Gravitational wave solutions to linearized Jordan Brans Dicke theory on a cosmological background

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Received: 10 June 2018 / Accepted: 18 July 2019 / Published online: 23 July 2019
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Abstract
Approximate solutions of Jordan–Brans–Dicke (JBD) theory for perturbed scalar field and perturbed Robertson–Walker metric, were found for an era dominated by the scalar field. Solutions for the scale factor \(a(t)\) and the scalar field \(\phi(t)\) in unperturbed JBD theory are dependent on the \(\omega\) parameter which determines how the scalar field is coupled to geometry of space–time. After adding metric perturbation \(h_{\mu\nu}(x)\) to Robertson–Walker metric and perturbation \(\delta\phi(x)\) to the scalar field \(\phi(t)\), we solved the JBD equations such that the scale factor and the scalar field solutions are \(a \propto t\) and \(\phi \propto t^{-2}\) with \(\omega = -3/2\). These results are necessary conditions for ordinary and scalar gravitational waves to exist in the scalar field-dominated era. Despite our result contradicts the value \(\omega > 10^4\) which is favored by the current solar system environment observations, \(\omega = -3/2\) makes JBD theory conformally invariant and fits recent supernovae type Ia data.

Keywords Gravitational waves · Jordan Brans Dicke theory · Scalar tensor theories

1 Introduction
The Standard Model of Cosmology had great success in explaining some cosmological issues at the fundamental level such as proton–neutron ratio in the early universe, primordial nucleosynthesis of the basic elements and cosmic microwave background radiation. Besides its successes, it also has some problems like horizon, flatness and relic particle abundance (or monopole) problems which had remained unsolved until the inflationary cosmology [1] idea came. However, one of the major problems which is the cosmological constant problem [2], continues its presence in cosmology. As a
dark energy candidate, its value which has been inferred from the Friedmann equation, is so much smaller than the value that has been calculated in elementary particle physics. On the other hand, scalar tensor theories which are considered as alternative gravity theories, suggest some explanations for the late-time evolution of the universe. Jordan–Brans–Dicke (JBD) theory [3] is one of them and it is mostly influenced from Mach Principle which simply states that inertial forces on a body are originating from gravitational effects of matter distribution of the rest of the universe. For many different cosmic scenarios, solutions of JBD theory exist in literature. So we intend to find its gravitational wave solutions in vacuum case in this article. Thanks to gravitational wave astronomy, which has been started with recent detection of gravitational waves [4] by the LIGO Scientific Collaboration, testing alternative theories of gravity [5] could be possible in the near future.

In JBD theory, the gravitational constant $G$ is not a constant but a parameter and it is related to a scalar field $\phi$. So the JBD action contains the Lagrangian of this scalar field $\phi$ and looks like

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left(\phi R + 16\pi L_M - \frac{\omega g^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi}{\phi} \right) \tag{1}$$

where $R$ is the Ricci scalar, $L_M$ is the matter Lagrangian and $\omega$ is the JBD parameter. To get the JBD equations, we should vary the action with respect to $g_{\mu\nu}$ and $\phi$. Variation operations give us the following equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi \phi T_{\mu\nu} + \frac{1}{\phi} \left(\nabla_{\mu} \partial_{\nu} \phi - g_{\mu\nu} g^{\alpha\beta} \nabla_{\alpha} \partial_{\beta} \phi\right)$$

$$+ \frac{\omega}{\phi^2} \left(\partial_{\mu} \phi \partial_{\nu} \phi - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \partial_{\alpha} \phi \partial_{\beta} \phi\right) \tag{2}$$

$$R + 2 \frac{\omega}{\phi^2} g^{\mu\nu} \nabla_{\mu} \partial_{\nu} \phi - \frac{\omega}{\phi^2} g^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi = 0 \tag{3}$$

where

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}} \tag{4}$$

is the energy momentum tensor of matter. Contracting (2) with $g^{\mu\nu}$ gives

$$- R = \frac{8\pi}{\phi} T - \frac{3}{\phi} \nabla_{\mu} \partial^{\mu} \phi - \frac{\omega}{\phi^2} \partial_{\mu} \phi \partial^{\mu} \phi \tag{5}$$

and substituting (5) into (3) yields

$$\frac{(3 + 2\omega)}{\phi} g^{\mu\nu} \nabla_{\mu} \partial_{\nu} \phi = \frac{8\pi}{\phi} T. \tag{6}$$

Equations (2) and (6) are the basic equations of the theory. Also the predictions of the theory are same with the predictions of Einstein field equation when $\omega \rightarrow \infty$. Current
observational data for solar system environment show that $\omega > 10^4$, so the theory is indistinguishable from general relativity or deviations are so small [6].

We focus on the vacuum solutions of the equations on a cosmological background. With word “vacuum”, it is meant that there is nothing in the environment which we are interested in, no matter, no radiation and no cosmological constant, only the scalar field. Besides, of course, the universe we live in, is not a steady state universe but it is expanding with the scale factor, so we will use Robertson–Walker (RW) metric which is

$$ds^2 = -(dt)^2 + a^2(t)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2]$$  \hspace{1cm} (7)

where $a$ is the scale factor of the universe and a function of time. At that point space is also considered as flat which means curvature parameter $k = 0$. For the vacuum case, (2) and (6) transform into

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{1}{\phi} \left( \nabla_\mu \partial_\nu \phi - g_{\mu\nu} g^{\alpha\beta} \nabla_\alpha \partial_\beta \phi \right) + \frac{\omega}{\phi^2} \left( \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi \right)$$  \hspace{1cm} (8)

and

$$g^{\mu\nu} \nabla_\mu \partial_\nu \phi = 0.$$  \hspace{1cm} (9)

As one can see, (9) can be substituted into (8), and the final form of the basic JBD equation for vacuum is

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{1}{\phi} \nabla_\mu \partial_\nu \phi + \frac{\omega}{\phi^2} \left( \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi \right).$$  \hspace{1cm} (10)

Now, all should be done is to place the Ricci tensor components and the Ricci scalar of RW metric into the equation in order to construct equations for different $\mu$ and $\nu$ values. The Ricci tensor and the Ricci scalar of the metric for $k = 0$ in cartesian coordinates, are

$$R_{00} = -3 \frac{\ddot{a}}{a},$$  \hspace{1cm} (11)

$$R_{11} = R_{22} = R_{33} = (a \ddot{a} + 2 \dot{a}^2),$$  \hspace{1cm} (12)

$$R = 6 \left( \frac{\dddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right).$$  \hspace{1cm} (13)

where dots represent time derivatives. We assume that the scalar field $\phi$ is only function of time, not dependent on spatial coordinates. After substituting the Ricci tensor components and the Ricci scalar into (10), we get for $\mu = 0$ and $\nu = 0$

$$3 \frac{\dddot{a}}{a} - \frac{\ddot{a}^2}{\phi} - \frac{\omega (\partial_0 \phi)^2}{2 \phi^2} = 0$$  \hspace{1cm} (14)
and for $\mu = 1$ and $\nu = 1$,
\[ -2 \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} + \frac{\dot{a}}{a} \frac{\partial_0 \phi}{\phi} - \frac{\omega (\partial_0 \phi)^2}{2 \phi^2} = 0. \] (15)

Other two equations for the cases $\mu = 2$, $\nu = 2$ and $\mu = 3$, $\nu = 3$ are not different from the equation of $\mu = 1$, $\nu = 1$, so they do not give any new information about the vacuum case solutions. Subtracting (15) from (14) yields
\[ 4 \frac{\ddot{a}^2}{a^2} + 2 \frac{\dot{a}^2}{a^2} - \frac{\partial_0^2 \phi}{\phi} - \frac{\dot{a}}{a} \frac{\partial_0 \phi}{\phi} = 0. \] (16)

By assuming that the scalar field and the scale factor have power law solutions, Eqs. (15) and (16) give us the powers which depend on the JBD parameter $\omega$. If $\phi \propto t^s$ and $a \propto t^q$, we get
\[ -3q^2 + 2q + sq - \frac{\omega}{2} s^2 = 0, \] (17)
\[ 6q^2 - 2q - sq + s - s^2 = 0. \] (18)

Solutions to these equations for $\omega \geq -3/2$ and $\omega \neq -4/3$ are
\[ q_\pm = \frac{1}{3\omega + 4} \left( \omega + 1 \pm \sqrt{\frac{2\omega + 3}{3}} \right), \] (19)
\[ s_\pm = \frac{1 \mp \sqrt{3(2\omega + 3)}}{3\omega + 4}. \] (20)

They also satisfy the relation
\[ 3q + s = 1. \] (21)

These solutions are same with that of O’Hanlon and Tupper [7]. Once we get the exact value of $\omega$ observationally, values of $q$ and $s$ can be determined.

## 2 Vacuum solutions to linearized JBD theory

In this section, approximate solutions for the scalar field, the scale factor, ordinary gravitational wave and scalar gravitational wave, are found by regarding perturbed RW metric and perturbed scalar field. We add first order perturbations to the metric and the scalar field, and neglect all the higher order perturbations in calculations. Since RW metric is function of time, the scalar field $\phi$ is chosen to be function of time for ansatz. In addition, the first order perturbations of the metric and the scalar field are functions of time and spatial coordinates. So perturbed metric and perturbed scalar field are
\[ g_{\mu\nu}(x) = f_{\mu\nu}(t) + h_{\mu\nu}(x) \] (22)
and

\[ \Phi(x) = \phi(t) + \delta \phi(x) \]  

(23)

where

- \( f_{\mu \nu} \) is the RW metric,
- \( h_{\mu \nu} \) is perturbation to the metric and \( |h_{\mu \nu}| \ll |f_{\mu \nu}| \),
- \( \delta \phi \) is perturbation to the scalar field and \( |\delta \phi| \ll |\phi| \).

Since we are dealing with the vacuum solutions, the energy momentum tensor and its trace are equal to zero. If the equations are written with explicit form of the field \( \Phi \) and the metric \( g_{\mu \nu} \), they look like

\[
R_{\mu\nu} - \frac{1}{2} R(f_{\mu\nu} + h_{\mu\nu}) = \frac{1}{(\phi + \delta \phi)} [\nabla_\mu \partial_\nu (\phi + \delta \phi)] \\
+ \frac{\omega}{(\phi + \delta \phi)^2} [\partial_\mu (\phi + \delta \phi) \partial_\nu (\phi + \delta \phi)] \\
- \frac{1}{2} (f_{\mu\nu} + h_{\mu\nu}) (f^{\alpha\beta} - h^{\alpha\beta}) \partial_\alpha (\phi + \delta \phi) \partial_\beta (\phi + \delta \phi) \\
\]  

(24)

and

\[
\frac{1}{(\phi + \delta \phi)} (f^{\mu\nu} - h^{\mu\nu}) \nabla_\mu \partial_\nu (\phi + \delta \phi) = 0. 
\]  

(25)

We have used inverse of the metric in (24) and (25) as

\[ g^{\mu\nu}(x) = f^{\mu\nu}(t) - h^{\mu\nu}(x). \]  

(26)

2.1 Ricci tensor and Ricci scalar for perturbed RW metric

The general forms [8]\(^1\) of first order components of the Ricci tensor are

\[
\delta R_{00} = -\frac{1}{2a^2} \left[ \partial_0^2 h_{kk} - \frac{\dot{a}}{a} \partial_0 h_{kk} + 2 \left( \frac{\dot{a}^2}{a^2} - \ddot{a} \right) h_{kk} \right],  
\]  

(27)

\[
\delta R_{0i} = -\frac{1}{2} \partial_0 \left[ \frac{1}{a^2} (\partial_i h_{kk} - \partial_k h_{ki}) \right],  
\]  

(28)

\[
\delta R_{ij} = -\frac{1}{2a^2} \left[ \nabla^2 h_{ij} - \partial_j \partial_k h_{ik} - \partial_i \partial_k h_{jk} + \partial_i \partial_j h_{kk} \right] \\
+ \frac{1}{2} \partial_0^2 h_{ij} - \frac{\dot{a}}{2a} [\partial_0 h_{ij} - \delta_{ij} \partial_0 h_{kk}] \\
+ \frac{\dot{a}^2}{a^2} [2h_{ij} - \delta_{ij} h_{kk}].  
\]  

(29)

Each of these components can be considered as summation of zeroth and first order parts like

\[ R_{\mu\nu} = \bar{R}_{\mu\nu} + \delta R_{\mu\nu} \]  

(30)

\(^1\) Since there is a minus sign difference for the definition of the Riemann tensor in Weinberg’s book, we multiply first order components of the Ricci tensor with a minus sign.
where

- $R_{\mu\nu}$ is the Ricci tensor for the metric $g_{\mu\nu},$
- $\tilde{R}_{\mu\nu}$ is the Ricci tensor for RW metric,
- $\delta R_{\mu\nu}$ is perturbation of the Ricci tensor.

Before proceeding to compute Ricci tensor components, we can make some simplifications for our sake. As is known, for Einstein field equation in Minkowski space-time, transverse-traceless perturbation $h_{TT}^{\mu\nu}$ represents plane wave solution in cartesian coordinates and it is composed of plus and cross polarized waves. For a plane wave which is propagating in $x^3$ direction, it looks like

$$h_{TT}^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_{11} & h_{12} & 0 \\ 0 & h_{21} & h_{22} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(31)

where

$$h_{11}(t - x^3) = h_+ e^{ik_\sigma x^\sigma}$$

and $h_{22} = -h_{11}$

(32)

and

$$h_{12}(t - x^3) = h_\times e^{ik_\sigma x^\sigma}$$

and $h_{12} = h_{21}.$

(33)

Also, if we wanted to solve the JBD equations for perturbed Minkowski metric, we should choose our scalar field as

$$\Phi(x) = \phi_0 + \delta\phi(x)$$

(34)

where $\phi_0$ is constant and $\delta\phi$ is a function of time and spatial coordinates. Then the solution [9] would be

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & \delta\phi & 0 \\ 0 & A^{(+)} - \frac{\delta\phi}{\phi_0} & A^{(\times)} & 0 \\ 0 & A^{(\times)} & -A^{(+)} - \frac{\delta\phi}{\phi_0} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(35)

where $A^{(+)}$ and $A^{(\times)}$ are ordinary gravitational waves and $\delta\phi/\phi_0$ is a scalar gravitational wave. Furthermore this metric perturbation has trace $\eta^{\mu\nu} h_{\mu\nu} = -2 (\delta\phi/\phi_0).$

Taking the solutions of the JBD equations for perturbed Minkowski metric into consideration, we can assume that perturbation of RW metric for JBD theory has trace

$$f^{\mu\nu} h_{\mu\nu} = -2 \frac{\delta\phi}{\phi}$$

(36)

and it is transverse to propagation direction of the wave. For a wave which is propagating in $x^3$ direction, it can be regarded as
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\[ h_{\mu\nu} = a^2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & (A - \frac{\delta \phi}{\phi}) & B & 0 \\ 0 & B & (-A - \frac{\delta \phi}{\phi}) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]  \tag{37}

where \( A, B \) and \( (\delta \phi/\phi) \) are plane waves. As one can see, we can take \( h_{0\nu} \) and \( h_{3\nu} \) components of the metric perturbation as zero. This assumption will simplify our calculations and by using (27), (28) and (29), components of the Ricci tensor can be written as

\[ R_{00} = \bar{R}_{00} + \delta R_{00} = -\frac{3\ddot{a}}{a} - \frac{1}{2a^2} \partial_0^2 (h_{11} + h_{22}) \]
\[ + \frac{\dot{a}}{a^3} \partial_0 (h_{11} + h_{22}) \]
\[ + \left( \frac{\ddot{a}}{a^3} - \frac{\dot{a}^2}{a^6} \right) (h_{11} + h_{22}), \]
\[ R_{11} = \bar{R}_{11} + \delta R_{11} = (a\ddot{a} + 2a^2) + \frac{1}{2} \partial_0^2 h_{11} - \frac{1}{2a^2} \partial_3^2 h_{11} \]
\[ + \frac{\dot{a}}{2a} \partial_0 h_{22} + \frac{\dot{a}^2}{a^2} (h_{11} - h_{22}), \]
\[ R_{22} = \bar{R}_{22} + \delta R_{22} = (a\ddot{a} + 2a^2) + \frac{1}{2} \partial_0^2 h_{22} - \frac{1}{2a^2} \partial_3^2 h_{22} \]
\[ + \frac{\dot{a}}{2a} \partial_0 h_{11} + \frac{\dot{a}^2}{a^2} (h_{22} - h_{11}), \]
\[ R_{33} = \bar{R}_{33} + \delta R_{33} = (a\ddot{a} + 2a^2) - \frac{1}{2a^2} \partial_3^2 (h_{11} + h_{22}) \]
\[ + \frac{\dot{a}}{2a} \partial_0 (h_{11} + h_{22}) - \frac{\dot{a}^2}{a^2} (h_{11} + h_{22}), \]
\[ R_{03} = \bar{R}_{03} + \delta R_{03} = -\frac{1}{2a^2} \partial_3 \partial_0 (h_{11} + h_{22}) \]
\[ + \frac{\dot{a}}{a^3} \partial_3 (h_{11} + h_{22}), \]
\[ R_{12} = \bar{R}_{12} + \delta R_{12} = \frac{1}{2} \partial_0^2 h_{12} - \frac{1}{2a^2} \partial_3^2 h_{12} \]
\[ - \frac{\dot{a}}{2a} \partial_0 h_{12} + \frac{2\dot{a}^2}{a^2} h_{12}, \]
\[ R_{01} = R_{02} = R_{13} = R_{23} = 0. \]

The Ricci scalar can be easily computed by contracting the Ricci tensor with the inverse of the metric and it can be regarded as summation of zeroth and first order parts like

\[ R = \bar{R} + \delta R \]  \tag{45}

where
\( R \) is the Ricci scalar for the metric \( g_{\mu\nu} \),
- \( \bar{R} \) is the Ricci scalar for Robertson–Walker metric,
- \( \delta R \) is perturbation of the Ricci scalar.

Contracting (30) with (26) yields
\[
R = R_{\mu\nu} g^{\mu\nu} = (\bar{R}_{\mu\nu} + \delta R_{\mu\nu})(f^{\mu\nu} - h^{\mu\nu})
= \bar{R}_{\mu\nu} f^{\mu\nu} - \bar{R}_{\mu\nu} h^{\mu\nu} + \delta R_{\mu\nu} f^{\mu\nu}
= \bar{R} - \bar{R}_{\mu\nu} h^{\mu\nu} + \delta R_{\mu\nu} f^{\mu\nu}.
\] (46)

Finally, the Ricci scalar for perturbed RW metric is found as
\[
R = 6 \left( \frac{\ddot{a}}{a} + \frac{a^2}{a^2} \right) + \frac{1}{a^2} \dot{\partial}_0^2 (h_{11} + h_{22})
- \frac{1}{a^4} \partial_3^2 (h_{11} + h_{22}) - 2 \left( \frac{\ddot{a}}{a^3} + \frac{a^2}{a^4} \right) (h_{11} + h_{22}).
\] (47)

### 2.2 Solutions to linearized JBD equations

In this section, our plan is to construct and solve perturbed JBD equations. Since we have found necessary elements in previous sections, they can be now placed into the equations. Then solutions of \( a, \phi \) and the perturbations can be obtained. We have two basic equations, however the first one which is Eq. (24), will yield more than one due to different components of the Einstein tensor which is \( G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \). Let us begin with the second JBD equation which is (25). There is summation in this relation between metric components and derivatives of \( \Phi \), and it can be expanded as
\[
\frac{1}{(\phi + \delta \phi)} [f^{00} \partial_0 (\phi + \delta \phi) + (f^{11} - h_{11}) \partial_1 (\phi + \delta \phi)]
+ (f^{22} - h_{22}) \partial_2 (\phi + \delta \phi) + f^{33} \partial_3 (\phi + \delta \phi)] = 0.
\] (48)

Then inserting metric components and covariant derivatives of partial derivatives of \( \Phi \) into (48) gives
\[
- \frac{\partial_0^2 \phi}{(\phi + \delta \phi)} - \frac{3\dot{a}}{a} \frac{\partial_0 \phi}{(\phi + \delta \phi)} - \frac{1}{2a^2} \frac{\partial_0 \phi}{\phi} \dot{\partial}_0 (h_{11} + h_{22})
+ \frac{a^3}{a^3} \frac{\partial_0 \phi}{\phi} (h_{11} + h_{22}) - \frac{\partial_0^2 \delta \phi}{\phi} - \frac{3\dot{a}}{a} \frac{\partial_0 \delta \phi}{\phi} + \frac{1}{a^2} \frac{\partial_0^2 \delta \phi}{\phi} = 0.
\] (49)

To separate zeroth and first order terms, we need one more arrangement like
\[
(\phi + \delta \phi)^{-1} \approx \frac{1}{\phi} \left( 1 - \frac{\delta \phi}{\phi} \right).
\] (50)
After placing (50) into (49), the first order equation is

\[ -\frac{\partial_0^2 \delta \phi}{\phi} - \frac{3\dot{a}}{a} \frac{\partial_0 \delta \phi}{\phi} + \frac{1}{2a^2} \frac{\partial_0^2 \delta \phi}{\phi} - \frac{1}{2a^2} \frac{\partial_0 (h_{11} + h_{22})}{\phi} + \frac{\dot{a}}{a^3} \frac{\partial_0 \delta \phi}{\phi} (h_{11} + h_{22}) + \frac{\partial_0^2 \phi \delta \phi}{\phi^2} + \frac{3\dot{a}}{a} \frac{\partial_0 \delta \phi}{\phi^2} = 0. \] \tag{51}

Since we know \( h_{11} + h_{22} = -2a^2(\delta \phi/\phi) \), by using this relation and integration by parts method, (51) can be written as

\[ -\partial_0^2 \left( \frac{\delta \phi}{\phi} \right) - \frac{3\dot{a}}{a} \partial_0 \left( \frac{\delta \phi}{\phi} \right) + \frac{1}{a^2} \partial_0^2 \frac{\delta \phi}{\phi} - \frac{\partial_0 \phi}{\phi} \partial_0 \left( \frac{\delta \phi}{\phi} \right) = 0. \] \tag{52}

This is the final form of the first order part of (48) and it has the form of a wave equation for \((\delta \phi/\phi)\). While first three terms come from \( f_{\mu\nu} \nabla_\mu \nabla_\nu (\delta \phi/\phi) \), the fourth one looks like an extra friction term.

Now, there is one more equation to solve for perturbed JBD theory. In total, this one gives six equations because of six different nonzero components of the Einstein tensor \( G_{\mu\nu} \) for perturbed RW metric. We start with inserting (38) and (47) into (24) to get equation of \( \mu = 0, \nu = 0 \) as

\[ 3\dot{a}^2 - \frac{1}{2a^2} \partial_3^2 (h_{11} + h_{22}) + \frac{\dot{a}}{a} \partial_0 (h_{11} + h_{22}) - \frac{2\dot{a}^2}{a^2} (h_{11} + h_{22}) = a^2 \frac{\partial_0^2 \phi \delta \phi}{\phi^2} + a^2 \frac{\partial_0^2 \delta \phi}{\phi} + \frac{\omega a^2}{2} \left[ \frac{(\partial_0 \phi)^2}{\phi^3} - \frac{(\partial_0 \phi)^2 \delta \phi}{\phi^2} + \frac{\partial_0 \phi \partial_0 \delta \phi}{\phi^2} \right]. \] \tag{53}

Since we are interested in first order terms in \( h_{\mu\nu} \) and \( \delta \phi \), Eq. (53) can be reduced to

\[ -a^2 \frac{\partial_0^2 \delta \phi}{\phi} - \frac{1}{2a^2} \partial_3^2 (h_{11} + h_{22}) + \frac{\dot{a}}{a} \partial_0 (h_{11} + h_{22}) - \frac{2\dot{a}^2}{a^2} (h_{11} + h_{22}) = \frac{\partial_0^2 \phi \delta \phi}{\phi^2} \] \tag{54}

By using \( h_{11} + h_{22} = -2a^2(\delta \phi/\phi) \), (52) and the following relations that are obtained via integration by parts method

\[ \frac{\partial_0^2 \delta \phi}{\phi} = \partial_0^2 \left( \frac{\delta \phi}{\phi} \right) + 2\frac{\partial_0 \phi}{\phi} \partial_0 \left( \frac{\delta \phi}{\phi} \right) + \frac{\partial_0^2 \phi \delta \phi}{\phi^2}, \] \tag{55}

\[ \omega a^2 \left[ -\frac{(\partial_0 \phi)^2 \delta \phi}{\phi^3} + \frac{\partial_0 \phi \partial_0 \delta \phi}{\phi^2} \right] = \omega a^2 \frac{\partial_0 \phi}{\phi} \partial_0 \left( \frac{\delta \phi}{\phi} \right). \] \tag{56}
Equation (54) can be simplified firstly as

\[ a \dot{a} \frac{\delta \phi}{\phi} - a^2 \frac{\partial_0 \delta \phi}{\phi} \partial_0 \left( \frac{\delta \phi}{\phi} \right) = \omega a^2 \frac{\partial_0 \delta \phi}{\phi} \partial_0 \left( \frac{\delta \phi}{\phi} \right), \]

then as

\[ \frac{\dot{a}}{a} - \frac{\partial_0 \phi}{\phi} = \omega \frac{\partial_0 \phi}{\phi}. \]

We have assumed that \( \phi \) and \( a \) have power-law solutions like \( \phi \propto t^s \) and \( a \propto t^q \). So, after we put them into (58), it turns into

\[ q = s(\omega + 1). \]

This is a new relation of \( q, s \) and \( \omega \). When equations for different \( \mu \) and \( \nu \) cases are checked, this relation consistently appears. Now, values of \( q, s \) and \( \omega \) can be found by using the relations

\[
\begin{align*}
-3q^2 + 2q + sq - \frac{\omega}{2}s^2 &= 0, \\
6q^2 - 2q - sq + s - s^2 &= 0, \\
q &= s(\omega + 1),
\end{align*}
\]

which are (17), (18) and (59) respectively. The solutions that satisfy these relations are \( q = 1, s = -2 \) with \( \omega = -3/2 \). Thus, we can write

\[ a(t) = a_0 \left( \frac{t}{t_0} \right) \]

and

\[ \phi(t) = \phi_0 \left( \frac{t}{t_0} \right)^{-2}. \]

Our finding for \( \omega \) may seem unpleasant because it is a negative coupling parameter and solar system observations showed \( \omega > 10^4 \). However, JBD theory with negative \( \omega \) value can explain accelerating expansion of the universe without any necessity of cosmological constant [10–12]. In addition, \( \omega = -3/2 \) is the value which makes JBD theory conformally invariant [13] and fits recent data of type Ia supernovae [14].

Since we obtained power law solutions for the scale factor and the scalar field, it is easy to put them into the wave equation and look for the solution. Wave equation form of a scalar gravitational wave in (52) is exactly the same for ordinary gravitational waves \( A \) and \( B \). Inserting (60) and (61) into the wave equation of \( A \) yields

\[ \frac{1}{a^2} \frac{\partial^2 A}{\partial z^2} - \frac{\partial^2 A}{\partial t^2} - \frac{1}{t} \frac{\partial A}{\partial t} = 0. \]
This equation can be arranged as

$$\frac{\partial^2 A}{\partial z^2} = \left( \frac{a_0}{t_0} \right)^2 t^2 \frac{\partial^2 A}{\partial t^2} + \left( \frac{a_0}{t_0} \right)^2 t \frac{\partial A}{\partial t}. \quad (63)$$

Simplifying the form of the wave equation by defining conformal time $\tau$, will help us to figure it out. We define

$$\frac{\partial t}{t} = \partial \tau \quad (64)$$

so

$$\ln t = \tau. \quad (65)$$

The wave equation with respect to $\tau$ is

$$\frac{\partial^2 A}{\partial z^2} = \left( \frac{a_0}{t_0} \right)^2 \frac{\partial^2 A}{\partial \tau^2} + \left( \frac{a_0}{t_0} \right)^2 \frac{\partial A}{\partial \tau}. \quad (66)$$

At that point, to be able to guess the form of the wave equation, the distance relation on a null geodesic for the scale factor can be written as

$$d = \int \frac{\partial t}{a} = \frac{t_0}{a_0} \int \frac{\partial t}{t} = t_0 \ln t = t_0 \tau \quad (67)$$

where $a_0 = 1$. This and the form of the wave equation in (66) encourage us to write a wave function like

$$A(z, \tau) = A_0 e^{i(\tilde{k}az - kt_0\tau)} \quad (68)$$

where $\tilde{k}$ is a complex wave number. Substituting this into (66) gives

$$\tilde{k}^2 = \left( \frac{t_0}{t} \right)^2 k^2 + i \left( \frac{t_0}{t} \right)^2 \frac{k}{t_0} \quad (69)$$

and so

$$\tilde{k} = \frac{t_0}{t} k \left( 1 + \frac{i}{kt_0} \right)^{1/2} = \frac{t_0}{t} k + \frac{i}{2t}. \quad (70)$$

After inserting this back to the wave function, it becomes

$$A(z, \tau) = A_0 e^{i(\tilde{k}z - kt_0\tau)} e^{-\frac{z}{t_0^2}}. \quad (71)$$

This is the form of the wave function for ordinary and scalar gravitational waves that we are looking for. As is expected, it decays exponentially at large distances.

### 3 Conclusion

So far, we have added perturbations to the metric and the scalar field. Perturbations are dependent on time and spatial coordinates because the metric perturbation and
$\delta \phi / \phi$ should have wave solutions. In order to simplify our calculations, we have made a gauge choice for the metric perturbation $h_{\mu \nu}$ such that it is transverse to the preferred propagation direction of gravitational wave. After we have constructed the JBD equations for perturbed metric and perturbed scalar field, we have separated the equations as the zeroth and the first order in $h_{\mu \nu}$ and $\delta \phi$. The first order equations have produced two important results. One of them is the form of the wave equation for scalar and ordinary gravitational waves. The other one is the new relation of $q$, $s$ and $\omega$ values. We have had three independent equations for three unknowns. Finally, we have found values of $q$, $s$, and $\omega$ as $1$, $-2$ and $-3/2$ respectively. These constitute the only solution for all the equations to be satisfied and determine how the scale factor and the scalar field evolve with time for ordinary gravitational and scalar gravitational waves to exist. Also $\omega = -3/2$ is another necessary condition for JBD theory to be conformally invariant and compatible with supernovae type Ia data. As it is indicated in [14], based on the recent observational data, the best fitting value of $\omega$ is $-1.477$. Although local tests indicate that $\omega$ is a large positive number, JBD theory with a negative coupling parameter is viable scenario for an era dominated by the scalar field.

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