UNITARY REPRESENTATIONS WITH DIRAC COHOMOLOGY: FINITENESS IN THE REAL CASE

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Abstract. Let $G$ be a complex connected simple algebraic group with a fixed real form $\sigma$. Let $G(\mathbb{R}) = G^\sigma$ be the corresponding group of real points. This paper reports a finiteness theorem for the classification of irreducible unitary Harish-Chandra modules of $G(\mathbb{R})$ (up to equivalence) having non-vanishing Dirac cohomology. Moreover, we study the distribution of the spin norm along Vogan pencils for certain $G(\mathbb{R})$, with particular attention paid to the unitarily small convex hull introduced by Salamanca-Riba and Vogan.

1. Introduction

In representation theory of Lie groups, the Dirac operator was firstly introduced by Parthasarathy to give geometric construction for most of the discrete series representations [23], whose algebraic parametrization was achieved by Harish-Chandra [11, 12]. This project was completed by Atiyah and Schmid: they realized all the discrete series in the kernel of the Dirac operator [2].

To understand the unitary dual of a real reductive Lie group better, Vogan [32] formulated the notion of Dirac cohomology in 1997, and conjectured that whenever non-zero, Dirac cohomology should reveal the infinitesimal character of the original module. This conjecture was confirmed by Huang and Pandžić [15] in 2002, see Theorem 2.1. Since then, Dirac cohomology became a new invariant for unitary representations of real reductive Lie groups, and classifying all the irreducible unitary representations (up to equivalence) with non-vanishing Dirac cohomology became an interesting problem which remained open. Among the entire unitary dual, as we shall see from [2], these representations are exactly the extreme ones in the sense of Parthasarathy’s Dirac operator inequality [23, 24]. Thus understanding them thoroughly should also be important.

An effective way to construct unitary representations is using cohomological induction. For instance, Salamanca-Riba [26] proved that any irreducible unitary representation with strongly regular infinitesimal character is cohomologically induced from a one-dimensional representation. Inspired by the work of Huang, Kang and Pandžić [17], a formula for Dirac cohomology of cohomologically induced modules was obtained whenever the inducing modules are weakly good [10]. However, it seems rather hard to get a similar unifying formula when the weakly good range condition is dropped. This point has perplexed us for quite a long time. Recently, in the special case of complex Lie groups (viewed as real Lie groups),

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we have proved that beyond the good range, there are at most finitely many irreducible unitary modules with non-zero Dirac cohomology, see Theorem A of [5]. The first aim of the current paper is to generalize this result to real reductive Lie groups.

Now let us be more precise. Let \( G \) be a complex connected simple algebraic group with finite center. Let \( \sigma : G \rightarrow G \) be a real form of \( G \). That is, \( \sigma \) is an antiholomorphic Lie group automorphism and \( \sigma^2 = \text{Id} \). Let \( \theta : G \rightarrow G \) be the involutive algebraic automorphism of \( G \) corresponding to \( \sigma \) via Cartan theorem (see Theorem 3.2 of [1]). Put \( G(\mathbb{R}) = G^\sigma \) as the group of real points. Denote by \( K \) a maximal compact subgroup of \( G \), and put \( K(\mathbb{R}) := K^\theta \). Denote by \( \mathfrak{g}_0 \) the Lie algebra of \( G(\mathbb{R}) \), and let \( \mathfrak{g}_0 = \mathfrak{t}_0 \oplus \mathfrak{p}_0 \) be the Cartan decomposition corresponding to \( \theta \) on the Lie algebra Level. Denote by \( \mathfrak{h}_{f,0} = \mathfrak{t}_{f,0} \oplus \mathfrak{a}_{f,0} \) the unique \( \theta \)-stable fundamental Cartan subalgebra of \( \mathfrak{g}_0 \). That is, \( \mathfrak{t}_{f,0} \subseteq \mathfrak{t}_0 \) is maximal abelian.

As usual, we drop the subscripts to stand for the complexified Lie algebras. For example, \( \mathfrak{g} = \mathfrak{g}_0 \otimes _\mathbb{R} \mathbb{C}, \mathfrak{h}_f = \mathfrak{h}_{f,0} \otimes _\mathbb{R} \mathbb{C} \) and so on. We fix a non-degenerate invariant symmetric bilinear form \( \langle \cdot, \cdot \rangle \) on \( \mathfrak{g} \). Its restrictions to \( \mathfrak{t}, \mathfrak{p} \), etc., will also be denoted by the same symbol. Then the Dirac cohomology of an irreducible \((\mathfrak{g}, K(\mathbb{R}))\) module \( \pi \) is defined as the \( \hat{K}(\mathbb{R}) \)-module

\[
\text{\( H_D(\pi) = \text{Ker } D/(\text{Im } D \cap \text{Ker } D), \)}
\]

where \( \hat{K}(\mathbb{R}) \) is the pin double covering group of \( K(\mathbb{R}) \). Here the Dirac operator \( D \) acts on \( \pi \otimes S_G \), and \( S_G \) is a spin module of the Clifford algebra \( C(\mathfrak{p}) \). We care the most about the case that \( \pi \) is unitary. Then \( D \) is symmetric with respect to a natural inner product on \( \pi \otimes S_G, \text{Ker } D \cap \text{Im } D = 0 \), and

\[
\text{\( H_D(\pi) = \text{Ker } D = \text{Ker } D^2. \)}
\]

Parthasarathy’s Dirac operator inequality now says that \( D^2 \) has non-negative eigenvalue on any \( \hat{K}(\mathbb{R}) \)-type of \( \pi \otimes S_G \). Moreover, by Theorem 3.5.2 of [16], it becomes equality on some \( \hat{K}(\mathbb{R}) \)-types of \( \pi \otimes S_G \) if and only if \( H_D(\pi) \) is non-vanishing.

Let \( \hat{G}(\mathbb{R})^d \) be the set of all equivalence classes of irreducible unitary representations of \( G(\mathbb{R}) \) with non-zero Dirac cohomology. Our main result is the following.

**Theorem A.** For all but finitely many exceptions, any member \( \pi \) in \( \hat{G}(\mathbb{R})^d \) is cohomologically induced from a member \( \pi_{L(\mathbb{R})} \) in \( \hat{L}(\mathbb{R})^d \) which is in the good range. Here \( L(\mathbb{R}) \) is a proper \( \theta \)-stable Levi subgroup of \( G(\mathbb{R}) \).

In the setting of the above theorem, we call the finitely many exceptions the scattered part of \( \hat{G}(\mathbb{R})^d \). For a fixed \( G(\mathbb{R}) \), our proof of Theorem A will actually give a method to pin down the scattered part of \( \hat{G}(\mathbb{R})^d \), see Remark 4.4. To figure out the other members of \( \hat{G}(\mathbb{R})^d \), Theorem A says that it boils down to look at \( \hat{L}(\mathbb{R})^d \) for the finitely many \( \theta \)-stable Levi of \( G(\mathbb{R}) \). Working within the good range, we can do cohomological induction in stages, see Corollary 11.173 of Knapp and Vogan [19]. Thus, like the special case of complex Lie groups [5], [8], we have that \( \hat{L}(\mathbb{R})^d \) is built up with central unitary characters and the scattered part of \( \hat{L}(\mathbb{R})^d_{ss} \) which contains only finitely many members. Here \( L(\mathbb{R})_{ss} \) denotes the derived
group of \( L(\mathbb{R}) \). Therefore, Theorem A actually leads to a finite algorithm for pinning down \( \hat{G}(\mathbb{R})^d \). From this aspect, we view it as a finiteness result.

We expect some applications of Theorem A in the theory of automorphic forms via the approach of [16], Chapter 8, where Huang and Pandžić sharpened the results of Langlands [21] and Hotta-Parthasarathy [14] by using the index theorem for the Dirac operator. It is also conceivable that Theorem A will be helpful for studying the Dirac index polynomial [21] and Hotta-Parthasarathy [14] by using the index theorem for the Dirac operator. It is this approach of [16, Chapter 8], where Huang and Pandžić sharpened the results of Langlands [21] for the key notion of \( u\text{-}small \). It will also be recalled under certain assumptions soon.

Among the entire set \( \hat{G}(\mathbb{R})^d \), its scattered part should be viewed as the “kernel” thus deserves particular attention. Conjecture 5.7 of Salamanca-Riba and Vogan [27] tries to reduce the classification of unitary representations to the classification of those containing a unitarily small (\( u\text{-}small \) for short) \( K(\mathbb{R}) \)-type. The reader can refer to Definition 6.1 of [27] for the key notion of \( u\text{-}small \). It will also be recalled under certain assumptions soon. Inspired by this unified conjecture, and based on our previous calculations [5, 8], we propose the following.

**Conjecture B.** Take any \( \pi \) in the scattered part of \( \hat{G}(\mathbb{R})^d \). Then any \( K(\mathbb{R}) \)-type of \( \pi \) contributing to \( H_D(\pi) \) must be unitarily small.

Our next result aims to collect some evidence for the above conjecture. More precisely, we will investigate the distribution of the spin norm [6] along Vogan pencils [28], with particular attention paid to the \( u\text{-}small \) convex hull. By specifying a Vogan diagram for \( g_0 \), we have chosen a positive root system \( \Delta^+(g, h_f) \). When restricted to \( t_f \), we have

\[
\Delta^+(g, t_f) = \Delta^+(t, t_f) \cup \Delta^+(p, t_f).
\]

Let \( \rho \) (resp., \( \rho_c \)) be the half sum of the positive roots in \( \Delta^+(g, t_f) \) (resp., \( \Delta^+(t, t_f) \)). Throughout this paper, the positive root system \( \Delta^+(t, t_f) \) is fixed once for all.

For the remaining part of this section, we assume that \( \mathfrak{t} \) has no center. Let \( \{\gamma_1, \ldots, \gamma_l\} \) be the simple roots of \( \Delta^+(t, t_f) \), and let \( \{\varpi_1, \ldots, \varpi_l\} \) be the corresponding fundamental weights. We will refer to a \( \mathfrak{t} \text{-type} \) \( E_\mu \)—an irreducible finite-dimensional representation of \( \mathfrak{t} \)—by its highest weight \( \mu = [a_1, \ldots, a_l] \), which stands for \( a_1 \varpi_1 + \cdots + a_l \varpi_l \). We denote by \( \Lambda \) the weight lattice for \( \Delta^+(t, t_f) \), and collect the dominant weights as \( \Lambda^+ \). Let \( C \) be the dominant Weyl chamber for \( \Delta^+(t, t_f) \), and collect all the non-negative integer combinations of \( \gamma_1, \ldots, \gamma_l \) as \( \Pi \).

Let

\[
R(\Delta(p, t_f)) = \left\{ \sum_{\alpha \in \Delta(p, t_f)} b_\alpha \alpha \mid 0 \leq b_\alpha \leq 1 \right\}.
\]

This convex set is invariant under \( W(\mathfrak{t}, t_f) \), and it is the \( u\text{-}small \) convex hull introduced by Salamanca-Riba and Vogan in [27]. We call a \( \mathfrak{t} \text{-type} \) \( E_\mu \) \( u\text{-small} \) if its highest weight \( \mu \) lies in \( R(\Delta(p, t_f)) \); otherwise, we would say that \( E_\mu \) is \( u\text{-}large \). The notion spin norm will be recalled in [21]. Let \( \beta \) be the highest weight of the \( \mathfrak{t} \)-representation \( p \). According to Corollary 3.5 of Vogan [28], the \( \mathfrak{t} \)-types in any infinite-dimensional \( (g, K(\mathbb{R})) \)-module must be the union of pencils, which are of the forms

\[
P(\mu) := \{ \mu + n\beta \mid n \in \mathbb{Z}_{\geq 0} \}.
\]
These objects are illustrated for the $G$ case in Figure 1 below, where the shaded region is $R(\Delta(p, t_f)) \cap C$, while the dotted circles stand for $u$-small $\mathfrak{t}$-types. Note that $2\rho_n$, $2\rho_n^{(1)}$ and $2\rho_n^{(2)}$ are extremal points of the $u$-small convex hull.

**Figure 1.** The $G$ case, where $\beta = 3\varpi_1 + \varpi_2$.

**Theorem C.** Let $\mathfrak{g}_0$ be on the following list

$\mathfrak{sl}(2n, \mathbb{R})$, $n \geq 2$; $\mathfrak{sl}(2n + 1, \mathbb{R})$; $\mathfrak{sl}(n, \mathbb{H})$, $n \geq 2$; $E_I, E_{II}, E_{IV}, E_V, E_{VI}, E_{VIII}, E_{IX}, F_I, F_{II}, G$.

The spin norm increases strictly along any pencil once it goes beyond the $u$-small convex hull. Namely, for any $u$-large weight $\mu$ such that $\mu - \beta$ is dominant, we have

\[ \|\mu\|_{\text{spin}} > \|\mu - \beta\|_{\text{spin}}. \]  \hspace{1cm} (6)

The requirement that $\mu$ is $u$-large is key for (6) to hold. For example, in the $G$ case, (6) fails for the $u$-small $\mathfrak{t}$-type $\mu = \beta$. Indeed, in that case, we have

\[ \|\beta\|_{\text{spin}} = \|\rho_c\| < \|0\|_{\text{spin}} = \|\rho\|. \]

Moreover, we note that in Theorem C, the unitary dual is known by Vogan when $\mathfrak{g}_0$ is classical [30] or $G$ [31].

Earlier, Theorem 1.1 of [7]—the counterpart of Theorem C for complex Lie groups—turned out to be very effective in controlling the infinitesimal characters for unitary representations. For instance, in our determination of all the equivalence classes of irreducible unitary representations with nonvanishing Dirac cohomology for complex $E_6$ [8], applying Theorem 1.1 of [7] has reduced the number of candidate representations in an $s$-family from 124048 to 3, see Example 4.1 of [8]. Therefore, we expect that Theorem C will improve the efficiency of detecting non-unitarity of irreducible representations of the concerned real reductive Lie groups.
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The paper is organized as follows. We collect some preliminaries in Section 2, and recall the Langlands parameters adopted in atlas [3] from Adams, van Leeuwen, Trapa and Vogan’s paper [1] in Section 3. Section 4 is devoted to the proof of Theorem A. We illustrate our strategy for Theorem C in Section 5. Then we handle the classical Lie algebras for Theorem C in Section 6, and deal with the exceptional Lie algebras in Section 7. Section [S] studies Sp(4, R). Throughout this paper, all the data about root systems are adopted as in Appendix C of Knapp [18].

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Langlands parameters adopted in atlas case will be studied in Section 8. In particular, a neat analogue of Theorem C is not obvious.

2. Preliminaries for Theorem A

This section aims to collect some preliminaries for Theorem A. We keep the basic notation $G, G(\mathbb{R}), \theta$, etc., as in the introduction.

2.1. Dirac cohomology. Let $H(\mathbb{R})_f = T(\mathbb{R})_f A(\mathbb{R})_f$ be the $\theta$-stable fundamental Cartan subgroup for $G(\mathbb{R})$. Then $T(\mathbb{R})_f$ is a maximal torus of $K(\mathbb{R})$. Put $\mathfrak{h}_f = \mathfrak{t}_f + \mathfrak{a}_f$ as the Cartan decomposition on the complexified Lie algebras Level. We fix a nondegenerate invariant symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$. Its restrictions to $\mathfrak{t}, \mathfrak{p}$, etc., will also be denoted by $\langle \cdot, \cdot \rangle$.

We denote by $\Delta(\mathfrak{g}, \mathfrak{h}_f)$ (resp., $\Delta(\mathfrak{g}, \mathfrak{t}_f)$) the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}_f$ (resp., $\mathfrak{t}_f$). The root system of $\mathfrak{t}$ with respect to $\mathfrak{t}_f$ is denoted by $\Delta(\mathfrak{t}, \mathfrak{t}_f)$. Note that $\Delta(\mathfrak{g}, \mathfrak{h}_f)$ and $\Delta(\mathfrak{g}, \mathfrak{t}_f)$ are reduced, while $\Delta(\mathfrak{g}, \mathfrak{t}_f)$ is not reduced in general. The Weyl group corresponding to these root systems will be denoted by $W(\mathfrak{g}, \mathfrak{h}_f), W(\mathfrak{g}, \mathfrak{t}_f)$ and $W(\mathfrak{t}, \mathfrak{t}_f)$.

We fix compatible choices of positive roots $\Delta^+(\mathfrak{g}, \mathfrak{h}_f)$ and $\Delta^+(\mathfrak{g}, \mathfrak{t}_f)$ so that a positive root in $\Delta(\mathfrak{g}, \mathfrak{h}_f)$ restricts to a positive root in $\Delta(\mathfrak{g}, \mathfrak{t}_f)$. As in [3], $\Delta^+(\mathfrak{g}, \mathfrak{t}_f)$ is a union of the set of positive compact roots $\Delta^+(\mathfrak{t}, \mathfrak{t}_f)$ and the set of positive noncompact roots $\Delta^+(\mathfrak{p}, \mathfrak{t}_f)$.

As usual, we denote by $\rho$ the half sum of roots in $\Delta^+(\mathfrak{g}, \mathfrak{h}_f)$, by $\rho_c$ the half sum of roots in $\Delta^+(\mathfrak{t}, \mathfrak{t}_f)$, and by $\rho_n$ the half sum of roots in $\Delta^+(\mathfrak{p}, \mathfrak{t}_f)$. Then $\rho, \rho_c, \rho_n$ are in $i\mathfrak{t}_{f, 0}^*$ and $\rho_n = \rho - \rho_c$.

Fix an orthonormal basis $Z_1, \ldots, Z_n$ of $\mathfrak{p}_0$ with respect to the inner product induced by the form $\langle \cdot, \cdot \rangle$. Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$ and let $C(\mathfrak{p})$ be the Clifford algebra of $\mathfrak{p}$ with respect to $\langle \cdot, \cdot \rangle$. The Dirac operator $D \in U(\mathfrak{g}) \otimes C(\mathfrak{p})$ is defined as

$$D = \sum_{i=1}^{n} Z_i \otimes Z_i.$$ 

It is easy to check that $D$ does not depend on the choice of the orthonormal basis $Z_i$ and it is $K(\mathbb{R})$-invariant for the diagonal action of $K(\mathbb{R})$ given by adjoint actions on both factors.

Let $K(\mathbb{R})_p$ be the subgroup of $K(\mathbb{R}) \times \text{Pin}\, \mathfrak{p}_0$ consisting of all pairs $(k, s)$ such that $\text{Ad}(k) = p(s)$, where $\text{Ad} : K(\mathbb{R}) \to O(\mathfrak{p}_0)$ is the adjoint action, and $p : \text{Pin}\, \mathfrak{p}_0 \to O(\mathfrak{p}_0)$ is the pin...
double covering map. Namely, $\widehat{K(\mathbb{R})}$ is constructed from the following diagram:

$$
\begin{array}{ccc}
\hat{K}\hat{(\mathbb{R})} & \longrightarrow & \text{Pin}_{0} \\
\downarrow & & \downarrow_{p} \\
K(\mathbb{R}) & \xrightarrow{\text{Ad}} & O(p_{0})
\end{array}
$$

Let $S_G$ be a spin module for $C(p)$, then $S_G$ is a $\widehat{K(\mathbb{R})}$ module. Let $\pi$ be a $(g, K(\mathbb{R}))$ module. Then $U(g)\otimes C(p)$ acts on $\pi \otimes S_G$ in the obvious way. In particular, the Dirac operator $D$ acts on $\pi \otimes S_G$, and the Dirac cohomology of $\pi$ is the $\widehat{K(\mathbb{R})}$-module defined in [11]. Here we note that $\hat{K}(\mathbb{R})$ acts on $\pi$ through $K(\mathbb{R})$ and on $S_G$ through the pin group $\text{Pin}_{0}$. Moreover, since $\text{Ad}(k)(Z_1), \ldots, \text{Ad}(k)(Z_n)$ is still an orthonormal basis of $p_0$, it follows that $D$ is $\hat{K}(\mathbb{R})$ invariant. Therefore, Ker$D$, Im$D$, and $H_D(X)$ are once again $\hat{K}(\mathbb{R})$ modules.

By setting the linear functionals on $t_f$ to be zero on $a_f$, we embed $t_f^*$ as a subspace of $\mathfrak{h}_f^*$. The following foundational result, conjectured by Vogan, was proved by Huang and Pandžić in Theorem 2.3 of [15].

**Theorem 2.1.** (Huang and Pandžić) Let $\pi$ be an irreducible $(g, K(\mathbb{R}))$ module. Assume that the Dirac cohomology of $\pi$ is nonzero, and that it contains the $K(\mathbb{R})$-type $E_\gamma$ with highest weight $\gamma \in t_f^* \subset \mathfrak{h}_f^*$. Then the infinitesimal character of $\pi$ is conjugate to $\gamma + \rho_c$ under $W(g, \mathfrak{h}_f)$.

2.2. **Cohomological induction.** Fix a non-zero element $H \in it_{f,0}$, then a $\theta$-stable parabolic subalgebra $q = l \oplus u$ of $g$ can be defined as the sum of nonnegative eigenspaces of $\text{ad}(H)$. Here the Levi subalgebra $l$ of $q$ is the zero eigenspace of $\text{ad}(H)$, while the nilradical $u$ of $q$ is the sum of positive eigenspaces of $\text{ad}(H)$. Then it follows from $\theta(H) = H$ that $l, u$ and $q$ are all $\theta$-stable. Set $L(\mathbb{R}) = N_{G(\mathbb{R})}(q)$.

Let us arrange the positive root systems in a compatible way, that is, $\Delta(u, \mathfrak{h}_f) \subseteq \Delta^+(g, \mathfrak{h}_f)$ and set $\Delta^+(l, \mathfrak{h}_f) = \Delta(l, \mathfrak{h}_f) \cap \Delta^+(g, \mathfrak{h}_f)$. Let $\rho^L$ denote the half sum of roots in $\Delta^+(l, \mathfrak{h}_f)$, and denote by $\rho(u)$ (resp., $\rho(u \cap p)$) the half sum of roots in $\Delta(u, \mathfrak{h}_f)$ (resp., $\Delta(u \cap p, \mathfrak{h}_f)$). Then

$$
\rho = \rho^L + \rho(u).
$$

Let $Z$ be an $(l, L(\mathbb{R}) \cap K(\mathbb{R}))$ module. Cohomological induction functors attach to $Z$ certain $(g, K(\mathbb{R}))$-modules $\mathcal{L}_j(Z)$ and $\mathcal{R}_j(Z)$, where $j$ is a nonnegative integer. For a definition, see Chapter 2 of [19]. Suppose that $\lambda_L \in \mathfrak{h}_f^*$ is the infinitesimal character of $Z$. We say $Z$ or $\lambda_L$ is **good** or in **good range** if

$$
\text{Re}\langle \lambda_L + \rho(u), \alpha \rangle > 0, \quad \forall \alpha \in \Delta(u, \mathfrak{h}_f).
$$

We say $Z$ or $\lambda$ is **weakly good** if

$$
\text{Re}\langle \lambda_L + \rho(u), \alpha \rangle \geq 0, \quad \forall \alpha \in \Delta(u, \mathfrak{h}_f).
$$

Let us recall a theorem which is mainly due to Vogan.
**Theorem 2.2.** (29 Theorems 1.2 and 1.3, or 19 Theorems 0.50 and 0.51) Suppose the admissible \((I, L(\mathbb{R}) \cap K(\mathbb{R}))\)-module \(Z\) is weakly good. Then we have

(i) \(L_j(Z) = R^j(Z) = 0\) for \(j \neq S(= \dim (u \cap \mathfrak{t}))\).

(ii) \(L_S(Z) \cong R^S(Z)\) as \((g, K(\mathbb{R}))\)-modules.

(iii) if \(Z\) is irreducible, then \(L_S(Z)\) is either zero or an irreducible \((g, K(\mathbb{R}))\)-module with infinitesimal character \(\lambda_L + \rho(u)\).

(iv) if \(Z\) is unitary, then \(L_S(Z)\), if nonzero, is a unitary \((g, K(\mathbb{R}))\)-module.

(v) if \(Z\) is in good range, then \(L_S(Z)\) is non-zero, and it is unitary if and only if \(Z\) is unitary.

Recall that Dirac cohomology of cohomologically induced modules has been determined whenever the inducing modules are weakly good 10.

**Theorem 2.3.** (Theorem B of 10) Suppose that the irreducible unitary \((I, L(\mathbb{R}) \cap K(\mathbb{R}))\) module \(Z\) has a real infinitesimal character \(\lambda_L \in \iota f_\theta^*\) which is weakly good. Then there is a \(K(\mathbb{R})\)-module isomorphism

\[
H_D(L_S(Z)) \cong L_S^K(\mathbb{R})(H_D(Z) \otimes \mathbb{C}_{-\rho(u \cap p)}).
\]

Using the knowledge of the functor \(L_S^K(\mathbb{R})\) stated after Corollary 5.85 of 19, we have the following.

**Corollary 2.4.** Under the setting of Theorem 2.3 we have that \(H_D(L_S(Z))\) is non-zero if and only if \(H_D(Z)\) is non-zero and that there exists a highest weight \(\gamma_L\) in \(H_D(Z)\) such that \(\gamma_L + \rho(u \cap p)\) is \(\Delta^+(\mathfrak{k}, \mathfrak{t}_f)\) dominant.

3. **Atlas parameters**

Let us recall necessary notation from 1 regarding the Langlands parameters in atlas. Let \(H\) be a maximal torus of \(G\). That is, \(H\) is a maximal connected abelian subgroup of \(G\) consisting of diagonalizable matrices. Note that \(H\) is complex connected reductive algebraic. Its character lattice is the group of algebraic homomorphisms

\[
X^* := \text{Hom}_{\text{alg}}(H, \mathbb{C}^\times).
\]

Now assume further that \(H\) is defined over \(\mathbb{R}\) (with respect to \(\sigma\)). That is, \(\sigma(H) = H\). Then we put \(H(\mathbb{R}) = H(\mathbb{R}, \sigma) = H^\sigma\), put \(T(\mathbb{R}) = H(\mathbb{R})^\theta\). Write

\[
\mathfrak{h}_0 = \mathfrak{t}_0 + \mathfrak{a}_0
\]

for the decomposition of the real Lie algebra of \(H(\mathbb{R})\) into \(+1\) and \(-1\) eigenspaces of \(\theta\). Put \(A = \exp(\mathfrak{a}_0)\). The group \(A\) is isomorphic to its Lie algebra \(\mathfrak{a}_0\), and

\[
H(\mathbb{R}) \cong T(\mathbb{R})A.
\]

Then, as shown in Proposition 4.3 of 1, characters of \(H(\mathbb{R})\) can be parameterized by \(\gamma = (\lambda, \nu) \in \widehat{T(\mathbb{R})} \times \hat{A}\), where

\[
\lambda \in \widehat{T(\mathbb{R})} \cong \text{Hom}_{\text{alg}}(T, \mathbb{C}^\times) \cong X^*/(1 - \theta)X^*
\]
and
\[ \nu \in \hat{A} \cong a^* \cong (X^*)^{-\theta} \otimes_\mathbb{C} \mathbb{C}. \]

Choose a Borel subgroup \( B \supset H \). In atlas, an irreducible \((g, K(\mathbb{R}))\) module \( \pi \) is parameterized by \((x, \lambda, \nu)\) via the Langlands classification, where \( x \) is a \( K \)-orbit of the Borel variety \( G/B \), \( \lambda \in X^*/(1 - \theta)X^* \) and \( \nu \in (X^*)^{-\theta} \otimes_\mathbb{C} \mathbb{C} \). In such a case, the infinitesimal character of \( \pi \) is
\[
\frac{1}{2}(1 + \theta)\lambda + \nu \in h^*. \tag{11}
\]

See Theorem 6.1 of [1] for more.

4. PROOF OF THEOREM A

In this section, we continue to let \( G, \sigma, \theta, \) etc., be as in the introduction. We fix a maximal torus \( H \) of \( G \) which is defined over \( \mathbb{R} \) with respect to \( \sigma \). Whenever a Borel subgroup \( B \supset H \) is fixed, we will have a set of positive roots \( \Delta^+(g, h) \). Denote by \( \rho(G) \) the half sum of roots in \( \Delta^+(g, h) \). Then of course \( \|\rho(G)\| \) is independent of the choice of \( B \). Now let us prepare two propositions.

**Proposition 4.1.** Let \( \pi \) be an irreducible unitary \((g, K(\mathbb{R}))\) module with atlas parameter \((x, \lambda, \nu)\). Assume that \( \pi \) has real infinitesimal character. Then \( \|\nu\| \leq \|\rho(G)\| \).

The spherical case of the above proposition was due to Helgason and Johnson [13]. In general, one can refer to Theorem 5.2 of Chapter IV of Borel and Wallach [4].

**Proposition 4.2.** Let \( \pi \) be an irreducible unitary \((g, K(\mathbb{R}))\) module with atlas parameter \((x, \lambda, \nu)\). Assume that \( \pi \) has real infinitesimal character. Then either \( \|\lambda + \theta\lambda\| \) is bounded, or \( \pi \) is cohomologically induced from an irreducible unitary Harish-Chandra module \( \pi_{L(\mathbb{R})} \) of \( L(\mathbb{R}) \) which is in the good range. Here \( L(\mathbb{R}) \) is a proper \( \theta \)-stable parabolic subgroup of \( G(\mathbb{R}) \).

**Proof.** We will show that whenever \( \|\lambda + \theta\lambda\| \) is big enough, then \( \pi \) must be cohomologically induced from some irreducible unitary Harish-Chandra module \( \pi_{L(\mathbb{R})} \) of \( L(\mathbb{R}) \) which is in the good range. Here \( L(\mathbb{R}) \) is certain proper \( \theta \)-stable Levi subgroup of \( G(\mathbb{R}) \).

Since \( \|\nu\| \leq \|\rho(G)\| \) by Proposition 4.1, there is some constant \( N \) (independent of \( \pi \)) so that
\[
|\langle \nu, \alpha^\vee \rangle| \leq N \tag{12}
\]
for every root \( \alpha \). Choose a Borel subgroup \( B \) of \( G \) making \( \frac{\lambda + \theta\lambda}{2} \) weakly dominant. That is,
\[
\frac{1}{2}(\lambda + \theta\lambda), \beta^\vee \rangle \geq 0 \tag{13}
\]
for every positive root \( \beta \). Let \( L \) be the \( \theta \)-stable Levi subalgebra of \( g \) generated by all those positive roots \( \beta \) such that
\[
\langle \frac{1}{2}(\lambda + \theta\lambda), \beta^\vee \rangle \leq N \tag{14}
\]
We claim that if \( \| \frac{\lambda + \theta \lambda}{2} \| \) is large enough, then \( l \) must be proper. Indeed, in such a case, there must be a simple root \( \beta_0 \) so that

\[
\frac{1}{2} (\lambda + \theta \lambda, \beta_0^\vee) > N.
\]

The root \( \beta_0 \) is distinct from any of the \( \beta \) described in \((14)\). Thus the claim holds.

Let \( q = l + u \) be the corresponding \( \theta \)-stable parabolic subalgebra of \( g \). If \( \gamma \) is any root in \( u \), then

\[
\langle \frac{1}{2} (\lambda + \theta \lambda), \gamma^\vee \rangle > N.
\]

Thus by \((12)\), we have

\[
\langle \frac{1}{2} (\lambda + \theta \lambda) + \nu, \gamma^\vee \rangle > 0.
\]

Therefore cohomological induction from \( \pi_{L(R)} \) by \( q \) to \( G(R) \) is in the good range, and Theorem 2.2 says that \( \pi_{L(R)} \) must be irreducible and unitary. □

**Remark 4.3.** The above proposition was inspired by a theorem of Vogan, presented by Paul on page 23 of [25] in the 2017 *atlas* workshop. The above proof was kindly told to us by Professor Vogan.

Now we are ready to prove Theorem A.

**Proof of Theorem A.** Let \( \pi \) be an irreducible unitary \((g, K(R))\) module with non-zero Dirac cohomology. Let \( (x, \lambda, \nu) \) be an atlas parameter of \( \pi \). By Theorem 2.1, the infinitesimal character \((11)\) of \( \pi \) must be real. There are two cases.

The first case is that \( \pi \) is cohomologically induced from an irreducible unitary Harish-Chandra module \( \pi_{L(R)} \) of \( L(R) \) which is in the good range, where \( L(R) \) is a proper \( \theta \)-stable Levi subgroup of \( G(R) \). In this case, Corollary 2.4 says that \( \pi_{L(R)} \) must be a member of \( \hat{L}(R)^d \) and we are done.

Otherwise, by Proposition 4.2 we conclude that \( \| \frac{\lambda + \theta \lambda}{2} \| \) must be bounded. Note that \( \| \nu \| \) is also bounded since \( \pi \) is assumed to be unitary, see Proposition 4.1. Therefore, the infinitesimal character

\[
\frac{1}{2} (1 + \theta) \lambda + \nu \in \mathfrak{h}_*^\vee
\]

of \( \pi \) is bounded. Let \( \Lambda \in \mathfrak{h}_*^\vee \) be the conjugation of the above element to \( \mathfrak{h}_f^* \). Since \( H_D(\pi) \) is assumed to be non-zero, Theorem 2.1 tells us that

\[
\Lambda = w(\gamma_G + \rho_c)
\]

for some \( w \in W(g, \mathfrak{h}_f) \) and for some \( \hat{K}(R) \) highest weight \( \gamma_G \) of \( \pi \otimes S_G \). Note that \( \gamma_G + \rho_c \) in \((19)\) lives in a discrete set, then so do \( \Lambda \) in \((19)\) and the element in \((18)\). Being bounded and being discrete simultaneously allow us to conclude that the infinitesimal character \((18)\) of \( \pi \) has finitely many choices. Since there are finitely many irreducible Harish-Chandra modules with a given infinitesimal character, it follows directly that \( \pi \) has at most finitely many choices. This handles the remaining cases, and the proof finishes. □
Remark 4.4. The above proof actually gives us a method to exhaust all the possible infinitesimal characters of the representations coming from the scattered part of $\hat{G}(\mathbb{R})_d$. Therefore, for a fixed $G(\mathbb{R})$, we can eventually pin down the scattered part of $\hat{G}(\mathbb{R})_d$ after a finite calculation.

5. Strategy for Theorem C

This section aims to explain our strategy for Theorem C. We continue with the setting of the introduction and assume that $\mathfrak{g}$ has no center. In particular, we have chosen a positive restricted root system $\Delta^+(\mathfrak{g}, t_f)$ as in (3), and a positive system $\Delta^+(\mathfrak{t}, t_f)$ is fixed once for all. Note that there could be other choices of positive root systems for $\Delta(\mathfrak{p}, t_f)$ which are compatible with $\Delta^+(\mathfrak{t}, t_f)$. Indeed, our fixed form $\langle \cdot, \cdot \rangle$ induces inner products on $it^*_{f,0}$. Denote by $C_g(it^*_{f,0})$ the closed Weyl chamber corresponding to $\Delta^+(\mathfrak{g}, t_f)$. Recall that $C$ is the dominant Weyl chamber corresponding to $\Delta^+(\mathfrak{t}, t_f)$. Then $C_g(it^*_{f,0})$ is contained in $C$. Define

$$W(\mathfrak{g}, t_f)^1 = \{w \in W(\mathfrak{g}, t_f) \mid w(C_g(it^*_{f,0})) \subseteq C\}.$$  

It is well-known that the multiplication map gives a bijection from $W(\mathfrak{t}, t_f) \times W(\mathfrak{g}, t_f)^1$ onto $W(\mathfrak{g}, t_f)$, see Kostant [20]. Then

$$\{w\Delta^+(\mathfrak{p}, t_f) \mid w \in W(\mathfrak{g}, t_f)^1\}$$

are exactly all the choices of positive roots systems for $\Delta(\mathfrak{p}, t_f)$ which are compatible with $\Delta^+(\mathfrak{t}, t_f)$. Let us enumerate the elements of $W(\mathfrak{g}, t_f)^1$ as $w(0) = e, w(1), \ldots, w(s-1)$. For $0 \leq j \leq s - 1$, put

$$(\Delta^+(j)(\mathfrak{p}, t_f) = w(j)\Delta^+(\mathfrak{p}, t_f), \quad (\Delta^+(j)(\mathfrak{g}, t_f) = (\Delta^+(j)(\mathfrak{p}, t_f) \cup \Delta^+(\mathfrak{t}, t_f)).$$

Note that $(\Delta^+(0)(\mathfrak{p}, t_f) = \Delta^+(\mathfrak{p}, t_f)$ and $(\Delta^+(0)(\mathfrak{g}, t_f) = \Delta^+(\mathfrak{g}, t_f)$. Denote by $\rho_n^{(j)}$ the half sum of the positive roots in $(\Delta^+(j)(\mathfrak{p}, t_f)$. Then we have

$$w(j)\rho = \rho_c + \rho_n^{(j)}, \quad 0 \leq j \leq s - 1.$$  

Recall from Lemma 9.3.2 of [33] that the spin module decomposes into the following $\mathfrak{t}$-types:

$$S_G = \bigoplus_{j=0}^{s-1} \mathbb{Z}^{l_0/2}E_{\rho_n^{(j)}},$$

where $l_0 = \dim_\mathbb{C} \mathfrak{a}$. Now as in [4], the spin norm of the $\mathfrak{t}$-type $E_\mu$ is defined to be

$$\|\mu\|_{\text{spin}} := \min_{0 \leq j \leq s-1} \|\mu - \rho_n^{(j)}\| + \rho_c.$$  

Here, $\| \cdot \|$ is the norm on $it^*_{f,0}$ induced from $\langle \cdot, \cdot \rangle$, while $\{\mu - \rho_n^{(j)}\}$ denotes the unique dominant weight to which $\mu - \rho_n^{(j)}$ is conjugate under the action of $W(\mathfrak{t}, t_f)$. For instance, $\{-\rho_c\} = \rho_c$. Note that the lowest weights of $S_G$ (without multiplicity) are precisely $-\rho_n^{(j)}$, $0 \leq j \leq s - 1$. Thus we have

$$\|0\|_{\text{spin}} = \|\rho\|, \quad \|\rho_n^{(j)}\|_{\text{spin}} = \|\rho_c\|_0 \leq j \leq s - 1.$$
We begin with a few lemmas about the u-small convex hull.

**Lemma 5.1.** The convex set \( R(\frac{1}{2}\Delta(p, t_f)) \) is the convex hull of all the extremal weights of the spin module \( S_G \).

**Proof.** Since any extremal weight of \( S_G \) must belong to \( R(\frac{1}{2}\Delta(p, t_f)) \), that LHS contains RHS is obvious. Since \( R(\frac{1}{2}\Delta(p, t_f)) \) is a convex set invariant under \( W(k, t_f) \), to verify that LHS is contained in RHS, it suffices to show that all the extremal points of \( R(\frac{1}{2}\Delta(p, t_f)) \) which are dominant for \( \Delta^+(t, t_f) \) belong to the RHS. However, these points are exactly all the highest weights (without multiplicity) of \( S_G \), and we are done. \( \square \)

**Lemma 5.2.** For any point \( \mu \in R(\frac{1}{2}\Delta(p, t_f)) \), we always have 
\[ \|\mu + \rho_c\| \leq \|\rho\|. \]

**Proof.** By Lemma 5.1, we need only to consider the case that \( \mu \) is an extremal weight of \( S_G \). Since \( \rho_c \) is dominant for \( \Delta^+(g, t_f) \), to have an upper bound for \( \|\mu + \rho_c\| \), one can further focus on the case that \( \mu \) is a highest weight of the spin module \( S_G \), say \( \mu = w^{(j)}\rho - \rho_c \). Then \( \mu + \rho_c = w^{(j)}\rho \) and the conclusion is now obvious. \( \square \)

**Proposition 5.3.** Let \( E_\mu \) be any u-small \( k \)-type with highest weight \( \mu \). Then 
\[ \|\rho_c\| \leq \|\mu\|_{\text{spin}} \leq \|\rho\|. \]

**Proof.** The first inequality is obvious from (21). We note that \( \|\mu\|_{\text{spin}} = \|\rho_c\| \) if and only if \( \mu \) is a highest weight of the spin module \( S_G \). For the second inequality, we use Theorem 6.7(f) of [27], which says that there is a positive system \( (\Delta^+)^{(j)}(g, t_f) \) containing \( \Delta^+(t, t_f) \) such that 
\[ \mu = \sum_{\beta \in (\Delta^+)^{(j)}(p, t_f)} c_\beta \beta, \quad c_\beta \in [0, 1]. \]

Note that 
\[ \|\mu\|_{\text{spin}} \leq \|\{\mu - \rho_n^{(j)}\} + \rho_c\|. \]

Since \( \rho_n^{(j)} \) is the half sum of the positive roots in \( (\Delta^+)^{(j)}(p, t_f) \), we have that \( \mu - \rho_n^{(j)} \in R(\frac{1}{2}\Delta(p, t_f)) \), which is \( W(t, t_f) \) invariant. Hence \( \{\mu - \rho_n^{(j)}\} \in R(\frac{1}{2}\Delta(p, t_f)) \). Therefore, by Lemma 5.2 we have 
\[ \|\{\mu - \rho_n^{(j)}\} + \rho_c\| \leq \|\rho\|. \]

\( \square \)

To compute the u-small \( \mathfrak{t} \)-types more effectively, let us explicitly write down Theorem 6.7(d) of [27] under the current setting. We prepare a bit more notation. Let \( \{\alpha_1, \ldots, \alpha_l\} \) be the simple roots for \( \Delta^+(g, t_f) \), with \( \{\xi_1, \ldots, \xi_l\} \) the corresponding fundamental weights.

**Lemma 5.4.** Any \( \mu \in \Lambda^+ \) is u-small if and only if \( \langle \mu + 2\rho_c, w^{(j)}\xi_i \rangle \leq 2\langle \rho, \xi_i \rangle \), \( 1 \leq i \leq l \), \( 0 \leq j \leq s - 1 \).
Proof. Note that $w^{(j)}\alpha_i$, $1 \leq i \leq l$, are the simple roots of $(\Delta^+)^{(j)}(\mathfrak{g}, t_f)$, and that $w^{(j)}\xi_i$, $1 \leq i \leq l$, are the corresponding fundamental weights. Thus by Theorem 6.7(d) of [27], $\mu$ is u-small if and only if $\langle \mu - 2\rho_n, w^{(j)}\xi_i \rangle \leq 0$, which is further equivalent to that
\[
\langle \mu + 2\rho_c - 2w^{(j)}\rho, w^{(j)}\xi_i \rangle \leq 0, \quad 1 \leq i \leq l, \quad 0 \leq j \leq s - 1.
\]
Since $\langle w^{(j)}\rho, w^{(j)}\xi_i \rangle = \langle \rho, \xi_i \rangle$, the desired description follows. □

Now let us explain our strategy for Theorem C. For $0 \leq j \leq s - 1$, put
\[
(22) \quad \Delta(\mu, j) = \{\mu - \rho_n^{(j)}\} - \{\mu - \beta - \rho_n^{(j)}\}.
\]
Then
\[
\|\{\mu - \rho_n^{(j)}\} + \rho_c\|^2 = \|\rho_c\|^2 + \|\{\mu - \rho_n^{(j)}\}\|^2 + 2\langle \rho_c, \{\mu - \rho_n^{(j)}\} \rangle
\]
\[
= \|\rho_c\|^2 + \|\mu - \beta - \rho_n^{(j)}\|^2 + 2\langle \rho_c, \{\mu - \beta - \rho_n^{(j)}\} \rangle.
\]
Similarly,
\[
\|\{\mu - \beta - \rho_n^{(j)}\} + \rho_c\|^2 = \|\rho_c\|^2 + \|\mu - \beta - \rho_n^{(j)}\|^2 + 2\langle \rho_c, \{\mu - \beta - \rho_n^{(j)}\} \rangle.
\]
Thus
\[
(23) \quad \|\{\mu - \rho_n^{(j)}\} + \rho_c\|^2 - \|\{\mu - \beta - \rho_n^{(j)}\} + \rho_c\|^2 = I + II.
\]
where
\[
(24) \quad I = 2\langle \rho_c, \Delta(\mu, j) \rangle,
\]
and
\[
(25) \quad II = \|\mu - \rho_n^{(j)}\|^2 - \|\mu - \beta - \rho_n^{(j)}\|^2.
\]
The term II is relatively easier to analyze, while the term I is subtle. Indeed, note firstly that $\Delta(\mu, j)$ lies in $\Lambda$, the weight lattice for $\Delta(\mathfrak{k}, t_f)$. Secondly, let $w \in W(\mathfrak{k}, t_f)$ be such that
\[
(26) \quad \{\mu - \beta - \rho_n^{(j)}\} = w(\mu - \beta - \rho_n^{(j)}).
\]
Then we have
\[
\{\mu - \rho_n^{(j)}\} - \{\mu - \beta - \rho_n^{(j)}\} = \{\mu - \rho_n^{(j)}\} - w(\mu - \rho_n^{(j)}) + w(\mu - \beta - \rho_n^{(j)})
\]
\[
= \{\mu - \rho_n^{(j)}\} - w(\mu - \rho_n^{(j)}) + w\beta.
\]
By the highest weight theorem (see e.g. Theorem 5.5 of [18]), the first term above lies in $\Pi$. Thus its inner product with $\rho_c$ is always non-negative. Unlike the complex case studied in [7], we may no longer conclude that $w\beta \in \Delta^+(\mathfrak{k}, t_f)$. Indeed, the root $\beta$ itself may not lives in $\Delta(\mathfrak{k}, t_f)$. Therefore, we can not always have that $I > 0$. However, by the same proof for (28) below, we have that
\[
\langle \rho_c, w\beta \rangle \geq \langle \rho_c, w_0\beta \rangle = \langle w_0^{-1}\rho_c, \beta \rangle = \langle w_0\rho_c - \beta \rangle = -\langle \rho_c, \beta \rangle.
\]
Here $w_0$ is the longest element of $W(\mathfrak{k}, t_f)$. Therefore, there is a naive lower bound for I. Namely,
\[
(27) \quad I \geq -2\langle \rho_c, \beta \rangle.
\]

In actual calculation when $\mathfrak{g}_0$ is exceptional, we will adopt the parabolic subgroups of $W(\mathfrak{k}, t_f)$ to sharpen the above lower bound. To be more precise, let $W_k$ be the subgroup of
Moreover, we have presented in Fig. 2. In this case, one calculates that
And we can leave the latter job to a computer.

\[ \langle \rho_c, w^\beta \rangle \geq \langle \rho_c, w_{0,k} \beta \rangle, \quad \forall w \in W_k. \]

Indeed, let \( w^{-1} = s_{\delta_1} \cdots s_{\delta_n} \) be a reduced decomposition of \( w^{-1} \) into simple reflections, where \( \delta_i \in \{ \gamma_1, \ldots, \gamma_l \} \). Then by Lemma 5.5 of [9], we have

\[ \rho_c - w^{-1} \rho_c = \sum_{k=1}^{n} \langle \rho_c, \delta_k^\vee \rangle s_{\delta_1} s_{\delta_2} \cdots s_{\delta_{k-1}} (\delta_k), \]

where \( \delta_k^\vee \) is the dual root of \( \delta_k \) and each \( s_{\delta_1} s_{\delta_2} \cdots s_{\delta_{k-1}} (\delta_k) \) is a positive root. Since \( \beta \) is a dominant weight and each element \( w \) of \( W_k \) can be extended to \( w_{0,k} \) by adding simple reflections, now (28) follows from (29) and that

\[ \langle \rho_c, w^\beta \rangle = \langle w^{-1} \rho_c, \beta \rangle = \langle \rho_c, \beta \rangle - \langle \rho_c - w^{-1} \rho_c, \beta \rangle. \]

Eventually, we will be able to find positive integers \( N_k \) (1 \( \leq k \leq l \)) such that (6) holds for any dominant weight \( \mu = [a_1, \ldots, a_l] \) whenever \( a_1 \geq N_1 \), or \( a_2 \geq N_2 \), ..., or \( a_l \geq N_l \). Therefore, it remains to check (6) for those \( u \)-large \( \mathfrak{f} \)-types in the following finite set

\[ \{ \mu = [a_1, \ldots, a_l] \mid 0 \leq a_k \leq N_k - 1, 1 \leq k \leq l; \mu - \beta \text{ is dominant} \}. \]

And we can leave the latter job to a computer.

The case EI considered in [7] will be a typical example to illustrate our strategy for exceptional Lie algebras.

6. Classical Lie algebras

This section aims to deal with the classical Lie algebras for Theorem C.

6.1. \( \mathfrak{sl}(2n, \mathbb{R}) \). This subsection aims to handle \( \mathfrak{sl}(2n, \mathbb{R}) \) (\( n \geq 2 \)), whose Vogan diagram is presented in Fig. 2. In this case, one calculates that

\[ \Delta^+(\mathfrak{t}, \mathfrak{t}_f) = \{ e_i \pm e_j \mid 1 \leq i < j \leq n \}, \quad \Delta^+(\mathfrak{p}, \mathfrak{t}_f) = \Delta^+(\mathfrak{t}, \mathfrak{t}_f) \cup \{ 2e_i \mid 1 \leq i \leq n \}. \]

We have \( W(\mathfrak{g}, \mathfrak{t}_f)^1 = \{ e, s_{2e_n} \}, \beta = 2e_1, \rho_n = ne_1 + (n - 1)e_2 + \cdots + 2e_{n-1} + e_n, \rho_n^{(1)} = ne_1 + (n - 1)e_2 + \cdots + 2e_{n-1} - e_n. \)

Moreover, \( \xi_i = e_1 + \cdots + e_i, 1 \leq i \leq n; \varpi_i = \xi_i, 1 \leq i \leq n - 2, \varpi_{n-1} = \frac{1}{2}(e_1 + \cdots + e_{n-1} - e_n), \varpi_n = \frac{1}{2}(e_1 + \cdots + e_{n-1} + e_n). \)

![Figure 2. The Vogan diagram for \( \mathfrak{sl}(2n, \mathbb{R}) \)](image-url)
Let $\mu = [m_1, \ldots, m_n]$ be a dominant weight. Then $\mu = (a_1, \ldots, a_n)$, where
\[ a_i = \sum_{j=i}^{n-2} m_j + \frac{m_{n-1} + m_n}{2}, 1 \leq i \leq n-2; \quad a_{n-1} = \frac{m_{n-1} + m_n}{2}; \quad a_n = \frac{-m_{n-1} + m_n}{2}. \]

Note that $a_1 \geq \cdots \geq a_n$ are simultaneously integers or half-integers.

Guided by Lemma 5.4, we calculate that the \( \mathfrak{t} \)-type $\mu$ is u-small if and only if
\begin{equation}
(30) \quad a_1 + \cdots + a_k \leq 2nk - k^2 + k, \quad 1 \leq k \leq n; \quad a_1 + \cdots + a_{n-1} - a_n \leq n^2 + n.
\end{equation}

We claim that $\mu$ is u-small when $a_1 \leq n + 1$. Indeed, in this case, we would have that
\[ a_1 + \cdots + a_k \leq nk + k \leq 2nk - k^2 + k, \]
and that
\[ a_1 + \cdots + a_{n-1} - a_n \leq n^2 + n. \]

Therefore, the claim follows from (30).

Now let us assume that $\mu$ is u-large and verify (6). As mentioned above, we have $a_1 \geq n + \frac{3}{2}$. Firstly, set $j = 0$. Then
\[ \mu - \rho_n = (a_1 - n, a_2 - (n - 1), \ldots, a_n - 1), \]
\[ \mu - \beta - \rho_n = (a_1 - n - 2, a_2 - (n - 1), \ldots, a_n - 1), \]
and
\[ \|\mu - \rho_n\|^2 - \|\mu - \rho_n - \beta\|^2 = 4(a_1 - n - 1) > 0. \]

Since $a_1 \geq n + \frac{3}{2}$, we always have $a_1 - n > |a_1 - n - 2|$. Let $B$ be the multi-set consisting of $|a_2 - (n - 1)|$, ..., $|a_n - 1|$. Let $b_1 \geq \cdots \geq b_n$ be the re-ordering of $a_1 - n$ and $B$. Let $c_1 \geq \cdots \geq c_n$ be the re-ordering of $|a_1 - n - 2|$ and $B$. Recall that $W(\mathfrak{t}, t_f)$ consists of permutations as well as all even sign changes. Thus
\[ \{\mu - \rho_n\} = (b_1, \ldots, b_{n-1}, *), \]
\[ \{\mu - \beta - \rho_n\} = (c_1, \ldots, c_{n-1}, *), \]
where the last entries are omitted since they do not affect the parings with $\rho_c = (n - 1, \ldots, 1, 0)$. Since $b_1 \geq c_1, \ldots, b_{n-1} \geq c_{n-1}$, we have
\[ 2(\rho_c, \{\mu - \rho_n\} - \{\mu - \beta - \rho_n\}) \geq 0. \]

Therefore, the LHS of (23) is positive for $j = 0$. The same proof shows that the LHS of (23) is positive for $j = 1$ as well. Thus (6) holds.

6.2. $\mathfrak{sl}(2n + 1, \mathbb{R})$. This subsection aims to handle $\mathfrak{sl}(2n + 1, \mathbb{R})$, whose Vogan diagram is presented in Fig. 3. In this case, one calculates that
\[ \Delta^+(\mathfrak{t}, t_f) = \{e_i \pm e_j \mid 1 \leq i < j \leq n\} \cup \{e_1, \ldots, e_n\} \]
and that
\[ \Delta^+(\mathfrak{p}, t_f) = \Delta^+(\mathfrak{t}, t_f) \cup \{2e_i \mid 1 \leq i \leq n\}. \]

We have $W(\mathfrak{g}, t_f)^1 = \{e\}, \quad \beta = 2e_1,$
\[ \rho_c = (n - 1/2, n - 3/2, \ldots, 1/2), \quad \rho_n = (n + 1/2, n - 1/2, \ldots, 3/2). \]
Moreover, $\xi_i = \omega_i = e_1 + \cdots + e_i, 1 \leq i \leq n - 1$, and $\xi_n = \omega_n = \frac{1}{2}(e_1 + \cdots + e_n)$. 
Let $\mu = [m_1, \ldots, m_n]$ be a dominant weight. Then $\mu = (a_1, \ldots, a_n)$, where

$$a_i = \sum_{j=i}^{n-1} m_j + m_n/2, \quad 1 \leq i \leq n-1; \quad a_n = m_n/2.$$ 

Note that $a_1 \geq \cdots \geq a_n$ are simultaneously integers or half-integers.

Guided by Lemma 5.4, we calculate that the $\mathfrak{t}$-type $\mu$ is u-small if and only if

$$a_1 + \cdots + a_k \leq 2nk - k^2 + 2k, \quad 1 \leq k \leq n. \tag{31}$$

We claim that $\mu$ is u-small when $a_1 \leq n + 3/2$. Indeed, in this case, we would have that

$$a_1 + \cdots + a_k \leq nk + 3k/2 \leq 2nk - k^2 + 2k.$$ 

Therefore, the claim follows from (33).

Now let us assume that $\mu$ is u-large and verify (6). As mentioned above, we have $a_1 \geq n+2$. Then

$$\mu - \rho_n = (a_1 - n - 1/2, a_2 - n + 1/2, \ldots, a_n - 3/2),$$

$$\mu - \beta - \rho_n = (a_1 - n - 5/2, a_2 - n + 1/2, \ldots, a_n - 3/2),$$

and

$$\|\mu - \rho_n\|^2 - \|\mu - \rho_n - \beta\|^2 = 4(a_1 - n - 3/2) > 0. \tag{32}$$

Since $a_1 \geq n+2$, we always have $a_1 - n - 1/2 > |a_1 - n - 5/2|$. Let $B$ be the multi-set consisting of $|a_2 - n + 1/2|, \ldots, |a_n - 3/2|$. Let $b_1 \geq \cdots \geq b_n$ be the re-ordering of $a_1 - n - 1/2$ and $B$. Let $c_1 \geq \cdots \geq c_n$ be the re-ordering of $|a_1 - n - 5/2|$ and $B$. Recall that $W(t, t_f)$ consists of permutations as well as all sign changes. Thus

$$\{\mu - \rho_n\} = (b_1, \ldots, b_{n-1}, b_n),$$

$$\{\mu - \beta - \rho_n\} = (c_1, \ldots, c_{n-1}, c_n).$$

Since $b_1 \geq c_1, \ldots, b_n \geq c_n$ and these inequalities can not happen simultaneously, we have

$$2\langle \rho_c, \{\mu - \rho_n\} - \{\mu - \beta - \rho_n\} \rangle > 0.$$ 

Thus (6) follows from (32).
6.3. \(\mathfrak{sl}(n, \mathbb{H})\). This subsection aims to handle \(\mathfrak{sl}(n, \mathbb{H})\), \(n \geq 2\), whose Vogan diagram is presented in Fig. 4. In this case, one calculates that

\[
\Delta^+(p, t_f) = \{e_i \pm e_j \mid 1 \leq i < j \leq n\}
\]

and that

\[
\Delta^+(t, t_f) = \Delta^+(p, t_f) \cup \{2e_i \mid 1 \leq i \leq n\}.
\]

We have \(W(g, t_f) = 1 = \{e\}\), \(\beta = e_1 + e_2\), \(\rho_c = (n, n - 1, \ldots, 2, 1)\), \(\rho_n = (n - 1, n - 2, \ldots, 1, 0)\).

Moreover, \(\varpi_i = \xi_i = e_1 + \cdots + e_i\), 1 \(\leq i \leq n\).

\[\text{Figure 4. The Vogan diagram for } \mathfrak{sl}(n, \mathbb{H})\]

Let \(\mu = [m_1, \ldots, m_n]\) be a dominant weight. Then \(\mu = (a_1, \ldots, a_n)\), where \(a_i = \sum_{j=i}^n m_j\). Guided by Lemma 5.4, we calculate that the \(t\)-type \(\mu\) is u-small if and only if

\[
a_1 + \cdots + a_k \leq 2nk - k^2 - k, \quad 1 \leq k \leq n.
\]

We claim that \(\mu\) is u-small when \(a_1 \leq n - 1\). Indeed, in this case, we would have that

\[
a_1 + \cdots + a_k \leq (n - 1)k \leq 2nk - k^2 - k.
\]

Therefore, the claim follows from (33).

Now let us assume that \(\mu\) is u-large and verify (6). As mentioned above, we have \(a_1 \geq n\). Then

\[
\mu - \rho_n = (a_1 - n + 1, a_2 - n + 2, \ldots, a_{n-1} - 1, a_n),
\]

\[
\mu - \beta - \rho_n = (a_1 - n, a_2 - n + 1, \ldots, a_{n-1} - 1, a_n),
\]

and

\[
\|\mu - \rho_n\|^2 - \|\mu - \rho_n - \beta\|^2 = 2(a_1 + a_2 - 2n + 2).
\]

Since \(W(t, t_f)\) consists of permutations as well as all sign changes, one sees easily that

\[2\langle \rho_c, \{\mu - \rho_n\} - \{\mu - \beta - \rho_n\} \rangle > 0\]

when \(a_2 \geq n - 1\). Thus (6) follows from (34) whenever \(a_2 \geq n - 1\).

Now assume \(a_2 \leq n - 2\). Then we claim that \(a_1 \geq 2n - 1\). Indeed, otherwise,

\[
a_1 + \cdots + a_k \leq (2n - 2) + (n - 2)(k - 1) \leq 2nk - k^2 - k,
\]

and we would conclude that \(\mu\) is u-small. Thus the claim holds, and (34) says that

\[
\|\mu - \rho_n\|^2 - \|\mu - \rho_n - \beta\|^2 \geq 2.
\]
On the other hand, let \( B \) be the multi-set of \(|a_3 - n + 3|, \ldots, |a_{n-1} - 1|, |a_n|\). Denote the members of \( B \) which are greater than \( n - 2 - a_2 \) by \( b_1 \geq \cdots \geq b_k \), and collect the remaining members of \( B \) by \( c_1 \geq \cdots \geq c_t \). Here \( t + k = n - 2 \). Then
\[
\{\mu - \rho_n\} = (a_4 - n + 1, b_1, \ldots, b_k, n - 2 - a_2, c_1, \ldots, c_t),
\]
\[
\{\mu - \beta - \rho_n\} = (a_1 - n, b_1, \ldots, b_k, n - 1 - a_2, c_1, \ldots, c_t).
\]
Thus
\[
2\langle \rho_c, \{\mu - \rho_n\} - \{\mu - \beta - \rho_n\}\rangle = 2(n - (n - (k + 1))) = 2(k + 1) > 0.
\]
Therefore (6) follows from (35) whenever \( a_2 \leq n - 2 \).

To sum up, Theorem C holds for \( \mathfrak{sl}(n, \mathbb{H}), n \geq 2 \).

7. Exceptional Lie algebras

This section aims to deal with the exceptional Lie algebras for Theorem C.

7.1. EI. This subsection aims to handle EI, whose Vogan diagram is presented in Fig. 5.

The simple roots for \( \Delta^+ (\mathfrak{g}, t_f) \) are
\[
\alpha_4 := \beta_2, \quad \alpha_3 := \beta_4, \quad \alpha_2 := \frac{1}{2}(\beta_3 + \beta_5), \quad \alpha_1 := \frac{1}{2}(\beta_1 + \beta_6).
\]
The root system \( \Delta^+ (\mathfrak{g}, t_f) \) is \( F_4 \), with \( \alpha_1, \alpha_2 \) short and \( \alpha_3, \alpha_4 \) long. On the other hand, \( \Delta^+ (\mathfrak{k}, t_f) \) is \( C_4 \), and has simple roots
\[
\gamma_1 := \alpha_2 + \alpha_3 + \alpha_4, \quad \gamma_2 := \alpha_1, \quad \gamma_3 := \alpha_2, \quad \gamma_4 := \alpha_3.
\]
Here \( \gamma_4 \) is long. One calculates that \( W(\mathfrak{g}, t_f)^1 = \{e, s_{\alpha_4}, s_{\alpha_3+\alpha_4}s_{\alpha_4}\} \) and that
\[
\beta = [0, 0, 0, 1], \quad \rho_n = [5, 1, 1, 0], \quad \rho_n^{(1)} = [3, 1, 1, 1], \quad \rho_n^{(2)} = [1, 1, 3, 0].
\]

Figure 5. The Vogan diagram for EI

Guided by Lemma 5.4, we calculate that the \( \mathfrak{k} \)-type \( \mu = [a, b, c, d] \) is u-small if and only if
\[
a + 2b + 2c + 2d \leq 18, \quad 2a + 3b + 4c + 4d \leq 34, \quad 3a + 4b + 5c + 6d \leq 48, \quad a + b + c + d \leq 14, \quad a + 2b + 3c + 4d \leq 24.
\]
In particular, there are 922 u-small \( \mathfrak{k} \)-types in total.
Now let us consider the distribution of the spin norm along pencils. Let \( \mu = [a, b, c, d] \) be a u-large \( \mathfrak{g} \)-type such that \( \mu - \beta \) is dominant. Then \( a, b, c \geq 0 \) and \( d \geq 1 \). It is easy to calculate that
\[
\| \mu - \rho_n^{(j)} \|^2 - \| \mu - \beta - \rho_n^{(j)} \|^2 = \begin{cases} 
2(a + 2b + 3c + 4d - 12) & \text{if } j = 0; \\
2(a + 2b + 3c + 4d - 14) & \text{if } j = 1, 2.
\end{cases}
\]

Let us handle the term \( I \) defined in \([24]\). We claim that it suffices to use elements from \( W_1 \) to conjugate all the \( \mu - \beta - \rho_n^{(j)} \) to \( C \) when \( a \geq 9 \). Let us explain the details for \( j = 0 \). Note firstly that for any fixed \( b, c \geq 0 \) and \( d \geq 1 \), we can find \( w \in W_1 \) such that
\[
w(\mu - \beta - \rho_n) = \{ \mu - \beta - \rho_n \}
\]
when \( a \) is big enough. Secondly, let \( w_{0, 1} \) be the longest element of \( W_1 \), then
\[
\langle w_{0, 1}[a - 5, -1, -1, 0], \gamma_1 \rangle \leq \langle w[a - 5, -1, -1, 0], \gamma_1 \rangle \leq \langle w[a - 5, b - 1, c - 1, d - 1], \gamma_1 \rangle.
\]
The above first step uses Lemma 7.4 of \([7]\), while the second step uses Lemma 7.5 there. Moreover, when \( j = 1, 2 \), we have
\[
\langle w_{0, 1}[a - 3, -1, -1, 1], \gamma_1 \rangle = a - 9, \quad \langle w_{0, 1}[a - 1, -1, -3, 0], \gamma_1 \rangle = a - 9,
\]
respectively. Thus the claim holds. Similarly, it suffices to use elements from \( W_2 \) (resp., \( W_3, W_4 \)) to conjugate all the \( \mu - \beta - \rho_n^{(j)} \) to \( C \) when \( b \geq 8 \) (resp., \( c \geq 7, d \geq 8 \)).

It is direct to check that
\[
2\langle \rho_c, w\beta \rangle \geq -4, \quad \forall w \in W_1,
\]
and that
\[
2\langle \rho_c, w\beta \rangle > 0, \quad \forall w \in W_2, W_3, W_4.
\]
Note that the naive lower bound for \( I \) here is \(-2\langle \rho_c, \beta \rangle = -20\). Now in view of \([23]\) and \([36]\), the inequality \((8)\) holds whenever \( a \geq 13 \), or \( b \geq 8 \), or \( c \geq 7 \), or \( d \geq 8 \).

Finally, it remains to check \((10)\) for any u-large \( \mu = [a, b, c, d] \) such that \( 0 \leq a \leq 12, 0 \leq b \leq 7, 0 \leq c \leq 6, 1 \leq d \leq 7 \). This has been carried out on a computer. Thus Theorem C holds for EI.

7.2. EII. This subsection aims to handle EII, whose Vogan diagram is presented in Fig. 6. In this case, \( \Delta^+(\mathfrak{g}, t_f) \) is \( E_6 \), with simple roots \( \alpha_1, \ldots, \alpha_6 \). We have that \(|W(\mathfrak{g}, t_f)| = 36\). We set
\[
\gamma_i = \alpha_{7-i}, \quad 1 \leq i \leq 4; \quad \gamma_5 = \alpha_1; \quad \gamma_6 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6.
\]
Then \( \beta = [0, 0, 1, 0, 0, 1] \).
Guided by Lemma 5.4 we calculate that the $\mathfrak{t}$-type $\mu = [a, b, c, d, e, f]$ is u-small if and only if
\[
\begin{align*}
a + b + c + d + e &\leq 12, \quad 2a + 4b + 3c + 2d + e \leq 24, \\
a + 2b + 3c + 4d + 2e &\leq 24, \quad a + 2b + 3c + 4d + 5e + 3f \leq 60, \\
a + 2b + 3c + 2d + e + f &\leq 24, \quad 5a + 10b + 9c + 8d + 7e + 3f \leq 96, \\
7a + 8b + 9c + 10d + 5e + 3f &\leq 96, \quad 3a + 4b + 5c + 4d + 3c + f \leq 44.
\end{align*}
\]
In particular, there are 22122 u-small $\mathfrak{t}$-types in total.

Now let us consider the distribution of the spin norm along pencils. Let $\mu = [a, b, c, d, e, f]$ be a u-large $\mathfrak{t}$-type such that $\mu - \beta$ is dominant. Then $c, f \geq 1$ and $a, b, d, e \geq 0$. It is easy to calculate that
\[
||\mu - \rho_n^{(j)}||^2 - ||\mu - \beta - \rho_n^{(j)}||^2 \geq a + 2b + 3c + 2d + e + f - 14, \quad 0 \leq j \leq 35.
\]

Similar to the EI case, it suffices to use elements from $W_1$ (resp., $W_2, W_3, W_4, W_5, W_6$) to conjugate all these $\mu - \beta - \rho_n^{(j)}$ to $C$ when $a \geq 6$ (resp., $b \geq 6, c \geq 7, d \geq 6, e \geq 6, f \geq 11$). Moreover, we have that
\[
2\langle \rho_c, w\beta \rangle \geq -4, \forall w \in W_1, W_5; \quad 2\langle \rho_c, w\beta \rangle \geq -8, \forall w \in W_6,
\]
and that
\[
2\langle \rho_c, w\beta \rangle > 0, \forall w \in W_2, W_3, W_4.
\]
Now in view of (23) and (37), the inequality (3) holds whenever $a \geq 15, b \geq 6, c \geq 7$, or $d \geq 6, e \geq 15, f \geq 20$.

Finally, it remains to check (5) for any u-large $\mu = [a, b, c, d, e, f]$ such that $0 \leq a, e \leq 14$, $0 \leq b, d \leq 5, 1 \leq c \leq 6, 1 \leq f \leq 19$. This has been carried out on a computer. Thus Theorem C holds for EII.

7.3. EIV. This subsection aims to handle EIV, whose Vogan diagram is presented in Fig. 7. In this case, $\Delta^+ (p, t_f) \subset \Delta^+ (\mathfrak{g}, t_f)$. Thus both $\Delta^+ (\mathfrak{g}, t_f)$ and $\Delta^+ (\mathfrak{t}, t_f)$ are $F_4$, with simple roots
\[
\alpha_4 := \beta_2, \quad \alpha_3 := \beta_4, \quad \alpha_2 := \frac{1}{2} (\beta_3 + \beta_5), \quad \alpha_1 := \frac{1}{2} (\beta_1 + \beta_6).
\]
Here $\alpha_1, \alpha_2$ are short, while $\alpha_3, \alpha_4$ are long. We identify $\gamma_i$ with $\alpha_i, 1 \leq i \leq 4$. We have that $W(\mathfrak{g}, t_f)^1 = \{e\}$ and that
\[
\beta = [1, 0, 0, 0], \quad \rho_n = [1, 1, 0, 0].
\]
Guided by Lemma \ref{lem:5.4}, we calculate that the $t$-type $\mu = [a, b, c, d]$ is u-small if and only if
\[2a + 3b + 4c + 2d \leq 10, \quad a + 2b + 3c + 2d \leq 6.\]
In particular, there are 37 u-small $t$-types in total. As observed in the author’s thesis, any $t$-type whose spin norm is upper bounded by $||\rho||$ must be u-small.

Now let us consider the distribution of the spin norm along pencils. Let $\mu = [a, b, c, d]$ be a u-large $t$-type such that $\mu - \beta$ is dominant. Then $a \geq 1$ and $b, c, d \geq 0$. Similar to the EI case, it suffices to use elements from $W_1$ (resp., $W_2, W_3, W_4$) to conjugate $\mu - \beta - \rho_n$ to $C$ when $a \geq 5$ (resp., $b \geq 2, c \geq 2, d \geq 3$). Let $w \in W_k$ be an element such that
\[
\{\mu - \rho_n\} = w(\mu - \beta - \rho_n).
\]
Since $\beta = 2\alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4$, by the technique of Lemma 7.3 of \cite{7}, we have that $w\beta \in \Delta^+ (g, t_f)$ for any $w \in W_k, 1 \leq k \leq 4$. Moreover,
\[
\{\mu - \rho_n\} - \{\mu - \beta - \rho_n\} = (\{\mu - \rho_n\} - w(\mu - \rho_n)) + w\beta.
\]
Thus (6) holds whenever $a \geq 5$, or $b \geq 2$, or $c \geq 2$, or $d \geq 3$.

Finally, it remains to check (6) for any u-large $\mu = [a, b, c, d]$ such that $1 \leq a \leq 4, 0 \leq b \leq 1, 0 \leq c \leq 1, 0 \leq d \leq 2$. This has been carried out on a computer. Thus Theorem C holds for EIV.

### 7.4. EV
This subsection aims to handle EV, whose Vogan diagram is presented in Fig. 8.

In this case, $\Delta^+(g, t_f)$ is $E_7$, with simple roots $\alpha_1, \ldots, \alpha_7$. Moreover, $\Delta^+(t, t_f)$ is $A_7$, with simple roots
\[
\gamma_1 := \alpha_1; \quad \gamma_i := \alpha_{i+1}, 2 \leq i \leq 6; \quad \gamma_7 := \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6.
\]
We have that $|W(g, t_f)| = 72$ and that $\beta = [0, 0, 0, 1, 0, 0, 0]$.

Guided by Lemma \ref{lem:5.4}, we calculate that there are 187200 u-small $t$-types in total.
Now let us consider the distribution of the spin norm along pencils. Let
\[ \mu = [a, b, c, d, e, f, g] \]
be a u-large \( \mathfrak{k} \)-type such that \( \mu - \beta \) is dominant. Then \( d \geq 1 \) and \( a, b, c, e, f, g \geq 0 \). One calculates that
\[
\|\mu - \rho_n(j)\|^2 - \|\mu - \beta - \rho_n(j)\|^2 \geq a + 2b + 3c + 4d + 3e + 2f + g - 20, \quad 0 \leq j \leq 71.
\]

The entry \( a \geq 10 \) in the following table means that it suffices to use \( w \in W_1 \) to conjugate all the \( \mu - \beta - \rho_n(j) \) to \( C \) whenever \( a \geq 10 \). Other entries of the first line are interpreted similarly.

| \( W_k \)-bound \( \frac{2}{2(\rho_c, w_{0,k}\beta)} \) | \( a \geq 10 \) | \( b \geq 10 \) | \( c \geq 10 \) | \( d \geq 11 \) | \( e \geq 10 \) | \( f \geq 10 \) | \( g \geq 10 \) |
|---|---|---|---|---|---|---|---|
| -8 | 0 | 8 | 8 | 16 | 8 | 0 | -8 |

Now in view of the above table, (23), (28), and (38), the inequality (6) holds whenever \( a \geq 25 \), or \( b \geq 10 \), or \( c \geq 10 \), or \( d \geq 11 \), or \( e \geq 10 \), or \( f \geq 10 \), or \( g \geq 25 \).

Finally, it remains to check (6) for any u-large \( \mu = [a, b, c, d, e, f, g] \) such that \( 0 \leq a, g \leq 24 \), \( 0 \leq b, c, e, f \leq 9 \), \( 1 \leq d \leq 10 \). This has been carried out on a computer. Thus Theorem C holds for EV.

7.5. EVI. This subsection aims to handle EVI, whose Vogan diagram is presented in Fig. 9.

In this case, \( \Delta^+(\mathfrak{g}, \mathfrak{t}_f) \) is \( E_7 \), with simple roots \( \alpha_1, \ldots, \alpha_7 \). Moreover, \( \Delta^+(\mathfrak{k}, \mathfrak{t}_f) \) is \( D_6 \times A_1 \), with simple roots
\[
\gamma_i := \alpha_{8-i}, \quad 1 \leq i \leq 4; \quad \gamma_5 = \alpha_2; \quad \gamma_6 = \alpha_3; \quad \gamma_7 := 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7.
\]

We have that \( |W(\mathfrak{g}, \mathfrak{t}_f)^1| = 63 \) and that \( \beta = [0, 0, 0, 0, 1, 1, 1] \).

![Figure 9. The Vogan diagram for EVI](image)

Guided by Lemma 5.4, we calculate that there are 105495 u-small \( \mathfrak{k} \)-types in total.

Now let us consider the distribution of the spin norm along pencils. Let
\[ \mu = [a, b, c, d, e, f, g] \]
be a u-large \( \mathfrak{k} \)-type such that \( \mu - \beta \) is dominant. Then \( f, g \geq 1 \) and \( a, b, c, d, e \geq 0 \). One calculates that
\[
\|\mu - \rho_n(j)\|^2 - \|\mu - \beta - \rho_n(j)\|^2 \geq a + 2b + 3c + 4d + 3e + 2f + g - 20, \quad 0 \leq j \leq 62.
\]

The entry \( a \geq 8 \) in the following table means that it suffices to use \( w \in W_1 \) to conjugate all the \( \mu - \beta - \rho_n(j) \) to \( C \) whenever \( a \geq 8 \). Other entries of the first line are interpreted similarly.
Now in view of the above table, \((23), (28), \) and \((38),\) the inequality \((6)\) holds whenever 
\(a \geq 23,\) or \(b \geq 8,\) or \(c \geq 8,\) or \(d \geq 8,\) or \(e \geq 8,\) or \(f \geq 9,\) or \(g \geq 32.\)

Finally, it remains to check \((6)\) for any \(u\)-large \(\mu = [a, b, c, d, e, f, g]\) such that \(0 \leq a \leq 22,\) 
\(0 \leq b, c, d, e \leq 7,\) \(1 \leq f \leq 8,\) \(1 \leq g \leq 31.\) This has been carried out on a computer. Thus 
Theorem C holds for EVI.

7.6. EVIII. This subsection aims to handle EVIII, whose Vogan diagram is presented in 
Fig. 10. In this case, \(\Delta^+(g, tf)\) is \(E_8,\) with simple roots \(\alpha_1, \ldots, \alpha_8\). Moreover, \(\Delta^+(k, tf)\) is 
\(D_8,\) with simple roots 
\[\gamma_1 := 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7; \quad \gamma_i := \alpha_{10-i}, \quad 2 \leq i \leq 6; \quad \gamma_7 = \alpha_2; \quad \gamma_8 = \alpha_3.\]

We have that \(|W(g, tf)| = 135\) and that \(\beta = [0, 0, 0, 0, 0, 0, 1, 0].\)

\[\begin{array}{cccccccc}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 & \alpha_8 \\
\end{array}\]

**Figure 10.** The Vogan diagram for EVIII

Guided by Lemma 5.4, we calculate that there are 1379322 \(u\)-small \(\mathfrak{k}\)-types in total. 
Now let us consider the distribution of the spin norm along pencils. Let 
\[\mu = [a, b, c, d, e, f, g, h]\]
be a \(u\)-large \(\mathfrak{k}\)-type such that \(\mu - \beta\) is dominant. Then \(g \geq 1\) and \(a, b, c, d, e, f, h \geq 0.\) One 
calculates that

\[(40) \| \mu - \rho_n^{(j)} \|^2 - \| \mu - \beta - \rho_n^{(j)} \|^2 \geq a + 2b + 3c + 4d + 5e + 6f + 4g + 3h - 32, \quad 0 \leq j \leq 134.\]

The entry \(a \geq 16\) in the following table means that it suffices to use \(w \in W_1\) to conjugate 
all the \(\mu - \beta - \rho_n^{(j)}\) to \(C\) whenever \(a \geq 16.\) Other entries of the first line are interpreted 
similarly.

| \(W_k\)-bound | \(a \geq 16\) | \(b \geq 16\) | \(c \geq 16\) | \(d \geq 16\) | \(e \geq 16\) | \(f \geq 16\) | \(g \geq 17\) | \(h \geq 16\) |
|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| \(2\langle \rho_c, w_{0,k}\beta \rangle\) | \(-14\) | \(-2\) | \(8\) | \(16\) | \(22\) | \(26\) | \(28\) | \(14\) |

Now in view of the above table, \((23), (28), \) and \((10),\) the inequality \((6)\) holds whenever 
\(a \geq 43,\) or \(b \geq 16,\) or \(c \geq 16,\) or \(d \geq 16,\) or \(e \geq 16,\) or \(f \geq 16,\) or \(g \geq 17,\) or \(h \geq 16.\)

Finally, it remains to check \((6)\) for any \(u\)-large \(\mu = [a, b, c, d, e, f, g, h]\) such that \(0 \leq a \leq 42,\) 
\(0 \leq b, c, d, e, f, h \leq 15,\) \(1 \leq g \leq 16.\) This has been done by Mathematica in about 24 hours. 
Thus Theorem C holds for EVIII.
7.7. EIX. This subsection aims to handle EIX, whose Vogan diagram is presented in Fig. [11]. In this case, $\Delta^+(g, t_f)$ is $E_8$, with simple roots $\alpha_1, \ldots, \alpha_8$. Moreover, $\Delta^+(t, t_f)$ is $E_7 \times A_1$, with simple roots

$$\gamma_i := \alpha_i, \quad 1 \leq i \leq 7; \quad \gamma_8 := 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8.$$ 

We have that $|W(g, t_f)\rangle = 120$ and that $\beta = [0, 0, 0, 0, 0, 1, 1]$. 

![Figure 11. The Vogan diagram for EIX](image)

Guided by Lemma 7.4, we calculate that there are $577367$ u-small $t$-types in total. Now let us consider the distribution of the spin norm along pencils. Let $\mu = [a, b, c, d, e, f, g, h]$ be a u-large $t$-type such that $\mu - \beta$ is dominant. Then $g, h \geq 1$ and $a, b, c, d, e, f \geq 0$. One calculates that

$$\text{(41) } \|\mu - \rho_n^{(j)}\|^2 - \|\mu - \beta - \rho_n^{(j)}\|^2 \geq 2a + 3b + 4c + 6d + 5e + 4f + 3g + h - 32, \quad 0 \leq j \leq 119.$$ 

The entry $a \geq 12$ in the following table means that it suffices to use $w \in W_1$ to conjugate all the $\mu - \beta - \rho_n^{(j)}$ to $C$ whenever $a \geq 12$. Other entries of the first line are interpreted similarly.

| $W_k$-bound | $a \geq 12$ | $b \geq 12$ | $c \geq 10$ | $d \geq 10$ | $e \geq 10$ | $f \geq 12$ | $g \geq 13$ | $h \geq 29$ |
|-------------|------------|------------|------------|------------|------------|------------|------------|------------|
| $2(\rho_c, w_0, k, \beta)$ | 6 | 14 | 16 | 20 | 22 | 24 | 26 | -26 |

Now in view of the above table, (23), (28), and (11), the inequality (6) holds whenever $a \geq 12$, or $b \geq 12$, or $c \geq 10$, or $d \geq 10$, or $e \geq 10$, or $f \geq 12$, or $g \geq 13$, or $h \geq 56$.

Finally, it remains to check (23) for any u-large $\mu = [a, b, c, d, e, f, g, h]$ such that $0 \leq a \leq 11$, $0 \leq b, f \leq 11$, $0 \leq c, d, e \leq 9$, $1 \leq g \leq 12$, $1 \leq h \leq 55$. This has been done by Mathematica in about 22 hours. Thus Theorem C holds for EIX.

7.8. FI. This subsection aims to handle FI, whose Vogan diagram is presented in Fig. [12]. In this case, $\Delta^+(g, t_f)$ is $F_4$, with simple roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. Here $\alpha_1, \alpha_2$ are short, while $\alpha_3, \alpha_4$ are long. Let $s_i$ stand for $s_{\alpha_i}$. We have that

$$W(g, t_f)\rangle = \{e, s_4, s_{4s_3}, s_{4s_3s_2}, s_{4s_3s_2s_1}, s_{s_4s_3s_2s_1s_3}, s_{s_4s_3s_2s_1s_3s_4}, s_{s_4s_3s_2s_1s_3s_4}, s_{s_4s_3s_2s_1s_3s_4}, s_{s_4s_3s_2s_1s_3s_4}, s_{s_4s_3s_2s_1s_3s_4}, s_{s_4s_3s_2s_1s_3s_4}\}.$$ 

We set $\gamma_i = \alpha_i, \quad 1 \leq i \leq 3; \quad \gamma_4 = 2\alpha_1 + 4\alpha_2 + 3\alpha_3 + 2\alpha_4$. 
Then $\beta = [0, 0, 1, 1]$, and
\[
\rho_n = [0, 0, 0, 7], \quad \rho_n^{(1)} = [0, 0, 1, 6], \quad \rho_n^{(2)} = [0, 2, 0, 5], \quad \rho_n^{(3)} = [1, 2, 0, 4], \\
\rho_n^{(4)} = [0, 3, 0, 3], \quad \rho_n^{(5)} = [3, 0, 1, 3], \quad \rho_n^{(6)} = [2, 1, 1, 2], \quad \rho_n^{(7)} = [5, 0, 0, 2], \\
\rho_n^{(8)} = [2, 0, 2, 1], \quad \rho_n^{(9)} = [4, 1, 0, 1], \quad \rho_n^{(10)} = [0, 0, 3, 0], \quad \rho_n^{(11)} = [4, 0, 1, 0].
\]

![Figure 12. The Vogan diagram for FI](image)

Guided by Lemma \[5.4\], we calculate that the $\kappa$-type $\mu = [a, b, c, d]$ is $u$-small if and only if
\[
\begin{align*}
 a + b + c &\leq 10, & a + 2b + 2c &\leq 12, & a + b + c + d &\leq 14, \\
 3a + 4b + 5c + d &\leq 34, & 2a + 3b + 3c + d &\leq 24, & a + 2b + 3c + d &\leq 18.
\end{align*}
\]

In particular, there are 1045 $u$-small $\kappa$-types in total.

Now let us consider the distribution of the spin norm along pencils. Let $\mu = [a, b, c, d]$ be a $u$-large $\kappa$-type such that $\mu - \beta$ is dominant. Then $c, d \geq 1$ and $a, b \geq 0$. It is easy to calculate that
\[
\|\mu - \rho_n^{(j)}\|^2 - \|\mu - \beta - \rho_n^{(j)}\|^2 = \begin{cases} a + 2b + 3c + d - 9 & \text{if } j = 0, 7, 9, 11; \\
a + 2b + 3c + d - 11 & \text{otherwise.}
\end{cases}
\]

Similar to the EI case, it suffices to use elements from $W_1$ (resp., $W_2, W_3, W_4$) to conjugate all these $\mu - \beta - \rho_n^{(j)}$ to $C$ when $a \geq 8$ (resp., $b \geq 8, c \geq 6, d \geq 8$). Moreover, we have
\[
2\langle \rho_c, w\beta \rangle \geq -1, \forall w \in W_1; \quad 2\langle \rho_c, w\beta \rangle \geq -5, \forall w \in W_4; \quad 2\langle \rho_c, w\beta \rangle > 0, \forall w \in W_2, W_3.
\]

Now in view of (23) and (12), the inequality (6) holds whenever $a \geq 9$, or $b \geq 8$, or $c \geq 6$, or $d \geq 14$.

Finally, it remains to check (5) for any $u$-large $\mu = [a, b, c, d]$ such that $0 \leq a \leq 8, 0 \leq b \leq 7, 1 \leq c \leq 5, 1 \leq d \leq 13$. This has been carried out on a computer. Thus Theorem C holds for FI.

7.9. FII. This subsection aims to handle FII, whose Vogan diagram is presented in Fig. 13.

In this case, $\Delta^+(\mathfrak{g}, t_f)$ is $F_4$, with simple roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. Here $\alpha_1, \alpha_2$ are short, while $\alpha_3, \alpha_4$ are long. We have that $W(\mathfrak{g}, t_f)^1 = \{e, s_{\alpha_1}, s_{\alpha_3}s_{\alpha_2}\}$ and that
\[
\beta = [0, 0, 0, 1], \quad \rho_n = [2, 0, 0, 0], \quad \rho_n^{(1)} = [1, 0, 0, 1], \quad \rho_n^{(2)} = [0, 0, 1, 0].
\]

![Figure 13. The Vogan diagram for FII](image)
Guided by Lemma 5.4, we calculate that the \( t \)-type \( \mu = [a, b, c, d] \) is u-small if and only if
\[
a + 2b + 2c + d \leq 4, \quad a + 2b + 3c + 2d \leq 6.
\]
In particular, there are 27 u-small \( t \)-types in total. As observed in the author’s thesis, any \( t \)-type whose spin norm is upper bounded by \( \| \rho \| \) must be u-small.

Now let us consider the distribution of the spin norm along pencils. Let \( \mu = [a, b, c, d] \) be a u-large \( t \)-type such that \( \mu - \beta \) is dominant. Then \( d \geq 1 \) and \( a, b, c \geq 0 \). It is easy to calculate that
\[
||\mu - \rho_n^{(j)}||^2 - ||\mu - \beta - \rho_n^{(j)}||^2 = \begin{cases} 
   a + 2b + 3c + 2d - 3 & \text{if } j = 0; \\
   a + 2b + 3c + 2d - 4 & \text{if } j = 1, 2.
\end{cases}
\]

Similar to the EI case, it suffices to use elements from \( W_1 \) (resp., \( W_2, W_3, W_4 \)) to conjugate all these \( \mu - \beta - \rho_n^{(j)} \) to \( C \) when \( a \geq 3 \) (resp., \( b \geq 3, c \geq 3, d \geq 5 \)). Moreover, we have
\[
2\langle \rho_c, w\beta \rangle \geq -1, \quad \forall w \in W_1; \quad 2\langle \rho_c, w\beta \rangle > 0, \quad \forall w \in W_2, W_3, W_4.
\]

Now in view of (23) and (43), the inequality (6) holds whenever \( a \geq 4 \), or \( b \geq 3 \), or \( c \geq 3 \), or \( d \geq 5 \).

Finally, it remains to check (4) for any u-large \( \mu = [a, b, c, d] \) such that \( 0 \leq a \leq 3 \), \( 0 \leq b, c \leq 2 \), \( 1 \leq d \leq 4 \). This has been carried out on a computer. Thus Theorem C holds for FII.

7.10. \( G \). This subsection aims to handle \( G \), whose Vogan diagram is presented in Fig. 14, where \( \alpha_1 = (1, -1, 0) \) is short, while \( \alpha_2 = (-2, 1, 1) \) is long. In this case, \( \Delta^+(g, t_f) \) is \( G_2 \), while \( \Delta^+(t, t_f) \) is \( A_1 \times A_1 \). Indeed, \( \Delta^+(t, t_f) \) consists of two orthogonal roots: \( \gamma_1 := \alpha_1 \), \( \gamma_2 := 3\alpha_1 + 2\alpha_2 \). One calculates that
\[
\xi_1 = 2\alpha_1 + \alpha_2, \quad \xi_2 = 3\alpha_1 + 2\alpha_2,
\]
and that
\[
\varpi_1 = (1/2, -1/2, 0), \quad \varpi_2 = (-1/2, -1/2, 1).
\]
Moreover, we have that \( W(g, t_f)^1 = \{e, s_{\alpha_2}, s_{\alpha_1 + \alpha_2}, s_{\alpha_2}\} \), and that
\[
\beta = [3, 1], \quad \rho_n = [0, 2], \quad \rho_n^{(1)} = [3, 1], \quad \rho_n^{(2)} = [4, 0].
\]

![Figure 14. The Vogan diagram for G](image)

Guided by Lemma 5.4, we calculate that the \( t \)-type \( \mu = [a, b] \) is u-small if and only if
\[
a + 3b \leq 12, \quad a + b \leq 8.
\]
Thus we can draw the picture of the u-small convex hull as in Fig. [1]. In particular, there are 29 u-small \( t \)-types in total. Note that any \( t \)-type whose spin norm is upper bounded by \( \| \rho \| \) must be u-small.
Now let us consider the distribution of the spin norm along pencils. It is easy to calculate that

\[(44) \quad \|\mu - \rho_n^{(j)}\|^2 - \|\mu - \beta - \rho_n^{(j)}\|^2 = \begin{cases} 
3(a + b - 4) & \text{if } j = 0; \\
3(a + b - 6) & \text{if } j = 1, 2.
\end{cases}\]

Assume that \(\mu = [a, b]\) is u-large and that \(\mu - \beta = [a - 3, b - 1]\) is dominant. In particular, \(a \geq 3, b \geq 1\). Note that

\[\mu - \beta - \rho_n = [a - 3, b - 3], \quad \mu - \beta - \rho_n^{(1)} = [a - 6, b - 2], \quad \mu - \beta - \rho_n^{(2)} = [a - 7, b - 1].\]

The above expressions suggest that when \(a \geq 7\) (resp. \(b \geq 3\)), we need only to use elements in the parabolic subgroup \(W_1 = \{e, s_{\gamma_2}\}\) (resp., \(W_2 = \{e, s_{\gamma_1}\}\)) of \(W(k, t_f)\) to conjugate all these \(\mu - \beta - \rho_n^{(j)}\) to \(C\). It is direct to check that

\[2\langle \rho_c, w\beta \rangle \geq 0, \quad \forall w \in W_1, W_2.\]

Note that the naive lower bound for \(I\) here is \(-2\langle \rho_c, \beta \rangle = -6\). Now in view of (23) and (44), the inequality (6) holds whenever \(a \geq 7\), or \(b \geq 4\).

Finally, the inequality (6) has been checked for any u-large \(\mu = [a, b]\) such that \(3 \leq a \leq 6\) and \(1 \leq b \leq 3\). Thus Theorem C holds for \(G\).

8. \(\text{Sp}(4, \mathbb{R})\)

This section aims to consider \(G = \text{Sp}(4, \mathbb{R})\). Then \(K(\mathbb{R}) = U(2)\) and \(T(\mathbb{R})_f = U(1) \times U(1)\). Thus \(\mathfrak{t}\) has center. We fix

\[\Delta^+(\mathfrak{t}, t_f) = \{(1, -1)\}, \quad \Delta^+(\mathfrak{p}, t_f) = \{(2, 0), (0, 2), (1, 1)\}.
\]

Note that \(\beta_1 = (2, 0)\) and \(\beta_2 = (0, -2)\) are the two highest weights of the \(K(\mathbb{R})\)-representation of \(\mathfrak{p}\). Moreover, one calculates that

\[\rho_n = (3/2, 3/2), \quad \rho_n^{(1)} = (3/2, -1/2), \quad \rho_n^{(2)} = (1/2, -3/2), \quad \rho_n^{(3)} = (-3/2, -3/2).
\]

Let \((p, q)\) denote the highest weight of a \(K(\mathbb{R})\)-type, where \(p \geq q\) are two integers. Then by Example 6.3 of [27], we have that the \(K(\mathbb{R})\)-type \((p, q)\) is u-small if and only if

\[p \leq 3, \quad -3 \leq q, \quad p - q \leq 4.
\]

Thus we can draw \(R(\Delta(\mathfrak{p}, t_f)) \cap C\) in Fig. 15, where the dotted circles stand for u-small \(K(\mathbb{R})\)-types. Note that \(2\rho_n, 2\rho_n^{(1)}, 2\rho_n^{(2)}, 2\rho_n^{(3)}\) there are extremal points of the u-small convex hull. There are 25 u-small \(K(\mathbb{R})\)-types in total.
Take $\mu_m = (-m + 2, -m)$, where $m \in \mathbb{Z}_{>0}$. Assume that $m$ is big enough. Then $\mu_m$ is $u$-large. Moreover, $\mu_m - \beta_1 = (-m, -m)$ and $\mu_m - \beta_2 = (-m + 2, -m + 2)$ are both dominant. One calculates that
\[
\|\mu_m - \beta_2\|_{\text{spin}} = \sqrt{2m^2 - 14m + 25} < \|\mu_m\|_{\text{spin}} = \sqrt{2m^2 - 10m + 17} < \|\mu_m - \beta_1\|_{\text{spin}} = \sqrt{2m^2 - 6m + 5}.
\]

Similarly, take $\mu'_m = (m + 2, m)$, where $m \in \mathbb{Z}_{>0}$. Assume that $m$ is big enough. Then $\mu'_m$ is $u$-large. Moreover, $\mu'_m - \beta_1 = (m, m)$ and $\mu'_m - \beta_2 = (m + 2, m + 2)$ are both dominant. One calculates that
\[
\|\mu'_m - \beta_1\|_{\text{spin}} = \sqrt{2m^2 - 6m + 5} < \|\mu'_m\|_{\text{spin}} = \sqrt{2m^2 + 2} < \|\mu'_m - \beta_2\|_{\text{spin}} = \sqrt{2m^2 + 2m + 1}.
\]

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