Annuitization and Asset Allocation

Moshe A. Milevsky\textsuperscript{1} and Virginia R. Young\textsuperscript{2}
York University and University of Michigan

Current Version: 25 August 2006

\textsuperscript{1}Milevsky, the contact author, is an Associate Professor of Finance at the Schulich School of Business, York University, Toronto, Ontario, M3J 1P3, Canada, and the Director of the Individual Finance and Insurance Decisions (IFID) Centre at the Fields Institute. He can be reached at Tel: (416) 736-2100 ext 66014, Fax: (416) 763-5487, E-mail: milevsky@yorku.ca. This research is partially supported by a grant from the Social Sciences and Humanities Research Council of Canada and the Society of Actuaries.

\textsuperscript{2}Young is a Professor of Mathematics at the University of Michigan, Ann Arbor, Michigan, 48109, USA. She can be reached at Tel: (734) 764-7227, Fax: (734) 764-7048, E-mail: vryoung@umich.edu. The authors acknowledge helpful comments from two anonymous JEDC referees, Jeff Brown, Narat Charupat, Glenn Daily, Jerry Green, Mike Orszag, Mark Warshawsky, and seminar participants at York University, University of Michigan, University of Wisconsin, University of Cyprus, the American Economics Association and the Western Finance Association annual meeting.
Abstract

ANNUITIZATION AND ASSET ALLOCATION

This paper examines the optimal annuitization, investment and consumption strategies of a utility-maximizing retiree facing a stochastic time of death under a variety of institutional restrictions. We focus on the impact of aging on the optimal purchase of life annuities which form the basis of most Defined Benefit pension plans. Due to adverse selection, acquiring a lifetime payout annuity is an irreversible transaction that creates an incentive to delay. Under the institutional all-or-nothing arrangement where annuitization must take place at one distinct point in time (i.e. retirement), we derive the optimal age at which to annuitize and develop a metric to capture the loss from annuitizing prematurely. In contrast, under an open-market structure where individuals can annuitize any fraction of their wealth at anytime, we locate a general optimal annuity purchasing policy. In this case, we find that an individual will initially annuitize a lump sum and then buy annuities to keep wealth to one side of a separating ray in wealth-annuity space. We believe our paper is the first to integrate life annuity products into the portfolio choice literature while taking into account realistic institutional restrictions which are unique to the market for mortality-contingent claims.

JEL Classification: J26; G11

Keywords: Insurance; Mortality; Retirement; Contingent-Claims; Financial Economics
1 Introduction and Motivation

Asset allocation and consumption decisions towards the end of the human life cycle are complicated by the uncertainty associated with the length of life. Although this risk can be completely hedged in a perfect market with life annuities – or, more precisely, with continuously-renegotiated tontines – real world frictions and imperfections impede the ability to do so in practice. Indeed, empirical and anecdotal evidence suggests that voluntary annuitization amongst the public is not very common, nor is it well understood even amongst financial advisors. Therefore, in attempt to fill this gap and integrate mortality-contingent claims into the finance literature, this paper examines the optimal annuitization strategy of a utility-maximizing retiree facing a stochastic time of death under a variety of institutional pension and annuity arrangements. We also examine the usual investment and consumption dynamics but focus our attention on the impact of aging and the increase in the actuarial force of mortality on the optimal purchase of mortality-contingent annuities, which form the basis of most Defined Benefit (DB) pension plans. One of our main insights is that due to severe adverse selection concerns, acquiring a lifetime payout annuity is an irreversible transaction that, we argue, creates an incentive to delay. We appeal to the analogy of a classical American option which should only be exercised once the value from waiting is no more than the value from exercising.

For the most part of the paper, the focus of our attention is a life annuity that pays a fixed (real or nominal) continuous payout for the duration of the annuitant’s life. (The appendix extends our basic model to variable immediate annuities.) From a financial perspective, this product is akin to a coupon-bearing bond that defaults upon death of the holder and for which there is no secondary market. Under the institutional all-or-nothing arrangement, where annuitization (i.e. the purchase) must take place at one distinct point in time (i.e. retirement), we locate the optimal age at which to annuitize and develop a metric to capture the loss from annuitizing prematurely. This optimal age, which is linked to the actuarial force of mortality, occurs within retirement years and is obviously gender specific but also depends on the individual’s subjective health status. All of this will be explained in the body of the paper.

In contrast to the restrictive (yet not uncommon) all-or-nothing arrangement, under an open-market structure where individuals can annuitize a fraction of their wealth at distinct points in time, we locate a general optimal annuity purchasing policy. In this case, we find that an individual will initially annuitize a lump sum – if they do not already have this minimum level in pre-existing DB pensions – and then buy additional life annuities in order to keep wealth to one side of a separating ray in wealth-annuity space. This is a type of barrier control result that is common in the literature on asset allocation with transaction costs.

We believe our paper is the first to integrate life and pension annuity products into the portfolio

---

1The first (working paper) version of this paper was circulated and presented at the January 2001 meeting of the American Economics Association in New Orleans.
choice literature while taking into account realistic institutional restrictions which are unique to the market for mortality-contingent claims.

1.1 Agenda and Outline

The remainder of this paper is organized as follows. In Section 2, we provide a brief explanation of the mechanics of the life annuity market and review the existing literature involving asset allocation, personal pensions, and payout annuities. In Section 3, we present the general model for our financial and annuity markets. In Section 4, we consider the case for which the individual is required to annuitize all her pensionable wealth at one point in time. This is effectively an optimal retirement problem and is akin to the situation (up until recently) in the United Kingdom, where retirees can drawdown their pension but must annuitize the remaining balance by a certain age, or to the situation for which individuals have the choice of when to start their retirement (DB) pension but must do so at one point in time. In fact, most Variable Annuity contracts sold in the United States have an embedded option to annuitize that can only be exercised once. Our analysis would cover this too. In this restrictive (but common) framework, we locate the optimal age for her to do so, and then define a so-called option value metric as the gain in utility from annuitizing optimally. Section 5 provides a variety of numerical examples for the optimal time to annuitize and also pursues the option analogy as a way of illustrating the loss from annuitizing pre-maturely.

Then, in Section 6, we consider a less restrictive open-market arrangement whereby the individual may annuitize any portion of her wealth at any time. This arrangement is applicable to individuals with substantial discretionary wealth who can purchase small (or large) quantities of annuities on an ongoing basis. In this case, we find that the individual annuitizes a lump sum as soon as possible (the amount might be zero) and then acquires more annuities depending on the performance of her stochastic wealth process. If her wealth subsequently increases in value, she purchases more annuities by annuitizing additional wealth; otherwise, she refrains from additional purchases and consumes from her originally-purchased annuities, as well as from liquidating investments in her portfolio. Furthermore, we explicitly solve for the optimal annuity purchasing policy under this less restrictive case when the force of mortality is constant, which implies that the future lifetime is exponentially distributed. Section 7 provides a variety of numerical examples for the open-market framework and flushes-out and explores a number of insights. Non-essential proofs and theorems are relegated to an Appendix (Section 9), while Section 8 concludes the paper with our main qualitative insights.
2 The Annuity Market and Literature

2.1 Life Annuity Basics

Life annuities are purchased directly from insurance companies and form the basis of most DB pension plans. In exchange for a lump-sum premium, which the company invests in its general account, the company guarantees to pay the annuitant a fixed (monthly or quarterly) payout for the rest of his or her life. This payout rate – which depends on prevailing interest rates and mortality projections – is irrevocably determined at the time of purchase (a.k.a. annuitization) and does not change for the life of the contract. The following chart illustrates some sample quotes which were provided by a life-annuity broker (in Canada).

| Certain \ Age | m 55 f | m 60 f | m 65 f | m 70 f | m 75 f | m 80 f |
|--------------|--------|--------|--------|--------|--------|--------|
| 0 yrs        | 631    | 590    | 686    | 633    | 877    | 780    |
| 5 yrs        | 628    | 589    | 681    | 631    | 855    | 770    |
| 10 yrs       | 620    | 584    | 666    | 623    | 799    | 741    |
| 15 yrs       | 607    | 578    | 644    | 611    | 729    | 699    |
| 20 yrs       | 591    | 569    | 618    | 596    | 662    | 651    |
| 25 yrs       | 573    | 559    | 589    | 578    | 601    | 594    |

Sample monthly payout based on a $100,000 initial premium (purchase)

The term word certain refers to the guarantee period built into the annuity. If the guarantee period is \( n \) years, then the individual buying such an annuity (or his or her estate) will receive the stated income for \( n \) years; thereafter, the individual will receive the money only if he or she is alive.

For example, in exchange for a $100,000 initial premium, a 75-year-old female will receive $911 per month for the rest of her life. This life annuity has no guarantee period, which means that if she were to die one instant after purchasing the life annuity (technically it would have to be after the first payment), her beneficiaries or estate would receive nothing in return. The $911 monthly income for those who survive consists of a mix of principal and interest as well as the implicit funds of those who do not survive. A male would receive slightly more per month, namely $1,039, due to the lower life expectancy of males.

A few things should be obvious from the table. First, the higher the purchase age, all else being equal, the greater the annuity income. In this case, the future life expectancy is shorter and the initial premium must be amortized and returned over a shorter time period. Likewise, a longer guarantee period yields a lower annuity income. In fact, an 80-year-old female buying a 20-year guarantee will receive virtually the same amount ($664) as a male of the same age, since neither is likely to live past the 20-year certain period; hence, the annuity is essentially a portfolio of zero-coupon bonds. Some other points are in order, especially for those not familiar with insurance pricing concepts.
1. The law of large numbers and the ability to diversify mortality risk is central to the pricing of life annuities. The above-mentioned payouts are determined by expected *objective* annuitant mortality patterns together with prevailing interest rates of corresponding durations. Profits, fees, and commissions are built into these quotes by *loading* the pure actuarial factor on the order of 1% to 5%.

2. Payout rates fluctuate from week-to-week because most of the insurance companies’ assets backing these lifetime guarantees are invested in fixed-income instruments. This can sometimes cause quotes to change on a daily basis. It is therefore reasonable to model the evolution of these prices in continuous time.

3. Most of the existing open-purchase (a.k.a retail) annuity market in North America is based on fixed nominal (and not inflation-adjusted) payouts. Real annuities are quite rare, which is an ongoing puzzle to many economists. Consequently, most of the numerical examples in our paper focus on nominal values, although there is nothing in our model that precludes using inflation-adjusted prices and returns as long as they are not mixed in the same model.

4. An additional form of life annuity is the variable payout kind whose periodic income is linked to the performance of pre-selected equity and bond indices. In this case, the above-mentioned $100,000 premium would go towards purchasing a number of payout *units* (as opposed to dollars) whose value would fluctuate over time. These annuities are the foundation of the US-based TIAA-CREF’s pension plan for University workers, but are quite rare anywhere else in the world. As a result, the bulk of our paper addresses the fixed payout kind, but we refer the interested reader to Appendix C (namely, Section 9.3) in which these products are integrated into our model.

### 2.2 Literature Review

This paper merges a variety of distinct strands in the portfolio choice and annuity literature. First, our work sits squarely within the classical Merton (1971) optimal asset allocation and consumption framework. However, in contrast to extensions of this model by Kim and Omberg (1996), Koo (1998), Sorensen (1999), Wachter (2002), Bodie, Detemple, Otruba, and Walter (2004), or the recent book by Campbell and Viceira (2002), for example – which are concerned with relaxing the dynamics of the underlying state variables and/or investigating the impact of (retirement) time horizon on portfolio choice – our model attempts to realistically incorporate mortality-contingent payout annuities within this framework.

A life-contingent annuity is the building block of most DB pension plans – see Bodie, Marcus, and Merton (1988) for details – but can also be purchased in the retail market. The irreversibility of this purchase is due to the well-known adverse selection issues identified by Akerlof (1970) and Rothschild and Stiglitz (1976).
On its own, the topic of payout annuities has been investigated quite extensively within the public economics literature. In fact, a so-called annuity puzzle has been identified in this field. The puzzle relates to the incredibly low levels of voluntary annuitization exhibited by retirees who are given the choice of purchasing a mortality-contingent payout annuity. For example, holders of variable annuity saving policies in the U.S. have the option to convert their accumulated savings into a payout annuity, and yet less than 2% elect to do so according to the National Association of Variable Annuities and LIMRA International (www.limra.com). In the comprehensive Health and Retirement Survey (HRS) conducted in the U.S, only 1.57% of the HRS respondents reported life annuity income. Likewise, only 8.0% of respondents with a defined contribution pension plan selected an annuity payout.

Collectively, these low levels of voluntary annuitization stand in contrast to the implications of the Modigliani life-cycle hypothesis, as pointed out in Modigliani’s (1986) Nobel prize lecture. Indeed, as originally demonstrated by Yaari (1965), individuals with no utility of bequest should hold all their assets in mortality-contingent annuities since they stochastically dominate the payout from conventional asset classes. The result of Yaari (1965) has been the subject of much research in the public economics and insurance literature, and we refer the interested reader to a series of papers by Friedman and Warshawsky (1990), Brown (1999, 2001), Mitchell, Poterba, Warshawsky, and Brown (MPWB) (1999), Brown and Poterba (2000), and Brown and Warshawsky (2001). Collectively, these papers place some of the ‘blame’ for low annuitization rates on the high loads and fees that are embedded in annuity prices. Other economic-based explanations include Kotlikoff and Summers (1981), Kotlikoff and Spivak (1981), Hurd (1989), and Bernhiem (1991), which focus on the role of families and their bequest motives on the demand for annuitization. Other models that focus on market imperfections and adverse selection include Brugiavini (1993) and Yagi and Nishigaki (1993).

Thus, given the rich literature on dynamic asset allocation and the increasing interest in pension-related finance issues, our objective is to incorporate longevity-insurance products into a portfolio and asset allocation framework that properly captures the actuarial and insurance imperfections. Although Richard (1975) extended Merton’s (1971) model to obtain Yaari’s (1965) results in a continuous-time framework, the institutional set-up lacked the realism of current payout annuity markets.

We also argue that the decision of when to purchase an irreversible life annuity endows the holder with an incentive to delay that can be heuristically viewed as an option. Indeed, under many institutional pension arrangements (such as in the United Kingdom up until recently or with regards to the rules concerning variable annuity saving policies in the U.S.), individuals are allowed

---

2Recent papers that attempt a portfolio-based model for annuitization along the same lines – most written after the first draft of this paper was released – include Kapur and Orszag (1999), Blake, Cairns, and Dowd (2000), Cairns, Blake, and Dowd (2005), Neuberger (2003), Dushi and Webb (2003), Sinclair (2003), Stabile (2003), and Battocchio, Menoncin, and Scaillet (2003), Kooijen, Nijman and Werker (2006).
to drawdown their pension via discretionary consumption but must eventually annuitize at one point in time their remaining wealth. We refer to this system as an all-or-nothing arrangement and argue that this is similar to Stock and Wise’s (1990) option to retire and echoes the framework of Sundaresan and Zapatero (1997) who examine optimal behavior (and valuation) of various pension benefits. Other institutional structures allow for annuitization at any time and in small quantities as well, and we refer to these systems as anything anytime throughout the paper. We investigate the optimal annuitization policy in both of these cases and provide extensive numerical examples that compare the two.

3 Financial and Pension Annuity Markets

In this section, we describe our model for financial and annuity markets. We assume that an individual can invest in a riskless asset whose price at time $s$, $X_s$, follows the process $dX_s = rX_s ds, X_t = X > 0$, for some fixed $r \geq 0$. Also, the individual can invest in a risky asset whose price at time $s$, $S_s$, follows geometric Brownian motion given by

$$
\begin{align}
\left\{ \begin{array}{l}
    dS_s &= \mu S_s ds + \sigma S_s dB_s, \\
    S_t &= S > 0,
\end{array} \right. 
\tag{1}
$$

in which $\mu > r$, $\sigma > 0$, and $B$ is a standard Brownian motion with respect to a filtration $\{F_s\}$ of the probability space $(\Omega, \mathcal{F}, P)$. Let $W_s$ be the wealth at time $s$ of the individual, and let $\pi_s$ be the amount that the decision maker invests in the risky asset at time $s$. Also, the decision maker consumes at a rate of $c_s$ at time $s$. Then, the amount in the riskless asset is $W_s - \pi_s$, and when the individual buys no annuities, wealth follows the process

$$
\begin{align}
\left\{ \begin{array}{l}
    dW_s &= d(W_s - \pi_s) + d\pi_s - c_s ds \\
    &= r (W_s - \pi_s) dt + \pi_s (\mu ds + \sigma dB_s) - c_s ds \\
    &= [rW_s + (\mu - r) \pi_s - c_s] ds + \sigma \pi_s dB_s, \\
    W_t &= w > 0.
\end{array} \right. 
\tag{2}
$$

In Sections 4 and 5, we assume that the decision maker seeks to maximize (over admissible $\{c_s, \pi_s\}$ and over times of annuitizing all his or her wealth, $\tau$) the expected utility of discounted consumption. Admissible $\{c_s, \pi_s\}$ are those that are measurable with respect to the information available at time $s$, namely $\mathcal{F}_s$, that restrict consumption to be non-negative, and that result in (2) having a unique solution; see Karatzas and Shreve (1998). We also allow the individual to value expected utility via a subjective hazard rate (or force of mortality), while the annuity is priced by using an objective hazard rate; which may or may not be the same.

Our financial economy is based on the (simpler) geometric Brownian motion plus risk-free rate model originally pioneered by Merton (1971), as opposed to the more recent and richer models developed by Kim and Omberg (1996), Sorensen (1999), Wachter (2002), or Campbell and Viceira.
(2002) for example. The reason is that we are primarily interested in the implications of introducing a mortality-contingent claim into the portfolio choice framework, as opposed to studying the impact of stochastic interest rates or mean-reverting equity premiums per se. By avoiding the computational price of a more complex set-up, we are able to obtain analytical solutions to our annuitization problems.

We now move on to the insurance and actuarial assumptions. We let $t_p^S_x$ denote the subjective conditional probability that an individual aged $x$ believes he or she will survive to age $x + t$. It is defined via the subjective hazard function, $\lambda^S_{x+t}$, by the formula

$$t_p^S_x = \exp\left(-\int_0^t \lambda^S_{x+s} \, ds\right).$$

We have a similar formula for the objective conditional probability of survival, $t_p^O_x$, in terms of the objective hazard function, $\lambda^O_{x+t}$.

The actuarial present value of a life annuity that pays $1 per year continuously to an individual who is age $x$ at the time of purchase is written $\bar{a}_x$. It is defined by

$$\bar{a}_x = \int_0^\infty e^{-rt} \, t_p^x \, dt.$$  \hspace{1cm} (4)

We deliberately use the risk-free rate $r$ in our annuity pricing because most of the recent empirical evidence suggests that the money’s worth of annuities relative to the risk-free Government yield curve is relatively close to one. In other words, the expected present value of payouts using the risk-free rate is equal to the premium paid for that benefit. Thus, it appears that the additional credit risk that the insurance company might take on by investing in higher risk bonds is offset by any insurance loads and commissions they charge. We refer the interested reader to the paper by MPWB (1999) for a greater discussion of the precise curve that is used for pricing in practice.

In terms of notation, if we use the subjective hazard rate to calculate the survival probabilities in equation (4), then we write $\bar{a}_x^S$, while if we use the objective (pricing) hazard rate to calculate the survival probabilities, then we write $\bar{a}_x^O$. Just to clarify, by objective $\bar{a}_x^O$, we mean the actual market prices of the annuity net of any insurance loading, whereas $\bar{a}_x^S$ denotes what the market price ‘would have been’ had the insurance company used the individual’s personal and subjective assessment of her mortality.

We refer the interested reader to Hurd and McGarry (1995, 1997) for a discussion of experiments involving “subjective” versus “objective” assessments of survival probabilities.\footnote{Also, more recently, Smith, Taylor, and Sloan (2001) claim that their “findings leave little doubt that subjective perceptions of mortality should be taken seriously.” They state that individuals’ “longevity expectations are reasonably good predictions of future mortality.” Other researchers, such as Bhattacharya, Goldman and Sood (2003) claim that individuals are biased in their estimates of mortality as evidenced by the viatical and life settlement market. We do not take a position on whether individuals estimate and discount mortality using the same forward curve as the insurance company and therefore allow for the two functions to be distinct.} We will
demonstrate that asymmetry of mortality beliefs might go a long way towards explaining why individuals who believe themselves to be less healthy than average are more likely to avoid annuities, despite having no declared bequest motive. In the classical perfect market Yaari (1965) framework, subjective survival rates do not play a role in the optimal policy. We will show that if the consumer disagrees with the insurance company’s pricing basis regarding her subjective hazard rate - or personal health status - she will delay annuitization in an all-or-nothing environment.

In Section 6, we start by assuming that the decision maker maximizes (over admissible \( \{c_s, \pi_s, A_s\} \)) the expected utility of discounted lifetime consumption as well as bequest, in which \( A_s \) is the annuity purchasing process. \( A_s \) denotes the non-negative annuity income rate at time \( s \) after any annuity purchases at that time; we assume that \( A_s \) is right-continuous with left limits. The source of this income could be previous annuity purchases or a pre-existing annuity, such as Social Security or pension income. We assume that the individual can purchase an annuity at the price of \( \bar{a}_{x+s}^O \) per dollar of annuity income at time \( s \), or equivalently, at age \( x + s \). In that case, the dynamics of the wealth process are given by

\[
\left\{ \begin{array}{l}
dW_s = [rW_{s-} + (\mu - r)\pi_s - c_s + A_{s-}]ds + \sigma\pi_s dB_s - \bar{a}_{x+s}^O dA_s, \\
W_{t-} = w > 0.
\end{array} \right.
\]  

The negative sign on the subscripts for wealth and annuities denotes the left-hand limit of those quantities before any (lump-sum) annuity purchases.

4 Restricted Market: All or Nothing

In this section, we examine the institutional arrangement where the individual is required to annuitize all her wealth in a lump sum at some (retirement) time \( \tau \). If the volatility of investment return \( \sigma = 0 \), and assuming the objective hazard rate increases over time, then we show that the individual annuitizes all her wealth at a time \( T \) for which \( \mu = r + \lambda_{x+T}^O \), which is the time at which the hazard rate (a.k.a. mortality credits) plus the risk-free rate is equal to the expected return from the asset. Furthermore, if \( \lambda_{x+t}^S = \lambda_{x+t}^O \) for all \( t > 0 \), then the individual will optimally consume exactly the annuity income after time \( T \). Therefore, for small values of \( \sigma \) and for \( \lambda_{x+t}^S \approx \lambda_{x+t}^O \) for all \( t \geq 0 \), the individual will consume approximately the annuity income after annuitizing her wealth, at least for time soon after the annuitization time. In fact, the classical annuity results, for example Yaari (1965), prove that consuming the entire annuity income is optimal in the absence of bequest motives. This is why, to simplify our work we assume that at some time \( \tau \), the individual annuitizes all her wealth \( W_\tau \) and thereafter consumes at a rate of \( \bar{a}_{x+\tau}^O \), the annuity income.

It follows that the associated value function of this problem is given by
\[
U(w, t)
= \sup_{\{c_s, \pi_s, \tau\}} \mathbb{E}^{w,t}[\int_{t}^{\tau} e^{-r(s-t)} s-t \ p_{x+t}^{S} u(c_s) \ ds + \int_{\tau}^{\infty} e^{-r(s-t)} s-t \ p_{x+t}^{S} u\left(\frac{W_{t}}{a_{x+t}^{O}}\right) \ ds]
= \sup_{\{c_s, \pi_s, \tau\}} \mathbb{E}^{w,t}[\int_{t}^{\tau} e^{-r(s-t)} s-t \ p_{x+t}^{S} u(c_s) \ ds + e^{-r(\tau-t)} \tau-t \ p_{x+t}^{S} u\left(\frac{W^{O}_{\tau}}{a_{x+t}^{O}}\right) a^{S}_{x+t}],
\]

in which \(u\) is an increasing, concave utility function of consumption, and \(\mathbb{E}^{w,t}\) denotes the expectation conditional on \(W_t = w\). Note that the individual discounts consumption at the riskless rate \(r\). If we were to model with a subjective discount rate of say \(\rho\), then this is equivalent to using \(r\) as in (6) and adding \(\rho - r\) to the subjective hazard rate. Thus, there is no effective loss of generality in setting the subjective discount rate equal to the riskless rate \(r\). In other words, while some life-cycle models in the literature adjust the discount rate for perceived risk and other subjective factors, we remind the reader that our underlying hazard rate \(\lambda_{x+t}^{S}\) effectively adjusts the discount rate for the probability of survival and, thus, takes these risks into account implicitly.

Note that in this section, we do not account for pre-existing annuities, such as state and corporate pensions for example. We anticipate that such annuities will change the optimal time of annuitization, but we defer this problem to Section 6. Also, note that we take the annuity prices as exogenously given. We are not creating an equilibrium (positive) model of pricing as in the adverse selection literature of Akerlof (1970) or Rothschild and Stiglitz (1976), but rather a normative model of how people should behave in the presence of these given market prices. Expanding to equilibrium considerations is beyond the scope of this (normative) paper, although we do make some statements regarding equilibrium pricing of annuities in the concluding remarks.

We also restrict our attention to the case in which the utility function exhibits constant relative risk aversion (CRRA), \(\gamma = -cu''(c)/u'(c)\). That is, \(u\) is given by
\[
u(c) = \frac{1}{1-\gamma}, \quad \gamma > 0, \gamma \neq 1.
\]
For this utility function, the relative risk aversion equals \(\gamma\), a constant. The utility function that corresponds to relative risk aversion 1 is logarithmic utility.

We next show that for CRRA utility, solving the problem in (6) is equivalent to assuming that the optimal stopping annuitization time is some fixed time in the future, say \(T\). Based on that value of \(T\), one finds the optimal consumption and investment policies. Finally, one finds the optimal value of \(T \geq 0\).

The feature of CRRA utility that drives this result is that wealth factors out of the value function; therefore, the stopping time \(\tau\) is not random, but rather deterministic. For other utility functions, \(\tau\) will be a random stopping time that depends on stochastic (state variable of) wealth, as is the case for the exercise time of American options.

To show that the optimal stopping time is deterministic for CRRA utility, we use the fact that if we can find a smooth solution \(V\) to the following variational inequality, then that smooth solution
equals the value function $U$ in (6); see Øksendal (1998, Chapter 10), for example.

$$(r + \lambda^S_{x+t}) V \geq V_t + rwV_w + \max_c \left[ u(c) - cV_w \right] + \max_{\pi} \left[ (\mu - r)\pi V_w + \frac{1}{2}\sigma^2\pi^2 V_{ww} \right],$$

and

$$V(w, t) \geq \bar{a}^S_{x+t} u \left( \frac{w}{\bar{a}^O_{x+t}} \right),$$

with equality in at least one of (8) and (9), and with $u$ given by (7).

We look for a solution of this variational inequality of the form

$$V(w, t) = \frac{1}{1-\gamma} w^{1-\gamma} \psi(t).$$

If $V$ is of this form, then $\psi$ necessarily solves

$$\frac{1}{1-\gamma} (r + \lambda^S_{x+t}) \psi \geq \frac{\gamma}{1-\gamma} \psi' + \delta \psi + \frac{\gamma}{1-\gamma},$$

and

$$\frac{1}{1-\gamma} \psi(t) \geq \frac{1}{1-\gamma} \left( \frac{\bar{a}^S_{x+t}}{\bar{a}^O_{x+t}} \right)^{1-\gamma},$$

with equality in at least one of (10) and (11). So, if we find a smooth solution $\psi$ to this variational inequality in time $t$, then we are done, and we can assert that $U(w, t) = \frac{1}{1-\gamma} w^{1-\gamma} \psi(t)$. The key feature to note is that wealth $w$ and time $t$ are multiplicatively separable in $U$; thus, the optimal time to annuitize one’s wealth is independent of wealth and is, therefore, deterministic.

To solve the variational inequality for $\psi$, we hypothesize that the “continuation region” (i.e. the time when one does not annuitize) is of the form $(0, T)$. In other words, one does not annuitize one’s wealth until time $T$. If our hypothesis is correct, then (10) holds with equality on $(0, T)$, while (11) holds with inequality on $(0, T)$ and with equality at $t = T$. Given any value of $T$, we can solve this boundary-value problem; write $\phi = \phi(t; T)$ as the solution of this problem. For $t < T$, the function $\phi$ is given by

$$\phi(t; T) = \left( \frac{\bar{a}^S_{x+T}}{\bar{a}^O_{x+T}} \right)^{1-\gamma} e^{-\frac{r-(1-\gamma)s}{\gamma}(T-t)} (T-t)^{\frac{1}{\gamma}} + \int_t^T e^{-\frac{r-(1-\gamma)s}{\gamma}(s-t)} (s-t)^{\frac{1}{\gamma}} ds,$$

with $\delta = r + \frac{1}{2\gamma} (\frac{\mu - r}{\sigma})^2$. For $t \geq T$, we have

$$\phi(t; T) = \left( \frac{\bar{a}^S_{x+T}}{\bar{a}^O_{x+T}} \right)^{1-\gamma}.$$

The function $\psi$ is the smooth solution of the variational inequality in (10) and (11), for which there will be a unique $T$. (That is, $\phi$ will have a continuous first derivative at $t = T \geq 0$.) This optimal value of $T \geq 0$ is the one such that $\frac{1}{1-\gamma} \phi^{\gamma}(t; T)$ is maximized. Note that it is also possible that the optimal time to annuitize is right now (i.e. $T = 0$).
Define the function $\tilde{U}$ by $\tilde{U}(w,t;T) = \frac{1}{1-\gamma}w^{1-\gamma}\phi(t;T)$. Then, the value function $U$ in (3) is given by $U(w,t) = \max_{T \geq 0} \tilde{U}(w,t;T)$, in which the optimal value of $T$ is independent of wealth $w$ because $w$ factors from the expression for $\tilde{U}$. In other words, the optimal time to annuitize is deterministic at time $t$.

One can obtain the optimal consumption and investment policies from the first-order necessary conditions in (8); they are given in feedback form by

$$C^*_t = c^*(W^*_t, t) = \frac{W^*_t}{\psi(t)}, \quad (14)$$

and

$$\Pi^*_t = \pi^*(W^*_t, t) = \frac{\mu - T}{\sigma^2} W^*_t, \quad (15)$$

respectively, in which $W^*_t$ is the optimally controlled wealth before annuitization (time $T$). If we are in the case of logarithmic utility one can show that the optimal consumption rate is $C^*_t = \frac{W^*_t}{\tilde{a}_{x+t}}$.

It is interesting to note that if $r = 0$, in which case the denominator of the optimal consumption, $\psi$, collapses to a (subjective) life expectancy, then the consumption rate is precisely the minimum rate mandated by the U.S. Internal Revenue Service for annual consumption withdrawals from IRAs after age 71. Specifically, the proportion required to be withdrawn from one’s annuity each year equals the start-of-year balance divided by the future expectation of life. Because $r > 0$ in reality, the minimum IRS-mandated consumption rate is less than what is optimal for individuals with logarithmic utility and with mortality equal to that in the IRS tables.

To find the optimal time of annuitization, differentiate $\tilde{U} = \tilde{U}(w,t;T)$ with respect to $T$, while assuming $t < T$. One can show that

$$\frac{\delta \tilde{U}}{\delta T} \propto \left[ \frac{\gamma}{\gamma - 1} \left( \frac{\tilde{a}_{x+t}^S}{\tilde{a}_{x+t}^O} \right)^{-\frac{1}{\gamma}} - \frac{1}{\gamma - 1} + \frac{\tilde{a}_{x+t}^O}{\tilde{a}_{x+t}^O} \right] + \tilde{a}_{x+t}^O \left[ \delta - (r + \lambda_{x+t}^O) \right]. \quad (16)$$

Thus, if the expression on the right-hand side of (16) is negative for all $T \geq 0$, then it is optimal to annuitize one’s wealth immediately, and we have $U(w,t) = \tilde{U}(w,t;0)$. However, if there exists a value $T^* > 0$ such that the right-hand side of (16) is positive for all $0 \leq T < T^*$ and is negative for all $T > T^*$, then it is optimal to annuitize one’s wealth at time $T^*$, and we have $U(w,t) = \tilde{U}(w,t;T^*)$. In all the examples we present below, one of these two conditions holds. We repeat that the decision to annuitize is independent of one’s wealth, an artifact of CRRA utility.

It is straightforward to show that $\tilde{U}(w,t;T)$ having a continuous derivative at $t = T > 0$ means that $T$ is a critical point of $\tilde{U}$; that is, the right-hand side of (16) is zero at that value of $T$. Moreover, if that critical point $T = T^*$ maximizes $\tilde{U}(w,t;T)$, then (11) holds with strict inequality on $(0,T^*)$, as desired.

---

4See the earlier working paper version of this paper, Milevsky and Young (2002a) for a proof and exploration of this fact.
If the subjective and objective forces of mortality are equal, then we have

$$\frac{\delta U}{\delta T} \propto [\delta - (r + \lambda_{x+t})].$$  \hspace{1cm} (17)

In this case, if the hazard rate $\lambda_{x+t}$ is increasing with respect to time $t$, then either $\delta \leq (r + \lambda_{x})$, from which it follows that it is optimal to annuitize one’s wealth immediately, or $\delta > (r + \lambda_{x})$, from which it follows that there exists a time $T$ in the future (possibly infinity) at which it is optimal to annuitize one’s wealth. The optimal age to purchase a fixed life annuity is when the force of mortality $\lambda_{x}$ is greater than a constant (reminiscent of Merton’s constant) defined by:

$$M := \frac{1}{2\gamma}(\frac{\mu - r}{\sigma})^2.$$  \hspace{1cm} (18)

One can then think of the hazard rate as a form of excess return on the annuity due to the embedded mortality credits and the fact that liquid wealth reverts to the insurance company when the buyer of the annuity dies. This annuity purchase condition leads to a number of appealing insights. Namely, higher levels of risk aversion ($\gamma$) and higher levels of investment volatility ($\sigma$) lead to lower annuitization ages, since the constant $M$ decreases under larger $\gamma, \sigma$ and increases under higher levels of $\mu$.

We observe that if the subjective force of mortality is different than the objective force of mortality, then the optimal time of annuitization increases from the $T$ given by the zero of the right-hand side of (16). We can show mathematically that this is true if the subjective force of mortality varies from the objective force to the extent that $\bar{a}_S^S < 2\bar{a}_O^O$ for all ages $x$ (see Appendix A), and we conjecture that it is true in general. Note that this inequality is automatically true for people who are less healthy because in this case $\bar{a}_S^S < \bar{a}_O^O$ for all $x$. For an individual who is less healthy than the average person, the annuity will be too expensive, and the person will want to delay annuitizing her wealth.

On the other hand, for an individual who is healthier than the average person, the annuity will be relatively cheap. However, such a healthy person will live longer on average and will be interested in receiving a larger annuity benefit by consuming less now and by waiting to buy the annuity later in life. Therefore, a healthy person is also willing to delay annuitizing her wealth in exchange for a larger annuity benefit (for a longer time). Note that this result is likely driven by the fact that in this model, we force the investor to consume the totality of the income from the annuity. If the investor were allowed to save some of that annuity income and invest it in the stock market, this result would not necessarily hold.

Of course, by following the optimal policies of investment, consumption, and annuitizing one’s wealth, an individual runs the risk of being able to consume less after annuitizing wealth than if she had annuitized wealth immediately at time $t = 0$. Naturally, there is the chance of the exact opposite, namely that the lifetime annuity stream will be higher. Therefore, to quantify this risk, we calculate the probability associated with various consumption outcomes. See Appendix B for
the formula of this probability. We include calculations of it in a numerical example below.

Finally, we define a metric for measuring the loss in value from annuitizing prematurely by computing the additional wealth that would be required to compensate the utility maximizer for forced annuitization. This is akin to the annuity equivalent wealth used by MPWB (1999), which we prefer to label a subjective option value. Technically, it is defined to be the least amount of money \( h \) that when added to current wealth \( w \) makes the person indifferent between annuitizing now (with the extra wealth) and annuitizing at time \( T \) (without the extra wealth). Thus, \( h \) is given by

\[
U(w, t; T) = U(w + h, t; 0),
\]

in which \( T \) is the optimal time of annuitization. In the examples in the next section, we express \( h \) as a percentage of wealth \( w \). This is appropriate because \( U \) exhibits CRRA with respect to \( w \).

5 Numerical Examples: Annuitize All or Nothing

In this section, we present two numerical examples to illustrate the results from the previous section. To start, although most mortality tables are discretized, we require a continuous-time mortality law. We use a Gompertz force of mortality, which is common in the actuarial literature for annuity pricing. See Frees, Carriere, and Valdez (1996) for examples of this model in annuity pricing. This model for mortality has also been employed in the economics literature for pricing insurance; see Johansson (1996), for example. The force of mortality is written

\[
\lambda_x = \exp\left(\frac{x - m}{b}\right) / b
\]

in which \( m \) is a modal value and \( b \) is a scale parameter. Note how the force of mortality itself increases exponentially with age.

In this paper, we fit the parameters of the Gompertz, namely \( m \) and \( b \), to the Individual Annuity Mortality 2000 (basic) Table with projection scale G. For males, we fit parameters \((m, b) = (88.18, 10.5);\) for females, \((92.63, 8.78)\). Initially, we assume that the subjective and objective forces of mortality are equal. Throughout this section, we assume that the seller of the annuity uses the female hazard rate to price annuities for women; similarly, for men. Figure 1 shows the graph of the probability density function of the future-lifetime random variable under a Gompertz hazard rate that is fitted to the discrete mortality table.

As for the capital market parameters, in both our examples, the risky stock is assumed to have drift \( \mu = 0.12 \) and volatility \( \sigma = 0.20 \). This is roughly in line with numbers provided by Ibbotson Associates (2001), which are widely used by practitioners when simulating long-term investment

\[
\text{Figure 1 about here.}
\]

We actually fit a Makeham hazard rate, or force of mortality, namely \( \lambda + \exp(\left((x - m)/b\right)/b \) in which \( \lambda \geq 0 \) is a constant that models an accident rate. However, the fitted value of \( \lambda \) was 0, so the effective form of the hazard rate is Gompertz (Bowers et al., 1997).
returns. We assume that the nominal rate of return of the riskless bond is $r = 0.06$. We display values for the option to delay annuitization $h$, for three different levels of risk aversion, $\gamma = 1$ (logarithmic utility) and $\gamma = 2$ and $\gamma = 5$. A variety of studies have estimated the value of $\gamma$ to lie between 1 and 2. See, the paper by Friend and Blume (1975) that provides an empirical justification for constant relative risk aversion, as well as the more recent MPWB (1999) paper in which the CRRA value is taken between 1 and 2. In the context of estimating the present value of a variable annuity for Social Security, Feldstein and Ranguelova (2001) provide some qualitative arguments that the value of CRRA is less than 3 and probably even less than 2. On the other hand, some of the equity premium literature, see Campbell and Viceira (2002) suggests that risk aversion levels might be much higher, which is why we have also displayed results for $\gamma = 5$.

5.1 Example #1

Table 1 provides the optimal age of annuitization – and what we have labeled the value of the option to delay as a percentage of initial wealth – as well as the probability of consuming less at the optimal time of annuitization than if one had annuitized one’s wealth immediately. We refer to this as the probability of a deferral failure. We provide numerical results for both males and females under very low ($\gamma = 1$), low ($\gamma = 2$) and high ($\gamma = 5$) coefficients of relative risk aversion.

Note that females annuitize at older ages compared to males because the mortality rate of females is lower at each given age. Also, note that more risk averse individuals wish to annuitize sooner, an intuitively pleasing result. However, notice that even at relatively high ($\gamma = 5$) levels of risk aversion, males do not annuitize prior to age 63 and females do not annuitize prior to age 70. Finally, the value of the option to delay annuitization – which is effectively the certainty equivalent of the welfare loss from annuitizing immediately – decreases as one gets closer to the optimal age of annuitization, as one expects.

The probability of deferral failure reported in Table 1a, although seemingly high, is balanced by the probability of ending up with more than, say, 20% of the original annuity amount. For example, for a 70-year-old female with $\gamma = 2$, the probability of consuming at least 20% more at the optimal age of annuitization than if she were to annuitize immediately is 0.474. Obviously, on a utility-adjusted basis this is a worthwhile trade-off as evidenced by the behavior of the value function. See Table 2 for tabulations of the probability that the individual consumes at least 20% more at the optimal age of annuitization than if he or she were to annuitize immediately, for various ages and for $\gamma = 1$ and 2.
These “upside” probabilities decrease as the optimal age of annuitization approaches. Also, for a given age, they decrease as the CRRA increases. This makes sense because a less risk-averse person is less willing to face a distribution with a higher variance.

5.2 Example #2

We continue the assumptions in the previous example as to the financial market. We have a male aged 60 with $\gamma = 2$, whose objective mortality follows that from the previous example; that is, annuity prices are determined based on the hazard rate given there. For this example, suppose that the subjective force of mortality is a multiple of the objective force of mortality; specifically, $\lambda^S_x = (1 + f) \lambda^O_x$, in which $f$ ranges from $-1$ (immortal) to infinity (at death’s door). This transformation is called the proportional hazard transformation in actuarial science introduced by Wang (1996), and it is similar to the transformation examined by Johansson (1996) in the economic context of the economic value of increasing one’s life expectancy.

In Table 3, we present the imputed value of the option to delay annuitization, the optimal age of annuitization, the optimal rate of consumption before annuitization (as a percentage of current wealth), and the rate of consumption after annuitization (also, as a percentage of current wealth). For comparison, if the male were to annuitize his wealth at age 60, the rate of consumption would be 8.34%. Also, the optimal proportion invested in the risky stock before annuitization is 75%.

Note that as the 60-year-old male’s subjective mortality gets closer to the objective (pricing) mortality, then the optimal age of annuitization decreases. It seems that the optimal age of annuitization will be a minimum when the subjective and objective forces of mortality equal, at least for increasing forces of mortality. We conjecture that this result is true in general, but we only have a proof of it when $\ddot{a}^S_x < 2\ddot{a}^O_x$; see Appendix A. Also, note that the consumption rate before annuitization increases as the person becomes less healthy, as expected.

Compare these rates of consumption with 8.34%, the rate of consumption if the male were to annuitize his wealth immediately. We see that if the male is healthy relative to the pricing force of mortality, then he is willing to forego current consumption in exchange for greater consumption when he annuitizes, at least up to $f = -0.4$. Past that point, the optimal rate of consumption before annuitization is greater than 8.34%. For a 60-year-old male with $f = -0.2$ (20% more healthy than average), see Figure 2 for a graph of the expected consumption rate as a percentage of initial wealth. We also graph the 25th and 75th percentiles of his random consumption. This individual expects to live to age 84.4. Note that the annuitant has roughly a 70% chance of consuming more throughout the remaining life compared to annuitizing at age 60.
6 Unrestricted Market: Annuitize Anything Anytime

In this section, we consider the optimal annuity-purchasing problem for an individual who seeks to maximize her expected utility of lifetime consumption and bequest. In Section 6.1, we allow the individual to have rather general preferences, while in Section 6.2, we specialize to the case for which preferences exhibit constant relative risk aversion. We allow the individual to buy annuities in lump sums or continuously, whichever is optimal. Our results are similar to those of Dixit and Pindyck (1994, pp 359ff). They consider the problem of a firm’s irreversible capacity expansion. For our individual, annuity purchases are also irreversible, and this leads to the similarity in results. Specifically, a discrete jump in wealth can only occur at the initial instant (in our case, with a lump-sum purchase of an annuity; in their case, with an initial investment of capital); thereafter, the annuity income remains constant or increases incrementally to keep wealth below a given barrier (for Dixit and Pindyck, capital stock was either constant or changed incrementally). In other words, the optimal control is a “barrier control” policy.

In Section 6.2.1, we continue with CRRA preferences and linearize the HJB equation in the region of no-annuity purchasing via a convex dual transformation in the case for which there is no bequest motive. In Section 6.2.1.1, we provide an implicit analytical solution to the optimal annuity purchasing problem developed in Section 6.2.1 in the case for which the force of mortality is constant. This leads us to Section 7, which provides a full set of numerical results.

6.1 General Utility of Consumption and Bequest

In this section, we show that the individual’s optimal annuity purchasing is given by a barrier policy in that she will annuitize just enough of her wealth to stay on one side of the barrier in wealth-annuity space. In equation (5), we described the dynamics of the wealth for this individual. Denote the random time of death of our individual by $\tau_d$. We assume that $\tau_d$ is independent of the randomness in the financial market, namely the Brownian motion $B$ driving the stock price. Thus, her value function at time $t$, or at age $x+t$, is given by

$$U(w, A, t) = \sup_{\{c_s, \pi_s, A_s\}} E^{w, A, t} \left[ \int_t^\infty e^{-r(s-t)} s-t p_{x+t}^S u_1(c_s) ds + e^{-r(\tau_d-t)} u_2(W_{\tau_d}) \right]$$

$$= \sup_{\{c_s, \pi_s, A_s\}} E^{w, A, t} \left[ \int_t^\infty e^{-r(s-t)} s-t p_{x+t}^S \left\{ u_1(c_s) + \lambda_{x+s}^S u_2(W_{s-}) \right\} ds \right],$$  \hspace{1cm} (20)

in which $u_1$ and $u_2$ are strictly increasing, concave utility functions of consumption and bequest, respectively. Also, $E^{w, A, t}$ denotes the expectation conditional on $W_{t-} = w$ and $A_{t-} = A$. In the last
equality, we used the independence of $\tau_d$ from the Brownian motion $B$ to simplify the expression for $U$. Note that we assume the individual discounts future consumption at the riskless rate $r$ since the mortality discounting – which increases the effective discount rate – is incorporated separately. The value function $U$ is jointly concave in $w$ and $A$.

We continue with a formal discussion of the derivation of the associated HJB equation. Suppose that at the point $(w, A, t)$, it is optimal not to purchase any annuities. It follows from Itô’s lemma that $U$ satisfies the equation at $(w, A, t)$ given by

$$(r + \lambda_{x,t}^S)U$$

$$= U_t + (rw + A)U_w + \max_{\pi} \left[ \frac{1}{2} \sigma^2 \pi^2 U_{ww} + (\mu - r) \pi U_w \right] + \max_{c \geq 0} \left[ -cU_w + u_1(c) \right] + \lambda_{x+t}^S u_2(w).$$

(21)

Because the above policy is in general suboptimal, (21) holds as an inequality; that is, for all $(w, A, t)$,

$$(r + \lambda_{x,t}^S)U$$

$$\geq U_t + (rw + A)U_w + \max_{\pi} \left[ \frac{1}{2} \sigma^2 \pi^2 U_{ww} + (\mu - r) \pi U_w \right] + \max_{c \geq 0} \left[ -cU_w + u_1(c) \right] + \lambda_{x+t}^S u_2(w).$$

(22)

Next, assume that at the point $(w, A, t)$ it is optimal to buy an annuity instantaneously. In other words, assume that the investor moves instantly from $(w, A, t)$ to $(w - \bar{a}^O_{x+t} \Delta A, A + \Delta A, t)$. Then, the optimality of this decision implies that

$$U(w, A, t) = U(w - \bar{a}^O_{x+t} \Delta A, A + \Delta A, t),$$

(23)

which in turns yields

$$U_A(w, A, t) - \bar{a}^O_{x+t} U_w(w, A, t) = 0.$$ 

(24)

Note that the lump-sum purchase is such that the marginal utility of annuity income equals the adjusted marginal utility of wealth, in which we adjust the marginal utility of wealth by multiplying by the cost of $\$1$ of annuity income. This result parallels many such in economics. Indeed, the marginal utility of annuity income is the marginal utility of the benefit, while the adjusted marginal utility of wealth is the marginal utility of the cost. Thus, the lump-sum purchase is such that the marginal utilities are equated.

\footnote{Due to space constraints, we refer the interested reader to the earlier working paper version by Milevsky and Young (2002b) for a detailed discussion of this and other properties of $U$.}
However, such a policy is in general suboptimal; therefore, (23) holds as an inequality and (24) becomes
\[
\bar{a}^O_{x+t}U_w(w, A, t) - U_A(w, A, t) \geq 0.
\] (25)

By combining (22) and (25), we obtain the HJB equation (26) below associated with the value function \(U\) given in (20). The following result can be proved as in Zariphopoulou (1992), for example.

**Proposition 6.1:** The value function \(U\) is a constrained viscosity solution of the Hamilton-Jacobi-Belman equation
\[
\min \left[ (r + \lambda^S_{x+t})U - U_t - (rw + A)U_w - \max_{\pi} \left( \frac{1}{2}\sigma^2 \pi^2 U_{ww} + (\mu - r)\pi U_w \right)\right.
- \max_{c \geq 0} (cu_1(c)) - \lambda^S_{x+t}u_1(w), \bar{a}^O_{x+t}U_w - U_A \left. \right] = 0.
\] (26)

Equation (24) defines a “barrier” in wealth-annuity income space. If wealth and annuity income lie to the right of the barrier at time \(t\), then the individual will immediately spend a lump sum of wealth to move diagonally to the barrier (up and to the left). The move is diagonal because as wealth decreases to purchase more annuities, annuity income increases. Thereafter, annuity income is either constant if wealth is low enough to keep to the left of the barrier, or annuity income responds continuously to infinitesimally small changes of wealth at the barrier.

Thus, as in Dixit and Pindyck (1994, pp 359ff) or in Zariphopoulou (1992), we have discovered that the optimal annuity-purchasing scheme is a type of barrier control. Other barrier control policies appear in finance and insurance. In finance, Zariphopoulou (1999, 2001) reviews the role of barrier policies in optimal investment in the presence of transaction costs; also see the references within her two articles. See Gerber (1979) for a classic text on risk theory in which he includes a section on optimal dividend payout and shows that it follows a type of barrier control.

### 6.2 Constant Relative Risk Aversion Preferences

In this subsection, we specialize the results of the previous subsection to the case for which the individual’s preferences exhibit CRRA. For this case, we can reduce the problem by one dimension, and we show that the barrier given in the previous section is a ray emanating from the origin in wealth-annuity space. Let
\[
u_1(c) = \frac{c^{1-\gamma}}{1-\gamma}, \quad \text{and} \quad u_2(w) = ku_1(w), \quad \gamma > 0, \gamma \neq 1, k \geq 0.
\] (27)

The parameter \(k \geq 0\) weights the utility of bequest relative to the utility of consumption. Davis and Norman (1990) and Shreve and Soner (1994) show that for CRRA preferences in the problem of consumption and investment in the presence of transaction costs, the value function \(U\) is a solution of its HJB equation in the classical sense, not just in the viscosity sense. Generally, if the force
of mortality is “eventually” large enough to make the value function well-defined, then this result holds for our problem, too.

For the utility functions in \( 27 \), it turns out that the value function \( U \) is homogeneous of degree \( 1 - \gamma \) with respect to wealth \( w \) and annuity income \( A \). That is, \( U(bw, bA, t) = b^{1-\gamma}U(w, A, t) \) for \( b > 0 \). Thus, if we define \( V \) by \( V(z, t) = U(z, 1, t) \), then we can recover \( U \) from \( V \) by

\[
U(w, A, t) = A^{1-\gamma}V(w/A, t), \quad A > 0.
\]

(28)

It follows that the HJB equation for \( U \) from Proposition 6.1 becomes the following equation for \( V \):

\[
\min \left[ (r + \lambda^S_{x+t}) V - V_t - (rz + 1)V_z - \max_{\hat{c}} \left( \frac{1}{2} \sigma^2 \hat{\pi}^2 V_{zz} + (\mu - r)\hat{\pi}V_z \right) \right]
- \max_{\hat{c} \geq 0} \left( -\hat{c}V_z + \frac{\hat{c}^{1-\gamma}}{1-\gamma} \right) - k\lambda^S_{x+t} \frac{z^{1-\gamma}}{1-\gamma}, \quad (z + \bar{a}_O^O) V_z - (1 - \gamma) V = 0,
\]

(29)

in which \( \hat{c} = c/A \), and \( \hat{\pi} = \pi/A \). Davis and Norman (1990) and Shreve and Soner (1994) use the same transformation in the problem of consumption and investment in the presence of transaction costs. Also, Duffie and Zariphopoulou (1993) and Koo (1998) use this transformation to study optimal consumption and investment with stochastic income.

Due to space considerations we simply refer to the working paper version by Milevsky and Young (2002b) which study properties of the optimal consumption and investment policies. Please refer to that work for details on the proof of the following proposition that describes the actions of the individual.

**Proposition 6.2:** For each value of \( t \geq 0 \), there exists a value of the wealth-to-income ratio \( z_0(t) \) that solves

\[
(z_0(t) + \bar{a}_O^O) V_z(z_0(t), t) = (1 - \gamma) V(z_0(t), t),
\]

(30)

such that

(i) If \( z = w/A > z_0(t) \), then the individual immediately buys an annuity so that

\[
\frac{w - \Delta A \bar{a}_O^O}{A + \Delta A} = z_0(t);
\]

(31)

Thus, \( V(z, t) = V(z_0(t), t) \) in this case.

(ii) If \( z = w/A < z_0(t) \), then the individual buys no annuity; i.e., she is in the region of inaction. Thus, in this case, \( V \) solves

\[
(r + \lambda^S_{x+t}) V
= V_t + (rz + 1)V_z + \max_{\hat{c}} \left( \frac{1}{2} \sigma^2 \hat{\pi}^2 V_{zz} + (\mu - r)\hat{\pi}V_z \right) + \max_{\hat{c} \geq 0} \left( -\hat{c}V_z + \frac{\hat{c}^{1-\gamma}}{1-\gamma} \right) + k\lambda^S_{x+t} \frac{z^{1-\gamma}}{1-\gamma}.
\]

(32)

It follows that at each time point, the barrier \( w = z_0(t)A \) is a ray emanating from the origin and lying in the first quadrant of \( (w, A) \) space.
Note that if $z_0(t) < \infty$, then it is optimal for the individual to have positive annuity income because the positive $w$ axis lies in the region \{(w, A, t) : w/A < z_0(t)\}.

Davis and Norman (1990) and Shreve and Soner (1994) find results similar to those in Proposition 6.2 for the problem of optimal consumption and investment in the presence of proportional transaction costs. In the next subsection, we show how to linearize the HJB equation of the individual who has no bequest motive.

### 6.2.1 Zero Bequest Motive: Linearization of the HJB Equation

Up until now we have assumed both utility of bequest and consumption in our specification. In this subsection, we linearize the nonlinear partial differential equation for $V$ in the region of inaction given by equation (32) with no bequest motive ($k = 0$). To this end, we consider the convex dual of $V$ defined by

$$\tilde{V}(y, t) = \max_{z > 0} [V(z, t) - zy].$$

The critical value $z^*$ solves the equation $0 = V_z(z, t) - y$; thus, $z^* = I(y, t)$, in which $I$ is the inverse of $V_z$ with respect to $z$. Note that one can retrieve the function $V$ from $\tilde{V}$ by the relationship

$$V(z, t) = \min_{y > 0} \left[ \tilde{V}(y, t) + zy \right].$$

Indeed, the critical value $y^*$ solves the equation $0 = \tilde{V}_y(y, t) + z = -I(y, t) + z$; thus, $y^* = V_z(z, t)$.

In the partial differential equation for $V$ with no bequest motive ($k = 0$), let $z = I(y, t)$ and rewrite the equation in terms of $\tilde{V}$ to obtain

$$\tilde{V}_t - (r + \lambda^S_{x+t})\tilde{V} + \lambda^S_{x+t}y\tilde{V}_y + my^2\tilde{V}_{yy} = -y - \frac{\gamma}{1 - \gamma} y^{1 - \frac{2}{\gamma}},$$

in which $m = \frac{1}{2} (\frac{\mu - r}{\sigma})^2$. Note that (35) is a linear partial differential equation.

Next, consider the boundary condition $U_A(w, A, t) = \tilde{a}^Q_{x+t} U_w(w, A, t)$ from equation (24). In terms of $V$, this condition can be written as in equation (27), and we repeat it here for convenience

$$-(1 - \gamma)V(z_0(t), t) + (z_0(t) + \tilde{a}^Q_{x+t})V_z(z_0(t), t) = 0.$$  \hfill (36)

Smooth pasting at the boundary implies that the derivative of this boundary condition with respect to $z$ evaluated at $z = z_0(t)$ holds and is given by

$$\gamma V_z(z_0(t), t) + (z_0(t) + \tilde{a}^Q_{x+t})V_{zz}(z_0(t), t) = 0.$$  \hfill (37)

We also have a boundary condition at $z = 0$ because at that point, the individual has no wealth to invest in the risky asset. Write $\tilde{\pi}^*$ in terms of $\tilde{V}$: $\tilde{\pi}^*(y, t) = \frac{\mu - r}{\sigma^2} y \tilde{V}_{yy}$. Thus, for $z = 0$ (with the corresponding value for $y$ written $y_a(t)$), we have that either $y_a(t) = 0$ or $\tilde{V}_{yy}(y_a(t), t) = 0$. 

20
Because \( V_z > 0 \) is strictly decreasing with respect to \( z \), we have \( y_a(t) > y_0(t) \geq 0 \) for all \( t \geq 0 \), in which \( y_a(t) \) and \( y_0(t) \) are defined by

\[
y_a(t) = V_z(0,t), \quad \text{and} \quad y_0(t) = V_z(z_0(t),t).
\] (38)

Thus, because \( y_a(t) > 0 \), in terms of \( \tilde{V} \), the boundary conditions become

\[
\tilde{V}_y(y_a(t),t) = 0,
\] (39)

for

\[
\tilde{V}_{yy}(y_a(t),t) = 0,
\] (40)

and

\[
(1 - \gamma)\tilde{V}(y_0(t),t) + \gamma y_0(t)\tilde{V}_y(y_0(t),t) = \tilde{a}^{O}_{x+t} y_0(t),
\] (41)

for

\[
\tilde{V}_y(y_0(t),t) + \gamma y_0(t)\tilde{V}_{yy}(y_0(t),t) = \tilde{a}^{O}_{x+t}.
\] (42)

6.2.1.1 Constant Force of Mortality

While still operating within the zero bequest world, if we assume that the forces of mortality are constant, that is, \( \lambda_{x+t}^S \equiv \lambda^S \) and \( \lambda_{x+t}^O \equiv \lambda^O \) for all \( t \geq 0 \), then we can obtain an “implicit” analytical solution of the value function \( V \) via the boundary-value problem given by (35) and (39) - (42). See Neuberger (2003) for recent and related work. In this case, \( V, \tilde{V}, y_a, \) and \( y_0 \) are independent of time, so (35) becomes the ordinary differential equation

\[
-(r + \lambda^S)\tilde{V}(y) + \lambda^S y \tilde{V}'(y) + y^2 \tilde{V}''(y) = -y - \frac{\gamma y^{1 - \frac{1}{\gamma}}}{1 - \gamma},
\] (43)

with boundary conditions

\[
\tilde{V}''(y_a) = 0,
\] (44)

for

\[
\tilde{V}'(y_a) = 0,
\] (45)

and

\[
(1 - \gamma)\tilde{V}(y_0) + \gamma y_0 \tilde{V}'(y_0) = \frac{y_0}{r + \lambda^O},
\] (46)

for
\[ \tilde{V}'(y_0) + \gamma y_0 \tilde{V}''(y_0) = \frac{1}{r + \lambda^O}. \]  

(47)

The general solution of (43) is

\[ \tilde{V}(y) = D_1 y^B_1 + D_2 y^B_2 + \frac{y}{r} + C_2 y^{1-\frac{1}{\gamma}}, \]

with \( D_1 \) and \( D_2 \) constants to be determined by the boundary conditions, with \( C_2 \) given by

\[ C_2 = r + \frac{\lambda^S}{\gamma} - m \frac{1 - \gamma}{\gamma^2}, \]

(49)

with \( B_1 \) and \( B_2 \) given by

\[ B_1 = \frac{1}{2m} \left[ (m - \lambda^S) + \sqrt{(m - \lambda^S)^2 + 4m(r + \lambda^S)} \right] > 1, \]

(50)

and

\[ B_2 = \frac{1}{2m} \left[ (m - \lambda^S) - \sqrt{(m - \lambda^S)^2 + 4m(r + \lambda^S)} \right] < 0. \]

(51)

The boundary conditions at \( y_0 \) give us

\[ D_1 \{1 + \gamma(B_1 - 1)\} y_0^{B_1} + D_2 \{1 + \gamma(B_2 - 1)\} y_0^{B_2} + \frac{y_0}{r} = \frac{y_0}{r + \lambda^O}, \]

(52)

and

\[ D_1 B_1 \{1 + \gamma(B_1 - 1)\} y_0^{B_1} + D_2 B_2 \{1 + \gamma(B_2 - 1)\} y_0^{B_2} + \frac{y_0}{r} = \frac{y_0}{r + \lambda^O}. \]

(53)

Solve equations (52) and (53) to get \( D_1 \) and \( D_2 \) in terms of \( y_0 \):

\[ D_1 = -\frac{\lambda^O}{r(r + \lambda^O)} \frac{B_1 - 1 - B_2}{B_1 - B_2} \left( \frac{y_0}{y_0} \right)^{B_1 - 1} \]

(54)

and

\[ D_2 = -\frac{\lambda^O}{r(r + \lambda^O)} \frac{B_1 - 1}{B_1 - B_2} \left( \frac{y_0}{y_0} \right)^{B_2 - 1}. \]

(55)

Next, substitute for \( D_1 \) and \( D_2 \) in \( \tilde{V}'(y_a) + \gamma y_a \tilde{V}''(y_a) = 0 \) from (44) and (45) to get

\[ \frac{\lambda^O}{r + \lambda^O} B_1 (B_1 - B_2) \left( \frac{y_a}{y_0} \right)^{B_1 - 1} + \frac{\lambda^O}{r + \lambda^O} B_2 (B_2 - 1) \left( \frac{y_a}{y_0} \right)^{B_2 - 1} = 1. \]

(56)

(56) gives us an equation for the ratio \( y_a/y_0 > 1 \). To check that (56) has a unique solution greater than 1, note that the left-hand side (i) equals \( \lambda^O/(r + \lambda^O) < 1 \) when we set \( y_a/y_0 = 1 \), (ii) goes to infinity as \( y_a/y_0 \) goes to infinity, and (iii) is strictly increasing with respect to \( y_a/y_0 \).

Next, substitute for \( D_1 \) and \( D_2 \) in \( \tilde{V}'(y_a) = 0 \) from (44) to get
Substitute for \( \frac{y_a}{y_0} \) in equation (57), and solve for \( y_a \). Finally, we can get \( y_0 \) from

\[
y_0 = \frac{y_a}{y_a/y_0},
\]

and \( D_1 \) and \( D_2 \) from equations (54) and (55), respectively.

Once we have the solution for \( \tilde{V} \), we can recover \( V \) from

\[
V(z) = \max_{y > 0} \left[ \tilde{V}(y) + zy \right]
= \max_{y > 0} \left[ D_1 y^{B_1} + D_2 y^{B_2} + \frac{y}{r} + C_2 y^{1-\frac{1}{\gamma}} + zy \right],
\]

in which the critical value \( y^* \) solves

\[
D_1 B_1 y^{B_1-1} + D_2 B_2 y^{B_2-1} + \frac{1}{r} + C_2 \left( 1 - \frac{1}{\gamma} \right) y^{-\frac{1}{\gamma}} + z = 0.
\]

Thus, for a given value of \( z = w/A \), solve (60) for \( y \) and substitute that value of \( y \) into (59) to get \( U(w, A) = V(z) \). Perhaps more importantly, we are interested in the critical value \( z_0 \) above which an individual spends a lump sum to purchase more annuity income. We pursue this in the examples in the next section.

## 7 Numerical Examples: Annuitize Anything Anytime

In this section, we provide a variety of numerical examples to illustrate the results of our anything anytime model. We focus attention on the impact of risk aversion, investment volatility, and insurance fees on the optimal amount annuitized.

In the first set of results, we assumed the following values for the hazard rate parameters: \( \lambda^S = \lambda^O = 0.04 \). That is, the force of mortality is constant and therefore the expected future lifetime is: \( 1/\lambda = 25 \) years. Furthermore, we set the risk-free interest rate to be \( r = 0.04 \), the drift of the risky asset is \( \mu = 0.08 \), and its volatility is \( \sigma = 0.20 \). We have selected these numbers – which are lower than those used in the earlier examples – to better capture a real (after-inflation) case in which Social Security benefits would be considered as part of the pre-existing annuity.

In Table 4a, for various values of \( \gamma \), we give the critical value of the ratio of wealth to annuity income \( z_0 = w/A \) above which the individual will spend a lump sum of wealth to increase her annuity income. We also include the amount that the individual will spend on annuities for a given (pre-existing) annuity income of \( A = $25,000 \), namely \( (w - z_0 A) / (1 + (r + \lambda^O)z_0) \).
For example, a retiree with $1,000,000 in liquid investable assets and with $25,000 in pre-existing annuity income would immediately (and irreversibly) annuitize between $727,620 and $914,176 depending on the coefficient of relative risk aversion. Notice from the same Table 4a that the amount spent on annuities increases for a given level of wealth as the individual becomes more risk averse, which is an intuitively pleasing result. Also, for a given level of risk aversion, the amount spent on annuities decreases as wealth decreases. In Table 4b, we present the results for the case when $A = 50,000$. Notice that when the pre-existing annuity income increases from $40,000 to $50,000, but under the same level of wealth, the amount spent immediately on additional annuity purchases is less.

In fact, in this case someone with $100,000 of wealth, or less, will not annuitize any additional wealth when their coefficient of relative risk aversion is $\gamma = 2$ or less. We emphasize – once again – that these numerical results assume that there is absolutely no bequest motive and no importance placed in inheritance, a spouse or any other survivors. In the presence of some weight on bequest (which would be expected in the real world) the amount annuitized would not be any higher.

Table 4c looks at a different aspect of the problem, namely the impact of investment volatility $\sigma$ on the optimal amount annuitized in the anything anytime environment. In this case, we assume that same $\lambda^O = \lambda^S = 0.04$ hazard rate, $r = 0.05$ risk free rate and $\mu = 0.12$ drift for the risky asset. We provide results under two differing levels of risk aversion: high ($\gamma = 5$) and low ($\gamma = 2$). In both of these cases, we assume a retiree/investor with $1,000,000 of liquid investable wealth and $40,000 in pre-existing annuity income.

Notice from Table 4c that as the investment volatility $\sigma$ increases from 0.12 to 0.20, the amount annuitized declines under both (and we can show, all) levels of risk aversion. In fact, under high levels of volatility the retiree/investor annuitizes $472,871 for $\gamma = 2$ and $768,568 for \gamma = 5$. These numbers drop to $12,692 and $469,789, respectively, when the investment volatility is reduced from 0.20 to 0.12. The economic intuition for this result is quite clear. As the relative risk of investing in the high-return alternative declines, it becomes much less appealing, on a risk-adjusted basis, to annuitize one’s wealth.

Table 4d investigates the impact of subjective vs. objective health status on the amount annuitized in the same anything anytime framework. In this case we assume an objective (annuity
pricing) hazard rate of $\lambda^O = 0.04$, but vary the subjective $\lambda^S$ from a value of 0.03 to 0.055. Thus, the insurance company believes the annuitant has a life expectancy of $1/0.04 = 25.0$ years, while the annuitant believes they are healthier (with a life expectancy increased to $1/0.03 = 33.3$ years) or unhealthier (with a life expectancy decreased to $1/0.055 = 18.2$ years). One can think of this as representing the impact of adverse selection or asymmetry of information on the amount annuitized. As in Table 4c, we assume the retiree/investor has $1,000,000$ in liquid wealth and pre-existing annuity income of $40,000$ per annum. The market parameters are $r = 0.05$ for the risk free rate, $\mu = 0.10$ drift of the risky asset and $\sigma = 0.16$ for the investment volatility.

In this case, and in contrast to the results from the all-or-nothing results, an increasing hazard rate has a uniform impact on the amount annuitized. Namely, people who consider themselves to be in worse health annuitize less, all else being equal. More specifically, a $\gamma = 2$ retiree/investor who believes he has a life expectancy of (only) 18.2 years will annuitize $514,496$ in contrast to $574,840$ when he believes his life expectancy is a higher 33.3 years. The same result is observed at higher levels of risk aversion, although it is less extreme in nominal (and marginal) terms. Notice that a 15-year difference in perceived life expectancy reduces the amount annuitized by $60,000$ under low ($\gamma = 2$) levels of risk aversion. But, at higher levels of ($\gamma = 5$) risk aversion, the difference is a mere $28,000$.

We remind the reader that these results have been obtained under a variety of assumptions, namely a constant hazard rate (exponential future lifetime distribution) and zero bequest motive, as well as constant risk-free rate and risk-premia. To eliminate these restrictions is the subject of ongoing research.

8 Conclusion and Main Insights

Our paper locates the optimal dynamic policy of a utility-maximizing individual who is interested in incorporating lifetime payout annuities (or defined benefit pension income) into his or her retirement portfolio. We investigated a variety of institutional arrangements and market structures with differing restrictions and constraints. Our main results can be stated as follows.

- An individual who is faced with the irreversible decision of when to start a fixed (nominal or real) lifetime pension annuity – with the proviso that annuitization must take place in an ‘all-or-nothing’ format – is endowed with an incentive to delay that is quite valuable at younger ages.

- There is an incentive to delay annuitization even in the absence of bequest motives. This is because market imperfections do not allow retirees to purchase payout annuities that offer complete asset allocation flexibility to match ones subjective consumption preferences vis à vis their health status. Stated differently, a market in which instantaneously-renegotiated life-contingent tontines are available would not give rise to our ‘option to delay’ result.
• Under an all-or-nothing framework, which is a feature of many public and private pension systems, the optimal age to annuitize is the age at which the option to delay has zero time value. This value is defined equal to the loss in utility that comes from not being able to behave optimally. This value depends on a person’s coefficient of relative risk aversion, as well as their subjective health status.

• Using historical market parameters and realistic mortality estimates, we conclude that in this all-or-nothing framework the optimal age to purchase a pure life-contingent annuity does not occur prior to age 70. This result is consistent with a variety of probabilistic-based models which are based on the relationship between mortality credits and the returns from competing asset classes.\footnote{This after-age-70 result has also been advocated in a variety of popular-press article, such as a recent piece in the Wall Street Journal, 3 September 2003, page C1, “It Pays to Delay: The Longer You Wait To Buy an Annuity, the More You Get,” by Jonathan Clements.}

• In the event that complete asset allocation flexibility is available within the payout annuity, which is akin to variable immediate annuities that are available in some countries and jurisdictions, the optimal age to annuitize is indeed earlier. Of course, high management fees and expenses relative to non-annuitized wealth can have a strong mitigating impact on the benefits from annuitization.

• When we move towards an open institutional system in which annuitization can take place in small portions and at anytime, we find that utility-maximizing investors should acquire a base amount of annuity income (i.e. Social Security or a DB pension) and then annuitize additional amounts if and when their wealth-to-income ratio exceeds a certain level. In this case which we label anything anytime, individuals annuitize a fraction of wealth as soon as they have opportunity to do so – i.e. they do not wait – and they then purchase more annuities as they become wealthier.

• Thus, in contrast to the all-or-nothing pension structure, in the case of an open system where annuities can be purchased on an ongoing basis we find that individuals prior to age 70 should have a minimal amount of annuity income and should immediately annuitize a fraction of wealth to create this base level of lifetime income if they do not already have this from pre-existing DB pensions. We reiterate that individuals should always hold some annuities, even in the presence of a bequest motive, as long as \( z_0(t) \) in Proposition 6.2 is less than infinity.

• Finally, our anything anytime model indicates that ceteris paribus, a larger amount of wealth relative to pre-existing annuity income, higher levels of risk aversion, greater investment volatility \( \sigma \) and better health status (i.e. lower subjective mortality rates) will all serve to increase the amount that is voluntarily annuitized. And, although the magnitude of these...
results depend on the specific parameters involved, all of these comparative statics are consistent with our qualitative intuition and should therefore re-enforce normative investment advice.

8.1 Directions for Future Research

That our paper raises a number of questions that should be considered in any future research on the topic, which we will now elaborate on.

First, by far the strongest assumption we have made in our modeling for both the restricted all-or-nothing market and the anything-anytime environment, is that the risk-free rate and the market risk premium are assumed constant. Thus, we have abstracted from any term structure effects, or predictability in the evolution of interest rates as well as stochasticity of investment volatility $\sigma$ and the market risk premium. Yet, recent models of asset allocation and portfolio choice have gone well beyond the classic Mertonian framework, which was the foundation of our analysis. The next step, from our perspective, is to enhance the financial chassis of our model and allow for more complicated market dynamics.

Indeed, a number of practitioners have advocated that people refrain from annuitizing when interest rates are low and that annuitization is more appealing when interest rates are high. Other commentators have advocated a dollar-cost averaging approach to annuitization to smooth out the interest rate risk, which resonates with the results from our anything-anytime analysis. Obviously, one might question the precise definition of high and low interest rates in this context, but it would certainly be interesting to investigate the impact of assuming a mean-reverting process for interest rates and possibly the yield curve as a whole. This would necessitate modeling the evolution of nominal versus real interest rates as well as the behavior of inflation. We, thus, envision an advanced model in which real (inflation-adjusted) and nominal life annuities co-exist in the optimal portfolio. This complicates our basic model by introducing at least one more state variable in the PDE/ODE, which is why we leave for future research.

Likewise, a recent innovation in the U.S. retirement income market is the introduction of guarantee living benefits, which are essentially staggered put options on a portfolio that promise a minimal level of income for as long the annuitant lives. These products which fall under the industry label of Guaranteed Minimum Withdrawal Benefits (GMWB), have some longevity insurance features and some derivative securities features. These products which are part of the trillion dollar Variable Annuity (VA) industry in the U.S. are growing in popularity and might actually compete with conventional life annuities as a way of generating a sustainable retirement income. Further research would examine the optimal demand and asset allocation including these hybrid GMWB products.

Furthermore, given the normative focus on this paper, we have ignored the positive equilibrium implications. The main question in this case would be how annuity prices would be affected by individuals’ desire to annuitize at an optimal time. Indeed, the payoff from conventional financial
assets are not age- or mortality-dependent and thus do not depend on the demographic structure or health status of the marginal investor. Under the discretionary and voluntary life annuities that we have analyzed, one can envision a situation in which very few people purchase a life annuity at age 50, which thus reduces the pool of individuals across which mortality risk can be diversified via the law of large numbers. This will have immediate implication on the pricing curve and the objective hazard rate, which we have taken as given. A detailed equilibrium analysis would attempt to derive the market’s objective hazard as a function of the heterogeneous mix of participant’s subjective hazard rates. But, this is far beyond the scope of the current paper.

On a related note, our model implicitly assumes that mortality rates are fully predictable in the future and that we are able to specify a survival function and annuity pricing equation during the entire horizon, conditional on the value of interest rates. In other words, we assume that the objective hazard rate is deterministic. However, there is a growing body of empirical and theoretical literature that argues that mortality risk is priced in equilibrium. In the extreme, this would imply that if one delays annuitization one runs the risk that annuity prices will actually increase, even though the individual has aged. This, of course, would introduce yet another variable in the decision and further complicate the analysis of the optimal age at which to annuitize. That said, with the impending retirement of American baby boomers and the industrial shift from Defined Benefit (DB) to Defined Contribution (DC) pension plans, we believe these issues will demand further academic attention as they take on greater practical importance.
9 Appendix

9.1 Appendix A: Impact of Objective vs. Subjective Health

In this appendix, we show that if the subjective force of mortality varies slightly from the objective force to the extent that \( \bar{a}_x^S < 2\bar{a}_x^O \) for all \( x \), then the optimal time of annuitization increases from the \( T \) given by the zero of the right-hand side of (16). In particular, if the individual is less healthy than normal (\( \lambda_x^S > \lambda_x^O \) for all \( x \)), then \( \bar{a}_x^S < \bar{a}_x^O \) for all \( x \), from which it follows that the optimal time of annuitization is delayed. Also, if the individual is more healthy than normal but only to the extent that \( \bar{a}_x^S < 2\bar{a}_x^O \) for all \( x \), then the optimal time of annuitization is delayed.

Suppose \( \bar{a}_{x+T}^S = \bar{a}_{x+T}^O + \varepsilon \) for some small \( \varepsilon \), not necessarily positive. Then, equation (16) at the critical value \( T \) becomes

\[
0 = \left[ \frac{\gamma}{1 - \gamma} \left( \frac{\bar{a}_{x+T}^O + \varepsilon}{\bar{a}_{x+T}^O} \right)^{-1} - \frac{1}{1 - \gamma} \frac{\bar{a}_{x+T}^O + \varepsilon}{\bar{a}_{x+T}^O} \right] + \left( \frac{\varepsilon}{\bar{a}_{x+T}^O} \right) \left[ \delta - (r + \lambda_x^O) \right].
\]

(61)

We can simplify this equation to

\[
0 = \left( \frac{\varepsilon}{\bar{a}_{x+T}^O} \right) \left[ \delta - (r + \lambda_x^O) \right] - \frac{1}{2} \left( \frac{\varepsilon}{\bar{a}_{x+T}^O} \right)^2 \\
- \frac{1}{6} \left( \frac{\varepsilon}{\bar{a}_{x+T}^O} \right)^3 + \ldots,
\]

(62)

if \( \frac{\varepsilon}{\bar{a}_{x+T}^O} \) lies between \(-1\) and \(1\). Thus, by the mean value theorem, there exists \( \varepsilon^* \) between 0 and \( \frac{\varepsilon}{\bar{a}_{x+T}^O} \) such that

\[
0 = \left( \frac{\varepsilon}{\bar{a}_{x+T}^O} \right) \left[ \delta - (r + \lambda_x^O) \right] + \frac{1}{2\gamma} (\varepsilon^*)^2,
\]

or equivalently,

\[
0 = \left[ \delta - (r + \lambda_x^O) \right] + \frac{(\varepsilon^*)^2}{2\gamma \bar{a}_{x+T}^O (1 + \frac{\varepsilon}{\bar{a}_{x+T}^O})}.
\]

(63)

The second term of the above equation is positive (and small) regardless of the sign of \( \varepsilon \). Thus, \( T \) is determined by setting \( \left[ \delta - (r + \lambda_x^O) \right] \) equal to a negative number. It follows that, for \( \lambda_x^O \) increasing with respect to \( x \), this value of \( T \) will be larger than the zero of (17).

Note that a sufficient condition for the above result is that \( \frac{\varepsilon}{\bar{a}_{x+T}^O} \) lie between \(-1\) and \(1\). Without difficulty, one can show that this requirement is equivalent to \( \bar{a}_{x+T}^S < 2\bar{a}_{x+T}^O \). For less healthy people (\( \lambda_x^S > \lambda_x^O \) for all \( x \)), we have \( \bar{a}_{x+T}^S < \bar{a}_{x+T}^O \) for all \( x \), so \( \bar{a}_{x+T}^S < 2\bar{a}_{x+T}^O \) holds automatically. Also, there is some leeway in this inequality, so that even healthier people might have that \( \bar{a}_{x+T}^S < 2\bar{a}_{x+T}^O \). Even when this inequality does not hold, we conjecture that we still have a delay in the time of
annuitization beyond that given by the zero of the right-hand side of (17), as we see in one of the examples in Section 5.

9.2 Appendix B: Probability of Reduced Payments

The probability that consumption (as a percentage of initial wealth) at optimal time of annuitization is \( p\% \) less than the consumption if one annuitizes immediately equals

\[
P \left( \frac{W^*_T}{a^O_{x+T}} < (1 - 0.01p) \frac{a^O_x}{\bar{a}^O_x} \mid W_0 = w \right)
\]

\[
= P \left( we^{2\delta - r \frac{(\mu - r)^2}{2\sigma^2}} T - \int_0^T k(s) ds + \frac{\mu - r}{\sigma^2} B_T \frac{\bar{a}^O_{x+T}}{\bar{a}^O_x} < (1 - 0.01p) \frac{a^O_{x+T}}{a^O_x} \right)
\]

\[
= P \left( e^{(1 - 0.01p)\frac{\mu - r}{\sigma^2}} B_T < \left( 1 - \frac{\mu - r}{2\sigma^2} \right) T + \int_0^T k(s) ds \right)
\]

\[
= \Phi \left( \ln \left( \frac{(1 - 0.01p)\bar{a}^O_{x+T}}{\bar{a}^O_x} \right) - \left( 2\delta - r - \frac{(\mu - r)^2}{2\sigma^2} \right) T + \int_0^T k(s) ds \right)
\]

Here, \( \Phi \) denotes the cumulative distribution function of the standard normal.

9.3 Appendix C: Variable and Fixed Annuity Benefits

In the body of the paper, we assumed that the only life annuities available upon annuitization provided a fixed payout. In this appendix, we examine a market in which annuity ‘wrappers’ are available on all asset classes with full asset allocation mobility. We investigate the optimal consumption, investment, and annuitization policies in this market. We introduce the symbol \( \beta \) to represent the proportion of wealth at the time of annuitization that is invested in the variable annuity, so that \( 1 - \beta \) is the proportion invested in the fixed annuity. We assume that the mix between the variable and fixed annuities, namely \( \beta \) versus \( 1 - \beta \), stays fixed throughout the remaining life of the annuitant, which is a so-called money mix plan and has certain optimality features as shown by Charupat and Milevsky (2002). Again, we consider CRRA utility and provide formulas for the power utility case. One can easily deal with the logarithmic case by letting \( \lambda \) approach 1 in the consumption, investment, and annuitization policies under power utility \( u(c) = \frac{1}{1-\gamma}c^{1-\gamma}, \gamma > 0, \gamma \neq 1 \).

To further capture the salient features of this product, we assume that the provider of the annuity has a nonzero insurance load on the fixed annuity such that the effective ‘return’ on the
fixed annuity is \( r' \), with \( r' \leq r \). Similarly, there is a non-zero insurance load on the variable annuity such that the drift on the variable annuity is \( \mu' \) with \( \mu' \leq \mu \) and \( \mu' - r' \leq \mu - r \).

For the mixture of a variable and a fixed annuity, define the value function \( V \) by

\[
V(w, t; T) = \sup_{\{c_s, \pi_s, \beta\}} E^{w,t} \left[ \int_t^T e^{-r(s-t)} s^{-t} P_{x+t}^S \frac{c_{s-\gamma}}{1-\gamma} ds \right. \\
+ \left. \int_T^\infty e^{-r(s-t)} s^{-t} P_{x+t}^S \frac{1}{1-\gamma} \left( \frac{W_{r'}}{a_{x+t}^{O,r'}} e^{\beta \left( \mu' - r' - \frac{\beta \sigma^2}{2} \right)(s-T) + \beta (B_s - B_T)} \right)^{1-\gamma} ds \right]
\]  

(65)
in which the second superscript on \( a_{x+t}^{O,r'} \), namely \( r' \), is the rate of discount used to calculate the actuarial present value of the annuity.

We can deal with the choice of \( \beta \) by noting that (Björk, 1998)

\[
E^{w,t} \left[ W_T^{1-\gamma} e^{\beta(1-\gamma) \left( \mu' - r' - \frac{\beta \gamma \sigma^2}{2} \right)(s-T) + \beta (B_s - B_T)} \right] = E^{w,t} \left[ W_T^{1-\gamma} e^{\beta(1-\gamma) \left( \mu' - r' - \frac{\beta \gamma \sigma^2}{2} \right)(s-T)} \right].
\]  

(66)

Thus, the expectation is maximized if we maximize \( \beta \left( \mu' - r' - \frac{\beta \gamma \sigma^2}{2} \right) \). It follows that the optimal value of \( \beta \) equals

\[
\beta^* = \frac{\mu' - r'}{\sigma^2}.
\]  

(67)

Note that the optimal choice of \( \beta \) is independent of the optimal time \( T \) of annuitization. Naturally, if \( \beta^* \) is greater than one, the holdings are truncated by the seller of the annuity at 100% of the risky stock.

It follows that \( V \) solves the HJB equation

\[
\begin{align*}
(r + \lambda^S_{x+t}) V &= V_t + \max_{\pi} \left[ \frac{1}{2} \sigma^2 p^2 V_{ww} + (\mu - r) \pi V_w \right] + rw V_w + \max_{c \geq 0} \left[ -c V_w + \frac{1}{1-\gamma} c^{1-\gamma} \right], \\
V(w, T; T) &= \frac{1}{1-\gamma} \left( \frac{w}{a_{x+t}^{S,r}} \right)^{1-\gamma} \frac{x^{(1-\gamma)(\mu' - r')^2}}{\sigma^2 \gamma^2}.
\end{align*}
\]  

(68)

Note that this is the same as the previous HJB equation, except that the boundary condition reflects the mixture of the variable and fixed annuities. The second superscript on the actuarial present value denotes the rate at which the annuity payments are discounted. It follows that \( V \) has the form as that given in equation (12), except that \( a_{x+t}^S = a_{x+t}^{S,r} \) is replaced with

\[
\frac{x^{(1-\gamma)(\mu' - r')^2}}{\sigma^2 \gamma^2}.
\]  

(69)
and $\tilde{a}_{x+T}^{O} = \tilde{a}_{x+T}^{O,r'}$ is replaced with $\tilde{a}_{x+T}^{O,r}$. Also, note that the optimal investment policy is similar in form to the one given in the body of the paper for the fixed-only payout case. If there are no loads on the fixed and variable annuities, that is, if $r' = r$ and $\mu' = \mu$, then the proportion of wealth invested in the risky asset from before annuitization equals the proportion after annuitization; however, in this case, the optimal strategy of the individual is to annuitize immediately. This latter result follows from the work of Yaari (1965). For a CRRA investor (with no bequest motives) with no insurance loads on the annuities, the optimal mixture between risky and risk-free assets is invariant to whether the portfolio is annuitized or not.

The derivative of $V$ with respect to $T$ is proportional to

$$
\frac{\delta V}{\delta T} \propto \left[ \frac{\gamma}{1 - \gamma} \left( \frac{S_{x+T} (1 - \gamma)(\mu' - r')^2}{\tilde{a}_{x+T}^{O,r'}} \right)^{\frac{1}{1 - \gamma}} \right] - \frac{1}{1 - \gamma} + \frac{S_{x+T} (1 - \gamma)(\mu' - r')^2}{\tilde{a}_{x+T}^{O,r'}} 
$$

in which $\delta' = r' + \frac{(\mu' - r)^2}{2\sigma^2}$. We can use this equation to determine the optimal value of $T$ in any given situation. In Table 5, we compare the optimal ages of annuitization and the imputed value of delaying when the individual can only buy a fixed annuity (compare with Table 1) and when the individual can buy a money mix of variable and fixed annuities.

**Table 5a about here.**

We assume that the financial market and mortality are as described in Section 5, except that for the variable annuity, the insurer has a 100-basis-point Mortality and Expense Risk Charge load on the return, so that the modified drift is $\mu' = 0.11$, and for the fixed annuity, the insurer has a 50-basis-point spread on the return, so that the modified rate of return is $r' = 0.055$.

**Table 5b about here.**

Assume that the individual has a CRRA of $\gamma = 2$, from which it follows that the individual will invest 75.0% in the risky stock before annuitization and 68.7% in the variable annuity after annuitization.

### 9.4 Appendix D: Escalating Annuity Benefit

Suppose that the individual can buy an escalating annuity. An escalating annuity is one for which the payments increase at a (constant) rate $g$. These are known as COLA (Cost Of Living Adjustment) annuities and are available from vendors that sell fixed annuities. These escalating annuities
are popular as a hedge against (expected) inflation, since it is virtually impossible to obtain true inflation-linked annuities in the U.S. The actuarial present value of an escalating annuity can be written $a^{T-g}_x$; that is, the rate of discount $r$ is reduced by the rate of increase of the payments $g$. As in the previous two sections, we consider CRRA utility and provide formulas for the power utility case. Thus, we can define the corresponding value function by

$$V(w, t; T) = \sup_{\{c_s, \pi_s, g\}} \left[ E \int_t^T e^{-r(s-t)} s-t P^S_{x+t} \frac{1}{1 - \gamma} s^{1-\gamma} ds \right.$$ 

$$+ \int_T^\infty e^{-r(s-t)} s-t P^S_{x+t} \frac{1}{1 - \gamma} \left( \frac{W_T}{\bar{a}_x^R} e^{g(s-T)} \right)^{1-\gamma} ds \bigg| W_t = w \right]$$

This expression is maximized with respect to $g$ if

$$\frac{1}{1 - \gamma} \left[ \int_0^\infty e^{-(r-(1-\gamma)g)s} s P^O_{x+T} ds \right]^{1-\gamma}$$

is maximized. The derivative of this expression with respect to $g$ is proportional to

$$\frac{\int_0^\infty s e^{-(r-(1-\gamma)g)s} s P^S_{x+T} ds}{\int_0^\infty e^{-(r-(1-\gamma)g)s} s P^S_{x+T} ds} \cdot \frac{\int_0^\infty s e^{-(r-g)s} s P^O_{x+T} ds}{\int_0^\infty e^{-(r-g)s} s P^O_{x+T} ds}.$$

Note that this is a difference of expectations of “$s$” with respect to two probability distributions. Also, note that if $\lambda^S_x = \lambda^O_x - c$ for all $x$ and for some constant $c$, then the optimal value of $g$ is $g^* = \frac{c}{\gamma}$. For example, if the individual is healthier to the extent that the subjective hazard rate is 0.01 less than the objective (pricing) hazard rate, then optimal rate of increase of the annuity payments (once the individual annuitizes his or her wealth) is $0.01\gamma$. Note that in general, healthier people want to buy escalating annuities with a positive $g$, while sicker people will want to buy escalating annuities with a negative $g$. This makes sense because healthier people anticipate living longer than normal, so they will be able to enjoy those larger annuity payments in the future. On the other hand, less healthy people will not live as long, so they demand higher payments now.

Table 6 provides a numerical example for an individual who believes he or she is healthier than normal with $f = -0.5$; that is, the person has one-half the mortality rate of the average person. Suppose that the CRRA is $\gamma = 1.5$. We compare these numbers with those when the individual can buy only a fixed annuity. It turns out that the optimal rate of escalation $g = 2\%$ (to two decimal places) for all ages and for both genders.

| Table 6 about here. | }

Note that the individual is willing to annuitize earlier if there is a 2% escalating annuity available; however, there is still an advantage to wait.
Table 1a: Optimal Age to Annuitize and Value of Delay under Low Risk Aversion

| Age | Optimal Age | Value of Delay | Prob. Lower Annuity | Optimal Age | Value of Delay | Prob. Lower Annuity |
|-----|-------------|----------------|---------------------|-------------|----------------|---------------------|
| 60  | 84.5 (80.3) | 44.0 (32.0)%   | 0.311 (0.353)       | 78.4 (73.0) | 15.3 (8.9)%  | 0.268 (0.321)       |
| 65  | 84.5 (80.3) | 33.4 (21.9)    | 0.346 (0.391)       | 78.4 (73.0) | 10.3 (4.3)   | 0.310 (0.372)       |
| 70  | 84.5 (80.3) | 22.7 (12.3)    | 0.385 (0.431)       | 78.4 (73.0) | 5.2 (0.8)    | 0.362 (0.435)       |
| 75  | 84.5 (80.3) | 12.3 (4.2)     | 0.429 (0.470)       | 78.4 (Now) | 1.2 (0.0)    | 0.428 (N/a)         |
| 80  | 84.5 (80.3) | 3.7 (0.02)     | 0.473 (0.500)       | Now (Now)  | neg. (neg.)  | N/a (N/a)           |
| 85  | Now (Now)   | neg. (neg.)    | N/a (N/a)           | Now (Now)  | neg. (neg.)  | N/a (N/a)           |

Notes: All-or-Nothing Market: The value of the option to delay annuitization for males and females with a coefficient of relative risk aversion of $\gamma = 1$ and $\gamma = 2$. We assume the non-annuitized funds are invested in a risky asset with drift $\mu = 0.12$ and volatility $\sigma = 0.20$. The risk-free rate is $r = 0.06$. The mortality is assumed to be Gompertz-Makeham fit to the IAM2000 table with projection Scale G. For example, a 70-year-old female with a coefficient of relative risk aversion of $\gamma = 2$, will effectively suffer a utility-equivalent 5.2% loss of her wealth if she chooses to annuitize immediately. The optimal time for her to annuitize is at age 78.4. The table also gives the probability of deferral failure, namely the probability that the annuity purchased at the optimal age will provide less income than an annuity bought right now.
Table 1b: Optimal Age to Annuitize and Value of Delay under Higher Risk Aversion

| Current Age | Optimal Age | Value of Delay | Optimal Age | Value of Delay |
|-------------|-------------|----------------|-------------|----------------|
| MALE        | FEMALE      |                |             |                |
| 60          | 63.4        | 0.41%          | 70.4        | 2.94%          |
| 65          | Now         | neg.           | 70.4        | 1.04%          |
| 70          | Now         | neg.           | 70.4        | 0.01%          |
| 75          | Now         | neg.           | Now         | neg.           |

Notes: All-or-nothing market. The risk-free rate is $r = 0.06$, the drift is $\mu = 0.12$, and the volatility is $\sigma = 0.20$. We use Gompertz-Makeham mortality with $m = 88.15$, $b = 10.5$ for males and $m = 92.63$, $b = 8.78$ for females. Notice that even at higher levels of risk aversion, males (and especially females) do not annuitize prior to age 60.
Table 2: Assuming Optimal Behavior, What is the Probability of Higher Consumption Upon Delay?

| Age | γ = 1, FEMALE (MALE) | γ = 2, FEMALE (MALE) |
|-----|----------------------|----------------------|
| 60  | 0.644 (0.596)        | 0.631 (0.551)        |
| 65  | 0.602 (0.549)        | 0.565 (0.459)        |
| 70  | 0.552 (0.494)        | 0.474 (0.296)        |
| 75  | 0.493 (0.425)        | 0.316 (0.133)        |
| 80  | 0.414 (0.137)        | N/a (N/a)            |
| 85  | N/a (N/a)            | N/a (N/a)            |

Notes: All-or-nothing market. Assuming the individual self-annuitizes and defers the purchase of a life annuity to the optimal age, this table indicates the probability of consuming at least 20% more at the time of annuitization, compared to if one annuitizes immediately. Thus, for example, a female (male) at age 65 with a coefficient of relative risk aversion of \( \gamma = 1 \) has a 64.4% (59.6%) chance of creating a 20% larger annuity flow.
### Table 3: How Does Subjective Health Status Impact Optimal Behavior?

| \( f \)  | More/Less Healthy | Optimal Age of Annuitization | Value of Delay Before Annuitization | Consumption Rate Before Annuitization | Consumption Rate After Annuitization |
|------|-------------------|-----------------------------|-----------------------------------|---------------------------------------|--------------------------------------|
| -1.0 |                   | 78.28                       | 13.79%                            | 7.55%                                 | 13.38%                               |
| -0.8 |                   | 74.58                       | 10.54                             | 7.95                                  | 11.79                                |
| -0.6 |                   | 73.71                       | 9.68                              | 8.18                                  | 11.47                                |
| -0.4 |                   | 73.29                       | 9.23                              | 8.37                                  | 11.33                                |
| -0.2 |                   | 73.09                       | 8.99                              | 8.54                                  | 11.26                                |
| 0.0  |                   | 73.03                       | 8.87                              | 8.70                                  | 11.24                                |
| 0.2  |                   | 73.08                       | 8.84                              | 8.85                                  | 11.26                                |
| 0.5  |                   | 73.31                       | 8.93                              | 9.06                                  | 11.33                                |
| 1.0  |                   | 74.04                       | 9.34                              | 9.38                                  | 11.59                                |
| 1.5  |                   | 75.21                       | 10.00                             | 9.68                                  | 12.03                                |
| 2.0  |                   | 76.96                       | 10.89                             | 9.98                                  | 12.76                                |
| 2.5  |                   | 79.71                       | 12.01                             | 10.26                                 | 14.12                                |
| 3.0  |                   | 85.38                       | 13.38                             | 10.55                                 | 18.01                                |

Notes: All-or-Nothing Market: The imputed value of delaying annuitization for a male, aged 60 with CRRA of \( \gamma = 2 \). We assume the funds are invested in a risky asset with drift \( \mu = 0.12 \) and volatility \( \sigma = 0.20 \). The risk-free rate is \( r = 0.06 \). The mortality is assumed to be Gompertz-Makeham fit to the IAM2000 table with projection Scale G. Thus, for example, if the individual’s subjective hazard rate is 20% higher (i.e. less healthy) than the mortality table used by the insurance company to price annuities, the optimal annuitization point is at age 73.1, and the value of the option is 8.84% of his wealth at age 60. While the 60-year-old male waits to annuitize, he consumes at the rate of 8.85% of assets, and once he purchases the fixed annuity, his consumption rate - and standard of living - will increase to 11.26% of assets.
Table 4a: How Does Wealth and Risk Aversion Affect Annuitization?
Amount of Money Spent on Annuities for Various Levels of Wealth and Risk Aversion ($A = 25,000$)

| Wealth (z₀)   | \(\gamma = 1.5\) | \(\gamma = 2.0\) | \(\gamma = 2.5\) | \(\gamma = 3.0\) | \(\gamma = 5.0\) |
|--------------|------------------|------------------|------------------|------------------|------------------|
| $1,000,000$  | $727,620$        | $792,020$        | $831,852$        | $858,901$        | $914,176$        |
| $500,000$    | $331,384$        | $371,251$        | $395,909$        | $412,653$        | $446,871$        |
| $250,000$    | $133,266$        | $160,866$        | $177,937$        | $189,529$        | $213,218$        |
| $100,000$    | $14,395$         | $34,635$         | $47,154$         | $55,655$         | $73,027$         |
| $50,000$     | $0$              | $0$              | $3559$           | $11,030$         | $26,296$         |

Notes: Anything Anytime Market: Table 4 illustrates for various levels of relative risk aversion \(\gamma\), the critical value of the ratio of liquid wealth to pre-existing annuity income \(z₀ = w/A\) above which the individual will spend a lump sum to increase her annuity income. We also include the amount the individual will spend on annuities, namely \((w - z₀A)/(1 + (r + \lambda^O)z₀)\). We assume the following parameter values: The force of mortality \(\lambda^S = \lambda^O = 0.04\), which implies a life expectancy of 25 years, the riskless rate of return is \(r = 0.04\), the risky asset’s drift is \(\mu = 0.08\), and the risky asset’s volatility is \(\sigma = 0.20\). Note that in contrast to the restricted all-or-nothing market, the individual immediately annuitizes a base level of income and then gradually annuitizes more as his wealth breaches higher levels.
Table 4b: How Does Wealth and Risk Aversion Affect Annuitization?

| Wealth     | γ = 1.5 | γ = 2.0 | γ = 2.5 | γ = 3.0 | γ = 5.0 |
|------------|---------|---------|---------|---------|---------|
| $1,000,000 | $662,802| $742,477| $791,789| $825,271| $893,741|
| $500,000   | $266,555| $321,715| $355,854| $379,034| $426,436|
| $250,000   | $68,432 | $111,334| $137,886| $155,915| $192,784|
| $100,000   | $0      | $0      | $7,106  | $22,044 | $52,592 |
| $50,000    | $0      | $0      | $0      | $0      | $5,862  |

Notes: Anything anytime market. In Table 4a, the pre-existing annuity income $A$ equals $25,000, while in Table 4b, $A$ is doubled to $50,000. Intuitively, the greater the level of pre-existing annuity income the less liquid wealth must be annuitized to provide the optimal annuitized consumption stream.
Table 4c: How Does Investment Volatility Affect Annuitization?

| Investment Volatility | Wealth of $1,000,000 and Initial Annuity Income of $40,000 |
|-----------------------|-----------------------------------------------------------|
|                       | Low Risk Aversion (\(\gamma = 2\)) | High Risk Aversion (\(\gamma = 5\)) |
| \(\sigma = 0.12\)    | $12,692                                      | $496,789                            |
| \(\sigma = 0.14\)    | $164,292                                     | $598,755                            |
| \(\sigma = 0.16\)    | $289,253                                     | $672,235                            |
| \(\sigma = 0.18\)    | $390,628                                     | $726,853                            |
| \(\sigma = 0.20\)    | $472,871                                     | $768,568                            |

Notes: \(r = 0.05\) and \(\mu = 0.12\), with \(\lambda^O = \lambda^S = 0.04\). At higher levels of investment volatility (risk), a greater amount is annuitized immediately.
Table 4d: How Does Subjective Health Status Affect Annuitization?

| Subjective Hazard Rate | Wealth of $1,000,000 and Initial Annuity Income of $40,000 |
|------------------------|----------------------------------------------------------|
| $\lambda^S = 0.030$   | Low Risk Aversion ($\gamma = 2$) $\rightarrow$ $574,840$ | High Risk Aversion ($\gamma = 5$) $\rightarrow$ $817,383$ |
| $\lambda^S = 0.035$   | $563,603$                                                 | $812,222$                                                 |
| $\lambda^S = 0.040$   | $551,941$                                                 | $806,842$                                                 |
| $\lambda^S = 0.045$   | $539,862$                                                 | $801,242$                                                 |
| $\lambda^S = 0.050$   | $527,375$                                                 | $795,423$                                                 |
| $\lambda^S = 0.055$   | $514,496$                                                 | $789,388$                                                 |

Notes: $r = 0.05$, $\mu = 0.10$, and $\sigma = 0.16$, with $\lambda^O = 0.04$. The higher (less healthy) hazard rate leads to reduced levels of annuitization.
| Age | Optimal age of annuitization | Value of annuitization delay | Optimal age of Annuitization | Value of Annuitization delay |
|-----|-----------------------------|-----------------------------|------------------------------|-----------------------------|
| 60  | 80.2 (75.2)                 | 21.0 (13.4)%                | 70.8 (64.1)                  | 3.4 (0.6)%                  |
| 65  | 80.2 (75.2)                 | 14.8 (7.5)                  | 70.8 (Now)                   | 1.3 (neg.)                  |
| 70  | 80.2 (75.2)                 | 8.5 (2.5)                   | 70.8 (Now)                   | 0.04 (neg.)                 |
| 75  | 0.2 (75.2)                  | 2.9 (0.003)                 | Now (Now)                    | neg. (neg.)                 |

Notes: In an all-or-nothing annuitization environment – where both fixed and variable annuities are available with complete asset allocation flexibility – this table illustrates the imputed value of delaying annuitization for males and females with a CRRA of $\gamma = 2$. We assume that the non-annuitized funds are invested in a risky asset with drift $\mu = 0.12$ and volatility $\sigma = 0.20$. The risk-free rate is $r = 0.06$. We introduce insurance loads on the variable and fixed annuities on the order of 100 basis points and 50 basis points, respectively. The mortality is assumed to be Gompertz fit to the IAM2000 table with projection Scale G.
Table 5b: All-or-Nothing Decision under Various Levels of Insurance Fees

| Insurance Fees Deducted | Low Risk Aversion (γ = 2) | High Risk Aversion (γ = 5) |
|-------------------------|--------------------------|---------------------------|
|                         | FEMALE | MALE | FEMALE | MALE | FEMALE | MALE |
| 50 b.p.                 | 67.2    | 60.0 | 66.9    | 60.0 |
| 100 b.p.                | 70.8    | 64.0 | 68.4    | 61.2 |
| 125 b.p.                | 72.1    | 65.6 | 69.1    | 61.9 |
| 150 b.p.                | 73.2    | 66.9 | 69.6    | 62.6 |
| 200 b.p.                | 74.9    | 69.0 | 70.6    | 63.8 |

Notes: Assumes the same parameter values as Table 5a.
Table 6: All-or-Nothing Decision with Variable and Fixed Immediate Escalating Annuities

| Age | Fixed Annuity | 2% Escalating Annuity |
|-----|---------------|------------------------|
|     | FEMALE (MALE) | FEMALE (MALE)          |
| 60  | 80.9 (76.1)   | 78.5 (73.2)            |
|     | 23.68 (15.59)%| 17.41 (9.61)%          |
| 65  | 80.9 (76.1)   | 78.5 (73.2)            |
|     | 17.05 (9.24)  | 11.50 (4.88)           |
| 70  | 80.9 (76.1)   | 78.5 (73.2)            |
|     | 10.22 (3.57)  | 5.80 (0.96)            |
| 75  | 80.9 (76.1)   | 78.5 (Now)             |
|     | 3.96 (0.15)   | 1.29 (0.00)            |

Notes: In an all-or-nothing annuitization environment with escalating annuities available, this table illustrates the value of delay for males and females with a CRRA of $\gamma = 1.5$. We assume the liquid funds are invested in a risky asset with drift $\mu = 0.12$ and volatility $\sigma = 0.20$. The risk-free rate is $r = 0.06$. The mortality is Gompertz-Makeham fit to the IAM2000 table with projection Scale G, while the individual has subjective mortality beliefs equal to one-half of the objective mortality. Note that the availability of an increasing annuity – which better matches the desired consumption profile – accelerates the optimal age of annuitization and reduces the option value to wait.
Notes: This figure shows the probability density function of the future-lifetime random variable under an analytic Gompertz-Makeham hazard rate that is fitted to the Individual Annuity Mortality Table 2000 with projection scale G. For males, the ‘best fitting’ parameters are \((m, b) = (88.18, 10.5)\) and for females they are \((92.63, 8.78)\).
Notes: This figure shows the expected consumption of a 60-year-old male who believes he is 20% more healthy than average population rates; 8.34% is the rate of consumption if he annuitizes his wealth at age 60. We also display the 25th and 75th percentiles of the distribution of consumption between ages 60 and 75.
References

[1] Akerlof, G. A. (1970), The market for lemons: Quality uncertainty and the market mechanism, *Quarterly Journal of Economics*, 84: 488-50

[2] Battocchio, P., F. Menoncin, and O. Scaillet (2003), Optimal asset allocation for pension funds under mortality risk during the accumulation and decumulation phases, working paper, FAME, DP 66.

[3] Björk, T. (1998), *Arbitrage Theory in Continuous Time*, Oxford University Press, New York, New York.

[4] Blake, D., A. J. Cairns, and K. Dowd (2000), PensionMetrics: Stochastic pension plan design during the distribution phase, Pensions Institute Working Paper.

[5] Bhattacharya, J., D.P. Goldman and N. Sood (2003), Market evidence of misperceived and mistaken mortality risks, NBER working paper #9863

[6] Bodie, Z., A. Marcus, and R. Merton (1988), Defined benefit versus defined contribution plans, in Z. Bodie, J. Shove, and D. Wise (editors), *Pensions in the U.S. Economy*, NBER Research Project Report.

[7] Bodie, Z., J. B. Detemple, S. Otruba, and S. Walter (2004) Optimal consumption–portfolio choices and retirement planning, *Journal of Economic Dynamics and Control*, 28 (6): 1013-1226.

[8] Bowers, N. L., H. U. Gerber, J. C. Hickman, D. A. Jones, and C. J. Nesbitt (1997), *Actuarial Mathematics*, Society of Actuaries, Schaumburg, Illinois.

[9] Brown, J. R. (1999), Are the elderly really over-annuitized? New evidence on life insurance and bequests, NBER Working Paper 7193.

[10] Brown, J. R. (2001), Private pensions, mortality risk, and the decision to annuitize, *Journal of Public Economics*, 82 (1): 29-62.

[11] Brown, J. R. and J. Poterba (2000), Joint life annuities and annuity demand by married couples, *Journal of Risk and Insurance*, 67 (4): 527-554.

[12] Brown, J. R. and M. J. Warshawsky (2001), Longevity-insured retirement distributions from pension plans: Market and regulatory issues, NBER Working Paper 8064.

[13] Brugiavini, A. (1993), Uncertainty resolution and the timing of annuity purchases, *Journal of Public Economics*, 50: 31-62.
[14] Cairns, A., D. Blake, and K. Dowd (2005), Stochastic lifestyling: Optimal dynamic asset allocation for defined contribution pension plans, *Journal of Economic Dynamics and Control*, in press.

[15] Campbell, J. Y. and L. M. Viceira (2002), *Strategic Asset Allocation - Portfolio Choice for the Long-Term Investor*, Oxford University Press, New York.

[16] Charupat, N. and M. A. Milevsky (2002), Optimal asset allocation in life annuities: a note, *Insurance: Mathematics and Economics*, 30: 199-209.

[17] Davis, M. H. A. and A. R. Norman (1990), Portfolio selection with transaction cost, *Mathematics of Operations Research*, 15: 676-713.

[18] Dixit, A. K. and R. S. Pindyck (1994), *Investment under Uncertainty*, Princeton University Press, Princeton, New Jersey.

[19] Duffie, D. and T. Zariphopoulou (1993), Optimal investment with undiversifiable income risk, *Mathematical Finance*, 3 (2): 135-148.

[20] Dushi, I. and A. Webb (2003), Annuitzation: Keeping your options open, working paper, International Longevity Center.

[21] Feldstein, M. and E. Ranguelova (2001), Individual risk in an investment-based social security system, *American Economic Review*, 91 (4): 1116-1125.

[22] Frees, E. W., J. Carriere, and E. Valdez (1996), Annuity valuation with dependent mortality, *Journal of Risk and Insurance*, 63 (2): 229-261.

[23] Friedman, B. and M. Warshawsky (1990), The cost of annuities: Implications for saving behavior and bequests, *Quarterly Journal of Economics*, 105 (1): 135-154.

[24] Friend, I. and M. E. Blume (1975), The demand for risky assets, *American Economic Review*, 65: 900-922.

[25] Gerber, H. U. (1979), *An Introduction to Mathematical Risk Theory*, S. S. Heubner Foundation Monograph Series 8, Philadelphia.

[26] Hurd, M. D. (1989), Mortality risk and bequest, *Econometrica*, 57 (July, 4): 779-813.

[27] Hurd, M. D. and K. McGarry (1995), Evaluation of the subjective probabilities of survival in the health and retirement study, *Journal of Human Resources*, 30: 268-291.

[28] Hurd, M. D. and K. McGarry (1997), The predictive validity of subjective probabilities of survival, NBER Working Paper 6193.
[29] Johansson, P. O. (1996), The value of changes in life expectancy, *Journal of Health Economics*, 15: 105-113.

[30] Kapur, S. and M. Orszag (1999), A portfolio approach to investment and annuitization during retirement, Birkbeck College (University of London) Mimeo, May 1999.

[31] Karatzas, I. and S. Shreve (1998), *Methods of Mathematical Finance*, Springer-Verlag, New York.

[32] Kim, T. S. and E. Omberg (1996), Dynamic nonmyopic portfolio behavior, *Review of Financial Studies*, 9: 141-161.

[33] Koijen, R.S.J., T.E. Nijman and B.J.M. Werker (2006), Optimal portfolio choice with annuitization, *working paper*, Tilberg University

[34] Koo, H. K. (1998), Consumption and portfolio selection with labor income: A continuous time approach, *Mathematical Finance*, 8: 49-65.

[35] Kotlikoff, L. J. and A. Spivak (1981), The family as an incomplete annuities market, *Journal of Political Economy*, 89 (2): 373-391.

[36] Kotlikoff, L. J. and L. H. Summers (1981), The role of intergenerational transfers in aggregate capital accumulation, *Journal of Political Economy*, 89 (3): 706-732.

[37] Merton, R. C. (1971), Optimum consumption and portfolio rules in a continuous-time model, *Journal of Economic Theory*, 3 (December): 373-413.

[38] Milevsky, M. A. and V. R. Young (2002a), Optimal asset allocation and the real option to delay annuitization: It’s not now-or-never, Schulich School of Business Working Paper.

[39] Milevsky, M. A. and V. R. Young (2002b), Optimal annuity purchasing, Schulich School of Business Working Paper.

[40] Mitchell, O. S., J. M. Poterba, M. J. Warshawsky, and J. R. Brown (1999), New evidence on the moneys worth of individual annuities, *American Economic Review*, 89 (December, 5): 1299-1318.

[41] Modigliani, F. (1986), Life cycle, individual thrift, and the wealth of nations, *American Economic Review*, 76 (3): 297-313

[42] Neuberger, A. (2003), Annuities and the optimal investment decision, London Business School Working Paper.

[43] Øksendal, B. (1998), *Stochastic Differential Equations: An Introduction with Applications*, 5th edition, Springer-Verlag, Berlin.
[44] Richard, S. (1975), Optimal consumption, portfolio and life insurance rules for an uncertain lived individual in a continuous time model, *Journal of Financial Economics*, 2: 187-203.

[45] Rothschild, M. and J. Stiglitz (1976), Equilibrium in competitive insurance markets: An essay on the economics of imperfect information, *Quarterly Journal of Economics*, 90: 629-649.

[46] Shreve, S. E. and H. M. Soner (1994), Optimal investment and consumption with transaction costs, *Annals of Applied Probability*, 4 (3): 206-236.

[47] Sinclair, S. (2003), Optimal annuitization with consumption shocks correlated with mortality, working paper, University of Waterloo.

[48] Smith, V. K., D. H. Taylor, and F. A. Sloan (2001), Longevity expectations and death: Can people predict their own demise?, *American Economic Review*, 91 (4): 1126-1134.

[49] Sorensen, C. (1999), Dynamic asset allocation and fixed income management, *Journal of Financial and Quantitative Analysis*, 34 (4): 513-532.

[50] Stabile, G. (2003), Optimal timing of annuity purchases: a combined stochastic control and optimal stopping problem, working paper, Universita degli Studi di Roma “La Sapienza.”

[51] Stock, J. H. and D. A. Wise (1990), Pensions, the option value of work, and retirement, *Econometrica*, 58 (5): 1151-1180.

[52] Sundaresan, S. and F. Zapatero (1997), Valuation, optimal asset allocation, and retirement incentives of pension plans, *Review of Financial Studies*, 10 (3): 631-660.

[53] Wachter, J. A. (2002), Portfolio and consumption decisions under mean-reverting returns: An exact solution for complete markets, *Journal of Financial and Quantitative Analysis*, 37 (1): 63-92.

[54] Wang, S. S. (1996), Premium calculation by transforming the layer premium density, *ASTIN Bulletin*, 26 (1): 71-92

[55] Yaari, M. E. (1965), Uncertain lifetime, life insurance and the theory of the consumer, *Review of Economic Studies*, 32: 137-150.

[56] Yagi, T. and Y. Nishigaki (1993), The inefficiency of private constant annuities, *Journal of Risk and Insurance*, 60 (September, 3): 385-412.

[57] Zariphopoulou, T. (1992), Investment/consumption models with transaction costs and Markov-chain parameters, *SIAM Journal on Control and Optimization*, 30: 613-636.
[58] Zariphopoulou, T. (1999), Transaction costs in portfolio management and derivative pricing, *Introduction to Mathematical Finance*, D. C. Heath and G. Swindle (editors), American Mathematical Society, Providence, Rhode Island. *Proceedings of Symposia in Applied Mathematics*, 57: 101-163.

[59] Zariphopoulou, T. (2001), Stochastic control methods in asset pricing, *Handbook of Stochastic Analysis and Applications*, D. Kannan and V. Lakshmikantham (editors), Marcel Dekker, New York.