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ISOTROPIC CURVE FLOWS ON $\mathbb{R}^{n+1,n}$

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1. Introduction

Let $\mathbb{R}^{n+1,n}$ be the vector space $\mathbb{R}^{2n+1}$ equipped with the index $n$, non-degenerate bilinear form $\langle X, Y \rangle = X^t C_n Y$, where

$$C_n = \sum_{i=1}^{2n+1} (-1)^{n+i-1} e_{i,2n+2-i}. \tag{1.1}$$

Let $O(n+1,n)$ denote the group of linear isomorphisms on $\mathbb{R}^{n+1,n}$ preserving $\langle , \rangle$. A subspace $I \subset \mathbb{R}^{n+1,n}$ is called isotropic if $\langle X, Y \rangle = 0$ for all $X, Y \in I$. A maximal isotropic subspace in $\mathbb{R}^{n+1,n}$ has dimension $n$.

A smooth map $\gamma : \mathbb{R} \rightarrow \mathbb{R}^{n+1,n}$ is called an isotropic curve if $\gamma, \gamma_s, \dot \gamma_s, \ldots, \gamma_s^{(2n)}$ are linearly independent and the span of $\gamma, \gamma_s, \ldots, \gamma_s^{(n-1)}$ is a maximal isotropic subspace of $\mathbb{R}^{n+1,n}$ for all $s \in \mathbb{R}$. Note that a curve being isotropic is independent of the choice of parameter. It is easy to see that there is an orientation preserving parameter $x$ (unique up to translation) for an isotropic curve such that $\langle \gamma_x^{(n)} \rangle \equiv 1$. We call such $x$ the isotropic parameter of $\gamma$.

Set $M_{n+1,n} = \{ \gamma : \mathbb{R} \rightarrow \mathbb{R}^{n+1,n} \mid \gamma \text{ isotropic, } \langle \gamma_x^{(n)} \rangle \equiv 1 \}.$

We prove that given $\gamma \in M_{n+1,n}$, there exists a unique smooth map $g : \mathbb{R} \rightarrow O(n+1,n)$ such that the $i$-th column is $\gamma_x^{(i-1)}$ for $1 \leq i \leq n+1$ and $g^{-1}g_x$ is of the form

$$g^{-1}g_x = b + \sum_{i=1}^{n} u_i \beta_i,$$

where

$$b = \sum_{i=1}^{2n} e_{i+1,i}, \quad \beta_i = e_{n+1-i,n+i} + e_{n+2-i,n+1+i}. \tag{1.2}$$

We call $g$, $u_i$, and $u = \sum_{i=1}^{n} u_i \beta_i$ the isotropic moving frame, the $i$-th isotropic curvature, and the isotropic curvature along $\gamma$ respectively.

Set

$$V_n = \bigoplus_{i=1}^{n} \mathbb{R} \beta_i. \tag{1.3}$$

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Let $\Psi : \mathcal{M}_{n+1,n} \rightarrow C^\infty(\mathbb{R}, V_n)$ be the isotropy curvature map defined by

$$\Psi(\gamma) = u = g^{-1}g_x - b = \sum_{i=1}^{n} u_i \beta_i,$$

where $u$ is the isotropy curvature of $\gamma$. Then $\Psi$ is onto and $\Psi^{-1}(u)$ is an $O(n+1,n)$-orbit for all $u \in C^\infty(\mathbb{R}, V_n)$. Hence $u_1, \ldots, u_n$ form a complete set of differential invariants for $\gamma \in \mathcal{M}_{n+1,n}$.

An isotropic curve flow on $\mathcal{M}_{n+1,n}$ is an evolution equation on $\mathcal{M}_{n+1,n}$ of the form

$$\gamma_t = g\xi(u),$$

where $g(\cdot,t)$, $u(\cdot,t)$ are the isotropic moving frame and isotropic curvature along $\gamma(\cdot,t)$ respectively and $\xi(u)$ is a $\mathbb{R}^{(2n+1)\times 1}$-valued differential polynomial of $u$ in $x$ variable. Hence isotropic curve flows are invariant under the action of $O(n+1,n)$.

We note that a map $\gamma : \mathbb{R} \rightarrow \mathbb{R}^{2,1}$ lies in $\mathcal{M}_{2,1}$ if and only if $\gamma$ lies in the light cone, is space-like, and is parametrized by its arc-length. The isotropic curvature of such curve is the standard curvature in differential geometry. Hence an isotropic curve flow on $\mathcal{M}_{2,1}$ is a geometric curve flow for space-like curves on the light cone of $\mathbb{R}^{2,1}$ preserving arc-length. We show that an isotropic curve flow on $\mathcal{M}_{2,1}$ is of the form

$$\gamma_t = (h(q))_x \gamma - h(q)\gamma_x$$

for some differential polynomial $h$ of the isotropic curvature $q(\cdot,t)$ of $\gamma(\cdot,t)$. In particular,

$$\gamma_t = q_x \gamma - q\gamma_x,$$  \hspace{1cm} (1.5)

$$\gamma_t = -\frac{1}{9}(q_{xxx} - 8qq_x)\gamma + \frac{1}{9}(q_{xx} - 4q^2)\gamma_x,$$  \hspace{1cm} (1.6)

are isotropic curve flows. We show that if $\gamma$ is a solution of (1.5) or (1.6), then the isotropic curvature $q$ is a solution of KdV

$$q_t = q_{xxx} - 3qq_x,$$

or the Kupershmidt-Kaup (KK) equation

$$q_t = -\frac{1}{9}(q^{(5)} - 10qq_{xxx} - 25q_xq_{xx} + 20q^2q_x)$$  \hspace{1cm} (1.7)

respectively. So these two curve flows give natural geometric interpretations of the KdV and KK equations respectively.

In this paper, we prove the following results:

(a) For general $n$, we construct two sequences of isotropic curve flows on $\mathcal{M}_{n+1,n}$ of B-type and A-type that map to the $\hat{B}_n^{(1)}$-KdV and $\hat{A}_2^{(2)}$-KdV hierarchies under the isotropic curvature map $\Psi$. The third
flows of B-type and A-type are
\begin{align}
\frac{\partial \gamma}{\partial t} &= -\frac{3}{n} u_1 \gamma_x + \gamma^{(3)}_x, \quad (1.8) \\
\frac{\partial \gamma}{\partial t} &= -\frac{3}{2n+1} (u_1)_x \gamma - \frac{6}{2n+1} u_1 \gamma_x + \gamma^{(3)}_x, \quad (1.9)
\end{align}
where $u_1(\cdot, t)$ is the first isotropic curvature along $\gamma(\cdot, t)$. Note that the flows (1.5) and (1.6) are the third and fifth flows of these two sequence when $n = 1$.

(b) Algorithms to compute the Bi-Hamiltonian structures and conservation laws for isotropic curve flows of B- and A-types are given.

(c) We construct Bäcklund transformations and Permutability formulas.

In particular, isotropic curve flows give natural geometric interpretations of the $\tilde{B}_n^{(1)}$-KdV and $\tilde{A}_{2n}^{(2)}$-KdV flows and techniques from soliton theory give bi-Hamiltonian, conservation laws, and BTs for these isotropic curve flows. We note that the relation between central curve flows on $\mathbb{R}^n \setminus 0$ and the soliton theory of the $\tilde{A}_{n-1}^{(1)}$-KdV hierarchy were considered in [7] and [10] for $n = 2$, in [11] for $n = 3$, and for general $n$ in [11].

This paper is organized as follows. We construct moving frames and isotropic curvatures for isotropic curves in section 2 give an explicit description of the tangent space of $\mathcal{M}_{n+1,n}$ at $\gamma$ in section 3 and review Drinfeld-Sokolov’s construction of KdV type equations associated to the affine Kac-Moody algebras $\tilde{B}_n^{(1)}$ and $\tilde{A}_{2n}^{(2)}$ in section 4. We prove results (a) and solve the Cauchy problem for these curve flows in section 5. Result (b) is proved in section 6. We construct Bäcklund transformations for isotropic curve flows of B-type and A-type and give an algorithm to construct infinitely many families of explicit solutions of these isotropic curve flows in sections 7 and 8 respectively.

2. MOVING FRAMES ALONG ISOTROPIC CURVES

In this section, we prove the existence of isotropic parameter and construct isotropic moving frames and curvatures along isotropic curves. We also give some properties of the isotropic curvature map $\Psi$.

The Lie algebra of $O(n + 1, n)$ is
\[
o(n + 1, n) = \{ A \in sl(2n + 1, \mathbb{R}) \mid A^t C_n + C_n A = 0 \}
= \{ (A_{ij}) \mid A_{ij} + (-1)^{i-j} A_{2n+2-j, 2n+2-i} = 0, \quad 1 \leq i \leq 2n+1 \}.
\]
Note that $A = (A_{ij}) \in o(n + 1, n)$ if and only if $A_{ij}$’s are symmetric (skew-symmetric resp.) with respect to the skew diagonal line $x + y = 2n + 2$ if $i + j$ is odd (even resp.) and $A_{ij} = 0$ if $i + j = 2n + 2$. Let
\[
\mathcal{G}_i = \text{span}\{ e_{j,j+i} \mid 1 \leq i + j \leq 2n + 1 \} \cap o(n + 1, n).
\]
Then
\[ o(n + 1, n) = d_{i=−2n}^2 \mathcal{G}_i, \quad \mathcal{G}_{−2n} = \mathcal{G}_{2n} = 0, \quad [\mathcal{G}_i; \mathcal{G}_j] \subset \mathcal{G}_{i+j}. \]

**Proposition 2.1.**

(i) The dimension of a maximal isotropic subspace of \(\mathbb{R}^{n+1,n}\) is \(n\).

(ii) The \(O(n + 1, n)\)-action on the space of ordered isotropic bases of \(\mathbb{R}^{n+1,n}\) defined by \(g \cdot (v_1, \ldots, v_n) = (gv_1, \ldots, gv_n)\) is transitive.

**Proof.** (i) Let \(\{e_i, 1 \leq i \leq 2n + 1\}\) denote the standard basis of \(\mathbb{R}^{2n+1}\). Then \(A = \text{span}\{e_1, e_2, \ldots, e_n\}\) is an isotropic space in \(\mathbb{R}^{n+1,n}\).

Let \(V = \text{span}\{v_1, \ldots, v_n\}\) be another \(n\)-dimension isotropic subspace, \(g_1 = (e_1, \ldots, e_n)\), and \(g_2 = (v_1, \ldots, v_n)\). We claim that there exists \(C \in O(n + 1, n)\) such that \(g_2 = Cg_1\). From linear algebra, we can extend \(\{v_1, \ldots, v_n\}\) to a basis \(\{v_1, \ldots, v_n, v_{n+1}, \ldots, v_{2n+1}\}\) in \(\mathbb{R}^{n+1,n}\) and denote \(\tilde{g}_2 = (v_1, \ldots, v_{2n+1}) \in O(n + 1, n)\). Then choose \(C = \tilde{g}_2\).

(ii) Suppose \(B = \text{span}\{w_1, \ldots, w_{n+1}\}\) is an isotropic subspace in \(\mathbb{R}^{n+1,n}\) of dimension \(n+1\). According to (i), there exists \(C \in O(n + 1, n)\), such that \((w_1, \ldots, w_n) = C(e_1, \ldots, e_n)\). Therefore, we may assume \(w_i = e_i, 1 \leq i \leq n\). Then from \(\langle e_i, w_{n+1}\rangle = 0\) for \(1 \leq i \leq n\), we have \(w_{n+1} = 0\), which is a contradiction. This proves (ii). \(\square\)

**Proposition 2.2.** If \(\gamma(s)\) is isotropic in \(\mathbb{R}^{n+1,n}\) for all \(s \in \mathbb{R}\), then there exists an orientation preserving parameter \(x = x(s)\) unique up to translation such that \(\langle \gamma_{x}^{(n)}, \gamma_{x}^{(n)}\rangle = 1\), i.e., \(x\) is the isotropic parameter of \(\gamma\).

**Proof.** Since \(\gamma\) is isotropic,
\[ \langle \gamma_{s}^{(n−1)}, \gamma_{s}^{(i)}\rangle = 0, \quad 0 \leq i \leq n−1. \]

Take the derivative with respect to \(s\) of both sides to get
\[ \langle \gamma_{s}^{(n−1)}, \gamma_{s}^{(i)}\rangle_s = \langle \gamma_{s}^{(n)}, \gamma_{s}^{(i)}\rangle + \langle \gamma_{s}^{(n−1)}, \gamma_{s}^{(i−1)}\rangle = 0 \]

So \(\langle \gamma_{s}^{(n)}, \gamma_{s}^{(i)}\rangle = 0\) for any \(0 \leq i \leq n−1\). Since the span of \(\{\gamma_{s}, \ldots, \gamma_{s}^{(n−1)}\}\) is maximal isotropic, \(\langle \gamma_{s}^{(n)}, \gamma_{s}^{(n)}\rangle \neq 0\).

We claim that \(\langle \gamma_{s}^{(n)}, \gamma_{s}^{(n)}\rangle > 0\) for all \(s \in \mathbb{R}\). To see this, we first note that Proposition 2.1 (ii) implies that there exists \(C \in O(n + 1, n)\) such that
\[ C(\gamma_{s}, \ldots, \gamma_{s}^{(n−1)}) = (e_1, e_2, \ldots, e_n), \]

where \(e_i\) is the \(i\)-th standard basis of \(\mathbb{R}^{2n+1}\). Let \(c = (c_1, c_2, \ldots, c_{2n+1})^t = C\gamma_{s}^{(n)}\). Then we have
\[ \langle C\gamma_{s}^{(n)}, C\gamma_{s}^{(i)}\rangle = \langle c, e_i\rangle = c_{2n+2−i} = \langle \gamma_{s}^{(n)}, \gamma_{s}^{(i)}\rangle = 0 \]

for \(1 \leq i \leq n\). So \(c_{2n+2−i} = 0\) for \(1 \leq i \leq n\). This implies that
\[ \langle \gamma_{s}^{(n)}, \gamma_{s}^{(n)}\rangle = \langle C\gamma_{s}^{(n)}, C\gamma_{s}^{(n)}\rangle = c_i^t C_n c = c_{n+1}. \]

But \(\langle \gamma_{s}^{(n)}, \gamma_{s}^{(n)}\rangle \neq 0\). This proves the claim.

Choose \(x\) such that \(\frac{dx}{ds} = (\gamma_{s}^{(n)}, \gamma_{s}^{(n)})^{1/2n}\) and the proposition follows. \(\square\)
Theorem 2.3. Given \( \gamma \in \mathcal{M}_{n+1,n} \), there exists a unique isotropic moving frame \( g = (p_1, \ldots, p_{2n+1}) : \mathbb{R} \to O(n+1,n) \) along \( \gamma \), i.e., \( p_i = \gamma^{(i-1)}_x \) for \( 1 \leq i \leq n+1 \) and \( g^{-1}g_x = b + \sum_{i=1}^n u_i \beta_i \) for some \( n \) smooth functions \( u_1, \ldots, u_n \), where \( b \) and \( \beta_i \)'s are given in (1.2).

Proof. We claim that \( p_{n+2}, \ldots, p_{2n+1} \) and \( u_1, \ldots, u_n \) can be constructed from recursive formulas. From \( \langle \gamma^{(n-1)}_x, \gamma^{(n)}_x \rangle = 0 \), we have \( \langle \gamma^{(n-1)}_x, \gamma^{(n+1)}_x \rangle = -\langle \gamma^{(n)}_x, \gamma^{(n)}_x \rangle = -1 \).

Let
\[
 u_1 = -\frac{1}{2} \langle \gamma^{(n+1)}_x, \gamma^{(n+1)}_x \rangle, \quad p_{n+2} = \gamma^{(n+1)}_x - u_1 \gamma^{(n-1)}_x. \tag{2.2}
\]

Then
\[
\begin{cases}
\langle \gamma^{(i)}_x, p_{n+2} \rangle = 0, & 0 \leq i \leq n - 2, \\
\langle \gamma^{(n-1)}_x, p_{n+2} \rangle = -1, & \langle \gamma^{(n)}_x, p_{n+2} \rangle = 0, \quad \langle p_{n+2}, p_{n+2} \rangle = 0.
\end{cases}
\]

Since \( \langle p_{n+2}, \gamma^{(n)}_x \rangle = 0 \), \( \langle (p_{n+2})_x, \gamma^{(n)}_x \rangle + \langle p_{n+2}, \gamma^{(n+1)}_x \rangle = 0 \). From (2.2),
\[
\langle (p_{n+2})_x, \gamma^{(n)}_x \rangle = u_1.
\]

On the other hand, \( \langle \gamma^{(n-2)}_x, p_{n+2} \rangle = 0 \) implies that
\[
\langle \gamma^{(n-2)}_x, (p_{n+2})_x \rangle = -\langle \gamma^{(n-1)}_x, p_{n+2} \rangle = 1.
\]

Set
\[
 u_2 = \frac{1}{2} \langle (p_{n+2})_x, (p_{n+2})_x \rangle - \frac{1}{2} u_1^2, \quad p_{n+3} = (p_{n+2})_x - u_2 \gamma^{(n-2)}_x - u_1 \gamma^{(n)}_x. \tag{2.3}
\]

Then
\[
\begin{cases}
\langle \gamma^{(i)}_x, p_{n+3} \rangle = 0, & 0 \leq i \leq n, i \neq n - 2, \\
\langle \gamma^{(n-2)}_x, p_{n+3} \rangle = 1, & \langle p_{n+2}, p_{n+3} \rangle = 0.
\end{cases}
\]

Suppose we have already found \( p_{n+2}, \ldots, p_{n+j} \) and \( u_1, \ldots, u_{j-1} \) for \( j \geq 3 \) satisfying
\[
\begin{cases}
\langle p_i, p_{n+j} \rangle = 0, & 1 \leq i \leq n + j, \quad i \neq n = 2 - j, \\
\langle p_{n+2-j}, p_{n+j} \rangle = (-1)^{j-1}, \\
\langle p_{n-1+j}, x \rangle = p_{n+j} + u_{j-1} p_{n-2+j} + u_{j-2} p_{n-4+j}.
\end{cases}
\]

Set
\[
\begin{cases}
 u_j = (-1)^{j} \langle (p_{n+j})_x, (p_{n+j})_x \rangle, \\
p_{n+j+1} = (p_{n+j})_x - u_j p_{n+1-j} - u_{j-1} p_{n+3-j}.
\end{cases}
\]

It follows from a direct computation that \( \langle p_i, p_{n+j+1} \rangle = (-1)^j \delta_{i,n+1-j} \).

Hence we get \( p_{n+j+1} \) and \( u_j \). The uniqueness of \( g \) and \( u_i \)'s comes from the way they are constructed. \( \square \)

The map \( g, u_i \), and \( u = \sum_{i=1}^n u_i \beta_i \) in Theorem 2.3 are the isotropic moving frame, the \( i \)-th isotropic curvature, and the isotropic curvature along \( \gamma \).

If follows from the Existence and Uniqueness Theorem of ordinary differential equations that we have
Proposition 2.4. Let $\Psi : M_{n+1,n} \to V_n$ be the isotropic curvature map. Then $\Psi$ is onto and $\Psi^{-1}(u)$ is an $O(n+1,n)$-orbit for all $u \in C^\infty(\mathbb{R}, V_n)$.

Example 2.5. Isotropic curves in $\mathbb{R}^{n+1,n}$ with zero isotropic curvatures are of the form

\[
\gamma = C(1, x, \frac{x^2}{2!}, \cdots, \frac{x^{2n-1}}{(2n-1)!}, \frac{x^{2n}}{2n!})^T, \quad C \in O(n+1,n). \tag{2.4}
\]

Example 2.6. The isotropic moving frame $g = (\gamma, \gamma_x, p_3)$ and isotropic curvature $q := u_1$ along $\gamma \in M_{2,1}$ are

\[
q = -\frac{1}{2} \langle \gamma_{xx}, \gamma_{xx} \rangle, \quad p_3 = \gamma_{xx} - q \gamma,
\]

\[
g^{-1}g_x = \begin{pmatrix}
0 & q & 0 \\
1 & 0 & q \\
0 & 1 & 0
\end{pmatrix}.
\]

Example 2.7. For $\gamma \in M_{3,2}$, let $u_1 = -\frac{1}{2} (\gamma_x^{(3)}, \gamma_x^{(2)}, \gamma_x), p_4 = \gamma_x^{(3)} - u_1 \gamma_x, u_2 = \frac{1}{2} ((p_4)_x, (p_4)_x - u_2^2)$, and $p_5 = (p_4)_x - u_2 \gamma - u_1 \gamma_{xx}$. Then $g = (\gamma, \gamma_x, \gamma_{xx}, p_4, p_5)$ is the isotropic moving frame along $\gamma$, i.e.,

\[
g^{-1}g_x = \begin{pmatrix}
0 & 0 & 0 & u_2 & 0 \\
1 & 0 & u_1 & 0 & u_2 \\
0 & 1 & 0 & u_1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix},
\]

and $u_1, u_2$ are the isotropic curvatures of $\gamma$.

Proposition 2.8. Let $\Psi : M_{n+1,n} \to C^\infty(\mathbb{R}, V_n)$ be the isotropic curvature map. Then the differential of $\Psi$ at $\gamma$ is

\[
d\Psi(\delta \gamma) = \delta u = [\partial_x + b + u, g^{-1} \delta g], \tag{2.5}
\]

where $g$, $u$, and $\delta g$ are the isotropic moving frame, isotropic curvature, and the variation of $g$ when we vary $\gamma$ by $\delta \gamma$ respectively.

Proof. It follows from $g^{-1}g_x = b + u$ that we have

\[
\delta u = -g^{-1} \delta gg^{-1}g_x + g^{-1}(\delta g)_x = -g^{-1} \delta g(b + u) + g^{-1}(\delta g)_x,
\]

On the other hand,

\[
(g^{-1} \delta g)_x = -g^{-1}g_xg^{-1} \delta g + g^{-1}(\delta g)_x = -(b + u)g^{-1} \delta g + g^{-1}(\delta g)_x,
\]

Therefore

\[
\delta u = -g^{-1} \delta g(b + u) + (g^{-1} \delta g)_x + (b + u)g^{-1}(\delta g) = [\partial_x + b + u, g^{-1} \delta g].
\]

As a consequence of Propositions 2.4 and 2.8 we have

Corollary 2.9. Given $u, \eta \in C^\infty(\mathbb{R}, V_n)$, then there exists $\xi \in C^\infty(\mathbb{R}, o(n+1,n))$ such that $[\partial_x + b + u, \xi] = \eta$, where $b$ is as in (1.2).
3. THE TANGENT SPACE OF \( \mathcal{M}_{n+1,n} \) AT \( \gamma \)

In this section, we identify the tangent space of \( \mathcal{M}_{n+1,n} \) at \( \gamma \) as \( \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^n) \).

Henceforth in this paper we set

\[
e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^{n+1,n}.
\]

Below we give a useful description of \( T_\gamma \mathcal{M}_{n+1,n} \).

**Proposition 3.1.** Let \( g \) and \( u \) be the isotropic moving frame and isotropic curvature along \( \gamma \in \mathcal{M}_{n+1,n} \). Suppose \( C : \mathbb{R} \to O(n+1,n) \) satisfies

\[
[\partial_x + b + u, C] \in \mathcal{C}^\infty(\mathbb{R}, V_n).
\]

Then \( \xi(\gamma) = gCe_1 \) is tangent to \( \mathcal{M}_{n+1,n} \) at \( \gamma \). Conversely, all tangent vectors of \( \mathcal{M}_{n+1,n} \) at \( \gamma \) arise this way.

**Proof.** It follows from the definition of \( \mathcal{M}_{n+1,n} \) that \( \delta \gamma \) is tangent to \( \mathcal{M}_{n+1,n} \) at \( \gamma \) if and only if

\[
\begin{aligned}
\langle (\delta \gamma)_x^{(i)}, \gamma_x^{(j)} \rangle + \langle \gamma_x^{(i)}, (\delta \gamma)_x^{(j)} \rangle &= 0, \quad 0 \leq i, j \leq n - 1, \\
\langle (\delta \gamma)_x^{(n+1)}, \gamma_x^{(n)} \rangle &= 0.
\end{aligned}
\]

Let \( \eta_j \) denote the \( j \)-th column of \( gC \) for \( 1 \leq j \leq 2n + 1 \). To prove \( gCe_1 \) is tangent to \( \mathcal{M}_{n+1,n} \), it suffices to prove that \( \eta_1 \) satisfies \([3.2]\). Let \( \rho = [\partial_x + b + u, C] \). A direct computation gives

\[
(gCe_1)_x = g_x e_1 + gCe_x = gC(b + u) + gp.
\]

Since the first \( n \) columns of \( \rho \) are zero, the first \( n + 1 \) columns of \( gC \) are related by

\[
\eta_2 = (\eta_1)_x, \ldots, \eta_{n+1} = (\eta_1)_x^{(n)}.
\]

Hence, for \( 0 \leq i, j \leq n - 1 \), we have

\[
\begin{aligned}
\langle (\eta_1)_x^{(i)}, \gamma_x^{(j)} \rangle + \langle \gamma_x^{(i)}, (\eta_1)_x^{(j)} \rangle &= \langle gCe_{i+1}, ge_{j+1} \rangle + \langle ge_{i+1}, gCe_{j+1} \rangle \\
&= \langle C_{e_{i+1}, e_{j+1}} + e_{i+1}, C_{e_{j+1}} \rangle = e_{i+1}^{(j)}(C^tC_e + C_n^tC_n)e_{j+1} \\
&= 0.
\end{aligned}
\]

Since \( C = (C_{ij}) \in o(n+1,n) \), \( \langle (\eta_1)_x^{(n)}, \gamma_x^{(n)} \rangle = C_{n+1,n} = 0 \). So \( \xi(\gamma) = \eta_1 \) is tangent to \( \mathcal{M}_{n+1,n} \) at \( \gamma \).

The converse follows from Proposition \([2.8]\) \qed

Next we prove that if \( C = (C_{ij}) \) satisfying \([3.1]\), then \( C \) is determined by \( \{C_{n+i,n+1-i}, 1 \leq i \leq n\} \) or \( \{C_{2i,1}, 1 \leq i \leq n\} \).

**Theorem 3.2.** Let \( u \in \mathcal{C}^\infty(\mathbb{R}, V_n) \), \( C = (C_{ij}) \in \mathcal{C}^\infty(\mathbb{R}, o(n+1,n)) \) and \( v_i = C_{n+i,n+1-i} \) for \( 1 \leq i \leq n \). Suppose \( [\partial_x + b + u, C] \in V_n \). Then we have the following.

(i) \( C_{ij} \)'s are differential polynomials in \( u, v_1, \ldots, v_n \).

(ii) \( C_{2n-2i,1} = v_{n-i} + \phi_i \) for \( 0 \leq i \leq n - 1 \), where \( \phi_i \) is a differential polynomial in \( u, v_{n+1-i}, \ldots, v_n \).
(iii) There exist differential polynomials $h_{2i+1}$ such that
\[ C_{2i+1,1} = h_{2i+1}(u, C_{2i+2,1}, \ldots, C_{2n,1}), \quad 0 \leq i \leq n - 1. \]

(iv) $C_{ij}$'s are differential polynomials of $u, C_{21}, C_{41}, \ldots, C_{2k,1}, \ldots, C_{2n,1}.$

Proof. Since $C \in o(n + 1, n), C_{n+1,i,n+2-i} = C_{n+i,n+1-i} = v_i$. Let $v = \sum_{i=1}^{n} v_i \beta_i \in V_{n}^t$, where $V_n$ and $\beta_i$'s are defined in (1.3). Let $G_{b}$ be as in (2.1). Then $[G_{i}, G_{j}] = G_{i+j}$. For $\xi \in o(n+1,n)$, let $\xi_{G_{j}}$ denote the $G_{j}$-component of $\xi$ w.r.t. $o(n + 1, n) = \oplus_{i=1}^{n-1} G_{i}$.

Suppose $[\partial_x + b + u, C] = \sum_{i=1}^{n} \eta_i \beta_i$. Write $C = \sum_{i=1}^{2n-1} C_i$ with $C_i \in G_{i}$. Then
\[ C_{j}^{\prime} + [b, C_{j+1}] + [u, C] G_{j} = \begin{cases} \eta_i \beta_i, & j = 2i - 1, \\ 0, & \text{else.} \end{cases} \tag{3.3} \]

We claim that $C_{j}$ are differential polynomials in $v$ and $u$. For $j = 1 - 2n$, we have $C_{2n,1} = C_{2n+1,2} = v_n$. For $j < 0$, if $j$ is even, $ad(b) : G_{j} \rightarrow G_{j-1}$ is a bijection. If $j$ is odd, then $dim(Im(ad(b)(G_{j}))) = dim(G_{j-1}) = dim(G_{j}) - 1$. Then from (3.3), for both cases, entries of $C_{j}$ are differential polynomials in $v_n, \ldots, v_{-1}$. Then by induction, the claim is true for $j < 0$.

Note that $ad(b)$ is a bijection from $G_{0}$ to $G_{-1}$, and we have the $G_{j}$ component $[u, C] G_{j}$ depends only on $u, v_{1}, \cdots, v_{n}$. So $C_{0}$ is a differential polynomial in $u$ and $v$.

For $j > 0$, we see that when $j$ is odd, $ad(b) : G_{j} \rightarrow G_{j-1}$ is again a bijection. When $j$ is even, we have $dim(Im(ad(b)(G_{j}))) = dim(G_{j}) = dim(G_{j-1}) - 1$. Therefore, in both cases, $C_{j}$ can be solved uniquely from $C_{j-1}$ and $\eta_i$'s are differential polynomials in entries of $C_{2i-1}$. By induction, the claim is true for $j > 0$. This proves the statement (i).

To prove (ii), let $j = 2i + 1 - 2n$ in (3.3). Then the linear system implies that $C_{2n-2i,1} = v_{n-i} + \phi_i, 0 \leq i \leq n - 1$, where $\phi_i$ is a differential polynomial in $u, v_{n+1-i}, \cdots, v_{n}$.

Statement (iii) and (iv) are consequence from (i) and (ii).

Corollary 3.3. Let $g = (p_{1}, \ldots, p_{2n+1})$ and $u$ denote the isotropic moving frame and isotropic curvature along $\gamma \in M_{n+1,n}$. Then $\xi = \sum_{i=1}^{2n} \xi_i p_i$ is tangent to $M_{n+1,n}$ at $\gamma$ if and only if
\[ \xi_{2i+1} = h_{2i+1}(u, \xi_{2i+2}, \cdots, \xi_{2n}), \quad 0 \leq i \leq n - 1, \tag{3.4} \]
where $h_{2i+1}$'s are the differential polynomials given in Theorem 3.2. In particular, we identify $T_{\gamma} M_{n+1,n}$ as $C^{\infty}(\mathbb{R}, \mathbb{R}^n)$.

The proof of Theorem 3.2 gives the following.

Corollary 3.4. Let $u \in C^{\infty}(\mathbb{R}, V_{n})$, and $v_{1}, \cdots, v_{n} \in C^{\infty}(\mathbb{R}, \mathbb{R})$. Then there exists a unique $C = (C_{ij}) \subset C^{\infty}(\mathbb{R}, o(n+1,n))$ such that $[\partial_x + b + u, C] \subset V_{n}$ and $C_{n+i,n+1-i} = v_{i}$ for $1 \leq i \leq n$.

Definition 3.5. Given $u \in C^{\infty}(\mathbb{R}, V_{n})$, let
\[ P_u : C^{\infty}(\mathbb{R}, V_{n}^t) \rightarrow C^{\infty}(\mathbb{R}, o(n+1,n)) \]
be the map defined by $P_u(v) = C$ for $v = \sum_{i=1}^n v_i \beta_i^t$, where $C$ is the unique $o(n + 1, n)$-value map defined by $v_1, \ldots, v_n$ in Corollary 3.4.

**Corollary 3.6.** Let $g$ and $u$ be the isotropic moving frame and the isotropic curvature along $\gamma \in M_{n+1,n}$. Then the following statements are equivalent for $C : \mathbb{R} \to o(n + 1, n)$:

1. $gC \xi_1$ is tangent to $M_{n+1,n}$ at $\gamma$.
2. $[\partial_x + b + u, C] \in V_u$.
3. $C = g^{-1} \delta g$, where $\delta g$ is the variation of $g$ when we vary $\gamma$ by $\delta \gamma$.
4. $C = P_u(\pi_0(C))$,

where $\pi_0 : o(n + 1, n) \to V_n^t$ be the projection defined by

$$
\pi_0((y_{ij})) = \sum_{i=1}^n y_{n+i,n+1-i} \beta_i^t, \quad \beta_i^t = e_{n+i,n+1-i} + e_{n+1+i,n+2-i}.
$$ (3.5)

**Corollary 3.7.** Given $\gamma \in M_{n+1,n}$ and $\xi = (\xi_1, \ldots, \xi_{2n}, 0)^t$ satisfying (3.4), then there is a unique $o(n + 1, n)$-valued differential polynomial $\Theta_\gamma(\xi) \in C^\infty(\mathbb{R}, o(n + 1, n))$ such that the $i$-th entry is $\xi_i$ for $1 \leq i \leq 2n + 1$ and $[\partial_x + b + u, \Theta_\gamma(\xi)] \in C^\infty(\mathbb{R}, V_n)$. Moreover, $\Theta_\gamma(\xi) = P_u(\pi_0(\Theta_\gamma(\xi)))$.

Below we give some examples of $T(M_{n+1,n})_\gamma$ when $n$ is small.

**Example 3.8.** $T_\gamma M_{2,1}$

When $n = 1$, $b = e_21 + e_32$, $\beta_1 = e_12 + e_23$, $V_1 = \mathbb{R} \beta_1$, $u = q \beta_1$. A direct computation implies that $[\partial_x + b + u, C] \in V_1$ if and only if

$$
C = \begin{pmatrix}
-\xi_x & -\xi_{xx} + q \xi & 0 \\
\xi & 0 & -\xi_{xx} + q \xi \\
0 & \xi & \xi
\end{pmatrix}.
$$

Moreover, $[\partial_x + b + u, C] = (-\xi_x^{(3)} + 2q \xi_x + (q)_{xx}) \beta_1$ and

$$
T(M_{2,1})_\gamma = \{-\xi_x \gamma + \xi \gamma_x | \xi \in C^\infty(\mathbb{R}, \mathbb{R})\}.
$$

**Example 3.9.** $T_\gamma M_{3,2}$

Here $b = \sum_{i=1}^4 e_{i+1,i}$, $\beta_1 = e_23 + e_34$, $\beta_2 = e_14 + e_25$, $V_2 = \mathbb{R} \beta_1 \oplus \mathbb{R} \beta_2$, and $u = u_1 \beta_1 + u_2 \beta_2$. Then $C \in C^\infty(\mathbb{R}, o(3,2))$ satisfying $[\partial_x + b + u, C] \in V_2$ if and only if

$$
C = \begin{pmatrix}
\eta_x^{(3)} - 2\xi_x - (u_1)_{x}\eta & * & * & 0 \\
\xi & a & * & 0 \\
-\eta_x & \zeta & 0 & * \\
0 & \eta & \zeta & -a \\
0 & \eta & (\eta)_x & \xi - \eta_x^{(3)} + 2\xi_x + (u_1)_{x}\eta
\end{pmatrix}
$$

for some real valued functions $\xi, \eta$, where $a = \eta_x^{(3)} - \xi_x - (u_1)_{x}\eta$ and $\zeta = \xi - \eta_{xx} + u_1 \eta$. So

$$
T(M_{3,2})_\gamma = \{(\eta_x^{(3)} - 2\xi_x - (u_1)_{x}\eta) \gamma + \xi \gamma_x - \eta_x \gamma_{xx} + \eta p_4 | \xi, \eta \in C^\infty(\mathbb{R}, \mathbb{R})\},
$$
where \( g = (\gamma, \gamma_x, \gamma_{xx}, p_4, p_5) \in O(3, 2) \) is the isotropic moving frame along \( \gamma \).

In particular, we see that \( X_1(\gamma) = -\frac{1}{2}u_1\gamma_x + p_4 \) and \( X_2(\gamma) = -\frac{3}{5}(u_1)_x\gamma - \frac{1}{5}u_1\gamma_x + p_4 \) are tangent vector fields of \( \mathcal{M}_{3,2} \). Hence

\[
\gamma_t = -\frac{1}{2}u_1\gamma_x + p_4, \tag{3.6}
\]

\[
\gamma_t = -\frac{3}{5}(u_1)_x\gamma - \frac{1}{5}u_1\gamma_x + p_4, \tag{3.7}
\]

are isotropic curve flows on \( \mathcal{M}_{3,2} \). By Example 2.7, we have \( p_4 = \gamma^{(3)}_x - u_1\gamma_x \).

So (3.6) and (3.7) are (1.8) and (1.9) with \( n = 2 \) respectively.

**Example 3.10.** \( \mathcal{M}_{n+1,n}, (n \geq 3) \)

For \( \gamma \in \mathcal{M}_{n+1,n} \), a direct computation implies that

\[
X(\gamma) = -(n\xi_x + 3(u_1)_x)\gamma + \xi\gamma_x + \gamma^{(3)}_x
\]

is tangent to \( \mathcal{M}_{n+1,n} \) at \( \gamma \). So

\[
\gamma_t = -(n\xi_x + 3(u_1)_x)\gamma + \xi\gamma_x + \gamma^{(3)}_x \tag{3.8}
\]

is an isotropic curve flow on \( \mathcal{M}_{n+1,n} \). For example, if we choose \( \xi \) to be \( -\frac{3}{n}u_1 \) and \( -\frac{6}{2n+1}u_1 \) then we get (1.8) and (1.9) given in the introduction respectively.

4. **The \( \hat{B}_{2n}^{(1)} \)-KdV and \( \hat{A}_{2n}^{(2)} \)-KdV hierarchies**

Drinfeld-Sokolov associated to each affine Kac-Moody algebra a KdV-type soliton hierarchy (cf. [4]). These hierarchies are constructed as quotient flows for some gauge group actions. Different cross sections of the gauge group action give different but equivalent hierarchies. In this paper, we construct a suitable cross section for the gauge action so that the differential invariants for the isotropic curves lies in this cross section. We also prove that the \( \hat{A}_{2n}^{(2)} \)-KdV hierarchy is the constraint KP hierarchy in [2].

Let \( \mathbb{C}^{n+1,n} \) be the vector space of \( \mathbb{C}^{2n+1} \) equipped with the bilinear form \( \langle X, Y \rangle = X^tC_nY \), where \( C_n \) is as in (1.1). Let \( O_{\mathbb{C}}(n+1,n) \) be the Lie group preserving \( \langle , \rangle \) on \( \mathbb{C}^{n+1,n} \), i.e.

\[
O_{\mathbb{C}}(n+1,n) = \{ A \in SL(2n+1, \mathbb{C}) \mid A^tC_nA = C_n \}. \tag{4.1}
\]

Its Lie algebra is \( o_{\mathbb{C}}(n+1,n) = \{ A \in sl(2n+1, \mathbb{C}) \mid A^tC_n + C_nA = 0 \} \).

Let \( B_n^+, N_n^+, T_n \) denote the subalgebras of upper triangular, strictly upper triangular, and diagonal matrices in \( o(n+1,n) \) respectively. Let \( B_n^+ \) and \( N_n^+ \) be the connected subgroups of \( O(n+1,n) \) with Lie algebras \( B_n^+ \) and \( N_n^+ \).
First we review the construction of the $\hat{B}_n^{(1)}$-KdV hierarchy. Set
\[
\hat{B}_n^{(1)} = \left\{ \xi(\lambda) = \sum_{i \leq m_0} \xi_i \lambda^i \mid \xi_i \in \mathcal{O}_n(n+1,n), \xi(\lambda) = \xi_0(\lambda), m_0 \in \mathbb{Z} \right\},
\]
\[
(\hat{B}_n^{(1)})_+ = \left\{ \xi(\lambda) = \sum_{i \geq 0} \xi_i \lambda^i \in \hat{B}_n^{(1)} \right\},
(\hat{B}_n^{(1)})_- = \left\{ \xi(\lambda) = \sum_{i < 0} \xi_i \lambda^i \in \hat{B}_n^{(1)} \right\}.
\]
Note that

(i) $\xi(\lambda) = \sum_i \xi_i \lambda^i \in \hat{B}_n^{(1)}$ if and only if $\xi_i \in \mathcal{O}_n(n+1,n)$ for all $i$,

(ii) $\hat{B}_n^{(1)} = (\hat{B}_n^{(1)})_+ \oplus (\hat{B}_n^{(1)})_-$ is a direct sum of linear subspaces.

Let
\[
\beta = \frac{1}{2}(e_{1,2n} + e_{2,2n+1})
\]  
and
\[
J_B = \sum_{i=1}^{2n} e_{i+1,i} + \frac{1}{2} \lambda(e_{1,2n} + e_{2,2n+1}) = b + \beta \lambda.
\]  
Note that $J_B^{2j} \not\in \hat{B}_n^{(1)}$ and
\[
J_B^{2j-1}(j \geq 1) \in (\hat{B}_n^{(1)})_+, \quad J_B^{2n+1} = \lambda J_B.
\]

So we can use the splitting $\hat{B}_n^{(1)} = (\hat{B}_n^{(1)})_+ \oplus (\hat{B}_n^{(1)})_-$ and the commuting sequence $\{J_B^{2j-1} \mid j \geq 1\}$ in $(\hat{B}_n^{(1)})_+$ to construct a hierarchy of soliton equations (cf. [9]). A direct computation implies that given $q \in C^\infty(\mathbb{R}, \mathcal{B}_n^\pm)$, there exists a unique $P(q, \lambda)$ conjugate to $J_B$ and satisfying
\[
[\partial_x + J_B + q, P(q, \lambda)] = 0,
\]

Expand $P^{2j-1}(q, \lambda)$ as a power series in $\lambda$,
\[
P^{2j-1}(q, \lambda) = \sum_{i \leq \left[\frac{2j-1}{2}\right]+1} P_{2j-1,i}(q) \lambda^i.
\]

It can be checked that all $P_{2j-1,i}(q)$’s are differential polynomials of $q$.

The $(2j-1)$-th flow $(j \geq 1)$ on $C^\infty(\mathbb{R}, \mathcal{B}_n^\pm)$ is
\[
q_{t_{2j-1}} = [\partial_x + b + q, P_{2j-1,0}(q)].
\]

These flows commute. Moreover, $q$ is a solution of $[4.7]$ if and only if
\[
[\partial_x + J_B + q, \partial_{t_{2j-1}} + (P^{2j-1}(q, \lambda))_+] = 0
\]

for any $\lambda \in \mathbb{C}$. Here $\xi_+$ is the projection of $\hat{B}_n^{(1)}$ onto $(\hat{B}_n^{(1)})_+$ along $(\hat{B}_n^{(1)})_-$. 

**Definition 4.1.** We call $F(x, t, \lambda) \in O_{\mathcal{C}}(n+1, n)$ a frame of the solution $q$ of $[4.7]$ if $F$ is holomorphic for $\lambda \in \mathbb{C}$ and satisfies
\[
F^{-1} F_x = J_B + q, \quad F^{-1} F_t = (P^{2j-1}(q, \lambda))_+, \quad F(x, t, \lambda) = F(x, t, \lambda).
\]
The group $C^\infty(\mathbb{R}, N_n^+)$ acts on $C^\infty(\mathbb{R}, \mathcal{B}_n^+)$ by gauge transformations, 
\[
\partial_x + J_B + \triangle * q = \triangle(\partial_x + J_B + q)\triangle^{-1},
\] (4.8)
or equivalently,
\[
\triangle * q = \triangle(J_B + q)\triangle^{-1} - \partial_x\triangle^{-1} - J_B \in \mathcal{B}_n^+.
\] (4.9)

A direct computation similar to the one given in [9] implies the following.

**Proposition 4.2.**

(i) Let $q \in C^\infty(\mathbb{R}, \mathcal{B}_n^+)$, and $\triangle \in C^\infty(\mathbb{R}, N_n^+)$. Then
\[
\triangle P(q, \lambda)\triangle^{-1} = P(\triangle * q, \lambda),
\]
where $P(q, \lambda)$ is defined by (4.5).

(ii) Let $q$ be a solution of (4.1), $\triangle \in C^\infty(\mathbb{R}, N_n^+)$, and $\tilde{q}(. , t) = \triangle * q(. , t)$. Then $\tilde{q}$ is again a solution of (4.1).

So (4.1) is invariant under the action of $C^\infty(\mathbb{R}, N_n^+)$ and it induces a quotient flow on the orbit space $\frac{C^\infty(\mathbb{R}, \mathcal{B}_n^+)}{C^\infty(\mathbb{R}, N_n^+)}$. The next Proposition, which can be proved by a direct computation, shows that $C^\infty(\mathbb{R}, V_n)$ is a cross section of this gauge action.

**Proposition 4.3.** Given $q \in C^\infty(\mathbb{R}, \mathcal{B}_n^+)$, then there exist a unique $\triangle \in C^\infty(\mathbb{R}, N_n^+)$ and $u \in C^\infty(\mathbb{R}, V_n)$ such that
\[
\triangle(\partial_x + J_B + q)\triangle^{-1} = \partial_x + J_B + u,
\] (4.10)
where $V_n$ is as in (1.3) and $J_B$ is given by (4.3). Moreover, entries of $\triangle$ and $u$ are differential polynomials of $q$.

**Definition 4.4.** Let $\Gamma : C^\infty(\mathbb{R}, \mathcal{B}_n^+) \to C^\infty(\mathbb{R}, V_n)$ and $D : C^\infty(\mathbb{R}, \mathcal{B}_n^+) \to C^\infty(\mathbb{R}, N_n^+)$ be the maps defined by $\Gamma(q) = u$ and $D(q) = \triangle$, where $q, u$, and $\triangle$ are related by (1.10) as in Proposition 4.3.

**Corollary 4.5.** (4.1) Let $u \in C^\infty(\mathbb{R}, V_n)$, $j \ge 1$, and $P_{2j-1,0}(u)$ be defined by (4.9). Then there exists a unique differential polynomial $\eta_j \in C^\infty(\mathbb{R}, N_n^+)$ of $u \in C^\infty(\mathbb{R}, V_n)$ such that
\[
[\partial_x + b + u, P_{2j-1,0}(u) - \eta_j(u)] \in C^\infty(\mathbb{R}, V_n).
\] (4.11)

The $(2j - 1)$-th flow (4.1) induces a quotient flow on the cross section $C^\infty(\mathbb{R}, V_n)$ by projecting the solutions to the cross section along orbits. This quotient flow is the $(2j - 1)$-th $\hat{\mathcal{B}}_n^{(1)}$-KdV flow,
\[
ut_{2j-1} = [\partial_x + b + u, P_{2j-1,0}(u) - \eta_j(u)]
\] (4.12)

**Corollary 4.6.** Let $\pi_0 : o(n + 1, n) \to V_n^t$ be the projection defined by (3.5), and $P_u$ the operator defined in Definition 3.3. Then

(i) $P_{2j-1,0}(u) - \eta_j(u) = P_u(\pi_0(P_{2j-1,0}(u))), \pi_0(P_{2j-1,0}(u) - \eta_j(u)) = \pi_0(P_{2j-1,0}(u))$. 

(ii) the $(2j - 1)$-th $\hat{B}_n^{(1)}$-KdV flow (4.12) can be written as

\[ u_{t_{2j-1}} = [\partial_x + b + u, P_u(\pi_0(P_{2j-1,0}(u)))] \tag{4.13} \]

where $P_{2j-1,0}(u)$ is given in (4.10).

Proof. Since $\eta_j(u)$ is strictly upper triangular, we have

\[
\pi_0(P_{2j-1,0}(u) - \eta_j(u)) = \pi_0(P_{2j-1,0}(u)).
\]

Then (i) and (ii) follows from (4.11) and Corollary 3.6. □

It follows from the construction of the $\hat{B}_n^{(1)}$-KdV flows that we have the following.

Proposition 4.7. Let $u, P(u, \lambda), P_{2j-1,0}(u)$, and $\eta_j(u)$ be the differential polynomials given in Corollary 4.5. Then the following statements are equivalent:

(i) $u$ is a solution of the $(2j - 1)$-th $\hat{B}_n^{(1)}$-KdV flow (4.12).

(ii) $[\partial_x + b + u, \partial_{t_{2j-1}} + P_{2j-1,0}(u) - \eta_j(u)] = 0$.

(iii) $[\partial_x + J_B + u, \partial_{t_{2j-1}} + (P^{2j-1}(u, \lambda))_+ - \eta_j(u)] = 0$ for all parameters $\lambda \in \mathbb{C}$ (this is the Lax pair for (4.12)).

(iv) The linear system

\[
\begin{cases}
g^{-1}g_x = b + u, \\
g^{-1}g_t = P_{2j-1,0}(u) - \eta_j(u),
\end{cases}
\tag{4.14}
\]

is solvable for $g : \mathbb{R}^2 \to O(n + 1, n)$.

(v) For $\lambda \in \mathbb{C}$, the linear system

\[
\begin{cases}
E^{-1}E_x = J_B + u, \\
E^{-1}E_t = (P^{2j-1}(u, \lambda))_+ - \eta_j(u),
\end{cases}
\tag{4.15}
\]

is solvable for $E(\cdot, \cdot, \lambda) : \mathbb{R}^2 \to O_C(n + 1, n)$.

Definition 4.8. We call $E(x, t, \lambda) \in O_C(n + 1, n)$ a frame of the solution $u$ of (4.12) if $E(x, t, \lambda)$ is holomorphic for $\lambda \in \mathbb{C}$ and is a solution of (4.15) satisfying

\[
(E(x, t, \lambda)) = E(x, t, \lambda).
\]

The next two Propositions follow from the constructions of flows (4.7) and (4.12).

Proposition 4.9. Let $q : \mathbb{R}^2 \to B^+_n$ be a solution of (4.7), and $\Delta(\cdot, t) = D(q(\cdot, t))$, where $D$ is the operator defined in Definition 4.4. Then $u = \Delta \ast q$ is a solution of the $(2j - 1)$-th $\hat{B}_n^{(1)}$-KdV flow (4.12), where the action $\ast$ is defined by (4.9). Moreover, if $F$ is a frame of the solution $q$ of (4.7), then $E = F \Delta^{-1}$ is a frame of the solution $u$ of (4.12).
Proposition 4.10. Let $u$ be a solution of (4.12). Suppose $\triangle : \mathbb{R}^2 \rightarrow N_n^+$ satisfying $\Delta_t\triangle^{-1} = \eta_j(u)$. Then $q(\cdot, t) = \Delta(\cdot, t)^{-1} * u(\cdot, t)$ is a solution of (4.7), where the action $*$ is defined by (4.9). Moreover, if $E$ is a frame of the solution $u$ of (4.12), then $F = E\triangle$ is a frame for the solution $q$ of (4.7).

Note that if $\triangle_t(x, t)\Delta^{-1}(x, t) = \eta_j(u)$ and $f(x) \in C^\infty(\mathbb{R}, N_n^+)$, then $\tilde{\Delta}(x, t) = \Delta(x, t)f(x)$ also satisfies $\tilde{\Delta}_t\tilde{\Delta}^{-1} = \eta_j(u)$ and $\tilde{q} = \tilde{\Delta}^{-1} * u = f * q$ is again a solution of (4.7).

Next we use (4.10) and (4.11) to write down the $(2j-1)$-th $\tilde{B}_n^{(1)}$-KdV flows for small $n$ and $j$.

Example 4.11. The $\tilde{B}_1^{(1)}$-KdV hierarchy

In this case, $J_B = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix}$ and $u = \begin{pmatrix} 0 & q & 0 \\ 0 & 0 & q \\ 0 & 0 & 0 \end{pmatrix}$. A direct computation implies that

$$P_{3,0}(u) = \begin{pmatrix} q_x & 0 & 0 \\ -q & 0 & 0 \\ 0 & -q & -q_x \end{pmatrix}, \quad \eta_2(u) = \begin{pmatrix} 0 & q^2 - q_{xx} & 0 \\ 0 & 0 & q^2 - q_{xx} \\ 0 & 0 & 0 \end{pmatrix},$$

$$\left(P^2(u, \lambda)\right)_+ - \eta_2(u) = J_B^3 + \begin{pmatrix} q_x & q_{xx} - q^2 & 0 \\ -q & 0 & q_{xx} - q^2 \\ 0 & -q & -q_x \end{pmatrix}.$$ 

The third $\tilde{B}_1^{(1)}$-KdV flow is the KdV $q_t = q_{xxx} - 3qq_x$. Since $sl(2, \mathbb{R})$ is isomorphic to $o(2, 1)$, the algebra $\tilde{B}_1^{(1)}$ is isomorphic to $\tilde{A}_1^{(1)}$. So the $\tilde{B}_1^{(1)}$-KdV hierarchy is the KdV hierarchy under this isomorphism.

Example 4.12. The $\tilde{B}_2^{(1)}$-KdV hierarchy

We have $J_B = \sum_{i=1}^4 e_{i+1, i} + \frac{\lambda}{2}(e_{14} + e_{25})$ and $u = u_1(e_{23} + e_{34}) + u_2(e_{14} + e_{25})$. Then

$$P_{3,0}(u) = \begin{pmatrix} 0 & u_2 & u_2' \\ -\frac{1}{2}u_1 & -\frac{1}{2}u_1' & \frac{1}{2}(u_1^2 - u_1') + 2u_2 \\ 0 & \frac{1}{2}u_1 & \frac{1}{2}(u_1^2 - u_1') + 2u_2 \\ 1 & 0 & \frac{1}{2}u_1' \\ 0 & 1 & 0 \end{pmatrix},$$

and

$$P_{3,0}(u) - \eta_2(u) = \begin{pmatrix} 0 & u_2 & u_2' \\ -\frac{1}{2}u_1 & -\frac{1}{2}u_1' & \frac{1}{2}(u_1^2 - u_1') + 2u_2 \\ 0 & \frac{1}{2}u_1 & \frac{1}{2}(u_1^2 - u_1') + 2u_2 \\ 1 & 0 & \frac{1}{2}u_1' \\ 0 & 1 & 0 \end{pmatrix}.$$
So the third flow is
\[
\begin{aligned}
(u_1)_t &= -\frac{1}{2}u_1^{(3)} + \frac{3}{2}u_1(u_1)_x + 3(u_2)_x, \\
(u_2)_t &= u_2^{(3)} - \frac{3}{2}u_1(u_2)_x.
\end{aligned}
\]

**Example 4.13.** Although we do not have the explicit formula for \( P_{3,0}(u) \) for general \( n \geq 3 \), a direct computation implies that the first column of \( P_{3,0}(u) \) is \((0, -\frac{3}{n}u_1, 0, 1, 0, \ldots, 0)^t \).

Next we construct the \( \hat{A}_{2n}^{(2)} \)-KdV hierarchy. Let
\[
\hat{A}_{2n}^{(1)} = \left\{ \xi(\lambda) = \sum_{i \leq n_0} \xi_i \lambda^i \mid \xi_i \in \mathfrak{sl}(2n + 1, \mathbb{C}), \xi(\lambda) = \xi(\lambda) \right\},
\]
\[
(\hat{A}_{2n}^{(1)})_+ = \left\{ \sum_{i \geq 0} \xi_i \lambda^i \in \hat{A}_{2n}^{(1)} \right\}, \quad (\hat{A}_{2n}^{(1)})_- = \left\{ \sum_{i < 0} \xi_i \lambda^i \in \hat{A}_{2n}^{(1)} \right\}.
\]
Let \( C_n \) be as in (1.1). Set
\[
\hat{A}_{2n}^{(2)} = \{ \xi(\lambda) \in \hat{A}_{2n}^{(1)} \mid C_n \xi^t(-\lambda)C_n + \xi(\lambda) = 0 \},
\]
\[
(\hat{A}_{2n}^{(2)})_+ = \hat{A}_{2n}^{(2)} \cap (\hat{A}_{2n}^{(1)})_+.
\]
Then \( \hat{A}_{2n}^{(2)} = (\hat{A}_{2n}^{(2)})_+ \oplus (\hat{A}_{2n}^{(2)})_- \). Note that \( \xi(\lambda) = \sum_{i \leq n_0} \xi_i \lambda^i \in \hat{A}_{2n}^{(2)} \) if and only if \( \xi_{2i} \in o_\mathbb{C}(n + 1, n) \) and \( C_n \xi_{2i+1} \xi_{2i+1} = \xi_{2i+1} \).

Set
\[
J = \left( \sum_{i=1}^{2n} e_{i+1,i} \right) + e_{1,2n+1} \lambda = b + e_{1,2n+1} \lambda.
\]
Then \( J^{2n+1} = \lambda e_{2n+1,1}, J^2 \not\in \hat{A}_{2n}^{(2)}, \) and \( J^{2j-1} \in (\hat{A}_{2n}^{(2)})_+ \). So the splitting of \( \hat{A}_{2n}^{(2)} \) and \( J^{2j-1} \) produce \((2j - 1)\)-th flow with \((2j - 1) \not\equiv 0 \pmod{2n+1}\).

A direct computation implies that given \( q \in C^\infty(\mathbb{R}, \mathcal{B}_n^+) \), there exists a unique \( S(q, \lambda) \in \hat{A}_{2n}^{(2)} \) such that
\[
\begin{aligned}
[\partial_x + J + q, S(q, \lambda)] &= 0, \\
S(q, \lambda) &= \text{conjugate to } J.
\end{aligned}
\]
Write \( S(u, \lambda)^{2j-1} \) as a power series of \( \lambda \),
\[
S^{2j-1}(q, \lambda) = \sum_{i \leq [\frac{2j-1}{2n+1}]+1} S_{2j-1,i}(q) \lambda^i.
\]
Then coefficients \( S_{2j-1,i}(q) \)'s are differential polynomials in \( q \).

Assume \( j \geq 0 \) and \((2j - 1) \not\equiv 0(\pmod{2n+1})\). Then
\[
q_{t_{2j-1}} = [\partial_x + b + q, S_{2j-1,0}(q)],
\]
is the flow on \( C^\infty(\mathbb{R}, \mathcal{B}_n^+) \) constructed from the splitting \((\hat{A}_{2n}^{(2)})_\pm \) and \( J^{2j-1} \).
These flows commute.
Again the group $C^\infty(\mathbb{R}, N^+_n)$ acts on $\partial + J + C^\infty(\mathbb{R}, B^+_n)$ by gauge transformations with $C^\infty(\mathbb{R}, V_n)$ a cross section, and (4.19) is invariant under this action. Hence flow (4.19) induces a quotient flow on $C^\infty(\mathbb{R}, V_n)$. We call this quotient flow the $(2j - 1)$-th $A^{(2j)}_{2n}$-KdV flow. In fact, given $u \in C^\infty(\mathbb{R}, V_n)$, and $(2j - 1) \not\equiv 0 (\mod(2n + 1))$, there exists a unique differential polynomial $\tilde{\eta}_j(u) \in C^\infty(\mathbb{R}, N^+_n)$ such that
\[
[\partial_x + J + u, S_{2j-1,0}(u) - \tilde{\eta}_j(u)] \in C^\infty(\mathbb{R}, V_n),
\] (4.20)
where $S_{2j-1,0}(u)$ is defined by (4.18).

The $(2j - 1)$-th $A^{(2j)}_{2n}$-KdV flow is
\[
u_{2j-1} = [\partial_x + b + u, S_{2j-1}(u) - \tilde{\eta}_j(u)],
\] (4.21)
where $S_{2j-1,0}(u)$ is defined as by (4.18) and $\tilde{\eta}_j(u)$ is as in (4.20).

Note that the $(2j - 1)$-th $A^{(2j)}_{2n}$-KdV flow can be written as
\[
u_{2j-1} = [\partial_x + b + u, P_u(\pi_0(S_{2j-1,0}(u))],
\] (4.22)
where $P_u$ is as defined in Definition 3.5 and $\pi_0$ is the projection as in (3.5).

**Proposition 4.14.**

The following statements are equivalent for $u \in C^\infty(\mathbb{R}^2, V_n)$:

1. $u$ is a solution of (4.21).

2. \[
[\partial_x + J + u, \partial_{2j-1} + (S^{2j-1}(u, \lambda))_+ - \tilde{\eta}_j(u)] = 0,
\] (4.23)
for all parameter $\lambda \in \mathbb{C}$.

3. \[
[\partial_x + b + u, \partial_{2j-1} + S_{2j-1,0}(u) - \tilde{\eta}_j(u)] = 0,
\] (4.24)

4. The linear system
\[
\begin{align*}
g^{-1}g_t &= b + u, \\
g^{-1}g_t &= S_{2j-1,0}(u) - \tilde{\eta}_j(u),
\end{align*}
\] (4.25)
is solvable for $g \in C^\infty(\mathbb{R}^2, O(n + 1, n))$.

5. The linear system
\[
\begin{align*}
E^{-1}E_x &= J + u, \\
E^{-1}E_t &= (S^{2j-1}(u, \lambda))_+ - \tilde{\eta}_j(u),
\end{align*}
\] (4.26)
is solvable for $E(\cdot, \cdot, \lambda) \in C^\infty(\mathbb{R}^2, O_C(n + 1, n))$ for all parameter $\lambda \in \mathbb{C}$.

We call a solution $E(x, t, \lambda)$ of (4.26) a frame of the solution $u$ of the $(2j - 1)$-th $A^{(2j)}_{2n}$-KdV flow if $E(x, t, \lambda)$ is holomorphic for $\lambda \in \mathbb{C}$ and satisfies
\[E(x, t, \lambda)^{-1} = C_n E^t(x, t, -\lambda)C_n, \quad \overline{E(x, t, \lambda)} = E(x, t, \lambda).\]

Similarly, we have the following.
Proposition 4.15. If \( q : \mathbb{R}^2 \to \mathbb{R}_n^+ \) is a solution of \((4.19)\), then \( u = \Delta \ast q \) is a solution of the \((2j - 1)\)-th \( \hat{A}_2^{(2)} \)-KdV flow \((4.21)\), where \( \Delta(\cdot, t) = D(q(\cdot, t)) \) and \( D \) is the operator defined in Definition 4.4. Moreover, if \( F \) is a frame of the solution \( q \) of \((4.19)\), then \( E = F \Delta^{-1} \) is a frame of the solution \( u \) of \((4.21)\).

Proposition 4.16. Let \( u \) be a solution of \((4.21)\), and \( \Delta : \mathbb{R}^2 \to \mathbb{R}_n^+ \) satisfying \( \Delta_t \Delta^{-1} = \tilde{\eta}_j(u) \). Then \( \eta(\cdot, t) = \Delta(\cdot, t)^{-1} \ast u(\cdot, t) \) is a solution of \((4.19)\). Moreover, if \( E \) is a frame of the solution \( u \) of \((4.21)\), then \( F = E \Delta \) is a frame for the solution \( q \) of \((4.19)\).

If \( \Delta_t \Delta^{-1} = \tilde{\eta}_j(u) \) and \( f \in C^\infty(\mathbb{R}, \mathbb{R}_n^+) \), then \( \Delta_1(x, t) = \Delta(x, t)f(x) \) also satisfies \( (\Delta_1)_t \Delta_1^{-1} = \tilde{\eta}_j(u) \) and \( q_1 = \Delta_1 \ast u \) is also a solution of \((4.19)\).

Below we write down \( S_{2j - 1, 0}(u) \) and \( \tilde{\eta}_j(u) \) for small \( n \) and \( j \).

Example 4.17. The \( \hat{A}_2^{(2)} \)-KdV hierarchy

Here \( u = q(e_{12} + e_{23}) \), and

\[
S_{5, 0}(u) = \begin{pmatrix}
-\frac{1}{5}(q_{xxx} - 8qq_x) & a & 0 \\
\frac{1}{5}(q_{xx} - 4q^2) & 0 & a \\
0 & \frac{1}{5}(q_{xx} - 4q^2) & \frac{1}{5}(q_{xxx} - 8qq_x)
\end{pmatrix},
\]

\[
\tilde{\eta}_3(u) = \begin{pmatrix}
0 & \theta & 0 \\
0 & 0 & \theta \\
0 & 0 & 0
\end{pmatrix},
\]

where
\[
a = -\frac{1}{27} q^{(4)} - \frac{7}{27} q q_{xx} + \frac{1}{3} q_x^2 + \frac{4}{81} q^3,
\]
\[
\theta = \frac{1}{27} (2q^{(4)} - 34qq_{xx} - 15q_x^2 + \frac{40}{3} q^3).
\]

Therefore,
\[
(S^5(u, \lambda))_+ - \tilde{\eta}_3(u)
\]
\[
= J^5 + \begin{pmatrix}
-\frac{9}{5} \lambda - \frac{1}{5}(q^{(3)} - 8qq_x) & -\frac{9}{5} \lambda + \zeta & -\frac{1}{5}(2qq_x - 5q^2) \lambda \\
\frac{12}{5} q x - 4q^2) & \frac{12}{5} q x - 4q^2) & \frac{12}{5} q x - 4q^2) \\
0 & \frac{9}{5} q x - 4q^2) & -\frac{9}{5} q x - 4q^2) + \frac{9}{5}(q^{(3)} - 8qq_x) \lambda
\end{pmatrix},
\]

where \( \zeta = -\frac{1}{5}(q^{(4)} - 9qq_x - 8q_x^2 + 4q^3) \). It follows that the fifth \( \hat{A}_2^{(2)} \)-KdV flow is the KK equation \((1.7)\) given in \([5]\) and \([6]\).

Example 4.18. The \( \hat{A}_4^{(2)} \)-KdV hierarchy

We have \( u = u_1(e_{23} + e_{35}) + u_2(e_{14} + e_{25}) \), the third \( \hat{A}_4^{(2)} \)-KdV flow is
\[
\begin{align*}
(u_1)_t &= -2(u_1)^{(3)} + \frac{7}{5} u_2' + \frac{12}{5} u_1 u_1', \\
(u_2)_t &= -\frac{5}{4} u_2^{(3)} - \frac{3}{5} (u_1^{(5)} - 6u_1 u_1^{(3)}) + 2u_1 u_2' - 3u_1 u_2).
\end{align*}
\]
\[(4.27)\]
And
\[
(S^3(u, \lambda))_+ - \tilde{\eta}_2(u) = \frac{3}{5}(u)_x + \frac{4}{5}(u)_x + \cdots + \eta^2(u) = J^3 + \begin{pmatrix}
\frac{3}{5}(u)_x & * & * & * \\
\frac{4}{5}(u)_x & \frac{3}{5}(u)_x & 0 & * \\
0 & \frac{3}{5}(u)_x & \frac{4}{5}(u)_x & -\frac{1}{5}(u)_x \\
0 & 0 & * & \frac{3}{5}(u)_x \\
\end{pmatrix}.
\]

Recall that the \(j\)-th Gelfand-Dickey (GD\(_n\)) flow is
\[
L_{t_j} = [(L^n_+)_+, L] \tag{4.28}
\]
for \(L = \partial^n + \sum_{i=1}^{n-1} v_i \partial^{i-1}\), where \((L^n_+)_+\) is the differential operator component of the pseudo differential operator \(L^n_+\) (cf. [3]). Next we show that the \(\hat{A}^{(2)}_{2n}\)-KdV hierarchy is a reduction of the GD\(_n\) hierarchy (cf. [3]).

Let \(L^\ast\) be the formal adjoint of \(L\) under the \(L^2\)-norm. For example,
\[
\partial^\ast = -\partial, \quad f^\ast = f, \quad f \in C^\infty(\mathbb{R}, \mathbb{R}). \tag{4.29a}
\]

It can be checked that
\[
D^a_{2n+1} = \{ L = \partial^{2n+1} + \sum_{i=1}^{2n} v_i \partial^{i-1} \mid L^\ast = -L \}, \tag{4.30}
\]
is invariant under the \((2j - 1)\)-th GD\(_{2n+1}\) flow, and the induced constraint flow is called the \((2j - 1)\)-th constraint KP flow (cf. [3]). Note that \(D^a_{2n+1}\) is equal to
\[
\{ L = \partial^{2n+1} - \sum_{i=1}^{n} (\partial^{n+1-i} u_i \partial^{n-i} + \partial^{n-i} u_i \partial^{n+1-i}) \mid u_i \in C^\infty(\mathbb{R}, \mathbb{R}) \}. \tag{4.31}
\]

Consider the matrix eigenvalue problem,
\[
(\partial_x + J + u) y = 0, \tag{4.32}
\]
where \(J\) is as \((4.16)\), \(u = \sum_{i=1}^{n} u_i \beta_i\) and \(y = (y_1, \cdots, y_{2n+1})^t\). Then the equation for \(y_{2n+1}\) is
\[
(\partial^{2n+1} - \sum_{i=1}^{n} (\partial^{n+1-i} u_i \partial^{n-i} + \partial^{n-i} u_i \partial^{n+1-i})) y_{2n+1} + \lambda y_{2n+1} = 0.
\]

A \(\partial\)-module structure on \(\hat{A}^{(1)}_{2n}\) was introduced in [4] to show that \(q = \sum_{i=1}^{2n} q_i e_{2n+1}\) is a solution of the \(j\)-th \(\hat{A}^{(1)}_{2n}\)-KdV flow if and only if \(\partial^{2n+1} - \sum_{i=1}^{2n} q_i \partial^{i-1}\) is a solution of the \(j\)-th GD\(_{2n+1}\) flow. It can be checked that \(\hat{A}^{(2)}_{2n}\) is a sub \(\partial\)-module of \(\hat{A}^{(1)}_{2n}\) and a similar proof as in [4] gives the following.
Proposition 4.19. $u = \sum_{i=1}^{n} u_i \partial_i$ is a solution of the $(2j-1)$-th $\hat{A}_{2n}^{(2)}$-flow (4.21) if and only if $\partial^{2n+1} - \sum_{i=1}^{n}(\partial^{n+1-i}u_i \partial^{n-i} + \partial^{n-i}u_i \partial^{n+1-i})$ is a solution of the $(2j-1)$-th constraint KP flow.

5. Hierarchies of isotropic curve flows

In this section, we prove result (a) stated in the introduction.

Proposition 5.1. Let $P_{2j-1,0}(u)$ be as in (4.6), and $S_{2j-1,0}(u)$ as in (4.18). Then

\[
\gamma_t = gP_{2j-1,0}(u)e_1, \quad (5.1)
\]

\[
\gamma_t = gS_{2j-1,0}(u)e_1. \quad (5.2)
\]

are isotropic curve flows on $\mathcal{M}_{n+1,n}$, where $g(\cdot, t)$ and $u(\cdot, t)$ are the isotropic moving frame and curvature of $\gamma(\cdot, t)$. We call (5.3) and (5.4) the $(2j-1)$-th isotropic curve flow on $\mathcal{M}_{n+1,n}$ of B-type and A-type respectively.

Proof. Let $Q_{2j-1}(u) = P_{2j-1,0}(u) - \eta_j(u)$, where $\eta_j(u)$ is as in (4.11). By (4.11), $[\partial_x + b + u, Q_{2j-1}(u)] \in V_n$. It follows from Proposition 3.1 that $\xi(\gamma) = gQ_{2j-1}(u)e_1$ is tangent to $\mathcal{M}_{n+1,n}$ at $\gamma$. Since $\eta_j(u)$ is strictly upper triangular, $Q_{2j-1}(u)e_1 = P_{2j-1,0}(u)e_1$. Therefore (5.3) is an isotropic curve flow on $\mathcal{M}_{n+1,n}$. A similar proof shows that (5.4) is an isotropic flow on $\mathcal{M}_{n+1,n}$.

Since $\eta_j(u)$ and $\tilde{\eta}_j(u)$ are strictly upper triangular, (5.1) and (5.2) can be written as

\[
\gamma_t = g(P_{2j-1,0}(u) - \eta_j(u))e_1, \quad (5.3)
\]

\[
\gamma_t = g(S_{2j-1,0}(u) - \tilde{\eta}_j(u))e_1. \quad (5.4)
\]

Example 5.2. Isotropic curve flows of B-type

(i) It follows from Example 4.13 that the third isotropic curve flow of B-type on $\mathcal{M}_{n+1,n}$ is (1.8), i.e.,

\[
\gamma_t = -\frac{3}{n}u_1 \gamma_x + \gamma_x^{(3)}.
\]

In particular, it is (1.5) when $n = 1$, and is (3.6) when $n = 2$.

(ii) By Example 4.11, the fifth isotropic curve flow of B-type on $\mathcal{M}_{3,2}$ is

\[
\gamma_t = (q_{xxx} - 3qq_x)\gamma + \left(\frac{3}{2}q^2 - qxx\right)\gamma_x. \quad (5.5)
\]

Example 5.3. Isotropic curve flows of A-Type

(i) The third isotropic curve flow of A-type on $\mathcal{M}_{n+1,n}(n \geq 3)$ is (1.9),

\[
\gamma_t = -\frac{3}{2n+1}(u_1)x\gamma - \frac{6}{2n+1}u_1 \gamma_x + \gamma_x^{(3)}. \quad (5.6)
\]
We prove this theorem for (5.3). The same proof works for (5.4).

**Proof.**

**Corollary 5.5.** Let

\[ \gamma_0 = \frac{3}{5} (u_1)_x \gamma - \frac{1}{5} u_1 \gamma_x + p_4 \quad (5.7) \]

is the third isotropic curve flow of A-type on \( M_{3,2} \). But \( p_4 = \gamma^{(3)}_x - u_1 \gamma_x \) (given in Example 2.7). So (5.7) can be also written as (5.6) with \( n = 2 \).

(iii) It follows from the formula of \( S_{5,0}(u) \) given in Example 4.17 that the fifth isotropic curve flow of A-type on \( M_{2,1} \) is (1.6).

\[ \text{Theorem 5.4.} \]

(i) If \( \gamma \) is a solution of the \((2j - 1)\)-th isotropic curve flow (5.3) of B-type \((5.3)\) \(\) of A-type resp.) on \( M_{n+1,n} \), then its isotropic curvature \( u \) is a solution of the \((2j - 1)\)-th \( B_{n}^{(1)} \)-KdV flow \((4.12)\) \(\) resp.).

(ii) Let \( u \) be a solution of \((4.12)\) \(\) resp.), \( c_0 \in O(n+1,n) \) a constant, and \( g \) the solution of \((4.14)\) \(\) resp.) with \( g(0,0) = c_0 \). Then \( \gamma(x,t) := g(x,t)e_1 \) is solution of the \((2j - 1)\)-th isotropic curve flow of B-type (5.3) \(\) resp.) with isotropic curvature \( u(\cdot, t) \) and \( g(\cdot, t) \) is the isotropic moving frame along \( \gamma(\cdot, t) \).

**Proof.** We prove this theorem for (5.3). The same proof works for (5.4).

(i) Let \( g(t) \) be the isotropic moving frame, and \( u(\cdot, t) \) the isotropic curvature along \( \gamma(\cdot, t) \). Then \( g^{-1} g_x = b + u \). Set \( \xi = g^{-1} g_t \). Hence \( b + u \) and \( \xi \) satisfy the zero curvature condition, i.e., \( [\partial_x + b + u, \partial_t + \xi] = 0 \). So we have \( u_t = [\partial_x + b + u, \xi] \). Since \( u_t \) lies in \( V_n \), \( \xi \) satisfies condition (3.1). Note that the first columns of \( \xi \) and \( Q_{2j-1}(u) := P_{2j-1,0}(u) - \eta_j(u) \) are the same. So by Corollary 3.4 \( \xi = Q_{2j-1}(u) \). By Proposition 4.7 \( u \) is a solution of (4.12).

(ii) Since \( g^{-1} g_x = b + u \), \( g \) is the isotropic moving frame along \( \gamma(\cdot, t) \). By assumption, \( g^{-1} g_t = Q_{2j-1}(u) \). Hence \( g_t = gQ_{2j-1}(u) \). Note that the first column of \( g \) is \( \gamma \). So we have \( \gamma_t = gQ_{2j-1}(u)e_1 \).

**Corollary 5.5.** Let \( \Psi : M_{n+1,n} \to C^\infty(\mathbb{R}, V_n) \) be the isotropic curvature map. Then \( \Psi \) maps the space of solutions of (5.3) \(\) resp.) modulo \( O(n+1,n) \) bijectively onto the space of solutions of (4.12) \(\) resp.) respectively.

The following Theorem is a consequence of Theorem 5.4.

\[ \text{Theorem 5.6.} \]

Let \( \gamma_0 \in M_{n+1,n} \) with rapidly decaying isotropic curvature \( u_0 \), and \( g_0 \in O(n+1,n) \) the isotropic moving frame along \( \gamma_0 \). Let \( u(x,t) \) be the solution of the \((2j - 1)\)-th \( B_{n}^{(1)} \)-KdV flow \((4.12)\) \(\) resp.) with initial date \( u(x,0) = u_0(x) \), and \( g \) the solution of (4.14) \(\) resp.) with \( g(0,0) = g_0(0) \). Then \( \gamma(x,t) = g(x,t)e_1 \) is a solution of (5.3) \(\) resp.)
respectively) with $\gamma(x, 0) = \gamma_0(x)$ and $\gamma(\cdot, t)$ has rapidly decaying isotropic curvatures.

If the solution of the periodic Cauchy problem for (4.12) is solved, then we can use a similar proof as for the $\hat{A}_n^{(1)}$-KdV flows (cf. [11]) to solve the periodic Cauchy problem for (4.14), i.e., we have the following.

**Theorem 5.7. [Cauchy problem with periodic initial data]**

Suppose $\gamma_0 \in M_{n+1, n}$ is periodic, $g_0$ and $u_0$ are the isotropic moving frame and curvature along $\gamma_0$. Let $u(x, t)$ be the solution of the $(2j-1)$th $\hat{B}_n^{(1)}$-KdV flow (4.12) ($\hat{A}_n^{(2)}$-KdV flow (4.21) resp.) periodic in $x$ such that $u(x, 0) = u_0(x)$, and $g(x, t)$ the solution of (4.14) (4.25) resp.) with $g(0, 0) = g_0(0)$. Then $\gamma(x, t) = g(x, t)e_1$ is a solution of (5.3) (5.3) resp.) with $\gamma(x, 0) = \gamma_0(x)$. Moreover, $\gamma(\cdot, t)$ is periodic in $x$ with isotropic curvature $u(\cdot, t)$.

### 6. Bi-Hamiltonian structure for isotropic curve flows

In this section, we first explain how to compute the bi-Hamiltonian structure and conservation laws for the $\hat{B}_n^{(1)}$-KdV and $\hat{A}_n^{(2)}$-KdV hierarchies. Then we pull back these bi-Hamiltonian structures and conservation laws by the isotropic curvature map $\Psi$ to get bi-Hamiltonian structures and conservation laws for isotropic curve flows.

The gradient $\nabla F(u) \in C^\infty(S^1, V_n^1)$ for a functional $F$ on $C^\infty(S^1, V_n)$ is defined by

$$dF_u(v) = \langle \nabla F(u), v \rangle = \int_{S^1} \text{tr}(v \nabla F(u))dx$$

for all $v \in C^\infty(S^1, V_n)$.

If $\{\cdot, \cdot\}$ is a Poisson structure on $C^\infty(S^1, V_n)$, then the Hamiltonian vector field $X_F$ for $F$ with respect to $\{\cdot, \cdot\}$ is defined by

$$\{F, H\}(u) = -\langle X_F(u), \nabla H(u) \rangle$$

(6.1)

for all functionals $H$.

The bi-Hamiltonian structure on $C^\infty(S^1, V_n)$ for the $\hat{B}_n^{(1)}$-KdV hierarchy given in [4] can be written as follows:

$$\{F_1, F_2\}_1(u) = \langle [\beta, P_u(\nabla F_1(u))], P_u(\nabla F_2(u)) \rangle,$$

$$\{F_1, F_2\}_2(u) = \langle [\partial_x + b + u, P_u(\nabla F_1(u))], P_u(\nabla F_2(u)) \rangle,$$

where $\beta$ is defined by (4.2) and $P_u : C^\infty(S^1, V_n^1) \to C^\infty(S^1, o(n+1, n))$ is the linear operator defined in Definition 5.5. Use (6.1) to see that the Hamiltonian vector field $Y_F$ of a functional $F$ with respect to $\{\cdot, \cdot\}_2$ is

$$Y_F(u) = [\partial_x + b + u, P_u(\nabla F(u))].$$

(6.2)

We have explained how to compute $P_u(\xi)$ in section 3. So we can compute the Hamiltonian vector field $X_F$ of $F$ with respect $\{\cdot, \cdot\}_1$. 
Example 6.1. Bi-Hamiltonian structure for the $\dot{B}_1^{(1)}$-KdV hierarchy

Write $\dot{\xi} = \nabla F_1(u) = \xi(e_{21} + e_{32})$, $\dot{\eta} = \eta(e_{21} + e_{32})$, $C = P_u(\xi) = (C_{ij})$ and $D = P_u(\eta) = (D_{ij})$. We use Example 3.3 to get $C$ and $D$ in terms of $\xi$ and $\eta$ respectively. So we have

$$\{F_1, F_2\}_1(u) = \langle [C, \beta], D \rangle = -2 \oint \xi_x \eta \, dx,$$

$$\{F_1, F_2\}_2(u) = \langle \partial_x + b + u, C \rangle, D \rangle = -2 \oint (\xi_x - 2u_1 \xi - (u_1)_x \xi) \eta \, dx.$$ This is the standard bi-Hamiltonian structure for the KdV-hierarchy [3].

Example 6.2. Bi-Hamiltonian structure for the $\dot{B}_2^{(1)}$-KdV hierarchy

Write $\dot{\xi} = \xi_1(e_{32} + e_{43}) + \xi_2(e_{41} + e_{52})$, $\eta = \eta_1(e_{32} + e_{43}) + \eta_2(e_{41} + e_{52})$, $C = P_u(\xi) = (C_{ij})$ and $D = P_u(\eta) = (D_{ij})$. We use Example 3.3 to get $C$ and $D$ in terms of $\xi$ and $\eta$ respectively, and obtain

$$\{F_1, F_2\}_1(u) = \langle [C, \beta], D \rangle = (C_{11} + C_{22}) \eta_2 + C_{31} \eta_1 - \xi_1 D_{31} - \xi_2 D_{11} + D_{22}),$$

$$\{F_1, F_2\}_2(u) = \langle \partial_x + b + u, C \rangle, D \rangle = -2 \oint ((\xi_2)_x^3 + 2(\xi_1)_x - u_1 \xi_2)_x - u_1 (\xi_2)_x \eta_2 + 2(\xi_2)_x \eta_1 \, dx,$$

where $p_1 = [\partial_x + b + u, C]_{14}$ is a 7-th ordered differential polynomial in $\xi_1, \xi_2$, and $p_2 = [\partial_x + b + u, C]_{23}$ is a 5-th ordered differential polynomial in $\xi_1, \xi_2$.

The following theorem can be proved the same way as for the $\dot{A}_n^{(1)}$-KdV hierarchy (cf. [11]).

Theorem 6.3. Let $u \in C^\infty(\mathbb{R}, V_n)$, $\beta$ as in (4.2), and $P(u, \lambda)$ defined by (4.5). Then we have

$$\langle \frac{\partial}{\partial \lambda} (\lambda^{-1} P^{2j-1}(u, \lambda)), \delta u \rangle = \delta (\lambda^{-1} P^{2j-1}(u, \lambda), \beta) = \langle \delta P^{2j-1}(u, \lambda), \beta \lambda^{-1} \rangle.$$

(6.3)

Theorem 6.4. Let $u, \beta, P(u, \lambda)$ be as in Theorem 6.3. Write $P^{2j-1}(u, \lambda) = \sum_i P_{2j-1,i}(u) \lambda^i$ as a power series in $\lambda$. Set

$$F_{2j-1}(u) = -\oint \text{tr}(P_{2j-1,-1}(u) \beta) \, dx.$$ (6.4)

Then $\nabla F_{2j-1}(u) = \pi_0(P_{2j-1,0}(u))$, where $\pi_0$ is the projection onto $V_n^\perp$ defined by (3.3). Moreover, the Hamiltonian equation for $F_{2j-1}$ with respect to $\{1, \}, 2$ ({}, 1 resp.) is the $(2j - 1)$-th ((2j - n) - 1)-th resp.) $\hat{B}_n^{(1)}$-KdV flow.

Proof. Compare the coefficient of $\lambda^{-2}$ of (6.3) to obtain the formula for $\nabla F_{2j-1}$.

By Corollary 4.6 we have $P_u(\pi_0(P_{2j-1,0}(u))) = P_{2j-1,0}(u) - \eta_j(u).$ So

$$P_u(\nabla F_{2j-1}(u)) = P_{2j-1,0}(u, \lambda) - \eta_j(u).$$ (6.5)
It follows from \((6.2)\) that the Hamiltonian flow for \(F_{2j-1}\) with respect to \(\{\}_2\) is the \((2j - 1)\)-th \(\hat{B}_n^{(1)}\)-KdV flow.

Let \(X_{2n+2j-1}\) denote the Hamiltonian vector field of \(F_{2n+2j-1}\) with respect to \(\{\}_1\). Compute directly to get
\[
\{F, F_{2n+2j-1}\}_1(u) = -\langle X_{2n+2j-1}(u), \nabla F(u) \rangle \\
= -\langle [\beta, P_u(\nabla F_{2n+2j-1}(u))], P_u(\nabla F(u)) \rangle, \text{ by } \begin{equation} \tag{6.5} \end{equation}
= -\langle [\beta, P_{2n+2j-1,0}(u) - \eta_{n+j}(u)], P_u(\nabla F(u)) \rangle, \quad \text{since } [\beta, \eta_j(u)] = 0,
= -\langle [\beta, P_{2n+2j-1,0}(u)], P_u(\nabla F(u)) \rangle.
\]

Compare coefficient of \(\lambda\) of the equation \((6.3)\) to get
\[
[\partial_x + b + u, P_{2n+2j-1,1}(u)] = [P_{2n+2j-1,0}(u), \beta].
\]

It follows from \((6.4)\) that we have
\[
P^{2n+2j-1}(u, \lambda) = \lambda P^{2j-1}(u, \lambda). \tag{6.6}
\]

Compare coefficient \(\lambda\) of \((6.6)\) to get \(P_{2n+2j-1,1}(u) = P_{2j-1,0}(u)\). So we have
\[
\{F, F_{2n+2j-1}\}_1(u) = \langle [\partial_x + b + u, P_{2j-1,0}(u)], P_u(\nabla F(u)) \rangle.
\]

Since \(\eta_j(u) \in \mathcal{N}_u^\ast\) and \([\partial_x + b + u, P_u(\nabla F(u))] \in V_n\),
\[
\langle [\partial_x + b + u, \eta_j(u)], P_u(\nabla F(u)) \rangle = -\langle \eta_j(u), [\partial_x + b + u, P_u(\nabla F(u))] \rangle = 0.
\]

This implies that
\[
\{F, F_{2n+2j-1}\}_1(u) = \langle [\partial_x + b + u, P_u(\nabla F_{2j-1}(u))], P_u(\nabla F(u)) \rangle.
\]

By the definition of \(P_u\), we have \([\partial_x + b + u, P_u(\nabla F_{2j-1}(u))] \in V_n\) and \(\pi_0(P_u(\nabla F(u))) = \nabla F(u)\). Hence
\[
\{F, F_{2n+2j-1}\}_1(u) = \langle [\partial_x + b + u, P_u(\nabla F_{2j-1}(u))], \nabla F(u) \rangle.
\]

This proves that the Hamiltonian flow for \(F_{2(j+n)-1}\) is the \((2j - 1)\)-th \(\hat{B}_n^{(1)}\)-KdV flow. \(\square\)

**Example 6.5. Conservation laws for the \(\hat{B}_n^{(1)}\)-KdV hierarchy**

Let \(h_{2j-1}(u) = \text{tr}(P_{2j-1,1}(u)\beta)\) denote the density of the conservation law \(F_{2j-1}\).

1. For \(n = 1\), we have \(u = q(e_{12} + e_{23})\) and
\[
h_1 = 2q, \quad h_3 = q^2, \quad h_5 = \frac{1}{2}(q^3 - qq_{xx}).
\]

2. For \(n = 2\), we have \(u = u_1(e_{23} + e_{34}) + u_2(e_{14} + e_{25})\) and
\[
h_1 = \frac{1}{2}u_1, \quad h_3 = \frac{1}{8}u_1^2 + \frac{1}{2}u_2.
\]

3. For general \(n\), we have
\[
h_1 = \frac{1}{n}u_1, \quad h_3 = \frac{2n - 3}{2n^2}u_1^2 + \frac{1}{n}u_2.
\]
We use the same proofs for the $\hat{A}^{(1)}_n$–KdV hierarchy to prove the following results for the $\hat{A}^{(2)}_{2n}$–KdV.

**Theorem 6.6.** Let $u \in C^\infty(\mathbb{R}, V_n)$, and $S(u, \lambda)$ defined by (4.18). Then
\[
\frac{\partial}{\partial \lambda} (\lambda^{-1} S^{2j-1}(u, \lambda), \delta u) = \lambda^{-1} (\delta S^{2j-1}(u, \lambda), e_{1,2n+1}).
\]

**Theorem 6.7.** Let $u, S(u, \lambda)$ be as in Theorem 6.6, and $S^{2j-1}(u, \lambda) = \sum_i S_{2j-1,i}(u)\lambda^i$. Set $h_{2j-1}(u) = tr(S_{2j-1,1}(u)e_{1,2n+1})$, and
\[
G_{2j-1}(u) = - \int h_{2j-1}(u)dx.
\]
(6.7)

Then $\nabla G_{2j-1}(u) = \pi_0(S_{2j-1,0}(u))$, where $\pi_0$ is the projection defined by (3.5). Moreover, the Hamiltonian flow for $G_{2j-1}$ with respect to $\{\cdot, \cdot\}$ resp.) is the $(2j-1)$-th $\hat{A}^{(2)}_{2n}$–KdV flow.

**Example 6.8.** Conservation laws for the $\hat{A}^{(2)}_{2n}$–KdV hierarchy

(1) For the $\hat{A}^{(2)}_{2}$–KdV hierarchy, $u = q(e_{12} + e_{23})$, we have
\[
h_1(u) = \frac{2}{3} q, \quad h_3(u) = \frac{5}{27} (q_x^2 + \frac{8}{3} q^3).
\]

(2) For general $n$, the first two conservation densities are
\[
h_1(u) = \frac{2}{2n+1} u_1, \quad h_3(u) = \frac{2}{2n+1} u_2 + \frac{4(n-1)}{(2n+1)^2} u_1^2.
\]

Let $\mathcal{M}_{n+1,n}(S^1)$ denote the space of $\gamma \in \mathcal{M}_{n+1,n}$ that is periodic with period $2\pi$. Then we have the following.

(i) The isotropic curvature map $\Psi : \mathcal{M}_{n+1,n}(S^1) \to C^\infty(S^1, V_n)$ induces an injective map from the orbit space $\mathcal{M}_{O(n+1,n)}(S^1)$ to $C^\infty(S^1, V_n)$.

(ii) The isotropic curve flows are invariant under $O(n+1,n)$.

(iii) Suppose $\gamma_t = g\xi(u)$ is an isotropic flow on $\mathcal{M}_{n+1,n}$. By Proposition 2.3, its curvature evolves as $u_t = [\partial_x + b + u, g^{-1} g_t]$, where $g(\cdot, t)$ is the isotropic moving frame. Moreover, it follows from Corollaries 3.6 and 3.7 that $g^{-1} g_t$ can be computed from $\xi(u)$.

We pull back the bi-Hamiltonian structure and conservation laws for the $\hat{B}^{(1)}_n$–KdV and $\hat{A}^{(2)}_{2n}$–KdV hierarchies on $C^\infty(S^1, V_n)$ to $\mathcal{M}_{n+1,n}(S^1)$ by the isotropic curvature map $\Psi$ for the isotropic curve flows. In particular, we have the following:

(a) A functional on $\mathcal{M}_{O(n+1,n)}(S^1)$ can be viewed as an $O(n+1,n)$-invariant functional on $\mathcal{M}_{n+1,n}(S^1)$. So it is of the form $F(\gamma) = F(\Psi(\gamma))$ for some functional $F$ on $C^\infty(S^1, V_n)$.

(b) Given functionals $F, H$ on $C^\infty(S^1, V_n)$, let $\tilde{F} = F \circ \Psi$ and $\tilde{H} = H \circ \Psi$. Then the induced Poisson structure on $O(n+1,n)$-invariant...
functionals on $\mathcal{M}_{n+1,n}(S^1)$ is
\[
\{\hat{F}, \hat{H}\}^\gamma_i(\gamma) = \{F, H\}_i(\Psi(u)), \quad i = 1, 2.
\]

The following is a consequence of Proposition 2.8, Corollary 2.9, and Corollary 3.6.

**Theorem 6.9.** Let $\Psi$ be the isotropic curvature map, and $X_F$ the Hamiltonian vector field of $F : C^\infty(S^1, V_{n}) \to \mathbb{R}$ with respect to $\{,\}_1$. Then

(i) there exists $\xi_F(u) \in C^\infty(S^1, o(n + 1, n))$ satisfying
\[
[\partial_x + b + u, \xi_F(u)] = X_F(u),
\]

(ii) the Hamiltonian equation for $\hat{F} = F \circ \Psi$ on $\mathcal{M}_{n+1,n}$ with respect to $\{,\}_1$ ($\{,\}_2$ resp.) is $\gamma_t = g\xi_F(u)e_1$ ($\gamma_t = gP_u(\nabla F(u))e_1$ resp.), where $g(\cdot, t)$ and $u(\cdot, t)$ are the isotropic curvature frame and isotropic curvature along $\gamma(\cdot, t)$.

In particular, the Hamiltonian flow for $\hat{F}_{2j-1}$ with respect to $\{,\}_1$ ($\{,\}_2$ resp.) on $\mathcal{M}_{n+1,n}$ is the $(2j - 1)$-th $(2(j - n) - 1)$-th resp.) isotropic curve flow of B-type. Similar statements hold for isotropic curve flow of A-type.

### 7. Bäcklund Transformations for the $\hat{B}_n^{(1)}$-KdV Flows

In this section, we first construct Bäcklund transformations (BTs) and a Permutability formula for flow (4.7) on $C^\infty(\mathbb{R}, B_n^+)$.

Then we use the gauge equivalence to construct BTs for the corresponding quotient flow (4.12), i.e., the $(2j - 1)$-th $\hat{B}_n^{(1)}$-KdV flow. Since we also obtain the formula of the frame of the new solution constructed from BTs for (4.7), we can construct BTs for isotropic curve flows of B-type. If we apply BTs to the trivial solution of the isotropic curve flow (i.e., the solution whose isotropic curvatures are zero) repeatedly, then we can obtain infinitely many families of explicit soliton solutions.

Let $\hat{\mathbb{B}}_n^{(1)}$ denote the group of smooth maps $f : S^1 \to SL(2n + 1, \mathbb{C})$ satisfying
\[
\begin{aligned}
\hat{f}(\lambda) &= f(\lambda), \\
\hat{f}(\lambda)^4 C_n f(\lambda) &= C_n,
\end{aligned}
\tag{7.1}
\]

and $(\hat{\mathbb{B}}_n^{(1)})_+$ the subgroup of $f \in \hat{\mathbb{B}}_n^{(1)}$ that is the restriction of a holomorphic map on $\mathbb{C}$ to $S^1$, and $(\hat{\mathbb{B}}_n^{(1)})_-$ the subgroup of $f \in \hat{\mathbb{B}}_n^{(1)}$ that is the boundary value of a holomorphic map $\hat{f}$ on $\epsilon^{-1} \leq |\lambda| \leq \infty$ for some small $\epsilon > 0$ and $\hat{f}(\infty) = 1$. Then the Lie algebras of $\hat{\mathbb{B}}_n^{(1)}$ and $(\hat{\mathbb{B}}_n^{(1)})_\pm$ are $\hat{B}_n^{(1)}$ and $(\hat{B}_n^{(1)})_\pm$ respectively. If a soliton hierarchy is constructed from a splitting $\mathcal{L}_\pm$ of a loop algebra $\mathcal{L}$, then we can use the loop group factorization to constructing BTs (cf. [8]) as follows:

(1) Find simple elements (i.e., rational maps) $f \in (\hat{\mathbb{B}}_n^{(1)})_-$ that have minimum number of poles.
(2) Given \( f \in (\hat{\mathbb{B}}(1)_n)^{-} \) and a frame \( F(x,t,\cdot) \in (\hat{\mathbb{B}}(1)_n)^{+} \) of a solution of (4.7), if we can factor \( fF(x,t,\cdot) = \tilde{F}(x,t,\cdot)f(x,t,\cdot) \) with \( \tilde{F}(x,t,\cdot) \in (\hat{\mathbb{B}}(1)_n)^{+} \) and \( f(x,t,\cdot) \in (\hat{\mathbb{B}}(1)_n)^{-} \), then it was proved in [8] that \( \tilde{F} \) is a frame of a new solution of (4.7).

We need the following Lemmas to construct rational elements in \((\hat{\mathbb{B}}(1)_n)^{-}\).

**Lemma 7.1.** Let \( \mathbb{R}^{n+1,n} = V_1 \oplus V_2, \) and \( V_i^{\perp} = \{ v \in \mathbb{R}^{n+1,n} \mid \langle v, V_i \rangle = 0 \} \). Let \( \pi \) be the projection of \( \mathbb{R}^{n+1,n} \) onto \( V_1 \) along \( V_2 \), and \( \pi^\sharp \) the projection of \( \mathbb{R}^{n+1,n} \) onto \( V_2^{\perp} \) along \( V_1^{\perp} \). Then \( \pi^\sharp = C_n\pi^\parallel C_n \).

**Proof.** Note that \( \pi^\sharp \) is the conjugate of \( \pi \) with respect to \( (\cdot, \cdot) \). First, if \( \langle \pi^\sharp X, Y \rangle = 0 \) for all \( X \in \mathbb{R}^{n+1,n} \), then \( \langle X, \pi Y \rangle = 0 \). Hence \( \pi Y = \mathbf{0} \). Therefore, \( \text{Im}(\pi^\sharp)^{\perp} \subset \ker(\pi) = V_2 \). Since \( \pi^\sharp = C_n\pi^\parallel C_n \) and \( \dim(\text{Im}(\pi^\sharp)) = \dim(\ker(\pi)) = V_2^{\perp} \). Hence \( \text{Im}(\pi^\sharp) = V_2^{\perp} \).

If \( Y \in \ker(\pi^\sharp) \), then \( \langle \pi X, Y \rangle = (X, \pi^\sharp Y) = 0 \) for all \( X \in \mathbb{R}^{n+1,n} \). Hence \( Y \in V_1^{\perp} \). Therefore, \( \ker(\pi^\sharp) = V_1^{\perp} \). \( \Box \)

**Lemma 7.2.** Let \( V_i, \pi, \pi^\sharp \) be as in Lemma 7.1 Then \( \pi \pi^\sharp = \pi^\sharp \pi = \mathbf{0} \) if and only if

\[ V_1 \subset V_1^{\perp}, \quad V_2^{\perp} \subset V_2. \]  

(7.2)

**Proof.** Since \( V_2^{\perp} \subset V_2 \), \( \pi \pi^\sharp = \mathbf{0} \). It follows from \( V_1 \subset V_1^{\perp} \) that we have \( \pi^\sharp \pi = \mathbf{0} \). \( \Box \)

Let \( \alpha_1 \neq \alpha_2 \in \mathbb{R} \), and \( \pi \) a projection of \( \mathbb{R}^{n+1,n} \). Set

\[ h_{\alpha_1,\alpha_2,\pi} = I + \frac{\alpha_1 - \alpha_2}{\lambda - \alpha_1}(I - \pi) = \frac{\lambda - \alpha_2}{\lambda - \alpha_1}I - \frac{\alpha_1 - \alpha_2}{\lambda - \alpha_1} \pi. \]  

(7.3)

**Proposition 7.3.** If \( \pi \pi^\sharp = \pi^\sharp \pi = \mathbf{0} \), then

\[ g_{\alpha_1,\alpha_2,\pi} = h_{\alpha_2,\alpha_1,\pi^\sharp} h_{\alpha_1,\alpha_2,\pi}. \]  

(7.4)

is in \((\hat{\mathbb{B}}(1)_n)^{-}\).

**Proof.** By Lemma 7.2 we have \( \pi \pi^\sharp = \pi^\sharp \pi = \mathbf{0} \). So we obtain

\[ g_{\alpha_1,\alpha_2,\pi} = I + \frac{\alpha_2 - \alpha_1}{\lambda - \alpha_2} \pi^\sharp + \frac{\alpha_1 - \alpha_2}{\lambda - \alpha_1} \pi, \]

\[ g_{\alpha_1,\alpha_2,\pi}^{-1} = I + \frac{\alpha_2 - \alpha_1}{\lambda - \alpha_2} \pi^\sharp + \frac{\alpha_1 - \alpha_2}{\lambda - \alpha_1} \pi = g_{\alpha_1,\alpha_2,\pi^\sharp}. \]

A direct computation implies that \( C_ng_{\alpha_1,\alpha_2,\pi}^{-1} = g_{\alpha_1,\alpha_2,\pi^\sharp} C_n \). \( \Box \)

**Theorem 7.4.** Let \( F(x,t,\cdot) \in (\hat{\mathbb{B}}(1)_n)^{+} \) be a frame of a solution \( \mathbf{q} \) of (4.7). Assume that \( \mathbb{R}^{n+1,n} = V_1 \oplus V_2 \) and \( V_1, V_2 \) satisfy (7.2). Let \( \pi \) be the projection of \( \mathbb{R}^{n+1,n} \) onto \( V_1 \) along \( V_2 \), and \( \pi^\sharp = C_n\pi^\parallel C_n \). Let \( \alpha_1 \neq \alpha_2 \in \mathbb{R} \), \( g_{\alpha_1,\alpha_2,\pi} \) as in (7.2). \( \pi \) is a projection of \( \mathbb{R}^{n+1,n} \) onto \( V_1(x,t) \) along \( V_2(x,t) \), and

\[ \tilde{F}(x,t,\lambda) = g_{\alpha_1,\alpha_2,\pi}(\lambda)F(x,t,\lambda)g_{\alpha_1,\alpha_2,\pi}(\lambda)^{-1}. \]
Then
\[ \tilde{q} = q + (\alpha_1 - \alpha_2)[\beta, \tilde{\pi} - \tilde{\pi}^2] \]
is a new solution of (1.7) and \( \tilde{F}(x, t, \cdot) \in (\hat{\mathbb{B}}^{(1)}_n)_+ \) is a frame for \( \tilde{q} \). (We will use \( g_{\alpha_1, \alpha_2, \tilde{\pi}^2}q \) to denote the new solution \( \tilde{q} \)).

**Proof.** Since \( F(x, t, \cdot) \in (\hat{\mathbb{B}}^{(1)}_n)_+ \) and \( \alpha_i \in \mathbb{R} \), \( F(x, t, \alpha_i) \in O(n + 1, n) \). So
\[ \langle \tilde{V}_1, \tilde{V}_1 \rangle = \langle F(x, t, \alpha_1)^{-1}V_1, F(x, t, \alpha_1)^{-1}V_1 \rangle = \langle V_1, V_1 \rangle = 0. \]
Therefore, \( \tilde{V}_1 \subset \tilde{V}_1^\perp \).

If \( v \in \tilde{V}_2 \), then
\[ \langle F(x, t, \alpha_2)v, Y \rangle = \langle v, F(x, t, \alpha_2)^{-1}Y \rangle = 0, \quad \forall Y \in V_2. \]
Hence \( F(x, t, \alpha_2)v \in V_2^\perp \). This implies that
\[ \tilde{V}_2^\perp \subset F(x, t, \alpha_2)^{-1}(V_2^\perp) \subset F(x, t, \alpha_2)^{-1}(V_2) = \tilde{V}_2. \]
By Lemma 7.2 and Proposition 7.3 \( g_{\alpha_1, \alpha_2, \tilde{\pi}^2(x, t)} \in (\hat{\mathbb{B}}^{(1)}_n)_- \).

Next we claim that \( \tilde{F}(x, t, \lambda) \) is holomorphic for \( \lambda \in \mathbb{C} \). It follows from the formula of \( \tilde{F} \) that \( \tilde{F}(x, t, \lambda) \) is holomorphic for \( \lambda \neq \alpha_1, \alpha_2 \). The residue of \( \tilde{F}(x, t, \lambda) \) at \( \alpha_1 \) is:
\[ (\alpha_1 - \alpha_2)\langle \tilde{\pi}^2F(x, t, \alpha_1)(I - \tilde{\pi}^2) + (I - \pi)F(x, t, \alpha_1)\tilde{\pi} \rangle. \]
Since \( \tilde{\pi} \) is the projection of \( \mathbb{R}^{n+1, n} \) onto \( \tilde{V}_2 \) along \( \tilde{V}_1^\perp \), we have:
\[ \text{Im}(F(x, t, \alpha_1)(I - \tilde{\pi}^2)) \subset \tilde{V}_2^\perp, \quad \text{Im}(F(x, t, \alpha_1)\tilde{\pi}) \subset V_1. \]
Hence \( \tilde{\pi}^2F(x, t, \alpha_1)(I - \tilde{\pi}^2) + (I - \pi)F(x, t, \alpha_1)\tilde{\pi} = 0 \). So \( \tilde{F}(x, t, \lambda) \) is holomorphic at \( \lambda = \alpha_1 \).

A similar computation implies that the residue of \( \tilde{F}(x, t, \lambda) \) at \( \alpha_2 \),
\[ (\alpha_2 - \alpha_1)(\pi F(x, t, \alpha_2)(I - \tilde{\pi}) + (I - \pi^2)F(x, t, \alpha_2)\tilde{\pi}^2) = 0. \]
So we have proved that \( g_{\alpha_1, \alpha_2, \pi^2}F(x, t, \cdot) = \tilde{F}(x, t, \cdot)g_{\alpha_1, \alpha_2, \tilde{\pi}(x, t)} \) with \( \tilde{F}(x, t, \cdot) \) in \( (\hat{\mathbb{B}}^{(1)}_n)_+ \) and \( g_{\alpha_1, \alpha_2, \tilde{\pi}(x, t)} \) in \( (\hat{\mathbb{B}}^{(1)}_n)_- \). It follows from [8] that \( \tilde{F} \) is a frame of a new solution \( \tilde{q} \) of (1.7). Note that \( \tilde{F}^{-1}\tilde{F}_x = b + \tilde{q} + \lambda \beta, \quad F^{-1}F_x = b + q + \lambda \beta \), and
\[ \tilde{F}^{-1}\tilde{F}_x = g_{\alpha_1, \alpha_2, \tilde{\pi}}F^{-1}F_xg_{\alpha_1, \alpha_2, \tilde{\pi}}^{-1} - (g_{\alpha_1, \alpha_2, \tilde{\pi}}xg_{\alpha_1, \alpha_2, \tilde{\pi}}^{-1})_x. \]
So we have
\[ (b + \tilde{q} + \lambda \beta)g_{\alpha_1, \alpha_2, \tilde{\pi}} = g_{\alpha_1, \alpha_2, \tilde{\pi}}(b + q + \lambda \beta) - (g_{\alpha_1, \alpha_2, \tilde{\pi}}x)_x. \]
Equate the constant term of the power series expansion of the above equation to get the formula for \( \tilde{q} \). \( \square \)

As a consequence of Proposition 4.9, Proposition 4.10, and Theorem 7.4, we obtain BTs for the \((2j - 1)\)-th \( \hat{B}^{(1)}_n \)-KdV flow.
Theorem 7.5. Let \( E(x, t, \cdot) \in \tilde{\mathcal{B}}_{n+1}^{(2)} \) be a frame of a solution \( u \) of (4.12), \( u_t = [\partial_x + b + u, P_{2j-1,0}(u) - \eta_j(u)] \), and \( \Delta(x, t) \in N_{n}^+ \) satisfying \( \Delta \Delta^{-1} = \eta_j(u) \). Let \( q = \Delta^{-1} * u, \alpha_1, \alpha_2, \pi, \tilde{\pi} \) as in Theorem 7.4 and \( \tilde{\Delta} = D(\tilde{q}) \), where \( * \) is the action defined by (4.9) and \( D \) is the operator given in Definition 4.4. Then

(i) \( q \) is a solution of (4.7),

(ii) let \( \tilde{q} = g_{\alpha_1, \alpha_2, \pi, \tilde{\pi}}^* q \) be as in Theorem 7.4, and

\[
\tilde{E} = g_{\alpha_1, \alpha_2, \pi} E \triangle g_{\alpha_1, \alpha_2, \tilde{\pi}}^{-1} \tilde{\Delta}^{-1}.
\]

Then \( \tilde{u} = \tilde{\Delta} * (g_{\alpha_1, \alpha_2, \pi^2}^*(\Delta^{-1} * u)) \) is a solution of (4.12) and \( \tilde{E} \) is a frame of \( \tilde{u} \).

Theorem 7.6. Let \( \gamma : \mathbb{R}^2 \to \mathcal{M}_{n+1,n} \) be a solution of the \((2j-1)\)-th isotropic curve flow (5.3) of B-type, \( g(\cdot, t) \) the isotropic moving frame along \( \gamma(\cdot, t) \), and \( u(\cdot, t) = (g^{-1} g_x - b) \) the solution of the \((2j-1)\)-th isotropic \( B^1_0 \)-KdV flow (4.12) as in Theorem 7.4. Let \( \alpha_1, \alpha_2, \pi, \tilde{\pi}, \tilde{\pi}^2, \Delta, \tilde{u} \) and \( \tilde{E} \) be as in Theorem 7.5. Then \( \gamma(x, t) = \tilde{E}(x, t, 0)e_1 \) is a solution of (5.3) with isotropic curvature \( \tilde{u} \), where \( e_1 = (1, 0, \cdots, 0)^t \).

Proof. Let \( E(x, t, \lambda) \) be the frame of the solution \( u \) of (4.12) satisfying \( E(0, 0, \lambda) = g(0, 0) \). Note that \( E(x, t, 0) \) and \( g(x, t) \) satisfy the same linear system,

\[
\begin{cases}
    g^{-1} g_x = b + u, \\
    g^{-1} g_t = P_{2j-1,0}(u) - \eta_j(u),
\end{cases}
\]

and have the same initial data. So \( E(x, t, 0) = g(x, t) \) for all \( x, t \). By Theorem 7.5, \( \tilde{E} \) is a frame of a new solution \( \tilde{u} \). It follows from Theorem 5.4 (ii) that \( \tilde{\gamma} \) is a solution of (5.3). \( \square \)

Next we write down the formula of BT for \( \gamma \) in terms of \( \gamma \).

Corollary 7.7. Let \( \gamma, u, g, \alpha_1, \alpha_2, \pi, \Delta, \tilde{\pi} \) be as in Theorem 7.6. If \( \alpha_1 \alpha_2 \neq 0 \), then

\[
\tilde{\gamma}(x, t) = g(x, t) \Delta(x, t) \left( I + \frac{\alpha_1 - \alpha_2}{\alpha_2} \tilde{\pi}^2(x, t) + \frac{\alpha_2 - \alpha_1}{\alpha_1} \tilde{\pi}(x, t) \right) e_1
\]

is a solution of (5.3).

Proof. By Theorem 7.6, \( \tilde{\gamma}(x, t) = \tilde{E}(x, t, 0)e_1 \) is a solution of the (5.3). Note that \( g_{\alpha_1, \alpha_2, \pi}(\lambda) \) is holomorphic at \( \lambda = 0 \) and \( g_{\alpha_1, \alpha_2, \pi}(0) \in O(n + 1, n) \). So

\[
\tilde{\gamma} = E(x, t, 0) \Delta(x, t) g_{\alpha_1, \alpha_2, \tilde{\pi}(x, t)}^{-1}(0) \tilde{\Delta}(x, t)^{-1} e_1
\]

is also a solution of (5.3). But \( \tilde{\Delta} \in N_{n}^+ \) implies that \( \tilde{\Delta}(x, t)e_1 = e_1 \). This proves the Corollary. \( \square \)
Remark 7.8. If $\alpha_1 \neq 0$ and $\alpha_2 = 0$, the formula for the new solution $\tilde{\gamma}$ obtained in Theorem 7.6 is more complicated. We use the same notation as in Theorem 7.6 and set
\[ \pi' = I - \pi, \quad (\pi^2)' = I - \pi^2. \]
Write $E(x, t, \lambda)$ as a power series in $\lambda$,
\[ E(x, t, \lambda) = E_0(x, t) + E_1(x, t)\lambda + E_2(x, t)\lambda^2 + \cdots. \]
Then the new solution $\tilde{\gamma}$ obtained in Theorem 7.6 is
\[ \tilde{\gamma}(x, t) = ((\pi^2)'(E_0(x, t)\Delta \pi' - \alpha_1 E_1(x, t)\Delta \pi^2)e_1 + (-\alpha_1 \pi E_1(x, t)\Delta \pi' + \alpha_2^2 \pi E_2(x, t)\Delta \pi^2)e_1. \]

Example 7.9. 1-soliton of the third isotropic curve flow of $B$-type on $M_{2,1}$

Note that $\gamma = (1, x, x^2/2)^t$ is a solution of the third isotropic curve flow (1.5) with isotropic curvature $u = 0$ and $E(x, t, \lambda) = \exp(J_B x + J_B^2 t)$ is a frame of $u = 0$. Next we apply BTs to write down explicit solutions for the isotropic curve flow (1.5),
\[ \gamma_t = u_x \gamma - u \gamma_x. \]

Note that $\eta_3(0) = 0$, so we can choose $\Delta$ to be the identity.
Set $\lambda = z^2$. Then $E(x, t, z^2)$ is equal to
\[ \left( \begin{array}{ccc} \frac{1}{2} \cosh(zx + z^3 t) + 1 & \frac{1}{2} \sinh(zx + z^3 t) & \frac{1}{4} \cosh(zx + z^3 t) - 1 \\ \frac{1}{2} \sinh(zx + z^3 t) & \cosh(zx + z^3 t) & \frac{1}{4} \sinh(zx + z^3 t) \\ \frac{1}{2} \sinh(zx + z^3 t) - 1 & \frac{1}{2} \sinh(zx + z^3 t) & \frac{1}{4} \cosh(zx + z^3 t) + 1 \end{array} \right). \]

Let $\{e_1, e_2, e_3\}$ be the standard basis on $\mathbb{R}^3$, and $V_1 = \mathbb{R} v_1$, $V_2 = \mathbb{R} e_2 \oplus \mathbb{R} e_3$. Then $V_1 \subset V_1^\perp$ and $V_2^\perp \subset V_2$. Let $\alpha_1 \neq \alpha_2 \in \mathbb{R}$. Set $\tilde{p}_1 = E^{-1}(x, t, \alpha_1^2)e_1$, $\tilde{p}_2 = E^{-1}(x, t, \alpha_2^2)e_2$, $\tilde{p}_3 = E^{-1}(x, t, \alpha_3^2)e_3$. And
\[ \tilde{V}_1 = \mathbb{R} \tilde{p}_1, \quad \tilde{V}_2 = \mathbb{R} \tilde{p}_2 \oplus \mathbb{R} \tilde{p}_3. \]

The projection $\tilde{\pi}$ of $\mathbb{R}^{2,1}$ onto $\tilde{V}_1$ with respect to $\mathbb{R}^{2,1} = \tilde{V}_1 \oplus \tilde{V}_2$ is
\[ \tilde{\pi} = (\tilde{p}_1 \quad 0 \quad 0) (\tilde{p}_1 \quad \tilde{p}_2 \quad \tilde{p}_3)^{-1}. \]

To simplify the result, we introduce some notation:
\[ c(x, t, \alpha) = \cosh(\alpha x + \alpha^3 t), \quad s(x, t, \alpha) = \sinh(\alpha x + \alpha^3 t). \quad (7.5) \]
Let $D$ denote the determinant of $(\tilde{p}_1 \quad \tilde{p}_2 \quad \tilde{p}_3)$. A direct computation implies that
\[ D = \frac{1}{4}(c(\alpha_1) + 1)(c(\alpha_2) + 1) - \frac{\alpha_2}{2\alpha_1} s(\alpha_1) s(\alpha_2) + \frac{\alpha_2^2}{4\alpha_1^2} (c(\alpha_1) - 1)(c(\alpha_2) - 1). \]
Apply BT for (4.7) to get a new solution \( \tilde{q}(x,t) = \begin{pmatrix} q_1 & q_2 & 0 \\ 0 & 0 & q_2 \\ 0 & 0 & -q_1 \end{pmatrix} \) of (4.7) (with \( j = 2 \)), where

\[
q_1 = \frac{(\alpha_1^2 - \alpha_2^2)(\alpha_2(c(\alpha_1) - 1)s(\alpha_2) - \alpha_1s(\alpha_1)(c(\alpha_2) + 1))}{4\alpha_1^2D},
\]

\[
q_2 = \frac{(\alpha_1^2 - \alpha_2^2)(\alpha_2(c(\alpha_1) - 1)(c(\alpha_2) - 1) - \alpha_1^2(\alpha_1 + 1)(c(\alpha_2) + 1))}{8\alpha_1^2D}.
\]

Here we use \( \alpha(c) \) and \( s(c) \) to denote functions \( c(c, c, \alpha) \) and \( s(c, c, \alpha) \) respectively. So the new solution is \( \tilde{u} = y(e_{12} + e_{21}) \) for (4.12), where \( y = q_2 + \frac{1}{2}q_1^2 + (q_1)x \). And the corresponding curve flow solution for (1.5) is:

\[
\tilde{\gamma}(x,t) = \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ x^2 & x & 1 \end{pmatrix} \begin{pmatrix} 2\alpha_1^2D - c(\alpha_1) - c(\alpha_2) \\ \frac{1}{\alpha_1}s(\alpha_1)(c(\alpha_2) + 1) - \frac{1}{\alpha_2}(c(\alpha_1) - 1)s(\alpha_2) \\ \frac{2\alpha_1^2D}{\alpha_1^2}\alpha_2(c(\alpha_1) - 1)(c(\alpha_2) + 1) \end{pmatrix}
\]

In particular, let \( \alpha_2 = 0 \). We get the following smooth solution of (1.5),

\[
\tilde{\gamma}(x,t) = \begin{pmatrix} 1 - \frac{\alpha s(x,t,\alpha)}{c(x,t,\alpha) + 1} + \frac{\alpha^2(c(x,t,\alpha) - 1)}{4(c(x,t,\alpha) + 1)}x^2 \\ \frac{2\alpha(c(x,t,\alpha) - 1)}{\alpha^2(c(x,t,\alpha) + 1)} - \frac{\alpha^2(c(x,t,\alpha) - 1)}{2(c(x,t,\alpha) + 1)}x \end{pmatrix},
\]

where \( \alpha \in \mathbb{R} \), and \( c(x,t,\alpha) \) is defined by (7.5). The isotropic curvature of \( \tilde{\gamma} \) is the one-soliton of the KdV,

\[
\tilde{u} = -\alpha_1^2\text{sech}^2\left(\frac{q_1}{2}x + \frac{q_2}{2}t\right).
\]

Next we give a Permutability formula for BTs of (4.7). First we need some Lemmas.

**Lemma 7.10.** Let \( \alpha_i, \beta_i, i = 1, 2 \) be four distinct constants in \( \mathbb{R}\setminus\{0\} \), and \( V_i, W_i \) linear subspaces of \( \mathbb{R}^{n+1} \) for \( i = 1, 2 \) such that

\[
\mathbb{R}^{n+1} = V_1 \oplus V_2, \quad V_1 \subset V_1^\perp, \quad V_2^\perp \subset V_2,
\]

\[
\mathbb{R}^{n+1} = W_1 \oplus W_2, \quad W_1 \subset W_1^\perp, \quad W_2^\perp \subset W_2.
\]

Let \( \pi_1 \) be the projection of \( \mathbb{R}^{n+1} \) onto \( V_1 \) along \( V_2 \), and \( \pi_2 \) the projection of \( \mathbb{R}^{n+1} \) onto \( W_1 \) along \( W_2 \). Set

\[
\begin{align*}
\hat{W}_1 &= g_{\beta_1, \beta_2, \pi_2}(\alpha_1)(V_1), \\
\hat{W}_2 &= g_{\beta_1, \beta_2, \pi_2}(\alpha_2)(V_2).
\end{align*}
\]

Then

(i) \( \hat{W}_1 \subset \hat{W}_1^\perp, \hat{W}_2^\perp \subset \hat{W}_2 \),

(ii) \( \hat{W}_2^\perp = g_{\beta_1, \beta_2, \pi_2}(\alpha_2)(V_2^\perp) \).
Proof. Note that $g_{\beta_1,\beta_2,\pi_2}(\alpha_i) \in O(n+1,n)$, for $i = 1, 2$. So for any $X, Y \in \mathbb{R}^{n+1,n},$
\[
\langle g_{\beta_1,\beta_2,\pi_2}(\alpha_i), g_{\beta_1,\beta_2,\pi_2}(\alpha_i)Y \rangle = \langle X, Y \rangle, \quad i = 1, 2.
\] This proves (i). To prove (ii), note that $X \in \tilde{V}_2^\perp$ if and only if \[
\langle X, g_{\beta_1,\beta_2,\pi_2}(\alpha_2)Y \rangle = \langle g_{\beta_1,\beta_2,\pi_2}(\alpha_1)^{-1}X, Y \rangle = 0, \quad \forall \ Y \in V_2.
\] Hence $g_{\beta_1,\beta_2,\pi_2}(\alpha_i)^{-1}X \in V_2^\perp$. This proves (ii). \[\square\]

**Proposition 7.11.** Let $\alpha_1, \alpha_2, \beta_1, \beta_2$ be four distinct non-zero real numbers, and $\pi_i, \tau_i$ projections of $\mathbb{R}^{n+1,n}$ satisfying $\pi_i \pi_i^\perp = \pi_i^\perp \pi_i = \tau_i \tau_i^\perp = \tau_i^\perp \tau_i = 0$ for $1 \leq i \leq 2$. Set
\[
V_1 = \text{Im}(\pi_1), \quad V_2 = \text{Ker}(\pi_1), \quad W_1 = \text{Im}(\pi_2), \quad W_2 = \text{Ker}(\pi_2),
\]
\[
\hat{V}_1 = \text{Im}(\tau_1), \quad \hat{V}_2 = \text{Ker}(\tau_1), \quad \hat{W}_1 = \text{Im}(\tau_1), \quad \hat{W}_2 = \text{Ker}(\tau_1).
\]
Then
\[
g_{\beta_1,\beta_2,\pi_2}g_{\alpha_1,\alpha_2,\pi_1} = \alpha_1, \alpha_2, \pi_1, \pi_2
\] (7.6)
if and only if
\[
\begin{cases}
\hat{V}_1 = g_{\alpha_1,\alpha_2,\tau_1}(\beta_2)W_1, & \hat{V}_2 = g_{\alpha_1,\alpha_2,\tau_1}(\beta_1)W_2, \\
\hat{W}_1 = g_{\beta_1,\beta_2,\alpha_1}(\hat{V}_1), & \hat{W}_2 = g_{\beta_1,\beta_2,\alpha_1}(\hat{V}_2).
\end{cases}
\] (7.7)

Proof. We first prove (7.7) is sufficient. The proof of Lemma 7.10 implies that $\hat{V}_1 \subset \hat{V}_1^\perp, \hat{V}_2^\perp \subset \hat{V}_2^\perp$. Since $g_{\alpha_1,\alpha_2,\pi_1} = g_{\alpha_1,\alpha_2,\pi_1}^\perp$, it suffices to prove that
\[
g_{\beta_1,\beta_2,\pi_2}g_{\alpha_1,\alpha_2,\pi_1} = \alpha_1, \alpha_2, \pi_1, \pi_2
\] (7.8)

It is equivalent to prove that the residues at $\alpha_1, \beta_2$ on both sides of (7.8) are equal, i.e.,
\[
\begin{align*}
g_{\beta_1,\beta_2,\pi_2}(\alpha_1)\pi_1 &= \tau_1 g_{\beta_1,\beta_2,\pi_2}(\alpha_1), & g_{\beta_1,\beta_2,\pi_2}(\alpha_2)\pi_1 &= \tau_1 g_{\beta_1,\beta_2,\pi_2}(\alpha_2), \\
g_{\beta_1,\beta_2,\pi_2}(\alpha_1)\pi_2 &= \tau_1^\perp g_{\beta_1,\beta_2,\pi_2}(\alpha_1), & g_{\beta_1,\beta_2,\pi_2}(\alpha_2)\pi_2 &= \tau_1^\perp g_{\beta_1,\beta_2,\pi_2}(\alpha_2).
\end{align*}
\]
These are true because we have
\[
\begin{align*}
\text{Im}(\tau_1) &= \text{g}_{\beta_1,\beta_2,\pi_2}(\alpha_1)\text{Im}(\pi_1), & \text{Im}(I - \tau_1) &= \text{g}_{\beta_1,\beta_2,\pi_2}(\alpha_2)\text{Im}(I - \pi_1), \\
\text{Im}(\tau_2) &= \text{g}_{\alpha_1,\alpha_2,\tau_1}(\beta_2)\text{Im}(\pi_2), & \text{Im}(I - \tau_2) &= \text{g}_{\alpha_1,\alpha_2,\tau_1}(\beta_1)\text{Im}(I - \pi_2).
\end{align*}
\]

The computation given for the sufficient part also proves necessary part. \[\square\]

**Theorem 7.12.** [Permutability]
Let $q$ be a solution of (4.7), and $\alpha_i, \beta_i, \pi_i, \tau_i$ as in Proposition 7.11 for $i = 1, 2$ satisfying (7.6). Let
\[
q_1 = g_{\alpha_1,\alpha_2,\pi_1}q = q + (\alpha_1 - \alpha_2)[\beta_1, \pi_1 - \pi_1^\perp],
\]
\[
q_2 = g_{\beta_1,\beta_2,\pi_2}q = q + (\beta_1 - \beta_2)[\beta_1, \pi_2 - \pi_2^\perp].
\]
Let the projections of $R$ be frames of $q$. Since $F$ is as in (4.2), given Theorem 7.12 is an algebraic function of $q, \tilde{\pi}_1, \tilde{\tau}_2$. Then

$$q_{12} = g_{\beta_1, \beta_2, \tau_2}(g_{\alpha_1, \alpha_2, \tau_1} \tilde{\tau}_2 q)$$

and $q_{21} = g_{\alpha_1, \alpha_2, \tau_1} g_{\beta_1, \beta_2, \tau_2} q$ solutions of (4.7) constructed from Bäcklund transformations (Theorem 7.4). Let $\tilde{\tau}_1$ and $\tilde{\tau}_2$ be the projections of $\mathbb{R}^{n+1,n}$ with

$$\text{Im}(\tilde{\tau}_1) = g_{\beta_1, \beta_2, \tilde{\tau}_2}(\alpha_1)\text{Im}(\tilde{\pi}_1), \quad \text{Ker}(\tilde{\tau}_1) = g_{\beta_1, \beta_2, \tilde{\tau}_2}(\alpha_2)\text{Ker}(\tilde{\pi}_1), \quad (7.9)$$

$$\text{Im}(\tilde{\tau}_2) = g_{\alpha_1, \alpha_2, \tilde{\tau}_1}(\beta_2)\text{Im}(\tilde{\pi}_2), \quad \text{Ker}(\tilde{\tau}_2) = g_{\alpha_1, \alpha_2, \tilde{\tau}_1}(\beta_1)\text{Ker}(\tilde{\pi}_2). \quad (7.10)$$

Then

$$q_{12} = q_{21} = q_1 + (\beta_1 - \beta_2)[\beta, \tilde{\tau}_2 - \tilde{\tau}_1^2] = q_2 + (\alpha_1 - \alpha_2)[\beta, \tilde{\tau}_1 - \tilde{\tau}_1^2], \quad (7.11)$$

where $\beta$ is as in (4.2).

**Proof.** Let $F$ be a frame of the solution $q$ of (4.7). Theorem 7.4 implies that

$$F_1 = g_{\alpha_1, \alpha_2, \pi_1} F_1 g_{\alpha_1, \alpha_2, \pi_1}^{-1},$$

$$F_2 = g_{\beta_1, \beta_2, \pi_2} F_2 g_{\beta_1, \beta_2, \pi_2}^{-1},$$

are frames of $q_1$ and $q_2$ respectively. Apply Theorem 7.4 to $q_2$ and $q_1$ to see that there are projections $\tilde{\tau}_1(x, t)$ and $\tilde{\tau}_2(x, t)$ such that

$$F_{12} = g_{\beta_1, \beta_2, \tau_2} g_{\alpha_1, \alpha_2, \tau_1} F_1 g_{\alpha_1, \alpha_2, \tau_1}^{-1} g_{\beta_1, \beta_2, \tau_2}^{-1},$$

$$F_{21} = g_{\alpha_1, \alpha_2, \tau_1} g_{\beta_1, \beta_2, \tau_2} F_2 g_{\beta_1, \beta_2, \tau_2}^{-1} g_{\alpha_1, \alpha_2, \tau_1}^{-1},$$

are frames of $q_{12}$ and $q_{21}$ respectively. Let $f = g_{\beta_1, \beta_2, \tau_2} g_{\alpha_1, \alpha_2, \tau_1}$. By assumption, $f = g_{\alpha_1, \alpha_2, \tau_1} g_{\beta_1, \beta_2, \tau_2}$. So we obtain

$$f F = F_{12} g_{\beta_1, \beta_2, \tau_2} g_{\alpha_1, \alpha_2, \tau_1} = F_{21} g_{\alpha_1, \alpha_2, \tau_1} g_{\beta_1, \beta_2, \tau_2}.$$ 

This gives two factorizations of $f F$ as the product of elements in $(\hat{\mathbb{B}}_n^{(1)})_+$ and $(\hat{\mathbb{B}}_n^{(1)})_-$. Since the factorization of $f F$ in $(\hat{\mathbb{B}}_n^{(1)})_+(\hat{\mathbb{B}}_n^{(1)})_-$ is unique, we get $F_{12} = F_{21}$ and

$$g_{\beta_1, \beta_2, \tau_2} g_{\alpha_1, \alpha_2, \tau_1} = g_{\alpha_1, \alpha_2, \tau_1} g_{\beta_1, \beta_2, \tau_2}.$$ 

(7.12)

It follows from (7.12) and Proposition 7.11 that $\hat{\tau}$ satisfies (7.9) and (7.10). Since $F_{12} = F_{21}$, we have $q_{12} = q_{21}$. Formula (7.11) follows from Theorem 7.4.

**Remark 7.13.**

(i) The solution $q_{12}$ given Theorem 7.12 is an algebraic function of $q, \tilde{\pi}_1, \tilde{\tau}_2$.

(ii) We apply Theorem 7.4 to the trivial solution $q = 0$ to get $k$ 1-soliton solutions $q_i$ of (4.7) and their frames $F_i$ for $1 \leq i \leq k$. Apply the Permutability Theorem 7.12 to construct 2-soliton solutions $q_{ij}$ of (4.7). Apply Theorem 7.12 to $q_2, q_{12}$ and $q_{23}$ to get the 3-soliton solution $q_{123}$. Continue this way to get explicit formulas for $k$-soliton solutions of (4.7) and their frames algebraically from one-solitons. Suppose $\hat{F}$ is a frame of a $k$-soliton solution $\hat{q}$ of (4.7), and $\hat{\Delta} = D(\hat{q})$, where $D$ is the operator given in Definition 3.4. Then $\hat{E} = \hat{F} \hat{\Delta}^{-1}$ is a frame of the solution $\hat{u} = \hat{\Delta} * \hat{q}$ of the $(2j - 1)$-th $\hat{B}_n^{(1)}$-KdV.
flow \[1.12\] and \(\tilde{\gamma}(x, t) = \tilde{E}(x, t, 0)e_1\) is an explicit solution of the isotropic curve flow \[5.3\].

8. Bäcklund transformations for the \(\hat{A}^{(2)}_{2n}\)-KdV flows

We proceed the same way as for the \(\hat{B}^{(1)}_n\)-KdV flows in section 7 to construct BTs for isotropic curve flows of A-type on \(\mathcal{M}_{n+1,n}\).

Let \(\hat{A}^{(2)}_{2n}\) denote the group of smooth maps \(f : S^1 \to SL(2n + 1, \mathbb{C})\) satisfying
\[
\overline{f(\lambda)} = f(\lambda), \quad f(-\lambda)C_n f(\lambda) = C_n,
\]
and \((\hat{A}^{(2)}_{2n})^+\) the subgroup of \(f \in \hat{A}^{(2)}_{2n}\) that can be extended to a holomorphic map on \(\mathbb{C}\), and \((\hat{A}^{(2)}_{2n})^-\) the subgroup of \(f \in \hat{A}^{(2)}_{2n}\) that is the boundary value of a holomorphic map \(\hat{f}\) defined on \(\epsilon^{-1} \leq |\lambda| \leq \infty\) such that \(\hat{f}(\infty) = I\) for some small \(\epsilon > 0\). Then the Lie algebras of \(\hat{A}^{(2)}_{2n}\) and \((\hat{A}^{(2)}_{2n})^\pm\) are \(A^{(2)}_{2n}\) and \((\hat{A}^{(2)}_{2n})^\pm\) respectively.

**Definition 8.1.** If \(V\) is a linear subspace of \(\mathbb{R}^{n+1,n}\) such that \(\mathbb{R}^{n+1,n} = V \oplus V^\perp\), then we call the projection of \(\mathbb{R}^{n+1,n}\) onto \(V\) along \(V^\perp\) an \(O(n+1,n)\)-projection.

The proofs of the following two Propositions are straight forward.

**Proposition 8.2.**
(a) \(\pi\) is an \(O(n+1,n)\)-projection if and only if \(\pi^2 = \pi = \pi^\sharp\).
(b) If \(V\) is a linear subspace of \(\mathbb{R}^{n+1,n}\), then the restriction of \(\langle , \rangle\) to \(V\) is non-degenerate if and only if \(\mathbb{R}^{n+1,n} = V \oplus V^\perp\).

**Proposition 8.3.** Let \(\pi\) be an \(O(n+1,n)\)-projection of \(\mathbb{R}^{n+1,n}\) onto \(V\) along \(V^\perp\), and \(\alpha \in \mathbb{R}\{0\}\). Set
\[
g_{\alpha,\pi}(\lambda) = I + \frac{2\alpha}{\lambda - \alpha}(I - \pi),
\]
Then \(g_{\alpha,\pi} \in (\hat{A}^{(2)}_{2n})^-\).

**Theorem 8.4.** Let \(F(x, t, \cdot) \in (\hat{A}^{(2)}_{2n})^+\) be a frame of a solution \(q\) of \[4.19\], \(\pi\) be an \(O(n+1,n)\)-projection onto \(V\) along \(V^\perp\), \(\alpha \in \mathbb{R}\{0\}\) a constant, and \(g_{\alpha,\pi}\) defined as in \[8.2\]. Set \(\tilde{V}(x, t) = F(x, t, \alpha)^{-1}(V)\), \(\tilde{\pi}(x, t)\) the \(O(n+1,n)\)-projection onto \(\tilde{V}(x, t)\) and \(\tilde{F}(x, t, \lambda) = g_{\alpha,\pi}(\lambda)F(x, t, \lambda)g_{\alpha,\tilde{\pi}(x,t)}(\lambda)^{-1}\). Then
1. \(\mathbb{R}^{n+1,n} = \tilde{V}(x, t) \oplus \tilde{V}(x, t)^\perp\),
2. \(\tilde{q} = q + 2\alpha[\tilde{\pi}, e_{1,2n+1}]\) is a solution of \[4.19\] and \(\tilde{F}\) is a frame of \(\tilde{q}\).
(We will use \(g_{\alpha,\pi} \cdot q\) to denote the new solution \(\tilde{q}\).)
Proof. Since $F(x,t, \cdot)$ satisfies (8.1) and $\alpha \in \mathbb{R}$, $F(x,t, \alpha) \in O(n + 1, n)$. Hence the restriction to $\tilde{V}(x,t)$ is non-degenerate. This proves statement (1).

By Proposition 8.3, $g_{\alpha, \tilde{\pi}}(x,t) \in (\hat{A}_{2n}^{(2)})_-$. So to prove (2), it suffices to prove that $\tilde{F}(x,t, \lambda)$ is holomorphic in $\lambda$. To prove this, we only need to show that the residues of $\tilde{F}(x,t, \lambda)$ at $\pm \alpha$ are zero. Note that the residue of $\tilde{F}(x,t, \lambda)$ at $\alpha$ is

$$2\alpha(I - \pi)F(x,t, \alpha)\tilde{\pi}.$$ 

Since $\text{Im}(\tilde{\pi}(x,t)) = \tilde{V}(x,t)$ and $F(x,t, \alpha)\tilde{V}(x,t) = V$, we have

$$F(x,t, \alpha)\text{Im}(\tilde{V}(x,t)) = V.$$ 

Therefore the residue of $\tilde{F}(x,t, \lambda)$ at $\lambda = \alpha$ is zero. This proves $\tilde{F}(x,t, \lambda)$ is holomorphic for $\lambda \in \mathbb{C}$. 

**Theorem 8.5.** Let $E(x,t, \cdot) \in (\hat{A}_{2n}^{(2)})_+$ be a frame of a solution $u$ of (4.21), i.e., $u_t = [\partial_x + b + u, S_{2j-1,0}(u) - \tilde{\eta}_j(u)]$. Let $\Delta(x,t) \in N_n^+$ satisfying $\Delta_t\Delta^{-1} = \tilde{\eta}_j(u)$. Then

(i) $q = \Delta^{-1} * u.$ is a solution of (4.19) and $F = E\Delta$ a frame for $q$.

(ii) Let $\alpha, \pi, \tilde{\pi}$ be as in Theorem 8.4 and $\tilde{q} = g_{\alpha, \tilde{\pi}} \cdot q$. Then

$$\tilde{E} = g_{\alpha, \pi}E\Delta g_{\alpha, \tilde{\pi}}^{-1} \Delta^{-1}$$

is a frame of a new solution $\tilde{u} = \tilde{\Delta} \ast (g_{\alpha, \pi} \ast (\Delta \ast u))$ of (4.21).

**Theorem 8.6.** Let $\gamma : \mathbb{R}^2 \to M_{n+1,n}$ be a solution of the $(2j-1)$-th isotropic curve flow (5.3) of $A$-type, $g(\cdot, t)$ the isotropic moving frame along $\gamma(\cdot, t)$, and $u(\cdot, t) = (g^{-1}g_x - b)$ the solution of the $(2j-1)$-th $\hat{A}_{2n}^{(2)}$-KdV flow (4.21) as in Theorem 7.4. Let $\alpha, \pi, \tilde{\pi}, \Delta, \tilde{E}$ and $\tilde{u}$ be as in Theorem 8.4. Then $\tilde{\gamma}(x,t) = \tilde{E}(x,t,0)e_1$ is is a solution of (5.3) and $\tilde{u}$ is its isotropic curvature, where $e_1 = (1, 0, \cdots, 0)^t$.

**Corollary 8.7.** Let $\gamma, g, \alpha_1, \alpha_2, \pi, \tilde{\pi}, \Delta$ be as in Theorem 8.6. If $\alpha_1\alpha_2 \neq 0$, then

$$\tilde{\gamma}(x,t) = g(x,t)\Delta(x,t)(2\tilde{\pi}(x,t) - I_{2n+1})e_1$$

is a solution of (5.3).
Example 8.8. 1-soliton of the isotropic curve flow of A-type on $\mathcal{M}_{2,1}$ of A-type

Note that $\gamma = (1, x, \frac{q}{x})^t \in \mathcal{M}_{2,1}$ is a solution of (1.6) with isotropic curvature $q = 0$. Set $\lambda = z^3$,

$$D(z) = \text{diag}(1, z, z^2), \quad \Xi = (\alpha(i-1)(j-1))_{3 \times 3}, \quad \sigma = e_{12} + e_{23} + e_{31},$$

$$A_i(x, t, z) = \exp(\alpha^{i-1} z x + (\alpha^{i-1} z^5 t), \quad i = 1, 2, 3,$$

$$(m_1(x, t, z), m_2(x, t, z), m_3(x, t, z)) = (e^{A_1}, \ldots, e^{A_3})\Xi.$$

Then the extended frame $E(x, t, z^3)$ of $q = 0$ is

$$E(x, t, z^3) = \frac{1}{3}D(z)^{-1} \begin{pmatrix} m_1(x, t, z) & m_2(x, t, z) & m_3(x, t, z) \\ m_3(x, t, z) & m_1(x, t, z) & m_2(x, t, z) \\ m_2(x, t, z) & m_3(x, t, z) & m_1(x, t, z) \end{pmatrix} D(z).$$

Apply Theorem [8.4] by choosing $V = R_v$ for some constant $v \in \mathbb{R}^{2,1}$ with $\langle v, v \rangle \neq 0$. Then we have

$$\ddot{v} = E(-x, -t, k^3)v, \quad \dddot{v} = \frac{1}{3}(e^{A_1}, e^{A_2}, e^{A_3})\Xi^{-1}D(k)(v_1, v_2, v_3)^t.$$

In particular, we can choose $v$ such that

$$\ddot{v} = (e^{A_1} + \alpha^2 e^{A_2}, \frac{1}{k}(e^{A_1} + \alpha e^{A_2}), \frac{1}{k^2}(e^{A_1} + e^{A_2}))^t.$$

Apply BT for (4.19) (Theorem 8.4), we get a new solution $\tilde{q} = \tilde{q}_1(e_{11} - e_{33}) + \tilde{q}_2(e_{12} + e_{23})$, where

$$\tilde{q}_1 = 2k \frac{(e^{A_1} + e^{A_2})^2}{4\alpha e^{A_1} + A_2 - e^{2A_1} - \alpha^2 e^{2A_2}}$$

$$\tilde{q}_2 = -2k \frac{e^{2A_1} + \alpha e^{2A_2} + (1 + \alpha)e^{A_1} + A_2}{4\alpha e^{A_1} + A_2 - e^{2A_1} - \alpha^2 e^{2A_2}}.$$

Hence the new solution $\ddot{v} = y(e_{12} + e_{21})$ of (1.6), where

$$y = \tilde{q}_2 + \frac{1}{2}\tilde{q}_1^2 + (\tilde{q}_1)x.$$

Let $M = 4\alpha e^{A_1} + A_2 - e^{2A_1} - \alpha^2 e^{2A_2}$, the corresponding curve flow solution for (1.6) is

$$\tilde{\gamma} = \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ \frac{x^2}{2} & x & 1 \end{pmatrix} \begin{pmatrix} \frac{2}{3}(e^{2A_1} + (1 + \alpha^2)e^{A_1} + A_2 + \alpha^2 e^{2A_2} + \frac{1}{2k^2}\tilde{q}_2^2) \\ \frac{1}{2k}\tilde{q}_2 \\ -\frac{1}{2k}\tilde{q}_1 \end{pmatrix}.$$

Next we derive the permutability formula for BTs of (1.19). First we need a Proposition.

**Proposition 8.9.** Let $\alpha_1, \alpha_2 \in \mathbb{R}\setminus\{0\}$, and $\pi_i$ an $O(n + 1, n)$-projection of $\mathbb{R}^{n+1,n}$ for $i = 1, 2$. If $|\alpha_1| \neq |\alpha_2|$, then

$$\phi = \alpha_1 - \alpha_2 - 2\alpha_2 \pi_2 - 2\alpha_1 \pi_1$$

is invertible. Moreover,
(i) $\tau_i = \phi \pi_i \phi^{-1}$ is an $O(n + 1, n)$-projection for $i = 1, 2$,

(ii) $g_{\alpha_2, \tau_2} g_{\alpha_1, \tau_1} = g_{\alpha_1, \tau_1} g_{\alpha_2, \tau_2}$ if and only if $\tau_i = \phi \pi_i \phi^{-1}$ for $i = 1, 2$.

Proof. Since $\pi_i^2 = \pi_i$, the eigenvalues of $\pi_i$ are $0, \pm 1$. Consider the eigenvalue of $\phi$, if $\pi_1 = \pi_2$, then the eigenvalue of $\phi$ is $\alpha_2 - \alpha_1$. If $\pi_1 \neq \pi_2$, then the possible eigenvalues for $\pi_i$ are $\pm (\alpha_1 + \alpha_2)$. Therefore, as long as $|\alpha_1| \neq |\alpha_2|$, $\phi$ is invertible.

From the definition of $\tau_i$, we have $\tau_i^2 = \tau_i$ for $i = 1, 2$. To prove (i), we only need to show that $\tau_i^2 C_n = C_n \tau_i$. It is equivalent to prove that

$$\pi_i^2 \phi^i C_n \phi = \phi^i C_n \phi \pi_i, \quad i = 1, 2.$$  \hspace{1cm} (8.3)

It follows from direct computation and the fact that both $\pi_1$ and $\pi_2$ are $O(n + 1, n)$-projections, we have

$$\phi^i C_n \phi = (\alpha_1 - \alpha_2)^2 C_n + 4\alpha_1 \alpha_2 C_n (\pi_1 + \pi_2 - \pi_2 \pi_1 - \pi_1 \pi_2).$$

Hence

$$\pi_i^2 \phi^i C_n \phi = (\alpha_1 - \alpha_2)^2 C_n \pi_1 + 4\alpha_1 \alpha_2 C_n (\pi_1 + \pi_2 - \pi_2 \pi_1 - \pi_1 \pi_2)$$

$$= \phi^i C_n \phi \pi_1$$

Similarly, $\pi_2^2 \phi^i C_n \phi = \phi^i C_n \phi \pi_2$.

To prove (ii), let

$$Y_1 = \alpha_1 - 2\alpha_1 \pi_1, \quad Y_2 = \alpha_2 - 2\alpha_2 \tau_2,$$

$$Z_1 = \alpha_1 - 2\alpha_1 \tau_1, \quad Z_2 = \alpha_2 - 2\alpha_2 \tau_2.$$  

We claim that $(\lambda + Y_2)(\lambda + Y_1) = (\lambda + Z_1)(\lambda + Z_2)$. Compare coefficients as an expansion of $\lambda$ to get

$$\begin{cases}
Y_1 + Y_2 = Z_1 + Z_2, \\
Y_2 Y_1 = Z_1 Z_2.
\end{cases}$$

So we get

$$Z_1 = (Y_1 - Z_2)Y_1(Y_1 - Z_2)^{-1}, \quad Y_2 = (Y_1 - Z_2)Z_2(Y_1 - Z_2)^{-1}.$$  

Hence $\tau_i = \phi \pi_i \phi^{-1}, i = 1, 2$. \hfill $\Box$

**Theorem 8.10. [Permutability]**

Let $\alpha_1, \alpha_2 \in \mathbb{R} \setminus \{0\}$ with $|\alpha_1| \neq |\alpha_2|$, and $\pi_i$ an $O(n + 1, n)$-projection for $i = 1, 2$. Let $q$ be a solution of (4.19), $q_i = g_{\alpha_i, \tau_i} \cdot q = q + 2\alpha_i [\tilde{\pi}_i, e_{1, 2n+1}]$, and

$$q_{12} = g_{\alpha_2, \tau_2} \cdot (g_{\alpha_1, \tau_1} \cdot q), \quad q_{21} = g_{\alpha_1, \tau_1} \cdot (g_{\alpha_2, \tau_2} \cdot q)$$

solutions of (4.19) obtained from Bäcklund transformations. Set

$$\phi = \alpha_1 - \alpha_2 + 2\alpha_2 \tilde{\pi}_2 - 2\alpha_2 \pi_1, \quad \tilde{\phi} = \alpha_1 - \alpha_2 + 2\alpha_2 \tilde{\pi}_2 - 2\alpha_2 \pi_1,$$

$$\tau_i = \phi \pi_i \phi^{-1}, \quad \tilde{\tau}_i = \phi \pi_i \phi^{-1}, \quad i = 1, 2.$$  

Then

$$q_{12} = q_{21} = q_1 + 2\alpha_2 [\tilde{\tau}_2, e_{1, 2n+1}] = q_2 + 2\alpha_1 [\tilde{\tau}_1, e_{1, 2n+1}].$$
Similarly, we can use Theorems 8.4 and 8.6 to construct explicit $k$-soliton solutions for isotropic curve flows of A-type.

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