NON-EXISTENCE OF CERTAIN CM ABELIAN VARIETIES WITH PRIME POWER TORSION

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Abstract. In this paper, we study a conjecture of Rasmussen and Tamagawa, on the finiteness of the set of isomorphism classes of abelian varieties with constrained prime power torsion. Our result is related with abelian varieties which have complex multiplication over their fields of definition.

1. Introduction. Let \( K \) be a finite extension of \( \mathbb{Q} \), \( \bar{K} \) an algebraic closure of \( K \) and \( G_K = \text{Gal}(\bar{K}/K) \) the absolute Galois group of \( K \). For a prime number \( l \), we denote by \( K(\mu_l) \) the field generated over \( K \) by the \( l \)-th roots of unity. If \( A \) is an abelian variety over \( K \), denote by \( K(A[l]) \) the field generated by \( K \) and the coordinates of all \( l \)-torsion points of \( A \). We denote by \( \mathcal{A}(K, g, l) \) the set of \( K \)-isomorphism classes of \( g \)-dimensional abelian varieties \( A \) over \( K \) which satisfy the following conditions.

\begin{itemize}
  \item[(RT_1)] \( K(A[l]) \) is an \( l \)-extension of \( K(\mu_l) \), that is, \( [K(A[l]) : K(\mu_l)] = l^a \) for some integer \( a \geq 0 \).
  \item[(RT_{red})] The abelian variety \( A \) has good reduction away from \( l \) over \( K \).
\end{itemize}

It follows from the condition (RT_{red}) and Faltings’ result on the Shafarevich Conjecture that \( \mathcal{A}(K, g, l) \) is a finite set. Rasmussen and Tamagawa suggested that such finiteness should hold if we take the union of these sets for \( l \) varying over all primes.

**Conjecture 1.1 ([18, Conjecture 1]).** The set \( \mathcal{A}(K, g) := \{([A], l) ; [A] \in \mathcal{A}(K, g, l), l : \text{prime number} \} \) is finite, that is, the set \( \mathcal{A}(K, g, l) \) is empty for any prime number \( l \) large enough.

This conjecture is proved only in a few cases. For example, Conjecture 1.1 in the case where \( K \) is the rational number field or a certain quadratic field, with \( g = 1 \) is proved by Rasmussen and Tamagawa in [18]. Their proof is based on results on \( K \)-rational points on modular curves in [14] and [15]. Arguments for algebraic points on Shimura curves (cf. [2], [3]) also give results on Conjecture 1.1 for QM-abelian surfaces over certain quadratic fields \( K \).

In this paper, we prove Conjecture 1.1 for abelian varieties in \( \mathcal{A}(K, g, l) \) which satisfy the condition that the representations associated with their \( l \)-adic Tate modules are abelian; we

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are interested in the subset $\mathcal{A}(K, g, l)_{ab}$ of $\mathcal{A}(K, g, l)$ consisting of $K$-isomorphism classes of $g$-dimensional abelian varieties $A$ over $K$ which satisfy the additional condition:

$\text{(RT}_{ab})$ The representation $\rho_{A, l}: G_K \to GL(T_l(A))$ associated with the $l$-adic Tate module $T_l(A)$ of $A$ has an abelian image.

In fact, we prove a more general result as follows. Denote by $\mathcal{A}'(K, g, l)_{ab}$ the set of $K$-isomorphism classes of $g$-dimensional abelian varieties $A$ over $K$ which satisfy $\text{(RT}_{ab})$ and the condition:

$\text{(RT}_{l})'$ For some finite extension $L$ of $K$ which is unramified at all places of $K$ above $l$, $L(A[l])$ is an $l$-extension of $L(\mu_l)$.

Remark that, differently from elements in $\mathcal{A}(K, g, l)_{ab}$, the reduction hypothesis $\text{(RT}_{red})$ is not imposed on elements in $\mathcal{A}'(K, g, l)_{ab}$. Our main result in this paper is the following theorem.

**Theorem 1.2.** The set $\mathcal{A}'(K, g, l)_{ab}$ is empty for any prime number $l$ large enough.

Since $\mathcal{A}(K, g, l)_{ab} \subset \mathcal{A}'(K, g, l)_{ab}$, we have the following corollary.

**Corollary 1.3.** The set $\mathcal{A}(K, g, l)_{ab}$ is empty for any prime number $l$ large enough.

Keys to the proof of Theorem 1.2 are to construct a compatible system of Galois representations which has a strong condition, Faltings’ trick in his proof of the Shafarevich Conjecture and Raynaud’s criterion of semistable reduction. We hope that this study will be a first step to solve Conjecture 1.1 for abelian varieties with complex multiplication. If an abelian variety $A$ over $K$ has complex multiplication over $K$ (in the sense of [20, Section 4]), then it is well-known that $\rho_{A, l}$ is abelian (cf. [loc. cit., Section 4, Corollary 2]). Thus we obtain

**Corollary 1.4.** The set of $K$-isomorphism classes of abelian varieties in $\mathcal{A}(K, g, l)$ which have complex multiplication over $K$ is empty for any prime number $l$ large enough.

We want to replace “an abelian image” in the statement of $\text{(RT}_{ab})$ with “a potential abelian image”. If Theorem 1.2 with this replacement is proved, then Conjecture 1.1 holds for CM abelian varieties, that is, if we denote by $\mathcal{A}(K, g, l)_{CM}$ the set of $K$-isomorphism classes of abelian varieties in $\mathcal{A}(K, g, l)$ which have complex multiplication over $\bar{K}$, then the set

$$\mathcal{A}(K, g)_{CM} := \{([A], l); [A] \in \mathcal{A}(K, g, l)_{CM}, \ l: \text{prime number}\}$$

is finite.

The paper proceeds as follows. Section 2 is devoted to a study of compatible systems. In Section 3, we recall some facts about Conjecture 1.1. Finally we prove our main theorem in Section 4.

**2. Compatible systems.** In this section, we use the notation given in Introduction. It is important for the proof of Theorem 1.2 to find conditions for the compatible systems to be of a simple form.

**2.1. Basic notions.** Let $E$ be a finite extension of $\mathbb{Q}$. For a finite place $\lambda$ of $E$, we denote by $l_\lambda$ the prime number below $\lambda$, $E_\lambda$ the completion of $E$ at $\lambda$ and $F_\lambda$ the residue field
of \(\lambda\). We denote by \(K_v\) the completion of \(K\) at a finite place \(v\) of \(K\). Let \(S\) be a finite set of finite places of \(K\) and \(T\) a finite set of finite places of \(E\). Put \(S_l = S \cup \{\text{places of } K \text{ above } l\}\).

A representation \(\rho : G_K \to GL_n(E_\lambda)\) is said to be \(E\)-rational (resp. \(E\)-integral) with ramification set \(S\) if \(\rho\) is unramified outside \(S_\lambda\) and the characteristic polynomial \(\det(XI_n - \rho(\text{Fr}_v))\) of \(\text{Fr}_v\) has coefficients in \(E\) (resp. the integer ring of \(E\)) for each finite place \(v \notin S_\lambda\) of \(K\), where \(\text{Fr}_v\) is an arithmetic Frobenius of \(v\) and \(I_n\) is the identity matrix of size \(n\).

Now we give definitions of compatible systems of \(\lambda\)-adic (resp. mod \(\lambda\)) representations, which mainly follow those in [10] and [11].

**DEFINITION 2.1.** (1) An \(E\)-rational (resp. \(E\)-integral) strictly compatible system \((\rho_\lambda)_{\lambda}\) of \(n\)-dimensional \(\lambda\)-adic representations of \(G_K\) with defect set \(T\) and ramification set \(S\) is a family of continuous representations \(\rho_\lambda : G_K \to GL_n(E_\lambda)\) for finite places \(\lambda\) of \(E\) not in \(T\) such that

(i) \(\rho_\lambda\) is unramified outside \(S_\lambda\);

(ii) for any finite place \(v \notin S\) of \(K\), there exists a monic polynomial \(f_v(X) \in E[X]\) (resp. \(f_v(X) \in E[X]\) with coefficients in the integer ring of \(E\)) such that for all finite places \(\lambda \notin T\) of \(E\) coprime to the residue characteristic of \(v\), the characteristic polynomial \(\det(XI_n - \rho_\lambda(\text{Fr}_v))\) of \(\text{Fr}_v\) is equal to \(f_v(X)\).

(2) An \(E\)-rational (resp. \(E\)-integral) strictly compatible system \((\bar{\rho}_\lambda)_{\lambda}\) of \(n\)-dimensional mod \(\lambda\) representations of \(G_K\) with defect set \(T\) and ramification set \(S\) is a family of continuous representations \(\bar{\rho}_\lambda : G_K \to GL_n(F_\lambda)\) for finite places \(\lambda\) of \(E\) not in \(T\) such that

(i) \(\bar{\rho}_\lambda\) is unramified outside \(S_\lambda\);

(ii) for any finite place \(v \notin S\) of \(K\), there exists a monic polynomial \(f_v(X) \in E[X]\) (resp. \(f_v(X) \in E[X]\) with coefficients in the integer ring of \(E\)) such that for all finite places \(\lambda \notin T\) of \(E\) coprime to the residue characteristic of \(v\), \(f_v(X)\) is integral at \(\lambda\) and the characteristic polynomial \(\det(XI_n - \bar{\rho}_\lambda(\text{Fr}_v))\) of \(\text{Fr}_v\) is equal to the reduction of \(f_v(X)\) mod \(\lambda\).

We will often suppress the sets \(S\) and \(T\) from the notations.

**EXAMPLE 2.2.** Let \(X\) be a proper smooth variety over \(K\). Let \(V_l := H^r_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_l)^\vee\) be the dual of the \(l\)-adic étale cohomology group \(H^r_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_l)\) of \(X\). Then the system \((V_l)_l\) is a \(\mathbb{Q}\)-rational strictly compatible system whose defect set is empty and whose ramification set is the set of finite places of \(K\) where \(X\) has bad reduction. This fact follows from the Weil Conjecture which is proved by Deligne (cf. [5], [6]).

It is conjectured that every \(E\)-rational strictly compatible system arises motivically.

**CONJECTURE 2.3** ([10, Conjecture 1]). Any \(E\)-rational strictly compatible system of \(\lambda\)-adic (resp. mod \(\lambda\)) representations arises motivically.

In fact, this conjecture is true if the representations are abelian.
THEOREM 2.4 ([11, Theorem 2, Corollary 1]). An $E$-rational strictly compatible system of abelian semisimple $\lambda$-adic (resp. mod $\lambda$) representations of $G_K$ arises from Hecke characters.

For informations of Hecke characters, see the next subsection.

For a finite place $v$ of $K$, we denote by $G_v$ the decomposition group at $v$ (where we fix an embedding $\bar{K} \hookrightarrow \bar{K}_v$) and by $I_v$ the inertia subgroup of $G_v$. An inertial level $\mathcal{L}$ of $K$ is a collection $(\mathcal{L}_v)_v$ of open normal subgroups $\mathcal{L}_v$ of $I_v$ for each finite place $v$ of $K$ such that $\mathcal{L}_v = I_v$ for almost all $v$. We say that a $\lambda$-adic representation of $G_K$ is geometric (cf. [8]) if

(i) it is unramified outside a finite set of places of $K$;
(ii) its restriction to $G_v$, for any finite place $v$ of $K$ is potentially semistable.

The inertial level $\mathcal{L}$ of a geometric $\lambda$-adic representation $\rho_\lambda$ of $G_K$ is the collection $(\mathcal{L}_v(\rho_\lambda))_v$ of open normal subgroups $\mathcal{L}_v(\rho_\lambda)$ of $I_v$ for each finite place $v$ of $K$, where $\mathcal{L}_v(\rho_\lambda)$ is the largest open subgroup of $I_v$ such that the restriction of $\rho_\lambda$ to $\mathcal{L}_v(\rho_\lambda)$ is semistable. By the definition of geometric Galois representations, we have $\mathcal{L}_v(\rho_\lambda) = I_v$ for almost all $v$. A compatible system $(\rho_\lambda)_{\lambda}$ of geometric $\lambda$-adic representations of $G_K$ has bounded inertial level if there exists an inertial level $\mathcal{L} = (\mathcal{L}_v)_v$ such that $\mathcal{L}_v \subseteq \mathcal{L}_v(\rho_\lambda)$ for all $\lambda$ and $v$.

Let $w_1, w_2, \ldots, w_n$ be integers. A $\lambda$-adic representation $\rho_\lambda$ is $E$-rational (resp. $E$-integral) with Frobenius weights $w_1, w_2, \ldots, w_n$ outside $S$ if $\rho_\lambda$ is $E$-rational (resp. $E$-integral) with ramification set $S$ and, for all finite places $v \not\in S_\lambda$, of $K$, the complex roots of the characteristic polynomial $\det(XI_v - \rho_\lambda(\text{Fr}_v))$ of $\text{Fr}_v$, for a chosen embedding of $E$ into the field of complex numbers $C$, have complex absolute values $q_v^{w_1/2}, q_v^{w_2/2}, \ldots, q_v^{w_n/2}$, where $q_v$ is the cardinality of the residue field of $v$. An $E$-rational strictly compatible system $\rho_\lambda$ is said to be an $E$-rational (resp. $E$-integral) strictly compatible system with Frobenius weights $w_1, w_2, \ldots, w_n$ if each $\rho_\lambda$ is $E$-rational (resp. $E$-integral) with Frobenius weights $w_1, w_2, \ldots, w_n$ outside the ramification set of $(\rho_\lambda)_{\lambda}$. We call $w_1, w_2, \ldots, w_n$ the Frobenius weights of $\rho_\lambda$ (resp. $(\rho_\lambda)_{\lambda}$), and $\rho_\lambda$ (resp. $(\rho_\lambda)_{\lambda}$) is said to be pure if $w_1 = w_2 = \cdots = w_n$. A compatible system $(\rho_\lambda)_{\lambda}$ of geometric $\lambda$-adic representations of $G_K$ has bounded Hodge-Tate weights if there exist integers $a$ and $b$ with $a \leq b$ such that, for any $\lambda$ and any finite place $v$ of $K$ above $l_\lambda$, all the Hodge-Tate weights of $\rho_\lambda|_{G_v}$ viewed as a $Q_{l_\lambda}$-representation are in $[a, b]$. Finally, a compatible system $(\tilde{\rho}_\lambda)_{\lambda}$ of mod $\lambda$ representations of $G_K$ is of bounded Artin conductor if there exists a non-zero ideal $n$ of $O_K$ such that, for any $\lambda$, the Artin conductor outside $l_\lambda$ of $\tilde{\rho}_\lambda$ divides $n$.

PROPOSITION 2.5. (1) An $E$-rational strictly compatible system $(\rho_\lambda)_{\lambda}$ of abelian semisimple $\lambda$-adic representations of $G_K$ has bounded inertial level and bounded Hodge-Tate weights.

(2) An $E$-rational strictly compatible system $(\tilde{\rho}_\lambda)_{\lambda}$ of abelian semisimple mod $\lambda$ representations of $G_K$ is of bounded Artin conductor.

PROOF. By Theorem 2.4, such compatible systems arise from Hecke characters. Hence the proposition follows from standard properties of a representation arising from Hecke characters (see Proposition 2.7 below).
2.2. Compatible systems arising from Hecke characters. In this subsection, we recall the construction of Galois representations arising from Hecke characters and their standard properties (cf. [10, Section 4.1], [17, Chapter I], [19, Chapter II] and [22]). We index as usual the real (resp. complex) places of $K$ by an embedding $\sigma$ of $K$ into $R$ (resp. pairs of elements $\{\sigma, c\sigma\}$ where $\sigma$ is an embedding of $K$ into $C$ with $\sigma(K) \not\subset R$ and $c$ is complex conjugation). We denote by $I_K$ the idele group of $K$ and by $(K^\infty_\infty)^0 \subset I_K$ the connected component of $K^\infty_\infty$ which contains the identity, where $K^\infty_\infty$ is the product of the unit groups of the completions of $K$ at the infinite places. For any $z \in C$, denote by $\bar{z}$ the complex conjugation of $z$.

**DEFINITION 2.6.** A Hecke character is a continuous homomorphism $\psi : I_K \to C^\times$ such that

(i) $\psi(K^\times) = \{1\}$;

(ii) there exist integers $n_{\sigma}, n_{c\sigma}$ which satisfy

$$\psi|_{(K^\infty_\infty)^0}(x) = \prod_{\sigma \text{ real}} x_{\sigma}^{n_{\sigma}} \prod_{\sigma \text{ complex}} x_{\sigma}^{-n_{\sigma} \bar{x}_{\sigma}}$$

with $x_{\sigma}$ the components of $x$.

We say that the family of integers $(n_{\sigma})_{\sigma}$ is the infinity type of $\psi$. The conductor of $\psi$ is the largest ideal $n$ such that elements of the finite ideles $I^{(\infty)}$ of $K$ congruent to $1 \mod n$ are in the kernel of $\psi$.

Now we construct Galois representations arising from a Hecke character $\psi$. We want to use class field theory: $G_{K}^{ab} \simeq I_K/K^\times(K^\infty_\infty)^0$, where $G_{K}^{ab}$ is the Galois group of the maximal abelian extension of $K$ over $K$ and $K^\times(K^\infty_\infty)^0$ is the topological closure of $K^\times(K^\infty_\infty)^0$. Let $l$ be a prime number. Let $\psi_0 : I_K \to C^\times$ be the homomorphism defined by

$$\psi_0(x) = \psi(x) \prod_{\sigma \text{ real}} x_{\sigma}^{-n_{\sigma}} \prod_{\sigma \text{ complex}} x_{\sigma}^{-n_{\sigma} \bar{x}_{\sigma}} ,$$

then its kernel is open and takes values in a sufficiently large finite extension $E$ (in $C$) of $Q$. and thus we may regard $\psi_0$ as a homomorphism $I_K \to E^\times$. By definition, $\psi_0$ factors through the quotient $I_K/(K^\infty_\infty)^0$, however, $\psi_0$ is not trivial on $K^\times$. We modify $\psi_0$ by changing the image of the $l$-part $K^\times_l := (K \otimes Q)(l)^\times \simeq \prod_{\ell \mid l} \mathbb{K}_\ell^\times$ of $I_K$. Suppose that $E$ contains the Galois closure of $K$ over $Q$ and take any finite place $\ell$ of $E$ above $l$. Let $\eta : K^\times \to E^\times$ be the homomorphism defined by $\eta(x) = \prod_{\sigma}(x)^{n_{\sigma}}$, where $\sigma$ runs through all embeddings $K \hookrightarrow C$, and extend $\eta$ to $\eta_l : (K \otimes Q)(l)^\times \to E^\times_l$. Then we have the continuous homomorphism $\psi_\lambda := \psi_0 \circ (\eta_l \circ \alpha_l)$, where $\alpha_l : I_K \to (K \otimes Q)(l)^\times$ is the projection. Using the isomorphism of class field theory $G_{K}^{ab} \simeq I_K/K^\times(K^\infty_\infty)^0$, we obtain a continuous character $\psi_\lambda : G_{K} \to E^\times_\lambda$. Since $\psi_\lambda$ is continuous, we know that $\psi_\lambda$ has values in the group of units of $E_\lambda$ and thus we obtain a mod $\lambda$ representation $\tilde{\psi}_\lambda : G_K \to F^\times_\lambda$. Denote by $\pi_\lambda$ a uniformizer of $K_v$ for any finite place $v$. By construction, we see the following standard properties for Galois representations arising from Hecke characters.
Proposition 2.7. Let the notation be as above.

1. The character $\psi_\lambda$ is unramified away from $ln$ where $n$ is the conductor of $\psi$. Moreover, $\psi_\lambda$ is locally algebraic in the sense of [19, Chapter III] (See also [17, Chapter I, Section 5]), and hence it is geometric (cf. [8, Section 6, Proposition]).

2. For any finite place $v$ away from $ln$, we have equalities

$$\psi_\lambda(F_v) = \psi_\lambda(\pi_v) = \psi(\pi_v) = \psi_0(\pi_v) \in E.$$  

In particular, $\psi_\lambda(F_v)$ is independent of the choice of $\lambda$ and has values in $E$.

3. For any finite place $v$ away from $ln$, we have equalities

$$\psi_\lambda(F_v) = \psi_\lambda(\pi_v) = \psi(\pi_v) = \psi_0(\pi_v) \in E.$$  

In particular, $\psi_\lambda(F_v)$ is independent of the choice of $\lambda$ and has values in $E$.

4. The system $\overline{\psi}_\lambda$ forms an $E$-rational strictly compatible system of 1-dimensional mod $\lambda$ representations of $G_K$ of bounded Artin conductor.

Proof. We only give a very rough sketch: To see locally algebraicity of (1), for example, use [17, Proposition (1.5.4)]. Boundednesses of the inertial level and the Artin conductor in (3) and (4) follow from the existence of the conductor of Hecke characters. The boundedness of Hodge-Tate weights in (3) can be seen from the infinite type of Hecke characters, or, for example, we can also check the boundedness by combining locally algebraicity of (1) and the arguments given in the last part (around p. 482) of the proof of [21, Theorem]. All the other statements follow from direct calculations.  

2.3. Structures of certain compatible systems. Choose an algebraic closure $\overline{F}_\lambda$ of $F_\lambda$. Put $\chi_\lambda: G_K \rightarrow Z_{l_\lambda}^\times \leftarrow E_{l_\lambda}^\times$ and $\bar{\chi}_\lambda: G_K \rightarrow \overline{F}_{l_\lambda}^\times \leftarrow F_{l_\lambda}^\times$, where $\chi_\lambda$ and $\bar{\chi}_\lambda$ are the $l_\lambda$-adic cyclotomic character and the mod $l_\lambda$ cyclotomic character, respectively. For a representation $\overline{\rho}_\lambda: G_K \rightarrow GL_n(F_\lambda)$ with abelian semisimplification, the representation $(\overline{\rho}_\lambda)^{\text{ss}} \otimes \overline{F}_\lambda$ is conjugate to the direct sum of $n$ characters, where the subscript “ss” means the semisimplification, and we call these $n$ characters the characters associated with $\overline{\rho}_\lambda$ (remark that Schur’s lemma implies that any irreducible abelian $\overline{F}_\lambda$-representation is of dimension one). For a $\lambda$-adic representation $\rho_\lambda$, we denote by $\overline{\rho}_\lambda$ a residual representation of $\rho_\lambda$ (for a chosen lattice). Note that the isomorphism class of $(\overline{\rho}_\lambda)^{\text{ss}}$ is independent of the choice of a lattice by the Brauer-Nesbitt theorem.

Theorem 2.8. Let $(\rho_\lambda)_\lambda$ be an $E$-rational strictly compatible system of $n$-dimensional geometric semisimple $\lambda$-adic representations of $G_K$. Suppose that there exists an infinite set $\Lambda$ of finite places of $E$ which satisfies the following conditions.

1. For any $\lambda \in \Lambda$, there exists a place $v_\lambda$ of $K$ above $l_\lambda$ such that

   (a) $\rho_\lambda$ is semistable at $v_\lambda$;

   (b) there exist integers $w_1 \leq w_2$ such that the Hodge-Tate weights of $\rho_\lambda|_{G_{v_\lambda}}$ are in $[w_1, w_2]$ for any pair $(\lambda, v_\lambda)$.

2. For any $\lambda \in \Lambda$, $(\overline{\rho}_\lambda)^{\text{ss}}$ is abelian and any character associated with $\overline{\rho}_\lambda$ has the form $\varepsilon \bar{\chi}_\lambda^a$, where $a$ is an integer and $\varepsilon: G_K \rightarrow \overline{F}_\lambda^\times$ is a character unramified at all places of $K$ above $l_\lambda$. 


(3) The Artin conductor of \((\tilde{\rho}_\lambda)^{ss}\) is bounded independently of the choice of \(\lambda \in \Lambda\).

That is, there exists a non-zero ideal \(n\) of \(O_K\) such that, for any \(\lambda \in \Lambda\), the Artin conductor outside \(l_\lambda\) of \((\tilde{\rho}_\lambda)^{ss}\) divides \(n\).

Then there exist integers \(m_1, m_2, \ldots, m_n\) and a finite extension \(L\) of \(K\) such that, for any \(\lambda\), the representation \(\rho_\lambda\) is isomorphic to \(X_{\lambda,1}^{m_1} \oplus X_{\lambda,2}^{m_2} \oplus \cdots \oplus X_{\lambda,n}^{m_n}\) on \(G_L\).

**Remark 2.9.** If Conjecture 2.3 holds, then we can remove the conditions (1) and (3) of Theorem 2.8 since these conditions are automatically satisfied.

**Lemma 2.10.** Let \(F\) be a field of characteristic \(l > 0\). Let \(\rho\) and \(\rho'\) be \(n\)-dimensional semisimple \(F\)-representations of a group \(G\). Assume that \(l > n\). If \(\text{Tr}(\rho(g)) = \text{Tr}(\rho'(g))\) for any \(g \in G\), then \(\rho\) is isomorphic to \(\rho'\).

**Proof.** Let \(V\) and \(V'\) be underlying \(F\)-vector spaces of \(\rho\) and \(\rho'\), respectively. Let \(\mathcal{S}\) be the category of isomorphism classes of simple \(F[G]\)-modules. Then there exist a finite set \(H\) of objects of \(\mathcal{S}\) and integers \(n_h, n'_h \geq 0\) for each \(h \in H\) such that

\[
V \simeq \bigoplus_{h \in H} W_h^{\oplus n_h}, \quad V' \simeq \bigoplus_{h \in H} W_h^{\oplus n'_h},
\]

as \(F[G]\)-modules. Here, \(W_h\) is a (chosen) representative of \(h\). By a method similar to the proof of [1, Section 12, Proposition 3], we obtain \((n_h - n'_h)\dim F(W_h) = 0\) in \(F\). Since \(l > n\), we know that \(\dim F(W_h) \in F^\times\) and thus \(n_h - n'_h \equiv 0 \mod l\). Since \(-n \leq n_h - n'_h \leq n\), by using the assumption \(l > n\) again, we obtain \(n_h = n'_h\).

**Proof of Theorem 2.8.** By taking a positive integer \(m\) large enough and twisting \(X_{\lambda}^m\) to \(\rho_\lambda\) for all \(\lambda\), we may assume \(w_1 \geq 0\). Furthermore, by replacing \(\Lambda\) with its infinite subset, we may suppose that \(l_\lambda\) does not divide the discriminant of \(K\) and \(l_\lambda > |E : Q| \cdot n\) for any \(\lambda \in \Lambda\). By the condition (3), there exists a non-zero ideal \(n\) of \(O_K\) such that, for any \(\lambda \in \Lambda\), the Artin conductor outside \(l_\lambda\) of \((\tilde{\rho}_\lambda)^{ss}\) divides \(n\). If we denote by \(\psi\) a character associated with \((\tilde{\rho}_\lambda)^{ss}\) for \(\lambda \in \Lambda\) and decompose \(\psi = \varepsilon X_{\lambda}^a\), where \(\varepsilon\) is as in the condition (2), then the Artin conductor outside \(l_\lambda\) of \(\varepsilon\) also divides \(n\). Hence, replacing the field \(K\) with the strict ray class field of \(K\) associated with \(n\), we may replace the condition (2) with the following condition (2)’:

(2)’ For any \(\lambda \in \Lambda\), \((\tilde{\rho}_\lambda)^{ss}\) is abelian and any character associated with \(\tilde{\rho}_\lambda\) has the form \(X_{\lambda}^a\).

Now take any \(\lambda \in \Lambda\). Let \(X_{\lambda,1}^a, X_{\lambda,2}^a, \ldots, X_{\lambda,n}^a\) be all the characters associated with \(\tilde{\rho}_\lambda\). By Lemma 2.10 together with the condition (2)’ and \(l_\lambda > |E : Q| \cdot n\), we see that the representation \((\tilde{\rho}_\lambda)^{ss}\) is conjugate to the direct sum of \(n\) characters (over \(F_{l_\lambda}\)) of the form \(X_{\lambda}^a\) which has values in \(F_{l_\lambda}^\times\). Hence if we regard the \(F_{l_\lambda}\)-representation \(\tilde{\rho}_\lambda\) as an \(F_{l_\lambda}\)-representation, its semisimplification is of a diagonal form whose diagonal components are the copies of \(X_{\lambda,1}^a, X_{\lambda,2}^a, \ldots, X_{\lambda,n}^a\) (here we note that \(l_\lambda > |F_{l_\lambda} : F_{l_\lambda}| \cdot n\)). Furthermore, it is a direct summand of the semisimplification of a residual representation of \(\rho_\lambda\) viewed as a \(Q_{l_\lambda}\)-representation. Therefore, by the condition (1) and applying Caruso’s result on an upper bound for tame inertia weights (cf. [4] or [16, Theorem 2.5 and Remark 2.6]) to \(\tilde{\rho}_\lambda|_{G_{l_\lambda}}\).
(viewed as an $F_{l_\lambda}$-representation), there exists an integer $0 \leq b_{\lambda,i} \leq w_2$ such that
\[
(\#) \quad b_{\lambda,i} \equiv a_{\lambda,i} \mod l_\lambda - 1
\]
for any $i$ (recall that $l_\lambda$ does not divide the discriminant of $K$). Now we claim that the set \{\(b_{\lambda,1}, b_{\lambda,2}, \ldots, b_{\lambda,n}\)\} is independent of the choice of $\lambda \in \Lambda$ large enough. Denote by $S$ the ramification set of $(\rho_{\lambda})_\lambda$. Take a $v_0 \notin S$ and decompose $\text{det}(XI_n - \rho_{\lambda}(F_{v_0})) = \prod_{j=1}^n (X - \alpha_{v_0,j})$. By the conditions $(2')$ and $(\natural)$, we have the congruence $\prod_{j=1}^n (X - \alpha_{v_0,j}) \equiv \prod_{j=1}^n (X - q_{v_0,j})$ in $\bar{F}_{l_\lambda}[X]$. If $l_\lambda$ is large enough (note that $\Lambda$ is an infinite set), then we obtain that this congruence is in fact an equality in $E[X]$: $\prod_{j=1}^n (X - \alpha_{v_0,j}) = \prod_{j=1}^n (X - q_{v_0,j})$. Therefore, the set \{\(b_{\lambda,1}, b_{\lambda,2}, \ldots, b_{\lambda,n}\)\} is independent of the choice of $\lambda \in \Lambda$ with $l_\lambda$ large enough. This proves the claim. We denote \{\(b_{\lambda,1}, b_{\lambda,2}, \ldots, b_{\lambda,n}\)\} by \{\(m_1, m_2, \ldots, m_n\)\} for such $\lambda$'s. Since $(\rho_{\lambda})_\lambda$ is a compatible system, we obtain the equation $\text{det}(XI_n - \rho_{\lambda}(F_{v})) = \prod_{j=1}^n (X - q_{v,j}^{m_j})$ for any $\lambda$ and any $v \notin S_{l_\lambda}$. Therefore, the representation $\rho_{\lambda}$ is isomorphic to $\bigoplus_{\lambda,1} \chi_{\lambda}^{m_1} \oplus \chi_{\lambda}^{m_2} \oplus \cdots \oplus \chi_{\lambda}^{m_n}$. This finishes the proof.

**COROLLARY 2.11.** Let $(\bar{\rho}_{\lambda})_\lambda$ be an $E$-rational strictly compatible system of abelian semisimple representations of $G_K$. Suppose that, for infinitely many finite places $\lambda$ of $E$, any character associated with $\bar{\rho}_{\lambda}$ has the form $\varepsilon \check{\chi}_{\lambda}^a$, where $\varepsilon: G_K \to \bar{F}_{l_{\lambda}}^\times$ is a character unramified at all places of $K$ above $l_{\lambda}$. Then there exist a finite extension $L$ of $K$ and integers $m_1, m_2, \ldots, m_n$ such that, for any $\lambda$, the representation $\rho_{\lambda}$ is isomorphic to $\bigoplus_{\lambda,1} \chi_{\lambda}^{m_1} \oplus \chi_{\lambda}^{m_2} \oplus \cdots \oplus \chi_{\lambda}^{m_n}$ on $G_L$.

**PROOF.** By Theorem 2.4, we know that there exist a finite extension $E'$ of $E$ and an $E'$-rational abelian semisimple compatible system $(\rho_{\lambda}')_\lambda'$ of $\lambda'$-adic representations of $G_K$ which arises from Hecke characters such that $(\rho_{\lambda}')_\lambda'$ is a lift of $(\bar{\rho}_{\lambda})_\lambda$, that is, a residual representation $\bar{\rho}_{\lambda'}$ of $\rho_{\lambda'}$ is isomorphic to $\bar{\rho}_{\lambda} \otimes F_{l_{\lambda}}'$ for any $\lambda$ and any finite place $\lambda'$ of $E'$ above $\lambda$. By Proposition 2.7, we see that $(\rho_{\lambda}')_\lambda'$ satisfies all the assumptions (1), (2) and (3) in Theorem 2.8. Consequently we obtain the desired result.

**COROLLARY 2.12.** Let $(\rho_{\lambda})_\lambda$ be an $E$-rational strictly compatible system of $n$-dimensional semisimple $\lambda$-adic representations of $G_K$. Suppose that

(i) $(\bar{\rho}_{\lambda})^{ss}_{\lambda}$ is abelian for almost all $\lambda$;

(ii) for infinitely many $\lambda$, any character associated with $(\bar{\rho}_{\lambda})^{ss}_{\lambda}$ has the form $\varepsilon \check{\chi}_{\lambda}^a$, where $\varepsilon: G_K \to F_{l_{\lambda}}^\times$ is a character unramified at all places of $K$ above $l_{\lambda}$.

Then there exist integers $m_1, m_2, \ldots, m_n$ and a finite extension $L$ of $K$ such that, for any $\lambda$, the representation $\rho_{\lambda}$ is isomorphic to $\bigoplus_{\lambda,1} \chi_{\lambda}^{m_1} \oplus \chi_{\lambda}^{m_2} \oplus \cdots \oplus \chi_{\lambda}^{m_n}$ on $G_L$.

**PROOF.** For any finite place $v$ of $K$ not in the ramification set of $(\rho_{\lambda})_\lambda$, let $f_v(X)$ be as in Definition 2.1 (1). Applying Corollary 2.11 to the compatible system $((\bar{\rho}_{\lambda})^{ss}_{\lambda})_\lambda$, we see that there exist a finite extension $L$ of $K$ and integers $m_1, m_2, \ldots, m_n$ such that, for any $\lambda$, the representation $(\bar{\rho}_{\lambda})^{ss}_{\lambda}$ is isomorphic to $\bigoplus_{\lambda,1} \chi_{\lambda}^{m_1} \oplus \chi_{\lambda}^{m_2} \oplus \cdots \oplus \chi_{\lambda}^{m_n}$ on $G_L$. Thus we have $f_v(X) = \prod_{j=1}^n (X - q_{v,j}^{m_j})$ in $F_{l_{\lambda}}[X]$ for any $\lambda$ coprime to $v$. Taking $\lambda$ with $l_{\lambda}$ large enough, we see that $f_v(X) = \prod_{j=1}^n (X - q_{v,j}^{m_j})$ in $E[X]$ and the result follows.
Let $\lambda$ and $\lambda'$ be finite places of $E$ of different residual characteristics. Let $\rho_{\lambda}$ be an $E$-rational $n$-dimensional semisimple $\lambda$-adic representation of $G_{\overline{K}}$ with ramification set $S$. Suppose that there exists a semisimple $\lambda'$-adic representation $\rho_{\lambda'}$ of $G_{\overline{K}}$ such that
\[
\det(XI_n - \rho_{\lambda}(Fr_v)) = \det(XI_n - \rho_{\lambda'}(Fr_v))
\]
for any $v \notin S_{\lambda} \cup S_{\lambda'}$. In the spirit of Fontaine-Mazur’s “Main Conjecture”, we hope that $\rho_{\lambda'}$ is crystalline for any finite place $v'$ of $K$ above $l_{\lambda'}$; when the residual characteristic of $\lambda'$ is prime to that of any place in $S$. However, to prove this hope seems not to be easy. If we consider representations which are pure, we can improve the statement (1) of Theorem 2.8 as below. (If the hope is true, it is not difficult to prove the proposition below without the assumption of purity by a method similar to the proof of Theorem 2.8.)

**Proposition 2.13.** Let $(\rho_{\lambda})_{\lambda}$ be an $E$-rational strictly compatible system of $n$-dimensional geometric semisimple $\lambda$-adic representations of $G_{\overline{K}}$. Suppose that $(\rho_{\lambda})_{\lambda}$ is pure. Suppose that there exists an infinite set $\Lambda$ of finite places of $K$ which satisfies the following conditions.

1. For any $\lambda \in \Lambda$, there exists a place $v_\lambda$ of $K$ above $l_\lambda$ such that
   a. there exists a constant $C > 0$ such that $[L_{v_\lambda} : \Sigma_{v_\lambda}(\rho_\lambda)] < C$ for any pair $(\lambda, v_\lambda)$. Here $\Sigma_{v_\lambda}(\rho_\lambda)$ is the inertial level of $\rho_\lambda$ at $v_\lambda$;
   b. there exist integers $w_1 \leq w_2$ such that the Hodge-Tate weights of $\rho_\lambda|_{G_{v_\lambda}}$ are in $[w_1, w_2]$ for any pair $(\lambda, v_\lambda)$.

2. For any $\lambda \in \Lambda$, $(\rho_{\lambda})^{ss}$ is abelian and any character associated with $\tilde{\rho}_\lambda$ has the form $\varepsilon \tilde{\chi}_{\lambda}^a$, where $\varepsilon : G_K \rightarrow \overline{\mathbb{F}}_{\lambda}^\times$ is a character unramified at all places of $K$ above $l_\lambda$.

3. The Artin conductor of $(\tilde{\rho}_\lambda)^{ss}$ is bounded independently of the choice of $\lambda \in \Lambda$ in the sense of (3) of Theorem 2.8.

Then there exist an integer $m$ and a finite extension $L$ of $K$ such that, for any $\lambda$, the representation $\rho_{\lambda}$ is isomorphic to $(\chi_{\lambda}^m)^{\otimes n}$ on $G_L$.

**Proof.** Most parts of the first paragraph of this proof will proceed by a method similar to the proof of Theorem 2.8 and hence we will often omit precise arguments. First we may assume that, for any $\lambda \in \Lambda$,

1. any character associated with $\tilde{\rho}_\lambda$ has the form $\tilde{\chi}_{\lambda}^a$,

and furthermore, there exists a positive integer $r$ such that $\rho_{\lambda}|_{G_{v_\lambda}}$ has Hodge-Tate weights in $[0, r]$ for any pair $(\lambda, v_\lambda)$ as in the condition (1). Suppose $\lambda$ is a finite place in $\Lambda$. Let $X_{\lambda}^{a_{\lambda,1}}, X_{\lambda}^{a_{\lambda,2}}, \ldots, X_{\lambda}^{a_{\lambda,n}}$ be all the characters associated with $\tilde{\rho}_\lambda$. Taking a finite place $v_\lambda$ as in the condition (1), there exists a finite extension $L_{w(\lambda)}$ of $K_{v_\lambda}$ such that $\rho_{\lambda}|_{G_{L_{w(\lambda)}}}$ is semistable and $[L_{w(\lambda)} : K_{v_\lambda}] \leq C$. If we denote by $e_{w(\lambda)}$ the absolute ramification index of $L_{w(\lambda)}$, then we have $e_{w(\lambda)} \leq C[K : Q]$, and Caruso’s result on an upper bound for tame inertia weights of $\rho_{\lambda}|_{G_{L_{w(\lambda)}}}$ (viewed as an $F_{l_\lambda}$-representation) implies that there exists an integer $0 \leq b_{\lambda,i} \leq e_{w(\lambda)}$ which satisfies $b_{\lambda,i} \equiv e_{w(\lambda)}a_{\lambda,i} \mod l_\lambda - 1$. Putting $e = lcm_{\lambda \in \Lambda}(e_{w(\lambda)})$ and $b_{\lambda,i} = b_{\lambda,i}^e/e_{w(\lambda)}$, we have $e \leq (C[K : Q])!$, $b_{\lambda,i} \in [0, e]$, and $b_{\lambda,i} \equiv ea_{\lambda,i} \mod l_\lambda$.
1. Note that $e$ is independent of the choice of $\lambda \in \Lambda$. Take any $v \notin S_{\lambda_0}$ and decompose \( \det(XI_n - \rho_{\lambda_0}(Fr_v)) = \prod_{j=1}^n(X - \alpha_{v,j}). \) Then, by an argument similar to that in the proof of Theorem 2.8, we can show that \( \prod_{j=1}^n(X - \alpha_{v,j}) = \prod_{j=1}^n(X - q_{\lambda,1}^b) \) if we take $\lambda \in \Lambda$ with $l_\lambda$ large enough. Since $(\rho_{\lambda})_\lambda$ is pure, we know that $b_{\lambda,1} = \cdots = b_{\lambda,n}$ and furthermore they are independent of the choice of $\lambda \in \Lambda$. Putting $b := b_{\lambda,1} = \cdots = b_{\lambda,n}$, we have

\[
\prod_{j=1}^n(X - \alpha_{v,j}^e) = \prod_{j=1}^n(X - q_{\lambda}^b). \quad (*)
\]

Fix $\lambda \in \Lambda$ and denote it by $\lambda_0$. By taking a finite extension $K'$ of $K$ large enough, we can define a continuous character $\chi_{\lambda_0}^{1/e} : G_{K'} \to E_{\lambda_0}^\times$ which has values in the integer ring of $E_{\lambda_0}$ and $(\chi_{\lambda_0}^{1/e})^e = \chi_{\lambda_0}$. In fact, this can be checked as follows: Let $m_{\lambda_0}$ be the maximal ideal of the integer ring of $E_{\lambda_0}$. Let $e_{\lambda_0}$ be the absolute ramification index of $E_{\lambda_0}$. Fix an integer $k$ which satisfies $k > e_{\lambda_0}/(l_{\lambda_0} - 1)$, and take a finite extension $K'$ of $K$ such that $\chi_{\lambda_0}(G_{K'}) \subset 1 + em_{\lambda_0}^k$. Then we obtain the desired character $\chi_{\lambda_0}^{1/e}$ by the composite

\[
G_{K'} \to 1 + em_{\lambda_0}^k \xrightarrow{\log} em_{\lambda_0}^k \xrightarrow{1/e} m_{\lambda_0}^k \xrightarrow{\exp} 1 + m_{\lambda_0}^k \subset E_{\lambda_0}^\times.
\]

In the argument below, we use the method of the proof of [12, Proposition 1.2]. Let $\lambda_0$ and $K'$ be as above and replace $K$ with this $K'$. The equation $(*)$ implies that, for any $v \notin S_{\lambda_0}$, all the roots of $\det(XI_n - \rho_{\lambda_0}^e(Fr_v))$ are $e$-th roots of unity, where $\rho_{\lambda_0}^e$ is the twist of $\rho_{\lambda_0}$ by $(\chi_{\lambda_0}^{1/e})^{-b}$. In particular, there are only finitely many possibilities for the characteristic polynomial of $Fr_v$. Now the function which takes $g \in G_K$ to $\det(XI_n - \rho_{\lambda_0}^e(g)) \in E_{\lambda_0}[X]$ is continuous (here, we equip $E_{\lambda_0}[X] \simeq \prod_{i \geq 0} E_{\lambda_0}X^i$ with the direct product topology of the $\lambda_0$-adic topology on $E_{\lambda_0}X^i$), and takes only finitely many values by Chebotarev’s density theorem. It follows that the set \( \{g \in G_K; \det(XI_n - \rho_{\lambda_0}^e(g)) = (X - 1)^n\} \) is an open subset of $G_K$, which contains the identity map of $K$. Hence there exists a finite extension $L$ of $K$ such that $G_L \subset \{g \in G_K; \det(XI_n - \rho_{\lambda_0}^e(g)) = (X - 1)^n\}$. Then we see that $\rho_{\lambda_0}$ is geometric, $\rho_{\lambda_0}$ is Hodge-Tate at any place of $L$ above $l_{\lambda_0}$ and thus we know that $b/e = m$ is an integer. This finishes the proof.

3. Rasmussen-Tamagawa conjecture. We continue to use the same notation as in the previous section. Let $g \geq 0$ be an integer. For any abelian variety $A$ over $K$, denote by $A[l]$ the group of $K$-valued $l$-torsion points of $A$ and by $K(A[l])$ the field generated by $K$ and the coordinates of $A[l]$.

**Definition 3.1.** We denote by $\mathcal{A}(K, g, l)$ the set of $K$-isomorphism classes of $g$-dimensional abelian varieties $A$ over $K$ which satisfy the following conditions.

- (RT) $K(A[l])$ is an $l$-extension of $K(\mu_l)$.
- (RT_{red}) The abelian variety $A$ has good reduction away from $l$ over $K$.

By (RT_{red}), the set $\mathcal{A}(K, g, l)$ is a finite set (cf. [7, Theorem 5] and [23, 1. Theorem]). Rasmussen and Tamagawa conjectured in [18] that for any $l$ large enough, this set is in
The condition \((RT)\) may be infinite (but the author does not know an example such that \(A\)). It is clear that \((RT)\) is not imposed on abelian varieties in \(\mathcal{A}(K, g, l)\) with semistable reduction everywhere. Then there exists an integer \(C = C([K : \mathbb{Q}], g)\), depending only on \([K : \mathbb{Q}]\) and \(g\), such that \(\mathcal{A}(K, g, l)_{\text{st}}\) is empty for any \(l > C\) with \(l \nmid d_K\). Here \(d_K\) is the discriminant of \(K\).

For a \(g\)-dimensional abelian variety \(A\) over \(K\), denote by \(\rho_{A,l} : G_K \to GL(T_l(A)) \simeq GL_{2g}(\mathbb{Z}_p)\) the representation determined by the action of \(G_K\) on the \(l\)-adic Tate module \(T_l(A)\) of \(A\). Consider the following conditions.

\((RT)\)' For some finite extension \(L\) of \(K\) which is unramified at all places of \(K\) above \(l\), \(L(A[l])\) is an \(l\)-extension of \(L(\mu_l)\).

\((RT_{\text{ab}})\) The representation \(\rho_{A,l}\) has an abelian image.

It is clear that \((RT)\) implies \((RT)\)' . We recall the definitions of \(\mathcal{A}(K, g, l)_{\text{ab}}\) and \(\mathcal{A}'(K, g, l)_{\text{ab}}\).

**Definition 3.2.** We define the sets \(\mathcal{A}(K, g, l)_{\text{ab}}\) and \(\mathcal{A}'(K, g, l)_{\text{ab}}\) of isomorphism classes of \(g\)-dimensional abelian varieties \(A\) over \(K\) as follows:

1. \([A] \in \mathcal{A}(K, g, l)_{\text{ab}}\) if and only if \(A\) satisfies \((RT)\), \((RT_{\text{red}})\) and \((RT_{\text{ab}})\).
2. \([A] \in \mathcal{A}'(K, g, l)_{\text{ab}}\) if and only if \(A\) satisfies \((RT)\)' and \((RT_{\text{ab}})\).

Clearly, we have \(\mathcal{A}(K, g, l) \supset \mathcal{A}(K, g, l)_{\text{ab}} \subset \mathcal{A}'(K, g, l)_{\text{ab}}\). Note that the reduction hypothesis \((RT_{\text{red}})\) is not imposed on abelian varieties in \(\mathcal{A}'(K, g, l)_{\text{ab}}\). Hence \(\mathcal{A}'(K, g, l)_{\text{ab}}\) may be infinite (but the author does not know an example such that \(\mathcal{A}'(K, g, l)_{\text{ab}}\) is infinite).

### 4. Proof of Theorem 1.2.

In this section, we use the same notation as in previous sections. First we study the structure of \(A[l]\) for an abelian variety \(A\) in \(\mathcal{A}'(K, g, l)_{\text{ab}}\). Let \(A\) be any \(g\)-dimensional abelian variety over \(K\). We denote by \(\tilde{\rho}_{A,l} : G_K \to GL(A[l]) \simeq GL_{2g}(\mathbb{F}_p)\) the representation determined by the action of \(G_K\) on \(A[l]\). Consider the following conditions.

\((RT_{\text{mod}})\) \((\tilde{\rho}_{A,l})^{ss}\) is conjugate to the direct sum of \(2g\) characters which are of the form \(\tilde{\chi}^a\).

\((RT_{\text{mod}})'(\tilde{\rho}_{A,l})^{ss}\) is abelian and the characters associated with \(\tilde{\rho}_{A,l}\) are of the form \(\varepsilon \tilde{\chi}^a\), where \(\varepsilon : G_K \to \mathbb{F}_p^\times\) is a continuous character which is unramified at all places above \(l\). The condition \((RT)\)' is equivalent to the condition \((RT_{\text{mod}})\) by the lemma below. Hence the \(K\)-isomorphism class \([A]\) of \(g\)-dimensional abelian variety \(A\) over \(K\) is in \(\mathcal{A}(K, g, l)\) if and only if \(A\) satisfies \((RT_{\text{mod}})\) and \((RT_{\text{red}})\).
LEMMA 4.1. Let $A$ be a $g$-dimensional abelian variety over $K$.

(1) The abelian variety $A$ satisfies $(\text{RT}_l)$ if and only if $A$ satisfies $(\text{RT}_{\text{mod}})$.

(2) Suppose that $(\bar{\rho}_{A,l})^\text{ss}$ is abelian. Then $A$ satisfies $(\text{RT}_l)'$ if and only if $A$ satisfies $(\text{RT}_{\text{mod}})'$.

PROOF. The assertion (1) is proved by the arguments of the proof of [18, Lemma 3] and thus we omit the proof. Suppose that $(\bar{\rho}_{A,l})^\text{ss}$ is abelian and denote by $\psi_1, \ldots, \psi_{2g}$ the characters associated with $\bar{\rho}_{A,l}$. If $A$ satisfies $(\text{RT}_{\text{mod}})'$, then we have $\psi_i = \epsilon_i \bar{\chi}_{l}^{a_i}$ for some integer $a_i$ where $\epsilon_i : G_K \to \bar{F}_l^{\times}$ is a continuous character which is unramified at all places of $K$ above $l$. Let $L$ be the composition field of all fields $\bar{K}^{\ker \epsilon_i}$ for all $i$. Then $L$ is unramified at all places of $K$ above $l$. Since each $\psi_i|_{G_{L(\mu_l)}}$ is trivial, we obtain $(\text{RT}_l)'$. Conversely, suppose that $(\text{RT}_l)'$ holds and take a field $L$ as in the statement of $(\text{RT}_l)'$. By (1), we know that each $\psi_i|_{G_L}$ is equal to $\bar{\chi}_{l}^{a_i}$ for some integer $a_i$. Hence $\epsilon_i := \psi_i \cdot \bar{\chi}_{l}^{-a_i} : G_K \to \bar{F}_l^{\times}$ is unramified at all places above $l$ and this implies $(\text{RT}_{\text{mod}})'$. $\square$

We recall the following two propositions.

PROPOSITION 4.2 (Faltings). Fix an integer $w$. The set of isomorphism classes of semisimple $n$-dimensional $l$-adic representations $G_K \to GL_n(\mathbb{Q}_l)$ which are $\mathbb{Q}$-integral with Frobenius weights less than or equal to $w$ outside $S$, is finite.

PROOF. The Proposition follows from the proof of [7, Theorem 5]. See also [13, Chapter VIII, Section 5, Theorem 11]. $\square$

PROPOSITION 4.3 (Raynaud’s criterion of semistable reduction. [9, Proposition 4.7]). Suppose $A$ is an abelian variety over a field $F$ with a discrete valuation $v$, $n$ is a positive integer not divisible by the residue characteristic, and the points of $A[n]$ are defined over an extension of $F$ which is unramified over $v$. If $n \geq 3$ then $A$ has semistable reduction at $v$.

Here is one consequence of Proposition 4.3: Let $A$ be an arbitrary abelian variety over a number field $K$. For any positive integer $n$, denote by $K(A[n])$ the field generated by $K$ and the coordinates of all $n$-torsion points of $A$. Then $A$ has semistable reduction everywhere over $K(A[12]) = K(A[3])K(A[4])$. In fact, the abelian variety $A$ has semistable reduction over $K(A[3])$ outside 3 (apply Proposition 4.3 for $F = K(A[3])$) and has semistable reduction over $K(A[4])$ outside 2 (apply Proposition 4.3 for $F = K(A[4])$). For later use, we put $K(A[l^\infty]) = \bigcup_{n \geq 0} K(A[l^n])$ for any prime number $l$.

For an integer $g > 0$, put

$$D_g := \mathbb{Z}GL_{2g}(\mathbb{Z}/3\mathbb{Z}) \cdot \mathbb{Z}GL_{2g}(\mathbb{Z}/4\mathbb{Z}).$$

If $\rho : G_K \to GL_{2g}(\mathbb{Q}_l)$ is an abelian representation, then, for any integer $k$, we denote by $\rho^k$ the representation $G_K \to GL_{2g}(\mathbb{Q}_l)$ which is defined by $\rho^k(s) := (\rho(s))^k$ for any $s \in G_K$. With this notation, we obtain the following lemma which plays an important role in the proof of Theorem 1.2 to construct a good compatible system.
LEMMA 4.4. Let \( g > 0 \) be an integer and \( l_0 \) a prime number. Let \( \mathcal{A}_{l_0} \) be the set of isomorphism classes of representations \( \rho: G_K \to GL_{2g}(Q_{l_0}) \) which are isomorphic to \( \rho_{A,l_0}^{D_g} \) for some \( g \)-dimensional abelian variety \( A \) over \( K \) such that \( K(A[I_0^{\infty}]) \) is an abelian extension of \( K \). Then \( \mathcal{A}_{l_0} \) is finite.

PROOF. If \( A \) is an abelian variety over \( K \) such that \( K(A[I_0^{\infty}]) \) is an abelian extension of \( K \), then \( A \) has potential good reduction everywhere (cf. [20, Section 2, Corollary 1]).

Putting \( L := K(A[12]) \), such an abelian variety \( A \) has good reduction everywhere over \( L \) by Proposition 4.3. Since \( [L : K] \) divides \( D_g \), the representation \( \rho_{A,l_0}^{D_g} \) is unramified outside \( l_0 \) for any \( g \)-dimensional abelian variety \( A \) over \( K \) such that \( K(A[I_0^{\infty}]) \) is an abelian extension of \( K \). Take any finite place \( v \) of \( K \) not above \( l_0 \). Let \( v_L \) be a finite place of \( L \) above \( v \) and denote by \( f \) the extension degree of \( F_{v_L} \) over \( F_v \), where \( F_{v_L} \) and \( F_v \) are residue fields of \( v_L \) and \( v \), respectively. Remark that \( D_g/f \) is an integer since \( L \) is a Galois extension of \( K \), and \( \rho_{A,l_0} (Fr_{v_L}) \) is well-defined since \( A \) has good reduction everywhere over \( L \) (cf. [20, Theorem 1]). It is not difficult to obtain the equation

\[
\text{det}(X I_{2g} - \rho_{A,l_0}^{D_g} (Fr_v)) = \text{det}(X I_{2g} - (\rho_{A,l_0} (Fr_{v_L}))^{D_g/f})
\]

Since \( A \) has good reduction everywhere over \( L \), the polynomial \( \text{det}(X I_{2g} - \rho_{A,l_0} (Fr_{v_L})) \) has rational integer coefficients and hence so is \( \text{det}(X I_{2g} - (\rho_{A,l_0} (Fr_{v_L}))^{D_g/f}) \). Consequently, the representation \( \rho_{A,l_0}^{D_g} \) is \( \mathcal{Q} \)-integral with Frobenius weight \( D_g/2 \) outside the set of finite places of \( K \) above \( l_0 \). Therefore, by Proposition 4.2, it is enough to prove that the representation \( \rho_{A,l_0}^{D_g} \) is semisimple. Note that it has been already known that \( \rho_{A,l_0} \) is semisimple (cf. [7, Theorem 3]). Since \( \rho_{A,l_0} \) is abelian and geometric, the representation \( \rho_{A,l_0} \) is locally algebraic in the sense of [19] (see also [8, Section 6, Proposition]). Therefore, by [17, (MT 1)], there exists a modulus of definition \( m \) and an algebraic homomorphism \( \phi: S_m \to GL_{2g} \) over \( \mathcal{Q} \) such that the \( l_0 \)-adic representation induced by \( \phi \) is isomorphic to \( \rho_{A,l_0} \). Here, the definition of the commutative algebraic group \( S_m \) over \( \mathcal{Q} \) is given in [19, Chapter II]. Note that any \( l_0 \)-adic representation arising from an algebraic morphism \( S_m \to GL_{2g} \) is automatically semisimple. Since \( \rho_{A,l_0}^{D_g} \) comes from the composition \( S_m \xrightarrow{D_g} S_m \xrightarrow{\phi} GL_{2g} \) where \( S_m \xrightarrow{D_g} S_m \) is the multiplication by \( D_g \) map, we obtain the fact that \( \rho_{A,l_0}^{D_g} \) is semisimple. \( \square \)

PROOF OF THEOREM 1.2. First we note that, if an abelian variety \( A \) over \( K \) satisfies (RT\(_{ab}\)), then \( \rho_{A,l} \) is abelian for any prime number \( l' \) (cf. [19, Chapter III, Section 2.3, Corollary 1]). Fix a prime number \( l_0 \) and denote by \( \mathcal{A}_{l_0} \) the set as in Lemma 4.4. Assume that there exist infinitely many prime numbers \( l \) such that \( \mathcal{A}'(K, g, l)_{ab} \) is not empty. For every such \( l \), we obtain the \( l_0 \)-adic representation \( \rho_{A,l_0}^{D_g} \) which is in the set \( \mathcal{A}_{l_0} \), where \( A \) is an abelian variety whose isomorphism class is in the set \( \mathcal{A}'(K, g, l)_{ab} \). By Lemma 4.4, we see that there exists a representation \( \rho_{l_0} \) in \( \mathcal{A}_{l_0} \) satisfying the following: For infinitely many \( l \), there exists an element \([A] \in \mathcal{A}'(K, g, l)_{ab} \) such that the representation \( \rho_{A,l_0}^{D_g} \) is isomorphic to \( \rho_{l_0} \). Thus we know that the representation \( \rho_{l_0} \) extends to a \( \mathcal{Q} \)-integral strictly compatible
system \((\rho_l)_{l}\) of \(2g\)-dimensional abelian semisimple \(l\)-adic representations of \(G_K\). Furthermore, for infinitely many prime numbers \(l\), the characters associated with a residual representation \(\bar{\rho}_l\) are of the form \(\bar{\chi}^{a}\) by Lemma 4.1, where \(\bar{\chi} : G_K \to \bar{\mathbb{F}}_{l}^\times\) is a continuous character which is unramified at all places of \(K\) above \(l\). Applying Corollary 2.12, we see that there exist integers \(m_1, \ldots, m_{2g}\) and a finite extension \(L\) of \(K\) such that \(\rho_{l_0}\) is isomorphic to \(\chi_{l_0}^{m_1} \oplus \chi_{l_0}^{m_2} \oplus \cdots \oplus \chi_{l_0}^{m_{2g}}\) on \(G_L\). In particular, for some prime number \(l\) and some \([A] \in \mathcal{A}(K, g, l)_{ab}\), \(\rho_{A,l_0}^{Dg}\) is isomorphic to \(\chi_{l_0}^{m_1} \oplus \chi_{l_0}^{m_2} \oplus \cdots \oplus \chi_{l_0}^{m_{2g}}\) on \(G_L\). Therefore, looking at the eigenvalues of the images of a Frobenius element (at some place) of \(\rho_{A,l_0}^{Dg}|_{G_L}\) and \(\chi_{l_0}^{m_1} \oplus \chi_{l_0}^{m_2} \oplus \cdots \oplus \chi_{l_0}^{m_{2g}}\), we know that \(D_g/2 = m_1 = m_2 = \cdots = m_{2g}\). Since \(\rho_{A,l_0}^{Dg}|_{G_L}\) has Hodge-Tate weights 0 and \(D_g\) at a place of \(L\) above \(l_0\) (cf. [19, III-7]), this is a contradiction.

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