Asymptotics of a Brownian ratchet for Protein Translocation

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Abstract

Protein translocation in cells has been modelled by Brownian ratchets. In such models, the protein diffuses through a nanopore. On one side of the pore, ratcheting molecules bind to the protein and hinder it to diffuse out of the pore. We study a Brownian ratchet by means of a reflected Brownian motion \( (X_t)_{t \geq 0} \) with a changing reflection point \( (R_t)_{t \geq 0} \). The rate of change of \( R_t \) is \( \gamma (X_t - R_t) \) and the new reflection boundary is distributed uniformly between \( R_t \) and \( X_t \). The asymptotic speed of the ratchet scales with \( \gamma^{1/3} \) and the asymptotic variance is independent of \( \gamma \).

1 Introduction

Brownian motion is a model of thermal fluctuations of small particles. If a particle moves according to such fluctuations, it is known to undergo undirected movement, i.e. Brownian motion is a martingale. However, the idea to use random fluctuations in order to force a particle in one direction is tempting and led to the paradox of the Brownian ratchet as introduced by Feynman et al. (1963, Chapter 46).

Quantitative models of Brownian ratchets for biological mechanisms were introduced in Simon et al. (1992) and Peskin et al. (1993). While the polymerisation ratchet serves as a model of growth of polymers against a barrier, we focus on the translocation ratchet, a model for protein transport. Here, a polymer moves in and out through a nanopore by thermal fluctuations. On the in-side of the nanopore, molecules may bind to the polymer and bound sites along the polymer are forbidden to move through the nanopore; see Figure 1 for an illustration.

A prominent example of a translocation ratchet was suggested on the basis of empirical data for Prepro-α Factor as protein and BiP as the ratcheting molecule by Matlack et al. (1999): after translation, proteins have to be transported into the lumen of the endoplasmatic reticulum (ER) where they are e.g. cut and folded in order to function properly. In general, such ratcheting molecules can be chaperones which bind to the translocated protein in-side...
Figure 1: (A) Illustration of the translocation ratchet. A protein moves through a nanopore by thermal fluctuations. On the in-side of the pore, molecules may bind to the protein which prevent the protein from moving out again. (B) The state of the protein at time \( t \). The total length of the protein on the in-side is \( X_t \) whereas the last bound molecule is located at \( R_t \).

The ER lumen. The resulting translocation ratchet has been studied and compared to data from [Matlack et al. (1999)] by [Liebermeister et al. (2001)] and [Elston (2002)]. In these models, a discrete set of sites in-side the ER lumen is either bound or unbound with BiP. The protein diffuses to either side of the nanopore with equal chances. Each unbound site of the protein may become bound by BiP and each bound BiP molecule may dissociate from the prepro-\( \alpha \) Factor. Finally, when the protein is completely in-side the ER lumen, it is released.

In mathematical models of translocation ratchets, several parameters have to be specified: the distance of sites along the protein which can be either bound or unbound with ratcheting molecules; the rate and location of association and rate of dissociation of ratcheting molecules. The case of a large association rate and a fixed distance of possible ratcheting sites was studied in [Budhiraja and Fricks (2006)]. (This ratchet is similar to the polymerisation ratchet of [Peskin et al. (1993)].) Budhiraja and Fricks obtain a Law of Large Numbers and a Central Limit Theorem for the speed of protein translocation if the protein diffuses through a Brownian motion with drift. In the present paper, we study a translocation ratchet for a continuum of possible ratcheting sites, in the limit of small ratcheting molecules, small dissociation rates and long proteins. In this model, we can compute a Law of Large Numbers (Theorem 1) and a Central Limit Theorem (Theorem 2) for the speed of protein translocation.

The paper is organised as follows: in Section 2 we introduce our model and give the main results, the Law of Large Numbers (Theorem 1) and the Central Limit Theorem (Theorem 2). In Section 3 we interpret our results with respect to existing literature on ratchet models. In Sections 4, 5 and 6 we provide the three main techniques used for the proof of Theorems 1 and 2; see also Remark 2.2. We conclude with a formal proof of both theorems in Section 7.
2 Model and Results

We study a translocation ratchet similar to the model by Liebermeister et al. (2001) and Elston (2002) using the following assumptions: (i) the protein moves in and out with equal probabilities; (ii) the protein movement is reflected at bound ratcheting molecules; (iii) the protein has a continuum of sites to which ratcheting molecules can bind; (iv) the dissociation rate of ratcheting molecules from the protein is much smaller than their binding rate to the protein; (v) the ratcheting molecules are infinitely small; (vi) the protein is infinitely long.

In mathematical terms, we consider a time-homogeneous Markov dynamics \((\mathcal{X}, \mathcal{R}) = (X_t, R_t)_{t\geq 0}\), starting in \((X_0, R_0) = (x, 0), x \geq 0\). Here, at time \(t\), \(X_t\) is the length of the protein on the in-side and \(R_t\) is the distance of the molecule closest to the boundary as measured from the end point of the protein that is inside the cell. The process \(\mathcal{X} = (X_t)_{t\geq 0}\) is reflected Brownian motion, where the reflection point at time \(t\) is \(R_t\). This reflection point \(R_t\) increases with \(t\): given that the value of \((\mathcal{X}, \mathcal{R})\) at time \(t\) is \((X_t, R_t)\), \(R_t\) jumps to a uniformly chosen value \(r \in [R_t, X_t]\) at rate \(\gamma(X_t - R_t)dr\) for some \(\gamma > 0\). In other words, at rate \(\gamma(X_t - R_t)\) a new ratcheting molecule binds uniformly on \([R_t; X_t]\) which provides a new reflection point for \(\mathcal{X}\). By this dynamics, \(R_t \leq X_t\) for all \(t \geq 0\), almost surely. We refer to \((\mathcal{X}, \mathcal{R})\) as the \(\gamma\)-Brownian ratchet started in \(x\). If we want to stress the dependence on \(\gamma\), we write \((\mathcal{X}^\gamma, \mathcal{R}^\gamma) = (X_t^\gamma, R_t^\gamma)_{t\geq 0}\) for the \(\gamma\)-Brownian ratchet. For a realization of the processes \((\mathcal{X}, \mathcal{R})\) see Figure 2.

The rate of jumps of \(\mathcal{R}\) interacts closely with the distance of \(\mathcal{X}\) and \(\mathcal{R}\). Since \(\mathcal{R}\) is non-decreasing and \(\mathcal{R} \leq \mathcal{X}\), the process \(\mathcal{X}\) also tends to grow. We immediately formulate our main results concerning the Law of Large Numbers and the Central Limit Theorem for the speed of \(\mathcal{X}\):

**Theorem 1** (Law of Large Numbers). Let \(\mathcal{X} = (X_t)_{t\geq 0}\) be the \(\gamma\)-Brownian ratchet started in \(x \geq 0\). Then,

\[
\frac{X_t}{t} \xrightarrow{t \to \infty} C_{\gamma}
\]
almost surely, where

\[ C_\gamma = \frac{\Gamma(2/3)}{\Gamma(1/3)} \left( \frac{3\gamma}{4} \right)^{1/3} \]  

(2.1)

and \( \Gamma(.) \) is the gamma function.

**Theorem 2** (Central Limit Theorem). Let \( \gamma > 0 \) and \( \mathcal{X} = (X_t)_{t \geq 0} \) be the \( \gamma \)-Brownian ratchet started in \( x \geq 0 \), \( C_\gamma \) as in (2.1) and \( X \) some \( N(0,1) \)-distributed random variable. Then, there is \( \sigma > 0 \) which is independent of \( \gamma \) such that

\[ \frac{X_t - C_\gamma t}{\sigma \sqrt{t}} \overset{t \to \infty}{\Rightarrow} X, \]

where ‘\( \Rightarrow \)’ denotes convergence in distribution.

**Remark 2.1** (Numerical values for \( C_\gamma \) and \( \sigma \)). The numerical value for the speed of the \( \gamma \)-Brownian ratchet is

\[ C_\gamma = \frac{\Gamma(2/3)}{\Gamma(1/3)} \left( \frac{3\gamma}{4} \right)^{1/3} \approx 0.459248\gamma^{1/3}. \]

For the numerical value of \( \sigma \), note that \( X_t \) is a reflected Brownian motion for \( \gamma = 0 \). Since \( \sigma \) does not depend on \( \gamma \) (as long as \( \gamma > 0 \)), it is tempting to conjecture that \( \sigma = \sqrt{1 - 2/\pi} \approx 0.60281 \), the asymptotic standard deviation for reflected Brownian motion. Clearly, this heuristics would require an interchange of limits, \( t \to \infty \) and \( \gamma \to 0 \). While we could not make this interchange rigorous, simulations support this conjecture for the value of \( \sigma \).

**Remark 2.2** (Main steps in the proofs). The proof of Theorems 1 and 2 relies on several ingredients which we will provide in Sections 4, 5 and 6, respectively.

First (see Section 4), the process \((\mathcal{X}, \mathcal{R})\) can be constructed graphically by a one-dimensional (non-reflecting) Brownian motion and an independent Poisson process on \([0; \infty) \times \mathbb{R}\) with intensity \( \gamma \). From this graphical construction, it becomes clear that the speed of \( X_t \), i.e., the limit of \( \frac{X_t}{\sqrt{t}} \), must be proportional to \( \gamma^{1/3} \), if it exists. In addition, the graphical construction shows that \( \text{Var}[X_t] \) does not depend on \( \gamma \); see Remark 4.6.

Second, to compute a candidate for the asymptotic speed \( \frac{X_t}{t} \), we study the Markov chain \((X_{\tau_n} - R_{\tau_n}, R_{\tau_n} - R_{\tau_n-1}, \tau_n - \tau_{n-1})_{n=1,2,...}\), where \( \tau_0 = 0 \) and \( \tau_1, \tau_2, ... \) are the jump times of \( \mathcal{R} \) (see Section 5). In particular, we show that this Markov chain has a unique invariant distribution and (as shown in Section 7) the asymptotic speed is the ratio of expectations of \( X_{\tau_n} - R_{\tau_n} \) and \( \tau_n - \tau_{n-1} \) under the invariant distribution, which are computed in Proposition 5.8.

Third, we find a renewal structure for the process \((\mathcal{X}, \mathcal{R})\) where renewal points \( \rho_1, \rho_2, ... \) are given by times \( t \) where \( X_t = R_t \) and between \( \rho_n \) and \( \rho_{n+1} \) a jump of \( \mathcal{R} \) has occurred (see Section 6). Using this renewal structure, we can show existence for the speed of \( X_t \) and use a Central Limit Theorem for cumulative processes in order to prove Theorem 2.

3 Applications in biology and extensions

We describe possible extensions of the \( \gamma \)-Brownian ratchet some of which already appeared in the literature. Moreover, we give biological interpretations of our findings.

**Remark 3.1** (Review of published ratchet models and extensions). Modelling protein translocation by ratcheting mechanisms has started with Simon et al. (1992) and Peskin et al. (1993).
We review several features of published models for protein translocation and hint to extensions of our mathematical model. Throughout, we assume that $X_t$ is the length of the protein in-side and $R_t$ is the position of the reflection point at time $t$.

**Dissociation of ratcheting molecules:** In the original approach of Simon et al. (1992), there is a finite set of equally spaced ratcheting sites along the protein. There are two rates which describe binding and dissociation of ratcheting molecules from the protein. In particular, the case of infinite rates with a constant ratio can be studied, which leads to an effective speed by assuming that each ratcheting site has a certain probability of being bound independent of all others. This is also the limiting case of fast binding and unbinding as studied in Ambjörnsson et al. (2005). These approaches lead to the following mathematical description:

1. If $R_t$ is the set of positions of bound ratcheting molecules at time $t$, let the reflection point of $X_t$ be $R_t := \max R_t$. In addition, a point $r$ with $0 \leq r \leq X_t$ is added at rate $\rho dr$ and an existing point $r \in R_t$ is taken from $R_t$ at a rate $\sigma$. Here, $\sigma$ describes the rate of dissociation of ratcheting molecules and the $\gamma$-Brownian ratchet is the special case $\sigma = 0$.

**Protein movement against a force:** Usually, proteins are present in folded states inside a cell. Travelling through a nanopore requires unfolding of these proteins. In particular, the part of the protein on the out-side might still be in a folded state while the part on the in-side is unfolded. Hence, moving in takes place against a force, as modelled by Liebermeister et al. (2001); see also Budhiraja and Fricks (2006). The mathematical description of such ideas extends the $\gamma$-Brownian ratchet as follows:

2. Between reflection events, $(X_t)_{t \geq 0}$ is a stochastic process with continuous paths, not necessarily a Brownian motion. In particular, if proteins have to unfold out-side the nanopore, $(X_t)_{t \geq 0}$ is best chosen as a reflected Brownian motion with negative drift.

**State-dependent binding rate:** The binding of ratcheting molecules to the protein might depend on various parameters. In the model of Liebermeister et al. (2001) ratcheting sites along the protein are occupied only if they are close to the nanopore. The physical reason for this preferred binding close to the nanopore is an interaction with the protein forming the nanopore. Such a phenomenon has been described for the translocation of Prepro-$\alpha$ Factor (the protein) with BiP (the ratcheting molecule) and Sec61p as the protein forming the nanopore (Matlack et al., 1999). A compromise between ratcheting molecules which bind with uniform rates along the protein and only in proximity to the nanopore was considered by Elston (2002). Finally, binding of ratcheting molecules might depend on the amino acid sequence of the protein as described in Abdolvahab et al. (2008). We suggest the following extension of the $\gamma$-Brownian ratchet:

3. Given $X_t$ and $R_t$, let the reflection point change to $r$, $R_t \leq r \leq X_t$ at rate $\rho(d(X_t - r))$ for some $\sigma$-finite measure $\rho$ on $\mathbb{R}_+$. The $\gamma$-Brownian ratchet is the special case that $\rho$ is Lebesgue measure on $\mathbb{R}_+$.

Some special cases of the extensions (i), (ii), (iii) are straight forward to analyse. Consider the extension (i) as an example. As will become clear in Section 4, the average time between jumps of $\mathcal{R}$ in the $\gamma$-Brownian ratchet scales with $\gamma^{-2/3}$. This implies that if $\sigma \ll \gamma^{2/3}$ the
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extension (i) is approximately equal to the $\gamma$-Brownian ratchet but in other cases, the speed of the ratchet in (i) remains to be solved. Another example is the extension (iii) in the case $
abla = \gamma \cdot \delta_0$ for some $\gamma > 0$, i.e., binding of ratcheting molecules to the protein occur at constant rate directly at the nanopore. For the case of BiP as ratcheting molecule and Prepro-$\alpha$ as protein, parameters estimated in [Elston 2002] indicate that this is a realistic scenario. In this case, jump times of $\mathcal{R} = (R_t)_{t \geq 0}$ are a rate-$\gamma$ Poisson process. At each jump time, we set $R_t = X_t$ which is also the new reflection point of the Brownian motion. Therefore, jump times are renewal points of such a Brownian ratchet. Moreover, starting a reflected Brownian motion $(B_t)_{t \geq 0}$ at 0 and if $T$ is an exponential time with rate $\gamma$, it can be computed [Borodin and Salminen 2002] that $B_T$ is exponentially distributed with rate $(2\gamma)^{1/2}$. This leads to an asymptotic velocity of $(2\gamma)^{-1/2} \cdot \gamma = (\gamma/2)^{1/2}$. A Central Limit Theorem can be proved as well.

Remark 3.2 (Biological interpretation of our results). In our model we use the asymptotics of an infinitely long protein and study properties of the velocity of the Brownian ratchet. In practice more relevant are of course a finitely long protein and the time the protein is completely on the in-side. For long proteins both approaches are similar: if $T_x := \inf\{t : X_t \geq x\}$ in our model, $T_x/x \approx 1/C_\gamma$ with $C_\gamma$ from (2.1) for large $x$.

Consider the case that the rate $\gamma$ for binding of ratcheting molecules to the protein, is proportional to the concentration $a$ of ratcheting molecules on the in-side of the nanopore. As Theorem 1 shows, the speed of translocation, $X_t/t$, is mainly determined by $\gamma^{1/3}$ which is proportional to $a^{1/3}$. That is, to double the speed of translocation requires an eight times higher concentration of ratcheting molecules. In contrast, consider a ratchet model described at the end of Remark 3.1 where ratcheting molecules bind to the protein preferably in close proximity to the nanopore. We observed that the speed of the ratchet is $(\gamma/2)^{1/2}$, so the speed of the ratchet is proportional to $a^{1/2}$. This means that the latter ratchet model uses the existing ratcheting molecules more efficient (i.e. protein translocation is faster) if the concentration $a$ is large but the $\gamma$-Brownian ratchet is more efficient when $a$ is small. It will be interesting to see if real biological systems tend to behave like the more efficient model.

Remark 3.3 (Edwards model). One mathematical model similar in spirit to the Brownian ratchet is the Edwards model (see [van der Hofstad et al. 1997]). Let $(B_t)_{t \geq 0}$ be standard Brownian motion and let $P$ denote its distribution on path space. Edwards’ model is a transformed measure on the path space defined for $\gamma > 0$ by

$$
\frac{dP_T^\gamma}{dP} = \frac{1}{Z_T^\gamma} \exp \left\{ - \gamma \int_\mathbb{R} L(T, x)^2 \, dx \right\}, \quad T \geq 0,
$$

where $L(T, x)$ is the local time in $x$ up to time $T$ and $Z_T^\gamma$ is the normalising constant. It can be seen that $\int_\mathbb{R} L(T, x)^2 \, dx$ is the self intersection local time. Thus, the new measure discourages self intersections of paths. For the Brownian ratchet, such intersections are also discouraged by the jumping reflection boundary.

The weak Law of Large Numbers is proven in [Westwater 1985]. In [van der Hofstad et al. 1997] the Central Limit Theorem in the following form is proven: For every $\gamma \in (0, \infty)$

$$
\lim_{T \to \infty} P_T^\gamma \left( \frac{|B_T| - C_\gamma T}{\sigma^* \sqrt{T}} \leq C \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^C e^{-x^2/2}, \quad \text{for all } C \in \mathbb{R}.
$$

As in the case of the Brownian ratchet, the asymptotic speed is of the form $C_\gamma^* = b^* \gamma^{1/3}$ for some absolute constant $b^*$. In addition, $\sigma^*$ does not depend on $\gamma$ as well. The constants
Figure 3: One realization of the graphical construction. The shown realization of the graphical construction leads to the same path of $(X, R)$ as shown in Figure 2.

$b^*$ and $\sigma^*$ can be expressed in terms of the largest eigenvalue of a certain Sturm-Liouville operator. In van der Hofstad (1998), Theorem 3, it is shown that $b^* \in [1.104, 1.124]$ and $\sigma^* \in [0.60, 0.66]$. In the case of Brownian ratchet we have $C_\gamma = b\gamma^{1/3}$ where $b \approx 0.459248$ and the conjectured value of $\sigma$ is $\sigma \approx 0.60281$.

4 Graphical construction and some applications

In this section we give a graphical construction of the Brownian ratchet; see also Figure 3. From that construction we deduce a useful scaling (Proposition 4.5) that will be needed in the sequel.

4.1 The construction

Definition 4.1. For $\gamma > 0$ let $N^\gamma$ be a Poisson point process on $[0; \infty) \times \mathbb{R}$ with intensity measure $\gamma \lambda^2(dt, dx)$, where $\lambda^2$ denotes the Lebesgue measure on $[0; \infty) \times \mathbb{R}$. Let $B = (B_t)_{t \geq 0}$ be an independent Brownian motion on $\mathbb{R}$ starting in $x \in \mathbb{R}$. We define $\tilde{\tau}_0, S_0, \tilde{\tau}_1, S_1, \tilde{\tau}_2, S_2, \ldots$ recursively by $\tilde{\tau}_0 = 0$, $S_0 = 0$,

$$\tilde{\tau}_{n+1} := \inf\{t > \tilde{\tau}_n : (\{t\} \times [B_t \wedge S_n, B_t \vee S_n]) \cap N^\gamma \neq \emptyset\}$$ (4.1)

and

$$S_{n+1} := \text{second coordinate of the unique element of } \left(\{\tilde{\tau}_{n+1}\} \times [B_{\tilde{\tau}_{n+1}} \wedge S_n, B_{\tilde{\tau}_{n+1}} \vee S_n]\right) \cap N^\gamma.$$ (4.2)

That is, $\tau_{n+1}$ is the first time after $\tau_n$ when there is a point of the Poisson process between the graph of the Brownian motion and $S_n$, and $S_{n+1}$ is the second coordinate (the space coordinate) of that point.
For $t \in [\tilde{\tau}_n, \tilde{\tau}_{n+1})$ we set

$$\tilde{R}_t = \sum_{i=1}^{n} |S_i - S_{i-1}| \quad \text{and} \quad \tilde{X}_t = \tilde{R}_t + |B_t - S_n|$$

and $(\tilde{X}, \tilde{R}) := (\tilde{X}_t, \tilde{R}_t)_{t \geq 0}$.

**Lemma 4.2.** The process $(\tilde{X}, \tilde{R})$ is a $\gamma$-Brownian ratchet started in $[x]$.

**Proof.** First note that $\tilde{R}$ is almost surely non-decreasing and $\tilde{X}_t \geq \tilde{R}_t$ for all $t \geq 0$ by construction. Consider any time $t \in [\tilde{\tau}_n, \tilde{\tau}_{n+1})$. The rate of occurrence of $\tilde{\tau}_{n+1}$ is $\gamma|B_t - S_n|$ if and only if $|S_n + 1 - S_n| = |r - \tilde{R}_t|$. By homogeneity of the Poisson process $\hat{N}^\gamma$, $r$ is uniform on $[\tilde{R}_t, \tilde{X}_t]$. In addition, $\tilde{X}_t$ behaves like a Brownian motion, reflected at $\tilde{R}_t$.

**Definition 4.3.** We denote that process $(\tilde{X}, \tilde{R}) = (\tilde{X}_t, \tilde{R}_t)_{t \geq 0}$ constructed in Definition 4.1 by the Brownian ratchet read off from $(\mathcal{B}, \hat{N}^\gamma)$. To stress the dependence on $\gamma$ we also write $(\tilde{X}^\gamma, \tilde{R}^\gamma) = (\tilde{X}_t^\gamma, \tilde{R}_t^\gamma)_{t \geq 0}$ for this process.

**Remark 4.4.** It is intuitively clear from the graphical construction that the long-time behaviour of $\mathcal{X}$, as given in Theorems 1 and 2, and $\mathcal{R}$, are identical. We will prove this fact in Proposition 6.1.

### 4.2 Scaling property

From the graphical construction we can deduce the following scaling property.

**Proposition 4.5.** Let $(X^\gamma, R^\gamma) = (X^\gamma_t, R^\gamma_t)_{t \geq 0}$ and $(X^1, R^1) = (X^1_t, R^1_t)_{t \geq 0}$ be Brownian ratchets with rates $\gamma > 0$ and 1, respectively, both starting in $x = 0$. Then,

$$(X^\gamma_t, R^\gamma_t)_{t \geq 0} \overset{d}{=} \gamma^{-1/3}(X^1_{\gamma^2/3t}, R^1_{\gamma^2/3t})_{t \geq 0}.$$  

**Proof.** We use the same notation as in Definition 4.1. By Lemma 4.2 it suffices to show (4.4) for the Brownian ratchet $(\mathcal{X}, \mathcal{R})$ read off from $\mathcal{B}$ and $N^\gamma$. Setting

$$g : \left\{ \begin{array}{ll} \mathbb{R}^2 & \rightarrow \mathbb{R}^2 \\ (t, x) & \rightarrow (\gamma^{-2/3}t, \gamma^{-1/3}x) \end{array} \right.$$ we find that $g(N^\gamma) \overset{d}{=} N^1$. Moreover, for the space-time version of the Brownian motion $\mathcal{B}$, i.e. $\hat{B} = (t, B_t)_{t \geq 0}$, started in 0, we have $g(\hat{B}) := (g(t, B_t))_{t \geq 0} \overset{d}{=} \hat{B}$ by the Brownian rescaling. Applying $g$ to each space-time point we obtain a path of a Brownian ratchet based on $g(\hat{B}) \overset{d}{=} \hat{B}$ and $g(N^\gamma) \overset{d}{=} N^1$. In other words, (4.4) holds for $(X^\gamma_t, R^\gamma_t)$ and we are done.

**Remark 4.6.** 1. From the scaling property of the Brownian ratchet, Proposition 4.5, we have

$$\lim_{t \to \infty} \frac{\mathbb{E}[X^\gamma_t]}{t} = \lim_{t \to \infty} \gamma^{-1/3} \frac{\mathbb{E}[X^1_{\gamma^2/3t}]}{t} = \lim_{t \to \infty} \gamma^{1/3} \frac{\mathbb{E}[X^1_{\gamma^2/3t}]}{\gamma^{2/3}t} = \gamma^{1/3} \lim_{t \to \infty} \frac{\mathbb{E}[X^1_t]}{t}.$$
In particular, the second limit does not depend on $\gamma$. The existence of the limits will be proven in Section 7.

2. Thanks to the scaling property of the Brownian ratchet, we may choose the most convenient value of $\gamma = \frac{1}{2}$ in most proofs below. Afterwards (i.e. in the proofs of Theorems 1 and 2) we use Proposition 4.5 to obtain results for general $\gamma$.

5 The Brownian ratchet at jump times

In this section we prove existence and uniqueness of the invariant distribution of a Markov chain representing the Brownian ratchet at jump times (see Definition 5.1). For this, we start in Lemma 5.4 with studying the time to the next jump and the increment of the Brownian ratchet during that time starting in $x$ in expectation. We then derive properties of the tails of the waiting time between jumps (Lemma 5.5). Afterwards we come to existence (Proposition 5.6) and uniqueness (Proposition 5.7) of an invariant distribution of the Brownian ratchet at jump times. We conclude the section by computing the waiting time to the next jump and increments for a Brownian ratchet in equilibrium (Proposition 5.8). In the whole section we restrict ourselves to the notationally convenient case $\gamma = \frac{1}{2}$. Our results can easily be extended to general $\gamma$ using the scaling property, Proposition 4.5.

Definition 5.1. For the Brownian ratchet $(X, R)$, set $\tau_0 = 0$ and denote the sequence of jump times of $R$ by $\tau_1, \tau_2, \ldots$. We set $(Y, W, \eta)_n = (Y_n, W_n, \eta_n)_{n=1,2,\ldots}$ by

$$Y_n = X_{\tau_n} - R_{\tau_n}, \quad W_n = R_{\tau_n} - R_{\tau_n-1} \quad \text{and} \quad \eta_n = \tau_n - \tau_{n-1}. \quad (5.1)$$

Observe that for any $k$, $(Y_n, W_n, \eta_n)_{n=k+1,k+2,\ldots}$ depends on $(Y_n, W_n, \eta_n)_{n=1,\ldots,k}$ only through $Y_k$. In particular, $(Y, W, \eta)$ is a Markov chain.

In this section we will frequently use Airy functions. We recall basic facts concerning these functions first.

Remark 5.2 (Airy functions). The Airy functions $Ai$ and $Bi$ are two linearly independent solutions of the differential equations

$$u''(x) - xu(x) = 0 \quad (5.2)$$

with $Ai(x) \xrightarrow{x \to \infty} 0$ and $Bi(x) \xrightarrow{x \to \infty} \infty$. We only need properties of the Airy functions on the non-negative real line. For further properties and explicit definitions of the Airy functions in terms of integrals or Bessel functions we refer the reader to Abramowitz and Stegun (1972). Denoting the gamma function by $\Gamma$ we have (see Abramowitz and Stegun 1972, 10.4.4 and 10.4.5)

$$Ai(0) = \frac{Bi(0)}{\sqrt{3}} = \frac{1}{3^{2/3}\Gamma(2/3)} \quad (5.3)$$

$$-Ai'(0) = \frac{Bi'(0)}{\sqrt{3}} = \frac{1}{3^{1/3}\Gamma(1/3)}. \quad (5.4)$$
The Wronskian,

\[ Bi'(x)Ai(x) - Ai'(x)Bi(x) \tag{5.5} \]

which is the determinant of the fundamental matrix of the differential equation \( (5.2) \), does not depend on \( x \) and is given by

\[ w := Bi'(0)Ai(0) - Ai'(0)Bi(0) = \frac{2}{\sqrt{3}\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})} = \frac{1}{\pi}. \tag{5.6} \]

(The last equality follows from Euler’s reflection formula \( \Gamma(z)\Gamma(1-z) = \pi/\sin \pi z. \))

The asymptotics of the Airy functions are well known, see e.g. Abramowitz and Stegun \( (1972, \text{p. 448–449}) \) or Janson \( (2007, \text{p. 161}) \). As \( x \to \infty \) we have

\[ Ai(x) \sim \frac{\pi^{-1/2}}{2} x^{-1/4} e^{-2x^{3/2}/3}. \tag{5.7} \]

(Here, as usual, \( a(x) \sim b(x) \) as \( x \to \infty \) means that \( a(x)/b(x) \to 1 \) as \( x \to \infty \).) The integral of \( Ai \) is given by

\[ \int_{0}^{\infty} Ai(u) \, du = \frac{1}{3}. \tag{5.8} \]

We note that

\[ Gi(x) := Ai(x) \int_{0}^{x} Bi(y) \, dy + Bi(x) \int_{x}^{\infty} Ai(y) \, dy. \tag{5.9} \]

is solution of the inhomogeneous equation

\[ u''(x) - xu(x) = -\frac{1}{\pi}. \]

5.1 Increments for single jumps of the Brownian ratchet

Between jumps of \( \mathcal{R} \), the process \( \mathcal{X} \) behaves like reflected Brownian motion. A jump time of the ratchet can be seen as a killing time of the reflected Brownian motion. Hence, we study a reflected Brownian motion, killed at rate \( \gamma \) times its value. Recall our assumption \( \gamma = \frac{1}{2} \).

**Definition 5.3** (Killed Brownian motion). Let \( (B_t)_{t \geq 0} \) denote Brownian motion started in \( x \geq 0 \) and let

\[ \tau = \inf \left\{ t > 0 : \frac{1}{2} \int_{0}^{t} |B_s| \, ds \geq \xi \right\}, \]

where \( \xi \) is an exponentially distributed, independent random variable with rate 1. Then, \( \tilde{B} := (\tilde{B}_t)_{t \geq 0} \) with \( \tilde{B}_t := |B_t| \) for \( 0 \leq t < \tau \) and \( \tilde{B}_t = \Delta \) for \( t \geq \tau \) and some \( \Delta \notin \mathbb{R} \) is a reflected Brownian motion, killed at rate \( \frac{1}{2} |B| \). We denote the probability measure of the Brownian motion, started in \( x \), by \( \mathbb{P}_x \).

To compute the speed of the ratchet we need the expectations of \( \tau \) and \( B_\tau \).
Lemma 5.4. Let $\tilde{B}$ be reflected Brownian motion, killed at rate $\frac{1}{2}|B|$. Then,

$$E_x[\tilde{B}_{\tau-}] = x + \frac{2\pi Ai(x)}{3^{1/6}\Gamma(2/3)}$$

(5.10)

and

$$E_x[\tau] = 2\pi (Gi(x) + 3^{-1/2}Ai(x)).$$

(5.11)

Proof. First we compute the Green function of $\tilde{B}$ following the scheme outlined on p. 18–19 in Borodin and Salminen (2002). Set $\sigma_y := \inf\{t > 0 : \tilde{B}_t = y\}$. The diffusion process $\tilde{B}$ is regular in the sense that for each $x, y \geq 0$

$$P_x(\sigma_y < \infty) > 0.$$

Such a diffusion is called transient if for some (and then for all) $x, y, x \neq y$

$$P_x(\sigma_y = \infty) > 0.$$

As $\tilde{B}$ can be killed in each interval with positive probability it is transient. Let $p(t; x, y)$ be the transition density of $\tilde{B}$ with respect to the speed measure, which is given by $m(dx) = 2dx$. In the transient case the Green function, defined by

$$G(x, y) := \int_0^\infty p(t; x, y) dt,$$

is finite for each $x, y \geq 0$ and can be computed in terms of solutions of the differential equation (5.2) with appropriate boundary conditions at 0 and $\infty$. Two linearly independent solutions are given by the Airy functions $Ai$ and $Bi$, see Remark 5.2. The solutions of (5.2) that we need are obtained as follows. For $x \geq 0$ let

$$\phi(x) = Ai(x)$$

$$\psi(x) = C_1Bi(x) + C_2Ai(x).$$

The function $\phi$ is a decreasing solution of (5.2) and satisfies $\lim_{x \to \infty} \phi(x) = 0$. The constants $C_1$ and $C_2$ have to be chosen such that the function $\psi$ is increasing and (this is the requirement for reflecting boundary at zero)

$$\psi'(0) = 0$$

(5.12)

It is easy to see, that in order to satisfy (5.12) we must have $C_2 = -C_1Bi'(0)/Ai'(0) = C_1\sqrt{3}$ (see (5.4) for the last equality). We may take any positive number $C_1$ in the definition of $\psi$ because, as one can see in (5.14), multiplication of $\phi$ and $\psi$ by some factors does not change the Green function. So we set $C_1 = 1$ and $C_2 = \sqrt{3}$. The Wronskian (which is independent of $x$) is given as in (5.6) by

$$w = \psi'(0)\phi(0) - \psi(0)\phi'(0) = Bi'(0)Ai(0) - Ai'(0)Bi(0) = \frac{1}{\pi},$$

(5.13)
Now the Green function is given by
\[
G(x, y) := \begin{cases} 
  w^{-1} \psi(x) \phi(y) & : 0 \leq x \leq y \\
  w^{-1} \psi(y) \phi(x) & : 0 \leq y \leq x.
\end{cases}
\] (5.14)

The density of the killing measure with respect to the speed measure is \( \frac{1}{2} y \), hence it is \( k(y) = y \) with respect to Lebesgue measure (see Borodin and Salminen [2002, p. 17]). Furthermore the killing position of the Brownian motion started in \( x \) has density
\[
G(x, y)k(y)
\]
with respect to Lebesgue measure (see Borodin and Salminen [2002, p. 14]). Now we can compute the expected killing position starting from \( x \):
\[
\mathbb{E}_x[\tilde{B}_{\tau-}] = \int_0^\infty y^2 G(x, y) \, dy = \frac{1}{w} \left( \phi(x) \int_0^x y^2 \psi(y) \, dy + \psi(x) \int_x^\infty y^2 \phi(y) \, dy \right). \tag{5.15}
\]

We compute the integrals separately. Using the fact that Airy functions are solutions of the differential equation (5.2) we have
\[
\int_x^\infty u^2 Ai(u) \, du = \int_x^\infty u Ai''(u) \, du = x Ai'(x) - Ai(x) + C
\]
and
\[
\int_x^\infty u^2 Bi(u) \, du = \int_x^\infty u Bi''(u) = x Bi'(x) - Bi(x) + C.
\]

Thus,
\[
\phi(x) \int_0^x y^2 \psi(y) \, dy = Ai(x) \left( x Bi'(x) - Bi(x) + Bi(0) + \sqrt{3} (xAi'(x) - Ai(x) + Ai(0)) \right)
\]
and
\[
\psi(x) \int_x^\infty y^2 \phi(y) \, dy = (Bi(x) + \sqrt{3} Ai(x)) \left( -x Ai'(x) + Ai(x) \right).
\]

In the last equality we have used the asymptotic relation (5.7). Substituting in (5.15) and simplifying we get
\[
\mathbb{E}_x[\tilde{B}_{\tau}] = \frac{1}{w} \left( xh(x) + Ai(x)(Bi(0) + \sqrt{3}Ai(0)) \right),
\]
where
\[
h(x) = (Ai(x)Bi'(x) - Ai'(x)Bi(x)) = w.
\]

Using (5.3) we arrive at
\[
\mathbb{E}_x[\tilde{B}_{\tau}] = x + \frac{2Ai(x)Bi(0)}{w} = x + \frac{2\pi Ai(x)}{3^{1/6} \Gamma(2/3)} \tag{5.16}
\]
which shows (5.10). For (5.11), the expected jump time of the reflection boundary if started from \( x \) is given by

\[
\mathbb{E}_x[\tau] = \int_0^\infty \int_0^\infty p(t; x, y) \, dt \, m(dy)
\]

\[
= 2 \int_0^\infty G(x, y) \, dy
\]

\[
= \frac{2}{w} \left( \phi(x) \int_y^x \psi(y) \, dy + \psi(x) \int_x^\infty \phi(y) \, dy \right)
\]

\[
= \frac{2}{w} \left( G(x) + 3^{-1/2} Ai(x) \right)
\]

\[
= 2 \pi \left( G(x) + 3^{1/2} Ai(x) \right)
\]

(5.17)

where we have used (5.8) for the next to last equality and the definition of \( G \) in (5.9).

In the next section we will need boundedness of second moments of \( \tau \), which is a consequence of the following lemma. Note that one also could use Kac’s moment formula (see e.g. Fitzsimmons and Pitman [1999]) to compute the second moment of \( \tau \) directly.

**Lemma 5.5** (Bound on tail distribution of \( \tau \)). There exist finite positive constants \( c \) and \( C \) such that for any \( x, t \geq 0 \)

\[
P_x(\tau > t) \leq Ce^{-ct}.
\]

**Proof.** We use the notation of Definition 5.3. Clearly,

\[
P_x(\tau > t) \leq P_0(\tau > t),
\]

so it is enough to prove the assertion in the case \( x = 0 \) and we omit the subscript 0 in the proof. Using the Brownian scaling property \((B_s)_{s \geq 0} \overset{d}{=} (t^{1/2}B_{s/t})_{s \geq 0}\) for any \( t > 0 \) and a usual change of variables, we have

\[
\int_0^t |B_s| \, ds \overset{d}{=} t^{3/2} \int_0^1 |B_s| \, ds.
\]

We set \( \mathcal{B} = \int_0^1 |B_s| \, ds \) and use \( f_{\mathcal{B}} \) to denote the density of that random variable. By the definition of \( \tau \)

\[
P(\tau > t) = P\left( \frac{1}{2} t^{3/2} \mathcal{B} < \xi \right) = \int_0^\infty f_{\mathcal{B}}(x) e^{-\frac{1}{2} t^{3/2} x} \, dx = \mathbb{E}\left[ e^{-\frac{1}{2} t^{3/2} \mathcal{B}} \right],
\]

(5.19)

i.e., the tail probability \( P(\tau > t) \) is given by the Laplace transform of \( \mathcal{B} \) evaluated at \( \frac{1}{2} t^{3/2} \).

We consider the Laplace transform evaluated at \( \lambda > 0 \) first and then set \( \lambda = \frac{1}{2} t^{3/2} \). A series representation of the Laplace transform of \( \mathcal{B} \) in terms of Airy functions goes back to Kac [1946]. Instead of using the formula for the Laplace transform directly we use the asymptotic expansions of \( f_{\mathcal{B}} \) to study the behaviour of the Laplace transform for large \( t \). We write

\[
\mathbb{E}[e^{-\lambda \mathcal{B}}] = \int_0^t e^{-\lambda x} f_{\mathcal{B}}(x) \, dx + \int_t^\infty e^{-\lambda x} f_{\mathcal{B}}(x) \, dx
\]

(5.20)
and estimate both integrals on the right hand side for appropriately chosen \(r\). The asymptotic expansions of \(f_B\) are (see [Janson 2007], p. 108)

\[
f_B(x) = \frac{\sqrt{6}}{\sqrt{\pi}} e^{-3x^2/2} (1 + \mathcal{O}(x^{-2})) \quad \text{as} \quad x \to \infty.
\]

and

\[
f_B(x) = Ae^{-a/x^2} (x^{-2} + \mathcal{O}(1)) \quad \text{as} \quad x \to 0.
\]

Here, \(A\) is a positive constant, \(a = 2|a_1'|^3/27 \approx 0.0783\) and \(a'\) is the largest real zero of the derivative of the Airy function \(Ai\).

Set \(g(x) := 2x^2 \log x\) for \(x > 0\) and \(g(0) := 0\). Note that \(g\) is continuous and negative on \((0, 1)\). We choose \(r > 0\) such that \(g(x) \geq -a/2\) for \(x \in [0, r]\). Then for some \(\tilde{A} \geq A\) we have for all \(x \in [0, r]\)

\[
f_B(x) \leq \tilde{A} e^{-\frac{a}{2}x^2} = \tilde{A} e^{-\frac{a}{2}\left(\frac{a}{2}\right)} \leq \tilde{A} e^{-\frac{a}{2}x^2}.
\]

It is easy to see that the function \(x \mapsto \lambda x + \frac{a}{2x^2}\) attains its maximum in \(x^* = \left(\frac{a}{\lambda}\right)^{1/3}\). For \(\lambda \geq a/r^3\) we have \(x^* \leq r\) and therefore

\[
\int_0^r e^{-\lambda x} f_B(x) \, dx \leq \tilde{A} \int_0^r e^{-\lambda x - \frac{a}{2x^2}} \, dx \\
\leq \tilde{A}r e^{-\lambda x^* - \frac{a}{2(x^*)^2}} = \tilde{A}r e^{-3a^{1/3}\lambda^{2/3}/2}.
\]

Now we estimate the second integral in (5.20). From the asymptotic expansion (5.21) and continuity of \(f_B\) it follows that there exists a finite positive constant \(\tilde{A}\) such that

\[
f_B(x) \leq \tilde{A} \frac{\sqrt{6}}{\sqrt{\pi}} e^{-3x^2/2}.
\]

Completion of the square in the exponent followed by a change of variables yield

\[
\frac{\sqrt{6}}{\sqrt{\pi}} \int_r^\infty e^{-\lambda x - \frac{a}{2x}} \, dx = \frac{\sqrt{6}}{\sqrt{\pi}} \int_0^\infty e^{-\frac{(3\lambda + \lambda^2 x^2)^2}{6} + \frac{\lambda^2}{3}} \, dx \\
= \frac{\sqrt{6}}{\sqrt{\pi}} e^{\frac{\lambda^2}{3}} \int_0^\infty e^{-\frac{y^2}{2}} \, dy = e^{\frac{\lambda^2}{3}} \left(1 - \Phi\left(\frac{3r + \lambda}{\sqrt{3}}\right)\right) \\
\leq e^{\frac{\lambda^2}{3}} \frac{\sqrt{3}}{3r + \lambda} e^{-\frac{1}{2} \left(\frac{3r + \lambda}{\sqrt{3}}\right)^2} = \frac{\tilde{A}}{3r + \lambda} e^{-\lambda r}
\]

for some constant \(\tilde{A}\). Here \(\Phi\) denotes the distribution function of the standard Gaussian distribution. In the next to last step we used the well known inequality \(1 - \Phi(x) \leq x^{-1} e^{-x^2/2}\). Combining the last two displays we obtain

\[
\int_r^\infty e^{-\lambda x} f_B(x) \, dx \leq \frac{\tilde{A}}{3r + \lambda} e^{-\lambda r}.
\]

Substitution of the estimates (5.22) and (5.23) in (5.20) shows that there exists finite positive constants \(c\) and \(C\) such that

\[
\mathbb{E}[e^{-\lambda B}] \leq Ce^{c(2\lambda)^{2/3}}, \quad t \geq 0.
\]
Now the assertion of the lemma follows from the last equation and (5.19),

$$\mathbb{P}(\tau > t) = \mathbb{E}[e^{-\frac{1}{2}t^{3/2}B}] \leq Ce^{-ct}.$$ 

\[ \Box \]

### 5.2 Existence and uniqueness of the invariant distribution at jump times

The Markov chain \((Y, W, \eta)\) describes the Brownian ratchet at jump times. Next, we show existence (Proposition 5.6) and uniqueness (Proposition 5.7) of the invariant distribution for this Markov chain.

**Proposition 5.6.** Denote by \((P^n_x)_{n \in \mathbb{N}}\) the distribution of \((Y, W, \eta)\) (introduced in Definition 5.1) induced by a Brownian ratchet starting with \((X_0, R_0) = (x, 0)\). For each \(x \geq 0\) there exists \(i_1, i_2, \ldots\) and a distribution \(P_x^*\) that is an invariant distribution for \((Y, W, \eta)\) and

$$\frac{1}{i_n} \sum_{k=1}^{i_n} P^n_x \xrightarrow{n \to \infty} P_x^*,$$

(5.24)

where \(\Rightarrow\) denotes weak convergence of probability measures.

**Proof.** If the limit of a subsequence of \(n^{-1} \sum_{k=0}^{n-1} P^n_x\) exists then it must be an invariant distribution of \((Y_n, W_n, \eta_n)_{n \geq 1}\). A proof of this fact in the continuous time case, that can be easily adapted to the discrete time case, can be found in (Liggett, 1985, p. 11). It remains to prove that \(n^{-1} \sum_{k=0}^{n-1} P^n_x\) is a tight sequence. This follows immediately once we prove tightness of \((P^n_x)_{n=1, 2, \ldots}\). To this end it is enough to show that the first moments of \(Y_n, W_n\) and \(\eta_n\) are bounded uniformly in \(n\).

Boundedness of the first moments of \(\eta_n\) follows from Lemma 5.5. For the boundedness of the first moments of \(Y_1, Y_2, \ldots\) and \(W_1, W_2, \ldots\), recall that \(R_\tau\) is distributed uniformly on \([R_\tau; X_\tau]\). Hence \(Y_n\) and \(W_n\) have the same distribution and it suffices to show boundedness of the first moment of \(Y_1, Y_2, \ldots\). For this, we recall from (5.10) that

$$\mathbb{E}_x[X_{\tau_1}] = \mathbb{E}_x[\tilde{B}_{\tau_1}] = x + \frac{2\pi Ai(x)}{3^{1/6} \Gamma(2/3)} \leq x + c$$

(5.25)

for

$$c = \frac{2\pi Ai(0)}{3^{1/6} \Gamma(2/3)},$$

since \(Ai\) is decreasing in \([0; \infty)\). Let \(\mathcal{F}_n = \sigma(X_{\tau_m}, R_{\tau_m} : m \leq n)\) and let \((U_n)\) be a sequence of iid random variables uniformly distributed on \((0, 1)\) and independent of \((\mathcal{X}, \mathcal{R})\). Using that sequence we have

$$X_{\tau_{n+1}} - R_{\tau_{n+1}} \overset{d}{=} X_{\tau_{n+1}} - (R_{\tau} + (X_{\tau_{n+1}} - R_{\tau})U_{n+1}) = (X_{\tau_{n+1}} - R_{\tau})(1 - U_{n+1}).$$
and hence
\[
\mathbb{E}_x[Y_{n+1}] = \mathbb{E}_x[\mathbb{E}_x[(X_{\tau_{n+1}} - R_{\tau_n})(1 - U_{n+1})|\mathcal{F}_n]] \\
= \frac{1}{2} \mathbb{E}_x[\mathbb{E}_x[(X_{\tau_{n+1}} - R_{\tau_n})] \\
= \frac{1}{2} \mathbb{E}_x[X_{\tau_n} - R_{\tau_n}] \\
\leq \frac{1}{2} \mathbb{E}_x[Y_n + c] \\
= \frac{1}{2} \mathbb{E}_x[Y_n] + \frac{1}{2} c
\]
where we have used (5.25). In other words, \( \mathbb{E}_x[Y_n] \leq x + c \) for all \( n = 1, 2, \ldots \) which implies boundedness of the first moments of \( Y_1, Y_2, \ldots \).

The following lemma shows uniqueness of the invariant distribution of the Markov chain \((\mathcal{Y}, \mathcal{W}, \eta)\).

**Proposition 5.7.** Let \((\mathcal{Y}, \mathcal{W}, \eta)\) be the Markov chain at jump times as introduced in Definition 5.1 based on a Brownian ratchet started in \( x \geq 0 \). If an invariant distribution of \((\mathcal{Y}, \mathcal{W}, \eta)\) exists, then it is unique.

**Proof.** We use the graphical construction and a coupling argument for the proof. Let \( \mathcal{N} \) be a Poisson process on \( \mathbb{R}_+ \times \mathbb{R} \) with intensity measure \( \gamma \cdot \lambda^2 \). In addition, \( \mathcal{B}^i = (B^i_t)_{t \geq 0}, i = 1, 2 \) are two Brownian motions, started in \( x_1 \leq x_2 \), such that \( \mathcal{B}^1 \) and \( \mathcal{B}^2 \) are independent before \( T := \inf\{s \geq 0 : B^1_s = B^2_s\} \) and \( B^i_T = B^2_T \) for \( t \geq T \). Moreover, \((\mathcal{X}^i, \mathcal{R}^i)\) is the Brownian ratchet read off from \((\mathcal{B}^i, \mathcal{N})\), \( i = 1, 2 \). Let \( \tau^i_n \) denote the jump times of \( \mathcal{R}^i \), \( i = 1, 2 \) and set
\[
Y^i_n = X^i_{\tau^i_n} - R^i_{\tau^i_n}, \quad W^i_n = R^i_{\tau^i_n} - R^i_{\tau^i_{n-1}}, \quad \eta^i_n = \tau^i_n - \tau^i_{n-1}
\]
for \( i = 1, 2 \). In order to show uniqueness of the invariant distribution of \((\mathcal{Y}, \mathcal{W}, \eta)\) it suffices to show that there is an almost surely finite stopping time \( T' \) such that \( (X^1_{t}, X^2_{t}, R^1_{t} - R^2_{t})_{t \geq T'} = (X^2_{t}, X^2_{t}, R^2_{t} - R^2_{t})_{t \geq T'}, \) almost surely.

We denote by \( \mathcal{S}^i_t \) the reflection point of \( \mathcal{B}^i \) in the graphical construction, \( i = 1, 2 \), i.e., given \( t \in [\tau^i_n; \tau^i_{n+1}) \), \( \mathcal{S}^i_t := \mathcal{S}^i_n \) where \( \mathcal{S}^1_1, \mathcal{S}^2_2, \ldots \) are as in (4.1) and (4.2) for \( i = 1, 2 \). Since the hitting time of \( \mathcal{B}^1 \) and \( \mathcal{B}^2 \) is almost surely finite, it is enough to construct the coupling using only one Brownian motion \( B_t \) starting in \( x = x_1 = x_2 \). So, assuming \( B^1_0 = B^2_0 \), we need to show that \( T := \inf\{t : R^1_t = R^2_t\} \) is almost surely finite. We set \( s_i := \mathcal{S}^i_0, i = 1, 2 \) and (without loss of generality) \( s_1 \geq s_2 \). In the case \( s_1 = s_2 \) we have \( T = 0 \) and we are done. Otherwise, there are three possibilities:

\[
x \leq s_2 < s_1, \quad s_2 < x < s_1, \quad s_2 < s_1 \leq x,
\]
where the first and the last are symmetric. As the Brownian motion is recurrent and the reflection boundaries tend to jump towards the Brownian motion, it is clear that the time until the Brownian motion hits one of the moving boundaries is a.s. finite. Thus, it suffices to consider the cases

\[
x = s_2 \leq s_1 \quad \text{and} \quad s_2 \leq x = s_1.
\]

(5.26)
Given \( t \in [\tau_n^i; \tau_{n+1}^i) \), recall that \( S_t^i \) jumps at \( \tau_{n+1}^i \) to the next point in \( N^\gamma \) between \( S_t^i \) and \( B_t^i \), \( i = 1, 2 \). Hence, the processes couple if and only if both components of \( (B_t, S_t^1) \) and \( (B_t, S_t^2) \) use the same point of the Poisson process \( N^\gamma \). This happens if the Poisson point, say \( N(t, y) \), that is used in the construction satisfies

\[
B_t < N(t, y) < S_t^2 \leq S_t^1 \quad \text{or} \quad S_t^2 \leq S_t^1 < N(t, y) < B_t.
\]

Consider the two cases of (5.26). In the first case we consider the (almost surely finite time of the) first jump of \( S_t^2 \) and in the second the (almost surely finite time of the) first jump of \( S_t^1 \). Due to the symmetry of the Brownian motion and the homogeneity of the Poisson process, this point is below \( x \) in the first case and above \( x \) in the second case, with probability at least \( \frac{1}{2} \). In other words, (5.27) is satisfied at that jump time with probability at least \( \frac{1}{2} \). If coupling did not occur, we wait an almost surely finite amount of time until either of (5.26) is fulfilled again. Of course it is possible that one or both boundaries jump in-between. But at each jump time the boundaries get closer together. Then the processes try to couple again upon the next jump times of either \( S_t^1 \) or \( S_t^2 \) with success probability larger or equal \( \frac{1}{2} \) and so on. Thus, the coupling time is bounded by a geometric number of finite times and hence is almost surely finite.

### 5.3 Increments under the invariant distribution

Propositions 5.6 and 5.7 guarantee existence and uniqueness of the invariant distribution of the Brownian ratchet at jump times. Next, we derive the increment of the Brownian ratchet and waiting times between jumps in equilibrium.

**Proposition 5.8.** Let \( \pi \) be the invariant distribution for \( Y_1, Y_2, \ldots \). Start the Brownian ratchet in \( \pi \) and denote the integral with respect to the resulting distribution by \( \mathbb{E}_\pi \). Then,

\[
\mathbb{E}_\pi[Y_1] = -3Ai'(0)
\]

and

\[
\mathbb{E}_\pi[\tau_1] = 6Ai(0).
\]

**Proof.** Let \( Y \) be distributed according to \( \pi \). Let a Brownian ratchet start in \( X_0 = Y, R_0 = 0 \). Since \( Y \) is in equilibrium, we find that the distribution of a Brownian ratchet at the first jump time is distributed like \( Y \). To be more precise, let \( \overline{B}^Y = (\overline{B}^Y_t)_{t \geq 0} \) be a reflected Brownian motion, killed at rate \( \frac{1}{2}|B| \), started in \( Y \) with increments independent of \( Y \). Furthermore \( \tau \) is the killing time of \( \overline{B}^Y \) as in Definition 5.3. Then,

\[
Y \overset{d}{=} \overline{B}^Y_{\tau-}U,
\]

where \( U \) is independent and uniformly distributed on \((0, 1)\), since the ratchet jumps to a uniformly distributed point in \([0; \overline{B}^Y_{\tau-}]\).
We denote by \( f_H \) the density of a random variable \( H \). The requirement \[5.30\] implies

\[
f_Y(z) = f_{B^*U}(z) = \int_0^\infty f_Y(x) \int_0^\infty f_{B_x}(u) \frac{1}{u} \frac{z}{u} du dx
= \int_0^\infty f_Y(x) \int_0^\infty f_{B_x}(u) \cdot \frac{1}{u} du dx
= \int_0^\infty f_Y(x) \int_0^\infty G(x, u) du dx
= \frac{1}{w} \int_0^\infty f_Y(x) \left[ \mathbb{1}_{\{z < x\}} \left( \int_x^\infty \phi(x) \psi(u) du + \int_x^\infty \psi(x) \phi(u) du \right) + \mathbb{1}_{\{z \geq x\}} \int_x^\infty \psi(x) \phi(u) du \right] dx
= \frac{1}{w} \left[ \int_z^\infty f_Y(x) \phi(x) \int_x^\infty \psi(u) du dx + \int_z^\infty f_Y(x) \psi(x) \int_x^\infty \phi(u) du dx + \int_0^z f_Y(x) \psi(x) \int_x^\infty \phi(u) du dx \right]
=: \frac{1}{w} \left[ I_1(z) + I_2(z) + I_3(z) \right].
\]

We differentiate with respect to \( z \) and obtain

\[
I_1'(z) = -\psi(z) \int_z^\infty f_Y(x) \phi(x) dx,
I_2'(z) = f_Y(z) \psi(z) \int_z^\infty \phi(u) du,
I_3'(z) = -\phi(z) \int_0^z f_Y(x) \psi(x) dx - f_Y(z) \psi(z) \int_z^\infty \phi(u) du.
\]

Thus,

\[
f_Y'(z) = -\frac{1}{w} \left[ \psi(z) \int_z^\infty f_Y(x) \phi(x) dx + \phi(z) \int_0^z f_Y(x) \psi(x) dx \right]. \tag{5.31}
\]

At this point we see that \( f_Y \) is strictly decreasing on \([0, \infty)\) since \( \phi \) and \( \psi \) are positive. We differentiate this equation two more times and obtain

\[
f_Y''(z) = -\frac{1}{w} \left[ \psi'(z) \int_z^\infty f_Y(x) \phi(x) dx - \psi(z) f_Y(z) \phi(z)
+ \phi'(z) \int_0^z f_Y(x) \psi(x) dx + \psi(z) f_Y(z) \phi(z) \right] \tag{5.32}
= -\frac{1}{w} \left[ \psi'(z) \int_z^\infty f_Y(x) \phi(x) dx + \phi'(z) \int_0^z f_Y(x) \psi(x) dx \right]
\]
and
\[ f''_Y(z) = -\frac{1}{w} \left[ \psi''(z) \int_z^\infty f_Y(x) \phi(x) \, dx - \psi'(z) f_Y(z) \phi(z) \right. \]
\[ \left. + \phi''(z) \int_0^z f_Y(x) \psi(x) \, dx + \phi'(z) f_Y(z) \psi(z) \right] \]
\[ = -\frac{1}{w} \left[ z \psi'(z) \int_z^\infty f_Y(x) \phi(x) \, dx + z \phi'(z) \int_0^z f_Y(x) \psi(x) \, dx \right. \]
\[ \left. - f_Y(z) (\psi'(z) \phi(z) - \phi'(z) \psi(z)) \right] \]  
(5.33)

Here we have used (5.31) for the next to last equality. Integrating the last equation we get
\[ f''_Y(z) = z f_Y(z) + c \]  
(5.34)

for some constant c. Using \( \psi'(0) = 0 \), which holds by the definition of \( \psi \), and (5.32) we obtain
\( c = f''_Y(0) = 0 \). From (5.31) we know that \( f_Y \) is decreasing. The unique (up to a constant factor) decreasing non-negative solution of (5.34) with \( c = 0 \) is given by \( \phi = Ai \). Thus, using (5.8), we have
\[ f_Y(z) = 3\phi(z) = 3Ai(z), \, z \geq 0. \]  
(5.35)

The expectation of \( Y \) is given by
\[ \mathbb{E}_\pi[Y_1] = \int y \pi(dy) = \int y f_Y(y) \, dy = 3 \int_0^\infty y Ai(y) \, dy = 3 \int_0^\infty Ai''(y) \, dy = -3Ai'(0). \]  
(5.36)

Using (5.11) we have
\[ \mathbb{E}_\pi[\tau_1] = \mathbb{E}[\mathbb{E}_Y[\tau]] = 2\pi \int_0^\infty f_Y(x) (Gi(x) + 3^{-1/2}Ai(x)) \, dx. \]  
(5.37)

For evaluating the last integral we need
\[ \int_0^\infty Ai^2(x) \, dx = \left[ xAi^2(x) - (Ai'(x))^2 \right]_0^\infty = (Ai'(0))^2, \]
\[ \int_0^\infty Ai^2(x) \int_0^x Bi(u) \, du \, dx = \int_0^\infty Bi(u) \int_u^\infty Ai^2(x) \, dx \, du \]
\[ = \int_0^\infty Bi(u) \left[ xAi^2(x) - (Ai'(x))^2 \right]_u^\infty \, du \]
\[ = \int_0^\infty -uAi^2(u)Bi(u) + (Ai'(u))^2Bi(u) \, du, \]
\[ \int_0^\infty Ai(x)Bi(x) \int_x^\infty Ai(u) \, du \, dx = \int_0^\infty Ai(u) \int_0^u Ai(x)Bi(x) \, dx \, du \]
\[ = \int_0^\infty Ai(u) \left[ xAi(x)Bi(x) - Ai'(x)Bi'(x) \right]_0^u \, dx \]
\[ = \int_0^\infty uAi^2(u)Bi(u) - Ai(u)Ai'(u)Bi'(u) + Ai(u)Ai'(0)Bi'(0) \, du. \]
Plugging \((5.35)\) into \((5.37)\) and combining the last equations, we obtain, using \((5.4)\), \((5.8)\) and the fact that the Wronskian \((5.5)\) does not depend on \(x\),

\[
\mathbb{E}_n[\tau_1] = 6\pi \left( \int_0^\infty Ai(x)Gi(x) \, dx + 3^{-1/2} \int_0^\infty (Ai(x))^2 \, dx \right)
\]

\[
= 6\pi \left( \int_0^\infty Ai'(u)(Ai'(u)Bi(u) - Bi'(u)Ai(u)) \, du + Ai'(0)Bi'(0) \int_0^\infty Ai(u) \, du + 3^{-1/2}(Ai'(0))^2 \right)
\]

\[
= 6\pi \left( - w \int_0^\infty Ai'(u) \, du + \frac{1}{3} Ai'(0)Bi'(0) - \frac{1}{3} Ai'(0)Bi'(0) \right)
\]

\[
= 6 Ai(0).
\]

\[
\square
\]

6 Regeneration structure

In this section we show that the Brownian ratchet has a renewal structure and can be seen as a cumulative process with a remainder term (see Definition 6.2). In order to use this structure we will have to bound moments of several quantities, i.e. of the time between renewal points, (Proposition 6.4), and the increment of the Brownian ratchet at renewal points (Proposition 6.6). As in the last section, we fix \(\gamma = \frac{1}{2}\).

6.1 The Brownian ratchet as a cumulative process

Remark 6.1 (Cumulative processes). We recall a definition of cumulative processes from Roginsky (1994). Let \((T_n, V_n)_{n \geq 1}\) be a sequence of bivariate iid random variables with \(T_1 > 0\) almost surely. The times \(T_1, T_2, \ldots\) are called regeneration times. Define a renewal process \((S_{M_t})_{t \geq 0}\) by

\[
M_t = \min \left\{ n : \sum_{i=1}^n T_i > t \right\} \quad \text{and} \quad S_n = \sum_{i=1}^n V_i.
\]

The process \((S_{M_t})_{t \geq 0}\) is called a type A cumulative process. Cumulative processes are well studied (see e.g. Smith (1955), Roginsky (1994), and references therein). Most importantly, it is known that finite second moments of \(T_1\) and \(V_1\) are sufficient for the strong Law of Large Numbers and the Central Limit Theorem for \((S_{M_t})_{t \geq 0}\).

Definition 6.2 (The Brownian ratchet as a cumulative process). Given a Brownian ratchet \((\mathcal{X}, \mathcal{R}) = (X_t, R_t)_{t \geq 0}\) with \((X_0, R_0) = (x,0)\), \(x \geq 0\), we define a sequence of regeneration times using times when \(X_t = R_t\) as well as jump times of \(\mathcal{R}\). We set

\[
\rho_0 := \inf \{ t \geq 0 : X_t = R_t \},
\]

\[
\bar{\rho}_0 := \inf \{ t > \rho_0 : R_t \neq R_{t-} \},
\]

and for \(n = 1, 2, \ldots\), using \(R_{t-} = \lim_{s \to t, s < t} R_s\)

\[
\rho_n := \inf \{ t > \bar{\rho}_{n-1} : X_t = R_t \},
\]

\[
\bar{\rho}_n := \inf \{ t > \rho_n : R_t \neq R_{t-} \}
\]
such that \( \rho_0 \leq \tilde{\rho}_0 \leq \rho_1 \leq \tilde{\rho}_1 \leq \ldots \). Note that if \( x = 0 \), then \( \rho_0 = 0 \) and \( X_{\rho_0} = 0 \). It is clear that \( (\rho_n - \rho_{n-1}, X_{\rho_n} - X_{\rho_{n-1}})_{n \geq 1} \) is a sequence of bivariate iid random variables. We define

\[
M_t := \min\{n : \rho_n > t\},
\]

\[
S_n := \sum_{i=1}^{n} (X_{\rho_i} - X_{\rho_{i-1}}),
\]

\[
A_t := X_{\rho_0} + X_t - X_{\rho_{M_t}}.
\]

Then we have

\[
X_t = S_{M_t} + A_t, \quad (6.3)
\]

that is, \( \mathcal{X} \) is a type A cumulative process with remainder \( A_t \).

**Remark 6.3** (The regenerative structure in the proof of Theorem 1 and 2). As mentioned in Remark 6.1, finite second moments of \( \rho_1 - \rho_0 \) and \( X_{\rho_1} - X_{\rho_0} \) are sufficient for the strong Law of Large Numbers and the Central Limit Theorem for \( S_{M_t} \). To see that the same holds for \( \mathcal{X} \), showing Theorems 1 and 2, we shall show in Proposition 6.7 that the remainder divided by \( \sqrt{t} \) converges to 0 almost surely as \( t \to \infty \).

### 6.2 Tail estimates and moment bounds

In the rest of the section, we denote the law of a Brownian ratchet \( (\mathcal{X}, \mathcal{R}) \), started in \( (x, 0) \) by \( \mathbb{P}_x \) (with expectation operator \( \mathbb{E}_x \)). We use the notation of Definition 6.2. We can use the Law of Large Numbers and the Central Limit Theorem for cumulative processes given \( \rho_1 - \rho_0 \) and \( X_{\rho_1} - X_{\rho_0} \) have a finite second moment. These are the main results of this section (Proposition 6.4 and 6.6).

**Proposition 6.4** (Moment bounds for \( \rho_1 - \rho_0 \)). Let \( \rho_0, \rho_1 \) be as in (6.1) and (6.2). Then,

\[
\mathbb{E}_x[(\rho_1 - \rho_0)^2] < \infty. \quad (6.4)
\]

The proof is based on Lemma 5.3 and the following result.

**Lemma 6.5.** Let \( \sigma \) be the stopping time when the Brownian ratchet hits the moving reflection boundary for the first time, i.e.

\[
\sigma := \inf\{t > 0 : X_t = R_t\}. \quad (6.5)
\]

Then, there exists a finite positive constant \( C \) such that

\[
\sup_x \mathbb{E}_x[\sigma^2] < C.
\]

**Proof of Proposition 6.4.** Clearly, \( \rho_1 - \rho_0 = (\hat{\rho}_0 - \rho_0) + (\rho_1 - \hat{\rho}_0) \). Since \( \hat{\rho}_0 - \rho_0 \buildrel d \over = \tau \) with \( \tau \) from Definition 5.3, finiteness of the second moment of the first summand follows immediately from Lemma 5.5. Moreover, given \( X_{\hat{\rho}_0} - R_{\hat{\rho}_0} = z \), we have that \( \rho_1 - \hat{\rho}_0 \) is distributed as \( \sigma \) for a Brownian ratchet started in \( z \) with \( \sigma \) from (6.5). Hence, finiteness of the second moment of the second summand follows from Lemma 6.5. Thus, the proposition is proved.
Proof of Lemma 6.3. Recall the jump times $\tau_1, \tau_2, \ldots$ of $R$. We consider the process $Z := (Z_{t+\sigma})_{t \geq 0}$ for $Z_t := X_t - R_t$ which is started in $x$, and has discontinuities at $\tau_1, \tau_2, \ldots$. Note that $\sigma$ is the time at which $Z_t$ hits 0 for the first time.

Note that $Z$ is an autonomous Markov process: locally it behaves like Brownian motion, reflected at 0 and each time there is a point in a Poisson point process on $[0; \infty) \times \mathbb{R}$ in $[0, Z_t]$ the process $Z_t$ restarts from this Poisson point, i.e. at discontinuity points the process $Z_t$ jumps down.

Our goal is to bound $\sigma$ from above by a simpler random variable $\hat{\sigma}$. To this end we couple $Z$ to a process $\hat{Z} = (\hat{Z}_t)_{t \geq 0}$ which starts in $x$ and has the following dynamics: the local increments of $\hat{Z}$ follow exactly those of $Z$, i.e. $\hat{Z}$ and $Z$ use the same underlying Brownian motion. At jump times $\tau_n$ of $Z$ with $Z_{\tau_n} \leq 1$, we set $\hat{Z}_{\tau_n} = 1$. In addition, $\hat{Z}$ jumps at rate $(1 - Z_t)^+$ to 1. At jump times $\tau_n$ with $Z_{\tau_n} > 1$, the process $\hat{Z}$ does not jump. We denote the first hitting time of $\hat{Z}$ of 0 by $\hat{\sigma}$. By this coupling, we have achieved the following:

1. At joint jump times $\tau$ of $\hat{Z}$ and $Z$ we have $Z_\tau \leq \hat{Z}_\tau = 1$. Jump times $\tau$ exclusive to $\hat{Z}$ must satisfy $Z_\tau \leq 1 = \hat{Z}_\tau$. For a jump time $\tau$ exclusive to $\hat{Z}$, and if $Z_{\tau_-} \leq \hat{Z}_{\tau_-}$, we have $Z_\tau \leq Z_{\tau_-} \leq \hat{Z}_{\tau_-} = \hat{Z}_\tau$. In particular, for all $t \geq 0$ we have $Z_t \leq \hat{Z}_t$.

2. The process $\hat{Z}$ jumps at rate 1. If $\tau$ is a jump time of $\hat{Z}$, then $\hat{Z}_t = 1$.

In particular, $\sigma \leq \hat{\sigma}$, almost surely, by 1., and it suffices to show that $\hat{\sigma}$ has a finite second moment. Set $\hat{\tau}_0 = 0$ and denote the jump times of $\hat{Z}$ by $\hat{\tau}_1, \hat{\tau}_2, \ldots$. In addition, let $(B_t)_{t \geq 0}$ be Brownian motion starting in 1, set

$$\tau = \inf \{ t > 0 : B_t = 0 \}$$

and

$$T_i = \hat{\tau}_{i+1} - \hat{\tau}_i, \; i \geq 0.$$ 

It is clear that $T_0, T_1, \ldots$ are iid random variables with rate one independent of $\hat{Z}_t$. Independent of initial position $\hat{Z}_0$ the process $\hat{Z}$ restarts from 1 at time $T_1$. Define $p := \mathbb{P}_1(\tau < T_1)$ and let $M$ be geometrically distributed with parameter $p$ independent of $T_1, T_2, \ldots$, i.e. $\mathbb{P}(M = n) = (1 - p)p^n, \; n \geq 0$. Note that $M$ is the number of fail trials for the process $\hat{Z}$ to reach 0, started in 1, before the next jump. It follows that

$$\hat{\sigma} \leq \sum_{i=0}^{M} T_i + \bar{\tau},$$

where $\bar{\tau}$ is distributed as $\tau$, conditioned to be smaller than $T_1$ and hence

$$\mathbb{E}_1[\bar{\tau}^2] = \frac{\mathbb{E}_1[\tau^2, \tau \leq T_1]}{\mathbb{P}_1(\tau < T_1)} \leq \frac{\mathbb{E}_1[T_1^2]}{p} = \frac{2}{p}.$$ 

Combining these results,

$$\mathbb{E}_x[\sigma^2] \leq \mathbb{E}_x[\hat{\sigma}^2] \leq \mathbb{E}[\left( \mathbb{E}\left( \sum_{i=0}^{M} T_i + \bar{\tau} \bigg| M \right) \right)^2] \leq \mathbb{E}\left( (2(M + 1) + (M + 1)M + \frac{1}{p}(3 + M)) \right) =: C < \infty.$$

The right hand side is independent of $x$, and we are done. \qed
Proposition 6.6 (Moment bounds for $X_{\rho_1} - X_{\rho_0}$). We have
\[
E_x[X_{\rho_0}^2] < \infty, \tag{6.6}
\]
\[
E_x[(X_{\rho_1} - X_{\rho_0})^2] < \infty \tag{6.7}
\]
and for each $t \geq 0$
\[
E_x[(X_t - X_{\rho_{Mt}})^2] \leq 2(E_x[(X_{\rho_1} - X_{\rho_0})^2] + E_x[X_{\rho_0}^2]). \tag{6.8}
\]

Proof. Note that $|X_{\rho_0}| \sim |B_{\rho_0}|$, which already shows that
\[
E_x[X_{\rho_0}^2] = E_x[B_{\rho_0}^2] = E_x[\rho_0] < \infty
\]
by the second Wald identity where we have used that $E_x[\rho_0] < \infty$ by Lemma 6.5.

Now, using the strong Markov property at time $\rho_0$ we have
\[
E_x[(X_{\rho_1} - X_{\rho_0})^2] = E_0[X_{\rho_0}^2] = E_0[B_{\rho_0}^2] = E_0[\rho_1] = E_x[\rho_1 - \rho_0] < \infty,
\]
where we have again used Wald’s second identity that is applicable by Proposition 6.4. This proves (6.7).

It remains to show (6.8). We have
\[
E_x[(X_t - X_{\rho_{Mt}})^2] = E_x[(X_t - X_{\rho_{Mt}})^21_{\{t<\rho_0\}}] + E_x[(X_t - X_{\rho_{Mt}})^21_{\{t\geq\rho_0\}}]. \tag{6.9}
\]
The first summand on the right hand side can be estimated as follows
\[
E_x[(X_t - X_{\rho_{Mt}})^21_{\{t<\rho_0\}}] \leq E_x[X_t^21_{\{t<\rho_0\}}] + E_x[X_{\rho_0}^2] \leq E_x[X_t^2] + E_x[X_{\rho_0}^2] \leq 2E_x[X_{\rho_0}^2], \tag{6.10}
\]
where for the last inequality we used the fact that the stopped process $(X_{\rho_{Mt}}^2)$ is a submartingale (as usual $a \wedge b = \min\{a, b\}$). Using a similar argument we have for the second summand on the right hand side of (6.9)
\[
E_x[(X_t - X_{\rho_{Mt}})^21_{\{t\geq\rho_0\}}] = E_0[(X_t - X_{\rho_1})^21_{\{t<\rho_1\}}] \leq E_0[X_t^21_{\{t<\rho_1\}}] + E_0[X_{\rho_1}^2] \leq E_0[X_t^2] + E_0[X_{\rho_1}^2] \leq 2E_0[X_{\rho_1}^2] = 2E_x[(X_{\rho_1} - X_{\rho_0})^2]. \tag{6.11}
\]
Combining (6.9), (6.10) and (6.11) we obtain (6.8).

Proposition 6.7 (Asymptotics of $A_t$ and $X_t - R_t$). We have
\[
\frac{A_t}{\sqrt{t}} \to 0 \quad \text{and} \quad \frac{X_t - R_t}{\sqrt{t}} \to 0 \quad \text{a.s. as } t \to \infty.
\]

Proof. The proof of the first assertion is an adaptation of the proof of Lemma 8 in [Smith 1955, p. 26]. In view of (6.6) it is enough to prove the first assertion for $X_0 = 0$. Then we need to show that
\[
\frac{X_t - X_{\rho_{Mt}}}{\sqrt{t}} \to 0 \quad \text{a.s. as } t \to \infty. \tag{6.12}
\]
We set
\[ Y_n = \sup_{t \in [\rho_{n-1}, \rho_n]} |X_t - X_{\rho_n}|. \]
Then \( Y_1, Y_2, \ldots \) is a sequence of independent identically distributed random variables with finite second moments. To prove this we note that
\[ Y_1 \leq X_{\rho_1} + \sup_{t \geq 0} |X_{t \wedge \rho_1}|. \]
By Doob’s maximal inequality
\[ \mathbb{E}_0 \left[ \sup_{t \geq 0} |X_{t \wedge \rho_1}|^2 \right] \leq 4 \mathbb{E}_0[X_{\rho_1}^2]. \]
Thus, finiteness of second moments of \( Y_1 \) follows from (6.7). Now
\[ \frac{|X_t - X_{\rho_{M_t}}|}{\sqrt{t}} \leq \frac{Y_{M_t}}{\sqrt{t}} = \frac{Y_{M_t}}{\sqrt{M_t}} \sqrt{t}/\sqrt{t}. \] (6.13)
Note that by Theorem 1 in [Doob (1948)] we have
\[ \lim_{t \to \infty} \frac{M_t}{t} = \frac{1}{\mathbb{E}[\rho_1 - \rho_0]} > 0, \text{ a.s.} \] (6.14)
Using Lemma 7 in [Smith (1955), p. 26] we obtain \( Y_{M_t}/\sqrt{M_t} \to 0 \) almost surely and therefore (6.12) holds.

For the second assertion we have
\[ \frac{(X_t - R_t)1_{\{t < \rho_0\}}}{t} \leq \frac{X_{t \wedge \rho_0}}{t} \to 0 \text{ a.s. as } t \to \infty. \]
For \( t \in [\rho_{n-1}, \rho_n], n \geq 1 \) we have
\[ 0 \leq X_t - R_t \leq Y_n. \]
Thus,
\[ \frac{(X_t - R_t)1_{\{t \geq \rho_0\}}}{t} \to 0 \text{ a.s. as } t \to \infty \]
follows by the same argument as before. \( \square \)

7 Proofs of Theorem 1 and 2

In the case \( \gamma = 0 \), Theorem 1 is true since \( X_t \overset{d}{=} |B_t| \) for a Brownian motion, started in \( x \). Hence, we can assume \( \gamma > 0 \) in the rest of the proof.

We make use of the regeneration structure, set out in Definition 6.2. We set
\[ r = \mathbb{E}_x[\rho_1 - \rho_0], \quad \mu = \mathbb{E}_x[X_{\rho_1} - X_{\rho_0}] \quad \text{and} \quad \beta^2 = \text{Var}_x \left[ X_{\rho_1} - X_{\rho_0} - \frac{(\rho_1 - \rho_0)\mu}{r} \right]. \] (7.1)
Here, \( r, \mu \) and \( \beta^2 \) are independent of \( x \) due to the regeneration structure. By Propositions 6.4 and 6.6 the time and space increments \( \rho_1 - \rho_0 \) and \( X_{\rho_1} - X_{\rho_0} \) have finite second moments. As in (6.14) we have \( \lim_{t \to \infty} \frac{M_t}{t} = \frac{1}{r} \) almost surely.
From Proposition 6.7 (6.14) and Law of Large Numbers applied to $S_n$ it follows that
\[
\frac{X_t}{t} = \frac{A_t}{t} + \frac{M_t S_{M_t}}{t M_t} \to \frac{\mu}{r} \text{ a.s. as } t \to \infty.
\]

We will compute $\frac{\mu}{r}$ using the ratio limit theorem for Harris recurrent Markov chains (see e.g. Revuz, 1984). Let $\pi$ denote the invariant distribution for $(Y, W, \eta)$ determined by Propositions 5.6 and 5.7. To show that this Markov chain is Harris recurrent we have to show that each Borel subset of $\mathbb{R}^3_{\geq 0}$, say $B$, having positive Lebesgue measure is visited infinitely often with probability 1. We cannot apply the Borel-Cantelli lemma directly, because the events $\{(Y_n, W_n, \eta_n) \in B\}_{n \geq 1}$ are not independent. But events lying between different regeneration points are independent. For $n \geq 0$ we set
\[
B_n = \{\exists m \geq 1 \text{ s.th. } \rho_{n-1} < \tau_{m-1}, \tau_m < \rho_n \text{ and } (Y_m, W_m, \eta_m) \in B\}.
\]

The events $B_n$ are independent and have the same positive probability. Thus, applying the Borel-Cantelli lemma to the sequence $(B_n)_{n \geq 0}$ we obtain
\[
P_x((Y_n, W_n, \eta_n) \in B \text{ i.o.}) \geq P_x(B_n \text{ i.o.}) = 1.
\]

Now using the ratio limit theorem we obtain that
\[
\frac{\mu}{r} = \lim_{t \to \infty} \frac{X_t}{t} = \lim_{t \to \infty} \frac{R_t}{t} = \lim_{n \to \infty} \frac{R_{\tau_n}}{\tau_n} = \lim_{n \to \infty} \frac{\sum_{k=1}^n W_k}{\sum_{k=1}^n \eta_k} = \frac{E_\pi[W_1]}{E_\pi[\eta_1]} \text{ a.s. } n \to \infty,
\]

where for the second equality we have used Proposition 6.7. We recall that $E_\pi[W_1] = E_\pi[Y_1]$. Furthermore, recalling that we have used $\gamma = \frac{1}{2}$ in Section 5 and using (5.3), (5.4) as well as Proposition 5.8 we obtain
\[
\frac{\mu}{r} = \frac{E_\pi[Y_1]}{E_\pi[\tau_1]} = -(2\gamma)^{1/3} \frac{3A'(0)}{6A(0)} = \frac{\gamma^{1/3} 3^{1/3} \Gamma(2/3)}{2^{2/3} \Gamma(1/3)}.
\]

by the scaling property, Proposition 4.5. This proves Theorem 1.

Using the CLT for cumulative processes (see e.g. Smith, 1955; Roginsky, 1994) and Proposition 6.7 we obtain that for all $x \in \mathbb{R}$
\[
\lim_{t \to \infty} \mathbb{P}\left(\frac{X_t - t\mu/r}{\beta(t/r)^{1/2}} \leq x\right) = \lim_{t \to \infty} \mathbb{P}\left(\frac{A_t}{\beta(t/r)^{1/2}} + \frac{S_{M_t} - t\mu/r}{\beta(t/r)^{1/2}} \leq x\right)
\]
\[
= \lim_{t \to \infty} \mathbb{P}\left(\frac{S_{M_t} - t\mu/r}{\beta(t/r)^{1/2}} \leq x\right) = \Phi(x),
\]

where $\Phi$ denotes the distribution function of the standard normal distribution, and $\beta^2$ and $r$ are as defined in (7.1). Hence, Theorem 2 holds for $\sigma = \beta/\sqrt{r}$.

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