The orthogonal projection of a fixed point on the tangent lines of a given curve yields a pedal curve of that curve. The aim of this study is to examine some special curves, such as pedal curves, which have singular points even for regular curves, in the Minkowski plane. For this, we investigate an anti-pedal and a primitive of a curve, which is closely related to the pedal curve. The primitive of a curve is a curve that is provided by the inverse construction to make pedal. Using the envelope of a family of functions, we obtain the notion of primitive for the curves in the Minkowski plane. Then, we show that an anti-pedal of the original curve is equal to the inversion image of the pedal curve. Moreover, we analyze the relationships between primitive and anti-pedal of the curve using the inversion. We also present examples that provide our results.
of primitoids, which are relatives of the primitive. In another study, they examined pedal, anti-pedal and primitive for quadratic curves (Izumiya and Takeuchi 2019b).

In this study, we look at anti-pedals and primitives, both of which have singularities even for regular curves. Especially, we consider curves in the Minkowski plane. Our conclusions are Lorentzian analogue to the results of Izumiya and Takeuchi (2020). Then, we define the notions of anti-pedal, primitive in the Minkowski plane and examine the relationships between them.

2. Material and Method

It is well known that the Minkowski plane $\mathbb{R}^2_1$ is the plane $\mathbb{R}^2$ allowing the metric produced by the scalar product $\langle u, v \rangle = -u_1v_1 + u_2v_2$ where $u = (u_1, u_2)$ and $v = (v_1, v_2)$. The vectors in $\mathbb{R}^2_1$ are classified as follows by this product:

If $(u, u) > 0$ or $u = 0$, then $u$ is spacelike. If $(u, u) = 0$ or $(u, u) < 0$ for a non-zero vector, then $u$ is lightlike or timelike, respectively (O’Neill 1983).

The norm of a vector $u = (u_1, u_2) \in \mathbb{R}^2_1$ is given by $\|u\| = \sqrt{\langle u, u \rangle}$ and the vector $u^\perp = (u_2, u_1)$, which is orthogonal to $u$ (Izumiya et al. 2018). Furthermore, the signature of $u$ is indicated by $\varepsilon$ and so $\frac{(u, u)}{\|u\|^2} = \varepsilon$.

Let $\gamma: I \to \mathbb{R}^2_1$ be a regular curve, which is parametrized by an open interval $I$. For any $s \in I$, the curve is a spacelike curve, a timelike curve, a lightlike curve if $\langle \gamma'(s), \gamma'(s) \rangle > 0$, $\langle \gamma'(s), \gamma'(s) \rangle < 0$, $\langle \gamma'(s), \gamma'(s) \rangle = 0$, respectively. In addition, $\gamma'(s)$ is the velocity vector of $\gamma$ and is written as $\gamma'(s) = \frac{d\gamma}{ds}(s)$. If a curve $\gamma$ is timelike or spacelike, we call it a non-lightlike curve (Li and Sun 2019).

Assume that $\gamma: I \to \mathbb{R}^2_1$ is a non-lightlike curve with an arc-length parameter $s$ such that $\|\gamma'(s)\| = 1$. In this situation, $T(s) = \gamma'(s)$ is the unit tangent vector with $sgn T(s) = \langle T(s), T(s) \rangle = \varepsilon$ and $N(s)$ is the unit normal vector with $sgn N(s) = \langle N(s), N(s) \rangle = -\varepsilon$.

Hence, we write the Frenet formula:

\[
\frac{d}{ds} \begin{bmatrix} T(s) \\ N(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) \\ -\kappa(s) & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \end{bmatrix}
\]

where $\kappa(s)$ is the curvature of $\gamma$ (Li and Sun 2019).

**Definition 2.1.** For a fixed $\phi$, the envelope of the family of lines defined by $F(s, x)$ consists of the points $x$ in the plane where $s$ exists with (Giblin and Warder 2014)

\[
F(s, x) = \frac{dF}{ds}(s, x) = 0.
\]

**Definition 2.2.** Let $\gamma$ be a non-lightlike regular curve in Minkowski plane. A pedal of curve $\gamma$ is given by (Aydin Şekerci and Izumiya 2021)

\[
Pe_{\gamma}(s) = -\varepsilon(\gamma(s), N(s))N(s).
\]

3. Results and Discussion

Let $\gamma: I \to \mathbb{R}^2_1$ be a non-lightlike curve with arc-parameter in Minkowski plane and there are no lightlike points. We define a family of functions $H: I \times (\mathbb{R}^2_1 \setminus \{0\}) \to \mathbb{R}$

\[
(s, x) \mapsto H(s, x) = (x - \gamma(s), \gamma(s)).
\]

For fixed $s \in I, H(s, x) = 0$ is an equation of the line through $\gamma(s)$ and orthogonal to the position vector $\gamma(s)$.

The envelope of family of lines is the primitive of curve $\gamma$. According to that, we obtain

\[
\frac{dH}{ds}(s, x) = (-\gamma'(s), \gamma'(s)) + (x - \gamma(s), \gamma'(s)) = (x - 2\gamma(s), T(s)).
\]

Any vector in $\mathbb{R}^2_1$ is represented by a linear combination as $\lambda T(s) + \xi N(s)$. Using this linear combination for the vector $x - \gamma(s)$ and substituting $\frac{dH}{ds}(s, x) = 0$, we obtain

\[
\langle \lambda T(s) + \xi N(s) - \gamma(s), T(s) \rangle = 0.
\]

Then, we have the following equation for $\lambda$:

\[
\lambda = \varepsilon(T(s), \gamma(s)).
\]

Also, considering $H(s, x) = 0$, we get

\[
H(s, x) = (x - \gamma(s), \gamma(s)) = 0.
\]

Moreover, from $x - \gamma(s) = \lambda T(s) + \xi N(s)$, we have

\[
\lambda(T(s), \gamma(s)) + \xi(N(s), \gamma(s)) = 0.
\]
Using \( \lambda \) we write
\[
\varepsilon(T(s), y(s))^2 + \xi(N(s), y(s)) = 0. \tag{10}
\]

Thus, we find
\[
\xi = -\varepsilon \frac{(T(s), y(s))^2}{(N(s), y(s))}. \tag{11}
\]

Taking into account \( \lambda \) and \( \xi \),
\[
x = y(s) + \varepsilon(T(s), y(s))T(s) - \frac{\varepsilon(T(s), y(s))^2}{(N(s), y(s))}N(s).
\]

From \( y(s) = \varepsilon(T(s), y(s))T(s) - \varepsilon(N(s), y(s))N(s) \), we write
\[
x = y(s) + y(s) + \varepsilon(N(s), y(s))N(s)
- \frac{\varepsilon(T(s), y(s))^2}{(N(s), y(s))}N(s)
= 2y(s) + \varepsilon \left[ \frac{(N(s), y(s))^2}{(N(s), y(s))} \right] N(s)
- \varepsilon \left[ \frac{(T(s), y(s))^2}{(N(s), y(s))^2} \right] N(s).
\]

Using the norm of \( y(s) \), which is given as
\[
\|y(s)\|^2 = sgn y(s) \left[ \frac{\varepsilon(T(s), y(s))^2}{(N(s), y(s))} - \varepsilon(N(s), y(s))^2 \right],
\]
there exists
\[
x = 2y(s) - sgn y(s) \frac{\|y(s)\|^2}{(N(s), y(s))} N(s).
\]

**Definition 3.1.** The primitive \( Pr_Y: I \to \mathbb{R}^2_+ / \{0\} \) of Minkowski plane curve \( y \) is given by
\[
Pr_Y(s) = 2y(s) - sgn y(s) \frac{\|y(s)\|^2}{(N(s), y(s))} N(s). \tag{12}
\]

Now, let us define the anti-pedal curve in the Minkowski plane. The anti-pedal curve is defined by the pedal curve and inversion. It is known that the pedal is given as the envelope of a family of functions (Aydin Şekerci and Izumiya 2021):
\[
G: I \times (\mathbb{R}^2_+ / \{0\}) \to \mathbb{R}
(s, x) \mapsto G(s, x) = (x - y(s), x).
\]

We use the definition of inversion, which is given by
\[
\Psi: \mathbb{R}^2_+ / \{0\} \to \mathbb{R}^2_+ / \{0\}
\]
\[
x \mapsto \Psi(x) = \frac{x}{\|x\|^2} = sgn x \frac{x}{(x, x)},
\]
to define an antipedal curve. Then we have
\[
\Psi(g_s^{-1}(0)) = \{x: (x, y(s)) = sgn x \} \tag{13}
\]
for \( g_s(x) = G(s, x) \). Here, \( G(s, x) = 0 \) means that
\[
sgn x \frac{x}{\|x\|^2}, y(s) = 1 \tag{14}
\]
since \( x - y(s), x = 0 \). Thus, we define a family of functions:
\[
F: I \times (\mathbb{R}^2_+ / \{0\}) \to \mathbb{R}
(s, x) \mapsto F(s, x) = (x, y(s)) - sgn x.
\]

The envelope of the family of lines is the anti-pedal curve of \( y \). According to that, we obtain
\[
\frac{\partial F}{\partial s}(s, x) = (x, y'(s)) = 0. \tag{15}
\]

Any vector in \( \mathbb{R}^2_+ \) is represented by a linear combination as \( \lambda T(s) + \xi N(s) \). Using this linear combination for the vector \( x - y(s) \) and substituting to \( \frac{\partial F}{\partial s}(s, x) = 0 \), we have the following equation for \( \lambda \):
\[
\lambda = 0. \tag{16}
\]

Also, using \( \lambda \) in \( F(s, x) = 0 \), we get
\[
\xi(N(s), y(s)) = sgn x. \tag{17}
\]

Then, we obtain
\[
\xi = sgn x \frac{y(s)}{(N(s), y(s))} \tag{18}
\]
and so, \( x \) can be written as follows:
\[
x = sgn x \frac{y(s)}{(N(s), y(s))} N(s). \tag{19}
\]

Moreover, the signature of \( x \) is equal to the signature of \( N(s) \). In that case, the following definition is expressed.

**Definition 3.2.** An anti-pedal \( APe_Y: I \to \mathbb{R}^2_+ / \{0\} \) of the curve \( y \) is given by
\[
APe_Y(s) = \frac{-\varepsilon}{(N(s), y(s))} N(s). \tag{20}
\]

**Proposition 3.3.** For any unit speed non-lightlike regular curve \( y: I \to \mathbb{R}^2_+ / \{0\} \), there exists
\[ \Psi \circ \text{APe}_y = \text{Pe}_y \]

where \( \Psi \) is an inversion, \( \text{APe}_y \) is an anti-pedal curve of \( y \) and \( \text{Pe}_y \) is a pedal curve of \( y \).

**Proof.** We obtain the equation with direct calculations using the anti-pedal of curve and inversion as follows:

\[
(\Psi \circ \text{APe}_y)(s) = \Psi \left( \text{APe}_y(s) \right)
\]

\[
= \Psi \left( -\varepsilon \frac{(N(s), y(s))}{\|N(s), y(s)\|^2} N(s) \right)
\]

\[
= -\varepsilon \frac{(N(s), y(s))}{\|N(s), y(s)\|^2} N(s)
\]

\[
= -\varepsilon N(s), y(s)) N(s)
\]

\[
= \text{Pe}_y(s)
\]

**Proposition 3.4.** For any unit speed non-lightlike regular curve \( y : I \to \mathbb{R}^2 \), there exists

\[ \Psi \circ \text{Pe}_y = \text{APe}_y \]

where \( \Psi \) is an inversion, \( \text{APe}_y \) is an anti-pedal curve of \( y \) and \( \text{Pe}_y \) is a pedal curve of \( y \).

**Proof.** Similar to the proof of Proposition 3.3, we obtain the equation with direct calculations using the pedal curve of curve and inversion as follows:

\[
(\Psi \circ\text{Pe}_y)(s) = \Psi \left( \text{Pe}_y(s) \right)
\]

\[
= \Psi \left( -\varepsilon N(s), y(s)) N(s) \right)
\]

\[
= -\varepsilon N(s), y(s)) N(s)
\]

\[
= \text{APe}_y(s)
\]

**Theorem 3.5.** Let \( y \) be a unit speed non-lightlike curve in Minkowski plane. Assume that \( y \) does not pass through the origin and there are no lightlike points. Then, the primitive and the anti-pedal of \( y \) have the following relationship:

\[ \text{Pr}_y(s) = -\varepsilon \text{sgn} \ y(s) \text{APe}_{\psi, y}(s) \]

where \( \varepsilon \) is the signature of the tangent vector field of \( y, \text{sgn} \ y(s) \) is the signature of \( y \) and \( \Psi \) is an inversion.

**Proof.** Firstly, we find \( \text{APe}_{\psi, y}(s) \). For this, the family of functions is \( F(s, x) = (x, y(s)) - \text{sgn} \ x \). Since \( F(s, x) = 0 \), we get

\[
(x, (\Psi \circ y)(s)) - \text{sgn} \ x = 0,
\]

\[
\left\{ x, \frac{y(s)}{\|y(s)\|^2} \right\} = \text{sgn} \ x.
\]

Any vector in \( \mathbb{R}^2 \) is represented by a linear combination as \( \lambda \text{T}(s) + \xi N(s) \). Using this linear combination for the vector \( x \),

\[
\frac{\lambda}{\|y(s)\|^2} \text{T}(s) + \frac{\xi}{\|y(s)\|^2} (N(s), y(s)) = \text{sgn} \ x.
\]

Moreover, we obtain

\[
\frac{\partial F}{\partial s}(s, x) = \langle x, \left( \frac{y(s)}{\|y(s)\|^2} \right) ' \rangle = 0.
\]

Since

\[
\left( \frac{y(s)}{\|y(s)\|^2} \right)' = \frac{1}{\|y(s)\|^2} \text{T}(s) - \frac{2}{\|y(s)\|^4} \text{sgn} \ y(s) (T(s), y(s)) y(s),
\]

Eq. (26) is written as

\[
\left\{ \lambda \text{T}(s) + \xi N(s), \frac{1}{\|y(s)\|^2} \text{T}(s) \right\} - \left\{ \frac{\lambda}{\|y(s)\|^2} \text{T}(s), \frac{2}{\|y(s)\|^4} \text{sgn} \ y(s) (T(s), y(s)) y(s) \right\} = 0.
\]

According to that, we have

\[
\frac{\lambda}{\|y(s)\|^2} \frac{2}{\|y(s)\|^4} \text{sgn} \ y(s) (T(s), y(s)) y(s) = 0.
\]
If we multiply Eq.(25) by $\frac{2\text{sgn} \mathbf{y}(s)\langle\mathbf{y}(s), \mathbf{T}(s)\rangle}{||\mathbf{y}(s)||^2}$ and add Eq.(28) side by side, then we can write as
\[
\lambda = 2\varepsilon \text{sgn} \mathbf{y}(s) \text{sgn} \mathbf{x} \langle \mathbf{y}(s), \mathbf{T}(s)\rangle.
\]
(29)

Using this $\lambda$, which we have obtained, we find $\xi$ as follows:
\[
\xi = \frac{\text{sgn} \mathbf{y}(s) \text{sgn} \mathbf{x}}{\langle \mathbf{y}(s), \mathbf{N}(s) \rangle} \left[ \langle \mathbf{y}(s), \mathbf{y}(s) \rangle - 2\varepsilon \langle \mathbf{y}(s), \mathbf{T}(s) \rangle^2 \right].
\]
(30)

Therefore, we get
\[
\mathbf{x} = 2\varepsilon \text{sgn} \mathbf{y}(s) \text{sgn} \mathbf{x} \langle \mathbf{y}(s), \mathbf{T}(s)\rangle \mathbf{T}(s)
\]
\[
+ \frac{\text{sgn} \mathbf{y}(s) \text{sgn} \mathbf{x}}{\langle \mathbf{y}(s), \mathbf{N}(s) \rangle} \left[ \langle \mathbf{y}(s), \mathbf{y}(s) \rangle - 2\varepsilon \langle \mathbf{y}(s), \mathbf{T}(s) \rangle^2 \right]\mathbf{N}(s)
\]
\[
- 2\varepsilon \langle \mathbf{y}(s), \mathbf{T}(s) \rangle^2 \mathbf{N}(s)
\]
\[
= 2 \text{sgn} \mathbf{y}(s) \text{sgn} \mathbf{x} \mathbf{y}(s)
\]
\[
+ \frac{\text{sgn} \mathbf{x} \text{sgn} \mathbf{y}(s) \langle \mathbf{y}(s), \mathbf{y}(s) \rangle}{\langle \mathbf{y}(s), \mathbf{N}(s) \rangle} \mathbf{N}(s)
\]
\[
- \frac{2\varepsilon \text{sgn} \mathbf{x}}{\langle \mathbf{y}(s), \mathbf{N}(s) \rangle} \left[ \varepsilon \text{sgn} \mathbf{y}(s) \langle \mathbf{y}(s), \mathbf{T}(s) \rangle^2 \right]\mathbf{N}(s)
\]
\[
- \varepsilon \text{sgn} \mathbf{y}(s) \langle \mathbf{y}(s), \mathbf{N}(s) \rangle^2 \mathbf{N}(s)
\]
\[
= 2 \text{sgn} \mathbf{y}(s) \text{sgn} \mathbf{x} \mathbf{y}(s)
\]
\[
+ \frac{\text{sgn} \mathbf{x} \||\mathbf{y}(s)||^2}{\langle \mathbf{y}(s), \mathbf{N}(s) \rangle} \mathbf{N}(s) - \frac{2\varepsilon \text{sgn} \mathbf{x}||\mathbf{y}(s)||^2}{\langle \mathbf{y}(s), \mathbf{N}(s) \rangle} \mathbf{N}(s)
\]
\[
= \text{sgn} \mathbf{y}(s) \text{sgn} \mathbf{x} \mathbf{y}(s)
\]
\[
- \text{sgn} \mathbf{y}(s) \frac{||\mathbf{y}(s)||^2}{\langle \mathbf{y}(s), \mathbf{N}(s) \rangle} \mathbf{N}(s)
\]
\[
= \text{sgn} \mathbf{y}(s) \text{sgn} \mathbf{x} \text{Pr}_y(s)
\]

Moreover, since $\text{sgn} \mathbf{x} = \text{sgn} \text{Pr}_y(s)$, we get
\[
\langle \text{Pr}_y(s), \text{Pr}_y(s) \rangle = \frac{||\mathbf{y}(s)||^4}{\langle \mathbf{y}(s), \mathbf{N}(s) \rangle^2} \langle \mathbf{N}(s), \mathbf{N}(s) \rangle
\]
\[
= -\varepsilon \frac{||\mathbf{y}(s)||^4}{\langle \mathbf{y}(s), \mathbf{N}(s) \rangle^2}.
\]

This means that the signature of $\mathbf{N}(s)$ and the signature of $\text{Pr}_y(s)$ are the same. According to that, we obtain
\[
\text{APe}_{\psi, y}(s) = -\varepsilon \text{sgn} \mathbf{y}(s) \text{Pr}_y(s).
\]

**Theorem 3.6.** Let $\mathbf{y}$ be a unit speed non-lightlike curve in Minkowski plane. Assume that $\mathbf{y}$ does not pass through the origin and there are no lightlike points. Then, the primitive and the anti-pedal of $\mathbf{y}$ have the following relationship:
\[
\text{Pr}_y = -\varepsilon \text{sgn} \mathbf{y} \text{APe}_y
\]
(31)

**Proof.** Firstly, we find $\text{Pr}_y$. For this, the family of functions is $H(s, x) = (x - \mathbf{y}(s), \mathbf{y}(s))$. Since $H(s, x) = 0$, we get
\[
(x - (\psi \circ \mathbf{y})(s), (\psi \circ \mathbf{y})(s)) = 0,
\]
\[
(x - \mathbf{y}(s), \frac{\mathbf{y}(s)}{||\mathbf{y}(s)||^2}, \frac{\mathbf{y}(s)}{||\mathbf{y}(s)||^2}) = 0.
\]

Any vector in $\mathbb{R}^2_1$ is represented by a linear combination as $\lambda \mathbf{T}(s) + \xi \mathbf{N}(s)$. Taking this linear combination for the vector $x - (\psi \circ \mathbf{y})(s)$,
\[
\langle \lambda \mathbf{T}(s) + \xi \mathbf{N}(s), \frac{\mathbf{y}(s)}{||\mathbf{y}(s)||^2} \rangle = 0,
\]
\[
\lambda \frac{1}{||\mathbf{y}(s)||^2} \langle \mathbf{T}(s), \mathbf{y}(s) \rangle + \xi \frac{\mathbf{y}(s)}{||\mathbf{y}(s)||^2} \langle \mathbf{N}(s), \mathbf{y}(s) \rangle = 0.
\]
(32)

Moreover, we obtain
\[
\frac{\partial H}{\partial s}(s, x) = -\left( \frac{\mathbf{y}(s)}{||\mathbf{y}(s)||^2}, \frac{\mathbf{y}(s)}{||\mathbf{y}(s)||^2} \right)^\prime + \left[ x - \frac{\mathbf{y}(s)}{||\mathbf{y}(s)||^2}, \left( \frac{\mathbf{y}(s)}{||\mathbf{y}(s)||^2} \right)^\prime \right] = 0.
\]
(33)

Since
\[
\left( \frac{\mathbf{y}(s)}{||\mathbf{y}(s)||^2} \right)^\prime = \frac{1}{||\mathbf{y}(s)||^2} \mathbf{T}(s)
\]
\[
- \frac{2 \text{sgn} \mathbf{y}(s) \langle \mathbf{T}(s), \mathbf{y}(s) \rangle}{||\mathbf{y}(s)||^4} \mathbf{y}(s),
\]

Eq. (33) is written as
\[
\left[ x - \frac{\mathbf{y}(s)}{||\mathbf{y}(s)||^2}, \frac{\mathbf{y}(s)}{||\mathbf{y}(s)||^2}, \left( \frac{\mathbf{y}(s)}{||\mathbf{y}(s)||^2} \right)^\prime \right] = 0.
\]

96
According to that, using linear combination $\lambda T(s) + \xi N(s)$ for $x - \frac{y(s)}{||y(s)||^2}$ we have

$$\begin{align*}
\lambda T(s) + \xi N(s) &= \frac{y(s)}{||y(s)||^2} - \frac{1}{||y(s)||^2} \langle y(s), T(s) \rangle T(s) \\
&\quad - 2 \text{sgn} \frac{y(s)}{||y(s)||^2} \langle y(s), T(s) \rangle \frac{||y(s)||}{4} y(s)
\end{align*}$$

Therefore, we obtain

$$\begin{align*}
\lambda \frac{\varepsilon}{||y(s)||^2} &= \frac{2 \text{sgn} \frac{y(s)}{||y(s)||^2} \langle y(s), T(s) \rangle^2}{||y(s)||^4} \\
&\quad - \frac{2 \xi \text{sgn} \frac{y(s)}{||y(s)||^2} \langle y(s), T(s) \rangle \langle y(s), N(s) \rangle}{||y(s)||^4} \\
&\quad + \frac{2 \text{sgn} \frac{y(s)}{||y(s)||^2} \langle y(s), T(s) \rangle \langle y(s), y(s) \rangle}{||y(s)||^6} = 0.
\end{align*}$$

From that equation, we get

$$\begin{align*}
\lambda &= \frac{\varepsilon}{||y(s)||^2} \frac{2 \text{sgn} \frac{y(s)}{||y(s)||^2} \langle y(s), T(s) \rangle^2}{||y(s)||^4} \\
&\quad - \frac{2 \xi \text{sgn} \frac{y(s)}{||y(s)||^2} \langle y(s), T(s) \rangle \langle y(s), N(s) \rangle}{||y(s)||^4} \\
&\quad + \frac{2 \text{sgn} \frac{y(s)}{||y(s)||^2} \langle y(s), T(s) \rangle \langle y(s), y(s) \rangle}{||y(s)||^6}.
\end{align*}$$

Thus, it can be written as

$$\begin{align*}
x &= \frac{\varepsilon \langle y(s), T(s) \rangle^2}{\langle y(s), N(s) \rangle ||y(s)||^2} - \frac{\varepsilon \langle y(s), N(s) \rangle^2}{\langle y(s), N(s) \rangle ||y(s)||^2} N(s) \\
&= \frac{\text{sgn} \frac{y(s)}{||y(s)||^2} \langle y(s), T(s) \rangle}{\langle y(s), N(s) \rangle} N(s).
\end{align*}$$

According to that, we obtain

$$x = -\varepsilon \frac{\text{sgn} \frac{y(s)}{||y(s)||^2} \langle y(s), T(s) \rangle}{\langle y(s), N(s) \rangle} N(s).$$

Therefore, the proof is completed.

4. Applications

Example 4.1. Let $y: I \rightarrow \mathbb{R}^2_0 \setminus \{0\}$, $I \subset \mathbb{R}$, be a curve which is expressed by $y(s) = (2 \sinh s, \cosh s)$ and is shown in Figure 1.

![Figure 1. The curve $y(s) = (2 \sinh s, \cosh s)$](image)

Therefore, $y(s)$ is a regular curve, there are no lightlike points and $y'(s) = (2 \cosh s, \sinh s)$. Then, we say that the curve is a timelike curve. The tangent vector field and the normal vector field of the curve $y(s)$ are given in the following forms:

$$\begin{align*}
T(s) &= \left(\frac{2 \cosh s}{\sqrt{1 + 3 \cosh^2 s}}, \frac{\sinh s}{\sqrt{1 + 3 \cosh^2 s}}\right).
\end{align*}$$
\[ N(s) = \frac{\sinh s}{\sqrt{1 + 3\cosh^2 s}} \cdot \frac{2\cosh s}{\sqrt{1 + 3\cosh^2 s}} \]

Using (12), the primitive of \( y \) is obtained as

\[ \text{Pr}_y(s) = \left( \frac{\sinh s}{2} (3\cosh^2 s + 4) \right) \]

\[ \cosh s \left( 3\sinh^2 s + 1 \right) \]

and is shown in Figure 2.

**Figure 2.** The primitive of curve \( y(s) \)

Also, from (20), the anti-pedal of \( y(s) \) is given by

\[ \text{APe}_y(s) = \frac{1}{2} (\sinh s, 2\cosh s) \]

and is shown in Figure 3.

**Figure 3.** The anti-pedal of curve \( y(s) \)

**Example 4.2.** Let \( y : I \to \mathbb{R}^2 \setminus \{0\}, I \subset \mathbb{R} \), be a curve which is expressed by \( y(s) = (\cosh s, s + \sinh s) \) and is shown in Figure 4.

**Figure 4.** The curve \( y(s) = (\cosh s, s + \sinh s) \)

Therefore, \( y(s) \) is a regular curve, there are no lightlike points and \( y'(s) = (\sinh s, 1 + \cosh s) \). Then, we say that the curve is a spacelike curve.

Using (12), the primitive of \( y \) is obtained as

\[ \text{Pr}_y(s) = \left( 2\cosh s - \frac{(1 + \cosh s)(s^2 - 1 + 2\sinh s)}{s \sinh s - 1 - \cosh s} \right) \]

\[ \frac{1}{2} \left( \sinh s - 1 - \cosh s \right) (1 + \cosh s, \sinh s) \]

and is shown in Figure 5.

**Figure 5.** The primitive of curve \( y(s) \)

Also, from (20), the anti-pedal of \( y(s) \) is given by

\[ \text{APe}_y(s) = \frac{1}{s \sinh s - 1 - \cosh s} (1 + \cosh s, \sinh s) \]

and is shown in Figure 6.

**Figure 6.** The anti-pedal of curve \( y(s) \)

### 5. Conclusions

In this study, the following conclusions are obtained about Minkowski plane curves:

(1) To obtain primitive and anti-pedal of the Minkowski plane curves, which does not pass through the origin and does not have lightlike point, the families of functions are defined and their envelopes are used. Thus, the primitive of curve is defined as

\[ \text{Pr}_y(s) = 2y(s) - s\text{gn} \frac{\|y(s)\|^2}{(N(s), y(s))} N(s), \]

and the anti-pedal of curve is defined as
Anti-pedals and Primitives of Curves in Minkowski Plane, Aydın Şekerci.

\[ \text{APe}_\gamma(s) = \frac{-\varepsilon}{\langle \mathbf{N}(s), \gamma(s) \rangle} \mathbf{N}(s). \]

(2) The pedal of the curve, which is non-lightlike Minkowski plane curve, does not pass through the origin and there is no lightlike point, is equivalent to the composition of an anti-pedal of the curve and inversion.

(3) The anti-pedal of the curve, which is non-lightlike Minkowski plane curve, does not pass through the origin and there is no lightlike point, is equivalent to the composition of an anti-pedal of the curve and inversion.

(4) Assume that \( \gamma \) is a curve, which is a non-lightlike Minkowski plane curve, does not pass through the origin and there is no lightlike point, and \( \Psi \) is an inversion. Then, the primitive of the curve \( \gamma \) is written as the product of a constant and the anti-pedal of \( \Psi \circ \gamma \) where this constant is calculated by multiplying the signature of the normal vector field of the curve and the signature of the curve.

(5) Similar to the previous result, assume that \( \gamma \) is a curve, which is a non-lightlike Minkowski plane curve, does not pass through the origin and there is no lightlike point, and \( \Psi \) is an inversion. Then, the primitive of \( \Psi \circ \gamma \) is written as the product of a constant and the anti-pedal of \( \gamma \) where this constant is calculated by multiplying the signature of the normal vector field of the curve and the signature of the curve.

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