Abstract

It is shown that two dimensional (2d) topological gravity in the conformal gauge has a larger symmetry than has been hitherto recognized; in the formulation of Labastida, Pernici and Witten it contains a twisted “small” $N = 4$ superconformal symmetry. There are in fact two distinct twisted $N = 2$ structures within this $N = 4$, one of which is shown to be isomorphic to the algebra discussed by the Verlindes and the other corresponds, through bosonization, to $c_M \leq 1$ string theory discussed by Bershadsky et.al. As a byproduct, we find a twisted $N = 4$ structure in $c_M \leq 1$ string theory. We also study the “mirror” of this twisted $N = 4$ algebra and find that it corresponds, through another bosonization, to a constrained topological sigma model in complex dimension one.
Our understanding of the underlying symmetry structure of string theory will be greatly facilitated in its formulation in a phase where the basic symmetries are unbroken. It has been suggested that topological gravity [1] and topological sigma models [2] provide such a framework namely, they describe the unbroken phase of ordinary gravity and the usual sigma models. This exotic gravitational theory where the general covariance remains unbroken has been formulated and clarified in two space–time dimensions by several groups [3–5]. In the original formulation of 2d topological gravity Labastida, Pernici and Witten started with a Lagrangian which is identically zero and therefore contains all the symmetries of the world. The basic field is the 2d metric, and the symmetries are chosen to be the combined diffeomorphism and an arbitrary shift of the metric denoted by a symmetric local GL(2,R) parameter. These symmetries are then fixed by introducing ghosts and ghosts of ghosts to obtain topological gravity action as a free conformal field theory.

In a somewhat different formulation Verlinde and Verlinde [6] wrote down 2d topological gravity action as a topological gauge theory where the gauge group is the 2d Poincare group ISO(2). In their formulation supersymmetry plays a more fundamental role. It has also been shown in [7] that 2d topological gravity in the conformal gauge possesses a large class of symmetries where the associated symmetry generators satisfy a twisted $N = 2$ superconformal algebra. The associated superconformal model has central charge $c^{N=2} = -9$ indicating that the superconformal model is non-unitary.

In this paper, we consider the topological gravity action of Labastida, Pernici and Witten (LPW). In the conformal gauge, we show that this action has a twisted $N = 2$ superconformal symmetry which is isomorphic to the symmetry algebra discussed by Verlinde and Verlinde. We also show, that the topological gravity action of LPW contains another distinct twisted $N = 2$ superconformal symmetry where the topological charge is the BRST charge associated with the diffeomorphism invariance. These two twisted $N = 2$ superconformal algebras are then shown to combine to form a bigger algebra — a twisted “small” $N = 4$ superconformal algebra [8,9] with the associated central charge $-9$. The topological gravity action, in fact, is invariant under these twisted $N = 4$ symmetry charges. In order to find an interpretation of the latter twisted $N = 2$ structure we make use of a bosonization [10] by which we relate the fields in the topological gravity to those of $c_M \leq 1$ string theory. We find that this $N = 2$ structure is precisely the same as appeared in the context of $c_M \leq 1$ string theory in ref.[11–13] with a particular central charge. In
this way we also recover the full twisted “small” $N = 4$ structure in $c_M \leq 1$ string theory as well. We then study the “mirror” [14] of this twisted $N = 4$ algebra. We find that by properly identifying the fields and by making use of another bosonization [15] they represent the generators of a constrained topological sigma model in complex dimension one. Thus, twisted $N = 4$ superconformal algebra provides a unifying framework for 2d topological gravity, $c_M \leq 1$ string theory and constrained topological sigma model.

The gauge fixed action of 2d topological gravity formulated in [3] has the form,

$$S = 2 \int_{\Sigma} d^2 \sigma \sqrt{-g} \left( -i b^{ij} D_i c_j - B^{ij} D_i \phi_j \right)$$

(1)

Here integration is over the 2d compact, boundaryless manifold $\Sigma$. $i, j$ are the 2d indices, $g_{ij}$ is a fixed metric on $\Sigma$ and covariant derivatives are with respect to this metric. $(b^{ij}, c^i)$ is the fermionic reparametrization ghost system whereas $(B^{ij}, \phi^i)$ is the bosonic deformation ghost of ghost system. The fields $b^{ij}$ and $B^{ij}$ are symmetric and traceless. The above action is invariant under the following BRST transformations,

$$\delta_Q b^{ij} = \epsilon_Q \left( ib^{k(i} D_{k} c^{j)} - 2i b^{ij} D_k c^k - i (D_k b^{ij}) c^k + B^{k(i} D_{k} \phi^{j)} - 2B^{ij} D_k \phi^k \right)$$

$$- (D_k B^{ij}) \phi^k - ig^{ij} b^{kl} D_k c_l - g^{ij} B^{kl} D_k \phi_l \right)$$

$$\delta_Q c^i = \epsilon_Q \left( ic^k D_k c^i + \phi^i \right)$$

$$\delta_Q B^{ij} = \epsilon_Q \left( 2B^{ij} D_k c^k - B^{k(i} D_{k} c^{j)} + (D_k B^{ij}) c^k + g^{ij} B^{kl} D_k c_l + b^{ij} \right)$$

$$\delta_Q \phi^i = \epsilon_Q \left( c^k D_k \phi^i - (D_k c^i) \phi^k \right)$$

(2)

(3)

(4)

(5)

where $\epsilon_Q$ is an infinitesimal fermionic parameter and we have defined $A^{i(B^j)} \equiv A^i B^j + A^j B^i$. It is also straightforward to check from (1) that the right hand side of (2) is the energy–momentum tensor defined as,

$$T^{ij} = -\frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g_{ij}}$$

(6)

Therefore, Eq.(2) tells us that the energy–momentum tensor is BRST-exact, one of the basic properties of a topological field theory [16]. Since (1) defines a free conformal system we rewrite the action in the conformal gauge $g_{zz} = g_{\bar{z}\bar{z}} = \frac{1}{2}$, $g_{z\bar{z}} = g_{\bar{z}z} = 0$. With the redefinitions $b_{zz} \equiv b$, $c^z \equiv -ic$, $B_{zz} \equiv -i\beta$, $\phi^z \equiv -i\gamma$ and similarly $b_{\bar{z}\bar{z}} \equiv \bar{b}$, $c^\bar{z} \equiv -i\bar{c}$, $B_{\bar{z}\bar{z}} \equiv -i\bar{\beta}$, $\phi^{\bar{z}} \equiv -i\bar{\gamma}$, the action (1) takes the more familiar form [17],

$$S = \int_{\Sigma} d^2 z ( -b \bar{\partial} c + \beta \bar{\partial} \gamma + \text{h.c.})$$

(7)
The BRST transformations (2–5) in this gauge reduces to,

\[ \delta_Q b(z) = \epsilon_Q T(z) \quad (8) \]
\[ \delta_Q c(z) = \epsilon_Q \left( c(z) \partial c(z) + \gamma(z) \right) \quad (9) \]
\[ \delta_Q \beta(z) = \epsilon_Q \left( G(z) - 2b(z) \right) \quad (10) \]
\[ \delta_Q \gamma(z) = \epsilon_Q \left( c(z) \partial \gamma(z) - \partial c(z) \gamma(z) \right) \quad (11) \]

and similar expressions for the fields \( \bar{b}(\bar{z}), \bar{c}(\bar{z}), \bar{\beta}(\bar{z}) \) and \( \bar{\gamma}(\bar{z}) \). Since here we are dealing with a free conformal field theory, the fields split up into holomorphic and antiholomorphic sectors. We only concentrate on the holomorphic sector. In (8) \( T(z) \) is given by,

\[ T(z) = 2 : \beta(z) \partial \gamma(z) : + : \partial \beta(z) \gamma(z) : - 2 : b(z) \partial c(z) : - \partial b(z) c(z) : \quad (12) \]

and also \( G(z) \) in (10) is defined as

\[ G(z) = G_s(z) + G_v(z) \quad (13) \]

where,

\[ G_s(z) = 2\beta(z) \partial c(z) + \partial \beta(z) c(z) \]
\[ G_v(z) = b(z) \quad (14) \]

Here \( T(z) \) is the energy-momentum tensor and \( G(z) \) its \( Q \)-partner in the conformal gauge. It has been noted by Verlinde and Verlinde that \( 2d \) topological gravity in the conformal gauge possesses a large class of symmetries \([6,7]\). In fact, all known topological conformal field theories contain a twisted \( N = 2 \) superconformal algebra (TCA) generated by two bosonic \( (T(z) \text{ and } J(z)) \) and two fermionic \( (G(z) \text{ and } Q(z)) \) currents. The operator product expansions (OPE) among the currents are given by:

\[ T(z)T(w) \sim \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} \quad (15) \]
\[ T(z)Q(w) \sim \frac{Q(w)}{(z-w)^2} + \frac{\partial Q(w)}{(z-w)} \quad (16) \]
\[ T(z)G(w) \sim \frac{2G(w)}{(z-w)^2} + \frac{\partial G(w)}{(z-w)} \quad (17) \]
\[ T(z)J(w) \sim -\frac{c^{N=2}/3}{(z-w)^2} + \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{(z-w)} \quad (18) \]
\[ J(z)Q(w) \sim \frac{Q(w)}{(z-w)} \quad (19) \]
The fermionic currents $Q(z)$ and $G(z)$ have conformal weights 1 and 2 respectively. Also, the $U(1)$ current $J(z)$ is a quasi-primary field. We observe as well that the $U(1)$ charges of $Q(z)$ and $G(z)$ are +1 and −1 respectively. Finally, we note from (22) that the energy-momentum tensor is in fact $Q$-exact. In order to find this structure in the topological gravity formulation of LPW, we first use the $Q$-transformations given in (8–11) and then derive the corresponding current by Noether’s method. The resulting current is:

$$
\hat{Q}(z) = 2c(z) : \beta(z) \partial \gamma(z) : + c(z) : \partial \beta(z) \gamma(z) : - c(z)b(z) \partial c(z) : \\
- \partial \left( : \beta(z) \gamma(z) : c(z) \right) + b(z) \gamma(z).
$$

(24)

It is easy to verify that this current $\hat{Q}(z)$ is not a good candidate for the twisted $N = 2$ generator $Q(z)$, since the OPE of $\hat{Q}(z)$ with itself is not regular (note (23)), but given as:

$$
\hat{Q}(z)\hat{Q}(w) \sim \frac{2\gamma(w)}{(z-w)^3} + \frac{\partial \gamma(w)}{(z-w)^2} + \frac{\partial \left( \gamma(w) : c(w)b(w) : + \frac{3}{2} \partial \gamma(w) \right)}{(z-w)}.
$$

(25)

We observe that it is possible to modify $\hat{Q}(z)$ by adding total derivative terms such that the OPE becomes regular. The new current in that case takes the form:

$$
Q(z) = \hat{Q}(z) - \partial \left( : \beta(z) \gamma(z) : c(z) \right) - \frac{3}{2} \partial^2 c(z).
$$

(26)

Note that with this modification the basic relations (8–11) remain untouched.

It is also clear from (10), (13) and (14) that, since the BRST charge is nilpotent, we have:

$$
\delta_Q b(z) = \delta_Q \left( 2\beta(z) \partial c(z) + \partial \beta(z) c(z) \right) = \epsilon_Q T(z)
$$

(27)

So, comparing with the TCA (15–23), we find two different candidates for the $G(z)$ current, namely, $G_s(z)$ and $G_v(z)$ as given in (14). Let us first consider $G_s(z)$ as our

** The basic OPE's we use are $\beta(z)\gamma(w) \sim \frac{1}{z-w}$ and $b(z)c(w) \sim \frac{1}{z-w}$.
candidate current. This current appeared in the formulation of Verlinde and Verlinde [6,7]. We note from (22) that the pole order 2 of the OPE with $Q(z)$ and $G(w)$ will generate the $U(1)$ current and it has the form:

$$J(z) = :c(z)b(z): + 2 :\beta(z)\gamma(z):.$$  \hspace{1cm} (28)

One can easily verify that the currents $T(z)$ in (12), $G_s(z)$ in (14), $Q_s(z)$ in (26) and $J(z)$ in (28) indeed satisfy a TCA (15–23) with central charge $c_{N=2} = -9$. This algebra can be shown to be isomorphic with the symmetry algebra discussed by Verlinde and Verlinde, since the generators in both formulations are related by a similarity transformation as follows:

$$T(z) = UT(z)U^{-1}$$
$$Q_s(z) = UQ(z)U^{-1} \equiv U\left(Q_v(z) + Q_s(z)\right)U^{-1}$$
$$G_s(z) = UG_s(z)U^{-1}$$
$$J(z) = UJ(z)U^{-1}$$

where we have defined

$$Q_v(z) = 2c(z) :\beta(z)\partial\gamma(z) : + c(z) :\partial\beta(z)\gamma(z) : - :c(z)b(z)\partial c(z): - 2\partial\left(:\beta(z)\gamma(z) : c(z)\right) - \frac{3}{2}\partial^2 c(z)$$

$$Q_s(z) = b(z)\gamma(z)$$  \hspace{1cm} (30)

and $U =: \exp\left[-\frac{1}{2} \oint dz c(z) G_s(z)\right] :$ is a unitary operator. It is evident from (29) that the action (1) is individually invariant under $Q_s$ and $Q_v$. This, in fact, helps us to recover another distinct set of generators which form a TCA with the same central charge $c_{N=2} = -9$. This second set consists of $T(z)$ and $J(z)$ as given in (12) and (28), but the fermionic currents have the form:

$$Q_v(z) = 2c(z) :\beta(z)\partial\gamma(z) : + c(z) :\partial\beta(z)\gamma(z) : - :c(z)b(z)\partial c(z): - 2\partial\left(:\beta(z)\gamma(z) : c(z)\right) - \frac{3}{2}\partial^2 c(z)$$

$$G_v(z) = b(z).$$  \hspace{1cm} (31)

Previous analysis, therefore, brings out two distinct $N = 2$ superconformal structures in the topological gravity formulation of LPW. At this point, it is natural to ask whether these two $N = 2$ structures are parts of a bigger algebra present in 2d topological gravity.
We find the answer in the positive. With the introduction of two more additional currents, we find that the following eight generators:

\[
T(z) = 2 : \beta(z) \partial \gamma(z) : + : \partial \beta(z) \gamma(z) : - 2 : b(z) \partial c(z) : - \partial b(z) c(z) :
\]

\[
Q_s(z) \equiv G_1^+(z) = : b(z) \gamma(z) :
\]

\[
G_s(z) \equiv G_1^-(z) = 2 \beta(z) \partial c(z) + \partial \beta(z) c(z)
\]

\[
Q_v(z) \equiv G_2^+(z) = 2c(z) : \beta(z) \partial \gamma(z) : + c(z) : \partial \beta(z) \gamma(z) : - : c(z) b(z) \partial c(z) :
\]

\[
-2 \partial \left( : \beta(z) \gamma(z) : c(z) \right) - \frac{3}{2} \partial^2 c(z)
\]

\[
G_v(z) \equiv G_2^-(z) = b(z)
\]

\[
J(z) = : c(z) b(z) : + 2 : \beta(z) \gamma(z) :
\]

\[
J^{++}(z) = : b(z) c(z) : \gamma(z) - : \gamma^2(z) \beta(z) : - \frac{3}{2} \partial \gamma(z)
\]

\[
J^{--}(z) = \beta(z)
\]

form a twisted “small” $N = 4$ superconformal algebra. Here we have made changes of notation according to the $U(1)$ charge. For example, $G^\pm(z)$ have $U(1)$ charges $\pm 1$ and $J^{\pm\pm}(z)$ have $\pm 2$. It is easy to check that all the $N = 4$ generators in (33) are in fact currents associated to symmetries of the action (1). The rest of the relevant OPEs among the currents are given below:

\[
G_1^+(z) G_2^+(w) \sim \frac{2 J^{++}(w)}{(z-w)^2} + \frac{\partial J^{++}(w)}{(z-w)}
\]

\[
G_1^-(z) G_2^-(w) \sim - \frac{2 J^{--}(w)}{(z-w)^2} - \frac{\partial J^{--}(w)}{(z-w)}
\]

\[
G_1^+(z) G_2^-(w) \sim G_1^-(z) G_2^+(w) \sim 0
\]

\[
J^{--}(z) J^{++}(w) \sim - \frac{3}{2} \frac{J(w)}{(z-w)^2} - \frac{J(w)}{(z-w)}
\]

\[
G_1^+(z) J^{--}(w) \sim - \frac{G_2^-(w)}{(z-w)}
\]

\[
G_2^+(z) J^{--}(w) \sim \frac{G_1^+(w)}{(z-w)}
\]

\[
G_1^-(z) J^{++}(w) \sim \frac{G_2^+(w)}{(z-w)}
\]

\[
G_2^-(z) J^{++}(w) \sim - \frac{G_1^+(w)}{(z-w)}
\]

\[
J^{++}(z) J^{++}(w) \sim J^{--}(z) J^{--}(w) \sim 0
\]

\[
G_1^+(z) J^{++}(w) \sim G_2^+(z) J^{++}(w) \sim 0
\]
Figure 1: Here we depict the different subalgebras present in the twisted “small” $N = 4$ algebra found in 2d Topological Gravity. The dashed lines contains the twisted $N = 2$ subalgebra isomorphic to the symmetry algebra discussed before by the Verlindes. Inside the continuous line we present the twisted $N = 2$ algebra which, through bosonization, have been shown to correspond to the $c_M \leq 1$ string theory. Finally, the subalgebra within the dotted line corresponds to an $SL(2, R)$ Kac-Moody algebra of level $-\frac{3}{2}$.

\[ G_1^-(z) J^{--}(w) \sim G_2^-(z) J^{--}(w) \sim 0 \]

Note that the currents $J(z)$, $J^{++}(z)$ and $J^{--}(z)$ form an $SL(2, R)$ Kac-Moody algebra at level $-3/2$. Thus, we found a bigger symmetry algebra, namely, a twisted “small” $N = 4$ superconformal algebra in the topological gravity of LPW (see Figure 1). A similar twisted $N = 4$ superconformal structure have recently been observed in the formulation of Verlinde and Verlinde in ref. [19].

It should be remarked here that although we find two twisted $N = 2$ structures in 2d topological gravity, the currents appeared in the second set (31) and (32) do not have a natural interpretation in terms of the fields $\beta(z)$, $\gamma(z)$, $b(z)$ and $c(z)$. A natural interpretation can be given if we bosonize the deformation ghost of ghost system as follows:

\[
\beta(z) = \frac{1}{2} \left[ \left( \lambda - \frac{1}{\lambda} \right) \partial \phi_L(z) + i \left( \lambda + \frac{1}{\lambda} \right) \partial \phi_M(z) \right] e^{-i\lambda \left( \phi_M(z) - i\phi_L(z) \right)} ;
\gamma(z) = e^{i\lambda \left( \phi_M(z) - i\phi_L(z) \right)} ; \tag{35} \]

where $\phi_M(z)$ and $\phi_L(z)$ are two bosonic fields and $\lambda$ is a constant. With this bosonization (31) becomes precisely the BRST current of $c_M \leq 1$ string theory once we identify $\phi_L(z)$
and \( \phi_M(z) \) as the Liouville and matter field of \((p, q)\) minimal models coupled to gravity system and \( \lambda = \sqrt{q/2p} \). This bosonization have been used in [10] in order to describe topological gravity structure in any \((p, q)\) minimal model coupled to gravity.

Using this bosonization, we write down all the eight generators (33) in terms of the matter, Liouville and the reparametrization ghosts \((b(z), c(z))\) of \(c_M \leq 1\) string theory as follows:

\[
T(z) = -\frac{1}{2} : (\partial \phi_M(z))^2 : + iQ_M \partial^2 \phi_M(z) - \frac{1}{2} : (\partial \phi_L(z))^2 : + iQ_L \partial^2 \phi_L(z)
\]

\[-2 : b(z) \partial c(z) : - : \partial b(z) c(z) :\]

\[
G_1^+(z) = : b(z) e^{i\lambda (\phi_M(z) - i\phi_L(z))} : 
\]

\[
G_1^-(z) = : c(z) \left[ -\frac{1}{2} (\partial \phi_M(z))^2 + iQ_M \partial^2 \phi_M(z) - \frac{1}{2} (\partial \phi_L(z))^2 + iQ_L \partial^2 \phi_L(z) - 2b(z) \partial c(z) \right. 
\]

\[-\partial b(z) c(z) : + \partial \left[ (\lambda + 1) c(z) \partial \phi_L(z) + i (\lambda - 1) c(z) \partial \phi_M(z) \right] - \frac{3}{2}(\partial^2 c(z) \right)
\]

\[
G_2^+(z) = : b(z) c(z) - \left( \lambda + \frac{1}{\lambda} \right) \partial \phi_L(z) - i \left( \lambda - \frac{1}{\lambda} \right) \partial \phi_M(z) 
\]

\[
J^{++}(z) = : b(z) c(z) - \frac{1}{2\lambda} \left[ i \partial \phi_M(z) - \partial \phi_L(z) \right] : e^{i\lambda (\phi_M(z) - i\phi_L(z))} : 
\]

\[
J^{--}(z) = : \frac{1}{2} \left[ (\lambda - \frac{1}{\lambda}) \partial \phi_L(z) + i (\lambda + \frac{1}{\lambda}) \partial \phi_M(z) \right] e^{-i\lambda (\phi_M(z) - i\phi_L(z))} : 
\]

Here \( Q_L = i \left( \lambda + \frac{1}{2\lambda} \right) \) and \( Q_M = - \left( \lambda - \frac{1}{2\lambda} \right) \) are the background charges of Liouville and matter sectors respectively. Also we note that \( J^{++}(z) \) is precisely one of the ground ring generators of \(c_M \leq 1\) string theory. One can explicitly verify that the generators (36) form a twisted “small” \(N = 4\) algebra with central charge \(-9\), part of which \((T(z), G^+_2(z), G^-_2(z) and J(z))\) has been discussed in ref. [12,13].

In ref. [13], a one parameter family of twisted \(N = 2\) superconformal algebra has been observed in \(c_M \leq 1\) string theory. We here note that for a particular value of this parameter \( a_3 = -3/2\), the symmetry algebra gets enlarged and becomes a part of the twisted \(N = 4\) superconformal algebra. Thus, \(N = 4\) algebra provides a unifying
framework for 2d topological gravity and $c_M \leq 1$ string theory.

Once we have a twisted $N = 4$ superconformal algebra generated by $T(z), G_1^+(z), G_1^-(z), G_2^+(z), G_2^-(z), J(z), J^{++}(z)$ and $J^{--}(z)$, we can find another independent set of generators by applying the so-called “mirror” transformation as follows [20]:

$$
T^*(z) = T(z) - \partial J(z), \quad J^*(z) = -J(z)
$$

$$
G_1^{++}(z) = G_1^-(z), \quad G_2^{++}(z) = G_2^-(z)
$$

$$
G_1^{--}(z) = G_1^+(z), \quad G_2^{--}(z) = G_2^+(z)
$$

$$
J^{++}(z) = J^{--}(z), \quad J^{--}(z) = J^{++}(z)
$$

Written explicitly in terms of the fields of topological gravity, the $N = 4$ generators in the mirror representation take the form:

$$
T^*(z) = -: \partial \beta(z) \gamma(z) : -: b(z) \partial c(z) :
$$

$$
G_1^{++}(z) = 2 \beta(z) \partial c(z) + \partial \beta(z) c(z)
$$

$$
G_2^{++}(z) = b(z)
$$

$$
G_1^{--}(z) = b(z) \gamma(z)
$$

$$
G_2^{--}(z) = 2 : \beta(z) \gamma(z) : \partial c(z) - : \partial \beta(z) \gamma(z) : c(z) - : c(z) b(z) \partial c(z) : -\frac{3}{2} \partial^2 c(z)
$$

$$
J^*(z) = -: c(z) b(z) : -2 : \beta(z) \gamma(z) :
$$

$$
J^{++}(z) = \beta(z)
$$

$$
J^{--}(z) = : b(z) c(z) : \gamma(z) - : \gamma^2(z) \beta(z) : -\frac{3}{2} \partial \gamma(z)
$$

It should be pointed out that the generators (38) have precisely the same form as the $N = 4$ generators of 2d topological gravity (33) except $T^*(z)$, which makes the physical interpretation of the theory completely different. We note that with respect to $T^*(z)$, $\beta(z)$, $\gamma(z)$, $b(z)$ and $c(z)$ have conformal weights 0, 1, 1, and 0 respectively instead of 2, $-1$, 2, and $-1$ as in the topological gravity. Identifying $\beta(z) \equiv x(z)$, $\gamma(z) \equiv \partial \overline{\pi}(z)$, $b(z) \equiv B(z)$ and $c(z) \equiv C(z)$, we note that $T^*(z)$ in (38) becomes the energy-momentum tensor of a constrained topological sigma model in complex dimension $d = 1$, where $x(z)$ denotes one of the complex target space coordinates of the topological sigma model and $\overline{\pi}(z)$ denotes the holomorphic part of the complex conjugate of $x(z)$. $B(z)$ and $C(z)$ are the fermionic fields. With the above identifications, the generators in the “mirror” theory takes the form:

$$
T^*(z) = -: \partial x(z) \partial \overline{\pi}(z) : -: B(z) \partial C(z) :
$$
\[ G^+_{1}(z) = 2x(z)\partial C(z) + \partial x(z)C(z) \]
\[ G^+_{2}(z) = B(z) \]
\[ G^-_{1}(z) = B(z)\partial \tau(z) \]
\[ G^-_{2}(z) = 2 : x(z)\partial \tau(z) : \partial C(z) - C(z) : \partial x(z)\partial \tau(z) : - : C(z)B(z)\partial C(z) : -\frac{3}{2}\partial^2 C(z) \]
\[ J^+(z) = - : C(z)B(z) : -2 : x(z)\partial \tau(z) : \]
\[ J^{++}(z) = x(z) \]
\[ J^{--}(z) = : B(z)C(z) : \partial \tau(z) - : x(z)(\partial \tau(z))^2 : -\frac{3}{2}\partial^2 \tau(z) \]

In this way, we also uncover a twisted \( N = 4 \) structure in the constrained topological sigma model in complex dimension one. This constrained topological sigma model has also been shown in [15] to be related with the \( c_M \leq 1 \) string through the following bosonization:

\[ x(z) = : b(z)c(z) - \frac{i}{2\lambda}\left(\partial \phi_M(z) + i\partial \phi_L(z)\right) : e^{i\lambda\left(\phi_M(z) - i\phi_L(z)\right)} : \]
\[ \partial \tau(z) = : e^{-i\lambda\left(\phi_M(z) - i\phi_L(z)\right)} : \]
\[ B(z) = : b(z)e^{i\lambda\left(\phi_M(z) - i\phi_L(z)\right)} : \]
\[ C(z) = : c(z)e^{-i\lambda\left(\phi_M(z) - i\phi_L(z)\right)} : \]

Substituting (40) into (39) one recovers easily the \( N = 4 \) generators of the \( c_M \leq 1 \) string theory (36) with \( G^\pm_{1}(z) \) and \( G^\pm_{2}(z) \) interchanged.

To conclude, we have shown that 2d topological gravity as formulated by Labastida, Pernici and Witten possesses a bigger symmetry in the conformal gauge than what was known before. We have explicitly shown that it contains two separate twisted \( N = 2 \) structures one of which is isomorphic to the symmetry algebra discussed before by Verlinde and Verlinde and the other, although does not have a natural interpretation in the topological gravity, can be related through bosonization to the \( N = 2 \) structure discussed in the context of \( c_M \leq 1 \) string theory in ref. [12,13]. These algebras are then found to be part of a larger algebra, namely they form a twisted “small” \( N = 4 \) algebra with the addition of two more generators. Topological gravity in fact is invariant under the full twisted \( N = 4 \) symmetry. We thus find a twisted \( N = 4 \) structure also in \( c_M \leq 1 \) string theory. Finally, we studied the “mirror” of this twisted \( N = 4 \) algebra. By using another bosonization we find that the “mirror” theory corresponds to the constrained
topological sigma model in one complex dimension. It would be interesting to investigate whether a similar extended $N = 4$ superconformal structure exists in $\hat{c}_M \leq 1$ fermionic string theory as well as in W-string theory.

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