Matrix integrals over unitary groups: An application of Schur-Weyl duality

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Abstract

The integral formulae over the unitary group $U(d)$ are reviewed with new results and new proofs. The normalization and the bi-invariance of the uniform Haar measure play the key role for these computations. These facts are based on Schur-Weyl duality, a powerful tool from representation theory of group.

1 Introduction

This review article is mainly concerned with some useful matrix integrals over unitary group. To compute the some integrals over the unitary group $U(d)$, we use frequently Schur-Weyl duality, a technique from representation theory of group. Before proceeding to give the specific details of Schur-Weyl duality, we need briefly introduce Schur-Weyl duality with its applications in quantum information theory.

In classical information theory, the method of types can be used to carry out some tasks such as estimating probability distribution, randomness concentration and data compression. It has been shown that Schur basis can be used to generalize classical method of types, thus allowing us to perform quantum counterparts of the previously mentioned tasks. In fact Schur basis is a natural choice if we want to study systems with permutation symmetry. Schur transformation can be used to carry out several tasks such as estimation of the spectrum of an unknown mixed state, universal distortion-free entanglement concentration using only local operations, and encoding into decoherence free subsystems, etc. Another applications of Schur transformation include communication without shared reference frame and universal compression of quantum date.

More explicit results related to Schur-Weyl duality can be mentioned. For example, Keyl and Werner using Schur-Weyl duality to estimate the spectrum of an unknown ($d$-level) mixed state $\rho$ from its $k$-fold product state [1]. Harrow gave efficient quantum circuits for Schur and Clebsch-Gordan transforms from computational point of view [2] [3]. Christandl employing Schur-Weyl duality in [4] [5] investigated the structure of multipartite quantum states, and obtained a group-theoretic proof for some entropy inequalities concerning von Neumann entropy, such as (strong) subadditivity of von Neumann entropy. Gour use

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this duality to classify the multipartite entanglement of quantum state in the finite dimensional setting [6]. A generalization of Schur-Weyl duality with applications in quantum estimation can be found in [7, 8].

2 Schur-Weyl duality

In this section, we give the details of Schur-Weyl duality. A generalization of this duality is obtained within the framework of the infinite-dimensional $C^*$-algebras [9]. In order to arrive at Schur-Weyl duality, we need the following ancillary results, well-known facts in representation theory. We assume familiarity with knowledge of representation theory of a compact Lie group or finite group [10, 11].

Proposition 2.1. Let $V$ and $W$ be finite dimensional complex vector spaces. If $\mathcal{M} \subseteq \text{End}(V)$ and $\mathcal{N} \subseteq \text{End}(W)$ are von Neumann algebras, then $(\mathcal{M} \otimes \mathcal{N})' = \mathcal{M}' \otimes \mathcal{N}'$.

Proof. Apparently, $\mathcal{M}' \otimes \mathcal{N}' \subseteq (\mathcal{M} \otimes \mathcal{N})'$. It suffices to show that $(\mathcal{M} \otimes \mathcal{N})' \subseteq \mathcal{M}' \otimes \mathcal{N}'$. For arbitrary $T \in (\mathcal{M} \otimes \mathcal{N})'$, by the Operator-Schmidt Decomposition,

$$T = \sum_j \lambda_j A_j \otimes B_j,$$

where $\lambda_j \geq 0$, and $A_j$ and $B_j$ are orthonormal bases of $\text{End}(\mathbb{C}^m)$ and $\text{End}(\mathbb{C}^n)$, respectively. Now for arbitrary $M \in \mathcal{M}$ and $N \in \mathcal{N}$, $M \otimes 1_W, 1_V \otimes N \in \mathcal{M} \otimes \mathcal{N}$, it follows that

$$[T, M \otimes 1_W] = 0 = [T, 1_V \otimes N].$$

That is,

$$\sum_j \lambda_j [A_j, M] \otimes B_j = 0 \quad \text{and} \quad \sum_j \lambda_j A_j \otimes [B_j, N] = 0.$$ 

We drop those terms for which $\lambda_j$ are zero. Thus $\lambda_j$ is positive for all $j$ in the above two equations. Since $\{A_j\}$ and $\{B_j\}$ are linearly independent, respectively, it follows that

$$[A_j, M] = 0 \quad \text{and} \quad [B_j, N] = 0.$$ 

This implies that $A_j \in \mathcal{M}'$ and $B_j \in \mathcal{N}'$. Therefore $T \in \mathcal{M}' \otimes \mathcal{N}'$. $\Box$

Proposition 2.2 (The dual theorem). Let $V$ be a representation of a finite group $G$ with decomposition $V \cong \bigoplus_{a \in G} n_a V_a \cong \bigoplus_{a \in G} V_a \otimes \mathbb{C}^{n_a}$. Let $\mathcal{A}$ be the algebra generated by $V$ and $\mathcal{B} = \mathcal{A}'$ its commutant. Then

$$\mathcal{A} \cong \bigoplus_{a \in \hat{G}} \text{End}(V_a) \otimes 1_{\mathbb{C}^{n_a}},$$

$$\mathcal{B} \cong \bigoplus_{a \in \hat{G}} 1_{V_a} \otimes \text{End}(\mathbb{C}^{n_a}).$$

Furthermore we have $\mathcal{B}' = \mathcal{A}$, where $\mathcal{B}'$ is the commutant of $\mathcal{B}$. That is $\mathcal{A} = \mathcal{A}''$ and $\mathcal{B} = \mathcal{B}''$. Thus both $\mathcal{A}$ and $\mathcal{B}$ are von Neumann algebras.
Proof. It is easy to see that 
\[ \frac{d_a}{|G|} \sum_{g \in G} V_{a,i}(g) V(g) \in \mathcal{A}. \]

By the orthogonality of the functions \( V_{a,i} \) and the decomposition of \( V \) into irreducible components, we get
\[
\frac{d_a}{|G|} \sum_{g \in G} V_{a,i}(g) V(g) = \frac{d_a}{|G|} \sum_{g \in G} V_{a,i}(g) \left( \bigoplus_{\beta \in \hat{G}} V_\beta(g) \otimes \mathbb{1}_{C^{n_\beta}} \right)
\]
\[
= \bigoplus_{\beta \in \hat{G}} \left( \frac{d_a}{|G|} \sum_{g \in G} V_{a,i}(g) V_\beta(g) \right) \otimes \mathbb{1}_{C^{n_\beta}}
\]
\[
= \bigoplus_{\beta \in \hat{G}} \sum_{k,l} \left( \frac{d_a}{|G|} \sum_{g \in G} V_{a,i}(g) V_{\beta,kl}(g) \right) E_{\beta,kl} \otimes \mathbb{1}_{C^{n_\beta}}
\]
\[
= E_{a,i} \otimes \mathbb{1}_{C^{n_a}},
\]

implying \( E_{a,i} \otimes \mathbb{1}_{C^{n_a}} \in \mathcal{A} \), hence \( \text{End}(V_a) \otimes \mathbb{1}_{C^{n_a}} \subseteq \mathcal{A} \). Now
\[
\mathcal{A} = \text{span}\{ V(g) : g \in G \} \cong \bigoplus_{a \in \hat{G}} \text{span}\{ V_a(g) \otimes \mathbb{1}_{C^{n_a}} \} = \bigoplus_{a \in \hat{G}} \text{End}(V_a) \otimes \mathbb{1}_{C^{n_a}}.
\]

Clearly \( \bigoplus_{a \in \hat{G}} \mathbb{1}_{V_a} \otimes \text{End}(C^{n_a}) \subseteq \mathcal{A}' = B \). To see that every element in \( B \) is of this form, i.e. \( B \subseteq \bigoplus_{a \in \hat{G}} \mathbb{1}_{V_a} \otimes \text{End}(C^{n_a}) \). Consider a projection \( c_a \) onto \( V_a \otimes C^{n_a} \). The projectors \( c_a \) form a resolution of the identity and \( c_a \in \mathcal{A} \). Since \( \mathcal{A}' = B \), it follows that any \( B \in \mathcal{A}' \) must commute with \( P_a \): \( c_a B = B c_a \). This leads to
\[
B = \left( \sum_a c_a \right) B = \sum_a c_a B c_a = \sum_a B_a.
\]
Moreover, \( B_a \in (\text{End}(V_a) \otimes \mathbb{1}_{C^{n_a}})' = 1_{V_a} \otimes \text{End}(C^{n_a}) \), thus it must be of the form \( B_a = 1_{V_a} \otimes b_a \). \( \square \)

Remark 2.3. Let \( \hat{G} \) be a complete set of inequivalent irreps of \( G \). Then for any reducible representation \( V \), there is a basis under which the action of \( V(g) \) can be expressed as
\[
V(g) \cong \bigoplus_{a \in \hat{G}} \bigoplus_{i=1}^{n_a} V_a(g) = \bigoplus_{a \in \hat{G}} V_a(g) \otimes \mathbb{1}_{n_a}, \quad (2.6)
\]

where \( a \in \hat{G} \) labels an irrep \( V_a \) and \( n_a \) is the multiplicity of the irrep \( V_a \) in the representation \( V \). Here we use \( \cong \) to indicate that there exists a unitary change of basis relating the left-hand side to the right-hand side. Under this change of basis we obtain a similar decomposition of the representation space \( V \) (known as the isotypic decomposition):
\[
V \cong \bigoplus_{a \in \hat{G}} V_a \otimes \text{Hom}_G(V_a, V), \quad (2.7)
\]

Since \( G \) acts trivially on \( \text{Hom}_G(V_a, V) \), Eq. (2.6) remains the same.
The value of Eq. (2.7) is that the unitary mapping from the right-hand side (RHS) to the left-hand side (LHS) has a simple explicit expression: it corresponds to the canonical map \( \varphi : \mathcal{X} \otimes \text{Hom}(\mathcal{X} \otimes \mathcal{Y}) \rightarrow \mathcal{Y} \) given by \( \varphi(x \otimes f) = f(x) \). Of course, this doesn’t tell us how to describe \( \text{Hom}_G(V_h, V) \), or how to specify an orthonormal basis for the space, but we will later find this form of the decomposition useful.

Consider a system of \( k \) qudits, each with a standard local computational basis \( \{|i\}, i = 1, \ldots, d\} \). The Schur-Weyl duality relates transforms on the system performed by local \( d \)-dimensional unitary operators to those performed by permutation of the qudits. Recall that the symmetric group \( S_k \) is the group of all permutations of \( k \) objects. This group is naturally represented in our system by

\[
P(\pi)|i_1 \cdots i_k\rangle := |i_{\pi^{-1}(1)} \cdots i_{\pi^{-1}(k)}\rangle, \tag{2.8}
\]

where \( \pi \in S_k \) is a permutation and \( |i_1 \cdots i_k\rangle \) is shorthand for \( |i_1\rangle \otimes \cdots \otimes |i_k\rangle \). Let \( U(d) \) denote the group of \( d \times d \) unitary operators. This group is naturally represented in our system by

\[
Q(U)|i_1 \cdots i_k\rangle := U|i_1\rangle \otimes \cdots \otimes U|i_k\rangle, \tag{2.9}
\]

where \( U \in U(d) \). Thus we have the following famous result:

**Theorem 2.4** (Schur). Let \( \mathcal{A} = \text{span}\{P(\pi) : \pi \in S_k\} \) and \( \mathcal{B} = \text{span}\{Q(U) : U \in U(d)\} \). Then:

\[
\mathcal{A}' = \mathcal{B} \quad \text{and} \quad \mathcal{A} = \mathcal{B}'. \tag{2.10}
\]

**First proof.** The proof is separated into two steps: 1) \( \mathcal{A}' = \text{span}\{A^\otimes k : A \in \text{End}(C^d)\} \). 2) \( \text{span}\{A^\otimes k : A \in \text{End}(C^d)\} = \mathcal{B} \).

In order to show that 1) holds, note that \( \text{End}((C^d)^{\otimes k}) = \text{End}(C^d)^{\otimes k} \). Firstly we show that

\[
\mathcal{A}' = \text{End}(C^d)^{\otimes k} \cap P(S_k)' = \text{End}((C^d)^{\otimes k}) \cap P(S_k)' = \text{span}\{Q(A) : A \in \text{End}(C^d)\}. \tag{2.11}
\]

We need only show that LHS is contained in RHS since the reverse inclusion is trivial.

For arbitrary \( \Gamma \in \mathcal{A}' = \text{End}(C^d)^{\otimes k} \cap P(S_k) \), we have \( \Gamma = T_{S_k}(\Gamma) \) and \( \Gamma \in \text{End}(C^d)^{\otimes k} \), where \( T_{S_k} = \frac{1}{k!} \sum_{\pi \in S_k} \text{Ad}_P(\pi) \). It suffices to show that

\[
\Gamma = T_{S_k}(A_1 \otimes \cdots \otimes A_k) \in \text{span}\{Q(A) : A \in \text{End}(C^d)\},
\]

where \( A_j \in \text{End}(C^d) \). In what follows, we show that each such \( \Gamma \) can be written in terms of tensor products \( A^{\otimes k} \). Since

\[
\frac{d}{dt}(M + tN)^{\otimes k} = \sum_{j=0}^{k-1} (M + tN)^{\otimes j}N(M + tN)^{\otimes (k-j-1)}, \tag{2.12}
\]

it follows that

\[
\left. \frac{d}{dt} \right|_{t=0} (M + tN)^{\otimes k} = \sum_{j=0}^{k-1} M^{\otimes j}NM^{\otimes (k-j-1)}. \tag{2.13}
\]

Consider the following partial derivative

\[
\left. \frac{\partial^{k-1}}{\partial t_2 \cdots \partial t_k} \right|_{t_2 = \cdots = t_k = 0} \left( A_1 + \sum_{j=2}^{k} t_j A_j \right)^{\otimes k}, \tag{2.14}
\]
which can be realized by subsequently applying
\[
\left. \frac{\partial}{\partial t_j} \right|_{t_j=0} (A + t_jA_j)^\otimes_k = \lim_{t_j \to 0} \frac{(A + t_jA_j)^\otimes_k - A^\otimes_k}{t_j},
\]  
(2.15)
itertatively going from \( j = k \) all the way to \( j = 2 \). The \( (2.14) \) takes the form of a limit of sums of tensor powers. Since \( \text{span}\{Q(A) : A \in \text{End}(\mathbb{C}^d)\} \) is a finite dimensional vector space, this limit is contained in \( \text{span}\{Q(A) : A \in \text{End}(\mathbb{C}^d)\} \). On the other hand, a direct calculation shows that

\[
k!\Gamma = \left. \frac{\partial^{k-1}}{\partial t_2 \ldots \partial t_k} \right|_{t_2 = \cdots = t_k = 0} \left( A_1 + \sum_{j=2}^k t_jA_j \right)^\otimes_k
\]  
(2.16)
and hence all operators \( \Gamma \) are contained in \( \text{span}\{Q(A) : A \in \text{End}(\mathbb{C}^d)\} \).

Next, we turn to prove that 2) holds. Firstly we show that

\[
\text{span}\{U^\otimes_k : U \in U(d)\} = \text{span}\{T^\otimes_k : T \in \text{GL}(d, \mathbb{C})\}.
\]  
(2.17)
For any \( T \in \text{GL}(d, \mathbb{C}) \), there exists \( M \in \text{End}(\mathbb{C}^d) \) such that

\[
T = e^M.
\]
(The elementary proof of this fact is shifted to the following remark.) Then

\[
T^\otimes_n = (e^M)^\otimes_n = \exp \left( \sum_{j=1}^n 1^\otimes j^{-1} \otimes M \otimes 1^{n-j} \right) = \exp(Q_*(M)),
\]  
(2.18)
where

\[
Q_*(M) := \left. \frac{d}{dt} \right|_{t=0} Q(e^{tM}).
\]
Clearly \( Q(e^{tM}) = Q(e^{tQ_*(M)}) \) for any real \( t \in \mathbb{R} \). In fact, \( Q \) is a Lie group representation of \( U(d) \) or \( \text{GL}(d, \mathbb{C}) \). \( Q_*(M) \) is a Lie algebra representation induced by \( Q \). If we can show that \( Q_*(M) \in \text{span}\{U^\otimes_n : U \in U(d)\} \), then by \( (2.18) \), it follows that \( T^\otimes_n \in \text{span}\{U^\otimes_n : U \in U(d)\} \).

Next, we show that \( Q_*(M) \in \text{span}\{U^\otimes_n : U \in U(d)\} \). For any skew-Hermitian operator \( X \), \( e^{tX} \) is a unitary, thus \( Q(e^{tX}) \in \text{span}\{U^\otimes_n : U \in U(d)\} \). By the connection of \( Q \) and \( Q_* \), we have \( Q(e^{tX}) = e^{tQ_*(X)} \), implying that \( Q_*(X) \in \text{span}\{U^\otimes_n : U \in U(d)\} \), where \( X \in u(d) \), a Lie algebra of \( U(d) \). Let \( M = X + \sqrt{-1}Y \) for \( X, Y \in u(d) \). Thus by the complex-linearity of \( Q_* \), it follows that

\[
Q_*(M) = Q_*(X) + \sqrt{-1}Q_*(Y).
\]
Since \( \text{span}\{U^\otimes_n : U \in U(d)\} \) is a complex-linear space, it follows that

\[
Q_*(X) + \sqrt{-1}Q_*(Y) \in \text{span}\{U^\otimes_n : U \in U(d)\}
\]  
(2.19)
whenever \( Q_*(X), Q_*(Y) \in \text{span}\{U^\otimes_n : U \in U(d)\} \). Therefore \( Q_*(M) \in \text{span}\{U^\otimes_n : U \in U(d)\} \).

Up to now, we established the fact that

\[
\text{span}\{U^\otimes_n : U \in U(d)\} = \text{span}\{T^\otimes_n : T \in \text{GL}(d, \mathbb{C})\}.
\]  
(2.20)
Secondly, we show that
\[ \text{span}\{T^\otimes n : T \in \text{GL}(d, \mathbb{C})\} = \text{span}\{A^\otimes n : A \in \text{End}(\mathbb{C}^d)\}. \] (2.21)
We use the fact that GL$(d, \mathbb{C})$ is dense in End$(\mathbb{C}^d)$. Indeed, for any $A \in \text{End}(\mathbb{C}^d)$, by the Singular Value Decomposition, we have
\[ A = UDV^t, \]
where $U, V \in U(d)$ and $D$ is a diagonal matrix whose diagonal entries are nonnegative. Define
\[ T_\epsilon = U \left( D + \frac{\epsilon}{1+\epsilon} I \right) V^t \]
for very small positive real $\epsilon$. Apparently $T_\epsilon \in \text{GL}(d, \mathbb{C})$ and $\|A - T_\epsilon\| < \epsilon$. This indicates that GL$(d, \mathbb{C})$ is dense in End$(\mathbb{C}^d)$ in norm topology.

For any fixed $A \in \text{End}(\mathbb{C}^d)$, we take $T \in \text{GL}(d, \mathbb{C})$ such that $\|A - T\|$ is very small. Since
\[ \|Q(A) - Q(T)\| \leq n\Delta^{n-1} \|A - T\|, \]
where $\Delta := \max\{\|A\|, \|T\|\}$, it follows, from the fact that span$\{T^\otimes n : T \in \text{GL}(d, \mathbb{C})\}$ is closed (in the finite-dimensional setting), that (2.21) is true. Therefore the proof is complete.

**Remark 2.5.** In this Remark, we will show that, for every $T \in \text{GL}(d, \mathbb{C})$, there exists $M \in \text{End}(\mathbb{C}^d)$ such that $T = e^M$. This result is a famous one in Lie theory. A general method for its proof is rather involved. To avoid usage of advanced tools in Lie theory. We give here an elementarily proof of it. We just use the matrix technique.

Indeed, it is easy to show that if $T$ is a diagonalizable matrix, then the conclusion is true. For a general case, we separate the proof into two cases:

**Case 1.** There is a sequence of diagonalizable matrices $T_k$ satisfying that
(i) $\lim_k T_k = T,$
(ii) If $T_k = e^{M_k}$, then there is a constant $c > 0$ such that $\|M_k\| \leq c$ holds for every $k$.

Now we show that the existence of $T_k$. Consider the Jordan canonical decomposition of $T$ for $T = PJP^{-1}$. Let $t_j$ be the diagonal entries of $J$. Note that $T$ is an invertible matrix, so $t_j \neq 0$ for every $1 \leq j \leq d$. Let
\[ T_k := P(J + \Lambda_k)P^{-1}, \]
where $\Lambda_k := \text{diag}(\lambda_{1}^{k}, \lambda_{2}^{k}, \ldots, \lambda_{d}^{k})$. Then $T_k$ meets the conditions (i) and (ii) in Case 1 if
(a) $\lim_k \lambda_{j}^{k} = 0$ for $j = 1, \ldots, d$;
(b) $t_j + \lambda_{j}^{k}$ are all different when $j$ runs from 1 to $d$ for every given $k$. Thus $T_k$ has $d$ different eigenvalues $t_j + \lambda_{j}^{k}$, and of course $T_k$ is diagonalizable;
(c) there is a constant $c$ such that $|\ln(t_j + \lambda_{j}^{k})| \leq c$ for every $k$ and $j$. Note that if (b) is true, then
\[ \|M_k\| = \max_j \left| \ln(t_j + \lambda_{j}^{k}) \right|. \]
The construction of $\lambda^k$ satisfying (a)–(c) is described as follows: For any given $k$, let $\lambda_1^k = \frac{t_1}{t_1 + 1}$, and $\lambda_j^k$ be one of $\frac{t_j}{t_1 + 1}$, $\frac{t_j}{t_2 + 1}$, \ldots, $\frac{t_j}{t_k + 1}$ such that $t_i + \lambda_i^k \neq t_j + \lambda_j^k$ whenever $i < j$. Apparently (a) and (b) are satisfied. To check (c), we have
$$\left| \ln(t_j + \lambda_j^k) \right| = \left| \ln(t_j) + \ln(1 + \lambda_j^k/t_j) \right| \leq \left| \ln(t_j) \right| + \left| \ln(1 + \lambda_j^k/t_j) \right|,$$
taking $c = \max_j \left| \ln(t_j) \right| + \ln 2$ is enough. That is $\|M_k\| = \max_j \left| \ln(t_j + \lambda_j^k) \right| \leq c$ for all $k$.

Case 2. When Case 1 holds, since the exponential function is a smooth and continuous function, so the image of the compact set $\exp(B(0,c))$ must be closed, where $B(0,c)$ is the closed ball with radius $c$, thus the limit $T$ of $e^{M_k}$ is also in $\exp(B(0,c))$. This means that there exists $M \in B(0,c)$ such that $T = e^M$. The proof is finished.

We remark here that the above first proof of Schur-Weyl duality makes reference to PhD thesis of Christandl [4]. The following second proof is taken from the book of Goodman and Wallach [11].

Second proof. Let $\{ |1\rangle, \ldots, |d\rangle \}$ be the standard basis for $\mathbb{C}^d$. For an ordered $k$-tuple $I = (i_1, \ldots, i_k)$ with $i_1, \ldots, i_k \in [d]$, where $|d| := \{1, \ldots, d\}$, define $|I| = k$ and $|I| := |i_1 \cdots i_k|$. The tensors $\{|I| : I \in [d]^k\}$, with $I$ ranging over all such $k$-tuples, give a basis for $(\mathbb{C}^d)^{\otimes k}$. The group $S_k$ permutes this basis by the action $P(\pi)|I\rangle = |\pi \cdot I\rangle$, where for $I = (i_1, \ldots, i_k)$ and $\pi \in S_k$, we define
$$\pi \cdot (i_1, \ldots, i_k) := (i_{\pi^{-1}(1)}, \ldots, i_{\pi^{-1}(k)}).$$
Note that $\pi$ changes the positions (1 to $k$) of the indices, not their values (1 to $d$), and we have $(\sigma \pi) \cdot I = \sigma \cdot (\pi \cdot I)$ for $\sigma, \pi \in S_k$.

Suppose $B \in \text{End}((\mathbb{C}^d)^{\otimes k})$ has matrix $[b_{I,J}]$ relative to the basis $\{|I| : I \in [d]^k\}$: $\langle I | B | J \rangle = b_{I,J}$ and
$$B|I\rangle = \sum_{I \in [d]^k} b_{I,J} |I\rangle.$$ We have
$$BP(\pi)|I\rangle = B|\pi \cdot I\rangle = \sum_{I} b_{I,\pi \cdot I} |I\rangle$$
for $\pi \in S_k$, whereas
$$P(\pi)B|I\rangle = \sum_{I} b_{I,J} |\pi \cdot I\rangle = \sum_{I} b_{\pi^{-1}I,J} |I\rangle.$$ (2.23)
Thus $B \in \mathcal{A}'$ if and only if $b_{I,\pi \cdot J} = b_{\pi^{-1}I,J}$ for all multi-indices $I, J$ and all $\pi \in S_k$. Replacing $I$ by $\pi \cdot I$, we can write this condition as
$$b_{\pi^{-1}I,J} = b_{I,J}, \quad \forall I, J, \pi \in S_k.$$ (2.24)
Consider the non-degenerate bilinear form $\langle X, Y \rangle := \text{Tr}(XY)$ on $\text{End}((\mathbb{C}^d)^{\otimes k})$. We claim that the restriction of this form to $\mathcal{A}'$ is non-degenerate. Indeed, we have a projection $X \mapsto X^\#$ of $\text{End}((\mathbb{C}^d)^{\otimes k})$ onto $\mathcal{A}'$ given by averaging over $S_k$:
$$X^\# = \frac{1}{k!} \sum_{\pi \in S_k} P(\pi)XP(\pi)^{-1}. \quad (2.25)$$
If \( B \in \mathcal{A}' \), then
\[
\langle X^\#, B \rangle = \frac{1}{k!} \sum_{\pi \in S_k} \operatorname{Tr} \left( \mathbf{P}(\pi) X \mathbf{P}(\pi)^{-1} B \right) = \langle X, B \rangle,
\]

since \( \mathbf{P}(\pi) B = B \mathbf{P}(\pi) \). Thus \( \langle \mathcal{A}', B \rangle = 0 \) implies that \( \langle X, B \rangle = 0 \) for all \( X \in \text{End}(\mathbb{C}^d) \), and so \( B = 0 \). Hence the trace form on \( \mathcal{A}' \) is non-degenerate.

To show that \( \mathcal{A}' = B \), it thus suffices to show that if \( B \in \mathcal{A}' \) is orthogonal to \( B \), then \( B = 0 \). Now if \( g = [g_{ij}] \in \text{GL}(d, C) \), then \( \mathbf{Q}(g) \) has matrix \( g_{I,J} = g_{ij} \cdots g_{ik} \) relative to the basis \( \{|I| : I \in [d]^k\} \). Thus we assume that

\[
\langle B, \mathbf{Q}(g) \rangle = \sum_{I,J} b_{I,J} g_{ij} \cdots g_{ik} = 0
\]

for all \( g \in \text{GL}(d, C) \), where \( b_{I,J} \) is the matrix of \( B \). Define a polynomial function \( p_B \) on \( M(C^d) \) by

\[
p_B(X) = \sum_{I,J} b_{I,J} x_{ij} \cdots x_{ik}
\]

for \( X = [x_{ij}] \in M(C^d) \). Clearly \( p_B \) is vanished over \( \text{GL}(d, C) \), a dense subset of \( M(C^d) \); and \( p_B \) is a continuous function on \( M(C^d) \), therefore \( p_B \equiv 0 \), so for all \( [x_{ij}] \in M(C^d) \), we have

\[
\sum_{I,J} b_{I,J} x_{ij} \cdots x_{ik} = 0.
\]

In what follows, we show that \( b_{I,J} = 0 \) for all \( I, J \). We begin by grouping the terms in the above equation according distinct monomials in the matrix entries \( \{x_{ij}\} \). Introduce the notation \( x_{I,J} = x_{ij} \cdots x_{ik}, \) and view these monomials as polynomial functions on \( M(C^d) \). Let \( \Theta \) be the set of all ordered pairs \( (I, J) \) of multi-indices with \( |I| = |J| = k \). The group \( S_k \) acts on \( \Theta \) by

\[
\pi \cdot (I, J) = (\pi \cdot I, \pi \cdot J).
\]

From Eq. (2.24), we see that \( B \) commutes with \( S_k \) if and only if the function \( (I, J) \mapsto b_{I,J} \) is constant on the orbits of \( S_k \) in \( \Theta \).

The action of \( S_k \) on \( \Theta \) defines an equivalence relation on \( \Theta \), where \( (I, J) \sim (I', J') \) if \( (I', J') = (\pi \cdot I, \pi \cdot J) \) for some \( \pi \in S_k \). This gives a decomposition of \( \Theta \) into disjoint equivalence classes. Choose a set \( \Gamma \) of representatives for the equivalence classes. Then every monomial \( x_{I,J} \) with \( |I| = |J| = k \) can be written as \( x_\gamma \) for some \( \gamma \in \Gamma \). Indeed, since the variables \( x_{ij} \) mutually commute, we have

\[
x_\gamma = x_{\pi \cdot \gamma}, \quad \forall \pi \in S_k; \gamma \in \Gamma.
\]

Suppose \( x_{I,J} = x_{I', J'} \). Then there must be an integer \( p \) such that \( x_{ij}^{(p)} = x_{ij'}^{(p)} \). Call \( p = 1' \). Similarly, there must be an integer \( q \neq p \) such that \( x_{ij}^{(q)} = x_{ij'}^{(q)} \). Call \( q = 2' \). Continuing this way, we obtain a permutation

\[
\pi : (1, 2, \ldots, k) \rightarrow (1', 2', \ldots, k')
\]

such that \( I = \pi \cdot I' \) and \( J = \pi \cdot J' \). This proves that \( \gamma \) is uniquely determined by \( x_\gamma \). For \( \gamma \in \Gamma \), let \( n_\gamma = |S_k \cdot \gamma| \) be the cardinality of the corresponding orbit.

Assume that the coefficients \( b_{I,J} \) satisfy Eqs. (2.24) and (2.27). Since \( b_{I,J} = b_\gamma \) for all \( (I, J) \in S_k \cdot \gamma \), it follows from Eq. (2.27) that

\[
\sum_{\gamma \in \Gamma} n_\gamma b_\gamma x_\gamma = 0.
\]
Since the set of monomials \( \{ x_\gamma : \gamma \in \Gamma \} \) is linearly independent, this implies that \( b_{I,J} = 0 \) for all \( (I,J) \in \Theta \). This proves that \( B = 0 \). Hence \( B = A' \).

The following result concerns with a wonderful decomposition of the representations on \( k \)-fold tensor space \( (C^d)^\otimes k \) of \( U(d) \) and \( S_k \), respectively, using their corresponding irreps accordingly. The proof is taken from \([4]\).

**Theorem 2.6 (Schur-Weyl duality).** There exist a basis, known as Schur basis, in which representation \( (Q, (C^d)^\otimes k) \) of \( U(d) \times S_k \) decomposes into irreducible representations \( Q_\lambda \) and \( P_\lambda \) of \( U(d) \) and \( S_k \), respectively:  

(i) \( (C^d)^\otimes k \cong \bigoplus_{\lambda \vdash (k,d)} Q_\lambda \otimes P_\lambda; \)
(ii) \( P(\pi) \cong \bigoplus_{\lambda \vdash (k,d)} \mathbb{1}_{Q_\lambda} \otimes P(\pi); \)
(iii) \( Q(U) \cong \bigoplus_{\lambda \vdash (k,d)} Q_\lambda(U) \otimes \mathbb{1}_{P_\lambda}. \)

Since \( Q \) and \( P \) commute, we can define representation \( (QP, (C^d)^\otimes k) \) of \( U(d) \times S_k \) as

\[
QP(U, \pi) = Q(U)P(\pi) = P(\pi)Q(U) \quad \forall (U, \pi) \in U(d) \times S_k. \tag{2.28}
\]

Then:

\[
QP(U, \pi) = U^\otimes k P(\pi) = P(\pi)U^\otimes k \cong \bigoplus_{\lambda \vdash (k,d)} Q_\lambda(U) \otimes P(\pi). \tag{2.29}
\]

In order to prove the above theorem, we first observe that algebras generated by \( P \) and \( Q \) centralize each other. Then we can apply double commutant theorem to get expression Eq. (2.29) only with unspecified range of \( \lambda \). In order to specify the range, we find a correspondence between irreducible representations of \( S_k \) and \( U(d) \) and partitions \( \lambda \vdash (k,d) \).

We call the unitary transformation performing the basis change from standard basis to Schur basis, Schur transform and denote by \( U_{\text{sch}} \). It has been shown that Schur transform can be implemented efficiently on a quantum computer.

**Proof.** The application of the Duality Theorem \([2,2]\) to \( G = S_k \) (and to its dual partner \( U(d) \)), Theorem \([2,4]\) shows the above three equations, where \( P_\lambda \) are irreducible representations of \( S_k \). The representation of \( U(d) \) that is paired with \( P_\lambda \) is denoted by \( Q_\lambda \).

In the following, we show that \( Q_\lambda \)'s are irreducible. A brief but elegant argument is follows: \( Q_\lambda \) is irreducible if and only if its extension to \( GL(d) \) is irreducible. That is \( Q_\lambda(U(d)) \) is irreducible if and only if \( Q_\lambda(GL(d,C)) \) is irreducible. So it suffices to show that \( Q_\lambda \) is indecomposable under \( GL(d,C) \). By Schur’s Lemma this is equivalent to showing that End\(_{GL(d,C)}(Q_\lambda) \cong C \). That is, the maps in End\(_{GL(d,C)}(Q_\lambda) \) that commute with the action of \( GL(d,C) \) are proportional to the identity.

In what follows, we show that End\(_{GL(d,C)}(Q_\lambda) \cong C \). From Schur’s Lemma, we have

\[
\text{End}_{S_k} \left( (C^d)^\otimes k \right) \cong \bigoplus_{\lambda} \text{End}(Q_\lambda) \otimes \mathbb{1}_{P_\lambda} \cong \bigoplus_{\lambda} \text{End}(Q_\lambda).
\]

Thus

\[
\text{End}_{GL(d,C)\times S_k} \left( (C^d)^\otimes k \right) \cong \bigoplus_{\lambda} \text{End}_{GL(d,C)}(Q_\lambda).
\]
By the dual theorem, \( GL(d, C) \) and \( S_k \) are double commutants,

\[
\text{End}_{S_k} \left( (C^d)^{\otimes k} \right) = \text{span} \left\{ T^{\otimes k} : T \in GL(d, C) \right\},
\]

and thus \( \text{End}_{GL(d, C) \times S_k} \left( (C^d)^{\otimes k} \right) \) is clearly contained in the center of \( \text{End}_{S_k} \left( (C^d)^{\otimes k} \right) \). Therefore \( \text{End}_{GL(d, C)}(Q_{\lambda}) \) is contained in the center of \( \text{End}(Q_{\lambda}) \cong C \). Finally

\[
\text{End}_{GL(d, C)}(Q_{\lambda}) \cong C.
\]

For the proof of \( \lambda \vdash (k, d) \), it is rather involved since we need the notion of highest weight classification of a compact Lie group. It is omitted here. We are done. \( \square \)

**Remark 2.7.** By the Duality Theorem 2.2 and Theorem 2.6 it follows that \( Q(X) \in B \) for \( X \in \text{End}(C^d) \). Furthermore the decomposition of \( Q(X) \) is of the form:

\[
Q(X) \cong \bigoplus_{\lambda \vdash (k, d)} Q_{\lambda}(X) \otimes P_{\lambda}.
\] (2.30)

Therefore

\[
X^{\otimes k} P(\pi) = P(\pi) X^{\otimes k} \cong \bigoplus_{\lambda \vdash (k, d)} Q_{\lambda}(X) \otimes P_{\lambda}(\pi).
\] (2.31)

The dimensions of pairing irreps for \( U(d) \) and \( S_k \), respectively, in Schur-Weyl duality can be computed by so-called hook length formulae. The hook of box \((i, j)\) in a Young diagram determined by a partition \( \lambda \) is given by the box itself, the boxes to its right and below. The hook length is the number of boxes in a hook. Specifically, we have the following result without its proof:

**Theorem 2.8** (Hook length formulae). The dimensions of pairing irreps for \( U(d) \) and \( S_k \), respectively, in Schur-Weyl duality can be given as follows:

\[
\dim(Q_{\lambda}) = \prod_{(i, j) \in \lambda} \frac{d + j - i}{h(i, j)} = \prod_{1 \leq i < j \leq d} \frac{\lambda_i - \lambda_j + j - i}{j - i},
\] (2.32)

\[
\dim(P_{\lambda}) = \frac{k!}{\prod_{(i, j) \in \lambda} h(i, j)}.
\] (2.33)

We will see the concrete example which is the most simple one:

**Example 2.9.** Suppose that \( k = 2 \) and \( d \) is greater than one. Then the Schur-Weyl duality is the statement that the space of two-tensors decomposes into symmetric and antisymmetric parts, each of which is also an irreducible module for \( GL(d, C) \):

\[
C^d \otimes C^d = \bigwedge^2 C^d \oplus \vee^2 C^d.
\] (2.34)

The symmetric group \( S_2 \) consists of two elements and has two irreducible representations, the trivial representation and the sign representation. The trivial representation of \( S_2 \) gives rise to the symmetric tensors, which are invariant (i.e. do not change) under the permutation of the factors, and the sign representation corresponds to the skew-symmetric tensors, which flip the sign.
3 Matrix integrals over unitary groups

In this section, we will give the proofs on some integrals over unitary matrix group. We will use the uniform Haar-measure $\mu$ over unitary matrix group $U(d)$. We also use the vec-operator correspondence. The vec mapping is defined as follows:

$$\text{vec}(|i\rangle\langle j|) = |ij\rangle. \quad (3.1)$$

Thus $\text{vec}(1_d) = \sum_{j=1}^{d} |jj\rangle$. Clearly

$$\text{vec}(AXB) = A \otimes B^T \text{vec}(X). \quad (3.2)$$

The vec mapping for bipartite case is still valid:

$$\text{vec}(|m\rangle\langle n| \otimes |\mu\rangle\langle \nu|) = \text{vec}(|m\mu\rangle\langle n\nu|) = |m\mu n\nu\rangle. \quad (3.3)$$

In what follows, we will employ Schur-Weyl duality to give the computations about the integrals of the following forms:

$$\int_{U(d)} U \otimes^k A(U \otimes^k)^\dagger d\mu(U) \text{ or } \int_{U(d)} U \otimes^k \otimes (U \otimes^k)^\dagger d\mu(U). \quad (3.4)$$

We demonstrate the integral formulae for the special cases where $k = 1, 2$ with detailed proofs since they have extremely important applications in quantum information theory. Analogously, we also obtain the explicit computations about the integrals of the following forms:

$$\int_{U(d)} U^k A(U^k)^\dagger d\mu(U) \text{ or } \int_{U(d)} U^k \otimes (U^k)^\dagger d\mu(U). \quad (3.5)$$

3.1 The $k = 1$ case

**Proposition 3.1** (Completely depolarizing channel). It holds that

$$\int_{U(d)} UAU^\dagger d\mu(U) = \frac{\text{Tr} (A)}{d} 1_d, \quad (3.6)$$

where $A \in M_d(C)$.

**Proof.** For any $V \in U(d)$, we have

$$V \left( \int_{U(d)} UAU^\dagger d\mu(U) \right) V^\dagger = \int_{U(d)} (VU) A(VU)^\dagger d\mu(U)$$

$$= \int_{U(d)} (VU) A(VU)^\dagger d\mu(VU)$$

$$= \int_{U(d)} WAW^\dagger d\mu(W) = \int_{U(d)} UAU^\dagger d\mu(U),$$

implying that $\int_{U(d)} UAU^\dagger d\mu(U)$ commutes with $U(d)$. Thus $\int_{U(d)} UAU^\dagger d\mu(U) = \lambda A 1_d$. By taking trace over both sides, we get $\lambda = \frac{\text{Tr}(A)}{d}$. Therefore the desired conclusion is obtained. $\square$

The application of Proposition 3.1 can be found in [12].
Corollary 3.2. It holds that
\[\int_{U(d_A)} (U_A \otimes 1_B)X_{AB}(U_A \otimes 1_B)^\dagger d\mu(U_A) = \frac{1_A}{d_A} \otimes \text{Tr}_A (X_{AB}). \tag{3.7}\]

Proof. We chose an orthonormal base \{1 \otimes |\mu\rangle : \mu = 1, \ldots, d_B\} for the second Hilbert space \(B\). Then \(X_{AB} = \sum_{\mu,\nu=1}^{d_B} X_{\mu\nu}^A \otimes |\mu\rangle\langle \nu|\) such that
\[\int_{U(d_A)} (U_A \otimes 1_B)X_{AB}(U_A \otimes 1_B)^\dagger d\mu(U_A) = \sum_{\mu,\nu=1}^{d_B} \left( \int_{U(d_A)} U_A X_{\mu\nu}^A U_A^\dagger d\mu(U_A) \right) \otimes |\mu\rangle\langle \nu| \]
\[= \sum_{\mu,\nu=1}^{d_B} \left( \text{Tr} \left( X_{\mu\nu}^A \frac{1_A}{d_A} \right) \otimes |\mu\rangle\langle \nu| \right) = \frac{1_A}{d_A} \otimes \text{Tr}_A (X_{AB}). \tag{3.9}\]
This completes the proof. \(\square\)

Corollary 3.3. It holds that
\[\int_{U(d_A)} \int_{U(d_B)} (U_A \otimes U_B)X_{AB}(U_A \otimes U_B)^\dagger d\mu(U_A)d\mu(U_B) = \text{Tr}_{AB} (X_{AB}) \frac{1_A}{d_A} \otimes \frac{1_B}{d_B}. \tag{3.10}\]

Corollary 3.4. It holds that
\[\int_{U(d)} U \otimes \text{vec}(U) d\mu(U) = \frac{1}{d} |\text{vec}(1_d)\rangle \langle \text{vec}(1_d)|. \tag{3.11}\]

Proof. Since
\[\text{vec} \left( \int_{U(d)} UAU^\dagger d\mu(U) \right) = \left( \int_{U(d)} U \otimes \text{vec}(U) d\mu(U) \right) |\text{vec}(A)\rangle, \]
\[\text{vec} \left( \frac{1}{d} |\text{vec}(1_d)\rangle \langle \text{vec}(1_d)| \right) = \frac{1}{d} |\text{vec}(1_d)\rangle \langle \text{vec}(1_d)| \langle \text{vec}(1_d), \text{vec}(A)\rangle. \]
Using Proposition 3.1 it follows that
\[\int_{U(d)} U \otimes \text{vec}(U) d\mu(U) = \frac{1}{d} |\text{vec}(1_d)\rangle \langle \text{vec}(1_d)|, \]
implying the result. \(\square\)

Corollary 3.5. It holds that
\[\int_{U(d)} U \otimes U^\dagger d\mu(U) = F, \tag{3.12}\]
where \(F\) is the swap operator defined as \(F = \sum_{i,j=1}^{d} |ij\rangle \langle ji|\).

The first proof. By taking partial transposes relative to second subsystems over both sides in Corollary 3.4 we get the desired identity. \(\square\)

The second proof. Let \(M = \int_{U(d)} U \otimes U^\dagger d\mu(U)\). Since Haar-measure \(\mu\) is uniform over the unitary group \(U(d)\), it follows that \(\mu(U) = \mu(V)\) for any \(U, V \in U(d)\). In particular, \(\mu(U) = \mu(U^\dagger)\). Thus \(M^\dagger = M\).
From the elementary fact that $\text{Tr} ((A \otimes B)F) = \text{Tr} (AB)$, we have $\text{Tr} (MF) = d$. Since Haar-measure is left-regular, it follows that

$$(V \otimes 1)M(1 \otimes V^\dagger) = \int_{U(d)} VU \otimes U^\dagger V^\dagger d\mu(U)$$

$$= \int_{U(d)} VU \otimes (VU)^\dagger d\mu(VU) = M.$$ 

That is $(V \otimes 1)M(1 \otimes V^\dagger) = M$ for all $V \in U(d)$. By taking traces over both sides, we have

$$\text{Tr} (M) = \text{Tr} \left( (V \otimes 1)M(1 \otimes V^\dagger) \right) = \text{Tr} \left( M(V \otimes V^\dagger) \right).$$

By taking integrals over both sides, we have

$$\int_{U(d)} \text{Tr} (M) d\mu(V) = \int_{U(d)} \text{Tr} \left( M(V \otimes V^\dagger) \right) d\mu(V),$$

which means that $\text{Tr} (M) = \text{Tr} (M^2)$. By Cauchy-Schwartz inequality, we get

$$d^2 = [\text{Tr} (MF)]^2 \leq \text{Tr} (M^2) \text{Tr} (F^2) = d^2 \text{Tr} (M),$$

implies that $\text{Tr} (M) \geq 1$. In what follows, we show that $\text{Tr} (M) = 1$. By the definition of $M$, we have

$$\text{Tr} (M) = \int_{U(d)} |\text{Tr} (U)|^2 d\mu(U)$$

$$= \int_{U(d)} \langle \text{vec}(1_d), \text{vec}(U) \rangle \langle \text{vec}(U), \text{vec}(1_d) \rangle d\mu(U)$$

$$= \left\langle \text{vec}(1_d) \left| \int_{U(d)} \text{vec}(U) \langle \text{vec}(U) | d\mu(U) \right| \text{vec}(1_d) \right\rangle.$$ 

Define a unital quantum channel $\Gamma$ as follows:

$$\Gamma = \int_{U(d)} \text{Ad}_U d\mu(U).$$

Thus by Proposition 3.3 we have $\Gamma(X) = \text{Tr} (X) \frac{1}{d} I_d$. By Choi-Jamiołkowski isomorphism, it follows that

$$J(\Gamma) = (\Gamma \otimes 1)(| \text{vec}(1_d) \rangle \langle \text{vec}(1_d) |) = \int_{U(d)} | \text{vec}(U) \rangle \langle \text{vec}(U) | d\mu(U).$$

For the completely depolarizing channel $\Gamma(X) = \text{Tr} (X) \frac{1}{d} I_d$, we already know that $J(\Gamma) = \frac{1}{d} I_d \otimes 1_d$. Therefore

$$\int_{U(d)} | \text{vec}(U) \rangle \langle \text{vec}(U) | d\mu(U) = \frac{1}{d} I_d \otimes 1_d. \quad (3.13)$$

Finally $\text{Tr} (M) = \frac{1}{d} \langle \text{vec}(1_d), \text{vec}(1_d) \rangle = 1$. This indicates that Cauchy-Schwartz inequality is saturated, and moreover the saturation happens if and only if $M \propto F$. Let $M = \lambda F$. By taking traces over both sides, we have $\lambda = \frac{1}{d}$. The desired conclusion is obtained.

**The third proof.** We derive directly the integral formula from the Schur Orthogonality Relations of a compact Lie group. See the Section 5.
Corollary 3.6. It holds that
\[
\int_{U(d)} |\text{Tr} (AU)|^2 d\mu(U) = \frac{1}{d} \text{Tr}(A^\dagger A),
\]  
(3.14)
where \( A \in M_d(C) \).

Proof. In fact,
\[
|\text{Tr} (AU)|^2 = \text{Tr} (AU \overline{\text{Tr} (AU)}) = \text{Tr} \left( (A \otimes A^\dagger) (U \otimes U^\dagger) \right).
\]
It follows that
\[
\int_{U(d)} |\text{Tr} (AU)|^2 d\mu(U) = \text{Tr} \left( (A \otimes A^\dagger) \int_{U(d)} U \otimes U^\dagger d\mu(U) \right) = \frac{1}{d} \text{Tr} \left( (A \otimes A^\dagger) F \right) = \frac{1}{d} \text{Tr} \left( AA^\dagger \right),
\]
implying the result. \( \square \)

Corollary 3.7. It holds that
\[
\int_{U(d_1)} \int_{U(d_2)} |\text{Tr} (A(U \otimes V))|^2 d\mu(U)d\mu(V) = \frac{1}{d_1d_2} \text{Tr} \left( A^\dagger A \right),
\]  
(3.15)
where \( A \in M_{d_1d_2}(C) \).

Proof. By the SVD of a matrix, we have
\[
A = \sum_j s_j |\Phi_j\rangle \langle \Psi_j|,
\]  
(3.16)
where \( s_j := s_j(A) \) is the singular values of the matrix \( A \) and \( |\Phi_j\rangle, |\Psi_j\rangle \in C^{d_1} \otimes C^{d_2} \). From the properties of the vec mapping for a matrix, we see that there exist \( d_2 \times d_1 \) matrices \( X_j \) and \( Y_j \), respectively, such that
\[
|\Phi_j\rangle = \text{vec}(X_j), \quad |\Psi\rangle = \text{vec}(Y_j).
\]  
(3.17)
This indicates that
\[
|\text{Tr} (A(U \otimes V))|^2 = \left| \sum_j s_j \langle \Psi_j | U \otimes V | \Phi_j \rangle \right|^2 = \sum_{i,j} s_is_j \langle \Psi_i | U \otimes V | \Phi_i \rangle \overline{\langle \Psi_j | U \otimes V | \Phi_j \rangle} = \sum_{i,j} s_is_j \langle \Psi_i | U \otimes V | \Phi_i \rangle \langle \Phi_j | U^\dagger \otimes V^\dagger | \Psi_j \rangle,
\]
which implies that
\[
|\text{Tr} (A(U \otimes V))|^2 = \sum_{i,j} s_is_j \langle Y_i, UX_iV^T \rangle \langle X_j, U^\dagger Y_j(V^T)^\dagger \rangle = \sum_{i,j} s_is_j \langle Y_i \otimes X_j, (U \otimes U^\dagger)(X_i \otimes Y_j)(V^T \otimes (V^T)^\dagger) \rangle.
\]
By substituting both operators into the above expression, it follows that

$$\int_{U(d_1)} \int_{U(d_2)} |\text{Tr}(A(U \otimes V))|^2 d\mu(U)d\mu(V)$$

$$= \sum_{i,j} s_i s_j \left( \sum_{i,j} \langle Y_i \otimes X_j | \left( \int_{U(d_1)} U \otimes U^\dagger d\mu(U) \right) \left( \int_{U(d_2)} V^\dagger \otimes (V^T)^\dagger d\mu(V) \right) \right)$$

$$= \frac{1}{d_1 d_2} \sum_{i,j} s_i s_j \left( \langle Y_i \otimes X_j | F_{11} (X_i \otimes Y_j) F_{22} \right)$$

$$= \frac{1}{d_1 d_2} \sum_{i,j} s_i s_j \left( \left( Y_i \otimes X_j \right)^\dagger F_{11} (X_i \otimes Y_j) F_{22} \right),$$

where $F_{11}$ is the swap operator on $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_1}$, $F_{22}$ is the swap operator on $\mathbb{C}^{d_2} \otimes \mathbb{C}^{d_2}$.

Taking orthonormal base $|\mu\rangle$ and $|m\rangle$ of $\mathbb{C}^{d_1}$ and $\mathbb{C}^{d_2}$, respectively, gives rise to

$$F_{11} = \sum_{\mu,\nu=1}^{d_1} |\mu\rangle \langle \nu|, \quad F_{22} = \sum_{m,n=1}^{d_2} |mn\rangle \langle nm|.$$

By substituting both operators into the above expression, it follows that

$$\int_{U(d_1)} \int_{U(d_2)} |\text{Tr}(A(U \otimes V))|^2 d\mu(U)d\mu(V) = \frac{1}{d_1 d_2} \sum_{i,j} s_i s_j \text{Tr} \left( X_i X_i^\dagger \right) \text{Tr} \left( Y_i Y_i^\dagger \right)$$

$$= \frac{1}{d_1 d_2} \text{Tr} \left( A^\dagger A \right).$$

The proof is complete. \hfill \Box

Note that Corollaries 3.5, 3.6, and 3.7 are used in the recent paper [13] to establish an interesting relationship between quantum correlation and interference visibility. In what follows, we obtain a general result:

**Proposition 3.8.** It holds that

$$\int_{U(d_1)} \int_{U(d_2)} \cdots \int_{U(d_n)} |\text{Tr}(A(U_1 \otimes U_2 \otimes \cdots \otimes U_n))|^2 d\mu(U_1) d\mu(U_2) \cdots d\mu(U_n)$$

$$= \frac{1}{d} \text{Tr} \left( A^\dagger A \right),$$

(3.18)

where $A \in M_d(\mathbb{C})$ for $d = \prod_{j=1}^{n} d_j$.

This result will be useful in the investigation of multipartite quantum correlation. The detail of its proof is as follows.

**Proof.** Firstly we note from Corollary 3.2 that

$$\int_{U(d_1)} (U_1 \otimes \mathbb{1}_{2\ldots n}) X_{12\ldots n} (U_1 \otimes \mathbb{1}_{2\ldots n})^\dagger d\mu(U_1) = \frac{1}{d_1} \otimes \text{Tr}_1 (X_{12\ldots n}).$$

(3.19)

Furthermore, we have

$$\int_{U(d_1)} \int_{U(d_2)} (U_1 \otimes U_2 \otimes \mathbb{1}_{3\ldots n}) X_{12\ldots n} (U_1 \otimes U_2 \otimes \mathbb{1}_{3\ldots n})^\dagger d\mu(U_1) d\mu(U_2) = \frac{1}{d_1} \otimes \frac{1}{d_2} \otimes \text{Tr}_{12} (X_{12\ldots n}).$$

(3.20)
By induction, we have
\[
\int_{U(d_1)} \int_{U(d_2)} \cdots \int_{U(d_n)} (U_1 \otimes U_2 \otimes \cdots \otimes U_n) X_{12\ldots n} (U_1 \otimes U_2 \otimes \cdots \otimes U_n)^\dagger d\mu(U_1)d\mu(U_2) \cdots d\mu(U_n)
= \text{Tr}_{12\ldots n} (X_{12\ldots n}) \frac{1}{d_1} \otimes \frac{1}{d_2} \otimes \cdots \otimes \frac{1}{d_n}.
\]
(3.21)

This implies that
\[
\int_{U(d_1)} \int_{U(d_2)} \cdots \int_{U(d_n)} |U_1 \otimes U_2 \otimes \cdots \otimes U_n, \langle U_1 \otimes U_2 \otimes \cdots \otimes U_n | d\mu(U_1)d\mu(U_2) \cdots d\mu(U_n)
= \frac{1}{d} 1_{12\ldots n} \otimes 1_{12\ldots n},
\]
(3.22)

where \(d = \prod_{j=1}^n d_j\). Now
\[
| \text{Tr} (A(U_1 \otimes U_2 \otimes \cdots \otimes U_n)) |^2 = \left\langle A^\dagger, U_1 \otimes U_2 \otimes \cdots \otimes U_n \right\rangle \left\langle U_1 \otimes U_2 \otimes \cdots \otimes U_n, A^\dagger \right\rangle
\]
implying
\[
\int_{U(d_1)} \cdots \int_{U(d_n)} | \text{Tr} (A(U_1 \otimes U_2 \otimes \cdots \otimes U_n)) |^2
= \left\langle A^\dagger \left[ \int_{U(d_1)} \cdots \int_{U(d_n)} | U_1 \otimes U_2 \otimes \cdots \otimes U_n \rangle \langle U_1 \otimes U_2 \otimes \cdots \otimes U_n | d\mu(U_1)d\mu(U_2) \cdots d\mu(U_n) \right] A^\dagger \right\rangle
= \frac{1}{d} \langle A^\dagger, A^\dagger \rangle = \frac{1}{d} \text{Tr} (A^\dagger A).
\]
(3.26)

We are done. \(\square\)

3.2 The \(k = 2\) case

**Proposition 3.9.** It holds that
\[
\int_{U(d)} (U \otimes U) A(U \otimes U)^\dagger d\mu(U)
= \left( \frac{\text{Tr} (A)}{d^2 - 1} - \frac{\text{Tr} (AF)}{d(d^2 - 1)} \right) 1_d^2 - \left( \frac{\text{Tr} (A)}{d(d^2 - 1)} - \frac{\text{Tr} (AF)}{d^2 - 1} \right) F,
\]
(3.27)

where \(A \in M_d(C)\) and the swap operator \(F\) is defined by \(F|ij\rangle = |ji\rangle\) for all \(i, j = 1, \ldots, d\).

**Proof.** Analogously, we have \(\int_{U(d)} (U \otimes U)A(U \otimes U)^\dagger d\mu(U)\) commutes with \(\{V \otimes V : V \in U(d)\}\). Denote \(P_\wedge := \frac{1}{2}(1_d^2 - F)\) and \(P_\vee := \frac{1}{2}(1_d^2 + F)\). It is easy to see that \(\text{Tr} (P_\wedge) = \frac{1}{2}(d^2 - d)\) and \(\text{Tr} (P_\vee) = \frac{1}{2}(d^2 + d)\).

Since \(F = \sum_{i,j} |ij\rangle \langle ji|\), it follows that \(F^\dagger = F\) and \(F^2 = 1_d^2\). Thus both \(P_\wedge\) and \(P_\vee\) are projectors and \(P_\wedge + P_\vee = 1_d^2\).

Because \(C^d \otimes C^d = \wedge^2 C^d \oplus \vee^2 C^d\), we have \(P_\wedge(C^d \otimes C^d)P_\wedge = \wedge^2 C^d\) and \(P_\vee(C^d \otimes C^d)P_\vee = \vee^2 C^d\).

Besides, for any \(V \in U(d)\),
\[
V \otimes V \equiv \begin{bmatrix}
P_\wedge(V \otimes V) & 0 \\
0 & P_\vee(V \otimes V)
\end{bmatrix}
\]

Now write
\[
\int_{U(d)} (U \otimes U) A(U \otimes U)^\dagger d\mu(U) = \begin{bmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{bmatrix}
\]

16
is a block matrix, where $M_{11} \in \text{End}(\wedge^2 \mathbb{C}^d)$, $M_{22} \in \text{End}(\vee^2 \mathbb{C}^d)$ and
\[ M_{12} \in \text{Hom}_{U(d)}(\vee^2 \mathbb{C}^d, \wedge^2 \mathbb{C}^d), \quad M_{21} \in \text{Hom}_{U(d)}(\wedge^2 \mathbb{C}^d, \vee^2 \mathbb{C}^d). \]

Thus
\[
\begin{bmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{bmatrix}
\begin{bmatrix}
P_\wedge(V \otimes V)P_\wedge & 0 \\
0 & P_\vee(V \otimes V)P_\vee
\end{bmatrix}
= \begin{bmatrix}
P_\wedge(V \otimes V)P_\wedge & 0 \\
0 & P_\vee(V \otimes V)P_\vee
\end{bmatrix}
\begin{bmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{bmatrix}.
\]

We get that, for all $V \in U(d)$,
\[
\begin{cases}
M_{11}(\wedge^2 V) = (\wedge^2 V)M_{11}, \\
M_{22}(\vee^2 V) = (\vee^2 V)M_{22}, \\
M_{12}(\vee^2 V) = (\wedge^2 V)M_{12}, \\
M_{21}(\wedge^2 V) = (\vee^2 V)M_{21}.
\end{cases}
\]

Therefore we obtained that
\[ M_{11} = \lambda(A)P_\wedge, \quad M_{22} = \mu(A)P_\vee, \quad M_{12} = 0, \quad M_{21} = 0. \]

That is
\[ \int_{U(d)} (U \otimes U)A(U \otimes U)^\dagger d\mu(U) = \begin{bmatrix}
\lambda(A)P_\wedge & 0 \\
0 & \mu(A)P_\vee
\end{bmatrix} = \lambda(A)P_\wedge + \mu(A)P_\vee. \] (3.28)

If $A = 1_{d^2}$ in Eq. (3.28), then $1_{d^2} = \lambda(1_{d^2})P_\wedge + \mu(1_{d^2})P_\vee$. Thus $\lambda(1_{d^2}) = \mu(1_{d^2}) = 1$ since $1_{d^2} = P_\wedge + P_\vee$ and $P_\wedge \perp P_\vee$.

If $A = P_\wedge$ in Eq. (3.28), then $P_\wedge = \lambda(P_\wedge)P_\wedge + \mu(P_\wedge)P_\vee$ since $U \otimes U$ commutes with $P_\wedge$. Thus $\lambda(P_\wedge) = 1$ and $\mu(P_\wedge) = 0$. Note that $\lambda(A), \mu(A)$ are two linear functionals. Thus we have: $\lambda(F) = -1$ and $\mu(F) = 1$. This indicates that
\[ \int_{U(d)} (U \otimes U)F(U \otimes U)^\dagger d\mu(U) = \lambda(F)P_\wedge + \mu(F)P_\vee = P_\vee - P_\wedge = F. \]

More simpler approach to this identity can be described as follows: Since $F(M \otimes N)F = N \otimes M$, it follows that $F(M \otimes N) = (N \otimes M)F$. Thus
\[
\int_{U(d)} (U \otimes U)F(U \otimes U)^\dagger d\mu(U) = \int_{U(d)} F(U \otimes U)(U \otimes U)^\dagger d\mu(U) = F \int_{U(d)} d\mu(U) = F = P_\vee - P_\wedge.
\]

Apparently
\[ \int_{U(d)} (U \otimes U)^\dagger F(U \otimes U)d\mu(U) = F = P_\vee - P_\wedge. \]

By taking trace over both sides above, we get
\[ \text{Tr}(A) = \lambda(A) \text{Tr}(P_\wedge) + \mu(A) \text{Tr}(P_\vee). \]
Now by multiplying $F$ on both sides in Eq. (3.28) and then taking trace again, we get

$$\int_{U(d)} \text{Tr} \left( (U \otimes U)^\dagger F(U \otimes U)A \right) d\mu(U)$$

$$= \lambda(A) \text{Tr} (P_\wedge F) + \mu(A) \text{Tr} (P_\vee F)$$

$$= \mu(A) \text{Tr} (P_\vee) - \lambda(A) \text{Tr} (P_\wedge),$$

where we used the fact that $P_\wedge F = -P_\wedge$ and $P_\vee F = P_\vee$. Thus we have

$$\begin{cases}
\frac{d(d-1)}{2} \lambda(A) + \frac{d(d+1)}{2} \mu(A) = \text{Tr} (A), \\
\frac{d(d+1)}{2} \mu(A) - \frac{d(d-1)}{2} \lambda(A) = \text{Tr} (AF).
\end{cases}$$

Solving this group of two binary equations gives rise to

$$\begin{cases}
\lambda(A) = \frac{\text{Tr}(A) - \text{Tr}(AF)}{d(d-1)}, \\
\mu(A) = \frac{\text{Tr}(A) + \text{Tr}(AF)}{d(d+1)}.
\end{cases}$$

Finally we obtained the desired conclusion as follows:

$$\int_{U(d)} (U \otimes U)A(U \otimes U)^\dagger d\mu(U) = \frac{\text{Tr} (A) - \text{Tr} (AF)}{d(d-1)} P_\wedge + \frac{\text{Tr} (A) + \text{Tr} (AF)}{d(d+1)} P_\vee.$$

We are done. \hfill \Box

The applications of Proposition 3.9 in quantum information theory can be found in [14] [15].

**Corollary 3.10.** It holds that

$$\int_{U(d)} U^\dagger A U X U^\dagger B U d\mu(U)$$

$$= \frac{d \text{Tr} (AB) - \text{Tr} (A) \text{Tr} (B)}{d(d^2 - 1)} \text{Tr} (X) 1_d + \frac{d \text{Tr} (A) \text{Tr} (B) - \text{Tr} (AB)}{d(d^2 - 1)} X. \quad (3.29)$$

**Proof.** It suffices to compute the integral $\int_{U(d)} (U^\dagger AU) \otimes (U^\dagger BU) d\mu(U)$ since

$$\text{vec} \left( \int_{U(d)} (U^\dagger AU) X (U^\dagger BU) d\mu(U) \right)$$

$$= \int_{U(d)} (U^\dagger AU) \otimes (U^\dagger BU) \text{vec}(X). \quad (3.30)$$

Once we get the formula for $\int_{U(d)} (U^\dagger AU) \otimes (U^\dagger BU) d\mu(U)$, taking partial transpose relative to the second factor in the tensor product, we get the formula for $\int_{U(d)} (U^\dagger AU) \otimes (U^\dagger BU)^\dagger d\mu(U)$.

Now by Proposition 3.9 we have

$$\int_{U(d)} (U^\dagger AU) \otimes (U^\dagger BU) d\mu(U) = \int_{U(d)} (U \otimes U)^\dagger (A \otimes B) (U \otimes U) d\mu(U)$$

$$= \left( \frac{\text{Tr} (A) \text{Tr} (B)}{d^2 - 1} - \frac{\text{Tr} (AB)}{d(d^2 - 1)} \right) 1_d^2 - \left( \frac{\text{Tr} (A) \text{Tr} (B)}{d(d^2 - 1)} - \frac{\text{Tr} (AB)}{d^2 - 1} \right) F$$

$$= \frac{d \text{Tr} (A) \text{Tr} (B) - \text{Tr} (AB)}{d(d^2 - 1)} 1_d^2 + \frac{d \text{Tr} (AB) - \text{Tr} (A) \text{Tr} (B)}{d(d^2 - 1)} F,$$
implying that
\[
\int_{U(d)} \left( U^t A U \right) \otimes \left( U^t B U \right)^T d\mu(U)
\]
\[
= \frac{d \Tr (A \Tr(B) - \Tr(AB))}{d(d^2 - 1)} \mathbb{1}_d + \frac{d \Tr (AB - A \Tr(B))}{d(d^2 - 1)} |\vec{1}_d \rangle \langle \vec{1}_d |.
\]
Substituting this identity into (3.30) gives the desired result.

Recall that a super-operator $\Phi$ is unitarily invariant if $\text{Ad}_{U^t} \circ \Phi \circ \text{Ad}_U = \Phi$ for all $U \in U(d)$. We also note that an super-operator $\Phi$ on End($\mathcal{H}_d$) can be represented as
\[
\Phi(X) = \sum_j A_j X B_j^t.
\]
Now we may give the specific form of any unitarily invariant super-operator in the following corollary.

**Corollary 3.11.** Let $\Phi$ be a unitarily invariant super-operator on End($\mathcal{H}_d$). Then
\[
\Phi(X) = \frac{d \Tr (\Phi(\mathbb{1}_d)) - \Tr (\Phi) \Tr (X) \mathbb{1}_d}{d(d^2 - 1)} + \frac{d \Tr (\Phi - \Phi(\mathbb{1}_d))}{d(d^2 - 1)} X,
\]
where $\Tr (\Phi)$ is the trace of super-operator $\Phi$, defined by $\Tr (\Phi) := \sum_{\mu,\nu} \langle \mu | \Phi(|\mu\rangle \langle \nu|) | \nu \rangle$.

**Proof.** Apparently $\text{Ad}_{U^t} \circ \Phi \circ \text{Ad}_U = \Phi$ for all $U \in U(d)$. This implies that, for the uniform Haar measure $d\mu(U)$ over the unitary group,
\[
\Phi(X) = \int_{U(d)} \Phi(X) d\mu(U) = \int_{U(d)} U^t \Phi(U X U^t) U d\mu(U)
\]
(3.33)
\[
= \sum_j \int_{U(d)} U^t A_j U X U^t B_j^t U d\mu(U).
\]
(3.34)

By Corollary 3.10
\[
\int_{U(d)} U^t A_j U X U^t B_j^t U d\mu(U)
\]
(3.35)
\[
= \frac{d \Tr (A_j B_j^t) - \Tr (A_j) \Tr(B_j^t)}{d(d^2 - 1)} \Tr (X) \mathbb{1}_d + \frac{d \Tr (A_j) \Tr(B_j^t) - \Tr (A_j B_j^t)}{d(d^2 - 1)} X.
\]
(3.36)

Thus
\[
\Phi(X) = \frac{d \sum_j \Tr (A_j B_j^t) - \sum_j \Tr (A_j) \Tr(B_j^t)}{d(d^2 - 1)} \Tr (X) \mathbb{1}_d
\]
(3.37)
\[
+ \frac{d \sum_j \Tr (A_j) \Tr(B_j^t) - \sum_j \Tr (A_j B_j^t)}{d(d^2 - 1)} X
\]
(3.38)
\[
= \frac{d \Tr (\Phi(\mathbb{1}_d)) - \Tr (\Phi) \Tr (\Phi(\mathbb{1}_d))}{d(d^2 - 1)} \Tr (X) \mathbb{1}_d + \frac{d \Tr (\Phi - \Phi(\mathbb{1}_d))}{d(d^2 - 1)} X,
\]
(3.39)
where we have used the fact that
\[
\Tr (\Phi) = \sum_{\mu, \nu} \langle \mu | \langle \nu | \Phi(|\mu\rangle \langle \nu|) | \nu \rangle = \sum_{\mu, \nu} \langle \mu | \Phi(|\mu\rangle \langle \nu|) | \nu \rangle
\]
(3.40)
\[
= \sum_j \sum_{\mu, \nu} \langle \mu | A_j | \mu \rangle \langle \nu | B_j^t | \nu \rangle = \sum_j \left( \sum_{\mu} \langle \mu | A_j | \mu \rangle \right) \left( \sum_{\nu} \langle \nu | B_j^t | \nu \rangle \right)
\]
(3.41)
\[
= \sum_j \Tr (A_j) \Tr (B_j^t).
\]
(3.42)
There is a caution that the trace of super-operator \( \Phi \) is different from the trace of operator \( \Phi(\mathbb{1}_d) \).

We can simplify this expression if we assume more structure on the super-operator. A trace-preserving unitarily invariant quantum operation \( \Lambda \) is a depolarizing channel: for \( \rho \in \mathcal{D}(\mathcal{H}_d) \),

\[
\Lambda(\rho) = pp + (1 - p) \frac{\mathbb{1}_d}{d}, \quad \left( p = \frac{\text{Tr} (\Phi) - 1}{d^2 - 1} \right).
\]

Indeed, this easily follows from the facts that \( \text{Tr} (\Phi(\mathbb{1}_d)) = d \) and \( \text{Tr} (\rho) = 1 \).

Let \( \Phi \) be a super-operator on \( \text{End}(\mathcal{H}_d) \). Define the twisted super-operator

\[
\Phi_T = \int_{U(d)} \text{Ad}_{U^T} \circ \Phi \circ \text{Ad}_{U} d\mu(U).
\]

Clearly twisted super-operator \( \Phi_T \) is unitarily invariant.

**Corollary 3.12.** Let \( X, Y \in \text{End}(\mathbb{C}^d) \). Then the uniform average of \( \langle \psi | X | \psi \rangle \langle \psi | Y | \psi \rangle \) over state vectors \( |\psi\rangle \) on the unit sphere \( S^{2d-1} \) in \( \mathbb{C}^d \) is given by

\[
\int_{S^{2d-1}} \langle \psi | X | \psi \rangle \langle \psi | Y | \psi \rangle \, d|\psi\rangle = \frac{\text{Tr} (XY) + \text{Tr} (X) \text{Tr} (Y)}{d(d + 1)}.
\]  

**Proof.** The original integral can be reduced to the computing of the following integral:

\[
\int_{U(d)} (U \otimes U)(X \otimes Y)(U \otimes U)^* \, d\mu(U)
\]

\[
= \frac{\text{Tr} (X \otimes Y) - \text{Tr} ((X \otimes Y)F)}{d(d - 1)} P_\wedge + \frac{\text{Tr} (X \otimes Y) + \text{Tr} ((X \otimes Y)F)}{d(d + 1)} P_\vee.
\]

Since \( P_\vee |\psi_0\psi_0\rangle = |\psi_0\psi_0\rangle \) and \( P_\wedge |\psi_0\psi_0\rangle = 0 \), it follows that

\[
\int_{S^{2d-1}} \langle \psi | X | \psi \rangle \langle \psi | Y | \psi \rangle \, d|\psi\rangle
\]

\[
= \langle \psi_0\psi_0 \bigg| \frac{\text{Tr} (X) \text{Tr} (Y) - \text{Tr} (XY)}{d(d - 1)} P_\wedge + \frac{\text{Tr} (X) \text{Tr} (Y) + \text{Tr} (XY)}{d(d + 1)} P_\vee \bigg| \psi_0\psi_0 \rangle
\]

\[
= \frac{\text{Tr} (XY) + \text{Tr} (X) \text{Tr} (Y)}{d(d + 1)}.
\]

We are done.

As a direct consequence of the above Corollary, it follows that for any super-operator \( \Phi \) on \( \text{End}(\mathcal{H}_d) \),

\[
\int_{S^{2d-1}} \langle \psi | \Phi(|\psi\rangle \langle \psi|) | \psi \rangle \, d|\psi\rangle = \frac{\text{Tr} (\Phi(\mathbb{1}_d)) + \text{Tr} (\Phi)}{d(d + 1)}.
\]  

**Corollary 3.13.** It holds that

\[
\int_{U(d)} U \otimes U \otimes U \otimes U \, d\mu(U)
\]

\[
= \frac{1}{d^2 - 1} \left( |\text{vec}(\mathbb{1}_d\mathbb{1}_d)| \langle \text{vec}(\mathbb{1}_d\mathbb{1}_d) | + |\text{vec}(F)\rangle \langle \text{vec}(F)| \right)
\]

\[
- \frac{1}{d(d^2 - 1)} \left( |\text{vec}(\mathbb{1}_d\mathbb{1}_d)| \langle \text{vec}(F) | + |\text{vec}(F)\rangle \langle \text{vec}(\mathbb{1}_d\mathbb{1}_d)| \right).
\]

**Proof.** Apparently, this result can be derived from Proposition 3.9.
Corollary 3.14. It holds that
\[
\int_{U(d)} U \otimes U \otimes U^t \otimes U^t d\mu(U) = \frac{P_{(13)(24)} + P_{(14)(23)}}{d^2 - 1} - \frac{P_{(1423)} + P_{(1324)}}{d(d^2 - 1)}.
\] (3.48)

Proof. By taking partial transposes relative to the third and fourth subsystems, respectively, over both sides in Corollary 3.13, it suffices to show that
\[
\begin{align*}
|\text{vec}(1_{d^2})\rangle\langle\text{vec}(1_{d^2})|^{T_{3,4}} &= P_{(13)(24)}, \\
|\text{vec}(F)\rangle\langle\text{vec}(F)|^{T_{3,4}} &= P_{(14)(23)}, \\
|\text{vec}(1_{d^2})\rangle\langle\text{vec}(F)|^{T_{3,4}} &= P_{(1423)}, \\
|\text{vec}(F)\rangle\langle\text{vec}(1_{d^2})|^{T_{3,4}} &= P_{(1324)}.
\end{align*}
\]

Note that
\[
\text{vec}(1_{d^2}) = \sum_{i,j=1}^{d} |iji\rangle, \quad \text{vec}(F) = \sum_{i,j=1}^{d} |iji\rangle.
\]

It follows that
\[
\begin{align*}
|\text{vec}(1_{d^2})\rangle\langle\text{vec}(1_{d^2})| &= \sum_{i,j,k,l=1}^{d} |iji\rangle \langle kll|, \\
|\text{vec}(F)\rangle\langle\text{vec}(F)| &= \sum_{i,j,k,l=1}^{d} |iji\rangle \langle kll|, \\
|\text{vec}(1_{d^2})\rangle\langle\text{vec}(F)| &= \sum_{i,j,k,l=1}^{d} |iji\rangle \langle kll|, \\
|\text{vec}(F)\rangle\langle\text{vec}(1_{d^2})| &= \sum_{i,j,k,l=1}^{d} |iji\rangle \langle kll|.
\end{align*}
\]

Therefore we have
\[
\begin{align*}
|\text{vec}(1_{d^2})\rangle\langle\text{vec}(1_{d^2})|^{T_{3,4}} &= \sum_{i,j,k,l=1}^{d} |ijkl\rangle \langle klji| = P_{(13)(24)}, \\
|\text{vec}(F)\rangle\langle\text{vec}(F)|^{T_{3,4}} &= \sum_{i,j,k,l=1}^{d} |ijkl\rangle \langle klji| = P_{(14)(23)}, \\
|\text{vec}(1_{d^2})\rangle\langle\text{vec}(F)|^{T_{3,4}} &= \sum_{i,j,k,l=1}^{d} |ijkl\rangle \langle klji| = P_{(1423)}, \\
|\text{vec}(F)\rangle\langle\text{vec}(1_{d^2})|^{T_{3,4}} &= \sum_{i,j,k,l=1}^{d} |ijkl\rangle \langle klji| = P_{(1324)}.
\end{align*}
\]

The proof is complete. \(\Box\)

Corollary 3.15. It holds that
\[
\int_{U(d)} |\text{Tr}(AU)\rangle\langle\text{Tr}(AU)|^{\dagger} d\mu(U) = \frac{2}{d^2 - 1} \left[\text{Tr} \left( A^\dagger A \right) \right]^2 - \frac{2}{d(d^2 - 1)} \text{Tr} \left( (A^\dagger A)^2 \right),
\] (3.49)

where \(A \in M_d(C)\).
Proof. Note that
\[
|\text{Tr}(AU)|^4 = \text{Tr}\left(\left[A^\otimes 2 \otimes (A^\otimes 2)^\dagger\right]\left[U^\otimes 2 \otimes (U^\otimes 2)^\dagger\right]\right).
\]
By Corollary 3.14 we obtain the final result. □

3.3 The general case

The partial materials in this subsection are written based on the results in [17, 18].

We recall that for an algebra inclusion \(\mathcal{M} \subset \mathcal{N}\), a conditional expectation is a \(\mathcal{M}\)-bimodule map \(\delta : \mathcal{N} \to \mathcal{M}\) such that \(\delta(1_N) = 1_M\).

For \(A \in \text{End}((C^d)^{\otimes k})\), we define
\[
\delta_k(A) = \int_{U(d)} U^{\otimes k} A \left(U^{\otimes k}\right)^\dagger d\mu(U). \tag{3.50}
\]
Clearly \(\delta_k : \text{End}((C^d)^{\otimes k}) \to P(C[S_k])\) is a conditional expectation. Moreover \(\delta_k\) is an orthogonal projection onto \(P(C[S_k])\). It is compatible with the trace in the sense that \(\text{Tr} \circ \delta_k = \text{Tr}\).

For \(A \in \text{End}((C^d)^{\otimes k})\), we set
\[
\Delta(A) = \sum_{\pi \in S_k} \langle P(\pi), A \rangle P(\pi)
= \sum_{\pi \in S_k} \text{Tr}\left(\text{AP}(\pi^{-1})\right) P(\pi) \in P(C[S_k]). \tag{3.51}
\]

Proposition 3.16. \(\Delta\) embraces the following properties:

(i) \(\Delta\) is a \(P(C[S_k])\)-\(P(C[S_k])\) bimodule morphism in the sense that
\[
\Delta(\text{AP}(\sigma)) = \Delta(A)\text{P}(\sigma), \quad \Delta(\text{P}(\sigma)A) = \text{P}(\sigma)\Delta(A).
\]

(ii) \(\Delta(1)\) coincides with the character of \(P\) hence it is equal to
\[
\Delta(1) = k! \sum_{\lambda \vdash k} \frac{s_\lambda(1^{\times d})}{f_\lambda} C_\lambda
\]
and is an invertible element of \(C[S_k]\); its inverse will be called Weingarten function and is equal to
\[
Wg = \frac{1}{(k!)^2} \sum_{\lambda \vdash (k,d)} \frac{(f_\lambda)^2}{s_\lambda(1^{\times d})} \chi_\lambda
\]

(iii) the relation between \(\Delta(A)\) and \(\delta_k(A)\) is explicitly given by
\[
\Delta(A) = \delta_k(A)\Delta(1)
\]

(iv) the range of \(\Delta\) is equal to \(P(C[S_k])\);

(v) the following holds true in \(P(C[S_k])\):
\[
\Delta(A\delta_k(B)) = \Delta(A)\Delta(B)\Delta(1)^{-1}.
\]
Proof. (i). Clearly we have:
\[
\Delta(\mathbf{A}^\mathbf{P}(\sigma)) = \sum_{\pi \in S_k} \text{Tr} \left( [\mathbf{A}^\mathbf{P}(\sigma)] \mathbf{P}(\pi^{-1}) \right) \mathbf{P}(\pi) \\
= \sum_{\pi \in S_k} \text{Tr} \left( \mathbf{A}^\mathbf{P}(\sigma \pi^{-1}) \right) \mathbf{P}(\sigma \pi^{-1}) \mathbf{P}(\sigma) \\
= \Delta(\mathbf{A}) \mathbf{P}(\sigma).
\]

Similarly, we also have: \(\Delta(\mathbf{P}(\sigma) A) = \mathbf{P}(\sigma) \Delta(A)\). Furthermore we get
\[
\Delta(\mathbf{P}(\sigma_l) A \mathbf{P}(\sigma_r)) = \mathbf{P}(\sigma_l) \Delta(A) \mathbf{P}(\sigma_r),
\]
where \(\sigma_l, \sigma_r \in S_k\). Therefore \(\Delta\) is bimodule morphism.

(ii). Let \(A = 1\) in the definition of \(\Delta\). We get that
\[
\Delta(1) = \sum_{\pi \in S_k} \text{Tr} \left( \mathbf{P}(\pi^{-1}) \right) \mathbf{P}(\pi) = \sum_{\pi \in S_k} \chi(\pi^{-1}) \mathbf{P}(\pi).
\]

By Schur-Weyl duality, we have
\[
(C^d)^{\otimes k} \cong \bigoplus_{\lambda \vdash (k,d)} Q_{\lambda} \otimes \mathbf{P}_{\lambda}
\]
and
\[
\chi = \sum_{\lambda \vdash (k,d)} d_{\lambda} \chi_{\lambda},
\]
where \(d_{\lambda}\) is the multiplicities of \(\mathbf{P}_{\lambda}\), i.e. \(d_{\lambda} = \dim(Q_{\lambda}) = s_{\lambda}(1^x d)\). Hence
\[
\chi(\pi^{-1}) = \sum_{\lambda \vdash (k,d)} d_{\lambda} \chi_{\lambda}(\pi^{-1}) = \sum_{\lambda \vdash (k,d)} s_{\lambda}(1^x d) \chi_{\lambda}(\pi^{-1}),
\]
which is substituted into the rhs of expression of \(\Delta(1)\) above, gives rise to
\[
\Delta(1) = \sum_{\pi \in S_k} \left( \sum_{\lambda \vdash (k,d)} s_{\lambda}(1^x d) \chi_{\lambda}(\pi^{-1}) \right) \mathbf{P}(\pi) \\
= \sum_{\lambda \vdash (k,d)} s_{\lambda}(1^x d) \left( \sum_{\pi \in S_k} \chi_{\lambda}(\pi^{-1}) \mathbf{P}(\pi) \right).
\]

Since the minimal central projection \(C_{\lambda}\) in \(\mathbf{P}(\mathbb{C}[S_k])\) must be of the following form:
\[
C_{\lambda} = \frac{f_{\lambda}}{k!} \sum_{\pi \in S_k} \chi_{\lambda}(\pi^{-1}) \mathbf{P}(\pi),
\]
it follows that
\[
\sum_{\pi \in S_k} \chi_{\lambda}(\pi^{-1}) \mathbf{P}(\pi) = \frac{k!}{f_{\lambda}} C_{\lambda}
\]
Thus
\[
\Delta(1) = k! \sum_{\lambda \vdash (k,d)} \frac{s_{\lambda}(1^x d)}{f_{\lambda}} C_{\lambda}.
\]
Moreover $\Delta(1)$ is invertible and
\[
\Delta(1)^{-1} = \frac{1}{k!} \sum_{\lambda \vdash k} \frac{f^\lambda}{s^\lambda(1 \times d)} C^\lambda.
\]

We denote by $W_g$ the function corresponding to $\Delta(1)^{-1}$, i.e.
\[
W_g = \frac{1}{(k!)^2} \sum_{\lambda \vdash (k,d)} \frac{(f^\lambda)^2}{s^\lambda(1 \times d)} \chi^\lambda.
\]

(iii). Since $Q(U)$ commutes with $P(\pi)$, it follows that
\[
\Delta(\delta_k(A)) = \sum_{\pi \in S_k} \text{Tr} \left( \delta_k(A) P(\pi^{-1}) \right) P(\pi)
\]
\[
= \sum_{\pi \in S_k} \text{Tr} \left( \int_{U(d)} Q(U) A Q(U)^\tau d\mu(U) P(\pi^{-1}) \right) P(\pi)
\]
\[
= \sum_{\pi \in S_k} \text{Tr} \left( A \int_{U(d)} Q(U)^\tau P(\pi^{-1}) Q(U) d\mu(U) \right) P(\pi)
\]
\[
= \sum_{\pi \in S_k} \text{Tr} \left( A P(\pi^{-1}) \right) P(\pi) = \Delta(A),
\]

implying
\[
\Delta(A) = \Delta(\delta_k(A)) = \delta_k(A) \Delta(1) = \Delta(1) \delta_k(A).
\]

This indicates that
\[
\delta_k(A) = \Delta(A) \Delta(1)^{-1} = \Delta(1)^{-1} \Delta(A)
\]
\[
= \frac{1}{k!} \left( \sum_{\pi \in S_k} \text{Tr} \left( A P(\pi^{-1}) \right) P(\pi) \right) \left( \sum_{\lambda \vdash k} \frac{f^\lambda}{s^\lambda(1 \times d)} C^\lambda \right).
\]

(iv). It is trivially from (ii) and (iii).

(v). It is easily seen that
\[
\Delta(A \delta_k(B)) = \Delta(A) \delta_k(B) = \Delta(A) \Delta(B) \Delta(1)^{-1}.
\]

We are done.

\[\square\]

**Corollary 3.17.** Let $k$ be a positive integer and $i = (i_1, \ldots, i_k)$, $i' = (i'_1, \ldots, i'_k)$, $j = (j_1, \ldots, j_k)$, $j' = (j'_1, \ldots, j'_k)$ be $k$-tuples of positive integers. Then
\[
\int_{U(d)} U_{i_1h} \cdots U_{i_kh} \bar{U}_{i'_1h} \cdots \bar{U}_{i'_kh} \ d\mu(U)
\]
\[
= \sum_{\sigma, \tau \in S_k} W_g(\sigma \tau^{-1}) \langle i_1 | j'_1(\sigma(1)) \rangle \cdots \langle i_k | j'_k(\sigma(k)) \rangle \langle j_1 | j'_1(\tau(1)) \rangle \cdots \langle j_k | j'_k(\tau(k)) \rangle
\]
Proof. Note that
\[ \Delta(A \otimes_k (B)) = \Delta(A) \Delta(B) \Delta((I)^{-1}). \]
In order to show our result, it is enough to take appropriate \( A = |i\rangle \langle i| \) and \( B = |j\rangle \langle j'| \), where \( |i\rangle = |i_1 \cdots i_k \rangle \), etc. Now that
\[
\int_{U(d)} U_{i_1j_1} \cdots U_{i_kj_k} U_{i'1} \cdots U_{i'k} d\mu(U) = \text{Tr} (A \otimes_k (B)).
\]
(3.59)
By the definition of \( \Delta \), we get
\[
\Delta(A \otimes_k (B)) = \sum_{\pi \in S_k} \text{Tr} \left( A \otimes_k (B) \Pi(\pi^{-1}) \right) \Pi(\pi)
\]
\[
= \text{Tr} (A \otimes_k (B)) I + \sum_{\pi \in S_k \setminus \{e\}} \text{Tr} \left( A \otimes_k (B) \Pi(\pi^{-1}) \right) \Pi(\pi)
\]
and
\[
\Delta(A) = \sum_{\sigma \in S_k} \text{Tr} \left( A \Pi(\sigma) \right) \Pi(\sigma)
\]
\[
= \sum_{\sigma \in S_k} \langle \pi | A \Pi(\sigma) | \sigma' \rangle \langle \sigma' | \Pi(\sigma^{-1}) \Pi(\sigma) | \sigma \rangle \Pi(\sigma)
\]
\[
= \sum_{\sigma \in S_k} \Pi(\sigma)|i_1 \cdots i_k \rangle \langle i_1 \cdots i_k | \Pi(\sigma^{-1}) \Pi(\sigma),
\]
where \( \Pi(\sigma)|i_1 \cdots i_k \rangle = |i_{\sigma^{-1}(1)} \cdots i_{\sigma^{-1}(k)} \rangle \) or \( \Pi(\sigma)|i_{\tau(1)} \cdots i_{\tau(k)} \rangle = |i_1 \cdots i_k \rangle \). That is,
\[
\Pi(\sigma) = \sum_{i_1, \ldots, i_k \in [d]} |i_1 \cdots i_k \rangle \langle i_{\tau(1)} \cdots i_{\tau(k)} |.
\]
Note also that \( \Pi(\sigma)^\dagger = \Pi(\sigma^{-1}) \). Therefore
\[
\Pi(\sigma^{-1}) = \Pi(\sigma)^\dagger = \sum_{i_1, \ldots, i_k \in [d]} |i_{\sigma^{-1}(1)} \cdots i_{\sigma^{-1}(k)} \rangle \langle i_1 \cdots i_k |.
\]
Similarly
\[
\Delta(B) = \sum_{\tau \in S_k} \langle j' | \Pi(\tau^{-1}) | j \rangle \Pi(\tau) = \sum_{\nu \in S_k} \langle j_{\tau^{-1}(1)} | \Pi(\tau) | j_{\tau^{-1}(1)} \rangle \cdots \langle j_{\tau^{-1}(k)} | \Pi(\tau) | j_{\tau^{-1}(k)} \rangle \Pi(\tau)
\]
\[
= \sum_{\tau \in S_k} \langle j_{\tau(1)} | j_{\tau^{-1}} \rangle \cdots \langle j_{\tau(k)} | j_{\tau^{-1}} \rangle \Pi(\tau^{-1}) \Pi(\tau)
\]
\[
= \sum_{\tau \in S_k} \langle j_{\tau(1)} | j_{\tau^{-1}(1)} \rangle \cdots \langle j_{\tau(k)} | j_{\tau^{-1}(k)} \rangle \Pi(\tau^{-1}).
\]
Note that
\[
\Delta((I)^{-1}) = \left( \frac{k!}{k^k s_{k/(1 \times d)} C_{\lambda}} \right)^{-1} = \frac{1}{k!} \sum_{\lambda \in \mathbb{P}} f_{\lambda} \left( \sum_{\lambda \in \mathbb{P}} \frac{s_{\lambda/(1 \times d)} C_{\lambda}}{f_{\lambda}^2} \right) \Pi(\pi)
\]
\[
= \sum_{\pi \in S_k} Wg(\pi^{-1}) \Pi(\pi),
\]
25
where
\[ C_\lambda := \sum_{\pi \in S_k} \frac{f^\lambda}{k!} \chi_\lambda(\pi^{-1})P(\pi) \]
is the minimal central projection and
\[ W_g := \frac{1}{(k!)^2} \sum_{\lambda \in S_k} \frac{(f^\lambda)^2}{s_\lambda(1 \times d)} \chi_\lambda \]
is the Weingarten function.

Up to now, we can get
\[
\Delta(A)\Delta(B)\Delta(1)^{-1} = \sum_{\sigma,\tau,\pi \in S_k} \langle i_1 i_2 \cdots i_k | j_1 j_2 \cdots j_k \rangle U_{i_1j_1} U_{i_2j_2} \cdots U_{i_kj_k} W_g(\pi^{-1})P(\sigma\tau^{-1}\pi) \]
\[
= \sum_{\sigma,\tau,\pi \in S_k} \langle i_1 i_2 \cdots i_k | j_1 j_2 \cdots j_k \rangle U_{i_1j_1} U_{i_2j_2} \cdots U_{i_kj_k} W_g(\sigma\tau^{-1})1 + \sum_{\sigma,\tau,\pi \in S_k, \sigma\tau^{-1} \pi \neq e} \langle i_1 i_2 \cdots i_k | j_1 j_2 \cdots j_k \rangle U_{i_1j_1} U_{i_2j_2} \cdots U_{i_kj_k} W_g(\pi^{-1})P(\sigma\tau^{-1}\pi). \]
Comparing both sides, we get
\[
\int_{U(d)} U_{i_1j_1} \cdots U_{i_kj_k} W_g(\sigma\tau^{-1})1 \int_{U(d)} U_{i_1j_1} U_{i_2j_2} \cdots U_{i_kj_k} d\mu(U) = \operatorname{Tr}(A\delta_k(B))
\]
\[
= \sum_{\sigma,\tau \in S_k} W_g(\sigma\tau^{-1}) \langle i_1 i_2 \cdots i_k | j_1 j_2 \cdots j_k \rangle U_{i_1j_1} U_{i_2j_2} \cdots U_{i_kj_k} d\mu(U). \]
This completes the proof. \qed

**Corollary 3.18.** If \( k \neq 1 \), then
\[
\int_{U(d)} U_{i_1j_1} \cdots U_{i_kj_k} d\mu(U) = 0.
\]

**Proof.** For every \( z \in U(1) \), the map \( U(d) \ni U \mapsto zU \in U(d) \) is measure-preserving, therefore
\[
\int_{U(d)} U_{i_1j_1} \cdots U_{i_kj_k} U_{i_1j_1} U_{i_2j_2} \cdots U_{i_kj_k} d\mu(U)
\]
\[
= \int_{U(d)} zU_{i_1j_1} \cdots zU_{i_kj_k} zU_{i_1j_1} \cdots zU_{i_kj_k} d\mu(U)
\]
\[
= z^{k-1} \int_{U(d)} U_{i_1j_1} \cdots U_{i_kj_k} U_{i_1j_1} U_{i_2j_2} \cdots U_{i_kj_k} d\mu(U),
\]
implicating that
\[
(1 - z^{k-1}) \int_{U(d)} U_{i_1j_1} \cdots U_{i_kj_k} d\mu(U) = 0.
\]
By the arbitrariness of \( z \in U(1) \), there exists a \( z_0 \in U(1) \) such that \( z_0^{k-1} \neq 1 \) since \( k \neq 1 \). \qed

**Remark 3.19.** What is \( \int_{U(d)} U d\mu(U) \)? One approach to see this is to form a \( d \times d \) matrix \( M \) whose \((i,j)\)-th entry is the \( \int_{U(d)} U_{ij} d\mu(U) \), for \( 1 \leq i, j \leq d \). Writing this in terms of matrix form, we have for any fixed \( V \in U(d) \),
\[
M = \int_{U(d)} U d\mu(U) = \int_{U(d)} V U d\mu(U) = V \int_{U(d)} U d\mu(U) = VM,
\]

26
where we used the fact that $d\mu(U)$ is regular. But $VM = M$ for all unitary $V$ can only hold if $M = 0$. That is,
\[
\int_{U(d)} Ud\mu(U) = 0.
\]

**Corollary 3.20.** It holds that
\[
\int_{U(d)} U^\otimes k \otimes (U^\otimes k)^\dagger d\mu(U) = \sum_{\sigma, \tau \in S_k} Wg(\sigma\tau^{-1})P_{\tau+k,\sigma^{-1}},
\]  
(3.63)

where, for any $\pi_1, \pi_2 \in S_k$,
\[
P_{\pi_1+k,\pi_2} |j_1 \cdots j_k i_1' \cdots i_k'\rangle := |i_2'_{\pi_1(1)} \cdots i_2'_{\pi_1(1)} j_1'_{\pi_1^{-1}(1)} \cdots j_1'_{\pi_1^{-1}(1)} \rangle.
\]  
(3.64)

**Proof.** Clearly
\[
\int_{U(d)} U^\otimes k \otimes (U^\otimes k)^\dagger d\mu(U) = \sum_{\sigma, \tau \in S_k} Wg(\sigma\tau^{-1})\langle i_1| j_1' \rangle \cdots \langle i_k| j_k' \rangle \langle j_1'| j_1^{-1}(1) \rangle \cdots \langle j_k'| j_k^{-1}(k) \rangle.
\]
Next, by the definition of $P_{\pi_1+k,\pi_2}$, hence we get
\[
P_{\tau+k,\sigma^{-1}} |j_1 \cdots j_k i_1' \cdots i_k'\rangle = |i_2'_{\tau^{-1}(1)} \cdots i_2'_{\tau^{-1}(1)} j_1'_{\tau^{-1}(1)} \cdots j_1'_{\tau^{-1}(1)} \rangle,
\]  
(3.65)

where $\pi + k$ means
\[
\begin{pmatrix} 1 & \cdots & k \\ \pi(1)+k & \cdots & \pi(k)+k \end{pmatrix},
\]

implying that
\[
\int_{U(d)} U^\otimes k \otimes (U^\otimes k)^\dagger d\mu(U) |j_1' \rangle = \sum_{\sigma, \tau \in S_k} Wg(\sigma\tau^{-1})P_{\tau+k,\sigma^{-1}} |j_1' \rangle.
\]  
(3.66)

The proof is complete. \(\blacksquare\)

**Remark 3.21.** In recent papers [19], the authors modified the Schur-Weyl duality in the sense that the commutant of $U^\otimes k-1 \otimes U$ can be specifically computed. They make an attempt in [20, 21] to use the obtained new commutant theorem investigate some questions in quantum information theory.

**Corollary 3.22.** The uniform average of $|\psi \rangle \langle \psi|^\otimes k$ over state vectors $|\psi \rangle$ on the unit sphere $S^{2d-1}$ in $\mathbb{C}^d$ is given by
\[
\int_{S^{2d-1}} |\psi \rangle \langle \psi|^\otimes k d|\psi \rangle = \frac{1}{S(k)(1 \times d)} C(k) = \frac{1}{\binom{k+d-1}{k}} C(k),
\]  
(3.67)

where the meaning of $C_\lambda$ can be referred to (6.11), here $\lambda = (k)$.

**Proof.** In fact, this result is a direct consequence of (3.54). More explicitly, let us fix a vector $|\psi_0 \rangle$. Then every $|\psi \rangle$ can be generated by a uniform unitary $U$ such that $|\psi \rangle = U|\psi_0 \rangle$. Thus
\[
\int_{S^{2d-1}} |\psi \rangle \langle \psi|^\otimes k d|\psi \rangle = \int_{U(d)} U^\otimes k|\psi_0 \rangle \langle \psi_0|^\otimes k U^\otimes k+d\mu(U).
\]  
(3.68)
Proposition 3.23. Let $\mathcal{H}_{in}$ and $\mathcal{H}_{out}$ be two copies of the Hilbert space $\mathcal{H} = (\mathbb{C}^d)^{\otimes k}$. Let $C^\vee_{(k)}$ be the projector on the totally symmetric subspace of $\mathcal{H}_{out} \otimes \mathcal{H}_{in}$. Then it holds that
\[
\int_{U(d)} |U^{\otimes k}\rangle \langle U^{\otimes k}| \, d\mu(U) = C^\vee_{(k)} \left( \left[ \text{Tr} \left( C^\vee_{(k)} \right) \right]^{-1} \otimes \mathbb{1}_{in} \right).
\] (3.71)

Corollary 3.24. Let $A \in \text{End}( (\mathbb{C}^d)^{\otimes k} )$. Then it holds that
\[
\int_{U(d)} \left( U^{\otimes k} \right) A \left( U^{\otimes k} \right)^\dagger \, d\mu(U) = \frac{1}{k!} \left[ \sum_{\pi \in S_k} \text{Tr} \left( A P(\pi^{-1}) P(\pi) \right) \right] \left[ \text{Tr} \left( C^\vee_{(k)} \right) \right]^{-1}.
\] (3.72)

Corollary 3.25. Assume $X \in \text{End}(\mathbb{C}^d)$ with spectrum $\{ x_j : 1, \ldots, d \}$. It holds that
\[
\int_{U(d)} \left( U^{\otimes k} \right) X^{\otimes k} \left( U^{\otimes k} \right)^\dagger \, d\mu(U) = \sum_{\lambda \vdash (k,d)} \frac{s_{\lambda}(x_1, \ldots, x_d)}{s_{\lambda}(1^{\times d})} C_\lambda = \sum_{\lambda \vdash (k,d)} \frac{\text{Tr} \left( C_\lambda X^{\otimes k} \right)}{\text{Tr} \left( C_\lambda \right)} C_\lambda.
\] (3.73)

Proof. We give a very simple derivation of this identity via Schur-Weyl duality, i.e. Theorem 2.6. Indeed, the mentioned integral can be rewritten as
\[
\int_{U(d)} \left( U^{\otimes k} \right) X^{\otimes k} \left( U^{\otimes k} \right)^\dagger \, d\mu(U) = \int_{U(d)} Q(U) Q(X) Q^\dagger(U) \, d\mu(U).
\] (3.74)

Now by Schur-Weyl duality, we have
\[
Q(U) = \bigoplus_{\lambda \vdash (k,d)} Q_\lambda(U) \otimes 1_{1^{\times k}}, \quad Q(X) = \bigoplus_{\lambda \vdash (k,d)} Q_\lambda(X) \otimes 1_{1^{\times k}} \quad Q^\dagger(U) = \bigoplus_{\lambda \vdash (k,d)} Q_\lambda^\dagger(U) \otimes 1_{1^{\times k}}.
\]

Thus
\[
\int_{U(d)} Q(U) Q(X) Q^\dagger(U) \, d\mu(U) = \sum_{\lambda \vdash (k,d)} \left( \int_{U(d)} Q_\lambda(U) Q_\lambda(X) Q_\lambda^\dagger(U) \, d\mu(U) \right) \otimes 1_{1^{\times k}}
\] (3.75)
\[
= \sum_{\lambda \vdash (k,d)} \left( \frac{1}{\dim(Q_\lambda)} \text{Tr} \left( Q_\lambda(X) \right) \mathbb{1}_{Q_\lambda} \right) \otimes 1_{1^{\times k}}
\] (3.76)
\[
= \sum_{\lambda \vdash (k,d)} \frac{1}{\dim(Q_\lambda)} \text{Tr} \left( Q_\lambda(X) \right) \mathbb{1}_{Q_\lambda} \otimes 1_{1^{\times k}}
\] (3.77)

which implies the desired result, where we have used the facts that
\[
\text{Tr} \left( Q_\lambda(X) \right) = s_\lambda(x_1, \ldots, x_d),
\]
\[
\dim(Q_\lambda) = s_\lambda(1^{\times d}), \quad C_\lambda = \mathbb{1}_{Q_\lambda} \otimes 1_{1^{\times k}}.
\]

This completes the proof. \qed

Taking $A = |\psi_0\rangle \langle \psi_0|^{\otimes k}$ in (3.34) gives rise to
\[
\int_{U(d)} U^{\otimes k} |\psi_0\rangle \langle \psi_0| U^{\otimes k} \, d\mu(U) = \left( \frac{1}{k!} \sum_{\pi \in S_k} P(\pi) \right) \left( \sum_{\lambda \vdash (k,d)} \frac{f_{\lambda}^{(k)}}{s(\lambda)(1^{\times d})} C_\lambda \right)
\] (3.69)
\[
= C(n) \left( \sum_{\lambda \vdash (k,d)} \frac{f_{\lambda}^{(k)}}{s(\lambda)(1^{\times d})} C_\lambda \right) = \frac{f^{(k)}}{s(k)(1^{\times d})} C(k),
\] (3.70)
 implying the desired result. \qed
Some matrix integrals related to random matrix theory

This section is written based on Taylor’s Lectures on Lie groups [22]. In this section, we give a direct derivation of a formula for

\[ \int |\text{Tr}(U^k)|^2 dU, \]  

(4.1)

of usage in random matrix theory. We also calculate a more refined object,

\[ \int U^k \otimes (U^k)^\dagger dU = \int U^k \otimes U^{-k} dU, \]  

(4.2)

which in turn yields a formula for

\[ \int f(U) \otimes g(U) dU. \]  

(4.3)

Let \( f : S^1 \to \mathbb{C} \) be a bounded Borel function, where \( S^1 = \{ e^{\sqrt{-1} \theta} : \theta \in (0, 2\pi) \} = \{ z \in \mathbb{C} : |z| = 1 \} \). Given \( U \in U(d) \), we define \( f(U) \in \text{End}(\mathbb{C}^d) \) by the spectral decomposition: If \( U = \sum_{j=1}^d e^{\sqrt{-1} \theta_j} |u_j\rangle \langle u_j| \) with \( \{ u_j : j = 1, \ldots, d \} \) being an orthonormal basis for \( \mathbb{C}^d \), then \( f(U) \) is defined as

\[ f(U) := \sum_{j=1}^d f(e^{\sqrt{-1} \theta_j}) |u_j\rangle \langle u_j|. \]  

(4.4)

For instance, \( U^k = \sum_{j=1}^d e^{\sqrt{-1}k \theta_j} |u_j\rangle \langle u_j| \).

We are interested in formulae for

\[ \int \text{Tr} (f(U)) \text{Tr} (g(U)) dU. \]  

(4.5)

Note that the above is equal to the trace of

\[ \int f(U) \otimes g(U) dU. \]  

(4.6)

The notion of Fourier series will be used here. On \( S^1 \), let \( d\mu(z) \) is the uniform and normalized Haar measure. Thus for \( z = e^{\sqrt{-1} \theta} \in S^1 \),

\[ d\mu(z) = \frac{d\theta}{2\pi}. \]

The functions \( \chi_m(z) = z^m \) or \( \theta \mapsto e^{\sqrt{-1} m \theta} \) for \( m \in \mathbb{Z} \) are just irreducible characters of \( U(1) = S^1 \), and thus form an orthonormal basis of complex \( L^2(S^1, \mu) \). For \( f \in L^1(S^1, \mu) \), defining the coefficients

\[ \hat{f}(k) := \int_{S^1} f(z) z^{-k} d\mu(z), \]  

(4.7)

yields the formal series

\[ f \sim \sum_{k \in \mathbb{Z}} \hat{f}(k) z^k. \]  

(4.8)

If \( f \in L^2(S^1, \mu) \), the series converges unconditionally to \( f \) in \( L^2(S^1, \mu) \). For general \( f \in L^1(S^1, \mu) \) and \( z \in S^1 \), let

\[ S_m f(z) := \sum_{k=-m}^m \hat{f}(k) z^k, \quad m = 0, 1, 2, \ldots \]  

(4.9)

29
Thus we have

\[ F(\theta) \sim \sum_{k \in \mathbb{Z}} \hat{f}(k)e^{\sqrt{-1}k\theta}, \quad (4.10) \]

which has been called the exponential Fourier series of \( F \). In terms of trigonometric functions we get another series

\[ F(\theta) \sim c_0 + \sum_{k=1}^{\infty} a_k \cos(k\theta) + b_k \sin(k\theta), \quad (4.11) \]

called the Fourier series of \( F \) (or of \( f \)). Here \( a_k := \hat{f}(k) + \hat{f}(-k) \) and \( b_k = \sqrt{-1}(\hat{f}(k) - \hat{f}(-k)) \).

Specifically, the Fourier series of \( F \) converges to \( F \) at a given \( \theta \) if and only if \( \lim_{m \to \infty} S_m f(e^{\sqrt{-1}\theta}) = f(e^{\sqrt{-1}\theta}). \)

Now we find that

\[ f(U) = \sum_{k=-\infty}^{+\infty} \hat{f}(k)U^k, \quad (4.12) \]

where

\[ \hat{f}(k) := \frac{1}{2\pi} \int_0^{2\pi} F(\theta)e^{-\sqrt{-1}k\theta}d\theta = \frac{1}{2\pi} \int_0^{2\pi} f(e^{\sqrt{-1}\theta})e^{-\sqrt{-1}k\theta}d\theta. \]

and

\[ F(\theta) = \sum_{k \in \mathbb{Z}} \hat{f}(k)e^{\sqrt{-1}k\theta} \iff f(e^{\sqrt{-1}\theta}) = \sum_{k \in \mathbb{Z}} \hat{f}(k)e^{\sqrt{-1}k\theta}. \]

Thus we have

\[ \int_{U(d)} f(U) \otimes g(U)dU = \int_{U(d)} \left( \sum_{i=-\infty}^{+\infty} \hat{f}(i)U^i \right) \otimes \left( \sum_{j=-\infty}^{+\infty} \hat{g}(j)U^j \right) dU \]

\[ = \sum_{i,j \in \mathbb{Z}} \hat{f}(i)\hat{g}(j) \int_{U(d)} U^i \otimes U^jdU \]

\[ = \sum_{i,j \in \mathbb{Z}} \hat{f}(i)\hat{g}(j)M_{ij}, \quad (4.15) \]

where \( M_{ij} = \int_{U(d)} U^i \otimes U^jdU \). Performing the measure-preserving transformation \( U \mapsto e^{\sqrt{-1}\psi}U \) on \( U(d) \), we see that

\[ M_{ij} = e^{\sqrt{-1}(i+j)\psi}M_{ij} \text{ for all } \psi \in \mathbb{R}. \quad (4.16) \]

Thus \( M_{ij} = 0 \) for \( i \neq -j \). Hence

\[ \int_{U(d)} f(U) \otimes g(U)dU = \sum_{k \in \mathbb{Z}} \hat{f}(k)\hat{g}(-k)M_k, \quad (4.17) \]

where \( M_k := \int_{U(d)} U^k \otimes U^{-k}dU \), which implies that

\[ \int_{U(d)} \text{Tr}(f(U))\text{Tr}(g(U))dU = \sum_{k \in \mathbb{Z}} \hat{f}(k)\hat{g}(-k)\text{Tr}(M_k). \quad (4.18) \]
It remains to compute the following integral
\[
\text{Tr} (M_k) = \int_{U(d)} \left| \text{Tr}(U^k) \right|^2 dU. \tag{4.19}
\]

Here we establish the following identity:

**Proposition 4.1.** It holds that
\[
\int_{U(d)} \left| \text{Tr}(U^k) \right|^2 dU = \min(k,d). \tag{4.20}
\]

**Proof.** Here we give a natural proof, based on Weyl’s integration formula, which implies that whenever \(\varphi : U(d) \to \mathbb{C}\) invariant under conjugation, then
\[
\int_{U(d)} \varphi(U)dU = C_d \int_{T^d} \varphi(D(\theta))J(\theta)d\theta = \frac{C_d}{(2\pi)^d} \int_0^{2\pi} \cdots \int_0^{2\pi} \varphi(D(\theta))J(\theta)d\theta
\]  
(4.21)

where \(D(\theta) = \text{diag}(e^{\sqrt{-1}\theta_1}, \ldots, e^{\sqrt{-1}\theta_d})\), and \(J(\theta) = \prod_{i<j} \left| e^{\sqrt{-1}\theta_i} - e^{\sqrt{-1}\theta_j} \right|^2\). We will verify in calculations below that \(C_d = 1/d!\).

Now
\[
\int_{U(d)} \left| \text{Tr}(U^k) \right|^2 dU = \frac{C_d}{(2\pi)^d} \int_0^{2\pi} \cdots \int_0^{2\pi} \left| e^{\sqrt{-1}\theta_1} + \cdots + e^{\sqrt{-1}\theta_d} \right|^2 J(\theta)d\theta.
\]  
(4.22)

We restate this as follows. Set \(\xi_j = e^{\sqrt{-1}\theta_j}\), so
\[
\left| e^{\sqrt{-1}\theta_1} + \cdots + e^{\sqrt{-1}\theta_d} \right|^2 = \left| \xi_1^k + \cdots + \xi_d^k \right|^2 = \sum_{p,q=1}^d \xi_p^{k-p} \xi_q^{k-q}
\]  
(4.23)

and
\[
J(\theta) = \prod_{i<j} \left| \xi_i - \xi_j \right|^2 = \prod_{i<j} (\xi_i - \xi_j)(\xi_i^{-1} - \xi_j^{-1})
\]  
(4.24)

\[
= (\text{sign } \tau)(\xi_1 \cdots \xi_d)^{-(d-1)} \prod_{i<j} (\xi_i - \xi_j)^2,
\]  
(4.25)

where \(\tau = (d \cdots 21)\), i.e. \(\tau(j) = d + 1 - j\) or \(\tau\) is written as
\[
\tau := \begin{pmatrix} 1 & 2 & \cdots & d \\ d & d-1 & \cdots & 1 \end{pmatrix}
\]

Note that \(\text{sign } \tau = (-1)^{\frac{d(d-1)}{2}}\). We see that \(\int_{U(d)} \left| \text{Tr}(U^k) \right|^2 dU\) is the constant term in
\[
C_d(\text{sign } \tau)(\xi_1 \cdots \xi_d)^{-(d-1)} \left( \sum_{p,q=1}^d \xi_p^{k-p} \xi_q^{k-q} \right) \prod_{i<j} (\xi_i - \xi_j)^2.
\]  
(4.26)

Thus our task is to identify the constant term in this Laurent polynomial. To work on the last factor, we recognize
\[
V(\xi) = \prod_{i<j} (\xi_i - \xi_j)
\]  
(4.27)

31
as a Vandermonde determinant; hence
\[ V(\zeta) = \sum_{\sigma \in S_d} (\text{sign } \sigma) \zeta_1^{\sigma(1)-1} \cdots \zeta_d^{\sigma(d)-1}. \]  
(4.28)

Hence
\[ \prod_{i<j}(\zeta_i - \zeta_j)^2 = V(\zeta)^2 = \sum_{\sigma, \pi \in S_d} (\text{sign } \sigma) (\text{sign } \pi) \zeta_1^{\sigma(1)+\pi(1)-2} \cdots \zeta_d^{\sigma(d)+\pi(d)-2}. \]  
(4.29)

Let us first identify the constant term in
\[ J(\theta) = (\text{sign } \tau)(\zeta_1 \cdots \zeta_d)^{-(d-1)}V(\zeta)^2. \]  
(4.30)

We see this constant term is equal to
\[
\frac{1}{(2\pi)^d} \int_0^{2\pi} \cdots \int_0^{2\pi} J(\theta) d\theta = (\text{sign } \tau) \frac{1}{(2\pi)^d} \int_0^{2\pi} \cdots \int_0^{2\pi} \left( \sum_{\sigma, \pi \in S_d} (\text{sign } \sigma) (\text{sign } \pi) \zeta_1^{\sigma(1)+\pi(1)-d-1} \cdots \zeta_d^{\sigma(d)+\pi(d)-d-1} \right) d\theta
\]
\[ = (\text{sign } \tau) \sum_{\sigma, \pi \in S_d} (\text{sign } \sigma) (\text{sign } \pi) \left( \frac{1}{2\pi} \int_0^{2\pi} \zeta_1^{\sigma(1)+\pi(1)-d-1} d\theta_1 \right) \times \cdots \times \left( \frac{1}{2\pi} \int_0^{2\pi} \zeta_d^{\sigma(d)+\pi(d)-d-1} d\theta_d \right)
\]
\[ = (\text{sign } \tau) \sum_{(\sigma, \pi) \in S_d \times S_d \forall \sigma(j) + \pi(j) = d+1} (\text{sign } \sigma) (\text{sign } \pi) = (\text{sign } \tau) \sum_{(\sigma, \pi) \in S_d \times S_d \forall \pi = \tau \sigma} (\text{sign } \sigma) (\text{sign } \pi). \]

Note that the sum is over all \((\sigma, \pi) \in S_d \times S_d\) such that \(\sigma(j) + \pi(j) = d+1\) for each \(j \in \{1, \ldots, d\}\). In other words, we get \(\pi(j) = d+1 - \sigma(j) = \tau(\sigma(j))\) for all \(j \in \{1, \ldots, d\}\), i.e. \(\pi = \tau \sigma\). Thus the sum is equal to
\[ (\text{sign } \tau) \sum_{\sigma \in S_d} (\text{sign } \sigma) (\text{sign } \tau \sigma) = d!, \]
which gives rise to \(C_d = 1/d!\).

Clearly
\[
C_d(\text{sign } \tau)(\zeta_1 \cdots \zeta_d)^{-(d-1)} \left( \sum_{p,q=1}^{d} \zeta_p \zeta_q^{-k} \right) \prod_{i<j}(\zeta_i - \zeta_j)^2
\]
(4.31)
\[ = C_d(\text{sign } \tau)(\zeta_1 \cdots \zeta_d)^{-(d-1)} \sum_{p,q=1}^{d} \sum_{(\sigma, \pi) \in S_d \times S_d} (\text{sign } \sigma) (\text{sign } \pi) \zeta_1^{\sigma(1)+\pi(1)-d-1} \cdots \zeta_d^{\sigma(d)+\pi(d)-d-1}, \]
(4.32)
\[ = C_d(\text{sign } \tau)(V_1(\zeta) + V_2(\zeta)), \]
(4.33)
where
\[ V_1(\zeta) := (\zeta_1 \cdots \zeta_d)^{-(d-1)} \sum_{p=1}^{d} \sum_{(\sigma, \pi) \in S_d \times S_d} (\text{sign } \sigma) (\text{sign } \pi) \zeta_1^{\sigma(1)+\pi(1)-d-1} \cdots \zeta_d^{\sigma(d)+\pi(d)-d-1}, \]
(4.35)
\[ V_2(\zeta) := (\zeta_1 \cdots \zeta_d)^{-(d-1)} \sum_{p \neq q} \sum_{(\sigma, \pi) \in S_d \times S_d} (\text{sign } \sigma) (\text{sign } \pi) \zeta_1^{\sigma(1)+\pi(1)-d-1} \cdots \zeta_d^{\sigma(d)+\pi(d)-d-1}. \]
(4.36)
Now
\[
V_1(\zeta) := d \times \sum_{(\sigma, \pi) \in S_d \times S_d} (\text{sign } \sigma)(\text{sign } \pi) \zeta_1^{\sigma(1)+\pi(1)-d-1} \cdots \zeta_d^{\sigma(d)+\pi(d) - d - 1}, \tag{4.37}
\]
implies
\[
\frac{1}{(2\pi)^d} \int_0^{2\pi} \cdots \int_0^{2\pi} V_1(\zeta) d\theta := d \cdot d!(\text{sign } \tau). \tag{4.38}
\]
It remains to consider the integral involved in \( V_2(\zeta) \). That is,
\[
\frac{1}{(2\pi)^d} \int_0^{2\pi} \cdots \int_0^{2\pi} V_2(\zeta) d\theta. \tag{4.39}
\]
We see that for a given \( p \neq q \), a pair \((\sigma, \pi) \in S_d \times S_d\) contributes to the constant term in \( V_2(\zeta) \) if and only if
\[
\sigma(j) + \pi(j) = \begin{cases} 
   d + 1, & \text{if } j \in \{1, \ldots, d\} \setminus \{p, q\} \\
   d + 1 - k, & \text{if } j = p \\
   d + 1 + k, & \text{if } j = q.
\end{cases} \tag{4.40}
\]
That is,
\[
\pi(j) = \begin{cases} 
   d + 1 - \sigma(j), & \text{if } j \in \{1, \ldots, d\} \setminus \{p, q\} \\
   d + 1 - \sigma(j) - k, & \text{if } j = p \\
   d + 1 - \sigma(j) + k, & \text{if } j = q.
\end{cases} \tag{4.41}
\]
By the definition of \( \tau \), \( d + 1 - \sigma(j) = \tau(\sigma(j)) \) for all \( j \). Thus
\[
\pi(j) = \begin{cases} 
   \tau(\sigma(j)), & \text{if } j \in \{1, \ldots, d\} \setminus \{p, q\} \\
   \tau(\sigma(j)) - k, & \text{if } j = p \\
   \tau(\sigma(j)) + k, & \text{if } j = q.
\end{cases} \tag{4.42}
\]
Define
\[
\omega_{pq}(j) = \begin{cases} 
   j, & \text{if } j \in \{1, \ldots, d\} \setminus \{j_p, j_q\} \\
   j - k, & \text{if } j = j_p \\
   j + k, & \text{if } j = j_q,
\end{cases} \tag{4.43}
\]
where \( j_p = \tau(\sigma(p)) \) and \( j_q = \tau(\sigma(q)) \). Therefore \( \pi = \omega_{pq} \tau \sigma \), where \( \omega_{pq} = (j_p j_q) \) with \( j_p - j_q = k \). Note that all the possible choices of \( \omega_{pq} \) depends on all the possible values of positive integer \( j_p \), i.e. totally \( d - k \) since \( k + 1 \leq j_p \leq d \).

Now
\[
V_2(\zeta) := \sum_{p \neq q} \sum_{(\sigma, \pi) \in S_d \times S_d} (\text{sign } \sigma)(\text{sign } \pi) \zeta_1^{\omega_{pq}(1)+\pi(1)-d-1} \cdots \zeta_d^{\omega_{pq}(d)+\pi(d) - d - 1}, \tag{4.44}
\]
33
implying that if $1 \leq k \leq d - 1$,

$$\frac{1}{(2\pi)^d} \int_0^{2\pi} \cdots \int_0^{2\pi} V_2(\xi) d\theta = \sum_{p \neq q} \sum_{(\sigma, \pi) \in S_d \times S_d: \pi = \omega_{pq} \tau \sigma} (\text{sign } \sigma)(\text{sign } \pi)$$

$$= \sum_{p \neq q} \sum_{\nu \in S_d} (\text{sign } \sigma)(\text{sign } \omega_{pq} \tau \nu)$$

$$= d! (\text{sign } \tau) \sum_{p \neq q} \text{sign } \omega_{pq} = -(d - k) \cdot d! (\text{sign } \tau),$$

where we used the fact that $\sum_{p \neq q} \text{sign } \omega_{pq} = -(d - k)$. The reason is for some pairs $(p, q)$ with $p \neq q$, $\omega_{pq}$ does not exist, but just exist for $d - k$ pairs $(p, q)$ with $p \neq q$. We also note that if $k \geq d$, the choice of $\omega_{pq}$ is empty. Therefore the integral involved in $V_2(\xi)$ is zero.

**Corollary 4.2.** For $k \geq 1, d \geq 2$, we have

$$\int_{U(d)} U^k \otimes U^{-k} dU = \frac{\min(k, d) - 1}{d^2 - 1} \mathbb{1}_{d^2} + \frac{d^2 - \min(k, d)}{d(d^2 - 1)} F,$$

where $F = \sum_{i,j=1}^d |ij\rangle \langle ji|$ is the swap operator on $\mathbb{C}^d \otimes \mathbb{C}^d$.

**Proof.** Apparently $[M_k, V \otimes V] = 0$ for all $V \in U(d)$. It follows from Proposition 3.9 that

$$M_k = \int_{U(d)} (V \otimes V) M_k (V \otimes V)^{-1} dV$$

$$= \left( \frac{\text{Tr} (M_k)}{d^2 - 1} - \frac{\text{Tr} (M_k F)}{d(d^2 - 1)} \right) \mathbb{1}_{d^2} + \left( \frac{\text{Tr} (M_k F)}{d^2 - 1} - \frac{\text{Tr} (M_k)}{d(d^2 - 1)} \right) F.$$

It suffices to compute $\text{Tr}(M_k)$ and $\text{Tr}(M_k F)$. Clearly $\text{Tr}(M_k F) = n$. By Proposition 4.1, we have $\text{Tr}(M_k) = \min(k, d)$. Therefore the desired conclusion is obtained.

**Corollary 4.3.** For $k \geq 1, d \geq 2$, we have

$$\int_{U(d)} U^k A(U^k)^\dagger dU = \frac{\min(k, d) - 1}{d^2 - 1} A + \frac{d^2 - \min(k, d)}{d(d^2 - 1)} \text{Tr} (A) \mathbb{1}_d,$$

where $F = \sum_{i,j=1}^d |ij\rangle \langle ji|$ is the swap operator on $\mathbb{C}^d \otimes \mathbb{C}^d$.

**Proof.** Firstly we have

$$\int_{U(d)} U^k \otimes \overline{U}^k dU = \frac{\min(k, d) - 1}{d^2 - 1} \mathbb{1}_{d^2} + \frac{d^2 - \min(k, d)}{d(d^2 - 1)} |\text{vec}(\mathbb{1}_d)\rangle \langle \text{vec}(\mathbb{1}_d)|,$$

which indicates that

$$\left( \int_{U(d)} U^k \otimes \overline{U}^k dU \right) |\text{vec}(A)\rangle$$

$$= \frac{\min(k, d) - 1}{d^2 - 1} |\text{vec}(A)\rangle + \frac{d^2 - \min(k, d)}{d(d^2 - 1)} |\text{vec}(\mathbb{1}_d)\rangle \langle \text{vec}(\mathbb{1}_d), \text{vec}(A)|.$$

Therefore

$$\int_{U(d)} U^k A(U^k)^\dagger dU = \frac{\min(k, d) - 1}{d^2 - 1} A + \frac{d^2 - \min(k, d)}{d(d^2 - 1)} \text{Tr} (A) \mathbb{1}_d,$$

implying the desired result. When $k = 1$, the result of the present proposition is reduced to Proposition 3.1.
Proposition 4.4. It holds that
\begin{equation}
\int_{U(d)} f(U) \otimes g(U) dU = \frac{h(0) - \mathcal{F}_d h(0)}{d^2 - 1} \left(1_{n^2} - \frac{1}{d} F\right) - \frac{h(0) - \hat{h}(0)}{d^2 - 1} (1_{d^2} - dF) + \hat{h}(0) 1_{d^2},
\end{equation}
where
\begin{equation}
h(\theta) = \frac{1}{2\pi} \int_0^{2\pi} f(t) g(t - \theta) dt
\end{equation}
and $\mathcal{F}_d h$ denotes the $d$-th Fejér mean of the Fourier series of $h$:
\begin{equation}
\mathcal{F}_d h(\theta) = \sum_{j=-d}^d \left(1 - \frac{\dim(|j|,d)}{d}\right) \hat{h}(j) e^{\sqrt{-1} \theta j}.
\end{equation}

5 Discussion and concluding remarks

We see that the integrals considered in this paper, where all the underlying domain of integrals are just $U(d)$. As a matter of fact, analogous problems can be considered when the unitary group $U(d)$ can be replaced by a compact Lie group $G$ of some particular property, for instance, we may assume that $G$ is a gauge group (see [7, 8]), a some kind of subgroup of $U(d)$.

In addition, we can derive some similar results from Schur Orthogonality Relations. Recall that for a compact Lie group $G$, let $\{g \rightarrow V^{(\mu)}(g)\}$ be the set of all inequivalent unitary irreps on the underlying vector space $V$. Consider the matrix entries of all these unitary matrices as a set of functions from $G$ to $\mathbb{C}$, denoted by $\{V_{ij}^{(\mu)}\}$. Then, they satisfy the following Schur-Orthogonality Relations:
\begin{equation}
\int_G V_{ij}^{(\mu)}(g) \overline{V_{kl}^{(\nu)}(g)} dg = \frac{1}{d_{\mu}} \delta_{\mu \nu} \delta_{ik} \delta_{jl},
\end{equation}
where $dg$ is the uniform probability Haar measure on $G$, bar means the complex conjugate and $d_{\mu}$ is the dimension of irrep $\mu$. We can make analysis about (5.1) as follows: For the orthonormal base $\{|i\} : i = 1, \ldots, d_{\mu}\}$ and $\{|k\} : k = 1, \ldots, d_{\nu}\}$, we have
\begin{equation}
V_{ij}^{(\mu)}(g) = \langle i | V^{(\mu)}(g) | j \rangle, \quad \overline{V_{ij}^{(\mu)}(g)} = \langle k | V^{(\nu)}(g) | l \rangle.
\end{equation}
Then
\begin{equation}
\int_G V^{(\mu)}(g) \otimes \overline{V^{(\nu)}(g)} dg = \frac{1}{d_{\mu}} \delta_{\mu \nu} \sum_{i,j=1}^{d_{\mu}} \sum_{k,l=1}^{d_{\nu}} \delta_{ik} \delta_{jl} |i l\rangle.
\end{equation}
That is
\begin{equation}
\int_G V^{(\mu)}(g) \otimes \overline{V^{(\nu)}(g)} dg = \begin{cases} 0, & \text{if } \mu \neq \nu, \\ \frac{1}{d_{\mu}} \overline{\text{vec}(1_{\mu})} \langle \text{vec}(1_{\mu}) |, & \text{if } \mu = \nu. \end{cases}
\end{equation}
Here $\text{vec}(1_{\mu}) := \sum_{i,j=1}^{d_{\mu}} |i j\rangle$. This indicates that
\begin{equation}
\int_G V^{(\mu)}(g) X V^{(\mu)^*}(g) dg = \frac{1}{d_{\mu}} \text{Tr} (X) 1_{\mu}.
\end{equation}
is a completely depolarizing channel. Therefore for \( \mu \neq \nu \),
\[
\int_G V^{(\mu)}(g) \otimes V^{(\nu),*}(g) dg = 0,
\]
and
\[
\int_G V^{(\mu)}(g) \otimes V^{(\mu),*}(g) dg = \frac{1}{d_\mu} F^{(\mu)},
\]
where \( F^{(\mu)} \) is the swap operator on the 2-fold tensor space of irrep \( \mu \). In view of this point, we naturally want to know if the integral
\[
\int_G V(g) \otimes V^*(g) dg
\]
can be computed explicitly, where \( \{ g \to V(g) \} \) is any unitary representation of \( G \). In particular, when \( G = U(d) \) and \( V(g) = Q(g) \), the integral (5.6) is reduced to the form:
\[
\int_{U(d)} Q(g) \otimes Q^*(g) dg,
\]
(5.7)
for which we have derived explicit formula in the present paper. We leave these topics for future research.

6 Appendix
To better understand Schur-Weyl duality, i.e. irreps of unitary group and permutation group, we collect some relevant materials. The details presented in the Appendix are written based on Notes of Audenaert [23].

6.1 Partitions
A partition is a sequence \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r, \ldots) \) of non-negative integers in non-increasing order
\[
\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq \cdots
\]
and containing finitely many non-zero terms. The non-vanishing terms \( \lambda_j \) are called the parts of \( \lambda \). The length of \( \lambda \), denoted \( \ell(\lambda) \), is the number of parts of \( \lambda \). The weight of \( \lambda \), denoted \( |\lambda| \), is the sum of the parts:
\[
|\lambda| := \sum_j \lambda_j.
\]
A partition \( \lambda \) with weight \( |\lambda| = k \) is also called a partition of \( k \), and this is denoted \( \lambda \vdash k \). We will also use the notation \( \lambda \vdash (k, d) \) to indicate that \( \lambda \vdash k \) and \( \ell(\lambda) \leq d \) in one statement.

For \( \lambda \vdash k \), we use the shorthand \( \tilde{\lambda} := \frac{\lambda}{k} \). For \( j \geq 1 \), the \( j \)-th element of \( \lambda \) is denoted by \( \lambda_j \). This element is a part if \( j \leq \ell(\lambda) \), otherwise it is 0. It is frequently convenient to use a different notation that indicates the number of times each integer \( j = 1, 2, \ldots, |\lambda| \) occurs as a part, the so-called multiplicity \( m_j \) of \( j \):
\[
\lambda = (1^{m_1} 2^{m_2} \ldots r^{m_r}, \ldots).
\]
As a shorthand we will use a superscripted index: \( \lambda_j = m_j(\lambda) \).

Now one has the relations
\[
\begin{cases}
\sum_{j=1}^k \lambda_j^i &= \ell(\lambda), \\
\sum_{j=1}^k j\lambda_j^j &= |\lambda| = k.
\end{cases}
\]

When dealing with numerical calculations it is necessary to impose an ordering on the set of partitions. We will adhere here to the lexicographic ordering, in which \( \lambda \) precedes \( \mu \), denoted \( \lambda > \mu \), if and only if the first non-zero difference \( \lambda_j - \mu_j \) is positive.
Example 6.1. With the above convention, the partitions of 5 are ordered as follows:

\[(5), (41), (32), (31^2), (2^21), (21^3), (1^5).\]

It is seen easily that lexicographic ordering is a total order.

6.2 Young frames and Young tableaux

Partitions can be graphically represented by Young diagrams, which are Young tableaux with empty boxes. The \(j\)-th part \(\lambda_j\) corresponds to the \(j\)-th row of the diagram, consisting of \(\lambda_j\) boxes. Conversely, the Young diagrams of \(k\) boxes can be uniquely labeled by a partition \(\lambda \vdash k\). We will therefore identify a Young diagram with the partition labeling it.

A Young tableau (YT) of \(d\) objects and of shape \(\lambda \vdash k\) is a Young diagram \(\lambda\) in which the boxes are labeled by numbers \(\{1, \ldots, d\}\).

A standard Young tableau (SYT) of shape \(\lambda \vdash k\) is a Young tableau of \(d = k\) objects such that the labels appear increasing in every row from left to right, and increasing in every column downwards; hence every number occurs exactly once.

A Semi-standard Young tableau (SSYT) of shape \(\lambda \vdash k\) is a Young tableau such that the labels appear non-decreasing in every row from left to right, and increasing in every column downwards.

The number of SSYTs of \(d\) objects and of shape \(\lambda \vdash k\) (imposing the condition \(\ell(\lambda) \leq d\)) is given by \(s_{\lambda}(1 \times d) \equiv s_{\lambda,d}(1)\); see below for an explanation.

The number \(f^\lambda\) of SYTs of shape \(\lambda \vdash (k,d)\) is

\[f^\lambda = k! \frac{\Delta(\mu_1, \ldots, \mu_d)}{\mu_1! \cdots \mu_d!}, \quad d = \ell(\lambda),\]  \hspace{1cm} (6.1)

where \(\Delta(\mu_1, \ldots, \mu_d)\) denotes the difference product of a non-increasing sequence

\[\Delta(\mu_1, \ldots, \mu_d) := \prod_{1 \leq i < j \leq d} (\mu_i - \mu_j),\]

and the numbers \(\mu_j = \mu_j(\lambda)\) are defined by

\[\mu_j(\lambda) := \lambda_j + \ell(\lambda) - j, \quad \text{for } j = 1, 2, \ldots, \ell(\lambda).\]

6.3 Permutations

We can display a permutation \(\pi\) using cycle notation. Given \(j \in \{1, \ldots, k\} := [k]\), the elements of the sequence \(j, \pi(j), \ldots\) cannot be distinct. Taking the first power \(n\) such that \(\pi^n(j) = j\), we have the cycle

\[(j, \pi(j), \ldots, \pi^{n-1}(j)).\]

Equivalently, the cycle \((i, j, \ldots, l)\) means that \(\pi\) sends \(i\) to \(j\), \ldots, and \(l\) back to \(i\). Now pick an element not in the cycle containing \(i\) and iterate this process until all members of \([k]\) have been used. For example \(\pi \in S_5, \pi = (1, 2, 3)(4)(5)\) in cycle notation. Note that cyclically permuting the elements within a cycle or reordering the cycles themselves does not change the permutation. Thus

\[(1, 2, 3)(4)(5) = (2, 3, 1)(4)(5) = (4)(2, 3, 1)(5) = (4)(5)(3, 1, 2).\]
A $k$-cycle, or cycle of length $k$, is a cycle containing $k$ elements. The cycle type, or simply the type, of $\pi$ is an expression of the form
\[(1^{m_1}, 2^{m_2}, \ldots, k^{m_k}),\]
where $m_k$ is the number of cycles of length $k$ in $\pi$. A 1-cycle of $\pi$ is called a fixed-point. Fixed-points are usually dropped from the cycle notation if no confusion will result. It is easy to see that a permutation $\pi$ such that $\pi^2 = 1$ if and only if all of $\pi$’s cycles have length 1 or 2.

Another way to give the cycle type is as a partition. A partition $\lambda$ of the cycles, sorted in non-increasing order, determines the cycle type of the permutation. Evidently, the cycle type of a permutation $\pi \in S_k$ such that $\pi \gamma = \gamma \pi$ for a given $\gamma \in S_k$ depends only on $\gamma$ and
\[|K_\gamma| = \frac{|G|}{|Z_\gamma|}.
\]

We can compute the size of a conjugacy class in the following manner. Let $G$ be any group and consider the centralizer of $g \in G$ defined by
\[Z_g := \{ h \in G : hgh^{-1} = g \},\]
i.e., the set of all elements that commute with $g$. Now, there is a bijection between the cosets of $Z_g$ and the elements of $K_g$, where $K_g$ is the conjugate class of $g$—the set of all elements conjugate to a given $g$—so that
\[|K_g| = \frac{|G|}{|Z_g|}.
\]

Now let $G = S_k$ and use $K_\gamma$ for $K_g$ when $g$ has type $\gamma$. Thus if $\gamma = (1^{m_1}, 2^{m_2}, \ldots, k^{m_k})$ and $g \in S_k$ has type $\gamma$, then $|Z_\gamma|$ depends only on $\gamma$ and
\[z_\gamma \eqdef |Z_\gamma| = 1^{m_1} m_1! 2^{m_2} m_2! \cdots k^{m_k} m_k!.
\]

The number $|K_\gamma|$ of elements in a conjugacy class $\gamma$ of $S_k$, denoted $h_\gamma$, is given by
\[h_\gamma = \frac{k!}{z_\gamma}.
\]

We know that every permutation $\pi \in S_k$ decomposes uniquely as a product of disjoint cycles. The orders of the cycles, sorted in non-increasing order, determine the cycle type of the permutation. Evidently, the cycle type of a permutation $\pi \in S_k$ is a partition of $k$. We will denote the cycle type of a permutation $\pi \in S_k$ by $\gamma = \gamma(\pi) \vdash k$. We shall identify the conjugacy classes with their cycle type, and even write $\pi \in \gamma$ for a permutation $\pi$ with cycle type $\gamma$.

For instance, $h_{(k)} = (k - 1)!$ and $h_{(1^k)} = 1$. Obviously, we need to have \(\sum_{\gamma \vdash k} \frac{1}{z_\gamma} = 1.\)
6.4 Products of power sums

For an integer \( r \geq 1 \), the \( r \)-th power sum in the variables \( x_j \) is \( p_r = \sum_j x_j^r \). For a partition \( \gamma \vdash (k, r) \), the power sum products \( p_\gamma \) are defined by

\[
p_\gamma := p_{\gamma_1} p_{\gamma_2} \cdots p_{\gamma_r} = \left( \sum_j x_j^{\gamma_1} \right) \left( \sum_j x_j^{\gamma_2} \right) \cdots \left( \sum_j x_j^{\gamma_r} \right).
\]

(6.2)

As a special case, \( p_\gamma(1 \times d) = d^r \), where \( r = \ell(\gamma) \) is nothing but the number of cycles in \( \gamma \).

6.5 Schur functions

To define the Schur symmetric functions, or S-functions, it is best to start with the polynomial case, i.e. with a finite number \( d \) of variables \( x_1, \ldots, x_d \). The complete set of S-functions is obtained by letting \( d \) tend to infinity. The S-functions \( s_\lambda \) of \( d \) variables and of homogeneity order \( k \) are labeled by partitions \( \lambda \vdash (k, d) \), and are defined by

\[
s_\lambda(x_1, \ldots, x_d) := \frac{\text{Det} \left( x_i^{\lambda_j + d-j} \right)_{i,j=1}^d}{\text{Det} \left( x_i^{d-j} \right)_{i,j=1}^d}
\]

(6.3)

(recall again that for \( j > \ell(\lambda), \lambda_j = 0 \)). For \( \ell(\lambda) > d \), one again has \( s_\lambda(x_1, \ldots, x_d) = 0 \). If some variables assume equal values, a limit has to be taken, since both numerator and denominator vanish in that case.

The denominator in the definition of the S-function is a Vandermonde determinant and is thus equal to \( \Delta(x_1, \ldots, x_d) \). The numerator is divisible (in the ring of polynomials) by each of the differences \( x_i - x_j \), and therefore also by the denominator; hence the S-functions in a finite number of variables really are polynomials.

For the important case where all \( d \) variables assume the value 1 (i.e. giving the number of semi-standard Young tableaux of \( d \) objects and of shape \( \lambda \)), we get, for \( \ell(\lambda) \leq d \):

\[
s_\lambda(1 \times d) = \frac{\Delta(\lambda_1 + d - 1, \lambda_2 + d - 2, \ldots, \lambda_d)}{\Delta(d - 1, d - 2, \ldots, 0)},
\]

(6.4)

and, again, \( s_\lambda(1 \times d) = 0 \) for \( \ell(\lambda) > d \). Note that \( \Delta(d - 1, d - 2, \ldots, 0) = 1!2! \cdots (d - 1)! \). In particular, if \( \lambda = (k) \), one finds that \( s_{(k)}(1 \times d) = (k+d-1)k \).

6.6 Characters of the symmetric group and unitary group

In the case of the symmetric group, the irreps are labeled by Young diagrams \( \lambda \). The character of a permutation \( \pi \in S_k \) in irrep \( \lambda \) is denoted \( \chi_\lambda(\pi) \). Since characters are class functions, one only needs to find the characters of any representative of a conjugacy class, so that one can use the symbol \( \chi_{\lambda,\gamma} \), with

\[
\chi_{\lambda,\gamma} = \chi_\lambda(\pi), \quad \forall \pi \in \gamma.
\]

The character table is the matrix with elements \( \chi_{\lambda,\gamma} \), where \( \lambda \) is the row index and \( \gamma \) the column index (assuming lexicographic ordering for both). As the conjugacy classes of \( S_k \) are labeled by partitions of \( k \), there are as many rows as columns, hence the character table is a square matrix.
The character of the identity permutation \( \epsilon \) equals the degree of the representation in the given irrep. One can show that this degree is equal to the number of standard Young tableaux of shape \( \lambda \)

\[
\chi_\lambda(\epsilon) = f^\lambda.
\]

The characters in irrep \( \lambda = (k) \) are all 1:

\[
\chi_{\lambda, (k)} = 1, \quad \forall \gamma \vdash k.
\]

Thus \( f^{(k)} = 1 \). For \( \gamma \) consisting of one cycle, \( \gamma = (k) \), the characters are

\[
\chi_{\lambda, (k)} = \begin{cases} 
(-1)^d, & \lambda = (k - d, 1^d), 0 \leq d \leq k \\
0, & \text{otherwise}.
\end{cases}
\]

In what follows, We now briefly consider the irreducible polynomial representations of the full linear group \( \text{GL}(d, \mathbb{C}) \) (note that both the full linear group \( \text{GL}(d, \mathbb{C}) \) and the unitary group \( \text{U}(d) \) embrace the same irreps). These representations get their name from the fact that their matrix elements are polynomials in the elements of the represented matrix. Just like the irreps of the symmetric group, the polynomial irreps of \( \text{GL}(d, \mathbb{C}) \) are labeled by Young diagrams. The conjugacy classes of \( \text{GL}(d, \mathbb{C}) \) consist of all matrices \( A \in \text{GL}(d, \mathbb{C}) \) have the same eigenvalues \( \lambda(a_1, \ldots, a_d) \) and thus can be labeled by these eigenvalues. The simple characters (known, in this context, as characteristics) are denoted \( \phi_\lambda(A) = \phi_\lambda(a_1, \ldots, a_d) \). According to a famous result by Schur, these characters are the Schur functions (polynomials) of the eigenvalues

\[
\phi_\lambda(a_1, \ldots, a_d) = s_\lambda(a_1, \ldots, a_d).
\]

### 6.7 Representations of \( S_k \) and \( \text{GL}(d, \mathbb{C}) \) on the tensor product space \( (\mathbb{C}^d)^{\otimes k} \)

Here we have denoted the dimension of the subspace \( Q_\lambda \) by \( t^\lambda(d) \), and the dimension of \( P_\lambda \) by \( f^\lambda \). The matrix \( Q_\lambda(A) \) is an irrep of \( A \in \text{GL}(d, \mathbb{C}) \) of degree \( t^\lambda(d) \), operating on \( Q_\lambda \). The matrix \( P_\lambda(\pi) \) is an irrep of \( \pi \in S_k \) of degree \( f^\lambda \), operating on \( P_\lambda \).

Taking traces yields the corresponding simple characters

\[
\begin{cases}
\text{Tr}(Q_\lambda(A)) = s_\lambda(a_1, \ldots, a_d), \\
\text{Tr}(P_\lambda(\pi)) = \chi_\lambda(\pi) = \chi_{\lambda, \gamma}(\pi),
\end{cases}
\]

where \( a_1, \ldots, a_d \) are the eigenvalues of \( A \). For the dimensions one finds

\[
\begin{cases}
t^\lambda(d) = \text{Tr}(Q_\lambda(1_d)) = s_\lambda(1^{\times d}), \\
f^\lambda = \text{Tr}(P_\lambda(\epsilon)) = \chi_\lambda(\epsilon),
\end{cases}
\]

i.e. \( t^\lambda(d) \) is the number of semi-standard Young tableaux \( \lambda \) of \( d \) objects, and \( f^\lambda \) is the number of standard Young tableaux \( \lambda \).

In accordance with these decompositions, the tensor space \( (\mathbb{C}^d)^{\otimes k} \) splits up into invariant subspaces. The subspaces \( Q_\lambda \otimes P_\lambda \) are invariant under all \( A^{\otimes k} \) and all \( P(\pi) \). They are further reducible into direct sums of \( f^\lambda \) subspaces of dimension \( t^\lambda(d) \), invariant under the transformations \( A^{\otimes k} \) but no longer invariant.
under permutations $P(\pi)$. These irreducible invariant subspaces are called the symmetry classes of the
tensor space. They are labeled by standard Young tableaux of shape $\lambda$.

We now consider the invariant subspaces $Q_\lambda \otimes P_\lambda$ corresponding to the Young diagrams $\lambda$. Their
dimension is $f^\lambda s_\lambda (1 \times d)$. We will denote the projectors on these subspaces by $C_\lambda$. They are the sum of the
Young projectors corresponding to the standard Young tableaux $\lambda$. We will consider the Young projectors
themselves in the next subsection. The projectors $C_\lambda$ form an orthogonal set and add up to the identity on
the full tensor space:

$$C_\lambda C_{\lambda'} = \delta_{\lambda \lambda'} C_{\lambda}, \quad \sum_{\lambda \vdash k} C_\lambda = 1_{(C^\lambda) \otimes k}, \quad \text{Tr} (C_\lambda) = f^\lambda s_\lambda (1 \times d). \quad (6.7)$$

Consider the conjugacy classes $\gamma$ of $S_k$ with cycle type $\gamma \vdash k$. We define the "class average" of all permu-
tation matrices with cycle type $\gamma$ as

$$C^\gamma := \frac{1}{n^\gamma} \sum_{\pi \in \gamma} P(\pi). \quad (6.8)$$

Note the distinction between the notations $C_\lambda$, where the subscript $\lambda$ labels an irrep, and $C^\gamma$, where the
superscript $\gamma$ labels a conjugacy class. Alternatively, we can write

$$C^\gamma = \frac{1}{k!} \sum_{\sigma \in S_k} P(\sigma \pi \sigma^{-1}). \quad (6.9)$$

The projectors $C_\lambda$ can be expressed in terms of the permutations $P(\pi)$, according to a general relation, as:

$$C_\lambda = \frac{f^\lambda}{k!} \sum_{\pi \in S_k} \chi_\lambda (\pi) P(\pi), \quad (6.10)$$

and in terms of $p^\gamma$ as:

$$C_\lambda = \frac{f^\lambda}{k!} \sum_{\gamma \vdash k} \frac{1}{z^\gamma} \chi_{\lambda, \gamma} C^\gamma. \quad (6.11)$$

Let $A$ be a matrix with eigenvalues $(a_1, \ldots, a_d)$. Taking the trace of one $\lambda$-term in the following expression:

$$A^\otimes k = \bigoplus_{\lambda \vdash k} Q_\lambda (A) \otimes 1_{P_\lambda}$$

immediately yields $C_\lambda A^\otimes k C_\lambda = Q_\lambda (A) \otimes 1_{P_\lambda}$, and

$$\text{Tr} \left( C_\lambda A^\otimes k \right) = f^\lambda s_\lambda (a_1, \ldots, a_d). \quad (6.12)$$

For $\pi \in \gamma \vdash k$, it is easy to see that

$$\text{Tr} \left( P(\pi) A^\otimes k \right) = \text{Tr} \left( C^\gamma A^\otimes k \right) = p^\gamma (a_1, \ldots, a_d). \quad (6.13)$$

Combining this with (6.11) gives the famous Frobenius formula, relating the characteristics of the full
linear group to the characters of the symmetric group

$$s_\lambda (a_1, \ldots, a_d) = \sum_{\gamma \vdash k} \frac{1}{z^\gamma} \chi_{\lambda, \gamma} p^\gamma (a_1, \ldots, a_d). \quad (6.14)$$
As this holds for any $A$, and thus for any set of values $a_i$ of whatever dimension, it yields the transition matrix from the $p_\gamma$ symmetric functions to the $S$-functions

$$s_\lambda = \sum_{\gamma \vdash k} \frac{1}{Z_\gamma} \chi_{\lambda, \gamma} p_\gamma.$$  \hfill (6.15)

Using the orthogonality relations of the characters, we find

$$C_\gamma = \sum_{\lambda \vdash k} \frac{1}{f_\lambda} \chi_{\lambda, \gamma} C_\lambda, \quad p_\gamma = \sum_{\lambda \vdash k} \frac{1}{f_\lambda} \chi_{\lambda, \gamma} s_\lambda.$$  \hfill (6.16)

### 6.8 Symmetric functions and representations of tensor products

A property of index permutation matrices that is both simple and powerful is that index permutation matrices over tensor products of Hilbert spaces are tensor products themselves. With a minor abuse of notation we identify $(H_A \otimes H_B)^{\otimes k}$ with $H_A^{\otimes k} \otimes H_B^{\otimes k}$ and write

$$P(\pi)(H_A \otimes H_B) = P(\pi)(H_A) \otimes P(\pi)(H_B).$$  \hfill (6.17)

Here $P(\pi)(H_A)$ acts on $H_A^{\otimes k}$, and $P(\pi)(H_B)$ acts on $H_B^{\otimes k}$. Clearly $P(\pi)(H_A \otimes H_B)$ acts on $(H_A \otimes H_B)^{\otimes k}$. As a short hand, the above equation can be written as

$$P^{AB}(\pi) = P^A(\pi) \otimes P^B(\pi).$$

This corresponds to considering symmetric functions of tensor products of variables. If $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$, then their tensor product, which is customarily denote $xy$ rather than $x \otimes y$, consists of all possible products $x_i y_j$. For power product sums one immediately sees

$$p_\gamma(xy) = p_\gamma(x)p_\gamma(y).$$  \hfill (6.18)

This yields for Schur functions

$$s_\lambda(xy) = \sum_{\mu, \nu \vdash k} g_{\lambda \mu \nu} s_\mu(x)s_\nu(y),$$  \hfill (6.19)

where $g_{\lambda \mu \nu}$ are the so-called **Kronecker coefficients**

$$g_{\lambda \mu \nu} := \frac{1}{k!} \sum_{\pi \in S_k} \chi_\lambda(\pi)\chi_\mu(\pi)\chi_\nu(\pi) = \sum_{\gamma \vdash k} \frac{1}{Z_\gamma} \chi_{\lambda, \gamma} \chi_{\mu, \gamma} \chi_{\nu, \gamma}.$$  \hfill (6.20)

One of the rare cases in which a closed formula can be given for the Kronecker coefficients, is $\lambda = (k)$. One finds

$$g_{(k) \mu \nu} = \delta_{\mu \nu} \quad \text{and} \quad s_{(k)}(xy) = \sum_{\lambda \vdash k} s_\lambda(x)s_\lambda(y).$$

A consequence of (6.19) is that for $X$ and $Y$, acting on $H_A$ and $H_B$, respectively,

$$\frac{1}{f_\lambda} \text{Tr} \left( C_\lambda(X \otimes Y)^{\otimes k} \right) = \sum_{\mu, \nu \vdash k} g_{\lambda \mu \nu} \left( \frac{1}{f_\mu} \text{Tr} \left( C_\mu X^{\otimes k} \right) \right) \left( \frac{1}{f_\nu} \text{Tr} \left( C_\nu Y^{\otimes k} \right) \right),$$  \hfill (6.21)

where $C_\lambda$ acts on $(H_A \otimes H_B)^{\otimes k}$, $C_\mu$ on $H_A^{\otimes k}$, and $C_\nu$ on $H_B^{\otimes k}$. In terms of the irreps of $GL(d, \mathbb{C})$ we have

$$Q_\lambda(X \otimes Y) \cong \bigoplus_{\mu, \nu \vdash k} g_{\lambda \mu \nu} Q_\mu(X) \otimes Q_\nu(Y).$$  \hfill (6.22)
where $g_{\lambda \mu \nu}$ counts the number of copies of $Q_\mu(X) \otimes Q_\nu(Y)$ in the direct sum.

Consider the computation about the partial trace of

$$\text{Tr}_B \left( C^A_{\lambda} (1 \otimes H^k_A \otimes C^B_v) \right),$$

where $C^A_{\lambda}$ acts on $(H_A \otimes H_B)^{\otimes k}$ and $C^B_v$ on $H^k_B$.

Since

$$C^A_{\lambda} = \frac{f^A}{k!} \sum_{\pi \in S_k} \chi_\lambda(\pi) P^A(\pi) = \frac{f^A}{k!} \sum_{\pi \in S_k} \chi_\lambda(\pi) P^A(\pi) \otimes P^B(\pi), \quad (6.23)$$

which, together with $C^B_v P^B(\pi) C^B_v = 1 \otimes P_v(\pi)$, implies that

$$\text{Tr}_B \left( C^A_{\lambda} (1 \otimes H^k_A \otimes C^B_v) \right) = \frac{f^A}{k!} \sum_{\pi \in S_k} \chi_\lambda(\pi) P^A(\pi) \text{Tr} \left( P^B(\pi) C^B_v \right) \quad (6.24)$$

$$= \frac{f^A}{k!} \sum_{\pi \in S_k} \chi_\lambda(\pi) P^A(\pi) S_\nu(1 \times d_\mu) \chi_v(\pi) \quad (6.25)$$

$$= \frac{f^A S_\nu(1 \times d_\mu)}{k!} \sum_{\pi \in S_k} \chi_\lambda(\pi) \chi_v(\pi) P^A(\pi) \quad (6.26)$$

$$= f^A S_\nu(1 \times d_\mu) \sum_{\gamma \in k} \frac{1}{\gamma} \chi_{\lambda, \gamma} \chi_{v, \gamma} C^A \quad (6.27)$$

$$= f^A S_\nu(1 \times d_\mu) \sum_{\mu \in \gamma} \frac{1}{f^\mu} \left( \sum_{\gamma \in k} \frac{1}{\gamma} \chi_{\lambda, \gamma} \chi_{v, \gamma} \chi_{\mu, \gamma} \right) C^A \quad (6.28)$$

$$= f^A S_\nu(1 \times d_\mu) \sum_{\mu \in \gamma} \frac{g_{\lambda \mu \nu}}{f^\mu} C^A. \quad (6.29)$$

Therefore we have

$$\text{Tr}_B \left( C^A_{\lambda} (1 \otimes H^k_A \otimes C^B_v) \right) = f^A S_\nu(1 \times d_\mu) \sum_{\mu \in \gamma} \frac{g_{\lambda \mu \nu}}{f^\mu} C^A. \quad (6.30)$$

This fact implies that

$$\text{Tr}_B \left( C^A_{\lambda} \right) = \sum_{\nu \in k} \text{Tr}_B \left( C^A_{\lambda} (1 \otimes H^k_A \otimes C^B_v) \right) = \sum_{\nu \in k} f^A S_\nu(1 \times d_\mu) \sum_{\mu \in \gamma} \frac{g_{\lambda \mu \nu}}{f^\mu} C^A. \quad (6.31)$$

In particular, for $\lambda = (k)$,

$$\text{Tr}_B \left( C^A_{(k)} \right) = \sum_{\mu \in \gamma} \frac{S_\mu(1 \times d_\mu)}{f^\mu} C^A. \quad (6.32)$$

In addition, we also have

$$\text{Tr}_B \left( C^A_{\lambda} (C^A_\mu \otimes C^B_v) \right) = f^A S_\nu(1 \times d_\mu) \frac{g_{\lambda \mu \nu}}{f^\mu} C^A. \quad (6.33)$$

43
implying
$$\text{Tr} \left( C^A_{\lambda} (C^A_{\mu} \otimes C^B_{\nu}) \right) = f^A_{\lambda \mu} s_{\mu} (1 \times d_A) s_{\nu} (1 \times d_B).$$
(6.35)

Summing over all $\lambda \vdash k$ gives rise to
$$\left( f^\mu s_{\mu} (1 \times d_A) \right) \left( f^\nu s_{\nu} (1 \times d_B) \right) = \text{Tr} \left( C^A_{\mu} \otimes C^B_{\nu} \right) = \sum_{\lambda \vdash k} \text{Tr} \left( C^A_{\lambda} (C^A_{\mu} \otimes C^B_{\nu}) \right)$$
$$= \left( \sum_{\lambda \vdash k} f^A_{\lambda \mu} \right) s_{\mu} (1 \times d_A) s_{\nu} (1 \times d_B),$$
(6.36)
implying that $f^\mu f^\nu = \sum_{\lambda \vdash k} f^A_{\lambda \mu \nu}$.

7 Weyl integration formula

This section is written based on Bump’s book \cite{24}.

7.1 Haar measure

If $G$ is a locally compact group, there is, up to a constant multiple, a unique regular Borel measure $\mu_L$ that is invariant under left translation. Here left translation invariance means that $\mu(M) = \mu(g M)$ for all measurable sets $M$. Regularity means that
$$\mu(M) = \inf \{ \mu(O) : M \subseteq O, O \text{ open} \}$$
(7.1)
$$= \sup \{ \mu(C) : M \supseteq C, C \text{ compact} \}$$
(7.2)

Such a measure is called a left Haar measure. It has the properties that any compact set has finite measure and any nonempty open set has positive measure.

I will not prove the existence and uniqueness of the Haar measure, which has already established. Left-invariance of the measure amounts to left-invariance of the corresponding integral,
$$\int_G f(g' g) d\mu_L(g) = \int_G f(g) d\mu_L(g),$$
(7.3)
for any Haar integral function $g$ on $G$.

There is also a right-invariant measure $\mu_R$, unique up to constant multiple, called a right Haar measure. Left and right Haar measures may or may not coincide. For example, if
$$G = \left\{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} : x, y \in \mathbb{R}, y > 0 \right\},$$
then it is easy to see that the left- and right-invariant measures are, respectively,
$$d\mu_L = y^{-2} dx dy, \quad d\mu_R = y^{-1} dx dy.$$ They are not the same. However, there are many cases where they do coincide, and if the left Haar measure is also right-invariant, we call $G$ unimodular.

44
Conjugation is an automorphism of $G$, and so it takes a left Haar measure to another left Haar measure, which must be a constant multiple of the first. Indeed,

$$
\int_G f(x^{-1}gx)d\mu_L(g) = \int_G f(g)d\mu_L(xgx^{-1}) = \int_G f(g)d\mu_L(gx^{-1}).
$$

(7.4)

Clearly $d\mu_L^\#(g) := d\mu_L(gx^{-1})$ defines a new left Haar measure. By the uniqueness of left Haar measure, up to constant multiple, $d\mu_L^\#(g) = \delta(x)d\mu_L(g)$, which implies that

$$
\int_G f(x^{-1}gx)d\mu_L(g) = \delta(x)\int_G f(g)d\mu_L(g).
$$

(7.5)

**Proposition 7.1.** The function $\delta : G \to \mathbb{R}_+^\times$ is a continuous homomorphism. The measure $\delta(g)d\mu_L(g)$ is a right-invariant, denoted $\mu_R(g)$.

**Proof.** Conjugation by first $x_1$ and then $x_2$ is the same as conjugation by $x_1x_2$ in one step. This can be seen from the following reasoning: Let $x = x_1x_2$ in (7.5), we have

$$
\int_G f(x_2^{-1}x_1^{-1}gx_1x_2)d\mu_L(g) = \delta(x_1x_2)\int_G f(g)d\mu_L(g)
$$

(7.6)

and

$$
\int_G f(x_2^{-1}x_1^{-1}gx_1x_2)d\mu_L(g) = \int_G f_2(x_2^{-1}x_1^{-1}gx_1)d\mu_L(g) = \delta(x_1)\int_G f_2(g)d\mu_L(g)
$$

(7.7)

$$
= \delta(x_1)\int_G f(x_2^{-1}gx_2)d\mu_L(g) = \delta(x_1)\delta(x_2)\int_G f(g)d\mu_L(g)
$$

(7.8)

where $f_2(g) := f(x_2^{-1}gx_2)$. That is

$$
\delta(x_1x_2) = \delta(x_1)\delta(x_2).
$$

Replace $f$ by $f\delta$ in the following

$$
\int_G f(gx)d\mu_L(g) = \delta(x)\int_G f(g)d\mu_L(g)
$$

(7.9)

we get

$$
\int_G f(gx)\delta(gx)d\mu_L(g) = \delta(x)\int_G f(g)\delta(g)d\mu_L(g),
$$

(7.10)

which gives rise to

$$
\int_G f(gx)\delta(g)d\mu_L(g) = \int_G f(g)\delta(g)d\mu_L(g),
$$

(7.11)

that is

$$
\int_G f(gx)d\mu_R(g) = \int_G f(g)d\mu_R(g),
$$

(7.12)

completing the proof.

**Proposition 7.2.** If $G$ is compact, then $G$ is unimodular and $\mu_L(G) < \infty$.

**Proof.** Since $\delta$ is a homomorphism, the image of $\delta$ is a subgroup of $\mathbb{R}_+^\times$. Since $G$ is compact, $\delta(G)$ is also compact, and the only compact subgroup of $\mathbb{R}_+^\times$ is just $\{1\}$. Thus $\delta$ is trivial, so a left Haar measure is right-invariant. We have mentioned as assumed fact that the Haar volume of any compact subset of a locally compact group is finite, so if $G$ is finite, its Haar volume is finite.
If \( G \) is compact, then it is natural to normalize the Haar measure so that \( G \) has volume 1. To simplify our notation, we will denote \( \int_G f(g) d\mu_L(g) \) by \( \int_G f(g) dg \).

**Proposition 7.3.** If \( G \) is unimodular, then the map \( g \to g^{-1} \) is an isometry.

**Proof.** It is easy to see that \( g \to g^{-1} \) turns a left Haar measure into a right Haar measure. If left and right Haar measures agree, then \( g \to g^{-1} \) multiplies the left Haar measure by a positive constant, which must be 1 since the map has order 2. \( \square \)

### 7.2 Weyl integration formula

Let \( G \) be a compact, connected Lie group, and let \( T \) be a maximal torus. It is already known that every conjugacy class meets \( T \). Thus we should be able to compute the Haar integral over \( G \). The following formula that allows this, the Weyl integration Formula, is therefore fundamental in representation theory and in other areas, such as random matrix theory.

\[
\int_G f(g) dg = \frac{1}{|W(G)|} \int_T \left( \int_{G/T} f(gtg^{-1}) |\text{Det}(1 - \text{Ad}(t))| d(gT) \right) dt. \tag{7.13}
\]

If \( G \) is a locally compact group and \( H \) a closed subgroup, then the quotient space \( G/H \) consisting of all cosets \( gH \) with \( g \in G \), given the quotient topology, is a locally compact Hausdorff space.

If \( X \) is a locally compact Hausdorff space let \( C_c(X) \) be the space of continuous, compactly supported functions on \( X \). If \( X \) is a locally compact Hausdorff space, a linear functional \( I \) on \( C_c(X) \) is called **positive** if \( I(f) \geq 0 \) if \( f \) is nonnegative. According to the Riesz representation theorem, every such \( I \) is of the form \( I(f) = \int_X f d\mu \) (7.14) for some regular Borel measure \( d\mu \).

**Proposition 7.4.** Let \( G \) be a locally compact group, and let \( H \) be a compact subgroup. Let \( d\mu_G \) and \( d\mu_H \) be left Haar measures on \( G \) and \( H \), respectively. Then there exists a regular Borel measure \( d\mu_{G/H} \) on \( G/H \) which is invariant under the action of \( G \) by left translation. The measure \( d\mu_{G/H} \) may be normalized so that, for \( f \in C_c(G) \), we have

\[
\int_{G/H} \left( \int_H f(gh) d\mu_H(h) \right) d\mu_{G/H}(gH). \tag{7.15}
\]

Here the function \( g \to \int_H f(gh) d\mu_H(h) \) is constant on the cosets \( gH \), and we are therefore identifying it with a function on \( G/H \).

**Proof.** We may choose the normalization of \( d\mu_H \) so that \( H \) has total volume 1. We define a map \( \Lambda : C_c(G) \to C_c(G/H) \) by

\[
(\Lambda f)(g) = \int_H f(gh) d\mu_H(h). \tag{7.16}
\]

Note that \( \Lambda f \) is a function on \( G \) which is right invariant under translation by elements of \( H \), so it may be regarded as a function on \( G/H \). Since \( H \) is compact, \( \Lambda f \) is compactly supported. If \( \phi \in C_c(G/H) \), regarding \( \phi \) as a function on \( G \), we have \( \Lambda \phi = \phi \) because

\[
(\Lambda \phi)(g) = \int_H \phi(gh) d\mu_H(h) = \int_H \phi(g) d\mu_H(h) = \phi(g). \tag{7.17}
\]
This shows that \( \Lambda \) is surjective. We may therefore define a linear functional \( I \) on \( C_c(G/H) \) by

\[
I(\Lambda f) = \int_G f(g) d\mu_G(g), \quad f \in C_c(G)
\]  

(7.18)

provided we check that this is well-defined. We must show that if \( \Lambda f = 0 \), then

\[
I(\Lambda f) = 0,
\]

(7.19)
i.e. \( \int_G f(g) d\mu_G(g) = 0 \). We note that the function \( (g, h) \mapsto f(gh) \) is compactly supported and continuous on \( G \times H \), so if \( \Lambda f = 0 \), we may use Fubini’s theorem to write

\[
0 = \int_G (\Lambda f)(g) d\mu_G(g) = \int_G \left( \int_H f(gh) d\mu_H(h) \right) d\mu_G(g)
\]

(7.20)

\[
= \int_H \left( \int_G f(gh) d\mu_G(g) \right) d\mu_H(h).
\]

(7.21)

In the inner integral on the right-hand side we make the variable change \( g \mapsto gh^{-1} \). Recalling that \( d\mu_G(g) \) is left Haar measure, this produces a factor of \( \delta_G(h) \), where \( \delta_G(h) \) is the modular homomorphism. Thus

\[
0 = \int_H \delta_G(h) \left( \int_G f(g) d\mu_G(g) \right) d\mu_H(h).
\]

Now the group \( H \) is compact, so its image under \( \delta_G \) is a compact subgroup of \( \mathbb{R}_+^n \), which must be \( \{1\} \). Thus \( \delta_G(h) = 1 \) for all \( h \in H \), and we obtain \( \int_G f(g) d\mu_G(g) = 0 \), justifying the definition of the functional \( I \). The existence of the measure on \( G/H \) now follows from the Riesz representation theorem.

Example 7.5. Suppose that \( G = U(n) \). A maximal torus is

\[
T = \{ \text{diag}(t_1, \ldots, t_n) : |t_1| = \cdots = |t_n| = 1 \}.
\]

Its normalizer \( N(T) \) consists of all monomial matrices (matrices with a single nonzero entry in each row and column) so that the quotient \( N(T)/T \cong S_n \).

Proposition 7.6. Let \( T \) be a maximal torus in the compact connected Lie group \( G \), and let \( t, g \) be the Lie algebras of \( T \) and \( G \), respectively.

(i) Any vector in \( g \) fixed by \( \text{Ad}(T) \) is in \( t \).

(ii) We have \( g = t \oplus t^\perp \), where \( t^\perp \) is invariant under \( \text{Ad}(T) \). Under the restriction of \( \text{Ad} \) to \( T \), \( t^\perp \) decomposes into a direct sum of two-dimensional real irreps of \( T \).

Let \( W(G) \) be the Weyl group of \( G \). The Weyl group acts on \( T \) by conjugation. Indeed, the elements of the Weyl group are cosets \( w = nT \) for \( n \in N(T) \). If \( t \in T \), the elements \( ntn^{-1} \) depends only on \( w \) so by abuse of notation we denote it \( wtw^{-1} \).

Theorem 7.7. (i) Two elements of \( T \) are conjugate in \( G \) if and only if they are conjugate in \( N(T) \).

(ii) The inclusion \( T \to G \) induces a bijection between the orbits of \( W(G) \) on \( T \) and the conjugacy classes of \( G \).
Proof. Suppose that \( t, u \in T \) are conjugate in \( G \), say \( gtg^{-1} = u \). Let \( H \) be the connected component of the identity in the centralizer of \( u \) in \( G \). It is a closed Lie subgroup of \( G \). Both \( T \) and \( gTg^{-1} \) are contained in \( H \) since they are connected commutative groups containing \( u \). As they are maximal tori in \( G \), they are maximal tori in \( H \), and so they are conjugate in the compact connected group \( H \). If \( h \in H \) such that \( hTh^{-1} = gTg^{-1} \), then \( w = h^{-1}g \in N(T) \). Since \( wtw^{-1} = h^{-1}uh = u \), we see that \( t \) and \( u \) are conjugate in \( N(T) \).

Since \( G \) is the union of the conjugates of \( T \), (ii) is a restatement of (i). \( \square \)

**Proposition 7.8.** The centralizer \( C(T) = T \).

**Proposition 7.9.** There exists a dense open set \( \Omega \) of \( T \) such that the \(|W(G)|\) elements \( wtw^{-1}(w \in W(G)) \) are all distinct for \( t \in \Omega \).

**Proof.** If \( w \in W(G) \), let

\[
\Omega_w = \{ t \in T : wtw^{-1} \neq t \}.
\]

It is an open subset of \( T \) since its complement is evidently closed. If \( w \neq 1 \) and \( t \) is a generator of \( T \), then \( t \in \Omega_w \) because otherwise if \( n \in N(T) \) represents \( w \), then \( n \in C(t) = C(T) \), so \( n \in T \). This is a contradiction since \( w \neq 1 \). By Kronecker Theorem, it follows that \( \Omega_w \) is a dense open set. The finite intersection \( \Omega = \cap_{w \neq 1} \Omega_w \) thus fits our requirements. \( \square \)

**Theorem 7.10** (Weyl). If \( f \) is a class function, and if \( dg \) and \( dt \) are Haar measures on \( G \) and \( T \) (normalized so that \( G \) and \( T \) have volume 1), then

\[
\int_G f(g)dg = \frac{1}{|W(G)|} \int_T f(t) \text{Det} \left( \left[ \text{Ad}(t^{-1}) - 1_{t^\perp} \right] |_{t^\perp} \right) dt.
\]

(7.22)

**Proof.** Let \( X = G/T \). We give \( X \) the measure \( d_X \) invariant under left translation by \( G \) such that \( X \) has volume 1. Consider the map

\[
\phi : X \times T \to G, \quad \phi(xT, t) = xtx^{-1}.
\]

Both \( X \times T \) and \( G \) are orientable manifolds of the same dimension. Of course, \( G \) and \( T \) both are given the Haar measures such that \( G \) and \( T \) have volume 1.

We choose volume elements on the Lie algebras \( g \) and \( t \) of \( G \) and \( T \), respectively, so that the Jacobians of the exponential maps \( g \to G \) and \( t \to T \) at the identity are \( 1 \).

We compute the Jacobian \( J\phi \) of \( \phi \). Parameterize a neighborhood of \( xT \) in \( X \) by a chart based on a neighborhood of the origin in \( t^\perp \). This chart is the map

\[
t^\perp \ni A \mapsto xe^AT.
\]

We also make use of the exponential map to parameterize a neighborhood of \( t \in T \). This is the chart \( t \ni B \mapsto te^B \). We therefore have the chart near the point \((xT, t)\) in \( X \times T \) mapping

\[
t^\perp \times t \ni (A, B) \mapsto (xe^AT, te^B) \in X \times T
\]

and, in these coordinates, \( \phi \) is the map

\[
(A, B) \mapsto xe^Ate^Be^{-A}x^{-1}.
\]

48
To compute the Jacobian of this map, we translate on the left by $t^{-1}x^{-1}$ and on the right by $x$. There is no harm in this because these maps are Haar isometries. We are reduced to computing the Jacobian of the map

$$ (A, B) \mapsto t^{-1}e^{A}t_{e}B e^{-A} = e^{\text{Ad}(t^{-1}A)}e_{B}e^{-A}. $$

Identifying the tangent space of the real vector space $\mathfrak{t} \times \mathfrak{t}$ with itself (that is, with $\mathfrak{g} = \mathfrak{t} \perp \mathfrak{t}$), the differential of this map is

$$ A \oplus B \mapsto \left( \text{Ad}(t^{-1}) - 1_{\mathfrak{t} \perp} \right) A \oplus B. $$

The Jacobian is the determinant of the differential, so

$$ (J\phi)(xT, t) = \text{Det} \left( \left[ \text{Ad}(t^{-1}) - 1_{\mathfrak{t} \perp} \right] |_{\mathfrak{t} \perp} \right). \tag{7.23} $$

The map $\phi : \mathcal{X} \times T \to G$ is a $|W(G)|$-fold cover over a dense open set and so, for any function $f$ on $G$, we have

$$ \int_{G} f(g) dg = \frac{1}{|W(G)|} \int_{\mathcal{X} \times T} f(\phi(xT, t)) J(\phi(xT, t)) d\mathcal{X} \times dt. \tag{7.24} $$

The integrand $f(\phi(xT, t)) J(\phi(xT, t)) = f(t) \text{Det} \left( \left[ \text{Ad}(t^{-1}) - 1_{\mathfrak{t} \perp} \right] |_{\mathfrak{t} \perp} \right)$ is independent of $x$ since $f$ is a class function, and the result follows.

\textbf{Remark 7.11.} Let $G$ be a Lie group and $\mathfrak{g}$ its Lie algebra. Identify both $T_{0}\mathfrak{g}$ and $T_{e}G$ with $\mathfrak{g}$. Then, $(d\exp)_{0} : T_{0}\mathfrak{g} \to T_{e}G$ is the identity map. Indeed,

$$ (d\exp)_{0}(A) = \frac{d}{dt} \bigg|_{t=0} \exp(0 + tA) = A. $$

That is $(d\exp)_{0}$ is the identity map over $\mathfrak{g}$.

\textbf{Proposition 7.12.} Let $G = U(n)$, and let $T$ be the diagonal torus. Writing

$$ t = \text{diag}(t_{1}, \ldots, t_{n}) \in T, $$

and letting $\int_{T} dt$ be the Haar measure on $T$ normalized so that its volume is 1, we have

$$ \int_{G} f(g) dg = \frac{1}{n!} \int_{T} f(t) \prod_{i<j} |t_{i} - t_{j}|^{2} dt. \tag{7.25} $$

\textbf{Proof.} We need to check that

$$ \text{Det} \left( \left[ \text{Ad}(t^{-1}) - 1_{\mathfrak{t} \perp} \right] |_{\mathfrak{t} \perp} \right) = \prod_{i<j} |t_{i} - t_{j}|^{2}. $$

To compute this determinant, we may as well consider the linear transformation induced by $\text{Ad}(t^{-1}) - 1_{\mathfrak{t} \perp}$ on the complexified vector space $\mathbb{C} \otimes \mathfrak{t} \perp$. We may identify $\mathbb{C} \otimes u(n)$ with $\text{gl}(n, \mathbb{C}) = M_{n}(\mathbb{C})$. We recall that $\mathbb{C} \otimes \mathfrak{t} \perp$ is spanned by the $T$-eigenspaces in $\mathbb{C} \otimes u(n)$ corresponding to nontrivial characters of $T$. There are spanned by the elementary matrices $E_{ij}$ with a 1 in the $(i,j)$-th position and zeros elsewhere, where $1 \leq i, j \leq n$ and $i \neq j$. The eigenvalue of $t$ on $E_{ij}$ is $t_{i}t_{j}^{-1}$. Hence

$$ \text{Det} \left( \left[ \text{Ad}(t^{-1}) - 1_{\mathfrak{t} \perp} \right] |_{\mathfrak{t} \perp} \right) = \prod_{i,j} (t_{i}t_{j}^{-1} - 1) = \prod_{i<j} (t_{i}t_{j}^{-1} - 1) (t_{j}t_{i}^{-1} - 1). \tag{7.26} $$

49
Since $|t_i| = |t_j| = 1$, we have

$$(t_i t_j^{-1} - 1)(t_j t_i^{-1} - 1) = (t_i - t_j)(t_i^{-1} - t_j) = |t_i - t_j|^2.$$  

This completes the proof. \(\square\)

**Remark 7.13.** Let $G = U(1) = S^1$, $\rho_n : S^1 \to \text{GL}(1, \mathbb{C})$ be given by $\rho_n(e^{\sqrt{-1} \theta}) = e^{\sqrt{-1} n \theta}$. Then $dg = d\theta/2\pi$ and

$$\frac{1}{2\pi} \int_0^{2\pi} e^{\sqrt{-1} m \theta} e^{-\sqrt{-1} n \theta} = \delta_{mn}.$$  

**Corollary 7.14.** If $f$ is a class function over $U(n)$, then

$$\int_{U(n)} f(u) du = \frac{1}{n!} \int_{T^n} f(D(\theta)) J(\theta) dD(\theta) = \frac{1}{(2\pi)^n n!} \int_0^{2\pi} \cdots \int_0^{2\pi} f(D(\theta)) J(\theta) d\theta_1 \cdots d\theta_n, \tag{7.27}$$

where

$$D(\theta) := \text{diag} \left( e^{\sqrt{-1} \theta_1}, \ldots, e^{\sqrt{-1} \theta_n} \right) \text{ and } J(\theta) := \prod_{i < j} |e^{\sqrt{-1} \theta_i} - e^{\sqrt{-1} \theta_j}|^2.$$  

**Remark 7.15.** We know that for one-dimensional torus, the normalized Haar measure is defined as $du := d\theta/2\pi$ over $U(1)$. This implies that for $n$-dimensional torus of $U(n)$:

$$T^n = U(1) \times \cdots \times U(1),$$

the normalized Haar measure is given by the product measure of $n$ one-dimensional measures of $U(1)$. Thus for $D(\theta) \in T^n$, described by $D(\theta) = u_1(\theta) \times \cdots \times u_n(\theta)$ with $du_j(\theta) = d\theta_j/2\pi$, the normalized Haar measure is defined as

$$dD(\theta) := du_1 \times \cdots \times du_n = \frac{d\theta_1}{2\pi} \times \cdots \times \frac{d\theta_n}{2\pi} = \frac{1}{(2\pi)^n} d\theta_1 \cdots d\theta_n = \frac{1}{(2\pi)^n} d\theta,$$  

where $d\theta := d\theta_1 \cdots d\theta_n$.

In Corollary 7.14 assume that $f \equiv 1$, then we have

$$1 = \int_{U(n)} f(u) du = \frac{1}{n!} \int_{T^n} f(\theta) dD(\theta) = \frac{1}{(2\pi)^n n!} \int_0^{2\pi} \cdots \int_0^{2\pi} f(\theta) d\theta_1 \cdots d\theta_n, \tag{7.29}$$

implying

$$n! = \int_{T^n} f(\theta) dD(\theta) = \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} f(\theta) d\theta. \tag{7.30}$$

In what follows, we give a check on the identity in (7.31). Here is another way of writing $J(\theta)$, which is useful. Set $e^{\sqrt{-1} \theta_j} = \zeta_j$. Then

$$J(\theta) = V(\zeta) V(\zeta), \tag{7.32}$$
where

\[ V(\zeta) := V(\zeta_1, \ldots, \zeta_n) = \prod_{1 \leq i < j \leq n} (\zeta_j - \zeta_i). \]

Now \( V(\zeta) \) is a Vandermonde determinant:

\[
V(\zeta) = \text{Det} \begin{pmatrix}
1 & 1 & \cdots & 1 \\
\zeta_1 & \zeta_2 & \cdots & \zeta_n \\
\vdots & \vdots & \ddots & \vdots \\
\zeta_1^{n-1} & \zeta_2^{n-1} & \cdots & \zeta_n^{n-1}
\end{pmatrix}.
\]

Define \( a_{ij} := \zeta_j^{i-1} \) \((i, j \in \{1, \ldots, n\})\). We can form a \( n \times n \) matrix \( A = [a_{ij}] \) in terms of \( a_{ij} \). Apparently, \( V(\zeta) = \text{Det}(A) \). According to the definition of determinant, the expansion of a determinant can be given by

\[
\text{Det}(A) = \sum_{\pi \in S_n} \text{sign}(\pi) a_{\pi(1)1} \cdots a_{\pi(n)n}. \tag{7.33}
\]

Now that \( a_{\pi(i)j} = \zeta_j^{\pi(i)-1} \). We thus obtain that

\[
V(\zeta) = \sum_{\pi \in S_n} \text{sign}(\pi) \zeta_1^{\pi(1)-1} \zeta_2^{\pi(2)-1} \cdots \zeta_n^{\pi(n)-1}. \tag{7.34}
\]

Now \( \zeta_j = \zeta_j^{-1} \) for \( \zeta_j \in U(1) \), so

\[
J(\theta) = \sum_{(\pi, \sigma) \in S_n \times S_n} \text{sign}(\pi) \text{sign}(\sigma) \zeta_1^{\pi(1)-\sigma(1)} \zeta_2^{\pi(2)-\sigma(2)} \cdots \zeta_n^{\pi(n)-\sigma(n)}. \tag{7.35}
\]

Hence

\[
\int_{T^n} J(\theta) dD(\theta) = \sum_{(\pi, \sigma) \in S_n \times S_n} \text{sign}(\pi) \text{sign}(\sigma) \int_{T^n} dD(\theta) \left( \zeta_1^{\pi(1)-\sigma(1)} \zeta_2^{\pi(2)-\sigma(2)} \cdots \zeta_n^{\pi(n)-\sigma(n)} \right)
\]

\[
= \sum_{(\pi, \sigma) \in S_n \times S_n} \text{sign}(\pi) \text{sign}(\sigma) \left( \frac{1}{2\pi} \int_0^{2\pi} e^{\sqrt{-1}(\pi(1)-\sigma(1))\theta} d\theta_1 \right) \times \cdots \times \left( \frac{1}{2\pi} \int_0^{2\pi} e^{\sqrt{-1}(\pi(n)-\sigma(n))\theta} d\theta_n \right)
\]

\[
= \sum_{(\pi, \sigma) \in S_n \times S_n} \text{sign}(\pi) \text{sign}(\sigma) \delta_{\pi(1)\sigma(1)} \cdots \delta_{\pi(n)\sigma(n)} = \sum_{\pi \in S_n} 1 = n!,
\]

where we used the fact that

\[
\frac{1}{2\pi} \int_0^{2\pi} e^{\sqrt{-1}(\pi(k)-\sigma(k))\theta} d\theta_k = \delta_{\pi(k)\sigma(k)}.
\]

Denote \( \theta = (\theta_1, \ldots, \theta_n) \) and define functionals \( \alpha_{ij}(\theta) = \theta_i - \theta_j \). We mention another way of writing \( J(\theta) \), i.e.

\[
J(\theta) = A(\theta) \overline{A(\theta)}, \tag{7.36}
\]

where

\[
A(\theta) := \prod_{i<j} \left( 1 - e^{-\sqrt{-1}\alpha_{ij}(\theta)} \right).
\]
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