The logic of quantum programs

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Abstract

We present a logical calculus for reasoning about information flow in quantum programs. In particular we introduce a dynamic logic that is capable of dealing with quantum measurements, unitary evolutions and entanglements in compound quantum systems. We give a syntax and a relational semantics in which we abstract away from phases and probabilities. We present a sound proof system for this logic, and we show how to characterize by logical means various forms of entanglement (e.g. the Bell states) and various linear operators. As an example we sketch an analysis of the teleportation protocol.

1 Introduction

In this paper we elaborate on the ideas presented in [2, 3, 9] and give a full-fledged dynamic Logic for Quantum Programs LQP. It is well-known that PDL (Propositional Dynamic Logic) and its fragment Hoare Logic are among the main logical formalisms used in program verification for classical programs, i.e. in checking that a given (classical) program meets the required specification. It is natural to ask for a quantum version of PDL, to be used in the verification of quantum programs. In our past work [3], we presented several such logical systems, and later extending this system into a dynamic logic LQA of quantum actions (i.e. compositions of measurements and unitary evolutions). In this paper, we extend LQA into a logic for compound quantum systems. We present a self-contained version of LQP such that no knowledge of LQA or LQM is necessary to understand the basic concepts. Note the difference between our logic and the approach with a similar name in [4]: our dynamic logic goes much further in capturing essential properties of quantum systems and quantum programs, as well as in recovering the ideas of traditional quantum logic [6, 7].

2 Quantum Frames

In this section we introduce quantum frames for single quantum systems and quantum frames for compound quantum systems; in the later case we restrict our attention to n compound qubits.

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2.1 Single System Quantum Frames

A modal frame is a set of states, together with a family of binary relations between states. A (generalized) PDL frame is a modal frame \((\Sigma, \{ S^? \}_{S \in \mathcal{L}}, \{ a \rightarrow \}_{a \in \mathcal{A}})\), in which the relations on the set of states \(\Sigma\) are of two types: the first, called tests and denoted by \(S^?\), are labelled with subsets \(S\) of \(\Sigma\), coming from a given family \(\mathcal{L} \subseteq P(\Sigma)\) of sets, called testable properties; the others, called actions, are labelled with action labels \(a\) from a given set \(\mathcal{A}\).

Given a PDL frame, there exists a standard way to give a semantics to the usual language of propositional dynamic logic. Classical PDL can be considered as a special case of such a logic, in which tests are given by classical tests: \(s S^? \rightarrow t\) if and only if \(s = t \in S\). Observe that classical tests, if executable, do not change the current state.

In the context of quantum systems, a natural idea is to replace classical tests by “quantum tests”, given by quantum measurements of a given property. Such tests will obviously change the state of the system. To model them, we introduce a special kind of PDL frames: quantum frames. The “tests” are essentially given by projectors in a Hilbert space. In [3], we considered PDL with the above-mentioned standard semantics, having the same clauses in the classical case, but interpreted in quantum frames. What we obtained is a quantum PDL, whose negation-free part with dynamic modalities for quantum tests is equivalent to what is traditionally called “(orthomodular) quantum logic” [6, 7]. In this paper, we extend the syntax of this logic to deal with unitary evolutions, entanglements and some quantum protocols.

**Definition 1. (Quantum Frames)**

Given a Hilbert space \(\mathcal{H}\), the following steps construct a Quantum (PDL) Frame

\[
\Sigma(\mathcal{H}) := (\Sigma, \{ S^? \}_{S \in \mathcal{L}}, \{ U \rightarrow \}_{U \in \mathcal{U}})
\]

1. Let \(\Sigma\) be the set of one dimensional subspaces of \(\mathcal{H}\), called the set of states. We denote a state \(s = \pi\) of \(\mathcal{H}\) using any of the non-zero vectors \(x \in \mathcal{H}\) that generate them. Note that any two vectors that differ only in phase (i.e. \(x = \lambda y\), with \(\lambda \in \mathbb{C}\) with \(|\lambda| = 1\)) will generate the same state \(s = \pi \in \Sigma\).

2. Call two states \(s\) and \(t\) in \(\Sigma\) orthogonal, and write \(s \perp t\), if and only if \(\forall x \in s \land \forall y \in t: x \text{ is orthogonal to } y\), i.e. if \(\langle x | y \rangle = 0\). Or, equivalently, we can state that \(s \perp t\) if and only if \(\exists x \in s, y \in t\) with \(x \neq 0\), \(y \neq 0\) and \(\langle x | y \rangle = 0\). We put \(S^\perp := \{ t \in \Sigma | t \perp s \text{ for all } s \in S\}\); and we denote by \(\overline{S} = S^{\perp \perp} := (S^\perp)^\perp\) the biorthogonal closure of \(S\). In particular, for a singleton \(\{x\}\), we just write \(\pi\) for \(\overline{\{x\}}\), which agrees with the notation \(\pi\) used above to denote the state generated by \(x\).

3. A set of states \(S \subseteq \Sigma\) is called a (quantum) testable property iff it is biorthogonally closed, i.e. if \(\overline{S} = S\). (Note that \(S \subseteq \overline{S}\) is always the case.) We denote by \(\mathcal{L} \subseteq P(\Sigma)\) the family of all quantum testable properties. All the other sets \(\overline{S} \in P(\Sigma) \setminus \mathcal{L}\) are called non-testable properties.
4. There is a natural bijective correspondence between the family \( \mathcal{L} \) of all testable properties and the family \( \mathcal{W} \) of all closed linear subspaces \( W \) of \( \mathcal{H} \), bijection given by \( S \mapsto W_S =: \bigcup S \). Observe that, under this correspondence, the image of the biorthogonal closure \( \overline{S} \) of any arbitrary set \( S \subseteq \Sigma \) is the closed linear subspace \( \bigcup S \subseteq \mathcal{H} \) generated by the union \( \bigcup S \) of all states in \( S \).

5. For each testable property \( S \in \mathcal{L} \), there exists a partial map \( S? \) on \( \Sigma \), called a quantum test. If \( W = W_S = \bigcup S \) is the corresponding subspace of \( \mathcal{H} \), then the quantum test is the map induced on states by the projector \( P_W \) onto the subspace \( W \). In other words, it’s given by:

\[
S?(x) := \begin{cases} P_W(x) \in \Sigma, & \text{if } x \not\in S^\perp \text{ (i.e. if } P_W(x) \neq 0) \\ \text{undefined} , & \text{otherwise} . \end{cases}
\]

We denote by \( S? \subseteq \Sigma \times \Sigma \) the binary relation corresponding to the partial map \( S? \), i.e. given by: \( s \rightarrow t \) if and only if \( S?(s) = t \). So we have a family of binary relations indexed by the testable properties \( S \in \mathcal{L} \).

6. For each unitary transformation \( U \) on \( \mathcal{H} \), consider the corresponding binary relation \( U \subseteq \Sigma \times \Sigma \), given by: \( s \rightarrow t \) if and only if \( U(x) = y \) for some non-zero vectors \( x \in s, y \in t \). So we obtain a family of binary relations indexed by the unitary transformations \( U \in \mathcal{U} \) (where \( \mathcal{U} \) is the set of unitary transformations on \( \mathcal{H} \)).

So a quantum frame is just a PDL frame built on top of a given Hilbert space \( \mathcal{H} \), using projectors as “tests” and unitary evolutions as “actions”. Our notion of “state” in this paper is closely connected to the way quantum logicians approach quantum systems, i.e., contrary to identifying states with unitary vectors (as customary in quantum computation), we took them to be one dimensional subspaces generated by these vectors. This imposes some limits to our approach, mainly that we will not be able to express phase-related properties. While it is possible to build up a quantum frame starting from unitary vectors as the states, the resulting logical system will be much more complex\footnote{It would require the introduction of a propositional tensor operator.} and so we do not elaborate on it in this paper.

**Operators on states, adjoints and generalized tests.** To generalize our notations introduced earlier, observe that every linear operator \( F : \mathcal{H} \rightarrow \mathcal{H} \) induces a partial map \( F : \Sigma \rightarrow \Sigma \) on states (i.e. subspaces), given by \( F(\pi) = F(\pi) \). (Note that linearity ensures that this map on states is well-defined.) In particular, every map \( F : \Sigma \rightarrow \Sigma \) obtained in this way has an adjoint \( F^\dagger : \Sigma \rightarrow \Sigma \), defined as the map on states induced by the adjoint ("Hermitian conjugate") of the linear operator \( F \) on \( \mathcal{H} \). Observe that, for unitary transformations \( U \), the adjoint is the inverse: \( U^\dagger = U^{-1} \). Also, one can naturally generalize quantum tests to arbitrary, possibly non-testable properties, \( S \subseteq \Sigma \), by putting: \( S? := \overline{S}? \).
So we identify a test of a “non-testable” property $S$ with the quantum test of its biorthogonal closure. Observe that $S^?\dagger = S?^\dagger$ (since projectors are self-adjoint).

**Definition 2. (Non-orthogonality, or Measurement Relation)**

For all $s, t \in \Sigma$, let $s \rightarrow t$ if and only if $s \overset{S?}{\rightarrow} t$ for some property $S \in \mathcal{L}$. In other words, $s \rightarrow t$ means that one can reach state $t$ by doing some measurement on state $s$. An important observation is that the measurement relation is the same as non-orthogonality: $s \rightarrow t$ iff $s \not\rightarrow t$. The non-orthogonality relation has indeed been used to introduce an accessibility relation in the orthoframe semantics within quantum logic [4].

**Definition 3. (Dynamic Modalities and Measurement Modalities)**

For any property $T \subseteq \Sigma$ and any partial map $F : \Sigma \rightarrow \Sigma$ induced on states by a linear operator $F$, let $[F]T := F^{-1}(T) = \{ s \in \Sigma : F(s) \in T \text{ if defined} \}$ and $\langle F \rangle T := \Sigma \setminus ((F)\Sigma \setminus F)$. Similarly, put $\Box T := \{ s \in \Sigma : \forall t(s \rightarrow t \Rightarrow t \in T) \}$ and $\Diamond T := \Sigma \setminus (\Box(\Sigma \setminus T))$.

Observe that $[F]T$ expresses the weakest precondition for the “program” $F$ and post-condition $T$. In particular, $[S?]T$ expresses the weakest precondition ensuring the satisfaction of property $T$ in any state after the system passes a quantum test of property $S$. Similarly, $\langle S? \rangle T$ means that one can perform a quantum test of property $S$ on the current state, ending up in a state having property $T$. $\Box T$ means that property $T$ will hold after any measurement (quantum test) performed on the current state. Finally, $\Diamond T$ means that property $T$ is potentially satisfied, in the sense that one can do some quantum test to reach a state with property $T$.

**Lemma 1.** For every property $S \subseteq \Sigma$, we have $S^\perp = [S?]\emptyset = \Sigma \setminus \Diamond S$ and $S = \Box \Diamond S$.

**Proposition 1.** For every property $S \subseteq \Sigma$, if $T \in \mathcal{L}$ (i.e. is testable), then $\Box S, S^\perp, [S?]T \in \mathcal{L}$ (are testable), and more generally $[F]T \in \mathcal{L}$, for every (map on states induced by a) linear operator $F$.

**Proposition 2. (Testable Properties)** A property $S \subseteq \Sigma$ is testable if and only if any of the following equivalent conditions hold: (1) $S = \overline{S}$; (2) $S = \Box \Diamond P$; (3) $\exists T \in \Sigma$ such that $P = T^\perp$; (4) $\exists T \in \Sigma$ such that $P = \Box T$. The family $\mathcal{L}$ of testable properties is a complete lattice with respect to inclusion, having as its meet set-intersection $S \cap T$, and as its join the biorthogonal closure of set-union $S \cup T := S^\perp \cup T$, called the quantum join of $S$ and $T$. For every state $s \in \Sigma$, the singleton $\{s\} \in \mathcal{L}$ is testable. For any arbitrary property $S \subseteq \Sigma$, we have $\overline{S} = \bigcup \{ \{s\} : s \in S \} = \bigcap \{ T \in \mathcal{L} : S \subseteq T \}$, so the biorthogonal closure of $S$ is the strongest testable property implied by (the property) $S$.

**Theorem 1.** In every quantum frame $\Sigma = \Sigma(\mathcal{H})$ the following properties for quantum tests are provable:

1. Partial functionality: If $s \overset{S?}{\rightarrow} t$ and $s \overset{S?}{\rightarrow} v$ then $t = v$.

2. Trivial tests: $\emptyset \overset{\emptyset}{\rightarrow} \emptyset$ and $\Sigma \overset{\Sigma}{\rightarrow} \Delta_\Sigma$, where $\Delta_\Sigma = \{(s, s) : s \in \Sigma\}$ is the identity relation on $\Sigma \times \Sigma$. 


3. **Adequacy**: if \( s \in S \) then \( S \xrightarrow{S} s \)

4. **Repeatability**: If \( S \in \mathcal{L} \) is testable and \( S \xrightarrow{S} t \), then \( t \in S \)

5. **Compatibility**: If \( S, T \in \mathcal{L} \) are testable and \( S ?; T ? = T ?; S ? \) then \( S ?; T ? = (S \cap T) ? \).

6. **Self-Adjointness**: If \( S \xrightarrow{S} w \xrightarrow{T} t \) then \( t \xrightarrow{T} v \xrightarrow{W} s \), for some \( v \in \Sigma \) and \( W \in \mathcal{L} \). In other words: if \( S \xrightarrow{S} w \rightarrow t \) then \( t \xrightarrow{T} v \rightarrow s \), for some \( v \in \Sigma \).

7. **Universal Accessibility**: For all \( s, t \in \Sigma \), there exists a state \( w \in \Sigma \) such that \( s \rightarrow w \rightarrow t \).

**Proofs**: Partial functionality follows from the fact that projectors correspond to partially defined maps in \( \mathcal{H} \). Trivial tests follows from the fact that projecting on the empty space yields the empty space and that projecting on the total space doesn’t change anything. Adequacy follows from the fact that for every \( x \in W \) we have that \( P_W(x) = x \). Repeatability follows from the fact that \( P_W(x) \in W \) for every \( x \in \mathcal{H} \). Compatibility follows from the fact that if two projectors commute, i.e. \( P_W \circ P_V = P_V \circ P_W \), then \( P_W \circ P_V = P_{W \cap V} \). Self-Adjointness follows from the more general Adjointness theorem stated below, together with the fact \( S^! = S ? \). Universal Accessibility can be proved by cases: If \( s \not\in t \), i.e. let \( s \rightarrow t \), then \( w = s \Rightarrow s \rightarrow s \rightarrow t \). If \( s \perp t \), i.e. let \( s \not\rightarrow t \) then let \( s = \bar{x}, t = \bar{y} \) with \( x, y \in \mathcal{H} \). Take the superposition \( x + y \in \mathcal{H} \) of \( x \) and \( y \) and note that \( x+y \neq 0 \) (since \( x + y = 0 \Rightarrow x = -y \Rightarrow s = t \) which contradicts \( s \not\in t \)). Next observe that \( x \not\in (x+y) \) (Indeed, suppose \( x \perp (x+y) \) then \( (x | x+y) = 0 \) and then \( (x | x) + (x | y) = 0 \); but \( x \perp y \) implies \( (x | x) = 0 \). So \( (x | x) = 0 \) follows that \( x = 0 \), which yields a contradiction). Similarly, we get \( y \not\in (x+y) \). Taking now \( w = x + y \), we can see that \( w \in \Sigma, s \rightarrow w \) and \( w \rightarrow t \).

**Theorem 2.** In every quantum frame \( \Sigma(\mathcal{H}) \) the following properties for unitary transformation (stated for all \( U, U^! \in \mathcal{U} \)) are provable:

1. **Functionality**: For every state \( s \in \Sigma \) we have \( \exists t : s \xrightarrow{U} t \)

2. **Inverse-adjoint (bijectivity)**: \( s \xrightarrow{U} t \xrightarrow{U^!} w \) implies \( s = w \). Similarly, \( s \xrightarrow{U^!} t \xrightarrow{U} w \) implies \( s = w \)

**Proofs**: Functionality follows from the fact that unitary transformations are well-defined on all states, i.e. the kernel of the linear map encoding the transformation is \( \emptyset \). Inverse-adjoint follows from the fact that unitary operators on a Hilbert space have the property that \( U^! = U^{-1} \).

**Theorem 3. (Adjointness)** Let \( F \) be a linear transformation and let \( s, w, t \in \Sigma \) be states: If \( s \xrightarrow{F} w \rightarrow t \) then there exists some state \( v \in \Sigma \) such that \( t \xrightarrow{F} v \rightarrow s \).
Proof: To prove this theorem we use the definition of adjointness in a Hilbert space: \( \langle Fx \mid y \rangle = \langle x \mid F^\dagger y \rangle \). From this, we get the equivalence: \( \langle Fx \mid y \rangle = 0 \) iff \( \langle x, F^\dagger y \rangle = 0 \); or, otherwise stated, \( Fx \perp y \) iff \( x \perp F^\dagger y \). Taking the negation of both sides and using the fact that the measurement relation \( s \rightarrow t \) is the same as non-orthogonality \( s \perp t \), we obtain the equivalence: \( \exists w(x \rightarrow F \mid w \rightarrow w) \) iff \( \exists w(x \rightarrow F \mid w \rightarrow w) \). This proves the adjointness property. As a consequence:

**Corollary 1.** For every property \( P \subseteq \Sigma \) and every linear map \( F \) we have: \( P \subseteq [F] \square (F^\dagger) \Diamond P \)

### 2.2 Compound System Quantum Frames

In this subsection we like to extend the quantum frame presented above for single systems into a quantum frame for compound systems. Let \( H \) be a Hilbert space of dimension 2 with basis \( \{ |0\rangle, |1\rangle \} \). We fix a natural number \( n \geq 2 \) (although later we will restrict to the case \( n \geq 4 \)), and we put \( N = \{1, 2, \ldots, n\} \). A compound-system quantum frame will be the quantum frame \( \Sigma(H_n) \) build on a Hilbert space \( H_n = H \otimes H \otimes \cdots \otimes H \) (\( n \) times).

**Notation.** In fact, we consider all the \( n \) copies of \( H \) as distinct (although isomorphic) and denote by \( H^{(i)} \) the \( i \)-th component of the tensor \( H^{\otimes n} \). Also, for any set of indices \( I \subseteq N \), we put \( H_I = H^{\otimes I} = \bigotimes_{i \in I} H^{(i)} \). (So, in particular, \( H_N = H_n = H \)). We denote by \( \epsilon_i : H \rightarrow H^{(i)} \) the canonical isomorphism between \( H \) and \( H^{(i)} \). This notation can be extended to sets \( I \subseteq N \) of indices of length \( |I| = k \), by putting \( \epsilon_I : H^{\otimes k} \rightarrow H_I \) to be the canonical isomorphism between these two spaces. For any vector \( |x\rangle \in H \), we denote by \( |x\rangle^{\otimes I} = \bigotimes_{i \in I} |x\rangle \) the corresponding vector in \( H_I \) (obtained by tensoring \(|I|\) copies of \(|x\rangle\)). Given a set \( I \subseteq N \), we say that a state \( s \in \Sigma(H) \) has its \( I \)-qubits in state \( s' \in \Sigma(H_I) \), and write \( s_I = s' \), if there exist vectors \( \psi \in s, \psi' \in H_I \) and \( \psi'' \in H_{N \setminus I} \) such that \( \psi = \mu_I(\psi' \otimes \psi'') \). Note that the state \( s_I \), if it exists, then it is unique (having the above property). In particular, when \( I = \{i\} \), we say that state \( s \) has as its \( i \)-th coordinate the state \( s_i \in H_{\{i\}} = H^{(i)} \).

We will further denote the vector \( |0\rangle^+ |1\rangle^+ \) by \(|+\rangle\), and similarly denote \( |0\rangle^- |1\rangle^- \) by \(|-\rangle\). For the states generated by the vectors in a two dimensional Hilbert space we introduce the following abbreviations: \( + := |+\rangle \), \(- := |-\rangle \), \( 0 := |0\rangle \), \( 1 := |1\rangle \). In order to refer to the state corresponding to a pair of qubits, we similarly delete the Dirac notation, e.g. \( 00 := |00\rangle = |0\rangle \otimes |0\rangle \).

The Bell states will be abbreviated as follows: \( \beta_{00} := |00\rangle + |11\rangle \), \( \beta_{01} := |01\rangle + |10\rangle \), \( \beta_{10} := |00\rangle - |11\rangle \), \( \beta_{11} := |01\rangle - |10\rangle \) and
\[ \gamma := \{|00\rangle + |01\rangle + |11\rangle + |10\rangle \}. \]

The following two results are well-known:

**Proposition 3.** Let \( H^{(i)} \) and \( H^{(j)} \) be two Hilbert spaces. There exists a bijective correspondence \( \psi \) between the linear maps \( F : H^{(i)} \to H^{(j)} \) and the states of \( H^{(i)} \otimes H^{(j)} \). Given the bases \( \{\epsilon^{(i)}_\alpha\}_\alpha \) and \( \{\epsilon^{(j)}_\beta\}_\beta \) of these spaces, the correspondence \( \psi \) is given by the mapping \( F \mapsto \psi(F) = \sum_{\alpha,\beta} m_{\alpha\beta} \langle \epsilon^{(i)}_\alpha | \cdot \rangle \epsilon^{(j)}_\beta \).

**Proposition 4.** Let \( H = H^{\otimes n} \) and let \( W = \{x \otimes |0\rangle^{\otimes (n-1)} : x \in H\} \) be given. Any linear map \( F : H \to H \) induces a linear map \( F_{(1)} : H \to H \) in a canonical manner: it is defined as the unique map on \( H \) satisfying \( F_{(1)}(x) = P_W \circ F(x \otimes |0\rangle^{\otimes (n-1)}) \). Conversely, any linear map \( G : H \to H \) can be represented as \( G = F_{(1)} \) for some linear map \( F : H \to H \).

**Notation.** The above results allow us to specify a compound state in \( H^{(i)} \otimes H^{(j)} \) via some linear map \( F \) on \( H \). Indeed, if \( F : H \to H \) is any such linear map, let \( F_{(1)} : H \to H \) be the map in the above proposition; this induces a corresponding map \( F^{(ij)} : H^{(i)} \to H^{(j)} \), by putting \( F^{(ij)} := \epsilon^i_1 \circ F_{(1)} \circ \epsilon^j_1 \), where \( \epsilon_i \) is the canonical isomorphism introduced above (between \( H \) and the \( i \)-th component \( H^{(i)} \) of \( H^{\otimes n} \)). Then we denote by \( F_{(ij)} \) the state

\[ F_{(ij)} := \psi(F^{(ij)}_{(1)}) \]

given by the above mentioned bijective correspondence \( \psi \) between \( H^{(i)} \to H^{(j)} \) and \( H^{(i)} \otimes H^{(j)} \).

The following result is also known from the literature:

**Proposition 5.** Let \( F : H \to H \) be a linear map. Then the state \( F_{(ij)} \) is “entangled according to \( F \)”; i.e. if \( F_{(1)}(|x\rangle) = |y\rangle \) and if the state of a 2-qubit system is \( F_{(ij)} \in H^{(i)} \otimes H^{(j)} \), then any measurement of qubit \( i \) resulting in a state \( x_i \) collapses the qubit \( j \) to state \( y_j \).

**Notation.** The notation \( F_{(ij)} \) can be further extended to define a property (set of states) \( F_{ij} \subseteq \Sigma = \Sigma(H) \), by defining it as the set of all states having the \( (i,j) \)-qubits in the state \( F_{(ij)} \):

\[ F_{ij} = \{ s \in \Sigma : s_{(i,j)} = F_{(ij)} \} = \{ \mu_{(i,j)}(\psi \otimes \psi') : \psi \in F_{(ij)}, \psi' \in H_N \setminus \{i,j\} \} \subseteq \Sigma \]

where \( \mu_{(i,j)} \) is as above the canonical isomorphism between \( H_{\{i,j\}} \otimes H_{N \setminus \{i,j\}} \).

In other words, \( F_{ij} \) is simply the property of an \( n \)-qubit compound state of having its \( i \)-th and \( j \)-th qubits (separated from the others, and) in a state that is “entangled according to \( F_{(ij)} \)”.

**Local properties.** Given a set \( I \subseteq N \), a property \( S \subseteq \Sigma \) is local in \( I \) if it corresponds to a property of the subsystem formed by the qubits in \( I \); in other words, if there exists some property \( S' \subseteq \Sigma(H_I) \) such that:

\[ S' = \{ s \in \Sigma : s_I \in S' \} \]
or, more explicitly: \( S' = \{ \mu_I(\psi \otimes \psi') : \psi \in S', \psi' \in \mathcal{H}_{N\setminus I} \} \). An example is the property \( F_{ij} \), which is \( \{i,j\}\)-local. The family of local properties is closed under union, intersection but not under complementation.

**Local transformations.** Given \( I \subseteq N \), a linear map \( F : \mathcal{H} \rightarrow \mathcal{H} \) is \( I\)-local if it “affects only the qubits in \( I \)”; in other words, if there exists a map \( G : \mathcal{H}_I \rightarrow \mathcal{H}_I \) such that:

\[
F \circ \mu_I (\psi \otimes \psi') = \mu_I (G(\psi) \otimes \psi')
\]

A map \( F : \Sigma \rightarrow \Sigma \) is \( I\)-local if it is the map induced on \( \Sigma \) by an \( I\)-local linear map on \( \mathcal{H} \). Examples are: all the tests \( S_I \) of \( I\)-local properties; logic gates that affect only the qubits in \( I \), i.e. (maps on \( \Sigma \) induced by) unitary transformations \( U_I : \mathcal{H} \rightarrow \mathcal{H} \) such that for all \( \psi, \psi' \in \mathcal{H}_I \), we have \( U_I \circ \mu_I (\psi \otimes \psi') = \mu_I (U(\psi) \otimes \psi') \), for some \( U : \mathcal{H}_I \rightarrow \mathcal{H}_I \). The family of local maps is closed under composition.

**Lemma 2.** The main lemma in [5] states (in our notation) that, given a quadruple of distinct indices \( i,j,k,l \), let \( F,G,H,U,V : \mathcal{H} \rightarrow \mathcal{H} \) be single-qubit linear maps, then we have:

\[
G_{jk} \circ V_k \circ U_j (F_{ij} \cap H_{kl}) \subseteq (H \circ U^\dagger \circ G \circ V \circ F)_{il}
\]

Using the formalism of entanglement specification networks introduced in [3], this can be encoded in the following diagrammatic representation:

[Diagram]

[5] and [1] use this as the main tool in explaining teleportation, quantum gate teleportation and many other quantum protocols. We will use this work in our logical treatment of such protocols, by taking this lemma as one of our main axioms.

Observe that in the above Lemma, the order in which the operations \( U_j \) and \( V_k \) are applied is in fact irrelevant. This is a consequence of the following important property of local transformations:

**Proposition 6.** (Compatibility of local transformations affecting different sets of qubits)

If \( I \cap J = \emptyset \), \( F_I \) is an \( I\)-local map and \( G_J \) is a \( J\)-local map, then we have:

\[
F_I \circ G_J = G_J \circ F_I
\]

Another important property of local maps (on states) is:
Proposition 7. ("Agreement Property") Let $F, G : \Sigma \to \Sigma$ be two $I$-local maps on states, having the same domain. $\text{dom}(F) = \text{dom}(G)$. Then their output-states agree on all non-$I$ qubits, i.e.:

$$F(s)_J = G(s)_J$$

for all $s \in \Sigma$ and all $J$ such that $I \cap J = \emptyset$. (We take this equality to imply in particular that the right-hand is defined iff the left-hand is also defined.)

Dynamic Characterizations of Main Unitary Transformations.

It is well-known that a linear operator on a vector space in a given Hilbert space is uniquely determined by the values it takes on the vectors of an (orthonormal) basis. An important observation is that this fact is no longer "literally true" when we move to "states" as one-dimensional subspaces instead of vectors. The reason is that "phase"-aspects (or, in particular, the signs $+$ and $-$) are not "state" properties in our setting. In other words, two vectors that differ only in phase, i.e. $x = \lambda y$ where $\lambda$ is a complex number with $|\lambda| = 1$, belong to the same subspaces, so they correspond to the same state $x = y$.

Example 1. (Counterexample) Consider a 2 dimensional Hilbert space in which we denote the basis vectors by $\ket{0}$ and $\ket{1}$, a transformation $I$ is given by $I(\alpha \ket{0} + \beta \ket{1}) = \alpha \ket{0} + \beta \ket{1}$; and a transformation $J$ is given by $J(\alpha \ket{0} + \beta \ket{1}) = \alpha \ket{0} - \beta \ket{1}$. Although $I$ and $J$ induce different operators on states, these operators map the basis states to the same images: $I(0) = I(\ket{0}) = 0 = J(\ket{0}) = J(0), I(1) = I(\ket{1}) = 1 = -\ket{1} = J(\ket{1}) = J(1)$. But of course we do distinguish the subspaces generated by different superpositions: $I(+) = \ket{0} + \ket{1} = + \neq -\ket{0} - \ket{1} = J(+)$.\footnote{The domain of a map is defined by $\text{dom}(F) = \{s \in \Sigma : F(s) \text{ is defined} \}$. If $F'$ is the corresponding linear map on $\mathcal{H}$, this means that $\text{dom}(F) = \{ \psi : F'(\psi) \neq 0 \}$.}

Proposition 8. A linear operator on the state space $\Sigma(\mathcal{H}_1)$ of a 2 dimensional Hilbert space is uniquely determined by its images on the states: $\{ \ket{0}, \ket{1}, \ket{+} \}$.

Corollary 2. A linear operator on the state space $\Sigma(\mathcal{H}_n)$ of the space $\mathcal{H}_n$ is uniquely determined by its images on the states:

$$\{ \ket{x}_1 \otimes \ldots \otimes \ket{x}_n : |x|_i \in \{ \ket{1}_i, \ket{0}_i, \ket{+}_i \} \}$$

In the definition of a quantum frame given above, we introduced the set $\mathcal{U}$ as the set of unitary transformations for single systems. For compound systems the set $\mathcal{U}$ will be extended with the kind of operators that are active on compound systems. Following the quantum computation literature, we take $\mathcal{U} = \{ X, Z, H, CNOT, \ldots \}$ where $X, Z$ and $H$ are defined by the following table:

|   | 0 | 1 | + |
|---|---|---|---|
| X | 1 | 0 | + |
| Z | 0 | 1 | - |
| H | + | - | 0 |
The transformation \( CNOT \) is given by the table:

| \( CNOT \) | 00 | 01 | 0+ | 10 | 1+ | +0 | +1 | ++ |
|---|---|---|---|---|---|---|---|---|
| 00 | 00 | 01 | 0+ | 10 | 1+ | +0 | +1 | ++ |

3 Syntax of LQP

The Basic Language of LQP

To build up the language of LQP, we are given a natural number \( n \), and we put \( N = \{1,2,\ldots,n\} \). We start from a set \( Q \) of \textit{propositional variables}, together with an \textit{arity map}, i.e. every \( p \in Q \) has an arity \( k \leq n \); a set \( C = \{+,1,\ldots\} \) of \textit{propositional constants}; and a set \( U = \{CNOT_2,X_1,H_1,Z_1,\ldots\} \) of constants, denoting \textit{basic programs}, to be interpreted as \textit{unitary transformations}; each such program comes also with an arity \( k \leq n \). The syntax of LQP is an extension of the classical syntax for PDL, with a set of propositional \textit{formulas} and a set of \textit{programs}, defined by mutual induction:

\[
\varphi ::= p_I \mid c_i \mid \overline{\pi}_{i,j} \mid \neg \varphi \mid \varphi \land \varphi \mid [\pi]\varphi
\]

\[
\pi ::= \top \mid \varphi ? \mid U_I \mid \pi^\dagger \mid \pi \cup \pi \mid \pi;\pi \mid \pi^*
\]

Here, we take \( I \) to denote sequences of distinct indices in \( N = \{1,2,\ldots,n\} \). In the above syntax, \( p_I \) and \( U_I \) are well-formed terms iff the arity \( k \) of \( p \), or of \( U \), matches the length of the sequence, i.e. \( k = |I| \). In the semantics we will interpret \( p \) to be a physical property of the system of \( |I| \) qubits, and the sentence \( p_I \) as saying that the qubits with indices in \( I \) have the property \( p \) consisting of \( k = |I| \) relevant basic states which are specifically the ones labeled corresponding to the numbers in the subset \( I \). Similarly, in the semantics it will become clear that every member of \( U \) encodes a specific quantum logical gate and the subscript \( I \) in \( U_I \) will then indicate on which qubits the gate is active. When the arity of a variable \( p \) is \( n \), then we skip the subscript, and simply write \( p \) instead of \( p_n \).

For a given propositional constant \( c \in C \), we interpret the sentence \( c_i \) as saying that “the \( i \)-th-qubit is in the state \( |c>\)”. Note that 1 as a logical (characterizing the qubit \( |1> \)) is different from the propositional formula \( \top \) (verum) which we formally introduce later in this section, to denote the “top” element of the lattice of properties. This, in its turn, is also different from the \textit{program} \( \top \), introduced in the syntax above, which will simply denote the trivial program, relating any two states.

Extending the Basic Language of LQP.

We extend our language by defining the operations for a \textit{classical disjunction} and a \textit{classical implication} in the usual way, i.e. \( \varphi \lor \psi := \neg (\neg \varphi \land \neg \psi) \), \( \varphi \rightarrow \psi := \neg \varphi \lor \psi \). We introduce constants \textit{verum} \( \top := 1_1 \lor \neg 1_1 \), and \textit{falsum} \( \bot := 1_1 \land \neg 1_1 \). We define the classical dual of \( [\pi]\varphi \) in the usual way as \( \langle \pi \rangle \varphi := \neg [\pi] \neg \varphi \); the \textit{measurement modalities} \( \Box \) and \( \Diamond \) that are known in the quantum logic literature can be defined in LQP by putting \( \Diamond \varphi := \langle \varphi ? \rangle \top \) and \( \Box \varphi := \neg \Diamond \neg \varphi \). The \textit{orthocomplement} is defined as \( \sim \varphi := \Box \neg \varphi \), or equivalently as
$\sim \phi := [\phi?]\perp$. By means of the orthocomplement we define new propositional constants $0 := \sim 1$, and $-i := \sim t_i$, and a binary operation for quantum join $\phi \sqcup \psi := \sim (\sim \phi \land \sim \psi)$. This expresses superpositions: $\phi \sqcup \psi$ is true at any state which is a superposition of states satisfying $\phi$ or $\psi$. We can also define the quantum dual of a modality $[\pi]\psi$ as $< \pi^{-1} \psi := \sim [\pi] \sim \psi$. Finally we put $< \pi > \psi := < (\pi^1)^{-1} > \psi$. As we’ll see, this captures the strongest post-condition ensured by applying program $\pi$ on a state satisfying (a precondition) $\psi$.

**Testable formulas.** We call a program $\pi$ deterministic if $\pi$ is constructed without the use of non-deterministic choice $\cup$ or iteration $\ast$. Next we define the set of testable formulas $\varphi_t$ of LQP to be a subset of the above given language, constructed by induction in the following way:

$$
\varphi_t := \perp \mid c_i \mid \pi_{i,j} \mid \varphi_t \land \varphi_t \mid [\pi] \varphi_t
$$

where $\pi$ is any deterministic program. Observe that the construction of $\pi$ might involve non-testable formulas. In particular, for an arbitrary (not necessarily testable) formula $\varphi$, remark that $[\varphi?] \psi_t$ is a testable formula.

**Proposition 9.** For any formula $\varphi$ in LQP, $\sim \varphi$ and $\square \varphi$ are testable formulas.

**Local formulas and local programs.** We would like to isolate local formulas and programs, i.e. the ones that “affect only the qubits in a given set $I \subseteq N$”. These formulas will express local properties (in the sense defined above). When we want to stress that a formula or program is local, we denote them with $\varphi_I$ or $\pi_I$. The definition is:

$$
\begin{align*}
\varphi_I & := p_j \mid c_i \mid \pi_{i,j} \mid \varphi_I \lor \varphi_I \mid \varphi_I \land \neg \varphi_I \mid \varphi_I \land [\pi] \varphi_I \\
\pi_I & := \varphi_I ? \mid \cup \mid \pi_I ; \pi_I \mid \pi_I \lor \pi_I \mid \pi_I = \pi_I
\end{align*}
$$

with $i, j \in I, J \subseteq I$. Observe that local formulas are not closed under negation: this is because the complement of a local property is not necessarily a local property. But instead they are closed under set-theoretic difference, disjunction, and also conjunction: this is because $\varphi \land \psi$ is equivalent to $\varphi \land (\neg \varphi \lor \neg \psi)$.

**Relabeling local formulas and programs.** When we label a local formula $\varphi_J$ or a local program $\pi_I$ with a sequence of indices $I$, we can of course take any other sequence $J$ of indices, with $|J| = |I|$, and substitute all the $I$ indices in our formula (program) with the corresponding $J$ indices; we denote by $\varphi_J$ , and respectively $\pi_J$ , the corresponding formula, or program.

**Notation.** The unary map induced by a program: We want to capture in our syntax the construction $F_{(i)}$, by which a linear map $F$ on $H^\otimes n$ was used to describe a unary map $F_{(i)}$ on $H$. For this, we put: $0_i := 0_i \cup (1_i \sim X_i)$, and $0_I := 0_i \cup 0_i \cup \ldots \cup 0_i$, where $I = (i_1, i_2, \ldots, i_k)$. This maps any qubit in $I$ to 0. Similarly, we put: $0_I := 0_i \land 0_i \land \ldots \land 0_i$. Finally we define:

$$
\pi_{(i)} := +0_{N \setminus \{i\}} \land \pi ; 0_{N \setminus \{i\}}\
$$
This is the map we need (which encodes a single qubit transformation). In fact, we shall only use \( \pi_{(1)} \) in the rest of this paper.

4 Semantics of LQP

An LQP-model is a quantum frame equipped with a valuation function, mapping each propositional variable \( p \) with arity \( k \) into a set \( \| p \| \subseteq \Sigma(H^{\otimes k}) \) of \( k \)-qubit states. Give a sequence \( I \) of length \( i \) of indices, let \( \epsilon \) be the canonical isomorphism between \( H^{\otimes k} \) and \( H^{\otimes I} \).

We will use the valuation map to give an interpretation \( \| \varphi \| \subseteq \Sigma \) to all our formulas, in terms of properties of our \( n \) qubit system, i.e. sets of states in \( \Sigma = \Sigma(H) \). In the same time, we give an interpretation \( \| \pi \| \subseteq \Sigma \times \Sigma \) to all our programs, in terms of binary relations between states. The two interpretations are defined by mutual recursion.

Interpretation of Programs: The basic programs \( U_I \), with \( |I| = k \), come from a list of corresponding \( k \)-bit unitary transformations \( U : H^{\otimes k} \rightarrow H^{\otimes k} \). We take \( \| U_I \| \) to be the (map on states induced by the) unique linear map on \( H \) such that:

\[
\| U_I \| \circ \mu_I(\psi \otimes \psi') := \mu_I(\epsilon_I \circ U \circ \epsilon_I^{-1}(\psi) \otimes \psi')
\]

for every \( \psi \in H_I, \psi' \in H_{N \setminus I} \). Here, recall that \( \epsilon_I \) is the canonical isomorphism between \( H^{\otimes k} \) and \( H_I \), and \( \mu_I \) is the canonical isomorphism between \( H_I \otimes H_{N \setminus I} \) and \( H \).

As for the others:

\[
\begin{align*}
\| \top \| & := \Sigma \times \Sigma \\
\| \pi_1 \cup \pi_2 \| & := \| \pi_1 \| \cup \| \pi_2 \| \\
\| \pi_1 \pi_2 \| & := \| \pi_1 \| \circ \| \pi_2 \| \\
\| (\pi^I)^+ \| & := \| \pi \| \\
\| (\pi \cup \pi_2)^I \| & := \| (\pi_1)^I \cup (\pi_2)^I \|
\end{align*}
\]

where \( R^* \) is the reflexive-transitive closure of relation \( R \). Note that deterministic programs have as interpretations \( \| \pi \| \) (maps on states which are induced by) linear maps on \( H \). The interpretation \( \| \pi \| \) allows us to extend the notation \( \pi \rightarrow \) to all programs, by putting:

\[
s \xrightarrow{\pi} t \text{ iff } (s, t) \in \| \pi \|.
\]

Interpretation of Formulas. We give the interpretation here first for all except propositional variables \( p_i \) and entangled state formulas \( \pi_{ij} \):

\[
\begin{align*}
\| \varphi \land \psi \| & = \| \varphi \| \cap \| \psi \|, \quad \| \neg \varphi \| = \Sigma \setminus \| \varphi \| \\
\| 1_i \| & = 1_i, \quad \| + \| = +_i
\end{align*}
\]

and finally \( \| \pi \varphi \| = \{ s \in \Sigma | \forall t : s \xrightarrow{\pi} t \Rightarrow t \in \| \varphi \| \} \).
The last clause obviously defines the weakest precondition $[\pi] \varphi$ ensuring that (postcondition) $\varphi$ will be satisfied after executing program $\pi$. As for the propositional variables, we put:

$$|| p_I || = \{ s \in H : s_I \in \epsilon_I(|| p ||) \}$$

$$= \{ \mu_I(\epsilon_I(\psi) \otimes \psi') : \psi \in || p ||, \psi' \in H_{N \setminus I} \}$$

where $\epsilon_I$ and $\mu_I$ are the above-mentioned canonical isomorphisms, and $s_I$ is (as defined above) the state of the qubits in $I$. So the meaning of $p_I$ is that the system of qubits with indices in $I$ is separated from (i.e. non-entangled with) the rest of the system, and that moreover this system has the property expressed by $p$.

The interpretation of $\pi_{ij}$, for deterministic programs $\pi$, is given by the construction $F_{ij}$ above. Since the interpretation $||\pi||$ of a deterministic program is a linear map on $H$, we know, by the results mentioned above, that the map $F_{(i)}$ can be used to specify a set of compound states $F_{ij} \subseteq H$. This is our intended interpretation for $\pi_{ij}$:

$$|| \pi_{ij} || := || \pi ||_{ij}$$

For the program $\top$, we put: $||\top|| := \{ s \in \Sigma : s_{i,j} \text{ is defined} \} = \{ \mu_{\{i,j\}}(\psi \otimes \psi') : \psi \in H_{\{i,j\}}, \psi' \in H_{N \setminus \{i,j\}} \}$, i.e. the property of having the $i,j$-qubits in a separated state from the others. This can be extended to other programs in the natural way, by putting e.g.

$$|| \pi \cup \pi_{ij} || := || \pi || \cup \pi_{ij} || \text{ etc.}$$

**Proposition 10.** The interpretation of any testable formula is a testable property. The interpretation of an $I$-local formula (or deterministic program) is an $I$-local formula (or $I$-local linear map on states).

**Lemma 3.** $|| \sim \varphi || = || \varphi ||$, $|| [\varphi?] \psi || = || [\varphi] ||$, $|| \varphi || = || \varphi ||$, $|| \Box \varphi || = \Box || \varphi ||$.

**Proposition 11.** The following are equivalent, for every formula $\varphi$:

1. $|| \varphi ||$ is testable
2. $\varphi$ is semantically equivalent to $\Box \Diamond \varphi$
3. $\varphi$ is semantically equivalent to some formula $\Diamond \varphi$
4. $\varphi$ is equivalent to some formula $\sim \psi$

5 **Axioms for LQP**

First, we admit all the axioms and rules of classical PDL, except for the ones concerning tests $\varphi$? In particular, we have the basic axiom and rule for sentence involving modalities $[\pi]$, stated for elementary sentences and basic programs:

**Kripke Axiom.** $\vdash [\pi](p \rightarrow q) \rightarrow ([\pi]p \rightarrow [\pi]q)$

**Necessitation Rule.** if $\vdash p$ then $\vdash [\pi]p$

Considering $\Box p$, we introduce the following axioms:

**Test Generalization Rule.** if $p \rightarrow [\varphi?]r$ for all $q$, then $\vdash p \rightarrow \Box r$
Testability Axiom. \( \vdash \square p \rightarrow [q?]p \)

Testability can be stated in its dual form by means of \( \langle q? \rangle p \rightarrow \Diamond p \) or equivalently as \( \langle q? \rangle p \rightarrow (p?)T \). This dual formulation of Testability allows us to give a straightforward interpretation: if the property associated to \( p \) can be actualized by a measurement (yielding an output state satisfying \( p \)), then we can directly test the property \( p \) (by doing a measurement for \( p \)). The Test Generalization Rule encodes the fact that \( \square \) is a universal quantifier over all possible measurements.

Other LQP-axioms are:

- **Partial Functionality.** \( \vdash \neg \langle q? \rangle q \rightarrow \langle p? \rangle \neg q \)
- **Adequacy.** \( \vdash p \land q \rightarrow \langle p? \rangle q \)
- **Repeatability.** \( \vdash [\psi?]q \rightarrow [p?]\neg q \) for all testable formulas \( \psi \),
- **Universal Accessibility.** \( \vdash \langle \pi \rangle \square q \rightarrow [\pi']p \)
- **Unitary Functionality.** \( \vdash \langle U \rangle q \leftrightarrow [U]q \)
- **Unitary Bijectivity 1.** \( \vdash [U;U']p \leftrightarrow [U;U']p \)
- **Unitary Bijectivity 2.** \( \vdash p \leftrightarrow [U;U']p \)
- **Adjointness.** \( \vdash p \rightarrow [\pi][\pi]\Diamond p \)

**Substitution Rule.** From \( \vdash \Theta \) infer \( \vdash \Theta[p_1 \backslash \varphi_1] \)

**Compatibility Rule.** For all testable formulas \( \psi, \varphi \) and every variable \( p \notin \varphi, \psi \):

\[ \vdash \Theta[p_1 \backslash \varphi_1] \]

From \( \vdash \langle \varphi?; \psi? \rangle p \rightarrow \langle \psi?; \varphi? \rangle p \) infer \( \vdash \langle \varphi?; \psi? \rangle p \rightarrow \langle (\varphi \land \psi)? \rangle p \)

**Proposition 12.** (Quantum Logic, Weak Modularity or Quantum Modus Ponens) All the axioms and rules of traditional Quantum Logic are satisfied by our testable formulas. In particular, from our axioms one can prove “Quantum Modus Ponens” \( \vdash \square \varphi \land [\varphi?] \psi \rightarrow \psi \). In its turn, this rule is equivalent to the condition known in quantum logic as Weak Modularity, stated as follows: \( \varphi \land (\neg \varphi \lor (\varphi \land \psi)) \vdash \psi \).

**Theorem 4.** (Soundness, Expressivity, Completeness of the above axioms with respect to PDL frames) In the presence of (axioms of classical logic, plus) Kripke’s Axioms, Necessitation, Test Generalization, Testability and Substitution Rule, all the other axioms above are sound and expressive with respect to the corresponding semantic conditions mentioned in Section 2 above. More precisely: any of these axioms is valid on a PDL frame iff the corresponding semantic condition is satisfied by the frame. Moreover, the system given by the above axioms is complete for the class of PDL frames satisfying all the corresponding semantic conditions.

**Proposition 13.** The formula \( \langle \pi \rangle^{-1} \varphi \) expresses the strongest testable post-condition ensured by executing program \( \pi \) on any state satisfying (precondition) \( \varphi \). In other words: for every testable \( \psi \) the following are equivalent:

1. \( \vdash \langle \pi \rangle^{-1} \varphi \rightarrow \psi \)

---

3 This explains why the weakest precondition \( [\varphi?] \psi \) has been taken as the basic implicational connective in traditional Quantum Logic, under the name of “Sasaki hook”, denoted by \( \varphi \xrightarrow{S} \psi \).
2. \( \vdash \varphi \rightarrow [\pi] \psi \)

Moreover, in the context of the other axioms, this equivalence is itself equivalent to the Adjointness Axiom. 

**Basic Axioms for constants** \((0, 1, +, -)\)

The first axiom says that \(c_i\)'s are “states” in the \(i\)-th part of the system, i.e. they are atomic properties, which determine completely whether any other property is jointly satisfied. We state in a weak, as well as in stronger version:

**Atomicity** (weak version). For all \(c \in \{0, 1, +, -\} \): \( \vdash c_i \land p_i \rightarrow [\square] (c_i \rightarrow p_i) \)

**Atomicity** (strong version). For all \(c \in \{0, 1, +, -\} \):
\( \vdash \bigwedge_{i \in I} c_i \land p_1 \rightarrow [\square] (\bigwedge_{i \in I} c_i \rightarrow p_I) \)

The following axioms state that \(+, \cdot\) and \(-, \text{local}\) have the same domain as \(\text{id}\):

**Proper Superposition Axioms:** \( \vdash +_i \rightarrow 0_i \land [\square] 1_i \) and \( \vdash -_i \rightarrow 0_i \land 1_i \).

Next two axioms assert that 1 and + are testable properties:

**Constants are testable.** \( \vdash [\square] 1_i \rightarrow 1_i \) and \( \vdash [\square] +_i \rightarrow +_i \).

**Determinacy Axiom of Deterministic Programs.** For deterministic programs \(\pi, \pi'\):
\( \vdash ([\square] \bigwedge_{i \in I; n \in \{0, 1, +, -\}^n} (\langle \pi > 1 \land ... \land c_i^{(n)}) \leftrightarrow \langle \pi' > 1 \land ... \land c_i^{(n)}) \rightarrow (\langle \pi > p \leftrightarrow \langle \pi' > p) \)

This expresses the above-mentioned property of linear operators on \(\mathcal{H}\) of being uniquely determined by their values on all the states \(|x >_1 \otimes ... |x >_n\rangle\), with \(|x >_i \in \{0 >_i, 1 >_i, + >_i\}\)

**Agreement Axiom.** If two \(I\)-local programs \(\pi, \pi'\) have the same domain, then their output states agree on all non-\(I\) qubits: i.e. if \(I \cap J = \emptyset\) then \(\boxdot (\langle \pi >_I \land \pi' >_I \leftrightarrow \langle \pi >_J \land \pi' >_J) \rightarrow (\langle \pi >_I >_J \leftrightarrow \langle \pi' >_I >_J)\)

**Compatibility of programs affecting different sets of qubits.** If \(I \cap J = \emptyset\) then \(\vdash [\pi_i; \pi_j] p \leftrightarrow [\pi_i; \pi_j] p\)

**Entanglement Rule.** From \(\vdash p_1 \rightarrow [\pi_{(1)}] q_1\) infer \(\vdash [\pi_i] q_i \rightarrow [\pi_{(1)}] q_1\)

**Entanglement Composition Axiom.** For distinct indices \(i, j, k, l\), programs \(\pi, \pi', \pi''\) and local \(\{1\}\)-programs \(\sigma_1, \rho_1\) we have:
\(\vdash \pi_{ij} \land \pi'_{kl} \rightarrow [\sigma_{ij}; \rho_{ij}: \pi''_{ik}] (\pi; \sigma_1; \pi''; \rho_1; \pi')_{il}\)

**Trivial Entanglement** \(\vdash p_{i,j} \rightarrow [\pi]_{ij}\). This says that separation of the \(i, j\)-qubits implies their trivial entanglement.

**Theorem 5.** (Teleportation Property). If \(\varphi_1\) is a 1-local testable property and if \(\vdash \varphi_1 \rightarrow [\pi_{(1)}] q_1\), then \(\vdash \varphi_1 \land [\sigma] q_1 \rightarrow [\pi_{(2)}] q_3\).

**Proof:** We apply the Entanglement Composition Axiom, taking \(i = 4, j = 1, k = 2, l = 3\), and substituting the programs \(\top\) for \(\pi, \sigma\) for \(\pi', \pi\) for \(\pi''\), \(\varphi_1\) for \(\sigma_1\), and \(id_1 = X_1; X_1\) for \(p_1\). We obtain:
\(\vdash \top_{41} \land [\sigma] q_3 \rightarrow [\varphi_1] q_4 (\top; p_1; \pi_{(1)}; \pi_{(2)}; \sigma)_{43}\).

On the other hand, we have \(\vdash \varphi_1 \land [\sigma] q_3 \rightarrow [0_4] (p_4 \land \top_{41} \land [\sigma] q_3)\) (since \(0_4\) is 4-local and has the same domain as \(id_4\), so by Agreement Axiom it agrees with
id₄ on non-4 qubits, thus preserving ϕ₁ and σ₂₃; but also ⊢ [0₄!]|0₄ and using the Trivial Entanglement Axiom, we get the conclusion). From these two together, we obtain: ⊢ ϕ₁ ∧ σ₂₃ → [0₄!]|[π₁₂?]([T; ϕ₁?; π; id_; σ]; 4₃). But on the other hand, we have ⊢ (T; ϕ₁?; π; id; σ); 4₃ → [0₄?]|q₃. (This is because we assumed ⊢ ϕ₁ → [π₁; σ₁]|q₁, from which it follows that ⊢ 0₁ → [T; ϕ₁?; σ₁]; id; σ; q₁, using the fact that id = id and ⊢ [ϕ₁?]ϕ₁, by Repeatability axiom and the testability of ϕ₁. Apply now Entanglement Rule, obtaining the above conclusion.) From these two we get that: ⊢ ϕ₁ ∧ σ₂₃ → [0₄!]|π₁₂?; 0₄?|q₃. The desired conclusion follows from the Agreement Axiom and the fact that 0₄!; π₁₂?; 0₄? and π₁₂? are {1, 2, 4}-local programs with the same domain.

**Characteristic Formulas.** In order to formulate our next axioms (dealing with special logic gates), we give some characteristic formulas for binary states, considering two qubits indexed by i and j:

| States          | Characteristic Formulas                      |
|-----------------|---------------------------------------------|
| | 00⟩ij = | 0⟩i ⊗ | 0⟩j | (0?)0j ∧ [1?] ⊥ |

Bell states:

\[ \beta_{xy}^{ij} = \begin{cases} 0⟩i ⊗ | y⟩j + (-1)^{x} | 1⟩i ⊗ | \tilde{y}⟩j & \text{with } \tilde{0} = 1 \text{ and } \tilde{1} = 0, x, y \in \{0, 1\} \\ \end{cases} \]

\[ (0?)yj \land (1?)\tilde{y}j \land (+?)\tilde{y}j \]

where \((-?)^x = -1 \text{ if } x = 1 \text{ and } (-?)^x = +1 \text{ if } x = 0 \]

\[ \gamma_{ij} = \beta_{00}^{ij} + \beta_{11}^{ij} = \begin{cases} 0⟩0i + | 01⟩ij + | 10⟩ij + | 11⟩ij & (0?) + j \land (1?) + j \land (+?) + j j \end{cases} \]

**Characteristic Axioms for Quantum Gates X and Z.** In general, for all unitary transformations U ⊂ SU, we have as a consequence of the previous axioms that: ⊢ p_K → |U⟩|p_K, for I ∩ K = ∅.

In addition to this, we require for X, Z, H:

\[ ⊢ 0_i \rightarrow [X_i]_i \quad ; \quad ⊢ 1_i \rightarrow [X_i]_i \quad ; \quad ⊢ +_i \rightarrow [X_i]_i \]
\[ ⊢ 0_i \rightarrow [Z_i]₀ \quad ; \quad ⊢ 1_i \rightarrow [Z_i]₁ \quad ; \quad ⊢ +_i \rightarrow [Z_i]₀ \]
\[ ⊢ 0_i \rightarrow [H_i]⁺ \quad ; \quad ⊢ 1_i \rightarrow [H_i]⁻ \quad ; \quad ⊢ +_i \rightarrow [H_i]₀ \]

**Notation.** For x, y ∈ {0, 1} and distinct indices i, j ∈ N, we make the following abbreviations for “Bell formulas”:

\[ β_{xy}^{ij} := \langle Z_i^x; X_i^y⟩_{ij} \]

**Proposition 14.** The Bell states β_{xy}^{ij} are characterized by the logic Bell formulas β_{xy}^{ij}. In other words, a state satisfies one of these formulas iff it coincides with the corresponding Bell state.

**Proof:** It is enough to check that the formulas β_{xy}^{ij} imply the corresponding characteristic formulas in the above table. For this, we use the Entanglement Axiom and the following (easily checked) theorems: ⊢ 0_i ↔ Z_i^x; X_i^y > y_i, ⊢ 1_j ↔ Z_i^x; X_i^y > y_i, ⊢ +_i ↔ Z_i^x; X_i^y > (−)?x.
Characteristic Axioms for CNOT. With the above notations, we put:

\[ \vdash 0_i \land c_j \rightarrow [\text{CNOT}_{ij}]c_j \] ; \[ \vdash 1_i \land 0_j \rightarrow [\text{CNOT}_{ij}]1_j \]

\[ \vdash +_i \land 0_j \rightarrow [\text{CNOT}_{ij}]\beta^{ij}_{00} \] ; \[ \vdash +_i \land 1_j \rightarrow [\text{CNOT}_{ij}]\beta^{ij}_{01} \]

\[ \vdash +_i \land +_j \rightarrow [\text{CNOT}_{ij}]\gamma^{ij} \quad \text{where} \quad \gamma^{ij} = \langle 0,? \rangle +_j \land \langle 1,? \rangle +_j \land \langle +,? \rangle +_j \]

Proposition 15. For all \( x,y \in \{0,1\} \):

\[ \vdash (H_i; \text{CNOT}_{i,j})(x_i \land y_j) = \beta^{ij}_{xy} \]

Corollary. If \( i,j,k \) are all distinct then

\[ \vdash (\text{CNOT}_{ij}; H_j; (x_i \land y_j)?(p)) = k \beta^{ij}_{xy}?(p) \]

Proof: From the above and \( H^\dagger = H, \text{CNOT}^\dagger = \text{CNOT} \), we get

\[ \vdash \beta^{ij}_{xy} \rightarrow [\text{CNOT}_{i,j}; H_i](x_i \land y_i) \]

and so

\[ \vdash (\text{CNOT}_{ij}; H_i; (x_i \land y_i)?) \top \leftrightarrow \langle \beta^{ij}_{xy} \rangle \top \]

The conclusion follows from this, together with the Agreement Axiom.

6 Correctness of the Teleportation Protocol

Following [8], quantum teleportation is the name of a technique that makes it possible to teleport the state of a quantum system without using a channel that allows for quantum communication, but with a channel that allows for classical communication. We are working in \( H \otimes H \otimes H \), with \( H \) being the two-dimensional (qubit) space, and so \( n = 3 \). We assume two agents, Alice and Bob who are separated in space and each has one qubit of an entangled EPR pair that is represented by \( \beta^{23}_{00} \in H(2) \otimes H(3) \). Alice holds in addition to her part of the EPR pair also a qubit \( q_1 \in H(1) \), in an unknown local state \( q_1 \). Alice “teleports” this state to Bob, i.e. she performs a program that will output a state satisfying \( \varphi_3 \). To do this, she first entangles \( q_1 \) with her part \( q_2 \) of the EPR pair (i.e. she performs a \( \text{CNOT}_{1,2} \) gate on the two qubits and then a Hadamard transformation \( H_1 \) on the first component). Bob’s qubit has suffered during the actions of Alice and when Alice will measure her qubits she will destroy the entanglement of the EPR pair that she shares with Bob. The initial state of Bob’s qubit is known and we can calculate which changes it has gone through when we know the result that Alice obtains from the two measurements. Moreover, the result that Alice obtains from the two measurements indicate the actions that Bob has to perform in order to transfer his qubit \( q_3 \) into the state \( q_1 \) was before the protocol. It is enough for Alice to send Bob two classical bits encoding the result \( x_1 \) of the first measurement and the result \( y_2 \) of the second measurement. This means that Bob will have to apply \( y \) times the \( X \)-gate followed by \( x \) times the \( Z \)-gate, if he wants to force his qubit \( q_3 \) into the state \( \varphi_3 \).
In our syntax, the quantum program described here is:

$$\pi = \bigcup_{x,y \in \{0,1\}} \text{CNOT}_{12}; H_1; (x_1 \land y_2)?; X^y_3; Z^x_3$$

and the validity expressing the correctness of teleportation is

$$\vdash \pi(\phi_1 \land \beta_{00}^{23}) \rightarrow [\pi] \phi_3$$

for all testable 1-local formulas $\phi_1$. To show this, observe that by applying the above Corollary (at the end of the last section) in which we take $i = 1, j = 2, k = 3$ and substitute $p_3$ with $[X^y_3; X^x_3] \phi_3$, we obtain that the validity above (to be proved) is equivalent to:

$$\vdash \phi_1 \land \beta_{00}^{23} \rightarrow [\beta_{xy}^{12}] [X^y_3; Z^x_3] \phi_3.$$ 

Replacing the logical Bell formulas with their definitions $\beta_{xy}^{ij} := (Z^i_1; X^j_1)_{ij}$, we obtain the following equivalent validity:

$$\vdash q_1 \land m_{23} \rightarrow [(Z^i_1; X^j_1)_{23}] [X^y_3; Z^x_3] \phi_3,$$

where $id = Z^y_0; X^x_0$ is the identity. This last validity follows from applying the Teleportation Property and the validity $\vdash \phi_1 \rightarrow [Z^i_1; X^y_3; X^y_1; Z^x_3] \phi_1$ (due to $X^{-1} = X, Z^{-1} = Z$).

**Note.** This proof of correctness can be easily adapted to cover Logic-Gate Teleportation. Moreover, the whole range of quantum programs covered by the “entanglement networks” in [5] can be similarly treated using our logic.

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