We study the medial axis of a set $K$ in Euclidean space (the set of points in space with more than one closest point in $K$) from a “coarse” and “quantitative” perspective. We show that on “most” balls $B(x, r)$ in the complement of $K$, the set of almost-closest points to $x$ in $K$ takes up a small angle as seen from $x$. In other words, most locations and scales in the complement of $K$ “appear” to fall outside the medial axis if one looks with only a certain finite resolution. The word “most” involves a Carleson packing condition, and our bounds are independent of the set $K$.

1. Introduction

If $K \subseteq \mathbb{R}^k$, the distance from a point $p \in \mathbb{R}^k$ to the set $K$ is
\[ d(p, K) = \inf \{ d(p, x) : x \in K \}, \]
where $d(p, x)$ denotes the Euclidean distance $|p - x|$. If $K$ is closed and $p \in \mathbb{R}^k$, then there is always a point $x \in K$ such that $d(p, x) = d(p, K)$, but this point $x$ may not be unique.

The set of points $p \in \mathbb{R}^k$ for which this closest point $x \in K$ is not unique is called the medial axis of $K$, which we denote Med($K$):

Definition 1.1. Given $K \subseteq \mathbb{R}^k$, let
\[ \text{Med}(K) = \{ p \in \mathbb{R}^k : \text{there exist } x, y \in K \text{ with } x \neq y \text{ and } d(p, K) = d(p, x) = d(p, y) \}. \]

The medial axis has a fairly long history in both pure and applied mathematics; the results of Erdös [6] appear before even the name “medial axis” was coined by Blum [2]. A good overview from the pure mathematical perspective can be found in the introduction and references of [9].

For one thing, it is well-known that the medial axis of any closed set $K$ has measure zero. A short proof of this fact can be given by applying Rademacher’s theorem on the differentiability of Lipschitz functions to the Lipschitz function $x \mapsto \text{dist}(x, K)$. See [12] or [9, Remark 13] for details. In fact, much stronger results than this can be proven on the smallness of the medial axis: see [6, 7].

In this paper, we prove that the medial axis is small from a “coarse” or “quantitative” perspective. In essence, our main result (Theorem 1.3 below) says that given a compact set $K \subseteq \mathbb{R}^k$, the set of locations and scales in the complement of $K$ that “appear” to be in the medial axis is “small”, with a control which is independent of the set $K$. The words “small” and “appear” need some further elaboration.
To measure the size of a collection of locations and scales in $\mathbb{R}^k$, we use the notion of a Carleson set:

**Definition 1.2.** Let $D \subseteq \mathbb{R}^k \times \mathbb{R}^+$ be measurable. Let $D_r = \{x : (x, r) \in D\}$. We say that $D$ is a Carleson set if there is a $C \geq 0$ such that for every $L > 0$ and every ball $B \subseteq \mathbb{R}^k$ of radius $L$,

$$\int_0^L |D_r \cap B| \frac{dr}{r} \leq C|B|,$$

where $|\cdot|$ denotes the Lebesgue measure of a set in $\mathbb{R}^k$. We call the minimal $C$ for which this is satisfied the Carleson constant of $D$.

This definition plays a major role in the area of quantitative geometric measure theory developed by David and Semmes [4]. Roughly speaking, if one thinks of $D \subseteq \mathbb{R}^k \times \mathbb{R}^+$ as a collection of balls in $\mathbb{R}^k$ (centers and radii), the Carleson condition says that this is a “small” collection: most points of $\mathbb{R}^k$ are not contained in too many balls of $D$ of very different radii. In particular, if $D$ is Carleson then every ball in $\mathbb{R}^k$ contains a ball of comparable size lying outside the collection $D$. A nice discussion of this concept is given by Semmes in [8, B.29].

Our main theorem considers the set of all balls $B(x, r)$ in the complement of a given set $K \subseteq \mathbb{R}^k$. Some of these balls may have the property that there are two points $z_1$ and $z_2$ in $K$ such that

- $z_1$ and $z_2$ both “almost” minimize the distance to $x$ in $K$, i.e.,
  $$d(x, z_1) \leq d(x, K) + \epsilon r$$
  for some small $\epsilon > 0$, and
- the angle between the segments $[x, z_1]$ and $[x, z_2]$ is “large”, i.e., bounded away from zero in some quantitative way.

We think of a ball satisfying these conditions as representing a point that “appears” to be in the medial axis if one looks with only some finite degree of resolution.

Our main theorem says that such balls are rare: they form a Carleson set. Moreover, the Carleson constant $C$ is completely independent of the set $K$, depending only on the dimension and the chosen parameters.

**Theorem 1.3.** Let $K \subseteq \mathbb{R}^k$ and let $\epsilon \geq 0$ and $\delta > 0$ satisfy $2\delta + \epsilon < 1$. Let

$$G = \{(x, r) : x \in \mathbb{R}^k, 0 < r < d(x, K), \text{ and there exist } z_1, z_2 \in K \text{ such that} \}
$$

$$d(x, z_1), d(x, z_2) \leq d(x, K) + \epsilon r$$

but $|\theta| > \cos^{-1} \left( 2 \left( \frac{1 - (2\delta + \epsilon)}{1 + 2\delta} \right)^2 - 1 \right)$

where $\theta$ is the angle between $[x, z_1]$ and $[x, z_2]$. Then $G$ is a Carleson set whose Carleson constant can be bounded above depending only on $\delta$ and $k$.

Note that the quantity $\cos^{-1} \left( 2 \left( \frac{1 - (2\delta + \epsilon)}{1 + 2\delta} \right)^2 - 1 \right)$ is positive if $\delta > 0$ and tends to 0 as $\delta, \epsilon \to 0$. The theorem is already of interest in the case $\epsilon = 0$, in which case it more directly concerns the medial axis of $K$. 

Thus, Theorem 1.3 gives a precise sense in which every medial axis is seen only at a small collection of locations and scales, in a way which is robust and independent of the base set $K$.

The proof of Theorem 1.3 uses the philosophy of the proof mentioned above that the medial axis has measure zero, which applies Rademacher’s theorem to the distance function to $K$ [9, Remark 13]. However, instead of using Rademacher’s theorem, which provides only infinitesimal and not “coarse” information, we use a result from the theory of “quantitative differentiation”, explained in Section 2. In Section 3 we apply this theorem along with some quantitative estimates on the distance function to prove Theorem 1.3.

To conclude the introduction, we emphasize that Theorem 1.3 is not implied by the fact that the medial axis has measure zero, nor even by the stronger results of [6, 7] mentioned earlier, because those results make no statements about the “large scale” structure of the medial axis.

2. Background on Quantitative Differentiation

The theory of “quantitative differentiation” considers approximation by Lipschitz functions at “large” scales, not just infinitesimal ones. It originates in work of Dorronsoro [5] and Jones [11], with extensions by many others since. Good overviews of the material we need can be found in Appendix B by Semmes in [8] (especially Section B.29), the notes of Young [13], or the master’s thesis of the second named author [10].

With the dimension $k$ understood from context, let $B(x, r)$ denote the closed ball of radius $r$ centered at $x \in \mathbb{R}^k$. We use $\mathbb{R}^+$ below to denote the positive real numbers.

**Definition 2.1.** Let $f : \mathbb{R}^k \to \mathbb{R}$ and $\epsilon > 0$. We say that $f$ is $\epsilon$-coarsely-differentiable on $B(x, r)$ if there is an affine function $\lambda : \mathbb{R}^k \to \mathbb{R}$ such that

$$|f(p) - \lambda(p)| \leq \epsilon r$$

for all $p \in B(x, r)$.

The main result we need is that every 1-Lipschitz function is coarsely differentiable on all balls outside of a Carleson set.

**Theorem 2.2.** Let $\epsilon > 0$. Let $f : \mathbb{R}^k \to \mathbb{R}$ be 1-Lipschitz. Let

$$G = \{(x, r) \in \mathbb{R}^k \times \mathbb{R}^+ : f \text{ is not } \epsilon\text{-coarsely differentiable on } B(x, r)\}.$$

Then $G$ is Carleson, and its Carleson constant can be bounded above depending only on $\epsilon$ and $k$.

As Semmes remarks in [8, B.29], it is difficult to trace the attribution of this precise result. It follows from the main results of [5], as discussed in [4]. It is stated explicitly as [8, Theorem B.29.10] and as [13, Theorem 2.4]. An exposition of the proof (following that of Young [13]) is given in [10], and a generalization to metric space targets is given in [11]. Further generalizations and analogs are discussed in [3].

3. Proof of the Main Theorem

The main work in the proof of Theorem 1.3 lies in the following result.
Theorem 3.1. Let $K \subseteq \mathbb{R}^k$ be compact. Let $f : \mathbb{R}^k \to \mathbb{R}$ be $f(x) = d(x, K)$. Let $\epsilon \geq 0$ and $\delta > 0$ be such that $2\delta + \epsilon < 1$. Let $x \in \mathbb{R}^k$ be such that there are $z_1, z_2 \in K$, where

$$d(x, z_1) \leq d(x, K) + \epsilon r,$$

$$d(x, z_2) \leq d(x, K) + \epsilon r,$$

and $0 < r < d(x, K)$. Suppose that $f$ is $\delta$-coarsely differentiable on $B(x, r)$. Let $\theta$ be the angle between $[x, z_1]$ and $[x, z_2]$. Then

$$|\theta| \leq \cos^{-1} \left( 2 \left( \frac{1 - (2\delta + \epsilon)}{1 + 2\delta} \right)^2 - 1 \right).$$

This theorem says that if $f$ is $\delta$-coarsely differentiable on some ball $B(x, r)$, then the set of “almost-closest” points to $x$ in $K$ must occupy a small angle as seen from $x$.

To prove this, we first need a few lemmas.

Lemma 3.2. Let $K \subseteq \mathbb{R}^k$. Let $f : \mathbb{R}^k \to \mathbb{R}$ be $f(x) = d(x, K)$. Then $f$ is 1-Lipschitz.

Proof. This is a well-known consequence of the triangle inequality, so we omit the simple proof. □

Lemma 3.3. Let $K \subseteq \mathbb{R}^k$ and $f : \mathbb{R}^k \to \mathbb{R}$ be $f(x) = d(x, K)$. Suppose that $x \in \mathbb{R}^k, z \in K, \epsilon \geq 0,$ and $r > 0$ satisfy

$$d(x, z) \leq d(x, K) + \epsilon r.$$

Let $y \in [x, z]$. Then

$$f(x) - f(y) \geq d(x, y) - \epsilon r.$$

Proof. It suffices to show that

$$f(y) \leq f(x) + \epsilon r - d(x, y).$$

As $x, y, z \in [x, z],$

$$d(x, z) = d(x, y) + d(y, z),$$

and so

$$d(y, z) = d(x, z) - d(x, y).$$

Thus,

$$f(y) = d(y, K) \leq d(y, z) = d(x, z) - d(x, y) \leq d(x, K) + \epsilon r - d(x, y) = f(x) + \epsilon r - d(x, y).$$

□
Lemma 3.4. Let $f : \mathbb{R}^k \to \mathbb{R}$ be 1-Lipschitz. Assume that there is a ball $B(x, r)$ and an affine function $A : \mathbb{R}^k \to \mathbb{R}$ such that

\begin{equation}
|f(t) - A(t)| \leq \delta r \quad \text{for all } t \in B(x, r).
\end{equation}

Write

\begin{equation}
A(t) = L(t) + C,
\end{equation}

where $L$ is linear and $C \in \mathbb{R}$.

Then for all vectors $v \in \mathbb{R}^k$,

$$|L(v)| \leq (1 + 2\delta)||v||.$$  

It suffices to prove this for all unit vectors $v \in \mathbb{R}^k$. To see why, assume the statement holds for all unit vectors and let $v \in \mathbb{R}^k$. Then as $L$ is linear

$$\frac{|L(v)|}{||v||} = L \left( \frac{v}{||v||} \right) \leq (1 + 2\delta).$$

Thus

$$|L(v)| \leq (1 + 2\delta)||v||.$$  

\textbf{Proof.} Let $v \in \mathbb{R}^k$ be a unit vector. Thus

$$|A(x) - A(x + rv)| = |A(x) - f(x)| + |f(x) - f(x + rv)| + |f(x + rv) - A(x + rv)|$$

$$\leq \delta r + r + \delta r$$

$$= r(1 + 2\delta).$$

But as $L$ is linear

$$|A(x) - A(x + rv)| = |L(x) + C - (L(x + rv) + C)|$$

$$= |L(x) - L(x + rv)|$$

$$= |L(rv)|$$

$$= r|L(v)|.$$  

Thus

$$r|L(v)| \leq r(1 + 2\delta)$$

so

$$|L(v)| \leq 1 + 2\delta.$$  

\textbf{Proof of Theorem 3.1} Let

$$D = \{ p \in \mathbb{R}^k : d(p, x) = r \}.$$  

Let $y_1 \in D \cap [x, z_1]$ and $y_2 \in D \cap [x, z_2]$. Note that these points exist because

$$d(x, z_i) \geq d(x, K) > r$$

by assumption.
Denote the vectors from $y_i$ to $x$ by
\[ w_1 = x - y_1 \quad \text{and} \quad w_2 = x - y_2. \]
We wish to find bounds for
\[ \left| L \left( \frac{w_1 + w_2}{2} \right) \right| \quad \text{and} \quad \left| \frac{w_1 + w_2}{2} \right|. \]
By (3.1) and as $y_1 \in [x, z_1]$, we can say by Lemma 3.3 that
\[
\begin{align*}
    f(x) - f(y_1) &\geq d(x, y_1) - \epsilon r \\
    &\geq r - \epsilon r \\
    &\geq r(1 - \epsilon) \\
    &> 0.
\end{align*}
\]
Thus
(3.4) \hspace{1cm} f(y_1) \leq f(x) - r(1 - \epsilon).
Similarly,
\[
    f(y_2) \leq f(x) - r(1 - \epsilon).
\]
As $f$ is 1-Lipschitz and by our work from above,
\[
\begin{align*}
    f(x) - f(y_1) &= |f(x) - f(y_1)| \\
    &\leq d(x, y_1) \\
    &= r.
\end{align*}
\]
So
(3.5) \hspace{1cm} f(y_1) \geq f(x) - r.
Similarly,
\[
    f(y_2) \geq f(x) - r.
\]
Let $A$ and $L$ be the same functions from Lemma 3.4. By (3.2),
\[
    f(y_1) - \delta r \leq A(y_1) \leq f(y_1) + \delta r.
\]
By (3.4) and (3.5),
\[
    f(x) - r - \delta r \leq A(y_1) \leq f(x) - r(1 - \epsilon) + \delta r.
\]
By (3.3),
(3.6) \hspace{1cm} f(x) - r - \delta r - C \leq L(y_1) \leq f(x) - r + r(\delta + \epsilon) - C.
Similarly
\[
    f(x) - r - \delta r - C \leq L(y_2) \leq f(x) - r + r(\delta + \epsilon) - C.
\]
By (3.2) and (3.3)
\[
    -\delta r \leq f(x) - (L(x) + C) \leq \delta r.
\]
Thus

\[(3.7) \quad f(x) - C - \delta r \leq L(x) \leq f(x) - C + \delta r.\]

Thus as \(L\) is linear and by \((3.6), (3.7)\)

\[
L(w_1) = L(x - y_1) \\
= L(x) - L(y_1) \\
\geq f(x) - C - \delta r - (f(x) - r + r(\epsilon + \delta) - C) \\
= -2\delta r - \epsilon r + r \\
= r(1 - (2\delta + \epsilon)).
\]

Similarly

\[
L(w_2) \geq r(1 - (2\delta + \epsilon)).
\]

Thus as \(L\) is linear

\[(3.8) \quad \left| L \left( \frac{w_1 + w_2}{2} \right) \right| \geq L \left( \frac{w_1 + w_2}{2} \right) = \frac{L(w_1) + L(w_2)}{2} \geq r(1 - (2\delta + \epsilon)).\]

Now, to obtain a bound for \(\left\| \frac{w_1 + w_2}{2} \right\|^2\), note first that \(\|w_1\| = \|w_2\| = r\). Thus,

\[
\left\| \frac{w_1 + w_2}{2} \right\|^2 = \frac{1}{4} \|w_1 + w_2\|^2 \\
= \frac{1}{4} ((w_1 + w_2) \cdot (w_1 + w_2)) \\
= \frac{1}{4} (\|w_1\|^2 + 2(w_1 \cdot w_2) + \|w_2\|^2) \\
= \frac{1}{4} (r^2 + 2(w_1 \cdot w_2) + r^2) \\
= \frac{r^2}{2} + \frac{1}{2} (w_1 \cdot w_2) \\
= \frac{r^2}{2} + \frac{1}{2} \|w_1\|\|w_2\| \cos(\theta) \\
= \frac{r^2}{2} + \frac{r^2}{2} \cos(\theta) \\
= r^2 \left( \frac{1 + \cos(\theta)}{2} \right).
\]

Thus,

\[(3.9) \quad \left\| \frac{w_1 + w_2}{2} \right\| = r \sqrt{\frac{1 + \cos(\theta)}{2}}.\]
Thus, by (3.8), (3.9) and Lemma 3.4,

\[ r \left( 1 - (2\delta + \epsilon) \right) \leq \left| L \left( \frac{w_1 + w_2}{2} \right) \right| \leq (1 + 2\delta) \left| \frac{w_1 + w_2}{2} \right| \leq r(1 + 2\delta) \sqrt{\frac{1 + \cos(\theta)}{2}}. \]

Thus,

\[ (1 - (2\delta + \epsilon)) \leq (1 + 2\delta) \sqrt{\frac{1 + \cos(\theta)}{2}}. \]

Now solving for \( \theta \)

\[ \left( 2 \left( \frac{1 - (2\delta + \epsilon)}{1 + 2\delta} \right)^2 - 1 \right) \leq \cos(\theta), \]

i.e.,

\[ |\theta| \leq \cos^{-1} \left( 2 \left( \frac{1 - (2\delta + \epsilon)}{1 + 2\delta} \right)^2 - 1 \right). \]

Now Theorem 1.3 follows directly from Theorem 2.2 and Theorem 3.1.

**Proof of Theorem 1.3.** Let \( f : \mathbb{R}^k \to \mathbb{R} \) be \( f(x) = d(x, K) \).

The measurability of the set \( G \subseteq \mathbb{R}^k \times \mathbb{R}^+ \) in Theorem 1.3 follows by standard arguments. Briefly, given \( t > 0 \), let

\[ G_t = \{(x, r) : x \in \mathbb{R}^k, 0 < r < d(x, K), \text{ and there exist } z_1, z_2 \in K \text{ such that} \]

\[ d(x, z_1), d(x, z_2) < d(x, K) + \epsilon r + t \]

\[ \text{but } |\theta| > \cos^{-1} \left( 2 \left( \frac{1 - (2\delta + \epsilon)}{1 + 2\delta} \right)^2 - 1 \right) \}, \]

where \( \theta \) again denotes the angle of \([x, z_1]\) and \([x, z_2]\). Then each \( G_t \) is open in \( \mathbb{R}^k \times \mathbb{R}^+ \) and

\[ G = \bigcap_{n=1}^{\infty} G_{1/n}, \]

so is therefore measurable.

To see the Carleson condition, it follows from Theorem 3.1 that if \( (x, r) \in G \) then \( f \) is not \( \delta \)-coarsely differentiable on \((x, r)\). Let \( H \) denote the collection of \((x, r)\) such that \( f \) is not \( \delta \)-coarsely differentiable on \( B(x, r) \). Then \( G \subseteq H \). By Lemma 3.2, \( f \) is 1-Lipschitz. By Theorem 2.2, \( H \) is Carleson, with Carleson constant bounded above depending only on \( \delta \). Thus, \( G \) is Carleson with Carleson constant bounded above by that of \( H \).
REFERENCES

[1] Jonas Azzam and Raanan Schul. “A quantitative metric differentiation theorem”. In: Proc. Amer. Math. Soc. 142.4 (2014), pp. 1351–1357.

[2] H. Blum. “A transformation for extracting new descriptors of shape”. In: Models for Perception of Speech and Visual Form. Ed. by W. Wathen-Dunn. Cambridge, MA: MIT Press, 1967.

[3] Jeff Cheeger. “Quantitative differentiation: a general formulation”. In: Comm. Pure Appl. Math. 65.12 (2012), pp. 1641–1670.

[4] Guy David and Stephen Semmes. Analysis of and on uniformly rectifiable sets. Vol. 38. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1993, pp. xii+356.

[5] José R. Dorronsoro. “A characterization of potential spaces”. In: Proc. Amer. Math. Soc. 95.1 (1985), pp. 21–31.

[6] Paul Erdős. “On the Hausdorff dimension of some sets in Euclidean space”. In: Bull. Amer. Math. Soc. 52 (1946), pp. 107–109.

[7] D. H. Fremlin. “Skeletons and central sets”. In: Proc. London Math. Soc. (3) 74.3 (1997), pp. 701–720.

[8] Misha Gromov. Metric structures for Riemannian and non-Riemannian spaces. English. Modern Birkhäuser Classics. Based on the 1981 French original, With appendices by M. Katz, P. Pansu and S. Semmes, Translated from the French by Sean Michael Bates. Birkhäuser Boston, Inc., Boston, MA, 2007, pp. xx+585.

[9] Piotr Hajłasz. “On an old theorem of Erdős about ambiguous locus”. In: Colloquium Mathematicum (2022).

[10] Kevin Hook. Quantitative differentiation and its applications. MS Thesis, Ball State University. 2022.

[11] Peter W. Jones. “Lipschitz and bi-Lipschitz functions”. In: Rev. Mat. Iberoamericana 4.1 (1988), pp. 115–121.

[12] Pietro Majer. Distance function to $\Omega \subset \mathbb{R}^n$ differentiable at $y \notin \Omega$ implies $\exists$ unique closest point. MathOverflow. URL:https://mathoverflow.net/q/299066 (version: 2018-04-30). eprint: https://mathoverflow.net/q/299066.

[13] Robert Young. Notes on quantitative rectifiability and differentiability. https://math.nyu.edu/~ryoung /courses/qdiff/diffNotes.pdf. Apr. 2020.

DEPARTMENT OF MATHEMATICAL SCIENCES, BALL STATE UNIVERSITY, MUNCIE, IN 47306
Email address: gcdavid@bsu.edu

DEPARTMENT OF MATHEMATICAL SCIENCES, BALL STATE UNIVERSITY, MUNCIE, IN 47306
Email address: kmhook2@bsu.edu