A FORMULA FOR THE EQUIVARIANT LOCAL INDEX OF THE REDUCED SPACE IN THE SYMPLECTIC CUTTING

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Abstract. We compute the equivariant local index for the reduced space in a symplectic cut space, provided that the reduced space is compact.

1. Introduction

In the joint work [2, 3, 4] with Fujita and Furuta we developed an index theory for Dirac-type operators on possibly non-compact Riemannian manifolds. We call the index in our theory the local index and also call its equivariant version the equivariant local index. The local index has several properties, such as deformation invariance and excision property. In particular, when the manifold is closed, the local index is nothing but the usual index of the Dirac-type operator. Using these properties, for Hamiltonian $S^1$-actions on prequantizable closed symplectic manifolds we can show that the equivariant Riemann-Roch index is obtained as the sum of the equivariant local indices for the inverse images of the integer lattice points by the moment map. When the lattice point is a regular value of the moment map we can compute its equivariant local index, see [4, 7]. So the problem is how to compute the equivariant local index when the lattice point is a singular value.

In this paper, as a special case of singular lattice points we compute the equivariant local index for the reduced space in a symplectic cut space. Let $(M, \omega)$ be a Hamiltonian $S^1$-manifold with moment map $\mu: M \to \mathbb{R}$ and $n \in \mathbb{R}$ a regular value of $\mu$. Suppose the $S^1$ acts on the level set $\mu^{-1}(n)$ freely. We denote the reduced space $\mu^{-1}(n)/S^1$ by $M_n$. The symplectic cut is a procedure to make a new Hamiltonian $S^1$-manifold $\overline{M}_{\mu \leq n}$, called the cut space, from the given $(M, \omega)$ and $n$. This method is introduced by Lerman in [5] and the reduced space $M_n$ is naturally contained in the cut space $\overline{M}_{\mu \leq n}$ as a fixed point set. In particular, $M_n$ is a singular fiber of the moment map of the $S^1$-action on $\overline{M}_{\mu \leq n}$. Moreover, if the Hamiltonian $S^1$-manifold $(M, \omega)$ is prequantizable and the regular value $n$ is integer, then the cut space $\overline{M}_{\mu \leq n}$ is also prequantizable, and a prequantum line bundle $(L, \nabla^L)$ on $(M, \omega)$ induces the prequantum line bundle $\left(\overline{L}, \nabla^{\overline{L}}\right)$ on $\overline{M}_{\mu \leq n}$.

Suppose also that $\mu^{-1}(n)$ is compact. Let $O$ be an $S^1$-invariant open neighborhood of $M_n$ in $\overline{M}_{\mu \leq n}$. Then the equivariant local index $\text{ind}_{S^1} (O, O \setminus M_n; \overline{L}|_{(O)})$ is defined. The purpose of this paper is to give a formula for $\text{ind}_{S^1} (O, O \setminus M_n; \overline{L}|_{(O)})$ in Theorem 4.1. This is an equivariant refinement of a partial result of [1, Section 5], in which we give a formula for the local index for a reduced space in a symplectic cut space in order to determine the local index of the zero section of the cotangent bundle of the $n$-dimensional sphere $S^n$ by using the geodesic flow.

This paper is organized as follows. In Section 2 we recall the equivariant local index in the case of the Hamiltonian $S^1$-actions. After that we briefly recall the
symplectic cut in Section 3. The main theorem (Theorem 4.1) is given in Section 4. Finally examples are given in Section 5.

**Notation.** In this paper we use the notation $C_{(n)}$ for the irreducible representation of $S^1$ with weight $n$.

## 2. Equivariant Local Index

Let $(M, \omega)$ be a prequantizable Hamiltonian $S^1$-manifold and $(L, \nabla^L)$ an $S^1$-equivariant prequantum line bundle on $(M, \omega)$ with the associated moment map $\mu : M \to \mathbb{R}$. We do not assume $M$ is compact. Since all orbits are isotropic the restriction of $(L, \nabla^L)$ to each orbit is flat.

**Definition 2.1.** An orbit $O$ is said to be $L$-acyclic if $H^0(\mathcal{O} ; (L, \nabla^L)|_O) = 0$.

Let $V$ be an $S^1$-invariant open set whose complement is compact and which contains only $L$-acyclic orbits. For these data we give the following theorem.

**Theorem 2.2** ([2, 3, 4]). There exists an element $\text{ind}_{S^1}(M, V ; L) \in R(S^1)$ of the representation ring such that $\text{ind}_{S^1}(M, V ; L)$ satisfies the following properties:

1. $\text{ind}_{S^1}(M, V ; L)$ is invariant under continuous deformation of the data.
2. If $M$ is closed, then $\text{ind}_{S^1}(M, V ; L)$ is equal to the equivariant Riemann-Roch index $\text{ind}_{S^1}(M ; L)$.
3. If $M'$ is an $S^1$-invariant open neighborhood of $M \setminus V$, then $\text{ind}_{S^1}(M, V ; L)$ satisfies the following excision property

$$\text{ind}_{S^1}(M, V ; L) = \text{ind}_{S^1}(M', M' \cap V ; L|_{M'}) .$$

4. $\text{ind}_{S^1}(M, V ; L)$ satisfies a product formula.

We call $\text{ind}_{S^1}(M, V ; L)$ the equivariant local index.

**Example 2.3.** For small positive real number $\varepsilon > 0$ which is less than 1 let $D_\varepsilon(C_{(1)}) = \{ z \in C_{(1)} \mid |z| < \varepsilon \}$ be the 2-dimensional disc of radius $\varepsilon$. As $(L, \nabla^L) \to (M, \omega)$ we consider

$$\left( D_\varepsilon(C_{(1)}) \times C_{(m)}, d + \frac{1}{2}(z d\bar{z} - \bar{z} dz) \right) \to \left( D_\varepsilon(C_{(1)}), \frac{\sqrt{-1}}{2\pi} dz \land d\bar{z} \right).$$

First let us detect non $L$-acyclic orbits. Suppose the orbit $O$ through $z \in D_\varepsilon(C_{(1)})$ has a non-trivial parallel section $s \in H^0(\mathcal{O} ; (L, \nabla)|_O)$. Then $s$ satisfies the following equation

$$0 = \nabla^L_{\partial_s} s = \frac{\partial s}{\partial \theta} - 2\pi \sqrt{-1} r^2 s,$$

where we use the polar coordinates $z = r e^{2\pi \sqrt{-1} \theta}$. Hence $s$ is of the form $s(\theta) = s_0 e^{2\pi \sqrt{-1} r^2 \theta}$ for some non-zero constant $s_0$. Since $s$ is a global section on $O$ $s$ satisfies $s(0) = s(1)$. This implies $r = 0$.

Next, we put $V = D_\varepsilon(C_{(1)}) \setminus \{ 0 \}$ and let us compute $\text{ind}_{S^1}(M, V ; L)$. We recall the definition of $\text{ind}_{S^1}(N, V ; L)$. For $t \geq 0$ consider the following perturbation of the Spin$^c$ Dirac operator $D : \Gamma \left( (\Lambda^{0,*}T^* M \otimes L) \to \Gamma \left( (\Lambda^{0,*}T^* M \otimes L) \right. \right.$ associated with the standard Hermitian structure on $M = D_\varepsilon(C_{(1)})$

$$D_t = D + t \rho D_{S^1},$$

where $\rho$ is a cut-off function of $V$ and $D_{S^1}$ is a first order formally self-adjoint differential operator of degree-one that satisfies the following conditions:

1. $D_{S^1}$ contains only derivatives along orbits.
(2) The restriction \( D_{S^1} \mid_O \) to an orbit \( O \) is the de Rham operator with coefficients in \( L \mid_O \).

(3) For any \( S^1 \)-equivariant section \( u \) of the normal bundle \( \nu_O \) of \( O \) in \( M \), \( D_{S^1} \) anti-commutes with the Clifford multiplication of \( u \).

See [2, 3, 4] for more details. From the second condition and \( \{0\} \) is the unique non-\( L \)-acyclic orbit we can see \( \ker (D_{S^1} \mid_O) = 0 \) for all orbits \( O \neq \{0\} \). Extend the complement of a neighborhood of 0 in \( D_{\varepsilon} (\mathbb{C}(1)) \) cylindrically so that all the data are translationally invariant. Then we showed in [2, 3] that for a sufficiently large \( t \) \( D_t \) is Fredholm, namely, \( \ker D_t \cap L^2 \) is finite dimensional and its super-dimension is independent of a sufficiently large \( t \). So we define

\[
\text{ind}_{S^1}(M, V; L) = \ker D^0_t \cap L^2 - \ker D^1_t \cap L^2
\]

for a sufficiently large \( t \). In this case, by the direct computation using the Fourier expansion of \( s \) with respect to \( \theta \), we can show that

\[
\ker D^0_t \cap L^2 \cong \mathbb{C}, \quad \ker D^1_t \cap L^2 = 0,
\]

and \( \ker D^0_t \cap L^2 \) is spanned by a certain \( L^2 \)-function \( a_0(r) \) on \( D_{\varepsilon} (\mathbb{C}(1)) \) which depends only on \( r = |z| \). Since the \( S^1 \)-action on \( \ker D^0_t \cap L^2 \) is given by pull-back and the \( S^1 \)-action on the fiber is given by \( C(m) \), we obtain

\[
\text{ind}_{S^1}(M, V; L) = \text{ind}_{S^1}(D_{\varepsilon} (\mathbb{C}(1)) ; D_{\varepsilon} (\mathbb{C}(1)) \setminus \{0\} ; D_{\varepsilon} (\mathbb{C}(1)) \times C(m)) = \mathbb{C}(-m).
\]

For more details see [2, Remark 6.10], or [6, Section 5.3.2].

It is well-known that the lift of \( S^1 \)-action on \( M \) to \( L \) defines the moment map \( \mu : M \to \mathbb{R} \) by the Kostant formula

\[
L_X s = \nabla^L_X s + 2\pi \sqrt{-1} \mu s,
\]

where \( s \) is a section of \( L \), \( X \) is the vector field which generates the \( S^1 \)-action on \( (M, \omega) \), and \( L_X s \) is the Lie derivative which is defined by

\[
L_X s(x) = \frac{d}{d\theta} \bigg|_{\theta=0} e^{-2\pi \sqrt{-1} \theta} s(e^{2\pi \sqrt{-1} \theta} x).
\]

**Lemma 2.4.** If an orbit \( O \) is not \( L \)-acyclic, namely, \( H^0(O; (L, \nabla^L) \mid_O) \neq 0 \), then, \( \mu(O) \in \mathbb{Z} \).

If \( M \) is closed, then we have the following localization formula for the equivariant Riemann-Roch index.

**Corollary 2.5.** Suppose \( M \) is closed. For \( i \in \mu(M) \cap \mathbb{Z} \) let \( V_i \) be an \( S^1 \)-invariant open neighborhood of \( \mu^{-1}(i) \) such that they are mutually disjoint, namely, \( V_i \cap V_j \neq \emptyset \) for all \( i \neq j \). Then,

\[
\text{ind}_{S^1}(M, L) = \bigoplus_{i \in \mu(M) \cap \mathbb{Z}} \text{ind}_{S^1}(V_i, V_i \cap V; L|_{V_i}).
\]

**3. Symplectic cut**

In this section let us briefly recall the symplectic cut and its properties we use in this paper. Let \( (M, \omega) \) be a Hamiltonian \( S^1 \)-space with moment map \( \mu : M \to \mathbb{R} \). For a real number \( n \) the cut space \( \overline{M}_{\mu \leq n} \) of \( (M, \omega) \) by the symplectic cutting [5] is the reduced space of the diagonal \( S^1 \)-action on \( (M, \omega) \times (\mathbb{C}(1), \frac{\sqrt{-1}}{2\pi} dz \wedge d\bar{z}) \), namely,

\[
\overline{M}_{\mu \leq n} = \left\{ (x, z) \in (M, \omega) \times (\mathbb{C}(1), \frac{\sqrt{-1}}{2\pi} dz \wedge d\bar{z}) \mid \mu(x) + |z|^2 = n \right\} / S^1.
\]

We denote the reduced space \( \mu^{-1}(n)/S^1 \) by \( M_n \).
Proposition 3.1. (1) If $S^1$ acts on $\mu^{-1}(n)$ freely, then, $\overline{M}_{\mu \leq n}$ is a smooth Hamiltonian $S^1$-space. The $S^1$-action is given as
\begin{equation}
\tau t[x, z] = [tx, z]
\end{equation}
for $t \in S^1$ and $[x, z] \in \overline{M}_{\mu \leq n}$.
(2) Under the assumption in (1), the reduced space $M_n$ and $\{ x \in M \mid \mu(x) \leq n \}$ are symplectically embedded into $\overline{M}_{\mu \leq n}$ by $M_n \ni [x] \mapsto [x, 0] \in \overline{M}_{\mu \leq n}$ and $\{ x \in M \mid \mu(x) \leq n \}$ is identified with the disjoint union $\{ x \in M \mid \mu(x) \leq n \}$ $\coprod M_n$ and with this identification $M_n$ is fixed by the $S^1$-action (3.1).

Suppose that $(M, \omega)$ is equipped with a prequantum line bundle $(L, \nabla^L) \to (M, \omega)$ and the $S^1$-action lifts to $(L, \nabla^L)$ in such a way that $\mu$ satisfies the Kostant formula (2.1).

Proposition 3.2. If $n$ is an integer and the $S^1$-action on $\mu^{-1}(n)$ is free, then $\overline{M}_{\mu \leq n}$ is prequantizable. In this case a prequantum line bundle $(\overline{\mathcal{L}}, \nabla^{\overline{\mathcal{L}}})$ on $\overline{M}_{\mu \leq n}$ is given by
\begin{equation}
(\overline{\mathcal{L}}, \nabla^{\overline{\mathcal{L}}}) = ((L, \nabla^L) \otimes \mathbb{C}(n)) \boxtimes \left( \mathbb{C}_1 \times \mathbb{C}(0), d + \frac{1}{2}(zd\overline{z} - \overline{z}dz) \right) \big|_{\Phi^{-1}(0)} / S^1,
\end{equation}
where $\Phi$ is the moment map $\Phi : M \times \mathbb{C}_1 \to \mathbb{R}$ associated to the lift of the diagonal $S^1$-action which is written as $\Phi(x, z) = \mu(x) + |z|^2 - n$, and the lift of the $S^1$-action (3.1) on $\overline{M}_{\mu \leq n}$ to $(\overline{\mathcal{L}}, \nabla^{\overline{\mathcal{L}}})$ is given by
\begin{equation}
t[u \otimes v \mathbb{E}(z, w)] = [(tu) \otimes v \mathbb{E}(z, w)]
\end{equation}
for $t \in S^1$ and $[u \otimes v \mathbb{E}(z, w)] \in \overline{\mathcal{L}}$. The moment map $\overline{\mu} : \overline{M}_{\mu \leq n} \to \mathbb{R}$ associated with the lift (3.2) is written as $\overline{\mu}(x, z) = \mu(x) = n - |z|^2$.

Remark 3.3. We denote the restriction of $(\overline{\mathcal{L}}, \nabla^{\overline{\mathcal{L}}})$ to $M_n$ by $(L_n, \nabla^{L_n})$. $(L_n, \nabla^{L_n})$ is a prequantum line bundle on $M_n$. The $S^1$-action (3.2) on $L_n$ is given by the fiberwise multiplication with weight $n$. Recall that $M_n$ is fixed by the $S^1$-action (3.1). See Proposition 3.1.

4. Main theorem

Let $(M, \omega)$ be a prequantizable Hamiltonian $S^1$-manifold and $(L, \nabla^L)$ an $S^1$-equivariant prequantum line bundle on $(M, \omega)$ with the associated moment map $\mu : M \to \mathbb{R}$. Let $n$ be an integer and we assume the $S^1$-action on $\mu^{-1}(n)$ is free. Then, the cut space $\overline{M}_{\mu \leq n}$ becomes a prequantizable Hamiltonian $S^1$-manifold and the $S^1$-equivariant prequantum line bundle $(\overline{\mathcal{L}}, \nabla^{\overline{\mathcal{L}}})$ is given by Proposition 3.1 and Proposition 3.2.

Suppose that $\mu^{-1}(n)$ is compact. We take a sufficiently small $S^1$-invariant open neighborhood $O$ of $M_n$ in $\overline{M}_{\mu \leq n}$ so that the intersection $\overline{\mu}(O) \cap \mathbb{Z}$ consists of the unique point $n$. Then we can define the equivariant local index $\text{ind}_{S^1} \left( O, O \setminus M_n; \mathcal{L}_O \right)$ of $M_n$ in $\overline{M}_{\mu \leq n}$. We give the following formula for $\text{ind}_{S^1} \left( O, O \setminus M_n; \mathcal{L}_O \right)$.

Theorem 4.1. Let $(M, \omega)$, $(L, \nabla^L)$, and $\mu$ be as above. Let $n$ be an integer. Suppose $S^1$ acts on $\mu^{-1}(n)$ freely and $\mu^{-1}(n)$ is compact. Let $O$ be a sufficiently small $S^1$-invariant open neighborhood of $M_n$ in $\overline{M}_{\mu \leq n}$ which satisfies $\overline{\mu}(O) \cap \mathbb{Z} = \{ n \}$. Then, the equivariant local index is given as
\begin{equation}
\text{ind}_{S^1} \left( O, O \setminus M_n; \mathcal{L}_O \right) = \text{ind} \left( M_n; L_n \right) \mathbb{C}(n),
\end{equation}
where $L_n$ is the prequantum line bundle $L_n = (L|_{\mu^{-1}(n)} \otimes \mathbb{C}(n))/S^1$ on $M_n$ and $\text{ind} \left( M_n; L_n \right)$ is the Riemann-Roch number of $M_n$. 
**Remark 4.2.** By replacing \( \mathbb{C}_{(1)} \) with \( \mathbb{C}_{(-1)} \) in the above construction we obtain the other cut space \( \overline{M}_{\mu \leq n} = \{ (x, z) \in M \times \mathbb{C}_{-1} : \mu(x) - |z|^2 = n \} / S^1 \). Theorem 4.1 also holds for \( \overline{M}_{\mu \geq n} \).

To prove the theorem we need some preliminaries. Let \( \text{grad}(\mu) \) be the negative gradient vector field of \( \mu \) with respect to an \( S^1 \)-invariant Riemannian metric \( g \) of \( M \), namely, the vector field determined by

\[-d\mu = g(\text{grad}(\mu), \cdot).\]

Since \( n \) is a regular value of \( \mu, \text{grad}(\mu) \) does not vanish near \( \mu^{-1}(n) \). Let \( \phi_\tau \) be the flow of the vector field \( \frac{1}{\| \text{grad}(\mu) \|^2} \text{grad}(\mu) \). Note that \( \phi_\tau \) exists on \( \mu^{-1}((n - \varepsilon, n + \varepsilon)) \) for a sufficiently small \( \varepsilon > 0 \) since \( \mu^{-1}(n) \) is compact by assumption.

**Lemma 4.3.** \( \phi_\tau \) commutes with the \( S^1 \)-action on \( M \) and satisfies the following property

\[\mu(\phi_\tau(x)) = \mu(x) - \tau\]

for \( x \in \mu^{-1}((n - \varepsilon, n + \varepsilon)) \).

**Proof.**

\[
\frac{d}{d\tau} \mu(\phi_\tau(x)) = d\mu \left( \frac{1}{\| \text{grad}(\mu) \|^2} \text{grad}(\mu) \right)
= -g \left( \text{grad}(\mu), \frac{1}{\| \text{grad}(\mu) \|^2} \text{grad}(\mu) \right)
= -1.
\]

\[\square\]

By the definition of the symplectic cutting, the normal bundle \( \nu \) of \( M_n \) in \( \overline{M}_{\mu \leq n} \) is given by

\[\nu = \mu^{-1}(N) \times_{S^1} \mathbb{C}_{(1)}.\]

For a sufficiently small \( \varepsilon > 0 \) let \( D_\varepsilon(\mathbb{C}_{(1)}) = \{ z \in \mathbb{C}_{(1)} : |z| < \varepsilon \} \) be the open disc of radius \( \varepsilon \). We put \( D_\varepsilon(\nu) = \mu^{-1}(n) \times_{S^1} D_\varepsilon(\mathbb{C}_{(1)}) \), and define an \( S^1 \)-action on \( D_\varepsilon(\nu) \) by

\[t[x, z] = [tx, z].\]

Let \( p : D_\varepsilon(\nu) \to M_n \) be the natural projection. We define a complex line bundle \( L_{D_\varepsilon(\nu)} \) on \( D_\varepsilon(\nu) \) by

\[L_{D_\varepsilon(\nu)} = p^* L_n \otimes (\mu^{-1}(n) \times_{S^1} (D_\varepsilon(\mathbb{C}_{(1)}) \times \mathbb{C}_{(0)})),\]

and define an lift of the \( S^1 \)-action (4.1) to \( L_{D_\varepsilon(\nu)} \) by

\[t([x, z], [u \otimes v]) \otimes [x', z', w]) = ([tx, z], [(tu) \otimes v]) \otimes [tx', z', w].\]

Then we can show the following lemma.

**Lemma 4.4.** (1) For a sufficiently small \( \varepsilon > 0 \) there exists an \( S^1 \)-equivariant embedding \( f_M : D_\varepsilon(\nu) \to \overline{M}_{\mu \leq n} \) with respect to the actions (4.1) and (3.1).

(2) Under the assumption in (1) there exists an \( S^1 \)-equivariant bundle map \( f_L : L_{D_\varepsilon(\nu)} \to \overline{M} \) with respect to the actions (4.2) and (3.2) such that \( f_L \) covers \( f_M \).

**Proof.** We define \( f_M \) and \( f_L \) by

\[f_M([x, z]) = [\phi_{|z|^2}(x), z],\]

\[f_L([x, z], [u \otimes v]) \otimes [x', z', w]) = \left( [\phi_{|z|^2}(t_0 u) \otimes t_0^0 v] \otimes (z, w) \right),\]
where \( t_0 \in S^1 \) is the unique element that satisfies \( t_0u \in L_x \) and \( \tilde{\phi}_x(t_0u) \) is the horizontal lift of \( \phi_x(x) \) starting from \( t_0u \) and \( \tilde{\phi}_{[z]^2}(t_0u) \) is its value at \( \tau = |z|^2 \). Then they are required maps.

\[ \square \]

**Proof of Theorem 4.1.** By Lemma 4.4 and the equivariant version of the product formula [3, Theorem 5.8] we obtain

\[
\begin{align*}
\ind_{S^1} \left( O, O \setminus M_n ; \bar{\mathcal{L}}_0 \right) &= \ind_{S^1} \left( D_\varepsilon(\nu), D_\varepsilon(\nu) \setminus M_n ; L_{D_\varepsilon(\nu)} \right) \\
&= \ind_{S^1} \left( M_n ; L_n \otimes \mu^{-1}(n) \times S^1 \ind_{S^1}(D_\varepsilon(C_{(1)}), D_\varepsilon(C_{(1)}) \setminus \{0\}; D_\varepsilon(C_{(1)}) \times C_{(0)}) \right). \\
&= \ind_{S^1} \left( M_n ; L_n \setminus \mu^{-1}(n) \times S^1 \ind_{S^1}(D_\varepsilon(C_{(1)}), D_\varepsilon(C_{(1)}) \setminus \{0\}; D_\varepsilon(C_{(1)}) \times C_{(0)}) \right) \\
&\quad \text{From Example 2.3 the equivariant local index } \ind_{S^1}(D_\varepsilon(C_{(1)}), D_\varepsilon(C_{(1)}) \setminus \{0\}; D_\varepsilon(C_{(1)}) \times C_{(0)}) \text{ is equal to } C_{(0)}.
\end{align*}
\]

By definition, \( L_n \) is naturally identified with the restriction of \( \bar{\mathcal{L}}_0 \) to \( M_n \). With this identification we can see that the restriction of the \( S^1 \)-action (3.2) to \( L_n \to M_n \) is nothing but the fiberwise multiplication of \( t^n \).

Since the \( S^1 \)-action on \( \ind_{S^1}(M_n ; L_n) \) is defined by the pull-back, the \( S^1 \)-action on \( \ind_{S^1}(M_n ; L_n) \) is given by the multiplication of \( t^n \) as we mentioned in Remark 3.3. This proves the theorem.

\[ \square \]

5. **Examples**

**Example 5.1** (Complex projective space). As \((L, \nabla) \to (M, \omega)\) we adopt

\[
\left( (C_{(1)})^m \times C_{(0)}, d + \frac{1}{2} \sum_{i=1}^m (\bar{z}_i d\bar{z}_i - \bar{z}_i dz_i) \right) \to \left( (C_{(1)})^m \frac{\sqrt{-1}}{2\pi} \sum_{i=1}^m dz_i \wedge d\bar{z}_i \right).
\]

For \( n = 1 \) the obtained \( \mathcal{M}_{\mu \leq n}, \bar{\mathcal{L}} \) and \( M_n \) are \( \mathbb{C}P^m, \mathcal{O}(1), \) and \( \mathbb{C}P^{m-1} \), respectively. The induced \( S^1 \)-actions on \( \mathbb{C}P^m \) and \( \mathcal{O}(1) \) are given by

\[
t([z_0 : z_1 : \cdots : z_m]) = [z_0 : tz_1 : \cdots : tz_m, w], \quad t[z_0 : z_1 : \cdots : z_m, w] = [z_0 : tz_1 : \cdots : tz_m, w].
\]

The moment map \( \mathbf{\mu} \) associated to the \( S^1 \)-action (5.1) is given by \( \mathbf{\mu}([z_0 : \cdots : z_m]) = \sum_{i=1}^m |z_i|^2 \). For \( k = 0 \), let \( O_k \) be a sufficiently small \( S^1 \)-invariant open neighborhood of \( \mathbf{\mu}^{-1}(k) \). Then the equivariant local index \( \ind_{S^1}(O_k, O_k \setminus \mathbf{\mu}^{-1}(k); \bar{\mathcal{L}}|_{O_k}) \) is defined and by Corollary 2.2 the equivariant Riemann-Roch index \( \ind_{S^1}(\mathcal{M}_{\mu \leq n}, \bar{\mathcal{L}}) \) satisfies following equality

\[
\ind_{S^1}(\mathcal{M}_{\mu \leq n}, \bar{\mathcal{L}}) = \ind_{S^1}(O_0, O_0 \setminus \mathbf{\mu}^{-1}(0); \bar{\mathcal{L}}|_{O_0}) + \ind_{S^1}(O_1, O_1 \setminus \mathbf{\mu}^{-1}(1); \bar{\mathcal{L}}|_{O_1}).
\]

The left hand side is computed as

\[
\ind_{S^1}(\mathcal{M}_{\mu \leq n}, \bar{\mathcal{L}}) = \ind_{S^1}(\mathbb{C}P^m, \mathcal{O}(1)) = C_{(0)} \oplus mC_{(1)}.
\]

For \( k = 1 \), since \( \mathbf{\mu}^{-1}(1) = M_n \), by Theorem 4.1 \( \ources{4.1}{\sources{5.1}{\ind_{S^1}(O_1, O_1 \setminus \mathbf{\mu}^{-1}(1); \bar{\mathcal{L}}|_{O_1})}} \) is given as

\[
\begin{align*}
\ind_{S^1}(O_1, O_1 \setminus \mathbf{\mu}^{-1}(1); \bar{\mathcal{L}}|_{O_1}) &= \ind_{S^1}(O_1, O_1 \setminus M_n; \bar{\mathcal{L}}|_{O_1}) \\
&= \ind(C\mathbb{C}P^{m-1}; C(1))C_{(1)} \\
&= mC_{(1)}.
\end{align*}
\]

For \( k = 0 \), it is easy to see that \( \mathbf{\mu}^{-1}(0) = \{[z_0 : 0 : \cdots : 0]\} \) and \( (\bar{\mathcal{L}}, \nabla_{\bar{\mathcal{L}}})|_{\mathbb{C}^m} \cong (\mathbb{C}^m, d + \bar{\mathcal{L}}(\bar{\nabla}z - zd\bar{z})). \) We can take \( O_0 \) in such a way that \( O_0 \) is identified with a sufficiently small open disc \( D = \{[z_1, \ldots, z_m] \in \mathbb{C}^m : \sum_{i=1}^m |z_i|^2 \leq \varepsilon \} \) with \( S^1 \)-action \( t(z_1, \ldots, z_m) = (tz_1, \ldots, tz_m) \). Then, by (5.2), (5.3), and (5.4) we obtain the following formula

\[
\ind_{S^1}(D, D \setminus \{0\}; D \times C_{(0)}) = C_{(0)}.
\]
In the case of \( m = 1 \) this formula can be obtained in \([2, \text{Remark 6.10}]\) and \([6, \text{Section 5.3.2}]\).

**Example 5.2** (Exceptional divisor). Let \( n \) and \((L, \nabla) \to (M, \omega)\) be as in Example 5.1. Then the obtained cut space \( M_{\mu \geq n} \) is the blow-up \( \tilde{C}^m \) of the origin in \( C^n \), and \( M_n \) and \( L_n \) are the exceptional divisor \( \mathbb{C}P^{m-1} \) and \( O(n) \), respectively. We take a sufficiently small invariant open neighborhood \( O \) of \( M_n \). Then, by Theorem 4.1 the equivariant local index \( \text{ind}_{S^1} (O, O \setminus M_n; \mathcal{L}_{|O}) \) is given by

\[
\text{ind}_{S^1} (O, O \setminus M_n; \mathcal{L}_{|O}) = \text{ind} \left( \mathbb{C}P^{m-1}; O(n) \right) \mathcal{C}(n) = \left( \frac{m-1+n}{m-1} \right) \mathcal{C}(n).
\]

**REFERENCES**

1. H. Fujita, M. Furuta, and T. Yoshida, *Geodesic flows on the sphere and Riemann-Roch numbers*, arXiv:1209.2924.
2. , *Torus fibrations and localization of index I*, J. Math. Sci. Univ. Tokyo 17 (2010), no. 1, 1–26.
3. , *Torus fibrations and localization of index II*, arXiv:0910.0358. To appear in Comm. Math. Phys..
4. , *Torus fibrations and localization of index III*, arXiv:1008.5007. To appear in Comm. Math. Phys..
5. E. Lerman, *Symplectic cuts*, Math. Res. Lett. 2 (1995), no. 3, 247–258.
6. T. Yoshida, *RR = #BS via localization of index*, Trends in Math. 12 (2010), no. 1, 1–41.
7. , *Equivariant local index*, RIMS Kôkyûroku Bessatsu B39 (2013), 215–232.

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