THE BOHR PHENOMENON FOR ANALYTIC FUNCTIONS ON SIMPLY CONNECTED DOMAINS

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Abstract. In this paper, we investigate the Bohr phenomenon for the class of analytic functions defined on the simply connected domain

\[ \Omega_\gamma = \{ z \in \mathbb{C} : |z + \gamma| < \frac{1}{1 - \gamma} \} \text{ for } 0 \leq \gamma < 1. \]

We study improved Bohr radius, Bohr-Rogosinski radius and refined Bohr radius for the class of analytic functions defined in \( \Omega_\gamma \), and obtain several sharp results.

1. Introduction and Preliminaries

Let \( B(\mathbb{D}) \) be the class of analytic functions in unit disk \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \) such that \( f(\mathbb{D}) \subseteq \overline{\mathbb{D}} \). The classical Bohr theorem for functions \( f \in B(\mathbb{D}) \) says that if \( f(z) = \sum_{n=0}^{\infty} a_n z^n \), then its associated majorant series \( M_f(r) \) satisfies the following inequality

\[
M_f(r) := \sum_{n=0}^{\infty} |a_n|r^n \leq 1 \quad \text{for } |z| = r \leq \frac{1}{3}
\]

and the constant \( 1/3 \), called Bohr radius for the class \( B(\mathbb{D}) \), cannot be improved. The inequality (1.1) is known as classical Bohr inequality (1.1) for the class \( B(\mathbb{D}) \). The Bohr inequality was first obtained by Harald Bohr [24] in 1914 with the constant \( 1/6 \). The optimal value \( 1/3 \), which is called the Bohr radius for disk case was later established independently by Weiner, Riesz and Schur. For the proofs we refer to [40] and [42]. The notion of Bohr inequality has been generalized to several complex variables by finding the multidimensional Bohr radius. We refer the reader to the articles [6, 7, 23, 37]. For more information and intriguing aspects on Bohr phenomenon, we suggest the reader to glance through the articles [1]–[5], [8]–[9], and [15]–[18]. Bohr phenomenon for operator valued functions have been extensively studied by Bhowmik and Das (see [21, 22]).

The main aim of this article is to study the Bohr inequality for the class of analytic functions that are defined in a general simply connected domain in the complex plain. Let \( \Omega \) be a simply connected domain containing \( \mathbb{D} \) and \( B(\Omega) \) be the class of analytic

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functions in $\Omega$ such that $f(\Omega) \subseteq \overline{D}$. We define the Bohr radius $B = B_\Omega$ for the class $B(\Omega)$ by

$$B := \sup \left\{ r \in (0, 1) : \sum_{n=0}^{\infty} |a_n|r^n \leq 1 \text{ for all } f \in B(\Omega) \text{ with } f(z) = \sum_{n=0}^{\infty} a_n z^n, \ z \in \overline{D} \right\}.$$ 

In particular, if $\Omega = \mathbb{D}$, then $B_\mathbb{D} = 1/3$, which is the classical Bohr radius for the class $B(\mathbb{D})$. Let $\mathbb{D}(a, r) := \{ z \in \mathbb{C} : |z - a| < r \}$. Clearly, $\mathbb{D} := \mathbb{D}(0, 1)$. Let $0 \leq \gamma < 1$. We consider the disk $\Omega_\gamma$ defined by

$$\Omega_\gamma := \left\{ z \in \mathbb{C} : \left| z + \frac{\gamma}{1-\gamma} \right| < \frac{1}{1-\gamma} \right\}.$$ 

It is easy to see that $\Omega_\gamma$ always contains the unit disk $\mathbb{D}$. In 2010, the notion of classical Bohr inequality \cite{Fournier2010} has been generalized by Fournier and Ruscheweyh \cite{Fournier2010} to the class $B(\Omega)$. More precisely,

**Theorem 1.2.** \cite{Fournier2010} For $0 \leq \gamma < 1$, let $f \in B(\Omega_\gamma)$, with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $\mathbb{D}$. Then,

$$\sum_{n=0}^{\infty} |a_n|r^n \leq 1 \text{ for } r \leq \rho := \frac{1+\gamma}{3+\gamma}.$$ 

Moreover, $\sum_{n=0}^{\infty} |a_n|\rho^n = 1$ holds for a function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $B(\Omega_\gamma)$ if, and only if, $f(z) = c$ with $|c| = 1$.

In this article, we study the Bohr-Rogosinski radius for the class $B(\Omega)$, In 2017, Kayumov and Ponnusamy \cite{Kayumov2017} introduced Bohr-Rogosinski radius motivated from Rogosinski radius for bounded analytic functions in $\mathbb{D}$. Rogosinski radius is defined as follows: Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in $\mathbb{D}$ and its corresponding partial sum of $f$ is defined by $S_N(z) := \sum_{n=0}^{N-1} a_n z^n$. Then, for every $N \geq 1$, we have $|\sum_{n=0}^{N-1} a_n z^n| < 1$ in the disk $|z| < 1/2$ and the radius $1/2$ is sharp. Motivated by Rogosinski radius, Kayumov and Ponnusamy have considered the Bohr-Rogosinski sum $R_N^f(z)$ is defined by

$$R_N^f(z) := |f(z)| + \sum_{n=N}^{\infty} |a_n||z|^n. \hspace{1cm} (1.3)$$ 

It is worth to point out that $|S_N(z)| = |f(z) - \sum_{n=N}^{\infty} a_n z^n| \leq |R_N^f(z)|$. Thus, it is easy to see that the validity of Bohr-type radius for $R_N^f(z)$, which is related to the classical Bohr sum (Majorant series) in which $f(0)$ is replaced by $f(z)$, gives Rogosinski radius in the case of bounded analytic functions in $\mathbb{D}$. There has been significant and extensive research carried out on Improved-Bohr inequality and Bohr-Rogosinski radius (see \cite{10, 29, 30, 31, 32, 33, 34, 35, 36, 38}).

**Lemma 1.4.** \cite{Molla2018} Let $a \in \mathbb{D}$ and $f \in B(\mathbb{D})$ with

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n, \ |z-a| \leq 1-|a|.$$
Then,

$$|a_n| \leq (1 + |a|)^{n-1} \frac{1 - |a_0|^2}{(1 - |a|^2)^n}, \quad n \geq 1.$$  

Recently, Evdoridis et al. [26] obtained the following coefficient bounds for functions defined in $\Omega_\gamma$.

**Lemma 1.5.** [26] For $\gamma \in [0, 1)$, let

$$\Omega_\gamma = \left\{ z \in \mathbb{C} : \left| z + \frac{\gamma}{1 - \gamma} \right| < \frac{1}{1 - \gamma} \right\},$$

and let $f$ be an analytic function in $\Omega_\gamma$, bounded by 1, with the series representation $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in the unit disk $\mathbb{D}$. Then

$$|a_n| \leq \frac{1 - |a_0|^2}{1 + \gamma} \text{ for } n \geq 1.$$

### 2. Main Results

Before we state an improved version of inequality of Theorem 1.2, we prove the following lemma.

**Lemma 2.1.** Let $g : \mathbb{D} \rightarrow \mathbb{D}$ be an analytic function, $m(\geq 2)$ be an integer, and let $\gamma \in \mathbb{D}$ be such that $g(z) = \sum_{n=0}^{\infty} a_n (z - \gamma)^n$ for $|z - \gamma| \leq 1 - |\gamma|$. Then

$$\sum_{n=0}^{\infty} (|\alpha_n| + \beta |\alpha_n|^m) \rho^n \leq 1 \text{ for } \rho \leq \rho_0 := (1 - \gamma^2)/(3 + \gamma),$$

where

$$\beta = \frac{(1 - \gamma)^m(3 + \gamma) - (1 - \gamma^2)}{8(m - 1)} \text{ for } 0 \leq \gamma \leq \gamma_*, < 1,$$

where $\gamma_*$ is the smallest root of the equation $(1 - \gamma)^m(3 + \gamma) + \gamma^2 - 1 = 0$.

Using Lemma 2.1, we obtain the following improved version of Theorem 1.2 for the class $B(\Omega_\gamma)$.

**Theorem 2.3.** For $0 \leq \gamma < 1$, and integer $m (\geq 2)$, let $f \in B(\Omega_\gamma)$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $z \in \mathbb{D}$, then we have

$$|a_0| + \sum_{n=1}^{\infty} \left( |a_n| + \beta \frac{|a_n|^m}{(1 - \gamma)^{(m-1)n}} \right) r^n \leq 1 \text{ for } r \leq r_0 = \frac{1 + \gamma}{3 + \gamma},$$

where $\beta$ as in Lemma 2.1. Furthermore, the quantities $\beta$ and $(1 + \gamma)/(3 + \gamma)$ cannot be improved.

Figure 1 demonstrates values of $\gamma_*$ in $[0, 1)$ for which $\beta(\gamma) > 0$ with $0 \leq \gamma \leq \gamma_* < 1$. The values of $\gamma_*$ are $\gamma_*(10) = 0.1083$, $\gamma_*(21) = 0.0519$, $\gamma_*(50) = 0.0219$ and $\gamma_*(100) = 0.011$. 
Figure 1. The roots $\gamma_*(m)$ of the equation $(1-\gamma)^m(3+\gamma) + \gamma^2 - 1 = 0$.

**Lemma 2.4.** Let $g : \mathbb{D} \to \overline{\mathbb{D}}$ be an analytic function, $\lambda \in [0, 512/243]$ and let $\gamma \in \mathbb{D}$ be such that $g(z) = \sum_{n=0}^{\infty} \alpha_n (z - \gamma)^n$ for $|z - \gamma| < 1 - |\gamma|$. Then

$$\sum_{n=0}^{\infty} |\alpha_n| |\rho|^n + \left(\frac{8}{9} - \frac{27}{64} \lambda\right) \left(\frac{S_{\rho}^\gamma}{\pi}\right) + \lambda \left(\frac{S_{\rho}^\gamma}{\pi}\right)^2 \leq 1 \quad \text{for} \quad \rho \leq \rho_0 = \frac{1 - |\gamma|^2}{3 + |\gamma|},$$

where $S_{\rho}^\gamma$ denotes the area of the image of the disk $D(\gamma; r(1-|\gamma|))$ under the mapping $g$.

By applying Lemma 2.4, we obtain the following improved version of Theorem 1.2.

**Theorem 2.5.** For $0 \leq \gamma < 1$ and $0 \leq \lambda \leq 512/243$, let $f \in B(\Omega_{\gamma})$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$, for $|z - \gamma| \leq 1 - |\gamma|$, then we have

$$\sum_{n=0}^{\infty} |a_n| r^n + \left(\frac{8}{9} - \frac{27}{64} \lambda\right) \left(\frac{S_{r(1-\gamma)}^\gamma}{\pi}\right) + \lambda \left(\frac{S_{r(1-\gamma)}^\gamma}{\pi}\right)^2 \leq 1 \quad \text{for} \quad r \leq r_0 = \frac{1 + \gamma}{3 + \gamma}.$$

Furthermore, the quantities $8/9 - 27\lambda/64$, $\lambda$ and $(1+\gamma)/(3+\gamma)$ cannot be improved.

**Lemma 2.6.** For $\gamma \in \mathbb{D}$, let $g \in B(\mathbb{D})$ with $g(z) = \sum_{n=0}^{\infty} \alpha_n (z - \gamma)^n$, for $|z - \gamma| \leq 1 - |\gamma|$, then

$$|g(z)| + \sum_{n=N}^{\infty} |\alpha_n| |\rho|^n \leq 1, \quad \text{for} \quad \rho \leq \rho_N,$$

where $\rho_N$ is the root of

$$2(1+\gamma)\rho^N + (1+\gamma)(1-\gamma)^{N-1}(\rho - 1)(1 - \gamma - \rho) = 0$$

in $(0, 1)$.

Using Lemma 2.6, we obtain the following Bohr-Rogosinski radius for the class $B(\Omega_{\gamma})$. 
Theorem 2.7. For $0 \leq \gamma < 1$ and integer $N \geq 1$, let $f \in \mathcal{B}(\Omega_\gamma)$ with $f(z) = \sum_{n=0}^\infty a_n z^n$ for $z \in \mathbb{D}$. Then, we have
\[
|f(\frac{z - \gamma}{1 - \gamma})| + \sum_{n=N}^\infty |a_n| r^n \leq 1 \quad \text{for} \quad r \leq r_0 = \frac{\rho_N}{1 - \gamma},
\]
where $\rho_N$ is the root of the equation
\[(2.8) \quad 2(1 + \rho)\rho^N + (1 + \gamma)(1 - \gamma)^{N-1}(\rho - 1)(1 - \gamma - \rho) = 0.
\]
Furthermore, the constant $\rho_N/(1 - \gamma)$ cannot be improved.

Using Lemma 1.5, we establish the following refined Bohr inequality for the class $\mathcal{B}(\Omega_\gamma)$.

Theorem 2.9. For $0 \leq \gamma < 1$, let $f \in \mathcal{B}(\Omega_\gamma)$ with $f(z) = \sum_{n=1}^\infty a_n z^n$ for $z \in \mathbb{D}$. Then we have
\[
\sum_{n=0}^\infty |a_{n+1}| r^n + \left(\frac{1}{1 + |a_1|} + \frac{r}{1 - r}\right) \sum_{n=2}^\infty |a_n|^2 r^{2(n-1)} \leq 1 \quad \text{for} \quad r \leq r_0 = \frac{1 + \gamma}{3 + \gamma}.
\]
The constant $r_0$ cannot be improved.

3. Proofs of the Main Results

Proof of the Lemma 2.1. Without loss of generality, we may assume that $\gamma \in [0,1)$. Using Lemma 1.4, we obtain
\[(3.1) \quad \sum_{n=1}^\infty |a_n| \rho^n \leq \frac{1 - |a_0|^2}{1 + \gamma} \sum_{n=1}^\infty \left(\frac{\rho}{1 - \gamma}\right)^n = \frac{(1 - |a_0|^2) \rho}{(1 + \gamma)(1 - \gamma - \rho)}.
\]
Further, we have
\[(3.2) \quad \sum_{n=1}^\infty |a_n|^m \rho^n \leq \frac{(1 - |a_0|^2)^m}{(1 + \gamma)^m} \sum_{n=1}^\infty \left(\frac{\rho}{1 - \gamma}\right)^n = \frac{(1 - |a_0|^2)^m \rho}{(1 + \gamma)((1 - \gamma)^m - \rho)}.
\]
The series in (2.2) contains positive terms for $\beta \geq 0$. Our aim is to find the smallest value of $\gamma$ in $[0,1)$ for which $\beta \geq 0$. That is
\[
\beta = \frac{(1 - \gamma)^m(3 + \gamma) - (1 - \gamma)^2}{8(m - 1)} := \frac{Q(\gamma)}{8(1 - m)} \geq 0,
\]
where $Q(\gamma) = (1 - \gamma)^m(3 + \gamma) - (1 - \gamma)^2$. Clearly, $\gamma = 1$ is a root of $Q(\gamma)$. Since $Q(\gamma)$ is a polynomial such that $Q(0) = 2 > 0$ and $m \geq 2$, we have
\[
Q\left(\frac{9}{10}\right) = \frac{3.9}{10^m} + \frac{81}{100} - 1 \leq \frac{84.9}{100} - 1 = \frac{15.1}{100} < 0.
\]
Therefore, there exists at least one root of $Q(\gamma)$ in $(0,1)$. Let $\gamma_*$ be the smallest root of $Q(\gamma)$. Then, it is easy to see that $Q(\gamma) \geq 0$, and hence $\beta \geq 0$ for all $\gamma \in [0,\gamma_*]$. 
A simple computation using (3.1) and (3.2) shows that

\[
|\alpha_0| + \sum_{n=1}^{\infty} |\alpha_n|\rho^n + \beta \sum_{n=1}^{\infty} |\alpha_n|^m\rho^n \\
\leq |\alpha_0| + \frac{(1 - |\alpha_0|^2) \rho}{(1 + \gamma)(1 - \gamma - \rho)} + \beta \frac{(1 - |\alpha_0|^2)^m \rho}{(1 + \gamma)((1 - \gamma)^m - \rho)} \\
= 1 + \Psi_\gamma(\rho) \\
\]

provided \( \Psi_\gamma(\rho) \leq 0 \), where

\[
\Psi_\gamma(\rho) = 1 - \frac{|\alpha_0|^2}{1 + \gamma} \left( \frac{\rho}{1 - \gamma - \rho} \right) + \beta \left( \frac{1 - |\alpha_0|^2}{1 + \gamma} \right)^m \left( \frac{\rho}{(1 - \gamma)^2 - \rho} \right) - (1 - |\alpha_0|). \\
\]

Since \((1 - \gamma) - \rho > (1 - \gamma)^m - \rho\), it is easy to see that \( \Psi_\gamma(\rho) \) is an increasing function of \( r \) for \( r < (1 - \gamma)^m \). A simplification shows that

\[
\Psi_\gamma(\rho) \\
= K \left( 1 + (1 - |\alpha_0|^2)^{m-1} \left( \frac{2\beta \rho}{(1 + \gamma)((1 - \gamma)^m - \rho)} + \frac{\phi_\gamma(\rho)}{(1 - |\alpha_0|^2)^{m-1}} \right) - \frac{2}{1 + |\alpha_0|} \right), \\
\]

where

\[
K = 1 - \frac{|\alpha_0|^2}{2} \quad \text{and} \quad \phi_\gamma(\rho) = \frac{2r}{(1 + \gamma)(1 - \gamma - \rho)} - 1. \\
\]

Let \( \rho \leq \rho_0 \) be such that \( \Psi_\gamma(\rho) \leq \Psi_\gamma(\rho_0) \), and \( \phi_\gamma(\rho_0) = 0 \). Then, it is easy to see that \( \phi_\gamma(\rho_0) = 0 \) if, and only if, \( \rho_0 = (1 - \gamma^2)/(3 + \gamma) \). Therefore, it is enough to prove that \( \Psi_\gamma(\rho_0) \leq 0 \) for \( |\alpha_0| \leq 1 \). Let \( \beta = \eta ((1 - \gamma)^m(3 + \gamma) - (1 - \gamma^2)) \), then it is easy to see

\[
\Psi_\gamma(\rho_0) = K \left( 1 + 2\eta(1 - |\alpha_0|^2)^{m-1} \left( \frac{1 - \gamma^2}{(1 + \gamma)^m} - \frac{2}{1 + |\alpha_0|} \right) \right) \\
:= KG_\gamma(|\alpha_0|), \\
\]

where

\[
G_\gamma(x) = 1 + 2\eta A(\gamma)(1 - x^2)^{m-1} - \frac{2}{x + 1} \\
\]

and

\[
A(\gamma) = \frac{1 - \gamma^2}{(1 + \gamma)^m} > 0 \quad \text{for} \quad \gamma \in [0, 1). \\
\]

It now remains to show that \( G_\gamma(x) \leq 0 \) for \( \gamma \in [0, 1) \) and \( x \in [0, 1] \). Since

\[
A'(\gamma) = -\frac{2(1 + \gamma)\gamma + m(1 - \gamma^2)}{(1 + \gamma)^{m+1}} \leq 0, \quad \text{for} \quad \gamma \in [0, 1) \\
\]

and \( A(0) = 1, \ A(1) = 0 \), it follows that \( A(\gamma) \) is a decreasing function and hence \( A(\gamma) \leq A(0) = 1 \). Since \( x \leq 1 \) and \( 0 < A(\gamma) \leq 1 \), we have

\[
-A(\gamma)x(1 + x^2)(1 - x^2)^{m-2} > -4. \\
\]
From (3.4), we have
\[
(G_\gamma(x))' = \frac{2}{(1 + x)^2} (1 - 2\eta A(\gamma)(m - 1)x(1 + x)^2(1 - x^2)^{m-2})
\]
\[
\geq \frac{2(1 - 8(m - 1)\eta)}{(1 + x)^2}.
\]
Clearly, \((G_\gamma(x))' > 0\) for \(x \in (0, 1)\) whenever \(\eta \leq 1/(8(m - 1))\). Therefore, \(G_\gamma(x)\) is an increasing function on \([0, 1]\) for \(\eta \leq 1/(8(m - 1))\). Equivalently,
\[
\beta \leq \frac{(1 - \gamma)^m(3 + \gamma) - (1 - \gamma^2)}{8(m - 1)}.
\]
In particular, \(G_\gamma(x) \leq 0\) for \(\gamma \in [0, \gamma_*]\) and \(x \in [0, 1]\), where \(\gamma_*\) is the smallest root of the equation \((1 - \gamma)^m(3 + \gamma) - (1 - \gamma^2) = 0\). This completes the proof. 

**Proof of Theorem 2.3.** For \(0 \leq \gamma < 1\), let
\[
\Omega_\gamma = \left\{ z \in \mathbb{C} : \left| z + \frac{\gamma}{1 - \gamma} \right| < \frac{1}{1 - \gamma} \right\}
\]
and the function \(f : \Omega_\gamma \to \mathbb{D}\) be given by \(f(z) = \sum_{n=0}^{\infty} a_n z^n\). Then the function \(g\) defined by
\[
g(z) = f \left( \frac{z - \gamma}{1 - \gamma} \right) = \sum_{n=0}^{\infty} \frac{a_n}{(1 - \gamma)^n} (z - \gamma)^n \text{ for } |z - \gamma| < 1 - \gamma
\]
belongs to \(B(\mathbb{D})\). Applying Lemma 2.1 to the function \(g\), we obtain
\[
|a_0| + \sum_{n=1}^{\infty} \left( \frac{|a_n|}{(1 - \gamma)^n} + \beta \left( \frac{|a_n|}{(1 - \gamma)^n} \right)^m \right) \rho^n \leq 1 \text{ for } \rho \leq \rho_0 = \frac{1 - \gamma^2}{3 + \gamma}.
\]
That is
\[
|a_0| + \sum_{n=1}^{\infty} \left( |a_n| + \beta \frac{|a_n|^m}{(1 - \gamma)^{(m-1)n}} \right) \left( \frac{\rho}{1 - \gamma} \right)^n \leq 1 \text{ for } \rho \leq \rho_0 = \frac{1 - \gamma^2}{3 + \gamma}
\]
which is equivalent to
\[
|a_0| + \sum_{n=1}^{\infty} \left( |a_n| + \beta \frac{|a_n|^m}{(1 - \gamma)^{(m-1)n}} \right) r^n \leq 1 \text{ for } r \leq r_0 = \frac{1 + \gamma}{3 + \gamma},
\]
where \(\rho = r(1 - \gamma)\) and
\[
\beta = \frac{(1 - \gamma)^m(3 + \gamma) - (1 - \gamma^2)}{8(m - 1)} \text{ for } 0 \leq \gamma \leq \gamma_* < 1.
\]
Here \(\gamma_*\) is the smallest root of the equation \((1 - \gamma)^m(3 + \gamma) + \gamma^2 - 1 = 0\).

In order to prove the sharpness of the radius, we consider the composition function \(f_a = h \circ H\) which maps \(\Omega_\gamma\), univalently onto \(\mathbb{D}\), where \(H : \Omega_\gamma \to \mathbb{D}\) defined by
Therefore, \( \Phi(z) = (1 - \gamma)z + \gamma \) and \( h : \mathbb{D} \to \mathbb{D} \) with \( h(z) = (a - z)/(1 - az) \), for \( a \in (0, 1) \). A simple computation shows that

\[
f_a(z) = \frac{a - \gamma - (1 - \gamma)z}{1 - a\gamma - a(1 - \gamma)} = C_0 - \sum_{n=1}^{\infty} C_n z^n \text{ for } z \in \mathbb{D},
\]

where \( a \in (0, 1) \) and

\[
C_0 = \frac{a - \gamma}{1 - a\gamma} \quad \text{and} \quad C_n = \frac{1 - a^2}{a(1 - a\gamma)} \left( \frac{a(1 - \gamma)}{1 - a\gamma} \right)^n.
\]

A simple computation shows that

\[
|a_0| + \sum_{n=1}^{\infty} \left( |a_n| + \beta \frac{|a_n|^m}{(1 - \gamma)^{m-1}n} \right) r^n
= \frac{a - \gamma}{1 - a\gamma} + \sum_{n=1}^{\infty} \left( \frac{1 - a^2}{a(1 - a\gamma)} \left( \frac{a(1 - \gamma)}{1 - a\gamma} \right)^n + \beta \frac{(1 - a^2)^m}{(1 - \gamma)^{m-1}n a^n(1 - a\gamma)^m} \left( \frac{a(1 - \gamma)}{1 - a\gamma} \right)^m \right) r^n
= \frac{a - \gamma}{1 - a\gamma} + \frac{(1 + a)(1 - a)(1 - \gamma)r}{(1 - a\gamma)(1 - a\gamma - ar(1 - \gamma))} + \frac{\beta(1 - a)^m(1 + a)m r}{(1 - \gamma)^{mn-n-m}(1 - a\gamma)^m}
= 1 - (1 - a)\Phi_\gamma(r),
\]

where

\[
\Phi_\gamma(r) = \frac{(1 + a)(1 - \gamma)r}{(1 - a\gamma)(1 - a\gamma - ar(1 - \gamma))} + \frac{\beta(1 - a)^{m-1}(1 + a)m r}{(1 - \gamma)^{mn-n-m}(1 - a\gamma)^m} + \frac{1}{1 - a - \frac{1}{1 - a\gamma}} - 1
= \frac{(1 + a)(1 - \gamma)r}{(1 - a\gamma)(1 - a\gamma - ar(1 - \gamma))} - \frac{\beta(1 - a)^{m-1}(1 + a)m r}{(1 - \gamma)^{mn-n-m}(1 - a\gamma)^m} - \frac{1}{1 - 1/a - \frac{1}{1 - a\gamma}} + 1 + \frac{2r_0}{(1 - \gamma)(1 - r_0)} + 1 + \frac{1}{1 - \gamma} = 0.
\]

Thus, \( \Phi_\gamma(r) < 0 \) for \( r > r_0 \). Hence, \( 1 - (1 - a)\Phi_\gamma(r) > 1 \) for \( r > r_0 \), which shows that \( r_0 \) is the best possible. This completes the proof.

**Proof of Lemma 2.4.** Without loss of generality, we assume that \( \gamma \in [0, 1) \). Also let \( z \in \mathbb{D}_\gamma := \mathbb{D}(\gamma, 1 - \gamma) \) if, and only if, \( w = (z - \gamma)/(1 - \gamma) \in \mathbb{D} \). Then we have

\[
g(z) = \sum_{n=0}^{\infty} \alpha_n (1 - \gamma)^n \phi^n(z) = \sum_{n=0}^{\infty} b_n \phi^n(z) := G(\phi(z))
\]

for \( z \in \mathbb{D}_\gamma \), where \( b_n = \alpha_n (1 - \gamma)^n \). A simple computation shows that

\[
(3.5) \quad \frac{S_\gamma}{\pi} = \frac{1}{\pi} \text{Area}(G(\mathbb{D}(0, \rho))) \leq (1 - |b_0|^2)^2 \frac{\rho^2}{(1 - \rho^2)^2} = (1 - |\alpha_0|^2)^2 \frac{\rho^2}{(1 - \rho^2)^2}.
\]
A simple computation shows that
\begin{equation}
\sum_{n=1}^{\infty} |\alpha_n|^2 \rho^n \leq \frac{1 - |\alpha_0|^2}{1 + \gamma} \sum_{n=1}^{\infty} \left( \frac{\rho}{1 - \gamma} \right)^n = \frac{1 - |\alpha_0|^2}{1 + \gamma} \frac{\rho}{1 - \gamma - \rho}.
\end{equation}

In view of (3.5) and (3.6), we obtain
\begin{align*}
|\alpha_0| + \sum_{n=1}^{\infty} |\alpha_n|^2 \rho^n & + k \left( \frac{S_2^\gamma}{\pi} \right)^2 + \lambda \left( \frac{S_\gamma^2}{\pi} \right)^2 \\
& = |\alpha_0| + \frac{(1 - |\alpha_0|^2) \rho}{(1 + \gamma)(1 - \gamma - \rho)} + k \frac{(1 - |\alpha_0|^2)^2 \rho^2}{(1 - \rho^2)^2} + \lambda \frac{(1 - |\alpha_0|^2)^4 \rho^4}{(1 - \rho^2)^4} \\
& = 1 + \Psi_1^\gamma(\rho),
\end{align*}

where
\begin{align*}
\Psi_1^\gamma(\rho) &= \frac{(1 - |\alpha_0|^2) \rho}{(1 + \gamma)(1 - \gamma - \rho)} + k \frac{(1 - |\alpha_0|^2)^2 \rho^2}{(1 - \rho^2)^2} + \lambda \frac{(1 - |\alpha_0|^2)^4 \rho^4}{(1 - \rho^2)^4} - (1 - |\alpha_0|) \\
& \quad \text{which can be written as} \\
& = \frac{1}{2} \left( 1 + 2\lambda(1 - |\alpha_0|^2)^3 \left( \frac{\rho^4}{(1 - \rho^2)^4} + \frac{k}{\lambda} \frac{\rho^2}{(1 - \rho^2)^2(1 - |\alpha_0|^2)^2} \right) \right) \\
& \quad + \frac{2\rho}{2\lambda(1 - |\alpha_0|^2)^3} \left( \frac{2\rho}{(1 + \gamma)(1 - \gamma - \rho) - 1} - \frac{2}{1 + |\alpha_0|} \right).
\end{align*}

Let \( \rho \leq \rho_0 \). Then, it is easy to see that \( \Psi_1^\gamma(\rho) \) is an increasing function and hence \( \Psi_1^\gamma(\rho) \leq \Psi_1^\gamma(\rho_0) \), where
\[ \frac{2\rho_0}{(1 + \gamma)(1 - \gamma - \rho_0)} = 1, \quad \text{i.e.,} \quad \rho_0 = \frac{1 - \gamma^2}{3 + \gamma}. \]

A simple computation shows that
\begin{align*}
\Psi_1^\gamma(\rho_0) &= \frac{1 - |\alpha_0|^2}{2} \left( 1 + 2\lambda(1 - |\alpha_0|^2)^3 A^4(\gamma) + 2k(1 - |\alpha_0|^2)A^2(\gamma) - \frac{2}{1 + |\alpha_0|} \right) \\
& = \frac{1 - |\alpha_0|^2}{2} J(|\alpha_0|),
\end{align*}

where
\[ J(x) = 1 + 2\lambda(1 - x^2)^3 A^4(\gamma) + 2k(1 - x^2)A^2(\gamma) - \frac{2}{1 + x} \quad \text{for} \quad x \in [0, 1] \]

and \( A(\gamma) = \frac{(3 + \gamma)(1 - \gamma^2)}{(3 + \gamma)^2 - (1 - \gamma^2)^2} \).

It is enough to show that \( J(x) \leq 0 \) for \( x \in [0, 1] \) and \( \gamma \in [0, 1) \) so that \( \Psi_1^\gamma(\rho_0) \leq 0 \). We note that \( A(\gamma) > 0 \) for \( \gamma \in [0, 1) \). Further,
\[ J(0) = 2\lambda A^4(\gamma) + 2kA^2(\gamma) - 1, \quad \text{and} \quad \lim_{x \to 1^-} J(x) = 0. \]
It can be seen that $A(\gamma) = (f_1 \circ f_2)(\gamma)$, where $f_1(\rho) = \rho/(1 - \rho^2)$ and $f_2(\gamma) = (1 - \gamma^2)/(3 + \gamma)$. Since $A'(\gamma) = f_1'(f_2(\gamma))f_2'(\gamma)$, where

\[(3.7)\quad f_2'(\gamma) = -\left(\frac{\gamma^2 + 6\gamma + 1}{(3 + \gamma)^2}\right) < 0\]

which implies that $f_1(\rho)$ is an increasing function of $\rho$ in $(0, 1)$, and $f_2$ is a decreasing function of $\gamma$ in $[0, 1)$. Hence, it follows that $A(\gamma)$ is a decreasing function of $\gamma$ in $[0, 1)$, with $A(0) = 3/8$ and $A(1) = 0$. It can be seen that $A^2(\gamma)$ and $A^4(\gamma)$ are decreasing functions on $[0, 1)$. Therefore, we have

\[A^2(\gamma) \leq A^2(0) = \frac{9}{64} \quad \text{and} \quad A^4(\gamma) \leq A^4(0) = \frac{81}{4096}.\]

Since $x \in [0, 1]$, we have

\[x(1 + x)^2A^2(\gamma) \leq \frac{9}{16} \quad \text{and} \quad x(1 + x)^2(1 - x^2)^2A^4(\gamma) \leq \frac{81}{1024}.\]

As a consequence, we obtain

\[J'(x) = \frac{2}{(1 + x)^2} \left( 1 - 2kx(1 + x)^2A^2(\gamma) - 6\lambda x(1 + x)^2(1 - x^2)^2A^4(\gamma) \right) \geq 0, \quad \text{if} \quad k + 27\lambda/64 \leq 8/9.\]

Therefore, $J(x)$ is an increasing function in $[0, 1]$ for $k + 27\lambda/64 \leq 8/9$. Hence, $J(x) \leq 0$ for all $x \in [0, 1]$ and $\gamma \in [0, 1)$. This completes the proof. \qed

**Proof of Theorem 2.5.** Let $f \in B(\Omega)$ and $g(z) = f((z - \gamma)/(1 - \gamma))$. Then, it is easy to see that $g \in B(\mathbb{D})$ and

\[g(z) = \sum_{n=0}^{\infty} \frac{a_n}{(1 - \gamma)^n}(z - \gamma)^n.\]

Using Lemma 2.4, we obtain

\[\sum_{n=0}^{\infty} \frac{|a_n|}{(1 - \gamma)^n} \rho^n + \left(\frac{8}{9} - \frac{27}{64}\lambda\right) \left(\frac{S^\gamma_\rho}{\pi}\right) + \lambda \left(\frac{S^\gamma_\rho}{\pi}\right)^2 \leq 1 \quad \text{for} \quad \rho \leq \frac{1 - \gamma^2}{3 + \gamma}\]

which is equivalent to

\[(3.8)\quad \sum_{n=0}^{\infty} |a_n| \left(\frac{\rho}{1 - \gamma}\right)^n + \left(\frac{8}{9} - \frac{27}{64}\lambda\right) \left(\frac{S^\gamma_\rho}{\pi}\right) + \lambda \left(\frac{S^\gamma_\rho}{\pi}\right)^2 \leq 1 \quad \text{for} \quad \rho \leq \frac{1 - \gamma^2}{3 + \gamma}.\]

Set $\rho = r(1 - \gamma)$, then in view of (3.8), we obtain

\[\sum_{n=0}^{\infty} |a_n|r^n + \left(\frac{8}{9} - \frac{27}{64}\lambda\right) \left(\frac{S^\gamma_{r(1 - \gamma)}}{\pi}\right) + \lambda \left(\frac{S^\gamma_{r(1 - \gamma)}}{\pi}\right)^2 \leq 1 \quad \text{for} \quad r \leq \frac{1 + \gamma}{3 + \gamma}.\]
To show the sharpness of the result, we consider the following function

\[ f_a(z) = \frac{a - \gamma - (1 - \gamma)z}{1 - a\gamma - a(1 - \gamma)} \quad \text{for } z \in \Omega_\gamma \text{ and } a \in (0, 1). \]

Define \( \phi : \mathbb{D} \to \mathbb{D} \) by \( \phi(z) = (a - z)/(1 - az) \) and \( H : \Omega_\gamma \to \mathbb{D} \) by \( H(z) = (1 - \gamma)z + \gamma. \) Then, the function \( f_a = \phi \circ H \) maps \( \Omega_\gamma \), univalently onto \( \mathbb{D} \). A simple computation shows that

\[ f_a(z) = \frac{a - \gamma - (1 - \gamma)z}{1 - a\gamma - a(1 - \gamma)} = C_0 - \sum_{n=1}^{\infty} C_n z^n \quad \text{for } z \in \mathbb{D}, \]

where \( a \in (0, 1) \) and

\[ C_0 = \frac{a - \gamma}{1 - a\gamma} \quad \text{and} \quad C_n = \frac{1 - a^2}{a(1 - a\gamma)} \left( \frac{a(1 - \gamma)}{1 - a\gamma} \right)^n. \]

A simple computation using (3.9) shows that

\[
\sum_{n=0}^{\infty} |a_n| r^n + \left( \frac{8}{9} - \frac{27}{64} \lambda \right) \left( \frac{S_{r(1-\gamma)}^\gamma}{\pi} \right) + \lambda \left( \frac{S_{r(1-\gamma)}^\gamma}{\pi} \right)^2 \left( \frac{r^4(1-a)^4(1-\gamma)^8}{((1-a\gamma)^2 - a^2r^2(1-\gamma)^4)^4} \right) \\
= \frac{a - \gamma}{1 - a\gamma} + \left( \frac{1 - a^2}{1 - a\gamma} \right) \frac{(1 - \gamma)r}{1 - a\gamma - ar(1 - \gamma)} + \left( \frac{8}{9} - \frac{27}{64} \lambda \right) \frac{r^2(1-a^2)^2(1-\gamma)^4}{((1-a\gamma)^2 - a^2r^2(1-\gamma)^4)^2} \\
+ \lambda \frac{r^4(1-a)^4(1-\gamma)^8}{((1-a\gamma)^2 - a^2r^2(1-\gamma)^4)^4} - \frac{1}{1-a} \frac{a - \gamma}{1 + a\gamma - 1}.
\]

where

\[ \Phi_1^\gamma(r) = -\frac{(1 + a)(1 - \gamma)r}{(1 - a\gamma - ar(1 - \gamma))(1 - a\gamma)} - \left( \frac{8}{9} - \frac{27}{64} \lambda \right) \frac{r^2(1-a)^2(1-\gamma)^4}{((1-a\gamma)^2 - a^2r^2(1-\gamma)^4)^2} \\
- \lambda \frac{r^4(1-a)^4(1-\gamma)^8}{((1-a\gamma)^2 - a^2r^2(1-\gamma)^4)^4} - \frac{1}{1-a} \left( \frac{a - \gamma}{1 + a\gamma - 1} \right). \]

It is easy to see that \( \Phi_1^\gamma(r) \) is strictly decreasing function of \( r \) in \((0, 1)\). Therefore, for \( r > r_0 = (1+\gamma)/(3+\gamma) \), we have \( \Phi_1^\gamma(r) < \Phi_1^\gamma(r_0) \). An elementary calculation shows that

\[
\lim_{a \to 1} \Phi_1^\gamma(r_0) = -\frac{2r_0}{1 - \gamma(1-r_0)} + \frac{1 + \gamma}{1 - \gamma} = 0.
\]

Therefore, \( \Phi_1^\gamma(r) < 0 \) for \( r > r_0 \). Hence, \( 1 - (1 - a)\Phi_1^\gamma(r) > 1 \) for \( r > r_0 \), which shows that \( r_0 \) is the best possible."
For functions $g \in B(D)$, from Lemma 1.4 we have

$$|a_n| \leq (1 + |\gamma|)^{n-1} \frac{1 - |a_0|^2}{(1 - |\gamma|^2)^n} \quad \text{for} \quad n \geq 1. \quad (3.11)$$

A simple computation using (3.11) gives

$$\sum_{n=N}^{\infty} |a_n| \rho^n \leq \frac{1 - |a_0|^2}{1 + \gamma} \sum_{n=N}^{\infty} \left( \frac{\rho}{1 - \gamma} \right)^n = \frac{(1 - |a_0|^2)}{(1 + \gamma)(1 - \gamma)^{N-1}} \left( \frac{\rho^N}{1 - \gamma - \rho} \right). \quad (3.12)$$

From (3.10) and (3.12) we obtain

$$|g(z)| + \sum_{n=N}^{\infty} |a_n| \rho^n \leq \frac{\rho + |g(0)|}{1 + \rho|g(0)|} + \frac{(1 - |a_0|^2)}{(1 + \gamma)(1 - \gamma)^{N-1}} \left( \frac{\rho^N}{1 - \gamma - \rho} \right) \leq 1 + \frac{\Phi_N(\rho)}{(1 + \rho|a_0|)(1 + \gamma)(1 - \gamma)^{N-1}(1 - \gamma - \rho)},$$

where

$$\Phi_N(\rho) = (\rho + |a_0|)A(\gamma)(1 - \gamma - \rho) + (1 + |a_0|)(1 - |a_0|)(1 + \rho|a_0|)\rho^N - (1 + \rho|a_0|)A(\gamma)(1 - \gamma - \rho) = (1 - |a_0|) \left( (1 + |a_0|)(1 + \rho|a_0|)\rho^N + A(\gamma)(\rho - 1)(1 - \gamma - \rho) \right) \leq (1 - |a_0|) \left( 2(1 + \gamma)\rho^N + A(\gamma)(\rho - 1)(1 - \gamma - \rho) \right),$$

where $A(\gamma) = (1 + \gamma)(1 - \gamma)^{N-1}$ and $|a_0| \leq 1$. An observation shows that $\Phi_N(\rho) \leq 0$ if $2(1 + \gamma)\rho^N + A(\gamma)(\rho - 1)(1 - \gamma - \rho) \leq 0$, and this holds for $\rho \leq \rho_N$, where $\rho_N$ is the root of

$$F_N(\gamma, \rho) = 2(1 + \gamma)\rho^N + A(\gamma)(\rho - 1)(1 - \gamma - \rho) = 0.$$

The existence of the root $\rho_N$ in $(0, 1)$ follows from the fact that $F_N(\gamma, \rho)$ is continuous and $F_N(\gamma, 0)F_N(\gamma, 1) < 0$. \hfill \Box

**Proof of Theorem 2.7** For $0 \leq \gamma < 1$, let $f \in B(\Omega_\gamma)$ such that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $z \in D$. Then, it is easy to see that

$$g(z) = f \left( \frac{z - \gamma}{1 - \gamma} \right) \in B(D) \quad \text{for} \quad |z - \gamma| < 1 - |\gamma|.$$

Further, $g(z) = f \left( \frac{z - \gamma}{1 - \gamma} \right) = \sum_{n=0}^{\infty} \frac{a_n}{(1 - \gamma)^n} (z - \gamma)^n.$

An application of Lemma 2.6 shows that

$$|f \left( \frac{z - \gamma}{1 - \gamma} \right)| + \sum_{n=N}^{\infty} \frac{|a_n|}{(1 - \gamma)^n} \rho^n \leq 1 \quad \text{for} \quad \rho \leq \rho_N. \quad (3.13)$$
Since $|z - \gamma| < 1 - \gamma$, we set $z - \gamma = w(1 - \gamma)$ for some $w \in \mathbb{D}$ and $\rho = r(1 - \gamma)$. Then, from (3.13), we obtain

\[
|f(w)| + \sum_{n=N}^{\infty} |a_n|r^n \leq 1 \quad \text{for} \quad r \leq \frac{\rho_N}{1 - \gamma},
\]

where $\rho_N$ as in Lemma 2.6. That is, $\rho_N$ is the smallest root of the equation $2(1 + \rho)\rho^N + A(\gamma)(\rho - 1)(1 - \gamma - \rho) = 0$.

In order to show the sharpness of the result, we consider the following function $f_a$ defined by

\[
f_a(z) = \frac{1 - \gamma - (1 - \gamma)z}{(1 - a\gamma) - (1 - \gamma)z} = B_0 - \sum_{n=1}^{\infty} B_n z^n \quad \text{for} \quad z \in \mathbb{D}.
\]

For $\gamma \in [0, 1)$, $a > \gamma$ and $\rho = r(1 - \gamma)$, we obtain

(3.14)

\[
M := |f_a(-\rho)| + \sum_{n=N}^{\infty} |a_n|\rho^n
\]

\[
= \frac{(a - \gamma) + (1 - \gamma)\rho}{(1 - a\gamma) + (1 - \gamma)\rho} + \frac{1 - a^2}{a(1 - a\gamma)} \left( \frac{a(1 - \gamma)}{1 - a\gamma} \right)^N \rho^N \left( \frac{1 - a\gamma}{(1 - a\gamma) - (1 - \gamma)\rho} \right)
\]

\[
= \frac{(a - \gamma) + (1 - \gamma)\rho}{(1 - a\gamma) + (1 - \gamma)\rho} + \frac{(1 - a^2)B^N\rho^N}{a((1 - a\gamma) - (1 - \gamma)\rho)}, \quad \text{where} \quad B = \frac{a(1 - \gamma)}{1 - a\gamma}
\]

\[
= \frac{(a - \gamma) + (1 - \gamma)\rho)((1 - a\gamma) - (1 - \gamma)\rho) + d\rho^N(1 - a^2)((1 - a\gamma) + (1 - \gamma)\rho)
\]

\[
+ ((1 - a\gamma) + (1 - \gamma)\rho)((1 - a\gamma) - (1 - \gamma)\rho)
\]

From (3.14), it is easy to see that $M > 1$ if $V(\rho) > 0$, where

\[
V(\rho) = ((a - \gamma) + (1 - \gamma)\rho)((1 - a\gamma) - (1 - \gamma)\rho) + d\rho^N(1 - a^2)((1 - a\gamma) + (1 - \gamma)\rho)
\]

\[
- ((1 - a\gamma) + (1 - \gamma)\rho)((1 - a\gamma) - (1 - \gamma)\rho)
\]

\[
= (1 - a)\left( (1 + a)((1 - a\gamma) + (1 - \gamma)\rho) d\rho^N
\]

\[
+ \left( (1 - a\gamma) - (1 - \gamma)\rho \right) \left( \rho(1 - \gamma) - (1 + \gamma) \right) \right).
\]

Note that $V(\rho) > 0$ if

(3.15) \[ W(\rho) := (1 + a)((1 - a\gamma) + (1 - \gamma)\rho) d\rho^N \]

\[
+ \left( (1 - a\gamma) - (1 - \gamma)\rho \right) \left( \rho(1 - \gamma) - (1 + \gamma) \right) > 0.
\]

Therefore, $M \leq 1$ for all $a \in [0, 1)$, only in the case when $\rho \leq \rho_N$. Finally, allowing $a \to 1$, from the inequality (3.15), it can be seen that $M > 1$ if $\rho > \rho_N$. Thus, $M > 1$ if $r > \rho_N/(1 - \gamma)$. This proves the sharpness. \qed
Proof of Theorem 2.9. Let \( f \in \mathcal{B}(\Omega_\gamma) \) be given by \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) for \( z \in \mathbb{D} \). Then, \( f \) can be expressed as \( f(z) = zh(z) \), where \( h \in \mathcal{B}(\Omega_\gamma) \) with \( h(z) = \sum_{n=0}^{\infty} b_n z^n \) and \( b_n = a_{n+1} \). Let \( |b_0| = |a_1| = a \), and \( h_0(z) = g(z) - b_0 \). Using Lemma 1.5, we obtain

\[
\begin{align*}
\sum_{n=0}^{\infty} |b_n|^r^n &= \left( \frac{1}{1 + |b_0|} + \frac{r}{1 + r} \right) \sum_{n=1}^{\infty} |b_n|^2 r^{2n} \\
&\leq a + \frac{1 - a^2}{1 + \gamma} \frac{r}{1 - r} + \left( \frac{1}{1 + a} + \frac{r}{1 - r} \right) \left( \frac{1 - a^2}{1 + \gamma} \right)^2 \frac{r^2}{1 - r^2}. \\
\intertext{That is,}
\sum_{n=0}^{\infty} |b_n|^r^n &\leq a + \frac{1 - a^2}{1 + \gamma} \frac{r}{1 - r} + \left( \frac{1}{1 + a} + \frac{r}{1 - r} \right) \left( \frac{1 - a^2}{1 + \gamma} \right)^2 \frac{r^2}{1 - r^2} \\
&\quad - \left( \frac{1}{1 + |b_0|} + \frac{r}{1 + r} \right) \sum_{n=1}^{\infty} |b_n|^2 r^{2n}.
\end{align*}
\]

Since

\[
\sum_{n=1}^{\infty} |a_n|^r^n = \sum_{n=0}^{\infty} |b_n|^r^{n+1} = r \sum_{n=0}^{\infty} |b_n|^r^n,
\]
in view of \((3.17)\) and \((3.18)\), we obtain

\[
\begin{align*}
\sum_{n=1}^{\infty} |a_n|^r^n &\leq r \left( a + \frac{1 - a^2}{1 + \gamma} \frac{r}{1 - r} \right) + \left( \frac{1}{1 + a} + \frac{r}{1 - r} \right) \left( \frac{1 - a^2}{1 + \gamma} \right)^2 \frac{r^2}{1 - r^2} \\
&\quad - \left( \frac{1}{1 + a} + \frac{r}{1 - r} \right) \sum_{n=1}^{\infty} |a_{n+1}|^2 r^{2n+1} \\
&= ra + \left( \frac{1 - a^2}{1 + \gamma} \right) \frac{r^2}{1 - r} + \left( \frac{1}{1 + a} + \frac{r}{1 - r} \right) \left( \frac{1 - a^2}{1 + \gamma} \right)^2 \frac{r^3}{1 - r^2} \\
&\quad - \left( \frac{1}{1 + a} + \frac{r}{1 - r} \right) \sum_{n=2}^{\infty} |a_n|^2 r^{2n-1}.
\end{align*}
\]

Further simplification shows that

\[
\begin{align*}
\sum_{n=1}^{\infty} |a_n|^r^n + \left( \frac{1}{1 + a} + \frac{r}{1 - r} \right) \sum_{n=2}^{\infty} |a_n|^2 r^{2n-1} \\
&\leq ra + \left( \frac{1 - a^2}{1 + \gamma} \right) \frac{r^2}{1 - r} + \left( \frac{1}{1 + a} + \frac{r}{1 - r} \right) \left( \frac{1 - a^2}{1 + \gamma} \right)^2 \frac{r^3}{1 - r^2} \\
&:= \mathcal{T}(a).
\end{align*}
\]

It is easy to see that \( \mathcal{T} \) can be represented as

\[
\mathcal{T}(a) = ar + A(1 - a^2) + B(1 - a)(1 - a^2) + C(1 - a^2)^2,
\]
where

\[ A = A(r) = \frac{r^2}{(1 + \gamma)(1 - r)}, \]
\[ B = B(r) = \frac{r^3}{(1 + \gamma)^2(1 - r^2)} \quad \text{and} \]
\[ C = C(r) = \frac{r^4}{(1 + \gamma)(1 - r)(1 - r^2)}. \]

Clearly, \( B \) and \( C \) are positive. We note that,

\[ T'(a) = r - 2Aa + B(3a^2 - 2a - 1) + 4C(a^3 - a), \]
\[ T''(a) = -2A + 2B(3a - 1) + 4C(3a^2 - 1) \quad \text{and} \]
\[ T'''(a) = 6B + 24Ca. \]

Since \( B \) and \( C \) are positive, it follows that \( T'''(a) > 0 \) for \( a \in [0, 1] \). In other words, \( T'' \) is an increasing function of \( a \) in \( [0, 1] \). Therefore, \( T''(a) \leq T''(1) = -2A + 4B + 8C = \frac{2r^2}{(1 + \gamma)^2(1 - r)(1 - r^2)} L(r), \)

where

\[ L(r) = 4r^2 + 2r(1 - r) - (1 + \gamma)(1 - r^2) = (1 + r)(r(3 + \gamma) - (1 + \gamma)). \]

It is easy to see that \( L(r) \leq 0 \) for \( r \leq r_0 = (1 + \gamma)/(3 + \gamma) \). Hence, \( T''(a) \leq 0 \) for \( a \in [0, 1] \) which implies that \( T' \) is decreasing in \( [0, 1] \). Therefore, for \( r \leq r_0 = (1 + \gamma)/(3 + \gamma) \), we obtain

\[ T'(a) > T'(1) = 1 - 2Aa = r \frac{1 + \gamma - r(3 + \gamma)}{(1 + \gamma)(1 - r)}. \]

Clearly, for \( r \leq r_0 \), we have \( T'(1) \geq 0 \) for all \( a \in [0, 1] \). Since \( T'(a) \geq 0 \) in \([0, 1]\), \( T \) is an increasing function in \([0, 1]\), and hence, we have \( T(a) \leq T(1) = r \). A simple computation shows that

\[ \sum_{n=0}^{\infty} |a_{n+1}| r^n + \left( \frac{1}{1 + |a_1|} + \frac{r}{1 - r} \right) \sum_{n=2}^{\infty} |a_n|^2 r^{2(n-1)} \leq 1 \quad \text{for} \quad r \leq r_0 = \frac{1 + \gamma}{3 + \gamma}. \]

To show that the sharpness of the radius we consider the function \( f_a \) by

\[ f_a(z) = z \left( \frac{a - \gamma - (1 - \gamma)z}{1 - a\gamma - a(1 - \gamma)z} \right) = B_0 z - \sum_{n=1}^{\infty} B_n z^{n+1} \quad \text{for} \quad z \in \mathbb{D}, \]

where

\[ B_0 = \frac{a - \gamma}{1 - a\gamma} \quad \text{and} \quad B_n = \frac{(1 - a^2)}{a(1 - a\gamma)} \left( \frac{a(1 - \gamma)}{1 - a\gamma} \right)^n. \]
It is easy to see that $a_1(f_a) = B_0$ and $a_n(f_a) = -B_{n-1}$. For $n \geq 2$, $\gamma \in [0, 1]$, and $a > \gamma$, a simple calculation shows that

$$D(r) := \sum_{n=1}^{\infty} |a_n|r^n + \left(\frac{1}{1 + |a_1|} + \frac{r}{1 - r}\right) \sum_{n=2}^{\infty} |a_n|^2 r^{2n-1}$$

$$= \left(\frac{a - \gamma}{1 - a\gamma}\right) r + \sum_{n=2}^{\infty} (1 - a^2) \left(\frac{a(1 - \gamma)}{1 - a\gamma}\right)^{n-1} r^n$$

$$+ \left(\frac{1}{1 + |B_0|} + \frac{r}{1 - r}\right) \sum_{n=2}^{\infty} \frac{(1 - a^2)^2}{a^2(1 - a\gamma)^2} \left(\frac{a(1 - \gamma)}{1 - a\gamma}\right)^{2(n-1)} r^{2n-1}$$

$$= \left(1 - \frac{1 - a}{1 - a\gamma}\chi(r)\right) r,$$

where

$$\chi(r) := 1 + r - \frac{(1 + a)(1 - \gamma)r}{1 - a\gamma - a(1 - \gamma)r} - \left(\frac{1 - a\gamma}{1 + a(1 - \gamma)} + \frac{r}{1 - r}\right) \frac{(1 + a)(1 - a^2)}{1 - a\gamma} \frac{(1 - \gamma)^2 r^2}{(1 - a\gamma)^2 - a^2(1 - \gamma)^2 r^2}.$$

It is not difficult to show that $\chi$ is strictly decreasing function in $r \in (0, 1)$. Hence, for $r > r_0$, we have $\chi(r) < \chi(x_0)$. It is worth to point out that

$$\lim_{a \to 1} \chi(r_0) = 1 + \gamma - \frac{2(1 - \gamma)r_0}{1 - \gamma - (1 - \gamma)r_0} = 1 + \gamma - \frac{2r_0}{1 - r_0} = 0.$$

This shows that $\chi(r) \leq 0$ for $r > r_0$ as $a \to 1$, and hence $D(r) > r$ for $r > r_0$. Therefore

$$\sum_{n=1}^{\infty} |a_n|r^n + \left(\frac{1}{1 + a} + \frac{r}{1 - r}\right) \sum_{n=2}^{\infty} |a_n|^2 r^{2n-1} > 1$$

and hence $r_0$ is the best possible. This completes the proof. \hfill \Box

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