Transport Equations for Driven Many-body Quantum Systems

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Transport equations for autonomous driven fermionic quantum systems are derived with the help of statistical assumptions, and of the Markov approximation. The statistical assumptions hold if the system consists of subsystems within which equilibration is sufficiently fast. The Markov approximation holds if the level density in each subsystem is sufficiently smooth in energy. The transport equation describes both, relaxation of occupation probability among subsystems at equal energy that leads to thermalization, and the transport of the system to higher energy caused by the driving force. The laser-nucleus interaction serves as an example for the applicability and flexibility of the approach.

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I. INTRODUCTION

We consider autonomous strongly interacting fermion systems under the influence of an external driving force. Examples are atomic nuclei hit by a strong laser pulse carrying MeV photons, and atoms in a trap where the confining potential oscillates with time. In both cases the motion of the constituents (nucleons or fermionic atoms) is primarily determined by a stationary potential (the mean field of the nuclear shell model or the stationary confining potential of the trap) which binds the particles. The external time-dependent force (the laser pulse or the oscillating potential) drives either system towards higher excitation energies. The interaction between the constituents (nucleons or atoms) drives the system towards statistical equilibrium.

Normally, a physical system equilibrates because it is coupled to an external reservoir (a heat bath). If driven by an external force, such a system is described in terms of a transport equation (a standard tool of nonequilibrium statistical quantum mechanics). The justification for that equation, based upon the presence of a heat bath, fails for autonomous systems. Transport equations also hold for autonomous systems. A prime example is the quantum Boltzmann equation that describes a quantum gas of interacting particles. Here equilibration is caused by the collision term describing particle interaction. The standard derivation of the quantum Boltzmann equation uses methods of quantum field theory. The systems we consider differ from the Boltzmann gas by the presence of a strong stationary potential. Transport equations have been used also in that context, for instance for the laser-nucleus interaction, albeit without firm theoretical justification. In the present paper we supply that justification. We show that statistical assumptions on coupling matrix elements between constituent parts of the system take the role of the heat bath. We derive the transport equations and, thereby, display the conditions of validity of these equations for autonomous systems.

In Section II we give two examples where transport equations have been or may be useful. In Section III we introduce and justify our statistical assumptions, and we define the conditions that allow us to use the Markov approximation in deriving the transport equation. In Section IV we derive and discuss the transport equation for internal relaxation. In Section V we generalize the approach to the description of driven systems. Section VI is devoted specifically to nuclear dipole absorption. Section VII contains a brief summary. Technical details are relegated to an appendix.

II. TWO EXAMPLES

The following two examples serve to set the stage and to give physical substance to the theoretical developments in the following Sections.

A. Laser-Nucleus Interaction

A laser pulse carrying photons with energy $\hbar \omega_0 \approx 10$ MeV that hits a target nucleus predominantly causes dipole excitation because the product of wave number $k$ and nuclear radius $R$ obeys $kR \ll 1$. When the target nucleus is in the ground state, dipole absorption excites the nuclear Giant Dipole Resonance (GDR). Pictorially speaking, the GDR is an oscillation of the center of mass of the protons against that of the neutrons. The Brink-Axel hypothesis postulates that this process is universal: Dipole excitation of any initial nuclear state $|i\rangle$ (ground or excited) with energy $E_i$, predominantly populates the nuclear Giant Dipole Resonance (GDR) built upon that state. The GDR is a mode $|d(i)\rangle$ of excitation given by the normalized product of the dipole operator and the initial state $|i\rangle$, with mean excitation energy $E_d - E_i$ where $E_d = \langle d(i)| H_N |d(i)\rangle$. The GDR is not an eigenstate of the nuclear Hamiltonian $H_N$. Rather, the GDR is distributed over the eigenstates of $H_N$ with an approximately Lorentzian distribution with width $\Gamma \approx 5$ MeV. In a time-dependent picture, the “spreading width” of the approach.
\( \Gamma^\dagger \) describes the damping of the GDR due to its rapid mixing with the eigenstates of \( H_N \). We refer to such mixing as to (nuclear) equilibration.

A sufficiently intense laser pulse may cause multiple dipole excitation, each dipole absorption process populating the GDR built upon the excited state(s) reached in the previous absorption process. Every such photon absorption process is followed by partial or complete equilibration, depending on the ratio of the rate for dipole absorption (derived in Ref. \[2\]) and the rate \( \Gamma^\dagger/\hbar \) for equilibration. For the theoretical description of the sequence of alternating absorption and equilibration processes, the Schrödinger equation is useless (nuclear level densities attain enormous values already several 10 MeV above the ground state, so that a numerical treatment is out of the question), and transport equations are called for. These describe the evolution in time of mean occupation probabilities of (classes of) nuclear states, see Refs. \[2–4\].

Intense pulses carrying photons with energy in the MeV range are under development both at the Nuclear Pillar of the Extreme Light Infrastructure \[8\] and at the Gamma Factory of the Large Hadron Collider at CERN \[8\]. The theoretical results in Refs. \[2–4\] should soon encounter experimental tests.

**B. Fermionic Atoms in a Trap**

We consider \( N \gg 1 \) fermionic atoms in a two-dimensional harmonic oscillator. That situation can be realized experimentally by capturing atoms with half-integer spin in a trap \[10\]. Two-dimensionality of the trap is effectively achieved by making the level spacing in the third spacial direction sufficiently large. The confining potential oscillates harmonically. That gives rise to a time-dependent dipole-type driving force leading to multiple dipole excitation, similar to the nuclear case. The atom-atom interaction is controlled via a Feshbach resonance and may be either attractive or repulsive \[11\]. Provided it is sufficiently strong and the excitation energy is sufficiently large, we expect the atom-atom interaction to cause equilibration, independently of its sign. Again, we deal with the interplay of a dipole interaction driving the system towards higher energies, and of the tendency of the system to equilibrate. The arguments in favor of a theoretical description in terms of transport equations are the same as in the nuclear case. Even for a small number of particles in the trap, the level density grows with excitation energy so strongly that there is no alternative to that approach. To the best of our knowledge, equilibration processes as investigated theoretically in this paper have not yet been studied experimentally for atoms in a trap. The system we study is close to equilibrium and, thus, differs from the ones studied in Refs. \[12–14\].

**III. STATISTICAL ASSUMPTIONS. MARKOV APPROXIMATION**

In this Section we give the general argument used to derive time-dependent transport equations for a time-reversal-invariant fermionic many-body Hamiltonian \( H \) of the form

\[
H = H_0 + V + W(t) .
\]

Here \( H_0 \) is a single-particle Hamiltonian containing the kinetic energy and a stationary central potential, \( V \) is a two-body interaction, and \( W(t) \) is the time-dependent external dipole interaction that drives the system. For nuclei, the central potential is the shell-model potential. For atoms in a trap, it is the external potential defining the trap plus possibly a mean-field contribution due to the atom-atom interaction. In nuclei, the two-body interaction \( V \) is the residual interaction of the shell model. For atoms in a trap, it is the external potential defining the trap, and of the system to equilibrate. The arguments in favor of a theoretical description in terms of transport equations are the same as in the nuclear case. Even for a small number of particles in the trap, the level density grows with excitation energy so strongly that there is no alternative to that approach. To the best of our knowledge, equilibration processes as investigated theoretically in this paper have not yet been studied experimentally for atoms in a trap. The system we study is close to equilibrium and, thus, differs from the ones studied in Refs. \[12–14\].

**A. Statistical Assumptions**

The eigenstates of \( H_0 \) are Slater determinants. For simplicity we assume that the ground-state Slater determinant \( S_0 \) corresponds to a closed shell. Then \( S_0 \) has spin zero, positive parity, and defines the Fermi energy. We define classes of Slater determinants of excited states \((\alpha, \mu)\). The class label \( \alpha \) stands for spin, parity, particle-hole number (defined with respect to the Fermi energy), and excitation energy while \( \mu \) is a running index for the states within the class.

Our basic statistical assumption is formulated in terms of the time-independent part \( H_0 + V \) of \( H \). We choose a representation where \( H_0 + V \) is diagonalized within each class, with eigenvalues \( E_{\alpha \mu} \) and eigenfunctions \( |\alpha \mu\rangle \). If the number of Slater determinants in each class is sufficiently large, and if the mixing of determinants in that class due to \( V \) is sufficiently thorough, it is justified to assume that within that class, \( H_0 + V \) acts like a random Hamiltonian, a member of the Gaussian Orthogonal Ensemble (GOE) of random matrices. Then for each class \( \alpha \) the projections of the eigenfunctions \( |\alpha \mu\rangle \) onto some fixed state are Gaussian-distributed random variables, and the
eigenvalues $E_{\alpha\mu}$ obey Wigner-Dyson statistics. That is our basic statistical assumption. It plays the role of the heat bath for open systems.

Our assumption neglects the existence of specific, non-statistical modes of excitation that occur both in nuclei and for atoms in a trap. In nuclei, these are, for instance, collective modes associated with quadrupole deformations of the ground state $|0\rangle$. In traps, the Higgs mode is an example. However, such modes are expected to be washed out at the higher excitation energies relevant for the present paper and are, therefore, disregarded.

The most thorough numerical test of our statistical assumption that we are aware of was performed in Ref. [17]. Spectra and eigenfunctions of nuclei in the middle of the sd-shell were calculated in the framework of the nuclear shell model with a residual two-body interaction. Typical dimensions of the Hamiltonian matrices were of the order of $10^3$. In the centres of the spectra so obtained, level spacing distribution and eigenfunction statistics agreed well with GOE predictions. We assume that the same statements hold for the eigenvalues $E_{\alpha\mu}$ and eigenfunctions $|\alpha\mu\rangle$ within each class $\alpha$.

In the context of our time-dependent transport equation, we distinguish equilibration and relaxation. Our statistical assumption implies that within each class, equilibration is instantaneous or, physically more accurately, that the equilibration time within each class is short in comparison with the other characteristic time scales of the system. These are the time scale for on-shell relaxation of occupation probabilities in different classes, and the time scale for dipole excitation. Both time scales enter the transport equation(s). Obviously, in every application of the formalism the classes $\alpha$ must be chosen judiciously so as to approximately satisfy that assumption. In nuclear physics empirical evidence supports our scheme. Precompound reactions initiated by light projectiles (protons or alpha particles) are described successfully as a sequence of nucleon-nucleon collisions, each collision populating the class of next-higher-particle-hole number $18$.

States in different classes $\alpha \neq \beta$ are connected by the nondiagonal elements $\langle \beta\nu|V|\alpha\mu\rangle$ of $H$. Because of time-reversal invariance these can be chosen real and symmetric. Obviously, the matrix elements of $V$ vanish unless the classes $\alpha$ and $\beta$ carry the same conserved quantum numbers. Since $V$ is a two-body interaction, the elements $\langle \beta\nu|V|\alpha\mu\rangle$ also vanish if $\alpha$ and $\beta$ differ in particle-hole number by more than one unit. Because of our statistical assumption, the non-vanishing non-diagonal matrix elements $\langle \beta\nu|V|\alpha\mu\rangle$ are zero-centered Gaussian-distributed random variables with second moments given by

$$
\left(\langle \beta\nu|V|\alpha\mu\rangle\langle \beta'\nu'|V|\alpha'\mu'\rangle\right) = \left(\delta_{\alpha\alpha'}\delta_{\mu\mu'}\delta_{\beta\beta'}\delta_{\nu\nu'}
+\delta_{\alpha\beta}\delta_{\mu\nu'}\delta_{\beta\alpha'}\delta_{\nu\mu'}\right)V^2_{\beta\alpha}.
$$

Here and in what follows, big angular brackets denote the average over the Gaussian distribution of matrix elements. The right-hand side of Eq. (2) defines the second moment $V^2_{\alpha\beta} = \langle \beta\nu|V^2|\alpha\mu\rangle$. The second moment is equal to the mean square matrix element of $V$ connecting states in classes $\alpha$ and $\beta$ and, therefore, measures the strength of the coupling between the two classes. The matrix elements $\langle \alpha\mu|W(t)|\beta\nu\rangle$ of the dipole interaction vanish unless the classes $\alpha$ and $\beta$ differ in parity. These elements, too, are zero-centered Gaussian-distributed random variables. They are addressed in Section V below.

**B. Markov Approximation**

We introduce the Markov approximation for the time-independent part $H_0 + V$ of the Hamiltonian $H$. To that end we calculate the time dependence of the mean total occupation probability $P_\alpha(t)$ of the states in class $\alpha$. The mean value is taken over the Gaussian distribution of the coupling matrix elements $V_{\alpha\beta}$, such that the equilibration time within each class is short in comparison with the other characteristic time scales of the system.

The solution $\Psi(t)$ of the time-dependent Schrödinger equation

$$
i\hbar \frac{\partial}{\partial t} \Psi(t) = (H_0 + V)\Psi(t)
$$

conserves probability, $(d/dt)|\Psi(t)|^2 = 0$, because $H_0 + V$ is Hermitean. In view of the statistical assumptions Eq. (2), the elements of the Hamiltonian $H_0 + V$ form a random-matrix ensemble, and $\Psi(t)$ is a random variable. For every realization of $H_0 + V$ the solution $\Psi(t)$ conserves probability. The same is then true of the average probability,

$$
\frac{d}{dt} \langle |\Psi(t)|^2 \rangle = 0.
$$

We use the interaction representation and expand $\Psi(t)$ in the basis of states $|\alpha\mu\rangle$,

$$
|\Psi(t)\rangle = \sum_{\alpha\mu} c_{\alpha\mu}(t) \exp\{-iE_{\alpha\mu}t/\hbar\}|\alpha\mu\rangle.
$$

We recall that the states $|\alpha\mu\rangle$ are eigenstates of the projection of $(H_0 + V)$ onto the space spanned by the states in class $\alpha$. The time dependence of $c_{\alpha\mu}(t)$ is due to elements of $V$ in Eq. (3) that connect states in classes $\alpha \neq \beta$. The average total occupation probability $P_\alpha(t)$ of states in group $\alpha$ at time $t$ is defined as

$$
P_\alpha(t) = \sum_{\mu} \langle |c_{\alpha\mu}(t)|^2 \rangle.
$$

From Eq. (4) we have $\langle \Psi(t)|\Psi(t)\rangle = \sum_{\alpha\mu} |c_{\alpha\mu}(t)|^2$, and Eq. (4) combined with the definition Eq. (6) yields

$$
\frac{d}{dt} \sum_{\alpha} P_\alpha(t) = 0.
$$
Total probability (i.e., the sum of the average total occupation probabilities of the states \((\alpha, \mu)\) is conserved.

In the interaction representation, the time-dependent Schrödinger equation reads

\[
\imath \hbar \dot{c}_{\alpha \mu}(t) = \sum_{\beta \nu} \hat{V}_{\alpha \mu; \beta \nu}(t)c_{\beta \nu}(t).
\]

(8)

The elements of the Hermitian matrix \(\hat{V}\) are given by

\[
\hat{V}_{\alpha \mu; \beta \nu}(t) = \langle \alpha \mu | \hat{V} | \beta \nu \rangle \exp\{\imath(E_{\alpha \mu} - E_{\beta \nu})t/\hbar\}.
\]

(9)

Here \(\alpha \neq \beta\). The Gaussian distribution of the matrix elements of \(\hat{V}\) implies that the matrix elements \(\hat{V}_{\alpha \mu; \beta \nu}\) are likewise zero-centered Gaussian random variables with second moment (see Eq. 2)

\[
\langle \hat{V}_{\alpha \mu; \beta \nu}(t_1)\hat{V}_{\alpha' \mu'; \beta' \nu'}(t_2) \rangle = V_{\alpha \beta}^2
\]

\[
\times \exp\{\imath(E_{\alpha \mu} - E_{\beta \nu})t_1\} \exp\{\imath(E_{\alpha' \mu'} - E_{\beta' \nu'})t_2\}
\]

\[
\times \left( \delta_{\alpha \alpha'} \delta_{\mu \mu'} \delta_{\beta \beta'} \delta_{\nu \nu'} + \delta_{\alpha \beta} \delta_{\mu \mu'} \delta_{\beta \alpha'} \delta_{\nu \nu'} \right).
\]

(10)

If at time \(t = 0\) the nucleus is in state \(|\alpha_0 \mu_0\rangle\), Eq. 8 is supplemented by the initial condition

\[
c_{\alpha \mu}(0) = \delta_{\alpha_0 \alpha}\delta_{\mu_0 \mu}.
\]

(11)

Expanding the solution of Eq. 8 in a formal series in powers of \(\hat{V}\) and using Eq. 11 yields

\[
c_{\alpha \mu}(t) = \sum_{n=0}^{\infty} T \int_0^t \prod_{k=1}^n \frac{dt_k}{\hbar} \left( \prod_{l=1}^n \hat{V}(t_l) \right)_{\alpha \mu; \alpha_0 \mu_0}.
\]

(12)

The symbol \(T\) denotes time ordering: The integration variables obey \(t \geq t_1 \geq t_2 \geq \ldots \geq t_n\). The product in big round brackets is understood as the matrix product. For \(n = 2\), for instance, the round bracket reads \(\sum_{\beta \nu} V_{\alpha \mu; \beta \nu}(t_1) V_{\beta \nu; \alpha_0 \mu_0}(t_2)\). The transport equations are differential equations in time for the mean occupation probabilities \(P_{\alpha}(t)\), subject to the initial condition 11.

They are obtained by using the expansion 12 for \(c_{\alpha \mu}\) and the corresponding expansion for \(c^*_{\alpha \mu}\), carrying out the average over the Gaussian distribution of the matrix elements \(\hat{V}\), and by taking the time derivative of the result. Because of the Gaussian distribution, the average is obtained by summing over all ways of averaging pairs of matrix elements of \(\hat{V}\) (“contracting” pairs). Thus, only terms containing even powers of \(\hat{V}\) contribute to the average. The two members of a contracted pair may both belong to \(c_{\alpha \mu}\), may both belong to \(c^*_{\alpha \mu}\), or one member may belong to \(c_{\alpha \mu}\) and the other to \(c^*_{\alpha \mu}\). The number of possible contraction patterns increases dramatically with the number \(n\) of factors \(\hat{V}\) in the series 12. In pracxis the average can be carried out only if some approximation is made.

Our central assumption is that for every class \(\alpha\), the average level density \(\rho_{\alpha}\) is very smooth. More precisely: The energy interval over which the average level density \(\rho_{\alpha}\) of any of the classes \(\alpha\) changes significantly, is large compared to \(\hbar\) times the inverse characteristic time \(t\) for either relaxation or for dipole absorption. That means

\[
\frac{\hbar^n}{t^n} \frac{\partial^n \rho_{\alpha}}{\partial E^n} \ll \rho_{\alpha}
\]

(13)

for \(n = 1, 2, \ldots\). The leading-order contraction patterns are obtained by neglecting all energy derivatives of \(\rho_{\alpha}\) for all classes \(\alpha\). That neglect leads to transport equations that relate the time derivatives of the occupation probabilities \(P_{\alpha}(t)\) with the values of the occupation probabilities \(P_{\beta}(t)\) at the same time \(t\) and, thus, do not carry any memory effects. That is why we refer to that neglect as to the Markov approximation.

IV. TRANSPORT EQUATION FOR INTERNAL RELAXATION

Using conditions 13, i.e., the Markov approximation, we now derive the transport equation for internal relaxation, i.e., for a system without external driving force, and we discuss the result.

A. Derivation

From Eq. 13 we have

\[
\dot{P}_{\alpha} = \sum_{\mu} \left\langle c^*_{\alpha \mu} \dot{c}_{\alpha \mu} \right\rangle + c.c.
\]

(14)

The dots indicate time derivatives. For \(\dot{c}\) we use Eq. 12 and obtain

\[
\dot{c}_{\alpha \mu} = \frac{1}{\hbar} \sum_{\beta \nu} \dot{V}_{\alpha \mu; \beta \nu}(t) \sum_{n=0}^{\infty} T \int_0^t \prod_{k=1}^n \frac{dt_k}{\hbar} \left( \prod_{l=1}^n \hat{V}(t_l) \right)_{\beta \nu; \alpha_0 \mu_0}.
\]

(15)

For \(c^*_{\alpha \mu}\) we take the complex conjugate of the expansion 12 and use \(\dot{V}^\dagger = \dot{V}\) or, equivalently, \(\dot{V} = \dot{V}^T\). The resulting expression has the form of Eq. 12 except that the sequence of factors in the big round brackets and of the indices on these brackets is reversed, and that \(i \rightarrow -i\). Thus, the last factor on the right of these brackets is \(V(t_1)_{\beta \nu; \alpha \mu}\). We use that form of \(c^*_{\alpha \mu}\) in the first term on the right-hand side of Eq. 14. To calculate the average (angular brackets in Eq. 14), we start with the first factor \(\hat{V}(t)\) in the expansion 12. According to the rules established in Section 11B that factor must be contracted with some other factor \(\hat{V}\) in the expansion of either \(\dot{c}\) or of \(c^*_{\alpha \mu}\). In the appendix we show that the terms of leading order in the sense of the inequalities 13 are obtained by contracting that first factor \(\hat{V}(t)\) either with the factor \(\hat{V}\) immediately to its right in the expansion of \(\dot{c}\), or with the factor \(\hat{V}\) immediately to its left in
the expansion of \( c_{\alpha \mu} \). Using Eqs. (19) we calculate both contributions.

Contraction of the factor \( \tilde{V}(t) \) with the factor immediately to its right gives for the first term on the right-hand side of Eq. (14)

\[
\frac{1}{(\hbar)^2} \sum_{\beta} V_{\alpha \beta} \sum_{\mu \nu} \int_0^t dt_1 \exp \{ i(E_{\alpha \mu} - E_{\beta \nu})(t - t_1) \} \times \left\langle c_{\alpha \mu}(t_1) c_{\alpha \mu}(t_1) \right\rangle .
\]

Here, \( V_{\alpha \beta} \) is defined in Eq. (22). We expand \( c_{\alpha \mu}(t_1) \) in a Taylor series in \( t_1 \). We define \( \Delta_{\nu} = -i(E_{\alpha \mu} - E_{\beta \nu})/\hbar \).

The integral over the term of order \( t_1^k \) is

\[
\sum_{\nu} \int_0^t dt_1 t_1^k \exp \{ \Delta_{\nu} t_1 \} = \sum_{\nu} \frac{t_1^k}{(\Delta_{\nu})^k} \int_0^t dt_1 \exp \{ \Delta_{\nu} t_1 \} = \sum_{\nu} \frac{t_1^k}{(\Delta_{\nu})^k} \left[ \frac{1}{\Delta_{\nu}} \exp \{ \Delta_{\nu} t \} - 1 \right] = \sum_{\nu} \frac{t_1^k}{\Delta_{\nu}} \exp \{ \Delta_{\nu} t \} \left[ 1 - \frac{k}{\Delta_{\nu}} + \ldots \right] .
\]

The dots indicate terms that contain factors \( (\Delta_{\nu} t)^{-l} \) with \( l \geq 2 \). As shown below, criterion (13) implies that all terms in the big straight brackets in the last line containing factors \( t^{-l} \Delta_{\nu}^{-l} \) with \( l \geq 2 \) are negligible. Thus, the big straight brackets in the last line are replaced by unity. Inserting the result into expression (16) we see that the exponential factors cancel. We use

\[
\sum_{\nu} \frac{1}{\Delta_{\nu}} = \sum_{\nu} \frac{i \hbar}{E_{\alpha \mu} - E_{\beta \nu}} \approx \pi \hbar \rho_{\beta}(E_{\alpha \mu}).
\]

We have replaced the energy denominator by a delta function (and thereby neglected the principal-value contribution). Expression (15) links only classes at the same energy. That holds for all terms generated in the perturbation expansion. As a consequence, the resulting transport equation relates only the mean occupation probabilities \( P_{E_{\alpha \mu}}(t) \) of states in class \( \alpha \) at energy \( E \) with each other, with \( E = E_{\alpha \mu} \) defined by the initial condition (1).

Resuming the Taylor expansion of \( c_{\alpha \mu}(t_1) \) gives for expression (16)

\[
-\frac{1}{2 \hbar} \Gamma_{E_{\alpha \mu}} P_{E_{\alpha \mu}}(t) .
\]

Here \( \Gamma_{E_{\alpha \mu}} \) is the “spreading width” for states in class \( \alpha \) at energy \( E \) defined by

\[
\Gamma_{E_{\alpha \mu}} = 2\pi \sum_{\beta} V_{\alpha \beta}^2 \rho_{\beta}(E) .
\]

with \( \rho_{\beta}(E) \) the average level density of states in class \( \beta \) at energy \( E \). Combining the result (19) with the corresponding contribution from the last term in Eq. (14) gives \(- (\Gamma_{E_{\alpha \mu}}/\hbar) P_{E_{\alpha \mu}}(t) \). That is the “loss term” which describes loss of occupation probability in class \( \alpha \) due to scattering into classes \( \beta \neq \alpha \).

To show that terms involving \( \sum_{\nu} t^{-l} \Delta_{\nu}^{-l} \) with \( l \geq 1 \) in Eq. (17) are negligible, we compare the terms with \( l = 0 \) and \( l = 1 \). According to Eq. (18) the term with \( l = 0 \) is given by \( \pi \hbar \rho_{\beta} \). For the term with \( l = 1 \) we have

\[
\frac{1}{t} \left\langle \sum_{\nu} \frac{1}{\Delta_{\nu}} \right\rangle = \frac{1}{t} \left\langle \sum_{\nu} \frac{i \hbar^2}{(E_{\alpha \mu} - E_{\beta \nu})^2} \right\rangle \approx -i \frac{\pi \hbar^2}{t} \left\langle \left( \frac{d \rho_{\beta}(E)}{dE} \right) |_{E_{\alpha \mu}} \right\rangle .
\]

The angular bracket in the last line denotes the average over \( E_{\alpha \mu} \). Criterion (13) implies that the term (21) is negligible compared to the term with \( l = 0 \). Terms of higher order \( l \) yield higher-order derivatives of \( \rho_{\beta} \) and are negligible, too.

Contraction of the factor \( \tilde{V}(t) \) with the factor immediately to its left gives for the first term on the right-hand side of Eq. (14)

\[
\frac{1}{\hbar^2} \sum_{\beta} V_{\alpha \beta}^2 \sum_{\mu \nu} \int_0^t dt_1 \exp \{ i(E_{\alpha \mu} - E_{\beta \nu})(t - t_1) \} \times \left\langle c_{\beta \nu}(t_1) c_{\beta \nu}(t_1) \right\rangle .
\]

Proceeding as in Eq. (17) we obtain

\[
\frac{1}{2 \hbar} \rho_{\alpha}(E) \sum_{\beta} V_{\alpha \beta}^2 P_{E_{\beta}}(t) .
\]

Combined with the corresponding contribution from the last term in Eq. (14) that gives \( (2\pi/\hbar) \rho_{\alpha} \sum_{\beta} V_{\alpha \beta}^2 P_{E_{\beta}}(t) \). That is the “gain term” which describes gain of occupation probability in class \( \alpha \) due to scattering out of classes \( \beta \neq \alpha \).

The sum of gain and loss terms gives the Markovian transport equation

\[
P_{E_{\alpha \mu}} = \sum_{\beta} \frac{2\pi}{\hbar} \rho_{\alpha}(E) V_{\alpha \beta}^2 P_{E_{\beta}} + \sum_{\beta} \left\langle \Gamma_{E_{\alpha \mu}} P_{E_{\alpha \mu}} \right\rangle .
\]

We have used \( V_{\alpha \beta}^2 = V_{\beta \alpha}^2 \). Eq. (24) describes the time evolution of the mean occupation probabilities \( P_{E_{\alpha \mu}}(t) \) of states in class \( \alpha \) at excitation energy \( E = E_{\alpha \mu} \) defined by the initial condition (1). It is easily checked that total probability is conserved, i.e., that Eq. (17) holds. Equilibrium, characterized by \( P_{\alpha} = 0 \) for all \( \alpha \), is attained when \( P_{\alpha} = C \rho_{\alpha} \) for all classes \( \alpha \), with \( C \) a normalization constant independent of \( \alpha \). On physical grounds we expect the system to attain equilibrium for large times. Whether and how quickly that actually happens depends upon the couplings between classes of states and is not investigated here.
B. Discussion

Eq. (24) is Markovian, i.e., the time evolution of the occupation probabilities $P_{E_\alpha}(t)$ is independent of the previous history of the system and depends only on the values of the occupation probabilities $P_{E_\alpha}(t)$ at time $t$. That is a consequence of criterion (13). Moreover, the transport equation (24) is not only Markovian but also completely on-shell. That follows from Eq. (18).

In the probability transport equation (24), the gain (loss) of occupation probability in class $\alpha$ due to transport into (out of) that class from (into) other classes is given by rates (changes of occupation probability per unit time). These rates are

$$R_{\alpha\rightarrow\beta} = (2\pi/h)V_{\alpha\beta}^2 \rho_\beta,$$

$$R_{\beta\rightarrow\alpha} = (2\pi/h)V_{\beta\alpha}^2 \rho_\alpha. \quad (25)$$

Here $R_{\alpha\rightarrow\beta}$ is the rate for decay of the states in group $\alpha$ into states in group $\beta$, and conversely for $R_{\beta\rightarrow\alpha}$. The two rates are related by detailed balance, $\rho_\alpha R_{\alpha\rightarrow\beta} = \rho_\beta R_{\beta\rightarrow\alpha}$. The expressions (25) for the rates have the form of Fermi’s Golden Rule, suggesting that they can be calculated in lowest order of perturbation theory (even though the transport equation (24) has been derived to all orders). We now show that this is indeed the case.

In lowest order of time-dependent perturbation theory, the average time-dependent probability $P_{\alpha\mu;\beta}(t)$ for decay of a given state $(\alpha, \mu)$ into any one of the states $(\beta, \nu)$ is given by

$$P_{\alpha\mu;\beta}(t) = \frac{1}{\mathcal{H}} \int_0^t dt_1 \int_0^t dt_2$$

$$\times \sum_{\nu} \left\langle \tilde{V}_{\beta\nu;\alpha\mu}(t_1) \tilde{V}^*_{\beta\nu;\alpha\mu}(t_2) \right\rangle$$

$$= V_{\beta\alpha}^2 \sum_{\nu} 4 \left\langle \sin^2\left|\frac{E_{\beta\nu} - E_{\alpha\mu}}{2\hbar}\right| \left(\frac{\sqrt{\rho_\beta}}{\sqrt{\rho_\alpha}}\right)^2 \right\rangle$$

$$= V_{\beta\alpha}^2 \int dE_{\beta\nu} \rho_\beta$$

$$\times 4 \left\langle \sin^2\left|\frac{E_{\beta\nu} - E_{\alpha\mu}}{2\hbar}\right| \left(\frac{\sqrt{\rho_\beta}}{\sqrt{\rho_\alpha}}\right)^2 \right\rangle. \quad (26)$$

In the last line we have used the continuum approximation and have replaced $\sum_{\nu} \rightarrow \int dE_{\beta\nu} \rho_\beta$.

Two scenarios exist for evaluating the last line of Eq. (26). (i) For sufficiently large values of $t$, the integrand differs essentially from zero only in a narrow energy interval centered at $E_{\alpha\mu}$. In that interval, $\rho_\beta$ is approximately constant. The remaining integral can be done and yields $2\pi t/\hbar$. Thus,

$$\lim_{t\rightarrow\infty} \frac{1}{t} P_{\alpha\mu;\beta}(t) = \frac{2\pi}{\hbar} V_{\alpha\beta}^2 \rho_\beta. \quad (27)$$

The right-hand side is equal to $R_{\alpha\rightarrow\beta}$ as given in Eq. (25). However, expression (27) holds only for $t \rightarrow \infty$. (ii) For values of $t$ relevant for the transport equation (24), condition (13) states that the level density $\rho_\alpha$ can be considered to be independent of energy and can, thus, be taken out from under the integration. That gives

$$\frac{1}{t} P_{\alpha\mu;\beta}(t) = \frac{2\pi}{\hbar} V_{\alpha\beta}^2 \rho_\beta, \quad (28)$$

in full accord with Eq. (25). Thus, while the standard calculation of the rate (Eq. (24)) applies for very large times only, expression (25) holds for values of $t$ that are relevant for the transport equation. Needless to say, expression (25) does not hold for very small times (where it is not actually used either). Put differently, expression (25) would apply for all values of $t$ only if $\rho_\beta$ were simply constant. That is impossible. The actual spectrum of the states $|\beta\nu\rangle$ has a finite range $\Delta E$. Condition (13) amounts to the requirement that $\Delta E \gg \hbar/t$ for the times characteristic of the transport equation, and that $\rho_\beta$ be very smooth within the energy interval $\Delta E$.

The characteristic times of the transport equation (24) are given by the rates (25). Condition (13) can, therefore, be formulated simply by saying that for all classes $\alpha$, the spreading width $\Gamma^\alpha_{\beta\nu}$ must be small compared to the energy interval $E_\alpha$ over which the average level density $\rho_\alpha$ changes significantly.

In summary, the combination of our statistical assumptions (4) and of the conditions (13) implies the Markovian, on-shell transport equation (24). It is legitimate to calculate the rates $R_{\alpha\rightarrow\beta}$ in that equation perturbatively and for $t \rightarrow \infty$, even though the rate equation (24) is not perturbative, and the rates are actually used for finite times.

Obviously the transport equation (24) holds separately for classes of states that carry different conserved quantum numbers.

V. TRANSPORT EQUATION FOR DRIVEN SYSTEMS

In addition to the residual interaction $V$ we now account for the dipole interaction $W(t)$ in Eq. (1). The time dependence of $W(t)$ has two sources. First, $W(t)$ sets in at time $t = 0$ and terminates at time $T$. Second, $W(t)$ oscillates harmonically with frequency $\omega_0$. We illustrate these statements for the examples in Section II.

For the laser-nucleus interaction, $T$ is the duration time of the laser pulse, and $\hbar \omega_0$ equals the mean photon energy in the laser pulse. We confine ourselves to the essentials and suppress all details due to the interaction with the electromagnetic field. These are given in Ref. [5]. We also suppress the (small) fluctuations of the actual photon frequencies in the laser pulse around the mean value $\omega_0$. In the interaction representation, the time dependence of the dipole matrix element is given by Eq. (16) of Ref. [5],

$$\theta(t) \Theta(T - t) \langle \alpha \mu | W | \beta \nu \rangle \exp\{i(E_{\alpha \mu} \pm \hbar \omega_0 - E_{\beta \nu})t/\hbar\}. \quad (29)$$
Here \( \Theta(t) \) and \( \Theta(T - t) \) are Heaviside functions describing beginning and end in time of the laser-nucleus interaction, and \( W \) is independent of time. The term \( \pm \hbar \omega_0 \) in the exponent (absent in Eq. (11)) corresponds, respectively, to dipole absorption (plus sign) or to induced dipole emission (minus sign) in the transition \( (\alpha \mu) \rightarrow (\beta \nu) \).

For atoms, we assume that the two-dimensional trap has the shape of a harmonic oscillator, with confining potential proportional to \( \sum_k (x_k^2 + y_k^2) \). Here \( (x_k, y_k) \) with \( k = 1, \ldots, N \) are the Cartesian coordinates of the \( N \) atoms. During the time \( T \) the confining potential is subject to externally imposed dipole oscillations, \( \sum_k (x_k^2 + y_k^2) \rightarrow \sum_k [(x_k^2 + y_k^2) + \xi(x_k - y_k) \sin(\omega_0 t)] \) with strength \( \xi \). In the interaction representation, the time dependence of the dipole matrix element \( W(t) \) also has the form of Eq. (29).

In both cases, the presence of an external driving force manifests itself in the time dependence of the matrix elements 29, which differs from that of the matrix elements of \( V \) in Eq. (11). The increment \( \hbar \omega_0 \) is positive or negative depending on whether energy is added to or taken from the system in the transition \( \alpha \rightarrow \beta \), and conversely for \( \beta \rightarrow \alpha \). The time-independent part of the operator \( W \) with matrix elements \( \langle \alpha \mu | W | \beta \nu \rangle \) represents the parity-changing dipole interaction, a sum of single-particle operators. The arguments of Section III show that the nonvanishing matrix elements of \( W \) are zero-centered Gaussian random variables with second moments given by

\[
\langle \langle \beta \nu | W | \alpha \mu \rangle \langle \beta' \nu' | W | \alpha' \mu' \rangle \rangle = \left( \delta_{\alpha \alpha'} \delta_{\mu \mu'} \delta_{\beta \beta'} \delta_{\nu \nu'} + \delta_{\alpha \beta'} \delta_{\mu \nu'} \delta_{\beta \alpha'} \delta_{\nu \mu'} \right) W_{\beta \alpha}^2 .
\]

(30)

Again, we have \( W_{\beta \alpha}^2 = W_{\alpha \beta}^2 \).

In deriving the transport equation we follow the steps taken in Sections IIIA and in the Appendix. The elements of \( V \) and of \( W(t) \) are uncorrelated zero-centered Gaussian random variables. The contraction rules apply separately for the elements of \( V \) and of \( W \). The time dependence of \( W(t) \) causes a change of Eq. (15) which now reads

\[
\left\langle \sum_{\nu} \frac{i \hbar}{E_{\alpha \mu} \pm \hbar \omega_0 - E_{\beta \nu}} \right\rangle \approx \pi \hbar \rho_\beta(E_{\alpha \mu} \pm \hbar \omega_0) .
\]

(31)

That shows that the second moment \( W_{\alpha \beta}^2 \) connects classes of states that differ in energy by \( \pm \hbar \omega_0 \). In averaging over the Gaussian-distributed matrix elements of \( W \), we again use conditions 13. The resulting contraction rules are the same as for the matrix elements of \( V \).

To derive the transport equations, we start from Eq. (14). The perturbation series for \( \varepsilon_\alpha(t) \) has the form of Eq. (15), with the replacement \( \tilde{V}(t) \rightarrow \tilde{V}(t) + W(t) \) in every term of the series, and correspondingly for \( \varepsilon^* \). The first factor in the expansion of \( \varepsilon_{\alpha \mu}(t) \) may either be a matrix element of \( V \) or a matrix element of \( W \). In the first case, conditions 13 imply that a non-vanishing result is obtained only if either the second factor in \( \varepsilon \) or the first factor in \( c^* \) is also a matrix element of \( V \). Proceeding as in Section IVA we arrive at the right-hand side of Eq. (24). In the second case, a nonvanishing result is obtained only if either the second factor in \( \varepsilon \) or the first factor in \( c^* \) is also a matrix element of \( V \). Proceeding as in Section IVA but now with regard to the matrix elements of \( W \), and taking account of Eq. (31), we find that the resulting contribution to the transport equation connects states that differ in energy by \( \pm \hbar \omega_0 \). As in Eq. (24) we account in the resulting transport equation explicitly for the energy \( E \), denoting the mean occupation probability at energy \( E \) of the states in class \( \alpha \) at time \( t \) by \( P_{E,\alpha}(t) \). The transport equation is

\[
\dot{P}_{E,\alpha} = \sum_{\beta} \frac{2 \pi}{\hbar} \rho_{E,\alpha} V_{E,\alpha,E,\beta}^2 P_{E,\beta} - P_{E,\alpha} \sum_{\beta} \frac{2 \pi}{\hbar} V_{E,\alpha,E,\beta}^2 P_{E,\alpha}^2,
\]

\[+ \Theta(t) (\Theta(T - t) - \sum_{\beta} \sum_{j=\pm1} \frac{2 \pi}{\hbar} \rho_{E,\alpha} W_{E,\alpha,E,\beta}^2 P_{E,\alpha,E,\beta} \times P_{(E+j\hbar \omega_0),\beta}) \]

\[- P_{E,\alpha} \sum_{\beta} \sum_{j=\pm1} \frac{2 \pi}{\hbar} \rho_{E,\alpha,E,\beta} W_{E,\alpha,E,\beta} \times P_{E,\alpha,E,\beta} \text{.}
\]

(32)

The terms in the first line correspond to Eq. (24). The terms in the second and third line feed the occupation probability in class \( (E, \alpha) \) either because of dipole excitation of states in class \( (E - \hbar \omega_0, \beta) \) or because of induced dipole emission of states in class \( (E + \hbar \omega_0, \beta) \), and conversely for the loss terms in the fourth line. Again, the rates are related by detailed balance, and Eq. (32) conserves occupation probability.

The rates for internal equilibration and for driven excitation must be determined individually for each system. Two examples show how to proceed. In Ref. [4] the imaginary part of the optical model for elastic scattering of nucleons by nuclei was used to determine the strength of the two-body interaction. In combination with level densities obtained from the nuclear shell model, that yielded the rates for internal equilibration. Similarly, in Ref. [7] the rate for laser-induced dipole excitation was calculated for a laser pulse carrying \( N \) photons with mean energy \( \hbar \omega_0 \) and duration time \( \tau \) using nuclear level densities and the dipole operator. It is important to realize that quite generally, rates may be calculated perturbatively using Fermi’s Golden Rule even though Eq. (32) is nonperturbative. That was demonstrated in Section IVA and in Ref. [7].

In fermionic many-body systems, the average level density rises steeply with excitation energy [10]. The steep rise, valid for an infinitely deep single-particle potential, is modified when the finite depth of the potential is taken into account. For the nuclear case that is demonstrated in Refs. [20, 21]. With increasing energy, the level density rises up to a maximum value \( E_0 \) of the energy whereupon it decreases. As long as the level density increases, detailed balance implies that dipole absorption is stronger...
than stimulated dipole emission, and the transport equa-
tion (32) drives the system to higher excitation energy. At \( E = E_0 \), however, the rates for induced dipole absorption and for stimulated dipole emission are equal while for \( E > E_0 \) stimulated emission prevails. Thus, \( E_0 \) marks the maximum energy the system can absorb. The occupation probability of the states is driven towards a saturation value at energy \( E_0 \) where it becomes stationary. That is illustrated in Refs. 3, 4 where variants of Eqs. (22) have been applied to the laser-nucleus interaction. Thermodynamically, \( E_0 \) corresponds to infinite temperature.

Actually, dipole absorption is limited not by thermody-
namic saturation but by evaporation. The system spills particles (nuclei or atoms) and, thereby, loses energy. The process is caused by the finite depth of the single-particle potential. Evaporation sets in at energies much lower than \( E_0 \). For nuclei, evaporation of \( s \)-wave neutrons dominates. It is described by the Weisskopf esti-
mate \[22\]. The estimate is based on statistical arguments involving average level densities. To account for evapo-
ration in the present framework, Eqs. (32) are, in effect, replaced by a chain of coupled equations, each describ-
ing the system after the loss of \( j = 0, 1, 2, \ldots \) particles. Each such equation has the form of Eqs. (22) but carries additional terms involving the Weisskopf estimate that account for the feeding (loss) of probability due to particle evaporation from the previous (from the present) member of the chain. Such equations can also be derived in the present framework but that is not shown here. Equations of that type have been used, for instance, in Refs. 3, 4. For atoms the trap potential has finite depth \( \alpha \) can be large.

In Eqs. (32) we have not displayed explicitly quantum numbers like parity, spin, or total angular momentum. The dipole interaction changes parity, and it changes spin or angular momentum by one unit. Thus, Eqs. (22) actually couple all classes with these quantum numbers. The number of coupled equations is large, especially when particle evaporation is also taken into account. For that reason, it may be advisable to use transport equations for occupation probabilities averaged over some or all these quantum numbers as done in Refs. 3, 4.

VI. NUCLEAR DIPOLE ABSORPTION

In applications, the formalism developed in previous sections may require some modification. We demonstrate that for the case of laser-induced nuclear dipole absorption. We consider two cases that differ in the intensity of the laser pulse or, equivalently, in the rate of dipole excitation. If that rate is small in comparison with the rate for on-shell relaxation, each photon absorption process is followed by complete internal relaxation. That defines the “quasiadiabatic” regime investigated in Refs. 2, 3. The rate for dipole absorption used in these papers was later shown \[7\] to follow from the the Brink-Axel hypo-
thesis \[5, 6\]. If, on the other hand, the rates for dipole absorption and for internal relaxation are approximately equal, both processes occur simultaneously. That defines the “sudden” regime studied in Ref. 4. Then the Brink-Axel hypothesis must be modified. We show how these alternatives fit into our general framework.

A. Quasiadiabatic Regime

Here the rate for photon absorption is small compared to the rate for equilibration. After every absorption of a photon, the nucleus equilibrates. It is not necessary to consider classes of states carrying different particle-hole numbers. The class label \( \alpha \) stands for spin \( J \) and parity \( \pi \). The states in class \((J, \pi)\) are eigenstates \( |\mu\rangle_{J\pi} \) of the nuclear Hamiltonian \( H_N \). The level density is \( \rho_{J\pi}(E) \).

In the transport equations (32), the terms on the right-hand side of the first line are absent. The role of the operator \( W(t) \) in Eq. (1) is taken by the dipole approximation to the full photon-nucleus interaction Hamiltonian \( H(t) \), the product of the nuclear current operator and the quantized radiation field.

For a laser pulse carrying \( N \) photons with mean energy \( \hbar \omega_0 \), the rate \( R \) for photon absorption from an arbitrary initial nuclear state \( |iJ_i\rangle \) with energy \( E_i \) and spin \( J_i \) has been calculated perturbatively in Ref. 7 with the help of the Brink-Axel hypothesis. As stated in Section II A the hypothesis says that dipole absorption by the state \( |iJ_i\rangle \) populates the normalized dipole modes \( |d(i)J_f\rangle \) pertaining to that state. These carry spins \( J_f = J_i, J_i \pm 1 \) and, assuming degeneracy, have energy \( E_d = \langle d(i)|H_N|d(i)\rangle \). The rate \( R \) is

\[
R = N \frac{c e^2}{9\pi \hbar c (E_i + \hbar \omega_0 - E_d)^2 + (1/4)\Gamma^2} \times 4\pi \alpha \left(\frac{\hbar \omega_0}{\hbar c}\right)^3 \sum_{J_f} |\langle iJ_i|Y_1|d(i)J_f\rangle|^2 . \tag{33}
\]

Here \( \alpha \ll 1 \) is the aperture of the laser pulse, \( \sigma \approx 10 \) keV is the spread in energy of the laser pulse around its mean value \( \hbar \omega_0 \), the sum over \( J_f \) extends over all spin values that can be reached via dipole absorption from the initial state \( |i\rangle \) with spin \( J_i \), and \( |\langle iJ_i|Y_1|d(i)J_f\rangle|^2 \) is the square of the reduced dipole matrix element. The operator \( rY_1 \) comprises a summation over neutrons and protons with their effective charges, written in units of \( e \). The Lorentzian in Eq. (33) stems from that fact that the dipole mode is spread over the eigenstates \( |\mu\rangle \) of \( H_N \) with a Lorentzian distribution centered at \( E_d \). That follows from averaging over a random-matrix model for \( H_N \), see, for instance, Ref. 22. The spreading width is

\[
\Gamma^\downarrow = 2\pi \left| \langle \mu|H_N|i(d)\rangle \right|^2 \rho(E_d) . \tag{34}
\]

Here \( \rho(E_d) \) is the nuclear level density at energy \( E_d \). We have suppressed the dependence on initial and final spins.
That is in line with the use made of Eqs. (33) and (34) in Refs. [7]. There the square of the reduced dipole matrix element in Eq. (35) is estimated via the dipole sum rule. Then the rate $R$ in Eq. (35) becomes independent of the initial state $|i\rangle$ and depends only on the difference $E_d - E_i$. It is assumed that within some domain of excitation energies and nuclear mass numbers, that difference is approximately constant, i.e., independent of the initial nuclear state $|i\rangle$. Likewise it is assumed that the spreading width $\Gamma^\downarrow$ is independent of excitation energy and mass number. These assumptions are in line with the assumed universality of the Brink-Axel hypothesis. As a result the rate $R$ for dipole excitation from any initial nuclear state is universal. The rates for stimulated absorption follow from detailed balance. In Refs. [2, 3] to calculate laser-induced multi-photon absorption in nuclei in the quasiadiabatic regime.

The dipole strength is spread over an energy interval of Lorentzian form with width $\Gamma^\downarrow$. In a time-dependent picture, spreading happens during a time of order $\hbar/\Gamma^\downarrow$. We refer to that process as to equilibration because it happens within a single class of states defined by spin and parity. In the quasiadiabatic regime, the time scale $R^{-1}$ for dipole absorption must be larger than that time, $R^{-1} \geq 2\pi\hbar/\Gamma^\downarrow$. Introducing the dipole width $\Gamma_{dip} = \hbar R$ we write that condition in intuitively appealing form as $\Gamma^\downarrow \geq 2\pi\Gamma_{dip}$. That is consistent with both, the general discussion of the derivation and applicability of rates in Section IV B and with a similar, more restricted derivation of expression (35) for $R$ in Ref. [7]. In both cases it is shown that Eq. (35) actually applies for times $t \geq 2\pi\hbar/\Gamma^\downarrow$.

To see that the contraction rules and the ensuing Markov approximation also apply in the present case we note that the role played by the level density in the rate expressions (29) is, in the case of Eq. (33), taken by the Lorentzian. Condition (34) for the validity of the Markov approximation, summarily written as $\hbar/(t\Delta E) \ll 1$ where $\Delta E$ is the energy interval over which any level density changes significantly, changes to $\hbar/(t\Gamma^\downarrow) \ll 1$. Replacing $\hbar/t \rightarrow \Gamma_{dip}$ we obtain $2\pi\Gamma_{dip} < \Gamma^\downarrow$. That agrees with the condition derived in the previous paragraph. It implies that the contraction rules do apply and that the Markov form of the transport equation approximately holds in the quasiadiabatic regime. The conditions for validity of the Markov approximation and of the adiabatic regime are seen to coincide. That is true even though $\Gamma^\downarrow$ is, as a rule, significantly smaller than $\Delta E$, and the condition $\hbar/(t\Gamma^\downarrow) \ll 1$ is more stringent.

The definition $\Gamma^\downarrow \geq 2\pi\Gamma_{dip}$ of the adiabatic regime must obviously be understood as an asymptotic condition. The more closely $\Gamma_{dip}$ approaches $\Gamma^\downarrow$ from below, the bigger are the expected deviations from the Markov approximation. The ensuing limitation of the quasiadiabatic approach is circumvented by going to the sudden regime in Section V B where the Markov approximation is reestablished albeit on a different time scale. The calculations in Refs. [2, 3] have used the transport equation for the quasiadiabatic regime but have been done, for instance, for $\Gamma^\downarrow = 5$ MeV, $\Gamma_{dip} = 5$ MeV, violating the condition $\Gamma^\downarrow \geq 2\pi\Gamma_{dip}$. The comparison with results [4] for the sudden regime shows that deviations from the Markov approximation set in rather slowly.

The Brink-Axel hypothesis is well established for excitation energies in the MeV range [24]. It is not clear, however, to what extent it actually holds for very highly excited initial states. It is conceivable that with increasing excitation energy, the hypothesis gradually loses validity. Then the spread $\Delta E$ of the dipole strength would increase. At large excitation energies the GDR is primarily observed experimentally via gamma decay of compound nuclei formed in heavy-ion collisions. It is found that the GDR is substantially broadened or disappears altogether [25, 26]. That is ascribed either to the large angular-momentum values involved in a heavy-ion induced reaction, to neutron evaporation, or to a significant intrinsic dynamical broadening of the GDR. The first cause is irrelevant here because laser-induced nuclear dipole excitation does not populate states with large spin values. Neutron evaporation is accounted for explicitly in the coupled transport equations of Ref. [3, 4]. We, therefore, focus attention on the possibility that the GDR is significantly broadened dynamically.

To account for that case we calculate the dipole absorption rate under a weaker and more general assumption than used in the Brink-Axel hypothesis. As in Section V we assume that the reduced nuclear dipole matrix elements connecting any initial state $|iJ_i\rangle$ with any final state $|fJ_f\rangle$ are zero-centered Gaussian-distributed random variables, and that the dipole strength of any initial state $|iJ_i\rangle$ is on average distributed uniformly over the final states $|fJ_f\rangle$ in an energy interval $\Delta E$ that is considerably larger than the spreading width $\Gamma^\downarrow$, characteristic of the ground-state regime. Then equilibration, governed by the time scale $\hbar/\Delta E$, is much more rapid than in the case of the Lorentzian in Eq. (33), and the conditions for the validity of the Markov approximation are fulfilled more readily. It is straightforward to repeat the calculation in Section VI of Ref. [7] for that case. The result for the rate is

$$R = N \frac{e^2}{9\pi} \frac{\epsilon^2}{\hbar c} 4\pi \alpha \left( \frac{\hbar \omega}{\hbar c} \right)^2 \sigma \times 2\pi\rho(E_i + \hbar \omega) \sum_{J_f} \langle |iJ_i||rY_1||fJ_f\rangle^2 . \quad (35)$$

Here $\rho(E_i + \hbar \omega)$ is the density of states at energy $E_i + \hbar \omega$, assumed to be the same for all three spin values $J_f$. The last line in Eq. (35) has the standard form of a rate. We compare the rates (33) and (35). The sum over $J_f$ in expression (33) exhausts the dipole sum rule. To
estimate the reduced matrix elements in Eq. (35), we observe that the sum \( \sum_{f} |\langle iJ_i | r J_f | fJ_f \rangle|^2 \) exhausts the dipole sum rule if it extends over states \( f \) within the energy interval \( \Delta E \). Thus \( \rho(E_i + \hbar \omega) \sum_{f} |\langle iJ_i | r J_f | fJ_f \rangle|^2 \) is approximately equal to \( \sum_{f} |\langle iJ_i | r J_1 | fJ_f \rangle|^2 \Delta E \), and we obtain

\[
\frac{\dot{R}}{R} \approx \frac{2\pi \Gamma}{\Delta E} . \tag{36}
\]

Significant deviations from the Brink-Axel hypothesis occur if that ratio is of order \( 10^{-1} \) or less. The photon absorption probability is reduced by that factor. A possible experimental signal for that to happen is a deceleration of the laser-induced photon absorption process with increasing nuclear excitation energy.

### B. Sudden Regime

In the sudden regime the dipole rate is at least as large as or even bigger than the rate for nuclear equilibration. The statistical approach of Section III A is based on the assumption that there exists a set of quickly equilibrating subsystems. The time scale for internal equilibration must be short in comparison with both, the dipole rate and the rate for nuclear relaxation.

To identify these subsystems we recall that in the ground-state domain, the dynamical description of the GDR involves a two-step process. First, dipole absorption populates a set of particle-hole states. These are not completely degenerate and, thus, contribute to the spreading of the GDR. That one-body effect is referred to as Landau damping \([28]\). The particle-hole states are not eigenstates of the nuclear Hamiltonian and mix with mp-mh states with \( m > 1 \). That mixing amounts to relaxation and gives rise to the total spreading of the GDR.

In line with that picture, the subsystems in Ref. \([4]\) are taken as classes of particle-hole states at fixed total energy. It is assumed that within each subsystem, equilibration is much more rapid than the mixing of different subsystems leading to relaxation. Such an assumption is similarly used in the theory of precompound reactions \([18]\). Here classes of states populated in the reaction are also classified according to the number of particles and holes. For instance, the collision of an incident proton with a nucleon in the target nucleus creates a two-particle one-hole state, the next collision leads to a three-particle two-hole state, etc. It is assumed that after each collision the \( (m + 1)p-mh \) states equilibrate rapidly, so that level densities may be used for describing the reaction. As a result, the sequence of collisions is described in terms of rates, each rate connecting a class of \( (m + 1)p-mh \) states with the next one. The success of that model \([18]\) supports the assumption of rapid equilibration. A similar physical picture underlies the approach developed in Ref. \([18]\).

Implementation of these ideas leads straightforwardly to the transport equations \([22]\). The class label \( \alpha \) is identified with spin, parity, and particle-hole number combined. Equilibration within each class is instantaneous. The remaining part of the two-body residual nuclear interaction connects only classes with identical quantum numbers, and it changes particle-hole number only by one unit. That yields the terms in the first line of Eq. (32). Emission and absorption of photons is described by the dipole approximation to the full photon-nucleus interaction Hamiltonian \( H(t) \) mentioned above. Dipole absorption either leaves particle-hole number unchanged or increases it by unity. Calculation of the rates for dipole absorption yields expressions like in Eq. (35) but specified for fixed particle-hole number, see Ref. \([4]\). That yields the remaining two lines in Eq. (32). The rates are normalized to the dipole sum rule.

### VII. SUMMARY AND DISCUSSION

Transport equations for autonomous driven fermionic quantum systems are derived with the help of statistical assumptions, and of the Markov approximation. The statistical assumptions hold if the system consists of subsystems within which equilibration is sufficiently fast. The Markov approximation holds if the level density in each subsystem is sufficiently smooth in energy. From a formal point of view, Eqs. (2) and (3) constitute sufficient conditions for the validity of Eqs. (24) and (32). The transport equations describe both, internal equilibration among subsystems at equal energy and the transport of the system to higher energy caused by the driving force. The result puts the use of transport equations for the laser-nucleus interaction on a firm theoretical basis. We are hopeful that such equations prove useful also for atoms in driven traps.

We end with some general remarks. The result (24) for internal relaxation has the form of the Pauli master equation. That equation goes back to the beginning of quantum mechanics. Nonetheless, derivation and validity of Eq. (24) remain a frequently discussed and timely topic, especially for autonomous systems. The derivation of Eq. (24) usually employs coupling to a reservoir. In that framework, relevant time scales are the decoherence time \( \tau_{\text{dec}} \) and the dissipation time \( \tau_{\text{diss}} \), see, for instance, Ref. \([29]\). Here \( \tau_{\text{dec}} \) is the time scale for exponential decay of the off-diagonal matrix elements of the density matrix, and \( \tau_{\text{diss}} \) is the time it takes the diagonal elements of the density matrix (i.e., our occupation probabilities \( P_{E\alpha}(t) \)) to attain equilibrium. Although we deal with an autonomous system, analogous time scales appear in our work. In Section XV nondiagonal elements of the density matrix have been shown to disappear upon averaging over the random-matrix ensemble. Such averaging is justified for times larger than the internal equilibration time within each class \( \alpha \) of states. Thus, our internal equilibration time bears a close analogy to \( \tau_{\text{dec}} \).

Our relaxation time is identical with \( \tau_{\text{diss}} \) because it is determined by the same equation.
Our derivation of the transport equations \([24]\) and \([30]\) uses the decomposition \([11]\) of the Hamiltonian \(H\) into a shell-model part \(H_0\) and the remainder. On that basis, we define classes of states \(\alpha\) and assume that within each class, equilibration is fast. Actually, the transport equations have a wider scope. Any system for which the states in Hilbert space can be grouped into classes such that the states within each class interact more strongly with each other than with the rest, so strongly, in fact, that states within each class are thoroughly mixed prior to overall relaxation, obeys the assumptions used in our derivation. The time evolution of the system is then described by equations like \([24]\) and \([30]\). The label \(\alpha\) refers to the classes of strongly interacting states.

Our Eq. \([24]\) holds for autonomous many-body quantum systems (systems not coupled to a reservoir). For such systems, relaxation towards statistical equilibrium has been experimentally observed in several areas of physics and has been interpreted with the help of a master equation \([11]\) in Eq. \([24]\). Precompound reactions provide an early example in nuclear physics. These show relaxation towards the equilibrated compound nucleus, accompanied by emission of particles \([12]\) \([30]\). A recent example for quantum thermalization in atomic physics is Ref. \([31]\). In such cases, the derivation of a master equation without recourse to a reservoir and, more generally, the understanding of quantum thermalization in isolated systems, pose a problem. The derivation of the quantum Boltzmann equation using field theory shows one solution to the problem. Here we have shown that insights gained in random-matrix theory also offer a solution. Arguing that equilibration within each class \(\alpha\) of states results in strongly mixed states as described by random-matrix theory, we have obtained Eq. \([24]\) by averaging over the resulting ensemble of random matrices. We believe that our approach clearly identifies the physical assumptions needed for Eq. \([24]\) and for the more general Eq. \([32]\) to hold.

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APPENDIX

We show that contraction of the first factor \(\tilde{V}_{\alpha \mu, \beta \nu}(t)\) in the expansion \([15]\) of \(c_{\alpha \mu}\) with any other factor \(\tilde{V}\) in that expansion except the second, yields terms that are negligible by the condition \([13]\). With \(c_{\alpha \mu}\) written in the manner described below Eq. \([15]\), the same arguments can be used to show that contraction of the first factor \(\tilde{V}_{\alpha \mu, \beta \nu}(t)\) in \(c_{\alpha \mu}\) with any factor \(\tilde{V}\) in \(c_{\beta \nu}\) except the rightmost one, are likewise negligible. That is not done here, and we confine ourselves to the expansion \([15]\) of \(c_{\alpha \mu}\).

In carrying out the time integrations we make use of Eq. \([17]\) where we replace the last straight bracket by unity. For every function \(F(t)\) that possesses a Taylor expansion in \(t\), we then have

\[
\sum_\nu \left[ \int_0^t dt_1 \exp\{\Delta_\nu t_1\} \right] F(t_1) \approx \sum_\nu \frac{1}{\Delta_\nu} \exp\{\Delta_\nu t\} F(t) .
\]

(37)

We use Eq. \([37]\) also in cases where the summation over \(\nu\) is performed somewhere later in the calculation, perhaps in conjunction with additional factors \(1/\Delta_\nu\).

We contract the first factor \(\tilde{V}(t)_{\alpha \mu, \beta \nu}\) in the expansion \([16]\) with the factor \(\tilde{V}_{\alpha \mu, \beta \nu}(t_n)\) for \(n \geq 2\), prior to any contraction affecting the other factors \(\tilde{V}\) in that expansion. Among the two options of Eq. \([10]\) we first consider the case \(\beta_n = \alpha, \nu_n = \mu\). That gives

\[
\frac{1}{(i\hbar)^n+1} T \int_0^t \prod_{k=1}^{n-1} dt_k \sum_{\beta \nu} V_{\alpha \beta}^2 \sum_{\gamma \rho} \tilde{V}(t_1)_{\beta \nu, \gamma \rho} \times \exp\{i(E_{\alpha \mu} - E_{\beta \nu})/(t-t_n)/\hbar\} c_{\alpha \mu}(t_n) .
\]

(38)

Here we must have \(n \geq 3\) because \(\tilde{V}\) is nondiagonal in class index. Carrying out the time integration over \(t_n\) we obtain

\[
\sum_\nu \int_0^t dt_1 \frac{1}{1} \sum_{\beta \nu} V_{\alpha \beta}^2 \sum_{\gamma \rho} \tilde{V}(t_1)_{\beta \nu, \gamma \rho} \times \exp\{i(E_{\alpha \mu} - E_{\beta \nu})/(t-t_n)/\hbar\} \times \exp\{i(E_{\gamma \rho} - E_{\alpha \mu})/(t-t_n-1)/\hbar\}
\]

\[\times \frac{1}{E_{\alpha \mu} - E_{\beta \nu}} c_{\alpha \mu}(t_n-1) .
\]

(39)

We contract \(\tilde{V}(t_1)\) with \(\tilde{V}(t_n-1)\). We show presently that contracting either of these factors with any other factor \(\tilde{V}\) in \(c\) or in \(c^*\) leads to higher derivatives than the first of the average level density \(\rho_\beta\). We obtain

\[
\frac{1}{(i\hbar)^n+1} T \int_0^t \prod_{k=1}^{n-1} dt_k \sum_{\beta \nu} V_{\alpha \beta}^2 \sum_{\gamma \rho} \tilde{V}(t_1)_{\beta \nu, \gamma \rho} \times \exp\{i(E_{\alpha \mu} - E_{\beta \nu})/(t-t_n)/\hbar\} \exp\{i(E_{\gamma \rho} - E_{\alpha \mu})/(t-t_n-1)/\hbar\}
\]

\[\times \frac{1}{E_{\alpha \mu} - E_{\beta \nu}} c_{\alpha \mu}(t_n-1) .
\]

(40)

We use Eq. \([9]\) and successively perform the integrations over \(t_n-1, t_n-2, \ldots, t_2\), using Eq. \([37]\). That gives

\[
\frac{1}{(i\hbar)^2} \int_0^t dt_1 \sum_{\gamma \rho} V_{\alpha \beta}^2 V_{\beta \gamma}^2 \sum_{\gamma \rho} \Delta \times \exp\{i(E_{\alpha \mu} - E_{\beta \nu})/(t-t_1)/\hbar\} \exp\{i(E_{\gamma \rho} - E_{\alpha \mu})/t_1/\hbar\}
\]

\[\times \frac{1}{E_{\alpha \mu} - E_{\beta \nu}} c_{\alpha \mu}(t_1) .
\]

(41)

The factor \(\Delta\) is the product of sums over inverse energy differences. These arise in the time integrations. None of these carries \(E_{\beta \nu}\). The big round bracket is the product of the time-independent matrices \(V\) introduced in Eq. \([2]\). We integrate over \(t_1\) and use Eq. \([37]\). In Eq. \([11]\) that yields the factor \(\sum_\nu (E_{\alpha \mu} - E_{\beta \nu})^{-2} \propto (d/dE)\rho_\beta\) which
is negligible. Contraction of $\tilde{V}(t_1)$ and of $\tilde{V}(t_{n-1})$ in Eq. (38) not with each other but with other factors $\tilde{V}$ in $\dot{c}$ or in $e^*$ causes the occurrence of additional terms $E_{\beta\nu}$ in exponential factors. The associated time integrations produce higher-order derivatives of $\rho_\beta$.

We turn to the second option in the contraction of $\tilde{V}_{\alpha\mu,\beta\nu}(t)$ with $\tilde{V}_{\alpha\mu,\beta\nu}(t_n)$ and put $\alpha_n = \alpha, \mu_n = \mu$. In the expansion of $\dot{c}$ we now carry explicitly all matrix elements $\tilde{V}(t_i)$ up to and including the one with $l = n+1$. That gives

\[
\frac{1}{(\hbar)^{n+2}} T \int_0^t \prod_{k=1}^{n+1} dt_k \sum_{\beta\gamma\rho} V_{\alpha\beta}^2 \left( \prod_{l=1}^{n-1} \tilde{V}(t_l) \right)_{\beta\nu,\alpha\mu} \\
\times \exp\{i(E_{\alpha\mu} - E_{\beta\nu})(t + t_n)/\hbar\} \tilde{V}_{\beta\nu,\gamma\rho}(t_{n+1}) \\
\times c_{\gamma\rho}(t_{n+1}) \, .
\] (42)

We contract $\tilde{V}(t_1)$ with $\tilde{V}(t_{n+1})$. Other contraction patterns generate additional terms $E_{\beta\nu}$ in exponentials and, thus, derivatives of higher order than the first of $\rho_\beta$.

These will not be considered. The result is

\[
\frac{1}{(\hbar)^{n+2}} T \int_0^t \prod_{k=1}^{n+1} dt_k \sum_{\beta\gamma\rho} V_{\alpha\beta}^2 V_{\beta\gamma}^2 \left( \prod_{l=2}^{n-1} \tilde{V}(t_l) \right)_{\gamma\rho,\alpha\mu} \\
\times \exp\{i(E_{\alpha\mu} - E_{\beta\nu})(t + t_n)/\hbar\} \exp\{i(E_{\beta\nu} - E_{\gamma\rho})t_1/\hbar\} \\
\times c_{\gamma\rho}(t_1) \, .
\] (43)

Integration over $t_{n+1}$ gives

\[
\frac{1}{(\hbar)^{n+1}} T \int_0^t \prod_{k=1}^{n+1} dt_k \sum_{\beta\gamma\rho} V_{\alpha\beta}^2 V_{\beta\gamma}^2 \left( \prod_{l=2}^{n-1} \tilde{V}(t_l) \right)_{\gamma\rho,\alpha\mu} \\
\times \exp\{i(E_{\alpha\mu} - E_{\beta\nu})t/\hbar\} \exp\{i(E_{\beta\nu} - E_{\gamma\rho})t_1/\hbar\} \\
\times c_{\gamma\rho}(t_1) \, .
\] (44)

Proceeding as before we see that the integration over $t_1$ eventually yields another factor $1/(E_{\beta\nu} - E_{\gamma\rho})$. Combining the two denominators and summing over $\nu$ we obtain the derivative of the level density $\rho_\beta$ which is negligible.

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