \to M \to Q \to L \to 0,
\]
where \( \text{pd}
Q < \infty \) and \( L \) is maximal Cohen-Macaulay. Tensoring with \( \mathfrak{m}R \) we obtain the sequence
\[
0 \to \text{Tor}_1^R(L, \mathfrak{m}R) \to F^n(M) \xrightarrow{g} F^n(Q) \to F^n(L) \to 0.
\]
Let \( C = \text{im}(g) \), then, since \( \text{Tor}_1^R(L, \mathfrak{m}R) \) has finite length, we have the following exact sequence of local cohomology
\[
0 \to \text{Tor}_1^R(L, \mathfrak{m}R) \to H^0_m(F^n(M)) \to H^0_m(C) \to 0.
\]
Note that depth \( F^n(Q) = \text{depth} Q > 0 \) and \( C \) injects into \( F^n(Q) \), thus \( H^0_m(C) = 0 \). Hence it is enough to prove the statement replacing \( f^M_{gHK} \) by \( \ell(\text{Tor}_1^R(L, \mathfrak{m}R)) \).

There is a \( \mathfrak{m} \)-primary ideal \( I \) (see [11]) that kills all functors \( \text{Tor}_i^R(L, -) \) for \( i > 0 \), so we can choose an element \( x \) in such ideal which is \( L \)-regular. Then the sequence \( 0 \to L \xrightarrow{x} L \to L/xL \to 0 \) gives that:
\[
0 \to \text{Tor}^R_{i+1}(L, \mathfrak{m}R) \to \text{Tor}^R_{i+1}(L/xL, \mathfrak{m}R) \to \text{Tor}^R_i(L, \mathfrak{m}R) \xrightarrow{x} 0
\]
for \( i \geq 1 \).

By the previous lemma and the fact that the minimal resolution of \( L \) is 2-periodic, we get that \( \ell(\text{Tor}^R_{i+1}(L/xL, \mathfrak{m}R)) = 2\ell(\text{Tor}^R_i(L, \mathfrak{m}R)) \). Now, \( L/xL \) still has finite projective dimension on the punctured spectrum and its minimal resolution, thus \( \text{Tor}_i \), are still 2-periodic for \( i > 1 \), so we can choose another element \( y \in I \) regular on \( L/xL \) and apply the same argument. This way, we continue choosing \( L \)-regular sequence \( x \) in \( I \) to get that \( \ell(\text{Tor}^R_{i+d}(L/xL, \mathfrak{m}R)) = 2^d\ell(\text{Tor}^R_i(L, \mathfrak{m}R)) \) and the result follows from the next lemma. \( \square \)
Lemma 5.3. Let $R$ be a Cohen-Macaulay local ring and $M$ be a module of finite length. Then for each $i > 0$ there are finite length modules $M_1, M_2$ such that:
\[
\ell(\text{Tor}_i^R(M, {}^nR)) = f_{gHK}^M(n) - f_{gHK}^{M_2}(n).
\]

Proof. Let $x$ be a regular sequence in the annihilator of $M$. Then there is an exact sequence
\[
0 \to N \to (R/xR)^m \to M \to 0.
\]
If $i = 1$, tensoring with $^nR$, we obtain the exact sequence
\[
0 \to \text{Tor}_1^R(M, {}^nR) \to F^m(N) \to F^m((R/xR)^m) \to F^m(M) \to 0,
\]
so we can take $M_1 = M \oplus N$ and $M_2 = (R/xR)^m$. For $i > 1$, we get that $\text{Tor}_i^R(M, {}^nR) \cong \text{Tor}_{i-1}^R(N, {}^nR)$, and the result follows by induction.

Example 5.4. Let $R = k[[x, y, z]]/(x^3 + y^3 + z^3)$ and $M = R/I$ with $I = (x, y + z)$. We know that $z^2$ kills all Tor and Ext of MCM modules. We have an exact sequence
\[
0 \to M \to Q \to I \to 0
\]
where $\text{pd}_R Q < \infty$. Thus
\[
f_{gHK}^M = \ell(\text{Tor}_1^R(I, {}^nR)) = \ell(\text{Tor}_2^R(R/I, {}^nR)) = \frac{\ell(\text{Tor}_1^R(R/(I, z^2), {}^nR))}{2}.
\]
The short exact sequence $0 \to R/(x, z^2) : y + z \to R/(x, z^2) \to R/(I, z^2) \to 0$ implies then that:
\[
f_{gHK}^M = \frac{f_{gHK}^R(R/(x, z^2) \cdot (I, z^2)) + f_{gHK}^R(R/(x, z^2))}{2}.
\]
Macaulay 2 calculation indicates that the functions are $4(q^2-1)/3$, $16q^2-4/3$, $10q^2-4/3$, $6q^2$ respectively.

6. Numerical evidences and examples

In this section we collect and comment on some numerical evidences involving $e_{gHK}(M)$.

Example 6.1. Let $R = k[[x, y, u, v]]/(xy-uv)$ where $k$ is algebraically closed of characteristic $p > 2$. For a module $M$ of positive depth, We can compute $f_{gHK}^M(n)$ as follows:

Let $M_1 = (x, y)$ and $M_2 = (x, v)$. It is known that (see for example [29]) $M_1$ and $M_2$ are the only nonfree indecomposable maximal Cohen-Macaulay modules. Moreover, there exists a decomposition
\[
^nR = R^{a_1} \oplus M_1^{a_2} \oplus M_2^b,
\]
where $a = b = \frac{q^d-a_q}{2}$.

One can check that $\text{Tor}_1^R(M_1, M_2) = 0$ and $\text{Tor}_1^R(M_1, M_1) = \text{Tor}_1^R(M_2, M_2) = 1$. Let $N = R^m \oplus M_1^c \oplus M_2^d$ be a decomposition of a maximal Cohen-Macaulay module $N$ obtained from a maximal Cohen-Macaulay approximation of $M$. We know that
\[
\ell(\text{Tor}_m^R(F^n(M))) = \ell(\text{Tor}_1^R(N, {}^nR)) = ca + db = (c + d)\frac{q^d-a_q}{2}.
\]

Example 6.2. There is some numerical evidence that over a normal graded domain of dimension two, $f_{gHK}^M(n) = cq^2 + \gamma(n)$ where $\gamma$ is a bounded (even periodic) function. For example:
(1) Consider \( R = k[[x, y, z]]/(x^3+y^3+z^3) \) and \( M = R/(x, y+z). \) Macaulay2 calculations suggest that \( f_{gHK}^M \) is \( \frac{4(q^2-1)}{3} \) for \( k = \mathbb{F}_2, \mathbb{F}_5, \mathbb{F}_7, \mathbb{F}_{11}. \) Compare with \( f_{gHK}^k = \frac{9q^2-5}{4}. \)

(2) For \( R = \mathbb{F}_3[[x, y, z]]/(x^4+y^4-z^4), \) \( M = R/(x, y^2-z^2), \) and \( N = R/(x, y-z) \) Macaulay2 suggests that \( f_{gHK}^M = 3q^2 - 3 \) and \( f_{gHK}^N = \frac{9(q^2-1)}{4}. \)

(3) For \( R = \mathbb{F}_3[[x, y, z]]/(x^5+y^5-z^5) \) (of \( \mathbb{F}_2 \)) and \( M = R/(x, y-z), \) computations show that \( f_{gHK}^M = \frac{16q^2-\gamma(n)}{5} \) with \( \gamma(n) = 24 \) for even \( n \) and \( = 16 \) for odd.

**Example 6.3.** Here are some examples showing that the behavior of \( f_{gHK}^M \) is not similar to the finite length case without the assumption on IPD(M). The point is that the second coefficient is not 0 as is the case for dimension 2 and old HK functions.

(1) When \( R = k[[x, y, t]]/(x^4+tx^2y^2+y^4), \) and \( M = R/(x, y), \) \( f_{gHK}^M = q-2. \)

(2) When \( R = k[[x, y, t]]/(x^4+txy^2+y^4), \) and \( M = R/(x, y), \) \( f_{gHK}^M = q^2/4 + q - 2 \) for \( k = \mathbb{F}_2 \) and \( q > 2. \) For \( k = \mathbb{F}_3, \mathbb{F}_5, \) it is \( (q^2-13)/4 + q. \)

(3) Let \( R = k[[x, y, t]]/(x^3+txy+y^3), \) and \( M = R/(x, y). \) Then for \( k = \mathbb{F}_2, \) \( f_{gHK}^M = \frac{q^2+2q-3}{3}. \) For \( k = \mathbb{F}_3, \) it is \( \frac{q^2+2q-3-\gamma(n)}{3}. \) For \( k = \mathbb{F}_5, \) it is \( \frac{q^2+2q-3}{3}. \) For \( k = \mathbb{F}_7, \) it is \( \frac{q^2+2q-3-\gamma(n)}{3}. \) Where \( \gamma(n) = 3 \) for odd \( n \) and \( = 5 \) for even.

The formula seems to depend on whether \( q = 1 \) mod 3.

(4) When \( R = \mathbb{F}_2[[x, y, t]]/(x^3+txy+y^3), \) and \( M = R/(x^3, y^3), \) we get \( 3q^2+2q-1. \) For \( M = R/(x^2, y^2, xy) \) we get \( q^2+q-2. \)

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