Criticality and surface tension in rotating horizon thermodynamics

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Abstract

We study a modified horizon thermodynamics and the associated criticality for rotating black hole spacetimes. Namely, we show that under a virtual displacement of the black hole horizon accompanied by an independent variation of the rotation parameter, the radial Einstein equation takes a form of a ‘cohomogeneity two’ horizon first law, \( \delta E = TS + \Omega \delta J - \sigma \delta A \), where \( E \) and \( J \) are the horizon energy (an analogue of the Misner–Sharp mass) and the horizon angular momentum, \( \Omega \) is the horizon angular velocity, \( A \) is the horizon area, and \( \sigma \) is the surface tension induced by the matter fields. For fixed angular momentum, the above equation simplifies and the more familiar (cohomogeneity one) horizon first law \( \delta E = T S - P \delta V \) is obtained, where \( P \) is the pressure of matter fields and \( V \) is the horizon volume. A universal equation of state is obtained in each case and the corresponding critical behavior is studied.

Keywords: black holes, thermodynamics, criticality

(Some figures may appear in colour only in the online journal)

1. Introduction

The thermodynamic interpretation of gravitational field equations has been a subject of intensive study for several decades. Amongst the various approaches, horizon thermodynamics \cite{1} provides a very simple and concrete manifestation of the idea that the Einstein equations can be rewritten as a thermodynamic identity \cite{2}. The original observation was that...
the radial Einstein equation can be rewritten as a first law of horizon thermodynamics for spherically symmetric black hole spacetimes. This was subsequently extended to more general situations [3–7].

For spherically symmetric black hole spacetimes, horizon thermodynamics essentially identifies the thermodynamic pressure $P$ with the $T'$, component of the energy–momentum tensor of matter fields. It is then easy to show that the radial Einstein equation, evaluated on the black hole horizon located at $r = r_+$, can be rewritten as an horizon equation of State (HES)

$$P = P(V, T),$$

which under an infinitesimal virtual displacement of the horizon becomes a (cohomogeneity one) horizon First Law (HFL)

$$\delta E = T\delta S - P\delta V,$$

where $T$ and $S$ are the horizon temperature and entropy identified through standard thermodynamic arguments. The quantity $E = r_+/2$ is an horizon energy (equal to the Misner–Sharp energy evaluated on the horizon), and $V$ is the geometric volume associated with the black hole,

$$P = T'_{|r=r_+}, \quad V = \frac{\Sigma_{d-2} r_+^{d-1}}{d - 1},$$

where for $d$ spacetime dimensions, $\Sigma_{d-2}$ denotes a finite volume of the $(d-2)$-dimensional ‘unit sphere’. We emphasize that both horizon equations (1) and (2) are universal, that is, independent of matter content and entirely fixed by the gravitational theory under consideration. (The actual matter dependence enters entirely through the pressure term $P$.)

Interestingly, similar equations have been recently studied for asymptotically AdS black holes in the context of extended phase space thermodynamics, e.g. [8–11], where one identifies the thermodynamic pressure $P_{\Lambda}$ with the (negative) cosmological constant $\Lambda$ (which is allowed to vary in the first law) and defines thermodynamic volume $V_{TD}$ as the quantity thermodynamically conjugate to $P_{\Lambda}$. For stationary black hole spacetimes in Einstein gravity with angular momentum $J$ and horizon angular velocity $\Omega$ that are coupled to a $U(1)$ charge $Q$ (with a corresponding chemical potential $\Phi$) this results in the following equation of state and the extended first law:

$$P_{\Lambda} = P_{\Lambda}(V_{TD}, T, Q, J), \quad \delta M = T\delta S + V_{TD}\delta P_{\Lambda} + \Phi\delta Q + \Omega\delta J,$$

where $M$ stands for the black hole mass, interpreted now as a gravitational enthalpy [8], and

$$P_{\Lambda} = -\frac{\Lambda}{8\pi}, \quad V_{TD} = \left(\frac{\partial M}{\partial P_{\Lambda}}\right)_{E, Q, J}.$$  \hspace{1cm} (4)

Contrary to (2), the extended first law (4) is typically of maximal cohomogeneity. The two approaches were, for spherically symmetric spacetimes, compared in [12, 13], where universality of the $P-V$ criticality of the horizon equation of state (1) was also demonstrated.

Surprisingly, despite its relative success for black holes with spherical symmetry, horizon thermodynamics has not really been fully extended to rotating black hole spacetimes. Indeed, only a few studies exist in this direction. It was shown in [5] for the Kerr–Newmann case and in [6] for the charged BTZ black hole that the radial Einstein equation can be rewritten as a ‘standard’ thermodynamic first law. We critically comment on the corresponding procedure, which explicitly uses the properties of a given solution, in appendix B. An implicit study of the rotating case is also contained in [7] where an HFL (2) is obtained for an arbitrary null
surface. However, in all studies of horizon thermodynamics so far the obtained first law is only of cohomogeneity one. The only admissible variation is due to the virtual displacement of the horizon, which clearly neglects additional features that are present due to rotation.

The aim of this paper is to formulate an extension of horizon thermodynamics to rotating black hole spacetimes that has the following ‘natural’ features: (i) it maintains the ‘universality’ regarding the matter content of the theory, that is, no particular properties of a given solution to the field equations are exploited and (ii) additional degrees of freedom associated with rotation are properly captured and allowed to vary in the HFL, leading naturally to a description of higher rank cohomogeneity.

As we shall see for a given ansatz for the rotating black hole geometry below, see equation (B1), the radial Einstein equation naturally results in the following modified HES and HFL:

\[
\sigma = \sigma (A, T, J),
\]

\[
\delta E = T \delta S + \Omega \delta J - \sigma \delta A,
\]

where \(E\) is the horizon energy analogous to the Misner–Sharp mass, \(J\) and \(\Omega\) are the horizon angular momentum and velocity, \(A\) is the horizon area, and \(\sigma\) is the surface tension induced by the matter fields.

Contrary to the first law (2), the modified HFL is a cohomogeneity two identity, as both the horizon radius \(r_h\) and the associated rotation parameter \(a\) are allowed to vary independently. However, although universal in the sense that all matter dependence is encapsulated in the surface tension \(\sigma\), the modified HFL depends crucially on the applicability of the given metric ansatz. (Only a limited number of solutions to Einstein-matter equations can be written in the form we consider.) This is in strong contrast to the spherically symmetric case, in which metric ansatz is general (within the symmetry requirements) and the horizon first law is completely universal, dependent only on the type of gravitational theory but not its matter content [13].

Finally, we show that in the case when an additional horizon structure—a black hole volume \(V\)—is independently identified, the HES (6) and the HFL (7) can be rewritten in a more familiar form

\[
P = P (V, T, J),
\]

\[
\delta E = T \delta S + \Omega \delta J - P \delta V,
\]

where pressure \(P\) depends on the matter content and is defined as a quantity thermodynamically conjugate to \(V\). However, to obtain these relations, either a very specific form of the volume has to be considered, or one has to restrict to a fixed angular momentum ensemble and accept the consequence that the HFL (9) is only a cohomogeneity one equation.

Our paper is organized as follows. In the next section we derive the modified HES and HFL (6) and (7), and study their criticality. In section 4 an additional structure of horizon volume is assumed and the horizon equations are rewritten in the more familiar (generically cohomogeneity one) form (8) and (9). Section 5 is devoted to discussion and conclusions. Appendix A describes an alternate derivation of horizon equations (6) and (7) and appendix B demonstrates that these equations also apply to black holes with cosmological constant. Finally, in appendix C we critically revise an argument showing the ‘equivalence’ of the

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4 In fact, the HES (6) and the HFL (7) are more general than their derivation employing the ansatz (B1) may suggest. We expect them to equally hold for asymptotically flat rotating black holes in higher dimensions, or, as demonstrated in appendix B, for black holes with a cosmological constant.
radial Einstein equation with the standard first law of black hole thermodynamics for the Kerr–Newmann black hole.

2. Rotating horizon thermodynamics

For concreteness and simplicity we work with the following ansatz for a rotating black hole geometry

$$\text{d}s^2 = -\frac{\Delta}{\rho^2}(\text{d}t - a \sin^2 \theta \text{d}\phi)^2 + \frac{\rho^2}{\Delta} \text{d}r^2 + \rho^2 \text{d}\theta^2 + \frac{\sin^2 \theta}{\rho^2} [\text{d}t - (r^2 + a^2) \text{d}\phi]^2,$$

(10)

generalizing the Kerr metric, where

$$\rho^2 = r^2 + a^2 \cos^2 \theta,$$

(11)

and we assume that the metric function $\Delta = \Delta(r)$ determines the position of the (non-extremal) black hole horizon located at the largest root of $\Delta(r_+) = 0$.

2.1. Modified horizon equations

We begin by deriving the modified HFL and HES (6) and (7), assuming Einstein gravity minimally coupled to matter. From the geometry we can immediately identify the black hole horizon area

$$A = 4\pi (r_+^2 + a^2) = 4S,$$

(12)

in terms of the entropy $S$. The horizon angular velocity

$$\Omega = -\frac{g_{t\phi}}{g_{\phi\phi}} \bigg|_{r_+} = \frac{a}{r_+^2 + a^2}$$

(13)

can likewise be identified, as can the black hole temperature

$$T = \frac{\Delta'(r_+)}{4\pi (r_+^2 + a^2)},$$

(14)

via standard Wick-rotation arguments. Note that no field equations are required up to this point, though the latter relation in (12) employs the assumption of Einstein gravity.

Let us next consider the radial Einstein equation, evaluated on the black hole horizon

$$8\pi T'_{r r_+} = G'_{r r_+} = \frac{a^2 - r_+^2 + r_+ \Delta'(r_+)}{\rho_+^2},$$

(15)

where $\rho_+^2 = r_+^2 + a^2 \cos^2 \theta$. Using (B7) we obtain

$$T = \frac{8\pi \rho_+^2 T'_{r r_+} + r_+^2 - a^2}{4\pi r_+ (r_+^2 + a^2)},$$

(16)
which yields
\[ T \delta S = \frac{2 \rho_0^2 T' \rho |_{\rho_c}}{r_c (r_c^2 + a^2)} \delta S + \frac{r_c^2 - a^2}{4 \pi r_c (r_c^2 + a^2)} \delta S \] (17)

upon multiplication by \( \delta S = 2 \pi (r_c \delta r_c + a \delta a) \). Note that the first term on the right-hand-side of (B9) depends on the matter content, whereas the second term is universal and completely fixed in terms of \( r_c \) and \( a \).

Now we make the following interesting observation. This latter term in (B9) can be written as
\[ \frac{r_c^2 - a^2}{4 \pi r_c (r_c^2 + a^2)} \delta S = \delta E - \Omega \delta J, \] (18)

upon defining
\[ E = \frac{r_c^2 + a^2}{2 r_c}, \quad J = E a, \] (19)

the former quantity being defined only up to a total variation. We see that the expressions for \( E \) and \( J \) are formally identical to those for the mass and angular momentum of vacuum Kerr black hole, respectively. Furthermore, in the absence of rotation, \( a \to 0 \), \( E \) reduces to the Misner–Sharp energy of a spherically symmetric spacetime evaluated on the black hole horizon. We therefore identify \( E \) as the horizon energy \( E \) and \( J \) as the horizon angular momentum of the black hole described by (B1).

We have thus found the following relation:
\[ \delta E = T \delta S + \Omega \delta J - \frac{2 \rho_0^2 T' \rho |_{\rho_c}}{r_c (r_c^2 + a^2)} \delta S. \] (20)

Since \( T \) must be constant on the horizon [14], the last equation is consistent only when \( \rho_0^2 T' \rho |_{\rho_c} \) is independent of coordinate \( \theta \). We therefore introduce the surface tension
\[ \sigma = \sigma (r_c, a) = \frac{\rho_0^2 T' \rho |_{\rho_c}}{2 r_c (r_c^2 + a^2)}, \] (21)

and so obtain (7) for the modified HFL
\[ \delta E = T \delta S + \Omega \delta J - \sigma \delta A. \] (22)

We pause to comment that a surface tension \( \sigma \) and a first law of the type \( \delta E = T \delta S - \sigma \delta A \) were first considered by York [15] in the context of the thermodynamics of black holes in a cavity. However there is a fundamental difference. York’s ‘surface tension’ is conjugate to the area of a cavity enclosing the ensemble and so the cavity area \( A \) and the black hole entropy \( S \) can vary independently. In our case we have no cavity. Instead the surface tension \( \sigma \) is conjugate to the area of black hole horizon itself and is entirely induced by the matter fields present in the spacetime. In particular, in vacuum we have \( \sigma = 0 \) and recover the standard 1st law of black hole thermodynamics
\[ \delta E = T \delta S + \Omega \delta J, \] (23)
whereas in the electrovacuum (Kerr–Newman) case, we have

\[ T'_{r_h} = -\frac{Q^2}{8\pi\rho} \Rightarrow \sigma = -\frac{Q^2}{16\pi r_+ (r_+^2 + a^2)}. \]

(24)

The HFL (22) is cohomogeneity two as both the horizon radius \( r_+ \) and the rotation parameter \( a \) can vary independently. Moreover, equation (16) together with (21) yields

\[ \sigma = \sigma(A, J, T) = \frac{T}{4} + \frac{a^2 - r_+^2}{16\pi r_+ (r_+^2 + a^2)}, \]

(25)

which is the surface tension HES (6). Here \( r_+ \) and \( a \) are implicitly given in terms of \( J \) and \( A \) through relations (19) and (12).

Equations (22) and (25) are together with the definition of the surface tension (21) the most important results of this section. Note that in order to write these equations down, no new quantities, apart from \( E \) and \( J \), had to be defined and the expressions are entirely given in terms of geometric horizon properties such as the area \( A \), temperature \( T \), and angular velocity \( \Omega \).

### 2.2. Surface tension criticality

Let us now study the possible critical behavior associated with the generalized horizon thermodynamics derived in the previous subsection. For concreteness, we do this in a canonical (fixed \( J \)) ensemble.

Since according to the HFL (22), the quantity \( E \) in (19) plays the role of thermodynamic energy (that is a thermodynamic potential expressed in terms of extensive thermodynamic
variables $S$, $J$ and $A$), we define

$$G_\sigma = G_\sigma (T, \sigma, J) = E - TS + \sigma A,$$

(26)

which is the corresponding surface tension Gibbs free energy $G_\sigma$. This quantity formally satisfies

$$\delta G = -S\delta T + \Omega \delta J + A\delta \sigma.$$

(27)

The behavior of $G = G(T, \sigma, J)$ is displayed in figure 1 for fixed $J = 1$ and three representative values of $\sigma$. For any $\sigma$ we observe two branches of black holes, meeting at a characteristic cusp. For negative $\sigma$ both branches on the other end terminate at finite $G$ and $T = 0$, whereas for positive $\sigma$ the upper branch eventually asymptotes to $G \to \infty$ at $T = 4\sigma$, with a divergence at $T = 0$ occurring for $\sigma = 0$. Apart from the presence of a cusp, no interesting thermodynamic behavior is observed for any values of $J$.

As with the spherically symmetric case \[13\], an interpretation of the concrete thermodynamic behavior depends on the actual matter content. For example, in vacuum, $\sigma = 0$ and only the black dashed curve applies. Similarly, for the electrovacuum case with nontrivial charge $\sigma < 0$ and behavior similar to the thin black curve in figure 1 is realized. We expect that our ansatz could be suitably generalized to accommodate rotating black hole with some type of a scalar hair \[16\], with free energy plots similar to the positive $\sigma$ curve.

2.3. Effective temperature

The modified HFL (22) has three terms on its right-hand-side but inherently is only cohomogeneity two. Furthermore, variation of $S$ is not independent of the variation of $A$. This suggests that we introduce an effective temperature

$$T_{\text{eff}} = T - 4\sigma = \frac{1}{4\pi r_s} \frac{r_s^2 - a^2}{r_s(r_s^2 + a^2)},$$

(28)

which is easily obtained by grouping the $T\delta S$ and $-\sigma\delta A$ terms together. Note that this quantity has no explicit dependence on matter, is constant on the horizon, and is positive for $r_s > a$. With this identification, the modified HFL (22) becomes manifestly of cohomogeneity two and reads

$$\delta E = T_{\text{eff}} \delta S + \Omega \delta J,$$

(29)

which is equivalent to equation (18). In fact, since $E$ and $J$ coincide with the mass and angular momentum of the vacuum Kerr black hole, the effective temperature $T_{\text{eff}}$ is nothing other than the temperature of the Kerr solution and (29) is the corresponding first law.

Stated this way, horizon thermodynamics is recast in universal form that is completely independent of the matter content and represented by the thermodynamics of a vacuum solution. Note that the same is true in the case of spherical symmetry upon absorbing the $-PdV$ term into $TdS$ in (2), which then simply reads $\delta E = T_{\text{eff}} \delta S$, with $T_{\text{eff}}$ being the temperature of the Schwarzschild black hole. This interpretation of horizon thermodynamics also opens a new way of deriving the horizon equations (22) and (25), as we demonstrate in appendix A.

In the light of previous discussion, it is obvious that the criticality of the HFL (29) coincides with that of the Kerr solution. Namely, the associated Gibbs free energy reads
The corresponding graph is displayed in figure 2. For non-trivial angular momentum $J$, we observe a characteristic cusp, completely independent of the matter content of the theory.

To summarize this section, we stress that both the surface tension and the effective temperature approaches are very natural in the horizon thermodynamics of rotating black holes. Both permit study of cohomogeneity two HFLs since variations of both $\delta a$ and $\delta r_+$ are allowed. Furthermore, there is no need to identify any extra structure beyond the horizon energy $E$ and angular momentum $J$ in (19). We shall now consider an alternate approach in which an additional structure, the black hole volume $V$, is defined.

3. $P-V$ criticality

We now consider the implications of rewriting the last term in (22) as a pressure-volume term

$$\sigma \delta A = P \delta V,$$

an equality that is only possible when $\delta V \propto \delta A$. Specifically we shall consider two approaches. Motivated by scaling properties, we consider $V \propto A^{3/2}$, which yields cohomogeneity two HFL in which $a$ and $r_+$ can be independently varied. We then consider the alternate possibility in which

Figure 2. Universal criticality. The $G_{\text{Tr}} - T_{\text{eff}}$ phase diagram is displayed for $J = 1$. We observe a characteristic cusp that is completely independent of the matter content of the theory.

$$G_{\text{Tr}} = G_{\text{Tr}} (T_{\text{eff}}, J) = E - T_{\text{eff}} S = \frac{r_+^2 + 3a^2}{4r_+},$$

and obeys

$$\delta G_{\text{Tr}} = -S \delta T_{\text{eff}} + \Omega \delta J.$$
while \( V \) is ‘independently specified’ by other criteria, e.g. identified with the geometric/thermodynamic volume of the black hole.

### 3.1. Volume as power of area

To keep the HFL of cohomogeneity two we impose

\[
V = \frac{4}{3\pi} \left( \frac{A}{4\pi} \right)^{3/2},
\]

where the constant of proportionality has been chosen to yield \( V = \frac{4}{3\pi} r_+^3 \) in the limit of zero rotation [13]. This then yields

\[
P = P(V, T, J), \tag{35}
\]

\[
\delta E = T\delta S + \Omega \delta J - P \delta V \tag{36}
\]

for the HES and HFL, respectively. The associated Gibbs free energy is defined through

\[
G_P = E - TS + PV, \tag{37}
\]

It obeys

\[
\delta G_P = -S\delta T + \Omega \delta J + V \delta P, \tag{38}
\]

and encodes information about possible thermodynamic phase transitions.

Specifically, the identification (32) implies

\[
P = \frac{\sigma A}{2\pi} \left( \frac{4\pi}{A} \right)^{3/2}, \tag{39}
\]

for the pressure. Upon using (25), the HES (35) then becomes

\[
P = \frac{T}{2r_+^2} \frac{1}{\sqrt{r_+^2 + a^2}} + \frac{a^2 - r_+^2}{8\pi r_+(r_+^2 + a^2)^{1/2}}, \tag{40}
\]

where \( r_+ \) and \( a \) are implicit functions of \( V \) and \( J \), according to (34) and (19).

Note that in the limit \( a \to 0 \) we recover that \( P = T' \rceil_{r_+} \), as identified in the spherically symmetric scenario [13], and the HES reduces to the HES for spherical black holes in Einstein gravity

\[
P = \frac{T}{2r_+} - \frac{1}{8\pi r_+^2}. \tag{41}
\]

(The same will be true for the HESs derived in the next subsection.)

The corresponding \( G-T \) diagram is displayed in figure 3. We observe that for positive \( P < P_c \), there is a characteristic swallow tail indicating the presence of a Van der Waals-like phase transition, e.g. [11].

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5 Note that such a ‘power law rewriting’ is similar in spirit to what has recently been discussed in [17] in the context of extended phase space thermodynamics.
3.2. Fixed angular momentum

Let us now study the situation when variations $\delta a$ and $\delta r_+$ are not independent. Specifically, let us consider the canonical ensemble where the horizon angular momentum is held fixed, $\delta J = 0$. This implies that $a = a(r_+)$ and

$$\delta a = \frac{a(a^2 - r_+^2)}{r_+(r_+^2 + 3a^2)}\delta r_+. \quad (42)$$

Upon using (32), the HES and the HFL take the form

$$P = P(V, T), \quad (43)$$

$$\delta E = T\delta S - P\delta V. \quad (44)$$

The novel feature in these expressions is that the black hole volume is a new thermodynamic quantity that is independently specified by some other criteria. In what follows we consider two examples: (i) the geometric volume of the Kerr black hole and (ii) the thermodynamic volume of the Kerr black hole, e.g. [18].

Considering first the geometric volume

$$V = \frac{r_+ A}{3}. \quad (45)$$

Figure 3. $P-V$ criticality: $V = \frac{4}{3}\pi[\Lambda/(4\pi)]^{3/2}$ case. The $G_p-T$ diagram is displayed for $J = 1$ and fixed pressure. Namely, the red curve corresponds to positive pressure $P = 0.007$, the thick black curve to $P = 0.003$ and shows the characteristic swallow tail, the dashed black curve to $P = 0$, and the thin black curve to negative $P = -0.003$.

Note that it is a standard practice in horizon thermodynamics that only one thermodynamic parameter is allowed to vary while other parameters are held constant. Namely, one is allowed to ‘virtually displace the horizon’, changing $r_+ [1, 3-7]$. In this subsection we essentially return back to a single parameter variation. However, contrary to the ‘standard’ practice, we consider ‘induced’ variation, $\delta a = \delta a(r_+)$, as required by the canonical ensemble.
Equation (32) yields

\[ P = \frac{6\sigma (r_+^2 + a^2)}{r_+ (3r_+^2 + 5a^2)} \]

which upon using (25) becomes

\[ P = \frac{3}{2} \frac{T (r_+^2 + a^2)}{r_+ (3r_+^2 + 5a^2)} + \frac{3}{8\pi} \frac{a^2 - r_+^2}{r_+^2 (3r_+^2 + 5a^2)}, \]

where \( a \) and \( r_+ \) are implicit functions of \( V \) and \( J \) through (45) and (19). The corresponding behavior of \( G = E - TS + PV = G(P, T) \), for various values of fixed \( P \) is illustrated in figure 4 and is qualitatively similar to behavior displayed in figure 3.

Another choice, motivated by extended phase space thermodynamics [11] is to set \( V \) equal to the thermodynamic volume of Kerr black hole

\[ V = \frac{r_+ A}{3} \left( 1 + \frac{a^2}{2r_+^2} \right) \]

in which case we recover

\[ P = \frac{12\sigma r_+ (r_+^2 + a^2)}{6r_+^4 + 9r_+^2 a^2 + a^4}, \]

or

\[ P = \frac{3Tr_+ (r_+^2 + a^2)}{6r_+^4 + 9r_+^2 a^2 + a^4} + \frac{3}{4\pi} \frac{a^2 - r_+^2}{6r_+^4 + 9r_+^2 a^2 + a^4}, \]

upon using (25). The corresponding \( G = G(P, T) \) behavior is qualitatively similar to that displayed in figure 4.

4. Summary

We have extended horizon thermodynamics from its traditional spherically symmetric ansatz [1] to rotating black hole spacetimes. The horizon area, temperature, and horizon angular velocity of the black hole are all straightforwardly identified. However to rewrite the Einstein equations for the axially symmetric metric ansatz (B1) as a first law of horizon thermodynamics it is necessary to make identifications for the angular momentum \( J \) and energy \( E \). These can be well motivated from the radial Einstein equation, but they are not uniquely defined (see however the discussion in appendix A). We expect that provided these quantities are appropriately re-defined, the same HFL (7) will hold for other rotating black holes, not necessarily described by the ansatz (B1), including black holes in higher dimensions or those with cosmological constant (see appendix B).

Our approach goes beyond traditional horizon thermodynamics, where one varies only the horizon radius. Our construction allows for both the horizon radius and the angular momentum to vary independently of one another, thus allowing for a fuller spectrum of thermodynamic possibilities. This involved identifying a ‘surface tension’ \( \sigma \), writing the work term as \( \delta W = -\sigma \delta A \). This is a provocative identification, as the term has the correct units and is dependent on matter content. However, any further reason for making such an identification remains elusive at the moment and will be a subject of further study.
We have also shown that horizon thermodynamics naturally identifies an effective temperature \( T_{\text{eff}} = T + T_m \), where \( T_m = -4\sigma \). This relationship follows from recognizing that the ‘work’ term \( \delta W = -\sigma \delta A = T_m \delta S \). In other words Hawking temperature \( T \) of a solution is the difference between the vacuum Hawking temperature \( T_{\text{eff}} \) and the matter field contribution \( T_m \). Making this identification yields a universal first law that is independent of matter content.

While it is tempting to see if we can eliminate the notion of surface tension in terms of pressure by introducing the notion of volume, we have found that this is somewhat problematic. As we demonstrated in section 3, this entails either a very specific choice of the black hole volume or restricting to the canonical ensemble, yielding a cohomogeneity-one first law. In the latter case the volume can be freely specified by other criteria and is not restricted by horizon thermodynamics. This stands in stark contrast to the extended phase space approach, in which volume is uniquely defined as the conjugate of thermodynamic pressure \([8–11]\), the latter being proportional to a cosmological constant.

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Appendix A. Alternate derivation of horizon equations

Although precisely in the spirit of horizon thermodynamics [1] generalized to the rotating case, the derivation of the horizon equations (22) and (25) in the main text suffers from non-uniqueness of the definition of horizon energy \( E \) and horizon angular momentum \( J \), (19). Although their definition is motivated by (18), the possibility of redefining \( E \) by a total derivative (accompanied by a proper modification of \( J \)) remains. For this reason in this appendix we give an alternate derivation of these equations, turning around the logic of the reasoning. Namely, we start again with the ansatz (B1) but consider the vacuum solution first\(^7\). This allows us to identify \( E \) and \( J \). We then carry the analysis to the non-vacuum case, keeping the same \( E \) and \( J \) to rederive equations (22) and (25) in a different fashion.

Let us start again with the ansatz (B1) and specify to the vacuum Kerr case, setting \( \Delta = r^2 - 2mr + a^2 \). The thermodynamic quantities then read

\[
E = m = \frac{r_+^2 + a^2}{2r_+}, \quad J = Ea, \\
\Omega = \frac{a}{a^2 + r_+^2}, \quad S = \frac{A}{4} = \pi (r_+^2 + a^2), \\
T_0 = T_{\text{eff}} = \frac{r_+^2 - a^2}{4\pi r_+(r_+^2 + a^2)}, \tag{A1}
\]

and obey the standard first law

\[
\delta E = T_0 \delta S + \Omega \delta J, \tag{A2}
\]

which is of course identical to the effective first law (29).

We next consider the spacetime with matter, keeping the same ansatz (B1) and general \( \Delta = \Delta(r) \) that determines the position of the horizon. The derivation of the horizon equations (22) and (25) then consists of the following four steps:

- We insist that even in the presence of matter the horizon energy \( E \) and the horizon angular momentum \( J \) are given by the vacuum expressions (A1). (This in some sense directly generalizes the idea of Misner–Sharp quantities to the case with rotation.)
- We employ the Euclidean trick to identify the actual temperature of the black hole horizon according to

\[
T = \frac{\Delta(r_+)}{4\pi (r_+^2 + a^2)}. \tag{A3}
\]

- We impose the radial Einstein equation evaluated on the horizon, to relate \( T \) and \( T_0 \)

\[
8\pi T^r_{\ r\ r} = G^r_{\ r\ r} = \frac{a^2 - r_+^2 + r_+ \Delta(r_+)}{\rho_+^3}, \tag{A4}
\]

which rewrites, upon using (A1), as

\[
T = T_0 + 4\sigma, \quad \sigma = \frac{8\pi \rho_+^3 T^r_{\ r\ r= r_+}}{4\pi r_+(r_+^2 + a^2)}. \tag{A5}
\]

\(^7\) This goes directly against the spirit of horizon thermodynamics that essentially tries to avoid working with concrete solutions of field equations.
So we identified the matter contribution to the temperature called surface tension \( \sigma \) in the main text, and recovered the HES (25).

- The final step is to rewrite the standard first law (A2) in terms of the actual temperature in the presence of matter

\[
\delta E = T_0 \delta S + \Omega H_0 \delta J = T \delta S + \Omega \delta J - \sigma \delta A, \tag{A6}
\]

which is the HFL (22).

We believe that this derivation in some sense reveals the true nature of horizon thermodynamics. It describes the standard vacuum first law from the perspective of an observer who measures the actual black hole temperature \( T \) and the surface tension \( \sigma \) associated with matter fields present in the spacetime. This is the origin of universality of horizon thermodynamics: all black holes satisfy ‘an equivalence class’ of first laws (A2) irrespective of the matter content of the theory. Specific features of a given black hole emerge only after the actual matter content and associated conserved charges are identified, along with their respective contributions to the first law.

Of course, exactly the same derivation would apply to the spherically symmetric case.

### Appendix B. Asymptotically AdS rotating horizon thermodynamics

Let us now show that we recover the same HFL (7) also in the asymptotically AdS case, showing hence that the result is more general than ‘its derivation’ through the ansatz (B1).

To this end we consider a more general ansatz for an asymptotically AdS rotating black hole geometry given by

\[
\begin{aligned}
\text{ds}^2 &= -\frac{\Delta}{\rho^2}(dt - a \sin^2 \theta \frac{d\phi}{\Xi})^2 + \frac{\rho^2}{\Delta} dr^2 \\
&\quad + \rho^2 d\theta^2 + \frac{\Sigma \sin^2 \theta}{\rho^2} \left[ a dt - (r^2 + a^2) \frac{d\phi}{\Xi} \right] ^2, \\
\end{aligned}
\tag{B1}
\]

generalizing the Kerr-AdS metric, where

\[
S = 1 - \frac{a^2}{l^2} \cos^2 \theta, \quad \Xi = 1 - \frac{a^2}{l^2},
\]

\[
\rho^2 = r^2 + a^2 \cos^2 \theta, \tag{B2}
\]

and we assume that the metric function \( \Delta = \Delta(r) \) determines the position of the (non-extremal) black hole horizon located at the largest root of \( \Delta(r_+) = 0 \). To ensure the AdS asymptotics we further assume the following large-\( r \) expansion:

\[
\Delta = \frac{r^4}{l^4} + \alpha(r^4). \tag{B3}
\]

We now follow the same procedure as for the asymptotically flat case. We identify the entropy

\[
S = \frac{A}{4} = \pi \frac{r_+^2 + a^2}{\Xi}, \tag{B4}
\]
and the horizon angular velocity
\[ \Omega_H = \left. \frac{g_{\phi\phi}}{g_{\phi r}} \right|_{r_+} = \frac{a \Xi}{r_+^2 + a^2}. \] (B5)

Since the solution is written in rotating coordinates, one has to subtract the rotation at infinity. Using (B3), we find \( \Omega_\infty = -a/l^2 \), and hence the angular velocity that enters the first law is
\[ \Omega = \Omega_H - \Omega_\infty = \frac{a (r_+^2 + l^2)}{l^2(r_+^2 + a^2)}. \] (B6)

We also identify the black hole temperature
\[ T = \frac{\Delta'(r_+)}{4\pi(r_+^2 + a^2)}. \] (B7)

Employing the radial Einstein equation, evaluated on the black hole horizon, we have
\[ 8\pi T_r^r |_{r_+} = \left(G_{rr} - \frac{3}{l^2} \delta_r^r \right) |_{r_+} = \frac{1}{r_+^4} \left[ a^2 - r_+^2 + r_+ \Delta' \left( r_+ \right) - \frac{3r_+^4}{l^2} - \frac{r_+^2 a^2}{l^2} \right], \]
where \( \rho_+^2 = r_+^2 + a^2 \cos^2 \theta \). Using (B7) we obtain
\[ T = \frac{8\pi \rho_+^4 T_r^r |_{r_+} + r_+^2 - a^2 + 3r_+^4/l^2 + r_+^2 a^2/l^2}{4\pi r_+ (r_+^2 + a^2)}, \] (B8)

which yields
\[ T \delta S = \frac{2 \rho_+^4 T_r^r |_{r_+} \delta S + \frac{r_+^2 - a^2 + 3r_+^4/l^2 + r_+^2 a^2/l^2}{4\pi r_+ (r_+^2 + a^2)} \delta S}{r_+ (r_+^2 + a^2)} \] (B9)

upon multiplying by \( \delta S \). The final step is to realize that the latter term can be re-written as \( \delta E - \Omega \delta J \), in terms of ‘Kerr-AdS’ quantities
\[ E = \frac{(r_+^2 + a^2)(l^2 + r_+^2)}{2l^2 r_+ \Xi^2}, \quad J = aE. \] (B10)

Hence we recover the HFL
\[ \delta E = T \delta S + \Omega \delta J - \sigma \delta A, \] (B11)
where, as before
\[ \sigma = \sigma(r_+, a) = \frac{\rho_+^4 T_r^r |_{r_+}}{2r_+ (r_+^2 + a^2)}. \] (B12)

Equation (B8) gives a modified HES which now reads
\[ \sigma = \sigma(A, J, T) = \frac{T}{4} + \frac{a^2 - r_+^2 - 3r_+^4/l^2 - r_+^2 a^2/l^2}{16\pi r_+ (r_+^2 + a^2)}. \] (B13)

Here \( r_+ \) and \( a \) are implicitly given in terms of \( J \) and \( A \) through relations (B10) and (B4).

Furthermore, in the asymptotically AdS case one can extend HFL (B11) to include variations of the cosmological constant, obtaining so a cohomogeneity-3 relation. Namely,
upon identifying the cosmological pressure as
\[ P_\Lambda = -\frac{\Lambda}{8\pi} = \frac{3}{8\pi l^2}, \]  
and varying \( l \) in all the above expressions, we recover an extended HFL
\[ \delta E = T\delta S + \Omega \delta J - \sigma \delta A + \mathcal{V} \delta P_\Lambda, \]  
where
\[ \mathcal{V} = \frac{r_+ A}{3} \left( 1 + \frac{1 + r_+^2/l^2}{2r_+^2} \frac{a^2}{\Xi} \right) \]  
is the thermodynamic volume of Kerr-AdS black hole, e.g. [11]. An alternative derivation of
this expression ala previous appendix is of course also possible.

We finally note that the above derivation can be repeated for asymptotically de Sitter
spacetimes by everywhere reversing the sign of \( l \). This will yield an HFL for the de Sitter
black hole horizon. An analogous HFL for the cosmological horizon remains to be
understood.

**Appendix C. Tautological argument for recovering the standard first law**

It has been claimed in [5] that for the Kerr–Newmann solution the radial Einstein equation is
equivalent to the standard first law of black hole thermodynamics. In this appendix we briefly
review this argument and comment on its tautological character.

The argument in [5] approximately goes as follows. (In fact we describe a slightly
generalized argument where variations of rotation parameter \( a \) are allowed.) One starts again
with the ansatz (B1), and identifies \( T, S, \) and \( \Omega \) according to (B7), (12), and (B5). The radial
Einstein equation then implies equation (B9)
\[ T\delta S = \left( \frac{2\rho^i T^i r_+}{r_+(r_+^2 + a^2)} + \frac{r_+^2 - a^2}{4\pi r_+(r_+^2 + a^2)} \right) \delta S. \]  
Next, the following specific properties of the Kerr–Newmann solution are used:
\[ a = J/M, \quad \Delta = r^2 + a^2 - 2mr + e^2, \]  
where the latter is used to write
\[ r_+ = M + \sqrt{M^2 - (J/M)^2 - Q^2}, \]  
upon identifying \( M = m \) and \( Q = e \). Differentiating these relations we get \( \delta r_+ \) and \( \delta a \) in terms
of variations of \( \delta M, \delta J, \) and \( \delta Q \). Inserting the additional expression
\[ T^r_{\mid r=r_+} = -\frac{Q^2}{8\pi \rho^i} \]  
(valid for the Kerr–Newman solution) into (C1), one can easily verify from these relations that
\[ TdS = dM - \Omega dJ - \Phi dQ, \]  
where \( \Phi = \frac{e}{a^2 + r^2} \). This concludes the proof in [5] showing that the radial Einstein equation
‘implies’ the first law of black hole thermodynamics.
However, as obvious from this derivation, one needs to identify the correct thermo-
dynamic charges
\[ M = m, \quad J = Ma, \quad Q = e, \]
in order to write (C3). Once this is known, together with identification of \( T \) and \( S \), one does not need to invoke the Einstein equation to write the first law (C5). This is simply given as a ‘unique’ linear combination of differentials of these charges. In other words, in the above argument the radial Einstein equation is not truly needed to write the first law.

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