Untwisting algebras with van den Bergh duality into Calabi-Yau algebras

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Recently, Jake Goodman and Ulrich Krähmer [4] have shown that a twisted Calabi-Yau algebra $A$ with modular automorphism $\sigma$ and dimension $d$ can be “untwisted,” in the sense that the Ore extensions $A[X; \sigma]$ and $A[X^\pm 1; \sigma]$ are Calabi-Yau algebras of dimension $d + 1$. The purpose of this note is to record the observation that this result holds in greater generality:

**Theorem.** If $A$ is an algebra that satisfies the conditions for van den Bergh duality of dimension $d$ and $U = \Ext^d_A(A, A \otimes A)$ is its dualizing bimodule, then the tensor algebra $A[U] = \bigoplus_{n \in \mathbb{N}_0} U \otimes A^n$ is a Calabi-Yau algebra of dimension $d + 1$.

If in this statement we suppose that $A$ is a twisted Calabi-Yau algebra with modular automorphism $\sigma$, then we recover the result of Goodman and Krähmer, for in that case $U = A\sigma$ is a twisted $A$-bimodule and $A[U]$ is isomorphic to the Ore extension $A[X; \sigma]$. We refer to the papers of van den Bergh [1] and Ginzburg [3] for the little information about duality and Calabi-Yau algebras that we need. We work over a fixed field, over which unadorned tensor products are taken, or over an arbitrary commutative ring, provided we add the hypothesis that $A$ be projective. All our complexes are cochain complexes, we underline their components of degree zero and, for brevity, we say that a complex of $A$-bimodules is good if it is of finite length and its components are finitely generated as bimodules.

**Proof.** Let us write, for simplicity, $B = A[U]$. The kernel $I$ of the obvious augmentation map $B \to A$ is finitely generated and projective as a $B$-module both on the left and on the right; indeed, restriction along the inclusion $U \hookrightarrow I$ gives an isomorphism of functors $\hom_B(I, -) \cong \hom_A(U, -)$ of left or right $B$-modules, and $U$ is finitely generated and projective as an $A$-module both on the left and on the right. Let $K$ be the complex

$$I \otimes B \longrightarrow B \otimes B$$

of $B^\otimes$-bimodules, and let us make the convention that the left and right actions of $B^\otimes$ are the outer and the inner ones, respectively; it is clear that $K$ is a
complex of projective and finitely generated right $B$-modules, and that its homology is $H(K) = A \otimes B$ concentrated in degree zero.

Let now $P$ be a good resolution of $A$ by projective $A$-bimodules. The complex of right $B$-modules $P \otimes_A K$ has finite length, and all its components are finitely generated and projective. To compute its homology, we can use a spectral sequence. Taking first homology with respect to the differential of $P$ we obtain—because $K$ is a complex of projective left $A$-modules—the complex $A \otimes_A K$. This can be identified with

$$B \otimes_A I \longrightarrow B \otimes_A B$$

and the homology of this is $B$ concentrated in degree zero. We conclude in this way that $P \otimes_A K$ is a good resolution of $B$ by projective $B$-bimodules.

We want to compute $\text{Ext}_{B^e}(B, B \otimes B)$. We have

$$\text{hom}_{B^e}(P \otimes_A K, B \otimes B) \cong \text{hom}_{A^e}(P, \text{hom}_{B^e}(K, B \otimes B))$$

and, since $A$ satisfies van den Bergh duality of dimension $d$ and dualizing module $U$, this has the same homology as

$$P \otimes_{A^e} \left(U \otimes_A \text{hom}_{B^e}(K, B \otimes B)\right)[-d]. \tag{1}$$

We use, as before, a spectral sequence to compute the homology of this complex. As $U$ is finitely generated and projective as an $A$-module,

$$\text{hom}_{B^e}(I \otimes B, B \otimes B) \cong \text{hom}_B(I, B \otimes B) \cong \text{hom}_A(U, B \otimes B)$$

$$\cong U^* \otimes_A B \otimes B,$$

with $U^* = \text{hom}_A(U, A)$. Now $U \otimes_A U^* \cong A$ as $A$-bimodules, so the complex $U \otimes_A \text{hom}_{B^e}(K, B \otimes B)$ can be identified with

$$U \otimes_A B \otimes B \longrightarrow B \otimes B$$

and its homology is $A \otimes B$ concentrated in degree one. It follows that taking homology with respect to the differential induced by that of $K$ in the complex (1), we get $P \otimes_{A^e} (A \otimes B)[-d - 1]$, and the homology of this, in turn, is clearly $B$ concentrated in degree $d + 1$. This proves the theorem.

**Corollary.** In the conditions of the theorem, the algebra $C = \bigoplus_{n \in \mathbb{Z}} U \otimes A^n$ is also Calabi-Yau of dimension $d + 1$.

Notice that this makes sense because the $A$-bimodule $U$ is invertible.

**Proof.** One can check at once that the multiplication in $C$ induces an isomorphism $C \otimes_B C \rightarrow B$. On the other hand, $C$ is flat as a left and as a right $B$-module: it is the colimit of the chain of its $B$-submodules of the form $\bigoplus_{n \geq n_0} U \otimes_A^n$ with $n_0 \in \mathbb{Z}$, each of which is projective, being isomorphic to $U \otimes_A^{n_0} \otimes_B B$ or $B \otimes_A U \otimes_A^{n_0}$. The corollary follows then from the theorem and the localization result [5, Theorem 6] of Farinati.
The theorem above and its corollary are unsatisfying in that they untwist the algebra $A$, which is of dimension $d$, into an algebra of dimension $d + 1$. In general, this seems to be as much as one can hope for. There is a case in which we can do better, though, and we end this note explaining this.

We put ourselves in the situation of the theorem again and suppose additionally that $U$ is a bimodule of finite order, so that there exist $\ell \in \mathbb{N}$ and an isomorphism $\phi : U \otimes A^\ell \to A$ of $A$-bimodules, and that the isomorphism $\phi$ is associative, in the sense that the diagram

$$
\begin{array}{cc}
U \otimes A^\ell \\
\downarrow \phi \otimes \text{id}_U
\end{array}
\begin{array}{c}
A \otimes A U
\end{array}
\begin{array}{c}
A \otimes A U
\end{array}
\begin{array}{c}
U \otimes_A A
\end{array}
\begin{array}{c}
U
\end{array}
$$

in which the unlabelled arrows are canonical isomorphisms commutes. We consider the $A$-submodule $R = \{ x - \phi(x) : x \in U \otimes A^\ell \} \subseteq B$. Since the isomorphism $\phi$ is associative, the left ideal $J = BR$ coincides with the right ideal $RB$ and it is a bilateral ideal: we can therefore consider the algebra $D = B/J$. There is clearly a direct sum decomposition $D \cong \bigoplus_{i=0}^{\ell-1} U \otimes A^i$ as $A$-bimodules, and this construction should remind us of the classical construction of cyclic algebras over a field.

**Proposition.** In the situation of the theorem, if the dualizing bimodule $U$ is of finite order and admits an associative isomorphism $\phi : U \otimes A^\ell \to A$ and $\ell$ is invertible in $A$, then the algebra $D = A[U]/J$ with $J$ generated by $R$ as above is a Calabi-Yau algebra of dimension $d$.

**Proof.** Let $\xi \in U \otimes A^\ell$ be such that $\phi(\xi) = 1$; since $\phi$ is associative, $\xi$ is central in $B$. The ideal $J$ is generated by $\xi - 1$ and, if $\rho : B \to B$ is right multiplication by $\xi - 1$, the complex

$$
B \overset{\rho}{\to} B
$$

is a resolution of $D$ as a left $B$-module, and we can use it to compute

$$
\text{Tor}_p^B(D, D) \cong \begin{cases} 
D, & \text{if } p \in \{0, 1\}; \\
0, & \text{if } p \geq 2.
\end{cases} \tag{3}
$$

It is immediate that these isomorphisms are of left $D$-modules and by lifting the multiplication on the right of $D$ on $D$ to an endomorphism of the resolution (2), we see that they are actually isomorphisms of $D$-bimodules.

We write $\xi = \xi_1 \otimes \cdots \otimes \xi_\ell$, with $\xi_1, \ldots, \xi_\ell \in U$ and omitting a sum à la Sweedler, and consider the element

$$
e = \frac{1}{\ell} \left( 1 \otimes 1 + \sum_{r=1}^{\ell-1} (\xi_1 \otimes \cdots \otimes \xi_r) \otimes (\xi_{r+1} \otimes \cdots \otimes \xi_\ell) \right) \in D \otimes_A D.
$$
If \( \mu : D \otimes_A D \to D \) is induced by the multiplication of \( D \), then \( \phi(e) = 1 \) and, because of the associativity of \( \phi \), \( de = ed \) for all \( d \in D \). It follows that there is a \( D^e \)-linear morphism \( s : D \to D \otimes_A D \) such that \( s(1) = e \) which splits \( \mu \) and, in particular, that \( D \) is a direct summand of the \( D^e \)-module \( D \otimes_A D \); one says in this situation that \( D/A \) is a separable extension of algebras, as in [6, §10.8]. If now \( P \) is a good resolution of \( A \) by projective \( A \)-bimodules, then \( D \otimes_A P \otimes_A D \) is a good resolution of \( D \) by \( D \)-bimodules and, since \( D \) is a direct summand of \( D \otimes_A D \), we see that \( D \) itself has a good resolution by projective \( D \)-bimodules.

The construction done by Cartan and Eilenberg in [2, XVI, §5, Eq. (2)] specialized for the obvious morphism \( B^e \to D^e \), together with the natural isomorphism \( \text{Tor}^B_r(D^e, B) \cong \text{Tor}^D_r(D, D) \) of [2, IX, Prop. 4.4], gives us a change-of-rings spectral sequence with \( E^{p,q}_{2} = \text{Ext}^p_{D^e}(\text{Tor}_q^B(D, D), D \otimes D) \) converging to \( \text{Ext}^B_{2}(D, D \otimes D) \). Since \( B \) is Calabi-Yau of dimension \( d + 1 \),

\[
\text{Ext}^r_{p}((B, D \otimes D) \cong \text{Tor}^B_{d+1-r}(B, D \otimes D) \cong \text{Tor}^D_{d+1-r}(D, D),
\]

so that we know the limit of the spectral sequence from (3). From (3) we also know that \( E^{p,q}_{2} = 0 \) if \( q \notin \{0, 1\} \), and that \( E^{p,0} \cong E^{p,1} \cong \text{Ext}^p_{D^e}(D, D \otimes D) \) for all \( p \). A standard argument with the spectral sequence —using the fact that \( \text{pdim}_D \; D < \infty \) — shows now that \( D \) is Calabi-Yau of dimension \( d \). \( \square \)

An easy and probably important remark to be made is that the dualizing bimodule \( U \) for an algebra \( A \) with van den Bergh duality is always central: the actions of the center \( Z(A) \) of \( A \) on the left and on the right on \( U \) coincide. It follows from this that when \( U \) is of finite order, so that there is an isomorphism of bimodules \( \phi : U \otimes_A U \cong A \), the fact that \( \phi \) be associative or not does not depend on the particular choice of \( \phi \); it is a property of \( A \). It is natural to ask:

**Question.** If the dualizing bimodule of an algebra with van den Bergh duality is of finite order, is it necessarily associative?

If the algebra is twisted Calabi-Yau, the answer is affirmative.

References

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