Asymptotics of radiation fields in asymptotically Minkowski spacetimes

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Radiation fields in Minkowski space

- Suppose $u$ solves $\Box u = 0$ with smooth, compactly supported initial data in $\mathbb{R} \times \mathbb{R}^n$. ($\Box u = f \in C^\infty_c(\mathbb{R}^{n+1})$ with $u = 0$ for $t \ll 0$ works as well.)
- In polar coordinates $(t, r, \omega)$, introduce $s = t - r \rho = \frac{1}{r}$, and introduce
  \[ v(\rho, s, \omega) = \rho^{-\frac{n-1}{2}} u \left( s + \frac{1}{\rho}, \frac{1}{\rho\omega} \right) \]

**Fact**

$v$ is smooth down to $\rho = 0$, i.e., to null infinity.

**Definition**

The forward radiation field is the function given by

\[ \mathcal{R}_+[u](s, \omega) = \partial_s v(0, s, \omega) \]

- In 1-d, these are the waves moving to the left and right.
The radiation field is of independent interest: $\mathcal{R}_+$ is

- an FIO
- a unitary isomorphism $\dot{H}^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \to L^2(\mathbb{R} \times S^{n-1})$
- a translation representation
- related to the Radon transform
- a concrete realization of the wave operators in Lax-Phillips scattering theory

The radiation field is understood in a variety of geometric contexts. See Friedlander, Sá Barreto, Wang, Melrose–Wang, Sá Barreto–Wunsch, . . . .
Motivating question

How does $R_+$ behave as $s \to \infty$?

- On Minkowski space $\mathbb{R} \times \mathbb{R}^n$,
  \[
  |R_+[u](s, \omega)| \lesssim \begin{cases} 
  (1 + s)^{-\infty} & n \text{ odd} \\
  (1 + s)^{-\frac{n+1}{2}} & n \text{ even}
  \end{cases}
  \]

- Klainerman–Sobolev inequalities yield
  \[
  |R_+[u](s, \omega)| \lesssim (1 + s)^{-1/2}
  \]
  on perturbations of Minkowski space.
Where does the radiation field live?

• Take the radial compactification of Minkowski space 
  \((\rho = (t^2 + r^2)^{-1/2}, \theta = (t, r)/\rho \in S^1)\):

  \[
dt^2 - \sum dz_j^2 = \cos 2\theta \frac{d\rho^2}{\rho^4} - \cos 2\theta \frac{d\theta}{\rho^2} + 2 \sin 2\theta \frac{d\rho}{\rho^2} \frac{d\theta}{\rho} - \sin^2 \theta \frac{d\omega^2}{\rho^2}.
  \]

• Introduce \(v = \cos 2\theta\) and metric becomes

  \[
  v \frac{d\rho^2}{\rho^4} - \frac{v}{4(1 - v^2)} \frac{dv^2}{\rho^2} - \frac{d\rho}{\rho^2} \frac{dv}{\rho} - \frac{1 - v}{2} \frac{d\omega^2}{\rho^2}
  \]

• The radiation field is the (rescaled) restriction of the solution \(u\) to the front face of the blow up of \(\{v = \rho = 0\}\).
Asymptotically Minkowski spaces

- Suppose \((M, g)\) is an \((n + 1)\)-dimensional compact manifold with connected boundary, \(g\) a time-oriented Lorentzian metric on \(M\) that extends to a nondegenerate quadratic form on \(scTM\).

**Definition**

\(g\) is a Lorentzian scattering metric if there is a boundary defining function \(\rho\) and a Morse-Bott function \(v \in C^\infty(M)\) so that 0 is a regular value for \(v\) and, in a neighborhood of \(\partial M\),

\[
g = v \frac{d\rho^2}{\rho^4} - 2f \frac{d\rho}{\rho^2} \frac{dv}{\rho} - \frac{h}{\rho^2},
\]

where \(f = \frac{1}{2} + O(v) + O(\rho)\) near \(v = \rho = 0\), and \(h|_{\text{Ann}(d\rho, dv)}\) is positive definite near \(\partial M\).

- Also impose a non-trapping assumption on the light rays.
Proposition

The radiation field exists for metrics of this form.

The radiation field blow-up:
Asymptotics of radiation fields

**Theorem**

Suppose \((M, g)\) is as above (non-trapping Lorentzian scattering), \(u\) is a tempered solution of \(\Box_g u = f \in C_c^\infty(M^\circ)\). Then \(R_+[u]\) has an asymptotic expansion of the form

\[
R_+[u] \sim \sum_j \sum_{\kappa \leq m_j} s^{-i\sigma_j} |\log s|^\kappa a_{j\kappa}
\]

**Note**

This is really a full asymptotic expansion for \(u\) in terms of \(\rho\) and \(s\).

**Note**

This is not an existence theorem!
Some remarks

- The $\sigma_j$ and $m_j$ in the expansion are related to the resonances of an asymptotically hyperbolic problem in the region of $\partial M$ where $\{ v > 0 \}$ (and in particular are independent of $u$).

  - This region inherits an AH metric: $k(X, Y) = -\frac{1}{v} g(\rho \tilde{X}, \rho \tilde{Y})$, where $\tilde{X}$, $\tilde{Y} \perp \rho^2 \partial \rho$. The $\sigma_j$ are the locations of the poles of an operator related to $(\Delta_k - \sigma^2)^{-1}$.

- Resonance gap (known) for $k$ yields rate of decay for $\mathcal{R}_+[u]$.

- In Minkowski space, $k$ is the hyperbolic metric, and the expansion for $u$ is of the form

  $$u \sim \begin{cases} O(\rho^{\frac{n-1}{2}} s^{-\infty}) & n \text{ odd} \\ \sum_j \rho^{\frac{n-1}{2}} s^{-\frac{n-1}{2} - j} a_j & n \text{ even} \end{cases}$$
Ideas in the proof

- Much heavy lifting done in recent paper of Vasy.
- Mellin transform reduces to problem on $\partial M$.
- $P_\sigma$ fits into framework of Vasy paper, yielding a preliminary asymptotic expansion.
- Propagation of singularities estimate implies remainder term is lower order.
- Work of Haber-Vasy implies the coefficients are $L^2$-based conormal distributions.
- Coefficients are classical conormal, so have expansions in $v$.
- Blow-up turns $v$ expansion into $s$ expansion (since $v = s\rho$).
Thank you