Discrete Space-Time Volume for 3-Dimensional BF Theory and Quantum Gravity

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Abstract

The Turaev-Viro state sum invariant is known to give the transition amplitude for the three dimensional BF theory with cosmological term, and its deformation parameter $\hbar$ is related with the cosmological constant via $\hbar = \sqrt{\Lambda}$. This suggests a way to find the expectation value of the spacetime volume by differentiating the Turaev-Viro amplitude with respect to the cosmological constant. Using this idea, we find an explicit expression for the spacetime volume in BF theory. According to our results, each labelled triangulation carries a volume that depends on the labelling spins. This volume is explicitly discrete. We also show how the Turaev-Viro model can be used to obtain the spacetime volume for (2+1) dimensional quantum gravity.

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The loop approach to quantum gravity [1] leads to a fascinating picture in which the structure of geometry at the Planck scale is fundamentally different from that of the usual Riemannian geometry. The elementary excitations of the quantum geometry which arises are one-dimensional, “loop-like”, rather than particle-like. The spectra of operators corresponding to such geometrical quantities as length, area and volume become quantized [2]. The “size” of the corresponding quanta of geometry is proportional to the Planck length.

However, so far, almost all the progress that has been made within this approach concerns space rather than spacetime aspects of quantum geometry. In this paper we propose a new set of ideas that can be used to study the spacetime quantum geometry. More precisely, using the path integral techniques, we propose a way to define the spacetime volume in the quantum theory. We do this in the context of BF theory and Euclidean quantum gravity in three spacetime dimensions. The simplicity of these (classically) closely related theories allows us to obtain an explicit expression for the spacetime volume.

There exists many ways to quantize 3-dimensional gravity (for a review see [3]). In this paper we concentrate on the approach pioneered by Ponzano and Regge [4]. This approach is intimately related with the loop quantum gravity approach. Indeed, using the ideas described by Baez [5] (see also [6]), one can think of the Ponzano-Regge model as a covariant (spacetime) version of loop quantum gravity in (2+1) dimensions. The Ponzano-Regge approach, as we explain below, is very close in spirit to the usual path integral approach, and, thus, is best suited for a study of spacetime aspects of the theory.

Although the Ponzano-Regge model gives a quantization of BF theory, not gravity, the two are closely related in three dimensions, at least as classical theories. Thus, one can hope to use BF theory to study certain aspects of 3-d quantum gravity. We will first derive an expression for the spacetime volume in BF theory, and comment on how BF theory can be used to obtain the spacetime volume for gravity. Let us describe the Ponzano-Regge model (or, more precisely, its generalization proposed by Turaev and Viro, see e.g. [7]) in more details. Let us consider the theory whose action is given by

$$S_A[A, E] = - \int_M \text{Tr} \left( E \wedge F + \frac{\Lambda}{12} E \wedge E \wedge E \right),$$

where $M$ is assumed to be a three-dimensional orientable manifold. The action is a functional of an SU(2) connection $A$, whose curvature form is denoted by $F$, and a 1-form $E$, which takes values in the Lie algebra of SU(2). Thus, the action (1) is that of BF theory in 3d, with $E$ field playing the role of $B$, and with an additional “cosmological term” added to the usual BF action. The relation to gravity in 3d is as follows. Having the one-form $E$, one can construct from it a real metric of Euclidean signature

$$g_{ab} = -\frac{1}{2} \text{Tr}(E_a E_b).$$

For our conventions on indices and others see the Appendix. Thus, the $E$ field in (1) has the interpretation of the triad field. One of the equations of motion that follows from (1) states that $A$ is the spin connection compatible with the triad $E$. Taking the triad $E$ to be non-degenerate and “right-handed”, i.e., giving a nowhere zero positive volume form $|E|$, and substituting into (1) the spin connection instead of $A$, one gets the Euclidean Einstein-Hilbert action
\[ \frac{1}{2} \int_{\mathcal{M}} d^3x \sqrt{g} ( R - 2\Lambda ). \]  

(3)

We use units in which \( 8\pi G = 1 \). The coefficient in front of (1) is chosen to yield precisely the Einstein-Hilbert action after the elimination of \( A \). Thus, on configurations of \( E \) field that are non-degenerate and right-handed, (1) is equivalent to Einstein-Hilbert action, and \( \Lambda \) in (1) is precisely the cosmological constant. This means that all solutions of Einstein’s equations can be obtained from the solutions of BF theory. However, BF theory has more solutions than gravity. For example, in BF theory one can have configurations of \( E \) field that give negative or zero value of the volume form at some points in spacetime. This does not cause any problems classically, for one can always restrict oneself to the sector where metric is non-degenerate and the volume form is positive. However, as we shall see below, this does cause problems in the quantum theory: the amplitudes of configurations with positive volume form interfere with the amplitudes of negative volume configurations. Thus, as we shall see, the quantum models corresponding to BF theory and gravity are rather different.

Let us first consider the quantization of BF theory. The Turaev-Viro model gives a way to calculate the vacuum-vacuum transition amplitude of this theory, i.e., the path integral

\[ Z(\Lambda, \mathcal{M}) = \int \mathcal{D}E \mathcal{D}A e^{iS[A,E]} . \]  

(4)

Note that we consider the transition amplitude of BF theory, not the partition function, which would be given by (1) without the \( i \) in the exponential. As we shall see below, this is the transition amplitude that is related to the Turaev-Viro state sum invariant. In this paper we consider only vacuum-vacuum amplitudes. Although the Turaev-Viro model can be used to calculate more general amplitudes between non-trivial initial and final states, we will not use this aspect of the model here. We consider the version of the model formulated on a triangulated manifold. Thus, let us fix a triangulation \( \Delta \) of \( \mathcal{M} \). Let us label the edges, for which we will employ the notation \( e \), by irreducible representations of the quantum group \( (SU(2))^q \), where \( q \) is a root of unity

\[ q = e^{\frac{2\pi i}{k}} \equiv e^{i\hbar} . \]  

(5)

Later we will relate the parameter \( \hbar \) with the cosmological constant \( \Lambda \). The irreducible representations of \( (SU(2))^q \) are labelled by half-integers (spins) \( j \) satisfying \( j \leq (k - 2)/2 \). Thus, we associate a spin \( j_e \) to each edge \( e \). The vacuum-vacuum transition amplitude of the theory is then given by the following expression (see, e.g. [7]):

\[ \text{TV}(q, \Delta) = \kappa V \sum_{j_e} \prod_{e} \dim_q(j_e) \prod_t (6j)_q , \]  

(6)

where \( \kappa \) and \( \dim_q(j) \) are defined by (A2),(A3) correspondingly, and \( V \) is the number of vertices in \( \Delta \). The last product in (6) is taken over tetrahedra \( t \) of \( \Delta \), and \( (6j)_q \) is the normalized quantum (6j)-symbol constructed from the 6 spins labelling the edges of \( t \). It turns out that (6) is independent of the triangulation \( \Delta \) and gives a topological invariant of \( \mathcal{M} \): \( \text{TV}(q, \Delta) = \text{TV}(q, \mathcal{M}) \).

The construction that interprets the Turaev-Viro invariant (6) as the vacuum-vacuum transition amplitude of the theory defined by (1) is as follows. It has been proved (see
e.g. \[8\] and the works cited therein) that (6) is equal to the squared absolute value of the Chern-Simons amplitude

\[
TV(q, \mathcal{M}) = |CS(k, \mathcal{M})|^2,
\]

with the level of Chern-Simons theory being equal to \(k\) from (3). It is known, however, that the action (1) can be written as a difference of two copies of Chern-Simons action

\[
S_{CS}(A) = \frac{k}{4\pi} \int_{\mathcal{M}} \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right).
\]

Indeed, note that

\[
S(A + \lambda E) - S(A - \lambda E) = \frac{k\lambda}{\pi} \int_{\mathcal{M}} \text{Tr} \left( E \wedge F + \frac{\lambda^2}{3} E \wedge E \wedge E \right),
\]

where \(\lambda\) is a real parameter. Thus, (8) is equal to (1) if

\[
\lambda = -\sqrt{\Lambda}/2, \quad k = \frac{2\pi}{\sqrt{\Lambda}}, \quad \text{or} \quad \hbar = \sqrt{\Lambda}.
\]

This relates the deformation parameter \(q\) of the Turaev-Viro model to the cosmological constant \(\Lambda\), in the case of positive \(\Lambda\), and proves that the Turaev-Viro amplitude is proportional to the vacuum-vacuum transition amplitude of the theory defined by (1):

\[
TV(q, \mathcal{M}) \sim Z(\Lambda, \mathcal{M}).
\]

Let us now discuss the relation between quantum BF theory and gravity. We note that, although classically BF theory and gravity are equivalent (they lead to the same equations of motion), the transition amplitude of BF theory (4) is not the transition amplitude for gravity. Indeed, in the case of gravity one has to perform the path integral of the exponentiated action only over non-degenerate metric configurations that define a positive spacetime volume:

\[
Z_{gr}(\Lambda, \mathcal{M}) = \int_{\text{Vol}(E) > 0} \mathcal{D}E \mathcal{D}A e^{iS[A,E]}.
\]

However, the path integral in (1) is taken over all configurations of \(E\) field, even those that are degenerate or define a negative volume. To see this, let us note that if the field \(E\) defines an everywhere positive volume, then \(-E\) gives a negative volume, and both are summed over in the path integral (4). Thus, the path integral includes contributions from both configurations of \(E\) field that give an everywhere positive and the ones that given an everywhere negative volume. Moreover, the path integral (1) takes into account also degenerate configurations of \(E\) in which the volume form is negative at some points in spacetime and positive at other. As we further discuss below, these are these degenerate configurations that may cause the quantum BF theory to be drastically different from gravity, if it turns out that they dominate the path integral. To further illustrate the difference between the quantum BF theory and gravity, let us write a heuristic expressions for the path integrals corresponding to the two theories. For the transition amplitude in gravity one can formally write:
\[
\int \mathcal{D}g \prod_{x \in \mathcal{M}} e^{i \mathcal{L}_{gr}(x)},
\]
where the path integral is taken over non-degenerate configurations of metric \(g\) and \(\mathcal{L}_{gr}(x)\) is the Lagrangian of gravity. In the path integral of the quantum BF theory, one takes into account all, even highly degenerate configurations of the \(E\) field. Thus, heuristically, its transition amplitude is given by
\[
\int_{\text{Vol}(E) \geq 0} \mathcal{D}E \mathcal{D}A \prod_{x \in \mathcal{M}} \cos(\mathcal{L}_{BF}(x)) = \int \mathcal{D}g \prod_{x \in \mathcal{M}} \cos(i \mathcal{L}_{gr}(x)).
\]
(14)

Here the integral is taken over configurations of \(E\) field that give a non-negative volume, and the presence of \(\cos\) accounts for the fact that one sums at each point both over positive and negative volumes. The presence of \(\cos\) here is also reminiscent of the fact (see [4]) that the Ponzano-Regge amplitude, which is given by the product of (6j)-symbols (see (22) below), in the limit of large spins \(j_e\) has the asymptotics of the cosine of the Regge calculus version of the Einstein-Hilbert action. Thus, to summarize, the quantum BF theory may well be drastically different from gravity because it takes into account highly degenerate, classically forbidden field configurations. However, the two theories would be related if there exists a phase of the quantum BF theory in which the path integral is dominated by non-degenerate configurations of \(E\) field, that is, configurations that have everywhere positive or negative volume. In this phase one would effectively have to consider only everywhere positive or everywhere negative volume configurations. In the transformation \(E \rightarrow -E\) the BF action (14) changes its sign. Thus, in this phase
\[
Z(\Lambda, \mathcal{M}) = Z_{gr}(\Lambda, \mathcal{M}) + \overline{Z_{gr}(\Lambda, \mathcal{M})},
\]
(15)

where overline denotes complex conjugation. In this phase there exists a relation between the spacetime volume in BF theory and gravity. We shall discuss this after we obtain an expression for the volume in the quantum BF theory.

Let us now study the spacetime volume in BF theory. Note that the expectation value of the volume is given simply by the derivative of the amplitude (4) with respect to \((-i\Lambda)\)
\[
\langle \text{Vol} \rangle = \frac{\int \mathcal{D}E \mathcal{D}A \text{Vol}(\mathcal{M}) e^{iS}}{\int \mathcal{D}E \mathcal{D}A e^{iS}} = i \frac{\partial \ln Z(\Lambda)}{\partial \Lambda},
\]
(16)

Thus, the expectation value of the volume of \(\mathcal{M}\) in BF theory can be obtained by differentiating the Turaev-Viro amplitude with respect to \((-i\Lambda)\). Here we calculate this derivative and find an explicit expression for the spacetime volume. For simplicity, we will study only the volume in the quantum theory with zero cosmological constant. This can be obtained by first differentiating (4) with respect to \(\Lambda\), and then evaluating the result at \(\Lambda = 0\). Thus, interestingly, the deformation parameter of the Turaev-Viro model \(q\) (related with \(\Lambda\) via (10)) serves as the quantity conjugate to the spacetime volume.

Since the Turaev-Viro amplitude is explicitly real, and, to find the volume, we differentiate its logarithm with respect to \(i\Lambda\), the expectation value of the volume is purely imaginary.
This is, at the first sight, surprising, but can be easily understood by taking into account the fact that both positive and negative volume configurations are summed over in (1), and that the action (2) changes its sign when $E \to -E$.

Let us now find this volume. An important subtlety arises, however, if one follows the procedure (16). To find the derivative (16) at $\Lambda = 0$ we have to find the first order term in $\Lambda$ in the expansion of (3). It is not hard to show, however, that (3) has the following asymptotic expansion in $\hbar$

$$
\left(\frac{\hbar^3}{4\pi}\right)^V \text{PR}(\Delta) \left(1 - \hbar^2 i\langle\text{Vol}\rangle\right),
$$

(17)

where PR is the amplitude of the Ponzano-Regge model

$$
\text{PR}(\Delta) = \sum_{j_e} \prod_e \text{dim}(j_e) \prod_t (6j),
$$

(18)

$V$ is the number of vertices in $\Delta$, and $i\langle\text{Vol}\rangle$ is a real quantity independent of $\hbar$. Thus, apparently there is no term proportional to $\hbar^2$ in this expansion. The resolution of this is that the deformation level $k$ of the Turaev-Viro model plays two distinct roles in the theory. First, it serves as a regulator for the Ponzano-Regge model. Indeed, the amplitude (18) diverges. It is only the combination $(\hbar^3 \text{PR})$, defined as the limit of $(\text{TV})$ as $\hbar \to 0$, that is finite. Second, the parameter $k$ (and the related to it $q$) also serves as a deformation parameter of the Ponzano-Regge model. The terms of the order $O(\hbar^2)$ in (17) are the ones that appear as the result of this deformation. To derive an expression for the spacetime volume we must be concerned only with these terms, for they describe the “deformation” of the vacuum-vacuum transition amplitude occurring due to the introduction of the cosmological constant term into the action functional. Thus, the expectation value of the spacetime volume is what is denoted by $\langle\text{Vol}\rangle$ in (17).

There is another, equivalent way to justify the absence of terms proportional to $\hbar^2$ in the decomposition (17). As we have mentioned above, the Turaev-Viro amplitude (3) is proportional to the transition amplitude (1). However, the proportionality coefficient depends on $\hbar$. Indeed, the integration over $(A+\lambda E), (A-\lambda E)$, which is carried out to obtain $|\text{CS}(k, \mathcal{M})|^2$ in (7) and thus the Turaev-Viro amplitude, is different from the integration over $A, E$ one has to perform to obtain (1). The difference in the integration measures is a power of $\hbar$. Thus, the amplitude (4) and the squared absolute value of the amplitude of the Chern-Simons theory are proportional to each other with the coefficient of proportionality being a power of $\hbar$. In the discretized version of the theory, given by the Turaev-Viro model, this power of $\hbar$ is replaced by $\hbar^{3V}$. Thus, this is the (divergent) Ponzano-Regge amplitude (18) that gives the amplitude for BF theory without the cosmological constant. Turaev-Viro amplitude, in the limit of small cosmological constant, differs from the BF amplitude by a power of $\hbar$.

These remarks being made, it is straightforward to write down an expression for the expectation value of the volume:

$$
i\langle\text{Vol}\rangle_\Delta = -\frac{\partial}{\partial \Lambda} \left(\frac{\text{TV}(\Lambda, \Delta)}{\text{PR}(\Delta)(\hbar^3/4\pi)^V}\right)_{\Lambda=0} =
$$
\[
\frac{1}{\text{PR}(\Delta)} \sum_{j_e} i \text{Vol}(\Delta, j) \left( \prod_e \dim(j_e) \prod_t (6j) \right),
\]  

(19)

where the function \( \text{Vol}(\Delta, j) \) of the triangulation \( \Delta \) and the labels \( j = \{j_e\} \) is given by

\[
i \text{Vol}(\Delta, j) = \sum_v \left( -\frac{\partial}{\partial \Lambda} \left( \frac{\kappa}{(\hbar^2/4\pi)} \right) \right)_{\Lambda=0} + \sum_e \left( -\frac{\partial \ln(\dim_q(j_e))}{\partial \Lambda} \right)_{\Lambda=0} + \sum_t \left( -\frac{\partial \ln((6j)_q)}{\partial \Lambda} \right)_{\Lambda=0}. \]

(20)

Here \( v \) stands for vertices of \( \Delta \), \( e \) stands for edges and \( t \) stands for tetrahedra. We intentionally wrote the expectation value of the volume in the form (19) to introduce the volume \( \text{Vol}(\Delta, j) \) of a labelled triangulation, which is our main object of interest. Indeed, (19) has the form

\[
\sum_{j_e} i \text{Vol}(\Delta, j) \frac{\text{Amplitude}(\Delta, j)}{\sum_{j_e} \text{Amplitude}(\Delta, j)},
\]

(21)

where

\[
\text{Amplitude}(\Delta, j) = \prod_e \dim(j_e) \prod_t (6j)
\]

(22)

is the amplitude of Ponzano-Regge model. This shows that \( \text{Vol}(\Delta, j) \) indeed has the interpretation of the volume of a labelled triangulation.

The volume (20) has three types of contributions: (i) from vertices; (ii) from edges; (iii) from tetrahedra. It is not hard to calculate the first two types of them. One finds that each vertex contributes exactly \( 1/12 \), and each edge contributes \( j_e(j_e + 1)/6 \), where \( j_e \) is the spin that labels the edge \( e \). It is much more complicated to find the tetrahedron contribution to the volume, that is, the derivative of \( \ln((6j)_q) \) with respect to \( \Lambda \). Here we simply give the result of this calculation. The details will appear elsewhere [9]. The simplest way to describe the result is graphical. First, let us, for each tetrahedron \( t \), introduce a special graph \( \Gamma \) living on the boundary of \( t \). The boundary of \( t \) is a triangulated 2-manifold with the topology of a sphere, and we define the graph \( \Gamma \) to be the one dual to that triangulation. The graph \( \Gamma \) has four vertices and six edges, and, as a piecewise linear complex, it is a tetrahedron. Let us label the edges of this graph, which are dual to the edges of the original graph, by the same spins as those labelling the edges of \( t \). Let us then construct the spin network corresponding to the labelled \( \Gamma \) (for more details on spin networks, see e.g. [10]). Recall that a spin network corresponding to a labelled graph is a function on a certain number of copies of the gauge group – \( G_E \), where \( E \) is the number of edges in the graph – which is constructed by taking the matrix elements of the group elements in the representations labelling the corresponding edges, and contracting these matrix elements with each other at vertices using intertwiners. In our case of \( \Gamma \) being a tetrahedron, all vertices are trivalent, that is, there are exactly three edges meeting at each vertex. In this case intertwiners are unique (up to an overall multiplicative constant), and given by the usual Clebsch-Gordan coefficients. We choose the normalized intertwiners; then the evaluation of the spin network on all group elements equal
to the unity in SU(2) gives just the normalized classical \((6j)\)-symbol, as the one in \([8]\). This can be represented graphically by:

\[
\left( \begin{array}{c}
\end{array} \right)_0 = (6j).
\]

(23)

Here the picture represents the spin network constructed above, and \((\cdot)_0\) represents its evaluation on the unity group elements. One gets a number that depends on the six spins labelling the edges of the spin network. This number is exactly the normalized classical \((6j)\)-symbol.

Let us now introduce an operation that can be called grasping. In this paper we introduce only certain basic graspings; for more details see e.g. \([1]\). Grasping will always be represented by a dashed line with open ends. The basic grasping is given by a line with two open ends. Each open end of this dashed line can be thought of as representing the Pauli matrices \(\sigma^i, i = 1, 2, 3\) and the line connecting the ends represents the contraction \(\sigma^i \otimes \sigma^i\).

In this paper we use a normalization such that each open end stands for \(\sigma^i / \sqrt{2}\), and each dashed line stands for \(\delta^{ij}\). The basic operation of grasping is that each open end can grasp any of the edges of a spin network state. The result of this grasping is the “insertion” of \(\sigma^i / \sqrt{2}\) into the edge of the spin network, in the same representation as the one labelling the edge that is being grasped. Thus, for example, the basic dashed line with two open ends can grasp with both its ends one and the same edge of a certain spin network. Let \(j\) denote the spin labelling this edge. Then the grasping simply multiplies the spin network by the value of the Casimir in the representation \(j\):

\[
\begin{array}{c}
\end{array} = 2j(j + 1) \begin{array}{c}
\end{array}
\]

(24)

One can also construct a more complicated grasping with three open ends by commuting the two basic 2-open-end graspings

\[
\begin{array}{c}
\end{array} = \begin{array}{c}
\end{array}
\]

(25)

Thus, with our normalizations, the vertex of this triple grasping stands for \(i\sqrt{2}\varepsilon^{ijk}\). This triple grasping is one of our main objects in what follows.

We are now in the position to describe the result of the derivative of \((6j)_q\) symbol with respect to \(\Lambda\). The result can be represented as a sum over graspings of the edges of the spin network constructed above:

\[
\frac{1}{16} \left( \frac{1}{24} \sum_{e,e',e''} \langle e' | \begin{array}{c}
\end{array} | e'' \rangle - \frac{1}{4} \sum_e \langle \begin{array}{c}
\end{array} | \rangle_0 \right) = \left( -\frac{\partial (6j)_q}{\partial \Lambda} \right)_{\Lambda=0}.
\]

(26)
Here the first sum is taken over all the graspings of the triples of distinct edges of the tetrahedron $t$ (there are 20 different graspings of this type, but only 16 of them are non-zero), and the second sum is over all graspings of the edges of $t$ (there are 6 graspings of this type – equal to the number of edges of $t$). The result of this graspings is then evaluated on the unity group elements to get a number. The derivative (26) is a real number depending only on the spins labelling the edges of $t$.

Thus, the final result for the spacetime volume in BF theory is

$$i\text{Vol}(\Delta, j) = \sum_v \frac{1}{12} + \sum_e \frac{j_e(j_e + 1)}{6} +$$

$$\sum_t \frac{1}{16} \frac{1}{24} \left( \langle e' \varepsilon \varepsilon e\rangle - \frac{1}{4} \sum_e \langle \varepsilon \varepsilon e\rangle \right).$$

(27)

Here $\varepsilon$ stands for the classical $(6j)$-symbol.

It is interesting to note that not only tetrahedra $t$ of $\Delta$ contribute to the volume, but also the edges $e$ and the vertices $v$. The contribution from the vertices is somewhat trivial – it is constant for each vertex. Nevertheless, when thinking about the triangulated manifold $\mathcal{M}$ one is forced to assign the spacetime volume to every vertex. The contribution from edges depends on the spins labelling the edges. Again, this implies that each edge of the triangulation $\Delta$ carries an intrinsic volume that depends on its spin. The contribution from tetrahedra is more complicated. It is given by a function that depends on the spins labelling the edges of each tetrahedron. It is interesting that this picture of the spacetime volume being split into contributions from vertices, edges and tetrahedra can be understood in terms of Heegard splitting of $\mathcal{M}$. Recall, that Heegard splitting of a three-dimensional manifold $\mathcal{M}$ decomposes $\mathcal{M}$ into three dimensional manifolds with boundaries. Then the original manifold can be obtained by gluing these manifolds along the boundaries. For the case of a triangulated manifold $\mathcal{M}$, as we have now, the Heegard splitting proceeds as follows. First, one constructs balls centered at the vertices of $\Delta$. Then one connects these balls with cylinders, whose axes of cylindrical symmetry coincide with the edges of $\Delta$. Removing from $\mathcal{M}$ the obtained balls and cylinders, one obtains a three-dimensional manifold with a complicated boundary. One has to further cut this manifold along the faces of $\Delta$. One obtains three types of “building blocks” that are needed to reconstruct the original manifold: (i) balls; (ii) cylinders; (iii) spheres with four discs removed. Each of this manifolds carries a part of the original volume of $\mathcal{M}$. Our result (27) provides one with exactly the same picture: the volume of $\mathcal{M}$ is concentrated in vertices (balls of the Heegard splitting), edges (cylinders), and tetrahedra (4-holed spheres).

The volume of each labelled triangulation is explicitly discrete, and is given by the function (27) of the labelling spins. This gives an important insight as to the nature of quantum spacetime geometry. Indeed, one of the results of the canonical approach to quantum gravity in three spacetime dimensions is that lengths of curves and areas of regions are quantized. Our results also imply that the spacetime volume is quantized.

An important drawback of the present approach is that the spacetime volume we have obtained is that of BF theory, not gravity. In particular, this is the reason why the spacetime
volume turns out to be purely imaginary. Let us now discuss a relation of the obtained spacetime volume of BF theory with the volume in gravity. As we have discussed above, the quantum BF theory may develop a phase in which non-degenerate configurations of $E$ field dominate the path integral. In this phase it is possible to obtain some information about the spacetime volume in gravity from the quantum BF theory. As we described above, in such a phase the BF theory transition amplitude is related to that in gravity according to (15). One can obtain a similar relation between the expectation value of the volume in the two theories. Thus, if BF theory is in the phase in which non-degenerate solutions dominate, one has:

$$\langle \text{Vol} \rangle_{\text{gr}} = \frac{\text{Re} Z_{\text{gr}}(\Lambda, \mathcal{M})}{i \text{Im} Z_{\text{gr}}(\Lambda, \mathcal{M})} \langle \text{Vol} \rangle.$$  \hfill (28)

To get this relation we have assumed that the expectation value $\langle \text{Vol} \rangle_{\text{gr}}$ of the volume in gravity is real. Thus, even in this phase, to relate the volume in gravity with the one in BF theory one has to know the imaginary part of the gravity amplitude. Unfortunately, it is not possible to extract this information from the Turaev-Viro model, which knows only about the real part of $Z_{\text{gr}}$. Thus, one cannot extract the expectation value of spacetime volume of 3d gravity from BF theory. However, it turns out to be possible to extract the expectation value of the volume squared. Indeed, using the same arguments as in (15), and assuming that the BF theory is in the non-degenerate phase, one can write

$$\langle \text{Vol}^2 \rangle_{\text{gr}} Z(\Lambda, \mathcal{M}) = \langle \text{Vol}^2 \rangle_{\text{gr}} Z_{\text{gr}}(\Lambda, \mathcal{M}) + \langle \text{Vol}^2 \rangle_{\text{gr}} Z_{\text{gr}}(\Lambda, \mathcal{M}).$$  \hfill (29)

Thus, assuming that $\langle \text{Vol}^2 \rangle_{\text{gr}}$ is a real quantity, we obtain

$$\langle \text{Vol}^2 \rangle_{\text{gr}} = \langle \text{Vol}^2 \rangle.$$  \hfill (30)

In other words, when non-degenerate fields $E$ dominate the path integral (4), the expectation value of the volume squared in Turaev-Viro model is the same as in gravity. Thus, it can be obtained by the methods of this paper by differentiating the Turaev-Viro amplitude with respect to $\Lambda$:

$$\langle \text{Vol}^2 \rangle_\Delta = - \frac{\partial^2}{\partial \Lambda^2} \left( \frac{\text{TV}(\Delta)}{\text{PR}(\Delta)(\hbar^3/4\pi)^V} \right)_{\Lambda=0}.$$  \hfill (31)

This also allows one to obtain an expression for the squared volume of a labelled triangulation. The results of this paper indicate that this squared volume will also be discrete, and depend only on the spins labelling the triangulation. Thus, if there indeed exist a phase of the quantum BF theory in which non-degenerate $E$ fields dominate, then one can gain some control over the spacetime volume in (2+1) quantum gravity using the results from BF theory. If, on the other hand, the path integral of BF theory is always dominated by highly degenerate solutions, as may well be the case, then the two theories have very little to do with each other, and the spacetime volume of BF theory is not the same as the volume in gravity. At the present stage of our understanding of quantum gravity it is hard to tell which of this two possibilities is realized.
Let us conclude by pointing out a possible application of the results of this paper. One of the main conceptual problems of quantum gravity is the absence of time in the corresponding quantum theory. In fact, this comes about not just for gravity, but for any generally covariant theory, as is, for example, BF theory discussed above. In the canonical theory this manifests itself in the fact that the Hamiltonian becomes a constraint; in the path integral approach a manifestation of this is that the path integral gives not the transition amplitude between states at different times, but the matrix element of the projection operator on the subspace of solutions of the Hamiltonian constraint. In the path integral approach this arises because one is summing over all spacetime geometries that can be put between the initial and final hypersurfaces. This is the basic reason why one loses track of the time: one is summing amplitudes of all spacetime geometries, even those which give a different proper time separation between the initial and final hypersurfaces. However, a possible alternative to this arises if one constructs the path integral as a sum over labelled triangulations, as we did above for the case of BF theory in three dimensions. In this case, as we found, each labelled triangulation can be assigned the spacetime volume carried by it. Thus, instead of summing over all triangulations, one can consider only the sum over triangulations having a fixed spacetime volume contained between the initial and final hypersurfaces. The sum over amplitudes of such triangulations would depend on the fixed spacetime volume. One can interpret this spacetime volume as a measure of the time elapsed between the initial and final hypersurfaces. Thus, an expression for spacetime volume, as, for example, the one obtained in this paper, allows one to introduce a natural time in the quantum theory.

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APPENDIX A:

All the traces we use in this paper are traces in the fundamental representation. Lower case latin letters stand for spacetime indices: $a, b, ... = 1, 2, 3$. The $E$ field we use is assumed to be anti-hermitian. This explains the minus sign in (2). Note the factor of 1/2 in (2). This is not the standard choice in the gravity literature, but turns out to be very convenient in 3d, for it allows one to get rid of the ugly factors of $\sqrt{2}$ in some formulas. In case $E$ has an interpretation of the triad field, the volume form is given by

$$\frac{1}{12} \varepsilon^{abc} \text{Tr}(E_a E_b E_c).$$  \hfill (A1)

Note that the volume form defined by $E$ can be both positive and negative, and only a configuration of $E$ giving the positive volume is a triad field of gravity.

The quantity $\kappa$ in (3) that, in the limit of $\hbar \to 0$, serves as a regulator of the Ponzano-Regge model, is defined by:

$$\kappa = -\frac{(q^{1/2} - q^{-1/2})^2}{2k}. \hfill (A2)$$
The quantum dimension $\dim_q(j) = [2j + 1]_q$, where $[n]_q$ is the quantum number

$$[n]_q = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}.$$  \hspace{1cm} (A3)
REFERENCES

[1] C. Rovelli, Loop quantum gravity, available as gr-qc/9710008.
[2] C. Rovelli and L. Smolin, Discreteness of area and volume in quantum gravity, Nucl. Phys. B442, 593 (1995); Erratum: Nucl. Phys. B456, 734 (1995).
A. Ashtekar and J. Lewandowski, Quantum theory of geometry I: Area operators, Class. Quant. Grav. 14, A55-A81 (1997).
A. Ashtekar and J. Lewandowski, Quantum theory of geometry II: Volume Operators, Adv. Theo. Math. Phys. 1, 388-429 (1997).
T. Thiemann, A length operator for canonical quantum gravity, gr-qc/9606092.
R. Loll, Spectrum of the volume operator in quantum gravity, Nucl. Phys. B460 143-154 (1996).
[3] S. Carlip, Lectures in (2+1)-dimensional gravity, available as gr-qc/9503024.
[4] G. Ponzano and T. Regge, Semiclassical limits of Racah coefficients, In Spectroscopic and theoretical methods in physics, ed. F. Block, North-Holland Amsterdam, 1968.
[5] J. Baez, Spin foam models, available as gr-qc/9709052.
[6] C. Rovelli, The basis of the Ponzano-Regge-Turaev-Viro-Ooguri quantum gravity model is the loop representation basis, Phys. Rev. D48, 2702-2707 (1993).
[7] N. Reshetikhin and V. G. Turaev, Invariants of three manifolds via link polynomials and quantum groups, Invent. Math. 103, 547-597 (1991).
[8] J. Roberts, Skein theory and Turaev-Viro invariants, Topology 34, 771-787 (1995).
[9] L. Freidel, manuscript in preparation.
[10] J. Baez, Spin networks in non-perturbative quantum gravity, in The Interface of Knots and Physics, ed. Louis Kauffman, A.M.S., Providence, 1996, pp. 167-203.
[11] D. Ban-Natan, On the Vassiliev knot invariants, Topology 34, 423-472 (1995).