MEASURE THEORETIC ASPECTS OF OSCILLATIONS OF ERROR TERMS

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Abstract. In this paper, we obtain Ω and Ω± estimates for a wide class of error terms ∆(x) appearing in the asymptotic formula for a summatory function of coefficients of the Dirichlet series. We revisit some classical Ω and Ω± bounds on ∆(x), and obtain Ω bounds for Lebesgue measure of the sets

\[ A_T := \{ T \leq x \leq 2T : |\Delta(x)| > \lambda x^{\alpha} \} \]

for some \( \alpha, \lambda > 0 \). We also prove that if the Lebesgue measure of \( A_T \) is \( \Omega(T^{1-\delta}) \), then

\[ \Delta(x) = \Omega(\pm x^{\alpha-\delta}), \]

for any \( 0 < \delta < \alpha \).

1. Introduction

Analysis of error terms in asymptotic formulas is of considerable importance in various fields of mathematics. For example, consider the von-Mangoldt function

\[ \Lambda(n) = \begin{cases} \log p & \text{if } n = p^r, \ r \in \mathbb{N}, \text{ and } p \text{ prime}, \\ 0 & \text{otherwise}. \end{cases} \]

The Prime Number Theorem says that

\[ \sum_{n \leq x} \Lambda(n) = x + \Delta(x), \]

where \( \Delta(x) \) is \( o(x) \). It is also known that the famous Riemann Hypothesis is equivalent to (see \[31\], also Theorem 3 below)

\[ \Delta(x) = O\left(x^{\frac{1}{2}} \log x\right). \]

The following result, proved by Hardy and Littlewood [12], shows that such an upper bound for \( \Delta(x) \) is optimal in terms of the power of \( x \):

\[ \limsup_{x \to \infty} \frac{\Delta(x)}{x^{\frac{1}{2}} \log \log \log x} > 0 \quad \text{and} \quad \liminf_{x \to \infty} \frac{\Delta(x)}{x^{\frac{1}{2}} \log \log \log x} < 0. \]

A weaker result by Landau [21] gives

\[ \limsup_{x \to \infty} \frac{\Delta(x)}{x^{\frac{1}{2}}} > 0 \quad \text{and} \quad \liminf_{x \to \infty} \frac{\Delta(x)}{x^{\frac{1}{2}}} < 0. \]

However, Landau’s method has wide applications, and it is flexible to obtain some measure theoretic results. In Landau’s method, the existence of a complex pole with real part \( \frac{1}{2} \) serves as a criterion for existence of the above limits. In this paper, we shall investigate on a quantitative version of Landau’s result by obtaining the Lebesgue measure of the sets where \( \Delta(x) > \lambda x^{1/2} \) and \( \Delta(x) < -\lambda x^{1/2} \), for
some $\lambda > 0$. We shall show that the large Lebesgue measure of the set where $|\Delta(x)| > \lambda x^{\frac{3}{2}}$, for some $\lambda > 0$, will replace the criterion for existence of a complex pole in Landau’s method. These ideas will become clear later in this paper.

**Outline.** In general, consider a sequence of real numbers $\{a_n\}_{n=1}^{\infty}$ having Dirichlet series

$$D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

that converges in some half-plane. The Perron summation formula [28 II.2.1] uses analytic properties of $D(s)$ to give

$$\sum_{n \leq x}^* a_n = \mathcal{M}(x) + \Delta(x),$$

where $\mathcal{M}(x)$ is the main term, $\Delta(x)$ is the error term (which would be specified later) and $\sum^*$ is defined as

$$\sum_{n \leq x}^* a_n = \begin{cases} \sum_{n \leq x} a_n & \text{if } x \notin \mathbb{N} \\ \sum_{n<x} a_n + \frac{1}{2}a_x & \text{if } x \in \mathbb{N}. \end{cases}$$

Standard measures of fluctuations (in this case, fluctuations of $\Delta(x)$) are $\Omega$ and $\Omega_\pm$ estimates, which are defined as follows.

**Definition 1.** Let $f_1(x)$ be a real valued function and $f_2(x)$ be a positive monotonically increasing function. We say that $f_1(x) = \Omega(f_2(x))$ if

$$\limsup_{x \to \infty} \frac{|f_1(x)|}{f_2(x)} > 0.$$  

Also $f_1(x) = \Omega_\pm(f_2(x))$ if

$$\limsup_{x \to \infty} \frac{f_1(x)}{f_2(x)} > 0 \quad \text{and} \quad \liminf_{x \to \infty} \frac{f_1(x)}{f_2(x)} < 0.$$  

In this paper, we obtain bounds for Lebesgue measures of the sets on which $\Omega$ and $\Omega_\pm$ results hold.

To obtain $\Omega$ and $\Omega_\pm$ estimates, we shall analyze the Mellin transform of $\Delta(x)$.

**Definition 2.** For a complex variable $s$, the Mellin transform $A(s)$ of $\Delta(x)$ is defined as:

$$A(s) = \int_1^{\infty} \frac{\Delta(x)}{x^{s+1}} dx.$$  

In general, $A(s)$ is holomorphic in some half-plane. In Section 2 we shall discuss a method to continue $A(s)$ meromorphically. In particular, we prove in Theorem 1 that under some natural assumptions

$$A(s) = \int_{\mathcal{C}} \frac{D(\eta)}{\eta(s-\eta)} d\eta,$$  

where the contour $\mathcal{C}$ is as in Definition 3 and $s$ lies to the right of $\mathcal{C}$. In Section 3 this result complements Theorem 3 in its applications.

In section 4 we revisit Landau’s method and obtain some measure theoretic results. If $A(s)$ has a pole at $\sigma_0 + it_0$ for some $t_0 \neq 0$, and has no real pole for $\Re(s) \geq \sigma_0$, then Landau’s method (Theorem 4) gives

$$\Delta(x) = \Omega_\pm(x^{\sigma_0}).$$
We also discuss a result of Kaczorowski and Szydło [19] on $E_2(x)$, where
\[
\int_0^x \left| \zeta \left( \frac{1}{2} + it \right) \right|^4 dt = xP(\log x) + E_2(x)
\]
and $P$ being a certain polynomial of degree 4. Motohashi [23] proved that
\[
E_2(x) \ll x^{2/3+\epsilon},
\]
and further in [24] he showed that
\[
E_2(x) = \Omega(\sqrt{x}).
\]
The result of Kaczorowski and Szydło mentioned above says that there exist constants $\lambda_0, \nu > 0$ such that
\[
\mu \{ 1 \leq x \leq T : E_2(x) > \lambda_0 \sqrt{x} \} = \Omega(\frac{T}{(\log T)^\nu})
\]
and
\[
\mu \{ 1 \leq x \leq T : E_2(x) < -\lambda_0 \sqrt{x} \} = \Omega(\frac{T}{(\log T)^\nu}) \quad \text{as } T \to \infty,
\]
and where $\mu$ is the Lebesgue measure [1]. These results not only prove $\Omega$ bounds, but also give quantitative estimates for the occurrences of such fluctuations. This result of Kaczorowski and Szydło has been generalized by Bhowmik, Ramaré and Schlage-Puchta [6] to localize fluctuations of $\Delta(x)$ to a dyadic range. Let
\[
\sum_{n \leq x} G_k(n) = \frac{x^k}{k!} - k \sum_{\rho} \frac{x^{k-1+\rho}}{\rho(1+\rho) \cdots (k-1+\rho)} + \Delta_k(x),
\]
where the Goldbach numbers $G_k(n)$ are defined as
\[
G_k(n) = \sum_{n_1, \ldots, n_k = n} \Lambda(n_1) \cdots \Lambda(n_k),
\]
and $\rho$ runs over nontrivial zeros of the Riemann zeta function $\zeta(s)$. Bhowmik, Ramaré and Schlage-Puchta proved that under Riemann Hypothesis
\[
\mu \{ T \leq x \leq 2T : \Delta_k(x) > (c_k + \epsilon'_k) x^{k-1} \} = \Omega(\frac{T}{(\log T)^6})
\]
and
\[
\mu \{ T \leq x \leq 2T : \Delta_k(x) < (c_k - \epsilon'_k) x^{k-1} \} = \Omega(\frac{T}{(\log T)^6}) \quad \text{as } T \to \infty,
\]
where $k \geq 2$ and $c_k, \epsilon'_k$ are well defined real number depending on $k$ with $\epsilon'_k > 0$.

In this paper, we obtain analogous results for other functions. In Theorem 6, we further generalize this theorem of Bhowmik, Ramaré and Schlage-Puchta so that it has more applications. Moreover, we carry forward this idea to study the influence of measures on the $\Omega$ and $\Omega_\pm$ results.

In Section 4, we obtain an $\Omega$ bound for the second moment of $\Delta(x)$ in a special case, namely
\[
\int_T^{2T} \Delta^2(x) dx = \Omega(T^{2\alpha+1+\epsilon}),
\]
for any $\epsilon > 0$ and for some $\alpha > 0$. This is an adaptation of a technique due to Balasubramanian, Ramachandra and Subbarao [5]. In particular, we obtain
\[
\Delta(x) = \Omega(x^\alpha).
\]
Also we derive an $\Omega$ bound for the measure of the set
\[
A(\alpha, T) := \{ x : x \in [T, 2T], |\Delta(x)| > x^\alpha \}.
\]

\footnote{Throughout this paper, $\mu$ will denote the Lebesgue measure on the real line $\mathbb{R}$.}
In Section 5, we establish a connection between \( \mu(A(\alpha, T)) \) and fluctuations of \( \Delta(x) \). In Proposition 5, we see that

\[
\mu(A(\alpha, T)) \ll T^{1-\delta} \implies \Delta(x) = \Omega(x^{\alpha+\delta/2}).
\]

However, Theorem 11 gives that \( \mu(A(\alpha, T)) = \Omega(T^{1-\delta}) \implies \Delta(x) = \Omega(\pm x^{\alpha-\delta}) \), provided \( A(s) \) does not have a real pole for \( \text{Re}(s) \geq \alpha - \delta \). In particular, this says that either we can improve on the \( \Omega \) result or we can obtain a tight \( \Omega \pm \) result for \( \Delta(x) \).

In this paper, we formulate our results in a way to be applicable in a wide generality. The nature of the problem on which the methods of this paper apply are formalized in various assumptions. A summary of the applications of these results obtained in this paper are given below.

**Applications.** We conclude the introduction to this paper by mentioning a few applications.

**Error Term of a Twisted Divisor Function.** For a fixed \( \theta \neq 0 \), we consider

\[
\tau(n, \theta) = \sum_{d|n} d^{i\theta}.
\]

This function is used in [10, Chapter 4] to measure the clustering of divisors. The Dirichlet series of \( |\tau(n, \theta)|^2 \) can be expressed in terms of the Riemann zeta function as

\[
D(s) = \sum_{n=1}^{\infty} \frac{|\tau(n, \theta)|^2}{n^s} = \frac{\zeta^2(s)\zeta(s+i\theta)\zeta(s-i\theta)}{\zeta(2s)} \quad \text{for} \quad \text{Re}(s) > 1.
\]

In [10, Theorem 33], Hall and Tenenbaum proved that

\[
\sum_{n \leq x} |\tau(n, \theta)|^2 = \omega_1(\theta)x \log x + \omega_2(\theta)x \cos(\theta \log x) + \omega_3(\theta)x + \Delta(x),
\]

where \( \omega_i(\theta) \)s are explicit constants depending only on \( \theta \). They also showed that

\[
\Delta(x) = O_\theta(x^{1/2} \log^6 x).
\]

Here the main term comes from the residues of \( D(s) \) at \( s = 1, 1 \pm i\theta \). All other poles of \( D(s) \) come from the zeros of \( \zeta(2s) \). Using a pole on the line \( \text{Re}(s) = 1/4 \), Landau’s method gives

\[
\Delta(x) = \Omega_\pm(x^{1/4}).
\]

In order to apply the method of Bhowmik, Ramaré and Schlage-Puchta, we need

\[
\int_T^{2T} \Delta^2(x)dx \ll T^{2\sigma_0+1+\epsilon},
\]

for any \( \epsilon > 0 \) and \( \sigma_0 = 1/4 \); such an estimate is not possible due to Corollary 5. Generalization of this method in Theorem 6 can be applied to get

\[
\mu(A_j \cap [T, 2T]) = \Omega(T^{1/2}(\log T)^{-12}) \quad \text{for} \quad j = 1, 2,
\]
and here $A_j$ for $\Delta(x)$ are defined as
\[ A_1 = \left\{ x : \Delta(x) > (\lambda(\theta) - \epsilon)x^{1/4} \right\}, \]
and  
\[ A_2 = \left\{ x : \Delta(x) < (-\lambda(\theta) + \epsilon)x^{1/4} \right\}, \]
for any $\epsilon > 0$ and $\lambda(\theta) > 0$ as in (25). But under Riemann Hypothesis, we show in (27) that the above $\Omega$ bounds can be improved to
\[ \mu(A_j) = \Omega\left(T^{3/4-\epsilon}\right), \quad \text{for } j = 1, 2 \]
and for any $\epsilon > 0$.

Fix a constant $c_1 > 0$ and define
\[ \alpha(T) = \frac{3}{8} - \frac{c_1}{(\log T)^{1/8}}. \]

In Corollary 1, we prove that
\[ \Delta(T) = \Omega\left(T^{\alpha(T)}\right). \]

In Proposition 3, we give an $\Omega$ estimate for the measure of the sets involved in the above bound:
\[ \mu(A \cap [T, 2T]) = \Omega\left(T^{2\alpha(T)}\right), \]
where
\[ A = \{ x : |\Delta(x)| \geq Mx^\alpha \} \]
for a positive constant $M > 0$. In Theorem 9, we show that
\[ \Delta(x) = \Omega\left(x^{\alpha(x)+\delta/2}\right) \quad \text{or} \quad \Delta(x) = \Omega_{\pm}\left(x^{3/8-\delta'}\right), \]
for $0 < \delta < \delta' < 1/8$. We may conjecture that
\[ \Delta(x) = O(x^{3/8+\epsilon}) \quad \text{for any } \epsilon > 0. \]

Theorem 9 and this conjecture imply that
\[ \Delta(x) = \Omega_{\pm}\left(x^{3/8-\epsilon}\right) \quad \text{for any } \epsilon > 0. \]

**Square Free Divisors.** Let $\Delta(x)$ be the error term in the asymptotic formula for partial sums of the square free divisors:
\[ \Delta(x) = \sum_{n \leq x} 2^{\omega(n)} \left( -\frac{x \log x}{\zeta(2)} + \left( -\frac{2\zeta'(2)}{\zeta^2(2)} + \frac{2\gamma - 1}{\zeta(2)} \right) x \right), \]
where $\omega(n)$ denotes the number of distinct primes divisors of $n$. It is known that $\Delta(x) \ll x^{1/2}$ (see [14]). Let $\lambda_1 > 0$ and the sets $A_j$, for $j = 1, 2$, be defined as in Section 3.3.2
\[ A_1 = \left\{ x : \Delta(x) > (\lambda_1 - \epsilon)x^{1/4} \right\}, \]
\[ A_2 = \left\{ x : \Delta(x) < (-\lambda_1 + \epsilon)x^{1/4} \right\}. \]

In (30), we show that
\[ \mu(A_j \cap [T, 2T]) = \Omega\left(T^{1/2}\right), \quad \text{for } j = 1, 2. \]
But under Riemann Hypothesis, we prove the following $\Omega$ bounds in (31):

$$
\mu(\mathcal{A}_j \cap [T, 2T]) = \Omega\left(T^{1-\epsilon}\right), \text{ for } j = 1, 2
$$

and for any $\epsilon > 0$.

*The Error Term in Prime Number Theorem.* Let $\Delta(x)$ be the error term in the Prime Number Theorem:

$$
\Delta(x) = \sum_{n \leq x} \Lambda(n) - x.
$$

We know from Landau’s theorem [21] that

$$
\Delta(x) = \Omega_{\pm}\left(x^{1/2}\right)
$$

and from the theorem of Hardy and Littlewood [12] that

$$
\Delta(x) = \Omega_{\pm}\left(x^{1/2}\log \log x\right).
$$

We define

$$
\mathcal{A}_1 = \left\{ x : \Delta(x) > (\lambda_2 - \epsilon)x^{1/2} \right\},
$$

$$
\mathcal{A}_2 = \left\{ x : \Delta(x) < (-\lambda_2 + \epsilon)x^{1/2} \right\},
$$

where $\lambda_2 > 0$ be as in Section 3.3.3. If we assume Riemann Hypothesis, then the theorem of Bhownik, Ramaré and Schlage-Puchta (see Theorem 5 below) along with (1) gives

$$
\mu(\mathcal{A}_j \cap [T, 2T]) = \Omega\left(\frac{T}{\log^4 T}\right) \text{ for } j = 1, 2.
$$

However, as an application of Corollary 1 we prove the following weaker bound unconditionally:

$$
\mu(\mathcal{A}_j \cap [T, 2T]) = \Omega\left(T^{1-\epsilon}\right), \text{ for } j = 1, 2
$$

and for any $\epsilon > 0$.

*Non-isomorphic Abelian Groups.* Let $a_n$ be the number of non-isomorphic abelian groups of order $n$, and the corresponding Dirichlet series is given by

$$
\sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \prod_{k=1}^{\infty} \zeta(k^s) \text{ for } \Re(s) > 1.
$$

Let $\Delta(x)$ be defined as

$$
\Delta(x) = \sum_{n \leq x}^{*} a_n - \sum_{k=1}^{6} \left(\prod_{j \neq k} \zeta(j/k)\right)x^{1/k}.
$$

It is an open problem to show that

(6) \hspace{1cm} \Delta(x) \ll x^{1/6+\epsilon} \text{ for any } \epsilon > 0.

The best result on upper bound of $\Delta(x)$ is due to O. Robert and P. Sargos [25], which gives

$$
\Delta(x) \ll x^{1/4+\epsilon}.
$$
Balasubramanian and Ramachandra [4] proved that
\[ \int_T^{2T} \Delta^2(x)dx = \Omega(T^{4/3}) \log T). \]
Following the proof of Proposition 3, we get
\[ \mu \left( \{ T \leq x \leq 2T : |\Delta(x)| \geq \lambda_3 x^{1/6} (\log x)^{1/2} \} \right) = \Omega(T^{5/6-\varepsilon}), \]
for some \( \lambda_3 > 0 \) and for any \( \epsilon > 0 \). Balasubramanian and Ramachandra [4] also obtained
\[ \Delta(x) = \Omega(\pm x^{9/122}) \]
Sankaranarayanan and Srinivas [26] improved this to
\[ \Delta(x) = \Omega(\pm x^{1/10} \exp \left( c \sqrt{\log x} \right)), \]
for some constant \( c > 0 \). It has been conjectured that
\[ \Delta(x) = \Omega(\pm x^{1/6-\delta}), \]
for any \( \delta > 0 \). In Proposition 6 we prove that either
\[ \int_T^{2T} \Delta^4(x)dx = \Omega(T^{5/3+\delta}) \text{ or } \Delta(x) = \Omega(\pm x^{1/6-\delta}), \]
for any \( 0 < \delta < 1/42 \). The conjectured upper bound (6) of \( \Delta(x) \) gives
\[ \int_T^{2T} \Delta^4(x)dx \ll T^{5/3+\delta}. \]
This along with Proposition 6 implies that
\[ \Delta(x) = \Omega(\pm x^{1/6-\delta}), \]
for any \( 0 < \delta < 1/42 \).

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2. **Mellin Transform Of The Error Term**

Recall that we have a sequence of real numbers \( \{ a_n \}_{n=1}^{\infty} \), with its Dirichlet series \( D(s) \). We also have
\[ \sum_{n \leq x} a_n = M(x) + \Delta(x), \]
where \( M(x) \) is the main term and \( \Delta(x) \) is the error term. The following set of assumptions will represent \( M(x) \) and \( D(s) \) in a wide generality.

**Assumptions 1.** Suppose there exist real numbers \( \sigma_1 \) and \( \sigma_2 \) satisfying \( 0 < \sigma_1 < \sigma_2 \), such that
(i) \( D(s) \) is absolutely convergent for \( \text{Re}(s) > \sigma_2 \).
(ii) $D(s)$ can be meromorphically continued to the half plane $\text{Re}(s) > \sigma_1$ with only finitely many poles $\rho$ of $D(s)$ satisfying

$$\sigma_1 < \text{Re}(\rho) \leq \sigma_2.$$ 

We shall denote this set of poles by $\mathcal{P}$.

(iii) The main term $M(x)$ is sum of residues of $D(s)x^s$ at poles in $\mathcal{P}$:

$$M(x) = \sum_{\rho \in \mathcal{P}} \text{Res}_{s=\rho} \left( \frac{D(s)x^s}{s} \right).$$

Note 1. We may also observe:

(i) For any $\epsilon > 0$, we have

$$|a_n|, |M(x)|, |\Delta(x)|, \sum_{n \leq x} a_n \ll x^{\sigma_2 + \epsilon}.$$ 

(ii) The main term $M(x)$ is a polynomial in $x$, and log $x$:

$$M(x) = \sum_{j \in \mathcal{J}} \nu_{1,j}x^{\nu_{2,j}}(\log x)^{\nu_{3,j}},$$

where $\nu_{1,j}$ are complex numbers, $\nu_{2,j}$ are real numbers with $\sigma_1 < \nu_{2,j} \leq \sigma_2$, $\nu_{3,j}$ are positive integers, and $\mathcal{J}$ is a finite index set.

Now we shall discuss a method to obtain a meromorphic continuation of $A(s)$ (see Definition 2) by expressing it as a contour integration involving $D(s)$. Below, we define our required contour $\mathcal{C}$.

**Definition 3.** Let $\sigma_1, \sigma_2$ and $T_0$ be as defined in Assumptions 1. Choose a positive real number $\sigma_3$ such that $\sigma_3 > \sigma_2$. We define the contour $\mathcal{C}$, as in Figure 2, as the union of the following five line segments:

$$\mathcal{C} = L_1 \cup L_2 \cup L_3 \cup L_4 \cup L_5,$$

where

$$L_1 = \{\sigma_3 + iv : T_0 \leq v < \infty\}, \quad L_2 = \{u + iT_0 : \sigma_1 \leq u \leq \sigma_3\},$$

$$L_3 = \{\sigma_1 + iv : -T_0 \leq v \leq -T_0\}, \quad L_4 = \{u - iT_0 : \sigma_1 \leq u \leq \sigma_3\},$$

$$L_5 = \{\sigma_3 + iv : -\infty < v \leq -T_0\}. $$

In the above definition, the set of poles of $D(s)$ that lie to the right of $\mathcal{C}$ is exactly the set $\mathcal{P}$. The main theorem of this section gives analytic continuation of $A(s)$ as follows:

**Theorem 1.** Under the conditions in Assumptions 1, we have

$$A(s) = \int_{\mathcal{C}} \frac{D(\eta)}{\eta(s-\eta)}d\eta,$$

when $s$ lies to the right of $\mathcal{C}$.

We shall use several preparatory lemmas to prove the above theorem. Our first lemma gives an integral expression for $\Delta(x)$. 
Lemma 1. The error term $\Delta(x)$ can be expressed as the following integral:

$$\Delta(x) = \oint_{\mathcal{C}} \frac{D(\eta)x^\eta}{\eta} \, d\eta.$$ 

Proof. Follows from the definition of $\mathcal{C}$ and $\Delta$, and using Perron’s formula.

As a consequence of the above lemma, we get:

$$A(s) = \int_1^\infty \int_{\mathcal{C}} \frac{D(\eta)x^\eta}{\eta} \frac{dx}{x^{s+1}}.$$ 

Now we shall justify interchange of the integrals of $\eta$ and $x$ in (7), which will help us to continue $A(s)$ meromorphically.

Definition 4. Define the following complex valued function $B(s)$:

$$B(s) := \oint_{\mathcal{C}} \frac{D(\eta)}{\eta} \int_1^\infty \frac{dx}{x^{s-\eta+1}} \, d\eta \quad = \int_{\mathcal{C}} \frac{D(\eta)d\eta}{(s-\eta)\eta}, \quad \text{for } Re(s) > Re(\eta).$$

Observe that $B(s)$ is well defined and analytic as the integral defining $B(s)$ is absolutely convergent.

Definition 5. For a positive integer $N$, define the contour $\mathcal{C}(N)$ as:

$$\mathcal{C}(N) = \{ \eta \in \mathcal{C} : |Im(\eta)| \leq N \}.$$ 

Definition 6. Integrating over $\mathcal{C}(N)$, define $B_N(s)$ as:

$$B_N(s) = \oint_{\mathcal{C}(N)} \frac{D(\eta)}{\eta} \int_1^\infty \frac{dx}{x^{s-\eta+1}} \, d\eta \quad = \int_{\mathcal{C}(N)} \frac{D(\eta)d\eta}{(s-\eta)\eta}, \quad \text{for } Re(s) > Re(\eta).$$
Lemma 2. The functions $B$ and $B_N$ satisfy the following identities:

\begin{align}
B(s) &= \lim_{N \to \infty} B_N(s) \\
(8) &= \lim_{N \to \infty} \int_1^\infty \int_{\gamma(N)} \frac{D(\eta)x^\eta}{\eta} \frac{dx}{x^{s+1}}, \\
(9) &= \lim_{N \to \infty} \int_1^\infty \int_{\gamma(N)} \frac{D(\eta)x^\eta}{\eta} \frac{dx}{x^{s+1}}.
\end{align}

Proof. Assume that $N > T_0$. To show (8), note:

$$|B(s) - B_N(s)| \leq \left| \int_{\gamma - \gamma(n)} \frac{D(\eta)d\eta}{(s-\eta)\eta} \right| \ll \left| \int_{\gamma_3+i\infty} D(\eta)d\eta + \int_{\gamma_3-i\infty} D(\eta)d\eta \right|.$$ 

This completes proof of (8).

We shall prove (9) using a theorem of Fubini and Tonelli [B.3.1, (b)]. To show that the integrals commute, we need to show that one of the iterated integrals in (9) converges absolutely. Note:

$$\int\int_{\gamma(N)} \frac{D(\eta)x^\eta}{\eta} \frac{dx}{x^{s+1}} = 0$$

as $\gamma_N$ is a finite contour. Thus (9) follows. \(\square\)

Define

$$B'_{N}(s) := \int_1^\infty \int_{\gamma(N)} \frac{D(\eta)x^\eta}{\eta} \frac{dx}{x^{s+1}}.$$ 

Hence, (9) of Lemma 2 can be restated as

$$\lim_{N \to \infty} B'_{N}(s) = B(s).$$

Now to show $A(s) = B(s)$, it is enough to show that

$$\lim_{N \to \infty} \int_1^\infty \int_{\gamma-\gamma(N)} \frac{D(\eta)x^\eta}{\eta} \frac{dx}{x^{s+1}} = 0.$$ 

Observe that the uniform convergence of the integrand is required to interchange the integral of $x$ with the limit, which in turn force the above limit to be zero. However, we do not have this. It is easy to see from Perron’s formula that the problem arises when $x$ is an integer. To handle this problem, we shall divide the integral in two parts, with one part being a neighborhood of integers.

Definition 7. For $\delta = \frac{1}{\sqrt{N}}$ (where $N \geq 2$), we construct the following set as a neighborhood of integers:

$$S(\delta) := [1, 1 + \delta] \cup (\cup_{m \geq 2}[m - \delta, m + \delta]).$$

Write

$$A(s) - B'_{N}(s) = J_{1,N}(s) + J_{2,N}(s) - J_{3,N}(s),$$

(10)
where
\[
J_{1,N}(s) = \int_{S(\delta)} \int_{|\eta| < \epsilon N} \frac{D(\eta) x^{\eta}}{\eta} \frac{dx}{x^{s+1}},
\]
\[
J_{2,N}(s) = \int_{S(\delta)} \int_{\sigma_3 - i \infty}^{\sigma_3 + i \infty} \frac{D(\eta) x^{\eta}}{\eta} \frac{dx}{x^{s+1}},
\]
\[
J_{3,N}(s) = \int_{S(\delta)} \int_{\sigma_3 - i N}^{\sigma_3 + i N} \frac{D(\eta) x^{\eta}}{\eta} \frac{dx}{x^{s+1}}.
\]

In the next three lemmas, we shall show that each of \( J_{i,N}(s) \to 0 \) as \( N \to \infty \).

**Lemma 3.** For \( \text{Re}(s) = \sigma > \sigma_3 + 1 \), we have the limit
\[
\lim_{N \to \infty} J_{1,N}(s) = 0.
\]

**Proof.** Using Perron’s formula [28, Theorem II.2.2] for \( x \in S(\delta)^c \), we have
\[
\left| \int_{\epsilon < \eta < \epsilon N} \frac{D(\eta) x^{\eta}}{\eta} \, d\eta \right| \ll \sum_{n=1}^{\infty} \frac{|a_n|}{n^{\sigma_3}(1 + N|\log(x/n)|)} \ll \frac{x^{\sigma_3}}{N} \sum_{n=1}^{\infty} \frac{|a_n|}{n^{\sigma_3}} + \frac{1}{N} \sum_{x/2 \leq n \leq 2x} \frac{x|a_n|}{|x-n|} \left( x^{-\sigma_3} \right),
\]
as \( \delta = \frac{1}{\sqrt{N}} \). From the above calculation, we see that
\[
|J_{1,N}| \ll \frac{1}{\sqrt{N}} \int_{1}^{\infty} x^{\sigma_3 - \sigma + \epsilon} \, dx \ll \frac{1}{\sqrt{N}},
\]
as \( \sigma = \text{Re}(s) > \sigma_3 + 1 + \epsilon \). This proves our required result. \( \square \)

**Lemma 4.** For \( \text{Re}(s) = \sigma > \sigma_3 \),
\[
\lim_{N \to \infty} J_{2,N}(s) = 0.
\]

**Proof.** Recall that
\[
\sum_{n \leq x}^* a_n = \begin{cases} \sum_{x \leq n} a_n + a_x/2 & \text{if } x \in \mathbb{N}, \\ \sum_{n \leq x} a_n & \text{if } x \notin \mathbb{N}. \end{cases}
\]
By Note \( \square \)
\[
\sum_{n \leq x}^* a_n \ll x^{\sigma_3}.
\]

Using this bound, we calculate an upper bound for \( J_{2,N} \) as follows:
\[
\left| \int_{S(\delta)} \int_{\sigma_3 - i \infty}^{\sigma_3 + i \infty} \frac{D(\eta) x^{\eta}}{\eta} \frac{dx}{x^{s+1}} \right| \leq \int_{S(\delta)} \frac{\sum_{n \leq x}^* a_n}{x^{\sigma_3 - \sigma - 1}} \frac{dx}{x^{s+1}} \ll \int_{S(\delta)} x^{\sigma_3 - \sigma - 1} \, dx \ll \int_{1}^{1+\delta} x^{\sigma_3 - \sigma - 1} \, dx + \sum_{m=2}^{\infty} \int_{m-\delta}^{m+\delta} x^{\sigma_3 - \sigma - 1} \, dx.
\]
This gives
\[ |J_{2,N}(s)| \ll \delta + \sum_{m \geq 2} \left( \frac{1}{(m - \delta)^{\sigma - \sigma_3}} - \frac{1}{(m + \delta)^{\sigma - \sigma_3}} \right). \]

Using the mean value theorem, for all \( m \geq 2 \) there exists a real number \( \overline{m} \in [m - \delta, m + \delta] \) such that
\[ |J_{2,N}(s)| \ll \delta + \sum_{m \geq 2} \frac{\delta}{\overline{m}^{\sigma - \sigma_3 + 1}} \ll \delta = \frac{1}{\sqrt{N}} \] by choosing \( \sigma > \sigma_3 \).

This implies that \( J_{2,N} \) goes to zero as \( N \to \infty \).

**Lemma 5.** For \( \sigma > \sigma_3 \), we have
\[ \lim_{N \to \infty} J_{3,N}(s) = 0. \]

**Proof.** Consider
\[ J_{3,N}(s) = \int_{S(\delta)} \int_{\sigma_3 - iN}^{\sigma_3 + iN} D(\eta) \frac{x^\eta}{\eta} d\eta \frac{dx}{x^{s+1}}. \]

This double integral is absolutely convergent for \( \text{Re}(s) > \sigma_3 \). Using the Theorem of Fubini-Tonelli [8, Theorem B.3.1, (b)], we can interchange the integrals:
\[ J_{3,N}(s) = \int_{\sigma_3 - iN}^{\sigma_3 + iN} D(\eta) \frac{x^\eta}{\eta} d\eta \left\{ \int_1^{1 + \delta} \frac{x^\eta}{x^{s+1}} dx + \sum_{m \geq 2} \int_{m - \delta}^{m + \delta} \frac{x^\eta}{x^{s+1}} dx \right\} d\eta. \]

For any \( \theta_1, \theta_2 \) such that \( 0 < \theta_1 < \theta_2 < \infty \), we have
\[ \int_{\theta_1}^{\theta_2} \frac{x^{\eta - s - 1}}{x^{s+1}} dx = \frac{1}{s - \eta} \left\{ \theta_2^{s - \eta} - \frac{1}{\theta_1^{s - \eta}} \right\} = \frac{\theta_2 - \theta_1}{\overline{\theta}^{s - \eta} + 1}, \]
for some \( \overline{\theta} \in [\theta_1, \theta_2] \). Applying the above formula to \( J_{3,N}(s) \), we get
\[ J_{3,N}(s) = \int_{\sigma_3 - iN}^{\sigma_3 + iN} D(\eta) \frac{2\delta}{\overline{m}^{s - \eta + 1}} d\eta = 2\delta \sum_{m \geq 1} \int_{\sigma_3 - iN}^{\sigma_3 + iN} \frac{D(\eta)}{m^{s - \eta + 1}} d\eta, \]
where \( 1/2 \in [1, 1 + \delta] \) and \( \overline{m} \in [m - \delta, m + \delta] \) for all integers \( m \geq 2 \). In the above calculation, we can interchange the series and the integral as the series is absolutely convergent. So we have
\[ J_{3,N}(s) \ll \delta \sum_{m \geq 1} \int_{-N}^N \frac{1}{(1 + |v|)m^{\sigma - \sigma_3 + 1}} dv \quad \text{(substituting } \eta = \sigma_3 + iv \text{)} \]
\[ \ll \delta \log N \sum_{m \geq 1} \frac{1}{m^{\sigma - \sigma_3 + 1}} \ll \frac{\log N}{\sqrt{N}}. \]

Here we used the fact that for \( \sigma > \sigma_3 \), the series
\[ \sum_{m \geq 1} \frac{1}{m^{\sigma - \eta + 1}} \]
is absolutely convergent. This proves our required result. \( \square \)
Proof of Theorem 1. From equation (10) and Lemma 3, 4 and 5, we get

\[ A(s) = \lim_{N \to \infty} B'_N(s), \]

when \( \text{Re}(s) > \sigma_3 + 1 \). From Lemma 2, we have

\[ B(s) = \lim_{N \to \infty} B'_N(s). \]

This gives \( A(s) \) and \( B(s) \) are equal for \( \text{Re}(s) > \sigma_3 + 1 \). By analytic continuation, \( A(s) \) and \( B(s) \) are equal for any \( s \) that lies right to \( \mathcal{C} \). \( \square \)

Remark 1. Though Theorem 1 has its significance in terms of its elegance and generality, there are alternative and easier ways to meromorphically continue \( A(s) \) in many special cases (see [1]).

In the next section, we shall use the meromorphic continuation of \( A(s) \) derived in Theorem 1 to obtain \( \Omega_\pm \) results for \( \Delta(x) \).

3. The Oscillation Theorem of Landau

We begin with a criterion for functions that do not change sign. This theorem appears in [1] and attributed to Landau [21].

Theorem 2 (Landau). Let \( f(x) \) be a piecewise continuous function defined on \( [1, \infty) \), bounded on every compact intervals and does not change sign when \( x > x_0 \) for some \( 1 < x_0 < \infty \). Define

\[ F(s) := \int_1^\infty \frac{f(x)}{x^{s+1}} dx, \]

and assume that the above integral is absolutely convergent in some half plane. Further, assume that we have an analytic continuation of \( F(s) \) in a region containing the following part of the real line:

\[ l(\sigma_0, \infty) := \{ \sigma + i0 : \sigma > \sigma_0 \}. \]

Then the integral representing \( F(s) \) is absolutely convergent for \( \text{Re}(s) > \sigma_0 \), and hence \( F(s) \) is an analytic function in this region.

Landau’s theorem gives a criteria when a function does not oscillate. We shall use Landau’s theorem indirectly by method of contradiction to show the sign changes of \( \Delta(x) \).

Consider the Mellin transformation \( A(s) \) of \( \Delta(x) \). We need the following situation to apply Landau’s theorem.

Assumptions 2. Suppose there exists a real number \( \sigma_0 \), \( 0 < \sigma_0 < \sigma_1 \), such that \( A(s) \) has the following properties:

(i) There exists \( t_0 \neq 0 \) such that

\[ \lambda := \limsup_{\sigma \searrow \sigma_0} (\sigma - \sigma_0) |A(\sigma + it_0)| > 0. \]

(ii) We also have

\[ l_s := \limsup_{\sigma \searrow \sigma_0} (\sigma - \sigma_0) A(\sigma) < \infty, \]

\[ l_i := \liminf_{\sigma \searrow \sigma_0} (\sigma - \sigma_0) A(\sigma) > -\infty. \]
(iii) The limits $l_i, l_s$ and $\lambda$ satisfy
$$l_i + \lambda > 0 \quad \text{and} \quad l_s - \lambda < 0.$$  
(iv) We can analytically continue $A(s)$ in a region containing the real line $l(\sigma_0, \infty)$.

**Remark 2.** From Assumptions 2 (i), we see that $\sigma_0 + it_0$ is a singularity of $A(s)$.

We construct the following sets for further use.

**Definition 8.** With $l_s, l_i$ and $\lambda$ as in Assumptions 2, and for an $\epsilon$ such that $0 < \epsilon < \min(\lambda + l_i, \lambda - l_s)$, define
$$A_1 := \{x : x \in [1, \infty), \Delta(x) > (l_i + \lambda - \epsilon)x^{\sigma_0}\}$$
and
$$A_2 := \{x : x \in [1, \infty), \Delta(x) < (l_s - \lambda + \epsilon)x^{\sigma_0}\}.$$  

3.1. $\Omega_\pm$ Results. Under Assumptions 2 and using methods from [19], we can derive the following measure theoretic theorem.

**Theorem 3.** Let the conditions in Assumptions 2 hold. Then for any real number $M > 1$, we have
$$\mu(A_j \cap [M, \infty]) > 0, \quad \text{for } j = 1, 2.$$  
In particular, we have
$$\Delta(x) = \Omega_\pm(x^{\sigma_0}).$$

**Proof.** We prove the Theorem only for $A_1$, as the other part is similar. Define
$$g(x) := \Delta(x) - (l_i + \lambda - \epsilon)x^{\sigma_0}, \quad G(s) := \int_1^\infty \frac{g(x)}{x^{s+1}} dx;$$
$$g^+(x) := \max(g(x), 0), \quad G^+(s) := \int_1^\infty \frac{g^+(x)}{x^{s+1}} dx;$$
$$g^-(x) := \max(-g(x), 0), \quad G^-(s) := \int_1^\infty \frac{g^-(x)}{x^{s+1}} dx.$$  
With the above notations, we have
$$g(x) = g^+(x) - g^-(x)$$
and
$$G(s) = G^+(s) - G^-(s).$$

Note that
$$G(s) = A(s) - \int_1^\infty (l_i + \lambda - \epsilon)x^{\sigma_0-s-1} dx$$
$$= A(s) + \frac{l_i + \lambda - \epsilon}{\sigma_0 - s}, \quad \text{for } \text{Re}(s) > \sigma_0.$$  
So $G(s)$ is analytic wherever $A(s)$ is, except possibly for a pole at $\sigma_0$. This gives
$$\limsup_{\sigma \downarrow \sigma_0} |G(\sigma + it_0)| = \limsup_{\sigma \downarrow \sigma_0} |A(\sigma + it_0)| = \lambda.$$  
(11) We shall use the above limit to prove our theorem. We proceed by method of contradiction. Assume that there exists an $M$ such that
$$\mu(A_1 \cap [M, \infty)) = 0.$$
This implies
\[ G^+(\sigma) = \int_1^\infty \frac{g^+(x)}{x^{s+1}} \, dx = \int_1^M \frac{g^+(x)}{x^{s+1}} \, dx \]
is bounded for any \( s \), and so is an entire function. By Assumptions 2, \( A(s) \) and \( G(s) \) can be analytically continued on the line \( l(\sigma_0, \infty) \). As \( G(s) \) and \( G^+(s) \) are analytic on \( l(\sigma_0, \infty) \), \( G^-(s) \) is also analytic on \( l(\sigma_0, \infty) \). The integral for \( G^-(s) \) is absolutely convergent for \( \text{Re}(s) > \sigma_3 + 1 \), and \( g^- (x) \) is a piecewise continuous function bounded on every compact sets. This suggests that we can apply Theorem 2 to \( G^-(s) \), and conclude that
\[ G^-(s) = \int_1^\infty \frac{g^-(x)}{x^{s+1}} \, dx \]
is absolutely convergent for \( \text{Re}(s) > \sigma_0 \).

From the above discussion, we summarize that the Mellin transformations of \( g, g^+ \) and \( g^− \) converge absolutely for \( \text{Re}(s) > \sigma_0 \). As a consequence, we see that \( G(\sigma), G^+(\sigma) \) and \( G^−(\sigma) \) are finite real numbers for \( \sigma > \sigma_0 \). For \( \sigma > \sigma_0 \), we compare \( G^+(\sigma) \) and \( G^−(\sigma) \) in the following two cases.

**Case 1:** \( G^+(\sigma) < G^−(\sigma) \).

In this case,
\[
(\sigma - \sigma_0)|G(\sigma + it_0)| \leq (\sigma - \sigma_0)|G(\sigma)| = -(\sigma - \sigma_0)G(\sigma) = -(\sigma - \sigma_0)A(\sigma) + l_i + \lambda - \epsilon.
\]

So we have
\[
\limsup_{\sigma \searrow \sigma_0} (\sigma - \sigma_0)|G(\sigma + it_0)| \leq l_i + \lambda - \epsilon - \liminf_{\sigma \searrow \sigma_0} (\sigma - \sigma_0)A(\sigma) \leq \lambda - \epsilon.
\]

This contradicts (11).

**Case 2:** \( G^+(\sigma) \geq G^−(\sigma) \).

We have,
\[
(\sigma - \sigma_0)|G(\sigma + it_0)| \leq (\sigma - \sigma_0)G^+(\sigma) = O(\sigma - \sigma_0) \quad (G^+(\sigma) \text{ being a bounded integral}).
\]

Thus
\[
\limsup_{\sigma \searrow \sigma_0} (\sigma - \sigma_0)|G(\sigma + it_0)| = 0.
\]

This contradicts (11) again.

Thus \( \mu(\mathcal{A}_1 \cap [M, \infty)) > 0 \) for any \( M > 1 \), which completes the proof. \( \square \)

### 3.2. Measure Theoretic \( \Omega_\pm \) Results

Results in last section shows that \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) are unbounded. But we do not know how the size of these sets grow. An answer to this question was given by Kaczorowski and Szyd\l{}o in [19] Theorem 4.

**Theorem 4** (Kaczorowski and Szydlo [19]). Let the conditions in Assumptions 2 hold. Also assume that for a non-decreasing positive continuous function \( h \) satisfying
\[ h(x) \ll x^\epsilon, \]
we have
\[
\int_T^{2T} \Delta^2(x) \, dx \ll T^{2\sigma_0 + 1} h(T).
\]
Then as $T \to \infty$,

$$
\mu (A_j \cap [1, T]) = \Omega \left( \frac{T}{h(T)} \right) \quad \text{for } j = 1, 2.
$$

The above theorem of Kaczorowski and Szydló has been generalized by Bhowmik, Ramaré and Schlage-Puchta by localizing the fluctuations of $\Delta(x)$ to $[T, 2T]$. Proof of this theorem follows from [6, Theorem 2].

**Theorem 5** (Bhowmik, Ramaré and Schlage-Puchta [6]). Let the assumptions in Theorem 4 hold. Then as $T \to \infty$,

$$
\mu (A_j \cap [T, 2T]) = \Omega \left( \frac{T}{h(T)} \right) \quad \text{for } j = 1, 2.
$$

In the above two theorems, (12) is a very strong condition to hold. For example, if $\Delta(x)$ is the error term in approximating $\sum_{n \leq x} |\tau(n, \theta)|^2$, we can not apply Theorem 5. In our next theorem, we generalize Theorem 5 by replacing the condition (12) by bounds that help to choose $h$.

**Theorem 6.** Let the conditions in Assumptions 3 hold. Assume that there is an analytic continuation of $A(s)$ in a region containing the real line $\mathbf{R}_{\sigma_0}$, Let $h_1$ and $h_2$ be two positive functions such that

$$
\int_{[T, 2T] \cap A_j} \frac{\Delta^2(x)}{x^{2\sigma_0+1}} \, dx \ll h_j(T), \quad \text{for } j = 1, 2.
$$

Then as $T \to \infty$,

$$
\mu (A_j \cap [T, 2T]) = \Omega \left( \frac{T}{h_j(T)} \right), \quad \text{for } j = 1, 2.
$$

**Proof.** We prove the theorem for the measure of $A_1$; the proof is similar for $A_2$. We define $g, g^+, g^-, G, G^+$ and $G^-$, as in Theorem 3 of Section 3. Assume that

$$
\mu (A_1 \cap [T, 2T]) = o \left( \frac{T}{h_1(T)} \right).
$$

This implies that for any $\varepsilon > 0$, there exists an integer $k(\varepsilon) > 0$ such that

$$
\frac{h_1(2^k)\mu(A_1 \cap [2^k, 2^{k+1}])}{2^k} < \varepsilon.
$$
for all $k > k(\varepsilon)$. Using the above assumption, we may obtain an upper bound for $G^+(\sigma)$ as follows:

\[
\int_{A_1} \frac{g^+(x)dx}{x^{\sigma+1}} \leq \sum_{k \geq 0} \int_{A_1 \cap [2^k, 2^{k+1}]} \frac{\Delta(x)dx}{x^{\sigma+1}}
\]

(as $\Delta(x) \geq g(x)$ on $A_1$ by Assumptions 2(iii))

\[
\leq \sum_{k \geq 0} \left( \int_{A_1 \cap [2^k, 2^{k+1}]} \frac{\Delta^2(x)dx}{x^{2\sigma_0+1}} \right)^{1/2} - \left( \frac{\mu(A_1 \cap [2^k, 2^{k+1}])}{2^{k(2(\sigma-\sigma_0)+1)}} \right)^{1/2}
\]

\[
\leq c_2 \sum_{k \geq 0} \left( \frac{h_1(2^k)\mu(A_1 \cap [2^k, 2^{k+1}])}{2^{k(2(\sigma-\sigma_0)+1)}} \right)^{1/2}
\]

\[
\leq c_3(\varepsilon) + \varepsilon^{1/2} \sum_{k \geq k(\varepsilon)} \frac{1}{2^{k(2(\sigma-\sigma_0))}} \quad \text{(Using (16)).}
\]

In the above inequalities, $c_2$ and $c_3(\varepsilon)$ are some constants, and $c_3(\varepsilon)$ depends on $\varepsilon$.

We summarize the above calculation to

\[
G^+(\sigma) \leq c_3(\varepsilon) + \varepsilon \int_{1}^{\infty} \frac{g^+(x)dx}{x^{\sigma+1}}
\]

is absolutely convergent for $\text{Re}(s) > \sigma_0$, and so it is analytic in this region. But

\[
G^-(s) = G(s) - G^+(s),
\]

and $G$ is analytic on $l(\sigma_0, \infty)$. So $G^-$ is also analytic on $l(\sigma_0, \infty)$. Using Theorem 2, we get that

\[
G^+(s) = \int_{1}^{\infty} \frac{g^+(x)dx}{x^{\sigma+1}}
\]

is absolutely convergent for $\text{Re}(s) > \sigma_0$. Absolute convergence of the integrals of $G$ and $G^+$ implies that the Mellin transformation of $g^-(x)$, given by

\[
G^-(s) = \int_{1}^{\infty} \frac{g^-(x)dx}{x^{\sigma+1}},
\]

is also absolutely convergent for $\text{Re}(s) > \sigma_0$. As a consequence, we get $G(\sigma), G^+(\sigma)$, and $G^-(\sigma)$ are real numbers for $\sigma > \sigma_0$. As indicated in Case-1 of Theorem 3, we can not have

\[
G^+(\sigma) < G^-(\sigma),
\]

when $\sigma > \sigma_0$. So we always have

\[
G^+(\sigma) \geq G^-(\sigma).
\]

From this inequality, we shall deduce a contradiction to (11). Using (17) and the form of $G^+$, we get

\[
(|\sigma-\sigma_0|G(\sigma+it)| \leq (\sigma-\sigma_0)G^+(\sigma) \leq (\sigma-\sigma_0)c_3(\varepsilon) + \frac{\varepsilon}{2^\sigma-\sigma_0} \log 2.
\]
From the above inequality, for $t = t_0$, we get
\[
\limsup_{\sigma \rightarrow \sigma_0} (\sigma - \sigma_0) |G(\sigma + it_0)| < \frac{\varepsilon}{\log 2},
\]
for any $\varepsilon > 0$. This is a contradiction to (11); thus, our assumption (15) is wrong. Hence
\[
\mu(A_1 \cap [x, 2x)) = \Omega \left( \frac{T}{h_1(T)} \right).
\]

Corollary 1. Let the conditions of Theorem 6 hold. If we have a monotonic positive function $h$ such that
\[
\Delta(x) = O(h(x)),
\]
then
\[
\mu(A_j \cap [T, 2T]) = \Omega \left( \frac{T^{1+2\sigma_0}}{h(T) + h^2(2T)} \right) \quad \text{for } j = 1, 2.
\]

Corollary 2. Similar to Corollary 1, we assume the conditions of Theorem 6. Then we have
\[
\int_{[T, 2T] \cap A_1} \Delta^2(x)dx = \Omega(T^{2\sigma_0+1}) \quad \text{for } j = 1, 2.
\]

Proof. The proof of this Corollary follows from an observation in the proof of Theorem 6. We shall prove this Corollary for $A_1$, and the proof for $A_2$ is similar. Note that as an important part of the proof of Theorem 6, we showed that the integral for $G^+(s)$ is absolutely convergent for $\Re(s) > \sigma_0$, by assuming (14) is false. Then we got a contradiction that proves (14). Now we proceed in a similar manner by assuming (21) is false. So we have
\[
\int_{[T, 2T] \cap A_1} \Delta^2(x)dx = o(T^{2\sigma_0+1}).
\]

So for an arbitrarily small constant $\varepsilon$, we have
\[
|G^+(s)| \leq \int_{A_1} g^+(x)dx \leq \sum_{k \geq 0} \int_{A_1 \cap [2^k, 2^{k+1}]} \frac{\Delta(x)dx}{x^{\sigma+1}} \leq \sum_{k \geq 0} \frac{1}{2k(\sigma - \sigma_0)} \left( \int_{A_1 \cap [2^k, 2^{k+1}]} \frac{\Delta^2(x)dx}{x^{2\sigma_0+1}} \right)^{1/2} \leq c_4(\varepsilon) + \varepsilon \sum_{k \geq k(\varepsilon)} \frac{1}{2k(\sigma - \sigma_0)},
\]

where $c_4(\varepsilon)$ is a positive constant depending on $\varepsilon$. From this, we obtain that $G^+(s)$ is absolutely convergent for $\Re(s) > \sigma_0$. Now, arguments similar to the proof of Theorem 6 yield a contradiction to (22). \qed

We may also verify that Corollary 2 implies Corollary 1.
Remark 3. Observe that in Theorem 5, the analytic continuation of $A(s)$, for $\text{Re}(s) > \sigma_0$, is implied by (12), while in Theorem 6 we need to assume an analytic continuation. For analytic continuation of $A(s)$, we shall use Theorem 1 of the previous section. Demonstrations of these techniques are given in the following applications.

3.3. Applications. Here we give three applications of Theorem 6. In the first application, we consider the error term that appears in an asymptotic formula for $\sum_{n \leq x} |\tau(n, \theta)|^2$. Theorem 5 is not applicable to this example. In the second application, we consider an error term that appears in an asymptotic formula for average order of the square-free divisors $d(2)(n)$. In this example, Theorem 5 is applicable under Riemann Hypothesis, whereas Theorem 6 gives a weaker measure theoretic $\Omega_\pm$ result unconditionally. In the third example, we obtain some results on the error term of the Prime Number Theorem. Though Theorem 5 is applicable in this case under Riemann hypothesis, we prove a slightly weaker result unconditionally by applying Corollary 1.

3.3.1. The Twisted Divisor Function. Let us write $a_n = |\tau(n, \theta)|^2$ for $\theta \neq 0$, where $\tau(n, \theta)$'s are defined in (2), and

$$D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \frac{\zeta^2(s)\zeta(s+i\theta)\zeta(s-i\theta)}{\zeta(2s)}$$

is its Dirichlet series that converges absolutely for $\text{Re}(s) > 1$. We define $\Delta(x)$ as in (4). An upper bound for $\Delta(x)$ (as in (5)) can be computed using Perron’s formula and fourth moment estimates of $\zeta(s)$ at $\text{Re}(s) = \frac{1}{2}$ (see [10, Theorem 33]). Define a contour $\mathcal{C}$ as given in Figure 2:

$$\mathcal{C} = \left(\frac{5}{4} - i\infty, \frac{5}{4} - i(\theta + 1)\right) \cup \left[\frac{5}{4} - i(\theta + 1), \frac{3}{4} - i(\theta + 1)\right] \cup \left[\frac{3}{4} - i(\theta + 1), \frac{3}{4} + i(\theta + 1)\right] \cup \left[\frac{3}{4} + i(\theta + 1), \frac{5}{4} + i(\theta + 1)\right] \cup \left[\frac{5}{4} + i(\theta + 1), \frac{5}{4} + i\infty\right].$$

From the Perron’s formula, we can show that

$$\Delta(x) = \int_{\mathcal{C}} \frac{D(\eta)x^\eta}{\eta} d\eta.$$

Using Theorem 4 we have

$$A(s) = \int_{1}^{\infty} \frac{\Delta(x)}{x^{s+1}} dx = \int_{\mathcal{C}} \frac{D(\eta)}{\eta(s-\eta)} d\eta,$$

when $s$ lies right to the contour $\mathcal{C}$. Denote the first nontrivial zero of $\zeta(s)$ with least positive imaginary part by $2s_0$, which is approximately

$$2s_0 = \frac{1}{2} + i14.134 \ldots$$
Define the contour $\mathcal{C}(s_0)$, as in Figure 3, such that $s_0$ and any real number $s \geq 1/4$ lie in the right side of this contour. A meromorphic continuation of $A(s)$ to all $s$ that lies right side of $\mathcal{C}(s_0)$ is given by

$$A(s) = \int_{\mathcal{C}(s_0)} \frac{D(\eta)}{\eta(s-\eta)} d\eta + \frac{\text{Res}_{\eta=s_0} D(\eta)}{s_0(s-s_0)}.$$  

From (24) we calculate the following two limits:

$$\lambda(\theta) := \lim_{\sigma \to 0} \sigma |A(\sigma + s_0)| = |s_0|^{-1} \left| \text{Res}_{\eta=s_0} D(\eta) \right| > 0,$$

and

$$\lim_{\sigma \to 0} \sigma A(\sigma + 1/4) = 0.$$  

For a fixed $\epsilon_0 > 0$, let

$$\mathcal{A}_1 = \left\{ x : \Delta(x) > (\lambda(\theta) - \epsilon_0)x^{1/4} \right\}$$

and

$$\mathcal{A}_2 = \left\{ x : \Delta(x) < (-\lambda(\theta) + \epsilon_0)x^{1/4} \right\}.$$  

Upper-bound of $\Delta$ from (5) and Corollary 1 give

$$\mu (\mathcal{A}_j \cap [T, 2T]) = \Omega \left( T^{1/2}(\log T)^{-12} \right) \text{ for } j = 1, 2.$$  

Under Riemann Hypothesis, Theorem 6 and Proposition 4 give

$$\mu (\mathcal{A}_j \cap [T, 2T]) = \Omega \left( T^{3/4-\epsilon} \right) \text{ for } j = 1, 2.$$
From Corollary 2 we get
\begin{equation}
\int_{A_j \cap [T, 2T]} \Delta^2(x) dx = \Omega \left(T^{3/2}\right) \text{ for } j = 1, 2.
\end{equation}

3.3.2. Square Free Divisors. Let \( a_n = 2^{\omega(n)} \), where \( \omega(n) \) denotes the number of distinct prime factors of \( n \); equivalently, \( a_n \) denotes the number of square free divisors of \( n \). We write
\[ \sum_{n \leq x} 2^{\omega(n)} = \mathcal{M}(x) + \Delta(x), \]
where
\[ \mathcal{M}(x) = x \log x \frac{\zeta(2)}{\zeta(2)} + \left( -\frac{2\zeta'(2)}{\zeta^2(2)} + \frac{2\gamma - 1}{\zeta(2)} \right) x, \]
and by a theorem of Hölder [15]
\begin{equation}
\Delta(x) \ll x^{1/2}.
\end{equation}
Under Riemann Hypothesis, Baker [2] has improved the above upper bound to
\[ \Delta(x) \ll x^{4/11}. \]
We may check that the Dirichlet series \( D(s) \) has the following meromorphic continuation:
\[ D(s) = \sum_{n=1}^{\infty} \frac{2^{\omega(n)}}{n^s} = \frac{\zeta^2(s)}{\zeta(2s)}. \]
Let \( A(s) \) be the Mellin transform of \( \Delta(x) \) at \( s \), and let \( s_0 \) be as in [23]. Similar arguments as in the previous application assure that \( A(s) \) has no real pole for \( \text{Re}(s) \geq 1/4 \), and yield the following limits:
\[ \lambda_1 := \lim_{\sigma \searrow 0} \sigma |A(\sigma + s_0)| = |s_0|^{-1} \left| \text{Res}_{\eta=s_0} D(\eta) \right| > 0. \]
and
\[ \lim_{\sigma \searrow 0} \sigma A(\sigma + 1/4) = 0. \]
For a fixed \( \epsilon_0 > 0 \), let
\[ A_1 = \left\{ x : \Delta(x) > (\lambda_1 - \epsilon_0)x^{1/4} \right\} \]
and \( A_2 = \left\{ x : \Delta(x) < (-\lambda_1 + \epsilon_0)x^{1/4} \right\} \).

Using Corollary 1 and (29), we get
\[ (30) \mu(A_j \cap [T, 2T]) = \Omega \left( T^{1/2} \right) \quad \text{for} \ j = 1, 2. \]

However, assuming the Riemann Hypothesis and arguing similarly as in Proposition 4, we may show that
\[ \int_T^{2T} \Delta^2(x) \ll T^{3/2 + \epsilon} \quad \text{for any} \ \epsilon > 0. \]
This upper bound along with Theorem 6 gives
\[ (31) \mu(A_j \cap [T, 2T]) = \Omega \left( T^{1-\epsilon} \right) , \quad \text{for} \ j = 1, 2 \]
and for any \( \epsilon > 0 \).

3.3.3. The Prime Number Theorem Error. Now we consider the error term in the Prime Number Theorem:
\[ \Delta(x) = \sum_{n \leq x} \Lambda(n) - x. \]

Let
\[ \lambda_2 = |2s_0|^{-1}, \]
where \( 2s_0 \) is the first nontrivial zero of \( \zeta(s) \) and is same as in the previous applications. As an application of Corollary 1 we shall prove the following proposition:

**Proposition 1.** We denote
\[ A_1 = \left\{ x : \Delta(x) > (\lambda_2 - \epsilon_0)x^{1/2} \right\} \]
and \( A_2 = \left\{ x : \Delta(x) < (-\lambda_2 + \epsilon_0)x^{1/2} \right\} \),
for a fixed \( \epsilon_0 \) such that \( 0 < \epsilon_0 < \lambda_2 \). Then
\[ \mu(A_j \cap [T, 2T]) = \Omega \left( T^{1-\epsilon} \right) , \quad \text{for} \ j = 1, 2 \]
and for any \( \epsilon > 0 \).

**Proof.** Here we apply Corollary 1 in a similar way as in the previous applications, so we shall skip the details.

The Riemann Hypothesis, Theorem 5 and (1) give
\[ \mu(A_j \cap [T, 2T]) = \Omega \left( \frac{T}{\log^4 T} \right) , \quad \text{for} \ j = 1, 2; \]
this implies the proposition. But if the Riemann Hypothesis is false, then there exists a constant \( a \), with \( 1/2 < a \leq 1 \), such that
\[ a = \sup\{ \sigma : \zeta(\sigma + it) = 0 \}. \]
Using Perron summation formula, we may show that
\[ \Delta(x) \ll x^{a+\epsilon}, \]
for any \(\epsilon > 0\). Also for any arbitrarily small \(\delta\), we have \(a - \delta < \sigma' < a\) such that \(\zeta(\sigma' + it') = 0\) for some real number \(t'\). If \(\lambda'' := |\sigma'+it'|^{-1}\), then by Corollary 1 we get
\[
\mu \left( \left\{ x \in [T,2T] : \Delta(x) > (\lambda''/2)x^{\sigma'} \right\} \right) = \Omega \left( T^{1-2\delta-2\epsilon} \right)
\]
and
\[
\mu \left( \left\{ x \in [T,2T] : \Delta(x) < -(\lambda''/2)x^{\sigma'} \right\} \right) = \Omega \left( T^{1-2\delta-2\epsilon} \right).
\]
As \(\delta\) and \(\epsilon\) are arbitrarily small and \(\sigma' > 1/2\), the above \(\Omega\) bounds imply the proposition. \(\square\)

Remark 4. Results similar to Proposition 1 can be obtained for error terms in asymptotic formulas for partial sums of Mobius function and for partial sums of the indicator function of square-free numbers.

Remark 5. In Section 3.3.2 and 3.3.3, we saw that \(\mu(A_j)\) are large. Now suppose that \(\mu(A_1 \cup A_2)\) is large, then what can we say about the individual sizes of \(A_j\)? We may guess that \(\mu(A_1)\) and \(\mu(A_2)\) are both large and almost equal. But this may be very difficult to prove. In Section 3 we shall show that if \(\mu(A_1 \cup A_2)\) is large, then both \(A_1\) and \(A_2\) are nonempty. In the next section, we obtain an \(\Omega\) bound for \(\mu(A_1 \cup A_2)\), with \(\sigma_0 = 3/8\) and \(\Delta(x)\) being the error term in \([4]\).

4. An Omega Theorem For The Twisted Divisor Function

In \([3]\) and \([4]\), Balasubramanian and Ramachandra introduced a method to obtain a lower bound for
\[ \int_T^{T+b} \frac{\Delta(x)^2}{x^{2\alpha+1}} \, dx \]
in terms of the second moment of \(D(s)\), for some \(b > 0\) and \(\alpha > 0\). A nondecreasing lower bound gives
\[ \Delta(x) = \Omega(x^{\alpha-\epsilon}), \text{ for any } \epsilon > 0. \]
In these papers, they considered the error terms in asymptotic formulas for partial sums of certain arithmetic functions such as sum of square-free divisors and counting function for non-isomorphic abelian groups. This method requires the Riemann Hypothesis to be assumed in certain cases. Balasubramanian, Ramachandra and Subbarao \([5]\) modified this technique to apply on error term in the asymptotic formula for the counting function of \(k\)-full numbers without assuming Riemann Hypothesis. This method has been used by several authors including \([20]\) and \([26]\).

In this section, we consider the Dirichlet series
\[ D(s) = \sum_{n \geq 1} \frac{\left| \tau(n, \theta) \right|^2}{n^s} = \frac{\zeta^2(s)\zeta(s+\theta)\zeta(s-\theta)}{\zeta(2s)}, \]
for \(\text{Re}(s) > 1\). In accordance with the notation of the last section, we write
\[ \sum_{n \leq x} \left| \tau(n, \theta) \right|^2 = M(x) + \Delta(x), \]
where the main term \( M(x) = \omega_1(\theta)x\log x + \omega_2(\theta)x\cos(\theta \log x) + \omega_3(\theta)x \) comes from the poles of \( D(s) \) at \( s = 1, 1 + i\theta \) and \( s = 1 - i\theta \). Adopting the method of Balasubramanian, Ramachandra and Subbarao, we derive the following theorem.

**Theorem 7.** For any \( c > 0 \) and for a sufficiently large \( T \) depending on \( c \), we get

\[
\int_T^{\infty} \frac{|\Delta(x)|^2}{x^{2\alpha+1}} e^{-2x/y} \, du \gg c \exp \left( c(\log T)^{7/8} \right),
\]

where

\[
\alpha(T) = \frac{3}{8} - \frac{c}{(\log T)^{1/8}}.
\]

In particular, this implies

\[
\Delta(x) = \Omega(x^{3/8}\exp(-c(\log x)^{7/8})),
\]

for some suitable \( c > 0 \).

In order to prove the theorem, we need several lemmas, which form the content of this section. We begin with a fixed \( \delta_0 \in (0, 1/16] \) for which we would choose a numerical value at the end of this section.

**Definition 9.** For \( T > 1 \), let \( Z(T) \) be the set of all \( \gamma \) such that

1. \( T \leq \gamma \leq 2T \),
2. either \( \zeta(\beta_1 + i\gamma) = 0 \) for some \( \beta_1 \geq \frac{1}{2} + \delta_0 \)
   or \( \zeta(\beta_2 + i2\gamma) = 0 \) for some \( \beta_2 \geq \frac{1}{2} + \delta_0 \).

Let

\[
I_{\gamma,k} = \{ T \leq t \leq 2T : |t - \gamma| \leq k\log^2 T \} \text{ for } k = 1, 2.
\]

We finally define

\[
J_k(T) = [T, 2T] \setminus \bigcup_{\gamma \in Z(T)} I_{\gamma,k}.
\]

**Lemma 6.** With the above definition, we have for \( k = 1, 2 \)

\[
\mu(J_k(T)) = T + O(T^{1-\delta_0/4}\log^3 T).
\]

**Proof.** We shall use an estimate on the function \( N(\sigma, T) \), which is defined as

\[
N(\sigma, T) := |\{ \sigma' + it : \sigma' \geq \sigma, \ 0 < t \leq T, \ \zeta(\sigma' + it) = 0 \}|.
\]

Selberg [29, Page 237] proved that

\[
N(\sigma, T) \ll T^{1-\frac{1}{4}(\sigma - \frac{1}{2})}\log T, \text{ for } \sigma > 1/2.
\]

Now the lemma follows from the above upper bound on \( N(\sigma, t) \), and the observation that

\[
|\bigcup_{\gamma \in Z(T)} I_{\gamma,k}| \ll N \left( \frac{1}{2} + \delta_0, T \right) \log^2 T.
\]

The next lemma closely follows Theorem 14.2 of [29], but does not depend on Riemann Hypothesis.

**Lemma 7.** For \( t \in J_1(T) \) and \( \sigma = 1/2 + \delta \) with \( \delta_0 < \delta < 1/4 - \delta_0/2 \), we have

\[
|\zeta(\sigma + it)|^{1\pm 1} \ll \exp \left( \log \log t \left( \frac{\log t}{\delta_0} \right)^{\frac{1-2\delta}{1-2\delta_0}} \right).
\]
\[ |\zeta(\sigma + 2it)|^{\pm 1} \ll \exp \left( \log \log t \left( \frac{\log t}{\delta_0} \right)^{\frac{1-2\delta}{1-2\delta_0}} \right). \]

**Proof.** We provide a proof of the first statement, and the second statement can be similarly proved.

Let \( 1 < \sigma' \leq \log t \). We consider two concentric circles centered at \( \sigma' + it \), with radius \( \sigma' - 1/2 - \delta_0/2 \) and \( \sigma' - 1/2 - \delta_0 \). Since \( t \in J_1(T) \) and the radius of the circle is \( \ll \log t \), we conclude that

\[ \zeta(z) \neq 0 \quad \text{for} \quad |z - \sigma' - it| \leq \sigma' - \frac{1}{2} - \frac{\delta_0}{2} \]

and also \( \zeta(z) \) has polynomial growth in this region. Thus on the larger circle, \( \log |\zeta(z)| \leq c_5 \log t \), for some constant \( c_5 > 0 \). By Borel-Carathéodory theorem,

\[ |z - \sigma' - it| \leq \sigma' - \frac{1}{2} - \delta_0 \implies \left| \log \zeta(z) \right| \leq \frac{c_6 \sigma'}{\delta_0} \log t, \]

for some \( c_6 > 0 \). Let \( 1/2 + \delta_0 < \sigma < 1 \), and \( \xi > 0 \) be such that \( 1 + \xi < \sigma' \). We consider three concentric circles centered at \( \sigma' + it \) with radius \( r_1 = \sigma' - 1 - \xi \), \( r_2 = \sigma' - \sigma \) and \( r_3 = \sigma' - 1/2 - \delta_0 \), and call them \( C_1, C_2 \) and \( C_3 \) respectively. Let

\[ M_i = \sup_{z \in C_i} |\log \zeta(z)|. \]

From the above bound on \( |\log \zeta(z)| \), we get

\[ M_3 \leq \frac{c_6 \sigma'}{\delta_0} \log t. \]

Suitably enlarging \( c_6 \), we see that

\[ M_1 \leq \frac{c_6}{\xi}. \]

Hence we can apply the Hadamard’s three circle theorem to conclude that

\[ M_2 \leq M_1^{1-\nu} M_3^{\nu}, \quad \text{for} \quad \nu = \frac{\log(r_2/r_1)}{\log(r_3/r_1)}. \]

Thus

\[ M_2 \leq \left( \frac{c_6}{\xi} \right)^{1-\nu} \left( \frac{c_6 \sigma' \log t}{\delta_0} \right)^{\nu}. \]

It is easy to see that

\[ \nu = 2 - 2\sigma + \frac{4\delta_0(1 - \sigma)}{1 + 2\xi - 2\delta_0} + O(\xi) + O \left( \frac{1}{\sigma'} \right). \]

Now we put

\[ \xi = \frac{1}{\sigma'} = \frac{1}{\log \log t}. \]

Hence

\[ M_2 \leq \frac{c_6 \log \nu \cdot \log \log t}{\delta_0} = \frac{c_7 \log \log t}{\sigma'} (\log t)^{2 - 2\sigma + \frac{4\delta_0(1 - \sigma)}{1 + 2\xi - 2\delta_0}}, \]

for some \( c_7 > 0 \). We observe that

\[ 2 - 2\sigma + \frac{4\delta_0(1 - \sigma)}{1 + 2\xi - 2\delta_0} < 2 - 2\sigma + \frac{4\delta_0(1 - \sigma)}{1 - 2\delta_0} = 1 - 2\delta. \]


So we get

$$|\log \zeta(\sigma + it)| \leq c_7 \log \log t \left( \frac{\log t}{\delta_0} \right)^{\frac{1-\delta_0}{2}} ,$$

and hence the lemma.

We put \( y = T^b \), for a constant \( b \geq 8 \). Now suppose that

$$\int_T^\infty \frac{\Delta(u)^2}{u^{2\alpha+1}} e^{-u/y} du \geq \log^2 T,$$

for sufficiently large \( T \). Then clearly

$$\Delta(u) = \Omega(u^\alpha).$$

Our next result explores the situation when such an inequality does not hold.

**Proposition 2.** Let \( \delta_0 < \delta < \frac{1}{4} - \frac{\delta_0}{2} \). For \( \frac{1}{4} + \delta/2 < \alpha < \frac{1}{2} \), suppose that

$$\int_T^\infty \frac{\Delta(u)^2}{u^{2\alpha+1}} e^{-u/y} du \leq \log^2 T,$$

for a sufficiently large \( T \). Then we have

$$\int_{Re(z)=\alpha} \frac{|D(z)|^2}{|s|^2} \ll 1 + \int_T^\infty \frac{\Delta(u)^2}{u^{2\alpha+1}} e^{-2u/y} du.$$

Before embarking on a proof, we need the following technical lemmas.

**Lemma 8.** For \( 0 \leq Re(z) \leq 1 \) and \( |Im(z)| \geq \log^2 T \), we have

$$\int_T^\infty e^{-u/y} u^{-z} du = \frac{T^{1-z}}{1-z} + O(T^{-b'})$$

and

$$\int_T^\infty e^{-u/y} u^{-z} \log u \ du = \frac{T^{1-z}}{1-z} \log T + O(T^{-b'}) ,$$

where \( b' > 0 \) depends only on \( b \).

**Proof.** By changing variable by \( v = u/y \), we get

$$\int_T^\infty \frac{e^{-u/y}}{u^z} du = \int_T^\infty e^{-v} v^{-z} dv .$$

Integrating the right hand side by parts

$$\int_T^\infty e^{-v} v^{-z} dv = \frac{e^{-T/y}}{1-z} \left( \frac{T}{y} \right)^{1-z} + \frac{1}{1-z} \int_T^\infty e^{-v} v^{1-z} dv .$$

It is easy to see that

$$\int_T^\infty e^{-v} v^{1-z} dv = \Gamma(2-z) + O \left( \left( \frac{T}{y} \right)^{2-\text{Re}(z)} \log T \right) .$$

Hence (34) follows using \( e^{-T/y} = 1 + O(T/y) \) and Stirling’s formula along with the assumption that \( |Im(z)| \geq \log^2 T \).

Proof of (35) proceeds in the same line and uses the fact that

$$\int_T^\infty e^{-v} v^{1-z} \log v \ dv = \Gamma'(2-z) + O \left( \left( \frac{T}{y} \right)^{2-\text{Re}(z)} \log T \right) .$$
Then we apply Stirling’s formula for $\Gamma'(s)$ instead of $\Gamma(s)$. □

**Lemma 9.** Under the assumption \([33]\), there exists $T_0$ with $T \leq T_0 \leq 2T$ such that

$$\frac{\Delta(T_0)e^{-T_0/y}}{T_0^{\alpha}} \ll \log^2 T,$$

and

$$\frac{1}{y} \int_{T_0}^{\infty} \frac{\Delta(u)e^{-u/y}}{u^\alpha} du \ll \log T.$$  

**Proof.** The assumption \([33]\) implies that

$$\log^2 T \geq \int_{T}^{2T} \frac{|\Delta(u)|^2}{u^{2\alpha+1}} e^{-u/y} du$$

$$= \int_{T}^{2T} \frac{|\Delta(u)|^2}{u^{2\alpha}} e^{-2u/y} \frac{u^{\alpha}}{u} du$$

$$\geq \min_{T \leq u \leq 2T} \left( \frac{|\Delta(u)|}{u^\alpha} e^{-u/y} \right)^2,$$

which proves the first assertion. To prove the second assertion, we use the previous assertion and Cauchy-Schwarz inequality along with assumption \([33]\) to get

$$\left( \int_{T_0}^{\infty} \frac{|\Delta(u)|^2}{u^\alpha} e^{-u/y} du \right)^2 \leq \left( \int_{T_0}^{\infty} \frac{|\Delta(u)|}{u^{2\alpha+1}} e^{-u/y} du \right) \left( \int_{T_0}^{\infty} u e^{-u/y} du \right)$$

$$\ll y^2 \log^2 T.$$

This completes the proof of this lemma. □

We now recall a mean value theorem due to Montgomery and Vaughan \([22]\).

**Notation.** For a real number $\theta$, let $\| \theta \| := \min_{n \in \mathbb{Z}} |\theta - n|.$

**Theorem 8** (Montgomery and Vaughan [22]). Let $a_1, \cdots, a_N$ be arbitrary complex numbers, and let $\lambda_1, \cdots, \lambda_N$ be distinct real numbers such that

$$\delta = \min_{m,n} \| \lambda_m - \lambda_n \| > 0.$$

Then

$$\int_{0}^{T} \left| \sum_{n \leq N} a_n \exp(i\lambda_n t) \right|^2 dt = \left( T + O \left( \frac{1}{\delta} \right) \right) \sum_{n \leq N} |a_n|^2.$$  

**Lemma 10.** For $T \leq T_0 \leq 2T$ and $Re(s) = \alpha$, we have

$$\int_{T}^{2T} \left| \sum_{n \leq T_0} \frac{|\tau(n, \theta)|^2}{n^s} e^{-n/y} \right|^2 t^{-2} dt \ll 1.$$  

**Proof.** Using theorem \([8]\) we get

$$\int_{T}^{2T} \left| \sum_{n \leq T_0} \frac{|\tau(n, \theta)|^2}{n^s} e^{-n/y} \right|^2 t^{-2} dt$$

$$\leq \frac{1}{T^2} \left( T \sum_{n \leq T_0} |b(n)|^2 + O \left( \sum_{n \leq T_0} n|b(n)|^2 \right) \right),$$
Thus
\[
\sum_{n \leq T_0} |b(n)|^2 \leq \sum_{n \leq T_0} \frac{d(n)^4}{n^{2\alpha}} \ll T_0^{1-2\alpha+\epsilon}
\]
and
\[
\sum_{n \leq T_0} n|b(n)|^2 \leq \sum_{n \leq T_0} \frac{d(n)^4}{n^{2\alpha-1}} \ll T_0^{2-2\alpha+\epsilon}
\]
for any \(\epsilon > 0\), since the divisor function \(d(n) \ll n^\epsilon\) for any \(\epsilon > 0\). This completes the proof as \(\alpha > 0\).

**Lemma 11.** For \(\text{Re}(s) = \alpha\) and \(T \leq T_0 \leq 2T\), we have
\[
\left| \sum_{n \geq 0} 1^\infty \left( \frac{\Delta(n + x + T_0)}{(n + x + T_0)^{s+1}} e^{-(n+x+T_0)/y} \right) dt \right|^2 \ll \int_T^{2T} \left| \sum_{n \geq 0} 1^\infty \left( \frac{\Delta(n + x + T_0)}{(n + x + T_0)^{s+1}} e^{-(n+x+T_0)/y} \right) \right|^2 dx.
\]

**Proof.** Using Cauchy-Schwarz inequality, we get
\[
\left| \sum_{n \geq 0} 1^1 \left( \frac{\Delta(n + x + T_0)}{(n + x + T_0)^{s+1}} e^{-(n+x+T_0)/y} \right) \right|^2 \leq \int_0^1 \left( \sum_{n \geq 0} \frac{\Delta(n + x + T_0)}{(n + x + T_0)^{s+1}} e^{-(n+x+T_0)/y} \right)^2 dx.
\]
Hence
\[
\int_T^{2T} \left| \int_0^1 \left( \sum_{n \geq 0} \frac{\Delta(n + x + T_0)}{(n + x + T_0)^{s+1}} e^{-(n+x+T_0)/y} \right) dx \right|^2 dt \\leq \int_T^{2T} \int_0^1 \left( \sum_{n \geq 0} \frac{\Delta(n + x + T_0)}{(n + x + T_0)^{s+1}} e^{-(n+x+T_0)/y} \right)^2 dx dt \\leq \int_T^{2T} \int_0^1 \left( \sum_{n \geq 0} \frac{\Delta(n + x + T_0)}{(n + x + T_0)^{s+1}} e^{-(n+x+T_0)/y} \right)^2 dt dx.
\]
From Theorem 8, we can get
\[
\int_T^{2T} \left| \sum_{n \geq 0} \frac{\Delta(n + x + T_0)}{(n + x + T_0)^{s+1}} e^{-(n+x+T_0)/y} \right|^2 dt = T \sum_{n \geq 0} \frac{\left| \Delta(n + x + T_0) \right|^2}{(n + x + T_0)^{2\alpha+2}} e^{-2(n+x+T_0)/y} + O\left( \sum_{n \geq 0} \frac{\left| \Delta(n + x + T_0) \right|^2}{(n + x + T_0)^{2\alpha+1}} \right) \ll \sum_{n \geq 0} \frac{\left| \Delta(n + x + T_0) \right|^2}{(n + x + T_0)^{2\alpha+1}} e^{-2(n+x+T_0)/y}.
\]
Lemma 7 to conclude that
\[ \delta \]
Since \( \delta \)
To estimate the second sum, we write
\[ \int_{\delta}^{1} \sum_{n > 0} \frac{\Delta(n + x + T_0)^2}{(n + x + T_0)^{s+1}} \, dx = \int_{T}^{1} \left| \sum_{n \geq 0} \frac{\Delta(n + x + T_0)^2}{(n + x + T_0)^{s+1}} \, dx \right|^2 \, dt \]
\[ \ll \int_{0}^{1} \sum_{n \geq 0} \frac{|\Delta(n + x + T_0)^2|}{(n + x + T_0)^{2s+1}} e^{-2(x+n+T_0)/y} \, dx \ll \int_{T}^{\infty} \frac{|\Delta(x)|^2}{x^{2s+1}} e^{-2x/y} \, dx, \]
completing the proof.

**Proof of Proposition** [2] For \( s = \alpha + it \) with \( 1/4 + \delta/2 < \alpha < 1/2 \) and \( t \in J_2(T) \), we have
\[ \sum_{n=1}^{\infty} \frac{\tau(n, \theta)|^2}{n^s} e^{-n/y} = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} D(s + w) \Gamma(w) y^w \, dw \]
\[ = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\log^2 T} + O \left( y^2 \int_{\log^2 T}^{\infty} |D(s + w)||\Gamma(w)| \, dw \right). \]
The above error term is estimated to be \( o(1) \). We move the integral to
\[ \left[ \frac{1}{4} + \frac{\delta}{2} - \alpha - i \log^2 T, \frac{1}{4} + \frac{\delta}{2} - \alpha + i \log^2 T \right]. \]
Let \( \delta' = 1/4 + \delta/2 - \alpha \). In this region \( \text{Re}(2s + 2w) = 1/2 + \delta \). So we can apply Lemma [7] to conclude that \( D(s + w) = O(T^\kappa) \), for some constant \( \kappa > 0 \). Thus the integrals along horizontal lines are \( o(1) \). Since the only pole inside this contour is at \( w = 0 \), we get
\[ \sum_{n=1}^{\infty} \frac{\tau(n, \theta)|^2}{n^s} e^{-n/y} = D(s) + \frac{1}{2\pi i} \int_{\delta' - i\log^2 T}^{\delta' + i\log^2 T} D(s + w) \Gamma(w) y^w \, dw + o(1). \]
Since \( \delta' < 0 \), the remaining integral can be shown to be \( o(1) \) for \( b \geq 8 \). Using \( T_0 \) as in Lemma [9] we now divide the sum into two parts:
\[ D(s) = \sum_{n \leq T_0} \frac{\tau(n, \theta)|^2}{n^s} e^{-n/y} + \sum_{n > T_0} \frac{\tau(n, \theta)|^2}{n^s} e^{-n/y} + o(1). \]
To estimate the second sum, we write
\[ \sum_{n > T_0} \frac{\tau(n, \theta)|^2}{n^s} e^{-n/y} = \int_{T_0}^{\infty} \frac{e^{-x/y}}{x^s} \left( \sum_{n \leq x} |\tau(n, \theta)|^2 \right) \, dx \]
\[ = \int_{T_0}^{\infty} \frac{e^{-x/y}}{x^s} d(M(x) + \Delta(x)) \]
\[ = \int_{T_0}^{\infty} \frac{e^{-x/y}}{x^s} M'(x) \, dx + \int_{T_0}^{\infty} \frac{e^{-x/y}}{x^s} d(\Delta(x)). \]
Recall that
\[ M(x) = \omega_1(\theta) x \log x + \omega_2(\theta) x \cos(\theta \log x) + \omega_3(\theta) x, \]
thus
\[ M'(x) = \omega_1(\theta) \log x + \omega_2(\theta) \cos(\theta \log x) - \theta \omega_2(\theta) \sin(\theta \log x) + \omega_1(\theta) + \omega_3(\theta). \]
Observe that
\[
\int_T^\infty \frac{e^{-x/y}}{x^s} \cos(\theta \log x) \, dx = \frac{1}{2} \int_T^\infty \frac{e^{-x/y}}{x^{s+i\theta}} \, dx + \frac{1}{2} \int_T^\infty \frac{e^{-x/y}}{x^{s-i\theta}} \, dx.
\]
Applying Lemma 8, we conclude that
\[
\int_T^\infty \frac{e^{-x/y}}{x^s} \mathcal{M}(x) \, dx = o(1).
\]
Integrating the second integral by parts:
\[
\int_T^\infty \frac{e^{-x/y}}{x^s} \, d(\Delta(x)) = \frac{e^{-T_0/y} \Delta(T_0)}{T_0^s} + \frac{1}{y} \int_T^\infty \frac{e^{-x/y}}{x^{s+1}} \Delta(x) \, dx - s \int_T^\infty \frac{e^{-x/y}}{x^{s+1}} \Delta(x) \, dx.
\]
Applying Lemma 9, we get
\[
\sum_{n > T_0} \frac{|\tau(n, \theta)|^2}{n^s} e^{-n/y} = s \int_T^\infty \frac{\Delta(x)e^{-x/y}}{x^{s+1}} \, dx + O(\log T)
\]
\[= s \sum_{n > T_0} \int_0^1 \frac{\Delta(n + x + T_0)e^{-(n+x+T_0)/y}}{(n + x + T_0)^{s+1}} \, dx + O(\log T).
\]
Hence we have
\[
D(s) = \sum_{n \leq T_0} \frac{|\tau(n, \theta)|^2}{n^s} e^{-n/y} + s \sum_{n \geq 0} \int_0^1 \frac{\Delta(n + x + T_0)e^{-(n+x+T_0)/y}}{(n + x + T_0)^{s+1}} \, dx + O(\log T).
\]
Squaring both sides and then integrating on \(J_2(T)\), we get
\[
\int_{J_2(T)} \left| \frac{D(\alpha + it)}{\alpha + it} \right|^2 \, dt \ll \int_T^{2T} \left| \sum_{n \leq T_0} \frac{|\tau(n, \theta)|^2}{n^s} e^{-n/y} \right|^2 \, dt \, \frac{dt}{t^2}
\]
\[+ \int_T^{2T} \left| \sum_{n \geq 0} \int_0^1 \frac{\Delta(n + x + T_0)e^{-(n+x+T_0)/y}}{(n + x + T_0)^{s+1}} \, dx \right|^2 \, dt.
\]
The proposition now follows using Lemma 10 and Lemma 11.

Proof of Theorem 7. We prove by contradiction. Suppose that (32) does not hold. Then, given any \(N_0 > 1\), there exists \(T > N_0\) such that
\[
\int_T^\infty \frac{\Delta(x)^2}{x^{2s+1}} e^{-2x/y} \, dx \ll \exp \left( c(\log T)^{7/8} \right),
\]
for all \(c > 0\). This gives
\[
\int_T^\infty \frac{\Delta(x)^2}{x^{2\beta+1}} e^{-2x/y} \, dx \ll 1,
\]
where
\[
\beta = \frac{3}{8} - \frac{c}{2(\log T)^{1/8}}.
\]
We apply Proposition 2 to get

\[ \int_{J_2(T)} |D(\beta + it)|^2 \, dt \ll 1. \]

Now we compute a lower bound for the last integral over \( J_2(T) \). Write the functional equation for \( \zeta(s) \) as

\[ \zeta(s) = \pi^{1/2 - s} \frac{\Gamma((1-s)/2)}{\Gamma(s/2)} \zeta(1 - s). \]

Using the Stilrings formula for \( \Gamma \) function, we get

\[ |\zeta(s)| = \pi^{1/2 - \beta} t^{1/2 - \beta} |\zeta(1 - s)| \left(1 + O \left( \frac{1}{T} \right) \right), \]

for \( s = \beta + it \). This implies

\[ |D(\beta + it)| = t^{2-4\beta} \frac{|\zeta(1-\beta+it)^2 \zeta(1-\beta-it-i\theta) \zeta(1-\beta-it+i\theta)|}{|\zeta(2\beta + 2it)|}. \]

Let \( \delta_0 = 1/16 \), and

\[ \beta = \frac{3}{8} - \frac{c}{2(\log T)^{1/8}} = \frac{1}{2} - \delta \]

with

\[ \delta = \frac{1}{8} + \frac{c}{2(\log T)^{1/8}}. \]

Then using Lemma 7 we get

\[ |\zeta(1-\beta+it)| = |\zeta \left( \frac{1}{2} + \delta + it \right) | \gg \exp \left( \log \log t \left( \frac{\log t}{\delta_0} \right)^{\frac{1-2\delta}{1-2\delta_0}} \right). \]

For \( t \in J_2(T) \) we observe that \( t \pm \theta \in J_1(T) \), and so the same bounds hold for \( \zeta(1-\beta+it+i\theta) \) and \( \zeta(1-\beta+it-i\theta) \). Further

\[ |\zeta(2\beta + 2it)| = |\zeta \left( \frac{1}{2} + \left( \frac{1}{2} - 2\delta \right) + i2t \right) | \ll \exp \left( \log \log t \left( \frac{\log t}{\delta_0} \right)^{\frac{4\delta_0}{1-2\delta_0}} \right). \]

Combining these bounds, we get

\[ |D(\beta + it)| \gg t^{2-4\beta} \exp \left( -5 \log \log t \left( \frac{\log t}{\delta_0} \right)^{\frac{1-2\delta}{1-2\delta_0}} \right). \]

Therefore

\[ \int_{J_2(T)} |D(\beta + it)|^2 \, dt \gg \int_{J_2(T)} |D(\beta + it)|^2 \, dt \gg \int_{J_1(T)} |D(\beta + it)|^2 \, dt \]

\[ \gg T^{4-8\beta} \exp \left( -10 \log \log T \left( \frac{\log T}{\delta_0} \right)^{\frac{1-2\delta}{1-2\delta_0}} \mu(J_2(T)) \right) \]

\[ \gg T^{5-8\beta} \exp \left( -10 \log \log T \left( \frac{\log T}{\delta_0} \right)^{\frac{1-2\delta}{1-2\delta_0}} \right), \]
where we use Lemma 6 to show that $\mu(J_2(T)) \gg T$. Now putting the values of $\delta$ and $\delta_0$ as chosen above, we get

$$\int_{J(T)} \frac{|D(\beta + it)|^2}{|\beta + it|^2} dt \gg \exp \left( 3c(\log T)^{7/8} \right),$$

since $\frac{1-2\delta}{1-2\delta_0} < 7/8$. This contradicts \( \text{[96]} \), and hence the theorem follows. $\square$

The following definition is required to state the corollaries.

**Definition 10.** An infinite unbounded subset $S$ of non-negative real numbers is called an $X$-Set.

The following two corollaries are immediate.

**Corollary 3.** For any $c > 0$ there exists an $X$-Set $S$, such that for sufficiently large $T$ depending on $c$ there exists an

$$X \in \left[ T, \frac{Tb}{2} \log^2 T \right] \cap S,$$

for which we have

$$\int_X^{2X} \frac{\Delta(x)^2}{x^{2\alpha+1}} dx \geq \exp \left( (c-\epsilon)(\log X)^{7/8} \right)$$

with $\alpha$ as in Theorem 7 and for any $\epsilon > 0$.

**Corollary 4.** For any $c > 0$ there exists an $X$-Set $S$, such that for sufficiently large $T$ depending on $c$ there exists an

$$x \in \left[ T, \frac{Tb}{2} \log^2 T \right] \cap S,$$

for which we have

$$\Delta(x) \geq x^{3/8} \exp \left( -c(\log x)^{7/8} \right).$$

We can now prove a "measure version" of the above result.

**Proposition 3.** For any $c > 0$, let

$$\alpha(x) = \frac{3}{8} - \frac{c}{(\log x)^{1/8}}$$

and $A = \{ x : |\Delta(x)| \gg x^{\alpha(x)} \}$. Then for every sufficiently large $X$ depending on $c$, we have

$$\mu(A \cap [X, 2X]) = \Omega(X^{2\alpha(X)}).$$

**Proof.** Suppose that the conclusion does not hold, hence

$$\mu(A \cap [X, 2X]) \ll X^{2\alpha(X)}.$$

Thus for every sufficiently large $X$, we get

$$\int_{A \cap [X, 2X]} \frac{|\Delta(x)|^2}{x^{2\alpha+1}} dx \ll X^{2\alpha} \frac{M(X)}{X^{2\alpha+1}} = \frac{M(X)}{X},$$

where $\alpha = \alpha(X)$ and $M(X) = \sup_{X \leq x \leq 2X} |\Delta(x)|^2$. Using dyadic partition, we can prove

$$\int_{A \cap [T, y]} \frac{|\Delta(x)|^2}{x^{2\alpha+1}} dx \ll \frac{M_0(T)}{T \log T},$$
where
\[ M_0(T) = \sup_{T \leq x \leq y} |\Delta(x)|^2 \]
and \( y = T^b \) for some \( b > 0 \) and \( T \) sufficiently large. This gives
\[ \int_T^{\infty} \frac{\Delta(x)^2}{x^{2\alpha+1}} e^{-2x/y} dx \ll \frac{M_0(T)}{T} \log T. \]
Along with (32), this implies
\[ M_0(T) \gg T \exp \left( \frac{C}{2} (\log T)^{7/8} \right). \]
Thus
\[ |\Delta(x)| \gg x^\frac{7}{8} \exp \left( \frac{C}{4} (\log x)^{7/8} \right), \]
for some \( x \in [T, y] \). This contradicts the fact that \( |\Delta(x)| \ll x^\frac{7}{8} (\log x)^6 \). \( \square \)

4.1. **Optimality of the Omega Bound for the Second Moment.** The following proposition shows the optimality of the omega bound in Corollary 3.

**Proposition 4.** Under Riemann Hypothesis (RH), we have
\[ \int_X^{2X} \Delta^2(x) dx \ll X^{7/4+\epsilon}, \]
for any \( \epsilon > 0 \).

**Proof.** The Perron’s formula gives
\[ \Delta(x) = \frac{1}{2\pi i} \int_{-T}^{T} \frac{D(3/8 + it)x^{3/8+it}}{3/8 + it} dt + O(x^\epsilon), \]
for any \( \epsilon > 0 \) and for \( T = X^2 \) with \( x \in [X, 2X] \). Using this expression for \( \Delta(x) \), we write its second moment as
\[ \int_X^{2X} \Delta^2(x) dx = \int_X^{2X} \int_{-T}^{T} \frac{D(3/8 + it_1)D(3/8 + it_2)x^{3/4+i(t_1+t_2)}}{(3/8 + it_1)(3/8 + it_2)} dt_1 dt_2 \]
\[ + O \left( X^{1+\epsilon} + |\Delta(x)| \right) \]
\[ \ll X^{7/4} \int_{-T}^{T} \int_{-T}^{T} \frac{D(3/8 + it_1)D(3/8 + it_2)}{(3/8 + it_1)(3/8 + it_2)(7/4 + i(t_1 + t_2))} dt_1 dt_2 + O(X^{3/2+\epsilon}). \]

In the above calculation, we have used the fact that \( \Delta(x) \ll x^{\frac{7}{8}+\epsilon} \) as in (5). Also note that for complex numbers \( a, b \), we have \( |ab| \leq \frac{1}{2}(|a|^2 + |b|^2) \). We use this inequality with
\[ a = \frac{|D(3/8 + it_1)|}{|3/8 + it_1| \sqrt{7/4 + i(t_1 + t_2)}} \quad \text{and} \quad b = \frac{|D(3/8 + it_2)|}{|3/8 + it_2| \sqrt{7/4 + i(t_1 + t_2)}} \]
to get
\[ \int_X^{2X} \Delta^2(x) dx \ll X^{7/4} \int_{-T}^{T} \int_{-T}^{T} \frac{|D(3/8 + it_1)|^2 dt_1}{|7/4 + i(t_1 + t_2)|} \frac{|D(3/8 + it_2)|^2 dt_2}{|7/4 + i(t_1 + t_2)|} + O(X^{3/2+\epsilon}) \]
\[ \ll X^{7/4} \log X \int_{-T}^{T} \frac{|D(3/8 + it_2)|^2 dt_2}{|7/4 + i(t_1 + t_2)|} + O(X^{3/2+\epsilon}). \]
Under RH, $|D(3/8 + it_2)| \ll |t_2|^{1+\epsilon}$. So we have
\[
\int_X^{2X} \Delta^2(x)dx \ll X^{7/4 + \epsilon} \text{ for any } \epsilon > 0.
\]

5. Influence Of Measure

In this section, we study the influence of measure of the set where $\Omega$-results hold. The following theorem is an illustration of the methods of this section, which will be proved in 5.3.2.

**Theorem 9.** Let $\Delta(x)$ be the error term of the summatory function of the twisted divisor function as defined in (3). For $c > 0$, let
\[
\alpha(x) = \frac{3}{8} - \frac{c}{(\log x)^{1/8}}.
\]

Let $\delta$ and $\delta'$ be such that
\[
0 < \delta < \delta' < \frac{1}{8}.
\]

Then either
\[
\Delta(x) = \Omega\left(x^{\alpha(x) + \frac{1}{2}}\right) \text{ or } \Delta(x) = \Omega_+\left(x^{\frac{3}{8} - \delta'}\right).
\]

Throughout this section, we assume the conditions and notations given in Assumptions 1. Further we have the following notations for this section.

**Notations.** For $i = 0, 1, 2$, let $\alpha_i(T)$ denote a positive monotonic function such that $\alpha_i(T)$ converges to a constant as $T \to \infty$. For example, in some cases $\alpha_i(T)$ could be $1 - 1/\log(T)$, which tend to 1 as $T \to \infty$.

For $i = 0, 1$, let $h_i(T)$ be positive monotonically increasing functions such that $h_i(T) \to \infty$ as $T \to \infty$.

For a real valued and non-negative function $f$, we denote
\[
A(f(x)) := \{x \geq 1 : |\Delta(x)| > f(x)\}.
\]

5.1. Refining Omega Result from Measure. Now we hypothesize a situation when there is a lower bound estimate for the second moment of the error term.

**Assumptions 3.** Let $\mathcal{S}$ be an $X$-Set. Define a real valued positive bounded function $\alpha(T)$ on $\mathcal{S}$, such that
\[
0 \leq \alpha(T) < M < \infty
\]
for some constant $M$. For a fixed $T$, we write
\[
\mathcal{A}_T := [T/2, T] \cap A(c_8x^{\alpha(x)}), \quad \text{for } c_8 > 0.
\]

For all $T \in \mathcal{S}$ and for constants $c_9$, $c_{10} > 0$, assume the following bounds hold:

(i) $\int_{\mathcal{A}_T} \frac{\Delta^2(x)}{x^{2\alpha + 1}}dx > c_9$,

(ii) $\mu(\mathcal{A}_T) < c_{10}h_0(T)$, and

(iii) the function
\[
x^{\alpha + 1/2}h_0^{-1/2}(x)
\]

is monotonically increasing for $x \in [T/2, T]$. 


We note that the first assumption indicates an $\Omega$-estimate. The next two assumptions indicate that the measure of the set on which the $\Omega$ estimate holds is not ‘too big’.

**Proposition 5.** Suppose there exists an $X$-Set $S$ having properties as described in Assumptions 3. Let the constant $c_{11}$ be given by

$$c_{11} := \sqrt{\frac{c_9}{2^{2M+1}c_{10}}}.$$  

Then there exists a $T_0$ such that for all $T > T_0$ and $T \in S$, we have

$$|\Delta(x)| > c_{11}x^{\alpha+1/2}h_0^{-1/2}(x)$$

for some $x \in [T/2, T]$.

In particular

$$\Delta(x) = \Omega(x^{\alpha+1/2}h_0^{-1/2}(x)).$$

**Proof.** If the statement of the above proposition is not true, then for all $x \in [T/2, T]$ we have

$$\Delta(x) \leq c_{11}x^{\alpha+1/2}h_0^{-1/2}(x).$$

From this, we may derive an upper bound for second moment of $\Delta(x)$:

$$\int_{A_T} \frac{\Delta^2(x)}{x^{2\alpha+1}} \, dx \leq \frac{c_{11}^2 T^{2\alpha+1} \mu(A_T \cap [T/2, T])}{h_0(T)(T/2)^{2\alpha+1}}$$

$$\leq c_{11}^2 2^{2M+1}c_{10} \leq c_9.$$  

The above bound contradicts (i) of Assumptions 3. This proves the proposition. □

### 5.2. Omega Plus-Minus Result from Measure.

In this section, we prove an $\Omega_{\pm}$ result for $\Delta(x)$ when $\mu(A_T)$ is big. We formalize the conditions in the following assumptions.

**Assumptions 4.** Suppose the conditions in Assumptions 3 hold. Let $l$ be an integer such that

$$l > \max(\sigma_2, 1),$$

and let $\alpha_1(T)$ be a monotonic function satisfying the inequality

$$0 < \alpha_1(T) \leq \sigma_1.$$  

Furthermore:

(i) the Dirichlet series $D(\sigma + it)$ has no pole when $\alpha_1(T) \leq \sigma \leq \sigma_1$;

(ii) if $|t| \leq T^{2l}$ and $\alpha_1(T) \leq \sigma \leq \sigma_1$, we have

$$|D(\sigma + it)| \leq c_{12}(|t| + 1)^{l-1}$$

for some constant $c_{12} > 0$.

**Assumptions 5.** Suppose there exists $\epsilon > 0$ such that the following holds:

if $D(\sigma + it)$ has no pole for $\alpha_1(T) - \epsilon < \sigma \leq \sigma_1$ and $|t| \leq 2T^{2l}$, then there exists a constant $c_{13} > 0$ depending on $\epsilon$ such that

$$|D(\sigma + it)| \leq c_{13}(|t| + 1)^{l-1},$$

when $\alpha_1(T) \leq \sigma \leq \sigma_1$ and $|t| \leq T^{2l}$.

Assumptions 5 says that if there are no poles of $D(s)$ in $\alpha_1(T) - \epsilon < \sigma \leq \sigma_1$, then it has polynomial growth in a certain region.
Lemma 12. Under the conditions in Assumptions, we have
\[ \Delta(x) = \int_{T/2}^{5T/2} D(\eta)^{-1} \frac{x^\eta}{\eta} d\eta + O(T^{-1}), \]
for all \( x \in [T/2, 5T/2] \).

Proof. Follows from Perron summation formula. \( \square \)

Lemma 13 (Balasubramanian and Ramachandra). Let \( T \geq 1, \delta_0 > 0 \) and \( f(x) \) be a real-valued integrable function such that
\[ f(x) \geq 0 \quad \text{for } x \in [T - \delta_0 T, 2T + \delta_0 T]. \]

Then for \( \delta > 0 \) and for a positive integer \( l \) satisfying \( \delta l \leq \delta_0 \), we have
\[ \int_{T}^{2T} f(x) dx \leq \frac{1}{(\delta T)^l} \int_{0}^{\delta T} \cdots \int_{0}^{\delta T} f(x) dx \, dy_1 \cdots dy_l. \]

Proof. For \( 0 \leq y_i \leq \delta T, i = 1, 2, \ldots, l \)
\[ \int_{T}^{2T} f(x) dx \leq \int_{T - \sum y_i}^{2T + \sum y_i} f(x) dx, \]
as \( f(x) \geq 0 \) in
\[ \left[ T - \sum_{i=1}^{l} y_i, 2T + \sum_{i=1}^{l} y_i \right] \subseteq [T - \delta_0 T, 2T + \delta_0 T]. \]

This gives
\[ \frac{1}{(\delta T)^l} \int_{0}^{\delta T} \cdots \int_{0}^{\delta T} f(x) dx \, dy_1 \cdots dy_l \]
\[ \geq \frac{1}{(\delta T)^l} \int_{0}^{\delta T} \cdots \int_{0}^{\delta T} f(x) dx \, dy_1 \cdots dy_l = \int_{T}^{2T} f(x) dx. \]
\( \square \)

The next theorem shows that if \( \Delta(x) \) does not change sign then the set on which \( \Omega \)-estimate holds can not be ‘too big’.

Theorem 10. Suppose the conditions in Assumptions hold. Let \( h_1(T) \) be a monotonically increasing function such that \( h_1(T) \to \infty \). Let \( \alpha_2(T) \) be a bounded positive monotonic function, such that
\[ 0 < \alpha_1(T) < \alpha_2(T) \leq \sigma_1, \]
and
\[ \frac{h_1(T)}{T^{\alpha_1}} \to \infty \text{ as } T \to \infty. \]

If there exist a constant \( x_0 \) such that \( \Delta(x) \) does not change sign on \( \mathcal{A}(h_1(x)) \cap [x_0, \infty) \), then
\[ \mu(\mathcal{A}(x^{\alpha_2}(x)) \cap [T, 2T]) \leq 4h_1(5T/2)T^{1-\alpha_2(T)} + O(1 + T^{1-\alpha_2(T)+\alpha_1(T)}) \]
for \( T \geq 2x_0. \)
Proof. Trivially we have

\[ \mu(A(x^{\alpha_2}) \cap [T, 2T]) \leq \int_T^{2T} \frac{|\Delta(x)|}{x^{\alpha_2}} \, dx. \]

Using Lemma \[\ref{lemma} \] on the above inequality, we get

\[ \mu(A(x^{\alpha_2}) \cap [T, 2T]) \leq \frac{1}{(\delta T)^l} \int_0^{\delta T} \cdots \int_0^{\delta T} \int_{T - \sum_i y_i}^{2T + \sum_i y_i} \frac{|\Delta(x)|}{x^{\alpha_2}} \, dx \, dy_1 \cdots dy_l, \]

where \( \delta = \frac{1}{2T}. \)

Let \( \chi \) denote the characteristic function of the complement of \( A(h_1(x)) \):

\[ \chi(x) = \begin{cases} 1 & \text{if } x \notin A(h_1(x)), \\ 0 & \text{if } x \in A(h_1(x)). \end{cases} \]

For \( T \geq 2x_0 \), \( \Delta(x) \) does not change sign on

\[ \left[ T - \sum_1^l y_i, 2T + \sum_1^l y_i \right] \cap A(h_1(x)), \]

as \( 0 \leq y_i \leq \delta T \) for all \( i = 1, \ldots, l \). So we can write the above inequality as

\[ \mu(A(x^{\alpha_2}) \cap [T, 2T]) \leq \frac{2}{(\delta T)^l} \int_0^{\delta T} \cdots \int_0^{\delta T} \int_{T - \sum_i y_i}^{2T + \sum_i y_i} \frac{|\Delta(x)|}{x^{\alpha_2}} \chi(x) \, dx \, dy_1 \cdots dy_l \\
\quad \quad \quad \quad \quad \quad + \frac{1}{(\delta T)^l} \left| \int_0^{\delta T} \cdots \int_0^{\delta T} \int_{T - \sum_i y_i}^{2T + \sum_i y_i} \frac{\Delta(x)}{x^{\alpha_2}} \, dx \, dy_1 \cdots dy_l \right|. \]

(37)

Since \( x \notin A(h_1(x)) \) implies \( |\Delta(x)| \leq h_1(x) \), we get

\[ \frac{2}{(\delta T)^l} \int_0^{\delta T} \cdots \int_0^{\delta T} \int_{T - \sum_i y_i}^{2T + \sum_i y_i} \frac{|\Delta(x)|}{x^{\alpha_2}} \chi(x) \, dx \, dy_1 \cdots dy_l \]

\[ \leq \frac{1}{(\delta T)^l} \left| \int_0^{\delta T} \cdots \int_0^{\delta T} \int_{T - \sum_i y_i}^{2T + \sum_i y_i} \frac{\Delta(x)}{x^{\alpha_2}} \, dx \, dy_1 \cdots dy_l \right|. \]

(38)

\[ \leq 4h_1(5T/2)T^{1-\alpha_2}. \]
We use the integral expression for $\Delta(x)$ as given in Lemma 12 and get
\[
\frac{1}{(\delta T)^l} \left| \int_0^{\delta T} \cdots \int_0^{\delta T} \int_{T-\sum_1^i y_i}^{2T+\sum_1^i y_i} \frac{\Delta(x)}{x^{\sigma_2}} dx \, dy_1 \cdots dy_l \right| = \frac{1}{(\delta T)^l} \left| \int_0^{\delta T} \cdots \int_0^{\delta T} \int_{T-\sum_1^i y_i}^{2T+\sum_1^i y_i} \frac{D(\eta)x^{\eta-\alpha_2}}{\eta} d\eta \, dx \, dy_1 \cdots dy_l \right| + O(1)
\]
\[
\ll 1 + \frac{1}{(\delta T)^l} \left| \int_{\alpha_1-iT^{2l}}^{\alpha_1+iT^{2l}} \frac{D(\eta)}{\eta} \int_0^{\delta T} \cdots \int_0^{\delta T} \int_{T-\sum_1^i y_i}^{2T+\sum_1^i y_i} x^{\eta-\alpha_2} dx \, dy_1 \cdots dy_l \, d\eta \right|
\]
\[
\ll 1 + \frac{1}{(\delta T)^l} \left| \int_{\alpha_1-iT^{2l}}^{\alpha_1+iT^{2l}} \frac{D(\eta)(2T+i\delta T)^{\eta-\alpha_2+i+1}}{\eta \prod_{j=1}^{l+1} (\eta - \alpha_2 + j)} d\eta \right|
\]
\[
\ll 1 + \frac{T^{\alpha_2-\alpha_1+i+1}}{(\delta T)^l} \int_{-T^{2l}}^{T^{2l}} (1+|t|)^{l-1} (1+|t|)^{l+2} dt \ll 1 + T^{1-\alpha_2+\alpha_1}.
\]
\[(39)\]

The theorem follows from \[(37), (38)\] and \[(39)\].

**Theorem 11.** Consider $\alpha_1(T), \alpha_2(T), \sigma_1, h_1(T)$ as in Theorem 10 and $\mathcal{P}$ as in Assumptions 7. Let $D(s)$ does not have a real pole in $[\alpha_1 - \epsilon_0, \infty) - \mathcal{P}$, for some $\epsilon_0 > 0$. Suppose there exists an $X$-Set $S$ such that for all $T \in S$

$$\mu(\mathcal{A}(x^{\sigma_2}) \cap [T, 2T]) > 5h_1(5T/2)T^{1-\sigma_2}.$$ 

Then:

(i) under Assumptions 4, we have

$$\Delta(x) = \Omega_{\pm}(h_1(x))$$

(In this case $\Delta(x)$ changes sign in $[T/2, 5T/2] \cap \mathcal{A}(h_1(x))$ for $T \in S$ and $T$ is sufficiently large);

(ii) under Assumptions 5, we have

$$\Delta(x) = \Omega_{\pm}(x^{\alpha_1-\epsilon}), \text{ for any } \epsilon > 0.$$ 

**Proof.** If the conditions in Assumptions 4 hold, then (i) follows from Theorem 10. To prove (ii), choose an $\epsilon$ such that $0 < \epsilon < \epsilon_0$. Now suppose $\eta_0$ is a pole of $D$ for $\Re(\eta) \geq \alpha_1(T) - \epsilon$ and $t \leq 2T^{2l}$, then by Theorem 3

$$\Delta(x) = \Omega_{\pm}(T^{\alpha_1-\epsilon}).$$

If there are no poles in the above described region of $\sigma + it$, then we are in the set-up of Assumptions 4 and get

$$\Delta(x) = \Omega_{\pm}(h_1(x)).$$

We have

$$T^{\alpha_1(T)} = o(h_1(T)),$$

which gives

$$\Delta(x) = \Omega_{\pm}(x^{\alpha_1-\epsilon}).$$

This completes the proof of (ii).
5.3. Applications. Now we shall see some examples demonstrating applications of Theorem 11.

5.3.1. Error term of the divisor function. Let \( d(n) \) denote the number of divisors of \( n \):

\[
d(n) = \sum_{d \mid n} 1.
\]

Dirichlet [13, Theorem 320] showed that

\[
\sum_{n \leq x} \tau(n) = x \log x + (2\gamma - 1)x + \Delta(x),
\]

where \( \gamma \) is the Euler constant and

\[
\Delta(x) = O(\sqrt{x}).
\]

Latest result on \( \Delta(x) \) is due to Huxley [16], which is

\[
\Delta(x) = O(x^{3/416}).
\]

On the other hand, Hardy [11] showed that

\[
\Delta(x) = \Omega^+ ((x \log x)^{1/4} \log \log x),
= \Omega^- (x^{1/4}).
\]

There are many improvements of Hardy’s result. Some notable results are due to K. Corrádi and I. KátaI [7], J. L. Hafner [9], and K. Soundararajan [27]. Below, we shall show that \( \Delta(x) \) is \( \Omega^\pm \left( x^{1/4} \right) \) as a consequence of Theorem 11 and results of Ivić and Tsang (see below). Moreover, we shall show that such fluctuations occur in \([T, 2T] \) for every sufficiently large \( T \).

Ivić [17] proved that for a positive constant \( c_{14} \),

\[
\int_T^{2T} \Delta^2(x) dx \sim c_{14} T^{3/2}.
\]

A similar result for fourth moment of \( \Delta(x) \) was proved by Tsang [30]:

\[
\int_T^{2T} \Delta^4(x) dx \sim c_{15} T^2,
\]

for a positive constant \( c_{15} \). Let \( \mathcal{A} \) denote the following set:

\[
\mathcal{A} := \left\{ x : |\Delta(x)| > \frac{c_{14} x^{1/4}}{6} \right\}.
\]

For sufficiently large \( T \), using the result of Ivić [17], we get

\[
\int_{[T, 2T] \cap \mathcal{A}} \frac{\Delta^2(x)}{x^{3/2}} dx = \int_T^{2T} \frac{\Delta(x)^2}{x^{3/2}} dx - \int_{[T, 2T] \cap \mathcal{A}} \frac{\Delta^2(x)}{x^{3/2}} dx
\geq \frac{1}{4T^{3/2}} \int_T^{2T} \Delta^2(x) dx - \frac{c_{14}}{6}
\geq \frac{c_{14}}{5} \geq \frac{c_{14}}{30}.
\]
Using Cauchy-Schwarz inequality and the result due to Tsang [30] we get
\[
\int_{[T,2T] \cap A} \frac{\Delta^2(x)}{x^{3/2}} \, dx \leq \left( \int_{[T,2T] \cap A} \frac{\Delta^4(x)}{x^2} \, dx \right)^{1/2} \left( \int_{[T,2T] \cap A} \frac{1}{x} \, dx \right)^{1/2}
\]
\[
\leq \left( c_{15} \mu([T,2T] \cap A) \right)^{1/2}.
\]

The above lower and upper bounds on second moment of \( \Delta \) gives the following lower bound for measure of \( A \):
\[
\mu([T,2T] \cap A) > \frac{c_{14}}{901c_{15}} T,
\]
for some \( T \geq T_0 \). Now, Theorem 11 applies with the following choices:
\[
\alpha_1(T) = \frac{1}{5}, \quad \alpha_2(T) = \frac{1}{4}, \quad h_1(T) = \frac{c_{14}}{9000c_{15}} T^{1/4}.
\]

Finally using Theorem 11 we get that for all \( T \geq T_0 \) there exists \( x_1, x_2 \in [T,2T] \) such that
\[
\Delta(x_1) > h_1(x_1) \quad \text{and} \quad \Delta(x_2) < -h_1(x_2).
\]
In particular we get
\[
\Delta(x) = \Omega_{\pm}(x^{1/4}).
\]

5.3.2. Error term of a twisted divisor function.

Recall that in (4) and (5), we have defined \( \Delta(x) \) as the error term that occurs while approximating \( \sum_{n \leq x} |\tau(n, \theta)|^2 \). Also recall that the corresponding Dirichlet series is given by
\[
D(s) = \sum_{n \leq x} \frac{|\tau(n, \theta)|^2}{n^s} = \frac{\zeta^2(s)\zeta(s+i\theta)\zeta(s-i\theta)}{\zeta(2s)}.
\]
Here the main term \( M(x) \) comes from the poles at \( 1, 1 \pm i\theta \). Now we assume a zero free region for \( D(\sigma+it) \), and estimate the growth of \( D(\sigma+it) \) in that region.

**Lemma 14.** Let \( \delta \) and \( \sigma \) be such that
\[
0 < \delta < \frac{1}{8}, \quad \text{and} \quad \frac{3}{8} - \delta \leq \sigma < \frac{1}{2}.
\]

If \( D(\sigma+it) \) does not have a pole in the above mentioned range of \( \sigma \), then for
\[
\frac{3}{8} - \delta + \frac{\delta}{2(1 + \log \log(3 + |t|))} < \sigma < \frac{1}{2},
\]
we have
\[
D(\sigma+it) \ll_{\delta, \theta} |t|^{2-2\sigma+\epsilon}
\]
for any positive constant \( \epsilon \).

**Proof.** Let \( s = \sigma + t \) with \( 3/8 - \delta \leq \sigma < 1/2 \). Recall that
\[
D(s) = \frac{\zeta^2(s)\zeta(s+i\theta)\zeta(s-i\theta)}{\zeta(2s)}.
\]
Using functional equation, we write
\[
D(s) = X(s) \frac{\zeta^2(1-s)\zeta(1-s-i\theta)\zeta(1-s+i\theta)}{\zeta(2s)},
\]
with
where $X(s)$ is of order (can be obtained from Sterling’s formula for $\Gamma$)

$$X(\sigma + it) \asymp t^{2-4\sigma}.$$  (41)

Using Sterling’s formula and Phragmen-Lindelof principle, we get

$$\zeta(1-s) \ll |t|^{\sigma/2} \log t.$$  

So we get

$$\zeta(2s) \ll |t|^{\sigma(\log \log t)4}.$$  (42)

An upper bound for $|\zeta(2s)|^{-1}$ can be calculated in a similar way as in Lemma 7.

$$|\zeta(2s)|^{-1} \ll \exp\left(c_{16}(\log \log t)(\log t)^{\frac{4(1-2\sigma)}{1+8\delta}}\right).$$  (43)

Now we complete the proof of Theorem 9.

**Proof of Theorem 9.** Let $M$ be any large positive constant, and define

$$A := A(Mx^{\alpha(x)}).$$

Then from Corollary 3, we have

$$\int_{[T,2T] \cap A} \frac{\Delta^2(x)}{x^{2\alpha(x)} + 1} \, dx \gg \exp\left(c(\log T)^{7/8}\right).$$

Assuming

$$\mu([T,2T] \cap A) \leq T^{1-\delta} \quad \text{for } T > T_0,$$  (44)

Proposition 5 gives

$$\Delta(x) = \Omega(x^{\alpha(x)/2})$$

as $h_0(T) = T^{1-\delta}$, which is the first part of the theorem. But if (44) does not hold, then we have

$$\mu([T,2T] \cap A) > T^{1-\delta}$$

for $T$ in an $X$-Set. We choose

$$h_1(T) = T^{\frac{3}{8} - \frac{3\epsilon}{(\log T)^{1/8}} - \delta}, \quad \alpha_1(T) = \frac{3}{8} - \frac{3\epsilon}{(\log T)^{1/8}} - \delta, \quad \alpha_2(T) = \alpha(T).$$

Let $\delta''$ be such that $\delta < \delta'' < \delta'$. If $D(\sigma + it)$ does not have pole for $\sigma > 3/8 - \delta''$ then by Lemma 14, $D(\alpha_1(T) + it)$ has polynomial growth. So Assumptions 5 is valid. Since

$$T^{1-\delta} > 5h_1(5T/2)T^{1-\alpha_2(T)}$$

by case (ii) of Theorem 11, we have

$$\Delta(T) = \Omega_{\pm}\left(T^{\frac{3}{8} - \frac{3\epsilon}{(\log T)^{1/8}} - \delta''}\right).$$

The second part of the above theorem follows by the choice $\delta' > \delta''$.  \qed
5.3.3. Average order of Non-Isomorphic abelian Groups.

Let \( a_n \) denote the number of non-isomorphic abelian groups of order \( n \). The Dirichlet series \( D(s) \) is given by

\[
D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_{k=1}^{\infty} \zeta(ks), \quad \text{Re}(s) > 1.
\]

The meromorphic continuation of \( D(s) \) has poles at \( 1/k \), for all positive integer \( k \geq 1 \). Let the main term \( M(x) \) be

\[
M(x) = \sum_{k=1}^{6} \left( \prod_{j \neq k} \zeta(j/k) \right) x^{1/k},
\]

and the error term \( \Delta(x) \) be

\[
\sum_{n \leq x} a_n - M(x).
\]

Balasubramanian and Ramachandra [4] proved that

\[
\int_{2T}^{T} \Delta^2(x) dx = \Omega(T^{4/3} \log T), \quad \text{and} \quad \Delta(x) = \Omega_{\pm}(x^{92/1221}).
\]

Sankaranarayanan and Srinivas [26] improved the \( \Omega_{\pm} \) bound to

\[
\Delta(x) = \Omega_{\pm} \left( x^{1/10} \exp \left( c \sqrt{\log x} \right) \right)
\]

for some constant \( c > 0 \). An upper bound for the second moment of \( \Delta(x) \) was first given by Ivić [18], and then improved by Heath-Brown [14] to

\[
\int_{T}^{2T} \Delta^2(x) dx \ll T^{4/3}(\log T)^{89}.
\]

This bound of Heath-Brown is best possible in terms of power of \( T \). But for the fourth moment, the similar statement

\[
\int_{T}^{2T} \Delta^4(x) dx \ll T^{5/3}(\log T)^C,
\]

which is best possible in terms of power of \( T \), is an open problem. Another open problem is to show that

\[
\Delta(x) = \Omega_{\pm}(x^{1/6-\delta}),
\]

for any \( \delta > 0 \). In the next Proposition, we shall show that at least one of the statement is true.

**Proposition 6.** Let \( \delta \) be such that \( 0 < \delta < 1/42 \). Then either

\[
\int_{T}^{2T} \Delta^4(x) dx = \Omega(T^{5/3+\delta}),
\]

or

\[
\Delta(x) = \Omega_{\pm}(x^{1/6-\delta}).
\]

**Proof.** If the first statement is false, then we have

\[
\int_{T}^{2T} \Delta^4(x) dx \leq c_{17} T^{5/3+\delta},
\]
for some constant $c_{17}$ depending on $\delta$ and for all $T \geq T_0$. Let $\mathcal{A}$ be defined by:

$$\mathcal{A} = \{ x : |\Delta(x)| > c_{18}x^{1/6} \}, \quad c_{18} > 0.$$ 

By the result of Balasubramanian and Ramachandra [11], we have an $X$-Set $\mathcal{S}$, such that

$$\int_{[T, 2T] \cap \mathcal{A}} \Delta^2(x)dx \geq c_{19}T^{4/3}(\log T)$$

for $T \in \mathcal{S}$. Using Cauchy-Schwartz inequality, we get

$$c_{19}T^{4/3}(\log T) \leq \int_{[T, 2T] \cap \mathcal{A}} \Delta^2(x)dx$$

$$\leq \left( \int_T^{2T} \Delta^4(x)dx \right)^{1/2} \left( \mu(\mathcal{A} \cap [T, 2T]) \right)^{1/2}$$

$$\leq \frac{c_{17}^2}{2}T^{5/6+\delta/2} \left( \mu(\mathcal{A} \cap [T, 2T]) \right)^{1/2}.$$ 

This gives, for a suitable positive constant $c_{20},$

$$\mu(\mathcal{A} \cap [T, 2T]) \geq c_{20}T^{1-\delta}(\log T)^2.$$ 

Now we use Theorem [11] (i), with

$$\alpha_2 = \frac{1}{6}, \quad \alpha_1 = \frac{13}{84} - \frac{\delta}{2}, \quad \text{and} \quad h_1(T) = T^{1/6-\delta}.$$ 

So we get

$$\Delta(x) = \Omega_{\pm}(x^{1/6-\delta}).$$

This completes the proof. \qed

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