A FULL-MODIFIED-NEWTON STEP $O(n)$ INFEASIBLE INTERIOR-POINT METHOD FOR THE SPECIAL WEIGHTED LINEAR COMPLEMENTARITY PROBLEM

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(Communicated by Wenxun Xing)

Abstract. The weighted complementarity problem (wCP) can be applied to a large variety of equilibrium problems in science, economics and engineering. Since formulating an equilibrium problem as a wCP may lead to highly efficient algorithms for its numerical solution, wCP is a nontrivial generalization of the complementarity problem. In this paper we consider a special weighted linear complementarity problem (wLCP), which is the more general optimization of the Fisher market equilibrium problem. A full-modified-Newton infeasible interior-point method (IIPM) for the special wLCP is proposed. The algorithm reformulates the central path of the perturbed problem as an equivalent system of equations, and uses only full-Newton steps at each iteration, so-called a feasibility step (i.e., a full-modified-Newton step) and several usual centering steps. The polynomial complexity of the algorithm is as good as the best known iteration bound for these types of IIPMs in linear optimization.

1. Introduction. The notion of a weighted complementarity problem (wCP), introduced by Potra [14] in 2012, significantly extends the notion of a complementarity problem (CP) [5, 11]. wCP is to find a pair of vectors belonging to the intersection of a manifold with a cone, such that the product of the vectors in a certain algebra equals a given weight vector. With a zero weight vector, wCP reduces to a CP. wCP can be used to model a larger class of equilibrium problems from science, economics and engineering, such as the linear programming and weighted centering problem [1], the Fisher market equilibrium problem [24], and so on.

2020 Mathematics Subject Classification. Primary: 90C33; Secondary: 90C51.
Key words and phrases. Weighted linear complementarity problem, infeasible interior-point method, full-modified-Newton step, polynomial complexity.

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The Fisher market equilibrium problem [14, 24] can be reformulated as the following weighted linear complementarity problem (wLCP)

\[
\begin{align*}
Ax &= b, \quad x \geq 0, \\
A^Ty + s &= c, \quad s \geq 0, \\
x^s &= w,
\end{align*}
\]

(1)

where \(A \in \mathbb{R}^{m \times n}\) is a given matrix, \(b \in \mathbb{R}^m\) is a given vector, and \(w \in \mathbb{R}^n_+\) is a given weight vector (the data of the problem).

In this paper, we consider the more general problem of the form (1), which consists of finding vectors \((x, s, y) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m\) such that

\[
\begin{align*}
Ax &= b, \quad x \geq 0, \\
A^Ty + s &= c, \quad s \geq 0, \\
x^s &= w,
\end{align*}
\]

(2)

with arbitrary \(A \in \mathbb{R}^{m \times n}\), \(b \in \mathbb{R}^m\), \(c \in \mathbb{R}^n\) and \(w \in \mathbb{R}^n_+\). We assume that \(A\) has full row rank. Obviously, (2) is a special wLCP over the nonnegative orthant.

The special wLCP (2) is called feasible (or strictly feasible) if \(\mathcal{F} \neq \emptyset\) (or \(\mathcal{F}^0 \neq \emptyset\)), where

\[
\begin{align*}
\mathcal{F} &= \{z = (x, s, y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \times \mathbb{R}^m : Ax = b, A^Ty + s = c\}, \\
\mathcal{F}^0 &= \{z = (x, s, y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \times \mathbb{R}^m : Ax = b, A^Ty + s = c\},
\end{align*}
\]

respectively. The solution set of wLCP (2) is denoted by

\[
\mathcal{F}^* = \{z = (x, s, y) \in \mathcal{F} : x^s = w\}.
\]

Throughout this paper, we always assume \(\mathcal{F}^0 \neq \emptyset\). Then the special wLCP (2) is skew-symmetric and strictly feasible, and hence by Theorem 4 in [15] it has a solution.

wCP is in connection with interior-point methods (IPM) [10]. In 2008, Ye [24] developed an IPM for solving the Fisher problem with linear utilities, which followed a nonsmooth central path. Potra [14] defined a different smooth central path and then proposed two IPMs for solving general monotone wLCP. This [14] also showed that compared to the nonlinear CP model, the wLCP model can lead to more efficient algorithms for solving the Fisher market equilibrium problem. In 2016, Potra [15] introduced the notion of a sufficient wLCP, generalized the characterization of the sufficient linear complementarity problem [5] to the sufficient wLCP, and proposed a corrector-predictor IPM for its numerical solution. In 2019, Chi, Gowda and Tao [4] considered the weighted horizontal linear complementarity problem on a Euclidean Jordan algebra, and established some existence and uniqueness results.

Moreover, smoothing Newton methods [13, 16, 21] can also be used to solve wCP, since wCP can be reformulated as a system of equations based on a smoothing function [3, 7]. Zhang [25] presented a smoothing Newton algorithm for solving monotone wLCP and proved its global and local quadratic convergence. Tang [19] proposed a variant nonmonotone smoothing algorithm for solving monotone wLCP with improved numerical results.

The first IPM with polynomial complexity was proposed by Karmarkar [9] for solving linear optimization (LO). A feasible IPM [2, 6] can be started only if a strictly feasible point is known. Wang, Yu and Teo [20] presented a full-Newton step
feasible interior-point algorithm for $P_*(\kappa)$-linear complementarity problems. Usually finding such a starting point is difficult. In that case an infeasible interior-point method (IIPM) [12] with an arbitrary positive starting point may be a good alternative. Roos [17] first proposed a full-Newton step IIPM for solving LO with the best known iteration bound for IIPMs, which uses only full-Newton steps. Subsequently, Gu, Mansouri, Zangiabadi, Bai and Roos [8] presented an improved full-Newton step IIPM for LO. Xu, Zhang, and Zhang [22] proposed a full-modified-Newton step IIPM for LO.

Motivated by the recent work of Potra and Roos [14, 17], in this paper we present a full-modified-Newton step IIPM for the special wLCP (2). By extending the full-Newton step IIPM for LO, our algorithm can start with an arbitrary “large” positive point. Each main iteration consists of only full-Newton steps, so-called a feasibility step (i.e., a full-modified-Newton step) and several centering steps. The algorithm is shown to possess the polynomial complexity as good as the best known iteration bound for these types of methods in LO.

The paper is organized as follows. In Section 2, we introduce the perturbed problem of the special wLCP, its central path, as well as the induced Newton direction, and then propose a full-modified-Newton step IIPM for the special wLCP. In Section 3, we analyze the centering step, the feasibility step, and the convergence of the algorithm. The polynomial complexity is derived in Section 4. Numerical results are reported in Section 5. Concluding remarks are presented in Section 6.

Conventions. The 2-norm and the infinity norm are denoted by $\| \cdot \|$ and $\| \cdot \|_\infty$, respectively. The symbol $e$ represents the vector of all-ones with dimension $n$. We denote the componentwise operations on vectors by the usual notation for real numbers. Thus, if $x, s \in \mathbb{R}^n$, $xs$ will denote the vector with components $x_is_i$, and $x$ will denote the vector whose $i$th is equal to $x_is_i$, for $i = 1, 2, \ldots, n$. $\min v$ denotes the minimum component of a vector $v$.

2. Full-modified-Newton step IIPM. In this section, we outline a full-modified-Newton step IIPM for solving the special wLCP (2).

We start with choosing an arbitrary $z^0 := (x^0, s^0, y^0)$ with $x^0 > 0$ and $s^0 > 0$ such that $x^0s^0 \geq w$. Denote

$$\kappa = x^0s^0, \quad t_0 = \frac{(x^0)^T(s^0)}{n},$$

$$w(t) = \left(1 - \frac{t}{t_0}\right)w + \frac{t}{t_0}\kappa, \quad t \in (0, t_0].$$

The central path of wLCP (2) is the curve given by the set of all points $(z; t) = (x, s, y; t)$, with $t \in (0, t_0]$, satisfying

$$Ax = b, \quad x \geq 0,$$

$$A^T y + s = c, \quad s \geq 0,$$

$$xs = w(t).$$

As $t$ goes to zero, the solutions $(x(t), s(t), y(t))$ of (5) converge to solutions of the special wLCP (2). The weighted complementarity condition $xs = w(t)$ is nonlinear, which makes it hard to directly solve the system (5). However, by applying Newton’s method we can find an approximate solution.
2.1. Perturbed problem and equivalent reformulation. For any $0 < \nu \leq 1$, we consider the perturbed problem $(wLCP_\nu)$ of the special $wLCP$ (2)

$$b - Ax = \nu r^0_b, \quad x \geq 0,$$

$$c - A^T y - s = \nu r^0_c, \quad s \geq 0,$$

$$xs = w,$$  \hspace{1cm} (6)

where the initial residual vectors $r^0_b$ and $r^0_c$ are defined by

$$r^0_b := b - Ax^0, \quad r^0_c := c - A^T y^0 - s^0.$$  

Note that if $\nu = 1$, then $(x, s, y) = (x^0, s^0, y^0)$ is a strictly feasible solution of wLCP$_\nu$ (6). We conclude that if $\nu = 1$, then wLCP$_\nu$ (6) satisfies the interior-point condition (IPC), i.e., wLCP$_\nu$ (6) has a feasible solution $(x(\nu), s(\nu), y(\nu))$ with $x(\nu) > 0$ and $s(\nu) > 0$. By following the proof of Theorem 5.13 in LO [23], we obtain the following result.

**Lemma 2.1.** The original special wLCP (2) is feasible, i.e., $\mathcal{F} \neq \emptyset$, if and only if for any $\nu$ satisfying $0 < \nu \leq 1$ the perturbed problem wLCP$_\nu$ (6) satisfies the IPC.

We define the central path of wLCP$_\nu$ (6) emanating from $z^0$ as the set of all points $(z; t) = (x, s, y; t)$ with $t \in (0, t_0]$, satisfying

$$b - Ax = \nu r^0_b, \quad x \geq 0,$$

$$c - A^T y - s = \nu r^0_c, \quad s \geq 0,$$

$$xs = w(t).$$  \hspace{1cm} (7)

From Lemma 2.1 and Proposition 3 in [15], (7) is strictly feasible with the skew-symmetric triplet, and hence has a unique solution $(x(t, \nu), s(t, \nu), y(t, \nu))$ for any $t \in (0, t_0]$. By construction, $(z^0, t_0) = (x^0, s^0, y^0, t_0) = (x(t_0, 1), s(t_0, 1), y(t_0, 1); t_0)$ belongs to this path.

A proximity measure of a point $z = (x, s, y)$ to the central path (7) is given by

$$\delta(x, s; t) := \delta(v) := \|e - v\|, \quad \text{where} \quad v := \sqrt{\frac{xs}{w(t)}} > 0.$$  \hspace{1cm} (8)

By (8),

$$xs = w(t) \iff v^2 = e \iff v = e \iff \delta(x, s; t) = 0 \iff v^2 = v.$$  

Substituting the last relation into (7) yields a new perturbed system for the special wLCP (2)

$$b - Ax = \nu r^0_b, \quad x \geq 0,$$

$$c - A^T y - s = \nu r^0_c, \quad s \geq 0,$$

$$xs = w(t)v.$$  \hspace{1cm} (9)

In the IIPM, we call the triple $(x, s, y)$ an $\varepsilon$–approximate solution of wLCP (2) if the 2-norms of the residual vectors $w(t) - w$, $b - Ax$ and $c - A^T y - s$ do not exceed $\varepsilon$, and also the proximity measure $\delta(x, s; t)$ satisfies $\delta(x, s; t) \leq \varepsilon$. The IIPM does not require a feasible starting point and generates an $\varepsilon$–approximate solution of wLCP (2).
2.2. The algorithm for the special wLCP. Now we outline the method. At a main iteration of the algorithm, suppose that for some $t \in (0, t_0]$ and a threshold parameter $\tau \in (0, 1)$ we have an iterate $(x, s, y)$ satisfying the feasibility conditions, i.e., the first two equations in (7) for $\nu = t/t_0$, and such that $\|w(t) - w\| = \nu\|x^0s^0 - w\|$ and $\delta(x, s; t) \leq \tau$. At the start of the first iteration this is certainly true, since $x^0 = e$ and $\delta(x^0, s^0; t_0) = 0$. We reduce $t$ to $t_+ = (1 - \theta)t$, with $\theta \in (0, 1)$. And then we find a new iterate $(x^+, s^+, y^+)$ that satisfies the first two equations in (7), with $t$ replaced by $t_+$ and $\nu$ by $\nu_+ = (1 - \theta)\nu = t_+/t_0$, and such that $\|w(t_+) - w\| = \nu_+\|x^0s^0 - w\|$ and $\delta(x^+, s^+; t_+) \leq \tau$.

More precisely, each main iteration consists of a feasibility step and several centering steps. The feasibility step serves to get an iterate $(x^f, s^f, y^f)$ that is strictly feasible for wLCP and closes to its $t_+$-center $(x(t_+, \nu_+), s(t_+, \nu_+), y(t_+, \nu_+))$. In fact, the feasibility step is designed in such a way that $\delta(x^f, s^f; t_+ \leq 1/4$, i.e., $(x^f, s^f, y^f)$ belongs to the quadratic convergence neighborhood with respect to the $t_+$-center of wLCP (see Corollary 1). By performing several centering steps starting at $(x^f, s^f, y^f)$ and targeting at the $t_+$-center of wLCP, we can get a new iterate $(x^+, s^+, y^+)$ that is strictly feasible for wLCP and $\delta(x^+, s^+; t_+) \leq \tau < 1/4$.

In the following, we describe the feasibility step. By extending the full-modified-Newton step for LO [22], we obtain from (9) the following linear system of equations in the search direction $(\Delta^f x, \Delta^f s, \Delta^f y)$ for the special wLCP (2):

$$
\begin{align*}
A \Delta^f x &= \theta \nu r_0^0, \quad x \geq 0, \\
A^T \Delta^f y + \Delta^f s &= \theta \nu r_0^0, \quad s \geq 0, \\
s \Delta^f x + x \Delta^f s &= w(t_+) - xs,
\end{align*}
$$

with $\theta \in (0, 1)$. Since $A$ has full row rank, and the vectors $x$ and $s$ are positive, the system (10) uniquely defines $(\Delta^f x, \Delta^f s, \Delta^f y)$. After the feasibility step, the new iterate is given by

$$
(x^f, s^f, y^f) = (x + \Delta^f x, s + \Delta^f s, y + \Delta^f y),
$$

which satisfies the first two equations in (6) with $\nu_+ = (1 - \theta)\nu$. Notice that in LO the duality gap and feasibility residuals are usually reduced at the same rate, and here we reduce $t$ to $t_+ = (1 - \theta)t$ and $\nu$ to $\nu_+ = (1 - \theta)\nu$ for the special wLCP (2).

In a centering step, we solve the following system for the search direction $(\Delta x, \Delta s, \Delta y)$

$$
\begin{align*}
A \Delta x &= 0, \quad x \geq 0, \\
A^T \Delta y + \Delta s &= 0, \quad s \geq 0, \\
s \Delta x + x \Delta s &= w(t) - xs,
\end{align*}
$$

which (uniquely) gives the usual search direction for feasible primal-dual IPMs. The new iterate after a centering step is given by

$$
(x^+, s^+, y^+) = (x + \Delta x, s + \Delta s, y + \Delta y).
$$

---

**Algorithm 2.1** (A full-modified-Newton step IIPM for the special wLCP)

**Input:**

- Accuracy parameter $\varepsilon > 0$;
- barrier update parameter $\theta$, $0 < \theta < 1$;
threshold parameter $\tau > 0$. 

\textbf{begin} 

\begin{align*}
  x &:= x^0 > 0; \\
  s &:= s^0 > 0; \\
  y &:= y^0; \\
  \kappa &= x^0s^0; \\
  t &:= t_0 = \frac{(x^0)^T(s^0)}{n}; \\
  \nu &= 1;
\end{align*}

\textbf{while} \( \max\{\delta(x,s;t), ||w(t) - w||, ||b - Ax||, ||c - A^Ty - s||\} > \varepsilon \) \textbf{do} 

\textbf{begin} 

feasibility step: 

\begin{align*}
  (x, s, y) &:= (x, s, y) + (\Delta f x, \Delta f s, \Delta f y); \\ \\
  t &:= (1 - \theta) t; \\ \\
  \nu &:= (1 - \theta) \nu;
\end{align*}

centering steps: 

\begin{align*}
  \textbf{while} \ \delta(x,s;t) > \tau \ \textbf{do} \ \\
  (x, s, y) &:= (x, s, y) + (\Delta x, \Delta s, \Delta y);
\end{align*}

\textbf{endwhile} 

\textbf{end} 
\textbf{end} 

The following two lemmas describe the lower and upper bounds on the components of the vector $v$, and estimate the norm of $v$. They will be used in the subsequent analysis of Algorithm 2.1.

\textbf{Lemma 2.2.} \[26\] For any $v \in \mathbb{R}^n$, one has 

$$1 - \delta(v) \leq v_i \leq 1 + \delta(v), \quad i = 1, \ldots, n.$$ 

\textbf{Lemma 2.3.} \[22\] For any $v \in \mathbb{R}^n$, one has 

$$\|v\| \leq \delta(v) + \sqrt{n}.$$ 

\textbf{3. Analysis of the algorithm.} We start Algorithm 2.1 with the specified initial point 

$$x^0 = s^0 = \zeta e, \quad y^0 = 0,$$

where $\zeta > 0$ is a constant such that $\max\{\|x^*\|_{\infty}, \|s^*\|_{\infty}\} \leq \zeta$ for some optimal solution $(x^*, s^*, y^*)$ of wLCP (2). Then 

$$0 \leq x^0 - x^* \leq x^0 = \zeta e, \quad 0 \leq s^0 - s^* \leq s^0 = \zeta e.$$ 

By (3) and (4), 

$$\kappa = x^0s^0 = \zeta^2 e \geq w, \quad t_0 = \frac{(x^0)^T(s^0)}{n} = \zeta^2,$$

$$\min w(t) = \min \left[ \left(1 - \frac{t}{t_0}\right) w + \frac{t}{t_0} \zeta^2 e \right] \geq \min w.$$ 

\textbf{3.1. Analysis of the centering step.} The centering step starts with an iterate $(x, s, y)$ that is strictly feasible for wLCP$_{\nu}$ (6) with $\nu = t/t_0$ and such that $\delta(x, s; t) \leq 1/4$. This implies that $(x, s, y)$ is within the quadratic convergence neighborhood of the central path (see Corollary 1). We perform several centering steps to find a new iterate $(x^+, s^+, y^+)$ that is still strictly feasible for wLCP$_{\nu}$ and satisfies $\delta(x^+, s^+, t) \leq \tau$ with $\tau < 1/4$. 
By (8), let us denote
\[ d_x := \frac{v\Delta x}{x}, \quad d_s := \frac{v\Delta s}{s}, \quad v^+ := \sqrt{\frac{x^+ s^+}{w(t)}}, \] (13)
where the search direction \((\Delta x, \Delta s, \Delta y)\) is defined by (12). Let
\[ A := AV^{-1}, \quad V := \text{diag}(v), \quad X := \text{diag}(x), \quad W(t) := \text{diag}(w(t)). \] (15)

Then the system (12) can be reformulated as
\[ Ad_x = 0, \]
\[ W(t)^{-1}A^T \Delta y + d_s = 0, \]
\[ d_x + d_s = v^{-1} - v. \] (16)

**Lemma 3.1.** Let \(\delta := \delta(v) < 2 - \sqrt{2} \approx 0.5858\), then the centering step is strictly feasible, and
\[ \delta(v^+) \leq \frac{\sqrt{2} (2 - \delta^2)}{1 + \sqrt{1 - \frac{2 - \delta^2}{1 - \delta} \delta^2}}. \] (17)

**Proof.** For the proof, we introduce a centering step length \(\beta \in [0,1]\) and define
\[ x(\beta) = x + \beta \Delta x, \quad s(\beta) = s + \beta \Delta s. \]

Then we obtain from (8), (12) and (13) that
\[ x(\beta)s(\beta) = (x + \beta \Delta x)(s + \beta \Delta s) \]
\[ = xs + \beta(x \Delta s + s \Delta x) + \beta^2 \Delta x \Delta s \]
\[ = xs + \beta(w(t) - x) + \beta^2 \frac{x^s}{y^2} d_x d_s \]
\[ = (1 - \beta)x + \beta(e + \beta d_x d_s)w(t). \] (18)

Since it follows from (16) that \(d_x^T d_s = 0\), we have by Lemma C.4 in [18] that
\[ ||d_x d_s|| \leq \frac{1}{4} ||d_x + d_s||^2, \quad ||d_x d_s|| \leq \frac{\sqrt{2}}{4} ||d_x + d_s||^2. \] (19)

From Lemma 2.2, (8) and (16), we obtain
\[ ||d_x + d_s|| = ||v^{-1} - v|| = ||(e + v^{-1})(e - v)|| \]
\[ \leq \left(1 + \frac{1}{1 - \delta} \right) ||e - v|| \]
\[ = \left(\frac{2 - \delta}{1 - \delta} \right) \delta. \] (20)
If $\delta < 2 - \sqrt{2}$, it follows from (18), (19) and (20) that
\[
x(\beta)s(\beta) \geq (1 - \beta)xs + \beta(1 - \|d_x + d_s\|)w(t)
\geq (1 - \beta)xs + \beta \left(1 - \frac{1}{4} \left(\frac{2 - \delta}{1 - \delta}\right)^2 \delta^2\right)w(t)
\geq (1 - \beta)xs + \beta \left[1 - \frac{1}{4} \left(\frac{2 - \delta}{1 - \delta}\right)^2 \delta^2\right]w(t)
> 0.
\]
Thus none of the entries of $x(\beta)$ and $s(\beta)$ vanishes for $0 \leq \beta \leq 1$. Since $x(0) = x > 0$, $s(0) = s > 0$, and $x(\beta)$ and $s(\beta)$ depend linearly on $\beta$, we have $x(\beta) > 0$ and $s(\beta) > 0$ for $0 \leq \beta \leq 1$. Hence $x^+ = x(1) > 0$ and $s^+ = s(1) > 0$.

Since by (13) and (18) we have
\[
(v^+)^2 = \frac{x^+ s^+}{w(t)} = e + d_x d_s,
\]
it follows from (14), (19) and (20) that
\[
\delta(v^+) = \|e - v^+\| \leq \frac{\|e - (v^+)^2\|}{1 + \min v^+}
\leq \frac{\|d_x d_s\|}{1 + \min \sqrt{e + d_x d_s}}
\leq \frac{\|d_x d_s\|}{1 + \sqrt{1 - \|d_x d_s\|}}
\leq \frac{\sqrt{2} \|d_x + d_s\|^2}{1 + \sqrt{1 - \frac{1}{4} \|d_x + d_s\|^2}}
\leq \frac{\sqrt{2} \left(\frac{2 - \delta}{1 - \delta}\right)^2 \delta^2}{1 + \sqrt{1 - \frac{1}{4} \left(\frac{2 - \delta}{1 - \delta}\right)^2 \delta^2}}.
\]

**Corollary 1.** If $\delta := \delta(v) \leq \frac{1}{4}$, then $\delta(v^+) \leq \delta^2$.

Indeed, assuming $\delta(x^f, s^f; t^+_n) \leq \frac{1}{4}$, Corollary 1 implies that the Newton process targeting at the $t^+_n$–center of wLCP $\nu^+_n$ is quadratically convergent. After $k$ centering steps, we will obtain the feasible iterate $(x^+, s^+; t^+_n)$ for wLCP $\nu^+_n$ such that
\[
\delta(x^+, s^+; t^+_n) \leq \left(\frac{1}{4}\right)^k.
\]
Thus $\delta(x^+, s^+; t^+_n) \leq \tau$ holds after at most
\[
\left\lfloor \log_2 \left(\log_2 \frac{1}{\tau}\right) \right\rfloor - 1 \tag{21}
\]
centering steps.
3.2. Analysis of the feasibility step. At the beginning of the feasibility step, we have an iterate \((x, s, y)\) that is strictly feasible for \(\text{WLCP}_\nu\) (6) with \(\nu = t/t_0\) and such that \(\|w(t) - w\| = \nu\|x^0s^0 - w\|\) and \(\delta(x, s; t) \leq \tau < 1/4\). We reduce \(t\) and \(\nu\) to \(t_+ = (1 - \theta)t\) and \(\nu_+ = (1 - \theta)\nu\), respectively, with \(\theta \in (0, 1)\). The feasibility step produces a new iterate \((x^f, s^f, y^f)\) that is strictly feasible for \(\text{WLCP}_{\nu_+}\) and satisfies \(\|w(t_+) - w\| = \nu_+\|x^0s^0 - w\|\) and \(\delta(x^f, s^f; t_+) \leq 1/4\). The hard part in this section is to guarantee that \(x^f\) and \(s^f\) are positive and satisfy \(\delta(x^f, s^f; t_+) \leq 1/4\).

Define

\[
d_x^f := \frac{\nu \Delta^f x}{x}, \quad d_s^f := \frac{\nu \Delta^f s}{s}, \quad v^f := \frac{x^f s^f}{w(t_+)} \tag{22}
\]

where the search direction \((\Delta^f x, \Delta^f s, \Delta^f y)\) is defined by (10). It follows from (8), (15) and (22) that the system of equations (10) can be expressed in terms of the scaled search directions \(d_x^f\) and \(d_s^f\) as

\[
\begin{align*}
\mathcal{A}d_x^f &= \theta \nu \nu_+^0, \\
W(t)^{-1}A^T \Delta^f y + d_s^f &= \theta \nu \nu_+ s^{-1} \nu_+^0, \\
d_x^f + d_s^f &= \frac{w(t_+)}{w(t)} e - v,
\end{align*}
\tag{23}
\]

with \(\theta \in (0, 1)\).

**Lemma 3.2.** One has

\[
(v^f)^2 = v + \frac{w(t)}{w(t_+)} d_x^f d_s^f.
\]

**Proof.** Using (11) and (22), one has

\[
\begin{align*}
x^f &= x + \Delta^f x = x \left( e + \frac{\nu d_x^f}{v} \right) = \frac{x}{v} (v + d_x^f), \tag{24} \\
s^f &= s + \Delta^f s = s \left( e + \frac{\nu d_s^f}{v} \right) = \frac{s}{v} (v + d_s^f). \tag{25}
\end{align*}
\]

We obtain from (22), (23), (24) and (25) that

\[
(v^f)^2 = \frac{x^f s^f}{w(t_+)} = \frac{x s}{w(t_+)} v^2 (v + d_x^f) (v + d_s^f) = \frac{x s}{w(t_+)} v^2 + v (d_x^f + d_s^f) + d_x^f d_s^f = \frac{w(t)}{w(t_+)} v^2 + v \left( \frac{w(t_+)}{w(t)} e - v \right) + d_x^f d_s^f = v + \frac{w(t)}{w(t_+)} d_x^f d_s^f.
\]

\[\square\]

**Lemma 3.3.** If

\[
\|d_x^f d_s^f\|_\infty < (1 - \delta(v)) \left( (1 - \theta) + \theta \min \frac{w}{\kappa} \right),
\]

then the iterate \((x^f, s^f, y^f)\) is strictly feasible such that

\[
(x^f)^T s^f \leq 2e^T w(t_+).
\]
Proof. We introduce a feasibility step length \( \alpha \in [0,1] \), and define
\[
x^f(\alpha) = x + \alpha \Delta^f x, \quad s^f(\alpha) = s + \alpha \Delta^f s.
\]
By (22) and (23), we have
\[
\frac{x^f(\alpha)s^f(\alpha)}{w(t)} = (v + \alpha d^f_x)(v + \alpha d^f_s)
\]
\[
= v^2 + \alpha v \left( \frac{w(t_+)}{w(t)} e - v \right) + \alpha^2 d^f_x d^f_s
\]
\[
= (1 - \alpha)v^2 + \alpha \left( \frac{w(t_+)}{w(t)} v + \alpha d^f_x d^f_s \right). \tag{26}
\]
Since \( t_+ = (1 - \theta)t \), we obtain from (4) that
\[
w(t_+) = [1 - t(1 - \theta)/t_0]w + [t(1 - \theta)/t_0]\kappa - \theta w + \theta w
\]
\[
= (1 - \theta)w(t) + \theta w. \tag{27}
\]
Now suppose \( \|d^f_x d^f_s\|_\infty < (1 - \delta(v)) \left[ (1 - \theta) + \theta \min \frac{w}{\kappa} \right] \), which implies
\[
d^f_x d^f_s > -(1 - \delta(v)) \left[ (1 - \theta) + \theta \min \frac{w}{\kappa} \right] e. \tag{28}
\]
Substituting (27) and (28) into (26) gives
\[
\frac{x^f(\alpha)s^f(\alpha)}{w(t)} > (1 - \alpha)v^2 + \alpha \left\{ \frac{(1 - \theta)w(t) + \theta w}{w(t)} v - \alpha (1 - \delta(v)) \right\}
\]
\[
\cdot \left[ (1 - \theta) + \theta \min \frac{w}{\kappa} \right] e
\]
\[
\geq (1 - \alpha)v^2 + \alpha [v - \alpha(1 - \delta(v))e] \left[ (1 - \theta) + \theta \min \frac{w}{\kappa} \right] \geq 0
\]
for \( 0 \leq \alpha \leq 1 \), where the last inequality is due to Lemma 2.2. Then using the same argument as Lemma 3.1, it is not difficult to show that \( x^f = x^f(1) > 0 \) and \( s^f = s^f(1) > 0 \).
Moreover, since \( \|d^f_x d^f_s\|_\infty < (1 - \delta(v)) \left[ (1 - \theta) + \theta \min \frac{w}{\kappa} \right] \), it follows from Lemma 2.2, Lemma 3.2, (22) and (27) that
\[
(x^f)^T s^f = e^T (x^f s^f) = e^T [w(t_+)(v^f)^2]
\]
\[
= w(t_+)^T \left( v + \frac{w(t)}{w(t_+)} d^f_x d^f_s \right)
\]
\[
\leq w(t_+)^T \left\{ (1 + \delta(v))e + (1 - \delta(v)) \left[ (1 - \theta) + \theta \min \frac{w}{\kappa} \right] \frac{w(t)}{w(t_+)(1 - \theta)w(t) + \theta w} \right\}
\]
\[
\leq w(t_+)^T [1 + \delta(v)]e + (1 - \delta(v))e
\]
\[
= 2e^T w(t_+). \tag{29}
\]
Let
\[
\omega(v) := \frac{1}{2} \sqrt{\|d^f_x\|^2 + \|d^f_s\|^2}.
\]
Then \( \|d^f_x\| \leq 2\omega(v), \|d^f_s\| \leq 2\omega(v) \), and
\[
(d^f_x)^T d^f_s \leq \|d^f_x\| \|d^f_s\| \leq 2\omega(v)^2, \tag{30}
\]
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Combining Lemma 3.3 with (31) yields the following result to guarantee strict feasibility of the iterate $(x^f, s^f, y^f)$ in Algorithm 2.1.

**Corollary 2.** If

$$
\omega(v) < \sqrt{\frac{(1 - \delta(v))(1 - \theta \min \frac{w}{\kappa})}{2}},
$$

then the iterate $(x^f, s^f, y^f)$ is strictly feasible.

By (8) and (22), we have

$$
\delta(v^f) := \delta(x^f, s^f; t_f) := \|e - v^f\|.
$$

(32)

To obtain an upper bound for $\delta(v^f)$, we present the following lemmas.

**Lemma 3.4.** If the iterate $(x^f, s^f, y^f)$ is strictly feasible, then

$$
\min v^f \geq 1 - \delta(v) - \frac{2\omega(v)^2}{(1 - \theta) + \theta \min \frac{w}{\kappa}}.
$$

*Proof.* By Lemma 2.2, Lemma 3.2, (27) and (31), we have

$$
\min (v^f)^2 = \min \left(v + \frac{w(t)}{w(t_f)} d^f_x d^f_s\right)
$$

$$
\geq \min v - \left\| \frac{w(t)}{(1 - \theta)w(t_f) + \theta w} \right\| \cdot \|d^f_x d^f_s\|_{\infty}
$$

$$
\geq 1 - \delta(v) - \frac{2\omega(v)^2}{(1 - \theta) + \theta \min \frac{w}{\kappa}}
$$

which completes the proof.

**Lemma 3.5.** One has

$$
\|e - (v^f)^2\| \leq \|e - v\| + \frac{2\omega(v)^2}{(1 - \theta) + \theta \min \frac{w}{\kappa}}.
$$

*Proof.* From Lemma 3.2, (27) and (31), we obtain

$$
\|e - (v^f)^2\| = \left\| e - v - \frac{w(t)}{w(t_f)} d^f_x d^f_s \right\|
$$

$$
\leq \|e - v\| + \left\| \frac{w(t)}{(1 - \theta)w(t_f) + \theta w} \right\| \cdot \|d^f_x d^f_s\|_{\infty}
$$

$$
\leq \|e - v\| + \frac{2\omega(v)^2}{(1 - \theta) + \theta \min \frac{w}{\kappa}}
$$

$$
\leq \|e - v\| + \frac{2\omega(v)^2}{(1 - \theta) + \theta \min \frac{w}{\kappa}}.
$$

□
Lemma 3.6. If the iterate \((x^f, s^f, y^f)\) is strictly feasible, then

\[
\|e - v^f\| \leq \frac{\|e - (v^f)^2\|}{1 + \sqrt{\frac{1 - \delta(v) - \frac{2\omega(v)^2}{(1 - \theta) + \theta \min \frac{w}{\kappa}}}{1 - \delta(v) - \frac{2\omega(v)^2}{(1 - \theta) + \theta \min \frac{w}{\kappa}}}.
\]

Proof. By Lemma 3.4, we have

\[
\|e - v^f\| \leq \frac{\|e - (v^f)^2\|}{1 + \min v^f}
\]

\[
\leq \frac{\|e - (v^f)^2\|}{1 + \sqrt{\frac{1 - \delta(v) - \frac{2\omega(v)^2}{(1 - \theta) + \theta \min \frac{w}{\kappa}}}{1 - \delta(v) - \frac{2\omega(v)^2}{(1 - \theta) + \theta \min \frac{w}{\kappa}}}.
\]

\]

Now we derive an upper bound for \(\delta(v^f)\), and then give a threshold value of \(\omega(v)^2\) such that \(\delta(v^f) \leq \frac{1}{4}\).

Lemma 3.7. If the iterate \((x^f, s^f, y^f)\) is strictly feasible, one has

\[
\delta(v^f) \leq \frac{\chi(v)}{1 + \sqrt{1 - \chi(v)}},
\]

where \(\chi(v) := \delta(v) + \frac{2\omega(v)^2}{(1 - \theta) + \theta \min \frac{w}{\kappa}}\). Assuming \(\delta(v) \leq \tau < \frac{1}{4}\), if

\[
\omega(v)^2 \leq \left(\frac{7}{32} - \frac{\tau}{2}\right) \left[(1 - \theta) + \theta \min \frac{w}{\kappa}\right],
\]

we have \(\delta(v^f) \leq \frac{1}{4}\).

Proof. It follows from (8), (32), Lemma 3.5 and Lemma 3.6 that

\[
\delta(v^f) = \|e - v^f\|
\]

\[
\leq \frac{\|e - (v^f)^2\|}{1 + \sqrt{\frac{1 - \delta(v) - \frac{2\omega(v)^2}{(1 - \theta) + \theta \min \frac{w}{\kappa}}}{1 - \delta(v) - \frac{2\omega(v)^2}{(1 - \theta) + \theta \min \frac{w}{\kappa}}}.
\]

\[
\leq \frac{\|e - v\| + \frac{2\omega(v)^2}{(1 - \theta) + \theta \min \frac{w}{\kappa}}}{1 + \sqrt{\frac{1 - \delta(v) - \frac{2\omega(v)^2}{(1 - \theta) + \theta \min \frac{w}{\kappa}}}{1 - \delta(v) - \frac{2\omega(v)^2}{(1 - \theta) + \theta \min \frac{w}{\kappa}}}.
\]

\[
= \frac{\chi(v)}{1 + \sqrt{1 - \chi(v)}},
\]
To guarantee $\delta(v') \leq \frac{1}{4}$, the inequality (33) implies that it suffices to have
\[
\chi(v) := \delta(v) + \frac{2\omega(v)^2}{(1 - \theta) + \theta \min \frac{w}{\kappa}} \leq \frac{7}{16}.
\]

Assuming $\delta(v) \leq \tau < \frac{1}{4}$, we obtain from the last relation that
\[
\omega(v)^2 \leq \left( \frac{7}{32} - \frac{\tau}{2} \right) \left[ (1 - \theta) + \theta \min \frac{w}{\kappa} \right].
\]

\Box

3.3. **An upper bound for $\|d^f\|^2 + \|d_s^f\|^2$.** In order to obtain an upper bound for $\omega(v)$, we consider the vectors $d^f$ and $d_s^f$ and estimate the value of $\|d^f\|^2 + \|d_s^f\|^2$.

Suppose that $w > 0$ in the subsequent analysis. Let $(\bar{x}, \bar{y}, \bar{z}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$ be a solution of wLCP (2) such that
\[
A\bar{x} = b, \quad 0 \leq \bar{x} \leq \zeta e,
\]
\[
A^T\bar{y} + \bar{z} = c, \quad 0 \leq \bar{z} \leq \zeta e,
\]
\[
\bar{x} \bar{y} = w. \tag{34}
\]

Since each iterate $(x, s, y)$ generated by Algorithm 2.1 is feasible for wLCP$_\nu$ (6) with $0 < \nu \leq 1$, we have
\[
b - Ax = \nu(b - A\zeta e), \quad x \geq 0,
\]
\[
c - A^T y - s = \nu(c - \zeta e), \quad s \geq 0,
\]
and hence
\[
A[x - (1 - \nu)\bar{x} - \nu\zeta e] = 0, \quad x \geq 0,
\]
\[
A^T[y - (1 - \nu)\bar{y}] + [s - (1 - \nu)\bar{z} - \nu\zeta e] = 0, \quad s \geq 0.
\]

By the last system, we have
\[
[x - (1 - \nu)\bar{x} - \nu\zeta e]^T[s - (1 - \nu)\bar{z} - \nu\zeta e] = 0,
\]
which implies
\[
[(1 - \nu)\bar{x} + \nu\zeta e]^T[(1 - \nu)\bar{y} + \nu\zeta e] + x^Ts = [(1 - \nu)\bar{x} + \nu\zeta e]^Ts + x^T[(1 - \nu)\bar{y} + \nu\zeta e] \geq \nu \zeta e^T(x + s).
\]

Since $\max\{\|x^*\|_\infty, \|s^*\|_\infty\} \leq \zeta$, we also obtain from Lemma 3.3, (4), (34) and (35) that
\[
[(1 - \nu)\bar{x} + \nu\zeta e]^T((1 - \nu)\bar{y} + \nu\zeta e) + x^Ts \\
\leq (1 - \nu)^2 \zeta^2 e^T w + \nu^2 \zeta^2 e^T e + \nu(1 - \nu)\zeta e^T(\bar{x} + \bar{y}) + 2e^T w(\nu t_0) \\
\leq (1 - \nu)^2 \zeta^2 e^T w + \nu^2 \zeta^2 + 2n(1 - \nu)^2 + 2e^T [(1 - \nu)w + \nu\zeta^2 e] \\
= [(1 - \nu)^2 + 2(1 - \nu)\nu + \nu^2 + 2\nu(1 - \nu) + 2
\]
\[
\leq (\nu^2 - 4\nu + 3)\zeta^2 + (-\nu^2 + 4\nu)\zeta^2 \\
= 3n\zeta^2.
\]

Hence
\[
e^T(x + s) \leq \frac{3n\zeta}{\nu}. \tag{36}
\]
In what follows we derive an upper bound for \( \|d^f_x\|^2 + \|d^f_s\|^2 \). By (15), we have 

\[ A = AD\sqrt{W(t)}, \]

where

\[ D := \text{diag} \left( \frac{xv^{-1}}{\sqrt{w(t)}} \right) = \text{diag} \left( \sqrt{\frac{x}{s}} \right) = \text{diag} \left( \sqrt{w(t)}vs^{-1} \right). \quad (37) \]

Then the system of equations (23) could be reformulated as

\[ AD\sqrt{W(t)}d^f_x = \theta vs^0, \]

\[ (\sqrt{W(t)})^{-1}DA^T \Delta^f y + d^f_s = \theta vvs^{-1}r^0, \]

\[ d^f_x + d^f_s = \frac{w(t_\nu)}{w(t)} e - v. \]

**Lemma 3.8.** Let \( \delta := \delta(v) \). Then

\[ \|d^f_x\|^2 + \|d^f_s\|^2 \leq 4(1 - \theta)^2 \delta^2 + 4\theta^2 \left[ n + (\delta + \sqrt{n})^2 - 2 \left( \frac{w}{\kappa} \right)^T v \right] + \frac{27\theta^2n^2\zeta^4}{(1 - \delta)^2 \min w^2}. \]

**Proof.** Let \( u_x := \theta v \left( \sqrt{W(t)} \right)^{-1} D^{-1}(\bar{x} - x^0), u_s := \theta v \left( \sqrt{W(t)} \right)^{-1} D(\bar{s} - s^0), \text{ and} r := \frac{w(t_\nu)}{w(t)} e - v. \) By following the proof in LO \[8\], we have

\[ \|d^f_x\|^2 + \|d^f_s\|^2 \leq 2\|r\|^2 + 3(\|u_x\|^2 + \|u_s\|^2). \quad (38) \]

From the definition of \( r \), (8) and (27),

\[
\|r\|^2 = \left\| \frac{w(t_\nu)}{w(t)} e - v \right\|^2 = \left\| (1 - \theta)w(t) + \theta w \left( \frac{w}{w(t)} - v \right) \right\|^2 \\
= \left\| (1 - \theta)(e - v) + \theta \left( \frac{w}{w(t)} - v \right) \right\|^2 \\
= (1 - \theta)^2\|e - v\|^2 + \theta^2 \left\| \frac{w}{w(t)} - v \right\|^2 + 2\theta(1 - \theta)(e - v)^T \left( \frac{w}{w(t)} - v \right) \\
\leq 2(1 - \theta)^2\|e - v\|^2 + 2\theta^2 \left\| \frac{w}{w(t)} - v \right\|^2 \\
= 2(1 - \theta)^2 \delta^2 + 2\theta^2 \left[ \left\| \frac{w}{w(t)} \right\|^2 + \|v\|^2 - 2 \left( \frac{w}{w(t)} \right)^T v \right] \\
\leq 2(1 - \theta)^2 \delta^2 + 2\theta^2 \left[ n + (\delta + \sqrt{n})^2 - 2 \left( \frac{w}{\kappa} \right)^T v \right],
\]

where the last inequality is due to Lemma 2.3 and (4). Since it follows from Lemma 2.2 and (36) that

\[ e^T \left( \frac{x}{sw(t)} + \frac{s}{xw(t)} \right) = e^T \left( \frac{x^2 + s^2}{xsw(t)} \right) \\
\leq \frac{[e^T(x + s)]^2}{\min w^2 \min w^2} \\
\leq \frac{\nu^2 \min w^2}{9n^2\zeta^2} \leq \frac{\nu^2(1 - \delta)^2 \min w^2}{9n^2\zeta^2}.
\]
we have by the definitions of \( u_x \) and \( u_s \), and (37) that
\[
\|u_x\|^2 + \|u_s\|^2
= \vartheta^2 \nu^2 \left( \left\| \left( \sqrt{W(t)} \right)^{-1} \bar{D} (\bar{x} - x^0) \right\|^2 + \left\| \left( \sqrt{W(t)} \right)^{-1} \bar{D} (\bar{s} - s^0) \right\|^2 \right)
\leq \vartheta^2 \nu^2 \zeta^2 \left( \left\| \left( \sqrt{W(t)} \right)^{-1} \bar{D} \right\|^2 e^T \left( \frac{x}{sw(t)} + \frac{s}{xw(t)} \right) \right)
\leq \frac{9 \vartheta^2 \nu^2 \zeta^4}{(1 - \delta)^2 \min \omega^2}.
\]

Thus, from (38),
\[
\|d^f_x\|^2 + \|d^f_s\|^2 \leq 2\|r\|^2 + 3(\|u_x\|^2 + \|u_s\|^2)
\leq 4(1 - \theta)^2 \delta^2 + 4\theta^2 \left[ n + (\delta + \sqrt{n})^2 - 2 \left( \frac{w}{\kappa} \right)^T v \right] + \frac{27 \vartheta^2 \nu^2 \zeta^4}{(1 - \delta)^2 \min \omega^2}.
\]

Now we estimate the value of the barrier parameter \( \theta \) such that the feasibility step is well-defined, i.e., \((x^f, s^f, y^f)\) is strictly feasible and in the quadratic convergence neighborhood of the \( t_+ \)-center of wLCP\( v_+ \).

**Theorem 3.9.** Let \( \beta := \frac{\zeta^2}{\min \omega} \), where \( x^0 = s^0 = \zeta e > 0, y^0 = 0 \) and \( \zeta^2 e \geq \omega \). If
\( \delta := (v) \leq \tau \) with \( \tau = \frac{1}{16} \) and \( \theta = \frac{2}{15 \beta \nu} \), the iterate \((x^f, s^f, y^f)\) is strictly feasible such that \( \delta(v^f) \leq \frac{1}{4} \).

**Proof.** If \( \tau = \frac{1}{16} \) and \( \theta = \frac{2}{15 \beta \nu} \) with \( \beta = \frac{\zeta^2}{\min \omega} \), it follows from Lemma 2.2, Lemma 3.8 and (29) that
\[
4 \omega(v)^2 = \|d^f_x\|^2 + \|d^f_s\|^2
\leq 4(1 - \theta)^2 \delta^2 + 4\theta^2 \left[ n + (\delta + \sqrt{n})^2 - 2 \left( \frac{w}{\kappa} \right)^T v \right] + \frac{27 \vartheta^2 \nu^2 \zeta^4}{(1 - \delta)^2 \min \omega^2}
\leq \frac{3}{4} \left[ (1 - \theta) + \theta \min \left( \frac{w}{\kappa} \right) \right]
\leq 2 \left( \beta(v) \right) \left[ (1 - \theta) + \theta \min \left( \frac{w}{\kappa} \right) \right].
\]
The last relation, together with Corollary 2 and Lemma 3.7, implies that the iterate \((x^f, s^f, y^f)\) is strictly feasible and satisfies \( \delta(v^f) \leq \frac{1}{4} \). \( \square \)

4. **Complexity analysis.** In this section, the polynomial-time complexity for Algorithm 2.1 is derived. By Theorem 3.9, if at the start of a main iteration the iterate satisfies \( \delta(x, s; t) \leq \tau \) with \( \tau = \frac{1}{16} \), then after the feasibility step, with \( \theta = \frac{2 \min \omega}{15 \nu \zeta^2} \),
the iterate satisfies $\delta(x^f, s^f; t_+) \leq \frac{1}{4}$. According to (21), at most
\[
\left\lfloor \log_2 \left( \log_2 \frac{1}{\tau} \right) \right\rfloor - 1 = \log_2(\log_2 16) - 1 \leq 1
\]
centering step suffices to get the iterate $(x^+, s^+, y^+)$ satisfying $\delta(x^+, s^+; t_+) \leq \tau$. So, each main iteration consists of at most 2 inner iterations, one feasibility step and at most one centering step.

Since it follows from Algorithm 2.1 and (4) that
\[
\|w(t_+) - w\| = \frac{t_+}{t_0} \|\kappa - w\| = (1 - \theta) \frac{t}{t_0} \|\zeta^2 e - w\|,
\]
the norms of the residual vectors and $t \in (0, t_0]$ are reduced by the factor $1 - \theta$ at each main iteration. Then we can find a solution such that $\max\{\|w(t) - w\|, \|b - Ax\|, \|c - A^T y - s\|\} \leq \varepsilon$ after at most
\[
\frac{2}{\theta} \log \frac{\max\{\|\zeta^2 e - w\|, \|r_0^b\|, \|r_0^c\|\}}{\varepsilon}
\]
inner iterations. If $\delta(x, s; t) > \varepsilon$ still holds, after one feasibility step and $k$ centering steps, it follows from Corollary 1 that the iterate $(x^+, s^+, y^+)$ satisfies
\[
\delta(x^+, s^+; t_+) \leq \left( \frac{1}{4} \right)^2 \leq \varepsilon,
\]
which gives the smallest integer
\[
k := \left\lfloor \log_2 \left( \log_2 \frac{1}{\varepsilon} \right) \right\rfloor - 1 \geq \log_2 \left( \log_2 \frac{1}{\varepsilon} \right) - 1.
\]
Hence, the total number of the inner iterations is bounded above by
\[
\frac{2}{\theta} \log \frac{\max\{\|\zeta^2 e - w\|, \|r_0^b\|, \|r_0^c\|\}}{\varepsilon} + \left\lfloor \log_2 \left( \log_2 \frac{1}{\varepsilon} \right) \right\rfloor.
\]
Since $\theta = 2 \min w 15n\zeta^2$, the main result can be stated in the following theorem.

**Theorem 4.1.** Suppose that the special $wLCP$ (2) is strictly feasible with $w > 0$, $A$ has full row rank, and $\zeta > 0$ is such that $\max\{\|x^s\|_\infty, \|s^s\|_\infty\} \leq \zeta$ for some solution $(x^s, s^s, y^s)$ of $wLCP$ (2). If the parameters $\tau$ and $\theta$ are defined as $\tau = \frac{1}{16}$ and $\theta = \frac{2 \min w}{15n\zeta^2}$, then Algorithm 2.1 finds an $\varepsilon$-approximate solution of the special $wLCP$ (2) after at most
\[
\frac{15n\zeta^2}{\min w} \log \frac{\max\{\|\zeta^2 e - w\|, \|r_0^b\|, \|r_0^c\|\}}{\varepsilon} + \left\lfloor \log_2 \left( \log_2 \frac{1}{\varepsilon} \right) \right\rfloor
\]
inner iterations.

5. **Numerical examples.** In this section, we have conducted some numerical experiments to evaluate the efficiency of Algorithm 2.1 for solving the special $wLCP$ (2). All the experiments were performed on a personal computer with Intel(R) Xeon(R) CPU E3-1226 v3 @ 3.30GHz 3.30GHz and 4.00GB memory. The operating system was Windows 10 and the implementations were done in MATLAB 9.4.0 (R2018a).
The algorithm was tested on some randomly generated special wLCPs. In detail, we generated a random matrix $A \in \mathbb{R}^{m \times n}$ with full row rank, a random weight vector $\omega > 0$, and random vectors $x \in \mathbb{R}^n_+$, $s \in \mathbb{R}^n_+$, $y \in \mathbb{R}^m$, and then let $b := Ax$, $c := A^T y + s$. Thus the generated special wLCPs (2) are strictly feasible, and therefore have solutions. Let $x^0 = s^0 = \zeta e > 0$ and $y^0 = 0 \in \mathbb{R}^m$ be initial points, where $\zeta^2 e \geq \omega$. The parameters were chosen as follows:

$$\theta = \frac{2 \min \omega}{15n\zeta^2}, \quad \tau = \frac{1}{16}.$$  

We used $\max\{\delta(v), \|\omega(t) - \omega\|, \|b - Ax\|, \|c - A^T y - s\|\} \leq 10^{-5}$ as the stopping criterion.

**Table 1.** Numerical results for the special wLCPs

| Time(s) | Iter | $t$   | $\delta(v)$ |
|---------|------|-------|-------------|
| 0.027   | 156  | 4.3155e-05 | 1.1389e-11 |
| 0.032   | 172  | 8.8496e-06 | 1.7788e-13 |
| 0.019   | 109  | 1.3410e-05 | 7.6732e-13 |
| 0.105   | 206  | 3.9157e-06 | 3.3555e-14 |
| 0.084   | 97   | 6.7376e-05 | 1.0484e-11 |
| 0.081   | 118  | 3.1367e-06 | 1.9156e-14 |
| 0.128   | 218  | 2.7493e-05 | 2.0044e-12 |
| 0.096   | 168  | 2.1576e-05 | 4.5414e-12 |

Table 1 shows numerical results for the special wLCP instances of size $m = 10$ and $n = 20$, with the CPU time in seconds (Time(s)), the number of iterations (Iter), and the values of $t$ and $\delta(v)$ at the $\varepsilon$–approximate solutions of the special wLCPs.

**Figure 1.** The value of $rb := \|b - Ax\|$
Then we generate random wLCP instances with each size of $m = 20, n = 40; m = 50, n = 100; m = 100, n = 200; \text{ and } m = 200, n = 400$. Figures 1-3 show that the norm of the residual vectors $\|b - Ax\|, \|c - A^Ty - s\|$ and $\|\omega(t) - \omega\|$ depend linearly on $t$. Figure 4 indicates that at the start of the first iteration $\delta(v^0) = 0$. Moreover, as $t \to 0$, all the residuals $\|b - Ax\|, \|c - A^Ty - s\|, \|\omega(t) - \omega\|$ and the proximity measure $\delta(v)$ converge to zero.
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6. Conclusions. We have presented a full-modified-Newton step IIPM for the special wLCP. As a generalization of CP, wCP [14] has a wider range of applications in economics, engineering and science. By extending the full-modified-Newton step IIPM for LO [22], we apply Newton's method to the transformed system of the central path for the special wLCP to get the new search directions. The algorithm possesses some good properties. Firstly, the algorithm can start from an arbitrary “large” positive point, without a strictly feasible starting point. Secondly, the algorithm uses only full-Newton steps at each iteration, and does not require any line search. Thirdly, under suitable assumptions, the iterates are shown to stay in the quadratic convergence neighborhood of the central path. Finally, the polynomial complexity in the algorithm is as good as the best known iteration bound for these types of methods in LO.

Acknowledgments. The authors are grateful to the editor and the anonymous referees for their valuable suggestions which have greatly improved this paper. This research is supported by the National Natural Science Foundation of China (Nos. 11861026, 11871383, 11661002), and Guangxi Natural Science Foundation (No. 2016GXNSFBA380102), China.

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Received September 2020; revised February 2021.

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