SIMPLE SEISMIC TESTS OF THE SOLAR CORE

DALLAS C. KENNEDY
Institute for Nuclear Theory, University of Washington, Box 35150, Seattle, WA 98195 and Department of Physics, University of Florida, Box 118440, Gainesville, FL 32611; kennedy@phys.ufl.edu
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ABSTRACT

A model-independent reconstruction of mechanical profiles (density, pressure) of the solar interior is outlined using the adiabatic sound speed and buoyancy frequency profiles. These can be inferred from helioseismology if both \( p \)- and \( g \)-mode frequencies are measured. A simulated reconstruction is presented using a solar model buoyancy frequency and available sound speed data.

Subject headings: Sun: interior — Sun: oscillations

1. INTRODUCTION

Standard solar models (SSMs) are now very accurate, exemplified by the work of Bahcall et al. (Bahcall, Basu, & Pinsonneault 1998; M. H. Pinsonneault 1998, private communication) and Turk-Chièze et al. (Brun, Turck-Chièze, & Morel 1998). But confidence in predicted solar neutrino fluxes is enhanced by checks of the solar core depending on only a few general assumptions, independent of detailed models. This can be done with helioseismology, although the \( g \)-modes are needed. Serious claims of \( g \)-modes have been put forward (Hill & Gu 1990; Thomson et al. 1995) but remain controversial (Guenther & Sills 1995; Lou 1996). Nonetheless, the \( g \)-modes will undoubtedly be observed at some point, despite their attenuation by the convective zone.

We assume negligible radiation pressure \( P_r \ll P_{\text{gas}} \) and no significant departures from an ideal gas \((P = \rho RT/\mu)\).

\[
\Gamma = d \ln P/d \ln \rho , \quad \Gamma_{\text{ad}} = (\partial \ln P/\partial \rho)_{\text{ad}} ,
\]

\[
V = d \ln T/d \ln P , \quad V_{\text{ad}} = (\partial \ln T/\partial \ln P)_{\text{ad}} .
\]

\[
\Gamma_{\text{ad}} = 5/3, \quad V_{\text{ad}} = 2/5 \quad \text{for a monatomic gas. The Brunt-Väisälä (buoyancy) frequency } N(r),
\]

\[
N^2 = g \left( \frac{1}{\Gamma} - \frac{1}{\Gamma_{\text{ad}}} \right) = \left( \rho g^2/P \right)(V_{\text{ad}} - V + V_n) ,
\]

\[
\lambda_p^{-1} = -d \ln P/\ln r , \quad \lambda_g^{-1} = -d \ln N/r , \quad \text{where } V_n = d \ln \mu/d \ln \rho , \quad \text{is real for propagating } g\text{-waves, outside any convective zones. The base of the solar convective zone } r_{\text{BCZ}} = (0.709) R_{\odot} \quad \text{(Gough et al. 1996). The adiabatic squared sound speed } c_{\text{ad}}^2 = \Gamma_{\text{ad}} P/\rho .
\]

The adiabatic seismology formalism here follows Unno et al. (1989), with some differences in notation. The seismic modes are labeled by radial index \( n \) (number of radial nodes) and angular indices \( l \) and \( m \); frequencies degenerate in the last if rotation is ignored. The spectrum exhibits “middle” \( f \)-modes (fundamental or \( n = 0, l > 1 \)) and the \( p \)-(\( g \)-) modes rising above (falling below) the \( f \)-mode in \( \nu \) for \( n > 0, l > 0 \). For large \( n \), the \( p \)- and \( g \)-modes are concentrated near the surface and the center, respectively. Their frequencies \( \nu_p \) and \( \nu_g \) are

\[
\nu_p^{-1}(n, l) \approx \frac{2}{n + (1/2)} \int_0^{R_{\odot}} \frac{dr}{c_{\text{ad}}(r)} , \quad \nu_g^{-1}(n, l) \approx \frac{\sqrt{l(l + 1)}}{2\pi^2} \int_0^{r_{\text{BCZ}}} \frac{dr}{r} N(r) , \quad \text{for large } n \text{ cannot be large in } \nu_g \text{.}
\]

A large set of measured frequencies allow reconstruction of the \( c_{\text{ad}}(r) \) and \( N(r) \) profiles. The Bahcall-Pinsonneault 1998 SSM (hereafter BP98) is used here when specific numerical results are needed (Bahcall et al. 1998).

2. TESTS OF HYDROSTATIC EQUILIBRIUM

Present helioseismic data rule out the \( \Gamma = 5/3 \) convective polytrope deep into the core (Basu et al. 1997; Bahcall et al. 1997). Hydrostatic equilibrium is tested by correct prediction of the \( \nu_p(n, l), n > 1 \). Success here would limit any remaining possibility of convection at the center (otherwise \( N^2 < 0 \)), although the Sun is not far from this state \((M \approx 1.1 M_\odot \text{ in standard calculations; see Kippenhahn & Weigert 1990). An equivalent test of hydrostatic equilibrium at the center is provided by the sound speed slope, measurable in principle with \( p \)-modes alone:

\[
\frac{dc_{\text{ad}}}{dr} = -g_{\Gamma_{\text{ad}}} \left( \frac{1}{\Gamma} - \frac{1}{\Gamma_{\text{ad}}} \right) + \frac{P}{\rho} \frac{d\Gamma_{\text{ad}}}{dr} , \quad 1 - \frac{1}{\Gamma} = V - V_n . \quad (4)
\]

Assume \( d\Gamma_{\text{ad}}/dr = 0 \) in the core and a nonsingular mass distribution, so that

\[
g = Gm/r^2 = (4\pi G/3)\rho , \quad r \to 0 \quad \text{as } r \to 0 . \quad (5)
\]

Thus, \( dc_{\text{ad}}^2/\ln r \) must vanish.

The measured sound speed slope places a limit on an unperturbed state with nonzero velocity field \( v_\theta \), taken here as radial for simplicity. This test checks for the presence of convective motion that does not efficiently transfer heat. Such convection would not directly affect \( \Gamma \) and the \( P/\rho \) profile and thus is not ruled out in general by current \( p \)-mode helioseismic data. Hydrostatic equilibrium is modified to

\[
-\frac{dP}{dr} = pg + \rho v_\theta^2 \frac{dv_\theta}{dr} , \quad (6)
\]

ignoring spherical symmetry breaking rotation and magnetic effects, also calculable. Assume again \( d\Gamma_{\text{ad}}/dr = 0 \) in
the core and \( g(0) = 0 \). Equation (4) no longer vanishes at the center and instead yields
\[
\frac{dc^2_{ad}(0)}{dr} = -\Gamma_{ad} v_0(0) \frac{dv_0(0)}{dr} \left[ 1 - \frac{1}{\Gamma(0)} \right].
\]
Equivalently, in terms of \( g \)-modes, \( v_g(n, l) \) is corrected (to linear order in \( v_g \)) by the “expectation value” in the \((n, l)\)-mode of \( k_g v_g/2\pi \), where \( k_g \) = radial wavenumber. The related length scale \( \approx R_0/\langle \text{few} \rangle \) for low-\( n \)-modes; but for high-\( n \) \( g \)-modes concentrated near the center, it is the central scale \( R_0; R_0^6 = 3P/c^2\pi G\rho_c^2 \), with \( R_0^6/R_0 \approx 0.12 \) for the present SSM (Kennedy & Bludman 1999). For large \( n \),
\[
k_r \rightarrow \frac{n}{R_0 \Omega_g} \sqrt{\frac{8\pi^2 \rho_c G}{3 \left[ \frac{1}{\Gamma(0)} - 1 \right]}} \quad \Omega_g = \int_0^{r_{02}} \frac{r^2 \, dr}{r} N(r).
\]
A conservative relative observational error for \( v_g \) is 0.003, while the relative theoretical uncertainty \( \approx 0.001 \) (Hill & Gu 1990; Guenther & Sills 1995; Bahcall, Basu, & Pinsonneault 1998). Thus, a direct bound on the core \( v_0 \) \( \leq 2\pi v_g(n, l)/k_r \sim (7.5 \times 10^7 \text{ cm s}^{-1})/[(l + 1)/n^2]; 1.2 \times 10^5 \text{ cm s}^{-1} \) for \( l = 1, n = 30 \), is feasible, independent of azimuthal splitting by \( m \).

3. MECHANICAL PROFILES

The \( N(r) \) and \( c_{ad}(r) \) profiles are sufficient to reconstruct uniquely the mechanical (hydrostatic) profiles. \( N(r) \) and \( dc_{ad}(r)/dr \) are proportional, either controlling the \( v_g(n, l) \):
\[
\frac{dc_{ad}(r)}{dr} = \frac{N(r)/2}{\sqrt{\Gamma_{ad} \Gamma(r) - 1}} \left[ 1 - \frac{1}{\Gamma(r)} \right] \Gamma_{ad}.
\]
Gravity modes are a result of the inhomogeneity of \( c_{ad}(r) \). Assuming \( d\Gamma_{ad}/dr = 0 \), \( \Gamma \) satisfies a quadratic equation,
\[
(1 + \mathcal{D}) \Gamma^2 - (2 + \mathcal{D}) \Gamma + 1 = 0, \quad \mathcal{D}(r) = \frac{2 dc_{ad}(r)/dr}{\Gamma_{ad} N(r)},
\]
with one physical root,
\[
\Gamma(r) = \frac{2 + \mathcal{D}(r)}{\Gamma_{ad} + \sqrt{[2 + \mathcal{D}(r)\Gamma_{ad}]^2 - 4[1 + \mathcal{D}(r)]}}.
\]
Thus, the \( \Gamma(\mathcal{D}) \) profile is reconstructible with \( N(r) \) and a numerical derivative of \( c_{ad}(r) \). Note that \( \Gamma = 1 \) when \( dc_{ad}/dr \) and \( \mathcal{D} = 0 \).

The mass and pressure profiles then follow:
\[
Gm(r) = -\frac{r^2 dc^2_{ad}(r)/dr}{\Gamma_{ad} [1 - 1/\Gamma(r)]} \quad 4\pi G\rho_c = -\frac{6 dc^2_{ad}(0)/dr^2}{\Gamma_{ad} [1 - 1/\Gamma(0)]},
\]
\[
P(r) = \frac{\rho(r)c^2_{ad}(r)}{\Gamma_{ad}},
\]
where \( \rho \) is obtained from \( m(r) \). Note that \( m > 0 \), because \( \Gamma > 1 \) if \( dc_{ad}/dr < 0 \) and \( \Gamma < 1 \) if \( dc_{ad}/dr > 0 \). (The \( dc_{ad}/dr = 0 \) and \( r \) \~ 0 \) limits are well defined in the ratios.)

The present SSM inner core is an example of the \( \Gamma < 1 \) case. Dimensionless structure in terms of homology variables can also be derived (Kennedy & Bludman 1999).

Direct methods of inversion without a reference SSM, based on \( p \)-modes alone and including use of the sound speed slope as a diagnostic of the equation of state, are discussed in the older literature (Gough 1984; Christensen-Dalsgaard et al. 1985; Gough & Toomre 1991; Gough, Kosovichev, & Toutain 1995) and, since the advent of Solar and Heliospheric Observatory (SOHO), Global Oscillation Network Group (GONG), and Birmingham Solar Oscillations Network Low-Degree(\( l \)) Oscillation Experiment (BiSON/LOWL), by Gough et al. (1996) and Basu et al. (1997).

Since the \( \nu \), especially at long wavelength, are affected by the whole Sun, errors in the inferred spatial structure are strongly correlated. An inversion for \( N(r) \), like the inversion for \( c_{ad}(r) \), will have uncertainties in both \( N \) and the position \( r \). The derived profiles (eq. [11]) will have errors compounded from all these sources—\( c_{ad}(r) \), \( N(r) \), and \( r \)—and thus will have errors larger than the errors in any one source.

4. NUMERICAL TEST OF THE METHOD

As the SSM evolves, \( V \) falls while \( V_a \) increases from zero, the core becoming more concentrated. Helioseismology is adiabatic and non-ideal, and cannot give the thermal or chemical structure without further assumptions or a full SSM. Many SSM core features can be understood via mechanical-thermal core homology without a detailed model (Bludman & Kennedy 1996). Homology applied to the core only is valid for power-law opacity and luminosity generation and does not require a polytrope. A polytrope is not even approximately valid for the present SSM core in any case (Kennedy & Bludman 1999). The homology in reality is violated somewhat, as the exponents are not constant and the luminosity is not produced exclusively by one reaction chain. But an accurate thermal and chemical

\[ \begin{align*}
\text{If the specific luminosity production is } e = \varepsilon_0 \rho^2 T^2, \\
\text{and the specific opacity is } \kappa = \kappa_0 \rho^2 T^{-2}, \\
\text{the homology yields } \frac{\rho^2 x^+ h^+}{x^+ h^+} = \varepsilon_0 \kappa_0 x^+ h^+ T^{2-4}, \quad (\text{correcting an error in Bludman & Kennedy 1996}).
\end{align*} \]
BP98 present Sun model profiles from center to base of convective zone at \( r/R_\odot = 0.71 \): (a) stiffness \( \Gamma \), with \( \Gamma \rightarrow 5/3 \) at base of convective zone; (b) Brunt-Väisälä buoyancy frequency \( N \), which vanishes at center and base of convective zone. (Model tables courtesy of M. H. Pinsonneault.)
reconstruction is not much less complicated and requires no fewer assumptions than a full SSM. On the other hand, the mechanical structure of a SSM can be specified by the model-independent reconstruction of &2, rather than calculated.

As & and & are in principle inferable from helioseismology, the numerical precision of the reconstruction can be estimated. No &-mode measurements are now available, so the SSM buoyancy frequency is used here. Combining &ad, &ad/dr, and the SSM & proﬁle, the profile & is obtained (eq. [9]) and, hence, so is &.

Figure 1 shows the measured sound speed below the convection zone, the points being ﬁxed by helioseismic inversion (Bahcall et al. 1997). The BP98 SSM & and & proﬁles, respectively, are shown in Figures 2a and 2b. In Figure 3 is displayed the ratio of the reconstructed to the SSM &. The helioseismic inversion points are separated by steps of &/& = 0.01, and we should thus expect agreement with the SSM at a level of about 1%, consistent with Figure 3.

The agreement of a true reconstruction with the SSM should be better than Figure 3, but the overall uncertainty would be somewhat larger than a percent, reflecting the &ad uncertainty; the additional uncertainties due to a measured, rather than a model, & proﬁle; and the errors induced by spatial correlation of the helioseismic inversion mesh points.

5. SUMMARY

Some solar core tests are proposed, independent of model details. Measured & and &-modes are needed to reconstruct &ad and & uniquely and independently of a reference SSM, with only general physical assumptions: hydrostatic equilibrium, the ideal gas law, and known, constant &ad. Absence of core convection implies &[0 in equation (2) and the existence of & for high & (eq. [3]). A non-singular, static mass distribution implies vanishing of the sound speed slope (eq. [4]) at the center (eq. [5]). The measured slope and/or high- & can set a limit on a central velocity ﬁeld. Reconstruction of the solar interior mechanical structure follows equations (10-11) from &ad, &ad/dr, and &. A test of the method is presented, using the available, measured &ad and a model & profile.3

The & and & proﬁles to the origin are essential for testing the SSM in the inner core, the critical region for the

3 Numerical analysis code in Mathematica 3.0 with data and model tables can be downloaded at http://www.phys.ufl.edu/~kennedy/soft/cover.html.
solar neutrino problem, where the higher energy Be and B neutrinos are produced and where solar structure becomes strongly nonpolytropic (Bahcall 1988; Kennedy & Bludman 1999). No p-mode data on a finer radial mesh and extending to the center are presently available, but they might become available soon, even if g-mode measurements remain elusive.

Truly independent tests of thermal structure and chemical evolution would require comparison with other Sun-like stars, in particular via asteroseismology (Deubner, Christensen-Dalsgaard, & Kurtz 1998). With the appropriate changes, the method outlined here can be used to analyze such stars as well.

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4 For the Bahcall-Pinsonneault present SSM, see http://www.sns.ias.edu/~jnb/SNdata/solarmodels.html.

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