About the $x$-$y$ symmetry of the $F_g$ algebraic invariants.

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Abstract: We complete the proof of the $x$ – $y$ symmetry of symplectic invariants of \cite{3}. We recall the main steps of the proof of \cite{4}, and we include the integration constants absent in \cite{4}.

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1 Introduction

In [3] we introduced some “invariants” $F_g(S)$ associated to an algebraic curve $S = \{(x, y) \mid P(x, y) = 0\}$ immersed in $\mathbb{C} \times \mathbb{C}$. We claimed in [4] that the $F_g$'s were invariant under the symmetry $x \leftrightarrow y$, i.e. if $\tilde{S} = \{(y, x) \mid P(x, y) = 0\} \subset \mathbb{C} \times \mathbb{C}$, we had $F_g(S) = F_g(\tilde{S})$. In fact, the identity $F_g(S) = F_g(\tilde{S})$ holds only for certain classes of spectral curves $S$, typically those appearing in the 2-matrix model as in [4]. However, as noticed by V. Bouchard and P. Sulkowski in [2] and later developed in [1], it can be wrong for more general classes of curves, because of some integration constants which were disregarded in [4]. The actual invariance, valid for any algebraic spectral curve is:

\[
\hat{F}_g(S) = F_g(S) + \frac{1}{2 - 2g} \sum_i t_i \int_{\alpha_i} \omega_{g,1}(S),
\]

for which we have:

\[
\hat{F}_g(S) = \hat{F}_g(\tilde{S}).
\]

In this article we recall the main steps of [4], and we include the integration constants missing in [4] in order to prove the symmetry property for the corrected $\hat{F}_g$.

2 Spectral curves and their invariants

Definition 2.1 An algebraic spectral curve $S = (C, x, y)$ is the data of a compact Riemann surface $C$ of genus $g$, together with a choice of $2g$ independent non-contractible cycles $A_1, \ldots, A_g, B_1, \ldots, B_g$ on it with symplectic intersections:

\[
A_i \cap B_j = \delta_{i,j}, \quad A_i \cap A_j = 0, \quad B_i \cap B_j = 0,
\]

and $x$ and $y$ are two meromorphic functions $C \to \mathbb{C}P^1$.

Moreover, we say that $S = (C, x, y)$ is a regular spectral curve, if $dx$ has only simple zeroes on $C$, and the zeroes of $dx$ are distinct from the zeroes of $dy$ and from the poles of $x$ and $y$.

The map $C \to \mathbb{C} \times \mathbb{C}, z \mapsto (x(z), y(z))$ defines an algebraic curve immersed in $\mathbb{C} \times \mathbb{C}$.

Definition 2.2 The zeroes of $dx$ are called the ”branchpoints”: 

\[
a = \{a_1, \ldots, a_s\} \quad , \quad dx(a_i) = 0.
\]

In a vicinity $U_a$ of a branchpoint $a$, a good local coordinate is $\zeta_a(z) = \sqrt{(x(z) - x(a))}$. 

The local Galois involution $s_a : U_a \to U_a$ is such that $x \circ s_a = x$, and $s_a \neq \text{Id}$. In the local coordinate $\zeta_a(z) = \sqrt{(x(z) - x(a))}$, the local Galois involution is simply:

\[
s_a \left( \sqrt{(x(z) - x(a))} \right) = -\sqrt{(x(z) - x(a))}.
\]
Following [3], to a regular spectral curve \( S = (\mathcal{C}, x, y) \) we associate its invariants:

**Definition 2.3** The invariants \( \omega_{g,n}(S) \) are symmetric meromorphic differentials \( \in K(\mathcal{C})^{\otimes n} \) (where \( K(\mathcal{C}) \) is the canonical bundle of \( \mathcal{C} \)), such that:

- \( \omega_{0,1} = ydx \),
- \( \omega_{0,2} \) is the fundamental second kind form on \( \mathcal{C} \) [?], i.e. the unique bilinear differential on \( \mathcal{C} \times \mathcal{C} \), with a normalized double pole on the diagonal, and no other poles:

\[
\omega_{0,2}(z_1, z_2) \sim \frac{dz_1 dz_2}{(z_1 - z_2)^2} + \text{analytical}
\]  

and normalized on the \( A \)-cycles:

\[
\forall z_1 \in \mathcal{C}, \quad \oint_{z_2 \in A_i} \omega_{0,2}(z_1, z_2) = 0;
\]  

- for \( n \geq 1 \) and \((g, n) \neq (0, 1), (0, 2)\), the \( \omega_{g,n}(S) \) are computed by the topological recursion of [3]:

\[
\omega_{g,n}(z_1, z_2, \ldots, z_n) = \sum_{a \in A} \text{Res}_{z \to a} K_a(z_1, z) \left( \omega_{g-1,n,1}(z, s_a(z), J) \right. \\
\left. + \sum'_{h+h'=g, I \cup I'=J} \omega_{h,1+|I|}(z, I) \omega_{h',1+|I'|}(s_a(z), I') \right)
\]  

where the recursion kernel is

\[
K_a(z_1, z) = -\frac{1}{2} \frac{\int_{z'=s_a(z)}^z \omega_{0,2}(z_1, z')}{\omega_{0,1}(z) - \omega_{0,1}(s_a(z))}
\]  

and the \( \sum' \) means that we exclude the terms \((h, I) = (0, \emptyset)\) and \((h, I) = (g, J)\).

The scalar invariants \( \omega_{g,0}(S) = F_g(S) \in \mathbb{C} \), are given by:

\[
\forall g \geq 2, \quad F_g = \frac{1}{2 - 2g} \sum_{a \in A} \text{Res}_{z \to a} \omega_{g,1}(z) \Phi(z)
\]  

where \( d\Phi = \omega_{0,1} = ydx \) in a vicinity of each \( a \in A \).

**Remark 2.1** We shall not consider \( F_0 \) and \( F_1 \) in this article, their \( x-y \) symmetry properties have already been established.

### 3 The x-y symmetry

Now, consider the two spectral curves

\[
S = (\mathcal{C}, x, y), \quad \tilde{S} = (\mathcal{C}, y, x)
\]  

where \( \xrightarrow{\text{x-y}} \Phi \) are given by:

\[
F_g = \frac{1}{2 - 2g} \sum_{a \in A} \text{Res}_{z \to a} \omega_{g,1}(z) \Phi(z)
\]  

where \( d\Phi = \omega_{0,1} = ydx \) in a vicinity of each \( a \in A \).
with the same compact Riemann surface $C$ and the same choice of independent contours $(A_i, B_i)_{i=1,\ldots,g}$, and which we assume are both regular. Let:
\[ a = \{a_1,\ldots,a_s\} = \text{zeros of } dx, \quad b = \{b_1,\ldots,b_t\} = \text{zeros of } dy \]  
(3-2)

We shall need to consider the poles of $x$ and $y$, we call them:
\[ \alpha = \{\alpha_1,\ldots,\alpha_r\} = \text{poles of } x \text{ and } y. \]  
(3-3)

\[ d_i = \deg_{\alpha_i} x, \quad \tilde{d}_i = \deg_{\alpha_i} y. \]  
(3-4)

We shall define the times:
\[ t_i = \Res_{z \to \alpha_i} ydx = -\tilde{t}_i = -\Res_{z \to \alpha_i} xdy. \]  
(3-5)

We shall denote:
\[ \omega_{g,n} \equiv \omega_{g,n}(S) \quad \tilde{\omega}_{g,n} \equiv \omega_{g,n}(\tilde{S}). \]  
(3-6)

and
\[ F_g = F_g(S), \quad \tilde{F}_g = F_g(\tilde{S}). \]  
(3-7)

Our goal is to compare the invariants, i.e. compute
\[ F_g - \tilde{F}_g = ? \]  
(3-8)

3.1 Sketch of the construction of [4]

The main idea in [4] is to define by a recursion a sequence of differentials for any $g \geq 0$ and any $n + m > 0$:
\[ \omega_{g,n,m}(S) = \omega_{g,m,n}(\tilde{S}) \in K(C)^{\otimes m+n}, \]  
(3-9)

which are by construction manifestly symmetric in the exchange of $x$ and $y$.

(These definitions in [4] may look complicated, but they are simply obtained by mimicking the loop equations in a 2 matrix model).

It was proved in [4] that

Proposition 3.1 ([4]) The differential forms $\omega_{g,n,m}$ satisfy:

- For any $n \geq 1$,
\[ \omega_{g,n,0} = \omega_{g,n}, \quad \omega_{g,0,n} = \tilde{\omega}_{g,n}; \]  
(3-10)

In particular $\omega_{0,1,0} = ydx$, $\omega_{0,0,1} = xdy$, $\omega_{0,2,0} = \omega_{0,0,2}$ = fundamental second kind differential.

- if $2g - 2 + n + m > 0$, $\omega_{g,n,m}(z_1,\ldots,z_n; \tilde{z}_1,\ldots,\tilde{z}_m)$ has poles only when $z_i \in a$, $\tilde{z}_j \in b$, and whenever $x(z_i) = x(\tilde{z}_j)$ or $y(z_i) = y(\tilde{z}_j)$.
Let \( q = \{q_1, \ldots, q_n\} \) and \( p = \{p_1, \ldots, p_m\} \), the following form is exact (with respect to the variable \( z \in \mathbb{C} \)):

\[
\omega_{g,n+1,m}(z, q; p) + \omega_{g,n,m+1}(q; z, p) = d_z \left( \frac{A_{g,n,m}(z; q; p)}{dx(z) \, dy(z)} \right)
\]  

(3-11)

where \( A_{g,n,m}(z; q; p) \) is a quadratic differential of \( z \in \mathbb{C} \), which has poles at \( z \in a \) and \( z \in b \) and when \( x(z) = x(q_i) \) or \( y(z) = y(q_i) \) or \( x(z) = x(p_j) \) or \( y(z) = y(p_j) \). It may also have poles at the poles \( z \to \alpha_i \);

- \( d_z(A_{g,n,m}(z; q; p)/dx(z)dy(z)) \) vanishes to order \( d_i + \tilde{d}_i \) at a pole \( z = \alpha_i \).

In particular, if \( n = m = 0 \) we have:

\[
\omega_{g,1}(z) + \omega_{g,1} = \omega_{g,0,0}(z) + \omega_{g,0,1}(z) = d_z \left( \frac{A_{g,0,0}(z)}{dx(z) \, dy(z)} \right)
\]  

(3-12)

where \( A_{g,0,0}(z) \) is a quadratic differential on \( \mathbb{C} \), whose only poles are at \( z \in a \cup b \). \( d_z(A_{g,0,0}(z)/dx(z)dy(z)) \) vanishes to order \( d_i + \tilde{d}_i \) at a pole \( z = \alpha_i \).

This implies, that, in the vicinity of \( \alpha_i \), there exists a choice of integration constant \( C_{g;i} \), such that:

\[
A_{g,0,0}(z) - C_{g;i} \, dx(z) \, dy(z) = D_{g,i}(z) \, dx(z) \, dy(z)
\]  

(3-13)

where \( D_{g,i}(z) \) vanishes to order \( d_i + \tilde{d}_i + 1 \) at \( \alpha_i \).

The integration constants satisfy

\[
C_{g;i} - C_{g;j} = \int_{\alpha_i}^{\alpha_j} \left[ \omega_{g,1}(z) + \tilde{\omega}_{g,1}(z) \right].
\]  

(3-14)

Since the sum of residues of a 1-form must be zero, we have:

\[
\sum_i t_i = \sum_i \text{Res} \ ydx = 0.
\]  

(3-15)

Therefore

\[
\sum_i t_i C_{g;i} = \sum_i t_i \int_{\alpha_i}^{\alpha_j} \left[ \omega_{g,1}(z) + \tilde{\omega}_{g,1}(z) \right]
\]  

(3-16)

is independent of a choice of origin \( o \in \mathbb{C} \).

### 3.2 Symmetry of the \( F_g \)'s

Let us define \( \Phi \) and \( \tilde{\Phi} = xy - \Phi \) as functions on some vicinity of the branchpoints, such that:

\[
d\Phi = ydx \quad , \quad d\tilde{\Phi} = xdy.
\]  

(3-17)
Therefore we have:

\[ F_g = \frac{1}{2 - 2g} \sum_{a \in \alpha} \text{Res} \omega_{g,1}(z) \Phi(z) \quad , \quad \tilde{F}_g = \frac{1}{2 - 2g} \sum_{b \in \beta} \text{Res} \tilde{\omega}_{g,1}(z) \tilde{\Phi}(z). \quad (3-18) \]

This implies that

\[ (2 - 2g)(F_g - \tilde{F}_g) = \sum_{z \rightarrow \alpha} \text{Res} \omega_{g,1}(z) \Phi(z) - \sum_{z \rightarrow b} \text{Res} \tilde{\omega}_{g,1}(z) \tilde{\Phi}(z). \quad (3-19) \]

Notice that, since \( \omega_{g,1} \) and \( \Phi \) have no pole at the \( b_i \)'s (zeroes of \( dy \)) and \( \tilde{\omega}_{g,1} \) and \( \tilde{\Phi} \) have no pole at the \( a_i \)'s (zeroes of \( dx \)), we may write:

\[ (2 - 2g)(F_g - \tilde{F}_g) = \sum_{a \in \alpha \cup \beta} \text{Res} \omega_{g,1}(z) \Phi(z) - \sum_{a \in \alpha \cup \beta} \text{Res} \tilde{\omega}_{g,1}(z) \tilde{\Phi}(z) \]

\[ = \sum_{z \rightarrow \alpha} \text{Res} \omega_{g,1}(z) \Phi(z) - \sum_{z \rightarrow \alpha} \text{Res} \tilde{\omega}_{g,1}(z) \tilde{\Phi}(z) \]

\[ = \sum_{z \rightarrow \alpha} \text{Res} (\omega_{g,1}(z) + \tilde{\omega}_{g,1}(z)) \Phi(z) - \tilde{\omega}_{g,1}(z) x(z)y(z). \quad (3-20) \]

It was proved in \[3\] that, for any spectral curve,

\[ \text{Res}_{z \rightarrow \alpha} \omega_{g,1}(z) x(z) y(z) = 0. \quad (3-21) \]

Therefore we have:

\[ (2 - 2g)(F_g - \tilde{F}_g) = \sum_{a \in \alpha \cup \beta} \text{Res}_{z \rightarrow \alpha} (\omega_{g,1}(z) + \tilde{\omega}_{g,1}(z)) \Phi(z) \]

\[ = \sum_{a \in \alpha \cup \beta} \text{Res}_{z \rightarrow \alpha} \Phi(z) d \left( \frac{A_{g,0,0}(z)}{dx(z) dy(z)} \right) \quad (3-22) \]

and, by integrating by parts,

\[ (2 - 2g)(F_g - \tilde{F}_g) = -\sum_{a \in \alpha \cup \beta} \text{Res}_{z \rightarrow \alpha} \frac{A_{g,0,0}(z)}{dx(z) dy(z)} d\Phi(z) \]

\[ = -\sum_{a \in \alpha \cup \beta} \text{Res}_{z \rightarrow \alpha} \frac{A_{g,0,0}(z)}{dx(z) dy(z)} y(z) dx(z) \]

\[ = -\sum_{a \in \alpha \cup \beta} \text{Res}_{z \rightarrow \alpha} A_{g,0,0}(z) \frac{y(z)}{dy(z)}. \quad (3-23) \]

Now, let us move the integration contour, so that we enclose all the other poles of \( A_{g,0,0} \), i.e. the \( \alpha_i \)'s. We have:

\[ (2 - 2g)(F_g - \tilde{F}_g) = \sum_{i} \text{Res}_{z \rightarrow \alpha_i} A_{g,0,0}(z) \frac{y(z)}{dy(z)} \]
\[ \sum_i \text{Res}_z \left( C_{g;i} dx(z) dy(z) + D_{g;i}(z) dx(z) dy(z) \right) \frac{y(z)}{dy(z)} \]  

(3-24)

Since \( D_{g;i}(z) \) vanishes to order \( d_i + \tilde{d}_i + 1 \) while \( y(z)dx(z) \) has a pole of order \( d_i + \tilde{d}_i + 1 \), the second term \( D_{g;i}(z)y(z)dx(z) \) is regular at the pole \( \alpha_i \), so that:

\[ (2 - 2g) (F_g - \tilde{F}_g) = \sum_i C_{g;i} \text{Res}_{z \to \alpha_i} y(z)dx(z) \]

(3-25)

and, according to eq. (3-16)

\[ \sum_i t_i C_{g;i} = \sum_i t_i \int_0^{\alpha_i} \left[ \omega_{g,1}(z) + \tilde{\omega}_{g,1}(z) \right]. \]  

(3-26)

We find

\[ (2 - 2g) (F_g - \tilde{F}_g) = \sum_i t_i \int_0^{\alpha_i} \left[ \omega_{g,1}(z) + \tilde{\omega}_{g,1}(z) \right] \]

\[ = \sum_i t_i \int_0^{\alpha_i} \omega_{g,1}(z) - \sum_i \tilde{t}_i \int_0^{\alpha_i} \tilde{\omega}_{g,1}(z). \]  

(3-27)

This implies that

\[ F_g - \frac{1}{2 - 2g} \sum_i t_i \int_0^{\alpha_i} \omega_{g,1} = \tilde{F}_g - \frac{1}{2 - 2g} \sum_i \tilde{t}_i \int_0^{\alpha_i} \tilde{\omega}_{g,1} \]  

(3-28)

and thus:

**Theorem 3.1** The following quantity:

\[ \hat{F}_g(S) = F_g(S) - \frac{1}{2 - 2g} \sum_i \left( \text{Res}_{\alpha_i} \omega_{0,1}(S) \right) \left( \int_0^{\alpha_i} \omega_{g,1}(S) \right) \]  

(3-29)

(which is independent of a choice of a generic basepoint \( o \in C \)) is invariant under the exchange \( (x, y) \leftrightarrow (y, x) \):

\[ \hat{F}_g(S) = \hat{F}_g(S). \]  

(3-30)

## 4 Conclusion

We have completed the proof of the \( (x \leftrightarrow y) \) symmetry of [4], by including the integration constants. We see that \( \hat{F}_g = F_g + \) integration constants, is symplectic invariant, rather than \( F_g \).

Remark that in the context of the 2-matrix model, and their scaling limit which is the \((p,q)\) minimal models for which \( t_i = \text{Res}_y dy = 0 \), the integration constants were absent, and thus the \( F_g \)'s were indeed invariant. This proves that

\[ F_g((p,q)\text{ minimal model}) = F_g((q,p)\text{ minimal model}). \]  

(4-1)
Aknowledgements

This work was motivated by the remarks of Vincent Bouchard, who did check numerically (in particular in [2] with Piotr Sulkowski and in [1] with J. Hutchinson, P. Loliencar, M. Meiers and M. Rupert) the \((x \leftrightarrow y)\) symmetry on many examples, and empirically observed that the \(\hat{F}_g\)'s were invariant. After his remarks we quickly found the missing integration constants \(C_{g,i}\) from [4], and here is the correction. We are grateful to him. The work of B.E. is supported by the Quebec government by the FQRNT fund, and by the ERC starting grant Field-knots with P. Sulkowski.

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