Superconductor Josephson junctions are a paradigmatic example of macroscopic quantum coherence. The underlying physical mechanism is the Josephson effect \cite{1}: two superconductors connected by a weak link have coherent dynamical behavior determined by the relative macroscopic phase of the superconducting condensates. Josephson junctions have been used to discuss fundamental concepts in quantum mechanics \cite{2}, perform precision measurements \cite{3}, and are now promising candidates to implement quantum information devices \cite{4}. One important feature of superconducting Josephson junctions is the possibility to precisely control and adjust the state of the system by varying external parameters, e.g., gate-voltage or magnetic flux.

Bose Josephson junctions have been only recently proposed \cite{5}, and realized experimentally \cite{6}, and many issues remain open. In the simplest configuration a Bose Josephson junction is realized by confining an ultracold Bose gas in a double-well potential. This configuration can be described using a two-mode model in which the bosons occupy the lowest level in each well. In the “classical” regime of large particle numbers and weak repulsive interactions the gas is well described by the Gross-Pitaevskii equation which, within the two-mode approximation, can be cast in the form of generalized Josephson equations for the time evolution of the relative phase and population imbalance among the two wells \cite{5}. These equations differ from the original ones used for superconductor Josephson junctions \cite{2} by the presence of a nonlinear coupling among the phase and population imbalance variables. This term originates from boson-boson interactions in the mean-field approximation and gives rise to a rich dynamical behavior, displaying self-trapping and π oscillations \cite{5}.

In this Letter we focus on the mesoscopic “quantum” regime beyond the Gross-Pitaevskii equation, in the limit of strong interactions and/or smaller values of $N$. This gets within reach of current experiments \cite{5}. As interactions are increased phase fluctuations become more and more important while number fluctuations are suppressed; the ground state of the system approaches a regime which can be viewed as a mesoscopic Mott insulator. During the time evolution the phase coherence first degrades (“phase diffusion” \cite{6}), but, in a closed quantum system, periodically revives, as it has been experimentally demonstrated \cite{4}. At intermediate times between phase collapse and revival, Schroedinger cat states are predicted to form \cite{11}, but are not easily observable in superconducting Josephson junctions \cite{12}.

The quantum behavior of superconducting Josephson junctions is usually accounted for by the standard phase model \cite{3,13}. This model has been proposed to study the quantum fluctuations in a Bose Josephson junction \cite{13}, and has been extended for large particle numbers $N$ with subleading $1/N$ corrections \cite{10}; however it does not account for large population imbalance among the two sides of the junction. In this work we overcome this limitation. Using the quantum two-mode Bose-Hubbard Hamiltonian we investigate the ground state properties of the junction which we summarize in a “phase diagram” obtained by studying number fluctuations at varying well asymmetry and interaction strength. We characterize the phase-coherence of the junction by calculating the momentum distribution. We then focus on the quantum dynamical behavior of the junction in a regime where Schroedinger cat states occur and show how they affect the time-dependent momentum distribution of the gas.

Model We start by the two-mode Bose-Hubbard Hamiltonian

$$
H = E_1 \hat{a}_1^\dagger \hat{a}_1 + E_2 \hat{a}_2^\dagger \hat{a}_2 + \frac{U_1}{2} \hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 + \frac{U_2}{2} \hat{a}_2^\dagger \hat{a}_2^\dagger \hat{a}_2 \hat{a}_2 - K(\hat{a}_2^\dagger \hat{a}_1 + \hat{a}_1^\dagger \hat{a}_2)
$$

(1)

where $\hat{a}_i, \hat{a}_i^\dagger$ with $i = 1, 2$ are bosonic field operators satisfying $[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}$, $E_i^0$ are the energies of the two wells, $U_i > 0$ are the boson-boson repulsive interactions and $K$ is the tunnel matrix element, i.e, the Rabi oscillation en-
ergy in the case of a non-interacting model. The Heisenberg equations of motion for this model yield

\begin{align}
\dot{n}_1 &= i\dot{\hat{n}}_1 = E_1^0 \hat{n}_1 + U_1 \hat{n}_1 \hat{n}_2 - K \hat{n}_1, \\
\dot{n}_2 &= i\dot{\hat{n}}_2 = E_2^0 \hat{n}_2 + U_2 \hat{n}_2 \hat{n}_1 - K \hat{n}_2,
\end{align}

where \(\hat{n}_i = \hat{a}_i^\dagger \hat{a}_i\). This is the quantum equivalent of the two-mode model used in the mean-field approximation \(\langle \hat{n}_i \rangle = \sqrt{N_i} \exp(i\theta_i)\) and \(\langle \hat{n}_i \rangle = N_t\) to describe Bose-Josephson junctions \cite{[17]} . Indeed, by defining \(n = (N_1 - N_2)/2\) and \(\phi = \theta_2 - \theta_1\) the above equations are readily transformed into the Josephson-like equations

\[\hbar \dot{\phi} = \Delta E + nU_s + Kn \cos \phi / \sqrt{(N/2)^2 - n^2}\]

where \(\Delta E = |E_1^0 + U_1(N - 1)/2 - E_2^0 - U_2(N - 1)/2|\) and \(U_s = (U_1 + U_2)\). To go beyond the mean field model, we now transform the Hamiltonian \(H_0\) into the exact quantum phase one by defining first the operators \(\hat{a}_i = \sqrt{N_i} \exp(i\theta_i)\) and \(\hat{n}_i = \hat{a}_i^\dagger \hat{a}_i\) with \(\hat{a}_i^\dagger \hat{a}_i = \hat{n}_i\) and then the relative-phase and relative-number operators \(\hat{\epsilon}^\dagger = \hat{\epsilon}^\dagger \hat{\epsilon} \hat{\epsilon} \hat{\epsilon}^\dagger\) and \(\hat{n} = (\hat{n}_1 - \hat{n}_2)/2\). Throughout the paper we work at fixed total particle number \(N = \hat{n}_1 + \hat{n}_2 = N\). In the new variables the Hamiltonian reads (up to a constant term)

\[H = \frac{1}{2}\left[\frac{U_s}{2}(\hat{n} - n_0)^2 - K \sqrt{N/2 - \hat{n}} + 1 \hat{\epsilon}^\dagger \hat{\epsilon} + N/2 + 1\right] - K \sqrt{N/2 + \hat{n}} - 1\hat{\epsilon}^\dagger \hat{\epsilon}^\dagger / \sqrt{N/2 - \hat{n}} + 1 \hat{\epsilon} - \hat{\epsilon}^\dagger / \sqrt{N/2 + \hat{n}} + 1 \hat{\epsilon}^\dagger \hat{\epsilon}^\dagger + N/2 + 1\]

with \(n_0 = -\Delta E/U_s\), related to the well asymmetry.

*Quasi-classical limit.* By using the commutation relations \(\sqrt{N/2 + \hat{n}} + 1 \hat{\epsilon}^\dagger \hat{\epsilon} = (\sqrt{N/2 + \hat{n}} + 1 - \sqrt{N/2 + \hat{n}} + 1 \hat{\epsilon}^\dagger \hat{\epsilon}) / \sqrt{N/2 + \hat{n}} + 1\) and \(\sqrt{N/2 - \hat{n}} + 1 \hat{\epsilon} - \hat{\epsilon}^\dagger = \sqrt{N/2 - \hat{n}} + 1 \hat{\epsilon}^\dagger \hat{\epsilon}^\dagger / \sqrt{N/2 + \hat{n}} + 1 \hat{\epsilon} - \hat{\epsilon}^\dagger / \sqrt{N/2 - \hat{n}} + 1 \hat{\epsilon}^\dagger \hat{\epsilon}^\dagger\) and by expanding the Hamiltonian \(H_0\) for large \(n\) we obtain:

\[H = \frac{1}{2}\left[\frac{U_s}{2}(\hat{n} - n_0)^2 - K N \left(1 + \frac{2n^2 + 1/2}{N^2}\right) \cos \phi \right] - K \left(\frac{2\hat{n}}{N} - \sin \phi\right),\]

where \(\cos \phi = (\hat{\epsilon}^\dagger + \hat{\epsilon})/2\) and \(\sin \phi = -i(\hat{\epsilon}^\dagger - \hat{\epsilon})/2\). The leading term in this expansion corresponds to the standard phase model used for superconductors, which reads \(H_{S,1} = E_C(\hat{n} - n_0)^2 - E_J \cos \phi\) with \(E_C\) the charging energy, \(n_0\) the dimensionless gate charge, and \(E_J\) the Josephson energy. Hence we have \(E_C \to U_s\) and \(E_J \to KN\). Also note that the well asymmetry \(n_0\) plays the role of \(n_0\), commonly used as an external parameter to control the state of superconducting junctions. The subleading terms in Eq. \(\text{[5]}\) correspond to a renormalization of the plasma-Josephson frequency \(\sqrt{U_s KN}\), such that in the limit \(U_s \to 0\) tends to the Rabi frequency \(\omega_R = E_J/N = K\) also present in the mean field solution. Eq. \(\text{[5]}\) shows how mesoscopic effects occur in Bose-Josephson junctions, involving a nonlinear coupling between \(\dot{n}\) and \(\phi\).

*Ground state of the quantum Hamiltonian.* The quantum Hamiltonian \(H_0\) contains two very different regimes depending on the ratio \(\gamma = KN/U_s\); for \(\gamma \gg 1\) it yields a quasi-classical "superfluid" regime, where phase fluctuations are suppressed and the mean field approximation applies, while for \(\gamma \ll 1\) it yields the fully quantum "Mott-insulator like" regime where number fluctuations are suppressed (number-squeezed states).

As we work in the two-mode approximation, the quantum Hamiltonian \(H_0\) can be represented on a finite \((N + 1) \times (N + 1)\) matrix. In the Fock basis for relative-number states \(|j\rangle\) the Hamiltonian \(H_0\) has a tridiagonal form with matrix elements \(\langle j | H | j\rangle = U_s(j - n_0)^2/2, |j + 1 | H | j\rangle = -K \sqrt{N/2 + j + 1} \sqrt{N/2 - j} - 1\) and \(\langle j - 1 | H | j\rangle = -K \sqrt{N/2 - j + 1} \sqrt{N/2 + j}\). We have evaluated the number fluctuations on the ground state of the system numerically, as represented in Fig\(\text{[1]}\) in the plane \((n_0, \gamma)\). The regions where number squeezing occurs are reminiscent of the Mott-insulator lobes of the phase diagram of the Bose-Hubbard model \(\text{[14]}\); however we find only a smooth crossover between the two regimes since we are in the two-mode case. Note that for half-integer values of \(n_0\) the number squeezed regions are strongly suppressed even in the regime \(\gamma \ll 1\). At these points the interaction energies of states \(j\) and \(j + 1\) coincide favoring particle number fluctuations even if \(K\) is small.

While in the limit \(\gamma \ll 1\) we have represented the Hamiltonian \(H_0\) in the Fock basis \(|j\rangle\) because its eigenvectors are very close to the Fock states, in the opposite limit \(\gamma \gg 1\), where large number fluctuations occur, it is more useful to represent the Hamiltonian by mapping it onto angular momentum variables in the subspace at fixed \(J^2 = N/2(N + 1)\) \(\text{[18]}\) (let us choose for simplicity \(N\) even). By setting \(\hat{J}_x = (\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1)/2\), \(\hat{J}_y = -i(\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1)/2\), \(\hat{J}_z = (\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2)/2\) we
We rewrite the Hamiltonian as:

$$\hat{H} = U_s(\hat{J}_z - n_0)^2/2 - 2K\hat{J}_x. \quad (6)$$

Its ground-state eigenvector is close to the angular-momentum coherent state [18]

$$|\alpha\rangle = \sum_{m=-N/2}^{N/2} \left( \frac{N}{m + N/2} \right)^{1/2} \frac{\alpha^{m+N/2}}{(1 + |\alpha|^2)^{N/2}} |m\rangle \quad (7)$$

with $\alpha = \tan(\theta/2) \exp(-i\phi)$ and $\hat{J}_z|m\rangle = m|m\rangle$. Interestingly, the average energy on the state $|\alpha\rangle$ is given by

$$\langle \alpha | \hat{H} | \alpha \rangle = U_s(1 - 1/N)n^2/2 - 2K \sqrt{(N/2)^2 - n^2} \cos \phi \quad (8)$$

with $n = -(N/2) \cos \theta$, which corresponds to $O(1/N)$ the mean field result for the energy, and at $n_0 = 0$ has a minimal value for $\theta = \pi/2$ and $\phi = 0$.

Momentum distribution. The momentum distribution is one of the most accessible experimental observables. Here we consider the ground-state momentum distribution for the Bose-Josephson junction using the exact quantum model. The field operator in the two-mode approximation reads $\hat{\Psi}(x) = \sum_{i,j=1}^{2} \Phi_i(x) a_j$, where $\Phi_i(x)$ denotes the ground state wavefunction of the well $i$. The one-body density matrix, defined as $\rho_i(x,x') = \langle \hat{\Psi}^\dagger(x)\hat{\Psi}(x')\rangle$, where the average is intended over the quantum state of the system, for the two-mode model reads $\rho_i(x,x') = \sum_{i,j=1}^{2} \Phi_i^\dagger(x) \Phi_j(x') |a_j a_i\rangle$. The momentum distribution is the Fourier transform with respect to the relative variable of the one-body density matrix, $n(p) = \int dx \int dx' \exp(-ip(x-x'))\rho_1(x,x')$ and becomes then $n(p) = \sum_{i,j=1}^{2} \Phi_i^\dagger(p) \Phi_j(p) |a_j a_i\rangle$, with $\Phi_i(p)$ being the Fourier transform of $\Phi_i(x)$. For a symmetric well or a weakly asymmetric situation we choose $\Phi_i(x) = \Phi_0(x - d/2), \Phi_j(x) = \Phi_0(x + d/2)$, with $d$ being the interwell distance, and hence we obtain for the momentum distribution

$$n(p) = |\Phi_0(p)|^2(N + e^{-ipd}(\hat{J}_+ + e^{ipd}(\hat{J}_-)). \quad (9)$$

This is the generalization of the result derived by Pitaevskii and Stringari [15] for a Bose-Josephson junction using the standard phase model, and has also been used by Gati et al. [7] to quantify thermal decoherence in the experiment. In the quasi-classical regime $\gamma \gg 1$ we can evaluate the average in Eq. (9) using the coherent state [7], which yields $n(p) = |\Phi_0(p)|^2(N + 2\sqrt{(N/2)^2 - n^2} \cos(pd + \phi))$; here we put $n_0 = 0$ such that $n = 0$ for the ground state. Hence for $\gamma \gg 1$ we expect interference fringes in momentum space [13] while in the fully quantum regime $\gamma \rightarrow 0$ the matrix elements $\langle \hat{J}_+ \hat{J}_- \rangle$ are vanishingly small, and the interferences are washed out as illustrated in Fig. 2. Notice that this implies averaging over repeated measurements, as a single measurement would still yield interference fringes [20].

Schrödinger cat states. Schrödinger cat states are quantum superpositions of macroscopic states. We suggest that such states might be realized as a result of the time evolution following a sudden rise of the barrier between the two wells, starting from an initially coherent state (ie in the regime $\gamma \gg 1$). This procedure has the advantage of starting from a quasi-classical state which is what is currently realized in experiments. For simplicity, we consider the symmetric case $n_0 = 0$. If at time $t = N_\phi$ we set the inter-well coupling $K$ in the Hamiltonian to zero, then the time evolution is governed by the term $U_s J_z^2/2$ in the Hamiltonian. For each basis vector $|m\rangle$ of the coherent state, the time evolution is given by $|m(t)\rangle = \exp(-i2\pi m^2 t/T) |m\rangle$, where $T = 4\pi \hbar/U_s$ is the revival period [21] such that $|\alpha(T)\rangle = |\alpha\rangle$. Consider now the special times $T/2q$, $q$ integer. In this case the phase factor governing the time evolution of the state $|m\rangle$ becomes $\exp(-i\pi m^2/q)$, which has the (anti-)periodicity property $\exp(-i\pi (m+q)^2/q) = (-1)^q \exp(-i\pi m^2/q)$ depending on the parity of $q$. Hence for even $q$ we can perform a discrete Fourier transform to obtain a cat state given by a superposition of $q$ coherent states,

$$|\alpha(T/2q)\rangle = \sum_{k=0}^{q-1} u_k e^{i\pi k N/q} e^{-i2\pi k q|\alpha\rangle} \quad (10)$$

where $u_k = (1/q) \sum_{m=0}^{q-1} e^{-i\pi m^2/q} e^{i2\pi km/q},$ and similar cat states exist for odd values of $q$.

In particular, for $q = 2$ we have $|\alpha(T/4)\rangle = |\exp(-i\pi/4)|\alpha\rangle + \exp(i\pi/4)(-1)^{N/2} - |\alpha\rangle|/\sqrt{2}$, which is a cat state given by the superposition of two coherent states. This kind of state was proposed for light coherent states by Yurke and Stoler [11] and for Josephson junctions by Gerry [12]. However, in the latter case it was not easy to probe it. Here, we show that the cat states affect
as discussed in [22]. On the other side, the time required for the unitary time evolution into the Schroedinger cat state is experimentally feasible as demonstrated already with the experiments in optical lattices [7].

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We have also verified that these features are robust with respect to small tunneling among the two wells.

The time-dependent momentum distribution. In particular, by direct evaluation, we obtain that the contrast in the momentum distribution vanishes exactly for the two-component cat state. Furthermore, with increasing $N$, the contrast gets reduced on larger time regions as higher-order cat states develop at times different from $t = T/4$ (Fig. 3a). For such cat states the phase distribution $P(\chi) = \langle \alpha(0) \exp(i\chi) | \alpha(t) \rangle$ has equidistant equal peaks in the interval $[0, 2\pi]$ (Fig. 3b) which upon averaging strongly reduce the momentum-distribution contrast. We have also verified that these features are robust with respect to small tunneling among the two wells.

Applications and experimental perspectives. Number-squeezed states are particularly important for atom-optics applications, as their phase-diffusion time is longer than coherent states (ie usual Bose-condensates). The measurement of the momentum distribution seems a promising route for the observation of quantum fluctuations and Schroedinger cat states on Bose Josephson junctions. One difficulty to overcome in the experimental procedure is the precise control over the number of atoms in the junction, which should stay constant during the experiment. Atom losses may induce dephasing as discussed in [22]. On the other side, the time required for the unitary time evolution into the Schroedinger cat state is experimentally feasible as demonstrated already with the experiments in optical lattices [7].

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