Quadrilateral reptiles

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Abstract
A polygon $P$ is called a reptile, if it can be decomposed into $k \geq 2$ nonoverlapping and congruent polygons similar to $P$. We prove that if a cyclic quadrilateral is a reptile, then it is a trapezoid. Comparing with results of Betke and Osburg we find that every convex reptile is a triangle or a trapezoid.

Keywords Tilings · Reptiles · Quadrilaterals

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1 Introduction and main results
A polygon $P$ is called a $k$-reptile, if it can be decomposed into $k$ nonoverlapping and congruent polygons similar to $P$. For example, every triangle and every parallelogram is a 4-reptile. A polygon $P$ is a reptile, if it is a $k$-reptile for some $k \geq 2$.

C. D. Langford presented several reptiles, including three 4-reptile trapezoids, and raised the problem of classifying all reptiles (Langford 1940). It was shown in Valette and Zamfirescu (1974) that every convex 4-reptile is a triangle, a parallelogram or one of Langford’s trapezoids.

Non-convex reptiles are abundant, and their characterization seems to be hopeless (see Osburg 2004, Sect 3.2). As for convex reptiles, it is known that they must be triangles or quadrilaterals (see Betke (1976) and Osburg 2004, Satz 2.23). A large step towards the classification of convex reptiles was made by Osburg. She proved in (Osburg 2004, Satz 2.9 and Folgerung 3.2) that every quadrilateral reptile (convex or not) is either a trapezoid or a cyclic quadrilateral. In this note our aim is to prove the following.

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Theorem 1.1 If a cyclic quadrilateral is a reptile, then it is a trapezoid.

Comparing with Osburg’s results we obtain the following corollaries (the first one was conjectured in Doyen and Landuyt 1983).

Corollary 1.2 Every convex reptile is a triangle or a trapezoid.

Corollary 1.3 Every quadrilateral reptile (convex or not) is a trapezoid.

The characterization of reptile trapezoids remains open.

It follows from Corollary 1.2 that the class of self-similar quadrilaterals is strictly larger than the class of quadrilateral reptiles. (A polygon $P$ is self-similar, if it can be dissected into a finite number $k \geq 2$ of similar copies of $P$. Indeed, one can show that every right kite (a kite with two right angles) is self-similar. On the other hand, according to Corollary 1.2 a kite is not a reptile (!), unless it is a rhombus. Therefore, a right kite is self-similar but is not a reptile, unless it is a square.

The characterization of self-similar quadrilaterals seems to be open as well. It is also unsolved if self-similar convex pentagons exist or not. More is known about the even larger class of self-affine polygons. It is proved in Hertel and Richter (2010) that every convex quadrilateral is self-affine, and that self-affine convex pentagons do exist.

The structure of our proof of Theorem 1.1 is the following. Let $Q$ be a cyclic quadrilateral, and suppose that $Q$ is a reptile. Then there is a polygon $Q'$ similar to $Q$ such that $Q'$ can be tiled with congruent copies of $Q$. In the next section we show that already the existence of a tiling in a neighbourhood of the smallest angle of $Q'$ implies that either $Q$ is a trapezoid, or $Q$ belongs to one of three families, each one depending on a single continuous parameter (see Theorem 2.1).

In Sect. 3 we show that the quadrilaterals belonging to the first two families are not reptiles unless they are trapezoids. The proofs are relatively easy, using either number theoretical conditions (similar to the argument of Snover et al. 1991) or a local argument similar to the proof of Theorem 2.1.

The third family, denoted by $\mathcal{F}_3$, consists of the quadrilaterals with sides $a, b, c, d$ such that $c = d$ and the angles included between the sides $a, b$ and between $c, d$ are right angles. Assuming that $c = d = 1$, the elements of $\mathcal{F}_3$ are determined by the length of $a$. The proof that the elements of $\mathcal{F}_3$ are not reptiles (apart from the square) occupies the last two sections.

There are two reasons why this proof is more difficult than ruling out the other two families. First, the quadrilaterals belonging to $\mathcal{F}_3$ tile the square, as Fig. 1 shows.

Consequently, they tile the whole plane in such a way that several local tilings can be extended indefinitely, so the argument used in Theorem 2.1 doesn’t work. Also, the number theoretical arguments fail in the case when the side lengths of the quadrilateral are rational. There are infinitely many such quadrilaterals, corresponding to the rational solutions of $a^2 + b^2 = 2$. The special case $a = 7/5, b = 1/5$ (when $a = 1 + 2b$) is particularly difficult to handle.

In order to deal with the family $\mathcal{F}_3$ we need a global view of the possible tilings. Let $\Sigma$ denote the quadrant $\{(x, y): x \geq 0, y \geq 0\}$. Since $\Sigma$ is tiled with the squares $\sigma_{i, j} = [(i(a + b), (i + 1)(a + b)] \times [(j(a + b), (j + 1)(a + b)]$
Fig. 1

(i, j = 0, 1, . . .), it follows that Σ can be tiled with congruent copies of Q. We say that a tiling of Σ with congruent copies of Q is trivial, if every square σ_{i,j} is tiled by four tiles of the tiling.

**Theorem 1.4** Suppose $Q \in \mathcal{F}_3$, and the angle included between its sides a and d differs from $\pi/3$. Then every tiling of $\Sigma$ with congruent copies of Q is trivial.

From Theorem 1.4 it is easy to infer that the elements of $\mathcal{F}_3$ are not reptiles (except the square). This reduction, together with the proof of Theorem 1.4 will be given in Sect. 5. We mention that if $Q \in \mathcal{F}_3$ has angle $\pi/3$ between its sides a and d, then there is a nontrivial tiling of $\Sigma$ with congruent copies, as shown in Fig. 2. The half-strips $S_i (i \in \mathbb{Z})$ in Fig. 2 have width $a + b$. They can be tiled with squares of size $(a + b) \times (a + b)$; therefore, they can be tiled with congruent copies of Q.

We conclude this section with some preliminary remarks.

Let $P$ be a $k$-reptile, and suppose $P$ is dissected into the congruent polygons $P_1, \ldots, P_k$ that are similar to $P$. Then we can dissect each $P_i$ into $k$ congruent polygons similar to $P$, and we find that $P$ is also a $k^2$-reptile. Continuing this process we obtain that if $P$ is a $k$-reptile, then it is also a $k^j$-reptile for every $j = 1, 2, \ldots$.

The segment with endpoints A and B is denoted by AB. We denote the length of the segment AB by AB.

Let Q be a cyclic quadrilateral, and suppose that Q is not a trapezoid. Let $\alpha, \beta, \gamma, \delta$ be the angles of Q listed counterclockwise and such that $\alpha$ is (one of) the smallest of the angles. Since $\delta = \pi - \beta$ and $\gamma = \pi - \alpha$, this implies that $\gamma$ is (one of) the largest of the angles; that is, $\alpha \leq \beta \leq \gamma$ and $\alpha \leq \delta \leq \gamma$.

Since Q is not a trapezoid, we have $\alpha \neq \beta$ and $\alpha \neq \delta$. Therefore, we have $\alpha < \beta < \gamma$ and $\alpha < \delta < \gamma$.

In the sequel we fix the labeling of the vertices, angles and sides of Q as follows. The vertices $A, B, C, D$ of Q are listed such that the angles of Q at the vertices $A, B, C, D$ are $\alpha, \beta, \gamma, \delta$. The lengths of the sides of Q are $a, b, c, d$ such that $\overline{AB} = a$, $\overline{BC} = b$, $\overline{CD} = c$ and $\overline{DA} = d$. We also assume that $a \geq d$. (The case $d \geq a$ can be reduced to $a \geq d$ if we turn to a reflected copy of Q instead of Q, and swap $\beta$ and $\delta$.)

**Lemma 1.5** We have $a \geq d > b$ and $a > c$. 

![Figure 1](image-url)
Proof Since $\alpha < \gamma = \pi - \alpha$, we have $\alpha < \pi / 2 < \gamma$. If $d \geq BD$, then, as the angle of the triangle $BCD$ at the vertex $C$ equals $\gamma > \pi / 2$, we have $b < BD \leq d$.

Next suppose $d < BD$. Let $K$ denote the circumscribed circle of $Q$. Let $ABD \angle = \varepsilon$. Since $d < BD$, we have $\varepsilon < \alpha$. On the other hand we have $\beta > \alpha$, and thus there is a point $E$ in the subarc of $K$ with end points $C$ and $D$ such that $ABE \angle = \alpha$. Then $ABED$ is a trapezoid, and thus $BE = d$. Now the angle of the triangle $BCE$ at the vertex $C$ is greater than $\gamma > \pi / 2$, and thus $b < BE = d$.

The proof of $a > c$ is similar. (Note that the proof of $d > b$ did not use $a \geq d$.) \(\square\)

Lemma 1.6 If $c \geq d$, then $\beta < 2\alpha$.

Proof Let $ABD \angle = \varepsilon$ and $DBC \angle = \zeta$. Since $Q$ is cyclic, we have $DAC \angle = \zeta$. Also, $c \geq d$ implies $\varepsilon \leq \zeta$, and thus $\beta = \varepsilon + \zeta \leq 2\zeta < 2\alpha$. \(\square\)

2 Reduction to one dimensional families

Let $Q$ be a cyclic quadrilateral, and suppose that $Q$ is not a trapezoid. We use the labeling of the angles, vertices and sides of $Q$ as described in the previous section. In this section we prove the following.

Theorem 2.1 If $Q$ is a reptile, then one of the following statements is true.

(i) $\alpha = \pi / 3$, $\beta = \delta = \pi / 2$ and $\gamma = 2\pi / 3$.
(ii) $b = c$, $\alpha = \pi / 3$ and $\gamma = 2\pi / 3$.
(iii) $c = d$, $\beta = \pi / 2$ and $\delta = \pi / 2$.
Proof. Suppose $Q$ is a $k$-reptile, where $k \geq 2$. Then, as we remarked earlier, $Q$ is a $k^j$-reptile for every $j = 1, 2, \ldots$. We choose $j$ so large that in every dissection of $Q$ into $k^j$ congruent polygons similar to $Q$, every tile has at least one vertex in the interior of $Q$.

Let $Q'$ be a quadrilateral similar to $Q$ such that $Q'$ can be tiled with $k^j$ congruent copies of $Q$. We fix such a tiling, and fix the labeling of the angles and sides of the tiles. If $T$ is a tile with vertices $X, Y, V, W$, then $|XY|, |YV|, |VW|, |WX|$ denote the labels of the sides of $T$. That is, if $|XY| = a$ then $XY = a$ and there are two possibilities: either $|YV| = b$, $|VW| = c$, $|WX| = d$ and the angles of $T$ at $X, Y, V, W$ are $\alpha, \beta, \gamma, \delta$, or $|YV| = d, |VW| = c, |WX| = b$ and the angles of $T$ at $X, Y, V, W$ are $\alpha, \delta, \gamma, \beta$. Note that if $|XY| = a$ then $XY = a$, but the converse is not necessarily true; it can happen, e.g., that $XY = a$ but $|XY| = d$. In this case, however, we must have $a = d$.

Labeling the angles has similar consequences. When we say, e.g., that a tile $T$ has angle $\beta$ at the vertex $Y$, then it implies that either $|XY| = a$ and $|YV| = b$, or $|XY| = b$ and $|YV| = a$.

The angle $\alpha$ of $Q'$ is packed with one single tile. We may assume that $Q$ itself is this tile; that is, $A$ is a vertex of $Q'$, and the sides $AB$ and $AD$ lie on consecutive sides of $Q'$. By the choice of $j$, the vertex $C$ of $Q$ lies in the interior of $Q'$.

Let $U$ denote the union of the boundaries of the tiles. We say that a segment $S \subset U$ is maximal if, for every segment $S'$ with $S \subset S' \subset U$ we have $S' = S$. We consider three cases and several subcases.

Case I: the segment $DC$ is not maximal. Then there is a segment $DE \subset U$ such that $C$ is an inner point of $DE$, and there is a tile $T$ having $C$ as a vertex, having angle $\pi - \gamma = \alpha$ at $C$, and having a side $CF$ on the segment $CB$. However, since $T$ and $Q$ are congruent, $CF$ equals either $a$ or $d$, and thus $d \leq CB = b$, which contradicts Lemma 1.5.

Case II: the segment $DC$ is maximal, and the segment $BC$ is also maximal. Then $C$ is a common vertex of the tiles $Q_0, Q_1, \ldots, Q_s$ such that $Q_0 = Q$, and the sum of the angles of $Q_i$ at $C$ equals $2\pi$.

Let $Q_1$ be the tile with a vertex at $C$ and having a side $CE \subset CB$. Since $a \geq d > b$ by Lemma 1.5, we have $|CE| = b$ or $c$.

Case II.1: $|CE| = c$ and $c < b$. In this case the segment $BC$ is packed with $m \geq 2$ segments of length $c$, and each of them is a side of a tile. That is, there is a partition $C = E_0, E = E_1, \ldots, E_m = B$ and there are tiles $T_1 = Q_1, T_2, \ldots, T_m$ such that $E_{i-1}E_i$ is a side of $T_i$ labeled as $c$. The angle of $T_i$ at the vertices $E_{i-1}$ and $E_i$ equals either $\delta$ or $\gamma$. Since $\delta + \gamma > \alpha + \gamma = \pi$, $T_i$ cannot have angle $\gamma$ at $E_{i-1}$ if $i \geq 2$, and cannot have angle $\gamma$ at $E_i$ if $i < m$. This is possible only if $m = 2$, $T_1 = Q_1$ has angle $\gamma$ at $C = E_0$ and $T_2$ has angle $\gamma$ at $B = E_2$. In this case, however, the sum of the angles of $Q_0$ and $T_2$ at $B$ equals $\beta + \gamma > \pi$, which is impossible.

Case II.2: $|CE| = b$, or $|CE| = c$ and $c = b$. Then $E = B$, and the angle of $Q_1$ at $B$ is $\beta, \delta$ or $\gamma$. Since the angle of $Q_0$ at $B$ equals $\beta$ and $\beta + \gamma > \alpha + \gamma = \pi$, the angle of $Q_1$ at $B$ is $\beta$ or $\delta$. Therefore, the angle of $Q_1$ at $C$ is $\gamma$. Note that the angle of $Q_1$ at $B$ can be $\delta$ only if $c = b$. 

\[ \square \] Springer
The sum of the angles of $Q_i$ ($i = 2, \ldots, s$) at $C$ equals $2\pi - 2\gamma = 2\alpha$. Since $\alpha$ is the smallest angle, this implies $s = 2$ or $s = 3$. In the latter case the angles of $Q_2$ and $Q_3$ at $C$ equal $\alpha$. This, however, contradicts the condition that $BC$ is maximal.

Thus $s = 2$, and we have $2\gamma + \eta = 2\pi$, where $\eta$ is the angle of $Q_2$ at $C$. Then $\eta = 2\alpha$, and $\eta$ equals one of $\beta, \gamma, \delta$. Let $FC$ be the side of $Q_2$ lying on the segment $DC$. Since $\overline{FC} < c < a$, we have $|FC| = b, c$ or $d$.

**Case II.2.1:** $\eta = \beta = 2\alpha$. Then the only possible label for $FC$ is $b$. By Lemma 1.6, $\beta = 2\alpha$ implies $c < d$. Now $|FC| = b$ implies that the angle of $Q_2$ at $F$ is $\gamma$. Since $\gamma > \beta$, it follows that $F \neq D$, and there is a tile $Q_3$ having a vertex at $F$ and angle $\alpha$ at $F$. If $GF$ is the side of $Q_3$ lying on the segment $DF$, then $\overline{GF} = a$ or $d$. However, we have $\overline{DF} < c < d \leq a$, a contradiction.

**Case II.2.2:** $\eta = \gamma$. Then we have $3\gamma = 2\pi, \gamma = 2\pi/3$ and $\alpha = \pi/3$. Now $2\beta + \alpha = 3\alpha = \pi$ and $\beta + \delta + \alpha > 3\alpha = \pi$, and thus the sum of the angles of $Q_0$ and $Q_1$ at $B$ equals $\pi$. If the angle of $Q_1$ at $B$ equals $\delta$, then $\beta = \pi/2$, $\delta = \pi/2$, and we have case (i) of the lemma.

If the angle of $Q_1$ at $B$ equals $\delta$, then necessarily $c = b$, and we have case (ii) of the lemma.

**Case II.2.3:** $\eta = \delta = 2\alpha$. Then we have $|FC| = c$ or $d$. By $\alpha < \beta = \pi - \delta = \pi - 2\alpha$ we obtain $\alpha < \pi/3$.

**Case II.2.3.1:** $c \geq d$. Then $c > b$, and thus the angle of $Q_1$ at $B$ equals $\beta$. We have $2\beta + \alpha = 2\pi - 4\alpha + \alpha = 2\pi - 3\alpha > \pi$, and thus the sum of the angles of $Q_0$ and $Q_1$ at $B$ equals $\pi$. Thus $2\beta = \pi, \beta = \delta = \pi/2, \alpha = \pi/4$. In this case, however, we have $d > c$, a contradiction (see Fig. 3).

**Case II.2.3.2:** $c < d$. Then $|FC| = c, F = D$, and the angle of $Q_2$ at $D$ must be $\gamma$. However, the sum of the two angles at $D$ equals $\delta + \gamma > \delta + \beta = \pi$, which is impossible.

**Case III:** the segment $DC$ is maximal, and the segment $BC$ is not maximal. Then $C$ is a common vertex of the tiles $Q$ and $R$ such that $R$ has angle $\alpha$ at $C$, and there is a point $F$ in the segment $DC$ such that $FC$ is a side of $R$. Thus $|FC| = a$ or $d$. Since $a > c$ by Lemma 1.5, $|FC| = a$ is impossible. Thus $|FC| = d$ and the angle of $R$ at $F$ equals $\delta$. We also have

$$c \geq d \text{ and } \beta < 2\alpha. \quad (1)$$

The second inequality follows from Lemma 1.6.

**Case III.1:** $F = D$. Then $c = d$ and the angle of $R$ at $D$ equals $\delta$. Now the angle of $Q$ at $D$ is also $\delta$, hence $2\delta \leq \pi$. If $2\delta = \pi$, then $\delta = \beta = \pi/2$ and we get (iii) of the lemma.

If $2\delta < \pi$, then there are other tiles at $D$. However, by (1) we have $2\delta + \alpha > \delta + 2\alpha > \delta + \beta = \pi$, which is impossible.

**Case III.2:** $F \neq D$. Then there is a partition $D = F_0, \ldots, F_n = C$ of the segment $DC$ and there are tiles $R_1, \ldots, R_n$ such that $n \geq 2$, $F_{i-1}F_i$ is a side of $R_i$ for every
\[ i = 1, \ldots, n, \; F_{n-1} = F \; \text{and} \; R_n = R. \] It is clear that \( |F_{i-1}F_i| \) is \( b \) or \( d \) for every \( i \). Since \( |F_{n-1}C| = d \), we have \( d < c \). Comparing with Lemma 1.5 we obtain
\[
a > c > d > b. \tag{2}
\]

We denote by \( \lambda_i \) and \( \mu_i \) the angle of \( R_i \) at \( F_{i-1} \) and at \( F_i \), respectively.

We prove that \( \beta = \delta = \pi/2 \). Suppose this is not true. Since \( |F_{n-1}F_n| = |FC| = d \) and \( \mu_n = \alpha \), it follows that \( \lambda_n = \delta \). Thus \( \mu_{n-1} \neq \gamma \) by \( \delta + \gamma > \pi \). Suppose \( \mu_{n-1} = \alpha \). Since \( \alpha + \delta < \pi \), there must be a third tile having a vertex at \( F_{n-1} \).

However, \( 2\alpha + \delta > \delta + \beta = \pi \), so this case is impossible. Next suppose \( \mu_{n-1} = \delta \). Then \( 2\delta \leq \pi \). Since \( \delta \neq \pi/2 \), there are other tiles with a vertex at \( F_{n-1} \). However, we have \( 2\delta + \alpha > \delta + 2\alpha > \delta + \beta = \pi \), which is impossible.

Thus the only possibility is \( \mu_{n-1} = \beta \). Since \( |F_{n-2}F_{n-1}| = b \) or \( d \), we have \( |F_{n-2}F_{n-1}| = b \) and \( \lambda_{n-1} = \gamma \). Then \( \mu_{n-2} = \alpha \), \( |F_{n-3}F_{n-2}| = d \) and \( \mu_{n-2} = \delta \).

Continuing this argument we obtain \( |F_{n-i-1}F_{n-i}| = d, \; \lambda_{n-i} = \delta, \; \mu_{n-i} = \alpha \) if \( i \) is even, and \( |F_{n-i-1}F_{n-i}| = b, \; \lambda_{n-i} = \gamma, \; \mu_{n-i} = \beta \) if \( i \) is odd.

Thus \( \lambda_1 = \gamma \) or \( \delta \). Since the angle of \( Q \) at \( D \) is \( \delta \), we have \( \delta + \lambda_1 \leq \pi \). Then \( \lambda_1 = \delta \) and \( 2\delta \leq \pi \). Then, by \( \delta \neq \pi/2 \), there must be other tiles with a vertex at \( D = F_0 \).

However, as we saw above, we have \( 2\delta + \alpha > \pi \), which is impossible.

This contradiction proves \( \delta = \beta = \pi/2 \). Then, by \( \beta < 2\alpha \) we get \( \pi/4 < \alpha < \pi/2 \).

The hypotenuse of the right triangles \( ABC \) and \( ACD \) is \( AC \). Therefore, we have \( a < AC < c + d \), and thus
\[
a - c < d. \tag{3}
\]

In the sequel we assume that none of (i)–(iii) of the lemma holds. In particular, we have \( \alpha \neq \pi/3 \) and \( \gamma \neq 2\alpha \). Then \( \gamma \) is not the linear combination of \( \alpha \) and \( \pi/2 \) with nonnegative integer coefficients. Indeed, this follows from \( \gamma = \pi - \alpha < 3\pi/4 < \alpha + \pi/2 < 3\alpha \) and from \( \gamma \neq 2\alpha \).
We have \( \lambda_1 = \alpha, \pi/2 \) or \( \gamma \). Since the angle of \( Q \) at \( D \) equals \( \delta = \pi/2 \), we have \( \pi/2 + \lambda_1 \leq \pi \), and thus \( \lambda_1 = \alpha \) or \( \pi/2 \). Now \( \lambda_1 = \alpha \) is impossible, as \( \alpha + \pi/2 < \pi < 2\alpha + \pi/2 \). Thus \( \lambda_1 = \pi/2 \) and \( \mu_1 = \alpha \) or \( \gamma \). If \( \mu_1 = \gamma \), then \( \lambda_2 = \alpha \). The converse is also true: if \( \mu_1 = \alpha \), then \( \lambda_2 = \gamma \), since \( \gamma \) is not the linear combination of \( \alpha \) and \( \pi/2 \) with nonnegative integer coefficients. Therefore, we have \( \mu_1 = \gamma \) and \( \lambda_2 = \alpha \), or \( \mu_1 = \alpha \) and \( \lambda_2 = \gamma \).

Let the vertices of \( R_1 \) and \( R_2 \) be \( D = F_0, F_1, G_1, H_1 \), and \( F_1, F_2, G_2, H_2 \), respectively. Since \( \mu_1 = \alpha \) or \( \gamma \), the angle of \( R_1 \) at \( H_1 \) equals \( \gamma \) or \( \alpha \). Therefore, the point \( H_1 \) is not a vertex of \( Q' \). Indeed, \( A \) is a vertex of \( Q' \), and if \( H_1 \) was also a vertex, then the angle of \( Q' \) at \( H_1 \) would be \( \pi/2 \), which is impossible by \( \gamma > \pi/2 \) and \( \alpha < \pi/2 < 2\alpha \).

Thus there is a tile \( P \) having a vertex at \( H_1 \) and having angle \( \mu_1 \) at \( H_1 \). The point \( H_1 \) is the endpoint of two sides of \( P \), one of them lies on the boundary of \( Q' \), the other one, \( H_1J \) lies in the interior of \( Q' \) (apart from the point \( H_1 \)).

**Case III.2.1:** \( \mu_1 = \alpha \) (see Fig. 4). Then \( \overline{H_1J} = a \) or \( d \). Since \( a > d > b \) by (2), the point \( G_1 \) is in the interior of the segment \( H_1J \). The segments \( G_1J \) and \( H_2G_2 \) are parallel to each other and perpendicular to the segment \( G_1H_2 \).

Then there is a partition \( G_1 = K_0, \ldots, K_m = H_2 \) of the segment \( G_1H_2 \) and there are tiles \( P_1, \ldots, P_m \) such that \( K_{i-1}K_i \) is a side of \( P_i \) for every \( i = 1, \ldots, m \). We have \( m \geq 2 \), since the tile \( P_1 \) cannot have angle \( \pi/2 \) at two consecutive vertices.

Now \( \overline{G_1H_2} = \overline{G_1F_1} - \overline{H_2F_1} = a - c \). By (3) we have \( a - c < d < c \), and thus \( |K_{i-1}K_i| = b \) for every \( i = 1, \ldots, m \).

Let \( \rho_1 \) and \( \sigma_1 \) denote the angle of \( P_i \) at \( K_{i-1} \) and at \( K_i \), respectively. Since \( \rho_1 = \pi/2 \), we have \( \sigma_1 = \gamma \), and thus \( \rho_2 = \alpha \). However, this implies \( |K_1K_2| \neq b \), which is impossible.

**Case III.2.2:** \( \mu_1 = \gamma \) (see Fig. 5). In this case \( \overline{F_1H_2} = a > c = \overline{F_1G_1} \). Then there is a partition \( H_1 = L_0, \ldots, L_t = G_1 \) of the segment \( H_1G_1 \) and there are tiles \( S_1, \ldots, S_t \) such that \( L_{i-1}L_i \) is a side of \( S_i \) for every \( i = 1, \ldots, t \). If \( t = 1 \), then \( H_1G_1 \) is a side of length \( d \) of the tile \( S_1 \), and then the angles of \( S_1 \) at \( H_1 \) and \( G_1 \) are \( \alpha \) and \( \pi/2 \) in
some order. However, since the angle of $R_1$ at $H_1$ equals $\alpha$ and $\gamma$ is not the linear combination of $\alpha$ and $\pi/2$ with nonnegative integer coefficients, the angle of $S_1$ at $H_1$ must be $\gamma$, which is impossible.

Therefore, we have $t \geq 2$, and thus $|L_{i-1}L_i| = b$ for every $i$ by (2). Then the angles of $S_i$ at $L_{i-1}$ and $L_i$ are $\gamma$ and $\pi/2$ in some order. Since $2\gamma > \gamma + \pi/2 > \pi$, this is possible only if $t = 2$ and $S_2$ has angle $\gamma$ at $G_1$. However, $S_2$ must have a right angle at $G_1$, which is a contradiction. \hfill \Box

### 3 Ruling out families (i) and (ii)

**Lemma 3.1** Suppose $Q$ satisfies the conditions of (i) of Theorem 2.1. Then $Q$ is not a reptile.

**Proof** We may assume $c = 1$. Suppose $Q$ is a $k$-reptile, where $k \geq 2$. Then $Q$ is also a $k^j$-reptile for every $j$. We choose $j$ such that $k^j b > a$. There is a quadrilateral $Q'$ similar to $Q$ such that $Q'$ can be dissected into $k^{2j}$ congruent copies of $Q$. Then the sides of $Q'$ have length $ma$, $mb$, $m$ and $md$, where $m = k^j$. We have

$$b = d \frac{\sqrt{3}}{2} - \frac{1}{2} \quad \text{and} \quad a = \frac{d}{2} + \frac{\sqrt{3}}{2}. \quad (4)$$

(See Fig. 6.) Let $X$, $Y$ be consecutive vertices of $Q'$ such that $XY = mb$, and $Q'$ has a right angle at $X$. Let $p$, $q$, $r$, $s$ be the number of tiles having a side of length $a$, $b$, $1$, $d$ lying on $XY$.

We show that $\max(p, s) > 0$. Let $X = X_0$, $\ldots$, $X_n = Y$ be a partition of $XY$ such that each $X_{i-1}X_i$ is a side of the tile $T_i$ ($i = 1, \ldots, n$). Note that $n \geq 2$ by $mb > a$. \hfill \square
Fig. 6

If $X_0X_1 = a$ or $d$, then we have $\max(p, s) \geq 1$. If, however, $X_0X_1 = b$ or $1$ then the angle of $T_1$ at $X_1$ equals $\gamma$. Then the angle of $T_2$ at $X_1$ equals $\alpha$, and thus $X_1X_2 = a$ or $d$, proving $\max(p, s) > 0$ in this case as well.

We have $mb = pa + qb + r + sd$, and thus $2(m - q)b = 2pa + 2r + 2sd$. From $\max(p, s) > 0$ it follows that $m - q > 0$. Substituting the values given by (4) we get

$$(m - q)d\sqrt{3} - (m - q) = pd + p\sqrt{3} + 2r + 2sd.$$  

Therefore, we have

$$[(m - q)\sqrt{3} - (p + 2s)]d = p\sqrt{3} + (m - q + 2r).$$  

Multiplying both sides by $(m - q)\sqrt{3} + (p + 2s)$ we obtain $Ad = B + C\sqrt{3}$, where $A, B, C \in \mathbb{Z}$, and $B > 0, C > 0$. Indeed, $B = 3(m - q)p + (p + 2s)(m - q + 2r) > 0$ and $C = (m - q)(m - q + 2r) + (p + 2s)p > 0$ by $m - q > 0$ and $\max(p, s) > 0$.

Since $d > 0$, it follows that $A > 0$. We find that $d = x + y\sqrt{3}$, where $x, y$ are positive rational numbers.

Let $ZY$ be the side of $Q'$ of length $m$, and let $t, u, v, w$ be the number of tiles having a side of length $a, b, 1, d$ lying on $ZY$. One can show, similarly to the previous case, that $\max(t, w) > 0$. We have $m - v = ta + ub + wd$, and thus

$$2(m - v) = 2ta + 2ub + 2wd$$  

$$= (tx + ty\sqrt{3} + t\sqrt{3}) + (ux\sqrt{3} + 3uy - u) + (2wx + 2wy\sqrt{3}).$$  

Since $\sqrt{3}$ is irrational, we have $t = w = 0$, which is impossible. \hfill \Box

**Lemma 3.2** Suppose $Q$ satisfies the conditions of (ii) of Theorem 2.1. If $Q$ is not a trapezoid, then $Q$ is not a reptile.

**Proof** We may assume that $b = c = 1$. Then the triangle $BCD$ is isosceles, the length of its legs is 1 and the angle at the apex is $2\pi/3$. Therefore, the circumscribed circle $M$ of the triangle $BCD$ has radius 1. Since $K$ is the circumscribed circle of $Q$ as well, we have $2 \geq a \geq d > 1$. Now $a = 2$ happens only when $a$ is the diameter of $K$, and...
then $Q$ is a trapezoid. Therefore, we have $2 > a \geq d > 1$. It is easy to check that if $a = d$, then $\beta = \delta = \pi/2$ when, by Lemma 3.1, $Q$ is not a reptile. So we may assume that $2 > a > d > 1$.

Since $Q$ is not a trapezoid, $\alpha < \beta < \gamma$ and $\alpha < \delta < \gamma$.

Suppose $Q$ is a reptile. Then there is a quadrilateral $Q'$ similar to $Q$ such that $Q'$ can be tiled with $k \geq 2$ congruent copies of $Q$. The angle $\beta$ of $Q'$ is packed with one single tile, since $\alpha < \beta$ and $2\alpha = 2\pi/3 = \gamma > \beta$. We may assume that $Q$ itself is this tile; that is, $B$ is a vertex of $Q'$, and the sides $AB$ and $BC$ lie on consecutive sides of $Q'$.

There is a tile $P$ having a vertex at $C$ and having angle $\alpha$ at $C$. The point $C$ is the endpoint of two sides of $P$, one of them lies on the boundary of $Q'$, the other one, $CE$ lies in the interior of $Q'$ (apart from the point $C$). The length of $CE$ is $a$ or $d$, and both are greater than $BC = 1$ by Lemma 1.5.

Then it follows that there is a partition $A = A_0, \ldots, A_n = D$ of the segment $AD$ and there are tiles $P_1, \ldots, P_n$ such that $A_{i-1}A_i$ is a side of $P_i$ for every $i = 1, \ldots, n$. Since $\overline{AD} = d$ and $2 > a > d > 1$, we must have $n = 1$; that is, $Q$ shares its side $AD$ with another tile $R$.

Let $\lambda$ be the angle of $R$ at the vertex $D$. Since $\overline{AD} = d$, we have $\lambda = \alpha$ or $\delta$. The angle of $Q$ at $D$ is $\delta$. If $\lambda = \alpha$, then $\delta + \alpha < \delta + \beta = \pi$ implies that there are other tiles with a vertex at $D$, and thus $\delta + 2\alpha \leq \pi$. However, $\delta > \alpha = \pi - 2\alpha$, so this case is impossible.

If $\lambda = \delta$, then $2\delta \leq \pi$. If $2\delta = \pi$, then $\delta = \beta = \pi/2$. In this case $Q$ is not a reptile by Lemma 3.1. If $2\delta < \pi$, then there are other tiles with a vertex at $D$, and thus $2\delta + \alpha \leq \pi$. However, $2\delta + \alpha > \delta + 2\alpha > \pi$, so this case is also impossible. \hfill $\Box$

### 4 Ruling out family (iii): preliminaries

In the next two sections our aim is to prove the following.

**Lemma 4.1** Suppose $Q$ satisfies the conditions of (iii) of Theorem 2.1. If $Q$ is not a square, then $Q$ is not a reptile.

We may assume that $c = d = 1$. Then we have $a > 1 > b$. From $a^2 + b^2 = 2$ we get $a < \sqrt{2}$. Also, we have $a + b < 2$, since $(a + b)/2 < \sqrt{(a^2 + b^2)/2} = 1$.

If $\alpha = \pi/3$, then $a = 1 + b$ by (4). The converse is also true, since the system of equations $a = 1 + b$, $a^2 + b^2 = 2$ determines $a$ and $b$. If $\alpha = \pi/3$, then $Q$ is not a reptile by Lemma 3.1. Therefore, in the sequel we may assume that $\alpha \neq \pi/3$ and $a \neq 1 + b$.

Note that $\beta = \pi/2 < 2\alpha$ by Lemma 1.6, and thus $\pi/4 < \alpha < \pi/2$. Then $\gamma$ is not the linear combination of $\alpha$ and $\pi/2$ with nonnegative integer coefficients. Indeed, this follows from the inequalities $3\alpha > \pi - \alpha = \gamma$, $\alpha + \pi/2 > 3\pi/4 > \gamma$ and from $2\alpha \neq \pi - \alpha = \gamma$.

Suppose a convex subset $\Sigma \subset \mathbb{R}^2$ is tiled with congruent copies of $Q$. We denote by $U$ the union of the boundaries of the tiles. By a barrier we mean a broken line $X_1X_2X_3X_4$ covered by $U$ such that the segments $X_1X_2$ and $X_3X_4$ lie on the same
side of the line going through $X_2X_3$. We say that $X_2X_3$ is the base of the barrier, and the angles $X_1X_2X_3 \angle$ and $X_2X_3X_4 \angle$ are the angles of the barrier.

If $X_1X_2X_3X_4$ is a barrier, then there are tiles $T_1, \ldots, T_n$ and there is a partition $X_2 = Y_0, Y_1, \ldots, Y_n = X_3$ of the base of the barrier such that $Y_iY_{i+1}$ is a side of $T_i$ ($i = 1, \ldots, n$). We say that $T_1, \ldots, T_n$ is the tiling of the barrier, and $X_2 = Y_0, Y_1, \ldots, Y_n = X_3$ is the partition of the barrier corresponding to the tiling.

**Lemma 4.2** Let $X_1X_2X_3X_4$ be a barrier with right angles.

(i) If $T_1, \ldots, T_n$ is a tiling of the barrier and $X_2 = Y_0, Y_1, \ldots, Y_n = X_3$ is the partition corresponding to the tiling, then $n$ is even, and $Y_iY_{i+2}$ equals one of $1 + b, a + b, 2, 1 + a$ for every $i = 1, \ldots, n/2$.

(ii) We have $X_2X_3 > 1$.

(iii) If $X_2X_3 = 1 + b$, then $\min(X_1X_2, X_3X_4) \leq 1$.

(iv) If $X_2X_3 = a + b$, then there are only two possible tilings as shown by Fig. 8.

**Proof** (i) Since $\pi/2 < 2\alpha$, $T_1$ has a right angle at $Y_0$. Therefore, its angle at $Y_1$ is $\alpha$ or $\gamma$. If it is $\alpha$, then the angle of $T_2$ at $Y_1$ is $\alpha$. Then $\overline{Y_0Y_1} \in \{1, b\}, \overline{Y_1Y_2} \in \{1, a\}$ and $\overline{Y_0Y_2} \in \{1 + b, a + b, 2, 1 + a\}$.

If the angle at $Y_1$ is $\alpha$, then the angle of $T_2$ at $Y_1$ is $\gamma$, because $\gamma$ is not the linear combination of $\alpha$ and $\pi/2$ with nonnegative integer coefficients. Then we obtain $\overline{Y_0Y_2} \in \{1 + b, a + b, 2, 1 + a\}$ again. Since the angle of $T_2$ at $Y_2$ is $\pi/2$, $Y_2Y_n$ is the base of a barrier with right angles, and we can argue by induction.

(ii) This follows from (i).

(iii) Suppose $\min(X_1X_2, X_3X_4) > 1$. There are two tilings of the barrier. One is shown by Fig. 7, the other is obtained by reflecting this tiling about the perpendicular bisector of the base. By symmetry, we may assume that the tiling is as in Fig. 7.

Since $X_1X_2 > 1$, there is a tile $T_3$ having the point $Y_3$ as vertex, and having angle $\alpha$ at $Y_3$. The tile $T_3$ has a side $Y_3Y_6$ of length 1 or $a$, and thus the point $Y_2$ is an inner...
point of the segment $Y_3Y_6$. Then $Y_6Y_2Y_5Y_4$ is a barrier with two right angles and base length $a - 1 < 1$, which is impossible by (ii).

(iv) Let $T_1, \ldots, T_n$ be a tiling, and let $X_2 = Y_0, Y_1, \ldots, Y_n = X_3$ be the corresponding partition. Since $a + b < 2$, we have either $Y_0Y_1 = b$ or $Y_{n-1}Y_n = b$. Suppose the former. Then $T_1$ has angle $\gamma$ at $Y_1$, and thus $T_2$ has angle $\alpha$ at $Y_1$. If $\overline{Y_1Y_2} = 1$, then $Y_3, \ldots, Y_n$ constitute a tiling of a barrier with two right angles and base length $a - 1 < 1$, contradicting (ii). Therefore, we have $Y_1Y_2 = a$. Then $n = 2$, and the tiling looks like the first tiling in Fig. 8. We obtain the second, if $\overline{Y_{n-1}Y_n} = b$.

Lemma 4.3 Let $X_1X_2X_3X_4$ be a barrier such that $X_1X_2X_3 \angle = \pi/2$, $X_2X_3X_4 \angle = \gamma$, and $X_2X_3 = 1$. Then the tiling of the barrier consists of one single tile having $X_2X_3$ as a side.

Proof Let $T_1, \ldots, T_n$ be a tiling of the barrier, and let $X_2 = Y_0, Y_1, \ldots, Y_n = X_3$ be the partition corresponding to the tiling. Since the angle of $T_n$ at $X_3$ equals $\gamma$, the angle of $T_n$ at $Y_{n-1}$ equals $\pi/2$. If $Y_{n-1} \neq X_2$, then $\overline{Y_{n-1}Y_n} = b$, and $1 - b = X_2Y_{n-1}$ is the base length of a barrier with two right angles. This, however, contradicts (ii) of Lemma 4.2.

Lemma 4.4 Let $X_1X_2X_3X_4$ be a barrier such that $X_1X_2X_3 \angle = \pi/2$, $X_2X_3X_4 \angle = \gamma$, and $X_2X_3 = a$. Then $a = 1 + 2b$, the tiling of the barrier is unique, and consists of three tiles $T_1, T_2, T_3$ such that $\overline{Y_0Y_1} = 1$ and $\overline{Y_1Y_2} = \overline{Y_2Y_3} = b$, where $X_2 = Y_0, Y_1, Y_2, Y_3 = X_3$ is the partition corresponding to the tiling.

Proof Let $T_1, \ldots, T_n$ be a tiling of the barrier, and let $X_2 = Y_0, Y_1, \ldots, Y_n = X_3$ be the partition corresponding to the tiling. Since the angle of $T_n$ at $X_3$ equals $\gamma$, the angle of $T_n$ at $Y_{n-1}$ equals $\pi/2$. Therefore, $X_2Y_{n-1}$ is the base of a barrier with two right angles. By Lemma 4.2, $X_2Y_{n-1} > 1$. Then $\overline{Y_{n-1}X_3} = \overline{X_2X_3} - \overline{X_2Y_{n-1}} < a - 1 < 1$, and thus $Y_{n-1}X_3 = b$ and $X_2Y_{n-1} = a - b$. Since $a - b < 2(1 + b)$, we have $a - b \in \{1 + b, a + b, 2, 1 + a\}$ by (i) of Lemma 4.2. We obtain $a - b = 1 + b$, $a = 1 + 2b$ and $n = 3$.

In order to complete the proof we have to show that $\overline{Y_0Y_1} = 1$ and $\overline{Y_1Y_2} = b$. If this is not true, then $\overline{Y_0Y_1} = b$ and $\overline{Y_1Y_2} = 1$. Figure 9 shows the arrangement of the tiles in this case. Then $Z_2X_2Y_2Z_6$ is a barrier with two right angles and base length $1 + b$. Since $Z_2X_2 = a > 1$ and $Z_6Y_2 = a > 1$, this contradicts (iii) of Lemma 4.2. $\square$
Lemma 4.5 Let the segments $XY$ and $YZ$ be covered by $U$, and suppose $XYZ \angle = \pi/2$ and $XY > 1$, $YZ > 1$. Let $\ell_X$ and $\ell_Z$ denote the halflines that start from $Y$ and go through $X$ and $Z$, respectively. Let $T$ be the (unique) tile having a vertex at $Y$ and lying in the quadrant bounded by the halflines $\ell_X$ and $\ell_Y$. Then the sides of $T$ with endpoint $Y$ have lengths $a$ and $b$.

Proof Since $XY, YZ \subset U$ and $\alpha > \pi/4$, it follows that $T$ has a right angle at $Y$. If the statement of the lemma is not true, then both sides of $T$ with endpoint $Y$ have length 1. We may assume that $T = Q$ with the usual labeling. Then $Y = D$, and by symmetry we may also assume that $A$ is an inner point of the segment $DZ$ and $C$ is an inner point of the segment $DX$ (see Fig. 10).

Since $XD > 1$, there is a tile $R_1$ having a vertex at $C$ and having angle $\alpha$ at $C$, and such that the point $B$ is an inner point of the side $CE$ of $R_1$. Note that $CE$ equals 1 or $a$, and that $R_1$ has a right angle at $E$.

Now $EBAZ$ is a barrier with angle $\pi/2$ and $\gamma$ and with base length $a$. By Lemma 4.4, we have $a = 1 + 2b$, and the tiling of $EBAZ$ is unique, and consists of three tiles $S_1, S_2, S_3$ as in Fig. 10. Let $B, X_1, X_2, A$ be the partition corresponding to the tiling of $EBAZ$; then $BX_1 = 1$ and $X_1X_2 = X_2A = b$. Let $B, X_1, X_3, F$ be the vertices of $S_1$, and let $X_1, X_2, X_4, X_5$ be the vertices of $S_2$.

Let $\ell$ denote the line going through the points $C, B$ and $E$. We prove that there is a point $G$ on $\ell$ such that $CG \subset U$ and $F$ is an inner point of the segment $CG$.

If $CE = a = 1 + 2b$, then $CE > 1 + b = CF$, and we can take $G = E$.

Suppose $CE = 1$, and let $R_2$ be the tile having a vertex at $E$ and different from $R_1$. Then $R_2$ has a side $EH$ lying on $\ell$. If $EH > b$, then we can take $G = H$. If $EH = b$, then $H = F$, and the angle of $R_2$ at $F$ is $\gamma$. Thus $R_2$ and $S_1$ both have an angle $\gamma$ at $F$. Since $2\pi - 2\gamma = 2\alpha$, it follows that there are two other tiles having a vertex at $F$, and both tiles have an angle $\alpha$ at $F$. Since $\gamma + \alpha = \pi$, we can see that the segment $CF$ can be continued in $U$; that is, there is a point $G$ with the required properties.

Then $GBX_2X_4$ is a barrier having right angles and base length $1 + b$. Since $GB > 1$ and $X_2X_4 = a > 1$, this contradicts (iii) of Lemma 4.2. This contradiction completes the proof.

Lemma 4.6 Let $X_1X_2X_3X_4$ be a barrier such that $X_1X_2X_3 \angle = \pi/2$, $X_2X_3X_4 \angle = \gamma$, and $X_2X_3 = 2a + b$. Then $X_1X_2 \leq 1$. 

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Proof Let $T_1, \ldots, T_n$ be a tiling of the barrier, and let $X_2 = Y_0, Y_1, \ldots, Y_n = X_3$ be the corresponding partition. Since $T_n$ has angle $\gamma$ at $X_3$, we have either $Y_{n-1}Y_n = 1$ or $Y_{n-1}Y_n = b$. We consider the two cases separately.

**Case I:** $Y_{n-1}Y_n = 1$. The tiles $T_1, \ldots, T_{n-1}$ constitute a tiling of a barrier with two right angles and with base length $2a + b - 1$. By (i) of Lemma 4.2, $n - 1$ is even, and $2a + b - 1$ equals a sum $c_1 + \ldots + c_k$, where $k = (n - 1)/2$, and each $c_i$ equals one of $1 + b, a + b, 2, 1 + a$. Now $a < \sqrt{2}$ implies $2a < 3$ and $2a + b - 1 < 2 + b < 2(1 + b)$. Therefore, we have $k = 1$ and $2a + b - 1 \in \{1 + b, a + b, 2, 1 + a\}$. By $a + b < 2a + b - 1 < 1 + a$, the only possibility is $2a + b - 1 = 2$, and thus we have $n = 3$ and $Y_0Y_1 = Y_1Y_2 = Y_2Y_3 = 1$.

Let $X_2, Y_1, Z_1, Z_2$ be the vertices of $T_1$. Then $X_2Y_1 = X_2Z_2 = 1$. Now $X_2X_3 = 2a + b > 1$. Therefore, $X_1X_2 \leq 1$ follows from Lemma 4.5.

**Case II:** $Y_{n-1}Y_n = b$. If the vertices of $T_n$ are $Y_{n-1}, X_3, W_1, W_2$, then $W_2Y_{n-1} = a$. The tiles $T_1, \ldots, T_{n-1}$ constitute a tiling of a barrier with two right angles and with base length $2a$. By (i) of Lemma 4.2, $n - 1$ is even, and $2a$ equals a sum $c_1 + \ldots + c_k$, where $k = (n - 1)/2$, and each $c_i$ equals one of $1 + b, a + b, 2, 1 + a$. Now $2a > 1 + a$ implies that $k \geq 2$. On the other hand, $a < \sqrt{2}$ implies $2a < 3$, hence $k \leq 2$. Therefore, we have $k = 2$, and $2a$ is the sum of two elements of the set $\{1 + b, a + b, 2, 1 + a\}$. Since the case $a = 1 + b$ has been excluded and $2a < 2(a + b)$, the only possibility is $2a = (1 + b) + (a + b)$. Then $Y_0Y_2 = 1 + b$ and $Y_2Y_4 = a + b$, or the other way around.

**Case II.1:** $Y_0Y_2 = 1 + b$ and $Y_2Y_4 = a + b$. Suppose $X_1X_2 > 1$. It follows from Lemma 4.5 that $Y_0Y_1 = b$, $Y_1Y_2 = 1$, and the arrangement of $T_1$ and $T_2$ is as in Fig. 11. By (iii) of Lemma 4.2, the segment $Y_2Z_3$ cannot be continued in $U$. Then, by (iv) of Lemma 4.2, the tiles $T_3$ and $T_4$ are as in Fig. 11. Then there is a tile $R$ having a vertex at $Z_5$. Then $R$ has a side $Z_5Z_6$ such that $Z_3$ is an inner point of $Z_5Z_6$, contradicting (iii) of Lemma 4.2.
Case II.2: \( Y_0Y_2 = a + b \) and \( Y_2Y_4 = 1 + b \). The proof in this case is the same as in Case II.1. The only difference is that we work from right to left; that is, from \( Y_4 \) to \( X_2 \), using \( W_2Y_4 = a > 1 \).

Lemma 4.7 Let \( T \) be a tile with vertices \( X_1, X_2, X_3, X_4 \) such that \( \overline{X_2X_3} = b \) and \( \overline{X_3X_4} = 1 \) (see Fig. 12). Suppose that there is a segment \( X_1X_5 \subset U \) such that \( X_2 \) is an inner point of \( X_1X_5 \). Then there is a segment \( X_2X_6 \subset U \) such that \( X_3 \) is an inner point of \( X_2X_6 \).

Proof Let \( R \) be the tile having a vertex at \( X_2 \), different from \( T \) and lying on the same side of \( X_1X_5 \) as \( T \). Then \( R \) has a side \( X_2X_7 \) lying on the line \( \ell \) going through \( X_2 \) and \( X_3 \).

If \( \overline{X_2X_7} > b \), then we can take \( X_6 = X_7 \). If \( \overline{X_2X_7} = b \), then \( X_7 = X_3 \), and the angle of \( R \) at \( X_3 \) is \( \gamma \). Thus \( T \) and \( R \) both have an angle \( \gamma \) at \( X_3 \). Since \( 2\pi - 2\gamma = 2\alpha \), it follows that there are two other tiles having a vertex at \( X_3 \), and both tiles have an angle \( \alpha \) at \( X_3 \). Since \( \gamma + \alpha = \pi \), we can see that the segment \( X_2X_3 \) can be continued in \( U \); that is, there is a point \( X_6 \) on \( \ell \) such that \( X_3 \) is an inner point of a segment \( X_2X_6 \subset U \).

5 Ruling out family (iii): conclusion

First we deduce Lemma 4.1 from Theorem 1.4.

Suppose \( Q \) is a reptile. Then there is a quadrilateral \( Q' \) similar to \( Q \) such that \( Q' \) can be tiled with \( k \geq 2 \) congruent copies of \( Q \). Let the vertices of \( Q' \) be \( X, Y, V, W \) such that \( \overline{XY} = \lambda a, \overline{YV} = \lambda, \overline{WV} = \lambda \) and \( \overline{WX} = \lambda b \), where \( \lambda = \sqrt{k} > 1 \). We may assume that \( X \) is the origin, \( Y \) is the point \( (\lambda a, 0) \) and \( W \) is the point \( (0, \lambda b) \). The square \( \sigma_{0,0}' = [0, \lambda(a+b)] \times [0, \lambda(a+b)] \) can be tiled with four copies of \( Q' \), and thus by \( 4k \) congruent copies of \( Q \). Now \( \Sigma \) is tiled with congruent copies of \( \sigma_{0,0}' \). Translating the tiling of \( \sigma_{0,0}' \) into each of these squares we obtain a tiling of \( \Sigma \) with congruent copies of \( Q \).
By Theorem 1.4, this tiling must be trivial. Let $i$ be the largest integer with $i(a + b) < \lambda a$. Then the line $\ell$ going through the vertices $Y$ and $V$ cuts the square $\sigma_{i,0}$ into two parts. Since the tiling is trivial and $\ell \cap \sigma_{i,0}$ is covered by the boundaries of the tiles, $\ell$ must intersect the $x$ axis at the point $i(a + b) + a$, and must intersect the line $y = a + b$ at the point $(i(a + b) + a, b + a + b)$. Then $\ell$ also cuts the square $\sigma_{i,1}$ into two parts. However, $\ell \cap \sigma_{i,1}$ cannot be covered by the boundaries of the tiles, since $b < a$, and thus $\ell$ enters the interior of a tile in $\sigma_{i,1}$, which is impossible.

The rest of the section is devoted to the proof of Theorem 1.4. Suppose a tiling of $\Sigma$ is given.

**Lemma 5.1** Let $X_1, \ldots, X_8$ be points such that $X_3, X_4, X_2, X_1$ are the vertices of a square $\sigma_1$ of side length $a + b$, $X_6, X_7, X_5, X_4$ are vertices of a square $\sigma_2$ of side length $a + b$, and $X_7X_8 > 1$. (See Fig. 13.) Suppose that at least one of the broken lines $X_1X_3X_4X_6X_8$ and $X_2X_6X_8$ is covered by $U$. Then $\sigma_2$ is tiled with four tiles.

**Proof** There is a tile $T_1 \subset \sigma_2$ having a vertex at $X_6$. Let the vertices of $T_1$ be $X_6, Y_1, Y_2, Y_3$ such that $Y_1$ is an inner point of the segment $X_6X_7$ and $Y_3$ is an inner point of the segment $X_6X_4$. By Lemma 4.5 we have $X_6Y_1 = a$ and $X_6Y_3 = b$, or the other way around. We consider these two cases separately.

**Case I:** $X_6Y_1 = a$ and $X_6Y_3 = b$. Then there is a tile $T_2$ with vertices $Y_3, Y_4, Y_5, Y_6$ such that $Y_4 \in \sigma_2$ and $Y_6$ is on the segment $X_6X_2$. Since the angle of $T_2$ at $Y_3$ equals $\alpha$, we have $Y_3Y_4 = a$ or $1$. We consider the two cases separately. Let $\ell$ denote the line going through the points $Y_3$ and $Y_2$.

**Case I.1:** $Y_3Y_4 = a$. Then $Y_3Y_2Y_1X_7$ is a barrier, and then, by Lemma 4.3, there is a tile $T_3$ with vertices $Y_1, X_7, Y_7, Y_2$. (See Fig. 14.) We have $Y_2Y_4 = a - 1 < Y_2Y_7 = 1$, and $Y_4Y_7 = Y_3Y_7 - Y_3Y_4 = 2 - a$.

We show that there is a point $Z$ on $\ell$ such that $Y_3Z \subset U$ and $Y_7$ is an inner point of the segment $Z_3Z$.

There is a tile $T_4$ with vertices $Y_4, Y_8, Y_9, Y_{10}$ such that $Y_8$ is on the line $\ell$. If $Y_4Y_8 \geq 1$ then we take $Z = Y_8$. On the other hand, if $Y_4Y_8 = b$ then, by $2 - a > b$, the point $Y_8$ is an inner point of the segment $Y_4Y_7$. In this case, however, there is a tile $T_5$ with vertices $Y_8, Y_{11}, Y_{12}, Y_{13}$ such that $Y_{11}$ is on the line $\ell$. Since the angle of $T_5$ at $Y_8$ is $\alpha$, we have $Y_8Y_{11} \geq 1$, and thus we can take $Z = Y_{11}$.

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In both cases, $ZY_7X_7X_8$ is a barrier as in Lemma 4.4, so we have a tile $T_8$ having sides of length 1 on each of the segments $Y_7X_7$ and $X_7X_8$ (see Fig. 14). Since $Y_7X_7 = a > 1$ and $X_7X_8 > 1$ by assumption, this contradicts Lemma 4.5.

**Case I.2:** $Y_3Y_4 = 1$. Then the vertices of $T_2$ are $Y_3$, $Y_2$, $Y_5$, $X_4$. (That is, we have $Y_4 = Y_2$ and $Y_6 = X_4$; see Fig. 15.)

Suppose that there is a segment $X_4Z$ in $U$ such that $Y_5$ is an inner point of $X_4Z$. Then there is a tile $T_4$ with vertices $Y_5$, $Y_{10}$, $Y_{11}$, $Y_{12}$ such that $Y_{10}$ is on the segment $Y_1Y_5$. Since the angle of $T_4$ at $Y_5$ is $\alpha$, we have $Y_5Y_{10} = a$ or 1. If it equals $a$, then $Y_{11}Y_{10}Y_1X_7$ is a barrier with $Y_{10}Y_1 = 2 - a < 1$. Thus the tiling of this barrier has more than one tile, since $2 - a$ is different from each of $a$, $b$ and 1. Thus $2 - a \geq 1 + b$, which is also impossible.
Then we have $Y_5Y_{10} = 1$, and thus $Y_{10} = Y_2$ and $Y_{12} = X_5$. Then $Y_{11}Y_2Y_1X_7$ is a barrier as in Lemma 4.3. Thus there is a tile with vertices $Y_2, Y_1, X_7, Y_{11}$ showing that $\sigma_2$ is tiled with four tiles.

Therefore, it is enough to show that there is a segment $X_4Z$ in $U$ containing $Y_5$ as an inner point. Suppose this is not true. Then there is a tile $T_3$ with vertices $Y_5, Y_7, Y_8, Y_9$ as in Fig. 15.

If the angle of $T_3$ at $Y_5$ equals $\gamma$ then, as the angle of $T_2$ at $Y_5$ is also $\gamma$, it follows that there are other tiles having a vertex at $Y_5$. Since $2\pi - 2\gamma = 2\alpha > \pi/2$, there must be two such tiles, each of them having angle $\alpha$ at $Y_5$. From this it is clear that there is a segment $X_4Z \subset U$ containing $Y_5$ as an inner point, contradicting our assumption. So the angle of $T_3$ at $Y_5$ cannot be $\gamma$.

If the broken line $X_2X_6X_8$ is in $U$, then by $X_4Y_5 = b$ we have $Y_9 = X_4$, and the angle of $T_3$ at $Y_5$ is $\gamma$, which is impossible. Therefore, the broken line $X_1X_3X_4X_6X_8$ is in $U$.

Suppose that the angle of $T_3$ at $Y_5$ is $\alpha$. Then the angle of $T_3$ at $Y_9$ equals $\pi/2$, and $X_1X_3Y_9Y_8$ is a barrier of two right angles. Since $Y_9Y_5 = 1$ or $a$, we have $X_3Y_9 = a + 2b - 1$ or $2b$. However, we have $a + 2b - 1 < 1 + b$ and $2b < 1 + b$, contradicting (i) of Lemma 4.2.

Thus the angle of $T_3$ at $Y_5$ equals $\pi/2$, and $X_1X_3Y_5Y_7$ is a barrier with two right angles and $X_3Y_5 = a + 2b$. By (i) of Lemma 4.2, $X_3Y_5 = a + 2b$ equals a sum $c_1 + \ldots + c_n$, where each $c_i$ equals one of $1 + b, a + b, 2, 1 + a$. Now $a + 2b < 2(1 + b)$, hence $n = 1$, and thus $a + 2b \in \{1 + b, a + b, 2, 1 + a\}$. The only possibilities are $a + 2b = 2$ and $a + 2b = 1 + a$.

We found that the barrier $X_1X_3Y_5Y_7$ is tiled with two tiles, $S$ and $T_3$, the corresponding partition is $X_3, Y_9, Y_5$, and either $X_3Y_9 = Y_9Y_5 = 1$, or $\{X_3Y_9, Y_9Y_5\} = \{1, a\}$. If $X_3Y_9 = 1$, then the side of $S$ lying on the segment $X_1X_3$ is of length 1. This, however, contradicts Lemma 4.5. Therefore, the only possibility is that $X_3Y_9 = a$ and $Y_9Y_5 = 1$. Then $a + 2b = a + 1, b = 1/2$, and the tiling of $X_1X_3Y_5Y_7$ is as shown by Fig. 16.
We have $Y_7Y_5Y_1 \angle = \alpha + \pi/2$. Since $\gamma < \alpha + \pi/2 < \alpha + \gamma$, it follows that, apart from $T_2$ and $T_3$, there are two more tiles with a vertex at $Y_5$, and one of them has angle $\alpha$ and the other has angle $\pi/2$ at $Y_5$. Let these tiles be $T_5$ and $T_6$, where $T_5$ has a vertex on the halfline starting from $Y_5$ and going through $Y_7$, and $T_6$ has a vertex on $Y_5Y_1$. Then $T_6$ cannot have angle $\alpha$ at $Y_5$, because in that case $T_6$ would have a vertex $Z$ such that the segment $X_4Z$ contains $Y_1$ as an inner point, which is impossible.

Therefore, $T_5$ has angle $\alpha$ and $T_6$ has angle $\pi/2$ at $Y_5$ (see Fig. 16). Let $Y_5, Z_1, Z_2, Z_3$ be the vertices of $T_5$ such that $Y_5, Z_3, Y_7$ are collinear. Then we have either $Y_5Z_3 = a$ or $Y_5Z_1 = a$.

If $Y_5Z_3 = a$, then the barrier $WY_8Y_7Z_3$ violates Lemma 4.4, since $Y_8Y_7 = a$, but $a < 2 = 1 + 2b$.

If, however, $Y_5Z_1 = a$, then $Z_1Y_5Y_1X_8$ is a barrier such that $Y_5Z_1 > 1$ and $Y_5Y_1 = 2$. Let $S_1, \ldots, S_n$ be the tiling of this barrier, and let $W_1, Y_1, W_2, W_3$ be the vertices of $S_n$, where $W_1$ is on the segment $Y_5Y_1$. Then $W_1Y_1 = 1$ or $b$. Therefore, $Z_1Y_5W_1W_3$ is a barrier with two right angles and base length $1$ or $2 - b$. By (ii) of Lemma 4.2, the first case is excluded. Thus $W_1Y_1 = b$, and then $W_1W_3 = a > 1$. Then $Z_1Y_5W_1W_3$ is a barrier with two right angles such that $Y_5W_1 = 2 - b = 1 + b$ and $Y_5Z_1 > 1$ and $W_1W_3 > 1$, which contradicts (iii) of Lemma 4.2.

**Case II:** $X_6Y_1 = b$ and $X_6Y_3 = a$. Then there is a tile $T_2$ with vertices $Y_3, Y_4, Y_5, Y_6$ such that $Y_4$ is on the line going through the points $Y_3$ and $Y_2$. Since the angle of $T_2$ at $Y_3$ equals $\gamma$, we have $Y_3Y_4 = b$ or $1$. We consider the two cases separately.

**Case II.1:** $Y_3Y_4 = b$ (see Fig. 17). There is a tile $T_3 \subset \sigma_2$ with vertices $Y_1, Y_7, Y_8, Y_9$ such that $Y_7$ is on the line $X_6X_7$. Since the angle of $T_3$ at $Y_1$ equals $\alpha$, we have $Y_1Y_9 = a$ or $1$. If $Y_1Y_9 = a$, then $Y_5Y_4Y_2Y_9$ is a barrier with two right angles and $Y_4Y_2 = 1 - b < 1$, which is impossible. Therefore, we have $Y_1Y_9 = 1$, and then $Y_7 = X_7$ and $Y_9 = Y_2$.

By Lemma 4.7, there exists a point $Y_{10}$ such that $Y_7Y_{10} \subset U$ and $Y_8$ is an inner point of the segment $Y_7Y_{10}$. Then $Y_5Y_4Y_8Y_{10}$ is a barrier with angles $\pi/2$ and $\alpha$, and
with $Y_4Y_8 = 2 - b$. Since $2 - b$ is different from each of $b$, 1 and $a$, the tiling of the barrier consists of more than one tile. One of the tiles has angle $\alpha$ at $Y_8$, so its side on the base has length $a$ or 1. The other tiles tile a barrier with two right angles, and thus $2 - b = Y_4Y_8 \geq (1 + b) + 1$, which is absurd. So this case cannot happen.

**Case II.2:** $Y_3Y_4 = 1$. Then $Y_4 = Y_2$ and $Y_6 = X_4$. There is a tile $T_3 \subset \sigma_2$ with vertices $Y_1, Y_7, Y_8, Y_9$ such that $Y_7$ is on the line $X_6X_7$. Let $\ell$ denote the line going through the points $Y_1, Y_2, Y_5$. Since the angle of $T_3$ at $Y_1$ equals $\alpha$, we have $Y_1Y_7 = a$ or 1.

**Case II.2.1:** $Y_1Y_7 = 1$ (see Fig. 18). We prove that there exists a point $Z$ on the line $\ell$ such that $Y_1Z \subset U$ and $Y_5$ is an inner point of the segment $Y_1Z$.

There is a tile $T_4$ having a vertex at $Y_9$. Let the vertices of $T_4$ be $Y_9, Y_{10}, Y_{11}, Y_{12}$, where $Y_{12}$ is on the line $\ell$. If $Y_9Y_{12} \geq 1$, then we take $Z = Y_{12}$. If $Y_9Y_{12} = b$, then $T_4$ has angle $\gamma$ at $Y_{12}$, and then there is a tile $T_5$ having angle $\alpha$ at $Y_{12}$. In this case let $Z$ be the vertex of $T_5$ lying on $\ell$.

If the broken line $X_2X_6X_8$ is covered by $U$ then, by Lemma 4.4, there is a tile $T_6$ such that $T_6$ has a vertex at $X_4$, and has sides of length 1 on both segments $X_2X_4$ and $X_4Y_5$. However, these segments are longer than 1, which contradicts Lemma 4.5.

If the broken line $X_1X_3X_4X_6X_8$ is covered by $U$, then $X_1X_3Y_5Z$ is a barrier such that $X_1X_3Y_5Z \angle = \pi/2, X_3Y_5Z \angle = \gamma$ and $X_3Y_5 = 2a + b$. By Lemma 4.6, this implies $a + b = X_1X_3 \leq 1$, which is not true. Therefore, this case cannot happen.

**Case II.2.2:** $Y_1Y_7 = a$. Applying Lemma 4.7 we find a point $W$ on the line going through the points $X_7, Y_8$ and $X_5$ and such that $X_7W \subset U$ and $Y_8$ is an inner point of the segment $X_7W$. Then $Y_5Y_9Y_8W$ is a barrier such that $Y_5Y_9Y_8W \angle = \pi/2, Y_9Y_8W \angle = \alpha$, and $Y_5Y_8 = 1$. Clearly, $Y_5Y_9Y_8W$ is tiled with a single tile $T_4$ such that $T_1, T_2, T_3$ and $T_4$ tile $\sigma_2$. □
Now we turn to the last step of the proof of Theorem 1.4. Let \( X_{i,j} \) denote the point with coordinates \((i(a+b), j(a+b))\). We denote by \( \Sigma \) the set of squares \( \sigma_{i,j} \) that are tiled with four tiles. We have to prove that \( \sigma_{i,j} \in \Sigma \) for every \( i, j \geq 0 \).

We prove this statement by induction on \( k = i + j \). By Lemma 5.1, \( \sigma_{0,0} \in \Sigma \), since the segments \( X_{0,0}X_{0,2} \) and \( X_{0,0}X_{2,0} \) belong to \( U \).

We have \( \sigma_{1,0} \in \Sigma \), since the broken line \( X_{0,2}X_{1,1}X_{1,0}X_{3,0} \) belongs to \( U \), and Lemma 5.1 applies.

We also have \( \sigma_{0,1} \in \Sigma \), since the broken line \( X_{0,3}X_{0,1}X_{2,1} \) belongs to \( U \), and Lemma 5.1 applies.

Let \( k \geq 2 \), and suppose that \( \sigma_{i,j} \in \Sigma \) for every \( i, j \) such that \( i + j < k \).

We prove \( \sigma_{k-j,j} \in \Sigma \) by induction on \( j \). As for \( j = 0 \), \( \sigma_{k,0} \in \Sigma \) follows from Lemma 5.1, since \( \sigma_{k-1,0}, \sigma_{k-2,1} \in \Sigma \) by the induction hypotheses, and thus the broken line \( X_{k-1,2}X_{k-1,1}X_{k,0}X_{k,0}X_{k+2,0} \) is covered by \( U \).

Let \( 0 < j \leq k \), and suppose that \( \sigma_{k-j+1,j-1} \in \Sigma \). Then there are three cases.

If \( j \leq k-2 \), then \( \sigma_{k-j-2,j+1}, \sigma_{k-j-1,j}, \sigma_{k-j,j-1}, \sigma_{k-j+1,j-1} \in \Sigma \) by the induction hypotheses. Therefore, the broken line

\[
X_{k-j-1,j+2}X_{k-j-1,j}X_{k-j,j+1}X_{k-j,j}X_{k-j+2,j}
\]

is covered by \( U \), and Lemma 5.1 applies.

If \( j = k-1 \), then \( \sigma_{0,k-1}, \sigma_{1,k-2}, \sigma_{2,k-2} \in \Sigma \) by the induction hypotheses. Therefore, the broken line \( X_{0,k+1}X_{0,k}X_{1,k}X_{1,k-1}X_{3,k-1} \) is covered by \( U \), and Lemma 5.1 applies.

Finally, If \( j = k \), then \( \sigma_{0,k-1}, \sigma_{1,k-1} \in \Sigma \) by the induction hypotheses. In this case the broken line \( X_{0,k+2}X_{0,k}X_{2,k} \) is covered by \( U \), and Lemma 5.1 applies.

This completes the proof of Theorem 1.4 and that of Theorem 1.1. \( \square \)

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