Method of moments estimators for the extremal index of a stationary time series

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Abstract: The extremal index \( \theta \), a number in the interval \([0,1]\), is known to be a measure of primal importance for analyzing the extremes of a stationary time series. New rank-based estimators for \( \theta \) are proposed which rely on the construction of approximate samples from the exponential distribution with parameter \( \theta \) that is then to be fitted via the method of moments. The new estimators are analyzed both theoretically as well as empirically through a large-scale simulation study. In specific scenarios, in particular for time series models with \( \theta \approx 1 \), they are found to be superior to recent competitors from the literature.

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1. Introduction

The statistical analysis of the extremal behavior of a stationary time series is important in many fields of application, such as in hydrology, meteorology, finance or actuarial science (Beirlant et al., 2004). Such an analysis typically consists of two steps: (1) assessing the tail of the marginal law and (2) assessing the serial dependence of the extremes, that is, the tendency that extreme observations occur in clusters. The present work is concerned with step (2). The most common and simplest mathematical object capturing the serial dependence between the extremes is provided by the extremal index \( \theta \in [0, 1] \). In a suitable asymptotic framework, the extremal index can be interpreted as the reciprocal of the expected size of a cluster of extreme observations. The underlying probabilistic theory was worked out in Leadbetter (1983); Leadbetter et al. (1983); O’Brien (1987); Hsing et al. (1988); Leadbetter and Rootzén (1988).

Estimating the extremal index based on a finite stretch of observations from the time series has been extensively studied in the literature. An early overview is provided in Section 10.3.4 in Beirlant et al. (2004), where the estimators are classified into three groups: estimators based on the blocks method, the runs method or the inter-exceedance time method. Respective references are Hsing (1993); Smith and Weissman (1994); Ferro and Segers (2003); Süveges (2007); Robert (2009); Northrop (2015); Cai (2019), among many others. The proposed estimators typically depend on two or, arguably preferable, one parameter to be chosen by the statistician. The present paper is on a class of method of moments estimators (based on the blocks method), which improves upon a recent estimator proposed in Northrop (2015) and analyzed theoretically in Berghaus and Bührer (2018).

Some notations and assumptions are necessary for the motivation of the new class of estimators. Throughout the paper, \( X_1, X_2, \ldots \) denotes a stationary sequence of real-valued random variables with continuous cumulative distribution function (c.d.f.) \( F \). The sequence is assumed to have an extremal index \( \theta \in (0, 1] \), i.e., for any \( \tau > 0 \), there exists a sequence \( u_b = u_b(\tau), b \in \mathbb{N} \), such that \( \lim_{b \to \infty} \bar{b} F(b) = \tau \) and

\[
\lim_{b \to \infty} \mathbb{P}(M_{1,b} \leq u_b) = e^{-\theta \tau},
\]

where \( \bar{F} = 1 - F \) and \( M_{1,b} = \max\{X_1, \ldots, X_b\} \). Next, define a sequence of standard uniform random variables by \( U_s = F(X_s) \) and let

\[
Y_{1,b} = -b \log(N_{1,b}), \quad N_{1,b} = F(M_{1,b}) = \max\{U_1, \ldots, U_b\}.
\]

Since \( \bar{b} F\left( e^{-x/b} \right) = b (1 - e^{-x/b}) \to x \) for \( b \to \infty \), it follows from (1.1) that, for any \( x > 0 \),

\[
\mathbb{P}(Y_{1,b} \geq x) = \mathbb{P}(M_{1,b} \leq e^{-x/b}) \to e^{-\theta x}.
\]

In other words, for large block length \( b \), \( Y_{1,b} \) approximately follows an exponential distribution with parameter \( \theta \), denoted by \( \text{Exp}(\theta) \) throughout. This inspired Northrop (2015) and Berghaus and Bührer (2018) to estimate \( \theta \) by the maximum likelihood estimator for the exponential distribution; see Section 2 below for details on how to arrive at an observable (rank-based) approximate sample from the \( \text{Exp}(\theta) \)-distribution based on an observed stretch of length \( n \) from the time series \( (X_n)_{n \in \mathbb{N}} \).

The idea of transforming observations into a sample of exponentially distributed observations is actually not new within extreme value statistics: it is also, among many others, the main motivation for the Pickands estimator in multivariate extremes (Pickands, 1981; Genest and Segers, 2009). More precisely, if \( (X, Y) \) is a bivariate random vector from a multivariate extreme value distribution with Pickands function \( A = (A(w))_{w \in [0, 1]} \), then \( \xi(w) = \min\{-\log F_X(X)/(1- \)
$w$), $-\log F_Y(Y)/w$ is exponentially distributed with parameter $A(w)$. Given a sample of size $n$ from $(X, Y)$, we may replace $F_X$ and $F_Y$ by their empirical counterparts and arrive at an approximate sample of size $n$ from the Exp$(A(w))$-distribution, to be, for instance, estimated by the maximum likelihood method.

The present paper is now motivated by the following observation: while the maximum likelihood estimator is asymptotically efficient in the ideal situation of observing an i.i.d. sample from the exponential distribution, it was shown in Genest and Segers (2009) for rank-based estimators of the Pickands function that it is in fact more efficient to consider alternative estimators based on the method of moments, such as a rank-based version of the CFG-estimator (Capéraà et al., 1997). Given that Northrop’s blocks estimator is also rank-based, the main motivation of this work is to consider CFG-type estimators for the extremal index $\theta$. Alongside, we will also investigate other moment-based estimators, among which is one that is closely connected to the madogram (Naveau et al., 2009). We will show that the new estimators may exhibit a substantially smaller asymptotic variance than Northrop’s maximum likelihood estimator, in particular the CFG-type estimator in the case where $\theta$ is close to one.

The remaining parts of this paper are organized as follows: in Section 2, we collect some results about certain useful moments of the exponential distribution and use those to introduce the new estimators for $\theta$. Regularity assumptions needed to prove asymptotic results are summarized and discussed in Section 3. The paper’s main results are then presented in Section 4, alongside with a discussion of certain aspects of the derived asymptotic variance formulas. Section 5 is about a particular time series model, for which we show that all regularity conditions imposed in Section 3 are met. The finite-sample performance of the new estimators is investigated in a Monte-Carlo simulation study in Section 6. Finally, all proofs are postponed to Section 7.

2. Definition of estimators

Recall the definition of $Y_{1:b}$ in (1.2), where $b \in \mathbb{N}$. Similarly, let

$$Z_{1:b} = b(1 - N_{1:b}), \quad N_{1:b} = F(M_{1:b}) = \max\{U_1, \ldots, U_b\},$$

and note that, as $b \to \infty$ and for any $x > 0$,

$$\mathbb{P}(Z_{1:b} \geq x) = \mathbb{P}(M_{1:b} \leq F^{-1}(1 - x/b)) \to e^{-\theta x}$$

(2.1)

by similar arguments as for $Y_{1:b}$. The convergence relations in (1.3) and (2.1) serve as a basis for the method of moments estimators defined below.

Subsequently, let $X_1, \ldots, X_n$ denote a finite stretch of observations from the stationary sequence $(X_s)_{s \in \mathbb{N}}$. Within Section 2.1 and 2.2, we start by using (1.3) and (2.1) to derive some observable, approximate samples from the Exp$(\theta)$-distribution. In Section 2.3, we collect some moment equations for the exponential distribution, which will then be used to motivate new estimators for the extremal index in Section 2.4.

2.1. Two approximate Exp($\theta$)-samples based on disjoint blocks maxima

Divide the sample $X_1, \ldots, X_n$ into $k_n$ successive blocks of size $b_n$, and for simplicity assume that $n = b_n k_n$ (otherwise, the last block of less than $b_n$ observations should be deleted). For $i = 1, \ldots, k_n$, let

$$M_{ni} = \max\{X_{(i-1)b_n+1}, \ldots, X_{ib_n}\}$$
denote the maximum of the $X_s$ in the $i$th block of observations and let

$$Y_{ni} = -b_n \log N_{ni}, \quad Z_{ni} = b_n(1 - N_{ni}), \quad N_{ni} = F(M_{ni}).$$

Due to relations (1.3) and (2.1), if the block size $b = b_n$ is sufficiently large, the (unobservable) random variables $Y_{ni}$ and $Z_{ni}$ are approximately exponentially distributed with parameter $\theta$. Observable counterparts are obtained by replacing $F$ by the (slightly adjusted) empirical c.d.f. $\hat{F}_n(x) = (n + 1)^{-1} \sum_{i=1}^n I(X_i \leq x)$, giving rise to the definitions

$$\hat{Y}_{ni} = -b_n \log \hat{N}_{ni}, \quad \hat{Z}_{ni} = b_n(1 - \hat{N}_{ni}), \quad \hat{N}_{ni} = \hat{F}_n(M_{ni}).$$

Both the samples $Y_{n}^{ab} = \{\hat{Y}_{ni} : i = 1, \ldots, k_n\}$ and $Z_{n}^{ab} = \{\hat{Z}_{ni} : i = 1, \ldots, k_n\}$ will be used later to define disjoint blocks estimators for $\theta$ (note that both samples are dependent over $i$ due to the use of $\hat{F}_n$, which complicates the asymptotic analysis).

### 2.2. Two approximate $\text{Exp}(\theta)$-samples based on sliding blocks maxima

As in the previous paragraph, let $n$ denote the sample size and $b_n$ denote a block length parameter (the assumption that $k_n = n/b_n \in \mathbb{N}$ is not needed, no discarding is necessary). For $t = 1, \ldots, n - b_n + 1$, let

$$M_{nt}^{ab} = M_{t, t+b_n-1} = \max\{X_t, \ldots, X_{t+b_n-1}\}$$

denote the maximum of the $X_s$ in a block of length $b_n$ starting at observation $t$. Define

$$Y_{nt}^{ab} = -b_n \log N_{nt}^{ab}, \quad Z_{nt}^{ab} = b_n(1 - N_{nt}^{ab}), \quad N_{nt}^{ab} = F(M_{nt}^{ab}),$$

$$\hat{Y}_{nt}^{ab} = -b_n \log \hat{N}_{nt}^{ab}, \quad \hat{Z}_{nt}^{ab} = b_n(1 - \hat{N}_{nt}^{ab}), \quad \hat{N}_{nt}^{ab} = \hat{F}_n(M_{nt}^{ab}).$$

By the same heuristics as before, the observable samples $Y_{n}^{ab} = \{\hat{Y}_{nt}^{ab} : t = 1, \ldots, n - b_n + 1\}$ and $Z_{n}^{ab} = \{\hat{Z}_{nt}^{ab} : t = 1, \ldots, n - b_n + 1\}$ are approximate samples from the exponential distribution and will be used later to define sliding blocks estimators for $\theta$ (both samples are heavily dependent over $i$ due to the use of $\hat{F}_n$ and the use of overlapping blocks).

### 2.3. Preliminaries on the exponential distribution

Some important moment equations, valid for a random variable $\xi$, which is $\text{Exp}(\theta)$-distributed, are collected. First,

$$\mathbb{E}\log \xi = -\log \theta - \gamma =: \varphi_{(C)}(\theta), \quad \text{(CFG)}$$

where $\gamma = -\int_0^\infty \log(x)e^{-x} \, dx \approx 0.577$ denotes the Euler-Mascheroni-constant. Equation (CFG) is the basis for motivating the CFG-estimator, see Capéraà et al. (1997); Genest and Segers (2009) and the details in Section 1. Next, note that

$$\mathbb{E}\exp(-\xi) = \frac{\theta}{1 + \theta} =: \varphi_{(M)}(\theta), \quad \text{(MAD)}$$

which serves as a basis for the madogram, see Naveau et al. (2009). A further choice, including (CFG) as a limit, is provided by

$$\mathbb{E}\xi^{1/p} = \theta^{-1/p}\Gamma(1 + 1/p) =: \varphi_{(R)}(\theta), \quad \text{(ROOT)}$$
where $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ denotes the Gamma function and where $p > 0$. The moment estimator in case of $p = 1$ will turn out to coincide with Northrop’s maximum likelihood estimator. Also note that the previous equation is equivalent to

$$E\left[\frac{\xi^{1/p} - 1}{1/p}\right] = \frac{\theta^{-1/p} \Gamma(1 + 1/p) - 1}{1/p} =: \tilde{\varphi}(R, p)(\theta),$$

and taking the limits for $p \to \infty$ on both sides (interchanging the limit and the expectation on the left) exactly yields Equation (CFG).

2.4. Definition of the estimators

Let $\chi_m = \{\xi_1, \ldots, \xi_m\}$ denote a generic sample (not necessarily independent) from the Exp($\theta$)-distribution. Replacing the moments in Equations (CFG), (MAD) and (ROOT) by their empirical counterparts and solving the equation for $\theta$, we obtain the following three estimators for $\theta$:

$$\hat{\theta}_{CFG}(\chi_m) = e^{-\gamma} \exp\left\{-\frac{1}{m} \sum_{i=1}^m \log(\xi_i)\right\},$$

$$\hat{\theta}_{MAD}(\chi_m) = \frac{1}{m} \sum_{i=1}^m \exp(-\xi_i),$$

$$\hat{\theta}_{R,p}(\chi_m) = \frac{\Gamma(1 + 1/p)}{1 - \frac{1}{m} \sum_{i=1}^m \exp(-\xi_i)} \left(\frac{1}{m} \sum_{i=1}^m \xi_i^{1/p}\right)^{-p},$$

where $p > 0$. It may be verified that $\lim_{p \to \infty} \hat{\theta}_{R,p}(\chi_m) = \hat{\theta}_{CFG}(\chi_m)$, see also (2.2) for another relationship between the two estimators. Next, replacing $\chi_m$ by any of the four samples $Y^db_n, Z^db_n, Y^sb_n$ or $Z^sb_n$ defined in Sections 2.1 and 2.2, we finally arrive at 12 method of moments estimators for $\theta$. We use the suggestive notations

$$\hat{\theta}^db_{CFG} = \hat{\theta}_{CFG}(Y^db_n), \quad \hat{\theta}^sb_{MAD} = \hat{\theta}_{MAD}(Z^sb_n)$$

to, e.g., denote the disjoint blocks CFG-estimator based on the $Y_{ni}$ and the sliding blocks madogram-estimator based on the $Z_{ni}$, respectively. Note that the four estimators of the form $\hat{\theta}^m_{R,1}, \hat{\theta}^m_{R,1}, m \in \{db, sb\}$, are the (pseudo) maximum likelihood (PML) estimators considered in Berghaus and Bücher (2018).

3. Mathematical preliminaries

Further mathematical details are necessary before we can state asymptotic results about the estimators defined in the previous section. The asymptotic framework and the conditions are mostly similar as in Section 2 in Berghaus and Bücher (2018), but will be repeated here for the sake of completeness.

The serial dependence of the time series $(X_s)_{s \in \mathbb{N}}$ will be controlled via mixing coefficients. For two sigma-fields $\mathcal{F}_1, \mathcal{F}_2$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let

$$\alpha(\mathcal{F}_1, \mathcal{F}_2) = \sup_{A \in \mathcal{F}_1, B \in \mathcal{F}_2} |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|.$$

In time series extremes, one usually imposes assumptions on the decay of the mixing coefficients between sigma-fields generated by $\{X_s I(X_s > F^\alpha(1-\varepsilon_n)) : s \leq \ell\}$ and $\{X_s I(X_s > F^\alpha(1-\varepsilon_n)) :$
s ≥ ℓ + k}, where ε_n → 0 is some sequence reflecting the fact that only the dependence in the tail needs to be restricted (see, e.g., Rootzén, 2009). As in Berghaus and Bücher (2018), we need a slightly stronger condition, that also controls the dependence between the smallest of all block maxima. More precisely, for −∞ ≤ p < q ≤ ∞ and ε ∈ (0, 1], let B^ε_p,q denote the sigma algebra generated by U^ε_p := U_p 1(U_p > 1 − ε) with s ∈ {p, . . . , q} and define, for ℓ ≥ 1,
\[ α_ε(ℓ) = \sup_{k \in \mathbb{N}} α(B^k_{1+k}, B^k_{1+ℓ+ε, ∞}). \]
In Condition 3.1(iii) below, we will impose a condition on the decay of the mixing coefficients for small values of ε. Note that the coefficients are bounded by the standard alpha-mixing coefficients of the sequence U_s, which can be retrieved for ε = 1.

The extremes of a time series may be conveniently described by the point process of normalized exceedances. The latter is defined, for a Borel set \( A \subset E := (0, 1] \) and a number \( x \in [0, \infty) \), by
\[ N_n^{(x)}(A) = \sum_{s=1}^{n} \mathbbm{1}(s/n \in A, U_s > 1 - x/n). \]
Note that \( N_n^{(x)}(E) = 0 \) iff \( N_{1,n} \leq 1 - x/n \); the probability of that event converging to \( e^{-θx} \) under the assumption of the existence of the extremal index θ.

Fix \( m \geq 1 \) and \( x_1 > \cdots > x_m > 0 \). For \( 1 ≤ p < q ≤ n \), let \( F_{p,q,n}^{(x_1, . . . , x_m)} \) denote the sigma-algebra generated by the events \( \{U_i > 1 - x_j/n\} \) for \( p ≤ i ≤ q \) and \( 1 ≤ j ≤ m \). For \( 1 ≤ ℓ ≤ n \), define
\[ α_{n,ℓ}(x_1, . . . , x_m) = \sup\{||\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|| : A \in F^{(x_1, . . . , x_m)}_{1:s,n}, B \in F^{(x_1, . . . , x_m)}_{s+ℓ,n,n}, 1 ≤ s ≤ n - ℓ\}. \]
The condition \( Δ_n\left(\{u_n(x_j)\}_{1 ≤ j ≤ m}\right) \) is said to hold if there exists a sequence \( (ℓ_n) \) with \( ℓ_n = o(n) \) such that \( α_{n,ℓ_n}(x_1, . . . , x_m) = o(1) \) as \( n \to ∞ \). A sequence \( (q_n) \) with \( q_n = o(n) \) is said to be \( Δ_n\left(\{u_n(x_j)\}_{1 ≤ j ≤ m}\right) - \) separating if there exists a sequence \( (ℓ_n) \) with \( ℓ_n = o(q_n) \) such that \( q_n^{-1}α_{n,ℓ_n}(x_1, . . . , x_m) = o(1) \) as \( n \to ∞ \). If \( Δ_n\left(\{u_n(x_j)\}_{1 ≤ j ≤ m}\right) \) is met, then such a sequence always exists, simply take \( q_n = \max\{n_0^{1/δ}, (nℓ_n)^{1/2}\} \).

By Theorems 4.1 and 4.2 in Hsing et al. (1988), if the extremal index exists and the \( Δ(u_n(x)) \) condition is met \( (m = 1) \), then a necessary and sufficient condition for weak convergence of \( N_n^{(x)} \) is convergence of the conditional distribution of \( N_n^{(x)}(B_n) \) with \( B_n = (0, q_n/n] \) given that there is at least one exceedance of \( 1 - x/n \) in \( \{1, . . . , q_n\} \) to a probability distribution \( π \) on \( Π \), that is,
\[ \lim_{n→∞} \mathbb{P}(N_n^{(x)}(B_n) = j \mid N_n^{(x)}(B_n) > 0) = π(j) \quad ∀ j ≥ 1, \]
where \( q_n \) is some \( Δ(u_n(x)) \) - separating sequence. Moreover, in that case, the convergence in the last display holds for any \( Δ(u_n(x)) \) - separating sequence \( q_n \). If the \( Δ(u_n(x)) \) - condition holds for any \( x > 0 \), then \( π \) does not depend on \( x \) (Hsing et al., 1988, Theorem 5.1).

A multivariate version of the latter results is stated in Perfekt (1994), see also the summary in Robert (2009), page 278, and the thesis Hsing (1984). Suppose that the extremal index exists and that the \( Δ(u_n(x_1), u_n(x_2) \) - condition is met for any \( x_1 ≥ x_2 ≥ 0, x_1 ≠ 0 \). Moreover, assume that there exists a family of probability measures \( \{π^x(σ) : σ ∈ [0, 1]\} \) on \( J = \{(i, j) : i ≥ j ≥ 0, i ≥ 1\} \), such that, for all \( (i, j) ∈ J \),
\[ \lim_{n→∞} \mathbb{P}(N_n^{(x)}(B_n) = i, N_n^{(x_2)}(B_n) = j \mid N_n^{(x)}(B_n) > 0) = \pi^{x_2/x_1}(i, j), \]
where \( q_n \) is some \( \Delta(u_n(x_1), u_n(x_2)) \)-separating sequence. In that case, the two-level point process \( N_n^{(x_1, x_2)} = (N_n^{(x_1)}, N_n^{(x_2)}) \) converges in distribution to a point process with characterizing Laplace transform explicitly stated in Robert (2009) on top of page 278. Note that

\[
\pi_2^{(1)}(i, j) = \pi(i) \mathbb{I}(i = j), \quad \pi_2^{(0)}(i, j) = \pi(i) \mathbb{I}(j = 0).
\]

Finally, we will need the tail empirical process

\[
e_n(x) = \frac{1}{\sqrt{k_n}} \sum_{s=1}^{n} \left\{ \mathbb{I}\left(U_s > 1 - \frac{x}{b_n}\right) - \frac{x}{b_n} \right\}, \quad x \geq 0, \tag{3.1}
\]

where \( U_s = F(X_s) \), see, e.g., Drees (2000); Rootzén (2009).

The following set of conditions will be imposed to establish asymptotic normality of the estimators.

**Condition 3.1.**

(i) The stationary time series \( (X_s)_{s \in \mathbb{N}} \) has an extremal index \( \theta \in (0, 1] \) and the above assumptions guaranteeing convergence of the one- and two-level point process of exceedances are satisfied.

(ii) There exists \( \delta > 0 \) such that, for any \( m > 0 \), there exists a constant \( \tilde{C}_m \) such that

\[
E \left[ |X_n^{(x_1)}(E) - N_n^{(x_2)}(E)|^{2+\delta} \right] \leq \tilde{C}_m (x_2 - x_1) \quad \text{for all } 0 \leq x_1 \leq x_2 \leq m, n \in \mathbb{N}.
\]

(iii) There exist constants \( c_2 \in (0, 1) \) and \( C_2 > 0 \) such that

\[
o_{c_2}(m) \leq C_2 m^{-\eta}
\]

for some \( \eta \geq 3(2 + \delta)/(\delta - \mu) > 3 \), where \( 0 < \mu < \min(\delta, 1/2) \) and \( \delta > 0 \) is from Condition (ii). The block size \( b_n \) converges to infinity and satisfies

\[
k_n = o(b_n^2), \quad n \to \infty.
\]

Further, there exists a sequence \( \ell_n \to \infty \) with \( \ell_n = o(b_n^{2/(2+\delta)}) \) and \( k_n \alpha_{c_2}(\ell_n) = o(1) \) as \( n \to \infty \).

(iv) There exist constants \( c_1 \in (0, 1) \) and \( C_1 > 0 \) such that, for any \( y \in (0, c_1) \) and \( n \in \mathbb{N} \),

\[
\text{Var}\left\{ \sum_{s=1}^{n} \mathbb{I}(U_s > 1 - y) \right\} \leq C_1 (ny + n^2y^2).
\]

(v) For any \( c \in (0, 1) \), one has

\[
\lim_{n \to \infty} \mathbb{P}\left( \min_{i=1, \ldots, 2k_n} N'_{ni} \leq c \right) = 0,
\]

where \( N'_{ni} = \max\{U_s, s \in [(i-1)b_n/2 + 1, \ldots, ib_n/2]\} \) for \( i = 1, \ldots, 2k_n \).

(vi) For any \( x > 0 \),

\[
\lim_{m \to \infty} \limsup_{n \to \infty} \mathbb{P}\left( N_{m, b_n} > 1 - \frac{x}{n} \bigg| U_1 \geq 1 - \frac{x}{n} \right) = 0.
\]

**Condition 3.2 (Integrability).**

(i) With \( \delta > 0 \) from Condition 3.1(ii), one has

\[
\limsup_{n \to \infty} E\left[ |\log(Z_{1, n})|^{2+\delta} \right] < \infty.
\]
(ii) Fix \( p > 0 \). With \( \delta > 0 \) from Condition 3.1(ii), one has
\[
\limsup_{n \to \infty} E\left[ Z_{1,n}^{(2+\delta)/p} \right] < \infty.
\]

**Condition 3.3** (Bias Condition). Recall \( \varphi(C), \varphi(M) \) and \( \varphi(R,p) \) defined in (CFG), (MAD) and (ROOT), respectively.

(i) As \( n \to \infty \), \( E[\log(Z_{1,b_n})] = \varphi(C)(\theta) + o(k_n^{-1/2}) \).

(ii) As \( n \to \infty \), \( E[\exp(-Z_{1,b_n})] = \varphi(M)(\theta) + o(k_n^{-1/2}) \).

(iii) Fix \( p > 0 \). As \( n \to \infty \), \( E\left[ Z_{1,b_n}^{1/p} \right] = \varphi(R,p)(\theta) + o(k_n^{-1/2}) \).

**Condition 3.4** (Technical Condition for the CFG-type estimator).

(i) For some \( q > 1/2 \), we have \( b_n = O(k_n^q) \) as \( n \to \infty \).

(ii) For some \( \tau \in (0,1/2) \), we have, as \( n \to \infty \),
\[
\{ e_n(x) \over x^{\tau} \}_{x \in [0,1]} \overset{d}= \{ e(x) \over x^{\tau} \}_{x \in [0,1]} \quad \text{in } D([0,1]),
\]
where \( e_n \) denotes the tail empirical process defined in (3.1) and where \( e \) is a centered Gaussian process with continuous sample paths and covariance as given in Lemma 7.3.

(iii) For any \( c > 0 \), we have, as \( n \to \infty \),
\[
\max_{Z_n \geq c} \left| {e_n(Z_n)} \over Z_n \sqrt{k_n} \right| = o_P(1).
\]

(iv) For any \( c > 0 \) and some \( \mu \in (1/2,1/2(1-\tau)) \) with \( \tau \) from (ii), as \( n \to \infty \),
\[
\mathbb{P}(Z_{n1} < ck_{n}^{-\mu}) - \mathbb{P}(\xi < ck_{n}^{-\mu}) = o\left( \log(n)^{-1} k_{n}^{-1/2} \right), \quad \text{where } \xi \sim \text{Exp}(\theta).
\]

The items of Condition 3.1 are the same as Condition 2.1(i)-(v) and (2.2) in Berghaus and Bücher (2018) and are discussed in great detail in that reference. Condition 3.2 is needed for uniform integrability of the sequences \( Z_{n1}^{1/p} \) and \( \log Z_{n1} \), respectively. It implies
\[
\lim_{n \to \infty} \text{Var}(Z_{n1}^{1/p}) = \text{Var}(\xi^{1/p}), \quad \lim_{n \to \infty} \text{Var}(\log Z_{n1}) = \text{Var}(\log \xi),
\]
respectively, where \( \xi \) denotes an exponentially distributed random variable with parameter \( \theta \).

Condition 3.3 is a bias condition requiring the approximation of the first moment of \( f(Z_n) \) by \( E[f(\xi)] \) to be sufficiently accurate, where \( f(x) \in \{ x^{1/p}, \exp(-x), \log x \} \).

Condition 3.4 is a technical condition which is only needed for deriving the asymptotics of the CFG-estimator. The Condition 3.4(i) requires \( b \) to be not too large. Sufficient conditions for Condition 3.4(ii) in terms of beta mixing coefficients can be found in Drees (2000). A sufficient condition for Condition 3.4(iii) is for instance strong mixing with polynomial rate \( \alpha_1(n) = O(n^{-1+\sqrt{2}-\varepsilon}) \), \( n \to \infty \), for some \( \varepsilon > 0 \), together with Condition 3.4(i) being met with \( q < 1/(\sqrt{2} - 1) \approx 2.41 \). Indeed, for any \( x \geq c \) and \( \eta > 0 \), one can write
\[
{e_n(x) \over x} = \frac{1}{\sqrt{k_n}} \sum_{s=1}^{n} \left( \mathbb{1}\{ U_s > 1 - {x \over b_n} \} - {x \over b_n} \right) \frac{1}{x} = -b_n^{1/2-\eta} \mathbb{U}_{n,\eta} \left( 1 - {b_n \over \sqrt{k_n}} \right),
\]
where
\[
\mathbb{U}_{n,\eta}(u) = \frac{1}{\sqrt{n}} \sum_{s=1}^{n} \left( \mathbb{1}\{ U_s \leq u \} - u \right) \frac{1}{(1-u)^\eta} \mathbb{1}_{[0,1]}(u).
\]
By Theorem 2.2 in Shao and Yu (1996), we have \( \sup_{x \geq 0} |U_{n, \eta}(1 - x/b_n)| = O_p(1) \) for all \( \eta \leq 1 - 2^{-1/2} \approx 0.29 \). Hence, by Condition 3.4(i),

\[
\max_{Z_{n} \geq c} \frac{\epsilon_n(Z_{m1})}{Z_n \sqrt{k_n}} = O_p \left( \frac{b_n^{1/2 - \eta}}{\sqrt{k_n}} \right) = O_p \left( k_n^{(1/2 - \eta) - 1/2} \right).
\]

The expression on the right-hand side is \( o_p(1) \) if we choose \( \eta \in (1/2 - 1/2q, 1 - 2^{-1/2}) \); note that the latter interval is non-empty since \( q < 1/(\sqrt{2} - 1) \). Finally, Condition 3.4(iv) is another technical condition requiring the approximation of the law of \( Z_{n1} \) by the exponential distribution to be sufficiently accurate in the lower tail.

### 4. Asymptotic Results

We present asymptotic results on all estimators defined in Section 2. For simplicity, all results are stated and proved for the \( \hat{Z}_{ni} \)-versions only. As in Theorem 4.1 in Berghaus and Buecher (2018), it may be verified that the respective versions based on \( \hat{Y}_{ni} \) show the same asymptotic behavior as the \( \hat{Z}_{ni} \)-versions. Throughout, for \( z \in (0, 1) \), let \( (\xi_1^{(z)}, \xi_2^{(z)}) \sim \xi_2^{(z)} \).

**Theorem 4.1.** Under Condition 3.1, 3.2(i), 3.3(i) and 3.4, we have

\[
\sqrt{k_n}(\bar{\theta}_{m, CFG}^{Z_n} - \theta) \xrightarrow{d} N(0, \sigma_{m, C}^2)
\]

for \( m \in \{db, sb\} \) and as \( n \to \infty \), where

\[
\sigma_{db,c}^2 = 2\theta^3 \int_0^1 \frac{\theta E[\xi_1^{(z)} \xi_2^{(z)}] - E[\xi_1^{(z)} I(\xi_1^{(z)} > 0)]}{z(1 + z)} \, dz + \{ \pi^2/6 - 2 \log(2) \} \theta^2,
\]

\[
\sigma_{db,c}^2 = \sigma_{db,c}^2 - \{ \pi^2/6 - 8 \log(2) + 4 \} \theta^2.
\]

**Theorem 4.2.** Under Condition 3.1 and 3.3(ii), we have

\[
\sqrt{k_n}(\bar{\theta}_{m, MAD}^{Z_n} - \theta) \xrightarrow{d} N(0, \sigma_{m, M}^2)
\]

for \( m \in \{db, sb\} \) and as \( n \to \infty \), where

\[
\sigma_{db,m}^2 = 4\theta^2(1 + \theta) \int_0^1 \frac{\theta E[\xi_1^{(z)} \xi_2^{(z)}] - E[\xi_1^{(z)} I(\xi_1^{(z)} > 0)]}{(1 + z)^3} \, dz + \theta^2(1 + \theta) \frac{2(1 + \theta)}{2(2 + \theta)}
\]

\[
\sigma_{db,m}^2 = \sigma_{db,m}^2 - \frac{3\theta^2 + 4\theta - 4(1 + \theta)(2 + \theta) \log(2) + 4(1 + \theta)/(2 + \theta)}{\theta(2 + \theta)(1 + \theta)^2}.
\]

**Theorem 4.3.** Fix \( p > 0 \). Under Condition 3.1, 3.2(ii) and 3.3(iii),

\[
\sqrt{k_n}(\bar{\theta}_{m, R,p}^{Z_n} - \theta) \xrightarrow{d} N(0, \sigma_{m, p}^2)
\]

for \( m \in \{db, sb\} \) and as \( n \to \infty \), where

\[
\sigma_{db,p}^2 = \frac{4p \theta^3}{B(1/p, 1/p)} \int_0^1 \frac{\theta E[\xi_1^{(z)} \xi_2^{(z)}] + E[\xi_1^{(z)} I(\xi_1^{(z)} > 0)]}{(1 + z)^{1 + \frac{3}{5}}} \, dz
\]

\[
+ \left\{ \frac{2p^3}{B(1/p, 1/p)} - p^2 - 2p \right\} \theta^2.
\]
Example 4.4. In the case that the time series is serially independent, the cluster size distributions are given by \( \pi(i) = 1(i = 1) \) and \( \pi^{(z)}_2(i,j) = (1-z)1(i = 1, j = 0) + z1(i = 1, j = 1) \), which implies

\[
\theta = 1, \quad E[\xi^{(z)}_1|\xi^{(z)}_2 = z] = z \quad \text{and} \quad E[\xi^{(z)}_1|\xi^{(z)}_2 = 0] = 1 - z.
\]

It can be seen that these formulas hold true whenever \( \theta = 1 \). Consequently, the limiting variances in Theorem 4.1 and 4.2 are equal to

\[
\sigma^2_{db,c} = \frac{\pi^2}{6} - 2 \log(2) \approx 0.2586, \quad \sigma^2_{db,c} = 6 \log(2) - 4 \approx 0.1588, \\
\sigma^2_{db,M} = 1/3, \quad \sigma^2_{db,M} \approx 0.32536.
\]
It is remarkable that the asymptotic variances are substantially smaller than those of the maximum likelihood estimator, see Example 3.1 in Berghaus and Bücher (2018), which are equal to 1/2 and 0.2726 for the disjoint and sliding blocks version, respectively. The limiting variance in the case of the Root-estimator is given by

\[
\sigma_{db,p}^2 = \frac{2p}{B(\frac{1}{p}, \frac{1}{p})} \left( \frac{p^2 + 2^{-2/p}}{p^2} - p^2 \right),
\]

\[
\sigma_{db,p}^2 = \sigma_{db,p}^2 - \frac{2p^3}{B(1/p, 1/p)} - \frac{4p}{\Gamma(1/p)^2} \int_0^\infty (1 - e^{-z}) z^{1/p-2} \Gamma(1/p, z) \, dz.
\]

Some values are

\[
\sigma_{db,1/2}^2 = \frac{15}{16}, \quad \sigma_{db,1}^2 = \frac{1}{2}, \quad \sigma_{db,2}^2 \approx 0.3662,
\]

\[
\sigma_{db,1/2}^2 = \frac{7}{16}, \quad \sigma_{db,2}^2 \approx 0.2726, \quad \sigma_{db,2}^2 \approx 0.212909.
\]

It can further be shown that \( \lim_{p \to \infty} \sigma_{m,p}^2 = \sigma_{m,C}^2 \) for \( m \in \{db, sb\} \).

**Remark 4.5.** Instead of working with \( \hat{F}_n \) in the definition of \( \hat{Z}_{ni} = b_n \{1 - \hat{F}_n(M_{ni})\} \), one may alternatively use the empirical c.d.f. of \( (X_s)_{s \in I_i} \), multiplied by \( (n - b_n)/(n - b_n + 1) \) for \( I_i = \{(i - 1)b_n + 1, \ldots, ib_n\} \), denoted by \( \hat{F}_{n,-i} \), and define \( \tilde{Z}_{ni} = b_n \{1 - \hat{F}_{n,-i}(M_{ni})\} \) and \( \tilde{\theta} = \tilde{\theta}(\tilde{Z}_{n1}, \ldots, \tilde{Z}_{nk_n}) \). This modification has been motivated as a bias reduction scheme in Northrop (2015). Since

\[
\tilde{Z}_{ni} = b_n \{1 - \hat{F}_{n,-i}(M_{ni})\} = b_n \{1 - \hat{F}_n(M_{ni})\} \frac{n + 1}{n - b_n + 1} = \hat{Z}_{ni} \frac{n + 1}{n - b_n + 1},
\]

some simple calculations show that, for instance for the CFG-estimator,

\[
e^{-\gamma} \exp \left\{ - \frac{1}{k_n} \sum_{i=1}^{k_n} \log(\tilde{Z}_{ni}) \right\} = \frac{n - b_n + 1}{n + 1} \tilde{\theta}_{db,CFG}^z,
\]

showing that the modification is asymptotically negligible. It is however beneficial in finite-sample situations, whence it has been applied throughout the finite-sample situations considered in Section 6. Obviously, similar adaptions can be applied to the sliding blocks version and the other moment based estimators.

5. **Example: max-autoregressive process**

In this section, we exemplarily discuss the new estimators when applied to a max-autoregressive process, defined by the recursion

\[
X_s = \max \{ \alpha X_{s-1}, (1 - \alpha)Z_s \}, \quad s \in \mathbb{Z},
\]

where \( \alpha \in [0, 1) \) and where \( (Z_s)_{s \in \mathbb{Z}} \) is an i.i.d. sequence of Fréchet(1)-distributed random variables. A stationary solution of the above recursion is

\[
X_s = \max_{j \geq 0} (1 - \alpha)^j Z_{s-j},
\]

such that the stationary solution is again Fréchet(1)-distributed. Note that a model with an arbitrary stationary c.d.f. \( F \) may be obtained by considering \( X_s = F^{\alpha \gamma}(\exp(-1/X_s)) \) and that all subsequent results are also valid for \( \{X_s\}_s \).

We start by explicitly calculating the asymptotic variances of the estimators in Section 5.1, and then show in Section 5.2 that all regularity conditions from Section 3 are met.
5.1. Asymptotic variances for the ARMAX-model

Recall that the ARMAX-model has extremal index \( \theta = 1 - \alpha \) and that the corresponding cluster size distribution is geometric, that is, \( \pi(j) = \alpha^{j-1}(1-\alpha), j \geq 1 \), see, e.g., Chapter 10 in Beirlant et al. (2004). From Example 6.1 in Berghaus and Bücher (2018), one further has

\[
E[\xi(z) \xi(z)^2] = \alpha^{w+1} + z + zw(1-\alpha), \quad E[\xi(z)] = 1 - \frac{\alpha^{w+1}}{1-\alpha} - z(w+1),
\]

where \( w = \lfloor \log(z)/\log(\alpha) \rfloor \) and \( (\xi(z), \xi(z)^2) \sim \pi(z) \). This allows to calculate the limiting variances in Theorem 4.1–4.3 explicitly. For the CFG-type estimator, some tedious but straightforward calculations imply

\[
\frac{\sigma_{db,C}^2}{\theta^2} = \pi^2/6 + 2 \log(2)(\alpha - 1) \quad \text{and} \quad \frac{\sigma_{sb,C}^2}{\theta^2} = 2 \log(2)(3+\alpha) - 4,
\]

see also Figure 2 for a picture of the graph of these functions. Next, we compare these variances with the disjoint and sliding blocks variances of the PML-estimator in Berghaus and Bücher (2018), which are given by \( \sigma_{db,1}^2 \) and \( \sigma_{sb,1}^2 \) and satisfy

\[
\frac{\sigma_{db,1}^2}{\theta^2} = \frac{1}{2}(1+\alpha) \quad \text{and} \quad \frac{\sigma_{sb,1}^2}{\theta^2} = \frac{8 \log(2) - 5 + \alpha}{2},
\]

respectively. Thus, \( \sigma_{db,C}^2 \leq \sigma_{db,1}^2 \) iff \( \alpha \leq \{1 + 4 \log(2) - \pi^2/3\}/\{4 \log(2) - 1\} \approx 0.2723 \) and \( \sigma_{sb,C}^2 \leq \sigma_{sb,1}^2 \) iff \( \alpha \leq \{3 - 4 \log(2)\}/\{4 \log(2) - 1\} \approx 0.128 \). Further comparisons can be drawn from Figure 2, where the asymptotic variances of \( \sqrt{n}(\hat{\theta}_n/\theta - 1) \) are additionally illustrated for the Madogram- and the Root-estimators.

5.2. Regularity Conditions for the ARMAX-model

Recall that \( X_s \) is Fréchet(1)-distributed, i.e., the stationary c.d.f. \( F \) is given by \( F(x) = \exp(-1/x) \) for \( x > 0 \), with inverse \( F^{-1}(x) = -\log(x)^{-1} \).
The assumptions in Condition 3.1 are satisfied as shown in Berghaus and Bücher (2018), page 2322, provided $b_n$ and $k_n$ are chosen to satisfy the conditions in Item (iii). Next, by induction,
\[ \mathbb{P} \left( \max_{s=1,\ldots,b} X_s \leq x \right) = F(x)^{1+\theta(b-1)}, \]
which implies that the c.d.f. of $Z_{1,b} = b\{1 - F(M_{1,b})\}$ is given by
\[ \mathbb{P}(Z_{1,b} \leq x) = 1 - \mathbb{P} \left( \max_{s=1,\ldots,b} X_s \leq F^{-1}(1 - x/b) \right) = \begin{cases} 1, & x \geq b, \\ 1 - \left( 1 - \frac{x}{b} \right)^{1+\theta(b-1)}, & x \in [0,b], \\ 0, & b \leq 0. \end{cases} \tag{5.1} \]

A tedious but straightforward calculation then shows that the assumptions in Condition 3.2 and 3.3 are met, provided $k_n/b_n^2 = o(1)$, cf. Condition 3.1(iii). Condition 3.4(i) is a condition on the choice of $b_n$, that is under the control of the statistician. Conditions 3.4(ii) and 3.4(iii) are consequences of mixing properties of $(X_s)_s$ as argued at the end of Section 3. It remains to show that Condition 3.4(iv) is satisfied. By (5.1) and with $\xi \sim \text{Exp}(\theta)$, we have
\[ \mathbb{P}(Z_{n1} < ck_n^{-\mu}) - \mathbb{P}(\xi < ck_n^{-\mu}) = \exp(-\theta ck_n^{-\mu}) - \left( 1 - \frac{ck_n^{-\mu}}{b_n} \right)^{1+\theta(b_n-1)} = o(k_n^{-1/2}(\log n)^{-1}), \quad n \to \infty, \]
for any $\mu > 1/2$, where the final estimate follows from Taylor’s theorem and Condition 3.4(i).

6. Finite-sample results

A Monte-Carlo simulation study was performed to assess the finite-sample performance of the introduced estimators and to compare them with competing estimators from the literature. The data is simulated from the following four time series models that were also investigated in Berghaus and Bücher (2018):

- The **ARMAX-model**:
  \[ X_s = \max\{\alpha X_{s-1}, (1 - \alpha)Z_s\}, \quad s \in \mathbb{Z}, \]
  where $\alpha \in [0,1)$ and where $(Z_s)_s$ is an i.i.d. sequence of standard Fréchet random variables. We consider $\alpha = 0, 0.25, 0.5, 0.75$ resulting in $\theta = 1, 0.75, 0.5, 0.25$.

- The **squared ARCH-model**:
  \[ X_s = (2 \times 10^{-5} + \lambda X_{s-1})Z_s^2, \quad s \in \mathbb{Z}, \]
  where $\lambda \in (0,1)$ and where $(Z_s)_s$ denotes an i.i.d. sequence of standard normal random variables. We consider $\lambda = 0.1, 0.727, 0.460, 0.422$, respectively (Table 3.1 in de Haan et al., 1989).

- The **ARCH-model**:
  \[ X_s = (2 \times 10^{-5} + \lambda X_{s-1}^2)^{1/2} Z_s, \quad s \in \mathbb{Z}, \]
  where $\lambda \in (0,1)$ and where $(Z_s)_s$ denotes an i.i.d. sequence of standard normal random variables. We consider $\lambda = 0.1, 0.727, 0.460, 0.422$, respectively (Table 3.2 in de Haan et al., 1989).
• The Markovian Copula-model (Darsow et al., 1992):

\[ X_s = F^+(U_s), \quad (U_s, U_{s-1}) \sim C_\vartheta, \quad s \in \mathbb{Z}. \]

Here, \( F^+ \) is the left-continuous quantile function of some arbitrary continuous c.d.f. \( F \), \( (U_s)_s \) is a stationary Markovian time series of order 1 and \( C_\vartheta \) denotes the Survival Clayton Copula with parameter \( \vartheta > 0 \). We consider choices \( \vartheta = 0.23, 0.41, 0.68, 1.06, 1.90 \) such that (approximately) \( \theta = 0.2, 0.4, 0.6, 0.8, 0.95 \) Berghaus and B"ucher (2018) and fix \( F \) as the standard uniform c.d.f. (the results are independent of this choice, as the estimators are rank-based). Algorithm 2 in Rémillard et al. (2012) allows to simulate from this model.

In each case, the sample size is fixed to \( n = 2^{13} = 8192 \) and the block size is chosen from \( b = b_n \in \{2^2, \ldots, 2^9\} \). The performance is assessed based on \( N = 3000 \) simulation runs each.

### 6.1. Comparison of the introduced estimators

We start by comparing the finite-sample properties of the proposed sliding blocks estimators \( \hat{\theta}_{\text{sb,CFG}}, \hat{\theta}_{\text{sb,MAD}} \) and \( \hat{\theta}_{\text{sb,R},p} \) for \( p \in \{0.5, 0.75, 1, 2, 4, 8, 16\} \) and for \( x \in \{s_n, y_n\} \). Respective results for the corresponding disjoint blocks version are omitted, as the latter are always outperformed by their sliding blocks counterparts. Subsequently, the index \( \text{sb} \) is therefore omitted.

As the simulation results are mostly similar among the different models and estimators, they are only partially reported, with a particular view on highlighting interesting qualitative features. We begin by a detailed investigation of the variance, the squared bias and the mean squared error (MSE) as a function of the block size parameter \( b \). For illustrative purposes, we restrict the presentation to the \( z_n \)-versions and the ARCH-model. The corresponding results are depicted in Figure 3 (for the CFG-, the Madogram- and three selected Root-estimators).

In general, as to be expected from the underlying theory, the variance curves are increasing in \( b \), while the squared bias curves are (mostly) decreasing in \( b \), resulting in a typical U-shape for the MSE curves. The hierarchy of the estimators with regard to the considered performance measures is similar among the considered values of \( \theta \). In terms of the MSE, up to an intermediate block size, the CFG- and Madogram-estimator are superior to the other estimators (especially to the PML-estimator), while for large block sizes the Madogram-estimator has a relatively high MSE, but the CFG-estimator partly remains superior. The Root-estimators are, as expected, ordered in \( p \) and located between the PML- and CFG-estimator.

Next, a comparison between the \( z_n \)- and \( y_n \)-versions of the estimators is drawn in Figure 4; for illustrative purposes, attention is restricted to six different models and two estimators. Remarkably, there are many models, especially for smaller values of \( \theta \), in which the MSE-curves of the \( y_n \)-versions lie uniformly below the ones of the \( z_n \)-versions. In the remaining models, neither version can be said to be strictly preferable. Furthermore, it is remarkable that, for \( \vartheta \) close to one, the MSE-curves of the \( y_n \)-versions are often no longer U-shaped, but increasing in the block size instead. The latter behavior may be explained by the proximity to the i.i.d. case, since in that case we have

\[
P(Y_{1:b} \geq y) = P(N_{1:b} \leq e^{-y/b}) = P(U_1 \leq e^{-y/b})^b = e^{-y}
\]

for all \( b \in \mathbb{N} \), such that there is real equality in relation (1.3), resulting in a vanishing bias.

Next, we investigate the dependence of the performance of the Root-estimators on the parameter \( p \); recall that \( p = 1 \) yields the PML-estimator, while ‘\( p = \infty \)’ yields the CFG-estimator. In Figure 5, the MSE-curves are depicted as a function of \( p \) for various fixed block sizes and for three selected models. It can be seen that choices of \( p < 1 \) lead to a poor behavior of the corresponding
estimators. At the same time, the results do not allow to identify some ‘optimal’ choice of $p \geq 1$ which is valid uniformly over all models. A similar conclusion can be drawn from Table 1, which presents, for the ARCH- and ARMAX-model and every block size $b$, the value of $p$ for which the Root-estimator attains the minimal MSE ($p = \infty$ corresponds to the CFG-estimator). One can see that most values of $p$ are represented, with $p = \infty$ appearing most often, but that there is no optimal choice of $p$ universally over all models.

6.2. Comparison with other estimators for the extremal index

In this section, we compare the performance of the introduced estimators with the following estimators: the bias-reduced sliding blocks estimator from Robert et al. (2009) (with a data-driven choice of the threshold as outlined in Section 7.1 of that paper), the integrated version of the blocks estimator from Robert (2009), the intervals estimator from Ferro and Segers (2003) and the ML-estimator from Süveges (2007). The parameters $\sigma$ and $\phi$ for the Robert-estimator (cf. page 276 of Robert, 2009) are chosen as $\sigma = 0.7$ and $\phi = 1.3$. In the case of the intervals- and Süveges-estimator, the choice of a threshold $u$ is required, which is here chosen as the $1 - 1/b_n$ empirical quantile of the observed data. With regard to our estimators, we present results for the sliding-blocks, bias-reduced and $z_n$-versions, if not indicated otherwise.
In Figure 6, we depict the MSE as a function of the block size $b$. For most models, the MSE-curves of the estimators from the literature are again U-shaped due to the bias-variance tradeoff already described in section 6.1. It can further be seen that no estimator is uniformly best in any model under consideration. The method-of-moment estimators do however compare quite well to the competitors.

The minimum values of the MSE-curves in Figure 6 are of particular interest. Due to the large amount of estimators and models under consideration (in total 26 estimators and 17 models) we try to simplify possible comparisons by the following aggregation, summarized in Table 2. First, in the first four columns of the table, we calculate, for each time series model and each estimator under consideration, the sum (sum over all values of $\theta$ considered for the specific model) of the

| Model | ARCH | ARMAX |
|-------|------|-------|
| Theta | 0.999 | 0.835 | 0.721 | 0.571 | 0.75 | 0.5 | 0.25 |
| $b = 4$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ |
| 8 | $\infty$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ |
| 16 | $\infty$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ |
| 32 | 2 | $\infty$ | $\infty$ | $\infty$ | $\infty$ | 16 | 8 | 1.5 | 2 |
| 64 | 2 | 2 | $\infty$ | $\infty$ | 16 | 8 | 4 | 1 | 1 |
| 128 | 2 | 1.5 | 4 | 4 | 8 | 4 | 1 | 1 | 0.75 |
| 256 | 2 | 4 | $\infty$ | $\infty$ | 4 | $\infty$ | 1 | 0.75 |
| 512 | 2 | 8 | $\infty$ | $\infty$ | 4 | $\infty$ | 1 | 0.75 |
| $\min_b$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ | 1.5 | 1 |

Table 1

Identification of the Root-estimator $p$ with the minimum MSE for the ARCH- and ARMAX-model and every considered block size $b$. The $p$ with the minimum MSE over all blocksizes is presented in the last line.
minimum MSE-values (minimum over $b$). Second, in the last four columns of the table, we present the sum of the minimum MSE-values (minimum over $b$) over all models, for which the extremal index $\theta$ lies in the interval $(0, 0.3]$, $(0.3, 0.6]$, $(0.6, 0.8]$ or $(0.8, 1]$, respectively. It can be seen that both the CFG- and the PML-estimator wins twice, the Madogram- and Søveges-estimator wins once, and that the remaining smallest values are covered by a version of the Root-estimator. Also note that for large values of $\theta \in (0.8, 1]$ (last column), the CFG-estimator and the Root-estimator
for $p \in \{8, 16\}$ are the best performing estimators.

7. Proof of Theorem 4.1–4.3

The proofs of Theorem 4.1–4.3 are actually quite similar in that each proof will be decomposed into a sequence of similar intermediate lemmas. Occasionally, those lemmas will be hardest to prove for Theorem 4.1 and easiest to prove for Theorem 4.2; this is also reflected by the larger number of conditions required for the proof of Theorem 4.1. The proof of Theorem 4.3 in turn is quite similar to the one in Berghaus and Bücher (2018), and of intermediate difficulty. For the above reasons, we will carry out the proof of Theorem 4.1 in great detail, and skip parts of the technical arguments needed for Theorem 4.2 and 4.3 where possible.

All convergences are for $n \to \infty$ if not stated otherwise.

7.1. Proof of Theorem 4.1

The following notations will be used throughout:

$$S_n = \frac{1}{k_n} \sum_{i=1}^{k_n} \log(\hat{Z}_{ni}),$$

$$S_{sb}^n = \frac{1}{n - b_n + 1} \sum_{i=1}^{n - b_n + 1} \log(\hat{Z}_{ni}^{sb}),$$

$$S_n = \frac{1}{k_n} \sum_{i=1}^{k_n} \log(Z_{ni}),$$

$$S_{sb}^n = \frac{1}{n - b_n + 1} \sum_{i=1}^{n - b_n + 1} \log(Z_{ni}^{sb}).$$

| Estimator | ARM | ARCH | sqARCH | Markov | (0, 0.3) | (0.3, 0.6) | (0.6, 0.8) | (0.8, 1) |
|-----------|-----|------|--------|--------|----------|-----------|-----------|---------|
| CFG, Z    | 4.80 | 8.54 | 8.46   | 5.22   | 1.80     | 16.71     | 5.22      | 3.29    |
| CFG, Y    | 2.56 | **6.98** | 8.41   | 7.06   | 1.25     | 12.59     | 3.45      | 7.73    |
| Madogram, Z | 5.17 | 8.87 | 7.92   | 6.95   | 2.36     | 16.91     | 5.51      | 4.15    |
| Madogram, Y | 3.00 | **7.08** | 8.62   | 8.80   | 2.73     | 12.79     | 3.69      | 8.30    |
| PML, Z    | 6.18 | 11.74 | 7.99   | 4.74   | 1.44     | 14.86     | 8.18      | 6.16    |
| PML, Y    | 1.96 | 8.40  | 7.45   | 6.17   | 0.85     | **12.16** | 3.59      | 7.38    |
| Root, $p = 0.5$, Z | 9.64 | 17.37 | 11.57  | 4.92   | 1.36     | 20.76     | 10.59     | 10.79   |
| Root, $p = 0.5$, Y | 2.33 | 11.99 | 8.49   | 5.95   | 1.43     | 14.64     | 5.67      | 7.01    |
| Root, $p = 0.75$, Z | 7.08 | 13.33 | 8.63   | 6.25   | 1.20     | 17.22     | 8.68      | 8.39    |
| Root, $p = 0.75$, Y | 2.03 | 9.26  | 7.63   | 5.89   | 1.12     | 12.45     | 4.07      | 7.17    |
| Root, $p = 1.25$, Z | 5.77 | 11.02 | 7.82   | 4.63   | 1.72     | 14.22     | 8.14      | 5.15    |
| Root, $p = 1.25$, Y | 1.96 | 8.06  | **7.37** | 6.55 | **0.76** | **12.24** | **3.42**  | 7.53    |
| Root, $p = 1.5$, Z | 5.54 | 10.48 | 7.86   | **4.10** | 1.95     | 14.17     | 7.30      | 4.55    |
| Root, $p = 1.5$, Y | 1.98 | 7.93  | **7.32** | 6.89 | **0.74** | 12.40     | **3.35**  | 7.63    |
| Root, $p = 2$, Z | 5.22 | 9.62  | 8.11   | **3.88** | 2.32     | 14.57     | 6.22      | 3.91    |
| Root, $p = 2$, Y | 2.03 | 7.88  | **7.34** | 7.35 | **0.76** | 12.76     | **3.32**  | 7.76    |
| Root, $p = 4$, Z | 4.84 | 9.10  | 8.40   | **4.19** | 2.12     | 15.65     | 5.29      | 3.46    |
| Root, $p = 4$, Y | 2.20 | 7.52  | 7.64   | 7.05   | 0.91     | **12.29** | 3.43      | 7.78    |
| Root, $p = 8$, Z | 4.76 | 8.88  | 8.42   | 4.78   | 1.90     | 16.48     | 5.15      | **3.32** |
| Root, $p = 8$, Y | 2.35 | 7.31  | 7.95   | 7.01   | 1.05     | 12.33     | 3.51      | 7.73    |
| Root, $p = 16$, Z | 4.76 | 8.69  | 8.41   | 5.26   | 1.83     | 16.88     | 5.15      | **3.25** |
| Root, $p = 16$, Y | 2.45 | **7.14** | 8.16   | 7.02   | 1.14     | 12.43     | 3.48      | 7.72    |
| Intervals | 3.49 | 12.53 | 11.72  | 58.71  | 9.25     | 32.91     | 22.33     | 21.96   |
| ML Sőveges | **1.90** | 22.67 | 8.70   | 10.76  | 1.44     | 28.24     | 6.09      | 8.27    |
| Robert | 8.54 | 12.45 | 9.97   | 7.24   | 3.62     | 17.72     | 8.31      | 8.55    |
| RSP | 8.09 | 11.68 | 9.77   | 7.07   | 3.61     | 17.68     | 7.86      | 7.45    |

Table 2

Sum of minimal Mean Squared Error multiplied by $10^3$ over different models and $\theta \in (0, 0.3], \theta \in (0.3, 0.6], \theta \in (0.6, 0.8]$ and $\theta \in (0.8, 1]$. The three smallest values per column are in boldface.
Finally, let \( \hat{\theta}_{db,CFG}^n = \varphi_{(C)}^{-1}(\hat{S}_n) \) and \( \hat{\theta}_{db}^n = \varphi_{(C)}^{-1}(\hat{S}_{db}) \), where \( \varphi_{(C)}^{-1}(x) = \exp(-(x + \gamma)) \). Observing that \( (\varphi_{(C)})'\{\varphi_{(C)}(\theta)\} = \theta \), the two assertions of the theorem are a consequence of the delta-method and Proposition 7.1 and Proposition 7.2, respectively.

**Proposition 7.1.** Under Condition 3.1, 3.2(i), 3.3(i) and 3.4, we have
\[
\sqrt{k_n}\{\hat{S}_n - \varphi_{(C)}(\theta)\} \overset{d}{\longrightarrow} \mathcal{N}(0, \sigma_{db,C}^2/\theta^2) \quad \text{as } n \to \infty.
\]

**Proof.** We may decompose
\[
\sqrt{k_n}\{\hat{S}_n - \varphi_{(C)}(\theta)\} = A_n + B_n + C_n,
\]
where
\[
A_n = \sqrt{k_n}\{\hat{S}_n - S_n\}, \quad B_n = \sqrt{k_n}\{S_n - E(S_n)\}, \quad C_n = \sqrt{k_n}\{E(S_n) - \varphi_{(C)}(\theta)\}.
\]
We have \( C_n = o(1) \) by Condition 3.3(i). For the treatment of \( A_n \), recall the tail empirical process defined in (3.1). Further, let \( \tilde{N}_{ni} = (n + 1)/n \times N_{mi} \), and note that
\[
1 - \tilde{N}_{mi} = 1 - \frac{1}{n} \sum_{s=1}^{n} \mathbb{1}(X_s > M_{ni}) = \frac{1}{n} \sum_{s=1}^{n} \mathbb{1}(U_s > 1 - \frac{Z_{mi}}{b_n})
\]
\[
= \frac{\sqrt{k_n}}{n} \frac{1}{n} \sum_{s=1}^{n} \mathbb{1}(U_s > 1 - \frac{Z_{mi}}{b_n}) - \frac{Z_{mi}}{b_n} + \frac{Z_{mi}}{b_n}
\]
\[
= \frac{\sqrt{k_n}}{n} e_n(Z_{ni}) + \frac{Z_{mi}}{b_n}.
\]
Finally, let
\[
\hat{H}_{k_n}(x) := \frac{1}{k_n} \sum_{i=1}^{k_n} \mathbb{1}(Z_{mi} \leq x)
\]
denote the empirical c.d.f. of \( Z_{n1}, \ldots, Z_{nk_n} \). By Equation (7.1), we obtain
\[
A_n = \frac{1}{\sqrt{k_n}} \sum_{i=1}^{k_n} \log\left(1 - \tilde{N}_{mi}\right) - \log\left(Z_{mi} b_n^{-1}\right)
\]
\[
= \frac{1}{\sqrt{k_n}} \sum_{i=1}^{k_n} \log\left\{\frac{n}{n+1} \left(\frac{1}{n} + 1 - \tilde{N}_{mi}\right)\right\} - \log\left(\frac{Z_{mi}}{b_n}\right)
\]
\[
= \frac{1}{\sqrt{k_n}} \sum_{i=1}^{k_n} \log\left\{\frac{1}{n} + \sqrt{\frac{e_n}{n}} e_n(Z_{ni}) + \frac{Z_{mi}}{b_n}\right\} - \log\left(\frac{Z_{mi}}{b_n}\right) + \log\left(\frac{n}{n+1}\right)
\]
\[
= \frac{1}{\sqrt{k_n}} \sum_{i=1}^{k_n} \log\left\{1 + \sqrt{\frac{e_n}{n}} \frac{e_n(Z_{ni})}{Z_{ni}} + \frac{b_n}{nZ_{ni}}\right\} + \sqrt{k_n} \log\left(\frac{n}{n+1}\right)
\]
\[
= \int_0^\infty W_n(x) \, d\hat{H}_{k_n}(x) + o(1),
\]
where
\[
W_n(x) = \sqrt{k_n} \log\left\{1 + \frac{1}{\sqrt{k_n}} \left(\frac{e_n(x)}{x} + \frac{1}{\sqrt{k_n}}\right)\right\}.
\]
Heuristically, \( \hat{H}_{b_n}(x) \approx 1 - \exp(-\theta x) \) and \( W_n(x) \approx e(x)/x \) (where \( e \) denotes the limit of the tail empirical process), whence the tentative limit of \( A_n \) should be

\[
A = \int_0^\infty \frac{e(x)}{x} \theta e^{-\theta x} \, dx.
\]

For a rigorous treatment of \( A_n + B_n \), let

\[
E_n = \int_0^\infty W_n(x) \, d\hat{H}_{b_n}(x), \quad E_{n,m} = \int_{1/m}^m W_n(x) \, d\hat{H}_{b_n}(x), \quad E'_m = \int_{1/m}^m \frac{e(x)}{x} \theta e^{-\theta x} \, dx
\]

and let \( B \) be defined as in Lemma 7.3 below. As shown above, \( A_n = E_n + o(1) \). The proposition is hence a consequence of Wichura’s theorem (Billingsley (1979), Theorem 25.5) and the following items:

(i) For all \( m \in \mathbb{N} \): \( E_{n,m} + B_n \xrightarrow{d} E'_m + B \) as \( n \to \infty \).

(ii) \( E'_m + B \xrightarrow{d} A + B \sim \mathcal{N}(0, \sigma^2_{ab,C}/\theta^2) \) as \( m \to \infty \).

(iii) For all \( \delta > 0 \): \( \lim_{m \to \infty} \limsup_{n \to \infty} \mathbb{P}(|E_n - E_{n,m}| > \delta) = 0 \).

The assertion in (i) is proven in Lemma 7.6. The assertion in (ii) follows from the fact that \( E'_m + B \) is normally distributed with variance \( \tau^2_m \) as specified in Lemma 7.6, and the fact that \( \tau^2_m \to \sigma^2_{ab,C}/\theta^2 \) as \( m \to \infty \) by Lemma 7.7. Finally, Lemma 7.8 proves (iii).

**Proposition 7.2.** Under Condition 3.1, 3.2(i), 3.3(i) and 3.4, we have

\[
\sqrt{k_n} \{ \hat{S}_n - \varphi(C)(\theta) \} \xrightarrow{d} \mathcal{N}(0, \sigma^2_{ab,C}/\theta^2) \quad \text{as} \quad n \to \infty.
\]

**Proof.** The proof is very similar to the proof of Proposition 7.1. Decompose

\[
\sqrt{k_n} \{ \hat{S}_n - g(\theta) \} = A_n^{ab} + B_n^{ab} + C_n^{ab},
\]

where

\[
A_n^{ab} := \sqrt{k_n} \{ \hat{S}_n - S_n^{ab} \}, \quad B_n^{ab} := \sqrt{k_n} \{ S_n^{ab} - E[S_n^{ab}] \}, \quad C_n^{ab} := \sqrt{k_n} \{ E[\hat{S}_n^{ab}] - \varphi(C)(\theta) \}.
\]

Again, we have \( C_n^{ab} = o(1) \) by Condition 3.3(i). A similar calculation as in (7.3) in the case of the disjoint blocks shows that \( A_n^{ab} \) can be written in the following way

\[
A_n^{ab} = \int_0^\infty W_n(x) \, d\hat{H}_n^{ab}(x) + o(1),
\]

where

\[
\hat{H}_n^{ab}(x) = \frac{1}{n-b_n+1} \sum_{t=1}^{n-b_n+1} 1(Z_{nt}^{ab} \leq x)
\]

denotes the empirical c.d.f. of \( Z_{n}^{ab}, \ldots, Z_{n,n-b_n+1}^{ab} \). We may now treat \( A_n^{ab} + B_n^{ab} \) exactly as \( A_n + B_n \) in the proof of Proposition 7.1, with \( E_n, E_{n,m} \) and Lemma 7.6, 7.7 and 7.8 replaced by

\[
E_n^{ab} = \int_0^\infty W_n(x) \, d\hat{H}_n^{ab}(x), \quad E_{n,m}^{ab} = \int_{1/m}^m W_n(x) \, d\hat{H}_n^{ab}(x),
\]

and Lemma 7.12, 7.13 and 7.14, respectively. \( \square \)
7.2. Auxiliary lemmas for the proof of Theorem 4.1 – Disjoint blocks

Throughout this section, we assume that Condition 3.1, 3.2(i) and 3.3(i) are met.

**Lemma 7.3.** For any \( x_1, \ldots, x_m \in [0, \infty) \) and \( m \in \mathbb{N} \), we have

\[
(e_n(x_1), \ldots, e_n(x_m), B_n)' \xrightarrow{d} (e(x_1), \ldots, e(x_m), B)',
\]

where \((e(x_1), \ldots, e(x_m), B)' \sim \mathcal{N}_{m+1}(0, \Sigma_{m+1})\) with

\[
\Sigma_{m+1} = \begin{pmatrix}
r(x_1, x_1) & \cdots & r(x_1, x_m) & f(x_1) \\
\vdots & \ddots & \vdots & \vdots \\
r(x_m, x_1) & \cdots & r(x_m, x_m) & f(x_m) \\
f(x_1) & \cdots & f(x_m) & \pi^2/6
\end{pmatrix}.
\]

Here, \( r(0,0) = 0 \) and, for \( x \geq y \geq 0 \) with \( x \neq 0 \),

\[
r(x,y) = \theta x \sum_{i=1}^{\infty} \sum_{j=0}^{i} ij \pi_2^{(y/x)}(i,j), \quad f(x) = h(x) - x \varphi(c)(\theta),
\]

\[
h(x) = \sum_{i=1}^{\infty} i \left[ \int_{0}^{\infty} 1(e^y \leq x)p_{g(x,y)}^{(x,e^y)}(i,0) \, dy - \int_{-\infty}^{0} p_{g(x,y)}^{(x,e^y)}(i,0) \, dy \right]
\]

and where, for \( i \geq j \geq 0, \ i \geq 1, \)

\[
p_{g(x,y)}^{(x,y)}(i,j) = \mathbb{P}(N_{E}^{(x,y)} = (i,j)), \quad N_{E}^{(x,y)} = \sum_{i=1}^{n} (\xi_{x_1}^{(y/x)}, \xi_{x_2}^{(y/x)}),
\]

with \( \eta \sim \text{Poisson}(\theta x) \) independent of i.i.d. random vectors \((\xi_{x_1}^{(y/x)}, \xi_{x_2}^{(y/x)}) \sim \pi_2^{(y/x)}, i \in \mathbb{N} \) and

\[
p_{g(x,y)}^{(x)}(i) = \mathbb{P}(N_{E}^{(x)} = i), \quad N_{E}^{(x)} = \sum_{i=1}^{\eta_2} \xi_i
\]

with \( \eta_2 \sim \text{Poisson}(\theta x) \) independent of i.i.d. random variables \( \xi_i \sim \pi, i \in \mathbb{N} \).

**Lemma 7.4.** For any \( m \in \mathbb{N} \), we have

\[
\{(W_n(x), B_n)\}_{x \in [1/m, m]} \xrightarrow{d} \left\{ \left( \frac{e(x)}{x}, B \right) \right\}_{x \in [1/m, m]} \quad \text{in} \ D([1/m, m]) \times \mathbb{R},
\]

where \((e, B)'\) is a centered Gaussian process with continuous sample paths and with covariance functional as specified in Lemma 7.3.

**Lemma 7.5.** For any \( m \in \mathbb{N} \), we have

\[
E_{n,m} = E_{n,m} + o_p(1) \quad \text{as} \ n \to \infty,
\]

where \( E_{n,m} = \int_{1/m}^{m} W_n(x) \theta e^{-\theta x} \, dx \).
Lemma 7.6. For any \( m \in \mathbb{N} \), we have
\[
E_{n,m} + B_n \xrightarrow{d} E'_m + B \sim \mathcal{N}(0, \sigma_m^2) \quad \text{as } n \to \infty,
\]
where, with \( r \) and \( f \) defined as in Lemma 7.3,
\[
x_m^2 = \theta^2 \int_{1/m}^{m} \int_{1/m}^{m} r(x, y) \frac{1}{xy} e^{-\theta(x+y)} \, dx \, dy + 2\theta \int_{1/m}^{m} f(x) \frac{1}{x} e^{-\theta x} \, dx + \frac{\pi^2}{6}.
\]

Lemma 7.7. As \( m \to \infty \), \( \tau_m^2 \to \sigma_{db,(C)}^2/\theta^2 \), where \( \sigma_{db,(C)}^2 \) is specified in Theorem 4.1.

Lemma 7.8. If, in addition to Condition 3.1, 3.2(i) and 3.3(i), Condition 3.4 holds, then, for all \( \delta > 0 \),
\[
\lim_{m \to \infty} \limsup_{n \to \infty} P(|E_{n,m} - E_n| > \delta) = 0.
\]

Proof of Lemma 7.3. We proceed similarly as in the proof of Lemma 9.3 in Berghaus and Bücher (2018). Weak convergence of the vector \((e_n(x_1), \ldots, e_n(x_m))\)' is a consequence of Theorem 4.1 in Robert (2009). For the treatment of the joint convergence with \( B_n \), we only consider the case \( m = 1 \) and set \( x_1 = x \); the general case can be treated analogously. For \( i = 1, \ldots, k_n \), we decompose a block \( I_i = \{(i-1)b_n + 1, \ldots, ib_n\} \) into a big block \( I_i^+ \) and a small block \( I_i^- \), where, recalling \( \epsilon_n \) from Condition 3.1(iii),
\[
I_i^+ = \{(i-1)b_n + 1, \ldots, ib_n - \epsilon_n\}, \quad I_i^- = \{ib_n - \epsilon_n + 1, \ldots, ib_n\},
\]
and set
\[
e_n^+(x) = \frac{1}{\sqrt{k_n}} \sum_{i=1}^{k_n} \sum_{x \in I_i^+} \left\{ 1 \left(U_i > 1 - \frac{x}{b_n}\right) - \frac{x}{b_n} \right\}, \quad B_n^+ = \frac{1}{\sqrt{k_n}} \sum_{i=1}^{k_n} \left\{ \log(Z_{ni}^+) - E[\log(Z_{ni}^+)] \right\},
\]
where \( Z_{ni}^+ = b_n(1 - N_{ni}^+), N_{ni}^+ = \max_{x \in I_i^+} U_x \). Next, according to Lemma 6.6 in Robert (2009),
\[
e_n^-(x) := e_n(x) - e_n^+(x) = o_P(1).
\]
It can further be shown by the same arguments as in the proof of Lemma 9.3 in Berghaus and Bücher (2018) that
\[
B_n^- := B_n - B_n^+ = o_P(1).
\]

Finally, for \( \varepsilon \in (0, c_1 \wedge c_2) \), define \( A_n^+ = \{ \min_{i=1, \ldots, k_n} N_{ni}^+ > 1 - \varepsilon \} \), and note that \( \mathbb{P}(A_n^+) \to 1 \) by Condition 3.1(v). As a consequence of the previous three statements, it suffices to show that, using the Cramér-Wold device,
\[
\{ \lambda_1 e_n^+(x) + \lambda_2 B_n^+ \} 1_{A_n^+} \xrightarrow{d} \lambda_1 e(x) + \lambda_2 B,
\]
(7.4)
for any \( \lambda_1, \lambda_2 \in \mathbb{R} \).

Now, the left-hand side of (7.4) can be written as
\[
\{ \lambda_1 e_n^+(x) + \lambda_2 B_n^+ \} 1_{A_n^+} = \frac{1}{\sqrt{k_n}} \sum_{i=1}^{k_n} \tilde{g}_{i,n} + o_P(1),
\]
where \( \tilde{g}_{i,n} = g_{i,n} 1_{Z_{ni}^+ < \varepsilon b_n} \) and where
\[
g_{i,n} = \lambda_1 \sum_{x \in I_i^+} \left\{ 1 \left(U_i > 1 - \frac{x}{b_n}\right) - \frac{x}{b_n} \right\} + \lambda_2 \left\{ \log(Z_{ni}^+) - E[\log(Z_{ni}^+)] \right\}.
\]
Note, that \( \tilde{g}_{i,n} \) only depends on the block \( I_i^n \) and is \( B_{(l_{i-1}+1),\ldots, l_i} \)-measurable. In particular, the \( (\tilde{g}_{i,n})_{i=1,\ldots,n} \) are each separated by a small block of length \( \ell_n \). A standard argument based on characteristic functions and the assumption on alpha mixing may then be used to show that the weak limit of \( k_n^{-1/2} \sum_{i=1}^{k_n} \tilde{g}_{i,n} \) is the same as if the \( \tilde{g}_{i,n} \) were independent.

Next, we show that Ljapunov’s condition (Billingsley (1979), Theorem 27.3) is satisfied. By Minkowski’s inequality, for any \( p \in (2, 2 + \delta) \), \( C_{\infty} = \sup_{n \in \mathbb{N}} E[|\tilde{g}_{1,n}|^p] < \infty \) by Condition 3.1(ii) and 3.2(i). Further, by stationarity and independence, we get

\[
\frac{\sum_{i=1}^{k_n} E[|\tilde{g}_{i,n}|^p]}{\text{Var} \left( \sum_{i=1}^{k_n} \tilde{g}_{i,n} \right)^{p/2}} = k_n^{1-p/2} \frac{E[|\tilde{g}_{1,n}|^p]}{(E[\tilde{g}_{1,n}^2])^{p/2}} \leq C_{\infty} \times k_n^{1-p/2} E[\tilde{g}_{1,n}^2]^{-p/2}.
\]

Hence, provided \( \lim_{n \to \infty} E[\tilde{g}_{1,n}^2] \) exists, the last expression converges to 0 and hence Ljapunov’s condition is met. As a consequence, \( k_n^{-1/2} \sum_{i=1}^{k_n} \tilde{g}_{i,n} \) weakly converges to a centered normal distribution with variance \( \lim_{n \to \infty} E[\tilde{g}_{1,n}^2] \).

Finally, since \( \lim_{n \to \infty} E[\tilde{g}_{1,n}^2] = \lim_{n \to \infty} E[\tilde{g}_{1,n}^2] \), it remains to be shown that

\[
\lim_{n \to \infty} E[\tilde{g}_{1,n}^2] = \lambda_1^2 r(x, x) + 2\lambda_1 \lambda_2 b(x) + \lambda_2^2 \pi^2/6.
\]

Since we may replace \( I_i^n \) by \( I_1 \) and then \( b_n \) by \( n \), this in turn is a consequence of

\[
\begin{align*}
\lim_{n \to \infty} & \text{Var} \left( N_n^{(x)} \right) = r(x, x), \\
\lim_{n \to \infty} & \text{Cov} \left\{ N_n^{(x)}, \log(Z_{1:n}) \right\} = f(x), \\
\lim_{n \to \infty} & \text{Var} \left\{ \log(Z_{1:n}) \right\} = \pi^2/6.
\end{align*}
\]

The assertion in (7.5) follows from Theorem 4.1 in Robert (2009). Further, since \( Z_{1:n} \overset{d}{\to} \xi \sim \text{Exp}(\theta) \) and since \( |\log(Z_{1:n})|^2 \) is uniformly integrable by Condition 3.2(i), we have

\[
\lim_{n \to \infty} \text{Var} \left\{ \log(Z_{1:n}) \right\} = \text{Var} \{ \log(\xi) \} = \frac{\pi^2}{6},
\]

which is (7.7). Finally, note that \( E[N_n^{(x)}] = x \) and \( E[\log(Z_{1:n})] \to \varphi(C)(\theta) \) by similar arguments as given above. As a consequence, (7.6) follows from \( \lim_{n \to \infty} E \left[ N_n^{(x)} \log(Z_{1:n}) \right] = h(x) \). The latter in turn can be seen as follows: first,

\[
E \left[ N_n^{(x)} \log(Z_{1:n}) \right] = \sum_{i=1}^{n} i \ E \left[ \mathbb{I} \left( N_n^{(x)} = i \right) \log(Z_{1:n}) \right].
\]

The expected value on the right-hand side can be written as

\[
\begin{align*}
\int_0^\infty & \mathbb{P} \left( \mathbb{I} \left( N_n^{(x)} = i \right) \log(Z_{1:n}) > y \right) \, dy - \int_{-\infty}^0 1 - \mathbb{P} \left( \mathbb{I} \left( N_n^{(x)} = i \right) \log(Z_{1:n}) > y \right) \, dy \\
= & \int_0^\infty \mathbb{P} \left( N_n^{(x)} = i, Z_{1:n} > e^y \right) \, dy - \int_{-\infty}^0 \mathbb{P} \left( N_n^{(x)} = i \right) - \mathbb{P} \left( N_n^{(x)} = i, Z_{1:n} > e^y \right) \, dy.
\end{align*}
\]

Now,

\[
\mathbb{P} \left( N_n^{(x)} = i, Z_{1:n} > e^y \right) = \mathbb{P} (N_n^{(x)} = i, N_n^{(e^y)} = 0) \rightarrow \begin{cases} p_2^{(x,e^y)}(i, 0), & x \geq e^y \geq 0, \\
0, & e^y > x \geq 0,
\end{cases}
\]

where
and \( \mathbb{P}(N_n^{(x)} = i) \to p^{(x)}(i) \), see Perfekt (1994) and Robert (2009). By uniform integrability we obtain that the expected value on the right-hand side of (7.8) converges to
\[
\sum_{i=1}^{\infty} \left[ \int_0^\infty 1(e^y \leq x) p_2^{(x,e^y)}(i,0) \, dy - \int_{-\infty}^{0} p^{(x)}(i) - 1(e^y \leq x) p_2^{(x,e^y)}(i,0) \, dy \right] = h(x).
\]
The proof is finished.

**Proof of Lemma 7.4.** For fixed \( x > 0 \), consider the function
\[
f_n : \mathbb{R} \to \mathbb{R}, \quad f_n(z) = \sqrt{k_n} \log \left\{ 1 + \frac{1}{\sqrt{k_n}} \left( \frac{z}{x} + \frac{1}{\sqrt{k_n}} \right) \right\}.
\]
For \( z_n \to z \), one has \( f_n(z_n) \to e(z)/z \). Hence, since \( (e_n(x_1), \ldots, e_n(x_m), B_n)^{\prime} \overset{d}{\to} (e(x_1), \ldots, e(x_m), B)^{\prime} \) for any \( x_1, \ldots, x_m > 0 \) and \( m \in \mathbb{N} \) by Lemma 7.3, we can apply the extended continuous mapping theorem (Theorem 18.11 in van der Vaart (1998)) to obtain \( (W_n(x_1), \ldots, W_n(x_m), B_n)^{\prime} \overset{d}{\to} (e(x_1)/x_1, \ldots, e(x_m)/x_m, B)^{\prime} \). This is the fidis-convergence needed to prove Lemma 7.4. Asymptotic tightness follows directly from asymptotic tightness of \( e_n \) and \( B_n \).

**Proof of Lemma 7.5.** Let \( H(x) = 1 - e^{-\theta x} \) be the c.d.f. of the \( \text{Exp}(\theta) \)-distribution. From the proof of Lemma 9.2 in Berghaus and Bücher (2018), we have, for any \( m \in \mathbb{N} \),
\[
\sup_{x \in [1/m, m]} |\hat{H}_{ka}(x) - H(x)| = o_p(1), \quad n \to \infty.
\]
Since
\[
E_{n,m} - E'_{n,m} = \int_{1/m}^{m} W_n(x) \, d(\hat{H}_{ka} - H)(x),
\]
the assertion follows from Lemma 7.4, Lemma C.8 in Berghaus and Bücher (2017) and the continuous mapping theorem.

**Proof of Lemma 7.6.** As a consequence of Lemma 7.5, Lemma 7.4 and the continuous mapping theorem, we have
\[
E_{n,m} + B_n = \int_{1/m}^{m} W_n(x) \, \theta e^{-\theta x} \, dx + B_n + o_p(1)
\]
\[
\overset{d}{\to} \int_{1/m}^{m} \frac{e(x)}{x} \theta e^{-\theta x} \, dx + B = E'_m + B \sim \mathcal{N}(0, \tau^2_m),
\]
where the variance \( \tau^2_m \) is given by
\[
\tau^2_m = \text{Var} \left\{ \int_{1/m}^{m} e(x) \frac{1}{x} \theta e^{-\theta x} \, dx \right\} + 2 \text{Cov} \left\{ \int_{1/m}^{m} e(x) \frac{1}{x} \theta e^{-\theta x} \, dx, B \right\} + \text{Var}(B)
\]
\[
= \theta^2 \int_{1/m}^{m} \int_{1/m}^{m} r(x,y) \frac{1}{xy} e^{-\theta(x+y)} \, dx \, dy + 2\theta \int_{1/m}^{m} f(x) \frac{1}{x} e^{-\theta x} \, dx + \frac{\pi^2}{6}
\]
as asserted.

**Proof of Lemma 7.7.** By the definition of \( \tau^2_m \) in Lemma 7.6
\[
\lim_{m \to \infty} \tau^2_m = \theta^2 \int_{0}^{\infty} \int_{0}^{\infty} r(x,y) \frac{1}{xy} e^{-\theta(x+y)} \, dx \, dy + 2\theta \int_{0}^{\infty} f(x) \frac{1}{x} e^{-\theta x} \, dx + \frac{\pi^2}{6}. \quad (7.9)
\]
For $x > y$, we have $r(x, y) = \theta x E[\xi_1(y/x)\xi_2(y/x)]$ with $(\xi_1(y/x), \xi_2(y/x)) \sim \pi^{(y/x)}$. Hence, applying the transformation $z = y/x$, the first summand on the right-hand side of (7.9) can be written as

$$\theta^2 \int_0^\infty \int_0^\infty r(x, y) \frac{1}{xy} e^{-\theta(x+y)} \, dx \, dy = 2\theta^3 \int_0^\infty \int_0^x E[\xi_1(z) \xi_2(z)] \frac{1}{y} e^{-\theta(x+y)} \, dy \, dx \equiv 2\theta^3 \int_0^\infty \int_0^1 \frac{E[\xi_1(z) \xi_2(z)]}{z} e^{-\theta(x+y)} \, dz \, dx$$

$$= 2\theta^2 \int_0^\infty \frac{E[\xi_1(z) \xi_2(z)]}{z(z+1)} \, dz. \quad (7.10)$$

For the second summand on the right-hand side of (7.9), note that

$$\sum_{i=1}^\infty ip_{2}(x, e^y)(i, 0) = E[\xi_1(e^y) I(\xi_2(e^y) = 0)] \theta xe^{-\theta e^y}, \quad (7.11)$$

see Formula (A.7) in the proof of Lemma 9.6 in Berghaus and Bücher (2018) and $\sum_{i=1}^\infty ip_{2}(x)(i) = E[N_E^{x}] = x$, see Robert (2009). Therefore, we can rewrite $h$ from Lemma 7.3 as follows

$$h(x) = \int_0^\infty I(e^y \leq x) E[\xi_1(e^y) I(\xi_2(e^y) = 0)] \theta xe^{-\theta e^y} \, dy$$

$$- \int_{-\infty}^0 x - I(e^y \leq x) E[\xi_1(e^y) I(\xi_2(e^y) = 0)] \theta xe^{-\theta e^y} \, dy.$$

$$= x \int_{1/x}^\infty I(z \leq 1) E[\xi_1(z) I(\xi_2(z) = 0)] \theta e^{-\theta x z} \, dz$$

$$- x \int_{0}^{1/xe} \frac{1}{x} - I(z \leq 1) E[\xi_1(z) I(\xi_2(z) = 0)] \theta e^{-\theta x z} \, dz,$$

where we have used the transformation $z = y/x$. For $0 < x \leq 1$, the first integral is zero and we obtain

$$h(x) = \int_{1/x}^1 \frac{1}{z} - E[\xi_1(z) I(\xi_2(z) = 0)] \theta e^{-\theta x z} \, dz - x \int_{1}^{1/xe} \frac{1}{z} \, dz$$

$$= -x \int_{0}^{1} \frac{1}{z} - E[\xi_1(z) I(\xi_2(z) = 0)] \theta e^{-\theta x z} \, dz + x \log(x),$$

while for $x > 1$,

$$h(x) = x \int_{1/x}^1 E[\xi_1(z) I(\xi_2(z) = 0)] \theta e^{-\theta x z} \, dz - x \int_{0}^{1/xe} \frac{1}{z} - E[\xi_1(z) I(\xi_2(z) = 0)] \theta e^{-\theta x z} \, dz.$$

As a consequence, writing $g(z) = E[\xi_1(z) I(\xi_2(z) = 0)]$, we obtain

$$\int_0^\infty h(x) \frac{1}{x} e^{-\theta x} \, dx = \int_0^1 \log(x) e^{-\theta x} \, dx - \int_0^1 e^{-\theta x} \int_0^{1/xe} \frac{1}{z} - g(z) \theta e^{-\theta x z} \, dz \, dx$$

$$+ \int_{1/x}^\infty e^{-\theta x} \int_0^{1/xe} g(z) \theta e^{-\theta x z} \, dz \, dx$$

$$- \int_{0}^{1/xe} e^{-\theta x} \int_0^{1/xe} \frac{1}{z} - g(z) \theta e^{-\theta x z} \, dz \, dx.$$
Next, some tedious calculations based on Fubini’s theorem allow to rewrite the sum of the last three double integrals as
\[
s = \int_0^1 e^{-\theta z} - 1 \cdot \frac{g(z)}{z(1 + z)} \, dz.
\]
Using the fact that \( g(z) = \frac{1}{\theta} - E[Z(\xi(z) > 0)] \), we thus obtain
\[
\int_0^\infty h(x) \frac{1}{x} e^{-\theta z} \, dx = \int_0^1 \log(z) e^{-\theta z} + \frac{e^{-\theta z} - 1}{\theta z} - \log(\theta + z) + E[Z(\xi(z) > 0)] \, dz
\]
\[
= \int_0^1 \log(z) e^{-\theta z} + \frac{1}{\theta z} - \frac{E[Z(\xi(z) > 0)]}{z(1 + z)} \, dz.
\]
Finally, one can show
\[
\int_0^1 \log(z) e^{-\theta z} + \frac{e^{-\theta z}}{\theta z} \, dz = -\log(\theta + z) + \varphi\gamma(\theta)/\theta,
\]
such that, assembling terms and recalling \( f(x) = h(x) - x\varphi\gamma(\theta) \),
\[
\int_0^1 f(x) \frac{1}{x} e^{-\theta x} \, dx = \int_0^1 h(x) \frac{1}{x} e^{-\theta x} \, dx - \varphi\gamma(\theta) \int_0^1 e^{-\theta x} \, dx
\]
\[
= -\log(2)/\theta - \int_0^1 \frac{E[Z(\xi(z) > 0)]}{z(1 + z)} \, dz. \quad (7.12)
\]
The lemma is now an immediate consequence of (7.9), (7.10) and (7.12).

**Proof of Lemma 7.8.** By Lemma 7.5, it suffices to show the assertion with \( E_{n,m} \) replaced by \( E'_{n,m} \). Define \( \hat{e}_n(x) := e_n(x) + k_n^{-1/2} \), such that, by Condition 3.4(iii), we have
\[
\max_{Z_{ni} \geq c} \left| \frac{\hat{e}(Z_{ni})}{Z_{ni} \sqrt{k_n}} \right| = o_P(1)
\]
for any constant \( c > 0 \). Fix \( m \in \mathbb{N} \). By the previous display, for any \( \varepsilon > 0 \), the event
\[
B_n = B_n(m, \varepsilon) = \left\{ \max_{Z_{ni} \geq m} \left| \frac{\hat{e}(Z_{ni})}{Z_{ni} \sqrt{k_n}} \right| \leq \varepsilon \right\}
\]
satisfies \( P(B_n) \rightarrow 1 \). Next,
\[
|E_{n,m} - E_n| \leq \left| \int_0^\infty \log \left( 1 + \frac{\hat{e}(x)}{x \sqrt{k_n}} \right) \sqrt{k_n} \right| I_{\{0,1/m\}}(x) \, dH_{k_n}(x)
\]
\[
\quad + \left| \int_0^\infty \log \left( 1 + \frac{\hat{e}(x)}{x \sqrt{k_n}} \right) \sqrt{k_n} \right| I_{\{m,\infty\}}(x) \, dH_{k_n}(x)
\]
\[
=: |V_{n1}| + |V_{n2}|,
\]
such that
\[
|E_{n,m} - E_n| = |E_{n,m} - E_n| I_{B_n} + o_P(1) \leq |V_{n1}| + |V_{n2}| I_{B_n} + o_P(1). \quad (7.13)
\]
We begin by treating the term $|V_{n2}|1_{B_n}$. Since $\log(1 + x) = \int_0^1 x/(1 + sx) \, ds$ for any $x > -1$, we have

$$V_{n2}1_{B_n} = \int_0^\infty \frac{\hat{e}_n(x)}{x} \int_0^1 \frac{1}{1 + s\frac{\hat{e}_n(x)}{x\sqrt{k_n}}} \, ds \, 1(x \geq m) \, d\hat{H}_{\hat{\kappa}_n}(x) \, 1_{B_n}$$

$$= \frac{1}{k_n} \sum_{i=1}^{k_n} \frac{\hat{e}_n(Z_{ni})}{Z_{ni}} 1(Z_{ni} \geq m) \int_0^1 \frac{1}{1 + s\frac{\hat{e}_n(Z_{ni})}{Z_{ni}\sqrt{k_n}}} \, ds \, 1_{B_n}$$

$$= k_n^{-3/2} \sum_{i=1}^{k_n} \frac{1}{Z_{ni}^2} 1(Z_{ni} \geq m) \int_0^1 \frac{1}{1 + s\frac{\hat{e}_n(Z_{ni})}{Z_{ni}\sqrt{k_n}}} \, ds \, \left\{ \sum_{t=1}^n f(U_t, Z_{ni}) + 1 \right\} 1_{B_n},$$

where

$$f(U_t, Z_{ni}) = 1(U_t > 1 - Z_{ni}/b_n) - Z_{ni}/b_n. \quad (7.14)$$

For given $\varepsilon \in (0, c_1 \wedge c_2)$ with $c_j$ as in Condition 3.1, let $C_n = C_n(\varepsilon)$ denote the event

$$\{\min_{i=1, \ldots, k_n} N_{ni} > 1 - \varepsilon/2\} = \{\max_{i=1, \ldots, k_n} Z_{ni} < \varepsilon b_n/2\},$$

which satisfies $P(C_n) \to 1$ by Condition 3.1(v). Hence, we can write $V_{n2}1_{B_n} = V_{n2}1_{C_n} + o_p(1)$, where

$$\tilde{V}_{n2} = k_n^{-3/2} \sum_{i=1}^{k_n} \frac{1}{Z_{ni}^2} 1(\varepsilon b_n/2 > Z_{ni} \geq m) \int_0^1 \frac{1}{1 + s\frac{\hat{e}_n(Z_{ni})}{Z_{ni}\sqrt{k_n}}} \, ds \, \left\{ \sum_{t=1}^n f(U_t, Z_{ni}) + 1 \right\} 1_{B_n}.$$ 

We obtain

$$|V_{n2}| \leq \frac{1}{m} k_n^{-3/2} \sum_{i=1}^{k_n} 1(\varepsilon b_n/2 > Z_{ni} \geq m) \int_0^1 \frac{1}{1 + s\frac{\hat{e}_n(Z_{ni})}{Z_{ni}\sqrt{k_n}}} \, ds \, \left\{ \sum_{t=1}^n f(U_t, Z_{ni}) + 1 \right\} 1_{B_n}.$$ 

On the event $B_n$ the integral over $s$ can be bounded as follows

$$\int_0^1 \frac{1}{1 + s\frac{\hat{e}_n(Z_{ni})}{Z_{ni}\sqrt{k_n}}} \, ds \, 1_{B_n} \leq \int_0^1 \frac{1}{1 - 8\varepsilon} \, ds \, 1_{B_n} \leq \frac{1}{1 - \varepsilon}.$$ 

The previous two displays imply that $|\tilde{V}_{n2}|$ is bounded by

$$\frac{1}{m} \frac{1}{1 - \varepsilon} k_n^{-3/2} \sum_{i=1}^{k_n} 1(\varepsilon b_n/2 > Z_{ni} \geq m) \left\{ \sum_{t=1}^n f(U_t, Z_{ni}) \right\} + O_p(k_n^{-1/2}).$$

The upper bound can now be treated exactly as in the proof of Lemma 9.1 in Berghaus and B"ucher (2018), finally yielding

$$\lim_{m \to \infty} \limsup_{n \to \infty} P(|V_{n2}1_{B_n}| > \delta) = 0. \quad (7.15)$$

It remains to treat $|V_{n1}|$. Fix $\mu \in (1/2, 1/\{2(1 - \tau)\})$ with $\tau \in (0, 1/2)$ from Condition 3.4(ii). Then

$$|V_{n1}| \leq T_n(0, dk_n^{-1}) + T_n(dk_n^{-1}, dk_n^{-\mu}) + T_n(dk_n^{-\mu}, 1/m)$$
\[
\begin{align*}
\text{Then,} & \quad T_n(a, b) = \sqrt{k_n} \int_0^\infty \mathbb{I}(x \in (a, b)] \left| \log \left( 1 + \frac{\tilde{c}_n(x)}{x \sqrt{k_n}} \right) \right| d\hat{H}_{k_n}(x). \\
\text{We start by covering the term } T_{n1} = T_n(0, dk_n^{-1}) \text{ and determining the constant } d. & \text{ Note that for the event } J_n = \{ \min_{i=1,\ldots,k_n} Z_{ni} > dk_n^{-1} \} \text{ one has } \\
\mathbb{P}(J_n) = \mathbb{P}(k_n \min_{i=1,\ldots,k_n} Z_{ni} > d) = \mathbb{P}(n (1 - \max_{i=1,\ldots,k_n} N_{ni}) > d) = \mathbb{P}(Z_{1,n} > d) \to e^{-d\theta}. \quad (7.16)
\end{align*}
\]

Then,
\[
\begin{align*}
\mathbb{P}(T_{n1} > \delta) & = \mathbb{P}(T_{n1} 1_{J_n} + T_{n1} 1_{J_n^c} > \delta) \\
& \leq \mathbb{P}(T_{n1} 1_{J_n} > \delta/2) + \mathbb{P}(T_{n1} 1_{J_n^c} > \delta/2) \\
& \leq \mathbb{P}(J_n^c) \to 1 - \exp(-d\theta).
\end{align*}
\]

Hence, for any given \( \varepsilon > 0 \) we can choose \( d = d(\varepsilon) < -\log(1 - \varepsilon)/\theta, \) such that
\[
\limsup_{n \to \infty} \mathbb{P}(T_{n1} > \delta) \leq \limsup_{n \to \infty} \mathbb{P}(J_n^c) = 1 - \exp(-d\theta) < \varepsilon. \quad (7.17)
\]

Next, consider \( T_{n3} = T_n(dk_n^{-\mu}, 1/m) \) and note that, for \( x \in (dk_n^{-\mu}, 1/m] \), we have
\[
\left| \frac{\tilde{c}_n(x)}{x^{1-\tau}} \right| \leq \frac{1}{d^{1-\tau}} \leq \frac{1}{d^{1-\tau} \sqrt{k_n}} \leq \frac{1}{d^{1-\tau} \sqrt{k_n}} \leq \frac{1}{d^{1-\tau} \sqrt{k_n}} k_n^{1/(1-\tau)-1/2} = o_p(1)
\]
uniformly in \( x \), by Condition 3.4(ii). As a consequence, the event
\[
D_n = \left\{ \left| \frac{\tilde{c}_n(x)}{x^{1-\tau}} \right| \leq \frac{1}{2} \right\}
\]
satisfies \( 1_{D_n} = o_p(1) \), whence, recalling that \( x/(1+x) \leq \log(1+x) \leq x \) for any \( x > -1 \), we have
\[
\begin{align*}
T_{n3} & = \sqrt{k_n} \int_{(dk_n^{-\mu}, 1/m]} \left| \log \left( 1 + \frac{\tilde{c}_n(x)}{x \sqrt{k_n}} \right) \right| 1_{D_n} d\hat{H}_{k_n}(x) + o_p(1) \\
& \leq \int_{(dk_n^{-\mu}, 1/m]} \max \left\{ \left| \frac{\tilde{c}_n(x)}{x} \right|, \left| \frac{\tilde{c}_n(x)}{x} \right| \left( 1 + \frac{\tilde{c}_n(x)}{x \sqrt{k_n}} \right)^{-1} \right\} 1_{D_n} d\hat{H}_{k_n}(x) + o_p(1) \\
& \leq 2 \int_{(dk_n^{-\mu}, 1/m]} \left| \frac{\tilde{c}_n(x)}{x} \right| \frac{1}{x^{1-\tau}} 1_{D_n} d\hat{H}_{k_n}(x) + o_p(1).
\end{align*}
\]

By Lemma 7.25, Condition 3.4(ii) and the continuous mapping theorem, the last expression converges weakly to
\[
T_3(m) = 2 \int_0^{1/m} \left| \frac{e(x)}{x^\tau} \right| \frac{1}{x^{1-\tau}} \ dH(x).
\]

As a consequence,
\[
\lim_{m \to \infty} \limsup_{n \to \infty} \mathbb{P}(T_{n3} \geq \delta) \leq \lim_{m \to \infty} \mathbb{P}(T_3(m) > \delta) = 0. \quad (7.18)
\]
Finally, regarding $T_{n_2}$, note that, for $x \in (dk^{-1}, dk^{-\mu})$,

\[
\frac{\tilde{e}_n(x)}{x\sqrt{k_n}} \geq \frac{1}{x} \left( \frac{1}{k_n} - \frac{1}{k_n} \sum_{i=1}^{n} x/b_n \right) \geq \frac{1}{dk_n^{1-\mu}} - 1,
\]

which implies

\[
\left| \log \left( 1 + \frac{\tilde{e}_n(x)}{x\sqrt{k_n}} \right) \right| = \log \left( 1 + \frac{\tilde{e}_n(x)}{x\sqrt{k_n}} \right) 1 \{ \tilde{e}_n(x)/x\sqrt{k_n} > 0 \} - \log \left( 1 + \frac{\tilde{e}_n(x)}{x\sqrt{k_n}} \right) 1 \{ \tilde{e}_n(x)/x\sqrt{k_n} < 0 \}
\leq \log \left( (n+1)d^{-1} + 1 \right) + \log \left( dk_1^{-\mu} \right)
\leq \log(n).
\]

As a consequence, the term $T_{n_2} = T_n(dk^{-1}, dk_n^{-\mu})$ can be bounded as follows

\[
T_{n_2} \lesssim \log(n) \sqrt{k_n} \int_{(dk^{-1}, dk_n^{-\mu}]} d\hat{H}_{k_n}(x) = \frac{\log(n)}{\sqrt{k_n}} \sum_{i=1}^{k_n} 1 \{ Z_{ni} \in (dk^{-1}, dk_n^{-\mu}] \}.
\]

Hence, by Condition 3.4(iv),

\[
E \left[ T_{n_2} \right] \lesssim \log(n) \sqrt{k_n}\mathbb{P}(Z_{n_1} < dk_n^{-\mu})
\]

\[
= \log(n) \sqrt{k_n} \{ 1 - \exp(-\theta dk_n^{-\mu}) \} + o(1)
\]

\[
= \theta \log(n) k_n^{1/2-\mu} \{ 1 + o(1) \} + o(1)
\]

\[
= O(\log(k_n)k_n^{1/2-\mu}) = o(1),
\]

(7.19)

where the last line follows from $\log n = \log k_n + \log b_n \lesssim (1 + q) \log k_n$ by Condition 3.4(i).

The assertion follows from (7.13), combined with (7.15), (7.16), (7.17), (7.18) and (7.19). \(\square\)

### 7.3. Auxiliary lemmas for the proof of Theorem 4.1 – Sliding blocks

Throughout this section, we assume that Condition 3.1, 3.2(i) and 3.3(i) are met.

**Lemma 7.9.** For any $x_1, \ldots, x_m \in [0, \infty)$ and $m \in \mathbb{N}$, we have

\[
(e_n(x_1), \ldots, e_n(x_m), B_n^{sh})' \xrightarrow{d} (e(x_1), \ldots, e(x_m), B_n^{sh})',
\]

where $(e(x_1), \ldots, e(x_m), B_n^{sh})' \sim \mathcal{N}_{m+1}(0, \Sigma_{m+1}^{sh})$ with

\[
\Sigma_{m+1}^{sh} = \begin{pmatrix}
  r(x_1, x_1) & \cdots & r(x_1, x_m) & f(x_1) \\
  \vdots & \ddots & \vdots & \vdots \\
  r(x_m, x_1) & \cdots & r(x_m, x_m) & f(x_m) \\
  f(x_1) & \cdots & f(x_m) & 8 \log(2) - 4
\end{pmatrix}
\]

Here, the functions $r$ and $f$ are defined as in Lemma 7.3.
Lemma 7.10. For any $m \in \mathbb{N}$, we have
\[
\{ (W_n(x), B_n(x)^{k_n}) \}_{x \in [1/m, m]} \xrightarrow{d} \left\{ \left( \frac{e(x)}{x}, B_n(x)^{k_n} \right) \right\}_{x \in [1/m, m]} \quad \text{in } D([1/m, m]) \times \mathbb{R},
\]
where $(e, B_n^{k_n})'$ is a centered Gaussian process with continuous sample paths and with covariance functional as specified in Lemma 7.9.

Lemma 7.11. For any $m \in \mathbb{N}$, we have
\[
E_{n,m}^{n,\sigma} = E_{n,m}' + o_p(1) \quad \text{as } n \to \infty,
\]
where $E_{n,m}' = \int_{1/m}^{m} W_n(x) \theta e^{-\theta x} \, dx$ is as in Lemma 7.5.

Lemma 7.12. For any $m \in \mathbb{N}$, we have
\[
E_{n,m}^{n,\sigma} + B_n^{k_n} \xrightarrow{d} E_{n,m}' + B_n^{k_n} \sim \mathcal{N}(0, \tau_{sb,m}^2) \quad \text{as } n \to \infty,
\]
where, with $r$ and $f$ defined as in Lemma 7.3,
\[
\tau_{sb,m}^2 = \theta^2 \int_{1/m}^{m} \int_{1/m}^{m} r(x,y) \frac{1}{xy} e^{-\theta(x+y)} \, dx \, dy + 2\theta \int_{1/m}^{m} f(x) \frac{1}{x} e^{-\theta x} \, dx + 8 \log(2) - 4.
\]

Lemma 7.13. As $m \to \infty$, $\tau_{sb,m}^2 \to \sigma_{sb,m}^2/\theta^2$, where $\sigma_{sb,m}^2$ is specified in Theorem 4.1.

Lemma 7.14. If in addition, Condition 3.4 holds, then, for all $\delta > 0$,
\[
\lim_{m \to \infty} \sup_{n \to \infty} P(\{ |E_{n,m}^{n,\sigma} - E_{n,m}^{b} | > \delta \}) = 0.
\]

Proof of Lemma 7.9. As in the proof of Lemma 7.3 we only show joint weak convergence of $(e_n(x), B_n(x))$ for some fixed $x > 0$; the general case can be shown analogously. For given $\varepsilon \in (0, c_1 \wedge c_2)$ let $A'_n = \{ \min_{n=1, \ldots, n-b_n+1} N_{nt}^{k_n} > 1 - \varepsilon \}$, such that $P(A_n) \to 1$ by Condition 3.1(v).

By the Crâmer-Wold device, it suffices to prove weak convergence of
\[
\lambda_1 e_n(x) + \lambda_2 B_n(x) = \sum_{j=1}^{k_n-1} \sum_{s \in I_j} \left\{ \frac{\lambda_1}{\sqrt{b_n}} \left( 1 - \frac{x}{b_n} \right) - \frac{x}{b_n} \right\} + \frac{\lambda_2 \sqrt{b_n}}{n-b_n+1} \left\{ \log(Z_n^{k_n}) - E[\log(Z_n^{k_n})] \right\} + o_p(1),
\]
for some arbitrary $\lambda_1, \lambda_2 \in \mathbb{R}$, where the negligible term stems from omitting a negligible number of summands.

We are going to apply a big block-small block argument, based on a suitable ‘blocking of blocks’ to take care of the serial dependence introduced through the use of sliding blocks. For that purpose, let $k_n > k_n$ be an integer sequence with $k_n \to \infty$ and $k_n = o(k_n^{(2+(1+\delta))})$, where $\delta$ is from Condition 3.1(ii). For $q_n = \lfloor k_n/(k_n + 2) \rfloor$ and $j = 1, \ldots, q_n$, define
\[
J^+_j = \bigcup_{i = (j-1)(k_n + 2) + 1}^{j(k_n + 2) - 2} I_i \quad \text{and} \quad J^-_j = I_{j(k_n + 2) - 1} \cup I_{j(k_n + 2)},
\]
Thus we have $q^*_n$ big blocks $J^+_{kn}$ of size $k^*_n b_n$, which are separated by a small block $J^-_{jn}$ of size $2b_n$, just as in the construction in the proof of Lemma 10.3 in Berghaus and Bührer (2018). Consequently, we have $\lambda_1 e_n(x) + \lambda_2 B^b_n = L_n^+ + L_n^− + o_p(1)$, where

$$L_n^\pm = \frac{1}{\sqrt{q_n}} \sum_{j=1}^{q_n} W_{nj}^\pm$$

with

$$W_{nj}^\pm = \sqrt{\frac{q_n}{k_n} } \sum_{s \in J^n_j} \lambda_1 \left\{ \mathbb{I} \left(U_s > 1 - \frac{x}{b_n}\right) - \frac{x}{b_n}\right\} + \frac{\lambda_2 n}{n - b_n + 1} b_n \left\{ \log(Z_{ns}^b) - E[\log(Z_{ns}^b)] \right\}$$

for $j = 1, \ldots, q^*_n$. In the following, we show that, on the one hand, $L_n^− L_n^+ \mathbb{I}_{A_n^+} = o_p(1)$ and that, on the other hand, $L_n^+ \mathbb{I}_{A_n^+}$ converges to the claimed normal distribution. First, we cover $L_n^− L_n^+ \mathbb{I}_{A_n^+}$. As in the proof of Lemma 7.3, we have

$$Z_{ns}^b = b_n \left( 1 - \max_{t = s, \ldots, s + b_n - 1} U_t \right) = b_n \left( 1 - \max_{t = s, \ldots, s + b_n - 1} U_1^{n, j} \right) =: Z_{ns}^{c, b}$$

on the event $A_n^+$, where $U_1^{n, j} = U_1 \mathbb{I}(U_1 > 1 - \epsilon)$. Hence, we can write $L_n^− L_n^+ \mathbb{I}_{A_n^+} = \tilde{L}_n^− + o_p(1)$ with

$$\tilde{L}_n^− = \frac{1}{\sqrt{q_n}} \sum_{j=1}^{\tilde{q}_n} W_{nj}^−,$$

where

$$W_{nj}^− = \sqrt{\frac{q_n}{k_n} } \sum_{s \in J^n_j} \lambda_1 \left\{ \mathbb{I} \left(U_s^c > 1 - \frac{x}{b_n}\right) - \frac{x}{b_n}\right\} + \frac{\lambda_2 n}{n - b_n + 1} b_n \left\{ \log(Z_{ns}^{c, b}) - E[\log(Z_{ns}^{c, b})] \right\}.$$

We proceed by showing that $\text{Var}[\tilde{L}_n^−] = o(1)$. By stationarity, one has

$$\text{Var}[\tilde{L}_n^−] = \text{Var}[W_{n1}^−] + \frac{2}{q_n} \sum_{j=1}^{\tilde{q}_n} \left(q_n^* - j\right) \text{Cov}(W_{n1}^−, W_{n,j+1}^−),$$

which is bounded by $3 \text{Var}[W_{n1}^−] + 2 \sum_{j=2}^{\tilde{q}_n} |\text{Cov}(W_{n1}^−, W_{n,j+1}^−)|$ in absolute value. First, we show $\text{Var}[W_{n1}^−] = o(1)$, for which it suffices to show that $||W_{n1}^−||_p = o(1)$ for some $p \in (2, 2 + \delta)$. By Minkowski’s inequality, one has

$$||W_{n1}^−||_p \leq 2 \sqrt{\frac{q_n}{k_n}} \left[ ||\lambda_1||_p N_{b_n}(x)||_p + ||\lambda_2||_p \log(Z_{n1}^{c, b}) - E[\log(Z_{n1}^{c, b})] ||_p \right]$$

$$= O(\sqrt{q_n/k_n}) = o(1)$$

by Condition 3.1(ii) and 3.2(i). It remains to treat the sum over the covariances. Since $W_{nj}^−$ is $\mathcal{B}(\mathcal{J}(k^*_n + 2b_n + 1); \mathcal{J}(k^*_n + 2b_n))$-measurable, we may apply Lemma 3.11 in Dehling and Philipp (2002) to obtain

$$\left| \text{Cov}(W_{n1}^−, W_{n,j+1}^−) \right| \leq 10 ||W_{n1}^−||_p^2 \alpha_2 (j k^*_n b_n)^{1-2/p}. $$
By Condition 3.1(iii), the sum \( \sum_{j=2}^{q_n} \alpha_{c_j}(j k_{n}^* b_n)^{1-2/p} \) converges to zero, hence \( ||W_{n1}^+||_p = o(1) \) as asserted.

Let us now treat the term \( L_n^+ I_{A_n'} \) and show weak convergence to the asserted normal distribution. One can write

\[
L_n^+ I_{A_n'} = \frac{1}{\sqrt{n}} \sum_{j=1}^{q_n} W_{nj}^+ + o_p(1), \quad W_{nj}^+ = W_{nj}^+ \mathbb{1} \left( \max_{t \in J_n} Z_{nt}^{sb} < \varepsilon b_n \right).
\]

A standard argument based on characteristic functions shows that the weak limit of \( \sum_{j=1}^{q_n} W_{nj}^+ \) is the same as if the summands were independent. By arguments as before, we may also pass back to an independent sample \( W_{nj}^+ \), \( j = 1, \ldots, q_n \). The assertion then follows from Ljapunov’s central limit theorem, once we have shown the Ljapunov condition.

For that purpose, note that \( ||W_{nj}^+||_{2+\delta} = O(\sqrt{q_n k_n}) = O(\sqrt{k_n^*}) \) by similar arguments as in (7.20) such that \( E||W_{nj}^+||_{2+\delta} = O(k_n^{(2+\delta)/2}) \). As a consequence,

\[
\frac{\sum_{j=1}^{q_n} E||W_{nj}^+||_{2+\delta}^2}{\text{Var} \left( \sum_{j=1}^{q_n} W_{nj}^+ \right)} = k_n^{-\frac{1}{2}} E\left[ \frac{||W_{nj}^+||_{2+\delta}^2}{E||W_{nj}^+||_{2+\delta}^2} \right] = O(k_n^{-\delta/2} k_n^{1+\delta}) = o(1),
\]

since \( k_n^* = o(k_n^{(2(1+\delta)/2)}) \) by construction and provided that the limit of \( E||W_{nj}^+||_{2+\delta} \) exists. If it does, we can conclude that \( L_n^+ \xrightarrow{d} \mathcal{N}(0, \lim_{n \to \infty} E||W_{nj}^+||_{2}^2) \) and it suffices to show that

\[
\lim_{n \to \infty} E||W_{nj}^+||_{2}^2 = \lambda_1^2 \tau(x, x) + 2 \lambda_1 \lambda_2 f(x) + \lambda_2^2 (8 \log(2) - 4).
\]

To this, note that \( W_{nj}^+ = \lambda_1 e_n^* (x) + \lambda_2 B_{n}^{sb} + o_p(1) \), where \( e_n^* \) and \( B_{n}^{sb} \) are defined as \( e_n \) and \( B_{n}^{sb} \) with \( n^* = k_n^* b_n \) and \( k_n \) by \( k_n^* \); and our general conditions still hold with this replacement. The result follows from Lemma 7.15 and Lemma 7.16 and the proof of Theorem 4.1 in Robert (2009).

**Proof of Lemma 7.10.** Up to notation, the proof is exactly the same as the one of Lemma 7.4 in the disjoint blocks case.

**Proof of Lemma 7.11.** The result follows immediately from the argument in the proof of Lemma 7.5 and the proof of Lemma 10.2 in Berghaus and Buecher (2018).

**Proof of Lemma 7.12.** Up to notation, the proof is exactly the same as the one of Lemma 7.6 in the disjoint blocks case.

**Proof of Lemma 7.13.** By the definition of \( \tau_{m}^2 \) and \( \tau_{sb,m}^2 \) in Lemma 7.6 and 7.12, we have

\[
\tau_{sb,m}^2 = \tau_{m}^2 - \pi^2/6 + 8 \log(2) - 4.
\]

Hence, by the proof of Lemma 7.7 and the definition of \( \sigma_{sb,C}^2 \) in Theorem 4.1,

\[
\lim_{m \to \infty} \tau_{sb,m}^2 = \sigma_{sb,C}^2 / \theta^2 - \pi^2/6 + 8 \log 2 - 4 = \sigma_{sb,C}^2 / \theta^2.
\]

**Proof of Lemma 7.14.** The proof is similar to the one of Lemma 7.8, which is why we keep it short. Write \( E_{n1}^{sb} - E_{n2}^{sb} \leq |V_{n1}| + |V_{n2}| \) with

\[
V_{n1} = \int_0^\infty \log \left( 1 + \frac{\tilde{e}_n(x)}{x \sqrt{k_n}} \right) \sqrt{k_n} \mathbb{1}_{(0,1/n]}(x) \, d\tilde{H}_n(x),
\]
\[ V_{n2} = \int_0^\infty \log \left( 1 + \frac{\tilde{e}_n(x)}{x\sqrt{k_n}} \right) \sqrt{k_n} \mathbb{I}_{[m,\infty)}(x) \, d\tilde{H}_n^{sb}(x), \]

where \( \tilde{e}_n(x) = e_n(x) + k_n^{-1/2} \). For some \( \varepsilon > 0 \) define the event

\[ B_n = \left\{ \max_{Z_{nm}^{sb} \geq m} \left| \frac{\tilde{e}_n(Z_{nm}^{sb})}{Z_{nm}^{sb} \sqrt{k_n}} \right| \leq \varepsilon \right\}, \]

such that \( \mathbb{P}(B_n) \to 1 \) by Condition 3.4(iii). As in the proof of Lemma 7.8, with \( f \) defined in (7.14), we can write

\[ V_{n2} \mathbb{I}_{B_n} = k_n^{-3/2} \sum_{i=1}^{k_n-1} \sum_{w \in I_i} \frac{1}{Z_{nw}^{sb}} \mathbb{I}(Z_{nw}^{sb} \geq m) \int_0^1 \frac{1}{1 + s} \frac{\tilde{e}_n(Z_{nw}^{sb})}{Z_{nw}^{sb} \sqrt{k_n}} \, ds \times b_n^{-1} \left\{ \sum_{j=1}^{k_n} \sum_{t \in I_j} f(U_t, Z_{nw}^{sb}) + 1 \right\} \mathbb{I}_{B_n} + o_p(1). \]

By Condition 3.1(v), \( \mathbb{P}(C_n) \to 1 \) where \( C_n = \{ \min_{i=1,\ldots,n-b_n+1} N_{ni}^{sb} > 1 - \varepsilon/2 \} \). Hence,

\[ \hat{V}_{n2} = k_n^{-3/2} \sum_{i=1}^{k_n-1} \sum_{w \in I_i} \frac{1}{Z_{nw}^{sb}} \mathbb{I}(\varepsilon b_n/2 > Z_{nw}^{sb} \geq m) \int_0^1 \frac{1}{1 + s} \frac{\tilde{e}_n(Z_{nw}^{sb})}{Z_{nw}^{sb} \sqrt{k_n}} \, ds \times b_n^{-1} \left\{ \sum_{j=1}^{k_n} \sum_{t \in I_j} f(U_t, Z_{nw}^{sb}) + 1 \right\}, \]

such that \( \hat{V}_{n2} \) can be bounded as in the proof of Lemma 7.8 as follows

\[ |\hat{V}_{n2} \mathbb{I}_{B_n}| \leq \frac{1}{m} \frac{1}{1 - \varepsilon} k_n^{-3/2} \sum_{i=1}^{k_n-1} \sum_{w \in I_i} \mathbb{I}(\varepsilon b_n/2 > Z_{nw}^{sb} \geq m) b_n^{-1} \left| \sum_{j=1}^{k_n} \sum_{t \in I_j} f(U_t, Z_{nw}^{sb}) \right| + o_p(1). \]

This expression can be handled as in the proof of Lemma 10.1 in Berghaus and B"{u}cher (2018), such that

\[ \lim_{m \to \infty} \limsup_{n \to \infty} \mathbb{P}(|\hat{V}_{n2} \mathbb{I}_{B_n} \mathbb{I}_{C_n}| > \delta) = 0. \]

The remaining term \( |V_{n1}| \) can be treated analogously to the eponymous term in the proof of Lemma 7.8.

**Lemma 7.15.** (a) For \( x \geq 0 \), as \( n \to \infty \),

\[ \text{Cov}(e_n(x), B_n^{sb}) \to 2 \int_0^1 h_{sb,x}(\xi) \, d\xi - 2x\varphi(\cdot)(\theta), \]

where

\[ h_{sb,x}(\xi) = \sum_{i=1}^\infty \int_0^\infty \mathbb{I}(y \leq \log(x)) \sum_{l=0}^i p^{(\xi)}(l) p_2^{((1-\xi)x, (1-\xi)e^\xi)}(i-l,0) e^{-\xi e^y} + \mathbb{I}(y > \log(x)) p^{(\xi)}(i) e^{-\xi e^y} \, dy. \]
We have as in the proof of Lemma 7.9. Let $A$ sigma-algebra; the general case can be treated by multiplying with suitable indicator functions.

Proof. (a) We assume that both $U_s$ and $Z_{nt}^{ab}$ are measurable with respect to the appropriate $B_{\xi}$ sigma-algebra; the general case can be treated by multiplying with suitable indicator functions as in the proof of Lemma 7.9. Let $A_j = \sum_{s \in I_j} 1(U_s > 1 - \frac{\ell}{b_n})$ and $D_j = \sum_{s \in I_j} \log(Z_{nt})$. Then

$$\text{Cov}(e_n(x), B_{nt}^{ab}) = \frac{1}{n-b_n} \text{Cov}(A_2, D_1 + D_2) + o(1)$$

$$= \frac{1}{b_n} \sum_{t=1}^{2b_n} \text{Cov} \left\{ \sum_{s \in I_2} 1(U_s > 1 - \frac{x}{b_n}), \log(Z_{nt}^{ab}) \right\} + o(1)$$

$$= \int_0^1 f_n(x) + g_n(x) \xi \, dx - 2x E \left[ \log(Z_{nt}^{ab}) \right] + o(1),$$

where

$$f_n(x) = \sum_{t=1}^{b_n} E \left[ \sum_{s \in I_2} 1(U_s > 1 - \frac{x}{b_n}) \log(Z_{nt}^{ab}) \right] 1\{x \in [\frac{t-1}{b_n}, \frac{t}{b_n}]\},$$

$$g_n(x) = \sum_{t=b_n+1}^{2b_n} E \left[ \sum_{s \in I_2} 1(U_s > 1 - \frac{x}{b_n}) \log(Z_{nt}^{ab}) \right] 1\{x \in [\frac{t-b_n-1}{b_n}, \frac{t-b_n}{b_n}]\}. $$

Note that $\lim_{n \to \infty} E[\log(Z_{nt}^{ab})] = \varphi(\xi)(\theta)$ by uniform integrability of $\log(Z_{1,n})$, and that $\sup_{t \in \mathbb{N}} \|f_n\| + \|g_n\| < \infty$ as a consequence of $\|\sum_{s \in I_2} 1(U_s > 1 - \frac{x}{b_n}) \|_2 \times \|\log(Z_{nt}^{ab})\|_2 < \infty$ by Condition 3.1(ii) and 3.2(i). Hence, the lemma is proven if we show that, for any $\xi \in (0, 1)$,

$$\lim_{n \to \infty} f_n(1 - \xi) = \lim_{n \to \infty} g_n(\xi) = h_{ab,x}(\xi).$$

Since the proof for $f_n(1 - \xi)$ is similar, we only treat $g_n(\xi)$, which can be written as

$$g_n(\xi) = E \left[ \sum_{s \in I_2} 1(U_s > 1 - \frac{x}{b_n}) \log(Z_{nt}^{ab}) \right] 1\{x \in [\frac{t-1}{b_n}, \frac{t}{b_n}]\}. $$
Let us next show joint weak convergence of \( \sum_{s \in I_2} \mathbb{1}(U_s > 1 - \frac{x}{b_n}) \) and \( \log(Z_{n,1(1+\xi)b_n+1}^{ab}) \). For that purpose, note that
\[
G_n(i, y) := \mathbb{P}\left( \sum_{s \in I_2} \mathbb{1}(U_s > 1 - \frac{x}{b_n}) = i, \log(Z_{n,1(1+\xi)b_n+1}^{ab}) \geq y \right)
\]

\[
= \mathbb{P}\left( \sum_{s \in I_2} \mathbb{1}(U_s > 1 - \frac{x}{b_n}) = i, \ Z_{n,1(1+\xi)b_n+1}^{ab} \geq e^y \right),
\]

coincides with \( F_n(i, e^y) \) in the proof of Lemma B.1 in Berghaus and B"{u}cher (2018). Hence, by that proof, we have
\[
\lim_{n \to \infty} G_n(i, y) = \sum_{i=0}^{i_0} p((\xi x)(l)) p_2((1-\xi) x, (1-\xi)e^y) (i - 1, 0) e^{-\theta e^y}
\]
for \( y \leq \log x \) and
\[
\lim_{n \to \infty} G_n(i, y) = p((\xi x)(i)) e^{-\theta e^y}
\]
for \( y > \log x \). Further, note that
\[
\lim_{n \to \infty} \mathbb{P}(N_{b_n}^{(x)}(E) = i) = p^{(x)}(i).
\]
As a consequence of the previous three displays, and since weak convergence and uniform integrability imply convergence of moments, we have
\[
g_n(\xi) = \sum_{i=1}^{\infty} i \int_{0}^{\infty} \mathbb{P}\left( \sum_{s = b_n+1}^{2b_n} \mathbb{1}(U_s > 1 - \frac{x}{b_n}) = i, \log(Z_{n,1(1+\xi)b_n+1}^{ab}) \geq y \right) dy
\]
\[
- i \int_{-\infty}^{0} \mathbb{P}\left( \sum_{s = b_n+1}^{2b_n} \mathbb{1}(U_s > 1 - \frac{x}{b_n}) = i, \log(Z_{n,1(1+\xi)b_n+1}^{ab}) \leq y \right) dy
\]
\[
= \sum_{i=1}^{\infty} i \int_{0}^{\infty} G_n(i, y) dy - i \int_{-\infty}^{0} \mathbb{P}(N_{b_n}^{(x)}(E) = i) - G_n(i, y) dy
\]
\[
\to h_{ab, x}(\xi)
\]
as asserted, which implies part (a) of the lemma.

(b) In the proof of Lemma B.3 in Berghaus and B"{u}cher (2018) it is shown that, for \( y \leq \log(x) \),
\[
S(x, y, \xi) = e^{-\theta e^y} \sum_{i=1}^{\infty} i \int_{0}^{\infty} p((\xi x)(l)) p_2((1-\xi) x, (1-\xi)e^y) (i - 1, 0)
\]
\[
= \xi x e^{-\theta e^y} + \mathbb{E}\left[ (\xi_{11}^{(y/x)} \mathbb{1}(\xi_{12}^{(y/x)} = 0) \right] \theta(1 - \xi) x e^{-\theta e^y},
\]
where \( (\xi_{11}^{(y/x)}, \xi_{12}^{(y/x)}) \sim \chi_2^{(y/x)} \). Equation (7.11) then allows to rewrite
\[
S(x, y, \xi) = \xi x e^{-\theta e^y} + (1 - \xi) \sum_{i=1}^{\infty} i p_2^{(x,e^y)} (i, 0) \equiv \xi x e^{-\theta e^y} + (1 - \xi) T(x, y).
\]
As a consequence, further noting that \(\sum_{i=1}^{\infty} i p^{(i)}(x) = \xi x\), we obtain

\[
h_{sb,x}(\xi) = \int_0^\infty \xi x e^{-\theta e^y} + 1(y \leq \log(x))(1 - \xi)T(x,y) \, dy.
\]

Then, by Fubini’s theorem,

\[
2 \int_0^1 h_{sb,x}(\xi) \, d\xi = \int_0^\infty xe^{-\theta e^y} + 1(y \leq \log x)T(x,y) \, dy.
\]

The assertion now follows from the fact that

\[
\int_0^\infty e^{-\theta e^y} \, dy = \int_0^\infty e^{-z} \frac{dz}{z} = -\text{Ei}(-\theta)
\]

and

\[
\int_{-\infty}^0 1 - e^{-\theta e^y} \, dy = \int_0^\theta 1 - e^{-z} \log(z) \, dz \right]
\]
By Condition 3.2(i), we have $\text{E}[\log(Z_{n1}^{sb})] \rightarrow \phi_{(C)}(\theta)$. Further,

$$\sup_{n \in \mathbb{N}} ||f_n||_\infty \leq \sup_{n \in \mathbb{N}} \text{E}[\log(Z_{n1}^{sb})]^2 < \infty,$$

whence convergence of the integral over $f_n$ in (7.21) may be concluded from the dominated convergence theorem, once we have shown pointwise convergence of $f_n$. To this end we show that, for any fixed $\xi \in (0, 1),$

$$\left( \log(Z_{n1}^{sb}), \log(Z_{n1}^{sb}(n, b, \xi) + 1) \right) \overset{d}{\rightarrow} (X^{(\xi)}, Y^{(\xi)})$$

(7.22)

for some random vector $(X^{(\xi)}, Y^{(\xi)})$. This in turn will imply

$$\lim_{n \to \infty} f_n(\xi) = \lim_{n \to \infty} \text{E}[log(Z_{n1}^{sb})] \log(Z_{n1}^{sb}(n, b, \xi) + 1)] = \text{E}[X^{(\xi)}Y^{(\xi)}]$$

by Condition 3.2(i) and therefore

$$\lim_{n \to \infty} \text{Var}(P_{n1}^{sb}) = 2 \int_0^1 \text{E}[X^{(\xi)}Y^{(\xi)}] \, d\xi - 2\phi_{(C)}(\theta)^2 = 2 \int_0^1 \text{Cov}(X^{(\xi)}, Y^{(\xi)}) \, d\xi.$$ (7.23)

For the proof of (7.22), define, for $x, y \in \mathbb{R},$

$$G_{n, \xi}(x, y) = \mathbb{P}\{\log(Z_{n1}^{sb}) > x, \log(Z_{n1}^{sb}(n, b, \xi) + 1) > y\} = \mathbb{P}(Z_{n1}^{sb} > e^x, Z_{n1}^{sb}(n, b, \xi) + 1 > e^y),$$

which converges to

$$G_{\xi}(x, y) = \exp\left(-\theta[\xi(e^x \wedge e^y) + (e^x \vee e^y)]\right)$$

by the proof of Lemma B.2 in Berghaus and Bächer (2018). Hence, (7.22), where the random vector $(X^{(\xi)}, Y^{(\xi)})$ has joint c.d.f.

$$F_{\xi}(x, y) = \mathbb{P}(X^{(\xi)} \leq x, Y^{(\xi)} \leq y) = 1 - \mathbb{P}(X^{(\xi)} > x) - \mathbb{P}(Y^{(\xi)} > y) + G_{\xi}(x, y),$$

$$= 1 - \exp(-\theta e^x) - \exp(-\theta e^y) + G_{\xi}(x, y).$$

We are left with calculating the right-hand side of (7.23). By Lemma 7.26, we have

$$V \equiv \int_0^1 \text{Cov}(X^{(\xi)}, Y^{(\xi)}) \, d\xi$$

$$= \int_0^1 \int_0^\infty \int_0^\infty G_{\xi}(x, y) - e^{-\theta e^x}e^{-\theta e^y} \, dx \, dy \, d\xi$$

$$+ \int_0^1 \int_0^\infty \int_0^\infty F_{\xi}(x, y) - (1 - e^{-\theta e^x})(1 - e^{-\theta e^y}) \, dx \, dy \, d\xi$$

$$\equiv A + B - 2 \cdot C.$$ (7.24)

We start with the first summand $A$. Recall the exponential integral $\text{Ei}(x) = -\int_{-\infty}^x e^{-t}/t \, dt$ for $x > 0$, and note that $\int_0^\infty e^{-\theta e^x} \, dx = -\text{Ei}(-\theta e^x)$ for $y \in \mathbb{R}$ and $\int_0^\infty e^{-\alpha \xi} \, d\xi = (1 - e^{-\alpha})/\alpha$ for $a > 0$. Fubini’s theorem allows to write

$$A = \int_0^\infty \int_0^\infty e^{-\theta e^y} \left\{ \int_0^1 e^{-\theta e^x} \, d\xi - e^{-\theta e^x} \right\} \, dx + \int_0^\infty e^{-\theta e^y} \left\{ \int_0^1 e^{-\theta e^x} \, d\xi - e^{-\theta e^x} \right\} \, dx \, dy$$
Finally, regarding the term $C$, we have

$$C = \int_{0}^{\infty} e^{-\theta e^y} e^{-\theta e^y} \, dx \, dy \, d\xi = \int_{-\infty}^{0} \int_{0}^{\infty} e^{-\theta e^y} \left( e^{-\theta e^y} - \int_{0}^{1} e^{-\theta \xi e^y} d\xi \right) \, dx \, dy$$

and

$$\{ - \text{Ei}(\theta) \} \int_{-\infty}^{0} e^{-\theta e^y} - \frac{1 - e^{-\theta e^y}}{\theta e^y} \, dy = \text{Ei}(\theta) \int_{0}^{\theta} \left( \frac{1}{z} - \frac{e^{-z}}{z} \right) \frac{1}{z} \, dz.$$ (7.27)

Next, the expressions in (7.25), (7.26) and (7.27) may be plugged-into (7.24). Using the notations

$$g(z) = \left\{ \frac{1}{z} - \frac{e^{-z}}{z} \right\} \frac{1}{z}, \quad h(z) = \left\{ \frac{e^{-z}}{z} - \frac{1}{z} + 1 \right\} \frac{e^{-z}}{z},$$

we obtain that

$$V = \int_{0}^{\theta} \left\{ \frac{1 - e^{-\theta}}{\theta} - 1 \right\} \frac{e^{-z}}{z} \, dz \, + h(z) + \left\{ - \text{Ei}(-z) \right\} g(z) \, dz$$

$$+ \int_{0}^{\theta} \left\{ \text{Ei}(\theta) - \text{Ei}(-z) \right\} g(z) + h(z) - 2 \text{Ei}(\theta) g(z) \, dz$$

$$= \int_{0}^{\theta} h(z) + \left\{ - \text{Ei}(-z) \right\} g(z) \, dz \, + \frac{1 - e^{-\theta}}{\theta} \left\{ \frac{1}{z} - \frac{e^{-z}}{z} \right\} \frac{-\text{Ei}(-\theta)}{-\text{Ei}(\theta)} \, dz.$$ (7.26)

and

$$A = \int_{0}^{\theta} \left\{ \frac{e^{-z}}{z} - \frac{1}{z} + 1 - \frac{e^{-z}}{z} \right\} \frac{e^{-z}}{z} \, dz - \text{Ei}(\theta) \left\{ \frac{1}{z} - \frac{e^{-z}}{z} \right\} \frac{1}{z} \, dz.$$ (7.25)

A similar calculation allows to rewrite

$$B = \int_{0}^{\theta} \int_{-\infty}^{\infty} G_{\xi}(x, y) - e^{-\theta e^y} \, dx \, dy \, d\xi$$

and

$$\text{Ei}(\theta) \int_{0}^{\theta} \left( \frac{1}{z} - \frac{e^{-z}}{z} \right) \frac{1}{z} \, dz.$$

Finally, regarding the term $C$, we have

$$C = \int_{-\infty}^{0} \int_{0}^{\infty} e^{-\theta e^y} e^{-\theta e^y} - G_{\xi}(x, y) \, dx \, dy \, d\xi$$

and

$$\{ - \text{Ei}(\theta) \} \int_{-\infty}^{0} e^{-\theta e^y} - \frac{1 - e^{-\theta e^y}}{\theta e^y} \, dy = \text{Ei}(\theta) \int_{0}^{\theta} \left( \frac{1}{z} - \frac{e^{-z}}{z} \right) \frac{1}{z} \, dz.$$ (7.27)
7.4. Proof of Theorem 4.2

The following notation will be used throughout:

\[
\begin{align*}
\hat{S}_n &= \frac{1}{k_n} \sum_{i=1}^{k_n} \exp(-\hat{Z}_{ni}), \\
S_n &= \frac{1}{k_n} \sum_{i=1}^{k_n} \exp(-Z_{ni}), \\
\hat{S}^{sb}_n &= \frac{1}{n-b_n+1} \sum_{i=1}^{n-b_n+1} \exp(-\hat{Z}^{sb}_{ni}), \\
S^{sb}_n &= \frac{1}{n-b_n+1} \sum_{i=1}^{n-b_n+1} \exp(-Z^{sb}_{ni}).
\end{align*}
\]

Note that \(\hat{\varphi}^{-1}_{\text{db,MAD}} = \varphi^{-1}_M(\hat{S}_n)\) and \(\hat{\varphi}^{-1}_{\text{sb,MAD}} = \varphi^{-1}_M(\hat{S}^{sb}_n)\), where \(\varphi_M(x) = x/(1+x)\). The assertion follows from the delta-method and Proposition 7.17 and 7.19.

Proposition 7.17. Under Condition 3.1 and 3.3(ii), we have
\[
\sqrt{k_n}(\hat{S}_n - \varphi_M(\theta)) \overset{d}{\to} N(0, \sigma^2_{\text{db,MAD}}/(1+\theta)^4) \text{ as } n \to \infty.
\]

Proof. Write \(\sqrt{k_n}(\hat{S}_n - \varphi_M(\theta)) = A_n + B_n + C_n\), where

\[
A_n = \sqrt{k_n}\{S_n - S_n\}, \quad B_n = \sqrt{k_n}\{S_n - E[S_n]\}, \quad C_n = \sqrt{k_n}\{E[S_n] - \varphi_M(\theta)\}.
\]

The term \(C_n\) is asymptotically negligible by Condition 3.3(ii). A straightforward calculation shows that the summand \(A_n\) can be written in terms of the tail empirical process \(e_n\) as

\[
A_n = \int_0^\infty W_n(x) \, d\hat{H}_n(x), \quad W_n(x) = \sqrt{k_n} e^{-x} \left[\exp\left(-e_n(x)k_n^{-1/2}\right) - 1\right],
\]

where \(\hat{H}_n\) is the empirical c.d.f. of \(Z_{n1}, \ldots, Z_{nk_n}\), see (7.2). The asymptotic normality of \(A_n + B_n\) can now be shown as in the proof of Proposition 7.1. The corresponding key result is given by Lemma 7.18; whose proof is similar (but easier) as for the CFG-estimator (Lemma 7.3) and is omitted for the sake of brevity.

Lemma 7.18. (a) For any \(x_1, \ldots, x_m \in [0, \infty)\), as \(n \to \infty\),

\[
(e_n(x_1), \ldots, e_n(x_m), B_n) \overset{d}{\to} (e(x_1), \ldots, e(x_m), B) \sim N_{m+1}(0, \Sigma_{m+1}),
\]

with

\[
\Sigma_{m+1} = \begin{pmatrix}
  r(x_1, x_1) & \cdots & r(x_1, x_m) & f(x_1) \\
  \vdots & \ddots & \vdots & \vdots \\
  r(x_m, x_1) & \cdots & r(x_m, x_m) & f(x_m) \\
  f(x_1) & \cdots & f(x_m) & \frac{g}{\theta+2} - \frac{p^2}{(\theta+1)^2}
\end{pmatrix},
\]

where the covariance function \(r\) is given as in Lemma 7.3 and

\[
f(x) = \sum_{i=1}^{\infty} \int_0^1 p^2(i) \left( p(\log(x) - \log(y)) \right)_{i,0} \mathbb{1}(x \geq -\log(y)) \, dy - x\varphi_M(\theta).
\]

(b) For any \(x_1, \ldots, x_m \in [0, \infty)\), as \(n \to \infty\),

\[
(W_n(x_1), \ldots, W_n(x_m), B_n) \overset{d}{\to} (-e^{-x_1}e(x_1), \ldots, -e^{-x_m}e(x_m), B).
\]
Proposition 7.19. Under Condition 3.1 and 3.3(ii), we have

\[
\sqrt{k_n} \left( \hat{S}_{n}^{ab} - \varphi(M)(\theta) \right) \xrightarrow{d} N(0, \sigma_{n}^{2} \Delta_n / (1 + \theta)^4) \quad \text{as } n \to \infty.
\]

Proof. The proof is similar to the proof of Proposition 7.17. Decompose \( \sqrt{k_n} \{ \hat{S}_{n}^{ab} - \varphi(M)(\theta) \} = A_{n}^{ab} + B_{n}^{ab} + C_{n}^{ab} \), where

\[
A_{n}^{ab} = \sqrt{k_n} \{ \hat{S}_{n}^{ab} - \varphi_{ab}(\theta) \}, \quad B_{n}^{ab} = \sqrt{k_n} \{ \hat{S}_{n}^{ab} - \varphi_{ab}(\theta) \}, \quad C_{n}^{ab} = \sqrt{k_n} \{ \varphi_{ab}(\theta) - \varphi(M)(\theta) \}.
\]

Again, we have \( C_{n}^{ab} = o(1) \) by Condition 3.3(ii) and

\[
A_{n}^{ab} = \int_{0}^{\infty} W_{n}(x) \, d\hat{H}_{n}^{ab}(x),
\]

where \( \hat{H}_{n}^{ab} \) denotes the empirical c.d.f. of \( Z_{n+1}^{ab}, \ldots, Z_{n,n-b_{n}+1}^{ab} \). The sum \( A_{n}^{ab} + B_{n}^{ab} \) can now be treated as in proof of Proposition 7.2. The corresponding key result, Lemma 7.9, needs to be replaced by Lemma 7.20; whose proof is again omitted for the sake of brevity.

Lemma 7.20. (a) For any \( x_1, \ldots, x_m \in [0, \infty) \), as \( n \to \infty \),

\[
(e_{n}(x_1), \ldots, e_{n}(x_m), B_{n}^{ab}) \xrightarrow{d} (e(x_1), \ldots, e(x_m), B^{ab}) \sim \mathcal{N}_{m+1}(0, \Sigma_{m+1}^{ab}),
\]

where all entries of \( \Sigma_{m+1}^{ab} \) are the same as those of \( \Sigma_{m+1} \) in Lemma 7.18 except for the entry at position \((m + 1, m + 1)\), which needs to be replaced by

\[
v(\theta) = 2 - \frac{4}{\theta + 1} + 4 \frac{\log(\theta + 1) - \log(\theta + 2) + \log(2)}{\theta(\theta + 1)} = \frac{2\theta^2}{(\theta + 1)^2}.
\]

(b) For any \( x_1, \ldots, x_m \in [0, \infty) \), as \( n \to \infty \),

\[
(W_{n}(x_1), \ldots, W_{n}(x_m), B_{n}^{ab}) \xrightarrow{d} (-e^{-x_1} e(x_1), \ldots, -e^{-x_m} e(x_m), B^{ab}).
\]

7.5. Proof of Theorem 4.3

For fixed \( p > 0 \), define

\[
\hat{S}_{n} = \frac{1}{k_n} \sum_{i=1}^{k_n} \hat{Z}_{ni}^{1/p}, \quad S_{n} = \frac{1}{k_n} \sum_{i=1}^{k_n} Z_{ni}^{1/p},
\]

\[
\hat{S}_{n}^{ab} = \frac{1}{n - b_{n} + 1} \sum_{i=1}^{n - b_{n} + 1} \hat{Z}_{ni}^{1/p}, \quad S_{n}^{ab} = \frac{1}{n - b_{n} + 1} \sum_{i=1}^{n - b_{n} + 1} Z_{ni}^{1/p}.
\]

Note that \( \hat{\theta}_{nb,R,p} = \varphi_{(R)}^{-1}(\hat{S}_{n}) \) and \( \hat{\theta}_{nb,R,p}^{ab} = \varphi_{(R)}^{-1}(\hat{S}_{n}^{ab}) \), where \( \varphi_{(R)}(x) = x^{-1/p} \Gamma(1 + 1/p) \). By the delta-method, the assertion follows from Proposition 7.21 and 7.23.

Proposition 7.21. Under Condition 3.1, 3.2(ii) and 3.3(iii), we have

\[
\sqrt{k_n} \left( \hat{S}_{n} - \varphi_{(R),p}(\theta) \right) \xrightarrow{d} N(0, \sigma_{n}^{2} \psi_{p}(\theta)) \quad \text{as } n \to \infty,
\]

where \( \psi_{p}(\theta) = \Gamma(1 + 1/p)^2 p^{-2} \theta^{-(2+2/p)} \).
Proposition 7.23. Under Condition 3.1, 3.2(ii) and 3.3(iii), we have
\[ \psi \]

The asymptotic normality of the term \( A \) by Condition 3.3(iii), the term \( C \) converges to zero. A straightforward calculation shows that the term \( A \) can be written as
\[ A_n = \int_0^\infty W_n(x) \, dH_n(x), \quad W_n(x) = \sqrt{k_n} \left\{ \frac{c_n(x)}{\sqrt{\theta}} + x \right\}^{1/p} - x^{1/p}. \]

The asymptotic normality of \( A_n + B_n \) can be shown as in the proof of Proposition 7.1 by an application of Wichura’s theorem. Here, Lemma 7.3 needs to be replaced by Lemma 7.22, whose proof is similar but easier and therefore omitted for the sake of brevity.

Lemma 7.22. (a) For any \( x_1, \ldots, x_m \in (0, \infty) \), as \( n \to \infty \),
\[ (e_n(x_1), \ldots, e_n(x_m), B_n) \overset{d}{\to} (e(x_1), \ldots, e(x_m), B) \sim \mathcal{N}_{m+1}(0, \Sigma_{m+1}) \]

with
\[ \Sigma_{m+1} = \begin{pmatrix} r(x_1, x_1) & \ldots & r(x_1, x_m) & f_p(x_1) \\ \vdots & \ddots & \vdots & \vdots \\ r(x_m, x_1) & \ldots & r(x_m, x_m) & f_p(x_m) \\ f_p(x_1) & \ldots & f_p(x_m) & \psi_p(\theta) \end{pmatrix}, \]

where the covariance function \( r \) is defined as in Lemma 7.3 and
\[ f_p(x) = \sum_{i=1}^{\infty} i \int_0^\infty p_x^{(x,y^p)}(i,0) \mathbf{1}(x \geq y^p) \, dy - x \varphi(\cdot) \psi_p(\theta), \]
\[ \psi_p(\theta) = \theta^{2p} \left\{ \Gamma(1+2/p) - \Gamma(1+1/p)^2 \right\}. \]

(b) For any \( x_1, \ldots, x_m \in (0, \infty) \), as \( n \to \infty \),
\[ (W_n(x_1), \ldots, W_n(x_m), B_n) \overset{d}{\to} (e(x_1)x_1^{\frac{1}{2}-1}p^{-1}, \ldots, e(x_m)x_m^{\frac{1}{2}-1}p^{-1}, B). \]

Proposition 7.23. Under Condition 3.1, 3.2(ii) and 3.3(iii), we have
\[ \sqrt{k_n} \{ S^\text{eb}_n - \varphi(\cdot) \psi_p(\theta) \} \overset{d}{\to} \mathcal{N}(0, \sigma^2, \psi_p(\theta)) \text{ as } n \to \infty, \]
where \( \psi_p(\theta) = \Gamma(1+1/p)^2p^{-2}2^{-2-2/p} \).

Proof. The proof is similar to the proof of Proposition 7.21. Write \( \sqrt{k_n} \{ S^\text{eb}_n - \varphi(\cdot) \psi_p(\theta) \} = A_n^\text{eb} + B_n^\text{eb} + C_n^\text{eb} \), where
\[ A_n^\text{eb} = \sqrt{k_n} \{ S_n^\text{eb} - c_n^\text{eb} \}, \quad B_n^\text{eb} = \sqrt{k_n} \{ S_n^\text{eb} - E[S_n^\text{eb}] \} \quad \text{and} \quad C_n^\text{eb} = \sqrt{k_n} \{ E[S_n^\text{eb}] - \varphi(\cdot) \psi_p(\theta) \}. \]

By Condition 3.3(iii), \( C_n^\text{eb} = o(1) \), and a straightforward calculation yields
\[ A_n^\text{eb} = \int_0^\infty W_n(x) \, d\hat{H}_n^\text{eb}(x), \]
where \( \hat{H}_n^\text{eb} \) denotes the empirical c.d.f. of \( Z_n^1, \ldots, Z_n^{n-b_n+1} \). The sum \( A_n^\text{eb} + B_n^\text{eb} \) can be treated as in the proof of Proposition 7.2, where the main result, Lemma 7.9, needs to be replaced by Lemma 7.24, whose proof is omitted for the sake of brevity. \( \Box \)
Lemma 7.24. (a) For any $x_1, \ldots, x_m \in (0, \infty)$, as $n \to \infty$,
\[
(e_n(x_1), \ldots, e_n(x_m), B_{n}^{\text{sh}}) \xrightarrow{d} (e(x_1), \ldots, e(x_m), B^{\text{sh}}) \sim N_{m+1}(0, \Sigma^{\text{sh}}_{m+1}),
\]
where all entries of $\Sigma^{\text{sh}}_{m+1}$ are the same as those of $\Sigma_{m+1}$ in Lemma 7.22 except for the entry at position $(m+1, m+1)$, which needs to be replaced by
\[
v_p^{\text{sh}}(\theta) = 4 p^{-2} \theta^{-2/p} \int_{0}^{\infty} (1 - e^{-z}) z^{1/p - 2 \theta} \Gamma(1/p, z) \, dz - 2 \theta^{-2/p} \Gamma(1 + 1/p)^2.
\]
(b) For any $x_1, \ldots, x_m \in (0, \infty)$, as $n \to \infty$,
\[
(W_n(x_1), \ldots, W_n(x_m), B_{n}^{\text{sh}}) \xrightarrow{d} (e(x_1)x_1^{\frac{1}{p} - 1}, \ldots, e(x_m)x_m^{\frac{1}{p} - 1}, B^{\text{sh}}).
\]

7.6. Further auxiliary lemmas

Lemma 7.25. Let $A$ be a continuous function on $[0, 1]$ such that $\lim_{x \to 0} A(x)/x^\eta = 0$ for some $\eta \in (0, 1/2)$. Further, let $H_n$ and $H$ be monotone and non-negative functions on $[0, 1]$ with
\[
\lim_{n \to \infty} \sup_{x \in [0, 1]} \frac{1}{x^{1-\eta}} \, dH_n(x) < \infty \quad \text{and} \quad \int_{[0, 1]} \frac{1}{x^{1-\eta}} \, dH(x) < \infty.
\]
If $\lim_{n \to \infty} \sup_{x \in [0, 1]} |B_n(x)| = 0$, where $B_n := H_n - H$, and if there is a sequence of measurable functions $A_n$ such that
\[
\lim_{n \to \infty} \sup_{x \in [0, 1]} \frac{|A_n(x) - A(x)|}{x^{\eta}} = 0,
\]
then we have
\[
\lim_{n \to \infty} \int_{[0, 1]} \frac{A_n(x)}{x} \, dB_n(x) = 0.
\]

Proof. For $r \in \mathbb{N}$ define the piecewise constant function
\[
\bar{A}_r(x) := \sum_{k=1}^{r} I_{[\frac{k}{r}, \frac{k+1}{r})} A(k/r)
\]
as an approximation of $A(x)/x$. We write $\int_{[0, 1]} A_n(x)/x \, dB_n(x) = I_{n1} + I_{n2} + I_{n3}$, where
\[
I_{n1} = \int_{[0, 1]} \frac{A_n(x) - A(x)}{x} \, dB_n(x), \quad I_{n2} = \int_{[0, 1]} \frac{A(x)}{x} - \bar{A}_r(x) \, dB_n(x),
\]
\[
I_{n3} = \int_{[0, 1]} \bar{A}_r(x) \, dB_n(x).
\]
The first integral is bounded by
\[
\int_{[0, 1]} \left| \frac{A_n(x) - A(x)}{x} \right| \, d(H_n + H)(x) \leq \sup_{x \in [0, 1]} \left| \frac{A_n(x) - A(x)}{x^{\eta}} \right| \int_{[0, 1]} \frac{1}{x^{1-\eta}} \, d(H_n + H)(x),
\]
which converges to zero by assumption. Regarding $I_{n2}$, we obtain
\[
|I_{n2}| = \left| \int_{[0, 1]} A(x)/x - \bar{A}_r(x)/x^{\eta} \, \frac{1}{x^{1-\eta}} \, dB_n(x) \right|
\]
\[ \leq \sup_{x \in [0,1]} \left| \frac{A(x) - \tilde{A}_r(x)x}{x^\eta} \right| \int_{[0,1]} \frac{1}{x^{1-\eta}} \, d(H_n + H)(x). \tag{7.28} \]

By uniform continuity of \( x \mapsto A(x)/x^\eta \) on \([0,1]\), we have
\[ \sup_{x \in [0,1]} \left| \frac{A(x) - \tilde{A}_r(x)x}{x^\eta} \right| \to 0 \text{ for } r \to \infty. \]

Thus, the limes superior (for \( n \to \infty \)) of the expression on the right-hand side of (7.28) can be made arbitrarily small by increasing \( r \). Finally, we can bound \(|I_{n3}|\) as follows
\[ |I_{n3}| \leq \sum_{k=1}^{r} \frac{|A(k/r)|}{k/r} \int_{[0,1]} \mathbb{I}\left(\frac{x-1}{r} \right) dB_n(x) = \sum_{k=1}^{r} \frac{|A(k/r)|}{k/r} \left| B_n\left(\frac{k}{r}\right) - B_n\left(\frac{k-1}{r}\right) \right| \]
\[ \leq 2r^2 \sup_{x \in [0,1]} |A(x)| \sup_{x \in [0,1]} |B_n(x)|, \]
which converges to zero by assumption. \( \square \)

Lemma 7.26. Let \( X \) and \( Y \) be real-valued random variables such that \( XY \) is integrable. Then,
\[ E[XY] = \int_{0}^{\infty} \int_{0}^{\infty} P(X > x, Y > y) \, dx \, dy + \int_{-\infty}^{0} \int_{-\infty}^{0} P(X \leq x, Y \leq y) \, dx \, dy \]
\[ - \int_{-\infty}^{0} \int_{0}^{\infty} P(X > x, Y \leq y) \, dx \, dy - \int_{0}^{\infty} \int_{-\infty}^{0} P(X \leq x, Y > y) \, dx \, dy. \]

Proof. This is a standard calculation based on Fubini’s theorem. \( \square \)

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