Finite size scaling analysis of intermittency moments in the two dimensional Ising model

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Abstract

Finite size scaling is shown to work very well for the block variables used in intermittency studies on a 2-d Ising lattice. The intermittency exponents so derived exhibit the expected relations to the magnetic critical exponent of the model.

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**Introduction.** Intermittency studies have developed into an active branch of particle physics. In spite of much effort, however, the origin of intermittency in multiparticle production processes is still controversial. In the original papers [1] two possible mechanisms have been pointed out. Intermittency could be: a result of particle production by long random cascades, or a signal of a phase transition. It was soon suggested (for early reference cf. [2]) that it also could result from suitably conspiring ordinary processes like resonance production and decay, Bose-Einstein interference etc. Standard QCD cascade was also shown to lead to intermittency at very high energies [3]. In order to shed some light on possible mechanisms, it is instructive to consider solvable systems where intermittency is present and its origin is unambiguous.

It has been pointed out by one of us [4] that intermittency should be observable at the phase transition point of the two-dimensional (2d) Ising model. Indeed the transition is of second order, hence the fluctuations are selfsimilar at all length scales. This is usually the condition for the occurrence of intermittency. In order to define it quantitatively, the $L \times L$ Ising lattice is divided into cells of size $l \times l$. For each cell a block variable $k_l$ is defined, and the thermodynamical averages of moments of $k_l$ are computed using standard Monte Carlo methods. If the moments $M_p$ show a power dependence on the size of a cell, i.e. if

$$\log M_p = -\lambda_p \log l + c_p,$$

where $\lambda_p$ and $c_p$ do not depend on $l$, the system is said to be intermittent with the intermittency exponents $\lambda_p$.

Such a formulation of the problem bears a striking resemblance to the renormalization group (RG) approach used to study critical phenomena [5]. Indeed, employing results of the RG analysis, Satz [6] derived a relation between intermittency exponents and the known critical magnetic exponent $\delta$ of the Ising model. Let us choose as the block variable

$$k_l = |\overline{s}_l|,$$

where $\overline{s}_l = \frac{1}{l^2} \sum_{i \in C_l} s_i$ denotes the average spin in an $l \times l$ cell $C_l$. Satz used $\overline{s}_l$ without the absolute value, which leads to some difficulties, but the idea is the same. According to the RG [7], when calculating thermodynamical averages, the dynamical variables $\overline{s}_l$ can be replaced by

$$\tilde{s}_l = l^x \overline{s}_l,$$

where the new variables $\tilde{s}_l = \pm 1$ just like the original spins. The constant $x$ is known as the anomalous dimension of the spin field. For the two-dimensional Ising model $x$ can be expressed by the critical magnetic exponent $\delta = 15$ (cf. [6]). One finds

$$x = \frac{2}{\delta + 1} = \frac{1}{8}$$
Using Eq. (3) one obtains the power dependence on the block size

\[ \langle |\mathbf{s}|^p \rangle = l^{-p/8} g\left(\frac{l}{L}, \frac{L}{\xi(\beta)}\right). \]  

(5)

Coefficient \( g \) depends in general on all independent ratios of the dimensional parameters entering the problem. In particular the dependence on temperature enters through the ratio of the infinite volume correlation length \( \xi(\beta) \) to any other relevant scale parameter. Assuming that \( g(0, L/\xi(\beta)) \neq 0 \), the above finite size scaling Ansatz gives Eq. (4), for \( 1 << l << L \) with the intermittency exponents

\[ \lambda_p = \frac{1}{8} p. \]  

(6)

Let us stress that Eq. (5) represents only the leading, in \( l \), result of the RG analysis. In general nonleading terms may affect this simple behaviour.

A convincing numerical confirmation of Eq. (6) was, however, more difficult. In Ref. [4] the number of “up” spins in a cell \( \frac{1}{2}(1 + s_l) \) was chosen as a block variable. In the vicinity of the critical temperature the increase of the normalized moments with decreasing cell size was demonstrated qualitatively. Later [8, 9] it was pointed out that \( Z_2 \) symmetric observables may be more convenient for quantitative determination of the intermittency indices, since they are insensitive to fluctuations between the two ordered phases. Such transitions, which occur in finite systems, may mask the true critical fluctuations.

The authors of Refs [8, 9] suggested the block variable \( \frac{1}{2}(1 + \overline{S}_l) \), where

\[ \overline{S}_l = \mathbf{s}_l \text{ sign}(\mathbf{s}_L). \]  

(7)

They also choose to work at the quasicritical temperature where, at given \( L \), the magnetic susceptibility is the largest. At first they did not get a clear signal for the power dependence [8], but an improved analysis [9] uncovered the selfsimilar behaviour; however, rather surprisingly, with intermittency indices derived from the percolation critical exponent and not from the magnetic one. A possible explanation of this result was suggested in [10]. Moments of this particular block variables are linear combinations of averages of various powers of \( \overline{S}_l \). Assuming for each power of \( \overline{S}_l \) a scaling law analogous to (1), one indeed finds, for the lattice sizes considered, the effective exponents similar to the percolation ones. Whether this is a numerical coincidence, or evidence for a genuine relation to percolation, is an open problem [11, 12]. Very recently Leroyer [13] used \( \overline{S}_l \) as the block variable and for \( \beta = \beta_c = \frac{1}{2} \log (1 + \sqrt{2}) \) got a clear confirmation of the power behaviour with intermittency exponents derived from the magnetic one. We shall comment on his results later.

**Results.** The main points of the present work are: the choice of \( |\mathbf{s}_l| \) as the block variable, and a consistent finite size scaling (FSS) analysis of our Monte Carlo data.
In the FSS approach the problem of choosing the best \( \beta \) is less severe, since one is studying the dependence on \( L \) and \( \beta \) in a small scaling window around \( \beta_c \). In the 2d Ising model the critical exponent \( \nu \) is equal to 1, hence the FSS variable is

\[
\frac{L}{\xi(\beta)} \simeq L\Delta\beta \equiv y, \quad \Delta\beta = \beta - \beta_c. \tag{8}
\]

Our runs were done for four values of \( y \) \((0.0, 0.105, 0.21, 0.42)\), i.e. in the range which corresponds to \( \beta \) between \( \beta_c \) and the pseudocritical one for given \( L \). We find numerically that the results depend weakly on the choice of \( y \) in this region.

Figure 1: Dependence of the second moment, for \( y = 0.105 \), on the cell size \( l \). Continuous lines correspond to fixed \( \frac{l}{L} \) (see text). The dashed line connects points for \( L = 1024 \).

MC simulations were performed using the Swendsen-Wang algorithm \([4]\). The values \( L = 64, 128, 256, 512, 1024 \) and \( 4 \leq l \leq L \) were used. For each lattice size about \( 10^5 \) measurements were done. In order to reduce autocorrelations a measurement was taken every sixth sweep, for smaller lattices, and every eight sweep for larger ones. We took advantage of the very slow growth of the autocorrelation time in the Swendsen-Wang method. The critical exponent characterizing this growth is estimated to be about .35. For each new lattice the first \( 10^4 \) sweeps were discarded to allow the system to reach equilibrium. Statistical errors of our MC results are always less than the sizes of the symbols used to denote points on the figures. At fixed \( l/L \) and \( y \), to a very good accuracy, the dependences of \( \log < |s_l|^p > \) on \( \log l \) are found to be linear in all the regions considered. The slopes agree with predictions (3) within small errors, which can be ascribed to the finite \( L \) corrections.
In Fig. 1 we show as an example the dependence of \( \log \langle s_l^2 \rangle \) on \( \log l \) for fixed \( y = 0.105 \) and fixed \( l/L = 2^{-m}, \ m = 0, 1, \ldots, 6 \). The dotted line joins the points with fixed \( L = 1024 \). The usual approach was to use such lines for \( l << L \) to find the slopes. Clearly, the slopes of lines evaluated at constant \( \frac{l}{L} \) are much better defined in accordance with Eq. (5). The slopes for the six lines in the figure range from 0.239 to 0.272 to be compared with \( \frac{1}{4} \) obtained from the magnetic exponent, Eq. (6), and \( \frac{10}{96} \simeq 0.104 \) resulting from the percolation exponent. Certainly the magnetic exponent is preferred. In Fig. 2 we show the results for \( y = 0.42, l/L = 1/8 \) and \( p = 1, 2, 3, 4, 5 \). The lines are normalized at \( l = 32 \) and have slopes \( p/8 \). The agreement is quite satisfactory.

![Figure 2: Dependence of the moments \( p = 1, \ldots, 5 \) for \( y = 0.42 \) on the cell size \( l \). The lines are calculated from \( \log_2 M_p(l) = \log_2 M_p(32) - \frac{p}{8}(\log_2 l - 5) \).](image)

The normalized moments [15] are of little interest for the present choice of the block variable, because from Eq. (5)

\[
F_p \equiv \frac{\langle |s_l|^p \rangle}{\langle |s_l| \rangle^p} \sim l^0, \quad l/L = \text{const},
\]  

where \( \sim \) means equal up to a scale independent factor. At fixed \( L \) there could be some \( l \) dependence from the argument of the scale invariant coefficients. However, since our numerical results show that these coefficients tend to nonzero limits as \( l/L \to 0 \), there are no relations analogous to Eq. (1) with non-zero \( \lambda_p \) for \( l << L \).

It is a property of the Ising model, and in fact of many field theories, that the first power of a field has a nontrivial anomalous dimension. In the multiparticle
language single particle inclusive distributions are not smooth. The anomalous dimension of the \( n \)-th power of the field is in our case simply \( n \) times the anomalous dimension of the field itself. Thus the anomalous dimensions in the numerator and denominator of Eq.(2) cancel. Fortunately in order to study selfsimilarity or intermittency it is enough to normalize moments, Eq.(5), by appropriate powers of the volume of a cell \([16]\), which looses no information about critical indices.

There is also no motivation to use the factorial \([1]\) or binomial \([17]\) moments, since, although we always consider the \( l \ll L \) limit, one should keep \( 1 \ll l \) at the same time, i.e. the “number of particles” in a cell must be large. This means that the bias introduced by the noise is not important, and tricks invented to filter it out are not necessary.

We discuss now moments of the block variable \( \bar{S}_l \), Eq.(7). For \( p \) even and/or \( l = L \) we have \( \bar{S}_l^p = |\bar{s}_l|^p \). Thus only the moments with \( p \) odd and \( l < L \) require a separate discussion. As we will see shortly, these moments have a different scaling behaviour, which is also confirmed by our data. The main difference with the previous block variable is that, due to the translational invariance and additivity, the first moment of \( \bar{S}_l \) is independent of the cell size \([\bar{1}]\). Indeed we have

\[
< \bar{S}_l > = < < \bar{S}_l > >= < \bar{S}_L > \sim L^{-1/8},
\]

(10)

Where the double brackets denote average over cells and over the ensemble, and \( \sim \) means, as usual, equal up to a scale invariant coefficient. Assuming that the RG result, Eq.(3), holds for the new variable

\[
< \bar{S}_l^p > \sim l^{-p/8} < \tilde{S}_l >, \quad p - odd,
\]

(11)

where we have used \( \tilde{S}_l^p = \tilde{S}_l \) for odd \( p \). By definition we have \( \tilde{S}_L = |\tilde{s}_L| = 1 \). Equations (10) and (11) imply

\[
< \tilde{S}_l > \sim \left( \frac{l}{L} \right)^{1/8},
\]

(12)

while for the normalized moments, studied also by Leroyer \([\bar{13}]\) one gets

\[
F_p \equiv \frac{< \bar{S}_l^p >}{< \bar{S}_l >^p} = \left( \frac{l}{L} \right)^{1-p} h(\frac{l}{L}, y), \quad p - odd,
\]

(13)

where the scale invariant function has been written explicitly. Again, for \( h(z, y) \) regular and non-vanishing at \( z = 0 \), the above formula gives the power behaviour with the intermittency exponents

\[
\lambda_p = \frac{p - 1}{8}, \quad p - odd.
\]

(14)

\footnote{One may say that \( \bar{S}_l \) has zero anomalous dimension, however any higher power does not.}
The behaviour (13) has been numerically confirmed by Ref.[13]. On the other hand for the unnormalized moments we get

\[ < S_l^p >= l^{(p-1)/2} H(l/L, y). \]  

At fixed \( L >> l \) these moments scale as \( < |S_l|^p >= l^{-1} \) while for \( l \approx L \) they coincide with \( < |S_l| >= l^{p-1} \). Both these features are clearly seen in Fig.3, where log \( < S_l > \) (\( p = 1, 3, 5 \)) and log \( < |S_l|^p > \) (\( p = 1, 2, 3, 4, 5 \)) are plotted versus log \( l \) at fixed \( L = 512 \). Note that also \( < S_l > \) for \( l << L \) is parallel to \( < |S_l|^0 > = 1 \).

**Figure 3**: Dependence of the moments, for \( y = 0.21 \), on the cell size \( l \). The dashed lines correspond to the block variable \( |S_l| \) for \( p = 1, 2, 3, 4, 5 \) and solid lines to \( S_l \) for \( p = 1, 3, 5 \). Note that the solid lines are parallel to the dashed ones with \( p \to p - 1 \).

**Summary.** It is seen that the intermittency of the Ising model is yet another consequence of the well known scale invariance of the system in the critical region. The renormalization group formula, together with the finite size scaling (5), give a good description of this phenomenon in the vicinity of \( \beta_c \) (\( 0 \leq L \Delta \beta \leq .42 \)), for lattice sizes \( L = 64 \) to 1024 and cell sizes \( l = 4 \) to \( L \). The power behaviour, Eq.(5), was confirmed with the intermittency exponents determined by the magnetic critical exponent \( \delta \) when \( |S_l| \) or \( S_l \) are used as block variables. Moments of both variables were shown to obey simple relations which are also confirmed by our data. On the other hand, it is still an open question whether it is possible to find a block variable, which would yield intermittency exponents related to the percolation exponent. In particular the proposal [12] to consider the largest connected cluster in a cell remains to be studied.
We would like to stress a few points which considerably simplify both the theoretical and the numerical studies.

- The absolute value of the resultant spin of a cell $|\vec{s}|$ is $Z_2$ symmetric and seems to be a convenient choice for the block variable.
- An advantage of the finite size scaling approach is that instead of selecting a particular value of $\beta$ one can study the simultaneous dependence on $\beta$ and $L$ by fixing $y = L\Delta\beta$ at several not too large values. This is both simpler and less restrictive.
- The finite size scaling analysis at fixed $l/L$ is simpler conceptually, and gives better defined slopes, than the analysis at fixed $L$. This implies that (nearly) exact power behaviour (1) at fixed $L$ can be expected only at $l << L$ (keeping still $1 << l$).

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