Aspects of Chern-Simons Theory

Gerald V. Dunne
Department of Physics
University of Connecticut
Storrs, CT 06269
USA
dunne@hep.phys.uconn.edu

Abstract
Lectures at the 1998 Les Houches Summer School: Topological Aspects of Low Dimensional Systems. These lectures contain an introduction to various aspects of Chern-Simons gauge theory: (i) basics of planar field theory, (ii) canonical quantization of Chern-Simons theory, (iii) Chern-Simons vortices, and (iv) radiatively induced Chern-Simons terms.
## Contents

1 Introduction 3

2 Basics of Planar Field Theory 5
   2.1 Chern-Simons Coupled to Matter Fields - “Anyons” 5
   2.2 Maxwell-Chern-Simons: topologically massive gauge theory 9
   2.3 Fermions in 2 + 1-dimensions 11
   2.4 Discrete Symmetries: \( P \), \( C \) and \( T \) 12
   2.5 Poincaré Algebra in 2 + 1-dimensions 14
   2.6 Nonabelian Chern-Simons Theories 15

3 Canonical Quantization of Chern-Simons Theories 17
   3.1 Canonical Structure of Chern-Simons Theories 17
   3.2 Chern-Simons Quantum Mechanics 19
   3.3 Canonical Quantization of Abelian Chern-Simons Theories 23
   3.4 Quantization on the Torus and Magnetic Translations 25
   3.5 Canonical Quantization of Nonabelian Chern-Simons Theories 28
   3.6 Chern-Simons Theories with Boundary 31

4 Chern-Simons Vortices 32
   4.1 Abelian-Higgs Model and Abrikosov-Nielsen-Olesen Vortices 33
   4.2 Relativistic Chern-Simons Vortices 36
   4.3 Nonabelian Relativistic Chern-Simons Vortices 40
   4.4 Nonrelativistic Chern-Simons Vortices: Jackiw-Pi Model 41
   4.5 Nonabelian Nonrelativistic Chern-Simons Vortices 45
   4.6 Vortices in the Zhang-Hansson-Kivelson Model for FQHE 46
   4.7 Vortex Dynamics 49

5 Induced Chern-Simons Terms 52
   5.1 Perturbatively Induced Chern-Simons Terms: Fermion Loop 52
   5.2 Induced Currents and Chern-Simons Terms 55
   5.3 Induced Chern-Simons Terms Without Fermions 56
   5.4 A Finite Temperature Puzzle 59
   5.5 Quantum Mechanical Finite Temperature Model 61
   5.6 Exact Finite Temperature 2 + 1 Effective Actions 64
   5.7 Finite Temperature Perturbation Theory and Chern-Simons Terms 67

6 Bibliography 69
1 Introduction

Planar physics – physics in two spatial dimensions – presents many interesting surprises, both experimentally and theoretically. The behaviour of electrons and photons [or more generally: fermions and gauge fields] differs in interesting ways from the standard behaviour we are used to in classical and quantum electrodynamics. For example, there exists a new type of gauge theory, completely different from Maxwell theory, in $2 + 1$ dimensions. This new type of gauge theory is known as a “Chern-Simons theory” [the origin of this name is discussed below in Section 2.6 on nonabelian theories]. These Chern-Simons theories are interesting both for their theoretical novelty, and for their practical application for certain planar condensed matter phenomena, such as the fractional quantum Hall effect [see Steve Girvin’s lectures at this School].

In these lectures I concentrate on field theoretic properties of Chern-Simons theories. I have attempted to be relatively self-contained, and accessible to someone with a basic knowledge of field theory. Actually, several important new aspects of Chern-Simons theory rely only on quantum mechanics and classical electrodynamics. Given the strong emphasis of this Summer School on condensed matter phenomena, I have chosen, wherever possible, to phrase the discussion in terms of quantum mechanical and solid state physics examples. For example, in discussing the canonical quantization of Chern-Simons theories, rather than delving deeply into conformal field theory, instead I have expressed things in terms of the Landau problem [quantum mechanical charged particles in a magnetic field] and the magnetic translation group.

In Section 2, I introduce the basic kinematical and dynamical features of planar field theories, such as anyons, topologically massive gauge fields and planar fermions. I also discuss the discrete symmetries $P$, $C$ and $T$, and nonabelian Chern-Simons gauge theories. Section 3 is devoted to the canonical structure and canonical quantization of Chern-Simons theories. This is phrased in quantum mechanical language using a deep analogy between Chern-Simons gauge theories and quantum mechanical Landau levels [which are so important in the understanding of the fractional quantum Hall effect]. For example, this connection gives a very simple understanding of the origin of massive gauge excitations in Chern-Simons theories. In Section 4, I consider the self-dual vortices that arise when Chern-Simons gauge fields are coupled to scalar matter fields, with either relativistic or nonrelativistic dynamics. Such vortices are interesting examples of self-dual field theoretic structures with anyonic properties, and also arise in models for the fractional quantum Hall effect where they correspond to Laughlin’s quasiparticle excitations. The final Section concerns Chern-Simons terms that are induced radiatively by quantum effects. These can appear in fermionic theories, in Maxwell-Chern-Simons models and in Chern-Simons models with spontaneous symmetry breaking. The topological nature of the induced term has interesting consequences, especially at finite temperature.

We begin by establishing some gauge theory notation. The familiar Maxwell (or, in the nonabelian case, Yang-Mills) gauge theory is defined in terms of the fundamental gauge field (connection) $A_\mu = (A_0, \vec{A})$. Here $A_0$ is the scalar potential and $\vec{A}$ is the vector potential. The Maxwell Lagrangian

$$\mathcal{L}_M = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - A_\mu J^\mu$$

is expressed in terms of the field strength tensor (curvature) $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and a matter
current $J^\mu$ that is conserved: $\partial_\mu J^\mu = 0$. This Maxwell Lagrangian is manifestly invariant under the gauge transformation $A_\mu \to A_\mu + \partial_\mu \Lambda$; and, correspondingly, the classical Euler-Lagrange equations of motion

$$\partial_\mu F^{\mu\nu} = J^\nu$$

are gauge invariant. Observe that current conservation $\partial_\nu J^\nu = 0$ follows from the antisymmetry of $F^{\mu\nu}$.

Now note that this Maxwell theory could easily be defined in any space-time dimension $d$ simply by taking the range of the space-time index $\mu$ on the gauge field $A_\mu$ to be $\mu = 0, 1, 2, \ldots, (d - 1)$ in $d$-dimensional space-time. The field strength tensor is still the antisymmetric tensor $F^{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and the Maxwell Lagrangian (1) and the equations of motion (2) do not change their form. The only real difference is that the number of independent fields contained in the field strength tensor $F^{\mu\nu}$ is different in different dimensions. [Since $F^{\mu\nu}$ can be regarded as a $d \times d$ antisymmetric matrix, the number of fields is equal to $\frac{1}{2} d(d-1)$.] So at this level, planar (i.e. $2 + 1$ dimensional ) Maxwell theory is quite similar to the familiar $3 + 1$ dimensional Maxwell theory. The main difference is simply that the magnetic field is a (pseudo-) scalar $\vec{B} = \epsilon^{ij} \partial_i A_j$ in $2 + 1$ dimensions, rather than a (pseudo-) vector $\vec{B} = \vec{\nabla} \times \vec{A}$ in $3 + 1$ dimensions. This is just because in $2 + 1$ dimensions the vector potential $\vec{A}$ is a two-dimensional vector, and the curl in two dimensions produces a scalar. On the other hand, the electric field $\vec{E} = -\nabla A_0 - \vec{A}$ is a two dimensional vector. So the antisymmetric $3 \times 3$ field strength tensor has three nonzero field components: two for the electric field $\vec{E}$ and one for the magnetic field $\vec{B}$.

The real novelty of $2 + 1$ dimensions is that instead of considering this ‘reduced’ form of Maxwell theory, we can also define a completely different type of gauge theory: a Chern-Simons theory. It satisfies our usual criteria for a sensible gauge theory – it is Lorentz invariant, gauge invariant, and local. The Chern-Simons Lagrangian is

$$\mathcal{L}_{CS} = \frac{\kappa}{2} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho - A_\mu J^\mu$$

There are several comments to make about this Chern-Simons Lagrangian. First, it does not look gauge invariant, because it involves the gauge field $A_\mu$ itself, rather than just the (manifestly gauge invariant) field strength $F^{\mu\nu}$. Nevertheless, under a gauge transformation, the Chern-Simons Lagrangian changes by a total space-time derivative

$$\delta \mathcal{L}_{CS} = \frac{\kappa}{2} \partial_\mu (\lambda \epsilon^{\mu\nu\rho} \partial_\nu A_\rho).$$

Therefore, if we can neglect boundary terms (later we shall encounter important examples where this is not true) then the corresponding Chern-Simons action, $S_{CS} = \int d^3x \mathcal{L}_{CS}$, is gauge invariant. This is reflected in the fact that the classical Euler-Lagrange equations

$$\frac{\kappa}{2} \epsilon^{\mu\nu\rho} F_{\nu\rho} = J^\mu; \quad \text{or equivalently} : \quad F^{\mu\nu} = \frac{1}{\kappa} \epsilon^{\mu\nu\rho} J^\rho$$

are clearly gauge invariant. Note that the Bianchi identity, $\epsilon^{\mu\nu\rho} \partial_\mu F_{\nu\rho} = 0$, is compatible with current conservation: $\partial_\mu J^\mu = 0$. 

4
A second important feature of the Chern-Simons Lagrangian is that it is \textit{first-order} in spacetime derivatives. This makes the canonical structure of these theories significantly different from that of Maxwell theory. A related property is that the Chern-Simons Lagrangian is particular to $2 + 1$ dimensions, in the sense that we cannot write down such a term in $3 + 1$ dimensions – the indices simply do not match up. Actually, it is possible to write down a “Chern-Simons theory” in any odd space-time dimension, but it is only in $2 + 1$ dimensions that the Lagrangian is quadratic in the gauge field. For example, the Chern-Simons Lagrangian in five-dimensional space-time is

$$
\mathcal{L} = \epsilon^{\mu\nu\rho\sigma\tau} A_\mu \partial_\nu A_\rho \partial_\sigma A_\tau.
$$

At first sight, pure Chern-Simons theory looks rather boring, and possibly trivial, because the source-free classical equations of motion reduce to $F_{\mu\nu} = 0$, the solutions of which are just pure gauges or “flat connections”. This is in contrast to pure Maxwell theory, where even the source–free theory has interesting, and physically important, solutions: plane-waves. Nevertheless, Chern-Simons theory can be made interesting and nontrivial in a number of ways:

(i) coupling to dynamical matter fields (charged scalars or fermions)
(ii) coupling to a Maxwell term
(iii) taking the space-time to have nontrivial topology
(iv) nonabelian gauge fields
(v) gravity

I do not discuss $2 + 1$ dimensional gravity in these lectures as it is far from the topic of this School, but I stress that it is a rich subject that has taught us a great deal about both classical and quantum gravity.

2 Basics of Planar Field Theory

2.1 Chern-Simons Coupled to Matter Fields - “Anyons”

In order to understand the significance of coupling a matter current $J^\mu = (\rho, \vec{J})$ to a Chern-Simons gauge field, consider the Chern-Simons equations in terms of components:

$$
\begin{align*}
\rho &= \kappa B \\
J^i &= \kappa \epsilon^{ij} E_j
\end{align*}
$$

The first of these equations tells us that the charge density is locally proportional to the magnetic field – thus the effect of a Chern-Simons field is to tie magnetic flux to electric charge. Wherever there is one, there is the other, and they are locally proportional, with the proportionality constant given by the Chern-Simons coupling parameter $\kappa$. This is illustrated in Figure for a collection of point charges. The second equation in (6) ensures that this charge-flux relation is preserved under time evolution because the time derivative of the first equation

$$
\dot{\rho} = \kappa \dot{B} = \kappa \epsilon^{ij} \partial_i \dot{A}_j
$$

together with current conservation, $\dot{\rho} + \partial_i J^i = 0$, implies that

$$
J^i = -\kappa \epsilon^{ij} \dot{A}_j + \epsilon^{ij} \partial_j \chi
$$
which is just the second equation in (3), with the transverse piece $\chi$ identified with $\kappa A_0$.

Thus, the Chern-Simons coupling at this level is pure constraint – we can regard the matter fields as having their own dynamics, and the effect of the Chern-Simons coupling is to attach magnetic flux to the matter charge density in such a way that it follows the matter charge density wherever it goes. Clearly, this applies either to relativistic or nonrelativistic dynamics for the matter fields. (A word of caution here - although the Chern-Simons term is Lorentz invariant, we can regard this simply as a convenient shorthand for expressing the constraint equations (3), in much the same way as we can always express a continuity equation $\dot{\rho} + \partial_i J^i = 0$ in a relativistic-looking way as $\partial_\mu J^\mu = 0$. Thus, there is no problem mixing nonrelativistic dynamics for the matter fields with a ‘relativistic-looking’ Chern-Simons term. The actual dynamics is always inherited from the matter fields.)

This tying of flux to charge provides an explicit realization of “anyons” [2, 3]. (For more details on anyons, see Jan Myrheim’s lectures at this school). Consider, for example, nonrelativistic point charged particles moving in the plane, with magnetic flux lines attached to them. The charge density

$$\rho(\vec{x}, t) = e \sum_{a=1}^{N} \delta(\vec{x} - \vec{x}_a(t))$$

(9)

describes $N$ such particles, with the $a^{th}$ particle following the trajectory $\vec{x}_a(t)$. The corresponding

Figure 1: A collection of point anyons with charge $e$, and with magnetic flux lines of strength $\frac{e}{\kappa}$ tied to the charges. The charge and flux are tied together throughout the motion of the particles as a result of the Chern-Simons equations (3).
current density is $\vec{j}(\vec{x}, t) = e \sum_{a=1}^{N} \dot{x}_a(t) \delta(\vec{x} - \vec{x}_a(t))$. The Chern-Simons equations (11) attach magnetic flux [see Figure 1]

$$B(\vec{x}, t) = \frac{1}{\kappa} e \sum_{a=1}^{N} \delta(\vec{x} - \vec{x}_a(t))$$

which follows each point particle throughout its motion.

If each particle has mass $m$, the net action is

$$S = \frac{m}{2} \sum_{a=1}^{N} \int dt \vec{v}_a^2 + \frac{\kappa}{2} \int d^3 x e^{\mu\nu\rho} A_\mu \partial_\nu A_\rho - \int d^3 x A_\mu J^\mu$$

The Chern-Simons equations of motion (11) determine the gauge field $A_\mu(\vec{x}, t)$ in terms of the particle current. The gauge freedom may be fixed in a Hamiltonian formulation by taking $A_0 = 0$ and imposing $\nabla \cdot \vec{A} = 0$. Then

$$A_i(\vec{x}, t) = \frac{1}{2\pi \kappa} \int d^2 y \epsilon_{ij} (x^j - y^j) \rho(\vec{y}, t) = \frac{e}{2\pi \kappa} \sum_{a=1}^{N} \epsilon_{ij} (x^j - x^j_a(t)) |\vec{x} - \vec{x}_a(t)|^2$$

where we have used the two dimensional Green’s function

$$\nabla^2 \left( \frac{1}{2\pi} \log |\vec{x} - \vec{y}| \right) = \delta^{(2)}(\vec{x} - \vec{y})$$

As an aside, note that using the identity $\partial_i arg(\vec{x}) = -\epsilon_{ij} x^j / |\vec{x}|$, where the argument function is $arg(\vec{x}) = \arctan(\frac{y}{x})$, we can express this vector potential (12) as

$$A_i(\vec{x}) = \frac{e}{2\pi \kappa} \sum_{a=1}^{N} \partial_i arg(\vec{x} - \vec{x}_a)$$

Naively, this looks like a pure gauge vector potential, which could presumably therefore be removed by a gauge transformation. However, under such a gauge transformation the corresponding nonrelativistic field $\psi(\vec{x})$ would acquire a phase factor

$$\psi(\vec{x}) \to \tilde{\psi}(\vec{x}) = \exp \left( -i \frac{e^2}{2\pi \kappa} \sum_{a=1}^{N} arg(\vec{x} - \vec{x}_a) \right) \psi(\vec{x})$$

which makes the field non-single-valued for general values of the Chern-Simons coupling parameter $\kappa$. This lack of single-valuedness is the nontrivial remnant of the Chern-Simons gauge field coupling. Thus, even though it looks as though the gauge field has been gauged away, leaving a ‘free’ system, the complicated statistical interaction is hidden in the nontrivial boundary conditions for the non-single-valued field $\tilde{\psi}$.

Returning to the point-anyon action (11), the Hamiltonian for this system is

$$H = \frac{m}{2} \sum_{a=1}^{N} \vec{v}_a^2 = \frac{1}{2m} \sum_{a=1}^{N} [\vec{p}_a - e\vec{A}(\vec{x}_a)]^2$$
where

\[ A^i(\vec{x}_a) = \frac{e}{2\pi\kappa} \sum_{b \neq a}^{N} \epsilon_{ij} \left( \frac{x^j_a - x^j_b}{|\vec{x}_a - \vec{x}_b|^2} \right) \]  

(17)

The corresponding magnetic field is

\[ B(\vec{x}_a) = \frac{e}{\kappa} \sum_{b \neq a}^{N} \delta(\vec{x}_a - \vec{x}_b) \]  

(18)

so that each particle sees each of the \( N - 1 \) others as a point vortex of flux \( \Phi = \frac{e}{\kappa} \), as expected. Note that the gauge field in (17) excludes the self-interaction \( a = b \) term, with suitable regularization [3, 4].

Figure 2: Aharonov-Bohm interaction between the charge \( e \) of an anyon and the flux \( \frac{e}{\kappa} \) of another anyon under double-interchange. Under such an adiabatic transport, the multi-anyon wavefunction acquires an Aharonov-Bohm phase (19).

An important consequence of this charge-flux coupling is that it leads to new Aharonov-Bohm-type interactions. For example, when one such particle moves adiabatically around another [as shown in Figure 2], in addition to whatever electrical interactions mediate between them, at the quantum level the nonrelativistic wavefunction acquires an Aharonov-Bohm phase

\[ \exp \left( ie \oint_C \vec{A} \cdot d\vec{x} \right) = \exp \left( \frac{ie^2}{\kappa} \right) \]  

(19)

If this adiabatic excursion is interpreted as a double interchange of two such identical particles (each with flux attached), then this gives an “anyonic” exchange phase

\[ 2\pi \Delta \theta = \frac{e^2}{2\kappa} \]  

(20)
which can be tuned to any value by specifying the value of the Chern-Simons coupling coefficient $\kappa$. This is the origin of anyonic statistics in point-particle language.

This is a first-quantized description of anyons as point particles. However, N-anyon quantum mechanics can be treated, in the usual manner of nonrelativistic many-body quantum mechanics, as the N-particle sector of a nonrelativistic quantum field theory $[3, 4]$. In this case, a Chern-Simons field is required to ensure that the appropriate magnetic flux is always attached to the (smeared-out) charged particle fields $\varphi(\vec{x}, t)$. This, together with the above-mentioned statistics transmutation, explains the appearance of Chern-Simons fields in the “composite boson” or “composite fermion” models for the fractional quantum Hall effect, which involve quasiparticles that have magnetic fluxes attached to charged particles $[5, 6, 7, 8]$. In such field theories there is a generalized spin-statistics relation similar to (20) – see later in Eq. (100). By choosing $\kappa$ appropriately, the anyonic exchange phase (20) can be chosen so that the particles behave either as fermions or as bosons. An explicit example of this statistical transmutation will be used in Section 4.6 on the Zhang-Hansson-Kivelson model $[5]$ for the fractional quantum Hall effect.

2.2 Maxwell-Chern-Simons: topologically massive gauge theory

Since both the Maxwell and Chern-Simons Lagrangians produce viable gauge theories in $2 + 1$ dimensions, it is natural to consider coupling them together. The result is a surprising new form of gauge field mass generation. Consider the Lagrangian

$$L_{\text{MCS}} = -\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + \frac{\kappa}{2} \epsilon^{\mu\nu\rho}A_\mu \partial_\nu A_\rho$$

(21)

The resulting classical field equations are

$$\partial_\mu F^{\mu\nu} + \frac{\kappa e^2}{2} \epsilon^{\nu\alpha\beta} F_{\alpha\beta} = 0$$

(22)

which describe the propagation of a single (transverse) degree of freedom with mass (note that $e^2$ has dimensions of mass in $2 + 1$ dimensions, while $\kappa$ is dimensionless):

$$m_{\text{MCS}} = \kappa e^2$$

(23)

This has resulted in the terminology “topologically massive gauge theory” $[3]$, where the term “topological” is motivated by the nonabelian Chern-Simons theory (see Section 2.6).

The most direct way to see the origin of this mass is to re-write the equation of motion (22) in terms of the pseudovector “dual” field $\tilde{F}^\mu \equiv \frac{1}{2} \epsilon^{\mu\nu\rho} F_{\nu\rho}$:

$$\left[ \partial_\mu \partial^\mu + (\kappa e^2)^2 \right] \tilde{F}^\nu = 0$$

(24)

Note that this dual field $\tilde{F}^\mu$ is manifestly gauge invariant, and it also satisfies $\partial_\mu \tilde{F}^\mu = 0$. The MCS mass can also be identified from the corresponding representation theory of the Poincaré algebra in $2 + 1$-dimensions, which also yields the spin of the massive excitation as

$$s_{\text{MCS}} = \frac{\kappa}{|\kappa|} = \pm 1$$

(25)
We shall discuss these mass and spin properties further in Section 2.5.

**Exercise 2.2.1**: Another useful way to understand the origin of the massive gauge excitation is to compute the gauge field propagator in (for example) a covariant gauge with gauge fixing term $\mathcal{L}_{gf} = -\frac{1}{2\xi e^2} (\partial_\mu A^\mu)^2$. By inverting the quadratic part of the momentum space lagrangian, show that the gauge field propagator is

$$\Delta_{\mu\nu} = e^2 \left( \frac{p^\mu g_{\nu\rho} - p_\mu p_\nu - i\kappa e^2 \epsilon_{\mu\rho\sigma} p^\sigma}{p^2(p^2 - \kappa^2 e^4)} \right) + \frac{\xi p_\mu p_\nu}{(p^2)^2}$$

This clearly identifies the gauge field mass via the pole at $p^2 = (\kappa e^2)^2$.

I emphasize that this gauge field mass is completely independent of the standard Higgs mechanism for generating masses for gauge fields through a nonzero expectation value of a Higgs field. Indeed, we can also consider the Higgs mechanism in a Maxwell-Chern-Simons theory, in which case we find two independent gauge field masses. For example, couple this Maxwell-Chern-Simons theory to a complex scalar field $\phi$ with a symmetry breaking potential $V(|\phi|)$

$$\mathcal{L}_{MCSH} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \frac{\kappa}{2} \epsilon_{\mu\rho\sigma} A_\mu \partial_\nu A_\rho + (D_\mu \phi)^* D^\mu \phi - V(|\phi|)$$

where $V(|\phi|)$ has some nontrivial minimum with $<\phi> = v$. In this broken vacuum there is an additional quadratic term $v^2 A_\mu A^\mu$ in the gauge field Lagrangian which leads to the momentum space propagator (with a covariant gauge fixing term)

$$\Delta_{\mu\nu} = \frac{e^2 (p^2 - m_H^2)}{(p^2 - m^2_+) (p^2 - m^2_-)} \left[ g_{\mu\nu} - \frac{p_\mu p_\nu}{(p^2 - \kappa^2 e^4 - m_H^2)} - \frac{i\kappa e^2 \epsilon_{\mu\rho\sigma} p^\rho}{(p^2 - m_H^2)^2} \right] + e^2 \xi \frac{p_\mu p_\nu (p^2 - \kappa^2 e^4 - m_H^2)}{(p^2 - m^2_+) (p^2 - m^2_-) (p^2 - \kappa e^2)}$$

where $m_H^2 = 2e^2 v^2$ is the usual Higgs mass scale (squared) and the other masses are

$$m^2_\pm = m_H^2 + \frac{(\kappa e^2)^2}{2} \pm \frac{\kappa e^2}{2} \sqrt{\kappa^2 e^4 + 4m_H^2}$$

or

$$m_\pm = \frac{m_{MCS}}{2} \left( \sqrt{1 + \frac{4m_H^2}{m_{MCS}^2}} \pm 1 \right)$$

From the propagator we identify two physical mass poles at $p^2 = m^2_\pm$.

The counting of degrees of freedom goes as follows. In the unbroken vacuum, the complex scalar field has two real massive degrees of freedom and the gauge field has one massive excitation (with mass coming from the Chern-Simons term). In the broken vacuum, one component of the scalar
field (the “Goldstone boson”) combines with the longitudinal part of the gauge field to produce a new massive gauge degree of freedom. Thus, in the broken vacuum there is one real massive scalar degree of freedom (the “Higgs boson”) and two massive gauge degrees of freedom.

The Higgs mechanism also occurs, albeit somewhat differently, if the gauge field has just a Chern-Simons term, and no Maxwell term [11]. The Maxwell term can be decoupled from the Maxwell-Chern-Simons-Higgs Lagrangian (27) by taking the limit

\[ e^2 \to \infty \quad \kappa = \text{fixed} \]  

which leads to the Chern-Simons-Higgs Lagrangian

\[ L_{CSH} = \frac{\kappa}{2} \epsilon^{\mu \nu \rho} A_\mu \partial_\nu A_\rho + (D_\mu \phi)^\dagger D^\mu \phi - V(|\phi|) \]  

Exercise 2.2.2: Show that the propagator (28) reduces in the limit (31) to

\[ \Delta_{\mu \nu} = \frac{1}{p^2 - (\frac{2v^2}{\kappa})^2} \left[ \frac{2v^2}{\kappa} g_{\mu \nu} - \frac{1}{2v^2} p_\mu p_\nu + \frac{i}{\kappa} \epsilon_{\mu \nu \rho} p^\rho \right] \]  

which has a single massive pole at \( p^2 = (\frac{2v^2}{\kappa})^2 \).

The counting of degrees of freedom is different in this Chern-Simons-Higgs model. In the unbroken vacuum the gauge field is nonpropagating, and so there are just the two real scalar modes of the scalar field \( \phi \). In the broken vacuum, one component of the scalar field (the “Goldstone boson”) combines with the longitudinal part of the gauge field to produce a massive gauge degree of freedom. Thus, in the broken vacuum there is one real massive scalar degree of freedom (the “Higgs boson”) and one massive gauge degree of freedom. This may also be deduced from the mass formulae (30) for the Maxwell-Chern-Simons-Higgs model, which in the limit (31) tend to

\[ m^+ \to \infty \quad m^- \to \frac{2v^2}{\kappa} \]  

so that one mass \( m^+ \) decouples to infinity, while the other mass \( m^- \) agrees with the mass pole found in (33). In Section 3.2 we shall see that there is a simple way to understand these various gauge masses in terms of the characteristic frequencies of the familiar quantum mechanical Landau problem.

2.3 Fermions in 2 + 1-dimensions

Fermion fields also have some new and interesting features when restricted to the plane. The most obvious difference is that the irreducible set of Dirac matrices consists of \( 2 \times 2 \) matrices, rather
than $4 \times 4$. Correspondingly, the irreducible fermion fields are 2-component spinors. The Dirac equation is

$$(i\gamma^\mu \partial_\mu - e\gamma^\mu A_\mu - m) \psi = 0, \quad \text{or} \quad i\frac{\partial}{\partial t} \psi = \left(-i\vec{\alpha} \cdot \vec{\nabla} + m\beta\right) \psi$$

(35)

where $\vec{\alpha} = \gamma^0\gamma^1$ and $\beta = \gamma^0$. The Dirac gamma matrices satisfy the anticommutation relations: $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, where we use the Minkowski metric $g^{\mu\nu} = \text{diag}(1, -1, -1)$. One natural representation is a ‘Dirac’ representation:

$$\gamma^0 = \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\gamma^1 = i\sigma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$\gamma^2 = i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(36)

while a ‘Majorana’ representation (in which $\beta$ is imaginary while the $\vec{\alpha}$ are real) is:

$$\gamma^0 = \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\gamma^1 = i\sigma^3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$\gamma^2 = i\sigma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

(37)

These $2 \times 2$ Dirac matrices satisfy the identities:

$$\gamma^\mu \gamma^\nu = g^{\mu\nu} \mathbf{1} - i\epsilon^{\mu\nu\rho} \gamma_\rho$$

(38)

$$\text{tr}(\gamma^\mu \gamma^\nu \gamma^\rho) = -2i \epsilon^{\mu\nu\rho}$$

(39)

Note that in familiar $3 + 1$ dimensional theories, the trace of an odd number of gamma matrices vanishes. In $2+1$ dimensions, the trace of three gamma matrices produces the totally antisymmetric $\epsilon^{\mu\nu\rho}$ symbol. This fact plays a crucial role in the appearance of induced Chern-Simons terms in quantized planar fermion theories, as will be discussed in detail in Section 5. Another important novel feature of $2 + 1$ dimensions is that there is no “$\gamma^5$” matrix that anticommutes with all the Dirac matrices - note that $i\gamma^0\gamma^1\gamma^2 = 1$. Thus, there is no notion of chirality in the usual sense.

2.4 Discrete Symmetries: $P$, $C$ and $T$

The discrete symmetries of parity, charge conjugation and time reversal act very differently in $2 + 1$-dimensions. Our usual notion of a parity transformation is a reflection $\vec{x} \rightarrow -\vec{x}$ of the spatial coordinates. However, in the plane, such a transformation is equivalent to a rotation (this Lorentz transformation has $\det(\Lambda) = (-1)^2 = +1$ instead of $\det(\Lambda) = (-1)^3 = -1$). So the improper
discrete ‘parity’ transformation should be taken to be reflection in just one of the spatial axes (it
doesn’t matter which we choose):

\[
\begin{align*}
x^1 &\rightarrow -x^1 \\
x^2 &\rightarrow x^2
\end{align*}
\]

From the kinetic part of the Dirac Lagrangian we see that the spinor field \( \psi \) transforms as

\[
\psi \rightarrow \gamma^1 \psi
\]

(where we have suppressed an arbitrary unimportant phase). But this means that a fermion mass
term breaks parity

\[
\bar{\psi} \psi \rightarrow -\bar{\psi} \psi
\]

Under \( P \), the gauge field transforms as

\[
A^1 \rightarrow -A^1, \quad A^2 \rightarrow A^2, \quad A^0 \rightarrow A^0
\]

which means that while the standard Maxwell kinetic term is \( P \)-invariant, the Chern-Simons term
changes sign under \( P \):

\[
\epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \rightarrow -\epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho
\]

Charge conjugation converts the “electron” Dirac equation (35) into the “positron” equation:

\[
(i \gamma^\mu \partial_\mu + e \gamma^\mu A_\mu - m) \psi_c = 0
\]

As is standard, this is achieved by the definition \( \psi_c \equiv C \gamma^0 \psi^* \), where the charge conjugation matrix
\( C \) must satisfy

\[
(\gamma^\mu)^T = -C^{-1} \gamma^\mu C
\]

In the Dirac representation we can choose \( C = \gamma^2 \). Note then that the fermion mass term is
invariant under \( C \) (recall the anticommuting nature of the fermion fields), as is the Chern-Simons
term for the gauge field.

Time reversal is an anti-unitary operation \( (T: i \rightarrow -i) \) in order to implement \( x^0 \rightarrow -x^0 \)
without taking \( P^0 \rightarrow -P^0 \). The action on spinor and gauge fields is [using the Dirac representation
(36)]

\[
\psi \rightarrow \gamma^2 \psi, \quad \vec{A} \rightarrow -\vec{A}, \quad A^0 \rightarrow A^0
\]

From this we see that both the fermion mass term and the gauge field Chern-Simons term change
sign under time reversal.

The fact that the fermion mass term and the Chern-Simons term have the same transformation
properties under the discrete symmetries of \( P, C \) and \( T \) will be important later in Section 5 when
we consider radiative corrections in planar gauge and fermion theories. One way to understand
this connection is that these two terms are supersymmetric partners in \( 2 + 1 \) dimensions [3].
2.5 Poincaré Algebra in 2 + 1-dimensions

The novel features of fermion and gauge fields in 2 + 1-dimensions, as well as the anyonic fields, can be understood better by considering the representation theory of the Poincaré algebra. Our underlying guide is Wigner’s Principle: that in quantum mechanics the relativistic single-particle states should carry a unitary, irreducible representation of the universal covering group of the Poincaré group \[14\].

The Poincaré group \(ISO(2,1)\) combines the proper Lorentz group \(SO(2,1)\) with space-time translations \[15, 16\]. The Lorentz generators \(L^{\mu\nu}\) and translation generators \(P^\mu\) satisfy the standard Poincaré algebra commutation relations, which can be re-expressed in 2 + 1-dimensions as

\[
\begin{align*}
[J^\mu, J^\nu] &= i\epsilon^{\mu\nu\rho} J^\rho, \\
[J^\mu, P^\nu] &= i\epsilon^{\mu\nu\rho} P^\rho, \\
[P^\mu, P^\nu] &= 0
\end{align*}
\]

where the pseudovector generator \(J^\mu\) is \(J^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho} L^{\nu\rho}\). Irreducible representations of this algebra may be characterized by the eigenvalues of the two Casimirs:

\[
P^2 = P_\mu P^\mu, \quad W = P_\mu J^\mu
\]  \hspace{1cm} (48)

Here, \(W\) is the Pauli-Lubanski pseudoscalar, the 2 + 1 dimensional analogue of the familiar Pauli-Lubanski pseudovector in 3 + 1 dimensions. We define single-particle representations \(\Phi\) by

\[
P^2 \Phi = m^2 \Phi, \quad W \Phi = -s m \Phi
\]  \hspace{1cm} (49)

defining the mass \(m\) and spin \(s\).

For example, a spin 0 scalar field may be represented by a momentum space field \(\phi(p)\) on which \(P^\mu\) acts by multiplication and \(J^\mu\) as an orbital angular momentum operator:

\[
P^\mu \phi = p^\mu \phi, \quad J^\mu \phi = -i\epsilon^{\mu\nu\rho} p_\nu \frac{\partial}{\partial p^\rho} \phi
\]  \hspace{1cm} (50)

Then the eigenvalue conditions \[49\] simply reduce to the Klein-Gordon equation \((p^2 - m^2)\phi = 0\) for a spin 0 field since \(P \cdot J\phi = 0\).

For a two-component spinor field \(\psi\) we take

\[
J^\mu = -i\epsilon^{\mu\nu\rho} p_\nu \frac{\partial}{\partial p^\rho} 1 - \frac{1}{2} \gamma^\mu
\]  \hspace{1cm} (51)

so the eigenvalue conditions \[49\] become a Dirac equation of motion \((i\gamma^\mu \partial_\mu - m)\psi = 0\), corresponding to spin \(s = \pm \frac{1}{2}\).

For a vector field \(A_\mu\), whose gauge invariant content may be represented through the pseudovector dual \(F^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho} F_{\nu\rho}\), we take

\[
(J^\mu)_{\alpha\beta} = -i\epsilon^{\mu\nu\rho} p_\nu \frac{\partial}{\partial p^\rho} \delta_{\alpha\beta} + i\epsilon^\mu_{\alpha\beta}
\]  \hspace{1cm} (52)
Then the eigenvalue condition
\[(P \cdot J)_{\alpha \beta} \tilde{F}^\beta = \imath \epsilon_{\alpha \beta \mu} p_\mu \tilde{F}^\beta = -s m \tilde{F}_\alpha\] (53)
has the form of the topologically massive gauge field equation of motion (22). We therefore deduce a mass \(m = \kappa e^2\) and a spin \(s = \text{sign}(\kappa) = \pm 1\). This agrees with the Maxwell-Chern-Simons mass found earlier in (23), and is the source of the Maxwell-Chern-Simons spin quoted in (25).

In general, it is possible to modify the standard “orbital” form of \(J^\mu\) appearing in the scalar field case (50) without affecting the Poincaré algebra:
\[J^\mu = -\imath \epsilon^{\mu \nu \rho} p_\nu \frac{\partial}{\partial p^\rho} - s \left( \frac{p^\mu + m \eta^\mu}{p \cdot \eta + m} \right), \quad \eta^\mu = (1, 0, 0)\] (54)
It is easy to see that this gives \(W = P \cdot J = -sm\), so that the spin can be arbitrary. This is one way of understanding the possibility of anyonic spins in 2 + 1 dimensions. Actually, the real question is how this form of \(J^\mu\) can be realized in terms of a local equation of motion for a field. If \(s\) is an integer or a half-integer then this can be achieved with a \((2s+1)\)-component field, but for arbitrary spin \(s\) we require infinite component fields [17].

### 2.6 Nonabelian Chern-Simons Theories

It is possible to write a nonabelian version of the Chern-Simons Lagrangian (3):
\[\mathcal{L}_{CS} = \kappa \epsilon^{\mu \nu \rho} \text{tr} \left(A_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho \right)\] (55)
The gauge field \(A_\mu\) takes values in a finite dimensional representation of the (semi-simple) gauge Lie algebra \(\mathcal{G}\). In these lectures we take \(\mathcal{G} = su(N)\). In an abelian theory, the gauge fields \(A_\mu\) commute, and so the trilinear term in (55) vanishes due to the antisymmetry of the \(\epsilon^{\mu \nu \rho}\) symbol. In the nonabelian case [just as in Yang-Mills theory] we write \(A_\mu = A_\mu^a T^a\) where the \(T^a\) are the generators of \(\mathcal{G}\) [for \(a = 1, \ldots, \text{dim}(\mathcal{G})\)], satisfying the commutation relations \([T^a, T^b] = f^{abc} T^c\), and the normalization \(\text{tr}(T^a T^b) = -\frac{1}{2} \delta^{ab}\).

**Exercise 2.6.1:** Show that under infinitesimal variations \(\delta A_\mu\) of the gauge field the change in the nonabelian Chern-Simons Lagrangian is
\[\delta \mathcal{L}_{CS} = \kappa \epsilon^{\mu \nu \rho} \text{tr} \left( \delta A_\mu F_{\nu \rho} \right)\] (56)
where \(F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\) is the nonabelian field strength.

From the variation (56) we see that the nonabelian equations of motion have the same form as the abelian ones: \(\kappa \epsilon^{\mu \nu \rho} F_{\nu \rho} = J^\mu\). Note also that the Bianchi identity, \(\epsilon^{\mu \nu \rho} D_\mu F_{\nu \rho} = 0\), is compatible with covariant current conservation: \(D_\mu J^\mu = 0\). The source-free equations are once again \(F_{\mu \nu} = 0\), for which the solutions are pure gauges (flat connections) \(A_\mu = g^{-1} \partial_\mu g\), with \(g\) in the gauge group.
An important difference, however, lies in the behaviour of the nonabelian Chern-Simons Lagrangian (55) under a gauge transformation. The nonabelian gauge transformation \( g \) (which is an element of the gauge group) transforms the gauge field as

\[
A_\mu \rightarrow A_\mu^g \equiv g^{-1}A_\mu g + g^{-1}\partial_\mu g
\]

(57)

**Exercise 2.6.2**: Show that under the gauge transformation (57), the Chern-Simons Lagrangian \( L_{CS} \) in (55) transforms as

\[
L_{CS} \rightarrow L_{CS} - \kappa \epsilon^{\mu\nu\rho} \partial_\mu \text{tr} (\partial_\nu g g^{-1} A_\rho) - \frac{\kappa}{3} \epsilon^{\mu\nu\rho} \text{tr} (g^{-1} \partial_\mu gg^{-1} \partial_\nu gg^{-1} \partial_\rho g)
\]

(58)

We recognize, as in the abelian case, a total space-time derivative term, which vanishes in the action with suitable boundary conditions. However, in the nonabelian case there is a new term in (58), known as the winding number density of the group element \( g \):

\[
w(g) = \frac{1}{24\pi^2} \epsilon^{\mu\nu\rho} \text{tr} (g^{-1} \partial_\mu gg^{-1} \partial_\nu gg^{-1} \partial_\rho g)
\]

(59)

With appropriate boundary conditions, the integral of \( w(g) \) is an integer - see Exercise 2.6.3. Thus, the Chern-Simons action changes by an additive constant under a large gauge transformation (i.e., one with nontrivial winding number \( N \)):

\[
S_{CS} \rightarrow S_{CS} - 8\pi^2 \kappa N
\]

(60)

This has important implications for the development of a quantum nonabelian Chern-Simons theory. To ensure that the quantum amplitude \( \exp(i S) \) remains gauge invariant, the Chern-Simons coupling parameter \( \kappa \) must assume discrete values

\[
\kappa = \text{integer} \frac{4\pi}{\kappa}
\]

(61)

This is analogous to Dirac’s quantization condition for a magnetic monopole [18]. We shall revisit this Chern-Simons discreteness condition in more detail in later Sections.

**Exercise 2.6.3**: In three dimensional Euclidean space, take the \( SU(2) \) group element

\[
g = \exp(i\pi N \frac{\vec{x} \cdot \vec{\sigma}}{\sqrt{\vec{x}^2 + R^2}})
\]

(62)

where \( \vec{\sigma} \) are the Pauli matrices, and \( R \) is an arbitrary scale parameter. Show that the winding number for this \( g \) is equal to \( N \). Why must \( N \) be an integer?
To conclude this brief review of the properties of nonabelian Chern-Simons terms, I mention the original source of the name “Chern-Simons”. S. S. Chern and J. Simons were studying a combinatorial approach to the Pontryagin density $\varepsilon^{\mu\nu\rho\sigma} \text{tr} (F_{\mu\nu} F_{\rho\sigma})$ in four dimensions and noticed that it could be written as a total derivative:

$$
\varepsilon^{\mu\nu\rho\sigma} \text{tr} (F_{\mu\nu} F_{\rho\sigma}) = 4 \partial_\sigma \left[ \varepsilon^{\mu\nu\rho\sigma} \text{tr} \left( A_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho \right) \right]
$$

(63)

Their combinatorial approach “got stuck by the emergence of a boundary term which did not yield to a simple combinatorial analysis. The boundary term seemed interesting in its own right, and it and its generalizations are the subject of this paper” [19]. We recognize this interesting boundary term as the Chern-Simons Lagrangian (55).

3 Canonical Quantization of Chern-Simons Theories

There are many ways to discuss the quantization of Chern-Simons theories. Here I focus on canonical quantization because it has the most direct relationship with the condensed matter applications which form the primary subject of this School. Indeed, the well-known Landau and Hofstadter problems of solid state physics provide crucial physical insight into the canonical quantization of Chern-Simons theories.

3.1 Canonical Structure of Chern-Simons Theories

In this Section we consider the classical canonical structure and Hamiltonian formulation of Chern-Simons theories, in preparation for a discussion of their quantization. We shall discover an extremely useful quantum mechanical analogy to the classic Landau problem of charged electrons moving in the plane in the presence of an external uniform magnetic field perpendicular to the plane. I begin with the abelian theory because it contains the essential physics, and return to the nonabelian case later.

The Hamiltonian formulation of Maxwell (or Yang-Mills) theory is standard. In the Weyl gauge ($A_0 = 0$) the spatial components of the gauge field $\vec{A}$ are canonically conjugate to the electric field components $\vec{E}$, and Gauss’s law $\vec{\nabla} \cdot \vec{E} = \rho$ appears as a constraint, for which the nondynamical field $A_0$ is a Lagrange multiplier. If you wish to remind yourself of the Maxwell case, simply set the Chern-Simons coupling $\kappa$ to zero in the following equations (64) - (69).

Now consider instead the canonical structure of the Maxwell-Chern-Simons theory with Lagrangian (21):

$$
\mathcal{L}_{\text{MCS}} = \frac{1}{2e^2} E_i^2 - \frac{1}{2e^2} B^2 + \frac{\kappa}{2} \epsilon^{ij} \dot{A}_i A_j + \kappa A_0 B
$$

(64)

The $A^0$ field is once again nondynamical, and can be regarded as a Lagrange multiplier enforcing the Gauss law constraint

$$
\partial_i F^{i0} + \kappa e^2 \epsilon^{ij} \partial_j A_j = 0
$$

(65)
This is simply the \( \nu = 0 \) component of the Euler-Lagrange equations \((22)\). In the \( A_0 = 0 \) gauge we identify the \( A_i \) as ‘coordinate’ fields, with corresponding ‘momentum’ fields

\[
\Pi^i = \frac{\partial L}{\partial \dot{A}_i} = \frac{1}{e^2} \dot{A}_i + \frac{\kappa}{2} \epsilon^{ij} A_j
\]

The Hamiltonian is obtained from the Lagrangian by a Legendre transformation

\[
\mathcal{H}_{MCS} = \Pi^i \dot{A}_i - L = \frac{e^2}{2} \left(\Pi^i - \frac{\kappa}{2} \epsilon^{ij} A_j\right)^2 + \frac{1}{2e^2} B^2 + A_0 \left(\partial_i \Pi^i + \kappa B\right)
\]

At the classical level, the fields \( A_i(\vec{x},t) \) and \( \Pi^i(\vec{x},t) \) satisfy canonical equal-time Poisson brackets. These become equal-time canonical commutation relations in the quantum theory:

\[
[A_i(\vec{x}), \Pi^j(\vec{y})] = i \delta^j_\ell \delta(\vec{x} - \vec{y})
\]

Notice that this implies that the electric fields do not commute (for \( \kappa \neq 0 \))

\[
[E_i(\vec{x}), E_j(\vec{y})] = -i \kappa e^4 \epsilon_{ij} \delta(\vec{x} - \vec{y})
\]

The Hamiltonian \((67)\) still takes the standard Maxwell form \( \mathcal{H} = \frac{1}{2e^2}(\vec{E}^2 + B^2) \) when expressed in terms of the electric and magnetic fields. This is because the Chern-Simons term does not modify the energy – it is, after all, first order in time derivatives. But it does modify the relation between momenta and velocity fields. This is already very suggestive of the effect of an external magnetic field on the dynamics of a charged particle.

Now consider a pure Chern-Simons theory, with no Maxwell term in the Lagrangian.

\[
\mathcal{L}_{CS} = \frac{\kappa}{2} \epsilon^{ij} \dot{A}_i A_j + \kappa A_0 B
\]

Once again, \( A_0 \) is a Lagrange multiplier field, imposing the Gauss law: \( B = 0 \). But the Lagrangian is first order in time derivatives, so it is already in the Legendre transformed form \( L = p \dot{x} - H \), with \( H = 0 \). So there is no dynamics – indeed, the only dynamics would be inherited from coupling to dynamical matter fields. Another way to see this is to notice that the pure Chern-Simons energy momentum tensor

\[
T^{\mu\nu} = \frac{2}{\sqrt{\det g}} \frac{\delta S_{CS}}{\delta g^{\mu\nu}}
\]

vanishes identically because the Chern-Simons action is independent of the metric, since the Lagrange density is a three-form \( L = \text{tr}(AdA + \frac{2}{3} AAA) \).

Another important fact about the pure Chern-Simons system \((70)\) is that the components of the gauge field are canonically conjugate to one another:

\[
[A_i(\vec{x}), A_j(\vec{y})] = i \frac{\kappa}{\epsilon} \epsilon_{ij} \delta(\vec{x} - \vec{y})
\]

This is certainly very different from the Maxwell theory, for which the components of the gauge field commute, and it is the \( A_i \) and \( E_i \) fields that are canonically conjugate. So pure Chern-Simons
is a strange new type of gauge theory, with the components $A_i$ of the gauge field not commuting with one another.

We can recover this noncommutativity property from the Maxwell-Chern-Simons case by taking the limit $e^2 \to \infty$, with $\kappa$ kept fixed. Then, from the Hamiltonian (67) we see that we are forced to impose the constraint

$$\Pi^i = \frac{\kappa}{2} \epsilon^{ij} A_j$$

then the Maxwell-Chern-Simons Hamiltonian (67) vanishes and the Lagrangian (64) reduces to the pure Chern-Simons Lagrangian (70). The canonical commutation relations (72) arise because of the constraints (73), noting that these are second-class constraints so we must use Dirac brackets to find the canonical relations between $A_i$ and $A_j$ [23].

### 3.2 Chern-Simons Quantum Mechanics

To understand more deeply this somewhat unusual projection from a Maxwell-Chern-Simons theory to a pure Chern-Simons theory we appeal to the following quantum mechanical analogy [24, 23]. Consider the long wavelength limit of the Maxwell-Chern-Simons Lagrangian, in which we drop all spatial derivatives. (This is sufficient for identifying the masses of excitations.) Then the resulting Lagrangian

$$L = \frac{1}{2e^2} \dot{A}_i^2 + \frac{\kappa}{2} \epsilon^{ij} \dot{A}_i A_j$$

has exactly the same form as the Lagrangian for a nonrelativistic charged particle moving in the plane in the presence of a uniform external magnetic field $b$ perpendicular to the plane

$$L = \frac{1}{2} m \dot{x}_i^2 + b \epsilon^{ij} \dot{x}_i x_j$$

The canonical analysis of this mechanical model is a simple undergraduate physics exercise. The momenta

$$p_i = \frac{\partial L}{\partial \dot{x}_i} = m \dot{x}_i + \frac{b}{2} \epsilon^{ij} \dot{x}_j$$

are shifted from the velocities and the Hamiltonian is

$$H = p_i \dot{x}_i - L = \frac{1}{2m} (p_i - \frac{b}{2} \epsilon^{ij} \dot{x}_j)^2 = \frac{m}{2} v_i^2$$

At the quantum level the canonical commutation relations, $[x_i, p_j] = i\delta_{ij}$, imply that the velocities do not commute: $[v_i, v_j] = -i \frac{b}{m^2} \epsilon_{ij}$. It is clear that these features of the Landau problem mirror precisely the canonical structure of the Maxwell-Chern-Simons system, for both the Hamiltonian (67) and the canonical commutation relations (68) and (69).

\[
\begin{align*}
\text{MCS field theory} & \quad \leftrightarrow \quad \text{Landau problem} \\
\frac{e^2}{\kappa} & \quad \leftrightarrow \quad \frac{1}{m} \\
\frac{\kappa e^2}{m} & \quad \leftrightarrow \quad \frac{b}{m} \\
MCS & \quad \leftrightarrow \quad \omega_c = \frac{b}{m}
\end{align*}
\]
This correspondence is especially useful because the quantization of the Landau system is well understood. The quantum mechanical spectrum consists of equally spaced energy levels (Landau levels), spaced by $\hbar \omega_c$ where the cyclotron frequency is $\omega_c = \frac{b}{m}$. See Figure 3. Each Landau level is infinitely degenerate in the open plane, while for a finite area the degeneracy is related to the net magnetic flux

$$N_{\text{deg}} = \frac{bA}{2\pi}$$

where $A$ is the area [20, 21].

![Figure 3](image-url)

Figure 3: The energy spectrum for charged particles in a uniform magnetic field consists of equally spaced ‘Landau levels’, separated by $\hbar \omega_c$ where $\omega_c$ is the cyclotron frequency. Each Landau level has degeneracy given by the total magnetic flux through the sample.

The pure Chern-Simons limit is when $e^2 \to \infty$, with $\kappa$ fixed. In the quantum mechanical case this corresponds to taking the mass $m \to 0$, with $b$ fixed. Thus, the cyclotron frequency $\omega_c = \frac{b}{m}$ becomes infinite and so the energy gap between Landau levels becomes infinite, isolating each level from the others. We therefore have a formal projection onto a highly degenerate ground state - the lowest Landau level (LLL). Interestingly, this is exactly the type of limit that is of physical interest in quantum Hall systems – see Steve Girvin’s lectures at this School for more details on the importance of the lowest Landau level. In the limit $m \to 0$ of projecting to the lowest Landau level, the Lagrangian (75) becomes $L = \frac{b}{2} \epsilon^{ij} \dot{x}_i x_j$. This is first order in time derivatives, so the two coordinates $x_1$ and $x_2$ are in fact canonically conjugate to one another, with commutation relations [compare with (72)]:

$$[x_i, x_j] = \frac{i}{b} \epsilon_{ij}$$

Thus, the two-dimensional coordinate space has become [in the LLL projection limit] a two-dimensional phase space, with $\frac{1}{b}$ playing the role of “$h$”. Applying the Bohr-Sommerfeld estimate of the number of quantum states in terms of the area of phase space, we find

$$N_{\text{deg}} \approx \frac{A}{\pi \hbar^2} = \frac{bA}{2\pi}$$

(80)
which is precisely Landau’s estimate (78) of the degeneracy of the lowest Landau level.

This projection explains the physical nature of the pure Chern-Simons theory. The pure Chern-Simons theory can be viewed as the $e^2 \to \infty$ limit of the topologically massive Maxwell-Chern-Simons theory, in which one truncates the Hilbert space onto the ground state by isolating it from the rest of the spectrum by an infinite gap. The Chern-Simons analogue of the cyclotron frequency $\omega_c$ is the Chern-Simons mass $\kappa e^2$. So the inclusion of a Chern-Simons term in a gauge theory Lagrangian is analogous to the inclusion of a Lorentz force term in a mechanical system. This explains how it was possible to obtain a mass (23) for the gauge field in the Maxwell-Chern-Simons theory without the Higgs mechanism - in the mechanical analogue, the Higgs mechanism corresponds to introducing a harmonic binding term $\frac{1}{2} m \omega^2 x^2$, which gives a characteristic frequency in the most obvious way. But the Landau system shows how to obtain a characteristic frequency (the cyclotron frequency) without introducing a harmonic binding term. We can view the Chern-Simons theory as a gauge field realization of this mechanism.

To clarify the distinction between these two different mass generation mechanisms for the gauge field, consider them both acting together, as we did in Section 2.2. That is, consider the broken (Higgs) phase of a Maxwell-Chern-Simons theory coupled to a scalar field (27). If we are only interested in the masses of the excitations it is sufficient to make a zeroth-order (spatial) derivative expansion, neglecting all spatial derivatives, in which case the functional Schrödinger representation reduces to the familiar Schrödinger representation of quantum mechanics. Physical masses of the field theory appear as physical frequencies of the corresponding quantum mechanical system. In the Higgs phase, the quadratic Lagrange density becomes

$$L = \frac{1}{2e^2} \dot{A}_i^2 + \frac{\kappa}{2} e^{ij} \dot{A}_i A_j - v^2 A_i A_i$$

This is the Maxwell-Chern-Simons Lagrangian with a Proca mass term $v^2 A_i^2$. In the analogue quantum mechanical system this corresponds to a charged particle of mass $\frac{1}{e^2}$ moving in a uniform magnetic field of strength $\kappa$, and a harmonic potential well of frequency $\omega = \sqrt{2 e v}$. Such a quantum mechanical model is exactly solvable, and is well-known [see Exercise 3.2.1] to separate into two distinct harmonic oscillator systems of characteristic frequencies

$$\omega_{\pm} = \frac{\omega_c}{2} \left( \sqrt{1 + \frac{4 \omega^2}{\omega_c^2}} \pm 1 \right)$$

where $\omega_c$ is the cyclotron frequency corresponding to the magnetic field and $\omega$ is the harmonic well frequency. Taking $\omega_c = \kappa e^2$ and $\omega = \sqrt{2 e v}$, we see that these characteristic frequencies are exactly the mass poles $m_{\pm}$ in (30) of the Maxwell-Chern-Simons Higgs system, identified from the covariant gauge propagator. The pure Chern-Simons Higgs limit corresponds to the physical limit in which the cyclotron frequency dominates, so that

$$\omega_- \to \frac{\omega^2}{\omega_c} = \frac{2 v^2}{\kappa} = m_- \quad \omega_+ \to \infty$$

The remaining finite frequency $\omega_-$ is exactly the mass $m_-$ found in the covariant propagator (33) for the Higgs phase of a pure Chern-Simons Higgs theory.
So we see that in $2 + 1$ dimensions, the gauge field can acquire one massive mode via the standard Higgs mechanism [no Chern-Simons term], or via the Chern-Simons-Higgs mechanism [no Maxwell term]; or the gauge field can acquire two massive modes [both Chern-Simons and Maxwell term].

**Exercise 3.2.1:** Consider the planar quantum mechanical system with Hamiltonian $H = \frac{1}{2m}(p^2 + \frac{b}{2} e^{ij} x^i)^2 + \frac{1}{2} m \omega^2 x^2$. Show that the definitions

$$
p_{\pm} = \sqrt{\frac{\omega_{\pm}}{2m\Omega}} p^1 \pm \sqrt{\frac{m\Omega \omega_{\pm}}{2}} x^2
$$

$$
x_{\pm} = \sqrt{\frac{m\Omega}{2\omega_{\pm}}} x^1 \mp \frac{1}{\sqrt{2m\Omega \omega_{\pm}}} p^2
$$

where $\Omega = \sqrt{\frac{b^2}{4m^2} + \omega^2}$, and $\omega_{\pm} = \Omega \pm \frac{b}{2m}$ are as in (82), separate $H$ into two distinct harmonic oscillators of frequency $\omega_{\pm}$.

A natural way to describe the lowest Landau level (LLL) projection is in terms of coherent states $|23, 24\rangle$. To see how these enter the picture, consider the quantum mechanical Lagrangian, which includes a harmonic binding term

$$
L = \frac{1}{2} m x_i^2 + \frac{b}{2} e^{ij} \dot{x}_i \dot{x}_j - \frac{1}{2} m \omega^2 x_i^2
$$

This quantum mechanics problem can be solved exactly. Converting to polar coordinates, the wavefunctions can be labelled by two integers, $N$ and $n$, with

$$
<\vec{x}|N, n> = \frac{N!}{\pi (N + |n|)!} (m\Omega)^{1+|n|} \frac{1}{r} e^{in\theta} e^{-\frac{1}{2} m\Omega r^2} L_N^{|n|} (m\Omega r^2)
$$

where $L_N^{|n|}$ is an associated Laguerre polynomial, $\Omega = \sqrt{\frac{b^2}{4m^2} + \omega^2}$, and the energy is $E(N, n) = (2N + |n| + 1)\Omega - \frac{b}{2m} n$. In the $m \to 0$ limit, the $N = 0$ and $n \geq 0$ states decouple from the rest, and the corresponding wavefunctions behave as

$$
<\vec{x}|0, n> = \frac{1}{\sqrt{\pi n!}} (m\Omega)^{\frac{1+n}{2}} r^n e^{in\theta} e^{-\frac{1}{2} m\Omega r^2}
$$

$$
\rightarrow \sqrt{\frac{b}{2\pi}} \frac{z^n}{\sqrt{n!}} e^{-\frac{1}{2} |z|^2}
$$

where we have defined the complex coordinate $z = \sqrt{\frac{b}{2}} (x_1 + ix_2)$. The norms of these states transform under this LLL projection limit as

$$
\int d^2 x |<\vec{x}|0, n>|^2 \rightarrow \int \frac{dz dz^*}{2\pi} e^{-|z|^2} |<z|n>|^2
$$
We recognize the RHS as the norm in the coherent state representation of a one-dimensional quantum system. Thus, the natural description of the LLL is in terms of coherent state wavefunctions $\langle z | n \rangle = \frac{z^n}{\sqrt{n!}}$. The exponential factor $e^{-\frac{1}{2}m|z|^2}$ becomes, in the $m \to 0$ limit, part of the coherent state measure factor. This explains how the original two-dimensional system reduces to a one-dimensional system, and how it is possible to have $z$ and $\bar{z}$ being conjugate to one another, as required by the commutation relations (79).

Another way to find the lowest Landau level wavefunctions is to express the single-particle Hamiltonian as
\[
H = -\frac{1}{2m} \left( D_1^2 + D_2^2 \right)
\]
where $D_1 = \partial_1 + i\frac{b}{2}x_2$ and $D_2 = \partial_2 - i\frac{b}{2}x_1$. Then, define the complex combinations $D_\pm = D_1 \pm iD_2$ as:
\[
D_+ = 2\partial_\bar{z} + \frac{b}{2}\bar{z}, \quad D_- = 2\partial_z - \frac{b}{2}z
\]
The Hamiltonian \((89)\) factorizes as
\[
H = -\frac{1}{2m} D_- D_+ + \frac{b}{2m}
\]
so that the lowest Landau level states [which all have energy $\frac{1}{2} \omega_c = \frac{b}{2m}$] satisfy
\[
D_+ \psi = 0, \quad \text{or} \quad \psi = f(z)e^{-\frac{b}{4}|z|^2}
\]
We recognize the exponential factor (after absorbing $\sqrt{\frac{b}{2}}$ into the definition of $z$ as before) as the factor in \((87)\) which contributes to the coherent state measure factor. Thus, the lowest Landau level Hilbert space consists of holomorphic wavefunctions $f(z)$, with coherent state norm as defined in \((88)\) [25]. This is a standard feature of the analysis of the fractional quantum Hall effect – see Steve Girvin’s lectures for further applications.

When the original Landau Hamiltonian contains also a potential term, this leads to interesting effects under the LLL projection. With finite $m$, a potential $V(x_1, x_2)$ depends on two commuting coordinates. But in the LLL limit (i.e., $m \to 0$ limit) the coordinates become non-commuting [see \((79)\)] and $V(x_1, x_2)$ becomes the projected Hamiltonian on the projected phase space. Clearly, this leads to possible operator-ordering problems. However, these can be resolved [25, 24] by insisting that the projected Hamiltonian is ordered in such a way that the coherent state matrix elements computed within the LLL agree with the $m \to 0$ limit of the matrix elements of the potential, computed with $m$ nonzero.

### 3.3 Canonical Quantization of Abelian Chern-Simons Theories

Motivated by the coherent state formulation of the lowest Landau level projection of the quantum mechanical systems in the previous Section, we now formulate the canonical quantization of abelian Chern-Simons theories in terms of functional coherent states. Begin with the Maxwell-Chern-Simons Lagrangian in the $A_0 = 0$ gauge:
\[
L_{\text{MCS}} = \frac{1}{2e^2} \dot{A}_i^2 + \frac{\kappa}{2} \epsilon^{ij} \dot{A}_i A_j - \frac{1}{2e^2} (\epsilon^{ij} \partial_i A_j)^2
\]
\[(93)\]
This is a quadratic Lagrangian, so we expect we can find the groundstate wavefunctional. Physical states must also satisfy the Gauss law constraint: $\nabla \cdot \vec{P} - \kappa B = 0$. This Gauss law is satisfied by functionals of the form

$$\Psi[A_1, A_2] = e^{-i \frac{\kappa}{2} \int B \lambda \Psi[A_T]}$$

where we have decomposed $\vec{A}$ into its longitudinal and transverse parts: $A_i = \partial_i \lambda + A_i^T$. Using the Hamiltonian (67), the ground state wavefunctional is

$$\Psi_0[A_1, A_2] = e^{-i \frac{\kappa}{2} \int B \lambda e^{-\frac{1}{2\kappa^2} \int A_i^T \sqrt{\kappa^2 - \nabla^2} A_i^T}}$$

The pure Chern-Simons limit corresponds to taking $\kappa^2 \to \infty$, so that the wavefunctional becomes

$$\Psi_0[A_1, A_2] \to e^{-\frac{1}{2} \int A \frac{\partial}{\delta A} A^*} e^{-\frac{1}{2} \int |A|^2}$$

where we have defined $A = \sqrt{\frac{\kappa}{2}}(A_1 + i A_2)$ [in analogy to the definition of $z$ in the previous section], and $\partial_{\pm} = (\partial_1 \mp i \partial_2)$. From this form of the groundstate wavefunctional we recognize the functional coherent state measure factor $e^{-\frac{1}{2} \int |A|^2}$, multiplying a functional $\Psi[A] = e^{-\int A \frac{\partial}{\delta A} A^*}$ that depends only on $A$, and not on $A^*$. This is the functional analogue of the fact that the LLL wavefunctions have the form $\psi = f(z)e^{-\frac{1}{2} |z|^2}$, as in (87) and (92). The fact that the Chern-Simons theory has a single ground state, rather than a highly degenerate LLL, is a consequence of the Gauss law constraint, for which there was no analogue in the quantum mechanical model. These pure Chern-Simons wavefunctionals have a functional coherent state inner product [compare with (88)]

$$< \Psi | \Phi > = \int DAD A^* e^{-\int |A|^2} (\Psi[A])^* \Phi[A]$$

Actually, we needn’t have gone through the process of taking the $\kappa^2 \to \infty$ limit of the Maxwell-Chern-Simons theory. It is much more direct simply to adopt the functional coherent state picture. This is like going directly to the lowest Landau level using coherent states, instead of projecting down from the full Hilbert space of all the Landau levels. The canonical commutation relations (72) imply $[A(z), A^*(w)] = \delta(z - w)$, so that we can represent $A^*$ as a functional derivative operator: $A^* = -\frac{\delta}{\delta \lambda}$. Then the pure Chern-Simons Gauss law constraint $F_{12} = 0$ acts on states as

$$\left( \frac{\partial}{\partial A} - \frac{\delta}{\delta \lambda} A \right) \Psi[A] = 0$$

with solution

$$\Psi_0[A] = e^{-\frac{1}{2} \int A \frac{\partial}{\delta A} A^*}$$

as in (106).

If this pure Chern-Simons theory is coupled to some charged matter fields with a rotationally covariant current, then the physical state (109) is an eigenstate of the conserved angular momentum operator $M = -\frac{\kappa}{2} \int x^i \epsilon^{ij} (A_j B + BA_j)$:

$$M \Psi_0[A] = \frac{\alpha^2}{4\pi \kappa} \Psi_0[A]$$
where $Q = \int d^2x \rho$. Comparing with the Aharonov-Bohm exchange phase $\Delta \theta = \frac{e^2}{4\pi \kappa}$ in (21) we see that the statistics phase $s$ coincides with the spin eigenvalue $M$. This is the essence of the generalized spin-statistics relation for extended (field theoretic) anyons.

### 3.4 Quantization on the Torus and Magnetic Translations

The quantization of pure Chern-Simons theories on the plane is somewhat boring because there is just a unique physical state (99). To make this more interesting we could include external sources, which appear in the canonical formalism as point delta-function sources on the fixed-time surface. The appearance of these singularities makes the projection to flat connections satisfying Gauss’s law more intricate, and leads to important connections with knot theory and the braid group. Alternatively, we could consider the spatial surface to have nontrivial topology, rather than simply being the open plane $\mathbb{R}^2$. For example, take the spatial manifold to be a Riemann surface $\Sigma$ of genus $g$. This introduces extra degrees of freedom, associated with the nontrivial closed loops around the handles of $\Sigma$ [29, 30, 31, 32, 33]. Interestingly, the quantization of this type of Chern-Simons theory reduces once again to an effective quantum mechanics problem, with a new feature that has also been treated long ago in the solid state literature under the name of the “magnetic translation group”.

To begin, it is useful to reconsider the case of $\mathbb{R}^2$. To make connection with the coherent state representation, we express the longitudinal-transverse decomposition of the vector potential, $A_i = \partial_i \omega + \epsilon_{ij} \partial_j \sigma$, in terms of the holomorphic fields $A = \frac{1}{2}(A_1 + iA_2)$ and $A^* = \frac{1}{2}(A_1 - iA_2)$. Thus, with $z = x^1 + ix^2$ and $A_i dx^i = A^* dz + Ad\bar{z}$, we have

$$A = \partial_\bar{z} \chi, \quad A^* = \partial_z \chi^*$$

(101)

where $\chi = \omega - i\sigma$ is a complex field. If $\chi$ were real, then $A$ would be purely longitudinal - i.e. pure gauge. But with a complex field $\chi$, this representation spans all fields. A gauge transformation is realized as a shift in the real part of $\chi$: $\chi \to \chi + \lambda$, where $\lambda$ is real.

On a nontrivial surface this type of longitudinal-tranverse decomposition is not sufficient, as we know from elementary vector calculus on surfaces. The gauge field is decomposed using a Hodge decomposition, which incorporates the windings around the $2g$ independent noncontractible loops on $\Sigma$. For simplicity, consider the $g = 1$ case: i.e., the torus. (The generalization to higher genus is quite straightforward). The torus can be parametrized as a parallelogram with sides 1 and $\tau$, as illustrated in Figure 4. The area of the parallelogram is $Im(\tau)$, and the field $A$ can be expressed as

$$A = \partial_\bar{z} \chi + i \frac{\pi}{Im(\tau)} \omega(z) a$$

(102)

where $\omega(z)$ is a holomorphic one-form normalized according to $\int |\omega(z)|^2 = Im(\tau)$. This holomorphic form has integrals $\oint_\alpha \omega = 1$ and $\oint_\beta \omega = \tau$ around the homology basis cycles $\alpha$ and $\beta$. For the torus, we can simply take $\omega(z) = 1$.

The complex parameter $a$ appearing in (102) is just a function of time, independent of the spatial coordinates. Thus the $A_0 = 0$ gauge Chern-Simons Lagrangian decouples into two pieces

$$L_{CS} = \frac{i\kappa \pi^2}{Im(\tau)} (\dot{a} a^* - \dot{a}^* a) + i\kappa \int_\Sigma (\partial_\bar{z} \chi \partial_z \chi^* - \partial_z \chi^* \partial_\bar{z} \chi)$$

(103)
Figure 4: The torus can be parametrized as a parallelogram with sides $\tau$ and $1$. There are two cycles $\alpha$ and $\beta$ representing the two independent non-contractible loops on the surface.

So the coherent state wavefunctionals factorize as $\Psi[A] = \Psi[\chi]\psi(a)$, with the $\chi$ dependence exactly as discussed in the previous section. On the other hand, the $a$ dependence corresponds exactly to a quantum mechanical LLL problem, with “magnetic field” $B = \frac{4e^2}{im\tau}$. So the quantum mechanical wavefunctions $\psi(a)$ have inner product

$$<\psi|\phi> = \int dada^* e^{-\frac{2e^2}{im\tau}|a|^2} (\psi(a))^* \phi(a)$$  \hspace{1cm} (104)

But we have neglected the issue of gauge invariance. Small gauge transformations, $\chi \rightarrow \chi + \lambda$, do not affect the $a$ variables. But because of the nontrivial loops on the spatial manifold there are also “large” gauge transformations, which only affect the $a$’s:

$$a \rightarrow a + p + q\tau, \quad p, q \in \mathbb{Z}$$  \hspace{1cm} (105)

To understand how these large gauge transformations act on the wavefunctions $\psi(a)$, we recall the notion of the “magnetic translation group”. That is, in a uniform magnetic field, while the magnetic field is uniform, the corresponding vector potential, which is what appears in the Hamiltonian, is not! Take, for example, $A_i = -\frac{B}{2}\epsilon_{ij}x^j$. Then there are magnetic translation operators

$$T(\vec{R}) \equiv e^{-i\vec{R} \cdot (\vec{p} - e\vec{A})}$$  \hspace{1cm} (106)

which commute with the particle Hamiltonian $H = \frac{1}{2m}(\vec{p} + e\vec{A})^2$, but do not commute with one another:

$$T(\vec{R}_1)T(\vec{R}_2) = T(\vec{R}_2)T(\vec{R}_1)e^{-ie\vec{B} \cdot (\vec{R}_1 \times \vec{R}_2)}$$  \hspace{1cm} (107)

The exponential factor here involves the magnetic flux through the parallelogram spanned by $\vec{R}_1$ and $\vec{R}_2$. In solid state applications, a crystal lattice establishes a periodic potential for the electrons. If, in addition, there is a magnetic field, then we can ask how the spectrum of Landau levels is modified by the periodic potential, or alternatively we can ask how the Bloch band structure of the
periodic potential is modified by the presence of the magnetic field \( \mathbf{B} \). The important quantity
in answering this question is the magnetic flux through one unit cell of the periodic lattice. It is known \(^{26, 27, 28}\) that the magnetic translation group has finite dimensional representations if the magnetic field is related to a primitive lattice vector \( \mathbf{e} \) by

\[
\mathbf{B} = 2\pi \frac{1}{\epsilon \Omega} \frac{N}{M} \mathbf{e}
\]  

(108)

where \( \Omega \) is the area of the unit cell, and \( N \) and \( M \) are integers. These representations are constructed by finding an invariant subgroup of magnetic translation operators; the rationality condition arises because all members of this invariant subgroup must commute, which places restrictions on the phase factors in \((107)\). Since we are considering a two-dimensional system, with the magnetic field perpendicular to the two-dimensional surface, the condition \((108)\) simplifies to :

\[
\frac{eB\Omega}{2\pi} = \frac{N}{M}
\]  

(109)

The case \( M = 1 \) is special; here the magnetic translations act as one-dimensional ray representations on the Hilbert space, transforming the wavefunction with a phase. Consistency of this ray representation gives the number of states as \( N = \frac{eB\Omega}{2\pi} \), which is just Landau’s estimate \((78)\) of the degeneracy of the LLL. But when \( \frac{N}{M} \) is rational, we still have a consistent finite dimensional action of the magnetic translation group on the wavefunctions. The invariant subgroup consists of ‘superlattice’ translations, where the superlattice is obtained by enlarging each length dimension of the unit cell by a factor of \( M \). This produces an enlarged unit cell with effective flux \( MN \) on which the magnetic translation group acts one-dimensionally. Thus the total dimension is \( MN \). Finally, if \( \frac{N}{M} \) is irrational, then the magnetic translation group has infinite dimensional representations.

These results can be mapped directly to the quantization of the abelian Chern-Simons theory on the torus. The quantum mechanical degrees of freedom, \( a \), have a LLL Lagrangian with magnetic field \( eB = \frac{4\pi^2 \kappa}{Im\tau} \). The large gauge transformations \((105)\) are precisely magnetic translations across a parallelogram unit cell. The area of the unit cell is \( \Omega = Im\tau \), the area of the torus. Thus

\[
\frac{eB\Omega}{2\pi} = \frac{1}{2\pi} \left( \frac{4\pi^2 \kappa}{Im\tau} \right) Im\tau = 2\pi\kappa
\]  

(110)

and the condition for finite dimensional representations of the action of the large gauge transformations becomes

\[
2\pi\kappa = \frac{N}{M}
\]  

(111)

If we require states to transform as a one-dimensional ray representation under large gauge transformations then we must have \( 2\pi\kappa = \text{integer} \). But if \( 2\pi\kappa \) is rational, then we still have a perfectly good quantization, provided we identify the physical states with irreducible representations of the global gauge transformations \(i.e., \) the magnetic translations). These states transform according to a finite dimensional irreducible representation of the global gauge transformations, and any element of a given irreducible representation may be used to evaluate matrix elements of a gauge invariant operator, because physical gauge invariant operators commute with the generators of large gauge
transformations. The dimension of the Hilbert space is $MN$. If $2\pi \kappa$ is irrational, there is still nothing wrong with the Chern-Simons theory – it simply means that there are an infinite number of states in the Hilbert space. These results are consistent with the connection between abelian Chern-Simons theories and two dimensional conformal field theories. Chern-Simons theories with rational $2\pi \kappa$ correspond to what are known as “rational CFT’s”, which have a finite number of conformal blocks, and these conformal blocks are in one-to-one correspondence with the Hilbert space of the Chern-Simons theory [29, 30, 31, 32].

3.5 Canonical Quantization of Nonabelian Chern-Simons Theories

The canonical quantization of the nonabelian Chern-Simons theory with Lagrangian (55) is similar in spirit to the abelian case discussed in the previous Section. There are, however, some interesting new features [29, 22, 31, 32, 34]. As before, we specialize to the case where space-time has the form $\mathbb{R} \times \Sigma$, where $\Sigma$ is a torus. With $\Sigma = T^2$, the spatial manifold has two noncontractible loops and these provide gauge invariant holonomies. The problem reduces to an effective quantum mechanics problem for these holonomies. Just as in the abelian case, it is also possible to treat holonomies due to sources (which carry a representation of the gauge algebra), and to consider spatial manifolds with boundaries. These two approaches lead to deep connections with two-dimensional conformal field theories, which are beyond the scope of these lectures – the interested reader is referred to [29, 31, 32, 34] for details.

We begin as in the abelian case by choosing a functional coherent state representation for the holomorphic wavefunctionals $\Psi = \Psi[A]$, where $A = \frac{1}{2}(A_1 + iA_2)$. The coherent state inner product is

$$< \Psi | \Phi > = \int DAD^* e^{4\kappa \int \text{tr} (AA^*)} (\Psi[A])^* \Phi[A]$$

(112)

Note that with our Lie algebra conventions (see Section 2.6) $\text{tr} (AA^*) = -\frac{1}{2} A^a (A^a)^*$. Physical states are annihilated by the Gauss law generator $F_{12} = -2iF_{z\bar{z}}$. Remarkably, we can solve this constraint explicitly using the properties of the Wess-Zumino-Witten (WZW) functionals:

$$S^\pm[g] = \frac{1}{2\pi} \int_{\Sigma} \text{tr} (g^{-1} \partial_z g g^{-1} \partial_{\bar{z}} g) \pm \frac{i}{12\pi} \int_{(3)} \epsilon^{\mu\nu\rho} \text{tr} (g^{-1} \partial_\mu g g^{-1} \partial_\nu g g^{-1} \partial_\rho g)$$

(113)

where in the second term the integral is over a three dimensional manifold with a two dimensional boundary equal to the two dimensional space $\Sigma$.

**Exercise 3.5.1:** Show that the WZW functionals (113) have the fundamental variations

$$\delta S^\pm[g] = \left\{ \begin{array}{ll}
-\frac{i}{4\pi} \int \text{tr} (g^{-1} \delta g \partial_z (g^{-1} \partial_{\bar{z}} g)) \\
-\frac{i}{4\pi} \int \text{tr} (g^{-1} \delta g \partial_{\bar{z}} (g^{-1} \partial_z g))
\end{array} \right\}$$

(114)

Consider first of all quantization on the spatial manifold $\Sigma = \mathbb{R}^2$. To solve the Gauss law constraint we express the holomorphic field $A$, using Yang’s representation [35], as

$$A = -\partial_z U U^{-1}, \quad U \in \mathcal{G}^C$$

(115)
This is the nonabelian analogue of the complexified longitudinal-transverse decomposition (101) $A = \partial_2 \chi$ for the abelian theory on the plane. $U$ belongs to the complexification of the gauge group, which, roughly speaking, is the exponentiation of the gauge algebra, with complex parameters.

With $A$ parametrized in this manner, the Gauss law constraint $F_{z \bar{z}} \Psi = 0$ is solved by the functional

$$\Psi_0[A] = e^{-4\pi \kappa S^{-}[U]}$$

(116)

To verify this, note that the results of Exercise 3.5.1 imply that

$$\delta \Psi_0 = 4\kappa \left[ \int \text{tr}(\delta A \partial_z UU^{-1}) \right] \Psi_0$$

(117)

From the canonical commutation relations (72), the field $A^a_z = \frac{1}{2}(A^a_1 - i A^a_2)$ acts on a wavefunctional $\Psi[A]$ as a functional derivative operator

$$A^a_z = \frac{1}{2\kappa} \delta \frac{\delta}{\delta A^a}$$

(118)

Thus, acting on the state $\Psi_0[A]$ in (116):

$$A^a_z \Psi_0[A] = -(\partial_z UU^{-1})^a \Psi_0[A]$$

(119)

Since $A^a_z$ acts on $\Psi_0$ by multiplication, it immediately follows that $F_{z \bar{z}} \Psi_0[A] = 0$, as required.

The physical state (116) transforms with a cocycle phase factor under a gauge transformation. We could determine this cocycle from the variation (58) of the nonabelian Lagrangian [22]. But a more direct way here is to use the fundamental Polyakov-Wiegmann transformation property [36] of the WZW functionals:

$$S[g_1g_2] = S[g_1] + S[g_2] + \frac{1}{\pi} \int \text{tr}(g_1^{-1} \partial_z g_1 \partial_{\bar{z}} g_2 g_2^{-1})$$

(120)

With the representation $A = -\partial_2 U U^{-1}$ of the holomorphic field $A$, the gauge transformation $A \rightarrow A^g = g^{-1} A g + g^{-1} \partial_z g$ is implemented by $U \rightarrow g^{-1} U$, with $g$ in the gauge group. Then

$$\Psi_0[A^g] = e^{-4\pi \kappa S^{-}[g^{-1}U]}$$

$$= e^{-4\pi \kappa S^+[g]-4\pi \kappa \int \text{tr}(A \partial_z gg^{-1})} \Psi_0[A]$$

(121)

**Exercise 3.5.2:** Check that the transformation law (121) is consistent under composition, and that it combines properly with the measure factor to make the coherent state inner product (112) gauge invariant.

Furthermore, note that the WZW factors in (116) and (121) are only well defined provided $4\pi \kappa = \text{integer}$. This is the origin of the discreteness condition (61) on the Chern-Simons coefficient in canonical quantization.
This describes the quantum pure Chern-Simons theory with spatial manifold being the open plane $\mathbb{R}^2$. There is a unique physical state \([116]\). To make things more interesting we can introduce sources, boundaries, or handles on the spatial surface. As in the abelian case, here we just consider the effect of higher genus spatial surfaces, and for simplicity we concentrate on the torus. Then the nonabelian analogue of the abelian Hodge decomposition \([102]\) is \([31, 32]\)

$$A = -\partial_\zbar U U^{-1} + \frac{i\pi}{Im\tau} U a U^{-1}$$  \hspace{1cm}  (122)

where $U \in \mathcal{G}^C$, and $a$ can be chosen to be in the Cartan subalgebra of the gauge Lie algebra. This is the nonabelian generalization of the abelian torus Hodge decomposition \([102]\). To motivate this decomposition, we note that when $U \in \mathcal{G}$ \([not \mathcal{G}^C]!\), this is the most general pure gauge (flat connection) on the torus. The $a$ degrees of freedom represent the nontrivial content of $A$ that cannot be gauged away, due to the noncontractible loops on the spatial manifold. By a gauge transformation, $a$ can be taken in the Cartan subalgebra (indeed, there is further redundancy due to the action of Weyl reflections on the Cartan subalgebra). Then, extending $U$ from $\mathcal{G}$ to $\mathcal{G}^C$, the representation \([122]\) spans out to cover all connections, just as in Yang’s representation \([113]\) on $\mathbb{R}^2$.

Combining the representation \([122]\) with the transformation law \([121]\), we see that the physical state wavefunctionals on the torus are

$$\Psi[A] = e^{-4\pi\kappa S^{-1}[U] + \frac{i\pi}{Im\tau} \int \text{tr}(aU^{-1}\partial_\zbar U)} \psi(a)$$  \hspace{1cm}  (123)

In the inner product \([112]\), we can change field variables from $A$ to $U$ and $a$. But this introduces nontrivial Jacobian factors \([31, 32]\). The corresponding determinant is another Polyakov-Weigmann factor \([37]\), with a coefficient $c$ arising from the adjoint representation normalization ($c$ is called the dual Coxeter number of the gauge algebra, and for $SU(N)$ it is $N$). The remaining functional integral over the gauge invariant combination $U^\dagger U$ may be performed (it is the generating functional of the gauged WZW model on the torus \([37]\)). The final result is an effective quantum mechanical model with coherent state inner product

$$<\Psi | \Phi > = \int dada^* e^{\frac{i\pi}{4\kappa}(4\pi\kappa+c)\text{tr}(aa^*)} (\psi(a))^* \phi(a)$$  \hspace{1cm}  (124)

This looks like the abelian case, except for the shift of the Chern-Simons coefficient $\kappa$ by $4\pi\kappa \rightarrow 4\pi\kappa +c$. In fact, we can represent the quantum mechanical Cartan subalgebra degrees of freedom as $r$-component vectors $\vec{a}$, where $r$ is the rank of the gauge algebra. Then large gauge transformations act on these vectors as $\vec{a} \rightarrow \vec{a} + \vec{m} + \tau \vec{n}$, where $\vec{m}$ and $\vec{n}$ belong to the root lattice $\Lambda_\mathbb{R}$ of the gauge algebra. The wavefunction with the correct transformation properties under these large gauge transformation shifts is a generalized theta function, which is labelled by an element $\vec{\lambda}$ of the weight lattice $\Lambda_\mathbb{W}$ of the algebra. These are identified under translations by root vectors, and also under Weyl reflections. Thus the physical Hilbert space of the nonabelian Chern-Simons theory on the torus corresponds to

$$\Lambda_\mathbb{W} / W \times (4\pi\kappa + c)\Lambda_\mathbb{R}$$  \hspace{1cm}  (125)

This parametrization of states is a familiar construction in the theory of Kac-Moody algebras and conformal field theories.
3.6 Chern-Simons Theories with Boundary

We conclude this review of basic facts about the canonical structure of Chern-Simons theories by commenting briefly on the manifestation of boundary degrees of freedom in Chern-Simons theories defined on spatial manifolds which have a boundary. We have seen in the previous Sections that the canonical quantization of pure Chern-Simons theory on the space-time $\Sigma \times \mathbb{R}$, where $\Sigma$ is a compact Riemann surface, leads to a Hilbert space that is in one-to-one correspondence with the conformal blocks of a conformal field theory defined on $\Sigma$. But there is another important connection between Chern-Simons theories and CFT – namely, if the spatial manifold $\Sigma$ has a boundary $\partial \Sigma$, then the Hilbert space of the Chern-Simons theory is infinite dimensional, and provides a representation of the chiral current algebra of the CFT defined on $\partial \Sigma$ [29, 31, 32].

The source of these boundary effects is the fact that when we checked the variation of the Chern-Simons action in (56) we dropped a surface term. Retaining the surface term, the variation of the Chern-Simons action splits naturally into a bulk and a surface piece [29, 32]:

$$\delta S_{CS} = \kappa \int d^3x \epsilon^{\mu\nu\rho} \text{tr}(\delta A_\mu F_{\nu\rho}) + \kappa \int d^3x \partial_\nu [\epsilon^{\mu\nu\rho} \text{tr}(A_\mu \delta A_\rho)] \quad (126)$$

The boundary conditions must be such that $\int_{\text{bndy}} \text{tr}(A \delta A) = 0$. When it is the spatial manifold $\Sigma$ that has a boundary $\partial \Sigma$, we can impose the boundary condition that $A_0 = 0$. The remaining local symmetry corresponds to gauge transformations that reduce to the identity on $\partial \Sigma$, while the time independent gauge transformations on the boundary are global gauge transformations.

With this boundary condition we can write

$$S_{CS} = -\kappa \int_{\Sigma \times \mathbb{R}} d^3x \epsilon^{ij} \text{tr}(A_i \dot{A}_j) + \kappa \int_{\Sigma \times \mathbb{R}} d^3x \epsilon^{ij} \text{tr}(A_0 F_{ij}) \quad (127)$$

Variation with respect to the Lagrange multiplier field $A_0$ imposes the constraint $F_{ij} = 0$, which has as its solution the pure gauges $A_i = g^{-1} \partial_i g$. Then it follows that the Chern-Simons action becomes

$$S = -\kappa \int_{\partial \Sigma \times \mathbb{R}} d\theta dt \text{tr}(g^{-1} \partial_\theta gg^{-1} \partial_\theta g) + \frac{\kappa}{3} \int_{\Sigma \times \mathbb{R}} \epsilon^{\mu\nu\rho} \text{tr} \left( g^{-1} \partial_\mu gg^{-1} \partial_\nu gg^{-1} \partial_\rho g \right) \quad (128)$$

This is the chiral WZW action. The quantization of this system leads to a chiral current algebra of the gauge group, with the boundary values of the gauge field $A_\theta = g^{-1} \partial_\theta g$ being identified with the chiral Kac-Moody currents. This relation gives another important connection between Chern-Simons theories (here, defined on a manifold with a spatial boundary) and conformal field theories [29, 32].

Boundary effects also play an important role in the theory of the quantum Hall effect [38, 39], where there are gapless edge excitations which are crucial for explaining the conduction properties of a quantum Hall liquid. Consider the variation of the abelian Chern-Simons action

$$\delta \left( \int d^3x \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \right) = 2 \int d^3x \epsilon^{\mu\nu\rho} \delta A_\mu \partial_\nu A_\rho + \int d^3x \epsilon^{\mu\nu\rho} \partial_\nu (A_\mu \delta A_\rho) \quad (129)$$
For an infinitesimal gauge variation, $\delta A_\mu = \partial_\mu \lambda$, this becomes a purely surface term

$$\delta \left( \int d^3 x \epsilon^{\mu \nu \rho} A_\mu \partial_\nu A_\rho \right) = \int d^3 x \epsilon^{\mu \nu \rho} \partial_\mu (\lambda \partial_\nu A_\rho)$$

(130)

For a space-time $D \times R$, where $D$ is a disc with boundary $S^1$

$$\delta \left( \int d^3 x \epsilon^{\mu \nu \rho} A_\mu \partial_\nu A_\rho \right) = \int_{S^1 \times R} \lambda (\partial_0 A_\theta - \partial_\theta A_0)$$

(131)

Thus, the Chern-Simons action is not gauge invariant. Another way to say this is that the current

$$J^\mu = \frac{\delta S_{CS}}{\delta A^\mu} = \frac{\kappa}{2} \epsilon^{\mu \nu \rho} F_{\nu \rho}$$

is conserved within the bulk, but not on the boundary. For a disc-like spatial surface, this noninvariance leads to an accumulation of charge density at the boundary at a rate given by the radial current:

$$J_r = \kappa E_\theta$$

(132)

where $E_\theta$ is the tangential electric field at the boundary. However, we recognize this noninvariance as exactly that of a $1 + 1$ dimensional Weyl fermion theory defined on the boundary $S^1 \times R$. Due to the $1 + 1$ dimensional chiral anomaly, an electric field (which must of course point along the boundary) leads to the anomalous creation of charge at the rate (with $n$ flavours of fermions):

$$\frac{\partial}{\partial t} Q = \frac{n}{2\pi} E$$

(133)

Therefore, when $2\pi \kappa$ is an integer [recall the abelian discreteness condition (111)] the noninvariance of the Chern-Simons theory matches precisely the noninvariance of the anomalous boundary chiral fermion theory. This corresponds to a flow of charge from the bulk to the edge and vice versa. This gives a beautiful picture of a quantum Hall droplet, with integer filling fraction, as an actual physical realization of the chiral anomaly phenomenon. Indeed, when $2\pi \kappa = n$, we can view the Hall droplet as an actual coordinate space realization of the Dirac sea of the edge fermions [40]. This also provides a simple effective description of the integer quantum Hall effect as a quantized flow of charge onto the edge of the Hall droplet. For the fractional quantum Hall effect we need more sophisticated treatments on the edge, such as bosonization of the $1 + 1$ dimensional chiral fermion edge theory in terms of chiral boson fields [38], or representations of $W_{1+\infty}$, the quantum algebra of area preserving diffeomorphisms associated with the incompressibility of the quantum Hall droplet [41].

4 Chern-Simons Vortices

Chern-Simons models acquire dynamics via coupling to other fields. In this Section we consider the dynamical consequences of coupling Chern-Simons fields to scalar fields that have either relativistic or nonrelativistic dynamics. These theories have vortex solutions, similar to (in some respects) but different from (in other respects) familiar vortex models such as arise in Landau-Ginzburg theory or the Abelian Higgs model. The notion of Bogomol’nyi self-duality is ubiquitous, with some interesting new features owing to the Chern-Simons charge-flux relation $\rho = \kappa B$. 

32
4.1 Abelian-Higgs Model and Abrikosov-Nielsen-Olesen Vortices

I begin by reviewing briefly the Abelian-Higgs model in 2 + 1 dimensions. This model describes a charged scalar field interacting with a $U(1)$ gauge field, and exhibits vortex solutions carrying magnetic flux, but no electric charge. These vortex solutions are important in the Landau-Ginzburg theory of superconductivity because the static energy functional [see (135) below] for the relativistic Abelian-Higgs model coincides with the nonrelativistic Landau-Ginzburg free energy in the theory of type II superconductors, for which vortex solutions were first studied by Abrikosov [42].

Consider the Abelian-Higgs Lagrangian [43]

$$
\mathcal{L}_{AH} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + |D_\mu \phi|^2 - \frac{\lambda}{4} \left( |\phi|^2 - v^2 \right)^2
$$

(134)

where the covariant derivative is $D_\mu \phi = \partial_\mu \phi + ie A_\mu \phi$, and the quartic potential has the standard symmetry breaking form as shown in Figure 5.

![Diagram of self-dual quartic potential](image_url)

Figure 5: The self-dual quartic potential $\frac{\lambda}{4} \left( |\phi|^2 - v^2 \right)^2$ for the Abelian-Higgs model. The vacuum manifold is $|\phi| = v$.

The static energy functional of the Abelian-Higgs model is

$$
\mathcal{E}_{AH} = \int d^2 x \left[ \frac{1}{2} B^2 + |\vec{D}\phi|^2 + \frac{\lambda}{4} \left( |\phi|^2 - v^2 \right)^2 \right]
$$

(135)

where $B = F_{12}$. The potential minimum has constant solutions $\phi = e^{i\alpha} v$, where $\alpha$ is a real phase. Thus the vacuum manifold is isomorphic to the circle $S^1$. Furthermore, any finite energy solution must have $\phi(\vec{x})$ tending to an element of this vacuum manifold at infinity. Therefore, finite energy solutions are classified by their winding number or vorticity $N$, which counts the number of times the phase of $\phi$ winds around the circle at spatial infinity:

$$
\phi(\vec{x})|_{|\vec{x}|=\infty} = v e^{iN\theta}
$$

(136)

The vorticity is also related to the magnetic flux because finite energy solutions also require $|\vec{D}\phi| \to 0$ as $|\vec{x}| \to \infty$. This implies that

$$
e A_i \sim -i \partial_i \ln \phi \sim N \partial_i \theta \quad \text{as} \quad |\vec{x}| \to \infty
$$

(137)
Therefore, the dimensionless magnetic flux is
\[
\Phi = e \int d^2 x B = e \oint_{|\vec{x}|=\infty} A_i dx^i = 2\pi N
\] (138)

A brute-force approach to vortex solutions would be to make, for example in the 1-vortex case, a radial ansatz:
\[
\phi(\vec{x}) = f(r)e^{i\theta}, \quad \vec{A}(\vec{x}) = a(r)\hat{\theta}
\] (139)
The field equations then reduce to coupled nonlinear ordinary differential equations for \(f(r)\) and \(a(r)\). One can seek numerical solutions with the appropriate boundary conditions: \(f(r) \to v\) and \(a(r) \to \frac{1}{er}\) as \(r \to \infty\); and \(f(r) \to 0\) and \(a(r) \to 0\) as \(r \to 0\). No exact solutions are known, but approximate solutions can be found numerically. The solutions are localized vortices in the sense that the fields approach their asymptotic vacuum values exponentially, with characteristic decay lengths set by the mass scales of the theory. Note that \(\lambda, e^2\) and \(v^2\) each has dimensions of mass; and the Lagrangian (134) has a Higgs phase with a massive gauge field of mass \(m_g = \sqrt{2} ev\), together with a massive real scalar field of mass \(m_s = \sqrt{\lambda}v\). In general, these two mass scales are independent, but the Abelian-Higgs model displays very different behavior depending on the relative magnitude of these two mass scales. Numerically, it has been shown that two vortices (or two antivortices) repel if \(m_s > m_g\), but attract if \(m_s < m_g\). When the masses are equal
\[
m_s = m_g
\] (140)
then the forces between vortices vanish and it is possible to find stable static multivortex configurations. When translated back into the Landau-Ginzburg model for superconductivity, this critical point, \(m_s = m_g\), corresponds to the boundary between type-I and type-II superconductivity. In terms of the Abelian-Higgs model (134), this critical point is known as the Bogomol’nyi [44] self-dual point where
\[
\lambda = 2e^2
\] (141)
With this relation between the charge \(e\) and the potential strength \(\lambda\), special things happen.

To proceed, we need a fundamental identity – one that will appear many times throughout our study of vortex solutions in planar gauge theories.
\[
|\vec{D}\phi|^2 = |(D_1 \pm iD_2)\phi|^2 \mp eB|\phi|^2 \pm i\jmath D_i J_j
\] (142)
where \(J_j = \frac{1}{2i}[\phi^* D_j \phi - \phi(D_j \phi)^*]\). Using this identity, the energy functional (135) becomes [44]
\[
\mathcal{E}_{\text{AH}} = \int d^2 x \left[ \frac{1}{2} \left( B + e(|\phi|^2 - v^2) \right)^2 + |D_\pm \phi|^2 + \left( \frac{\lambda}{4} - \frac{e^2}{2} \right) \left(|\phi|^2 - v^2\right)^2 + ev^2 B \right]
\] (143)
where \(D_\pm \equiv (D_1 \pm iD_2)\), and we have dropped a surface term. At the self-dual point (141) the potential terms cancel, and we see that the energy is bounded below by a multiple of the magnitude of the magnetic flux (for positive flux we choose the lower signs, and for negative flux we choose the upper signs):
This bound is saturated by fields satisfying the first-order Bogomol’nyi self-duality equations \([44]\):

\[
\begin{align*}
D_\pm \phi &= 0 \\
B &= \pm e(|\phi|^2 - v^2)
\end{align*}
\] (145)

The self-dual point \([141]\) is also the point at which the \(2 + 1\) dimensional Abelian-Higgs model \([134]\) can be extended to an \(N = 2\) supersymmetric (SUSY) model \([45, 46]\). That is, first construct an \(N = 1\) SUSY Lagrangian of which \([134]\) is the bosonic part. This SUSY can then be extended to \(N = 2\) SUSY only when the \(\phi\) potential is of the form in \([134]\) and the self-duality condition \([141]\) is satisfied. This is clearly related to the mass degeneracy condition \([140]\) because for \(N = 2\) SUSY we need \textit{pairs} of bosonic particles with equal masses (in fact, the extension to \(N = 2\) SUSY requires an additional neutral scalar field to pair with the gauge field \(A_\mu\)). This feature of \(N = 2\) SUSY corresponding to the self-dual point is a generic property of self-dual models \([47, 48]\), and we will see it again in our study of Chern-Simons vortices.

The self-duality equations \([145]\) are not solvable, or even integrable, but a great deal is known about the solutions. To bring them to a more manageable form, we decompose the scalar field \(\phi\) into its phase and magnitude:

\[\phi = e^{i\omega} \rho^{1/2}\] (146)

Then the first of the self-duality equations \([145]\) determines the gauge field

\[eA_i = -\partial_i \omega \mp \frac{1}{2} \epsilon_{ij} \partial_j \ln \rho\] (147)

everywhere away from the zeros of the scalar field. The second self-duality equation in \([145]\) then reduces to a nonlinear elliptic equation for the scalar field density \(\rho\):

\[
\nabla^2 \ln \rho = 2e^2 \left(\rho - v^2\right)
\] (148)

No exact solutions are known for this equation, even when reduced to an ordinary differential equation by the condition of radial symmetry. However, it is easy to find (numerically) vortex-like solutions with \(\phi = f(r)e^{\pm iN\theta}\) where \(f(r)\) satisfies

\[
\frac{1}{r} \frac{d}{dr} \left( r \frac{d}{dr} f^2(r) \right) = 2e^2 (f^2 - v^2)
\] (149)

Many interesting theorems have been proved concerning the general solutions to the self-dual Abelian-Higgs equations \([142]\). These are paraphrased below. Readers interested in all the fine-print should consult \([49]\) and the original papers.

\textit{Existence and Uniqueness}: Let \((\phi, \vec{A})\) be a smooth finite energy solution to the Abelian-Higgs self-duality equations \([145]\). Then

\begin{enumerate}
  \item \(\phi\) has a finite number of zeros \(z_1, \ldots, z_m\);
  \item around each zero, \(\phi \sim (z - z_k)^{n_k} h_k(z)\), where \(h_k(z)\) is smooth and \(h_k(z_k) \neq 0\);
  \item the vorticity is given by the net multiplicity of zeros: \(N = \sum_{k=1}^{m} n_k\);
  \item given any set of zeros, \(z_1, \ldots, z_m\), the solution is unique, up to gauge equivalence;
\end{enumerate}
Furthermore, it has been shown that all finite energy solutions to the full second-order static equations of motion are solutions to the first-order self-duality equations. Thus, the solutions described in the above theorem cover all finite energy static solutions.

These results mean that the moduli space of static multivortex solutions is $2N$ dimensional, and these $2N$ parameters can be associated with the locations of the zeros of the Higgs field $\phi$. This counting is confirmed by an index-theorem fluctuation analysis \[50\]. We shall return to this moduli space later in Section 4.7 when we discuss the dynamics of vortices.

To conclude this review of the Abelian-Higgs model I mention that this model has also been studied on spatial manifolds that are compact Riemann surfaces. This is of interest for making comparisons with numerical simulations, which are necessarily finite, and also for studying the thermodynamics of vortices \[71\]. The main new feature is that there is an upper limit, known as Bradlow’s bound \[52\], on the vorticity for a given area of the surface. The appearance of such a bound is easy to see by integrating the second of the self-duality equations (145) over the surface (assuming positive flux, we take the lower signs):

$$\int d^2 x e B = e^2 v^2 \int d^2 x - e^2 \int d^2 x |\phi|^2$$

(150)

Since $\int d^2 x e B = 2\pi N$, and $\int d^2 x |\phi|^2$ is positive, this implies that

$$N \leq \frac{e^2 v^2}{2\pi} \text{ area}$$

(151)

[In the mathematics literature $v^2$ and $\lambda$ are usually scaled to 1, so that the self-dual value of $e^2$ is $\frac{1}{2}$, in which case the bound reads: $4\pi N \leq \text{area.}$] A similar bound applies when considering the Abelian-Higgs vortex solutions with periodic ‘t Hooft boundary conditions defined on a unit cell of finite area \[53\].

### 4.2 Relativistic Chern-Simons Vortices

A natural generalization of the Abelian-Higgs model of the previous section is to consider the effect of taking the gauge field to be governed by a Chern-Simons Lagrangian rather than a Maxwell Lagrangian. The name “relativistic” Chern-Simons vortices comes from the fact that a Chern-Simons gauge field inherits its dynamics from the matter fields to which it is coupled, and here it is coupled to a relativistic scalar field – later we shall consider vortices arising from a Chern-Simons gauge field coupled to matter fields with nonrelativistic dynamics. Numerous studies were made of vortex solutions in models with Chern-Simons and/or Maxwell terms, with symmetry breaking scalar field potentials \[54, 55\]. However, no analogue of the Bogomol’nyi self-dual structure of the Abelian-Higgs model was found until a particular sixth-order scalar potential was chosen in a model with a pure Chern-Simons term \[56, 57\].

Consider the Lagrangian

$$\mathcal{L}_{\text{RCS}} = \frac{\kappa}{2} \epsilon^\mu\nu\rho A_\mu \partial_\nu A_\rho + |D_\mu \phi|^2 - V(|\phi|)$$

(152)
where $V(|\phi|)$ is the scalar field potential, to be specified below. The associated energy functional is

$$\mathcal{E}_{\text{RCS}} = \int d^2 x \left[ |D_0 \phi|^2 + |\bar{D} \phi|^2 + V(|\phi|) \right]$$

(153)

Before looking for self-dual vortices we note a fundamental difference between vortices in a Chern-Simons model and those in the Abelian-Higgs model, where the gauge field is governed by a Maxwell term. The Abelian-Higgs vortices carry magnetic flux but are electrically neutral. In contrast, in a Chern-Simons model the Chern-Simons Gauss law constraint relates the magnetic field $B$ to the conserved $U(1)$ charge density as

$$B = \frac{1}{\kappa} J_0 = \frac{i}{\kappa} (\phi^* D_0 \phi - (D_0 \phi)^*)$$

(154)

Thus, if there is magnetic flux there is also electric charge:

$$Q = \int d^2 x J^0 = \kappa \int d^2 x B = \kappa \Phi$$

(155)

So solutions of vorticity $N$ necessarily carry both magnetic flux $\Phi$ and electric charge $Q$. They are therefore excellent candidates for anyons.

To uncover the Bogomol’nyi-style self-duality, we use the factorization identity (142), together with the Chern-Simons Gauss law constraint (154), to express the energy functional as

$$\mathcal{E}_{\text{RCS}} = \int d^2 x \left[ |D_0 \phi \pm i \frac{\kappa}{\kappa} (|\phi|^2 - v^2) \phi|^2 + |D_{\pm} \phi|^2 + V(|\phi|) \right] - \frac{1}{\kappa^2} |\phi|^2 (|\phi|^2 - v^2)^2 \pm v^2 B$$

(156)

Thus, if the potential is chosen to take the self-dual form

$$V(|\phi|) = \frac{1}{\kappa^2} |\phi|^2 (|\phi|^2 - v^2)^2$$

(157)

then the energy is bounded below [choosing signs depending on the sign of the flux]

$$\mathcal{E}_{\text{RCS}} \geq v^2 |\Phi|$$

(158)

The bound (158) is saturated by solutions to the first-order equations

$$D_{\pm} \phi = 0, \quad D_0 \phi = \mp i \frac{\kappa}{\kappa} (|\phi|^2 - v^2) \phi$$

(159)

which, when combined with the Gauss law constraint (154), become the self-duality equations:

$$D_{\pm} \phi = 0$$
$$B = \pm \frac{2}{\kappa^2} |\phi|^2 (|\phi|^2 - v^2)$$

(160)

These are clearly very similar to the self-duality equations (145) obtained in the Abelian-Higgs model. However, there are some significant differences. Before discussing the properties of solutions,
Figure 6: The self-dual potential \( \frac{1}{\kappa^2} |\phi|^2(|\phi|^2-v^2)^2 \) for the relativistic self-dual Chern-Simons system. Note the existence of two degenerate vacua: \( \phi = 0 \) and \( |\phi| = v \).

a few comments are in order. First, as is illustrated in Figure 6, the self-dual potential (157) is sixth-order, rather than the more commonly considered case of fourth-order.

Such a potential is still power-counting renormalizable in 2 + 1 dimensions. Furthermore, the potential is such that the minima at \( \phi = 0 \) and at \( |\phi|^2 = v^2 \) are degenerate. Correspondingly, there are domain wall solutions that interpolate between the two vacua [58]. In the Higgs vacuum, the Chern-Simons-Higgs mechanism leads to a massive gauge field [recall (33)] and a massive real scalar field. With the particular form of the self-dual potential (157) these masses are equal:

\[
m_s = \frac{2v^2}{\kappa} = m_g
\]

Just as in the Abelian-Higgs case, the relativistic Chern-Simons vortex model has an associated \( N = 2 \) SUSY, in the sense that the Lagrangian (152), with scalar potential (157), is the bosonic part of a SUSY model with extended \( N = 2 \) SUSY [59].

**Exercise 4.2.1**: The \( N = 2 \) SUSY extension of the relativistic Chern-Simons vortex system (152) has Lagrangian

\[
\mathcal{L}_{\text{SUSY}} = \frac{\kappa}{2} e^{\mu\nu\rho} A_\mu \partial_\rho A_\nu + |D_\mu \phi|^2 + i \bar{\psi} D \psi - \frac{1}{\kappa^2} |\phi|^2(|\phi|^2-v^2)^2 + \frac{1}{\kappa}(3|\phi|^2-v^2) \bar{\psi} \psi
\]

Show that there are pairs of bosonic fields degenerate with pairs of fermionic fields, in both the symmetric and asymmetric phases.

To investigate vortex solutions, we decompose the scalar field \( \phi \) into its magnitude \( \sqrt{\rho} \) and phase \( \omega \) as in (146). The gauge field is once again determined by the first self-duality equation to be \( A_i = -\partial_i \omega + \frac{1}{2} \epsilon_{ij} \partial_j \ln \rho \), as in (147), away from the zeros of the scalar field. The second self-duality equation then reduces to a nonlinear elliptic equation:

\[
\nabla^2 \ln |\phi|^2 = \frac{4}{\kappa^2} |\phi|^2(|\phi|^2-v^2)
\]
Just as in the Abelian-Higgs case (148), this equation is neither solvable nor integrable. However, numerical solutions can be found using a radial vortex-like ansatz. A significant difference from the Abelian-Higgs case is that while the Abelian-Higgs vortices have magnetic flux strings located at the zeros of the scalar field \( \phi \), in the Chern-Simons case we see from (160) that the magnetic field vanishes at the zeros of \( \phi \). The magnetic field actually forms rings centred on the zeros of \( \phi \).

Numerical studies lead to two different types of solutions, distinguished by their behaviour at spatial infinity:

1. **Topological solutions**: \( |\phi| \to v \) as \( |\vec{x}| \to \infty \).

2. **Nontopological solutions**: \( |\phi| \to 0 \) as \( |\vec{x}| \to \infty \).

In case 1, the solutions are topologically stable because they interpolate between the unbroken vacuum \( \phi = 0 \) at the origin and the broken vacuum \( |\phi| = v \) at infinity. For these solutions, existence has been proven using similar complex analytic and variational techniques to those used for the Abelian-Higgs model [60].

**Existence**: There exist smooth finite energy solutions \((\phi, \vec{A})\) to the relativistic Chern-Simons self-duality equations (160) such that

(i) \( |\phi| \to v \) as \( |\vec{x}| \to \infty \);

(ii) \( \phi \) has a finite number of zeros \( z_1, \ldots, z_m \);

(iii) around each zero, \( \phi \sim (z - z_k)^{n_k} h_k(z) \), where \( h_k(z) \) is smooth and \( h_k(z_k) \neq 0 \);

(iv) the vorticity is given by the net multiplicity of zeros: \( N = \sum_{k=1}^{m} n_k \).

Interestingly, the uniqueness of these solutions has not been rigorously proved. Nor has the equivalence of these self-dual solutions to all finite energy solutions of the full second-order equations of motion.

The topological vortex solutions have flux, charge, energy:

\[
\Phi = 2\pi N, \quad Q = \kappa \Phi, \quad \mathcal{E} = v^2|\Phi| \tag{164}
\]

Furthermore, they have nonzero angular momentum. For \( N \) superimposed vortices, the angular momentum can be evaluated as

\[
J = -\pi\kappa N^2 = -\frac{Q^2}{4\pi\kappa} \tag{165}
\]

which is the anyonic relation (20).

The nontopological solutions, with asymptotic behaviour \( |\phi| \to 0 \) as \( |\vec{x}| \to \infty \), are more complicated. The only existence proof so far is for superimposed solutions [61]. However, numerical studies are quite convincing, and show that

\[
\Phi = 2\pi(N + \alpha), \quad Q = \kappa \Phi, \quad \mathcal{E} = v^2|\Phi| \tag{166}
\]

where \( \alpha \) is a continuous parameter. They have nonzero angular momentum, and for \( N \) superimposed vortices

\[
J = -\pi\kappa(N^2 - \alpha^2) = -\frac{Q^2}{4\pi\kappa^2} + NQ \tag{167}
\]
There is an analogue of Bradlow’s bound (151) for the relativistic Chern-Simons vortices. Integrating the second self-duality equation in (160), we get
\[
\int d^2 x B = \frac{v^4}{2\kappa^2} \int d^2 x - \frac{2}{\kappa^2} \int d^2 x \left( |\phi|^2 - \frac{v^2}{2} \right)^2
\]
which implies that the vorticity is bounded above by
\[
N \leq \frac{v^4}{4\pi\kappa^2} \text{ area}
\]
A related bound has been found in the study of periodic solutions to the relativistic Chern-Simons equations [62, 64].

**Exercise 4.2.2**: The self-dual model (152) may be generalized to include also a Maxwell term for the gauge field, but this requires an additional neutral scalar field \(N\):
\[
L_{\text{MCS}} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \frac{\kappa}{2} \mu^{\mu\rho} A_\mu \partial_\nu A_\rho + |D_\mu \phi|^2 + \frac{1}{2e^2} (\partial_\mu N)^2 - V(|\phi|, N)
\]
with self-dual potential
\[
V = |\phi|^2 (N - \frac{v^2}{\kappa})^2 + \frac{e^2}{2} (|\phi|^2 - \kappa N)^2
\]
Show that in the symmetric phase the neutral scalar field \(N\) is degenerate with the massive gauge field. Show that in the asymmetric phase the \(N\) field and the real part of \(\phi\) have masses equal to the two masses of the gauge field. Check that
(i) the limit \(e^2 \to \infty\) reduces to the relativistic Chern-Simons vortex model of (152)
(ii) the limit \(\kappa \to 0\) reduces to the Abelian-Higgs model (134).

### 4.3 Nonabelian Relativistic Chern-Simons Vortices

The self-dual Chern-Simons vortex systems studied in the previous section can be generalized to incorporate nonabelian local gauge symmetry [65, 66]. This can be done with the matter fields and gauge fields in different representations, but the most natural and interesting case seems to be with adjoint coupling, with the matter fields and gauge fields in the same Lie algebra representation. Then the gauge covariant derivative is \(D_\mu \phi = \partial_\mu \phi + [A_\mu, \phi]\) and the Lagrangian is
\[
\mathcal{L} = \kappa e^{\mu\nu\rho} tr \left( A_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho \right) + tr \left( |D_\mu \phi|^2 \right) - \frac{1}{4\kappa^2} tr \left( |[\phi, \phi^\dagger], \phi | - v^2 \phi|^2 \right)
\]
where we have used the short-hand notation \(|D_\mu \phi|^2 = (D_\mu \phi)^\dagger D_\mu \phi\). There is a nonabelian version of the factorization identity (142) which with adjoint coupling reads
\[
tr \left( |\bar{D}\phi|^2 \right) = tr \left( |D_\pm \phi|^2 \right) \pm i tr \left( \phi^\dagger [F_{12}, \phi] \right) \pm \varepsilon_{ij} \partial_i tr \left( \phi^\dagger D_j \phi - (D_j \phi)^\dagger \phi \right)
\]
By the same argument as in the abelian case, we can show that with the potential as in (172), the associated energy functional is bounded below by an abelian magnetic flux. This Bogomol’nyi bound is saturated by solutions to the nonabelian self-duality equations

\[
D_\pm \phi = 0 \\
F_{\pm} = \frac{1}{\kappa^2} [v^2 \phi - [[\phi, \phi^\dagger], \phi], \phi^\dagger]
\]

(174)

Once again, the self-dual point is the point at which the model becomes the bosonic part of an \(N = 2\) SUSY model. The self-dual potential has an intricate pattern of degenerate minima, given by solutions to the embedding equation

\[
[[\phi, \phi^\dagger], \phi] = v^2 \phi
\]

(175)

This equation describes the embedding of \(SU(2)\) into the gauge Lie algebra, as can be seen by making the identifications:

\[
\phi = \frac{1}{\sqrt{v}} J_+; \quad \phi^\dagger = \frac{1}{\sqrt{v}} J_-; \quad [[\phi, \phi^\dagger], \phi] = \frac{1}{v} [J_+, J_-] = \frac{1}{v} J_3
\]

(176)

in which case the vacuum condition (173) reduces to the standard \(SU(2)\) commutation relations. Therefore, for \(SU(N)\), the number of gauge inequivalent vacua is given by the number of inequivalent ways of embedding \(SU(2)\) into \(SU(N)\). This number is in fact equal to the number \(P(N)\) of partitions of the integer \(N\). In each of these vacua, the masses of the gauge and scalar fields pair up in degenerate pairs, reflecting the \(N = 2\) SUSY of the extended model including fermions. The masses are given by universal formulae in terms of the exponents of the gauge algebra [66].

Not many rigorous mathematical results are known concerning solutions to the nonabelian self-duality equations, although partial results have been found [63]. Physically, we expect many different classes of solutions, with asymptotic behaviour of the solutions corresponding to the various gauge inequivalent vacua.

4.4 Nonrelativistic Chern-Simons Vortices : Jackiw-Pi Model

As mentioned before, Chern-Simons gauge fields acquire their dynamics from the matter fields to which they couple, and so they can be coupled to either relativistic or nonrelativistic matter fields. The nonrelativistic couplings discussed in this and subsequent sections are presumably more immediately relevant for applications in condensed matter systems. We shall see that Bogomol’nyi self-duality is still realizable in the nonrelativistic systems.

We begin with the abelian Jackiw-Pi model [67]

\[
\mathcal{L}_{JP} = \frac{\kappa}{2} \epsilon^{\mu \nu \rho} A_\mu \partial_\nu A_\rho + \frac{1}{i} \psi^* D_0 \psi - \frac{1}{2m} |\bar{D} \psi|^2 + \frac{g}{2} |\psi|^4
\]

(177)

The quartic term represents a self-coupling contact term of the type commonly found in nonlinear Schrödinger systems. The Euler-Lagrange equations are

\[
i D_0 \psi = -\frac{1}{2m} \bar{D}^2 \psi - g |\psi|^2 \psi
\]

41
\[ F_{\mu\nu} = \frac{1}{\kappa} \epsilon_{\mu
u\rho} J^\rho \quad (178) \]

where \( J^\mu \equiv (\rho, \tilde{J}) \) is a Lorentz covariant short-hand notation for the conserved nonrelativistic charge and current densities: \( \rho = |\psi|^2 \), and \( J^j = -\frac{i}{2m} \left( \psi^* D^j \psi - (D^j \psi)^* \psi \right) \). This system is Galilean invariant, and there are corresponding conserved quantities: energy, momentum, angular momentum and Galilean boost generators. There is, in fact, an addition dynamical symmetry [67] involving dilations, with generator

\[ D = tE - \frac{1}{2} \int d^2 x \vec{x} \cdot \vec{P} \quad (179) \]

and special conformal transformations, with generator

\[ K = -t^2 E + 2tD + \frac{m}{2} \int d^2 x \vec{x}^2 \rho \quad (180) \]

Here \( E \) is the energy and \( \vec{P} \) is the momentum density.

The static energy functional for the Jackiw-Pi Lagrangian \([177]\) is

\[ \mathcal{E}_{JP} = \int d^2 x \left[ \frac{1}{2m} |\tilde{D} \psi|^2 - \frac{g^2}{2} |\psi|^4 \right] \quad (181) \]

Using the factorization identity \([142]\), together with the Chern-Simons Gauss law constraint \( F_{12} = \frac{1}{2} |\psi|^2 \), the energy becomes

\[ \mathcal{E}_{JP} = \int d^2 x \left[ \frac{1}{2m} |D \pm \psi|^2 - \left( \frac{g}{2} \pm \frac{1}{2m \kappa} \right) |\psi|^4 \right] \quad (182) \]

Thus, with the self-dual coupling

\[ g = \mp \frac{1}{m \kappa} \quad (183) \]

the energy is bounded below by zero, and this lower bound is saturated by solutions to the first-order self-duality equations

\[ D \pm \psi = 0 \]

\[ B = \frac{1}{\kappa} |\psi|^2 \quad (184) \]

Note that with the self-dual coupling \([183]\), the original quartic interaction term, \(-\frac{g^2}{2} |\psi|^4 = \pm \frac{1}{2m \kappa} |\psi|^4 \), can be understood as a Pauli interaction term \( \pm \frac{B}{2m} |\psi|^2 \), owing to the Chern-Simons constraint \( |\psi|^2 = \kappa B \).

The self-duality equations \([184]\) can be disentangled as before, by decomposing the scalar field \( \psi \) into a phase and a magnitude \([146]\), resulting in a nonlinear elliptic equation for the density \( \rho \):

\[ \nabla^2 \ln \rho = \pm \frac{2}{\kappa} \rho \quad (185) \]

42
Surprisingly [unlike the previous nonlinear elliptic equations (148, 163) in the Abelian-Higgs and relativistic Chern-Simons vortex models], this elliptic equation is exactly solvable! It is known as the Liouville equation [68], and has the general real solution

\[ \rho = \kappa \nabla^2 \ln \left(1 + |f|^2\right) \]  

(186)

where \( f = f(z) \) is a holomorphic function of \( z = x^1 + ix^2 \) only.

**Exercise 4.4.1:** Verify that the density \( \rho \) in (186) satisfies the Liouville equation (185). Show that only one sign is allowed for physical solutions, and show that this corresponds to an attractive quartic potential in the original Lagrangian (177).

As a consequence of the Chern-Simons Gauss law, these vortices carry both magnetic and electric charge: \( Q = \kappa \Phi \). The net matter charge \( Q \) is

\[ Q = \kappa \int d^2x \nabla^2 \ln \left(1 + |f|^2\right) = 2\pi \kappa \left[r \frac{d}{dr} \ln \left(1 + |f|^2\right)\right]_0^\infty \]  

(187)

Explicit radially symmetric solutions may be obtained by taking \( f(z) = (\frac{z}{z_0})^N \). The corresponding charge density is

\[ \rho = \frac{4\kappa N^2}{r_0^3} \frac{\left(\frac{r}{r_0}\right)^{2(N-1)}}{\left[1 + \left(\frac{r}{r_0}\right)^{2N}\right]^2} \]  

(188)

Figure 7: Density \( \rho \) for a radially symmetric solution (188) representing one vortex with \( N = 2 \).
As \( r \to 0 \), the charge density behaves as \( \rho \sim r^{2(N-1)} \), while as \( r \to \infty \), \( \rho \sim r^{-2-2N} \). At the origin, the vector potential behaves as \( A_i(r) \sim -\partial_i \omega \mp (N-1)\epsilon_{ij} \frac{z_j}{r^2} \). We can therefore avoid singularities in the vector potential at the origin if we choose the phase of \( \psi \) to be \( \omega = \pm (N-1)\theta \). Thus the self-dual \( \psi \) field is

\[
\psi = \frac{2N\sqrt{\kappa}}{r_0} \left( \frac{r}{r_0} \right)^{N-1} \left( 1 + \frac{r}{r_0} \right)^{-2N} e^{\pm i(N-1)\theta} \tag{189}
\]

Requiring that \( \psi \) be single-valued we find that \( N \) must be an integer, and for \( \rho \) to decay at infinity we require that \( N \) be positive. For \( N > 1 \) the \( \psi \) solution has vorticity \( N-1 \) at the origin and \( \rho \) goes to zero at the origin. See Figure 7 for a plot of the density for the \( N = 2 \) case. Note the ring-like form of the magnetic field for these Chern-Simons vortices, as the magnetic field is proportional to \( \rho \) and so \( B \) vanishes where the field \( \psi \) does.

For the radial solution (188) the net matter charge is \( Q = \int d^2x \rho = 4\pi \kappa N \); and the corresponding flux is \( \Phi = 4\pi N \), which represents an even number of flux units. This quantized character of the flux is a general feature and is not particular to the radially symmetric solutions.

The radial solution (188) arose from choosing the holomorphic function \( f(z) = \left( \frac{z}{z_0} \right)^N \), and corresponds to \( N \) vortices superimposed at the origin. A solution corresponding to \( N \) separated vortices may be obtained by taking

\[
f(z) = \sum_{a=1}^{N} \frac{c_a}{z - z_a} \tag{190}
\]

Figure 8: Density \( \rho \) for a solution (190) representing two separated vortices.

There are \( 4N \) real parameters involved in this solution: \( 2N \) real parameters \( z_a \ (a = 1, \ldots, N) \) describing the locations of the vortices, and \( 2N \) real parameters \( c_a \ (a = 1, \ldots, N) \) corresponding to the scale and phase of each vortex. See Figure 8 for a plot of the two vortex case. The solution in
(190) is in fact the most general finite multi-soliton solution on the plane. Solutions with a periodic matter density $\rho$ may be obtained by choosing the function $f$ in (186) to be a doubly periodic function [69].

I conclude this section by noting that the dynamical symmetry of the Jackiw-Pi system guarantees that static solutions are necessarily self-dual. This follows from the generators (179) and (180). Consider the dilation generator $D$ in (179). It is conserved, but so is $\vec{P}$ for static solutions. This implies that $E$ must vanish, which is only true for self-dual solutions.

4.5 Nonabelian Nonrelativistic Chern-Simons Vortices

Just as the relativistic Chern-Simons vortices of Section 4.2 could be generalized to incorporate local nonabelian gauge symmetry, so too can the nonrelativistic models discussed in the previous section. We consider the case of adjoint coupling, with Lagrangian

$$\mathcal{L} = \kappa \epsilon^{\mu\nu\rho} \text{tr} \left( A_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho \right) + i \text{tr} \left( \psi^\dagger D_0 \psi \right) - \frac{1}{2m} \text{tr} \left( |\vec{D} \psi|^2 \right) + \frac{1}{4m\kappa} \text{tr} \left( [\psi, \psi^\dagger]^2 \right)$$

(191)

Using the nonabelian factorization identity (173), together with the Gauss law constraint, $F_{+\mp} = \frac{1}{\kappa} [\psi, \psi^\dagger]$, the static energy functional can be written as

$$\mathcal{E} = \frac{1}{2m} \int d^2 x \text{tr} \left( |D_\mp \psi|^2 \right)$$

(192)

which is clearly bounded below by 0. The solutions saturating this lower bound satisfy the first-order self-duality equations

$$D_\mp \psi = 0$$

$$F_{+\mp} = \frac{1}{\kappa} [\psi, \psi^\dagger]$$

(193)

These self-duality equations have been studied before in a different context, as they are the dimensional reduction of the four-dimensional self-dual Yang-Mills equations

$$F^{\mu\nu} = \pm \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$$

(194)

**Exercise 4.5.1**: Show that the self-dual Yang-Mills equations, with signature $(2, 2)$, reduce to the self-dual Chern-Simons equations (193) if we take fields independent of two of the coordinates, say $x^3$ and $x^4$, and combine the gauge fields $A_3$ and $A_4$ to form the fields $\psi$ and $\psi^\dagger$.

The self-duality equations (193) are integrable, as they can be expressed as a zero curvature condition in the following way. Define a spectral connection [with spectral parameter $\lambda$]

$$A_+ = A_+ - \lambda \sqrt{\frac{\kappa}{\lambda}} \psi; \quad A_- = A_- + \frac{1}{\lambda} \sqrt{\frac{\kappa}{\lambda}} \psi^\dagger$$

(195)
Then the corresponding curvature is
\[
\mathcal{F}_{++} = \partial_+ A_- - \partial_- A_+ + [A_+, A_-]
\]
\[
= \left\{ F_{+-} - \frac{1}{\kappa} [\psi, \psi^\dagger] \right\} + \sqrt{\frac{\Gamma}{\kappa}} \lambda D_- \psi - \sqrt{\frac{\Gamma}{\kappa \lambda}} D_+ \psi^\dagger
\]

Therefore, the condition of zero curvature, \(\mathcal{F}_{++} = 0\), for arbitrary spectral parameter \(\lambda\), encodes the self-dual Chern-Simons equations (193). Explicit exact solutions can also be obtained by making simplifying a\textsuperscript{ge}t\textsuperscript{i}cal ansätze which reduce the self-duality equations to the Toda equations, which are coupled analogues of the Liouville equation (185) and which are still integrable [70, 71].

In fact, all finite charge solutions can be found by mapping the self-duality equations (193) into the chiral model equations, which can then be integrated exactly in terms of \textit{unitons}. To see this, set the spectral parameter \(\lambda = 1\) in (195) and use the zero curvature \(\mathcal{F}_{++} = 0\) to define
\[
A_{\pm} = g^{-1} \partial_{\pm} g
\]

Then the conjugation \(\chi = \frac{1}{\sqrt{\kappa}} g \psi g^{-1}\) transforms the self-duality equations (193) into a single equation
\[
\partial_- \chi = [\chi^\dagger, \chi]
\]
Furthermore, if we define \(\chi = \frac{1}{2} h^{-1} \partial_+ h\), with \(h\) in the gauge group, then (198) becomes the chiral model equation
\[
\partial_+ (h^{-1} \partial_- h) + \partial_- (h^{-1} \partial_+ h) = 0
\]
All solutions to the chiral model equations with finite \(\int \text{tr}(h^{-1} \partial_- hh^{-1} \partial_+ h)\) can be constructed in terms of Uhlenbeck’s \textit{unitons} [72, 73]. These are solutions of the form
\[
h = 2p - 1
\]
where \(p\) is a holomorphic projector satisfying: (i) \(p^\dagger = p\), (ii) \(p^2 = p\), and (iii) \((1 - p) \partial_+ p = 0\). This means that all finite charge solutions of the self-dual Chern-Simons vortex equations (193) can be constructed in terms of \textit{unitons} [66].

**Exercise 4.5.2:** Show that a holomorphic projector \(p\) can be expressed as \(p = M(M^\dagger M)^{-1} M^\dagger\), where \(M = M(x)\) is any rectangular matrix. For \(SU(2)\) show that the \textit{uniton solution} leads to a charge density \([\psi, \psi^\dagger]\) which, when diagonalized, is just the Liouville solution (186) times the Pauli matrix \(\sigma^3\).

### 4.6 Vortices in the Zhang-Hansson-Kivelson Model for FQHE

There have been many applications of Chern-Simons theories to the description of the quantum Hall effect, and the fractional quantum Hall effect in particular (see e.g. [74, 38, 3, 99, 8]). In this section I describe one such model, and show how it is related to our discussion of Chern-Simons vortices.
Zhang, Hansson and Kivelson [5] reformulated the problem of interacting fermions in an external magnetic field as a problem of interacting bosons with an extra Chern-Simons interaction describing the statistical transmutation of the fermions into bosons. This transmutation requires a particular choice for the Chern-Simons coupling constant, as we shall see below. The Chern-Simons coupling is such that an odd number of flux quanta are ‘tied’ to the fermions [recall Figure 1]; thus the fermions acquire an additional statistics parameter [given by (20)] and so become effective bosons. The ZHK model is basically a Landau-Ginzburg effective field theory description of these boson fields, coupled to a Chern-Simons field that takes care of the statistical transmutation. It looks like a fairly innocent variation on the Jackiw-Pi model of Section 4.4, but the minor change makes a big difference to the vortex solutions. The ZHK Lagrangian is

\[ \mathcal{L}_{ZHK} = -\frac{\kappa}{2} \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho + i \psi^* (\partial_0 + ia_0) \psi - \frac{1}{2m} |(\partial_i + i(a_i + A_i^{ext})) \psi|^2 \]

\[ -\frac{1}{2} \int d^2 x' \left( |\psi(x)|^2 - n \right) V(x - x') \left( |\psi(x')|^2 - n \right) \] (201)

where we have adopted the notation that the statistical Chern-Simons gauge field is \( a_\mu \), while the external gauge field describing the external magnetic field is \( A_i^{ext} \). We have also, for convenience in some of the subsequent equations, written the Chern-Simons coupling as \(-\kappa\). The constant \( n \) appearing in the potential term denotes a uniform condensate charge density.

Normally a complex scalar field \( \psi \) is used to describe bosons. But when the Chern-Simons coupling takes the values

\[ \kappa = \frac{1}{2\pi(2k-1)}; \quad k \geq 1 \] (202)

the anyonic statistics phase (20) of the \( \psi \) fields is \( \frac{1}{2\kappa} = (2k-1)\pi \); that is, the fields are antisymmetric under interchange. Thus the fields are actually fermionic. We can alternatively view this as the condensing of the fundamental fermionic fields into bosons by the attachment of an odd number of fluxes through the Chern-Simons coupling [5]. Consider a delta-function contact interaction with

\[ V(x - x') = \frac{1}{m\kappa} \delta(x - x') \] (203)

in which case we can simply express the potential as

\[ V(\rho) = \frac{1}{2m\kappa} (\rho - n)^2 \] (204)

The static energy functional for this model is

\[ \mathcal{E}_{ZHK} = \int d^2 x \left[ \frac{1}{2m} |\partial_i + i(a_i + A_i^{ext}) \psi|^2 + \frac{1}{2m\kappa} (\rho - n)^2 \right] \] (205)

Clearly, the minimum energy solution corresponds to the constant field solutions

\[ \psi = \sqrt{n}, \quad a_i = -A_i^{ext}, \quad a_0 = 0 \] (206)
for which the Chern-Simons gauge field opposes and cancels the external field. Since the Chern-Simons constraint is \( b = -\frac{1}{\kappa} \rho \), we learn that these minimum energy solutions have density

\[
\rho = n = \kappa B^{\text{ext}}
\]

(207)

With the values of \( \kappa \) in (202), these are exactly the conditions for the uniform Laughlin states of filling fraction

\[
\nu = \frac{1}{2k - 1}
\]

(208)

To describe excitations about these ground states, we re-express the energy using the factorization identity (142).

\[
\mathcal{E}_{\text{ZHK}} = \int d^2x \left[ \frac{1}{2m} |D_\pm \psi|^2 + \frac{1}{2m} \left( B^{\text{ext}} - \frac{1}{\kappa} \rho \right) \rho + \frac{1}{2m\kappa} (\rho - n)^2 \right]
\]

\[
= \int d^2x \left[ \frac{1}{2m} |D_\pm \psi|^2 + \frac{1}{2m\kappa} \left( \rho - \kappa B^{\text{ext}} \right)^2 + \frac{\kappa}{2m} B^{\text{ext}} B + \frac{1}{2m\kappa} (\rho - n)^2 \right]
\]

(209)

In the last step we have chosen the lower sign, and used the relation \( n = \kappa B^{\text{ext}} \) to cancel the potential terms. Note that in the last line, \( B \) is the total magnetic field \( B = B^{\text{ext}} + b \), where \( b \) is the Chern-Simons magnetic field.

Thus, the energy is bounded below by a multiple of the total magnetic flux. This bound is saturated by solutions to the first-order equations

\[
D^- \psi = 0 \quad \quad \quad B = B^{\text{ext}} - \frac{1}{\kappa} \rho
\]

(210)

As before, the first equation allows us to express the total gauge field \( A_i = a_i + A_i^{\text{ext}} \) in terms of the phase and the density, and the second equation reduces to a nonlinear elliptic equation for the density:

\[
\nabla^2 \ln \rho = \frac{2}{\kappa} (\rho - n)
\]

(211)

Comparing this with the corresponding equation (185) in the Jackiw-Pi model, we see that the effect of the external field and the modified potential (204) is to include a constant term on the RHS. But this converts the Liouville equation back into the vortex equation (148) for the Abelian-Higgs model! This can be viewed as both good and bad news – bad in the sense that we no longer have the explicit exact solutions to the Liouville equation (185) that we had in the Jackiw-Pi model, but good because we know a great deal about the Abelian-Higgs models vortices, even though we do not have any explicit exact solutions. First, we learn that there are indeed well-behaved vortex solutions in the ZHK model, and that their magnetic charge is related to their vorticity. But now, because of the Chern-Simons relation, these vortices also have electric charge, proportional to their magnetic charge. In particular these vortices have the correct quantum numbers for the quasi-particles in the Laughlin model for the FQHE [5].
Exercise 4.6.1 : Show that if we modify the Jackiw-Pi model by including a background charge density \( \rho_0 \) (instead of an external magnetic field) \[75\]

\[ L = \frac{\kappa}{2} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + i \psi^* D_0 \psi - \frac{1}{2m} |D\psi|^2 - V(\rho) + \rho_0 A_0 \] 

(212)

then with the potential \( V(\rho) = \frac{1}{2m\kappa} (\rho - \rho_0)^2 \), the self-dual vortex equations also reduce to a nonlinear elliptic equation of the Abelian-Higgs form \([148]\):

\[ \nabla^2 \ln \rho = \frac{2}{\kappa} (\rho - \rho_0) \] 

(213)

4.7 Vortex Dynamics

So far, we have only dealt with static properties of vortices in various \( 2 + 1 \)-dimensional field theories. However, the more interesting question concerns their dynamics; and beyond that, we are ultimately interested in their quantization. Various different approaches have been developed over the years for studying vortex dynamics. Particle physicists and field theorists, motivated largely by Manton’s work \([76]\) on the low energy dynamics of solitons (of which these planar Bogomol’nyi vortices are an example), have studied the dynamics of vortices in the Abelian-Higgs model, which is governed by relativistic dynamics for the scalar field. Condensed matter physicists have developed techniques for studying vortices in superconductors and in Helium systems, where the dynamics is nonrelativistic \([77]\). The Chern-Simons vortices are particularly interesting, because in addition to introducing the new feature of anyon statistics of vortices, they appear to require methods from both the particle physics and condensed matter physics approaches. Having said that, there is, as yet, no clear and detailed understanding of the dynamics of Chern-Simons vortices. This is a major unsolved problem in the field.

Consider first of all the dynamics of vortices in the Abelian-Higgs model of Section \[4.1\]. Since no exact vortex solutions are known, even for the static case, we must be content with approximate analytic work and/or numerical simulations. As mentioned earlier, it is known from numerical work \([78]\) that the vortices in the Abelian-Higgs model repel one another when the scalar mass exceeds the gauge mass, and attract when the gauge mass exceeds the scalar mass. When these two mass scales are equal \([140]\) we are in the self-dual case, and there are no forces between static vortices. Manton’s approach to the dynamics of solitons provides an effective description of the dynamics at low energies when most of the field theoretic degrees of freedom are frozen out. Suppose we have static multi-soliton solutions parametrized by a finite dimensional ‘moduli space’ – the space consisting of the minima of the static energy functional \([133]\). We assume that the true dynamics of the full field theory is in some sense “close to” this moduli space of static solutions. Then the full dynamics should be approximated well by a projection onto a finite dimensional problem of dynamics on the moduli space. This is an adiabatic approximation in which one assumes that at each moment of time the field is a static solution, but that the parameters of the static solution [in the vortex case we can loosely think of these parameters as the locations of the vortices] vary slowly with time.

This approach has been applied successfully to the Abelian-Higgs vortices \([79]\), with the \( N \)-vortex parameters taken to be the zeros \( z_1, \ldots, z_N \) of the scalar \( \phi \) field \[recall the theorem in\]
Section 4.1. For well separated zeros we can think of these zeros as specifying the locations of the vortices. Indeed, the exponential approach of the fields to their asymptotic values motivates and supports the approximation of well separated vortices as a superposition of single vortices, with only exponentially small errors. (Actually, to be a bit more precise, the N-vortex moduli space is not really \( \mathbb{C}^N \); we need to take into account the identical nature of the vortices and factor out by the permutation group \( S_N \). Thus the true N-vortex moduli space is \( \mathbb{C}^N/S_N \), for which a good set of global coordinates is given by the symmetric polynomials in the zeros \( z_1, \ldots, z_N \).)

The total energy functional is

\[
H = T + V
\]

where the kinetic energy is

\[
T = \int d^2 x \left[ \frac{1}{2} \dot{A}_i \dot{A}_i + |\dot{\phi}|^2 \right]
\]

and the potential energy \( V \) is the static energy functional (135). There is also the Gauss law constraint, \( \vec{\nabla} \cdot \vec{E} = J^0 \), to be imposed. In the adiabatic approximation, the potential energy remains fixed at \( v^2 |\Phi|^2 \), given by the saturated Bogomol’nyi bound (144). But when the moduli space parameters become time dependent, we can insert these adiabatic fields

\[
\phi = \phi(\vec{x}; z_1(t), \ldots, z_N(t)), \quad \vec{A} = \vec{A}(\vec{x}; z_1(t), \ldots, z_N(t))
\]

into the kinetic energy (215), integrate over position \( \vec{x} \), and obtain an effective kinetic energy for the moduli parameters \( z_a(t) \), for \( a = 1, \ldots, N \). In terms of real coordinates \( \vec{x}_a \) on the plane, this kinetic energy takes the form

\[
T = \frac{1}{2} g_{ab} \dot{\vec{x}}_a \cdot \dot{\vec{x}}_b
\]

where the metric \( g_{ab} \) is a (complicated) function depending on the positions and properties of all the vortices. Samols [79] has shown that this construction has a beautiful geometric interpretation, with the metric \( g_{ab} \) being hermitean and Kähler. Furthermore, the dynamics of the slowly moving vortices corresponds to geodesic motion on the moduli space. While the metric cannot be derived in closed form, much is known about it, and it can be expressed solely in terms of the local properties of the vortices. It should be mentioned that the step of performing the spatial integrations to reduce the field theoretic kinetic energy (215) to the finite dimensional moduli space kinetic energy (217) involves some careful manipulations due to the nature of vortex solutions in the neighbourhood of the zeros of the scalar field \( \phi \). The essential procedure is first to excise small discs surrounding the zeros. The contributions from the interior of the discs can be shown to be negligible as the size of the disc shrinks to zero. The contribution from the outside of the discs can be projected onto a line integral around each disc, using Stokes’s theorem and the linearized Bogomol’nyi self-duality equations. These line integrals may then be expressed in terms of the local data of the scalar fields in the neighbourhood of each disc:

\[
\ln |\phi|^2 \approx \ln |z - z_k|^2 + a_k + \frac{1}{2} \left\{ b_k(z - z_k) + b_k^*(z^* - z_k^*) \right\} + \ldots
\]

There are several important differences complicating the direct application of this ‘geodesic approximation’ to the dynamics of the relativistic Chern-Simons vortices described in Section 4.2.
While it is still true that the static multi-vortex solutions can be characterized by the zeros of the scalar field [although no rigorous proof of uniqueness has been given so far], the fact that the vortices appear to be anyonic means that we cannot simply factor out by the symmetric group $S_N$ to obtain the true moduli space. Presumably the true moduli space would need to account for braidings of the vortex zeros. Second, the gauge field makes no contribution to the kinetic energy in the case of Chern-Simons vortices – all the dynamics comes from the scalar field. Correspondingly, even though there are no repulsive or attractive forces between the static self-dual vortices, there may still be velocity dependent forces that we do not see in the completely static limit. Thus, it is more convenient to consider the effective action (rather than the energy) for motion on the moduli space. Both these considerations suggest that we should expect a term linear in the velocities, in addition to a quadratic kinetic term like that in (217).

To see how these velocity dependent forces might arise, consider the relativistic Chern-Simons vortex model (152):

$$L_{RCS} = |D_0 \phi|^2 + \kappa A_0 B - \frac{\kappa}{2} \epsilon^{ij} A_i \dot{A}_j - |\dot{D}_i \phi|^2 - \frac{1}{\kappa^2} |\phi|^2 (|\phi|^2 - v^2)^2$$

(219)

Decomposing $\phi = \sqrt{\rho} e^{i\omega}$ as in (146), Gauss’s law determines the nondynamical field $A_0$ to be: $A_0 = -\dot{\omega} - \frac{\kappa B}{2\rho}$. Then the Lagrangian (219) can be re-expressed as

$$L_{RCS} = \frac{1}{4} \dot{\rho}^2 - \kappa B \dot{\omega} - \frac{\kappa}{2} \epsilon^{ij} A_i \dot{A}_j - \left[ |D_\pm \phi|^2 + \frac{\kappa^2}{4\rho} \left( B \mp \frac{2}{\kappa^2} \rho (\rho - v^2) \right)^2 \right] \pm v^2 B$$

(220)

To implement Manton’s procedure, we take fields that solve the static self-dual equations (160), but with adiabatically time-dependent parameters. As moduli parameters we take the zeros $\vec{q}_a(t)$ of the $\phi$ field. Then the term in the square brackets in (220) vanishes for self-dual solutions. Furthermore, for an $N$ vortex solution the vorticity is such that $\omega = \sum_{a=1}^N \arg(\vec{x} - \vec{q}_a(t))$. Then we can integrate over $\vec{x}$ to obtain an effective quantum mechanical Lagrangian for the vortex zeros:

$$L(t) = \int d^2x L = \frac{1}{2} g^{ij} \dot{q}_a^i q_b^j + \dot{A}_a^j(q_a) q_a^j \pm 2\pi v^2 N$$

(221)

The term linear in the velocities comes from the $B \dot{\omega}$ term in (220), while the $\epsilon^{ij} A_i \dot{A}_j$ term integrates to zero [51]. The coefficient of the linear term is

$$A_a^i = 2\pi \kappa \epsilon_{ij} \sum_{b \neq a} \frac{q_a^j - q_b^j}{|q_a^i - q_b^i|^2} + \text{local}$$

(222)

where the first term is responsible for the anyonic nature of the vortices, while the ‘local’ term is only known approximately in terms of the local expansion (218) in the neighbourhood of each vortex, and is a complicated function of the positions of all the vortices. The linear coefficient $A_a^i$ is interpreted as a linear connection on the moduli space. But, despite a number of attempts [80, 51], we still do not have a good understanding of the quadratic metric term $g^{ij}$ in the effective Lagrangian (221). This is an interesting outstanding problem.
Another important problem concerns the implementation of this adiabatic approximation for the description of vortex dynamics in *nonrelativistic* Chern-Simons theories, such as the Jackiw-Pi model or the Zhang-Hansson-Kivelson model. In these cases the field Lagrangian has only first-order time derivatives, so the nature of the adiabatic approximation is somewhat different \[82, 83\].

5 Induced Chern-Simons Terms

An important feature of Chern-Simons theories is that Chern-Simons terms can be induced by radiative quantum effects, even if they are not present as bare terms in the original Lagrangian. The simplest manifestation of this phenomenon occurs in $2 + 1$ dimensional QED, where a Chern-Simons term is induced in a simple one-loop computation of the fermion effective action \[84\]. Such a term breaks parity and time-reversal symmetry, as does a fermion mass term $m \bar{\psi} \psi$. There are two complementary ways to investigate this effective action — the first is a direct perturbative expansion in powers of the coupling for an arbitrary background gauge field, and the second is based on a Schwinger-style calculation of the induced current $< J^\mu >$ (from which the form of the effective action may be deduced) in the presence of a special background with constant field strength $F_{\mu \nu}$. Chern-Simons terms can also be induced in gauge theories without fermions, and in the broken phases of Chern-Simons-Higgs theories. Interesting new features arise when we consider induced Chern-Simons terms at finite temperature.

5.1 Perturbatively Induced Chern-Simons Terms : Fermion Loop

We begin with the perturbative effective action. To facilitate later comparison with the finite temperature case, we work in Euclidean space. The one fermion loop effective action is

$$S_{\text{eff}}[A, m] = N_f \log \det (i\partial + A + m)$$ (223)

where $m$ is a fermion mass. The physical significance of this fermion mass will be addressed below. We have also included the overall factor of $N_f$ corresponding to the number of fermion flavours. This, too, will be important later. For now, simply regard $N_f$ and $m$ as parameters.

A straightforward perturbative expansion yields

$$S_{\text{eff}}[A, m] = N_f \text{tr} \log (i\partial + m) + N_f \text{tr} \left( \frac{1}{i\partial + m} A \right) + \frac{N_f}{2} \text{tr} \left( \frac{1}{i\partial + m} \nabla^\mu A - \frac{1}{i\partial + m} A \right) + \ldots$$ (224)

The first term is just the free ($A = 0$) case, which is subtracted, while the second term is the tadpole. Since we are seeking an induced Chern-Simons term, and the abelian Chern-Simons term is quadratic in the gauge field $A_\mu$, we restrict our attention to the quadratic term in the effective action (interestingly, we shall see later that this step is not justified at finite temperature)

$$S_{\text{eff}}^{\text{quad}}[A, m] = \frac{N_f}{2} \int \frac{d^3 p}{(2\pi)^3} \left[ A^\mu ( -p ) \Gamma^{\mu \nu} (p) A^\nu (p) \right]$$ (225)
where the kernel is
\[ \Gamma^{\mu\nu}(p, m) = \int \frac{d^3 k}{(2\pi)^3} \text{tr} \left[ \gamma^\mu \frac{\not{p} + \not{k} - m}{(p + k)^2 + m^2} \gamma^\nu \frac{\not{k} - m}{k^2 + m^2} \right] \]
(226)
corresponding to the one-fermion-loop self-energy diagram shown in Figure 9 (a). Furthermore, since the Chern-Simons term involves the parity-odd Levi-Civita tensor \( \epsilon^{\mu\nu\rho} \), we consider only the \( \epsilon^{\mu\nu\rho} \) contribution to the fermion self-energy. This can arise because of the special property of the gamma matrices (here, Euclidean) in 2 + 1 dimensions
\[ \text{tr}(\gamma^\mu \gamma^\nu \gamma^\rho) = -2 \epsilon^{\mu\nu\rho} \]
(227)
(Note that this may be somewhat unfamiliar because in 3 + 1 dimensions we are used to the fact that the trace of an odd number of gamma matrices is zero). It is then easy to see from (226) that the parity odd part of the kernel has the form
\[ \Gamma^{\mu\nu}_{\text{odd}}(p, m) = \epsilon^{\mu\nu\rho} p^\rho \Pi^{\text{odd}}(p^2, m) \]
(228)
where
\[ \Pi^{\text{odd}}(p^2, m) = 2m \int \frac{d^3 k}{(2\pi)^3} \frac{1}{[(p + k)^2 + m^2][k^2 + m^2]} = \frac{1}{2\pi} \frac{m}{|p|} \text{arcsin} \left( \frac{|p|}{\sqrt{p^2 + 4m^2}} \right) \]
(229)
In the long wavelength (\( p \to 0 \)) and large mass (\( m \to \infty \)) limit we find
\[ \Gamma^{\mu\nu}_{\text{odd}}(p, m) \sim -\frac{i}{4\pi} \frac{m}{|m|} \epsilon^{\mu\nu\rho} p^\rho + O\left( \frac{p^2}{m^2} \right) \]
(230)
Inserting the leading term into the quadratic effective action (225) and returning to coordinate space, we find an induced Chern-Simons term
\[ S^{\text{CS}}_{\text{eff}} = -i \frac{N_f}{2} \frac{1}{4\pi} \frac{m}{|m|} \int d^3 x \epsilon^{\mu\nu\rho} A^\mu \partial_\nu A_\rho \]
(231)
**Exercise 5.1.1 :** Consider the three-photon leg diagram in Figure 9 (b), and show that in the large mass limit (\( m \gg p_1, p_2 \)):
\[ \Gamma^{\mu\nu\rho}_{\text{odd}}(p_1, p_2, m) \sim -i \frac{1}{4\pi} \frac{m}{|m|} \epsilon^{\mu\nu\rho} + O\left( \frac{p^2}{m^2} \right) \]
(232)
Hence show that in the nonabelian theory a nonabelian Chern-Simons term is induced at one-loop (note that the Chern-Simons coefficient is imaginary in Euclidean space):
\[ S^{\text{CS}}_{\text{eff}} = -i \frac{N_f}{2} \frac{1}{4\pi} \frac{m}{|m|} \int d^3 x \epsilon^{\mu\nu\rho} \text{tr} \left( A^\mu \partial_\nu A_\rho + \frac{2}{3} A^\mu A_\nu A_\rho \right) \]
(233)
Figure 9: The one-loop Feynman diagrams used in the calculation of the induced Chern-Simons term (at zero temperature). The self-energy diagram (a) is computed in (226), while the three-photon-leg diagram (b) is treated in Exercise 5.1.1

We now come to the physical interpretation of these results [84]. Consider the evaluation of the QED effective action (223) at zero fermion mass. The computation of $S_{\text{eff}}[A,m=0]$ requires regularization because of ultraviolet ($p \to \infty$) divergences. This regularization may be achieved, for example, by the standard Pauli-Villars method:

$$S_{\text{eff}}^{\text{reg}}[A,m=0] = S_{\text{eff}}[A,m=0] - \lim_{M \to \infty} S_{\text{eff}}[A,M] \quad (234)$$

The Pauli-Villars technique respects gauge invariance. But the $M \to \infty$ limit of the second term in (234) produces an induced Chern-Simons term, because of the perturbative large mass result (230). Therefore, in the process of maintaining gauge invariance we have broken parity symmetry – this is initiated by the introduction of the Pauli-Villars mass term $M \bar{\psi} \psi$ which breaks parity, and survives the $M \to \infty$ limit in the form of an induced Chern-Simons term (231). This is the “parity anomaly” of 2+1 dimensional QED [84]. It is strongly reminiscent of the well known axial anomaly in 3+1 dimensions, where we can maintain gauge invariance only at the expense of the discrete axial symmetry. There, Pauli-Villars regularization introduces a fermion mass which violates the axial symmetry. Recall that there is no analogous notion of chirality in 2+1 dimensions because of the different Dirac gamma matrix algebra; in particular, there is no “$\gamma^5$” matrix that anticommutes with all the gamma matrices $\gamma^\mu$. Nevertheless, there is a parity anomaly that is similar in many respects to the 3+1 dimensional axial anomaly.

In the nonabelian case, the induced Chern-Simons term (233) violates parity but restores invariance under large gauge transformations. It is known from a nonperturbative spectral flow argument [84] that $S_{\text{eff}}[A,m=0]$ for a single flavour of fermion ($N_f = 1$) is not gauge invariant, because the determinant (of which $S_{\text{eff}}$ is the logarithm) changes by a factor $(-1)^N$ under a large gauge transformation with winding number $N$. Thus $S_{\text{eff}}[A,m=0]$ is shifted by $N\pi i$. But the induced Chern-Simons term (233) also shifts by $N\pi i$, when $N_f = 1$, under a large gauge transformation with winding number $N$. These two shifts cancel, and the regulated effective action (234) is gauge invariant. This is reminiscent of Witten’s “SU(2) anomaly” in 3+1 dimensions [85]. This is a situation where the chiral fermion determinant changes sign under a large gauge transformation.
with odd winding number, so that the corresponding effective action is not invariant under such a gauge transformation. As is well known, this anomaly is avoided in theories having an even number $N_f$ of fermion flavours, because the shift in the effective action is $N_f N \pi i$, which is always an integer multiple of $2\pi i$ if $N_f$ is even (here $N$ is the integer winding number of the large gauge transformation). The same is true here for the parity anomaly in the nonabelian 2 + 1 dimensional case; if $N_f$ is even then both $S_{\text{eff}}[A, m = 0]$ and the induced Chern-Simons term separately shift by a multiple of $2\pi i$ under any large gauge transformation.

These results are from one-loop calculations. Nevertheless, owing to the topological origin of the Chern-Simons term, there is a strong expectation that the induced Chern-Simons terms should receive no further corrections at higher loops. This expectation is based on the observation that in a nonabelian theory the Chern-Simons coefficient must take discrete quantized values in order to preserve large gauge invariance. At one loop we have seen that the induced coefficient is $\frac{N_f L}{2}$, which is an integer for even numbers of fermion flavours, and reflects the parity anomaly in theories with an odd number of fermion flavours. At higher loops, if there were further corrections they would necessarily destroy the quantized nature of the one-loop coefficient. This suggests that there should be no further corrections at higher loops. This expectation has strong circumstantial evidence from various higher order calculations. Indeed, an explicit calculation \[26\] of the two-loop induced Chern-Simons coefficient for fermions showed that the two-loop contribution vanishes, in both the abelian and nonabelian theories. This is a highly nontrivial result, with the zero result arising from cancellations between different diagrams. This led to a recursive diagrammatic proof by Coleman and Hill \[27\] that in the abelian theory there are no contributions to the induced Chern-Simons term beyond those coming from the one fermion loop self-energy diagram. This has come to be known as the ‘Coleman-Hill theorem’. There is, however, some important fine-print – the Coleman-Hill proof only applies to abelian theories (and zero temperature) because it relies on manifest Lorentz invariance and the absence of massless particles.

5.2 Induced Currents and Chern-Simons Terms

Another way to compute the induced Chern-Simons term in the fermionic effective action \[223\] is to use Schwinger’s proper time method to calculate the induced current $< J^\mu >$, and deduce information about the effective action from the relation

$$< J^\mu > = \frac{\delta}{\delta A_\mu} S_{\text{eff}}[A] \tag{235}$$

Schwinger’s famous ‘proper-time’ computation \[28\] showed that the 3+1 dimensional QED effective action can be computed exactly for the special case of a background gauge field $A_\mu$ whose corresponding field strength $F_{\mu\nu}$ is constant. The corresponding calculation in 2 + 1 dimensions \[29\] is actually slightly easier because there is only a single Lorentz invariant combination of $F_{\mu\nu}$, namely $F_{\mu\nu} F^{\mu\nu}$. (In 3 + 1 dimensions there is also $F_{\mu\nu} \tilde{F}^{\mu\nu}$.) Schwinger’s ‘proper-time’ technique is also well-suited for computing the induced current $< J^\mu >$ in the presence of a constant background field strength.

A constant field strength may be represented by a gauge field linear in the space-time coordinates: $A_\mu = \frac{1}{2} x^\nu F_{\nu\mu}$, with $F_{\mu\nu}$ being the constant field strength. Since $A$ is linear in $x$, finding the
spectrum of the Dirac operator $\partial + iA$ reduces to finding the spectrum of a harmonic oscillator. This spectrum is simple and discrete, thereby permitting an explicit exact solution. This computation does, however, require the introduction of a regulator mass $m$ for the fermions. The result for the induced current is

$$< J^\mu > = \frac{1}{2} \frac{m}{|m|} \frac{1}{4\pi} \epsilon^{\mu\nu\rho} F_{\nu\rho}$$

(236)

By Lorentz invariance, we conclude that this result should hold for nonconstant background fields, at least to leading order in a derivative expansion. This is the result for a single flavour of fermions. For $N_f$ flavours the result is simply multiplied by $N_f$. Integrating back to get the effective action, we deduce that the effective action must have the form

$$S_{\text{eff}}[A] = S_{\text{eff}}^{NA}[A] + \frac{N_f}{2} \frac{m}{|m|} \frac{1}{4\pi} S_{\text{CS}}$$

(237)

where $S_{\text{eff}}^{NA}[A]$ is parity even but nonanalytic in the background field. This agrees with the perturbative calculation described in the previous section. Furthermore, we can also do this same computation for special nonabelian backgrounds with constant field strength (note that a constant nonabelian field strength, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$, can be obtained by taking commuting gauge fields that are linear in the space-time coordinates, as in the abelian case, or by taking constant but non-commuting gauge fields).

**Exercise 5.2.1**: Illustrate the appearance of terms in the $2+1$ dimensional effective action that are parity preserving but nonanalytic in the background field strength, by computing the effective energy of $2+1$ dimensional fermions in a constant background magnetic field $B$. Make things explicitly parity preserving by computing $\frac{1}{2}(S_{\text{eff}}[B, m] + S_{\text{eff}}[B, -m])$.

### 5.3 Induced Chern-Simons Terms Without Fermions

The issue of induced Chern-Simons terms becomes even more interesting when bare Chern-Simons terms are present in the original Lagrangian. Then Chern-Simons terms may be radiatively induced even in theories without fermions. In a classic calculation, Pisarski and Rao [10] showed that a gauge theory of $2+1$ dimensional $SU(N)$ Yang-Mills coupled to a Chern-Simons term has, at one-loop order, a finite additive renormalization of the bare Chern-Simons coupling coefficient:

$$4\pi \kappa_{\text{ren}} = 4\pi \kappa_{\text{bare}} + N$$

(238)

where the $N$ corresponds to the $N$ of the $SU(N)$ gauge group. This radiative correction is consistent with the discretization condition [recall (61)] that the Chern-Simons coefficient $4\pi\kappa$ must be an integer for consistency with large gauge invariance at the quantum level. As such, this integer-valued finite shift is a startling result, since it arises from a one-loop perturbative computation, which *a priori* we would not expect to ‘know’ anything about the nonperturbative large gauge transformations.
Here I briefly outline the computation of the renormalized Chern-Simons coefficient in such a Chern-Simons-Yang-Mills (CSYM) theory \[10\]. The Euclidean space bare Lagrangian is

\[
\mathcal{L}_{\text{CSYM}} = -\frac{1}{2} \text{tr}(F_{\mu\nu} F^{\mu\nu}) - i m e^{\mu\rho\nu} \text{tr}(A_\mu \partial_\nu A_\rho + \frac{2}{3} e A_\mu A_\nu A_\rho)
\]  

\text{(239)}

where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + e [A_\mu, A_\nu] \). Note that the Chern-Simons coefficient is imaginary in Euclidean space. The discreteness condition \(61\) requires

\[
4\pi \frac{m}{e^2} = \text{integer} \quad \text{(240)}
\]

where \( m \) is the mass of the gauge field. The bare gauge propagator (with covariant gauge fixing) is

\[
\Delta_{\text{bare}}^{\mu\nu}(p) = \frac{1}{p^2 + m^2} \left( \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} - m \epsilon_{\mu\nu\rho} \frac{p_\rho}{p^2} \right) + \xi p_\mu p_\nu \left( \frac{p^2}{p^2} \right)
\]  

\text{(241)}

The gauge self-energy \( \Pi_{\mu\nu} \) comes from the relation \( \Delta_{\mu\nu}^{-1} = (\Delta_{\mu\nu}^{\text{bare}})^{-1} + \Pi_{\mu\nu} \), and may be decomposed as

\[
\Pi_{\mu\nu}(p) = (\delta_{\mu\nu} p^2 - p_\mu p_\nu) \Pi_{\text{even}}(p^2) + m \epsilon_{\mu\nu\rho} p^\rho \Pi_{\text{odd}}(p^2)
\]  

\text{(242)}

Then the renormalized gauge propagator is defined as

\[
\Delta_{\mu\nu}(p) = \frac{1}{Z(p^2)[p^2 + m_{\text{ren}}^2(p^2)]} \left( \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} - m_{\text{ren}}^2(p^2) \epsilon_{\mu\nu\rho} \frac{p_\rho}{p^2} \right) + \xi p_\mu p_\nu \left( \frac{p^2}{p^2} \right)
\]  

\text{(243)}

where \( Z(p^2) \) is a wavefunction renormalization factor and the renormalized mass is

\[
m_{\text{ren}}(p^2) = \frac{Z_m(p^2)}{Z(p^2)} m
\]  

\text{(244)}

with

\[
Z(p^2) = 1 + \Pi_{\text{even}}(p^2), \quad Z_m(p^2) = 1 + \Pi_{\text{odd}}(p^2)
\]  

\text{(245)}

The important divergences are in the infrared \( (p^2 \to 0) \), and we define the renormalized Chern-Simons mass to be

\[
m_{\text{ren}} = m_{\text{ren}}(0) = \frac{Z_m(0)}{Z(0)} m
\]  

\text{(246)}

There is also, of course, charge renormalization to be considered. The renormalized charge is

\[
e_{\text{ren}}^2 = \frac{e^2}{Z(0)(Z(0))^2}
\]  

\text{(247)}

where \( Z(p^2) \) comes from the renormalization of the ghost propagator. In writing this expression for the renormalized charge we have used the standard perturbative Ward-Takahashi identities for (infinitesimal) gauge invariance (note, however, that the Chern-Simons term introduces new vertices; but this is only a minor change). The important thing is that none of the Ward-Takahashi identities places any constraint on \( Z_m(0) \), which comes from the odd part of the gauge self-energy.
at zero external momentum \((245)\). The renormalization factors \(Z(0)\) and \(\tilde{Z}(0)\) are finite in Landau gauge, and a straightforward (but messy) one-loop calculation \([10]\) leads to the results

\[
Z_m(0) = 1 + \frac{7}{12\pi} N \frac{e^2}{m}, \quad \tilde{Z}(0) = 1 - \frac{1}{6\pi} N \frac{e^2}{m}
\]  

(248)

Putting these together with the renormalized mass \((246)\) and charge \((247)\) we find that, to one-loop order:

\[
\left(\frac{m}{e^2}\right)_{\text{ren}} = \left(\frac{m}{e^2}\right) Z_m(0)(\tilde{Z}(0))^2 = \left(\frac{m}{e^2}\right) \left\{ 1 + \left(\frac{7}{12\pi} - \frac{1}{3\pi}\right) N \frac{e^2}{m}\right\} = \left(\frac{m}{e^2}\right) + \frac{N}{4\pi}
\]

(249)

But this is just the claimed result:

\[
4\pi \kappa_{\text{ren}} = 4\pi \kappa_{\text{bare}} + N
\]

(250)

It is widely believed that this is in fact an all-orders result, although no rigorous proof has been given. This expectation is motivated by the observation that if there \textit{were} further contributions to \(Z_m(0)\) and \(\tilde{Z}(0)\) at two loops, for example,

\[
\left(\frac{m}{e^2}\right)_{\text{ren}} = \left(\frac{m}{e^2}\right) + \frac{N}{4\pi} + \alpha \frac{N^2}{(m/e^2)^2}
\]

(251)

(where \(\alpha\) is some numerical coefficient) then the renormalized combination \(4\pi \left(\frac{m}{e^2}\right)_{\text{ren}}\) could no longer be an integer. Explicit two-loop calculations have shown that there is indeed no two-loop contribution \([89]\), and there has been much work done (too much to review here) investigating this finite renormalization shift to all orders. Nevertheless, from the point of view of perturbation theory, the result \(4\pi \kappa_{\text{ren}} = 4\pi \kappa_{\text{bare}} + N\) seems almost too good. We will acquire a deeper appreciation of the significance of this result when we consider the computation of induced Chern-Simons terms using finite temperature perturbation theory in Section 5.4. I should also mention that there are nontrivial subtleties concerning regularization schemes in renormalizing these Chern-Simons theories \([90]\), in part due to the presence of the antisymmetric \(\epsilon^{\mu\nu\rho}\) tensor which does not yield easily to dimensional regularization. These issues are particularly acute in the renormalization of \textit{pure} Chern-Simons theories (no Yang-Mills term).

The story of induced Chern-Simons terms becomes even more interesting when we include scalar (Higgs) fields and spontaneous symmetry breaking. In a theory with a Higgs scalar coupled to a gauge field with a bare Chern-Simons term, there is a radiatively induced Chern-Simons term at one loop. If this Higgs theory has a nonabelian symmetry that is completely broken, say \(SU(2) \to U(1)\), then the computation of the zero momentum limit of the odd part of the gauge self-energy suggests the shift

\[
4\pi \kappa_{\text{ren}} = 4\pi \kappa_{\text{bare}} + f \left(\frac{m_{\text{Higgs}}}{m_{\text{CS}}}\right)
\]

(252)

where \(f\) is some complicated (noninteger!) function of the dimensionless ratio of the Higgs and Chern-Simons masses \([91]\). So \(4\pi \kappa_{\text{ren}}\) is not integer valued. But this is not a problem here because there is no residual nonabelian symmetry in the broken phase, since the \(SU(2)\) symmetry has
been completely broken. However, consider instead a partial breaking of the original nonabelian symmetry [say from $SU(3)$ to $SU(2)$] so that the broken phase does have a residual nonabelian symmetry. Then, remarkably, we find \[92, 93\] that the complicated function $f$ reduces to an integer: $4\pi\kappa_{\text{ren}} = 4\pi\kappa_{\text{bare}} + 2$, (the 2 corresponds to the residual $SU(2)$ symmetry in this case). This result indicates a surprising robustness at the perturbative level of the nonperturbative discreteness condition on the Chern-Simons coefficient, when there is a nonabelian symmetry present.

Actually, in the case of complete symmetry breaking, the shift \[252\] should really be interpreted as the appearance of “would be” Chern-Simons terms in the effective action. For example, a term $\epsilon^\mu\nu\rho tr(D_\mu\phi F_{\nu\rho})$ in the effective action is gauge invariant, and in the Higgs phase in which $\phi \rightarrow <\phi>$ at large distances, this term looks exactly like a Chern-Simons term. This is because we extracted the Chern-Simons coefficient in the large distance ($p^2 \rightarrow 0$) limit where $\phi$ could be replaced by its asymptotic expectation value $<\phi>$. This observation has led to an interesting extension of the Coleman-Hill theorem to include the case of spontaneous symmetry breaking \[94\]. However, in the partial symmetry breaking case no such terms can be written down with the appropriate symmetry behaviour, so this effect does not apply in a phase with residual nonabelian symmetry. Correspondingly, we find that the integer shift property does hold in such a phase.

### 5.4 A Finite Temperature Puzzle

In this section we turn to the question of induced Chern-Simons terms at nonzero temperature. All the results mentioned above are for $T = 0$. The case of $T > 0$ turns out to be significant both for practical and fundamental reasons. In the study of anyon superconductivity \[15\] one of the key steps involves a cancellation between the bare Chern-Simons term and an induced Chern-Simons term. While this cancellation was demonstrated at $T = 0$, it was soon realised that at $T > 0$ this same cancellation does not work because the finite $T$ induced Chern-Simons coefficient is temperature dependent. The resolution of this puzzle is not immediately obvious. This strange $T$ dependent Chern-Simons coefficient has also caused significant confusion regarding the Chern-Simons discreteness condition: $4\pi\kappa = \text{integer}$. It seems impossible for a temperature dependent Chern-Simons coefficient $\kappa(T)$ to satisfy this consistency condition. However, recent work \[96, 97, 98, 99\] has led to a new understanding and appreciation of this issue, with some important lessons about finite temperature perturbation theory in general.

We concentrate on the induced Chern-Simons terms arising from the fermion loop, as discussed in Sections 5.1 and 5.2, but now generalized to nonzero temperature. Recall from \[229\] and \[230\] that the induced Chern-Simons coefficient is essentially determined by

$$
\kappa_{\text{ind}} = \frac{N_f}{2} \Pi_{\text{odd}}(p^2 = 0, m)
= \frac{N_f}{2} \int \frac{d^3k}{(2\pi)^3} \frac{2m}{(k^2 + m^2)^2}
= \frac{N_f}{2} \frac{1}{4\pi} \frac{m}{|m|}
$$

(253)

If we simply generalize this one loop calculation to finite temperature (using the imaginary time
formalism) then we arrive at
\[ \kappa_{\text{ind}}^{(T)} = \frac{N_f}{2} T \sum_{n=-\infty}^{\infty} \int \frac{d^2 k}{(2\pi)^2} \frac{2m}{[(2n + 1)\pi T]^2 + \vec{k}^2 + m^2]^2} \] (254)
where we have used the fact that at finite temperature the ‘energy’ \( k_0 \) takes discrete values \((2n + 1)\pi T\), for all integers \( n \in \mathbb{Z} \).

**Exercise 5.4.1**: Take the expression (254) and do the \( \vec{k} \) integrals and then the \( k_0 \) summation, to show that
\[ \kappa_{\text{ind}}^{(T)} = \frac{N_f}{2} \frac{T}{4\pi} \sum_{n=-\infty}^{\infty} \frac{2m}{[(2n + 1)\pi T]^2 + m^2]} \]
\[ = \frac{N_f}{2} \frac{1}{4\pi} \tanh\left(\frac{\beta m}{2}\right) \]
\[ = \frac{N_f}{2} \frac{m}{4\pi |m|} \tanh\left(\frac{\beta |m|}{2}\right) \] (255)
where \( \beta = \frac{1}{T} \).

Thus, it looks as though the induced Chern-Simons coefficient is temperature dependent. Note that the result (255) reduces correctly to the zero \( T \) result (253) because \( \tanh\left(\frac{\beta |m|}{2}\right) \to 1 \) as \( T \to 0 \) (i.e., as \( \beta \to \infty \)). Indeed, the \( T > 0 \) result is just the \( T = 0 \) result multiplied by the smooth function \( \tanh\left(\frac{\beta |m|}{2}\right) \). This result has been derived in many different ways \([100]\), in both abelian and nonabelian theories, and in both the real time and imaginary time formulations of finite temperature field theory. The essence of the calculation is as summarized above.

On the face of it, a temperature dependent induced Chern-Simons term would seem to violate large gauge invariance. However, the nonperturbative (spectral flow) argument for the response of the fermion determinant to large gauge transformations at zero \( T \) \([84]\) is unchanged when generalized to \( T > 0 \). The same is true for the hamiltonian argument for the discreteness of \( 4\pi\kappa \) in the canonical formalism. Thus the puzzle. Is large gauge invariance really broken at finite \( T \), or is there something wrong with the application of finite \( T \) perturbation theory? We answer these questions in the next Sections. The essential new feature is that at finite temperature, other parity violating terms (other than the Chern-Simons term) can and do appear in the effective action; and if one takes into account all such terms to all orders (in the field variable) correctly, the full effective action maintains gauge invariance even though it contains a Chern-Simons term with a temperature dependent coefficient. In fact, it is clear that if there are higher order terms present (which are not individually gauge invariant), one cannot ignore them in discussing the question of invariance of the effective action under a large gauge transformation. Remarkably, this mechanism requires the existence of nonextensive terms (i.e., terms that are not simply space-time integrals of a density) in the finite temperature effective action, although only extensive terms survive in the zero temperature limit.
5.5 Quantum Mechanical Finite Temperature Model

The key to understanding this finite temperature puzzle can be illustrated with a simple exactly solvable 0 + 1 dimensional Chern-Simons theory \[96\]. This is a quantum mechanical model, which at first sight might seem to be a drastic over-simplification, but in fact it captures the essential points of the 2 + 1 dimensional computation. Moreover, since it is solvable we can test various perturbative approaches precisely.

Consider a 0+1 dimensional field theory with \(N_f\) flavours of fermions \(\psi_j\), \(j = 1 \ldots N_f\), minimally coupled to a \(U(1)\) gauge field \(A\). It is not possible to write a Maxwell-like kinetic term for the gauge field in 0 + 1 dimensions, but we can write a Chern-Simons term - it is linear in \(A\). [Recall that it is possible to define a Chern-Simons term in any odd dimensional space-time]. We formulate the theory in Euclidean space (i.e., imaginary time \(\tau\), with \(\tau \in [0, \beta]\)) so that we can go smoothly between nonzero and zero temperature using the imaginary time formalism. The Lagrangian is

\[
\mathcal{L} = \sum_{j=1}^{N_f} \psi_j^\dagger (\partial_\tau - iA + m) \psi_j - i\kappa A
\]  

There are many similarities between this model and the 2+1 dimensional model of fermions coupled to a nonabelian Chern-Simons gauge field. First, this model supports gauge transformations with nontrivial winding number. This may look peculiar since it is an abelian theory, but under the \(U(1)\) gauge transformation \(\psi \rightarrow e^{i\lambda}\psi\), \(A \rightarrow A + \partial_\tau\lambda\), the Lagrange density changes by a total derivative and the action changes by

\[
\Delta S = -i\kappa \int_0^\beta d\tau \partial_\tau \lambda = -2\pi i\kappa N
\]  

where \(N \equiv \frac{1}{2\pi} \int_0^\beta d\tau \partial_\tau \lambda\) is the integer-valued winding number of the topologically nontrivial gauge transformation.

**Exercise 5.5.1** : Show that, in the imaginary time formalism, such a nontrivial gauge transformation is \(\lambda(\tau) = \frac{2N\pi}{\beta}(\tau - \frac{\beta}{2})\); while, in real time, a nontrivial gauge transformation is \(\lambda(t) = 2N \arctan(t)\). In each case, explain why the winding number \(N\) must be an integer.

From \[257\] we see that choosing \(\kappa\) to be an integer, the action changes by an integer multiple of \(2\pi i\), so that the Euclidean quantum path integral \(e^{-S}\) is invariant. This is the analogue of the discreteness condition \[61\] on the Chern-Simons coefficient in three dimensional nonabelian Chern-Simons theories. (The extra \(4\pi\) factor in the 2 + 1 dimensional case is simply a solid angle normalization factor.)

Another important similarity of this quantum mechanical model to its three dimensional counterpart is its behaviour under discrete symmetries. Under naive charge conjugation \(C : \psi \rightarrow \psi^\dagger\), \(A \rightarrow -A\), both the fermion mass term and the Chern-Simons term change sign. This mirrors the situation in three dimensions where the fermion mass term and the Chern-Simons term each change...
sign under a discrete parity transformation. In that case, introducing an equal number of fermions of opposite sign mass, the fermion mass term can be made invariant under a generalized parity transformation. Similarly, with an equal number of fermion fields of opposite sign mass, one can generalize charge conjugation to make the mass term invariant in our 0 + 1 dimensional model.

Induced Chern-Simons terms appear when we compute the fermion effective action for this theory:

$$S[A] = \log \left[ \frac{\det (\partial_\tau - iA + m)}{\det (\partial_\tau + m)} \right]^{N_f}$$

(258)

The eigenvalues of the operator $\partial_\tau - iA + m$ are fixed by imposing the boundary condition that the fermion fields be antiperiodic on the imaginary time interval, $\psi(0) = -\psi(\beta)$, as is standard at finite temperature. Since the eigenfunctions are

$$\psi(\tau) = e^{(A-m)\tau + i \int^\tau A(\tau') d\tau'}$$

(259)

the antiperiodicity condition determines the eigenvalues to be

$$\Lambda_n = m - i \frac{a}{\beta} + \frac{(2n-1)\pi i}{\beta}, \quad n = -\infty, \ldots, +\infty$$

(260)

where we have defined

$$a \equiv \int_0^\beta d\tau A(\tau)$$

(261)

which is just the 0 + 1 dimensional Chern-Simons term.

Given the eigenvalues (260), the determinants in (258) are simply

$$\frac{\det (\partial_\tau - iA + m)}{\det (\partial_\tau + m)} = \prod_{n=-\infty}^{\infty} \left[ \frac{m - i \frac{a}{\beta} + \frac{(2n-1)\pi i}{\beta}}{m + \frac{(2n-1)\pi i}{\beta}} \right] = \frac{\cosh \left( \frac{\beta m - i \frac{a}{2}}{2} \right)}{\cosh \left( \frac{\beta m}{2} \right)}$$

(262)

where we have used the standard infinite product representation of the cosh function. Thus, the exact finite temperature effective action is

$$S[A] = N_f \log \left[ \cos \left( \frac{a}{2} \right) - i \tanh \left( \frac{\beta m}{2} \right) \sin \left( \frac{a}{2} \right) \right]$$

(263)

Several comments are in order. First, notice that the effective action $S[A]$ is not an extensive quantity (i.e., it is not an integral of a density). Rather, it is a complicated function of the Chern-Simons action: $a = \int d\tau A$. We will have more to say about this later. Second, in the zero temperature limit, the effective action reduces to

$$S[A]_{T=0} = -i \frac{N_f}{2} \frac{m}{|m|} \int d\tau A(\tau)$$

(264)

which [compare with (233)] is an induced Chern-Simons term, with coefficient $\pm \frac{N_f}{2}$. This mirrors precisely the zero $T$ result (231) for the induced Chern-Simons term in three dimensions [the factor
of $\frac{1}{4\pi}$ is irrelevant because with our $2+1$ dimensional normalizations it is $4\pi\kappa$ that should be an integer, while in the $0+1$ dimensional model it is $\kappa$ itself that should be an integer. This extra $4\pi$ is just a solid angle factor.

At nonzero temperature the effective action is much more complicated. A formal perturbative expansion of the exact result (263) in powers of the gauge field yields

$$S[A] = -i\frac{N_f}{2} \left[ \tanh\left(\frac{\beta m}{2}\right) a - \frac{i}{4} \text{sech}^2\left(\frac{\beta m}{2}\right) a^2 + \frac{1}{12} \tanh\left(\frac{\beta m}{2}\right) \text{sech}^2\left(\frac{\beta m}{2}\right) a^3 + \ldots \right] \quad (265)$$

The first term in this perturbative expansion

$$S^{(1)}[A] = -i\frac{N_f}{2} \tanh\left(\frac{\beta m}{2}\right) \int A \quad (266)$$

is precisely the Chern-Simons action, but with a temperature dependent coefficient. Moreover, this $T$ dependent coefficient is simply the zero $T$ coefficient from (264), multiplied by the smooth function $\tanh(\frac{\beta|m|}{2})$. Once again, this mirrors exactly what we found in the $2+1$ dimensional case in the previous Section – see (253) and (255).

If the computation stopped here, then we would arrive at the apparent contradiction mentioned earlier – namely, the “renormalized” Chern-Simons coefficient

$$\kappa_{\text{ren}} = \kappa_{\text{bare}} - \frac{N_f}{2} \tanh\left(\frac{\beta m}{2}\right) \quad (267)$$

would be temperature dependent, and so could not take discrete values. Thus, it would seem that the effective action cannot be invariant under large gauge transformations.

The flaw in this argument is clear. At nonzero temperature there are other terms in the effective action, besides the Chern-Simons term, which cannot be ignored; and these must all be taken into account when considering the question of the large gauge invariance of the effective action. Indeed, it is easy to check that the exact effective action (263) shifts by $(N_f N)\pi i$, independent of the temperature, under a large gauge transformation, for which $a \to a + 2\pi N$. But if the perturbative expansion (265) is truncated to any order in perturbation theory, then the result cannot be invariant under large gauge transformations: large gauge invariance is only restored once we resum all orders. The important point is that the full finite $T$ effective action transforms under a large gauge transformation in exactly the same way as the zero $T$ effective action. When $N_f N$ is odd, this is just the familiar global anomaly, which can be removed (for example) by taking an even number of flavours, and is not directly related to the issue of the temperature dependence of the Chern-Simons coefficient. The clearest way to understand this global anomaly is through zeta function regularization of the theory [77], as is illustrated in the following exercise.

**Exercise 5.5.2**: Recall the zeta function regularization definition of the fermion determinant, $\det(O) = \exp(-\zeta'(0))$, where the zeta function $\zeta(s)$ for the operator $O$ is

$$\zeta(s) = \sum_{\lambda} \lambda^{-s} \quad (268)$$
where the sum is over the entire spectrum of $O$. Using the eigenvalues in (260), express this zeta function for the $0+1$ dimensional Dirac operator in terms of the Hurwitz zeta function $\zeta_H(s,v) \equiv \sum_{n=0}^{\infty}(n+v)^{-s}$. Hence show that the zeta function regularized effective action is

$$S_{\text{zeta}}[A] = \pm i \frac{N_f}{2} a + N_f \log \left[ \cos \left( \frac{a}{2} \right) - i \tanh \left( \frac{\beta m}{2} \right) \sin \left( \frac{a}{2} \right) \right]$$

(269)

[You will need the Hurwitz zeta function properties: $\zeta_H(0,v) = \frac{1}{2} - v$, and $\zeta_H'(0,v) = \log \Gamma(v) - \frac{1}{2} \log(2\pi)$]. The sign ambiguity in the first term corresponds to the ambiguity in defining $(\lambda)^{-a}$. The effect of this additional term is that the zeta function regularized effective action changes by an integer multiple of $2\pi i$ under the large gauge transformation $a \rightarrow a + 2\pi N$, even when $N_f$ is odd. Show that this is consistent with the fact that this large gauge transformation simply permutes the eigenvalues in (260) and so should not affect the determinant. (Note that this explanation of the global anomaly [101] is independent of the temperature, so it is somewhat beside the point for the resolution of the problem of an apparently $T$ dependent Chern-Simons coefficient.)

To conclude this section, note that only the first term in the perturbative expansion (265) survives in the zero temperature limit. The higher order terms all vanish because they have factors of $\text{sech}^2(\frac{\beta m}{2})$. This is significant because all these higher order terms are nonextensive — they are powers of the Chern-Simons action. We therefore do not expect to see them at zero temperature. Indeed, the corresponding Feynman diagrams vanish identically at zero temperature. This is usually understood by noting that they must vanish because there is no gauge invariant (even under infinitesimal gauge transformations) term involving more than one factor of $A(\tau)$ that can be written down. This argument, however, assumes that we only look for extensive terms; at nonzero temperature, this assumption breaks down and correspondingly we shall see that our notion of perturbation theory must be enlarged to incorporate nonextensive contributions to the effective action. For example, let us consider an action quadratic in the gauge fields which can have the general form

$$S^{(2)}[A] = \frac{1}{2} \int d\tau_1 \ d\tau_2 \ A(\tau_1) F(\tau_1 - \tau_2) A(\tau_2)$$

(270)

where, by symmetry, $F(\tau_1 - \tau_2) = F(\tau_2 - \tau_1)$. Under an infinitesimal gauge transformation, $A \rightarrow A + \partial_\tau \lambda$, this action changes by: $\delta S^{(2)}[A] = - \int d\tau_1 \ d\tau_2 \ \lambda(\tau_1) \partial_\tau_2 F(\tau_1 - \tau_2) A(\tau_2)$. Clearly, the action (270) will be invariant under an infinitesimal gauge transformation if $F = 0$. This corresponds to excluding such a quadratic term from the effective action. But the action can also be invariant under infinitesimal gauge transformations if $F = \text{constant}$, which would make the quadratic action (270) nonextensive, and in fact proportional to the square of the Chern-Simons action. The origin of such nonextensive terms will be discussed in more detail in Section 5.7 in the context of finite temperature perturbation theory.

5.6 Exact Finite Temperature $2+1$ Effective Actions

Based on the results for the $0+1$ dimensional model described in the previous Section, it is possible to compute exactly the parity violating part of the $2+1$ dimensional QED effective action when
the background gauge field $A_\mu(\vec{x}, \tau)$ takes the following special form:

$$A_0(\vec{x}, \tau) = A_0, \quad \vec{A}(\vec{x}, \tau) = \vec{A}(\vec{x})$$

(271)

and the static background vector potential $\vec{A}(\vec{x})$ has quantized flux:

$$\int d^2 x e^{i j} \partial_i A_j = \int d^2 x B = 2\pi N, \quad N \in \mathbb{Z}$$

(272)

Under these circumstances, the three dimensional finite temperature effective action breaks up into an infinite sum of two dimensional effective actions for the two dimensional background $\vec{A}(\vec{x})$. To see this, choose Euclidean gamma matrices in three dimensions to be:

$$\gamma_0 = i\sigma_3, \quad \gamma_1 = i\sigma_1, \quad \gamma_2 = i\sigma_2.$$  

Then the Dirac operator appearing in the three dimensional effective action is

$$-i(\partial - iA) + m = \begin{pmatrix} \partial_0 - iA_0 + m & D_- \\ D_+ & -\partial_0 + iA_0 + m \end{pmatrix}$$

(273)

where $D_\pm = D_1 \pm iD_2$ are independent of $\tau$ by virtue of the ansatz (271). Recalling that at finite $T$ the operator $\partial_0$ has eigenvalues $(\frac{2n+1}{\beta} \pi i)$ for $n \in \mathbb{Z}$, we see that the problem is reduced to an infinite set of Euclidean two dimensional problems.

To proceed, consider the eigenfunctions $\begin{pmatrix} f \\ g \end{pmatrix}$, and eigenvalues $\mu$, of the massless two dimensional Dirac operator

$$\begin{pmatrix} 0 & \frac{D_-}{D_+} \\ D_+ & \frac{D_-}{D_+} \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = \mu \begin{pmatrix} f \\ g \end{pmatrix}$$

(274)

It is a straightforward (but messy) algebraic exercise to show that given such an eigenfunction corresponding to a nonzero eigenvalue, $\mu \neq 0$, it is possible to construct two independent eigenfunctions $\phi_\pm$ of the three dimensional Dirac operator [97]:

$$\begin{pmatrix} m - iA_0 + \frac{(2n+1)\pi i}{\beta} \\ D_+ \end{pmatrix} \begin{pmatrix} D_- \\ m + iA_0 - \frac{(2n+1)\pi i}{\beta} \end{pmatrix} \phi_\pm = \lambda_\pm \phi_\pm$$

(275)

where

$$\lambda_\pm = m \pm i \sqrt{\mu^2 + (A_0 - \frac{(2n+1)\pi i}{\beta})^2}$$

(276)

and $\phi_\pm = \begin{pmatrix} f \\ \alpha_\pm g \end{pmatrix}$, with

$$\alpha_\pm = \frac{i}{\mu} (A_0 - \frac{(2n+1)\pi i}{\beta}) \pm i \sqrt{1 + \frac{1}{4\mu^2} (A_0 - \frac{(2n+1)\pi i}{\beta})^2}$$

(277)

So, for each nonzero eigenvalue $\mu$ of the two dimensional problem, there are two eigenvalues $\lambda_\pm$ of the three dimensional Dirac operator. But from the form (276) of these eigenvalues, we see that their contribution to the three dimensional determinant is even in the mass $m$; and therefore these
eigenvalues (coming from nonzero eigenvalues of the two-dimensional problem) do not contribute to the parity odd part of the three dimensional effective action.

In fact, the only contribution to the parity odd part comes from the zero eigenvalues of the two dimensional problem. From the work of Landau [20] (and Aharonov and Casher [21]) we know that there are \( \mathcal{N} = \frac{1}{\pi} \int d^2 x B \) of these zero eigenvalues. This ‘lowest Landau level’ can be defined by the condition \( D_+ g = 0 \), so that the eigenfunctions of the three dimensional Dirac operator are

\[
\phi_0 = \left( \begin{array}{c} 0 \\ g \end{array} \right), \quad \text{where} \quad D_+ g = 0
\]

Thus the relevant eigenvalues of the three dimensional Dirac operator are

\[
\lambda_0^{(n)} = m + \frac{i A_0}{\beta} - \frac{(2n + 1) \pi i}{\beta}, \quad n \in \mathbb{Z}
\]

each with degeneracy \( \mathcal{N} \).

There is no paired eigenvalue, so to compute the parity odd part of the finite temperature three dimensional effective action we simply trace over these eigenvalues, and multiply by \( \mathcal{N} \). But this is exactly the same problem that we solved in the last section [see (260)], with \( \mathcal{N} \) playing the role of \( N_f \), the number of fermion flavours. Thus, we see immediately that

\[
S_{\text{eff}}^{\text{odd}} [A] = \frac{\mathcal{N}}{2} \left( \log \left[ \cos \left( \frac{a}{2} \right) - i \tanh \left( \frac{\beta m}{2} \right) \sin \left( \frac{a}{2} \right) \right] - \log \left[ \cos \left( \frac{a}{2} \right) + i \tanh \left( \frac{\beta m}{2} \right) \sin \left( \frac{a}{2} \right) \right] \right)
\]

\[
= -i \mathcal{N} \arctan \left[ \frac{\tanh \left( \frac{\beta m}{2} \right) \tan \left( \frac{a}{2} \right) \tan \left( \frac{a}{2} \right) \tan \left( \frac{a}{2} \right)}{\tan \left( \frac{a}{2} \right) \tan \left( \frac{a}{2} \right) \tan \left( \frac{a}{2} \right)} \right]
\]

where \( a \equiv \beta A_0 = \int_0^\beta A_0 \). This is simply the imaginary part of the \( 0 + 1 \) exact effective action (263).

A more rigorous zeta function analysis of this problem has been given in [27], along the lines outlined in the Exercise from the last section. But the key idea is the same – when the three dimensional gauge background has the restricted static form of (271), the problem reduces to a set of two dimensional problems; and moreover, only the zero modes of this two dimensional system contribute to the parity odd part of the three dimensional effective action. This can also be phrased in terms of chiral Jacobians of the two dimensional system [98].

The background in (271) supports large gauge transformations at finite temperature as a consequence of the \( S^1 \) of the Euclidean time direction. So, if \( \lambda(\tau) = \frac{2N\pi}{\beta} (\tau - \frac{\beta}{2}) \), independent of \( \vec{x} \), then the gauge transformation \( A_\mu \rightarrow A_\mu + \partial_\mu \lambda \), does not affect \( \vec{A} \), but \( A_0 \rightarrow A_0 + \frac{2N\pi}{\beta} \). In the notation of (281) this means \( a \rightarrow a + 2N\pi \). Thus our discussion of these large gauge transformations reduces exactly to the discussion of the previous section for the \( 0 + 1 \) dimensional model.

While this is a nice result, it is still a bit unsatisfying because these are not the nonabelian large gauge transformations in three dimensions that we were originally considering. In fact, if we adopt the static ansatz (271) then the abelian Chern-Simons term reduces to

\[
\int d^3 x e^{\mu \nu \rho} A_\mu \partial_\nu A_\rho = 4\pi \mathcal{N} \int_0^\beta A_0 \tag{281}
\]
which is just the $0 + 1$ dimensional Chern-Simons term. So transformations that stay within this ansatz are simply the nontrivial winding number transformations of the $0 + 1$ dimensional model.

We can make a similar static ansatz in the nonabelian case. For static fields, the nonabelian Chern-Simons term simplifies to

$$
\int d^3 x \, \epsilon^{\mu \nu \rho} \mathrm{tr} \left( A_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho \right) = \int_0^\beta d\tau \, \mathrm{tr} \left[ A_0 \left( \int d^2 x \, \epsilon^{ij} F_{ij} \right) \right] \tag{282}
$$

where $\epsilon^{ij} F_{ij}$ is the (Lie algebra valued) nonabelian covariant anomaly in two dimensions. It is possible to make gauge transformations that shift this Chern-Simons action by a constant, and by choosing $\vec{A}$ appropriately (for example, in terms of unitons) this constant shift can be made integer multiple of $2\pi i$. But this constant shift is not due to the winding number term in the change of the nonabelian Chern-Simons Lagrangian under a gauge transformation – rather, it is due to the total derivative term. Therefore, the simple nonabelian generalization of (280), with a static nonabelian ansatz, does not really answer the question of what happens to the discreteness condition (61) at finite temperature.

### 5.7 Finite Temperature Perturbation Theory and Chern-Simons Terms

These results for the finite temperature effective action contain some interesting lessons concerning finite temperature perturbation theory. The exact results of the previous sections are clearly very special. For general $2 + 1$ dimensional backgrounds we cannot compute the effective action exactly. Nor can we do so in truly nonabelian backgrounds that support large gauge transformations with nonvanishing winding number. Furthermore, Chern-Simons terms may be induced not only in fermionic systems, but also in Chern-Simons-Yang-Mills [10] and in gauge-Higgs models with spontaneous symmetry breaking [91, 92]. In such models there are no known exact results, even at zero temperature. At finite $T$, perturbation theory is one of the few tools we have.

An important lesson we learn is that there is an inherent incompatibility between large gauge invariance and finite temperature perturbation theory. We are accustomed to perturbation theory being gauge invariant order-by-order in the coupling $\epsilon$, but this is not true for large gauge invariance at finite temperature. We see this explicitly in the perturbative expansion (265) [note that since we had absorbed $\epsilon$ into the gauge field $A$, the order of perturbation is effectively counted by the number of $A$ factors]. If we truncate this expansion at any finite order, then the result is invariant under small gauge transformations, but it transforms under a large gauge transformation in a $T$ dependent manner. It is only when we re-sum all orders, to obtain the exact effective action (263), that the response of the effective action to a large gauge transformation becomes $T$ independent, as it should be. There is actually a simple way to understand this breakdown of large gauge invariance at any finite order of perturbation theory [97]. A gauge transformation (with factors of $\epsilon$ restored) is

$$
A_\mu \rightarrow A_\mu + \frac{1}{\epsilon} \partial_\mu \Lambda \tag{283}
$$

For an infinitesimal gauge transformation, the $\frac{1}{\epsilon}$ factor can be absorbed harmlessly into a redefinition of the gauge function $\Lambda$. But such a rescaling does not remove the $\epsilon$ dependence for a large gauge transformation, because such a gauge transformation must satisfy special boundary
conditions at $\tau = 0$ and $\tau = \beta$ (in the imaginary time formalism). A rescaling of $\Lambda$ simply moves $e$ into the boundary conditions. The effect is that a large gauge transformation can mix all orders in a perturbative expansion in powers of $e$, thus destroying the large gauge invariance order-by-order.

Diagrammatically, the appearance of higher order terms, other than the Chern-Simons term, in the perturbative expansion (265) means that at finite temperature the diagrams with many external 'photon' legs contribute to the parity odd part of the effective action. This is in contrast to the case at $T = 0$ where only a single graph contributes – in $0 + 1$ dimensions it is the one-leg graph, and in $2 + 1$ dimensions it is the two-leg self-energy graph. Actually, these higher-leg graphs are perfectly compatible with infinitesimal gauge invariance, but they violate the zero temperature requirement of only have extensive quantities in the effective action. In the $0+1$ dimensional model, the standard Ward identities for infinitesimal gauge invariance $[p_\mu \Gamma^{\mu\nu\cdots} = 0$, etc ...] simplify (because there is no contraction of indices) to imply that the diagram is proportional to a product of delta functions in the external energies. In position space this simply means that each term is proportional to a nonextensive term like $(f A)^n$. But at zero temperature such nonextensive terms are excluded for $n > 1$, and indeed one finds, reassuringly, that the corresponding diagrams vanish identically. At finite temperature we cannot exclude terms that are nonextensive in time, and so these terms can appear; and correspondingly we discover that these diagrams are indeed nonvanishing at $T > 0$.

Accepting the possibility of nonextensive terms, the requirement that the fermion determinant change by at most a sign under a large gauge transformation, $a \to a + 2\pi N$, leads to the general form:

$$\exp \left[ -\Gamma(a)/N_f \right] = i \sum_{j=0}^{\infty} \left( d_j \cos\left(\frac{(2j+1)a}{2}\right) + f_j \sin\left(\frac{(2j+1)a}{2}\right) \right)$$

(284)

The actual answer (263) gives as the only nonzero coefficients: $d_0 = 1$ and $f_0 = i \tanh(\beta m)$. This fact can only be deduced by computation, not solely from gauge invariance requirements.

These same comments apply to the $2 + 1$ dimensional case when the background is restricted by the static ansatz (271). The static nature of the background once again makes a multi-leg diagram proportional to a product of delta functions in the external energies. For the answer to be extensive in space (but possibly nonextensive in time) we can only have one external spatial index, say $i$, and then invariance under infinitesimal static gauge transformations requires this diagram to be proportional to $\epsilon^{ij} p_j$. Factoring this out, the remaining diagrams are just like the multi-leg diagrams of the $0 + 1$ dimensional model, and can be computed exactly. [There is a slight infrared subtlety due to the difficulty in Fourier transforming a finite flux static background, but this is easily handled.] So, not surprisingly, the perturbative computation in the static ansatz reduces to that of the $0 + 1$ case, just as happens in the exact evaluation.

As soon as we attempt to go beyond the static ansatz, or consider induced Chern-Simons terms in non-fermionic theories, we strike some critical problems. The most significant is that the zero momentum limit (230), via which we identified the induced Chern-Simons terms, is no longer well defined at finite temperature. This is a physics problem, not just a mathematical complication. At finite $T$, Lorentz invariance is broken by the thermal bath and so a self-energy function $\Pi(p) = \Pi(p^0, \vec{p})$ is separately a function of energy $p^0$ and momentum $\vec{p}$. Thus, as is well known even in scalar field theories [102], the limits of $p^0 \to 0$ and $\vec{p} \to 0$ do not commute. The original computations of the finite temperature induced Chern-Simons coefficient [see (255)]
explicitly employed the “static limit”

\[ \lim_{|\vec{p}| \to 0} \Pi(p^0 = 0, \vec{p}) \]  

(285)

It is easy to see that the ‘opposite’ limit \( \lim_{p^0 \to 0} \Pi(p^0, |\vec{p}| = 0) \) gives a different answer at finite \( T \). This ambiguity simply does not arise in the \( 0 + 1 \) dimensional model, and the exact \( 2 + 1 \) dimensional results of the previous section avoided this ambiguity because the static ansatz (271) corresponds explicitly to the static limit (285).

Finally, another important issue that is not addressed by our \( 0 + 1 \) dimensional model, or the corresponding static \( 2 + 1 \) dimensional results, is the Coleman-Hill theorem [87], which essentially states that only one-loop graphs contribute to the induced Chern-Simons term. This is an explicitly zero temperature result, as the proof assumes manifest Lorentz covariance. But the question of higher loops does not even come up in the \( 0 + 1 \) dimensional model, or the static \( 2 + 1 \) dimensional backgrounds, because the ‘photon’ does not propagate; thus, there are no higher loop diagrams to consider.

It would be interesting to learn more about finite temperature effective actions whose zero temperature forms have induced Chern-Simons terms. There is undoubtedly more to discover.

Acknowledgement: I thank the Les Houches organizers, A. Comtet, Th. Jolicoeur, S. Ouvry and F. David, for the opportunity to participate in this Summer School. This work has been supported by the U.S. Department of Energy grant DE-FG02-92ER40716.00, and by the University of Connecticut Research Foundation.

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