Quasi regular polygons and their duals with Coxeter symmetries $D_n$ represented by complex numbers

To cite this article: M Koca and N O Koca 2011 J. Phys.: Conf. Ser. 284 012039

View the article online for updates and enhancements.
Quasi regular polygons and their duals with Coxeter symmetries $D_n$, represented by complex numbers

M Koca$^1$ and N O Koca$^2$

$^1, 2$ Department of Physics, College of Science, Sultan Qaboos University, P.O. Box 36, Al- Khoud, 123 Muscat, Sultanate of Oman

E-mail: $^1$kocam@squ.edu.om; $^2$nazife@squ.edu.om

Abstract. This paper deals with tiling of the plane by quasi regular polygons and their duals. The problem is motivated from the fact that the graphene, infinite number of carbon molecules forming a honeycomb lattice, may have states with two bond lengths and equal bond angles or one bond length and different bond angles. We prove that the Euclidean plane can be tiled with two tiles consisting of quasi regular hexagons with two different lengths (isogonal hexagons) and regular hexagons. The dual lattice is constructed with the isotoxal hexagons (equal edges but two different interior angles) and regular hexagons. We also give similar tilings of the plane with the quasi regular polygons along with the regular polygons possessing the Coxeter symmetries $D_n$, $n=2,3,4,5$. The group elements as well as the vertices of the polygons are represented by the complex numbers.

1. Introduction

The graphene [1], an infinite sheet of carbon atoms [2], tiled with regular hexagons has attracted much attention. In this paper we approach the problem from the mathematical point of view and deal with tiling of the plane by quasi regular polygons possessing dihedral symmetries. We use the rank-2 Coxeter diagrams to describe the symmetries of the polygons. The polygons are of two types: the isogonal polygons consisting of two alternating unequal edges with equal interior angles; the isotoxal polygons consisting of equal edges but with alternating unequal interior angles. The isogonal polygon with $2n$ sides is vertex transitive under the dihedral group $D_n$. Its dual polygon, the isotoxal polygon, is edge transitive under the same symmetry. The tiling of the plane with the isogonal and the isotoxal hexagons is an interesting problem by itself. The problem can be extended to the tiling of the plane by the isogonal polygons with the Coxeter symmetries $D_n$, $n=4$ and $5$. We also give the dual tilings of the plane consisting of isotoxal polygons as well as corresponding regular polygons. The tiling of the plane with two isotoxal hexagons and one regular hexagon at the same vertex suggests that the graphene may have a state with equal bond lengths but with different bond angles. This is a slightly modified version of the honeycomb model of graphene. One can find some quasi regular 2D tiles in the reference [3]. In three dimensions a superspace-group approach has been formulated for the description of the composite crystals [4].

In section 2 we introduce rank-2 Coxeter diagrams [5] describing the root spaces as well as dual spaces by complex numbers. In section 3 we construct some even-sided polygons with alternating two edge lengths (isogonal polygons) having equal interior angles. We also construct their dual polygons (isotoxal polygons) with equal edge lengths but with alternating two interior angles. With this
approach we observe that in the case of $W(A_2) \approx D_3$ symmetry the Euclidean plane can be tiled by two isogonal hexagons having two different edge lengths and a regular hexagon all sharing the same vertex. The quasi regular polygons with dihedral symmetries $W(B_2) \approx D_4$ and $W(H_2) \approx D_3$ are also constructed and the tiling of the plane with the isogonal and isotoxal octagons and decagons are discussed. Section 4 is devoted to the concluding remarks.

2. Coxeter diagrams with complex numbers

All Coxeter diagrams are represented with one type of simple roots [6] contrary to the Dynkin diagrams representing the Lie algebra root systems having long and short roots. In 2D space we have an infinite number of Coxeter diagrams shown in figure 1. It is customary to use the notations $I_2(n)$ for the rank-2 Coxeter diagrams but we will continue using the notations $A_i \oplus A_1$, $A_2$, $B_2$ and $H_2$ for $I_2(n)$, when $n=2,3,4,5$ respectively.

Figure 1. Rank-2 Coxeter diagram $I_2(n)$.

Figure 1 implies that the angle between the simple roots $\alpha_i$ and $\alpha_2$ is $\frac{(n-1)\pi}{n}$. We choose the norm of the roots $\sqrt{2}$ to be consistent with the Dynkin diagrams when they coincide. The Cartan matrix defined by the scalar product $C_{ij} = (\alpha_i, \alpha_j)$ of the rank-2 diagrams and its inverse $(C^{-1})_{ij} = (\omega_i, \omega_j)$, $(i,j=1,2)$ will read

$$C = \begin{bmatrix}
2 & 2 \cos \frac{(n-1)\pi}{n} \\
2 \cos \frac{(n-1)\pi}{n} & 2 
\end{bmatrix},$$

$$C^{-1} = \frac{1}{4 \sin^2 \frac{(n-1)\pi}{n}} \begin{bmatrix}
2 & -2 \cos \frac{(n-1)\pi}{n} \\
-2 \cos \frac{(n-1)\pi}{n} & 2 
\end{bmatrix}. \quad (1)$$

The fundamental weights $\omega_i$ are the basis vectors of the dual space defined by the relation $(\alpha_i, \omega_j) = \delta_{ij}$ [6] where $\delta_{ij}$ is the Kronecker delta. The simple roots and the fundamental weights are related to each other by the relations:

$$\omega_i = (C^{-1})_{ij} \alpha_j, \quad \alpha_i = C_{ij} \omega_j. \quad (2)$$

Summation over the repeated index is implicit. Action of the reflection generator $r_i$ on an arbitrary vector $\Lambda$ is defined by the relation

$$r_i \Lambda = \Lambda - (\Lambda, \alpha_i) \alpha_i \quad (\text{no summation over } i). \quad (3)$$

They generate the dihedral group $D_n$ of order $2n$ satisfying the relations $r_i^2 = r_2^2 = (r_i r_2)^n = 1$. When $\Lambda$ and $\alpha_i$ are represented by complex numbers, equation (3) can be written as
Here the roots are complex numbers for rank-2 Coxeter diagrams, a subset of quaternions, where the unit complex numbers $\alpha_1' = \alpha_1 / \sqrt{2}$ and $\alpha_2' = \alpha_2 / \sqrt{2}$ generate the cyclic group of order $n$; however, the generators in equation (4) generate the dihedral group of order $2n$. Let $\beta = \exp(i \frac{\pi}{n})$ be the unit complex number. Then all integer powers $\{\beta^k, k = 1, 2, ..., 2n\}$ constitute a scaled copy of the root system of rank-2 Coxeter diagrams. If $\Lambda$ represents any complex number, the generators act on $\Lambda$ as follows:

$$r_1\Lambda = -\overline{\Lambda} ; \quad r_2\Lambda = -e^{\frac{2(n-1)\pi}{n}} \overline{\Lambda}.$$  (5)

Let us follow the Lie algebraic technique to obtain the orbits of the Coxeter groups. If $g$ is the Lie algebra of rank $l$ then the highest weight

$$\Lambda = a_1 \omega_1 + a_2 \omega_2 + ... + a_l \omega_l \equiv (a_1, a_2, ..., a_l)$$  (6)

is represented by the $l$ non-negative integers [7]. Applying the Coxeter-Weyl group $W(g)$ on the highest weight one can generate the $\Lambda$-orbit $\equiv O(\Lambda) = W(g) \Lambda$. In 2D space the orbits of the Coxeter group is either regular polygons or even-sided quasi regular polygons. In what follows, we will discuss the orbits of the Coxeter groups with $n = 2, 3, 4, 5$. The fundamental weights in terms of complex numbers can be written as

$$\omega_k = \frac{1}{\sqrt{2}} \left(1 - i \frac{\cos (\frac{(n-1)\pi}{n})}{\sin (\frac{(n-1)\pi}{n})}\right), \quad \omega_k = \frac{i}{\sqrt{2} \sin \frac{(n-1)\pi}{n}}.$$  (7)

A general vertex $\Lambda = a_1 \omega_1 + a_2 \omega_2$ is then an arbitrary complex number which can be written as

$$\Lambda = \frac{1}{\sqrt{2}} \left[a_1 + i \frac{1}{\sin (\frac{(n-1)\pi}{n})} \left(-a_1 \cos \frac{(n-1)\pi}{n} + a_2\right)\right]$$  (8a)

and the generators act as follows,

$$r_i \Lambda = -\overline{\Lambda} ; \quad r_2 \Lambda = -e^{\frac{2(n-1)\pi}{n}} \overline{\Lambda} ; \quad r_2r_2^* \Lambda = e^{\frac{2\pi}{n}} \Lambda ; \quad r_2r_2 \Lambda = e^{\frac{2\pi}{n}} \Lambda .$$  (8b)

It is clear from equation (8b) that $r_1^2 = r_2^2 = (r_1r_2)^n = (r_2r_1)^n = 1$.

The $2n$ vertices of the polygon can be determined as

$$(r_2)^{2k} \Lambda = e^{\frac{2\pi k}{n}} \Lambda , \quad (r_2)^{2k+1} \Lambda = -e^{\frac{2\pi k}{n}} \Lambda , \quad k = 1, 2, ..., n .$$  (9)

For $a_1 = a_2 = a$ the $2n$ vertices of a regular polygon of edge length $\sqrt{2}a$ are given in terms of complex numbers by a simple formula

$$\frac{a}{\sqrt{2} \cos (\frac{(n-1)\pi}{n})} e^{\frac{2\pi i}{2n} \sin (\frac{(n-1)\pi}{n})}, \quad \frac{-a}{\sqrt{2} \cos (\frac{(n-1)\pi}{n})} e^{\frac{2\pi i}{2n} \sin (\frac{(n-1)\pi}{n})}, \quad k = 1, 2, ..., n .$$  (10)
3. Construction of the quasi regular polygons and the tiling of the plane

3.1. \( n = 2 \) with \( D_2 = C_2 \times C_2 \) symmetry

Two simple roots \( \alpha_1 = \sqrt{2} \) and \( \alpha_2 = \sqrt{2}i \), representing the Coxeter diagram \( A_1 \oplus A_1 \), are orthogonal to each other where the generators form the Klein’s four-group \( D_2 = C_2 \times C_2 \). A general orbit is obtained by acting the group elements on the vector \( \Lambda = \frac{1}{\sqrt{2}}(\alpha_1 + i\alpha_2) \). For \( \Lambda = \omega_1 \equiv (10) \) and \( \Lambda = \omega_2 \equiv (01) \), the orbits are two segments of straight lines perpendicular to each other. The orbit \( O(\Lambda = \omega_1 + \omega_2) \equiv O(11) \) involves the vectors \( \pm \omega_1 = \pm \frac{1}{\sqrt{2}} \) and \( \pm \omega_2 = \pm \frac{1}{\sqrt{2}}i \) which form a square. For all rank-2 Coxeter groups when the orbit is derived from the vector \( \Lambda = a(\omega_1 + \omega_2) \equiv a(11) \), where \( a \) is an arbitrary real number, then the polygon has an additional symmetry. It is the symmetry of the Coxeter-Dynkin diagram which can be defined by the generator \( \gamma : \alpha_1 \leftrightarrow \alpha_2 \) leading to a larger symmetry \( W(g) : C_2 \) where \( (\cdot) \) denotes the semi-direct product of two groups. In the above case the group is \( D_4 \approx (C_2 \times C_2) : C_2 \approx C_4 : C_4' \) of order 8.

When we consider the most general case, namely, \( \Lambda = a_1\omega_1 + a_2\omega_2 \) the orbit represents a rectangle of sides \( \sqrt{2}a_1, \sqrt{2}a_2 \). The rectangle is an isogonal polygon with the \( D_2 \) symmetry. The dual of the rectangle is a rhombus (an isotoxal polygon) whose vertices can be determined by taking the mid point of one of the edge of the rectangle, say, the vector \( a_1\omega_1 \). We take the other vector \( \lambda\omega_2 \) bisecting the edge of length \( \sqrt{2}a_1 \) of the rectangle. To determine the dual of the rectangle the line joining these vectors \( \lambda\omega_2 - a_1\omega_1 \) must be orthogonal to the vector \( \Lambda = a_1\omega_1 + a_2\omega_2 \) which determines the scale factor \( \lambda = \frac{a_1^2}{a_2} \) the vertices of which consist of two fundamental orbits \( O(a_10) = \{ \pm a_1\omega_1 \} \) and \( \lambda O(10) = \{ \pm \lambda\omega_2 \} \) representing a rhombus. The rectangle is vertex transitive under the Klein’s group \( C_2 \times C_2 \) and its dual rhombus is edge transitive. For the values \( a_1 = 1 \) and \( a_2 = 2 \) the rectangle and its dual rhombus are given in figure 2.

**Figure 2.** The rectangle (a) and its dual rhombus (b) possessing the symmetry \( D_2 = C_2 \times C_2 \).

3.2 \( n = 3 \) with \( W(A_3) \approx D_3 \approx S_4 \), symmetry

The Coxeter group \( W(A_3) \approx D_3 \approx S_4 \) consists of six elements. Three elements \( r_1, r_2, r_1r_2r_1 = r_2r_1r_2 \) represent reflections with respect to the lines orthogonal to the roots \( \alpha_1, \alpha_2, \alpha_1 + \alpha_2 \) respectively and the rotational group elements \( 1, r_1r_2 \) and \( (r_1r_2)^2 \) represent the cyclic group \( C_3 \) which rotates the system by \( 120^\circ \). We have two fundamental orbits \( O(10) = \{ \omega_1, \omega_2 - \omega_1, -\omega_2 \} \) and \( O(01) = \{ \omega_2, -\omega_1, \omega_1 - \omega_2 \} \). Each orbit represents an equilateral
triangle, dual to each other which are transformed to each other by the Coxeter-Dynkin diagram symmetry.

The orbit \( O(11) = O(\omega_1 + \omega_2) \) represents a regular hexagon which has a larger symmetry \( D_3 \approx D_6 \) because of the diagram symmetry \( \gamma : \alpha_i \leftrightarrow \alpha_2 \). Now we discuss the orbit obtained from the vector \( \Lambda = a_1\omega_1 + a_2\omega_2 \) where \( a_1 \neq a_2 \neq 0 \). The Coxeter group \( W(A_2) \) generates the six elements of the orbit \( O(\Lambda) = O(a_1, a_2) \) arranged in the counter clockwise order as follows:

\[
O(a_1, a_2) = \{ \Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4, \Lambda_5, \Lambda_6 \}
\]

where

\[
\begin{align*}
\Lambda_1 &= a_1\omega_1 + a_2\omega_2, \\
\Lambda_2 &= -a_1\omega_1 + (a_1 + a_2)\omega_2, \\
\Lambda_3 &= -(a_1 + a_2)\omega_1 + a_1\omega_2, \\
\Lambda_4 &= -a_2\omega_1 - a_1\omega_2, \\
\Lambda_5 &= a_2\omega_1 - (a_1 + a_2)\omega_2, \\
\Lambda_6 &= (a_1 + a_2)\omega_1 - a_2\omega_2.
\end{align*}
\]  

They can also be obtained as complex numbers from equation (10) by substituting \( n=3 \). These vectors represent the vertices of an isogonal hexagon with \( 120^\circ \) interior angles and the alternating edge lengths \( \sqrt{2}a_i \) and \( \sqrt{2}a_2 \).

The dual of the isogonal hexagon is an isotoxal hexagon in which the edges are equal, however, it has two different alternating interior angles \( \alpha \) and \( \beta \) such that \( \alpha + \beta = 240^\circ \). The vertices of the isotoxal hexagon lie on two fundamental orbits \( O(a_i, 0) \) and \( \lambda O(0,1) \). The scale factor \( \lambda \) is determined by the relation

\[
(\omega_2 - a_1\omega_1)(a_1\omega_1 + a_2\omega_2) = 0 \quad \Rightarrow \quad \lambda = \frac{a_1(a_1 + a_2)}{a_i + 2a_2}.
\]  

The vertices of the isotoxal hexagon can then be written in the counter clockwise order as follows:

\[
\{ B_1 = a_2\omega_2, B_2 = \lambda\omega_2, B_3 = a_1(-\omega_1 + \omega_2), B_4 = -\lambda\omega_1, B_5 = -a_1\omega_2, B_6 = \lambda(\omega_1 - \omega_2) \}.
\]  

Defining \( \eta = \frac{\lambda}{a_i} \) one can check that the edge length of the isotoxal hexagon is given by

\[
a_i[\frac{2}{3}(\eta^2 - \eta + 1)]^{\frac{1}{3}}
\]  

with two different interior angles given by

\[
\alpha = \cos^{-1} \left( \frac{2\eta^2 - 2\eta - 1}{2(\eta^2 - \eta + 1)} \right), \quad \beta = \cos^{-1} \left( \frac{-\eta^2 - 2\eta + 2}{2(\eta^2 - \eta + 1)} \right).
\]  

One can prove that for any value of \( \eta, \alpha + \beta = 240^\circ \). The isogonal hexagon and its dual isotoxal hexagon are shown in figure 3 for the values \( a_1 = 1 \) and \( a_2 = 2 \).

![Figure 3](attachment:image.png)
One can tile the plane with regular and isogonal hexagons provided two isogonal hexagons and one regular hexagon share the same vertex regardless of the values of $a_1$ and $a_2$. One type of tiling of the plane is shown in figure 4(a) with the values $a_1 = 1$ and $a_2 = 2$. Another tiling is shown in figure 4(b) for the isogonal hexagon for $a_1 = 2$ and $a_2 = 1$. In the limit $a_1 \to 0$ and $a_2 = 1$ isogonal hexagon turns out to be an equilateral triangle and such tiling is depicted in figure 4(c). The other extreme limit is shown in figure 4(d) where $a_1 = 1$ and $a_2 \to 0$. The honeycomb lattice corresponds to the case where $a_1 = a_2$ which is shown in the figure 4(e). The honeycomb lattice representing the tiling of the plane with regular hexagons possesses translational invariance, that is to say, it is invariant under the affine Coxeter group $\tilde{W}(A_2)$. We anticipate that a neutral state of the graphene consisting of infinite number of carbon atoms can be represented by a tiling similar to the one in figure 4(a) provided one of the parameters represents the double bonds (say $a_1$) and the other is representing the single bonds ($a_2$). Experimentally one expects $a_1 \ll a_2$, but nevertheless they are nearly equal to each other contrary to the isogonal hexagon in figure 4(a) which is an exaggerated version of this quasi-crystal lattice. As we will discuss in the paper [8] the $C_{60}$ molecule represents tiling of the sphere with isogonal hexagons and the pentagons for it has two bond lengths.

![Figure 4](image_url)

**Figure 4.** Tiling of the plane with isogonal hexagons-regular hexagons (a-b), regular hexagons and triangles (c), tiling with triangles (d) and the honeycomb lattice (e).

Tiling of the plane can also be made with the isotoxal hexagons by joining its two vertices with two alternating angles $\alpha$ and $\beta$ so that one obtains an exterior angle of $120^\circ$ at each vertex. This way we create a regular hexagon surrounded by six isotoxal hexagons as shown in figure 5. This tiling is the dual of the tiling in figure 4(a). In this tiling all hexagons have the same edge lengths. However at each vertex the three angles satisfy, as expected, the relation $\alpha + \beta + 120^\circ = 360^\circ$. Here again when
\( \eta \to 1 \) we obtain the honeycomb lattice of figure 4(a). This tiling can be considered as a conducting graphene model allowing slight modifications of the bond angles.

![Figure 5. Tiling of the plane with isotoxal and regular hexagons.](image)

3.3. \( n = 4 \) with \( W(B_2) \approx D_4 \) symmetry

We can repeat similar arguments raised for the case \( n=3 \). Here also we have two fundamental orbits

\[
O(10) = \{ \omega_1, -\omega_1 + \sqrt{2} \omega_2, -\omega_1, \omega_1 - \sqrt{2} \omega_1 \} \tag{16a}
\]

\[
O(01) = \{ \omega_2, -\omega_2 + \sqrt{2} \omega_1, -\omega_2, \omega_2 - \sqrt{2} \omega_1 \} .\tag{16b}
\]

Each orbit represents a square. The orbit \( O(\omega_1 + \omega_2) = O(11) \) is a regular octagon possessing the symmetry \( D_4 : C_2 \approx D_8 \). One can tile the plane with regular octagon and the square as shown in figure 6.

![Figure 6. Tiling of the plane by regular octagons and squares.](image)

The general orbit \( O(\Lambda) = O(a_1, a_2) \) with \( a_1 \neq a_2 \neq 0 \) is an isogonal octagon with interior angles 135°. The vertices of an isogonal octagon can be generated from the complex number \( \Lambda = a_1 \omega_1 + a_2 \omega_2 = \frac{1}{\sqrt{2}} [a_i + (a_i + \sqrt{2} a_i) i] \). The vertices of the isotoxal octagons are determined by finding the factor \( \lambda = \frac{a_i (\sqrt{2} a_i + a_i)}{a_i + \sqrt{2} a_i} \). The vertices of isotoxal octagon is the union of the orbits \( O(a_i, 0) \) and \( \lambda O(01) \). The alternating angles satisfy the relation \( \alpha + \beta = 270^\circ \). The isogonal octagon with values \( a_1 = 1 \) and \( a_2 = 2 \) and its dual isotoxal octagon are depicted in figure 7.
The isogonal octagon (a) and its dual isotoxal octagon (b).

Figure 7. The isogonal octagon (a) and its dual isotoxal octagon (b).

The tiling of the plane with two isogonal octagons and one square at each vertex is shown in figure 8(a). The tiling of the plane with two isotoxal octagons and one square at one vertex is shown in figure 8(b).

Figure 8. Aperiodic tiling of the plane by isogonal octagons (a) and isotoxal octagons (b) with squares.

3.4 \( n = 5 \) with \( W(H_2) \approx D_5 \) symmetry

This case corresponds to the Coxeter group \( W(H_2) \approx D_5 \), the dihedral group of order 10. Here the Cartan matrix and its inverse corresponding to the Coxeter diagram \( H_2 \) can be written in terms of the golden ratio \( \tau = \frac{1 + \sqrt{5}}{2} \) and \( \sigma = \frac{1 - \sqrt{5}}{2} \) as

\[
C = \begin{bmatrix}
2 & \tau \\
-\tau & 2
\end{bmatrix}
\quad \text{and} \quad
C^{-1} = \frac{1}{2 + \sigma} \begin{bmatrix} 2 & \tau \\ \tau & 2 \end{bmatrix}.
\tag{17}
\]

The orbit of a general vector \( \Lambda = a_1 \omega_1 + a_2 \omega_2 \) can be obtained as the set of 10 vectors \( O(a_1, a_2) = D_5 (a_1, a_2) \). The fundamental orbits are the regular pentagons which are the duals of each other. The regular decagon is obtained by letting \( a_1 = 1 \) and \( a_2 = 1 \). The regular decagon is both vertex and edge transitive under the dihedral group \( W(H_2) : C_5 \approx D_{10} \) because of the diagram symmetry.

The tiling of the plane with five-fold symmetry is introduced by Penrose [9] since the plane cannot be tiled only with regular pentagons. It has been recently shown that the Islamic tiling of the plane [10] with five-fold symmetry dates back to the medieval time. The Islamic architecture used five different tiles, decagon, pentagon, rhombus, nonregular hexagon and bow tie. Here we give in figure 9 one of those tilings of the plane with regular decagons and bow ties.
One can also construct an isogonal decagon represented by alternating edge lengths $\sqrt{2}a_1$ and $\sqrt{2}a_2$ with $a_1 \neq a_2 \neq 0$. Its interior angles are all equal to $144^\circ$. One such isogonal decagon is shown in figure 10(a) corresponding to the alternating edge lengths for $a_1 = 1$ and $a_2 = 2$. The dual of the isogonal decagon is the isotoxal decagon with equal edge lengths but with alternating angles $\alpha + \beta = 288^\circ$. The isotoxal decagon is shown in figure 10(b).

One type of aperiodic tiling of the plane with isogonal decagons is shown in figure 11.

4. Conclusion
We have displayed a method to construct the quasi regular polygons and their duals using the rank-2 Coxeter diagrams. The isogonal polygons with $2n$ sides and their duals isotoxal polygons possess the dihedral symmetry $D_n$. It is tempting to suggest that the tiling of the plane by the isogonal hexagon and regular hexagon may represent a state of graphene where double bond and single bonds can be represented by two edges of the isogonal hexagons. We have constructed a number of isogonal...
polygons with their duals possessing various dihedral symmetries. The corresponding tilings of the plane with the tiles chosen as isogonal and isotoxal polygons are studied.

References
[1] Novoselov K S, Geim A K, Morozov S V, Jiang D, Zhang Y, Dubonos SV, Gregorieva I V and Forsov AA 2004 *Science*, 306 666; Geim A K and Novoselov K S 2007 *Nature Materials* 6 183
[2] Wallace P R 1947 *Phys. Rev.* 71 622
[3] Senechal M 1989 *Introduction to the Mathematics of Quasicrystals* ed Marko V. Jaric (Academic Press) p 1
[4] Janner A and Janssen T 1980 *Acta Cryst.* A36 399; ibid A36 408
[5] Coxeter H S M and Moser W O J 1965 *Generators and relations for discrete groups* (Springer Verlag); Coxeter H S M 1973 *Regular complex polytopes* (Cambridge: Cambridge University Press)
[6] Carter R W 1972 *Simple groups of lie type* (John Wiley & Sons Ltd.); Humphreys J E 1990 *Reflection groups and coxeter groups* (Cambridge University Press, Cambridge)
[7] Slansky R 1981 *Phys. Rep.* 79, 1
[8] Koca M, Al-Ajmi M and Shidani S 2010 Quasi regular polyhedra and their duals with Coxeter symmetries represented by quaternions II *arXiv:1006.2973*
[9] Penrose R 1974 *Bull.Inst. Math. Appl.* 10, 266; Penrose R 1989 *Introduction to the Mathematics of Quasicrystals* vol 2 ed Marko V. Jaric (Academic Press) p 53
[10] Lu P J and Steinhardt PJ 2007 *Science* 315, 1106