UPPER AND LOWER BOUNDS FOR THE ITERATES OF ORDER-PRESERVING HOMOGENEOUS MAPS ON CONES

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Abstract. We define upper bound and lower bounds for order-preserving homogeneous of degree one maps on a proper closed cone in \( \mathbb{R}^n \) in terms of the cone spectral radius. We also define weak upper and lower bounds for these maps. For a proper closed cone \( C \subset \mathbb{R}^n \), we prove that any order-preserving homogeneous of degree one map \( f : \text{int} C \to \text{int} C \) has a lower bound. If \( C \) is polyhedral, we prove that the map \( f \) has a weak upper bound. We give examples of weak upper bounds for certain order-preserving homogeneous of degree one maps defined on the interior of \( \mathbb{R}^n_+ \).

1. Introduction

A closed cone \( C \subset \mathbb{R}^n \) is a closed convex set such that (i) \( C \cap (-C) = \{0\} \) and (ii) \( \lambda C = C \) for all \( \lambda \geq 0 \). If \( C \) has nonempty interior, we say that \( C \) is a proper closed cone. For any proper closed cone, the dual cone \( C^* = \{ x \in \mathbb{R}^n : \langle x, y \rangle \geq 0 \text{ } \forall y \in C \} \) is a proper closed cone.

Any closed cone \( C \) defines a partial ordering on \( \mathbb{R}^n \) by \( x \leq_C y \) if and only if \( y - x \in C \). When the cone \( C \) is understood, we will write \( \leq \) instead of \( \leq_C \). Let \( D \subset \mathbb{R}^n \) be a domain in \( \mathbb{R}^n \). A map \( f : D \to \mathbb{R}^n \) is said to be order-preserving if and only if \( f(x) \leq f(y) \) whenever \( x \leq y \). It is called order-reversing if \( f(x) \geq f(y) \) whenever \( x \leq y \). We say that \( f : D \to \mathbb{R}^n \) is homogeneous of degree \( \alpha \) if \( f(\lambda x) = \lambda^\alpha f(x) \) for all \( \lambda > 0 \) and \( x \in D \). Order-preserving homogeneous of degree one maps from a cone into itself have been extensively studied (see e.g., \cite{6}). They are a natural extension of the nonnegative matrices, and there are many examples of such maps in applications \cite{9}. Many important properties of nonnegative matrices generalize to order-preserving homogeneous of degree one maps on cones.

Let \( C \) be a proper closed cone and let \( f : C \to C \) be a continuous order-preserving homogeneous of degree one map. We define the cone spectral radius of \( f \) to be

\[
\rho_C = \rho_C(f) = \lim_{k \to \infty} \|f^k(x)\|^{1/k},
\]

for any \( x \in \text{int} C \). The value of \( \rho_C \) is independent of \( x \) \cite[Proposition 5.3.6]{6}. Once again, when the cone \( C \) is understood, we will write \( \rho \) instead of \( \rho_C \). The well known Krein-Rutman theorem \cite[Corollary 5.4.2]{6} asserts that any continuous order-preserving homogeneous of degree one map \( f : C \to C \) has an eigenvector \( x \in C \) such that \( f(x) = \rho x \). Note that any such eigenvector will serve as a lower bound on the iterates of \( f \) in the following sense. If \( x \leq y \), then \( f^k(x) \leq f^k(y) \) for all \( k \in \mathbb{N} \) because \( f \) is order-preserving. Thus \( f^k(y) \geq \rho^k x \) for all \( k \).

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Suppose now that homogeneous of degree one maps that are defined only on the interior of the cone, the entire closed cone. There are several examples of important order-preserving homogeneous of degree one map in a finite dimensional space, it is normal \([6, \text{Lemma 1.2.5}]\), so there exists a constant \(c > 0\) such that \(y \geq x\) implies \(||y|| \geq c||x||\). In particular \(||f^k(x)|| \geq c||x||\), so \(\rho(f) \geq c > 0\).

We say that \(y \in C\setminus\{0\}\) is a lower bound for \(f\) if for all \(x \in \text{int} C\), \(x \geq y\) implies that \(f(x) \geq \rho y\). Note that any eigenvector of \(f\) is a lower bound. Unfortunately, the Krein-Rutman theorem does not apply in general if \(f\) is only defined on the interior of the cone. Without an eigenvector corresponding to the spectral radius, it is not clear that a lower bound must exist. We address this question in the next section, but for now, let us introduce a weaker notion. We say that \(w \in C^*\) is a weak lower bound for \(f\) if there exists \(x \in \text{int} C\) such that \(\langle f^k(x), w \rangle \geq \rho^k \langle x, w \rangle\) for all \(k \geq 0\).

For \(f : \text{int} C \to \text{int} C\) order-preserving and homogeneous of degree one we say that \(y \in \text{int} C\) is an upper bound if \(x \leq y\) implies that \(f(x) \leq \rho y\). Unlike lower bounds, we do not allow upper bounds on the boundary of \(C\). After all, if \(y\) is contained in the boundary of \(C\), then there is no \(x \in \text{int} C\) such that \(x \leq y\) so it does not make sense to refer to \(y\) as an upper bound. We say that \(w \in C^*\) is a weak upper bound for \(f\) if there exists \(x \in \text{int} C\) such that \(\langle f^k(x), w \rangle \leq \rho^k \langle x, w \rangle\). A weak upper or lower bound is uniform in the following sense.

**Lemma 1.** Let \(C\) be a proper closed cone in \(\mathbb{R}^n\) and \(f : \text{int} C \to \text{int} C\) be order-preserving and homogeneous of degree one. If \(w\) is a weak lower (upper) bound for \(f\) and \(y \in \text{int} C\), then there exists a constant \(c > 0\) such that \(\langle f^k(y), w \rangle \geq c \rho^k\) (respectively, \(\langle f^k(y), w \rangle \leq c \rho^k\)).

**Proof.** If \(w\) is a weak lower bound for \(f\), then there exists \(x \in \text{int} C\) such that \(\langle f^k(x), w \rangle \geq \rho^k \langle x, w \rangle\). Since both \(x, y \in \text{int} C\), there exists \(\alpha > 0\) such that \(\alpha x \leq y\).

Applying the map \(f^k\),

\[
\alpha f^k(x) \leq f^k(y)
\]

\[
\alpha \langle f^k(x), w \rangle \leq \langle f^k(y), w \rangle
\]

Therefore

\[
\alpha \langle x, w \rangle \leq \alpha \langle f^k(x), w \rangle \leq \langle f^k(y), w \rangle \leq \beta \langle f^k(x), w \rangle
\]

Letting \(c = \alpha \langle x, w \rangle\) completes the proof if \(w\) is a weak lower bound. The proof for weak upper bounds is essentially the same.

As the following lemma shows, the notion of a weak upper or lower bound is indeed weaker than an upper or lower bound.

**Lemma 2.** Let \(C\) be a proper closed cone in \(\mathbb{R}^n\) and \(f : \text{int} C \to \text{int} C\) be order-preserving and homogeneous of degree one. If \(f\) has a lower (upper) bound, then there is a weak lower (weak upper) bound on the iterates of \(f\).

**Proof.** Suppose that \(y\) is a lower bound for \(f\). Then for every \(x \in \text{int} C\), there is a maximal \(\lambda > 0\) such that \(x \geq \lambda y\) and it is clear that \(\lambda y\) is also a lower bound. Thus \(x - \lambda y \in \partial C\) where \(\partial C\) denotes the boundary of \(C\). We may choose \(w \in C^*\setminus\{0\}\) such
that \( \langle x - \lambda y, w \rangle = 0 \) and \( \langle x, w \rangle = \langle \lambda y, w \rangle \). Since \( \lambda \) is a lower bound, \( f^k(x) \geq \rho^k \lambda y \) for all \( k \in \mathbb{N} \). Therefore \( \langle f^k(x), w \rangle \geq \rho^k \langle \lambda y, w \rangle = \rho^k \langle x, w \rangle \). The proof for upper bounds is essentially the same. \( \square \)

We will prove that for any proper closed cone \( C \) and any order-preserving homogeneous of degree one map \( f : \text{int} C \to \text{int} C \), the map \( f \) has a lower bound. For order-preserving homogeneous of degree one map on the standard cone \( \mathbb{R}^n_+ = \{ x \in \mathbb{R}^n : x_i \geq 0 \text{ for all } i \in \{1, ..., n\} \} \), we show in section 3 that there is a formal eigenvector that is almost an upper bound for the map. In particular this will establish a weak upper bound for the iterates of any such map on the standard cone. We then extend this result to show that on any polyhedral cone there is always a weak upper bound for the iterates of any order-preserving homogeneous of degree one map defined on the interior.

2. Lower Bounds

Let \( C \) be a proper cone in \( \mathbb{R}^n \) and let \( f : \text{int} C \to \text{int} C \) be order-preserving and homogeneous of degree one. It is known [4, Theorem 2.10] that if \( C \) is a polyhedral cone, then \( f \) has a continuous extension to \( C \) that is order-preserving and homogeneous of degree one. By the Krein-Rutman theorem this extension has an eigenvector \( y \in C \backslash \{0\} \) with eigenvalue equal to the cone spectral radius \( \rho(f) \). This proves that order-preserving, homogeneous of degree one self-maps of the interior of a closed polyhedral cone must have a lower bound.

When the cone \( C \) is not polyhedral, the map \( f \) might not extend continuously to the boundary of \( C \). In this case, however, there must still be a lower bound.

**Theorem 1.** Let \( C \) be any proper closed cone in \( \mathbb{R}^n \) and suppose \( f : \text{int} C \to \text{int} C \) is order-preserving and homogeneous of degree one with cone spectral radius \( \rho(f) \) and suppose there exists \( y \in C \backslash \{0\} \) such that for any \( x \in \text{int} C \) with \( x \geq y \), \( f^k(x) \geq \rho^k y \) for all \( k \in \mathbb{N} \).

**Proof.** Fix \( v \in \text{int} C^* \) and \( x_0 \in \text{int} C \) and let \( f_\epsilon(x) = f(x) + \epsilon \langle x, v \rangle x_0 \) for \( \epsilon > 0 \). By [4, Theorem 5.4.1], each map \( f_\epsilon \) has an eigenvector \( y_\epsilon \in C \) with \( \|y_\epsilon\| = 1 \) and \( f_\epsilon(y_\epsilon) = \rho_\epsilon y_\epsilon \), where \( \rho_\epsilon \) is the cone spectral radius of \( f_\epsilon \). Furthermore, \( \lim_{\epsilon \to 0} \rho_\epsilon = r \) exists. Since \( \rho_\epsilon > \rho \) for every \( \epsilon \), it follows that \( r \geq \rho \). We may choose a sequence \( \{\epsilon_i\}_{i \in \mathbb{N}} \) such that \( \epsilon_i \to 0 \) and \( y_{\epsilon_i} \to y \) where \( y \in C \).

For a fixed \( x \in \text{int} C \) with \( x \geq y \), and for each \( y_{\epsilon_i} \), there exists a maximal \( \lambda_{\epsilon_i} > 0 \) such that \( x \geq \lambda_{\epsilon_i} y_{\epsilon_i} \). In particular, \( x - \lambda_{\epsilon_i} y_{\epsilon_i} \in \partial C \), where \( \partial C \) denotes the boundary of \( C \). We claim that \( \inf_{\epsilon > 0} \lambda_{\epsilon} > 0 \). Since \( x \in \text{int} C \), there exists \( \delta > 0 \) such that \( x - \delta z \in \text{int} C \) for all \( z \in \mathbb{R}^n \) with \( \|z\| = 1 \). Thus \( x - \delta y_{\epsilon_i} \in \text{int} C \) for all \( \epsilon > 0 \). So \( x \geq \delta y_{\epsilon_i} \) for all \( \epsilon \) and therefore \( \inf \lambda_{\epsilon_i} \geq \delta \). By taking a refinement if necessary, we may assume that \( \lambda_{\epsilon_i} \to \lambda \) where \( \lambda \geq \inf_{\epsilon > 0} \lambda_{\epsilon_i} > 0 \). Then \( x - \lambda y_{\epsilon_i} \) converges to \( x - \lambda y \) and since \( \partial C \) is a closed set, \( x - \lambda y \in \partial C \). Given that \( x \geq y \), \( \lambda \geq 1 \).

Since \( x \geq \lambda_{\epsilon_i} y_{\epsilon_i} \) for each \( \epsilon \), \( f^k(x) \geq \lambda_{\epsilon_i} f^k(y_{\epsilon_i}) = \lambda_{\epsilon_i} \rho_{\epsilon_i}^k y_{\epsilon_i} \) for any \( k \in \mathbb{N} \). Taking a limit, we see that \( f^k(x) \geq \lambda \rho^k y \geq \rho^k y \) for all \( k \in \mathbb{N} \). \( \square \)

It has been noted that order-preserving homogeneous of degree one maps on the interior of symmetric cones have weak lower bounds [4, Corollary 21]. The result above implies that both weak lower bounds and lower bounds will exist for order-preserving homogeneous of degree one self-maps of the interior of any proper closed cone.
3. Upper Bounds on the Standard Cone

Even in the standard cone an order-preserving homogeneous of degree one map $f : \text{int} \mathbb{R}^n_+ \to \text{int} \mathbb{R}^n_+$ might not have an upper bound. For example, the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ defines a linear transformation on the standard cone $\mathbb{R}^2_+$ with spectral radius 1, but the iterates of any vector with positive entries under powers of $A$ forms an unbounded sequence. The following theorem shows that if we relax the definition of an upper bound slightly, then we get a kind of upper bound that is not in $C$ but has all of the other properties of an upper bound for the map $f$.

**Theorem 2.** Let $f : \text{int} \mathbb{R}^n_+ \to \text{int} \mathbb{R}^n_+$ be order-preserving, homogeneous of degree one. Let $\rho = \rho(f)$ be the cone spectral radius of $f$. The map $f$ extends continuously to an order-preserving map on $(0, \infty)^n$ and there exists $z \in (0, \infty)^n$ such that $z$ has at least one finite entry and $f(z) = \tilde{\rho} z$ where $\tilde{\rho} \leq \rho$. In particular, if $x \leq z$, then $f(x) \leq \rho z$.

We refer to $z$ in the theorem above as a *formal eigenvector* of $f$. A formal eigenvector satisfies all of the properties of an upper bound for $f$, except that it is not an element of $\mathbb{R}^n_+$. Before we prove Theorem 2, we need some definitions and a lemma.

In what follows, let $L : [0, \infty]^n$ be the entry-wise reciprocal map:

$$
(Lx)_i = \begin{cases} 
\frac{1}{x_i} & \text{if } x_i \in (0, \infty) \\
\infty & \text{if } x_i = 0 \\
0 & \text{if } x_i = \infty
\end{cases}
$$

(2)

The set $[0, \infty]^n$ has the obvious partial order $x \leq y$ if $x_i \leq y_i$ for all $i \in \{1, \ldots, n\}$. Note that $L$ is order-reversing with respect to this partial ordering. Furthermore, $L(\lambda x) = \lambda^{-1} x$ for all $\lambda \in (0, \infty)$ so $L$ is homogeneous of degree $-1$ on $\text{int} \mathbb{R}^n_+$.

**Lemma 3.** Let $f : \text{int} \mathbb{R}^n_+ \to \text{int} \mathbb{R}^n_+$ be an order-preserving homogeneous of degree one map and let $L$ be given by (2). Then $\rho(L \circ f \circ L)^{-1} \leq \rho(f)$.

**Proof.** For any $x \in \text{int} \mathbb{R}^n_+$ and $k \in \mathbb{N}$, $(L \circ f \circ L)^k(x) = L \circ f^k \circ L(x)$. By the Cauchy-Schwarz inequality, $||Lz|||z|| \geq n$ for all $z \in \text{int} \mathbb{R}^n_+$. Thus $||L \circ f^k \circ L(x)||^{1/k} \geq n^{1/k}$. Taking the limit as $k \to \infty$ we see that $\rho(f) \geq \rho(L \circ f \circ L)^{-1}$. $\square$

**Proof of Theorem 2.** Let $L : \text{int} \mathbb{R}^n_+ \to \text{int} \mathbb{R}^n_+$ be given by (2). The map $L \circ f \circ L$ is an order-preserving homogeneous of degree one map on $\text{int} \mathbb{R}^n_+$. By [2, Theorem 2.10], $L \circ f \circ L$ extends continuously to an order-preserving map on the entire cone $\mathbb{R}^n_+$. By the Krein-Rutman theorem, the continuous extension of $L \circ f \circ L$ must have an eigenvector $y \in \mathbb{R}^n_+ \setminus \{0\}$ with eigenvalue $\hat{\rho}^{-1}$ where $\hat{\rho} = \rho(L \circ f \circ L)^{-1}$. By Lemma 3 $\hat{\rho} \leq \rho = \rho(f)$. Since $L$ is a continuous order-reversing bijection from $\mathbb{R}^n_+$ onto $(0, \infty]^n$, it follows that $f$ extends continuously to an order-preserving map on $(0, \infty)^n$. Let $z = L(y)$. Since $y \neq 0$, at least one entry of $z$ is finite and $f(z) = L \circ L \circ f(L(y)) = L(L \circ f \circ L(y)) = L(\hat{\rho}^{-1}(y)) = \hat{\rho} z$. $\square$

The continuous extension of an order-preserving homogeneous of degree one map to the set $(0, \infty)^n$ is the crucial insight in the proof above. This extension is noted in the context of order-preserving additively homogeneous maps in [1].
Example 1. Let $J_n(\lambda)$ denote the $n$-by-$n$ Jordan matrix with eigenvalue $\lambda > 0$,

$$J_n(\lambda) = \begin{bmatrix} \lambda & 1 & 0 \\ \vdots & \ddots & \ddots \\ 0 & \cdots & \lambda \end{bmatrix}$$

The linear transformation corresponding to $J_n(\lambda)$ maps the interior of $\mathbb{R}^n_+$ into itself. Up to scalar multiplication, the unique Perron eigenvector is $e_1 = [1, 0, \ldots, 0]^T$, and the spectral radius of $J_n(\lambda)$ is $\lambda$. The formal eigenvector corresponding to $J_n(\lambda)$ (which is unique up to scalar multiplication) is $[\infty, \ldots, \infty, 1]$. Thus, Theorem 2 tells us that $(J_n(\lambda)^k(x))^n \leq \lambda^k x^n$ for all $k \in \mathbb{N}$.

Example 2. The “DAD maps” were introduced in [8] to solve the problem of trying to find diagonal matrices $D_1$ and $D_2$ for a nonnegative $m$-by-$n$ matrix $A$ such that $D_1 A D_2$ is doubly stochastic. Such matrices exist if and only if the nonlinear map $f : x \mapsto L A^T L A x$ has an eigenvector with all positive entries. This in turn occurs if and only if the matrix $A$ is a direct sum of fully indecomposable matrices [2].

Note that the map $f$ is order-preserving and homogenous of degree one and it is defined on the interior of $\mathbb{R}^n_+$. Therefore it must extend continuously to the boundary of $\mathbb{R}^n_+$, and this continuous extension can be computed using the convention that $(0)^{-1} = \infty$ and $(\infty)^{-1} = 0$.

Assume that $A$ is an $m$-by-$n$ nonnegative matrix such that every row and column contains at least one non-zero entry. By permuting the rows and columns of $A$, we may assume [5] that $A$ has the following form,

$$A = \begin{bmatrix} A_1 & B_{12} & \cdots & B_{1\sigma} \\ A_2 & \cdots & B_{2\sigma} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_{\sigma} \end{bmatrix}$$

where each $A_i$ is an $m_i$-by-$n_i$ matrix with a corresponding DAD-map $f_i(x) = L A_i^T L A_i x$ with a unique eigenvector $v_i$ (up to scaling) with all positive entries in $\mathbb{R}^{m_i}$ and eigenvalue $\lambda_i = m_i/m_i$. Furthermore $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{\sigma}$. If $A$ is indecomposable (that is, there are no permutation matrices $P$ and $Q$ such that $PAQ$ is a direct sum of two matrices), then the form above is unique. Note that the eigenvector $v_{\sigma}$ can be extended to a formal eigenvector for the map $f(x) = L A^T L A x$ by letting

$$v = \begin{bmatrix} \infty \\ \vdots \\ \infty \\ v_{\sigma} \end{bmatrix}$$

Then $f(v) = L A^T L A v = \lambda_{\sigma} v$. Note that the cone spectral radius of $f$ is $\lambda_1$ [5] so the eigenvalue corresponding to the formal eigenvector may be strictly less than the cone spectral radius.
4. Weak Upper Bounds

**Theorem 3.** Let \( f : \text{int} \, C \to \text{int} \, C \) be order-preserving and homogeneous of degree one with \( \rho(f) = \rho \). If \( C \) is a polyhedral cone and \( x \in \text{int} \, C \), then there is a \( w \in \mathbb{C}^n \setminus \{0\} \) such that \( \langle f^k(x), w \rangle \leq \rho^k \langle x, w \rangle \) for all \( k \in \mathbb{N} \).

**Proof.** Fix \( x_0 \in \text{int} \, C \), and a linear functional \( v \in \text{int} \, C^* \). Let \( f_\epsilon(x) = f(x) + \epsilon \langle x, v \rangle x_0 \) for any \( x \in \text{int} \, C \). Let \( \rho_\epsilon = \rho(f_\epsilon) \). By [6] Theorem 5.4.1, each \( f_\epsilon \) has an eigenvector \( x_\epsilon \in \text{int} \, C \) with \( f_\epsilon(x_\epsilon) = \rho_\epsilon x_\epsilon \). Since we are free to scale the eigenvectors \( x_\epsilon \), we will require that \( x_\epsilon \) be the smallest scalar multiple of \( x_\epsilon \) with the property that \( x_\epsilon \geq x \). For each \( \epsilon > 0 \) there is a unit vector \( w_\epsilon \in \text{extr} \, C^* \) such that \( \langle x_\epsilon, w_\epsilon \rangle = \langle x, w_\epsilon \rangle \). Since \( C \) is polyhedral, there are only finitely many extreme rays of \( C^* \). We may choose a sequence \( \epsilon_i \) such that \( \epsilon_i \to 0 \) and for all \( i \in \mathbb{N} \), \( w_{\epsilon_i} = w \) where \( w \) is a unit vector in \( \text{extr} \, C^* \). Then, \( \langle x_{\epsilon_i}, w \rangle = \langle x, w \rangle \) for all \( i \). We see that

\[
\langle f^k(x), w \rangle \leq \langle f^k_{\epsilon_i}(x_{\epsilon_i}), w \rangle = \rho^k_{\epsilon_i} \langle x, w \rangle.
\]

Since \( C \) is polyhedral, it follows that the cone spectral radius \( \rho_\epsilon \) is continuous [6 Corollary 5.5.4]. This means that \( \rho_\epsilon \to \rho \) as \( \epsilon \to 0 \). Therefore, \( \langle f^k(x), w \rangle \leq \rho^k \langle x, w \rangle \) for all \( k \in \mathbb{N} \).

A stronger result than Theorem 3 can be obtained for linear maps with an invariant closed cone in \( \mathbb{R}^n \) by using the adjoint.

**Theorem 4.** Let \( C \) be a proper closed cone in \( \mathbb{R}^n \). Let \( T : C \to C \) be a linear map and let \( T^* : C^* \to C^* \) denote the adjoint of \( T \). Let \( \rho = \rho(T) \) be the spectral radius of \( T \). Then there is a \( w \in C^* \setminus \{0\} \) such that \( \langle T(x), w \rangle = \rho \langle x, w \rangle \) for all \( x \in C \). In particular, \( w \) is a weak upper bound for the iterates of \( T \).

**Proof.** Note that the spectral radius of the adjoint \( \rho(T^*) = \rho(T) \). Let \( w \) be an eigenvector of \( T^* \) with eigenvalue \( \rho \), as is guaranteed to exist by the Krein-Rutman theorem. For any \( x \in C \), \( \langle T(x), w \rangle = \langle x, T^*(w) \rangle = \rho \langle x, w \rangle \).

For an arbitrary proper closed cone \( C \subset \mathbb{R}^n \), it is not yet known whether any order-preserving homogeneous of degree one map \( f : \text{int} \, C \to \text{int} \, C \) must have a weak upper bound.

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