PFISTER’S LOCAL–GLOBAL PRINCIPLE AND SYSTEMS OF QUADRATIC FORMS

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Abstract. Let \( q \) be a unimodular quadratic form over a field \( K \). Pfister’s famous local–global principle asserts that \( q \) represents a torsion class in the Witt group of \( K \) if and only if it has signature 0, and that in this case, the order of Witt class of \( q \) is a power of 2. We give two analogues of this result to systems of quadratic forms, the second of which applying only to nonsingular pairs. We also prove a counterpart of Pfister’s theorem for finite-dimensional \( K \)-algebras with involution, generalizing a result of Lewis and Unger.

Introduction

Let \( K \) be a field of characteristic different from 2 and let \( (V, q) \) be a unimodular (i.e. nondegenerate) quadratic space over \( K \). We write \( n \times q \) for the quadratic form \( (v_1, \ldots, v_n) \mapsto \sum_i q(v_i) : V^n \to K \).

Pfister’s celebrated local-global principle (see [17, Theorem 2.7.3], for instance) states that there exists \( n \in \mathbb{N} \) such that \( n \times q \) is hyperbolic if and only if the signature of \( q \) (relative to all orderings of the field \( K \)) is 0, and that in this case, \( n \) can be taken to be a power of 2. This work is concerned with analogues of this result to systems of quadratic forms, and in particular to pairs of forms.

To that end, we say that a system of quadratic forms \( \{q_i\}_{i \in I} \) on a \( K \)-vector space \( V \) is hyperbolic if \( V \) is the direct sum of two \( K \)-subspaces on which each of the forms \( q_i \) vanishes. This is one of several possible notions for a “trivial” system of quadratic forms listed by Pfister in [14, p. 133]; it is the most suitable for our purposes as it implies that every form in the \( K \)-span of the system has signature 0 (Proposition 1.1). This definition also appeared in [5, §4], and if it is applied to single non-unimodular quadratic forms, then Pfister’s local-global principle still holds as stated, see Proposition 1.1.

It is tempting to hope that if every quadratic form in \( \operatorname{span}_K \{q_i | i \in I\} \) has signature 0, then \( n \times \{q_i\}_{i \in I} \) is hyperbolic for some \( n \). However, as we demonstrate in Section 2 this is already false for pairs of forms. Therefore, one cannot expect Pfister’s local-global principle to generalize naïvely to systems of forms, and indeed, the analogues that we shall give here will take a more sophisticated form.

To phrase our results, let \( A = A(\{q_i\}_{i \in I}) \) denote the \( K \)-subalgebra of \( \operatorname{End}(V) \times \operatorname{End}(V)^{\text{op}} \) consisting of pairs \((\phi, \psi^{\text{op}})\) satisfying

\[
q_i(\psi x, y) = q_i(x, \phi y)
\]

for all \( x, y \in V \) and \( i \in I \), and let \( \sigma : A \to A \) denote the involution given by \((\phi, \psi^{\text{op}})^\sigma = (\psi, \phi^{\text{op}})\).

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Key words and phrases. quadratic form, system of quadratic forms, signature, ordered field, algebra with involution, hermitian category, hermitian form.
This construction had been utilized by many authors, e.g., Bayer-Fluckiger \[1\] §1.1 and Wilson \[18\] §4.3. Following the latter source, we call \(A\) the \(K\)-algebra of adjoints of \(\{q_i\}_{i \in I}\) and \(\sigma\) its canonical involution.

Denote the involution-trace quadratic form \(x \mapsto \text{Tr}_{A/K}(x^\sigma x) : A \to K\) by \(q_{A,\sigma}\).

Our first main results is:

**Theorem A.** In the previous notation, the following conditions are equivalent:

(a) \(n \times \{q_i\}_{i \in I}\) is hyperbolic for some \(n \in \mathbb{N}\).

(b) \(\text{sgn} q_{A,\sigma} = 0\).

When these conditions hold, the minimal \(n\) for which (a) holds is a power of 2.

This is a generalization of Pfister's local-global principle because, when \(\{q_i\}_{i \in I}\) consists of a single form \(q\), we have \(\text{sgn} q_{A,\sigma} = (\text{sgn} q)^2\) (Proposition 4.1(i)).

In the course of proving this result, we show that a finite-dimensional \(K\)-algebra with involution \((A, \sigma)\) admits \(n \in \mathbb{N}\) such that \((A, \sigma) \otimes_K (M_n(K), t)\) is hyperbolic (see Section 3) if and if \(\text{sgn} q_{A,\sigma} = 0\) (Theorem 3.3). This generalizes a theorem of Lewis and Unger \[12\] Theorem 3.2], who established the case where \((A, \sigma)\) is central simple over \(K\).

Our second generalization of Pfister's local-global principle applies only to pairs of quadratic forms, but is in the spirit of the naive statement we disqualified above.

Letting \((V, \{q_i\}_{i \in I})\) and \((A, \sigma)\) be as before, it can happen that adjoining a quadratic form \(q : V \to K\) to the system \(\{q_i\}_{i \in I}\) will not change the algebra of adjoints \(A\). For example, this always the case if \(q \in \text{span}_K \{q_i \mid i \in I\}\). We denote by \(\text{Cl}\{q_i \mid i \in I\}\) the \(K\)-vector space of all such forms; this construction was introduced to us by James Wilson, who also observed that \(\text{Cl}(\cdot)\) is a closure operator.

By virtue of Theorem A if some \(q \in \text{Cl}\{q_i \mid i \in I\}\) has nonzero signature, then there cannot exist an \(n \in \mathbb{N}\) such that \(n \times \{q_i\}_{i \in I}\) is hyperbolic. Our second main result asserts the converse of this statement for nonsingular pairs of quadratic forms, provided \(K\) is a number field or real closed. Here, a system of quadratic forms \(\{q_i\}_{i \in I}\) is called nonsingular if there is a \(K\)-field \(L\) such that \(\text{span}_L \{q_i \mid i \in I\}\) contains a unimodular quadratic form over \(L\) (one can take \(L = K\) if \(K\) is infinite).

**Theorem B.** Suppose that \(K\) is a number field or a real closed field and let \(\{q_i\}_{i=1,2}\) be a nonsingular pair of quadratic forms on a \(K\)-vector space \(V\). Then the following conditions are equivalent:

(a) \(n \times \{q_i\}_{i=1,2}\) is hyperbolic for some \(n \in \mathbb{N}\).

(b) Every \(q \in \text{Cl}\{q_1, q_2\}\) has signature 0.

If \(q_1 = q_2\), then \(\text{Cl}\{q_1, q_2\} = Kq_1\) (Proposition 4.1), so Theorem B also generalizes Pfister's local-global principle.

In fact, Theorem B holds for all fields \(K\) satisfying a certain condition (see Section 5), which we believe to hold for all fields. We conjecture that the nonsingularity assumption can be removed as well.

We further note that Theorem A implies that there exists \(n \in \mathbb{N}\) such that \(n \times \{q_i\}_{i \in I}\) is hyperbolic if and only if the same statement holds after base-changing to the real closure of \(K\) relative to each of its ordering. Thus, writing \(K_P\) for the real closure of \(K\) relative to an ordering \(P\), we have the following corollary, which holds over any field.

**Corollary C.** Let \(\{q_i\}_{i=1,2}\) be a nonsingular pair of quadratic forms on a \(K\)-vector space \(V\). Then the following conditions are equivalent:

(a) \(n \times \{q_i\}_{i=1,2}\) is hyperbolic for some \(n \in \mathbb{N}\).
For every ordering $P$ of $K$ and every $q \in \text{Cl}[(q_1)_{K_P},(q_2)_{K_P}]$, we have $\text{sgn}_q = 0$.

Finally, we ask:

**Question D.** Do Theorem 9 and Corollary 6 apply to systems consisting of more than 2 quadratic forms?

The paper is organized as follows: Section 1 is preliminary and recalls relevant definitions and facts. In Section 2, we give nontrivial examples of vector spaces of quadratic forms consisting of forms with signature 0, and demonstrate our main results on them. Section 3 concerns with generalizing Lewis and Ung's theorem stated above. This is used in Section 4 to prove Theorem A. The remaining two sections concern with proving Theorem B. Section 5 recalls necessary facts about hermitian categories, and the proof itself is given in Section 6.

We are grateful to Eva Bayer-Fluckiger and David Leep for several useful conversations. We also thank James Wilson for introducing to us the notion of closure of sets of quadratic forms used in Theorem B.

1. Preliminaries

Throughout this paper, $K$ denotes a field of characteristic not 2. All $K$-vector spaces and $K$-algebras are assumed to be finite-dimensional.

We refer the reader to [17] for necessary definitions concerning quadratic, bilinear and hermitian forms.

Our assumptions on the characteristic of $K$ allows us no to distinguish between quadratic and bilinear forms, and we will use the same letter to denote a quadratic form $q : V \to K$ and its associated bilinear form $(x,y) \mapsto \frac{1}{2}(q(x+y)-q(x)-q(y)) : V \times V \to K$. As usual, given $\alpha_1,\ldots,\alpha_n \in K$, the diagonal quadratic $(x_1,\ldots,x_n) \mapsto \sum_i \alpha_i x_i^2 : K^n \to K$ is denoted $(\alpha_1,\ldots,\alpha_n)$. We do not require quadratic forms to be unimodular (i.e. nondegenerate).

If $L$ is a $K$-field, $U$ and $V$ are $K$-vector spaces and $\phi \in \text{Hom}_K(U,V)$, then we write $U_L = U \otimes_K L$ and $\phi_L = \phi \otimes_K \text{id}_L : U_L \to V_L$. Similar notation will be applied to algebras, quadratic forms, involutions, etcetera.

Recall that an ordering $P$ of $K$ is a subset of $K^\times := K - \{0\}$ such that $P + P \subseteq P$, $P \cdot P \subseteq P$, $K^\times = P \cup -P$ and $P \cap -P = \emptyset$. In this case, given $\alpha,\beta \in K$, we write $\alpha <_P \beta$ if $\beta - \alpha \in P$, and set

$$\text{sgn}_P(\alpha) = \begin{cases} 1 & \alpha \in P \\ 0 & \alpha = 0 \\ -1 & \alpha \in -P \end{cases}.$$ 

If $P$ is clear from the context, we shall suppress it and write $\alpha < \beta$, resp. $\text{sgn}(\alpha)$. The real closure of $K$ relative to $P$ is denoted $K_P$.

Let $P$ be an ordering of $K$ and let $(V,q)$ be a quadratic space over $K$. Recall that $q$ is called positive (resp. negative) definite relative to $P$ if $q(v) > 0$ (resp. $q(v) < 0$) for all $v \in V - \{0\}$. The $P$-signature of $q$, denoted $\text{sgn}_P(q)$, is largest possible dimension of a subspace on which $q$ is positive definite minus the largest possible dimension of a subspace on which $q$ is negative definite. Note that this definition also makes sense for non-unimodular forms, and we have

$$\text{sgn}_P(\alpha_1,\ldots,\alpha_n) = \sum_i \text{sgn}_P(\alpha_i).$$

Let $\Theta$ denote the set of all orderings of $K$. The total signature of $q$, denoted $\text{sgn} q$, is the function $\Theta \to \mathbb{Z}$ mapping $P$ to $\text{sgn}_P(q)$. 

Let \((V,q)\) be a quadratic space. Recall from the introduction that \(q\) is called hyperbolic if there exist subspaces \(U, U' \subseteq V\) such that \(V = U \oplus U'\) and \(q\) vanishes on \(U\) and \(U'\). This agrees with the usual definition of hyperbolic quadratic forms when \(q\) is unimodular. The following proposition summarizes some properties of hyperbolic quadratic forms in the non-unimodular case. Notably, such forms have signature 0.

**Proposition 1.1.** Let \((V,q)\) be a quadratic space, possibly non-unimodular. Then:

(i) \(q\) is hyperbolic if and only if \(q \equiv q' \oplus (0, \ldots, 0)\) for some hyperbolic unimodular quadratic form \(q'\).

(ii) Pfister’s local-global principle holds for \((V,q)\): There exists \(n \in \mathbb{N}\) such that \(n \times q\) is hyperbolic if and only if \(\text{sgn} \, q = 0\), and in this case, the smallest such \(n\) is a power of 2.

**Proof.** (i) The “if” part is clear. For the “only if” part, write \(V = U \oplus W\) so that \(q\) vanishes on both \(U\) and \(W\). Let \(R\) denote the radical of \(q\), let \(\overline{V} = V/R\) and let \(q' : \overline{V} \to K\) denote the quadratic form given by \(q'(x + R) = q(x)\) (this is well-defined because \(R = V^\perp\)). Then \(q'\) is unimodular and vanishes on \(\overline{U} := (R + U)/R\) and \(\overline{W} := (R + W)/R\), hence \(q'\) is hyperbolic. Since \(q \equiv q' \oplus (0, \ldots, 0)\), we are done.

(ii) This follows from (i) and Pfister’s local-global principle. \(\square\)

Let \(V\) be a \(K\)-vector space and let \(I\) be a set. By an \(I\)-indexed system of quadratic forms on \(V\) we mean a collection \(\{q_i\}_{i \in I}\) consisting of quadratic forms on \(V\). We also say that \((V, \{q_i\}_{i \in I})\) is a vector space with a system of quadratic forms. Given \(n \in \mathbb{N}\), we write \(n \times \{q_i\}_{i \in I} = \{n \times q_i\}_{i \in I}\), which is an \(I\)-indexed system of forms on \(V^n\). Recall that \(\{q_i\}_{i \in I}\) is called nonsingular if there is a \(K\)-field \(L\) such that \(\text{span}_L \{q_i\}_{i \in I} \mid i \in I\) contains a unimodular form.

**Lemma 1.2.** Let \(\{q_i\}_{i \in I}\) be a nonsingular system of quadratic forms on a \(K\)-vector space \(V\). If \(K\) is infinite, then \(\text{span}_K \{q_i\}_{i \in I} \mid i \in I\) contains a unimodular form.

**Proof.** Let \(L\) be a field such that \(\text{span}_L \{q_i\}_{i \in I} \mid i \in I\) contains a unimodular form. Then there exist \(t \in \mathbb{N}\), \(q_1, \ldots, q_t \in \{q_i\}_{i \in I}\) and \(\alpha_1, \ldots, \alpha_t \in L\) such that \(\sum_{i=1}^t \alpha_i(q_i) L\) is a unimodular quadratic form over \(L\). Let \(B\) be a basis to \(V\) and let \(f \in K[x_1, \ldots, x_t]\) denote the determinant of \(\sum_i x_i q_i\) relative to \(B\). Then \(f(\alpha_1, \ldots, \alpha_t) \neq 0\), hence \(f \neq 0\). Since \(K\) is infinite, there are \(\beta_1, \ldots, \beta_t \in K\) such that \(f(\beta_1, \ldots, \beta_t) \neq 0\). Then \(\sum_i \beta_i q_i \in \text{span}_K \{q_i\}_{i \in I}\) is unimodular. \(\square\)

## 2. Examples

Before setting to prove Theorems A and B, we first exhibit nontrivial examples of systems of quadratic forms with \(K\)-span consisting of signature-0 forms. In particular, we shall see that the dimension of the \(K\)-span can be arbitrary large, even when the system is not hyperbolic, and that such systems \(\{q_i\}_{i \in I}\) may fail to admit \(n \in \mathbb{N}\) such that \(n \times \{q_i\}_{i \in I}\) is hyperbolic.

For the sake of brevity, quadratic forms on \(K^n\) will be given simply as their Gram matrix relative to the standard basis. In this setting, the ring of adjoints of a system of quadratic forms \(\{q_i\}_{i \in I}\) on \(K^n\) is the collection of pairs \((\phi, \psi^{op})\) \(\in \text{End}(K^n) \times \text{End}(K^n)^{op} = M_n(K) \times M_n(K)^{op}\) satisfying

\[
\psi^i q_i = q_i \phi \quad \forall i \in I.
\]

Moreover, a quadratic form \(q : K^n \to K\) (viewed as a symmetric matrix) lies in \(\text{Cl}\{q_i \mid i \in I\}\) if and only if

\[
\psi^i q = q \phi \quad \forall (\phi, \psi^{op}) \in A\{q_i\}_{i \in I}\).
\]
Example 2.1. Let \( \{q_i\}_{i=1}^{n^2+1} \) be a basis to the space of quadratic forms on \( K^{2n} \) taking the form

\[
\begin{bmatrix}
\alpha I_n & a \\
\alpha^t & -\alpha I_n
\end{bmatrix},
\]

where \( I_n \) denotes the \( n \times n \) identity matrix, \( a \in M_n(K) \) and \( \alpha \in K \). To see that every form in \( Q := \text{span}_K \{q_i\}_{i=1}^{n^2+1} \) has signature 0, let \( P \) be an ordering of \( K \) and consider \( U = K^n \times \{0\}^n \) and \( W = \{0\}^n \times K^n \). Then for every \( q \in Q \), exactly one of the following holds:

- \( q|_U \) is positive definite and \( q|_W \) is negative definite relative to \( P \);
- \( q|_U \) is negative definite and \( q|_W \) is positive definite relative to \( P \);
- \( q \) vanishes on both \( U \) and \( W \).

Each of these possibilities implies \( \text{sgn}_P q = 0 \), so \( Q \) is an \((n^2+1)\)-dimensional space consisting of 0-signature forms.

The system \( \{q_i\}_{i=1}^{n^2+1} \) is not hyperbolic because there is no nonzero vector which is annihilated by all forms in the system. However, \( 2 \times \{q_i\}_{i=1}^{n^2+1} \) is hyperbolic. Indeed, \( 2 \times q_i \) vanishes on

\[
V_1 := \{(u, v, -v, u) \mid u, v \in K^n\}
\quad \text{and} \quad
V_2 := \{(u, v, v, -u) \mid u, v \in K^n\}
\]

for all \( i \), and \( K^{4n} = V_1 \oplus V_2 \) (here we view \( K^{4n} \) as \( K^n \times K^n \times K^n \times K^n \)).

Example 2.2. Take \( K = \mathbb{R} \) and let \( \{q_1, q_2\} \) be a basis to the space \( Q \) of quadratic forms on \( K^4 \) of the form

\[
\begin{bmatrix}
-\alpha & \alpha & \alpha & \alpha \\
\alpha & \beta & \gamma & \alpha \\
\alpha & \beta & \alpha & \beta \\
\alpha & \gamma & \beta & \alpha
\end{bmatrix},
\]

It is easy to see that every \( q \in Q \) is hyperbolic, and thus has signature 0. However, there is no \( n \in \mathbb{N} \) such that \( n \times \{q_i\}_{i=1, 2} \) is hyperbolic. Indeed, straightforward computation shows that \( A = A(\{q_1, q_2\}) \) is the \( K \)-subalgebra of \( M_4(K) \times M_4(K)^{\text{op}} \) consisting of pairs of the form

\[
\begin{bmatrix}
x & y & -z & -z \\
y & \alpha & \beta & \gamma \\
z & \alpha & \beta & \alpha \\
-\alpha & \alpha & \beta & \gamma
\end{bmatrix}^{\text{op}}.
\]

This in turn implies that \( \text{Cl}\{q_1, q_2\} \) is the 3-dimensional space of quadratic forms of the form

\[
\begin{bmatrix}
-\alpha & \alpha & \alpha & \alpha \\
\alpha & \beta & \beta & \beta \\
\alpha & \beta & \gamma & \gamma \\
\alpha & \beta & \gamma & \gamma
\end{bmatrix},
\]

and this space contains forms of nonzero signature (take \( \alpha = \beta = 0 \) and \( \gamma \neq 0 \)).

Alternatively, one can check that \( \text{sgn} q_{A, \sigma} = 2 \), and reach the same conclusion using Theorem A.

Example 2.3. Let \( K = \mathbb{Q} \) and let \( \{q_1, q_2\} \) be a basis to the 2-dimensional \( K \)-vector space \( Q \) consisting of quadratic forms on \( K^4 \) of the form

\[
\begin{bmatrix}
\alpha & \beta & \beta & \beta \\
\alpha & 2\beta & \alpha & \beta \\
\beta & \alpha & \beta & \alpha
\end{bmatrix}.
\]
Then every \( q \in Q \) is either the zero form or a hyperbolic unimodular quadratic form, hence \( Q \) consists entirely of forms with signature 0. However, there exist forms in \( Q \otimes_{K} \mathbb{R} \) of nonzero signature (e.g. take \( \alpha = 2 \) and \( \beta = \sqrt{2} \)), so there is no \( n \in \mathbb{N} \) such that \( n \times \{ q_i \}_{i=1}^{n} \) is hyperbolic. Theorem \([\text{B}]\) guarantees that we can also find \( q \in Cl\{ q_1, q_2 \} \) with nonzero signature, and indeed, one can check that the diagonal form \( \langle 0, 0, 2, 1 \rangle \) is such an example.

We finish with a general method for producing high-dimensional vector spaces of quadratic forms consisting of forms with signature 0. Small-scale experiments suggest that applying it with a “generic” choice of parameters will result in a system having a form of nonzero signature in its closure.

**Example 2.4.** Let \( n \in \mathbb{N} \) and let \( S \) and \( E \) be \( K \)-subspaces of \( M_\mathbb{Q}(K) \) such that \( S \) consists of symmetric matrices representing quadratic forms of signature 0, and any nonzero matrix in \( E \) is invertible. Let

\[
Q = Q(S, E) := \left\{ \begin{bmatrix} 0 & e \\ e^t & s \end{bmatrix} \mid e \in E, s \in S \right\} \subseteq M_{2n}(K).
\]

We claim that the signature of any quadratic form in \( Q \) is 0. Indeed, if \( q = \begin{bmatrix} 0 & e \\ e^t & s \end{bmatrix} \in Q \), then \( e \) is either invertible or 0. In the first case, \( q \) is unimodular of dimension \( 2n \) and admits a totally isotropic subspace of dimension \( n \), so it is hyperbolic and has signature 0, whereas in the second case, \( \text{sgn}(q) = \text{sgn}(s) = 0 \).

In the case \( K = \mathbb{R} \), the largest possible dimension of \( E \) was determined by Adams \([\text{I}]\) and equals to the Hurwitz–Radon number \( \rho(n) \) given by \( 8a + 2b \) if \( n = 2^a + \beta \), with \( 0 \leq a, 0 \leq b \leq 3 \) and \( \beta \) odd.

### 3. Algebras with Involution and Involution-Trace Forms

By a \( K \)-algebra with involution we mean a pair \((A, \sigma)\) such that \( A \) is a \( K \)-algebra and \( \sigma \) is a \( K \)-involution. A \( K \)-algebra with involution is simple if it has no nonzero proper ideals stable under its involution. In this case, \( \text{Cent}(A)^{\{\sigma\}} := \{ a \in \text{Cent}(A) : a^\sigma = a \} \) is a field. Recall that the involution-trace form of a \( K \)-algebra with involution \((A, \sigma)\) is the quadratic form \( q_{A, \sigma} : A \to K \) given by

\[
q_{A, \sigma}(x) = Tr_{A/K}(x^\sigma x).
\]

It is not unimodular in general.

Following \([\text{I}]\), we say that \((A, \sigma)\) is a central simple \( K \)-algebra with involution if \((A, \sigma)\) is simple and \( \text{Cent}(A)^{\{\sigma\}} = K \). We alert the reader that in this case, it is common to define the involution trace form of \((A, \sigma)\) using the reduced trace \( \text{Trd}_{A/\text{Cent}(A)} \) instead of the trace; see \([\text{II}]\) and \([\text{III}]\).

For every \( n \in \mathbb{N} \), write \( n \times (A, \sigma) = (M_n(A), n \times \sigma) \), where \( n \times \sigma \) is the involution \((a_{ij}) \mapsto (a_{ji}^\sigma) : M_n(A) \to M_n(A) \). Then \( n \times (A, \sigma) \cong (A \otimes \sigma) \otimes_K (M_n(K), t) \), where \( t \) denotes the matrix transpose, and \( q_{n \times (A, \sigma)} \cong n^2 \times q_{A, \sigma} \).

We say that \((A, \sigma)\), or just \( \sigma \), is hyperbolic if there exists an idempotent \( e \in A \) such that \( e^\sigma + e = 1 \). The relation to hyperbolic hermitian forms is expressed in the following proposition.

**Proposition 3.1.** Let \((A, \sigma)\) be a \( K \)-algebra with involution, let \( n \in \mathbb{N} \) and let \( f_n : A^n \times A^n \to A \) denote the 1-hermitian form over \((A, \sigma)\) given by \( f_n((x_i)_{i=1}^n, (y_i)_{i=1}^n) = \sum_i x_i^\sigma y_i \). Then \( n \times \sigma \) is hyperbolic if and only if \( f_n \) is hyperbolic.

**Proof.** View the elements of the right \( A \)-module \( A^n \) as column vectors and identify \( \text{End}_A(A^n) \) with \( M_n(A) \). Writing \( \tau = n \times \sigma \), it is easy to see that \( f_n(ax, y) = f_n(x, a^\tau y) \) for all \( a \in M_n(A) \) and \( x, y \in A^n \).
Suppose that there exists an idempotent \( e \in M_n(A) \) such that \( e^2 + e = 1 \). Then \( f_n(x, y) = f_n(x, e^2 y) = f_n(x, 0) = 0 \), and similarly \( f_n((1 - e)x, (1 - e)y) = 0 \). Since \( A^n = \text{Im}(e) + \text{Im}(1 - e) \), it follows that \( f_n \) is hyperbolic.

Conversely, suppose that there exist \( A \)-submodules \( U, V \subseteq A \) such that \( A^n = U \oplus V \) and \( f(U, U) = f(V, V) = 0 \), and let \( e = \text{id}_U \oplus 0_V \in M_n(A) \). Then for all \( x, y \in A^n \), we have \( f_n(x, e^2 y) = f_n(x, y) = f_n(ex, ey + (1 - e)y) = f_n(ex, (1 - e)y) = f_n(ex + (1 - e)x, (1 - e)y) = f_n(x, (1 - e)y) \). Since \( f_n \) is unimodular, \( e^2 = 1 - e \) and \( \tau \) is hyperbolic.

We record the following corollary:

**Corollary 3.2.** Let \( (A, \sigma) \) be a \( K \)-algebra with involution and let \( n, m \in \mathbb{N} \). If \( n \times \sigma \) and \( m \times \sigma \) are hyperbolic, then so is \( \gcd(n, m) \times \sigma \).

**Proof.** By Proposition 3.1, we need to show that \( \gcd(n, m) \times f_1 \) is hyperbolic. By assumption, \( f_n = n \times f_1 \) and \( f_m = m \times f_1 \) represent the trivial class in the Witt group of \( (A, \sigma) \), so \( \gcd(n, m) \times f_1 \) also represents the trivial class. By [13, Proposition 5.12] (for instance), this means that \( \gcd(n, m, \sigma) \) is hyperbolic.

The purpose of this section is to prove the following theorem, which was established by Lewis and Unger [13, Theorem 3.2] for central simple \( K \)-algebras with involution.

**Theorem 3.3.** Let \( (A, \sigma) \) be a finite-dimensional \( K \)-algebra with involution. Then the following conditions are equivalent:

1. \( n \times (A, \sigma) \) is hyperbolic for some \( n \in \mathbb{N} \).
2. \( \text{sgn} \, q_{A, \sigma} = 0 \).

When these conditions hold, the minimal \( n \) for which (a) holds is a power of 2.

Similarly to [13], we first establish the theorem when \( K \) real-closed, and then use it to prove the general case.

**Lemma 3.4.** Suppose that \( K \) is real closed and \( (A, \sigma) \) is a simple \( K \)-algebra with involution. Then:

1. \( \text{sgn} \, q_{A, \sigma} \geq 0 \).
2. \( \text{sgn} \, q_{A, \sigma} = 0 \) if and only if \( 2 \times (A, \sigma) \) is hyperbolic.

**Proof.** The case where \( (A, \sigma) \) is central simple over \( K \) is contained in [6, Corollary 5.3]. It remains to consider the case where \( C := \text{Cent}(A)^{\sigma} \) is strictly larger than \( K \). Since \( K \) is real closed, \( C = K[\sqrt{-1}] \) and \( C \) is algebraically closed. In this case, it is well-known that there exists \( n \in \mathbb{N} \) such that one of the following hold:

1. \( (A, \sigma) \cong (M_n(C), t) \);
2. \( (A, \sigma) \cong (M_{2n}(C), s) \) with \( s \) given by \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d^t & -b^t \\ -c^t & a^t \end{pmatrix} \) \( a, b, c, d \in M_n(K) \);
3. \( (A, \sigma) \cong (M_n(C) \times M_n(C)^{op}, u) \) with \( u \) given by \( (a, b^{op}) \mapsto (h, a^{op}) \).

It is routine to check that in each of these cases \( 2 \times (A, \sigma) \) is hyperbolic and \( \text{sgn} \, q_{A, \sigma} = 0 \).

**Lemma 3.5.** Suppose that \( \text{char} \, K = 0 \). Let \( (A, \sigma) \) be a \( K \)-algebra with involution, let \( J \) denote its Jacobson radical, let \( \overline{A} = A/J \) and let \( \overline{\sigma} : \overline{A} \rightarrow \overline{A} \) denote the involution induced by \( \sigma \). Then:

1. \( \overline{\tau} \) factors as a product \( \prod_{i=1}^n (A_i, \sigma_i) \) of simple algebras with involution.
2. There exist \( \alpha_1, \ldots, \alpha_t \in \mathbb{Q}_{>0} \subseteq K \) such that

\[ q_{A, \sigma} \cong \alpha_1 q_{A_1, \sigma_1} \oplus \cdots \oplus \alpha_t q_{A_t, \sigma_t} \oplus (0, \ldots, 0). \]
Proof. (i) This follows from the fact that $\overline{A}$ is semisimple and $\overline{\sigma}$ permutes the central idempotents of $\overline{A}$. The details are left to the reader.

(ii) For every $a \in A$, write $a^{(i)}$ for the image of $a$ under $A \rightarrow A$. If $M$ is a left $A$-module, let $a|_M$ denote the linear operator $x \mapsto ax : M \rightarrow M$.

Since $J$ is a left submodule of $A$, we have $\text{Tr}_{A/K}(a) = \sum_{j \geq 0} \text{Tr}(a|_{J/j^2+1})$ for all $a \in A$. Write $J^j/J^{j+1} = \prod_{i=1}^t V_{i,j}$, where $V_{i,j}$ is a left $A_i$-module. Then

$$\text{Tr}_{A/K}(a) = \sum_{j \geq 0} \sum_{i=1}^t \text{Tr}(a^{(i)}|_{V_{i,j}}).$$

Choose $K$-subspaces $S_1, \ldots, S_t \subseteq A$ such that $A = J \oplus S_1 \oplus \cdots \oplus S_t$ and $\overline{\sigma} = A_i$ for all $i$. Then $S_i S_{i'} \subseteq J$ for $i \neq i'$. By (1), $q_{A,\sigma}$ vanishes on $J$, so $S_1, \ldots, S_t$ are pairwise orthogonal relative to $q_{A,\sigma}$.

We claim that for all $i \in \{1, \ldots, t\}$ and $j \geq 0$, there is $\alpha_i \in \mathbb{Q}_{>0}$ such that

$$\text{Tr}(x^n x|_{V_{i,j}}) = \alpha_i \text{Tr}_{A_i/K}(x^\sigma)$$

for all $x \in A_i$. Provided this holds, the map $a \mapsto a^{(i)} : S_i \rightarrow A_i$ defines an isometry from $q_{A,\sigma}|_{S_i}$ to $(\sum_i \alpha_i) \cdot q_{A_i,\sigma}$, and the proposition follows.

Fix $i$ and write $(B,\tau) = (A_i,\sigma_i)$. If $B$ is simple and $W$ is a simple left $B$-module, then $V_{i,j} \cong W^m$ and $\overline{B} \cong W^n$ for some $n,m$. Thus, for all $y \in B$, we have $\text{Tr}(y|_{V_{i,j}}) = \text{Tr}(y|_{W^m}) = m \text{Tr}(y|_W) = \frac{m}{n} \text{Tr}(y|_{W^n}) = \frac{m}{n} \text{Tr}_{B/K}(y)$, and we can take $\alpha_i = \frac{m}{n}$.

If $B$ is not simple, then we may assume that $B = C \times C^{op}$, where $C$ is a simple $K$-algebra, and $\tau$ is given by $(y, z^{op}) \mapsto (z, y^{op})$. Let $U$ denote a simple left $C$-module and let $W$ denote a simple left $C^{op}$-module. Then there is $n \in \mathbb{N}$ such that $\overline{B} \cong U^n \oplus W^n$. Writing $x = (y, z^{op}) \in C \times C^{op}$, it is easy to check that $\text{Tr}_{B/K}(x^\tau x) = 2 \text{Tr}_{C/K}(yz) = 2n \text{Tr}(yz|_U) = 2n \text{Tr}_{B/K}(x^\tau x)$, and similarly, $\text{Tr}_{B/K}(x^\tau x) = 2n \text{Tr}_{B/K}(x^\tau x)$ with $m = \text{length}_B V_{i,j}$. This completes the proof.

Lemma 3.6. Let $(R,\sigma)$ be a ring with involution such that $2 \in R^\times$, let $J \leq R$ be a nilpotent ideal such that $J^\sigma = J$, and let $\overline{\sigma}$ denote the involution $r + J \mapsto r^\sigma + J : R/J \rightarrow R/J$. Then every idempotent $e \in R/J$ satisfying $e^\sigma + e = 1$ is the image of an idempotent $e \in R$ satisfying $e^\sigma + e = 1$. In particular, $\sigma$ is hyperbolic if and only if $\overline{\sigma}$ is hyperbolic.

Proof. This is a special case of [7, Corollary 4.9.16]. We recall and streamline the proof for the sake of completeness.

It is enough to consider the case $J^2 = 0$. It is well-known that $e$ can be lifted to an idempotent $f \in R$. Write $a = \frac{1}{2} f f^\sigma$ and $b = \frac{1}{2} f^\sigma f$. Then $a, b \in J$ (because $\varepsilon e^\overline{\sigma} = e^\overline{\sigma} e = 0$, $a^\sigma = a$, $b^\sigma = b$ and $a^2 = b^2 = ab = ba = 0$ (because $J^2 = 0$).

In addition, $af = a f^\sigma f = ab = 0$, and similarly, $f^\sigma a = fb = b f^\sigma = 0$. Let $e = f - a - b$. The previous identities imply readily that $e^2 = e$ and $ee^\overline{\sigma} = e^\overline{\sigma} e = 0$. Thus, $e$ is an idempotent mapping onto $e$ and $e + e^\sigma$ is an idempotent mapping onto $1 + J$. Since $J$ is nilpotent, we must have $e + e^\sigma = 1$.

Lemma 3.7. Theorem 3.5 holds when $K$ is real closed. In fact, when the conditions hold, one can take $n = 2$ in (a).

Proof. Let $(\overline{A}, \overline{\sigma})$ and $(A_i, \sigma_i)_{i=1}^t$ be as in Lemma 3.5. By part (ii) of that lemma, we have $\text{sgn} q_{A,\sigma} = \sum \text{sgn} q_{A_i,\sigma_i}$. Since $\text{sgn} q_{A,\sigma} \geq 0$ for all $i$ (Lemma 3.4(i)), $\text{sgn} q_{A,\sigma} = 0$ if and only if $\text{sgn} q_{A_i,\sigma_i} = 0$ for all $i$. By Lemma 3.4(ii), this is equivalent to the hyperbolicity of $2 \times (\overline{A}, \overline{\sigma})$, which is in turn equivalent to $2 \times (A, \sigma)$ being hyperbolic, by Lemma 3.6.
Lemma 3.8. Let \((A, \sigma)\) be a \(K\)-algebra with involution and let \(\alpha \in K^\times\) be an element such \(\alpha\) and \(-\alpha\) are not squares in \(K\). If both \(\sigma_K[\sqrt{\alpha}]\) and \(\sigma_K[\sqrt{-\alpha}]\) are hyperbolic, then so is \(2 \times \sigma\).

Proof. The case where \((A, \sigma)\) is central simple is a result of Lewis and Unger, see \([13, p. 475]\) or \([6, \text{Lemma 6.1}]\). We will derive the general case from their result.

Let \((A, \sigma)\) and \((A_i, \sigma_i)\) be as in Lemma 3.5. By Lemma 3.6, it is enough to prove the lemma for \((A, \sigma)\), which in turn amounts to proving it for each factor \((A_i, \sigma_i)\). We may therefore assume that \((A, \sigma)\) is simple.

Write \(F = \text{Cent}(A)[\sigma]\) and recall that \(F\) is a field. If \(\alpha\) is a square in \(F\), then \(F \otimes_K K[\sqrt{\alpha}] \cong F \times F\). As a result, \((A_K[\sqrt{\alpha}], \sigma_K[\sqrt{\alpha}]) \cong (A, \sigma) \times (A, \sigma)\), so \((A, \sigma)\) is hyperbolic by our assumptions.

Similarly, \((A, \sigma)\) is hyperbolic if \(-\alpha\) is a square in \(F\).

Finally, if neither \(\alpha\) nor \(-\alpha\) are squares in \(F\), then we may regard \((A, \sigma)\) as a central simple \(F\)-algebra and finish by the result mentioned at the beginning. □

Proposition 3.9 (Bayer-Fluckiger, Lenstra). Let \((A, \sigma)\) be a \(K\)-algebra with involution and let \(L/K\) be a finite odd-degree field extension. Then \((A_L, \sigma_L)\) is hyperbolic if and only if \((A, \sigma)\) is hyperbolic.

Proof. By Proposition 3.1, it is enough to prove the corresponding statement for unimodular hermitian forms over \((A, \sigma)\). Write \(W(A, \sigma)\) for the Witt group of \(1\)-hermitian forms over \((A, \sigma)\). By a result of Bayer-Fluckiger and Lenstra \([2, \text{Proposition 1.2}]\), the restriction map \(W(A, \sigma) \to W(A_L, \sigma_L)\) is injective. By \([5, \text{Proposition 5.12}]\) (for instance), every hermitian form representing the trivial class in \(W(A, \sigma)\) is hyperbolic, so we are done. □

Now we can prove Theorem 3.8.

Proof. To see that \((a) \implies (b)\), take an ordering \(P\) of \(K\) and apply Lemma 3.7 after base changing from \(K\) to \(K_P\). We turn to show that \((b) \implies (a)\), and moreover, that \(n\) in condition (a) can be taken to be a power of 2. By Corollary 3.2 this will imply that the minimal possible \(n\) for which \((a)\) holds is a power of 2.

Let \(\mathcal{K}\) be an algebraic closure of \(K\) and suppose, for the sake of contradiction, that there is no \(k \in \mathbb{N} \cup \{0\}\) such that \(2^k \times (A, \sigma)\) is hyperbolic. By Zorn’s lemma, there exists a \(K\)-subfield \(L \subseteq \mathcal{K}\) which is maximal relative to the property that \(2^k \times (A_L, \sigma_L)\) is not hyperbolic for all \(k\). Proposition 3.9 implies that \(L\) has no proper odd-degree finite extensions, and by Lemma 3.8 for every \(\alpha \in K^\times\), at least one of \(\alpha\) or \(-\alpha\) is a square. In addition, \(-1\) is not a sum of squares in \(L\), otherwise there exists \(k \in \mathbb{N}\) such that \((M_{2^k}(L), t)\), and hence \(2^k \times \sigma_L\), is hyperbolic \([6, \text{Proposition 6.2}]\). We conclude that \(L\) is real closed, but this contradicts Lemma 3.7. □

Remark 3.10. The fact that the minimal \(n\) for which condition (a) in Theorem 3.8 holds is a power of 2 can also be derived from a theorem of Bayer-Fluckiger, Parimala and Serre \([3, \text{Theorem 3.1.1}]\), by arguing as in the proof of Corollary 3.2.

4. Proof of Theorem \(A\)

We use Theorem 3.8 to prove Theorem \(A\).

Proof of Theorem \(A\) Recall that we are given a \(K\)-vector space with a system of forms \((V, \{q_i\}_{i \in I})\) and \((A, \sigma)\) is its algebra of adjoints together with the canonical involution.

The theorem will follow from Theorem 3.8 if we verify the following two facts:

(i) The algebra of adjoints of \(n \times \{q_i\}_{i \in I}\) with its canonical involution is isomorphic to \(n \times (A, \sigma)\).
(ii) \{q_i\}_{i \in I}$ is hyperbolic if and only if \((A, \sigma)\) is hyperbolic.

The first statement is straightforward. As for the second, if \(e = (\phi, \psi^{op}) \in A\) is an idempotent such that \(e + e^\sigma = 1\), then \(\phi\) and \(\psi\) are idempotents in \(\text{End}(V)\) such that \(\phi + \psi = \text{id}_V\). Thus, \(V = \text{id} \oplus \text{id} \psi\). Since for all \(x, y \in V\), we have \(q_i(\psi x, \psi y) = q_i(x, \phi y) = 0\) and \(q_i(\psi x, \phi y) = q_i(\phi x, y) = 0\), this means that \(\{q_i\}_{i \in I}\) is hyperbolic. Conversely, if \(V = U \oplus W\) and each \(q_i\) vanishes on both \(U\) and \(W\), then \(e := (\text{id}_U \oplus \text{id}_W, (0_U \oplus \text{id}_W)^{op})\) is easily seen to be an idempotent in \(A\) satisfying \(e + e^\sigma = 1\).

\[\square\]

Theorem \[\overline{A}\] can be regarded as a generalization of Pfister’s local-global principle by means of part (i) of the following proposition, which is well-known when \(q\) is unimodular.

**Proposition 4.1.** Let \((V, q)\) be a quadratic space and let \((A, \sigma)\) denote the algebra-with-involution of adjoints of \(\{q\}\). Then:

(i) \(\text{sgn}_{A, \sigma} = (\text{sgn} q)^2\).

(ii) \(\text{Cl}(q) = Kq\).

**Proof.** We may assume without loss of generality that \(q = (\alpha_1, \ldots, \alpha_m) \oplus (n \times 0)\) with \(\alpha_1, \ldots, \alpha_m \in K^\times\). Writing \(g := \text{diag}(\alpha_1, \ldots, \alpha_m)\) and identifying \(\text{End}(V) = \text{End}(K^{m+n})(K)\), straightforward computation now shows that

\[A = \{[\begin{smallmatrix} a & 0 \\ 0 & t \end{smallmatrix}] \mid [\begin{smallmatrix} s & a \phi^{op} \\ \phi^{op} & t \end{smallmatrix}] \mid a \in M_m(K), t, z \in M_{m \times n}(K), y, w \in M_{n \times n}(K)\}\]

(i) It is routine to check that \(q_{A, \sigma} \cong [\bigoplus_{i,j \in \{1,\ldots,m\}} (\alpha_1 \alpha_j^{-1})] \oplus [n^2 \times (1, -1)] \oplus [2mn \times (0)]\). Thus, for every ordering \(P\) of \(K\), we have

\[\text{sgn}_{P, q_{A, \sigma}} = \sum_{i,j \in \{1,\ldots,m\}} \text{sgn}_{P}(\alpha_i) \text{sgn}_{P}(\alpha_j) = \left(\sum_{i=1}^{m} \text{sgn}(\alpha_i)\right)^2 = (\text{sgn}_{P, q})^2.\]

(ii) Let \(w \in \text{Cl}(q)\) and let \([\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}]\) denote its Gram matrix relative to the standard basis of \(K^{m+n}\), where \(a \in M_m(K)\). By the description of \(A\) given above, we have \([\begin{smallmatrix} 0 & t_n \end{smallmatrix}] [\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}] = [\begin{smallmatrix} a & b \end{smallmatrix}] [\begin{smallmatrix} 0 & t_n \end{smallmatrix}]\) and \([\begin{smallmatrix} s^{-1} & 0 \\ 0 & 1 \end{smallmatrix}] [\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}] = [\begin{smallmatrix} a & b \end{smallmatrix}] [\begin{smallmatrix} 0 & 0 \end{smallmatrix}]\) for all \(x \in M_m(K)\). These equalities imply that \(b = 0, c = 0, d = 0\) and \(g^{-1}a \in \text{Cent}(M_m(K))\). Consequently, \(w = aq\) for some \(a \in K\).

\[\square\]

5. Hermitian Categories

We recall some facts about hermitian categories that will be needed for the proof of Theorem \[\overline{B}\] in the next section. We refer the reader to [53 §2] for the relevant definitions, and to [37 §7] or [11] Chapter III for an extensive treatment.

All categories are tacitly assumed to be small. The composition symbol “\(\circ\)” will often be suppressed in formulas.

Let \(\mathcal{C} = (\mathcal{C}, \ast, \omega)\) be a hermitian category. The category of unimodular 1-hermitian spaces over \(\mathcal{C}\) is denoted \(\text{UH}(\mathcal{C})\).

Given a set \(I\), an \(I\)-indexed system of hermitian forms over \(\mathcal{C}\) is a pair \((C, \{h_i\}_{i \in I})\) consisting of an object \(C \in \mathcal{C}\) and a collection \(\{h_i\}_{i \in I}\) of (possibly non-unimodular) 1-hermitian forms on \(C\). An isometry from \((C, \{h_i\}_{i \in I})\) to another \(I\)-indexed system \((C', \{h'_i\}_{i \in I})\) is an isomorphism \(f : C \to C'\) such that \(f^i h'_i f = h_i\) for all \(i \in I\).

The category of \(I\)-indexed systems of hermitian forms with isometries as morphisms is denoted \(\text{Sys}_I(\mathcal{C})\).

Define the category \(\text{Aff}_I(\mathcal{C})\) as follows:

- Objects are triples \((U, V, \{f_i\}_{i \in I})\) with \(U, V \in \mathcal{C}\) and \(\{f_i\}_{i \in I} \subseteq \text{Hom}_\mathcal{C}(U, V')\).
- Morphisms from \((U, V, \{f_i\}_{i \in I})\) to \((U', V', \{f'_i\}_{i \in I})\) are formal symbols \((\phi, \psi^{op})\) such that \(\phi \in \text{Hom}_\mathcal{C}(U, U')\), \(\psi \in \text{Hom}_\mathcal{C}(V', V)\) and \(\psi^* f_i = f_i \phi\) for all \(i \in I\).
• Composition is defined by \( (\phi, \psi^{\text{op}}) \circ (\phi', \psi'^{\text{op}}) := (\phi \phi', (\psi' \psi)^{\text{op}}) \).

We call \( \text{A}I_{\text{I}}(\mathcal{C}) \) the category of \textit{twisted arrows} over \( \mathcal{C} \) and make it into a hermitian category by setting

\[
(U, V, \{f_i\}_{i \in I})^* = (V, U, \{f^*_i \circ \omega_{U} \}_{i \in I}),
(\phi, \psi^{\text{op}})^* = (\psi, \phi^{\text{op}}),
\omega(U, V, \{f_i\}) = (\text{id}_U, \text{id}_V^{\text{op}}).
\]

Note that if \( h = (\phi, \psi^{\text{op}}) \) is a hermitian form on \( Z \in \text{A}I_{\text{I}}(\mathcal{C}) \), then \( \phi = \psi \), because \( h = h^* \omega_Z \).

We alert the reader that \( \text{A}I_{\text{I}}(\mathcal{C}) \) is not the category of twisted \textit{double} \( I \)-arrows defined in [5, §4] and denoted \( \text{A}I_{\text{II}}(\mathcal{C}) \). Rather, \( (U, V, \{f_i\}) \mapsto (U, V, \{f_i\}, \{f_i\}) \) identifies \( \text{A}I_{\text{I}}(\mathcal{C}) \) as a full hermitian subcategory of \( \text{A}I_{\text{II}}(\mathcal{C}) \).

The following theorem is a variation of [5, Theorem 4.1]

\textbf{Theorem 5.1.} Define \( F : \text{Sys}_I(\mathcal{C}) \to \text{UH}(\text{A}I_{\text{I}}(\mathcal{C})) \) and \( G : \text{UH}(\text{A}I_{\text{I}}(\mathcal{C})) \to \text{Sys}_I(\mathcal{C}) \) by

\[
F(C, \{h_i\}) = ((C, C), \{h_i\}, (\text{id}_C, \text{id}_C^{\text{op}})), \quad F(\phi) = (\phi, (\phi^{-1})^{\text{op}}).
G(U, V, \{f_i\}, (\alpha, \alpha^{\text{op}})) = (U, \{\alpha^* f_i\}_{i \in I}), \quad G(\phi, \psi^{\text{op}}) = (\phi, \psi).
\]

Then \( F \) and \( G \) are well-defined functors which are mutually inverse. Moreover \( F \) and \( G \) respect orthogonal sums and preserve hyperbolicity.

\textit{Proof.} We already observed that \( \text{A}I_{\text{I}}(\mathcal{C}) \) is a full subcategory of \( \text{A}I_{\text{II}}(\mathcal{C}) \). With this observation at hand, the proof is the same as the proof of [5, Theorem 4.1, Proposition 4.2]. \( \square \)

\textbf{Remark 5.2.} Take \( \mathcal{C} \) to be the category of finite-dimensional \( K \)-vector spaces with the usual duality \( V^* = \text{Hom}_K(V, K) \), let \( (V, \{g_i\}_{i \in I}) \) be a system of quadratic forms over \( \mathcal{C} \), and let \( (Z, h) = F(V, \{g_i\}_{i \in I}) \). Then \( \text{End}_{\text{A}I_{\text{I}}(\mathcal{C})}(Z) \) is precisely the algebra of adjoints \( A = A((g_i)_{i \in I}) \), and the canonical involution \( \sigma : A \to A \) coincides with the involution \( f \mapsto h^{-1} f^* h \) on \( \text{End}(Z) \). The information that \( (A, \sigma) \) carries on the system \( \{g_i\}_{i \in I} \) is therefore a manifestation of Theorem 5.1.

We call the hermitian category \( \mathcal{C} \) a \textit{finite hermitian} \( K \)-category if the Hom-groups in \( \mathcal{C} \) are finite dimensional \( K \)-vector spaces, composition is \( K \)-bilinear and \( * : \mathcal{C} \to \mathcal{C} \) is \( K \)-linear on Hom-groups. Recall also that \( \mathcal{C} \) is \textit{pseudo-abelian} if every idempotent morphism has a kernel. For example, both properties are satisfied by the category of finite-dimensional \( K \)-vector spaces with the usual duality \( V^* = \text{Hom}_K(V, K) \). In addition, if \( \mathcal{C} \) satisfies either property, then so does \( \text{A}I_{\text{I}}(\mathcal{C}) \).

Suppose henceforth that \( \mathcal{C} \) is a semi-abelian finite hermitian \( K \)-category, and fix a collection \( \mathcal{Z} \) of objects in \( \mathcal{C} \) such that every indecomposable object \( C \in \mathcal{C} \) satisfies \( C \cong Z \) or \( C \cong Z^* \) for unique \( Z \in \mathcal{Z} \). Given \( Z \in \mathcal{Z} \), a hermitian space \( (C, h) \in \text{UH}(\mathcal{C}) \) is said to be of type \( Z \) if \( C \) is isomorphic to a summand of \( (Z \oplus Z^*)^n \) for some \( n \in \mathbb{N} \). The following theorems are due to Quebbemann, Scharlau and Schulte [15, Theorems 3.2, 3.3].

\textbf{Theorem 5.3.} Every \( (C, h) \in \text{UH}(\mathcal{C}) \) admits a factorization

\[
(C, h) \in \bigoplus_{Z \in \mathcal{Z}} (C_Z, h_Z)
\]

in which \( (C_Z, h_Z) \) is a unimodular hermitian space of type \( Z \) and \( C_Z = 0 \) for all but finitely many \( Z \in \mathcal{Z} \). The factors \( (C_Z, h_Z) \) are uniquely determined up to isometry.
**Theorem 5.4.** Let \( Z \in \mathcal{Z} \), let \( \varepsilon \in \{ \pm 1 \} \) and let \( h : Z \to Z \) be a unimodular \( \varepsilon \)-hermitian form. Write \( E = \text{End}_\varepsilon(Z) \) and \( \overline{E} = E / \text{Jac} E \), let \( \sigma : E \to E \) be given by \( \sigma' = h^{-1} \psi h \) and let \( \sigma : \overline{E} \to \overline{E} \) be given by \((\overline{\sigma})^\sigma = \overline{\sigma} \). Then there is a functor \( T \) from the category of unimodular 1-hermitian spaces of type \( Z \) over \( \mathcal{C} \) to the category of unimodular \( \varepsilon \)-hermitian spaces over \((\overline{E}, \overline{\sigma})\) having the following properties:

(i) \( T \) induces a bijection on isomorphism classes.

(ii) \( T \) respects orthogonal sums and hyperbolicity.

(iii) \( T \) maps 1-hermitian spaces over \( \mathcal{C} \) with underlying object \( Z^n \) to \( \varepsilon \)-hermitian spaces over \((\overline{E}, \overline{\sigma})\) with underlying module \( \overline{E}^n \).

6. Proof of Theorem 5

We finally prove Theorem 5. In fact, we shall prove a more general result. To state it, we introduce the following condition on the field \( K \):

(E) For every ordering \( P \) of \( K \) and every two disjoint finite subsets \( S, T \subseteq KP \), there exists \( f \in K[X] \) such that \( f(s) > P 0 \) for all \( s \in S \) and \( f(t) < P 0 \) for all \( t \in T \).

Fields satisfying (E) include all real closed fields and fields which are dense in their real closure relative to each of their orderings, e.g., number fields. (Indeed, take \( f \) to be an approximation of an appropriate interpolation polynomial in \( KP[X] \).) We conjecture that (E) holds for all fields.

Theorem 5 is a special case of:

**Theorem 6.1.** Suppose that \( K \) satisfies condition (E), e.g., \( K \) is a number field or a real-closed field. Let \( (V, \{ q_i \}_{i=1,2}) \) be a \( K \)-vector space with a nonsingular pair of quadratic forms. Then the following conditions are equivalent:

(a) \( n \times \{ q_i \}_{i=1,2} \) is hyperbolic form some \( n \in \mathbb{N} \).

(b) Every \( q \in \text{Cl}(q_1, q_2) \) has signature 0.

For the remainder of this section, we adopt the convention of Section 4 in which quadratic forms on \( K^n \) are identified with their Gram matrix relative to the standard basis. In addition, we let \( \mathcal{C} \) denote the category whose objects are the \( K \)-vector spaces \( \{ K^0, K^1, K^2, \ldots \} \), its morphisms are given by \( \text{Hom}(K^n, K^m) = M_{m \times n}(K) \), and its composition is matrix product. We make \( \mathcal{C} \) into a hermitian category by letting \( \ast \) fix all objects and act as the matrix transpose on morphisms, and setting \( \omega = \text{id} \). Write \( \text{Sys}_2(K) = \text{Sys}_2(\mathcal{C}) \). Then, under our conventions, objects of \( \text{Sys}_2(K) \) can be regarded as vector spaces with a pair of quadratic forms in the sense of Section 4 and all such a pairs are obtained in this manner, up to isomorphism.

Denote \( \overline{\text{Ar}}_{1,2}(\mathcal{C}) \) as \( \overline{\text{Ar}}_2(K) \) and write the objects \( (U, V, \{ f_i \}_{i=1,2}) \) of \( \overline{\text{Ar}}_2(K) \) as quadruples \( (U, f_1, f_2, V) \). The assignment \( (U, f_1, f_2, V) \mapsto (U, f_1, f_2, V^*) \) and \( (\phi, \psi)^{op} \mapsto (\phi, \psi^*) \) defines an equivalence between \( \overline{\text{Ar}}_2(K) \) and the category of *Kronecker modules*, i.e., pairs of vector spaces with a pair of linear maps from the first space to the second space.

If \( L \) is a \( K \)-field, then we have evident base change functors \( \text{Sys}_2(K) \to \text{Sys}_2(L) \) and \( \overline{\text{Ar}}_2(K) \to \overline{\text{Ar}}_2(L) \). It is routine to check that the equivalence of Theorem 6 is compatible these functors (see 3D, §3D, Remark 2.2 for a generalization).

Given \( n \in \mathbb{N} \) and \( \alpha \in F \), we define the following \( n \times n \) matrices

\[
J_n(\alpha) = \begin{bmatrix} \alpha & 1 & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & 1 \end{bmatrix}, \quad S_n(\alpha) = \begin{bmatrix} \alpha & 1 & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & 1 \end{bmatrix}, \quad T_n = \begin{bmatrix} 1 & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & 1 \end{bmatrix},
\]
There is a monic prime polynomial \( U \) such that \( \alpha \) is a root of \( U \). Let \( \tilde{A}_r \) be the object in \( \text{Aff}_2(K) \) corresponding to \( \alpha \). It is easy to see that \( Z(A) \cong Z(A') \) if and only if \( A \) and \( A' \) are conjugate. Furthermore, \( Z(A) \) is indecomposable in \( \text{Aff}_1(\mathcal{C}) \) if and only if \( K^n \) is an indecomposable \( K[A] \)-module. In this case, the characteristic polynomial of \( A \) coincides with its minimal polynomial and is a prime power, and every matrix which commutes with \( A \) belongs to \( K[A] \).

**Lemma 6.2.** Let \( (K^n, \{q_i\}_{i=1,2}) \in \text{Sys}_2(K) \) and let \( C = F(K^n, \{q_i\}_{i=1,2}) = (K^n, q_1, q_2, K^n) \) (see Theorem 5.7). If \( q_1 \) is unimodular, then \( C \) is a direct sum of indecomposable objects of the form \( Z(A) \).

**Proof.** We observed above that \( \text{Aff}_2(K) \) is equivalent to the category of Kronecker modules over \( K \). The indecomposable Kronecker modules were classified in the 19th century by Kronecker, see [8, p. 69], for instance. If \( Z = (U, f, g, V) \) is an indecomposable summand of \( C \), then \( f \) must be invertible, in which case it follows easily from the classification that \( Z \cong Z(A) \) for some \( A \in M_n(K) \), \( n \in \mathbb{N} \).

**Lemma 6.3.** Let \( Z = Z(A) \) be an indecomposable object in \( \text{Aff}_2(K) \). Then:

(i) There exists an isomorphism \( h : Z \to Z^* \) such that \( h = h^* \).

(ii) For every \( h \) as in (i), the involution \( f \mapsto h^{-1} f^* h : \text{End}(Z) \to \text{End}(Z) \) is the identity.

(iii) Every unimodular hermitian space of type \( Z \) over \( \text{Aff}_2(K) \) is the orthogonal sum of hermitian spaces with underlying object isomorphic to \( Z \).

**Proof.** (i) Saying that \( h = (\phi, \phi^p) : Z \to Z^* \) is an isomorphism amounts to saying that \( \phi \) is a symmetric invertible \( n \times n \) matrix such that \( \phi A \phi^{-1} = A^t \). It is well-known that such \( \phi \) exists, see [3] Theorem 66, for instance.

(ii) Write \( h = (\phi, \phi^p) \) with \( \phi \in \text{GL}_n(K) \). Then \( \phi = \phi^t \) and \( \phi^t A = A^t \phi \). Let \( f = (\xi, \psi^p) \in \text{End}(Z) \). Then \( \psi^t \xi = \xi \) and \( \psi^t A = A^t \xi \), hence \( \xi A = A^t \xi \). Since \( K^n \) is an indecomposable \( K[A] \)-module, \( \xi = p(A) \) for some \( p \in K[X] \), so \( \phi^{-1} \xi \phi = p(\phi^{-1} A^t \phi) = p(A) = \xi \). This implies readily that \( h^{-1} f^* h = f \).

(iii) Write \( E = \text{End}(Z) \). By (i) and Theorem 5.4, we reduce into showing that every unimodular 1-hermitian form over \( (E, \pi) \) is diagonalizable. By Fitting’s lemma, \( E \) is a field, and by (ii), \( \pi = \text{id} \). Since every quadratic form over a field of characteristic not 2 is diagonalizable, we are done. \( \square \)

**Lemma 6.4.** Let \( P \) be an ordering of \( K \) and let \( R = K_P \). Let \( A \in M_n(K) \) be a matrix such that \( K^n \) is an indecomposable \( K[A] \)-module, let \( f \) denote its characteristic polynomial and let \( \alpha_1, \ldots, \alpha_t \) be the roots of \( f \) in \( R \). Let \( h : Z(A) \to Z(A) \) be a unimodular hermitian form. Then there exist \( r \in \mathbb{N} \) and a decomposition

\[
(Z(A), h)_R = (Z_0, h_0) \oplus \bigoplus_{i=1}^t (Z_r(\alpha_i), h_i)
\]

in \( \text{UH}(\text{Aff}_2(R)) \) such that \( h_0 \oplus h_0 \) is hyperbolic.

**Proof.** There is a monic prime polynomial \( p \in K[X] \) and \( r \in \mathbb{N} \) such that \( f = p^r \). Since \( R \) is real-closed, there are distinct prime degree-2 polynomials \( q_1, \ldots, q_s \in R[X] \) such that \( p = (X - \alpha_1) \cdots (X - \alpha_t) q_1 \cdots q_s \). For each \( j \in \{1, \ldots, s\} \), choose \( B_j \in M_2(R) \) with minimal polynomial \( q_j^r \). Then \( A \) is conjugate to \( J_r(\alpha_1) \oplus \cdots \oplus J_r(\alpha_t) \oplus B_1 \oplus \cdots \oplus B_s \), and hence \( Z(A)_R \cong Z_r(\alpha_1) \oplus \cdots \oplus Z_r(\alpha_t) \oplus Z(B_1) \oplus \cdots \oplus Z(B_s) \).
Proof. (i) One readily checks that $\text{End}(Z(B_i))$ factors as $[\bigoplus_{i=1}^r (Z_i(a_i), h_i)] \oplus \bigoplus_{j=1}^s (Z(B_j), h'_j)$. We take $(Z_0, h_0) = \bigoplus_{i=1}^r (Z(B_i), h_i)$.

To complete the proof, we need to check that $h'_j \oplus h'_j$ is hyperbolic for all $j$. Write $E = \text{End}(Z(B_j))$. It is easy to see that the assignment $(\phi, \phi^{op}) \mapsto \phi$ defines a $K$-algebra isomorphism between $E$ and the centralizer of $B_j$ in $M_2(R)$, which is just $R[B_j]$. Thus, $E \cong R[X]/(q^0)$. By Theorem 5.3 and Lemma 6.3, it is enough to show that every 2-dimensional unimodular quadratic form over $\mathcal{F} := E/\text{Jac} E \cong R[X]/(q_j)$ is hyperbolic. This holds because $\mathcal{F}$ is a quadratic extension of the real-closed field $R$, and hence algebraically closed.

Lemma 6.5. Assume that $K$ is real closed, let $\alpha \in K$, let $h = (\phi, \phi^{op}) : Z_n(\alpha) \to Z_n(\alpha^*)$ be a unimodular hermitian form and write $\{q_i\}_{i=1,2} = G(Z, h)$ (see Theorem 5.4). Then:

(i) There exist $\beta_1 \in K^\times$ and $\beta_2, \ldots, \beta_n \in K$ such that 

$$
\phi = \begin{bmatrix} \beta_1 & \beta_2 & \cdots & \beta_n \\
\vdots & \vdots & \ddots & \vdots \\
\beta_1 & \beta_2 & \cdots & \beta_n \\
\end{bmatrix}
$$

(ii) $(Z, h) \cong (Z, (I_n, I_n^{op}))$ if $\beta_1 > 0$ and $(Z, h) \cong (Z, (-I_n, -I_n^{op}))$ if $\beta_1 < 0$.

Proof. (i) We have $\phi^* T_n = T_n^* \phi$ and $\phi^* S_n(\alpha) = S_n(\alpha) \phi$, hence $J_n(\alpha) \phi = T_n^{-1} S_n(\alpha) \phi = T_n^{-1} \phi^* S_n(\alpha) = \phi T_n^{-1} S_n(\alpha) = \phi J_n(\alpha)$, so $\phi$ commutes with $J_n(\alpha)$. Since $K^n$ is an indecomposable $K[J_n(\alpha)]$-module, this means that $\phi$ is a polynomial in $J_n(\alpha)$, and thus has the desired form.

(ii) It is easy to see that there exists $\eta \in GL_n(K)$, taking the same form as $\phi$ such that $\text{sgn}(\beta_1) \eta^2 = \phi$. It is routine to check that $\eta^* T_n = T_n \eta$ and $\eta^* S_n(\alpha) = S_n(\alpha) \eta$, from which it follows that $(\eta, \eta^{op})$ is an isometry from $(Z_n(\alpha), (\phi, \phi^{op}))$ to $(Z_n(\alpha), (\text{sgn}(\beta_1) I_n, \text{sgn}(\beta_1) I_n^{op}))$.

Lemma 6.6. Let $(K^n, \{q_i\}_{i=1,2}) \in \text{Sys}_2(K)$ and assume that $q_1$ is unimodular. Let $J = q_1^{-1} q_2$ and $Z = (K^n, q_1, q_2, K^n)$. Then:

(i) $(J, J^{op})$ is a central element of $\text{End}(Z)$.

(ii) For every $f \in K[X]$, the morphism $q_1 f(J) : K^n \to (K^n)^*$ in $\mathcal{C}$ defines a quadratic form in $C(q_1, q_2)$.

(iii) Continuing (ii), if $\{q_i\}_{i=1,2}$ is hyperbolic, then so is $q_1 f(J)$.

Proof. (i) One readily checks that $J q_1 = q_1 J$ and $J q_2 = q_2 J$, hence $(J, J^{op}) \in \text{End}(Z)$. Let $(\phi, \psi^{op}) \in \text{End}_K(Z)$. Then $\psi^* q_1 = q_1 \phi$ and $\psi^* q_2 = q_2 \phi$, from which it follows that $\phi J = \phi q_1^{-1} q_2 = q_1^{-1} \psi^* q_2 = q_1^{-1} q_2 \phi = J \phi$ and $\psi J = \psi q_1^{-1} q_2 = q_1^{-1} \phi^* q_2 = q_1^{-1} q_2 \psi = J \psi$, so $(\phi, \psi^{op})$ and $(J, J^{op})$ commute.

(ii) Let $(\phi, \psi^{op}) \in A(\{q_i\}_{i=1,2}) = \text{End}(Z)$ (see Remark 5.2). We need to check that $\psi^* q_1 f(J) = q_1 f(J) \phi$. We observed in the proof of (i) that $\phi J = J \phi$, so $\psi^* q_1 f(J) = q_1 \phi f(J) = q_1 f(J) \phi$.

(iii) This follows from (ii) and claim (ii) in the proof of Theorem A.

We are now ready to prove Theorem 6.7, thus establishing Theorem 6.8.

Proof of Theorem 6.7. By Theorem A the statement is vacuous if $K$ is not real. We may therefore assume that $K$ is real, and in particular infinite. Now, by Lemma 1.2 $\text{span}_K \{q_1, q_2\}$ contains a unimodular form. Replacing $\{q_i\}_{i=1,2}$ with another pair
spanning \( \text{span}_K\{q_1,q_2\} \), we may assume that \( q_1 \) is unimodular. The implication (a) \( \Rightarrow \) (b) was explained in the introduction, so we turn to prove the converse. In fact, by Theorem [A] it is enough to check that for every ordering \( P \) of \( K \), the system \( 2 \times (q_i)_{i=1,2} \) is hyperbolic.

We may assume that \( V = K^v \) for some \( v \); recall that we treat \( q_1, q_2 \) as symmetric \( v \times v \) matrices. Write \( R = K_p, J = q_i^{-1}q_2, Z = (K^v, q_1, q_2, K^v), h = (I_v, I_{op}) : Z \to Z^* \) and note that \((Z, h) = F(V, \{q_i\}_{i=1,2}) \) (see Theorem 5.1). Let \( \mathcal{P} \subseteq K[X] \) denote the monic prime factors of the characteristic polynomial of \( J \). For every \( p \in \mathcal{P} \) and \( n \in \mathbb{N} \), let \( A_{p,n} \) denote a fixed square matrix over \( K \) with characteristic polynomial \( p^n \). Then \( Z(A_{p,n}) \in \mathcal{A}_2(K) \) is indecomposable and isomorphic to its dual (Lemma 6.3(i)).

Note that if \( Z \cong (U, f, g, V) \) in \( \mathcal{A}_2(K) \), then \( f^{-1}g \) is conjugate to \( J \). Thus, by Theorem 5.3 and Lemma 6.2 we have a decomposition

\[
(Z, h) \cong \bigoplus_{p \in \mathcal{P}} \bigoplus_{n \in \{1, \ldots, N\}} (Z_{p,n}, h_{p,n}),
\]

where \( N \) is a sufficiently large integer and \( Z_{p,n} \) is a direct sum of copies of \( Z(A_{p,n}) \).

Fix some \( p \in \mathcal{P} \). We will prove that \( 2 \times (h_{p,n})_R \) is hyperbolic for all \( p \) and \( n \) by a decreasing inducting on \( n \). Thanks to Theorem 5.1 this will finish the proof. The case \( n = N + 1 \) holds vacuously, so assume that the claim has been established for all \( n > k \) for some \( k \leq N \).

Write \( W = Z(A_{p,k}) \) and let \( \alpha_1, \ldots, \alpha_t \) denote the roots of \( p \) in \( R \). Thanks to Lemma 6.3(iii), there is a decomposition

\[
(Z_{p,k}, h_{p,k}) = \bigoplus_{j=1}^s (W_j, w_j),
\]

and by Lemma 6.4 we further have

\[
(W, w_j)_R \cong (W'_j, w'_j) \oplus \bigoplus_{i=1}^t (Z_k(\alpha_i), w_{ji})
\]

with \( 2 \times w'_j \) being hyperbolic.

Fixing \( i \in \{1, \ldots, t\} \) and \( j \in \{1, \ldots, s\} \) and writing \( w_{ji} = (\phi_{ji}, \phi_{ji}^{op}) \), Lemma 6.5(i) implies that \( \phi_{ji} \) is an upper-triangular matrix with constant diagonal; denote the scalar occurring on the diagonal of \( \phi_{ji} \) by \( \beta_{ji} \). Write

\[
r_i = \#\{ j \in \{1, \ldots, s\} : \beta_{ji} > 0 \} \quad \text{and} \quad r'_i = \#\{ j \in \{1, \ldots, s\} : \beta_{ji} < 0 \}.
\]

Then, by Lemma 6.3(ii),

\[
(Z_{p,k}, h_{p,k})_R \cong \bigoplus_{j=1}^s (W'_j, w'_j) \oplus \bigoplus_{i=1}^t \left( Z_k(\alpha_i)^{r_i+r'_i}, r_i \times (I_k, I_{op}) \oplus r'_i \times (-I_k, I_{op}) \right).
\]

As a result, if \( r_i = 0 \) for all \( i \in \{1, \ldots, t\} \), then \( 2 \times (h_{p,k})_R \) is hyperbolic.

Let \( g \) denote the product of all primes in \( \mathcal{P} \setminus \{p\} \), and let \( m \in \{0, \ldots, t\} \). By virtue of condition [E], we can find a polynomial \( f_m \in K[X] \) such that for all \( i \in \{1, \ldots, t\} \),

\[
\varepsilon_{m,i} := f_m(\alpha_i) g^{N}(\alpha_i) \prod_{\ell \neq i} (\alpha_i - \alpha_\ell) < 0 \iff m = i.
\]

Let \( z_m = q_p^{k-1}(J) g^{N}(J) f_m(J) \). By Lemma 6.6(ii), \( z_0, \ldots, z_t \) are quadratic forms in \( \text{Cl}(q_1, q_2) \), so they all have \( P \)-signature 0 by assumption.

Observe that \((V, \{q_i\}_{i=1,2}) = GF(V, \{q_i\}_{i=1,2}) = G(Z, h) \) (see Theorem 5.1). Since \( G \) respects orthogonal sums, each of the morphisms \( q_1, q_2, J, z_0, \ldots, z_t \) in \( \mathcal{G} \) factors as a direct sum of components corresponding to the decomposition
Given \((q, n) \in \mathcal{P} \times \{1, \ldots, N\}\), write \(z_{m,q,n}\) for the component of \(z_m\) corresponding to \(Z_{q,n}\). By the definition of \(G\) in Theorem 5.1 we have

\[
(z_m) = (I^I) \cdot p^{-1}(I^I A_{q,n}) \cdot g^N(I^I A_{q,n}) \cdot f(I^I A_{q,n}) = p^{k-1}(A_{q,n}) g^N(A_{q,n}) f(A_{q,n}).
\]

From this we see that \(z_{m,q,n} = 0\) if \(n < k\) or \(q \neq p\). In addition, Lemma 6.9(iii) and the induction hypothesis imply that \(2 \times z_{m,p,n}\) is hyperbolic whenever \(n > k\). As a result, \(\text{sgn}_P z_{m,p,k} = \text{sgn}_P z_m = 0\).

Write \(b_m = (z_{m,p,k})_R\). Then \(b_m\) is isomorphic to an orthogonal sum of components corresponding to the decompositions \(3\) and \(4\): write \(b_{m,j_1}, \ldots, b_{m,j_t}\) and \(b'_{m,j_1}\) for the components corresponding to \(w_{j_1}, \ldots, w_{j_t}\) and \(w'_{j_1}\), respectively. As in the previous paragraph, \(b'_{m,j_1}\) is hyperbolic, and we have

\[
b_{m,j_1} = \phi_{j_1}^t T_n \cdot f_m(T_n^{-1} S_k(\alpha_i)) \cdot g^N(T_n^{-1} S_n(\alpha_i)) \cdot p^{k-1}(T_n^{-1} S_k(\alpha_i)) = \\
= \phi_{j_1}^t T_n \cdot f_m(J_n(\alpha_i)) \cdot g^N(J_k(\alpha_i)) \cdot p^{k-1}(J_k(\alpha_i)) = \\
\begin{bmatrix}
\cdots & 0 \\
0 & 0 \\
\cdots & 0 \beta_{j_1 \varepsilon m, i}
\end{bmatrix}
\]

As a result, for \(m \in \{1, \ldots, t\}\), we have

\[
0 = \text{sgn}_P z_{m,p,k} = \text{sgn}_P b_m = \sum_{i,j} \text{sgn}(\beta_{j_1 \varepsilon m, i}) = -(r_m - r'_m) + \sum_{i \neq m} (r_i - r'_i),
\]

whereas for \(m = 0\), we get

\[
0 = \text{sgn}_P z_{0,p,k} = \text{sgn}_P b_0 = \sum_{i,j} \text{sgn}(\beta_{j_1 \varepsilon 0, i}) = \sum_{i} (r_i - r'_i).
\]

Subtracting both equations gives \(2(r_m - r'_m) = 0\), so \(r_m = r'_m\) for all \(m \in \{1, \ldots, t\}\). This completes the proof. \(\Box\)

References

[1] J. F. Adams. Vector fields on spheres. Ann. of Math. (2), 75:603–632, 1962.
[2] E. Bayer-Fluckiger and H. W. Lenstra, Jr. Forms in odd degree extensions and self-dual normal bases. Amer. J. Math., 112(3):359–373, 1990.
[3] E. Bayer-Fluckiger, R. Parimala, and J-P. Serre. Hasse principle for G-trace forms. Izv. Ross. Akad. Nauk Ser. Mat., 77(3):5–28, 2013.
[4] Eva Bayer-Fluckiger. Principe de Hasse faible pour les systèmes de formes quadratiques. J. Reine Angew. Math., 378:53–59, 1987.
[5] Eva Bayer-Fluckiger, Uriya A. First, and Daniel A. Moldovan. Hermitian categories, extension of scalars and systems of sesquilinear forms. Pacific J. Math., 270(1):1–26, 2014.
[6] Karim Johannes Becher and Thomas Unger. Weakly hyperbolic involutions. Expo. Math., 36(1):78–97, 2018.
[7] Uriya A. First. Bilinear forms and rings with involution. 2012. Ph.D. dissertation. Submitted at Bar–Ilan University. Available on the author’s webpage.
[8] Peter Gabriel. Appendix: degenerate bilinear forms. J. Algebra, 31:67–72, 1974.
[9] Irving Kaplansky. Linear algebra and geometry. Dover Publications, Inc., Mineola, NY, revised edition, 2003. A second course.
[10] Max-Albert Knus. Quadratic and Hermitian forms over rings, volume 294 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1991. With a foreword by I. Bertuccconi.
[11] Max-Albert Knus, Alexander Merkurjev, Markus Rost, and Jean-Pierre Tignol. The book of involutions, volume 44 of American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI, 1998. With a preface in French by J. Tits.
[12] David W. Lewis and J-P. Tignol. On the signature of an involution. Arch. Math. (Basel), 60(2):128–135, 1993.
[13] David W. Lewis and Thomas Unger. A local-global principle for algebras with involution and Hermitian forms. Math. Z., 244(3):469–477, 2003.
[14] Albrecht Pflister. Quadratic forms with applications to algebraic geometry and topology, volume 217 of London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1995.
[15] Heinz-Georg Quebbemann, Winfried Scharlau, and Manfred Schulte. Quadratic and Hermitian forms in additive and abelian categories. *J. Algebra*, 59(2):264–289, 1979.

[16] Anne Quéguiner. Signature des involutions de deuxième espèce. *Arch. Math. (Basel)*, 65(5):408–412, 1995.

[17] Winfried Scharlau. *Quadratic and Hermitian forms*, volume 270 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1985.

[18] James B. Wilson. Decomposing $p$-groups via Jordan algebras. *J. Algebra*, 322(8):2642–2679, 2009.