Superderivations for Modular Graded Lie Superalgebras of Cartan-type

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Abstract Superderivations for the eight families of finite or infinite dimensional graded Lie superalgebras of Cartan-type over a field of characteristic \( p > 3 \) are completely determined by a uniform approach: The infinite dimensional case is reduced to the finite dimensional case and the latter is further reduced to the restrictedness case, which proves to be far more manageable. In particular, the outer superderivation algebras of those Lie superalgebras are completely determined.

Keywords Lie superalgebra; Cartan-type; superderivation
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0. Introduction

Eight families of \( \mathbb{Z} \)-graded Lie superalgebras of Cartan-type were constructed over a field of characteristic \( p > 3 \) \cite{2, 3, 6, 9, 10, 15}. These Lie superalgebras are subalgebras of the full superderivation algebras of the associative superalgebras—tensor products of the divided power algebras and the exterior superalgebras. The superderivation algebras were studied in one-by-one fashion for the finite dimensional and simple ones \cite{2, 3, 6, 9, 11, 14, 16}. The present paper aims to use a uniform method to determine the surperderivation algebras of all the eight families of graded Lie superalgebras of Cartan-type, including the infinite dimensional or non-simple ones. In particular, the outer superderivation algebras of those Lie superalgebras are completely determined. We should mention that we adopt a method for Lie algebras \cite[Lemma 6.1.3]{12} and benefit much from reading \cite{12, 13}. It should be also mentioned that the present paper covers some known results about superderivations for the finite dimensional simple graded Lie superalgebras of Cartan-type mentioned above \cite{2, 9, 11, 14, 16} and certain inaccuracies in the literature are corrected.

Throughout \( \mathbb{F} \) is an algebraically closed field of characteristic \( p > 3 \), \( \mathbb{Z}_2 := \{0, 1\} \) is the field of two elements. As in usual, \( \mathbb{Z}, \mathbb{N} \) and \( \mathbb{N}_0 \) are the sets of integers, nonnegative integers and positive integers, respectively. For a \( \mathbb{Z}_2 \)-graded vector space \( V \), denote by \( |x| = \alpha \) the parity of a homogeneous element \( x \in V_\alpha, \alpha \in \mathbb{Z}_2 \). If \( V \) is a \( \mathbb{Z} \)-graded vector space and \( x \in V \) is a \( \mathbb{Z} \)-homogeneous element, write \( zd(x) \)

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for the \(\mathbb{Z}\)-degree of \(x\). The symbol \(|x|\) (resp. \(zd(x)\)) always implies that \(x\) is a \(\mathbb{Z}_2\)-\(\mathbb{Z}\)-homogeneous element.

1. Basics

Fix two positive integers \(m\) and \(n > 1\). Let \(O(m)\) be the divided power algebra over \(\mathbb{F}\) with basis \(\{x^{(\alpha)} | \alpha \in \mathbb{N}^m\}\) and \(\Lambda(n)\) the exterior superalgebra over \(\mathbb{F}\) with \(n\) variables \(x_{m+1}, \ldots, x_{m+n}\). The tensor product \(O(m, n) := O(m) \otimes_{\mathbb{F}} \Lambda(n)\) is a supercommutative associative superalgebra in the usual way. For \(g \in O(m)\), \(f \in \Lambda(n)\), write \(gf\) for \(g \otimes f\). Fix two \(m\)-tuples of positive integers \(l := (t_1, t_2, \ldots, t_m)\) and \(\pi := (\pi_1, \pi_2, \ldots, \pi_m)\) such that \(\pi_i := p^i - 1\). The divided power algebra \(O(m)\) contains a finite dimensional subalgebra \(O(m; \underline{l}) := \text{span}_\mathbb{F}\{x^{(\alpha)} | \alpha \in \mathbb{N}^m\}\), where \(\mathbb{N}^m\) is the set of all \(m\)-tuples of non-negative integers. In particular, \(O(m, n)\) has a finite dimensional subalgebra \(O(m, n; \underline{l}) := O(m; \underline{l}) \otimes_{\mathbb{F}} \Lambda(n)\).

Let \(|a| := (\pi_1, \pi_2, \ldots, \pi_m)\) be a \(k\)-shuffle, that is, a strictly increasing sequence of \(k\) integers between \(m + 1\) and \(m + n\). Write \(x^{|a|} := x_1^{|a_1|} x_2^{|a_2|} \cdots x_k^{|a_k|}\). Notice that we also denote the set \(\{i_1, i_2, \ldots, i_k\}\) by the \(k\)-shuffle \(|a|\). The symbol \(|a|\) is a \(k\)-shuffle, and \(\pi := (\pi_1, \pi_2, \ldots, \pi_m)\). For short, put \(I_0 := \{m, \ldots, m+n\}\) and \(I := \{m, \ldots, m+n\}\). For a proposition \(P\), put \(\delta_P := 1\) if \(P\) is true and \(\delta_P := 0\) otherwise. For \(\varepsilon_i := (\delta_{i_1}, \ldots, \delta_{i_m})\), we abbreviate \(x^{(\varepsilon_i)}\) to \(x_i\) for \(i \in I_0\). Let \(\partial_i\) be the Special superderivation of \(O(m, n)\) such that \(\partial_i(x_j) = \delta_{ij}\) for \(i, j \in I\).

From now on, we adopt the convention \((m, n; \infty) = (m, n)\). For example, we have \(O(m, n; \infty) = O(m, n)\). Let us introduce eight families of \(\mathbb{Z}\)-graded Lie superalgebras of Cartan-type as follows:

(1.1) The generalized Witt superalgebra \(W(m, n; \underline{l})\) is spanned by all \(f_r \partial_r\), where \(f_r \in O(m, n; \underline{l}), r \in I\).

(1.2) Let \(\text{div} : W(m, n; \underline{l}) \to O(m, n; \underline{l})\) be the divergence, which is an even linear operator such that \(\text{div}(f \partial_k) = (-1)^{|\partial_k|} f\partial_k(f)\) for all \(k \in I\). For \(i, j \in I\), let \(D_{ij} : O(m, n; \underline{l}) \to W(m, n; \underline{l})\) be a linear operator such that for \(a \in O(m, n; \underline{l})\), \(D_{ij}(a) := (-1)^{|\partial_i|+|\partial_j|} \partial_i(a) \partial_j(a) - (-1)^{|\partial_i|}|\alpha| \partial_i(a) \partial_j\). The Special superalgebra is \(S(m, n; \underline{l}) := \{D \in W(m, n; \underline{l}) | \text{div}(D) = 0\}\).

It is the derived algebra of \(\mathbb{F}(m, n; \underline{l}) := \{D \in W(m, n; \underline{l}) | \text{div}(D) \in \mathbb{F}\}\).

Moreover, the derived algebra of \(S(m, n; \underline{l})\),
\[S(m, n; \underline{l})(1) := \text{span}_\mathbb{F}\{D_{ij}(a) | a \in O(m, n; \underline{l}), i, j \in I\},\]

is a simple Lie superalgebra.

Write \(m = 2r\) or \(2r + 1\). Let \(I\) be the involution of \(I\) such that \(i' = i + r\) for \(i \in I, r\) and \(i' = i\) for \(i \in I_1\). We also use the mapping \(\sigma : I \to \{1, -1\}\) given by \(\sigma(i) = -1\) for \(i \in r + 1, 2r\) and \(\sigma(i) = 1\) otherwise.

(1.3) Suppose \(m = 2r\) is even. Let \(D_H : O(m, n; \underline{l}) \to W(m, n; \underline{l})\) be an even linear operator given by \(D_H(a) := \sum_{i \in I} \sigma(i)(-1)^{|\partial_i|}\partial_i(a)\partial_i\). The Hamiltonian superalgebra is...
Superderivations for moular Graded Lie supralgebras

\[ H(m, n; \mathfrak{L}) = \text{span}_F \{ D_H(a) \mid a \in \mathcal{O}(m, n; \mathfrak{L}) \}. \]

Its derived algebra is simple. While \( H(m, n; \mathfrak{L}) \) is the derived algebra of the Lie superalgebra

\[ \mathfrak{H}(m, n; \mathfrak{L}) := \mathfrak{H}(m, n; \mathfrak{L})_0 \oplus \mathfrak{H}(m, n; \mathfrak{L})_1, \]

where for \( \alpha \in \mathbb{Z}_2 \),

\[ \mathfrak{H}(m, n; \mathfrak{L})_\alpha = \left\{ \sum_{i \in I} a_i \partial_i \in W(m, n; \mathfrak{L})_\alpha \mid \partial_i(a_{i'}) = (-1)^{|i||i'|+|i|+|i'|+|s(i)||s(j)|} \sigma(i)\sigma(j)\partial_j(a_{i'}), i, j \in I \right\}. \]

Write \( \mathcal{O}(m, n; \mathfrak{L}) \) for the quotient superspace \( \mathcal{O}(m, n; \mathfrak{L})/F \cdot 1 \) and view \( D_H \) as the linear operator of \( \mathcal{O}(m, n; \mathfrak{L}) \). One sees that \( H(m, n; \mathfrak{L}) \cong (\mathcal{O}(m, n; \mathfrak{L}), \cdot, \cdot_H) \), where the bracket is: \([a, b]_H := D_H(a)(b) \) for \( a, b \in \mathcal{O}(m, n; \mathfrak{L}) \).

(1.4) Suppose \( m = 2r + 1 \) is odd. The contact superalgebra is by definition

\[ K(m, n; \mathfrak{L}) := \text{span}_F \{ D_K(a) \mid a \in \mathcal{O}(m, n; \mathfrak{L}) \}. \]

\[ D_K(a) := - \sum_{i \in I \setminus \{m\}} (-1)^{|i||a|} \left( x_i \partial_m(a) + \sigma(i') \partial_{i'}(a) \right) \partial_i + \left( 2a - \sum_{i \in I \setminus \{m\}} x_i \partial_i(a) \right) \partial_m. \]

We have a Lie superalgebra isomorphism

\[ K(m, n; \mathfrak{L}) \cong (\mathcal{O}(m, n; \mathfrak{L}), \cdot, \cdot_K), \]

where the Lie bracket is: \([a, b]_K := D_K(a)(b) - 2\partial_m(f)(g) \). Note that \( K(m, n; \mathfrak{L})^{(1)} \) is simple.

In the below, we introduce the other four families of Lie superalgebras of Cartan-type. In these cases, suppose \( m > 2 \) and \( n = m \) or \( m + 1 \). Let \( \text{inv} \) be the involution of \( I \) such that \( i = i + m \) for \( i \in I_0 \). When \( n = m \), from \[ \[ \[ \] \] \] we have the following two families of Lie superalgebras.

(1.5) Define an odd linear operator \( T_H : \mathcal{O}(m, m; \mathfrak{L}) \rightarrow W(m, m; \mathfrak{L}) \) such that \( T_H(a) := - \sum_{i \in I} (-1)^{|i||a|} \partial_i(a) \partial_i \) for \( a \in \mathcal{O}(m, m; \mathfrak{L}) \). The odd Hamiltonian superalgebra is

\[ HO(m; \mathfrak{L}) := \text{span}_F \{ T_H(a) \mid a \in \mathcal{O}(m, m; \mathfrak{L}) \}, \]

which is simple. It is the derived algebra of the Lie superalgebra

\[ HO(m; \mathfrak{L}) := HO(m; \mathfrak{L})_0 \oplus HO(m; \mathfrak{L})_1, \]

where for \( \alpha \in \mathbb{Z}_2 \),

\[ HO(m; \mathfrak{L})_\alpha = \left\{ \sum_{i \in I} a_i \partial_i \in W(m, m; \mathfrak{L})_\alpha \mid \partial_i(a_{i'}) = (-1)^{|i||i'|+|i|+|i'|+|s(i)||s(j)|} \sigma(i)\sigma(j)\partial_j(a_{i'}), i, j \in I \right\}. \]

We have a Lie superalgebra isomorphism \( HO(m; \mathfrak{L}) \cong (\mathcal{O}(m, m; \mathfrak{L}), \cdot, \cdot_{HO}) \), where the Lie bracket is: \([a, b]_{HO} := T_H(a)(b) \) for \( a, b \in \mathcal{O}(m, m; \mathfrak{L}) \).
The odd Contact superalgebra is \( SHO(m;\mathfrak{l}) := S(m, m;\mathfrak{l}) \cap HO(m;\mathfrak{l}) \). Its second derived superalgebra is simple. Put \( SKO(m;\mathfrak{l}) := S(m, m;\mathfrak{l}) \cap HO(m;\mathfrak{l}) \).

When \( n = m+1 \), from \([2,10]\) we have the following two families of Lie superalgebras.

(1.7) The odd Contact superalgebra is
\[
KO(m;\mathfrak{l}) := \text{span}_F \{ D_{KO}(a) \mid a \in O(m, m + 1;\mathfrak{l}) \},
\]
where \( D_{KO} : O(m, m + 1;\mathfrak{l}) \longrightarrow W(m, m + 1;\mathfrak{l}) \) is given by
\[
D_{KO}(a) := T_H(a) + (-1)^{|a|} |\partial_{2m+1}(a)| D + (|D| - 2a) |\partial_{2m+1}|
\]
Hereafter, \( D := \sum_{i=1}^{2m} x_i \partial_i \). Note that \( KO(m;\mathfrak{l}) \) is simple and \( KO(m;\mathfrak{l}) \cong (O(m, m + 1;\mathfrak{l}),\cdot,|\cdot|_{KO}) \), where the bracket is
\[
[a, b]_{KO} = D_{KO}(a)(b) - (-1)^{|a|} 2 \partial_{2m+1} (a) b \quad \text{for } a, b \in O(m, m + 1;\mathfrak{l}).
\]

(1.8) Given \( \lambda \in \mathbb{F} \), for \( a \in O(m, m + 1;\mathfrak{l}) \) \( m > 3 \), consider the linear operator \( \text{div}_\lambda \):
\[
\text{div}_\lambda(a) := (-1)^{|a|} 2 \left( \sum_{i=1}^{m} \partial_i \partial_i (a) + D - m \lambda |\text{id}_{O(m, m + 1;\mathfrak{l})}| \partial_{2m+1} (a) \right).
\]
The kernel of \( \text{div}_\lambda \) is called the Special odd Contact superalgebra, denoted by \( SKO(m;\mathfrak{l}) \). Its second derived algebra is simple.

**Convention 1.1.** Hereafter \( X \) denotes \( W, S, H, K, HO, SHO, KO \) or \( SKO \). For simplicity we usually write \( X(\mathfrak{l}) \) for \( X(m, n;\mathfrak{l}) \) \( (X = W, S, H \ or \ K) \) and \( X(m;\mathfrak{l}) \) \( (X = HO, SHO, KO \ or \ SKO) \), where \( t = \infty \) or \( n \). It is convenience to identify \( X(\mathfrak{l}) \) with a finite dimensional subalgebra of \( X(\infty) \) for \( t \neq \infty \).

\( X(\mathfrak{l}) \) and its derived algebras are referred to as the graded Lie superalgebras of Cartan-type.

**Remark 1.2.** When \( t = \infty \), \( S(\mathfrak{l}), H(\mathfrak{l}), K(\mathfrak{l}), SHO(\mathfrak{l})^{(1)} \) and \( SKO(\mathfrak{l})^{(1)} \) are simple.

## 2. Reduction

In this section we establish some technical lemmas to simplify our consideration. Propositions \([2,3]\) and \([2,7]\) play an important role for determining the superderivations of Lie superalgebras of Cartan-type. For later use we first list the heights of the graded Lie superalgebras of Cartan type.

**Remark 2.1.** \( X(\mathfrak{l}) \) has a principal grading satisfying that
\[
zd(x_i) = -zd(\partial_i) = 1 + \delta_{X=K} \delta_{i=m} + \delta_{X=KO} \delta_{i=2m+1} + \delta_{X=SKO} \delta_{i=2m+1},
\]
Let \( h(X) \) denote the height of \( X(\mathfrak{l})^{(2)} \) and put \( \zeta(\mathfrak{l}) := \sum_{i=1}^{m} p^i - m + n \).
Superderivations for modular Graded Lie superalgebras

Heights of Lie superalgebras of Cartan type

| Height $h(X)$ | Lie superalgebra $X$ |
|--------------|------------------|
| $\xi(L) - 1$ | $W, KH$          |
| $\xi(L) - 2$ | $SHO, SKO$ with $m\lambda + 1 \neq 0$ in $\mathbb{F}$ |
| $\xi(L) - 3$ | $H, SKO$ with $m\lambda + 1 = 0$ in $\mathbb{F}$ |
| $\xi(L) - 4$ | $SHO$            |
| $\xi(L) - 5$ | $K$ with $n - m - 3 \neq 0$ in $\mathbb{F}$ |
| $\xi(L) - 6$ | $K$ with $n - m - 3 = 0$ in $\mathbb{F}$ |

Suppose $L$ is a Lie superalgebra and $V$ is an $L$-module. Denote by $\text{Der}(L, V)$ the superderivation space and $\text{Ind}(L, V)$ the inner derivation space. Clearly, $\text{Der}(L, V)$ is an $L$-submodule of $\text{Hom}_\mathbb{F}(L, V)$. Assume in addition that $L = \bigoplus_{r \in \mathbb{Z}} L_r$ is $\mathbb{Z}$-graded and finite-dimensional, and $V = \bigoplus_{r \in \mathbb{Z}} V_r$ is a $\mathbb{Z}$-graded $L$-module. Then the superderivation space inherits a $\mathbb{Z}$-graded $L$-module structure

$$\text{Der}(L, V) = \bigoplus_{r \in \mathbb{Z}} \text{Der}_r(L, V).$$

As in the usual, write

$$\text{Der}^-(L, V) := \text{span}_\mathbb{F}\{\phi \in \text{Der}_i(L, V) \mid i < 0\}.$$

Let $T \subseteq L_0 \cap L_0^*$ be a torus of $L$ with the weight space decompositions:

$$L = \bigoplus_{\alpha \in \Theta} L_\alpha, \quad V = \bigoplus_{\beta \in \Delta} V_\beta.$$

Then there exist subsets $\Theta_i \subseteq \Theta$ and $\Delta_i \subseteq \Delta$ such that $L_i = \bigoplus_{\alpha \in \Theta_i} L_\alpha \cap L_\alpha$ and $V_j = \bigoplus_{\beta \in \Delta_j} V_\beta \cap V_\beta$. Hence $L$ and $V$ have the corresponding $\mathbb{Z} \times T^*$-grading structures, respectively. Of course $\text{Der}(L, V)$ inherits a $\mathbb{Z} \times T^*$-grading from $L$ and $V$ as above. A superderivation $\phi \in \text{Der}(L, V)$ is called a weight-derivation if it is $T^*$-homogeneous. Write $\theta$ for the zero weight.

**Lemma 2.2.** A weight-derivation $\phi \in \text{Der}(L, V)$ is inner if it is a nonzero weight-derivation. In particular, any derivation $\psi \in \text{Der}(L, V)$ is inner modulo a derivation of zero weight-derivation.

**Proof.** Suppose $\phi \in \text{Der}_{(\alpha)}(L, V)$ and $\alpha \neq \theta$. Since $\text{Der}_{(\alpha)}(L, V)$ is $\mathbb{Z}$-graded, write $\phi = \sum_{\alpha \in \mathbb{Z}} \phi_\alpha$, where $\phi_\alpha \in \text{Der}_{(\alpha)}(L, V)$. Then there exists $t \in T$ with $\alpha(t) \neq 0$ such that for arbitrary $x \in L$,

$$\alpha(t)\phi_\alpha(x) = (t \cdot \phi_\alpha)(x) = t \cdot (\phi_\alpha(x)) - \phi_\alpha([t, x]) = x \cdot (\phi_\alpha(t)).$$

Hence $\phi_\alpha(x) = x \cdot (\alpha(t)^{-1}\phi_\alpha(t))$, which implies that $\phi_\alpha$ is inner. Then $\phi$ is inner. \(\square\)

Analogous to [13, Proposition 3.3.5 and Lemma 4.7.1], we have

**Lemma 2.3.** Let $L = \bigoplus_{i \neq 0} L_i$ be a finite dimensional $\mathbb{Z}$-graded simple Lie superalgebra. The following statements hold:

1. $L_{-r}$ and $L_h$ are irreducible $L_0$-modules.
2. $[L_0, L_h] = L_h$, \quad $[L_0, L_{-r}] = L_{-r}$.
3. $C_{L_{-h-1}}(L_1) = 0$, \quad $[L_{-h-1}, L_1] = L_h$.
4. $C_L(\oplus_{i \geq 0} L_i) = L_h$, \quad $C_L(\oplus_{i < 0} L_i) = L_{-r}$. 
If \( M \subseteq L \) is a subalgebra containing \( L_{-1} \oplus L_1 \) and if \( M \cap L_{-1} \neq 0 \), then \( M = L \). □

Analogous to Lemma 2.1.3, we have

**Lemma 2.4.** Let \( V \) be an arbitrary vector superspace over \( \mathbb{F} \). Suppose \( A_1, A_2, \ldots, A_k \in \text{End}_\mathbb{F}V \) span an abelian sub-Lie superalgebra of \( \mathfrak{gl}(V) \). Suppose further each \( A_i \) is generalized invertible, that is, there is \( B_i \in \text{End}_\mathbb{F}V \) with \( |B_i| = |A_i| \) such that

\[
A_i B_i A_i = A_i, \quad 1 \leq i \leq k,
\]

\[
A_i B_j = (-1)^{|A_i||B_j|} B_j A_i, \quad i \leq j \leq k.
\]

If \( v_1, v_2, \ldots, v_k \in V \) satisfy:

\[
A_i B_i (v_i) = v_i, \quad 1 \leq i \leq k,
\]

\[
A_i (v_j) = (-1)^{|A_i||A_j|} A_j (v_i), \quad 1 \leq i, j \leq k,
\]

then there exists \( v \in V \) such that \( A_i(v) = v_i \) for all \( 1 \leq i \leq k \). □

For \( i \in \mathbf{I} \), define a linear operator \( \Phi_i : W(\mathfrak{L}) \rightarrow W(\mathfrak{L}) \), such that

\[
\Phi_i(x_\varepsilon) := \left\{ \begin{array}{ll}
x^{(\alpha+\varepsilon)} x^\mu \partial_j, & \text{if } i \in I_0; \\
x^{(\alpha)} x^\mu \partial_j, & \text{if } i \in I_1.
\end{array} \right.
\]

When \( \mathfrak{L} \neq \infty \) we adopt the convention that \( x^{(\alpha+\varepsilon)} = 0 \) whenever \( \alpha + \varepsilon \notin \mathbb{Z}(m; \mathfrak{L}) \). Clearly, \( \Phi_i \) is of \( \mathbb{Z} \)-degree \( i \) and \( |\Phi_i| = |\partial_i| \). An element \( D \in W(\mathfrak{L}) \) is called \( i \)-integral if \( \partial_i \Phi_i(D) = D \).

**Lemma 2.5.** Let \( L \) be a \( \mathbb{Z} \)-graded subalgebra of \( W(\mathfrak{L}) \) with depth \( r \) such that \( L_{-r} = \text{span}_\mathbb{F}\{\partial_j \mid j \in J(k)\} \) for some \( k \in \mathbf{I} \), where \( J(k) = \{i_1, \ldots, i_k\} \) is the set of \( k \) integers in \( \mathbf{I} \). Then \( \phi(\partial_j) \) is \( j \)-integral for any \( \phi \in \text{Der}(L, W(\mathfrak{L})) \). Furthermore, there exists \( D \in W(\mathfrak{L}) \) such that \( \phi - \text{ad}D \) vanishes on \( L_{-r} \).

**Proof.** For \( j \in \mathbf{I}_1 \), we have \( [\partial_j, \phi(\partial_j)] = 0 \) and \( \phi(\partial_j) \) is \( j \)-integral. Suppose \( j \in \mathbf{I}_0 \). It is clear that the elements of \( W(\mathfrak{L}) \) are \( j \)-integral for all \( j \in \mathbf{I}_0 \). For \( \mathfrak{L} \neq \infty \), \( \text{Der}(W) \) is a restricted Lie superalgebra with respect to the \( p \)-power and, consequently,

\[
\text{ad}((\text{ad}\partial_j)^{p^{j-1}} \phi(\partial_j)) = [\phi, (\text{ad}\partial_j)^{p^j}] = 0.
\]

Since \( W(\mathfrak{L}) \) is simple, we have \( (\text{ad}\partial_j)^{p^{j-1}} \phi(\partial_j) = 0 \) and then \( \phi(\partial_j) \) is \( j \)-integral. Without loss of generality one may assume that \( \phi \) is \( \mathbb{Z}_2 \)-homogeneous. For \( \partial_j, \partial_s \in L_{-r} \), since \( [\partial_j, \partial_s] = 0 \), it follows that

\[
\text{ad}\partial_j((-1)^{|\phi||\partial_j|} \phi(\partial_s)) = (-1)^{|\phi||\partial_j|} \text{ad}\partial_s((-1)^{|\phi||\partial_j|} \phi(\partial_j))
\]

Put \( V = W(\mathfrak{g}) \), \( A_i = \text{ad}\partial_j \), \( B_i = \Phi_j \) and \( v_1 = (-1)^{|\phi||\partial_j|} \phi(\partial_j) \). Form Lemma 2.4, we can find \( D \in W(\mathfrak{g}) \) such that \( \phi - \text{ad}D \) vanishes on \( L_{-r} \).

For short, write \( L^- \) for \( \sum_{i \geq 0} L_i \) when \( L \) is a \( \mathbb{Z} \)-graded Lie superalgebra.

**Proposition 2.6.** For any \( \phi \in \text{Der}(X(\mathfrak{L}), W(\mathfrak{L})) \), there exists \( D \in W(\mathfrak{L}) \) such that \( \phi - \text{ad}D \) vanishes on \( X^-(\mathfrak{L}) \). If \( X(\mathfrak{L}) \) is finite dimensional and \( \phi \in \text{Der}(X(\mathfrak{L}), W(\mathfrak{L})) \) for \( k \geq -r + 1 \), where \( r \) is the depth of \( X(\mathfrak{L}) \), then there exists \( D \in W(\mathfrak{L}) \) such that \( \phi - \text{ad}D \) vanishes on \( X(\mathfrak{L}) \).
Proof. Without loss of generality one may assume that $\phi$ is $\mathbb{Z}_2$-homogeneous. Obviously, the conclusions hold for $X = W, S, H, HO$ or $SHO$.

For $X = K$, by Lemma 2.5, there exists $D_1 \in W(\mathfrak{l})$ such that $(\phi - adD_1)(1) = 0$. Put $\varphi = \phi - adD_1$. Then for any $r, q \in \mathcal{I}(m)$, we have $\partial_m(\varphi(x_r)) = 0$. It follows that

$$\sigma(r')\partial_r\bigl((-1)^{|r'|+1}|\partial_r|\varphi(x_{r'})\bigr) = (-1)^{|r'|+1}|\partial_r|\sigma(q')\partial_q\bigl((-1)^{|q'|+1}|\partial_q|\varphi(x_{q'})\bigr). \quad (2.1)$$

Put $A_r = \sigma(r')\partial_r$, $B_r = \Phi_r$, and $v_r = (-1)^{|(\phi)|r'}\varphi(x_{r'})$ for $r \in \mathcal{I} \setminus m$. From (2.1) we have by a direct computation that $v_r$ is $r$-integral. By Lemmas 2.4 and 2.5, there exists $D_2 \in W(\mathfrak{l})$ with $\partial_m(D_2) = 0$ such that $A_r(D_2) = v_r$. It is easy to prove that there exists $D \in W(\mathfrak{l})$ such that $\phi - adD$ vanishes on $K^{-} (\mathfrak{l})$.

For $X = KO$ or $SKO$, the proof is similar. When $X(\mathfrak{l})$ is finite dimensional, by induction on $i$ and Lemma 2.5, we obtain that $\phi - adD$ vanishes on $X(\mathfrak{l})$ and $D \in W(\mathfrak{l})\_{\mathcal{H}}$.

\begin{proposition}
Let $\phi \in \text{Der}(X(\infty))$. If $\phi(X^- (\infty)) = 0$, then $\phi$ leaves $X(\mathfrak{l})$ invariant for any $\mathfrak{l} \neq \infty$.
\end{proposition}

\begin{proof}
Let $E \in X(\infty)$. A sufficient and necessary condition for $E \in X(\mathfrak{l})$ is that

1. for $X = W, S, H, HO$ or $SHO$, $(\text{ad}\partial_i)^{p_i}(E) = 0$ for all $i \in \mathcal{I}_0$;
2. for $X = KO$ or $SKO$, $(\text{ad}\partial_i)^{p_i}(E) = 0$ for all $i \in \mathcal{I}_0 \setminus \{2m + 1\}$;
3. for $X = K$, $(\text{ad}\partial_i)^{p_i}(E) = (\text{ad}1)^{p^{m}_m}(E) = 0$ for all $i \in \mathcal{I}_0 \setminus \{m\}$.

Since $\phi(X^- (\infty)) = 0$ we obtain that

$\phi, \text{ad}\partial_i = 0$ for all $i \in \mathcal{I}_0$ and $X = W, S, H, HO$ or $SHO$;

$\phi, \text{ad}\partial_i = 0$ for all $i \in \mathcal{I}_0 \setminus \{2m + 1\}$ and $X = KO$ or $SKO$;

$\phi, \text{ad}\partial_i = 0$ for all $i \in \mathcal{I}_0 \setminus \{m\}$ and $X = K$.

Thus we have $\phi(X(\mathfrak{l})) \subset X(\mathfrak{l})$. \qed

In order to prove the next proposition, we establish a technical lemma.

\begin{lemma}
Let $L = HO, SHO, KO$ or $SKO$. Suppose $A_L$ is a subalgebra of $L(\infty)$ and $s \geq 1$. If $L(s)^{(2)} + \mathbb{F}x\{\sigma(p_r^{s+1}-1)\epsilon_i\} \subset A_L$, then $L(s + \epsilon_i)^{(2)} \subset A_L$.
\end{lemma}

\begin{proof}
Note that

$$f_{HO} := x^{(p_r^{s}-2)\epsilon_i}x^\omega \in HO(\mathfrak{k}); \quad f_{KO} := x^{(p_r^{s}-2)\epsilon_i}x^\omega \in KO(\mathfrak{k});$$

$$f_{SHO} := \sum_{j=1}^{m} x^{(p_r^{s}-2)\epsilon_i} \partial_j(x^\omega) \in SHO(\mathfrak{k});$$

$$f_{SKO} := \sum_{j=1}^{m} x^{(p_r^{s}-2)\epsilon_i} \partial_j(x^\omega)$$

$$+ (-1)^{m-1} m \lambda x^{(p_r^{s}-2)\epsilon_i} \partial_j(x^\omega - (2m+1)) \in SKO(\mathfrak{k}).$$

Computing $\left[x^{(p_r^{s+1}-1)\epsilon_i}, f_x\right]_X$, one gets
(1) \( x^{(p+1)p\epsilon_1}x^{\omega_i} \) is \( \in A_{HO} \cap H(\mathfrak{g} + \epsilon_1)_{h(H) - 1} \);

(2) \( x^{(p+1)p\epsilon_1}x^{\omega_i} \) is \( \in A_{KO} \cap K(\mathfrak{g} + \epsilon_1)_{h(K) - 1} \);

(3) \( \sum_{j=1}^m x^{(p+1)p\epsilon_1} \partial_i \partial_j x^{(\omega_i)} \) is \( \in A_{HO} \cap H(\mathfrak{g} + \epsilon_1)_{h(H) - 1} \);

(4) \( A_{SKO} \cap (SKO(\mathfrak{g} + \epsilon_1)_{h(SKO)} + SKO(\mathfrak{g} + \epsilon_1)_{h(SKO) - 1}) \) contains the element

\[ (-1)^m(m\lambda + 3)x^{(p+1)p\epsilon_1} \partial_i x^{(\omega_i)(2m+1)} - \sum_{j=1}^m x^{(p+1)p\epsilon_1} \partial_i \partial_j x^{(\omega_i)}, \]

where \( h(L) \) is the height of \( L(\mathfrak{g} + \epsilon_1) \) (see Remark 2.4). Now the conclusion follows from Lemma 2.3.

**Proposition 2.9.** Let \( A_X \) denote a subalgebra of \( X(\mathfrak{g}) \) and \( \mathfrak{g} \geq 1 \). Put

\[
E_W = x^{(p'\epsilon_1)}\partial_j \text{ for some } j \in I_0; \\
E_S = D_j x^{(p'\epsilon_1)} \text{ for some } j \in I_0 \setminus \{i\}; \\
E_H = E_{SKO} = x^{(p'\epsilon_1)}.
\]

If \( X(\mathfrak{g})_{h(X)} \subseteq A_X \), then \( X(\mathfrak{g})_{h(X)} \subseteq A_X \).

**Proof.** By Lemma 2.3, it is sufficient to show that \( A_X \cap X(\mathfrak{g} + \epsilon_1)_{h(X) - 1} \neq 0 \) or \( A_X \cap X(\mathfrak{g} + \epsilon_1)_{h(X)} \neq 0 \), where \( h(X) \) is the height of \( X(\mathfrak{g} + \epsilon_1)(2) \) (see Remark 2.1).

From [12, Lemma 5.2.6] it follows that

\[
f_W := x^{(p+1)p\epsilon_1} \partial_i \in A_W \cap W(\mathfrak{g} + \epsilon_1)_{h(W) - n}, \quad f_H := x^{(p+1)p\epsilon_1} \partial_i \in A_H \cap H(\mathfrak{g} + \epsilon_1)_{h(H) - n - 1};
\]

\[
f_K := x^{(p+1)p\epsilon_1} \partial_i \in A_K \cap (K(\mathfrak{g} + \epsilon_1)_{h(K) - n} + K(\mathfrak{g} + \epsilon_1)_{h(K) - n - 1});
\]

where \( \rho = i \) if \( i \neq m; \rho = 1 \) if \( i = m \).

Computing \( [f_W, x, x^{\omega_i} \partial_i], [f_H, x^{(2\epsilon_i)} x^{\omega_i}]_H \) and \( [f_K, x^{(2\epsilon_i)} x^{\omega_i}]_K \), respectively, one gets

(1) \( x^{(p+1)p\epsilon_1} x^{\omega_i} \partial_i \in A_W \cap W(\mathfrak{g} + \epsilon_1)_{h(W)} \), where \( l \in I_1 \);

(2) \( x^{(p+1)p\epsilon_1} x^{2\epsilon_1} \partial_i \in A_H \cap H(\mathfrak{g} + \epsilon_1)_{h(H) - 1};
\]

(3) \( x^{(p+1)p\epsilon_1} x^{\omega_i} \partial_i \in A_K \cap (K(\mathfrak{g} + \epsilon_1)_{h(K) - 1} + K(\mathfrak{g} + \epsilon_1)_{h(K) - 1});
\]

It follows that \( X(\mathfrak{g} + \epsilon_1)(2) \subset A_X \) for \( X = W, H \) or \( K \). For \( X = HO \) or \( SHO \), choosing \( j \in I_0 \setminus \{i\} \), we have

\[
x^{(p+1)p\epsilon_1}, x^{(p+1)p\epsilon_1}, x^{(p+1)p\epsilon_1} \in A_X.
\]

Then

\[
x^{(p+1)p\epsilon_1} = ((p-1)(\text{ad}_x)^{(p\epsilon_1)}x)^{p-1}(x^{(p+1)p\epsilon_1}) \in A_X.
\]

Observe that, for \( X = KO \) or \( SKO \),
As in the case $X = HO$, we have $x^{((p^i+1)\varepsilon_i)} \in A_X$. From Lemma 2.8 we obtain that $X(\mathfrak{g} + \varepsilon_i)(2) \subset A_X$ for $X = HO, SHO, KO$ or $SKO$.

For $X = S$, without loss of generality, we may assume inductively that $A_S$ contains $B_l := D_{ij}(x^{(\pi+(l-1)p^i\varepsilon_i)}x^u)$, where $1 \leq l \leq p - 1$. Note that for $1 \leq a, 1 \leq b \leq p - 1, 1 \leq l \leq p,$

\[
\binom{l p^a - b}{p^a} = \binom{1}{1} = l - 1.
\]

Then

\[
D_{ij}(x^{(\pi+(l p^a)\varepsilon_i-\varepsilon_j)}x^u) = l^{-1}[D_{ij}(x^{(p^i+1)\varepsilon_i)}),B_l] \in A_S.
\]

Choose any $k \in I_1$. One sees that $A_S$ contains

\[
[D_{ik}(x^{(\varepsilon_i-\varepsilon_j)}x_k),[D_{ij}(x^{(p^i+1)\varepsilon_i)}),B_l]] = -lB_{l+1},
\]

which implies that $0 \neq B_p = D_{ij}(x^{(\pi+(p-1)p^i\varepsilon_i)}x^u) \in A_S \cap S(\mathfrak{g} + \varepsilon_i)(2)$. The proof is complete.

3. Superderivations

As before, $X = W, S, H, K, HO, SHO, KO$ or $SKO$. Apparently, it is much easier to determine the superderivations of $X(\mathfrak{h})(2)$ than to determine the superderivations of $X(\mathfrak{l})$. On the other hand, $X(\mathfrak{h})(2)$ contains almost the whole elementary information of $X(\mathfrak{l})$ in a sense: $X(\mathfrak{h})(2)$ contains all the formal variables of the underlying superalgebras and all the partial derivatives $\partial_i$ of $X(\mathfrak{l})$. As expected, starting from the superderivation algebras of the “basic” subalgebra $X(\mathfrak{h})(2)$, we are able to determine the superderivations of the “big” algebra $X(\mathfrak{l})$ and its derived algebra no matter $X(\mathfrak{l})$ is finite dimensional or not.

We first introduce two “exceptional” superderivations for $HO$ and $SHO$.

1. By 3.4, $HO(\mathfrak{l})$ has an outer derivation

\[
\Phi : HO(\mathfrak{l}) \to HO(\mathfrak{l}), \quad f \mapsto \sum_{i \in I_0} \partial_i \partial_i(f).
\]

2. By a direct computation, we can show that $SHO(3,3;\mathfrak{l})(2)$ has an outer derivation

\[
\Theta : SHO(\mathfrak{l}) \to SHO(\mathfrak{l}), \quad f \mapsto \tau(f),
\]

where $\tau : \mathcal{O}(\mathfrak{l}) \to \mathcal{O}(\mathfrak{l})$ is a linear operator such that for $\alpha = \alpha_1 \varepsilon_1 + \alpha_2 \varepsilon_2 + \alpha_3 \varepsilon_3$,

\[
\tau(x^{(\alpha)}x^u) = (a_1 x^{(\varepsilon_1)}\partial_2 \partial_3 + a_2 x^{(\varepsilon_2)}\partial_1 \partial_3 + a_3 x^{(\varepsilon_3)}\partial_1 \partial_2)(x^{(\alpha)}x^u),
\]

where

\[
a_i = \begin{cases} 
[(1 + b_j)(\alpha_i + 1)]^{-1} & \alpha_i \not\equiv -1 \pmod{p}, \\
0 & \alpha_i \equiv -1 \pmod{p}, 
\end{cases}
\]

where $b_i = \sum_{j \in \{1,2,3\}} \delta_i, j \not= 0 \delta_j \in \mathfrak{w}$. 

Remark 3.1. In [4], \( \Theta \) mentioned in (2) is neglected by mistake when \( m = 3 \).

As in Lie algebra case, \( X(1)^{(2)} \) is generated by its local part and one may determine its superderivations by a direct computation. Here we single out certain conclusions on the negative superderivations from [1, 3, 5, 11, 14, 16]:

Remark 3.2. Der\(^{-} \) (\( X(1)^{(2)} \)) \( \cong X^{-}(1)^{(2)} \) \( \oplus \) \( \delta_{X=HO}F \Phi \) \( \oplus \) \( \delta_{X=SHO}m_{3}F \Theta \).

Furthermore,
\[ \{ \phi \in \text{Der}(X(1)^{(2)}) \mid \phi(X^{-}(1)^{(2)}) = 0 \} \cong X(1)^{(2)}_{-r} \oplus \delta_{X=HO}F \Phi \oplus \delta_{X=SHO}m_{3}F \Theta, \]

where \( r \) is the depth of \( X(1)^{(2)} \).

Note that \( T := \sum_{i \in I} k_{i}e_{i} \delta_{i} \) is abelian and acts diagonally on \( W(m, n) \). We call \( T_{X} := X \cap T \) the canonical torus of \( X \). Following [12, Lemma 6.1.3], we have

Lemma 3.3. Let \( \tilde{X} \) be a \( \mathbb{Z} \)-graded subalgebra of \( X(\ell) \) containing \( X(\ell)^{(2)} \) and \( Q := \{ \phi \in \text{Der}(\tilde{X}, X(\infty)) \mid \phi(\tilde{X}^{-}) = 0 \} \). Then
\[ Q \cong \tilde{X}^{-}_{r} \oplus \text{span}_{i \in I_{0}} \left\{ \sum_{j=1}^{\infty} F \delta_{i}^{j} \mid i \in I_{0} \right\} \oplus \delta_{X=HO}F \Phi \oplus \delta_{X=SHO(3,3,\ell)}F \Theta, \]

where \( r \) is the depth of \( \tilde{X} \).

Proof. For any \( \phi \in Q \), we can consider the following cases:

Case 1: \( \ell \neq \infty \). From Proposition 2.7 we know that \( \phi \) leaves \( X(\ell)^{(2)} \) invariant. In view of Remark 5.2 we may assume that \( X(\ell)^{(2)} \subset \text{ker} \phi \).

Suppose \( 1 \leq s \leq \ell \) to be maximal element satisfying \( X(s)^{(2)} \subset \text{ker} \phi \). Then
\[ X(s)^{(2)} \subset X^{-}(s)^{(2)} = 0. \]

In addition, for \( X = H, HO \) or \( SHO \),
\[ X(\ell)^{(2)} \subset X^{-}(s)^{(2)} = 0. \]

Whence \( \phi(\tilde{X} \cap X(s)) = 0 \) and \( \phi(\tilde{X} \cap X(s)) = 0 \). The conclusion holds if \( s = \ell \).

Suppose \( s < \ell \) and let \( i_{0} \) be an index such that \( s_{i_{0}} < t_{i_{0}} \). Fix any \( j \in I_{0} \) and consider the elements \( E_{X} \in \tilde{X} \) listed below:
\[ E_{W} = x(p^{s_{i_{0}}+1}e_{i_{0}}); \quad E_{H} = x(p^{s_{i_{0}}+1}e_{i_{0}}); \]
\[ E_{S} = D_{i_{0}j_{0}}(x(p^{s_{i_{0}}+1}e_{i_{0}})); \quad E_{K} = x(p^{s_{i_{0}}+1}e_{i_{0}}); \]
\[ E_{HO} = F_{SHO} = x(p^{s_{i_{0}}+1}e_{i_{0}}); \quad E_{KO} = E_{SKO} = x(p^{s_{i_{0}}+1}e_{i_{0}}). \]

By Proposition 2.7 \( E_{X} \notin \text{ker} \phi \). However, a computation shows that \( [E_{X}, X^{-}(s)] \subset \text{ker} \phi \), whence
\[ [\phi(E_{X}), X^{-}(s)] = 0. \]  \( \text{(3.1)} \)

Let \( T_{X} \) be the canonical torus of \( X \). By Lemma 2.2 we can assume that \( \phi \) is a zero weight-superderivation. Since \( X(\ell) \) is centerless, we have \( \phi(T_{X}) = 0 \). Thus we can obtain the following results.

Case 1: Suppose \( X = W, S, H, HO \) or \( SHO \). 3.3 means \( \phi(E_{X}) \in \sum_{i \in I} k_{i}F \delta_{i} \).

By a direct computation, we can find \( \beta_{X} \in F \) such that
Then $\phi - \beta_X \text{ad}\partial_t^{\beta_0}$ vanishes on $X(\mathfrak{g})^{(2)} + FE_X$.

**Case 2:** Suppose $X = K$. From (3.1) we have $\phi(E_K) \in K(\mathfrak{g})_2$ and therefore, $\phi(E_K) = 2\beta_K \partial_m$ for some $\beta_K \in \mathbb{F}$. Then $\phi - \beta_K \text{ad}\partial_t^{\beta_0}$ vanishes on $K(\mathfrak{g})^{(1)} + FE_K$.

**Case 3:** Suppose $X = KO$ or $SKO$. From (3.1) we have $\phi(E_X) \in X(\mathfrak{g})_2$ and $\phi(E_X) = -2\beta_X \partial_{2m+1}$ for some $\beta_X \in \mathbb{F}$. Then $\phi - \beta_X \text{ad}\partial_t^{\beta_0}$ vanishes on $X(\mathfrak{g})^{(2)} + FE_X$.

Summarizing, one sees from Proposition 2.7 that $\phi$ vanishes on $X(\mathfrak{g} + \varepsilon_{\omega})^{(2)}$ modulo $\beta_X \text{ad}\partial_t^{\beta_0}$. By induction on $\mathfrak{g}$ we may assume that $X(\mathfrak{g})^{(2)} \subset \ker \phi$. It follows that $\phi(X) = 0$ and the proof in this case is complete.

**Case 2:** Let $\mathfrak{g} = \infty$. From Proposition 2.7 we know that $\phi$ leaves $X(\mathfrak{g})^{(2)}$ invariant for any $\mathfrak{g} \neq \infty$. From Case 1 we can assume that

$$
\phi|_{X(\mathfrak{g})^{(2)}} = \text{ad}D_t + \sum_{i=1}^{m} \sum_{j=1}^{\infty} a(t)_{ij} \text{ad}\partial_t^p + \mu_t \delta_X = \text{HO} + \nu_t \delta_X = \text{SHO} \delta_{m=3} \Theta,
$$

where $D_t \in X(\mathfrak{g})^{(2)}_r$ and $r$ is the depth of $X(\mathfrak{g})^{(2)}$ and $a(t)_{ij}, \mu_t, \nu_t \in \mathbb{F}$. A direct computation shows that

$$
a(t)_{ij} = a(s)_{ij} \text{ for } \mathfrak{g} \leq \mathfrak{g};$$

$$
\mu_t = \mu_s, \quad \nu_t = \nu_s, \quad D_t = D_s \text{ for all } \mathfrak{g} \leq \mathfrak{g}_s \in \mathbb{N}_0^m.
$$

Put $D_0 := D_t, \mu_0 := \mu_t, \nu_0 := \nu_t$ and $\varphi := \phi - \text{ad}D_0 - \mu_0 \delta_X = \text{HO} - \nu_0 \delta_X = \text{SHO} \delta_{m=3} \Theta$. Then

$$
\varphi|_{X(\mathfrak{g})^{(2)}} = \sum_{i=1}^{m} \sum_{j=1}^{\infty} a(t)_{ij} \partial_t^p.
$$

For $i \in \overline{1, m}$ and $j > 0$, choose $\lambda^X_{ij} \in \mathbb{F}$ such that $\lambda^X_{ij}$ is the coefficient of $\mathfrak{B}_X$ in $\varphi(\mathfrak{E}_X)$, where $\mathfrak{B}_X, \mathfrak{E}_X \in X(\mathfrak{g})^{(2)}$. Further information is listed below:

| $\lambda^X_{ij}$ | $\mathfrak{B}_X$ | $\mathfrak{E}_X$ |
|------------------|------------------|------------------|
| $\lambda^X_{ij}$ | $x - \partial_t^j$ | $x(p' \varepsilon_i, x - \partial_t^j)$ |
| $\lambda^X_{ij}$ | $D_t(x^\omega)$ | $D_t(x(p' \varepsilon_i, x^\omega))$ |
| $\lambda^X_{ij}$ | $x^\omega$ | $x(p' \varepsilon_i, x^\omega)$ |
| $\lambda^X_{ij}$ | $1$ | $x(p' \varepsilon_i)$ |
| $\lambda^X_{ij}$ | $x^\omega$ | $x(p' \varepsilon_i, x^\omega)$ |
| $\lambda^X_{ij}$ | $x(p' \varepsilon_i, x^\omega)$ | $x(p' \varepsilon_i)$ |
| $\lambda^X_{ij}$ | $x^{\omega - (l)}$ | $x(p' \varepsilon_i, x^{\omega - (l)})$, where $l \neq i$ |
| $\lambda^X_{ij}$ | $x_{2m+1}$ | $x(p' \varepsilon_i, x_{2m+1})$ |
| $\lambda^X_{ij}$ | $x_{2m+1} + m x(x_i x_j)$ | $x(p' \varepsilon_i, x_{2m+1} + m x(x_i x_{2m+1})$ |

Put

$$
\delta^X = \sum_{i=1}^{m} \sum_{j=1}^{\infty} \lambda^X_{ij} \text{ad}\partial_t^p.
$$
Clearly, \((\varphi - \delta X)(X(2)) = 0\). It follows that \((\varphi - \delta X)(\tilde{X}) = 0\) and the proof in this case is complete.

By a computation we are able to show that
\[
\text{Nor}_{W(t)}X(t) = \text{Nor}_{W(t)}X(t^{(1)}) \quad \text{for} \quad X = S, H \text{ or } K;
\]
\[
\text{Nor}_{W(t)}X(t) = \text{Nor}_{W(t)}X(t^{(1)}) = \text{Nor}_{W(t)}X(t^{(2)}) \quad \text{for} \quad X = SHO \text{ or } SKO.
\]

Further information is listed below (c.f. [1, 2, 9–11, 14, 16]):

| Table 3.1: Normalizer of \(X(t)\) in \(W(t)\) |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \(X\)           | \(S\)           | \(H\)           | \(K\)           | \(HO\)          | \(SHO\)         | \(KO\)          | \(SKO\)         |
| \(\text{Nor}\)  | \(\text{S} \oplus F \mathcal{D}\) | \(\text{H} \oplus F \mathcal{D}\) | \(\text{K} \oplus F \mathcal{D}\) | \(\text{HO} \oplus F \mathcal{D}\) | \(\text{SHO} \oplus F \mathcal{D}\) | \(\text{KO} \oplus F \mathcal{D}\) | \(\text{SKO} \oplus F x_1 x_1\) |

Hereafter,
\[
\mathcal{D} = \sum_{i \in I} x_i \partial_i, \quad \text{the degree derivation of } X(t), \text{ where } X = S, H, HO \text{ or } SHO.
\]

Let \(M_{m \times \infty}\) be the vector space of all \(m \times \infty\) matrices over \(F\), that is
\[
M_{m \times \infty} := \left\{ \sum_{i=1}^{m} \sum_{j=1}^{\infty} F e_{ij} \mid e_{ij} \text{ is the unit of } m \times \infty \text{ matrix} \right\},
\]
which is regarded as an abelian subalgebra of \(\text{Der}(L(t))\) by letting
\[
[e_{ij}, D] = [\partial_{ij}^p, D], \quad \text{for any } D \in W(t),
\]
where \(L = X, X^{(1)}, \text{ or } X^{(2)}\).

Put \([e_{ij}, \Phi] = [e_{ij}, \Theta] = 0\).

**Theorem 3.4.**

\[
\text{Der}(L(t)) \cong \left( \text{Nor}_{W(t)}L(t) \right) \oplus \delta_{L=SHO} F \Phi \oplus \delta_{L=SKO} F \Theta \oplus M_{m \times \infty},
\]
where \(L = X, X^{(1)}, \text{ or } X^{(2)}\).

**Proof.** It is a direct result of Proposition 2.6 and Lemma 3.3.

\[\square\]

4. Outer superderivations

In this section, let \(X = W, S, H, K, HO, SHO, KO \text{ or } SKO\) and \(L = X(t), X(t^{(1)}) \text{ or } X(t^{(2)})\). Denote by \(\text{Der}_{\text{out}}(L) := \text{Der}(L)/\text{ad}(L)\) the outer superderivation algebra of \(L\). Write \(\delta'_{i,j} = 1 \text{ if } i \equiv j \pmod{p}; \delta'_{i,j} = 0 \text{ otherwise.}\)

For future reference, we establish the following Lie algebras.

1. Put \(\mathfrak{G}^X(t) := M_{m \times \infty} \text{ when } X = W, K \text{ or } KO. \text{ For } t \neq \infty, \text{ the direct sum of Lie algebras}
\[
\mathfrak{G}^X(t) := \mathfrak{G}^X(t) \oplus \delta'_{m-n,3} F K
\]
is an abelian Lie algebra.
In particular, they are all Lie algebras. Furthermore, from [2, 9, 11, 14, 16] we have

Proof.

(2) Write the direct sum of Lie algebras $\mathfrak{g}^S(L) := M_{m \times \infty} \oplus F^{S}$ and let $V^S := \text{span}_F \{ g^S_i \mid i \in \mathbb{T}, m \}$ be an abelian Lie algebra. Then, for $L \neq \infty$, the semi-direct sum

$$\mathfrak{g}^S(L) := \mathfrak{g}^S(L) \rtimes_{\text{ad}} V^S$$

is a Lie algebra with multiplication $[M_{m \times \infty}, V^S] = 0$ and $[f^S, g^S_i] = g^S_i$.

(3) Write the direct sum of Lie algebras $\mathfrak{g}^H(L) := M_{m \times \infty} \oplus F^{H}$ and let $V^H := \text{span}_F \{ g^H_i \mid i \in \mathbb{T}, m \}$ be an abelian Lie algebras. Then, the semi-direct sum

$$\mathfrak{g}^H(L) := \mathfrak{g}^H(L) \rtimes_{\text{ad}} \delta_{L \neq \infty} V^H$$

is a Lie algebra with multiplication $[M_{m \times \infty}, V^H] = 0$ and $[f^H, g^H_i] = -2g^H_i$. Moreover, for $L \neq \infty$, the semi-direct sum

$$\mathfrak{g}^H(L) := \mathfrak{g}^H(L) \rtimes_{\text{ad}} F^{H}$$

is a Lie algebra with multiplication $[M_{m \times \infty}, f^{H}] = [V^H, f^{H}] = 0$ and $[f^H, f^{H}] = (n - m - 2)f^{H}$.

(4) Let $V^{HO} := \text{span}_F \{ f_1^{HO}, f_2^{HO} \}$ be a Lie algebra given by $[f_1^{HO}, f_2^{HO}] = -2f_2^{HO}$. Then the direct sum of Lie algebras $\mathfrak{g}^{HO}(L) := M_{m \times \infty} \oplus V^{HO}$ is a Lie algebra. Let $V^{HO} := \text{span}_F \{ g_i^{HO} \mid i \in \mathbb{T}, m \}$ be an abelian Lie algebra. Then, the semi-direct sum

$$\mathfrak{g}^{HO}(L) := \mathfrak{g}^{HO}(L) \rtimes_{\text{ad}} \delta_{L \neq \infty} V^{HO}$$

is a Lie algebra with multiplication $[M_{m \times \infty}, V^{HO}] = [f_2^{HO}, V^{HO}] = 0$ and $[f_1^{HO}, \delta_i^{HO}] = -2g^{HO}_i$.

Theorem 4.1. Let $X = W, S, H, K, HO$ or $KO$. The outer superderivation algebras are as follows:

$$\text{Der}_{\text{out}}(X(L)) \cong \mathfrak{g}^X(L);$$
$$\text{Der}_{\text{out}}(X(L)^{(1)}) \cong \mathfrak{g}^X(L), \text{ where } X = S, H, K, \text{ or } \mathfrak{L} \neq \infty.$$ 

In particular, they are all Lie algebras. Furthermore, $\text{Der}_{\text{out}}(K(L)^{(1)}), \text{Der}_{\text{out}}(H(\infty))$ and $\text{Der}_{\text{out}}(X(L))$ are abelian when $X = W, S, H, K$ or $KO$.

Proof. From [2, 9, 11, 14, 16] we have

(1) $\mathfrak{S}(L) = S(L) \oplus F \mathfrak{D}$
$$= S(L)^{(1)} \oplus \delta_{L \neq \infty} \sum_{i \in I_0} F_{x}^{(\sigma, \pi, \varepsilon_i)} x^{w} \partial_{i} \oplus F \mathfrak{D};$$

(2) $\mathfrak{M}(L) = H(L) \oplus \delta_{L \neq \infty} \sum_{i \in I_0} F_{x}^{(\pi, \varepsilon_i)} \partial_{i}$
$$= H(L)^{(1)} \oplus \delta_{L \neq \infty} (F_{x}^{\pi} x^{w} \oplus \sum_{i \in I_0} F_{x}^{(\pi, \varepsilon_i)} \partial_{i});$$

(3) $K(L) = K(L)^{(1)} \oplus \delta_{L \neq \infty} \delta_{n - m, 3}^{\varepsilon} F_{x}^{(\pi)} x^{w};$

(4) $\mathfrak{M}O(L) = HO(L) \oplus \delta_{L \neq \infty} \sum_{i \in I_0} F_{x}^{(\pi, \varepsilon_i)} \partial_{i}$.
From Table 3.1 and Theorem 3.4, by a direct computation we have the desired results.

The structures of $SHO(\mathfrak{l})$ and $SKO(\mathfrak{l})$ are very complicated. Now we consider their outer superderivations.

(1) Let $V^{SHO} := \text{span}_F \{ f_1^{SHO}, f_2^{SHO} \}$ be an abelian Lie algebra and $V^{SHO} := \text{span}_F \{ g_i^{SHO} \mid i \in \mathbb{N} \}$ be an abelian Lie superalgebra with $V^{SHO}_0 = 0$.

Clearly, the direct sum of Lie algebra $\mathfrak{g}^{SHO}(\mathfrak{l}) := M_{m \times \infty} \oplus V^{SHO}$ is an abelian Lie algebra. Then the semi-direct sum

$$\mathfrak{g}^{SHO}(\mathfrak{l}) := \mathfrak{g}^{SHO}(\mathfrak{l}) \ltimes_{ad} \mathfrak{g}^{V^{SHO}}$$

is a Lie (super)algebra with multiplication $[M_{m \times \infty}, V^{SHO}] = [f_{SHO}^1, V^{SHO}] = 0$ and $[f_{SHO}^1, g_i^{SHO}] = -2 g_i^{SHO}$.

For $\mathfrak{l} = \infty$, the semi-direct sum

$$\mathfrak{g}_{1}^{SHO}(\mathfrak{l}) := \mathfrak{g}^{SHO}(\mathfrak{l}) \ltimes_{ad} F_{3}^{SHO}$$

is a Lie algebra with multiplication $[M_{m \times \infty}, f_3^{SHO}] = 0$; $[f_1^{SHO}, f_3^{SHO}] = (m - 2)f_3^{SHO}$; $[f_2^{SHO}, f_3^{SHO}] = f_3^{SHO}$.

For $\mathfrak{l} \neq \infty$, let $\Lambda(m) := \Lambda(m)$ be an abelian Lie superalgebra by letting $\Lambda(m)_i := \Lambda(m)_{i + \infty}$, $i = 0, 1$. Then the semi-direct sum

$$\mathfrak{g}_{1}^{SHO}(\mathfrak{l}) := \mathfrak{g}^{SHO}(\mathfrak{l}) \ltimes_{ad} \Lambda(m)$$

is a Lie superalgebra, having a $\mathbb{Z}_2$-grading structure induced by $\mathfrak{g}^{SHO}(\mathfrak{l})$ and $\Lambda(m)$, with multiplication

$$[f_1^{SHO}, x^u] = (2|u| - m - 2)x^u; \quad [f_2^{SHO}, x^u] = x^u; \quad [x^u, x^v] = 0;$$

$$[M_{m \times \infty}, x^u] = 0; \quad [g_i^{SHO}, x^u] = (-1)^{(i,u)} \delta_{i\in\infty} x^{u-(\bar{i})},$$

for all $x^u, x^v \in \Lambda(m)_i$, where $(-1)^{(i,u)}$ is determined by the equation $\partial_i(x^u) = (-1)^{(i,u)} x^{u-(\bar{i})}$.

Let $V^{SHO}_1 := \text{span}_F \{ f_3^{SHO}, \delta_{m = 3} f_3^{SHO} \}$ and

$$\mathfrak{g}_{2}^{SHO}(\mathfrak{l}) := \mathfrak{g}_{1}^{SHO}(\mathfrak{l}) \oplus V^{SHO}_1$$

be a $\mathbb{Z}_2$-grading space with

$$\mathfrak{g}_{2}^{SHO}(\mathfrak{l})_0 = \mathfrak{g}_{1}^{SHO}(\mathfrak{l})_0 \oplus V^{SHO}_0; \quad \mathfrak{g}_{2}^{SHO}(\mathfrak{l})_1 = \mathfrak{g}_{1}^{SHO}(\mathfrak{l})_1.$$

Then $\mathfrak{g}_{2}^{SHO}(\mathfrak{l})$ is a Lie superalgebra by letting

$$[f_1^{SHO}, f_2^{SHO}] = -4 f_3^{SHO}; \quad [f_2^{SHO}, f_1^{SHO}] = 2 f_3^{SHO};$$

$$[M_{m \times \infty}, f_4^{SHO}] = [V^{SHO}, f_4^{SHO}] = [\Lambda(m), f_4^{SHO}] = 0;$$

$$[f_1^{SHO}, f_5^{SHO}] = [f_2^{SHO}, f_5^{SHO}] = -f_5^{SHO};$$

$$[M_{m \times \infty}, f_5^{SHO}] = [V^{SHO}, f_5^{SHO}] = 0;$$

$$[\partial_j(x^u), f_3^{SHO}] = g_j^{SHO}, \quad [x^u, f_3^{SHO}] = 0; \quad \text{for } j = 1, 2, 3;$$

$$[1, f_5^{SHO}] = 0; \quad [x^u, f_5^{SHO}] = 2^{-1}(3 f_2^{SHO} + f_4^{SHO}); \quad [f_3^{SHO}, f_5^{SHO}] = 1.$$
(2) Write the direct sum of Lie algebras $\mathfrak{g}^{SKO}(\mathfrak{l}) := M_{m \times \infty} \oplus F^f_{SKO}$.

For $\mathfrak{l} = \infty$, the semi-direct sum

$$
\mathfrak{g}^{SKO}(\mathfrak{l}) = \big\{ \mathfrak{g}^{SKO}(\mathfrak{l}) \ltimes_{\text{ad}} F^f_{1} \big| m\lambda - m + 2 \equiv 0 \pmod{p} \text{ or } \lambda = 1; \text{ otherwise.} \big\}
$$

is a Lie algebra with multiplication $[M_{m \times \infty}, f^f_{1}] = 0$ and $[f^f_{SKO}, f^f_{1}] = f^f_{1}$.

For $\mathfrak{l} \neq \infty$, we introduce some symbols for simplicity. Let $(i_1, i_2, \ldots, i_k)$ be a $k$-tuple of pairwise distinct positive integers. Write the integer

$$
l(\lambda, m) := \sum_{k \in \mathfrak{g}^{(\lambda, m)}} \binom{n}{k} + \sum_{k \in \mathfrak{g}^{(\lambda, m)}} \binom{p}{k},
$$

where $\mathfrak{g}^{(\lambda, m)} := \{ k \in \mathbb{N} | m\lambda - m + 2k + l = 0 \in F \}$. Let $V^{SKO} := V_{0}^{SKO} \oplus V_{1}^{SKO}$ be a $\mathbb{Z}_2$-graded vector space where

$$
V_{0}^{SKO} := V_{01} \oplus V_{02}, \quad V_{1}^{SKO} := V_{11} \oplus V_{12};
$$

$$
V_{01} := \text{span}_{F} \{ X_{i_1, \ldots, i_r} | r \in \mathfrak{g}^{(\lambda, m)}, (i_1, \ldots, i_r) \in J(r), m - r \text{ is odd} \};
$$

$$
V_{02} := \text{span}_{F} \{ Y_{j_1, \ldots, j_l} | l \in \mathfrak{g}^{(\lambda, m)}, (j_1, \ldots, j_l) \in J(l), m - l \text{ is even} \};
$$

$$
V_{11} := \text{span}_{F} \{ X_{i_1, \ldots, i_r} | r \in \mathfrak{g}^{(\lambda, m)}, (i_1, \ldots, i_r) \in J(r), m - r \text{ is even} \};
$$

$$
V_{12} := \text{span}_{F} \{ Y_{j_1, \ldots, j_l} | l \in \mathfrak{g}^{(\lambda, m)}, (j_1, \ldots, j_l) \in J(l), m - l \text{ is odd} \};
$$

$$
J(0) := \emptyset, \quad J(r) := \{ (i_1, \ldots, i_r) | 1 \leq i_1 < \cdots < i_r \leq m \}.
$$

Moreover, $V^{SKO}$ is a Lie superalgebra by letting

$$
[V_{01} + V_{11}, V_{01} + V_{11}] = [V_{02} + V_{12}, V_{02} + V_{12}] = 0;
$$

$$
[X_{i_1, \ldots, i_r}, Y_{j_1, \ldots, j_l}] = (-1)^{r(m - r + 1)}[Y_{j_1, \ldots, j_l}, X_{i_1, \ldots, i_r}]
$$

$$
= \left\{ \begin{array}{ll}
0, & (i_1, \ldots, i_r) \neq (j_1 + 1, \ldots, j_m) \\
\delta_{m\lambda - 1} \text{sgn}(\tilde{i}_{r+1}, \ldots, \tilde{i}_m, \tilde{i}_2, \ldots, \tilde{i}_r) \cdot 1, & (i_1, \ldots, i_r) = (j_1 + 1, \ldots, j_m).
\end{array} \right.
$$

Then, the semi-direct sum

$$
\mathfrak{g}^{SKO}(\mathfrak{l}) := \mathfrak{g}^{SKO}(\mathfrak{l}) \ltimes_{\text{ad}} V^{SKO}
$$

is a Lie superalgebra, having a $\mathbb{Z}_2$-grading structure induced by $V^{SKO}$, with multiplication $[M_{m \times \infty}, V^{SKO}] = 0$, $\text{ad} f^f_{SKO} | V^{SKO} = \text{id}_{V^{SKO}}$.

Moreover, the semi-direct sum

$$
\mathfrak{g}^{SKO} := \mathfrak{g}^{SKO}(\mathfrak{l}) \ltimes_{\text{ad}} \delta_{m\lambda - 1} F^f_{1}
$$

is a Lie superalgebra, having a $\mathbb{Z}_2$-grading structure induced by $\mathfrak{g}^{SKO}$, with multiplication $[M_{m \times \infty}, F^f_{1}] = [V^{SKO}, F^f_{1}] = 0$; $[f^f_{SKO}, f^f_{1}] = 2 f^f_{1}$.

Note that $\mathfrak{g}^{SKO} = \mathfrak{g}^{SKO}$ is a Lie algebra when $\delta_{m\lambda - 1} = 0$.

**Theorem 4.2.** Let $X = SHO$ or $SKO$. The outer superderivation algebras are as follows:

$$
\text{Der}_{out}(X(\mathfrak{l})) \cong \mathfrak{g}^{X}(\mathfrak{l});
$$

$$
\text{Der}_{out}(X(\mathfrak{l})(1)) \cong \mathfrak{g}^{X}(\mathfrak{l}), \text{ when } \mathfrak{l} = \infty;
$$
\[ \text{Der}_\text{out}(X(t)^{(i)}) \cong \mathfrak{g}_i^X(t), \quad i = 1, 2 \quad \text{when} \quad t \neq \infty. \]

Moreover, \( \text{Der}_\text{out}(X(t)) \), \( \text{Der}_\text{out}(X(\infty)^{(1)}) \) and \( \text{Der}_\text{out}(SKO(t)^{(i)}) \), \( i = 1, 2 \) in the case \( \delta_{m\lambda,-1} = 0 \) are all Lie algebras. In addition, \( \text{Der}_\text{out}(X(\infty)) \) is abelian.

**Proof.** For \( t \neq \infty \), if \( m > 3 \), the conclusions follow directly from [1, 10]; if \( X = \text{SHO} \) and \( m = 3 \), the conclusions hold from a simple computation. For \( t = \infty \), \( \text{SHO}(t) \cap \mathfrak{s}(t) = \text{SHO}(t)^{(1)} \oplus \mathfrak{f}_{x_1x_1} \)

\[ \text{SKO}(t) = \begin{cases} \text{SKO}(t)^{(1)} \oplus \mathfrak{f}_{x_{1\omega}} & \text{if } m\lambda - m + 2 \equiv 0 \pmod{p} \text{ or } \lambda = 1; \\ \text{SKO}(t)^{(1)} & \text{otherwise}. \end{cases} \]

From Table 3.1 and Theorem 3.4, by a direct computation we have the desired results.

**Remark 4.3.** In the finite dimensional simple case Theorems 4.1 and 4.2 are known [1, 2, 8–10].

**Remark 4.4.** Note that there is an error in the formulation of Theorem 2.11 in [3].

**Remark 4.5.** When \( t < \infty \) denote \( \eta = \sum_{i=1}^{m} t_i \). The following dimension formulas hold:

| \( X \) | \( \text{dim} \left( \text{Der}_\text{out}(X(t)) \right) \) | \( \text{dim} \left( \text{Der}_\text{out}(X(t)^{(1)}) \right) \) | \( \text{dim} \left( \text{Der}_\text{out}(X(t)^{(2)}) \right) \) |
|---|---|---|---|
| \( W \) | \( \eta - m \) | \( \eta - m + 1 \) | \( \eta - m + 1 \) |
| \( S \) | \( \eta - m + 1 \) | \( \eta + 1 \) | \( \eta + 2 \) |
| \( H \) | \( \eta + 1 \) | \( \eta + 2 \) | \( \eta + 2 \) |
| \( K \) | \( \eta - m \) | \( \eta - m + \delta_{m,-3} \) | \( \eta - m + \delta_{m,-3} \) |
| \( HO \) | \( \eta + 2 \) | \( \eta + 2m + 2 \) | \( \eta + 2m + 3 + \delta_{m,3} \) |
| \( SHO \) | \( \eta - m \) | \( \eta - m + 1 \) | \( \eta - m + 1 \) |
| \( KO \) | \( \eta - m + 1 \) | \( \eta - m + 1 + l(\lambda,m) \) | \( \eta - m + 1 + l(\lambda,m) + \delta_{m\lambda,-1} \) |

**References**

[1] W. Bai, W.-D. Liu and L. Ni. Superderivations of the finite dimensional special odd Hamiltonian superalgebras. arXiv: 1007.1098

[2] J.-Y. Fu, Q.-C. Zhang and C.-P. Jing. The Cartan-type modular Lie superalgebra \( KO \). Commun. Algebra. 34(1) (2006): 107–128.

[3] X.-Y. Hua and W.-D. Liu. Derivations of infinite-dimensinal odd Hamiltonian modular Lie superalgebra. J. Math. Res. Expos. 27(4) (2007): 750–754.

[4] V. G. Kac. Lie superalgebras. Adv. Math. 26 (1977): 8–96.

[5] V. G. Kac. Classification of infinite dimensional simple linearly compact Lie superalgebras. Adv. Math. 139 (1998): 1–55.
[6] W.-D. Liu and Y.-H. He. Finite dimensional special odd Hamiltonian superalgebras in prime characteristic. Commun. Contemp. Math. 11(4) (2009): 523–546.

[7] W.-D. Liu and Y.-Z. Zhang. Derivations for the even part of modular Lie superalgebras $W$ and $S$ of Cartan-type. Internat. J. Algebra Computation 17(4) (2007): 661–714.

[8] W.-D. Liu and Y.-Z. Zhang. The outer derivation algebras of finite-dimensional Cartan-type modular Lie superalgebras. Commun. Algebra. 33 (2005): 2131–2146.

[9] W.-D. Liu, Y.-Z. Zhang, and X.-L. Wang. The derivation algebra of the Cartan-type Lie superalgebra $HO$. J. Algebra 273 (2004): 176–205.

[10] W.-D. Liu and J.-X. Yuan. Finite dimensional special odd contact superalgebras over a field of prime characteristic. arXiv: 0911.3466

[11] F.-M. Ma and Q.-C Zhang. Derivation algebra of modular Lie superalgebra $K$ of Cartan-type. J. Math. (Wuhan) 20(4) (2000): 431–435 (in Chinese).

[12] H. Strade. Simple Lie algebras over fields of positive characteristic, I. Structure theory. Walter de Gruyter, Berlin-New York, 2004.

[13] H. Strade and R. Farnsteiner. Modular Lie Algebras and Their Representations. Monographs and Textbooks in Pure and Applied Mathematics, 116, Marcel Dekker, New York, 1988.

[14] Y. Wang and Y.-Z. Zhang. Derivation algebra $\text{Der}(H)$ and central extensions of Lie superalgebras. Commun. Algebra 32 (2004): 4117–4131.

[15] Y.-Z. Zhang. Finite-dimensional Lie superalgebras of Cartan-type over a field of prime characteristic. Chin. Sci. Bull. 42 (1997): 720–724.

[16] Q.-C. Zhang and Y.-Z. Zhang. Derivation algebras of modular Lie superalgebras $W$ and $S$ of Cartan-type. Acta Math. Sci. 20(1) (2000): 137–144.