Stationary Axisymmetric Solutions and their Energy Contents in Teleparallel Equivalent of Einstein Theory

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We apply the energy-momentum tensor which is coordinate independent to calculate the energy content of the axisymmetric solutions. Our results are compared with what have been obtained before within the framework of Einstein general relativity and Møller’s tetrad theory of gravitation.

1. Introduction

General geometric arena of PGT, the Riemann-Cartan space $U_4$, may be a priori restricted by imposing certain conditions on the curvature and the torsion. Thus, Einstein’s GR is defined in Riemann space $V_4$, which is obtained from $U_4$ by the requirement of vanishing torsion. Another interesting limit of PGT is the teleparallel or Weitzenböck geometry $T_4$. The vanishing of the curvature means that parallel transport is path independent. The teleparallel geometry is, in sense, complementary to Riemannian: curvature vanishes, and torsion remains to characterize the parallel transport. Of particular importance for the physical interpretation of the teleparallel geometry the fact that there is a one-parameter family of teleparallel Lagrangians which is empirically equivalent to GR [8, 23, 9, 12]. For the parameter value $B = 1/2$ the Lagrangian of the theory coincides, modulo a four-divergence, with the Einstein-Hilbert Lagrangian, and defines (TEGR).

The teleparallel equivalent of general relativity (TEGR) is a viable alternative geometrical description of Einstein’s general relativity written in terms of the tetrad field [10] and continue to be object of thorough investigations [33, 34, 35, 36]. In the framework of the TEGR it has been possible to address the longstanding problem of defining the energy, momentum and angular momentum of the gravitational field [13, 14, 15]. The tetrad field seems...
to be a suitable field quantity to address this problem, because it yields the gravitational field and at the same time establishes a class of reference frames in space-time [16]. Moreover there are simple and clear indications that the gravitational energy-momentum defined in the context of the TEGR provides a unified picture of the concept of mass-energy in special and general relativity.

The tetrad formulation of gravitation was considered by Møller in connection with attempts to define the energy of gravitational field [5, 6]. For a satisfactory description of the total energy of an isolated system it is necessary that the energy-density of the gravitational field is given in terms of first- and/or second-order derivatives of the gravitational field variables. It is well-known that there exists no covariant, nontrivial expression constructed out of the metric tensor. However, covariant expressions that contain a quadratic form of first-order derivatives of the tetrad field are feasible. Thus it is legitimate to conjecture that the difficulties regarding the problem of defining the gravitational energy-momentum are related to the geometrical description of the gravitational field rather than are an intrinsic drawback of the theory [13, 17].

Definition of the angular momentum of the gravitational field is given in the framework of the TEGR [18]. In similarity to the definition of the gravitational energy-momentum, Maluf et al. [18] have interpreted the appropriate constraint equations as equations that defined the gravitational angular momentum. This definition turned out to be coordinate independent. The definition of $P^a$ is invariant under global $SO(3, 1)$ transformations. It has been argued elsewhere [19] that it makes sense to have a dependence of $P^a$ on the frame. The energy-momentum in classical theories of particles and fields does depend on the frame, and it has been asserted that such dependence is a natural property of the gravitational energy-momentum. The total energy of a relativistic body, depends on the frame.

A well posed and mathematically consistence expression for the gravitational energy has been developed [13]. It arises in the realm of the Hamiltonian formulation of the TEGR [20] and satisfies several crucial requirements for any acceptable definition of gravitational energy. The gravitational energy-momentum $P^a$ [13, 14] obtained in the framework of the TEGR has been investigated in the context of several distinct configuration of the gravitational filed. For asymptotically flat space-times $P^{(0)}$ yields the ADM energy [31]. In the context of tetrad theories of gravity, asymptotically flat space-times may be characterized by the asymptotic boundary condition

$$
\epsilon_{a\mu} \cong \eta_{a\mu} + \frac{1}{2} h_{a\mu}(1/r),
$$

and by the condition $\partial_{\mu} e_{a\mu} = O(1/r^2)$ in the asymptotic limit $r \to \infty$, with $\eta_{ab} = (-1, +1, +1, +1)$ is the metric of Minkowski space-time. An important property of tetrad fields that satisfy the above as that in the flat space-time limit one has $e_{a\mu}(t, x, y, z) = \delta_{a\mu}$, and therefore the torsion tensor $T_{a\mu\nu} = 0$. Maluf [16] has extended the definition of $P^a$ for the gravitational energy-momentum [13, 20] to any arbitrary tetrad fields, i.e., for the tetrad fields that satisfy $T_{a\mu\nu} \neq 0$ for the flat space-time. The redefinition is the only possible consistent extension of $P^a$, valid for the tetrad fields that do not satisfy the above equation.

Recently, Sharif and Amir (2007) have found the teleparallel (TP) version of the non-null Einstein Maxwell solutions [27]. Then, they have used the TP version of Møller (1978) to evaluate the energy-momentum distribution of these solutions. They have found that
the energy content in the TP theory is equal to the energy in GR (as found by Sharif and Fatima (2006)) plus some additional part. Also they have discussed three possibilities for the axial-vector field.

It is the aim of the present work to calculate the energy content of the axisymmetric solutions using the definition of the gravitational energy which is coordinate independent. In Sect. 2, a brief review of the derivation of the field equations of the gravitational field is given. A summary of the derivation of energy and angular momentum in TEGR is also given in Sect. 2. In Sect. 3, we derive the axially symmetric solutions in TEGR and then, calculate their energy content. The final section is devoted to discussion and conclusion.

2. The TEGR for gravitation

In a spacetime with absolute parallelism the parallel vector field \( e^a_\mu \) define the nonsymmetric affine connection

\[
\Gamma^\lambda_{\mu\nu} \overset{\text{def.}}{=} e^\lambda_a e^a_{\mu\nu}, \tag{1}
\]

where \( e^a_{\mu\nu} = \partial_\nu e^a_\mu \). The curvature tensor defined by \( \Gamma^\lambda_{\mu\nu} \) is identically vanishing, however. The metric tensor \( g_{\mu\nu} \) is given by

\[
g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu, \tag{2}
\]

with the Minkowski metric \( \eta_{ab} = \text{diag}(+1, -1, -1, -1) \) \( \dagger \).

The Lagrangian density for the gravitational field in the TEGR, in the presence of matter fields, is given by\( \overset{\ddagger}{\text{[13, 16]}} \)

\[
L_G = e L_G = -\frac{e}{16\pi} \left( \frac{T^{abc}T_{abc}}{4} + \frac{T^{abc}T_{bac}}{2} - T^aT_a \right) - L_m = -\frac{e}{16\pi} \Sigma^{abc}T_{abc} - L_m, \tag{3}
\]

where \( e = \text{det}(e^a_\mu) \). The tensor \( \Sigma^{abc} \) is defined by

\[
\Sigma^{abc} \overset{\text{def.}}{=} \frac{1}{4} \left( T^{abc} + T^{bac} - T^{cab} \right) + \frac{1}{2} \left( \eta^{ac}T^b - \eta^{ab}T^c \right). \tag{4}
\]

\( T^{abc} \) and \( T^a \) are the torsion tensor and the basic vector field defined by

\[
T^a_{\mu\nu} \overset{\text{def.}}{=} e^a_\lambda T^\lambda_{\mu\nu} = \partial_\mu e^a_\nu - \partial_\nu e^a_\mu, \tag{5}
\]

and

\[
T^\mu = T^\nu_{\nu\mu}, \quad T^a = e^a_\mu T^\mu = T^b_{b\mu}. \tag{6}
\]

*spacetime indices \( \mu, \nu, \cdots \) and SO(3,1) indices \( a, b, \cdots \) run from 0 to 3. Time and space indices are indicated to \( \mu = 0, i \), and \( a = (0), (i) \).

\( \overset{\ddagger}{\text{Latin indices are raising and lowering with the aid of } \eta_{ab} \text{ and } \eta^{ab}}. \)

\( \overset{\ddagger}{\text{Throughout this paper we use the relativistic units } , c = G = 1 \text{ and } \kappa = 8\pi}. \)
The quadratic combination $\Sigma^{abc}T_{abc}$ is proportional to the scalar curvature $R(e)$, except for a total divergence term [17]. $L_m$ represents the Lagrangian density for matter fields.

The gravitational field equations for the system described by $L_G$ are the following

$$e_a\lambda e_b\mu \partial_\nu \left( e^{abc} \right) - e \left( \Sigma^{b\nu}_{\phantom{b\nu}a} T_{b\nu\mu} - \frac{1}{4} e_{a\mu} T_{bcd} \Sigma^{bcd} \right) = \frac{1}{2} \kappa e T_{a\mu},$$

(7)

where

$$\frac{\delta L_m}{\delta e^{a\mu}} \equiv e T_{a\mu}.$$

It is possible to prove by explicit calculations that the left hand side of the symmetric part of the field equations (7) is exactly given by [13]

$$\frac{e}{2} \left[ R_{a\mu}(e) - \frac{1}{2} e_{a\mu} R(e) \right].$$

The axial-vector part of the torsion tensor $A_\mu$ is defined by

$$A_\mu \equiv \frac{1}{6} \epsilon_{\mu\rho\sigma} T^{\nu\rho\sigma} = \frac{1}{3} \epsilon_{\mu\rho\sigma} \gamma^{\nu\rho\sigma},$$

where

$$\epsilon_{\mu\rho\sigma} \equiv \sqrt{-g} \delta_{\mu\rho\sigma},$$

(8)

and $\gamma_{\mu\rho\sigma} = \eta^{ab} e_{a\nu} e_{b\rho} ; \sigma$ being the contorsion tensor and $\delta_{\mu\rho\sigma}$ is completely antisymmetric and normalized as $\delta_{0123} = -1$.

In the context of Einstein’s general relativity, rotational phenomena is certainly not a completely understood issue. The prominent manifestation of a purely relativistic rotation effect is the dragging of inertial frames. If the angular momentum of the gravitational field of isolated system has a meaningful notion, then it is reasonable to expect the latter to be somehow related to the rotational motion of the physical sources.

The angular momentum of the gravitational field has been addressed in the literature by means of different approaches. The oldest approach is based on pseudotensors [29, 26], out of which angular momentum superpotentials are constructed. An alternative approach assumes the existence of certain Killing vector fields that allow the construction of conserved integral quantities [1]. Finally, the gravitational angular momentum can also be considered in the context of Poincaré gauge theories of gravity [24], either in the Lagrangian or in the Hamiltonian formulation. In the latter case it is required that the generators of spatial rotations at infinity have a well defined functional derivatives. From this requirement a certain surface integral arises, whose value is interpreted as the gravitational angular momentum.

The Hamiltonian formulation of TEGR is obtained by establishing the phase space variables. The Lagrangian density does not contain the time derivative of the tetrad component $e_{a0}$. Therefore, this quantity will arise as a Lagrange multiplier [30]. The momentum canonically conjugated to $e_{ai}$ is given by $\Pi^{ai} = \delta L / \delta \dot{e}_{ai}$. The Hamiltonian formulation is obtained by rewriting the Lagrangian density in the form $L = p \dot{q} - H$, in terms of $e_{ai}$, $\Pi^{ai}$ and the Lagrange multipliers. The Legendre transformation can be successfully carried out and the final form of the Hamiltonian density has the form [20]

$$H = e_{a0} C^a + \alpha_{ik} \Gamma^{ik} + \beta_k \Gamma^k,$$

(9)
plus a surface term. Here $\alpha_{ik}$ and $\beta_k$ are Lagrange multipliers that are identified as

$$\alpha_{ik} = \frac{1}{2}(T_{i0k} - T_{k0i}) \quad \text{and} \quad \beta_k = T_{00k},$$

and $C^a$, $\Gamma^{ik}$ and $\Gamma^k$ are first class constraints. The Poisson brackets between any two field quantities $F$ and $G$ is given by

$$\{F, G\} = \int d^3x \left( \frac{\delta F}{\delta \epsilon_{ai}(x)} \frac{\delta G}{\delta \Pi^{ai}(x)} - \frac{\delta F}{\delta \Pi^{ai}(x)} \frac{\delta G}{\delta \epsilon_{ai}(x)} \right).$$

We recall that the Poisson brackets $\{\Gamma^{ij}(x), \Gamma^{kl}(x)\}$ reproduce the angular momentum algebra [17].

The constraint $C^a$ is written as $C^a = -\partial_i \Pi^{ai} + h^a$, where $h^a$ is an intricate expression of the field variables. The integral form of the constraint equation $C^a = 0$ motivates the definition of the gravitational energy-momentum $P^a$ four-vector [17]

$$P^a = -\int_V d^3x \partial_i \Pi^{ai},$$

where $V$ is an arbitrary volume of the three-dimensional space. In the configuration space we have

$$\Pi^{ai} = -\frac{2}{\kappa} \sqrt{-g} \Xi^{ai},$$

with

$$\partial_\nu(\sqrt{-g} \Xi^{\lambda\nu}) = \frac{\kappa}{2} \sqrt{-g} e^a_\mu (t^{\lambda\mu} + T^{\lambda\mu}), \quad \text{where} \quad t^{\lambda\mu} = \frac{1}{2\kappa} \left( 4 \Xi^{bc\lambda} T_{bc\mu} - g^{\lambda\mu} \Xi^{bcd} T_{bcd} \right).$$

The emergence of total divergences in the form of scalar or vector densities is possible in the framework of theories constructed out of the torsion tensor. Metric theories of gravity do not share this feature. By making $\lambda = 0$ in Eq. (13) and identifying $\Pi^{ai}$ in the left side of the latter, the integral form of Eq. (13) is written as

$$P^a = \int_V d^3x \sqrt{-g} e^a_\mu (t^{0\mu} + T^{0\mu}).$$

Eq. (14) suggests that $P^a$ is now understood as the gravitational energy-momentum [19]. The spatial component $P^{(i)}$ form a total three-momentum, while temporal component $P^{(0)}$ is the total energy [26].

It is possible to rewrite the Hamiltonian density of Eq. (9) in the equivalent form [18]

$$H = e_{a0} C^a + \frac{1}{2} \lambda_{ab} \Gamma^{ab}, \quad \text{with} \quad \lambda_{ab} = -\lambda_{ba},$$

(15)
are the Lagrangian multipliers that are identified as \( \lambda_{ik} = \alpha_{ik} \) and \( \lambda_{0k} = -\lambda_{k0} = \beta_k \). The constraints \( \Gamma^{ab} = -\Gamma^{ba} \) [20] embodies both constraints \( \Gamma^{ik} \) and \( \Gamma^k \) by means of the relation

\[
\Gamma^{ik} = \epsilon_a^i \epsilon_b^k \Gamma^{ab}, \quad \text{and} \quad \Gamma^k \equiv \Gamma^{0k} = \epsilon_a^0 \epsilon_b^k \Gamma^{ab}.
\] (16)

The constraint \( \Gamma^{ab} \) can be read as

\[
\Gamma^{ab} = M^{ab} + \frac{2}{\kappa} \sqrt{-g} e^0 \left( \Sigma^{acb} - \Sigma^{bca} \right).
\] (17)

In similarity to the definition of \( P^a \), the integral form of the constraint equation \( \Gamma^{ab} = 0 \) motivates the new definition of the space-time angular momentum. The equation \( \Gamma^{ab} = 0 \) implies

\[
M^{ab} = -\frac{2}{\kappa} \sqrt{-g} e^0 \left( \Sigma^{acb} - \Sigma^{bca} \right).
\] (18)

Maluf et al. [17, 18] defined

\[
L^{ab} = 2 \int_V d^3 x M^{[ab]},
\] (19)

as the 4-angular-momentum of the gravitational field for an arbitrary volume \( V \) of the three-dimensional space. In Einstein-Cartan type theories there also appear constraints that satisfy the Poisson bracket given by Eq. (11). However, such constraints arise in the form \( \Pi^{[ij]} = 0 \), and so a definition similar to Eq. (19), i.e., interpreting the constraint equation as an equation for the angular momentum of the field, is not possible. Definition (19) is three-dimensional integral. The quantities \( P^a \) and \( L^{ab} \) are separately invariant under general coordinate transformations of the three-dimensional space and under time reparametrizations, which is an expected feature since these definitions arise in the Hamiltonian formulation of the theory. Moreover, these quantities transform covariantly under global \( SO(3, 1) \) transformations [18].

3. Energy content of axisymmetric solutions

Now we are going to calculate the energy content of the axisymmetric tetrad field that has the form [27]

\[
(e_i^\mu) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
\frac{B(\rho, z)}{F(\rho)} \sin \phi & e^{-K(\rho, z)} \cos \phi & -\frac{1}{F(\rho)} \sin \phi & 0 \\
-\frac{B(\rho, z)}{F(\rho)} \cos \phi & e^{-K(\rho, z)} \sin \phi & \frac{1}{F(\rho)} \cos \phi & 0 \\
0 & 0 & 0 & e^{-K(\rho, z)}
\end{pmatrix}.
\] (20)
Using Eq. (2) the associated metric of the tetrad field given by Eq. (20) takes the well known form

\[
ds^2 = dt^2 - e^{2K(r,z)}d\rho^2 - \left(F^2(\rho) - B^2(\rho , z)\right) d\phi^2 - e^{2K(r,z)}dz^2 + 2B(\rho , z)dtd\phi,
\]

(21)

\[B(\rho , z), K(\rho , z)\text{ and } F(\rho)\text{ are unknown functions which satisfy the following relations}
\]

\[
\dot{B} = FW', \quad B' = -\frac{1}{4}aF\left(\dot{W}^2 - W'^2\right),
\]

\[
K' = -\frac{1}{2}aFWW', \quad \ddot{W} + \dot{F}F^{-1}\dot{W} + W'' = 0, \tag{22}
\]

where dot and prime denoting the derivatives w.r.t. \(\rho\) and \(z\) respectively. Here \(a\) is a constant and \(W\) is an arbitrary function of \(\rho\) and \(z\) in general [27]. McIntosh’s give a solution in the form \(W = -2Bz\) while McLenaghan et. al. solution’s has the form \(W = 2\ln \rho\) [2]. The above metric represents a five classes of non-null electromagnetic field and prefect fluid solutions which possesses a metric symmetry not inherited by the electromagnetic field and admits a homothetic vector field. Two of these classes contain electrovac solutions as special cases, while the other three necessarily contain some fluid. Generalization of metric given by Eq. (21) is given in [11].

Applying the tetrad field of Eq. (20) to the field equations (7) we get the non-vanishing components to have the form

\[
T^{0}_0 = \frac{-e^{-2K}}{64\pi F^3}\left(2B[F\dot{B} + FB' - \dot{B}\dot{F}] + 3F[\dot{B}^2 + B'^2] - 4F^2[F\dot{K} + FK'' + \ddot{F}]\right),
\]

\[
T^{0}_2 = \frac{-e^{-2K}}{32\pi F^3}\left(F^2[F\dot{B} + FB' - \dot{B}\dot{F} - 2B\dot{F}] - B^2[\dot{B}\dot{F} - F\dot{B} - FB''] + 2BF[\dot{B}^2 + B'^2]\right),
\]

\[
T^{1}_1 = \frac{e^{-2K}}{64\pi F^2}\left(\dot{B}^2 + 4F\dot{F}\dot{K} - B'^2\right), \quad T^{1}_3 = T^{3}_1 = \frac{e^{-2K}}{32\pi F^2}\left(\dot{B}B' + 2F\dot{F}K'\right),
\]

\[
T^{2}_0 = \frac{e^{-2K}}{32\pi F^3}\left(F[\dot{B} + B''] - \dot{B}\dot{F}\right), \quad T^{2}_2 = \frac{e^{-2K}}{64\pi F^3}\left(F[\dot{B}^2 + 2B\dot{B} + B'^2 + 2BB''] - 2B\dot{B}\dot{F} + 4F^3[\dot{K} + K'']\right),
\]

\[
T^{3}_3 = \frac{e^{-2K}}{64\pi F^2}\left(B'^2 - \dot{B}^2 + 4F[\ddot{F} - \dot{F}\dot{K}]\right), \tag{23}
\]

where \(T^{\mu\nu}\) is the energy-momentum tensor.

A special solution of the above non linear P.D.E can be obtained by choosing [27]

\[
B(\rho, z) = \frac{m}{n}e^{\rho}, \quad F(\rho) = e^{\rho}, \quad K(\rho, z) = 0, \tag{24}
\]

where \(m\) and \(n\) are constants. The above solution reproduce the well known solution which is known as the electromagnetic generalization of the Gödel solution [27, 2].

The second solution can be obtained by choosing

\[
B(\rho, z) = e^{\rho}, \quad F(\rho) = \frac{e^{\rho}}{\sqrt{2}}, \quad K(\rho, z) = 0, \tag{25}
\]

which known as the space time homogenous Gödel metric [27, 2].
Now let us calculate the energy content of the tetrad (20) using (12). To do so let us calculate the non-vanishing components of the torsion tensor. Using Eq. (20) in Eq. (5) we get

\[ T^{(0)}_{12} = \frac{1}{F} \left( (e^K - \dot{F}) [e^K B F \sin^2 \phi + B(F \cos \phi + 1)] - F \dot{B} \right), \quad T^{(0)}_{13} = -e^{2K} B K' \sin \phi \cos \phi \]

\[ T^{(0)}_{23} = B', \quad T^{(1)}_{12} = \sin \phi (\dot{F} - e^K), \quad T^{(1)}_{13} = e^K K' \cos \phi, \quad T^{(2)}_{12} = (e^K - \dot{F})(e^K \sin^2 \phi + \cos \phi), \]

\[ T^{(2)}_{13} = -e^{2K} K' \sin \phi \cos \phi, \quad T^{(3)}_{13} = -e^K \dot{K}. \]

The non-vanishing components of the basic vector \( T_\mu \) are

\[ T_1 = (\dot{F} - e^K)(e^K \sin^2 \phi + \cos \phi) + e^K \dot{K}, \quad T_2 = \sin \phi (\dot{F} - e^K), \quad T_3 = e^K K' \cos \phi. \] (27)

Pereira et al. [21] have proved that the axial vector tensor plays the role of the gravitomagnetic component of the gravitational field in the case of slow rotation and weak field approximations. The non-vanishing components of the axial vector tensor, defined by Eq. (8), associated with the tetrad field given by Eq. (20) are

\[ A^0 = \frac{K' \sin \phi \cos \phi (F^2 - 2B^2)}{3F}, \quad A^1 = \frac{e^{-2K} B'}{3F}, \quad A^2 = \frac{2BK' \sin \phi \cos \phi}{3F}, \]

\[ A^3 = \frac{1}{3F^2} \left( 2BF(1 - e^{-K} \dot{F}) \{ \sin^2 \phi + e^{-K} \cos \phi \} + Be^{-K} \right) - e^{-2K}(FB + B\dot{F}) \]. (28)

It is of interest to compare our results with that obtained before by Sharif and Amir (2007). They have calculated the axial vector of the tetrad given by Eq. (20) and wrote the non-vanishing components as \( A^1 \) which coincides with what we have obtained in Eq. (28) and the other component as \( A^3 \) that has the form

\[ A^3 = \frac{\dot{B}e^{-2K}}{3F}, \] (29)

which is completely different from that obtained in Eq. (28). Therefore, the analysis related to the axial vector part given in Ref. [27] will completely now need some modifications which we will do. The spacelike axial vector can now be written [27]

\[ A = \sqrt{-g_{11}} A^1 \hat{e}_\rho + \sqrt{-g_{22}} A^2 \hat{e}_\phi + \sqrt{-g_{33}} A^3 \hat{e}_z, \] (30)

where \( \hat{e}_\rho, \hat{e}_\phi \) and \( \hat{e}_z \) are unit vectors along the radial \( \rho \), \( \phi \) and z-directions respectively. Using Eq. (28) in (30) we get

\[ A = \frac{e^{-K} B'}{3F} \hat{e}_\rho + \frac{2\sqrt{F^2 - B^2} BK' \sin \phi \cos \phi}{3F} \hat{e}_\phi + \frac{e^K}{3F^2} \left( (1 - e^{-K} \dot{F}) \{ 2BF \{ \sin^2 \phi + e^{-K} \cos \phi \} + Be^{-K} \} - e^{-2K} F \dot{B} \right) \hat{e}_z. \] (31)

The spin precession of a Dirac particles in teleparallel gravity is related to the axial vector by [27, 25]

\[ \frac{dS}{dt} = -b \times S, \] (32)
where $S$ is the spin vector of a Dirac particles and $b = 3/2A$ where $A$ is given by Eq. (31).

The direct evidence for the coupling of intrinsic spin to the rotation of the Earth has become available [3]. According to the TEGR every spin $\frac{1}{2}$ particle in the laboratory has an additional interaction Hamiltonian. However, such intrinsic spin must precess in a sense opposite to the sense of rotation of the Earth as measured by the observer. The corresponding extra Hamiltonian associated with such motion would be of the form [4]

$$\delta H = -b \cdot \sigma,$$  \hspace{1cm} (33)

where $\sigma$ is the spin of the particle.

To calculate the energy density associated with the tetrad field given by Eq. (20) we must calculate $\Sigma^{\mu\nu\lambda}$ which defined in Eq. (4). The necessary non-vanishing components of $\Sigma^{\mu\nu\lambda}$ are

$$\Sigma^{001} = \frac{1}{2F^2} \left((1 - e^{-K}\dot{F})\{F(\sin^2 \phi + e^{-K}\cos \phi)(B^2 - F^2) + e^{-K}B^2\} - e^{-2K}B\frac{dF}{d\rho} \dot{B} + e^{-K}F^3\dot{K}\right),$$

$$\Sigma^{002} = \frac{-\sin \phi (e^{-K} - \dot{F})}{2F^2}, \quad \Sigma^{003} = \frac{-e^{-K}(e^{-K}BB' - \cos \phi F^2K')}{2F^2},$$

$$\Sigma^{101} = \frac{-B\sin \phi (e^{-K} - e^K - (e^{-2K} - 1)\dot{F})}{2F^2},$$

$$\Sigma^{102} = \frac{1}{4F^2} \left((1 - e^{-K}\dot{F})\{2BF[\sin^2 \phi + e^{-K}\cos \phi] + e^{-K}B\} - e^{-2K}F\dot{B}\right),$$

$$\Sigma^{103} = \frac{-1}{4}e^{-2K}BK'\sin \phi \cos \phi, \quad \Sigma^{201} = \frac{-e^{-K}}{4F^3}(B - Be^{-K}\dot{F} - e^{-K}F\dot{B}),$$

$$\Sigma^{202} = \frac{B\sin \phi}{2F^2}(e^{-K} - \dot{F}), \quad \Sigma^{203} = \frac{-e^{-2K}B'}{4F^2}, \quad \Sigma^{301} = \frac{-e^{-2K}BK'\sin \phi \cos \phi}{4},$$

$$\Sigma^{302} = \frac{-e^{-2K}B'}{4F^2}, \quad \Sigma^{303} = \frac{B\sin \phi}{2F^2}(e^K - \dot{F}).$$  \hspace{1cm} (34)

Using (34) in (13) we get

$$\Pi^{(0)0} = \frac{1}{\kappa F^2} \left(F \left\{ \frac{d}{d\rho} (BB') + \frac{d}{dz} (BB') + \sin^2 \phi \left[B^2 - F^2\right] \frac{d}{d\rho} (\dot{F}e^K) \right\} + B\dot{B}\dot{F}[1 + 2F \cos \phi] + B^2 \left[F \dot{F} - 2\dot{F}^2\right] - \cos \phi F^3 \frac{d}{dz} (K'e^K) + \left[\cos \phi \frac{d}{d\rho} (\dot{F}e^K) + e^K\dot{F}^2 \{1 + F \cos^2 \phi\} \right\} \{B^2 + F^2\} + 2e^K \left(\frac{B^3}{F}\right) \frac{d}{d\rho} (BF) + F^2 \frac{d}{d\rho} (Fe^{2K}) - e^K \dot{K} \left[B^2 + F^2\dot{F}\right] - F^2 \cos \phi \left[\frac{d}{d\rho} \left\{ Fe^K \left[1 + \cos \phi e^K\right]\right\} \right] + 2 \cos \phi \frac{d}{d\rho} (e^K B^2F) - F^3 \frac{d}{d\rho} (\dot{K}e^K) + e^K B \sin^2 \phi \left[2F \dot{F} \dot{B} - e^K \left\{ FB\dot{K} + 2F\dot{B} - \dot{F}\right\}\right] + e^{3K}F \cos \phi \left[1 - e^{-K}\dot{F}\right]\right).$$  \hspace{1cm} (35)

The non-vanishing components of the momentum density have the form

$$\Pi^{(1)0} = \frac{\sin \phi \cos \phi e^K}{\kappa F^3} \left(F \frac{d}{d\rho} (Be^K) - Be^K \dot{F} - F \frac{d}{d\rho} (e^{3K}B) + Be^{3K} \dot{F} + BF \frac{d}{d\rho} (e^{2K}\dot{F}) - Be^{2K}\dot{F}^2\right)$$
\( -F \frac{d}{d\rho} (BF) + Fe^{2K} BF - 2e^{2K} FB \cos \phi + 2FB e^K \cos \phi \dot{F} + FB e^K - FB \dot{F} + \frac{F^3}{2} \cos \phi \frac{d}{dz} (BK') \)

\[ \Pi^{(2)0} = \frac{\cos \phi}{\kappa F^2} \left( Be^K \dot{F} - B \dot{F}^2 - \frac{F}{2} (e^K \dot{K} - \dot{e^K} \dot{B} + B \ddot{F} + F \ddot{B} + 2B' + F^2 B e^{3K} \cos \phi - F^2 B e^{2K} \cos \phi \dot{F} \right) \]

\[ \Pi^{(3)0} = \frac{2 \sin \phi e^K}{\kappa F} \left( \dot{F} e^{2K} [B' + 3BK'] - \frac{F \cos \phi}{2} (FB \dot{K}' + B \ddot{F}K' + F \ddot{B}K' + FB \ddot{K}') \right. \]

\[ - Be^{3K} (B' + 4K'). \]

(36)

It is of interest to note that if \((F(\rho) = \text{const}, K(\rho, z) = K_1(\rho), B(\rho, z) = B(\rho))\) then the component of the momentum density \(\Pi^{(3)0} = 0\).

Now let us repeat the same calculations using the solution given by Eq.(24), in this case the basic vector has the non-vanishing components

\[ T_1 = (ne^{n\rho} - 1)(\sin^2 \phi + \cos \phi), \quad T_2 = \sin \phi (ne^{n\rho} - 1), \]

(37)

and the non-vanishing components of the axial vector field are

\[ A^3 = \frac{1}{3} \left( \left[ \frac{2m}{n} (1 - ne^{n\rho}) \{\sin^2 \phi + \cos \phi\} + \frac{m}{n} \right] \right) - 2m \right) \].

(38)

\[ A = \frac{1}{3} \left( \left[ \frac{2m}{n} (1 - ne^{n\rho}) \{\sin^2 \phi + \cos \phi\} + \frac{m}{n} \right] \right) - 2m \right) \hat{e}_z. \]

(39)

The corresponding extra Hamiltonian [7] is given by

\[ \delta H = - \mathbf{b} \cdot \sigma = - \frac{1}{2} \left( \left[ \frac{2m}{n} (1 - ne^{n\rho}) \{\sin^2 \phi + \cos \phi\} + \frac{m}{n} \right] \right) - 2m \right) \hat{e}_z \cdot \sigma, \]

(40)

and the component of the energy density is given by

\[ \Pi^{(0)0} = \frac{e^{n\rho}}{nk} \left( (n^2 - m^2) [\sin^2 \phi + \cos \phi] (1 - 2ne^{n\rho}) + ne^{n\rho} \cos \phi (e^{n\rho} - n) + 2m^2 n \right). \]

(41)

The non-vanishing components of the momentum density have the form

\[ \Pi^{(1)0} = \frac{m \sin \phi e^{n\rho}}{nk} \left( e^{n\rho} \{1 - 3 \cos^2 \phi - n\} + \frac{e^{-2n\rho}}{2} (1 - n) + n(3 \cos^2 \phi - 1) \right) \]

\[ \Pi^{(2)0} = \frac{m e^{n\rho}}{kn^2} \left( m^2 \left\{ 4ne^{n\rho} + 3ne^{2n\rho} - 2e^{n\rho} \cos \phi - 2e^{n\rho} \cos^2 \phi + 3n \cos \phi e^{2n\rho} (1 - \cos \phi) - 1 \right\} \right. \]

\[ + n^2 \left\{ 1 - 3ne^{2n\rho} + n \sin^2 \phi + 2e^{n\rho} - 2 \cos^2 \phi - \cos \phi + e^{n\rho} \cos \phi + 3n \cos \phi e^{2n\rho} (\cos \phi - 1) - n \cos \phi \right\} + e^{n\rho} \cos^2 \phi (4 \cos^2 \phi - 5) \right\} + n \left\{ 3 \cos^2 \phi - 4 \cos^4 \phi + e^{-n\rho} (\cos^2 \phi - \frac{1}{2}) + \cos \phi (1 + e^{-n\rho}) \right\} \]

(42)
For the solution given by Eq. (25) the basic vector has the non-vanishing components

\[ T_1 = \left( \frac{1}{\sqrt{2}} a e^{a \rho} - \sqrt{2} \right) (\sin^2 \phi + \cos \phi), \quad T_2 = \frac{\sin \phi}{\sqrt{2}} (a e^{a \rho} - \sqrt{2}), \]

and the non-vanishing components of the axial vector field are

\[ A^3 = \frac{1}{3} \left( 2 \sqrt{2} e^{-a \rho} + e^{2 \rho} (e^{2 \rho} + 1) (\sqrt{2} - a) \right) \{ \sin^2 \phi + \cos \phi \} - 2 e^{\rho(4-a)} (\sqrt{2} a - e^{-a \rho}) \hat{e}_z. \]

The corresponding extra Hamiltonian [7] is given by

\[ \delta H = -b \cdot \sigma = -\frac{1}{2} \left( \sqrt{2} e^{-a \rho} + e^{2 \rho} (e^{2 \rho} + 1) (\sqrt{2} - a) \right) \{ \sin^2 \phi + \cos \phi \} - 2 e^{\rho(4-a)} (\sqrt{2} a - e^{-a \rho}) \hat{e}_z \cdot \sigma, \]

and the component of the energy density has the form

\[ \Pi^{(0)0} = \frac{e^{a \rho}}{\kappa} \left( \sin^2 \phi (a^2 e^{a \rho} - \frac{a}{\sqrt{2}}) + \cos \phi (a^2 e^{a \rho} - \frac{a}{\sqrt{2}}) + \frac{a^2}{\sqrt{2}} + e^{-a \rho} \cos \phi (\sqrt{2} e^{-a \rho} - a) \right). \]

The non-vanishing components of the momentum density have the form

\[ \Pi^{(1)0} = \frac{\sin \phi e^{a \rho}}{\kappa} \left( e^{-2 a \rho} (1 - a) + e^{-a \rho} (1 - a \sqrt{2} - 3 \sqrt{2} \cos^2 \phi) + a (3 \cos^2 \phi - 1) \right) \]

\[ \Pi^{(2)0} = \frac{e^{a \rho}}{2 \kappa} \left( e^{-a \rho} (2 \sqrt{2} \cos \phi (1 + \cos \phi) - \sqrt{2}) + \sqrt{2} e^{a \rho} (8 a^2 - a \cos^2 \phi - 3 a \cos \phi + 4 a \cos^4 \phi - 2 a) \right) + 3 a^2 e^{2 a \rho} (\sin^2 \phi + \cos \phi) + 2 \cos^2 \phi (3 - 4 \cos^2 \phi - a^2 - 2 a) + 2 \cos \phi (1 - a - a^2) + 2 a (a - 1), \]

\[ \Pi^{(3)0} = 0. \]

4. Main results and discussion

We have applied the axisymmetric tetrad field given by Eq. (20) with three unknown functions of \( \rho \) and \( z \) to the field equations (7). We have obtained two special solutions by taking particular values of the unknown functions \( E, F \) and \( K \). The first one is given by Eq. (24) and is known as electromagnetic generalization of the Gödel solution [22], while the second one is given by Eq. (25) and is known as Gödel solution [22]. These solutions are special solutions of the non-linear P.D.E. given by Eq. (23) [27, 22].
We have calculated the basic vector $T^a$ defined by Eq. (6) for the tetrad field (20). The components we have obtained as given by Eq. (27) are different from the components obtained in Ref. [27] which are

$$T_1 = \frac{-1}{F}(\dot{F} - e^K) - \dot{K}, \quad T_3 = -K'. \quad (49)$$

We have calculated the axial vector part applied Eq. (20) to Eq. (8). Our results are completely different from the results obtained in Ref. [27]. Therefore, the analysis of the extra Hamilton used in [27] is again discussed. The extra Hamiltonian is now recalculated for the tetrad field of Eq. (20) and is given by Eq. (33).

The energy density and momentum density are calculated using the energy-momentum tensor of TEGR which is coordinate independent. For the energy momentum density and momentum density derived in Ref. [27] we have some comments:

i) The energy momentum density and momentum density, are not correct.

ii) In Ref. [27] a discussion is given for the case when the arbitrary parameter $\lambda = 1$. It is not clear why $\lambda = 1$ is discussed in spite that for this case still the theory of Møller deviates from GR.

iii) When the tetrad (21) of Ref. [27] applied to the superpotential (12) of [27], it is logic to obtain the energy density and momentum density with the arbitrary parameter $\lambda$. If we want to compare the result with GR we must take $\lambda = 0$ not $\lambda = 1$! Here in this work we have done the calculations of the energy density and momentum density using the gravitational energy-momentum tensor which is coordinate independent since it is constructed out from Hamilton structure.
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