HYPERBOLICITY OF THE MODULI OF CERTAIN FANO THREEFOLDS

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ABSTRACT. We prove the Shafarevich conjecture for Fano threefolds of Picard rank 1, index 1 and degree 4.

1. Introduction

In [7] Faltings proved that, given a number field $K$, a finite set of places $S$ on $K$ and a positive integer $g$, there are only finitely many isomorphism classes of abelian varieties of dimension $g$ over $K$ with good reduction outside $S$. Faltings’s finiteness result illustrates a more general phenomenon, commonly referred to as the Shafarevich conjecture, that the set of objects of fixed type over a number field $K'$ with good reduction outside $S$ should be finite. This far-reaching conjecture has been verified for K3 surfaces (and hyperkähler varieties) [1, 27, 32, 33], cyclic covers [21], ample hypersurfaces in abelian varieties [23] (building on [24]), polycurves [28, 15], flag varieties [13], complete intersections of Hodge level at most one [12], certain types of Fano threefolds [20], del Pezzo surfaces [31], and Enriques surfaces [36].

In this paper, we focus on the Shafarevich conjecture for Fano threefolds. Interestingly, the Shafarevich conjecture for Fano threefolds can fail; see [20, Theorem 1.4] for a precise statement. Our main result however verifies the Shafarevich conjecture for (smooth) Fano threefolds with Picard rank 1, index 1 and degree 4 (see Definition 2.1 and Definition 2.4). Note that this is one of the cases for which the Shafarevich conjecture was not handled by Loughran and Javanpeykar [20].

Theorem 1.1 (Shafarevich conjecture for Fano threefolds of type (1,1,4)). Let $K$ be a number field and let $S$ be a finite set of places on $K$. Then the set of $K$-isomorphism classes of Fano threefolds of Picard rank 1, index 1 and degree 4 over $K$ with good reduction outside $S$ is finite.

The Shafarevich conjecture is a statement about the finiteness of integral points on certain moduli stacks. The rough idea is that sets of integral points on moduli stacks should be finite by Lang’s conjecture [16, 22] if the stack is hyperbolic (in a suitable sense).

In fact, to prove Theorem 1.1 we translate the claimed finiteness statement into a property of the moduli stack of such Fano threefolds. It is therefore crucial to understand the structure of the moduli of such Fano threefolds. After Iskovskikh’s classification [11, table 3.5], the Fano threefolds of Picard rank 1, index 1 and degree 4 come in two types:

a) smooth quartics, and

b) double covers of smooth quadrics in $\mathbb{P}^4$.

The latter type is called hyperelliptic. We show the folklore fact that in families of such Fano varieties, the locus of hyperelliptic ones is closed.

Theorem 1.2 (Closedness of the hyperelliptic locus). Let $B$ be a scheme over $\mathbb{Z}[1/2]$. Let $X \to B$ be a Fano threefold of type $(1,1,4)$ over $B$. Then the set of $b \in B$ such that $X_b$ is hyperelliptic is closed in $B$.

The proof of Theorem 1.2 uses the fact that Fano threefolds of type $(1,1,4)$ are weighted complete intersections, locally for the Zariski topology. This is well-known for such Fano threefolds over algebraically closed fields, and our contribution is that this also persists over more general base schemes.

Theorem 1.3 (Zariski local presentation). Let $B$ be a scheme over $\mathbb{Z}[1/2]$ and let $f : X \to B$ be a Fano threefold of type $(1,1,4)$. Let $b \in B$ be a point. There is an affine open neighbourhood $U = \text{Spec } R \subseteq B$ of $b$ and polynomials $q_2, q_4 \in R[x_0, \ldots, x_4, z]$ of degree $\deg(q_2) = 2$ and $\deg(q_4) = 4$ with respect to the grading given by $\deg(x_i) = 1$ and $\deg(z) = 2$ such that the Fano threefold $f|U : f^{-1}(U) \to U$ is isomorphic to $V_4(q_2, q_4) \subseteq \mathbb{P}_R(1,1,1,1,1,2)$, and $\omega_{f^{-1}(U)/U}$ is identified with $O(1)$.

Let $\mathcal{F}$ be the moduli stack of Fano threefolds of type $(1,1,4)$, let $\mathcal{Q} \subseteq \mathcal{F}$ be the locus of smooth quartics and let $\mathcal{H} \subseteq \mathcal{F}$ be the locus of hyperelliptic Fano threefolds; see Section 2 and Section 4 for definitions. Theorem 1.2 implies that $\mathcal{F}$ is stratified via $\mathcal{H}$ and $\mathcal{Q}$.

Theorem 1.4 (Stratification of the moduli of Fano threefolds of type $(1,1,4)$). The inclusion $\mathcal{H} \hookrightarrow \mathcal{F}$ is a closed immersion, the inclusion $\mathcal{Q} \hookrightarrow \mathcal{F}$ is an open immersion, and for any field $k$ with $2 \in k^*$, we have $\mathcal{F}(k) = \mathcal{Q}(k) \sqcup \mathcal{H}(k)$.

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Theorem 1.1 is a consequence of the fact that $\mathcal{F}$ is an "arithmetically hyperbolic" stack. Here we follow [19] and say that a finitely presented algebraic stack $X$ over an algebraically closed field $k$ of characteristic $0$ is arithmetically hyperbolic over $k$ if there is a $\mathbb{Z}$-finitely generated subring $A \subseteq k$ and a (finitely presented) model $\mathcal{X}$ for $X$ over $A$ such that, for any $\mathbb{Z}$-finitely generated subring $A' \subseteq k$ containing $A$, the set
\[ \text{im} \left( \pi_0(\mathcal{X}(A')) \to \pi_0(\mathcal{X}(k)) \right) \]
is finite.

We will show that the stack $\mathcal{F}$ is arithmetically hyperbolic over any algebraically closed field of characteristic zero. We follow the terminology of [17] and say that a finitely presented algebraic stack $X$ over an algebraically closed field $k$ of characteristic $0$ is absolutely arithmetically hyperbolic if $X_k$ is arithmetically hyperbolic over $L$ for every algebraically closed field extension $L \supseteq k$.

For an arbitrary stack, there should be no difference between being arithmetically hyperbolic over $\mathbb{Q}$ and being absolutely arithmetically hyperbolic, i.e., arithmetic hyperbolicity should persist over all field extensions. This is formalized by the following conjecture for stacks alluded to in [19, Remark 4.13].

**Conjecture 1.5** (Persistence Conjecture). Let $k$ be an algebraically closed field of characteristic zero, and let $X$ be a finitely presented algebraic stack over $k$. If $X$ is arithmetically hyperbolic over $k$, then $X$ is absolutely arithmetically hyperbolic.

This is a stacky version of [14, Conjecture 1.1] (see also [16, Conjecture 17.5]). The conjecture says, in particular, that the finiteness of rational points over number fields on a projective variety over $\mathbb{Q}$ should imply the finiteness over all finitely generated fields; this was formulated as a precise question by Lang in [22, p. 202]. The main result of this paper is that $\mathcal{F}$ is absolutely arithmetically hyperbolic.

**Theorem 1.6** (Main result). The stack $\mathcal{F}$ of Fano threefolds of type $(1,1,4)$ is absolutely arithmetically hyperbolic.

The proof of this result uses the fact that the intermediate Jacobian defines a morphism of stacks
\[ p: (\mathcal{F}_c)^n \to (A_{30,c})^n, \]
where $A_{30}$ is the stack of 30-dimensional principally polarized abelian schemes. We show that this period map is quasi-finite by using the stratification of $\mathcal{F}$ (Theorem 1.4 and invoking our results on the infinitesimal Torelli property for Fano threefolds of type $(1,1,4)$ [23, Theorem 1.3]. The absolute arithmetic hyperbolicity of $\mathcal{F}$ will then follow from the absolute arithmetic hyperbolicity of $A_{30}$.

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2. The Stack of Fano Threefolds

Following [20, Section 2-3], we introduce the stack of Fano threefolds.

**Definition 2.1.** Let $k$ be a field. A Fano variety over $k$ is a smooth proper geometrically integral variety $S$ over $k$ such that its anticanonical bundle is ample.

Let $B$ be a scheme. A Fano scheme over $B$ (or family of Fano varieties over $B$) is a smooth proper morphism $X \to B$ of schemes whose fibres are Fano varieties. A Fano $n$-fold over $B$ is a Fano scheme of relative dimension $n$.

**Definition 2.2.** Let $B$ be a Dedekind scheme with function field $K$. A Fano variety $X$ over $K$ has good reduction over $B$ if there is a Fano scheme $X \to B$ and an isomorphism $X_K \cong X$.

**Remark 2.3.** Let $f: X \to B$ be a smooth proper morphism with geometrically connected fibres of relative dimension $n$. Then $\omega_{X/B} = \Lambda^n \Omega_{X/B}$ is a line bundle. Since $f$ is proper, the line bundle $\omega_{X/B}^{-1}$ is relatively ample if and only if it is fibre-wise ample; see [3, Theorem 4.7.1]. Hence $X \to B$ is a Fano scheme if and only if $\omega_{X/B}^{-1}$ is relatively ample.

**Definition 2.4.** Let $k$ be an algebraically closed field and let $X$ be a Fano threefold over $k$. Then we define:

1. The Picard rank of $X$ is $\rho(X) = \text{rank}_\mathbb{Z} \text{Pic}(X)$.
2. The index of $X$ is $r(X) = \max \{ m \in \mathbb{N} \mid \omega_X^{-1}/m \in \text{Pic}(X) \}$.
3. The degree of $X$ is the triple intersection number $d(X) = (\omega_X^{-1}/r(X))^3$.
4. The type of $X$ is the triple $(\rho(X), r(X), d(X))$.

**Definition 2.5.** We define a fibred category $p: \text{Fano} \to \text{Sch}$, where for a scheme $B$, the objects of $\text{Fano}(B)$ are Fano threefolds over $B$. A morphism $(f: X \to B) \to (f': X' \to B')$ of two Fano three in $\text{Fano}$ is given by a pair $(g, h)$, where $g: B \to B'$ and $h: X \to X'$ are morphisms of schemes such that the square
\[
\begin{array}{ccc}
X & \xrightarrow{h} & X' \\
\downarrow f & & \downarrow f' \\
B & \xrightarrow{g} & B'
\end{array}
\]
is cartesian. The functor $p$ is the forgetful functor, that remembers only the base scheme.

Given a triple of positive integers $(\rho, r, d) \in \mathbb{N}^3$, we define $\text{Fano}_{\rho, r, d} \to \text{Sch}$ to be the full fibred subcategory of Fano threefolds $f : X \to B$ such that all geometric fibres of $f$ are Fano varieties of Picard rank $\rho$, index $r$ and degree $d$. We define $\mathcal{F}$ to be the fibred category (Fano$_{1,1,4})_{\mathbb{Z}[1/2]}$.

**Proposition 2.6.**

1. The fibred category $\mathcal{F}$ is a finite type algebraic stack with an affine diagonal over $\mathbb{Z}[1/2]$.
2. The stack $\mathcal{F}_{\mathbb{Q}}$ is smooth over $\mathbb{Q}$.
3. There is an $N \in \mathbb{N}$ such that $\mathcal{F}_{\mathbb{Z}[1/N]}$ is separated over $\mathbb{Z}[1/N]$.
4. There is an $N \in \mathbb{N}$ such that for all schemes $B$ over $\mathbb{Z}[1/N]$ and $X, Y \in \mathcal{F}(B)$, the morphism $\text{Isom}_B(X, Y) \to B$ is finite.
5. The stack $\mathcal{F}_{\mathbb{Q}}$ is a Deligne-Mumford stack.

**Proof.** Note that (1), (2), (3) and (4) follows from [21, Lemma 3.5, 3.6, 3.7, 3.8]. Finally, (5) follows from the fact that a finite type separated algebraic stack with an affine diagonal over $\mathbb{Q}$ is Deligne-Mumford. $\square$

3. Fano threefolds of Picard number 1, index 1, degree 4 over fields

Fano threefolds over an algebraically closed field have been classified by Iskovskikh [11, Table 3.5] in characteristic 0. His result was later generalized by Shepherd-Barron [33] to positive characteristic in the case where the Picard rank is 1. In this section, we will generalize the characterization for Fano threefolds of type $(1, 1, 4)$ to the case of non algebraically closed fields of characteristic not equal to 2. In Section 4, we will then generalize it to families of Fano threefolds.

**Theorem 3.1** (Iskovskikh-Shepherd-Barron [33, Propositions 4.1–4.3]). Let $k$ be an algebraically closed field. Let $X$ be a Fano threefold of type $(1, 1, 4)$ over $k$.

a) Then $X$ is a smooth quartic in $\mathbb{P}^4$, or

b) $X$ is a double cover of a smooth quadric in $\mathbb{P}^4$ ramified along a smooth surface of degree 8.

Let $k$ be a field with $2 \in k^\times$. We consider the weighted projective space $\mathbb{P}_k(1, 1, 1, 1, 2) = \text{Proj} k[x_0, \ldots, x_4, z]$ with weights $\deg(x_i) = 1$ for $i \in \{0, \ldots, 4\}$ and $\deg(z) = 2$; see [6] for an introduction on weighted projective varieties. Varieties of both types a) and b) represent special cases of smooth weighted complete intersections of degree $(2, 4)$. Considering type a), let

$$X = V_+(q_4) \subseteq \mathbb{P}^4_k = \text{Proj} k[x_0, \ldots, x_4]$$

be a smooth quartic. Then $X$ is isomorphic to the complete intersection

$$X' = V_4(z, q_4) \subseteq \mathbb{P}_k(1, 1, 1, 1, 2).$$

Considering type b), let $X$ be a double cover of a smooth quadric $V_+(q_2) \subseteq \mathbb{P}^4$ which is ramified along a smooth surface $V_+(q_2, q_4)$ of degree 8. Then $X$ is isomorphic to

$$X' = V_+(q_2, q_4 - z^2) \subseteq \mathbb{P}_k(1, 1, 1, 1, 2).$$

The double cover map is given via projection onto the first 5 homogeneous coordinates. Note in both cases $X'$ does not contain the point $Q = (0 : 0 : 0 : 0 : 1)$. In fact, the following more general statement holds.

**Lemma 3.2.** If $Y \subseteq \mathbb{P}_k(1, 1, 1, 1, 2)$ be a positive-dimensional smooth complete intersection, then $Y$ does not contain $Q$.

**Proof.** The point $Q$ is contained in the open affine neighbourhood

$$U = D_+(z) = \text{Spec} k[x_0 \ldots x_4/z]_{[0 \leq i \leq j \leq 4]} \subseteq \mathbb{P}_k(1, 1, 1, 1, 2).$$

The generators $w_{i,j} = \frac{x_i x_j}{z}$ satisfy relations

$$q_{0,\sigma} = w_{0,\sigma_1,\sigma_2}w_{0,\sigma_3,\sigma_4} - w_{\sigma_1,\sigma_2,\sigma_3,\sigma_4}w_{\sigma_3,\sigma_4},$$

where $\alpha \in \{0, \ldots, 4\}^4$ with $\alpha_1 \leq \alpha_2$ and $\alpha_3 \leq \alpha_4$ and $\sigma$ is a permutation of $\{1, 2, 3, 4\}$ with $\alpha_1 \leq \alpha_2$ and $\alpha_3 \leq \alpha_4$. We see $U \cong \text{Spec} k[w_{i,j}]/(q_{0,\sigma})$. The point $Q$ corresponds to the point with coordinates $w_{i,j} = 0$. Note that $\frac{\partial q_{0,\sigma}}{\partial w_{i,j}}(Q) = 0$ for all $i, j, \alpha, \sigma$. Therefore it is not possible for $U \cap Y$ to satisfy the Jacobi criterion if $Y$ does contain $Q$. $\square$

In the following, we will find an explicit description for the anti-canonical bundle of the weighted complete intersection $X'$. By [6, Theorem 3.3.4], there is an isomorphism

$$\omega_{X'/k} \cong \mathcal{O}_{X'}(-1).$$

Note that for a closed subvariety $Y \subseteq \mathbb{P}_k(W_1, \ldots, W_r)$ with weighted coordinate ring $A$, the sheaf $\mathcal{O}_{Y}(m)$ is the graded $\mathcal{O}_Y$-module associated to the degree shifted graded module $A(m)$. In general, it is not true that $\mathcal{O}_{Y}(m)$ is a line bundle or that $\mathcal{O}_{Y}(m) \otimes \mathcal{O}_{Y}(l) \cong \mathcal{O}_{Y}(m+l)$; see [6, Section 1.5]. For example, if
Let \( P = \mathbb{P}_k(1, 1, 1, 1, 2) \) as above, then \( \mathcal{O}_P(1) \otimes \mathcal{O}_P(1) \cong \mathcal{O}_P(2) \). This is because in any neighbourhood of \( Q = (0 : 0 : 0 : 0 : 1) \), the section \( z \) of \( \mathcal{O}_P(2) \) is not a product of sections of \( \mathcal{O}_P(1) \). However, this is the only problematic point.

**Lemma 3.3.** Let \( k \) be a field, let \( X \subseteq \mathbb{P}_k(1, 1, 1, 1, 2) \) be a closed subvariety that does not contain the point \( Q = (0 : 0 : 0 : 0 : 1) \). Then for any \( m, l \in \mathbb{Z} \), the sheaf \( \mathcal{O}_X(m) \) is a line bundle and the multiplication map induces an isomorphism

\[
\mathcal{O}_X(m) \otimes \mathcal{O}_X(l) \cong \mathcal{O}_X(m + l).
\]

**Proof.** We have \( X \subseteq \mathbb{P}_k(1, 1, 1, 1, 2) \setminus \{Q\} = \bigcup \mathcal{D}_+(x_i) \), where

\[
\mathcal{D}_+(x_i) = \text{Spec } k \left[ \frac{x_0, \ldots, x_4}{x_0, \ldots, x_4} \right].
\]

Let \( U_i = X \cap \mathcal{D}_+(x_i) \). The assertion follows, since the multiplication map

\[
\mathcal{O}_X|_{U_i} \xrightarrow{\cdot \, x_i^m} \mathcal{O}_X(m)|_{U_i}
\]

is an isomorphism. \( \square \)

By Lemma 3.2 and Lemma 3.3, the isomorphism 3.1 induces an isomorphism

\[
(3.2)
\omega_{X/k}^{-1} \cong \mathcal{O}_X(i)
\]

for all \( i \in \mathbb{Z} \). The proof of the following proposition will utilize the fact that our Fano threefolds come with a canonical embedding into weighted projective space associated to the anticanonical bundle.

**Proposition 3.4.** Let \( k \) be a field with \( 2 \in k^\times \), let \( X \) be a Fano threefold of type \((1, 1, 4)\) over \( k \). Then the function \( \phi(i) = \dim_k H^0(X, \omega_{X/k}^{-i}) \) satisfies \( \phi(1) = 5, \phi(2) = 15, \phi(3) = 35 \) and \( \phi(4) = 69 \). Let \( \xi_0, \ldots, \xi_4 \) be a basis for \( H^0(X, \omega_{X/k}^{-4}) \). Then the following statements hold.

1. If \( X_k \) is of type \( a \), then the variety \( X \) is a smooth quartic in \( \mathbb{P}^4_k \). The monomials of degree 4, \( \xi_0^4, \xi_0 \xi_1, \ldots, \xi_4 \) satisfy a relation \( q_4 \). The map \( \xi_i \mapsto x_i \) induces an isomorphism of graded \( k \)-algebras

\[
\bigoplus_{i \geq 0} H^0(X, \omega_{X/k}^{-i}) \cong k[x_0, \ldots, x_4]/(q_4)
\]

and an isomorphism \( X \cong V_+(q_4) \subseteq \mathbb{P}^4_k \) of varieties.

2. If \( X_k \) is of type \( b \), then the variety \( X \) is a double cover of a smooth quadric in \( \mathbb{P}^4_k \) ramified along a smooth surface of degree 8. The monomials of degree 2, \( \xi_0^2, \xi_0 \xi_1, \ldots, \xi_4^2 \) satisfy a relation \( q_2 \) and span a 14-dimensional subspace of \( H^0(X, \omega_{X/k}^{-2}) \). Let \( \zeta \in H^0(X, \omega_{X/k}^{-2}) \) be an element completing those monomials to a generating set. There are polynomials \( q_2 \in k[x_0, \ldots, x_4]_2 \) and \( q_4 \in k[x_0, \ldots, x_4, z]_4 \) such that the map \( \xi_i \mapsto x_i, \zeta \mapsto z \) induces an isomorphism if graded \( k \)-algebras

\[
\bigoplus_{i \geq 0} H^0(X, \omega_{X/k}^{-i}) \cong k[x_0, \ldots, x_4, z]/(q_2, q_4 - z^2)
\]

and an isomorphism \( X \cong V_+(q_2, q_4 - z^2) \subseteq \mathbb{P}_k(1, 1, 1, 1, 2) \) of varieties.

**Proof.** We will prove (2). The proof of (1) is similar. Let \( X \) be a Fano threefold over \( k \) of type \((1,1,4)\) such that there is an isomorphism

\[
X_{\mathbb{P}} \cong V_+(q'_2, q'_4 - z^2) \subseteq \mathbb{P}_{\mathbb{P}}(1, 1, 1, 1, 2),
\]

where \( q'_j \in k[x_0, \ldots, x_4]_j \). Isomorphism 3.2 induces an isomorphism

\[
\bigoplus_{i \geq 0} H^0 \left( X_{\mathbb{P}}, \omega_{X_{\mathbb{P}}/\mathbb{P}}^{-i} \right) \cong k[x_0, \ldots, x_4, z]/(q'_2, q'_4 - z^2) =: R
\]

of graded \( k \)-algebras. Furthermore for each \( i \geq 0 \), there is an isomorphism

\[
H^0(X, \omega_{X/k}^{-i}) \otimes \mathbb{P} \cong H^0 \left( X_{\mathbb{P}}, \omega_{X_{\mathbb{P}}/\mathbb{P}}^{-i} \right).
\]

Hence, the value \( \phi(i) \) can be determined by counting the elements of a \( \mathbb{P} \)-vector space basis for the degree \( i \) part \( R_i \); see 2.2 for the computations. The basis \( \xi_0, \ldots, \xi_4 \) of \( H^0(X, \omega_{X/k}^{-1}) \) is also a basis for \( H^0(X, \omega_{X/k}^{-1}) \).

The image of the multiplication pairing \( R_1 \otimes R_1 \to R_2 \) is generated by the 15 monomials \( x_0^2, x_0 x_1, \ldots, x_4^2 \). These monomials satisfy the relation \( q_2 \). Therefore, the image has dimension 14. Hence, also the monomials \( \xi_0^2, \xi_0 \xi_1, \ldots, \xi_4^2 \) generate a 14-dimensional subspace of \( H^0(X, \omega_{X/k}^{-2}) \), and therefore satisfy a relation.

Let \( q_2 \in k[x_0, \ldots, x_4]_2 \setminus \{0\} \) such that \( q_2(\xi_0, \ldots, \xi_4) = 0 \), and let \( \zeta \in H^0(X, \omega_{X/k}^{-2}) \) be an element completing those monomials to a generating set. The vector space \( k[x_0, \ldots, x_4, z]/(q_2)_4 \) has dimension 70. Hence the map

\[
(k[x_0, \ldots, x_4, z]/(q_2))_4 \to H^0(X, \omega_{X/k}^{-4})
\]
given by $x_i \mapsto \xi_i, z \mapsto \zeta$ has a non-zero kernel. Let $q''_4 \neq 0$ be an element of the kernel. Then the maps

$$\bigoplus_{i \geq 0} H^0(X, \omega_X^{-1}) \to k[x_0, \ldots, x_4, z]/(q_2, q''_4)$$

and

$$X \to V_+(q_2, q''_4) \subseteq \mathbb{P}_k(1,1,1,1,1,2)$$

are isomorphisms after the flat base change to the algebraic closure. Therefore, they were isomorphisms, to begin with. By Lemma 3.2, the variety $V_+(q_2, q''_4)$ does not contain $Q$. Hence, $q''_4$ has a $z^2$-term and after a suitable change of coordinates, we get $q''_4 = q_4 - z^2$. \hfill \Box

4. Families of Fano threefolds of Picard number 1, index 1, degree 4

As seen in Section 3 over fields of characteristic unequal to 2, there are two mutually exclusive types of Fano threefolds of Picard rank 1, index 1 and degree 4, namely smooth quartics and hyperelliptic ones. In this section, we will study how these types behave in families of Fano threefolds over schemes $B$ over $\mathbb{Z}[1/2]$. The main tool in this study will be Theorem 1.3 which says that families of such Fano threefolds Zariski locally are weighted complete intersections.

**Local presentation as weighted complete intersection.** We need the following technical lemma, which is a combination of Proposition 3.4 with a "cohomology and basechange" argument.

**Lemma 4.1.** Let $B$ be a scheme over $\mathbb{Z}[1/2]$ and let $f: X \to B$ be a Fano threefold of type $(1,1,4)$ and let $l \in \mathbb{Z}_{>0}$. Then the following statements hold:

1. The formation of $f_*\omega_X^{-1}$ commutes with arbitrary base change.
2. The sheaf $f_*\omega_X^{-1}$ is locally free.
3. Let $r(l) = \text{rank } f_*\omega_X^{-1}$. We have $r(1) = 5, r(2) = 5, r(3) = 35$ and $r(4) = 69$.
4. Assume additionally that $B$ is affine. Let $b \in B$ be a point. Then the natural map

$$H^0(X, \omega_X^{-1}) \otimes k(b) \to H^0 \left( X_b, \omega_{X_b/k(b)}^{-1} \right) : \xi \otimes \lambda \mapsto \lambda \cdot \xi|_b$$

is an isomorphism.

**Proof.** The morphism $f$ is proper and smooth of relative dimension 3. Hence $f$ is locally of finite presentation and $\omega_X^{-1}$ is a sheaf of finite presentation on $X$ and flat over $B$. By Proposition 3.4 for any point $b \in B$, the fibre $X_b$ is a weighted complete intersection of degree $(2,4)$ in $\mathbb{P}_{k(b)}(1,1,1,1,1,2)$, and $\omega_X^{-1}|_{X_b}$ is isomorphic to $\mathcal{O}_{X_b}(l)$; see Isomorphism 3.2. Hence $h^1 \left( X_b, \omega_{X_b/k(b)}^{-1} \right) = 0$ and the function

$$b \mapsto h^0 \left( X_b, \omega_{X_b/k(b)}^{-1} \right)$$

is an isomorphism.

**Proof of Theorem 1.3.** We will successively shrink $B$ and thereby also $X$ to find a suitable $U$. By Lemma 1.1 (2), we can shrink $B$ such that $B = \text{Spec } R$ is affine and $f_*\omega_X^{-1}$ is free for $l \in \{1,2,3,4\}$. We choose a basis $\xi_0, \ldots, \xi_4$ for the free $R$-module $H^0(X, \omega_X^{-1}) = H^0(B, f_*\omega_X^{-1})$. Then the restrictions $\xi_0|_{k(b)}, \ldots, \xi_4|_{k(b)}$ form a basis for $H^0 \left( X_b, \omega_{X_b/k(b)}^{-1} \right)$ by Lemma 4.1 (4). By Proposition 3.4 we know that the 15 elements $\xi_0^2, \xi_0 \xi_1, \ldots, \xi_4^2$ generate a subspace of dimension at least 14 in the 15-dimensional vector space $H^0 \left( X_b, \omega_{X_b/k(b)}^{2} \right)$. We choose $\zeta \in H^0 \left( X_b, \omega_{X_b/k(b)}^{2} \right) \setminus \{0\}$ such that $\xi_0^2, \xi_0 \xi_1|_{k(b)}, \ldots, \xi_4^2, \zeta$ is a set of generators for $H^0 \left( X_b, \omega_{X_b/k(b)}^{2} \right)$. After shrinking $B$, we may find a $\zeta \in H^0(X, \omega_X^{2})$ such that $\zeta|_{b} = \zeta$. Shrinking $B$ again, we may assume that $\xi_0^2, \xi_0 \xi_1, \ldots, \xi_4^2, \zeta$ generates $H^0(X, \omega_X^{2})$. Hence these elements define a surjection $\phi: R^{16} \twoheadrightarrow H^0(X, \omega_X^{2})$. Let $K = \ker(\phi)$. As $H^0(X, \omega_X^{2})$ is a free $R$-module of rank 15, we get an exact sequence

$$0 \to K \otimes k(b) \to k(b)^{16} \to H^0(X_b, \omega_{X_b/k(b)}^{2}) \to 0,$$

where $K \otimes k(b)$ is a 1-dimensional $k(b)$ vector space. After shrinking $B$, we can find $\lambda \in K$ such that $\lambda|_{b}$ generates $K \otimes k(b)$. In particular $\lambda|_{b} \neq 0$. We shrink $B$ such that $\lambda|_{p} \neq 0$ for all $p \in B$. Then $\lambda|_{p}$ generates $K \otimes k(p)$ for all $p \in P$. The element $\lambda$ corresponds to a polynomial $q_2 \in R[x_0, \ldots, x_4, z]$ such that $q_2(\xi_0, \ldots, \xi_4, \zeta) = 0$.

By Proposition 3.4 the restrictions

$$\xi_0^4|_p, \xi_3^4|_p \xi_1|_p, \ldots, \xi_4^4|_p, \xi|_p \xi_1|_p \xi_2|_p, \xi|_p \xi_3^2$$
generate $H^0 \left( X_p, \omega_{X_p/k(p)}^{-4} \right)$. The map
\[
\left( \frac{k(p)[x_0, \ldots, x_4, z]}{q_2[p]} \right)_4 \to H^0 \left( X_p, \omega_{X_p/k(p)}^{-4} \right)
\]
that maps $x_i \mapsto \xi_i|_{p}$ and $z \mapsto \zeta|_{p}$ has a 1 dimensional kernel. As above, after shrinking $B$, we find a global generator $\nu$ for this kernel. The section $\nu$ corresponds to a polynomial $q_4 \in R[x_0, \ldots, x_4, z]$ such that $q_4(\xi_0, \ldots, \xi_4, \zeta) = 0$. The map $x_i \mapsto \xi_i$, $z \mapsto \zeta$ induces a morphism
\[
X \to V_4(q_2, q_4) \subseteq \mathbb{P}_R(1, 1, 1, 1, 2).
\]
Again by Proposition [3.4], this is an isomorphism fibre-wise. Hence it is an isomorphism.

The hyperelliptic and smooth quartic locus.

**Definition 4.2.** Let $B$ be a scheme over $\mathbb{Z}[1/2]$. A Fano scheme $f: X \to B$ of type $(1,1,4)$ is called hyperelliptic if there is an open affine cover $B = \bigcup_{i \in I} U_i$, $U_i = \text{Spec} R_i$ such that the polarized scheme
\[
(f|_{f^{-1}(U_i)}: f^{-1}(U_i) \to U_i, \omega_{f^{-1}(U_i)/U_i}^{-1})
\]
is isomorphic to a complete intersection $(V_+(q_2, q_4), \mathcal{O}(1))$ in $\mathbb{P}_R(1, 1, 1, 1, 2)$ with $q_2 \in R_i[x_0, \ldots, x_4]$.

**Remark 4.3.** Theorem [1.8] guarantees that there is an open affine cover on which a Fano scheme is a weighted complete intersection. The additional condition imposed by being hyperelliptic is that $q_2$ does not depend on $z$. Note that every geometric fibre of a hyperelliptic Fano threefold is hyperelliptic in the sense of Iskovskikh's classification. More generally, the following statement holds.

Let $B$ be a scheme over $\mathbb{Z}[1/2]$ and let $f: X \to B$ be a hyperelliptic Fano threefold of type $(1,1,4)$. Then for every field $k$ and every point $b \in B(k)$, the fibre $X_b$ is a double cover of a smooth quadric in $\mathbb{P}_k^4$ ramified along a smooth surface of degree 8. Indeed, since $f: X \to B$ is hyperelliptic, each fibre is a weighted complete intersection $V_+(q_2, q_4) \subseteq \mathbb{P}_k(1, 1, 1, 1, 2)$ with $q_2$ not depending on $z$. By Lemma [3.2], the polynomial $q_4$ has a $z^2$-term and we may assume that $q_4 = q_4^2 + z^2$ for some $q_4 \in k[x_0, \ldots, x_4]$ after a suitable change of coordinates. The variety $V_+(q_2, q_4^2 + z^2)$ is a double cover of $V_+(q_2, q_4) \subseteq \mathbb{P}_k^4$ ramified along $V_+(q_2, q_4)$.

The condition on $q_2$ in the definition of hyperelliptic does not depend on the choice of the open covering.

**Lemma 4.4.** Let $B$ be a scheme over $\mathbb{Z}[1/2]$. Let $f: X \to B$ be a hyperelliptic Fano scheme of type $(1,1,4)$. Let $U = \text{Spec} R \subseteq B$ be an open affine subscheme such that there is an isomorphism of polarized schemes
\[
(f|_U: f^{-1}(U), \omega_{f^{-1}(U)/U}^{-1}) \cong (V_+(q_2, q_4), \mathcal{O}(1))
\]
over $U$. Write $q_2 = q_2^* + cz$ with $q_2^* \in R[x_0, \ldots, x_4]$. Then we have $c = 0$.

**Proof.** Note that $c = 0$ if and only if there is an open affine covering $U = \bigcup V_j$ such that $c|_{V_j} = 0$. As $f: X \to B$ is hyperelliptic, after shrinking $U$ if necessary, we may assume that $f^{-1}(U)$ has a presentation $V_+(q_2^*, q_4^*) \subseteq \mathbb{P}_R(1, 1, 1, 1, 2)$ with $q_4^* \in R[x_0, \ldots, x_4]$. Thus, there is an isomorphism over $U$ of polarized schemes
\[
(V_+(q_2^* + c, q_4^*), \mathcal{O}(1)) \cong (V_+(q_2^*, q_4^*), \mathcal{O}(1)).
\]
The isomorphism of polarized schemes induces an isomorphism
\[
\alpha: R[x_0, \ldots, x_4, z]/(q_2^* + cz, q_4^*) \to R[x_0, \ldots, x_4, z]/(q_2^*, q_4^*)
\]
of graded $R$-algebras. Since $z$ is of degree 2, the image $\alpha(z)$ has to be of degree 2. We write $\alpha(z) = az + p$ with $a \in R$ and $p \in R[x_0, \ldots, x_4][2]$. As the degree 2 relation $q_4^*$ does not depend on $z$, it follows from the subjectivity of $\alpha$ that $a$ is invertible. Since $a \cdot c = 0$, it follows that $c = 0$, as required.

**Definition 4.5.** Let $B$ be a scheme over $\mathbb{Z}[1/2]$. A Fano scheme $f: X \to B$ of type $(1,1,4)$ is a smooth quartic if every geometric fibre is a smooth quartic.

One might ask why we did not define hyperelliptic Fano threefolds via a fibre-wise criterion as well. The problem with such a definition is that over non-reduced bases, it would allow for "hyperelliptic" Fano threefolds that are not double covers, as the following example shows.

**Example 4.6.** Let $k$ be a field with $2 \in k^\times$. Consider the family
\[
X = V_+(x_0^2 + \cdots + x_4^2 + \varepsilon x_0 \cdots + x_4^2 - z^2) \subseteq \mathbb{P}_{k[e]}(1, 1, 1, 1, 1, 2)
\]
over $k[e] := k[t]/(t^2)$. This family has just one geometric fibre, which is hyperelliptic. However, it is neither hyperelliptic nor a smooth quartic.

At the same time, this is an example of a first-order deformation of a hyperelliptic Fano threefold that is not hyperelliptic. This is in accordance with the fact that the locus of hyperelliptic Fanos is a closed substack of $\mathcal{F}$. 

**Proposition/Definition 4.7** (hyperelliptic locus). Let $B$ be a scheme over $\mathbb{Z}[1/2]$ and let $f: X \to B$ be a Fano threefold of type $(1,1,4)$. Then there is a closed subscheme $c: B_{hyp} \to B$ such that $X \times_B B_{hyp} \to B_{hyp}$ is hyperelliptic with the following universal property. If $g: T \to B$ is a morphism of schemes such that $X \times_B T$ is hyperelliptic, then there is a unique morphism $g': T \to B_{hyp}$ such that $g = c \circ g'$. Moreover, $B_{hyp}$ is the set of points $b \in B$ such that the geometric fibre $X_b$ over $\kappa(b)$ is hyperelliptic. We call $B_{hyp}$ the hyperelliptic locus of $f: X \to B$.

**Proof.** The universal property shows that if it exists, the hyperelliptic locus is unique up to a unique isomorphism. Therefore, the assertion is local on $B$, because the universality of the hyperelliptic locus allows us to construct the locus locally and glue it together afterwards. By Theorem 4.4 we may therefore assume that $B = \text{Spec } R$ is affine and $X = V_+(q_2, q_4) \subseteq \mathbb{P}_R(1,1,1,1,1,2)$ is a weighted complete intersection. We write $q_2 = q_2' + cz$, where $q_2', c \in R$ and $c \neq 0$. We set $B_{hyp} = V(c) \subseteq B$.

The assertion that a morphism $g: T \to B$ is a morphism of schemes such that $X \times_B T$ is hyperelliptic factors uniquely through $B_{hyp}$ is local on $T$. We may therefore assume that $T = \text{Spec } S$ is affine. Let $\alpha: R \to S$ be the morphism of rings corresponding to $g$. Now

$$X \times_B T \cong V_+(\alpha(q_2), \alpha(q_2') + \alpha(c)z) \subseteq \mathbb{P}_S(1,1,1,1,1,2)$$

is a presentation as weighted complete intersection. Therefore $\alpha(c) = 0$ by Lemma 4.3. Hence $g$ factors uniquely through $B_{hyp} = V(c) = \text{Spec } R/(c)$. \hfill $\square$

There is a similar statement for the smooth quartic locus that is proven completely analogous.

**Proposition/Definition 4.8.** (smooth quartic locus) Let $B$ be a scheme over $\mathbb{Z}[1/2]$ and let $f: X \to B$ be a Fano threefold of type $(1,1,4)$. There is an open subscheme $\alpha: B_{sq} \to B$ such that $f: f^{-1}(B_{sq}) \to B_{sq}$ is a smooth quartic with the following universal property. If $g: T \to B$ is a morphism of schemes such that $X \times_B T$ is a smooth quartic, then there is a unique morphism $g': T \to B_{sq}$ such that $g = c \circ g'$. Furthermore, $B_{sq}$ is the complement of $B_{hyp}$ in $B$.

**The involution on a hyperelliptic Fano scheme.** Next, we want to construct an involution on hyperelliptic Fano threefolds. Note that local presentations can be chosen in a certain form.

**Lemma 4.9.** Let $B$ be a scheme over $\mathbb{Z}[1/2]$ and let $f: X \to B$ be a hyperelliptic Fano threefold of type $(1,1,4)$. Let $b \in B$ be a point. There is an affine open neighbourhood $U = \text{Spec } R \subseteq B$ of $b$ and polynomials $q_2, q_4 \in R[x_0, \ldots, x_4]$ of degree $\deg(q_2) = 2$ and $\deg(q_4) = 4$ such that there is an isomorphism

$$f^{-1}(U), \omega_{f^{-1}(U)/U} \cong V_+(q_2, q_4 - z^2), \mathcal{O}(1))$$

of polarized schemes over $U$.

**Proof.** By Theorem 4.3 we can find a local presentation $V_+(q_2, q_4)$ with $q_2, q_4 \in R[x_0, \ldots, x_4]$. Write $q_4 = q_4' + p_2z + d\bar{z}^2$ with $q_4', p_2 \in R[x_0, \ldots, x_4], d \in R$. The coefficient $d$ is invertible. Otherwise there would have to be a fibre with $d \otimes k(b) = 0$. This is impossible as this fibre then would be singular by Lemma 4.2. After the change of coordinates $z \mapsto z + \frac{2}{d\bar{z}}$, we get the desired equations. \hfill $\square$

**Definition 4.10.** Let $B$ be a scheme over $\mathbb{Z}[1/2]$ and let $f: X \to B$ be a hyperelliptic Fano threefold of type $(1,1,4)$. The involution induced by the double cover is the map $\iota: \text{Aut}(B)(X)$ that over any open affine $U \subseteq B$ with $(f^{-1}(U), \omega_{f^{-1}(U)/U}) \cong (V_+(q_2, q_4 - z^2), \mathcal{O}(1))$ as in Lemma 4.9 is given by $z \mapsto -z$.

To see that the involution is well defined, we have to show that the defined maps agree on the intersection of two open affines over which $X$ has a presentation of the given form. Since we can cover the intersection with smaller open affines, it suffices to show the following statement. If $U = \text{Spec } R \subseteq B$ is open affine and $q_2, q_4, q_2', q_4' \in R[x_0, \ldots, x_4]$ are polynomials such that

$$V_+(q_2, q_4 - z^2), \mathcal{O}(1)) \cong (f^{-1}(U), \omega_{f^{-1}(U)/U}) \cong (V_+(q_2', q_4' - z^2), \mathcal{O}(1)),$$

then the maps that are given by $z \mapsto -z$ are identified under the isomorphisms. The isomorphism of polarized schemes induces an isomorphism of graded $R$-algebras

$$\alpha: R[x_0, \ldots, x_4, z]/(q_2, q_4 - z^2) \to R[x_0, \ldots, x_4, z]/(q_2', q_4' - z^2).$$

To preserve the equations, the image $\alpha(z) = az + p$ now has to satisfy $p = 0$. Hence $\alpha$ commutes with the map $z \mapsto -z$.

5. The stacks of hyperelliptic and smooth quartic Fanos

In this section, we define the stack of hyperelliptic and smooth quartic Fanos and prove Theorem 1.4.

**Definition 5.1.** Let $\mathcal{F}$ over $\mathbb{Z}[1/2]$ be the stack of Fano threefolds of type $(1,1,4)$ as defined in Section 2.

1. We define $\mathcal{H}$ to be the full fibre subcategory of $\mathcal{F}$ of those Fano threefolds $f: X \to B$ that are hyperelliptic.

2. We define $\mathcal{Q}$ to be the full fibre subcategory of $\mathcal{F}$ of those Fano threefolds $f: X \to B$ that are smooth quartic.
Lemma 5.2. The categories fibred in groupoids $\mathcal{H} \to \text{Spec } \mathbb{Z}[1/2]$ and $\mathcal{Q} \to \text{Spec } \mathbb{Z}[1/2]$ are stacks.

Proof. For the statement on smooth quartics, we refer to [2]. We prove the statement about $\mathcal{H}$. Since $\mathcal{H}$ is a full subcategory of the stack $\mathcal{F}$, it is a prestack. It remains to show that any descent datum for any given fppf cover is effective. To do so, we consider a cartesian square

$$
\begin{array}{ccc}
X' & \longrightarrow & X \\
\downarrow f' & & \downarrow f \\
B' & \longrightarrow & B
\end{array}
$$

where $f : X \to B$ and $f' : X' \to B'$ are Fano threefolds with the latter being hyperelliptic and $g$ is an fppf morphism. Let $B = \bigcup_{i \in I} U_i$, $U_i = \text{Spec } R_i$ be an open affine cover such that for each $i \in I$ there are $q_4 \in R_i[x_0, \ldots, x_4, z_4]$, $q_2 \in R_i[x_0, \ldots, x_4]/c$ and $c \in R_i$ such that

$$(V_+(q_2 + cz, q_4), \mathcal{O}(1)) \cong (f^{-1}(U_i), \omega_f^{-1}(U_i)).$$

Now we can find a cover $B' = \bigcup_{i \in I, j \in J} V_{i,j}$, where $V_{i,j}$ is quasi compact and $g(V_{i,j}) = U_i$. Fix some $i, j$. Let $V_{i,j} = \bigcup_{m=1}^n W_m$ be a finite open affine cover. Set $W = \prod_{m=1}^n W_m$. Then $W$ is affine, $X' \times B' W$ is hyperelliptic and the map $W \to U_i$ induced by $g$ is faithfully flat. The corresponding ring map

$$\alpha : R_i \to S := \Gamma(W, \mathcal{O}_W)$$

is faithfully flat, hence injective. As $X' \times B' W \cong V_+(\alpha(q_2) + \alpha(c)z, \alpha(q_4)) \subseteq \mathbb{P}_S(1, 1, 1, 1, 2)$ is hyperelliptic we see that $\alpha(c) = 0$ by Lemma 4.9. Hence $c = 0$. As $i, j$ were chosen arbitrarily, this proves that $f : X \to B$ is hyperelliptic.

Proof of Theorem 6.1. That $\mathcal{H} \to \mathcal{F}$ is a representable closed immersion and $\mathcal{Q} \to \mathcal{F}$ is a representable open immersion is a reformulation of Proposition 4.7 and Proposition 4.8 respectively. For details, see [20]. That for any field $k$ with $2 \in k^\times$, we have $\mathcal{F}(k) = \mathcal{Q}(k) \sqcup \mathcal{H}(k)$ follows from Proposition 5.7.

6. First-order deformations of hyperelliptic Fano threefolds of type $(1,1,4)$

In this section, we determine the space of first-order deformations of hyperelliptic Fano threefolds of type $(1,1,4)$. Let $k$ be a field with $2 \in k^\times$, let $k[\epsilon] := k[x]/(x^2)$ be the ring of dual numbers over $k$ and let $f : X \to \text{Spec } k$ be a hyperelliptic Fano threefold of type $(1,1,4)$. A first-order lift of $X$ is given by a pair of morphisms

$$(\tilde{f} : \tilde{X} \to \text{Spec } k[\epsilon], i : X \to \tilde{X})$$

such that the square

$$
\begin{array}{ccc}
X & \longrightarrow & \tilde{X} \\
\downarrow f & & \downarrow \tilde{f} \\
\text{Spec } k & \longrightarrow & \text{Spec } k[\epsilon]
\end{array}
$$

is cartesian, where $g$ is the closed immersion given by $\epsilon \to 0$. Two lifts $(\tilde{f}' : \tilde{X}' \to \text{Spec } k[\epsilon], i' : X \to \tilde{X})$ and $(\tilde{f} : \tilde{X} \to \text{Spec } k[\epsilon], i : X \to \tilde{X})$ are considered to be equivalent if there is an isomorphism $\phi : \tilde{X}' \to \tilde{X}$ over $k[\epsilon]$ such that $\phi \circ i' = i$. An equivalence class of first-order lifts is called a first-order deformation.

We are interested in characterising first-order deformations of $X$ which are again hyperelliptic. For this note that $X$ comes with an involution associated to the double cover $\iota : X \to X$; see Definition 4.10. If $(\tilde{f} : \tilde{X} \to \text{Spec } k[\epsilon], i : X \to \tilde{X})$ is a first-order lift such that $\tilde{f} : \tilde{X} \to \text{Spec } k[\epsilon]$ is hyperelliptic, then there is an involution $\iota : \tilde{X} \to \tilde{X}$ which extends $\iota$. From the explicit description of $X$ as a weighted complete intersection it follows that a lift is hyperelliptic if and only if the involution of $X$ extends to the lift. This observation allows us to determine the space of deformations that are hyperelliptic.

Theorem 6.1. If $X$ is a hyperelliptic Fano threefold of type $(1,1,4)$ over a field $k$ with $2 \in k^\times$ with involution $\iota$, then the space of first-order deformations of $X$ that are hyperelliptic is the subspace of $\iota$-invariants

$$\text{H}^1(X, TX)^\iota \subseteq \text{H}^1(X, TX)$$

of the space of all first-order deformations.

Proof. Let $G = \mathbb{Z}/2\mathbb{Z}$. Then $G$ acts on $X$ via $\iota$. By the observation above and [4, Proposition 3.2.7], the space of first-order deformations of $X$ that are hyperelliptic is given by the equivariant sheaf cohomology group $\text{H}^1(X; G, TX)$. By [23, Theorem 1.4], the action of $G$ on $TX$ is faithful. Therefore we can apply [10, Proposition 5.2.3] to see that $\text{H}^1(X; G, TX) \cong \text{H}^1(X, TX)^G = \text{H}^1(X, TX)^\iota$.

$\square$
Let $k$ be an algebraically closed field of characteristic zero. We follow [18, §2] and say that a finitely presented algebraic stack $X$ over $k$ is \textit{geometrically hyperbolic over} $k$ if, for every smooth integral curve $C$ over $k$, every point $c \in C(k)$ and object $x \in X(k)$, the set of isomorphism classes of morphisms $f: C \to X$ with $f(c) \cong x$ is finite. We will show that geometric hyperbolicity implies the persistence of arithmetic hyperbolicity. The proof of this fact uses an inductive argument. The following lemma gives us the induction step.

**Lemma 7.1.** Let $k \subseteq L$ be an extension of algebraically closed fields of characteristic zero such that $L$ is of transcendence degree $1$ over $k$. Let $X$ be a finite type separated arithmetically hyperbolic Deligne-Mumford stack over $k$. If $X$ is geometrically hyperbolic over $k$, then $X_L$ is arithmetically hyperbolic over $L$.\[\text{Proof.}\] (If $X$ is a variety, then this is [14, Lemma 4.2]. We adapt the proof of loc. cit. to the setting of stacks.)

The notion of arithmetic hyperbolicity is independent of the chosen model; check for [19, Lemma 4.8]. We choose a $\mathbb{Z}$-finitely generated subring $A \subseteq k$ and a model $X$ for $X$ over $A$. Since $X$ is separated and Deligne-Mumford, it has a finite diagonal. The property of having a finite diagonal spreads out; see [30, B.3]. Hence after possibly replacing $A$ with a finitely generated extension contained in $K$, we may and do assume that $X$ has a finite diagonal. Note that $X$ is also a model for $X_L$.

Let $B \subseteq L$ be a $\mathbb{Z}$-finitely generated subring containing $A$. We will find a $\mathbb{Z}$-finitely generated subring $B' \subseteq L$ containing $B$ such that $\pi_0(X(B'))$ is finite. We may assume $B$ is not contained in $k$. Otherwise, we are done since $X$ is arithmetically hyperbolic over $k$.

The morphisms $\text{Spec } B \to \text{Spec } A$ and $\text{Spec } A \to \text{Spec } \mathbb{Z}$ are generically smooth and finitely presented. As smoothness spreads out, we can find finitely generated extensions $A \subseteq A' \subseteq B \subseteq B' \subseteq L$ such that $A' \subseteq B'$ and both $\text{Spec } B' \to \text{Spec } A'$ and $\text{Spec } A' \to \text{Spec } \mathbb{Z}$ are smooth. The scheme $C = \text{Spec } B'$ is integral and smooth of relative dimension $1$ over $\text{Spec } A'$. As $k$ is algebraically closed, the affine curve $C = C_k$ has $k$-sections. Hence, after possibly replacing $A'$ and $B'$ by finitely generated extensions still satisfying $A' \subseteq k$ and $B' \subseteq L$ and the smoothness properties above, there is a section $c \in C(A')$. Since $A'$ is smooth over $\mathbb{Z}$, it is in particular integrally closed. Therefore, as $X$ is arithmetically hyperbolic over $k$, by applying [19, Theorem 4.23], we see that $\pi_0(X(A'))$ is finite. For any morphism $f: C \to X$, we have $f(c) \in X(A')$. Therefore, we get an inclusion
\[
\pi_0(X(B')) = \pi_0(X(C)) = \text{Hom}_C(C, X) \subseteq \bigcup_{x \in \pi_0(X(A'))} \text{Hom}_C((C, c), (X, x)).
\]

Furthermore, consider the diagram:
\[
\begin{array}{ccc}
\text{Hom}_C((C, c), (X, x)) & \xrightarrow{\alpha} & \text{Hom}_A((C, c_k), (X, x_k)) \\
\downarrow & & \downarrow \\
\text{Hom}_C(C, X) = \pi_0(X(A')) & \xrightarrow{\pi_0(X(L))} & \pi_0(X(L)) = \text{Hom}(L, X)
\end{array}
\]

The natural inclusion $\iota$ is injective and the pullback map $\beta$ has finite fibres by [19, Proposition 4.19]. Hence $\alpha$ has finite fibres. Since $X$ is geometrically hyperbolic over $k$, the set $\text{Hom}_A((C, c_k), (X, x_k))$ is finite. From this, we deduce that $\pi_0(X(A'))$ is finite.\[\square\]

**Lemma 7.2.** Let $k \subseteq L$ be an extension of algebraically closed fields of characteristic zero and let $X$ be a finite type separated arithmetically hyperbolic Deligne-Mumford stack over $k$ such that, for every algebraically closed subfield $K \subseteq L$ containing $k$, the stack $X_K$ is geometrically hyperbolic over $K$. Then $X_L$ is arithmetically hyperbolic over $L$.\[\text{Proof.}\] As arithmetic hyperbolicity expresses the finiteness of points valued in $\mathbb{Z}$-finitely generated integral domains of characteristic zero, the stack $X_L$ is arithmetically hyperbolic if and only if $X_K$ is arithmetically hyperbolic for all algebraically closed subfields $K \subseteq L$ that have a finite transcendence degree over $k$. Therefore, we may and do assume that $L/k$ has a finite transcendence degree. Let $k = K_0 \subseteq K_1 \subseteq \ldots \subseteq K_n = L$ be a chain of extensions of algebraically closed fields each having transcendence degree one. We apply Lemma 7.1 inductively to see that $X_{K_i}$ is arithmetically hyperbolic for all $i$.\[\square\]

**Theorem 7.3.** Let $k$ be an uncountable algebraically closed field of characteristic zero. Let $X$ be a finite type separated Deligne-Mumford stack over $k$. If $X$ is arithmetically hyperbolic over $k$ and geometrically hyperbolic over $k$, then $X_L/L$ is arithmetically hyperbolic over $L$ and geometrically hyperbolic over $L$ for any algebraically closed field extension $k \subseteq L$.\[\text{Proof.}\] This is a combination of Lemma 7.2 with the fact that geometric hyperbolicity over uncountable fields persists [18, Lemma 2.4].\[\square\]

The following two lemmata will be used to prove that $\mathcal{F}$ is geometrically hyperbolic.
Lemma 7.4. Let $X$ be a finite type separated Deligne-Mumford stack over a field $k$ and let $C$ be a smooth integral curve over $k$ with function field $K$. Then the map $\pi_0(X(C)) \to \pi_0(X(K))$ is injective.

Proof. Let $g_1, g_2 \in X(C)$ be objects. Since the diagonal $\Delta : X \to X \times_k X$ is finite, the Isom-sheaf $\text{Isom}_{X(C)}(g_1, g_2)$ is a finite scheme over $C$. In particular, every generic section of

$\text{Isom}_{X(C)}(g_1, g_2) \to C$

extends by the valuative criterion of properness (uniquely) to a section. I.e. if $g_1, g_2$ are isomorphic over $K$, then they are isomorphic over $C$. □

Lemma 7.5. Let $f : X \to Y$ be a quasi-finite representable morphism of separated finite type Deligne-Mumford stacks over a field $k$. If $Y$ is geometrically hyperbolic, then $X$ is geometrically hyperbolic.

Proof. Let $C$ be a smooth integral curve over $k$ and $c \in C(k), x \in X(k)$ points. Let $y = f(x) \in Y(k)$. The set $\text{Hom}_k((C, c), (Y, y))$ is finite. Therefore it suffices to show that for any fixed $g \in \text{Hom}_k((C, c), (Y, y))$, there are only finitely many isomorphism classes of morphisms $\phi : (C, c) \to (X, x)$ such that $f \circ \phi \cong g$. Consider the generic point $\eta : \text{Spec} k(C) \to C$. As seen in [20, Proposition 2.10], if $f$ is quasi-finite, there are only finitely many possibilities for the image of $\eta$ in $X$ up to isomorphism. Hence by Lemma 7.4 there are only finitely many possibilities for the isomorphism class of $\phi$. □

8. Period map and arithmetic hyperbolicity

In this section, we will prove Theorem 1.6 and Theorem 1.1. The following well-known lemma will be used to prove the arithmetic hyperbolicity of the stack $\mathcal{F}$.

Lemma 8.1. Let $f : X \to Y$ be a morphism of finite type separated Deligne-Mumford stacks. If $f$ is injective on tangent spaces, then $f$ is quasi-finite.

Proof. Let $Y' \to Y$ be an étale surjective morphism with $Y'$ a scheme and let $X' \to X \times_Y Y'$ be an étale surjective morphism with $X'$ a scheme. Then the morphism $X' \to Y'$ induced by $f$ is injective on tangent spaces. Hence it is unramified and therefore quasi-finite; see [34, Tag 02V5, Tag 02BG]. This shows $f$ is quasi-finite.

Let $\mathcal{A}_g$ be the stack of principally polarized abelian varieties of dimension $g$. Recall that $\mathcal{A}_g$ is a finite type separated Deligne-Mumford stack over $\mathbb{Z}$; see [29]. Over the complex numbers, the intermediate Jacobian of a Fano threefold defines a complex-analytic period map to $\mathcal{A}_g^{an, \mathbb{C}}$. Our main result is that this period map has finite fibres in the setting of Fano threefolds of type $(1, 1, 4)$.

Theorem 8.2. The period map

$p : \mathcal{F}_C^{an} \to \mathcal{A}_g^{an, \mathbb{C}}$

is quasi-finite.

Proof. First, we consider the period map restricted to the smooth quartic locus. The infinitesimal Torelli problem for hypersurfaces is completely understood; see [8]. In particular, if $X$ is a smooth quartic, the differential of the period map

$$(dp)_X : H^4(X, T_X) \to \bigoplus_{p+q=n} \text{Hom}_\mathbb{C} \left( H^p(X, \Omega_X^p, H^{p+1}(X, \Omega_X^{-1}) \right)$$

is injective. Therefore, the period map restricted to $\mathcal{Q}$ is quasi-finite by Lemma 8.1.

Furthermore, by Theorem 6.1 if $X$ is a hyperelliptic Fano threefold over $\mathbb{C}$ with involution $\iota$ associated to the double cover, then the tangent space of $\mathcal{H}$ at the object $X$ is $H^1(X, T_X)$. Therefore, by [25, Theorem 1.3], the differential of the period map restricted to $\mathcal{H}$ is injective. Again by Lemma 8.1 we conclude that $p|_{\mathcal{H}}$ is quasi-finite.

Since $\mathcal{F}$ is the union of $\mathcal{Q}$ and $\mathcal{H}$ (Theorem 1.6), the period map $p$ is quasi-finite. □

Proof of Theorem 1.6. By [8], the stack $(\mathcal{A}_g)^{an, \mathbb{C}}$ is absolutely arithmetically hyperbolic. Moreover, by [18, Theorem 1.7], the stack $(\mathcal{A}_g)^{an, \mathbb{C}}$ is geometrically hyperbolic. Note by Proposition 2.6 (5) the stack $\mathcal{F}_C$ is Deligne-Mumford. Hence by [19, Theorem 6.4], Lemma 7.5 and Theorem 8.2 the stack $\mathcal{F}_C$ is arithmetically and geometrically hyperbolic. Hence by Theorem 7.3 the stack $X_L$ is arithmetically and geometrically hyperbolic for any algebraically closed field extension $L \supseteq \mathbb{C}$. □

Proof of Theorem 1.1. Let $K$ be a number field and let $S$ be a finite set of places on $K$. We may assume that all places lying over 2 are contained in $S$. As seen in [24, Proposition 2.10], if $B$ is a connected scheme and $f : X \to B$ is a Fano threefold, then the type is constant in the fibres of $f$. Hence, the set of isomorphism classes of Fano threefolds $X$ of type $(1, 1, 4)$ over $K$ with good reduction outside $S$ is the image

$\pi_0(\mathcal{F}(O_K, S)) \to \pi_0(\mathcal{F}(K))$.

Since $\mathcal{F}_S$ is arithmetically hyperbolic by Theorem 1.6 this set is finite by [19, Theorem 4.22]. □
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