A DIOPHANTINE PROBLEM WITH PRIME VARIABLES

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Abstract. We study the distribution of the values of the form \(\lambda_1 p_1^{k_1} + \lambda_2 p_2^{k_2} + \lambda_3 p_3^{k_3}\), where \(\lambda_1, \lambda_2\) and \(\lambda_3\) are non-zero real number not all of the same sign, with \(\lambda_1/\lambda_2\) irrational, and \(p_1, p_2\) and \(p_3\) are prime numbers. We prove that, when \(1 < k < 4/3\), these value approximate rather closely any prescribed real number.

Dedicated to Prof. R. Balasubramanian on the occasion of his 60th birthday

1. Introduction

The problem that we want to study in this paper can be stated in general as follows: given \(r\) non-zero real numbers \(\lambda_1, \ldots, \lambda_r\), and positive real numbers \(k_1, \ldots, k_r\), approximate a given real number \(\varpi\) by means of values of the form

\[
\lambda_1 p_1^{k_1} + \cdots + \lambda_r p_r^{k_r},
\]

(1)

where \(p_1, \ldots, p_r\) denote primes. If \(\rho = 1/k_1 + 1/k_2 + \cdots + 1/k_r\) is “small” the goal is to show that

\[
|\lambda_1 p_1^{k_1} + \lambda_2 p_2^{k_2} + \cdots + \lambda_r p_r^{k_r} - \varpi| < \eta
\]

has infinitely many solutions for every fixed \(\eta > 0\). If \(\rho\) is “large” one expects to be able to prove the stronger result that, in fact, some \(\eta \to 0\) is admissible in (2): more precisely, it should be possible to take \(\eta\) as a small negative power of \(\max_j p_j\). The number of variables \(r\) also plays a role, of course. Some hypothesis on the irrationality of at least one ratio \(\lambda_i/\lambda_j\) is necessary, and also on signs, if one wants to approximate to all real numbers and not only some proper subset. We will make everything precise in due course.

Many such results are known, with various types of assumptions and conclusions, and we now give a brief description of a few among them. Vaughan [15] has \(r = 3\) and \(k_j = 1\) for all \(j\), the non-zero coefficients \(\lambda_j\) not all of the same sign with \(\lambda_1/\lambda_2\) irrational. In this case \(\eta\) is essentially \((\max_j p_j)^{-1/10}\). The paper [16] contains more elaborate results of the same kind, with the same integral exponent \(k \geq 1\) for all primes. Baker and Harman [11] and Harman [7] have a result similar to Vaughan’s [15] with \(\eta = (\max_j p_j)^{-1/6}\) and \(\eta = (\max_j p_j)^{-1/5}\) respectively. This has been recently improved to \(\eta = (\max_j p_j)^{-2/9}\) by Matomäki [12].

The papers by Brüdern, Cook and Perelli [2], Cook [3], Cook and Fox [4], Harman [8], Cook and Harman [5] all deal with the number of “exceptional” real numbers \(\varpi\) that can not be well approximated by values of type (1), but in this case there are many differences with the results quoted above. First, \(\eta\) does not depend on the primes \(p_j\) but rather on \(\varpi\) (it is a small negative power of \(\varpi\), in fact), but, in their setting, this is essentially equivalent to the alternative statement as we shall see presently. It is more important, of course, to define carefully what “exceptional” means. Actually, the results apply to suitable sequences of positive real numbers \(\varpi_n\) with limit \(+\infty\), and it is shown that the number of exceptional elements in the sequence, that is, elements that can not be approximated within the prescribed precision, is small in a strong quantitative sense.

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The assumption is that the coefficients $\lambda_j$ are all positive, which is not a restriction in this case, that the ratio $\lambda_1/\lambda_2$ is irrational and algebraic, and that $k_j$ is the same positive integer $k$ for all $j$. The assumption on $\lambda_1/\lambda_2$ is needed to deal with some exponential sums on the so-called “minor” arc.

Tolev [13] has $r = 3$, the coefficients $\lambda_j$ all equal to 1 and all the exponents $k_j$ equal to a constant $k \in (1, 15/14)$. The conclusion is that all sufficiently large real numbers $\varpi$ can be approximated, with $\eta$ a negative power of $\varpi$.

Parsell [13] considers two primes and a large number of powers of 2, so that in a sense $\rho = 2 + \varepsilon$, but $r$ is large and $\eta$ is arbitrary but fixed. This has been improved in [13] by the present authors, who showed that a smaller number of powers of 2 is needed. In a similar vein, Languasco and Zaccagnini [9] have the corresponding result with one prime, two squares of primes and a large number of powers of 2. Finally, the present authors [10] have a result with one prime, two squares of primes and a large number of powers of 2. In all of these papers, it is assumed that $\rho = 5/2$, $r = 4$ and $\eta = (\max_j p_j)^{-1/18 + \varepsilon}$, while in [11] they deal with one prime, the square of a prime and the $k$-th power of a prime with $k \in (1, 33/29)$ and $\eta = (\max_j p_j)^{-3(3-29k)/(72k) + \varepsilon}$. In all of these papers, it is assumed that one, carefully chosen, among the ratios $\lambda_i/\lambda_j$ is irrational.

Our main result is the following Theorem.

**Theorem 1.** Let $1 < k < 4/3$ be a real number and assume that $\lambda_1$, $\lambda_2$, and $\lambda_3$ are non-zero real numbers, not all of the same sign and that $\lambda_1/\lambda_2$ is irrational. Let $\varpi$ be any real number. For any $\varepsilon > 0$ the inequality

$$|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3^k - \varpi| \leq (\max_j p_j)^{3/10 - 2/(5k) + \varepsilon}$$

has infinitely many solutions in prime variables $p_1$, $p_2$ and $p_3$.

In the notation above, we have $r = 3$, $\rho = 2 + 1/k$ and $\eta = (\max_j p_j)^{3/10 - 2/(5k) + \varepsilon}$. We use the variant of the circle method introduced by Davenport and Heilbronn [6] to deal with these problems, where the variables are not necessarily integral. The following lemmas are the two key ingredients of the proof. They relate a suitable $L^2$-average of the error on the “major” arc to a generalized version of the so-called Selberg integral, which is a well-known and widely used tool in this context: see [2], [9], [10]. The same argument, with comparatively minor changes, can be used with $\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3^2$ (with the same hypothesis on the ratio $\lambda_1/\lambda_2$ and on signs as above), and $\eta = (\max_j p_j)^{-1/18 + \varepsilon}$.

Before the statement, we need to define the relevant quantities, beginning with the exponential sums. As usual, we write $e(\alpha) = e^{2\pi i \alpha}$. For any real $k \geq 1$ we let

$$S_k(\alpha) = \sum_{\delta X \leq p \leq X} \log p \ e(p^k \alpha) \quad \text{and} \quad U_k(\alpha) = \sum_{\delta X \leq n \leq X} e(n^k \alpha)$$

where $\delta$ is a small, fixed positive constant, which may depend on the coefficients $\lambda_j$. Then we set

$$J_k(X, h) = \int_{X}^{2X} \left( \theta((x+h)^{1/k}) - \theta(x^{1/k}) - ((x+h)^{1/k} - x^{1/k}) \right)^2 dx.$$ (5)

This is the generalized version of the Selberg integral referred to above: the classical function is $J_1(X, h)$.

**Lemma 1.** Let $k \geq 1$ be a real number. For $0 < Y \leq 1/2$ we have

$$\int_{-Y}^{Y} |S_k(\alpha) - U_k(\alpha)|^2 d\alpha \ll_k \frac{X^{2k-2} \log^2 X}{Y} + Y^2 X + Y^2 J_k \left( X, \frac{1}{2Y} \right),$$
where $J_k(X, h)$ is defined in \([5]\).

This is Theorem 1 of \([11]\).

**Lemma 2.** Let $k \geq 1$ be a real number and $\varepsilon$ be an arbitrarily small positive constant. There exists a positive constant $c_1 = c_1(\varepsilon)$, which does not depend on $k$, such that

$$J_k(X, h) \ll_k h^2 X^{2/k-1} \exp \left( - c_1 \left( \frac{\log X}{\log \log X} \right)^{1/3} \right)$$

uniformly for $X^{1-5/(6k+\varepsilon)} \leq h \leq X$.

This is the special case $C = 12/5$ of Theorem 2 of \([11]\).

### 2. Proof of Theorem 1

In order to prove that \([3]\) has infinitely many solutions, it is sufficient to show the existence of an increasing sequence $X_n$ with limit $+\infty$ such that \([3]\) has at least a solution with $\max_j p_j \in [\delta X_n, X_n]$. This sequence actually depends on rational approximations for $\lambda_1/\lambda_2$: more precisely, we recall that there are infinitely many pairs of integers $a$ and $q$ such that $(a, q) = 1$, $q > 0$ and

$$\frac{|\lambda_1|}{\lambda_2} - \frac{a}{q} \leq \frac{1}{q^2}. $$

We take the sequence $X = q^{5k/(k+2)}$ (dropping the useless suffix $n$) and then, as customary, define all of the circle-method parameters in terms of $X$. We may obviously assume that $q$ is sufficiently large. The choice of the exponent $5k/(k+2)$ is justified in the discussion following the proof of Lemma 4. As usual, we approximate to $S_k$ using the function

$$T_k(\alpha) = \int_{(\delta X)^{1/k}}^{X^{1/k}} e(t^k \alpha) \, dt$$

and notice the simple inequality

$$T_k(\alpha) \ll_{k, \delta} X^{1/k-1} \min(X, |\alpha|^{-1}). \quad (6)$$

Since the variables are not integers, we cannot count exact hits as in the standard applications of the circle method, only near misses, so that we need some measure of proximity. For $\eta > 0$, we detect solutions of \([3]\) by means of the function

$$\hat{K}_\eta(\alpha) = \max(0, \eta - |\alpha|),$$

which, as the notation suggests, is the Fourier transform of

$$K_\eta(\alpha) = \left( \frac{\sin(\pi \eta \alpha)}{\pi \alpha} \right)^2$$

for $\alpha \neq 0$, and, by continuity, $K_\eta(0) = \eta^2$. This relation transforms the problem of counting solutions of the inequality \([3]\) into estimating suitable integrals. We recall the trivial, but crucial, property

$$K_\eta(\alpha) \ll \min(\eta^2, |\alpha|^{-2}). \quad (7)$$

When $X$ is an interval, a half line, or the union of two such sets we let

$$I(\eta, \varpi, X) = \int_X S_1(\lambda_1 \alpha) S_1(\lambda_2 \alpha) S_k(\lambda_3 \alpha) K_\eta(\alpha) e(-\varpi \alpha) \, d\alpha.$$
The starting point of the method is the observation that

\[ I(\eta, \varpi, \mathbb{R}) = \sum_{\delta X \leq p_1, p_2 \leq X} \log p_1 \log p_2 \log p_3 \int_{\mathbb{R}} K_\eta(\alpha) e\left((\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3^k - \varpi)\alpha\right) \, d\alpha \]

\[ = \sum_{\delta X \leq p_1, p_2 \leq X} \log p_1 \log p_2 \log p_3 \max(0, \eta - |\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3^k - \varpi|) \]

\[ \leq \eta (\log X)^3 \mathcal{N}(X), \]

where \( \mathcal{N}(X) \) denotes the number of solutions of the inequality (3) with \( p_1, p_2 \in [\delta X, X] \) and \( p_3^k \in [\delta X, X] \). In other words, \( I(\eta, \varpi, \mathbb{R}) \) provides a lower bound for the quantity that we are interested in.

We now give the definitions that we need to set up the method. More definitions will be given at appropriate places later. We let \( P = P(X) = X^{5/6k - \varepsilon}, \eta = \eta(X) = X^{3/10 - 2/(5k) + \varepsilon} \), and \( R = R(X) = \eta^{-2}(\log X)^{3/2} \). The choice for \( P \) is justified at the end of (10); the one for \( \eta \) at the end of (17) and the one for \( R \) at the end of (18). See also (13) for a fuller discussion. We now decompose \( \mathbb{R} \) as \( \mathfrak{M} \cup \mathfrak{m} \cup \mathfrak{t} \) where

\[ \mathfrak{M} = \left[ -\frac{P}{X}, \frac{P}{X} \right], \quad \mathfrak{m} = (-R, -\frac{P}{X}) \cup \left( \frac{P}{X}, R \right), \quad \mathfrak{t} = \mathbb{R} \setminus (\mathfrak{M} \cup \mathfrak{m}), \]

so that

\[ I(\eta, \varpi, \mathbb{R}) = I(\eta, \varpi, \mathfrak{M}) + I(\eta, \varpi, \mathfrak{m}) + I(\eta, \varpi, \mathfrak{t}). \]

The sets \( \mathfrak{M}, \mathfrak{m} \) and \( \mathfrak{t} \) are called the major arc, the intermediate (or minor) arc and the trivial arc respectively. In (13) we prove that the major arc yields the main term for \( I(\eta, \varpi, \mathbb{R}) \). We show in (17) that the contribution of the intermediate arc does not cancel the main term, exploiting the hypothesis that \( \lambda_1/\lambda_2 \) is irrational to prove that \( |S_1(\lambda_1\alpha)| \) and \( |S_1(\lambda_2\alpha)| \) can not both be large for \( \alpha \in \mathfrak{m} \): see Lemma (4) for the details. The trivial arc, treated in (18) only gives a rather small contribution.

From now on, implicit constants may depend on the coefficients \( \lambda_j \), on \( \delta, k \) and \( \varpi \).

3. The major arc

We write

\[ I(\eta, \varpi, \mathfrak{M}) = \int_{\mathfrak{M}} S_1(\lambda_1\alpha)S_1(\lambda_2\alpha)S_k(\lambda_3\alpha)K_\eta(\alpha)e(-\varpi\alpha) \, d\alpha \]

\[ = \int_{\mathfrak{M}} T_1(\lambda_1\alpha)T_1(\lambda_2\alpha)T_k(\lambda_3\alpha)K_\eta(\alpha)e(-\varpi\alpha) \, d\alpha \]

\[ + \int_{\mathfrak{M}} (S_1(\lambda_1\alpha) - T_1(\lambda_1\alpha))T_1(\lambda_2\alpha)T_k(\lambda_3\alpha)K_\eta(\alpha)e(-\varpi\alpha) \, d\alpha \]

\[ + \int_{\mathfrak{M}} S_1(\lambda_1\alpha)(S_1(\lambda_2\alpha) - T_1(\lambda_2\alpha))T_k(\lambda_3\alpha)K_\eta(\alpha)e(-\varpi\alpha) \, d\alpha \]

\[ + \int_{\mathfrak{M}} S_1(\lambda_1\alpha)S_1(\lambda_2\alpha)(S_k(\lambda_3\alpha) - T_k(\lambda_3\alpha))K_\eta(\alpha)e(-\varpi\alpha) \, d\alpha \]

\[ = J_1 + J_2 + J_3 + J_4, \]

say. We will give a lower bound for \( J_1 \) and upper bounds for \( J_2, J_3 \) and \( J_4 \). For brevity, since the computations for \( J_3 \) are similar to, but simpler than, the corresponding ones for \( J_2 \) and \( J_4 \), we will skip them.
4. Lower Bound for $J_1$

The lower bound $J_1 \gg \eta^2 X^{1+1/k}$ is proved in a classical way. We have

$$J_1 = \int_{\mathbb{R}} T_1(\lambda_1 \alpha) T_1(\lambda_2 \alpha) T_k(\lambda_3 \alpha) K_\eta(\alpha)e(-\varpi \alpha) \, d\alpha$$

$$= \int_{\mathbb{R}} T_1(\lambda_1 \alpha) T_1(\lambda_2 \alpha) T_k(\lambda_3 \alpha) K_\eta(\alpha)e(-\varpi \alpha) \, d\alpha$$

$$+ O \left( \int_{P/X}^{+\infty} |T_1(\lambda_1 \alpha) T_1(\lambda_2 \alpha) T_k(\lambda_3 \alpha)| K_\eta(\alpha) \, d\alpha \right).$$

Using inequalities (6) and (7), we see that the error term is

$$\ll \eta^2 X^{1/k-1} \int_{P/X}^{+\infty} \frac{d\alpha}{\alpha^3} \ll \eta^2 X^{1+1/k} P^{-2} = o(\eta^2 X^{1+1/k}).$$

For brevity, we set $\mathcal{D} = [\delta X, X]^2 \times [\delta X]^{1/k}, X^{1/k}]$ and rewrite the main term in the form

$$\int \cdots \int_{\mathcal{D}} e((\lambda_1 t_1 + \lambda_2 t_2 + \lambda_3 t_3^k - \varpi)\alpha) K_\eta(\alpha) \, dt_1 \, dt_2 \, dt_3$$

$$= \int \cdots \int_{\mathcal{D}} \max(0, \eta - |\lambda_1 t_1 + \lambda_2 t_2 + \lambda_3 t_3^k - \varpi|) \, dt_1 \, dt_2 \, dt_3.$$

We now proceed to show that the last integral is $\gg \eta^2 X^{1+1/k}$. Apart from trivial permutations or changes of sign, there are essentially two cases:

1. $\lambda_1 > 0, \lambda_2 < 0, \lambda_3 < 0$;
2. $\lambda_1 > 0, \lambda_2 > 0, \lambda_3 < 0$.

We briefly deal with the second case, the other one being similar. A suitable change of variables shows that

$$J_1 \gg \int \cdots \int_{\mathcal{D}'} \max(0, \eta - |\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3|) \, du_1 \, du_2 \, du_3,$$

$$\gg X^{1/k-1} \int \cdots \int_{\mathcal{D}'} \max(0, \eta - |\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3|) \, du_1 \, du_2 \, du_3,$$

where $\mathcal{D}' = [\delta X, (1-\delta) X]^3$, for large $X$. For $j = 1, 2$, let $a_j = 2|\lambda_3|\delta/|\lambda_j|$, $b_j = 3a_j/2$ and $J_j = [a_j X, b_j X]$. Notice that if $u_j \in J_j$ for $j = 1, 2$, then

$$\lambda_1 u_1 + \lambda_2 u_2 \in [4|\lambda_3|\delta X, 6|\lambda_3|\delta X]$$

so that, for every such choice of $(u_1, u_2)$, the interval $[a, b]$ with endpoints $\pm \eta/|\lambda_3| + (\lambda_1 u_1 + \lambda_2 u_2)/|\lambda_3|$ is contained in $[\delta X, (1-\delta) X]$. In other words, for $u_3 \in [a, b]$ the values of $\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3$ cover the whole interval $[-\eta, \eta]$. Hence, for any $(u_1, u_2) \in J_1 \times J_2$ we have

$$\int_{\delta X}^{(1-\delta) X} \max(0, \eta - |\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3|) \, du_3 = |\lambda_3|^{-1} \int_{-\eta}^{\eta} \max(0, \eta - |u|) \, du \gg \eta^2.$$

Finally,

$$J_1 \gg \eta^2 X^{1/k-1} \int_{J_1 \times J_2} du_1 \, du_2 \gg \eta^2 X^{1+1/k},$$

which is the required lower bound.
5. Bound for $J_2$

We recall definition (1) and notice that the Euler summation formula implies that

$$T_k(\alpha) - U_k(\alpha) \ll 1 + |\alpha|X \quad \text{for any } k \geq 1. \quad (8)$$

Using (7) we see that

$$J_2 \ll \eta^2 \int_{a_1} a_2 |S_1(\alpha) - T_1(\alpha)| \, d\alpha$$

$$\leq \eta^2 \int_{a_1} a_2 |S_1(\alpha) - U_1(\alpha) + U_1(\alpha) - T_1(\alpha)| \, d\alpha$$

$$+ \eta^2 \int_{a_1} a_2 |U_1(\alpha) - T_1(\alpha)| \, d\alpha$$

$$= \eta^2 (A_2 + B_2),$$

say. In order to estimate $A_2$ we use Lemmas 1 and 2. By the Cauchy inequality and (6) above, for any fixed $A > 0$ we have

$$A_2 \ll \left( \int_{-P/X}^{P/X} |S_1(\alpha) - U_1(\alpha)|^2 \, d\alpha \right)^{1/2} \left( \int_{-P/X}^{P/X} |T_1(\alpha)|^2 |T_k(\alpha)|^2 \, d\alpha \right)^{1/2}$$

$$\ll \left( \frac{X}{(\log X)^4} \right)^{1/2} \left( \int_{0}^{P/X} X^{2+2/k} \, d\alpha + \int_{1/X}^{P/X} \frac{X^{2/k-2}}{\alpha^4} \, d\alpha \right)^{1/2} \ll A \frac{X^{1+1/k}}{(\log X)^{A/2}}$$

by Lemma 2 which we can use provided that $X/P \geq X^{1/6+\varepsilon}$, that is, $P \leq X^{5/6-\varepsilon}$. This proves that $\eta^2 A_2 = o(\eta^2 X^{1+1/k})$. Furthermore, using inequalities (6) and (8) we see that

$$B_2 \ll \int_{0}^{1/X} |T_1(\alpha)| |T_k(\alpha)| \, d\alpha + X \int_{1/X}^{P/X} \alpha |T_1(\alpha)| |T_k(\alpha)| \, d\alpha$$

$$\ll \frac{1}{X} X^{1+1/k} + X^{1/k} \int_{1/X}^{P/X} \frac{d\alpha}{\alpha} \ll X^{1/k} \log P,$$

so that $\eta^2 B_2 = o(\eta^2 X^{1+1/k})$.

6. Bound for $J_4$

Inequality (7) implies that

$$J_4 \ll \eta^2 \int_{a_1} a_2 |S_1(\alpha)| |S_1(\alpha)| |S_1(\lambda_3 \alpha) - T_k(\lambda_3 \alpha)| \, d\alpha$$

$$\ll \eta^2 \int_{a_1} a_2 |S_1(\alpha)| |S_1(\alpha)| |S_1(\lambda_3 \alpha) - U_k(\lambda_3 \alpha)| \, d\alpha$$

$$+ \eta^2 \int_{a_1} a_2 |S_1(\alpha)| |S_1(\lambda_3 \alpha) - T_k(\lambda_3 \alpha)| \, d\alpha$$

$$= \eta^2 (A_4 + B_4),$$

say. The Parseval inequality and trivial bounds yield, for any fixed $A > 0$,

$$A_4 \ll X \left( \int_{a_1} a_2 |S_1(\alpha)|^2 \, d\alpha \right)^{1/2} \left( \int_{a_1} a_2 |S_1(\lambda_3 \alpha) - U_k(\lambda_3 \alpha)|^2 \, d\alpha \right)^{1/2}$$

$$\ll X (X \log X)^{1/2} \frac{P}{X} J_k \left( X, \frac{X}{P} \right)^{1/2} \ll A X^{1+1/k} (\log X)^{1/2 - A/2}$$

for any fixed $A > 0$. 


by Lemmas 1 and 2 which we can use provided that \( X/P \geq X^{1-5/(6k)+\varepsilon} \), that is, \( P \leq X^{5/(6k)-\varepsilon} \). This proves that \( \eta^2 A_4 = o(\eta^2 X^{1+1/k}) \). Furthermore, using (3), the Cauchy inequality and trivial bounds we see that

\[
B_4 \ll \int_0^{1/X} |S(\lambda_1 \alpha)| |S(\lambda_2 \alpha)| \, d\alpha + X \int_{1/X}^{P/X} \alpha |S(\lambda_1 \alpha)| |S(\lambda_2 \alpha)| \, d\alpha
\]

\[
\ll X + P \left( \int_{1/X}^{P/X} |S(\lambda_1 \alpha)|^2 \, d\alpha \right)^{1/2} \ll PX \log X.
\]

Hence \( B_4 \ll PX \log X \), so that taking \( P = o(X^{1/k}(\log X)^{-1}) \) we get \( \eta^2 B_4 = o\left(\eta^2 X^{1+1/k}\right) \).

We may therefore choose

\[
P = X^{5/(6k)-\varepsilon}.
\]

7. The Intermediate Arc

We need to show that \(|S(\lambda_1 \alpha)|\) and \(|S(\lambda_2 \alpha)|\) can not both be large for \( \alpha \in \mathfrak{m} \), exploiting the fact that \( \lambda_1/\lambda_2 \) is irrational. We achieve this using a famous result by Vaughan about \( S(\alpha) \).

Lemma 3 (Vaughan [17], Theorem 3.1). Let \( \alpha \) be a real number and \( a, q \) be positive integers satisfying \((a, q) = 1\) and \(|\alpha - a/q| \leq q^{-2}\). Then

\[
S(\alpha) \ll \left( \frac{X}{\sqrt{q}} + \sqrt{Xq} + X^{4/5} \right) \log^4 X.
\]

Lemma 4. Let \( 1 \leq k < 4/3 \). Assume that \( \lambda_1/\lambda_2 \) is irrational and let \( X = q^{5k/(k+2)} \), where \( q \) is the denominator of a convergent of the continued fraction for \( \lambda_1/\lambda_2 \). Let \( V(\alpha) = \min\{|S(\lambda_1 \alpha)|, |S(\lambda_2 \alpha)|\} \). Then we have

\[
\sup_{\alpha \in \mathfrak{m}} V(\alpha) \ll X^{4/5+1/(10k)} \log^4 X.
\]

Proof. Let \( \alpha \in \mathfrak{m} \) and \( Q = X^{2/5-1/(5k)} \leq P \). By Dirichlet’s Theorem, there exist integers \( a_i, q_i \) with \( 1 \leq q_i \leq X/Q \) and \((a_i, q_i) = 1\), such that \(|\lambda_i a q_i - a_i| \leq Q/X \), for \( i = 1, 2 \). We remark that \( a_1 a_2 \neq 0 \) otherwise we would have \( \alpha \in \mathfrak{M} \). Now suppose that \( q_i \leq Q \) for \( i = 1, 2 \). In this case we get

\[
a_2 q_1 \lambda_1 - a_1 q_2 = (\lambda_1 a q_1 - a_1) \frac{a_2}{\lambda_2} - (\lambda_2 a q_2 - a_2) \frac{a_1}{\lambda_2}
\]

and hence

\[
\left| a_2 q_1 \lambda_1 - a_1 q_2 \right| \leq 2 \left( 1 + \left| \frac{\lambda_1}{\lambda_2} \right| \right) \frac{Q^2}{X} < \frac{1}{2q}
\]

for sufficiently large \( X \). Then, from the law of best approximation and the definition of \( \mathfrak{m} \), we obtain

\[
X^{(k+2)/(5k)} = q \leq |a_2 q_1| \ll q_1 q_2 R \leq Q^2 R \leq X^{(k+2)/(5k)-\varepsilon},
\]

which is absurd. Hence either \( q_1 > Q \) or \( q_2 > Q \). Assume that \( q_1 > Q \). Using Lemma 3 on \( S(\lambda_1 \alpha) \), we have

\[
V(\alpha) \leq |S(\lambda_1 \alpha)| \ll \sup_{Q < q_1 \leq X/Q} \left( \frac{X}{\sqrt{q_1}} + \sqrt{Xq_1} + X^{4/5} \right) \log^4 X
\]

\[
\ll X^{4/5+1/(10k)} (\log X)^4.
\]

The other case is totally similar and hence Lemma 4 follows. \( \square \)
Lemma 5. For $j = 1$ and 2 we have

$$
\int_{m} |S_1(\lambda_j \alpha)|^2 K_\eta(\alpha) \, d\alpha \ll \eta X \log X \quad \text{and} \quad \int_{m} |S_k(\lambda_3 \alpha)|^2 K_\eta(\alpha) \, d\alpha \ll \eta X^{1/k}(\log X)^3.
$$

Proof. We have to split the range $[P/X, R]$ into two intervals in order to use (7) efficiently. In the first case we have

$$
\int_{m} |S_1(\lambda_j \alpha)|^2 K_\eta(\alpha) \, d\alpha \ll \eta^2 \int_{P/X}^{1/\eta} |S_1(\lambda_j \alpha)|^2 \, d\alpha + \int_{1/\eta}^{R} |S_1(\lambda_j \alpha)|^2 \, \frac{d\alpha}{\alpha^2}
$$

by (7), for $j = 1, 2$. By periodicity

$$
\eta^2 \int_{P/X}^{1/\eta} |S_1(\lambda_j \alpha)|^2 \, d\alpha \ll \eta \int_{0}^{1} |S_1(\alpha)|^2 \, d\alpha \ll \eta X \log X,
$$

by the Prime Number Theorem (PNT). We also have

$$
\int_{1/\eta}^{R} |S_1(\lambda_j \alpha)|^2 \, \frac{d\alpha}{\alpha^2} \ll \int_{|\lambda_j|/\eta}^{+\infty} |S_1(\alpha)|^2 \, \frac{d\alpha}{\alpha^2} \ll \sum_{n \geq |\lambda_j|/\eta} \frac{1}{(n-1)^2} \int_{n-\eta}^{n} |S_1(\alpha)|^2 \, d\alpha \ll \eta X \log X,
$$

again by the PNT. This proves the first part of the statement. For the second part, we argue in a similar way, replacing the PNT by an appeal to (iii) of Lemma 7 in Tolev [14].

Now let

$$
\mathcal{X}_1 = \{\alpha \in [P/X, R] : |S_1(\lambda_1 \alpha)| \leq |S_1(\lambda_2 \alpha)|\}
$$

$$
\mathcal{X}_2 = \{\alpha \in [P/X, R] : |S_1(\lambda_1 \alpha)| \geq |S_1(\lambda_2 \alpha)|\}
$$

so that $[P/X, R] = \mathcal{X}_1 \cup \mathcal{X}_2$ and

$$
\left| I(\eta, \varpi, m) \right| \ll \left( \int_{\mathcal{X}_1} + \int_{\mathcal{X}_2} \right) |S_1(\lambda_1 \alpha) S_1(\lambda_2 \alpha) S_k(\lambda_3 \alpha)| K_\eta(\alpha) \, d\alpha.
$$

Cauchy’s inequality gives

$$
\int_{\mathcal{X}_1} \leq \max_{\alpha \in \mathcal{X}_1} |S_1(\lambda_1 \alpha)| \left( \int_{\mathcal{X}_1} |S_1(\lambda_2 \alpha)|^2 K_\eta(\alpha) \, d\alpha \right)^{1/2} \left( \int_{\mathcal{X}_1} |S_k(\lambda_3 \alpha)|^2 K_\eta(\alpha) \, d\alpha \right)^{1/2}
$$

$$
\ll X^{4/5+1/(10k)}(\log X)^4(\eta X \log X)^{1/2}(\eta X^{1/k}(\log X)^3)^{1/2}
$$

$$
\ll \eta X^{13/10+3/(5k)}(\log X)^6
$$

by Lemmas 4 and 5. The computation on $\mathcal{X}_2$ is similar and gives the same final result. Summing up,

$$
\left| I(\eta, \varpi, m) \right| \ll \eta X^{13/10+3/(5k)}(\log X)^6,
$$

and this is $o(\eta^2 X^{1+1/k})$ provided that

$$
\eta = \infty(X^{3/10-2/(5k)}(\log X)^6).
$$

8. The trivial arc

Using the Cauchy inequality and a trivial bound for $S_k(\lambda_3 \alpha)$ we see that

$$
\left| I(\eta, \varpi, t) \right| \leq 2 \int_{R}^{+\infty} |S_1(\lambda_1 \alpha)| |S_1(\lambda_2 \alpha)| |S_k(\lambda_3 \alpha)| K_\eta(\alpha) \, d\alpha
$$

$$
\ll X^{1/k} \left( \int_{R}^{+\infty} |S_1(\lambda_1 \alpha)|^2 K_\eta(\alpha) \, d\alpha \right)^{1/2} \left( \int_{R}^{+\infty} |S_1(\lambda_2 \alpha)|^2 K_\eta(\alpha) \, d\alpha \right)^{1/2}
$$
\[ \ll X^{1/k}C_1^{1/2}C_2^{1/2}, \]

say, where in the last but one line we used the inequality \((7)\), and, for \(j = 1, 2\), we set

\[ C_j = \int_{|\lambda_j|R}^{+\infty} \frac{|S_1(\alpha)|^2}{\alpha^2} \, d\alpha. \]

We argue as in the proof of Lemma 5. Using the PNT we have

\[ C_j \ll \sum_{n \geq |\lambda_j|R} \frac{1}{(n-1)^2} \int_{n-1}^n |S_1(\alpha)|^2 \, d\alpha \ll \frac{X \log X}{|\lambda_j|R}. \]

Collecting these estimates, we conclude that

\[ |I(\eta, \varpi, t)| \ll \frac{X^{1+1/k} \log X}{R}. \]

Hence, the choice

\[ R = \eta^{-2} (\log X)^{3/2} \]

is admissible.

9. Remark on the choice of the parameters

The choice \( X = q^{5k/(k+2)} \) with \( 1 \leq k < 4/3 \) arises from the bounds \((10)\) and \((11)\). Their combination prevents us from choosing the optimal value \( X = q^2 \). This is justified as follows: neglecting log-powers, let \( X = q^{a(k)}, Q = X^{b(k)}, \eta = X^{-c(k)} \), and recall the choices \( P = X^{5/(6k)-\varepsilon} \) in \((9)\) and \( R = \eta^{-2}(\log X)^{3/2} \) in \((13)\) which are due, respectively, to the bound for \( B_4 \) and for the trivial arc. Then, essentially, we have to maximize \( k \) subject to the constraints

\[
\begin{aligned}
 a(k) &\geq 1 \\
 0 &\leq b(k) \leq 5/(6k) \\
 c(k) &\geq 0 \\
 2b(k) - 1 &\leq -1/a(k) \quad \text{by (10)}, \\
 2b(k) + 2c(k) &\leq 1/a(k) \quad \text{by (11)}, \\
 -c(k) &\geq \frac{1}{2} - \frac{1}{3k} - \frac{1}{5k} \quad \text{by (12)},
\end{aligned}
\]

which is a linear optimization problem in the variables \( 1/a(k), b(k), c(k) \) and \( 1/k \). The solution for this problem is \( 1/a(k) = (k+2)/(5k), b(k) = (2k-1)/(5k), c(k) = (4-3k)/(10k) \), for \( 1/k \geq 3/4 \), and this is equivalent to the statement of the main Theorem.

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