Conjugate heat transfer in the unbounded flow of a viscoelastic fluid past a sphere

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Abstract

This work addresses the conjugate heat transfer of a simplified PTT fluid flowing past an unbounded sphere in the Stokes regime ($Re = 0.01$). The problem is numerically solved with the finite-volume method assuming axial symmetry, absence of natural convection and constant physical properties. The sphere generates heat at a constant and uniform rate, and the analysis is conducted for a range of Deborah ($0 \leq De \leq 100$), Prandtl ($10^0 \leq Pr \leq 10^5$), Brinkman ($0 \leq Br \leq 100$) and conductivity ratios ($0.1 \leq \kappa \leq 10$), in the presence or absence of thermal contact resistance at the solid-fluid interface. The drag coefficient shows a monotonic decrease with $De$, whereas the stresses on the sphere surface and in the wake first increase and then decrease with $De$. A negative wake was observed for the two solvent viscosity ratios tested ($\beta = 0.1$ and 0.5), being more intense for the more elastic fluid. In the absence of viscous dissipation, the average Nusselt number starts to decrease with $De$ after an initial increase. Heat transfer enhancement relative to an equivalent Newtonian fluid was observed for the whole range of conditions tested. The temperature of the sphere decreases and becomes more homogeneous when its thermal conductivity increases in relation to the conductivity of the fluid, although small changes are observed in the Nusselt number. The thermal contact resistance at the interface increases the average temperature of the sphere, without affecting significantly the shape of the temperature profiles inside the sphere. When viscous dissipation is considered, significant changes are observed in the heat transfer process as $Br$ increases. Overall, a simplified PTT fluid can enhance heat transfer compared to a Newtonian fluid, but increasing $De$ does not necessarily improve heat exchange.

Keywords: viscoelastic fluid, PTT, conjugate heat transfer, finite-volume method, sphere.

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1. Introduction

Heat transfer is a relevant process in several non-Newtonian fluid flows. The plastics industry is a good example of the critical influence of heat in such flows [1]. Indeed, in a typical flowchart the plastic feedstock is first melt from its solid state to flow across several equipment until it gains its final shape upon re-solidification. Heat needs to be either removed from or supplied to the polymer at each processing stage, and both the amount and rate of heat transferred affect the final product quality [1]. Each stage of the polymer processing line needs to take heat transfer into account to be correctly dimensioned. The temperature effect in the flow of Newtonian or non-Newtonian fluids can be also felt in everyday life, for example, in the form of an increased resistance when trying to flow highly viscous fluids like honey or shampoos in cold environments. From a theoretical standpoint, no flow is rigorously isothermal, although the isothermal assumption could be assumed with negligible error in several situations of practical interest.

Unless simplifications are undertaken, heat transfer processes often require the analysis of both solid and fluid phases, which interact between each other. In such conjugate heat transfer (CHT) problems, there is flow in the fluid and heat is exchanged between the different phases. Heat exchangers are perhaps the paradigmatic example of engineering equipment falling in the CHT category.

The non-isothermal flow past a sphere can be also considered a CHT problem when the temperature distribution inside the sphere is non-trivial and results from the solution of the energy equation inside the solid domain. This occurs, for example, in the transient cooling process of a hot sphere placed in a cold stream of fluid. In another example, the sphere could be generating heat at a constant rate and the interest could be in minimizing its surface temperature, as well as avoiding significant thermal hot spots. This is akin to the thermal management of electric circuits [2] and can be directly transposed to the thermal management in microbots. In fact, the non-isothermal flow of a complex fluid past a sphere with volumetric heat generation can be considered the simplistic analogue of an electric-powered microbot swimming inside a blood vessel. Besides, it shall be noted that the flow past a sphere is a canonical benchmark case for numerical methods [3-7] and a base model to obtain correlations for hydrodynamic (drag and lift coefficients) [6, 8, 9] and thermal (Nusselt number) [9, 10] parameters, which can be then generalized to bodies of arbitrary shape.
The non-isothermal flow past a bounded or unbounded sphere has been studied numerically and analytically for Newtonian [9, 10] and non-Newtonian fluids [11-22]. However, most of the studies with non-Newtonian fluids were limited to non-elastic fluids, as power-law [15, 17-20], Bingham [13, 21, 22] and Herschel-Bulkley [14] fluids, and only a few addressed viscoelastic fluids [11, 12, 16]. These studies encompass natural, forced and mixed heat convection, for spheres which are either stationary or in rotation about a fixed axis. However, the focus was given to the fluid, whereas the sphere was simply modeled as a fixed temperature or fixed heat flux boundary condition. Among the few studies concerning viscoelastic fluids [11, 12, 16], which date back to more than 30 years ago, the work of Westerberg and Finlayson [12] explored a larger number of variables. They studied the heat transfer and flow past a sphere for different fluids (Newtonian, Generalized-Newtonian, Phan-Thien-Tanner and Upper-Convected Maxwell), assigning a fixed temperature to the sphere surface and keeping the Reynolds number below $10^3$ [12]. However, there is still a generalized lack of knowledge on the non-isothermal flow of viscoelastic fluids past a sphere, independently of the boundary conditions applied at the sphere surface.

In this work, we aim to reduce this gap of the literature by addressing numerically the conjugate heat transfer between an unbounded sphere and a simplified Phan-Thien-Tanner (PTT) fluid flowing over its surface. A constant and uniform volumetric heat source is prescribed at the interior of the sphere, which heats up the fluid in contact. Heat is transferred to the fluid by conduction and forced convection – natural convection is not considered. The system under study includes both the fluid and the sphere. The analysis is carried out for steady conditions, axisymmetric creeping flow ($Re = 0.01$), for varying Prandtl ($10^0 \leq Pr \leq 10^5$), Brinkman ($0 \leq Br \leq 100$) and Deborah ($0 \leq De \leq 100$) numbers, in the presence or absence of thermal contact resistance at the solid-fluid interface and for different ratios of thermal conductivity between the solid and the fluid ($0.1 \leq \kappa \leq 10$). The two main objectives of this work are, firstly, to elucidate about the physics of the problem under study and, secondly, to provide benchmark data for a CHT problem involving a viscoelastic fluid, which, to the best of our knowledge, is among the first provided in the literature. Moreover, we also briefly discuss important aspects related with the application of a coupled-solution approach to CHT problems. Perhaps not less important is the release in open-source of the non-isothermal solver developed in this work [23], which can find applications in diverse and complex CHT problems involving non-Newtonian fluids [1, 24-26].
The remainder of this work is organized as follows. Section 2 describes the CHT problem under consideration in terms of geometry, mesh and boundary conditions. The governing equations are introduced in Section 3 and in Section 4 we present the finite-volume numerical method adopted to solve such equations. The dimensionless numbers controlling the CHT problem are identified in Section 5 and the main results obtained in this work are presented and discussed in Section 6. Finally, Section 7 ends the text with the main conclusions from this work.

2. Problem description: geometry, mesh and boundary conditions

Consider the unbounded flow around a sphere which is generating heat at a constant and uniform rate in its interior. The geometry for this problem is schematically depicted in Fig. 1. The sphere with radius $R$ is immersed in a circular domain which extends up to $200R$ around the sphere’s center. The computational domain is made large enough such that the surrounding boundaries have a minimum impact on the solution (Appendix A). The outer boundaries of the fluid domain are kept circular to facilitate the meshing procedure and to obtain cells in the fluid domain with low non-orthogonality.

Due to the flow symmetry around the $x$-axis, only a wedge of the total domain is effectively simulated, i.e. the meshes have a single cell in $\theta$ direction. Such simplification does not compromise the accuracy in the range of dimensionless numbers simulated and drastically reduces the computational cost compared to a full 3D simulation.

Three meshes were used to assess the dependency of the numerical solution on spatial resolution (Appendix A). The main characteristics of the meshes are specified in Table 1,
where $\Delta r_{\text{min}}$ and $\Delta \phi_{\text{min}}$ are the minimum cell spacing in the radial and azimuthal directions, respectively, and $n_{r,f}$ and $n_{\phi,f}$ are the number of cells in each of these directions, in the fluid domain. Cells are compressed toward the sphere surface in the radial direction and near the axis in the azimuthal direction. The meshes are block-structured in both the fluid and solid domains with the blocks arranged as shown in Fig. 2. Although optional, the distribution of cells was made perfectly continuous at the solid-fluid interface (Fig. 2).

Table 1 – Characteristics of the computational grids used to assess the numerical accuracy with mesh refinement.

| Mesh | $\Delta r_{\text{mid}}/R = \Delta \phi_{\text{mid}}/R$ | $n_{\phi,f}$ | $n_{r,f}$ | Number of cells |
|------|-----------------|-------------|-------------|-----------------|
|      |                 |             |             | Fluid domain | Solid domain   |
| M1   | 0.0032          | 300         | 234         | 70 200        | 32 200         |
| M2   | 0.0021          | 450         | 351         | 157 950       | 72 450         |
| M3   | 0.0016          | 600         | 468         | 280 800       | 128 800        |

Figure 2 – Distribution of blocks in the structured mesh (drawing not to scale). Roman numerals are for the fluid domain and Arabic numerals are for the solid domain. The distribution of blocks and cells is symmetric in relation to plane $x = 0$. The zoomed view at the bottom is for mesh M1.

The surface of the simulation domain was subdivided into five boundaries: the sphere surface, the axis of symmetry, the two sides of the wedge, a “virtual” inlet at
The following boundary conditions were applied:

- Sphere surface: \( \mathbf{u} = 0 \), \( \nabla p \cdot \mathbf{n} = 0 \), the components of \( \mathbf{\tau} \) are linearly extrapolated [27] and the temperature on each side of the surface obeys Eqs. (6) and (7);
- Axis of symmetry: axial symmetry;
- Sides of the wedge: rotational periodicity;
- Inlet: \( \mathbf{u} = (U, 0, 0) \), \( \nabla p \cdot \mathbf{n} = 0 \), \( \mathbf{\tau} = 0 \) and \( T = T_0 \);
- Outlet: \( \nabla u_i \cdot \mathbf{n} = 0 \), \( p = 0 \), \( \nabla \tau_{ij} \cdot \mathbf{n} = 0 \) and \( \nabla T \cdot \mathbf{n} = 0 \);

3. Governing equations

3.1. Fluid region

In the non-isothermal, incompressible, laminar flow of a viscoelastic fluid, the set of governing equations to be solved are mass conservation, momentum balance, energy conservation and the constitutive equation to evolve the polymeric extra-stresses. Each of these equations is discussed next.

3.1.1. Mass conservation and momentum balance

Mass conservation (Eq. 1) and momentum balance (Eq. 2) can be expressed as

\[
\nabla \cdot \mathbf{u} = 0
\]

\[
\rho_f \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \eta_s \nabla^2 \mathbf{u} + \nabla \cdot \mathbf{\tau}
\]

where \( \mathbf{u} \) is the velocity vector, \( t \) is the time, \( p \) is the pressure, \( \mathbf{\tau} \) is the polymeric extra-stresses tensor, \( \rho_f \) is the fluid density and \( \eta_s \) is the solvent viscosity. The total extra-stresses tensor (\( \mathbf{\tau}' \)) is composed of a solvent contribution (\( \mathbf{\tau}_s \)) and a polymeric contribution (\( \mathbf{\tau}_p \)), such that \( \mathbf{\tau}' = \mathbf{\tau}_s + \mathbf{\tau} = \eta_s \left( \nabla \mathbf{u} + \nabla \mathbf{u}^\top \right) + \mathbf{\tau} \). The viscosity (\( \eta_0 \)) is also split in solvent (\( \eta_s \)) and polymeric (\( \eta_p \)) contributions, such that \( \eta_0 = \eta_s + \eta_p \) is the zero shear-rate viscosity. A Newtonian fluid is obtained for \( \mathbf{\tau} = 0 \) and \( \eta_0 = \eta_s \).

3.1.2. Constitutive equation

Several constitutive equations are available to model the effect of velocity on the polymeric extra-stresses tensor of viscoelastic fluids. In this study, we use the simplified PTT model [28]:

\[
\exp \left( \frac{\lambda}{\eta_p} \text{tr}(\mathbf{\tau}) \right) \mathbf{\tau}^\vee + \lambda \mathbf{\tau} = \eta_p \left( \nabla \mathbf{u} + \nabla \mathbf{u}^\top \right)
\]
where $\varepsilon$ is the extensibility parameter, $\lambda$ is the relaxation time of the fluid and $\tau = \frac{\partial \tau}{\partial t} + \mathbf{u} \cdot \nabla \tau - \nabla \cdot \mathbf{u} - \nabla \mathbf{u}^T \cdot \tau$ represents the upper-convected time derivative. A PTT fluid has an inherent shear-thinning behavior in shear flow. The constant-viscosity Oldroyd-B model is recovered for $\varepsilon = 0$ and this model degenerates into the Upper-Convected Maxwell model if $\eta_s = 0$. It shall be noted that the shear-thinning behavior of the PTT model, besides realistic for several fluids (e.g. polymer melts), allowed us to reach higher Deborah numbers than the Oldroyd-B model.

### 3.1.3. Energy equation

The equation stating the conservation of energy, here expressed in terms of the temperature variable, can be written as (neglecting heat transfer by radiation)

$$
\rho_i c_{p,f} \left( \frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T \right) = \nabla \cdot (k_i \nabla T) + \tau : \nabla \mathbf{u}
$$

where $T$ is the temperature, $k_i$ represents the thermal conductivity of the fluid and $c_{p,f}$ is the specific heat capacity of the fluid.

It is worth noting that the viscous dissipation term, the second term in the RHS of Eq. (4), corresponds to the pure entropy elasticity case of the more general viscous dissipation term proposed by Peters [29]. Under such hypothesis, all mechanical energy is dissipated as heat [30]. The results presented in this work for a Brinkman number of 0 were obtained by removing the viscous dissipation term from Eq. (4).

### 3.1.4. Temperature-dependent properties

In non-isothermal flows, the temperature affects the physical properties of the fluid. However, for small temperature variations the properties can be assumed approximately constant. In addition, some physical properties of some fluids are weakly sensitive to temperature changes. This is, for example, the case of the isobaric specific heat capacity of water, which presents a maximum variation of less than 1% in the interval 0-100 °C. Moreover, considering temperature-dependent properties introduces additional degrees of freedom to the thermal analysis. Therefore, we carried out the simulations assuming constant physical properties. In general, i.e. considering real viscoelastic fluids, while such approximation is acceptable for $\rho$, $c_p$ and $k$, some differences in the results shall be expected regarding $\eta$ and $\lambda$ [7, 31].
3.2. Solid region

3.2.1. Energy equation

In the solid region, i.e. inside the sphere, only the energy equation is solved,

$$\rho_s c_{p,s} \frac{\partial T}{\partial t} = \nabla \cdot \left( k_s \nabla T \right) + S_{T,s}$$  \hspace{1cm} (5)

where $\rho_s$, $c_{p,s}$ and $k_s$ represent the density, specific heat capacity and thermal diffusivity of the solid, respectively, and $S_{T,s}$ is a volumetric energy source. The three physical properties are assumed temperature-independent and uniform over the whole sphere.

3.3. Solid-fluid interface

At the sphere surface, special care needs to be taken regarding the energy equation, which is solved on the both sides of the interface (Eqs. 4 and 5). The conservation of energy imposes that the heat flux crossing the sphere surface must be continuous and equal on both sides. Since at the sphere surface the heat flux is only by conduction, the previous condition is tantamount to

$$k_s \nabla T \big|_{s,j} \cdot \mathbf{n}_s = -k_i \nabla T \big|_{f,i} \cdot \mathbf{n}_i \iff -k_s \frac{\partial T}{\partial r} \bigg|_{R \rightarrow R^*} = -k_i \frac{\partial T}{\partial r} \bigg|_{r \rightarrow R^*}$$  \hspace{1cm} (6)

If contact resistance is considered [25], the temperature can be discontinuous at the interface. In this general case, the temperature also obeys the condition

$$-k_s \nabla T \big|_{s,j} \cdot \mathbf{n}_s = h_{res} \left( T_{s,j} - T_{f,i} \right) \iff -k_s \frac{\partial T}{\partial r} \bigg|_{R \rightarrow R^*} = h_{res} \left( T_{s,j} - T_{f,i} \right)$$  \hspace{1cm} (7)

where $T_{f,i}$ and $T_{s,i}$ represent the temperature at the interface on the fluid and solid sides, respectively, and $h_{res}$ is a heat transfer coefficient characterizing the contact resistance. The inverse of $h_{res}$ defines the thermal contact resistance, $R_C \equiv 1/h_{res}$. In order to keep the LHS of Eq. (7) defined, when $R_C \rightarrow 0$ then $\left( T_{s,j} - T_{f,i} \right) \rightarrow 0$, and this corresponds to perfect thermal contact.

4. Numerical method

The set of equations presented in the previous section was discretized and solved using rheoTool [23], an open-source toolbox implemented in the finite-volume framework of OpenFOAM®. The isothermal solvers available in the toolbox were modified in order to handle non-isothermal flows and multi-region domains in a coupled way. Only these two modifications will be discussed here. The details about the base viscoelastic fluid flow solver and coupled matrices were presented elsewhere [27, 32].
The resulting non-isothermal solver used to obtain the results presented in this work is freely available in rheoTool [23].

**4.1. Solid-fluid temperature coupling**

The numerical implementation of the boundary conditions in Eqs. (6) and (7) can assume several forms. When the energy equation for the solid and fluid phases are solved separately, a simple way is to derive explicit expressions for \( T_{f,i} \) and \( T_{s,i} \) from the discretized form of Eqs. (6) and (7). However, the resulting method is not accurate for transient simulations and may not converge in stiff cases. Imposing explicitly the flux, rather than the temperature itself, is also a possibility [26], but shares the same drawbacks as the previous approach. When a single energy equation including the multiple domains is solved, the method presented by Habla et al. [25] can be applied. The energy equation is conditionally volume-averaged throughout the solid and fluid phases and the boundary conditions arise as an additional diffusive term at interface cells. This method has a simple implementation, but is only first-order accurate [25].

The approach implemented in this work is based on coupled matrices [32] and on the discretization of the diffusive term \( \nabla \cdot (k\nabla T) \) for a varying conductivity [33]. Consider the 1D solid-fluid interface depicted in Fig. 3, where the boundary conditions of Eqs. (6) and (7) apply. It is assumed that the heat exchange between the two regions takes place by conduction, a contact resistance \((1/h_{\text{res}})\) can exist at the interface and, in such case, the temperature on each side of the interface may differ \((T_{i} \neq T_{s})\). The question to be solved is how to express the conductivity at the interface, such that the diffusive term at the interface cells could be expressed as in any interior cell, i.e. as a function of \((T_{s} - T_{i})\)? According to Patankar [33], equating the heat flux at the interface, as it appears in the discrete Laplace operator (using a Gaussian scheme), to the heat flux arising from a heat balance between points F and S (considering serial resistances) results in

\[
k_{i} \left( \frac{T_{s} - T_{f}}{\delta x_{f}} + \frac{T_{f} - T_{i}}{\delta x_{s}} \right) = \frac{T_{s} - T_{f}}{k_{f} + \frac{1}{h_{\text{res}}}} + \frac{T_{f} - T_{i}}{k_{s} + \frac{1}{h_{\text{res}}}} \Rightarrow k_{i} = \frac{\delta x_{f}}{k_{f} + \frac{1}{h_{\text{res}}}} + \frac{\delta x_{s}}{k_{s} + \frac{1}{h_{\text{res}}}}
\]

(8)

In the absence of contact resistance, Eq. (8) is the well-known harmonic average of the conductivities on each side. The standard Laplace operator can be used at cells F and S, as long as the conductivity at the interface is computed with Eq. (8). It should be noted that Eq. (8) assumes that the area normal to the heat transfer direction is constant between points F and S, which is not true for the problem under study since the area increases in
the radial direction. However, as the mesh is refined the distance between points F and S decreases and the variation of area between these two points tends to zero. We confirmed numerically, in a one-dimensional case, that the convergence with mesh refinement is of second-order, even neglecting this small area variation. Therefore, Eq. (8) was used in the form it is presented.

![Diagram](image)

**Figure 3** – Schematic representation of two cells at the solid(s)-fluid(f) interface. The temperature and thermal conductivity are uniform in each cell, but the interface temperature ($T_{i,i}$ and $T_{s,i}$) can be different on each side of the interface due to the serial contact resistance ($R_C$).

In practice, the solution method starts with the assembling of the energy equation in each region. Each matrix of coefficients is then inserted in a coupled matrix [32] and the coefficients of the Laplace operator at interface cells are then modified according to Eq. (8). These coefficients ensure the coupling between the two energy equations. If both the solid and fluid regions are composed exclusively of orthogonal cells and the time-derivatives are removed from the equations, the solution of the coupled matrix is obtained in a single iteration. This contrasts with the multiple iterations needed by the methods referred above to obtain the steady-state temperature field in both regions [25, 26]. Moreover, since the coupling between regions is only at the matrix level, there is no need for the duplication of fields and/or meshes used in some methods [34].

Although the temperature on each side of the interface ($T_{i,i}$ and $T_{s,i}$) could be determined based on the heat flux crossing the interface (equal on both sides), we instead derived expressions for each temperature based on the discretized form of Eqs. (6) and (7),

\[
\begin{align*}
-k_f \frac{T_{f,i} - T_s}{\delta x_f} &= -k_f \frac{T_{f,i} - T_f}{\delta x_f} \\
-k_s \frac{T_{s,i} - T_s}{\delta x_s} &= h_{res} \left( T_{s,i} - T_{f,i} \right) \quad (9)
\end{align*}
\]

After some basic algebraic manipulation of these expressions, which form a system of two equations on two unknowns ($T_{f,i}$ and $T_{s,i}$), we arrive at
\[ T_{s,i} = \frac{h_{res} \left( T_s k_s \delta x_s + T_i k_t \delta x_t \right) + T_s k_s k_t}{h_{res} \left( k_s \delta x_s + k_t \delta x_t \right) + k_s k_t} \] (10)

\[ T_{s,i} = \frac{T_s h_{res} \delta x_s + T_i k_t}{h_{res} \delta x_s + k_t} \] (11)

In the limit of no contact resistance \((h_{res} \rightarrow \infty)\), i.e. perfect thermal contact, the temperature on the two sides of the interface match,

\[ T_{s,i} = T_{s,i} = T_i = \frac{T_s k_s \delta x_s + T_i k_t \delta x_t}{k_s \delta x_s + k_t \delta x_t} \] (12)

It should be noted that Eqs. (9), (10) and (11) could be used instead of the interpolated thermal conductivity to impose the implicit coupling between the energy equation in the two regions.

In Fig. 3, the two cells represented have a matching face. However, the solver was implemented in such a way that non-matching faces are also allowed, using the so-called arbitrary mesh interface (AMI) of OpenFOAM®. In practice, area-weighted matrix coefficients are introduced to allow such interfaces.

**4.2. The log-conformation tensor approach for non-isothermal flows**

The log-conformation tensor approach has been proposed by Fattal and Kupferman [35] to mitigate the high Weissenberg number problem in isothermal flows, i.e. for constant viscosity coefficients and relaxation time. This technique was used in the present work without modifications, since we do not consider temperature-dependent properties. However, it is important to note, as general case, that when the viscosity and relaxation time are temperature-dependent properties and their ratio \( \frac{\eta_p(T)}{\lambda(T)} \) is not constant, the log-conformation tensor approach may need to be modified. It can be shown that an additional term involving the material derivative \( \frac{D}{Dt} \left( \frac{\lambda(T)}{\eta_p(T)} \right) \) needs to be taken into account to ensure the analytical equivalence between the \( \tau \)-based and the log-transformed constitutive equation.

**4.3. Discretization and solution method**

When the physical properties of the fluid are independent of the temperature, as considered in this work, there is a one-way coupling between flow and heat transfer. The flow affects the heat transfer through the viscous dissipation and convective terms in the
energy equation (Eq. 4), but the temperature field has no effect on the flow. In such cases, it is possible to decouple hydrodynamics from heat transfer, which, in practice, allows simulating different heat transfer scenarios from a single set of $p$-$u$-$\tau$ fields [10]. Therefore, the simulations are split into two stages. Firstly, we seek to solve the flow for a given pair of $(De, \beta)$ values, disregarding heat transfer. Secondly, we evaluate several heat transfer scenarios for the computed $p$-$u$-$\tau$ fields.

The solution method adopted to solve for the flow is segregated, using the SIMPLEC algorithm [36] for pressure-velocity coupling and the method presented in [27] to couple polymeric stresses and velocity. The method is inherently transient [27], but steady-state solutions are reached upon time evolution for steady flows. Since only steady-state solutions are sought in this work, the order of accuracy of the schemes used to discretize time-derivatives is of minor importance. A first-order implicit Euler scheme has been employed for such purpose. The dimensionless time-step used in the flow simulations was $\Delta \tilde{t} = 0.0025$ and the simulations converged at $\tilde{t} \approx 200$. As noted by Harlen et al. [37], the steadiness of the drag coefficient can not be used alone to assess convergence, since it typically stabilizes before the stresses in the wake of the sphere. Convective terms were discretized with the CUBISTA [38] high-resolution scheme and the Green-Gauss theorem was used to compute gradients and derivatives (except $\nabla \mathbf{u}$, computed with a least-squares method), where linear interpolation has been used to interpolate values from cell centers to face centers.

The coupled solution method applied to solve the energy equation was described in detail in [32]. The enhanced stability of the coupled solver allows to remove the time-derivatives from the equation. Convergence was typically achieved in less than 20 iterations depending on the conditions (more than one iteration is needed to converge the simulations due to the mesh non-orthogonality in the solid domain). The schemes mentioned for the flow equations were also used to discretize the convective and diffusive terms of the energy equation in the solid and fluid regions.

5. Dimensionless numbers

The governing equations can be rendered dimensionless using the following set of normalized variables (written with a tilde): $\tilde{r} = \frac{rU}{D}$, $\tilde{x} = \frac{x}{D}$, $\tilde{u} = \frac{u}{U}$, $\tilde{\tau} = \frac{\tau D}{\eta_0 U}$, $\tilde{p} = \frac{pD}{\eta_0 U}$ and $\tilde{T} = \frac{T - T_0}{\tilde{q}_w D / k_i} = 6k_i \frac{T - T_0}{S_{1,5} D^2}$. In the case of the dimensionless temperature,
the surface-averaged (non-uniform) heat flux across the interface \( \overline{q_w} \frac{S_{T,s} D}{6} \) is used in the normalization.

Once the dimensionless variables are replaced in the governing equations, a set of dimensionless numbers arise and completely define the problem under study. The dimensionless numbers governing the flow in our study are the Reynolds number, 
\[
Re = \frac{\rho_f U D}{\eta_0},
\]
the Deborah number, 
\[
De = \frac{\lambda U}{D},
\]
the solvent viscosity ratio, 
\[
\beta = \frac{\eta_s}{\eta_0},
\]
and the extensibility parameter of the PTT model (\( \varepsilon \), see Eq. 3). On the other hand, the thermal component of the problem is controlled by the Prandtl number, 
\[
Pr = \frac{\eta_0 c_{p,f}}{k_f},
\]
the Péclet number, 
\[
Pe = Re Pr = \frac{\rho_f U D c_{p,f}}{k_f},
\]
which arises as the product of two other dimensionless numbers, the Brinkman number, 
\[
Br = \frac{6 \eta_0}{S_{T,s} D^2},
\]
the thermal conductivity ratio, 
\[
\kappa = \frac{k_s}{k_f},
\]
and the dimensionless contact resistance, 
\[
\Omega = \frac{k_f}{h_{ns} D}.
\]

The Nusselt number, volume-averaged temperature of the sphere and drag coefficient are dimensionless quantities of interest in the post-processing stage. The Nusselt number corresponds to the dimensionless form of the heat transfer coefficient defined at the sphere surface (fluid side). It varies in the azimuthal direction, such that a local Nusselt number can be defined as
\[
Nu_{\phi} = \frac{D}{(T_{w,\phi} - T_0)} \left( - \frac{\partial T}{\partial r} \right)_{w,\phi},
\]
where subscript \( w,\phi \) denotes a variable evaluated at a given azimuth on the sphere surface, in the fluid side. A surface-averaged Nusselt number (\( \overline{Nu} \)) can be also defined by integrating \( Nu_{\phi} \) over the sphere surface,
\[
\overline{Nu} = \frac{1}{S_s} \int_{S_s} Nu_{\phi} dS
\]
Note that \( T_0 \) in Eq. (13) should be replaced by a bulk temperature to keep the definition consistent with an energy balance at the sphere surface. The definition adopted
for Nuφ, though not rigorous from that standpoint, is widely used in the literature and easy to apply. The existence of negative Nuφ values in this work is a consequence of this choice.

The volume-averaged temperature of the sphere is simply the integral of the temperature field over the sphere volume normalized by the total volume,

$$\overline{T_v} = \frac{1}{V_s} \int_T dV$$

(15)

The drag coefficient (Cd) represents the normalized component of the force exerted by the fluid on the sphere, in the flow direction \( \hat{i} = (1,0,0) \), and is given by

$$C_d = \frac{8}{\rho_t U^2 \pi D^2} \int_{S} \left[-pI + \eta_t \left(\nabla u + \nabla u^T\right) + \tau \right] \cdot \hat{n} \cdot \hat{i} dS$$

(16)

6. Results and discussion

Before applying the non-isothermal solver to the problem under study, several verification tests were carried out to ensure the correctness of the algorithm. Among other tests, we verified the analytical solution for the heat transfer between two slabs of different conductivities with or without contact resistance at the interface [26], the numerical solution for the conjugate heat transfer between a calibrator and a polymer layer [26] and the numerical solution for heat transfer in the laminar flow of a simplified PTT fluid in a pipe [31]. The solver showed a good agreement with the available analytical/numerical solutions in all the tested cases. The results are not shown for conciseness, but some of these tests have been made available as tutorials [23].

All the simulations were performed in the laminar (Stokesian) flow regime, for Re = 0.01. The extensibility parameter of the PTT model was also kept fixed at \( \varepsilon = 0.25 \), which in practice represents concentrated polymers or polymer melts; this is a common value used in the literature [39-42]. The remaining parameters \( (Pr, \kappa, \Omega \) and \( Br \) were varied independently and an individual section is dedicated to the effect of each one for different values of De and two values of \( \beta \) (0.1 and 0.5). However, prior to the thermal analysis we characterize the flow around the sphere in the next section.

6.1. Flow characteristics

The drag coefficient for the range of De tested is presented in Table 2 and plotted in Fig. 4. The drag coefficient decreases continuously with De and seems to approach an horizontal asymptote at high De. The horizontal lines, plotted as dashed lines in Fig. 4, correspond to the drag coefficient for the Newtonian component of the fluid, \( C_d = 24\beta/Re \).
assuming the correlation $C_d = 24/Re$ for Newtonian fluids in Stokes flow [43]. The results suggest that the elastic contribution to the drag coefficient decreases as $De$ increases. A natural consequence of this hypothesis is that the drag coefficient is lower for more elastic fluids, i.e. lower $\beta$ values, which is confirmed by Table 2 and Fig. 4. This is in contrast with the results obtained from a few tests carried out with an Oldroyd-B fluid ($\beta = 0.5$), Fig. 5 and Table 2. For this constant-viscosity viscoelastic fluid, the drag coefficient shows an initial decrease up to $De \approx 1$, followed by a small increase with $De$, apparently showing a positive contribution of elasticity to $C_d$. These results for the Oldroyd-B fluid are in reasonable agreement with the correlation recently proposed in [8] (plotted in Fig. 5), showing a maximum error $\sim 1\%$ for the set of $De$ tested.

Table 2 – Drag coefficient for the unbounded flow of PTT and Oldroyd-B fluids around a sphere.

| $De$ $\beta = 0.1$ | $C_d$ (PTT) | $De$ $\beta = 0.5$ | $C_d$ (Oldroyd-B) $\beta = 0.5$ |
|-----------------|-------------|-----------------|-----------------|
| 0               | 2423        | 0.5             | 2419            |
| 0.1             | 2419        | 1               | 2417            |
| 0.5             | 2360        | 1.5             | 2427            |
| 1               | 2218        | 2               | 2453            |
| 1.5             | 2047        | 2.5             | 2494            |
| 2               | 1877        | 3               | 2180            |
| 3               | 1597        | 4               | 2062            |
| 4               | 1402        | 5               | 1974            |
| 5               | 1264        | 7               | 1906            |
| 7               | 1082        | 10              | 1809            |
| 10              | 922         | 15              | 1715            |
| 15              | 774         | 25              | 1618            |
| 25              | 630         | 50              | 1515            |
| 50              | 490         | 75              | 1406            |
| 75              | 431         | 100             | 1357            |
| 100             | 397         |                 | 1329            |

The velocity and the total stress components aligned with the symmetry axis are plotted in Fig. 6 for the PTT fluid. The profiles are taken along the axis of the simulation domain ($x/R < -1$ and $x/R > 1$) and over the sphere surface ($-1 \leq x/R \leq 1$). The velocity profiles for the lowest $\beta$ value show the existence of a negative wake, i.e. a region of fluid in the wake of the sphere moving in the opposite direction to the sphere (considering the sphere moving in a fluid at rest) [37, 44]. In the laboratory frame of reference, this is seen
as the region of the velocity profile for which $\bar{u}_x > 1$. The negative wake is also present for $\beta = 0.5$, but at a lower extent, as shown in the insets of Figs. 6(a) and (b) containing the maximum axial velocity values reached at each $De$. Regarding the negative wake, the fluid with lower $\beta$ starts forming the wake at a lower $De$, the maximum velocity in the wake is higher, but it also starts decaying at lower $De$ ($De \approx 25$ for $\beta = 0.1$ and $De \approx 50$ for $\beta = 0.5$). Moreover, the negative wake seems to start farther apart from the sphere for increasing $De$, while also extending over a longer region. For the $De$ values not showing a negative wake, it is still interesting to note that the velocity in the wake recovers faster than in the Newtonian case. This is opposite to what is observed for constant-viscosity viscoelastic fluids, which present a slower decay (extended wake) as a result of strain-hardening in the wake of the sphere [37].

![Figure 4](image-url)  
**Figure 4** – Variation of the drag coefficient as a function of $De$. The dashed lines represent $C_d = 24\beta/Re$. The solid lines are only a guide to the eye.
Figure 5 – Comparison of the drag coefficient obtained for a PTT fluid and an Oldroyd-B fluid ($\beta = 0.5$ in both cases). The solid line represents the correlation in [8] for an Oldroyd-B fluid in similar conditions.

![Figure 5](image1.png)

Figure 6 – Velocity (a and b) and total stresses (c and d) profiles. The profiles are taken at the axis for $x/R < -1$ and $x/R > 1$, whereas they are sampled from the sphere surface for $-1 \leq x/R \leq 1$. The insets in (a) and (b) represent the maximum axial velocity as a function of $De$ (the line is a guide to the eye). The insets in (c) and (d) contain the maximum $\tilde{\tau}_{xx}$ component value over the sphere surface (circle symbols) and axis (cross symbols) for different $De$ (the lines are a guide to the eye).

The stress profiles in Figs. 6(c) and (d) display a minimum upstream of the sphere, before they vanish at the stagnation point $x/R = -1$. At the sphere surface, the profiles are bell-shaped and a peak value is reached for $\phi < 90^\circ$. A second peak is observed in the wake of the sphere, less than one radius downstream of the sphere. The insets in Figs. 6(c)
and (d) show the maximum $\tilde{\tau}_{xx}$ for each $De$, at the sphere surface and in the wake. Firstly, it can be seen that $\tilde{\tau}_{xx,\text{max}}$ increases up to a critical $De$ and starts decreasing beyond that point. The critical $De$ is, however, different for the sphere surface and the wake. Secondly, after the critical $De$ is reached, the difference between the two peak values decreases, whereas it increases before the critical $De$, with the peak at the sphere surface being higher than the peak in the wake.

The existence of a critical $De$ beyond which the normal stresses decrease is likely due to the slower elastic response of the fluid. Notwithstanding the non-monotonic behavior of the stress profiles with $De$, the drag coefficient shows a monotonic decrease with $De$, mostly due to shear-thinning and the dominant shear contribution to the drag coefficient.

The contours of velocity and extra-stresses are plotted in Fig. 7 for $De = 2$ and $De = 25$, and the two values of $\beta$. The negative wake can be clearly seen in the velocity contours for $De = 25$ and $\beta = 0.1$. These contours further show that all components of the polymeric stress tensor are higher (in absolute value) for $De = 2$ than for $De = 25$, being also higher for the more elastic fluid ($\beta = 0.1$). It is therefore expectable that the behavior seen in the insets of Figs. 6(a) and (b) for $\tilde{\tau}_{xx}$ can be extended to the remaining components of the polymeric stress tensor.

The results reported above are consistent with related works in the literature concerning PTT fluids. In fact, Afonso et al. [39] studied the flow past a confined cylinder and reported a monotonic decrease of the drag coefficient with $De$, the existence of negative wakes (more intense for lower values of $\varepsilon$) and a similar trend for the stresses over the cylinder (an initial increase with $De$, followed by a decrease from a critical $De$; for $\varepsilon = 0.25$). The monotonic decrease of the drag coefficient with $De$ and the presence of a negative wake was also verified in [40-42] for the confined flow past a sphere.
Figure 7 – Contours of velocity and polymeric stresses for $De = 2$ (left) and $De = 25$ (right). In each subfigure, the upper part is for $\beta = 0.1$ and the lower part is for $\beta = 0.5$. The flow is from left to right.
6.2. Heat transfer

The temperature profiles over the sphere surface presented in the following sections are equal on the solid and fluid side when $\Omega = 0$. In such cases, we will not differentiate between either sides of the sphere.

6.2.1. Prandtl number effect

The results presented in this section are for fixed $Br = 0$, $\kappa = 1$ and $\Omega = 0$. With the purpose of examining the effect of $Pr$ for different $De$ and $\beta$, the CHT problem was solved for $0 \leq De \leq 100$, $10^0 \leq Pr \leq 10^5$ and $\beta = 0.1$, 0.5. The surface-averaged Nusselt number is presented in Table 3 and the volume-averaged temperature of the sphere is shown in Table 4. The surface-averaged Nusselt number is also plotted in Fig. 8 for $\beta = 0.1$ as the ratio between the value at a given $De$ and the value obtained for a Newtonian fluid. This ratio is $\geq 1$ for all the conditions tested and the fluid with lower $\beta$ (more elastic fluid) presents higher values of $\overline{Nu}$. Thus, viscoelasticity enhances heat transfer, but in a non-monotonic manner. Indeed, similar to the stress profiles, the Nusselt number increases up to a critical $De$ and then decreases. In the range of conditions tested, this critical $De$ is always higher for the lower $\beta$ tested and tends to decrease as $Pr$ increases. For example, the critical $De$ is approximately 10-15 at $Pr = 10^2$ and 5-7 at $Pr = 10^5$. These values of critical $De$ are higher than those observed for the stress profiles (inset of Figs. 6c and d), but smaller than the $De$ at which the maximum wake velocity is reached (inset of Figs. 6a and b). Therefore, a direct cause-effect relationship can not be established, although it is clear that the inversion of behavior in the $\overline{Nu}$ – $De$ relation is related to the rearrangement of the velocity field around the sphere, via the convective term in the energy equation, which is itself related to the polymeric stresses. The maximum $De$ reached in the simulations is not high enough to disclose the asymptotic Nusselt number to which the profiles seem to tend for increasing $De$. The increase of the Nusselt number verified at low $De$ is consistent with the results in [12], where the same behavior was reported up to $De = 0.7$ (the maximum $De$ reached in that work for a PTT fluid with $\varepsilon = 0.015$, $\eta_s/\eta_p = 1/8$ and $Pe \leq 100$).

The results for the volume-averaged temperature of the sphere are consistent with the behavior of the Nusselt number. The temperature increase is lower when the fluid is viscoelastic, displaying a minimum at the same $De$ where the Nusselt number peaks.
### Table 3 – Surface-averaged Nusselt number (\( \bar{N}_u \)) for different \( De, \beta \) and \( Pr \) (\( Br = 0, \kappa = 1 \) and \( \Omega = 0 \)).

| \( De \) | \( Pr = 10^0 \) | \( Pr = 10^1 \) | \( Pr = 10^2 \) | \( Pr = 10^3 \) | \( Pr = 10^4 \) | \( Pr = 10^5 \) |
|-------|----------------|----------------|----------------|----------------|----------------|----------------|
|       | \( \beta = 0.1 \) | \( \beta = 0.1 \) | \( \beta = 0.1 \) | \( \beta = 0.1 \) | \( \beta = 0.1 \) | \( \beta = 0.1 \) |
|       | \( \beta = 0.5 \) | \( \beta = 0.5 \) | \( \beta = 0.5 \) | \( \beta = 0.5 \) | \( \beta = 0.5 \) | \( \beta = 0.5 \) |
| 0     | 2.007          | 2.007          | 2.004          | 2.044          | 3.207          | 3.336          |
| 0.1   | 2.007          | 2.007          | 2.004          | 2.044          | 3.207          | 3.336          |
| 0.5   | 2.007          | 2.007          | 2.004          | 2.044          | 3.207          | 3.336          |
| 1     | 2.007          | 2.007          | 2.004          | 2.044          | 3.207          | 3.336          |
| 1.5   | 2.007          | 2.007          | 2.004          | 2.044          | 3.207          | 3.336          |
| 2     | 2.007          | 2.007          | 2.004          | 2.044          | 3.207          | 3.336          |
| 3     | 2.007          | 2.007          | 2.004          | 2.044          | 3.207          | 3.336          |
| 4     | 2.007          | 2.007          | 2.004          | 2.044          | 3.207          | 3.336          |
| 5     | 2.007          | 2.007          | 2.004          | 2.044          | 3.207          | 3.336          |
| 7     | 2.007          | 2.007          | 2.004          | 2.044          | 3.207          | 3.336          |
| 10    | 2.007          | 2.007          | 2.004          | 2.044          | 3.207          | 3.336          |
| 15    | 2.007          | 2.007          | 2.004          | 2.044          | 3.207          | 3.336          |
| 25    | 2.007          | 2.007          | 2.004          | 2.044          | 3.207          | 3.336          |
| 50    | 2.007          | 2.007          | 2.004          | 2.044          | 3.207          | 3.336          |
| 75    | 2.007          | 2.007          | 2.004          | 2.044          | 3.207          | 3.336          |
| 100   | 2.007          | 2.007          | 2.004          | 2.044          | 3.207          | 3.336          |

### Table 4 – Volume-averaged sphere temperature (\( \bar{T} \)) for different \( De, \beta \) and \( Pr \) (\( Br = 0, \kappa = 1 \) and \( \Omega = 0 \)).

| \( De \) | \( Pr = 10^0 \) | \( Pr = 10^1 \) | \( Pr = 10^2 \) | \( Pr = 10^3 \) | \( Pr = 10^4 \) | \( Pr = 10^5 \) |
|-------|----------------|----------------|----------------|----------------|----------------|----------------|
|       | \( \beta = 0.1 \) | \( \beta = 0.1 \) | \( \beta = 0.1 \) | \( \beta = 0.1 \) | \( \beta = 0.1 \) | \( \beta = 0.1 \) |
|       | \( \beta = 0.5 \) | \( \beta = 0.5 \) | \( \beta = 0.5 \) | \( \beta = 0.5 \) | \( \beta = 0.5 \) | \( \beta = 0.5 \) |
| 0     | 0.598          | 0.598          | 0.598          | 0.598          | 0.598          | 0.598          |
| 0.1   | 0.598          | 0.598          | 0.598          | 0.598          | 0.598          | 0.598          |
| 0.5   | 0.598          | 0.598          | 0.598          | 0.598          | 0.598          | 0.598          |
| 1     | 0.598          | 0.598          | 0.598          | 0.598          | 0.598          | 0.598          |
| 1.5   | 0.598          | 0.598          | 0.598          | 0.598          | 0.598          | 0.598          |
| 2     | 0.598          | 0.598          | 0.598          | 0.598          | 0.598          | 0.598          |
| 3     | 0.598          | 0.598          | 0.598          | 0.598          | 0.598          | 0.598          |
| 4     | 0.598          | 0.598          | 0.598          | 0.598          | 0.598          | 0.598          |
| 5     | 0.598          | 0.598          | 0.598          | 0.598          | 0.598          | 0.598          |
| 7     | 0.598          | 0.598          | 0.598          | 0.598          | 0.598          | 0.598          |
| 10    | 0.598          | 0.598          | 0.598          | 0.598          | 0.598          | 0.598          |
| 15    | 0.598          | 0.598          | 0.598          | 0.598          | 0.598          | 0.598          |
| 25    | 0.598          | 0.598          | 0.598          | 0.598          | 0.598          | 0.598          |
| 50    | 0.598          | 0.598          | 0.598          | 0.598          | 0.598          | 0.598          |
| 75    | 0.598          | 0.598          | 0.598          | 0.598          | 0.598          | 0.598          |
Figure 8 – Variation of the surface-averaged Nusselt number as a function of $De$, for several $Pr$ ($\beta = 0.1$, $Br = 0$, $\kappa = 1$ and $\Omega = 0$). The Nusselt number is plotted as the ratio between $\bar{Nu}$ at a given $De$ and $\bar{Nu}$ for $De = 0$, representing the heat transfer enhancement relative to the Newtonian case. The lines are a guide to the eye.

It should be noted that the heat transfer is conduction-dominated up to $Pr \approx 10^2$ ($Pe \approx 1$). In this range, the effect of $De$ is small and the Nusselt number tends to its theoretical value for a pure-conduction problem ($\bar{Nu} \to 2$ as $Pr \to 0$).

Focusing now on a local analysis, Fig. 9 plots $Nu_\phi$ as a function of $\phi$, for $\beta = 0.1$, in the convection-dominated regime ($Pr = 10^5$). The local Nusselt number reaches its maximum in the upstream region of the sphere, and decreases as the fluid heats up and flows over the sphere surface (decreasing $\phi$). The difference between Newtonian and viscoelastic fluids also diminishes in the flow direction. The temperature at the centerline (the interval $-1 \leq x/R \leq 1$ is inside the solid sphere) is plotted in Fig. 10 together with the temperature along the sphere surface. As expected, the surface temperature increases in the flow direction, showing a rapid increase in the rear region of the sphere. At the sphere centerline, the temperature has a quasi-parabolic-like shape, with its center shifted downstream to the sphere’s center. It shall be noted that the theoretical profile expected for a pure-conduction problem ($Pr = 0$) would be a parabola centered with the sphere. In the limit $Pr \to 0$, the temperature only depends on the radial coordinate and the problem reduces to a single dimension. The detachment from this limiting case, as $Pr$ is increased, can be clearly seen in the contours of temperature plotted in Fig. 11, where there is a
break of the fore-aft symmetry (characteristic of heat diffusion) driven by the forced heat convection.

**Figure 9** – Local Nusselt number along the sphere surface for different $De$ ($\beta = 0.1$, $Pr = 10^5$, $Br = 0$, $\kappa = 1$ and $\Omega = 0$). The fluid flows from $\phi/\pi = 1$ to $\phi/\pi = 0$.

**Figure 10** – Temperature profile over the centerline and sphere surface for different $De$ ($\beta = 0.1$, $Pr = 10^5$, $Br = 0$, $\kappa = 1$ and $\Omega = 0$). For $-1 \leq x/R \leq 1$ the temperature is taken at the interior (centerline) and surface of the sphere, whereas outside this interval the temperature is taken over the centerline in the fluid region.
Figure 11 – Contours of temperature ($\tilde{T}$) for different $Pr$ ($\beta = 0.1, Br = 0, \kappa = 1$ and $\Omega = 0$). In each subfigure, the upper part is for $De = 0$ (Newtonian case) and the lower part is for $De = 7$. The black solid line represents the cut of the sphere surface. The flow is from left to right.

6.2.2. Thermal conductivity ratio effect

The results presented in this section are for fixed $Pr = 10^5, Br = 0$ and $\Omega = 0$. The high Prandtl number was selected (and kept in the remaining sections) to ensure a convection-dominated heat transfer regime, where the effect of $De$ is clearer, as seen in the previous section. The simulations with $\kappa = 1$ were conducted in the previous section and we now consider the cases $\kappa = 0.1$ and $\kappa = 10$, in order to assess the effect of this parameter on heat transfer.

The surface-averaged Nusselt number and volume-averaged sphere temperature are presented in Table 5. The differences in the average Nusselt number are small among the three values of $\kappa$ tested, although it is clear that a lower $\kappa$ improves the heat transfer, probably due to an increase of the driving temperature difference ($T_s - T_l$) induced from
the sphere side. The effect of \( \kappa \) is more evident in the volume-averaged temperature of the sphere, with a clear increase of the temperature as the sphere becomes less conductive (\( \kappa \) decreases). Again, the fluid with lower \( \beta \) displays a higher heat transfer coefficient.

**Table 5** – Surface-averaged Nusselt number and volume-averaged sphere temperature for different \( De, \beta \) and \( \kappa \) (\( Pr = 10^3, Br = 0 \) and \( \Omega = 0 \)). Note that the data for \( \kappa = 1 \) is repeated here from Tables 3 and 4 for the ease of comparison.

| \( De \) | \( \kappa = 0.1 \) | \( \kappa = 1 \) | \( \kappa = 10 \) | \( \kappa = 0.1 \) | \( \kappa = 1 \) | \( \kappa = 10 \) |
| --- | --- | --- | --- | --- | --- | --- |
| \( \bar{Nu} \) | \( \bar{\bar{T}}_s \) | \( \bar{Nu} \) | \( \bar{\bar{T}}_s \) | \( \bar{Nu} \) | \( \bar{\bar{T}}_s \) | \( \bar{Nu} \) | \( \bar{\bar{T}}_s \) |
| 0 | 12.044 1.087 | 11.956 0.187 | 11.454 0.099 | 12.044 1.087 | 11.956 0.187 | 11.454 0.099 |
| 0.1 | 12.059 1.087 | 11.971 0.187 | 11.467 0.099 | 12.053 1.087 | 11.965 0.187 | 11.462 0.099 |
| 0.5 | 12.370 1.085 | 12.284 0.185 | 11.770 0.097 | 12.205 1.086 | 12.118 0.186 | 11.611 0.098 |
| 1 | 12.981 1.080 | 12.900 0.181 | 12.387 0.093 | 12.433 1.084 | 12.348 0.184 | 11.843 0.096 |
| 1.5 | 13.512 1.077 | 13.437 0.177 | 12.938 0.089 | 12.592 1.083 | 12.508 0.183 | 12.007 0.095 |
| 2 | 13.889 1.074 | 13.818 0.175 | 13.333 0.087 | 12.691 1.082 | 12.608 0.182 | 12.110 0.094 |
| 3 | 14.318 1.072 | 14.249 0.172 | 13.775 0.084 | 12.788 1.081 | 12.706 0.182 | 12.211 0.094 |
| 4 | 14.519 1.071 | 14.451 0.171 | 13.977 0.083 | 12.823 1.081 | 12.740 0.181 | 12.246 0.093 |
| 5 | 14.616 1.070 | 14.548 0.171 | 14.071 0.082 | 12.830 1.081 | 12.748 0.181 | 12.254 0.093 |
| 7 | 14.672 1.070 | 14.603 0.171 | 14.122 0.082 | 12.814 1.081 | 12.732 0.182 | 12.237 0.094 |
| 10 | 14.628 1.071 | 14.559 0.171 | 14.073 0.082 | 12.768 1.081 | 12.685 0.182 | 12.190 0.094 |
| 15 | 14.479 1.071 | 14.409 0.172 | 13.921 0.083 | 12.691 1.082 | 12.607 0.182 | 12.111 0.094 |
| 25 | 14.195 1.073 | 14.123 0.173 | 13.634 0.085 | 12.575 1.083 | 12.490 0.183 | 11.992 0.095 |
| 50 | 13.730 1.075 | 13.654 0.176 | 13.162 0.088 | 12.416 1.084 | 12.330 0.184 | 11.831 0.097 |
| 75 | 13.448 1.077 | 13.370 0.178 | 12.877 0.089 | 12.334 1.085 | 12.248 0.185 | 11.748 0.097 |
| 100 | 13.267 1.078 | 13.188 0.179 | 12.693 0.091 | 12.283 1.085 | 12.197 0.185 | 11.696 0.098 |

The temperature profiles along the centerline are now discontinuous (same temperature at the interface but different temperature gradients on each side of the interface), as required to ensure the continuity of the heat flux when there is a mismatch in the thermal conductivities (Fig. 12). Moreover, a higher value of \( \kappa \) leads to a more homogeneous distribution of temperature inside the sphere and increases the deviation of the temperature profile at the centerline from the parabolic shape (Fig. 12).

### 6.2.3. Thermal contact resistance effect

The results presented in this section are for fixed \( Pr = 10^5, Br = 0 \) and \( \kappa = 1 \). The parameter tested was the thermal contact resistance. Three values of \( \Omega \) were tested, \( \Omega = 0.05, 0.1 \) and 0.5. We remember that \( \Omega = 0 \) represents perfect thermal contact, which has been addressed in the previous sections.
Figure 12 – Temperature profile along the centerline and sphere surface for (a) $\kappa = 0.1$ and (b) $\kappa = 10$ ($De = 7$, $\beta = 0.1$, $Pr = 10^5$, $Br = 0$ and $\Omega = 0$). For $-1 \leq x/R \leq 1$ the temperature is taken at the interior (centerline) and surface of the sphere, whereas outside this interval the temperature is taken along the centerline in the fluid region. The inset in (a) is a zoomed view of the main figure.

The surface-averaged Nusselt number and volume-averaged sphere temperature are listed in Table 6 for the different combinations of $De$ and $\beta$ tested. The higher the contact resistance, the higher the sphere temperature, as expected. Nonetheless, there is only a very slight variation of the average Nusselt number – $\overline{Nu}$ only increases marginally with $\Omega$ (Table 6).

The temperature profiles over the centerline are discontinuous at the interface, presenting a different value on each side, but equal derivative (Fig. 13). The temperature profiles in the fluid side (centerline and sphere surface) change very slightly with $\Omega$. Indeed, at the plotting scale of Fig. 13 it is difficult to distinguish these profiles for the different values of $\Omega$. On the other hand, the effect of $\Omega$ on the temperature profiles in the solid side (centerline and sphere surface) is more notorious (Fig. 13). These profiles are vertically shifted as $\Omega$ increases, with only minor changes of shape at the rear of the sphere.
Table 6 – Surface-averaged Nusselt number and volume-averaged sphere temperature for different $De$, $\beta$ and $\Omega$ ($Pr = 10^5$, $Br = 0$ and $\kappa = 1$).

| De    | $\Omega = 0.05$ | $\Omega = 0.1$ | $\Omega = 0.5$ | $\Omega = 0.05$ | $\Omega = 0.1$ | $\Omega = 0.5$ |
|-------|----------------|----------------|----------------|----------------|----------------|----------------|
|       | $\bar{Nu}$ | $\bar{\tilde{T}}_s$ | $\bar{Nu}$ | $\bar{\tilde{T}}_s$ | $\bar{Nu}$ | $\bar{\tilde{T}}_s$ | $\bar{Nu}$ | $\bar{\tilde{T}}_s$ | $\bar{Nu}$ | $\bar{\tilde{T}}_s$ | $\bar{Nu}$ | $\bar{\tilde{T}}_s$ |
| 0     | 11.962    | 0.237          | 11.968       | 0.287          | 11.998       | 0.687          | 11.962       | 0.237          | 11.968       | 0.287          | 11.998       | 0.687          |
| 0.1   | 11.978    | 0.237          | 11.983       | 0.287          | 12.013       | 0.687          | 11.972       | 0.237          | 11.977       | 0.287          | 12.007       | 0.687          |
| 0.5   | 12.290    | 0.235          | 12.295       | 0.285          | 12.324       | 0.685          | 12.124       | 0.236          | 12.130       | 0.286          | 12.159       | 0.686          |
| 1     | 12.906    | 0.231          | 12.911       | 0.281          | 12.937       | 0.681          | 12.354       | 0.234          | 12.360       | 0.284          | 12.388       | 0.684          |
| 1.5   | 13.441    | 0.227          | 13.446       | 0.277          | 13.470       | 0.677          | 12.514       | 0.233          | 12.519       | 0.283          | 12.547       | 0.683          |
| 2     | 13.822    | 0.225          | 13.826       | 0.275          | 13.849       | 0.675          | 12.614       | 0.232          | 12.619       | 0.282          | 12.646       | 0.682          |
| 3     | 14.253    | 0.222          | 14.257       | 0.272          | 14.279       | 0.672          | 12.712       | 0.232          | 12.717       | 0.282          | 12.744       | 0.682          |
| 4     | 14.455    | 0.221          | 14.458       | 0.271          | 14.480       | 0.671          | 12.746       | 0.231          | 12.752       | 0.281          | 12.779       | 0.681          |
| 5     | 14.551    | 0.221          | 14.555       | 0.271          | 14.577       | 0.671          | 12.754       | 0.231          | 12.759       | 0.281          | 12.786       | 0.681          |
| 7     | 14.607    | 0.221          | 14.610       | 0.271          | 14.632       | 0.671          | 12.738       | 0.232          | 12.743       | 0.282          | 12.770       | 0.681          |
| 10    | 14.562    | 0.221          | 14.566       | 0.271          | 14.588       | 0.671          | 12.691       | 0.232          | 12.697       | 0.282          | 12.724       | 0.682          |
| 15    | 14.413    | 0.222          | 14.417       | 0.272          | 14.439       | 0.672          | 12.613       | 0.232          | 12.619       | 0.282          | 12.647       | 0.682          |
| 25    | 14.127    | 0.223          | 14.131       | 0.273          | 14.155       | 0.673          | 12.496       | 0.233          | 12.502       | 0.283          | 12.530       | 0.683          |
| 50    | 13.659    | 0.226          | 13.663       | 0.276          | 13.688       | 0.676          | 12.336       | 0.234          | 12.342       | 0.284          | 12.371       | 0.684          |
| 75    | 13.375    | 0.228          | 13.380       | 0.278          | 13.405       | 0.677          | 12.254       | 0.235          | 12.260       | 0.285          | 12.288       | 0.685          |
| 100   | 13.193    | 0.229          | 13.198       | 0.279          | 13.224       | 0.679          | 12.203       | 0.235          | 12.209       | 0.285          | 12.238       | 0.685          |

Figure 13 – Temperature profile along the centerline and sphere surface for four different values of the contact resistance ($De = 7$, $\beta = 0.1$, $Pr = 10^5$, $Br = 0$ and $\kappa = 1$). In the range $-1 \leq x/R \leq 1$, three different profiles are plotted: the temperature along the sphere centerline (thick lines), the temperature along the sphere surface on the solid side (thick lines) and the temperature along the sphere surface on the fluid side (thin lines). For $\Omega = 0$, the profiles at the surface on the solid and fluid side coincide with each other. Note that the profiles of temperature over the sphere surface on the fluid side are visually indistinguishable from each other.
Similar to the effect observed for $\kappa$, varying the value of $\Omega$ does not influence the non-monotonic behavior of the $\overline{Nu} - De$ relation, and the maximum is still reached at the same $De$. The more elastic fluid ($\beta = 0.1$) also keeps showing a stronger heat transfer enhancement.

### 6.2.4. Brinkman number effect

The results presented in this section are for fixed $Pr = 10^5$, $\kappa = 1$ and $\Omega = 0$. This last set of simulations was performed taking into account viscous dissipation, i.e. $Br \neq 0$. Under these conditions, in addition to the heat generated inside the sphere, there is also heat generation in the fluid side due to viscous dissipation. The simulations were run for $Br = 1, 10$ and $100$.

The heat transfer enhancement relative to the Newtonian case is plotted in Figs. 14(a) and (b) for the several conditions tested. While the $\overline{Nu} - De$ relation is non-monotonic at low $Br$, there is an inversion of behavior at $Br \approx 1$; the curves for $Br = 10$ and $Br = 100$ show a continuous increase of $\overline{Nu}$ with $De$. This is accompanied by a continuous decrease of the average temperature of the sphere with $De$ at such high $Br$ (Table 7). As $Br$ increases, the local Nusselt number decreases over the whole surface of the sphere, even becoming negative in the rear region of the sphere ($\phi \lesssim 57^\circ$) for $Br = 100$ (Fig. 14c). The local temperature profile also suffers changes with $Br$, as shown in Fig. 14(d). While the maximum temperature along the centerline occurs inside the sphere at low $Br$, the peak temperature shifts to the fluid region in the rear of the sphere as $Br$ is increased to 100.

These results can be explained by distinguishing between the range of low $Br$ and the range of high $Br$. For low $Br$, the heat generated by viscous dissipation is negligible compared to the heat generated by the sphere. The sphere transfers heat to the fluid over its whole surface. The layer of fluid surrounding the sphere is colder than the surface of the sphere, but as $Br$ increases this layer becomes warmer due to viscous dissipation, and the average Nusselt number decreases. For high $Br$, the viscous dissipation generates larger amounts of heat and there are regions where the fluid layer adjacent to sphere is hotter than the sphere surface. In those regions, the sphere receives heat from the fluid and $Nu_\phi$ becomes locally negative. In the limit, when $Br \to +\infty$ the heat generated by the sphere is negligible and all the heat transferred arises from viscous dissipation. If further $Pr \to 0$, the temperature of the system becomes homogeneous and $\overline{Nu} \to 0$. 

Table 7 – Surface-averaged Nusselt number and volume-averaged sphere temperature for different De, β and Br (Pr = 10^5, κ = 1 and Ω = 0).

| De | Br = 1 | Br = 10 | Br = 100 | Br = 1 | Br = 10 | Br = 100 |
|----|--------|---------|----------|--------|---------|----------|
|    |  β = 0.1 |         |          | β = 0.5 |         |          |
|    |  Nu   | $\tilde{T}_s$ |  Nu   | $\tilde{T}_s$ |  Nu   | $\tilde{T}_s$ |  Nu   | $\tilde{T}_s$ |  Nu   | $\tilde{T}_s$ |  Nu   | $\tilde{T}_s$ |
| 0  | 9.429 | 0.212 | 3.439 | 0.434 | 0.700 | 2.649 | 9.429 | 0.212 | 3.439 | 0.434 | 0.700 | 2.649 |
| 0.1| 9.459 | 0.212 | 3.470 | 0.431 | 0.711 | 2.629 | 9.445 | 0.212 | 3.455 | 0.432 | 0.706 | 2.639 |
| 0.5| 9.730 | 0.209 | 3.600 | 0.422 | 0.748 | 2.553 | 9.580 | 0.210 | 3.523 | 0.427 | 0.726 | 2.599 |
| 1  | 10.224 | 0.203 | 3.798 | 0.407 | 0.792 | 2.441 | 9.777 | 0.208 | 3.612 | 0.420 | 0.748 | 2.543 |
| 1.5| 10.682 | 0.199 | 4.006 | 0.392 | 0.836 | 2.321 | 9.929 | 0.206 | 3.694 | 0.414 | 0.766 | 2.488 |
| 2  | 11.048 | 0.195 | 4.204 | 0.378 | 0.876 | 2.204 | 10.040 | 0.205 | 3.764 | 0.408 | 0.781 | 2.437 |
| 3  | 11.564 | 0.191 | 4.616 | 0.355 | 0.944 | 1.996 | 10.185 | 0.203 | 3.879 | 0.398 | 0.804 | 2.347 |
| 4  | 11.902 | 0.188 | 4.868 | 0.337 | 1.002 | 1.827 | 10.276 | 0.202 | 3.971 | 0.391 | 0.822 | 2.273 |
| 5  | 12.138 | 0.186 | 5.135 | 0.323 | 1.056 | 1.695 | 10.339 | 0.202 | 4.048 | 0.384 | 0.837 | 2.212 |
| 7  | 12.436 | 0.184 | 5.574 | 0.304 | 1.152 | 1.501 | 10.418 | 0.201 | 4.169 | 0.375 | 0.861 | 2.117 |
| 10 | 12.666 | 0.182 | 6.070 | 0.285 | 1.275 | 1.313 | 10.482 | 0.200 | 4.300 | 0.365 | 0.887 | 2.016 |
| 15 | 12.819 | 0.181 | 6.642 | 0.267 | 1.439 | 1.128 | 10.534 | 0.200 | 4.449 | 0.355 | 0.917 | 1.908 |
| 25 | 12.868 | 0.181 | 7.326 | 0.250 | 1.675 | 0.938 | 10.570 | 0.199 | 4.625 | 0.344 | 0.953 | 1.786 |
| 50 | 12.747 | 0.182 | 8.106 | 0.233 | 2.031 | 0.751 | 10.583 | 0.199 | 4.827 | 0.332 | 0.996 | 1.655 |
| 75 | 12.614 | 0.183 | 8.464 | 0.227 | 2.246 | 0.671 | 10.580 | 0.199 | 4.922 | 0.326 | 1.017 | 1.596 |
| 100| 12.515 | 0.183 | 8.666 | 0.224 | 2.391 | 0.627 | 10.576 | 0.199 | 4.979 | 0.323 | 1.030 | 1.562 |

The contours plotted in Fig. 15 offer a global view of the temperature distribution inside the sphere and in the fluid for $De = 7$. With the increase of $Br$, the sphere becomes hotter on its downstream side due to the cumulative effect of viscous dissipation. The isotherms inside the sphere are no longer spherical, but become nearly normal to the axis of symmetry (flow direction). In the fluid region, the volume of fluid with $\tilde{T} > 0$ increases with $Br$ due to viscous dissipation.
Figure 14 – Brinkman number effect on heat transfer: (a) Nusselt number ratio for $\beta = 0.1$; (b) Nusselt number ratio for $\beta = 0.5$; (c) local Nusselt number profile along the sphere surface for $\beta = 0.1$ and $De = 7$; (d) temperature profile along the centerline and sphere surface for $\beta = 0.1$ and $De = 7$ (in the range $-1 \leq x/R \leq 1$ the temperature is taken along the centerline and surface of the sphere, whereas outside this interval the temperature is taken along the centerline in the fluid region). In (a) and (b), the lines are only a guide to the eye. All the results plotted were obtained for $Pr = 10^5$, $\kappa = 1$ and $\Omega = 0$. 
7. Conclusions

The heat transfer and unbounded flow of a simplified PTT fluid past a sphere was investigated in this work. The sphere generates heat in its interior at a constant and uniform rate. Keeping the Reynolds number fixed ($Re = 0.01$), we investigated the effect of Prandtl number, Brinkman number, thermal conductivity ratio and thermal contact resistance for Deborah numbers in the interval $0 \leq De \leq 100$ and two different solvent viscosity ratios ($\beta = 0.1$ and 0.5). The assumption of temperature-independent parameters allowed to decouple the flow from the heat transfer process.

The drag coefficient of the sphere showed a monotonic decrease with $De$, attributed to the shear-thinning behavior of the PTT model. The fluid with higher $\beta$ presented a lower drag coefficient. The stresses acting on the sphere surface and in the wake of the sphere decrease with $De$ after an initial period of increase up to a critical $De$. A negative wake was observed behind the sphere for both values of $\beta$ after exceeding a threshold $De$, being more intense for the more elastic fluid (lower $\beta$).

In the absence of viscous dissipation, the Nusselt number increases with $De$ up to a critical $De$, above which it decreases. The more elastic fluid ($\beta = 0.1$) displays a higher Nusselt number and the average dimensionless temperature of the sphere is consequently lower.
A higher ratio of thermal conductivities (solid more conductive than fluid) lowers and homogenizes the temperature of the sphere, and decreases simultaneously the Nusselt number, albeit with much lower intensity.

The existence of thermal contact resistance at the solid-fluid interface increases the temperature on the solid side. As the resistance increases, the temperature profiles on the solid side are shifted to higher values, but their shape only suffers minor changes. The effect on the Nusselt number is small.

The introduction of viscous dissipation in the fluid can drastically change the heat transfer process. At a sufficiently high Brinkman number, part of the sphere receives heat from the fluid.

The present work can be complemented in several ways by future studies. One possible direction is to analyze the effect of the parameters that were held fixed in this work: the Reynolds number and the extensibility parameter of the PTT model, possibly accounting for temperature-dependent properties. Other variables can be added to the problem, as for example slip on the sphere surface. Another possible direction is the investigation of the transients of this problem. Due to the scarcity of results on the heat transfer around a sphere involving viscoelastic fluids, studying the flow side in isolation, by imposing a fixed heat flux or fixed temperature on the sphere surface, is still another possible direction of research.

Appendix A

Discrete numerical methods like finite-volumes present, in general, a dependency of the solution on the spatial resolution of the computational grid. Besides, the problem under study has additional dependency on the size of the simulation domain due to the artificial boundary conditions applied at the surrounding boundaries to simulate an unbounded flow domain. This last question was discussed several times in the literature and domains of different sizes have been employed in different works (e.g. [8, 10]). The final error affecting the solution depends on many factors, such as the Reynolds number, the nature of the fluid (Newtonian vs non-Newtonian), the type of artificial boundary conditions applied at the outer boundary (outflow, symmetry, periodicity, etc.), the type of numerical methods employed, among others. The error also affects differently each post-processing variable. Therefore, the existence of an optimal domain size and mesh resolution for all conditions is unrealistic if computational cost is to be taken into account.
Instead, these parameters should be selected individually for each specific problem as a compromise between accuracy and computational cost.

In order to assess the mesh and domain dependency of the solution, a set of simulations was carried out in grids of different resolutions and sizes. Keeping the domain size fixed at 200$R$, the mesh resolution was varied as indicated in Table 1, originating meshes M1, M2 and M3. A second set of meshes with fixed resolution, but different size was generated by varying the domain size of mesh M2, originating meshes M2$_{R50}$, M2$_{R100}$, M2$_{R150}$, M2$_{R250}$, M2$_{R300}$ and M2$_{R350}$ (it is implicit that M2 is in fact M2$_{R200}$). As the name suggest, the size of the domain in these meshes varies between 50$R$ and 350$R$ (the number of cells in the fluid domain increases with mesh size, since the resolution is kept constant). The simulations were carried out for $De = 10$, $\beta = 0.5$, $Pr = 10^5$, $Br = 0$, $\kappa = 1$ and $\Omega = 0$.

The results obtained are listed in Table I. It can be shown that the mesh resolution has a small effect on the parameters under analysis, which is indicative that the range of mesh resolutions tested is already appropriate to capture accurately the main characteristics of the flow. On the other hand, the domain size seems to have a more important influence on the solution, for this low $Re$ flow, vanishing as the domain size increases. However, a larger domain also signifies a higher computational cost, not only because of the higher number of cells in the mesh, but also due to the higher number of iterations taken to converge to steady-state. Based on these results, mesh M2 was selected and used throughout this work as a compromise between accuracy and computational cost (each simulation of the flow in mesh M2 takes approximately 24 h to complete running in parallel in 7 processors; the heat transfer simulation only takes a few minutes for each set of conditions).

It is worth mentioning that when the Oldroyd-B or Upper-Convected Maxwell models are used, the stress profiles typically show a high dependency on the mesh resolution in the wake of the sphere for increasing $De$ (e.g. [8]). This issue is common to the simulation of flow around cylinders [39, 45], where both type of flows form a strong and thin birefringent strand in the wake of the obstacle, due to the growth of normal stresses with $De$. This is not always reflected in the drag coefficient, or at least not with the same intensity, as the stresses acting on the sphere are typically less dependent on mesh resolution. In our case, this issue does not arise because the stresses decrease in the wake of the sphere with the increase of $De$. We verified the mesh independency of the stresses profiles at different $De$ values and it was confirmed that such profiles are in fact mesh-independent for all the resolutions tested (M1, M2 and M3).
Table I – Effect of mesh resolution and size of the simulation domain on the drag coefficient, surface-averaged Nusselt number, volume-averaged and maximum sphere temperature. The simulations were performed for $\text{De} = 10$, $\beta = 0.5$, $\text{Pr} = 10^5$, $\text{Br} = 0$, $\kappa = 1$ and $\Omega = 0$.

| Mesh  | $C_d$  | $\overline{N_{Mu}}$ | $\overline{T_s}$ | $\overline{T_{s,\text{max}}}$ |
|-------|--------|---------------------|------------------|------------------------|
| M1    | 1714.38| 12.6844             | 0.1819           | 0.3325                 |
| M2    | 1714.58| 12.6852             | 0.1819           | 0.3325                 |
| M3    | 1715.00| 12.6856             | 0.1819           | 0.3325                 |
| M2$_{R50}$ | 1746.03| 12.7673             | 0.1814           | 0.3320                 |
| M2$_{R100}$ | 1724.52| 12.7137             | 0.1817           | 0.3324                 |
| M2$_{R150}$ | 1717.52| 12.6944             | 0.1818           | 0.3325                 |
| M2$_{R250}$ | 1712.68| 12.6799             | 0.1819           | 0.3326                 |
| M2$_{R300}$ | 1711.43| 12.6764             | 0.1820           | 0.3326                 |
| M2$_{R350}$ | 1710.56| 12.6740             | 0.1820           | 0.3326                 |
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