Observability of the arrival time distribution using spin-rotator as a quantum clock

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Introduction.—Of late, the question of calculating the arrival or transit time distribution in quantum mechanics has been a topic of much interest. For comprehensive reviews see, for example, Muga and Leavens \textsuperscript{1}, and Muga et al. \textsuperscript{2}. A number of schemes \textsuperscript{3,4,5} have been suggested in the literature for calculating the arrival time distribution such as those based on axiomatic approaches, trajectory models of quantum mechanics, attempts to define and calculate the arrival time distribution using the consistent histories approach, and attempts of constructing the time of arrival operator, etc. Thus there is an inherent nonuniqueness within the formalism of quantum mechanics for calculating time distributions such as the arrival time distribution.

Against the backdrop of such studies, it remains an open question as to what extent these different quantum mechanical approaches for calculating the time distributions can be empirically discriminated. An effort along this direction was made by Damborenea et al. \textsuperscript{6} who considered the measurement of arrival time by the emission of a first photon from a two-level system moving into a laser-illuminated region. They had evaluated the probability for this emission of the first photon by using the quantum jump approach. The suitable approximations under which such calculated results could be related to Kijowski’s axiomatic arrival time distribution and the arrival time distribution defined in terms of the probability current density (not its modulus) were also discussed. Subsequently, further work was done along this direction by Hegerfeldt et al. \textsuperscript{7} who made more precise the connection of this approach with Kijowski’s distribution.

In this paper we address this question from a new perspective so that one can start from an axiomatically defined time distribution and then directly relate it to the actually testable results. In order to illustrate this approach, here in particular, we use a time distribution postulated in terms of the (normalised) modulus of the probability current density. Based on this, we derive a distribution of spin orientations along different directions for the spin-1/2 neutral particles emerging from a spin-rotator (SR) which contains a constant magnetic field. Such a calculated distribution function can then be tested by suitably using a Stern-Gerlach(SG) device, as explained later. Thus the scheme formulated in this paper can also be viewed as a verification of the observability of the quantum probability current density.

Unlike the position probability density, the status of probability current density in quantum mechanics as an observable quantity has remained a problematic issue; see, for example, Kan and Griffin \textsuperscript{8} who pointed out that for a many-particle system, any linear operator representation for velocity is inconsistent with a linear operator representation for the probability current density, such as the one constructed by Landau \textsuperscript{9} for the quantum theory of superfluid helium. Nevertheless, in the context of single-particle dynamics, the probability current density has been used in the quantum mechanical predictions of time distributions such as the arrival time \textsuperscript{10}, tunneling and reflection times \textsuperscript{11}.

Next, let us consider the analysis of the experimental techniques for measuring the “arrival time” or “time of flight”. We note that such analysis is usually done semi-classically or classically \textsuperscript{12}. Therefore it is curious that the question of a consistent quantum mechanical treatment of the measurement of time has remained murky ever since Pauli’s argument \textsuperscript{13} that “there cannot be self adjoint time operator conjugate to any Hamiltonian bounded from below”. Subsequently, a number of authors \textsuperscript{14} have pointed out various conceptual and mathematical problematic aspects of this question. On the other hand, several specific toy

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models\footnote{13} have also been proposed to investigate the feasibility of how actually the measurement of a time distribution can be performed in a way consistent with the basic principles of quantum mechanics.

Now, since such debates arise essentially if one considers how to directly measure time in quantum mechanics, here we bypass this vexed issue by adopting the following strategy. We consider the SR as a \textit{“quantum clock”} where the basic quantity which determines the actually observable result is the probability density function $\Pi(\phi)$ which corresponds to the probability distribution of spin orientations along different directions for the particles emerging from the SR, $\phi$ being the angle by which the spin orientation of a spin-1/2 neutral particle (say, a neutron) is rotated from its initial spin polarised direction. Note that this angle $\phi$ is determined by the transit time ($t$) within the SR. Hence the probability density function $\Pi(\phi)$ stems from $\Pi(t)$ which represents the distribution of times over which the particles interact with the constant magnetic field while passing through the SR. It is the evaluation of this quantity $\Pi(t)$ which critically depends on what quantum mechanical approach one adopts for calculating such a time distribution.

The plan of this paper is as follows. We will first elaborate on the relevant setup, with a discussion of how the estimation of the quantity $\Pi(\phi)$ can actually be tested by using a SG device. Subsequently, we will outline the key ingredients of the specific scheme adopted in this paper for calculating $\Pi(\phi)$ in terms of $\Pi(t)$ using the modulus of the probability current density. This will be followed by illustrative numerical estimates.

The setup.—Traditionally, a SR has been mainly used for the neutron interferometric studies\footnote{16}. Application of the Larmor precession of spin in a magnetic field has earlier been discussed, for example, in the context of the scattering of a plane wave from a potential barrier\footnote{17}. On the other hand, the scheme proposed in this paper explores an application of Larmor precession such that one can empirically test any given quantum mechanical formulation for calculating the arrival time distribution.

We consider an ensemble of spin 1/2 neutral particles, say, neutrons having magnetic moment $\mu$. The spatial part of the total wave function is represented by a localised narrow Gaussian wave packet $\psi(x, t = 0)$ (for simplicity, it is considered to be one dimensional) which is peaked at $x = 0$ at $t = 0$ and moves with the group velocity $u$. Thus the initial total wave function is given by $\Psi = \psi(x, t = 0) \otimes \chi(t = 0)$ where $\chi(t = 0)$ is the initial spin state which is taken to be same for all members of the ensemble.

The SR used in our setup (Fig. 1) has within it a constant magnetic field $B = B\hat{z}$ directed along the $+\hat{z}$ axis, confined between $x = 0$ and $x = d$. Within the SR, the spatial part of the total wave function is assumed to propagate freely, while its spin part interacts with the constant magnetic field. This assumption is justified in our setup because, for our choices of parameters, the magnitude of the Zeeman potential energy of the interaction of spin of the neutron with the constant magnetic field ($\simeq 0.01 \text{neV}$) is exceedingly small compared to the kinetic energy of the neutrons ($\simeq 0.01 \text{eV}$). Hence to a very high degree of accuracy we can consider the evolution of the spatial wave function within the SR to be free. Therefore the spatial and the spin parts of the total wave function can be considered to evolve independent of each other in a tensor product Hilbert space $H = H_1 \otimes H_2$ where $H_1$ and $H_2$ are the disjoint Hilbert spaces corresponding to the spatial and the spin parts of the total wave function respectively\footnote{16}.

Here in our analysis we assume that the spin part of the total wave function begins to interact with the constant magnetic field in the SR at the instant ($t = 0$) when the peak of the incoming wave packet is at the entry point ($x = 0$) of the SR. Thus the calculational procedure adopted here is essentially valid for a sufficiently narrow wave packet. Then it can be assumed that the entire ensemble of particles corresponding to the initial wave packet start interacting with the magnetic field at $t = 0$. This assumption is crucial in this scheme in order to enable the arrival / transit time distribution $\Pi(t)$ to be mapped onto $\Pi(\phi)$.

It is important to mention here that in the standard approach, the question as regards the initial instant at which a propagating wave packet of any arbitrary width starts interacting with a localized potential (e.g. the localized magnetic field in our setup) is intrinsically problematic since there is no unique criterion for fixing this instant. Implications of this nonuniqueness have hitherto remained unaddressed in the literature. Following the usual procedure (expected to be valid for narrow wave packets) we have assumed in our paper that the spin part of the wave function begins to evolve under a given localized potential essentially from the instant when the peak of the wave packet reaches the entry point of that potential. This criterion is of course not the only criterion one can use to define this initial instant and one can certainly use any other criterion to fix this initial instant.

Basically there are two different kinds of non-uniqueness in the quantum mechanical treatment of our problem; one is that there is no unique criterion available within the standard framework of quantum mechanics for fixing the initial instant when the spin part of the wave function starts to interact with the localized potential; the other non-uniqueness is inherent within the formalism of quantum mechanics is regarding the time-duration over which the wave packet interacts with the localized potential which in our paper is fixed by the arrival/transit time distribution. These are the crucial conditions rele-
FIG. 1: Spin-1/2 particles, say, neutrons with initial spin orientations polarised along the +x-axis and associated with a localized Gaussian wave packet (peaked at $x = 0, t = 0$) pass through a spin-rotator (SR) containing a constant magnetic field $B$ directed along the +x-axis. The particles emerging from the SR have a distribution of their spins oriented along different directions. Calculation of this distribution function is experimentally tested by measuring the spin observable along a direction $\hat{n}(\theta)$ in the xy-plane making an angle $\theta$ with the initial spin polarised along +x-axis. This is done by suitably orienting the direction $\hat{n}(\theta)$ of the inhomogeneous magnetic field in the Stern-Gerlach (SG) device.

vant to our work.

The important point is that using our scheme one can test experimentally any postulated quantum mechanical approach for calculating the arrival time distribution with different criteria for fixing the initial time of interaction. Note that a number of approaches have been suggested in the literature for calculating the arrival time distribution. We have adopted in our paper one particular approach, viz., the current density approach to calculate the arrival time distribution using a particular method of calculation (e.g. fixing the initial time of interaction in terms of the peak), as an example, to illustrate our general scheme.

Next, we recall that when a spin-polarised particle (say, a neutron) passes through the constant magnetic field within a SR, its spin orientation is rotated by an angle $\phi$ with respect to the initial spin polarised direction along +x axis. This angle is fixed by the time $t$ spent by the particle within the SR, given by the well known quantum mechanical relation $\phi = 2\omega t$ where $\omega = \mu B/\hbar$.

Now, let us consider an ensemble of particles, say, neutrons passing through the SR where initially all members of this ensemble have their spins polarised along, say, the +x axis. Given the same initial spin state they evolve over different times (characterised by $\Pi(t)$, the distribution of transit times within the SR) under the interaction with the constant magnetic field within the SR for many repetitions of the experiment.

Here we would like to stress that in our setup, the very fact that a distribution of spins emerges from the SR implies the existence of a distribution of transit times within the SR, since the spin rotation is proportional to the transit time. Hence in order that a distribution of spins emerges from the SR, the spin part of the wave function going through the SR must necessarily interact with the SR magnetic field over different times in each run of the experiment. It then follows that the unitary evolution operators $U = \exp(-iHt/\hbar)$ are different for each repetition of the experiment, although the spin interaction Hamiltonian $H$ is the same for all of them. Thus the emergent spin states get polarised along different directions and consequently the final ensemble of particles emerging from the SR is in a mixed state of spin states polarised along various directions with different respective probabilities.

Hence we can write the final density matrix of the total ensemble at any time which is large enough so that by which all the particles of the ensemble (total wave packet) have passed through the spin rotator (SR) to be given by

$$W_f = \sum_I \Pi(t) |\chi(t)\rangle \langle \chi(t)|$$  \quad (1)

where $|\chi(t)\rangle$'s occurring in the right hand side of Eq.(1) are the time evolved pure states which have evolved under the given potential within SR over different times (denoted by the symbol "+$\cdot$"). The final density matrix by combining these time evolved spin states is written at a sufficiently large time by which all the members of the ensemble have passed through the SR. Now, using the relation $t = \phi/2\omega$, one can rewrite the density matrix given by Eq.(1) in the following form

$$W_f = \sum_\phi \Pi(\phi)|\chi(\phi)\rangle \langle \chi(\phi)|$$  \quad (2)

where the summation in Eq. (2) is over the different values of $\phi$ corresponding to different values of transit time ($t$) within the SR, and $|\chi(\phi)\rangle$ is the normalised spin state which represents the spin polarization along any direction making an angle $\phi$ with the +x axis. Here $\Pi(\phi)$ is the normalised probability density of spin orientations which is obtained from $\Pi(t)$ through the relation $t = \phi/2\omega$. The quantity $\Pi(\phi)$ $d\phi$ represents the probability of spins emerging from the SR having their orientations within the angles $\phi$ and $\phi + d\phi$. Note that using Eq.(2), it follows that

$$Tr\{W_f\} = \sum_i \langle u_i | W_f | u_i \rangle = \sum_\phi \Pi(\phi) \sum_i |\langle u_i | \chi(\phi)\rangle|^2$$  \quad (3)

where we’ve introduced the states $\{|u_i\rangle\}$ as a complete set of orthonormal basis for any spin state $|\chi(\phi)\rangle$, and hence $\sum_i |\langle u_i | \chi(\phi)\rangle|^2 = 1$. Note that the above result is valid regardless of whether the spin states $|\chi(\phi)\rangle$ for the different values of $\phi$ are orthogonal or not.

Testability of $\Pi(\phi)$ using the Stern-Gerlach device.—Now, for testing the scheme we have outlined for calculating the probability density function $\Pi(\phi)$, let us consider
the measurement of a spin variable, say $\hat{S}_\theta$, by a SG device (Fig.1) in which the inhomogeneous magnetic field is oriented along a direction $\hat{n}(\theta)$ in the xy-plane making an angle $\theta$ with the initial spin-polarised direction (+$\hat{x}$ axis) of the particles. Then for the spins of the particles emerging from the SR polarised along different directions (with the probabilities $\Pi(\phi)$) making angles $\phi$ with the +$\hat{x}$ axis, the probabilities of finding the spin component along +$\theta$ direction and that along its opposite direction are respectively given by

$$P_+(\theta) = \int_0^{2\pi} \Pi(\phi) \cos^2\left(\frac{\theta - \phi}{2}\right) d\phi$$

$$P_-(\theta) = \int_0^{2\pi} \Pi(\phi) \sin^2\left(\frac{\theta - \phi}{2}\right) d\phi$$

where $P_+(\theta) + P_-(\theta) = 1$, and here we are essentially restricting to the situations in which the relevant parameters $d$, $u$ and $B$ are such that the spin rotation angles $\phi$ for all the particles emerging from the SR are restricted between $\phi = 0$ and $\phi = 2\pi$. To explain in more detail how Eqs.(4) and (5) are derived, let us first consider particles passing through the spin rotator (SR) with all their spins oriented along a definite direction, say, x-axis. The initial x-polarised spin state can be written in terms of the z-bases $|\uparrow\rangle_z$ and $|\downarrow\rangle_z$ as $\chi(0) = 1/\sqrt{2} (|\uparrow\rangle_z + |\downarrow\rangle_z)$. Then in such a case, the spin polarised state rotates only in the xy-plane. If $\phi$ is the rotation angle with respect to the initial spin orientation along -$\hat{x}$-axis, such a rotated spin state in the xy-plane can be typically written as $\chi(\phi) = 1/\sqrt{2} (|\uparrow\rangle_z e^{i\phi} |\downarrow\rangle_z)$. Now, for the purpose of measurement after the spins emerge from SR, if one applies SG-magnetic field along a direction $\hat{n}(\theta)$ in the xy-plane making an angle $\theta$ with the -$\hat{x}$-axis, then the bases states for the spin operator $\hat{S}_\theta$ are respectively $|\uparrow\rangle_\theta = 1/\sqrt{2} (|\uparrow\rangle_z + e^{i\theta} |\downarrow\rangle_z)$ and $|\downarrow\rangle_\theta = 1/\sqrt{2} (|\uparrow\rangle_z - e^{i\theta} |\downarrow\rangle_z)$. Then for this spin measurement the probabilities of getting $|\uparrow\rangle_\theta$ and $|\downarrow\rangle_\theta$ are $p_+(\theta) = |\langle\uparrow_\theta | \chi(\phi)\rangle|^2 = \cos^2(\theta - \phi)/2$ and $p_-(\theta) = |\langle\downarrow_\theta | \chi(\phi)\rangle|^2 = \sin^2(\theta - \phi)/2$ respectively. Now, since in our setup, instead of a definite spin polarised state, we have considered a distribution of spin orientations along different directions for the particles emerging from the SR characterised by the distribution function $\Pi(\phi)$, consequently we have the expressions for the probabilities $P_+(\theta)$ and $P_-(\theta)$ as given by Eq.(4) and Eq.(5) respectively. Note that the departures obtained from the prediction given by semiclassical approach (which assumes that, neglecting the effect due to the spreading of the wave packet, the spins of all members of the ensemble rotate by an amount which is determined by the time spend by the peak of the wave packet in traversing the region within the SR) can be observed by sensitive measurements of $P_+(\theta)$ and $P_-(\theta)$.

It is these probabilities $P_+(\theta)$ and $P_-(\theta)$ which constitute the basic observable quantities in this scheme which are determined by the distribution of spins $\Pi(\phi)$ of the particles emerging from the SR. The estimations of these probabilities crucially depend on how one calculates the quantity $\Pi(\phi)$ whose evaluation, in turn, is contingent on the procedure adopted for calculating the relevant time distribution $\Pi(t)$. As mentioned earlier, the specification of such a time distribution is not unique in quantum mechanics. For the setup indicated in Fig.1, $\Pi(t)$ represents the arrival time distribution at the exit point $(x=d)$ of the SR, which is also the distribution of transit times $(t)$ within the SR. In the specific scheme we are using, $\Pi(t)$ is taken to be represented by the modulus of the probability current density $|J(X,t)|$ (suitably normalised) evaluated at the spatial point $X=(x=d,y=0,z=0)$: i.e., we take $\Pi(t) = |J(X,t)|/\int_{-\infty}^{\infty} |J(X,t)|dt$. Leavens [11, 12] has justified the above interpretation of the modulus of probability current density $|J(X,t)|$ as an (unnormalized) arrival time distribution using the Bohm’s causal model of quantum mechanics. Hence $\Pi(\phi) = |J(X,\phi)|/\int_{-\infty}^{\infty} |J(X,\phi)|d\phi$. Thus in this scheme, the calculation of $\Pi(\phi)$ ultimately hinges on evaluating $J(X,\phi)$ from $J(X,t)$.

The evaluation of $\Pi(\phi)$.—Since we have to first calculate $J(X,t)$, we begin by recalling that the standard expression for the non-relativistic quantum probability current density is given by

$$J(x,t) = Re \left[ \psi^*(x,t) \left( -\frac{i\hbar}{m} \nabla \psi(x,t) \right) \right]$$

which satisfies the quantum mechanical equation of continuity given by

$$\frac{\partial \rho}{\partial t} + \nabla . J = 0$$

where the position probability density $\rho(x,t) = \psi^*(x,t) \psi(x,t)$.

Then comes a key point. If one adds any divergence-free term to the above expression for $J(x,t)$, then the new expression also satisfies the same equation of continuity. Hence there is a nonuniqueness inherent in the nonrelativistic expression for the probability current density. Curiously, this point has not been noted even in the premier textbooks like that by Landau and Lifshitz [19], Merzbacher [20]. However, relatively recently this problem of nonuniqueness has been highlighted [21] and it has been pointed out that the probability current density derived from the Dirac equation for any spin-1/2 particle is unique and even in the non-relativistic limit it contains a spin-dependent term which is present in addition to the expression for $J(x,t)$ given by Eq.(6). Interestingly, one can further argue that this property...
of the uniqueness of probability current density is not specific to the Dirac equation, but is a consequence of any relativistic quantum mechanical equation. The argument is as follows.

The probability current density obtained from any consistent relativistic quantum mechanical equation needs to satisfy a covariant form of the continuity equation of $j^\mu$ where the zeroth component of $j^\mu$ is associated with the position probability density. If one replaces $j^\mu$ by $\overrightarrow{j}$ which is also conserved, i.e., $\hat{\partial}_\mu j^\mu = 0$ where $\overrightarrow{j} = j^\mu + a^\mu$ ($a^\mu$ is an arbitrary 4-vector), then the zeroth component of $\overrightarrow{j}$ will have to be the same as the position probability density given by $j^0$. Hence it follows that $a^0 = 0$.

Next, we consider this current as seen from another Lorentz frame. This is given by $\overrightarrow{j}' = j^\mu + a^\mu$. Hence in this frame $\overrightarrow{j}' = j^0 + a^\mu$, and again if the position probability density has to remain unchanged, then one must have $a^0 = 0$. But we know that the only 4-vector whose fourth component vanishes in all frames is the null vector. Thus $a^\mu = 0$. It therefore follows that for any consistent relativistic quantum mechanical equation satisfying the covariant form of the continuity equation, the relativistic current is uniquely fixed. This uniqueness is also preserved in the non-relativistic limit of the relevant relativistic equation.

Now, in order to make this paper self-contained, we briefly recapitulate the key steps involved in deriving the expression for the probability current density in the non-relativistic limit from the Dirac equation in 3 + 1 dimension for a free spin-1/2 particle of rest mass $m_0$ given by

$$i\hbar \frac{\partial \Psi}{\partial t} = \left[ \frac{\hbar c}{i} \alpha_i \frac{\partial}{\partial x_i} + \beta m_0 c^2 \right] \Psi$$

(8)

where

$$\alpha_i = \left( \begin{array}{cc} 0 & \sigma_i \\ \sigma_i & 0 \end{array} \right), \beta = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right), \Psi = \left( \begin{array}{c} \Psi_1 \\ \Psi_2 \end{array} \right)$$

and $\Psi_1, \Psi_2$ are individually two component spinors. Subsequently, Eq. (8) leads to two coupled equations

$$\frac{\partial \Psi_1}{\partial t} = -\sigma_i \frac{\partial \Psi_2}{\partial x_i} - \frac{i}{\hbar} m_0 c^2 \Psi_1$$

(9)

$$\frac{\partial \Psi_2}{\partial t} = -\sigma_i \frac{\partial \Psi_1}{\partial x_i} + \frac{i}{\hbar} m_0 c^2 \Psi_2$$

(10)

Then taking the positive energy solution $\Psi_2 \propto \exp(-ixE/t)$, substituting it in Eq.(10) and putting $E \approx m_0 c^2$ in the non-relativistic regime, we get

$$\Psi_2 = -\frac{i}{2m_0 c} \sigma_i \frac{\partial \Psi_1}{\partial x_i}$$

(11)

Multiplying Eq. (9) by $\Psi_1^\dagger$ from the left and multiplying again the hermitian conjugate of Eq. (9) by $\Psi_1$ from the right, we add the two equations. Substituting the value of $\Psi_2$ from Eq.(11) in this resulting equation, one can then obtain the following equation given by

$$\frac{\partial}{\partial t} (\Psi_1^\dagger \Psi_1) +$$

$$\frac{\partial}{\partial x_i} \left[ \frac{i\hbar}{2m_0} \left( \Psi_1^\dagger \sigma_i \frac{\partial \Psi_1}{\partial x_i} - \left( \frac{\partial \Psi_1^\dagger}{\partial x_i} \sigma_i \right) \Psi_1 \right) \right] = 0$$

(12)

Now, comparing Eq.(12) with Eq.(7), it is seen that the Dirac current in 3 + 1 dimension for a free spin-1/2 neutral particle in the non-relativistic limit is of the form given by

$$\mathbf{J}(\mathbf{x}, t) =$$

$$- \frac{i\hbar}{2m_0} \left( \Psi_1^\dagger \sigma_i \frac{\partial \Psi_1}{\partial x_i} - \left( \frac{\partial \Psi_1^\dagger}{\partial x_i} \sigma_i \right) \Psi_1 \right)$$

(13)

where $\Psi_1$ is a two component spinor which can be written as $\Psi_1 = \psi(\mathbf{x}, t) \chi(t)$. Simplifying Eq.(13) by using the Gordon decomposition [21, 22], one finally obtains

$$\mathbf{J}(\mathbf{x}, t) =$$

$$= \text{Re} \left[ \psi^*(\mathbf{x}, t) \left( - \frac{i\hbar}{m_0} \nabla \psi(\mathbf{x}, t) \right) + \frac{1}{m_0} \left[ \nabla \rho \times \mathbf{s}(t) \right] \right]$$

(14)

$$= \mathbf{J}_{\text{Sch}}(\mathbf{x}, t) + \mathbf{J}_{\text{Spin}}(\mathbf{x}, t)$$

where $\rho = |\psi(\mathbf{x}, t)|^2$, $\mathbf{s}(t) = \frac{\hbar}{2} \chi^\dagger(t) \sigma \chi(t)$, $\sigma = \sigma_x \hat{x} + \sigma_y \hat{y} + \sigma_z \hat{z}$ and $\chi(t)^\dagger \chi(t) = 1$. In Eq.(14), the first term represents the usual Schrodinger current ($\mathbf{J}_{\text{Sch}}(\mathbf{x}, t)$) and the second term gives the contribution of the spin dependent current ($\mathbf{J}_{\text{Spin}}(\mathbf{x}, t)$). The above decomposition is possible because there is no spatial dependence on the spin state $\chi(t)$ which is only time dependent here in our case.

Next, in the context of our setup, in order to evaluate the quantity $\mathbf{J}(\mathbf{x} = \mathbf{X}, \phi)$, we first consider the spatial part of the total wave function. As mentioned earlier, it is taken to be a one dimensional Gaussian wave packet which is peaked at the entry point $(x = 0)$ of the SR at $t = 0$ (Fig. 1). The initial spatial wave function is then given by

$$\psi(x, t = 0) = \frac{1}{(2\pi\sigma_0^2)^{1/4}} \exp \left[-\frac{x^2}{4\sigma_0^2} + ikx \right]$$

(15)

where $\sigma_0$ is the initial width of the associated wave packet. The wave number $k = m_0 u / \hbar$ where $u$ is the group velocity of the wave packet moving along the $+\hat{x}$ axis. Since the spatial part of the total wave function propagates freely, being unaffected by the constant magnetic field confined within the SR, the Schrodinger time evolved spatial wave function calculated from the initial
wave function \( \psi(x,t=0) \) given by Eq. (15) is of the form
\[
\psi(x,t) = \frac{1}{(2\pi A_t^2)^{1/4}} \exp \left[ -\frac{(x-ut)^2}{4A_t\sigma_0} + ik(x - \frac{1}{2} ut) \right]
\]  
and hence the time evolved position probability density is given by
\[
\rho(x,t) = \frac{1}{(2\pi \sigma_t^2)^{1/2}} \exp \left[ -\frac{(x-ut)^2}{2\sigma_t^2} \right]
\]  
where \( A_t = \sigma_0 \left( 1 + \frac{\hbar t}{2m_0\sigma_0^2} \right) \) and \( \sigma_t = |A_t| = \sigma_0 \left( 1 + \frac{\hbar^2 t^2}{4m_0\sigma_0^2} \right)^{1/2} \); \( \sigma_t \) is the width of the wave packet at any instant \( t \).

Next, we consider the spin part of the total wave function. As mentioned earlier, the initial spin of a spin-1/2 neutral particle is taken to be polarized along the +\( \hat{x} \)-axis; i.e., the initial spin state is given by
\[
\chi(0) = |\uparrow_x \rangle = \frac{1}{\sqrt{2}} \left[ |\uparrow_z \rangle + |\downarrow_z \rangle \right]
\]  
and hence
\[
s(0) = \frac{\hbar}{2} \chi^\dagger(0) \sigma \chi(0) = \frac{\hbar}{2} \hat{x}
\]  
Then we proceed to calculate the time evolved spin part of the total wave function under the interaction Hamiltonian \( H = \mu \sigma \cdot \mathbf{B} \). For this purpose we note that the constant magnetic field \( \mathbf{B} = \vec{B} \hat{z} \) within the SR, confined between \( x = 0 \) and \( x = d \), is directed along the +\( \hat{z} \)-axis. Then the time evolved spin state \( \chi(t) \) is given by
\[
\chi(t) = \exp \left( -\frac{iHt}{\hbar} \right) \chi(0) = \frac{1}{\sqrt{2}} \left[ e^{-i\omega t} |\uparrow_z \rangle + e^{i\omega t} |\downarrow_z \rangle \right]
\]  
whence
\[
s(t) = \frac{\hbar}{2} \chi^\dagger(t) \sigma \chi(t) = \frac{\hbar}{2} \left( \cos 2\omega t \hat{x} + \sin 2\omega t \hat{y} \right)
\]  
where \( \omega = \mu B/\hbar \). Given the above expressions for \( \psi(x,t) \), \( \rho(x,t) \) and \( s(t) \) corresponding to Eqs. (16), (17) and (21) respectively, one can now calculate the total probability current density at any given point from Eq.(14). We specifically evaluate it at the exit point \( \mathbf{X}(x=d,y=0,z=0) \) of the SR. Then the expression for the total current density \( \mathbf{J}(\mathbf{X},t) \) reduces to the form given by
\[
\mathbf{J}(\mathbf{X},t) = \rho(x=d,t) \left\{ u + \frac{(d - ut)\hbar^2 t^2}{4m_0^2\sigma_0^4 + \hbar^2 t^2} \right\} \hat{x} + \rho(x=d,t) \left\{ \frac{h(ut - d)}{2m_0\sigma_0^2} \sin 2\omega t \right\} \hat{z}
\]  
The first term in Eq.(22) is the usual Schroedinger current, and the second term represents the additional contribution arising from the spin of the particle. Next, substituting \( t = \phi/2\omega \) in Eq.(22), one gets the following expression for the probability distribution of spin orientations for the particles emerging from the SR given by
\[
\mathbf{J}(\mathbf{X},\phi) = \rho(d,\phi) \left\{ u + \frac{(d - u\phi)\hbar^2}{4m_0^2\sigma_0^4 + \hbar^2 \phi^2} \right\} \hat{x} + \rho(d,\phi) \left\{ \frac{h(u\phi - d)}{2m_0\sigma_0^2} \sin \phi \right\} \hat{z}
\]  
whence
\[
\Pi(\phi) = \frac{\left| \mathbf{J}(\mathbf{X},\phi) \right|}{\int_0^{2\pi} \left| \mathbf{J}(\mathbf{X},\phi) \right| d\phi}
\]  
where \( |\mathbf{J}(\mathbf{X},\phi)| \) is calculated from Eq.(23). Next, we proceed to present the results of a few numerical estimates for the observable probabilities \( P_+(\theta) \) and \( P_-(\theta) \) determined by Eqs. (4) and (5) respectively, where \( \Pi(\phi) \) is calculated using Eqs. (23) and (24).

Numerical estimates for \( P_+(\theta) \) and \( P_-(\theta) \).—These estimates have been done using different values of the relevant parameters; viz. the initial width (\( \sigma_0 \)), the spatial extension (\( d \)) of the constant magnetic field confined within the SR, the group velocity (\( u \)) of the peak of the wave packet and the magnitude of the constant magnetic field (\( B \)).

The choices of these parameters in our calculations are constrained by the condition that the parameter \( d \) should be sufficiently large compared to the half width of the wave packet peaked at the exit point \( (x=d) \) of the SR; also, the relevant parameters \( d \), \( u \) and \( B \) are chosen such that the spin rotation angle \( \phi \) is effectively confined between \( \phi = 0 \) and \( \phi = \pi \) so that the SG magnet does not come in the way of the neutron beam.

Now, for presenting the results of our estimates, we first show a few representative curves (Figs. 2a and 2b) which correspond to the probability density functions \( \Pi(\phi) \) of the spin orientations of the particles emerging from the SR. Note that the curves in Figs. 2a and 2b are respectively associated with two different sets of choices (say, I and II) of the parameters \( d \), \( u \), and \( B \). While the two curves in Fig. 2a correspond to two different values of \( \sigma_0 \) (10^{-5} and 10^{-4}), the two curves in Fig. 2b correspond to those values of \( \sigma_0 \). The curves in Fig. 2a
The quantity $\Pi(\phi)$ denotes the probability density function which represents the distribution of spin orientations of the particles emerging from the SR. This quantity is plotted against the angle of spin rotation $\phi$. The two curves in Fig. 2a correspond to the set I of the choices $d = 1\text{cm}$, $u = 3 \times 10^5\text{cm/s}$ and $B = 10\text{ gauss}$; both these curves are peaked at $\phi = \phi_1 = 34.94767^\circ$. The two curves in Fig. 2b correspond to the set II of the choices $d = 2\text{cm}$, $u = 3 \times 10^5\text{cm/s}$ and $B = 10\text{ gauss}$; both these curves are peaked at $\phi = \phi_2 = 69.89534^\circ$.

Note that both the curves in Fig. 2a represent the probability density function $\Pi(\phi)$ peaked at $\phi = \phi_1 = 34.94767^\circ$, while the curves in Fig. 2b represent $\Pi(\phi)$ peaked at $\phi = \phi_2 = 69.89534^\circ$. For different choices of the initial width ($\sigma_0$) with the other relevant parameters $d$, $u$ and $B$ remaining fixed, it is seen from each of Figs. 2a and 2b that the qualitative nature of these curves and the location of their peak remain the same, while their variances differ, increasing with decreasing values of $\sigma_0$.

Next, a few representative results of the numerical computations based on using Eqs. (4), (5) and (24) are given in Tables 1 and 2 which correspond respectively to the sets of values I and II of the parameters $d$, $u$ and $B$ (corresponding to the Fig. 2a and Fig. 2b respectively). The results shown in any one of these Tables indicate how the observable quantities $P_+(\theta)$ and $P_-(\theta)$ vary with different initial widths $\sigma_0$, corresponding to a specific orientation ($\theta$) of the inhomogeneous magnetic field in the SG device (Fig. 1).

It is seen that for a given value of $\theta$, for both the sets I and II of the choices of the relevant parameters $d$, $u$ and $B$, the variation in the values of the probabilities $P_+(\theta)$ and $P_-(\theta)$ is very small, but is detectable for smaller values of $\sigma_0$. More comprehensive estimates of $P_+(\theta)$, $P_-(\theta)$ and fluctuations in the distribution of spins emerging from the SR, for a wider variation of $\theta$ and other relevant parameters $d$, $u$ and $B$, will be presented in a later study.

Table. 1. The quantities $P_+(\theta)$ and $P_-(\theta)$ denote probabilities of the spin measurement along different directions $\hat{n}(\theta)$ making angles $\theta$ with the initial spin polarised along $+\hat{x}$ - axis. The numerical values of $P_+(\theta)$ and $P_-(\theta)$ are calculated for different initial widths $\sigma_0$ of the Gaussian wave packet. In this Table, the results are presented for three different values of $\theta$ for the set I of the values of the other relevant parameters $d = 1\text{cm}$, $u = 3 \times 10^5\text{cm/s}$ and $B = 10\text{ gauss}$, while $\phi_1 = 34.94767^\circ$ at which the curve of $\Pi(\phi)$ is peaked.

| $\theta$ | $\sigma_0$ (cm) | $P_+(\theta)$ | $P_-(\theta)$ |
|----------|-----------------|----------------|----------------|
| $\phi_1$ | $10^{-8}$ | 1.00000 | 0.00000 |
| $\phi_1$ | $10^{-6}$ | 0.75000 | 0.25000 |
| $\phi_1$ | $10^{-4}$ | 0.50000 | 0.50000 |

Table. 2. The quantities $P_+(\theta)$ and $P_-(\theta)$ denote probabilities of the spin measurement along different directions $\hat{n}(\theta)$ making angles $\theta$ with the initial spin polarised along $+\hat{x}$ - axis. The numerical values of $P_+(\theta)$ and $P_-(\theta)$ are calculated for different initial widths $\sigma_0$ of the Gaussian wave packet. In this Table, the results are presented for three different values of $\theta$ for the set II of the values of the other relevant parameters $d = 2\text{cm}$, $u = 3 \times 10^5\text{cm/s}$ and $B = 10\text{ gauss}$, while $\phi_2 = 69.89534^\circ$ at which the curve of $\Pi(\phi)$ is peaked.

| $\theta$ | $\sigma_0$ (cm) | $P_+(\theta)$ | $P_-(\theta)$ |
|----------|-----------------|----------------|----------------|
| $\phi_2$ | $10^{-8}$ | 1.00000 | 0.00000 |
| $\phi_2$ | $10^{-6}$ | 0.75000 | 0.25000 |
| $\phi_2$ | $10^{-4}$ | 0.50000 | 0.50000 |

Summary and Outlook.—In the setup discussed in this paper, the observable quantities are the probabilities $P_+(\theta)$ and $P_-(\theta)$ which correspond to the measurement of a spin variable along any direction by a SG device performed on the particles emerging from the SR. Evaluations of these quantities crucially depend on the probability distribution $\Pi(\phi)$ of the orientations of spins of the particles emerging from the SR. This in turn depends on the quantity $\Pi(t)$ which corresponds to the distribution of transit times over which the particles interact with the magnetic field while passing through the SR.

The quantity $\Pi(t)$ is calculated in this paper for spin-1/2 particles in terms of the modulus of the probability
current density. Hence the estimates presented here for the observable probabilities $P_+ (\theta)$ and $P_- (\theta)$ are ultimately determined by the modulus of the probability current density. Thus if the experimental results for such a setup corroborate such predictions, this would constitute a verification of the observability of the probability current density.

As mentioned earlier, because of an inherent nonuniqueness, there are also other quantum mechanical approaches which can be used to evaluate the time distribution $\Pi (t)$, apart from the specific scheme we’ve used in this paper based on the modulus of the probability current density. In the context of our setup, it should be instructive to derive the respective pre-

It should definitely be instructive to study the situation taking a non-Gaussian wave function for the spatial part which is proposed to be done as a sequel to this work, apart from other sequels based on different criteria for fixing the initial time of interaction and the duration over which a propagating wave packet interacts with the localised potential.

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[1] J. G. Muga and C. R. Leavens, Phys. Rep. 338, 353 (2000).
[2] Time in Quantum Mechanics, edited by J. G. Muga, R. Sala Mayato and I. L. Egusquiza (Springer-Verlag, Berlin, 2002).

[3] J. Kijowski, Rept. Math. Phys. 6, 361 (1974); N. Yamada, S. Takagi, Prog. Theor. Phys. 86, 599 (1991); Y. Aharanov, D. Bohm, Phys. Rev. 122, 1649 (1961); P. Busch, M. Grabowski, P. J. Lahti, Phys. Lett. A, 357, 353 (1994); N. Grot, C. Rovelli, R. S. Tate, Phys. Rev. A, 54, 4676 (1996); V. Delgado, J. G. Muga, Phys. Rev. A, 56, 3425 (1997).

[4] C.R. Leavens, Phys. Lett. A 178, 27 (1993).
[5] W.R. McKinnon and C.R. Leavens, Phys. Rev. A 51, 2748 (1995).
[6] J. M. Damborenea, I. L. Egusquiza, G. C. Hegerfeldt, and J. G. Muga, Phys. Rev. A, 66, 052104 (2002).

[7] G.C.Hegerfeldt, D.Seidel and J.G.Muga, Phys. Rev. A, 68, 022111 (2003).
[8] K. K. Kan and J. J. Griffin, Phys. Rev. C, 15, 1126 (1977).
[9] L.Landau, J. Phys. 5, 71 (1941).
[10] C. R. Leavens, Phys Lett. A, 178, 27 (1993); J.G. Muga, S. Brouard, and D. Macias, Ann. Phys. 240, 351 (1995); C. R. Leavens, Phys. Rev. A, 58, 840 (1998); V. Delgado, Phys. Rev A, 59, 1010 (1999); C.R.Leavens, Phys. Lett A, 303, 154 (2002); Md. M. Ali, A.S. Majumdar, D. Home and S. Sengupta, Phys. Rev. A, 68, 042105 (2003).
[11] V.S. Olkhovsky and E. Recami, Phys. Rep. 214, 339 (1992); R. S. Dumont and T. L. Marchioro II, Phys. Rev. A, 47, 85 (1993); J. G. Muga, S. Brouard and R. Sala, Phys. Lett. A, 167, 24 (1992); J.G.Muga and H. Cruz, Physica B, 179, 326 (1992); S. Brouard, R. Sala, and J.G. Muga, Phys. Rev. A, 49, 4312 (1994); W.R. McKinnon and C.R. Leavens, Phys. Rev. A, 51, 2748 (1995); A. Chalminor, A. Lasenby, S. Somaroo, C. Doran and S. Gull Phys. Lett. A, 227, 143 (1997).
[12] Atomic and Molecular Beam Methods, edited by G. Scholes (Oxford University Press, Oxford, 1988), Chapters 7, 8, 9, 14, p. 268.
[13] W. Pauli, in Encyclopedia of Physics, edited by S. Flugge (Springer, Berlin, 1958), Vol. 5/1, p. 60.
[14] G. R. Alcock, Ann. Phys. (N.Y.), 53, 253 (1969); 53, 286 (1969); 53, 311 (1969); E. H. Hauge and J. A. Stonner, Rev. Mod. Phys. 61, 917 (1989); W.G. Unruh and R.M. Wald, Phys. Rev. D, 40, 2598 (1989); Y. Aharonov, J. Oppenheim, S. Popescu, B. Reznik and W. G. Unruh, Phys. Rev. A, 57, 4130 (1998); C. R. Leavens, Phys. Lett. A, 303, 154 (2002).
[15] H. Salecker and E. P. Wigner, Phys. Rev. 109, 571 (1958); A. Peres, Am. J. Phys. 48, 552 (1980); J. J. Halliwell, Prog. Theor. Phys. 102, 707 (1999); A. D. Baute, I. L. Eusquiza and J. G. Muga, Phys. Rev. A, 64, 041401 (2001); P.C.W. Davies, Class. Quantum Grav. 21, 2761 (2004).
[16] See, for example, the recent experiment by Y. Hasegawa, R. Loidl, G. Badurek, M.Baron and H. Rauch, Nature, 425, 45 (2003) and the relevant references cited therein.
[17] A. I. Baz, Sov. J. Nucl. Phys. 5, 161 (1967); M. Buttiker, Phys. Rev. B, 27, 6178 (1983).
[18] J. J. Sakurai, *Modern Quantum Mechanics* (Addison-Wesley, Reading, MA) pp. 76-78.

[19] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics - Nonrelativistic Theory* (Pergamon Press, Oxford, 1965), p. 57.

[20] E. Merzbacher, *Quantum Mechanics* (John Wiley and Sons, New York, 1961), pp. 26-27.

[21] P. Holland, Phys. Rev. A, 60, 4326 (1999); Ann. Phys. (Leipzig) 12, 446 (2003); W. Struyve, W. De Baere, J. De Neve and S. De Weirdt, Phys. Lett. A, 322, 84 (2004).

[22] W. Gordon, Z. Phys, 50, 630 (1928).