Running coupling at finite temperature and chiral symmetry restoration in QCD

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We analyze the running gauge coupling at finite temperature for QCD, using the functional renormalization group. The running of the coupling is calculated for all scales and temperatures. At finite temperature, the coupling is governed by a fixed point of the 3-dimensional theory for scales smaller than the corresponding temperature. The running coupling can drive the quark sector to criticality, resulting in chiral symmetry breaking. Our results provide for a quantitative determination of the phase boundary in the plane of temperature and number of massless flavors. Using the experimental value of the coupling at the \(T\) mass scale as the only input parameter, we obtain, e.g., for \(N_f = 3\) massless flavors a critical temperature of \(T_{ct} \approx 148\) MeV in good agreement with lattice simulations.

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I. INTRODUCTION

Strongly interacting matter is believed to have fundamentally different properties at high temperature than at low or zero temperature \([1]\). Whereas the latter can be described in terms of ordinary hadronic states, a hadronic picture at increasing temperature is eventually bound to fail; instead, a description in terms of quarks and gluons is expected to arise naturally owing to asymptotic freedom. In the transition region between these asymptotic descriptions, effective degrees of freedom, such as order parameters for the chiral or deconfining phase transition, may characterize the physical properties in simple terms, i.e., with a simple effective action \([2]\).

If a simple description at or above the phase transition does not exist and the system is strongly interacting in all conceivable sets of variables \([3]\), a formulation in terms of microscopic degrees of freedom has the greatest potential to bridge wide ranges in parameter space from first principles.

In this Letter, we report a nonperturbative study of finite-temperature QCD parameterized in terms of microscopic degrees of freedom: gluons and quarks. We use the functional renormalization group (RG) \([4, 5, 6]\) and concentrate on two problems which are accessible in microscopic language: first, we compute the running of the gauge coupling driven by quantum as well as thermal fluctuations, generalizing previous zero-temperature studies \([7]\). Second, we investigate the induced quark dynamics including its back-reactions on gluodynamics, in order to monitor the status of chiral symmetry at finite temperature. With this strategy, the critical temperature of chiral symmetry restoration can be computed. This Letter is particularly devoted to a presentation of the central physical mechanisms which our study reveals for the dynamics near the phase transition; all technical details can be found in a separate publication \([8]\).

The functional RG yields a flow equation for the effective average action \(\Gamma_k\)

\[
\partial_t \Gamma_k = \frac{1}{2} \text{STr} \partial_t R_k (\Gamma^{(2)}_k + R_k)^{-1}, \quad t = \frac{k}{\Lambda},
\]

where \(\Gamma_k\) interpolates between the bare action \(\Gamma_{k=\Lambda} = S\) and the full quantum effective action \(\Gamma = \Gamma_{k=0}\); \(\Gamma^{(2)}_k\) denotes the second functional derivative with respect to the fluctuating field. The regulator function \(R_k\) specifies the Wilsonian momentum-shell integration, such that the flow of \(\Gamma_k\) is dominated by fluctuations with momenta \(p^2 \approx k^2\).

An approximate solution to the flow equation can reliably describe also nonperturbative physics if the relevant degrees of freedom in the form of RG relevant operators are kept in the ansatz for the effective action. As the crucial ingredient, the choice of this ansatz has to be guided by all available physical information.

II. TRUNCATED RG FLOW FOR THERMAL GLUODYNAMICS

In this work, we truncate the space of possible action functionals to a tractable set of operators which is motivated from various sources and principles. For the principle of gauge invariance, we use the background-field formalism as developed in \([9]\), i.e., we work in the Landau-de Witt background-field gauge and follow the strategy of \([7, 10]\) for an approximate resolution of the gauge constraints \([11]\). Decomposing the gauge field into a background-field part and a fluctuation field, this strategy focusses on the flow in the background-field sector of the action and neglects an independent running of the fluctuation-field sector. In general, the solution of the strongly-coupled gauge sector represents the most delicate part of this study, owing to a lack of sufficient \(a \text{ priori}\) control of nonperturbative truncation schemes. A first and highly nontrivial check of a solution is already given by the stability of its RG flow, since oversimplifying truncations which miss the right degrees of freedom generically exhibit IR instabilities of Landau-pole type.
The IR stability of our solution arises from an important conceptual ingredient: we optimize our truncated flow with an adjustment of the regulator to the spectral flow of $\Gamma^{(2)}$ instead of a naive canonical momentum-shell regularization. For this, we integrate over shells of eigenvalues of $\Gamma^{(2)}$, by inserting $\Gamma^{(2)}$ into the regulator and accounting for the flow of these eigenvalues. More precisely, we use the exponential regulator $Z_k$ of the form $R_k(\Gamma^{(2)}) = \Gamma^{(2)}/[\exp(\Gamma^{(2)}/Z_k k^3) - 1]$, where $Z_k$ denotes the wave function renormalization of the corresponding field (gluons or ghost). In a perturbative language, the optimizing spectral adjustment allows for a resummation of a larger class of diagrams.

The main part of our truncation consists of an infinite set of operators given by powers of the Yang-Mills Lagrangian,

$$\Gamma_{\text{YM}}[A] = \int d^4 x \mathcal{W}_k(\theta), \quad \theta = \frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \quad (2)$$

In the function $\mathcal{W}_k(\theta) = W_1(\theta) + \frac{1}{2} W_2(\theta^2) + \frac{1}{3!} W_3(\theta^3) \ldots$, the coefficients $W_i$ form an infinite set of generalized couplings. This truncation represents a gauge-covariant gradient expansion in the field strength, neglecting higher-derivative terms and more complicated color and Lorentz structures. Hence, the truncation additionally relies entirely on gluonic correlators projected onto their small-momentum limit and onto the particular color and Lorentz structure arising from powers of $F^2$. Since perturbative gluons are certainly not the true degrees of freedom in the IR, an inclusion of infinitely many gluonic operators appears mandatory in order to have a chance to capture the relevant physics in this gluonic language. The covariant gradient expansion does not only facilitate a systematic classification of the gluonic operators, it is also a consistent expansion in the framework of the functional RG.\(^1\)

Furthermore, our truncation includes standard (bare) gauge-fixing and ghost terms, neglecting any nontrivial running in these sectors. We emphasize that we do not expect that this truncation reflects the true behavior in these sectors, but we assume that the non-trivial running in these sectors does not qualitatively modify the running of the background-field sector where we read off the physics.

A well-known problem of gauge-covariant gradient expansions in gluodynamics is the appearance of an IR unstable Nielsen-Olesen mode in the spectrum. At finite temperature $T$, this problem is severe, since such a mode will be strongly populated by thermal fluctuations, typically spoiling perturbative computations. Our flow equation allows us to resolve this problem with the aid of the IR regulator. We remove this mode’s unphysical thermal population by a $T$-dependent regulator. In this way, we obtain a strictly positive spectrum for the thermal fluctuations.

In the present truncation, the flow equation results in a differential equation for the function $W_k$ of the form

$$\partial_t W_k(\theta) = \mathcal{F}[\partial_0 W_k, \partial_0^2 W_k, \partial_0 \partial_0 W_k, \partial_0 \partial_0^2 W_k], \quad (3)$$

where the extensive functional $\mathcal{F}$ depends on derivatives of $W_k$, on the coupling $g$ and the temperature $T$; it is displayed in [5]. We use the nonrenormalization of the product of coupling and background field, $g A$, for a nonperturbative definition of the running coupling in terms of the background wave function renormalization $Z_k \equiv W_1 [13]$.

$$\beta g^2 \equiv \partial_t g^2 = \eta g^2, \quad \eta = -\frac{1}{Z_k} \partial_t Z_k. \quad (4)$$

The flow of $Z_k$, and thus the running of the coupling, is successively driven by all generalized couplings $W_i$. Keeping track of all contributions from the flows of the $W_i$, Eq. (3) boils down to a recursive relation,

$$\partial_t W_i = f_{ij}(g, T) \partial_t W_j, \quad (5)$$

with $f_{ij}(g, T)$ representing the expansion coefficients of the RHS of Eq. (3), which obey $f_{ij} = 0$ for $j > i + 1$. Solving Eq. (5) for $\partial_t Z_k \equiv \partial_t W_1$, we obtain a nonperturbative $\beta g^2$ function in terms of an infinite asymptotic but resumable power series,

$$\beta g^2 = \sum_{m=1}^{\infty} a_m(\frac{g}{m})^m, \quad (6)$$

with temperature-dependent coefficients $a_m$.\(^2\) For explicit representations of the $a_m$ and further details, we refer the reader to [5]. At zero $T$, the $\beta g^2$ function agrees well with perturbation theory for small coupling, reproducing one-loop exactly and two-loop within a few percent error. For larger coupling, the resummed integral representation of Eq. (6) reveals a second zero of the $\beta g^2$ function for finite $g^2$, corresponding to an IR attractive non-Gaussian fixed point $g_c^2 > 0$, which confirms the results of [2]. The resulting zero-temperature flow of the coupling is displayed by the (red) solid “$T=0$” line in Fig. 1. Deviations from perturbation theory become significant at scales below 1 GeV; the IR fixed-point behavior sets in on scales of order $O(100 \text{ MeV})$. As initial condition, we use the measured value of the coupling at the $\tau$ mass scale $\tau_0$, $\alpha_0 = 0.322$, which evolves to the world average of $\alpha_s$ at the $Z$ mass scale. We stress that

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\(^1\) The gluonic gradient expansion as a local expansion can, of course, not be expected to give reliable answers to all questions; for instance, bound-state phenomena such as glue balls are encoded in the nonlocal pole structure of higher-order vertices.

\(^2\) An iteration of the same procedure would result in $\beta$ functions for the other higher-order couplings $W_i$ with $i > 1$. We neglect these, since they do not exert a direct influence on the quark sector discussed below.
as expected. loop perturbative running coupling at zero temperature, affected for scales $k \gg T$ is larger than the extent of the compactified Euclidean time direction. Hence, these modes become effectively 3-dimensional and their limiting behavior is governed by the spatial 3d Yang-Mills theory. This dimensional reduction has been discussed for the running coupling in a perturbative weak-coupling framework in \cite{20}. Our results generalize this to arbitrary couplings. As a nontrivial new result, we observe the existence of a non-Gaussian IR fixed point $g_{3d,*}^2$ also in the reduced 3-dimensional theory. By virtue of a straightforward matching between the 4d and 3d coupling, the observed power law for the 4d coupling is a direct consequence of the strong-coupling IR behavior in the 3d theory, $g^2(k \ll T) \sim g_{3d,*}^2/k/T$. We find a 3d fixed-point value of $\alpha_{3d,*} \equiv g_{3d,*}^2/(4\pi) \approx 2.7$ which demonstrates that the system is strongly coupled despite the naive decrease of the 4d coupling; also the 3d background anomalous dimension is large, approaching $\eta_{3d} \to 1$ near the IR fixed point. This scenario is reminiscent to hot 4d $\phi^4$ theory which approaches a strongly coupled IR limit at high temperatures analogous to the 3d Wilson-Fisher fixed point \cite{21}. The observation of an IR fixed point in the 3d theory again agrees with recent results in the Landau gauge \cite{22}. Note that our flow to the 3d theory is driven by thermal as well as quantum fluctuations. This is different from a purely thermal flow \cite{23, 24} as used in \cite{25}, where the IR limit is four-dimensional, being characterized by a decoupling of fluctuation modes owing to thermal masses. The 3d IR fixed point and the perturbative UV behavior already qualitatively determine the momentum asymptotics of the running coupling. Phenomenologically, the behavior of the coupling in the transition region at mid-momenta is most important, which is quantitatively provided by the full 4d finite-temperature flow equation.

At finite temperature $T$, the UV behavior remains unaffected for scales $k \gg T$ and agrees well with the one-loop perturbative running coupling at zero temperature, as expected.\footnote{Our truncation does not reproduce higher orders in the high-temperature small-coupling expansion which proceeds with odd powers in $g$ beyond one loop.\cite{25}. These odd powers are a result of a resummation which, in the language of the effective action, requires non-local operators, being neglected so far. In any case, we do not expect the underlying quasi-particle picture of these operators to hold near the chiral phase transition, such that their omission should not qualitatively modify our low-temperature results.} In the IR, the running is strongly modified: The coupling increases towards lower scales until it develops a maximum near $k \sim T$. Below, the coupling decreases according to a power law $g^2 \sim k/T$, see Fig. 1. This behavior has a simple explanation: the wavelength of fluctuations with momenta $p^2 < T^2$ is smaller than the extent of the compactified Euclidean time direction. Hence, these modes become effectively 3-dimensional and their limiting behavior is governed by the spatial 3d Yang-Mills theory.\cite{23, 24} as used in \cite{25}. The determination of the critical temperature $T_c$, above which chiral symmetry is restored requires a second crucial ingredient for our truncation: we study the
the gauge dynamics. At these fixed points, the fermionic
self-interactions of order $d_4$ via the gluon-gluon exchange. We emphasize that this is an important
difference to, e.g., the Nambu-Jona-Lasinio (NJL) model, where the $\lambda$'s are independent input parameters.

We consider all linearly-independent four-quark interactions permitted by gauge and chiral symmetry. A priori,
these include color and flavor singlets and octets in the $(S-P)$, $(V-A)$ and $(V+A)$ channels. $U_4(1)$-violating
interactions are neglected, since they may become relevant only inside the $\chi$SB regime or for small $N_f$. We drop
any nontrivial momentum dependencies of the $\lambda$'s and study these couplings in the point-like limit $\lambda(p_1 \ll k)$.
This is a severe approximation, since it inhibits a study of QCD properties in the chirally broken regime; for instance,
mesons manifest themselves as momentum singularities in the $\lambda$'s. Nevertheless, the point-like truncation can be a reasonable approximation in the chirally symmetric regime, as has recently been quantitatively con-
ﬁrmed for the zero-temperature chiral phase transition in many-quark QCD. Our truncation is based on the assumption that quark dynamics both near the finite-$T$ phase boundary as well as near the many-flavor phase boundary is driven by similar mechanisms. Our restric-
tions on the four-quark interactions result in a total number of four linearly-independent $\lambda$ couplings; all others
channels are related to this minimal basis by means of Fierz transformations.

Introducing the dimensionless couplings $\lambda = k^2 \hat{\lambda}$, the $\beta$ functions for the $\lambda$ couplings are of the form

$$\beta_\lambda = 2\lambda - \lambda A \lambda - b \lambda g^2 - c g^4,$$  \hspace{1cm} (8)

where the coefficients $A$, $b$, $c$ are temperature dependent, $A$ being a matrix and $b$ a vector in the space of $\lambda$ couplings (for explicit representations, see [31, 32]). Within this truncation, a simple picture for the chiral dynamics arises: at weak gauge coupling, the RG flow
keeps $\lambda$ fixed and only $g$ is running. The $g$ running generates quark self-interactions of order $\lambda \sim g^4$ via the
last term in Eq. (8) with a negligible back-reaction on the gluonic RG flow. If the gauge coupling in the IR remains smaller than a critical value $g < g_{\text{cr}}$, the $\lambda$ self-interactions remain bounded, approaching fixed points $\lambda_s$ in the IR. Technically, the $\sim g^4$ term is balanced by the first term $\sim 2\lambda$ at these fixed points. The fixed points are the counter-parts of the Gaussian fixed point $\lambda_s^{\text{GauSB}} = 0$ in NJL-like models (at $g^2 = 0$), here being modiﬁed by the gauge dynamics. At these ﬁxed points, the fermionic subsystem remains in the chirally invariant phase which is indeed realized at high temperatures $T > T_{\text{cr}}$.

If the gauge coupling increases beyond the critical coupling $g > g_{\text{cr}}$, the IR fixed points $\lambda_s$ are destabilized and the $\chi$SB order parameter becomes flat and is about to develop a nonzero vacuum expectation value.

Whether or not chiral symmetry is preserved by the ground state therefore depends on the running QCD coupling $g$ relative to the critical coupling $g_{\text{cr}}$ which is required to trigger $\chi$SB. For instance, at zero temperature, the SU(3) critical coupling for the quark system is $\alpha_{\text{cr}} = g^2_{\text{cr}}/(4\pi) \sim 0.8$ in our RG scheme [32], being only weakly dependent on the number of colors $N_f$. Since the IR fixed point for the gauge coupling is much larger $\alpha_g > \alpha_{\text{cr}}$ for not too many massless flavors, the QCD vacuum is characterized by $\chi$SB. At ﬁnite temperature, the running of the gauge coupling is considerably modiﬁed in the IR. Moreover, the critical coupling is $T$ dependent, $g_{\text{cr}} = g_{\text{cr}}(T/k)$. This can be understood from the fact that all quark modes acquire thermal masses and, thus, stronger interactions are required to excite critical quark dynamics. This thermal decoupling is visible in the coef-

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Running QCD coupling $\alpha_s(k, T)$ for $N_f = 3$ massless quark flavors and $N_f = 3$ colors and the critical value of the running coupling $\alpha_s(k, T)$ as a function of $k$ for $T = 130$ MeV (upper panel) and $T = 220$ MeV (lower panel). The existence of the $(\alpha_s, \alpha_{\text{cr}})$ intersection point (marked by a circle) in the former indicates that the $\chi$SB quark dynamics can become critical.}
\end{figure}
coefficients $A$, $b$, and $c$ in Eq. (8), all of which vanish in the limit $T/k \to \infty$.

In Fig. 2 we show the running coupling $\alpha_s$ and its critical value $\alpha_{ct}$ for $T = 130\text{MeV}$ and $T = 220\text{MeV}$ as a function of the regulator scale $k$. The intersection point $k_{ct}$ between both marks the scale where the quark dynamics become critical. Below the scale $k_{ct}$, the system runs quickly into the $\chi$SB regime. We estimate the critical temperature $T_{ct}$ as the lowest temperature for which no intersection point between $\alpha_s$ and $\alpha_{ct}$ occurs.\footnote{Strictly speaking, this is a sufficient but not a necessary criterion for chiral-symmetry restoration. In this sense, our estimate for $T_{ct}$ is an upper bound for the true $T_{ct}$. Small corrections to this estimate could arise, if the quark dynamics becomes uncritical again by a strong decrease of the gauge coupling towards the IR.}

Compared to [8], we have further resolved the finite-$T$ Lorentz structure of the four-fermion couplings [29], resulting in a slightly improved estimate for $T_{ct}$: we find $T_{ct} \approx 172^{+34}_{-30}\text{MeV}$ for $N_f = 2$ and $T_{ct} \approx 148^{+31}_{-34}\text{MeV}$ for $N_f = 3$ massless quark flavors in good agreement with lattice simulations [8]. The errors arise from the experimental uncertainties on $\alpha_s$ [10]. Dimensionless ratios of observables are less contaminated by this uncertainty of $\alpha_s$. For instance, the relative difference for $T_{ct}$ for $N_f=2$ and $3$ flavors is $\frac{T_{ct}^{N_f=2}-T_{ct}^{N_f=3}}{(T_{ct}^{N_f=2}+T_{ct}^{N_f=3})/2} = 0.150 \ldots 0.165$ in reasonable agreement with the lattice value\footnote{The large uncertainty on the lattice value arises from the fact that the statistical errors on the $N_f = 2$ and $N_f = 3$ results for $T_{ct}$ are uncorrelated.} of $\sim 0.121 \pm 0.069$.

Furthermore, we compute the critical temperature for the case of many massless quark flavors $N_f$, see Fig. 3. We observe an almost linear decrease of the critical temperature for increasing $N_f$ with a slope of $\Delta T_{ct} = T(N_f) - T(N_f + 1) \approx 24\text{MeV}$ for small $N_f$. In addition, we find a critical number of quark flavors, $N_{ct}^{SB} = 12$, above which no chiral phase transition occurs. This result for $N_{ct}^{SB}$ agrees with other studies based on the 2-loop $\beta$ function [29]; however, the precise value of $N_{ct}^{SB}$ is exceptionally sensitive to the 3-loop coefficient which can bring $N_{ct}^{SB}$ down to $N_{ct}^{SB} \approx 10^{+3}_{-0.7}$ [27]. Since we do not consider our truncation to be sufficiently accurate for a precise estimate of this coefficient, our study does not contribute to a reduction of the current error on $N_{ct}^{SB}$. Instead, we would like to emphasize that the flattening of the phase boundary near $N_{ct}^{SB}$ is a generic prediction of the IR fixed-point scenario: here, the symmetry status of the system is governed by the fixed-point regime where dimensionful scales such as $\Lambda_{QCD}$ lose their importance [8]. In any case, since $N_{ct}^{SB}$ is smaller than $N_{ct}^{SB} = \frac{1}{3}N_c = 16.5$, our study provides further evidence for the existence of a regime where QCD is chiral symmetric but is still asymptotically free.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig3.png}
\caption{Chiral-phase-transition temperature $T_{ct}$ versus the number of massless quark flavors $N_f$. In the dashed-line region, we expect $U_A(1)$-violating operators to become quantitatively important. The flattening at $N_f \gtrsim 10$ is a consequence of the IR fixed-point structure [8].}
\end{figure}

\section{Conclusion}

In summary, we have determined the $\chi$SB phase boundary in QCD in the plane of temperature and flavor number. Our quantitative results are in accord with lattice simulations for $N_f = 2, 3$. For larger $N_f$, we observe a linear decrease of $T_{ct}$, leveling off near $N_{ct}^{SB}$ owing to the IR fixed-point structure of QCD. Our results are based on a consistent operator expansion of the QCD effective action that can systematically be generalized to higher orders.

The qualitative validity and the quantitative convergence of this expansion are naturally difficult to analyze in this strongly-coupled gauge system, particularly for the gluonic sector. The fact that our truncation results in a stable RG flow at strong interactions is already a highly non-trivial check that any ansatz which misses the true degrees of freedom generically fails. A more quantitative evaluation of the validity of our expansion will require the inclusion of higher-order operators in the covariant gradient expansion as well as higher-order ghost terms. An inclusion of operators that distinguish between electric and magnetic sectors at finite $T$, e.g., $(u_{\mu}F_{\mu\nu})^2$ with the heat-bath four-velocity $u_{\mu}$, should facilitate to distinguish between differing coupling strengths in the two sectors, as done in [27] using an ansatz inspired by hard thermal loop computations.

We observe an improved control over the truncation in the quark sector at least for the chirally symmetric phase, which suffices to trace out the phase boundary. Quantitatively, this has been confirmed by a stability analysis of universal quantities such as $N_{ct}^{SB}$ under a variation of the regulator in [27] which gives strong support to the point-like truncation of the quark self-interactions. Qualitatively, the reliability of the quark truncation can also
be understood by the fact that the feed-back of higher-order operators, such as \( \bar{\psi}\psi \), is generally suppressed by the one-loop structure of the flow equation.

Future extensions should include mesonic operators which can be treated by RG rebosonization techniques [32]. This would not only provide access to the broken phase and mesonic properties, but also permit a study of the order of the phase transition. For further phenomenology, the present quantitative results that rely on only one physical input parameter can serve as a promising starting point.

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