Canonicity of Bäcklund transformation: 
$r$-matrix approach. I.

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Abstract. For the Hamiltonian integrable systems governed by $SL(2)$-invariant $r$-matrix (such as Heisenberg magnet, Toda lattice, nonlinear Schrödinger equation) a general procedure for constructing Bäcklund transformation is proposed. The corresponding BT is shown to preserve the Poisson bracket. The proof is given by a direct calculation using the $r$-matrix expression for the Poisson bracket.

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In 1976 Flaschka and McLaughlin [1] has demonstrated that the standard Bäcklund transformation (BT) for the KdV equation is canonical with respect to the associated symplectic structure. Subsequently, the canonicity of BTs has been proved for some more integrable models, see e. g. Appendix to [2].

The aim of the present paper is to apply the $r$-matrix formalism [3] to the problem of proving the canonicity of BT. Our proof is quite general and requires only that the Poisson brackets of the corresponding Lax operator could be written down in the $r$-matrix form. In our treatment of Bäcklund transformation we adopt the approach of [4] where a program has been formulated of reexamining BTs from the Hamiltonian point of view.

Consider an integrable Hamiltonian system possessing a Lax matrix $L(u)$ depending on the dynamical variables and a complex parameter $u$ (spectral parameter). The spectral invariants of $L(u)$ are supposed to generate the commuting Hamiltonians of the system. Defining a Bäcklund transformation as a canonical transformation preserving the Hamiltonians of the system (see [4] for a more detailed list of properties of BT) we conclude that it has to preserve as well the spectral invariants of $L(u)$. As a consequence, the original matrix $L(u)$ and the transformed one $\Lambda(u)$ must be related by a similarity transformation

$$\Lambda(u) = M(u) L(u) M(u)^{-1}.$$

(1)

(see [4] for a detailed account of the theory of BT as gauge transformations).

An important practical question arising in the theory of BT is how to find, given a Lax matrix $L(u)$, such a matrix $M(u)$ which would generate a BT. To check that a matrix $M(u)$ is admissible one needs, first, to verify that the system of equations resulting from (1) is self-consistent, and, second, to proof that the resulting transformation of dynamical variables is canonical, that is preserves the Poisson bracket.

Below we solve the both problems for the class of integrable systems governed by the $SL(2)$-invariant $r$-matrix. Suppose that $L(u)$ is a matrix of order $2 \times 2$ and the Poisson brackets between the entries of $L(u)$ can be expressed in the $r$-matrix form [3]

$$\{\hat{L}(u), \hat{L}(v)\} = [r_{12}, \hat{L}(u) \hat{L}(v)]$$

(2)

where $\hat{L} = L \otimes 1, \hat{L} = 1 \otimes L$, and $r_{12} = \kappa(u-v)^{-1}P_{12}$ is the standard $SL(2)$-invariant solution to the classical Yang-Baxter equation [3], $\kappa$ being a constant and $P_{12}$ being the permutation operator in $\mathbb{C}^2 \otimes \mathbb{C}^2$. The class of integrable models thus defined includes such well-known models as the nonlinear Schrödinger equation, Heisenberg magnetic chain, Toda lattice [3].

When choosing an ansatz for the matrix $M(u)$ we shall follow [3, 5] where it was observed that, in the cases of Heisenberg magnetic chain and of the lattice Landau-Lifshitz equation, the matrix $M(u)$ happens to have the same form, as a function of $u$, as the corresponding elementary Lax matrix $L(u)$ for the chain consisting only of one atom. As shown below, such choice of $M(u)$ is valid for any integrable model governed by the $r$-matrix specified above.
Our ansatz for $M(u)$ mimics the form of elementary Lax matrix for the isotropic Heisenberg magnetic chain [3, 3]:
\[
M(u) = (u - \lambda)I + S
\]
where
\[
\text{tr} S = 0, \quad \det S = -\mu^2
\]
$\lambda$ and $\mu$ being free parameters. It is convenient to perform a reparametrization $\lambda_1 = \lambda + \mu$, $\lambda_2 = \lambda - \mu$, so that $\mu = (\lambda_1 - \lambda_2)/2$. The constraints (4) on the matrix $S$ leave two more undetermined parameters. Denoting them $p$ and $q$ and choosing a particular parametrization of $S$ we fix the following ansatz for $M(u)$
\[
M(u) = \begin{pmatrix}
  u - \lambda_1 + pq & p \\
  -pq^2 + 2\mu q & u - \lambda_2 - pq
\end{pmatrix}.
\]

We shall consider $\lambda_1$, $\lambda_2$ as the free parameters of BT. The parameters $p$, $q$ are then to be determined from the equations (1).

Let us introduce the eigenbasis of $M(u)$
\[
|1\rangle = \frac{1}{2\mu} \begin{pmatrix} 1 \\ -q \end{pmatrix}, \quad |2\rangle = \frac{1}{2\mu} \begin{pmatrix} p \\ 2\mu - pq \end{pmatrix}
\]
and the dual eigenbasis
\[
\langle 1| = (2\mu - pq, -p), \quad \langle 2| = (q, 1),
\]
as well as the corresponding spectral projectors
\[
P_{ij} = |i\rangle \langle j|, \quad i, j \in \{1, 2\}
\]
satisfying
\[
P_{ij} P_{kl} = P_{il} \delta_{jk}.
\]

In terms of the projectors $P_{ij}$ the matrix $M(u)$ and its inverse are written down, respectively, as
\[
M(u) = (u - \lambda_1)P_{11} + (u - \lambda_2)P_{22}
\]
and
\[
M(u)^{-1} = (u - \lambda_1)^{-1}P_{11} + (u - \lambda_2)^{-1}P_{22}.
\]

Note that
\[
\det M(u) = (u - \lambda_1)(u - \lambda_2).
\]

Let us derive now from (1) the equations determining the parameters $p$, $q$. Suppose that $L(u)$ is polynomial in $u$ (this covers all lattice models, the continuous models can then be obtained as appropriate limits). Requiring that the transformation (1) preserves the polynomiality of $L(u)$ we conclude that the apparent poles of the right-hand-side of (1) at $u = \lambda_{1,2}$ due to (11) should vanish,
\[
0 = \lim_{u \to \lambda_i} L(u) = (\lambda_i - \lambda_{1-i})P_{1-i,1-i}L(\lambda_i)P_{ii},
\]

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which gives us two equations for determining $p$ and $q$ as functions of the dynamical variables of the system:

$$\text{tr} \, P_{12} L(\lambda_1) = 0, \quad \text{tr} \, P_{21} L(\lambda_2) = 0.$$  \hspace{1cm} (14)

With the parameters $p$ and $q$ determined by the equations (14), the matrix $\Lambda(u)$ is defined by (1) as a function of the dynamical variables and free parameters $\lambda_{1,2}$. The next step is to show that $\Lambda(u)$ satisfies the same Poisson bracket relations (2) as $L(u)$:

$$\{\dot{\Lambda}(u), \dot{\Lambda}(v)\} = [r_{12}, \dot{\Lambda}(u)\dot{\Lambda}(v)].$$ \hspace{1cm} (15)

The calculation of the Poisson brackets for $\Lambda(u)$, though cumbersome, is quite straightforward, since all the necessary ingredients are already prepared. Substituting (14) into $\{\dot{\Lambda}(u), \dot{\Lambda}(v)\}$ and differentiating the products of matrices we obtain a rather long expression. To write it down in a more compact form let us introduce the following notation

$$\left\{ \begin{array}{l}
\langle \dot{M}^2 \rangle = M(u)\{ \dot{M}(u), M(v)\} M(u)^{-1} M(v)^{-1}, \\
\langle \dot{L}^2 \rangle = M(v)\{ \dot{L}(u), M(v)\} M(u)^{-1} M(v)^{-1}, \\
\langle \dot{L} \dot{M} \rangle = M(u)M(v)\{ \dot{M}(u), \dot{M}(v)\} M(u)^{-1} M(v)^{-1}, \\
\end{array} \right.$$ \hspace{1cm} (16)

$$\tilde{r}_{12} = M(u)\dot{M}(v) r_{12} M(u)^{-1} \dot{M}(v)^{-1}. \hspace{1cm} (17)$$

Using (16), (17) one can write down the left-hand-side of (15) as

$$\{\dot{\Lambda}(u), \dot{\Lambda}(v)\} = \langle \dot{M}^2 \rangle \dot{\Lambda}(u)\dot{\Lambda}(v) + \langle \dot{L}^2 \rangle \dot{\Lambda}(v) - \dot{\Lambda}(u) \langle \dot{M}^2 \rangle \dot{\Lambda}(v) + \langle \dot{L} \dot{M} \rangle \dot{\Lambda}(u) + [\tilde{r}_{12}, \dot{\Lambda}(u)\dot{\Lambda}(v)] - \dot{\Lambda}(u) \langle \dot{M}^2 \rangle - 2 \dot{\Lambda}(v) \langle \dot{M}^2 \rangle \dot{\Lambda}(u) - 2 \dot{\Lambda}(v) \langle \dot{L}^2 \rangle + \dot{\Lambda}(u) \dot{\Lambda}(v) \langle \dot{M}^2 \rangle. \hspace{1cm} (18)$$

Our aim is to show that the resulting expression is equal to $[r_{12}, \dot{\Lambda}(u)\dot{\Lambda}(v)]$. Note, first, that using the identity

$$P_{12} = \dot{P}_{11} \dot{P}_{11} + \dot{P}_{12} \dot{P}_{21} + \dot{P}_{21} \dot{P}_{12} + \dot{P}_{22} \dot{P}_{22}$$

one can show that

$$\tilde{r}_{12} = r_{12} + 2\mu \kappa \left( \frac{\dot{P}_{12} \dot{P}_{21}}{(u - \lambda_2)(v - \lambda_1)} - \frac{\dot{P}_{21} \dot{P}_{12}}{(u - \lambda_1)(v - \lambda_2)} \right). \hspace{1cm} (19)$$

It remains then to calculate the Poisson brackets between $L(u)$ and $M(u)$. To do it, we recollect the equations (14) defining implicitly $p$ and $q$. The calculate the Poisson
brackets for $p$ and $q$ we shall use the trick employed in a similar situation in [7]. For any function $f$ on the phase space we have

$$0 = \{ f, \text{tr} P_{12} L(\lambda_1) \}$$

$$= \{ f, p \} \text{tr} \frac{\partial P_{12}}{\partial p} L(\lambda_1) + \{ f, q \} \text{tr} \frac{\partial P_{12}}{\partial q} L(\lambda_1) + \text{tr} P_{12}\{ f, L(\lambda_1) \}, \quad (20a)$$

$$0 = \{ f, \text{tr} P_{21} L(\lambda_2) \}$$

$$= \{ f, p \} \text{tr} \frac{\partial P_{21}}{\partial p} L(\lambda_2) + \{ f, q \} \text{tr} \frac{\partial P_{21}}{\partial q} L(\lambda_2) + \text{tr} P_{21}\{ f, L(\lambda_2) \}. \quad (20b)$$

The Poisson brackets $\{ f, p \}$ and $\{ f, q \}$ are then determined by solving the linear system (20). Using the equalities

$$\frac{\partial P_{12}}{\partial p} = 0, \quad \frac{\partial P_{12}}{\partial q} = \frac{1}{2\mu}(P_{11} - P_{22}) + \frac{p}{\mu} P_{12}, \quad (21)$$

$$\frac{\partial P_{21}}{\partial p} = P_{11} - P_{22}, \quad \frac{\partial P_{21}}{\partial q} = \frac{p^2}{2\mu}(P_{11} - P_{22}) - \frac{p}{\mu} P_{21}, \quad (22)$$

$$\frac{\partial M(u)}{\partial p} = 2\mu p, \quad \frac{\partial M(u)}{\partial q} = p^2 P_{12} + P_{21} \quad (23)$$

and introducing the notation

$$w_i(\lambda) = \text{tr} P_{ii} L(\lambda), \quad w(\lambda) = w_1(\lambda) - w_2(\lambda) \quad (24)$$

one obtains then

$$\{ f, M(v) \} = -\frac{2\mu}{w(\lambda_1)} P_{21} \text{tr}\{ f, L(\lambda_1) \} P_{12} - \frac{2\mu}{w(\lambda_2)} P_{12} \text{tr}\{ f, L(\lambda_2) \} P_{21} \quad (25)$$

(note that the last formula does not depend on parametrization of $S$).

Now, using (24) it is easy to calculate the brackets

$$\{ \hat{L}(u), \hat{M}(v) \} = -\frac{2\mu\kappa}{(u - \lambda_1) w(\lambda_1)} \left( w_1(\lambda_1) \hat{P}_{12} \hat{L}(u) \hat{P}_{21} - w_2(\lambda_1) \hat{L}(u) \hat{P}_{12} \hat{P}_{21} \right)$$

$$-\frac{2\mu\kappa}{(u - \lambda_2) w(\lambda_2)} \left( w_2(\lambda_2) \hat{P}_{21} \hat{L}(u) \hat{P}_{12} - w_1(\lambda_2) \hat{L}(u) \hat{P}_{21} \hat{P}_{12} \right) \quad (26)$$

$$\{ \hat{M}(u), \hat{L}(v) \} = \frac{2\mu\kappa}{(v - \lambda_1) w(\lambda_1)} \left( w_1(\lambda_1) \hat{P}_{21} \hat{P}_{12} \hat{L}(v) - w_2(\lambda_1) \hat{L}(v) \hat{P}_{21} \hat{P}_{12} \right)$$

$$+ \frac{2\mu\kappa}{(v - \lambda_2) w(\lambda_2)} \left( w_2(\lambda_2) \hat{P}_{12} \hat{P}_{21} \hat{L}(v) - w_1(\lambda_2) \hat{L}(v) \hat{P}_{12} \hat{P}_{21} \right) \quad (27)$$

$$\{ \hat{M}(u), \hat{M}(v) \} = -\frac{2\mu\kappa (w_1(\lambda_1) w_2(\lambda_2) - w_1(\lambda_1) w_2(\lambda_2))}{w(\lambda_1) w(\lambda_2)} \left( \hat{P}_{12} \hat{P}_{21} - \hat{P}_{21} \hat{P}_{12} \right), \quad (28)$$
Recalling the notation (16) one arrives to the expressions

\[ \langle L^1 M^2 \rangle = -\frac{2\mu \kappa}{(u-\lambda_2)(v-\lambda_1)} \left( \frac{w_1(\lambda_1)}{w(\lambda_1)} \frac{1}{P_{12}} \Lambda(u) \frac{1}{P_{21}} - \frac{w_2(\lambda_1)}{w(\lambda_1)} \Lambda(u) \frac{1}{P_{12}} \frac{2}{P_{21}} \right) \]

\[ -\frac{2\mu \kappa}{(u-\lambda_1)(v-\lambda_2)} \left( \frac{w_2(\lambda_2)}{w(\lambda_2)} \frac{1}{P_{21}} \Lambda(u) \frac{2}{P_{12}} - \frac{w_1(\lambda_2)}{w(\lambda_2)} \Lambda(u) \frac{2}{P_{12}} \frac{1}{P_{21}} \right), \] (29)

\[ \langle M^1 L^2 \rangle = \frac{\mu \kappa}{(u-\lambda_1)(v-\lambda_2)} \left( \frac{w_1(\lambda_1)}{w(\lambda_1)} \frac{1}{P_{21}} \frac{2}{P_{12}} \Lambda(v) \frac{2}{P_{21}} - \frac{w_2(\lambda_1)}{w(\lambda_1)} \frac{2}{P_{21}} \Lambda(v) \frac{1}{P_{12}} \right) \]

\[ + \frac{\mu \kappa}{(u-\lambda_2)(v-\lambda_1)} \left( \frac{w_2(\lambda_2)}{w(\lambda_2)} \frac{1}{P_{12}} \frac{2}{P_{21}} \Lambda(v) \frac{2}{P_{12}} - \frac{w_1(\lambda_2)}{w(\lambda_2)} \frac{2}{P_{12}} \Lambda(v) \frac{1}{P_{21}} \right), \] (30)

\[ \langle M^1 M^2 \rangle = -\frac{2\mu \kappa(w_1(\lambda_1)w_2(\lambda_2) - w_2(\lambda_1)w_1(\lambda_2))}{w(\lambda_1)w(\lambda_2)} \]

\[ \times \left( \frac{\frac{1}{P_{12}} \frac{2}{P_{21}}}{(u-\lambda_2)(v-\lambda_1)} - \frac{\frac{1}{P_{21}} \frac{2}{P_{12}}}{(u-\lambda_1)(v-\lambda_2)} \right). \] (31)

Finally, substituting (29), (30), (31) together with (19) into (18), we obtain, after massive cancellations, the equality (15) which finishes the proof of the canonicity of Bäcklund transformation.

**Discussion**

In this paper, we have studied only the case of the XXX type $r$-matrix. There is little doubt that our results can be generalized to the cases of XXZ and XYZ type $r$-matrices. Taking the limit of linear Poisson brackets

\[ \{ \hat{L}(u), \hat{L}(v) \} = [r_{12}, \hat{L}(u) + \hat{L}(v)] \] (32)

one obtains, as a corollary, the canonicity of BT for the Gaudin model [7].

The proof of canonicity of BT given in the present paper, being quite simple and straightforward, does not explain, however, the reasons why we should choose for $M$ the same ansatz (3) as for $L$. An answer to this question is given in the second part of our paper which is being prepared for publication.

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