Abstract

We give asymptotically tight estimate of tangent space variation on a Riemannian submanifold of Euclidean space with respect to the local feature size of the submanifold. We show that the result is a consequence of structural properties of local feature size function of the Riemannian submanifold.

1 Introduction

Let \( M \) be a compact Riemannian submanifold of the Euclidean space \( \mathbb{R}^N \). For a point \( p \) in \( M \), we will denote by \( T_p M \) the tangent space of \( M \) at \( p \).

In this note we will be proving the following result:

**Theorem 1 (Tangent space variation)**  Let \( p, q \in M \) and \( \|p - q\| = t \\text{lf}_{s}(p) \) with \( t \leq 1/4 \).

Then

\[
\sin \angle(T_p M, T_q M) \leq \min\{1, t f(t)\}, \text{ where } f(t) = \frac{(2 + 3t + 2t^2)^2 + 4t + 5}{2 - 2t}.
\]

Observe that \( f(x) = \frac{9}{2} + O(t) \), and for all \( t \leq 1/10 \), then \( f(t) < 6 \).

We will define \( \text{lf}_{s}(\cdot) \) in the next section.

Special case of this result in the context of 2-dimensional Riemannian submanifolds in \( \mathbb{R}^3 \) was proved by Amenta and Dey [AD14, Thm. 2], and a weaker bound for the general case of Riemannian submanifolds of Euclidean space was proved by Niyogi, Smale and Weinberger [NSW08, Prop. 6.2 & 6.3].

One interesting feature of our result, other than the facts that it is more general and asymptotically tight, is that the proof of our result is extremely elementary and follows from structural properties of local feature size function of Riemannian submanifolds.

Notations. We will denote by \( \|x - y\| \) the standard Euclidean distance between \( x, y \in \mathbb{R}^N \), and for a point \( x \in \mathbb{R}^N \) and a set \( X \subseteq \mathbb{R}^N \), the distance between \( x \) and \( X \) will be denoted by \( d(x, X) = \inf_{y \in X} \|x - y\| \).

If \( U \) and \( V \) are vector subspaces of \( \mathbb{R}^d \), with \( \dim(U) \leq \dim(V) \), the angle between them is defined by

\[
\angle(U, V) = \max_{u \in U} \min_{v \in V} \angle(u, v) = \arccos \inf_{u \in U} \sup_{v \in V} \frac{\langle u; v \rangle}{\|u\| \|v\|},
\]

where \( u \) and \( v \) are vectors in \( U \) and \( V \) respectively. This is the largest principal angle between \( U \) and \( V \). Note that it is easy to show from the above definition that if \( \dim(U) = \dim(V) \) then \( \angle(U, V) = \angle(V, U) \), for a proof see [BGI14, Lem. 2.1].

The angle between affine subspaces is defined as the angle between the corresponding parallel vector subspaces.

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*This result first appeared in Chapter 4 of the second author’s PhD thesis [Gho12, Lem. 4.3.2].
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2 Medial axis and local feature size

The medial axis of $M$ is the closure of the set of points of $\mathbb{R}^N$ that have more than one nearest neighbor on $M$. The local feature size of $x \in M$, $lfs(x)$, is the distance of $x$ to the medial axis of $M$ [AB99]. As is well known and can be easily proved, $lfs$ is Lipschitz continuous, i.e.,

$$lfs(x) \leq lfs(y) + \|x - y\|.$$

Amenta and Dey [AD14, Thm. 2], proved the following tangent variation result for the case of 2-dimensional Riemannian submanifold of $\mathbb{R}^3$:

**Theorem 2 (Two-dimensional case)** Let $M$ be a 2-dimensional Riemannian submanifold of $\mathbb{R}^3$. Let $p, q \in M$ such that $\|p - q\| = t lfs(p)$ with $t \leq \frac{1}{3}$ then

$$\sin \angle(T_pM, T_qM) \leq \frac{t}{1 - t}.$$

Note that the proof of the above result is restricted to the case where the dimension of the submanifold is two and the codimension is one.

The infimum of $lfs$ over $M$ is called the reach $rch(M)$ of $M$. Niyogi, Smale and Weinberger [NSW08, Prop. 6.2 & 6.3] proved the following bound on the tangent variation on Riemannian submanifolds:

**Theorem 3** Let $M$ be a Riemannian submanifold of $\mathbb{R}^N$. Let $p, q \in M$ such that $\|p - q\| = t rch(M)$ with $t \leq \frac{1}{2}$. Then

$$\sin \angle(T_pM, T_qM) \leq 2\sqrt{t(1 - t)}.$$

Observe that the above bound of $O(\sqrt{t})$ is strictly weaker than the bound $O(t)$ obtained in Theorem [1] and the proof of their result uses tools from differential geometry, like parallel transport.

3 The proof

The following lemma, proved in [GW04, Lem. 6 & 7], states some basic properties of local feature size function.

**Lemma 4** 1. Let $p, q \in M$ such that $\|p - q\| = t lfs(p)$ with $t < 1$, then

$$d(q, T_pM) \leq \frac{t^2}{2} lfs(p).$$

2. Let $p \in M$ and $x \in T_pM$ such that $\|p - x\| \leq t lfs(p)$ with $t \leq \frac{1}{4}$, then

$$d(x, T_pM) \leq 2t^2 lfs(p).$$

We will give an extremely simple proof of Theorem 1 using the above result.

Let $t = \frac{\|p - q\|}{lfs(q)}$. Using the fact that $lfs$ is 1-Lipschitz, we have

$$(1 - t) lfs(q) \leq lfs(p) \leq (1 + t) lfs(q) \tag{2}$$

Let $u$ be an unit vector in $T_pM$. Let $p_u = p + t lfs(p) \cdot u$, and $p'_u$ denote the point closest to $p_u$ on $M$. Then, from Lemma 4(2), we have $\|p_u - p'_u\| \leq 2t^2 lfs(p)$. Therefore, using the fact that $lfs(p) \leq (1 + t) lfs(q)$ (see Eq. 2), we have

$$\|q - p'_u\| \leq \|q - p\| + \|p - p_u\| + \|p_u - p'_u\| \leq t lfs(q) + (t + 2t^2) lfs(p) \leq t lfs(q) + (t + 2t^2)(1 + t) lfs(q) = t (2 + 3t + 2t^2) lfs(q).$$
Using Lemmas 4 (1) and (2), and Eq. (2), we have
\[ d(p_u, T_q M) \leq d(p'_u, T_q M) + d(p_u, p'_u) \]
(3)
\[ \leq \frac{\ell^2}{2} (2 + 3t + 2t^2)^2 \text{lfs}(q) + 2\ell^2 \text{lfs}(p) \]
\[ \leq \frac{\ell^2}{2} (2 + 3t + 2t^2)^2 \text{lfs}(q) + 2\ell^2 (1 + t) \text{lfs}(q) \]
\[ = \frac{\ell^2}{2} ((2 + 3t + 2t^2)^2 + 4(1 + t)) \text{lfs}(q) \]
(4)

Figure 1: In the figure \( \alpha = \angle([r, r_u], T_q M) \) and \( \Delta = d(r_u, T_q M) \).

Let \( r \in T_q M \) be the point closest to \( p \) in \( T_q M \), i.e., \( \|p - r\| = d(p, T_q M) \), and let \( v \) be a unit vector in \( T_q M \) that makes the smallest angle with \( u \). Let \( r_u = r + \ell \text{lfs}(p) \cdot u \). Now observe that as \( d(p, r) = d(p_u, r_u) \), we have
\[ d(r_u, T_q M) \leq d(r_u, p_u) + d(p_u, T_q M) = d(p, T_q M) + d(p_u, T_q M) \]
(5)
and the projection of the line segment \([r, r_u]\) onto \( T_q M \) is parallel to \( v \) which implies \( \angle(u, v) = \angle([r, r_u], T_q M) \).

Using the fact that \( d(r, r_u) = \ell \text{lfs}(p) \geq t(1 - t) \text{lfs}(q) \) (from Eq. (2)), \( d(p, T_q M) \leq \frac{\ell^2}{2} \text{lfs}(q) \) (from Lemma 4 (1)), and Eq. (3), we get
\[ \sin \angle(u, v) = \sin \angle([r, r_u], T_q M) \]
\[ = \frac{d(r_u, T_q M)}{d(r, r_u)} \]
\[ \leq \frac{d(p, T_q M) + d(p_u, T_q M)}{d(r, r_u)} \]
\[ \leq \frac{\ell^2}{2d(r, r_u)} ((2 + 3t + 2t^2)^2 + 4(1 + t) + 1) \text{lfs}(q) \]
\[ \leq t \left( \frac{(2 + 3t + 2t^2)^2 + 4t + 5}{2 - 2t} \right) \]
\[ \overset{\text{def}}{=} tf(t) \]

This completes the proof of Theorem 1.
4 Discussion

Lower bound. It is easy to see that $O(t)$ bound on the tangent space variation on submanifolds is also tight. Consider a $N - 1$-dimensional unit sphere $S^{N-1}$ in $\mathbb{R}^N$. For all $p \in S^{N-1}$, $\text{lfs}(p) = 1$. Let $p, q \in S^{N-1}$ with $\|p - q\| = t$. Using elementary high school geometry, see Fig 2, we can show that

$$\sin \angle(N_p S^{N-1}, N_p S^{N-1}) = t \sqrt{1 - \frac{t^2}{4}} = \Omega(t)$$

Using the above equation, and the fact that $\angle(T_p S^{N-1}, T_p S^{N-1}) = \angle(N_p S^{N-1}, N_p S^{N-1})$, see [BG14, Lem. 2.1], we get the lower bound.

Regarding the constants. For $t \leq 1/10$, we have $f(t) < 6$, unlike the 2-dimensional case where it is close to 1, and we expect in the general case the constant to be closer to 1.

References

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