Standard models of abstract intersection theory for operators in Hilbert space

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Abstract. For an operator in a possibly infinite-dimensional Hilbert space of a certain class, we set down axioms of an abstract intersection theory, from which the Riemann hypothesis regarding the spectrum of that operator follows. In our previous paper [BU] we constructed a GNS (Gelfand-Naimark-Segal) model of abstract intersection theory. In this paper we propose another model, which we call a standard model of abstract intersection theory. We show that there is a standard model of abstract intersection theory for a given operator if and only if the Riemann hypothesis and semi-simplicity hold for that operator. (For the definition of semi-simplicity of an operator in Hilbert space, see the definition in Introduction.) We show this result under a condition for a given operator which is much weaker than the condition in the previous paper. The operator satisfying this condition can be constructed by the method of automorphic scattering in [U].

Combining this with a result from [U], we can show that an Dirichlet $L$-function, including the Riemann zeta-function, satisfies the Riemann hypothesis and its all nontrivial zeros are simple if and only if there is a corresponding standard model of abstract intersection theory. Similar results can be proven for GNS models since the same technique of proof for standard models can be applied.

1. Introduction

In the 1940s Weil [W1] developed an intersection theory on surfaces over finite fields to apply it to the proof of the Riemann hypothesis for curves over finite fields (one-variable function fields over finite fields).

In this paper we introduce axioms ((AIT1)–(AIT3) in §3) of abstract intersection theory for an operator in a possibly infinite-dimensional Hilbert space, which are analogous to Weil’s theory. We consider a collection $\mathcal{AIT}$ that consists of a vector space, its specific vectors and some maps, satisfying these axioms. From this collection one can derive the Riemann hypothesis regarding the spectrum of that operator. Therefore we call $\mathcal{AIT}$ an abstract intersection theory.

Let $H$ be a possibly infinite-dimensional $\mathbb{C}$-Hilbert space. Let $A: H \supset \text{dom}(A) \to H$ be a $\mathbb{C}$-linear operator acting on $H$. Here $\text{dom}(A)$ denotes the domain of the operator $A$. We assume that its spectrum $\sigma(A)$ consists only of the point spectrum $\sigma_p(A)$. That is, $\sigma(A)$ consists only of eigenvalues of $A$.

We say that the operator $A$ satisfies the Riemann hypothesis (RH, shortly) if

$$\text{Re}(s_i) = \frac{1}{2} \text{ for all } s_i \in \sigma(A) = \sigma_p(A).$$

We say that the operator $A$ is semi-simple if

$$\nu(s_i) = 1 \text{ for all } s_i \in \sigma(A) = \sigma_p(A).$$

Here $\nu(s_i)$ is the Riesz index of $s_i$. For its definition see the paragraph preceding the conditions (OP1)–(OP5) in §2, which $A$ is assumed to satisfy. All these conditions are satisfied by an
operator $A$ obtained from automorphic scattering theory [U], which gives a spectral interpretation of a certain Dirichlet $L$-function, including the Riemann zeta-function. See Remark 2.1 (4) in §2.

In our previous work [BU], we showed $\text{AIT} \Rightarrow \text{RH}$. We also constructed a model $\text{AIT}_{\text{GNS}}$ of abstract intersection theory based on an analogue of the GNS (Gelfand-Naimark-Segal) representation. We call $\text{AIT}_{\text{GNS}}$ a GNS model of abstract intersection theory. We showed $\text{AIT}_{\text{GNS}} \Leftrightarrow \text{RH}$, assuming the semi-simplicity of $A$ ([BU, Theorem 3.1]).

We observe that there is some freedom in constructing models of abstract intersection theory to investigate the spectrum of operators in Hilbert space and nontrivial zeros of corresponding Dirichlet $L$-functions. In this paper we propose another new model $\text{AIT}_m$, which we call a GNS model of abstract intersection theory. We showed $\text{AIT}_m \Leftrightarrow \text{RH}$ & semi-simplicity (Theorem 5.2 (2)). The technique for proving this statement can also be applied to $\text{AIT}_{\text{GNS}}$ in the previous paper and one can show $\text{AIT}_{\text{GNS}} \Leftrightarrow \text{RH}$ & semi-simplicity (Theorem 5.3). Therefore we significantly strengthen our previous results in [BU] for both GNS and standard models, dropping the semi-simplicity assumption (the condition (OP3-b) in [BU]). The condition (OP3-b) in this paper is much weaker and is satisfied by operators coming from scattering theory for Dirichlet $L$-functions [U].

As a consequence of Theorems 5.2 and 5.3 combined with the results in [U] from automorphic scattering theory, we can show that an Dirichlet $L$-function, including the Riemann zeta-function, satisfies the RH and its all nontrivial zeros are simple if and only if there is a corresponding standard model $\text{AIT}_m$ (or GNS model $\text{AIT}_{\text{GNS}}$) of abstract intersection theory (Theorem 5.4).

The plan of this paper is as follows.

In §2 we define an analogue of the classical Frobenius morphism for the operator $A$. The spectrum of this analogue is similar to that of the classical Frobenius morphism if the operator $A$ satisfies the Riemann hypothesis. The introduction of this analogue is also hinted by Weil’s explicit formulas [W2].

In §3 we introduce a general notion of abstract intersection theory $\text{AIT}$ and set down its axioms ((AIT1), (AIT2) and (AIT3)).

In §4 we construct a specific example of abstract intersection theory, which we call a standard model $\text{AIT}_m$, using analogy with the classical K"unneth formula for $\ell$-adic cohomology.

In §5 we state our main theorems (Theorems 5.2, 5.3 and 5.4).

In §6 we show that there is a strong analogy between Weil’s approach to zeta-functions for curves over finite fields and our approach to Dirichlet $L$-functions. For Weil’s intersection theory, see also Grothendieck [Gro], Monsky [Mon] and Serre [S].

We should note that there is a program by Connes and Marcolli (and Consani) [CM] to adapt Weil’s proof of RH for function fields to the case of number fields. See also Connes [C]. There is also a conjectural cohomological approach by Deninger [D] toward the interpretation of $L$-functions analogous to the etale cohomology theory of varieties over finite fields.

2. An analogue of the Frobenius morphism for the operator $A$

Let $H$ be a possibly infinite-dimensional $\mathbb{C}$-Hilbert space. If $H$ is infinite-dimensional we assume that $H$ is separable. Let $A: H \supset \text{dom}(A) \to H$ be a possibly unbounded operator on $H$.

If $s_i \in \sigma(A)$ is an isolated spectrum point, one can take a small enough bounded domain $\Delta$ of $\mathbb{C}$ such that $\{s_i\} \subseteq \Delta$ (i.e., $\{s_i\} \subset \Delta^c$) and $\overline{\Delta} \cap (\sigma(A) - \{s_i\}) = \emptyset$. Then one can define the
Riesz projection $P_{\{s_i\}}: H \to H$ by

$$P_{\{s_i\}} := \frac{1}{2\pi i} \int_{\partial \Delta} (sI - A)^{-1}ds.$$  

Here $I: H \to H$ is the identity operator on $H$. $P_{\{s_i\}}$ is a bounded operator on $H$.

For $s_i \in \sigma_p(A)$, the Riesz index $\nu(s_i)$ of $s_i$ is defined as the smallest positive number $\leq \infty$ such that

$$\text{Ker}((s_i I - A)^{\nu(s_i)}) = \text{Image}(P_{\{s_i\}}).$$

Let $\text{mult}(s_i) = \dim \text{Image}(P_{\{s_i\}})$, which we call the (algebraic) multiplicity of $s_i \in \sigma_p(A)$.

We assume the following properties of $A$.

**(OP1)** $A$ is closed.

**(OP2)** The spectrum $\sigma(A)$ consists only of the point spectrum (i.e., eigenvalues) $\sigma_p(A)$, i.e., $\sigma(A) = \sigma_p(A)$, which accumulates at most at infinity.

**(OP3)** (a) $\text{Image}(P_{\{s_i\}})$ is finite-dimensional for any $s_i \in \sigma_p(A)$.

(b) $\nu(s_i) = \text{mult}(s_i)$ for any $s_i \in \sigma_p(A)$.

**(OP4)** $\sigma(A) \subset \Omega_\infty$, where $\Omega_\infty := \{s \in \mathbb{C} | 0 < \text{Re}(s) < 1\}$.

**(OP5)** $\text{Re}(s_i) < \frac{1}{2}$ for some $s_i \in \sigma(A)$ if and only if there is $s_j \in \sigma(A)$ such that $\text{Re}(s_j) > \frac{1}{2}$.

**Remark 2.1.**

(1) **(OP1)** is needed when one applies Lemma 2.1 of [BU] on spectral decomposition. Lemma 2.1 of [BU] is taken from Gohberg, Goldberg and Kaashoek [GoGK, XV.2, Theorem 2.1, p. 326].

(2) In the previous paper [BU], the condition (OP3-b) was the simi-simplicity $\nu(s_i) = 1$ ($s_i \in \sigma(A)$). The above stated (OP3-b) is a much weaker condition. This condition says that each spectrum (eigenvalue) of $A$ has just one corresponding Jordan block. Actually this is satisfied in the construction using automorphic scattering theory [U]. See Remark 2.1(4) below.

(3) The above (OP5) is (OP5-a) in [BU]. (OP5-b) in [BU], which is necessary for the construction of GNS models of abstract intersection theory is not necessary for the construction of standard models in this paper. (OP5-b) in [BU] is used to keep the space $V$ an $\mathbb{R}$-linear space in the GNS model. In the standard model we apply the complexification $V_\mathbb{C}$ of $V$ instead (see §3). (OP5-b) in [BU] is satisfied by an operator $A$ constructed in [U] (see Remark 2.1(4) below).

(4) Let $\Gamma$ be a congruence subgroup of $SL_2(\mathbb{Z})$ such that $\Gamma \backslash \mathbb{H} (\simeq \Gamma \backslash SL_2(\mathbb{R})/SO(2))$ is noncompact and has one cusp at $i\infty$. Here $\mathbb{H}$ denotes the upper half-plane. In [U] the second author constructed a scattering theory for automorphic forms on $\Gamma \backslash \mathbb{H}$. Furthermore he constructed an operator $A$ satisfying **(OP)**–**(OP5)** whose (point) spectrum coincides with the nontrivial zeros of the Dirichlet $L$-function $L(s, \chi)$ associated to $\Gamma$, counted with multiplicity: $s_i \in \sigma(A) (= \sigma_p(A))$ and $\nu(s_i) (= \text{mult}(s_i)) = m_i$ if and only if $s_i$ is a nontrivial zero of $L(s, \chi)$ of order $m_i$. We call $s_i \in \mathbb{C}$ a nontrivial zero of the Dirichlet $L$-function $L(s, \chi)$ if $L(s_i, \chi) = 0$ and $0 < \text{Re}(s_i) < 1$. See Theorem 4.1(i) ($\Rightarrow$ (OP1)), (iii-a), (iii-b), (iii-c) ($\Rightarrow$ (OP2) and (OP3)) and (iv) ($\Rightarrow$ (OP4) and (OP5), and (OP5-b) in [BU]) of [U, p. 455]. The theory of automorphic scattering was initiated by Pavlov-Faddeev [PavF] and then Lax-Phillips
Now for $Y > 0$ let

$$\sigma_Y(A) := \{ s \in \sigma(A) \mid |\text{Im}(s)| < Y \}.$$  

Note that $\sigma_Y(A)$ is a finite set by (OP2). Let the parameter space $\mathcal{Y}$ be defined by

$$\mathcal{Y} := \{ Y > 0 \mid \sigma_Y(A) \neq \emptyset \} - \{ |\text{Im}(s)| \mid s \in \sigma(A) \}.$$  

Fix a function

$$q: \mathcal{Y} \to (0, 1) \cup (1, \infty).$$

Let $B(X)$ denote the set of bounded operators on a $\mathbb{C}$-Hilbert space $X$. By definition $T: X \supset \text{dom}(T) \to X$ is a bounded operator if $\text{dom}(T) = X$ and the operator norm $\|T\| < \infty$.

Let $\Sigma_H$ be the set of closed subspaces of $H$. We will construct maps

$$F_A: \mathcal{Y} \to B(H)$$

and

$$\mathcal{H}: \mathcal{Y} \to \Sigma_H$$

such that $F_A(Y): H \to H$ satisfies the following conditions for each $Y \in \mathcal{Y}$.

**(FROB-a)\)**

$$F_A(Y)\mathcal{H}(Y) \subset \mathcal{H}(Y)$$

(i.e., the subspace $\mathcal{H}(Y)$ is invariant for $F_A(Y)$).

**(FROB-b)\)**

$$\sigma(F_A(Y)|_{\mathcal{H}(Y)}) = \sigma_p(F_A(Y)|_{\mathcal{H}(Y)}) = \{ q(Y)^s | s \in \sigma_Y(A) \}$$

and

$$\sigma(F_A(Y)) = \sigma_p(F_A(Y)) = \sigma(F_A(Y)|_{\mathcal{H}(Y)}) \cup \{ 0 \}.$$  

Note that $\sigma(F_A(Y)|_{\mathcal{H}(Y)})$ is a finite set since $\sigma_Y(A)$ is.

The operator $F_A(Y)$ ($Y \in \mathcal{Y}$) is considered to be an analogue of the classical Frobenius morphism, since the spectrum of this analogue is similar to that of the classical Frobenius morphism if the operator $A$ satisfies the Riemann hypothesis. It is also hinted by the spectral side of Weil’s explicit formulas [W2] (see §6).

**Models $F_{A,m}$ and $\mathcal{H}_m$ of $F_A$ and $\mathcal{H}$:**

Now we construct the models $F_{A,m}: \mathcal{Y} \to B(H)$ and $\mathcal{H}_m: \mathcal{Y} \to \Sigma_H$ which satisfy (Frob-a) and (Frob-b). These models will constitute parts of a standard model $\text{AIT}_m$ constructed in §4. Let

$$\Omega_Y := \{ s \in \mathbb{C} | 0 < \text{Re}(s) < 1, |\text{Im}(s)| < Y \}$$

for $Y \in \mathcal{Y}$. Note that $\Omega_Y \cap \sigma(A) = \sigma_Y(A)$ for $Y \in \mathcal{Y}$ by (OP4). Note that for $Y \in \mathcal{Y},$

$$\sigma_Y(A) \subseteq \Omega_Y \quad (\text{i.e., } \overline{\sigma_Y(A)} = \sigma_Y(A) \subset \Omega^*_Y = \Omega_Y) \quad \text{and} \quad \overline{\Omega_Y} \cap (\sigma(A) - \sigma_Y(A)) = \emptyset.$$  

Therefore, for $Y \in \mathcal{Y}$, the Riesz projection

$$P_{\sigma_Y(A)}: H \to H$$

can be well-defined by

$$P_{\sigma_Y(A)} := \frac{1}{2\pi i} \oint_{\partial\Omega_Y} (sI - A)^{-1} ds.$$
$P_{\sigma_Y(A)}$ is a bounded operator on $H$. Let $\mathcal{H}_m: \mathcal{Y} \to \Sigma_H$ be defined by

\[ \mathcal{H}_m(Y) := \text{Image}(P_{\sigma_Y(A)}). \]

By (OP2) and (OP3-a), $\mathcal{H}_m(Y)$ is finite-dimensional for each $Y \in \mathcal{Y}$.

Given $Y \in \mathcal{Y}$, let

\[ F_{A,m}(Y): H \supset \text{dom}(F_{A,m}(Y)) \to H \]

be defined by

\[ F_{A,m}(Y)x := \frac{1}{2\pi i} \left( \oint_{\partial Y} q(Y)^s(sI - A)^{-1}ds \right)x \]

for

\[ x \in \text{dom}(F_{A,m}(Y)) := \{x \in H | F_{A,m}(Y)x \text{ exists in } H\}. \]

Note that $\sigma_Y(A)$ is a bounded set. Thus, by Lemma 2.1 of [BU], $\mathcal{H}_m(Y) \subset \text{dom}(A)$ and $A\mathcal{H}_m(Y) \subset \mathcal{H}_m(Y)$. Let $A|_{\mathcal{H}_m(Y)}: \mathcal{H}_m(Y) \to \mathcal{H}_m(Y) \subset H$ be the restriction of $A$ to $\mathcal{H}_m(Y)$. Since $\mathcal{H}_m(Y)$ is finite-dimensional, $A|_{\mathcal{H}_m(Y)}$ is a bounded operator, i.e., $A|_{\mathcal{H}_m(Y)} \in B(\mathcal{H}_m(Y))$.

Similarly, let $H(s_i) := \text{Image}(P_{\{s_i\}})$. By (OP2) and (OP3-a), $H(s_i)$ is finite-dimensional. Again by Lemma 2.1 of [BU], $H(s_i) \subset \text{dom}(A)$ and $AH(s_i) \subset H(s_i)$. Let $A|_{H(s_i)}: H(s_i) \to H(s_i) \subset H$ be the restriction of $A$ to $H(s_i)$. By the same argument for $A|_{\mathcal{H}_m(Y)}$, we have $A|_{H(s_i)} \in B(H(s_i))$.

**Lemma 2.1.** Suppose that $A$ satisfies (OP1), (OP2) and (OP3-a). Then

(i) For each $Y \in \mathcal{Y}$,

\[ \text{dom}(F_{A,m}(Y)) = H. \]

Furthermore, $F_{A,m}(Y)$ is a bounded operator on $H$, i.e., $F_{A,m}(Y) \in B(H)$.

(ii) The subspace $\mathcal{H}_m(Y)$ is $F_{A,m}(Y)$-invariant:

\[ F_{A,m}(Y)\mathcal{H}_m(Y) \subset \mathcal{H}_m(Y). \]

That is, $F_{A,m}$ satisfies (Frob-a).

(iii) For each $Y \in \mathcal{Y}$, we have

\[ \sigma(F_{A,m}(Y)|_{\mathcal{H}_m(Y)}) = \sigma_p(F_{A,m}(Y)|_{\mathcal{H}_m(Y)}) = \{q(Y)^s | s \in \sigma_Y(A)\} \]

and

\[ \sigma(F_{A,m}(Y)) = \sigma_p(F_{A,m}(Y)) = \sigma(F_{A,m}(Y)|_{\mathcal{H}_m(Y)}) \cup \{0\}. \]

That is, $F_{A,m}$ satisfies (Frob-b).

(iv) Let $t: \mathcal{Y} \to \mathbb{R} - \{0\}$ be defined by $t(Y) := \log q(Y)$ (i.e., $e^{t(Y)} = q(Y)$) for $Y \in \mathcal{Y}$. For each $Y \in \mathcal{Y}$, we have

\[ F_{A,m}(Y) = e^{t(Y)A|_{\mathcal{H}_m(Y)}}P_{\sigma_Y(A)} = \sum_{n=0}^{\infty} \frac{t(Y)^n}{n!} A|_{\mathcal{H}_m(Y)} P_{\sigma_Y(A)} = \sum_{s_i \in \sigma_Y(A)} e^{t(Y)A|_{H(s_i)}} P_{\{s_i\}}. \]

(v) Suppose further that $A$ satisfies (OP3-b). Then, with respect to an appropriate basis of $H(s_i)$, $e^{t(Y)A|_{H(s_i)}}$ is written as

\[ e^{t(Y)A|_{H(s_i)}} = N(s_i) \]

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with

\[
N(s_i) = \begin{pmatrix}
\frac{t(Y)^0 e^t(Y)_{s_i}}{0!} & \cdots & \frac{t(Y)^{m_i} e^t(Y)_{s_i}}{0!} \\
\frac{t(Y)^1 e^t(Y)_{s_i}}{1!} & \cdots & \frac{t(Y)^{m_i-1} e^t(Y)_{s_i}}{(m_i-1)!} \\
\vdots & \ddots & \vdots \\
\frac{t(Y)^{m_i} e^t(Y)_{s_i}}{m_i!} & \cdots & \frac{t(Y)^{m_i} e^t(Y)_{s_i}}{0!}
\end{pmatrix} \in M_{m_i}(\mathbb{C}).
\]

Here \(m_i = \nu(s_i) = \text{mult}(s_i)\).

Proof. Let \(K(Y) = \text{Ker}(P_{\sigma_Y(A)})\). Then by Lemma 2.1 of [BU], \(K(Y)\) is \(A\)-invariant in the sense that \(A(K(Y) \cap \text{dom}(A)) \subset K(Y)\). Thus one can define \(A|_{K(Y)}: K(Y) \supset \text{dom}(A|_{K(Y)}) \rightarrow K(Y)\), the restriction of \(A\) to \(K(Y)\). Then we have \(\sigma(A|_{K(Y)}) = \sigma(A) - \sigma_Y(A)\) (Lemma 2.1 [BU]). We also have

\[A = \begin{pmatrix} A|_{H_m(Y)} & 0 \\ 0 & A|_{K(Y)} \end{pmatrix}\]

on \(H = H_m(Y) \oplus K(Y)\). Note that the direct sum \(\oplus\) does not necessarily mean the orthogonal sum.

By (OP2) and (OP3-a), \((sI - A)^{-1}\) is meromorphic in the whole \(\mathbb{C}\)-plane. However, since \((sI - A|_{K(Y)})^{-1}\) is holomorphic in \(\Omega_Y\), we have by the functional calculus for the bounded operator \(A|_{H_m(Y)}\)

\[
F_{A,m}(Y) = \frac{1}{2\pi i} \oint_{\partial \Omega_Y} q(Y)^s (sI - A)^{-1} ds
\]

\[= \frac{1}{2\pi i} \oint_{\partial \Omega_Y} e^{t(Y)s} (sI - A)^{-1} ds
\]

\[= \frac{1}{2\pi i} \oint_{\partial \Omega_Y} e^{t(Y)s} \left( (sI - A|_{H_m(Y)})^{-1} 0 \right) (sI - A|_{K(Y)})^{-1} ds
\]

\[= \frac{1}{2\pi i} \oint_{\partial \Omega_Y} e^{t(Y)s} (sI - A|_{H_m(Y)})^{-1} ds
\]

\[= \frac{1}{2\pi i} \oint_{\partial \Omega_Y} e^{t(Y)s} A|_{H_m(Y)} P_{\sigma_Y(A)} ds
\]

which shows (i) and (ii). By Lemma 2.1 of [BU], we have \(\sigma(A|_{H_m(Y)}) = \sigma_Y(A)\). Applying the spectral mapping theorem to the bounded operator \(A|_{H_m(Y)}\) (recall that \(\text{dim}_\mathbb{C} H_m(Y) < \infty\)), this also shows (iii).

Note that

\[P_{\sigma_Y(A)} = \bigoplus_{s_i \in \sigma_Y(A)} P_{\{s_i\}}\]

and

\[H_m(Y) = \bigoplus_{s_i \in \sigma_Y(A)} H(s_i)\]
Here $\bigoplus$ denotes the (not necessarily orthogonal) direct sum. Therefore we have

$$F_{A,m}(Y) = \frac{1}{2\pi i} \sum_{s_i \in \sigma_Y(A)} \oint_{\partial \Omega_Y} q(Y)^s (sI - A|_{H(s_i)})^{-1} P_{s_i} ds = \sum_{s_i \in \sigma_Y(A)} e^{t(Y)A|_{H(s_i)}} P_{s_i}.$$  

From this (iv) follows. 

Note that by Lemma 2.1 of [BU] we have $\sigma(A|_{H(s_i)}) = \{s_i\}$. Thus, by (OP3-b), $A|_{H(s_i)}$ is written with respect to an appropriate basis of $H(s_i)$ as

$$A|_{H(s_i)} = M(s_i)$$

with

$$M(s_i) = \begin{pmatrix} s_i & 1 & 0 \\ s_i & 1 & 0 \\ \vdots & \vdots & \vdots \\ s_i & 1 & 0 \end{pmatrix} \in M_{m_i}(\mathbb{C}).$$

Here $m_i = \nu(s_i)$. 

Note that

$$(sI - M(s_i))^{-1} = \begin{pmatrix} 1 \frac{1}{s-s_i} \frac{1}{(s-s_i)^2} \cdots \frac{1}{(s-s_i)^m_i-1} \\ \frac{1}{s-s_i} \frac{1}{(s-s_i)^2} \cdots \frac{1}{(s-s_i)^m_i-1} \\ \vdots & \vdots & \vdots \\ \frac{1}{s-s_i} & \frac{1}{s-s_i} & \cdots \frac{1}{s-s_i} \end{pmatrix}.$$  

Note that $q(Y)^s = e^{t(Y)s} = \sum_{n=0}^{\infty} \frac{t(Y)^n e^{t(Y)s}}{n!} (s-s_i)^n$. From this (v) follows by using the residue theorem. 

\[\sq]\  

3. Abstract intersection theory and its axioms  

Let $V$ be an $\mathbb{R}$-linear space endowed with a symmetric $\mathbb{R}$-bilinear form $\beta: V \times V \to \mathbb{R}$. Denote by $V_\mathbb{C}$ the complexification of $V$ given by $V_\mathbb{C} = V \otimes_\mathbb{R} \mathbb{C}$. To simplify the notation, we identify $v \otimes \alpha$ with $\alpha v$ for $v \in V$ and $\alpha \in \mathbb{C}$. Therefore we have $V \subset V_\mathbb{C}$. Then one can define the complexification $\beta_\mathbb{C}: V_\mathbb{C} \times V_\mathbb{C} \to \mathbb{C}$ of $\beta$ by

$$\beta_\mathbb{C}(\alpha_1 v_1, \alpha_2 v_2) := \alpha_1 \overline{\alpha_2} \beta(v_1, v_2) \quad (v_1, v_2 \in V, \alpha_1, \alpha_2 \in \mathbb{C}).$$  

It is easy to check that $\beta_\mathbb{C}(\alpha w_1, w_2) = \alpha \cdot \beta_\mathbb{C}(w_1, w_2)$ and $\beta_\mathbb{C}(w_2, w_1) = \overline{\beta_\mathbb{C}(w_1, w_2)}$ for $w_1, w_2 \in V_\mathbb{C}$ and $\alpha \in \mathbb{C}$. 

Let $\text{End}_\mathbb{C}(V_\mathbb{C})$ denote the set of $\mathbb{C}$-linear operators $T: V_\mathbb{C} \supset \text{dom}(T) \to V_\mathbb{C}$ such that $\text{dom}(T) = V_\mathbb{C}$. Suppose that there are nonzero vectors $v_{01}, v_{10}$ and $h_\alpha$ in $V$, maps $v_\beta: \mathcal{Y} \to V_\mathbb{C}$ and $\Phi_A: \mathcal{Y} \to \text{End}_\mathbb{C}(V_\mathbb{C})$ which satisfy the conditions listed below ((AIT1)–(AIT3)). We call a collection

$$\mathbb{A}\mathbb{I}\mathbb{T} = (V, v_{01}, v_{10}, v_\beta, h_\alpha, \beta, \Phi_A, F_A, \mathcal{H})$$

which satisfies these conditions an abstract intersection theory. The map $\Phi_A$ is associated with the operator $A$ in $\S 2$. $F_A: \mathcal{Y} \to B(H)$ along with $\mathcal{H}: \mathcal{Y} \to \Sigma_H$ is an analogue of the Frobenius morphism defined in $\S 2$, which satisfies (Frob-a) and (Frob-b). $F_A$ is related with $\Phi_A$ by the axiom (AIT3).
(AIT1) (a) $\beta(y, x) = \beta(x, y) \in \mathbb{R}$ for $x, y \in V$. $\beta_C(y, x) = \overline{\beta_C(x, y)} \in \mathbb{C}$ for $x, y \in V_C$. (b) $\beta(v_01, v_01) = 0$. (c) $\beta(v_{10}, v_{10}) = 0$. (d) $\beta(v_01, v_{10}) = 1$.

For each $Y \in \mathcal{Y}$ and all $n \geq 0$: (e) $\beta_C(\Phi_A(Y)^n v_{01}(Y), v_{01}) = 1$. (f) $\beta_C(\Phi_A(Y)^n v_{01}(Y), v_{10}) = O(q(Y)^n)$. (g) $\beta_C(\Phi_A(Y)^n v_{01}(Y), \Phi_A(Y)^n v_{10}(Y)) = O(q(Y)^n)$.

(AIT2) For $x \in V$, if $\beta(x, h_a) = 0$ then $\beta(x, x) \leq 0$.

Note that (AIT1-e)–(AIT1-g) are assumed to hold for each $Y \in \mathcal{Y}$. The Bachmann-Landau notation $O(q(Y)^n)$ in (AIT1) is with respect to $n \gg 0$ for $q(Y)$ with $Y \in \mathcal{Y}$ fixed. We call (AIT2) the Hodge property, and $h_a$ a Hodge vector.

**Lemma 3.1.** Under the assumptions (AIT1-a)–(AIT1-d) and (AIT2), we have

$$\beta(x, x) \leq 2\beta(x, v_{01})\beta(x, v_{10}) \quad (x \in V).$$

**Proof.** See the proof of [BU, Lemma 3.1].

Let the $\mathbb{R}$-bilinear form $\langle \cdot, \cdot \rangle_V : V \times V \rightarrow \mathbb{R}$ be defined by

$$(*) \quad \langle x, y \rangle_V := \beta(x, v_{01})\beta(v_{10}, y) + \beta(x, v_{10})\beta(v_{01}, y) - \beta(x, y)$$

for $x, y \in V$. By Lemma 3.1, $\langle \cdot, \cdot \rangle_V$ is positive semidefinite, i.e., $\langle x, x \rangle_V \geq 0$ for $x \in V$. Indeed, as we will see soon below (IP-b), (IP-c)), this bilinear form must be positive semidefinite, not positive definite.

We obtain the complexification $\langle \cdot, \cdot \rangle_{V_C} : V_C \times V_C \rightarrow \mathbb{C}$ of $\langle \cdot, \cdot \rangle_V : V \times V \rightarrow \mathbb{R}$ by

$$\langle \alpha_1 v_1, \alpha_2 v_2 \rangle_{V_C} := \overline{\alpha_1} \overline{\alpha_2} \langle v_1, v_2 \rangle_V$$

for $v_1, v_2 \in V$ and $\alpha_1, \alpha_2 \in \mathbb{C}$.

**Lemma 3.2.** $\langle \cdot, \cdot \rangle_{V_C}$ is positive semidefinite, i.e., $\langle x, x \rangle_{V_C} \geq 0$ for all $x \in V_C$.

**Proof.** Since for $x, y \in V$ and $t \in \mathbb{R}$,

$$\langle tx + y, tx + y \rangle_V = \langle x, x \rangle_V t^2 + 2\langle x, y \rangle_V t + \langle y, y \rangle_V \geq 0,$$

we have the Cauchy-Schwarz inequality for $\langle \cdot, \cdot \rangle_V$

$$|\langle x, y \rangle_V| \leq \sqrt{\langle x, x \rangle_V \langle y, y \rangle_V} \quad (x, y \in V),$$

provided that $\langle x, x \rangle_V \neq 0$. If $\langle x, x \rangle_V = 0$ then $\langle x, y \rangle_V$ also must be zero. Therefore we have the Cauchy-Schwarz inequality for $\langle \cdot, \cdot \rangle_V$ for any $x, y \in V$.

Let $\mathcal{V}$ be a basis of $V$. Split $\mathcal{V}$ into two disjoint sets $\mathcal{V} = \{u_{i}\}_{i \in I} \cup \{v_{j}\}_{j \in J}$ so that $\langle u_{i}, u_{i} \rangle_{V} = 0$ and $\langle v_{j}, v_{j} \rangle_{V} \neq 0$. Note that $\mathcal{V}$ is also a basis of $V_C$ with the same properties that $\langle u_{i}, u_{i} \rangle_{V_C} = 0$ and $\langle v_{j}, v_{j} \rangle_{V_C} \neq 0$. Therefore any $x \in V_C$ can be written as

$$x = \sum_{i \in I} \alpha_{x,i} u_{i} + \sum_{j \in J} \alpha_{x,j} v_{j}$$

for some finite subsets $I_x \subset I$ and $J_x \subset J$ with $\alpha_{x,i}, \alpha_{x,j} \in \mathbb{C}$.
Apply the Gram-Schmidt process to \( \{v_j\}_{j \in I_x} \) in \( V \) to obtain an orthonormal set \( \{e_j\}_{j \in J_x} \) in \( V \). Then we have
\[
x = \sum_{i \in I_x} \alpha_{x,i} u_i + \sum_{j \in J_x} \alpha'_{x,j} e_j
\]
for some \( \alpha'_{x,j} \in \mathbb{C} \).

From the Cauchy-Schwarz inequality for \( \langle \cdot, \cdot \rangle_V \), we have \( \langle u_{i_1}, u_{i_2} \rangle_V = \langle u_{i_1}, u_{i_2} \rangle_{V_C} = 0 \) for \( i_1, i_2 \in I_x \) and \( \langle u_i, e_j \rangle_V = \langle u_i, e_j \rangle_{V_C} = 0 \) for \( i \in I_x \) and \( j \in J_x \). Thus it is easy to see that \( \langle x, x \rangle_{V_C} \geq 0 \).

Note that \( \langle \cdot, \cdot \rangle_{V_C} \) is compatible with \( \beta_C \), i.e., we have
\[
\langle x, y \rangle_{V_C} = \beta_C(x, v_{01}) \beta_C(v_{10}, y) + \beta_C(x, v_{10}) \beta_C(v_{01}, y) - \beta_C(x, y).
\]

It is easy to see that from (AIT1), (f) and (**) the following conditions follow for any \( Y \in \mathcal{Y} \).

\( \textbf{(IP)} \)

(a) \( \langle y, x \rangle_V = \overline{\langle x, y \rangle}_V \in \mathbb{R} \) for \( x, y \in V \). \( \langle y, x \rangle_{V_C} = \langle x, y \rangle_{V_C} \in \mathbb{C} \) for \( x, y \in V_C \).
(b) \( \langle v_{01}, v_{01} \rangle_V = 0 \). (c) \( \langle v_{10}, v_{10} \rangle_V = 0 \). (d) \( \langle v_{01}, v_{10} \rangle = 0 \).

For each \( Y \in \mathcal{Y} \) and all \( n \geq 0 \):
(e) \( \langle \Phi_A(Y)^n v_{01}(Y), v_{01} \rangle_{V_C} = 0 \). (f) \( \langle \Phi_A(Y)^n v_{01}(Y), v_{10} \rangle_{V_C} = 0 \).
(g) \( \langle \Phi_A(Y)^n v_{01}(Y), \Phi_A(Y)^n v_{01}(Y) \rangle_{V_C} = O(q(Y)^n) \).

**Lemma 3.3.** For \( \langle \cdot, \cdot \rangle_{V_C} \), we have the Cauchy-Schwarz inequality:
\[
|\langle x, y \rangle_{V_C}| \leq \sqrt{\langle x, x \rangle_{V_C} \langle y, y \rangle_{V_C}} \quad (x, y \in V_C).
\]

**Proof.** Let \( \lambda = \langle x, x \rangle_{V_C} \). By Lemma 3.2 we have \( \lambda \geq 0 \). Note that (e.g., MacCluer [Mac, Exercise 1.7, p. 24])
\[
0 \leq \langle \lambda y - \langle y, x \rangle_{V_C} x, \lambda y - \langle y, x \rangle_{V_C} x \rangle_{V_C} = \lambda\{(\lambda \langle y, y \rangle_{V_C} - |\langle x, x \rangle_{V_C}|)^2\}.
\]
Therefore if \( \lambda > 0 \) we have the inequality. Suppose \( \lambda = 0 \). For the basis \( V \) in the proof of Lemma 3.2,
\[
x = \sum_{i \in I_x} \alpha_{x,i} u_i + \sum_{j \in J_x} \alpha_{x,j} v_j
\]
for some finite subsets \( I_x \subset I \) and \( J_x \subset J \) with \( \alpha_{x,i}, \alpha_{x,j} \in \mathbb{C} \). Applying the Gram-Schmidt process to \( \{v_j\}_{j \in I_x} \) in \( V \), obtain an orthonormal set \( \{e_j\}_{j \in J_x} \) in \( V \). Then as in the proof of Lemma 3.2 we have for some \( \alpha'_{x,j} \)
\[
x = \sum_{i \in I_x} \alpha_{x,i} u_i + \sum_{j \in J_x} \alpha'_{x,j} e_j.
\]
Since \( \lambda = 0 \) we have \( \alpha'_{x,j} = 0 \). Therefore
\[
x = \sum_{i \in I_x} \alpha_{x,i} u_i.
\]
Similarly, \( y \) can be expressed as
\[
y = \sum_{i \in I_y} \alpha_{y,i} u_i + \sum_{j \in J_y} \alpha_{y,j} v_j.
\]
for some finite subsets $I_y \subset I$ and $J_y \subset J$ with $\alpha_{y,i}, \alpha_{y,j} \in \mathbb{C}$. Since $\langle u_i, u_i \rangle_V = 0$ for $i \in I_x$, we have, by the Cauchy-Schwarz inequality for $\langle \cdot, \cdot \rangle_V$, $\langle u_{i_1}, u_{i_2} \rangle_V = \langle u_{i_1}, u_{i_2} \rangle_{V_{C}} = 0$ for $i_1 \in I_x$ and $i_2 \in I_y$ and $\langle u_i, v_j \rangle_V = \langle u_i, v_j \rangle_{V_{C}} = 0$ for $i \in I_x$ and $j \in J_y$. Thus we have $\langle x, y \rangle_{V_{C}} = 0$.  

Now we introduce axiom (AIT3), which we call the Lefschetz type formula.

\textbf{(AIT3)} For each $Y \in \mathcal{Y}$ and all $n \geq 0$,

$$\text{tr}(F_A(Y)^n) = \langle \Phi_A(Y)^n v_{\delta}(Y), v_{\delta}(Y) \rangle_{V_{C}}.$$ 

Here $\text{tr}(F_A(Y)^n)$ denotes the trace of $F_A(Y)^n$.

4. Standard models of abstract intersection theory

In this section we construct a model

$$\text{AIT}_m = (V_m, v_{01,m}, v_{10,m}, v_{\delta,m}, h_{a,m}, \beta_m, \Phi_A,m, F_{A,m}, \mathcal{H}_m)$$

of an abstract intersection theory $\text{AIT}$. We call $\text{AIT}_m$ which satisfies (AIT1)--(AIT3) a standard model of abstract intersection theory.

Recall that we have constructed the models $F_{A,m}$ and $\mathcal{H}_m$ of $F_A$ and $\mathcal{H}$ in §2. We will construct the remaining elements of the model below.

Let $\{e_i\}_{i=1}^N$ ($1 \leq N := \dim_{\mathbb{C}} H \leq \infty$) be an orthonormal basis of the $\mathbb{C}$-Hilbert space $H$. Therefore

$$H = \left\{ \sum_{i=1}^N \alpha_i e_i \mid \alpha_i \in \mathbb{C}, \sum_{i=1}^N |\alpha_i|^2 < \infty \right\}.$$ 

Let $H^1$ be an $\mathbb{R}$-Hilbert space defined by

$$H^1 := \left\{ \sum_{i=1}^N \alpha_i e_i \mid \alpha_i \in \mathbb{R}, \sum_{i=1}^N |\alpha_i|^2 < \infty \right\}.$$ 

Then we have $H^1_\mathbb{C} := H^1 \otimes_{\mathbb{R}} \mathbb{C} = H$ by identifying $e_i \otimes \alpha$ with $\alpha e_i$ for $\alpha \in \mathbb{C}$.

Define $\mathbb{R}$-linear spaces $H^0$ and $H^2$ by

$$H^0 := \{ \alpha f \mid \alpha \in \mathbb{R} \} \quad \text{and} \quad H^2 := \{ \alpha g \mid \alpha \in \mathbb{R} \}$$

with

$$\langle f, f \rangle_{H^0} := 0 \quad \text{and} \quad \langle g, g \rangle_{H^2} := 0.$$ 

\textbf{Remark 4.1.} The reason why $f \in H^0$ and $g \in H^2$ are defined so that they satisfy the above conditions for degenerate inner product is that (IP-b) and (IP-c) in §3 must be satisfied. See (IP-b) and (IP-c) in the proof of Lemma 4.1 below. 

Then the complexifications $H^0_\mathbb{C} := H^0 \otimes_{\mathbb{R}} \mathbb{C}$ and $H^2_\mathbb{C} := H^2 \otimes_{\mathbb{R}} \mathbb{C}$ are regarded naturally as

$$H^0_\mathbb{C} = \{ \alpha f \mid \alpha \in \mathbb{C} \} \quad \text{and} \quad H^2_\mathbb{C} = \{ \alpha g \mid \alpha \in \mathbb{C} \}$$

by identifying $f \otimes \alpha$ (resp. $g \otimes \alpha$) with $\alpha f$ (resp. $\alpha g$) for $\alpha \in \mathbb{C}$. 

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Let
\[ H^\bullet := H^0 \oplus H^1 \oplus H^2. \]
Here \( \oplus \) means the orthogonal direct sum. That is, we assume that \( f \) and \( g \) are linearly independent and that \( \langle f, x \rangle_{H^\bullet} = \langle x, f \rangle_{H^\bullet} = 0 \) for \( x \in H^1 \oplus H^2 \) and \( \langle g, x \rangle_{H^\bullet} = \langle x, g \rangle_{H^\bullet} = 0 \) for \( x \in H^0 \oplus H^1 \). The inner product \( \langle \cdot, \cdot \rangle_{H^\bullet} \) on \( H^\bullet \) is inherited from \( \langle \cdot, \cdot \rangle_{H^i} \ (i = 0, 1, 2) \), that is \( \langle x_i, y_i \rangle_{H^\bullet} := \langle x_i, y_i \rangle_{H^i} \) for \( x_i, y_i \in H^i \).

Define an \( \mathbb{R} \)-linear space \( V_m \) by
\[ V_m := (H^0 \otimes \mathbb{R} H^2) \oplus (H^1 \otimes \mathbb{R} H^1) \oplus (H^2 \otimes \mathbb{R} H^0) \]
with
\[ \langle x_1 \otimes x_2, y_1 \otimes y_2 \rangle_{V_m} := \langle x_1, y_1 \rangle_{H^\bullet} \langle x_2, y_2 \rangle_{H^\bullet}. \]
Since
\[ H^1 \otimes \mathbb{R} H^1 = \left\{ \sum_{i,j=1}^N \alpha_{ij} e_i \otimes e_j \mid \alpha_{ij} \in \mathbb{R}, \sum_{i,j=1}^N |\alpha_{ij}|^2 < \infty \right\}, \]
we have
\[ (H^1 \otimes \mathbb{R} H^1)_C = \left\{ \sum_{i,j=1}^N \alpha_{ij} e_i \otimes e_j \mid \alpha_{ij} \in \mathbb{C}, \sum_{i,j=1}^N |\alpha_{ij}|^2 < \infty \right\} = H^1_C \otimes_C H^1_C \]
by identifying \((e_i \otimes e_j) \otimes \alpha\) with \(\alpha e_i \otimes e_j\) for \(\alpha \in \mathbb{C}\). Note that \(\{e_i \otimes e_j\}_{i,j=1}^N\) is an orthonormal basis of the tensor products \(H^1 \otimes \mathbb{R} H^1\) and \(H^1_C \otimes_C H^1_C\). Similarly, by identifying \((f \otimes g) \otimes \alpha\) (resp. \((g \otimes f) \otimes \alpha\)) with \(\alpha f \otimes g\) (resp. \(\alpha g \otimes f\)) for \(\alpha \in \mathbb{C}\), we have
\[ (H^0 \otimes \mathbb{R} H^2)_C = \{ \alpha f \otimes g \mid \alpha \in \mathbb{C} \} = H^0_C \otimes_C H^2_C \]
and
\[ (H^2 \otimes \mathbb{R} H^0)_C = \{ \alpha g \otimes f \mid \alpha \in \mathbb{C} \} = H^2_C \otimes_C H^0_C. \]
Note that generally we have \((X \otimes \mathbb{R} Y)_C = X_C \otimes_C Y_C\).

Therefore we now have
\[ (V_m)_C = V_m \otimes \mathbb{C} = (H^0_C \otimes_C H^2_C) \oplus (H^1_C \otimes_C H^1_C) \oplus (H^2_C \otimes_C H^0_C) \]
with
\[ \langle x_1 \otimes x_2, y_1 \otimes y_2 \rangle_{(V_m)_C} = \langle x_1, y_1 \rangle_{H^\bullet_C} \langle x_2, y_2 \rangle_{H^\bullet_C}, \]
where
\[ H^\bullet_C = H^0_C \oplus H^1_C \oplus H^2_C \]
as the orthogonal direct sum. Note that the complexification \(\langle \cdot, \cdot \rangle_{H^\bullet_C}\) of the inner product \(\langle \cdot, \cdot \rangle_{H^\bullet}\) is given by \(\langle \alpha_1 x_1, \alpha_2 x_2 \rangle_{H^\bullet_C} := \bar{\alpha}_1 \overline{\alpha_2} \langle x_1, x_2 \rangle_{H^\bullet}\) for \(x_1, x_2 \in H^\bullet\) and \(\alpha_1, \alpha_2 \in \mathbb{C}\).

Extend the operator \(A\) on \(H^1_C\) (\(= H\)) to the operator \(A\) on \(H^\bullet_C\) by
\[ Af = A|_{H^0_C} f := 0 \quad \text{and} \quad Ag = A|_{H^2_C} g := g. \]

Accordingly, we extend the map \(F_{A,m} : \mathcal{Y} \to B(H)\) to \(F_{A,m} : \mathcal{Y} \to \text{End}_C(H^\bullet_C)\) so that
\[ F_{A,m}(Y)f := e^{i(Y)A|_{H^0_C}} f = f \quad \text{and} \quad F_{A,m}(Y)g := e^{i(Y)A|_{H^2_C}} g = q(Y) g \]
for \(Y \in \mathcal{Y}\). Here \(\text{End}_C(H^\bullet_C)\) denotes the set of \(\mathbb{C}\)-linear operators \(T : H^\bullet_C \supset \text{dom}(T) \to H^\bullet_C\) with \(\text{dom}(T) = H^\bullet_C\).
Let
\[ v_{01,m} := f \otimes g \in H^0 \otimes_R H^2 \subset H^0_C \otimes_C H^2_C \quad \text{and} \quad v_{10,m} := g \otimes f \in H^2 \otimes_R H^0 \subset H^2_C \otimes_C H^0_C. \]

Recall that \( \mathcal{H}_m(Y) := \text{Image}(P_{\sigma_Y(A)}) \subset H^1_C \). Recall also that \( F_{A,m}(Y)\mathcal{H}_m(Y) \subset \mathcal{H}_m(Y) \) (i.e., (Frob-a)) by Lemma 2.1 (ii). Recall that, by (OP2) and (OP3-a), \( \mathcal{H}_m(Y) \) is finite-dimensional. Let \( g(Y) := \frac{1}{2} \dim_C \mathcal{H}_m(Y) \). Let \( \{e^Y_i\}_{i=1}^{2g(Y)} \) be an orthonormal basis of \( \mathcal{H}_m(Y) \).

For each \( Y \in \mathcal{Y} \) let
\[ v_{\delta,m}(Y) := \left( \sum_{i=1}^{2g(Y)} e^Y_i \otimes e^Y_i \right) + v_{01,m} + v_{10,m} \in (V_m)_C. \]

Let \( \Phi_{A,m}(Y) := I \otimes F_{A,m}(Y) \), where \( I \) denotes the identity operator on \( H^*_C = H^0_C \oplus H^1_C \oplus H^2_C \).

**Lemma 4.1.** Suppose that an operator \( A: H \supset \text{dom}(A) \to H \) that satisfies (OP1), (OP2) and (OP3-a) is given. Then the above construction satisfies
(i) The conditions (IP-a)–(IP-f).
(ii) The Lefschetz type formula (AIT3).

**Proof.** (i) (IP-a) is obvious from definition.
(IP-b): \( \langle v_{01,m}, v_{01,m} \rangle_{V_m} = \langle f \otimes g, f \otimes g \rangle_{V_m} = \langle f, f \rangle_{H^*} \langle g, g \rangle_{H^*} = 0. \)
(IP-c): \( \langle v_{10,m}, v_{10,m} \rangle_{V_m} = \langle g \otimes f, g \otimes f \rangle_{V_m} = \langle g, g \rangle_{H^*} \langle f, f \rangle_{H^*} = 0. \)
(IP-d): \( \langle v_{01,m}, v_{10,m} \rangle_{V_m} = \langle f \otimes g, g \otimes f \rangle_{V_m} = \langle f, g \rangle_{H^*} \langle g, f \rangle_{H^*} = 0. \)
Since \( F_{A,m}(Y)^nf = f \), \( F_{A,m}(Y)^ng = q(Y)^ng \) and \( \mathcal{H}_m(Y) \) is \( F_{A,m}(Y) \)-invariant, we have
\[
\Phi_{A,m}(Y)^n v_{\delta,m}(Y) = I \otimes F_{A,m}(Y)^n \left\{ \sum_{i=1}^{2g(Y)} e^Y_i \otimes e^Y_i + f \otimes g + g \otimes f \right\}
\]
\[
= \sum_{i=1}^{2g(Y)} e^Y_i \otimes F_{A,m}(Y)^n e^Y_i + f \otimes F_{A,m}(Y)^n g + g \otimes F_{A,m}(Y)^n f
\]
\[
= \sum_{i=1}^{2g(Y)} e^Y_i \otimes F_{A,m}(Y)^n e^Y_i + f \otimes q(Y)^ng + g \otimes f
\]
\[
= \sum_{i=1}^{2g(Y)} e^Y_i \otimes F_{A,m}(Y)^n e^Y_i + q(Y)^nf \otimes g + g \otimes f.
\]
(IP-e) and (IP-f) follow from this since \( H^0 \perp H^1 \perp H^2 \) and \( \langle f, f \rangle_{H^0} = \langle g, g \rangle_{H^2} = 0. \)
(ii) To show (AIT3) note that
\[
\langle \Phi_{A,m}(Y)^n v_{\delta,m}(Y), v_{\delta,m}(Y) \rangle_{(V_m)_C}
\]
\[
= \left\langle \sum_{i=1}^{2g(Y)} e^Y_i \otimes F_{A,m}(Y)^n e^Y_i + q(Y)^n f \otimes g + g \otimes f, \sum_{j=1}^{2g(Y)} e^Y_j \otimes e^Y_j + f \otimes g + g \otimes f \right\rangle_{(V_m)_C}
\]
\[
= \sum_{i=1}^{2g(Y)} \sum_{j=1}^{2g(Y)} \langle e^Y_i, e^Y_j \rangle_{H^1_C \otimes_C H^1_C}
\]
\[
= \sum_{i=1}^{2g(Y)} \langle F_{A,m}(Y)^n e^Y_i, e^Y_i \rangle_{H^1_C}
\]
\[
= \sum_{i=1}^{2g(Y)} \langle F_{A,m}(Y)^n e^Y_i, e^Y_i \rangle_{H^1_C}
\]
\[
= \sum_{i=1}^{2g(Y)} \langle F_{A,m}(Y)^n e^Y_i, e^Y_i \rangle_{H^1_C}
\]
This completes the proof of (AIT3). □

Lemma 4.2. In the same situation as in Lemma 4.1 and its proof, suppose that (IP-g) further holds. Then there is a bilinear form $\beta_m: V_m \times V_m \to \mathbb{R}$ and a Hodge vector $h_{a,m} \in V$ which satisfy (AIT1), (AIT2), (*) and (**).

Proof.  
Proof of (AIT1), (*), and (**) : Recall that 

$$V_m = (H^0 \otimes \mathbb{R} H^2) \oplus (H^1 \otimes \mathbb{R} H^1) \oplus (H^2 \otimes \mathbb{R} H^0),$$

$$v_{01,m} = f \otimes g \in H^0 \otimes \mathbb{R} H^2 \quad \text{and} \quad v_{10,m} = g \otimes f \in H^2 \otimes \mathbb{R} H^0.$$ 

Therefore we can set 

$$\beta_m(v_{01,m}, v_{01,m}) := 0, \quad \beta_m(v_{10,m}, v_{10,m}) := 0 \quad \text{and} \quad \beta_m(v_{01,m}, v_{10,m}) = \beta_m(v_{10,m}, v_{01,m}) := 1,$$

which are (AIT1-b), (AIT1-c) and (AIT1-d), respectively. Furthermore we can set 

$$\beta_m(x, v_{01,m}) = \beta_m(v_{01,m}, x) := 0 \quad \text{and} \quad \beta_m(x, v_{10,m}) = \beta_m(v_{10,m}, x) := 0$$

for all $x \in H^1 \otimes \mathbb{R} H^1$. Therefore we have 

$$(\beta_m)_c(x, v_{01,m}) = (\beta_m)_c(v_{01,m}, x) = 0 \quad \text{and} \quad (\beta_m)_c(x, v_{10,m}) = (\beta_m)_c(v_{10,m}, x) = 0$$

for all $x \in (H^1 \otimes \mathbb{R} H^1)_c = H^1_c \otimes \mathbb{C} H^1_c$.

Now for each $Y \in \mathcal{Y}$ let 

$$v_{\delta_1,m}(Y) := \sum_{i=1}^{2g(Y)} e_i^Y \otimes e_i^Y \in (H^1 \otimes \mathbb{R} H^1)_c = H^1_c \otimes \mathbb{C} H^1_c.$$ 

Note that $v_{\delta,m}(Y) = v_{\delta_1,m}(Y) + v_{01,m} + v_{10,m}$. Recall from the proof of Lemma 4.1 that 

$$\Phi_{A,m}(Y)^n v_{\delta,m}(Y) = \sum_{i=1}^{2g(Y)} e_i^Y \otimes F_{A,m}(Y)^n e_i^Y + q(Y)^n f \otimes g + g \otimes f$$

$$= \sum_{i=1}^{2g(Y)} e_i^Y \otimes F_{A,m}(Y)^n e_i^Y + q(Y)^n v_{01,m} + v_{10,m}$$

$$= \Phi_{A,m}(Y)^n v_{\delta_1,m}(Y) + q(Y)^n v_{01,m} + v_{10,m}.$$ 

Thus, since $\Phi_{A,m}(Y)^n v_{\delta_1,m}(Y) \in (H^1 \otimes \mathbb{R} H^1)_c$, we have 

$$(\beta_m)_c(\Phi_{A,m}(Y)^n v_{\delta,m}(Y), v_{01,m}) = 1 \quad \text{and} \quad (\beta_m)_c(\Phi_{A,m}(Y)^n v_{\delta,m}(Y), v_{10,m}) = q(Y)^n = O(q(Y)^n),$$

which are (AIT1-e) and (AIT1-f), respectively. Now that we are given $\beta_m(x, v_{01,m}), \beta_m(x, v_{10,m}), \beta_m(v_{10,m}, y)$ and $\beta_m(v_{01,m}, y)$, and $(x, y)_V$, we can define $\beta_m(x, y)$ for $x, y \in V_m$ by 

$$\beta_m(x, y) := \beta_m(x, v_{01,m}) \beta_m(v_{10,m}, y) + \beta_m(x, v_{10,m}) \beta_m(v_{01,m}, y) - (x, y)_V.$$
Then we see that (AIT1-a), (⋆) and (⋆⋆) are satisfied. Now we have
\[
(\beta_m)_C(\Phi_{A,m}(Y)^n v_{\delta,m}(Y), \Phi_{A,m}(Y)^n v_{\delta,m}(Y)) \\
= (\beta_m)_C(\Phi_{A,m}(Y)^n v_{\delta,m}(Y), v_{01,m}) \cdot (\beta_m)_C(v_{10,m}, \Phi_{A,m}(Y)^n v_{\delta,m}(Y)) \\
+ (\beta_m)_C(\Phi_{A,m}(Y)^n v_{\delta,m}(Y), v_{10,m}) \cdot (\beta_m)_C(v_{01,m}, \Phi_{A,m}(Y)^n v_{\delta,m}(Y)) \\
- \langle \Phi_{A,m}(Y)^n v_{\delta,m}(Y), \Phi_{A,m}(Y)^n v_{\delta,m}(Y) \rangle_{\langle v_m \rangle_C}.
\]

(AIT1-g) follows from this and (IP-g).

Proof of (AIT2): Let \( h_{a,m} := v_{01,m} + v_{10,m} \). If \( \beta_m(x, h_{a,m}) = 0 \), then \( \beta_m(x, v_{01,m}) = -\beta_m(x, v_{10,m}) \).

Hence we have
\[
\beta_m(x, x) = 2\beta_m(x, v_{01,m})\beta_m(x, v_{10,m}) - \langle x, x \rangle_{V_m} = -2\beta_m(x, v_{01,m})^2 - \langle x, x \rangle_{V_m} \leq 0.
\]

Therefore \( h_{a,m} \) is a Hodge vector. \( \square \)

Remark 4.2. Note that, given an inner product \( \langle \cdot, \cdot \rangle_{H^1_C} \) for \( H^1_C = H \), the choice of \( \beta_m \) is not unique in our construction of standard models. \( \square \)

5. Main theorems

We use the following lemma (see, e.g., [Mon, Lemma 2.2, p. 20]) in the proof of Theorem 5.2 below.

Lemma 5.1. Let \( \lambda_i \ (1 \leq i \leq N < \infty) \) be complex numbers. Then there exist infinitely many integers \( n \geq 1 \) such that \( |\lambda_1|^n \leq |\sum_{i=1}^N \lambda_i^n| \).

Theorem 5.2. Let \( A : H \supset \text{dom}(A) \to H \) be an operator satisfying (OP1), (OP2), (OP3-a), (OP4) and (OP5).

(1) If there exists an abstract intersection theory \( \text{AIT} \) (in the sense of §3) for \( A \), then the Riemann hypothesis holds for \( A \).

(2) Suppose further that \( A \) satisfies (OP3-b). Then, there exists a standard model \( \text{AIT}_m \) for \( A \) if and only if the Riemann hypothesis holds for \( A \) and \( A \) is semi-simple.

Proof. (1): Suppose that the RH for \( A \) does not hold. Then by (OP5) one can find and fix \( Y \in \mathcal{Y} \) so that \( \sigma_Y(A) \) contains \( s, s_\beta \in \sigma(A) \) with \( \text{Re}(s) < \frac{1}{2}, \text{Re}(s_\beta) > \frac{1}{2} \), respectively. Therefore \( \sigma_Y(A) \) contains \( s_1 \) such that \( q(Y)^{\text{Re}(s_1)} > q(Y)^{\frac{1}{2}} \). Actually, if \( 0 < q(Y) < 1 \) set \( s_1 = s \), while if \( q(Y) > 1 \) set \( s_1 = s_\beta \).

Recall that \( \sigma_Y(A) \) is a finite set. Let \( s_i \ (2 \leq i \leq 2g(Y) := \dim_C \mathcal{H}(Y)) \) be all the other eigenvalues of \( A \) in \( \sigma_Y(A) \), counted with algebraic multiplicities. Let \( \lambda_i = q(Y)^{s_i} \ (1 \leq i \leq 2g(Y)) \). Then by Lemma 5.1, \( \nu_\mu := \sum_{i=1}^{2g(Y)} \lambda_i^n \) is not \( O(q(Y)^{\frac{1}{2}}) \), since we could choose \( s_1 \) so that \( |\lambda_1|^n = |q(Y)^{s_1}|^n = q(Y)^{\frac{1}{2}}(1 + \epsilon)^n \) for some \( \epsilon > 0 \).

By (Frob-b), we have
\[
\sigma(F_A(Y)^n) = \sigma_p(F_A(Y)^n) = \{q(Y)^{s_1}|s \in \sigma_Y(A)\} \cup \{0\} = \{\lambda_i^n|1 \leq i \leq 2g(Y)\} \cup \{0\}.
\]

By (AIT3) and Lemma 3.3 (the Cauchy-Schwarz inequality), we have
\[
|\nu_\mu| = |\text{tr}(F_A(Y)^n)| = |\langle \Phi_A(Y)^n v_\delta(Y), v_\delta(Y) \rangle_{V_C}| \\
\leq \sqrt{|\langle v_\delta(Y), v_\delta(Y) \rangle_{V_C}| \cdot |\langle \Phi_A(Y)^n v_\delta(Y), \Phi_A(Y)^n v_\delta(Y) \rangle_{V_C}|}.
\]
Therefore, by (IP-g), we see that $\nu_n$ is $O(q(Y)^{\frac{3}{2}})$. However, this is a contradiction.

If part of (2): By Lemma 4.1, we have (IP-a)–(IP-f) and (AIT3) for $V_m$ and $\Phi_{A,m}(Y)$. Therefore all we have to do now is to verify (IP-g) to apply Lemma 4.2. Since the RH for the operator $A$ is assumed to hold, each eigenvalue $\lambda_\ell$ ($1 \leq \ell \leq 2g(Y)$), counted with algebraic multiplicities, of $F_{A,m}(Y)$ can be written as $\lambda_\ell = q(Y)^{\frac{1}{2}}e^{i\theta_\ell}$ ($\theta_\ell \in \mathbb{R}$). By the semi-simplicity assumption for $A$, one can choose eigenvectors $w_\ell$ associated with $\lambda_\ell$ so that $F_{A,m}(Y)w_\ell = \lambda_\ell w_\ell$.

Recall that $\{e_i(Y)\}_{i=1}^{2g(Y)}(g(Y) := \frac{1}{2}\dim \mathcal{H}_m(Y))$ is an orthonormal basis of $\mathcal{H}_m(Y)$ (see §4). Now one can write $e_i^\ell$ as $e_i^\ell = \sum_{\ell=1}^{2g(Y)} \alpha_{i\ell}w_\ell$ for some $\alpha_{i\ell} \in \mathbb{C}$. Then

$$\langle \Phi_{A,m}(Y)^n v_{\delta,m}(Y), \Phi_{A,m}(Y)^n v_{\delta,m}(Y) \rangle_{(V_m)_C}$$

$$= \langle \sum_{i=1}^{2g(Y)} e_i^\ell \otimes F_{A,m}(Y)^n e_i^\ell + q(Y)^n f \otimes g + g \otimes f, \sum_{j=1}^{2g(Y)} e_j^\ell \otimes F_{A,m}(Y)^n e_j^\ell + q(Y)^n f \otimes g + g \otimes f \rangle_{(V_m)_C}$$

$$= \langle \sum_{i=1}^{2g(Y)} e_i^\ell \otimes F_{A,m}(Y)^n e_i^\ell, \sum_{j=1}^{2g(Y)} e_j^\ell \otimes F_{A,m}(Y)^n e_j^\ell \rangle_{(H^1_{\mathbb{C}} \otimes_c H^1_{\mathbb{C}})}$$

$$= \sum_{i=1}^{2g(Y)} \sum_{j=1}^{2g(Y)} \langle e_i^\ell, e_j^\ell \rangle_{H^1_{\mathbb{C}}} \langle F_{A,m}(Y)^n e_i^\ell, F_{A,m}(Y)^n e_j^\ell \rangle_{H^1_{\mathbb{C}}}$$

$$= \sum_{i=1}^{2g(Y)} \langle F_{A,m}(Y)^n e_i^\ell, F_{A,m}(Y)^n e_i^\ell \rangle_{H^1_{\mathbb{C}}}$$

$$= \sum_{i=1}^{2g(Y)} \sum_{\ell=1}^{2g(Y)} \alpha_{i\ell} F_{A,m}(Y)^n w_\ell, \sum_{m=1}^{2g(Y)} \alpha_{i\ell} F_{A,m}(Y)^n w_\ell \rangle_{H^1_{\mathbb{C}}}$$

Since $F_{A,m}(Y)^n w_\ell = \lambda_\ell^n w_\ell$, we have (IP-g). Therefore by Lemma 4.2, we have (AIT1) and (AIT2) for $V_m$.

Only if part of (2): By Lemma 2.1 (i), (ii) and (iii), $\mathbb{A} \mathbb{T}_m \Rightarrow$ RH can be proved as (1).

Let us now show $\mathbb{A} \mathbb{T}_m \Rightarrow$ semi-simplicity. Suppose that we have $\mathbb{A} \mathbb{T}_m$ but $A$ is not semi-simple to the contrary. Then one can find and fix $Y \in \mathcal{Y}$ such that

$$\sigma_Y(A) = \{s_1, s_2, \ldots, s_{N-1}, s_N\}$$

which satisfies

$$|\text{Im}(s_1)| < |\text{Im}(s_2)| < \cdots < |\text{Im}(s_{N-1})| < |\text{Im}(s_N)|$$

with

$$\nu(s_1) = \nu(s_2) = \cdots = \nu(s_{N-1}) = 1 \quad \text{and} \quad \nu(s_N) > 1.$$  

What we want to do is to calculate

$$\langle \Phi_{A,m}(Y)^n v_{\delta,m}(Y), \Phi_{A,m}(Y)^n v_{\delta,m}(Y) \rangle_{(V_m)_C}$$

and show that it is not of order $O(q(Y)^n)$, which contradicts (IP-g).

We use the notation in Lemma 2.1 and its proof. Then $m_i = \dim \mathbb{C} H(s_i) = \nu(s_i) = \text{mult}(s_i)$ ($1 \leq i \leq N$) by (OP3-b). We regard $H(s_i)$ as $\mathbb{C}^m$, that is $H(s_i) \simeq \mathbb{C}^m$. Then we can take a
basis $w_{i,\ell} \in \mathbb{C}^{m_i}$ ($1 \leq \ell \leq m_i = \dim_{\mathbb{C}} H(s_i)$) of the form

$$w_{i,\ell} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \cdot \ell$$

so that $e^{t(Y)A|_{H(s_i)}}$ can be written in the matrix form $N(s_i)$ as in Lemma 2.1 (v). In other words, $w_{i,\ell}$ ($1 \leq \ell \leq m_i$) are generalized eigenvectors of $M(s_i) = A|_{H(s_i)}$. Let

$$J_i := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \cdots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \in M_{m_i}(\mathbb{C}).$$

Since $J_i^m = 0$ for $m \geq m_i$, $N(s_i)$ in Lemma 2.1 (v) can be written as

$$N(s_i) = e^{t(Y)s_i} \begin{pmatrix} \frac{t(Y)^0}{0!} & \frac{t(Y)^1}{1!} & \cdots & \frac{t(Y)^{m_i-1}}{(m_i-1)!} \\ \frac{t(Y)^0}{0!} & \frac{t(Y)^1}{1!} & \cdots & \frac{t(Y)^{m_i-1}}{(m_i-1)!} \\ \vdots & \ddots & \ddots & \cdots \\ \frac{t(Y)^0}{0!} & \frac{t(Y)^1}{1!} & \cdots & \frac{t(Y)^{m_i-1}}{(m_i-1)!} \end{pmatrix}$$

$$= e^{t(Y)s_i} \sum_{k=0}^{\infty} \frac{(t(Y)J_i)^k}{k!}$$

$$= e^{t(Y)s_i} \sum_{k=0}^{m_i-1} \frac{(t(Y)J_i)^k}{k!}$$

$$= e^{t(Y)s_i} e^{t(Y)J_i}.$$

Note that

$$e^{nt(Y)J_i} w_{i,\ell} = \begin{pmatrix} \frac{(nt(Y))^{\ell-1}}{(\ell-1)!} \\ \frac{(nt(Y))^{\ell-2}}{(\ell-2)!} \\ \vdots \\ \frac{(nt(Y))^0}{0!} \\ 0 \end{pmatrix} = \sum_{k=1}^{\ell} \frac{(nt(Y))^{k-1}}{(k-1)!} w_{i,\ell-k+1}$$

for $1 \leq i \leq N$ and $1 \leq \ell \leq m_i$.

Recall that $\{ e^Y \}_{\mu=1}^{2g(Y)}$ is an orthonormal basis of $\mathcal{H}_m(Y)$. Thus $e^Y_\mu$ can be written as

$$e^Y_\mu = \sum_{i=1}^{N} \sum_{\ell=1}^{m_i} \alpha_{i,\ell}^\mu w_{i,\ell}$$
for some $\alpha_{i,\ell}^\mu \in \mathbb{C}$. Then we have by Lemma 2.1 (iv) and (v)

$$F_{A,m}(Y)^n e^Y = \sum_{i=1}^N F_{A,m}(Y)^n \sum_{\ell=1}^{m_i} \alpha_{i,\ell}^\mu w_{i,\ell} = \sum_{i=1}^N N(s_i)^n \sum_{\ell=1}^{m_i} \alpha_{i,\ell}^\mu w_{i,\ell} = \sum_{i=1}^N \sum_{\ell=1}^{m_i} \alpha_{i,\ell}^\mu N(s_i)^n w_{i,\ell}.$$ 

Recall from the proof of the If part of (2) that

$$\langle \Phi_{A,m}(Y)^n \nu_{\delta,m}(Y), \Phi_{A,m}(Y)^n \nu_{\delta,m}(Y) \rangle_{(Y_m)_C} = \sum_{\mu=1}^{2g(Y)} \langle F_{A,m}(Y)^n e^Y, F_{A,m}(Y)^n e^Y \rangle_{H^1_C}.$$ 

Now using (5.1) and (5.2) we have

$$\langle F_{A,m}(Y)^n e^Y, F_{A,m}(Y)^n e^Y \rangle_{H^1_C} = \sum_{i=1}^N \sum_{\ell=1}^{m_i} \alpha_{i,\ell}^\mu \sum_{j=1}^N \sum_{m=1}^{m_i} \alpha_{j,m}^\mu (N(s_i)^n w_{i,\ell}, N(s_j)^n w_{j,m})_{H^1_C}.$$ 

Let $M_\mu := \max \{\ell | \alpha_{N,\ell}^\mu \neq 0\}$. Then $\alpha_{N,\ell}^\mu \alpha_{N,1}^\mu = 0$ if $\ell > M_\mu$ or $m > M_\mu$. Note that $\text{Re}(s_i) = \frac{1}{2}$ (\forall i) since the RH holds by $\mathbb{A} \mathbb{T}_m \Rightarrow \text{RH}$. Recall that $q(Y) = e^{t(Y)}$. Therefore we have

$$\langle F_{A,m}(Y)^n e^Y, F_{A,m}(Y)^n e^Y \rangle_{H^1_C} = \alpha_{N,M_\mu}^\mu \sum_{\ell=1}^{m_i} \alpha_{N,\ell}^\mu \sum_{j=1}^N \sum_{m=1}^{m_i} \alpha_{j,m}^\mu \sum_{a=1}^{\ell} (nt(Y))^{a-1} w_{i,\ell-a+1} + \sum_{b=1}^m (nt(Y))^{b-1} w_{j,m-b+1}.$$ 

Let $M := \max \{M_\mu | 1 \leq \mu \leq 2g(Y)\}$. Since $e^Y (1 \leq \mu \leq 2g(Y))$ are a basis of $\mathcal{H}_m(Y)$, $\alpha_{N,m}^\mu \neq 0$ for at least one $\mu$. Hence we have $M = m_N > 1$. Now we have

$$\sum_{\mu=1}^{2g(Y)} \langle F_{A,m}(Y)^n e^Y, F_{A,m}(Y)^n e^Y \rangle_{H^1_C} = \sum_{\mu=m_N}^{2g(Y)} \left( C_\mu q(Y)^n n^{2(m_N-1)} + O(q(Y)^n n^{2m_N-3}) \right) \neq O(q(Y)^n).$$
which contradicts (IP-g). This completes the proof. □

In our previous paper [BU] we constructed a model of abstract intersection theory based on an analogue of the GNS (Gelfand-Naimark-Segal) representation. Let us call this model which satisfies (INT1)–(INT3) in [BU] a GNS model and denote it as $\text{AIT}_{GNS}$. The method of the proof of the above theorem also applies to this model. Therefore we have the following theorem.

**Theorem 5.3.** Let $A: H \supset \text{dom}(A) \to H$ be an operator satisfying (OP1), (OP2), (OP3), (OP4) and (OP5). Suppose further that $A$ satisfies (OP5-b) in [BU]. Then there exists a GNS model $\text{AIT}_{GNS}$ for $A$ if and only if the Riemann hypothesis holds for $A$ and $A$ is semi-simple.

We say that $L(s, \chi)$ satisfies the Riemann hypothesis if any nontrivial zero $s_i$ of $L(s, \chi)$ satisfies $\text{Re}(s_i) = \frac{1}{2}$. We say that a nontrivial zero $s_i$ of $L(s, \chi)$ is simple if it is a zero of $L(s, \chi)$ of order one.

Combining Theorems 5.2 and 5.3 with Theorem 4.1 (iv) of [U] (see Remark 2.1 (4)) we obtain the following theorem.

**Theorem 5.4.** Let $A: H \supset \text{dom}(A) \to H$ be an operator constructed in [U] corresponding to the Dirichlet $L$-function $L(s, \chi)$ associated with a congruence subgroup $\Gamma$ of $SL_2(\mathbb{Z})$. Then

1. $L(s, \chi)$ satisfies the Riemann hypothesis and its all nontrivial zeros are simple if and only if there exists a standard model $\text{AIT}_m$ for $A$.

2. $L(s, \chi)$ satisfies the Riemann hypothesis and its all nontrivial zeros are simple if and only if there exists a GNS model $\text{AIT}_{GNS}$ for $A$.

**Remark 5.1.** In the above theorem if $\Gamma = SL_2(\mathbb{Z})$ then the Dirichlet $L$-function $L(s, \chi)$ reduces to the Riemann zeta-function $\zeta(s)$. □

6. Analogy with the classical theory

Recall that Weil’s explicit formula (according to Patterson [Pat]) reads as follows:

\[ \phi(0) + \phi(1) - \sum_{\rho} \phi(\rho) = W_\infty(f) + \sum_{p: \text{prime}} \log p \sum_{n=1}^{\infty} \{f(p^n) + f(p^{-n})\} p^{-\frac{s+\sigma}{2}}. \]

Here $f$ is a fast decreasing function on $\mathbb{R}_+$, $\phi$ is the Mellin transform of $f$, $W_\infty$ is an appropriate functional of $f$, and $\rho$ runs over nontrivial zeros of the Riemann zeta-function (or the $L$-function), counted with multiplicity. For the original work of Weil, see [1952b] and [1972] of [W2]. See also [C] and [CM, p. 344].

The idea of introducing the model $F_{A,m}(Y)$ of an analogue of the Frobenius morphism in this paper is hinted by the spectral side of the above formula. By Lemma 2.2 of [BU, p. 702] there is a function $\phi_Y(s)$ ($Y \in \mathcal{Y}$) which is analytic in an open set $\supset \Omega_\infty$ such that

i. $\phi_Y(0) = 1$,

ii. $\phi_Y(1) = q(Y)$,

iii. $\phi_Y(s_i) = q(Y)^{s_i}$ if $s_i \in \sigma_Y(A)$,

iv. $\lim_{s \to s_i} \frac{\phi_Y(s)}{(s - s_i)^{m_i}} = c_{Y,i} \in \mathbb{C}$ for some $c_{Y,i} \neq 0$ if $s_i \in \sigma(A) - \sigma_Y(A)$ with $\nu(s_i) = m_i$. 

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For this \( \phi_Y(s) \), let \( \phi_Y(A) : H \supset \text{dom}(\phi_Y(A)) \to H \) be defined by

\[
\phi_Y(A)x := \lim_{T \to \infty} \frac{1}{2\pi i} \left( \oint_{\partial \Omega_T} \phi_Y(s)(sI - A)^{-1}ds \right)x
\]

for

\[
x \in \text{dom}(\phi_Y(A)) := \{ x \in H \mid \text{the limit } \phi_Y(A)x \text{ exists in } H \}.
\]

Then it is easy to prove that \( \text{dom}(\phi_Y(A)) = H \) and that

\[
\phi_Y(A) = F_{A,m}(Y).
\]

For the proof use \( (sI - M(s_i))^{-1} \) in the proof of Lemma 2.1. It is also easy to see that

\[
\text{tr}(\phi_Y(A)) = \sum_{s_i \in \sigma_Y(A)} \text{mult}(s_i)\phi_Y(s_i).
\]

Let \( C \) be a smooth projective curve (one-dimensional scheme) over a finite field \( \mathbb{F}_q \). Let \( \text{Frob} \) be the Frobenius morphism on \( C \). Then \( F_A(Y) \) in §2 is an analogue of \( \text{Frob} \).

For \( S = C \times C \), the surface over \( \mathbb{F}_q \), let \( \text{Pic}(S) \) be its Picard group, which we regard as a \( \mathbb{Z} \)-module, so as to preserve the analogy with Weil divisors. The \( \mathbb{R} \)-linear space \( V \) in §3 is modeled on \( \text{Pic}(S) \otimes_{\mathbb{Z}} \mathbb{R} \). The \( \mathbb{R} \)-bilinear form \( \beta(\cdot, \cdot) \) in §3 is modeled on the \( \mathbb{R} \)-tensored intersection pairing \( \langle \cdot, \cdot \rangle \) on \( \text{Pic}(S) \otimes_{\mathbb{Z}} \mathbb{R} \).

The operator \( \Phi_A(Y) \) in (AIT1) is an analogue of the linear map on \( \text{Pic}(S) \otimes_{\mathbb{Z}} \mathbb{R} \) induced by the morphism \( \text{id} \times \text{Frob} \) on \( S \). Then one may regard \( v_{01}, v_{10}, v_\delta(Y) \) and \( \Phi_A(Y)^n v_\delta(Y) \) in (AIT1) as analogues of cycles \( pt \times C, C \times pt, \Delta \) and \( \Gamma_{\text{Frob}^n} \) in \( \text{Pic}(S) \), respectively. Here \( \Delta \) is the diagonal, and \( \Gamma_{\text{Frob}^n} \) is the graph of \( \text{Frob}^n \). So here is the dictionary.

| \( \text{Pic}(S) \otimes_{\mathbb{Z}} \mathbb{R} \) | \( V \) |
|-----------------|------|
| \( pt \times C \) | \( v_{01} \) |
| \( C \times pt \) | \( v_{10} \) |
| \( \Delta \) | \( v_\delta(Y) \) |
| \( \Gamma_{\text{Frob}^n} \) | \( \Phi_A(Y)^n v_\delta(Y) \) |

The cycles \( pt \times C, C \times pt, \Delta \) and \( \Gamma_{\text{Frob}^n} \) have the following properties.

(i) \( i(pt \times C, pt \times C) = 0 \). (ii) \( i(C \times pt, C \times pt) = 0 \). (iii) \( i(pt \times C, C \times pt) = 1 \).

(iv) \( i(\Gamma_{\text{Frob}^n}, pt \times C) = 0 \). (v) \( i(\Gamma_{\text{Frob}^n}, C \times pt) = q^n \). (vi) \( i(\Gamma_{\text{Frob}^n}, \Gamma_{\text{Frob}^n}) = q^n \).

The axioms of (AIT1) are analogues of these properties.

The Hodge property in (AIT2) comes from the classical Hodge index theorem. A Hodge vector \( h_a \) corresponds to an ample hyperplane section of \( S \), thereby \( \beta(\cdot, h_a) \) gives an analogue of the degree function \( \text{deg} \otimes_{\mathbb{Z}} 1 : \text{Pic}(S) \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{R} \). Lemma 3.1 is an analogue of the inequality of Castelnuovo-Severi.

The construction of \( V_m \) of a standard model in §4 is hinted by the Künneth formula for the étale cohomology. The Tate conjecture for \( S = C \times C \) and codimension one is equivalent to that the map \( \text{Pic}S \otimes \mathbb{Q}_\ell \to H^2_{et}(S, \mathbb{Q}_\ell(1)) \) is bijective (Proposition (4.3) of Tate [T2]). Note that \( H^2_{et}(S, \mathbb{Q}_\ell(1)) = H^2_{et}(S, \mathbb{Q}_\ell(1))^{\text{Gal}({\overline{F_q}}/F_q)} \), where \( S = S \times_{F_q} \overline{F_q} \) (see [T2]). Tate [T1] himself has proven his conjecture for abelian varieties over finite fields for the case of codimension one. From this the Tate conjecture follows also for \( S = C \times C \) in the codimension one case. By the Künneth formula for \( \ell \)-adic cohomology we have
Here $C = C \times_{F_q} \overline{\mathbb{F}_q}$. The definition of the $\mathbb{R}$-linear space $V_m$ is modeled on this. For the Künneth formula for $\ell$-adic cohomology see Chap. 6, §8 of Milne [Mil].

For a morphism $\varphi: C \to C$, the Lefschetz fixed-point formula for the $\ell$-adic étale cohomology group $H^i_{\text{ét}}(C, \mathbb{Q}_\ell)$ is given by

$$\text{tr}(\varphi^m|_{H^0_{\text{ét}}}) - \text{tr}(\varphi^m|_{H^1_{\text{ét}}}) + \text{tr}(\varphi^m|_{H^2_{\text{ét}}}) = i(\Gamma_{\varphi^m}, \Delta),$$

where $\Gamma_{\varphi^m}$ is the graph of $\varphi^m$. If $\varphi = \text{Frob}$, then it turns out that

$$\text{tr}(\varphi^m|_{H^0_{\text{ét}}}) = 1 = i(\Gamma_{\varphi^m}, \text{pt} \times C)i(\Delta, C \times \text{pt})$$

and

$$\text{tr}(\varphi^m|_{H^2_{\text{ét}}}) = q^n = i(\Gamma_{\varphi^m}, C \times \text{pt})i(\Delta, \text{pt} \times C).$$

So the Lefschetz fixed-point formula reads for $\varphi^n = \text{Frob}^n$ as

$$\text{tr}(\varphi^m|_{H^0_{\text{ét}}}) = i(\Gamma_{\varphi^n}, \text{pt} \times C)i(\Delta, C \times \text{pt}) + i(\Gamma_{\varphi^n}, C \times \text{pt})i(\Delta, \text{pt} \times C) - i(\Gamma_{\varphi^n}, \Delta) =: (\Gamma_{\varphi^n}, \Delta)_{\text{tr}(S)}(\otimes_{\mathbb{R}}),$$

(AIT3) is modeled on this. Consider the operators $A$ and $F_{A,m}(Y)$ ($Y \in \mathcal{Y}$) which are extended to $H^*_{\mathfrak{C}} = H^0_{\mathfrak{C}} \oplus H^1_{\mathfrak{C}} \oplus H^2_{\mathfrak{C}}$ as in §4. Then we have

$$\phi_Y(A)f = F_{A,m}(Y)f = f = \phi_Y(0)f \quad \text{and} \quad \phi_Y(A)g = F_{A,m}g = q(Y)g = \phi_Y(1)g$$

for $f \in H^0_{\mathfrak{C}}$ and $g \in H^2_{\mathfrak{C}}$. The operator $\phi_Y(A)$ acting on $H^i_{\mathfrak{C}}$ is an analogue of $\text{Frob}^*$ acting on $H^i_{\text{ét}}$ ($i = 0, 1, 2$). Since

$$\Phi_{A,m}(Y)^n v_{\delta,m}(Y) = \sum_{i=1}^{2g(Y)} e_i^Y \otimes F_{A,m}(Y)^n e_i^Y + f \otimes F_{A,m}(Y)^n g + g \otimes F_{A,m}(Y)^n f$$

$$= \sum_{i=1}^{2g(Y)} e_i^Y \otimes F_{A,m}(Y)^n e_i^Y + \phi_Y(1)^n v_{01,m} + \phi_Y(0)^n v_{10,m}$$

(see the proof of Lemma 4.2), we have by the setting of the proof of Lemma 4.2

$$\text{tr}(\phi_Y(A)^n|_{H^0_{\mathfrak{C}}}) = \phi_Y(0)^n = (\beta_m)_{\mathfrak{C}}(\Phi_{A,m}(Y)^n v_{\delta,m}(Y), v_{01,m}) \cdot (\beta_m)_{\mathfrak{C}}(v_{10,m}, v_{\delta,m}(Y))$$

and

$$\text{tr}(\phi_Y(A)^n|_{H^2_{\mathfrak{C}}}) = \phi_Y(1)^n = (\beta_m)_{\mathfrak{C}}(\Phi_{A,m}(Y)^n v_{\delta,m}(Y), v_{01,m}) \cdot (\beta_m)_{\mathfrak{C}}(v_{01,m}, v_{\delta,m}(Y)).$$

Therefore we have by (**)

$$\text{tr}(\phi_Y(A)^n|_{H^0_{\mathfrak{C}}}) - \text{tr}(\phi_Y(A)^n|_{H^2_{\mathfrak{C}}}) + \text{tr}(\phi_Y(A)^n|_{H^2_{\mathfrak{C}}}) = (\beta_m)_{\mathfrak{C}}(\Phi_{A,m}(Y)^n v_{\delta,m}(Y), v_{\delta,m}(Y)),$$

which is equivalent to (AIT3).
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