Capability Analysis of Abaqus Shell Element Based on Finite Element Method

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Abstract: Shell-like structures are widely used in many engineering structures. This work is forcing on the performance of the S4 element and S4R element, two types of conventional shell element in Abaqus, in different R/h ratios. In the scenario of a pipe-like configuration, both DKQ and MPT have different positions in different R/h ratios.

1. Introduction
Shell-like structures are widely used in many engineering structures. They are usually specified into three categories based on the radius-thickness ratio: R/h > 20, 6 < R/h < 20, and R/h < 6. They are considered as a thin shell, a moderately thick shell, and a thick shell, respectively. These categories introduce two well-known plate theories considered, subjected to both plate and shell structure, which are Reissner-Mindlin plate theory (MPT) (Reissner, 1943; Mindlin, 1951) and Kirchhoff plate theory (KPT). Kirchhoff theory is subjected to the thin plate or shell with the neglectable shear deformation. It opts to the Mindlin plate theory with only C0 functions inside when dealing with the thick plate. Meanwhile, KPT and MPT are highly used in commercial software (like ANSYS, Abaqus) to solve typical kinds of shell problem based on the Finite Element Analysis (FEA).

However, both KPT and MPT have significant shortcomings in some specific situations. For KPT, it fails to overcome the scenario with thick plate since the equations do not come with shear deformation. Also, MPT indeed is convenient and avoid the time consuming, while it is not satisfied with the Kirchhoff constraints when the thickness is thin and approaching zero. Motivated by this terrible performance called shear locking, commercial software, for example, Abaqus, have servals options for users to decline the influence of shear locking and dedicate to make a compatible model (S4 element or S4R element) to solve both thin and thick plate problem. In general, the S4 element is a fully integrated finite-membrane-strain element that is for all purposes of the shell problem. S4R element is the reduced integrated quadrilateral finite-membrane-strain element which adopts reduce integration theory (Zienkiewicz et al., 1971) advocated to treat the shear locking. The performance of these elements significantly changes in the theories participated. Consequently, it is essential to have a preliminary study of the behavior in advanced in order to optimize the selection when dealing with different models.

Nowadays the advanced computational software (such as MATLAB, Python, C++) allows us to build a more complex FE model to test the compatibility of the elements in Abaqus based on conventional theories in different circulations. This work is forcing on the performance of the S4 element and S4R element, two types of conventional shell element in Abaqus, in different R/h ratios. Above testing by the standard 2D patch, we use the MATLAB to simulate a more complex 3D shell structure (Intersection-like circle shell) and determine the difference of displacements between Abaqus results (S4 and S4R) and the MATLAB results (Kirchhoff theory and Mindlin Theory). For the MATLAB program, we decided to use degenerated 3D element that leads to the classical equation of shell structure.
They will compare with the results from Abaqus in the different type of thickness changed by the R/h ratio. They will be discussed in the following sections.

2. Definition of Intersection Compressing Test
In this scenario, there is the pipe-like structure with two circle shell orthogonal to each other. It is considering an isotropic, homogeneous, and elastic material used in this structure. With the pressure acting on the top and bottom edges, that is easy to treat as a symmetric structure, and one-eighth of the pipe are considered in FEA. The geometric dimensions are described in Fig 1.

![Fig 1. Geometry of Intersection](image)

Mechanical Property is shown as following:

| Property          | Value                      |
|-------------------|----------------------------|
| Young’s Modulus   | $2.7 \times 10^7$ Psi      |
| Poisson Ratio     | 0.3                        |
| Radius            | 13.55 Inch                 |
| Length (Height)   | 60 Inch                    |

The strategy working in the test is as mentioned below, three results from the MATLAB and the other two are from Abaqus by different R/h ratios. More details are as following:

| Element Type       | Description                                                                 |
|--------------------|-----------------------------------------------------------------------------|
| S4                 | S4-element structure created by Abaqus                                      |
| S4R                | S4R-element structure created by Abaqus                                      |
| Normal & DKQ       | Ordinary four-nodes membrane theory with an artificial drilling part combined with Discrete Kirchhoff Quadrilateral (DKQ) built in MATLAB |
| Drilling DOF & DKQ | Drilling Degrees of Freedom theory (Drilling DOF) (Batoz, J. and Tahar, M., 1982.) combines with DKQ built-in MATLAB |
| Drilling DOF & Mindlin | Drilling DOF combines with Mindlin Plate Theory (MPT) built in MATLAB          |

Since the Drilling DOF guarantees the non-singularity in the stiffness matrix, the first collocation is introduced to makes sure the Drilling DOF theory do not influence the results on off-plane DOF and comparably have an accurate result in the in-plane degrees of freedom (DOF). Also, it will be discussed when a significant difference happened between the results. The second and the third one will play as a touchstone in either thin or thick shell structure to test the S4 and S4R element. Also, other factors are contributing to the variation control: all the tests are using the same geometry and same size mesh grids, including the same elements labels and nodes labels. MATLAB mesh model will be built by extracting the mesh grids from S4 and S4R test in Abaqus. Besides, all the tests are using the same step length and initial value.
All the tests are using the 4-nodes element. Higher order element has been proved that can improve the accuracy of the analysis, including solving the shear-locking problem. Some theories even introduced 8-nodes elements to compare with the 4-nodes elements. That is irrational and incomparable. Furthermore, high order elements are less frequently to be used than 4-nodes elements due to its time consuming and the inefficiency. Hence, in this work, we are only talking about the 4-nodes elements for both in-plane and off-plane theories.

Besides, in order to get a general conclusion based on the test, there are six certain points represented the different situation, monitored during the procedure:

Node 5: Node on the free edge.
Node 7: Node on the center of the surface of the intersection configuration.
Node 13: Center node on the intersection line.
Node 14: Arbitrary node on the intersection line.
Node 16: Arbitrary node on the intersection line.
Node 43: Forced node on the top of the intersection.

| Node | Location |
|------|----------|
| 5    | Node on the free edge |
| 7    | Node on the center of the surface of the intersection configuration |
| 13   | Center node on the intersection line |
| 14   | Arbitrary node on the intersection line |
| 16   | Arbitrary node on the intersection line |
| 43   | Forced node on the top of the intersection |

### 3. Isoparametric Method and Gaussian Quadrature

The isoparametric formulation provides a simplification to investigate the irregular and non-linear shape models. For a distorted shape in a 2D coordinate (x, y), Isoparametric Method allows to change the non-linear element contour from the global coordinate to the linear shape configuration in nature coordinate (ξ, η). In nature coordinate, assume that

\[ u(\xi, \eta) = a \xi + b \eta + c \eta + d \eta \xi \]

By using the constraint

\[ u(1,1) = a + b + c + d; \quad u(1,-1) = a + b + c - d; \]
\[ u(-1,1) = a - b + c - d; \quad u(-1,-1) = a - b - c + d; \]  

(1a-d)

Thus, for all \( u(\xi, \eta) \)

\[ u(\xi, \eta) = \varphi_1(\xi, \eta)u_1 + \varphi_2(\xi, \eta)u_2 + \varphi_3(\xi, \eta)u_3 + \varphi_4(\xi, \eta)u_4 \]

(2)

Where

\[ \varphi_1(\xi, \eta) = \frac{1}{4}(1 - \xi)(1 - \eta) ; \quad \varphi_2(\xi, \eta) = \frac{1}{4}(1 + \xi)(1 - \eta) ; \]
\[ \varphi_3(\xi, \eta) = \frac{1}{4}(1 + \xi)(1 + \eta) ; \quad \varphi_4(\xi, \eta) = \frac{1}{4}(1 - \xi)(1 + \eta) ; \]

(3a-d)

Since there are functions in terms of \( \xi \) and \( \eta \), a transform equation is required to change the Cartesian coordinate to nature coordinate. Therefore, the Jacobian Matrix is taken into account.

\[ J = Jacobian = \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{vmatrix} \]

(4)

The Stiffness matrix should be as the following:

\[ K = \int_{-1}^{1} \int_{-1}^{1} \left( \begin{array}{c} \frac{\partial \varphi}{\partial \xi} \\ \frac{\partial \varphi}{\partial \eta} \end{array} \right)^T J^{-1} J J^{-1} \left( \begin{array}{c} \frac{\partial \varphi}{\partial \xi} \\ \frac{\partial \varphi}{\partial \eta} \end{array} \right) |J| d\xi d\eta \]

(5)

In addition, Gaussian quadrature plays a crucial role in the evaluation of any integration range from -1 to 1 for all dimensions in the coordinate. When deciding the quantity of Gauss points used for the element, the table attached summarize the criteria of choosing. In the table, the N refers to the order of the polynomial of degree that is used in the formula, 2N-1 is the number of Gauss points that should be used to approximate the integration.

For each Gauss point (2N-1), the corresponding \( W(\text{weight}) \) and \( \xi(\text{location}) \) are presented below.

### Table 1. Weight and location distribution for each Gauss point.

| 2N-1 | \( \xi \) | \( W \) |
Drilling Degrees of Freedom Theory

Drilling Degrees of Freedom (Drilling DOF) theory is accustomed in some simplifications of the structure. It manifests the conventional membrane element with the in-plane drilling degrees of freedom, reduced the integration by penalty formulation. Drilling DOF avoid the possibility of the singularity in the 3D stiffness matrix. Also, it has excellent performance on the load transformation that would diminish the impact of the inaccuracy of in-plane degrees of freedom on the off-plane degrees of freedom, which is formed by the two plate theories.

Now considering a plate element with thickness ‘t,’ the energy function should be as following:

\[
I = \int_{\Omega} Et \begin{pmatrix} \varepsilon_x & \varepsilon_y & \varepsilon_{xy} \end{pmatrix}^T \begin{pmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & 1-v \end{pmatrix} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{pmatrix} dxdy + \frac{1}{2} \gamma \int_{\Omega} \frac{1}{2} \left( \frac{\partial \varepsilon_x}{\partial x} - \frac{\partial \varepsilon_y}{\partial y} \right)^2 dxdy
\]

Where \(\gamma\) is the penalty constant, E is Young’s modulus, and \(v\) is the Poisson ratio.

From the equation above, because of the penalty term, Drilling DOF can successfully introduce \(\theta_z\) into our equation above, and that is how drilling DOF works with three degrees of freedom. The in-plane displacements employ Allman-type interpolation:

\[
\psi_i = \sum_{i=1}^{4} \psi_i(\xi, \eta) \begin{pmatrix} u_i \\ v_i \end{pmatrix} + \sum_{k=5}^{8} N_k(\xi, \eta) \frac{L_{ij}}{G} (\theta_{zj} - \theta_{zt}) \begin{pmatrix} C_{ij} \\ S_{ij} \end{pmatrix}
\]

Where (Zienkiewicz and Taylor,1989)

\[
\psi_i (i = 1,2,3,4) \text{ are the same shape function on (3a-d)}
\]

\[
N_k(\xi, \eta) = \frac{1}{2} \left( 1 - \xi^2 \right) \left( 1 + \eta_k \cdot \eta \right) \quad k = 5,7
\]

\[
N_k(\xi, \eta) = \frac{1}{2} \left( 1 - \xi^2 \right) \left( 1 + \xi_k \cdot \xi \right) \quad k = 6,8
\]

\[
X_{ij} = X_j - X_i; \quad Y_{ij} = Y_j - Y_i; \quad L_{ij} = \sqrt{(X_{ij})^2 + (Y_{ij})^2};
\]

\[
C_{ij} = \frac{Y_{ij}}{L_{ij}}; \quad S_{ij} = -\frac{X_{ij}}{L_{ij}};
\]

The first part of equation (7) in terms of \(\psi_i\) shows the regular shape function. The second part indicates the discretion of the middle node of each line of the element. Hence, as mentioned in the Gaussian Quadrature section, this element should wisely adopt 3-by-3 Gauss points to approach the approximation.

Discrete Kirchhoff Quadrilateral (DKQ) (Batoz, and Tahar, 1982) element is an efficient triangular element based on the classical Kirchhoff Plate Theory (KPT). DKQ is attributed to the discrete Kirchhoff theory and satisfied the Kirchhoff constraints along the entire boundary. The strain energy is as follows:
\[ u(w, \theta_x, \theta_y) = \frac{1}{2} D \int_{\Omega} \begin{vmatrix} \frac{\partial \theta_y}{\partial x} & -\frac{\partial \theta_x}{\partial y} \\ \frac{\partial \theta_x}{\partial y} & \frac{\partial \theta_x}{\partial x} \end{vmatrix}^T \begin{bmatrix} 1 & v \\ v & 1 \end{bmatrix} \left( \begin{bmatrix} \frac{\partial \theta_y}{\partial x} \\ \frac{\partial \theta_x}{\partial y} \end{bmatrix} - \begin{bmatrix} \frac{\partial \theta_x}{\partial y} \\ \frac{\partial \theta_y}{\partial x} \end{bmatrix} \right) \right) \, dx \, dy \] (10)

For each corner node \((i = 1,2,3,4)\)

\[
\begin{align*}
\theta_y &= - \sum_{i=1}^{4} \psi_i(\xi, \eta) \theta_{yi} - \sum_{k=5}^{8} N_k(\xi, \eta) \theta_{yk} \\
\theta_x &= \sum_{i=1}^{4} \psi_i(\xi, \eta) \theta_{xi} + \sum_{k=5}^{8} N_k(\xi, \eta) \theta_{xk}
\end{align*}
\]
Where

\[
\begin{align*}
\theta_{yk} &= \left( \frac{1}{4} \frac{S_{ij}}{L_{ij}} - \frac{1}{2} c_{ij} \right) \theta_{yi} + \frac{3}{4} c_{ij} S_{ij} \theta_{xi} + \left( \frac{1}{4} \frac{S_{ij}}{L_{ij}} - \frac{1}{2} c_{ij} \right) \theta_{yi} + \frac{3}{4} c_{ij} S_{ij} \theta_{xi} - \frac{3}{2} S_{ij} \theta_{xj} - \frac{3}{2} S_{ij} \theta_{yi} - \frac{3}{4} c_{ij} W_{ij} - W_{ij} \end{align*}
\]

\[
\begin{align*}
\theta_{xk} &= \left( \frac{1}{4} \frac{S_{ij}}{L_{ij}} - \frac{1}{2} c_{ij} \right) \theta_{xi} - \frac{3}{4} c_{ij} S_{ij} \theta_{yi} + \left( \frac{1}{4} \frac{S_{ij}}{L_{ij}} - \frac{1}{2} c_{ij} \right) \theta_{xi} - \frac{3}{4} c_{ij} S_{ij} \theta_{yi} - \frac{3}{4} c_{ij} W_{ij} - W_{ij}
\end{align*}
\]

The definitions of \(\psi_i, c_{ij}, S_{ij}, L_{ij}\) are obtained in equation (3a-d) and (9a-e). \(W\) refers to the deflection. Apparently, all the equations obey the Kirchhoff constraints when considering of thin plate:

\[
\begin{align*}
\frac{\partial w}{\partial x} + \theta_x &= 0; \\
\frac{\partial w}{\partial y} - \theta_y &= 0
\end{align*}
\] (13)

Moreover, rather than using the classical KPT, the DKQ remain \(C^0\) functions into the equation, holding the same efficiency as MPT.

### 6. Reissner-Mindlin Plate Theory

Reissner-Mindlin plate theory (MPT) was assumed by Reissner (1943) and Mindlin (1951).

The assumption is derived when the thickness is relatively large (R/h) and the shear stress cannot be neglected. Different from DKQ, MPT assume that the straight line perpendicular to the mid-surface remain straight but not necessarily normal to the mid-surface after deformation. In the MPT, the element is expressed by \(C^0\) function in terms of \(\theta_x\) and \(\theta_y\).

Therefore, in MPT, the strain energy:

\[
\begin{align*}
u(w, \theta_x, \theta_y) &= \frac{1}{2} D \int_{\Omega} \left( \frac{\partial \theta_y}{\partial x} \frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_x}{\partial y} \frac{\partial \theta_y}{\partial x} + \frac{1-v}{2} \left( \frac{\partial \theta_y}{\partial x} \frac{\partial \theta_x}{\partial y} - \frac{\partial \theta_x}{\partial y} \frac{\partial \theta_y}{\partial x} \right)^2 \right) \, dx \, dy + \alpha \int_{\Omega} \left( \frac{\partial w}{\partial x} + \theta_y \right)^2 + \left( \frac{\partial w}{\partial y} + \theta_x \right)^2 \, dx \, dy
\end{align*}
\]

Where

\[
D = \frac{E t^3}{12(1-v^2)}, \quad \alpha = \frac{6K(1-v)}{t^2}
\] (15a, b)

In matrix notation, the equation should be:

\[
\begin{align*}
u(w, \theta_x, \theta_y) &= \frac{1}{2} D \int_{\Omega} \begin{bmatrix} \frac{\partial \theta_y}{\partial x} & -\frac{\partial \theta_x}{\partial y} \\ \frac{\partial \theta_x}{\partial y} & \frac{\partial \theta_x}{\partial x} \end{bmatrix}^T \begin{bmatrix} 1 & v \\ v & 1 \end{bmatrix} \left( \begin{bmatrix} \frac{\partial \theta_y}{\partial x} \\ \frac{\partial \theta_x}{\partial y} \end{bmatrix} - \begin{bmatrix} \frac{\partial \theta_x}{\partial y} \\ \frac{\partial \theta_y}{\partial x} \end{bmatrix} \right) \right) \, dx \, dy + \alpha \int_{\Omega} \begin{bmatrix} \frac{\partial w}{\partial x} + \theta_y \\ \frac{\partial w}{\partial y} + \theta_x \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \left( \begin{bmatrix} \frac{\partial w}{\partial x} + \theta_y \\ \frac{\partial w}{\partial y} - \theta_x \end{bmatrix} \right) \, dx \, dy
\end{align*}
\]

In the equation, D is the flexural rigidity and \(\alpha\) is the penalty parameter.

### 7. Results

For the regular nodes (relatively small deformation on the corresponding element) which are not on the intersecting line (Node 5, 7, and 43), they are satisfied the rules of DKQ and MPT. Also, it should be
noticed that on node 5 and 7, the difference between DKQ and MPT is so tiny that both are on the same badge of Abaqus results. Thus, it could be concluded that in the static problem, both DKQ and MPT have the same performance on the regular free edge no matter how much the radius-thickness ratio is. Also, Abaqus may use either of the theories and have the same trajectory of MATLAB results. Furthermore, in the range of thin plate, the normal membrane caused the difference in the normal displacement of Z direction of node 5. It is further proved that Drilling DOF does improve the accuracy of the simulation result. The displacement vs. radius-thickness ratio plots of node 5 and 7 are shown below. Since the symmetry of the pipe configuration, there are zero displacement on the Dx(normal displacement on x-direction), Ry (rotational displacement on Y direction) and Rz (rotational displacement on Z direction) on node 5, zero displacement on the Dy, Dz, Rx, Ry, and Rz direction on node 7, respectively.

On node 43, because of the pressure acting on the edge of the top of the intersection, node 43 shows the rational result on the Dx, Dy, Dz, and Ry, whose all results are highly matching to each other as what we talked on node 5 and 7. However, it shows that when the result focuses on the Rx and Rz direction, Abaqus results are precisely the same as the DKQ theory. The result of MPT has a poor performance all way along with the range from thin to thick shell. The plots are shown below.
Hence, on the forced node, such as node 43, Abaqus are using the combination of Drilling DOF and DKQ all the time when dealing with both of thin and thick shell. For the nodes on the large distorted element along the intersecting line (node 13, 14 and 16), significant error came into different directions. On the normal direction and Rx of node 13, Abaqus results show a massive difference comparing to the MPT during the thin and moderately thick shell problem and came to close when the problem hits the thick shell problem. This phenomenon is reasonable since MPT has less accuracy due to the shear-locking. The same thing happened on the Rx of node 14 and 16. However, on the range of the moderately thick problem, Abaqus result went nowhere close to any of the result from MATLAB.
Besides, nodes on the intersecting line have shown a different trend in the direction of Ry and Rz. Based on the figures shown below, the S4 and S4R plots are slightly too far from the DKQ and closer to the MPT regarding all ratio of radius and thickness. Even the trend of Abaqus results are up to follow the MPT rule, there still have 30% error occurring on the thin and moderately thick shell problem. Hence, it could be noticed that on the Ry and Rz direction of the nodes on the large distorted element, the S4, and S4R are suspected to follow the MPT rule all the time while some other factors might need to be considered and cause the error.

Fig 16. Rotational Disp. on Node 16 Rx
Fig 17. Short-Range Scale Node 16 Rx
Fig 18. Rotational Disp. on Node 13 Ry
Fig 19. Short-Range Scale Node 13 Ry
Fig 20. Rotational Disp. on Node 14 Ry
Fig 21. Short-Range Scale Node 14 Ry
Fig 22. Rotational Disp. on Node 16 Ry

Fig 23. Short-Range Scale Node 16 Ry

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