NON-LINEAR FOURIER TRANSFORMS AND THE BRAVERMAN-KAZHDAN CONJECTURE

TSAO-HSIEN CHEN

Abstract. In this article we prove a conjecture of Braverman-Kazhdan in [BK] on the acyclicity of gamma sheaves in the de Rham setting. The proof relies on the techniques developed in [BFO] on Drinfeld center of Harish-Chandra bimodules and character $D$-modules. As an application, we show that the functors of convolution with gamma $D$-modules, which can be viewed as a version of non-linear Fourier transforms, commute with induction functors and are exact on the category of admissible $D$-modules on a reductive group.

1. Introduction

Let $k$ be an algebraic closure of a finite field. The Fourier-Deligne transform

$$F_V : D_c^b(V, \mathbb{Q}_\ell) \to D_c^b(V, \mathbb{Q}_\ell)$$

on the derived category of $\ell$-adic sheaves on a vector space $V$ over $k$ had found remarkable applications to number theory and representation theory. The Fourier-Deligne transform has the following remarkable properties: (1) $F_V$ is exact with respect to the perverse $t$-structure on $D_c^b(V, \mathbb{Q}_\ell)$ (see [KL]) and (2) when $V = g$ is a reductive Lie algebra, the functor $F_V$ commutes with induction functors (see [L]).

Let $G$ be a reductive group over $k$ and $\breve{G}$ be the dual group of $G$ over $\mathbb{C}$. In their work [BK] [BK1], Braverman and Kazhdan associated to each representation $\rho : \breve{G} \to \text{GL}(V_\rho)$ of the dual group $\breve{G}$, satisfying some mild technical conditions, a perverse sheaf $\Psi_{G,\rho}$ on $G$ called gamma sheaf and study the functor

$$F_{G,\rho} := (-) \ast \Psi_{G,\rho} : D_c^b(G, \mathbb{Q}_\ell) \to D_c^b(G, \mathbb{Q}_\ell)$$

of convolution with $\Psi_{G,\rho}$. The functor $F_{G,\rho}$ can be thought as a non-linear analogue of the Fourier-Deligne transform and they conjectured the following properties parallel to the properties (1) and (2) above:

Conjecture 1.1. (1) $F_{G,\rho}$ is exact with respect to the perverse $t$-structure. (2) $F_{G,\rho}$ commutes with induction functors.

It is shown in loc. cit. that the property (2) of the conjecture above follows from the following acyclicity of gamma sheaves:

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Conjecture 1.2 (Conjecture 9.2 [BK]). Let $B$ be a Borel subgroup and consider the quotient map $\pi_U : G \to G/U$, where $U$ is the unipotent radical of $B$. Then $(\pi_U)_{!*}\Psi_{G,\rho}$ is supported on $T = B/U \subset G/U$. In other words, for any $g \in G - B$ we have $H^*_c(gU, i^*\Psi_{G,\rho}) = 0$. Here $i : gU \to G$ denotes the inclusion map.

In [CN], Cheng and Ngô established Conjecture 1.2 for $G = GL_n$ by generalizing the argument in [BK] for $GL_2$.

The gamma sheaf $\Psi_{G,\rho}$ has an obvious analogue in the $D$-modules setting, which we call it gamma $D$-module, and the goal of this paper is to study $D$-modules analogue of Conjecture 1.1 and Conjecture 1.2.

We now state our main results. Fix a Borel subgroup $B$ and a maximal tours $T \subset B$. We begin with a construction, due to Braverman and Kazhdan, of gamma $D$-module $\Psi_{G,\lambda}$ attached to a collection of co-characters $\lambda = \{\lambda_1, ..., \lambda_r\}$ of the maximal torus $T$. The relation between $\Psi_{G,\lambda}$ and the gamma $D$-module attached to a representation of the dual group $\check{G}$ will be explained in §4.6. Let $\lambda = \{\lambda_1, ..., \lambda_r\}$ be a collection of co-characters of $T$. Consider the following maps $pr_{\lambda} := \prod \lambda_i : G^r_m \to T$, $tr : G^r_m \to G_a, (x_1, ..., x_r) \to \sum x_i$. Assume $pr_{\lambda}$ is stable under the action of the Weyl group $W$ and each $\lambda_i \in \lambda$ is $\sigma$-positive (see Definition 4.1), then Braverman and Kazhdan showed that

$$\Psi_{\lambda} := (pr_{\lambda})_{!*}tr^*(\mathbb{C}[x]e^x),$$

where $\mathbb{C}[x]e^x$ is the exponential $D$-module on $G_a = \text{Spec}(\mathbb{C}[x])$, is a (de Rham) local system on the image of $pr_{\lambda}$ equipped with a natural $W$-equivariant structure. Assume further that $pr_{\lambda}$ is onto. Then $\Psi_{\lambda}$ is a $W$-equivariant local system on $T$. Moreover, the $W$-equivariant structure on $\Psi_{\lambda}$ induces a $W$-action on the induction $\text{Ind}^G_T_{\check{C}B}(\Psi_{\lambda})$ and the gamma $D$-module $\Psi_{G,\lambda}$ is defined as the $W$-invariant factor of $\text{Ind}^G_T_{\check{C}B}(\Psi_{\lambda})$:

$$\Psi_{G,\lambda} := \text{Ind}^G_T_{\check{C}B}(\Psi_{\lambda})^W.$$ 

We prove the following equivalent form of Conjecture 1.2 in the $D$-module setting:

**Theorem 1.3.** (see Theorem 5.2) $\text{Av}_U(\Psi_{G,\lambda}) := (\pi_U)_{!*}\Psi_{G,\lambda}$ is supported on $T = B/U \subset G/U$.

In view of exactness property in Conjecture 1.1 we establish the following result. We call a holonomic $D$-module on $G$ admissible if the action of the center $Z$ of the universal enveloping algebra $U(g)$, viewing as invariant differential operators, is locally finite. We denote by $\mathcal{A}(G)$ the abelian category of admissible $D$-modules on $G$ and $D(\mathcal{A}(G))$ be the corresponding derived category. Denote by $D(G)_{\text{hol}}$ the derived category of holonomic $D$-modules on $G$. Consider the functor $F_{G,\lambda} := (-) \ast \Psi_{G,\lambda} : D(G)_{\text{hol}} \to D(G)_{\text{hol}}$ of convolution with gamma $D$-module $\Psi_{G,\lambda}$.

**Theorem 1.4.** (see Theorem 6.5) The functor $F_{G,\lambda}$ restricts to a functor

$$F_{G,\lambda} : D(\mathcal{A}(G)) \to D(\mathcal{A}(G))$$
which is exact with respect to the natural t-structure. That is, we have $F_{G,\Lambda}(M) \in \mathcal{A}(G)$ for $M \in \mathcal{A}(G)$.

The proofs of Theorem 1.3 and Theorem 1.4 make use of certain remarkable character $D$-module $M_{\theta}$ on $G$ with generalized central character $\theta \in \hat{T}/W$ and the results in [BFO] on the equivalence between Drinfeld center of Harish-Chandra bimodules and character $D$-modules. In more details, we construct for each $\theta \in \hat{T}/W$ a $W$-equivariant local system $\mathcal{E}_{\theta}$ on $T$ and consider the character $D$-module $M_{\theta} := \text{Ind}_{T \subset B}^{G}(\mathcal{E}_{\theta})^{W}$ (see §3.4 §3.5). Using the results in [BFO], we prove the acyclicity of $M_{\theta}$ similar to Conjecture 1.2 (see Theorem 3.8) and compute the convolution of $M_{\theta}$ with the gamma $D$-module $\Psi_{G,\Lambda}$ (see Theorem 4.5). A key step in the proof of the acyclicity of $M_{\theta}$ is the identification of the global section of $\mathcal{E}_{\theta}$ with certain element in the Drinfeld center of Harish-Chandra bimodules (see §3.6). Those results together with some simple vanishing lemmas (see §5.1) imply Theorem 1.3. Theorem 1.4 follows from a computation of the convolution of $\Psi_{\Lambda}$ with certain (pro-) local system on $T$ (see Lemma 4.6) and the exactness property of the (twisted) Harish-Chandra functor in [BFO, Corollary 3.4] (see also [CY]).

It seems that our methods may be applicable to the setting of $\ell$-adic sheaves. A new ingredient needed is an appropriate version of the results in [BFO] in the $\ell$-adic setting.

The paper is organized as follows. In Section 2 we recall some facts about algebraic groups and $D$-modules. In Section 3 we introduce the character $D$-module $M_{\theta}$ and prove the acyclicity of it using [BFO]. In Section 4 we recall the construction of gamma $D$-modules $\Psi_{G,\Lambda}$ and compute the convolution of $\Psi_{G,\Lambda}$ with $M_{\theta}$. In Section 5 we prove Theorem 5.2: the acyclicity of gamma $D$-module $\Psi_{G,\Lambda}$. In Section 6 we consider the functor $F_{G,\Delta} := (-) \ast \Psi_{G,\Lambda}$ of convolution with the gamma $D$-module and we prove that $F_{G,\Delta}$ commutes with induction functors (see Theorem 6.1) and is exact on the category of admissible $D$-modules on $G$ (see Theorem 6.5).

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2. Notations

2.1. Group data. Let $G$ be a reductive group over $\mathbb{C}$. Let $B \subset G$ be a Borel subgroup and $T \subset B$ be a maximal torus. Let by $\Lambda$ be the weight lattice. Let $\Phi$ be the root system determined by $(G, T)$ and $\Phi^+$ be the set of positive roots determined by $(G, B)$ and $\Pi$ be the set of simple roots. We denote by $W$ be the Weyl group and $W_{a} = W \rtimes \Lambda$ the affine Weyl group.

We denote by $\hat{G}$ the dual group of $G$ and $\hat{T}$ the dual maximal torus. We fix a non-degenerate $W$-invariant form $(\cdot, \cdot)$ on $t$ and use it to identify $t^*$ with $\mathfrak{t}$.

1The author learned the existence of $\mathcal{E}_{\theta}$ from R.Bezrukavnikov.
We denote by $\mathcal{B} = G/B$ the flag variety, $X = G/N$ the basic affine space, and $Y = X \times X$.

We denote by $\mathfrak{g}, \mathfrak{b}, \mathfrak{t}$ (resp. $\tilde{\mathfrak{g}}, \tilde{\mathfrak{b}}, \tilde{\mathfrak{t}}$) the Lie algebras of $G, B, T$ (resp. $\tilde{G}, \tilde{B}, \tilde{T}$). For any $\xi \in \tilde{T}$ (resp. $\mu \in \tilde{t}$) we write $[\xi] \in \tilde{T}/W$ (resp. $[\mu] \in \tilde{T} = \tilde{t}/\Lambda$) for its image in $\tilde{T}/W$ (resp. $\tilde{T}$).

We denote by $G_{rs}$ (resp. $T_{rs}$) be the open subset consisting of regular semi-simple elements in $G$ (resp. $T$).

2.2. $D$-modules. For any smooth variety $X$ over $\mathbb{C}$ we denote by $\mathcal{M}(X)$ the abelian category of $D$-modules on $X$ and $\mathcal{M}(X)_{hol}$ the full subcategory of holonomic $D$-modules. We write $D(X)$ for the bounded derived category $D$-modules on $X$ and $D(X)_{hol}$ for the bounded derived category of holonomic $D$-modules on $X$. We denote by $\Theta_X$ and $\mathcal{D}_X$ the sheaf of functors on $X$ and the sheaf of differential operators on $X$ respectively. For a $\mathcal{F} \in D(X)$, we denote by $\mathcal{H}^i(\mathcal{F}) \in \mathcal{M}(X)$ its $i$-th cohomology $D$-module.

Let $f : X \to Y$ be a map between smooth varieties. Then we have functors $f_*, f^!$ between $D(X)$ and $D(Y)$ and functors $f^*, f_!$, $f_*$ between $D(X)_{hol}$ and $D(Y)_{hol}$. Note all functors above are understood in the derived sense. We denote by $\mathcal{D}$ the duality functor on $D(X)_{hol}$. We define $f^0 := f^![\dim Y - \dim X]$. When $Y = \text{Spec}(\mathbb{C})$ is a point we sometimes write $R\Gamma_{\text{dr}}(\mathcal{M}) := f_*(\mathcal{M})$ and $H^i_{\text{dr}}(\mathcal{M}) := \mathcal{H}^i(f_*(\mathcal{M}))$.

For $\mathcal{M}, \mathcal{M}' \in D(X)_h$, we define $\mathcal{M} \otimes \mathcal{M} = \Delta^*(\mathcal{M} \boxtimes \mathcal{M}')$ and $\mathcal{M} \boxtimes^! \mathcal{M} = \Delta^!(\mathcal{M} \boxtimes \mathcal{M}')$ where $\Delta : X \to X \times X$ is the diagonal embedding.

For a $D$-module $\mathcal{M}$ on $X$ we denote by $\Gamma(\mathcal{M})$ (resp. $R\Gamma(\mathcal{M})$) the global sections (resp. derived global sections) of $\mathcal{M}$ regrading as quasi-coherent $\mathcal{O}_X$-module.

By a local system on $X$ we mean a $\mathcal{O}_X$-coherent $D$-module on $X$, a.k.a. a vector bundle on $X$ with a flat connection.

Assume $f : X \to Y$ is a principal $T$-bundle. A $D$-module $\mathcal{F}$ on $X$ is called $T$-monodromic if it is weakly $T$-equivariant (see [BB1, Section 2.5]). We denote by $\mathcal{M}(X)_{mon}$ the category consisting of $T$-monodromic $D$-modules on $X$. A object $\mathcal{F} \in D(X)$ is called $T$-monodromic if $\mathcal{H}^i(\mathcal{F}) \in \mathcal{M}(X)_{mon}$ for all $i$. We denote by $D(X)_{mon}$ the full subcategory consisting of $T$-monodromic objects. Let $\mathcal{F} \in \mathcal{M}(X)_{mon}$. For any $\mu \in \tilde{t}$ (or $\mu \in \mathfrak{t}$) we denote $\Gamma^\mu(\mathcal{F})$ (resp. $R\Gamma^\mu(\mathcal{F})$) the maximal summand of $\Gamma(X, \mathcal{F})$ (resp. $R\Gamma(X, \mathcal{F})$) where $U(\mathfrak{t})$ (acting as infinitesimal translations along the action of $T$) acts with the generalized eigenvalue $\mu$.

3. Drinfeld center and Character $D$-modules $\mathcal{M}_\theta$

In this section we attach to each $\theta \in \tilde{T}/W$ a $W$-equivariant local system $\mathcal{E}_\theta$ on $T$ and use it to construct a character $D$-module $\mathcal{M}_\theta$ on $G$. The main result of this section is
an acyclicity property of $M_\theta$ (see Theorem 3.8). The proof uses the results of Drinfeld center of Harish-Chandra bimodules and character $D$-modules in [BFO].

3.1. Functors between equivariant categories. Let $H$ be a smooth algebraic group acting on a smooth variety $Z$. We denote by $\mathcal{M}_H(Z)$ (resp. $\mathcal{M}_H(Z)_{hol}$) the category of $H$-equivariant $D$-modules (resp. holonomic $D$-modules) on $Z$. We denote by $D_H(Z)$ the $H$-equivariant derived category of $D$-modules on $Z$ and $D_H(Z)_{hol}$ the $H$-equivariant derived category of holonomic $D$-modules on $Z$.

Let $f : Z \to Z'$ be a map between two smooth varieties. Assume $H$ acts on $Z$ and $Z'$ and $f$ is compatible with those $H$-actions. Then the functors $f^*, f^!; f_*, f_!$ lift to functors between $D_H(Z)$ and $D_H(Z')$.

For any closed subgroup $H' \subset H$ the forgetful functor $\text{obl}_H^{H'} : D_H(Z) \to D_{H'}(Z)$ admits a right adjoint

\[ \text{Ind}_H^{H'} : D_{H'}(Z) \to D_H(Z) \]

Consider the quotient map $\pi_U : G \to X = G/U$. It induces functors

\[ \text{Av}_U : D_B(G) \to D_B(G) \xrightarrow{\text{obl}_B^G} D_B(\pi_U)^* \to D_B(X) \]

\[ \text{Av}_U^! : D_B(G)_{hol} \to D_B(G)_{hol} \xrightarrow{\text{obl}_B^G} D_B(X)_{hol} \]

between equivariant derived categories. Here $G$ acts on $G$ by the conjugation action. We call $\text{Av}_U$ (resp. $\text{Av}_U^!$) star averaging functor (resp. shriek averaging functor). The functor $\text{Av}_U$ admits a right adjoint

\[ \text{Av}_G := \text{Ind}_B^G \circ \pi_U^* : D_B(X) \to D_B(G) \to D_G(G). \]

We shall recall Lusztig’s induction and restriction functors. Consider

\[ T = B/U \xleftarrow{\tilde{c}} B \xrightarrow{u} G \]

We define

\[ \text{Ind}_{T \subset B}^G := \text{Ind}_B^G \circ u_* \circ r' : D_T(T) \to D_G(G), \quad \text{Res}_{T \subset B}^G := r_* \circ u' : D_G(G) \to D_T(T). \]

Here is an equivalent definition of $\text{Ind}_{T \subset B}^G$: consider the Grothendieck-Springer simultaneous resolution:

\[ \begin{array}{ccc}
\tilde{G} & \xrightarrow{\tilde{c}} & T \\
\downarrow{\tilde{q}} & & \downarrow{q} \\
G & \xrightarrow{c} & T/W
\end{array} \]

where $\tilde{G}$ consists of pairs $(g, hB) \in G \times G/B$ such that $h^{-1}gh \in B$, the map $\tilde{c}$ is given by $\tilde{c}(x, hB) = h^{-1}gh \mod U$, and $\tilde{q}, q$ are the natural projection maps. The group $G$ acts
on $\tilde{G}$ by the formula $x(g, hB) = (xgx^{-1}, xhB)$ and $\tilde{q}$ (resp. $\tilde{c}$) is $G$-equivariant where $G$ acts on $G$ (resp. $T$) via the conjugation action (resp. trivial action). We have

\begin{equation}
(3.4) \quad \text{Ind}_{T \subset B}^G \simeq \tilde{q}_* c^* : D_T(T) \to D_G(G).
\end{equation}

Consider the following maps

$G \xleftarrow{p} G \times G/B \xrightarrow{q} Y/T = (G/U \times G/U)/T$

where $p(g, xB) = g$ and $q(g, xB) = (gxU, xU)$ mod $T$. The group $G$ acts on $G$, $G \times B$ and $Y/T$ by the formulas $a \cdot g = aga^{-1}$, $a \cdot (g, xU) = (aga^{-1}, axB)$, $a(xU, yU) = (axU, ayU)$. One can check that $p$ and $q$ are compatible with those $G$-actions.

Following [MV], we consider the functor

\begin{equation}
(3.5) \quad \text{HC} = q_* p^![−\dim G/B] : D(G) \to D(Y/T).
\end{equation}

The functor above admits a right adjoint $\text{CH} = p_* q^*[\dim G/B] : D(Y/T) \to D(G)$. We use the same notations for the corresponding functors between $G$-equivariant derived categories $D_G(G)$ and $D_G(Y/T)$. Following [G], we call HC the Harish-Chandra functor.

Recall the following well-known fact:

**Lemma 3.1 (Theorem 3.6 [MV]).**

1. Let $\sigma : \mathcal{N} \to \mathcal{N}$ be the Springer resolution of the nilpotent cone $\mathcal{N}$ and let $Sp := \sigma_0 \mathcal{N}$ be the Springer $D$-module. For any $F \in D(G)$ there is canonical isomorphism

$\text{CH} \circ \text{HC}(F) \simeq F \ast Sp$.

2. We have a canonical isomorphism $\text{CH} \circ \text{HC} \simeq \text{Av}_G \circ \text{Av}_U[−]$.

3. The identity functor is a direct summand of $\text{CH} \circ \text{HC} \simeq \text{Av}_G \circ \text{Av}_U[−]$.

We will need the following properties of induction functors.

**Proposition 3.2 (Theorem 2.5 and Proposition 2.9 in [BK1]).**

1. For any local system $F$ on $T$, we have $\text{Ind}_{T \subset B}^G(F) \in M_G(G)$.

2. Let $F$ be a local system on $T$. For any $w \in W$, there is a canonical isomorphism

$\text{Ind}_{T \subset B}^G(F) \simeq \text{Ind}_{T \subset B}^G(w^*F)$.

3. Let $W' \subset W$ be a subgroup and $F$ be a $W'$-equivariant local system on $T$. There is a canonical $W'$-action on $\text{Ind}_{T \subset B}^G(F)$.

4. Let $F \in D(T)$ and $\mathcal{G} \in D_G(G)$. Assume $\mathcal{F}' := \text{Av}_U(\mathcal{G})$ is supported on $T = B/U$. There is an isomorphism

$\text{Ind}_{T \subset B}^G(F) \ast \mathcal{G} \to \text{Ind}_{T \subset B}^G(F \ast \mathcal{F}')$. 
3.2. **Hecke categories.** Consider the left $G$ and right $T \times T$ actions on $Y = G/U \times G/U$. For any $\xi, \xi' \in \hat{T} \simeq i/\Lambda$ we denote by $M_{\xi, \xi'}$ the category of $G$-equivariant $D$-modules on $G/U \times G/U$ which are $T \times T$-monodromic with generalized monodromy $(\xi, \xi')$, that is, $U(t) \otimes U(t)$ (acting as infinitesimal translations along the right action of $T \times T$) acts locally finite with generalized eigenvalues in $(\xi, \xi')$. Consider the quotient $Y/T$ where $T$ acts diagonally from the right. The group $T$ acts on $Y/T$ via the formula $t(xU, yU) \mod T = (xU, ytU) \mod T$. To every $\xi \in \hat{T}$ we denote by $M_\xi$ the category of $G$-equivariant $T$-monodromic $D$-modules on $Y/T$ with generalized monodromy $\xi$. We write $D(M_{\xi, \xi'})$ and $D(M_\xi)$ for the corresponding $G$-equivariant monodromic derived categories.

The groups $B$ and $T \times T$ act on $X = G/U$ by the formula $b(xU) = bxb^{-1}U$, $(t,t')(xU) = txt'U$. For any $(\xi_1, \xi_2) \in \hat{T} \times \hat{T}$ we write $H_{\xi_1, \xi_2}$ for the category of $U$-equivariant $T \times T$-monodromic $D$-modules on $X$ with generalized monodromy $(\xi_1, \xi_2)$. For any $\xi \in \hat{T}$ we write $H_\xi$ for the category of $B$-equivariant $T$-monodromic $D$-modules on $X$ with generalized monodromy $\xi$, where $B$ acts on $X$ by the same formula as before and $T$ acts on $X$ by the formula $t(xU) = txU$. We denote by $D(H_\xi)$ (resp. $D(H_{\xi_1, \xi_2})$) the corresponding $B$-equivariant (resp. $U$-equivariant) monodromic derived category.

Consider the embedding $i : X \to Y, gU \to (eU, gU)$.

**Lemma 3.3.** [MV]

1. The functor $i^0 = i^![\dim X] : D_G(Y) \to D_U(X)$ is an equivalence of categories with inverse given by $(i^0)^{-1} := \text{Ind}_B^G \circ i_*[\dim G - \dim B]$.

2. We have $i^0 HC \simeq Av_U$.

We have the convolution product $D_G(Y) \times D_G(Y) \to D_G(Y)$ given by $(\mathcal{F}, \mathcal{F}') \to (p_{13}_* (p_{12}^* \mathcal{F} \otimes p_{23}^* \mathcal{F}'))$. Here $p_{ij} : G/U \times G/U \times G/U \to Y = G/U \times G/U$ is the projection on the $(i, j)$-factors. The convolution product on $D_G(Y)$ restricts to a convolution product on $D(M_{\xi, \xi'-1})$. The equivalence $i^0 : D_G(Y) \simeq D_U(X)$ above induces convolution products on $D_U(X)$ and $D(H_\xi)$. In addition, there is an action of $D_U(X)$ on $D(X)$ by right convolution. The convolution operation will be denoted by $\ast$.

We will need the following lemma. Let $X$ be an algebraic variety with an action of an affine algebraic group $G$. Denote the action map by $a : G \times X \to X$.

**Lemma 3.4 (Lemma 2.1 [BFQ]).** For any $A \in D(G)$, $\mathcal{F} \in D(X)$ We have a canonical isomorphism

$$R\Gamma(a_*(A \boxtimes \mathcal{F})) \simeq R\Gamma(A) \otimes_{U(\mathfrak{g})} L R\Gamma(\mathcal{F}).$$

3.3. **Character $D$-modules.** We denote by $CS(G)$ the category of finitely generated $G$-equivaraint $D$-modules on $G$ such that the action of the center $Z \subset U(\mathfrak{g})$, embedding as left invariant differential operators, is locally finite. To every $\theta \in \hat{T}/W = \hat{i}/W_a$, we denote by $CS_\theta(G)$ the category of finitely generated $G$-equivaraint $D$-modules on
modules on $G$ such that the action of the center $Z \subset U(\mathfrak{g})$ is locally finite and has generalized eigenvalues in $\theta$. We denote by $D(CS(G))$ (resp. $D(CS_\theta)$) the minimal triangulated full subcategory of $D_G(G)$ containing all objects $\mathcal{M} \in D_G(G)$ such that $\mathcal{M}(\mathfrak{m}) \in CS(G)$ (resp. $\mathfrak{m}(\mathfrak{m}) \in CS_\theta(G)$). We call $CS(G)$ and $CS_\theta(G)$ (resp. $D(CS(G))$ and $D(CS_\theta)$) the category (resp. derived category) of character $D$-modules on $G$ and character $D$-modules on $G$ with generalized central character $\theta$.

We have the following:

**Proposition 3.5.** (1) Let $\xi \in \tilde{T}$ be a lifting of $\theta$. Then $D(CS_\theta(G))$ is generated by the image of $D(H_\xi)$ (resp. $D(M_\xi)$) under the functor $\text{Av}_G : D_B(X) \to D_G(G)$ (resp. $\text{CH} : D_G(Y/T) \to D_G(G)$).

(2) Let $\mathfrak{g} \in CS(G)_\theta$. We have

$$HC(\mathfrak{g}) \in \bigoplus_{\xi \in \tilde{T}, \xi \rightarrow \theta} D(M_\xi), \quad (\text{resp. } \text{Av}_U(\mathfrak{g}) \in \bigoplus_{\xi' \tilde{T}, [\xi'] = \theta} D(H_\xi).)$$

(3) The functors $\text{Ind}_{T \subset B}^G$ and $\text{Res}_{T \subset B}^G$ preserve the derived categories of character $D$-modules. Moreover, the resulting functors $\text{Ind}_{T \subset B}^G : D(CS(T)) \to D(CS(G))$, $\text{Res}_{T \subset B}^G : D(CS(G)) \to D(CS(T))$ are independent of the choice of the Borel subgroup $B$ and $t$-exact with respect to the natural $t$-structures on $D(CS(G))$ and $D(CS(T))$.

(4) Let $CS_{T}(G) \subset CS(G)$ be the full subcategory generated by the image of $\text{Ind}_{T \subset B}^G : CS(T) \to CS(G)$. For any $\mathfrak{g} \in CS_{T}(G)$, the local system $\mathcal{F} = \text{Res}_{T \subset B}^G(\mathfrak{g}) \in CS(T)$ carries a canonical $W$-equivariant structure, moreover, there is a canonical isomorphism

$$\text{Ind}_{T \subset B}^G(\mathcal{F})^W \simeq \mathfrak{g}.$$  

Here $\text{Ind}_{T \subset B}^G(\mathcal{F})^W$ is the $W$-invariant factor of $\text{Ind}_{T \subset B}^G(\mathcal{F})$ for the $W$-action constructed in Proposition 3.2.

**Proof.** Part (1), (2), (3) are proved in [G, L]. We now prove part (4). We first show that $\mathcal{F} = \text{Res}_{T \subset B}^G(\mathfrak{g})$ is canonically $W$-equivariant. Let $x \in N(T)$ and $w \in N(T)/T = W$ be its image in the Weyl group. Denote $B_x := \text{Ad}_x B$. Consider the following commutative diagram

$$\begin{array}{ccc}
T & \xrightarrow{w} & B \\
\downarrow & & \downarrow \text{Ad}_x \\
T & \xleftarrow{B_x} & G \\
\end{array}$$

where $w : T \to T$ the natural action of $w \in W$ on $T$ and the horizontal arrows are the natural inclusion and projection maps. The base change theorems and the fact that the functors $\text{Res}_{T \subset B}^G$ and $\text{Res}_{T \subset B}^{G_x}$ are canonical isomorphic (see part (3)) imply

$$(3.6) \quad \text{Res}_{T \subset B}^G(\text{Ad}_x^* \mathfrak{g}) \simeq w^* \text{Res}_{T \subset B}^G(\mathfrak{g}) \simeq w^* \text{Res}_{T \subset B}^G(\mathfrak{g}).$$
Since $\mathfrak{g}$ is $G$-conjugation equivariant, we have a canonical isomorphism $c_x : \mathfrak{g} \simeq \text{Ad}_x^* \mathfrak{g}$. Applying $\text{Res}^G_{T_C B}$ to $c_x$ and using (3.6) we get

$$
(3.7) \quad \mathcal{F} = \text{Res}^G_{T_C B}(\mathfrak{g}) \simeq \text{Res}^G_{T_C B}(\text{Ad}_x^* \mathfrak{g}) \simeq w^* \text{Res}^G_{T_C B}(\mathfrak{g}) = w^* \mathcal{F}.
$$

We claim that the isomorphism above depends only the image $w$ and we denote it by

$$
(3.8) \quad c_w : \mathcal{F} \simeq w^* \mathcal{F}.
$$

To prove the claim it is enough to check that for $x \in T$ the restriction of the isomorphism (3.7) to $T_{rs}$ is equal to the identity map. By [G], the restriction $\mathcal{F}|_{T_{rs}}$ is canonically isomorphic to $\mathfrak{g}|_{T_{rs}}$ and the map in (3.7) is equal to the restriction of $c_x$ to $T_{rs}$. Since the adjoint action $\text{Ad}_x : G \to G$ is trivial on $T$, the claim follows from the fact that any $T$-equivariant structure of a local system on $T$ is trivial. The $G$-conjugation equivariant structure on $\mathfrak{g}$ implies $\{c_w\}_{w \in W}$ satisfies the required cocycle condition, hence, the data $(\mathcal{F}, \{c_w\}_{w \in W})$ defines a $W$-equivariant structure on $\mathcal{F} = \text{Res}^G_{T_C B}(\mathfrak{g})$. We shall prove $\text{Ind}^G_{T_C B}(\mathcal{F})^W \simeq \mathfrak{g}$. Let $j : G_{rs} \to G$ the natural inclusion and $c_{rs} : G_{rs} \to T_{rs}/W$ the restriction of the Chevalley map $c : G \to T/W$ to $G_{rs}$. Note that we have $\mathfrak{g} \simeq \text{Ind}_{G_{rs}}^{G} (\mathfrak{g}|_{G_{rs}})$ and $\text{Ind}^G_{T_C B}(\mathcal{F})^W \simeq j_{rs}(q^*_{rs}(\mathcal{F}))$, where $\mathcal{F} \in \text{Loc}(T_{rs}/W)$ is the descent of $\mathcal{F}|_{T_{rs}}$ along the map $q_{rs} : T^{rs} \to T^{rs}/W$. So we reduce to show $\mathfrak{g}|_{G_{rs}} \simeq c_{rs}^* (\mathcal{F})$ and this follows again from the fact that $\mathfrak{g}|_{T_{rs}} \simeq \mathcal{F}|_{T_{rs}} \in \text{Loc}(W(T_{rs}))$.

\[ \square \]

3.4. Local systems $L_\xi$, $\tilde{L}_\xi$, $E_\xi$, and $E_\theta$. Let $\xi \in \tilde{T}$. It defines an one dimensional representation $\chi_\xi$ of $\pi_1(T)$ via $\tilde{T} \simeq \text{Hom}(\pi_1(T), \mathbb{G}_m)$, and the corresponding local system on $T$, denoted by $L_\xi$, is called the Kummer local system on $T$ associates to $\xi$. Let $W_\xi$ be the stabilizer of $\xi$ in $W$. Let $S := \text{Sym}(t)$ and $S_\pm$ denote the argumentation ideal of $S$. Define $S_n := S/S^n$, $n \in \mathbb{Z}_{\geq 0}$ and $S_\xi := S/S \cdot S^W_\xi$. Note the natural action of $W$ on $S$ induces an action of $W_\xi$ on $S_\xi$. We view $S_n$ and $S_\xi$ as $t$-module by restricting the natural $S = \text{Sym}(t)$-actions on $S_n$ and $S_\xi$ to $t$. Obviously, the $t$-actions are nilpotent.

Consider the following representations $\rho_n$ of $\pi_1(T)$ in the space $S_n$, by identifying $\pi_1(T)$ with a lattice in $t$ and defining

$$
(3.9) \quad \rho_n(t) \cdot s = \exp(t) \cdot s
$$

where $t \in t$, $s \in S_n$. Similarly, we consider the representation $\rho^\text{uni}_\xi$ of $W_\xi \ltimes \pi_1(T)$ in the space $S_\xi$ by setting

$$
(3.10) \quad \rho^\text{uni}_\xi(w,t) \cdot u = w(\exp(t) \cdot u)
$$

where $(w,t) \in W_\xi \ltimes \pi_1(T)$, $u \in S_\xi$.

Since the character $\chi_\xi$ is fixed by $W_\xi$, it extends to a character of $W_\xi \ltimes \pi_1(T)$ which we still denote by $\chi_\xi$. We define the following representation of $W_\xi \ltimes \pi_1(T)$:

$$
(3.11) \quad \rho_\xi := \rho^\text{uni}_\xi \otimes \chi_\xi.
$$
The induction $\text{Ind}_{W_{\xi} \ltimes \pi_1(T)}^{W \times \pi_1(T)} \rho_\xi$ of $\rho_\xi$ depends only on the image $\theta = [\xi] \in \check{T}/W_\xi$ and we denote the resulting representation by

$$\rho_\theta := \text{Ind}_{W_{\xi} \ltimes \pi_1(T)}^{W \times \pi_1(T)} \rho_\xi.$$ 

**Definition 3.6.**  
1. We denote by $\mathcal{E}_\theta$ the $W$-equivariant local systems on $T_{\theta}$ corresponding to the representation $\rho_\theta$. 
2. We denote by $\mathcal{E}_\xi$ and $\mathcal{E}_{\xi}^{\text{uni}}$ the $W_\xi$-equivariant local systems on $T$ corresponding to the representation $\rho_\xi$ and $\rho_\xi^{\text{uni}}$. 
3. We denote by $\mathcal{L}_n$ (resp $\mathcal{L}_n^{\text{uni}}$) the local systems on $T$ corresponding to the representation $\rho_n$ (resp. $\rho_n \otimes \chi_\xi$). Consider the projective system

$$\mathcal{L}_\xi^1 \leftarrow \mathcal{L}_\xi^2 \leftarrow \mathcal{L}_\xi^3 \cdots$$

and we define the following (pro-)local system on $T$

$$\hat{\mathcal{L}}_\xi = \varprojlim (\mathcal{L}_\xi^n).$$

3.5. **Character $D$-module $M_\theta$.** Let $\theta \in \check{T}/W$ and let $\mathcal{E}_\theta$ be the $W$-equivaraint local system constructed in §3.3. Consider $\text{Ind}_{T \subset B}(\mathcal{E}_\theta)$ which is a $G$-equivariant $D$-module on $G$. The $W$-equivariant structure on $\mathcal{E}_\theta$ defines a $W$-action on $\text{Ind}_{T \subset B}(\mathcal{E}_\theta)$.

**Definition 3.7.** We define $M_\theta$ to be the $W$-invariant factor of $\text{Ind}_{T \subset B}(\mathcal{E}_\theta)$

$$M_\theta := \text{Ind}_{T \subset B}(\mathcal{E}_\theta)^W.$$
3.6. Drinfeld center of Harish-Chandra bimodules. In this section we construct certain elements in the Drinfeld center of Harish-Chandra bimodules and identify them with the local systems $\mathcal{E}_\xi$ under the global section functor.

We first recall facts about Harish-Chandra bimodules following [BG, BFO]. Let $U = U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. Let $Z = Z(U)$ be the center of $U$. Consider the dot action of $W$ on $\mathfrak{t}$, $w \cdot t = w(t + \rho) - \rho$, where $\rho$ is the sum of of positive roots. We have the Harish-Chandra isomorphism $hc : Z \simeq \mathcal{O}(\mathfrak{t})^W$ such that for any $\lambda \in \mathfrak{t}$ the center $Z$ acts on the Verma module associated to $\lambda$ via $z \mapsto hc(z)(\lambda)$. For any $\lambda \in \mathfrak{t}$ we write $m_\lambda$ for the corresponding maximal ideal and denote by $I_\lambda$ the maximal ideal of $Z$ corresponding to $m_\lambda$ under the Harish-Chandra isomorphism.

Consider the extended universal enveloping algebra $\widetilde{U} = U \otimes_{\mathbb{Z}} \mathcal{O}(\mathfrak{t})$, where $Z$ acts on $\mathcal{O}(\mathfrak{t})$ via the Harish-Chandra isomorphism. We denote by $\widetilde{U}_\lambda = \widetilde{U} / \widetilde{U} m_\lambda$ and $\widetilde{U}_\lambda = \lim_{\to}(\widetilde{U} / \widetilde{U} m_\lambda)$.

We denote by $\mathcal{HC}_\lambda$ the category of finitely generated Harish-Chandra bimodules over $\widetilde{U}_\lambda$, that is, finitely generated continuous $\widetilde{U}_\lambda$-bimodules such that the diagonal action of $\mathfrak{g}$ is locally finite. We denote by $D(\mathcal{HC}_\lambda)$ the corresponding derived category. The tensor product $M \otimes M' := M \otimes_{\widetilde{U}_\lambda} M'$, $M, M' \in HC_\lambda$ (resp. $M \otimes^L M' := M \otimes^L_{\widetilde{U}_\lambda} M'$, $M, M' \in D(\mathcal{HC}_\lambda)$) defines a monoidal structure on $\mathcal{HC}_\lambda$ (resp. $D(\mathcal{HC}_\lambda)$).

Proposition 3.9 (BG, BFO). Let $\xi \in \mathfrak{t} = \mathfrak{g}/\Lambda$ and $\lambda \in \mathfrak{t}$ be a dominate regular lifting of $\xi$. The functor
$$R\Gamma^{\mathfrak{t}, -\lambda - 2\rho} : (D(M_{\xi, \lambda}^{-1}), \ast) \simeq (D(\mathcal{HC}_\lambda), \otimes^L)$$
is an equivalence of monoidal categories.

Let $\xi \in \mathfrak{t}$ and $\lambda \in \mathfrak{t}$ a dominant regular lifting of $\xi$. By Proposition 3.9 we have an equivalence of monoidal categories

(3.13) $M : (D(H_{\xi, \lambda}), \ast) \simeq (D(M_{\xi, \lambda}^{-1}), \ast) \simeq (D(\mathcal{HC}_\lambda), \otimes^L)$.

Consider the following full abelian subcategory $H_{\xi, \lambda}^t := M^{-1}(\mathcal{HC}_\lambda) \subset D(H_{\xi, \lambda})$. For any $M, M' \in H_{\xi, \lambda}^t$ we define
$$M \ast^t M' := M^{-1}(M(M) \otimes M(M')) \in H_{\xi, \lambda}^t.$$ One can check that $\ast^t$ defines a monoidal structure on $H_{\xi, \lambda}^t$ and the functor $M$ induces an equivalence of abelian monoidal categories

(3.14) $M^t : (H_{\xi, \lambda}^t, \ast^t) \simeq (\mathcal{HC}_\lambda, \otimes)$.

For each $\lambda \in \mathfrak{t}$ we consider the following $S$-module $S^\lambda_{\xi}$: we have $S^\lambda_{\xi} = S_{\xi}$ as vector spaces and the $S$-module structure is given by $s \cdot m = a_{\lambda}(s)m$, $s \in S, m \in S^\lambda_{\xi}$, where $a_{\lambda} : S \to S, f \mapsto (x \mapsto f(x + \lambda))$. We define

(3.15) $Z_{\lambda} := \widetilde{U}_\lambda \otimes_S S^\lambda_{\xi} \in \mathcal{HC}_\lambda$.
where $S$ acts on $\tilde{U}_\lambda$ via the the map $p_\lambda : S \simeq Z(\tilde{U}) \to Z(\tilde{U}_\lambda)$.

We denote by $Z(\mathcal{H}_\lambda, \otimes)$ (resp. $Z(H^i_{\xi} \otimes)$) the Drinfeld center of the monoidal category $Z(\mathcal{H}_\lambda, \otimes)$ (resp. $Z(H^i_{\xi} \otimes)$). Recall an element in $Z(\mathcal{H}_\lambda, \otimes)$ consists of an element $\mathcal{M} \in \mathcal{H}_\lambda$ together with family of compatible isomorphisms $b_{\mathcal{M}} : \mathcal{M} \otimes \mathcal{F} \simeq \mathcal{M} \otimes \mathcal{F}$ for $\mathcal{F} \in \mathcal{H}_\lambda$.

**Proposition 3.10.** Let $\lambda \in \hat{\xi}$ be a dominant regular weight and $\xi \in \hat{T}$ be its image.

1. To every $\mathcal{M} \in \mathcal{H}_\lambda$ there is a canonical isomorphism
   \[ b_\mathcal{M} : Z_\lambda \otimes \mathcal{M} \simeq \mathcal{M} \otimes Z_\lambda \]
   such that the data $(Z_\lambda, b_\mathcal{M})_{\mathcal{M} \in \mathcal{H}_\lambda}$ defines an element in the Drinfeld center $Z(\mathcal{H}_\lambda, \otimes)$.

2. We have $\mathcal{M}(\mathcal{E}_\xi) \simeq Z_\lambda$.

**Proof.** Proof of (1). Consider the map $m_\lambda : S \otimes S \xrightarrow{a_\lambda \otimes a_\lambda = \lambda^2} S \otimes S \xrightarrow{\rho \otimes \rho = \lambda^2} Z(\tilde{U}_\lambda \otimes \tilde{U}^{-\lambda}_{\lambda - 2\rho})$. To every $\mathcal{M} \in \mathcal{H}_\lambda$, the map above defines an action of $S \otimes S$ on $\mathcal{M}$ and the result in [3] implies that this action factors through $S \otimes S \to S \otimes S_{W_\xi} S$. Therefore, for every $\mathcal{M} \in \mathcal{H}_\lambda$, we have a canonical isomorphism $b_\mathcal{M} : Z_\lambda \otimes \mathcal{M} \simeq S/S \cdot S_{W_\xi} \otimes_S \mathcal{M} \simeq Z_\lambda \otimes \mathcal{M} \otimes S/S \cdot S_{W_\xi} \simeq Z_\lambda \otimes \mathcal{M}$. One can check that those isomorphisms satisfy the required compatibility conditions and the data $(Z_\lambda, b_\mathcal{M})$ defines an element in $Z(\mathcal{H}_\lambda, \otimes)$.

Proof of (2). Let $\tilde{E}_\xi \in M_\xi$ be the image of $E_\xi$ under the equivalence $(\iota^0)^{-1} : H_\xi \simeq M_\xi$ in Lemma 3. Theorem 3. Then by definition we have $\mathcal{M}(E_\xi) \simeq RT\Gamma^{\hat{\lambda}, \hat{-\lambda}}(\xi^+ \xi)$, where $\pi : Y \to Y/T$. Consider the map
   \[ a : T \times (G/U \times G/U)/T \to (G/U \times G/U)/T, \quad (t, gU, g'U) \to (gt^{-1}U, g'U). \]

Then it follows from the definition of $(\iota^0)^{-1}$ that we have $\tilde{E}_\xi = a_*(E_\xi \boxtimes \Delta_\xi \xi / G/B)$, here $\Delta : G/B \to (G/U \times G/U)/T$ is the embedding $gB \to (gU, g'U)$ mod $T$. Note that $RT(\Delta \xi \xi / G/B \otimes p_2^* \xi / G/B) \simeq \tilde{U} (p_2$ is the right projection map $(G/U \times G/U)/T \to G/B$) hence by Lemma 3.4 we get
   \[ RT\Gamma^{\hat{\lambda}, \hat{-\lambda}}(\xi^+ \xi) \simeq RT\Gamma^{\hat{\lambda}}(\tilde{E}_\xi \boxtimes p_2^* \xi / G/B) = RT\Gamma^{\hat{\lambda}}(a_*(\xi \boxtimes (\Delta \xi \xi / G/B \otimes p_2^* \xi / G/B)) \simeq \tilde{U} \boxtimes Sym(\xi) \Gamma^{\hat{\lambda}}(\xi_\xi) \]
   Since $\Gamma^{\hat{\lambda}}(\xi_\xi) \simeq S_\xi^\lambda$, part (2) follows.

**Corollary 3.11.**

1. We have $\mathcal{E}_\xi \in H^1_{\xi \xi}$.

2. To every $\mathcal{M} \in H^1_{\xi \xi}$ there is a canonical isomorphism
   \[ b_\mathcal{M} : \mathcal{E}_\xi \ast \mathcal{M} \simeq \mathcal{M} \ast \mathcal{E}_\xi \]
   such that the data $(\mathcal{E}_\xi, b_\mathcal{M})_{\mathcal{M} \in H^1_{\xi \xi}}$ defines an element in the Drinfeld center $Z(H^1_{\xi \xi}, \ast^i)$.

**Proof.** This follows immediately from above proposition and [3.14].
3.7. Proof of Theorem 3.8. Recall the notion of translation functor \( \theta^\mu_\lambda : \mathcal{H}c_\lambda \to \mathcal{H}c_\mu \) where \( \mu \in \lambda + \Lambda \). In \([BFO]\), they proved the following:

(1) There is a lifting \( \theta^\mu_\lambda : Z(\mathcal{H}c_\lambda, \otimes) \to Z(\mathcal{H}c_\mu, \otimes) \) such that the functor \( F : Z(\mathcal{H}c_\lambda, \otimes) \to Z(\mathcal{H}c_\xi, \otimes), L \to \bigoplus_{\mu \in (\lambda + \Lambda)/W} \theta^\mu_\lambda(L) \) define an equivalence of braided monoidal categories.

(2) For any \( \mathcal{M} \in \mathcal{M}_G(G) \) the global section \( \Gamma(\mathcal{M}) \) is naturally a Harish-Chandra bimodule, with a canonical central structure and the resulting functor \( \Gamma : \mathcal{M}_G(G) \to Z(\mathcal{H}c, \otimes) \) is an equivalence of abelian categories. Moreover, the equivalence above restricts to an equivalence \( CS_\theta \simeq Z(\mathcal{H}c_\xi, \otimes) \) and the composed equivalence

\[
CS_\theta \simeq Z(\mathcal{H}c_\xi, \otimes) \overset{F^{-1}}{\longrightarrow} Z(\mathcal{H}c_\lambda, \otimes)
\]

is isomorphic to \( R\Gamma^{\lambda, -\lambda-2\rho} \circ \pi^0 \mathcal{H}C \). Here \( \pi : Y \to \mathcal{Y}/T \) is the projection map.

Let \( Z_\lambda \in Z(\mathcal{H}c_\lambda, \otimes), \tilde{E}_\xi \in M_\xi \) be as in Proposition 3.10. Define \( \tilde{E}_\theta \simeq \bigoplus_{\xi \in T, \xi \to \theta} \tilde{E}_\xi \in \mathcal{M}_G(Y) \). By the discussion above there exists a character \( D \)-module \( M_\theta \in CS_\theta \) such that

\[
R\Gamma^{\lambda, -\lambda-2\rho} \circ \pi^0 \mathcal{H}C(M_\theta) = Z_\lambda.
\]

Hence by Proposition 3.10, we have

\[
R\Gamma^{\lambda, -\lambda-2\rho} \circ \pi^0 \mathcal{H}C(M_\theta) \simeq R\Gamma^{\lambda, -\lambda-2\rho}(\pi^0 \tilde{E}_\xi)
\]

for any regular dominant \( \lambda \in \mathfrak{t} \) mapping to \( \xi \). Since \( \pi^0 \mathcal{H}C : D(CS_\theta) \to \bigoplus_{\xi \in T, \xi \to \theta} D(M_\xi - \lambda) \) and \( R\Gamma^{\lambda, -\lambda-2\rho} : D(M_\xi - \lambda) \simeq \mathcal{H}c_\lambda \) is an equivalence of category for regular dominant \( \lambda \), this implies \( \mathcal{H}C(M_\theta) \simeq \tilde{E}_\theta \). Applying the equivalence \( i^0 : D(M_\xi) \simeq D(H_\xi) \) on both sides and using Lemma 3.3, we get

\[
(3.16) \quad \text{Av}_{\mathcal{U}}(M_\theta) = i^0(\mathcal{H}C(M_\theta)) \simeq i^0 \tilde{E}_\theta \simeq \mathcal{E}_\theta.
\]

The isomorphism above implies \( \mathcal{E}_\theta \simeq \text{Av}_{\mathcal{U}}(M_\theta) \simeq \text{Res}_{T \subset B}(M_\theta) \), hence by part (4) of Proposition 3.5, there is canonical \( W \)-equivariant structure on \( \mathcal{E}_\theta \) such that \( M_\theta \simeq \text{Ind}_{T \subset B}(\mathcal{E}_\theta)^W \). In the lemma below we will show that this \( W \)-equivariant structure on \( \mathcal{E}_\theta \) coincides with the one in Definition 3.6; therefore we have \( M_\theta \simeq M'_\theta \simeq \text{Ind}_{T \subset B}(\mathcal{E}_\theta)^W \) and the theorem follows from (3.16).

**Lemma 3.12.** The \( W \)-equivariant structure on \( \mathcal{E}_\theta \simeq \text{Res}_{T \subset B}(M'_\theta) \) constructed in Proposition 3.5 coincides with the one in Definition 3.6.

**Proof.** We give a proof in the case when \( \theta = [\xi] \in \mathfrak{t}/W \). The proof for the general cases are similar. By construction we have \( Z_\theta = U_0 \otimes (S/S_n^W) \), \( M_\theta \simeq \bigoplus_{\mu \in \Lambda \cap W} \theta^\mu_\theta(Z_\theta) \), and \( \mathcal{E}_\theta = \mathcal{E}_\xi = \mathcal{E}_e^{\text{sep}} \). Let \( x \in N(T) \) and \( w \in W \) its image in the Weyl group. Let

\[
c_x : M_\theta \simeq \text{Ad}^*_x M'_\theta
\]

be the isomorphism coming from the \( G \)-conjugation equivariant structure on \( M'_\theta \) and

\[
c_w : \mathcal{E}_\theta \simeq \text{Res}_{T \subset B}^G M'_\theta \simeq \text{Res}_{T \subset B}^G \text{Ad}^*_w M'_\theta \simeq w^* \mathcal{E}_\theta
\]

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the induced map in (3.8). By Lemma 3.4 we have
\[ R\Gamma^0(\mathbb{A}^0 U(M'_{\theta})) \simeq (U(\mathfrak{g})/U(\mathfrak{g}) n) \otimes_S S^{S/W}_+, \quad R\Gamma^0(\mathbb{A}^0 U(\text{Ad}^*_s M'_{\theta})) \simeq (U(\mathfrak{g})/U(\mathfrak{g}) n_x) \otimes_S S^{S/W}_+. \]
Here in the right isomorphism \( n_x = \text{Ad}_x n \) and the \( S \)-module structure on \( S^{S/W}_+ \) is given by \( s \cdot m = \text{Ad}_x(s)m, \) \( s \in S, m \in S^{S/W}_+ \). Since the map \( c_x \) comes from the adjoint \( G \)-action on \( Z_0 \), under the isomorphisms in (3.17), the map
\[ R\Gamma^0(\mathbb{A}^0 U(c_x)) : R\Gamma^0(\mathbb{A}^0 U(M'_{\theta})) \simeq R\Gamma^0(\mathbb{A}^0 U(\text{Ad}^*_s M'_{\theta})) \]
becomes
\[ (U(\mathfrak{g})/U(\mathfrak{g}) n) \otimes_S S^{S/W}_+ \to (U(\mathfrak{g})/U(\mathfrak{g}) n_x) \otimes_S S^{S/W}_+, \quad v \otimes s \to \text{Ad}_x(v) \otimes s. \]
This implies the map
\[ a_w : S^{S/W}_+ \simeq R\Gamma^0(\mathcal{E}_{\theta}) \xrightarrow{R\Gamma^0(c_w)} R\Gamma^0(w^* \mathcal{E}_{\theta}) \simeq S^{S/W}_+ \]
induced by \( c_w \) is equal to the natural \( W \)-action map \( s \to \text{Ad}_x s = w(s) \). Notice that the assignment \( w \to a_w \) is the representation of \( W \) on \( S^{S/W}_+ \simeq \mathcal{E}_{\theta}\) coming from the \( W \)-equivariant structure \( \{c_w\}_{w \in W} \) on \( \mathcal{E}_{\theta} \). Since the \( W \)-equivariant structure on \( \mathcal{E}_{\theta} \) constructed in Definition 3.6 uses the same representation of \( W \) on \( S^{S/W}_+ \), the lemma follows.

\[ \square \]

4. Gamma \( D \)-modules

In this section we recall the definition of gamma \( D \)-modules on a reductive group due to D.Kazhdan and A.Braverman. We compute the convolution of gamma \( D \)-modules with the \( D \)-modules \( \mathcal{E}_\xi \) and \( M_{\theta} \) introduced in (3.5) (see Proposition 4.3 and Theorem 4.5). Those computations play important roles in §5 for the proof of the acyclicity of Gamma \( D \)-modules and in §6 for the proof of exactness properties of non-linear Fourier transforms.

4.1. Gamma \( D \)-modules on \( T \). For any \( c \in \mathbb{C}^\times \) we consider the corresponding exponential \( D \)-module \( \mathbb{C}[x]e^{cx} := \{ f e^{cx} | f \in \mathbb{C}[x] \} \) on \( \mathbb{G}_a = \text{Spec} \mathbb{C}[x] \) with generator \( e^{cx} \) and \( \partial_x(f e^{cx}) = (f' + c) e^{cx}, \ f \in \mathbb{C}[x] \). For each nontrivial cocharacter \( \lambda : \mathbb{G}_m \to T \) we define \( \Psi(\lambda, c) := \lambda_*(j^* \mathbb{C}[x]e^{cx}) \) where \( j : \mathbb{G}_m \to \mathbb{G}_a \) is the natural inclusion. Note that since \( \lambda \) is finite we have
\[ \Psi(\lambda, c) = \lambda_*(j^* \mathbb{C}[x]e^{cx}) \simeq \lambda_!(j^* \mathbb{C}[x]e^{cx}). \]
Recall the convolution product \( \ast \) (resp. \( \ast \)) on \( D(T)_{\text{hol}} \)
\[ \mathcal{F} \ast \mathcal{F}' = m_*(\mathcal{F} \boxtimes \mathcal{F}'), \ (\text{resp. } \mathcal{F} \ast \mathcal{F}' = m_!(\mathcal{F} \boxtimes \mathcal{F}')) \]
Here \( m : T \times T \to T, (x, y) \to xy \) is the multiplication map. For every collection of possibly repeated nontrivial cocharacter \( \Delta = (\lambda_1, ..., \lambda_r) \) we define
\[ \Psi_{\Delta, c} := \Psi(\lambda_1, c) \ast \cdots \ast \Psi(\lambda_r, c) \]
\[ \Psi_{\Delta_c} := \Psi(\lambda_1, c) \ast \cdots \ast \Psi(\lambda_r, c). \]

Following [BK1], we call \( \Psi_{\Delta_c} \) and \( \Psi'_{\Delta_c} \) gamma \( D \)-modules on \( T \).

Let \( \text{pr}_\Delta : G_m^r \to T, (x_1, \ldots, x_r) \to \prod_{i=1}^r \lambda_i(x_i) \) and \( \text{tr} : G_m^r \to G, (x_1, \ldots, x_r) \to \sum x_i \). Then using base changes one can show that \( \Psi_{\Delta_c} = \text{pr}_\Delta^* \text{tr}^*(\mathbb{C}[x]e^{cx}) \) and \( \Psi'_{\Delta_c} = \text{pr}_{\Delta'} \text{tr}^*(\mathbb{C}[x]e^{cx}) \).

**Definition 4.1.** Let \( \sigma : T \to G_m \) be a character. A co-character \( \lambda \) is called \( \sigma \)-positive if \( \sigma \circ \lambda : G_m \to G_m \) has the form \( t \to t^n \) for some positive integer \( n \).

We have the following properties of gamma \( D \)-modules:

**Proposition 4.2.** Let \( \Delta = \{\lambda_1, \ldots, \lambda_r\} \) be a collection of \( \sigma \)-positive co-characters.

1) Then the natural map \( \Psi'_{\Delta,c} \to \Psi_{\Delta,c} \) is an isomorphism. Moreover, \( \Psi_{\Delta,c} \) is a local system on the image \( T_\Delta := \text{Im}(\text{pr}_\Delta) \subset T \).

2) We have \( \mathbb{D}(\Psi_{\Delta,c}) \simeq \Psi_{\Delta,-c} \).

3) Let \( \mathcal{L} \) be a Kummer local system on \( T \). We have
\[
H^i_{dR}(\Psi_{\Delta,c} \otimes \mathcal{L}) = 0
\]
for \( i \neq 0 \) and \( \dim H^0_{dR}(\Psi_{\Delta,c} \otimes \mathcal{L}) = 1 \). Moreover, we have a canonical isomorphism
\[
\Psi_{\Delta,c} \ast \mathcal{L} \simeq H^0_{dR}(\Psi_{\Delta,c} \otimes \mathcal{L}^{-1}) \otimes \mathcal{L}.
\]

4) Consider the functor of right convolution with \( \Psi_{\Delta,c} \in D_U(X) \):
\[
(-) \ast \Psi_{\Delta,c} : D(X) \to D(X).
\]
The functor above preserves the \( T \)-monodromic subcategory \( D(X)_{\text{mon}} \subset D(X) \), where \( T \) acts on \( X = G/U \) from the right.

**Proof.** Part 1), 2), 3) are proved in [BK1, Theorem 4.2, Theorem 4.8]. Part 4) follows from part 3). \( \square \)

Let \( \Delta = (\lambda_1, \ldots, \lambda_r) \) be a collection of \( \sigma \)-positive cocharacters. Recall that the Weyl group \( W \) acts naturally on \( X(T) \) and we assume that the set \( \{\lambda_i\}_{i=1, \ldots, r} \) are invariant under this action. Following [BK] (see also [CN]), we shall construct a \( W \)-equivariant structure on \( \Psi_{\Delta} \). Let \( (\lambda_{i_1}, \ldots, \lambda_{i_k}) \) be the distinct cocharacters appearing in \( \Delta \) and \( m_i \) be the multiplicity of \( \lambda_{i_i} \in \Delta \). Let \( A_i = \{\lambda_i| \lambda_i = \lambda_{i_i}\} \). Then we have \( \{\lambda_1, \ldots, \lambda_r\} = A_1 \cup \ldots \cup A_k \). The symmetric group on \( r \)-letters \( S_r \) acts naturally on \( \{\lambda_1, \ldots, \lambda_r\} \) and we define \( S_\Delta = \{\sigma \in S_r| \sigma(A_i) = A_i\} \). There is a canonical isomorphism
\[
S_\Delta \simeq S_{m_1} \times \cdots \times S_{m_k}.
\]
Define \( S'_\Delta = \{\eta \in S_r| \text{such that } \eta(A_i) = A_{r(i)} \text{ for } \tau \in S_k\} \). We have a natural map \( \pi_k : S'_\Delta \to S_k \) sending \( \eta \) to \( \tau \). The kernel of \( \pi_k \) is isomorphic to \( S_\Delta \) and its image, denote
by $\Sigma_k$, consists of $\tau \in \Sigma_k$ such that $m_i = m_{r(i)}$. In other words, there is a short exact sequence

$$0 \to S \to S' \xrightarrow{\pi_k} \Sigma_k \to 0.$$  

Notice that the Weyl group $W$ acts on $\{\lambda_i, \ldots, \lambda_n\}$ and the induced map $W \to \Sigma_k$ has image $\Sigma_k$. So we have a map $\rho : W \to \Sigma_k$. Pulling back the short exact sequence above along $\rho$, we get an extension of $W'$ of $W$ by $S$

$$0 \to S \to W' \to W \to 0$$

where an element in $w' \in W'$ consists of pair $(w, \eta) \in W \times S'$ such that $\rho(w) = \pi_k(\eta) \in \Sigma_k$.

The group $W'$ acts on $G_m$ (resp. $T$) via the composition of the action of $S_r$ (resp. $W$) with the natural projection $W' \to S'_r \subset S_r$ (resp. $W' \to W$) and the map $pr'_\Lambda : G_m \to T$ and $tr' : G_m \to G_a$ is $W'$-equivaraint where $W'$ acts trivially on $G_a$.

Since $\Psi_{\Lambda, c} \simeq (pr'_\Lambda)_! tr'((C[x]e^{tx}))$, the discussion above implies for each $w' = (w, \eta) \in W'$ there is an isomorphism

$$i_{w'} : \Psi_{\Lambda, c} \simeq w'\Psi_{\Lambda, c}.$$  

We define

$$i_{w'} = \text{sign}_w(\eta) \text{sign}_W(w) i_{w'} : \Psi_{\Lambda, c} \simeq w'\Psi_{\Lambda, c}$$

where sign$_w$ and sign$_W$ are the sign characters of $S_r$ and $W$. According to [BK1], the isomorphism $i_{w'}$ depends only on $w$. Denote the resting isomorphism by $i_w$, then the data $(\Psi_{\Lambda, c}, \{i_w\}_{w \in W})$ defines a $W$-equivariant structure on $\Psi_{\Lambda, c}$.

4.2. **Mellin transform.** Let $x_i \in \Lambda$ be a basis and consider the regular function $\Theta(T) \simeq C[x_i^{\pm 1}]$ and the algebra of differential operators $\Gamma(D_T) \simeq C[x_i^{\pm 1}] / \{v_i x_j - x_j (\delta_{ij} + v_i)\}$ where $v_i = x_i \partial_k$ and $t$ are a basis for the $T$-invariant vector fields. The Mellin transform functor

$$M : M(T) \to C[v_i] \text{mod}^{\Lambda}, \quad N \to \Gamma(N)$$

defined by considering $\Gamma(D_T)$ as algebra of difference operators $C[v_i][x_i^{\pm 1}] / \{v_i x_j - x_j (\delta_{ij} + v_i)\}$, is an equivalence of abelian categories between $D$-modules on $T$ and $\Lambda$-equivariant $C[v_i]$-modules. Consider the derived category of $\Lambda$-equivariant $C[v_i]$-modules $\mathbf{D}(C[v_i] \text{-mod}^{\Lambda})$ with the monoidal structure given by the derived tensor product over $C[v_i]$. We have $M(M \star N) \simeq M(M) \otimes^L_{C[v_i]} M(N)$.

Let $W' \subset W$ be a subgroup. Consider the category $M_{W'}(T)$ (resp. $C[v_i] \text{-mod}^{W' \ast \Lambda}$) of $W'$-equivariant $D$-module on $T$ (resp. $W' \ast \Lambda$-equivariant $C[v_i]$-modules). Then the functor $M$ extends naturally to an equivalence of categories

$$M_{W'} : M_{W'}(T) \to C[v_i] \text{-mod}^{W' \ast \Lambda}.$$  

Note that we have a canonical equivalence $C[v_i] \text{-mod}^{\Lambda} \simeq \text{QCoh}(t^*)^\Lambda$ (resp. $C[v_i] \text{-mod}^{W' \ast \Lambda} \simeq \text{QCoh}(t)^{W' \ast \Lambda}$) where $\text{QCoh}(t)^\Lambda$ (resp. $\text{QCoh}(t)^{W' \ast \Lambda}$) is the category of $\Lambda$-equivariant...
(resp. $W' \times \Lambda$-equivariant) quasi-coherent sheaves on $\mathfrak{t}$. We write $\mathcal{M} : \mathcal{M}(T) \to \text{Qcoh}(\mathfrak{t}^* \Lambda)$ (resp. $\mathcal{M}_W : \mathcal{M}_W(T) \to \text{Qcoh}(\mathfrak{t}^* W' \times \Lambda)$) for the composition of $\mathcal{M}$ (resp. $\mathcal{M}_W$) with the equivalence above.

Let $(N, c_w : N \simeq w^* N) \in \mathcal{M}_W(T)$. Then the $W'$-equivariant structure on $\mathcal{M}_W(N)$ is determined by the following $W'$-action on $\mathcal{M}(N)$:

$$a_w : \mathcal{M}(N) = \Gamma(N) \overset{\Gamma(c_w)}{\rightarrow} \Gamma(w^* N) \simeq \Gamma(N) = \mathcal{M}(N), \ w \in W'.$$

### 4.3. Mellin transform of $\Psi_{\Delta,c}$

We describe the Mellin transform of $\Psi_{\Delta,c}$. Recall

$$\Psi_{\Delta,c} \simeq (\text{pr}_{\Delta})_* \text{tr}^*(\mathbb{C}[x]e^{cx}) \simeq (\mathbb{C}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}]e^{\sum cx_i} \otimes_{O_{G_m}} \omega_{G_m}) \otimes_{O_{\mathbb{G}_m}} D_{G_m} \rightarrow T \otimes_{\mathbb{G}_m} \omega_T^{-1}.$$

Fix a nowhere vanishing $G_m^r$-invariant (resp. $T$-invariant) section $r_1 \in \Gamma(\omega_{G_m})$ (resp. $r_2 \in \Gamma(\omega_T)$). Then the trivialization $O_{G_m} \simeq \omega_{G_m}$ (resp. $O_T \simeq \omega_T$) induced by $r_1$ (resp. $r_2$) defines an isomorphism

$$t_{\Delta} : \mathcal{M}(\Psi_{\Delta,c}) \simeq \mathbb{C}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}]e^{\sum cx_i} \otimes_{O_{[v_1, \ldots, v_r]}} S$$

where $v_i = x_i \partial_{x_i}$ and the $\mathbb{C}[v_1, \ldots, v_r]$-module structure on $S = \text{Sym}(\mathfrak{t})$ is given by $v_i \cdot s = d\lambda_i(v_i)s, \ s \in S$, here $d\lambda_i : \mathbb{C}[v_i] \rightarrow S$ is the differential of the cocharacter $\lambda_i$.

Recall the $W$-action on $\mathcal{M}(\Psi_{\Delta,c})$:

$$a_w : \mathcal{M}(\Psi_{\Delta,c}) = \Gamma(\Psi_{\Delta,c}) \overset{\Gamma(c_w)}{\rightarrow} \Gamma(w^* \Psi_{\Delta,c}) \simeq \Gamma(\Psi_{\Delta,c}) = \mathcal{M}(\Psi_{\Delta,c}), \ w \in W.$$

Let $w' = (w, \eta) \in W'$. We have $\eta^*(r_1) = \text{sign}(\eta)r_1$ (resp. $w^* r_2 = \text{sign}(w)r_2$) and the construction of $t_{w'}$ (see (4.1)) implies the following description of $a_w$: the following diagram commutes

$$\begin{array}{ccc}
\mathcal{M}(\Psi_{\Delta,c}) & \overset{t_{\Delta}}{\rightarrow} & \mathbb{C}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}]e^{\sum cx_i} \otimes_{O_{[v_1, \ldots, v_r]}} S \\
& a_w & \downarrow b_{w'} \\
\mathcal{M}(\Psi_{\Delta,c}) & \overset{t_{\Delta}}{\rightarrow} & \mathbb{C}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}]e^{\sum cx_i} \otimes_{O_{[v_1, \ldots, v_r]}} S
\end{array}$$

where $b_{w'} : \mathbb{C}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}]e^{\sum cx_i} \otimes_{O_{[v_1, \ldots, v_r]}} S \rightarrow \mathbb{C}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}]e^{\sum cx_i} \otimes_{O_{[v_1, \ldots, v_r]}} S, f \otimes s \rightarrow \eta(f) \otimes w(s)$.

### 4.4. Mellin transform of $\mathcal{E}_\xi$ and $\mathcal{L}_n^\mathfrak{t}$

We describe the Mellin transforms of $\mathcal{E}_\xi$ and $\mathcal{L}_n^\mathfrak{t}$. To every $\mu \in \mathfrak{t}$ let $l_\mu : \mathfrak{t} \rightarrow \mathfrak{v}, v \rightarrow v - \mu$. Recall the $S$-module $S_\xi = S/S_{n, \mathfrak{t}}$, $S_n = S/S_n^n$ introduced in (3.4). Let $S_\xi := O_{\mathfrak{t}} \otimes S_\xi$, $S_n := O_{\mathfrak{t}} \otimes S_n$ be quasi-coherent sheaves on $\mathfrak{t}$ corresponding to $S_\xi = S/S_{n, \mathfrak{t}}$ and $S_n = S/S_n^n$. We define $S_\xi^\mu := l_\mu^* S_\xi, S_n^\mu := l_\mu^* S_n$ and write $S_\xi^\mu = \Gamma(S_\xi^\mu)$, $S_n^\mu = \Gamma(S_n^\mu)$. Note that the $S$-module $S_\xi^\mu$ here agrees with the one in (3.15). Clearly we have $l_\lambda^* S_\xi^\mu \simeq S_\xi^\mu \Lambda$ (resp. $l_\lambda^* S_n^\mu \simeq S_n^\mu + \Lambda$) for $\lambda \in \mathfrak{t}^*$. Consider the projection map $\pi_\xi : \mathfrak{t} \rightarrow \mathfrak{t}/\mathfrak{w}_\xi$. We have $S_\xi \simeq \pi_\xi^* \delta$, where $\delta$ the the skyscraper sheaf supported at $0 \in \mathfrak{t}^*/\mathfrak{w}_\xi$, and the equality $\pi_\xi \circ l_\mu \circ w = \pi_\xi \circ l_\mu$ defines an isomorphism $S_\xi \simeq w^* S_{\xi}^\mu$. 17
The isomorphisms \( l_\lambda^* S_n^\mu \simeq S_n^{\mu+\lambda} \) for \( \lambda \in \Lambda \) defines a \( \Lambda \)-equivariant structure on \( \bigoplus_{\mu \in t^*, \mu = \xi} \mathcal{S}_{\lambda}^\mu \) and we have

\[
\mathcal{M}(\mathcal{L}_\xi^n) \simeq \bigoplus_{\mu \in t^*, \mu = \xi} S_n^\mu \in \text{Qcoh}(\hat{t})^\Lambda, \quad (\text{resp. } \mathcal{M}(\mathcal{L}_\xi^n) \simeq \bigoplus_{\mu \in t^*, \mu = \xi} S_n^\mu \in S^\text{-mod}_\Lambda).
\]

Similarly, the isomorphisms \( S_n^{\mu+\lambda} \simeq l_\lambda^* S_n^\mu, \lambda \in \Lambda \) and \( S_n^\mu \simeq l^w S_n^w(\mu) \) define a \( W_\xi \times \Lambda \)-equivariant structure on \( \bigoplus_{\mu \in t^*, \mu = \xi} \mathcal{S}_{\lambda}^\mu \) and we have

\[
\mathcal{M}_{W_\xi}(\mathcal{E}_\xi) \simeq \bigoplus_{\mu \in t^*, \mu = \xi} \mathcal{S}_{\xi}^\mu \in \text{Qcoh}(\hat{t})^{W_\xi \times \Lambda}, \quad (\text{resp. } \mathcal{M}_{W_\xi}(\mathcal{E}_\xi) \simeq \bigoplus_{\mu \in t^*, \mu = \xi} \mathcal{S}_{\xi}^\mu \in S^\text{-mod}_{W_\xi \times \Lambda}).
\]

Let

\[
\mathcal{u}_w^\mu : S_n^\mu = \Gamma(S_n^\mu) \to \Gamma(w^* S_n^w(\mu)) \simeq \Gamma(S_n^w(\mu)) = S_n^w(\mu)
\]

be the map induced by the isomorphism \( S_n^\mu \simeq w^* S_n^w(\mu) \). Then the \( W_\xi \)-action on \( \mathcal{M}(\mathcal{E}_\xi) \), defined in (4.2), decomposes as

\[
(4.4) \quad \mathcal{u}_w = \bigoplus u_w^\mu : \mathcal{M}(\mathcal{E}_\xi) = \bigoplus_{\mu \in t^*, \mu = \xi} S_n^\mu \to \bigoplus_{\mu \in t^*, \mu = \xi} S_n^w(\mu) = \mathcal{M}(\mathcal{E}_\xi), \quad w \in W_\xi.
\]

4.5. We have the following key proposition whose proof will be given in section 4.7.

**Proposition 4.3.** There is an isomorphism

\[
\Psi_{\Delta, e} \ast \mathcal{E}_\xi \simeq \mathcal{E}_\xi
\]

of \( W_\xi \)-equivariant local systems on \( T \).

4.6. **Gamma \( D \)-modules on \( G \).** We preserve the setup in 4.1. Let \( \Delta = (\lambda_1, ..., \lambda_r) \) be a collection of \( W \)-invariant \( \sigma \)-positive co-characters. By Proposition 3.2, the \( W \)-equivariant structure on \( \Psi_{\Delta, e} \) defines a \( W \)-action on \( \text{Ind}_{T \subset B}^G(\Psi_{\Delta, e}) \).

**Definition 4.4.** The gamma \( D \)-module attached to \( \Delta \) is the \( W \)-invariant factor of the \( D \)-module \( \text{Ind}_{T \subset B}^G(\Psi_{\Delta, e}) \)

\[
\Psi_{G, \Delta, e} := \text{Ind}_{T \subset B}^G(\Psi_{\Delta, e})^W.
\]

Giving a representation \( \rho : \hat{G} \to GL(V_\rho) \), its restriction \( \rho|_{\hat{T}} \) is diagonalizable, i.e. there exist a collection of co-characters \( \Delta_\rho = \{ \lambda_1, ..., \lambda_r \} \) such that \( V_\rho = \bigoplus V_{\lambda_i} \) where \( \hat{T} \) acts on \( V_{\lambda_i} \) by the \( \lambda_i \in \text{Hom}(G_m, T) \simeq \text{Hom}(\hat{T}, G_m) \). Note that \( \Delta_\rho \) is automatically \( W \)-stable. Assume each \( \lambda_i \in \Delta_\rho \) is \( \sigma \)-positive and \( \text{pr}_{\Delta_\rho} \) is onto, then the gamma \( D \)-module (or rather, the corresponding gamma sheaf) \( \Psi_{G, \rho, e} := \Psi_{G, \Delta_\rho, e} \) attached to \( \Delta_\rho \) is the one studied in [BK, BK1].
The following property of gamma $D$-module follows from Proposition 4.2
(4.5) \[ D(\Psi_{G_\Delta c}) \simeq \Psi_{G_\Delta -c}. \]

Recall the character $D$-module $M_\theta$ in (3.5)

**Theorem 4.5.** There is an isomorphism
\[ \Psi_{G_\Delta c} \ast M_\theta \simeq M_\theta. \]

**Proof.** Fix a lifting of $\xi \in \hat{T}$ of $\theta$. Then we have $A\nu_U(M_\theta) \simeq E_\theta \simeq \text{Ind}^W_{\nu_\xi} E_\xi$, which is supported on $T = B/U$. Thus by Proposition 3.2 and Proposition 4.3 we have
\[ \text{Ind}^G_{T \subset B}(\Psi_{\Delta c}) \ast M_\theta \simeq \text{Ind}^G_{T \subset B}(\Psi_{\Delta c} \ast E_\theta) \simeq \text{Ind}^W_{\nu_\xi}(\text{Ind}^G_{T \subset B}(\Psi_{\Delta c} \ast E_\xi)) \simeq \text{Ind}^W_{\nu_\xi}(\text{Ind}^G_{T \subset B}(E_\xi)). \]

Now taking $W$-invariant on both sides of the isomorphism above and using (3.12), we arrive
\[ \Psi_{G_\Delta c} \ast M_\theta \simeq (\text{Ind}^G_{T \subset B}(\Psi_{\Delta c}) \ast M_\theta)^W \simeq (\text{Ind}^G_{T \subset B}(\Psi_{\Delta c} \ast E_\xi))^W \simeq \text{Ind}^G_{T \subset B}(E_\xi)^W \simeq M_\theta. \]

\[ \square \]

4.7. **Proof of Proposition 4.3.** We shall construct an isomorphism $\Psi_{\Delta c} \ast E_\xi \simeq E_\xi$. For this we will first construct an isomorphism $\Psi(\lambda, c) \ast E_\xi \simeq E_\xi$ for $\lambda \in \hat{T}$. For simplicity, we will write $\Psi_\Delta$ (resp. $\Psi(\lambda)$) for $\Psi_{\Delta c}$ (resp. $\Psi(\lambda, c)$). By (4.2 and 4.4) we have
\[ M(E_\xi) = \bigoplus_{\mu \in \nu_U, [\mu] = \xi} S^\mu_\xi \] and
(4.6) \[ M(\Psi(\lambda) \ast E_\xi) \simeq M(\Psi(\lambda)) \otimes S M(E_\xi) \simeq \bigoplus_{\mu \in \nu_U, [\mu] = \xi} C[x^{\pm1}]^{\nu_U} \otimes_{C[v]} S^\mu_\xi, \]
here $x$ is a coordinate of $G_m$, $v = x \partial_x$, and $C[v]$ acts on $S^\mu_\xi$ via the map $d\lambda : C[v] \to S$. Write $\lambda(\mu) = a_{\lambda, \mu} + n_{\lambda, \mu}$, with $a_{\lambda, \mu} \in [0, 1)$, $n_{\lambda, \mu} \in \mathbb{Z}$, and consider the free $C[v]$-submodule
\[ E_{\lambda, \mu} := C[v] \cdot x^{n_{\lambda, \mu}} e^{cx} \subset C[x^{\pm1}]^{\nu_U} \]

generated by $x^{n_{\lambda, \mu}} e^{cx}$. From the relation $v \cdot x^n e^{cx} = (nx^n + cx^{n+1}) e^{cx}$, we deduce that $C[x^{\pm1}]^{\nu_U} e^{cx} / E_{\lambda, \mu}$, as a quasi-coherent sheaf on Spec$C[v] \simeq C$, is supported away from $\lambda(\mu)$. Since $S^\mu_\xi$ is supported on $\lambda(\mu)$, we deduce that $(C[x^{\pm1}]^{\nu_U} e^{cx} / E_{\lambda, \mu}) \otimes_{C[v]} S^\mu_\xi = 0$ and
(4.7) \[ C[x^{\pm1}]^{\nu_U} \otimes_{C[v]} S^\mu_\xi \simeq E_{\lambda, \mu} \otimes_{C[v]} S^\mu_\xi \simeq S^\mu_\xi. \]

Combining (4.6) and (4.7) we get
\[ M(\Psi(\lambda) \ast E_\xi) \simeq \bigoplus_{\mu \in \nu_U, [\mu] = \xi} E_{\lambda, \mu} \otimes_{C[v]} S^\mu_\xi \simeq \bigoplus_{\mu \in \nu_U, [\mu] = \xi} S^\mu_\xi \simeq M(E_\xi) \]
and this gives
(4.8) \[ \Psi(\lambda) \ast E_\xi \simeq E_\xi. \]

The isomorphism above defines an isomorphism
(4.9) \[ \kappa : \Psi_\Delta \ast E_\xi \simeq \Psi(\lambda_1) \ast \cdots \ast \Psi(\lambda_r) \ast E_\xi \simeq E_\xi. \]
We shall show that $\kappa$ is compatible with the $W_\xi$-equivariant structures on both sides. According to (4.2), it suffices to show that the map $M(\kappa) : M(\Psi_\Delta \ast E_\xi) \simeq M(E_\xi)$ is compatible with the $W_\xi$-actions on both sides. Denote $x_{\Delta,\mu} := \prod_{i=1}^n x_i^{n_{\lambda_i,\mu}}$ and consider the free $\mathbb{C}[v_1, ..., v_r]$-submodule
\[ E_{\Delta,\mu} := \mathbb{C}[v_1, ..., v_r] : x_{\Delta,\mu} e^{\sum c_i} \subset \mathbb{C}[x_1^{\pm 1}, ..., x_r^{\pm 1}] e^{\sum c_i} \]
generated by $x_{\Delta,\mu} e^{\sum c_i}$. It follows from (4.7) that
\[ M(\Psi_\Delta \ast E_\xi) \simeq \bigoplus_{\mu \in \mathbb{T}, [\mu] = \xi} \mathbb{C}[x_1^{\pm 1}, ..., x_r^{\pm 1}] e^{\sum c_i} \otimes \mathbb{C}[v_1, ..., v_r] S_\xi^\mu \simeq \bigoplus_{\mu \in \mathbb{T}, [\mu] = \xi} E_{\Delta,\mu} \otimes \mathbb{C}[v_1, ..., v_r] S_\xi^\mu. \]
Moreover, under the isomorphism above, the map $M(\kappa)$ becomes
\[ (4.10) \quad \bigoplus_{\mu \in \mathbb{T}, [\mu] = \xi} E_{\Delta,\mu} \otimes \mathbb{C}[v_1, ..., v_r] S_\xi^\mu \simeq \bigoplus_{\mu \in \mathbb{T}, [\mu] = \xi} S_\xi^\mu, \quad x_{\Delta,\mu} \otimes s \mapsto s. \]

We now describe the $W_\xi$-action on $M(\Psi_\Delta \ast E_\xi)$. Let $w' = (w, \eta) \in W'$ with $w \in W_\xi$. Consider the map
\[ a_\eta : \mathbb{C}[x_1^{\pm 1}, ..., x_r^{\pm 1}] e^{\sum c_i} \rightarrow \mathbb{C}[x_1^{\pm 1}, ..., x_r^{\pm 1}] e^{\sum c_i}, \quad f \mapsto \eta(f) e^{\sum c_i}. \]
Since
\[ a_\eta(x_{\Delta,\mu} e^{\sum c_i}) = \left( \prod_{i=1}^n x_i^{n_{\lambda_i,\mu}} \right) e^{\sum c_i} = \left( \prod_{i=1}^r x_i^{n_{\lambda_i,w(\mu)}} \right) e^{\sum c_i} = \left( \prod_{i=1}^r x_i^{n_{\lambda_i,w(\mu)}} \right) e^{\sum c_i} = x_{\Delta,\mu} e^{\sum c_i} \]
the map $a_\eta$ restricts to a map $a_\eta : E_{\Delta,\mu} \rightarrow E_{\Delta,w(\mu)} \subset \mathbb{C}[x_1^{\pm 1}, ..., x_r^{\pm 1}] e^{\sum c_i}$. Consider the $W_\xi$-action on $M(\Psi_\Delta \ast E_\xi) : a_w : M(\Psi_\Delta \ast E_\xi) \rightarrow M(\Psi_\Delta \ast E_\xi), \quad w \in W_\xi$. It follows from the description for the $\tilde{W}_\xi$-action on $M(\Psi_\Delta)$ in (4.3) that we have the following commutative diagram
\[ (4.11) \quad \bigoplus_{\mu \in \mathbb{T}, [\mu] = \xi} E_{\Delta,\mu} \otimes \mathbb{C}[v_1, ..., v_r] S_\xi^\mu \xrightarrow{a_w} M(\Psi_\Delta \ast E_\xi) \]
\[ \bigoplus_{\mu \in \mathbb{T}, [\mu] = \xi} E_{\Delta,w(\mu)} \otimes \mathbb{C}[v_1, ..., v_r] S_\xi^{w(\mu)} \xrightarrow{a_w} M(\Psi_\Delta \ast E_\xi) \]
where
\[ (4.12) \quad u_w = \bigoplus_{\mu \in \mathbb{T}, [\mu] = \xi} S_\xi^\mu \mapsto \bigoplus_{\mu \in \mathbb{T}, [\mu] = \xi} S_\xi^{w(\mu)} = M(E_\xi), \quad w \in W_\xi \]
is map in (4.1) describing the $W_\xi$-action on $M(E_\xi)$. Since the map in (4.10) satisfies the following commutative diagram
\[ \bigoplus_{\mu \in \mathbb{T}, [\mu] = \xi} E_{\Delta,\mu} \otimes \mathbb{C}[v_1, ..., v_r] S_\xi^\mu \xrightarrow{\oplus a_\eta \otimes u_w} \bigoplus_{\mu \in \mathbb{T}, [\mu] = \xi} S_\xi^\mu \]
\[ \bigoplus_{\mu \in \mathbb{T}, [\mu] = \xi} E_{\Delta,w(\mu)} \otimes \mathbb{C}[v_1, ..., v_r] S_\xi^{w(\mu)} \xrightarrow{\oplus a_\eta \otimes u_w} \bigoplus_{\mu \in \mathbb{T}, [\mu] = \xi} S_\xi^{w(\mu)} \]
we deduce from (4.11) and (4.12) that $M(\kappa) : M(\Psi \ast \xi) \simeq M(\xi)$ is compatible with the $W_\xi$-action on both sides. This finishes the proof of the proposition.

4.8. Recall the local systems $L_\xi$ in §3.4. We have $M(L_\xi) = \bigoplus_{\mu \in \hat{t}, [\mu] = \xi} S^n_\mu$ (see (4.1)). Using the relation $v \cdot a^e = (nx^n + cx^{n+1})e^x$ and the fact that $S^n_\mu$, viewing as $C[v]$-module via $d\lambda : C[v] \to S$, is supported on $\lambda(\mu)$, the same argument as in the proof of (4.9) gives:

**Lemma 4.6.** There exists a projective system of isomorphisms $\Psi \ast L_\xi \simeq L^n_\xi$.

5. **Kazhdan-Braverman conjecture**

In [BK, Conjecture 9.2] A.Braverman and D.Kazhdan conjectured the following vanishing property of gamma $D$-module:

**Conjecture 5.1.** $Av_U(\Psi) \simeq T = B/U \subset G/U$. Here $Av_U : D_B(G)_{hol} \to D_B(G/U)_{hol}$ is the shriek averaging functor in (3.7).

Since $D(\Psi) \simeq Av_U(D(\Psi)) \simeq Av_U(\Psi_{G,\lambda})$ (see (4.5)), the conjecture above is equivalent to the following:

**Theorem 5.2.** $Av_U(\Psi) \in D_B(G/U)$ is supported on $T = B/U \subset G/U$.

**Proof.** It suffices to show that the natural map $r : \text{Res}_{T \subset B}^{G}(\Psi_{G,\lambda}) \to Av_U(\Psi_{G,\lambda})$ is an isomorphism. We claim that the convolution

$$r : \text{Res}_{T \subset B}^{G}(\Psi_{G,\lambda}) \ast \xi \to Av_U(\Psi_{G,\lambda}) \ast \xi$$

of $r$ with $\xi$ is an isomorphism for all $\theta \in \hat{T}/W$. For this, it is enough to show that $Av_U(\Psi_{G,\lambda}) \ast \xi$ is supported on $T$ and this follows from Theorem 3.8 and Theorem 4.5. Indeed, we have $Av_U(\Psi_{G,\lambda}) \ast \xi \simeq Av_U(\Psi_{G,\lambda}) \ast Av_U(M_{\xi}) \simeq Av_U(\Psi_{G,\lambda} \ast M_{\xi}) \simeq \text{Ind}_{W_\xi}^{W} \xi$. Note $\xi \simeq \text{Ind}_{W_\xi}^{W} \xi$ and (5.1) implies that cone($r$), the cone of $r$, satisfies cone($r$) $\ast \xi = 0$ for all $\xi \in \hat{T}$. Since $\xi$ is a local system on $T$ with generalized monodromy $\xi \in \hat{T}$, Lemma 5.4 and Lemma 5.5 below imply cone($r$) = 0. The theorem follows.

**Corollary 5.3.** We have $Av_U(\Psi) \simeq \Psi_{\lambda}$.

**Proof.** Indeed, by [BK, Theorem 6.6] we have $\text{Res}_{T \subset B}^{G}(\Psi_{G,\lambda}) \simeq \Psi_{\lambda}$. Thus the theorem above implies $\text{Res}_{T \subset B}^{G}(\Psi_{G,\lambda}) \simeq \Psi_{\lambda}$. □
5.1. Vanishing lemmas. Let $X$ be a smooth variety with a free $T$ action $a : T \times X \to X$. For $\mathcal{L} \in D(T)$ and $\mathcal{F} \in D(X)$ we define $\mathcal{L} * \mathcal{F} := a_*(\mathcal{L} \boxtimes \mathcal{F}) \in D(X)$.

**Lemma 5.4.** Let $\mathcal{L}$ be a local system on $T$ with generalized monodromy $\xi \in \hat{T}$, that is, $\mathcal{L} \otimes \mathcal{L}_{\xi}$ is an unipotent local system. Let $\mathcal{F} \in D(X)_{\text{hol}}$ and assume $\mathcal{L} * \mathcal{F} = 0$. Then we have $\mathcal{L}_{\xi} * \mathcal{F} = 0$.

**Proof.** There is a filtration $0 = \mathcal{L}^{(0)} \subset \mathcal{L}^{(1)} \subset \cdots \subset \mathcal{L}^{(k)} = \mathcal{L}$ such that

$$0 \to \mathcal{L}^{(i-1)} \to \mathcal{L}^{(i)} \to \mathcal{L}^{(i)} / \mathcal{L}^{(i-1)} \simeq \mathcal{L}_{\xi} \to 0.$$ 

Assume $\mathcal{L}_{\xi} * \mathcal{F} \neq 0$ and let $m$ be the smallest number such that $H^{\geq m}(\mathcal{L}_{\xi} * \mathcal{F}) = 0$. An induction argument, using above short exact sequence, shows that $H^{\geq m}(\mathcal{L}^{(i)} * \mathcal{F}) = 0$ for $i = 1, \ldots, k$. Now since $\mathcal{L} * \mathcal{F} = 0$, the distinguished triangle

$$\mathcal{L}^{(k-1)} * \mathcal{F} \to \mathcal{L} * \mathcal{F} \to \mathcal{L}_{\xi} * \mathcal{F} \to \mathcal{L}^{(k-1)} * \mathcal{F}[1]$$

implies

$$\mathcal{L}_{\xi} * \mathcal{F} \simeq \mathcal{L}^{(k-1)} * \mathcal{F}[1].$$

Therefore we have $H^{m-1}(\mathcal{L}_{\xi} * \mathcal{F}) \simeq H^{m-1}(\mathcal{L}^{(k-1)} * \mathcal{F}[1]) = H^m(\mathcal{L}(k-1) * \mathcal{F}) = 0$ which contradicts to the fact that $m$ is the smallest number such that $H^{m}(\mathcal{L}_{\xi} * \mathcal{F}) = 0$. We are done.  

**Lemma 5.5.** Let $\mathcal{F} \in D(X)_{\text{hol}}$. If $\mathcal{L}_{\xi} * \mathcal{F} = 0$ for all $\xi \in \hat{T}$, then $\mathcal{F} = 0$.

**Proof.** Since $T$ acts freely on $X$ we have an embedding $o_x : T \to X, t \to t \cdot x$. Moreover, by base change, we have

$$R\Gamma_{\text{dr}}(T, \mathcal{L}_{\xi} \boxtimes ! o_x^! \mathcal{F}) \simeq i^!_x (\mathcal{L}_{\xi} * \mathcal{F}) = 0$$

for all $\xi \in \hat{T}$. Here $i : x \to X$ is the natural inclusion map. By [GL, Proposition 3.4.5] it implies $o_x^! \mathcal{F} = 0$ for all $x \in X$. The lemma follows.

□

6. Non-linear Fourier transforms

In this section we fix a $c \in \mathbb{C}^\times$ and write $\Psi_{G,\Delta} = \Psi_{G,\Delta,c}, \Psi_{\Delta} = \Psi_{\Delta,c}$. Following Braverman-Kazhdan, we consider the functor of convolution with gamma $D$-module:

$$F_{G,\Delta} := (-) * \Psi_{G,\Delta} : D(G)_{\text{hol}} \to D(G)_{\text{hol}}, \mathcal{F} \to \mathcal{F} * \Psi_{G,\Delta}.$$ 

The result in [BK1] (see, for example, [BK1, Theorem 5.1]) suggests that the functor $F_{G,\Delta}$ can be thought as a version of non-linear Fourier transform on the derived category of holonomic $D$-modules.

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2 In [GL] they proved the vanishing result in the setting of $\ell$-adic sheaves, but the same proof works for the setting of $D$-modules.
The following property of $F_{G,\Delta}$ follows from Proposition 3.2, Proposition 4.2, and Theorem 5.2:

**Theorem 6.1.** ($F_{G,\Delta}$ commutes with induction functors) For every $\mathcal{F} \in D(T)_{\text{hol}}$ we have

$$F_{G,\Delta}(\text{Ind}^G_{T\subset B}(\mathcal{F})) \cong \text{Ind}^G_{T\subset B}(F_{T,\lambda}(\mathcal{F})).$$

Here $F_{T,\lambda}(\mathcal{F}) := \mathcal{F} \ast \Psi_{\lambda}$. In particular, for any Kummer local system $L_\xi$ on $T$ we have

$$F_{G,\Delta}(\text{Ind}^G_{T\subset B}(L_\xi)) \cong V_{\Delta,\xi} \otimes \text{Ind}^G_{T\subset B}(L_\xi).$$

Here $V_{\Delta,\xi} := H^0_{dR}(\Psi_{\lambda} \otimes L^{-1}_\xi)$.

We have the following conjecture:

**Conjecture 6.2** (see Conjecture 6.8 in [BK1]). $F_{G,\Delta}$ is an exact functor.

We shall prove a weaker statement which says that $F_{G,\Delta}$ is exact on the category of admissible $D$-modules. We first recall the definition of admissible modules following [G].

**Definition 6.3.** A holonomic $D$-module $\mathcal{F}$ on $G$ is called admissible if the action of the center $Z(U(G))$ of $U(G)$, viewing as invariant differential operators, is locally finite. We denote by $\mathcal{A}(G)$ the abelian category of admissible $D$-modules on $G$ and $D(\mathcal{A}(G))$ be the corresponding derived category.

**Remark 6.4.** We do not require admissible $D$-modules to be $G$-equivariant with respect to the conjugation action. So the definition of admissible modules here is more general than the one in [G].

We have the following characterization of admissible modules: a $\mathcal{F} \in \mathcal{M}(G)_{\text{hol}}$ is admissible if and only if $HC(\mathcal{F}) \in D(Y/T)$ is monodromic with respect to the right $T$-action, or equivalently, $Av_U(\mathcal{F}) \in D(X)$ is monodromic with respect to the right $T$-action.

To every $\theta \in \breve{T}/W$, let $\mathcal{A}(G)_\theta$ be the full subcategory of $\mathcal{A}(G)$ consisting of holonomic $D$-modules on $G$ such that $Z(U(G))$ acts locally finitely with generalized eigenvalues in $\theta$. The category $\mathcal{A}(G)$ decomposes as

$$\mathcal{A}(G) = \bigoplus_{\theta \in \breve{T}/W} \mathcal{A}(G)_\theta.$$  

**Theorem 6.5.** The functor $F_{G,\Delta} : D(G)_{\text{hol}} \rightarrow D(G)_{\text{hol}}$, $\mathcal{F} \mapsto \mathcal{F} \ast \Psi_{G,\Delta}$ preserves the subcategory $D(\mathcal{A}(G))$ and the resulting functor

$$F_{G,\Delta} : D(\mathcal{A}(G)) \rightarrow D(\mathcal{A}(G))$$

is exact with respect to the natural $t$-structure. That is, we have $F_{G,\Delta}(\mathcal{F}) \in \mathcal{A}(G)$ for $\mathcal{F} \in \mathcal{A}(G)$.  

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Proof. We show that $F_{G \Delta}$ preserves $D(A(G))$. Using the characterization of admissible modules above we have to show that $A\nu_U(F_{G \Delta}(\mathcal{M}))$ is monodromic for $\mathcal{M} \in D(A(G))$. Since $A\nu_U(\Psi_{G \Delta}) \simeq \Psi_{\Delta}$ by Corollary 5.3 we have

$$A\nu_U(F_{G \Delta}(\mathcal{M})) \simeq A\nu_U(\mathcal{M}) \ast A\nu_U(\Psi_{G \Delta}) \simeq A\nu_U(\mathcal{M}) \ast \Psi_{\Delta}$$

which is $T$-monodromic by Proposition 6.2. The claim follows.

We show that $F_{G \Delta}$ is exact on $A(G)$. Let $\mathcal{O}_Y$ (resp. $\mathcal{O}_X$) be pre-image of the open $G$-orbit (resp. $B$-orbit) in $B \times B$ (resp. $X \to B$) under the projection map $Y \to B \times B$ (resp. $X \to B$). The quotient $G \setminus \mathcal{O}_Y$ (resp. $U \setminus \mathcal{O}_X$) is a torsor over $T$, choosing a trivialization of the torsor, we get a map $p_Y : \mathcal{O}_Y \to T$ (resp. $p_X : \mathcal{O}_X \to T$). We denote by $j_Y : \mathcal{O}_Y \to Y$ (resp. $j_X : \mathcal{O}_X \to X$) the natural embedding. Consider the following pro-object in $M_{\xi,w_0(\xi^{-1})}$ (resp. $H_{\xi^{-1},w_0(\xi^{-1})}$):

$$I_Y := j_Y!(p_Y^0 \hat{\mathcal{L}}_{w_0(\xi^{-1})}) := \varprojlim \ j_Y!(p_Y^0 \hat{\mathcal{L}}_{w_0(\xi^{-1})}) \quad (\text{resp.} \ I_X := j_X!(p_X^0 \hat{\mathcal{L}}_{w_0(\xi^{-1})}) := \varprojlim \ j_X!(p_X^0 \hat{\mathcal{L}}_{w_0(\xi^{-1})}))$$

Recall the notion of intertwining functor (see [BB] [BG])

$$(-) \ast I_Y : D(Y)_{\xi^{-1}} \to D(Y)_{\xi,w_0(\xi^{-1})} \quad (\text{resp.} \ (-) \ast I_X : D(X)_{\xi^{-1},\xi^{-1}} \to D(X)_{\xi^{-1},w_0(\xi^{-1})})$$

According to [BFO, Corollary 3.4], the assignment $\mathcal{M} \to HC(M) \ast I_Y := \varprojlim HC(M) \ast j_Y!(p_Y^0 \hat{\mathcal{L}}_{\xi})$, $\mathcal{M} \in D(G)_{\text{hol}}$ restricts to a functor

$$HC(-) \ast I_Y : D(A(G)_{[\xi)}) \to D(Y)_{\xi,w_0(\xi^{-1})}$$

which is $t$-exact and conservative.

So to prove the exactness of $F_{G \Delta}$ it suffices to show that

$$HC(F_{G \Delta}(\mathcal{M})) \ast I_Y \in M(Y)_{\xi,w_0(\xi^{-1})}$$

for all $\mathcal{M} \in A(G)_{[\xi]}$. We claim that there is an isomorphism of pro-objects

$$HC(\Psi_{G \Delta}) \ast I_Y \simeq I_Y.$$  

Thus

$$HC(F_{G \Delta}(\mathcal{M})) \ast I_Y \simeq HC(M \ast \Psi_{G \Delta}) \ast I_Y \simeq HC(M) \ast HC(\Psi_{G \Delta}) \ast I_Y \simeq HC(M) \ast I_Y$$

which is in $M(Y)_{\xi,w_0(\xi^{-1})}$ by the exactness of the functor $HC(-) \ast I_Y$. We are done.

Proof of the claim. Applying the equivalence $i^0 : D_G(Y) \sim D_U(X)$ to (6.1) and using $i^0(HC(\Psi_{G \Delta})) \simeq A\nu_U(\Psi_{G \Delta}) \simeq \Psi_{\Delta}$, $i^0(I_Y) \simeq I_X$, we reduce to show that there is an isomorphism of pro-objects $\Psi_{\Delta} \ast I_X \simeq I_X$. Note that we have $\hat{\mathcal{L}}_{\xi^{-1}} \ast I_X \simeq I_X$ hence by Lemma 4.6

$$\Psi_{\Delta} \ast I_X \simeq \Psi_{\Delta} \ast \hat{\mathcal{L}}_{\xi^{-1}} \ast I_X \simeq \hat{\mathcal{L}}_{\xi^{-1}} \ast I_X \simeq I_X.$$

The claim follows.

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3The definition of intertwining functor here is different from that of [BFO], though one can show that the two definition are equivalent. In [BFO], the intertwining functor is described as a shriek convolution with certain $G$-equivariant $D$-module on $Y$.

4Indeed, it follows from the fact that the functor $\hat{\mathcal{L}}_{\xi}(-) : D_U(X) \to \text{pro}(D_U(X))$ (here pro$(D_U(X))$ is the category of pro-objects in $D_U(X)$), when restricts to the subcategory $D(H_{\xi,X})$ consisting of $T \times T$-monodromic complexes with generalized monodromy $(\xi,\xi')$, is isomorphic to the identity functor.
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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO, CHICAGO, IL 60637, USA.

E-mail address: chenth@math.uchicago.edu