Schwinger’s Principle and Gauge Fixing in the Free Electromagnetic Field

C. A. M. de Melo\textsuperscript{1,*}, B. M. Pimentel\textsuperscript{1,**} and P. J. Pompeia\textsuperscript{1,2,***}

\textsuperscript{1}Instituto de Física Teórica
Universidade Estadual Paulista
Rua Pamplona,145
01405-900 - São Paulo, S.P.
Brazil

\textsuperscript{2}Centro Técnico Aeroespacial
Instituto de Fomento e Coordenação Industrial
Divisão de Confiabilidade Metrológica
Praça Marechal Eduardo Gomes, 50
12228-901 - São José dos Campos, S.P.
Brazil

A manifestly covariant treatment of the free quantum electromagnetic field, in a linear covariant gauge, is implemented employing the Schwinger’s Variational Principle and the B-field formalism. It is also discussed the abelian Proca’s model as an example of a system without constraints.

§1. Introduction

The covariant quantization of the electromagnetic field is one of the most peculiar problems of Quantum Field Theory because of the masslessness of the photon. In spite of its vectorial nature, only the two transverse components of the photon are observable, and the third freedom yields the Coulomb interaction between charged particles. The first consistent manifestly covariant quantization of the electromagnetic field was formulated by Gupta\textsuperscript{1)} and independently by Bleuler,\textsuperscript{2)} describing the photon in the Fermi gauge and the Lorentz condition not being an operator identity but a restriction which is imposed on the physical states. As it is well known, in their theory, the state vectors do not necessarily have positive norm, and the space spanned by them is an indefinite metric Hilbert space. It is often convenient theoretically to use an gauge independent quantization, however, such a method is not known nowadays. Concerning to covariance, Schwinger has proposed a general principle that describes the Quantum Field Theory in a manifestly covariant way.

Schwinger’s Variational Principle was implemented in 1951\textsuperscript{3)} as a powerful method to extract the kinematical (commutation relations) and dynamical (equations of motion) qualities from only one principle in a self-consistent and covariant manner.

\textsuperscript{*} email: cassius@ift.unesp.br

\textsuperscript{**} email: pimentel@ift.unesp.br

\textsuperscript{***} email: pompeia@ift.unesp.br
With this method Schwinger wished to systematize the Quantum Electrodynamics (QED) in such a way that the various developments and results could be obtained from a theory of the fundamental processes involved.

On the other hand, QED is a Gauge Theory with non-trivial dynamic constraints among its degrees of freedom, and therefore it is necessary a careful analysis in the establishment of its degrees of freedom and the consequences of this symmetry. A particular and profitable way to broach the freedom of gauge of the free electromagnetic field was established by Nakanishi in 1966-67 for covariant linear gauges, through an auxiliary scalar field $B$. In a series of subsequent works, Nakanishi and collaborators applied this formalism to analyse several systems as General Relativity, Proca’s field and the Model of Higgs.

In this paper, we intend to do a brief analysis of the free electromagnetic field employing Schwinger’s Variational Principle, using Palatini’s variation method, and the formalism of Nakanishi’s auxiliary field to treat the freedom of gauge of the system. The Palatini’s method of variation was originally developed in the context of the General Relativity Theory as a manner of finding a coupled system of first order equations for the gravitational potential $g_{\mu\nu}$ and its “momentum” $\Gamma^\rho_{\mu\nu}$.

In section 2 we will give a brief description of Schwinger’s Principle. Next, in section 3 we make an application to free electromagnetic field using the formalism of auxiliary field for covariant linear gauges, obtaining the equations of motion and commutation relations. To make a comparison, we make the same analysis to the vectorial massive field in section 4. To determine the dynamical degrees of freedom of the system, we analyse the Cauchy Problem of the both systems in section 5. At last, we make some comments, showing our conclusions.

§2. Schwinger’s Variational Principle

In Schwinger’s approach, the problem of the relativistic quantum dynamic is to determine the transformation function $\langle a^2, \sigma^2 | a^1, \sigma^1 \rangle$, where $\sigma^1$ and $\sigma^2$ are two distinct space-like surfaces, and $a^1$ and $a^2$ are the eigenvalues of a complete set $A$ that describe the system, associated to each surface respectively.

The fundamental issue of this formulation of Quantum Field Theory is the Schwinger’s Variational Principle, that can be written as

$$\delta \langle a^2, \sigma^2 | a^1, \sigma^1 \rangle = i \langle a^2, \sigma^2 | \delta S | a^1, \sigma^1 \rangle = i \langle a^2, \sigma^2 | \delta \left[ \int_{\sigma^1}^{\sigma^2} d\Omega L(x_a(x), x) \right] | a^1, \sigma^1 \rangle =$$

$$= i \langle a^2, \sigma^2 | \left( \int_{\sigma^2} - \int_{\sigma^1} \right) d\sigma G^\mu(\sigma) | a^1, \sigma^1 \rangle ,$$

where $L$ is the Lagrangian density operator that describe the system of interest and $G^\mu(\sigma)$ is the infinitesimal generator of general variations on the states. This generator may contain information on kinematical and dynamical aspects of the system, farther on symmetries. Moreover, in virtue of the invariance of the structure of the Variational Principle under an addition of a total divergence,

$$L \rightarrow \bar{L} = L + \partial_\mu A^\mu$$
we can change the generator and therefore choose the most convenient representation for the calculations\(^3\). In practical employments of the above expression, it maybe necessary the use of boundary conditions for the fields on the limits of space-like surfaces, or, equivalently, on a time-like surface connecting both \(\sigma\) surfaces, that delimit the space-time volume of interest. This principle is the starting point to several developments, such as: commutation relations, equations of motion, perturbation theory, construction of propagators, scattering, etc. Moreover, it can be shown that the canonical commutation relations obtained from the variational principle are covariant, and also give us the infinitesimal generators of Lorentz transformations:\(^3, 10\)

\[ G^\mu \rightarrow \bar{G}^\mu = G^\mu + \Lambda^\mu \]

\section{B-Field Formalism and Free Electromagnetic Field}

To implement the gauge fixing of the free electromagnetic field in a covariant manner, we introduce an hermitian auxiliary scalar field \(B (x)\). The introduction of this new field was inspired in the generalized canonical dynamics proposed by Utiyama.\(^11\) Therefore, suppose the following Lagrangian density operator\(^12\) for the electromagnetic field with an hermitian auxiliary scalar field, \(B (x)\):

\[
L = -\frac{1}{2} \left\{ F^{\mu\nu}, \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu) \right\} + \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} \left\{ B, \partial_\mu A_\mu \right\} + \frac{1}{2} \alpha B^2
\]

where \(\{ a, b \}\) express the anti-commutator of the field operators \(a\) and \(b\); \(\alpha\) is a fixed scalar parameter, and it is implicit the hypothesis that the hermitian field operators \(A_\mu, F^{\mu\nu}\) and \(B\) are independent, as proposed by Palatini’s method.\(^8\) The field operators are assumed to be analytical applications of \(M^4 \equiv \mathbb{R}^{(3,1)}\) (where \(x^\mu\) is not quantized) in the functional space of operators, in such a way that the raising and lowering of indexes are made by the non-quantized Minkowski’s metric \(\eta^{\mu\nu}\) with signature \((+, -, -, -)\).

This Lagrangian density operator is gauge invariant since

\[
F^{\mu\nu} \rightarrow F^{\mu\nu}, \quad B \rightarrow B, \quad A_\mu \rightarrow A_\mu + \partial_\mu \Phi
\]

\[
L \rightarrow L + \Box \Phi
\]

where \(\partial_\mu \partial^\mu \Phi = \Box \Phi \equiv 0\) and \(\Phi (x)\) is a scalar field operator.

If we consider only variations at fixed point, we have

\[
\delta L = \frac{1}{2} \partial_\mu \left\{ F^{[\nu\rho]} + B \eta^{\mu\rho}, \delta A_\nu \right\} - \frac{1}{4} \left\{ \delta F^{\mu\nu}, (\partial_\mu A_\nu - \partial_\nu A_\mu) - F_{\mu\nu} \right\} +
\]

\[
+ \frac{1}{2} \left\{ \partial_\mu F^{[\nu\rho]} - \partial^\rho B, \delta A_\nu \right\} + \frac{1}{2} \left\{ \delta B, \eta^{\mu\rho} \partial_\nu A_\mu + \alpha B \right\},
\]

\(^3\) Naturally this choice alters our interpretation of the generator, and it results in different functional relations among the fundamental operators.

\(^**\) It is interesting to note that in his original paper Nakanishi has proposed a kinetic term proportional to \((\partial_\mu B) (\partial^\mu B)\) which had no effect in the canonical commutation relations and equations of motion, and therefore no kinematical or dynamical consequences.
where we use a compact notation, \( F^{[\nu \mu]} \equiv \frac{1}{2} (F^{\nu \mu} - F^{\mu \nu}) \). Integrating the variation of \( \delta L \) in a volume \( \Omega \) we see that the first term of the second member can be transformed in a surface’s integral with the support of Gauss’s theorem. The other terms give us the equations of motion for the fields

\[
\begin{align*}
\partial_\mu F^{[\mu \nu]} - \partial^\nu B &= 0 \\
\partial_\mu A_\nu - \partial_\nu A_\mu &= F_{\mu \nu} \\
\partial^\mu A_\mu + \alpha B &= 0
\end{align*}
\] (3.1-3.3)

We know that it is exactly the surface term that gives us the generator of infinitesimal field transformations, \( G \).

\[
G_2 - G_1 = \left( \int_{\sigma_1} - \int_{\sigma_2} \right) d\sigma^\mu \left[ \frac{1}{2} \left\{ F^{[\nu \mu]} (\vec{x}) + B (\vec{x}) \eta^{\nu \mu}, \delta A_\nu (\vec{x}) \right\} \right],
\]

where \( (x - \bar{x})^2 < 0 \).

With the supposition that \( \delta A_\nu \) is proportional to identity, which means that \( \delta A_\nu \) commutes with all other operators, we have

\[
\delta A_\lambda (x) = -i [A_\lambda (x), G],
\]

where \( \eta^{\nu \mu} \) is proportional to identity.

Making a choice of frame in such a way that \( x^0 = \bar{x}^0 \), we see that the only possible solution for these equations is that the fields satisfy, over the surface \( \sigma_0 \), the identities

\[
\left[ A_\lambda (x), \left( F^{[\nu 0]} (\bar{x}) + \eta^{\nu 0} B (\bar{x}) \right) \right] = i \delta_\nu^\lambda \delta (x - \bar{x}),
\]

the suffix 0 in the commutator indicates that the involved operators are both considered at the same instant of time, fixed by the choice of a space-like hypersurface. This expression show us that the canonical momentum conjugated to \( A_\lambda \) must be \( F^{[\nu 0]} + \eta^{\nu 0} B \).

Employing the generator of the functional variations we find the induced variation on the \( F^{\alpha \beta} \) and \( B \) operators. We can still consider the independence of symmetrical and antisymmetrical parts of \( F^{\alpha \beta} \), and see that

\[
\delta \pi^{\alpha \beta} (x) = -i \int d\sigma^\mu \left[ \delta^{\alpha \beta} (x), \pi^{\nu \mu} (\bar{x}) \right] \delta A_\nu (\bar{x}).
\]

On the other hand, adding a surface term we have the following generator of functional variations:

\[
\bar{G} = - \int d\sigma^\mu \frac{1}{2} \left\{ \delta F^{[\nu \mu]} + \delta B \eta^{\nu \mu}, A_\nu \right\} = - \int d\sigma^\mu \frac{1}{2} \left\{ \delta \pi^{\nu \mu}, A_\nu \right\}.
\]
The induced variations using the same procedure will become

\[
\bar{\delta}A_\lambda (x) = i \int d\sigma^\bar{x}_0 [A_\lambda (x), A_\nu (\bar{x})] \delta \pi^\nu (\bar{x}) \tag{3.6}
\]

\[
\bar{\delta}\pi^{\alpha \beta} (x) = i \int d\sigma^\bar{x}_0 [\pi^{\alpha \beta} (x), A_\nu (\bar{x})] \delta \pi^\nu (\bar{x}) \tag{3.7}
\]

The four equations (3.4-3.7) constitute a system of functional relations. If we take only the \(\beta = 0\) components, and impose a kinematical independence between \(\pi^\nu = \pi^{\nu 0} = F^{[\nu \bar{0}]} + B^{\nu \bar{0}}\) and \(A_\nu\),

\[
\delta A_\lambda (x) = -i \int d\sigma^\bar{x}_0 [A_\lambda (x), \pi^\nu (\bar{x})] \delta A_\nu (\bar{x})
\]

\[i \int d\sigma^\bar{x}_0 [A_\lambda (x), A_\nu (\bar{x})] \delta \pi^\nu (\bar{x}) = -i \int d\sigma^\bar{x}_0 [\pi^\alpha (x), \pi^\nu (\bar{x})] \delta A_\nu (\bar{x}) = 0
\]

\[\delta \pi^\alpha (x) = i \int d\sigma^\bar{x}_0 [\pi^\alpha (x), A_\nu (\bar{x})] \delta \pi^\nu (\bar{x})
\]

One of the possible solutions of this functional relation system is

\[
[\pi^\alpha (x), \pi^\nu (\bar{x})]_0 = [A_\lambda (x), A_\nu (\bar{x})]_0 = 0
\]

\[
[A_\lambda (x), \pi^\nu (\bar{x})]_0 = i \delta^\nu_\lambda \delta (x - \bar{x})
\]

§4. Proca’s Field

The massive term of the Lagrangian density operator can be written as

\[
L_P = \frac{m^2}{2} A_\mu A^\mu \quad \Rightarrow \quad \delta L_P = \frac{m^2}{2} (\delta A_\mu A^\mu + A_\mu \delta A^\mu) = \frac{m^2}{2} \{A^\mu, \delta A_\mu\}
\]

where \(m\) is the parameter of mass of Proca’s Field.

Note that this variation does not contain a surface term. Then, we see that the only equation of motion for the Proca’s Field that changes is

\[
\partial_\mu F^{[\mu \nu]} - \partial^\nu B + m^2 A^\nu = 0 \tag{4.1}
\]

and the generator of functional variations of fields is

\[
G = \int d\sigma_\mu \left( F^{[\mu \nu]} + B^{\nu \bar{\mu}} \right) \delta A_\nu
\]

The above generator is the same as we obtained before with the free electromagnetic field, then the commutation relations for Proca’s Field are also the same.

§5. Dynamical Variables

Now we will try to determine the dynamical quantities for both systems studied before using the analysis of Cauchy’s data. This is a powerful instrument to separate the dynamical and constrained sectors of a theory, making use only of the equations of motion of the system, and so it can be applied even in the case of a system of linear field operators, as it was considered here.
5.1. Proca’s Field

Let us look again the equations of motion of Proca’s Field. Substituting (3.2) in (4.1), we have
\[ \partial_{\mu} \partial^{\mu} A^{\nu} - \partial_{\nu} \partial^{\mu} A^{\mu} - \partial^{\nu} B + m^2 A^{\nu} = 0 \] (5.1)

Now, substituting (5.3) in the above equation,
\[ \Box A^{\nu} + m^2 A^{\nu} - (1 - \alpha) \partial^{\nu} B = 0. \] (5.2)

Taking the four-divergence of (5.1) we find
\[ \Box B - m^2 \partial^{\nu} A^{\nu} = 0 \] (5.3)

If we take the gradient of equation (8.3) and substitute in (5.1) multiplied by \( \alpha \) we see that
\[ \alpha (\Box A^{\nu} + m^2 A^{\nu}) + (1 - \alpha) \partial^{\nu} \partial_{\mu} A^{\mu} = 0 \] (5.4)

Multiplying (3.3) by \( m^2 \) and substituting in (5.3),
\[ \Box B + \alpha m^2 B = 0 \] (5.5)

From this last equation, we can see directly that the auxiliary scalar field \( B \) is a dynamical quantity. On the other hand, the 0 component of equation (5.4) can be written as
\[ \partial^0 \partial_0 A^0 = -(1 - \alpha) \partial^0 \partial_i A^i - \alpha (\partial^i \partial_i A^0 + m^2 A^0) \]

Once that the above equation involves temporal derivatives of second order of \( A^0 \), and derivatives of the components \( A^i \), we need to check if these last are dynamical quantities.

So, taking the \( k \) component of equation (5.4), we obtain
\[ \partial^0 \partial_0 A^k = -\partial^i \partial_i A^k - m^2 A^k - \frac{(1 - \alpha)}{\alpha} \partial^k (\partial_0 A^0 + \partial_i A^i) \]

that show us that, for \( \alpha \neq 0 \), we can determine the \( A^k \) as dynamical quantities.

Since the \( A^k \) are established, we may affirm that \( A^0 \) is also a dynamical quantity. Thus, it becomes clear that Proca’s Field is a system whithout constraints. This result could also be obtained in a more direct manner (which includes the case \( \alpha = 0 \)) making use of equations (5.2) and (5.5). Once that (5.5) tell us that \( B \) is a dynamical quantity, we can isolate the second temporal derivative of \( A^\nu \) in (5.2):
\[ \partial^0 \partial_0 A^\nu = -\partial^i \partial_i A^\nu + (1 - \alpha) \partial^\nu B - m^2 A^\nu \]

and with the knowledge of \( B, A^\nu, \) and their first order temporal derivatives in a given instant, we are able to determine these fields in any future moment.

5.2. Electromagnetic Field

Following the same steps used in the case of Proca’s Field, but using (3.1) instead of (4.1), we obtain
\[ \Box B = 0 \]

\[ \partial^0 \partial_0 A^0 = \alpha \partial_i (\partial^0 A^i - \partial^i A^0) - \partial^0 \partial_i A^i \]

\[ \partial^0 \partial_0 A^k = -\partial^\nu \partial_\nu A^k - \frac{(1 - \alpha)}{\alpha} \partial^k (\partial_0 A^0 + \partial_i A^i) \]

This system of equations show us that the evolution in time of any component of \( A^\mu \) is coupled to evolution of the others components, what is a consequence of the constraint given by the auxiliary field \( B \) - see equation (3.3). It shows us that the five quantities \( B, A^\nu \) are dynamical in the case \( \alpha \neq 0 \).

As we made before in the case of Proca’s Field, we can also obtain the second order temporal derivatives of all components of \( A^\mu \) as a function of derivatives of \( B \), directly from the equation

\[ \Box A^\nu - (1 - \alpha) \partial^\nu B = 0 \]

So, we see that Nakanishi’s formalism gives us a good limit for the massless field from Proca’s equations. We can affirm this once that all results for the free electromagnetic field may be obtained from the massive case, making \( m = 0 \).

§6. Final Remarks

The union of B-field, Schwinger and Palatini’s formalisms have become more fertile since that an equivalent form of Lorentz condition of Classical Electrodynamics can be found as a representative identity between operators \( (3.3) \), which implies in a wave equation for the expected values of the operator \( A^\nu \) between the physical states.\(^4\) We also see that the equations of motion, as well as the commutation relations for the field operators, can be obtained in a covariant form from an unique variational principle, being, in that way, automatically self-consistent.

Another advantage is that this procedure may be generalized in a natural manner by the inclusion of source terms or spinorial fields in the Lagrangian density operator, in order to construct a theory of Quantum Electrodynamics in a general linear covariant gauge.

The analysis of the constraints of the theory can be implemented through Cauchy’s Data,\(^13\) and indicate that the B-field formalism provide a good limit for the massless vectorial field from the Proca’s field.\(^6\)

In future perspectives, we intend to implement the same method to make a preliminary study of quantized gravitational field.

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References

1) Gupta, S. N. - *Proc. Phys. Soc. (London)* A63 (1950), 681.
2) Bleuler, K. - *Helv. Phys. Acta* 23 (1950), 567.
3) Schwinger, J. S. - *Phys. Rev.* 82 (1951), 914; *Phys. Rev.* 91 (1953), 713; *Phys. Rev.* 91 (1953), 728; *Phys. Rev.* 92 (1953), 1283; *Phys. Rev.* 93 (1954), 615; *Phys. Rev.* 94 (1954), 1362.
4) Nakanishi, N. - *Prog. Theor. Phys.* 35 (1966), 1111; *Prog. Theor. Phys.* 38 (1967), 881.
5) Nakanishi, N. - *Prog. Theor. Phys.* 59 (1978), 972.
6) Nakanishi, N. - *Suppl. of the Prog. Theor. Phys.* 51 (1972), 1; Ghose & Das, A. - *Nucl. Phys.* 84 (1972), 299.
7) Nakanishi, N. - *Prog. Theor. Phys.* 49 (1973), 640.
8) Palatini, A. - *Rendiconti del Circolo Matematico di Palermo* 43 (1919), 203; Misner, C.W., Thorne, K.S. & Wheeler, J.A. - *Gravitation*, W.H. Freeman & Company, New York, (1995) - chap. 21.
9) Milton, K. - *A Quantum Legacy - Seminal Papers of Julian Schwinger*, World Scientific, Singapore, (2000) - 8.
10) Schwinger, J. S. - *Phys. Rev.* 74 (1948), 1439.
11) Utiyama, R. - *Prog. Theor. Phys. Suppl.* No. 9 (1959), 19.
12) Nakanishi, N. & Ojima, I. - *Covariant Operator Formalism of Gauge Theories and Quantum Gravity*, World Scientific, Lecture Notes in Physics, Vol.27 (1991).
13) Felsager, B. - *Geometry, Particles, and Fields*, Springer-Verlag, New York, (2001).