Integrating Singular Functions on the Sphere

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Abstract

We obtain rigorous results concerning the evaluation of integrals on the two sphere using complex methods. It is shown that for regular as well as singular functions which admit poles, the integral can be reduced to the calculation of residues through a limiting procedure.

I. INTRODUCTION

Standard textbooks on mathematical physics state that integrals of regular functions on the sphere are easy to compute if one uses the spherical harmonics decomposition. These books, however, do not explain what to do if the integrands have singularities since in that case the functions do not admit an expansion in spherical harmonics.

In this paper we formulate a useful technique to evaluate integrals on the two sphere of integrands that might possess singular (pole-type) behavior. The method is based on the use of Stokes Theorem to convert the two-dimensional integrals on the complex plane (obtained by the stereographic projection of the sphere), into line integrals around singular points and then to evaluate the latter by a generalization

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of the Cauchy Residue Theorem. Our approach is inspired in the theory of Residue Currents developed in [1].

Our main result (stated in Theorem [IV.3]) is that principal values of integrals in the sphere of (not necessarily integrable) \( C^m \) functions \( g \) with poles (cf. Definition [IV.1]) can be explicitly evaluated as a sum of residue limits of a \( C^m \) solution with poles of the differential equation \( \frac{\partial}{\partial \bar{z}} f = \frac{g}{(1+\bar{z}z)^2} \). We also show how to construct an explicit \( C^m \) solution with poles to this equation (cf. Corollary [IV.2]). These results are useful in a wide range of different physical theories. One can apply these techniques to obtain explicit evaluations of Feynman propagators and Feynman graphs, to obtain solutions of differentials equation on the sphere, etc.

This paper is organized as follows: In section II we give some mathematical preliminaries needed for the present work. In section III we present the main result for functions that are \( C^m \) on \( \mathbb{C} \) except for a finite number of singularities, and in the last section we apply these results to integrals on the sphere.

II. PRELIMINARIES

For later reference we give some standard formulae, and review some results which contain the basic ideas that we shall use throughout this paper (for more details see [2]).

Let \( D \) be a closed disc in the complex plane \( \mathbb{C} \), bounded by the circle \( \gamma \). The Cauchy formula

\[
    h(z_0) = \frac{1}{2\pi i} \oint_{\gamma} \frac{h(z)dz}{z-z_0},
\]

(1)
gives the value of \( h(z_0) \) for any \( z_0 \) in the interior of \( D \) as an integral along \( \gamma \) with counterclockwise orientation when \( h \) is a holomorphic (complex differentiable) function on some open neighborhood of \( D \). But if \( h \) is not holomorphic but merely smooth (i.e. its real and imaginary parts are continuously differentiable in the real sense) there is a similar formula giving the value of \( h(z_0) \), which shall be given later in this section.

Let us write \( z = x + iy \), and let

\[
    h(z, \bar{z}) = h_1(x, y) + ih_2(x, y),
\]
where \( h_1 \) and \( h_2 \) are the real and imaginary parts of \( h \) respectively. We say that \( h \) is \( C^\infty (\mathbb{C}^n) \) if \( h_1 \) and \( h_2 \) are \( C^\infty (\mathbb{C}^n) \) in the usual sense for functions of two real variables \( x \) and \( y \). In other words, all partial derivatives of any order (all those up to the \( n \)-th order) of \( h_1 \) and \( h_2 \), exist and are continuous. We write \( h \in C^n(D) \) to mean that \( h \) is \( C^n \) on some open set containing \( D \).

For such functions we define

\[
\frac{\partial h}{\partial z} = \frac{1}{2} \left( \frac{\partial h}{\partial x} - i \frac{\partial h}{\partial y} \right) \quad \text{and} \quad \frac{\partial h}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial h}{\partial x} + i \frac{\partial h}{\partial y} \right).
\]

Thus, the **Cauchy-Riemann equations** can be formulated by saying that \( h \) is holomorphic if and only if

\[
\frac{\partial h}{\partial \bar{z}} = 0.
\]

Since \( z = x+iy \) and \( \bar{z} = x-iy \), a \( C^\infty \) or \( C^n \) complex function can be described as a function in the complex variables \((z, \bar{z})\), for which the following complex formulation of the Green-Stokes formula holds.

**Green-Stokes Formula.** Given a region \( B \subset \mathbb{C} \) bounded by a finite number of curves, oriented so that the region lies to the left of each curve, the Stokes-Green Formula in the variable \( z \) is

\[
\int_{\gamma} f \, dz + g \, d\bar{z} = \int_B \left( \frac{\partial g}{\partial z} - \frac{\partial f}{\partial \bar{z}} \right) \, dz \wedge d\bar{z},
\]

where \( \gamma \) is the boundary of \( B \), \( f \) and \( g \) have continuous first partial derivatives,

\[
dz = dx + idy \quad d\bar{z} = dx - idy
\]

and

\[
dz \wedge d\bar{z} = -2idx \wedge dy.
\]

Using (3), one can formulate the following version of the Cauchy Theorem for \( C^1(D) \) functions (see [2] and [3] for a proof).

**Cauchy Theorem for \( C^1(D) \) functions.** Let \( h \in C^1(D) \) and \( z_0 \) be a point in the interior of \( D \). Further, let \( \gamma \) be the circle around \( D \) with counterclockwise orientation, then
\[ h(z_0, \bar{z}_0) = \frac{1}{2\pi i} \oint_{\gamma} \frac{h}{z - z_0} \, dz + \frac{1}{2\pi i} \int_D \frac{\partial h}{\partial \bar{z}} \frac{dz \wedge d\bar{z}}{z - z_0}. \quad (5) \]

It is clear that if \( h \) is holomorphic, the double integral disappears obtaining the standard Cauchy formula.

In the proof of this Theorem the basic ingredient is the evaluation of the improper integral

\[
\int_D \frac{\partial h}{\partial \bar{z}} \frac{dz \wedge d\bar{z}}{z - z_0} = \lim_{\varepsilon \to 0} \int_{D^\varepsilon} \frac{\partial h}{\partial \bar{z}} \frac{dz \wedge d\bar{z}}{z - z_0},
\]

where \( D^\varepsilon \) is the region obtained from \( D \) by deleting a small disc of radius \( \varepsilon \) centered at the point \( z_0 \). The boundary of this region, \( \partial D^\varepsilon \), consists of two curves, \( \gamma \) and \(-\gamma_\varepsilon\), where the first one has counterclockwise orientation and the second one has clockwise orientation.

**Remark II.1** Note that since \( h \) is \( C^1 \) the l.h.s. of (6) exists.

Therefore, using the Stokes formula (3), we obtain

\[
\int_{D^\varepsilon} \frac{\partial h}{\partial \bar{z}} \frac{dz \wedge d\bar{z}}{z - z_0} = \oint_{\gamma_\varepsilon} h(z, \bar{z}) \frac{dz}{z - z_0} - \oint_{\gamma} h(z, \bar{z}) \frac{dz}{z - z_0}.
\]

The integral along \( \gamma_\varepsilon \) is evaluated replacing \( h(z, \bar{z}) \) by its 0-order Taylor expansion in the variables \((z, \bar{z})\), which exists since \( h \) is \( C^1 \), i.e.

\[
\oint_{\gamma_\varepsilon} h(z, \bar{z}) \frac{dz}{z - z_0} = \oint_{\gamma_\varepsilon} h(z_0, \bar{z}_0) \frac{dz}{z - z_0} + \oint_{\gamma_\varepsilon} \sum_{m+l=1} \Gamma_{m+l}(z, \bar{z}) (\bar{z} - \bar{z}_0)^m (z - z_0)^l,
\]

for some continuous functions \( \Gamma_{10} \) and \( \Gamma_{01} \). Taking the limit of (6) when \( \varepsilon \to 0 \), the second integral of the r.h.s. of (6) is proved to be zero, and the first one gives us the l.h.s. of (5).

Considering the special case when the function \( h \) vanishes on the boundary of the disc, the integral along the circle \( \gamma \) is equal to 0, thus the formula (6) becomes

\[ h(z_0, \bar{z}_0) = \frac{1}{2\pi i} \int_D \frac{\partial h}{\partial \bar{z}} \frac{dz \wedge d\bar{z}}{z - z_0}. \]

This allows us to recover the values of a function with compact support from its derivative \( \frac{\partial h}{\partial \bar{z}} \). Conversely, one gets the following result (c.f. [3]).
Theorem II.1 Let $h \in C^1(D)$ be a $C^1$ function on the closed disc $D$, with compact support contained in the interior of $D$. Then the function

$$f(z, \bar{z}) = \frac{1}{2\pi i} \int_D \frac{h(\zeta, \bar{\zeta})}{\zeta - z} d\zeta \wedge d\bar{\zeta}$$

is defined and is $C^1$ on $D$, and satisfies

$$\frac{\partial f}{\partial \bar{z}} = h(z, \bar{z}) \quad \text{for} \quad z \in D.$$

The proof is essentially a Corollary of the Cauchy Theorem by means of differentiation under the integral sign after the change of variable $\eta = \zeta - z$.

Remark II.2 In case $h$ is a function with compact support contained in the interior of $D$, which has continuous partial derivatives up to some order $n \geq 1$, $f$ is also a $C^n$ function on $D$.

III. SOME RESULTS IN COMPLEX ANALYSIS

In this section we shall extend the Cauchy Theorem and Theorem II.1 to functions that are $C^n$ in an open set obtained by removing a finite number of points in $\mathbb{C}$, with singularities of the following kind:

Definition III.1 Let $n \in \mathbb{N}$ and $F$ be a finite set of points in $\mathbb{C}$, i.e. $F = \{z_i\}_{i=0}^s$. We say that a function $f \in C^n(\mathbb{C} - F)$, is a $C^n$ function with poles if:

1. For each $z_i$ there exists a neighborhood $N_i$ of $z_i$ and a $C^n$ function $h$ on $N_i$ such that $f(z, \bar{z}) = \frac{h(z, \bar{z})}{(z - z_i)^m}$, for some integer $m$, $m \leq n$.

2. There exists a positive number $\varepsilon$ and a $C^n$ function $h$ on the ball of radius $\varepsilon$ around the origin such that for any $z$, $0 < |z| < \varepsilon$, $f(1/z, 1/\bar{z}) = \frac{h(z, \bar{z})}{z^m}$, for some integer $m$, $m \leq n - 2$.

Remark III.1 The above definition is in fact concerned with the singular behavior of the 1-form $f(z, \bar{z})dz$ on $\mathbb{C} \cup \{\infty\}$, which may be identified with the 2-sphere $S^2$. The subtle point on the behavior at infinity is that the function $h$, which is in principle defined in a punctured ball around the origin, has in fact a $C^n$ extension to the entire ball.
We now present the tools needed to generalize the Cauchy Theorem for $C^n$ functions with poles. This generalization is stated in Theorem III.2 and its proof is based on the same basic idea as the proof of the Cauchy Theorem for $C^1(D)$ functions given in the previous section. Let $f$ be a $C^n$ function with poles such that $\frac{\partial f}{\partial \bar{z}}$ is an integrable function on $\mathbb{C}$. We can then write

$$\int_{\mathbb{C}} \frac{\partial f}{\partial \bar{z}} \, dz \wedge d\bar{z} = \lim_{R \to \infty} \lim_{\varepsilon \to 0} \int_{D^\varepsilon_R} \frac{\partial f}{\partial \bar{z}} \, dz \wedge d\bar{z},$$

(8)

where $D^\varepsilon_R$ is obtained by first taking from the disc $D_R$, centered at the origin with radius $R$ sufficiently large so that the poles of $f$ remain inside $D_R$, and then deleting discs of radius $\varepsilon$ around the poles.

**Remark III.2** Even if $\frac{\partial f}{\partial \bar{z}}$ is not an integrable function over $\mathbb{C}$, the limit in the r.h.s of (8) does anyway exist (cf. [1]).

This Remark justifies the following Definition:

**Definition III.2** Let $f \in C^n(\mathbb{C} - \mathcal{F})$. We define the principal value (PV) of the integral of $f$ over $\mathbb{C}$ as

$$\text{PV} \int_{\mathbb{C}} f(z, \bar{z}) \, dz \wedge d\bar{z} := \lim_{R \to \infty} \lim_{\varepsilon \to 0} \int_{D^\varepsilon_R} f(z, \bar{z}) \, dz \wedge d\bar{z}.$$

Hence, in order to obtain a formula similar to (8), we shall evaluate the integral

$$\text{PV} \int_{\mathbb{C}} \frac{\partial}{\partial \bar{z}} f(z, \bar{z}) \, dz \wedge d\bar{z} = \lim_{R \to \infty} \lim_{\varepsilon \to 0} \int_{D^\varepsilon_R} \frac{\partial}{\partial \bar{z}} f(z, \bar{z}) \, dz \wedge d\bar{z}.$$

using the Stokes-Green formula (3) over $D^\varepsilon_R$:

$$\int_{D^\varepsilon_R} \frac{\partial}{\partial \bar{z}} f(z, \bar{z}) \, dz \wedge d\bar{z} = - \int_{\partial D^\varepsilon_R} f(z, \bar{z}) \, dz,$$

where $\partial D^\varepsilon_R$ is the oriented boundary of $D^\varepsilon_R$ given by $\partial D^\varepsilon_R = \bigcup_l (-\gamma^l) \cup \gamma_R$

and $\gamma^l$ and $\gamma_R$ are circles, the first ones centered at the poles $z = z_l$ with radius $\varepsilon$ and the second one being the boundary of $D_R$, all with counterclockwise orientation. Therefore,
\[ \int_{D_R} \frac{\partial}{\partial \bar{z}} f(z, \bar{z}) \, dz \wedge d\bar{z} = \sum_{i} \oint_{\gamma_{i}^R} f(z, \bar{z}) \, dz - \oint_{R} f(z, \bar{z}) \, dz. \] \tag{9}

The integrals along \( \gamma_{i}^\varepsilon \) can be evaluated in the following way. By Definition III.1, a function with a pole of multiplicity \( n \) at \( z_0 \) can be written as
\[ f(z, \bar{z}) = \frac{h(z, \bar{z})}{(z - z_0)^n}, \] \tag{10}

with \( h(z, \bar{z}) \in C^m(D) \) for some \( D \) containing \( z_0 \).

As \( h(z, \bar{z}) \in C^m(D) \), it has a Taylor expansion up to the \((n - 1)\)-th order. The integral containing the remainder along \( \gamma_{i}^\varepsilon \) becomes zero at the limit when \( \varepsilon \to 0 \) and the integral containing the Taylor polynomial gives rise to the definition of the residue of \( C^m \) functions with poles.

The above assertion is proved in the following Lemma and its proof is standard (c.f. [4]).

**Lemma III.1** Let \( f \) be a \( C^m \) function with poles and write \( f \) as in (10) near a pole \( z_0 \). Then
\[ \lim_{\varepsilon \to 0} \oint_{|z-z_0|=\varepsilon} f(z, \bar{z}) \, dz = \frac{2\pi i}{(n - 1)!} \partial_{\bar{z}}^{(n-1)} h(z_0, \bar{z}_0), \]

**Proof:** Since \( h \) is a \( C^n \) function in a neighborhood of \( z_0 \), we may consider its Taylor expansion
\[ h(z, \bar{z}) = \sum_{0 \leq m+l<n} \frac{\partial^m \partial^l}{m! \, l!} h(z_0, \bar{z}_0) (\bar{z} - \bar{z}_0)^m (z - z_0)^l + \sum_{m+l=n} \Gamma_{lm} (\bar{z} - \bar{z}_0)^m (z - z_0)^l, \]

where \( \Gamma_{ml} \) are continuous functions in \((z, \bar{z})\). It is obvious that
\[ \lim_{\varepsilon \to 0} \oint_{|z-z_0|=\varepsilon} \Gamma_{lm}(z, \bar{z}) (\bar{z} - \bar{z}_0)^m (z - z_0)^l \, dz = 0 \]

since \( \Gamma_{lm}(z, \bar{z}) (\bar{z} - \bar{z}_0)^m (z - z_0)^l \) is bounded near \( z_0 \) and the length of the path goes to zero when \( \varepsilon \to 0 \). Thus
\[ \lim_{\varepsilon \to 0} \oint_{|z-z_0|=\varepsilon} f(z, \bar{z}) \, dz = \sum_{0 \leq m+l<n} \frac{\partial^m \partial^l}{m! \, l!} h(z_0, \bar{z}_0) \lim_{\varepsilon \to 0} \oint_{|z-z_0|=\varepsilon} (\bar{z} - \bar{z}_0)^m (z - z_0)^l \, dz. \] \tag{11}
In case $m > 0$, we claim that all the line integrals are zero. In order to see this, consider the integral

$$I_\varepsilon = \oint_{|z-z_0|=\varepsilon} (\bar{z} - \bar{z}_0)^m (z - z_0)^{l-n} dz.$$ 

In polar coordinates this integral can be written as

$$I_\varepsilon = i \int_0^{2\pi} \varepsilon^{l-n+m+1} e^{i(l-n-m+1)t} dt.$$ 

It is clear that $I_\varepsilon = 0$ if $l-n-m+1 \neq 0$; otherwise $l-n+m+1 = 2m$ which implies that $\lim_{\varepsilon \to 0} I_\varepsilon = 0$. Thus, the only nonvanishing terms of (11) are those given by setting $m = 0$.

In this case the only non zero integral is when $l-n = -1$. Therefore

$$\lim_{\varepsilon \to 0} \oint_{|z-z_0|=\varepsilon} f(z, \bar{z}) dz = \frac{2\pi i}{(n-1)!} \partial_z^{(n-1)} h(z_0, \bar{z}_0).$$

This completes the proof. □

This result gives rise to the definition of local residue for $C^m$ functions with poles.

**Definition III.3** Let $f$ be a function with a pole at $z_0$. We define the residue of $f$ at $z_0$ by the limit

$$\text{Res}(f; z_0) := \frac{1}{2\pi i} \lim_{\varepsilon \to 0} \oint_{|z-z_0|=\varepsilon} f(z, \bar{z}) dz.$$ 

Note that when the function has a holomorphic numerator, this definition is the standard one for meromorphic functions.

Moreover, we define:

**Definition III.4** Given a $C^m$ function with poles, the residue of $f$ at infinity is the limit

$$\text{Res}(f; \infty) := -\frac{1}{2\pi i} \lim_{R \to \infty} \oint_{\gamma_R} f(z, \bar{z}) dz.$$ 

**Remark III.3** Given a $C^m$ function with poles $f$, similar arguments as in the proof of Lemma III.1 after the change of variable given by the inversion $1/z$, show that the above limit exists, i.e. the residue of $f$ at infinity is well defined.
We now generalize the Cauchy Theorem to $C^n$ functions with poles.

**Theorem III.2** Let $f$ be a $C^n$ function with poles, regular on $\mathbb{C} - \mathcal{F}$, where $\mathcal{F} = \{z_1, \ldots, z_s\}$. Then, the principal value

$$\text{PV} \int_{\mathbb{C}} \frac{\partial}{\partial \bar{z}} f(z, \bar{z}) \, dz \wedge d\bar{z}$$

exists and

$$\text{PV} \int_{\mathbb{C}} \frac{\partial}{\partial \bar{z}} f(z, \bar{z}) \, dz \wedge d\bar{z} = 2\pi i \left( \sum_l \text{Res}(f; z_l) + \text{Res}(f; \infty) \right).$$

**Proof:** By equation (9)

$$\text{PV} \int_{\mathbb{C}} \frac{\partial}{\partial \bar{z}} f(z, \bar{z}) \, dz \wedge d\bar{z} = \sum_l \lim_{\varepsilon \to 0} \oint_{\gamma_l \varepsilon} f(z, \bar{z}) \, dz - \lim_{R \to \infty} \oint_{\gamma_R} f(z, \bar{z}) \, dz,$$

and the limits in the r.h.s. exist by Lemma III.1 and Remark III.3, and equal $2\pi i$ times the corresponding residues of $f$. The statement follows at once. $\square$

Considering the special case when $f$ has a simple pole at $z_0$, i.e.

$$f(z, \bar{z}) = \frac{h(z, \bar{z})}{(z - z_0)}, \quad \text{with} \quad h \in C^n(D)$$

and $\text{Res}(f, \infty) = 0$, Theorem III.2 gives

$$h(z_0, \bar{z}_0) = \frac{1}{2\pi i} \text{PV} \int_{\mathbb{C}} \frac{\partial}{\partial \bar{z}} h(z, \bar{z}) \frac{dz \wedge d\bar{z}}{(z - z_0)}.$$ 

This allows to recover the values of the function $h$ from its derivatives. Conversely, as in the case of $C^1(D)$ functions with compact support in $D$, one has the following result.

**Theorem III.3** Let $g$ be a $C^1(\mathbb{C})$ function satisfying

$$\lim_{|z| \to \infty} |z|^{1+\alpha} |g(z, \bar{z})| = 0,$$

for some positive $\alpha$. Then the function

$$f(z, \bar{z}) = \frac{1}{2\pi i} \int_{\mathbb{C}} g(\zeta, \bar{\zeta}) \frac{d\zeta \wedge d\bar{\zeta}}{(\zeta - z)}$$

is in $C^1(\mathbb{C})$ and satisfies $\frac{\partial}{\partial \bar{z}} f = g$ for $z \in \mathbb{C}$. 

Proof: Since $\frac{1}{\zeta - z}$ is integrable near $z$ and tends to 0 for $|\zeta| \to \infty$ and $g$ has the decay given by (12), we deduce the existence of the integral. With the change of variables $\eta = \zeta - z$ the integral becomes

$$f(z, \bar{z}) = \frac{1}{2\pi i} \int_C g(z + \eta, \bar{z} + \bar{\eta}) \frac{d\eta \wedge d\bar{\eta}}{\eta}$$

and applying $\frac{\partial}{\partial \bar{z}}$ to $f$, we obtain

$$\frac{\partial}{\partial \bar{z}} f(z, \bar{z}) = \frac{1}{2\pi i} \int_C \frac{\partial g(z + \eta, \bar{z} + \bar{\eta})}{\partial \bar{z}} \frac{d\eta \wedge d\bar{\eta}}{\eta}$$

$$= \frac{1}{2\pi i} \int_C \frac{\partial g(\zeta, \bar{\zeta})}{\partial \bar{\zeta}} \frac{d\zeta \wedge d\bar{\zeta}}{(\zeta - z)}.$$

Because of the decay of $g$, the residue of $g(\zeta, \bar{\zeta}) (\zeta - z)$ at infinity is zero. Thus, applying Theorem III.2, we obtain $\frac{\partial}{\partial \bar{z}} f = g$. $\square$

This result can be extended to general $C^n$ functions with poles of any order. In this case the double integral does not necessarily exist, thus we shall use instead the notion of principal value.

It is not true in general that given a $C^n$ function with poles $g$, the equation $\frac{\partial}{\partial \bar{z}} f = g$ has a $C^n$ solution with poles. For instance, $g = 1$ does not have this property. Any solution of $\frac{\partial}{\partial \bar{z}} f = 1$ is of the form $f(z, \bar{z}) = \bar{z} + p(z)$, where $p$ is a holomorphic function, and so $z^m f(1/z, 1/\bar{z})$ is not $C^n$ at the origin for any $m \leq n-2$, i.e. $f$ is not a $C^n$ function with poles.

Theorem III.4 Let $g$ be a $C^n$ function with poles. Suppose moreover that $\bar{z}^2 g$ is also a $C^n$ function with poles. Then, there exists a $C^n$ function $f$ with poles such that $\frac{\partial}{\partial \bar{z}} f = g$.

Proof: Let $\mathcal{F} = \{z_1, \ldots, z_s\}$ be the poles of $g$ in $\mathbb{C}$, and suppose $(z - z_i)^m g(z, \bar{z})$ is a $C^n$ function around each pole $z_i$. Denote $p(z) := \prod_i (z - z_i)^{m_i}$. Then, $p \cdot g$ is a $C^n$ function in the plane.

Since $\bar{z}^2 g$ is a $C^n$ function with poles, there exist a positive number $R' > 0$ and a $C^n$ function $h$ on $B_{2/R'}$ such that $h(z, \bar{z}) := \frac{-z^m g(1/z, 1/\bar{z})}{\bar{z}^2}$, for some $m \leq n - 2$. Consider a $C^\infty$ function $\psi$ with support compact contained in $B_{2/R'}$ which is identically equal to 1 on the ball $B_{1/R'}$ of radius $1/R'$ around the origin. Then, by
Remark [II.2] after Theorem [II.1], there exists a $C^n$ function $H$ on $B_{2/R'}$ such that $\frac{\partial}{\partial \bar{z}} H = \psi \cdot h(z, \bar{z})$. Then, the restriction of $H$ to $B_{1/R'}$ is a $C^n$ function satisfying $\frac{\partial}{\partial \bar{z}} H = h(z, \bar{z})$. Set $h_1(z, \bar{z}) := z^m H(1/z, 1/\bar{z})$. Then, $h_1 \in C^n(|z| > R')$ satisfies condition 2 in the definition of $C^n$ functions with poles. Moreover, $\frac{\partial}{\partial \bar{z}} h_1 = -\frac{z^m}{z^2} \frac{\partial}{\partial \bar{z}} (H(1/z, 1/\bar{z})) = g$ on $|z| > R'$.

Let $R > R'$ and, similarly, take a $C^\infty$ function $\psi$ with support compact contained in the ball $B_{2R}$ of radius $2R$ around the origin, such that $\psi \equiv 1$ on the ball $B_R$ of radius $R$. Again by Remark [II.2] after Theorem [II.1], there exists a $C^n$ function $H'$ on $B_{2R}$ such that $\frac{\partial}{\partial \bar{z}} H' = \psi \cdot p \cdot g$. Then, $h_2 := H'/p$ is a $C^n$ function on $B_R - \mathcal{F}$ which satisfies condition 1 in the definition of $C^n$ functions with poles, and clearly $\frac{\partial}{\partial \bar{z}} h_2 = g$ on $B_R - \mathcal{F}$.

Then, $h_2 - h_1$ is a holomorphic function on the annulus $R' < |z| < R$, and therefore admits a Laurent expansion

$$h_2(z, \bar{z}) - h_1(z, \bar{z}) = \sum_{k=1}^{+\infty} a_k z^k.$$

Denote $H_2(z) := -\sum_{k=1}^{+\infty} a_k z^k$, $H_1(z) := \sum_{k=0}^{-\infty} a_k z^k$. Observe that $H_2$ is a Taylor series which is convergent for points with absolute value greater than $R'$ and thus it must be convergent in the whole ball of radius $R$ around the origin. Similarly, $H_1$ defines a holomorphic function on $|z| > R'$ (which is also holomorphic at infinity). Clearly, $H_1 + h_1 = H_2 + h_2$ on the annulus. Then, we have a well defined global function $f$, $f = H_2 + h_2$ for $|z| < R$ and $f = H_1 + h_1$ for $|z| > R'$, satisfying $\frac{\partial}{\partial \bar{z}} f = g$. Moreover, $f$ satisfies both conditions in Definition [III.1], i.e. $f$ is a $C^n$ function with poles. \]

**Corollary III.5** Let $g$ be a $C^n$ function with poles, regular outside the finite set $\mathcal{F}$, such that $z^2 g$ is also a $C^n$ function with poles. Then, the function defined in $C^n - \mathcal{F}$ by

$$f(z, \bar{z}) = \frac{1}{2\pi i} \text{PV} \int_{C} g(\zeta, \bar{\zeta}) \frac{d\zeta \wedge d\bar{\zeta}}{(\zeta - z)}$$

is a $C^n$ function with poles and satisfies that $\frac{\partial}{\partial \bar{z}} f = g$ for all $z \notin \mathcal{F}$.

**Proof:** Since $g(\zeta)$ is a $C^n$ function with poles, $\frac{\partial}{\partial \bar{z}} g(\zeta)$ is also a $C^n$ function with poles, which ensures the existence of the principal value. By Theorem [III.4], there
exists a $C^n$ function with poles $F$ such that $\frac{\partial}{\partial \zeta} F = g$. Then, $\frac{\partial}{\partial \zeta} \left( \frac{F(\zeta, \bar{\zeta})}{\zeta - z} \right) = \frac{g(\zeta, \bar{\zeta})}{\zeta - z}$. Note that we can deduce from the proof of Theorem II.4 that the poles of $F$ in the plane are also contained in $\mathcal{F}$. For any $z \in \mathbb{C} - \mathcal{F}$, $\text{Res} \left( \frac{F(\zeta, \bar{\zeta})(\zeta - z)}{(\zeta - z)}; z \right) = F(z, \bar{z})$.

By Theorem II.2 we obtain

$$f(z, \bar{z}) = \frac{1}{2\pi i} \text{PV} \int_{\mathbb{C}} g(\zeta, \bar{\zeta}) \frac{d\zeta \wedge d\bar{\zeta}}{(\zeta - z)} = \sum_{z_l \in \mathcal{F} \cup \{\infty\}} \text{Res} \left( \frac{F(\zeta, \bar{\zeta})}{(\zeta - z)}; z_l \right) + F(z, \bar{z}). \quad (14)$$

We deduce from Lemma III.1 that each residue in the above sum is a rational (meromorphic) function of $z$ (with poles contained in $\mathcal{F} \cup \{\infty\}$). Therefore, $f$ is a $C^n$ function with poles and $\frac{\partial}{\partial \bar{z}} f = g$. $\square$

Although the main motivation of this paper is to explicitly show how to compute the PV of integrals like (13), it may be useful to have an integral representation of a solution of the differential equation $\frac{\partial}{\partial \bar{z}} f = g$. Note that this solution is not unique since we can always add a meromorphic function to the function $f$ defined by (13).

**IV. INTEGRALS ON THE SPHERE**

In this section we shall apply the results obtained in section II in order to evaluate integrals on the two sphere whose integrands might possess singular behavior. The main result is stated in Theorem IV.1.

Let $(\theta, \phi)$ be the usual coordinates of the sphere, the complex stereographic coordinates transformation $z = e^{i\phi} \cot \left( \frac{\theta}{2} \right)$ entails that $S^2 = \mathbb{C} \cup \{\infty\}$.

Now let us assume that $g$ is an integrable function on the sphere. Using the coordinates $(z, \bar{z})$ defined above, we write

$$\int_{S^2} g(\theta, \phi) \sin \theta d\theta d\phi = 2i \int_{\mathbb{C}} g(z, \bar{z}) \frac{dz \wedge d\bar{z}}{(1 + z\bar{z})^2}. \quad (15)$$

The orientation that we use in the complex plane is the right-hand orientation, i.e. $dz \wedge d\bar{z} = -2i dx \wedge dy$. Note that the r.h.s. of (15) is an improper integral and since $g$ is integrable, the principal value of the integral is the value of the integral.

We now consider functions with a finite number of singularities on the sphere.
**Definition IV.1** Let $\mathcal{F}$ be a finite set of points in the sphere. We say that function $g \in C^n(S^2 - \mathcal{F})$ is a $C^n$ function with poles on $S^2$ if the restriction of $g$ to the complex plane is a $C^n$ function with poles according to Definition III.4.

**Remark IV.1** Consider for instance a $C^\infty$ function $g$ on $S^2$ which coincides with $1/z$ near infinity. Then, $g$ is regular at $\infty$ but $\text{Res}(g; \infty) \neq 0$.

Given a $C^n$ function $g$ with poles on $S^2$, let $D_R$ be a disk centered at the origin, with radius $R$ sufficiently large so that the singularities of $g$ remain inside $D_R$. We write

\[
\int\int_{S^2} g d\mu := 2i \int_C g(z, \bar{z}) \frac{dz \wedge d\bar{z}}{(1 + z\bar{z})^2} = \lim_{R \to \infty} \lim_{\varepsilon \to 0} 2i \int_{D_{\varepsilon}^R} g(z, \bar{z}) \frac{dz \wedge d\bar{z}}{(1 + z\bar{z})^2} \tag{16}
\]

where $D_{\varepsilon}^R$ is the region obtained deleting from $D_R$ discs of radius $\varepsilon$ around the singularities of $g$.

A key observation is that for any $C^m$ function with poles $g$, the function $G(z, \bar{z}) := \frac{g}{(1 + z\bar{z})^2}$ satisfies the additional hypothesis in Theorem III.4. In fact, given $m$ and a $C^n$ function $h$ around the origin such that $h(z, \bar{z}) = z^m g(1/z, 1/\bar{z})$, $\frac{z^m G(1/z, 1/\bar{z})}{\bar{z}} = h(z, \bar{z}) \frac{z^2}{(1 + z\bar{z})^2}$ is a $C^n$ function near the origin. Therefore, by Theorem III.4 (or Corollary III.3), there exists a $C^n$ solution with poles $f$ of the differential equation

\[
\frac{\partial}{\partial \bar{z}} f = \frac{g}{(1 + z\bar{z})^2}. \tag{17}
\]

**Remark IV.2** We can add to $f$ in (17) an arbitrary rational function $w$ (i.e. an arbitrary solution with pole-singularities of the homogeneous differential equation $\frac{\partial}{\partial \bar{z}} w = 0$).

Using the Green-Stokes formula (3), we obtain

\[
\int_{D_{\varepsilon}^R} \frac{\partial}{\partial \bar{z}} f(z, \bar{z}) dz \wedge d\bar{z} = \sum_i \oint_{\gamma_i^\varepsilon} f(z, \bar{z}) dz - \oint_{\gamma_R} f(z, \bar{z}) dz \tag{18}
\]

where $\gamma_i^\varepsilon$ and $\gamma_R$ are circles, the first ones centered at the singularities $z = \eta_i$ with radius $\varepsilon$ and the second one is the boundary of $D_R$, all with counterclockwise orientation.
Thus, the integral (16) becomes
\[
\int_{S^2} g \, d\mu = \lim_{R \to \infty} \lim_{\varepsilon \to 0} 2i \int_{D^\varepsilon_R} \frac{\partial}{\partial \bar{z}} f(z, \bar{z}) \, dz \wedge d\bar{z}
\]
\[
= 2i \sum_l \lim_{\varepsilon \to 0} \oint_{\gamma_l^\varepsilon} f(z, \bar{z}) \, dz - 2i \lim_{R \to \infty} \oint_{\gamma_R} f(z, \bar{z}) \, dz.
\]
(19)

As a consequence of Theorem III.2 we get the following result.

**Theorem IV.1** Assume that \(g\) is a \(C^n\) function with poles on \(S^2\) and let \(f\) be a \(C^n\) function with poles satisfying (17). Then the principal value
\[
\text{PV} \int_{S^2} g \, d\mu = 2i \text{PV} \int_{C} \frac{g(z, \bar{z})}{(1 + z\bar{z})^2} \, dz \wedge d\bar{z}
\]
exists and
\[
\text{PV} \int_{S^2} g \, d\mu = -4\pi \left( \sum_l \text{Res}(f; \eta_l) + \text{Res}(f; \infty) \right).
\]
where \(\eta_l\) are the poles of \(f\).

The proof follows at once from Theorem III.2. Besides if \(g\) is integrable, the principal value is the value of the integral.

Note that as we said in Remark IV.2 the function \(f\) is not unique, but the result given above is independent of the choice of the solution of (17) since the sum of the residues of a meromorphic function on \(S^2\) is zero.

The same ideas as in Corollary III.5 can be used to present an explicit solution with poles of the differential equation (17).

**Corollary IV.2** Let \(g\) be a \(C^n\) function with poles on \(S^2\), regular outside \(\mathcal{F} = \{z_1, \ldots, z_s\}\). Then the function defined in \(S^2 - \mathcal{F}\)
\[
f(z, \bar{z}) = \frac{1}{2\pi i} \text{PV} \int_{C} \frac{g(\zeta, \bar{\zeta})}{(1 + \zeta\bar{\zeta})^2} \, d\zeta \wedge d\bar{\zeta}.
\]
is a \(C^n\) function with poles on \(S^2\) and satisfies that \(\frac{\partial}{\partial \bar{z}} f = \frac{g}{(1 + z\bar{z})^2}\) for all \(z \in \mathbb{C}, z \neq z_j, j = 1, \ldots, s\).

Finally, we exemplify the result given in Theorem IV.1 for \(g = 1\) and different choices of \(f\). Consider the integral
\[
\iint_{S^2} \sin \theta d\theta d\phi = 2i \int_{C} \frac{dz \wedge d\bar{z}}{(1 + z\bar{z})^2}.
\]
and the solutions of (17) $f_0(z, \bar{z}) = -\frac{1}{z(1+z\bar{z})}$ and $f_1(z, \bar{z}) = \frac{z}{1+z\bar{z}}$, then
\[
\text{Res}(f_0; 0) = -1 \quad \text{and} \quad \text{Res}(f_0; \infty) = 0,
\]
thus the contribution to the integral comes from the pole of $f_0$ at zero. Whereas the contribution to the integral using $f_1$ comes from the residue at infinity. Note the function $f_0$ has a pole at $z = 0$ whereas $f_1$ is $C^\infty(C)$, moreover $f_1$ is the integral representation given by Theorem III.3.

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