Universality class for bootstrap percolation with $m = 3$ on the cubic lattice

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Abstract
We study the $m = 3$ bootstrap percolation model on a cubic lattice, using Monte Carlo simulation and finite-size scaling techniques. In bootstrap percolation, sites on a lattice are considered occupied (present) or vacant (absent) with probability $p$ or $1-p$, respectively. Occupied sites with less than $m$ occupied first-neighbours are then rendered unoccupied; this culling process is repeated until a stable configuration is reached. We evaluate the percolation critical probability, $p_c$, and both scaling powers, $y_p$ and $y_h$, and, contrarily to previous calculations, our results indicate that the model belongs to the same universality class as usual percolation (i.e., $m = 0$). The critical spanning probability, $R(p_c)$, is also numerically studied, for systems with linear sizes ranging from $L = 32$ up to $L = 480$: the value we found, $R(p_c) = 0.270 \pm 0.005$, is the same as for usual percolation with free boundary conditions.

Key words: Correlated percolation; phase transition; universality classes.

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1 Introduction
In many physical phenomena percolation effects play an important role. Particularly, some dilute magnets are well described, in what concerns magnetic phase transitions, by uncorrelated diluted models. In these models, magnetic sites on a lattice are randomly replaced by non-magnetic ones, and a bond links each pair of occupied (magnetic) first neighbours. At zero temperature, the problem is purely geometrical and is described in the following way. Sites on a lattice are randomly occupied with probability $p$, while all bonds are considered to be present. A cluster is then defined as a collection of present sites which are connected to each other through steps between occupied first-neighbours. As $p$ increases, an infinite cluster appears for the first time at a critical probability $p_c$, which is lattice dependent. In analogy with thermal critical phenomena, some quantities are singular at the critical point, following a power-law behavior near $p_c$. Typical examples are the probability $P(p)$ that a site belongs to the infinite cluster, which behaves as $P(p) \sim (p - p_c)^\beta$ near $p_c$, and the correlation length $\xi(p)$, which diverges at $p_c$, according to $\xi(p) \sim |p - p_c|^{-\nu}$.
Nevertheless, some systems were found to be better described by correlated percolation models, where the presence of sites (or bonds) depends also on their neighbourhood. Typical examples of correlated percolation models are bootstrap percolation \(^3\) and site-bond correlated percolation \(^4\). The physical motivation for the introduction of the former model comes from diluted magnetic systems where competition between exchange (which favour a magnetic ground state) and crystal-field (which leads to a non-magnetic ground state) interactions takes place. To mimic this competition at zero temperature, the bootstrap percolation model was introduced \(^5, 6\): in this model, sites on a lattice are randomly occupied with probability \(p\) but only those with at least \(m\) occupied first-neighbours remain occupied. In the stable final configuration, all occupied sites have at least \(m\) occupied first-neighbours or the whole lattice is empty.

As our purpose in this paper is to study one implementation of the bootstrap percolation model, we briefly review some of its properties in what follows (for a thorough discussion of the results up to 1990, see Reference \(^3\)).

The \(m = 0\) case regains the usual (uncorrelated) site percolation model, where the transition is continuous and \(p_c < 1\) in two and three dimensions. More specifically, the most precise evaluations of the critical exponents for three-dimensional lattices are \(\nu = 1/y_p = 0.875 \pm 0.008\) and \(\beta = 0.412 \pm 0.010\) from simulation \(^7\) and \(\nu = 1/y_p = 0.872 \pm 0.070\) and \(\beta = 0.405 \pm 0.025\) from series \(^8\). In the \(m = 1\) case, only isolated sites are removed by the culling process: these sites do not contribute to the critical behaviour and \(p_c\) and all critical exponents remain the same as for usual percolation. For \(m = 2\), on the other hand, isolated clusters with two sites are eliminated, as well as some dangling structures of more compact clusters. The elimination of these dangling structures, however, does not break the infinite cluster (whenever present) and, therefore, the critical probability is the same as for usual percolation. Moreover, as the exponent \(\nu\) is also connected to the formation of this infinite cluster, its value is the same for \(m = 2\) and \(m = 0\). In what concerns “field” exponents (\(\beta\) for instance), previous results for two-dimensional systems indicate a higher value of this exponent for \(m = 2\) than for usual percolation \(^9\). Nevertheless, it has been shown later, through simulations on bigger lattices in two dimensions \(^10\) and general arguments applied to both two and three dimensions \(^11\), that the exponent \(\beta\) is the same for uncorrelated and \(m = 2\) bootstrap percolation models.

Let us now turn our attention to higher values of \(m\). It is generally believed that, for any value of \(m\) where only infinite clusters can survive the culling process, \(p_c = 1\). This is indeed the case for \(m = 2d - 1\) on hypercubic lattices \((d\) is the dimension of the lattice\(^12\), for \(m = 4\) on cubic \(^12\) and triangular \(^3\) lattices and for \(m = 5\) on the triangular lattice \(^13\). Moreover, it has been shown that, for these cases, the usual finite-size scaling relation does not hold.

This finite-size scaling predicts that, if a suitable definition of a finite-lattice “critical” point, \(p_{av}\), is made, this point will approach the critical point in the thermodynamic limit, \(p_c\), as:

\[
p_c - p_{av} \sim L^{-1/\nu},
\]  

(1)
where $L$ is the linear size of the finite system, such that $L \gg 1$, and $\nu$ is the usual critical exponent. This finite-size behaviour is indeed observed for $m \leq 2$, but fails for high values of $m$. For $m = 2d - 1$ on hypercubic lattices, it has been proven that $p_c - p_{av} \sim 1/(\log^{d-1} L)$ \[14\], e.g., proportional to $1/\log(\log L)$ for $d = 3$ dimensions. Also for $m = 4$ on the cubic lattice, the correct finite-size behaviour is $p_c - p_{av} \sim 1/\log(\log L)$, with $p_c = 1$ \[15\]. These results for high $m$ have been conjectured or tested in numerical simulations \[16\].

The results stated in the previous paragraphs do not apply to the $m = 3$ case on the cubic lattice. For this model, one expects $p_c$ to be above the value for uncorrelated percolation (where $p_c = 0.311605 \pm 0.000005$ \[17\]), since the infinite cluster for usual percolation at $p_c$ is not stable with respect to the culling process for $m = 3$. On the other hand, since finite clusters are still stable for this value of $m$ on the cubic lattice, it is expected that $p_c < 1$. Numerical simulations confirmed this scenario \[13, 18\], although the relative small sizes used in those works indicate that the values might not be precise (we will return to this point later). In what concerns the critical exponents, previous results indicate that the exponent $\nu$ is the same as for usual percolation \[18\] but $\beta$ is higher than its uncorrelated counterpart \[13, 18\]. However, in neither of these works a extrapolation to the thermodynamic limit is attempted, leaving the possibility that finite-size effects might be the reason for the discrepancy in the values of $\beta$. This possibility was first proven to be right, in the context of two-dimensional bootstrap percolation models, in References \[10\] and later confirmed for $m = 3$ on the triangular lattice \[19\]. The possibility of a new universality class for bootstrap percolation with $m = 3$ on the cubic lattice is the problem we address in this work. We resort to numerical simulation methods, which, together with finite-size scaling analysis, allowed us to obtain more precise values for the critical parameters. We also study the so-called critical spanning probability, i.e., the probability that a given lattice has a cluster connecting its boundaries at criticality \[20, 21\]. This quantity shows some degree of universality: it depends on the dimension and shape of the system and on the specific boundary condition but not on the lattice type (simple cubic or f.c.c., for instance) and on the particular kind of percolation (site or bond) \[17, 21, 22\]. It is then interesting to see whether it remains invariant for percolation models such that long-range correlation is involved, like bootstrap percolation.

The remainder of the paper is organized as follows. In the next section we present the method and discuss some technical details, as well as the results for the critical parameters. In Section 3 we discuss some previous results concerning the critical spanning probability for percolation and our results for bootstrap percolation. Finally, we summarize our results in the last section.

### 2 Method and Results

The method we use is connected to real-space renormalization-group and finite-size scaling procedures \[23\]. The approach needs that precise values of the physical quantities are available; these are only obtained for high values of the
linear system size $L$. We then studied finite systems of size $L^3$, with $32 \leq L \leq 480$; from the results for $L \gg 1$, it is possible to use finite-size scaling techniques to extrapolate to the thermodynamic limit ($L = \infty$).

The critical probability $p_c$ and the critical exponent $\nu$ are calculated as follows. For a lattice of size $L$, we occupy each site with probability $p$, apply the bootstrap condition and test the lattice for percolation (here we define percolation as the presence of a cluster which connects the bottom and top planes of the finite cubic lattice; we discuss this and other technical points below). Our finite-size estimate of the critical probability, $p^*$, is taken as the value of $p$ at which the cell percolates for the first time, when $p$ is increased from zero; this procedure is made for $N$ different runs (which correspond to $N$ different seeds to the random number generator). Each run leads to a different value of $p^*$, since the lattice is finite. We take the average of the $N$ values of $p^*$ as our estimate of $p_{av}$ (see Section I). It is then assumed that $p_{av}$ will approach $p_c$ as given by Equation 1. Moreover, it is possible to calculate the width $\sigma = \sqrt{\langle p^*^2 \rangle - p_{av}^2}$, which behaves as ($\langle p^*^2 \rangle$ stands for the average of $p^*$ over the $N$ realizations):

$$\sigma \sim L^{-1/\nu}.$$  \hfill (2)

From the previous equation and a log $-\log$ plot of $\sigma \times L$, it is then possible to obtain the value of the critical exponent $\nu$. The data is depicted in Figure 1; from the slope of the straight line we obtain $\nu = 0.89 \pm 0.04$. We compare this value with other evaluations in Table 1, within the numerical precision, this exponent is the same for bootstrap percolation with $m = 3$ and ordinary percolation on the cubic lattice and agrees with previous evaluations of $\nu$ for bootstrap percolation with $m = 3$ on the cubic lattice. Note from the graph that the straight line regime is achieved for $L \geq 128$, while for uncorrelated percolation this regime sets in for smaller values of $L$. This is expected, since finite-size effects are stronger for correlated percolation problems than for their uncorrelated counterparts 4 [24]. Therefore, we neglect the data for $L \leq 96$ and used only lattices with $128 \leq L \leq 480$ in our linear regression analysis.

In order to calculate the critical threshold, we resort to Equations 1 and 2, which imply that, for $L \gg 1$:

$$p_c - p_{av} \sim \sigma.$$  \hfill (3)

This is a convenient way to calculate $p_c$, since it does not depend on the value of $\nu$; such dependence would appear if Equation 1 was used. It is shown in Figure 2 a plot of $p_{av} \times \sigma$: $p_c$ is given by the linear coefficient and the value obtained is $p_c = 0.57256 \pm 0.00006$. This value is slightly above the one calculated in Reference 18, in which the value of $L$ varied from 10 to 110. If we use the same range in our calculation the extrapolated value of $p_c$ is consistent with the result of 18 (see Table 1).

Let us now discuss some technical points. An important quantity is the value of $p$ at which the lattice percolates for the first time, when $p$ is increased from zero. One needs to define what “percolate” means for a finite lattice. In
Figure 1: Log-log plot $\sigma \times L$, for $32 \leq L \leq 480$. The linear size $L$ is always a multiple of 32. The dotted line is a linear fit of the data for $128 \leq L \leq 480$ only (see text).

Figure 2: Graph of $p_{av} \times \sigma$, for $32 \leq L \leq 480$. The linear size $L$ is always a multiple of 32. The dotted line is a linear fit of the data for $128 \leq L \leq 480$ only (see text). Although not depicted in the figure, there are error bars in the values of $\sigma$, which were taken into account in the evaluation of $p_c$. 
Table 1: Critical parameters for $m=3$ bootstrap percolation and comparison with previous results for this model and for usual percolation.

|                | $m=3$ bootstrap percolation | usual percolation |
|----------------|-----------------------------|-------------------|
| $p_c$          | $0.57256\pm0.00006^a$       | $0.311605\pm0.000005^d$ |
|                | $0.5717\pm0.0005^b$         |                   |
|                | $0.568\pm0.002^c$          |                   |
| $\nu$          | $0.89\pm0.04^a$            | $0.875\pm0.008^c$ |
|                | $0.876\pm0.010^b$          | $0.872\pm0.070^f$ |
| $\beta$        | $0.37\pm0.03^a$            | $0.41\pm0.02^e$   |
|                | $0.6\pm0.1^b$              | $0.40\pm0.06^f$   |
|                | $0.82\pm0.04^c$            |                   |

$^a$ Present work. $^b$ Reference [18]. $^c$ Reference [13]. $^d$ Reference [17]. $^e$ Reference [7]. $^f$ Reference [8].

In this work, we used the rule called $R_3$ in Reference [23], i.e., a lattice percolates if, after the bootstrap culling process, there is a path of present sites which links the boundaries of the lattice in a fixed direction (vertical, say). There are other possible definitions (see [23]) and it is expected that all of them lead to the same value of $p_c$ in the thermodynamic limit. Note, however, that the value of the critical spanning probability does depend on this definition, as we will discuss in Section 3. The numerical procedure we used to test for percolation is the Hoshen-Kopelman algorithm [25]: for usual percolation, it requires the storage of only one plane. For bootstrap percolation, on the other hand, the bootstrap iteration needs the storage of the whole lattice, due to correlation effects. To cope with this drawback, we store the lattice in bits, instead of words: this saves memory and time, since the updates connected to the bootstrap rule can be made in parallel for a set of 32 sites [24]. To define all six first-neighbours of sites at the boundaries of the lattice, periodic boundary conditions are used. It is expected that the boundary condition does not affect the values of the critical parameters in the thermodynamic limit, since it is a “surface” effect. Finally, let us mention that the number of realizations $N$ (see Section 1) varied from $\sim 32000$, for the smaller lattices, to $\sim 20000$, for the bigger ones. We use two procedures to obtain $y_h$ in the
with $y_h$ independent of $L$; this equation leads to a straight line in a log–log plot of $\lambda_h \times L$, for $L \gg 1$. As depicted in Figure 3, this is indeed the case for $128 \leq L \leq 480$. From a linear fitting of the data for this range of $L$, and resorting to the scaling relation $\beta = (d - y_h)\nu$, we obtained the value $\beta = 0.37 \pm 0.03$ (see Table 1). Alternatively, we could take into account correction-to-scaling terms, through:

$$\lambda_h = L^{y_h} (1 + B/L).$$

From this equation, we see that local slopes of log $\lambda_h \times \log L$ provide estimates of $y_h(L)$, which, when extrapolated to $L \to \infty$, leads to an evaluation of $y_h$ in the thermodynamic limit. We have applied this procedure for $32 \leq L \leq 480$, using three consecutive points to calculate the local slopes and extrapolating to $L \to \infty$ through a $y_h(L) \times 1/L$ graph. The value thus obtained for $\beta$ is the same as for the first procedure. In Table 1 we see that our estimate of $\beta$ disagrees with the values calculated in References [13] and [18]. In evaluating this exponent, the former uses lattices of linear size $L = 35$, while the latter uses $L = 80$; neither attempted to make an extrapolation to the thermodynamic limit. From Table 1, we can infer that our estimate of $\beta$ for the bootstrap model we study is, within the error bars, equal to the corresponding value of this exponent for usual (uncorrelated) percolation. Since we have already seen that $\nu$ is also the same for both models, we can draw the conclusion that usual percolation and bootstrap percolation with $m = 3$ on the cubic lattice belong to the same universality class. This result contradicts References [13] and [18]; we believe this is caused by the small lattices used in those works.

### 3 Critical spanning probability

It has been established some time ago that the critical spanning probability, $R(p_c)$, defined as the probability of spanning a lattice at the critical point, shows some degree of universality [20, 27]. More precisely, this quantity does not depend on the lattice type and on the kind of percolation. This result contradicts the assumption made on early applications of real-space renormalization group procedures to percolation. In those, it was assumed that the critical spanning probability is equal to the critical percolation threshold, $p_c$, and, therefore, lattice dependent [23]. However, later numerical tests confirmed and expanded the universality proposal [17, 21, 22].

Nevertheless, no study has been made on correlated models, to the best of our knowledge. While it is expected that short-range correlations do not change the universality scenario [4], it is not clear whether $R(p_c)$ changes when long-range correlations are introduced. A convenient model to test these possibilities is the bootstrap percolation one. While the behaviour of $R(p_c)$ is trivial for the cases where $p_c = 1$, for $m = 3$ on the simple cubic lattice the presence of
correlation may lead to a non trivial behavior. We study this possibility using numerical simulation on simple cubic lattices of size $L^3$, with $32 \leq L \leq 480$. The programs and algorithms used are essentially the same as the ones described in the previous section. The basic procedure is to generate $M$ independent runs for each lattice size and compute the fraction of those which percolate after the bootstrap condition is used and a stable configuration is reached. The values of $M$ are the same as those used in the calculation of $y_h$ (see previous section).

The results are depicted in Figure 4. We can infer that $R(p_c) = 0.270 \pm 0.005$, where the error bar is a rough estimate. There are two previous calculation of $R(p_c)$ for the uncorrelated site percolation on the cubic lattice with free boundary conditions: they lead to $R(p_c) = 0.265 \pm 0.005$ [22] and $R(p_c) = 0.28$ [17]. Our value agrees, within the numerical error, with the first one but the result of Reference [17] cannot be ruled out. It is then reasonable to infer that usual percolation and $m = 3$ bootstrap percolation on the cubic lattice belong to the same universality class, also in what regards the critical spanning probability.

4 Summary

We calculate, using numerical simulations and finite-size scaling techniques, the critical parameters of the bootstrap percolation model with $m = 3$ on the cubic lattice, using finite lattices of size $L^3$, with $32 \leq L \leq 480$. Our evaluations of $\nu$
and $\beta$ strongly support the conclusion that usual percolation and the model we study belong to the same universality class. This result disagrees with previous calculations [13, 18]; we believe this is due to the small sizes used in those works. To support this assumption, we note that the finite-size scaling assumptions hold for lattices of linear size $L \geq 128$, which are higher than the sizes studied in previous works.

The critical spanning probability, $R(p_c)$, is also calculated. It has been shown that this quantity shows some degree of universality, but, to the best of our knowledge, no study concerning correlated percolation models have been done so far. Our result for bootstrap percolation with $m = 3$ on the cubic lattice, $R(p_c) = 0.270 \pm 0.005$, is, within the numerical accuracy, the same value as for usual percolation with free boundary conditions. Therefore, we can infer that $R(p_c)$ is not sensitive to short range correlations and even to some long range correlations, like the one studied in this paper.

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