Birkhoff strata of the Grassmannian $\text{Gr}^{(2)}$: Algebraic curves

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Abstract

Algebraic varieties and curves arising in Birkhoff strata of the Sato Grassmannian $\text{Gr}^{(2)}$ are studied. It is shown that the big cell $\Sigma_0$ contains the tower of families of the normal rational curves of all odd orders. Strata $\Sigma_{2n}$, $n = 1, 2, 3, \ldots$ have hyperelliptic subsets $W_{2n}$, with the points containing hyperelliptic curves of genus $n$ and their coordinate rings. Strata $\Sigma_{2n+1}$, $n = 0, 1, 2, 3, \ldots$ contain $(2m+1, 2m+3)$--plane curves for $n = 2m, 2m+1$ ($m \geq 2$) and $(3, 4)$ and $(3, 5)$ curves in $\Sigma_3$, $\Sigma_5$ respectively. Curves in the strata $\Sigma_{2n+1}$ have zero genus.

1 Introduction

Grassmannian $\text{Gr}^{(2)}$ is a very important specialization of the universal Sato Grassmannian [1]. The most known its appearance is due to the connection with the theory of the KdV equation [2, 3]. The present paper is devoted to the study of the Grassmannian $\text{Gr}^{(2)}$ within the framework proposed recently in [4]. The main idea of this approach is to analyze algebro-geometric structures arising in Sato Grassmannian, in our case in the Birkhoff strata of $\text{Gr}^{(2)}$, without any a priori reference to any integrable system.

Recall that Sato Grassmannian $\text{Gr}$ can be viewed as the set of closed vector subspaces in the infinite dimensional set of all formal Laurent series with coefficients in $\mathbb{C}$ with certain special properties (see e.g. [2, 3]). Each subset $W \subset \text{Gr}$ contains points possessing an algebraic basis $(w_0(z), w_1(z), w_2(z), \ldots)$ where

$$w_n = \sum_{k=-\infty}^{n} a_k^2 z^k$$

of finite order $n$. Grassmannian $\text{Gr}$ is a connected Banach manifold which exhibits a stratified structure [2, 3], i.e. $\text{Gr} = \bigcup_S \Sigma_S$ where the stratum $\Sigma_S$ is a subset in $\text{Gr}$ formed by elements of the form [1] such that possible values $n$ are given by the infinite set $S = \{s_0, s_1, s_2, \ldots\}$ of integers $s_n$ with $s_0 < s_1 < s_2 < \ldots$ and $s_n = n$ for large $n$. Big cell $\Sigma_0$ corresponds to $S = \{0, 1, 2, \ldots\}$. Other strata are associated with the sets $S$ different from $S_0$.

$\text{Gr}^{(2)}$ is the subset of elements $W$ of $\text{Gr}$ obeying the condition $z^2 \cdot W \subset W$ [2, 3]. This condition imposes strong constraints on the Laurent series and on the structure of the strata. Namely, Birkhoff stratum $\Sigma_S$ in $\text{Gr}^{(2)}$ corresponds to the sets $S$ such that $S + 2 \subset S$, i.e. all possible $S$ having the form [2, 3]

$$S_m = \{-m, -m+2, -m+4, \ldots, m, m+1, m+2, \ldots\}$$

with $m = 0, 1, 2, \ldots$. Codimension of $\Sigma_m$ is $m(m+1)/2$. One has $\text{Gr}^{(2)} = \bigcup_{m \geq 0} \Sigma_m$.

In this paper, using the properties of the Birkhoff strata $\text{Gr}^{(2)}$, we show that the big cell $\Sigma_0$ contains a maximal closed subset $W_0$ which geometrically is a tower of infinite families of rational normal (Veronese) curves of all odd orders. It is demonstrated that the strata $\Sigma_{2n}$, $n = 1, 2, \ldots$ contain subsets $W_{2n}$ closed with respect to pointwise multiplication if the coefficients of Laurent series $w_n$ obey certain associativity constraints. Geometrically the subsets $W_{2n}$ represent infinite families of coordinate rings for the hyperelliptic curves of genus $n$. Each point of the subset $W_{2n}$ contains hyperelliptic curves and its
coordinate rings. Then it is shown that the strata $\Sigma_3$ and $\Sigma_5$ contain $(3,4)$ and $(3,5)$ degenerate plane
curves respectively. In the strata $\Sigma_{2m+1}$, $m \geq 2$ one has families of $(2m + 1, 2m + 3)$ plane curves of zero
genus.

In the second part of this work $[5]$ the tangent cohomology of the subsets $W_n$, and associated integrable
systems of hydrodynamical type will be studied.

The paper is organized as follows. The big cell is discussed in section 2. Stratum $\Sigma_1$ is considered in
section 3. Closed subsets $W_2$ in the stratum $\Sigma_2$ and corresponding elliptic curves are studied in section 4.
Stratum $\Sigma_3$ and associated $(3,4)$ curves are analysed in section 5. Section 6 is devoted to general strata
$\Sigma_{2n}$, $(n = 2, 3, 4, \ldots)$. Stratum $\Sigma_5$ and the generic strata $\Sigma_{2n+1}$, $(n = 3, 4, \ldots)$ are discussed in section
7.

2 Big cell

The principal stratum $\Sigma_0$ for which $S = \{0, 1, 2, \ldots\}$ (called also big cell) is a dense open set and it has
codimension zero$[2][3]$. It possesses a canonical basis $(p_0, p_1, p_2, \ldots)$ where

$$p_i(z) = z^i + \sum_{k \geq 1} \frac{H_k^i}{z^k}, \quad i = 0, 1, 2, \ldots \quad (3)$$

with arbitrary $H_k^i$.

Accordingly to the approach proposed in $[4]$ we first look for a subset $W_0 \subset \Sigma_0$ closed with respect
to multiplication. Similar to the big cell in the general Gr one has

Lemma 2.1 Laurent series $[4]$ at fixed $H_k^i$ obey the condition $z^2W_0 \subset W_0$ and the equations

$$p_j(z)p_k(z) = \sum_{l \geq 0} C^l_{jk}p_l(z) \quad (4)$$

if and only if

$$H_{2i}^{2n} = 0, \quad i = 1, 2, 3, \ldots, n = 0, 1, 2, \ldots, \quad (5)$$

and

$$H_{2i}^{2m+1} = H_{2k+1}^{2(m+n)+1} - \sum_{s=0}^{n-1} H_{2s+1}^{2m+1}H_{2k+1}^{2(n-s)-1} = 0, \quad (6)$$

$$H_{2i}^{2m+1} + H_{2k+1}^{2(m+n)+1} + \sum_{l=0}^{k-1} H_{2l+1}^{2m+1}H_{2(k-l)-1}^{2n+1} = 0.$$

The constants $C^l_{jk}$ are given by

$$C_{2m,2m}^{2l} = \delta_{m+n},$$

$$C_{2m+1,2m+1}^{2l+1} = \delta_{m+n} + H^{2m+1}_{2(n-l)-1},$$

$$C_{2m+1,2m+1}^{2l} = \delta_{m+n} + H^{2m+1}_{2(m-l)+1} + H^{2m+1}_{2(n-l)+1} \quad (7)$$

and $p_{2n} = p_2^2 = z^{2n}$, $n \geq 0$.

An immediate consequence of this lemma is given by the following

Proposition 2.2 The subset $W_0 \subset \Sigma_0$ the elements of which are given by vector spaces with basis
$\langle p_i(z) \rangle_i$ and parameters $H_k^i$ obeying the constraints $[4]$, is closed with respect to pointwise multiplication
$W_0 \cdot W_0 \subset W_0$. It is a maximal closed subset in the big cell. This subset $W_0$ is an infinite family of
infinite-dimensional associative commutative algebra with unity $p_0 = 1$. 

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The last statement follows from the equivalence of equations (8) to the associativity conditions
\[ \sum_s C_{ij}^s C_{ks}^r - C_{ik}^s C_{js}^r = 0 \]  
for the structure constants \( C_{jk}^l \).

Relations (8) written explicitly, i.e.
\[
\begin{align*}
    p_{2n}p_{2n} &= p_{2(m+n)}, \\
    p_{2n}p_{2m+1} &= p_{2(m+n)+1} + \sum_{s=0}^{n-1} H_{2s+1}^{2m+1} p_{2(n-s)-1}, \\
    p_{2n+1}p_{2m+1} &= p_{2(m+n+1)} + \sum_{s=0}^{m} H_{2s+1}^{2n+1} p_{2(m-s)} + \sum_{s=0}^{n} H_{2s+1}^{2m+1} p_{2(n-s)},
\end{align*}
\]

imply that
\[
\begin{align*}
    z^2 &= p_1^2 - 2H_{1}^1, \\
    p_3 &= p_1^3 - 3H_{1}^1 p_1, \\
    p_5 &= p_1^5 - 5H_{1}^1 p_1^3 + \frac{15}{2} H_{1}^{12} p_1, \\
    \ldots
\end{align*}
\]
or equivalently
\[
\begin{align*}
    \lambda &= p_1^2 - 2H_{1}^1, \\
    p_{2n+1} &= \alpha_n(\lambda)p_1
\end{align*}
\]
where \( \lambda = z^2 \) and \( \alpha_n(\lambda) = \prod_{s=1}^{n} \left( \lambda - \frac{H_{1}^{2(n-s)+1}}{2(n-s)^2+1} \right) \).

Similar to (9) one can treat \( \lambda, p_1, p_3, \ldots \) as the affine coordinates. So one has the following geometrical interpretation of the subset \( W_0 \).

**Proposition 2.3** Big cell \( \Sigma_0 \) contains an infinite-dimensional algebraic variety \( \Gamma_0 \) with the ideal
\[
\langle \lambda - p_1^2 + 2H_{1}^1, i_1^{(2)}, i_2^{(2)}, \ldots \rangle
\]
where \( i_1^{(2)} = p_{2n+1} - \alpha(\lambda)p_1 \) and the variables \( H_{1}^j \) obey the constraints (8). This variety \( \Gamma_0 \) is an infinite tower of infinite families of rational normal (Veronese) curves of all odd orders.

Formulas (10) represent a canonical parameterization of rational normal curves (see e.g. [6]). For instance, the curves defined by the first two equations (10) is the classical twisted cubic in the three-dimensional space with the coordinates \( \lambda, p_1, p_3 \).

There is an infinite set of independent variables among all \( H_{1}^j \) constrained by conditions (9). A natural set of independent \( H_{1}^j \) is given by \( H_{1}^1, H_{1}^3, H_{1}^5, \ldots \).

It is also easy to see using (11) that the ideal \( i_0^{(2)} \) contains singular “hyperelliptic” curves of genus zero given by the equations
\[
p_{2n+1}^2 = (\lambda + 2H_{1}^1)\alpha_n(\lambda)^2.
\]

Infinite family of algebraic varieties described in Proposition 2.3 is in its turn the algebraic variety in the affine space with coordinates \( p_i, \) \( (i = 1, 2, 3, \ldots) \) and \( H_{j}^k, (j, k = 1, 2, 3, \ldots) \) defined by the quadrics
\[
f_{jk} = p_jp_k - p_{j+k} - \sum_{s=0}^{k} H_{j}^s p_s - \sum_{s=0}^{j} H_{k}^s p_s = 0
\]
and equations (9).

We emphasize that an infinite tower of normal rational curves for fixed \( H_{1}^j \) is in correspondence with a point of the subset \( W_0 \).
3 Stratum $\Sigma_1$

The stratum $\Sigma_1$ is the lowest stratum different from the big cell and it corresponds to $m = 1$ and $S = \{-1, 1, 2, \ldots\}$. Due to the absence of zero order element $w_0$ the canonical basis is of the form

$$p_i(z) = z^i + H_0^i + \sum_{k \geq 1} H_k^i \frac{z^k}{k!}, \quad i = 1, 2, 3, \ldots . \quad (15)$$

Since $\langle p_i \rangle_{i=-1,1,2,\ldots}$ one should consider only $p_j$ with $j = 1, 2, 3, \ldots$

**Lemma 3.1** A set $W_1$ of Laurent series (15) obey the condition $z^2 \cdot W_1 \subset W_1$ and the equations

$$p_j(z)p_k(z) = \sum_{l \geq 1} C_{jk}^l p_l(z), \quad j, k = 1, 2, 3, \ldots \quad (16)$$

if and only if the parameters $H_k^i$ satisfy the constraints

$$H_{0}^{2j+k} + H_{0}^{2j}H_{0}^{2k} = 0,$$

$$H_{0}^{2(k+j)+1} + H_{0}^{2k+1}H_{0}^{2j+1} + \sum_{l=0}^{j-1} H_{2l+1}^{2k+1}H_{0}^{2(j-l)-1} = 0,$$

$$H_{2}^{2k+1} + H_{0}^{2k}H_{2}^{2j+1} - H_{0}^{2j}H_{0}^{2k+1} - H_{0}^{2j}H_{2}^{2k+1} - H_{0}^{2k}H_{2}^{2j+1} - \sum_{s=0}^{j-1} H_{2s+1}^{2k+1}H_{2s+1}^{2(j-s)-1} = 0,$$

$$H_{0}^{2j+1} + H_{0}^{2k}H_{2}^{2j+1} + \sum_{l=0}^{k-1} H_{0}^{2j+1}H_{0}^{2s+1} + \sum_{s=0}^{j-1} H_{0}^{2s}H_{2}^{2k+1} = 0,$$

$$H_{2}^{2j+1} + H_{2}^{2k+1}H_{2}^{2j+1} + \sum_{l=0}^{j-1} H_{2l+1}^{2j+1}H_{2s+1}^{2k+1} - \sum_{s=0}^{k-1} H_{2s+1}^{2j+1}H_{2s+1}^{2(l-s)-1} - \sum_{s=0}^{j-1} H_{2s+1}^{2k+1}H_{2s+1}^{2(l-s)-1} = 0$$

and

$$C_{2j,2k}^{2l} = \delta_{j+k}^l + H_{0}^{2j}H_{0}^{2k}H_{2}^{2l}, \quad C_{2j,2k+1}^{2l+1} = \delta_{j+k}^l + H_{0}^{2j}H_{0}^{2k}H_{2}^{2l+1}, \quad C_{2j+1,2k+1}^{2l+1} = H_{0}^{2k+1}H_{0}^{2j}H_{2}^{2l}, \quad C_{2j+1,2k+1}^{2l} = \delta_{j+k+1}^l + H_{0}^{2j+1}H_{0}^{2k+1}H_{2}^{2l} + H_{2}^{2k+1}H_{2}^{2l+1} \quad (18)$$

The analysis of the constraints (17) gives

$$H_{i}^{2n} = 0, \quad n, i = 1, 2, 3, \ldots \quad (19)$$

and

$$H_{0}^{2n} = -(-H_{0}^{2})^{n}, \quad n = 1, 2, 3, \ldots \quad (20)$$

i.e.

$$p_{2n}(z) = z^{2n} - (-H_{0}^{2})^{n}, \quad n = 1, 2, 3, \ldots \quad (21)$$

For the elements $p_{2n+1}$ one instead has

$$p_2 = p_1^2 - 2H_0^1 p_1, \quad p_3 = p_1^3 - 3H_0^1 p_1^2 - (3H_1^1 - 3H_0^1)^2 p_1, \quad \ldots . \quad (22)$$

Similar to the big cell one has a subset $W_1$ in $\Sigma_1$ closed with respect to multiplication which algebraically is an infinite-dimensional commutative associative algebra $A_1$ with the structure constants given by (18).
in the basis (17). Geometrically $W_1$ is an infinite tower of families of rational normal curves of all odd orders passing through the origin $p_1 = p_2 = p_3 = \cdots = 0$.

The fact that for the stratum $\Sigma_1$ one has results which are similar to those for big cell is not that surprising. Indeed, taking into account the relations (17), namely

$$2H_1^2 - H_0^2 + H_1^2 = 0, \quad H_0^n + H_0^1H_0^2 + H_0^1H_1^1 = 0$$

and the formula (21), i.e. $p_2 = z^2 + H_0^2$, one can rewrite equations (22) as

$$p_3 - H_0^3 = (p_1 - H_0^1)^3 - 3H_1^1(p_1 - H_0^1).$$

In the variables

$$\tilde{p}_1 = p_1 - H_0^1, \quad \tilde{p}_3 = p_3 - H_0^3$$

the equations (24) the first two equations (11) for the big cell. It is a direct check that in the variables

$$\tilde{p}_k = p_k - H_0^k, \quad k = 1, 2, 3, \ldots$$

all equations (22) coincide with equations (11) for the big cell.

Thus the result for the stratum $\Sigma_1$ and big cell are connected by a simple change of variables (26). Similar situation take place for other strata $\Sigma_m$ with odd $m$.

## 4 Stratum $\Sigma_2$ and elliptic curves

For the stratum $\Sigma_2$ with $S = \{-2, 0, 2, 3, 4, \ldots\}$ the positive order elements of the canonical basis are given by

$$p_0 = 1 + \sum_{k \geq 1} \frac{H_0^k}{z^k},

p_j = z^j + H_1^j z + \sum_{k \geq 1} \frac{H_k^j}{z^k}, \quad k, j = 2, 3, 4, \ldots.$$ (27)

First we note that $(p_2 - z)^2 \notin \langle p_i \rangle_{i=-2,0,2,3,\ldots}$ and the analogue of the Lemmas (2.1) and (3.1) is given by

**Lemma 4.1** A set $W_2$ of Laurent series (27) obey the equations

$$p_j(z)p_k(z) = \sum_{l=0,2,4,\ldots} C^l_{jk}p_l(z)$$

and the condition $z^2W_2 \subset W_2$ is satisfied if and only if

$$H_k^{2n} = 0, \quad k = -1, 1, 2, 3, \ldots, \quad n = 0, 1, 2, \ldots,$$

$$H_k^{2n+1} = 0, \quad n, k = 1, 2, 3, \ldots$$

and

$$H_k^{2m+1} - H_{2k+1}^{2(m+n+1)} - \sum_{s=1}^{n-2} H_{2s+1}^{2m+1}H_{2k+1}^{2(n-s)+1} = 0,$$

$$H_k^{2m+1} - H_{2k+1}^{2(n+k+1)} + \sum_{s=1}^{k} H_{2s+1}^{2m+1}H_{2(l-s)-1}^{2n+1} = 0.$$

(29) (30)

**Constants** $C^l_{jk}$ are given by

$$C^l_{2n, 2m} = \delta^l_{m+n},

C^l_{2n, 2m+1} = \delta^l_{m+n} + H_{2(n-l)-1}^{2m+1},

C^l_{2n+1, 2m+1} = \delta^l_{m+n+1} + H_{2(n-l)-1}^{2n+1} + H_{2(n-l)-1}^{2m+1} + H_{2(n-l)-1}^{2m+1} + H_{2(n-l)-1}^{2m+1} \delta^l_{2},$$

$$+ (H_{2(n-l)-1}^{2n+1} + H_{2(n-l)-1}^{2m+1}) \delta^l_{0}.$$ (31)
which imply

\[ p_{2n}p_{2m} = p_{2(m+n)}, \]
\[ p_{2n}p_{2m+1} = p_{2(m+n)+1} + \sum_{k=-1}^{n-2} H_{2k+1}^{2m+1} p_{2(n-k)-1}, \]
\[ p_{2n+1}p_{2m+1} = p_{2(m+n+1)} + \sum_{k=-1}^{m} H_{2k+1}^{2m+1} p_{2(m-k)} + \sum_{k=-1}^{n} H_{2k+1}^{2m+1} p_{2(n-k)} \]

\[ + H_{1}^{2m+1} H_{-1}^{2m+1} p_2 + (H_{-1}^{2n+1} H_{1}^{2m+1} + H_{1}^{2n+1} H_{-1}^{2m+1}) \]

and \( p_{2n} = z^{2n}, n \geq 0 \).

As a consequence, one has

**Proposition 4.2** The stratum \( \Sigma_2 \) contains a maximal closed subset \( W_2 \) whose elements are vector spaces with basis \( H_i^j \) obeying the constraints \( (24), (30) \) and such that \( z^2 W_2 \subset W_2 \).

The relations (28) readily imply that all \( p_i(z) \) are generated by two elements \( z^2 \) and \( p_3 \).

Using (32) one can show that the set of independent relations (28) is given by

\[ p_3^2 = \lambda^3 + 2H_{-1}^2 \lambda^2 + \left( H_{1}^{2} + 2H_{-1}^{2} \right) \lambda + 2H_{-1}^{1} H_{1}^{3} + 2H_{3}^{2} \]  

(33)

and

\[ p_{2n+1} = \left( \lambda^{n-1} - \sum_{i=0}^{n-2} H_{-1}^{2(n-i)-1} \lambda^i \right) p_3 \]

(34)

This relation is obtained using iteratively the formula

\[ p_{2n+1} = \lambda p_{2n-1} + H_{-1}^{2n-1} p_3. \]

(35)

**Proposition 4.3** Subset \( W_2 \) is an infinite family of infinite-dimensional commutative associative algebra with the basis \( 1, p_2, p_3, p_4, \ldots \) isomorphic to \( \mathbb{C}[\lambda, p_3]/C_6 \)

where

\[ C_6 = p_3^2 - \lambda^3 - 2H_{-1}^2 \lambda^2 - \left( H_{1}^{2} + 2H_{-1}^{2} \right) \lambda - (2H_{-1}^{1} H_{1}^{3} + 2H_{3}^{2}). \]

(36)

**Proof** Associativity follows from the fact that the conditions (29) and (30) are equivalent to the condition

\[ \sum_{s=0,2,3,\ldots} C_{jk}^m C_{m}^{s} = \sum_{s=0,2,3,\ldots} C_{jk}^{s} C_{s}^{r} \quad j, k, l, r = 0, 2, 3, \ldots \]

(37)

for the constants \( C_{jk}^m \) given by (32) \( \square \)

Treating now \( \lambda, p_3, p_5 \) and \( H_i^j \) as affine coordinates one has the following geometrical interpretation of the subset \( W_2 \).

**Proposition 4.4** The subset \( W_2 \) is an infinite dimensional algebraic variety \( \Gamma_2 \) in the affine space with coordinates \( p_j, (j = 2, 3, 4, \ldots), H_i^j, (j = 3, 5, 7, \ldots, k = -1, 1, 3, 5, \ldots) \) defined by the intersection of quadrics

\[ f_{jk} = p_j p_k - p_{j+k} - \sum_{l=0,2,3,\ldots} C_{jk}^{l} p_l(z) = 0 \]

(38)

and quadrics (37). An ideal \( I^{(2)} \) of this variety is

\[ I^{(2)} = \langle C_6, t_1^{(2)}, t_5^{(2)}, t_7^{(2)}, \ldots \rangle \]

(39)

where \( t_{2n+1}^{(2)} = p_{2n+1} - \left( \lambda^{n-1} - \sum_{i=0}^{n-2} H_{-1}^{2(n-i)-1} \lambda^i \right) p_3 \).

(36)
Since $W_2 \sim \mathbb{C}[\lambda, p_3]/C_6$ one can view $\Gamma_2$ as the infinite family of coordinate rings of the elliptic curve $C_6 = 0$ parameterized by the variables $H_k^i$ obeying the conditions (29) and (30). Analyzing these conditions one concludes that there is an infinite set of independent variables among all $H_k^i$, for example $H_1^3, H_2^3, H_3^3, H_5^3, \ldots$. 

It is a direct check that the curve $C_6 = 0$ has genus one. So the stratum $\Sigma_2$ contains an infinite family of elliptic curves parameterized by $H_1^3, H_2^3, H_3^3$. 

We emphasize that each of these elliptic curves belong to a point of the subset $W_2$. So, following [4], such points of $\Sigma_2$ will be called elliptic points and the whole subset $W_2$ an elliptic subset.

The ideal $I^{(2)}$ contains singular hyperelliptic curves of all orders and of genus 1 given by

$$p_{2n+1}^2 = \left(\lambda^{n-1} - \sum_{i=0}^{n-2} H_{-1}^{2(n-i)-1} \lambda^i \right)^2 \left(\lambda^3 + 2H_{-1}^3 \lambda^2 + \left(H_{-1}^3 + 2H_{1}^3\right) \lambda + 2H_{-1}^3H_{1}^3 + 2H_{3}^3 \right)$$  \hspace{1cm} (40)

5 \textbf{Stratum $\Sigma_3$: $(3,4)$ curves of zero genus}

Next case corresponds to $S = \{-3, -1, 1, 3, 4, 5, \ldots\}$. Due to the absence of elements of orders zero and two positive elements of the canonical basis are given by

$$p_1 = z + H_0^3 + \sum_{k \geq 1} \frac{H_k^4}{z^k},$$

$$p_j = z^j + H_{-2}^j z^2 + H_0^4 + \sum_{k \geq 1} \frac{H_k^4}{z^k}, \quad j = 3, 4, 5, \ldots.$$  \hspace{1cm} (41)

Since $p_i^2$ has order two a closed subspace can be generated only by the elements $p_3, p_4, p_5, \ldots$.

\textbf{Lemma 5.1} A set $W_3$ of Laurent series $p_j(z), j = 3, 4, 5, \ldots$ obey the equations

$$p_j(z) p_k(z) = \sum_{l=3,4,5,\ldots} C_{j,k}^{l} p_l(z), \quad j, k = 3, 4, 5, \ldots$$  \hspace{1cm} (42)

and the condition $z^2 W_3 \subset W_3$ if and only if

$$p_j = z^j + H_{-2}^j z^2 + H_0^4, \quad j \geq 5,$$  \hspace{1cm} (43)

$$H_{-2}^j + H_{-2}^{j-2} H_0^4 - H_0^{j-2} = 0,$$  \hspace{1cm} (44)

and

$$H_0^4 + 2H_0^3 H_{-2}^2 - H_{-2}^2 H_0^4 - H_{-2}^4 = 0,$$

$$H_0^{4j} - H_{-2}^{2j} H_0^4 - H_{-2}^{4j} H_0^4 = 0.$$  \hspace{1cm} (45)

\textbf{Proof} Let us begin with the condition $z^{2n} W_3 \subset W_3$. One has

$$z^{2n} p_m(z) = z^{2n+m} + \cdots + H_{2n-1}^m z^2 + \cdots$$  \hspace{1cm} (46)

In $W_3$ there is no element which contains the term of order one. Hence, with necessity $H_{2n-1}^m = 0$ for all $n = 1, 2, 3, \ldots$ and $m = 3, 4, 5, \ldots$, i.e.

$$p_j(z) = z^j + H_{-2}^j z^2 + H_0^4 + \sum_{n \geq 1} \frac{H_k^4}{z^{2n}}, \quad j = 3, 4, 5, \ldots.$$  \hspace{1cm} (47)

Then considering the product $p_{2k+1} p_j$ one has

$$p_{2k+1}(z)p_j(z) = z^{2k+j+1} + \cdots + H_{2k}^j z^2 + \cdots.$$  \hspace{1cm} (48)
The terms of the order $z^i$, $i \geq 3$ can be represented as a superposition of $p_3, p_4, \ldots, p_{2k+j+1}$ giving the constants $C^i_{jk}$ while the coefficient in front of $z$ should vanish. Hence $H^i_{2k} = 0$ for all $k = 1, 2, 3, \ldots$. So

$$p_j = z^j + H^i_{-2}z^2 + H^i_0 \quad j \geq 3. \quad (49)$$

The coefficients $H^i_{-2}$ and $H^i_0$ are not all independent. Indeed, the relations

$$z^2p_3 = p_5 + H^i_{-2}p_4,$$

$$z^2p_4 = p_6 + H^i_{-2}p_4,$$

$$z^4p_3 = pr + H^i_{-2}p_6 + H^i_0p_4,$$

$$\ldots$$

imply

$$H^5_{-2} - H^5_0 + H^3_{-2}H^4_{-2} = 0,$$

$$H^5_0 + H^3_{-2}H^4_0 = 0,$$

$$H^6_{-2} - H^4_0 + H^5_{-2} = 0,$$

$$H^6_0 + H^3_{-2}H^4_0 = 0,$$

$$H^7_{-2} + H^3_{-2}H^6_{-2} + H^3_0H^4_{-2} = 0,$$

$$H^7_0 + H^3_{-2}H^6_0 + H^3_0H^4_0 = 0,$$

$$\ldots$$

and so on. The relations $(51)$ are the lowest members of the relations $(44)$, Using these relations, one can express all $H^i_{-2}$, $H^i_0$ with $j = 5, 6, 7, \ldots$ in terms of $H^3_{-2}$, $H^3_0$ and $H^4_{-2}$, $H^4_0$.

Furthermore, the vanishing of the coefficients in front of $z^a$ and $z^0$ in the relation

$$p_3^2 - \left( p_6 + 2H^3_{-2}p_5 + H^5_{-2}p_4 + 2H^3_0p_3 \right) = 0 \quad (52)$$

is equivalent to the conditions

$$H^6_{-2} - H^6_{-2}H^5_{-2} - H^3_{-2}H^4_{-2} = 0,$$

$$H^6_0 - H^6_0 - 2H^3_{-2}H^3_0 - H^3_{-2}H^4_0 = 0. \quad (53)$$

Finally taking into account $(51)$, one gets the constraints $(55)$. So there are only two independent parameters among all coefficients $H^i_{-2}$ and $H^i_0$. The simplest choice is to take $H^3_{-2}$ and $H^4_{-2}$ as independent variables. At last, the direct calculation gives

$$C^i_{jk} = \delta^i_{j+k} + H^k_{-2}\delta^j_{j+2} + H^k_0\delta^j_{j} + H^k_{-2}\delta^j_{k+2} + H^i_0\delta^i_{k} + H^i_{-2}H^k_{-2}\delta^i_{j}. \quad (54)$$

An immediate consequence of the Lemma $(51)$ is given by

**Proposition 5.2** The stratum $\Sigma_3$ contains the subset $W_3$ closed with respect to pointwise multiplication $W_3 \cdot W_3 \subset W_3$. Elements of $W_3$ are vector spaces with basis $(p_i)_i$ of the form $(43)$ with $H^i_{-2}, H^i_0$ obeying the constraints $(44)$ and $(47)$. The subset $W_3$ is an infinite family of infinite-dimensional associative and commutative algebra $A_3$ with the basis $(43)$ and structure constants $(54)$.

A geometrical interpretation of $W_3$ is provided by

**Proposition 5.3** The subset $W_3$ can be viewed as the two parametric family of algebraic varieties defined by the relations

$$p_{j+k} - \sum_l C^l_{jk}p_l = p_{j+k} + H^k_{-2}p_{j+2} + H^k_0p_j + H^j_{-2}p_{k+2} + H^j_0p_k + H^j_{-2}H^k_{-2}p_4 = 0. \quad (55)$$
The ideal of this family contains the plane (3,4) curve (in the terminology of [7]) defined by the equation

\[
p_3^3 - p_3^4 + 4H^3L_{2p_3p_4} - \left(3H^3L_{2p_3} - 2H^3L_{2p_4}\right)p_3^2p_4 - \left(-4H^3L_{0} + 2H^3L_{2p_2}\right)p_3^3
\]

\[
- \left(3H^3L_{0} + 4H^3L_{2} + H^3L_{2} + H^3L_{2}H^3L_{2} - 2H^3L_{2}H^3L_{2}\right)p_3^2 - \left(4H^3L_{2p_3} + 8H^3L_{2p_4} - 2H^3L_{2p_2}\right)p_3^4
\]

\[
- 6H^3L_{2p_3} - 2H^3L_{2p_4} + H^3L_{2p_3}H^3L_{2p_4} - 2H^3L_{2p_2}H^3L_{2p_4} - 2H^3L_{2p_3}H^3L_{2p_2}
\]

\[
+ H^3L_{2p_3}H^3L_{2p_4} + H^3L_{2p_3} - 2H^3L_{2p_4}
\]

\[
p_3^2 - \left(3H^3L_{0} - 2H^3L_{2p_3} - 2H^3L_{2p_4} - 2H^3L_{2p_2}\right)p_3
\]

\[
+ 3H^3L_{2p_3} + 2H^3L_{2p_4} - 8H^3L_{2p_2} - 3H^3L_{2p_3}H^3L_{2p_4} - 2H^3L_{2p_2}H^3L_{2p_4} - 2H^3L_{2p_3}H^3L_{2p_2}
\]

\[
- 2H^3L_{2p_3}H^3L_{2p_4} - 2H^3L_{2p_3}H^3L_{2p_2}
\]

\[
= 1 + \sum_{k=1}^{\infty} \frac{H^3L_{k}}{z^k}
\]

(56)

where \(H^3L_{2},H^3L_{2},H^3L_{2}\) and \(H^3L_{0}\) obey the constraints [13]. The (3,4) curve \((56)\) have zero genus.

**Proof** By direct calculation with the use of polynomial form \((43)\) of \(p_j\).

Comparing the results of this and previous section, one observes an essential difference between the geometrical properties of the subspaces \(W_2\) and \(W_3\). This is due to the quite different form of the Laurent series belonging to \(W_l\) which is the consequence of a different situation with elements of the first order in \(z\). Namely, though in both cases \(W_l\) does not contain the element \(p_1(z)\). The absence in \(W_3\) of the terms of order \(z^j\) in \(p_j(z)\) leads to a strong constraints leading to the polinomiality of \(p_j(z)\).

We note also that due to the presence of the element \(p_0 = 1\) of zero order in \(W_0\) one has \(z^2 \in W_2\) while \(z^2 \notin W_3\). As a consequence, for instance, one can choose \(p_3\), \(p_4\) and \(z^2p_3\) as the generators of the algebra \(A_3\) instead of \(p_3\), \(p_4\) and \(p_5\).

A way to avoid the polinomiality of all \(p_j(z) \in W_0\) would be to relax the condition \(z^2W_3 \subset W_3\). Since \(z^2\) is not an element of \(W_3\) it would be natural not to require that the product of \(z^2\) and an element of \(W_3\) belongs to \(W_3\), but instead to require that \(z^2W_3 \subset W_3\). The presence of the element \(p_1(z)\) in \(\Sigma_0\), allows us to avoid immediate constraints on \(p_j(z)\) followed from the relations of the type \((46)\) and \((48)\). for instance, instead of the conditions \((50)\) one gets the following ones

\[
z^2p_3 - p_5 - H^3L_{2p_4} = H^3L_{p_1},
\]

\[
z^2p_4 - p_6 - H^3L_{2p_4} = H^3L_{p_1},
\]

and so on. In virtue of the equations of this type one obtains an infinite set of relations for \(H^3L_{k}\). Computer analysis strongly indicates that these conditions again lead to the constraint \(H^3L_{k} = 0\), \(k = 1,2,3,\ldots\), \(j = 3,4,5,\ldots\), i.e. to the polinomiality of all \(p_j(z)\).

### 6 Strata \(\Sigma_2n\). Hyperelliptic curves of genus \(n\)

Stratum \(\Sigma_2n\) with arbitray \(n\) is characterized by \(S = \{-2n,-2n+2,-2n+4,\ldots,0,2,4,\ldots,2n,2n+1,2n+2,\ldots\}\). So it does not contain, in particular, \(n\) elements of the order \(1,3,5,\ldots,2n-1\) and the positive order elements of the canonical basis are given by

\[
p_0 = 1 + \sum_{k=1}^{\infty} \frac{H^0L_{2k}}{z^k},
\]

\[
p_j = z^j + \sum_{k=0}^{j-1} H^3L_{2k-1}z^{2k+1} + \sum_{k=1}^{j} H^3L_{2k}, \quad j = 2,4,6,\ldots,2n-2,
\]

\[
p_j = z^j + \sum_{k=0}^{n-1} H^3L_{2k-1}z^{2k+1} + \sum_{k=1}^{\infty} H^3L_{2k}, \quad j = 2n,2n+1,2n+2,2n+3,\ldots.
\]

As in the previous cases the \(p_j\) with negative \(j\) do not should be taked into account and one has
Lemma 6.1 A set \( W_{2n} \) at fixed \( H_k \) of Laurent series obey the condition \( z^2 W_{2n} \subset W_{2n} \) and equations

\[
p_j(z) p_k(z) = \sum_l C_{j,k}^l p_l(z), \quad j, k, l = 0, 2, 4, \ldots, 2n, 2n+1, 2n+3, \ldots
\]

if and only if

\[
H_k^{2m} = 0, \quad m = 0, 1, 2, \ldots, k = -2n+2, -2n+4, \ldots, -2, 0, 2, 3, \ldots
\]

\[
H_{2k}^{2m+1} = 0, \quad m = 0, 1, 2, \ldots, k = -n, -n+1, -n+2, \ldots
\]

and

\[
H_{2(l+k)+1}^{2j+1} - H_{2(l+k)+1}^{2(j+k)+1} - \sum_{s=-n}^{s=-n} H_{2s+1}^{2j+1} H_{2s+1}^{2(k+s)+1} = 0,
\]

\[
H_{2(l+j)+1}^{2j+1} + H_{2(l+j)+1}^{2k+1} + \sum_{s=-n}^{s=-n} H_{2s+1}^{2j+1} H_{2s+1}^{2(k+s)+1} = 0.
\]

Rewriting equation (59) separately for \( p_j \) with even and odd \( j \), i.e.

\[p_{2j} p_{2k} = p_{2(j+k)},\]

\[p_{2j} p_{2k+1} = p_{2(j+k)+1} + \sum_{s=-n}^{s=-n} H_{2s+1}^{2j+1} p_{2(k+s)+1},\]

\[p_{2j+1} p_{2k+1} = p_{2(j+k+1)} + \sum_{s=-n}^{s=-n} H_{2s+1}^{2j+1} p_{2(j+s)+1} + \sum_{s=-n}^{s=-n} H_{2s+1}^{2k+1} p_{2(k+s)+1} + \sum_{s=-n}^{s=-n} H_{2s+1}^{2j+1} H_{2s+1}^{2k+1} p_{2(j+s)+1} + \sum_{s=-n}^{s=-n} H_{2s+1}^{2j+1} H_{2s+1}^{2k+1} p_{2(j+s)+1},\]

one concludes that

\[p_{2m} = (z^2)^m, \quad p_{2m+1} = \alpha(\lambda) p_{2m+1}, \quad m = n+1, n+2, \ldots, \lambda = z^2\]

for suitable \( \alpha(\lambda) \in \mathbb{C}[\lambda] \). Moreover

\[p_{2n+1}^2 = \lambda^{2n+1} + \sum_{k=0}^{2n} u_k \lambda^k\]

where the coefficients \( u_k \) can be obtained from

\[p_{2n+1}^2 = \lambda^{2n+1} + 2 \sum_{s=0}^{2n} H_{2(n-s)+1}^{2n+1} \lambda^s + \sum_{k=-n}^{k=-n} \sum_{s=0}^{n-k-1} H_{2(k+s)+1}^{2n+1} H_{2(s+k)+1}^{2n+1} \lambda^s.\]

Thus, one has

Proposition 6.2 The stratum \( \Sigma_{2n} \) for \( n = 2, 3, 4, \ldots \) contains maximal subset \( W_{2n} \) closed with respect to pointwise multiplication. Elements of \( W_{2n} \) are vector spaces with basis given by \( \langle p_i \rangle_{i=0, 2, 4, \ldots, 2n, 2n+1, 2n+3, \ldots} \) with parameters \( H_k \) obeying the constraints (60) and (61).

Proposition 6.3 The subset \( W_{2n} \) is the infinite family of infinite-dimensional commutative associative algebra \( A_{2n} \) isomorphic to \( \mathbb{C}[\lambda, p_{2n+1}]/C_{2n+1} \) where \( \lambda = z^2 \) and

\[C_{2n+1} = p_{2n+1}^2 - \lambda^{2n+1} - \sum_{k=0}^{n} u_k \lambda^k = 0\]

and \( u_k \) are given by (63).
Proposition 6.4 The subset $W_{2n}$ in $\Sigma_{2n}$ is an infinite family of infinite-dimensional algebraic variety $\Gamma_{2n}$ defined by the relations $(59), (60), (61), (66)$. Its ideal is

$$I_{2n+1} = (C_{2n+1}, I_{2n+3}, I_{2n+5}, \ldots)$$

where $I_{2n+1}^{(n)} = p_{2n+1} - a_m(\lambda)p_{2n+1}, m = n + 1, n + 2, n + 3, \ldots$.

In other words the variety $\Gamma_{2n}$ is the intersection of the cubic $C_{2n+1} = 0$ and infinite set of algebraic curves $I_{2n+1}^{(n)}, m = n + 1, n + 2, n + 3, \ldots$. One can easily see that the ideal $I_{2n}$ contains higher order hyperelliptic curves but all of them have genus $n$.

Thus stratum $\Sigma_{2n}$ is characterized by the presence of the plane hyperelliptic curves $C_{2n+1}$ of genus $n$ in every point of the closed subset $W_{2n}$. This is due to the presence of $n$ gaps (elements $p_1(z), p_3(z), \ldots, p_{2n-1}(z)$) in the basis of $W_{2n}$. The fact that for hyperelliptic curves (Riemann surfaces) of genus $n$ one has $n$ (Weierstrass) gaps in a generic point is well known in the theory of abelian functions (see e.g. [8]). Probably not that known observation is that these gaps and consequently the properties of corresponding algebraic curves are prescribed by the structure of the Birkhoff strata $\Sigma_{2n}$ in $Gr^{(2)}$. In different context an appearance of hyperelliptic curves in Birkhoff strata of $Gr^{(2)}$ has been observed in [9].

7 Strata $\Sigma_{2n+1}$

Stratum $\Sigma_{2n+1}, n = 2, 3, 4, \ldots$ is characterized by $S = \{-2n - 1, -2n + 1, \ldots, -1, 1, 3, \ldots, 2n + 1, 2n + 2, \ldots\}$. So, the positive order elements of the canonical basis in $\Sigma_{2n+1}$ are of the form

$$p_j(z) = z^j + H_j^{2j} z^{2j-2} + H_{j+2} z^{2j-2} + \cdots + H_0^j + \sum_{k \geq 1} \frac{H_j^k}{z^k}, \quad j = 1, 3, \ldots, 2n - 1$$

$$p_j(z) = z^j + H_{-2n} z^{2n} + H_{-2n+1} z^{2n-1} + \cdots + H_0^j + \sum_{k \geq 1} \frac{H_j^k}{z^k}, \quad j = 2n + 1, 2n + 2, \ldots.$$
As in the previous cases the $p_j$ with $j \leq 1$ do not should be taked into account.

Closed subsets in $\Sigma_{2n+1}$ have different structure for different $n$. In order to see this let us begin with $\Sigma_5$. In this case the elements \textbf{[11]} of the canonical basis are

\[ p_1 = z + H_1^0 + \sum_{k \geq 1} H_{2k}^1, \]
\[ p_3 = z^3 + H_{-2}^3 z^2 + H_{0}^3 + \sum_{k \geq 1} H_{2k}^1, \]
\[ p_j = z^j + H_{-4}^j z^4 + H_{-2}^j z^2 + H_{0}^j + \sum_{k \geq 1} H_{2k}^j, \quad j = 5, 6, 7, \ldots. \]

(72)

It is easy to see that the maximal closed subset $W_5$ in $\Sigma_5$ is the subset whose points are vector spaces with basis $(p_3, p_5, p_6, \ldots)$.

\textbf{Lemma 7.1} A set $W_5$ at fixed $H_k^i$ of the Laurent series $p_3, p_5, p_6, \ldots$ obey the equations

\[ p_j(z)p_k(z) = \sum_{l=3,5,6,\ldots} C_{jkl}p_l(z) \]

(73)

and the condition $z^2 W_5 \subset W_5$ if and only if $H_k^i = 0$, $j = 3, 5, 6, \ldots, k = 1, 2, 3, \ldots$, i.e. all $p_j$ are polynomials and $H_k^i$, $k = -4, -2, 0$ obey the constraints

\[ H_0^5 = 0, \quad H_{-2}^5 = H_0^3, \quad H_{-4}^5 = H_{-2}^3, \]
\[ H_0^6 = -H_{-2}^3, \quad H_{-2}^6 = -2H_0^3 H_{-2}^3, \quad H_{-4}^6 = -H_{-2}^3, \]
\[ \ldots \]

(74)

The proof of the polynomiality of $p_j(z)$ is exactly the same as for $W_3$ (Lemma \textbf{5.1}). The constraints \textbf{[74]} follow from equations \textbf{[73]} and the condition $z^5 W_5 \subset W_5$. For instance one has $z^2 p_3 = p_5, z^5 p_5 = p_7 + H_{-4}^5 p_6$ etc. . The constants $C_{jkl}^i$ are given by

\[ C_{jkl}^i = \delta_{j+k} \sum_{s=0}^{2} H_{-2s}^j H_{2s+k}^l + \sum_{s=0}^{2} H_{-2s}^j \delta_{2s+j}^l + \sum_{s,r=0}^{2} H_{-2s}^j H_{-2r}^k \delta_{2(s+r)}^{l}, \quad j, k \geq 3 \]

(75)

where, for sake of compactess, we use $H_{-2}^3 = 0$. As a consequence of this lemma one has

\textbf{Proposition 7.2} The stratum $\Sigma_5$ contains a maximal subset $W_5$ closed with respect to pointwise multiplication $W_5 \cdot W_5 \subset W_5$. Elements of $W_5$ are vector spaces $\langle p_i \rangle_{i=3,5,6,\ldots}$ and $H_{-4}^5, H_{-2}^5, H_0^5$ obeying the constraints \textbf{[74]}.

Algebraically $W_5$ is an infinite family of infinite-dimensional commutative associative algebra $A_5$ of polynomials with the structure constants given by \textbf{[75]}. Geometrically $W_5$ is the infinite algebraic variety $\Gamma_5$ defined by the equations \textbf{[73]} and \textbf{[51]}.

First equations of the set of equations \textbf{[73]} are

\[ p_3^2 = p_6 + 2H_{-2}^3 p_5 + 2H_0^3 p_3, \]
\[ p_3 p_5 = p_8 + 2H_{-2}^3 p_7 + H_{-2}^3 p_6 + 2H_0^3 p_5, \]

(76)

and so on. So the algebra $A_5$ is generated by $p_3, p_5$ and $p_7$.

It is not also difficult to show that an ideal of the variety $\Gamma_5$ contains the family of plane $(3,5)$ curve

\[ p_5^3 - p_3^5 + 2H_{-2}^3 p_3 p_5 - H_{-2}^3 p_5^2 + 2H_0^3 p_3^4 - 2H_0^3 H_{-2}^3 p_3^2 - H_0^3 p_3^3 = 0 \]

(77)

parameterized by two variables $H_{-2}^3$ and $H_0^3$. Due to the polinomiality of $p_3$ and $p_5$ in $z$, the genus of of curve \textbf{[77]} is obviously equal to zero. The ideal of the varieties contains another rational plane curve

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given by
\[ p_6^5 - p_5^6 + 6 H_{-2}^3 p_5^4 + 14 H_{-2}^2 p_5^2 p_6 + \left(-6 H_0^3 - 16 H_{-2}^3\right) p_5^3 p_6^2 - \left(-16 H_0^3 H_{-2}^2 - 9 H_{-2}^4\right) p_5^4 p_6 \]
\[ - \left(-10 H_0^3 H_{-2}^2 - 2 H_{-2}^5\right) p_5^5 + 2 H_0^2 p_5^6 + 8 H_0^3 H_{-2}^2 p_5 p_6 + 10 H_{-2}^2 H_0^2 p_5^2 p_6^2 \]
\[ - \left(-14 H_0^3 - 4 H_{-2}^3 H_0^2\right) p_5^3 p_6 + 20 H_0^3 H_{-2}^3 p_5^4 + H_0^4 p_6^3 + 2 H_0^4 H_{-2}^2 p_5 p_6^2 + 8 H_0^5 p_5^3 = 0. \] (78)

The stratum \( \Sigma_5 \) exhibits the main features of higher strata \( \Sigma_{4m-1} \), \( m = 1, 2, 3, \ldots \). The maximal closed subsets \( W_{4m-1} \) have the basis \( (p_{2m+1}, p_{2m+3}, \ldots, p_{4m-1}, p_{4m}, p_{4m+1}, \ldots) \) while the stratum \( \Sigma_{4m+1} \), \( m = 1, 2, 3, \ldots \) have the basis \( (p_{2m+1}, p_{2m+3}, \ldots, p_{4m-1}, p_{4m+1}, p_{4m+2}, \ldots) \) with the respective \( p_j \). Then one can demonstrate an analog of the Lemma 7.1 for \( \Sigma_{4m \pm 1} \) which in particular says that all \( p_j(z) \) are polynomials in \( z \) obeying the equations
\[ p_j(z) p_k(z) = \sum_l C^l_{jk} p_l(z), \quad j, k, l = 2m + 1, 2m + 3, \ldots \] (79)

together with certain constraints on \( H_j^k \).

Consequently one has closed subsets \( W_{4m \pm 1} \) in \( \Sigma_{4m \pm 1} \) which algebraically are commutative and associative algebras and geometrically they represent families of algebraic varieties \( \Gamma_{4m \pm 1} \) defined by the equation (79). Ideals of the varieties \( \Gamma_{4m \pm 1} \) contain plane \( (2m + 1, 2m + 3) \) curve
\[ p_{2m+1}^{2m+3} - p_{2m+3}^{2m+1} + \cdots = 0, \quad m = 1, 2, 3, \ldots \] (80)
of zero genus.

Properties of these rational curves will be discussed elsewhere.

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