Strong-coupling dynamics of Bose–Einstein condensate in a double-well trap

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Abstract

Dynamics of the repulsive Bose–Einstein condensate (BEC) in a double-well trap is explored within the 3D time-dependent Gross–Pitaevskii equation. The model avoids numerous common approximations (two-mode treatment, time-space factorization, fixed values of the chemical potential and barrier penetrability, etc) and thus may provide a realistic description of BEC dynamics, including both weak-coupling (sub-barrier) and strong-coupling (above-barrier) regimes and their crossover. The strong coupling regime is achieved by increasing the number $N$ of BEC atoms and thus the chemical potential. The evolution with $N$ of Josephson oscillations (JO) and macroscopic quantum self-trapping (MQST) is examined and the crucial impact of the BEC interaction is demonstrated. At weak coupling, the calculations well reproduce the JO/MQST experimental data. At strong coupling with a significant overlap of the left and right BECs, we observe a remarkable persistence of the Josephson-like dynamics: the JO and MQST converge to a high-frequency JO-like mode where both population imbalance and phase difference oscillate around the zero averages.

1. Introduction

Nowadays, trapped Bose–Einstein condensates (BEC) are widely recognized as a source of fascinating new physics, see monographs [1, 2] as well as early [3–5] and recent [6, 7] reviews. Among diverse aspects of this field, much attention is paid to the dynamics of bound condensates and relevant nonlinear effects caused by the interaction between BEC atoms [6, 8]. A boson Josephson junction in a double-well trap represents a typical example of such a relevant system. Its basic dynamical regimes, Josephson oscillations (JO) and macroscopic quantum self-trapping (MQST) were widely investigated in the weak coupling (WC) limit, both in theory [6, 8–10] and experiment [11–13]. At the same time, it yet remains uncertain how the JO/MQST dynamics would look in the strong coupling (SC) limit, when tunnelling through the barrier separating the wells becomes strong and the sub-barrier transfer is further transformed to the above-barrier one. At SC, the left and right BECs in a double-well trap may considerably overlap so that using the relative variables of Josephson dynamics, namely population imbalance $z$ and phase difference $\theta$, is no longer well justified. Natural questions then arise, whether the JO and MQST survive at SC and, if yes, then how they would look.

These questions are real for BEC with a strong interaction factor $UN$ (the product of the interaction strength $U$ and number of BEC atoms $N$), where both JO and MQST can exist. Further, SC appears in various BEC problems. Besides, SC is familiar in the explorations exploiting considerable changes of the trap, e.g. controlled BEC transport [14, 15], BEC interferometry [16–18], generation of solitons [17, 19], squeezing and entanglement [20] etc. In most of these problems, JO/MQST shows up as a detrimental factor. Knowledge of the JO/MQST behaviour at SC might thus help minimize this factor.

Numerous previous studies of Josephson dynamics in a double-well trap were mainly performed within the two-mode approximation (TMA) [21] (where only the two lowest energy
levels of the system contribute to the dynamics) and in the WC limit. In principle, the TMA is acceptable even in the SC case if the interaction factor $UN$ is small and the total order parameter is used. However, the TMA imposes a severe upper limit on the number of atoms (usually $N < 1000$) in interacting BEC [21]. The TMA modifications, like a variable tunnelling model [22] and a direct implementation of Wannier states [23] improve the TMA description but still keep this limitation. The exploration of the interaction-induced MQST at SC needs a large $UN$ and leads to exceeding the TMA limit. Hence, to be on the safe side, the JO/MQST exploration at SC should be carried out beyond the TMA. The theory should embrace the impact of all excited states and avoid, as much as possible, other standard approximations, e.g. the space-time factorization and partition (for every well) of the total order parameter.

In the present mean-field study, we solve the 3D (three-dimensional) time-dependent Gross–Pitaevskii equation [24]. The total order parameter is used to make the theory relevant for both WC and SC cases. The number of atoms, $1000 \leq N < 10000$, is taken sufficiently large to validate the mean-field treatment. The method does not use any of approximations mentioned above. Both the population imbalance $z$ and phase difference $\theta$ are examined.

The time-dependent Gross–Pitaevskii and nonlinear Schrödinger equations beyond TMA have already been used for the investigation of BEC dynamics, see e.g. [6, 8, 17, 25, 26]. In [17, 25, 26], the transformation of the trap between double- and single-well shapes was used and thus the SC was touched. For small BECs with $N \leq 100–200$, the many-body quantum analysis of JO/MQST was done [25, 26]. The important role of excited states beyond the TMA was pointed out. However, these studies have not especially addressed the evolution of JO/MQST at the WC–SC transition and have not thus provided a relevant analysis thereof. Moreover, the previous studies were mainly performed in the 1D limit [27], and their results thus cannot be directly applied to the 3D case, which is often met in practice, e.g. in the basic JO/MQST Heidelberg experiment [11, 12]. As compared to the studies mentioned above, our mean-field analysis of JO/MQST at SC is more systematic, complete and realistic: it is done beyond TMA, accesses both population imbalance and relative phase, implements a realistic initiation of the dynamics (see discussion below) and is suitable for 3D experiments.

Our calculations are performed for a double-well configuration and the JO/MQST initial conditions of the Heidelberg experiment [11, 12]. The repulsive condensate of $^{87}$Rb atoms is used. For $N = 1000$, we very well reproduce the JO/MQST experimental data [11, 12], which confirms the validity of our model in the WC limit. The WC to SC transition is obtained by increasing the number of BEC atoms from $N = 1000$ to 10 000. The increase of $UN$ results in the upshift of the chemical potential $\mu$ and in the growth of the coupling between the wells. The SC is attained at $N > 3000$. It is then shown that, despite an essential overlap, the left and right BECs still keep their individuality. Thus, using the relative Josephson conjugate variables $z$ and $\theta$ remains reasonable. We show that in this regime JO survive, but acquire a higher frequency. The MQST is transformed to a similar JO. Actually, JO and MQST merge to a similar mode. These findings constitute the main result of this study. A similar MQST $\rightarrow$ JO transfer was earlier predicted as a reappearance of tunnelling in the strong interaction limit [28], but for a 1D BEC.

In most of the previous studies (see, e.g., [8]), the initial conditions are obtained by constraining the system into a non-stationary state of the symmetric trap. This may be questionable in the SC regime. It is thus crucial to examine that point by building initial conditions with a more realistic technique [11, 12]. So, in this study, we first produce a stationary state with the proper initial conditions in an asymmetric trap. Then the trap is non-adiabatically transformed to the symmetric form. The advantage of such tuning is double. First, it is a realistic procedure matching experimental conditions. Second, it prepares the system in a true stationary state and induces no artificial initial kick.

The paper is organized as follows. The calculation scheme is sketched in section 2. Results of the calculations are discussed in section 3. The summary is given in section 4.

2. Calculation scheme

We solve the 3D time-dependent Gross–Pitaevskii equation [24]

$$i\hbar \frac{\partial \Psi}{\partial t}(\mathbf{r}, t) = \left[ -\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}) + g_0 |\Psi(\mathbf{r}, t)|^2 \right] \Psi(\mathbf{r}, t), \quad (1)$$

for the total order parameter $\Psi(\mathbf{r}, t)$ describing the BEC in both left and right wells of the trap. Here, $g_0 = 4\pi \hbar^2 a_s/m$ is the interaction parameter, $a_s$ is the scattering length and $m$ is the atomic mass. The trap potential

$$V(\mathbf{r}) = \frac{m}{2} \left( \omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2 \right) + V_0 \cos^2(\pi x/q_0), \quad (2)$$

includes the anisotropic harmonic confinement and the barrier in the $x$-direction, where $V_0$ is the barrier height and $q_0$ determines the barrier width. Note that the barrier parameters may depend on time at the stage of preparation of the initial conditions.

Following the experiment [11, 12], we use a BEC of $^{87}$Rb atoms with $a_s = 5.75$ nm. The trap frequencies are $\omega_x = 2\pi \times 78$ Hz, $\omega_y = 2\pi \times 66$ Hz, $\omega_z = 2\pi \times 90$ Hz i.e. $\omega_x + \omega_y = 2\omega_z$ [11, 12]. The barrier parameters are $V_0 = 420 \times h$ Hz and $q_0 = 5.2 \mu m$ [11, 12]. The distance between centres of the left and right wells is then $d = 4.4 \mu m$. For $N = 1000$ atoms, we reproduce the WC conditions of the JO/MQST Heidelberg experiment [11, 12].

The static solutions of (1) are found within the damped gradient method [29], while the time evolution is computed within the time-splitting [30] and fast Fourier-transformation techniques. The order parameter $\Psi(\mathbf{r}, t)$ is determined in a 3D Cartesian grid. The requirement $\int_{-\infty}^{+\infty} d\mathbf{r}^3 |\Psi(\mathbf{r}, t)|^2 = N$ in time is directly fulfilled by using an explicit unitary propagator.

Reflecting boundary conditions are used throughout, but have no impact on the dynamics because of the harmonic confinement, which makes them effectless. No time–space factorization is used. The conservation of the total energy $E$ and number of atoms $N$ is controlled. Note that the Gross–Pitaevskii equation is mathematically equivalent to the
nonlinear Schrödinger equation. In this sense, our approach is a counterpart of the time-dependent Hartree–Fock method for the system of interacting bosons.

The JO and MQST are studied in terms of the time-dependent normalized population imbalance $z$ and phase difference $\theta$,

$$z(t) = \frac{N_L(t) - N_R(t)}{N}, \quad \theta(t) = \phi_R(t) - \phi_L(t),$$

where $N_{L,R}(t)$ are respectively populations of the left and right wells (with $N_L(t) + N_R(t) = N$) and $\phi_{L,R}(t)$ are the corresponding BEC phases. The populations read

$$N_j(t) = \int_{-\infty}^{+\infty} dr^3 |\Psi_j(r,t)|^2,$$

with $j = L, R$ and $\Psi_L(r,t) = \Psi(x \leq 0, y, z, t), \Psi_R(r,t) = \Psi(x \geq 0, y, z, t)$.

The phases $\phi_j(t)$ are defined as

$$\phi_j(t) = \arctan \frac{\gamma_j(t)}{\zeta_j(t)},$$

with the averages

$$\gamma_j(t) = \frac{1}{N_j} \int_{-\infty}^{+\infty} dr^3 \text{Im}(\Psi_j(r,t))|\Psi_j(r,t)|^2,$$

$$\zeta_j(t) = \frac{1}{N_j} \int_{-\infty}^{+\infty} dr^3 \text{Re}(\Psi_j(r,t))|\Psi_j(r,t)|^2.$$

Computation of the phase time evolution through arctan may be cumbersome. So the time evolution is calculated through the phase increments $\delta \phi_j(t + \delta t) \approx \phi_j(t) + \delta \phi_j(t)$ for a small time step $\delta t$. Namely we use

$$\delta \phi_j(t) = \sqrt{\left[\frac{\delta \gamma_j(t)}{\gamma_j(t + \delta t)}\right]^2 + \left[\frac{\delta \zeta_j(t)}{\zeta_j(t + \delta t)}\right]^2},$$

with $\delta \gamma_j(t) = \gamma_j(t + \delta t) - \gamma_j(t)$, $\delta \zeta_j(t) = \zeta_j(t + \delta t) - \zeta_j(t)$.

The calculations are performed for $N = 1000$, 3000, 5000 and 10 000 atoms. For all cases, the initial population imbalance $\theta_0$ is 0.3 for JO and 0.6 for MQST [11, 12]. The initial phase difference is $\delta \theta = 0$ for both JO and MQST.

In most of the previous studies, the initial state is prepared as the lowest non-stationary state with the constrained $\theta_0$ in the symmetric trap, see, e.g., [8] (constrained technique, CT). The constraint is reasonable for WC ($N = 1000$ in our study), but may be questionable for SC where the chemical potential $\mu_s$ associated with the motion in the $x$-direction approaches or even exceeds the barrier height. Besides, the constraint procedure deviates from the actual experimental initialization [11, 12] of the JO/MQST dynamics. Hence we use here, in addition to the constraint calculations, a more reliable and realistic initialization following [11, 12]. We start from the asymmetric trap produced from the symmetric one by a right-shift of $d$ of the barrier. The value of the shift is adjusted to provide the given $\theta_0$ in the lowest stationary state of the asymmetric trap. In some cases, an additional widening of the barrier, $q_0 \to q$, in the asymmetric trap is applied. Then the trap is rapidly (for a time $\tau$) returned to the symmetric form and the JO/MQST time evolution starts. We call this procedure barrier shift technique (BST).

### Table 1. Parameters of the procedure for the JO/MQST initial conditions. See the text for details.

| $N$  | JO  | MQST | JO  | MQST | JO  | MQST |
|------|-----|------|-----|------|-----|------|
|      | $d$ (μm) | $\tau$ (ms) | $q$ (μm) | $\rho_0$ (μm) | $V_0$ (μm) |
| 1000 | 0.25 | 0.5  | 4  | 8  | 5.2 | 5.2 |
| 3000 | 0.45 | 1.05 | 5  | 9  | 5.2 | 5.2 |
| 5000 | 0.62 | 1.35 | 5  | 5  | 5.2 | 6.6 |
| 10 000 | 1.35 | 1.35 | 8  | 2  | 5.2 | 10.1 |

The return time $\tau$ is chosen to provide a reasonable initialization of JO/MQST. The calculations show that too rapid a return shakes the system too much and leads to a fuzzy and fragile JO/MQST. Instead, for too slow a return, the equilibration process notably modifies $z$ and $\theta$ from their initial values $\theta_0$ and $\delta \theta$. Altogether, this procedure is much more justified and realistic than the mere constraint. For $N = 1000$, it fully reproduces the formation of the initial conditions in the experiment [11, 12].

### 3. Results and discussion

#### 3.1. Static interaction effect

In our study, we swap from the WC to SC case and approach their crossover point by increasing the number of atoms $N$ from 1000 to 10 000. This results in increasing the integral energy and in subsequent growth of the chemical potential $\mu_s$. Thus, we naturally come from deeply sub-barrier to above barrier cases.

This is demonstrated in figure 1 where the static BEC density $\rho(x)$ and chemical potential $\mu_s$ (along the $x$-direction) are shown at different values of $N$ and compared to the relevant trap potential $V_t(x) = m\omega_z^2x^2/2 + V_0\cos^2(\pi x/q_0)$. As mentioned above, the trap parameters are the same as in the experiment [11, 12]. The density

$$\rho(x) = \int_{-\infty}^{+\infty} dy dz |\Psi(x, y, z)|^2$$

is determined from the solutions of (1) for interacting BEC.

Figure 1 shows that, for $N = 1000$, the overlap of the left and right BECs is small. The ratio of the densities at their maxima ($x = \pm 2.2 \mu m$) and centre of the trap ($x = 0$) is $\rho_l/\rho_c = 5.1$. This is the WC case used in experiment [11, 12]. Increasing $N$ results in increasing the densities and in a larger overlap in the barrier region. For $N = 10 000$, we already reach the SC case with $\rho_l/\rho_c = 2$. Here, the overlap of the left and right BECs cannot be neglected and the system should be treated with the total order parameter. At the same time, the left and right density bumps are still distinctive. So a physical view of the system in terms of two (strongly coupled) BECs is still reasonable and one may yet expect JO/MQST dynamics described by the relative variables $z$ and $\theta$.

To discriminate the sub-barrier and above-barrier cases, one should compare the chemical potential $\mu_s$ and the barrier height $V_0$. For $\mu_s$, only the $x$-direction has to be taken into

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The nonlinear (with interaction) chemical potential $\mu_{\alpha\beta}$ is equal to $\hbar (\alpha \omega_{0} + \omega_{z})$.

Since $\omega_{0} + \omega_{z} = 2 \alpha \omega_{t}$, the subtractive term in (10) is equal to $\hbar \alpha \omega_{t}$. The calculations give $\alpha = \mu_{\alpha\beta}/\mu_{0} = 3/4$, i.e. the $x$-direction along mainly contributes to the total chemical potential $\mu_{0}$. This is because the barrier separates the harmonic $x$-confinement into two narrower regions and thus effectively increases $\omega_{z}$.

In the nonlinear case ($\theta_{0} \neq 0$), the estimation of $\mu_{z}$ is straightforward for a 1D system but more complicated for a 3D one. Here, we roughly put $\mu_{z} = \alpha \mu$, where $\mu$ is the total nonlinear chemical potential. Hence, we suppose that in the linear and nonlinear cases the relative contribution of the $x$-direction to the chemical potential is the same.

Figure 1 compares the linear $\mu_{\alpha\beta}$ and nonlinear $\mu_{z}$ to the barrier height $V_{0}$. It is seen that $\mu_{\alpha\beta}$ does not depend on $N$. It is always much lower than $V_{0}$, thus leading to the deeply sub-barrier case. If the repulsive interaction is switched on, the nonlinear chemical potential $\mu_{z}$ increases with $N$. We see the sub-barrier case for $N = 1000$, the crossover region for $N = 3000–5000$ and the above-barrier case for $N = 10 000$.

3.2. JO and MQST evolution

The evolution of JO and MQST with $N$ is demonstrated in figures 2 and 3. In all cases, the initial ($t = 0$) conditions are $z_{0} = 0.3, \theta_{0} = 0$ for JO and $z_{0} = 0.6, \theta_{0} = 0$ for MQST [11, 12]. Both the CT and BST [11, 12], described in section 2, are used for the initialization of the dynamics. For the BST, the time $\tau$ when the asymmetric trap is fully reduced to the symmetric form is marked by a vertical line. It is seen that CT at $t = 0$ and BST at $t = \tau$ give somewhat different $z$ and $\theta$. The larger the $N$ and $\tau$ (and thus the equilibration time), the greater the difference. Nevertheless, the CT and BST usually initiate a similar (up to a constant time shift) dynamics, especially for JO. In what follows, we will consider the more realistic BST results.

First of all, note that for $N = 1000$ our calculations well reproduce the JO and MQST experimental data [11, 12]. Following figures 2(a) and (e), we obtain for JO the robust $z$- and $\theta$-oscillations with the frequency $\omega_{z} \approx 2 \pi \times 23 Hz$, which is very close to the experimental value $\omega_{z}^{\exp} = 2 \pi \times 25 Hz$. In figure 3(a) for MQST, the calculations give $z$-oscillations around $\langle z \rangle = 0.6$ with the frequency $\omega_{z}^{\exp} = 2 \pi \times 78 Hz$. A small underestimation of the experimental frequencies can be caused by using in our calculations a smaller number of atoms, $N = 1000$ instead of $N \approx 1150 \pm 150$ in the experiment. In figure 3(e) for MQST, the computed $\theta$ increases linearly with the rate $\Delta \theta = \theta_{\max} - \theta_{\min} = 1.4 \pi$ for JO and $\Delta \theta = 0.3$ for MQST also reproduce the experimental data. Good agreement of our results with the experiment [11, 12] proves the high relevance and realistic character of our method and justifies its application to the more sophisticated cases considered below.

The evolution of JO with $N$ is exhibited in figure 2. It is seen that, despite a significant change in the conditions from $N = 1000$ (WC, sub-barrier transfer) through $N = 3000–5000$ (significant coupling, crossover region) to $N = 10 000$ (SC, above-barrier transfer), the JO keep the main features: oscillations of $z$ and $\theta$ with the same frequency $\omega_{z}$ around zero average values. The time shift between $z$ and $\theta$ is about half a period. Amplitudes of the oscillations do not change with $N$. The evolution of JO with $N$ (and thus with the interaction factor $NU$) is mainly reduced to growth of the frequency $\omega_{z}$. 

Figure 1. The double-well trap potential $V_{0}(x)$ (bold curve), the linear (without interaction) chemical potential $\mu_{0}$ (dotted line), the nonlinear (with interaction) chemical potential $\mu_{i}$ (solid line) and BEC density $\rho(x)$ (dotted curve) in the stationary states with $N_{1} = N_{2} = N/2$ for $N = 1000$ (a), 3000 (b), 5000 (c) and 10 000 (d).
This trend is natural since the larger the $N$, the higher the chemical potential $\mu_x$ and the larger the barrier penetrability $K$. Furthermore, the larger the $K$, the higher the $\omega_{JO}$, as it should be in Rabi-like oscillations. Altogether, we see that JO survive (with a higher frequency) even at SC and above-barrier transfer. The crossover region $N = 3000–5000$ is passed monotonically.

Note that the above JO evolution is also supported by the WC arguments [9], though application of these arguments needs a word of caution. In the TMA within WC for the interacting BEC [9], JO dynamics is driven by the interaction/coupling ratio $\Lambda = NU/K$. With increasing $N$, the interaction part $NU$ grows linearly, while the barrier penetrability $K$ rises exponentially. Altogether, $\Lambda$ falls with $N$ which, following TMA calculations [9], should lead to decreasing $\omega_{JO}$. Instead, our calculations demonstrate the opposite trend. The point is that our approach is a counterpart of the time-dependent Hartree–Fock method for boson systems, where the many-body problem for interacting bosons is reduced to one-body problem for motion of a boson in an effective one-body potential involving the impact of the interaction. So the WC arguments [9] should be used in the noninteracting limit where $\omega_{JO} \propto K$. Then we get the JO trend displayed in figure 2.

In figure 3, the evolution of MQST with $N$ is demonstrated. Compared with JO, this evolution is more complicated and needs more time to be exhibited. Hence, we use here the larger time interval 80 ms. Figure 3 shows that MQST is transformed with $N$ to JO from figure 2. Namely, at $N = 1000$, there is an ordinary MQST in agreement with the experiment [11, 12]. At $N = 3000–5000$, $z$-oscillations around $\langle z \rangle = 0.6$ are gradually reduced to slower oscillations around $\langle z \rangle = 0$ and, at $N = 10000$, basically converge to JO in figure 2(d). The linear evolution of $\theta$ is turned into JO-like oscillations around the average $\langle \theta \rangle = 2\pi$. Since the shift $2\pi$ is irrelevant, one actually obtains the JO in figure 2(h).

A similar MQST evolution was earlier obtained in the strong interaction limit for 1D BEC [28]. It was shown that for a sufficiently strong interaction, the amplitude of MQST $z$-oscillations starts to grow with $NU_{1D}$, thus leading to the reappearance of tunnelling between the two wells. This effect was explained, within the modified TMA, by coupling the second and third modes. Our model takes into account all the modes and thus should cover the result [28], but now for a 3D system. Actually, the transfer MQST $\rightarrow$ JO in our calculations exhibits a similar effect: increasing the amplitude of $z$-oscillations and thus the reappearance of the tunnelling.

The most remarkable result of this study is that, despite a significant overlap and strong coupling in the SC case, the left and right BECs in the trap still keep their individuality and accept the Josephson-like dynamics in terms of the relative variables, population imbalance $z$ and phase difference $\theta$. In particular, the JO dynamics obtained at a modest initial $z_0$ remains very regular and robust.
4. Summary

Dynamics of the repulsive Bose–Einstein condensates (BECs) in a double-well trap was inspected for the cases of weak, intermediate and strong coupling between the left and right BECs. Weak coupling is characterized by a small overlap of BECs and deep sub-barrier tunnelling of atoms. Strong coupling assumes a considerable overlap of BECs and above-barrier transfer of atoms. The coupling increase is induced by raising the BEC chemical potential, which in turn is caused by increasing the number of BEC atoms and thus enforcing the interaction effect. The numerical analysis is performed with the 3D time-dependent Gross–Pitaevskii equation. The two-mode and other ordinary approximations are avoided. The main dynamical regimes, the Josephson oscillations (JO) and macroscopic quantum self-trapping (MQST) are inspected. As compared to previous sporadic studies, we provide a systematic, complete and realistic analysis of JO/MQST at SC conditions. The exploration inspects both population imbalance and relative phase and implements a realistic initialization of the dynamics using the prescription of the 3D Heidelberg experiment [11, 12].

The main result of this study is a remarkable persistence of the Josephson-like dynamics at the strong coupling when the tunnelling is actually replaced by the above-barrier transfer. The calculations show that, under these conditions, the JO successfully survive but acquire a higher frequency. Instead, the MQST is destroyed and finally reduced to the same high-frequency JO mode. This mode remains robust even at a significant overlap of the left and right BECs in the trap. The right and left BECs thus continue to keep their individuality and the Josephson relative variables, population imbalance \( z \) and phase difference \( \theta \) still remain relevant.

We consider these results as significant since JO and MQST are basic dynamical regimes for BEC in a double-well trap and knowledge of their behaviour in various limit cases is indeed important. Earlier, JO/MQST were mainly studied for weak coupling but it was not clear how they would look in the opposite strong-coupling limit which is also often met in various problems. We have clarified this point. What is important is that our analysis was done in terms of a rather realistic model, closely following the conditions and prescriptions of actual experiments with 3D geometry.

It would be interesting to check the findings of this study in experiment, e.g., by means of BEC interferometry.

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