Two-sided bounds on minimum-error quantum measurement, on the reversibility of quantum dynamics, and on maximum overlap using directional iterates

Jon Tyson*
Jefferson Lab, Harvard University

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Abstract

In a unified framework, we estimate the following quantities of interest in quantum information theory:

1. The minimum-error distinguishability of arbitrary ensembles of mixed quantum states.
2. The approximate reversibility of quantum dynamics in terms of entanglement fidelity. (This is referred to as "channel-adapted quantum error recovery" when applied to the composition of an encoding operation and a noise channel.)
3. The maximum overlap between a bipartite pure quantum state and a bipartite mixed state that may be achieved by applying a local quantum operation to one part of the mixed state.
4. The conditional min-entropy of bipartite quantum states.

A refined version of the author’s techniques [J. Math. Phys. 50, 032016] for bounding the first quantity is employed to give two-sided estimates of the remaining three quantities.

We obtain a closed-form approximate reversal channel. Using a state-dependent Kraus decomposition, our reversal may be interpreted as a quadratically-weighted version of that of Barnum and Knill [J. Math. Phys. 43, 2097]. The relationship between our reversal and Barnum and Knill’s is therefore similar to the relationship between Holevo’s asymptotically-optimal measurement [Theor. Probab. Appl. 23, 411] and the “pretty good” measurement of Belavkin [Stochastics 1, 315] and Hausladen & Wootters [J. Mod. Optic. 41, 2385]. In particular, we obtain relatively simple reversibility estimates without negative matrix powers at no cost in tightness of our bounds. Our recovery operation is found to significantly outperform the so-called “transpose channel” in the simple case of depolarizing noise acting on half of a maximally-entangled state. Furthermore, our overlap results allow the entangled input state and the output target state to differ, thus obtaining estimates in a somewhat more general setting.

Using a result of König, Renner, and Schaffner [IEEE. Trans. Inf. Th. 55, 4337], our maximum overlap estimate is used to bound the conditional min-entropy of arbitrary bipartite states.

Our primary tool is “small angle” initialization of an abstract generalization of the iterative schemes of Ježek-Řeháček-Fiurášek [Phys. Rev. A 65, 060301], Ježek-Fiurášek-Hradil [Phys. Rev. A 68, 012305], and Reimpell-Werner [Phys. Rev. Lett. 94, 080501]. The monotonicity result of Reimpell [Ph.D. Thesis, 2007] follows in greater generality.

*jonetyson@X.Y.Z, where X=post, Y=Harvard, Z=edu
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1 Introduction

This paper considers the following problem of relevance in quantum information theory:

The maximum overlap problem: Let $\mu_{KH}$ be a positive semidefinite trace-class operator on $K \otimes H$, and let $M_{LH}$ be positive semidefinite bounded operator on $L \otimes H$, where $H$, $K$, and $L$ are separable Hilbert spaces. What is maximum overlap

$$\text{MO} (\mu_{KH}, M_{LH}) = \sup_{R} \text{Tr}_{CH} (M_{LH} R_{K \to L} (\mu_{KH})),$$

where the supremum is over all quantum operations $R$ from $K$ to $L$?

The maximum-overlap problem has the following important special cases:

1. The minimum-error quantum detection problem [1-4]:

   If an unknown quantum state $\rho_k$ is randomly selected from given ensemble of such states, with what probability may the value of $k$ be determined by a carefully-chosen quantum measurement?

2. Approximate reversal of quantum dynamics [5-20]:

   Suppose that an arbitrary quantum operation $A$ acts on a given quantum state $\rho$. How well may the action of $A$ be reversed by application of a recovery channel $R$, so as to preserve the entanglement of the original system with the environment? This problem is one of “channel-adapted quantum error recovery” when the operation $A$ is of the form $A = N \circ E$, where $E$ is an encoding operation designed to protect against a known noise process $N$.

3. Estimation of conditional min-/max-entropy of bipartite quantum states [21]:

   Let $\rho_{AB}$ be a bipartite quantum mixed state. Estimate the conditional min-entropy $H_{\text{min}} (A|B)$ of $A$ given $B$.

Since all of these problems are believed to defy closed-form solution, the purpose of this paper is to provide estimates. In section 3 a refined proof of the two-sided “generalized Holevo-Curlander” bounds of [22, 23] for case 1 is given. This method is extended in sections 4 and 5 to yield simple two-sided estimates for cases 2-3 and for MO ($\mu, M$) in the case of rank-1 $M$.

We briefly introduce each of the cases 1-3 before outlining our approach and surveying closely-related work.

1.1 Minimum-error detection

The minimum-error quantum detection problem was first studied in the 1960’s in connection with the design of optical detectors [24], and it has since become of importance in quantum Shannon theory (for example [25-27]) and in the design of quantum algorithms [28-36]. A generalization to the theory of wave pattern recognition may be found in [37]. Various general upper and/or lower bounds on quantum distinguishability may be found in [14, 22, 23, 25, 32, 38-44].

The minimum-error quantum detection problem is precisely formulated by

Definition 1 Let

$$\mathcal{E} = \{\rho_k\}_{k \in K}$$

be an ensemble of quantum states, represented as positive semidefinite operators normalized by a-priori probability, setting

$$\text{Tr} \rho_k = p_k,$$
where \( p_k \) is the likelihood that \( \rho_k \) will be drawn from \( \mathcal{E} \). A quantum measurement \([45]\) is described by a \textbf{positive-operator-valued measure (POVM)}, which consists of a vector \( M = \{ M_k \}_{k \in K} \) of positive semidefinite operators satisfying \( \sum M_k \leq 1 \)(Throughout this paper the operator inequality \( A \leq B \) means \( B - A \) is positive semidefinite.) The probability that the value \( k \) is measured when \( M \) is applied to a unit-trace density matrix \( \rho \) is given by

\[
Pr_M (k | \rho) = \Tr M_k \rho.
\]

The \textbf{success rate} for the POVM \( M \) to correctly determine the value of \( k \) corresponding to a random element of the ensemble \( \mathcal{E} \) is given by

\[
P_{\text{succ}} (M) = \sum_k p_k Pr_M \left( k | \frac{\rho_k}{p_k} \right) = \Tr \sum_{k \in K} M_k \rho_k.
\]

(4)

The \textbf{minimum-error measurement problem} consists of finding a POVM maximizing (4).

1.1.1 The relationship to “worst-case” detection

Sometimes one is interested in the “worst-case” distinguishability

\[
\max_M \min \sum_k \Tr M_k \hat{\rho}_k,
\]

of a collection of unit states \( \hat{\rho}_k \). As pointed out in \([46]\), the minimax theorem \([47]\) implies that

\[
\max_M \min \sum_k \Tr (M_k \hat{\rho}_k) = \max_M \min \sum_k M_k p_k \hat{\rho}_k = \min \max_M \sum_k M_k p_k \hat{\rho}_k,
\]

(6)

where \( \{ p_k \} \) represents a probability distribution. In particular, single-instance bounds (for fixed \( \{ p_k \} \)) may in principle be minimized over all distributions \( \{ p_k \} \) to give corresponding “worst-case” bounds.

1.2 Channel-adapted quantum error recovery

The following problem is of importance in quantum information theory, quantum communication, and quantum computing:

\begin{quote}
Suppose that one wishes to store, process, or transmit quantum data using a process that is subject to noise or loss. How well may the effects of this noise be avoided, corrected, or eliminated by encoding the data into a protected form, from which it may be later recovered unharmed by this noise?
\end{quote}

This problem arises in any physical implementation of quantum communication or computation, since unmitigated interactions with the environment tend to corrupt quantum signals or memory. By the celebrated “threshold theorem” \([48–52]\), one may in principle use error correction and concatenated quantum codes to perform an arbitrary quantum computation in the presence of noise below a fixed “threshold” amount.

Standard quantum error correction seeks to design encoding and decoding maps which \textit{exactly} correct for a given class of errors. Early successes of this program were the first codes that could protect against arbitrary single-qubit errors \([53–55]\), followed by general theoretical advances of \([56]\), and by the construction of codes that correct for arbitrary single-qubit errors by encoding a single qubit into five \([57, 58]\).

Alternatively, one may consider approximate quantum error correction. For example, Leung \textit{et al} \([59]\) consider relaxed error correction criteria to allow for efficient correction of a known dominant noise process. Furthermore, Crépeau, Gottesman, and Smith \([60]\) construct approximate error correcting codes which asymptotically correct twice as many arbitrary local errors as would
be possible under exact error correction, even though they achieve fidelity exponentially close to 1 in the limit of long codes.

Under the banner of approximate channel adapted error correction, a number of authors [7-20] alternatively have sought to treat quantum encoding and/or recovery as optimization problems. Mathematically, given a “noise” channel \( \mathcal{N} \) one seeks an encoding operation \( \xi \) and a recovery operation \( \mathcal{R} \) so that the composition

\[
\Xi = \mathcal{R} \circ \mathcal{N} \circ \xi
\]

is as close to the identity channel as possible. Measures of “closeness” to the identity include

**Definition 2** Let \( \rho \) be a mixed quantum state over a Hilbert space \( \mathcal{H} \), which may be represented as a pure quantum state \( |\psi_\rho\rangle \) of the original system entangled with an environment \( \mathcal{E} \). The entanglement fidelity [61] of the quantum operation \( \Xi : B^1(\mathcal{H}) \rightarrow B^1(\mathcal{H}) \) is given by

\[
F_e(\rho, \Xi) = \langle \psi_\rho | \Xi (|\psi_\rho\rangle \langle \psi_\rho|) |\psi_\rho\rangle.
\]  

(Note that the choice of purification does not affect the defined quantity.) The channel fidelity is the entanglement fidelity when \( \rho \) is taken to be maximally-mixed. Given a collection of states \( \rho_k \) with a-priori probabilities \( p_k \), one defines the average entanglement fidelity [14]

\[
\bar{F}_e(\{ (\rho_k, p_k) \}, \Xi) = \sum p_k F_e(\rho_k, \Xi).
\]  

Following [8, 11, 14, 16], we shall fix the encoding operation \( \xi \) and the noise process \( \mathcal{N} \). In particular, we focus on the problem of finding an approximately optimal quantum recovery map, or channel reversal, for the composed map

\[
\mathcal{A} = \mathcal{N} \circ \xi,
\]

in the sense of entanglement fidelity.

### 1.2.1 Other metrics for error recovery

A number of works have considered other measures of reversibility of quantum channels. Kretschmann, Schlingermann, and Werner [62] have obtained two-sided bounds on the CB-norm reversibility of channels in terms of the CB-distance between the complementary channel and a depolarizing channel. Ng and Mandayam [20] have employed the transpose channel (a special case of Barnum and Knill’s [14] reversal) to study quantum error correction using the metric of worst-case (non-entanglement) fidelity. Yamamoto, Hara, and Tsumura [17] considered a fixed encoding operation \( \mathcal{E} \) and used semidefinite programing to find a sub-optimal channel \( \mathcal{R} \) to roughly optimize the “worst-case” entanglement fidelity

\[
\max_{\mathcal{R}} \min_{\rho} F_e(\rho, \mathcal{R} \circ \mathcal{N} \circ \mathcal{E}).
\]  

More will be said about worst-case bounds in section 1.5.4, below.

### 1.3 Quantum conditional min- and max-entropy

The following related quantities (and their \( \varepsilon \)-smooth counterparts) are of interest in quantum cryptography (for example [21, 63-68]) and/or in studies of non-identically distributed and/or non-asymptotic problems in quantum information theory (for example [69-73]):

**Definition 3** Let \( \rho_{AB} \) be a bipartite density operator on \( \mathcal{H}_A \otimes \mathcal{H}_B \). The min-entropy of \( A \) conditioned on \( B \) [21, 70] is defined by

\[
H_{\text{min}}(A|B)_{\rho} := -\log_2 \inf_{v_B} \{ \text{Tr} v_B | \rho_{AB} \leq \mathbb{I}_A \otimes v_B \},
\]  

where the infimum ranges over positive semidefinite \( v_B \). The max-entropy of \( A \) conditioned on \( B \) [21, 70] is defined by

\[
H_{\text{max}}(A|B)_{\rho} := -H_{\text{min}}(A|C)_{\rho},
\]  

\[
\text{Tr} v_B
\]
where the min-entropy on the RHS is evaluated for a purification $\rho_{ABC}$ of $\rho_{AB}$. The max-information \cite{73} that $B$ has about $A$ is given by

$$I_{\text{max}}(A : B)_\rho = H_{\text{min}}(A|B)_{\rho_{A}^{1/2} \rho_{AB} \rho_{A}^{1/2}},$$

where $\rho_A = \text{Tr}_B \rho_{AB}$.

Estimates of $H_{\text{min}}(A|B)_\rho$ are obtained as a corollary of our estimates for maximum overlap in conjunction with the following recent theorem:

**Theorem 4 (König, Renner, Schaffner \cite{70})** Let the Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ be finite-dimensional. Then the min-entropy of $A$ conditioned on $B$ for the state $\rho_{AB}$ may be expressed as

$$H_{\text{min}}(A|B)_\rho = -\log \left( \dim(\mathcal{H}_A) \sup_\mathcal{R} (\langle \Phi_{AA'} | \mathcal{R}_{B \to A'}(\rho_{AB}) | \Phi_{AA'} \rangle) \right),$$

where $\Phi_{AA'}$ is a bipartite maximally-entangled state between $A$ and reference system $A' \simeq A$, and where the supremum is over quantum operations from $B$ to $A'$.

### 1.4 Directional Iterates: An abstract approach for deriving estimates

The first step in proving our estimates will be to recast all of the problems of the first section as instances of

**Problem 5 (Maximal seminorm problem)** Let $S$ be a subset of a real or complex semidefinite inner product space $V$. Find a maximal-seminorm element of $S$. (A semidefinite inner product has all the usual properties of an inner product, except that one may have $\langle x, x \rangle = 0$ for nonzero $x$.)

The following generalization of the iterative schemes of \cite{6, 7, 10, 74}, will prove useful for analyzing this class of problems:

**Definition 6** An abstract Ježek-Řeháček-Fiurášek-Hradil-Reimpell-Werner iterate of $g \in V$ is an element $g^{(+)} \in S$ which maximizes $\text{Re} \langle g^{(+)}, g \rangle$. Such $g^{(+)}$ will also be called directional iterates.

---

**Fig 1a**: Iterates converging on $x_{\text{max}}$. (contours drawn orthogonal to $G$.)

**Fig 1b**: Note that $\|X_{\text{max}}\| \geq \|G^{(+)}\| \geq \|\Pi G^{(+)}\| \geq \|\Pi X_{\text{max}}\|$, with approximate equality for reasonably small $\theta$. Here $\Pi = |G \rangle \langle G |$, and $V$ has real scalars. A complex variant appears as inequality 13.
Useful properties of these iterates are given by

Lemma 7 (Geometric properties of directional iterates) Suppose that $S \subseteq V$ has a maximal-seminorm vector $x_{\text{max}}$, and assume that each $g \in V$ admits a directional iterate $g^{(+)}$. Then

G1. One has the following inequalities

$$
\|x_{\text{max}}\| \geq \|g^{(+)}\| \geq \Lambda(g) \geq \|x_{\text{max}}\| \cos(\theta),
$$

where

$$
\Lambda(g) := \text{Re} \left( \frac{g^{(+)} \cdot g}{\|g\|} \right),
$$

$$
\cos \theta := \text{Re} \left( \frac{g \cdot x_{\text{max}}}{\|g\| \|x_{\text{max}}\|} \right).
$$

G2. The map $g \mapsto g^{(+)}$ is seminorm-increasing on $S$. In particular, if $g \in S$ then

$$
\|g^{(+)}\|^2 \geq \|g\|^2 + \|g^{(+)} - g\|^2.
$$

Note: The importance of property G1 is this: If one can construct a guess $g$ subtending a reasonably small angle with $x_{\text{max}}$ then both $\Lambda(g)$ and $\|g^{(+)}\|$ are reasonably good estimates for $\|x_{\text{max}}\|$. (Note that although $\|g^{(+)}\|$ is a closer approximation to $\|x_{\text{max}}\|$, in our applications $\Lambda(g)$ will have a much simpler expression.)

Proof. To prove property G1, note that

$$
\|x_{\text{max}}\| \geq \|g^{(+)}\| \geq \frac{\text{Re} \left( g^{(+)} \cdot g \right)}{\|g\|} \geq \frac{\text{Re} \left( x_{\text{max}} \cdot g \right)}{\|g\|} = \|x_{\text{max}}\| \cos(\theta).
$$

The first inequality is trivial, the second is Schwarz’s, and the third is by the definition of $g^{(+)}$.

To prove property G2, write

$$
\|g^{(+)}\|^2 = \|g^{(+)} - g\|^2 + \|g\|^2 + 2 \text{Re} \left( \left( g^{(+)} , g \right) - \langle g, g \rangle \right).
$$

The last term on the RHS is nonnegative by the definition of $g^{(+)}$. $\blacksquare$

We now may set forth the following:

General strategy for estimating maximal seminorms:

1. Find a “small angle guess” $g$, such that the angle defined by (15) is provably small in some approximate sense.

2. Obtain two-sided bounds for $\|x_{\text{max}}\|$ using this bound on $\theta$ in conjunction with (13).

3. Make this bound explicit by computing $g^{(+)}$ and $\Lambda(g)$.

By property G2 of Lemma 7, one may have some hope of obtaining a maximal element as the limit of repeated iteration, as occurs in Fig. 1a. In sections 1.4.1-1.4.2 we review numerical schemes in the literature which may be seen as examples of this process. (These sections may be skimmed on first reading.)
1.4.1 Example 1: Ježek-Řeháček-Fiurášek iteration for POVMs

Ježek, Řeháček, and Fiurášek (JRF) [74, 75] proposed an unproven numerical method for computing optimal POVMs, using iteration of the mapping \( M \mapsto M^{(\oplus)} \) given by

**Definition 8** The Ježek-Řeháček-Fiurášek (JRF) iterate of a POVM \( M = \{ M_k \}_{k \in K} \) [74, 75] is the POVM defined by

\[
M^{(\oplus)}_k = \left( \sum_{\ell \in K} \rho_\ell M_\ell \rho_\ell \right)^{-1/2^+} - \frac{1}{2} \rho_k M_k \rho_k \left( \sum_{\ell \in K} \rho_\ell M_\ell \rho_\ell \right)^{-1/2^+}, \tag{17}
\]

Here the negative matrix power is defined by

\[
A^{-s^+} = \sum_{\lambda_j > 0} \lambda_j^{-s} \Pi_j \tag{18}
\]

for \( s \geq 0 \) and self-adjoint \( A \) with spectral decomposition \( A = \sum \lambda_j \Pi_j \).

Ježek, Řeháček, and Fiurášek made the following:

**Numerical Observation 9 (JRF [74, 75])** JRF iteration monotonically increases success rate:

\[
P_{\text{succ}} \left( M^{(\oplus)} \right) \geq P_{\text{succ}} \left( M \right).
\]

Furthermore, iteration of this map starting from \( \{ M_k = \mathbb{1} \} \) converges to an optimal measurement

\[
\lim_{j \to \infty} P_{\text{succ}} \left( M^{(\oplus)}_j \right) = P_{\text{succ}} \left( M^{\text{opt}} \right). \tag{19}
\]

In section 3.1, JRF iteration is exhibited as a disguised form of directional iteration. JRF’s numerically-observed monotonicity then follows immediately from property G2 of lemma 7.

1.4.2 Example 2: Ježek-Fiurášek-Hradil and Reimpell-Werner iterates

Ježek, Fiurášek, and Hradil (JFH) [6, 75] proposed an unproven numerical scheme for the maximum-likelihood problem [75–78] in quantum process tomography, which contains the maximum-overlap problem (1) as a special case.

Reimpell and Werner [7, 10] introduced a mild generalization of this special case of JRH’s algorithm, for use in finding maximizers of the following:

**Definition 10** A Reimpell-Werner functional [7, 10] \( \mathcal{R} \mapsto f(\mathcal{R}) \) is a linear functional of linear transformations \( \mathcal{R} : B^1(K) \to B^1(\mathcal{L}) \) such that \( f(\mathcal{R}) \geq 0 \) for all completely positive \( \mathcal{R} \).

Reimpell and Werner were interested in the special cases of approximate quantum error recovery and quantum encoding in the sense of channel fidelity. In particular, setting

\[
f_N(\mathcal{E}, \mathcal{R}) = F_\epsilon \left( \mathbb{1} / \dim \mathcal{H}, \mathcal{R} \circ N \circ \mathcal{E} \right), \tag{20}
\]

where \( N \) is a known noise map, they alternatively optimized the encoder \( \mathcal{E} \) and decoder \( \mathcal{R} \) in a seesaw fashion.

By analogy with the matrix-power method [7, 10], they proposed an unproven numerical method for maximizing \( f(\mathcal{R}) \) by iteration of the following map:

**Definition 11** Let \( \mathcal{L} \) and \( \mathcal{K} \) be finite-dimensional, and represent the Reimpell-Werner functional \( f \) as

\[
f(\mathcal{R}) = \frac{\text{Tr}}{\mathcal{L} \mathcal{K}^*} \left( F \tilde{\mathcal{R}} \right), \tag{21}
\]
where \( \tilde{R} \in B^1(\mathcal{L} \otimes \mathcal{K}^*) \) is the Choi matrix of \( R \) (see Definition 27) and \( F \) is a positive operator on \( \mathcal{L} \mathcal{K}^* \). The Reimpell-Werner iterate \( R^\oplus \) of \( R \) [7, 10] is the quantum operation with Choi matrix

\[
\tilde{R}^\oplus = \Gamma^{-1/2^+} F \tilde{R} F \Gamma^{-1/2^+},
\]

where \( \Gamma : \mathcal{L} \mathcal{K}^* \rightarrow \mathcal{L} \mathcal{K}^* \) is given by

\[
\Gamma = \mathbb{1}_\mathcal{L} \otimes \text{Tr}_\mathcal{L} (F \tilde{R} F).
\]

Reimpell [10] proved the monotonicity property \( f(R^\oplus) \geq f(R) \) using a clever matrix analysis argument. In particular, the optimal map \( R \) is a fixed point of this iteration.

In Appendix B we show that Reimpell-Werner iteration (and the special case of restricted JRH iteration) may be viewed as directional iteration on the corresponding space of Stinespring dilations. In particular, Reimpell’s monotonicity result is exhibited as a special case of Lemma 7.

### 1.5 Relevant existing bounds, suboptimal measurements, and approximate reversals

#### 1.5.1 Quadratic measurements and Generalized Holevo-Curlander bounds

**Definition 12** Let \( E = \{p_k | \psi_k \rangle \langle \psi_k|\} \) be an ensemble of pure states. Then Holevo’s pure state measurement [43] is given by \( M_k = |e_k \rangle \langle e_k| \), where

\[
e_k = e_k^{\text{Holevo}} := \left( \sum \rho_k^2 |\psi_k \rangle \langle \psi_k| \right)^{-1/2^+} p_k \psi_k.
\]

Holevo constructed this measurement using an approximate minimal principle, and proved

**Theorem 13 (Holevo's asymptotic optimality theorem [43])** Holevo’s measurement is asymptotically-optimal for distinguishing pure states in the sense that for fixed probabilities \( \{p_k\} \) one has

\[
\frac{P_{\text{fail}}(\{e_k^{\text{Holevo}}\})}{P_{\text{fail}}^{\text{optimal}}} \rightarrow 1
\]

as the \( \psi_k \) are varied so that \( \langle \psi_i, \psi_j \rangle \rightarrow \delta_{ij} \). Here \( P_{\text{fail}} = 1 - P_{\text{succ}} \) represents the failure rate.

A natural mixed-state generalization of Holevo’s measurement is given by

**Definition 14** The quadratically-weighted measurement [22, 74] for distinguishing the ensemble (2) is the first Ježek-Řeháček-Fiurášek iterate

\[
M_k^{QW} = \left( \sum \rho_l^2 \right)^{-1/2^+} \rho_k^2 \left( \sum \rho_l^2 \right)^{-1/2^+}.
\]

**Remark:** The quadratically-weighted measurement is an example of a Belavkin-Maslov measurement (see page 39 of [37]).

Generalizing the pure-state results of Holevo [43] and Curlander [44], the author proved the following:

**Theorem 15 (Generalized Holevo-Curlander bounds [22])** One has the following bounds on the success rate of the optimal measurement \( M^{\text{opt}} \) for distinguishing the ensemble \( E \) of Definition 1:

\[
\Lambda^2 \leq P_{\text{succ}}(M^{QW}) \leq P_{\text{succ}}(M^{\text{opt}}) \leq \Lambda,
\]

where

\[
\Lambda = \text{Tr} \sqrt{\sum \rho_k^2} \leq 1.
\]

**Note:** The upper bound of (27) was essentially a special case of a pre-existing bound of Ogawa and Nagaoka [23], which is a simple consequence of matrix monotonicity.
1.5.2 The “pretty good” measurement and Barnum & Knill’s distinguishability bound

Another approximately-optimal measurement is the linearly-weighted measurement given by

**Definition 16** The Belavkin-Hausladen-Wootters “pretty good” measurement (PGM) is given by

$$M^\text{PGM}_k = \left( \sum \rho_e \right)^{-1/2^+} \rho_k \left( \sum \rho_e \right)^{-1/2^+}. \quad (29)$$

A comparison of the PGM with Holevo’s pure state measurement was conducted in [83]. It was found that Holevo’s measurement outperforms the PGM for ensembles of two pure states, and that the PGM does NOT satisfy Holevo’s asymptotic optimality property (25).

The PGM is approximately-optimal for “reasonably-distinguishable” ensembles in the following precise sense:

**Theorem 17 (Barnum-Knill [14])** The success rate of the PGM satisfies

$$\frac{P\text{succ}(M^\text{PGM})}{P\text{succ}(M^{\text{opt}})} \geq P\text{succ}(M^{\text{opt}}), \quad (30)$$

where $M^{\text{opt}}$ is an optimal measurement.

Re-expressing this inequality in terms of $P\text{fail} = 1 - P\text{succ}$, one sees that the PGM has a failure rate within a factor of two of the optimal:

$$P\text{fail}(M^{\text{opt}}) \leq P\text{fail}(M^\text{PGM}) \leq 2 \times P\text{fail}(M^{\text{opt}}). \quad (31)$$

The relationship between Barnum and Knill’s bound (30) and the bounds of Theorem 15 is explained by the following proposition:

**Proposition 18 (Comparison with the Barnum-Knill bounds)** Both of the lower bounds of inequality 27 are sufficiently tight to also satisfy Barnum and Knill’s tightness relation (30):

$$\frac{P\text{succ}(M^{Q\text{W}})}{P\text{succ}(M^{\text{opt}})} \geq \frac{\Lambda^2}{P\text{succ}(M^{\text{opt}})} \geq P\text{succ}(M^{\text{opt}}). \quad (32)$$

In particular, $P\text{fail}(M^{Q\text{W}}), 2(1 - \Lambda), \text{ and } 1 - \Lambda^2$ all lie in the interval $[P\text{fail}(M^{\text{opt}}), 2 \times P\text{fail}(M^{\text{opt}})]$.

**Proof.** Equation 32 follows immediately by double application of inequality 27. The claimed inclusions follow as in inequality 31, where one additionally uses the inequality $1 - \Lambda^2 \leq 2(1 - \Lambda)$.

1.5.3 Barnum and Knill’s approximate reversal map

Generalizing the “pretty good” measurement[108], Barnum and Knill have constructed a reversal of an arbitrary quantum operation $A : B^1(H) \rightarrow B^1(K)$ that is approximately optimal for reasonably reversible $A$ in a precise sense:

**Theorem 19 (Barnum-Knill [14])** Assume that the density operators $\rho_k \in B^1(H)$ of equation 8 commute, set $\rho = \sum p_k \rho_k$, and let $A^\dag : B(K) \rightarrow B(H)$ be the adjoint of $A$ (see Def. 24, below). Then the recovery operation

$$R^BK(v) = \sqrt{\rho} A^\dag \left( (A(\rho))^{-1/2^+} v (A(\rho))^{-1/2^+} \right) \sqrt{\rho} \quad (33)$$

is approximately optimal in the sense that

$$\frac{F_e(\{\rho_k, p_k\}, R^BK \circ N)}{\max_R F_e(\{\rho_k, p_k\}, R \circ N)} \geq \max_R F_e(\{\rho_k, p_k\}, R \circ N), \quad (34)$$

where $F_e$ is the average entanglement fidelity of equation 8.
A special case of eq. 33 is of recent [20] interest in the literature:

**Definition 20** The transpose channel \([84]\) is the special case of the Barnum-Knill reversal \(R^{BK}\) for maximally-mixed \(\rho\).

A reversal of approximately optimal entanglement fidelity which is closely related to the quadratic measurement will be constructed in section 5.

### 1.5.4 The bounds of Bény and Oreshkov

Generalizing the problem of quantum error-recovery, Bény and Oreshkov [18] have more-generally considered channel simulation. In particular, they consider the “worst-case” entanglement fidelity

\[
\max_{R} \min_{\rho} F_{\rho}(R \mathcal{A}, \mathcal{M})
\]

with which the channel \(\mathcal{A}\) may be used to simulate the channel \(\mathcal{M}\). Here one has

\[
F_{\rho}(\mathcal{N}, \mathcal{M}) = \min_{\rho} f\left(\mathcal{N}_{H \rightarrow K}\left|\psi_{\rho}\right\rangle_{HH}, \left\langle\psi_{\rho}\right|_{HH}\right),
\]

where \(\psi_{\rho}\) is a purification of \(\rho\) and (changing their conventions slightly) \(f(\rho, \sigma) = (\text{Tr} \sqrt{\sqrt{\rho} \sigma \sqrt{\rho}})^2\) is the fidelity between the states \(\rho\) and \(\sigma\). Note that quantum error recovery is the \(\mathcal{M} = 1\) special case. Employing the min-max Theorem and a beautiful (and short!) duality argument involving complementary channels, they obtain the following theorem:

**Theorem 21** (Bény-Oreshkov [18]) One has the worst-case recovery bounds

\[
\left(\frac{3}{4} + \frac{1}{4} \tilde{\Lambda}_{\sigma}\right)^2 \geq \max_{R} \min_{\rho} F_{\rho}(\rho, R \circ \mathcal{A}) \geq \tilde{\Lambda}_{\sigma}^2,
\]

where the state \(\sigma\) is an adjustable parameter, \(\mathcal{A}\) has Kraus decomposition \(\mathcal{A}(\mu) = \sum E_i \mu E_i^\dagger\), and

\[
\tilde{\Lambda}_{\sigma} := \min_{\rho} \text{Tr} \sqrt{\sum E_i \rho^2 E_i^\dagger \times \text{Tr} \left(E_j \sigma E_j^\dagger\right)}.
\]

Furthermore, if \(\rho\) is fixed then one obtains

\[
\left(\frac{3}{4} + \frac{1}{4} F_{\rho}(\hat{\mathcal{A}}, S)\right)^2 \geq \max_{\hat{\mathcal{A}}} F_{\rho}(\rho, \mathcal{R} \circ \hat{\mathcal{A}}) \geq \left(F_{\rho}(\hat{\mathcal{A}}, S)\right)^2,
\]

where \(\hat{\mathcal{A}}\) is a channel complementary to \(\mathcal{A}\) and \(S(\sigma) = \hat{\mathcal{A}}(\rho) \times \text{Tr} \sigma\).

**Remarks:**

1. There is an apparent, but unexplained, relationship between our work below and the results of Bény-Oreshkov, which appeared in arXiv preprint form almost-simultaneously to ours. Further will be said on this matter in [19]. (See Theorem 44 and Proposition 45, below.)

2. It is important to note that in the finite-dimensional case that one has the identity

\[
\max_{R} \min_{\rho} F_{\rho}(\rho, \mathcal{R} \circ \mathcal{N} \circ \mathcal{E}) = \min_{\rho} \max_{R} F_{\rho}(\rho, \mathcal{R} \circ \mathcal{N} \circ \mathcal{E}).
\]

This follows from the min-max Theorem [85], where the convexity of the mapping \(\rho \mapsto F_{\rho}(\rho, \mathcal{R} \circ \mathcal{N} \circ \mathcal{E})\) is evident from equation 1.10 of [15] and where one may take the recovery \(\mathcal{R}\) to range over the convex set of quantum operations (trace non-increasing completely positive maps). In particular, one may obtain “worst-case” recovery bounds (albeit with unevaluated minimization over \(\rho\)) from single-instance bounds on \(F_{\rho}(\rho, \mathcal{R} \circ \mathcal{N})\), which we exclusively consider below.
1.6 Results
Section 1.4 has already introduced directional iteration as an abstract method for estimating solutions of maximal-seminorm problems. This incorporates several explicitly-defined numerical iterative schemes, including:

- The iteration of Ježek, Řeháček, and Fiurášek for computing optimal quantum measurements.
- The iteration of M. Ježek, J. Fiurášek, and Z. Hradil as restricted to the maximum-overlap problem.
- The iteration of Reimpell and Werner for numerically optimizing quantum error correction (both encoding and recovery).

Defined by a minimal-principle, directional iteration monotonically increases seminorm essentially by construction. In particular:

- Ježek, Řeháček, and Fiurášek’s numerical observation that their iteration only increases success rate is proven in greater generality.
- This gives a short proof of Reimpell’s monotonicity Theorem (pp. 39-42 of [10]) for iterative optimization of quantum error correction.

Section 3 introduces our techniques by presenting a new proof of the generalized Holevo-Curlander bounds (Theorem 15) on the distinguishability of arbitrary ensembles of mixed quantum states.

In section 4, Theorem 39 gives concise two-sided bounds for the maximum overlap problem (1), in the restricted case that \( M_{\mathcal{H}} \) is rank 1. Corollary 40 bounds the quantum conditional min-entropy. Appendix C shows how one may apply these bounds to recover the bounds of section 3.

Theorem 44 of section 5 applies our overlap bounds to estimate approximate channel reversibility in the sense of entanglement fidelity. (The bounds of section 4.2.5 more generally allow the entangled input and output states to differ, however.) Our channel-reversibility estimates apply to the case of channel-adapted approximate quantum error recovery.

Section 5.3 compares our reversibility estimates and approximate reversal map to those of Barnum and Knill. Although our bounds take a particularly simple form, they are still sufficiently accurate to satisfy the tightness relation (34) satisfied by the bounds of Barnum and Knill. The relationship between our recovery map and Barnum and Knill’s is found to be analogous to the relationship between Holevo’s asymptotically optimal measurement and the so-called “pretty good” measurement. Furthermore, our recovery operation is found to significantly outperform the transpose channel in the case of depolarizing noise acting on half of a maximally-entangled state.

The conclusion points out directions for future research.

2 Notation, conventions, and mathematical background
The reader who is only interested in minimum-error distinguishability bounds should proceed directly to section 3, referring back only as directed.

Def. 22 Let \( \mathcal{H} \) and \( \mathcal{K} \) be Hilbert spaces, and let \( A : \mathcal{H} \to \mathcal{K} \) be a bounded linear operator. The absolute value is \( |A| = \sqrt{A^\dagger A} \). The space \( B^1(\mathcal{H} \to \mathcal{K}) \) consists of all operators of finite trace norm \( \|A\|_1 = \text{Tr}|A| \). The space \( B^2(\mathcal{H} \to \mathcal{K}) \) consists of all operators of finite Hilbert-Schmidt norm \( \|A\|_2 = \sqrt{\text{Tr}A^\dagger A} \). This space has the inner product

\[ \langle A, B \rangle = \text{Tr}A^\dagger B. \] (39)
The space $B(\mathcal{H} \to \mathcal{K})$ consists of all operators of finite operator norm, given by

$$\|A\| = \|A\|_\infty = \sup_{0 \neq \psi \in \mathcal{H}} \frac{\|A\psi\|}{\|\psi\|}. \quad (40)$$

When $\mathcal{H} = \mathcal{K}$, these spaces will be denoted by $B(\mathcal{H})$, $B^1(\mathcal{H})$, and $B^2(\mathcal{H})$, for short. An operator $A$ is a contraction if $\|A\| \leq 1$.

It is assumed that the reader is familiar with the following trace-norm inequalities, which may be found in [86]:

$$|\text{Tr} A| \leq \|A\|_1 = \|A^\dagger\|_1 \quad \text{if } \mathcal{K} = \mathcal{H}. \quad (41)$$

$$\|WA\|_1 \leq \|W\|_\infty \times \|A\|_1. \quad (42)$$

Furthermore,

$$\sup_{\|U\| \leq 1} \text{Re} \left( \text{Tr} A^\dagger U \right) = \|A\|_1, \quad (43)$$

where $A : \mathcal{H} \to \mathcal{K}$ and the supremum is over contractions $U : \mathcal{H} \to \mathcal{K}$. It follows simply from the singular value decomposition that $U$ is a maximizer of (43) iff

$$U|_{\text{Ran}(A^\dagger A)} = A (A^\dagger A)^{-1/2+}, \quad (44)$$

where $(A^\dagger A)^{-1/2+}$ is defined by (18).

**Definition 23** Let $A$ be a self-adjoint operator. The **positive projection** $\Pi_+ (A)$ is the projection onto the closure of the range of the positive part of $A$. In particular, if $A$ has spectral decomposition $A = \sum \lambda_i |\psi_i\rangle \langle \psi_i|$ then

$$\Pi_+ (A) = \sum_{\lambda_i > 0} |\psi_i\rangle \langle \psi_i|. \quad (45)$$

A more thorough discussion of most of the following terms may be found in [45]:

**Definition 24** A **quantum state** is a trace-class positive semidefinite operator $\rho$ on a Hilbert space $\mathcal{H}$. (Generally states are of unit trace, although in section 3 it will be convenient to normalize them by a-priori probability.) The **support** $\text{supp} (A)$ of the transformation $A : \mathcal{H} \to \mathcal{K}$ is the closure of the range of $A^\dagger A$, or equivalently the orthogonal complement of the null-space of $A$. A **quantum channel** is a trace preserving completely positive map. A **quantum operation** is a trace non-increasing completely positive map. A linear operator $U : \mathcal{K} \to \mathcal{L} \otimes \mathcal{E}$ is a **Stinespring dilation** [87] of a completely positive map $\mathcal{R} : B^1(\mathcal{K}) \to B^1(\mathcal{L})$ if

$$\mathcal{R} (\rho) = \text{Tr}_{\mathcal{E}} \left( U \rho U^\dagger \right) \quad (46)$$

for all $\rho \in B^1(\mathcal{K})$. The **adjoint** $\mathcal{R}^\dagger : B(\mathcal{L}) \to B(\mathcal{K})$ has the defining property that

$$\text{Tr}_{\mathcal{E}} \left( X_\mathcal{L} \mathcal{R}_{\mathcal{K} \to \mathcal{E}} (Y_\mathcal{K}) \right) = \text{Tr}_{\mathcal{K}} \left( \mathcal{R}^\dagger_{\mathcal{L} \to \mathcal{K}} (X_\mathcal{L}) Y_\mathcal{K} \right) \quad (47)$$

for $X \in B(\mathcal{L})$ and $Y \in B^1(\mathcal{K})$.

It is important to note that if $\mathcal{R}$ and $U$ are related by (46) then $\mathcal{R}$ is a channel iff $U$ is an isometry ($U^\dagger U = 1$), and $\mathcal{R} : B^1(\mathcal{K}) \to B^1(\mathcal{L})$ is a quantum operation iff $U$ is a contraction. Furthermore, it is observed in Appendix A that each quantum operation $\mathcal{A}$ has a canonical dilation with the canonical environment

$$\mathcal{E} = \mathcal{L}_\mathcal{E}^* \otimes \mathcal{K}_\mathcal{E}, \quad (48)$$
where $\mathcal{K}_L$ and $\mathcal{L}_K^*$ are copies of $\mathcal{K}$ and the dual space of $\mathcal{L}$, respectively.

**Tensor product notation:** A linear operator $A : \mathcal{H} \rightarrow \mathcal{K}$ will often be denoted as $A_{\mathcal{H} \rightarrow \mathcal{K}}$, and will be identified without further comment with any operator of the form $A \otimes 1_L$ where $1_L$ is the identity operator on some other Hilbert space $\mathcal{L}$. If $|\psi\rangle \in \mathcal{L}$, the transformation $|\psi\rangle \otimes A : \mathcal{H} \rightarrow \mathcal{K} \otimes \mathcal{L}$ is defined by

$$\langle \psi \rangle_{\mathcal{L}} \otimes A_{\mathcal{H} \rightarrow \mathcal{K}} |\phi\rangle_{\mathcal{H}} = |\psi\rangle_{\mathcal{L}} \otimes |A\phi\rangle_{\mathcal{K}}$$

(49)

When $A : B^1(\mathcal{H}) \rightarrow B^1(\mathcal{K})$ is a quantum operation, it will often be denoted as $A_{\mathcal{H} \rightarrow \mathcal{K}}$.

### 2.1 Basis-free constructions using dual spaces and double kets

As in a number of previous works on channel-adapted quantum error recovery [15–17], in Section 5 and in the Appendices A and B it will prove convenient to treat Hilbert spaces and their duals on equal footing:

**Definition 25** The dual space $\mathcal{H}^*$ of the Hilbert space $\mathcal{H}$ is the set of linear functionals $\bar{\psi} : \mathcal{H} \rightarrow \mathbb{C}$ of the form

$$\bar{\psi}(\phi) := \langle \psi, \phi \rangle : \mathcal{H} \rightarrow \mathbb{C},$$

(50)

where $\psi \in \mathcal{H}$. The space $\mathcal{H}^*$ is a Hilbert space in its own right with inner product

$$\langle \bar{\psi}_1, \bar{\psi}_2 \rangle_{\mathcal{H}^*} := \overline{\langle \psi_1, \psi_2 \rangle_{\mathcal{H}}} = \langle \psi_2, \psi_1 \rangle_{\mathcal{H}},$$

(51)

where the bar in the middle denotes complex conjugation.

Use of the dual space as a Hilbert space in its own right has pleasant computational properties which are amenable to Dirac notation. For example, if $\psi \in \mathcal{H}$ has the coordinate expansion

$$\psi = \sum \psi_i |i\rangle_{\mathcal{H}},$$

(52)

then the dual vector $\bar{\psi} \in \mathcal{H}^*$ has the expansion

$$\bar{\psi} = \sum \bar{\psi}_i |i\rangle_{\mathcal{H}^*},$$

(53)

where the coordinates $\bar{\psi}_i := \overline{\langle i | \psi \rangle_{\mathcal{H}}}$. are simply the complex conjugates of the coordinates $\psi_i$:

$$\bar{\psi}_i = \overline{\psi}_i.$$ (54)

Given the linear transformation $A \in B^2(\mathcal{H} \rightarrow \mathcal{K})$

$$A_{\mathcal{H} \rightarrow \mathcal{K}} = \sum A_{ij} |i\rangle_{\mathcal{K}} \langle j|_{\mathcal{H}}$$

(55)

one may form the **conjugate operator** $\bar{A} : \mathcal{H}^* \rightarrow \mathcal{K}^*$, the **transpose** $A^\dagger : \mathcal{K}^* \rightarrow \mathcal{H}^*$, and the **basis-free double ket** $|A\rangle_{\mathcal{K} \mathcal{H}^*} \in \mathcal{K} \otimes \mathcal{H}^*$ by

$$\bar{A}_{\mathcal{H}^* \rightarrow \mathcal{K}^*} = \sum \bar{A}_{ij} |i\rangle_{\mathcal{K}^*} \langle j|_{\mathcal{H}^*},$$

(56)

$$A^\dagger_{\mathcal{K}^* \rightarrow \mathcal{H}^*} = \sum A_{ij} |j\rangle_{\mathcal{H}^*} \langle i|_{\mathcal{K}^*},$$

(57)

$$|A\rangle_{\mathcal{K} \mathcal{H}^*} = \sum A_{ij} |i\rangle_{\mathcal{K}} \langle j|_{\mathcal{H}^*}.$$ (58)

These equations may be replaced by basis-independent definitions, since they are uniquely-specified by the identities $\bar{A}\bar{\phi} = \langle A\phi \rangle$, $A^\dagger = A^\dagger$, and $\langle \psi_K | \langle \phi_{\mathcal{H}^*} | A\rangle_{\mathcal{K} \mathcal{H}^*} = \langle \psi, A\phi \rangle$, for $\phi \in \mathcal{H}$ and $\psi \in \mathcal{K}$, respectively.

The **basis-free double bra**

$$\langle A \rangle_{\mathcal{K} \mathcal{H}^*} = \sum \bar{A}_{ij} \langle i|_{\mathcal{K}} \langle j|_{\mathcal{H}^*}.$$ (59)
denotes the linear functional on $\mathcal{K} \otimes \mathcal{H}^*$ corresponding to $|A\rangle\rangle$. The partial transpose is the isometric extension of the mapping $A_{\mathcal{H} \rightarrow \mathcal{K} \otimes \mathcal{L} \rightarrow \mathcal{M} :} \rightarrow A_{\mathcal{K}^* \rightarrow \mathcal{H}^* \otimes \mathcal{L} \rightarrow \mathcal{M} :}$, i.e.

$$PT_{B^2(\mathcal{H} \rightarrow \mathcal{K}^* \rightarrow \mathcal{H}^*) \rightarrow B^2(\mathcal{K}^* \rightarrow \mathcal{H}^* \rightarrow \mathcal{M}^*)} \left( \sum_{m,k,h,l} X_{m,k,h,l} |m,k\rangle \langle h,l| \right) = \sum_{m,k,h,l} X_{m,k,h,l} |\bar{h},k\rangle \langle \bar{l},l|,$$  \hspace{1cm} (60)

which maps $B^2(\mathcal{H} \otimes \mathcal{L} \rightarrow \mathcal{K} \otimes \mathcal{M} ) \rightarrow B^2(\mathcal{K}^* \otimes \mathcal{L} \rightarrow \mathcal{H}^* \otimes \mathcal{M} )$, where $\mathcal{L}$ and $\mathcal{M}$ are arbitrary Hilbert spaces.

We collect some useful identities involving basis-free double-kets:

**Lemma 26**

1. If $A, B : \mathcal{H} \rightarrow \mathcal{K}$ then

$$\langle \langle A, B \rangle \rangle_{\mathcal{K} \mathcal{H}^*} = \text{Tr} A^\dagger B$$  \hspace{1cm} (61)

2. Let $A : \mathcal{K} \rightarrow \mathcal{L}$, $B : \mathcal{H} \rightarrow \mathcal{M}$, and $C : \mathcal{H} \rightarrow \mathcal{K}$. Then

$$\left( A_{\mathcal{K} \rightarrow \mathcal{L}} \otimes \bar{B}_{\mathcal{H}^* \rightarrow \mathcal{M}^*} \right) \left| C \right\rangle_{\mathcal{K} \mathcal{H}^*} = \left| A C B^\dagger \right\rangle_{\mathcal{L} \mathcal{M}^*}. $$  \hspace{1cm} (62)

3. Let $A : \mathcal{H} \rightarrow \mathcal{L}$ and $B : \mathcal{K} \rightarrow \mathcal{L}$. Then

$$\langle \langle A \left|_{\mathcal{L} \mathcal{H}^*} \right. \times \left| B \right\rangle_{\mathcal{L} \mathcal{K}^*} = \text{Tr}_\mathcal{L} \left| B \right\rangle_{\mathcal{L} \mathcal{K}^*} \langle \langle A \right|_{\mathcal{L} \mathcal{H}^*} = \left| B^\dagger A \right\rangle_{\mathcal{L} ^2}$$  \hspace{1cm} (63)

4. Let $A : \mathcal{H} \rightarrow \mathcal{K}$ and $B : \mathcal{H} \rightarrow \mathcal{L}$. Then

$$\text{Tr}_{\mathcal{H}^*} \left| A \right\rangle_{\mathcal{K} \mathcal{H}^*} \left. \langle \langle B \right|_{\mathcal{L} \mathcal{H}^*} = A_{\mathcal{H} \rightarrow \mathcal{K} \mathcal{L} \rightarrow \mathcal{H} \mathcal{K}} \left. \langle \langle B \right|_{\mathcal{L} \mathcal{H}^*} = B^\dagger_{\mathcal{L} \rightarrow \mathcal{H} \otimes A_{\mathcal{H} \rightarrow \mathcal{K} :} \mathcal{H} \otimes \mathcal{L} \rightarrow \mathcal{H} \otimes \mathcal{K}}$$  \hspace{1cm} (64)

Note that by multilinearity it is enough to check these identities for rank-1 operators.

**Definition 27** The canonical purification \cite{110} of a quantum state $\rho \in B^1(\mathcal{H})$ is given by

$$|\psi_\rho\rangle = |\sqrt{\rho}\rangle_{\mathcal{H} \mathcal{H}^*}.$$  \hspace{1cm} (66)

When $\mathcal{K}$ is finite-dimensional,\cite{111} the Choi matrix \cite{88} of a transformation $R : B^1(\mathcal{K}) \rightarrow B^1(\mathcal{L})$ is given by

$$\tilde{R} = R(\left| \mathbb{I} \right\rangle_{\mathcal{K} \mathcal{K}^*} \left.< \mathbb{I} \right|) \in B^1(\mathcal{L} \mathcal{K}^*). $$  \hspace{1cm} (67)

Note that by (64) and (63), $\psi_\rho$ has the standard defining property

$$\rho = \text{Tr}_{\mathcal{H}^*} |\psi_\rho\rangle \langle \psi_\rho| $$  \hspace{1cm} (68)

of a purification of $\rho$, and also of $\tilde{\rho}$

$$\tilde{\rho} = \text{Tr}_{\mathcal{H}^*} |\psi_{\tilde{\rho}}\rangle \langle \psi_{\tilde{\rho}}| $$  \hspace{1cm} (69)

In particular, if $\mathcal{H}$ is finite-dimensional then the state $(\dim \mathcal{H})^{-1/2} |\mathbb{I}\rangle_{\mathcal{H} \mathcal{H}^*}$ is maximally-entangled, and indeed the singular value decomposition of an operator

$$A = \sum \lambda_i |f_i\rangle \langle g_i|$$

corresponds precisely to the Schmidt decomposition of its double-ket

$$|A\rangle\rangle = \sum \lambda_i |f_i\rangle |\bar{g}_i\rangle.$$  

**Remark:** A basis-free construction of the Stinespring dilation may be found in Appendix A.
3 Minimum-error distinction as a maximal seminorm problem

The minimum-error quantum detection problem of Definition 1 may be reformulated as a maximal-seminorm problem using the identity

\[ P_{\text{succ}}(M) = \|E\|^2, \]

per the following definition:

**Definition 28** Let \( \mathcal{E} = \{\rho_k\}_{k \in K} \) be the ensemble of Definition 1. A vector of operators \( E = \{E_k : \mathcal{H} \rightarrow \mathcal{H}\}_{k \in K} \) is a **generalized measurement (GM)** [24] corresponding to the POVM \( M = \{M_k\}_{k \in K} \) if one has the decomposition

\[ M_k = E_k^\dagger E_k. \]  

The \( \mathcal{E} \)-**semi-inner product** is defined for vectors of operators \( F = \{F_k : \mathcal{H} \rightarrow \mathcal{H}\}_{k \in K} \) and \( G = \{G_k : \mathcal{H} \rightarrow \mathcal{H}\}_{k \in K} \) by

\[ \langle F,G \rangle_\mathcal{E} = \text{Tr} \sum_{k \in K} F_k^\dagger G_k \rho_k. \]

The \( \mathcal{E} \)-**semi-inner product space** is the space \( V_\mathcal{E} = \{E \ | \ \|E\|_\mathcal{E} < \infty\} \), on which \( \langle \cdot, \cdot \rangle_\mathcal{E} \) is well-defined. The set \( S_\mathcal{E} \subseteq V_\mathcal{E} \) will denote the set of generalized measurements of \( \mathcal{E} \).

**Remark:** It is important to note that if \( \mathcal{E} \) is a perfectly distinguishable ensemble of more than one element then \( \|\cdot\|_\mathcal{E} \) is only a seminorm. In particular, any cyclic permutation \( E' \) of a perfectly-distinguishing generalized measurement must satisfy \( \|E'\| = 0 \).

3.1 Computation of directional iterates

Our first step is to compute directional iterates for generalized measurements:

**Theorem 29 (Directional iteration for generalized measurements)** Take \( S = S_\mathcal{E} \) and \( V = V_\mathcal{E} \) as in Def. 6. Then a directional iterate of \( E \in V_\mathcal{E} \) is given by

\[ E_k^{(\pm)} = E_k \rho_k \left( \sum_{\ell} \rho_{\ell} E_{\ell}^\dagger E_{\ell} \rho_{\ell} \right)^{-1/2}, \]

where the exponent is given by equation 18. Furthermore, one has the identity

\[ \langle E^{(\pm)}, E \rangle_\mathcal{E} = \text{Tr} \sqrt{\sum_k M_k \rho_k}. \]

**Remark:** It follows by comparison of eq. (73) with eq. (17) that the iteration \( E \mapsto E^{(\pm)} \) for GMs corresponds to Ježek, Řeháček, and Fiurášek’s iteration \( M \mapsto M^{\oplus} \) for POVMs. In particular, \( P_{\text{succ}}(M^{\oplus}) \geq P_{\text{succ}}(M) \), as was observed numerically in [74].

**Proof.** The proof is an easy modification of that of Theorem 9 of [22], which employs the \( E_k = \mathbb{1} \) special case of (73). One has the identity

\[ \text{Re} \langle E, F \rangle_\mathcal{E} = \text{Re} \text{Tr} V_E^\dagger U_F, \]

where \( V_E, U_F : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathbb{C}^K \) are defined by

\[ V_E \psi = \sum_{k \in K} (E_k \rho_k \psi) \otimes |k\rangle_{\mathbb{C}^K}, \]

\[ U_F \psi = \sum_{k \in K} (F_k \psi) \otimes |k\rangle_{\mathbb{C}^K}, \]
where \( |k\rangle_{\mathcal{C}^k} \) is the standard basis of \( \mathcal{C}^k \). Then \( F \) is a generalized measurement iff \( U_F \) is a contraction, with \( \|U_F\| \leq 1 \). But a contraction \( U_F \) maximizing (75) is computed using equation 44

\[
U_F \psi = V_E \left( V_E^\dagger V_E \right)^{-1/2+} \psi = \sum |k\rangle_{\mathcal{C}^k} \otimes E_k \rho_k \left( \sum \rho_t E_t^\dagger E_t \rho_t \right)^{-1/2+} \psi.
\]

Equations (73) and (74) follow. ■

### 3.2 A “small-angle” guess

In order to use Lemma 7 to prove distinguishability bounds, one must construct a guess \( G \) subtending a provably-small angle with an optimal generalized measurement \( E^{\text{opt}} \). As a hint of how to proceed, consider the case that the ensemble \( \mathcal{E} = \mathcal{E}^{\text{PD}} \) is a perfectly-distinguishable ensemble, consisting of states \( \rho_k \) of mutually-orthogonal support. An optimal GM is simply given by

\[
E^{\text{opt}}_k = \Pi_+ (\rho_k),
\]

where the positive projection on the right was defined in (45). Use of spectral theory may be avoided, however, if one notes that the semi-inner product \( \langle E,F \rangle_\mathcal{E} \) of equation 72 is sensitive to the action of the \( E_k \) and \( F_k \) only on the ranges of the corresponding \( \rho_k \). In particular, the simplest-possible “guess”

\[
G_k = 1 \quad \text{for all } k
\]

satisfies \( G \equiv E^{\text{opt}} \mod \langle \cdot, \cdot \rangle_\mathcal{E} \), since

\[
\|E^{\text{opt}} - G\|_\mathcal{E}^\text{POVM} = \text{Tr} \sum (\Pi_+ (\rho_k) - 1)^2 \rho_k = 0.
\]

This equation suggests that the guess (77) will remain appropriate for “reasonably distinguishable” ensembles. Indeed, equation 73 shows that the iterate \( G^{(+)} \) corresponds to the mixed-state generalization of Holevo’s asymptotically-optimal measurement (24). Furthermore, one obtains the following “small angle” estimates:

**Lemma 30** Define \( G \in V_F \) by equation 77, and let \( M \) be a POVM of non-zero success rate. Then one can decompose \( M_k = E_k^\dagger E_k \) in such a way that \( \langle G,E \rangle_\mathcal{E} \in \mathbb{R} \) and

\[
\cos (\theta) := \frac{\langle G,E \rangle_\mathcal{E}}{\|G\|_\mathcal{E} \|E\|_\mathcal{E}} \geq \sqrt{P_{\text{succ}} (M)}.
\]

**Proof.** Chose \( M_k = E_k^\dagger E_k \) arbitrarily. By the polar decomposition, there exist unitary \( U_k : \mathcal{H} \to \mathcal{H} \) so that \( U_k E_k \rho_k \geq 0 \) for all \( k \). Setting

\[
E_k = U_k^\dagger \tilde{E}_k,
\]

it follows from H"{o}lder inequality’s (42) that

\[
\langle G,E \rangle_\mathcal{E} = \text{Tr} \sum E_k \rho_k = \sum \|E_k \rho_k\|_1 \geq \sum \left| \text{Tr} E_k^\dagger E_k \rho_k \right| = P_{\text{succ}} (M).
\]

Using the fact that \( \|G\|_\mathcal{E} = 1 \), the conclusion follows by dividing both sides by \( \|E\|_\mathcal{E} = \sqrt{P_{\text{succ}} (M)} \).

**Remark:** Note that if one rescales \( G \) into generalized measurement, as is only possible when the index set is finite, then one obtains the “random guessing” measurement \( \tilde{G}_k = 1 / \sqrt{|K|} \). The reader may therefore be surprised that the first iterate \( \tilde{G}^{(+)} \) (which is unaffected by rescaling of \( G \)) corresponds to Holevo’s asymptotically-optimal measurement, since this guess isn’t even “smart” enough to take into account the a-priori probabilities of the \( \rho_k \)!

The resolution of this paradox is that the unrescaled guess \( G_k = 1 \) does NOT correspond to random guessing: it guesses every value \( k \). Equation 78 is therefore analogous to the statement that the teacher who grades a multiple-choice test by means of a punched overlay (with holes only for the correct answers) will give top marks to the daring schoolboy who marks ALL of the ovals on his exam.

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3.3 A simple proof of the generalized Holevo-Curlander bounds

We have assembled all the pieces necessary to apply Lemma 7:

**Proof of Theorem 15.** Take \( V = V_E \) and \( S = S_E \) to be as in Definition 28, and let \( G \) be given by (77). By Theorem 29, one has

\[
\left( G(+) \right)^\dagger G(+) = M^{QW}. \tag{81}
\]

By Lemma 30 we may decompose \( M_{k}^{\text{opt}} = (E_{k}^{\text{opt}})^\dagger E_{k}^{\text{opt}} \) in such a way that the “small angle” estimate (79) holds in the equivalent form

\[
\langle E_{k}^{\text{opt}}, G \rangle_E \geq P_{\text{succ}}(M_{k}^{\text{opt}}) \tag{82}
\]
given by (80). Replacing \( x_{\text{max}} \) by \( E_{k}^{\text{opt}} \) in (13) gives

\[
\sqrt{P_{\text{succ}}(M_{k}^{\text{opt}})} \geq \sqrt{P_{\text{succ}}(M^{QW})} \geq \Lambda(G) \geq \langle E_{k}^{\text{opt}}, G \rangle_E, \tag{83}
\]

where we have used the fact that \( \|G\|_E = 1 \). But by equations (14), (74), and (28),

\[
\Lambda(G) = \text{Tr} \sqrt{\sum \rho_k^2} = \Lambda.
\]

The last inequality of (27) follows by appending (82) to (83). The remaining three inequalities of (27) follow by squaring (83). The inequality \( \Lambda \leq 1 \) follows by (27). \( \blacksquare \)

4 Maximum overlap as a maximal-seminorm problem

The maximum overlap problem of equation 1 may be expressed as a maximal-seminorm problem using the identity

\[
\text{Tr}_{L\mathcal{H}} (M_{\mathcal{L}} R_{K \to L} (\mu_{KH})) = \|U\|_{\mu,M}^2, \tag{84}
\]

where \( U \) is a Stinespring dilation of \( R \) and the seminorm is from the following definition:

**Definition 31** Let \( \mathcal{E} = \mathcal{L}_E^* \otimes K_E \) be the the canonical environment (48) of \( \mathcal{R}_{K \to \mathcal{L}} \). For operators \( A, B : \mathcal{H} \to \mathcal{L} \otimes \mathcal{E} \), the \( \mu \text{-}\mathcal{M} \) semi-inner product is defined by

\[
\langle A_{K \to \mathcal{L} \mathcal{E}}, B_{K \to \mathcal{L} \mathcal{E}} \rangle_{\mu,M} = \text{Tr}_{\mathcal{L} \mathcal{E}} (M_{\mathcal{L}} B_{K \to \mathcal{L} \mathcal{E}} \mu_{KH} (A^\dagger)_{\mathcal{L} \mathcal{E}}\to K). \tag{85}
\]

The \( \mu \text{-}\mathcal{M} \) semi-inner product space is the space \( V_{\mu,M} = \{ U : K \to \mathcal{L} \mathcal{E} \mid \|U\|_{\mu,M} < \infty \} \), on which \( \langle \bullet, \bullet \rangle_{\mu,M} \) is well-defined. The purification ball is the set \( S = \{ U : K \to \mathcal{L} \mathcal{E} \mid \|U\| \leq 1 \} \subseteq V_{\mu,M} \), where \( \|\bullet\| \) is the operator-norm.

4.1 Computation of directional iterates

As in the case of measurements, it is not difficult to compute directional iterates:

**Theorem 32** (Directional iteration for maximum overlap is JF H iteration) Let \( V = V_{\mu,M} \) and \( S \) be as in Definition 31. Then the operator \( U_{K \to \mathcal{L} \mathcal{E}} \in V_{\mu,M} \) has the directional iterate

\[
U_{K \to \mathcal{L} \mathcal{E}}^{(+)} = Q (Q^\dagger Q)^{-1/2^+}, \tag{86}
\]

where

\[
Q_{K \to \mathcal{L} \mathcal{E}} = \text{Tr}_{\mathcal{H}} (M_{\mathcal{L}} U_{K \to \mathcal{L} \mathcal{E}} \mu_{KH}). \tag{87}
\]

Furthermore, one has

\[
\langle U, U^{(+)} \rangle_{\mu,M} = \|Q_{K \to \mathcal{L} \mathcal{E}}\|_1. \tag{88}
\]
Remark: Let $U_{K \to LE}$ be a Stinespring dilation of a CP map $R_{K \to L}$. Then the operator $U_{K \to LE}^{(+)}$ of the above Theorem is a dilation of the Ježek-Fiurášek-Hradil iterate $[6, 75]$, already mentioned in section 1.4.2.

Proof. By cyclicity of the trace and equations (43)-(44),

$$\max_{W_{K \to LE} \in S} \text{Re} \langle U, W \rangle_{\mu, M} = \max_{\|W\| \leq 1} \text{Re} \text{Tr}_{\mathcal{LE}} (M_{\mathcal{LE}} W_{K \to LE} \mu_{KH} (U^\dagger)_{\mathcal{LE} \to K})$$

$$= \max_{W} \text{Tr}_{K} ((Q^\dagger)_{\mathcal{LE} \to K} W_{K \to LE})$$

$$= \|Q_{K \to LE}\|_1,$$  \hspace{1cm} (89)

with maximizer $W = U^{(+)}$ given by (86). \hfill \blacksquare

4.2 The restricted maximum-overlap problem

The remainder of this work restricts consideration to the case that

$$M_{\mathcal{LE}} = |\phi\rangle_{\mathcal{LE}} \langle \phi|$$

is a rank 1 projection, seeking to estimate

$$\text{MO} (\mu_{KH}, \phi_{\mathcal{LE}}) := \sup_{\mathcal{R}} \langle \phi_{\mathcal{LE}} | R_{K \to L} (\mu_{KH}) | \phi_{\mathcal{LE}} \rangle,$$  \hspace{1cm} (91)

where the supremum is over quantum operations $\mathcal{R}$ from $K$ to $L$. For convenience, we denote

$$\langle A_{K \to LE}, B_{K \to LE} \rangle_{\mu, \phi} := \langle A, B \rangle_{\mu, |\phi\rangle \langle \phi|} = \langle \phi_{\mathcal{LE}} | \text{Tr}_{\mathcal{LE}} (B_{K \to LE} \mu_{KH} (A^\dagger)_{\mathcal{LE} \to K}) | \phi_{\mathcal{LE}} \rangle$$  \hspace{1cm} (92)

and $V_{\mu, \phi} = V_{\mu, |\phi\rangle \langle \phi|}$.

It is worth mentioning that by Theorems 1 and 2 of [70] (see also equation 182 of the appendix), the minimum-error detection problem is a special case of the restricted maximum overlap problem. The importance of this fact for this work is as follows: One may use the study of quantum measurements as a testing ground to for techniques for the study of the maximum overlap problem and its special cases, including quantum error recovery. Furthermore, as we have already seen, Barnum and Knill [14] considered channel reversibility in the sense of average entanglement fidelity by generalizing the “pretty good” measurement.

4.2.1 A minor simplification

We use the following notation for the partial traces of $|\phi_{\mathcal{LE}}\rangle$:

$$\phi_{\mathcal{L}} = \text{Tr}_{\mathcal{H}} \phi_{\mathcal{LE}} \langle \phi|$$  \hspace{1cm} (93)

$$\phi_{\mathcal{H}} = \text{Tr}_{\mathcal{L}} \phi_{\mathcal{LE}} \langle \phi|$$  \hspace{1cm} (94)

Using the identity

$$|\phi_{\mathcal{LE}}\rangle = \Pi_+ (\phi_{\mathcal{H}}) \phi_{\mathcal{LE}} \rangle,$$  \hspace{1cm} (95)

where the positive projection $\Pi_+ (\phi_{\mathcal{H}})$ is given by equation 45, one has the following

Observation 33 One has the identity

$$\langle \phi_{\mathcal{LE}} | R_{K \to L} (\mu_{KH}) | \phi_{\mathcal{LE}} \rangle = \langle \phi_{\mathcal{L}} | R_{K \to L} (\mu_{KH}) | \phi_{\mathcal{L}} \rangle,$$  \hspace{1cm} (96)

for any $\mathcal{R}$, where $\mu$ is defined by

$$\mu_{KH} = \Pi_+ (\phi_{\mathcal{H}}) \times \mu_{KH} \times \Pi_+ (\phi_{\mathcal{H}}).$$  \hspace{1cm} (97)
4.2.2 A “small angle” guess

The strategy of Sec. 1.4 calls for construction of a guess $G \in V_{\mu,\phi}$ subtending a provably small angle with some dilation $W^{\text{opt}}: K \to L \otimes E$ of an optimal overlap operation $R^{\text{opt}}$. We will be most concerned with the “reasonably overlappable” case, for which one has the crude approximation

$$\sup_{R} \langle \phi_L H | R^{\text{opt}}_{K \to L} (\hat{\mu}_{K H}) | \phi_L H \rangle \approx \text{Tr} \hat{\mu}_{K H}. \quad (98)$$

Our choice of guess will therefore be motivated by the case in which exact equality is obtained:

**Proposition 34 (The perfectly overlappable case)** Let $|\phi_L H \rangle$ be a unit vector and let $R$ be a quantum operation. Then one has perfect overlap

$$\langle \phi_L H | R_{K \to L} (\hat{\mu}_{K H}) | \phi_L H \rangle = \text{Tr} \hat{\mu} \quad (99)$$

if and only if

$$R_{L \to K}^\dagger (|\phi_L H \rangle \langle \phi_L H |)_{\text{Ran} (\hat{\mu}_{K H})} = 1, \quad (100)$$

where the adjoint $R^\dagger$ is given by (47).

**Proof.** Since

$$R^\dagger (|\phi_L H \rangle \langle \phi_L H |) \leq R^\dagger (1) \leq 1,$$

the conclusion follows from (47) and (43)-(44). \qed

In section 3.2 we saw for finite ensembles that a properly-rescaled version of the “daring schoolboy’s” guess $\{ G_k = 1 \}$ could be implemented by “random guessing,” without use of any measurement apparatus. This suggests consideration of a guess $G_{K \to L \ell}$ for which the corresponding (possibly trace-increasing) CP map

$$R^{G} (\rho) = \text{Tr} (G\rho G^\dagger) \quad (101)$$

is independent of $\rho$.

The following Lemma shows that an analogue of equation 78 is satisfied by a guess of this kind:

**Lemma 35 (Construction of a guess)** Let $\phi_L H$ be a unit vector and let $G_{K \to L \ell}$ be a dilation (101) of the (usually trace-increasing) CP map

$$R^{G}_{K \to L} (\rho) := \phi_L^{1+} \times \text{Tr} (\rho), \quad (102)$$

where $E$ is the canonical environment (48). Here we use the notation introduced in equations 18 and 93. Then

1. One has the identity

$$R^{G} (\rho)_{L \to K}^\dagger (|\phi_L H \rangle \langle \phi_L H |) = \Pi_+ (\phi_H) \otimes 1_{K \to K}. \quad (103)$$

2. If $\mu$ and $\phi$ are “perfectly overlappable” by a quantum operation $R = R^{\text{opt}}$, as in equation 99, then $R^{\text{opt}}$ has a dilation $W^{\text{opt}}_{K \to L \otimes E}$ such that

$$\| G - W^{\text{opt}} \|_{\mu,\phi} = 0. \quad (104)$$

**Remark:** Note that the choice of dilation $G$ does not affect the operation

$$R^{(+)} (\rho) := \text{Tr} E G^{(+)} \rho (G^{(+)})^\dagger,$$
since the replacement $G \rightarrow U_{\mathcal{E} \rightarrow \mathcal{E}} G_{K \rightarrow \mathcal{E}}$, where $U_{\mathcal{E} \rightarrow \mathcal{E}}$ is unitary, simply induces the replacement $G^{(+)} \rightarrow U_{\mathcal{E} \rightarrow \mathcal{E}} G^{(+)}$.

**Proof.** Equation 103 is trivial. To prove the equation 104, note that equations 47 and 103 imply

$$G_{\mathcal{E} \rightarrow K}^\dagger |\phi_{\mathcal{E}H} \rangle \langle \phi_{\mathcal{E}H}| G_{K \rightarrow \mathcal{E}} = (R_{\mathcal{E}}^G)^\dagger_{L \rightarrow K} (|\phi_{\mathcal{E}H} \rangle \langle \phi_{\mathcal{E}H}|) = \Pi_+ (\phi_{\mathcal{E}H}) \otimes \mathbb{1}_{K \rightarrow K}. \quad (105)$$

In particular, $\langle \phi_{\mathcal{E}H}| G_{K \rightarrow \mathcal{E} \otimes \mathcal{E}}$ restricts to an isometry from $\text{ran} (\hat{\phi}_{\mathcal{E}H}) \otimes K \supseteq \text{ran} (\hat{\mu}_{KH})$ into $\mathcal{E}$. Let $W'_{K \rightarrow \mathcal{E}}$ be a dilation of $R^\text{opt}$. By Proposition 99 it similarly follows that $\langle \phi_{\mathcal{E}H}| W'_{K \rightarrow \mathcal{E}}$ is also an isometry on $\text{ran} (\hat{\mu}_{KH})$, implying that there exists a unitary $X_{\mathcal{E} \rightarrow \mathcal{E}}$ such that

$$X_{\mathcal{E} \rightarrow \mathcal{E}} \langle \phi_{\mathcal{E}H}| W'_{K \rightarrow \mathcal{E} \otimes \mathcal{E}} = \langle \phi_{\mathcal{E}H}| G_{K \rightarrow \mathcal{E} \otimes \mathcal{E}}$$

on $\text{ran} (\hat{\mu}_{KH})$. Equation 104 follows from (92) by setting $W^\text{opt}_{K \rightarrow \mathcal{E} \otimes \mathcal{E}} = X_{\mathcal{E} \rightarrow \mathcal{E}} W'_{K \rightarrow \mathcal{E} \otimes \mathcal{E}}$. ■

### 4.2.3 Angle Estimates

The following estimate shows that $G_{K \rightarrow \mathcal{E}}$ remains a reasonably-good guess when $\phi$ and $\mu$ are only reasonably-overlappable, c.f. inequality 79:

**Lemma 36 (Angle estimates)** Take $\phi_{\mathcal{E}H}$ to be a unit vector, take the CP map $R^G_{\mathcal{E}}$ and the guess $G_{K \rightarrow \mathcal{E} \otimes \mathcal{E}}$ to be as in Lemma 35, and let $R_{K \rightarrow \mathcal{E}}$ be any quantum operation for which

$$\langle \phi_{\mathcal{E}H}| R_{K \rightarrow \mathcal{E}} (\mu_{KH}) |\phi_{\mathcal{E}H} \rangle > 0.$$  

Then $R$ has a Stinespring dilation $W_{K \rightarrow \mathcal{E}}$ such that $(W, G)_{\mu, \phi} \in \mathbb{R}$ and

$$\cos (\theta) := \frac{\langle W, G \rangle_{\mu, \phi}}{\|W\|_{\mu, \phi} \|G\|_{\mu, \phi}} \geq \frac{\langle \phi_{\mathcal{E}H}| R_{K \rightarrow \mathcal{E}} (\hat{\mu}_{KH}) |\phi_{\mathcal{E}H} \rangle}{\|R^G_{L \rightarrow K} (|\phi_{\mathcal{E}H} \rangle \langle \phi_{\mathcal{E}H}|)\|_{\infty} \text{Tr} (\hat{\mu}_{KH})}. \quad (106)$$

Here $\hat{\mu}_{KH}$ is given by (97) and the adjoint $R^\dagger$ is from equation 47. Furthermore, one has the identities

$$\|G\|_{\mu, \phi}^2 = \text{Tr} (\hat{\mu}_{KH}) \quad (107)$$

$$\langle G^{(+)} , G \rangle_{\mu, \phi} = \text{Tr} \left( (R^G_{L \rightarrow K})^\dagger (|\phi_{\mathcal{E}H} \rangle \langle \phi_{\mathcal{E}H}|) \hat{\mu}_{KH} \right) \quad (108)$$

where $G^{(+)}$ is the iterate of $G$ given by Theorem 32.

**Remark:** Note that $\cos (\theta) = 1$ if perfect overlap $\langle \phi| R_{K \rightarrow \mathcal{E}} (\hat{\mu}_{KH}) |\phi \rangle_{\mathcal{E}H} = \text{Tr} \hat{\mu}_{KH}$ is achieved.

**Proof.** Equation 107 follows from part 1 of Lemma 35:

$$\|G\|_{\mu, \phi}^2 = \langle \phi_{\mathcal{E}H}| R^G_{K \rightarrow \mathcal{E}} (\hat{\mu}_{KH}) |\phi_{\mathcal{E}H} \rangle = \text{Tr} \left( (R^G_{L \rightarrow K})^\dagger (|\phi_{\mathcal{E}H} \rangle \langle \phi_{\mathcal{E}H}|) \hat{\mu}_{KH} \right) = \text{Tr} \hat{\mu}_{KH}$$

To prove the angle estimate (106), note that one has the identity

$$\langle W_{K \rightarrow \mathcal{E} \otimes \mathcal{E}}, G \rangle_{\mu, \phi} = \text{Tr} P_{\mathcal{E} \rightarrow \mathcal{E}}, \quad (109)$$

where

$$P_{\mathcal{E} \rightarrow \mathcal{E}} = \langle \phi_{\mathcal{E}H}| G_{K \rightarrow \mathcal{E} \otimes \mathcal{E} \mu_{KH}} (W^\dagger)_{L \rightarrow K} |\phi_{\mathcal{E}H} \rangle. \quad (110)$$

Starting with any dilation $W_{K \rightarrow \mathcal{E}}$ of $R$, we may assure that $P_{\mathcal{E} \rightarrow \mathcal{E}}$ is positive semidefinite by a replacement

$$W_{K \rightarrow \mathcal{E}} \rightarrow X_{\mathcal{E} \rightarrow \mathcal{E}} W_{K \rightarrow \mathcal{E}},$$
where the unitary operator $X_{\mathcal{E} \to \mathcal{E}}$ comes from the polar decomposition of $P_{\mathcal{E} \to \mathcal{E}}$. It follows that the LHS of (106) is real and maximized over the choice of dilation of $\mathcal{R}$. We claim that there exists an operator $Z : \mathcal{E} \to \mathcal{E}$ such that
\[
Z_{\mathcal{E} \to \mathcal{E}} P_{\mathcal{E} \to \mathcal{E}} = \langle \phi_{\mathcal{E} \mathcal{H}} | W_{K \to \mathcal{E} \mathcal{E}} \mu_{\mathcal{K} \mathcal{H}} (W^\dagger)_{\mathcal{L} \mathcal{L} \to \mathcal{K}} | \phi_{\mathcal{E} \mathcal{H}} \rangle
\] (111)
\[
\|Z_{\mathcal{E} \to \mathcal{E}}\|_\infty = \left\| R^\dagger_{\mathcal{L} \mathcal{L} \to \mathcal{K}} (\langle \phi_{\mathcal{E} \mathcal{H}} | \phi_{\mathcal{E} \mathcal{H}} \rangle) \right\|_\infty^{1/2}
\] (112)
Assuming this claim, Hölder’s inequality (42) implies that
\[
\langle W_{K \to \mathcal{L} \mathcal{E}}, G \rangle_{\mu, \phi} = \frac{1}{\|Z\|_\infty} \left| \frac{Z_{\mathcal{E} \to \mathcal{E}} P_{\mathcal{E} \to \mathcal{E}}}{Z_{\mathcal{E} \to \mathcal{E}}} \right| \geq \frac{\langle \phi_{\mathcal{E} \mathcal{H}} | R_{\mathcal{K} \to \mathcal{L}} (\mu_{\mathcal{K} \mathcal{H}}) | \phi_{\mathcal{E} \mathcal{H}} \rangle}{\left\| R^\dagger_{\mathcal{L} \mathcal{L} \to \mathcal{K}} (\langle \phi_{\mathcal{E} \mathcal{H}} | \phi_{\mathcal{E} \mathcal{H}} \rangle) \right\|_\infty^{1/2}}.
\] (113)
The angle estimate (106) follows by dividing both sides by
\[
\|W\|_{\mu, \phi} = \sqrt{\langle \phi_{\mathcal{E} \mathcal{H}} | R_{\mathcal{K} \to \mathcal{L}} (\mu_{\mathcal{K} \mathcal{H}}) | \phi_{\mathcal{E} \mathcal{H}} \rangle}
\]
and by the square root of equation 107.
To prove the claims (111) − (112), define
\[
Z_{\mathcal{E} \to \mathcal{E}} = \langle \phi_{\mathcal{E} \mathcal{H}} | W_{K \to \mathcal{L} \mathcal{E}} (G^\dagger)_{\mathcal{L} \mathcal{L} \to \mathcal{K}} | \phi_{\mathcal{E} \mathcal{H}} \rangle,
\] (114)
where $\mathcal{L}'$ is a copy of $\mathcal{L}$. Then
\[
\|Z_{\mathcal{E} \to \mathcal{E}}\|_\infty = \left\| (Z^\dagger Z)_{\mathcal{E} \to \mathcal{E}} \right\|_\infty^{1/2} = \left\| \langle \phi_{\mathcal{E} \mathcal{H}} | G_{K \to \mathcal{L} \mathcal{E}} W_{K \to \mathcal{L} \mathcal{E}} (G^\dagger)_{\mathcal{L} \mathcal{L} \to \mathcal{K}} | \phi_{\mathcal{E} \mathcal{H}} \rangle \right\|_\infty^{1/2}
\] = \left\| \langle \phi_{\mathcal{E} \mathcal{H}} | G_{K \to \mathcal{L} \mathcal{E}} R^\dagger_{\mathcal{L} \mathcal{L} \to \mathcal{K}} \langle \phi_{\mathcal{E} \mathcal{H}} | \phi_{\mathcal{E} \mathcal{H}} \rangle (G^\dagger)_{\mathcal{L} \mathcal{L} \to \mathcal{K}} | \phi_{\mathcal{E} \mathcal{H}} \rangle \right\|_\infty^{1/2}.
\] But as in the proof of Lemma 35, the operator $\langle \phi_{\mathcal{E} \mathcal{H}} | G_{K \to \mathcal{L} \mathcal{E}}$ is an isometry from $\mathcal{K} \otimes \text{ran} (\phi_{\mathcal{H}})$ into $\mathcal{E}$, proving (112).
To prove (111), note that equations 114, 105, & 95 imply that
\[
Z_{\mathcal{E} \to \mathcal{E}} \langle \phi_{\mathcal{E} \mathcal{H}} | G_{K \to \mathcal{L} \mathcal{E}} = \langle \phi_{\mathcal{E} \mathcal{H}} | W_{K \to \mathcal{L} \mathcal{E}} \Pi_+ (\phi_{\mathcal{H}}) = \langle \phi_{\mathcal{E} \mathcal{H}} | W_{K \to \mathcal{L} \mathcal{E}}.
\] (115)
Equation 111 follows, since by (110)
\[
Z_{\mathcal{E} \to \mathcal{E}} P_{\mathcal{E} \to \mathcal{E}} = Z_{\mathcal{E} \to \mathcal{E}} \langle \phi_{\mathcal{E} \mathcal{H}} | G_{K \to \mathcal{L} \mathcal{E}} \mu_{\mathcal{K} \mathcal{H}} (W^\dagger)_{\mathcal{L} \mathcal{L} \to \mathcal{K}} | \phi_{\mathcal{E} \mathcal{H}} \rangle
\] = \langle \phi_{\mathcal{E} \mathcal{H}} | W_{K \to \mathcal{L} \mathcal{E}} \mu_{\mathcal{K} \mathcal{H}} (W^\dagger)_{\mathcal{L} \mathcal{L} \to \mathcal{K}} | \phi_{\mathcal{E} \mathcal{H}} \rangle.
\] It remains to prove equation 108. By equation 88,
\[
\langle G^{(+)} | G \rangle_{\mu, \phi} = \|Q_{\mathcal{K} \to \mathcal{L} \mathcal{E}}\|_1,
\] (116)
where by equations 87 and 90,
\[
Q_{\mathcal{K} \to \mathcal{L} \mathcal{E}} = \text{Tr} (\phi_{\mathcal{E} \mathcal{H}} \langle \phi_{\mathcal{E} \mathcal{H}} | G_{K \to \mathcal{L} \mathcal{E}} \mu_{\mathcal{K} \mathcal{H}})
\] = \langle \phi_{\mathcal{E} \mathcal{H}} | G_{K \to \mathcal{L} \mathcal{E}} \mu_{\mathcal{K} \mathcal{H}} | \phi_{\mathcal{E} \mathcal{H}} \rangle,
\] (117)
where $\mathcal{L}'$ is a copy of $\mathcal{L}$. It follows by equations 105 and 95 that
\[
Q^\dagger Q = \langle \phi_{\mathcal{E} \mathcal{H}} | \mu_{\mathcal{K} \mathcal{H}} \chi_+ (\phi_{\mathcal{H}}) \mu_{\mathcal{K} \mathcal{H}} | \phi_{\mathcal{E} \mathcal{H}} \rangle
\] = \langle \phi_{\mathcal{E} \mathcal{H}} | \tilde{\mu}_{\mathcal{K} \mathcal{H}} | \phi_{\mathcal{E} \mathcal{H}} \rangle.
\] (118)
The conclusion follows. ■
4.2.4 The “Quadratic Overlapper”

The following operation is analogous to the quadratically-weighted measurement:

**Definition 37** The “quadratic overlapper” is the operation $R_{K→L}^{QO}: B^1(K) → B^1(L)$ defined by

$$R_{K→L}^{QO}(v_K) = \text{Tr} G_{K→L}^{(+)} v_K (G_{K→L}^{(+)})^\dagger,$$  (119)

where $G_{K→L}^{(+)} ∈ V_{μ,φ}$ is the “small-angle guess” of Lemma 35, with directional iterate $G_{K→L}^{(+)}$ given by equations 86-87 of Theorem 32.

Alternatively, one may express $R_{K→L}^{QO}$ using

**Theorem 38 (Computation of the quadratic overlapper)** One has the formula

$$R_{K→L}^{QO}(v_K) = \text{Tr}_{K→H} \left( \hat{μ}_{K→H}^2 \left( \left( Y_{-1/2} vY_{-1/2}^* \right)_{K→K} \otimes |φ⟩_L ⟨φ| \right) \right),$$  (120)

where

$$Y_{K→K} = ⟨φ_L| \hat{μ}_{K→K}^2 |φ_L⟩,$$  (121)

and $\hat{μ}_{K→K}$ is given by (97).

**Proof.** By equation 86, the guess $G$ has the iterate

$$G_{K→L}^{(+)} = Q \left( Q^\dagger Q \right)^{-1/2+},$$

where the operator $Q$ is defined by (87). By equations (117)-(118), one has

$$Q_{K→L} = ⟨φ|_{L→L} G_{K→L} |μ_{K→L}| \hat{φ}_{L→L}⟩,$$

$$Q^\dagger Q = ⟨φ|_{L→L} \hat{μ}_{K→K}^2 |μ_{K→L}| \hat{φ}_{L→L}⟩ = Y_{K→K}.$$

Setting

$$\hat{v}_{K→K} = Y_{-1/2} vY_{-1/2}^*,$$  (122)

it follows that

$$\text{Tr}_{K→L} G_{K→L}^{(+)} v_K \left( G_{K→L}^{(+)} \right)^\dagger = \text{Tr}_{K→L} \left( (φ|_{L→L} G_{K→L} |μ_{K→L}| \hat{φ}_{L→L}⟩) \hat{v}_{K→K} ⟨φ|_{L→L} G_{K→L} |μ_{K→L}| \hat{φ}_{L→L}⟩ \right),$$

$$= \text{Tr}_{K→L} \left( (μ_{K→L} |φ⟩_{L→L}) \hat{v}_{K→K} ⟨φ|_{L→L} (G^\dagger)_{K→L} |φ⟩_{L→L} G_{K→L} |φ⟩_{L→L} \right)$$

$$= \text{Tr}_{K→L} \left( μ_{K→L} \hat{v}_{K→K} ⟨φ|_{L→L} G_{K→L} |φ⟩_{L→L} \right)$$

$$= \text{Tr}_{K→L} \left( μ_{K→L} \hat{v}_{K→K} ⟨φ|_{L→L} G_{K→L} |φ⟩_{L→L} \right)$$

Here equation 123 uses cyclicity of the trace, equation 124 uses (105), and equation 125 uses (95), (97), and cyclicity of the trace.

4.2.5 Estimates for the restricted maximum overlap problem

On now may apply our angle estimates in a manner similar to that of section 3.3:

**Theorem 39 (Two-sided estimates for the maximum overlap problem)** Let $μ_{K→L}$ be positive semidefinite on $K ⊗ L$ and let $|φ_L⟩$ be a unit vector. Then

$$\frac{A^2}{\text{Tr} μ_{K→L}} ≤ ⟨φ|_{L→L} R_{K→L}^{QO} (μ_{K→L}) |φ⟩_{L→L} ≤ \max_{R_{K→L}} ⟨φ|_{L→L} R_{K→L} (μ_{K→L}) |φ⟩_{L→L} \leq \Lambda × \left\| (R_{K→L}^{opt})^\dagger \right\|_∞ \leq \Lambda,$$  (126)

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where the maximum is over quantum operations $\mathcal{R} : B^1(\mathcal{K}) \to B^1(\mathcal{L})$, where $\mathcal{R}^{\text{opt}}$ attains this maximum, and where
\[
\Lambda = \text{Tr}_{\mathcal{K}} \sqrt{\langle \phi_{\mathcal{LH}} | \hat{\mu}_{\mathcal{KH}}^2 | \phi_{\mathcal{LH}} \rangle}.
\] (127)

Here $\hat{\mu}_{\mathcal{KH}}$ is given by (97), $\mathcal{R}^\dagger$ is given by (47), and one interprets $0^2/0 = 0$.

**Remark:** It follows from (126) that
\[
\Lambda \leq \text{Tr}(\hat{\mu}_{\mathcal{KH}}) \times \left\| (\mathcal{R}^{\text{opt}})^\dagger_{\mathcal{L} \to \mathcal{K}} (| \phi_{\mathcal{LH}} \rangle \langle \phi_{\mathcal{LH}} |) \right\|^{1/2} \leq \text{Tr} \hat{\mu}_{\mathcal{KH}}.
\]

**Note:** Given an arbitrary invertible operator $X : \mathcal{H} \to \mathcal{H}$, one may obtain potentially sharper estimates from inequality (126) using a replacement of the form
\[
\hat{\phi}_{\mathcal{LH}} \to \frac{X_{\mathcal{H} \to \mathcal{H}}^\dagger \hat{\phi}_{\mathcal{LH}}}{\|X_{\mathcal{H} \to \mathcal{H}}^\dagger \hat{\phi}_{\mathcal{LH}}\|},
\] (128)

\[
\hat{\mu}_{\mathcal{KH}} \to \|X_{\mathcal{H} \to \mathcal{H}}^\dagger \hat{\phi}_{\mathcal{LH}}\|^2 (X^{-1})^\dagger \hat{\mu}_{\mathcal{KH}} X^{-1},
\] (129)

which does not change the overlap $\langle \phi_{\mathcal{LH}} | \mathcal{R}_{\mathcal{K} \to \mathcal{L}} (\hat{\mu}_{\mathcal{KH}}) | \phi_{\mathcal{LH}} \rangle$.

**Proof.** Since all quantities in (126) scale linearly in $\mu$, set $\text{Tr} \hat{\mu}_{\mathcal{KH}} = 1$. (The case $\hat{\mu} = 0$ is trivial.)

Let $\mathcal{R}^{\text{opt}}$ attain the maximum in (126). Take $G_{\mathcal{K} \to \mathcal{L}}$ to be the “small angle” guess of Lemma 35, with iterate $G_{\mathcal{K} \to \mathcal{L}}^{(+)}$ given by Theorem 32. By Lemma 36, there exists a Stinespring dilation $W_{\mathcal{K} \to \mathcal{L}}^{\text{opt}}$ of $\mathcal{R}^{\text{opt}}$ such that the angle estimate (106) holds in the equivalent form
\[
\langle W^{\text{opt}}, G \rangle_{\mu,\phi,} \geq \frac{\langle \phi_{\mathcal{LH}} | \mathcal{R}_{\mathcal{K} \to \mathcal{L}}^{\text{opt}} (\hat{\mu}_{\mathcal{KH}}) | \phi_{\mathcal{LH}} \rangle}{\| (\mathcal{R}^{\text{opt}})^\dagger_{\mathcal{L} \to \mathcal{K}} (| \phi_{\mathcal{LH}} \rangle \langle \phi_{\mathcal{LH}} |) \|^{1/2}}
\] (130)

given by (113). But by lemma 7 and (107),
\[
\| W^{\text{opt}} \|_{\mu,\phi,} \geq \| G^{(+)} \|_{\mu,\phi,} \geq \Lambda (G) \geq \langle W^{\text{opt}}, G \rangle_{\mu,\phi,},
\] (131)

where by (14) and (107) – (108),
\[
\Lambda (G) = \text{Re} \left( G^{(+)}, G/ \| G \|_{\mu,\phi,} \right)_{\mu,\phi,} = \Lambda.
\] (132)

The third inequality of (126) follows by appending (130) to (131). Squaring the first three quantities of (131) proves the first two inequalities of (126). The final inequality follows from the fact that $\mathcal{R}^{\text{opt}}$ is a quantum operation. ■

### 4.3 Estimates for quantum conditional min-entropy

Theorem 39 has the following corollary:

**Corollary 40** Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be finite-dimensional, and let $\rho_{AB}$ be a density matrix on $\mathcal{H}_A \otimes \mathcal{H}_B$. Then for any $s \in \mathbb{R}$ the conditional min-entropy (10) of $A$ given $B$ satisfies the bounds
\[
- \log_2 \left( \sqrt{\text{Tr} \rho_A^s} \times \text{Tr} \sqrt{\text{Tr} \rho_{AB} \rho_A^{s^*} \rho_{AB}} \right) \leq H_{\text{min}} (A|B)_\rho \leq - \log_2 \left( \frac{(\text{Tr} B \sqrt{\text{Tr} A \rho_{AB} \rho_A^{s^*} \rho_{AB}})^2}{\text{Tr} \rho_A^{1-s}} \right)
\] (133)

where $\rho_A = \text{Tr}_B \rho_{AB}$. Here any non-positive powers are evaluated as in equation 18. (In particular
\[
\rho_A^0 := \Pi_+ (\rho_A),
\] (134)

where the RHS is the positive projection (45).)

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Remarks:

1. The $s = 0$ case of (133) is particularly simple:
\[
- \log \left( \frac{\text{Tr}_B \sqrt{\text{Tr}_A \rho_{AB}^2}}{\text{Tr}_A} \right) - \frac{1}{2} \log (\text{rank} (\rho_A)) \leq H_{\min} (A|B)_{\rho} \leq -2 \log_2 \left( \frac{\text{Tr}_B \sqrt{\text{Tr}_A \rho_{AB}^2}}{\text{Tr}_A} \right). 
\] (135)

2. If $s = 1/2$ and $\rho_{AB} = |\psi_{AB}\rangle \langle \psi_{AB}|$ is pure then a simple calculation\footnote{113} shows that the upper and lower bounds of (133) agree, yielding the known \footnote{70} expression
\[
H_{\min} (A|B)_{|\psi_{AB}\rangle} = -2 \log_2 \left( \frac{\text{Tr}_B \sqrt{\text{Tr}_A \rho_{A}^2}}{\text{Tr}_A} \right). 
\] (136)

3. More generally, if $\rho_{AB} = |\psi_{AB}\rangle \langle \psi_{AB}|$ is pure then the upper bound of (133) is exact and independent of $s$.

4. If $\rho_{AB}$ is a maximally-entangled pure state then the lower bound of (133) is also exact and independent of $s$.

Proof. Let
\[
\rho_A = \sum \lambda_k |k\rangle_A \langle k|
\]
be a spectral decomposition. Then by equation 12 one has the identity
\[
2^{-H_{\min} (A|B)_s} = \langle \psi_s |_{AA^*} \max_{R} (R_{B \rightarrow A^*} (\mu_s)_{AB}) |\psi_s\rangle_{AA^*},
\] (137)
where
\[
|\psi_s\rangle_{AA^*} = \left| \frac{\sigma^{s/2}}{\rho_A^{s/2}} \right|_{AA^*} = \sum_{k \text{ with } \lambda_k > 0} \frac{\lambda_k^{s/2}}{\sqrt{\text{Tr}_A \rho_{A}^{s/2}}} |k\rangle_A \langle \bar{k}|_{A^*}, 
\] (138)
\[
(\mu_s)_{AB} = \left| \frac{\sigma^{s/2}}{\rho_A^{s/2}} \rho_{AB} \rho_{A}^{-s/2}. 
\] (139)
The bounds (133) follow by Theorem 39. \[\blacksquare\]

Remark: Using the fourth term of inequality 126, one may tighten the lower bound of (133) in cases where one can estimate
\[
\left\| (R_{opt}^{opt})_{B \rightarrow A^*} (|\psi_s\rangle \langle \psi_s|) \right\|_\infty,
\]
where $R_{opt}$ is a maximizer of (137). Appendix C shows that this works in the case that $\rho_{AB}$ is a “quantum-classical” state.

5 Approximate Channel Reversals

This section applies Theorem 39 to estimate the reversibility of an arbitrary quantum operation $A : B^1 (H) \rightarrow B^1 (K)$, as measured by entanglement fidelity $\max_{R_{K \rightarrow H}} F_{F_e} (\rho, R \circ A)$. (Note that more generally Theorem 39 gives estimates when the input state of $A$ and target output state of $R$ differ, but we focus on this special case.)

In order to express the our reversibility estimates in a more intuitive form (and to understand the relationship of the corresponding reversal with that of Barnum and Knill), it is useful to introduce a method for applying functions to CP maps.
5.1 The $\rho$-functional calculus for CP maps

One may tailor the Kraus decomposition [89] a CP map to a given input density matrix $\rho$:

**Definition 41** Let $\mathcal{A} : B^1(\mathcal{H}) \rightarrow B^1(\mathcal{K})$ be a completely-positive map and let $\rho$ be a density matrix on $\mathcal{H}$. A $\rho$-Kraus decomposition of the restriction of $\mathcal{A}$ to $B^1(\text{supp} (\rho))$ is a decomposition of the form

$$\mathcal{A}(\mu) = \sum p_k E_k^\dagger \mu E_k, \quad \text{supp} (\mu) \subseteq \text{supp} (\rho)$$

where

1. The vectors $E_k |\psi_\rho\rangle \in \mathcal{K} \otimes \mathcal{H}^*$ are orthonormal, where $|\psi_\rho\rangle$ is the purification (66).
2. The $p_k$ are non-negative.
3. The $\rho$-Kraus operators $E_k : \mathcal{H} \rightarrow \mathcal{K}$ have supports in $\text{supp} (\rho)$.

**Remarks:**

1. Existence of a $\rho$-Kraus decomposition follows by an easy modification of standard techniques. (See Proposition 43, below.)
2. When $\mathcal{A}$ is trace-preserving, one interprets $\mathcal{A}$ as acting on $\rho$ by randomly sending the purification $|\psi_\rho\rangle_{\mathcal{H}\mathcal{H}^*}$ into one of the orthonormal vectors $E_k |\psi_\rho\rangle$, which are classically-distinguishable by the observer with access to $\mathcal{K}\mathcal{H}^*$.
3. By equations 66 and 61, the orthonormality of the $E_k |\psi_\rho\rangle_{\mathcal{H}\mathcal{H}^*}$ is equivalent to the condition

$$\text{Tr} \left( E_k^\dagger E_\ell \rho \right) = \delta_{k\ell}. \quad (141)$$

If $\rho$ is maximally-mixed one therefore obtains the usual orthogonality conditions [90] sometimes required for the Kraus operators.

Given a state $\rho$, there is a natural notion of applying functions to completely positive maps:

**Definition 42 ($\rho$-functional calculus for CP maps)** Let $\mathcal{A} : B^1(\mathcal{H}) \rightarrow B^1(\mathcal{K})$ be a completely positive map with the $\rho$-Kraus decomposition (140). For $f : [0, \infty) \rightarrow [0, \infty)$ one defines the CP map $f_\rho(\mathcal{A}) : B^1(\text{supp} (\rho)) \rightarrow B^1(\mathcal{K})$ by

$$(f_\rho (\mathcal{A})) (\mu) = \sum f (p_k) E_k^\dagger \mu E_k. \quad (142)$$

The quadratic reweighting $\mathcal{A}^{(2,\rho)}$ of $\mathcal{A}$ corresponds to the case $f (p) = p^2$:

$$\mathcal{A}^{(2,\rho)} (\mu) = \sum p_k^2 E_k^\dagger \mu E_k. \quad (143)$$

The following proposition shows that the CP maps $f_\rho (\mathcal{A})$ and $\mathcal{A}^{(2,\rho)}$ are well-defined and independent of the decomposition (140):

**Proposition 43** Let $f$, $\mathcal{A}$, and $\rho$ be as in Definition 42. Then

1. A $\rho$-Kraus decomposition (140) of $\mathcal{A}$ exists. Furthermore,

$$\sum p_k = \text{Tr} \mathcal{A} (\rho),$$

so that $\{ p_k \}$ is a probability distribution if $\mathcal{A}$ is trace preserving.
2. One has the identity

\[
(f_\rho(A))(\mu) = \text{Tr}_{\mathcal{H}^*} \left( \left( \rho^{-1/2+} \mu^1 \rho^{-1/2+} \right) \mathcal{H}^* \to \mathcal{H}^* \right) \times f \left( A \left( \left| \psi_\rho \right\rangle_{\mathcal{H} \mathcal{H}^*} \langle \psi_\rho \right| \right) \right),
\]

for all \( \mu \in \mathcal{B}^1(\text{supp}(\rho)) \), where barred product of operator acts on \( \mathcal{H}^* \) as indicated by equations (56-60). Here one applies \( f \) to a self-adjoint operator using the functional calculus \([86]\): Given a spectral decomposition

\[
A = \sum \lambda_i \Pi_i,
\]

one writes

\[
f(A) = \sum f(\lambda_i) \Pi_i.
\]

3. In particular, the CP map \( f_\rho(A) \) is independent of the choice of decomposition (140), and

\[
A^{(2,\rho)}(\mu) = \text{Tr}_{\mathcal{H}^*} \left( \left( \rho^{-1/2+} \mu^1 \rho^{-1/2+} \right) \mathcal{H}^* \to \mathcal{H}^* \right) \times \left( A \left( \left| \psi_\rho \right\rangle_{\mathcal{H} \mathcal{H}^*} \langle \psi_\rho \right| \right)^2 \right),
\]

for \( \mu \in \mathcal{B}^1(\text{supp}(\rho)) \).

Remarks:

1. If \( f(p) = p \) is the identity function and if \( \rho \) is maximally mixed then equation (144) reduces to the usual procedure for recovering a channel from its Choi matrix (67).

2. Equation (147) gives the form of \( A^{(2,\rho)} \) which appears when one applies Theorem 39 to obtain reversibility estimates. The transpose of \( \mu \) becomes a partial transpose (60) when \( A^{(2,\rho)} \) is applied to the state of a composite quantum system.

3. A prescription for computing \( A^{(2,\rho)} \) from an arbitrary set of Kraus operators of \( A \) appears in the next section.

Proof. By equations (62, 66, 18, and 45),

\[
\mu^{1/2} \rho^{-1/2} \left| \psi_\rho \right\rangle_{\mathcal{H} \mathcal{H}^*} = \left| \rho^{1/2} \rho^{-1/2+} \mu^{1/2} \right\rangle_{\mathcal{H} \mathcal{H}^*} = \left| \Pi_+ (\rho) ^{1/2} \right\rangle_{\mathcal{H} \mathcal{H}^*} = \left| \psi_\rho \right\rangle_{\mathcal{H} \mathcal{H}^*}.
\]

It therefore follows by equation (68) that for densities \( \mu \in \mathcal{B}^1(\mathcal{H}) \),

\[
A(\mu) = \text{Tr}_{\mathcal{H}^*} A \left( \left| \psi_\rho \right\rangle_{\mathcal{H} \mathcal{H}^*} \langle \psi_\rho \right| \right) = \text{Tr}_{\mathcal{H}^*} \mu^{1/2} \rho^{-1/2+} A \left( \left| \psi_\rho \right\rangle_{\mathcal{H} \mathcal{H}^*} \langle \psi_\rho \right| \right) \rho^{-1/2+} \mu^{1/2}.
\]

Taking a spectral decomposition

\[
A \left( \left| \psi_\rho \right\rangle \langle \psi_\rho \right| \right) = \sum p_k \left| F_k \right\rangle_{\mathcal{K} \mathcal{H}^*} \langle F_k |,
\]

it follows from equations (149, 62, and 64) that

\[
A(\mu) = \text{Tr}_{\mathcal{H}^*} \sum p_k \left| F_k \rho^{-1/2+} \mu^{1/2} \right\rangle_{\mathcal{K} \mathcal{H}^*} \langle \left| F_k \rho^{-1/2+} \mu^{1/2} \right| \right| = \sum p_k F_k \rho^{-1/2+} \mu^{1/2} F_k^\dagger.
\]

Setting

\[
E_k = F_k \rho_k^{1/2+}
\]

gives the desired \( \rho \)-Kraus decomposition, where the desired condition (141) follows from the orthonormality of the \( \left| F_k \right\rangle \in \mathcal{K} \otimes \mathcal{H}^* \) using equation (61). If \( A \) is trace-preserving then

\[
1 = \text{Tr}_{\mathcal{K} \mathcal{H}^*} A \left( \left| \psi_\rho \right\rangle_{\mathcal{H} \mathcal{H}^*} \langle \psi_\rho \right| \right) = \text{Tr}_{\mathcal{K} \mathcal{H}^*} \sum p_k E_k \left| \psi_\rho \right\rangle_{\mathcal{H} \mathcal{H}^*} \langle \psi_\rho \right| E_k^\dagger = \sum p_k,
\]

by the defining orthonormality condition on the \( \{ E_k \} \), proving that \( \{ p_k \} \) is a probability distribution.
Now suppose that we are given an arbitrary $\rho$-Kraus decomposition (140) and that $\mu \in B^1 (\text{supp} (\rho))$. Note that since both sides of (144) are linear in $\mu$ we may assume without loss of generality that $\mu$ is positive semidefinite. Then by equations 68, 148, 146, 140, and cyclicity of the trace

$$\sum f (p_k) E_k \mu E_k^\dagger = \text{Tr} \sum f (p_k) E_k |\psi_\mu\rangle_{H H^*} \langle \psi_\mu| E_k^\dagger$$

$$= \text{Tr} \frac{\mu^{1/2} \rho^{-1/2}}{\mu^{1/2}} \left( \sum f (p_k) E_k |\psi_\mu\rangle_{H H^*} \langle \psi_\mu| E_k^\dagger \right) \rho^{-1/2} \mu^{1/2}$$

$$= \text{Tr} \frac{\mu^{1/2} \rho^{-1/2}}{\mu^{1/2}} f \left( \sum p_k E_k |\psi_\mu\rangle_{H H^*} \langle \psi_\mu| E_k^\dagger \right) \rho^{-1/2} \mu^{1/2}$$

$$= \text{Tr} \left( \rho^{-1/2} \mu^{1/2} \rho^{-1/2} \times f \left( A \left( |\psi_\mu\rangle_{H H^*} \langle \psi_\mu| \right) \right) \right),$$

as desired. ■

5.2 Quadratic quantum error recovery

**Theorem 44** Let $A : B^1 (H) \to B^1 (K)$ be a quantum operation, and let $\rho$ be a density matrix on $H$. Then one has the following bounds on the optimal entanglement fidelity of recovery

$$\frac{\Lambda^2}{\text{Tr} A (\rho)} \leq F_e (\rho, R^{QR} \circ A) \leq \sup_{R \in \mathcal{R}} F_e (\rho, R \circ A) \leq \Lambda,$$  \hspace{1cm} (153)

where the supremum is over quantum operations $R : B^1 (K) \to B^1 (H)$, where

$$\Lambda = \text{Tr} \sqrt{A^{(2, \rho)} (\rho^2)} \leq \text{Tr} A (\rho),$$  \hspace{1cm} (154)

where $A^{(2, \rho)}$ is given by Definition 42 (see also equations 147 and 172, below), and where quadratic recovery operation is given by

$$R^{QR} (\cdot) = \rho \left( A^{(2, \rho)} \right)^\dagger \left( \left( A^{(2, \rho)} (\rho^2) \right)^{-1/2} \right)^\dagger \left( A^{(2, \rho)} (\rho^2) \right)^{-1/2} \right)^\dagger \rho.$$  \hspace{1cm} (155)

Here the adjoint $(\bullet)^\dagger$ is from Definition 24.

**Remark:** In the case that $A$ is trace-preserving, one may plug the square of the last inequality of (153) into the first inequality (153), giving

$$\frac{F_e (\rho, R^{QR} \circ A)}{\sup_{R \in \mathcal{R}} F_e (\rho, R \circ A)} \geq \frac{\Lambda^2}{\sup_{R \in \mathcal{R}} F_e (\rho, R \circ A)} \geq \sup_{R \in \mathcal{R}} F_e (\rho, R \circ A).$$  \hspace{1cm} (156)

In particular, both of the lower bounds of (153) are sufficiently tight to also satisfy the tightness relation (34) of Barnum and Knill [14]. (Note, however, that Barnum and Knill also produce estimates for average entanglement fidelity, under certain commutativity assumptions.) Furthermore, by expressing the bounds (153) in terms of the infidelity $1 - F_e (\rho, R \circ A)$ one obtains the fact that $R^{QR}$, like $R^{BK}$, has an infidelity of recovery within a factor of two of the optimal.

**Proof.** Let $H_{in}$ be a copy of $H$, let $|\psi_\mu\rangle_{H H^*}$ be the canonical purification (66) of $\rho$, and set

$$\mu_{K H^*} = H_{in} \rightarrow K \left( |\psi_\mu\rangle_{H_{in} H^*} \langle \psi_\mu| \right).$$

Using the replacements $(H, K, L) \to (H^*, K, H)$ and $|\phi\rangle_{L H} \to |\psi_\mu\rangle_{H H^*}$, Theorem 39 gives estimates of the form

$$\frac{\Lambda^2}{\text{Tr} \mu_{K H^*}} \leq F_e (\rho, R^{QO} \circ A) \leq \sup_{R \in \mathcal{R}} F_e (\rho, R \circ A) \leq \Lambda.$$  \hspace{1cm} (157)

We claim that $\text{Tr} \mu_{K H^*} = \text{Tr} A (\rho)$, $\Lambda = \Lambda$, and $R^{QO} = R^{QR}$.
First claim: Note that
\[ \hat{\mu}_{KH^*} = \Pi_+ \left( \text{Tr}_H |\psi_\rho\rangle_{KH^*} \langle \psi_\rho| \right) \mu_{KH} \Pi_+ \left( \text{Tr}_H |\psi_\rho\rangle_{KH^*} \langle \psi_\rho| \right) \]

\[ = \Pi_+ (\hat{\rho}_{H^*}) \mathcal{A}_{H_{in} \rightarrow K} \left( \left| \sqrt{\mathcal{T}} \right\rangle_{H_{in} H^*} \langle \sqrt{\mathcal{T}} | \right) \Pi_+ (\hat{\rho}_{H^*}) \]

\[ = \mathcal{A}_{H_{in} \rightarrow K} \left( \left| \sqrt{\mathcal{T}} \Pi_+ (\rho) \right\rangle_{H_{in} H^*} \langle \sqrt{\mathcal{T}} \Pi_+ (\rho) | \right) \]

\[ = \mathcal{A}_{H_{in} \rightarrow K} \left( \left| \psi_\rho \right\rangle_{H_{in} H^*} \langle \psi_\rho | \right) = \mu_{KH^*}, \]

where the first three equalities used (97) \& (94), (69), and (62). It follows by equation 68 that
\[ \text{Tr}_{KH^*} \hat{\mu}_{KH^*} = \text{Tr}_K \mathcal{A}_{H_{in} \rightarrow K} \left( \text{Tr}_H |\psi_\rho\rangle_{KH^*} \langle \psi_\rho| \right) = \text{Tr} \mathcal{A} (\rho), \]
as desired.

Second claim: One computes
\[ \langle \psi_\rho |_{KH^*} (\hat{\mu}_{KH^*})^2 |\psi_\rho\rangle_{KH^*} \]

\[ = \langle \psi_\rho |_{KH^*} \left( \mathcal{A}_{H_{in} \rightarrow K} \left( \left| \psi_\rho \right\rangle_{H_{in} H^*} \langle \psi_\rho | \right) \right)^2 |\psi_\rho\rangle_{KH^*} \]

\[ = \text{Tr}_{KH^*} \left[ \left( \mathcal{A}_{H_{in} \rightarrow K} \left( \left| \psi_\rho \right\rangle_{H_{in} H^*} \langle \psi_\rho | \right) \right)^2 |\psi_\rho\rangle_{KH^*} \right] \]

\[ = \text{Tr}_{H^*} \left[ \left( \mathcal{A}_{H_{in} \rightarrow K} \left( \left| \psi_\rho \right\rangle_{H_{in} H^*} \langle \psi_\rho | \right) \right)^2 |\psi_\rho\rangle_{KH^*} \right] \]

\[ = \mathcal{A}_{H^*} (\rho^2), \]

where \( \mathcal{H}_{in} \) is a copy of \( \mathcal{H} \) and where our steps (in sequence) used (161), cyclicity of the trace, (69), and (147). That \( \Lambda = \Lambda \) now follows from equation 127.

Third claim: By equation 120
\[ R_{K \rightarrow L}^{\text{QO}} (\nu) = \text{Tr}_{KH^*} \left[ (\hat{\mu}_{KH^*})^2 \left( \left( \nu \nu^{-1/2} \nu \nu^{-1/2} \right)_{K \rightarrow K} \otimes |\psi_\rho\rangle_{KH^*} \langle \psi_\rho| \right) \right] \]

where by (121) and (165)
\[ Y_{K \rightarrow K} = \mathcal{A}^{(2,\rho)} (\nu^2), \]

But by (66) \& (62), (147), and (65) one has
\[ \text{Tr}_{H^*} \left[ (\hat{\mu}_{KH^*})^2 |\psi_\rho\rangle_{KH^*} \langle \psi_\rho| \right] \]

\[ = \text{Tr}_{H^*} \left[ \left( \mathcal{A}_{H_{in} \rightarrow K} \left( |\psi_\rho\rangle_{H_{in} H^*} \langle \psi_\rho| \right) \right)^2 \hat{\rho}_{H^*}^{-1/2} |\rho_\rangle_{H^*} \langle \rho| \hat{\rho}_{H^*}^{-1/2} \right] \]

\[ = \left( \mathcal{A}^{(2,\rho)}_{H_{in} \rightarrow K} \otimes \mathbb{1}_H \right) \left( B^{(\mathcal{H}^*)} \rightarrow B^{(\mathcal{H}_{in})} \right) (|\rho\rangle_{H^*} \langle \rho|) \]

\[ = \left( \mathcal{A}^{(2,\rho)}_{H_{in} \rightarrow K} \otimes \mathbb{1}_H \right) (\rho_{H \rightarrow H_{in}} \otimes \rho_{H_{in} \rightarrow H}), \]

where \( \mathcal{H}_{in} \) is a copy of \( \mathcal{H} \). So setting
\[ X_{K \rightarrow K} = Y^{-1/2} \nu Y^{-1/2} = \left( \mathcal{A}^{(2,\rho)} (\rho^2) \right)^{-1/2} \nu \left( \mathcal{A}^{(2,\rho)} (\rho^2) \right)^{-1/2}, \]
and substituting (168) into (166) gives

\[ \tilde{\mathcal{R}}_{K \rightarrow L}(\nu) = \text{Tr}_{K^{*}} \left( (\tilde{\mu}_{KH^{*}})^{2} (X_{K \rightarrow K} \otimes |\psi_{\rho}\rangle_{H^{*}} \langle \psi_{\rho}|) \right) \]

\[ = \text{Tr}_{K} \left( (\tilde{\mu}_{KH^{*}})^{2} |\psi_{\rho}\rangle_{H^{*}} \langle \psi_{\rho}| X_{K \rightarrow K} \right) \]

\[ = \text{Tr}_{K} \left( (A^{(2,\rho)}_{H_{in} \rightarrow K} (\rho_{H_{in} \rightarrow H_{in}} \otimes \rho_{H_{in} \rightarrow H}) X_{K \rightarrow K} \right) \]

\[ = \text{Tr}_{H_{in}} \left[ (\rho_{H_{in} \rightarrow H_{in}} \otimes \rho_{H_{in} \rightarrow H}) \left( (A^{(2,\rho)})_{K \rightarrow H_{in}}^{\dagger} (X) \right) \right] \]

\[ = \rho_{H_{in} \rightarrow H} \left( (A^{(2,\rho)})_{K \rightarrow H_{in}}^{\dagger} (X) \right) \rho_{H_{in} \rightarrow H_{in}}. \quad (170) \]

This proves the claim.

The inequality \( \Lambda \leq \text{Tr} A (\rho) \) follows from (153).

The following proposition puts our recovery bounds into a form closer to the nearly simultaneously-appearing bounds of Bény and Oreshkov (Theorem 21, above):

**Proposition 45** Suppose that \( \rho \) is a density on \( \mathcal{H} \) and that the quantum operation \( A: B^{1} (\mathcal{H}) \rightarrow B^{1} (\mathcal{K}) \) has a Kraus decomposition of the form

\[ A(\mu) = \sum F_{k} \mu F_{k}^{\dagger}, \quad (171) \]

where the \( F_{k} \) are not constrained to satisfy any orthogonality conditions. Then for \( \mu \in B^{1} (\text{supp} (\rho)) \) one has

\[ A^{(2,\rho)} (\mu) = \sum_{k\ell} F_{k} \mu F_{\ell}^{\dagger} \times \text{Tr} \left( F_{k}^{\dagger} F_{\ell} \rho \right) \quad (172) \]

**Proof.** Since both sides of (172) are linear in \( \mu \), we may assume without loss of generality that \( \mu \) is positive semidefinite. The conclusion follows using equations 147, 148, 66, 62, and 61 & 64 (in said order):

\[ A^{(2,\rho)} (\mu) = \text{Tr}_{H^{*}} \left( \mu^{1/2} \rho^{-1/2} \times \left( \sum_{k} F_{k} \langle \psi_{\rho}|_{H^{*}} \langle \psi_{\rho}| F_{k}^{\dagger} \right)^{2} \times \rho^{-1/2} \mu^{1/2} \right) \]

\[ = \text{Tr}_{H^{*}} \left( \sum_{k\ell} | F_{k} \rangle \langle F_{k}|_{K^{*}} \langle \psi_{\rho}|_{H^{*}} \langle \psi_{\rho}| F_{\ell}^{\dagger} \langle F_{k}| \right) \]

\[ = \text{Tr}_{H^{*}} \left( \sum_{k\ell} | F_{k} \rangle \langle F_{k}|_{K^{*}} \langle F_{k}| \langle F_{k} \rangle \langle F_{k}| \right) \]

\[ = \sum_{k\ell} F_{k} \mu F_{\ell}^{\dagger} \times \text{Tr} \left( F_{k}^{\dagger} F_{\ell} \rho \right). \]

\[ \blacksquare \]

5.3 The relationship between the Quadratic Recovery and Barnum and Knill’s reversal

As we have already seen in equation 156, the quadratic reversal \( \mathcal{R}^{QR} \) and the simple lower bound of (153) are both sufficiently accurate to also satisfy the tightness relation (34) of Barnum and Knill. This section makes a brief comparison of the quadratic reversal operation with the reversal map of Barnum and Knill (for the special case of non-average entanglement fidelity) in light of the relationship between the quadratic measurement and the PGM.
Re-expressing the elements of the ensemble (2) as \( \rho_k = p_k \hat{\rho}_k \), where \( \text{Tr}(\hat{\rho}_k) = 1 \) and \( p_k = \text{Tr} \rho_k \) is the chance that \( \hat{\rho}_k \) appears, the formulas for the “pretty good” and quadratically-weighted measurements become

\[
M_k^{\text{PGM}} = \left( \sum p_\ell \hat{\rho}_\ell \right)^{-1/2} \left[ p_k \hat{\rho}_k - \frac{1}{2} \left( \sum p_\ell \hat{\rho}_\ell \right)^{-1} \right] \\
M_k^{\text{QW}} = \left( \sum p_\ell \hat{\rho}_\ell^2 \right)^{-1/2} \left[ p_k \hat{\rho}_k^2 - \frac{1}{2} \left( \sum p_\ell \hat{\rho}_\ell^2 \right)^{-1} \right].
\]

In particular, to get from the pretty-good measurement to the quadratic measurement, one replaces all probabilities and density matrices by their squares.

A simple examination of the formulas (33) and (155) shows that a similar relationship exists between the entanglement fidelity case of the Barnum-Knill reversal \( R^{\text{BK}} \) and the quadratically-weighted reversal \( R^{\text{QR}} \). Note that the corresponding probabilities \( p_k \), which must be replaced by their squares, are viewed as being hidden in the \( \rho \)-Kraus decomposition (140) of the reversed map \( A \).

In [83] various weightings for Belavkin pure-state square-root measurements were compared, and it was argued that Holevo’s quadratically-weighted measurement had qualitative and quantitative advantages over the linearly weighted PGM. Based on analogy, we conjecture that \( R^{\text{QR}} \) will typically (but not always) outperform \( R^{\text{BK}} \).

5.3.1 Depolarizing noise and the quadratic transpose channel

It is perhaps interesting to quantitatively compare the actions of the Barnum-Knill reversal \( R^{\text{BK}} \) with the quadratic recovery \( R^{\text{QR}} \) in the simplest special case, in which depolarizing noise

\[
(A_p)_{\mathcal{H} \to \mathcal{H}} (\mu) = p \times \frac{1}{\dim \mathcal{H}} \text{Tr} (\mu) + (1 - p) \times \mu, \quad p \in [0, 1]
\]

acts on half of a maximally-entangled state. (Note that the \( \rho = 1/\dim \mathcal{H} \) case of \( R^{\text{BK}} \), also known as the transpose channel [84], has recently [20] been employed in the study of approximate quantum error correction.) For \( \dim \mathcal{H} > 1 \), one easily obtains

\[
R^{\text{BK}}_p = A_p \\
R^{\text{QR}}_p = A_{f(p, \dim \mathcal{H})},
\]

where

\[
f (p, \dim \mathcal{H}) := \frac{p^2}{(1 - p)^2 (\dim \mathcal{H})^2 + (2 - p) p}
\]

satisfies

\[
f (p, \dim \mathcal{H}) \leq f (p, 1) = p^2.
\]

In particular, both recovery operations “correct” depolarization errors by committing further depolarization. Fortunately, however, when \( p < 1 \) the quadratic recovery depolarizes with lower probability than the transpose channel, especially when \( \dim \mathcal{H} \) is large or \( p \) is small.

A more detailed quantitative comparison of \( R^{\text{BK}}_p, R^{\text{QR}}_p \), and of reversals of other possible weightings (perhaps generalizing the cubically-weighted measurement of [91, 92]) will be left for future work.

6 Conclusion and future directions

We have generalized the iterative schemes of Ježek-Řeháček-Fiurášek [74], Ježek-Fiurášek-Hradil [6, 75], and Reimpell-Werner [7, 10]. Using an abstract framework, “small angle” guesses were employed to construct concise two-sided bounds for minimum-error quantum detection, maximum overlap, quantum conditional min-entropy, and the reversibility of quantum dynamics. An
approximately-optimal channel reversal and overlap operation were derived. The resulting bounds were sufficiently tight to also satisfy the tightness relations of Barnum and Knill [14], although our methods more generally allowed the target state and the input state to differ. Our recovery operation was found to be a significant improvement of the transpose channel in the simple case of depolarizing noise acting on half of a maximally-entangled state.

As a direction for future study, we note that Barnum and Knill constructed an approximate reversal operation in the more general sense of average entanglement fidelity, albeit with commutativity assumptions of unknown necessity. A remaining open question is whether one can generalize our quadratic reversal construction to this case of average entanglement fidelity, and whether these commutativity assumptions may be removed. More generally, one may ask how to obtain estimates for the maximum overlap problem without our assumed purity of the target state. The principle difficulty in answering both of these questions is in finding an appropriate “small angle guess,” in the sense of lemma 7.

Another future direction, in which we have made recent progress [93], is to employ matrix monotonicity to obtain bounds for the maximum overlap problem, including its special cases of channel reversibility and quantum conditional min-entropy.

Appendix A: Canonical Stinespring dilations

Using only the square root function and the natural isomorphisms of Section 2.1, one may construct Stinespring dilations which are independent of any choice of a basis:

**Definition 46** Let \( \mathcal{R} : B^1(\mathcal{K}) \to B^1(\mathcal{L}) \) be a completely positive map, with \( \mathcal{K} \) finite-dimensional. The canonical environment is given by

\[
\mathcal{E} = \mathcal{L}^*_E \otimes \mathcal{K}_E,
\]

where \( \mathcal{L}^*_E \) and \( \mathcal{K}_E \) are copies of \( \mathcal{L}^* \) and \( \mathcal{K} \), respectively. The canonical Stinespring dilation \( U_\mathcal{R} \) of \( \mathcal{R} \) is the linear transformation \( U_\mathcal{R} : \mathcal{K} \to \mathcal{L} \otimes \mathcal{E} \) such that \( |U_\mathcal{R}\rangle \langle \mathcal{L}\mathcal{E}K^*| = |U\rangle \langle U| \mathcal{L}\mathcal{E}K^* \) is the canonical purification (66) of the Choi matrix \( \tilde{\mathcal{R}} = \mathcal{R} \left( \langle \mathcal{I}\rangle_{\mathcal{K}\mathcal{K}^*} \langle \mathcal{I}\rangle_{\mathcal{L}\mathcal{E}} \right) \).

That \( U_\mathcal{R} \) is a bona fide purification of \( \mathcal{R} \) follows from the following lemma:

**Lemma 47** Let \( \mathcal{R} : B^1(\mathcal{K}) \to B^1(\mathcal{L}) \) be a quantum operation, with \( \mathcal{K} \) finite-dimensional. Then \( U_{\mathcal{K}\to\mathcal{L}\mathcal{E}} \) is a Stinespring dilation of \( \mathcal{R} \) if \( |U\rangle \langle U| \mathcal{L}\mathcal{E}\mathcal{K}^* \) is a purification of the Choi matrix \( \tilde{\mathcal{R}} \).

**Proof.** Suppose that \( U \) dilates \( \mathcal{R} \). Then by equation 62

\[
\tilde{\mathcal{R}} = \text{Tr}_{\mathcal{E}} U_{\mathcal{K}\to\mathcal{L}\mathcal{E}} \langle \mathcal{I}\rangle_{\mathcal{K}\mathcal{K}^*} \langle \mathcal{I}\rangle_{\mathcal{L}\mathcal{E}} \langle \mathcal{I}\rangle_{\mathcal{K}\mathcal{K}^*} = \text{Tr}_{\mathcal{E}} |U\rangle \langle U| \mathcal{L}\mathcal{E}\mathcal{K}^*,
\]

so \( |U\rangle \) purifies \( \tilde{\mathcal{R}} \).

Conversely, suppose that \( |U\rangle \mathcal{L}\mathcal{E}\mathcal{K}^* \) purifies \( \tilde{\mathcal{R}} \), and let \( \upsilon \in B^1(\mathcal{K}) \) be a density matrix. Then by equations 64, 62, 66, and 68,

\[
\text{Tr}_{\mathcal{E}} \left( U_{\mathcal{K}\to\mathcal{L}\mathcal{E}} \upsilon_{\mathcal{K}} \langle U_{\mathcal{K}\to\mathcal{L}\mathcal{E}} \rangle^* \right) = \text{Tr}_{\mathcal{K}^*} \left( \left( \upsilon_{\mathcal{K}^*}^{1/2} \langle U \rangle \langle U| \upsilon_{\mathcal{K}^*}^{1/2} \right) \right) = \text{Tr}_{\mathcal{K}^*} \left( \left( \upsilon_{\mathcal{K}^*}^{1/2} \mathcal{R} \left( \langle \mathcal{I}\rangle_{\mathcal{K}\mathcal{K}^*} \langle \mathcal{I}\rangle_{\mathcal{L}\mathcal{E}} \langle \mathcal{I}\rangle_{\mathcal{K}\mathcal{K}^*} \right) \right) \right) = \mathcal{R} \left( \text{Tr}_{\mathcal{K}^*} \psi_{\upsilon} \langle \psi_{\upsilon}\rangle \right) = \mathcal{R} \left( \upsilon \right).
\]

\[\blacksquare\]
Appendix B: Reimpell-Werner iteration as directional iteration

The purpose of this section is to verify that Reimpell-Werner iteration (introduced in section 1.4.2) for CP maps corresponds to directional iteration of the corresponding Stinespring dilations.

One may re-express the maximized functional \( f(\mathcal{R}) \) of equation (21) as

\[
    f(\mathcal{R}) = \| U_{\mathcal{R}} \|_F^2,
\]

where \( U_{\mathcal{R}} \) is a Stinespring dilation of the CP map \( \mathcal{R} : B^1(K) \rightarrow B^1(L) \) and the seminorm is defined by

**Definition 48** Let \( \mathcal{E} = \mathcal{L}_K^* \otimes \mathcal{K}_E \) be the canonical environment (178) for quantum operations from \( \mathcal{K} \) to \( \mathcal{L} \). For operators \( U, W : \mathcal{K} \rightarrow \mathcal{L} \otimes \mathcal{E} \), define the semidefinite inner product

\[
    \langle U, W \rangle_F = \langle \langle U \rangle_{\mathcal{L}KE}^*, F_{LK} \cdot | \langle W \rangle_{\mathcal{L}KE}^* \rangle_{LK} \rangle_{LK}^{*},
\]

where \( A \mapsto |A\rangle \) is the isomorphism of equation 58. Let \( V_F = \{ U \mid \|U\|_F < \infty \} \), on which \( (\cdot, \cdot)_F \) is a well-defined semidefinite inner product. Let \( S = \{ U \mid \|U\| \leq 1 \} \).

**Theorem 49 (Reimpell-Werner iteration is directional iteration)** Suppose that \( U_{\mathcal{K} \rightarrow \mathcal{L}E} \) is a Stinespring dilation of a CP map \( \mathcal{R} : B^1(K) \rightarrow B^1(L) \). Then \( U \in V_F \) has a directional iterate \( U^{(+)} \in S \) which dilates the Reimpell-Werner iterate \( \mathcal{R} \oplus \) of Def. 11.

**Proof.** Let \( X_{\mathcal{K} \rightarrow \mathcal{L}E} \) be the operator defined by

\[
    |X_{\mathcal{K} \rightarrow \mathcal{L}E}\rangle \rangle_{\mathcal{L}KE} = F_{LK} \cdot | \langle U \rangle_{\mathcal{L}KE}^* \rangle_{LK} \rangle_{LK}^{*}.
\]

Then by equations 181, 43, and 44,

\[
    \max_{W \in S} \text{Re} \langle W, U \rangle_F = \max_{W \in S} \text{Re} \text{Tr} (W_{\mathcal{K} \rightarrow \mathcal{L}E})^\dagger X_{\mathcal{K} \rightarrow \mathcal{L}E} = \|X_{\mathcal{K} \rightarrow \mathcal{L}E}\|_1,
\]

with maximizer \( W = U^{(+)} \) given by

\[
    U^{(+)} = X (X^\dagger X)^{-1/2^+}.
\]

By equations 62, 63, and 23 one has

\[
    \left| U^{(+)} \right\rangle \rangle_{\mathcal{L}KE} = \left( \frac{X^\dagger X}{\mathcal{K} \rightarrow \mathcal{K}^*} \right)^{-1/2^+} \left| X \right\rangle \rangle_{\mathcal{L}KE},
\]

\[
    \left( \text{Tr}_{\mathcal{L}E} \left| X \right\rangle \rangle_{\mathcal{L}KE} \langle \langle X \rangle \rangle \right)^{-1/2} \left| X \right\rangle \rangle_{\mathcal{L}KE},
\]

\[
    \left( \text{Tr}_{\mathcal{L}E} F_{\mathcal{L}K} \cdot | U \rangle \rangle_{\mathcal{L}KE} \langle \langle U \rangle_{\mathcal{L}KE}^* \cdot F_{\mathcal{L}K} \cdot \right)^{-1/2^+} F_{\mathcal{L}K} \cdot | U \rangle \rangle_{\mathcal{L}KE},
\]

\[
    \Gamma^{-1/2^+} F_{\mathcal{L}K} \cdot U_{\mathcal{K} \rightarrow \mathcal{L}E} \left| 1 \right\rangle \rangle_{\mathcal{K}K}.
\]

It follows from Lemma 47 and Eq. 22 that \( U^{(+)} \) dilates \( \mathcal{R} \oplus \). ■
Appendix C: The relationship between overlap bounds and state distinguishability

As remarked in section 4.2, Theorems 1 and 2 of [70] (see also equations 186-189, below) imply that minimum-error distinguishability of a finite collection of quantum states $\mathcal{E} = \{\rho_k\}_{k=1,\ldots,m}$ may be expressed in terms of restricted maximum overlap:

$$
P_{\text{succ}} (M^{\text{opt}}) = m \times \max_{\mathcal{R}_{\mathcal{H} \to (\mathbb{C}^m)^*}} \langle \phi_{\mathbb{C}^m} (\mathbb{C}^m)^* \mid \mathcal{R}_{\mathcal{H} \to (\mathbb{C}^m)^*} (\mu_{\mathcal{H}\mathbb{C}^m}) \mid \phi_{\mathbb{C}^m} (\mathbb{C}^m)^* \rangle.
$$  \hspace{1cm} (182)

Here the vector $\phi \in \mathbb{C}^m \otimes (\mathbb{C}^m)^*$ is the maximally-mixed state

$$
\langle \phi_{\mathbb{C}^m} (\mathbb{C}^m)^* \rangle = \frac{1}{\sqrt{m}} \mathbb{I}_{\mathbb{C}^m \mathbb{C}^m^*} := \frac{1}{\sqrt{m}} \sum_{k} |k\rangle_{\mathbb{C}^m} \langle k|_{\mathbb{C}^m^*},
$$  \hspace{1cm} (183)

and $\mu \in B^1 (\mathcal{H} \otimes \mathbb{C}^m)$ is the “quantum-classical” state

$$
\mu_{\mathcal{H} \otimes \mathbb{C}^m} = \sum_{k=1}^{m} \rho_k \otimes |k\rangle_{\mathbb{C}^m} \langle k|,
$$  \hspace{1cm} (184)

where the $\rho_k$ are normalized as in Definition 1.

If one applies the overlap bounds of Theorem 39 (or the $s = 0$ case of Corollary 40 combined with Eq. 12), one obtains

$$
\left( \text{Tr} \sqrt{\sum_{k=1}^{m} \rho_k^2} \right)^2 \leq P_{\text{succ}} (M^{\text{QW}}) \leq P_{\text{succ}} (M^{\text{opt}}) \leq \sqrt{m} \left\| (\mathcal{R}^{\text{opt}})^\dagger (|\phi\rangle \langle \phi|) \right\|_{\infty} \times \text{Tr} \sqrt{\sum_{k=1}^{m} \rho_k^2} \leq \sqrt{m} \times \text{Tr} \sqrt{\sum_{k=1}^{m} \rho_k^2}.
$$  \hspace{1cm} (185)

In particular, if one neglects the $\left\| \mathcal{R}^{\dagger} \right\|_{\infty}$ factor in the fourth expression of this estimate then one picks up a spurious factor of $\sqrt{m}$ not appearing in the bounds of Theorem 15. (Weakness of the upper bound is not surprising, since $\phi$ and $\mu$ are generally not “reasonably overlappable.”)

In order to show how one may apply the fourth term of the overlap estimate (185), we give another proof of Theorem 15. It is hoped that similar methods may lead to sharper upper in other instances of maximum overlap or conditional min-entropy.

An “overlap proof” of Theorem 15. We restrict consideration to the case $\mathcal{E} = \{\rho_k\}_{k=1,\ldots,m}$. Given a quantum operation $\mathcal{R}_{\mathcal{H} \to \mathbb{C}^m}$ one has the identity

$$
m \times \langle \phi_{\mathbb{C}^m} (\mathbb{C}^m)^* \mid \mathcal{R}_{\mathcal{H} \to (\mathbb{C}^m)^*} (\mu_{\mathcal{H}\mathbb{C}^m}) \mid \phi_{\mathbb{C}^m} (\mathbb{C}^m)^* \rangle = P_{\text{succ}} (M^\mathcal{R}),
$$  \hspace{1cm} (186)

where $\mu \in B^1 (\mathcal{H} \otimes \mathbb{C}^m)$ and $\phi \in \mathbb{C}^m \otimes \mathbb{C}^m$ are as in equations (183)-(184) and where the POVM $M^\mathcal{R}$ corresponding to the operation $\mathcal{R}$ is given by

$$
M_k^\mathcal{R} := (\mathcal{R})^\dagger (|\bar{k}\rangle_{\mathbb{C}^m} \langle \bar{k}|), \quad k = 1, \ldots, m.
$$  \hspace{1cm} (187)

Since any given POVM $M$ may be expressed in the form of (187) for the quantum operation $\mathcal{R} = \mathcal{R}^M$ given by

$$
\mathcal{R}^{\mathcal{R}}_{\mathcal{H} \to (\mathbb{C}^m)^*} (\rho) := \sum_{k=1}^{m} |\bar{k}\rangle_{\mathbb{C}^m} \langle \bar{k}| \times \text{Tr} (M_k \rho),
$$  \hspace{1cm} (188)

maximization of (186) over operations $\mathcal{R}$ gives the identity (182). Taking $M^{\text{opt}}$ to be some optimal measurement, it follows that a maximizer of the LHS of (186) is given by

$$
\mathcal{R}^{\text{opt}} = \mathcal{R}^{M^{\text{opt}}},
$$  \hspace{1cm} (189)
where $M^{\text{opt}}$ is an optimal measurement. One estimates
\[
\left\| (R^{\text{opt}})^{\dagger}_{C_{m} \rightarrow H} (|\phi\rangle_{C_{m} C_{m}^{\ast}} \langle \phi|) \right\|_{\infty} = \left\| \frac{1}{m} \sum |k\rangle_{C_{m}} \langle k| \otimes M^{\text{opt}}_{k} \right\|_{\infty} \leq \frac{1}{m}.
\]
(190)

Applying the bounds (126) to (186) yields the chain of inequalities
\[
\left( \sqrt{\sum \rho_{k}^{2}} \right)^{2} \leq P_{\text{suc}} (M^{\text{QW}}) \leq P_{\text{suc}} (M^{\text{opt}}) \leq \sqrt{\sum \rho_{k}^{2}} \leq \sqrt{m} \times \sqrt{\sum \rho_{k}^{2}}.
\]
(191)

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Note Added: Private communication from the authors of [18] indicates that they have obtained the quadratic recovery channel by alternative means [19].

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\[ f(x) = x + x^3 \]
on \( \mathbb{R} \) satisfies Reimpell’s definition of “stable” about the fixed point \( x = 0 \), since  \( \| f'(0) \| \leq 1 \). Note, however, that the iterative sequence  \( x, f(x), f(f(x)), \ldots \) diverges unless \( x = 0 \). Furthermore, consideration of iterated rotations of the unit ball in \( \mathbb{R}^2 \) shows that an assumption of compactness is no remedy.

Note: See [108].
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\sum_j \pi_j \|\psi_j - e_j\|^2 = 2 \left(1 - \text{Re} \text{Tr} \left(U \Pi \Gamma^1/2 \right)\right). \]
The line just after equation (9) should read “where \(V^* = |\Pi \Gamma^{1/2} \left(\Pi \Gamma^{1/2}\right)^{-1}|.\) The final expression in the paper should be \[2 \left(1 - \text{Tr} \left|\Gamma^1/2 \Pi\right|\right).\]

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[105] One usually requires that a POVM satisfy \( \sum M_k = \mathbb{1} \). The relaxed condition \( \sum M_k \leq \mathbb{1} \) allows the possibility that the POVM may fail to return an result. (Alternatively, one may augment \( \{M_k\} \) with the operator \( M_{\text{error}} = \mathbb{1} - \sum M_k \). The operator \( M_{\text{error}} \) could perhaps serve as a useful error flag, i.e. an indication that a state orthogonal to span (\( E \)) has been detected.)

[106] Other numerical methods for computing optimal measurements exist [96–99].

[107] The maximum-overlap problem (1) is equivalent to the one-data-point special case of maximum-likelihood quantum process tomography. The specific iteration is defined by equations 14 through 17 of [6].

[108] Barnum and Knill incorrectly assert on page 2103 of [14] that the asymptotically-optimal measurement (24) introduced by Holevo in [43] is equal to the “pretty good” measurement (29).

[109] Indeed, in Quantum Field Theory it is natural to use dual states \( \tilde{\psi} \in \mathcal{H}^* \) to represent antiparticles. See, for example, [100].

[110] Basis-dependent versions may be found in [101, 102].

[111] See [104] for the infinite-dimensional case.
Indeed, if $\mathcal{E} = \{p_k |\psi_k\rangle \langle \psi_k|\}$ is an ensemble of linearly-independent pure states $\psi_k$ spanning $\mathcal{H}$ then the maximizer $E = G^{(+)}$ of $\text{Re} \langle G, E \rangle$ is of the form $E_k = |\psi_k\rangle \langle e_k|$, with $\{e_k\}$ orthonormal.\footnote{One may therefore express $\sup_E \text{Re} \langle G, E \rangle = \sup_{\{e_k\}} \text{Re} \left( \sum p_k \langle e_k, \psi_k \rangle \right) = 1 - \inf_{\{e_k\}} C_{\text{Holevo}}\left( \{e_k\} \right)$, where optimization is over orthonormal $\{e_k\}$. Here $C_{\text{Holevo}} = \frac{1}{2} \sum p_k \|\psi_k - e_k\|^2$ is the “approximate cost function” which was minimized by Holevo (\cite{43}, equation 8) in his construction of $M^{QW}$ in the case of pure states. (See also \cite{22, 103}.)}

One needs the identity

$$\text{Tr}_B \sqrt{\text{Tr}_A \rho_{AB} \rho_A^{-1/2}} \rho_{AB} = \sqrt{\langle \psi_{AB} | \psi_{AB} \rangle \rho_A^{-1/2}} \times \text{Tr}_B \sqrt{\rho_B} = \left( \text{Tr} \sqrt{\rho_A} \right)^{3/2}.$$ 

Note that if $\{ |e_k\rangle \langle e_k| \}_{k=1, \ldots, \dim \mathcal{H}}$ is a POVM then the $e_k$ are orthonormal. In particular, normality of the $e_k$ follows from the inequalities $\|e_k\|^2 \leq \| |\psi_k\rangle \langle e_k| \|_\infty = 1$ and the identity $\sum \|e_k\|^2 = \text{Tr} \sum |e_k\rangle \langle e_k| = \dim \mathcal{H}$. Orthogonality then follows from the identity $\langle e_k| \left( \sum |e_k\rangle \langle e_k| \right) |e_\ell\rangle = \|e_\ell\|^2$. 

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