**ABSTRACT**- Aim of the paper is to obtain 2d analogue of the backreaction equation which will be useful to study final state of quantum perturbed spherically symmetric curved space times. Thus we take Einstein-massless-scalar \( \psi \) tensor gravity model described on class of spherically symmetric curved space times. We rewrite the action functional in 2d analogue in terms of dimensionless dilaton-matter field \( (x = \Phi \psi) \) where dilaton field \( \Phi \) is conformal factor of 2-sphere. Then we seek renormalized expectation value of quantum dilaton-matter field stress tensor operator by applying Hadamard renormalization prescription. Singularity of the Green function is assumed to be logarithmic form. Covariantly conservation condition on the renormalized quantum dilaton-matter stress tensor demands to input a variable cosmological parameter \( \lambda(x) \). Energy conditions (weak, strong and null) is studied on the obtained renormalized stress tensor leading to dynamical equations for \( \lambda(x) \). Classical dilaton-matter field \( \Phi \) in spherical symmetric curved space times, Variable cosmological parameter and null-like apparent horizon equation. Setting null-like apparent horizon equation \( \nabla_{\alpha} \Phi \nabla^{\alpha} \Phi = 0 \), our procedure predicts that physically correct value of the parameter in the anomaly trace \( \int (\lambda - \frac{\alpha}{\lambda}) \) should be \( \alpha = 6 \). At last we solved the backreaction equation and obtained explicitly metric field solution in the slow varying limits of the quantum and dilaton fields which has black holes topology and its singularity is covered by apparent horizon hypersurface.

**Keywords**- Dilaton fields, Dimensional reduction, Hadamard renormalization, Spherically symmetric curved space times, Variable cosmological parameter

**I. INTRODUCTION**

In absence of a viable theory of pure quantum gravity, its semiclassical approximation readily yields particle creation in curved background space time (see [1,2] and references therein). In the latter approach the gravitational field is retained as a classical background, while the matter fields are quantized in the usual way. In the latter view the perturbed metric is obtained by the semi-classical Einstein backreaction equations.

\[
G_{\mu\nu} = 8\pi G \{ \mathcal{T}^{\text{class}}_{\mu\nu} + \langle \mathcal{T}_{\mu\nu} \rangle_{\text{ren}} \}
\]

where we used units \( c = \hbar = 1 \), \( G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \) with \( \mu, \nu = 0,1,2,3 \) is Einstein tensor in four dimensional curved space-time, \( R_{\mu\nu} \) (Ricci tensor (scalar)). \( T^{\text{class}}_{\mu\nu} \) is classical matter fields stress tensor. \( \langle \mathcal{T}_{\mu\nu} \rangle_{\text{ren}} \) is renormaized expectation value of quantum matter fields operator. According to wald’s axioms [3], \( \langle \mathcal{T}_{\mu\nu} \rangle_{\text{ren}} \) must be covariantly conserved \( \nabla_{\mu} \langle \mathcal{T}_{\mu\nu} \rangle_{\text{ren}} = 0 \), but in the presence of trace anomaly. For conformaly invariant fields the trace anomaly \( \langle \mathcal{T}^{\text{ren}}_{\mu\nu} \rangle_{\text{ren}} \) is nonzero, unlike its classical counterpart, and is independent of the quantum state where the expectation value is taken. It is completely expressed in terms of geometrical objects as

\[
\langle \mathcal{T}^{\text{ren}}_{\mu\nu} \rangle_{\text{ren}} = \frac{1}{2880\pi^2} \left\{ aC_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} + b(R_{\alpha\beta}R^{\alpha\beta} - R^2 / 3) + c\nabla_\gamma \nabla^\gamma R + dR^2 \right\}
\]

where \( a, b, c, d \) are known as depended on the spin of the quantum fields under consideration [1,2] and \( C_{\alpha\beta\gamma\delta} \) is Weyl tensor. Whether such an approach makes sense is subject to debate. Due to the non-linearity of gravity, it will certainly fail for effects that occur on the scale of Planck length \( (G\hbar/c^3)^{1/2} = 1.616 \times 10^{-33}\text{cm} \), or involve singularities. Thus it will certainly not be possible to correctly describe, among other things, the very final stage of black holes evaporation in a semiclassical model. On the other hand, one might expect meaningful results as long as one stays in the region exterior of a reasonably sized black hole. It is hopped that the semiclassical approximation in gravity works similarly to the quantum electrodynamics one which is able to describe quantum particles in exterior electromagnetic fields. Yet in the semiclassical approximation as well as in full quantum gravity, the equations describing the evolution of the system must be solved self-consistently. In four dimensions, this poses a problem: One is only able to calculate the Hawking radiation for a fixed spherically symmetric background metric. Even in the latter case, one obtain instead a relation constraining undetermined function [4] and so study of black hole Hawking radiation in four dimension exhibits with some little success. Hence the Hawking radiation and backreaction effects of created particles on the dynamical background metric is still as an open problem.

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In order to get a suitable answer to this problem, one takes two-dimensional analogue of the gravitational models from four dimensions by introducing a dilaton field which contains physical properties of tangential pressure of (classical and quantum) matter fields. The latter idea is a good proposal and zeta function regularization method is used to obtain effective action functionals and corresponding anomaly trace in literature [5,6,7,8]. In this paper we use other procedure called with Hadamard renormalization prescription. Our procedure inputs a variable cosmological parameter \( \Lambda(x) \) reaching to the covariantly conservation condition of the renormalized stress tensor. This variable cosmological parameter is really corrections of an essential effective cosmological constant \( \Lambda_{\text{eff}} = \frac{1}{16\pi G} \) for the covariantly conservation condition of the renormalized stress tensor. This variable cosmological parameter is described in terms of variable cosmological parameter. The suggested renormalization prescription makes ultraviolet singularities of all massless scalar matter fields leading to a nonsingular covariantly conserved stress tensor with anomaly trace in the presence of variable cosmological parameter. The suggested variable cosmological parameter is described in terms of derivatives of the dilaton field, Ricci scalar of induced 2d background metric and derivatives of quantum vacuum state \( W_0(x) \). In section IV we study energy conditions (weak, strong, null) on the obtained renormalized stress tensor which leads to dynamical equations of the fields \( \Phi, W_0(x), \lambda(x) \). Also our procedure in weak quantum field limits follows results of the one which obtained from zeta function regularization method. In section V we use apparent horizon property of the curved space times on the obtained anomaly trace of quantum matter field stress tensor expectation value. Section VI denotes to concluding remarks.

II. THE MODEL

We take Einstein-Hilbert gravity minimally coupled with mass-less scalar field propagating in s-mode. In section III we suggest symmetric two-point Hadamard Green function to be obtained respectively as \[ g(x^0) dx^a dx^b + \Phi^2(x^0)(d\theta^2 + \sin^2 \theta d\phi^2) \] where signature of the metric (8) is assumed to be \((-,-,+,+,-)\). Then we assume that the metric fields \( g_{ab}, \Phi \) and matter field \( \psi \) are independent of angular coordinates \( (\theta, \phi) \) propagating in spherically modes (S-channel) and integrate (3) with respect to angular coordinates \( \theta \) and \( \phi \) leading to [9]

\[
I = \frac{1}{4G} \int d^2x \sqrt{g} \{ 1 + g^{ab} \partial_a \Phi \partial_b \Phi + \frac{1}{2} \Phi^2 R \} - 2\pi \int d^2x \sqrt{g} \Phi^2 \partial_a \psi \partial^a \psi.
\]

\( \Phi \) is called geometrical dilaton field with length dimensions and it is in agreement with the status of boson particles in it is of view of field theory. \( g_{ab}(x^0, x^+1) \) is 2d induced metric on the hypersurface \( \theta = \phi = \text{constant} \). \( g \) is absolute value of determinant of 2d metric \( g_{ab} \) and \( \tilde{R} \) is corresponding 2d Ricci scalar. The matter field \( \psi \) has inverse of length dimensions. Varying (9), with respect to \( g_{ab}, \Phi, \) and \( \psi \), the corresponding field equations are obtained respectively as [9]

\[
\Phi^2 \tilde{G}_{ab} = -2\Phi \nabla_a \nabla_b \Phi + g_{ab} \{ 2\Phi \nabla_c \nabla^c \Phi + \partial_a \Phi \partial^a \Phi - 1 \} = 8\pi G \Phi^2 \tilde{T}_{ab}[\psi],
\]

\[
\tilde{G}_{\theta \theta} = \Phi \nabla_c \nabla^c \Phi - \frac{1}{2} \tilde{R} \Phi^2 = 8\pi G \tilde{T}_{\theta \theta} = -4\pi G \Phi^2 \partial_c \psi \partial^c \psi,
\]
and
\[
\nabla^a \nabla^c \psi = -2 J^a \nabla_a \psi, \quad J_a = \nabla_a \ln \Phi \tag{12}
\]
where we defined
\[
\nabla_a \nabla^a = \frac{1}{\sqrt{g}} \partial_a (\sqrt{g} g^{ab} \partial_b), \quad \partial_a = \frac{\partial}{\partial x^a} \tag{13}
\]
and non-angular components of the stress tensor (6) as
\[
\tilde{T}_{ab}[\psi] = \partial_a \psi \partial_b \psi - \frac{1}{2} g_{ab} \partial_c \psi \partial^c \psi. \tag{14}
\]
The matter stress tensor (14) is trace free but same as (7) dose not satisfy the covariant conservation condition in 2d space times. Applying (13) and (14) one can obtain
\[
\nabla^a T_{ab} = -2 J_a \partial^a \psi \partial_b \psi, \quad T^a_a = 0 \tag{15}
\]
where we are dropped over tilde \(\sim\). Violation of covariant conservation is caused because of non-vanishing dilaton current \(J_a\) and it is coupled with matter current \(\partial_a \psi\) as a source in RHS of the matter wave equation (12). The quantity \(J_a \partial^a \psi\) treats as scalar charge for the field \(\psi\) from view of string theory. Originally this charge comes from dynamical effects of reference frames. For instance in higher dimensional string theory of gravity the Brans-Dicke scalar tensor theory is charge-less and so a covariantly conserved model in Jordan frame but it is not in other frames (see Ref. [14] chapter 2). Hence the string theory accepts that the Bianchi identity no longer implies the covariant conservation of the stress tensors separately in 4+d dimensional curved space times. In other words stress tensors of matter and geometrical dilaton fields do not need follow covariant conservation conditions separately. Physically non-conservation condition of the stress tensor implies that the motion of a free test particle is no longer geodesic when the particle has an intrinsic scalar charge and the gravitational background contains a non-trivial dilaton component.

However we follow here other point of view: dimensional reduction of the space times causes to break covariant conservation condition. On the other hand we know that renormalization of the quantum matter fields breaks also the covariant conservation condition of the stress tensor (see [1,2] and references therein). Some applicable methods are presented to satisfy the covariant conservation condition but by inducing anomaly trace. What is correspondence between them to obtain both quantum matter stress tensor and its geometrical classical dilaton counterpart satisfying covariantly conservation condition separately in 2d gravity model (9)? In the following section we try to obtain a suitable answer to this question. We apply Hadamard renormalization prescription to evaluate regular expectation value of quantum matter stress tensor operator \(<\tilde{T}_{ab}[\psi]>_{\text{ren}}\) by presenting a variable cosmological parameter \(\lambda(x)\).

### III. Hadamard Renormalization

If \(\psi\) treats as massless quantum bosons. Then it will be linear operator \(\hat{\psi}\) operating on arbitrary state of Hilbert space. Corresponding stress energy tensor operator \(\tilde{T}_{ab}[\hat{\psi}]\) become bi-linear with respect to \(\psi\) and regular stress tensor counterpart \(<\tilde{T}_{ab}[\psi]>_{\text{ren}}\) (subscript ‘\(\text{ren}\)’ denotes to ‘renormalized’) is obtained by eliminating its ultraviolet divergence terms, in one loop level. With given \(<\tilde{T}_{ab}[\psi]>_{\text{ren}}\) one can write two dimensional analogue of the metric back reaction equation (1) by regarding (10) and (11) such as follows.

\[
G_{ab} = -\frac{2\nabla_a \nabla_b \Phi}{\Phi} + g_{ab} \left( \frac{2 \nabla_c \nabla^c \Phi}{\Phi} + \frac{\partial_a \partial_b \Phi}{\Phi^2} - \frac{1}{\Phi^2} \right)
\]

\[
= 8 \pi G \frac{\Phi^2 \tilde{T}_{ab}[\psi]>_{\text{ren}}}{\Phi^2} \tag{16}
\]

and

\[
G_{b\theta} = \Phi \nabla_a \nabla^b \Phi - \frac{1}{2} R \Phi^2 = -4 \pi G <\Phi^2 \partial_a \psi \partial^b \psi>_{\text{ren}} \tag{17}
\]

where \(g_{ab}\) and \(\Phi\) is still treats as classical geometrical fields whereas the matter field \(\psi\) is assumed to be treat as quantum field. Furthermore we would not move \(\Phi\) outside the expectation quantities \(<\Phi^2 \tilde{T}_{ab}[\psi]>_{\text{ren}}\) and \(<\Phi^2 \partial_a \psi \partial^b \psi>_{\text{ren}}\) because variable dilaton field causes to violation of covariantly conservation of matter stress tensor \(\tilde{T}_{ab}[\psi]\) in its classical regime (see Eq. (15)). Applying (16) and (17) the Bianchi identity \(\nabla^a G_{\mu\nu} = 0\) in 4d leads to the following constraint condition.

\[
\nabla^a <\Phi^2 \tilde{T}_{ab}[\psi]>_{\text{ren}} = \nabla_b \left( \frac{1}{\Phi^2} \right) <\Phi^2 \partial_a \psi \partial^b \psi>_{\text{ren}}. \tag{18}
\]

In 4d space times \(\Phi\) and \(1/\psi\) has length dimension and conformal invariance property of the matter action in (3) is broken in 2d analogue (9). Hence it will be useful we define a dimensionless dilaton-matter field as

\[
\chi = \Phi \psi \tag{19}
\]

before than that we proceed to apply renormalization prescription and evaluate expectation value of its stress tensor operator \(<\tilde{T}_{ab}[\chi]>_{\text{ren}}\). Applying (19), one can rewrite matter part of the action (9) as

\[
I_{\text{matter}}[\chi; g_{ab}, \Phi] = 2 \pi \int \sqrt{g} dx^2 g^{ab} \{ \nabla_a \chi \nabla_b \chi + \chi^2 J_a J_b - \chi J_b \nabla_a \chi - \chi J_a \nabla_b \chi \}. \tag{20}
\]

Dynamical equation of the field \(\chi\) is obtained by varying the above action with respect to \(\chi\) as

\[
\{ \nabla_c \nabla^c - \frac{\nabla_c \nabla^c \Phi}{\Phi} \} \chi = 0 \tag{21}
\]
Trace free $T_{ab}^\sigma[\chi]$ stress tensor of the field $\chi$ is obtained by varying (20) with respect to $g^{ab}$ such as follows.

$$T_{ab}[\chi] = \nabla_a \nabla_b \chi + \chi^2 J_a J_b - \chi(J_a \nabla_b \chi + J_b \nabla_a \chi)$$

$$\frac{g_{ab}}{2} \{ \nabla_c \nabla^c \chi + \chi^2 J_c J^c - 2 \chi J_c \nabla^c \chi \}$$ (22)

which is equivalent with $\Phi^2 T_{ab}[\psi]$ and so we can deduce

$$< \Phi^2 T_{ab}[\psi] > \equiv < \hat{T}_{ab}[\hat{\chi}] >$$ (23)

and

$$< \Phi^2 \partial_\psi \hat{\psi} \partial_\psi \hat{\psi} > \equiv$$

$$< \partial_\chi \hat{\chi} > - 2 J^c \hat{\chi} + J_c J^c < \hat{\chi} > - \hat{\chi}^2 > .$$ (24)

In the following we seek renormalized expectation values of the quantities (22) and (24) by applying the Hadamard renormalization prescription.

This approach is begun with definition of the expectation value of stress tensor (22) such as follows.

$$< \hat{T}_{ab}[\hat{\chi}] > = \lim_{x_1 \to x} D_{ab}(x,x') G^+ (x,x')$$ (25)

where a state of $\hat{\chi}$ is characterized by a hierarchy of Wightman function which for a symmetric two-point function we have

$$G^+(x,x') = \frac{1}{2} < \hat{\chi}(x) \hat{\chi}(x') + \hat{\chi}(x') \hat{\chi}(x) >$$ (26)

and

$$D_{ab}(x,x') = g_b^c \nabla_a \nabla_b - g_a^c \nabla_a \nabla_b + J_a J_b$$

$$- J_a \{ \nabla_b + g_b^c \nabla_c \} - J_b \{ \nabla_a + g_a^c \nabla_c \}$$

$$- g^{ab} \{ g_{c}^c \nabla_c \nabla^c - J_c (\nabla^c + \nabla^c) + \frac{J_c J^c}{2} \}$$ (27)

with the bivector of parallel transport $g_{c}^c$, is the bilocal differential operator. This expression makes explicit that the singular character of the operator $\hat{T}_{ab}$ emerges as a consequence of the short-distance singularity of the symmetric two-point function $G^+(x,x')$. Equivalence principle suggest that the leading singularity of $G^+(x,x')$ should have a close correspondence to singularity structure of the two-point function of massless fields in Minkowski space [13]. In general the entire singularity of $G^+(x,x')$ may have a more complicated structure. Usually one assumes that $G^+(x,x')$ has a singular structure represented by the Hadamard expansions. This means that in a normal neighborhood of a point $x$ in 2d curved space time, we can suggest logarithmic dependence (Hadamard Green functions in 4d curved space times have singularities same as $\sigma^{-1}$ and

$$\ln \sigma \mid 1,2,15,16,17 \rangle \text{ of the Green function } G^+(x,x') \text{ for a massless quantum scalar field } \chi$$

$$G^+(x,x') = V(x,x') \ln \sigma(x,x') + W(x,x')$$ (28)

where $2\sigma(x,x') = \sigma\sigma$ with $\sigma = \nabla_a \sigma$ is one-half square of the geodesic distance between $x$ and $x'$. Non-singular two point functions $V(x,x')$, and $W(x,x')$ have the following power series expansions

$$V(x,x') = \sum_{n=0}^{\infty} V_n (x,x') \sigma^n$$ (29)

and

$$W(x,x') = \sum_{n=0}^{\infty} W_n (x,x') \sigma^n$$ (30)

where $V(x,x')$ ($W(x,x')$) is state-independent (dependent) 2 point functions. The Green function (28) satisfies the field equation (21) with respect to both points $x$ and $x'$ as

$$\{ \nabla_c \nabla^c - \frac{\nabla_c \nabla^c \Phi}{\Phi} \} G^+(x,x') = g^{-1/2} \delta^2(x-x')$$ (31)

where $\delta^2(x-x')$ is well known Dirac delta function in 2 dimensions. Applying (28), (29), (30) and (31), with $x' \neq x$, the coefficients $V_n(x,x')$ and $W_n(x,x')$ satisfies the following recursion relations.

$$2(n+1)^2 V_{n+1} + 2(3n+1) \nabla_a V_{n+1} \sigma^a + (\nabla_c \nabla^c - \frac{\nabla_c \nabla^c \Phi}{\Phi}) V_n = 0,$$ (32)

$$(\nabla_c \nabla^c - \frac{\nabla_c \nabla^c \Phi}{\Phi}) W_n + 2(n+1) \nabla_a W_{n+1} \sigma^a + 2(n+1)^2 W_{n+1}$$

$$+ 4(n+1) V_{n+1} + 2 \nabla_a V_{n+1} \sigma^a = 0.$$ (33)

Covariant Taylor series expansion for symmetric two point functions is written as [15,16] (see also [17])

$$\Gamma(x,x') = \Gamma(x) - \frac{1}{2} \nabla_a \Gamma(x) \sigma^a + \frac{1}{2} \Gamma_{ab}(x) \sigma^a \sigma^b + \frac{1}{4} \Gamma_{ab}(x) \sigma^a \sigma^b \sigma^c + O(\sigma^2)$$ (34)

which for $W(x,x')$ we obtain from coincidence limits

$$W(x) = \lim_{x' \to x} W(x,x') = \lim_{x' \to x} W_0(x,x')$$

$$= W_0(x) = \langle \hat{\chi}^2 \rangle_{ren} = \langle \Phi^2 \hat{\psi}^2 \rangle_{ren}.$$ (35)

The above renormalized expectation value is called vacuum state of the quantum dilaton-matter field $\chi$. Also
one can obtain from coincidence limits of the equations (32), (33), (34) and (35)

\[ V_1(x) = \frac{1}{2} \left\{ \nabla_c \nabla^c \Phi V_0(x) - V_0^c(x) \right\} , \tag{36} \]

\[ W_1(x) = V_0^c(x) - \frac{W_0^c(x)}{2} + \left( \frac{W_0(x)}{2} - V_0(x) \right) \frac{\nabla_c \nabla^c \Phi}{\Phi} \tag{37} \]

and

\[ W_{ab}(x) = W_{0ab}(x) + W_1(x)g_{ab}. \tag{38} \]

Applying (28) and (34) for \( \Gamma(x, x') = V_0(x, x') \) the equation (31) with \( x' \neq x \) leads to

\[ \{ \nabla_c \nabla^c - \frac{\nabla_c \nabla^c \Phi}{\Phi}\} W(x, x') = \]

\[ \frac{V_0(x)}{3} \left\{ R_{ab} \frac{\sigma \sigma^b}{\sigma} - \frac{1}{4} \nabla_a R_{bc} \frac{\sigma \sigma^b \sigma^c}{\sigma} \right\} + O(\sigma) \tag{39} \]

Inserting (34) for \( \Gamma(x, x') = W(x, x') \), the above equation reduces to the following conditions.

\[ W^c(x) = \frac{\nabla_c \nabla^c \Phi}{\Phi} W_0(x) + \frac{V_0(x)}{3} R \tag{40} \]

and

\[ \nabla^b \left[ 3\widetilde{W}_{0ab}(x) + \frac{g_{ab}}{4} (V_0(x)R - 3 \nabla_c \nabla^c W_0(x) - \lambda(x)) \right] = R_{ae} \nabla^c W_0(x) \tag{41} \]

where

\[ \widetilde{W}_{0ab}(x) = W_{0ab}(x) - \frac{1}{2} g_{ab} W_0^c(x) \tag{42} \]

and we used identities

\[ \nabla_c \nabla^c \nabla^b W_0(x) = \nabla^b \nabla_c \nabla^c W_0(x) + R^{ab} \nabla_a W_0(x), \tag{43} \]

\[ \nabla^b \nabla_a \nabla_b W_0(x) = \nabla_a \nabla_c \nabla^c W_0(x) + R_{ab} \nabla^b W_0(x). \tag{44} \]

We defined effective variable cosmological parameter \( \lambda(x) \) satisfying the constraint condition

\[ R \nabla_a V_0(x) = \nabla_a \lambda(x) \tag{45} \]

and also applied

\[ V_0^a(x) = V_0(x) \left( \frac{R}{6} + \frac{\nabla_c \nabla^c \Phi}{\Phi} \right), \quad V_1(x) = - \frac{V_0(x)}{12} R \tag{46} \]

\[ W_{ab}(x) = \widetilde{W}_{0ab}(x) + g_{ab} \left( \frac{V_0(x)}{6} R + \frac{W_0(x)}{2} \frac{\nabla_c \nabla^c \Phi}{\Phi} \right) \tag{47} \]

which are obtained from (36), (37), (38), (40). Now we subtract from \( G^+(x, x') \) defined by (28), a local symmetric two point function \( G^+_L(x, x') \) with the same short-distance singularity of the Hadamard expansion. Then we make a renormalized expectation value of stress tensor (25) as

\[ < \hat{T}_{ab}[\chi] >_{ren} = \lim_{x' \to x} D^{ab}(x, x') \{ G^+(x, x') - G^+_L(x, x') \} \tag{48} \]

which by applying (28) can be rewritten as

\[ < \hat{T}_{ab}[\chi] >_{ren} = \lim_{x' \to x} D^{ab}(x, x') \{ W(x, x') \}. \tag{49} \]

Explicit form of the nonsingular stress tensor (49) is obtained by inserting (34) [with \( \Gamma(x, x') = W(x, x') \)], (47) and taking its coincidence limit as

\[ < \hat{T}_{ab}[\chi] >_{ren} = \nabla_a \nabla_b W_0(x) - 2 \widetilde{W}_{0ab}(x) - \]

\[ \frac{3}{2} (J_a \nabla_b + J_b \nabla_a) W_0(x) + J_a J_b W_0(x) + \]

\[ g_{ab} \{ J_c \nabla^c W_0(x) - \frac{\nabla_c \nabla^c W_0(x)}{2} - J^c J_c W_0(x) \} \tag{50} \]

where \( < \hat{T}_{ab}[\chi] >_{ren} = -J_c \nabla^c W_0(x) \). With same calculation one can obtain for (24):

\[ < g^{ab} \nabla_a \hat{\chi} \nabla_b \hat{\chi} >_{ren} = \lim_{x' \to x} D(x, x') \{ W(x, x') \} = \]

\[ \frac{\nabla_c \nabla^c W_0(x)}{2} - J_c \nabla^c W_0(x) + J_c J^c W_0(x) - \]

\[ \frac{V_0(x)}{3} R + W_0(x) \frac{\nabla_c \nabla^c \Phi}{\Phi} \tag{51} \]

where we defined

\[ D(x, x') = g^{a'b'} \nabla_a \nabla_{a'}. \tag{52} \]

Applying (50) and identity (43) one can obtain

\[ \nabla^b \{ < \hat{T}_{ab}[\chi] >_{ren} + 2 \widetilde{W}_{0ab}(x) + \frac{3}{2} (J_a \nabla_b + J_b \nabla_a) W_0(x) \]

\[ - J_a J_b W_0(x) + g_{ab} \left\{ \frac{1}{2} J_c J^c W_0(x) - J_c \nabla^c W_0(x) \right\} \]

\[ - \frac{1}{2} \nabla_c \nabla^c W_0(x) \} = R_{ae} \nabla^e W_0(x). \tag{53} \]

Subtracting (41) from (53) we obtain

\[ \nabla^a \Sigma_{ab} = 0 \tag{54} \]
where $\Sigma_{ab}$ is general state independent divergence-less stress tensor relating to $<T_{ab}>_{ren}$ as

$$<T_{ab}>_{ren} = -\Sigma_{ab} + \tilde{W}_{ab}(x) + J_a J_b W_0(x) - \frac{3}{2}(J_a \nabla_b + J_b \nabla_a)W_0(x) + g_{ab}\left\{\frac{V_0(x)R}{4} - \frac{\lambda(x)}{4} - \frac{\nabla_c \nabla^c W_0(x)}{4} - \frac{J_c J^c W_0(x)}{2} + J_c \nabla^c W_0(x)\right\}$$

(55)

with

$$<T^a_{a}>_{ren} = -\Sigma^a_{a} + \frac{V_0(x)R}{2} - \frac{\lambda(x)}{2} - J_c \nabla^c W_0(x) = \frac{\nabla_c \nabla^c W_0(x)}{2}.$$ (56)

Applying (56) and trace of the equation (16) we obtain

$$3 \left(J_a \nabla_b + J_b \nabla_a\right) W_0(x) + g_{ab}\left\{\frac{V_0(x)R}{4} - \frac{\lambda(x)}{4} - \frac{\nabla_c \nabla^c W_0(x)}{4} - \frac{J_c J^c W_0(x)}{2} + J_c \nabla^c W_0(x)\right\}$$

(18) leads to the following constraint condition.

$$V_0(x) = \frac{3\Phi^2}{4\pi G} - \frac{3\Phi \nabla_c \nabla^c \Phi}{2\pi G R} +$$

$$\frac{3\Phi^2 J^b J^c \Phi + 5\Phi \nabla_c \nabla^c \Phi - 2\Phi \nabla_c \nabla^c \Phi}{8\pi G R(J^b J^c)}.$$ (61)

Applying the above result one can obtain explicit form of the cosmological parameter $\lambda(x)$ from (45) as

$$\lambda(x) = \int R(x)\nabla_a V_0(x)dx^a + \text{Constant.}$$ (62)

This equation denotes to fluctuations of the variable cosmological parameter $\lambda(x)$ satisfying to the wave equation

$$\nabla_c \nabla^c \lambda(x) - \nabla^c \ln R \nabla_c \lambda = R \nabla_c \nabla^c V_0(x).$$ (63)

This wave equation is derived from constraint condition (45) and its RHS treats as geometrical source.

However for a fixed 2d background metric $g_{ab}dx^a dx^b$, we obtained 6 equations defined by (54), (57), (58), (59), (61) and (62) which are not enough to determine seven quantities $W_0(x)$, $\lambda(x)$, $V_0(x)$, $\Phi(x)$, $\Sigma_{ab}$, $\Sigma^a_{a}$ and $W_{0ab}(x)$. Explicit form of all these quantities are depended to form of the dilaton field $\Phi$. What is its dynamical equation? In particular spherically symmetric static space times with $\Phi(r) = r$ one can continue to solve the above equations and obtain the foregoing dynamical fields but this is a bad restriction on our procedure. For general dynamical 4d spherically symmetric curved space times we should be have other management. Usually energy conditions play important role on the physical sources. We study energy conditions on 4d counter part of quantum matter stress tensor given by (51), (55) and (56) in the following section.

**IV. ENERGY CONDITIONS**

In general we consider time-like curves whose tangent 4-vector $V^\mu = (V^a, 0, 0)$, with $V^\mu V_\mu = \beta > 0$, $a = 0, 1$ and background metric signature $(-, +, +, +)$ which represents the radial velocity vector of a family observer. In the latter case weak (WEC) and strong (SEC) energy conditions leads to

$$WEC: \quad <\Phi^2 \tilde{T}_{ab}[\dot{\psi}] >_{ren} V^a V^b = \eta \geq 0$$ (64)

and

$$SEC: \quad <\Phi^2 \tilde{T}_{ab}[\dot{\psi}] >_{ren} V^a V^b -$$

$$\frac{1}{2} \left\{<\Phi^2 \tilde{T}_{a}^a[\dot{\psi}] >_{ren} - <\Phi^2 \partial_c \dot{\psi} \partial^c \dot{\psi} >_{ren}\right\} V^a V_a = \delta \geq 0.$$ (65)
There is also a null energy condition (NEC) for radial null vector field $N^\mu = (N^0, 0, 0)$ with $N^\mu N_\mu = 0$ and $a = 0, 1$ as

$$NEC : \quad < \phi^2 \hat{T}_{\alpha \beta} \hat{\phi} >_{ren} N^a N^b = \sigma \geq 0. \quad (66)$$

Obviously, the above energy conditions emerge directly from the geodesic structure of the spherically symmetric space time (8).

Defining

$$V^a J_a = \alpha, \quad V^a V_a = \beta > 0, \quad N^a J_a = \gamma \quad (67)$$

and applying (51), (55), and (56) the energy conditions (64), (65) and (66) leads to the following relations respectively.

$$WEC : \quad (W_{0ab} - \Sigma_{ab}) V^a V^b + W_0(x) \left( \alpha^2 - \frac{\beta}{2} J^c J_c \right) +$$

$$\beta J_c (3 \alpha V_c) \nabla^c W_0(x) +$$

$$\frac{\beta}{4} \left( V_0(x) R - \lambda - \nabla_c \nabla^c W_0(x) - 2 W^c_{0c}(x) \right) = \eta \quad (68)$$

$$SEC : \quad \Sigma_c = \frac{2(\delta - \eta)}{\beta} \frac{\lambda(x)}{2} + \frac{5V_0(x)}{6} R - \nabla_c \nabla^c W_0(x) -$$

$$\left( J_c J^c + \frac{\nabla_c \nabla^c \phi}{\phi} \right) W_0(x) \quad (69)$$

and

$$NEC : \quad (W_{0ab} - \Sigma_{ab}) N^a N^b + \gamma^2 W_0(x) -$$

$$3 \gamma N^c \nabla_c W_0(x) = \sigma. \quad (70)$$

Applying (57) and (58), the SEC given by (69) leads to the following wave equation.

$$\nabla_c \nabla^c \phi^2 - \left( J_c J^c + \frac{R}{2} \right) \phi^2 = 1 + \frac{8 \pi G(\eta - \delta)}{\beta} \quad (71)$$

where we used identity $2 \phi \nabla_c \nabla^c \phi + 2 \phi^2 J_c J^c = \nabla_c \nabla^c \phi^2$. This equation describes evolutions of surface area of apparent horizon $S = 4\pi \phi^2$ of the 4d spherically symmetric space time (7) propagating in 2d induced space time $g_{ab} dx^a dx^b$. With (71), our strategy about formulation of 2d analogue of the backreaction equation (1) and the renormalized expectation value of the quantum matter-dilaton field stress tensor operator is finished. It will be useful now we imply apparent horizon property of the 4d spherically symmetric curved space time (8) on our derived equations.

V. APPARENT HORIZON

Assuming $S = 4\pi \phi^2$ to be surface area of apparent horizon of the spherically symmetric curved space time (8), one can obtain its position by the null condition

$$g^{ab} \nabla_a S \nabla_b S = 0 \quad (72)$$

which by defining $J_a = \nabla_a \ln \phi$ leads to the condition

$$J_a J^a = 0. \quad (73)$$

In this case we can use $J_a = N_a$ as a suitable null vector field in the NEC (66) for which $\gamma = 0$ (see (67)). In this case the NEC given by (70) leads to

$$(W_{0ab}(x) - \Sigma_{ab}) J^a J^b = \sigma \geq 0. \quad (74)$$

Setting $\sigma = 0$ we can choose

$$W_{0ab}(x) = \Sigma_{ab}(x) + \xi g_{ab} \quad (75)$$

where $\xi$ is arbitrary constant parameter. Using (73) and (75) the WEC (68) and SEC (69) leads to respectively

$$WEC : \quad W^c_{0c}(x) = 2\xi + \frac{V_0(x) R}{2} - \frac{\lambda(x)}{2} \frac{\nabla_c \nabla^c W_0(x)}{2} +$$

$$\frac{2}{\beta} \left\{ \alpha^2 W_0(x) + (\beta J_c - 3 \alpha V_c) \nabla^c W_0(x) - \eta \right\} \quad (76)$$

and

$$SEC : \quad \Sigma_c = \frac{2(\delta - \eta)}{\beta} - \frac{\lambda(x)}{2} + \frac{5V_0(x)}{6} R -$$

$$\nabla_c \nabla^c W_0(x) - \frac{\nabla_c \nabla^c \phi}{\phi} W_0(x). \quad (77)$$

One of trivial solutions of the equation (45) is slow varying regime of the cosmological parameter $\lambda(x)$ for which we can exclude its derivatives as

$$\lambda(x) = \frac{4(\delta - \eta)}{\beta} \equiv constant, \quad V_0(x) = \frac{1}{20 \pi}. \quad (78)$$

Under the latter assumptions the anomaly trace (5.6) become

$$\Sigma_c \approx \frac{R}{24 \pi} - \omega \frac{\nabla_c \nabla^c \phi}{\phi} \quad (79)$$

in weak quantum field (WQF) limits as

$$W_0(x) \approx constant = \omega > 0 \quad (80)$$

by excluding its derivatives. The anomaly trace (79) follows well known one which is derived from zeta function regularization method in 2d dilaton quantum field theory [5,6,7,8,18,19,20,21,22,23,24,25,26,27,28,29,30] as

$$\Sigma_c(x) = \frac{1}{24 \pi} \left\{ R - \alpha \frac{\nabla_c \nabla^c \phi}{\phi} + (\alpha - 6) \frac{\nabla_c \phi \nabla^c \phi}{\phi^2} \right\} \quad (81)$$
The arbitrary parameter \( \alpha \) is the coefficient in question [30]. \( \alpha = -2 \) proposed by R. Bousso and S. W. Hawking [18] which turned out to be a mistake. \( \alpha = 4 \) obtained by Kummar et al. [7,19,20] for the same setup of the two-dimensional model as was used by Bousso and Hawking. \( \alpha = 6 \) obtained by Elizalde et al. [21] and V. Mukhanov, A. Wipf and A. Zelnikov [5]. This result turned out to be correct physically satisfying our statement about apparent horizon induction on the anomaly. In other word (81) reduces to (79) by setting

\[
\alpha = 6, \quad \omega = \frac{1}{4\pi}.
\] (82)

In strong quantum field limits where we can not exclude fluctuations of the field \( W_0(x) \) and so its derivatives should be considered in procedure one should be follow exact equations given in the previous section. Generally, our procedure is useful to study final state of quantum perturbed 4d spherically symmetric curved space times. Asymptotically flat classical static metric solution of the model (3) was obtained previously by Jains-Newman-Winicour (JNW) [31,32]. As a future work one can use presented formalism to study physical effect of the obtained anomaly on the quantum perturbed JNW metric solution. However we seek here slow varying limits of the back reaction equation and obtain its vacuum sector metric solution such as follows.

VI. SLOW VARYING LIMITS

Setting

\[
(\lambda(x), W_0(x), V_0(x)) = (\Lambda, \omega, \mu) = \text{constant}
\] (83)

and

\[
J_a = N_a, \quad J_a N^a = J^a J_a = \gamma = 0
\] (84)

where we assumed \( \Lambda \) to be the cosmological constant parameter the equations (56), (58), (59), (61), (68), (69), (70) and (71) reduce to the following forms respectively.

\[
\left( \omega - \frac{\Phi^2}{4\pi G} \right) \frac{\nabla_c \nabla^c \Phi}{\Phi} = \left( \frac{2\mu}{3} + \frac{\Phi^2}{4\pi G} \right) \frac{R}{2} \tag{85}
\]

\[
\Sigma^c = \frac{1}{4\pi G} \frac{\Lambda}{2} + \frac{\mu R}{2} + \frac{\Phi \nabla_c \nabla^c \Phi}{4\pi G} \tag{86}
\]

\[
\tilde{W}_{0ab}(x) = \tilde{\Sigma}_{ab} - \omega J_a J_b - \frac{1}{4\pi G} \left( \Phi \nabla_a \nabla_b \Phi \frac{1}{2} g_{ab} \Phi \nabla_c \nabla^c \Phi \right) \tag{87}
\]

\[
J^b \nabla^c (2\Phi \nabla_a \nabla_b - 5\Phi \nabla_c \nabla^c \Phi) = 0 \tag{88}
\]

\[
\eta = (W_{0ab}(x) - \Sigma_{ab}) V^a V^b + \alpha^2 \omega + \frac{\beta}{2}[\mu R - \Lambda - 2W_{0c}(x)] \tag{89}
\]

\[
\Sigma^c = \frac{(\delta - \eta)}{\beta} - \frac{\Lambda}{2} + \frac{5\mu R}{6} - \frac{\omega \nabla_c \nabla^c \Phi}{\Phi} \tag{90}
\]

\[
\sigma = (W_{0ab}(x) - \Sigma_{ab}) J^a J^b \tag{91}
\]

\[
\frac{\nabla_c \nabla^c \Phi}{\Phi} - \frac{R}{4} = \frac{1}{\Phi^2} \left( \frac{1}{2} + \frac{4\pi G(\eta - \delta)}{\beta} \right) \tag{92}
\]

Applying (85), (86), (90) and (92) one can results

\[
\eta = \delta \tag{93}
\]

\[
R = \frac{2}{\Phi^2} \left( \frac{\omega + \frac{\Phi^2}{4\pi G}}{\frac{4\mu}{3} - \omega + \frac{\Phi^2}{4\pi G}} \right) \tag{94}
\]

\[
\frac{\nabla_c \nabla^c \Phi}{\Phi} = \frac{\nabla_c J^c}{\Phi} = \frac{1}{\Phi^2} \left( \frac{2\mu}{3} + \frac{\Phi^2}{4\pi G} \right) \tag{95}
\]

and

\[
\Sigma^c = \frac{\mu}{4\pi G} \left( \frac{3}{2} - \frac{\Phi^2}{4\pi G} \right) + \omega \left( \frac{\Phi}{4} - \frac{1}{\Phi^2} + \omega + \frac{\Phi^2}{4\pi G} \right) - \frac{\Phi^2}{4\pi G} \tag{96}
\]

Inserting (93), (94), (95), and (96) the equations defined by (87), (88), (89) and (90) leads to the following forms respectively.

\[
\tilde{W}_{0ab}(x) = \Sigma_{ab} + \left( \omega - \frac{\Phi^2}{4\pi G} \right) J_a J_b - \frac{\Phi^2 \nabla_a J_b}{4\pi G} + \frac{\Phi^2}{2} \left[ \frac{2\mu}{3} - \omega \right] + \frac{(\omega - 3\mu)}{4\pi G} - \frac{\mu \omega}{\Phi^2} \left( \frac{1}{4\pi G} + \frac{\Lambda}{2} \right) \frac{\Phi^2}{4\pi G} \tag{97}
\]

\[
J^b \nabla^c J_b + 3J^b \nabla^c J_a = 0, \quad J_a = \nabla_a \ln \Phi \tag{98}
\]

\[
\eta = \frac{1}{24\pi G} \left[ \beta (\frac{3\omega - 2\mu}{4\pi G} - \omega + \frac{\Phi^2}{4\pi G}) - 6\Phi^2 (\alpha^2 + V^a V^b \nabla_a J_b) \right] \tag{99}
\]

and

\[
\sigma = \frac{\Phi^2 J^b \nabla_c \nabla^c J_b}{12\pi G} \tag{100}
\]

The equation (97) determines only traceless part of the tensor \( W_{0ab}(x) \). One can obtain its trace part \( W_{0c}(x) \) under the assumption

\[
\sigma = 0 \tag{101}
\]

for which the NEC is still satisfied. In the latter case (98) and (100) is eliminated trivially leading to the condition

\[
J^b \nabla^c J_b = 0 \tag{102}
\]
as follows.

We are now in position to write explicit form of the Green function (28) in terms of Hadamard series expansion. However one can rewrite the non-linear dilaton wave equation (95) same as poisson equation by defining a suitable dilaton field density $\rho(\Phi)$ such as follows.

$$\nabla \rho J = - \frac{\rho(\Phi)}{4\pi G} \tag{104}$$

where we defined self-interaction dilaton field density as

$$\rho(\Phi) = \frac{4\pi G}{\Phi^2} \left( \frac{\omega}{3} - 3 \Phi^2 \right) \tag{105}$$

where asymptotically flat Minkowski region $\Phi \to \infty$ is free of dilaton field $\rho(\Phi) \to 0$. Also the above density has a singularity at particular scale $\Phi_s = \sqrt{4\pi G (\frac{2\mu}{3} - \omega)}$ where $\omega \leq \frac{2\mu}{3}$. We are now in position to solve the metric backreaction equation.

VII. BACKREACTION METRIC SOLUTION

It is useful to choose conformaly flat frame in 2d space times for which the 2d part of the background metric (8) is given by

$$g_{ab}(x)dx^a dx^b = e^{f(\Phi)} du dv \tag{106}$$

where $(u, v)$ are suitable null coordinates. In this case one can obtain corresponding 2d Ricci scalar as

$$R = e^{-f(\Phi)} \nabla_c \nabla^c f(\Phi) = e^{-f(\Phi)} f'(\Phi) \nabla_c \nabla^c \Phi \tag{107}$$

where over-prime $'$ denotes to differentiation with respect to the dilaton field $\Phi$ and we used $J_c J^c = 0$. Applying (94), (95) and the above relation we obtain differential equation about the conformal factor of the metric such as follows.

$$e^{-f(s)} \frac{df(s)}{ds} = \frac{1}{s} \left( \omega + s \right) \left( \frac{2\mu}{3} + s \right) \tag{108}$$

where we defined dimensionless apparent horizon surface area as

$$s(\Phi) = \frac{\Phi^2}{4\pi G} \tag{109}$$

Integrating (108) one can obtain

$$e^{f(s)} = \frac{1}{\left( \frac{2\mu}{3} - 1 \right) \ln \left( \frac{2\mu}{3} + s \right) - \frac{3\omega}{2\mu} \ln s + C} \tag{110}$$

where $C$ is a suitable integral constant which should be fixed by using boundary condition of the space time. We assume that the obtained metric (110) has apparent horizon from point of view of a particular frame with coordinates $(t, s, \theta, \varphi)$. In this case location of the apparent horizon is obtained from the null-like condition of its hypersurface $s_H = \text{constant}$ as $g^{ss} \partial_s s_H \partial_s s_H = 0$ leading to the condition $e^{-f(s_H)} = 0$. If we set $s_H = 1$ (dimensionless Plank length) and use $e^{-f(s_H)} = 0$ then (110) leads to

$$C = \left( 1 - \frac{3\omega}{2\mu} \right) \ln \left( 1 + \frac{2\mu}{3} \right). \tag{111}$$

Using (111) the metric solution (110) become

$$e^{f(s)} = \frac{1}{\left( \frac{2\mu}{3} - 1 \right) \ln \left( \frac{2\mu}{3} + s \right) - \frac{3\omega}{2\mu} \ln s}. \tag{112}$$

VIII. CONCLUDING REMARKS

In this article we used 2d analogue of the Einstein-massless scalar gravity to study 4d spherically symmetric quantum field theory. Hadamard renormalization prescription is used to obtain renormalized matter-dilaton stress tensor in the presence of variable cosmological parameter which has critical role to satisfy the stress tensor covariantly conservation condition. Singularity of the Hadamard Green function is assumed to be has logarithmic type same as the Green function in 2d Minkowski flat space time satisfying the general covariance condition. Our procedure has an advantage with respect to other methods such as zeta function regularization: Applying energy conditions (SEC, WEC, NEC) on the renormalized quantum matter dilaton field stress tensor we obtained dynamical equations of the dilaton field $\Phi$, quantum vacuum state $W_\Phi(x)$ and variable cosmological parameter $\lambda(x)$ respectively. This is still an important problem in the Hadamard renormalization prescription used in general form of background metric in higher than 2 dimensions. In slow varying limits of quantum fields our obtained anomaly trace satisfies the well known one which is obtained from zeta function regularization method. In the slow varying limits of the fields we solved the back reaction equation and obtained metric solution containing a horizon.

REFERENCES

1. N. D. Birrell and P. C. W. Davies, Quantum Fields in Curved space, Cambridge University press, Cambridge, England, (1982).

2. L. Parker and D. Toms Quantum Field Theory in Curved space-time, Cambridge University Press, Cambridge (2009).

3. R. M. Wald, Phys. Rev. D17, 1477 (1978).
4. S. M. Christensen and S. A. Fulling, Phys. Rev. D15, 2088 (1977).
5. V. Mukhanov, A. Wipe and A. Zelnikov, Phys. Lett B332, 283 (1994), [hep-th/9403018]
6. R. Balbinot and A. Fabbri, Phys. Rev. D59, 044031 (1999), [hep-th/9807123]
7. W. Kummer, H. Liebl and D. V. Vassilevich, Mod. Phys. Lett. A12, 2683 (1997), [hep-th/9707041]
8. R. Balbinot and A. Fabbri, Phys. Lett. B459, 112 (1999), [gr-qc/9904034]
9. P. Thomi, B. Isaak, and P. Hajicek, Phys. Rev. D30, 1168 (1984).
10. Shri Ram and M. K. Verma, ADVANCED RESEARCH in PHYSICS AND ENGINEERING, University of Cambridge, UK, February 20-22 (2010), Published by WSEAS press, ISBN: 1790-5117, ISSN:978-960-474-163-2, Pages:23-28.
11. Antonio Alfonso-Faus, Recent Researches in Artificial Intelligence, Knowledge Engineering and Data Bases, University of Cambridge, UK, February 20-22 (2011), Published by WSEAS press, ISBN:978-960-474-273-8, Pages:249-254.
12. Andrew Waloott Bechwith, Advances in Applied Mathematics, Gdansk University of Technology, Poland, May 15-17 (2014), Published by WSEAS press, ISBN:978-960-474-380-3, Pages:313-322.
13. R. Haag. H. Naruhofer and U. Stein, Commun. Math. Phys. 94, 219, (1984).
14. M. Gasperini, Elements of String Cosmology, Cambridge University press (2007).
15. M. R. Brown, J. Math. Phys.25(1), 136 (1984).
16. D. Bernard and A. Folacci, Phys. Rev. D34, 2286 (1986).
17. H. Ghafernejad and H. Salehi, Phys. Rev. D 56, 4633, (1997); 57, 5311 (E) (1998).
18. R. Bousso and S. W. Hawking, Phys. Rev. D56, 7788 (1997).
19. W. Kummer, H. Liebl and D. V. Vassilevich, Phys. Rev. D58, 108501 (1998), [hep-th/9801122]
20. S. Ichinose, Phys. Rev. D57, 6224, (1998), [hep-th/9707025]
21. E. Elizalde, S. Naftulim, S. O. Odintsov, Phys. Rev. D49, 2852 (1994), [hep-th/9308020]
22. S. Nojiri and S. D. Odintsov, Mod. Phys. Lett. A12, 2083 (1997), [hep-th/9706009]
23. S. Nojiri and S. D. Odintsov, Phys. Rev. D57, 2363 (1998), [hep-th/9706143]
24. S. Nojiri and S. D. Odintsov, Phys. Lett B416, 85 (1998), [hep-th/9708139]
25. S. Nojiri and S. D. Odintsov, Phys. Lett. B426, 29 (1998), [hep-th/9801052]
26. S. Nojiri and S. D. Odintsov, Phys. Rev. D57, 4847 (1998), [hep-th/9801180]
27. S. J. Gates Jr, T. Kadoyoshi and S. D. Odintsov, Phys. Rev. D58, 084026 (1998), [hep-th/9802139]
28. P. Van Nieuwenhuizen, S. Nojiri and S. D. Odintsov, Phys. Rev. D60, 084014 (1999), [hep-th/9901119]
29. S. Nojiri and S. D. Odintsov, Int. J. Mod. Phys. A16, 1015 (2001), [hep-th/0009202]
30. J. S. Dowker, Class. Quantum Grav 15, 1881 (1998), [hep-th/9802029]
31. A. I. Janis, E. T. Newman and J. Winicour, Phys. Rev. Lett. 20, 878, (1968).
32. K. S. Virbhadra, D. Narasimha and S. M. Chitre, Astron. Astrophys. 337, 1-8 (1998).