Global Well-posedness and Global Attractor for Two-dimensional Zakharov–Kuznetsov Equation

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Abstract The initial value problem for two-dimensional Zakharov–Kuznetsov equation is shown to be globally well-posed in $H^s(\mathbb{R}^2)$ for all $\frac{5}{3} < s < 1$ via using $I$-method in the context of atomic spaces. By means of the increment of modified energy, the existence of global attractor for the weakly damped, forced Zakharov–Kuznetsov equation is also established in $H^s(\mathbb{R}^2)$ for $\frac{10}{7} < s < 1$.

Keywords Zakharov–Kuznetsov equation, global well-posedness, global attractor, $I$-method, atomic spaces

MR(2010) Subject Classification 35Q53, 35B41, 35A01, 35Q35, 35B45

1 Introduction and Main Results

We consider the Cauchy problem for the symmetrized two-dimensional Zakharov–Kuznetsov (ZK) equation

$$\begin{cases}
  u_t + (\partial_x^3 + \partial_y^3)u + (\partial_x + \partial_y)u^2 = 0, & (x, y) \in \mathbb{R}^2, \quad t \geq 0, \\
  u(x, y, 0) = u_0(x, y) \in H^s(\mathbb{R}^2),
\end{cases} \quad (1.1)$$

where $u = u(x, y, t)$ is a real-valued function.

The ZK equation was initially deduced as a model of nonlinear unidirectional ion-acoustic wave propagation in a magnetized plasma by Zakharov and Kuznetsov [19]. It may be treated as a higher dimensional generalization of the Korteweg–de Vries (KdV) equation. For more details we refer to the papers [20, 21] about the two-dimensional ZK equation appearing here in physical circumstances.

Even though the ZK equation is not completely integrable, there still exist two conserved quantities for the flow of the ZK,

$$M(u)(t) = \int_{\mathbb{R}^2} u^2(x, y, t)dxdy = \int_{\mathbb{R}^2} u_0^2(x, y)dxdy = M(u_0) \quad (1.2)$$

and

$$E(u)(t) = \int_{\mathbb{R}^2} \frac{1}{2} |\nabla u|^2 - \frac{1}{2} \partial_x u \partial_y u - \frac{1}{3} u^3dxdy = E(u_0). \quad (1.3)$$
Faminskii [6] firstly obtained the local well-posedness for the two-dimensional ZK equation in the energy space $H^1(\mathbb{R}^2)$ by making use of local smoothing effects together with a maximal function estimate for the linearized equation. This method was inspired by Kenig, Ponce and Vega who dealt with the local well-posedness for the KdV equation in [13]. With the help of the $L^2$ and $H^1$ conservation laws, he proved global well-posedness for the ZK equation additionally. Following this idea, Linares and Pastor [22] optimized the proof of Faminskii to show local well-posedness in $H^s(\mathbb{R}^2)$ for $s > \frac{3}{4}$. Grünrock and Herr [9] along with Molinet and Pilod [24] proved local well-posedness in a larger data space $H^s(\mathbb{R}^2)$ for $s > \frac{1}{2}$ by taking advantage of the Fourier restriction norm method and a kind of sharp Strichartz estimates. On the basis of the method of Grünrock and Herr, in [28] we recently improved the local well-posedness result to $B^s_{2,1}(\mathbb{R}^2)$ via applying frequency decomposition as well as atomic spaces introduced by Koch and Tataru. Actually, it was Molinet and Pilod who originally applied this crucial tool of atomic spaces to the ZK equation. They obtained the global well-posedness for ZK equation in $H^s(\mathbb{R}^3)$ for $s > 1$ by using the conservation laws, doubling time, the argument of Bourgain (see [2]) and atomic spaces. As the other part of the same paper [28] we utilized the $I$-method to prove global well-posedness in $H^s(\mathbb{R}^2)$ for $\frac{11}{13} < s < 1$.

As mentioned above, ZK equation is an asymptotic description of the propagation of non-linear ion-acoustic wave in a magnetized plasma. However, one cannot totally neglect external excitation and the energy dissipation mechanism in reality (see [1], [26] and [32]). Therefore, we would like to consider the following Zakharov–Kuznetsov equation with weak dissipation and forcing terms

$$
\begin{cases}
  u_t + (\partial_x^2 + \partial_y^3)u + (\partial_x + \partial_y)u^2 + \gamma u = f, \quad (x, y) \in \mathbb{R}^2, \quad t \geq 0 \\
  u(x, y, 0) = u_0(x, y) \in H^s(\mathbb{R}^2),
\end{cases}
$$

(1.4)

where the damping parameter $\gamma$ is a positive constant and the external forcing term $f \in H^1(\mathbb{R}^2)$ is independent of $t$.

Global attractor is a bounded subset which is invariant by the flow and attracts all trajectories when $t$ approaches to $+\infty$ under the corresponding topology. Moise and Rosa studied the regularity of the global attractor of the weakly damped, forced KdV equation in [23]. Haraux [12] and Goubet [8] proved the asymptotic smoothing effect for dissipative wave equation and nonlinear Schrödinger equation respectively. As we know that the conservation laws will become invalid in low regularity situation, Colliander, Keel, Staffilani, Takaoka and Tao [3, 4] developed a theory of almost conserved quantities starting with energy in order to extend the local solutions globally in time for dispersive equations. Applying $I$-method to the weakly damped, forced KdV equation, Tsugawa [30] proved the existence of global attractor on Sobolev space of negative index. Prashant [27] lately showed an analogue result below energy space for the mKdV equation.

In this article, we refine the global well-posedness for the ZK equation to $\frac{11}{13} < s$. The new ingredient here is a closer investigation into how to use the symmetry and atomic space to acquire a better control of the modified energy. Whereafter, the increment of the modified energy helps us gain the existence of global weak attractor below $H^1$ for the weakly damped, forced ZK equation. The advantage of $U^2$, $V^2$ spaces is to obtain sharp estimates in the time
variable, which makes the proof simpler than when we use $X^{s,1/2+}$. For instance, as regards local well-posedness for the ZK equation in $H^{5/2}(\mathbb{R}^2)$, it corresponds to the non-admissible endpoint Strichartz estimates in some way. However this can be compensated partly by the decay brought from utilizing atomic spaces. Especially, the use of $U^2$, $V^2$ spaces will assist us to study the problem on the 2D torus or the $\mathbb{R} \times \mathbb{T}$, which may be nice as a future problem (see also [24] for ZK equation on $\mathbb{R} \times \mathbb{T}$).

We now state the main results of this paper.

**Theorem 1.1** The initial value problem (1.1) is globally well-posed in $H^s(\mathbb{R}^2)$ for $\frac{5}{7} < s < 1$. 

**Remark 1.2** The increment in $E(I_N u)$ on the right-hand of (3.24) is $N^{-\frac{1}{2}}$. One could repeat the almost conservation law argument to prove global well-posedness of (1.1) for all $\frac{1}{2} < s < 1$ upon gaining $N^{-1}$. 

**Theorem 1.3** Let $\frac{10}{11} < s < 1$. Then, there exists a semi-group $A(t)$ and maps $M_1$ and $M_2$ such that $A(t)u_0$ is the unique solution of (1.4) satisfying 

$$A(t)u_0 = M_1(t)u_0 + M_2(t)u_0,$$ 

and for $t > T_1$ 

$$\sup_{t > T_1} \|M_1(t)u_0\|_{H^1} < K$$ 

and 

$$\|M_2(t)u_0\|_{H^s} < Ke^{-\gamma(t-T_1)}$$ 

where $T_1$ depends on $\|u_0\|_{H^s}$, $\|f\|_{H^1}$ and $\gamma$, the constant $K$ depends only on $\|f\|_{H^1}$ and $\gamma$. 

From this theorem, we know that $M_1(t)$ is a bounded mapping and $M_2(t)$ converges uniformly to 0 in $H^s$. It means that the semi-group $A(t)$ is asymptotically compact in the sense of weak topology. Therefore we gain the existence of the global attractor in $H^s$ from Proposition 2.9.

**Corollary 1.4** In the sense of weak topology, the global attractor for (1.4) exists in $H^s$ for $\frac{10}{11} < s < 1$.

Note that 

$$\partial_t \int u^2 = -2 \int u(\partial_x^3 + \partial_y^3)u - 2 \int u(\partial_x + \partial_y)u^2 - 2\gamma \int u^2 + 2 \int fu$$ 

$$= -2\gamma \int u^2 + 2 \int fu.$$ 

Setting $h(t) = e^{2\gamma t}\|u\|_{L^2}^2$ and using Cauchy–Schwarz inequality in the last term, one can obtain 

$$h'(t) \leq 2e^{\gamma t}\|f\|_{L^2}\sqrt{h(t)}.$$ 

This implies that 

$$\partial_t \sqrt{h(t)} \leq e^{\gamma t}\|f\|_{L^2}.$$ 

Hence, we have 

$$\|u\|_{L^2} \leq e^{-\gamma t}\|u_0\|_{L^2} + \gamma^{-1}\|f\|_{L^2}. \quad (1.8)$$

**Remark 1.5** For the modified energy, we expect a similar exponential decay as (1.8) (see Proposition 4.4).
Remark 1.6 Yang [33] upgraded Tsugawa’s result through lowering $-\frac{3}{5}$ to $-\frac{1}{2}$ which can be seen as the critical Sobolev index for the KdV equation. However, for the ZK equation and the mKdV equation it is not easy to find good cancellation of resonant parts as the Schrödinger equation (see [5]). Concerning the existence of global attractor, there still are much work to do for these two kinds of equations.

Organization of the Paper In Section 2, we recall some useful propositions and estimates about $U^p$ and $V^p$. Then we prove global well-posedness by using $I$-method in atomic spaces in Section 3. Section 4 is devoted to a-priori estimate and the proof of Theorem 1.2. Finally, in Section 5 we prove well-posedness, the weakly continuity and Corollary 1.4.

We now list notations used throughout this paper. Let $c < 1$, $C > 3$ denote universal constants. The notation $c+$ stands for $c + \epsilon$ for some $0 < \epsilon \ll 1$. Similarly, we shall write $c− = c− \epsilon$. We put $(a) = (1 + a^2)^{\frac{1}{2}}$ for $a \in \mathbb{R}$ and fix a smooth cut-off function $\chi \in C^\infty_0([-2, 2])$ satisfying $\chi$ is even, nonnegative, and $\chi = 1$ on $[-1, 1]$. We denote spatial variables by $x, y$ and their dual Fourier variables by $\xi, \eta$. As usual, $\tau$ is the dual variable of the time $t$. Let $\tilde{f}$ denote the Fourier transform of $f$ in both time and spatial variables. Let $\hat{f}$ denote its Fourier transform only in space or in time. For $s \in \mathbb{R}$, $I_x^s$ and $I_y^s$ denote the one-dimensional Riesz-potential operators of order $-s$ with respect to spatial variable $x$ and $y$. We also write $\zeta = (\xi, \eta)$, $\lambda = (\xi, \eta, \tau)$ and $\mu = \tau - \xi^3 - \eta^3$ for brevity. We will make frequent use of the capital letters $N, N_1, N_2$ and $N_3$ which denote dyadic numbers and we write $\sum_{N \geq 1} a_N = \sum_{n \in N} a_2^n$, $\sum_{N \geq M} a_N = \sum_{n \in N; 2^n \geq M} a_2^n$ for dyadic summations.

2 Function Spaces and Estimates

In this section we introduce some properties of $U^p$ and $V^p$ spaces (see [11, 16–18]) which is another powerful tool to handle low regularity well-posedness for dispersive equations.

Let $1 \leq p < \infty$ and $\mathcal{Z}$ be the set of finite partitions $-\infty = t_0 < t_1 < \cdots < t_{K-1} < t_K = \infty$.

For any $\{t_k\}_{k=0}^K \subset \mathcal{Z}$ and $\{\phi_k\}_{k=0}^{K-1} \in L^2$ with $\sum_{k=0}^{K-1} \|\phi_k\|_2^p = 1$, $\phi_0 = 0$. We call the function $a : \mathbb{R} \rightarrow L^2$ given by

$$a = \sum_{k=1}^K 1_{[t_{k-1}, t_k)} \phi_{k-1}$$

a $U^p$-atom. The atomic space is

$$U^p = \left\{ u = \sum_{j=1}^\infty \lambda_j a_j : a_j \text{ $U^p$-atom}, \lambda_j \in \mathbb{C}, \sum_{j=1}^\infty |\lambda_j| < \infty \right\}$$

equipped with the norm

$$\|u\|_{U^p} = \inf \left\{ \sum_{j=1}^\infty |\lambda_j| : u = \sum_{j=1}^\infty \lambda_j a_j \text{ $U^p$-atom}, \lambda_j \in \mathbb{C} \right\}.$$
is finite, where we use the convention that \( v(-\infty) = \lim_{t \to -\infty} v(t) \) and \( v(\infty) = 0 \). We denote \( v \in V^p \) when \( v(\cdot) \in V^p \). Moreover, we define the closed subspace \( V^p_{rec} (V^p_{-rec}) \) of all right continuous functions in \( V^p (V^p) \).

The unitary operator \( e^{itS} : L^2 \to L^2 \) is defined to be the Fourier multiplier

\[
e^{itS} u_0(\xi, \eta) = e^{it(\xi^3 + \eta^3)} \hat{u}_0(\xi, \eta),
\]

where \( S = -\partial_x^3 - \partial_y^3 \). Let us define \( U^p_S = e^{itS} U^p \) with norm \( \|u\|_{U^p_S} = \|e^{-S} u\|_{U^p} \) and \( V^p_S = e^{itS} V^p \) with norm \( \|v\|_{V^p_S} = \|e^{-S} v\|_{V^p} \).

Given the Littlewood–Paley multipliers by

\[
\hat{P}_1 u = \chi(2|\xi|) \hat{u}
\]

and

\[
\hat{P}_N u = \psi_N(\|\xi\|) \hat{u} \quad \text{for } N \geq 2,
\]

where \( \psi(x) = \chi(x) - \chi(2x) \) and \( \psi_N = \psi(N^{-1} \cdot) \), the smooth projections according to the dispersive relationship are defined by

\[
\mathcal{F}(Q_M u)(\xi, \eta, \tau) = \psi_M(\tau) \mathcal{F} u(\xi, \eta, \tau),
\]

\[
\mathcal{F}(Q^S_M u)(\xi, \eta, \tau) = \psi_M(\tau - \xi^3 - \eta^3) \mathcal{F} u(\xi, \eta, \tau),
\]

as well as \( Q^S_{\geq M} = \sum_{N \geq M} Q^S_N \) and \( Q^S_{< M} = I - Q^S_{\geq M} \). Note that \( Q^S_M = e^{itS} Q_M e^{-itS} \).

Let’s recall some useful results in \( U^p \) and \( V^p \).

**Proposition 2.1** Let \( 1 < p < q < \infty \) and \( \frac{1}{p} + \frac{1}{q} = 1 \). We have

(i) \( U^p, V^p, V^p_{rc}, V^p_{-rc} \) are Banach spaces,

(ii) \( U^p \subset V^p_{-rc} \subset U^q \),

(iii) \( \|u\|_{U^p} = \sup_{\|v\|_{V^p} = 1} |\int (u'(t), v(t)) dt| \) if \( u \in V^p_{-rc} \) is absolutely continuous on compact interval.

**Lemma 2.2** We have

\[
\|Q^S_M u\|_{L^2(\mathbb{R}^3)} \lesssim M^{- \frac{1}{2}} \|u\|_{V^p_S} \quad (2.2)
\]

\[
\|Q^S_M u\|_{U^p_S} \lesssim \|u\|_{U^p_S}, \quad \|Q^S_{< M} u\|_{U^p_S} \lesssim \|u\|_{U^p_S} \quad (2.3)
\]

\[
\|Q^S_M u\|_{V^p_S} \lesssim \|u\|_{V^p_S}, \quad \|Q^S_{< M} u\|_{V^p_S} \lesssim \|u\|_{V^p_S} \quad (2.4)
\]

Similarly to [7, Lemma 2.3] and [31, Lemma 5.3], the extension principle also holds true for \( U^p_S \) spaces.

**Proposition 2.3** Let \( T_0 : L^2 \times \cdots \times L^2 \to L^1_{\text{loc}}(\mathbb{R}^2; \mathbb{C}) \) be an \( n \)-linear operator. Assume that for some \( 1 < p, q < \infty \)

\[
\|T_0(e^{it\phi_1}, \ldots, e^{it\phi_n})\|_{L^2_{\text{loc}}(\mathbb{R}^2)} \lesssim \prod_{j=1}^{n} \|\phi_j\|_{L^2}.
\]

Then, there exists \( T : U^p_S \times \cdots \times U^p_S \to L^q(\mathbb{R}; L^q_{x,y}(\mathbb{R}^2)) \) satisfying

\[
\|T(u_1, \ldots, u_n)\|_{L^q(\mathbb{R}; L^q_{x,y}(\mathbb{R}^2))} \lesssim \prod_{j=1}^{n} \|u_j\|_{U^p_S},
\]

such that \( T(u_1, \ldots, u_n) \)(\( t(x, y) = T_0(u_1(t), \ldots, u_n(t))(x, y) \) a.e.
Next we present the Strichartz estimates and bilinear estimates (see Section 5 in [28]).

**Lemma 2.4** Given $N_1 \geq N_2$, assume that $(q, r)$ satisfies $\frac{2}{q} + \frac{2}{r} = 1$ and $q > 3$. Let $I_{x,-}^*$ be the bilinear operator with symbol $|\xi_1 - \xi_2|^\alpha$, i.e.

$$
\mathcal{F}_{x,y}(I_{x,-}^*(f_1, f_2))(\xi, \eta) = \int_{\xi = \xi_1 + \xi_2} |\xi_1 - \xi_2|^\alpha \prod_{j=1}^2 \hat{f}_j(\xi_j, \eta_j).
$$

Then, we have

$$
\left\| \chi \left( \frac{t}{T} \right) u \right\|_{L^2(\mathbb{R}^3)} \lesssim T^{\frac{1}{2}} \| u \|_{V_3^2},
$$

(2.5)

$$
\| u \|_{L_t^1 L_x^q_y} \lesssim \| u \|_{L_t^2},
$$

(2.6)

$$
\| I_{x}^\frac{1}{2} I_y^\frac{1}{2} e^{tS} u_0 \|_{L^4(\mathbb{R}^3)} \lesssim \| u_0 \|_{L^2(\mathbb{R}^2)},
$$

(2.7)

$$
\| I_{x}^\frac{1}{2} I_y^\frac{1}{2} (P_{N_1} e^{tS} u_0, P_{N_2} e^{tS} v_0) \|_{L^2(\mathbb{R}^3)} \lesssim N_{2}^\frac{3}{2} \| P_{N_1} u_0 \|_{L^2(\mathbb{R}^2)} \| P_{N_2} v_0 \|_{L^2(\mathbb{R}^2)}.
$$

(2.8)

Moreover, (2.8) is equally valid with $x$ replaced by $y$.

**Lemma 2.5** Let $0 \leq \epsilon < \frac{1}{2}$ and $0 \leq \theta \leq 1$. Then,

$$
\| \tilde{D} e^{tS} u_0 \|_{L_t^q L_x^r} \lesssim \| u_0 \|_{L_t^2},
$$

(2.9)

where $\tilde{D} u = |\xi + \eta|^\epsilon \hat{u}(\xi, \eta)$, $q = \frac{6}{2 + (2 + \epsilon)}$ and $r = \frac{2}{1 - \epsilon}$.

**Proof** Set $\tilde{I}_t(x, y) = \int_{\mathbb{R}^2} |\xi + \eta|^{\epsilon + i\delta} e^{it(\xi^3 + \eta^3 + (x\xi + y\eta))} d\xi d\eta$. Let $\xi' = 4^{-\frac{1}{3}}(\xi + \eta)$ and $\eta' = \sqrt{3} 4^{-\frac{1}{3}}(\xi - \eta)$. Then we get

$$
\tilde{I}_t(x, y) = c I_t(2^{-\frac{1}{3}}(x + y), 2^{-\frac{1}{3}}3^{-\frac{1}{3}}(x - y)),
$$

where $I_t(x, y)$ is defined as [22, Lemma 2.1]. This implies

$$
\tilde{I}_t(x, y) \lesssim |t|^{-\frac{2\epsilon}{2 + \epsilon}}.
$$

Hence, (2.9) is a direct consequence of interpolation theorem and standard Stein–Thomas argument.

We can avail ourselves of the technique in [25] to decompose the time cut-off into low- and high-frequency parts.

For any $\delta > 0$, we write $1_\delta$ the characteristic function on $[0, \delta]$ and

$$
1_\delta = 1_{\delta, \kappa}^{\text{low}} + 1_{\delta, \kappa}^{\text{high}}, \quad \tilde{1}_{\delta, \kappa}^{\text{low}}(\tau) = \chi(\tau/\kappa) \tilde{1}_\delta(\tau)
$$

for some $\kappa > 0$. \qed

**Lemma 2.6** For any $\kappa, \delta > 0$, it holds

$$
\| 1_{\delta, \kappa}^{\text{high}} \|_{L^2(\mathbb{R})} \lesssim \delta^{\frac{1}{2}} \kappa^{-\frac{1}{4}},
$$

(2.10)

$$
\| 1_{\delta, \kappa}^{\text{high}} \|_{L^\infty} \lesssim 1,
$$

(2.11)

$$
\| 1_{\delta, \kappa}^{\text{low}} \|_{L^\infty} \lesssim 1.
$$

(2.12)

**Definition 2.7** ([29, Definition 1.2]) Let $X$ be a Banach space and $A(t)$ be a semigroup. An attractor is a set $\mathcal{A} \subset X$ that enjoys the following properties:

(i) $\mathcal{A}$ is an invariant set $(A(t)\mathcal{A} = \mathcal{A}, \forall t \geq 0)$,
(ii) $\mathcal{A}$ possesses an open neighborhood $\mathcal{U}$ such that, for every $u_0$ in $\mathcal{U}$, $A(t)u_0$ converges to $\mathcal{A}$ as $t \to \infty$:
\[ \text{dist}(A(t)u_0, \mathcal{A}) \to 0 \quad \text{as} \quad t \to \infty. \]

**Definition 2.8 ([29])** We say that $\mathcal{A} \subset X$ is a global (or universal) attractor for the semigroup $\{A(t)\}_{t \geq 0}$ if $\mathcal{A}$ is a compact attractor that attracts the bounded sets of $X$ (and its basin of attraction is then all of $X$).

**Proposition 2.9 ([29])** Let $X$ be a metric space. Suppose that the semi-group $A(t)$ is from $X$ into itself for $t \geq 0$ and it is asymptotically compact which means that for every bounded sequence $\{x_k\}$ in $X$ and every sequence $t_k \to \infty$, $\{A(t_k)x_k\}_k$ is relatively compact in $X$. Assume that there exists an open set $\mathcal{U}$ and a bounded set $\mathcal{B}$ of $\mathcal{U}$ such that $\mathcal{B}$ is absorbing (see [29, Definition 1.4]) in $\mathcal{U}$.

Then the $\omega$-limit set of $\mathcal{B}$, $\omega = \omega \mathcal{B}$, is a compact attractor which attracts the bounded sets of $\mathcal{U}$. It is the maximal bounded attractor in $\mathcal{U}$ (for the inclusion relation).

### 3 Global Well-posedness

We recall the $I$-method. Given $m : \mathbb{R}^2 \to \mathbb{C}$, $m$ is said to be symmetric if
\[ m(\zeta_1, \ldots, \zeta_k) = m(\sigma(\zeta_1, \ldots, \zeta_k)) \]
for all $\sigma \in S_k$, where $S_k$ is the permutation group for $k$ elements. The symbol $m$ can be symmetrized as follows
\[ [m]_{\text{sym}}(\zeta_1, \ldots, \zeta_k) = \frac{1}{k!} \sum_{\sigma \in S_k} m(\sigma(\zeta_1, \ldots, \zeta_k)). \]

We define a $k$-linear functional acting on $k$ functions $u_1, \ldots, u_k$ for each $m$
\[ \Lambda_k(m; u_1, \ldots, u_k) = \int_{\zeta_1 + \ldots + \zeta_k = 0} m(\zeta_1, \ldots, \zeta_k) \hat{u}_1(\zeta_k) \cdots \hat{u}_k(\zeta_k). \]

Usually we write $\Lambda_k(m)$ instead of $\Lambda_k(m; u, \ldots, u)$ for convenience. Note that $\Lambda_k(m) = \Lambda_k([m]_{\text{sym}})$ from symmetries.

Given $s < 1$, $N \gg 1$ and a smooth, radially symmetric, non-increasing function $m(\zeta)$ satisfying
\[ m(\zeta) = \begin{cases} 1, & |\zeta| \leq N, \\ \left( \frac{|\zeta|}{N} \right)^{s-1}, & |\zeta| \geq 2N, \end{cases} \]
we define the Fourier multiplier operator
\[ \hat{I}f(\zeta) = m(\zeta) \hat{f}(\zeta). \]

For $\sigma > 0$ and $N' = \frac{N}{\sigma}$, we define
\[ m'(\zeta) = \begin{cases} 1, & |\zeta| \leq N', \\ \left( \frac{|\zeta|}{N'} \right)^{s-1}, & |\zeta| \geq 2N', \end{cases} \]
and the scaling operator
\[ \hat{I}'f(\zeta) = m'(\zeta) \hat{f}(\zeta). \]
Without causing confusion, sometimes we denote \( m(N_j) = m(\zeta)|_{|\zeta|=N_j} \) for simplicity.

Note that from
\[
\left| \int \partial_x u \partial_y u \, dx \, dy \right| \leq \frac{1}{2} \int |\nabla u|^2 \, dx \, dy,
\]
it is easy to prove that \( \|u\|_{H^s} \) and \( E(Iu)(t) \) are comparable as [28, Proposition 3.2].

**Proposition 3.1** Let \( \frac{2}{5} < s < 1 \). Then
\[
\|E(Iu)(t)\| \lesssim N^{2(1-s)}\|u(t)\|_{H^s(\mathbb{R}^2)}^2 + \|u(t)\|_{L^3(\mathbb{R}^2)}^3,
\]
\[
\|u(t)\|_{H^s(\mathbb{R}^2)}^2 \lesssim |E(Iu)(t)| + \|u_0\|_{L^2(\mathbb{R}^2)}^2 + \|u_0\|_{L^2(\mathbb{R}^2)}.
\]

We start from the local existence theorem. The work space is denoted by \( Y^s \) which can be defined via the norm
\[
\|u\|_{Y^s} = \left( \sum_N N^{2s} \|P_N u\|_{U^s_3}^2 \right)^{\frac{1}{2}}.
\]

Obviously, \( Y^s \subset L^\infty H^s \) holds true.

**Lemma 3.2** Let \( N_1, N_2 \) and \( N_3 \) be dyadic numbers and \( N_2 \lesssim N_1 \). Assume that \( |\xi_k| \sim N_k \) \((k = 1, 2)\) and \( |\xi_1 + \xi_2| \sim N_3 \). Denote \( \Omega_1 = \{\lambda_1 + \lambda_2 = \lambda, |\eta| \leq |\xi|, |\xi| \lesssim |\xi_1 - \xi_2|\} \) and \( \Omega_2 = \{\lambda_1 + \lambda_2 = \lambda, |\eta| \leq |\xi|, |\xi| \sim |\xi_1| \lesssim |\eta_1| \sim |\eta_2|\} \). We have

(i) if \( N_1 \sim N_3 \gg N_2 \), then
\[
\left\| \int_{\Omega_1} \frac{(\xi + \eta) m(\xi_1 + \xi_2)}{m(\xi_1) m(\xi_2)} \tilde{u}_{N_1}(\lambda_1) \tilde{u}_{N_2}(\lambda_2) d\lambda_1 \right\|_{L^3_x(\mathbb{R}^3)} \lesssim N_2^{-s} \|u_{N_1}\|_{U^s_3} \|v_{N_2}\|_{U^s_3},
\]

(ii) if \( N_1 \sim N_2 \gg N_3 \), then
\[
\left\| \int_{\Omega_1} \frac{(\xi + \eta) m(\xi_1 + \xi_2)}{m(\xi_1) m(\xi_2)} \tilde{u}_{N_1}(\lambda_1) \tilde{u}_{N_2}(\lambda_2) d\lambda_1 \right\|_{L^3_x(\mathbb{R}^3)} \lesssim N_1^{\frac{1}{2}} C(N_1, N_3) \|u_{N_1}\|_{U^s_3} \|v_{N_2}\|_{U^s_3}
\]

for \( k = 1, 2 \), where
\[
C(N_1, N_3) = \begin{cases} 1, & N \gg N_1 \sim N_2 \gg N_3, \\ \left( \frac{N_1}{N} \right)^{2-2s}, & N_1 \sim N_2 \gg N \gg N_3, \\ N^{s-1} N_1^{2-2s} N_3^{-s}, & N_1 \sim N_2 \gg N_3 \gg N, \\ N_1^{-1-s}, & N_1 \sim N_2 \sim N_3 \gg N_3. \end{cases}
\]

**Proof** It suffices to show the estimates for free solutions by Proposition 2.3. To this end, we can prove it along the same line as the proof of [28, Lemma 5.3].

If \( N_1 \sim N_3 \gg N_2 \), it is obvious that
\[
\left| \frac{m(\xi_1 + \xi_2)}{m(\xi_1) m(\xi_2)} \right| \lesssim 1 \lor \left( \frac{N_2}{N} \right)^{1-s} \lesssim N_2^{1-s}.
\]

Besides, we have \( |\xi + \eta| \lesssim |\xi| \lesssim |\xi|^\frac{1}{2} |\xi_1 - \xi_2|^\frac{1}{2} \) on \( \Omega_1 \). Therefore, (2.8) gives
\[
\left\| \int_{\Omega_1} \frac{(\xi + \eta) m(\xi_1 + \xi_2)}{m(\xi_1) m(\xi_2)} \mathcal{F}(e^{iS}u_{0,N_1})(\lambda_1) \mathcal{F}(e^{iS}v_{0,N_2})(\lambda_2) d\lambda_1 \right\|_{L^3_x(\mathbb{R}^3)} \lesssim N_1^{\frac{1}{2}} \left\| \int_{\Omega_1} |\xi|^\frac{1}{2} |\xi_1 - \xi_2|^\frac{1}{2} \mathcal{F}(e^{iS}u_{0,N_1})(\lambda_1) \mathcal{F}(e^{iS}v_{0,N_2})(\lambda_2) d\lambda_1 \right\|_{L^3_x(\mathbb{R}^3)}
\]
Lemma 3.3 
Proof gives $G.W.P.$ and Attractor for $2$ | $0$

We assume that
\begin{align}
    \frac{m(\zeta_1 + \zeta_2)}{m(\zeta_1)m(\zeta_2)} &\lesssim \frac{1}{N} + R \lesssim S_2 \lesssim 1, \\
    \|v_0\|_{L^2} &\lesssim N, \\
    \int \left| \int_{\Omega_1} \xi \cdot \frac{\partial}{\partial x} \right| N \lesssim \frac{1}{N} + R \lesssim S_2 \lesssim 1.
\end{align}

From Hölder’s inequality and (2.7), we get
\begin{align}
    &\lesssim N^{\frac{3}{2}-s} \|u_0\|_{U^2_3} \|v_0\|_{U^2_3} \|w_0\|_{V^2_3}, \quad (3.3) \\
    &\lesssim \frac{1}{N} + R \lesssim S_2 \lesssim 1, \\
    &\lesssim \frac{1}{N} + R \lesssim S_2 \lesssim 1.
\end{align}

Additionally, the estimate holds true on $\Omega_1$ in a similar way. Hence, the proof is complete.

Lemma 3.3 
Let $N_1, N_2, N_3$ be dyadic numbers and $C(N_1, N_3)$ be given as in Lemma 3.2.

Then, for all $0 < \delta \leq 1$ it holds
\begin{enumerate}
    \item if $N_1 \sim N_3 \gg N_2$,
        \begin{align}
            \left| \int_{\mathbb{R}^3} \chi \left( \frac{t}{\delta} \right) I(u_{N_1} v_{N_2}) (\partial_x + \partial_y) w_{N_3} dxdydt \right| &\lesssim \delta^\frac{1}{2} N^{\frac{3}{2}-s} \|u_{N_1}\|_{U^2_3} \|v_{N_2}\|_{U^2_3} \|w_{N_3}\|_{V^2_3}, \quad (3.3) \\
            \left| \int_{\mathbb{R}^3} \chi \left( \frac{t}{\delta} \right) I(u_{N_1} v_{N_2}) (\partial_x + \partial_y) w_{N_3} dxdydt \right| &\lesssim \delta^\frac{1}{2} N^{\frac{3}{2}-s} \|u_{N_1}\|_{U^2_3} \|v_{N_2}\|_{U^2_3} \|w_{N_3}\|_{V^2_3}. \quad (3.4)
        \end{align}

    \item if $N_1 \sim N_2 \gg N_3$,
        \begin{align}
            \left| \int_{\mathbb{R}^3} \chi \left( \frac{t}{\delta} \right) I(u_{N_1} v_{N_2}) (\partial_x + \partial_y) w_{N_3} dxdydt \right| &\lesssim \delta^\frac{1}{2} N^{\frac{3}{2}-s} \|u_{N_1}\|_{U^2_3} \|v_{N_2}\|_{U^2_3} \|w_{N_3}\|_{V^2_3}, \quad (3.3) \\
            \left| \int_{\mathbb{R}^3} \chi \left( \frac{t}{\delta} \right) I(u_{N_1} v_{N_2}) (\partial_x + \partial_y) w_{N_3} dxdydt \right| &\lesssim \delta^\frac{1}{2} N^{\frac{3}{2}-s} \|u_{N_1}\|_{U^2_3} \|v_{N_2}\|_{U^2_3} \|w_{N_3}\|_{V^2_3}. \quad (3.4)
        \end{align}

Proof 
As the technique used in Proposition 5.4 of [28] is applicable here, we just give a sketch to avoid being lengthy and tedious.

 Parseval formula shows that it suffices to control
\begin{align}
    \int \Sigma \xi = \sum_{j=1}^{3} \chi \left( \frac{t}{\delta} \right) I_{\lambda_j}(\xi_j) \frac{m(\zeta_1 + \zeta_2)}{m(\zeta_1)m(\zeta_2)} \tilde{u}_N(\lambda_1) \tilde{v}_N(\lambda_2) \tilde{w}_N(\lambda_3) \chi w_N(\lambda_3)
\end{align}

by $\|u_{N_1}\|_{U^2_3}, \|v_{N_2}\|_{U^2_3}$ and $\|w_{N_3}\|_{V^2_3}$.

We assume that $|\eta_3| \leq |\xi_3|$ and denote the absolute value of the quantity above by $R$.

If $N_1 \sim N_3 \gg N_2$, then $|\xi_3| \lesssim |\xi_1 - \xi_2|$. Applying Cauchy–Schwarz inequality, (3.1) and (2.5) gives
\begin{align}
    R \lesssim \left| \int \Sigma \chi \left( \frac{t}{\delta} \right) I_{\lambda_j}(\xi_j) \frac{m(\zeta_1 + \zeta_2)}{m(\zeta_1)m(\zeta_2)} \tilde{u}_N(\lambda_1) \tilde{v}_N(\lambda_2) \tilde{w}_N(\lambda_3) |d\lambda_1| \right| \|w_N\|_{L^2(R^3)} \|w_{N_3}\|_{V^2_3}
\end{align}

So as to prove (3.4), we need to split the domain of the integration into five regions $R = R_1 + R_2 + R_3 + R_4 + R_5$. One can assume $|\eta_1| \geq |\eta_2|$ by symmetry.

Region 1 
$|\xi_3| \lesssim |\xi_1 - \xi_2|$. 


Using (3.2), we have

\[
R_1 \lesssim \left| \int_\Omega (\xi + \eta) \frac{m(\zeta_1 + \zeta_2)}{m(\zeta_1)m(\zeta_2)} \bar{u}_N(\lambda_1) \bar{v}_N(\lambda_2) d\lambda_1 \right|_{L^2(\mathbb{R}^3)} \left| \chi \left( \frac{t}{\delta} \right) w_N \right|_{L^2(\mathbb{R}^3)} \\
\lesssim \delta^\frac{1}{2} N_1^{\frac{3}{2}} C(N_1, N_3) \| u_{N_1} \|_{U_2^2} \| v_{N_2} \|_{U_2^2} \| w_{N_3} \|_{V_2^2}.
\]

Region 2 \quad |\xi_3| \gg |\xi_1 - \xi_2| \quad \text{and} \quad |\eta_1| \gg |\xi_3|.

This condition gives

\[
|\eta_1| \sim |\eta_2| \gg |\xi_3| \sim |\xi_1| \sim |\xi_2|.
\]

Applying Cauchy–Schwarz inequality, (3.2) and (2.5), we obtain

\[
R_2 \lesssim \left| \int_\Omega (\xi + \eta) \frac{m(\zeta_1 + \zeta_2)}{m(\zeta_1)m(\zeta_2)} \bar{u}_N(\lambda_1) \bar{v}_N(\lambda_2) d\lambda_1 \right|_{L^2(\mathbb{R}^3)} \left| \chi \left( \frac{t}{\delta} \right) w_N \right|_{L^2(\mathbb{R}^3)} \\
\lesssim \delta^\frac{1}{2} N_1^{\frac{3}{2}} C(N_1, N_3) \| u_{N_1} \|_{U_2^2} \| v_{N_2} \|_{U_2^2} \| w_{N_3} \|_{V_2^2}.
\]

Region 3 \quad |\xi_3| \gg |\xi_1 - \xi_2| \quad \text{and} \quad |\eta_1| \sim |\xi_3| \sim |\eta_2|.

We get the bound of \( R_3 \) like Region 2.

Region 4 \quad |\xi_3| \gg |\xi_1 - \xi_2| \quad \text{and} \quad |\eta_1| \sim |\xi_3| \gg |\eta_2|.

We decompose \( \text{Id} = Q_{<M}^S + Q_{\geq M}^S \) and divide the integral \( R_4 \) into eight pieces.

Case A \quad \( Q_j^S = Q_{<M}^S \) for \( j = 1, 2, 3 \).

We go a step further to decompose time into low- and high-frequency parts.

Case A (1) \quad All of these three are low-frequency.

The integral vanishes.

Case A (2) \quad At least one of these three is high-frequency.

We estimate when the first one is high-frequency

\[
\int_\Omega (\xi_3 + \eta_3) \frac{m(\zeta_1 + \zeta_2)}{m(\zeta_1)m(\zeta_2)} \mathcal{F} (1_{\delta, \kappa} u_{N_1}^S, v_{N_1}^S, v_{N_2}^S, w_{N_3}^S),
\]

where \( 1_v, 1_w \in \{ 1_{\delta, \kappa}, 1_{\delta, \kappa} \} \).

Hölder’s inequality, (2.6), Lemma 2.6 and (2.3) provide

\[
\left| \int_\Omega (\xi_3 + \eta_3) \frac{m(\zeta_1 + \zeta_2)}{m(\zeta_1)m(\zeta_2)} \mathcal{F} (1_{\delta, \kappa} u_{N_1}^S, v_{N_1}^S, v_{N_2}^S, w_{N_3}^S) \right| \\
\lesssim \| 1_{\delta, \kappa} u_{N_1}^S \|_{L^q(\mathbb{R}^3)} \| 1_v Q_{<M}^S v_{N_2}^S \|_{L^q(\mathbb{R}^3)} \| 1_w Q_{<M}^S w_{N_3}^S \|_{L^q(\mathbb{R}^3)} \\
\lesssim \| 1_{\delta, \kappa} \|_{L^q(\mathbb{R}^3)} \| Q_{<M}^S u_{N_1}^S \|_{L^q(\mathbb{R}^3)} \| Q_{<M}^S v_{N_2}^S \|_{L^q(\mathbb{R}^3)} \| Q_{<M}^S w_{N_3}^S \|_{L^q(\mathbb{R}^3)} \\
\lesssim \delta^\frac{1}{2} N_3^{-\frac{1}{2}} N_1^{-\frac{1}{2}} \| Q_{<M}^S u_{N_1}^S \|_{U_2^2} \| Q_{<M}^S v_{N_2}^S \|_{U_2^2} \| Q_{<M}^S w_{N_3}^S \|_{U_2^2} \\
\lesssim \delta^\frac{1}{2} N_3^{-\frac{1}{2}} N_1^{-\frac{1}{2}} \| u_{N_1}^s \|_{U_2^2} \| v_{N_2}^s \|_{U_2^2} \| w_{N_3}^s \|_{V_2^2},
\]

where \( u_{N_1}^s = \mathcal{F}^{-1} \hat{u}(\zeta_1), v_{N_2}^s = \mathcal{F}^{-1} \hat{v}(\zeta_2) \) and \( w_{N_3}^s = \mathcal{F}^{-1} (\xi_3 + \eta_3) m(\zeta_3) \hat{w}(\zeta_3) \).

From the definitions of \( U^2 \) and \( V^2 \), there hold the facts

\[
\| u_{N_1}^s \|_{U_2^2} \lesssim \frac{\| u_{N_1} \|_{U_2^2}}{m(N_1)}, \quad \| v_{N_2}^s \|_{U_2^2} \lesssim \frac{\| v_{N_2} \|_{U_2^2}}{m(N_2)}.
\]
and
\[ \|w^*_N\|_{V_3^2} \lesssim N_3 m(N_3) \|w_N\|_{V_3^2}. \]  
(3.7)

Notice that
\[ \frac{m(N_3)}{m(N_1) m(N_2)} \lesssim C(N_1, N_3). \]  
(3.8)

Combining (3.6)–(3.8) with (3.5), one has
\[ \left| \int (\xi_3 + \eta_3) \frac{m(\xi_1 + \xi_2)}{m(\xi_1) m(\xi_2)} \mathcal{F}(\chi \delta Q_3^S u_{N_1}) \mathcal{F}(\chi_3^S v_{N_2}) \mathcal{F}(\chi_{3}^S w_{N_3}) \right| \]
\[ \lesssim \delta^\frac{1}{2} N_3^{-\frac{1}{2}} N_1^{-\frac{1}{2}} N_3^{-\frac{3}{2}} m(N_3) \|u_{N_1}\|_{U_3^2} \|v_{N_2}\|_{U_3^2} \|w_{N_3}\|_{V_3^2} \]
\[ \lesssim \delta^\frac{1}{2} N_1^{-\frac{1}{2}} C(N_1, N_3) \|u_{N_1}\|_{U_3^2} \|v_{N_2}\|_{U_3^2} \|w_{N_3}\|_{V_3^2}. \]

**Case B** \( Q_3^S = Q_{\geq M}^S \) for some \( j = 1, 2, 3 \).

We take \( Q_3^S = Q_{\geq M}^S \) for instance to give the estimate.

By using Hölder’s inequality, (2.2), (2.6) and (2.3), we get
\[ \left| \int (\xi_3 + \eta_3) \frac{m(\xi_1 + \xi_2)}{m(\xi_1) m(\xi_2)} \mathcal{F}(\chi \delta Q_3^S u_{N_1}) \mathcal{F}(\chi_3^S v_{N_2}) \mathcal{F}(\chi_{3}^S w_{N_3}) \right| \]
\[ \lesssim \|\chi \delta Q_3^S u_{N_1}\|_{L^1(\mathbb{R}^3)} \|\chi \delta Q_3^S v_{N_2}\|_{L^1(\mathbb{R}^3)} \|\chi \delta Q_3^S w_{N_3}\|_{L^2(\mathbb{R}^3)} \]
\[ \lesssim M^{-\frac{1}{2}} \|\chi \delta\|_{L^2(\mathbb{R}^3)} \|Q_3^S u_{N_1}\|_{L^4_x L^4_y} \|Q_3^S v_{N_2}\|_{L^4_x L^4_y} \|Q_3^S w_{N_3}\|_{V_3^2} \]
\[ \lesssim \delta^\frac{1}{2} N_1^{-\frac{1}{2}} N_3^{-\frac{3}{2}} \|Q_3^S u_{N_1}\|_{U_3^2} \|Q_3^S v_{N_2}\|_{U_3^2} \|Q_3^S w_{N_3}\|_{V_3^2} \]
\[ \lesssim \delta^\frac{1}{2} N_1^{-\frac{1}{2}} C(N_1, N_3) \|u_{N_1}\|_{U_3^2} \|v_{N_2}\|_{U_3^2} \|w_{N_3}\|_{V_3^2}. \]

**Region 5** \( |\xi_3| \gg |\xi_1 - \xi_2| \) and \( |\xi_3| \gg |\eta_1| \geq |\eta_2|. \)

This can be dealt with in the same way as Region 4.

Next, we turn to the estimate of the nonlinear term.

**Proposition 3.4** Let \( \frac{5}{6} < s < 1 \) and \( 0 < \delta \leq 1 \). We have
\[ \left\| \int_0^t e^{(t-t')^s} \chi \left( \frac{t}{\delta} \right) (\partial_x + \partial_y) I(uv)(t') dt' \right\|_{Y^1} \lesssim \delta^\frac{1}{2} \|Iu\|_{Y^1} \|Iv\|_{Y^1}. \]  
(3.9)

**Proof** From symmetry and definition of \( Y^1 \) we need to consider the following two terms
\[ J_1 = \sum_{N_3} N_3^2 \left\| \sum_{N_1 \gg N_2} \int_0^t e^{(t-t')^s} \chi \left( \frac{t}{\delta} \right) (\partial_x + \partial_y) P_{N_3} I(u_{N_1} v_{N_2})(t') dt' \right\|^2_{U_3^2} \]
and
\[ J_2 = \sum_{N_3} N_3^2 \left\| \sum_{N_1 \sim N_2} \int_0^t e^{(t-t')^s} \chi \left( \frac{t}{\delta} \right) (\partial_x + \partial_y) P_{N_3} I(u_{N_1} v_{N_2})(t') dt' \right\|^2_{U_3^2}. \]

Combining Proposition 2.1 (iii) with (3.3), we obtain
\[ J_1 \lesssim \sum_{N_3} N_3^2 \left( \sup_{N_1 \gg N_2} \left| \int_{\mathbb{R}^3} \chi \left( \frac{t}{\delta} \right) I(u_{N_1} v_{N_2})(\partial_x + \partial_y) w_{N_3} dxdydt \right| \right)^2. \]
From Duhamel’s principle, we get

\[ \delta \sum_{N_3} N_3^2 \left( \sum_{N_1 \gg N_2} N_2^{\frac{3}{2} - s} \| Iu_{N_1} \|_{L^2} \| Iv_{N_2} \|_{L^2} \right)^2 \]

\[ \lesssim \delta \sum_{N_1} N_1^2 \| Iu_{N_1} \|_{L^2}^2 \sum_{N_2} N_2^2 \| Iv_{N_2} \|_{L^2}^2 \sum_{N_2} N_2^{1 - 2s} \]

\[ \lesssim \delta \| Iu \|_{Y^1}^2 \| Iv \|_{Y^1}^2. \]

The second term can be controlled via (3.4) and Minkowski’s inequality,

\[ J_2 \lesssim \sum_{N_3} N_3^5 \left( \sum_{N_1 \sim N_3} \sup_{w_{N_3} \in V_3} \left| \int_{\mathbb{R}^3} e^{\left( \frac{t}{\delta} \right)} I(u_{N_1} v_{N_2}) (\partial_x + \partial_y) w_{N_3} \, dx \, dy \, dt \right| \right)^2 \]

\[ \lesssim \delta^2 \left( \sum_{N_3 \sim N_1} N_3^2 \left( N_1^{\frac{3}{2} - s} \| Iu_{N_1} \|_{L^2} \| Iv_{N_1} \|_{L^2} \right)^2 \right. \]

\[ \left. + \sum_{N_3 \ll N} N_3^2 \left( \sum_{N \ll N_1} N_1^2 \left( \frac{N_1}{N} \right)^{2 - 2s} \| Iu_{N_1} \|_{L^2} \| Iv_{N_1} \|_{L^2} \right)^2 \right) \]

\[ + \sum_{N \ll N_3} N_3^2 \left( \sum_{N_3 \ll N_1} N_1^{\frac{2}{3} - 2s} \| Iu_{N_1} \|_{L^2} \| Iv_{N_1} \|_{L^2} \right)^2 \]

\[ \lesssim \delta^2 \left( \sum_{N_1} N_1^{5 - 2s} \| Iu_{N_1} \|_{L^2}^2 \| Iv_{N_1} \|_{L^2}^2 \right) \]

\[ + N_1^{4s - 2} \left( \sum_{N \ll N_1} N_1^{\frac{2}{3} - 2s} \| Iu_{N_1} \|_{L^2} \| Iv_{N_1} \|_{L^2} \right)^2 \]

\[ + \sum_{N_3 \ll N_1} N_3^{2s} \left( \sum_{N_1} N_1^{\frac{5}{2} - 2s} \| Iu_{N_1} \|_{L^2} \| Iv_{N_1} \|_{L^2} \right)^2 \]

\[ \lesssim \delta^2 \left( \sum_{N_1} N_1^4 \| Iu_{N_1} \|_{L^2}^2 \| Iv_{N_1} \|_{L^2}^2 \right) \]

\[ + \left( \sum_{N_1} N_1^{\frac{3}{2} - s} \| Iu_{N_1} \|_{L^2} \| Iv_{N_1} \|_{L^2} \right)^2 \]

\[ \lesssim \delta^2 \| Iu \|_{Y^1}^2 \| Iv \|_{Y^1}^2. \]

Hence, we complete the proof. \( \square \)

**Proposition 3.5** Let \( \frac{5}{2} < s < 1 \). Assume \( u_0 \) satisfies \( |E(Iu_0)| \leq 1 \). Then there is a constant \( \delta = \delta(\|u_0\|_{L^2(\mathbb{R}^2)}) \) and a unique solution \( u \) to (1.1) on \([0, \delta]\), such that

\[ \| Iu \|_{Y^1} \lesssim 1, \]

where the implicit constant is independent of \( \delta \).

**Proof** Acting multiplier operator \( I \) on both sides of (1.1), one knows

\[ \partial_t Iu + (\partial_x^2 + \partial_y^2) Iu + (\partial_x + \partial_y) I(u^2) = 0. \]  

(3.10)

From Duhamel’s principle, we get

\[ Iu = \chi \left( \frac{t}{\delta} \right) e^{tS} Iu_0 - \int_0^t e^{(t-t')S} \chi \left( \frac{t}{\delta} \right) (\partial_x + \partial_y) I(u^2)(t') \, dt'. \]
Letting $\epsilon$.

Applying Proposition 3.4, one obtains

From (3.13) and (3.14), we get

According to Gagliardo–Nirenberg's inequality and Hölder's inequality, it holds

setting $Y$

It follows from the definition of $Y$

According to Equation (3.10) and integration by parts, we have

Then, the contraction mapping principle ensures the existence of local solution. Moreover,

setting $Q(\delta) \equiv \|Iu\|_{Y^1}$, the bound (3.12) yields

According to Gagliardo–Nirenberg’s inequality and Hölder’s inequality, it holds

Letting $\epsilon \ll 1$, one can obtain

This gives us that

From (3.13) and (3.14), we get

As $Q(\delta)$ is continuous in the variable $\delta$, a bootstrap argument yields $Q(\delta) \lesssim 1$, i.e. $\|Iu\|_{Y^1} \lesssim 1$, and the implicit constant is independent of $\delta$.

According to Equation (3.10) and integration by parts, we have

$$
\frac{dE(Iu)(t)}{dt} = - \int_{\mathbb{R}^2} (\Delta Iu - \partial_x \partial_y Iu + (Iu)^2) \partial_t Iudxdy
$$
Lemma 3.6  Let $N_1, N_2, N_3$ be dyadic numbers. Then, for all $0 < \delta \leq 1$ we have if $N_1 \sim N_2 \sim N_3 \gtrsim N$,

$$\left| \int_{\sum_{j=1}^{3} \lambda_j = 0} \sum_{j=1}^{3} (\xi^3_j + \eta^3_j) \left(1 - \frac{m(\zeta_2 + \zeta_3)}{m(\zeta_2)m(\zeta_3)}\right) \prod_{j=1}^{3} \mathcal{F}(\chi_{\delta} u_j)(\lambda_j) \right| \lesssim \delta^\frac{s}{2} N_1^2 \sum_{j=1}^{3} \|u_j\|_{L^2}^2;$$

(3.16)

if $N_1 \sim N_2 \gtrsim N_3, N_1 \gtrsim N$,

$$\left| \int_{\sum_{j=1}^{3} \lambda_j = 0} \sum_{j=1}^{3} (\xi_1 \xi_2 \xi_3 + \eta_1 \eta_2 \eta_3) \prod_{j=1}^{3} \mathcal{F}(\chi_{\delta} u_j)(\lambda_j) \right| \lesssim \delta^\frac{s}{2} N_1 N_3^2 \prod_{j=1}^{3} \|u_j\|_{L^2}^2$$

(3.17)

and

$$\left| \int_{\sum_{j=1}^{3} \lambda_j = 0} \sum_{j=1}^{3} (\xi^3_j + \eta^3_j) m^2(\zeta_j) \prod_{j=1}^{3} \mathcal{F}(\chi_{\delta} u_j)(\lambda_j) \right| \lesssim \delta^\frac{s}{2} N_1 N_3^2 \left(1 \vee \left(\frac{N_3}{N}\right)^{1-s}\right) \prod_{j=1}^{3} \|u_j\|_{L^2}^2.$$

(3.18)

Proof  When $N_1 \sim N_2 \sim N_3 \gtrsim N$,

$$1 - \frac{m(\zeta_2 + \zeta_3)}{m(\zeta_2)m(\zeta_3)} \lesssim \frac{m(\zeta_1)}{m(\zeta_2)m(\zeta_3)} \lesssim \left(\frac{N_1}{N}\right)^{1-s}$$

and

$$|\xi^3_j + \eta^3_j| \lesssim |\xi_1 + \eta_1| N_1^2.$$

Then one can get (3.16) as in the second part of Lemma 3.3.

In order to prove (3.17), we assume $|\xi_1 \xi_2 \xi_3| \geq |\eta_1 \eta_2 \eta_3|$ by symmetry.

Case 1  $|\xi_2| \lesssim |\xi_1 - \xi_3|.$

From (2.8), we have

$$\left| \int_{\sum_{j=1}^{3} \lambda_j = 0} (\xi_1 \xi_2 \xi_3 + \eta_1 \eta_2 \eta_3) \prod_{j=1}^{3} \mathcal{F}(\chi_{\delta} u_j)(\lambda_j) \right|$$
\[ \lesssim N_1 N_3 \left| \int \sum_{j=1}^{3} \lambda_j = \frac{|\xi_2|^2}{|\xi_1 - \xi_3|^3} \prod_{j=1}^{3} \mathcal{F}(\chi_3 u_j)(\lambda_j) \right| \]
\[ \lesssim N_1 N_3 \left\| \int |\xi_1 + \xi_3| \frac{\xi_1 - \xi_3}{\xi_1 - \xi_3} \tilde{u}_1(\lambda_1) \tilde{u}_3(\lambda_3) \right\|_{L^2(\mathbb{R}^3)} \left\| \chi \left( \frac{t}{\delta} \right) u_2 \right\|_{L^2(\mathbb{R}^3)} \]
\[ \lesssim \delta \frac{\xi_3^3}{N_3} \prod_{j=1}^{3} \|u_j\|_{U^3}. \]

**Case 2** \(|\xi_2| \gg |\xi_1 - \xi_3|\).
In this case, \(|\xi_2| \sim |\xi_1| \sim |\xi_3| \lesssim N_3 \ll N_1 \sim N_2\). Hence,
\[ |\xi_1^2 \xi_3| \lesssim N_3^2 |\xi_1|^2 |\xi_2|^2 |\eta_1|^2 |\eta_2|^2. \]

From (2.7), we have
\[ \left| \int \sum_{j=1}^{3} \lambda_j = \frac{(\xi_1 \xi_2 \xi_3 + \eta_1 \eta_2 \eta_3) \prod_{j=1}^{3} \mathcal{F}(\chi_3 u_j)(\lambda_j)}{\sum_{j=1}^{3} \lambda_j} \right| \]
\[ \lesssim \left| \int \sum_{j=1}^{3} \lambda_j = \frac{|\xi_1 \xi_2 \xi_3| \prod_{j=1}^{3} \mathcal{F}(\chi_3 u_j)(\lambda_j)}{\sum_{j=1}^{3} \lambda_j} \right| \]
\[ \lesssim N_3^2 \left| \int \sum_{j=1}^{3} \lambda_j = \frac{|\xi_1|^2 |\eta_1|^2 |\xi_2|^2 |\eta_2|^2 \prod_{j=1}^{3} \mathcal{F}(\chi_3 u_j)(\lambda_j)}{\sum_{j=1}^{3} \lambda_j} \right| \]
\[ \lesssim N_3^2 \|I_{\xi_1}^\delta I_{\eta_1}^\delta u_1\|_{L^4(\mathbb{R}^3)} \|I_{\xi_2}^\delta I_{\eta_2}^\delta u_2\|_{L^4(\mathbb{R}^3)} \left\| \chi \left( \frac{t}{\delta} \right) u_3 \right\|_{L^2(\mathbb{R}^3)} \]
\[ \lesssim \delta^3 N_1 N_3^2 \prod_{j=1}^{3} \|u_j\|_{U^3}. \]

It’s easy to see that \(N_1 \sim N_2 \gg N_3\) implies \(|\xi_1| \sim |\xi_2| \sim N_1\) or \(|\eta_1| \sim |\eta_2| \sim N_1\). Without loss of generality one can assume \(|\xi_1| \sim |\xi_2| \sim N_1\). Then, the mean-value theorem gives the bound of the symbol
\[ \sum_{j=1}^{3} (\xi_j^3 + \eta_j^3) m^2(\xi_j) \lesssim N_1^2 \max \{|\xi_3|, |\eta_3|\} m^2(\xi_1). \]

Hence, one can show (3.18) in a similar way as above.

We complete the proof of this lemma. \(\square\)

**Proposition 3.7** Let \(\frac{5}{4} < s < 1\). We have
\[ \left| \int_0^\delta \mathcal{A}_3 \left( (\xi_1^3 + \eta_1^3) \left( 1 - \frac{m(\xi_2 + \xi_3)}{m(\xi_2) m(\xi_3)} \right) I_u \right) \right| \lesssim \delta \frac{\xi_1^3}{N_1} \left\| I_u \right\|_{Y^1}. \]

**Proof** The fact
\[ \sum_{N_1} N_1^{0} - N_1 \|P_{N_1} I_u\|_{U^2} \leq \left( \sum_{N_1} N_1^{0} \right)^{\frac{1}{2}} \|I_u\|_{Y^1} \]
implies that it suffices to prove
\[ \left| \int_0^\delta \int \sum_{j=1}^{3} \xi_j = 0 \left( (\xi_1^3 + \eta_1^3) \left( 1 - \frac{m(\xi_2 + \xi_3)}{m(\xi_2) m(\xi_3)} \right) \right) \prod_{j=1}^{3} \tilde{u}_j(\xi_j) \right| \]
Estimate (3.17) tells us that

\[ \| u_j \|_{U_3^3} \]

for any function \( u_j \) \((j = 1, 2, 3)\) with frequencies supported on \(|\xi_j| \sim N_j\).

We denote \( L_1 \) the left-hand side of (3.20) and \( M(\zeta_1, \zeta_2, \zeta_3) = [(\xi_3^3 + \eta_1^3)(1 - \frac{m(\zeta_2 + \zeta_3)}{m(\zeta_1)m(\zeta_2)m(\zeta_3)})]_{\text{sym}} \).

Furthermore, one can assume \( N_1 \sim N_2 \gtrsim N_3 \) and \( N_1 \gtrsim N \).

**Case 1** \( N_1 \sim N_2 \sim N_3 \gtrsim N \).

We obtain by (3.16)

\[ L_1 \lesssim \delta^\frac{s}{2} N^{s-1} N_1^{\frac{s}{2} - s} \prod_{j=1}^{3} \| u_j \|_{U_3^3} \]

\[ \lesssim \delta^\frac{s}{2} N^{s-1} N_1^{\frac{s}{2} - s} + N_3^{1-3} \prod_{j=1}^{3} \| u_j \|_{U_3^3} \]

\[ \lesssim \delta^\frac{s}{2} N^{s-1} + N_3^{1-3} \prod_{j=1}^{3} \| u_j \|_{U_3^3}. \]

**Case 2** \( N_1 \sim N_2 \gg N_3, N_1 \gtrsim N \).

We write \( M = M_1 - M_2 \), where \( M_1 = \sum_{j=1}^{3} (\xi_j^3 + \eta_j^3) \) and \( M_2 = \sum_{j=1}^{3} \frac{(\xi_j^3 + \eta_j^3)m^2(\zeta_j)}{m(\zeta_1)m(\zeta_2)m(\zeta_3)} \). The corresponding terms are

\[ L_{1,1} = \left| \int_0^\delta \int_{\Sigma_{j=1}^{3} \xi_j = 0} M_1(\zeta_1, \zeta_2, \zeta_3) \prod_{j=1}^{3} \hat{u}_j(\zeta_j) \right| \]

and

\[ L_{1,2} = \left| \int_0^\delta \int_{\Sigma_{j=1}^{3} \xi_j = 0} M_2(\zeta_1, \zeta_2, \zeta_3) \prod_{j=1}^{3} \hat{u}_j(\zeta_j) \right|. \]

Estimate (3.17) tells us that

\[ L_{1,1} \lesssim \delta^\frac{s}{2} N_1 N_3^{\frac{3}{2}} N_1^{1-3} N_1^{1-3} \prod_{j=1}^{3} N_j^{1-3} \| u_j \|_{U_3^3} \]

\[ \lesssim \delta^\frac{s}{2} N_1 N_3^{1-3} \prod_{j=1}^{3} N_j^{1-3} \| u_j \|_{U_3^3}. \]

Using (3.18), we get the contribution of \( L_{1,2} \),

\[ L_{1,2} \lesssim \delta^\frac{s}{2} N_1 N_3^{\frac{3}{2}} \left( 1 + \left( \frac{N_3}{N} \right)^{1-s} \right) N_1^{1-3} N_1^{1-3} \prod_{j=1}^{3} N_j^{1-3} \| u_j \|_{U_3^3} \]

\[ \lesssim \delta^\frac{s}{2} N_1 N_3^{1-3} \prod_{j=1}^{3} N_j^{1-3} \| u_j \|_{U_3^3}. \]

This complete the proof of (3.20), and hence (3.19). \qed

**Proposition 3.8** Let \( \frac{5}{2} < s < 1 \). We have

\[ \left| \int_0^\delta \Lambda_4 \left( (\xi_1 + \xi_2 + \eta_1 + \eta_2) \frac{m(\zeta_1 + \zeta_2)}{m(\zeta_1)m(\zeta_2)}; \xi \right) \right| \lesssim \delta^s N^{-1} \| u \|_{U_1^j}. \] (3.21)
Proof. As previous discussion, it suffices to show
\[
\left| \int_0^\delta \int_{\sum_{j=1}^4 \lambda_j = 0} (\xi_1 + \xi_2 + \eta_1 + \eta_2) \frac{m(\xi_1) + m(\xi_2)}{m(\xi_1)m(\xi_2)} \prod_{j=1}^4 F(\chi u_j)(\lambda_j) \right|
\approx \delta^{\frac{3}{2}} N^{-1 + \sum_{j=1}^4 N_j^{1 -} \|u_j\|_{U_3^j}}. \tag{3.22}
\]

If \( N \gg N_j \) (\( j = 1, 2, 3, 4 \)), then it is easy to see \( [(\xi_1 + \xi_2) \frac{m(\xi_1) + m(\xi_2)}{m(\xi_1)m(\xi_2)}] \text{sym} = 0 \), (3.22) holds trivially. We may assume \( N_1 \geq N_2, N_3 \geq N_4 \) by symmetry and let \( L_2 \) denote the left-hand side of (3.22).

From Hölder’s inequality and (2.6), we obtain
\[
L_2 \lesssim \left| \int_0^\delta \int_{\xi = \xi_1 + \xi_2} (\xi + \eta) \frac{m(\xi)}{m(\xi_1)m(\xi_2)} \tilde{u}_1(\xi_1) \tilde{u}_2(\xi_2) d\xi_1 dt \right| \|u_3\|_{L^4} \|u_4\|_{L^4}
\lesssim \delta^{\frac{3}{2}} N_1 \left\| \tilde{u}_1 \right\|_{L^6_{x,y}} \left\| \tilde{u}_2 \right\|_{L^6_{x,y}} \lesssim \delta^{\frac{3}{2}} N_1 \left\| u_3 \right\|_{L^6_{x,y}} \left\| u_4 \right\|_{L^6_{x,y}}
\lesssim \delta^{\frac{3}{2}} N_1 \left\| u_3 \right\|_{U_3^j} \left\| u_4 \right\|_{U_3^j}
\lesssim \delta^{\frac{3}{2}} N_1 \left\| u_3 \right\|_{U_3^j} \left\| u_4 \right\|_{U_3^j}
\lesssim \delta^{\frac{3}{2}} N_1 \left\| u_3 \right\|_{U_3^j} \left\| u_4 \right\|_{U_3^j}
\lesssim \delta^{\frac{3}{2}} \frac{N_1}{m(N_1)m(N_2)} \prod_{j=1}^4 \left\| u_j \right\|_{U_3^j}. \tag{3.22}
\]

Case 1. \( N_1 \ll N_3 \).

This implies \( N_3 \sim N_4 \gg N_2, N_3 \sim N_4 \gg N \). Combining the pointwise bound about the symbol
\[
\frac{N_j^{1 -}}{m(N_j)} \lesssim 1
\]
for all \( N_j \), we have
\[
L_2 \lesssim \delta^{\frac{3}{2}} N_1 N_3^{-2 +} \prod_{j=1}^4 N_j^{1 -} \left\| u_j \right\|_{U_3^j}
\lesssim \delta^{\frac{3}{2}} N^{-1 +} \prod_{j=1}^4 N_j^{1 -} \left\| u_j \right\|_{U_3^j}.
\]

Case 2. \( N_1 \gg N_3, N_1 \sim N_2 \gg N \).

Similarly to Case 1, we obtain that
\[
L_2 \lesssim \delta^{\frac{3}{2}} \frac{N_1}{m(N_1)m(N_2)} N_1^{-2 +} \prod_{j=1}^4 N_j^{1 -} \left\| u_j \right\|_{U_3^j}
\lesssim \delta^{\frac{3}{2}} N_1 \left( \frac{N_1}{N} \right)^{(1-s)} N_1^{-2 +} \prod_{j=1}^4 N_j^{1 -} \left\| u_j \right\|_{U_3^j}
\lesssim \delta^{\frac{3}{2}} N^{-1 +} \prod_{j=1}^4 N_j^{1 -} \left\| u_j \right\|_{U_3^j}.
\]
Case 3 \( N_1 \gtrsim N_3, N_1 \gg N_2 \).

In this case, \( N_1 \sim N_3 \gtrsim N \), we control \( L_2 \) by using (3.1),
\[
L_2 \lesssim \left\| \int_{\lambda = \lambda_1 + \lambda_2} (\xi + \eta) \frac{m(\zeta_1 + \zeta_2)}{m(\zeta_1)m(\zeta_2)} \bar{u}_1(\lambda_1) \bar{u}_2(\lambda_2) d\lambda_1 \right\|_{L^2} \left\| \chi_\delta u_3 \right\|_{L^4} \left\| \chi_\delta u_4 \right\|_{L^4}
\]
\[
\lesssim \delta^\frac{1}{4} N_2^\frac{1}{2} \prod_{j=1}^4 \| u_j \|_{U_3^j}
\]
\[
\lesssim \delta^\frac{1}{3} N_2^\frac{1}{2} \left( 1 \vee \left( \frac{N_2}{N} \right)^{1-s} \right) N_1^{-2} N_2^{-1} \prod_{j=1}^4 N_1^{-1} \| u_j \|_{U_3^j}
\]
\[
\lesssim \delta^\frac{1}{3} N^{-2} \prod_{j=1}^4 N_1^{-1} \| u_j \|_{U_3^j}.
\]
Therefore we complete the proof of (3.22). \( \square \)

It’s not hard to obtain the following proposition in a similar way.

**Proposition 3.9** Let \( \frac{5}{7} < s < 1 \). We have
\[
\left| \int_0^\delta \Lambda_3 \left( (\xi_3 + \eta_3) \frac{m(\zeta_1 + \zeta_2)}{m(\zeta_1)m(\zeta_2)}; iu \right) \right| \lesssim \delta^\frac{2}{7} N^{-2} \| iu \|_{Y_1}^3.
\]

**Proposition 3.10** Let \( \frac{5}{7} < s < 1, N \gg 1 \). Assume \( u_0 \) satisfies \( |E(Iu_0)| \leq 1 \). Then there is a constant \( \delta = \delta(\| u_0 \|_{L^2(\mathbb{R}^2)}) > 0 \) so that there exists a unique solution
\[
\text{for } (1.1) \text{ satisfying }
\]
\[
E(I_N u)(\delta) = E(I_N u)(0) + \delta^\frac{1}{7} O(N^{-\frac{1}{7}+}).
\]

**Proof** From Proposition 3.5, there exists a unique solution \( u \) to (1.1) on \([0, \delta] \) satisfying \( \| iu \|_{Y_1} \lesssim 1 \).

Combining (3.15), Proposition 3.7 and Proposition 3.8, one has
\[
|E(Iu)(\delta) - E(Iu)(0)| = \left| \int_0^\delta \Lambda_3 \left( (\xi_1^2 + \eta_1^2) \left( 1 - \frac{m(\zeta_1 + \zeta_3)}{m(\zeta_1)m(\zeta_3)} \right); iu \right) \right|
\]
\[
+ \left| \int_0^\delta \Lambda_4 \left( (\xi_1 + \xi_2 + \eta_1 + \eta_2) \frac{m(\zeta_1 + \zeta_2)}{m(\zeta_1)m(\zeta_2)}; iu \right) \right|
\]
\[
\lesssim \delta^\frac{1}{7} N^{-\frac{1}{7}+} \| iu \|_{Y_1}^3 + \delta^\frac{1}{7} N^{-\frac{1}{7}+} \| iu \|_{Y_1}^3
\]
\[
\lesssim \delta^\frac{1}{7} N^{-\frac{1}{7}+}.
\]

**Proof of Theorem 1.1** Our purpose is to construct a solution \( u \) on \([0, T] \) for any \( T > 0 \). Note that \( u \) is a solution to (1.1), then \( u_\lambda(x, y, t) = \lambda^2 u(\lambda x, \lambda y, \lambda^3 t) \) is also a solution with initial data \( u_{\lambda,0}(x, y) = \lambda^2 u_0(\lambda x, \lambda y) \). Hence it suffices to acquire the well-posedness for \( u_\lambda \) on \([0, \frac{T}{\lambda^3}] \).

From Proposition 3.1 the energy will be arbitrarily small by taking \( \lambda \) small,
\[
E(I_N u_{\lambda,0}) \lesssim N^{2(1-s)} \| u_{\lambda,0} \|^2_{H^s(\mathbb{R}^2)} + \| u_{\lambda,0} \|^2_{L^3(\mathbb{R}^2)}
\]
\[
\lesssim N^{2(1-s)} \lambda^{2(s+1)} \| u_0 \|^2_{H^s(\mathbb{R}^2)} + \lambda^4 \| u_0 \|^2_{L^3(\mathbb{R}^2)}
\]
\[
\lesssim (N^{2(1-s)} \lambda^{2(s+1)} + \lambda^4)(1 + \| u_0 \|^2_{H^s(\mathbb{R}^2)})^2.
\]
Assume $N \gg 1$ is given ($N$ will be chosen later), then we have

$$E(I_N u_{\lambda,0}) \leq \frac{1}{4}$$

by setting

$$\lambda = \lambda(N, \|u_0\|_{H^s(\mathbb{R}^2)}) \sim N^{\frac{4}{s+1}}.$$ 

Now applying Proposition 3.10 to $u_{\lambda,0}$, one gets

$$E(Iu_\lambda)(\delta) \leq \frac{1}{4} + CN^{-\frac{1}{2}+} < \frac{1}{2},$$

Thus from Proposition 3.5 the solution $u_\lambda$ can be extended to $t \in [0, 2\delta]$. Iterating this procedure $M$ steps, we have

$$E(Iu_\lambda)(t) \leq \frac{1}{4} + CMN^{-\frac{1}{2}+}$$

for $t \in [0, (M+1)\delta]$. That’s to say, as long as $MN^{-\frac{1}{2}+} \leq 1$ the solution $u_\lambda$ can be extended to $t \in [0, (M+1)\delta]$. Taking $N(T) \sim T^{2\left(\frac{1}{2} - \frac{1}{s+1}\right)} \gg 1$, then

$$(M+1)\delta \sim N^{\frac{1}{2}+} \sim TN^{\frac{3(s+1)}{2(s+3)}} \sim T \frac{1}{\lambda^3}.$$ 

Hence, we obtain global well-posedness for (1.1) when $s > \frac{5}{7}$. Moreover, from Proposition 3.1 we have

$$\|u(T)\|_{H^s(\mathbb{R}^2)} \lesssim \lambda^{-s-1} \left\| u_\lambda \left( \frac{T}{\lambda^3} \right) \right\|_{H^s(\mathbb{R}^2)}$$

$$\lesssim \lambda^{-s-1} \left( \left\| E(Iu_\lambda) \left( \frac{T}{\lambda^3} \right) \right\|_{H^s(\mathbb{R}^2)} + \|u_{\lambda,0}\|_{L^2(\mathbb{R}^2)}^2 + \|u_{\lambda,0}\|_{L^2(\mathbb{R}^2)}^2 \right)$$

$$\lesssim \lambda^{-s-1}(1 + \|u_0\|_{H^s(\mathbb{R}^2)})^2$$

$$\lesssim T^{\frac{2(1-s)(1+s)}{s+3}} (1 + \|u_0\|_{H^s(\mathbb{R}^2)})^2.$$ 

4 Global Attractor

This section follows Tsugawa’s idea which he applied to show the existence of global attractor for KdV equation on Sobolev spaces of negative index (see [30]).

We find a representation of the rescaled equation associated to (1.4),

$$\begin{cases}
\partial_t v + (\partial_x^2 + \partial_y^2)v + (\partial_x + \partial_y)v^2 + \gamma \lambda^{-3} v = \lambda^{-3} g, \\
v(x, y, 0) = v_0(x, y) \in H^s(\mathbb{R}^2)
\end{cases} (4.1)$$

with $v(x, y, t) = \lambda^{-2} u(\lambda^{-1} x, \lambda^{-1} y, \lambda^{-3} t), v_0(x, y) = \lambda^{-2} u_0(\lambda^{-1} x, \lambda^{-1} y)$ and $g(x, y) = \lambda^{-2} f(\lambda^{-1} x, \lambda^{-1} y)$.

From the definition of rescaled operator $I'$, one clearly knows

$$\|I'v\|_{L^2} = \lambda^{-1}\|Iu\|_{L^2}, \quad \|I'g\|_{L^2} = \lambda^{-1}\|If\|_{L^2}$$

and

$$\|I'v\|_{H^1} = \lambda^{-2}\|Iu\|_{H^1}, \quad \|I'g\|_{H^1} = \lambda^{-2}\|If\|_{H^1}.$$ 

Now, we give the time local result for weakly damped forced ZK equation.
Proposition 4.1 Let $\frac{5}{4} < s < 1$. Assume $I'v_0 \in H^1(\mathbb{R}^2)$ and $I'g \in H^1(\mathbb{R}^2)$, then there is a constant $\delta = \delta(||I'v_0||_{H^1(\mathbb{R}^2)}, \lambda^{-3}||I'g||_{H^1(\mathbb{R}^2)}, \gamma \lambda^{-3}) > 0$ so that there exists a unique solution
\[ v(x, y, t) \in C([0, \delta], H^s(\mathbb{R}^2)) \]
of (4.1) satisfying
\[ ||I'v||_{Y^1} \lesssim ||I'v_0||_{H^1(\mathbb{R}^2)} + \lambda^{-3}||I'g||_{H^1(\mathbb{R}^2)} \]
and
\[ \sup_{t \in [0, \delta]} ||I'v(t)||_{H^1} \lesssim ||I'v_0||_{H^1(\mathbb{R}^2)} + \lambda^{-3}||I'g||_{H^1(\mathbb{R}^2)}. \]

Proof Acting $I'$ on (4.1) gives
\[ \partial_t I'v + (\partial_x^2 + \partial_y^2)I'v + (\partial_x + \partial_y)I'v^2 + \gamma \lambda^{-3}I'v = \lambda^{-3}I'g, \tag{4.2} \]
which also can be written as an integral equation
\[ I'v = \mathcal{F} I'v, \]
where
\[ \mathcal{F} I'v = \chi \left( \frac{t}{\delta} \right) e^{t} I'v_0 - \int_{0}^{t} e^{(t-t')S} \chi \left( \frac{t'}{\delta} \right) ((\partial_x + \partial_y)I'v^2 + \gamma \lambda^{-3}I'v - \lambda^{-3}I'g)dt'. \]

By the duality of $U^p$, we have
\[ \left\| \int_{0}^{t} e^{(t-t')S} \chi \left( \frac{t'}{\delta} \right) (\gamma \lambda^{-3}I'v - \lambda^{-3}I'g)dt' \right\|_{Y^1} \]
\[ \lesssim \left( \sum_{N_1} N_1^2 \left( \sup_{||w||_{U^2_N} \leq 1} \int_{\mathbb{R}^2} \chi \left( \frac{t}{\delta} \right) (\gamma \lambda^{-3}P_{N_1} I'v - \lambda^{-3}P_{N_1} I'g)wdx y dt \right) \right)^{1/2} \]
\[ \lesssim \left( \sum_{N_1} N_1^2 \left( \sup_{||w||_{U^2_N} \leq 1} ||\gamma \lambda^{-3} P_{N_1} I'v - \lambda^{-3} P_{N_1} I'g||_{L^2_{t} L^2_{x,y}} \right) \right)^{1/2} \]
\[ \lesssim \delta \left( \sum_{N_1} N_1^2 (||\gamma \lambda^{-3} P_{N_1} I'v||_{U^2_N} + \lambda^{-3}||P_{N_1} I'g||_{L^2_{t} L^2_{x,y}}) \right)^{1/2} \]
\[ \lesssim \delta (\gamma \lambda^{-3} ||I'v||_{Y^1} + \lambda^{-3} ||I'g||_{H^1}). \tag{4.3} \]

Set
\[ B = \{ I'v \in Y^1 \mid ||I'v||_{Y^1} < C_0(||I'v_0||_{H^1} + \lambda^{-3}||I'g||_{H^1}) \}. \]

Applying Proposition 3.4 and (4.3), on $B$ one gets
\[ ||\mathcal{F} I'v||_{Y^1} \lesssim ||I'v_0||_{H^1} + \delta \hat{\tau} ||I'v||_{Y^1}^2 + \delta \gamma \lambda^{-3} ||I'v||_{Y^1} + \delta \lambda^{-3} ||I'g||_{H^1} \]
\[ \lesssim \delta \hat{\tau} C_0^2 (||I'v_0||_{H^1} + \lambda^{-3}||I'g||_{H^1})^2 + (1 + C_0 \delta \gamma \lambda^{-3})(||I'v_0||_{H^1} + \lambda^{-3}||I'g||_{H^1}) \]
and
\[ ||\mathcal{F} I'v_1 - \mathcal{F} I'v_2||_{Y^1} \lesssim (\delta \gamma \lambda^{-3} + \delta \hat{\tau} ||I'v_1||_{Y^1} + \delta \hat{\tau} ||I'v_2||_{Y^1}) ||I'v_1 - I'v_2||_{Y^1} \]
\[ \lesssim (\delta \gamma \lambda^{-3} + \delta \hat{\tau} C_0 (||I'v_0||_{H^1} + \lambda^{-3}||I'g||_{H^1})) ||I'v_1 - I'v_2||_{Y^1}. \]
If we assume
\[ \lambda^{-3} \gamma \ll 1, \quad \|I'v_0\|_{H^1} \ll 1, \quad \lambda^{-3}\|I'g\|_{H^1} \ll 1, \] (4.4)
then
\[ \mathcal{J} : B \to B \]
is a strict contraction mapping.

Finally, fixing \( \lambda = \lambda_0 \) and \( I'v_0, I'g \in H^1 \), we consider \( \sigma \)-scaling of \( v \)
\[ w(x, y, t) = \sigma^{-2}v(\sigma^{-1}x, \sigma^{-1}y, \sigma^{-3}t). \]
It’s equivalent to consider well-posedness on \([0, \sigma^3 \delta]\) for \( w \).
Observe that
\[ (\sigma \lambda_0)^{-3} \gamma \ll 1, \]
\[ \|I''w_0\|_{H^1} \lesssim \sigma^{-1}\|I'v_0\|_{H^1} \ll 1, \]
\[ (\sigma \lambda_0)^{-3}\|I'g\|_{H^1} \ll 1 \]
provided \( \sigma \) is chosen to be sufficiently large, which verifies (4.4). It means that the Cauchy problem for \( w \) is well-posed on the time interval \([0, 1]\). Hence, (4.2) is locally well-posed on \([0, \sigma^{-3}]\). We complete the proof. \( \square \)

In the next place, we explore the increment of \( I'v \) through modified energy \( E(I'v) \).
From (4.2), we obtain
\[
\frac{dE(I'v)(t)}{dt} = - \int_{\mathbb{R}^2} (\Delta I'v - \partial_x \partial_y I'v + (I'v)^2) \partial_t I'v dx dy
\]
\[ = \int_{\mathbb{R}^2} ((\partial^2_x + \partial^2_y)I'v + (\partial_x + \partial_y)I'v^2)((\Delta - \partial_x \partial_y)I'v + (I'v)^2) dx dy
\]
\[ + \int_{\mathbb{R}^2} (\gamma \lambda^{-3}I'v - \lambda^{-3}I'g)((\Delta - \partial_x \partial_y)I'v + (I'v)^2) dx dy
\]
\[ = -2\gamma \lambda^{-3}E(I'v) - \int_{\mathbb{R}^2} \lambda^{-3}I'g((\Delta - \partial_x \partial_y)I'v + (I'v)^2) dx dy
\]
\[ + \frac{1}{3} \gamma \lambda^{-3} \int_{\mathbb{R}^2} (I'v)^3 dx dy + \Lambda_3 \left( \xi_1^3 + \eta_1^2 \left( 1 - \frac{m(\zeta_2 + \zeta_3)}{m(\zeta_2)m(\zeta_3)} \right) ; I'v \right)
\]
\[ + \Lambda_4 \left( (\xi_1 + \xi_2 + \eta_1 + \eta_2) \frac{m(\zeta_1 + \zeta_2)}{m(\zeta_1)m(\zeta_2)} ; I'v \right). \]

This implies
\[
\frac{d}{dt} E(I'v)(t)e^{2\gamma \lambda^{-3}t} = - \int_{\mathbb{R}^2} \lambda^{-3}I'g((\Delta - \partial_x \partial_y)I'v + (I'v)^2)e^{2\gamma \lambda^{-3}t} dx dy
\]
\[ + \frac{1}{3} \gamma \lambda^{-3}\|I'v\|_{L^3}^3 e^{2\gamma \lambda^{-3}t} + (\Lambda_3 + \Lambda_4)e^{2\gamma \lambda^{-3}t}. \] (4.5)

Integrating (4.5) over \([0, T']\), one gets
\[
E(I'(T'))e^{2\gamma \lambda^{-3}T'} - E(I'(0))
\]
\[ = - \int_0^{T'} \int_{\mathbb{R}^2} \lambda^{-3}I'g((\Delta - \partial_x \partial_y)I'v + (I'v)^2)e^{2\gamma \lambda^{-3}t} dx dy dt
\]
\[ + \frac{1}{3} \gamma \lambda^{-3} \int_0^{T'} \|I'v\|_{L^3}^3 e^{2\gamma \lambda^{-3}t} dt + \int_0^{T'} (\Lambda_3 + \Lambda_4)e^{2\gamma \lambda^{-3}t} dt. \] (4.6)
Lemma 4.2  Assume that $v$ is a solution of (4.1) on $[0,T']$. Then, we have

$$\sup_{t \in [0,T']} \|I'v(t)\|_{L^2}^2 e^{2\gamma \lambda^{-3} t} \leq C_1 \left( \|I'v_0\|_{L^2}^2 + \frac{1}{\gamma^2} \|I'g\|_{L^2}^2 e^{2\gamma \lambda^{-3} T'} + \left| \int_0^{T'} e^{2\gamma \lambda^{-3} t} \Lambda_3' dt \right| \right)$$  \hspace{1cm} (4.7)

and

$$\sup_{t \in [0,T']} \|I'v(t)\|_{H^1}^2 e^{2\gamma \lambda^{-3} t} \leq C_1 \left( \|I'v_0\|_{H^1}^2 + \|I'v_0\|_{L^2}^2 + \frac{1}{\gamma^2} \|I'g\|_{L^2}^2 e^{2\gamma \lambda^{-3} T'} + \frac{1}{\gamma^2} \|I'g\|_{L^2}^2 e^{2\gamma \lambda^{-3} T'} + \frac{1}{\gamma^2} \|I'g\|_{L^2}^2 e^{2\gamma \lambda^{-3} T'} \right)
+ \left| \int_0^{T'} (\Lambda_3 + \Lambda_4) e^{2\gamma \lambda^{-3} t} dt \right| + \left| \int_0^{T'} \Lambda_3' e^{2\gamma \lambda^{-3} t} dt \right|^2$$ \hspace{1cm} (4.8)

where $\Lambda_3 = \Lambda_3((\xi_3 + \eta_3) \frac{m(\zeta_1 + \zeta_2)}{m(\zeta_1) m(\zeta_2)} ; I'v)$, $C_1 > 1$.

Proof  Similarly to (4.5), one gets

$$\frac{d}{dt} \|I'v(t)\|_{L^2(\mathbb{R}^2)}^2 e^{2\gamma \lambda^{-3} t} = 2\lambda^{-3} e^{2\gamma \lambda^{-3} t} \int_{\mathbb{R}^2} I'g I'v dx dy - 2e^{2\gamma \lambda^{-3} t} \Lambda_3,$$

which implies

$$\|I'v(t)\|_{L^2(\mathbb{R}^2)}^2 e^{2\gamma \lambda^{-3} t} \leq \|I'v_0\|_{L^2}^2 + \left| \int_0^{T'} e^{2\gamma \lambda^{-3} t} \Lambda_3' dt \right| + \frac{e^{\gamma \lambda^{-3} T'}}{\gamma} \|I'g\|_{L^2} \sup_{t \in [0,T']} \|I'v(t)\|_{L^2} e^{\gamma \lambda^{-3} t}$$

$$\|I'v(t)\|_{L^2(\mathbb{R}^2)}^2 e^{2\gamma \lambda^{-3} t} \leq \|I'v_0\|_{L^2}^2 + \left| \int_0^{T'} e^{2\gamma \lambda^{-3} t} \Lambda_3' dt \right| + \frac{C_\epsilon e^{2\gamma \lambda^{-3} T'}}{\gamma^2} \|I'g\|_{L^3}^2 \sup_{t \in [0,T']} \|I'v(t)\|_{L^2}^2 e^{\frac{4\gamma \lambda^{-3} t}{\gamma}}$$

$$\|I'v(t)\|_{L^2(\mathbb{R}^2)}^2 e^{2\gamma \lambda^{-3} t} \leq \|I'v_0\|_{L^2}^2 + \epsilon \sup_{t \in [0,T']} \|I'v(t)\|_{H^1}^2 e^{2\gamma \lambda^{-3} t}$$

for $t \in [0,T']$.

As the last term of (4.9) can be absorbed by the left side via taking $\epsilon$ sufficiently small, we obtain (4.7).

Using Hölder’s inequality, one has

$$\left| \int_0^{T'} \int_{\mathbb{R}^2} \lambda^{-3} I'g((\Delta - \partial_2 \partial_y) I'v + (I'v)^2) e^{2\gamma \lambda^{-3} t} dx dy dt \right| \leq \frac{e^{\gamma \lambda^{-3} T'}}{\gamma} \|\nabla I'g\|_{L^2} \sup_{t \in [0,T']} \|\nabla I'v(t)\|_{L^2} e^{\gamma \lambda^{-3} t}$$

$$\|I'v(t)\|_{L^2(\mathbb{R}^2)}^2 e^{2\gamma \lambda^{-3} t} \leq \|I'v_0\|_{L^2}^2 + \epsilon \sup_{t \in [0,T']} \|I'v(t)\|_{L^2}^2 e^{\frac{4\gamma \lambda^{-3} t}{\gamma}}$$

$$\|I'v(t)\|_{L^2(\mathbb{R}^2)}^2 e^{2\gamma \lambda^{-3} t} \leq \|I'v_0\|_{L^2}^2 + \epsilon \sup_{t \in [0,T']} \|I'v(t)\|_{L^2}^2 e^{\frac{4\gamma \lambda^{-3} t}{\gamma}}$$

$$\|I'v(t)\|_{L^2(\mathbb{R}^2)}^2 e^{2\gamma \lambda^{-3} t} \leq \|I'v_0\|_{L^2}^2 + \epsilon \sup_{t \in [0,T']} \|I'v(t)\|_{L^2}^2 e^{\frac{4\gamma \lambda^{-3} t}{\gamma}}.$$  \hspace{1cm} (4.10)

Hence, from (4.6) and (4.10) we obtain

$$\|I'v(t)\|_{H^1}^2 e^{2\gamma \lambda^{-3} t} \leq \|I'v_0\|_{H^1}^2.$$
Let Proposition 4.3 and \( G.W.P. \) and Attractor for (4.11), (4.12) and (4.13), we can get (4.8).

Then we have

\[
\|I'v\|_{L^3}^3 e^{2\gamma \lambda^{-3} T'} + \frac{e^{2\gamma \lambda^{-3} T'}}{\gamma^2} \|I'g\|_{H^1}^2 + \left| \int_0^{T'} (\Lambda_3 + \Lambda_4) e^{2\gamma \lambda^{-3} t} dt \right| + \frac{e^{2\gamma \lambda^{-3} T'}}{\gamma^3} \|I'g\|_{L^3}^3 + \sup_{t \in [0,T']} \|I'(t)\|_{L^3}^3 e^{2\gamma \lambda^{-3} t}
\]  

(4.11)

for \( t \in [0,T'] \).

Gagliardo–Nirenberg’s inequality and (4.7) give

\[
\|I'g\|_{L^3}^3 \lesssim \|I'g\|_{H^1} \|I'g\|_{L^2}^2 \lesssim \gamma \|I'g\|_{H^1}^2 + \frac{1}{\gamma} \|I'g\|_{L^2}^4
\]

(4.12)

and

\[
\sup_{t \in [0,T']} \|I'(t)\|_{L^3}^3 e^{2\gamma \lambda^{-3} t} \lesssim \sup_{t \in [0,T']} \|I'(t)\|_{H^1} \|I'(t)\|_{L^2}^2 e^{2\gamma \lambda^{-3} t}
\]

\[
\lesssim C_e \sup_{t \in [0,T']} \|I'(t)\|_{L^2}^2 e^{2\gamma \lambda^{-3} t} + \epsilon \sup_{t \in [0,T']} \|I'(t)\|_{H^1}^2 e^{2\gamma \lambda^{-3} t}
\]

\[
\lesssim C_e \|I'v_0\|_{L^2}^4 + \frac{C_e}{\gamma^3} e^{2\gamma \lambda^{-3} T'} \|I'g\|_{L^2}^4 + C_e \left| \int_0^{T'} e^{2\gamma \lambda^{-3} t} \Lambda_3' dt \right|^2 + \epsilon \sup_{t \in [0,T']} \|I'(t)\|_{H^1}^2 e^{2\gamma \lambda^{-3} t},
\]

(4.13)

Collecting (4.11), (4.12) and (4.13), we can get (4.8).

The following a priori estimate is impactful for controlling \( \|u\|_H^2 \).

**Proposition 4.3** Let \( C_2 \ll 1, C_3 > 1, C_4 \gg 1 \) and \( T' > 0 \) be given. Assume \( v \) is a solution of (4.1) on \( [0,T'] \). If \( \lambda^3 \geq \gamma, (N')^{\frac{1}{2}} \geq C_4 \lambda^2 T' \),

\[
1 \lesssim \lambda^2 \|I'v_0\|_{L^2}^2, \quad 1 \lesssim \lambda^2 \|I'g\|_{L^2}^2,
\]

\[
\|I'v_0\|_{L^2}^2 + \frac{1}{\gamma^2} \|I'g\|_{L^2}^2 e^{2\gamma \lambda^{-3} T'} \leq C_2,
\]

\[
\|I'v_0\|_{H^1}^2 + \frac{1}{\gamma^2} \|I'g\|_{H^1}^2 e^{2\gamma \lambda^{-3} T'} \leq \lambda^{-2} C_2,
\]

then we have

\[
\|I'(T')\|_{L^3}^3 e^{2\gamma \lambda^{-3} T'} \leq C_3 \left( \|I'v_0\|_{L^2}^2 + \frac{1}{\gamma^2} \|I'g\|_{L^2}^2 e^{2\gamma \lambda^{-3} T'} \right),
\]

\[
\|I'(T')\|_{H^1}^2 e^{2\gamma \lambda^{-3} T'} \leq C_3 \left( \|I'v_0\|_{H^1}^2 + \|I'v_0\|_{L^2}^2 + \frac{1}{\gamma^2} \|I'g\|_{H^1}^2 e^{2\gamma \lambda^{-3} T'} + \frac{1}{\gamma^2} \|I'g\|_{L^2}^2 e^{2\gamma \lambda^{-3} T'} \right).
\]
Proof. Actually choosing \( \lambda \) sufficiently large, from scaling we have
\[
\|I'v_0\|_{\mathcal{H}^1_l}^2 + \|I'v_0\|_{\mathcal{L}_2}^2 + \frac{1}{\gamma^2} \|I'g\|_{\mathcal{H}^1_l}^2 e^{2\gamma\lambda^{-3}T'} + \frac{1}{\gamma^4} \|I'g\|_{\mathcal{L}_2}^2 e^{2\gamma\lambda^{-3}T'} \\
\ll \|I'v_0\|_{\mathcal{L}_2}^2 + \frac{1}{\gamma^2} \|I'g\|_{\mathcal{L}_2}^2 e^{2\gamma\lambda^{-3}T'}.
\]

Take \( \delta = \delta(\|I'v_0\|_{\mathcal{H}^1_l}, \lambda^{-3}\|I'g\|_{\mathcal{H}^1_l}, \gamma\lambda^{-3}) \) as in Proposition 4.1 and \( j \in \mathbb{N} \) satisfying \( \delta j = T' \). For \( 0 \leq k \leq j, k \in \mathbb{N} \), we prove
\[
\|I'v(k\delta)\|_{\mathcal{L}_2}^2 e^{2\gamma\lambda^{-3}k\delta} \leq 2C_1 \left( \|I'v_0\|_{\mathcal{L}_2}^2 + \frac{1}{\gamma^2} \|I'g\|_{\mathcal{L}_2}^2 e^{2\gamma\lambda^{-3}k\delta} \right) \\
\leq 2C_1C_2
\]
and
\[
\|I'v(k\delta)\|_{\mathcal{H}^1_l}^2 e^{2\gamma\lambda^{-3}k\delta} \\
\leq 2C_1 \left( \|I'v_0\|_{\mathcal{H}^1_l}^2 + \|I'v_0\|_{\mathcal{L}_2}^2 + \frac{1}{\gamma^2} \|I'g\|_{\mathcal{H}^1_l}^2 e^{2\gamma\lambda^{-3}k\delta} + \frac{1}{\gamma^4} \|I'g\|_{\mathcal{L}_2}^2 e^{2\gamma\lambda^{-3}k\delta} \right) \\
\leq 4C_1C_2
\]
by induction.

For \( k = 0 \), (4.14) and (4.15) hold trivially. We assume (4.14) and (4.15) hold true for \( k = l \) where \( 0 \leq l \leq j - 1 \). From Lemma 4.2, one has
\[
\|I'v((l + 1)\delta)\|_{\mathcal{L}_2}^2 e^{2\gamma\lambda^{-3}(l+1)\delta} \\
\leq C_1 \left( \|I'v_0\|_{\mathcal{L}_2}^2 + \frac{1}{\gamma^2} \|I'g\|_{\mathcal{L}_2}^2 e^{2\gamma\lambda^{-3}(l+1)\delta} + \left| \int_0^{(l+1)\delta} e^{2\gamma\lambda^{-3}t} \Lambda_3 dt \right| \right)
\]
and
\[
\|I'v((l + 1)\delta)\|_{\mathcal{H}^1_l}^2 e^{2\gamma\lambda^{-3}(l+1)\delta} \\
\leq C_1 \left( \|I'v_0\|_{\mathcal{H}^1_l}^2 + \|I'v_0\|_{\mathcal{L}_2}^2 + \frac{1}{\gamma^2} \|I'g\|_{\mathcal{H}^1_l}^2 e^{2\gamma\lambda^{-3}(l+1)\delta} \\
+ \frac{1}{\gamma^4} \|I'g\|_{\mathcal{L}_2}^2 e^{2\gamma\lambda^{-3}(l+1)\delta} + \left| \int_0^{(l+1)\delta} (\Lambda_3 + \Lambda_4) e^{2\gamma\lambda^{-3}t} dt \right| \\
+ \left| \int_0^{(l+1)\delta} \Lambda_3' e^{2\gamma\lambda^{-3}t} dt \right|^2 \right).
\]
Therefore, it suffices to prove
\[
\left| \int_0^{(l+1)\delta} e^{2\gamma\lambda^{-3}t} \Lambda_3 dt \right| \leq \|I'v_0\|_{\mathcal{L}_2}^2 + \frac{1}{\gamma^2} \|I'g\|_{\mathcal{L}_2}^2 e^{2\gamma\lambda^{-3}(l+1)\delta}
\]
and
\[
\left| \int_0^{(l+1)\delta} (\Lambda_3 + \Lambda_4) e^{2\gamma\lambda^{-3}t} dt \right| + \left| \int_0^{(l+1)\delta} e^{2\gamma\lambda^{-3}t} \Lambda_3' dt \right|^2 \\
\leq \|I'v_0\|_{\mathcal{H}^1_l}^2 + \|I'v_0\|_{\mathcal{L}_2}^2 + \frac{1}{\gamma^2} \|I'g\|_{\mathcal{H}^1_l}^2 e^{2\gamma\lambda^{-3}(l+1)\delta} + \frac{1}{\gamma^4} \|I'g\|_{\mathcal{L}_2}^2 e^{2\gamma\lambda^{-3}(l+1)\delta}.
\]
From Proposition 3.9, we have
\[
\left| \int_0^{(l+1)\delta} e^{2\gamma\lambda^{-3}t} \Lambda_3' dt \right| = \left| \int_0^{(l+1)\delta} \Lambda_3 (\xi_3 + \eta_3) m(\xi_1 + \xi_2) e^{2\gamma\lambda^{-3}t} I'v \right|
\]
Proposition 4.1, (4.14) and (4.15) provide the upper bound of \( \| I'v \|_{Y^1_{\delta,(m+1)\delta}} \)

\[
\lesssim \delta \tilde{\Psi} (N')^{-2+} \sum_{m=0}^{l} \| e^{\tilde{\Psi} \gamma \lambda^{-3} t} I'v \|_{Y^1_{\delta,(m+1)\delta}}^3 \\
\lesssim \delta \tilde{\Psi} (N')^{-1} \sum_{m=0}^{l} \| I'v \|_{Y^1_{\delta,(m+1)\delta}}^3 e^{2 \gamma \lambda^{-3} (m+1) \delta}. 
\] (4.18)

Hence, from (4.18) and (4.19), one obtains

\[
\| I'v \|_{Y^1_{\delta,(m+1)\delta}}^3 \lesssim \| I'v(m\delta) \|_{H^1}^3 e^{2 \gamma \lambda^{-3} (m+1) \delta} + (\lambda^{-3} \| I'g \|_{H^1})^3 e^{2 \gamma \lambda^{-3} (m+1) \delta} \\
\leq (C_1 C_2)^N C_1 \left( \| I'v_0 \|_{L^2}^2 + \frac{1}{\gamma^2} \| I'g \|_{L^2}^2 e^{2 \gamma \lambda^{-3} (l+1) \delta} \right).
\] (4.19)

We can get (4.16) from (4.20) as \( C_2 \ll 1 \).

Moreover,

\[
\left| \int_0^{(l+1)\delta} e^{2 \gamma \lambda^{-3} t} \Lambda_3 dt \right|^2 \\
\lesssim \frac{1}{4} \left( \| I'v_0 \|_{H^1}^2 + \| I'v_0 \|_{L^2}^2 + \frac{1}{\gamma^2} \| I'g \|_{H^1}^2 e^{2 \gamma \lambda^{-3} (l+1) \delta} + \frac{1}{\gamma^2} \| I'g \|_{L^2}^2 e^{2 \gamma \lambda^{-3} (l+1) \delta} \right)
\]

which allows that one only needs to show

\[
\left| \int_0^{(l+1)\delta} (\Lambda_3 + \Lambda_4) e^{2 \gamma \lambda^{-3} t} dt \right| \\
\leq \frac{1}{2} \left( \| I'v_0 \|_{H^1}^2 + \| I'v_0 \|_{L^2}^2 + \frac{1}{\gamma^2} \| I'g \|_{H^1}^2 e^{2 \gamma \lambda^{-3} (l+1) \delta} + \frac{1}{\gamma^2} \| I'g \|_{L^2}^2 e^{2 \gamma \lambda^{-3} (l+1) \delta} \right). 
\] (4.21)

Similarly, from Propositions 3.7 and 3.8, we have

\[
\left| \int_0^{(l+1)\delta} (\Lambda_3 + \Lambda_4) e^{2 \gamma \lambda^{-3} t} dt \right| \\
\lesssim \delta \tilde{\Psi} (N')^{-1} \sum_{m=0}^{l} \| I'v \|_{Y^1_{\delta,(m+1)\delta}}^3 e^{2 \gamma \lambda^{-3} (m+1) \delta} \\
\lesssim \delta \tilde{\Psi} (C_4 \lambda^2 N')^{-1} \sum_{m=0}^{l} (\| I'v(m\delta) \|_{H^1}^3 + (\lambda^{-3} \| I'g \|_{H^1})^3 e^{2 \gamma \lambda^{-3} (m+1) \delta}). 
\] (4.22)

On one hand, (4.14) and (4.15) tell us that

\[
\| I'v(m\delta) \|_{H^1}^3 e^{2 \gamma \lambda^{-3} (m+1) \delta} \\
\lesssim C_1^2 \left( \| I'v_0 \|_{L^2}^3 + \frac{1}{\gamma^2} \| I'g \|_{L^2}^3 e^{2 \gamma \lambda^{-3} (m+1) \delta} \right)
\]
Collecting (4.22)–(4.25), one gets

\[
\lesssim C_1^2 \lambda^2 \left( \| I'v_0 \|_{L^2}^4 + \frac{1}{\gamma^4} \| I'g \|_{L^2}^4 e^{2\gamma\lambda^{-3}(m+1)\delta} \right)
\] (4.23)

and

\[
\| I'(m\delta) \|_{H^1}^3 e^{2\gamma\lambda^{-3}(m+1)\delta}
\lesssim (C_1 C_2)^2 \left( \| I'v_0 \|_{H^1}^2 + \| I'v_0 \|_{L^2}^4 + \frac{1}{\gamma^2} \| I'g \|_{H^1}^2 e^{2\gamma\lambda^{-3}(m+1)\delta} + \frac{1}{\gamma^3} \| I'g \|_{L^2}^2 e^{2\gamma\lambda^{-3}(m+1)\delta} \right).
\] (4.24)

On the other hand, we have

\[
(\lambda^{-3} \| I'g \|_{H^1})^3 e^{2\gamma\lambda^{-3}(m+1)\delta} \lesssim \frac{1}{\gamma^3} \| I'g \|_{L^2}^3 e^{2\gamma\lambda^{-3}(m+1)\delta} + \frac{1}{\gamma^2} \| I'g \|_{H^1}^2 e^{2\gamma\lambda^{-3}(m+1)\delta} \]
\[
\lesssim \lambda^2 \frac{1}{\gamma^3} \| I'g \|_{L^2}^3 e^{2\gamma\lambda^{-3}(m+1)\delta} + (C_2)^2 \frac{1}{\gamma^2} \| I'g \|_{H^1}^2 e^{2\gamma\lambda^{-3}(m+1)\delta} \]
\[
\lesssim \lambda^2 \left( \frac{1}{\gamma^4} \| I'g \|_{L^2}^4 e^{2\gamma\lambda^{-3}(m+1)\delta} + \frac{1}{\gamma^2} \| I'g \|_{H^1}^2 e^{2\gamma\lambda^{-3}(m+1)\delta} \right).
\] (4.25)

Collecting (4.22)–(4.25), one gets

\[
\left| \int_0^{(l+1)\delta} (\Lambda_3 + \Lambda_4) e^{2\gamma\lambda^{-3}t} dt \right|
\lesssim \delta^\frac{1}{\gamma^2} \left( C_4 \lambda^2 T' \right)^{-1} C_2^2 \lambda_2^2 \left( \| I'v_0 \|_{H^1}^2 + \| I'v_0 \|_{L^2}^4 \right) 
\]
\[
+ \frac{1}{\gamma^3} \| I'g \|_{H^1}^2 e^{2\gamma\lambda^{-3}(l+1)\delta} + \frac{1}{\gamma^2} \| I'g \|_{L^2}^2 e^{2\gamma\lambda^{-3}(l+1)\delta} \]
\[
\lesssim \delta^\frac{1}{\gamma^2} C_2^2 C_4^{-1} \left( \| I'v_0 \|_{H^1}^2 + \| I'v_0 \|_{L^2}^4 \right) 
\]
\[
+ \frac{1}{\gamma^3} \| I'g \|_{H^1}^2 e^{2\gamma\lambda^{-3}(l+1)\delta} + \frac{1}{\gamma^2} \| I'g \|_{L^2}^2 e^{2\gamma\lambda^{-3}(l+1)\delta} \),
\]

which gives (4.21) by taking \(C_4\) sufficiently large.

**Proposition 4.4** Let \(C_2 \ll 1\), \(C_3 > 1\), \(C_4 \gg 1\) and \(T > 0\) be given. Assume \(u\) is a solution of (1.4) on \([0, T]\). If \(N^\mathbb{R}_- \geq \gamma\), \(N^0+ \geq C_4 T\),

\[
\| Iu_0 \|_{L^2}^2 + \frac{1}{\gamma^2} \| If \|_{L^2}^2 e^{2\gamma T} \leq C_2 N^\mathbb{R}_-
\]

and

\[
\| Iu_0 \|_{H^1}^2 + \frac{1}{\gamma^2} \| If \|_{H^1}^2 e^{2\gamma T} \leq C_2 N^\mathbb{R}_-,\n\]

then we have

\[
\| Iu(T) \|_{L^2}^2 e^{2\gamma T} \leq C_3 \left( \| Iu_0 \|_{L^2}^2 + \frac{1}{\gamma^2} \| If \|_{L^2}^2 e^{2\gamma T} \right), \]
\[
\| Iu(T) \|_{H^1}^2 e^{2\gamma T} \leq C_3 \left( \| Iu_0 \|_{H^1}^2 + \| Iu_0 \|_{L^2}^2 + \frac{1}{\gamma^2} \| If \|_{H^1}^2 e^{2\gamma T} + \frac{1}{\gamma^2} \| If \|_{L^2}^2 e^{2\gamma T} \right). \]
Proof Noticing that \( \|I'v\|_{H^2}^2 = \lambda^{-2}\|Iu\|_{H^2}^2 \), \( \|I'g\|_{H^1}^2 = \lambda^{-4}\|Iv\|_{H^1}^2 \), and the fact \( \lambda^2\|I'v\|_{H^2}^2 = \|Iu\|_{H^2}^2 > \frac{1}{2}\|u\|_{H^2}^2 \) \((N \gg 1)\), Proposition 4.4 is acquired via taking \( \lambda = N^{\frac{\beta}{2}\gamma} - \), \( N' = N^2 \) and \( T' = \lambda^3 T \) in Proposition 4.3.

Finally, we show the existence of the global attractor.

Proof of Theorem 1.3 Let \( 0 < \epsilon \ll 1 \) be fixed. We choose \( T_1 > 0 \) so that

\[
e^{2\gamma T_1} > (\|u_0\|_{H^2} + \|u_0\|_{H^2}) \left( \frac{1}{\gamma^2}\|f\|_{H^1} + \frac{1}{\gamma^4}\|f\|_{L^2} \right)^{-1} \max \left\{ \frac{2^{2(1-s)}+}{\Lambda}, (C_4 T_1)^{2(1-s)+} \right\},
\]

which is possible as \( 11(1-s) < 1 \). \( T_1 \) depends only on \( \|u_0\|_{H^2}, \|f\|_{H^1} \) and \( \gamma \). Set

\[
N = \max \left\{ \frac{\gamma^{\frac{\beta}{2}\gamma} +}{}, (C_4 T_1)^{\frac{\beta}{2}\gamma}, \left( \frac{C_2}{2}\|u_0\|_{H^2} \right) \frac{2^{11(1-s)+}}{10^{(1-s)+}}, (2C_2^{-1}\gamma^{-2}\|f\|_{H^2} e^{2\gamma T_1})^{11(1-s)+} \right\}.
\]

From the choice of \( T_1 \) and \( N \), we know

\[
N^{\frac{\beta}{2}\gamma} \geq \gamma, \quad N^2 \geq C_4 T_1
\]

and

\[
\|Iu_0\|_{H^1}^2 \leq N^{2-2s}\|u_0\|_{H^2}^2 \leq \frac{C_2}{2} N^{2\frac{\beta}{2}\gamma},
\]

\[
\gamma^{-2}\|If\|_{H^2} e^{2\gamma T_1} \leq \frac{C_2}{2} N^{2\frac{\beta}{2}\gamma}.
\]

Hence, from Proposition 4.4, one gains

\[
\|u(T_1)\|_{H^2} \leq \|Iu(T_1)\|_{H^2}^2 \leq C_3 \left( \|Iu_0\|_{H^2}^2 e^{-2\gamma T_1} + \|u_0\|_{H^2} e^{-2\gamma T_1} + \frac{1}{\gamma^2}\|If\|_{H^1}^2 + \frac{1}{\gamma^4}\|f\|_{L^2}^4 \right)
\]

\[
\leq C_3 \left( N^{2(1-s)}\|u_0\|_{H^2} e^{-2\gamma T_1} + \|u_0\|_{L^2} e^{-2\gamma T_1} + \frac{1}{\gamma^2}\|f\|_{H^1}^2 + \frac{1}{\gamma^4}\|f\|_{L^2}^4 \right).
\]

From (4.26) and (4.27), we get

\[
N^{2(1-s)} e^{-2\gamma T_1}\left( \|u_0\|_{H^2}^2 + \|u_0\|_{L^2}^2 \right) < \frac{1}{\gamma^2}\|f\|_{H^1}^2 + \frac{1}{\gamma^4}\|f\|_{L^2}^4,
\]

which helps us give the bound

\[
\|u(T_1)\|_{H^2}^2 \leq 2C_3 \left( \frac{1}{\gamma^2}\|f\|_{H^1}^2 + \frac{1}{\gamma^4}\|f\|_{L^2}^4 \right) < K_1,
\]

where \( K_1 \) depends only on \( \|f\|_{H^2} \) and \( \gamma \).

In the next place, one can fix \( T_2 > 0 \) and solve (1.4) on time interval \([T_1, T_1 + T_2]\) with initial data replaced by \( u(T_1) \). Let \( K_2 > 0 \) be sufficiently large such that

\[
K_2 e^{2\gamma t} > \max \left\{ \frac{\gamma^{22(1-s)+}}{3}, (C_4 T_1)^{2(1-s)+}, (2C_2^{-1}K_1)^{11(1-s)+}, (2C_2^{-1}\gamma^{-2}\|f\|_{H^2} e^{2\gamma t})^{11(1-s)+} \right\}.
\]

(4.28) for any \( t > 0 \). Set \( N^{2(1-s)} = K_2 e^{2\gamma T_2} \), then Inequality (4.28) verifies the assumptions in Proposition 4.4 and hence we obtain

\[
\|Iu(T_1 + T_2)\|_{H^1}^2 \leq C_3 \left( N^{2(1-s)}\|u(T_1)\|_{H^2} e^{-2\gamma T_2} + \|u(T_1)\|_{L^2} e^{-2\gamma T_2} + \frac{1}{\gamma^2}\|f\|_{H^1}^2 + \frac{1}{\gamma^4}\|f\|_{L^2}^4 \right).
\]
\[ \leq C_3 \left( K_1 K_2 + K_1^2 + \frac{1}{\gamma^2} \| f \|_{H^1}^2 + \frac{1}{\gamma^4} \| f \|_{L^2}^4 \right) < K_3. \]

For \( t > T_1 \), we define the maps \( M_1(t) \) and \( M_2(t) \) as

\[ M_1(t)u_0 = \hat{A}(t)u_0|_{|\xi| < N_1}, \quad M_2(t)u_0 = \hat{A}(t)u_0|_{|\xi| > N_1}, \]

where \( \hat{A}(t)u_0 = u(t) \) and \( N_1 = (K_2 e^{2\gamma(t-T_1)})^{\frac{1}{2(1-\gamma)}}. \)

It’s easy to see that for \( t > T_1 \),

\[ \| M_1(t)u_0 \|_{H^1}^2 \leq \| Iu(t) \|_{H^1}^2 < K_3, \]
\[ \| M_2(t)u_0 \|_{H^s}^2 \leq N^{2s-2} \| Iu(t) \|_{H^1}^2 < K_2^{-1} K_3 e^{-2\gamma(t-T_1)}. \]

Hence we obtain Theorem 1.3 by taking \( K = \max\{K_3^\frac{1}{2}, K_2^{-\frac{1}{2}} K_3^\frac{1}{2}\} \).

5 Global Attractor in Weak Topology

In order to define an infinite-dimensional dynamical system from the evolution equation (1.4), first of all we should make sure that the corresponding initial value problem is well-posed in \( H^{s,0} \).

**Proposition 5.1** Let \( \frac{0}{0} < s < 1 \). Assume \( u_0 \in H^s \), then there exists \( T = T(\| u_0 \|_{H^s}) > 0 \) and

\[ u(x, y, t) \in C([0, T], H^s(\mathbb{R}^2)) \]

such that \( u(x, y, t) \) is the unique solution of (1.4).

**Proof** By Duhamel’s formulation

\[ u = \chi(t) e^{tS} u_0 - \int_0^t e^{(t-t')S} \chi(t) \left( \frac{1}{2} (\partial_x + \partial_y) u^2 + \gamma u - f \right) dt'. \]

From the definition of \( Y^s \) and the duality of \( U^p \), one knows that

\[ \left\| \int_0^t e^{(t-t')S} \chi(t)(\gamma u - f) dt' \right\|_{Y^s} \]
\[ \lesssim \left( \sum_{N_1} N_1^{2s} \left( \sup_{\| w \|_{Y^s} \leq 1} \left\| \int_{\mathbb{R}^2} \chi(t)(\gamma P_{N_1} u - P_{N_1} f) w dx dy dt \right\| \right) \right)^{\frac{1}{2}} \]
\[ \lesssim \left( \sum_{N_1} N_1^{2s} \left( \sup_{\| w \|_{Y^s} \leq 1} \| \gamma P_{N_1} u - \chi(t) P_{N_1} f \|_{L^2} \| \chi(t) w \|_{L^1(Y^s)} \right) \right)^{\frac{1}{2}} \]
\[ \lesssim \left( \sum_{N_1} N_1^{2s} \| \gamma P_{N_1} u \|_{L^2}^2 + \| P_{N_1} f \|_{L^2}^2 \right)^{\frac{1}{2}} \]
\[ \lesssim \gamma \| u \|_{Y^s} + \| f \|_{L^2}. \quad (5.1) \]

Set

\[ B = \{ u \in Y^s \mid \| u \|_{Y^s} < C_0(\| u_0 \|_{L^2} + \| f \|_{L^2}) \}; \]

From (5.1) we have

\[ \| u \|_{Y^s} \lesssim \| u_0 \|_{L^2} + \| u \|_{Y^s}^2 + \gamma \| u \|_{Y^s} + \| f \|_{L^2} \]
\[ \lesssim C_0^2 (\| u_0 \|_{L^2} + \| f \|_{L^2})^2 + (1 + C_0 \gamma) (\| u_0 \|_{L^2} + \| f \|_{L^2}) \quad (5.2) \]
and
\[ \|u_1 - u_2\|_{Y^s} \lesssim (\gamma + \|u_1\|_{Y^s} + \|u_2\|_{Y^s})\|u_1 - u_2\|_{Y^s} \lesssim (\gamma + C_0(\|u_0\|_{L^2} + \|f\|_{L^2}))\|u_1 - u_2\|_{Y^s}. \] (5.3)

We rescale Equation (1.4) to get (4.1) for constructing a strict contraction mapping on \( \tilde{B} \), where
\[ \tilde{B} = \{v \in Y^s \mid \|v\|_{Y^s} < C_0(\|v_0\|_{L^2} + \lambda^{-3}\|g\|_{L^2})\}. \]

Similarly to (5.2) and (5.3),
\[ \|v\|_{Y^s} \lesssim \|v_0\|_{L^2} + \|v\|_{Y^s}^2 + \gamma \lambda^{-3}\|v\|_{Y^s} + \lambda^{-3}\|g\|_{L^2} \lesssim C_0^2(\|v_0\|_{L^2} + \lambda^{-3}\|g\|_{L^2})^2 + (1 + C_0 \gamma \lambda^{-3})(\|v_0\|_{L^2} + \lambda^{-3}\|g\|_{L^2}) \] (5.4)

and
\[ \|v_1 - v_2\|_{Y^s} \lesssim (\gamma \lambda^{-3} + \|v_1\|_{Y^s} + \|v_2\|_{Y^s})\|v_1 - v_2\|_{Y^s} \lesssim (\gamma \lambda^{-3} + C_0(\|v_0\|_{L^2} + \lambda^{-3}\|g\|_{L^2}))\|u_1 - u_2\|_{Y^s}. \] (5.5)

By choosing \( \lambda \) sufficiently large, one gets
\[ \gamma \lambda^{-3} \ll 1, \quad \|v_0\|_{L^2} = \lambda^{-\frac{1}{2}}\|u_0\|_{L^2} \ll 1, \quad (\lambda)^{-3}\|g\|_{L^2} = \lambda^{-\frac{3}{2}}\|f\|_{L^2} \ll 1. \]

Hence there exists a unique solution \( v \) on \([0, 1]\) from the fixed point argument. Thus, (1.4) is locally well-posed on \([0, T]\) by taking \( T = \lambda^{-3} \).

Let’s consider the weakly continuous dependence of solution on initial data in \( H^s \) for the Cauchy problem of (1.4). Without loss of generality, one may assume \( \gamma = 0 \) and \( f = 0 \). We now explain how to show that if a sequence of initial data \( \{u_{0n}\} \) converges to \( u_0 \) weakly in \( H^s \) and a sequence of solutions \( \{u_n\} \) with \( u_n(0) = u_{0n} \) converges to a solution \( u \) with \( u(0) = u_0 \) weakly in \( X^{s,1/2}_T \) for \( s > \frac{14}{13} \), then \( u_n(t) \) converges to \( u(t) \) weakly in \( H^s \) for each \( t \in [0, T] \). Here we use \( X^{s,1/2}_T \) instead of \( Y^s \) to avoid being too lengthy and tedious. The prerequisite “if” part can be proved through the standard local existence theorem of solution (see Remark 5.2 below and see also [14, 15]).

Let \( \psi \in C_0^\infty(\mathbb{R}^2) \) and let \( T' \) be a fixed positive constant with \( 0 < T' < T \). The Duhamel formula yields
\[
(e^{-TS}(u_n(T') - u(T')), \psi)
= (u_{0n} - u_0, \psi) + 2\int_0^{T'} (e^{-tS}((u_n - u)(u_n + u))(t), (\partial_x + \partial_y)\psi) \, dt
= (u_{0n} - u_0, \psi) + 2((u_n - u)(u_n + u), f),
\] (5.6)
where we write \( u \) and \( u_n \) for \( \chi_{\{0 \leq t \leq T'\}} u \) and \( \chi_{\{0 \leq t \leq T'\}} u_n \), moreover we put \( f = \chi_{\{0 \leq t \leq T'\}} e^{tS}(\partial_x + \partial_y)\psi \). It suffices to prove that the last term on the right hand side of (5.6) converges to zero as \( n \to \infty \), since \( \{u_n\} \) is bounded in \( L^\infty(0, T; H^s) \).
By $P_N$, we denote the projection : $u \in L^2(\mathbb{R}^2) \mapsto \mathcal{F}^{-1}[\chi_{\{|\zeta| \leq N\}}(\zeta) \hat{u}(\zeta)]$. Set $u_N = P_N u$ and $u_{>N} = (I - P_N) u$. We define a cut-off function $\varphi(x, y, t) \in C^\infty(\mathbb{R}^3)$ as follows,

$$
\varphi(x, y, t) = \begin{cases} 
1, & x^2 + y^2 + t^2 \leq 1, \\
0, & x^2 + y^2 + t^2 \geq 4.
\end{cases}
$$

For $R > 1$, we put $\varphi_R(x, y, t) = \varphi(x/R, y/R, t/R)$. The quadratic nonlinear interaction of $u$ and $v$ is decomposed into the following four terms,

$$
uw = u_{>N}v_{>N} + u_NV_{>N} + u_NV_N + u_NV_N.
$$

It follows from (2.2) that the first three terms on the right hand side can be made arbitrarily small as $N$ gets larger and larger. So, we have only to consider the term $u_NV_N$. The contribution of the term $u_NV_N$ can be divided into the following two terms,

$$
\langle u_NV_N, f \rangle = \langle u_NV_N, (1 - \varphi_R)f \rangle \\
+ \langle u_NV_N, \varphi_Rf \rangle \\
=: I_1 + I_2.
$$

For $I_1$,

$$
|I_1| \leq C(R), \quad R > 1, \quad (5.7)
$$

where $C(R) \to 0$ as $R \to \infty$. From Hölder’s inequality,

$$
|I_2| \leq \|u_N\|_{L^2_x,y,t(\{|z| < 2R\})} \|v_N\|_{L^2_x,y,t(\{|z| < 2R\})} \|f\|_{L^\infty_x,y,t}, \quad (5.8)
$$

where $z = (x, y, t) \in \mathbb{R}^3$.

We now show that the mapping : $u \mapsto u_N$ is compact from $X^{0,1/2}$ to $L^2_{x,y,t}(\{|z| < 2R\})$ for each $R > 1$. Note that

$$
|\tau| \leq |\tau - \xi^3 - \eta^3| + |\xi + \eta||\xi^2 + \eta^2 - \xi\eta|.
$$

Hence, for $\varepsilon > 0$, we have

$$
\|D_x^\tau u\|_{L^2(\mathbb{R}^3)} \lesssim \|u\|_{X^{0,\varepsilon}} + \|\tilde{D}_x^\tau u\|_{L^2(\mathbb{R}^3)}.
$$

Let $\mathcal{R}$ be a mapping which restricts function $f$ on $\mathbb{R}^3$ to $f_{\{|z| < 2R\}}$. By the compact embedding

$$
\mathcal{R}(1 + \tilde{D} + D)_t^{-\varepsilon} L^6_t L^4_{x,y}(\mathbb{R}^3) \subset L^2_{x,y,t}(\{|z| < 2R\}), \quad \varepsilon > 0,
$$

we can conclude from Lemma 2.5 that the mapping : $u \mapsto u_N$ is compact from $X^{0,1/2}$ to $L^2_{x,y,t}(\{|z| < 2R\})$ for small $\varepsilon > 0$. Therefore, the weakly continuous dependence in $H^s$, $s > 11/13$ of solution on initial data follows from the above argument and (5.6).

**Remark 5.2** Let $T$ be a positive constant which denotes the existence time of solution to (1.4). For the weakly continuous dependence, we consider the integral equation associated with the Cauchy problem (1.4),

$$
u(t) = \alpha(t)e^{tS}u_0 - \alpha(t) \int_0^t e^{(t-t')S}(\partial_x + \partial_y)(\alpha_T(t')u(t'))^2 dt', \quad t \in \mathbb{R}, \quad (5.9)
$$

where $\alpha$ is a cut-off function in $C_0^\infty(\mathbb{R})$ such that $\alpha(t) = 1 (|t| < 1)$ and $\alpha(t) = 0 (|t| > 2)$ and $\alpha_T(t) = \alpha(t/T)$. All what we need to do is to prove that if a sequence of initial data $\{u_0n\}$
converges to $u_0$ weakly in $H^s$ for $s > 11/13$, a solution $u$ of (5.9) is given by a weak limit of the sequence of solutions $u_n$ of (5.9) with $u_0$ replaced by $u_{0n}$. In fact, we can extract a subsequence \{u_{n'}\} converging to some $u$ weakly in $X^{s,1/2+}$, since a sequence of solutions \{u_n\} of (5.9) with $u_0$ replaced by $u_{0n}$ is bounded in $X^{s,1/2+}$ for $s > 11/13$. Furthermore, by the duality argument similar to above, we can easily see that $u$ is a solution of (5.9), which is unique in $X^{s,1/2+}$ for $s > 11/13$ (see [10, Theorem 1.1 on page 6556]). Therefore, the whole sequence of \{u_n\} converges to $u$ weakly in $X^{s,1/2+}$.

**Proof of Corollary 1.4**  
By Proposition 5.1 the initial value problem is well-posed in $H^s$, hence one can define an infinite-dimensional dynamical system from the evolution equation (1.4). Moreover, we get the weakly continuous in $H^s$ by Remark 5.2. From Theorem 1.3, we know that $M_1(t)$ is a bounded mapping and $M_2(t)$ converges uniformly to 0 in $H^s$. It means that the semi-group $A(t)$ is asymptotically compact in the sense of weak topology. Therefore we gain the existence of the global attractor in $H^s$ by Proposition 2.9 (see [29, Remark 1.4 and Theorem 1.1]).

**Acknowledgements**  
The author would like to express his deep gratitude to Professor Yoshio Tsutsumi, Professor Baoxiang Wang and Professor Liqun Zhang for their guidance and help.

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