Extensions of Erdős-Gallai Theorem and Luo’s Theorem with Applications

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Abstract

The famous Erdős-Gallai Theorem states that every graph with \( n \) vertices and \( m \) edges contains a path of length at least \( \frac{2m}{n} \). In this note, we first establish a simple but novel extension of Erdős-Gallai Theorem by proving that every graph \( G \) contains a path of length at least \( \frac{(s+1)N_{s+1}(G)}{N_s(G)} + s - 1 \), where \( N_j(G) \) denotes the number of \( j \)-cliques in \( G \) for \( 1 \leq j \leq \omega(G) \). We also construct a family of graphs which shows our extension improves the estimate given by Erdős-Gallai Theorem. Among applications, we show, for example, that the main results of [20], which are on the maximum possible number of \( s \)-cliques in an \( n \)-vertex graph without \( P_k \) (and without cycles of length at least \( c \)) can be easily deduced from this extension. Indeed, to prove these results, Luo [20] generalized a classical theorem of Kopylov and established a tight upper bound on the number of \( s \)-cliques in an \( n \)-vertex 2-connected graph with circumference less than \( c \). We prove a similar result for an \( n \)-vertex 2-connected graph with circumference less than \( c \) and large minimum degree. We conclude this paper with an application of our results to a problem from spectral extremal graph theory on consecutive lengths of cycles in graphs.

Keywords: Clique; Cycle; Generalized Turán number; Erdős-Gallai Theorem; Kopylov’s theorem

1 Introduction

Let \( \mathcal{H} \) be a family of graphs. The Turán number \( ex(n, \mathcal{H}) \) is the largest possible number of edges in an \( n \)-vertex graph \( G \) which contains no member of \( \mathcal{H} \) as a subgraph. If \( \mathcal{H} = \{ H \} \), then we write \( ex(n, H) \) for \( ex(n, \mathcal{H}) \).

Erdős and Gallai [9] proved the following celebrated theorems on Turán numbers of cycles and paths.

**Theorem 1.1** (Erdős and Gallai [9]). \( ex(n, C_{\geq l}) = \frac{(l-1)(n-1)}{2}, \) where \( C_{\geq l} \) is the set of all cycles of length at least \( l \), where \( l \geq 3 \).

**Theorem 1.2** (Erdős and Gallai [9]). \( ex(n, P_l) = \frac{(l-2)n}{2}, \) where \( l \geq 2 \).

For the tightness of Theorem 1.1 one can check the graph consisting of \( \frac{n-1}{l-2} \) cliques of size \( l-1 \) with a common vertex, where \( n-1 \) is divisible by \( l-2 \). The tightness of Theorem 1.2 is shown by the graph with \( \frac{n}{l-1} \) disjoint \( K_{l-1} \), where \( n \) is divisible by \( l-1 \). For more

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improvements and extensions of Erdős-Gallai’s theorems, see [1, 22, 19, 13, 23, 11, 5, 6].

We refer the reader to an excellent survey on related topics by Füredi and Simonovits [14].

Let $T$ be a graph and $\mathcal{H}$ be a family of graphs. The generalized Turán number $e(n, T, \mathcal{H})$ is the maximum possible number of copies of $T$ in an $n$-vertex graph which is $\mathcal{H}$-free for each $H \in \mathcal{H}$. When $\mathcal{H} = \{H\}$, we write $e(n, T, H)$ instead of $e(n, T, \{H\})$. If $T = K_2$ then $e(n, K_2, H) = e(n, H)$ is the classical Turán number of $H$. The generalized Turán number has received a lot of attention recently. There are several notable and nice papers concerning the generalized Turán number $e(n, T, H)$ (see [8, 3, 15, 14, 1, 20, 10]). Erdős [8] first determined $e(n, K_t, K_r)$ for all $t < r$. Bollobás and Győri [3] determined the order of magnitude of $e(n, C_5, C_5)$. Their estimate was improved by Alon and Shikhelman [11] and recently by Ergemlidze et al. [10]. Alon and Shikhelman obtained a number of results on $e(n, T, H)$ for different $T$ and $H$ and posed several open problems in [11].

In the direction of the classical Erdős-Gallai Theorems (Theorems 1.1 and 1.2), Luo [20] recently determined the exact values of $e(n, K_s, C_{2l})$ and $e(n, K_s, P_l)$.

**Theorem 1.3** (Luo [20]). $e(n, K_s, C_{2l}) = \frac{n-1}{l-2} \binom{l-1}{s}$, where $l \geq 3$ and $s \geq 2$.

**Theorem 1.4** (Luo [20]). $e(n, K_s, P_l) = \frac{n}{l-1} \binom{l-1}{s}$, where $l \geq 2$ and $s \geq 2$.

Moreover, Luo’s result turned out to be useful for investigating Turán-type problems in hypergraphs. For example, Győri et al. [16] applied Theorem 1.4 to study the maximum number of hyperedges in an $r$-uniform connected $n$-vertex hypergraph without a Berge path of length $k$. (We will mention another one in the last section.)

For a graph $G$, let $\omega(G)$ be the clique number of $G$, i.e., the size of a largest clique in $G$. For $1 \leq j \leq \omega(G)$, we use $N_j(G)$ to denote the number of copies of $K_j$ in $G$. Recall Theorem 1.2 can be rephrased as each graph contains a path of length at least $\frac{2N_2}{N_1}$. Inspired by Luo’s work in [20], we first prove the following extension of Theorem 1.2.

**Theorem 1.5.** Let $G$ be a graph. For each positive integer $s$ with $1 \leq s \leq \omega(G)$, there is a path of length at least $\frac{(s+1)N_{s+1}(G)}{N_s(G)} + s - 1$ in $G$.

We are able to find a family of graphs which shows our extension improves the estimate given by Theorem 1.2. Let $G$ be an $n$-vertex graph which consists of a $K_{n-2}$ and two pendant edges sharing an endpoint from the $K_{n-2}$. Theorem 1.2 implies that $G$ contains a path of length at least $\frac{2N_2(G)}{N_1(G)} = n - 5 + \frac{10}{n}$; while Theorem 1.5 tells us that $G$ contains a path with length at least $\frac{(n-2)N_{n-2}(G)}{N_{n-3}(G)} + n - 4 = n - 3$, where we choose $s = n - 3$. So it is easy to see our extension gives a better estimate for this family of graphs.

Moreover, as applications of Theorem 1.5 we will show Theorems 1.3 and 1.4 (which are two main results of the paper [20]) can be easily derived from it. Finally, we will slightly improve a result in [21] on consecutive lengths of cycles in graphs.

In fact, to prove Theorems 1.3 and 1.4 Luo [20] extended some classical theorems due to Kopylov [18]. Let $H(n, k, c)$ be a graph obtained from $K_{c-k}$ by connecting each vertex of a set of $n-(c-k)$ isolated vertices to the same $k$ vertices choosing from $K_{c-k}$. Let $f_s(n, k, c)$ be the number of $K_s$ in $H(n, k, c)$. Namely, $f_s(n, k, c) = \binom{c-k}{s} + \binom{k}{s-1} (n-(c-k)).$ When $s = 2$, it equals the number of edges in $H(n, k, c)$. Improving Theorem 1.1 Kopylov [18] proved the following.

**Theorem 1.6** (Kopylov [18]). Let $n \geq c \geq 5$ and $G$ be a 2-connected graph on $n$ vertices with circumference less than $c$. Then $N_2(G) \leq \max\{f_2(n, 2, c), f_2(n, \left\lceil \frac{c-1}{2} \right\rceil, c)\}$.
Kopylov’s theorem was reproved by Fan, Lv and Wang in [12], who indeed proved a slightly stronger result with the aid of another result of Woodall [23]. In the same paper [23], Woodall posed a general conjecture which generalizes a previous result on nonhamiltonian graphs due to Erdős [7].

**Conjecture 1** (Woodall [23]). Let \( n \geq c \geq 5 \). If \( G \) is a 2-connected graph on \( n \) vertices with circumference less than \( c \) and minimum degree \( \delta(G) \geq k \), then \( N_2(G) \leq \max\{f_2(n, k, c), f_2(n, \lceil \frac{c-1}{2} \rceil, c)\} \).

One can easily find that Kopylov’s theorem confirmed Woodall’s conjecture for \( k = 2 \).

Generalizing Kopylov’s result, Luo [20] proved the following theorem.

**Theorem 1.7** (Luo [20]). Let \( n \geq c \geq 5 \) and \( s \geq 2 \). If \( G \) is a 2-connected graph on \( n \) vertices with circumference less than \( c \), then \( N_s(G) \leq \max\{f_s(n, 2, c), f_s(n, \lceil \frac{c-1}{2} \rceil, c)\} \).

Our second result is an extension of Theorem 1.7 which is in the spirit of Kopylov’s remark (see the footnote).

**Theorem 1.8.** Let \( n \geq c \geq 5 \) and \( s \geq 2 \). If \( G \) is a 2-connected graph on \( n \) vertices with circumference less than \( c \) and minimum degree \( \delta(G) \geq k \), then \( N_s(G) \leq \max\{f_s(n, k, c), f_s(n, \lceil \frac{c-1}{2} \rceil, c)\} \).

The paper is organized as follows. In Section 2, we will present the proof of Theorem 1.5 and its applications. We will sketch the proof of Theorem 1.8 in Section 3. In the last section, we will mention an application of Theorem 1.5 to spectral extremal graph theory.

## 2 Proof of Theorem 1.5 and its applications

We will need the following simple lemma to prove Theorem 1.5.

**Lemma 1.** Let \( z_i \) be a real number and let \( \left\{ \frac{x_i}{y_i} : 1 \leq i \leq s \right\} \) be a sequence of numbers, where \( x_i \geq 0 \) and \( y_i > 0 \). If \( z_i \geq \frac{x_i}{y_i} \) for every \( 1 \leq i \leq s \), then \( \max\{z_i : 1 \leq i \leq s\} \geq \sum_{i=1}^{s} \frac{x_i}{y_i} \).

**Proof.** We assume

\[
\frac{x_1}{y_1} = \max \left\{ \frac{x_i}{y_i} : 1 \leq i \leq s \right\}.
\]

Equivalently, \( x_1 y_i \geq x_i y_1 \) for \( 1 \leq i \leq s \). We add all inequalities up and get

\[
x_1 \sum_{i=1}^{s} y_i \geq y_1 \sum_{i=1}^{s} x_i, \text{ i.e., } \frac{x_1}{y_1} \geq \frac{\sum_{i=1}^{s} x_i}{\sum_{i=1}^{s} y_i}.
\]

Thus we have

\[
\max\{z_i : 1 \leq i \leq s\} \geq \frac{x_1}{y_1} \geq \frac{\sum_{i=1}^{s} x_i}{\sum_{i=1}^{s} y_i}.
\]

\[\Box\]
Proof of Theorem 1.5. We prove the theorem by induction on $s$. Theorem 1.2 gives the base case, where $s = 1$. Suppose the theorem is true when $s = l - 1$, where $s \leq \omega(G) - 1$. We have to show it is true when $s = l$. For each vertex $x \in V(G)$, we use $G_x$ to denote the subgraph induced by $N_G(x)$. By induction hypothesis, for each vertex $x \in V(G)$ with $N_{l-1}(G_x) \neq 0$, $G_x$ contains a path $P_x$ of length at least $\frac{(l-1)N_{l-1}(G_x)}{N_l(G_x)} + l - 2$. For $i = l - 1, l$, let $V_i := \{ x \in V(G) : N_i(G_x) \neq 0 \}$. Note that $\sum_{x \in V_{l-1}} N_{l-1}(G_x) = l N_l(G)$ and $\sum_{x \in V_l} N_i(G_x) = (l + 1) N_{l+1}(G)$. It is easy to observe $V_i \subseteq V_{i-1}$. By definition, $N_i(G_y) = 0$ for each $y \in V_{i-1} \setminus V_i$. Thus $\sum_{x \in V_i} N_i(G_x) = \sum_{x \in V_{l-1}} N_i(G_x)$. By Lemma 1 there exists a vertex $v \in V(G)$ such that the path $P_v$ in $G_v$ has length at least

$$\sum_{x \in V_{l-1}} N_l(G_x) + l - 2 = \frac{l \sum_{x \in V_{l-1}} N_l(G_x)}{\sum_{x \in V_{l-1}} N_{l-1}(G_x)} + l - 2 = \frac{(l+1)N_{l+1}(G)}{N_l(G)} + l - 2.$$ 

Since the path $P_v$ is in $G_v$, there is a path of length at least $\frac{(l+1)N_{l+1}(G)}{N_l(G)} + l - 1$ in $G$. The theorem holds for $s = l$ and we proved the theorem. \hfill \Box

We next mention a few applications of Theorem 1.5.

A short proof of Theorem 1.4. Suppose that $G$ is $P_l$-free. Let $P$ be a longest path in $G$. Then the length of $P$ is at most $l - 2$. By Theorem 1.5 we have $l - 2 \geq \frac{s N_s(G)}{N_{s-1}(G)} + s - 2$ whenever $N_s(G) \neq 0$. This implies that

$$N_s(G) \leq \frac{l - s}{s} N_{s-1}(G) \leq \frac{(l - s)(l - s + 1) \cdots (l - 3)}{s(s - 1) \cdots 3} N_2(G).$$

By Theorem 1.2, we have $N_2(G) \leq \frac{(l-2)n}{2}$. Thus we get $N_s(G) \leq \frac{n}{l-1} \frac{(l-1)}{s}$. It is obvious that Theorem 1.2 and Theorem 1.4 have the same extremal graph. This completes the proof. \hfill \Box

For two graphs $G$ and $H$, we write $G \vee H$ for their join which satisfies $V(G \vee H) = V(G) \cup V(H)$ and $E(G \vee H) = E(G) \cup E(H) \cup \{ xy : x \in V(G), y \in V(H) \}$. We point out that the proof of Theorem 1.5 implicitly implies the following lemma.

Lemma 2. Let $s \geq 2$ be an integer. Let $G$ be a graph with $N_s(G) \neq 0$. Then $G$ contains a subgraph $P_l \vee K_1$, where $l$ is at least $\frac{(s+1)N_{s+1}(G)}{N_s(G)} + s - 1$. In particular, $G$ contains a cycle of length at least $\frac{(s+1)N_{s+1}(G)}{N_s(G)} + s$.

Another main result in [20] can be deduced from Lemma 2.

A short proof of Theorem 1.3. The proof is similar to the proof of Theorem 1.4. Let $c$ be the circumference of $G$. By Lemma 2 and the condition in Theorem 1.3 we have $l - 1 \geq c \geq \frac{s N_s(G)}{N_{s-1}(G)} + s - 1$. This implies that

$$N_s(G) \leq \frac{l - s}{s} N_{s-1}(G) \leq \frac{(l - s)(l - s + 1) \cdots (l - 3)}{s(s - 1) \cdots 3} N_2(G).$$

By Theorem 1.2, we have $N_2(G) \leq \frac{(n-1)(l-1)}{2}$. Thus we get $N_s(G) \leq \frac{n-1}{l-2} \frac{(l-1)}{s}$. We note that the extremal graph for Theorem 1.1 is also the one for Theorem 1.3. This completes the proof. \hfill \Box

3 Proof of Theorem 1.8

We will need the following lemma, whose proof is omitted in [18]. We would like to mention that this generalizes Bondy’s lemma on longest cycles, whose proof is implicit in the proof of Lemma 1 in [4].
Lemma 3 (Kopylov [18]). Let $G$ be a 2-connected $n$-vertex graph with a path $P$ of $m$ edges with endpoints $x$ and $y$. For $v \in V(G)$, let $d_P(v) = |N(v) \cap V(P)|$. Then $G$ contains a cycle of length at least $\min\{m + 1, d_P(x) + d_P(y)\}$.

We also need a definition from Kopylov [18].

Definition 1 (α-disintegration of a graph, Kopylov [18]). Let $G$ be a graph and $\alpha$ be a natural number. Delete all vertices of degree $\leq \alpha$ from $G$; for the resulting graph $G'$, we again delete all vertices of degree $\leq \alpha$ from $G'$; until we finally get a graph, denoted by $H(G; \alpha)$, such that all vertices are of degree $> \alpha$.

Our proof is very similar to Kopylov’s proof [18] of Theorem 1.6 and the proof of Theorem 1.7 in [20]. We only give the sketch and omit the details. We split the proof into five steps.

A sketch of the proof of Theorem 1.8. Let $G$ be a counterexample. We assume $G$ is edge maximal, i.e., adding each nonedge creates a cycle of length at least $c$. Thus each pair of nonadjacent vertices is connected by a path of length at least $c - 1$. Let $t = \lceil \frac{c - 1}{2} \rceil$ and $H = H(G; t)$.

Claim 1 ([20]). $H$ is not empty.

Claim 2 ([18]). $H$ is a clique.

The main differences come from Claims 3 and 4, whose proofs need the minimum degree condition and a new function.

Claim 3. Let $r = |V(H)|$. Then $k \leq c - r \leq t$.

Proof. As $H = H(G; t)$ is a clique, we get $r \geq t + 2$. We claim $r \leq c - k$, where $\delta(G) \geq k$. Suppose $r \geq c - k + 1$. If $x \in V(G) \setminus V(H)$, then $x$ is not adjacent to at least one vertex in $H$. Otherwise, $x \in H$. We pick $x \in V(G) \setminus V(H)$ and $y \in V(H)$ satisfying the following two conditions: (a) $x$ and $y$ are not adjacent; and (b) A longest path in $G$ from $x$ to $y$ contains the largest number of edges among such nonadjacent pairs. Let $P$ be a longest path in $G$ from $x$ to $y$. Clearly, $|V(P)| \geq c$ as $G$ is edge maximal. We have $N_G(x) \subseteq V(P)$. Otherwise, let $z \in N_G(x)$ and $z \notin V(P)$. If $z$ and $y$ are not adjacent, then we get a longer path from $z$ to $y$, which is a contradiction to the selection of $x$ and $y$. If $z$ and $y$ are adjacent, then we get a cycle of length at least $c + 1$, which is also a contradiction to the assumption of $G$. Similarly, we can show $N_H(y) \subseteq V(P)$. Therefore, by Lemma 3 we get a cycle with length at least $\min\{c, d_P(x) + d_P(y)\} \geq \min\{c, k + c - k\} = c$, which is a contradiction. Thus we get $r \leq c - k$. Recall $t + 2 \leq r \leq c - k$. We get $k \leq c - r \leq c - t - 2 \leq t$.

Claim 4. Let $H' = H(G; c - r)$. Then $H \neq H'$.

Proof. If $H = H'$, then we have

$$N_s(G) \leq (n - r) \left( \frac{c - r}{s - 1} \right) + \binom{r}{s} = f_s(n, c - r, c) \leq \max\{f_s(n, k, c), f_s(n, t, c)\},$$

as the function $f_s(n, x, c)$ is convex for $x \in [k, t]$. This is a contradiction and completes the proof.

Claim 5. $G$ contains a cycle of length at least $c$.

The proof of the claim above is the same as Kopylov’s proof and we skip it.
4 Concluding remarks

For a graph $G$, let $\mu(G)$ be the largest eigenvalue of the adjacency matrix. Nikiforov [21] proved the following: If $G$ is a graph of sufficiently large order $n$ and the spectral radius $\mu(G) > \sqrt{\left\lceil \frac{n^2}{4} \right\rceil}$, then $G$ contains a cycle of length $t$ for every $t \leq n/320$.

Notice that Lemma 2 implies the following fact: A graph $G$ contains all cycles of length $t \in [3, l]$, where $l = \frac{3N_3(G)}{N_2(G)} + 2$. This fact can be used to show $G$ contains a cycle of length $t$ for every $t \leq n/160$, which slightly improves the result in [21].

We just give the sketch of the proof. Compared with the original proof in [21], the improvement comes from the fact mentioned above. In [21], it is shown that for $n$ sufficiently large, there exists an induced subgraph $H \subset G$ with $|H| > n/2$, satisfying one of the following conditions:

(i) $\mu(H) > (1/2 + 1/80)|H|$;

(ii) $\mu(H) > |H|/2$ and $\delta(H) > 2|H|/5$.

For case (i), it is shown in [21] that $N_3(H) \geq \frac{1}{960}|H|^3$. In this case, if $e(H) = N_3(H) > \frac{|H|^2}{4}$, then a theorem of Bollobás [2] implies there are cycles of lengths from 3 to $\frac{|H|}{2}$ in $H$. Thus there are cycles of length $t$ for each $3 \leq t \leq \frac{n}{2}$. So we assume $e(H) \leq \frac{|H|^2}{4}$. By the fact mentioned above, $H$ contains all cycles of length $l \in [3, \frac{\sqrt{\frac{3}{4}}|H|^3}}{4}]$. Since $\frac{\sqrt{\frac{3}{4}}|H|^3}{4} \geq \frac{1}{160}n$, we proved the result for the case (i). The proof of the result for case (ii) follows from Nikiforov’s proof.

Similar to Theorem 1.8, we have the following result and omit the details of the proof.

Theorem 4.1. If $G$ is an $n$-vertex connected graph containing no $P_l$ and having minimum degree $\delta(G) \geq k$, where $n \geq l \geq 4$, then $N_s(G) \leq \max\{f_s(n, k, l-1), f_s(n, \left\lceil \frac{l}{2} \right\rceil - 1, l-1}\}$.

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