Covers of Abelian varieties as analytic Zariski structure

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Abstract:
We use tools of mathematical logic to analyse the notion of a path on an complex algebraic variety, and are led to formulate a "rigidity" property of fundamental groups specific to algebraic varieties, as well as to define a bona fide topology closely related to etale topology. These appear as criteria for \( \aleph_1 \)-categoricity, or rather stability and homogeneity, of the formal countable language we propose to describe homotopy classes of paths on a variety, or equivalently, its universal covering space.

Technically, for a variety \( A \) defined over a finite field extension of the field \( \mathbb{Q} \) of rational numbers, we introduce a countable language \( L(A) \) describing the universal covering space of \( A(\mathbb{C}) \), or, equivalently, homotopy classes of paths in \( \pi_1(\mathbb{A}(\mathbb{C})) \). Under some assumptions on \( A \) we show that the universal covering space of \( A(\mathbb{C}) \) is an analytic Zariski structure \[19\], and present an \( L_{\omega_1\omega}(L(A)) \)-sentence axiomatising the class containing the structure and that is stable and homogeneous over elementary submodels. The "rigidity" condition on fundamental groups says that projection of of the fundamental group of a variety is the fundamental group of the projection, up to finite index and under some irreducibility assumptions, and is used to prove that the projection of an irreducible closed set is closed in the analytic Zariski structure.

In particular, we define an analytic Zariski structure on the universal covering space of an Abelian variety defined over a finite extension of the field \( \mathbb{Q} \) of rational numbers.

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1. Introduction

In §1 we describe our approach in a non-technical manner; §1.1.1 describes our philosophy behind the author’s thesis [3], the present paper and [4], and §1.1.2 announces our main results but not detailing definitions. A detailed exposition of our motivation is found in §1.2. In §2.1 we give the definitions and state the results in §2.2. The rest of the paper is devoted to the proof.

1.1. General Framework

1.1.1. Our philosophy

Is the notion of homotopy on a complex algebraic variety an algebraic notion? That is, can the notion of homotopy be characterised in a purely algebraic way, without reference to the complex topology?

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We can restrict to 1-dimensional homotopies only: a 1-dimensional homotopy is a path, so the question is now whether the notion of a path on a complex algebraic manifold, up to fixed point homotopy, can be characterised in a purely algebraic way.

We provide a partial positive answer to the following more precise question. Assume that one has an abstract notion of a path up to homotopy, so that one is able to speak about homotopy classes of paths, their endpoints, liftings along topological coverings, paths lying in a subvariety. Can this notion be described without recourse to the complex topology?

Is it true that one can axiomatise this notion in such a way that any of its realisations comes from a choice of an embedding of the underlying field into \( \mathbb{C} \), or equivalently, a choice of a locally compact Archimedean Hausdorff topology on the underlying field (if its cardinality is \( 2^{\aleph_0} \))?

Is the resulting formal theory “good” from a model-theoretic point of view?

Model theory allows a rigorous formulation of the question as the problem of proving categoricity of a structure related to the fundamental groupoid, or equivalently the universal covering space, of a complex algebraic variety. Such categoricity questions are extensively studied in model theory, specifically by Shelah [15, 16] and a short list of conditions sufficient for categoricity of an \( L_{\omega_1, \omega} \)-sentence is known (this is the notion of an excellent theory). Our model-theoretic analysis shows that the positive answer to our question is plausible and is essentially equivalent to deep geometric and arithmetic properties of the underlying variety. Some of the properties are known to hold, some others are conjectured.

We study the interaction between the model theory, arithmetic and geometry of complex algebraic varieties. Our main results state that certain basic model-theoretic conditions do indeed hold. In general the proofs require some technical finiteness and compactness conditions and assume some complex-analytic and arithmetic properties and conjectures. For some classes of varieties, for example Abelian varieties, these conditions are known to hold, and for these classes the results are unconditional. In particular we prove that there exists an \( \aleph_1 \)-categorical \( L_{\omega_1, 1} \)-axiomatisation of universal covering spaces in such classes.

In [3, Ch.V] (cf. also [4]) we consider a special case where the underlying variety is an elliptic curve, and prove that the natural \( L_{\omega_1, 1} \)-axiomatisation of the universal cover of an elliptic curve is \( \aleph_1 \)-categorical; analysis there shows that \( \aleph_1 \)-categoricity of that axiomatisation is essentially equivalent to a arithmetic conjecture on Galois representations known for elliptic curves.

Finally we would like to note that the model-theoretic analysis of universal covers falls very naturally into the framework of (analytic) Zariski geometries started by Hrushovski-Zilber in [8] and further developed by Zilber and his collaborators [23, 21, 1, 10, 19] around an expectation that many basic mathematical structures may be considered as a model-theoretic structure with nice properties, above all categoricity. Importantly, it has been understood that the model theory relevant here is essentially non-first-order. In fact, our main result is that the structures we consider are indeed analytic Zariski as defined in [19], thus providing a series of examples of analytic Zariski geometries.

1.1.2. Technical summary of results

In §3.1 we define a natural formal countable language \( L_\Lambda \) associated with the universal covering space \( p : U \rightarrow A(\mathbb{C}) \) of a complex projective algebraic variety \( A(\mathbb{C}) \) defined over \( \mathbb{Q} \) or \( \mathbb{C} \). Assuming subgroup separability of the fundamental group along with its Cartesian powers, we prove that

- the positively type-definable sets in \( L_\Lambda \) form a topology analogous to Zariski topology on the set of geometric points of a variety,

and, moreover, that

- the universal covering space \( U^{L_\Lambda} \), as an \( L_\Lambda \)-structure, is an analytic Zariski structure [19, Def.6.1.11]
By virtue of $U^{L_A}$ being analytic Zariski, we then know

- the structure $U^{L_A}$ is homogeneous over countable submodels ($\omega$-model homogeneity), and realises countably many types over a countable submodel.

We then consider in §5 a fragment of the $L_{\omega_1\omega}(L_A)$-theory $\text{Theory}_{\omega_1\omega}(U)$ of $U^{L_A}$ and introduce a natural set of axioms $\mathcal{X}$ of geometric, analytic Zariski flavour to show that

- the class of models defined by $\mathcal{X}$ is stable (in a non-elementary context) over countable models, and all its models are homogeneous over submodels.

These are prerequisites, by Shelah’s theory, of categoricity in uncountable cardinals. Notice that some of the properties, e.g. atomicity of every model, could, by Shelah’s theory, be obtained just by an $L_{\omega_1\omega}$-definable expansion of the theory $\mathcal{X}$. This, by Shelah’s theory, is enough to imply $\aleph_1$-categoricity of an $L_{\omega_1\omega}$-class $\Phi$ containing $U^{L_A}$, for an arbitrary smooth projective variety $A$ with certain conditions on the fundamental group. (Cf. Definition 2.1.2.1 for the exact definition of the class of algebraic varieties).

Finally we remark that our approach is essentially different from Zilber’s of [22] since our language $L_A$ is in general stronger than Zilber’s. In fact $L_A$ “adjusts” itself to the geometric properties of the covering of $A$, and is defined for any $A$ whereas [22] is restricted to the class of Abelian varieties. Our language allows us to produce a sentence in all cases, conjecturally categorical for suitably “self-sufficient” $A$ whereas [22] is restricted only to considering Abelian varieties, and those are sometimes obviously not “self-sufficient”, say Abelian varieties of dimension greater than 1. We refer to [3, IV§6] for details. Here we just remark that it is possible to consider the language $L_A$ corresponding to an ample homogeneous $\mathbb{C}^*$-bundle $A = L^*$ over $X$, and show that $L_A$ defines the 1st Chern class of $X(\mathbb{C})$ as an element $c_1 \in H^2(\pi_1(X(\mathbb{C}), 0), \mathbb{Z})$ or, equivalently, as an alternating bilinear Riemannian form $\Lambda \times \Lambda \to \mathbb{Z}$. 

1.2. Motivations and implications

In this section we discuss the motivations behind our choice of the language and explain our approach in greater detail. In our opinion the motivations here are more important than the proofs that follow.

We should add that we do not mention yet another motivation relating to category theory and Poincare groupoids ([3, §I.2.3], cf. also [4]), as it has no relation to the methods of this paper.

1.2.1. The Logic approach: What is an appropriate language to talk about paths?

Abstract algebraic geometry provides a language appropriate to talk about complex algebraic varieties; what language would be appropriate to talk about the homotopies on the algebraic varieties, in particular about paths, i.e. 1-dimensional homotopies? What is the right mathematical measure to judge appropriateness of the language for such a notion?

Abstract algebraic geometry over a field has no complete analogue of the notion. However, there is a strong intuition based on the naive notion of a path in complex topology; it is a well-known phenomenon that naive arguments based on the notion of a path quite often lead to statements which generalise, in one way or another, to, say, arbitrary schemes, but which are quite difficult to prove. There have been many attempts to develop substitute notions, starting from Grothendieck [SGA1,SGA2,SGA4½] who developed for this purpose the notion of a finite covering in the category of arbitrary schemes (étale morphism); see Grothendieck [5] for an attempt to provide an algebraic formalism to express homotopy properties of topological spaces, and Voevodsky-Kapranov [18] for exact definitions.

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Thus, from the point of view of philosophy of mathematics, it is natural to try to understand why the notion of a path is so fruitful and applicable, despite the fact that all attempts to generalise it to non-topological contexts have had only partial success.

We intend to propose in this work a model-theoretic structure which contains an abstract substitute for the notion of a path. The substitute must possess the familiar properties of paths appearing in the topological context, rich enough to imply a useful theory of paths; in particular they must determine the notion of a path on an abstract algebraic variety uniquely up to isomorphism.

Note that Grothendieck [5], cf. also Voevodsky-Kapranov [18], provides a natural algebraic setup to talk about paths thereby rather directly leading to a choice of a language (of 2-functors). Our approach is in fact based on a similar idea.

Model theory provides a framework to formulate the uniqueness property in a mathematically rigorous fashion. Following [20, 21] we use the notion of categoricity in uncountable cardinals (of non-elementary classes). In his philosophy categoricity is a model-theoretic criterion for determining when an algebraic formalisation of an object, of perhaps geometric character, is canonical and reflects the properties of the object in a complete way.

In this work we introduce a language $L_A$ which is appropriate for describing the basic homotopy properties of algebraic varieties in their complex topology, and prove some partial results towards categoricity and stability of associated structures in that language. The expressive power of $L_A$ is studied elsewhere; here we make the following remarks whose justification can be found in [3, Ch.II]. The language $L_A$ is capable of expressing properties of 1-dimensional homotopies, i.e. the properties of paths up to homotopies fixing the ends. We can speak in $L_A$ in terms of lifting paths to a topological covering, paths lying in closed algebraic subvarieties (i.e. a homotopy class has a representative which lies in the subvariety), paths in direct products and so on. These properties are sufficient to carry out many basic 1-dimensional homotopy theory constructions. Most notably, following a construction in Mumford [12] one can definably construct a bilinear form $\varphi_L : \pi_1(\mathbb{A}(\mathbb{C}), 0) \times \pi_1(\mathbb{A}(\mathbb{C}), 0) \to \pi_1(\mathbb{C}^*, 1)$ in the second homology group $H^2(\mathbb{A}(\mathbb{C}), \mathbb{Z}) \cong \bigwedge^2 H^1(\mathbb{A}(\mathbb{C}), \mathbb{Z})$ associated to an algebraic $\mathbb{C}^*$-bundle $L$ over a complex Abelian variety $X(\mathbb{C})$. Thus, generally the language has more expressive power than the one considered originally by Zilber in [22]; in particular, some Abelian varieties which are not categorical in Zilber’s language of [22] are expected to be categorical in our language. It would be interesting to know whether our language can interpret Hodge decomposition on cohomology groups, using the isomorphism $H^n(\mathbb{A}(\mathbb{C}), \mathbb{C}) \cong \bigwedge^n H^1(\mathbb{A}(\mathbb{C}), \mathbb{C}) = \bigwedge^n \text{Hom}(\pi_1(\mathbb{A}(\mathbb{C}), 0), \mathbb{C})$ (cf. [12]).

The results which we prove towards categoricity in uncountable cardinalities are partial. We prove categoricity in cardinality $\aleph_1$ for some special classes of algebraic varieties, e.g. for elliptic curves. We also prove important necessary conditions, such as stability and homogeneity over models, for much wider classes.

### 1.2.2. The Geometric approach: Analytic Zariski structures

The universal covering of an algebraic variety is one of the simplest analytic structures associated to an algebraic variety and which is more than an algebraic variety itself; the universal covering space inherits all the local structure the base space possesses; and in particular, for a complex algebraic variety it is a complex analytic space. Thus it is natural to consider it in the context of Zariski geometries [23]: one wants to define a Zariski-type topology on the universal covering space $\mathbb{U}$ of variety $\mathbb{A}(\mathbb{C})$ reflecting the connection between $\mathbb{U}$ and $\mathbb{A}$, and such that $\mathbb{U}$ possesses homogeneity, stability and categoricity properties, perhaps in a non-first order, $L_{\omega_1\omega}$, way, in a countable language related to the chosen topology on $\mathbb{U}$.

For this, consider the universal covering space $p : \mathbb{U} \to \mathbb{A}(\mathbb{C})$ of an algebraic variety $\mathbb{A}$. It is natural to assume that the covering map $p$ and the full algebraic variety structure on $\mathbb{A}(\mathbb{C})$ are definable. Then the analytic subsets of $\mathbb{U}$ which are the preimages $p^{-1}(Z(\mathbb{C}))$ of algebraic subvarieties...
2. A Zariski topology on a universal covering space of an Abelian variety

2.1. Definitions and background

2.1.1. Notations and some background

We briefly introduce basic notions of topology we require. Consult [13, Ch.4, §§2-4] or [9] for details.

For a Hausdorff, locally connected and locally linearly connected topological space $B$ with a distinguished base-point $b \in B$, the universal covering space $(U, u_0)$ of $(B, b)$ is the space of all paths starting at the base-point $b$, i.e. continuous maps $\gamma : [0, 1] \to B$, $\gamma(0) = b$, considered up to homotopy fixing the end-points, and endowed with the natural topology, and further equipped with the covering homeomorphism $p : U \to B$, $p(\gamma) = \gamma(1)$. Two paths $\gamma_1, \gamma_2 : [0, 1] \to B$ are end-point homotopic if there exists a homotopy $\Gamma : [0, 1] \times [0, 1] \to B$ such that $\Gamma(0, t) = \gamma_0(t), \Gamma(1, t) = \gamma_1(t), \Gamma(t, 0) = \gamma_0(0) = \gamma_1(0), \Gamma(t, 1) = \gamma_0(1) = \gamma_1(1)$. The fundamental groupoid $\pi_1(B)$ is the set of all paths considered up to end-point homotopy, equipped with the partial operation of concatenation. A concatenation $\gamma_0 \gamma_1, \gamma_0(1) = \gamma_1(1)$ of paths is a path which first follows the first path $\gamma_0$, and then goes along the second path $\gamma_1$; this defines concatenation up to homotopy. The fundamental group $\pi_1(B, b) = p^{-1}(b)$ is the group of all loops $\gamma \in \pi(B), \gamma(0) = \gamma(1) = b$. A deck transformation of $U$ is a homeomorphism $\tau : U \to U$ commuting with $p, \tau \circ p = p$. Deck transformation of $U$ form a group $\Gamma = \pi(U)$ called the deck transformation group. The deck transformation group $\pi(U)$ is canonically identified with the fundamental group $\pi_1(B, b)$: to an element $\gamma \in \pi(V')$ there corresponds path $p(\gamma \gamma_x \gamma_y)$ where $b' \in U$ is arbitrary such that $p(b') = b$. The covering map $p : U \to B$ is a local homeomorphism; a analytic space structure on $B$ induces a unique analytic space structure on $U$. There is a Galois correspondence between normal subgroups $H < \Gamma$ and covering spaces $B^H \doteq U/H \to B$. The map $U \to B^H$ is a universal covering map, and its deck transformation group is $H$; the map $B^H \to B$ is a covering and its deck transformation group is the factor group $\Gamma/H$.

A map $p : X \to Y$ is called a fibration iff for any space $Z$ any homotopy $F : Z \times I \to Y$ covered at the initial time $t = a$, can be covered at all times $a \leq t \leq b, I = [a, b]$ by some homotopy $G : Z \times I \to X$ so that $p \circ G(z, t) = F(z, t), G(z, a) = g(z)$. That is, if map $f(z) = F(z, a) : Z \to Y$ is covered by a map $g : Z \to X, f(z) = F(z, a) = p \circ g(z), z \in Z$, then there exist a homotopy $G : Z \times I \to X$ covering $F : Z \times I \to X$,

$$G(z, a) = g(z)$$

$$F(z, t) = p \circ G(z, t).$$
Homotopy $G$ is called a covering homotopy with initial condition $g$. We also say that homotopy $F$ lifts to homotopy $G$, and that fibration $p : X \to Y$ has lifting property. We will use extensively the case when $Z = I$ is an interval and $g = g(0, a)$ is a simply a point; this case is called the path-lifting property. A covering is a fibration with discrete fibres.

Often one modifies the definition by restricting $Z$ to a subclass of spaces, e.g. $Z = I^n$ is required to be a direct product of intervals (Serre fibration). This distinction is not important in this paper.

### 2.1.2. Our Assumptions.

The most interesting, and the only unconditional, example where our theorems apply, is that of $U/\Gamma$ an Abelian variety: $U = \mathbb{C}^2$, $\Gamma = \pi(U/\Gamma, 0)$ is a lattice in $U$.

However, our assumptions are geometric; in particular the assumptions do not mention the group structure of an Abelian variety. We call the corresponding class of varieties LERF.

We assume $U$ is a smooth complex analytic space equipped with an free cocompact action $\Gamma : U \to U$ of a subgroup separable (cf. 2.1.2) finitely generated group $\Gamma : U \to U$. Further we assume that all Cartesian powers of $\Gamma$ are subgroup separable, and that $U/\Gamma$ is a projective algebraic variety.

**Subgroup separability of $\pi(U)$**. A group $\Gamma$ is called subgroup separable, or locally extended residually finite, often abbreviated lerf, iff for any finitely generated subgroup $G < \Gamma$ and an element $g \notin G$ there exists a finite index subgroup $H$ such that $G < H$ and $g \notin H$. This is a non-trivial property rather hard to establish; it is known that the fundamental groups of complex curves ([14]) and $\mathbb{Z}^n$, $SL_2(\mathbb{Z})$ are subgroup separable; however, it is known that $F_2 \times F_2$ ([11]) is not subgroup separable, and so in general the products of subgroup separable groups are not subgroup separable. This property may be reformulated topologically: the group $\Gamma = \pi_1(A)$ is subgroup separable if and only if for any finitely generated $G < \Gamma$ and any compact subset $C \subset A^G = U/G$, the covering splits as $A^G \to A^H \to A$ such that $A^H \to A$ is a finite covering and the compact $C$ maps to $A^H$ by a homeomorphism. In fact, we need this property only when $G$ is the fundamental group of an algebraic subset of $A$.

**LERF varieties**. The above enables us to define the class of LERF varieties to which our theorems apply.

**Definition 2.1.2.1.** We call a smooth projective algebraic variety $A(\mathbb{C})$ LERF if all finite Cartesian powers of the group of deck transformations $\pi(U)$ are subgroup separable.

### 2.1.3. Co-etale topology, its core and inner core

We define topologies on $U$ and Cartesian powers on $U$.

**Definition of the co-etale topology**. We give 3 equivalent definitions of co-etale topology on $U$; we prove the equivalence in Decomposition Lemma 2.3.2.1.

**Definition 2.1.3.1.** (I) A subset of $U^n$, $n > 0$, is closed in co-etale topology $\mathcal{G}$ iff it is either (I)(i) an irreducible analytic component of a closed analytic set such that the set is set-wise invariant under the action of the fundamental group, or (I)(ii) a closed analytic set such that each of its analytic irreducible component satisfies (I)(i) above.

We call a closed analytic subset $Z$ of $U^n$ unfurled iff every connected component of $U$ is irreducible. It is known that every smooth closed analytic set is unfurled.

(C) A subset of $U^n$, $n > 0$, is closed in co-etale topology $\mathcal{G}$ iff it is either (C)(i) a connected component of an unfurled closed analytic set such that the set is set-wise invariant under action of a finite index subgroup of the fundamental group, or (C)(ii) a closed analytic set such that each of its analytic irreducible components satisfies (C)(i) above.
Definition 2.1.3.2. A subset of $U^n$, $n > 0$, is closed in co-etale topology $\mathcal{G}$ if it is either (C') $(i)$ a connected component of an unfurled closed analytic set such that the set is set-wise invariant under action of a finite index subgroup of the fundamental group, or (C') $(ii)$ a countable intersection of sets as in (C') $(i)$

**Countable core $C_0$.** By our assumptions, $A(\mathbb{C}) = U/\Gamma$ is a complete projective algebraic variety defined over $\overline{\mathbb{Q}}$, and therefore by Chow’s Lemma every closed analytic subset of $A(\mathbb{C})$ is in fact algebraic and defined over a finitely generated subfield of $\mathbb{C}$. This enables us to speak of the field of definition of a $\Gamma$-invariant closed analytic subset of $U/\Gamma$, as $\Gamma$-invariant closed analytic subsets are in 1-1 correspondence with closed analytic subsets of $A(\mathbb{C})$.

This enables us to define the following.

**Definition 2.1.3.3.** The countable core $C_0$ consists of closed sets that are unions of irreducible components of $\Gamma$-invariant closed sets defined over $\overline{\mathbb{Q}}$.

Note that a point $u \in U$ is in the countable core iff $p(x) \in A(\overline{\mathbb{Q}})$.

In Lemma 3.2.0.4 we prove that core sets are enough to define all sets; in the following way: that every irreducible co-etale closed subset $Z \subset U^n$ can be represented as a connected component $Z \times \{g\}$ of a hyperplane section $Z' \cap U^n \times \{g\}$ of a co-etale closed set $Z'$ in the countable core.

**Countable inner core $C_0$.** In fact, in our structure we may define analogs of sets over $\mathbb{Q}$ (or perhaps the maximal Abelian extension of $\mathbb{Q}$), and not just $\overline{\mathbb{Q}}$.

**Definition 2.1.3.3.** The countable inner core $C_0$ consists of the subsets of $U^n \times U^n$ defined by relations $x' \sim_H y'$ and $x' \sim_{Z,A} y'$ where $Z \subset A^n$ is a closed subvariety defined over the field of definition of $A$, $H$ a finite index subgroup of $\Gamma$, and the relation is defined as follows.

$$x' \sim_{Z,A} y' \iff \exists \tau \in H^n: \tau x' = y'.$$

We shall also consider

$$x' \sim_{Z,A,n} y' \text{ iff } x' \text{ and } y' \text{ lie in the same connected component of the preimage } p_H^{-1}(Z_i(\mathbb{C})), Z_i \subset A^H(\mathbb{C})^n \text{ an irreducible component of algebraic variety } p_H^{-1}(Z(\mathbb{C})) \subset A^H(\mathbb{C})^n.$$

2.2. Our Results: Definition of analytic Zariski structure, and the main theorem.

We have defined a topology on every Cartesian power of $U$, and the notion of countable core.

Every co-etale closed set is closed in analytic topology, and thus possesses the dimension; let this be the dimension function of the analytic Zariski structure.

**Theorem 2.2.0.4.** The data as defined above, form an analytic Zariski structure as defined in [19, Def.6.1.11]. Moreover, the analytic Zariski structure belongs to an explicitly axiomatised $L_{\omega_1,\omega}$-class $\mathfrak{X}(A(\mathbb{C}))$ that is $\omega$-stable over submodels, every model is $\omega$-homogeneous.

**Corollary 2.2.0.5.** Every countable model extends uniquely to a model of cardinality $\aleph_1$. It is consistent with ZFC that every countable model extends uniquely to a model of cardinality continuum.

The rest of paper is devoted to the proof of these claims; see §6, Theorem 6.0.4.7 and Theorem 6.0.4.8.

We also formulate a conjecture; see [3, §IV.6-§IV.7] or a forthcoming paper to clarify its relationship to a categoricity conjecture of [22].
Conjecture 2.2.0.6. For generic complex Abelian varieties $\mathbf{A}$ defined over a number field, an analogously defined $L_{\omega_1\omega_1}$-class $\mathcal{X}(\mathbf{A}(\mathbb{C}) \times \mathbb{C}^*)$ is analytic Zariski, excellent and categorical in uncountable cardinalities. A sufficient condition is that the Mumford-Tate group of $\mathbf{A}$ is the symplectic group, i.e. the largest possible.

2.3. Reduction to unfurled subsets: equivalence of the definitions

In this section we prove that the definitions 2.1.3.1$(I)$ and 2.1.3.1$(C)$ of the collection $\mathcal{S}$ do agree. It is the main prerequisite to prove that $\mathcal{S}$ is a topology.

2.3.1. Prerequisites on analytic irreducible decomposition and coverings in algebraic geometry

Irreducible Decomposition in smooth analytic spaces. To avoid confusion, below we say “an open ball” to mean a neighbourhood open in complex topology, not in the analytic Zariski topology.

Fact 2.3.1.1. Let $U$ be a smooth complex analytic space, and let $Y, Z \subset U$ be closed analytic subsets in $U$. Then

1. (irreducible decomposition) $Z$ admits a unique decomposition $Z = \bigcup_{i \in \mathbb{N}} Z_i$ into a countable union of analytic irreducible closed subsets $Z_i$’s.

2. (analyticity is a local property) a set $X \subset U$ is analytic iff for all $x \in X$, there exists an open ball $x \in B_x$ such that $X \cap B_x$ is an analytic subset of $B_x$.

3. (local identity principle) for an open ball $B \subset U$, if $Y$ is irreducible and $Y \cap B \subset Z \cap B$ then $Y \subset Z$.

4. (local identity principle; analytic continuation) for an open ball $B \subset U$, if $Y$ and $Z$ are irreducible and $Y \cap B$ and $Z \cap B$ have a common irreducible component, then $Y = Z$.

5. (density of smooth points) for an open ball $B \subset U$, if $Z_0 \subset Z \cap B$ is an irreducible component of $Z \cap B$, then there exist a point $z_0 \in Z_0$ and an open ball $z_0 \in B_0 \subset B$ such that $B_0 \cap Z \subset Z_0$.

6. (local finiteness) a compact set $C \subset U$ intersects only finitely many irreducible components of a closed analytic set $Z$.

7. (analyticity of a union of irreducible components) a union of, possibly infinitely many, irreducible components of an analytic set is analytic.

8. (irreducible decomposition) if $Y \subset Z$ and $Y$ is irreducible, then $Y$ is contained in an irreducible component of $Z$.

9. (smooth points of irreducible sets) the set of smooth points of an irreducible set is connected; consequently, the irreducible decomposition $Z = \bigcup_i Z_i$ of a closed analytic set $Z$ is determined by the decomposition $Z^{\text{sm}} = \bigcup_i (Z^{\text{sm}} \cap Z_i)$ into connected components of the set of its smooth points.

Proof. Those are well-known properties of smooth complex analytic spaces.

(1) is by [17, §5.4,Theorem, p.49]. By Prop. 5.3 of [17], Theorem 5.1 [ibid.] states (7) and (6). Corollary 2 of Prop. 5.3 [ibid.] implies (3) and (4). Theorem 5.4 [ibid.] implies (5). (2,3,4) together imply (8). (9) is by [17,§5.4,Theorem].
Finite topological coverings in algebraic geometry. We also need a form of Riemann existence theorem.

Fact 2.3.1.2 (Generalised Riemann existence theorem). Let $A(C)$ be a normal algebraic variety over $C$. If $q : T \to A(C)$ is a finite covering of topological spaces, then $T$ admits a structure of a complex algebraic variety such that $q_{\text{top}} : T \to A(C)$ becomes an algebraic morphism, i.e. there exists an algebraic variety $B(C)$ over $C$, an algebraic morphism $q_{\text{alg}} : B(C) \to A(C)$, and a homeomorphism $\varphi : T \to B(C)$ of topological spaces such that the diagramme of topological spaces commutes

$$
\begin{array}{ccc}
T & \xrightarrow{q_{\text{top}}} & A(C) \\
\varphi \downarrow & & \downarrow \text{id} \\
B(C) & \xrightarrow{q_{\text{alg}}} & A(C)
\end{array}
$$

Moreover, the homeomorphism $\varphi : T \to B(C)$ is well-defined up to an automorphism of $B$ commuting with the covering morphism $q_{\text{alg}}$.

Proof. Grothendieck [SGA1, Exp.XII, Th.5.1]; by a variety over $C$ we mean a Noetherian scheme of finite type over $C$. One may also look in [7, Appendix B, §3, Theorem 3.2] for some explanations.

\[\square\]

2.3.2. Reduction to unfurled subsets : the proof

For a subset $Z \subset U$, let $\Gamma Z = \bigcup_{\gamma \in \Gamma} \gamma Z'$ denote the $\Gamma$-orbit of set $Z$.

For $H \triangleleft \text{fin} \Gamma$, let $p_H : U \to U/\sim_H$ be the factorisation map since $A = U/\Gamma$; by Fact 2.3.1.2, we choose and fix isomorphisms $A^H(C) \cong U/\sim_H$ where $A^H(C)$ is an algebraic variety; the deck group of covering $A^H(C) \to A(C)$ is the finite group $\Gamma/H$.

Lemma 2.3.2.1 (First Decomposition lemma; Noetherian property; Reduction to Unfurled Subsets). Assume $A$ is LERF.

Every $\Gamma$-invariant analytic closed set has a decomposition as a finite union of unfurled closed analytic subsets invariant under the action of a finite index subgroup of $\Gamma$.

In other words, a $\Gamma$-invariant analytic closed set has an analytic decomposition of the form

$$W' = HZ'_1 \cup \ldots \cup HZ'_k,$$

where $H \triangleleft \text{fin} \Gamma$ is a finite index normal subgroup of $\Gamma$, the analytic closed sets $Z'_1, \ldots, Z'_k$ are irreducible, and for any $\tau \in H$ either $\tau Z'_i = Z'_i$ or $\tau Z'_i \cap Z'_i = \emptyset$.

Such decomposition also exists for closed analytic sets invariant under the action of a finite index subgroup of $\Gamma$.

Proof. Let us prove that (a) there exists a decomposition as above without the condition on intersections, and then prove (b) the irreducible components satisfy $\tau Z'_i = Z'_i$ or $\tau Z'_i \cap Z'_i = \emptyset$ for $\tau \in \Gamma$.

The proof of (a) is relatively simple, and follows from the Fact 2.3.1.1 in a rather straightforward way; we do it first.

The proof of the second claim (b) uses rather more delicate local analysis of the structure, and several local-to-global properties of analytic subsets of smooth complex analytic spaces as well as some finiteness properties of Zariski geometry of algebraic varieties.

So let us start to prove (a). Let $Z'$ be an irreducible component of $p^{-1}(Z(C))$; by $\Gamma$-invariance of $p^{-1}(Z(C))$, for any $\gamma \in \Gamma$, the set $\gamma Z'$ is also an irreducible component of $p^{-1}(Z(C))$, and so $\Gamma Z'$ is a union of irreducible components of $p^{-1}Z(C)$; thus, by Fact 2.3.1.1 above, $\Gamma Z' \subset p^{-1}(Z(C))$ is analytic.
The covering morphism $p : U \to A(\mathbb{C})$ is a local isomorphism, and analyticity is a local property; by $\Gamma$-invariance of $\Gamma Z'$, it implies $p(\Gamma Z')$ is analytic. For different irreducible components $Z'_1 \neq Z'_2$ of $p^{-1}(Z(\mathbb{C}))$ it can not hold that $p(Z'_1) \subseteq p(Z'_2)$; indeed, then $\Gamma Z'_1 = p^{-1}p(Z'_1) \subset \Gamma Z'_2 = p^{-1}p(Z'_2)$, and so $Z'_1 = \bigcup (Z'_1 \cap Z'_2), \gamma \in \Gamma$; thus, $Z'_1$ can not be irreducible unless $Z'_1 \subset \gamma Z'_2$, for some $\gamma \in \Gamma$. To conclude, closed sets $p(Z')$, $Z'$ vary among irreducible components of an algebraic subvariety $Z(\mathbb{C})$, cover the whole of $Z(\mathbb{C})$; they are also irreducible. Thus they are the analytic irreducible components of $Z$. The analytic irreducible components of an algebraic set are algebraic and irreducible by [6], and thus they are the algebraic irreducible components; in particular there are only finitely many of them. That gives the required decomposition.

Now let us start to prove $(b)$. First of all, note that we may suppose $Z$ to be irreducible.

Let $Z^{(n)} = \bigcup Z'_1 \cap \ldots \cap Z'_n$ be the union of all intersections of $n$-tuples of different irreducible components of $p^{-1}(Z(\mathbb{C}))$.

**Claim 2.3.2.2.** The set $p(Z^{(n)})$ is an algebraic subset of $Z(\mathbb{C})$, for $n > 0$. For $n$ sufficiently large, $Z^{(n)}$ is empty.

**Proof.** By the local finiteness (Fact 2.3.1.1) a compact subset intersects only finitely many of the irreducible components $\gamma Z'_i$'s; thus $Z^{(n)}$ is locally a finite union of intersections of analytic sets, and therefore is analytic. By the $\Gamma$-invariance of $\gamma Z'_i$'s it is $\Gamma$-invariant, and thus $p$ provides a local isomorphism of $Z^{(n)}$ and its image; therefore the image $p(Z^{(n)})$ is analytic. By Chow Lemma this implies it is in fact algebraic. If $n$ is greater then the number of local irreducible components at a point of $Z$ in $A$, then by Fact 2.3.1.1(local identity principle) $Z^{(n)}$ has to be empty.

The claim above implies $Z^{(n)}$ are co-etàle closed, for any $n$. By Claim $(a)$ of Lemma, we may choose finitely many points $z'_i$'s so that any irreducible component of $Z^{(n)}$, for each $n > 0$, contains a $\Gamma$-translate of one of $z'_i$'s.

By Fact 2.3.1.1(5) every point $z'_i$ is contained in only finitely many irreducible components of $p^{-1}(Z(\mathbb{C}))$. Let $Z'_1, \ldots, Z'_k$ be all the irreducible components of $p^{-1}(Z(\mathbb{C}))$ containing at least one of the points $z'_i$.

For a subset $V \subset U^n$, define the deck transformation group of $V$ as $\pi(V) = \{ \gamma \in U^n : \gamma V \subset V \}$. If $V$ is a connected component of $\Gamma$-invariant set $p^{-1}(p(V))$, then $\pi(V)$ is canonically identified with the fundamental group $\pi_1 \pi(V, x_0)$: to an element $\gamma \in \pi(V)$ there corresponds path $p(\gamma x_0, \gamma x_0)$ where $x_0 \in V$ is arbitrary such that $p(x'_0) = x_0$.

Notice that $\pi\left(Z_i\right) = \pi\left(Z_i \cap (\Gamma Z'_i)^{sm}\right)$ where $(\Gamma Z'_i)^{sm}$ is the set of smooth points of $\Gamma Z'_i$, and that by Fact 2.3.1.1(9) the set $Z'_i \cap (\Gamma Z'_i)^{sm}$ is a connected component of $\pi(Z'_i)^{sm}$. By the topological argument above, $\pi(Z'_i)$ is the fundamental group of a constructible algebraic set $p(Z'_i)^{sm}$. As a constructible algebraic set, it admits a finite triangulation into simplices, e.g. by o-minimal cell decomposition, and this implies that its fundamental group is finitely presented. In particular, it is finitely generated and we may apply subgroup separability of $\Gamma$ to find a normal finite index subgroup $H \subset \Gamma$ such that $HZ'_i \neq HZ'_j$ for $i \neq j$, i.e. $p_H(Z'_i) \neq p_H(Z'_j)$.

Consider $Z'_i \cap hZ'_i, h \in H$ and assume $\emptyset \subseteq Z'_i \cap hZ'_i \subseteq Z'_i$. Then there exists $\gamma^{-1} \in \Gamma$ such that $\gamma^{-1}z'_j \in Z'_i \cap hZ'_i$, i.e. $z'_j \in \gamma Z'_i \cap \gamma hZ'_i = \gamma Z'_i \cap h'\gamma Z'_i$. Both $\gamma Z'_i$ and $h'\gamma Z'_i$ are connected components containing $z'_j$ and by definition we have chosen $H$ small enough so that $H\gamma Z'_i \neq HH'\gamma Z'_i$, a contradiction.

In other words, we have proven that there exists a normal finite index subgroup $H < \pi(A(\mathbb{C}))$ such that $Z'_i$ is a connected component of $p_H^{-1} p_H (Z'_i)$, i.e. the connected components of the preimages of the irreducible components of $p_H^{-1}(Z(\mathbb{C}))$ are irreducible. 

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2.3.3. **Equivalence of the two definitions of co-etale topology**

The next corollary shows equivalence of the two definitions of co-etale topology.

Notice that the notion of an $H$-invariant set is essentially algebraic: an $H$-invariant set is a preimage of a closed algebraic subset in the finite cover $A^H(C)$. Thus, the meaning of the next corollary that in fact co-etale closed sets encode a mix of algebraic data and topological, homotopical data, not of analytic one.

**Corollary 2.3.3.1.** Definitions 2.1.3.1(I) and 2.1.3.1(C) are equivalent. In particular, an irreducible co-etale closed set is a connected component of a unfurled closed analytic set invariant under action of a finite index subgroup of the fundamental group.

**Proof.** Lemma 2.3.2.1 above implies that each co-etale irreducible closed set according to 2.1.3.1(I) is also closed according to 2.1.3.1(C), i.e. is a connected component of a a unfurled closed analytic set invariant under action of a finite index subgroup of the fundamental group.

On the other hand, the lemma implies that each $H$-invariant set is a finite union of sets of the form $HZ'_i$ where $Z'_i$ are irreducible. Then, $\Gamma Z'_i$ is also closed analytic as a finite union of translates of $HZ'_i$, and moreover, each translate of $Z'_i$ is an irreducible component of $\Gamma Z'_i$ and thus co-etale closed. This implies every (C)-closed set is also (I)-closed.

An algebraic reformulation. The Lemma has the following algebraic consequence. All the notions mentioned in the Corollary are preserved under replacing the ground field by another algebraically field; thus it holds for any characteristic 0 algebraically closed field instead of $C$. One may think of this property as a rather weak property of irreducible decomposition for the co-etale topology; it is also a statement about a resolution of non-normal singularities.

**Corollary 2.3.3.2.** Let $A$ be LERF. Then for any closed subvariety $Z \subset A(C)$, there exists a finite étale cover $q : A^H(C) \to A(C)$ such that, for any further étale cover $q' : A^G(C) \to A^H(C)$, the connected components of $q^{-1}(Z_i) \subset A^G(C)$ are irreducible, where $Z_i$’s are the irreducible components of $q^{-1}(Z)$.

**Proof.** Indeed, it is enough to take $H$ as in Decomposition Lemma.

Note that when $Z$ is normal, the corollary is a well-known geometric fact.

2.4. Co-etale topology is a topology.

**Lemma 2.4.0.3.** (a) The collection $\mathcal{S}$ of subsets of $U^n$ forms a topology, for every $n$. (a') Moreover, the collection $\mathcal{S}$ satisfies Axioms (L1)-(L8) of [19]. (b) An $\mathcal{S}$-irreducible $\mathcal{S}$-closed set is analytically irreducible closed set. (c) An analytically irreducible component of a $\mathcal{S}$-closed set is $\mathcal{S}$-closed $\mathcal{S}$-irreducible.

**Proof.** (b) By Definition 2.1.3.1(I), a co-etale irreducible co-etale closed set $W'$ is a countable union of irreducible component of $\Gamma$-invariant closed analytic sets. Those components are co-etale closed by definition, and thus co-etale irreducibility implies the union is necessarily trivial. Thus, the set is an analytic irreducible component of a $\Gamma$-invariant set, i.e. in particular irreducible as an analytic set.

(c) is immediate by Definition 2.1.3.1(I).

(a) As $\mathcal{S}$ consists only of closed analytic sets, an analytic irreducible component of a finite union of $\mathcal{S}$-closed sets is an analytic irreducible component of one of them; this shows that $\mathcal{S}$ is closed under finite union. To prove $\mathcal{S}$ is closed under infinite intersection, we first observe that an irreducible component of an infinite intersection (that is still a closed analytic set) is necessarily the intersection of irreducible closed analytic components of these sets; by the descending chain condition for analytic irreducible closed sets, the intersection is necessarily finite. Thus, by Definition 2.1.3.1(I) it is enough.
to show that each irreducible component of the intersection of irreducible \( \mathcal{S} \)-closed sets is \( \mathcal{S} \)-closed (irreducible).

Thus, it is enough to prove that the intersection of two irreducible \( \mathcal{S} \)-closed sets, say \( X \) and \( Y \), is \( \mathcal{S} \)-closed. Now, by Definition 2.1.3.1(C), \( X \) and \( Y \) are connected components of closed analytic sets \( X' \) and \( Y' \) invariant under action of finite index subgroups, say \( H \) and \( G \), of the fundamental group. Then, \( X \cap Y' \) is a connected component of the intersection \( X' \cap Y' \) that is invariant under action of \( H \cap G \), the latter also being a finite index subgroup. By Definition 2.1.3.1(C), this implies that \( X \cap Y \) is \( \mathcal{S} \)-closed. This proves (a); note the interplay between (I) and (C) of Definition 2.1.3.1.

(a') We have just proven (L1); axioms (L2-L7) are immediate by inspection of any of the definitions. (L8) requires 2.1.3.1(C): a hyperplane section of a connected component of a closed analytic set invariant under action of a finite index subgroup is a connected component of the intersection that is also invariant under a finite index subgroup. (This argument does not work for irreducible components, as they may intersect).

\[\square\]

2.5. Good dimension notion : (DP), (DU), (SI), (AF)

The following properties are defined in [19, §3.1]. Following notation there, \( S \subseteq cl \) \( S' \) reads \( S \) is a closed subset of \( S' \), \( S \subseteq an \) \( S' \) reads \( S \) is an analytic subset of \( S' \), and \( S \subseteq op \) \( S' \) reads \( S \) is an open subset of \( S' \).

**Lemma 2.5.0.4** (Good dimension). (DP) Dimension of a point is 0

(DU) Dimension of unions: \( \dim(S_1 \cup S_2) = \max(\dim S_1, \dim S_2) \)

(SI) Strong irreducibility: For \( S \subseteq cl V \subseteq op \mathbb{U}^n \), \( \dim S_1 < \dim S \), if \( S \) is irreducible and \( S_1 \subseteq cl S \) is closed, then \( S_1 = S \)

(AF) Addition formula: For any irreducible \( S \subseteq cl V \subseteq op \mathbb{U}^n \) and a projection map \( \pi : \mathbb{U}^n \rightarrow \mathbb{U}^m \)

\[ \dim S = \dim \pi(S) + \min_{a \in \pi(S)} \dim \pi^{-1}(a) \cap S \]

(PS) Presmoothness: For any closed irreducible \( S_1, S_2 \subseteq \mathbb{U}^n \), the dimension of any irreducible component of \( S_1 \cap S_2 \) is not less than

\[ \dim S_1 + \dim S_2 - \dim \mathbb{U}^n \]

**Proof.** These are inherited from complex analytic geometry.

\[\square\]

2.6. Analyticity (AS), (SI),(DP),(CU), (INT),(CMP), (CC)

Recall that [19, §6.1.2] distinguishes a class of sets in a topology that he calls ‘analytic’. Namely, in a topology \( T \) a locally closed set \( S \) is called analytic in an open set \( U \) iff \( S \) is a closed subset of \( U \) and for every \( a \in S \) there is an open \( a \in V_a \subseteq U \) such that \( S \cap V_a \) is the union of finitely many relatively closed irreducible subsets. Note that by Fact 2.3.1.1(6,7), a locally closed analytic set is analytic in this sense: take \( V_a \) to be the complement of the union of the irreducible components of \( S \) not containing \( a \). This argument also works for co-etale topology, i.e., in co-etale topology, each locally closed set is analytic in this sense.

Next Lemma establishes (INT), (CMP),(CC) and (AS) of [loc.cit., §6.1], and therefore, that \( \mathbb{U} \) is a topological structure with a good dimension theory [loc.cit.,Def.6.1.1].

**Lemma 2.6.0.5** (Analytic sets). (INT) (Intersections) If \( S_1, S_2 \subseteq an \mathbb{U}^n \) are irreducible and analytic in \( \mathbb{U}^n \), then \( S_1 \cap S_2 \) is analytic in \( \mathbb{U}^n \)

(CMP) (Components) If \( S \subseteq an \mathbb{U}^n \) and \( a \in S \) then there is \( S_a \subseteq an \mathbb{U}^n \), a finite union of irreducible analytic subsets of \( \mathbb{U}^n \), and some \( S_a' \subseteq an \mathbb{U}^n \) such that \( a \in S_a \setminus S_a' \) and \( S = S_a \cup S_a' \)

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(CC) (Countability of the number of components) Any \( S \subseteq \mathbb{U}^n \) is a union of at most countably many irreducible components.

(AS) [Analytic stratification] Every locally closed set is analytic.

(aPS) [Analytic Presmoothness] If \( S_1, S_2 \subseteq \mathbb{U}^n \) and both \( S_1, S_2 \) are irreducible, then for any irreducible component \( S_0 \) of \( S_1 \cap S_2 \)
\[
\dim S_0 \geq \dim S_1 + \dim S_2 - \dim \mathbb{U}^n.
\]

Proof. Immediate by Fact 2.3.1.1. \( \square \)

2.7. \( \Theta \)-definable sets, \( \Theta \)-generic points and \( \Theta \)-definable closure

Recall that \( U/\Gamma \cong \mathbf{A}(\mathbb{C}) \) has the structure of an algebraic variety over \( \mathbb{C} \) and that the \( \Gamma \)-invariant sets are in a bijective correspondence with the algebraic subvarieties of \( \mathbf{A}(\mathbb{C}) \). This suggests us that we may try to pull back to \( U \) the notion of a generic point in \( \mathbf{A}(\mathbb{C}) \).

The following definition behaves well only for \( \Theta \subseteq \mathbb{C} \) algebraically closed.

Definition 2.7.0.6. We say that a \( \Gamma \)-invariant co-etale closed subset \( W' \subset U \) is defined over an algebraically closed subfield \( \Theta \subset \mathbb{C} \) if \( p(W') \subset \mathbf{A}(\Theta) \) is a subvariety defined over \( \Theta \).

An co-etale closed set is defined over a subfield \( \Theta \subset \mathbb{C} \) iff it is a countable union of irreducible components of \( \Gamma \)-invariant co-etale closed subsets defined over \( \Theta \).

Definition 2.7.0.7. For a set \( V \subseteq \mathbb{U}^n \), let \( \text{Cl}_\Theta V \) be the intersection of all closed \( \Theta \)-definable sets containing \( V \):
\[
\text{Cl}_\Theta(V) = \bigcap_{V \subseteq W, W/\Theta \text{ is } \Theta \text{-definable closed}} W
\]
A point \( v \in V \) is called \( \Theta \)-generic iff \( V = \text{Cl}_\Theta(v) \), i.e. there does not exist a closed \( \Theta \)-definable proper subset of \( V \) containing \( v \).

Lemma 2.7.0.8. (a) \( \text{Cl}_\Theta(V) \) is \( \Theta \)-definable
(b) \( \text{Cl}_\Theta(V) = \bigcup_{v \in V} \text{Cl}_\Theta(v) = \bigcup_{S \subseteq V} \text{Cl}_\Theta(S) \) (union over all finite subsets)

Proof. (a) : By Decomposition Lemma, it is sufficient to consider only irreducible \( V \). However, for irreducible \( V \) we may assume that all sets appearing in the definition of \( \text{Cl}_\Theta(V) \) are again irreducible and therefore the intersection is finite. It is immediate that a finite intersection of \( \Theta \)-definable sets is \( \Theta \)-definable.

(b) : This follows from the Decomposition Lemma. If \( V \) is irreducible, then \( V = \text{Cl}_\Theta(v) \) for \( v \) a \( \Theta \)-generic point of \( V \). If not, by Decomposition Lemma, \( V \) decomposes as a union of translates of irreducible sets \( V_1, \ldots, V_n \). Thus the union \( \bigcup_{v \in V} \text{Cl}_\Theta(v) \) is the union of the corresponding translates of the closures \( \text{Cl}_\Theta(V_1), \ldots, \text{Cl}_\Theta(V_n) \) of the irreducible components \( V_1, \ldots, V_n \). By Lemma 2.4.0.3, \( \text{Cl}_\Theta(V) \) being closed implies any union of translates of \( \text{Cl}_\Theta(V_i) \) is closed; and thus \( \bigcup_{v \in V} \text{Cl}_\Theta(v) \) is a finite union of closed sets, therefore closed itself. But obviously \( V \subset \bigcup_{v \in V} \text{Cl}_\Theta(v) \) and therefore \( \text{Cl}_\Theta(V) \subset \bigcup_{v \in V} \text{Cl}_\Theta(v) \). On the other hand, for any \( v \in V \) \( \text{Cl}_\Theta(v) \subset \text{Cl}_\Theta(V) \) and thus \( \text{Cl}_\Theta(V) \supset \bigcup_{v \in V} \text{Cl}_\Theta(v) \). This implies the lemma. \( \square \)

Lemma 2.7.0.9. If a set \( W' \subset U \) is defined over \( \overline{\mathbb{Q}} \subset \mathbb{C} \) then \( W' \subset U \) is \( L_{\mathbb{A}} \)-defined with parameters from \( p^{-1}(\mathbf{A}(\overline{\mathbb{Q}})) \).

Proof. An irreducible component of the preimage of an algebraic variety \( W(\mathbb{C}) \subset \mathbf{A}(\mathbb{C}) \) defined over \( \overline{\mathbb{Q}} \) is an irreducible component of the preimage of the variety
\[
\bigcup_{\sigma \in \text{Gal}(\overline{\mathbb{Q}}/k)} \sigma W(\mathbb{C})
\]
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defined over \( k \). In order for the union to be finite, we use that \( W \) is defined over \( \overline{\mathbb{Q}} \), i.e., over a finite degree subfield of \( \overline{\mathbb{Q}} \). The relation \( \sim_W \) is in \( L_A(A) \), and \( W' \) can be defined by \( x \sim_W a_1 \& \ldots \& x \sim_W a_k \), for some set of \( \overline{\mathbb{Q}} \)-rational points \( a_1, \ldots, a_k \in W'(\overline{\mathbb{Q}}) \).

Recall we assume \( \Theta \) to be algebraically closed.

**Lemma 2.7.0.10.** For every finite index subgroup \( H \lhd_{\text{fin}} \Gamma \), if \( W' \) is irreducible co-etale closed, then \( w' \in W' \) is \( \Theta \)-generic iff \( w = p_H(w') \in W = p_H(W') \) is \( \Theta \)-generic in \( W \).

**Proof.** The point \( w' \in W' \) is not \( \Theta \)-generic iff there exists a \( \Theta \)-defined irreducible set \( w' \in V' \subset W' \); necessarily \( \dim V' < \dim W' \) and \( p_H(V') \neq p_H(W') \). \( \Box \)

We would rather avoid using this corollary due to its non-geometric character, but unfortunately we do use it.

**Lemma 2.7.0.11.** A connected component of a non-empty \( \Theta \)-generic fibre of a co-etale closed irreducible set defined over \( \Theta \) contains a \( \Theta \)-generic point. That is, if \( W' \subset U \times U \) is co-etale irreducible and \( \text{pr} : W' \to U \) is the projection, and \( g' \in \text{Clpr} W' \) is a \( \Theta \)-generic point of the co-etale closed set \( V' = \text{Clpr} W' \), then the \( \Theta \)-generic fibre \( W_{g'}' = \text{pr}^{-1}(g') \) contains a \( \Theta \)-generic point of \( W' \).

**Proof.** Basic properties of generic points of algebraic varieties imply this property for algebraic varieties. Let \( W_{g''}' \) be a connected component of a fibre of \( W' \) over a \( \Theta \)-generic point \( g' \) of \( \text{Clpr} W' \). Then \( p(W_{g''}') \) is a connected component of the fibre \( W_g \), where \( W = p_H(W') \), \( g = p(g') \) is such that \( W' \) is a connected component of \( p_H^{-1}(W) \); this may be seen with the help of the path-lifting property, for example. Genericity of \( g' \in \text{Clpr} W' \) implies that the point \( g \in \text{Clpr} W \) is \( \Theta \)-generic, and, as a connected component of the fibre \( W_g \) of an algebraic variety, \( p(W_{g''}') \) contains a \( \Theta \)-generic point, and then its preimage in \( W_{g''}' \) is also \( \Theta \)-generic. \( \Box \)

**2.8. (WP) Weak properness : Stein factorisation and fundamental groups**

Above establishes that \( U \) satisfies all but those axioms of an analytic Zariski structure that describe the image of a projection — (SP),(WP) and (FC). To prove these these axioms, we use that in algebraic geometry, all morphisms are topologically very simple: each morphism of complex smooth connected algebraic varieties is, excepting a closed subset of smaller dimension, a topological fibre bundle with connected fibres, followed by a finite topological covering (i.e., a fibre bundle with finite fibres). This is known as Stein factorisation. Via the long exact sequence of a fibration, this allows us to describe the behaviour of the fundamental group with respect to algebraic morphisms. We use this to prove (FC).

Let us give an idea behind the calculations. We need to exclude the counterexample of a finite non-closed spiral in \( \mathbb{C}^* \times \mathbb{C}^* \) projecting onto a circle in \( \mathbb{C}^* \). In the cover, the spiral \( S \) unwinds to a curve \( S' \) of finite length while the circle \( S^1 \) unwinds to an infinite line \( L \). As countably many deck translates of \( \text{pr} S' \) cover the whole of the line \( L \), their dimension must be the same in an analytic Zariski structure. Observe that for the counterexample it is essential that the projection \( \text{pr} \pi(S) \longrightarrow \pi(S') \) is not surjective, a possibility excluded by Proposition 2.8.3.2.

Let us remark that although the circle is not definable for obvious reasons, the variety \( \mathbb{C}^* \) is definable and homotopic to the circle, and so considerations above imply that we need to show there is no irreducible co-etale closed subset of \( \mathbb{C}^n \) with finite deck transformation group projecting surjectively onto \( \mathbb{C}^* \).

**2.8.1. Prerequisites: topological structure of algebraic morphisms**

*Exact sequence of fundamental groups of a fibration.*

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Fact 2.8.1.1. For a fibration $f : A \to B$ of (nice, e.g. Hausdorff, linearly connected, locally linearly connected) topological spaces, a pair of points $a \in A, b = f(a) \in B$, we have an exact sequence of homotopy groups

$$\pi_2(B, b) \to \pi_1(f^{-1}(b), a) \to \pi_1(A, a) \to \pi_0(B, b) \to \pi_0(A, a) \to \pi_0(f^{-1}(b), a) \to \pi_0(A, a) \to 0$$

Remark 2.8.1.2. In fact, fibrations are thought of as analogues of exact sequences of Abelian groups in 'the non-Abelian context' of topological spaces.

Normal closed analytic sets.

Definition 2.8.1.3. ([2, §7.2,Def.7.4]) A closed analytic subset $X$ is normal at a point $x \in X$ if the ring $O_{X,x}$ of germs of holomorphic functions over neighbourhoods of $x$ is integrally closed in its field of fractions. A closed analytic subset is normal iff it is normal at every point.

A normalisation morphism $n$ of variety $Y$ is a morphism $n : X \to Y$ from a normal variety $X$ such that any dominant, i.e. surjective on a Zariski open subset, morphism $f : Z \to Y$ lifts up to a unique morphism $\tilde{f} : Z \to X$ such that $f = \tilde{f} \circ n$.

Any smooth closed analytic set is normal ([2, §7.4]).

We only use the following two properties of a normal variety:

Fact 2.8.1.4. A normalisation morphism exists for any variety, and is functorial. Namely, for every variety $(Y, y), y \in Y$ with a base-point we may choose a normalisation morphism $n : (n(Y), n(y)) \to (Y, y)$ such that for every pair of morphisms $f : (X, x) \to (Y, y), g : (Y, y) \to (Z, z)$ it holds that $n(fg) = n(f)n(g)$.

Proof. Lemma §7.11 of [2] and Oka’s normalisation principle of [loc.cit.,§7.12].

Fact 2.8.1.5. Let $X$ be a closed analytic subset of a Stein manifold, or let $X$ be an algebraic variety. If $X$ is connected and normal, then $X$ is irreducible.

Proof. Implied by [2, §7.4].

Fundamental groups of open subsets of normal varieties.

Fact 2.8.1.6. Let $Y$ be a connected normal complex space and $Y^0 \subset Y$ be open. Then $\pi_1(Y^0(C), y_0) \to \pi_1(Y(C), y_0)$ is surjective, for every $y_0 \in Y^0(C)$.

Proof. Kollar, Prop.2.10.1

Stein factorisation.

Fact 2.8.1.7. Any projective morphism $f : Y \to X$ of algebraic varieties admits a factorisation $f = f_0 \circ f_1$ as a product of a finite morphism $f_0 : Y \to Y'$ and a morphism $f_1$ with connected fibres.

Proof. [7, Ch. III, Corollary 11.5]

A morphism of normal algebraic varieties is topologically a fibration on an Zariski open subset. For normal varieties we have a more precise statement:

Fact 2.8.1.8. Let $f : X \to Y$ be a morphism of irreducible normal algebraic complex varieties such that $Y \subset f(X)$.

Then there exist an open subset $Y^0 \subset Y$ and $X^0 = f^{-1}(Y^0)$, and a variety $Z^0$ such that $f$ factories as follows:

$$X^0 \to f^0 : Z^0 \to f_{\text{et}} : Y^0$$

where

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1. $Z^0 \rightarrow Y^0$ is a finite topological covering in complex topology (i.e. an étale morphism).

2. $X^0 \rightarrow Z^0$ is a topological fibre bundle (in complex topology) with connected fibres.

In particular,

3. $f : X^0 \rightarrow Y^0$ is a fibration, and its fibres are of boundedly many connected components.

4. we have a short exact sequence

$$\rightarrow \pi_2(Y^0(\mathbb{C}), y_0) \rightarrow \pi_1(f^{-1}_0(Y^0(\mathbb{C})), x_0) \rightarrow \pi_1(X^0(\mathbb{C}), x_0) \rightarrow \pi_1(Y^0(\mathbb{C}), y_0) \rightarrow \pi_0(f^{-1}_0(Y^0(\mathbb{C})), x_0)$$

**Proof.** Kollar, Proposition 2.8.1.

Note that while $f^0 : X^0 \rightarrow Y^0$ is interpretable in the theory of algebraic varieties and in $L_A$, as indeed any morphism of algebraic varieties is, the theory may not say anything about the induced morphism $(f^0)_* : U(X^0) \rightarrow U(Y^0)$ of the universal covering spaces of $X^0(\mathbb{C})$ and $Y^0(\mathbb{C})$.

**Morphisms of fundamental groups of normal varieties.** The Fact 2.8.1.8 above leads to a fact about fundamental groups specific to algebraic geometry.

**Fact 2.8.1.9.** Let $f : X \rightarrow Y$ be a morphism of normal algebraic connected complex varieties; assume that $f(X)$ is open in $Y$.

Then there is an open subset $Y^0 \subset Y$ defined over the same field as $Y$, such that for every point $g \in Y^0(\mathbb{C}) \subset Y(\mathbb{C})$, every point $g' \in X_g = f^{-1}(g)$ a generic fibre of $f$ over generic point $g \in Y(\mathbb{C})$, it holds that the sequence

$$f_* : \pi_1(X_g(\mathbb{C}), g') \rightarrow \pi_1(X(\mathbb{C}), g') \rightarrow \pi_1(Y(\mathbb{C}), g) \rightarrow 0$$

is exact up to finite index.

**Proof.** Follows from Facts 2.8.1.6 and 2.8.1.8 and 2.8.1.1 (the exact sequence of the fundamental groups of a fibration). That is, Kollar, Proposition 2.8.1 and Kollar, Proposition 2.10.1.

2.8.2. **Extending to non-normal subvarieties**

The above provides an explicit description of morphisms topologically, between normal algebraic varieties.

However, we need to deal with an *arbitrary* subvarieties, not necessarily normal. We do so by considering the image of the fundamental groups in the big ambient variety that is normal.

**Fundamental subgroups of non-normal subvarieties.**

**Fact 2.8.2.1.** Assume $A$ is LERF.

Let $p : U \rightarrow A(\mathbb{C})$ be the universal covering space, let $\iota : W \rightarrow A \times A$ be a closed subvariety, and let $Z = C1pr W$. Assume that $p^{-1}(W(\mathbb{C}))$ and $p^{-1}(Z(\mathbb{C}))$ are unfurled.

Then there is an open subset $Z^0 \subset Z$ defined over the same field as $Z$, such that for every point $g \in Z^0(\mathbb{C}) \subset Z(\mathbb{C})$, every point $(g, g') \in W_g = f^{-1}(g)$ a generic fibre of $f$ over generic point $g \in Z(\mathbb{C})$, it holds that the sequence of subgroups of $\pi_1(A(\mathbb{C})^2, (p(g'), p(g)))$.

$$\iota_*(\pi_1(W_g(\mathbb{C}), (g, g'))) \rightarrow \iota_*(\pi_1(W(\mathbb{C}), (g, g'))) \rightarrow \iota_*(\pi_1(Z(\mathbb{C}), g)) \rightarrow 0$$

which is exact up to finite index, and the homomorphisms are those of subgroups of $\pi_1(A(\mathbb{C})^2, (p(g'), p(g)))$. 

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Proof. We prove this by passing to the normalisation of varieties $W$ and $Z = \text{Cl} \text{pr} W$. The assumption about the irreducibility of connected components implies that the composite maps of fundamental groups $\pi_1(\hat{W}) \rightarrow \pi(W) \rightarrow \iota_\ast \pi_1(W)$ and $\pi_1(\hat{Z}) \rightarrow \pi_1(Z) \rightarrow \iota_\ast \pi_1(Z)$ are surjective.

To show this, first note that the universal covering spaces $\hat{W}(\mathbb{C})$ and $\hat{Z}(\mathbb{C})$ are irreducible as analytic spaces; indeed, normality is a local property, and so they are normal as analytic spaces; they are obviously connected, and for normal analytic spaces connectivity implies irreducibility.

By properties of covering maps, a morphism between analytic spaces lifts up to a morphism between their universal covering spaces (as analytic spaces); thus the normalisation map $\bar{n}_W : \hat{W} \to W$. The normalisation morphism $\bar{n}_W$ is finite and closed by Hartshorne [7, Ch.II,§3,Ex.3.5,3.8]; therefore $\bar{n}_W$ is also, and the image of an irreducible set is irreducible. Therefore $\bar{n}_W(\hat{W})$ is an irreducible subset of a connected component of $p^{-1}(W(\mathbb{C}))$. Moreover, if we choose different liftings $n_W$, we may cover $p^{-1}(W(\mathbb{C}))$ by a countable number of such sets. Now, we use the assumption that a connected component of $p^{-1}(W(\mathbb{C}))$ is irreducible to conclude that the image $\bar{n}_W(\hat{W})$ coincides with a connected component of $p^{-1}(W(\mathbb{C}))$. This implies that the map of fundamental groups is surjective; this may be easily seen if one thinks of a fundamental group as the group of deck transformations.

Let $n_W : W \to W$, $n_{W_g} : \hat{W}_g \to W_g$ and $n_Z : \hat{Z} \to Z$ be the normalisation of varieties $W, W_g$ and $Z$.

By the universality property of normalisation in §2.8.1 we may lift the normalisation morphism $n_{W_g} : \hat{W}_g \to W_g$ to construct a commutative diagram:

$$
\begin{array}{ccc}
\hat{W}_g & \to & \hat{W} \\
\downarrow & & \downarrow \\
W_g & \to & W \\
\end{array}
$$

By functoriality of $\pi_1$, this diagram and embedding $\iota : W \to A \times A$ gives us

$$
\begin{array}{ccc}
\pi_1(W_g) & \to & \pi_1(\hat{W}) \\
\downarrow & & \downarrow \\
\pi_1(W_g) & \to & \pi_1(W) \\
\downarrow & & \downarrow \\
\iota_\ast \pi_1(W_g) & \to & \iota_\ast \pi_1(W) \\
\end{array}
$$

Now, $g'$ is $\Theta$-generic in $\hat{W}_g$; We are almost finished now. By Fact 2.8.1.9 the upper row of the diagram is exact up to finite index, and $\pi_1(\hat{W}) \to \pi_1(\hat{Z})$ are surjective, up to finite index; by assumptions on $W$ and $Z$, the composite morphisms $\pi_1(\hat{Z}) \to \iota_\ast \pi_1(Z)$ and $\pi_1(\hat{W}) \to \iota_\ast \pi_1(W)$ are surjective. Diagram chasing now proves that the bottom row is also exact up to finite index, and the map $\iota_\ast \pi_1(\hat{W}) \to \iota_\ast \pi_1(\hat{Z})$ is surjective up to finite index.

2.8.3. Deck transformation groups of co-etale irreducible sets

Recall notation $\pi(V') = \{\gamma \in \Gamma^n : \gamma V' \subset V'\}$ for $V' \subset U^n$, and that if $V'$ is a connected component of $p^{-1}(V(\mathbb{C}))$, then the deck transformation group $\pi(V')$ is canonically identified with the fundamental group $\pi_1(V(\mathbb{C}), x_0), x_0 \in p(V')$: to an element $\gamma \in \pi(V')$ there corresponds path $p(\gamma x_0, \gamma x_0)$ where $x_0 \in V'$ is arbitrary such that $p(x_0') = x_0$.

Deck transformation group of a co-etale irreducible set is cocompact.

Corollary 2.8.3.1. In a co-etale irreducible set $W$, the deck transformation group $\pi(W)$ acts cocompactly, i.e. transitive up-to-compact.
That is, for every co-etale irreducible closed set \( W \) there is a compact subset \( W_0 \subset W \) such that every point \( w \in W \) there are \( \gamma w_0, \gamma \in \pi(W) \), \( w_0 \in W_0 \) and \( w = \gamma w_0 \).

**Proof.** By Decomposition Lemma, \( W \) is a connected component of \( HW = p_H^{-1} \pi H(W) \), for some finite index subgroup \( H \subset \Gamma \). As \( p_H \) is a local isomorphism, \( HW \) being closed analytic implies \( p_H(W) = p_H(HW) \subset U^n/\Gamma^n \) is closed analytic and therefore Zariski closed by Chow Lemma. This implies that \( W \) is a topological covering of a closed set in complex topology, and \( \pi(W) \cap H \) is its deck transformation group. This implies the corollary.

**Deck transformation group of the projection of an irreducible co-etale closed set.**

**Proposition 2.8.3.2** (Action of \( \pi(U) \) on \( U \)). Let \( W' \) and \( V' = C\text{pr} W' \) be co-etale irreducible closed sets. Then there is a finite index subgroup \( H <_{\text{fin}} \Gamma \) such that

1. \( \pi(W') \cap H = \{ \gamma \in H : \gamma W' \subset W' \} = \{ \gamma \in H : \gamma W' \cap W' \neq \emptyset \} = \{ \gamma \in H : \gamma x' \in W' \} \),
   for any point \( x' \in W' \)
2. \( \text{pr} [\pi(W') \cap H] = \pi(V') \cap H \).
3. for an open subset \( V' \subset V' \) it holds that for arbitrary connected component \( W_{g}' \) of fibre \( W_{g}' \)
   over \( g' \in V' \) there is a sequence exact up to finite index
   \[
   \pi(W_{g}') \longrightarrow \pi(W') \xrightarrow{\text{pr}} \pi(V') \longrightarrow 0,
   \]
   i.e. there exists a finite index subgroup \( H <_{\text{fin}} \Gamma \) independent of \( g \) and \( W_{g}' \) such that the sequence
   is exact:
   \[
   \pi(W_{g}^{'}) \cap H \longrightarrow \pi(W') \cap [H \times H] \xrightarrow{\text{pr}} \pi(V') \cap H \longrightarrow 0,
   \]
   Moreover, if \( W' \) and \( V' \) are defined over an algebraically closed field \( \Theta \), so is \( V - V' \). In particular, the above sequence is exact for \( g \) a \( \Theta \)-generic point of \( V' = C\text{pr} W' \).

**Proof of Proposition.** To prove (1), apply Decomposition Lemma to the co-etale closed set \( \Gamma W' \); by Decomposition Lemma, take \( H <_{\text{fin}} \Gamma \) to be such that the set \( \Gamma W' \) decomposes as a union of a finite number of \( \Gamma \)-invariant sets whose connected components are irreducible, and therefore they are translates of \( W' \). This implies (1). The item (2) is implied by (3).

Let us now prove item (3). Let \( H \) be such that \( W' \) and \( V' \) are connected components of \( p_H^{-1} \Gamma W(C) \), \( p_H^{-1}(V(C)) \), respectively, where \( W(C) = p_H(W') \), \( V(C) = p_H(V') \). Consider projection morphism \( \text{pr} : A \times A \to A \); it induces a morphism \( \text{pr} : W(C) \to V(C) \). By Lemma 2.8.2.1 it gives rise to a sequence exact up to finite index:

\[
\iota_* \pi_1(W_g^C(C), w) \to \iota_* \pi_1(W(C), w) \to \iota_* \pi_1(V(C), \text{pr} w) \to 0
\]

where \( W_g^C \) is a connected component of a fibre of \( W \) over \( g \in V \), and \( g \) varies in an open subset \( V^0 \) of \( V \), and \( w \) varies in \( W_g^C \). The index depends only on the Stein factorisation of the projection, and is therefore independent of \( g \) and fibre \( W_g^C \).

Recall that there is a canonical identification of \( \pi(W') \) and \( \iota_* \pi_1(W(C), w) \), and of \( \iota_* \pi_1(W_g^C(C), w) \) and \( \pi(W_g^C) \), etc. As a canonical identification respects morphisms, Proposition is implied.

**Corollary 2.8.3.3.** Let \( W' \) be a co-etale irreducible closed set, and let \( V' = C\text{pr} W' \). Then \( \pi(\text{pr} W') \) is a finite index subgroup of \( \pi(V') \).

**Proof.** By item (3) of Lemma 2.8.3.2.
Corollary 2.8.4.1 (Chevalley Lemma). For the co-etale topology, it holds:

(SP) Projections of closed irreducible sets are irreducible closed.

(SP)\text{alg} Projections of closed sets invariant under a finite index subgroup of the fundamental group, are closed.

(SP)\text{gen} Projection of an irreducible constructible set contains all generic points of the projection.

(WP) The projection of an irreducible set open in its closure contains an open subset of the closure of the projection.

Proof. It is easy to check that the projection of an $H$-invariant closed set is closed; indeed, say for $H = \Gamma$, note $\text{pr} (\Gamma W') = \text{pr} (W')$, and thus $\text{pr} \Gamma W' = \text{pr} (\Gamma (W')) = \text{pr} (\Gamma W) = \text{pr} W'$, where $V = \text{pr} W$. As $\mathcal{A}(\mathbb{C})$ is projective, $V$ is a closed algebraic subset of $\mathcal{A}(\mathbb{C})$, and thus $\text{pr} (\Gamma W)$ is a $\Gamma$-invariant closed subset of $U$. By definition of $\text{Et}$, it is co-etale closed. This proves $(SP)\text{gen}$.

To prove (SP), let $W'$ be a co-etale irreducible closed set which is a connected component of $HW'$. Let $V'$ be the closure of $\text{pr} W'$; we intend to apply item (3) of Proposition above.

The set $\text{pr} HW'$ is closed, and thus $V' \subset \text{pr} HW'$. The set $V'$ is closed, and thus it is contained in a connected component $V'_1$ of $\text{pr} HW'$. Take $v' \in V'_1 \subset V'_1$, and find $w' \in W'$ such that $\text{pr} (hw') = v'$; this is possible due to $V' \subset \text{pr} HW'$. Also $\text{pr} W' \subset V'$, and thus $\text{pr} (w') \in V'$, $\text{pr} (h) \text{pr} (w') = v' \in V'$. Then $v' \in \text{pr} (h) V'_1 \cap V'_1$.

We may further take $H$ sufficiently small so that $\pi(V'_1) \cap H = \{ \tau \in \Gamma : \tau (V'_1) \cap V'_1 \neq \emptyset \} = \{ \tau \in \Gamma : \tau V'_1 = V'_1 \}$. Then $\text{pr} (h) \in \pi(V'_1)$, and Proposition 2.8.3.2(2) implies there exists an element $h_1 \in \Gamma (W') \cap [H \times H]$ such that $\text{pr} (h) = \text{pr} h_1$. Then, $h_1 W' = W'$, and thus $\text{pr} (h_1 w') = \text{pr} (h) \text{pr} w' = v'$, as required.

This argument can be given topologically. We reprove $(SP)\text{alg}$ topologically.

First, we may assume that $W'$ is a connected component of $p_H^1 p_H(W') = HW'$, and by Chevalley Lemma for algebraic varieties there is a set $V_0 \subset \text{pr} p_H(W') \subset V$ such that $V_0 \subset V$ is open in $V$. Let $V'$ be the connected component of $p_H^1 (V)$ containing $\text{pr} W'$. Take $V' \subset V' \cap p_H^1 (V_0)$; then $V' \subset V'$ is open in $V'$ as an intersection with an open set.

Take $v' \in V' \cap V_0$, and take $w' \in W'$, $\text{pr} p_H (w') = \text{pr} (v') \in V_0 \subset \text{pr} W'$ such a point $w'$ in $W'$ exists by what we call the covering property of connected components. Now, $w' \in V'_1$, and thus $\gamma_0 \in \pi(V')$ where $\gamma_0$ is defined by $v' = \gamma_0 \text{pr} w'$. Condition $\text{pr} p_H (w') = \text{pr} (v') \in A_H (K)$ implies $\gamma_0 \in H$. Thus the inclusion $\text{pr} \pi (V') \cap H = \pi (V') \cap H$ implies there exists $\gamma_1 \in \pi (V')$, $\gamma_1 = \gamma_0$, and thus $v' = \gamma_0 \text{pr} w' = \gamma_0 \text{pr} (\gamma_1 w')$, and the Chevalley lemma is proven.

$(SP)\text{gen}$ is implied by (SP), as the projection is irreducible and every fibre above a generic point of $\text{pr} W$ contains a generic point of $W = \mathbb{C} S$ that is necessarily contained in $S$.

(WP) is also implied by (SP). Let $W' \subset U^n$ be irreducible, and let $W_i = W_i \cap W \subset W$ be closed irreducible subsets of $W$ such that $\bigcup_i W_i$ is closed. We need to prove that $\text{pr} (W \setminus \bigcup_i W_i) \subset U^m$ is open in its closure. It is easy to notice that that we may assume that $\bigcup_i W_i = \bigcap_{\gamma \in \Gamma_g} \bigcup_i \gamma W_i \subset \bigcup_{\gamma \in \Gamma_g} \bigcup_i \gamma W_i$ does not change: if $W_x = (\bigcup_i W_i) \cap W_x$ then $W_x = \bigcap_{\gamma \in \Gamma_g} \gamma (\bigcup_i W_i) \cap W_x = \bigcap_{\gamma \in \Gamma_g} \bigcap_i \gamma W_i \cap W_x = \bigcap_{\gamma \in \Gamma_g} \gamma W_x$.
The infinite intersection is closed in co-etale topology, and therefore every compact subset of $U^n$ intersects only finitely many of closed subsets of $\bigcap_{\gamma \in \Gamma_a} (\bigcup_{i} \gamma W_i)$.

Take an open ball $B \subset U^m$ such that its closure is compact, and take a finite index subgroup $H < \Gamma$ such that for every $W_i$ intersecting $\text{pr}^{-1}(B)$ and every $\gamma \in H^n$, either $\gamma W_i = W_i$ or $\gamma W_i \cap W_i = \emptyset$; we may do so by Proposition 2.8.3.2 taking into account that we only need to consider finitely many $W_i$’s by $\Gamma_g$-invariance. Finally consider the quotient $W_H = (W \setminus \bigcup_{\gamma \in \Gamma_{\gamma}, W_i \cap \text{pr}^{-1}(B) \neq \emptyset} \gamma W_i)/H^{n-m}$. It is a subset of an algebraic variety, and, in Zariski topology, is open in its closure. Therefore by [HartAG,Ch.I,Ex.3.10,Ch.II,Ex.3.22] $\text{pr} (W_H)$ is constructible and also $\text{pr} (W \setminus \bigcup_{i} W_i) \cap B = H^m \text{pr} (W_H) \cap B$. Thus, for every open ball $B$, inside of $B$ the set $V_b = pr W \setminus \text{pr} (W \setminus \bigcup_{i} W_i)$ coincides with a union of co-etale constructible sets, and, consequently, the closure of $V_b$ in complex topology coincides with a finite union of co-etale closed sets, locally in every open ball $B$. This implies $V_b$ is closed analytic, and then, in the analytic irreducible decomposition, every irreducible component coincides with an irreducible component of a co-etale closed set. By definition (I), this implies $V_b$ is closed in co-etale topology, as required.

**Remark 2.8.4.2.** It is not sufficient to show that locally in complex topology, $V_b$ contains an open subset of $pr W$, as the following counterexample shows. Let us explain the picture. Consider a countably infinite family of lines in $\mathbb{C}^2$ passing though a point, and take the union of countably many intervals lying on these lines. Then, the union is not contained in the complement of any closed analytic set; and it is easy to ensure that, on every compact subset, the union is contained in the complement of only finitely many of these lines, i.e. is in the complement of a closed analytic set.

2.8.5. **Corollary: (FC) Parametrising fibres of particular dimensions**

The proof of (FC)(min) is quite similar to that of (WP).

**Corollary 2.8.5.1.** (FC). For a locally closed irreducible set $S \subset U^n \times U^m$ and the projection $\text{pr} : U^n \times U^m \rightarrow U^m$, it holds

(FC)(min) there exists an open set $V$ such that $V \cap \text{pr} S \neq \emptyset$ and for every $v \in V \cap \text{pr} S$, $\text{dim}(\text{pr}^{-1}(v) \cap S) = \min \{ \text{dim}(\text{pr}^{-1}(a) \cap S) \}

(FC)(>) The set $a \in \text{pr} S : \text{dim}(\text{pr}^{-1}(a) \cap S) \geq k$ is of the form $T \cap \text{pr} S$ for some constructible $T$.

**Proof.** (FC)(min) Let $W = \text{Cl} S$ be the closure of $S$ and let $H < \Gamma$ a finite index subgroup as provided by Proposition 2.8.3.2. By [HartAG,Ch.I,Ex.3.10,Ch.II,Ex.3.22], for an open subset $V^0 \subset \text{pr} H^{n+m} W$, for every point $v \in V^0$, it holds that every irreducible component of fibre $(W/H^{n+m})_v$ of the algebraic morphism $\text{pr} : W/H^{n+m} \rightarrow U^m/H^m$ is of dimension $e = \text{dim} W - \text{dim} \text{pr} W = \text{dim}(W/H^{n+m}) - \text{dim} \text{pr} (W/H^{n+m})$. The latter that every irreducible component of $\text{pr}^{-1}(W/H^{n+m})_v$ is of dimension $e$ unless empty, and $V^0 \cap \text{pr} W$ is as required. The proof of (FC)(>) is similar.

2.8.6. **Uniformity of generic fibres**

Let $\pi_0(W')$ denote the set of irreducible components of $W'$.

**Corollary 2.8.6.1** (Generic Fibres). In notation of Proposition above, for a point $g' \in V' = \text{Cl} \text{pr} W'$ not contained in some proper $\Theta$-definable closed subset of $V$, the fibre $W_{g'}$ has finitely many connected components, and for any connected component $W_{g'}^{rc}$ of $W_{g'}$, it holds

\[
W' \cap g' \times H W_{g'}^{rc} = g' \times W_{g'}^{rc},
\]

\[
W' \cap g' \times H W_{g'} = g' \times W_{g'}.
\]

In particular, the formulae above hold for $g'$ a $\Theta$-generic point of $V$.  

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Proof. Let \( H \) be as in Proposition 2.8.3.2. The fibre \( W'_{g'} \) is the intersection of \( W_g \) with a coordinate plane, and therefore is co-etale closed. By Decomposition Lemma, the fibre \( W'_{g'} \) is a union of \( H \)-translates of a finite number of irreducible sets \( Z''_1, \ldots, Z''_k \).

To prove the claim, take \( h, h \in H \) such that \( Z'_{i}, h Z'_{i} \subset W_{g'} \). Then \( (id, h) \in H \times H \), and \( (id, h^{-1})W' \cap W' \supset g' \times Z'_{i} \neq \emptyset \), and by Proposition 2.8.3.2(1) this implies \( (id, h^{-1})W' = W' \) and \( (id, h^{-1}) \in \pi(W') \). However, by Proposition 2.8.3.2(2) \( \pi(W'_{g'}) \cap H = \ker(\text{pr}_{g'}(\pi(W') \rightarrow \text{pr}(V')) \cap H) \), and thus \( h \in \pi(W'_{g'}) \), \( h W'_{g'} = W'_{g'} \) for any connected component \( W'_{g'} \) of fibre \( W'_{g'} \).

To prove \( W' \cap g' \times HW'_{g'} = g' \times W'_{g'} \), take \( h \in H \) such that \( g' \times h W'_{g'} \cap W \neq \emptyset \). Then \( (id, h) \in H \times H \) and

\[(id, h)W' \cap W' \supset g' \times h W'_{g'} \cap W' \neq \emptyset,\]

by Proposition 2.8.3.2(1) this implies \( (id, h)W' = W' \) i.e. \( (id, h) \in \pi(W') \). Now Proposition 2.8.3.2(2), \( \pi(W'_{g'}) \cap H = \ker(\text{pr}_{g'}(\pi(W') \rightarrow \text{pr}(V')) \cap H) \) gives \( h W'_{g'} = W'_{g'} \), i.e \( h \in \pi(W'_{g'}) \), as required.

In particular, \( W' \cap HW'_{g'} = W'_{g'} \) and \( W' \cap W'_{g'} = g' \times W'_{g'} \)

3. Core sets: A language for the co-etale topology: \( k \)-definable sets

So far we haveanalysed the topology on \( U \) (and its Cartesian powers \( U^n \)'s) whose closed sets are rather easy to understand. Now, to put the considerations above in a framework of model-theory, we want to define a language able to define closed sets in the co-etale topology. From an algebraic point of view, that corresponds to defining an automorphism group of \( U \) with respect to the co-etale topology. The automorphism group is to be an analogue of a Galois group.

In the terminology of [19], this corresponds to a choice of core closed subsets. Our language is smaller than that: core closed subsets are definable with parameters (corresponding to core subsets).

Let us draw an analogy to the action of Galois group on \( \overline{\mathbb{Q}}\) as an algebraic variety defined over \( \mathbb{Q} \) endowed with Zariski topology. The Galois group may not be defined as the group of bijections continuous in Zariski topology: for example, all polynomial maps are continuous in Zariski topology; linear and affine maps \( x \rightarrow ax + b \) are such continuous bijections.

Thus we distinguish certain \( \mathbb{Q} \)-definable subsets among Zariski closed subsets of \( \overline{\mathbb{Q}}^3 \), and then define Galois group as the group of transformation (of \( \overline{\mathbb{Q}} \)) preserving the distinguished \( \mathbb{Q} \)-defined subsets (of \( \overline{\mathbb{Q}}^3 \)); in this case the graphs of addition and multiplication. It is then derived, rather trivially, that this implies that Galois group acts by transformation continuous in Zariski topology.

Recall the way this is derived: the \( \mathbb{Q} \)-definable subsets are given names, in this case addition and multiplication, and then each closed set (subvariety) is given a name by the equations defining the set of its points; in fact, in algebraic geometry the word variety means rather the name, the set of equations, rather that the set of points the equations define.

In order to define a useful automorphism group of the co-etale topology, we follow the same pattern.

Model theory provides us with means to give precise meaning to the argument above, and to define mathematically what is it exactly that we want. In these terms, the distinguished subsets form a language, and the Galois group is the group of automorphisms of the structure in that language. Model theory studies that group via the study of the structure.

3.1. Definition of a language \( L_A \) for universal covers in the co-etale topology

In this §, it becomes essential that \( A \) is defined over an algebraic field \( k \subset \overline{\mathbb{Q}} \subset \mathbb{C} \) embedded in \( \mathbb{C} \).

We consider \( p: U \rightarrow A(\mathbb{C}) \) as a one-sorted structure \( U \), in the language \( L_A \) which has the following symbols:

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the symbols \( \sim_{Z,A} \) for \( Z \) a closed subvariety of \( A(\mathbb{C})^n \) defined over number field \( k \), and,

the symbols \( \sim_H \), for each normal subgroup \( H <_{\text{fin}} \pi(U)^n \) of finite index

The symbols are interpreted as follows:

\[
x' \sim_{Z,A} y' \iff \text{ points } x' \in U^n \text{ and } y' \in U^n \text{ lie in the same (analytic) irreducible component of the } \Gamma\text{-invariant closed analytic set } \mathfrak{p}^{-1}(Z(\mathbb{C})) \subset U^n.
\]

\[
x' \sim_H y' \iff \exists \tau \in \pi(U)^n : \tau x' = y' \text{ and } \tau \in H.
\]

Note that we do not assume \( Z \) to be connected.

As justified by Corollary 3.1.0.3, we get a bi-interpretable language by considering the following predicates instead:

\[
x' \sim_{Z,i} y' \iff \text{ points } x' \in U^n \text{ and } y' \in U^n \text{ lie in the same connected component of the preimage } \mathfrak{p}_H^{-1}(Z_{i}(\mathbb{C})), Z_i \subset A^H(\mathbb{C})^n \text{ an irreducible component of algebraic variety } \mathfrak{p}_H^{-1}(Z(\mathbb{C})) \subset A^H(\mathbb{C})^n.
\]

Corollary 3.1.0.3. For every closed \( \Gamma\)-invariant analytic subset \( Z' \) of \( U^n \), there exist closed analytic subsets \( Z'_1, \ldots, Z'_n \) invariant under action of a finite index subgroup \( H \) of \( \Gamma \), such that

\[
x \sim_{Z,i} y \iff x \sim_{Z'_1} y \lor x \sim_{Z'_2} y \lor \ldots \lor x \sim_{Z'_n} y.
\]

Consequently, for every closed subvariety \( Z \) of \( A \), there exist subvarieties \( Z_1, \ldots, Z_n \) of a finite étale cover \( A^H \) such that

\[
x \sim Z, y \iff x \sim_{Z'_1} y \lor x \sim_{Z'_2} y \lor \ldots \lor x \sim_{Z'_n} y.
\]

Proof. Take \( H \) and \( Z'_1, \ldots, Z'_n \) as in Decomposition Lemma 2.3.2.1. \( \square \)

Note that the language \( L_A \) is countable. This is an essential property, from model-theory point of view; in technical, down-to-earth terms it is useful to make inductive constructions.

Let us use this opportunity to remind that we use symbols \( \sim_Z \) rather abusively to mean “lie in the same irreducible component of” either \( \Gamma Z, \mathfrak{p}_H^{-1}(Z) \), etc.

3.2. \( L_A \)-definability of \( \pi(U) \)-action etc

In the next lemma, a closed set means a co-etale closed set.

Lemma 3.2.0.4. For any normal finite index subgroup \( H <_{\text{fin}} \Gamma \) it holds

1. the relation

\[
\text{Aff}_H(x, y, z, t) = \exists \gamma \in H : \gamma x = y \land \gamma z = t
\]

is \( L_A(\emptyset) \)-definable

2. An \( H \)-invariant closed set is \( L_A \)-definable with parameters.

3. A connected component of a generic fibre of an \( L_A \)-definable irreducible closed set is uniformly \( L_A \)-definable; the definition is valid over an open subset of the projection, definable over the same set of parameters.

4. Any co-etale closed irreducible set is a connected component of a fibre of an \( L_A \)-definable set.

5. An irreducible closed set is \( L_A \)-definable.
Proof. To prove (1), note that
\[ p^{-1}(\Delta(C)) = \bigcup_{\gamma \in \Gamma} \{(x, \gamma x) : x \in U\} \]
where \( \Delta = \{(x, x) : x \in A\} \) is an algebraic closed subvariety defined over \( k \). The connected components \( \{(x, \gamma x) : x \in U\}, \gamma \in \Gamma \) are the equivalence classes of the relation \( \sim_\Delta \), and thus are definable with parameters.

Evidently \( \text{Aff}_e(x, y, s, t) \iff (x, y) \sim_\Delta (s, t) \) lie in the same connected component of \( p^{-1}(\Delta(C)) \subset U \times U \).

To prove (2), we consider two cases. \( \mathbb{Q} \)-case: An irreducible closed subvariety \( Z_{\mathbb{Q}} \subset A \) defined over \( \mathbb{Q} \) is an irreducible component of subvariety
\[ Z_k = \bigcup_{\sigma : k \rightarrow \mathbb{C}} \sigma(Z) \]
of \( A \), where \( k \) is the field of definition of \( Z \) of finite degree. The formula implies \( Z \) is \( L_A \)-definable with parameters with the help of symbol \( \sim_Z, \mathbb{Q}, A \); the parameters may be taken to lie in \( A(\mathbb{Q}) \) but not necessarily in \( A(k) \). A slightly more complicated argument could give a construction defining \( Z \) as a connected component.

For an analytic co-etale closed irreducible set \( Z' \subset U \), it holds that \( Z' \) is an irreducible component of \( \Gamma Z' \), i.e. it is an irreducible component of \( p^{-1}(Z) = p^{-1}p(Z') \). Thus the above argument gives that every co-etale irreducible subset of \( U \) defined over \( \mathbb{Q} \) is \( L_A \)-definable with parameters.

\( \mathbb{Q}(t_1, \ldots, t_n) \)-case: Thus we have to deal with the case when \( p(Z) \) is not \( \mathbb{Q} \)-definable. Our strategy is to show that any such set is a connected component of a \( \mathbb{Q} \)-generic fibre of a \( \mathbb{Q} \)-definable set, and then show that such connected components are uniformly definable. Uniformity will be important for us later in axiomatising \( U \).

Let us see first that each co-etale closed irreducible set is a connected component of a fibre of a co-etale closed irreducible set defined over \( \mathbb{Q} \).

Take a co-etale irreducible set \( Z' \) and take \( H <_{\text{fin}} \Gamma \) such that \( Z' \) is a connected component of \( HZ' = p_H^{-1}(Z) \), for an irreducible algebraic closed set \( Z = p_H(Z') \). By the theory of algebraically closed field, we know that \( Z \) can be defined as a Boolean combination, necessarily a positive one, of \( \mathbb{Q} \)-definable closed subsets and their fibres; by passing to a smaller subset if necessary, we see that the irreducibility of \( Z \) implies that algebraic subset \( Z \subset A(\mathbb{Q}) \) is a connected component of a \( \mathbb{Q} \)-generic fibre of a \( \mathbb{Q} \)-definable closed subset \( W \subset A(\mathbb{Q})^n \). Then \( HZ' \) is the corresponding fibre of \( p_H^{-1}(W) \). The closed set \( Z' \) is a union of the corresponding fibres of the irreducible components of \( p_H^{-1}(W) \), and irreducibility of \( Z' \) implies that union is necessarily trivial. Thus, we have that \( Z' \) is a connected component of a fibre of an irreducible co-etale closed set defined over \( \mathbb{Q} \). We may also ensure that \( Z' \) is a connected component of a \( \mathbb{Q} \)-generic fibre of \( W' \) by intersecting \( W' \) with the preimage of an irreducible \( \mathbb{Q} \)-definable set containing \( \text{pr} Z' \), and repeating the process if necessary.

Let us now prove that the connected components of the \( \mathbb{Q} \)-generic fibres of an irreducible \( \mathbb{Q} \)-definable set are \( \mathbb{Q} \)-definable.

Let \( W' \subset A(\mathbb{C})^2 \), and let \( V' = \text{Clpr} W' \) be as in Proposition 2.8.3.2 and Corollary 2.8.6.1. The morphism \( \text{pr} : W \to V \) admits a Stein factorisation (Fact 2.8.1.7) \( \text{pr} = f_0 \circ f_1 \) as a composition of a finite morphism \( f_0 : W \to V_1 \) and a morphism with connected fibres \( f_1 : V_1 \to V \). In particular, two points \( x_1, x_2 \in W_0 \) lie in the same connected component of fibre \( W_0 \) iff \( f_0(x_1) = f_0(x_2) \).

Now set
\[ x' \sim_{W_0} y' \iff x' \sim_W y' \& \text{pr} x' = \text{pr} y' \& f_0(p_H(x')) = f_0(p_H(y')) \] (3.1)
Definition 4.0.0.9. An irreducible constructible set is a set whose closure is irreducible.

Definition 4.0.0.10. We say that $w \in W$ is a $\Theta$-generic point of an irreducible constructible set iff $w$ does not lie in a proper $\Theta$-definable subset of $W$.

We say that a property holds for a uniform generic point of $W$ iff it holds for every point is some open $\Theta$-definable subset of $W$.

Lemma 4.0.0.10. A projection of an irreducible $\Theta$-constructible set is $\Theta$-constructible.
Proof. Let $W \subset U \times U$ be an irreducible set defined over $\Theta$, and let $W_0$ be the set of all $\Theta$-generic points of $W$; generally speaking, $W_0$ is not definable. We need to prove that $\text{pr } W$ is also $\Theta$-constructible. Let $g$ be a $\Theta$-generic point of the closure of $\text{pr } W$; we know $g \in \text{pr } W$ by (SP) of Lemma 2.8.4.1. By Lemma 2.8.6.1 we know that the (non-empty) fibre $W_g$ contains a $\Theta$-generic point of $W$, and thus $g \in \text{pr } W_0$, as required.

The set of realisations of a complete quantifier-free syntactic type $p/\Theta$ with parameter set $\Theta$ is $\Theta$-constructible; and conversely, every $\Theta$-constructible set can be represented in this form.

Thus, the above lemma is equivalent to $\omega$-homogeneity for such types.

Definition 4.0.0.11. We say that $U$ is homogeneous for irreducible closed sets over $\Theta$, or homogeneous for syntactic quantifier-free complete types over $\Theta$, or model homogeneous iff either of the following equivalent conditions holds

1. the projection of an irreducible $\Theta$-constructible set is $\Theta$-constructible;
2. for any tuples $a, b \in U^n$ and $c \in U^m$ if $\text{qftp}(a/\Theta) = \text{qftp}(b/\Theta)$ then there exists $d \in U^m$ such that $\text{qftp}(a, c/\Theta) = \text{qftp}(b, d/\Theta)$

To see that the conditions are equivalent, note that the set of realisations of a complete quantifier-free type $\text{qftp}(u/\Theta)$ is $\Theta$-constructible; its projection contains $a$ and also is $\Theta$-constructible; $a$ is its $\Theta$-generic point; then $\text{tp}(a/\Theta) = \text{tp}(b/\Theta)$ implies $b$ is also $\Theta$-generic, i.e. belongs to the projection.

The above proves the following result.

Property 4.0.0.12. The standard model $p : U \to A(\mathbb{C})$ in language $L_A$ is model homogeneous, i.e. it is $\omega$-homogeneous for closed sets over arbitrary algebraically closed subfield $\Theta \subseteq \mathbb{C}$.

Proof. Follows directly from Def. 4.0.0.11 and Lemma 4.0.0.10.

Corollary 4.0.0.13. The set of realisations of a quantifier-free type $\text{qftp}(x/\Theta)$ over $p^{-1}(A(\Theta))$ consists of $\Theta$-generic points of some co-étale irreducible closed subset of $U$.

Proof. Follows from the previous statements.

5. An $L_{\omega,1,\omega}$-axiomatisation $\mathcal{X}(A(\mathbb{C}))$ and stability of the corresponding $L_{\omega,1,\omega}$-class.

In this § we introduce an axiomatisation $\mathcal{X}(A(\mathbb{C}))$ for $L_{\omega,1,\omega}(L_A)$-class which contains the standard model $p : U \to A(\mathbb{C})$, and is stable over models and all models in it are model homogeneous. We then show that the class of models satisfies $(\exists_{\mathbb{R}_+,\mathbb{R}_+})$ of Theorem 6.0.4.8.

5.1. Algebraic $L_A(G)$-structures

We know that $U/G = A^G(\mathbb{C})$ carries the structure of an algebraic variety over field $\mathbb{C}$. The covering $A^G(\mathbb{C}) \to A(\mathbb{C})$ carries a structure in a reduct $L_A(G)$ of language $L_A$. In fact, similar interpretation works for an arbitrary algebraically closed field $K$ instead of $K = \mathbb{C}$.

For every finite index subgroup $G \triangleleft \text{Int } \Gamma$, there is a well-defined covering $A^G \to A$ of finite degree. The space $A(\mathbb{C})$ is projective, and thus $A^G(\mathbb{C})$ is also a complex projective manifold. By Fact 2.3.1.2, $A^G$ has the structure of an algebraic variety.

Recall that we use the following fact as the defining property of an étale covering: the morphism $B(K) \to A(K)$ of varieties over an algebraically closed field $K$ of char 0 is étale iff there exists an embedding $i : K' \to \mathbb{C}$ of the field $K'$ of definition of $A$ and $B$ into $\mathbb{C}$ such that the corresponding morphism $i(B)(\mathbb{C}) \to i(A)(\mathbb{C})$ is a covering of topological spaces.
5.2. Axiomatisation

Axiom 5.2.1.1. Let \( p_G : \mathbb{A}^G(K) \to \mathbb{A}(K) \) be a finite étale morphism. Let \( L_\mathbb{A}(G) \subset L_\mathbb{A} \) be the language consisting of all predicates of \( L_\mathbb{A} \) of form \( \sim_Z \) and symbols \( \sim_H \) for \( G \subset H \). Then \( \mathbb{A}^G(K) \to \mathbb{A}(K) \) carries an \( L_\mathbb{A}(G) \)-structure as follows:

1. \( x' \sim_Z y' \iff \) points \( x', y' \in \mathbb{A}^G(K)^n \) lie in the same irreducible component of algebraic closed subset \( p_G^{-1}(Z(K)) \) of \( \mathbb{A}^G(K)^n \).

2. \( x' \sim_H y' \iff \) there exist an algebraic morphism \( \tau : \mathbb{A}^G \to \mathbb{A}^G \) and a co-etale covering morphism \( q : \mathbb{A}^G \to \mathbb{A}^H \) such that \( \tau(x') = y' \) and \( \tau \circ q = q \):

\[
\begin{align*}
\mathbb{A}^G \xrightarrow{\tau} & \mathbb{A}^G \\
\downarrow q \text{ étale cover} & \downarrow q \\
\mathbb{A}^H \xrightarrow{id} & \mathbb{A}^H
\end{align*}
\]

For \( G = e \) the trivial group and \( K = \mathbb{C} \), the construction above would degenerate into the interpretation of \( U \to \mathbb{A} \) if it were well-defined.

For \( G = \Gamma \), \( \mathbb{A}^G = \mathbb{A} \), and thus \( L_\mathbb{A}(\Gamma) \) is just a form of the language for the algebraic variety \( \mathbb{A} \); here the point is that we have predicates for the relations for irreducible components of \( k \)-definable closed subsets only.

In general, the above is simply a variation of an ACF structure on \( \mathbb{A} \). In particular, all Zariski closed subsets of \( (\mathbb{A}^G)^n(K) \) are \( L_\mathbb{A}(G) \)-definable.

5.2. Axiomatisation \( \mathcal{X}(\mathbb{A}(\mathbb{C})) \) of the universal covering space \( U \)

We define the axiomatisation \( \mathcal{X} = \mathcal{X}(\mathbb{A}(\mathbb{C})) \) to be an \( L_{\omega,1}(L_\mathbb{A}) \)-sentence corresponding to Axiom 5.2.1.1 and Axioms 5.2.2.1-5.2.2.5 below.

5.2.1. Basic Axioms

These axiom describe quotations \( U/ \sim_H \) for \( H <_{\text{fin}} \Gamma \), and some properties of \( U \to U/ \sim_H \).

Axiom 5.2.1.1. All first-order statements valid in \( U \) and expressible in terms of \( L_\mathbb{A} \)-interpretable relations

\( x' \sim_{Z,\mathbb{A}^G} y' : = \exists x'' \exists y''(x'' \sim_Z y'' \& x'' \sim_G x' \& y'' \sim_G y') \), \( G <_{\text{fin}} \Gamma \)

and \( \sim_{G, G <_{\text{fin}} \Gamma} \).

Essentially, these axioms describe \( U_G \) as an algebraic variety.

5.2.2. Path-lifting Property Axiom, or the covering property Axiom

Axiom 5.2.2.1 (Path-lifting Property for \( W \); Covering Property for \( W \)). For every \( L_\mathbb{A} \)-predicate \( \sim_W \) and all \( G <_{\text{fin}} \Gamma \) small enough, we have an axiom

\( x' \sim_{W, \mathbb{A}, G} y' \implies \exists y''(y'' \sim_G y' \& x' \sim_W y'') \)

We also have a stronger axiom for fibres of \( W \); here we use that the relation “to lie in the same connected component of a fibre of a variety” is algebraic and therefore the corresponding \( G \)-invariant relation is \( L_\mathbb{A} \)-definable.

Axiom 5.2.2.2 (Lifting Property for fibres). For all \( G <_{\text{fin}} \Gamma \) sufficiently small, we have an axiom

\( (x_0', x_1') \sim_{W, \mathbb{A}, G} (y_0', y_1') \implies \exists y''(y'' \sim_G y_0' \& y'' \sim_G y_1' \& (x_0', x_1') \sim_W (x_0', y_1')) \)

in a slightly different notation

\( x' \sim_{W, \mathbb{A}, G} y' \implies \exists y''(y'' \sim_G y' \& \text{pr } x' = \text{pr } y'' \& x' \sim_W y'') \)
The relation \( x' \sim_{W,G} y' \) is defined by the formula (3.1) (cf. Claim 3.2.0.5).

**Axiom 5.2.2.3** (Fundamental group is residually finite).
\[
\forall x'y'(x' = y' \iff \bigwedge_{H < \text{fin}} \Gamma_H \sim_H y')
\]

Thus, it says that two elements of \( U \) separated by an element of \( H \) for every \( H < \text{fin} \), have to be equal.

The next property is strengthening of the previous one; namely, if an element \( b \) is \( \sim_H \)-equivalent to an element of a group generated by \( a_1, \ldots, a_n \), then it is actually in the group. In terms of paths, this has the following interpretation: take loops \( \gamma_1, \ldots, \gamma_n \) and a loop \( \lambda \). If for every \( H < \text{fin} \) it holds that \( \lambda \) is \( \sim_H \)-equivalent to some concatenation of paths \( \gamma_1, \ldots, \gamma_n \), then it is actually a concatenation of these paths.

**Axiom 5.2.2.4** ("Translations have finite length", subgroup separability). For all \( N \in \mathbb{N} \) we have an \( L_{\omega_1 \omega} \)-axiom
\[
\forall b \forall a_1 \ldots \forall a_N.
\]
\[
\bigwedge_{H < \text{fin}} \bigvee_{n \in \mathbb{N}} \exists h_1 \ldots h_n \left( b \sim_H h_n & h_1 = a_1 & \bigwedge_{1 \leq i \leq n} 1 \leq j < N (h_i, h_{i+1}) \sim_{\Delta} (a_j, a_{j+1}) \right)
\]
\[
\implies \bigvee_{n \in \mathbb{N}} \exists h_1 \ldots h_n \left( b = h_n & h_1 = a_1 & \bigwedge_{1 \leq i \leq n} 1 \leq j < N (h_i, h_{i+1}) \sim_{\Delta} (a_j, a_{j+1}) \right)
\]

The next axiom is needed to apply the axioms above. It reflects the fact that the fundamental groups of varieties are finitely generated, a fact we used and prove in the proof of Lemma 2.3.2.1. recall that this was proved as a corollary of the fact that topologically an algebraic variety can be triangulated into finitely many contractible pieces nicely glued together.

**Axiom 5.2.2.5** (Groups \( \pi(W_g) \) are finitely generated). For every symbol \( \sim_W \) and for each \( H \subset \Gamma \) small enough we have an \( L_{\omega_1 \omega} \)-axiom:
\[
\bigwedge_{N \in \mathbb{N}} \exists a_1 \ldots \exists a_N \forall b.
\]
\[
\bigwedge_{1 \leq i \neq j \leq N} (a_i \sim_W a_j & a_i \sim_H a_j & \text{pr } a_i = \text{pr } a_j) & \left( \bigwedge_{i=1}^N (b \sim_W a_i & \text{pr } b = \text{pr } a_i) \implies \bigvee_{n \in \mathbb{N}} \exists h_1 \ldots h_n \left( b = h_n & h_1 = a_1 & \bigwedge_{j=1}^{N-1} (h_i, h_{i+1}) \sim_{\Delta} (a_j, a_{j+1}) & \text{pr } h_i = \text{pr } h_{i+1} \right) \right)
\]

In fact, we may combine the two axioms above into one weaker axiom which would require subgroup separability with respect to the subgroups \( \pi(W) \).

### 5.2.3. Standard model \( U \) is a model of \( \mathcal{X} \)

The universal covering space \( p : U \to A(\mathbb{C}) \) satisfies the Axiom 5.2.1.1 by definition.

To prove \( U \) satisfies Axiom 5.2.2.1, note that for \( G < \text{fin} \) small enough, the relations \( x' \sim_{W,G} y' \) means that \( p_G(x') \) and \( p_G(y') \) lie in the same irreducible component \( W_i \) of the preimage of \( W \subset A(\mathbb{C})^n \) in \( A^G(\mathbb{C})^n \). Take a path \( \gamma \) connecting \( \gamma(0) = p_G(x') \) and \( \gamma(1) = p_G(y') \) lying in \( W_i \); by the lifting property it lifts to a path \( \gamma' \), \( \gamma'(0) = x' \) such that \( p_G(\gamma'(t)) = \gamma(t) \), \( 0 \leq t \leq 1 \). Then, \( p_G(\gamma'(1)) = p_G(y') \), and thus \( \gamma'(1) \sim_G y' \). On the other hand, \( \gamma'(1) \) and \( x' \) lie in the same connected component of the preimage of the irreducible component \( W_i \) in \( U \). Now note that by Decomposition Lemma 2.4.0.3 for \( G \) small enough such a connected component has to be irreducible, and thus Axiom 5.2.2.1 holds.
The Axiom 5.2.2.2 has a similar geometric meaning as Axiom 5.2.2.1; the assumption is that \( p_C(x') \) and \( p_C(y') \) lie in the same connected component of a fibre \( W_g \); it is enough to take \( \gamma \) to lie in fibre \( W_g \) to arrive to the conclusion of Axiom 5.2.2.1.

Axiom 5.2.2.3 follows from the condition 2 of the definition of a LERF variety.

Axioms 5.2.2.4 is condition 2 of the definition of a LERF variety.

The geometric meaning of \((h_i, h_{i+1}) \sim \Delta (a_i, a_{i+1})\) is as follows. The pair of points \( a_i, a_{i+1} \) determines a path \( \gamma \) in \( A(\mathbb{C}) \), \( \gamma(0) = \gamma(1) = p(a_i) = p(a_{i+1}) \). For points \( h_i, h_{i+1} \) such that \( p(h_i) = p(h_{i+1}) \), they can be joined by a lifting of \( \gamma \) iff \((h_i, h_{i+1}) \sim \Delta (a_i, a_{i+1})\). Thus the assumption in the axiom says that if any two points of fibre above \( p(b) = p(a_1) \) can be joined by a concatenation of liftings of finitely many paths \( \gamma_i \)'s in \( A(\mathbb{C}) \), \emph{up to a translate by an element of} \( H \), then they can in fact be just joined by such a sequence. In a way, this can be thought of as disallowing paths of infinite length.

On the other hand, the condition \((h_i, h_{i+1}) \sim \Delta (a_i, a_{i+1})\) can be interpreted as \( h_{i+1} = \tau_i h_i \) where \( \tau_i \) is the deck transformation taking \( a_i \) into \( a_{i+1} \). Thus, the assumption says that if \( b \in \pi(U) \) belongs to the group generated by \( \tau_i \)'s, up to \( \sim_H \), then \( b \) does belong to the subgroup generated by \( \tau_i \)'s.

The last remaining Axiom 5.2.2.5 means that the fundamental groups \( \pi(W_g) \) is finitely generated, and we already used this Fact in the proof of Lemma 2.3.2.1.

5.3. Analysis of models of \( \mathcal{X} \)

5.3.1. Models \( U/\sim_H \) as algebraic varieties

Let \( U \models \mathcal{X} \) be an \( L_A \)-structure modelling axiomatisation \( \mathcal{X}(A(\mathbb{C})) \), and let \( U \) be the standard model, i.e. the universal covering space of \( A(\mathbb{C}) \) considered as an \( L_A \)-structure.

We know that \( U/\sim_H \cong A^H(\mathbb{C}) \) for some algebraic varieties \( A^H(\mathbb{C}) \) defined over \( \mathbb{C} \). The relations \( \sim_H, \sim_H \) are essentially relations on \( U/\sim_H \), and thus Axiom 5.2.1.1 says that the first-order theories of \( U/\sim_H \) and that of standard model \( U/\sim_H \), in the language \( L_A(H) = \{ \sim_H, \sim_H, Z \} \) coincide. We know by properties of analytic covering maps that an irreducible co-etale closed subset of \( U \) covers an irreducible Zariski closed subset of \( A^H(\mathbb{C}) \), and thus the relation \( \sim_H \) on \( U/\sim_H \) interprets as saying that \( x, y \in A^H(K) \) lie in the same (Zariski) irreducible component of the preimage of \( Z(K) \) in \( A^H(K) \). In particular, every component is definable by \( g \sim z \) where \( g \) is taken to be its generic point. Since every \( \overline{\mathbb{Q}} \)-definable closed subvariety is an irreducible subvariety of a \( \overline{\mathbb{Q}} \)-definable subvariety, this implies that every \( \overline{\mathbb{Q}} \)-definable closed subvariety of \( A^H(\mathbb{C}) \) is \( L_A(H) \)-definable. Thus, full theory of an algebraically closed field is reconstructible in \( L_A(H) \) on \( U/\sim_H \); and thus, there is an algebraically closed field \( K = \overline{\mathbb{K}} \), \( \text{char} \, K = 0 \) such that \( U/\sim_H \cong A^H(K) \).

Fix these isomorphisms \( U/\sim_H \cong A^H(K) \), and let \( p_H : U \to A^H(K) \) be the projection morphism. Then the above considerations say

\[
x' \sim_{W,H} y' \iff p_H(x') \sim_{W,H} p_H(y') \iff x' \text{ and } y' \text{ lie the same (Zariski) irreducible component of the preimage of } Z(K) \text{ in } A^H(K).
\]

\[
x \sim_G y' \iff \text{ there exist an algebraic morphism } \tau : A^G \to A^H \text{ and a co-etale covering morphism } q : A^H \to A^G \text{ such that } \tau(x') = y' \text{ and } \tau \circ q = q:
\]

\[
\begin{array}{ccc}
A^H & \xrightarrow{\tau} & A^H \\
| \text{ etale cover} | & \downarrow q \\
A^G & \xrightarrow{id} & A^G
\end{array}
\]
An important corollary of above considerations is that any set of form $p_H^{-1}(Z(K)), Z(K) \subset A^H(K)$ is $L_A$-definable.

**Notation 5.3.1.1.** Let us introduce new relations on $U$; eventually we will prove that they are first-order definable. We introduce the relations below for every closed subvariety of $A(K)$, not necessarily defined over $k$ (those would be in $L_A$)

$$x' \sim_W y' \iff p_H(x') \sim_{W,H} p_H(y') \text{ for all } H \triangleleft_{\text{fin}} \Gamma.$$

An irreducible component of relation $\sim_W$ is a maximal set of points in $U$ pairwise $\sim_W$-related. A subset of $U$ is basic closed iff it is a union of irreducible components of relations $\sim_{W_1, \ldots, W_n}$, for some $W_1, \ldots, W_n$. An irreducible closed set is an irreducible component of a relation $\sim_W$ for some closed subvariety $W$. Let us call a subset of $U$ co-etale closed iff it is the intersection of basic closed sets. This defines an analogue of the co-etale topology on $U$.

**5.3.2. Group action of fibres of $p : U \to A(K)$ on $U$**

For a point $x_0 \in U$, let $\pi(U, x_0) = \{y : y \sim x_0\} = p^{-1} p(x_0)$ be the fibre of $p : U \to A(K)$. For every point $x_0' \in U$ and every point $y' \sim x_0'$, there exists a point $z'' \in U$ such that $p_G(z', z'') \sim z'' \in U$ such that $p_G(z'' \sim y')$; this follows from Axiom 5.2.1.1. Then, by lifting property for $\Delta \subset A^2(K)$, there exists $z''' \in U$ such that $z''' \sim y''' \sim_{\Delta} (z', z'') \sim \Delta (x_0', y')$. Moreover, such a point $z'''$ is unique. Indeed, by Axiom 5.2.1.1 the conditions $p_H(z''' \sim y'' \sim y''')$ and $(z', z''') \sim_{\Delta,H} (x_0', y')$ determine $p_H(z''' \sim y'' \sim y''')$ uniquely for every $H \triangleleft_{\text{fin}} \Gamma$. This implies that $z'''$ is unique by Axiom 5.2.2.3.

The above construction defines an action $\sigma$ of $\pi(U, x_0') = \{y : y \sim x_0'\} = p^{-1} p(x_0')$ on $U$: a point $y' \sim \tau x_0'$ sends $z'$ into $z''', \sigma y' z'''$. Axiom 5.2.1.1 and Axiom 5.2.2.1 imply that it is in fact a group action.

Let $\pi(U)$ be the group of transformations of $U$ induced by $\pi(U, x_0)$; the group does not depend on the choice of $x_0$. We refer to $\pi(U)$ as the group of deck transformations, or the fundamental group of $U$. This terminology is justified by the fact that $\pi \circ p = p$, for $p : U \to A(K)$ the covering map.

For a subset $W \subset U^n$, let $\pi(W) = \{\tau : U^n \to U^n : \tau(W) \subset W, \tau \in \pi(U)^n\}$.

**5.3.3. Decomposition Lemma for $U$**

We use a Corollary to Lemma 2.3.2.1.

**Lemma 5.3.3.1 (Decomposition lemma; Noetherian property).** Assume $A$ is LERF.

A subset $p^{-1}(W), W \subset A(K)$ has a decomposition of the form

$$W' = HZ'_1 \cup \ldots \cupHZ'_k,$$

where $H \triangleleft_{\text{fin}} \Gamma$ is a finite index normal subgroup of $\Gamma$, the co-etale closed sets $Z'_1, \ldots, Z'_k$ are irreducible components of relations $\sim_{Z'_i}$, for some algebraic subvarieties $Z_i$ of $A(K)$, and for any $\tau \in H$

either $\tau Z'_i = Z'_i$ or $\tau Z'_i \cap Z'_i = \emptyset$.

**Proof.** By a corollary to Decomposition Lemma 2.3.2.1 we may choose $H \triangleleft_{\text{fin}} \Gamma$ with the following property.

Let $Z_i \subset A^H(K)$’s be the irreducible components of $p_H p^{-1}(W)$. Then, they have the property that the connected components of $p_G Z_i \subset A^G(K)$ are irreducible. Choose $Z'_i$ to be an irreducible components of relations $\sim_{Z'_i}$, i.e. the closed sets $p_H^{-1}(Z_i)$. We claim that these $Z'_i$’s give rise to a decomposition as above.

Before we are able to prove this, let us prove the lifting property for $\sim_{Z_i}$, namely that the map $p_H : Z'_i \to Z_i(K)$ is surjective. For convenience, we drop the index $i$ below.

By passing to a smaller subgroup if necessary we may find a variety $V \subset A^H(K)^n$ defined over $\mathbb{Q}$ such that for some $g \in A^n(K)$, $Z_i$ is a connected component of fibre $V_g$ of $V$ over $g$, and it holds

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that if points \( x', y' \) are such that \( p_H(x'), p_H(y') \in Z_i \) and \( x' \sim_W y', p_H(\text{pr } x') = p_H(\text{pr } y') = g' \) lie in the same connected component of \( V' \) over \( g, p_H(g') = g \), then in fact \( x' \) and \( y' \) lie in the same connected component of the preimage of \( g \times Z_i, x' \sim_Z y' \).

Consider Axiom 5.2.2.2 for all \( G \triangleleft H \) sufficiently small

\[
x' \sim_{V_g} y' \implies \exists y''(y'' \sim_G y' \& \text{pr } x' = \text{pr } y'' \& x' \sim_V y'')
\]

Now take any point \( z' \in Z' \subset U \) and a point \( y \in Z(K) \). We want to prove \( p_H(Z') \supseteq Z(K) \), and thus it is enough to prove there exists \( y_1 \in U, \ p_H(y_1) = y, z' \sim_Z y_1 \). We know that there exist \( y_2 \in U, z' \sim_{Z,G} y_2 \), due to Axiom 5.2.1.1. Since \( Z = V_g \) for some \( g \in U^{n-1} \), we also have \( (g', z') \sim_{V_g} (g', y_2) \), and taking \( p_H(g') = g, x' = (g', z'), y' = (g', y_2) \), Axiom 5.2.2.2 gives the conclusion

\[
\exists y''(y'' \sim_G y' \& \text{pr } x' = \text{pr } y'' \& x' \sim_V y'').
\]

The conclusion says points \( x', y'' \in U^n, p_H(x'), p_H(y') \in Z_i \) lie in the same connected component of \( p_H^{-1}(V) \), are \( \sim_G \)-equivalent, and lie above the same point \( g', p_H(g') = g \). Then by Lemma 3.2.0.5 we know that \( p_H(x'), p_H(y') \) lie in the same connected component of the corresponding preimage of \( Z_i \). By definition of \( Z' \), this means \( \text{pr }_2 y' \in Z' \). Thus, we have proved that \( p_H(Z') = Z(K) \) is surjective.

Now the following by now standard argument concludes the proof.

The covering property implies that

\[
p_H^{-1}(Z(K)) = \bigcup_{h \in H} hZ' = HZ';
\]

indeed, by properties of \( Z \) we know that the relations \( x' \sim_{Z,G} y' \) are equivalence relations for all \( G \triangleleft H \). Moreover, we know that any two equivalence classes are conjugated by the action of an element of \( H \); this is so because the covering property implies that there is an element of each of the classes above each element of \( Z(K) \). This implies the lemma.

We single out the following part of the proof as a corollary.

Recall that \( \sim^{c} \) means “to lie in the same connected component of”.

**Corollary 5.3.3.2 (the covering property).** For a subvariety \( Z \subset A(K) \), \( x' \sim_{Z,G} y' \implies \exists y''(y'' \sim_G y' \& x' \sim_Z y'') \).

**Proof.** The proof of the lifting property above proves the corollary for \( Z \subset A^H(K) \) such that the relations \( \sim^{c}_Z \) and \( \sim_Z \) are equivalent. However, by Decomposition Lemma any set \( p_H^{-1}(Z) \) can be decomposed into a union of such sets; then going from one irreducible component to another one intersecting it gives the corollary.

**Corollary 5.3.3.3 (Topology on \( U \)).** The collection of co-etale closed subsets of \( U \) forms a topology with a descending chain conditions on irreducible sets. A basic co-etale closed set possesses an irreducible decomposition as a union of a finite number of basic co-etale closed sets whose co-etale connected components are co-etale irreducible. A union of irreducible components of a co-etale closed set is co-etale closed.

That is,

1. the collection of co-etale closed subsets on \( U^n, n > 0 \) forms a topology. The projection and inclusion maps \( \text{pr } : U^n \to U^{n'}, (x_1, \ldots, x_n) \to (x_{i_1}, \ldots, x_{i_m}) \) and \( i : U^n \hookrightarrow U^{n'}, (x_1, \ldots, x_n) \to (x_{i_1}, \ldots, x_{i_m}, c_{m'}, \ldots, c_m) \) are continuous.

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2. There is no infinite decreasing chain \( \subseteq U_{i+1} \subseteq U_i \subseteq \ldots \subseteq U_0 \) of co-etale closed irreducible sets.

3. A union of irreducible components of a co-etale closed set is co-etale closed.

4. A set is basic co-etale closed iff it is a union of connected components of a finite number of \( H \)-invariant sets, for some \( H <_{\text{fin}} \Gamma \) a finite index subgroup of \( \Gamma \).

5. A basic co-etale closed set is a union of a finite number of basic co-etale closed sets whose co-etale connected components are co-etale irreducible. Moreover, those sets may be taken so that their connected components within the same set are translates of each other by the action of a finite index subgroup \( H <_{\text{fin}} \Gamma \).

\textbf{Proof.} The last item is a reformulation of Decomposition Lemma. All the items but (1) trivially follow from (5).

Let us prove the intersection of two co-etale closed set \( W_1 \) and \( W_2 \) is co-etale closed.

Assume \( W_1 \) and \( W_2 \) are unions of connected component of \( H \)-invariant sets \( HW_1 \) and \( HV_2 \). The intersection \( HW_1 \cap HV_2 \) is \( H \)-invariant and the set \( W_1 \cap W_2 \) is a union of the connected components of \( HW_1 \cap HV_2 \). The intersection \( HW_1 \cap HV_2 = p_H^{-1}(p_H(W_1) \cap p_H(V_2)) \) is co-etale closed by definition, and thus its connected components are also co-etale closed. By definition this implies \( W_1 \cap W_2 \) is co-etale closed.

An infinite intersection is closed by definition.

The descending chain condition follows from the fact that an irreducible subset of an irreducible set necessarily has smaller dimension. \( \square \)

\subsection{5.3.4. Semi-Properness (SP)}

Let \( W' \subseteq U \) be an irreducible closed subset of \( U \), i.e. a subset of \( U \) defined by

\[ x \sim_W a_1 \& \ldots \& x \sim_W a_n \]

where \( a_1, \ldots, a_n \in U \) are such that

\[ \forall y \forall z \left( \bigwedge_{1 \leq i \leq n} y \sim a_i \& \bigwedge_{1 \leq i \leq n} z \sim a_i \implies y \sim_W z \right) . \]

Such a set \( W' \) we call an irreducible component of closed set defined by \( x \sim_W x \), or simply an irreducible component of relation \( \sim_W \).

\textbf{Lemma 5.3.4.1 (Chevalley Lemma, (SP)).} A projection of a co-etale irreducible closed set is co-etale closed.

\textbf{Proof.} Let \( W' \) be such an irreducible set, and let \( W' = \text{Cl}_{\text{pr}} W' \) be the least closed set containing its closure. By definition of \( V' \) \( p_H(\text{pr } W') \subseteq p_H(V') \); and by definition of closure \( V' \subseteq \text{pr } HW' = p_H^{-1}(p_H(W')) \); the set \( \text{pr } p_H(W') \) is closed by Chevalley Lemma for projective algebraic varieties. The inequalities imply \( p_H(\text{pr } W') = p_H(W') \) for every subgroup \( H <_{\text{fin}} \Gamma \).

A deck transformation leaving \( W' \) invariant, also leaves \( V' \) invariant, i.e. \( \text{pr } \pi(W') \subseteq \pi(V') \). On the other hand, the equality \( p_H(\text{pr } W') = p_H(V') \) implies for any \( H <_{\text{fin}} \Gamma \), \( \text{pr } \pi(W') / H = \pi(V') / H \).

Let us now use Axiom 5.2.2.4 to show that this implies that \( \text{pr } (\pi(W) \cap [H \times H]) = \pi(V') \cap H \).

Let us now prove that \( \pi(W') \cap H \times H \) is finitely generated for some \( H <_{\text{fin}} \Gamma \).

We know by Corollary to Lemma 2.8.6.1 that \( W' = Y'_{\mathbb{Q}} \) is a fibre of a \( \mathbb{Q} \)-defined set \( Y' \) over a point \( g' \) such that \( p_H(g') \in \text{pr } p_H(Y') \mathbb{Q} \)-generic.

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We know that for every $G \triangleleft_{\text{fin}} H$, for a connected component $Y_G$ of $p_G p_H^{-1}(Y)$, the intersection $Y_G \cap g' \times p_G p_H^{-1}(Y_g)$ is connected; geometrically, that means that a lifting of $W = Y_g \subset Y$ along the covering map $Y_G \rightarrow Y$ is a fibre of $Y$. This holding for every $G \triangleleft_{\text{fin}} H$, it implies that for $Y'$ a connected component of $p_H^{-1}(Y)$, the intersection $Y'_{g'} \cap g' \times p_H^{-1}(Y_g)$ is connected, and therefore it coincides with a connected component of $p_H^{-1}(Y_g) = p_H^{-1}(W)$. Moreover, this implies that if $h \in H$ is such that $h Y'_{g'} \subset p_H^{-1}(Y_g) = p_H^{-1}(W)$, then $h Y'_{g'} \subset Y'_{g'}$, i.e. $h \in \pi(Y_{g'}) \cap H = \pi(Y_g') \cap H$. Thus, to prove that $\pi(W) \cap H = \pi(Y_{g'}) \cap H$ is finitely generated, it is enough to prove that $\pi(Y_{g'}) \cap H$ is finitely generated. However, the latter is claimed by Axiom 5.2.2.4 for every variety $Y$ defined over $\mathbb{Q}$.

Let $g_1, \ldots, g_n$ be the generators of $\pi(W') \cap [H \times H]$. Now take $\tau \in \pi(V') \cap H$, $\tau(V') = V'$. We know that $\tau / G \in \text{pr} \pi(W')/G$, for every $G \triangleleft_{\text{fin}} H$, and therefore $\tau$, up to $\sim_G$, is expressible as a product of $g_1, \ldots, g_n$. In other words, that means that $x'$ and $\tau x'$ can be joined by a sequence of points $x' = h_1, h_2, \ldots, h_n = \tau x'$ such that $h_{i+1} = g_j h_i$ for all $1 \leq i \leq n$, and where $n = n(G)$ depends on subgroup $G$. By Axiom 5.2.2.5 there is a uniform bound on such $n = n(G)$, and $\tau$ is expressible as a product of $g_1, \ldots, g_n$, and therefore belongs to $\text{pr} \pi(W')$.

Now we finish the proof by the covering property argument similar to the topological proof of Chevalley Lemma in complex case.

Let $V_0 \subset \text{pr} p_H(W') \subset V$ where $V_0 \subseteq V$ is open in $V$; then $V$ is irreducible. Recall $V' = \text{Cl} \text{pr} W'$ and take $V_0' = V' \cap p_H^{-1}(V_0)$; we know $V_0' \subset V'$ is open in $V'$. We also know $V_0' \subset \text{Cl} \text{pr} W'$.

Take $v' \in V_0'$ and take $w' \in W'$, $\text{pr}_H(w') = p_H(v') \in V_0 \subset \text{pr} W'$; such a point $w'$ in $W'$ exists by the covering property. Now, $\text{pr}_H(w') \in V'$, and thus $\gamma_0 \in \pi(V')$ where $\gamma_0$ is defined by $v' = \gamma_0 \text{pr} w'$. Condition $\text{pr}_H(w') = p_H(v') \in A^H(K)$ implies $\gamma_0 \in H$. Thus the inclusion $\text{pr} \pi(W') \cap H = \pi(V') \cap H$ implies there exists $\gamma_1 \in \pi(W')$, $\text{pr} \gamma_1 = \gamma_0$, and thus $v' = \gamma_0 \text{pr} w' = \text{pr} (\gamma_1 w')$, and the Chevalley lemma is proven.

6. Homogeneity and stability over models

In the §§ above we have established the main properties of the co-etale topology on $U$ (and its Cartesian powers $U^n$). That allows us to define and prove the basic properties of $\Theta$-generic points, for $\Theta$ an algebraically closed subfield of $K$.

The notion of a $\Theta$-generic point extends to $U$ in a natural way. Recall that for a closed $\Theta$-defined set $V'$, the set $\Theta_{\text{cl}e} V'$ is the set of all $\Theta$-generic points of $V'$. Recall also that a set of $\Theta$-generic points of a $\Theta$-defined set is called $\Theta$-constructible.

**Lemma 6.0.4.2 (Homogeneity).** Any structure $U \models \mathcal{X}$ is model homogeneous, i.e. the projection of a $\Theta$-constructible set is $\Theta$-constructible, for any algebraically closed subfield $\Theta$ of the ground field.

**Proof.** First note that a point $w' \in W'$ in an irreducible set $W'$ is $\Theta$-generic iff $p(w') \in p(W')$ is $\Theta$-generic. By Chevalley Lemma, the fibre $W'_g$ is non-empty for $g' \in \text{pr} W' \Theta$-generic. Moreover, by Lemma 2.7.0.10 a connected component of fibre $W_g, g = p(g')$ always contains a $\Theta$-generic point $w \in W$ of $W$. The lifting $w', p(w') = w$ is always $\Theta$-generic, and we may find such a lifting in any connected component of a fibre over a generic point. This implies the lemma.

**Definition 6.0.4.3.** Let $U, U_1, U_2 \models \mathcal{X}$ be $L_{\mathbb{A}}$-models of $\mathcal{X}(\mathbb{A}(\mathbb{C}))$ and $U \subset U_1 \cap U_2$. We say that tuples $a \in U_1^n$ and $b \in U_2^n$ have the same syntactic quantifier-free type over $U$ in class $\mathcal{R}$ if $a$ and $b$ satisfy the same quantifier-free $L_{\mathbb{A}}$-formulae with parameters in $U$.

**Definition 6.0.4.4.** A class $\mathcal{R}$ of $L_{\mathbb{A}}$-structures is syntactically stable over countable submodels iff for any countable structure $U \models \mathcal{R}$, the set of complete $L_{\mathbb{A}}$-types over a structure $U$ realised in a structure $U' \models \mathcal{R}$ is at most countable.
Definition 6.0.4.5. A class $\mathcal{R}$ of $L_A$-structures is quantifier-free syntactically stable over countable submodels iff there are only countably many quantifier-free syntactic types in class $\mathcal{R}$ over any countable model $U \in \mathcal{R}$.

Lemma 6.0.4.6 (Stability over submodels). Assume $A$ is LERF. The class of $L_A$-models of $\mathcal{X}(A(\mathbb{C}))$ is quantifier-free syntactically stable over submodels.

Proof. If $U \prec U'$ is an elementary substructure, then $U = U'(\Theta) = \{ u \in U' : p(u) \in A(\Theta) \}$, for some algebraically closed subfield $\Theta$.

Every positive quantifier-free $L_A$-formula over $U$ determines a closed set defined over $\Theta$. For every tuple $v' \in U'$, there is a least closed set $V' = \text{Cl}_\Theta(v')$ containing $v'$ and defined over $\Theta$; it is irreducible, and is a connected component of an algebraic subvariety $V/\Theta$ of $A^H$ defined over $\Theta$, for some $H <_{\text{fin}} \Gamma$. Moreover, $\text{Cl}_\Theta(v')$ has a $\Theta$-point $v'_p$. Thus, the quantifier-free $L_A$-type of tuple $v'$ is determined by the point $v'_p \in U$ and a subvariety $V/\Theta$. Therefore, there are only countable number of such types, which implies that class $\mathcal{R}$ is quantifier-free syntactically stable over submodels.

Theorem 6.0.4.7 (Homogeneity and Stability of class $\mathcal{R}$). Assume $A$ is LERF.

All structures $L_A$-models of $\mathcal{X}(A(\mathbb{C}))$ are model homogeneous. The class of $L_A$-models of $\mathcal{X}(A(\mathbb{C}))$ is syntactically quantifier-free stable over countable submodels.

Proof. Implied by preceding two lemmata.

Finally, we may state Theorem 6.0.4.8, which was the goal of the paper.

Theorem 6.0.4.8 (Model Stability of $\mathcal{X}(U)$). Let $A$ be a smooth projective algebraic variety which is LERF. Let $L_A$ be the countable language defined in Def. 3.1.0.2. Then

(286) Any two models $U_1 \models \mathcal{X}$ and $U_2 \models \mathcal{X}$ of axiomatisation $\mathcal{X}$ and of cardinality $\aleph_1$, such that there exist a common countable submodel $U_0 \models \mathcal{X}$, $U_0 \subset U_1$ and $U_0 \subset U_1$

are isomorphic, $U_1 \cong_{L_A} U_2$, and, moreover, the isomorphism $\varphi$ is identity on $U_0$.

Proof. This is closely related to Proposition 6.0.4.7; however, let us prove this directly in an explicit manner; in this argument we try to put an emphasis on the properties of the topology, although this could also be treated as a very common model-theoretic argument.

We will prove that every partial $L_A$-isomorphism $f : U_1 \rightarrow U_2$, $f(a) = b$, $a \in U_1^n$, $f|_{U_0} = \text{id}|_{U_0}$, $n \in \mathbb{N}$ finite, defined on $U_0 \cup \{a_1, \ldots, a_n\}$, can be extended to $U_0 \cup \{a_1, \ldots, a_n\} \cup \{c\}$, $f(c) \in U_2$ for any element $c \in U_1$. This allows to extend a partial $L_A$-isomorphism from a countable model to its countable extension. This is enough: by taking unions of chains of countable submodels we get isomorphism between models of cardinality $\aleph_1$. Note that one cannot get isomorphism between models of cardinality $\aleph_2$ in this way.

Let $V_1 = \text{Cl}_{U_0}(a)$, $W_1 = \text{Cl}_{U_0}(a,c)$ be the minimal closed irreducible subsets containing points $a \in U_1^n$ and $(a,c) \in U_1^{n+1}$; let $V_2 = \text{Cl}_{U_0}(f(a))$ be the corresponding subset of $U_2$. Since $f$ is an $L$-isomorphism, sets $V_1$ and $V_2$ are defined by the same $L$-formulae with parameters in $U_0$.

Take a subgroup $H <_{\text{fin}} \Gamma$ sufficiently small such that $V_1, V_2, W_1, W_2$ are connected components of $p_H^{-1}p_H(V_1), p_H^{-1}p_H(V_2), p_H^{-1}p_H(W_1), p_H^{-1}p_H(W_2)$, respectively. Pick points $v_1, w_1 \in U_0$ such that $v_1 \in V_1, v_2 \in W_1, v_2 \in W_2, w_2 \in W_2$.

Now, by definition of $W_2$ we have $p_H W_2 = p_H W_2$, and also $w_2 \in V_2$; choose $c' \in U_2$ such that $(p_H(b), p_H(c')) \in p_H(W_2)$ is a $U_0$-generic point of $p_H(W_2)$. Then by the lifting property for $W_2$ there exists a point $(b', c'') \in W_2$ such that $p_H(b') = p_H(b), p_H(c'') = p_H(c')$. However, this implies that $b' \in p_H W_2 \subset V_2$ is a $U_0$-generic point of $V_2$. Therefore by the homogeneity properties in Lemma 6.0.4.2, or equivalently because the projection $p_H W_2$ is a closed set definable over $U_0$, this
implies $V_2 \subseteq \text{pr}_W W_2$, and, in particular, there exists $d \in U_1$ such that $(b, d) \in W_2$ is a $U_0$-generic point. Now set $f(c) = d$. By construction, the points $(a, c) \in U_1$ and $(b, d) \in U_2$ lie in the same $U_0$-definable closed sets, and, since every basic relation of $L_A$ defines a closed $U_0$-defined set, this implies that $f$ is indeed an $L_A$-isomorphism, as required. 

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