SET OF ALL DENSITIES OF EXPONENTIALLY S-NUMBERS

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Abstract. Let $G$ be the set of all finite or infinite increasing sequences of positive integers beginning with 1. For a sequence $S = \{s(n)\}, n \geq 1,$ from $G$ a positive number $N$ is called an exponentially $S$-number ($N \in E(S)$), if all exponents in its prime power factorization are in $S$. The author [2] proved that, for every sequence $S \in G$, the sequence of exponentially $S$-numbers has a density $h = h(E(S)) \in [\frac{6}{\pi^2}, 1]$. In this note we study the set $\{h(E(S))\}$ of all such densities.

1. Introduction

Let $G$ be the set of all finite or infinite increasing sequences of positive integers beginning with 1. For a sequence $S = \{s(n)\}, n \geq 1,$ from $G$, a positive number $N$ is called an exponentially $S$-number ($N \in E(S)$), if all exponents in its prime power factorization are in $S$. The author [2] proved that, for every sequence $S \in G$, the sequence of exponentially $S$-numbers has a density $h = h(E(S)) \in [\frac{6}{\pi^2}, 1]$. More exactly, the following theorem was proved in [2]:

Theorem 1. For every sequence $S \in G$ the sequence of exponentially $S$-numbers has a density $h = h(E(S))$ such that

$$\sum_{i \leq x, \ i \in E(S)} 1 = h(E(S))x + O(\sqrt{x \log x e^{\sqrt{\log \log x}}}),$$

with $c = 4\sqrt{\frac{2.4}{\log 2}} = 7.443083...$ and

$$h(E(S)) = \prod_{p} \left(1 + \sum_{i \geq 2} \frac{u(i) - u(i-1)}{p^i}\right),$$

where $u(n)$ is the characteristic function of sequence $S$: $u(n) = 1$, if $n \in S$ and $u(n) = 0$ otherwise.

Note that Sloane’s Online Encyclopedia of Integer Sequences [3] contains some sequences of exponentially $S$-numbers, $S \in G$. For example, A005117 ($S = \{1\}$), A004709 ($S = \{1, 2\}$), A268335 ($S = A005408$), A138302 ($S = \{2^n\}_{n\geq0}$), A197680 ($S = \{n^2\}_{n\geq1}$), A115063 ($S = \{F_n\}_{n\geq2}$), A209061 ($S = A005117$), etc.

Everywhere below we write $\{h(E(S))\}$, understanding $\{h(E(S))\}_{S \in G}$. In
(Section 6) the author posed the question: is the set \( \{ h(E(S)) \} \) dense in the interval \( [\frac{6}{\pi^2}, 1] \)? Berend [1] gave a negative answer by finding a gap in the set \( \{ h(E(S)) \} \) in the interval

\[
\left( \prod_p (1 - \frac{p - 1}{p^3}), \prod_p (1 - \frac{1}{p^3}) \right) \subset \left[ \frac{6}{\pi^2}, 1 \right].
\]

Berend’s idea consists of the partition of \( G \) into two subsets - of those sequences which contain 2 and those that do not contain 2 - and applying \( G \) Berend’s idea consists of the partition of \( G \) into two subsets - of those sequences which contain 2 and those that do not contain 2 - and applying

Lemma 1. \( G \) is uncountable.

Proof. Trivially \( G \) is equivalent to the set of all subsets of \( \{ 2, 3, 4, \ldots \} \). \( \square \)

Lemma 2. For every two distinct \( A, B \in G \), we have \( h(E(A)) \neq h(E(B)) \).

Proof. Let \( A = \{ a(i) \}_{i \geq 1}, \ B = \{ b(i) \}_{i \geq 1} \). Let \( n \geq 1 \) be maximal index such that \( a(i) = b(i), \ i = 1, \ldots, n \), while \( a(n + 1) \neq b(n + 1) \). Note that, if \( A_n = \{ a(1), \ldots, a(n) \}, \ A^* = \{ a(1), \ldots, a(n), a(n + 1), a(n + 1) + 1, a(n + 1) + 2, \ldots \} \), then

\[
h(E(A_{n+1})) \leq h(E(A)) \leq h(E(A^*))
\]

and analogously for sequence \( B \).

Distinguish four cases:

(i) \( a(n + 1) = a(n) + 1, \ b(n + 1) \geq a(n) + 2 \);

(ii) \( \text{for} \ k \geq 2, \ a(n + 1) \geq a(n) + k, \ b(n + 1) = a(n) + 1 \);

(iii) \( \text{for} \ k \geq 3, \ a(n + 1) = a(n) + k, \ a(n) + 2 \leq b(n + 1) \leq a(n) + k - 1 \);

(iv) \( \text{for} \ k \geq 2, \ a(n + 1) = a(n) + k, \ b(n + 1) \geq a(n) + k + 1 \).

(i) By (2) and (4), we have

\[
h(E(A)) = \prod_p \left( 1 + \sum_{i=2}^{a(n)} \frac{u(i) - u(i - 1)}{p^i} \right),
\]

where \( u(n) \) is the characteristic function of \( A \). Since here \( u(a(n + 1)) - u(a(n + 1) - 1) = 0 \), then in the right hand side we sum up to \( a(n) \). On the other hand,

\[
h(E(B^*)) \leq \prod_p \left( 1 + \sum_{i=2}^{a(n)} \frac{u(i) - u(i - 1)}{p^i} - \frac{1}{p^{a(n)+1}} + \frac{1}{p^{a(n)+2}} \right).
\]

By (5)-(6), \( h(E(B)) < h(E(A)) \).
(ii) Symmetrically to (i), we have

\[ h(E(B)) \geq \prod_p \left( 1 + \sum_{i=2}^{a(n)} \frac{u(i) - u(i-1)}{p^i} \right). \]

On the other hand,

\[ h(E(A^*)) \leq \prod_p \left( 1 + \sum_{i=2}^{a(n)} \frac{u(i) - u(i-1)}{p^i} - \frac{1}{p^{a(n)+1}} + \frac{1}{p^{a(n)+2}} \right). \]

So, \( h(E(A)) < h(E(B)) \).

(iii) Again, by (2) and (4), we have

\[ h(E(B)) \geq \prod_p \left( 1 + \sum_{i=2}^{a(n)} \frac{u(i) - u(i-1)}{p^i} - \frac{1}{p^{a(n)+1}} + \frac{1}{p^{a(n)+k-1}} \right), \]

while

\[ h(E(A^*)) \leq \prod_p \left( 1 + \sum_{i=2}^{a(n)} \frac{u(i) - u(i-1)}{p^i} - \frac{1}{p^{a(n)+1}} + \frac{1}{p^{a(n)+k}} \right). \]

Hence, \( h(E(A)) < h(E(B)) \).

(iv) Symmetrically,

\[ h(E(B^*)) \leq \prod_p \left( 1 + \sum_{i=2}^{a(n)} \frac{u(i) - u(i-1)}{p^i} - \frac{1}{p^{a(n)+1}} + \frac{1}{p^{a(n)+k+1}} \right), \]

while

\[ h(E(A)) \geq \prod_p \left( 1 + \sum_{i=2}^{a(n)} \frac{u(i) - u(i-1)}{p^i} - \frac{1}{p^{a(n)+1}} + \frac{1}{p^{a(n)+k}} - \frac{1}{p^{a(n)+k+1}} \right) \]

and since \( \frac{2}{p^{a(n)+k+1}} \leq \frac{1}{p^{a(n)+k}} \), where the equality holds only in case \( p = 2 \), then \( h(E(A)) > h(E(B)) \). \( \square \)

Lemmas 1 and 2 directly imply

**Theorem 2.** The set \( \{h(E(S))\}_{S \in G} \) is uncountable.

Denote by \( G(F) \) the subset of the finite sequences from \( G \). Since the set of all finite subsets of a countable set is countable, then \( G(F) \) is countable and then the set \( \{h(E(S))\}_{S \in G(F)} \) is also countable.

3. **Perfectness**

**Lemma 3.** Every point of the set \( h(E(S)) \) is an accumulation point.

**Proof.** Distinguish two cases: a) \( S \) is finite set; b) \( S \) is infinite set.
a) Let $S = \{s(1), ..., s(k)\} \in G(F)$. Let $n \geq s(k) + 2$. Denote by $S_n$ the sequence $S_n = \{s(1), ..., s(k), n\}$. Then, by (2),

\[
\prod_p \left( 1 + \sum_{i=2}^{s(k)} \frac{u(i) - u(i-1)}{p^i} - \frac{1}{p^{s(k)+1}} + \frac{1}{p^n} \right) - \prod_p \left( 1 + \sum_{i=2}^{s(k)} \frac{u(i) - u(i-1)}{p^i} - \frac{1}{p^{s(k)+1}} \right).
\]

For the first product $\prod_p (n)$,

\[
\prod_p (n) = \exp \left( \sum_p \log \left( 1 + \sum_{i=2}^{s(k)} \frac{u(i) - u(i-1)}{p^i} - \frac{1}{p^{s(k)+1}} + \frac{1}{p^n} \right) \right),
\]

the series over primes converges uniformly since

\[
\sum_p \sum_{i \geq 2} \frac{|u(i) - u(i-1)|}{p^i} \leq \sum_p \sum_{i \geq 2} \frac{1}{p^i} = \sum_p \frac{1}{(p-1)p}.
\]

Therefore, $\lim_{n \to \infty} (\prod_p (n)) = \prod_p (\lim_{n \to \infty} (\ldots))$ which coincides with the second product. So $\lim_{n \to \infty} h(E(S_n)) = h(E(S))$.

b) Let $S = \{s(1), ..., s(k), \ldots\} \in G$ be infinite sequence. Let $S_n = \{s(1), ..., s(n)\}$ be the $n$-partial sequence of $S$. In the same way, taking into account the uniform convergence of the product for density of $S_n$, we find that $\lim_{n \to \infty} h(E(S_n)) = h(E(S))$. \hfill \Box

**Theorem 3.** The set \(\{h(E(S))\}\) is a perfect set.

A proof we give in Section 5.

4. Gaps

Let us show that, for every finite $S \in G$, with the exception of $S = \{1\}$, there exists an $\varepsilon > 0$ such that the image of $h$ is disjoint from the interval $(h(E(S))) - \varepsilon, h(E(S))$.

We need a lemma.

**Lemma 4.** Let $A, B \in G$ be distinct sequences. Let $s^* = s^*(A, B)$ be the smallest number which is a term of one of them, but not in another. If, say, $s^* \in A$, then $h(E(A)) > h(E(B))$.

**Proof.** In fact, the lemma is a corollary of the proof of Lemma 2. Comparing with the proof of Lemma 2, we have $s^*(A, B) = n + 1$. We see that in all four cases in the proof of Lemma 2, the statement of Lemma 4 is confirmed. \hfill \Box
Lemma 5. \( h \) is the smallest of which is \( h(S) \in S \) that contradicts the condition: 2) if \( i \), \( 1 \leq i \leq k \), is the smallest for which \( S \) misses \( s(i) \), then, by the condition, all terms of \( S \) are more than \( s(i) \). So \( h(S_1, S_2) = s(i) \in S_2 \), if \( i < k \), while, if \( i = k \), since \( S \) differs from \( S_2 \), \( h(S, S_2) = s(k) + j \in S_2 \), where \( j \) is the smallest for which \( s_k + j \) is not in \( S \). Hence, by Lemma 3, \( h(E(S_2)) > h(E(S)) \) and again \( h(E(S)) \) is not in interval (14). \( \square \)

Lemma 5. Every gap in \( \{h(E(S))\} \) has the form described in Proposition 1.

Proof. Indeed, the gap (14) is in a right neighborhood of \( h(E(S_2)) \). Let a sequence \( S \in \mathcal{G} \) do not contain any infinite set of positive integers \( K \). Adding to \( S \) \( k \in K \), which goes to infinity, we obtain set \( S_k \) such that \( h(E(S_k)) > h(E(S)) \) and \( h(E(S_k)) \to h(E(S)) \). So, in a right neighborhood of \( h(E(S)) \) cannot be a gap of \( \{h(E(S))\} \). In opposite case, when \( S \in \mathcal{G} \) does not contain only a finite set of positive integers, in a right neighborhood of \( h(E(S)) \) a gap of \( \{h(E(S))\} \) is possible, but in this case \( S \) has the form of \( S_2 \) in Proposition 1. Also, if \( S \in \mathcal{G} \) is infinite, then in a left neighborhood of \( h(E(S)) \) cannot be a gap of \( \{h(E(S))\} \), since \( h(E(S)) \) is a limiting point of \( \{h(E(S_n))\} \), where \( S_n \) is the \( n \)-partial sequence of \( S \). \( \square \)

It is easy to see that, for distinct sequences \( S_1 \), the gaps (14) are disjoint.

From Propositions 1 and Lemma 5 we have the statement:

Theorem 4. The set \( \{h(E(S))\} \) has countably many gaps.

5. Proof of Theorem 3

Proof. By Lemma 3, the set \( \{h(E(S))\} \) does not contain isolated points. For a set \( A \subseteq \left[ \frac{6}{\pi^2}, 1 \right] \), let \( \overline{A} \) be \( \left[ \frac{6}{\pi^2}, 1 \right] \setminus A \). Let, further, \( \{g\} \) be the set of all gaps of \( \{h(E(S))\} \). Then we have
\[ \{h(E(S))\} = \bigcup g = \bigcap \overline{g}. \]

Since a gap \( g \) is an open interval, then \( \overline{g} \) is a closed set. But arbitrary intersections of closed sets are closed. Thus the set \( \{h(E(S))\} \) is closed without isolated points. So it is a perfect set. \( \square \)

6. Conclusion

Thus, by Theorems 2.4, the set \( \{h(E(S))\} \) is a perfect set with a countable set of gaps which associate with some left-sided neighborhoods of the densities of all exponentially finite \( S \)-sequences, \( S \in G \), except for \( S = \{1\} \). It is natural to conjecture that the sum of lengths of all gaps equals the length of the whole interval \( [\frac{\pi}{2}, 1] \), or, the same, that the set \( \{h(E(S))\} \) has zero measure. This important question we remain open.

Remark 1. Possible to solve this problem could help a remark that the deleting in (2) 0’s (when \( u_i = u_{i-1} \)) we obtain an alternative sequence of \(-1, 1\).

7. Acknowledgement

The author is grateful to Daniel Berend for very useful discussions.

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