Two weight commutators on spaces of homogeneous type and applications

Xuan Thinh Duong, Runming Gong, Marie-Jose S. Kuffner, Ji Li, Brett D. Wick and Dongyong Yang

Abstract: In this paper, we establish the two weight commutator of Calderón–Zygmund operators in the sense of Coifman–Weiss on spaces of homogeneous type, by studying the weighted Hardy and BMO space for $A_2$ weight and by proving the sparse operator domination of commutators. The main tool here is the Haar basis and the adjacent dyadic systems on spaces of homogeneous type, and the construction of a suitable version of a sparse operator on spaces of homogeneous type. As applications, we provide a two weight commutator theorem (including the high order commutator) for the following Calderón–Zygmund operators: Cauchy integral operator on $\mathbb{R}$, Cauchy–Szegö projection operator on Heisenberg groups, Szegö projection operators on a family of unbounded weakly pseudoconvex domains, Riesz transform associated with the sub-Laplacian on stratified Lie groups, as well as the Bessel Riesz transforms (one-dimension and high dimension).

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1 Introduction and Statement of Main Results

It is well-known that Coifman, Rochberg and Weiss [8] characterized the boundedness of the commutator \([b, R_i]\) acting on Lebesgue spaces in terms of BMO with \(R_j = \frac{\partial}{\partial x_j} \Delta^{-1/2}\) the \(j\)-th Riesz transform on the Euclidean space \(\mathbb{R}^n\), which extended the work of Nehari [41] about Hankel operators from complex setting to the real setting \(\mathbb{R}^n\). Later, Bloom in [3] established the characterisation of weighted BMO in terms of boundedness of commutators \([b, H]\) in the two weight setting, where \(H\) is the Hilbert transform on \(\mathbb{R}\).

Recent remarkable results were achieved by Holmes–Lacey–Wick [21] giving the characterisation of weighted BMO space on \(\mathbb{R}^n\) in terms of boundedness of commutators of Riesz transforms, and by Lerner–Ombrosi–Rivera-Ríos [32, 33] in terms of boundedness of commutators of Calderón–Zygmund operators with homogeneous kernels \(\Omega(\frac{x}{|x|^n})\frac{1}{|x|}\), and Hytönen [25] in terms of boundedness of commutators of a more general version of Calderón–Zygmund operators and weighted BMO functions on \(\mathbb{R}^n\). Meanwhile, two weight commutator has also been studied extensively in different settings, see for example [13, 15, 20, 27].

We note that to get the lower bound of the two weight commutator for Riesz transforms (for Hilbert in one dimension) in terms of weighted BMO space, the first proofs used spherical harmonic expansion for the Riesz (Hilbert) kernels, which relies on properties of the Fourier transform of the Riesz (Hilbert) kernels. A similar method of expansion of the Riesz transform associated with Neumann Laplacian was used in [13] for a larger class of \(A_p\) weights and for the BMO space associated with Neumann Laplacian which is strictly larger than classical BMO. In [32], concerning the two weight commutator for Calderón–Zygmund operators associated with homogeneous kernel \(\Omega(\frac{x}{|x|})\frac{1}{|x|}\), the proof of the lower bound was obtained by assuming suitable conditions on the homogeneous function \(\Omega\), see also [19, 20]. More recently, Hytönen [25] studied the two weight commutator for Calderón–Zygmund operators and proposed a condition denoted by the “non-degenerate Calderón–Zygmund kernel”, then the lower bound was obtained by a construction of factorisation. Also, in [14, 15], they established a version of a pointwise kernel lower bound for the Riesz transform associated to sub-Laplacian on stratified Lie groups, which covers the Heisenberg groups, and used this kernel lower bound to obtain the two weight commutator result following the idea in [32].

However, there are a few other important Calderón–Zygmund operators (not built on the Euclidean space setting) whose kernels do not have connection to the Fourier transform and are not of homogeneous type such as \(\Omega(\frac{x}{|x|^n})\frac{1}{|x|}\). Moreover, whether the kernels fall into Hytönen’s “non-degenerate Calderón–Zygmund kernel” has not been studied before and hence the two weight commutator estimates and higher order commutator are unknown.

For example, the Riesz transform from Muckenhoupt–Stein [40]: \(R_\lambda := -\frac{d}{dx}(\Delta_\lambda)^{-\frac{1}{2}}\), associated with the Bessel operator on \(\mathbb{R}_+\):

\[
\Delta_\lambda := -\frac{d^2}{dx^2} - \frac{2\lambda}{x} \frac{d}{dx}, \quad x > 0, \lambda > -\frac{1}{2},
\]

and the Riesz transform \(R_{\lambda,j} := \frac{d}{dx}(\Delta_\lambda)^{(n+1)}(\Delta_\lambda)^{-\frac{j}{2}}, j = 1, \ldots, n + 1\), associated with the Bessel operator \(\Delta_\lambda^{(n+1)}\) on \(\mathbb{R}_+^{n+1}\) studied in Huber [23]:

\[
\Delta_\lambda^{(n+1)} = -\frac{d^2}{dx_1^2} \cdots - \frac{d^2}{dx_n^2} - \frac{d^2}{dx_{n+1}^2} - \frac{2\lambda}{x_{n+1}} \frac{d}{dx_{n+1}}.
\]

Another example is the Cauchy–Szegö projection operator \(\mathcal{C}\) (for all the notation below we refer to Section 2 in Chapter XII in Stein [44]), which is the orthogonal projection from \(L^2(\mathbb{R}^n)\).
to the subspace of functions \( \{ F^b \} \) that are boundary values of functions \( F \in \mathcal{H}^2(U^n) \). The associated Cauchy–Szegő kernel is as follows.

\[
C(f)(x) = \int_{\mathbb{H}^n} K(y^{-1} \ast x) f(y) dy,
\]

where \( K(x) = -\frac{2}{\pi} \left( \frac{|t + i|}{|t + i|^{2-n}} \right) \) for \( x = [\zeta, t] \in \mathbb{H}^n = \mathbb{C}^n \times \mathbb{R} \).

Then it is natural to study the following question: is there a setting, by which the characterisation of two weight commutators and the related \( \text{BMO} \) space for Calderón–Zygmund operators \( T \) can be obtained, that can be applied to Calderón–Zygmund operators such as the Bessel Riesz transform, the Cauchy–Szegő projection operator on Heisenberg groups, and many other examples?

To address this question we work in a general setting: spaces of homogeneous type introduced by Coifman and Weiss in the early 1970s, in [9], see also [10]. We say that \((X, d, \mu)\) is a space of homogeneous type if \( d \) is a quasi-metric on \( X \) and \( \mu \) is a nonzero measure satisfying the doubling condition. A quasi-metric \( d \) on a set \( X \) is a function \( d : X \times X \rightarrow [0, \infty) \) satisfying (i) \( d(x, y) = d(y, x) \geq 0 \) for all \( x, y \in X \); (ii) \( d(x, y) = 0 \) if and only if \( x = y \); and (iii) the quasi-triangle inequality: there is a constant \( A_0 \in [1, \infty) \) such that for all \( x, y, z \in X \),

\[
d(x, y) \leq A_0[d(x, z) + d(z, y)].
\]

We say that a nonzero measure \( \mu \) satisfies the doubling condition if there is a constant \( C_\mu \) such that for all \( x \in X \) and \( r > 0 \),

\[
\mu(B(x, 2r)) \leq C_\mu \mu(B(x, r)) < \infty,
\]

where \( B(x, r) \) is the quasi-metric ball by \( B(x, r) := \{ y \in X : d(x, y) < r \} \) for \( x \in X \) and \( r > 0 \). We point out that the doubling condition (1.2) implies that there exists a positive constant \( n \) (the upper dimension of \( \mu \)) such that for all \( x \in X \), \( \lambda \geq 1 \) and \( r > 0 \),

\[
\mu(B(x, \lambda r)) \leq C_\mu \lambda^n \mu(B(x, r)).
\]

Throughout this paper we assume that \( \mu(X) = \infty \) and that \( \mu(\{x_0\}) = 0 \) for every \( x_0 \in X \).

We now recall the singular integral operator on spaces of homogeneous type in the sense of Coifman and Weiss.

**Definition 1.1.** We say that \( T \) is a Calderón–Zygmund operator on \((X, d, \mu)\) if \( T \) is bounded on \( L^2(X) \) and has the associated kernel \( K(x, y) \) such that \( T(f)(x) = \int_X K(x, y) f(y) d\mu(y) \) for any \( x \not\in \text{supp}\, f \), and \( K(x, y) \) satisfies the following estimates: for all \( x \neq y \),

\[
|K(x, y)| \leq \frac{C}{V(x, y)},
\]

and for \( d(x, x') \leq (2A_0)^{-1} d(x, y) \),

\[
|K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \leq \frac{C}{V(x, y)} \omega \left( \frac{d(x, x')}{d(x, y)} \right),
\]

where \( V(x, y) = \mu(B(x, d(x, y))) \) and by the doubling condition we have that \( V(x, y) \approx V(y, x) \), \( \omega : [0, 1] \rightarrow [0, \infty) \) is continuous, increasing, subadditive, \( \omega(0) = 0 \).
We say that $\omega$ satisfies the Dini condition if \( \int_0^1 \omega(t) \frac{dt}{t} < \infty \).

Let $T$ be a Calderón–Zygmund operator on $X$. Suppose $b \in L^1_{\text{loc}}(X)$ and $f \in L^p(X)$. Let $[b, T]$ be the commutator defined by

$$[b, T]f(x) := b(x)T(f)(x) - T(bf)(x).$$

The iterated commutators $T^m_b$, $m \in \mathbb{N}$, are defined inductively by

$$T^m_b f(x) := [b, T^{m-1}_b]f(x), \quad T^1_b f(x) := [b, T]f(x).$$

Next we use $A_p$, $1 \leq p \leq \infty$, to denote the Muckenhoupt weighted class on $X$ (see the precise definition of $A_p$ in Section 2), and the weighted BMO on $X$ is defined as follows (the Euclidean version of weighted BMO was first introduced by Muckenhoupt and Wheeden [39]).

**Definition 1.2.** Suppose $w \in A_\infty$. A function $b \in L^1_{\text{loc}}(X)$ belongs to $\text{BMO}_w(X)$ if

$$\|b\|_{\text{BMO}_w(X)} := \sup_B \frac{1}{w(B)} \int_B |b(x) - b_B| \, d\mu(x) < \infty,$$

where $b_B := \frac{1}{\mu(B)} \int_B b(x) \, d\mu(x)$ and the supremum is taken over all balls $B \subset X$.

Our first main result is the following theorem.

**Theorem 1.3.** Suppose $1 \leq p < \infty$, $\lambda_1, \lambda_2 \in A_p$, $\nu := \lambda_1^{1/p} \lambda_2^{-1} \nu$ and $m \in \mathbb{N}$. Suppose $b \in \text{BMO}_{\frac{\nu}{\lambda_m}}(X)$. Then for any Calderón–Zygmund operator $T$ as in Definition 1.1 with $\omega$ satisfying the Dini condition, there exists a positive constant $C_1$ such that

$$\|T^m_b : L^p_{\lambda_1}(X) \to L^p_{\lambda_2}(X)\| \leq C_1 \|b\|_{\text{BMO}_{\frac{\nu}{\lambda_m}}(X)}^m \left( [\lambda_1]_{A_p} [\lambda_2]_{A_p} \right) \frac{m+1}{2} \max\{1, \frac{1}{p-1}\}.$$ 

To obtain the upper bound, we characterise the sparse system and then use the idea from [32] to build a suitable version of a sparse operator on a space of homogeneous type. Here we apply the tool of adjacent dyadic systems from [26], the explicit construction of Haar basis from [29], and we have to allow suitable overlapping for the sparse sets due to the partition and covering of the whole space via quasi-metric balls.

To consider the lower bound of the commutator, we assume that the Calderón–Zygmund operator $T$ as in Definition 1.1 with $\omega$ satisfying $\omega(t) \to 0$ as $t \to 0$, and that $T$ satisfies the following “non-degenerate” condition:

**There exist positive constant $c_0$ and $C$ such that for every $x \in X$ and $r > 0$, there exists $y \in B(x, Cr) \setminus B(x, r)$ satisfying

$$|K(x, y)| \geq \frac{1}{c_0 \mu(B(x, r))}.$$**

Note that in $\mathbb{R}^n$, this “non-degenerated” condition was first proposed in [24], and a similar assumption on the behaviour of the kernel lower bound was proposed in [32]. On stratified Lie groups, a similar condition of the Riesz transform kernel lower bound was verified in [14]. Then we have the following lower bound.

**Theorem 1.4.** Suppose $1 < p < \infty$, $\lambda_1, \lambda_2 \in A_p$, $\nu := \lambda_1^{1/p} \lambda_2^{-1} \nu$ and $m \in \mathbb{N}$. Suppose $b \in L^1_{\text{loc}}(X)$ and that $T$ is a Calderón–Zygmund operator as in Definition 1.1 and satisfies the non-degenerated condition (1.6). Also suppose that $T^m_b$ is bounded from $L^p_{\lambda_1}(X)$ to $L^p_{\lambda_2}(X)$. Then $b \in \text{BMO}_{\frac{\nu}{\lambda_m}}(X)$, and there exists a positive constant $C_2$ such that

$$\|b\|_{\text{BMO}_{\frac{\nu}{\lambda_m}}(X)} \leq C_2 \left\|T^m_b : L^p_{\lambda_1}(X) \to L^p_{\lambda_2}(X)\right\|.$$
Based on the characterisation of $BMO_\nu(X)$ via commutators $T_0^1 = [b, T]$, we further have the weak factorisation for the weighted Hardy space $H^1_\nu(X)$ as follows.

**Theorem 1.5.** Suppose $1 < p < \infty$, $\lambda_1, \lambda_2 \in A_p$ and $\nu := \lambda_1^{1/p} \lambda_2^{-1/p}$. Let $p'$ be the conjugate of $p$ and $\nu' := \nu^{-1}$. For any $f \in H^1_{\nu'}(X)$, there exist numbers $\{\alpha_j^k\}_{k,j}$, functions $\{g_j^k\}_{k,j} \subset L^p_{\lambda_1}(X)$ and $\{h_j^k\}_{k,j} \subset L^p_{\lambda_2}(X)$ such that

$$f = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \alpha_j^k \Pi(g_j^k, h_j^k)$$

in $H^1_\nu(X)$, where the operator $\Pi$ is defined as follows: for $g \in L^p_{\lambda_1}(X)$ and $h \in L^p_{\lambda_2}(X)$,

$$\Pi(g, h) := gTh - hT^*g,$$

where $T$ is a Calderón–Zygmund operator as in Definition 1.1 and satisfies the non-degenerated condition (1.6). Moreover, we have

$$\|f\|_{H^1_\nu(X)} \approx \inf \left\{ \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \|\alpha_j^k\| \left\|g_j^k\right\|_{L^p_{\lambda_1}(X)} \left\|h_j^k\right\|_{L^p_{\lambda_2}(X)} : f = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \alpha_j^k \Pi(g_j^k, h_j^k) \right\},$$

(1.8)

where the implicit constants are independent of $f$.

As applications, besides the classical Hilbert transform, Riesz transform and the Calderón–Zygmund operators with homogeneous kernels $\Omega(\frac{x}{|x|}, \frac{1}{|x|^n})$ on $\mathbb{R}^n$ (studied in [21, 32]), we use our main theorems to obtain the two weight commutator result of the following operators:

1. the Cauchy integral operator $C_A$ along a Lipschitz curve $z := x + iA(x)$, $x \in (-\infty, \infty)$ and $A' \in L^\infty(\mathbb{R})$;
2. The Cauchy–Szegő projection operator on Heisenberg group $\mathbb{H}^n$;
3. The Szegő projection operator on a family of weakly pseudoconvex domains;
4. Riesz transforms associated with Sub-Laplacian on stratified Lie groups $\mathcal{G}$;
5. Riesz transforms associated with Bessel operator $\Delta_\lambda$ on $\mathbb{R}_+$ for $\lambda > -1/2$;
6. Riesz transforms associated with higher order Bessel operator $\Delta_{n,\lambda}$ on $\mathbb{R}^{n+1}_+$ for $\lambda > -1/2$.

The definitions of the above operators will be given in Section 7. We have the following result.

**Theorem 1.6.** Let $T$ be one of the operators listed above and let $(X, d, \mu)$ be the underlying space adapted to $T$. Suppose $1 < p < \infty$, $\lambda_1, \lambda_2 \in A_p$, $\nu := \lambda_1^{1/p} \lambda_2^{-1/p}$ and $m \in \mathbb{N}$. Suppose that $b \in L^1_{\text{loc}}(X)$. Then we have

$$\|b\|_{BMO}^{\nu, X} \approx \|T_b^{\nu, m} : \mathbb{L}^p_{\lambda_1}(X) \to \mathbb{L}^p_{\lambda_2}(X)\|.$$

(1.9)

Moreover, based on the result above for $m = 1$ and on the duality, the corresponding weighted Hardy space $H^1_\nu(X)$ has a weak factorisation as in (1.7).
To prove this theorem, the key step is to verify that all these operators listed above satisfy the conditions as in Definition 1.1 and the homogeneous condition as in (1.6). We point out that such verification for Cauchy integral operator $C_A$ is direct. The verifications of Cauchy–Szegő projection operator on Heisenberg group, the Szegő projection operator on a family of weakly pseudoconvex domains and the Riesz transforms associated with Sub-Laplacian on stratified Lie groups can be derived based on the results in [44, Chapter XII], [18] and [14], respectively. The verification for Riesz transforms associated with Bessel operator $\Delta_\lambda$ on $\mathbb{R}^+$ for $\lambda > 0$ can be derived from the result in [40], while for $\lambda \in (-1/2, 0)$ is new here. The verification for Riesz transforms associated with higher order Bessel operator is totally new, especially the pointwise kernel lower bound of this Riesz transform.

We now address our result Theorem 1.6 with respect to the 6 examples above, respectively:

1. The unweighted result was obtained in [34] when $m = 1$, and the two weight result is new here for $m \geq 1$;

2. This result is new, even the unweighted version is unknown;

3. This result is new, even the unweighted version is unknown;

4. This result was obtained in [15] when $m = 1$ and is new here when $m > 1$.

5. The unweighted result was obtained in [16] when $\lambda > 0$ and $m = 1$, the two weight result is new here for $m \geq 1$ and for all $\lambda > -1/2$;

6. This result is new, even the unweighted version is unknown;

This paper is organised as follows. In Section 2 we recall the necessary preliminaries on spaces of homogeneous type. In Section 3, we first characterise the sparse system equivalently via the $\Lambda$-Carleson packing condition and the $\eta$-sparse condition, and then borrowing the idea from [32], we study the sparse operators and its domination of commutator on spaces of homogeneous type, and by using this as a main tool, in Section 4 we obtain the upper bound of two weight commutator, i.e., Theorem 1.3. In Section 5 we provide the lower bound of two weight commutator, i.e., Theorem 1.4, by combining the ideas in [33] and [25]. In Section 6 we provide a study of weighted Hardy space and its duality on spaces of homogeneous type, and provide the proof of Theorem 1.5. In Section 7 we provide the applications where we address the new points in this paper. In the last section we also provide a new proof of the lower bound of two weight commutator in the product setting for little bmo space on spaces of homogeneous type. Note that in $\mathbb{R}^n \times \mathbb{R}^m$, this was first studied by [22] by using the Fourier transform for the Riesz transform kernel.

Throughout the paper, we denote by $C$ and $\tilde{C}$ positive constants which are independent of the main parameters, but they may vary from line to line. For every $p \in (1, \infty)$, we denote by $p'$ the conjugate of $p$, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$. If $f \leq Cg$ or $f \geq Cg$, we then write $f \lesssim g$ or $f \gtrsim g$; and if $f \lesssim g \lesssim f$, we write $f \approx g$.

2 Preliminaries on Spaces of Homogeneous Type

Let $(X, d, \mu)$ be a space of homogeneous type as mentioned in Section 1.
2.1 A System of Dyadic Cubes

In \((X, d, \mu)\), a countable family \(\mathcal{D} := \bigcup_{k \in \mathbb{Z}} \mathcal{D}_k\), \(\mathcal{D}_k := \{Q^k_\alpha : \alpha \in \mathcal{A}_k\}\), of Borel sets \(Q^k_\alpha \subseteq X\) is called a system of dyadic cubes with parameters \(\delta \in (0, 1)\) and \(0 < a_1 \leq A_1 < \infty\) if it has the following properties:

\[
X = \bigcup_{\alpha \in \mathcal{A}_k} Q^k_\alpha \quad \text{(disjoint union) for all } k \in \mathbb{Z};
\]

for each \((k, \alpha)\) and each \(\ell \leq k\), there exists a unique \(\beta\) such that \(Q^k_\alpha \subseteq Q^\ell_\beta\);

\[
Q^k_\beta + 1 \subseteq Q^k_\alpha, \quad Q^k_\alpha = \bigcup_{Q \in \mathcal{A}_{k+1}, Q \subseteq Q^k_\alpha} Q;
\]

\[
B(x^k_\alpha, a_1 \delta^k) \subseteq Q^k_\alpha \subseteq B(x^k_\alpha, A_1 \delta^k) =: B(Q^k_\alpha);
\]

\[
\mu(Q^k_{\beta + 1}) \leq \mu(Q^k_{\alpha}) \leq C_{\mu, 0}(Q^k_{\beta + 1}). \tag{2.2}
\]

The set \(Q^k_\alpha\) is called a dyadic cube of generation \(k\) with centre point \(x^k_\alpha \in Q^k_\alpha\) and sidelength \(\delta^k\).

From the properties of the dyadic system above and from the doubling measure, we can deduce that there exists a constant \(C_{\mu, 0}\) depending only on \(C_{\mu}\) as in (1.2) and \(a_1, A_1\) as above, such that for any \(Q^k_\alpha\) and \(Q^k_{\beta + 1}\) with \(Q^k_{\beta + 1} \subseteq Q^k_\alpha\),

\[
\mu(Q^k_{\beta + 1}) \leq \mu(Q^k_{\alpha}) \leq C_{\mu, 0}(Q^k_{\beta + 1}). \tag{2.2}
\]

We recall from [26] the following construction, which is a slight elaboration of seminal work by M. Christ [5], as well as Sawyer-Wheeden [43].

**Theorem 2.1.** On \((X, d, \mu)\), there exists a system of dyadic cubes with parameters \(0 < \delta \leq (12A_0^3)^{-1}\) and \(a_1 := (3A_0^3)^{-1}, A_1 := 2A_0\). The construction only depends on some fixed set of countably many centre points \(x^k_\alpha\), having the properties that \(d(x^k_\alpha, x^k_\beta) \geq \delta^k\) with \(\alpha \neq \beta\), \(\min_{x \in X} d(x, x^k_\alpha) < \delta^k\) for all \(x \in X\), and a certain partial order “\(\leq\)” among their index pairs \((k, \alpha)\). In fact, this system can be constructed in such a way that

\[
\mathcal{Q}_\alpha^k = \{x^k_\beta : (\ell, \beta) \leq (k, \alpha)\}, \quad \mathcal{Q}_\alpha^k := \int \mathcal{Q}_\alpha^k = \left( \bigcup_{\gamma \neq \alpha} \mathcal{Q}_\gamma^k \right)^c, \quad \mathcal{Q}_\alpha^k \subseteq Q^k_\alpha \subseteq \mathcal{Q}_\alpha^k,
\]

where \(Q^k_\alpha\) are obtained from the closed sets \(\overline{Q}_\alpha^k\) and the open sets \(\widetilde{Q}_\alpha^k\) by finitely many set operations.

We also recall the following remark from [29, Section 2.3]. The construction of dyadic cubes requires their centre points and an associated partial order be fixed \(a\ priori\). However, if either the centre points or the partial order is not given, their existence already follows from the assumptions; any given system of points and partial order can be used as a starting point. Moreover, if we are allowed to choose the centre points for the cubes, the collection can be chosen to satisfy the additional property that a fixed point becomes a centre point at all levels:

\[
\text{given a fixed point } x_0 \in X, \text{ for every } k \in \mathbb{Z}, \text{ there exists } \alpha \text{ such that } x_0 = x^k_\alpha, \text{ the centre point of } Q^k_\alpha \in \mathcal{D}_k. \tag{2.3}
\]
2.2 Adjacent Systems of Dyadic Cubes

On \((X, d, \mu)\), a finite collection \(\{\mathcal{D}^t : t = 1, 2, \ldots, T\}\) of the dyadic families is called a collection of adjacent systems of dyadic cubes with parameters \(\delta \in (0, 1), 0 < a_1 \leq A_1 < \infty\) and \(1 \leq C_{\text{adj}} < \infty\) if it has the following properties: individually, each \(\mathcal{D}^t\) is a system of dyadic cubes with parameters \(\delta \in (0, 1)\) and \(0 < a_1 \leq A_1 < \infty\); collectively, for each ball \(B(x, r) \subseteq X\) with \(\delta^{k+3} < r \leq \delta^{k+2}, k \in \mathbb{Z}\), there exist \(t \in \{1, 2, \ldots, T\}\) and \(Q \in \mathcal{D}^t\) of generation \(k\) and with centre point \(t^{-1}x^k\) such that \(d(x, t^{-1}x^k) < 2A_0\delta^k\) and

\[
B(x, r) \subseteq Q \subseteq B(x, C_{\text{adj}}r).
\]  

We recall from [26] the following construction.

**Theorem 2.2.** Let \((X, d, \mu)\) be a space of homogeneous type. Then there exists a collection \(\{\mathcal{D}^t : t = 1, 2, \ldots, T\}\) of adjacent systems of dyadic cubes with parameters \(\delta \in (0, (96A_0^5)^{-1}), a_1 := (12A_0^3)^{-1}, A_1 := 4A_0^2\) and \(C := 8A_0^3\delta^{-3}\). The centre points \(t^{-1}x^k\) of the cubes \(Q \in \mathcal{D}^t\) have, for each \(t \in \{1, 2, \ldots, T\}\), the two properties

\[
d(t^{-1}x^k, t^{-1}x^\beta) \geq (4A_0^2)^{-1}\delta^k \quad (\alpha \neq \beta), \quad \min_{\alpha} d(x, t^{-1}x^\alpha) < 2A_0\delta^k \quad \text{for all } x \in X.
\]

Moreover, these adjacent systems can be constructed in such a way that each \(\mathcal{D}^t\) satisfies the distinguished centre point property (2.3).

We recall from [29, Remark 2.8] that the number \(T\) of the adjacent systems of dyadic cubes as in the theorem above satisfies the estimate

\[
T = T(A_0, A_1, \delta) \leq \tilde{A}_1^6(4A_0^4/\delta)^{\log_2 \tilde{A}_1},
\]

where \(\tilde{A}_1\) is the geometrically doubling constant, see [29, Section 2].

2.3 An Explicit Haar Basis on Spaces of Homogeneous Type

Next we recall the explicit construction in [29] of a Haar basis \(\{h_Q^\epsilon : Q \in \mathcal{D}, \epsilon = 1, \ldots, M_Q - 1\}\) for \(L^p(X, \mu), 1 < p < \infty\), associated to the dyadic cubes \(Q \in \mathcal{D}\) as follows. Here \(M_Q := \#Q(Q) = \#\{R \in \mathcal{D}_k : R \subseteq Q\}\) denotes the number of dyadic sub-cubes (“children”) the cube \(Q \in \mathcal{D}_k\) has; namely \(Q\) is the collection of dyadic children of \(Q\).

**Theorem 2.3 ([29]).** Let \((X, d, \mu)\) be a space of homogeneous type and suppose \(\mu\) is a positive Borel measure on \(X\) with the property that \(\mu(B) < \infty\) for all balls \(B \subseteq X\). For \(1 < p < \infty\), for each \(f \in L^p(X, \mu)\), we have

\[
f(x) = \sum_{Q \in \mathcal{D}} \sum_{\epsilon = 1}^{M_Q - 1} (f, h_Q^\epsilon)h_Q^\epsilon(x),
\]

where the sum converges (unconditionally) both in the \(L^p(X, \mu)\)-norm and pointwise \(\mu\)-almost everywhere.

The following theorem collects several basic properties of the functions \(h_Q^\epsilon\).

**Theorem 2.4 ([29]).** The Haar functions \(h_Q^\epsilon, Q \in \mathcal{D}, \epsilon = 1, \ldots, M_Q - 1\), have the following properties:

(i) \(h_Q^\epsilon\) is a simple Borel-measurable real function on \(X\);

(ii) \(h_Q^\epsilon\) is supported on \(Q\);
(iii) \( h^0_Q \) is constant on each \( R \in \mathcal{H}(Q) \);
(iv) \( \int h^0_Q \, d\mu = 0 \) (cancellation);
(v) \( \langle h^0_Q, h^0_{Q'} \rangle = 0 \) for \( \epsilon \neq \epsilon' \), \( \epsilon, \epsilon' \in \{1, \ldots, M_Q - 1\} \);
(vi) the collection \( \{ \mu(Q)^{-1/2}1_Q \} \cup \{ h^0_Q : \epsilon = 1, \ldots, M_Q - 1 \} \) is an orthogonal basis for the vector space \( V(Q) \) of all functions on \( Q \) that are constant on each sub-cube \( R \in \mathcal{H}(Q) \);
(vii) if \( h^0_Q \neq 0 \) then \( \| h^0_Q \|_{L^p(X,\mu)} = \mu(Q)^{1/p - 1/2} \) for \( 1 \leq p \leq \infty \);
(viii) \( \| h^0_Q \|_{L^1(X,\mu)} \cdot \| h^0_Q \|_{L^\infty(X,\mu)} \approx 1 \).

As stated in [29], we also have \( h^0_Q := \mu(Q)^{-1/2}1_Q \) which is a non-cancellative Haar function. Moreover, the martingale associated with the Haar functions are as follows: for \( Q \in \mathcal{D}_k \),

\[
E_Q f = \langle f, h^0_Q \rangle h^0_Q \quad \text{and} \quad D_Q f = \sum_{\epsilon=1}^{M_Q-1} \mathbb{D}_Q^\epsilon f,
\]

where \( \mathbb{D}_Q^\epsilon = \langle f, h^\epsilon_Q \rangle h^\epsilon_Q \) is the martingale operator associated with the \( \epsilon \)-th subcube of \( Q \). Also we have

\[
E_k f = \sum_{Q \in \mathcal{D}_k} E_Q f \quad \text{and} \quad D_k f = E_{k+1} f - E_k f.
\]

Hence, based on the construction of Haar system \( \{ h^\epsilon_Q \} \) in [29] we obtain that for each \( R \in \mathcal{D} \),

\[
\sum_{Q: R \subset Q} \sum_{\epsilon=1}^{M_Q-1} \langle f, h^\epsilon_Q \rangle h^\epsilon_Q h^0_R = \sum_{Q: R \subset Q} D_Q f \cdot h^0_R = E_R f \cdot h^0_R = \langle f, h^0_R \rangle h^0_R h^0_R.
\]

### 2.4 Muckenhoupt \( A_p \) Weights

**Definition 2.5.** Let \( w(x) \) be a nonnegative locally integrable function on \( X \). For \( 1 < p < \infty \), we say \( w \) is an \( A_p \) weight, written \( w \in A_p \), if

\[
[w]_{A_p} := \sup_B \left( \int_B w \right) \left( \int_B \left( \frac{1}{w} \right)^{1/(p-1)} \right)^{p-1} < \infty.
\]

Here the supremum is taken over all balls \( B \subset X \). The quantity \([w]_{A_p}\) is called the \( A_p \) constant of \( w \). For \( p = 1 \), we say \( w \) is an \( A_1 \) weight, written \( w \in A_1 \), if \( M(w)(x) \leq w(x) \) for a.e. \( x \in X \), and let \( A_\infty := \bigcup_{1 \leq p < \infty} A_p \) and we have \([w]_{A_\infty} := \sup_B \left( \int_B w \right) \exp \left( \int_B \log \left( \frac{1}{w} \right) \right) \) < \( \infty \).

Next we note that for \( w \in A_p \) the measure \( w(x) \, d\mu(x) \) is a doubling measure on \( X \). To be more precise, we have that for all \( \lambda > 1 \) and all balls \( B \subset X \),

\[
w(\lambda B) \leq \lambda^{np} [w]_{A_p} w(B),
\]

where \( n \) is the upper dimension of the measure \( \mu \), as in (1.3).

We also point out that for \( w \in A_\infty \), there exists \( \gamma > 0 \) such that for every ball \( B \),

\[
\mu \left( \left\{ x \in B : w(x) \geq \gamma \int_B w \right\} \right) \geq \frac{1}{2} \mu(B).
\]

And this implies that for every ball \( B \) and for all \( \delta \in (0,1) \),

\[
\int_B w \leq C \left( \int_B w^\delta \right)^{1/\delta}; \quad (2.5)
\]

see also [33].
3 Sparse Operators and Domination of Commutators on Spaces of Homogeneous Type

Let $\mathcal{D}$ be a system of dyadic cubes on $X$ as in Section 2.1. As in the Euclidean setting, we have two competing versions of sparsity for a collection of sets, one geometric and the other a Carleson measure condition.

Definition 3.1. Given $0 < \eta < 1$, a collection $\mathcal{S} \subset \mathcal{D}$ of dyadic cubes is said to be $\eta$-sparse provided that for every $Q \in \mathcal{S}$, there is a measurable subset $E_Q \subset Q$ such that $\mu(E_Q) \geq \eta \mu(Q)$ and the sets $\{E_Q\}_{Q \in \mathcal{S}}$ have only finite overlap.

Definition 3.2. Given $\Lambda > 1$, a collection $\mathcal{S} \subset \mathcal{D}$ of dyadic cubes is said to be $\Lambda$-Carleson if for every cube $Q \in \mathcal{D}$,

$$\sum_{P \in \mathcal{S}, P \subseteq Q} \mu(P) \leq \Lambda \mu(Q).$$

We first show that the above two definitions are equivalent in a space of homogeneous type. The proof closely follows the original idea in [31] with some minor modifications.

Theorem 3.3. Given $0 < \eta < 1$ and a collection $\mathcal{S} \subset \mathcal{D}$ of dyadic cubes, the following statements hold:

- If $\mathcal{S}$ is $\eta$-sparse, then $\mathcal{S}$ is $\frac{\eta}{c}$-Carleson.
- If $\mathcal{S}$ is $\frac{1}{\eta}$-Carleson, then $\mathcal{S}$ is $\eta$-sparse.

The reason for the extra constant $c$ in the above, is that for later parts of our argument, to control the commutator, we need to allow the sets $E_Q$ to have finite overlap. If the sets $E_Q$ were exactly disjoint then one could take $c = 1$ in the above and the statement would be cleaner and more in line with that in [31].

Proof. Note that if a collection $\mathcal{S} \subset \mathcal{D}$ of dyadic cubes is $\eta$-sparse, that is for every $Q \in \mathcal{S}$, there is a measurable subset $E_Q \subset Q$ such that $\mu(E_Q) \geq \eta \mu(Q)$ and the sets $\{E_Q\}_{Q \in \mathcal{S}}$ have only finite overlap, we will have that $\mathcal{S}$ is $c\eta^{-1}$-Carleson according to Definition 3.2 (following from the standard computation and the constant $c$ denoting the amount of overlap of the sets $E_Q$).

Thus, it suffices to show that for $\Lambda > 1$, if a collection $\mathcal{S} \subset \mathcal{D}$ of dyadic cubes is $\Lambda$-Carleson, then it is $\Lambda^{-1}$-sparse. Here we will follow the proof in [31] but with some minor modifications. To see this, we first point out that if the collection $\mathcal{S} \subset \mathcal{D}$ of dyadic cubes $\{Q\}$ has a bottom layer $\mathcal{D}_K$ for some fixed integer $K$, then it is direct to construct the set $E_Q$. We begin with considering all dyadic cubes $\{Q\} \subset \mathcal{S} \cap \mathcal{D}_K$ and choose any measurable set $E_Q \subset Q$ of measure $\Lambda^{-1}\mu(Q)$ for them. We now just repeat this choice for each dyadic cube in upper layers one by one. To be more specific, for each $Q \in \mathcal{S} \cap \mathcal{D}_k$ with $k \leq K$, choose a set

$$E_Q \subset Q \setminus \bigcup_{R \in \mathcal{S}, R \subseteq Q} E_R$$

such that $\mu(E_Q) = \Lambda^{-1}\mu(Q)$. We now show that such choice of $E_Q$ is possible. In fact, note that for every $Q \in \mathcal{S}$, we have

$$\mu\left( \bigcup_{R \in \mathcal{S}, R \subseteq Q} E_R \right) \leq \Lambda^{-1} \sum_{R \in \mathcal{S}, R \subseteq Q} \mu(R) \leq \Lambda^{-1}(\Lambda - 1)\mu(Q) = (1 - \Lambda^{-1})\mu(Q),$$

so that
where the last inequality follows from the $\Lambda$-Carleson condition and from the fact that $R \subset Q$. This shows that

$$\mu\left(Q \setminus \bigcup_{R \in S, R \subset Q} E_R\right) \geq \mu(Q) - (1 - \Lambda^{-1})\mu(Q) = \Lambda^{-1}\mu(Q)$$

and so the choice of the top set $E_Q$ is always possible.

Next, we consider the case that there is no fixed bottom layer. We run the above construction with a particular choice for each $K = 0, 1, 2, \ldots$ and then pass to the limit. To begin with, fix $K \geq 0$. For each $Q \in S \cap (\cup_{k \leq K} D_k)$, we define the sets $\hat{E}_Q^{(K)}$ inductively as follows.

First, for each $Q \in S \cap D_k$ with $k \leq K$, we consider the auxiliary set

$$Q(t, Q) := B(x_Q, t\delta^k) \cap Q, \quad t \in (0, A_1),$$

where $x_Q$ is the centre point of $Q$ and $A_1, \delta$ are the constants as introduced in Section 2.1. From property (2.1), it is clear that when $0 < t < a_1$, then $B(x_Q, t\delta^k) \subset Q$ and when $t > A_1$, then $Q \subset B(x_Q, t\delta^k)$; moreover, we have $\mu(B(x_Q, t\delta^k)) \to 0$ as $t \to 0^+$.

Now for $Q \in S \cap D_k$, from the above observations together with the continuity and monotonicity of the function $t \mapsto Q(t, Q) = \mu(B(x_Q, t\delta^k)) \cap Q$, we conclude that there must be some $t_{\Lambda, K, K} \in (0, A_1)$ such that $\mu(B(x_Q, t_{\Lambda, K, K}\delta^k) \cap Q) = \Lambda^{-1}\mu(Q)$. Here and in what follows, we use the triple $(\Lambda, k, K)$ for the subscript of $t$, where $\Lambda$ denotes the value of such $t$ depends on $\Lambda, k$ denotes that $Q$ is in the layer $k$ and the last $K$ denotes that we start at the layer $K$. We set

$$\hat{E}_Q^{(K)} := Q(t_{\Lambda, K, K}, Q) = B(x_Q, t_{\Lambda, K, K}\delta^k) \cap Q.$$

Suppose now $\hat{E}_R^{(K)}$ are already defined for every $R \in S \cap (\cup_{k+1 \leq i \leq K} D_i)$. We now define $\hat{E}_Q^{(K)}$ for $Q \in S \cap D_k$ in the following manner. We set

$$\hat{E}_Q^{(K)} := Q(t_{\Lambda, K, K}, Q) \bigcup F_Q^{(K)};$$

where

$$F_Q^{(K)} := \bigcup_{R \in S \cap (\cup_{k+1 \leq i \leq K} D_i), R \subset Q} \hat{E}_R^{(K)}$$

and $t_{\Lambda, K, K} \in (0, A_1)$ is chosen such that the set

$$E_Q^{(K)} := Q(t_{\Lambda, K, K}, Q) \setminus F_Q^{(K)}$$

satisfies $\mu(E_Q^{(K)}) = \Lambda^{-1}\mu(Q)$.

Now we claim that $\hat{E}_Q^{(K)} \subset \hat{E}_Q^{(K+1)}$ for every $Q \in S \cap (\cup_{k \leq K} D_k)$. To see this, we note that for each $Q \in S \cap D_k$, $\hat{E}_Q^{(K)}$ is just the set $Q(t_{\Lambda, K, K}, Q)$, and the other hand, $\hat{E}_Q^{(K+1)}$ contains the set $Q(t_{\Lambda, K, K+1}, Q)$ which has the same centre point as $Q(t_{\Lambda, K, K}, Q)$, but with $t_{\Lambda, K, K+1} \geq t_{\Lambda, K, K}$ since

$$\mu\left(Q(t_{\Lambda, K, K+1}, Q) \setminus \bigcup_{R \in S \cap D_{k+1}, R \subset Q} \hat{E}_R^{(K+1)}\right) = \Lambda^{-1}\mu(Q) = \mu(Q(t_{\Lambda, K, K}, Q)).$$

Hence, we see that for each $Q \in S \cap D_k$, we have $\hat{E}_Q^{(K)} \subseteq \hat{E}_Q^{(K+1)}$. Then, we proceed via backward induction. Assume that $\hat{E}_Q^{(K)} \subseteq \hat{E}_Q^{(K+1)}$ for every $Q \in S \cap (\cup_{k \leq i \leq K} D_i)$. Take any $Q \in S \cap D_k$. Then the inductive hypothesis implies that $F_Q^{(K)} \subseteq F_Q^{(K+1)}$. Let $Q(t_{\Lambda, K, K}, Q)$ be the set added to $F_Q^{(K)}$ when constructing $\hat{E}_Q^{(K)}$. Then we have

$$\mu\left(Q(t_{\Lambda, K, K}, Q) \setminus F_Q^{(K+1)}\right) \leq \mu\left(Q(t_{\Lambda, K, K}, Q) \setminus F_Q^{(K)}\right) = \Lambda^{-1}\mu(Q),$$

where $\Lambda$ denotes that the value of such
which implies that $t_{\Lambda,k,K+1} \geq t_{\Lambda,k,K}$. Thus, we have $Q(t_{\Lambda,k,K},Q) \subset Q(t_{\Lambda,k,K+1},Q)$, which yields $\widehat{E}_Q^{(K)} \subseteq \widehat{E}_Q^{(K+1)}$, and hence the claim follows.

Now for $Q \in \mathcal{S} \cap \mathcal{D}_k$, we define

$$\hat{E}_Q := \lim_{K \to \infty} \widehat{E}_Q^{(K)},$$

which, by using the claim above, equals

$$\bigcup_{K=k}^{\infty} \widehat{E}_Q^{(K)} \subset Q.$$ 

Moreover, for each $K$ we have

$$\mu(\hat{E}_Q^{(K)}) = \mu(\hat{E}_Q \setminus F_Q^{(K)}) = \Lambda^{-1} \mu(Q).$$

Note that the sets $F_Q^{(K)}$ also form an increasing sequence (with respect to $K$), so for each $Q \in \mathcal{S}$, the limit set

$$E_Q := \lim_{K \to \infty} E_Q^{(K)} = \hat{E}_Q \setminus \left( \lim_{K \to \infty} F_Q^{(K)} \right) = \hat{E}_Q \setminus \left( \bigcup_{R \in \mathcal{S}, R \subset Q} \hat{E}_R \right)$$

exists, and is contained in $Q$ and has the required measure. Moreover, all $E_Q$ are disjoint. The proof of Theorem 3.3 is complete.

We now recall the well-known definition for sparse operators.

**Definition 3.4.** Given $0 < \eta < 1$ and an $\eta$-sparse family $\mathcal{S} \subset \mathcal{D}$ of dyadic cubes. The sparse operator $A_\mathcal{S}$ is defined by

$$A_\mathcal{S}f(x) := \sum_{Q \in \mathcal{S}} f_Q \chi_Q(x).$$

Following the proof of [38, Theorem 3.1], we obtain that

$$\|A_\mathcal{S}f\|_{L^p(X)} \leq C_{\eta,n,p}[\mu]_{A_p} \max\{1, \frac{1}{p-1}\} \|f\|_{L^p(X)}, \quad 1 < p < \infty.$$ 

Denote by $\Omega(b,B)$ the standard mean oscillation

$$\Omega(b,B) := \frac{1}{\mu(B)} \int_B |b(x) - b_B|d\mu(x). \quad (3.1)$$

**Lemma 3.5.** Given $0 < \gamma < 1$. Let $\mathcal{D}$ be a dyadic system in $X$ and let $\mathcal{S} \subset \mathcal{D}$ be a $\gamma$-sparse family. Assume that $b \in L^{1,\infty}(X)$. Then there exists a $\frac{\gamma}{2(\gamma+1)}$-sparse family $\tilde{\mathcal{S}} \subset \mathcal{D}$ such that $\mathcal{S} \subset \tilde{\mathcal{S}}$ and for every cube $Q \in \tilde{\mathcal{S}}$,

$$|b(x) - b_Q| \leq C \sum_{R \in \tilde{\mathcal{S}}, R \subset Q} \Omega(b,R) \chi_R(x) \quad (3.2)$$

for a.e. $x \in Q$.

**Proof.** Fix a dyadic cube $Q \in \mathcal{D}$. We now show that there exists a family of pairwise disjoint cubes $\{P_j\} \subset \mathcal{D}(Q)$ such that $\sum_j \mu(P_j) \leq \frac{1}{2} \mu(Q)$ and for a.e. $x \in Q$,

$$|b(x) - b_Q| \leq C \cdot C_{\mu,b} \Omega(b,Q) + \sum_j |b(x) - b_{P_j}| \chi_{P_j}(x). \quad (3.3)$$
Let $M^d_Q$ be the standard dyadic local maximal operator restricted to $D(Q)$ and $C_{M^d_Q}$ be the weak type $(1, 1)$-norm of $M^d_Q$. Then one can choose a constant $C$ depending on $C_{M^d_Q}$ such that the set $E := \{x \in Q : M^d_Q(b - b_Q)(x) > 4C_{\mu, 0} \cdot C \cdot \Omega(b, Q)\}$ satisfies that $\mu(E) \leq \frac{1}{4C_{\mu, 0}} \mu(Q)$, where $C_{\mu, 0}$ is the constant as in (2.2).

If $\mu(E) = 0$, then (3.3) holds trivially with the empty family $\{P_j\}_j$. If $\mu(E) > 0$, then we now apply the Calderón–Zygmund decomposition to the function $h(x) := \chi_E(x)$ on $Q$ at height $\lambda := \frac{\mu}{4C_{\mu, 0}}$ as follows: we begin by considering the descendants of $Q$ in $D(Q)$ since

$$\int_Q |h(x)| d\mu(x) < \lambda \mu(Q).$$

Let $\{Q_j^{(1)}\} \subset D(Q)$ be the children of $Q$. If

$$\int_{Q_j^{(1)}} |h(x)| d\mu(x) > \lambda \mu(Q_j^{(1)})$$

then we select it as our candidate cube. If

$$\int_{Q_j^{(1)}} |h(x)| d\mu(x) \leq \lambda \mu(Q_j^{(1)})$$

then we keep looking at the children of $Q_j^{(1)}$ in $D(Q)$ and then repeat the above selection criteria and we will stop only when we find some descendant of $Q_j^{(1)}$ in $D(Q)$ such that it meets the criteria (3.4).

Then it is direct to see that this produces pairwise disjoint cubes $\{P_j\} \subset D(Q)$ such that

$$\frac{1}{2C_{\mu, 0}} \mu(P_j) < \mu(P_j \cap E) \leq \frac{1}{2} \mu(P_j)$$

and $\mu(E \setminus \bigcup_j P_j) = 0$. It follows that $\sum_j \mu(P_j) \leq \frac{1}{2} \mu(Q)$ and $P_j \cap E^c \neq \emptyset$. Therefore, we get

$$|b_{P_j} - b_Q| \leq \frac{1}{\mu(P_j)} \int_{P_j} |b(x) - b_Q| d\mu(x) \leq 4C_{\mu, 0} \cdot C \cdot \Omega(b, Q)$$

and for a.e. $x \in Q$, $|b(x) - b_Q| \chi_{Q \setminus \bigcup_j P_j} \leq 4C_{\mu, 0} \cdot C \Omega(b, Q)$.

Then, we have

$$|b(x) - b_Q| \chi_Q(x) \leq |b(x) - b_Q| \chi_{Q \setminus \bigcup_j P_j}(x) + \sum_j |b_{P_j} - b_Q| \chi_{P_j}(x) + \sum_j |b(x) - b_{P_j}| \chi_{P_j}(x)$$

$$\leq 4C_{\mu, 0} \cdot C \Omega(b, Q) + \sum_j |b(x) - b_{P_j}| \chi_{P_j}(x),$$

which gives (3.3).

We observe that if $P_j \subset R$, where $R \in D(Q)$, then $R \cap E^c \neq \emptyset$. Hence $P_j$ in (3.5) can be replaced by $R$, namely, we have $|b_R - b_Q| \leq 4C_{\mu, 0} \cdot C \Omega(b, Q)$. Therefore, if $\cup_j P_j \subset \cup_i R_i$, where $R_i \in D(Q)$, and the cubes $\{R_i\}$ are pairwise disjoint, then we have

$$|b(x) - b_Q| \leq 4C_{\mu, 0} \cdot C \Omega(b, Q) + \sum_i |b(x) - b_{R_i}| \chi_{R_i}(x).$$
Iterating (3.3), from the selection of \( \{P_j \} \) and from Definition 3.2, we obtain that there exists a \( \frac{1}{2} \)-sparse family \( \mathcal{F}(Q) \subset \mathcal{D}(Q) \) such that for a.e. \( x \in Q \),
\[
|b(x) - b_Q|\chi_Q(x) \leq 4C_{\mu,0} \cdot C \sum_{P \in \mathcal{F}(Q)} \Omega(b, P)\chi_P(x).
\]

Now for each \( \mathcal{F}(Q) \), let \( \hat{\mathcal{F}}(Q) \) be the family that consists of all cubes \( \{P \} \subset \mathcal{F}(Q) \) that are not contained in any cube \( R \in \mathcal{S} \) with \( R \not\subset Q \). Then we define
\[
\hat{\mathcal{S}} := \bigcup_{Q \in \mathcal{S}} \mathcal{F}(Q).
\]
It is clear, by construction, that the augmented family \( \hat{\mathcal{S}} \) contains the original family \( \mathcal{S} \). Furthermore, if \( \mathcal{S} \) and each \( \mathcal{F}(Q) \) are sparse families, then the augmented family \( \hat{\mathcal{S}} \) is also sparse.

To be specific, we have that if \( \mathcal{S} \subset \mathcal{D} \) is an \( \gamma \)-sparse family then the augmented family \( \hat{\mathcal{S}} \) built upon \( \frac{1}{2} \)-sparse family \( \mathcal{F}(Q) \), \( Q \in \mathcal{S} \), is an \( \frac{\gamma}{2(\gamma + 1)} \)-sparse family.

We now show (3.2). Take an arbitrary cube \( Q \in \mathcal{S} \). Let \( P_j \) be the cubes appearing in (3.3). Denote by \( \mathcal{M}(Q) \) the family of the maximal pairwise disjoint cubes from \( \hat{\mathcal{S}} \) which are strictly contained in \( Q \). Then by the augmentation process, \( \cup_j P_j \subset \cup_{P \in \mathcal{M}(Q)} P \). Therefore, by (3.6), we have
\[
|b(x) - b_Q|\chi_Q(x) \leq 4C_{\mu,0} \cdot C \Omega(b, Q) + \sum_{P \in \mathcal{M}(Q)} |b(x) - b_P|\chi_P(x).
\]

Now split \( \hat{\mathcal{S}}(Q) := \{P \in \mathcal{S} : P \subset Q \} \) into the layers \( \hat{\mathcal{S}}(Q) = \bigcup_{k=0}^{\infty} \mathcal{M}_k \), where \( \mathcal{M}_0 := \{Q \} \), \( \mathcal{M}_1 := \mathcal{M}(Q) \) and \( \mathcal{M}_k \) is the family of the maximal elements of \( \mathcal{M}_{k-1} \). Iterating (3.7) \( k \) times, we get that
\[
|b(x) - b_Q|\chi_Q(x) \leq 4C_{\mu,0} \sum_{P \in \mathcal{M}(Q)} \Omega(b, P)\chi_P(x) + \sum_{P \in \mathcal{M}_k} |b(x) - b_P|\chi_P(x).
\]

Now we observe that since \( \hat{\mathcal{S}} \) is \( \frac{\gamma}{2(\gamma + 1)} \)-sparse,
\[
\sum_{P \in \mathcal{M}_k} \mu(P) \leq \frac{1}{k+1} \sum_{i=0}^{k} \sum_{P \in \mathcal{M}_i} \mu(P) \leq \frac{1}{k+1} \sum_{P \in \mathcal{S}(Q)} \mu(P) \leq \frac{2(\gamma + 1)}{\gamma(\gamma + 1)} \mu(Q).
\]

By letting \( k \to \infty \) in (3.8), we obtain (3.2).  

Let \( T \) be a Calderón–Zygmund operator as in Definition 1.1. We now have the maximal truncated operator \( T^* \) defined by
\[
T^* f(x) := \sup_{\epsilon > 0} \left| \int_{d(x,y) > \epsilon} K(x,y)f(y)d\mu(y) \right|.
\]

We recall the standard Hardy–Littlewood maximal function \( \mathcal{M}f(x) \) on \( X \), defined as
\[
\mathcal{M}f(x) := \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f(y)|d\mu(y),
\]
where the supremum is taken over all balls \( B \subset X \). We now have the grand maximal truncated operator \( \mathcal{M}_T \) defined by
\[
\mathcal{M}_T f(x) := \sup_{B \ni x} \sup_{\xi \in B} \left| T(\chi_{X \setminus C_{\lambda_0} B}) \xi \right|,
\]
where the supremum is taken over all balls \( B \subset X \) containing \( x \), \( \tilde{j}_0 \) is the smallest integer such that
\[
2^{\tilde{j}_0} > \max\{3A_0, 2A_0 \cdot C_{\text{adj}}\} \tag{3.9}
\]
and \( C_{\tilde{j}_0} := 2^{\tilde{j}_0+2}A_0 \), where \( C_{\text{adj}} \) is an absolute constant as mentioned in Section 2.2. Given a ball \( B_0 \subset X \), for \( x \in B_0 \) we define a local grand maximal truncated operator \( \mathcal{M}_{T,B_0} \) as follows:
\[
\mathcal{M}_{T,B_0}f(x) := \sup_{B \ni x, B \subset B_0} \text{ess sup}_{\xi \in B} \left| T(f \chi_{C_{\tilde{j}_0}B_0 \setminus C_{\tilde{j}_0}B})(\xi) \right|.
\]

Then we first claim that the following lemma holds.

**Lemma 3.6.** The following pointwise estimates hold:

(i) for a.e. \( x \in B_0 \), \( |T(f \chi_{C_{\tilde{j}_0}B_0})(x)| \leq C\|T\|_{L^1 \to L^{1,\infty}}|f(x)| + \mathcal{M}_{T,B_0}f(x) \).

(ii) for all \( x \in X \), \( \mathcal{M}_T f(x) \leq C_M f(x) + T^* f(x) \).

**Proof.** The result in the Euclidean setting is from [30, Lemma 3.2]. Here we can adapt the proof in [30] to our setting of spaces of homogeneous type. \( \square \)

Next we have the sparse domination for the higher order commutator.

**Theorem 3.7.** Let \( T \) be the Calderón–Zygmund operator as in Definition 1.1 and let \( b \in L^{1}_{\text{loc}}(X) \). For every \( f \in L^{\infty}(X) \) with bounded support, there exist \( T \) dyadic systems \( D^t \), \( t = 1, 2, \ldots, T \) and \( \eta \)-sparse families \( S_t \subset D^t \) such that for a.e. \( x \in X \),
\[
|T^n_b(f)(x)| \leq C \sum_{t=1}^{T} \sum_{k=0}^{m} C^k_m \sum_{Q \in S_t} |b(x) - b_Q|^{m-k} \left( \frac{1}{\mu(Q)} \int_Q |b(z) - b_Q|^k |f(z)|dz \right) \chi_Q(x), \tag{3.10}
\]
where \( C^k_m := \frac{m!}{(m-k)!k!} \).

**Proof.** We follow the idea as in [33, 27] for the domination, and adapt it to our setting of space of homogeneous type.

Suppose \( f \) is supported in a ball \( B_0 := B(x_0, r) \subset X \). We now consider a decomposition of \( X \) with respect to this ball \( B_0 \). We define the annuli \( U_j := 2^{j+1}B_0 \setminus 2^jB_0 \), \( j \geq 0 \) and we choose \( j_0 \) to be the smallest integer such that
\[
j_0 > \tilde{j}_0 \quad \text{and} \quad 2^{j_0} > 4A_0. \tag{3.11}
\]

Next, for each \( U_j \), we choose the balls
\[
\{\tilde{B}_{j,\ell}\}^{|U_j|}_{\ell=1} \tag{3.12}
\]
centred in \( U_j \) and with radius \( 2^{j - \tilde{j}_0}r \) to cover \( U_j \). From the geometric doubling property [9, p. 67], it is direct to see that
\[
\sup_j L_j \leq C_{A_0, \mu, \tilde{j}_0}, \tag{3.13}
\]
where \( C_{A_0, \mu, \tilde{j}_0} \) is an absolute constant depending only on \( A_0, \tilde{j}_0 \) and \( C_\mu \).
We now first study the properties of these $\tilde{B}_{j,\ell}$. Denote $\tilde{B}_{j,\ell} := B(x_{j,\ell}, 2^{-j_0}r)$, where $j_0$ is defined as in (3.9). Then we have $C_{adj} \tilde{B}_{j,\ell} := B(x_{j,\ell}, C_{adj}2^{-j_0}r)$, where $C_{adj}$ is an absolute constant as mentioned in Section 2.2. We claim that
\[ C_{adj} \tilde{B}_{j,\ell} \cap U_{j+j_0} = \emptyset, \quad \forall j \geq 0 \quad \text{and} \quad \forall \ell = 1, 2, \ldots, L_j; \] (3.14)
and that
\[ C_{adj} \tilde{B}_{j,\ell} \cap U_{j-j_0} = \emptyset, \quad \forall j \geq j_0 \quad \text{and} \quad \forall \ell = 1, 2, \ldots, L_j. \] (3.15)

Assume (3.14) and (3.15) at the moment. Now combining the properties as in (3.14) and (3.15), we see that each $C_{adj} \tilde{B}_{j,\ell}$ only intersects with at most $2j_0 + 1$ annuli $U_j$. Moreover, for every $j$ and $\ell$, $c_{j_0} \tilde{B}_{j,\ell}$ covers $B_0$.

Now for the given ball $B_0$ as above, we point out that from (2.4) we have that there exist an integer $t_0 \in \{1, 2, \ldots, T\}$ and $Q_0 \in \mathcal{D}^{t_0}$ such that $B_0 \subseteq Q_0 \subseteq C_{adj}B_0$. Moreover, for this $Q_0$, as in (2.1) we use $B(Q_0)$ to denote the ball that contains $Q_0$ and has measure comparable to $Q_0$. Then it is easy to see that $B(Q_0)$ covers $B_0$ and $\mu(B(Q_0)) \lesssim \mu(B_0)$, where the implicit constant depends only on $C_{adj}$, $C_\mu$ and $A_1$ as in (2.1).

We show that there exists a $\frac{1}{2}$-sparse family $\mathcal{F}^{t_0} \subset \mathcal{D}^{t_0}(Q_0)$, the set of all dyadic cubes in $t_0$-th dyadic system that are contained in $Q_0$, such that for a.e. $x \in B_0$,
\[ |T^m_b(f \chi_{C_{j_0}B(Q_0)})(x)| \leq C \sum_{k=0}^m C^k_m \sum_{Q \in \mathcal{F}^{t_0}} \left| \frac{|b(x) - b_{R_Q}|^m}{k} |f| \right| |b - b_{R_Q}|^k \chi_{C_{j_0}B(Q)}(x). \] (3.16)

Here, $R_Q$ is the dyadic cube in $\mathcal{D}^t$ for some $t \in \{1, 2, \ldots, T\}$ such that $C_{j_0}B(Q) \subset R_Q \subset C_{adj}C_{j_0}B(Q)$, where $B(Q)$ is defined as in (2.1), $j_0$ defined as in (3.11) and $j_0$ defined as in (3.9).

To prove the claim it suffices to prove the following recursive estimate: there exist pairwise disjoint cubes $P_j \in \mathcal{D}^{t_0}(Q_0)$ such that $\sum_j \mu(P_j) \leq \frac{1}{4} \mu(Q_0)$ and
\[ |T^m_b(f \chi_{C_{j_0}B(Q_0)})(x)| |\chi_{Q_0}(x)| \leq C \sum_{k=0}^m C^k_m \left| \frac{|b(x) - b_{R_Q}|^m}{k} |f| \right| |b - b_{R_Q}|^k \chi_{C_{j_0}B(Q)}(x). \] (3.17)

for a.e. $x \in B_0$.

Iterating this estimate we obtain (3.16) with $\mathcal{F}^{t_0}$ being the union of all the families $\{P^k_j\}$ where $\{P^0_j\} = \{Q_0\}$, $\{P^1_j\} = \{Q_j\}$ as mentioned above, and $\{P^k_j\}$ are the cubes obtained at the $k$-th stage of the iterative process. It is also clear that $\mathcal{F}^{t_0}$ is a $1/2$-sparse family.

Let us prove then the recursive estimate. We observe that for any arbitrary family of disjoint cubes $\{P_j\} \subset \mathcal{D}^{t_0}(Q_0)$, we have that
\[ |T^m_b(f \chi_{C_{j_0}B(Q_0)})(x)| |\chi_{Q_0}(x)| \]
\[ \leq |T^m_b(f \chi_{C_{j_0}B(Q_0)})(x)| |\chi_{Q_0 \cup_j P_j}(x)| + \sum_j |T^m_b(f \chi_{C_{j_0}B(Q_0)})(x)| |\chi_{P_j}(x)| \]
\[ \leq |T^m_b(f \chi_{C_{j_0}B(Q_0)})(x)| |\chi_{Q_0 \cup_j P_j}(x)| + \sum_j |T^m_b(f \chi_{C_{j_0}B(Q_0) \cup_j B(P_j)})(x)| |\chi_{P_j}(x)| \]
\[ + \sum_j |T^m_b(f \chi_{C_{j_0}B(P_j)})(x)| |\chi_{P_j}(x)|. \]
So it suffices to show that we can choose a family of pairwise disjoint cubes \( \{ P_j \} \subset D^0(Q_0) \) with \( \sum_j \mu(P_j) \leq \frac{1}{2} \mu(B_0) \) and such that for a.e. \( x \in B_0 \),

\[
|T^m_b (f \chi_{C_{j_0} B(Q_0)}) (x) | \chi_{Q_0 \cup_j P_j} (x) + \sum_j |T^m_b (f \chi_{C_{j_0} B(Q_0) \setminus C_{j_0} B(P_j)}) (x) | \chi_{P_j} (x)
\]

\[
\leq C \sum_{k=0}^m C^k_m |b(x) - b_{RQ_0}|^{|m-k|} \| f \| \| b - b_{RQ_0} \| \chi_{C_{j_0} B(Q_0)}.
\]

To see this, use the fact that

\[
T^m_b f = T^m_{b-RQ_0} f = \sum_{k=0}^m (-1)^k C^k_m T((b - b_{RQ_0})^k f)(b - b_{RQ_0})^{m-k},
\]

we obtain that

\[
|T^m_b (f \chi_{C_{j_0} B(Q_0)}) (x) | \chi_{Q_0 \cup_j P_j} (x) + \sum_j |T^m_b (f \chi_{C_{j_0} B(Q_0) \setminus C_{j_0} B(P_j)}) (x) | \chi_{P_j} (x)
\]

\[
\leq \sum_{k=0}^m C^k_m T((b - b_{RQ_0})^k f \chi_{C_{j_0} B(Q_0)}) (x) \| b(x) - b_{RQ_0} \| \chi_{Q_0 \cup_j P_j} (x)
\]

\[
+ \sum_{k=0}^m C^k_m T((b - b_{RQ_0})^k f \chi_{C_{j_0} B(Q_0) \setminus C_{j_0} B(P_j)}) (x) \| b(x) - b_{RQ_0} \| \chi_{P_j} (x)
\]

\[
=: I_1 + I_2.
\]

Now for \( k = 0, 1, \ldots, m \), we define the set \( E_k \) as

\[
E_k := \left\{ x \in B_0 : |b(x) - b_{RQ_0}|^k |f(x)| > \alpha \| b - b_{RQ_0} \| \chi_{C_{j_0} B(Q_0)} \right\}
\]

\[
\cup \left\{ x \in B_0 : \mathcal{M}_{T,B_0} (b - b_{RQ_0})^k f(x) > \alpha C_T \| b - b_{RQ_0} \| \chi_{C_{j_0} B(Q_0)} \right\}
\]

and \( E := \cup_{k=0}^m E_k \). Then, choosing \( \alpha \) big enough (depending on \( C_{j_0}, C_{adj}, C_{\mu} \) and \( A_1 \) as in (2.1)), we have that

\[
\mu(E) \leq \frac{1}{4 C_{\mu,0}} \mu(B_0),
\]

where \( C_{\mu,0} \) is the constant in (2.2). We now apply the Calderón–Zygmund decomposition to the function \( \chi_E \) on \( B_0 \) at the height \( \lambda := \frac{1}{2C_{\mu,0}} \), to obtain pairwise disjoint cubes \( \{ P_j \} \subset D^0(Q_0) \) such that

\[
\frac{1}{2C_{\mu,0}} \mu(P_j) \leq \mu(P_j \cap E) \leq \frac{1}{2} \mu(P_j)
\]

and \( \mu(E \setminus \cup_j P_j) = 0 \). It follows that

\[
\sum_j \mu(P_j) \leq \frac{1}{2} \mu(B_0) \quad \text{and} \quad P_j \cap E^c \neq \emptyset.
\]

Then we have

\[
\operatorname{ess sup}_{\xi \in P_j} \left| T \left( |b - b_{RQ_0}|^k |f| \chi_{C_{j_0} B(Q_0) \setminus C_{j_0} B(P_j)} \right)(\xi) \right| \leq C \| f \| \| b - b_{RQ_0} \| \chi_{C_{j_0} B(Q_0)},
\]

which allows us to control the summation in the term \( I_2 \) above.
Now from (i) in Lemma 3.6, we obtain that for a.e. $x \in B_0$,
\[
|T((b - b_{R_{Q_0}})^k f \chi_{\mathcal{C}_0 B(Q_0)})(x)| \leq C|x - b_{R_{Q_0}}|^k |f(x)| + \mathcal{M}_{T,B_0}((b - b_{R_{Q_0}})^k f \chi_{\mathcal{C}_0 B(Q_0)})(x).
\]
Since $\mu(E \cup P_j) = 0$, we have that from the definition of the set $E$, the following estimate
\[
|b(x) - b_{R_{Q_0}}|^k |f(x)| \leq \alpha \left| |b - b_{R_{Q_0}}|^k \right|_{\mathcal{C}_0 B(Q_0)}
\]
holds for a.e. $x \in B_0 \setminus \cup P_j$, and also
\[
\mathcal{M}_{T,B_0}((b - b_{R_{Q_0}})^k f \chi_{\mathcal{C}_0 B_0})(x) \leq \alpha C_T |b - b_{R_{Q_0}}|^k \left| \left| f \right| \right|_{\mathcal{C}_0 B(Q_0)}
\]
holds for a.e. $x \in B_0 \setminus \cup P_j$. These estimates allow us to control the summation in the term $I_1$ above. Thus, we obtain that (3.17) holds, which yields that (3.16) holds.

We now consider the partition of the space as follows. Suppose $f$ is supported in a ball $B_0 \subset X$. We have
\[
X = \bigcup_{j=0}^{\infty} 2^j B_0.
\]

We now consider the annuli $U_j := 2^j B_0 \setminus 2^{j+1} B_0$ for $j \geq 0$ and the covering $\{ \tilde{B}_{j,\ell} \}_{\ell=1}^{T_j}$ of $U_j$ as in (3.12). We note that for each $\tilde{B}_{j,\ell}$, there exist $t_{j,\ell} \in \{1, 2, \ldots, T\}$ and $\hat{Q}_{j,\ell} \in \mathcal{Q}^{j,\ell}$ such that $\tilde{B}_{j,\ell} \subseteq \hat{Q}_{j,\ell} \subseteq \mathcal{C}_0 B_{j,\ell}$. Moreover, we note that for each such $\tilde{B}_{j,\ell}$, the enlargement $\mathcal{C}_0 B(\hat{Q}_{j,\ell})$ covers $B_0$ since $\mathcal{C}_0 B_{j,\ell}$ covers $B_0$.

We now apply (3.16) to each $\tilde{B}_{j,\ell}$, then we obtain a $\frac{1}{2}$-sparse family $\tilde{F}_{j,\ell} \subset \mathcal{Q}^{j,\ell} (\hat{Q}_{j,\ell})$ such that (3.16) holds for a.e. $x \in \tilde{B}_{j,\ell}$.

Now we set $\mathcal{F} := \cup_{j,\ell} \tilde{F}_{j,\ell}$. Note that the balls $\mathcal{C}_0 \tilde{B}_{j,\ell}$ are overlapping at most $C_{\mathcal{A}_0,\mu,\mu,\mathcal{C}_0}^{-1} (2j_0 + 1)$ times, where $C_{\mathcal{A}_0,\mu,\mathcal{C}_0}$ is the constant in (3.13). Then we obtain that $\mathcal{F}$ is a $\frac{1}{2C_{\mathcal{A}_0,\mu,\mu,\mathcal{C}_0}^{-1} (2j_0 + 1)}$-sparse family and for a.e. $x \in X$,
\[
|T^m_k f(x)| \leq C \sum_{m=0}^{\infty} C_m \sum_{Q \in \mathcal{F}} \left| (b(x) - b_{Q_0})^m |f| |b - b_{Q_0}|^m \right|_{\mathcal{C}_0 B(Q)} \chi_Q(x).
\]

Since $\mathcal{C}_0 B(Q) \subset R_Q$, and it is clear that $\mu(R_Q) \leq \mathcal{C} \mu(\mathcal{C}_0 B(Q))$ ($\mathcal{C}$ depends only on $C_\mu$ and $\mathcal{C}_0$), we obtain that $|f|_{\mathcal{C}_0 B(Q)} \leq \mathcal{C} |f|_{R_Q}$. Next, we further set $\mathcal{S}_t := \{ R_Q \in \mathcal{D}^t : Q \in \mathcal{F}, \ t \in \{1, 2, \ldots, T\} \}$, and from the fact that $\mathcal{F}$ is $\frac{1}{2C_{\mathcal{A}_0,\mu,\mu,\mathcal{C}_0}^{-1} (2j_0 + 1)}$-sparse, we can obtain that each family $\mathcal{S}_t$ is $\frac{1}{2C_{\mathcal{A}_0,\mu,\mu,\mathcal{C}_0}^{-1} (2j_0 + 1)}$-sparse. Now we let
\[
\eta := \frac{1}{2C_{\mathcal{A}_0,\mu,\mu,\mathcal{C}_0}^{-1} (2j_0 + 1)}
\]
where $\bar{\tau}$ is a constant depending only on $\mathcal{C}$, $\mathcal{C}_0$ and the doubling constant $C_\mu$. Then it follows that (3.10) holds, which finishes the proof.

In the end, we show (3.14) and (3.15).

We first show (3.14) by contradiction. Suppose there exists some $\tilde{B}_{j,\ell} = B(x_{j,\ell}, 2^j y_{j,\ell})$ such that $C_{\mathcal{A}_0} \tilde{B}_{j,\ell} \cap U_{j+1} \neq \emptyset$. Then there exists at least one $y_{j,\ell} \in C_{\mathcal{A}_0} \tilde{B}_{j,\ell} \cap U_{j+1}$. Then from the definition of $U_{j+1}$ we see that
\[
d(x_0, y_{j,\ell}) \geq 2^{j+1} y_{j,\ell}.
\]
Moreover, from the definition of $x_{j,\ell}$ and the quasi triangular inequality (1.1) we get that
\[
d(x_0, y_0) \leq A_0(d(x_0, x_{j,\ell}) + d(x_{j,\ell}, y_0)) < A_0(2^{j+1}r + C_{adj}2^{j-\tilde{j}_0}r),
\]
which, together with the previous inequality, show that $2^{j_0}r \leq A_0(2^{j_0+1}r + C_{adj}2^{j_0-3}r)$, and hence we have
\[
2^{j_0} \leq A_0(2 + C_{adj}2^{j_0-3}) < 3A_0,
\]
which contradicts to (3.11). Hence, we see that (3.14) holds.

We now show (3.15), and again we will prove it by contradiction. Suppose there exists some $\tilde{B}_{j,\ell} \in B(x_{j,\ell}, 2^{j_0}r)$ such that $C_{adj} \tilde{B}_{j,\ell} \cap U_{j_0} \neq \emptyset$, where $j \geq j_0$. Then there exists at least one $y_0 \in C_{adj} \tilde{B}_{j,\ell} \cap U_{j_0}$. From the definition of $x_{j,\ell}$ and the quasi triangular inequality (1.1), we see that
\[
2^{j_0}r \leq d(x_0, x_{j,\ell}) \leq A_0(d(x_0, y_0) + d(y_0, x_{j,\ell})) < A_0(2^{j_0+1}r + C_{adj}2^{j_0-3}r),
\]
which implies that
\[
1 \leq A_0(2^{j_0+1} + C_{adj}2^{j_0-3}),
\]
which contradicts to (3.11) and (3.9). Hence, we see that (3.15) holds. \qed

4 Upper Bound of the Commutator $T_b^m$: Proof of Theorem 1.3

In this section we provide the proof of Theorem 1.3 following the idea in [33].

Let $\mathcal{D}$ be a dyadic system in $(X, d, \mu)$ and let $\mathcal{S}$ be a sparse family from $\mathcal{D}$. We now define
\[
A_b^{m,k}f(x) := \sum_{Q \in \mathcal{S}} |b(x) - b_Q|^{m-k}\left(\frac{1}{\mu(Q)}\int_Q |b(z) - b_Q|^k |f(z)|d\mu(z)\right)\chi_Q(x).
\]

By duality, we have that
\[
\|A_b^{m,k}f\|_{L^p_\lambda(X)} \leq \sup_{g \in L^p_\lambda(X)} \sum_{Q \in \mathcal{S}} \left(\int_Q |g(x)\lambda(x)||b(x) - b_Q|^{m-k}d\mu(x)\right)\left(\frac{1}{\mu(Q)}\int_Q |b(z) - b_Q|^k |f(z)|d\mu(z)\right).
\]

Now by Lemma 3.5, there exists a sparse family $\tilde{\mathcal{S}} \subset \mathcal{D}$ such that $\mathcal{S} \subset \tilde{\mathcal{S}}$ and for every cube $Q \in \tilde{\mathcal{S}}$, for a.e. $x \in Q$,
\[
|b(x) - b_Q| \leq C \sum_{P \in \tilde{\mathcal{S}}, P \subset Q} \Omega(b, P)\chi_P(x).
\]

Since $b$ is in $\text{BMO}_{\nu^\perp}(X)$, then we have for a.e. $x \in Q$
\[
|b(x) - b_Q| \leq C\|b\|_{\text{BMO}_{\nu^\perp}(X)} \sum_{P \in \tilde{\mathcal{S}}, P \subset Q} \frac{\nu^\perp(P)}{\mu(P)}\chi_P(x).
\]

Then combining this estimate and inequality (4.1), we further have
\[
\|A_b^{m,k}f\|_{L^p_\lambda(X)}
\]
\[
\leq C \|b\|_{BMO}^m \|x\|_{L^p(X)} \sup_{g \|g\|_L^p(x) = 1} \sum_{Q \subseteq S} \left( \frac{1}{\mu(Q)} \int_Q |g(x)\lambda(x)| \left( \sum_{P \in S \cap Q} \frac{\nu^P}{\mu(P)} \chi_P(x) \right)^{m-k} \, d\mu(x) \right) \\
\times \left( \frac{1}{\mu(Q)} \int_Q \left( \sum_{P \in S \cap Q} \frac{\nu^P}{\mu(P)} \chi_P(z) \right)^{k} |f(z)| \, d\mu(z) \right) \mu(Q).
\]

Next, note that for each \( \ell \in \mathbb{N} \), we have
\[
\left( \sum_{P \in S \cap Q} \frac{\nu^P}{\mu(P)} \chi_P(x) \right)^{\ell} = \sum_{P_1, P_2, \ldots, P_l \in S, P_1, P_2, \ldots, P_l \subseteq Q} \frac{\nu^{P_1}}{\mu(P_1)} \cdots \frac{\nu^{P_l}}{\mu(P_l)} |h|_{P_1 \cap \cdots \cap P_l}(x) \\
\leq \ell! \sum_{P_1, P_2, \ldots, P_l \in S, P_1 \cap \cdots \cap P_l \subseteq Q} \frac{\nu^{P_1}}{\mu(P_1)} \cdots \frac{\nu^{P_l}}{\mu(P_l)} |h|_{P_1 \cap \cdots \cap P_l}(x) \\
\leq C \sum_{P_1, P_2, \ldots, P_l \in S, P_1 \cap \cdots \cap P_l \subseteq Q} \frac{\nu^{P_1}}{\mu(P_1)} \cdots \frac{\nu^{P_l}}{\mu(P_l)} \int_{P_1 \cap \cdots \cap P_l} A_{S, \nu}^{P_1}(|h|(x)) \nu^{P_1}(x) \, d\mu(x) \\
= C \sum_{P_1, P_2, \ldots, P_l \in S, P_1 \cap \cdots \cap P_l \subseteq Q} \frac{\nu^{P_1}}{\mu(P_1)} \cdots \frac{\nu^{P_l}}{\mu(P_l)} \left( A_{S, \nu}^{P_1}(|h|(x)) \right)_{P_1 \cap \cdots \cap P_l} \mu(P_{l-1}),
\]

where \( A_{S, \nu}^{P_1}(|h|(x)) := A_{S}(|h|(x)) \nu^{P_1}(x) \) and \( A_{S}(h) := \sum_{Q \subseteq S} h_Q \chi_Q \).

By iteration, we obtain that
\[
\int_Q |h(x)| \left( \sum_{P \in S \cap Q} \frac{\nu^P}{\mu(P)} \chi_P(x) \right)^{\ell} \, d\mu(x) \leq C \int_Q A_{S, \nu}^{P_1}( |h|(x) ) \, d\mu(x),
\]

where \( A_{S, \nu}^{P_1}( |h|(x) ) \) denotes the \( \ell \)-fold iteration of \( A_{S, \nu}^{P_1} \). Then we have
\[
\| A_{b}^{m,k} f \|_{L^p(X)} \leq C \|b\|_{BMO}^m \|x\|_{L^p(X)} \sup_{g \|g\|_L^p(x) = 1} \sum_{Q \subseteq S} \left( \frac{1}{\mu(Q)} \int_Q A_{S, \nu}^{P_1}( |g| \lambda(x) ) \, d\mu(x) \right) \\
\times \left( \frac{1}{\mu(Q)} \int_Q A_{S, \nu}^{P_1}( |f| ) \, d\mu(z) \right) \mu(Q) \\
\leq C \|b\|_{BMO}^m \|x\|_{L^p(X)} \sup_{g \|g\|_L^p(x) = 1} \left( \int_X A_{S} \left( A_{S, \nu}^{P_1}( |f| ) \right)(x) A_{S, \nu}^{P_1}( |g| \lambda ) \, d\mu(x) \right),
\]
Observe that $A_S$ is self-adjoint. We have
\[
\int_X A_S \left( A_S^{k,\mu} \left( |f| \right) \right) (x) A_S^{m-k,\mu} \left( |g| \lambda \right)(x) d\mu(x)
= \int_X A_S \left( A_S^{k,\mu} \left( |f| \right) \right) (x) A_S^{m-k-1,\mu} \left( |g| \lambda \right)(x) d\mu(x)
= \ldots
= \int_X A_S \left( A_S^{m,\mu} \left( |f| \right) \right) (x) |g(x)| \lambda(x) d\mu(x).
\]
Then from Hölder’s inequality, we further have
\[
\| A_b^{m,k} f \|_{L^p(X)} \\
\leq C \| b \|_{\text{BMO}_{\nu^{\frac{m}{\nu+1}}}(X)} \| g \|_{L^p(X)} \sup_{1 \leq i \leq m} \left( \left( \int_X A_S \left( A_S^{m,\mu} \left( |f| \right) \right)(x) \lambda(x) d\mu(x) \right)^\frac{1}{p} \right)\| g \|_{L^p(X)}
\leq C \| b \|_{\text{BMO}_{\nu^{\frac{m}{\nu+1}}}(X)} \| A_{\mu} \| \left( |\lambda| A_{\mu} [\lambda \cdot \nu^{\frac{m}{\nu+1}}] A_{\mu} \right)^{\max\left\{ 1, \frac{1}{p-1} \right\}} \| A_S \left( A_S^{m-1,\mu} \left( |f| \right) \right) \|_{L^p_{\lambda,\nu^{\frac{m}{\nu+1}}}(X)}
\leq C \| b \|_{\text{BMO}_{\nu^{\frac{m}{\nu+1}}}(X)} \left( |\lambda| A_{\mu} [\lambda \cdot \nu^{\frac{m}{\nu+1}}] A_{\mu} \right)^{\max\left\{ 1, \frac{1}{p-1} \right\}} \| f \|_{L^p_{\lambda,\nu^{\frac{m}{\nu+1}}}(X)}
\]
Then by iteration we have that
\[
\| A_b^{m,k} f \|_{L^p(X)} \leq C \| b \|_{\text{BMO}_{\nu^{\frac{m}{\nu+1}}}(X)} \left( |\lambda| A_{\mu} [\lambda \cdot \nu^{\frac{m}{\nu+1}}] A_{\mu} \right)^{\max\left\{ 1, \frac{1}{p-1} \right\}} | f \|_{L^p_{\lambda,\nu^{\frac{m}{\nu+1}}}(X)}
\leq C \| b \|_{\text{BMO}_{\nu^{\frac{m}{\nu+1}}}(X)} \left( |\lambda| A_{\mu} [\mu] A_{\mu} \prod_{i=1}^{m-1} [\lambda^{1-\frac{i}{m}} \cdot \nu^{\frac{i}{m}}] A_{\mu} \right)^{\max\left\{ 1, \frac{1}{p-1} \right\}} | f \|_{L^p_{\lambda,\nu^{\frac{m}{\nu+1}}}(X)}
\]
We note that by Hölder’s inequality we have
\[
\prod_{i=1}^{m-1} [\lambda^{1-\frac{i}{m}} \cdot \nu^{\frac{i}{m}}] A_{\mu} \leq \left( |\lambda| A_{\mu} [\mu] A_{\mu} \right)^{\frac{m-1}{2}}
\]
As a consequence, we have that
\[
\| A_b^{m,k} f \|_{L^p(X)} \leq C \| b \|_{\text{BMO}_{\nu^{\frac{m}{\nu+1}}}(X)} \left( |\lambda| A_{\mu} [\mu] A_{\mu} \right)^{\frac{m+1}{2}} | f \|_{L^p_{\lambda,\nu^{\frac{m}{\nu+1}}}(X)}
\]
5 **Lower Bound of the Commutator $T_b^m$: Proof of Theorem 1.4**

In this section, we use some ideas from [25, 32, 33] and adapt them to our general setting with the aim to prove Theorem 1.4. To begin with, let $T$ be the Calderón–Zygmund operator as in Definition 1.1 with the kernel $K$ and $\omega$ satisfying $\omega(t) \to 0$ as $t \to 0$, and satisfy the homogeneous condition as in (1.6).

We first introduce another version of the homogeneous condition: There exist positive constants $3 \leq A_1 \leq A_2$ such that for any ball $B := B(x_0, r) \subset X$, there exist balls $\tilde{B} := B(y_0, r)$ such that $A_1 r \leq d(x_0, y_0) \leq A_2 r$, and for all $(x, y) \in (B \times \tilde{B})$, $K(x, y)$ does not change sign and
\[
|K(x, y)| \geq \frac{1}{\mu(B)}.
\]
(5.1)
If the kernel \( K(x, y) := K_1(x, y) + iK_2(x, y) \) is complex-valued, where \( i^2 = -1 \), then at least one of \( K_i \) satisfies (5.1).

Then we first point out that the homogeneous condition (1.6) implies (5.1).

**Proposition 5.1.** Let \( T \) be the Calderón–Zygmund operator as in Definition 1.1 with the kernel \( K \) and \( \omega \) satisfying \( \omega(t) \to 0 \) as \( t \to 0 \), and satisfy the homogeneous condition as in (1.6). Then \( T \) satisfies (5.1).

**Proof.** Let \( T \) be the Calderón–Zygmund operator as in Definition 1.1 with the kernel \( K \) and \( \omega \) satisfying \( \omega(t) \to 0 \) as \( t \to 0 \), and satisfy the homogeneous condition as in (1.6). Since \( \omega(t) \to 0 \) as \( t \to 0 \), there exists \( \delta \in (0, 1) \) such that when \( 0 < t < \delta \),

\[
\omega(t) < \frac{1}{20 \cdot 3^n \cdot C \cdot \mu \cdot c_0},
\]

where \( c_0 \) is from (1.6), \( C \) is from Definition 1.1 and \( \mu \) is from (1.3).

For all numbers \( A \) with

\[
A > \max \left\{ 3, 2 + \frac{1}{\delta}, 2A_0 \right\},
\]

and for any ball \( B := B(x_0, r) \subset X \), according to the homogeneous condition (1.6), there exists a point \( y_0 \in B(x_0, C Ar) \setminus B(x_0, Ar) \) such that

\[
|K(x_0, y_0)| \geq \frac{1}{c_0 \mu(B(x_0, Ar))}.
\]

Next, from the smoothness condition (1.5), we have that for every \( x \in B(x_0, r) \) and \( y \in B(y_0, r) \),

\[
|K(x, y) - K(x_0, y_0)| \leq |K(x, y) - K(x, y_0)| + |K(x_0, y) - K(x_0, y_0)|
\]

\[
\leq \frac{C}{\mu(B(x_0, (A - 2)r))} \omega \left( \frac{1}{A - 2} \right) \cdot \frac{1}{\mu(B(x_0, Ar))} \omega \left( \frac{r}{Ar} \right)
\]

where we use the fact that \( \omega(t) \) is increasing. Next, by (1.3), we obtain that

\[
|K(x, y) - K(x_0, y_0)| \leq 2CC_\mu \left( \frac{A}{A - 2} \right)^n \omega \left( \frac{1}{A - 2} \right) \frac{1}{\mu(B(x_0, Ar))} \leq \frac{1}{10c_0 \mu(B(x_0, Ar))},
\]

where the last inequality follows from the choice of \( A \) as in (5.2).

We now fix a positive number \( A_1 \) satisfying (5.2) and set \( A_2 := C A_1 \).

We first consider the kernel \( K(x, y) \) to be a real-valued function. If \( K(x_0, y_0) > 0 \), then for every \( x \in B(x_0, r) \) and \( y \in B(y_0, r) \) we have that

\[
K(x, y) = K(x, y_0) - (K(x_0, y_0) - K(x, y)) \geq K(x_0, y_0) - |K(x, y) - K(x_0, y_0)|
\]

\[
\geq \frac{1}{c_0 \mu(B(x_0, Ar))} - \frac{1}{10c_0 \mu(B(x_0, Ar))} > \frac{1}{2c_0 \mu(B(x_0, Ar))}.
\]

Similarly if \( K(x_0, y_0) < 0 \), then every \( x \in B(x_0, r) \) and \( y \in B(y_0, r) \) we have that

\[
K(x, y) < -\frac{1}{2c_0 \mu(B(x_0, Ar))}.
\]
Thus, combining these two cases we obtain that (5.1) holds.

Next we consider the kernel $K(x, y)$ to be a complex function. We write $K(x, y) = K_1(x, y) + iK_2(x, y)$, with $i^2 = -1$. Then (5.3) implies that

$$\text{either } |K_1(x_0, y_0)| \geq \frac{\sqrt{2}}{2c_0\mu(B(x_0, Ar))} \text{ or } |K_2(x_0, y_0)| \geq \frac{\sqrt{2}}{2c_0\mu(B(x_0, Ar))}.$$

Suppose $|K_j(x_0, y_0)| \geq \frac{\sqrt{2}}{2c_0\mu(B(x_0, Ar))}$ for some $j \in \{1, 2\}$. If $K_j(x_0, y_0) > 0$, then every $x \in B(x_0, r)$ and $y \in B(y_0, r)$ we have that

$$K_j(x, y) = K_j(x_0, y_0) - (K_j(x_0, y_0) - K_j(x, y)) \geq K_j(x_0, y_0) - |K(x, y) - K(x_0, y_0)| \geq \frac{\sqrt{2}}{2c_0\mu(B(x_0, Ar))} - \frac{1}{10c_0\mu(B(x_0, Ar))} > \frac{1}{2c_0\mu(B(x_0, Ar))}.$$

Similarly if $K_j(x_0, y_0) < 0$ for some $j \in \{1, 2\}$, then every $x \in B(x_0, r)$ and $y \in B(y_0, r)$ we have that

$$K_j(x, y) < -\frac{1}{2c_0\mu(B(x_0, Ar))}.$$

Thus, (5.1) holds for $K_j(x, y)$.

The proof of Proposition 5.1 is complete. □

**Definition 5.2.** By a median value of a real-valued measurable function $f$ over $B$ we mean a possibly non-unique, real number $\alpha_B(f)$ such that

$$\mu(\{x \in B : f(x) > \alpha_B(f)\}) \leq \frac{1}{2}\mu(B) \text{ and } \mu(\{x \in B : f(x) < \alpha_B(f)\}) \leq \frac{1}{2}\mu(B).$$

It is known that for a given function $f$ and ball $B$, the median value exists and may not be unique; see, for example, [28].

**Lemma 5.3.** Let $b$ be a real-valued measurable function. For any ball $B$, let $\tilde{B}$ be as in (5.1). Then there exist measurable sets $E_1, E_2 \subset B$, and $F_1, F_2 \subset \tilde{B}$, such that

(i) $B = E_1 \cup E_2$, $\tilde{B} = F_1 \cup F_2$ and $\mu(E_i) \geq \frac{1}{2}\mu(B)$, $i = 1, 2$;

(ii) $b(x) - b(y)$ does not change sign for all $(x, y)$ in $E_i \times F_i$, $i = 1, 2$;

(iii) $|b(x) - \alpha_{\tilde{B}}(b)| \leq |b(x) - b(y)|$ for all $(x, y)$ in $E_i \times F_i$, $i = 1, 2$.

**Proof.** For the given balls $B$ and $\tilde{B}$, following the idea in [33, Proposition 3.1] we set

$$F_1 := \{y \in \tilde{B} : b(y) \leq \alpha_{\tilde{B}}(b)\} \text{ and } F_2 := \{y \in \tilde{B} : b(y) \geq \alpha_{\tilde{B}}(b)\}.$$

Moreover, we define

$$E_1 := \{x \in B : b(x) \geq \alpha_{\tilde{B}}(b)\} \text{ and } E_2 := \{x \in B : b(x) \leq \alpha_{\tilde{B}}(b)\}.$$

Then by Definition 5.2, we see that $\mu(F_i) \geq \frac{1}{2}\mu(B)$, $i = 1, 2$. Moreover, for $(x, y) \in E_i \times F_i$, $i = 1, 2$,

$$|b(x) - b(y)| = |b(x) - \alpha_{\tilde{B}}(b) + \alpha_{\tilde{B}}(b) - b(y)| = |b(x) - \alpha_{\tilde{B}}(b)| + |\alpha_{\tilde{B}}(b) - b(y)| \geq |b(x) - \alpha_{\tilde{B}}(b)|.$$

This finishes the proof of Lemma 5.3. □
We now return to the proof of Theorem 1.4, following the approach and method in [33].

**Proof of Theorem 1.4.** For given \( b \in L^1_{\text{loc}}(X) \) and for any ball \( B \), let \( \Omega(b, B) \) be the oscillation as in (3.1). Under the assumptions of Theorem 1.4, we will show that for any ball \( B \),

\[
\Omega(b, B) \leq \frac{\nu^+(B)}{\mu(B)}, \tag{5.4}
\]

Without loss of generality, we assume that \( K(x, y) \) is real-valued. Let \( B \) be a ball. We apply the assumption (5.1) and Lemma 5.3 to get sets \( E_i, F_i, i = 1, 2 \).

On the one hand, by Lemma 5.3 and (5.1), we have that for \( f_i := \chi_{F_i}, i = 1, 2 \),

\[
\frac{1}{\mu(B)} \sum_{i=1}^{2} \int_B |T_b^m f_i(x)| \, d\mu(x) \geq \frac{1}{\mu(B)} \sum_{i=1}^{2} \int_{E_i} |T_b^m f_i(x)| \, d\mu(x) \\
= \frac{1}{\mu(B)} \sum_{i=1}^{2} \int_{E_i} \int_{F_i} |b(x) - b(y)|^m |K(x, y)| \, d\mu(y) \, d\mu(x) \\
\geq \frac{1}{\mu(B)} \sum_{i=1}^{2} \int_{E_i} \int_{F_i} |b(x) - \alpha_B(b)|^m \frac{\mu(B)}{\mu(B)} \, d\mu(y) \, d\mu(x) \\
\geq \frac{1}{\mu(B)} \int_B |b(x) - \alpha_B(b)|^m \, d\mu(x) \\
\geq \Omega(b; B)^m.
\]

On the other hand, from Hölder’s inequality and the boundedness of \( T_b^m \), we deduce that

\[
\frac{1}{\mu(B)} \sum_{i=1}^{2} \int_B |T_b^m f_i(x)| \, d\mu(x) \\
\leq \frac{1}{\mu(B)} \sum_{i=1}^{2} \left[ \int_B |T_b^m f_i(x)|^p \lambda_2(x) \, d\mu(x) \right]^{1/p} \left( \int_B \lambda_2(x)^{-\frac{1}{p-1}} \, d\mu(x) \right)^{1/p'} \\
\leq \frac{1}{\mu(B)} \sum_{i=1}^{2} [\lambda_1(F_i)]^{1/p} \left( \int_B \lambda_2(x)^{-\frac{1}{p-1}} \, d\mu(x) \right)^{1/p'} \\
\leq \frac{1}{\mu(B)} [\lambda_1(B)]^{1/p} \left( \int_B \lambda_2(x)^{-\frac{1}{p-1}} \, d\mu(x) \right)^{1/p'} \\
\leq \frac{1}{\mu(B)} [\lambda_1(B)]^{1/p} \left( \int_B \lambda_2(x)^{-\frac{1}{p-1}} \, d\mu(x) \right)^{1/p'},
\]

where in the last inequality, we use the fact that \( K_1 r_B \leq d(x_B, x_B) \leq K_2 r_B \) and \( \lambda_1 \in A_p \).

Combining the two inequalities above and invoking \( \lambda_i \in A_p \), we conclude that

\[
\Omega(b, B)^m \lesssim \frac{1}{\mu(B)} [\lambda_1(B)]^{1/p} \left( \int_B \lambda_2(x)^{-\frac{1}{p-1}} \, d\mu(x) \right)^{1/p'} \lesssim \left( \frac{\nu^+(B)}{\mu(B)} \right)^m,
\]

where the last inequality follows from the argument as in the proof of Theorem 1.1 in [33], by using (2.5). Thus, (5.4) holds and hence, the proof of Theorem 1.4 is complete. □
6 Weighted Hardy space, Duality and Weak Factorisation: Proof of Theorem 1.5

In this section we study the weighted Hardy, BMO spaces and duality, as well as their dyadic versions on spaces of homogeneous type.

6.1 Dyadic Littlewood–Paley Square Function

Following the form in [21] we now introduce the dyadic Littlewood–Paley square function on spaces of homogeneous type.

Definition 6.1. Given a dyadic grid \( \mathcal{D} \) on \( X \), the dyadic square function \( S_{\mathcal{D}} \) is defined by:

\[
S_{\mathcal{D}} f := \left[ \sum_{Q \in \mathcal{D}} \sum_{\epsilon = 1}^{M_Q - 1} |\langle f, h_{Q}^{\epsilon} \rangle|^2 \frac{\chi_{Q}}{\mu(Q)} \right]^{\frac{1}{2}}.
\]

Our main result in this subsection is:

Theorem 6.2. Suppose \( 1 < p < \infty \) and \( w \in A_p \). Then we have

\[
\|S_{\mathcal{D}} f\|_{L^p_w(X)} \leq C_p \left[ w \right]^{\max\{1, \frac{1}{p} - 1\}} \|f\|_{L^p_w(X)}.
\]

We prove this theorem by following the idea in [42, Theorem 3.1 and Corollary 3.2]. To begin with, we first introduce an auxiliary lemma.

Lemma 6.3. Let \( w \) be an \( A_2 \) weight in \( (X, \rho, \mu) \). Then

\[
\sum_{Q \in \mathcal{D}} \sum_{\epsilon = 1}^{M_Q - 1} |\langle f, h_{Q}^{\epsilon} \rangle|^2 \frac{1}{\langle w \rangle_Q} \lesssim \left[ w \right]_{A_2} \|f\|_{L^2_w(X)}^2
\]

for all \( f \in L^2_w(X) \), where

\[
\langle w \rangle_Q := \frac{1}{\mu(Q)} \int_Q w(x) d\mu(x).
\]

Proof. Recall from [29], we have \( h_{Q}^{\epsilon} = a_{\epsilon} \chi_{Q_{\epsilon}} - b_{\epsilon} \chi_{E_{\epsilon + 1}} \), where

\[
a_{\epsilon} := \sqrt{\frac{\mu(E_{\epsilon + 1})}{\mu(Q_{\epsilon})\mu(E_{\epsilon})}}, \quad b_{\epsilon} := \sqrt{\frac{\mu(Q_{\epsilon})}{\mu(E_{\epsilon})\mu(E_{\epsilon + 1})}} \quad \text{and} \quad E_{\epsilon} = Q_{\epsilon} \cup E_{\epsilon + 1},
\]

where \( Q_{\epsilon} \) and \( E_{\epsilon + 1} \) are disjoint. Now we introduce the weighted Haar system \( \{h_{Q}^{w,\epsilon}\}_{1 \leq \epsilon \leq M_Q - 1, Q \in \mathcal{D}} \) in \( L^2_w(X) \), where

\[
h_{Q}^{w,\epsilon} := \frac{1}{\sqrt{w(E_{\epsilon})}} \left( \frac{\sqrt{w(E_{\epsilon + 1})}}{\sqrt{w(Q_{\epsilon})}} \chi_{Q_{\epsilon}} - \frac{\sqrt{w(Q_{\epsilon})}}{\sqrt{w(E_{\epsilon + 1})}} \chi_{E_{\epsilon + 1}} \right).
\]

Note that when \( w = 1 \), we have

\[
h_{Q}^{1,\epsilon} := h_{Q}^{\epsilon} = \frac{1}{\sqrt{\mu(E_{\epsilon})}} \left( \frac{\sqrt{\mu(E_{\epsilon + 1})}}{\sqrt{\mu(Q_{\epsilon})}} \chi_{Q_{\epsilon}} - \frac{\sqrt{\mu(Q_{\epsilon})}}{\sqrt{\mu(E_{\epsilon + 1})}} \chi_{E_{\epsilon + 1}} \right).
\]

We set

\[
h_{E_{\epsilon}} := \frac{\chi_{E_{\epsilon}}}{\mu(E_{\epsilon})},
\]
and write $h_Q^\epsilon = C_Q(w, \epsilon)h^w_Q + D_Q(w, \epsilon)h^1_{E_\epsilon}$.

It is easy to see that $\int_Q h^w_Q dw = 0$ and $\int_Q (h^w_Q)^2 dw = 1$. This implies

$$D_Q(w, \epsilon) = \frac{\hat{w}(Q, \epsilon)}{(w)_{E_\epsilon}},$$

where $\hat{w}(Q, \epsilon) := \langle w, h^\epsilon_Q \rangle$

and, after some computation,

$$C_Q(w, \epsilon)^2 = \frac{\mu(E_{\epsilon+1})}{\mu(E_{\epsilon})} \langle w \rangle_{Q_\epsilon} + \frac{\mu(Q_\epsilon)}{\mu(E_{\epsilon})} \langle w \rangle_{E_{\epsilon+1}} - \frac{\mu(E_{\epsilon+1})}{\mu(E_{\epsilon})} \frac{w(Q_\epsilon)}{w(E_{\epsilon})} \langle w \rangle_{Q_{\epsilon}}$$

$$- \frac{\mu(Q_\epsilon)}{\mu(E_{\epsilon})} \frac{w(E_{\epsilon+1})}{w(E_{\epsilon})} \langle w \rangle_{E_{\epsilon+1}} + 2 \frac{w(E_{\epsilon+1})}{w(E_{\epsilon})} \frac{w(Q_\epsilon)}{\mu(E_{\epsilon})} \langle w \rangle_{Q_\epsilon}$$

Note that it doesn’t really matter what $C_Q(w, \epsilon)$ really is as long as we have some nice bound for it. In fact, from Lemma 4.6 in [29], we have that

$$C_Q(w, \epsilon)^2 \langle w \rangle_{Q_{\epsilon}}^{-1} \lesssim 1,$$

which implies that $C_Q(w, \epsilon)^2 \langle w \rangle_{Q_{\epsilon}}^{-1} \lesssim 1$.

Now,

$$\sum_{Q \in \mathcal{D}} \sum_{\epsilon=1}^{M_Q-1} \langle w \rangle_{Q_{\epsilon}}^{-1} |\langle f, h^\epsilon_Q \rangle|^2 = \sum_{Q \in \mathcal{D}} \sum_{\epsilon=1}^{M_Q-1} \langle w \rangle_{Q_{\epsilon}}^{-1} |\langle f, C_Q(w, \epsilon)h^w_Q + D_Q(w, \epsilon)h^1_{E_\epsilon} \rangle|^2$$

$$= \sum_{Q \in \mathcal{D}} \sum_{\epsilon=1}^{M_Q-1} \langle w \rangle_{Q_{\epsilon}}^{-1} |C_Q(w, \epsilon)\langle f, h^w_Q \rangle + D_Q(w, \epsilon)\langle f, h^1_{E_\epsilon} \rangle|^2$$

$$= \sum_{Q \in \mathcal{D}} \sum_{\epsilon=1}^{M_Q-1} \langle w \rangle_{Q_{\epsilon}}^{-1} C_Q(w, \epsilon)^2 |\langle f, h^w_Q \rangle|^2$$

$$+ 2 \sum_{Q \in \mathcal{D}} \sum_{\epsilon=1}^{M_Q-1} \langle w \rangle_{Q_{\epsilon}}^{-1} C_Q(w, \epsilon) D_Q(w, \epsilon) \langle f, h^w_Q \rangle \langle f, h^1_{E_\epsilon} \rangle$$

$$+ \sum_{Q \in \mathcal{D}} \sum_{\epsilon=1}^{M_Q-1} \langle w \rangle_{Q_{\epsilon}}^{-1} D_Q(w, \epsilon)^2 |\langle f, h^1_{E_\epsilon} \rangle|^2$$

$$=: S_1 + S_2 + S_3.$$ 

$S_2$ can be bounded by $\sqrt{S_1} \sqrt{S_3}$, so it suffices to bound $S_1$ and $S_3$. By using the bound on $C_Q(w, \epsilon)$, we have

$$S_1 \lesssim \sum_{Q \in \mathcal{D}} \sum_{\epsilon=1}^{M_Q-1} |\langle f, h^w_Q \rangle|^2 = \sum_{Q \in \mathcal{D}} \sum_{\epsilon=1}^{M_Q-1} |\langle w^{-1} f, h^w_Q \rangle_{L^2(X)}|^2 \leq \|f\|_{L^{w^{-1}}(X)}^2,$$

On the other hand,

$$S_3 = \sum_{Q \in \mathcal{D}} \sum_{\epsilon=1}^{M_Q-1} \langle w \rangle_{Q_{\epsilon}}^{-1} D_Q(w, \epsilon)^2 |\langle f, h^1_{E_\epsilon} \rangle|^2 = \sum_{Q \in \mathcal{D}} \sum_{\epsilon=1}^{M_Q-1} D_Q(w, \epsilon)^2 |\langle f \rangle_{E_\epsilon}^{-1} |w|^{-1} \rangle.$$
Now,

\[
\frac{1}{\mu(E_c)} \sum_{R \subset Q \eta \subset E} D_R(w, \eta)^2 \langle w \rangle_R^{-1} \langle w \rangle_{E_\eta}^2 = \frac{1}{\mu(E_c)} \sum_{R \subset Q \eta \subset E} \hat{w}(R, \eta)^2 \langle w \rangle_{E_\eta}^2 \langle w \rangle_R^{-1} \langle w \rangle_{E_\eta}^2 \lesssim \frac{1}{\mu(E_c)} \sum_{R \subset Q \eta \subset E} \hat{w}(R, \eta)^2 \langle w \rangle_R^{-1} \langle w \rangle_{E_\eta}^2 \lesssim [w]_{A_2} \langle w \rangle_{E_\eta}^2,
\]

where the last inequality follows from a Bellman function technique that can be found in [6]. Thus, by adopting the remark of Treil in [45, Section 5] on the dyadic Carleson Embedding Theorem on a general space of homogeneous type, we get:

\[
S_3 = \sum_{Q \in \mathcal{D}} \sum_{\epsilon=1}^{M_Q^{-1}} D_Q(w, \epsilon)^2 \langle f \rangle_{E_\epsilon}^2 \langle w \rangle_{E_\epsilon}^{-1} \lesssim [w]_{A_2} \langle w \rangle_{E_\epsilon}^2 \lesssim \|fw^{-\frac{1}{2}}\|_{L^2(X)}^2.
\]

The proof of Lemma 6.3 is complete. \qed

**Proof of Theorem 6.2.** Suppose \( w \in A_2 \). Following the argument in the proof of Theorem 3.1 in [42], we obtain that the Lemma 6.3 above implies that

\[
\|f\|_{L^2_w(X)} \lesssim [w]_{A_2} \|S_{\mathcal{D}} f\|_{L^2_w(X)},
\]

where the implicit constant is independent of \( f \) and \( w \).

Then following the argument in the proof of Corollary 3.2 in [42], we obtain that

\[
\|S_{\mathcal{D}} f\|_{L^2_w(X)} \lesssim [w]_{A_2} \|f\|_{L^2_w(X)}.
\]

Next, by the sharp form of Rubio de Francia’s extrapolation theorem (due to Dragičević, Grafakos, Pereyra and Petermichl [12] in the Euclidean space and due to Anderson and Damián [1] on spaces of homogeneous type), this implies the corresponding weighted \( L^p \) bound

\[
\|S_{\mathcal{D}} f\|_{L^p_w(X)} \leq C_p [w]^\max\{1, \frac{1}{p'}\} \|f\|_{L^p_w(X)}.
\]

The proof of Theorem 6.2 is complete. \qed

### 6.2 Weighted Hardy Spaces, Duality and Weak Factorisation

We now introduce the atoms for the weighted Hardy space.

**Definition 6.4.** Suppose \( w \in A_2 \). A function \( a \) is called a \((1, 2)\)-atom, if there exists a ball \( B \subset X \) such that

1. \( \text{supp}(a) \subset B; \)
2. \( \int_B a(x) \, d\mu(x) = 0; \)
3. \( \|a\|_{L^2_w(B)} \leq [w(B)]^{-\frac{1}{2}}. \)

**Definition 6.5.** Suppose \( w \in A_2 \). A function \( f \) is said to belong to the Hardy space \( H^1_{w, 2}(X) \), if \( f = \sum_{j=1}^\infty \lambda_j a_j \) with \( \sum_{j=1}^\infty |\lambda_j| < \infty \) and \( a_j \) is a \((1, 2)\)-atom for each \( j \). Moreover, the norm of \( f \) on \( H^1_{w, 2}(X) \) is defined by \( \|f\|_{H^1_{w, 2}(X)} = \inf \{ \sum_{j=1}^\infty |\lambda_j| \} \), where the infimum is taken over all possible decompositions of \( f \) as above.
We then have the duality between the weighted Hardy space and weighted BMO. We point out that for the sake of simplicity, we obtain this result for $p = 2$. For the Euclidean version of duality of weighted Hardy and BMO spaces, we refer to [13, Section 4] for the full range of $p \in [1, 2]$.

**Theorem 6.6.** Suppose $w \in A_2$. Then we have $(H^1_{w,2}(X))^\prime = \text{BMO}_w(X)$.

**Proof.** To prove $\text{BMO}_w(X) \subset (H^1_{w,p}(X))^\prime$, for any $g \in \text{BMO}_w(X)$, let

$$
\ell_g(a) = \int_X a(x)g(x)\,d\mu(x),
$$

where $a$ is an atom as in Definition 6.4.

Assume that $a$ is supported in a ball $B \subset X$. Then by Hölder’s inequality and $w \in A_2$, we see that

$$
\left| \int_X g(x)a(x)\,d\mu(x) \right| = \left| \int_B [g(x) - g_B]a(x)\,d\mu(x) \right|
\leq \left[ \int_B |g(x) - g_B|^2w^{-1}(x)\,d\mu(x) \right]^{\frac{1}{2}} \left[ \int_B |a(x)|^2w(x)\,d\mu(x) \right]^{\frac{1}{2}}
\leq \left[ \frac{1}{w(B)} \int_B |g(x) - g_B|^2w^{-1}(x)\,d\mu(x) \right]^{\frac{1}{2}}
\leq C\|g\|_{\text{BMO}_w(X)}.
$$

Thus $\ell_g$ can be extended to a bounded linear functional on $H^1_{w,2}(X)$.

Conversely, assume that $\ell \in (H^1_{w,p}(X))^\prime$. For any ball $B \subset X$, let

$$
L^2_{0,w}(B) = \left\{ f \in L^2_{w}(B) : \text{supp}(f) \subset B, \int_B f(x)\,d\mu(x) = 0 \right\}.
$$

Then we see that for any $f \in L^2_{0,w}(B)$, $a := \frac{1}{|w(B)|^{\frac{1}{2}}\|f\|_{L^2_{w}(B)}}f$ is an atom as in Definition 6.4. This implies that

$$
|\ell(a)| \leq \|\ell\|\|a\|_{H^1_{w,2}(X)} \leq \|\ell\|.
$$

Moreover, we see that

$$
|\ell(f)| \leq \|\ell\|\|w(B)|^{\frac{1}{2}}\|f\|_{L^2_{w}(B)}.
$$

From the Riesz Representation theorem, there exists $[\varphi] \in [L^2_{0,w}(B)]^\ast = L^2_{w-1}(B)/C$, and $\varphi \in [\varphi]$, such that for any $f \in L^2_{0,w}(B)$,

$$
\ell(f) = \int_B f(x)\varphi(x)\,d\mu(x)
$$

and

$$
\|\varphi\| = \inf_c \|\varphi + c\|_{L^2_{w-1}(B)} \leq \|\ell\|\|w(B)|^{\frac{1}{2}}.
$$

Now for a fixed ball $B$, we define $B_j = 2^jB$, $j \in \mathbb{N}$. And for $B_0$, we mean the ball $B$ itself. Then we have that for all $f \in L^2_{0,w}(B)$ and $j \in \mathbb{N},$

$$
\int_B f(x)\varphi_{B_j}(x)\,d\mu(x) = \int_B f(x)\varphi_{B_j}(x)\,d\mu(x).
$$
It follows that for almost every $x \in B$, $\varphi_B(x) - \varphi_{B_0}(x) = C_j$ for some constant $C_j$. From this we further deduce that for all $j, l \in \mathbb{N}$, $j \leq l$ and almost every $x \in B_j$,

$$\varphi_{B_j}(x) - C_j = \varphi_{B_0}(x) = \varphi_{B_l}(x) - C_l.$$  

Define $\varphi(x) = \varphi_j(x) - C_j$ on $B_j$ for $j \in \mathbb{N}$. Then $\varphi$ is well defined. Moreover, since $X = \bigcup_j B_j$, by Hölder’s inequality and $w \in A_p$, we see that for any $c$ and any ball $B \subset X$,

$$\left[ \int_B \left| \varphi(x) - \varphi_B \right|^2 w^{-1}(x) \, d\mu(x) \right]^{\frac{1}{2}} = \sup_{\|f\|_{L^2_w(B)} \leq 1} \left| \langle f, \varphi - \varphi_B \rangle \right|.$$

Taking the infimum over $c$, we have that $\varphi \in \text{BMO}_w(X)$ and $\|\varphi\|_{\text{BMO}_w(X)} \leq C\|\ell\|$. \hfill \Box

We now provide a sketch of the proof of Theorem 1.5, the details are similar to the weak factorisation result obtained in [8].

**Proof of Theorem 1.5.** Similar to [8] (see also Corollary 1.4 in [21] and its proof), as a consequence of the duality of weighted Hardy space $H^1_w(X)$ and BMO$_w(X)$ (Theorem 6.6 above), we see that if $f$ is of the form (1.7), then $f$ is in $H^1_w(X)$ with the $H^1_w(X)$-norm control by the right-hand side of (1.8). Conversely, based the characterisation of BMO$_w(X)$ via the commutator $[b, T]$ as in Theorem 1.4 (the case of $m = 1$) and the linear functional analysis argument as in [7], we get that every $f \in H^1_w(X)$ admits an factorisation as in (1.7). Hence, we obtain that Theorem 1.5 holds. \hfill \Box

7 Applications

The aim of this section is to show that Theorem 1.6 holds for each of the six operators listed in the introduction.

7.1 Cauchy’s Integral Operator

Let $A(x)$ be a Lipschitz function on $\mathbb{R}$. Consider the Lipschitz curve as $z = x + iA(x)$, $x \in (-\infty, \infty)$. Recall that the Cauchy integral adapted to this Lipschitz curve is:

$$C_A(f)(x) = p.v. \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(y) \, dy}{(x-y) + i(A(x) - A(y))}.$$  

The (unweighted version) commutator result was obtained in [34]. Here we point out that the two weight commutator and high order commutator results also hold for Cauchy’s Integral Operator.

**Proposition 7.1.** Theorem 1.6 holds for the Cauchy integral operator $C_A$ with the underlying setting $(\mathbb{R}, |\cdot|, dx)$.  


Proof. To see this, we point out that this operator has the associated kernel
\[ C_A(x, y) = \frac{1}{\pi (x - y) + i(A(x) - A(y))}, \]
which satisfies the size condition
\[ |C_A(x, y)| \leq \frac{1}{|x - y|} \]
and the smoothness condition
\[ |C_A(x, y) - C_A(x, y')| + |C_A(y, x) - C_A(y', x)| \leq 2(\|A'\|_\infty + 1) \frac{|y - y'|}{|x - y|^2} \]
for every \( x, y, y' \) such that \( |y - y'| \leq |x - y|/2 \). Moreover, for any interval \( I := I(x_0, r) \), we take \( y_0 = x_0 + 4r \). Then we see that \( \text{Re} C_A(x_0, y_0) \), the real part of \( C_A(x_0, y_0) \), satisfies that \( \text{Re} C_A(x_0, y_0) \leq 0 \) and
\[ |\text{Re} C_A(x_0, y_0)| = \frac{1}{\pi (x_0 - y_0)^2 + (A(x_0) - A(y_0))^2} \geq \frac{y_0 - x_0}{(\|A'\|_\infty + 1)(x_0 - y_0)^2} \geq \frac{1}{|I|}. \]
Therefore, (1.6) holds. As a consequence of this fact and Theorems 1.3 and 1.4, we see that Theorem 1.6 holds.

7.2 The Cauchy–Szegő Projection Operator on the Heisenberg Group \( \mathbb{H}^n \)

We recall all the related definitions for the Heisenberg group in [44, Chapter XII]. Recall that \( \mathbb{H}^n \) is the Lie group with underlying manifold \( \mathbb{C}^n \times \mathbb{R} = \{ [z, t] : z = (z_1, \ldots, z_n) \in \mathbb{C}^n, t \in \mathbb{R} \} \) and multiplication law
\[ [z, t] \circ [z', t'] := [z_1 + z'_1, \ldots, z_n + z'_n, t + t' + 2\text{Im}(\sum_{j=1}^n z_j z'_j)]. \]
The identity of \( \mathbb{H}^n \) is the origin and the inverse is given by \([z, t]^{-1} = [-z, -t]\). Hereafter, we identify \( \mathbb{C}^n \) with \( \mathbb{R}^{2n} \) and use the following notation to denote the points of \( \mathbb{C}^n \times \mathbb{R} \equiv \mathbb{R}^{2n+1} \): \( g = [z, t] \equiv [x, y, t] = [x_1, \ldots, x_n, y_1, \ldots, y_n, t] \) with \( z = [z_1, \ldots, z_n], z_j = x_j + iy_j \) and \( x_j, y_j, t \in \mathbb{R} \). Then, the composition law \( \circ \) can be explicitly written as \( g \circ g' = [x, y, t] \circ [x', y', t'] = [x + x', y + y', t + t' + 2(y, x') - 2(x, y')] \), where \( \langle \cdot , \cdot \rangle \) denotes the usual inner product in \( \mathbb{R}^n \).

We recall the upper half-space \( \mathcal{U}^n \) and its boundary \( \partial \mathcal{U}^n \) as follows:
\[ \mathcal{U}^n = \{ z \in \mathbb{C}^{n+1} : \text{Im}(z_{n+1}) > \sum_{j=1}^n |z_j|^2 \}, \quad \partial \mathcal{U}^n = \{ z \in \mathbb{C}^{n+1} : \text{Im}(z_{n+1}) = \sum_{j=1}^n |z_j|^2 \}. \]

For any function \( F \) defined on \( \mathcal{U}^n \), we write \( F_c \) for its vertical translate: \( F_c(z) = F(z + ci) \) with \( i = (0, \ldots, 0, i) \). We also recall the Hardy space \( \mathcal{H}^2(\mathcal{U}^n) \), which consists of all functions \( F \) holomorphic on \( \mathcal{U}^n \) for which \( \|F\|_{\mathcal{H}^2(\mathcal{U}^n)} = \left( \sup_{\varepsilon > 0} \int_{\partial \mathcal{U}^n} |F_c(z)|^2 d\beta(z) \right)^{1/2} < \infty \), where \( d\beta(z) \) is the surface measure on \( \partial \mathcal{U}^n \).

The Cauchy–Szegő projection operator \( C \) is the orthogonal projection from \( L^2(\partial \mathcal{U}^n) \) to the subspace of functions \( \{ F^b \} \) that are boundary values of functions \( F \in \mathcal{H}^2(\mathcal{U}^n) \). According to [44, Section 2.3, Section 2.4, Chapter XII], we get that for \( f \in L^2(\mathbb{H}^n) \),
\[ C(f)(x) = \int_{\mathbb{H}^n} K(x, y)f(y)dy, \]
where $K(x, y) = K(y^{-1} \circ x)$ for $x \neq y$ and

$$K(x) = -\frac{\partial}{\partial t} \left( \frac{c}{n} t + i|\zeta|^2 - n \right) \quad \text{for } x = [\zeta, t] \in \mathbb{H}^n = \mathbb{C}^n \times \mathbb{R},$$

and $c = 2^{n-1}n+1n!\pi^{-n-1}$.

**Proposition 7.2.** Theorem 1.6 holds for the Cauchy–Szegö projection operator $C$ with the underlying setting $(\mathbb{H}^n, \rho, dx)$, where $dx$ is the usual Lebesgue measure on $\mathbb{C}^n \times \mathbb{R}$ and $\rho$ is the norm on $\mathbb{H}^n$ defined by $\rho(x) := \max\{|\zeta|, |t|^\frac{1}{2}\}$ for $x = [\zeta, t] \in \mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$.

**Proof.** We begin by recalling that with this norm $\rho(x)$ as above, and we set $\rho(x, y) := \rho(y^{-1} \circ x)$. From [44, Section 2.5, Chapter XII] we obtain that the Cauchy–Szegö kernel $K(x, y)$ satisfies the following conditions:

$$|K(x, y)| \approx \rho(x, y)^{-2n-2}$$

$$|K(x, y) - K(x, y_0)| \lesssim \frac{\rho(y, y_0)}{\rho(x, y)} \frac{1}{\rho(x, y)^{2n+2}} \quad \text{whenever } \rho(x, y) \geq c \rho(y, y_0)$$

$$|K(x, y) - K(x_0, y)| \lesssim \frac{\rho(x, x_0)}{\rho(x, y)} \frac{1}{\rho(x, y)^{2n+2}} \quad \text{whenever } \rho(x, y) \geq c \rho(x, x_0).$$

Thus, it is straightforward to see that $K(x, y)$ satisfies (1.6). Hence, we see that Theorem 1.6 holds for the Cauchy–Szegö projection operator $C$. $\square$

### 7.3 The Szegö Projection Operator on a Family of Unbounded Weakly Pseudoconvex Domains

We now recall the weakly pseudoconvex domains $\Omega_k$ and their boundary $\partial \Omega_k$, $k \in \mathbb{Z}_+$, from Greiner and Stein [18]:

$$\Omega_k = \left\{ (z_1, z_2) \in \mathbb{C}^2 : \text{Im}(z_2) > |z_1|^{2k} \right\}, \quad \partial \Omega_k = \left\{ (z_1, z_2) \in \mathbb{C}^2 : \text{Im}(z_2) = |z_1|^{2k} \right\}.$$

Recall that $\partial \Omega_k$ is naturally parameterized by $z_1$ and $\text{Re}z_2$. We use the following notation. Points in $\partial \Omega_k$ are denoted by $\zeta, \omega, \nu$ etc.

$$\zeta = (z_1, z_2) \sim (z, t), \quad z = z_1 \in \mathbb{C}, \ t = \text{Re}(z_2) \in \mathbb{R};$$

$$\omega = (w_1, w_2) \sim (w, s), \quad w = w_1 \in \mathbb{C}, \ s = \text{Re}(w_2) \in \mathbb{R};$$

$$\nu = (u_1, u_2) \sim (u, r), \quad u = u_1 \in \mathbb{C}, \ r = \text{Re}(u_2) \in \mathbb{R}.$$

The Szegö projection $S$ on $\Omega_k$ is the orthogonal projection from $L^2(\partial \Omega_k)$ to the Hardy space $H^2(\Omega_k)$ of holomorphic functions on $\Omega_k$ with $L^2$ boundary values. The Szegö kernel $S(\zeta, \omega)$ is the kernel for which

$$S(f)(\zeta) = \int_{\partial \Omega_k} f(\omega) S(\zeta, \omega) dV(\omega),$$

where $dV(\omega) = dV(x, y, s) = dx dy ds$ with $\omega = (w_1, s) = (x + iy, s)$, which is Lebesgue measure on the parameter space $\mathbb{R}^3$. Greiner and Stein [18] have computed the Szegö kernel with Lebesgue measure on the parameter space with the formula

$$S(\zeta, \omega) = \frac{1}{4\pi^2} \left[ \left( \frac{i}{2} [s - t] + \frac{|z_1|^{2k} + |w_1|^{2k}}{2} + \frac{\mu + \eta}{2} \right)^{1/2} - \bar{z}_1 \bar{w}_1 \right]^2 \times \left( \frac{i}{2} [s - t] + \frac{|z_1|^{2k} + |w_1|^{2k}}{2} + \frac{\mu + \eta}{2} \right)^{k+1} \left[ \frac{k+1}{k} \right].$$
where $\mu = \text{Im}(z_2) - |z_1|^{2k}$ and $\eta = \text{Im}(w_2) - |w_1|^{2k}$.

In [11], Diaz defined and analyzed a pseudometric $d(\zeta, \omega)$ globally suited to the complex geometry of $\partial \Omega_k$, which was arrived at by the study of the Szegö kernel. This allows the treatment of the Szegö kernel as a singular integral kernel:

$$d(\zeta, \omega) = \left( \frac{i}{2} [s - t] + \frac{|z_1|^{2k} + |w_1|^{2k}}{2} \right)^\frac{1}{2} - zw.$$

Then the pseudometric balls are defined as $B_\zeta(\delta) = B^d_\zeta(\delta) = \{ \omega \in \partial \Omega_k : d(\zeta, \omega) < \delta \}$ and the volume of the balls is

$$V(B_\zeta(\delta)) = 4\pi \delta^2 \left( \frac{(\sin(\pi/k))^{2k-2}}{4} |z|^{2k-2} \delta^2 + \frac{1}{2} \delta^{2k} \right),$$

and it is shown that this measure is doubling.

**Proposition 7.3.** Theorem 1.6 holds for the Szegö projection $S$ with the underlying space of homogenous type $(\partial \Omega_k, d, dV)$, where $d$ and $dV$ are as introduced above.

**Proof.** We point out that it is proved in [11] that $S(\zeta, \omega)$ satisfies the following size and smoothness conditions:

$$|S(\zeta, \omega)| \approx \frac{1}{V(B_\zeta(d(\zeta, \omega)))},$$

$$|S(\zeta, \omega) - S(\zeta', \omega)| \lesssim \frac{d(\zeta, \zeta')}{d(\zeta, \omega)} \frac{1}{V(B_\zeta(d(\zeta, \omega)))}, \quad \text{for } cd(\zeta, \zeta') \leq d(\zeta, \omega),$$

$$|S(\zeta, \omega) - S(\zeta, \omega')| \lesssim \left( \frac{d(\omega, \omega')}{d(\zeta, \omega)} \right) \frac{1}{V(B_\zeta(d(\zeta, \omega)))}, \quad \text{for } cd(\omega, \omega') \leq d(\zeta, \omega).$$

Thus, from (7.1) it is direct to see that $S(\zeta, \omega)$ satisfies (1.6). Hence, we see that Theorem 1.6 holds for the Szegö projection operator $S$ on $\Omega_k$ for $k \in \mathbb{Z}_+$.

### 7.4 Riesz Transforms Associated with sub-Laplacian on Stratified Nilpotent Lie Groups

Recall that a connected, simply connected nilpotent Lie group $G$ is said to be stratified if its left-invariant Lie algebra $\mathfrak{g}$ (assumed real and of finite dimension) admits a direct sum decomposition

$$\mathfrak{g} = \bigoplus_{i=1}^k V_i \text{ where } [V_i, V_j] = V_{i+j} \text{ for } i \leq k - 1.$$  

One identifies $\mathfrak{g}$ and $G$ via the exponential map $\exp : \mathfrak{g} \rightarrow G$, which is a diffeomorphism. We fix once and for all a (bi-invariant) Haar measure $dg$ on $G$ (which is just the lift of Lebesgue measure on $\mathfrak{g}$ via exp). There is a natural family of dilations on $\mathfrak{g}$ defined for $r > 0$ as follows:

$$\delta_r \left( \sum_{i=1}^k v_i \right) = \sum_{i=1}^k r^i v_i, \quad \text{with } v_i \in V_i.$$  

This allows the definition of dilation on $G$, which we still denote by $\delta_r$. We choose once and for all a basis $\{X_1, \ldots, X_n\}$ for $V_1$ and consider the sub-Laplacian $\Delta = \sum_{j=1}^n X_j^2$. Observe that $X_j$ ($1 \leq j \leq n$) is homogeneous of degree 1 and $\Delta$ of degree 2 with respect to the dilations in...
Proposition 7.5. Theorem 1.6 holds for the Bessel Riesz transform $R_j$ setting (the unweighted version of commutator theorem for $R_j$) in $\mathcal{BMO}$ space via commutators in the two weight setting.

In [2], Betancor et al. further considered $\lambda$-dimension reduction and obtained the integral kernel with the convolution kernel) and $|p| = 0$ in the $\mathcal{BMO}$ setting.

The Bessel Riesz transform is defined as $R_j(x, r) = (x, r)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(y) \delta_{\frac{x}{r}}(y) dy$. For the Riesz transform kernel, we have the following lower bound estimate, obtained in [14]:

$$K_j = \frac{1}{\sqrt{\pi}} \int_{0}^{+\infty} h^{-\frac{1}{2}} X_j p_h(g) dh = \frac{1}{\sqrt{\pi}} \int_{0}^{+\infty} h^{-\frac{Q}{2} - 1} (X_j p_h(g)) dh.$$ \[7.2\]

Proposition 7.4. Theorem 1.6 holds for the Riesz transform $X_j(-\Delta)^{-\frac{1}{2}}$ ($1 \leq j \leq n$) with the underlying setting $(\mathcal{G}, \rho, dg)$, where $\rho$ is the homogeneous norm on $\mathcal{G}$ (see [14]).

Proof. For the Riesz transform kernel, we have the following lower bound estimate, obtained in [14]: Fix $j = 1, \ldots, n$. There exist $0 < \varepsilon_0 < 1$ and $C > 0$ such that for any $0 < \eta < \varepsilon_0$ and for all $g \in \mathcal{G}$ and $r > 0$, we can find $g_* = g_*(g, g, r) \in \mathcal{G}$ satisfying

$$\rho(g, g_*) = r, \quad |K_j(g_1, g_2)| \geq C r^{-Q}, \quad \forall g_1 \in B(g, \eta r), g_2 \in B(g_*, \eta r)$$

and all $K_j(g_1, g_2)$ have the same sign.

From this kernel lower bound estimate, it is direct to see that for each $j$, $K_j(g_1, g_2)$ satisfies (1.6). Hence, Theorem 1.6 holds for $X_j(-\Delta)^{-\frac{1}{2}}$ ($1 \leq j \leq n$).

7.5 Riesz Transform Associated with the Bessel Operator on $\mathbb{R}^+$

Consider $\mathbb{R}^+ = (0, \infty)$. For $\lambda > -\frac{1}{2}$, the Bessel operator $\Delta_\lambda$ on $\mathbb{R}^+$ ([40]) is defined by

$$\Delta_\lambda = -\frac{d^2}{dx^2} - \frac{2\lambda}{x} \frac{d}{dx}.$$ 

It is a formally self-adjoint operator in $L^2(\mathbb{R}^+, dm_\lambda)$, where $dm_\lambda(x) = x^{2\lambda} dx$. For any $x \in \mathbb{R}^+$ and $r > 0$, let $I(x, r) = (x - r, x + r) \cap \mathbb{R}^+$. Moreover, we assume that $r \leq x$ without loss of generality. Observe that for any $x \in \mathbb{R}^+$ and $r \in (0, x]$, $m_\lambda(I(x, r)) \sim x^{2\lambda} r$. Thus, $(\mathbb{R}^+, \rho, dm_\lambda)$ is a space of homogeneous type.

The Bessel Riesz transform is defined as $R_\lambda = \frac{d}{dx} (\Delta_\lambda)^{-\frac{1}{2}}$. In [40], Muckenhoupt–Stein introduced and obtained the $L^p(\mathbb{R}^+, dm_\lambda)$-boundedness of $R_\lambda$ for $\lambda / 0 = (0, \infty)$. Under this condition, the unweighted version of commutator theorem for $R_\lambda$ was obtained in [16] via weak factorisation. However, the two weight commutator and high order commutator are unknown, and the case when $\lambda \in (-1/2, 0)$ is totally unknown. Here we will establish the two weight commutator and high order commutator for $R_\lambda$ for all $\lambda = (-1/2, \infty)$.

Proposition 7.5. Theorem 1.6 holds for the Bessel Riesz transform $R_\lambda$ with the underlying setting $(\mathbb{R}^+, \rho, dm_\lambda)$.

Proof. In [2], Betancor et al. further considered $R_\lambda$ for the range $\lambda \in (-1/2, \infty)$. They showed that for $f \in C_c^\infty(\mathbb{R}^+)$ and $x \in (0, \infty)$,

$$R_\lambda f(x) = \text{p.v.} \int_{0}^{\infty} R_\lambda(x, y) f(y) dm_\lambda(y)$$
with the kernel
\[ R_\lambda(x, y) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{\partial}{\partial x} W^\lambda_t(x, y) \frac{dt}{\sqrt{t}} \]
for \( x, y \in (0, \infty) \) with \( x \neq y \). Here \( W^\lambda_t(x, y) \) is the heat kernel associated to \( \Delta_\lambda \)
\[ W^\lambda_t(x, y) = \frac{(xy)^{-\lambda+1/2}}{2t} e^{-(x^2+y^2)/4t} I_{\lambda-1/2} \left( \frac{xy}{2t} \right) \]  
(7.3)
with \( I_\nu \) being the modified Bessel function of the first kind and order \( \nu > -1 \). They also showed that \( R_\lambda \) is bounded on the space \( L^p(\mathbb{R}^+, x^\delta dx) \) if and only if \( p > 1 \) and \(-1 - p < \delta < (2\lambda + 1)p - 1\).
Moreover, the kernel \( R_\lambda(x, y) \) has the following estimates (see [2, Lemmas 4.3 and 4.4]):

(i) for \( x/2 < y < 2x \) and \( x \neq y \),
\[ R_\lambda(x, y) = \frac{1}{\pi} \frac{(xy)^{-\lambda}}{y-x} + O \left( y^{-2\lambda-1} \left( 1 + \log \frac{xy}{(y-x)^2} \right) \right) ; \]
(ii) in the off-diagonal region,
\[ |R_\lambda(x, y)| \lesssim \begin{cases} x^{-2\lambda-1}, & y \leq x/2; \\ xy^{-2\lambda-2}, & 2x \leq y. \end{cases} \]

From this fact, one deduces that \( R_\lambda(x, y) \) satisfies (1.4) and (1.5) (see [4, Theorem 2.2]). Moreover, there exist \( K_1 \in (0, 1/2) \) small enough, \( K_2 > 1 \) and \( C_\lambda > 0 \) such that

(i) for any \( x, y \in \mathbb{R}^+ \) with \( 0 < y/x - 1 < K_1 \),
\[ R_\lambda(x, y) \geq C_\lambda \frac{1}{x^\lambda y^\lambda} \frac{1}{y-x} ; \]  
(7.4)
(ii) for any \( x, y \in \mathbb{R}^+ \) with \( 0 < K_2 x \leq y \),
\[ R_\lambda(x, y) \geq C_\lambda xy^{-2\lambda-2} . \]  
(7.5)

Then an argument involving (7.4) and (7.5) shows that assumption (5.1) holds (see also [36, Lemma 2.3]). In fact, let \( I := I(x_0, r) \) with \( x_0 \geq r \) and \( K_0 := (K_1 + K_2 + 2)/2K_1 \). We consider the following two cases.

Case (a): \( x_0 \leq 2K_0r \). In this case, \( m_\lambda(I) \sim x_0^{2\lambda} r \sim K_0 x_0^{2\lambda+1} \). Let \( y_0 := x_0 + 4K_0 r \). Then \((2K_0 + 1)x_0 \leq y_0 \leq (4K_0^2 + 1)x_0 \). This via (7.5) implies that
\[ R_\lambda(x_0, y_0) \gtrsim \frac{x_0}{y_0^{2\lambda+2}} \sim \frac{1}{m_\lambda(I)} . \]

Case (b): \( x_0 > 2K_0r \). In this case, \( m_\lambda(I) \sim x_0^{2\lambda} r \). Let \( y_0 := x_0 + K_2 r \). Then \( 0 < y_0/x_0 - 1 < K_1 \) and
\[ R_\lambda(x_0, y_0) \gtrsim \frac{1}{x_0^{2\lambda}(y_0 - x_0)} \sim \frac{1}{m_\lambda(I)} , \]
which implies that Theorem 1.6 holds for \( R_\lambda \).
7.6 Riesz Transforms Associated with Bessel Operators on $\mathbb{R}^{n+1}$

We now recall the Bessel operator and the Bessel Riesz transform in high dimension from Huber [23]. Consider $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0, \infty)$. For $\lambda > \frac{1}{2}$,

$$
\Delta^{(n+1)}_\lambda = -\frac{d^2}{dx_1^2} \cdots - \frac{d^2}{dx_n^2} - \frac{d^2}{x_{n+1}^2} + \frac{d}{x_{n+1}} \quad (7.6)
$$

The operator $\Delta^{(n+1)}_\lambda$ is symmetric and non-negative in $C^\infty_c(\mathbb{R}^{n+1}_+) \subset L^2(\mathbb{R}^{n+1}, d\mu_\lambda)$, where

$$
d\mu_\lambda(x) := \prod_{j=1}^{n} dx_j x_{n+1}^{2\lambda} d\mu_{n+1}.
$$

The $j$-th Riesz transform is defined as

$$R_{\lambda,j} = \frac{d}{dx_j} (\Delta^{(n+1)}_\lambda)^{-\frac{1}{2}}, \quad j = 1, \ldots, n + 1.
$$

We point out that there is no known results for the commutator of $R_{\lambda,j}$. Here we provide an intensive study of the kernel of $R_{\lambda,j}$, especially for the lower bound, and then we obtain the two weight commutator and higher order commutator for $R_{\lambda,j}$.

**Proposition 7.6.** Theorem 1.6 holds for the Bessel Riesz transform $R_{\lambda,j}$, $j = 1, \ldots, n + 1$, with the underlying setting $(\mathbb{R}^{n+1}_+, |\cdot|, d\mu_\lambda)$.

**Proof.** To begin with, note that (7.6) can be written as $\Delta^{(n+1)}_\lambda = \Delta^{(n)} + \Delta_\lambda$, where $\Delta^{(n)}$ denotes the standard Laplacian on $\mathbb{R}^n$, and $\Delta_\lambda$ denotes the Bessel operator on $\mathbb{R}_+$, which is one-dimensional as shown in Section 7.5. Then it is clear that $e^{-t\Delta^{(n+1)}_\lambda} = e^{-t(\Delta^{(n)} + \Delta_\lambda)}$ and hence the heat kernel

$$p_{t,\Delta^{(n+1)}_\lambda}(x, y) = p_{t,\Delta^{(n)}_\lambda}(x', y') W_t^\lambda (x_{n+1}, y_{n+1})$$

for $x = (x', x_{n+1})$, $y = (y', y_{n+1}) \in \mathbb{R}^n \times (0, \infty)$, where $W_t^\lambda$ is the heat kernel of $\Delta_\lambda$ as in (7.3).

Then it is direct that for $1 \leq j \leq n$:

$$R_{\lambda,j}(x, y) = c_{n,\lambda} \frac{\partial}{\partial x_j} \int_0^\infty p_{t,\Delta^{(n+1)}_\lambda}(x, y) \frac{dt}{\sqrt{t}}$$

$$= c_{n,\lambda} \frac{\partial}{\partial x_j} \int_0^\infty \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x'-y'|^2}{4t}} W_t^\lambda (x_{n+1}, y_{n+1}) \frac{dt}{\sqrt{t}}$$

and for $j = n + 1$,

$$R_{\lambda,n+1}(x, y) = c_{n,\lambda} \frac{\partial}{\partial x_{n+1}} \int_0^\infty p_{t,\Delta^{(n+1)}_\lambda}(x, y) \frac{dt}{\sqrt{t}}$$

$$= c_{n,\lambda} \frac{\partial}{\partial x_{n+1}} \int_0^\infty \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x'-y'|^2}{4t}} W_t^\lambda (x_{n+1}, y_{n+1}) \frac{dt}{\sqrt{t}}.$$

By [4, Theorem 2.2], $\{R_{\lambda,j}\}_{j=1}^{n+1}$ are Calderón-Zygmund operators with kernel satisfying (1.4) and (1.5).

Moreover, we have the following estimates on $\{R_{\lambda,j}(x, y)\}_{j=1}^{n+1}$.

**Lemma 7.7.** Let $j \in \{1, \ldots, n\}$. The following statements hold:
(i) There exist positive constants \( \tilde{C}, \tilde{c} > 1 \) such that for any \((x, y)\) with \( 0 < x_{n+1} \leq y_{n+1}/\tilde{c} \), \( R_{\lambda,j}(x, y) \) does not change sign and
\[
|R_{\lambda,j}(x, y)| \geq \tilde{C} \frac{|x_j - y_j|}{(y_{n+1}^2 + |x' - y'|^2)^{\alpha + \lambda + 1}}.
\]

(ii) There exists a positive constant \( C_n \) such that for any \((x, y)\),
\[
R_{\lambda,j}(x, y) = C_n \frac{y_j - x_j}{x_{n+1}^{\lambda+1} y_{n+1}^{\lambda+1} |x - y|^{n+2}} + \mathcal{O} \left( \frac{1}{x_{n+1}^{\lambda+1} y_{n+1}^{\lambda+1} |x - y|^n} \right).
\]

**Proof.** By (7.3) and letting \( z > 0 \) and applying the Lebesgue Dominated Convergence Theorem, we have
\[
R_{\lambda,j}(x, y) = \int_0^\infty \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x' - y'|^2}{4t}} \frac{x_j - y_j}{2t} \left( y_{n+1}^{\lambda+1} + |x' - y'|^2 \right)^{-\frac{\lambda}{2} - \frac{1}{2}} dt.
\]

Recall that for any \( z > 0 \) and \( \nu > -1 \),
\[
\lim_{z \to 0^+} z^{-\nu} I_\nu(z) = \frac{1}{\sqrt{2\pi} \Gamma(\nu + 1)},
\]
and for any \( z > 0 \), \( \nu > -1 \) and \( n = 0, 1, 2, \ldots \),
\[
I_\nu(z) = \frac{e^z}{\sqrt{2\pi} z^{\nu+1}} \sum_{k=0}^n (-1)^k [\nu, k](2z)^{-k} + \mathcal{O}(z^{-n-1}),
\]
where \([\nu, k] := \frac{(4\nu^2 - 1)(4\nu^2 - 3^2) \cdots (4\nu^2 - (2k - 1)^2)}{2^{2k} \Gamma(k + 1)}\),

see [2, p. 109].

Then letting \( z \to 0 \) and applying the Lebesgue Dominated Convergence Theorem, we have
\[
R_{\lambda,j}(x, y) \to c_{n,\lambda}(x_j - y_j) y_{n+1}^{-\lambda-2} \int_0^\infty u^{-\frac{\nu}{2} - \lambda - 2} e^{-\frac{1}{2nu} (\frac{|x' - y'|^2}{y_{n+1}^2} + 1)} du.
\]

This shows (i).

Now for \( j = 1, 2, \ldots, n \), let
\[
R_j(x, y) := c_{n,\lambda} \int_0^\infty \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x' - y'|^2}{4t}} \frac{x_j - y_j}{2t} \left( y_{n+1}^{\lambda} + |x' - y'|^2 \right)^{-\frac{\lambda}{2} - \frac{1}{2}} W_t(x_{n+1}, y_{n+1}) \frac{dt}{\sqrt{t}}.
\]
where
\[ W_t(x_{n+1}, y_{n+1}) := \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x_{n+1} - y_{n+1})^2}{4t}}. \]

Then we see that for any \( x, y \in \mathbb{R}^n \) such that \( x_j \neq y_j \),
\[ R_j(x, y) = C_n \frac{y_j - x_j}{x_{n+1}^2 y_{n+1}^2 |x - y|^{n+2}}. \]

On the other hand, by using (7.8) for \( n = 0 \), we conclude that
\[
|R_{\lambda, j}(x, y) - R_j(x, y)| = \frac{c_{n, \lambda}}{x_{n+1}^2 y_{n+1}^2 |x - y|^{n+2}} \int_0^\infty \frac{x_{n+1}^2 y_{n+1}^2}{\lambda} \frac{e^{-\frac{(x_{n+1} - y_{n+1})^2}{4t}}}{\sqrt{4\pi t}} \left[ 1 + C \frac{2t}{x_{n+1}^2 y_{n+1}^2} e^{-\frac{(x_{n+1} - y_{n+1})^2}{4t}} \right] dt.
\]

The proof of Lemma 7.7 is complete.

Similarly, for \( R_{\lambda, n+1}(x, y) \), we have show the following lemma.

**Lemma 7.8.** The following statements hold:

(i) There exist positive constants \( \tilde{C}, \tilde{c} \geq 1 \) such that for any \((x, y)\) with \( 0 < x_{n+1} \leq y_{n+1}/\tilde{c} \) and \( \frac{\lambda^{\frac{1}{2}}}{\lambda + 1} < \frac{y_{n+1}^2}{y_{n+1}^2 + |x' - y'|^2} < 1 \), \( R_{\lambda, n+1}(x, y) \) does not change sign and
\[
|R_{\lambda, n+1}(x, y)| \geq \tilde{C} \frac{x_{n+1}}{(y_{n+1}^2 + |x' - y'|^2)^{\frac{3}{2} + \lambda + 1}}.
\]

(ii) There exists a positive constant \( C_n \) such that for any \((x, y)\),
\[ R_{\lambda, n+1}(x, y) = C_n \frac{y_{n+1} - x_{n+1}}{(x_{n+1} y_{n+1})^\lambda} \frac{1}{|x - y|^{n+2}} + O \left( \frac{1}{x_{n+1}^\lambda y_{n+1}^{\lambda+1}} \frac{1}{|x - y|^{n+2}} \right). \]

**Proof.** Observe that
\[
\frac{\partial}{\partial x_{n+1}} W_t(x_{n+1}, y_{n+1}) = \frac{1}{(2t)^{\lambda + \frac{1}{2}}} \left[ x_{n+1} \left( \frac{y_{n+1}}{2t} \right)^2 \left( \frac{x_{n+1} y_{n+1}}{2t} \right)^{-\lambda - \frac{3}{2}} I_{\lambda + \frac{3}{2}} \left( \frac{x_{n+1} y_{n+1}}{2t} \right) - \frac{x_{n+1}}{2t} \left( \frac{x_{n+1} y_{n+1}}{2t} \right)^{-\lambda + \frac{1}{2}} I_{\lambda - \frac{1}{2}} \left( \frac{x_{n+1} y_{n+1}}{2t} \right) \right] e^{-\frac{x_{n+1}^2 + y_{n+1}^2}{4t}}.
\]
see, [2]. Then we have

\[
R_{\lambda, n+1}(x, y) = c_{n, \lambda} \int_0^\infty \frac{1}{(4\pi t)^2} \frac{1}{(2t)^{\lambda + \frac{1}{2}}} x_{n+1} \left( \frac{y_{n+1}}{2t} \right)^2 \left( \frac{x_{n+1} y_{n+1}}{2t} \right)^{-\lambda - \frac{1}{2}}
\]

\[
\times I_{\lambda + \frac{1}{2}} \left( \frac{x_{n+1} y_{n+1}}{2t} \right) e^{-\frac{x_{n+1}^2 + y_{n+1}^2}{4t}} dt
\]

\[
- c_{n, \lambda} \int_0^\infty \frac{1}{(4\pi t)^2} \frac{1}{(2t)^{\lambda + \frac{1}{2}}} x_{n+1} \left( \frac{y_{n+1}}{2t} \right)^2 \left( \frac{x_{n+1} y_{n+1}}{2t} \right)^{-\lambda - \frac{1}{2}}
\]

\[
\times I_{\lambda - \frac{1}{2}} \left( \frac{x_{n+1} y_{n+1}}{2t} \right) e^{-\frac{x_{n+1}^2 + y_{n+1}^2}{4t}} dt.
\]

By change of variables, we have

\[
R_{\lambda, n+1}(x, y) = c_{n, \lambda} \left[ x_{n+1} \int_0^\infty \frac{1}{(4\pi t)^2} \frac{1}{(2t)^{\lambda + \frac{1}{2}}} \left( \frac{y_{n+1}}{2t} \right)^2 \left( \frac{x_{n+1} y_{n+1}}{2t} \right)^{-\lambda - \frac{1}{2}}
\]

\[
\times I_{\lambda + \frac{1}{2}} \left( \frac{y_{n+1}^2}{2t} \right) e^{-\frac{y_{n+1}^2 (1+\gamma)}{4t}} dt
\]

\[
- x_{n+1} \int_0^\infty \frac{1}{(4\pi t)^2} \frac{1}{(2t)^{\lambda + \frac{1}{2}}} \left( \frac{y_{n+1}}{2t} \right)^2 \left( \frac{x_{n+1} y_{n+1}}{2t} \right)^{-\lambda - \frac{1}{2}}
\]

\[
\times I_{\lambda - \frac{1}{2}} \left( \frac{y_{n+1}^2}{2t} \right) e^{-\frac{y_{n+1}^2 (1+\gamma)}{4t}} dt.
\]

By letting \( z \to 0 \) and applying (7.7), we see that

\[
R_{\lambda, n+1}(x, y) \frac{y_{n+1}^{\gamma+2}}{x_{n+1}} \to C_{n, \lambda} \left[ \frac{1}{2^{\lambda + \frac{3}{2}} \Gamma(\lambda + \frac{3}{2})} \int_0^\infty \frac{1}{u^{\lambda + 2}} e^{-\frac{1}{2u}(1+\frac{x'^2}{y_{n+1}^2})} du \right]
\]

\[
- \frac{1}{2^{\lambda + \frac{3}{2}} \Gamma(\lambda + \frac{1}{2})} \int_0^\infty \frac{1}{u^{\lambda + 2}} e^{-\frac{1}{2u}(1+\frac{x'^2}{y_{n+1}^2})} du \right]
\]

\[
= C_{n, \lambda} 2^{\frac{n+3}{2}} \frac{(y_{n+1}^{\gamma+2} + |x' - y'|^2)^{\frac{\lambda}{2} + 1}}{(y_{n+1}^{\gamma+2} + |x' - y'|^2)^{\frac{\lambda}{2} + 1}}
\]

\[
\times \left( \frac{y_{n+1}^{\gamma+2} + |x' - y'|^2}{y_{n+1}^{\gamma+2} + |x' - y'|^2} \right)^{\frac{\lambda}{2} + 1 - 1}
\]

\[
= C_{n, \lambda} \frac{y_{n+1}^{\gamma+2}}{y_{n+1}^{\gamma+2} + |x' - y'|^2} \left( \frac{y_{n+1}^{\gamma+2} + |x' - y'|^2}{y_{n+1}^{\gamma+2} + |x' - y'|^2} \right)^{\frac{\lambda}{2} + 1 - 1}
\]

Since

\[
\frac{y_{n+1}^{\gamma+2}}{y_{n+1}^{\gamma+2} + |x' - y'|^2} \left( \frac{y_{n+1}^{\gamma+2} + |x' - y'|^2}{y_{n+1}^{\gamma+2} + |x' - y'|^2} \right)^{\frac{\lambda}{2} + 1 - 1} - 1 > 0,
\]

the conclusion (i) holds.
Let
\[ H(x, y) := c_{n, \lambda} \int_0^\infty \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} \frac{1}{(2t)^{\lambda+1}} \left( \frac{x_{n+1}y_{n+1}}{2t} \right)^{-\lambda} \times \left[ x_{n+1} \left( \frac{y_{n+1}}{2t} \right)^2 \left( \frac{x_{n+1}y_{n+1}}{2t} \right)^{-1} - \frac{x_{n+1}}{2t} \right] \frac{dt}{\sqrt{t}}. \]

Then observe that
\[ \int_0^\infty \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} \frac{1}{(2t)^{\lambda+1}} \left( \frac{x_{n+1}y_{n+1}}{2t} \right)^{-\lambda} \left[ x_{n+1} \left( \frac{y_{n+1}}{2t} \right)^2 \left( \frac{x_{n+1}y_{n+1}}{2t} \right)^{-1} - \frac{x_{n+1}}{2t} \right] \frac{dt}{\sqrt{t}}. \]

By using (7.8) for \( n = 0 \),
\[ |R_{\lambda, n+1}(x, y) - H(x, y)| \lesssim \int_0^\infty \frac{1}{t^{\lambda+1}} e^{-\frac{|y-y'|^2}{4t}} \left( \frac{x_{n+1}y_{n+1}}{2t} \right)^{-\lambda} \frac{1}{y_{n+1}} \frac{dt}{\sqrt{t}} \lesssim \frac{1}{x_{n+1}y_{n+1}} \frac{1}{|x-y|^n}. \]

We then see that (ii) holds. The proof of Lemma 7.8 is complete.

We now return to the proof of Proposition 7.6. Based on Lemmas 7.7 and 7.8, we see that (5.1) holds. Indeed, for any \( x := (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}_+ \) and \( r \in (0, \infty) \), let
\[ Q(x, r) := \{ y := (y_1, \ldots, y_{n+1}) \in \mathbb{R}^{n+1}_+ : |x_j - y_j| \leq r/2, j \in \{1, \ldots, n+1\} \}. \]

Then we have \( \mu_\lambda(Q(x, r)) \sim r^{n+1} 2^{2\lambda} x_{n+1}^\lambda \). Let \( C_0 \gg \tilde{c} \). For any \( x := (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}_+ \) and \( r \in (0, \infty) \), if \( r > \frac{\tilde{c}}{C_0} x_{n+1} \), take \( y := (y_1, \ldots, y_{n+1}) \) such that \( y_i = x_i + C_0 r \) for \( i = j \) or \( i = n+1 \) and \( y_i := x_i \) otherwise. Then by Lemma 7.7(i), we see that \( y_{n+1} \geq \tilde{c} x_{n+1} \) and
\[ |R_{\lambda, j}(x, y)| \gtrsim \frac{C_0 r}{[(y_2^{2} + (C_0 r)^{2}]^{\lambda+1}} \geq \frac{1}{\mu_\lambda(Q(x, r))}. \]

If \( r \leq \frac{\tilde{c}}{C_0} x_{n+1} \), then there exists \( y \in \mathbb{R}^{n+1}_+ \) such that \( |y_{n+1} - x_{n+1}| = |y_j - x_j| \sim |y - x| \) and \( |y_{n+1} - x_{n+1}| \ll x_{n+1} \). Then by Lemma 7.7(ii), we also have
\[ |R_{\lambda, j}(x, y)| \gtrsim \frac{|y_j - x_j|}{x_{n+1}^{\lambda} y_{n+1}^{\lambda} |x - y|^{n+2}} \sim \frac{1}{\mu_\lambda(Q(x, r))}. \]

Therefore, (1.6) holds for \( R_{\lambda, j}(x, y) \), \( j \in \{1, \ldots, n\} \). The argument for \( R_{\lambda, n+1}(x, y) \) is similar and omitted. Then Theorem 1.6 holds for the Bessel Riesz transform \( R_{\lambda, j}, j = 1, \ldots, n+1 \).

The proof of Proposition 7.6 is complete.

8 A Digression to Product Setting: Little bmo Space

In this section we consider the weighted little bmo space on product spaces of homogeneous type. To begin with, let \( (X_1, d_1, \mu_1) \) and \( (X_2, d_2, \mu_2) \) be two copies of spaces of homogeneous type as stated in Section 2, and denote \( X := X_1 \times X_2, \bar{\mu} := \mu_1 \times \mu_2 \). Moreover, for the points in \( X \), we denote \( \bar{x} := (x_1, x_2) \in \bar{X} \).

Macias and Segovia [35] proved the following fundamental result on spaces of homogeneous type. Suppose that \( (X, d) \) is a space endowed with a quasi-metric \( d \) that may have no regularity.
Then there exists a quasi-metric \( d' \) that is pointwise equivalent to \( d \) such that \( d(x, y) \sim d'(x, y) \) for all \( x, y \in X \) and there exist constants \( \theta \in (0, 1) \) and \( C > 0 \) so that \( d' \) has the following regularity:

\[
|d'(x, y) - d'(x', y)| \leq C d'(x, x')^\theta [d'(x, y) + d'(x', y)]^{1-\theta}.
\]

for all \( x, x', y \in X \). Moreover, if the quasi-metric balls are defined by this new quasi-metric \( d' \), that is, \( B'(x, r) := \{ y \in X : d'(x, y) < r \} \) for \( r > 0 \), then these balls are open in the topology induced by \( d' \). See [35, Theorem 2, p.259]. So, without lost of generality, we assume that in our product setting, the quasi-metrics \( d_1 \) and \( d_2 \) have regularity with constants \( \theta_1 \) and \( \theta_2 \), respectively.

We now recall the product \( A_p(\tilde{X}) \) weights on product spaces of homogeneous type.

**Definition 8.1.** Let \( w(x_1, x_2) \) be a nonnegative locally integrable function on \( \tilde{X} \). For \( 1 < p < \infty \), we say \( w \) is a product \( A_p \) weight, written as \( w \in A_p(\tilde{X}) \), if

\[
[w]_{A_p(\tilde{X})} := \sup_{R} \left( \int_{R} w \right) \left( \int_{R} \left( \frac{1}{w} \right) \right)^{1/(p-1)} < \infty.
\]

Here the supremum is taken over all “rectangles” \( R := B_1 \times B_2 \subset \tilde{X} \), where \( B_i \) are balls in \( X_i \) for \( i = 1, 2 \). The quantity \([w]_{A_p(\tilde{X})}\) is called the \( A_p \) constant of \( w \).

Next we recall the weighted little \( \text{bmo} \) space on product spaces of homogeneous type.

**Definition 8.2.** For \( 1 < p < \infty \) and \( w \in A_p(\tilde{X}) \), the weighted little \( \text{bmo} \) space \( \text{bmo}_w(\tilde{X}) \) is the space of all locally integrable functions \( b \) on \( \tilde{X} \) such that

\[
\|b\|_{\text{bmo}_w(\tilde{X})} = \sup_{R} \frac{1}{w(R)} \int_{R} |b(x)| \sim \|b\|_{L_1(\tilde{X})} < \infty,
\]

where the supremum is taken over all “rectangles” \( R = B_1 \times B_2 \subset \tilde{X} \), where \( B_i \) are balls in \( X_i \) for \( i = 1, 2 \).

Similar to [22, Section 7.1], we introduce the bi-parameter Journé operator on \( \tilde{X} \) as follows. Let \( C_0^{\eta_1}(X_1) \), \( \eta_1 \in (0, \theta_1] \), denote the space of continuous functions \( f \) with bounded support such that

\[
\|f\|_{C_0^{\eta_1}(X_1)} := \sup_{x, y \in X_1, x \neq y} \frac{|f(x) - f(y)|}{d_1(x, y)^{\eta_1}} < \infty.
\]

Let \( C_0^{\eta_2}(X_2) \), \( \eta_2 \in (0, \theta_2] \) be defined similarly.

**I. Structural Assumptions:** Given \( f = f_1 \otimes f_2 \) and \( g = g_1 \otimes g_2 \), where \( f_i, g_i : X_i \to \mathbb{C} \), \( f_i, g_i \in C_0^{\eta_i}(X_i) \) satisfy \( \text{supp } f_i \cap \text{supp } g_i = \emptyset \) for \( i = 1, 2 \), we assume the kernel representation

\[
\langle Tf, g \rangle = \int_{\tilde{X}} \int_{\tilde{X}} K(\tilde{x}, \tilde{y}) f(\tilde{y}) g(\tilde{x}) d\mu(\tilde{y}) d\mu(\tilde{x}).
\]

The kernel \( K : \tilde{X} \times \tilde{X} \setminus \{ (\tilde{x}, \tilde{y}) \in \tilde{X} \times \tilde{X} : x_1 = y_1, \text{ or } x_2 = y_2 \} \to \mathbb{C} \) is assumed to satisfy:

1. **Size condition:**

\[
|K(\tilde{x}, \tilde{y})| \leq C \frac{1}{\mu_1(B(x_1, d_1(x_1, y_1))) \mu_2(B(x_2, d_2(x_2, y_2)))}.
\]

2. **Hölder conditions:**
2a. if $d_1(y_1, y_1') \leq \frac{1}{2A_0} d_1(x_1, y_1)$ and $d_2(y_2, y_2') \leq \frac{1}{2A_0} d_2(x_2, y_2)$:

$$|K(\vec{x}, \vec{y}) - K(\vec{x}, (y_1, y_2))| \
\leq C \frac{d_1(y_1, y_1')^\delta d_2(y_2, y_2')^\delta}{\mu_1(B(x_1, d_1(x_1, y_1))) d_1(x_1, y_1)^\delta \mu_2(B(x_2, d_2(x_2, y_2))) d_2(x_2, y_2)^\delta}.$$

2b. if $d_1(x_1, x_1') \leq \frac{1}{2A_0} d_1(x_1, y_1)$ and $d_2(x_2, x_2') \leq \frac{1}{2A_0} d_2(x_2, y_2)$:

$$|K(\vec{x}, \vec{y}) - K((x_1, x_2'), \vec{y})| \
\leq C \frac{d_1(x_1, x_1')^\delta d_2(x_2, x_2')^\delta}{\mu_1(B(x_1, d_1(x_1, y_1))) d_1(x_1, y_1)^\delta \mu_2(B(x_2, d_2(x_2, y_2))) d_2(x_2, y_2)^\delta}.$$

2c. if $d_1(y_1, y_1') \leq \frac{1}{2A_0} d_1(x_1, y_1)$ and $d_2(x_2, x_2') \leq \frac{1}{2A_0} d_2(x_2, y_2)$:

$$|K(\vec{x}, \vec{y}) - K((x_1, x_2'), (y_1, y_2))| \
\leq C \frac{d_1(y_1, y_1')^\delta d_2(x_2, x_2')^\delta}{\mu_1(B(x_1, d_1(x_1, y_1))) d_1(x_1, y_1)^\delta \mu_2(B(x_2, d_2(x_2, y_2))) d_2(x_2, y_2)^\delta}.$$

3. Mixed size and Hölder conditions:

3a. if $d_1(x_1, x_1') \leq \frac{1}{2A_0} d_1(x_1, y_1)$:

$$|K(\vec{x}, \vec{y}) - K((x_1', x_2), \vec{y})| \
\leq C \frac{d_1(x_1, x_1')^\delta}{\mu_1(B(x_1, d_1(x_1, y_1))) d_1(x_1, y_1)^\delta \mu_2(B(x_2, d_2(x_2, y_2)))}. $$

3b. if $d_1(y_1, y_1') \leq \frac{1}{2A_0} d_1(x_1, y_1)$:

$$|K(\vec{x}, \vec{y}) - K(\vec{x}, (y_1', y_2))| \
\leq C \frac{d_1(y_1, y_1')^\delta}{\mu_1(B(x_1, d_1(x_1, y_1))) d_1(x_1, y_1)^\delta \mu_2(B(x_2, d_2(x_2, y_2)))}. $$

3c. if $d_2(x_2, x_2') \leq \frac{1}{2A_0} d_2(x_2, y_2)$:

$$|K(\vec{x}, \vec{y}) - K((x_1, x_2'), \vec{y})| \
\leq C \frac{d_2(x_2, x_2')^\delta}{\mu_1(B(x_1, d_1(x_1, y_1))) \mu_2(B(x_2, d_2(x_2, y_2))) d_2(x_2, y_2)^\delta}. $$

3d. if $d_2(y_2, y_2') \leq \frac{1}{2A_0} d_2(x_2, y_2)$:

$$|K(\vec{x}, \vec{y}) - K(\vec{x}, (y_1, y_2'))| \
\leq C \frac{d_2(y_2, y_2')^\delta}{\mu_1(B(x_1, d_1(x_1, y_1))) \mu_2(B(x_2, d_2(x_2, y_2))) d_2(x_2, y_2)^\delta}. $$

4. Calderón–Zygmund structure in $X_1$ and $X_2$ separately: If $f = f_1 \otimes f_2$ and $g = g_1 \otimes g_2$ with $\text{supp} f_1 \cap \text{supp} g_1 = \emptyset$, we assume the kernel representation:

$$\langle Tf, g \rangle = \int_{X_1} \int_{X_1} K_{f_2 g_2}(x_1, y_1) f_1(y_1) g_1(x_1) d\mu_1(x_1) d\mu_1(y_1),$$
where the kernel $K_{f_2,g_2} : X_1 \times X_1 \setminus \{(x_1, y_1) \in X_1 \times X_1 : x_1 = y_1\}$ satisfies the following size condition:
\[
|K_{f_2,g_2}(x_1, y_1)| \leq C(f_2, g_2) \frac{1}{\mu_1(B(x_1, d_1(x_1, y_1)))}
\]
and Hölder conditions:
\[
|K_{f_2,g_2}(x_1, y_1) - K_{f_2,g_2}(x_1', y_1)| \leq \frac{C(f_2, g_2)}{\mu_1(B(x_1, d_1(x_1, y_1)))^\delta} d_1(x_1, x_1') \leq \frac{1}{2A_0} d_1(x_1, y_1),
\]
\[
|K_{f_2,g_2}(x_1, y_1) - K_{f_2,g_2}(x_1, y_1')| \leq \frac{C(f_2, g_2)}{\mu_1(B(x_1, d_1(x_1, y_1)))^\delta} d_1(x_1, x_1') \leq \frac{1}{2A_0} d_1(x_1, y_1).
\]

We only assume the above representation and a certain control over $C(f_2, g_2)$ on the diagonal, that is:
\[
C(\chi_{Q_2}, \chi_{Q_2}) + C(\chi_{Q_2}, \nu_{Q_2}) + C(\nu_{Q_2}, \chi_{Q_2}) \leq C\mu_2(Q_2)
\]
for all cubes $Q_2 \subset X_2$ and all "$Q_2$-adapted zero-mean" functions $\nu_{Q_2}$ that is, $\text{supp} \nu_{Q_2} \subset Q_2$, $|\nu_{Q_2}| \leq 1$ and $\int_{X_2} \nu_{Q_2}(x_2) d\mu_2(x_2) = 0$. We assume the symmetrical representation with kernel $K_{f_2,g_2}$ in the case $\text{supp} f_2 \cap \text{supp} g_2 = \emptyset$.

II. Boundedness and Cancellation Assumptions:
1. Assume $T_1, T_1^*, T_1$, and $T_1^* 1$ are in product BMO$(\mathcal{X})$, where $T_1$ is the partial adjoint of $T$ defined by $(T_1(f_1 \otimes f_2), g_1 \otimes g_2) = (T(f_1 \otimes f_2), (f_1 \otimes g_2))$.
2. Assume $|\langle T_1(\chi_{Q_1} \otimes \chi_{Q_2}), \chi_{Q_1} \otimes \chi_{Q_2}\rangle| \leq C\mu_1(Q_1)\mu_2(Q_2)$ for all cubes $Q_i \subset X_i$ (weak boundedness).
3. Diagonal BMO conditions: for all cubes $Q_i \subset X_i$ and all non-zero functions $a_{Q_i}$ and $b_{Q_i}$ that are $Q_i-1$ and $Q_i-2$ adapted, respectively, assume:
\[
|\langle T_1(a_{Q_1} \otimes \chi_{Q_2}), \chi_{Q_1} \otimes \chi_{Q_2}\rangle| \leq C\mu_1(Q_1)\mu_2(Q_2),
|\langle T_1(\chi_{Q_1} \otimes \chi_{Q_2}), a_{Q_i} \otimes \chi_{Q_2}\rangle| \leq C\mu_1(Q_1)\mu_2(Q_2),
|\langle T_1(\chi_{Q_1} \otimes b_{Q_2}), \chi_{Q_1} \otimes \chi_{Q_2}\rangle| \leq C\mu_1(Q_1)\mu_2(Q_2),
|\langle T_1(\chi_{Q_1} \otimes \chi_{Q_2}), \chi_{Q_1} \otimes b_{Q_2}\rangle| \leq C\mu_1(Q_1)\mu_2(Q_2).
\]

For the upper bound of the commutator of such operators $T$ and $b \in \text{bmo}_p(\mathcal{X})$, following the same approach as that in [22], and combining all necessary tools as recalled in Section 2 on spaces of homogeneous type (such as the adjacent dyadic systems, Haar basis, et al), we obtain that

**Theorem 8.3.** Let $1 < p < \infty$ and $\lambda_1, \lambda_2 \in A_p(\mathcal{X})$, and define $\nu = \lambda_1^{\frac{1}{p}} \lambda_2^{-\frac{1}{p}}$. Let $T$ be a bi-parameter Journé operator on $\mathcal{X}$ and $b \in \text{bmo}_p(\mathcal{X})$. Then we obtain that
\[
\|b, T| L^p_{\lambda_1}(\mathcal{X}) | L^p_{\nu}(\mathcal{X})\| \lesssim \|b\|_{\text{bmo}_p(\mathcal{X})}.
\]

We now provide a broader version of the lower bound. Note that in [22] the authors only considered the lower bound of commutator with respect to double Riesz transforms, and their proof relies on Fourier transform and hence can not be adapted to spaces of homogeneous type.

We assume that the bi-parameter Journé operator $T$ satisfies the following "homogeneous" condition:

**there exist positive constants** $c_0$ and $\mathcal{C}$ **such that for every** $x_1 \in X_1$, $x_2 \in X_2$ **and** $r_1, r_2 > 0$, **there exist** $y_1 \in B_1(x_1, C r_1) \setminus B_1(x_1, r_1)$ **and** $y_2 \in B_2(x_2, C r_2) \setminus B_2(x_2, r_2)$ **satisfying**
\[
|K(x_1, y_1; x_2, y_2)| \geq \frac{1}{c_0 \mu_1(B_1(x_1, r_1)) \mu_2(B_2(x_2, r_2))}.
\]

Then we have the following lower bound.
Theorem 8.4. Let $T$ be a bi-parameter Journé operator on $\tilde{X}$ and $T$ satisfies the following “homogeneous” condition as above. Let $1 < p < \infty$ and $\lambda_1, \lambda_2 \in A_p(\tilde{X})$, and define $\nu := \lambda_1^{\frac{1}{p}} \lambda_2^{\frac{1}{p}}$. Suppose that $b \in L^1_{\text{loc}}(\tilde{X})$ and that $\|[b, T] : L^{p}_{\lambda_1}(\tilde{X}) \to L^{p}_{\lambda_2}(\tilde{X})\| < \infty$. Then we obtain that $b \in \text{bmo}_{\nu}(\tilde{X})$ with

$$\|b\|_{\text{bmo}_{\nu}(\tilde{X})} \lesssim \|b, T\| : L^{p}_{\lambda_1}(\tilde{X}) \to L^{p}_{\lambda_2}(\tilde{X})\|.$$

To see this, we first point out that the homogeneous condition (8.1) implies the following condition: there exist positive constants $3 \leq A_1 \leq A_2$ such that for any ball $B_i := B_i(x_0, r_i) \subset X_i$, there exist balls $\tilde{B}_i := B_i(y_0^{(i)}, r_i)$ such that $A_1 r_i \leq d_i(x_0^{(i)}, y_0^{(i)}) \leq A_2 r_i$. Moreover, for all $(x_1, y_1; x_2, y_2) \in (B_1 \times \tilde{B}_1) \times (B_2 \times \tilde{B}_2)$, $K(x_1, y_1; x_2, y_2)$ does not change sign and

$$|K(x_1, y_1; x_2, y_2)| \lesssim \frac{1}{\mu_1(B_1)} \frac{1}{\mu_2(B_2)}.$$

If $K(x_1, y_1; x_2, y_2) := K_1(x_1, y_1; x_2, y_2) + iK_2(x_1, y_1; x_2, y_2)$ is complex-valued, where $i^2 = -1$, then at least one of $K_1$ satisfies the assumption above.

We next consider the median on “rectangles” $R = B_1 \times B_2 \subset \tilde{X}$. By a median value of a real-valued measurable function $f$ over $R$ we mean a possibly non-unique, real number $\alpha_{R}(f)$ such that $\bar{\mu}(\{(x_1, x_2) \in R : f(x_1, x_2) > \alpha_{R}(f)\}) \leq \frac{1}{2}\mu_1(B_1)\mu_2(B_2)$ and $\bar{\mu}(\{(x_1, x_2) \in R : f(x_1, x_2) < \alpha_{R}(f)\}) \leq \frac{1}{2}\mu_1(B_1)\mu_2(B_2)$.

Now following the idea in Lemma 5.3, for the given rectangle $R = B_1 \times B_2$, $\tilde{B}_1$ and $\tilde{B}_2$, set

$$E_1 := \{(x_1, x_2) \in B_1 \times B_2 : b(x_1, x_2) \geq \alpha_{\tilde{B}_1 \times \tilde{B}_2}(b)\}$$

and

$$E_2 := \{(x_1, x_2) \in B_1 \times B_2 : b(x_1, x_2) \leq \alpha_{\tilde{B}_1 \times \tilde{B}_2}(b)\}.$$

Then by the definition of $\alpha_{R}(f)$, we see that $\bar{\mu}(E_i) \geq \frac{1}{2}\mu_1(B_1)\mu_2(B_2)$ for $i = 1, 2$. Moreover, for $(x_1, x_2) \times (y_1, y_2) \in (E_1 \times F_1) \cup (E_2 \times F_2)$,

$$|b(x_1, x_2) - b(y_1, y_2)| = |b(x_1, x_2) - \alpha_{\tilde{B}_1 \times \tilde{B}_2}(b) + \alpha_{\tilde{B}_1 \times \tilde{B}_2}(b) - b(y_1, y_2)|$$

$$\geq |b(x_1, x_2) - \alpha_{\tilde{B}_1 \times \tilde{B}_2}(b)| \geq |b(x_1, x_2) - \alpha_{\tilde{B}_1 \times \tilde{B}_2}(b)|.$$

Proof of Theorem 8.4. For given $b \in L^1_{\text{loc}}(\tilde{X})$ and for any rectangle $R = B_1 \times B_2$, let

$$O(b; R) := \frac{1}{\bar{\mu}(R)} \int_R |b(x_1, x_2) - b_R| \, d\mu_1(x_1) d\mu_2(x_2).$$

Under the assumptions of Theorem 8.4, we will show that for any ball $B$,

$$O(b; R) \lesssim \frac{\nu(R)}{\bar{\mu}(R)} \tag{8.2}$$

Without loss of generality, we assume that $K(x_1, y_1; x_2, y_2)$ is real-valued. Let $R = B_1 \times B_2$ be a rectangle. Then we have two rectangles $B_1 \times B_2, B_1 \times B_2$ and sets $E_i, F_i, i = 1, 2$, as above.
On the one hand, we have that for $f_i := \chi_{F_i}$, $i = 1, 2$,
\[
\frac{1}{\mu(R)} \sum_{i=1}^{2} \int_{B_1 \times B_2} |[b, T]f_i(x_1, x_2)| \, d\mu_1(x_1)d\mu_2(x_2)
\geq \frac{1}{\mu(R)} \sum_{i=1}^{2} \int_{F_i} |[b, T]f_i(x_1, x_2)| \, d\mu_1(x_1)d\mu_2(x_2)
= \frac{1}{\mu(R)} \sum_{i=1}^{2} \int_{F_i} |b(x_1, x_2) - b(y_1, y_2)||K(x_1, y_1; x_2, y_2)| \, d\mu_1(y_1)d\mu_2(y_2)d\mu_1(x_1)d\mu_2(x_2)
\geq \frac{1}{\mu(R)} \sum_{i=1}^{2} \int_{F_i} |b(x_1, x_2) - \alpha_{\tilde{B}_1 \times \tilde{B}_2}(b)| \, d\mu_1(x_1)d\mu_2(x_2)
\geq \frac{1}{\mu(R)} \int_{B_1 \times B_2} |b(x_1, x_2) - \alpha_{\tilde{B}_1 \times \tilde{B}_2}(b)| \, d\mu_1(x_1)d\mu_2(x_2)
\geq \mathcal{O}(b; R).
\]

On the other hand, from Hölder’s inequality and the boundedness of $[b, T]$, we deduce that
\[
\frac{1}{\mu(R)} \sum_{i=1}^{2} \left[ \int_{B_1 \times B_2} |[b, T]f_i(x_1, x_2)|^p \lambda_2(x_1, x_2) \, d\mu_1(x_1)d\mu_2(x_2) \right]^{1/p}
\leq \frac{1}{\mu(R)} \left[ \int_{B_1 \times B_2} |[b, T]f_i(x_1, x_2)|^p \lambda_2(x_1, x_2)^{-1/p} \, d\mu_1(x_1)d\mu_2(x_2) \right]^{1/p'}
\leq \frac{1}{\mu(R)} \left[ \lambda_1(F_i) \right]^{1/p} \left( \int_{B_1 \times B_2} \lambda_2(x_1, x_2)^{-\frac{1}{p'}} \, d\mu_1(x_1)d\mu_2(x_2) \right)^{1/p'}
\leq \frac{1}{\mu(R)} \left[ \lambda_1(\tilde{B}_1 \times \tilde{B}_2) \right]^{1/p} \left( \int_{R} \lambda_2(x_1, x_2)^{-\frac{1}{p'}} \, d\mu_1(x_1)d\mu_2(x_2) \right)^{1/p'}
\leq \frac{1}{\mu(R)} \left[ \lambda_1(R) \right]^{1/p} \left( \int_{R} \lambda_2(x_1, x_2)^{-\frac{1}{p'}} \, d\mu_1(x_1)d\mu_2(x_2) \right)^{1/p'}
\leq \frac{\nu(R)}{\mu(R)}.
\]
where in the last inequality, we use the facts that $K_1 r_{B_1} \leq d(x_{B_1}, x_{\tilde{B}_1}) \leq K_2 r_{B_1}$ and $\lambda_1(x_1, x_2) \in A_p(\tilde{X})$.

Combining the two inequalities above and invoking $\lambda_i \in A_p(\tilde{X})$, we conclude that
\[
\mathcal{O}(b; R) \leq \frac{1}{\mu(R)} \left[ \lambda_1(R) \right]^{1/p} \left( \int_{R} \lambda_2(x_1, x_2)^{-\frac{1}{p'}} \, d\mu_1(x_1)d\mu_2(x_2) \right)^{1/p'} \leq \frac{\nu(R)}{\mu(R)}.
\]
Thus, (8.2) holds and hence, the proof of Theorem 8.4 is complete. 

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Xuan Thinh Duong, Department of Mathematics, Macquarie University, NSW, 2109, Australia.
E-mail: xuan.duong@mq.edu.au

Ruming Gong, School of Mathematical Sciences, Guangzhou University, China.
E-mail: gongruming@163.com

Marie-Jose S. Kuffner, Department of Mathematics, Washington University–St. Louis, St. Louis, MO 63130-4899 USA
E-mail: mariejose@wustl.edu

Ji Li, Department of Mathematics, Macquarie University, NSW, 2109, Australia.
E-mail: ji.li@mq.edu.au

Brett D. Wick, Department of Mathematics, Washington University–St. Louis, St. Louis, MO 63130-4899 USA
E-mail: wick@math.wustl.edu

Dongyong Yang, School of Mathematical Sciences, Xiamen University, Xiamen, China
E-mail: dyyang@xmu.edu.cn