ON OBSERVABILITY OF SIGNAL OVER BACKGROUND

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Abstract

Several statistics used by physicists to declare the signal observability over the background are compared. It is shown that the frequentist method of testing a precise hypothesis allows one to estimate the power value of criteria with specified level of significance for the considered statistics by Monte Carlo calculations. The application of this approach for the analysis of discovery potential of experiments is discussed.

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Introduction

One of the common tasks for searching experiments is the detection of a predicted new Phenomenon. As a rule the estimations of an expected mean \( N_s \) for the signal events of new Phenomenon and \( N_b \) for the background events are known. Then we want to know is the given experiment able to detect new Phenomenon or not. To check the statement about the observation of Phenomenon a researcher uses some function of the observed number of events – a statistic. The value of this statistic for detected \( x \) events allows one to find the degree of confidence of the discovery statement. After having drawn a conclusion on the observation of Phenomenon, two possibilities for mistake are available: to state that Phenomenon is absent but in real life it exists (Type I error), or to state that Phenomenon exists but it is absent (Type II error).

In this paper we compare the “signal significances” used by the researchers for the hypothesis testing about the observation of Phenomenon:

(a) “significance” \( S_1 = \frac{N_s}{\sqrt{N_b}} \),

(b) “significance” \( S_2 = \frac{N_s}{\sqrt{N_s + N_b}} \),

(c) “significance” \( S_{12} = \sqrt{N_s + N_b} - \sqrt{N_b} \),

(d) likelihood ratio as is defined in references [5, 6].

For this purpose we formulate the null and alternative hypotheses, construct the statistical test, determine the rejection region by Monte Carlo calculations, make the decision and find the power of test for the criteria with a specified level of significance. We also use an equal-tailed test to study the behaviour of Type I and Type II errors versus \( N_s \) and \( N_b \) for specified values of \( S_1 \) and \( S_2 \). The hypotheses testing results obtained by Monte Carlo calculations are compared with result obtained by the direct calculations of probability density functions.

1 Notations

Let us study a physical process during a fixed time. The estimations of the average number of signal events which indicate new Phenomenon (\( N_s \))
and of the average number of background events ($N_b$) in the experiment are given. We suppose that the events have the Poisson distributions with the parameters $N_s$ and $N_b$, i.e. the random variable $\xi \sim \text{Pois}(N_s)$ describes the signal events and the random variable $\eta \sim \text{Pois}(N_b)$ describes the background events. Say we observed $x$ events – the realization of the studying process $X = \xi + \eta$ ($x$ is the sum of signal and background events in the experiment). Here $N_s, N_b$ are non-negative real numbers and $x$ is an integer.

The classical frequentist methods of testing a precise hypothesis allow one to construct a rejection region and determine associated error probabilities for the following “simple” hypotheses:

\[ H_0 : X \sim \text{Pois}(N_s + N_b) \quad \text{versus} \quad H_1 : X \sim \text{Pois}(N_b) \]

where $\text{Pois}(N_s + N_b)$ and $\text{Pois}(N_b)$ have the probability density functions (p.d.f.’s)

\[ f_0(x) = \frac{(N_s + N_b)^x}{x!} e^{-(N_s + N_b)} \]

for the case of presence and

\[ f_1(x) = \frac{(N_b)^x}{x!} e^{-N_b} \]

for the case of absence of signal events in the universe population.

In Fig.1 the p.d.f.’s $f_0(x)$ (a) and $f_1(x)$ (b) for the case $N_s + N_b = 104$ and $N_b = 53$ ([3], Table.13, cut 6) are shown. As is seen the intersection of these p.d.f.’s takes place. Let us denote the threshold (critical value) that divides the abscissa in Fig.1 into the rejection region and the area of accepted hypothesis $H_0$ via $Nev$. The incorrect rejection of the null hypothesis $H_0$, the Type I error (the statement that Phenomenon is absent, but it is present), has the probability $\alpha = \sum_{x=0}^{Nev} f_0(x)$, and the incorrect acceptance of $H_0$, the Type II error (the statement that Phenomenon exists, but it is absent), has the probability $\beta = \sum_{x=Nev+1}^{\infty} f_1(x)$. The dependence of $\alpha$ and $\beta$ on the value of $Nev$ for above example is presented in Fig.2.

### 2 Hypothesis testing

In this Section we show the procedure of the rejection region construction for the likelihood ratio $B(x)$.

We denote by $B(x) = \frac{f_0(x)}{f_1(x)}$ the likelihood ratio of $H_0$ to $H_1$ in the area of existing $B(X)$. The decision to either reject or accept $H_0$ will depend on the observed value of $B(x)$, where small values of $B(x)$ correspond to the rejection of $H_0$. For the traditional frequentist the classical most powerful test of the simple hypothesis is determined by some critical value $c$ such that
if $B(x) \leq c$, reject $H_0$, 
if $B(x) > c$, accept $H_0$.

In compliance with this test, the frequentist reports Type I and Type II error probabilities as $\alpha = P_0(B(X) \leq c) \equiv F_0(c)$ and $\beta = P_1(B(X) > c) \equiv 1 - F_1(c)$, where $F_0$ and $F_1$ are cumulative density functions of $B(X)$ under $H_0$ and $H_1$, respectively. For a conventional equal-tailed test with $\alpha = \beta$, the critical value $c$ satisfies $F_0(c) \equiv 1 - F_1(c)$.

In the same way we can construct the rejection region, find the critical values $c_1$, $c_2$ and $c_{12}$, the probabilities $\alpha$ and $\beta$ for the statistics $s_1 = \frac{x - N_b}{\sqrt{N_b}}$ (for “significance” $S_1$), $s_2 = \frac{x - N_b}{\sqrt{x}}$ (for “significance” $S_2$) and $s_{12} = \sqrt{x} - \sqrt{N_b}$ (for “significance” $S_{12}$). Here, the value of $x - N_b$ is the estimation of the number of signal events. Note that “significance” $S_{12}$ depends on $S_1$ and $S_2$, namely, $S_{12} = \frac{S_1 \cdot S_2}{S_1 + S_2}$.

3 Determination of probability density functions for statistics

The probability density functions of statistics under consideration can be obtained in an analytical form. Another way to obtain the p.d.f. is the calculations by a Monte Carlo simulation of the results of a large number of experiments (see as an example [7, 8, 9]) for the given values $N_s$ and $N_b$. In this study we use the latter approach. The p.d.f.’s for $N_s + N_b = 104$ and $N_b = 53$ obtained by this way are shown in Fig.3 (these distributions are the result of $10^5$ simulation experiments for random variables $\xi$ and $\eta$). The difference between these p.d.f.’s and p.d.f.’s resulting from direct calculations of the probabilities (Fig.1) is extremely small.

In Fig.4 the p.d.f.’s of statistic $s_2$ for the case of $N_s = 51$, $N_b = 53$ (a) and the case of $N_s = 0$, $N_b = 53$ (b) are shown. The behaviour of probabilities $\alpha$ and $\beta$ versus the critical value $c_2$ for the statistic $s_2$ is also presented in Fig.4 (c).

It is worth to stress that this approach allows one to construct the p.d.f.’s and, correspondingly, the acceptance and the rejection regions for complicated statistics with account for the systematic errors and the uncertainties in $N_b$ and $N_s$ estimations.
4 Comparison of different statistics

We compare the statistic $s_1$, the statistic $s_2$, the statistic $s_{12}$ and the likelihood ratio ($B(x - N_b)$ in our case). The reason for the comparison is the existence of an opinion that the value of such type statistic ($s_1$, $s_2$, $s_{12}$) characterizes the difference between the samples with and without signal events in terms of “standard deviations” (1 $\sigma$, 2 $\sigma$, …, 5 $\sigma$). To anticipate a little, the values of $\alpha$ and $\beta$ corresponding to these “standard deviations” depend on the value of the sample and for $S_1$, for example, $\alpha$ and $\beta$ have a perceptible value even if $N_s$ and $N_b$ satisfy the condition $S_1 = 5$.

The Type I error $\alpha$ is also called a significance level of the test. The value for $\beta$ is meaningful only when it is related to an alternative hypothesis $H_1$. The dependence $1 - \beta$ is referred to as a power function that allows one to choose a preferable statistic for the hypothesis testing. It means that for the specified significance level we can determine the critical value $c$ (correspondingly, $c_1$, $c_2$, $c_{12}$) and find the power $1 - \beta$ of this criterion. The greater the value $1 - \beta$, the better statistic separates hypotheses for the specified value of $\alpha$.

In Table 1 the comparison result is shown. For several values of $N_s$ and $N_b$ (significance level $\alpha = 0.01$) the critical values $c_1$, $c_2$, $c_{12}$, $c$ and the corresponding values of power $1 - \beta$ of these criteria for the statistics $s_1$, $s_2$, $s_{12}$ and the likelihood ratio are presented. As is seen from Table I there is no visible difference in the power values for the considered statistics, i.e. we can use in an equivalent manner either of these statistics for the hypotheses testing.

5 Equal-tailed test

Of concern to us is the question: What is meant by the statement that

$$S_1 = \frac{N_s}{\sqrt{N_b}} = 5 \text{ or } S_2 = \frac{N_s}{\sqrt{N_s + N_b}} = 5 ?$$

Tables 2 and 3 give the answer to this question. In Tables 2 and 3 the values $N_s$ and $N_b$ corresponding to the above condition, the values $\alpha$ and $\beta$ determined by applying equal-tailed test (in this study we use the conditions

1If $f_i(x)$ is the standard normal distribution, then the 1 $\sigma$ deviation from 0 corresponds the area of tail that is equal to 0.1587, 2 $\sigma$ – 0.0228, 3 $\sigma$ – 0.00135, 4 $\sigma$ – 0.000032 and 5 $\sigma$ – 0.000003.

2The conditions $\min(0.01 - \alpha)$ and $\alpha \leq 0.01$ are performed.
Table 1: The comparison of power of criteria for different statistics. The values $c_1$, $c_2$, $c_{12}$ and $c$ are the critical values of statistics $s_1$, $s_2$, $s_{12}$ and likelihood ratio for $\alpha = 0.01$. The values $1 - \beta$ are the power for corresponding critical values.

| $N_s$ | $N_b$ | $s_1$  | $1 - \beta$ | $s_2$  | $1 - \beta$ | $s_{12}$ | $1 - \beta$ | $c$    | $1 - \beta$ |
|------|-------|-------|-------------|-------|-------------|---------|-------------|-------|-------------|
| 10   | 5     | 0.89  | 0.762       | 0.75  | 0.762       | 0.3     | 0.762       | 0.035 | 0.760       |
| 15   | 2.23  | 0.968 | 1.58        | 0.968 | 0.968       | 0.8     | 0.968       | 0.078 | 0.968       |
| 20   | 4.02  | 0.999 | 2.40        | 0.999 | 1.4         | 0.999   | 2.563       | 0.999 |             |
| 25   | 5.81  | 1.000 | 3.06        | 1.000 | 1.9         | 1.000   | 110.0       | 1.000 |             |
| 15   | 10    | 1.26  | 0.864       | 1.06  | 0.866       | 0.4     | 0.865       | 0.045 | 0.864       |
| 20   | 2.52  | 0.986 | 1.88        | 0.986 | 0.985       | 0.9     | 0.985       | 0.269 | 0.986       |
| 25   | 3.79  | 0.999 | 2.55        | 0.999 | 1.4         | 0.999   | 3.939       | 0.999 |             |
| 30   | 5.05  | 1.000 | 3.13        | 1.000 | 1.8         | 1.000   | 307.0       | 1.000 |             |
| 15   | 15    | 0.77  | 0.750       | 0.70  | 0.747       | 0.2     | 0.750       | 0.040 | 0.749       |
| 20   | 1.80  | 0.947 | 1.49        | 0.947 | 0.948       | 0.9     | 0.948       | 0.117 | 0.947       |
| 25   | 2.84  | 0.994 | 2.15        | 0.994 | 1.1         | 0.994   | 0.667       | 0.994 |             |
| 30   | 3.87  | 0.999 | 2.73        | 1.000 | 1.5         | 1.000   | 7.795       | 1.000 |             |
| 20   | 55    | 0.13  | 0.535       | 0.00  | 0.479       | -0.1    | 0.483       | 0.052 | 0.536       |
| 25   | 0.67  | 0.733 | 0.64        | 0.733 | 0.2         | 0.735   | 0.049       | 0.731 |             |
| 30   | 1.21  | 0.873 | 1.12        | 0.874 | 0.4         | 0.843   | 0.074       | 0.873 |             |
| 35   | 1.88  | 0.963 | 1.68        | 0.962 | 0.7         | 0.950   | 0.231       | 0.962 |             |
| 40   | 2.42  | 0.989 | 2.10        | 0.988 | 1.0         | 0.988   | 0.512       | 0.989 |             |
| 45   | 2.96  | 0.997 | 2.60        | 0.998 | 1.3         | 0.998   | 2.894       | 0.998 |             |
| 50   | 3.64  | 1.000 | 2.98        | 1.000 | 1.5         | 1.000   | 9.957       | 1.000 |             |

$\min(\beta - \alpha)$ and $\alpha \leq \beta$ are presented. One can see the dependence of $\alpha$ (or $\beta$) on the value of sample. The case of $N_s = 5$ and $N_b = 1$ for $S_1$ (Fig.5) is perhaps the most dramatic example. We have $5\sigma$ deviation, however, if we reject the hypothesis $H_0$, we are mistaken in 6.2% of cases and if we accept the hypothesis $H_0$ we are mistaken in 8.0% of cases.

One can point out that for a good deal of events the values of $\alpha$ for $S_1$ and $S_2$ approach each other. A simple argument explains such dependence. The $x - N_b$ has the variation equal to $\sqrt{N_s + N_b}$ for nonzero signal events, and to $\sqrt{N_b}$ if signal events are absent. Correspondingly, if $N_b \gg N_s$, the contribution of $N_s$ to the variation is very small. Therefore, the standard deviation tends to unity both for the distribution of $s_1$ (Fig.6) and for the distribution of $s_2$. It means that for the sufficiently large $N_b$, the val-
Table 2: The dependence of $\alpha$ and $\beta$ determined by using equal-tailed test on $N_s$ and $N_b$ for $S_1 = 5$. The $\kappa$ is the area of intersection of probability density functions $f_0(x)$ and $f_1(x)$.

| $N_s$ | $N_b$ | $\alpha$ | $\beta$ | $\kappa$ |
|-------|-------|----------|---------|----------|
| 5     | 1     | 0.0620   | 0.0803  | 0.1423   |
| 10    | 4     | 0.0316   | 0.0511  | 0.0828   |
| 15    | 9     | 0.0198   | 0.0415  | 0.0564   |
| 20    | 16    | 0.0141   | 0.0367  | 0.0448   |
| 25    | 25    | 0.0162   | 0.0225  | 0.0383   |
| 30    | 36    | 0.0125   | 0.0225  | 0.0333   |
| 35    | 49    | 0.0139   | 0.0164  | 0.0303   |
| 40    | 64    | 0.0114   | 0.0171  | 0.0278   |
| 45    | 81    | 0.0124   | 0.0136  | 0.0260   |
| 50    | 100   | 0.0106   | 0.0143  | 0.0245   |
| 55    | 121   | 0.0114   | 0.0120  | 0.0234   |
| 60    | 144   | 0.0100   | 0.0126  | 0.0224   |
| 65    | 169   | 0.0106   | 0.0109  | 0.0216   |
| 70    | 196   | 0.0095   | 0.0115  | 0.0209   |
| 75    | 225   | 0.0101   | 0.0102  | 0.0203   |
| 80    | 256   | 0.0091   | 0.0107  | 0.0198   |
| 85    | 289   | 0.0096   | 0.0097  | 0.0193   |
| 90    | 324   | 0.0088   | 0.0101  | 0.0189   |
| 95    | 361   | 0.0081   | 0.0106  | 0.0185   |
| 100   | 400   | 0.0086   | 0.0097  | 0.0182   |
| 150   | 900   | 0.0078   | 0.0084  | 0.0162   |
| 500   | $10^4$| 0.0068   | 0.0068  | 0.0136   |
| 5000  | $10^6$| 0.0062   | 0.0065  | 0.0125   |

Values of $\alpha$ and $\beta$ obtained by equal-tailed test have a constant value close to 0.0062. These distributions also can be approximated by a standard Gaussian $\mathcal{N}(0,1)$ for the pure background and Gaussian $\mathcal{N}(5,1)$ for the signal mixed with the background. Therefore, the equal-tailed test for the normal distributions gives $c_1 = 2.5$ and $\alpha = \beta = 0.0062$. These are the limiting values of $\alpha$ and $\beta$ for the requirement $S_1 = 5$ or $S_2 = 5$ (by the way $S_{12}$ equals 2.5 in this case).

In a similar way we can determine the behaviour of the Type I and Type II errors depending on $N_s$ and $N_b$ for a small number of events and we can

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3It is a conventional notation for normal distribution $\mathcal{N}(\text{mean, variance})$. 
Table 3: The dependence of $\alpha$ and $\beta$ determined by using equal-tailed test on $N_s$ and $N_b$ for $S_2 \approx 5$. The $\kappa$ is the area of intersection of probability density functions $f_0(x)$ and $f_1(x)$.

| $N_s$ | $N_b$ | $\alpha$   | $\beta$   | $\kappa$ |
|-------|-------|------------|------------|-----------|
| 26    | 1     | $0.519 \cdot 10^{-5}$ | $0.102 \cdot 10^{-4}$ | $0.154 \cdot 10^{-4}$ |
| 29    | 4     | $0.661 \cdot 10^{-4}$ | $0.764 \cdot 10^{-4}$ | $0.142 \cdot 10^{-3}$ |
| 33    | 9     | $0.127 \cdot 10^{-3}$ | $0.439 \cdot 10^{-3}$ | $0.440 \cdot 10^{-3}$ |
| 37    | 16    | $0.426 \cdot 10^{-3}$ | $0.567 \cdot 10^{-3}$ | $0.993 \cdot 10^{-3}$ |
| 41    | 25    | $0.648 \cdot 10^{-3}$ | $0.118 \cdot 10^{-2}$ | $0.172 \cdot 10^{-2}$ |
| 45    | 36    | $0.929 \cdot 10^{-2}$ | $0.193 \cdot 10^{-2}$ | $0.262 \cdot 10^{-2}$ |
| 50    | 49    | $0.133 \cdot 10^{-2}$ | $0.185 \cdot 10^{-2}$ | $0.314 \cdot 10^{-2}$ |
| 55    | 64    | $0.178 \cdot 10^{-2}$ | $0.179 \cdot 10^{-2}$ | $0.357 \cdot 10^{-2}$ |
| 100   | 300   | $0.317 \cdot 10^{-2}$ | $0.428 \cdot 10^{-2}$ | $0.735 \cdot 10^{-2}$ |
| 150   | 750   | $0.445 \cdot 10^{-2}$ | $0.450 \cdot 10^{-2}$ | $0.894 \cdot 10^{-2}$ |

predict the limiting values of $\alpha$ and $\beta$ for a large number of events in case of other statements about statistic $s_1$ (Table 4) or any other estimator.

Right column in Tables 2, 3 and 4 contains the value of probability $\kappa$.

The $\kappa$ is a characteristic of the observability of Phenomenon for the given $N_s$ and $N_b$. In particular, it is the fraction of p.d.f. $f_0(x)$ for statistic $x$ that can be described by the fluctuation of background in case of the absence of Phenomenon. The value of $\kappa$ equals the area of intersection of probability density functions $f_0(x)$ and $f_1(x)$ (Fig.1). Clearly, if we superimpose the p.d.f.’s $f_0(x)$ and $f_1(x)$ and choose the intersection point of curves (point $N_{ev} = \left[ \frac{N_s}{ln(1 + N_s/N_b)} \right]$) as a critical value for the hypotheses testing, we have $\kappa \equiv \alpha + \beta$. As is seen from Tables 2, 3 and 4 the value of $\kappa$ is also close to the sum $\alpha + \beta$ determined by using the equal-tailed test.

The accuracy of determination of the critical value by Monte Carlo calculations depends on the number of Monte Carlo trials and on the level of significance defined by the critical value. To illustrate, Fig.7 shows the

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Notice that in this point $f_0(N_{ev}) = f_1(N_{ev})$ (in our case conditions $\min(f_0(N_{ev}) - f_1(N_{ev}))$ and $f_1(N_{ev}) \leq f_0(N_{ev})$ are performed). By this is meant that this checking can be named as the equal probability test. Of course, if we use the hypotheses testing we can also determine $N_{ev}$ having found the minimum of the sum of $\alpha$ and $\beta$ or having found the minimum of the sum of weighted $\alpha$ and $\beta$ or having exploited any other condition in accordance with the requirements of experiment. The $\kappa$ may be thought of as independent of these requirements.
distribution of the estimations of the value $\frac{\alpha + \beta}{2}$ for the case $N_s = 100$, $N_b = 500$ and for the $10^5$ Monte Carlo trials in each estimation (equal-tailed test is used). The result obtained via the direct calculations of p.d.f.’s is also shown in this Figure. Thus, this method is accurate enough to give reliable results for estimation of the discovery potential of the experiment.

The approach to the determination of the critical region in the hypotheses testing by Monte Carlo calculation of p.d.f.’s can be used to estimate the integrated luminosity which is necessary for detection the predicted effects with sufficient accuracy. In Fig.8 (a) the dependence of $N_{ev}$ on integrated luminosity (Table.12, cut.5, $m_{\chi_1} = 85$ GeV, $N_s = 45$, $N_b = 45$) is shown. The corresponding values of $\alpha$ and $\beta$ are presented in Fig.8 (b). As evident from Figure the integrated luminosity $L = 8 \cdot 10^4 pb^{-1}$ is sufficient to detect sleptons under the requirement that the probability $\kappa \approx \alpha + \beta$ less than 1%.

**Conclusion**

In this paper the discussion on the observation of new Phenomenon is restricted to the testing of simple hypotheses in case of the predicted values $N_s$ and $N_b$ and the observed value $x$. As is stressed in [5], the precise hypothesis testing should not be done by forming a traditional confidence interval and simply checking whether or not the precise hypothesis is compatible with the confidence interval. A confidence interval [8] is usually of considerable importance in determining where the unknown parameter is likely to be, given that the alternative hypothesis is true, but it is not useful in determining whether or not a precise null hypothesis is true.

To compare several statistics used for the hypotheses testing, we employ the method that allows one to construct the rejection regions via the determination the probability density functions of these statistics by Monte Carlo calculations. As is shown, the considered statistics have close values of power for the specified significance level and can be used for the hypotheses testing in an equivalent manner. Also, it has been shown that the estimations of Type I and Type II errors obtained by this method have a reasonable accuracy. The method was used to make the inferences on the observability of some predicted phenomena.

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Figure 1: The probability density functions $f_0(x)$ (a) and $f_1(x)$ (b) for the case of 51 signal events and 53 background events obtained by direct calculations of the probabilities.
Figure 2: The dependence of Type I $\alpha$ and Type II $\beta$ errors on $N_{ev}$ for the case of 51 signal events and 53 background events.
Figure 3: The probability density functions $f_0(x)$ (a) and $f_1(x)$ (b) for the case of 51 signal events and 53 background events obtained by Monte Carlo simulation.
Figure 4: The probability density functions $f_0(x)$ (a) and $f_1(x)$ (b) of statistic $s_2$. The dependence of Type I and Type II errors on critical value $c_2$ (c) for the case of 51 signal events and 53 background events.
Figure 5: The probability density functions $f_0(x)$ (a) and $f_1(x)$ (b) of statistic $s_1$. The dependence of Type I and Type II errors on critical value $c_1$ (c) for the case of 5 signal events and 1 background events.
Figure 6: The probability density functions $f_0(x)$ (a) and $f_1(x)$ (b) of statistic $s_1$. The dependence of Type I and Type II errors on critical value $c_1$ (c) for the case of 5000 signal events and $10^6$ background events.
Table 4: The dependence of $\alpha$ and $\beta$ determined by using equal-tailed test on $N_s$ and $N_b$ for $S_1 = 2$, $S_1 = 3$, $S_1 = 4$, $S_1 = 6$ and $S_1 = 8$. The $\kappa$ is the area of intersection of probability density functions $f_0(x)$ and $f_1(x)$.

| $S_1$ | $N_s$ | $N_b$ | $\alpha$ | $\beta$ | $\kappa$ |
|-------|-------|-------|-----------|---------|----------|
| 2     | 2     | 1     | 0.199     | 0.265   | 0.4634   |
|       | 4     | 4     | 0.192     | 0.216   | 0.4061   |
| 6     | 9     | 1     | 0.184     | 0.199   | 0.3817   |
|       | 8     | 16    | 0.179     | 0.188   | 0.3680   |
|       | $\infty$ | $\infty$ | 0.1587 | 0.1587 | 0.3174 |
| 3     | 3     | 1     | 0.0906    | 0.263   | 0.3184   |
|       | 6     | 4     | 0.0687    | 0.216   | 0.2408   |
|       | 9     | 9     | 0.0917    | 0.123   | 0.2159   |
|       | 12    | 16    | 0.0722    | 0.131   | 0.1952   |
|       | $\infty$ | $\infty$ | 0.0668  | 0.0668  | 0.1336   |
| 4     | 4     | 1     | 0.0400    | 0.263   | 0.2050   |
|       | 8     | 4     | 0.0459    | 0.110   | 0.1406   |
|       | 12    | 9     | 0.0424    | 0.0735  | 0.1130   |
|       | 16    | 16    | 0.0407    | 0.0572  | 0.0977   |
|       | $\infty$ | $\infty$ | 0.0228  | 0.0228  | 0.0456   |
| 6     | 6     | 1     | 0.0301    | 0.0806  | 0.1008   |
|       | 12    | 4     | 0.0217    | 0.0217  | 0.0434   |
|       | 18    | 9     | 0.0089    | 0.0224  | 0.0271   |
|       | 24    | 16    | 0.00751   | 0.0132  | 0.0198   |
|       | $\infty$ | $\infty$ | 0.00135 | 0.00135 | 0.0027   |
| 8     | 8     | 1     | 0.0061    | 0.0822  | 0.0402   |
|       | 16    | 4     | 0.0049    | 0.0081  | 0.0131   |
|       | 24    | 9     | 0.0016    | 0.0052  | 0.00567  |
|       | 32    | 16    | 0.00128   | 0.00237 | 0.00331  |
|       | $\infty$ | $\infty$ | 0.000032 | 0.000032 | 0.000064 |
Figure 7: The variation of $\frac{\alpha + \beta}{2}$ in the equal-tailed hypotheses testing ($N_s = 100$, $N_b = 500$ and $N_s = 0$, $N_b = 500$ in 40 Monte Carlo simulations of probability density functions).
Figure 8: The dependence of the critical value $N_{ev}$ (a), Type I and Type II errors (b) on integrated luminosity $L$ for the case $N_s = N_b$ and $N_s = 45$ for $L = 10^5 pb^{-1}$ (equal-tailed test).