Metric for two equal Kerr black holes

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Abstract

We show that the exact solution of Einstein’s equations describing a system of two aligned identical Kerr black holes separated by a massless strut follows straightforwardly from the extended 2-soliton solution possessing equatorial symmetry, and we give its concise analytic representation in terms of physical parameters.

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I. INTRODUCTION

In our recent paper [1] we have considered in detail a vacuum specialization of the general 2-soliton electrovac metric [2] (henceforth referred to as the MMR solution) in application to the description of the exterior geometry of neutron stars [3, 4]. Although we gave in that paper three different representations of the vacuum MMR solution, still we had one more representation of the latter solution that had been constructed by us some time ago for treating the two-body configurations of spinning black holes of Kerr's type [5]; however, we left its consideration for the future because of the specific objectives of the paper [1]. The appearance of a preprint [6] devoted to a system of corotating Kerr sources yet motivates us to publish our results on the vacuum MMR solution not earlier included into the paper [1], especially taking into account that although the authors of the paper [6] solve correctly the axis condition for the binary system, they still offer only a complicated form of the resulting metric, in the absence of some important details of the derivation that might be interesting to the reader. Our present work will aim therefore at working out a concise representation of the 3-parameter subfamily of the MMR solution describing a system of two equal Kerr black holes kept apart from falling onto each other by a massless strut [7], that would be alternative to the representation obtained in Ref. [6]. To accomplish this goal, we will first rewrite, using the procedure we have developed in a series of papers devoted to the binary black-hole configurations [8–11], the vacuum MMR metric in terms of physical parameters by taking as a starting point the axis data of the extended equatorially symmetric 2-soliton solution in the form

$$e(z) = \frac{z^2 - b_1 z + b_2}{z^2 + b_1 z + b_2}, \quad (1)$$

where $b_1$ and $b_2$ are two arbitrary complex constants, and a bar over a symbol means complex conjugation. The particular 3-parameter case of two separated Kerr black holes will then arise after imposing the axis condition in the general 4-parameter metric.

The paper is organized as follows. In the next section we perform a reparametrization of the data (1) in terms of the quantities $M$, $a$, $\sigma$ and $R$ related, respectively, to the masses of the sources, their angular momenta, the horizons' half-lengths and the coordinate distance between the centers of the sources. The reparametrized axis data is then used for writing out the MMR solution in a new concise representation with the aid of the general formulas of Ref. [13]. In Sec. III we solve the axis condition for the MMR solution and analyze the
resulting 3-parameter configuration of corotating Kerr black holes, thus confirming some
of the results of Ref. [6]. By expanding the expression of the interaction force in inverse
powers of $R$, we show in particular that the leading spin-spin repulsion term has precisely
the same form as was given earlier by Dietz and Hoenselaers [14] through the analysis of
two limiting cases of spinning particles. In Sec. IV we give the reparametrized form of the
extended 2-soliton metric suitable for treating the case of two non-equal Kerr black holes.
Sec. V contains concluding remarks.

II. YET ANOTHER REPRESENTATION OF THE VACUUM MMR SOLUTION

We would like to recall that the extended vacuum soliton solutions [13] constructed with
the aid of Sibgatullin’s integral method [15] are written in terms of the parameters $\alpha_n$ and
$\beta_l$, the former parameters taking real values or forming complex conjugate pairs (these
determine the location of sources on the symmetry axis), and the latter being roots of the
denominator in the axis data, hence taking arbitrary complex values.

In the 2-soliton case with the additional equatorial symmetry we have $\alpha_1 = -\alpha_4$, $\alpha_2 =
-\alpha_3$, so that the $\alpha$’s can be parametrized as

$$\alpha_1 = \frac{1}{2}R + \sigma, \quad \alpha_2 = \frac{1}{2}R - \sigma, \quad \alpha_3 = -\frac{1}{2}R + \sigma, \quad \alpha_4 = -\frac{1}{2}R - \sigma,$$

(2)
or, inversely,

$$R = \frac{1}{2}(\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4), \quad \sigma = \frac{1}{2}(\alpha_1 - \alpha_2) = \frac{1}{2}(\alpha_3 - \alpha_4),$$

(3)

where $R$ is the coordinate distance between the centers of black holes, and $\sigma$ is the half-
length of the horizon of each black hole (see Fig. 1). Note that $\sigma$ in the above formulas (2)
and (3), as well as throughout this paper, can also take on pure imaginary values, in which
case the solution would describe a pair of equal hyperextreme objects. However, except for
some special occurrences, below we will restrict our analysis to the black-hole configurations
only.

To identify the complex parameters $\beta_1$ and $\beta_2$, one has to introduce explicitly the axis
data — the value of the Ernst complex potential [16] on the upper part of the symmetry
axis. In our case such data is given by formula (1), and obviously can be cast into the
equivalent form

$$e(z) = \frac{z^2 - 2(M + ia)z + c + id}{z^2 + 2(M - ia)z + c - id}.$$
involving four arbitrary real constants $M$, $a$, $c$ and $d$. Since $\beta_1$ and $\beta_2$ are roots of the denominator on the right-hand side of (1), it is clear that these verify the relation $\beta_1 + \beta_2 = -2(M - ia)$ and $\beta_1\beta_2 = c - id$, while the denominator itself can be formally written as $(z - \beta_1)(z - \beta_2)$.

We must bear in mind that the parameters $\alpha_n$ in Sibgatullin’s method satisfy the equation

$$e(z) + \bar{e}(z) = 0,$$  

(5)

which means that if we want to introduce these $\alpha_n$ into the 2-soliton solution as arbitrary parameters in the form (2), then we have to solve the equation

$$e(z) + \bar{e}(z) = \frac{2(z - \alpha_1)(z - \alpha_2)(z - \alpha_3)(z - \alpha_4)}{(z - \beta_1)(z - \beta_2)(z - \bar{\beta}_1)(z - \bar{\beta}_2)},$$  

(6)

for the constants $c$ and $d$ by equating the coefficients at the same powers of $z$ on both sides of (6). A simple algebra then yields

$$c = -\frac{1}{4}R^2 + 2M^2 - 2a^2 - \sigma^2, \quad d = \epsilon \sqrt{(R^2 - 4M^2 + 4a^2)(\sigma^2 - M^2 + a^2)}, \quad \epsilon = \pm 1,$$  

(7)

with which the axis data (4) finally takes the form

$$e(z) = \frac{z^2 - 2(M + ia)z - \frac{1}{4}R^2 + 2M^2 - 2a^2 - \sigma^2 + id}{z^2 + 2(M - ia)z - \frac{1}{4}R^2 + 2M^2 - 2a^2 - \sigma^2 - id},$$  

(8)

where the constant quantity $d$ has been defined in (7).

Therefore, we have rewritten the axis data (1) containing the parameters $M$, $a$, $c$ and $d$ in the equivalent form (8) involving the desired set of the parameters $M$, $a$, $R$ and $\sigma$. It is worth noting that while the physical meaning of the constants $R$ and $\sigma$ is transparent, the interpretation of the parameters $M$ and $a$ can be revealed by calculating the solution’s total mass $M_T$ and total angular momentum $J_T$ from (8) with the help of the Fodor et al. procedure [17] for the evaluation of Geroch-Hansen multipole moments [18, 19]. Thus we get

$$M_T = 2M, \quad J_T = 4Ma - d,$$  

(9)

whence it follows immediately that $M$ is half the total mass of the configuration, whereas $a$ is the rotational parameter. Observe that $M$ does not coincide exactly with the mass of each black-hole constituent because the intermediate region $\{\rho = 0, |z| < \alpha_2\}$ in Fig. 1 may in principle carry some mass, positive or negative.
Once the axis data is worked out, the corresponding potential \( \mathcal{E} \) satisfying the Ernst equation \([16]\),
\[
(\mathcal{E} + \bar{\mathcal{E}})\Delta \mathcal{E} = 2(\nabla \mathcal{E})^2,
\]
(10)
can be obtained from the formula \([13]\)
\[
\mathcal{E} = \frac{E_+}{E_-}, \quad E_{\pm} = \pm \frac{1 \begin{vmatrix} 1 & 1 & 1 & 1 \\ r_1 & r_2 & r_3 & r_4 \\ \alpha_1 - \beta_1 & \alpha_2 - \beta_1 & \alpha_3 - \beta_1 & \alpha_4 - \beta_1 \\ \alpha_1 - \beta_2 & \alpha_2 - \beta_2 & \alpha_3 - \beta_2 & \alpha_4 - \beta_2 \\ 0 & 1 & 1 & 1 \\ 0 & \alpha_1 - \beta_1 & \alpha_2 - \beta_1 & \alpha_3 - \beta_1 & \alpha_4 - \beta_1 \\ 0 & \alpha_1 - \beta_2 & \alpha_2 - \beta_2 & \alpha_3 - \beta_2 & \alpha_4 - \beta_2 \end{vmatrix}}{1 \begin{vmatrix} 1 & 1 & 1 & 1 \\ \bar{r}_1 & \bar{r}_2 & \bar{r}_3 & \bar{r}_4 \\ \bar{\alpha}_1 - \bar{\beta}_1 & \bar{\alpha}_2 - \bar{\beta}_1 & \bar{\alpha}_3 - \bar{\beta}_1 & \bar{\alpha}_4 - \bar{\beta}_1 \\ \bar{\alpha}_1 - \bar{\beta}_2 & \bar{\alpha}_2 - \bar{\beta}_2 & \bar{\alpha}_3 - \bar{\beta}_2 & \bar{\alpha}_4 - \bar{\beta}_2 \\ 0 & 1 & 1 & 1 \\ 0 & \bar{\alpha}_1 - \bar{\beta}_1 & \bar{\alpha}_2 - \bar{\beta}_1 & \bar{\alpha}_3 - \bar{\beta}_1 & \bar{\alpha}_4 - \bar{\beta}_1 \\ 0 & \bar{\alpha}_1 - \bar{\beta}_2 & \bar{\alpha}_2 - \bar{\beta}_2 & \bar{\alpha}_3 - \bar{\beta}_2 & \bar{\alpha}_4 - \bar{\beta}_2 \end{vmatrix}},
\]
(11)
by just substituting the expressions of \( \alpha 's \) and \( \beta 's \) determined by (2) and (8) into (11), and taking into account that the functions \( r_n \), which depend on the coordinates \( \rho \) and \( z \), have the form \( r_n = \sqrt{\rho^2 + (z - \alpha_n)^2} \).

In the Ernst formalism \([16]\), the knowledge of the potential \( \mathcal{E} \) is sufficient for the construction of the corresponding metric functions \( f, \gamma \) and \( \omega \) from the stationary axisymmetric line element
\[
ds^2 = f^{-1}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2] - f(dt - \omega d\varphi)^2,
\]
(12)
and the explicit expressions for these functions defined by the potential (11) can be found in Ref. \([13]\) both in the form of determinants and in the expanded form most suitable for concrete computations and presentation of the results. Our own evaluation of \( \mathcal{E}, f, \gamma \) and \( \omega \) for the axis data (8) yields the following final formulas:
\[
\mathcal{E} = \frac{A - B}{A + B}, \quad f = \frac{A\bar{A} - B\bar{B}}{(A + B)(\bar{A} + \bar{B})}, \quad e^{2\gamma} = \frac{A\bar{A} - B\bar{B}}{K_0^2 R_+ R_- r_+ r_-},
\]
\[
\omega = 4a - \frac{2\text{Im}[G(\bar{A} + \bar{B})]}{A\bar{A} - B\bar{B}},
\]
\[
A = R^2(R_+ - R_-)(r_+ - r_-) - 4\sigma^2(R_+ - r_+)(R_- - r_-),
\]
\[
B = 2R\sigma[(R + 2\sigma)(R_- - r_+) - (R - 2\sigma)(R_+ - r_-)],
\]
\[
G = -zB + R\sigma[2R(r_- - r_+) + 4\sigma(R_+ R_- - r_+ r_-)
- (R^2 - 4\sigma^2)(R_+ - R_- - r_+ + r_-)],
\]
(13)
where
\[
R_\pm = \frac{-M(\pm 2\sigma + R) + id}{2M^2 + (R + 2ia)(\pm \sigma + ia)} \sqrt{\rho^2 + \left( z + \frac{1}{2}R \pm \sigma \right)^2},
\]
\[
r_\pm = \frac{-M(\pm 2\sigma - R) + id}{2M^2 - (R - 2ia)(\pm \sigma + ia)} \sqrt{\rho^2 + \left( z - \frac{1}{2}R \pm \sigma \right)^2},
\]
and
\[
K_0 = \frac{4R^2\sigma^2(R^2 - 4\sigma^2)[(R^2 + 4a^2)(\sigma^2 + a^2) - 4M(M^3 + ad)]}{[M^2(R + 2\sigma)^2 + d^2][M^2(R - 2\sigma)^2 + d^2]}.
\]

Eqs. (13)-(15) and (7) fully determine the desired representation of the 4-parameter vacuum MMR solution which, as will be seen in the next section, is very suitable for treating the case of two separated Kerr black holes. One can check by direct calculation that on the upper part of the symmetry axis \(\{ \rho = 0, z > \frac{1}{2}R + \sigma \}\) the potential \(E\) in (13) reduces to the axis data (8).

III. TWO IDENTICAL KERR BLACK HOLES SEPARATED BY A STRUT

The MMR solution discussed in the previous section can be interpreted as describing a pair of corotating Kerr black holes after subjecting its parameters to the constraint
\[
\omega = 0 \quad \text{for} \quad \rho = 0, \ |z| < \frac{1}{2}R - \sigma, \quad (16)
\]
which is known as the axis condition; this being satisfied, converts the region \(\{ \rho = 0, |z| < \frac{1}{2}R - \sigma \}\) into a massless conical singularity, a strut \(\square\), which separates the two black-hole constituents and prevents them from falling onto each other. In this special case, the parameter \(M\) becomes equal to the Komar mass \(\square\) of each constituent exactly, while the individual angular momentum \(J\) of each black hole becomes equal to \(J_T/2\) because the strut does not make contribution into the mass and angular momentum of the configuration.

On the symmetry axis, the metric function \(\omega\) of the 2-soliton metric takes constant values generically \(\square\), so that from the condition (16) we get a (complicated) algebraic equation for the parameters \(M, a, \sigma\) and \(R\), which nonetheless factorizes and eventually leads to the quadratic equation for \(\sigma\),
\[
(R^2 + 2MR + 4a^2)^2\sigma^2 - M^2R^2(R + 2M)^2
+a^2(R^2 - 4M^2 + 4a^2)(R^2 + 4MR - 4M^2 + 4a^2) = 0,
\]
(17)
with the positive root

$$\sigma = \sqrt{M^2 - a^2 + \frac{4M^2a^2(R^2 - 4M^2 + 4a^2)}{(R^2 + 2MR + 4a^2)^2}}, \quad (18)$$

which coincides with the expression for \(\sigma\) obtained in Ref. [6].

Taking into account (18), the constant quantity \(d\) from (7) assumes the form

$$d = \frac{2Ma(R^2 - 4M^2 + 4a^2)}{R^2 + 2MR + 4a^2}, \quad (19)$$

and this is exactly the quantity \(\delta\) from the paper [6]. The constant \(K_0\) from (15) rewrites, with account of (18) and (19), as

$$K_0 = \frac{4\sigma^2[(R^2 + 2MR + 4a^2)^2 - 16M^2a^2]}{M^2[(R + 2M)^2 + 4a^2]} \quad (20)$$

Mention that the above expression for \(d\) can be also used for writing \(\sigma\) in a slightly simpler form

$$\sigma = \sqrt{M^2 - a^2 + d^2(R^2 - 4M^2 + 4a^2)^{-1}}. \quad (21)$$

Therefore, the 3-parameter specialization of the MMR solution describing two equal corotating Kerr black holes separated by a strut is defined concisely by the formulas (13), (14) and (18)-(20). Apparently, our expressions for the Ernst potential and for all metric functions defining this subfamily are a good deal simpler than the ones obtained in Ref. [6].

On the horizons (the null hypersurfaces \(\rho = 0, -\sigma < z - \frac{1}{2}R < \sigma\) and \(\rho = 0, -\sigma < z + \frac{1}{2}R < \sigma\) — two thick rods in Fig. 1), the black-hole constituents of this binary configuration are expected to verify the well-known Smarr mass formula [22]

$$M = \frac{1}{4\pi}\kappa S + 2\Omega J, \quad (22)$$

where \(\kappa\) is the surface gravity, \(S\) the area of the horizon, \(\Omega\) the horizon’s angular velocity and \(J\) the Komar angular momentum of a black hole. Apparently, because of the equatorial symmetry of the problem, the relation (22) should be checked only for one of the constituents, say, for the upper one. Since the black holes are corotating, their Komar masses and angular momenta are both halves the respective total values, \(M_T\) and \(J_T\), determined by (9); hence, the mass of each black hole is \(M\), while the corresponding individual angular momentum \(J\) is given, as it follows from (9) and (19), by the expression

$$J = \frac{Ma[(R + 2M)^2 + 4a^2]}{R^2 + 2MR + 4a^2}, \quad (23)$$
and one can see that the inverse dependence $a(J)$ is defined by a cubic equation.

For the calculation of the quantities $\kappa$, $\Omega$, and $\rho$, the following formulas should be used [23]:

$$\kappa = \sqrt{-\omega H^{-2} e^{-2\gamma H}}, \quad \Omega = \omega H^{-1}, \quad S = 4\pi \sigma \kappa^{-1},$$  \hspace{1cm} (24)

where $\omega H$ and $\gamma H$ denote the values of the metric functions $\omega$ and $\gamma$ on the horizon. The straightforward calculations carried out for the upper black hole yield the following expression for the horizon’s angular velocity:

$$\Omega = \frac{(M - \sigma)(R^2 + 2MR + 4a^2)}{2Ma[(R + 2M)^2 + 4a^2]},$$  \hspace{1cm} (25)

while the quantities $S$ and $\kappa$ are defined by the formula [6]

$$S = \frac{4\pi \sigma}{\kappa} = \frac{8\pi M[(R + 2M)^2 + 4a^2][(R + 2M)(M + \sigma) - 4a^2]}{(R + 2\sigma)(R^2 + 2MR + 4a^2)}.$$  \hspace{1cm} (26)

Then it is easy to see that Smarr’s relation (22) is indeed verified by virtue of (23), (25) and (26).

Let us briefly comment on the possibility of the equilibrium without a strut between two corotating Kerr sources. If we denote by $\gamma_0$ the constant value of the metric function $\gamma$ on the strut, then the interaction force in our binary system can be found by means of the formula $F = (e^{-\gamma_0} - 1)/4$ [7, 24], thus yielding [6]

$$F = \frac{M^2[(R + 2M)^2 - 4a^2]}{(R^2 - 4M^2 + 4a^2)[(R + 2M)^2 + 4a^2]}.$$  \hspace{1cm} (27)

This force becomes zero at infinite separation of the constituents, and also when $|a| = (R + 2M)/2$. In the latter case, $\sigma$ becomes a pure imaginary quantity, which means that balance at finite separation is only possible between two hyperextreme Kerr sources; the value of the angular momentum leading to the equilibrium is $|J| = M(R + 2M)^2/(R + M)$, being characteristic of the Dietz-Hoenselaers equilibrium configuration [14].

In order to have a somewhat better idea about the interaction force in the generic case, it seems plausible to resort to some approximations in (27) for introducing the angular momentum $J$ explicitly. Then we readily get from (23) and (27) the following approximate formula for $F$ as $R \to \infty$:

$$F \approx \frac{M^2}{R^2} + \frac{4M^4 - 12J^2}{R^4} + \frac{80MJ^2}{R^6} + O\left(\frac{1}{R^6}\right).$$  \hspace{1cm} (28)

The form of the leading term in (28) responsible for the spin-spin interaction of corotating Kerr sources coincides with the one already given by Dietz and Hoenselaers [14] through the analysis of two limiting cases of spinning particles in the double-Kerr solution [25].
IV. TOWARDS THE DESCRIPTION OF TWO NON-EQUAL KERR BLACK HOLES

We will now outline a possible approach to treating the general case of interacting non-equal Kerr black holes which is likely to provide new information in the future about the spin-spin repulsion force in binary systems of rotating bodies. This approach consists in reparametrizing the general extended 2-soliton solution in the manner similar to the one already applied to the equatorially symmetric case in the previous sections. The starting point of such a procedure is the axis data of the form

\[ e(z) = \frac{z^2 + a_1 z + a_2}{z^2 + b_1 z + b_2}, \]  

where \(a_1, a_2, b_1\) and \(b_2\) are four arbitrary complex constants, together with the choice of the parameters \(\alpha_n\) of the extended soliton solution in the form slightly different from (2) (see Fig. 2),

\[ \alpha_1 = \frac{1}{2}R + \sigma_1, \quad \alpha_2 = \frac{1}{2}R - \sigma_1, \quad \alpha_3 = -\frac{1}{2}R + \sigma_2, \quad \alpha_4 = -\frac{1}{2}R - \sigma_2, \]  

\(\sigma_1\) and \(\sigma_2\) taking real or pure imaginary values (real \(\sigma\)’s, as usual, define black holes, while pure imaginary \(\sigma\)’s — the hyperextreme objects). The elimination of the angular momentum monopole parameter in (29) with the aid of the Fodor et al. method [17] and fixing the origin of coordinates by means of (30) reduces the number of arbitrary real parameters in the data (29) to six overall, and the procedure of introducing the parameters \(\alpha_n\) into the axis data described in Sec. I then leads to the following expression for the reparametrized data (29):

\[ e(z) = \frac{z^2 - (M + ia)z + c + id}{z^2 + (M - ia)z + g + ih}, \]  

where \(M\) is the total mass, \(a\) is the rotational parameter, while the constant quantities \(c, d, g\) and \(h\) are defined as follows:

\[ c = s - \mu, \quad g = s + \mu, \quad d = \frac{1}{4a}(\tau + \delta), \quad h = \frac{1}{4a}(\tau - \delta), \]  

with

\[ s = \frac{1}{4}[R^2 + 2(\sigma_1^2 + \sigma_2^2 - M^2 + a^2)]; \]
\[ \delta = \epsilon\sqrt{\tau^2 - \kappa}, \quad \epsilon = \pm 1, \]
\[ \tau = 2R(\sigma_1^2 - \sigma_2^2) - 4M\mu, \]
\[ \kappa = a^2[16(\mu^2 - s^2) + (R^2 - 4\sigma_1^2)(R^2 - 4\sigma_2^2)]. \]
The six arbitrary real parameters involved in the axis data (31) are hence \( M, a, R, \sigma_1, \sigma_2, \mu \), and one see that in the particular case \( \mu = 0, \sigma_1 = \sigma_2 = \sigma \) the data (31) reduces to the equatorially symmetric data (8), albeit a formal redefinition \( M \rightarrow 2M, a \rightarrow 2a \).

Using the general formulas of the paper [13], we have worked out the Ernst potential and the whole metric determined by the axis data (31) in the following concise form:

\[
\mathcal{E} = \frac{A - B}{A + B}, \quad f = \frac{A\bar{A} - B\bar{B}}{(A + B)(A + B)}, \quad e^{2\gamma} = \frac{A\bar{A} - B\bar{B}}{K_0 R_+ R_- r_+ r_-},
\]

\[
\omega = 2a - \frac{2\text{Im}[G(\bar{A} + \bar{B})]}{A\bar{A} - BB},
\]

\[
A = [R^2 - (\sigma_1 + \sigma_2)](R_+ - R_-)(r_+ - r_-) - 4\sigma_1 \sigma_2 (R_+ - r_-)(R_- - r_+),
\]

\[
B = 2\sigma_1 (R^2 - \sigma_1^2 + \sigma_2^2)(R_- - R_+) + 2\sigma_2 (R^2 + \sigma_1^2 - \sigma_2^2)(r_+ - r_-)
\]

\[
+ 4R\sigma_1 \sigma_2 (R_+ + R_- - r_+ - r_-),
\]

\[
G = -zB + \sigma_1 (R^2 - \sigma_1^2 + \sigma_2^2)(R_- - R_+)(r_+ + r_- + R)
\]

\[
+ \sigma_2 (R^2 + \sigma_1^2 - \sigma_2^2)(r_+ - r_-)(R_+ + R_- - R)
\]

\[
- 2\sigma_1 \sigma_2 \{2R[r_+ + r_- - R_+ R_- - \sigma_1 (r_+ - r_-) + \sigma_2 (R_- - R_+)]
\]

\[
+ (\sigma_1^2 - \sigma_2^2)(r_+ + r_- - R_+ - R_-)\} ,
\]

(34)

where the functions \( R_\pm \) and \( r_\pm \) are given by the expressions

\[
R_\pm = \frac{\delta + 2ia[M(\pm 2\sigma_2 + R) - 2\mu]}{\tau - ia[(\pm 2\sigma_2 + R)(\pm 2\sigma_2 + R + 2ia) + 4s]} \sqrt{\rho^2 + \left(z + \frac{1}{2}R \pm \sigma_2\right)^2},
\]

\[
r_\pm = \frac{\delta + 2ia[M(\pm 2\sigma_1 - R) - 2\mu]}{\tau - ia[(\pm 2\sigma_1 - R)(\pm 2\sigma_1 - R + 2ia) + 4s]} \sqrt{\rho^2 + \left(z - \frac{1}{2}R \pm \sigma_1\right)^2},
\]

(35)

and the choice of the constant \( K_0 \) in the formula for \( \gamma \) must preserve the asymptotic flatness of the solution.

In order to interpret the metric (34) as describing two unequal Kerr black holes, it is necessary to solve the condition \( \omega = 0 \) on the part \( \{ \rho = 0, -\frac{1}{2}R + \sigma_2 < z < \frac{1}{2}R - \sigma_1 \} \) of the \( z \)-axis. However, the bad thing is that, compared to the equatorially symmetric case, the resulting explicit form of the axis condition in the general case is extremely cumbersome, so that really very powerful computers are needed for being able to perform the required calculations in the analytical form. In spite of that, the numerical analysis of the axis condition suggests that the analytical treatment of the general case is still possible in principle because this condition leads to the quartic algebraic equation for the parameter \( \mu \). We do
not exclude that some clever redefinitions of the parameters or fortunate substitutions might cause the factorization of the axis condition and the eventual resolution of the problem in a relatively compact form on the basis of the metric (34). But the accomplishment of this technically very complicated mission will remain a task for the future.

V. CONCLUSION

Therefore, we have shown that the vacuum MMR solution is very fit for the analytical description and study of the binary configuration of corotating identical Kerr black holes, for which we have worked out a concise representation that improves the one obtained in Ref. [6]. We have restricted our consideration exclusively to the case of the non-extreme constituents because the extreme case of two equal or non-equal Kerr black holes is described by a subclass of the well-known Kinnersley-Chitre solution [26] which was already identified and discussed in our earlier work [27].

We are convinced that in order to get a better insight into the nature of the spin-spin interaction, future research should be more concentrated on the configurations of non-equal spinning bodies because, apparently, the cases of identical constituents can be considered as degenerations of the respective generic cases and hence could in principle hide some important information about the real strength of the spin-spin repulsion or attraction. In this respect, a good understanding of the systems of identical spinning bodies is certainly necessary and brings us closer to the description of more sophisticated binary configurations that arise, for instance, within the framework of the general 2-soliton spacetime [34].

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[1] V. S. Manko and E. Ruiz, Phys. Rev. D 93, 104051 (2016).
[2] V. S. Manko, J. Martín, and E. Ruiz, J. Math. Phys. 36, 3063 (1995).
[3] N. Stergioulas, Living Rev. Relativ. 6, 3 (2003).
[4] G. Pappas and T. A. Apostolatos, Mon. Not. R. Astron. Soc. 429, 3007 (2013).
[5] R. P. Kerr, Phys. Rev. Lett. 11, 237 (1963).
[6] I. Cabrera-Munguia, V. E. Ceron, L. A. López, and O. Pedraza, arXiv:1702.02209.
[7] W. Israel, Phys. Rev. D 15, 935 (1977).
[8] V. S. Manko, Phys. Rev. D 76, 124032 (2007).
[9] V. S. Manko, E. D. Rodchenko, E. Ruiz, and B. I. Sadovnikov, Phys. Rev. D 78, 124014 (2008).
[10] V. S. Manko, R. I. Rabadán, and E. Ruiz, Class. Quantum Grav. 30, 145005 (2013).
[11] V. S. Manko, R. I. Rabadán, and J. D. Sanabria-Gómez, Phys. Rev. D 89, 064049 (2014).
[12] F. J. Ernst, V. S. Manko and E. Ruiz, Class. Quantum Grav. 24, 2193 (2007).
[13] V. S. Manko and E. Ruiz, Class. Quantum Grav. 15, 2007 (1998).
[14] W. Dietz and C. Hoenselaers, Ann. Phys. (NY) 165, 319 (1985).
[15] N. R. Sibgatullin, Oscillations and Waves in Strong Gravitational and Electromagnetic Fields (Berlin: Springer, 1991); V. S. Manko and N. R. Sibgatullin, Class. Quantum Grav. 10, 1383 (1993).
[16] F. J. Ernst, Phys. Rev. 167, 1175 (1968).
[17] G. Fodor, C. Hoenselaers, and Z. Perjés, J. Math. Phys. 30, 2252 (1989).
[18] R. Geroch, J. Math. Phys. 13, 394 (1972).
[19] R. O. Hansen, J. Math. Phys. 15, 46 (1974).
[20] A. Komar, Phys. Rev. 113, 934 (1959).
[21] A. Tomimatsu and M. Kihara, Prog. Theor. Phys. 67, 1406 (1982).
[22] L. Smarr, Phys. Rev. Lett. 30, 71 (1973).
[23] A. Tomimatsu, Prog. Theor. Phys. 72, 73 (1984).
[24] G. Weinstein, Comm. Pure Appl. Math. 43, 903 (1990).
[25] D. Kramer and G. Neugebauer, Phys. Lett. A 75, 259 (1980).
[26] W. Kinnersley and D. M. Chitre, J. Math. Phys. 19, 2037 (1978).
[27] V. S. Manko and E. Ruiz, Prog. Theor. Phys. 125, 1241 (2011).
FIG. 1: Location of two identical Kerr black holes on the symmetry axis: $\alpha_4 = -\alpha_1$, $\alpha_3 = -\alpha_2$. 
FIG. 2: Location of two non-equal Kerr black holes on the symmetry axis.