Non-symmetric convex domains have no basis of exponentials

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Abstract

A conjecture of Fuglede states that a bounded measurable set $\Omega \subset \mathbb{R}^d$, of measure 1, can tile $\mathbb{R}^d$ by translations if and only if the Hilbert space $L^2(\Omega)$ has an orthonormal basis consisting of exponentials $e_\lambda(x) = \exp 2\pi i \langle \lambda, x \rangle$. If $\Omega$ has the latter property it is called spectral. We generalize a result of Fuglede, that a triangle in the plane is not spectral, proving that every non-symmetric convex domain in $\mathbb{R}^d$ is not spectral.

§0. Introduction

Let $\Omega$ be a measurable subset of $\mathbb{R}^d$ of measure 1 and $\Lambda$ be a discrete subset of $\mathbb{R}^d$. We write $e_\lambda(x) = \exp 2\pi i \langle \lambda, x \rangle$, $(x \in \mathbb{R}^d)$, $E_\Lambda = \{e_\lambda : \lambda \in \Lambda\} \subset L^2(\Omega)$.

The inner product and norm on $L^2(\Omega)$ are

$$\langle f, g \rangle_\Omega = \int_\Omega f \overline{g}, \quad \|f\|_\Omega^2 = \int_\Omega |f|^2.$$

Definition 1  The pair $(\Omega, \Lambda)$ is called a spectral pair if $E_\Lambda$ is an orthonormal basis for $L^2(\Omega)$. A set $\Omega$ will be called spectral if there is $\Lambda \subset \mathbb{R}^d$ such that $(\Omega, \Lambda)$ is a spectral pair. The set $\Lambda$ is then called a spectrum of $\Omega$.

Example: If $Q_d = (-1/2, 1/2)^d$ is the cube of unit volume in $\mathbb{R}^d$ then $(Q_d, \mathbb{Z}^d)$ is a spectral pair. We write $B_R(x) = \{y \in \mathbb{R}^d : |x - y| < R\}$.

Definition 2  (Density)

(i) The set $\Lambda \subset \mathbb{R}^d$ has uniformly bounded density if for each $R > 0$ there exists a constant $C > 0$ such that $\Lambda$ has at most $C$ elements in each ball of radius $R$ in $\mathbb{R}^d$.

(ii) The set $\Lambda \subset \mathbb{R}^d$ has density $\rho$, and we write $\rho = \text{dens} \Lambda$, if we have

$$\rho = \lim_{R \to \infty} \frac{|\Lambda \cap B_R(x)|}{|B_R(x)|},$$

uniformly for all $x \in \mathbb{R}^d$.

We define translational tiling for complex-valued functions below.

Definition 3  Let $f : \mathbb{R}^d \to \mathbb{C}$ be measurable and $\Lambda \subset \mathbb{R}^d$ be a discrete set. We say that $f$ tiles with $\Lambda$ at level $w \in \mathbb{C}$, and sometimes write “$f + \Lambda = w \mathbb{R}^d$”, if

$$\sum_{\lambda \in \Lambda} f(x - \lambda) = w, \quad \text{for almost every (Lebesgue) } x \in \mathbb{R}^d,$$

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with the sum above converging absolutely a.e. If \( \Omega \subset \mathbb{R}^d \) is measurable we say that \( \Omega + \Lambda \) is a tiling when \( 1_\Omega + \Lambda = w \mathbb{R}^d \), for some \( w \). If \( w \) is not mentioned it is understood to be equal to 1.

**Remarks**

1. If \( f \in L^1(\mathbb{R}^d) \) and \( \Lambda \) has uniformly bounded density one can easily show (see KL96 for the proof in one dimension, which works in higher dimension as well) that the sum in (1) converges absolutely a.e. and defines a locally integrable function of \( x \).
2. In the very common case when \( f \in L^1(\mathbb{R}^d) \) and \( \int_{\mathbb{R}^d} f \neq 0 \) the condition that \( \Lambda \) has uniformly bounded density follows easily from (1) and need not be postulated a priori.
3. It is easy to see that if \( f \in L^1(\mathbb{R}^d) \), \( \int_{\mathbb{R}^d} f \neq 0 \) and \( f + \Lambda \) is a tiling then \( \Lambda \) has a density and the level of the tiling \( w \) is given by 
   \[
   w = \int_{\mathbb{R}^d} f \cdot \text{dens~} \Lambda.
   \]

From now on we restrict ourselves to tiling with functions in \( L^1 \) and sets of finite measure.

**Example:** \( Q_d + \mathbb{Z}^d \) is a tiling.

The following conjecture is still unresolved.

**Conjecture:** (Fuglede [F74]) If \( \Omega \subset \mathbb{R}^d \) is bounded and has Lebesgue measure 1 then \( L^2(\Omega) \) has an orthonormal basis of exponentials if and only if there exists \( \Lambda \subset \mathbb{R}^d \) such that \( \Omega + \Lambda = \mathbb{R}^d \) is a tiling.

**Remark:** It is not hard to show [F74] that \( L^2(\Omega) \) has a basis \( \Lambda \) which is a lattice (i.e., \( \Lambda = A\mathbb{Z}^d \), where \( A \) is a non-singular \( d \times d \) matrix) if and only if \( \Omega + \Lambda^* \) is a tiling. Here

\[
\Lambda^* = \left\{ \mu \in \mathbb{R}^d : \langle \mu, \lambda \rangle \in \mathbb{Z}, \forall \lambda \in \Lambda \right\}
\]

is the dual lattice of \( \Lambda \) (we have \( \Lambda^* = A^{-\top}\mathbb{Z}^d \)).

Fuglede [F74] showed that the disk and the triangle in \( \mathbb{R}^2 \) are not spectral domains.

In this note we prove the following generalization of Fuglede’s triangle result.

**Theorem 1** Let \( \Omega \) have measure 1 and be a convex, non-symmetric, bounded open set in \( \mathbb{R}^d \). Then \( \Omega \) is not spectral.

The set \( \Omega \) is called symmetric with respect to 0 if \( y \in \Omega \) implies \( -y \in \Omega \), and symmetric with respect to \( x_0 \in \mathbb{R}^d \) if \( y \in \Omega \) implies that \( 2x_0 - y \in \Omega \). It is called non-symmetric if it is not symmetric with respect to any \( x_0 \in \mathbb{R}^d \). For example, in any dimension a simplex is non-symmetric.

It is known [V54, M80] that every convex body that tiles \( \mathbb{R}^d \) by translation is a centrally symmetric polytope and that each such body also admits a lattice tiling and, therefore (see the remark after Fuglede’s conjecture above), its \( L^2 \) admits a lattice spectrum. Given Theorem [1] to prove Fuglede’s conjecture restricted to convex domains, one still has to prove that any symmetric convex body that is not a tile admits no orthonormal basis of exponentials for its \( L^2 \).

In \S \ref{sec:prelim} we derive some necessary and some sufficient conditions for \( f + \Lambda \) to be a tiling. These conditions roughly state that tiling is equivalent to a certain tempered distribution, associated with \( \Lambda \) being “supported” on the zero set of \( \hat{f} \) plus the origin. Similar conditions had been derived in KL96 but here we have to work with less smoothness for \( \hat{f} \). To compensate for the lack of smoothness we work with compactly supported \( \hat{f} \) and nonnegative \( f \) and \( \hat{f} \), conditions which are fulfilled for our problem.

In \S \ref{sec:tiling} we restate the property that \( \Omega \) is spectral as a tiling problem for \( \left| \hat{1}_\Omega \right|^2 \) and use the conditions derived in \S \ref{sec:prelim} to prove Theorem \ref{thm:main}. What makes the proof work is that when \( \Omega \) is a non-symmetric convex set the set \( \Omega - \Omega \) has volume strictly larger than \( 2^d \text{vol} \Omega \).
§1. Fourier-analytic conditions for tiling

Our method relies on a Fourier-analytic characterization of translational tiling, which is a variation of the one used in [KL96]. We define the (generally unbounded) measure
\[ \delta_\Lambda = \sum_{\lambda \in \Lambda} \delta_\lambda, \]
where \( \delta_\lambda \) represents a unit mass at \( \lambda \in \mathbb{R}^d \). If \( \Lambda \) has uniformly bounded density then \( \delta_\Lambda \) is a tempered distribution (see for example [R73]) and therefore its Fourier Transform \( \widehat{\delta_\Lambda} \) is defined and is itself a tempered distribution.

The action of a tempered distribution (see [R73]) \( \alpha \) on a Schwartz function \( \phi \) is denoted by \( \alpha(\phi) \). The Fourier Transform of \( \alpha \) is defined by the equation
\[ \widehat{\alpha}(\phi) = \alpha(\widehat{\phi}). \]
The support \( \text{supp} \alpha \) is the smallest closed set \( F \) such that for any smooth \( \phi \) of compact support contained in the open set \( F^c \) we have \( \alpha(\phi) = 0 \).

**Theorem 2** Suppose that \( f \geq 0 \) is not identically 0, that \( f \in L^1(\mathbb{R}^d) \), \( \widehat{f} \geq 0 \) has compact support and \( \Lambda \subset \mathbb{R}^d \). If \( f+\Lambda \) is a tiling then
\[ \text{supp} \widehat{\delta_\Lambda} \subseteq \{ x \in \mathbb{R}^d : \widehat{f}(x) = 0 \} \cup \{ 0 \}. \quad (2) \]

**Proof of Theorem 2.** Assume that \( f+\Lambda = w\mathbb{R}^d \) and let
\[ K = \{ \widehat{f} = 0 \} \cup \{ 0 \}. \]
We have to show that
\[ \widehat{\delta_\Lambda}(\phi) = 0, \quad \forall \phi \in C_0^\infty(K^c). \]
Since \( \widehat{\delta_\Lambda}(\phi) = \delta_\Lambda(\widehat{\phi}) \) this is equivalent to \( \sum_{\lambda \in \Lambda} \widehat{\phi}(\lambda) = 0 \), for each such \( \phi \). Notice that \( h = \phi/\widehat{f} \) is a continuous function, but not necessarily smooth. We shall need that \( \widehat{h} \in L^1 \). This is a consequence of a well-known theorem of Wiener [R73, Ch. 11]. We denote by \( \mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d \) the d-dimensional torus.

**Theorem (Wiener)** If \( g \in C(\mathbb{T}^d) \) has an absolutely convergent Fourier series
\[ g(x) = \sum_{n \in \mathbb{Z}^d} \widehat{g}(n)e^{2\pi i(n,x)}, \quad \widehat{g} \in \ell^1(\mathbb{Z}^d), \]
and if \( g \) does not vanish anywhere on \( \mathbb{T}^d \) then \( 1/g \) also has an absolutely convergent Fourier series.

Assume that
\[ \text{supp} \widehat{\phi}, \text{supp} \widehat{f} \subseteq \left( -\frac{L}{2}, \frac{L}{2} \right)^d. \]
Define the function \( F \) to be:
(i) periodic in \( \mathbb{R}^d \) with period lattice \( (L\mathbb{Z})^d \),
(ii) to agree with \( \widehat{f} \) on \( \text{supp} \phi \),
(iii) to be non-zero everywhere and,
(iv) to have \( \widehat{F} \in \ell^1(\mathbb{Z}^d) \), i.e.,
\[ \widehat{F} = \sum_{n \in \mathbb{Z}^d} \widehat{F}(n)\delta_{L^{-1}n}, \]
is a finite measure in $\mathbb{R}^d$.

One way to define such an $F$ is as follows. First, define the $(LZ)^d$-periodic function $g \geq 0$ to be $\hat{f}$ periodically extended. The Fourier coefficients of $g$ are $\hat{g}(n) = L^{-d}f(\frac{-n}{L}) \geq 0$. Since $g, \hat{g} \geq 0$ and $g$ is continuous at 0 it is easy to prove that $\sum_{n \in \mathbb{Z}^d} \hat{g}(n) = g(0)$, and therefore that $g$ has an absolutely convergent Fourier series.

Let $\epsilon$ be small enough to guarantee that $\hat{f}$ (and hence $g$) does not vanish on $(\text{supp } \phi) + B_\epsilon(0)$. Let $k$ be a smooth $(LZ)^d$-periodic function which is equal to 1 on $(\text{supp } \phi)(LZ^d)$ and equal to 0 off $(\text{supp } \phi + B_\epsilon(0))(LZ^d)$, and satisfies $0 \leq k \leq 1$ everywhere. Finally, define

$$F = kg + (1-k).$$

Since both $k$ and $g$ have absolutely summable Fourier series and this property is preserved under both sums and products, it follows that $F$ also has an absolutely summable Fourier series. And by the nonnegativity of $g$ we get that $F$ is never 0, since $k = 0$ on $Z(\hat{f})(LZ^d)$.

By Wiener's theorem, $\hat{F}^{-1} \in \ell^1(\mathbb{Z}^d)$, i.e., $\hat{F}^{-1}$ is a finite measure on $\mathbb{R}^d$. We now have that

$$\left(\frac{\phi}{f}\right) \wedge = \hat{\phi} \hat{F}^{-1} = \hat{\phi} \ast \hat{F}^{-1} \in L^1(\mathbb{R}^d).$$

This justifies the interchange of the summation and integration below:

$$\sum_{\lambda \in \Lambda} \hat{\phi}(\lambda) = \sum_{\lambda \in \Lambda} \left(\frac{\phi}{f}\right) \wedge (\lambda)$$
$$= \sum_{\lambda \in \Lambda} \left(\frac{\phi}{f}\right) \wedge \ast \hat{f} (\lambda)$$
$$= \sum_{\lambda \in \Lambda} \int_{\mathbb{R}^d} \left(\frac{\phi}{f}\right) \wedge (y) f(y - \lambda) \, dy$$
$$= \int_{\mathbb{R}^d} \left(\frac{\phi}{f}\right) \wedge (y) \sum_{\lambda \in \Lambda} f(y - \lambda) \, dy$$
$$= w \int_{\mathbb{R}^d} \left(\frac{\phi}{f}\right) \wedge (y) \, dy$$
$$= \frac{\phi}{f}(0)$$
$$= 0,$$

as we had to show.

For a set $A \subseteq \mathbb{R}^d$ and $\delta > 0$ we write

$$A_\delta = \left\{ x \in \mathbb{R}^d : \text{dist} (x, A) < \delta \right\}.$$ 

We shall need the following partial converse to Theorem 2.

**Theorem 3** Suppose that $f \in L^1(\mathbb{R}^d)$, and that $\Lambda \subset \mathbb{R}^d$ has uniformly bounded density. Suppose also that $O \subset \mathbb{R}^d$ is open and

$$\text{supp } \hat{\delta}_\Lambda \setminus \{0\} \subseteq O \quad \text{and} \quad O_\delta \subseteq \left\{ \hat{f} = 0 \right\},$$

for some $\delta > 0$. Then $f + \Lambda$ is a tiling at level $\hat{f}(0) \cdot \hat{\delta}_\Lambda(\{0\})$. 

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Proof. Let $\psi : \mathbb{R}^d \to \mathbb{R}$ be smooth, have support in $B_1(0)$ and $\hat{\psi}(0) = 1$ and for $\epsilon > 0$ define the approximate identity $\psi_{\epsilon}(x) = \epsilon^{-d} \psi(x/\epsilon)$. Let

$$f_{\epsilon} = \hat{\psi}_{\epsilon} f,$$

which has rapid decay.

First we show that $(\int f_{\epsilon})^{-1} f_{\epsilon} + \Lambda$ is a tiling. That is, we show that the convolution $f_{\epsilon} \ast \delta_{\Lambda}$ is a constant. Let $\phi$ be any Schwartz function. Then

$$f_{\epsilon} \ast \delta_{\Lambda}(\phi) = \hat{f}_{\epsilon}(0) \hat{\phi}(0) \hat{\delta}_{\Lambda}(\{0\}),$$

which implies

$$f_{\epsilon} \ast \delta_{\Lambda}(x) = \hat{f}_{\epsilon}(0) \hat{\delta}_{\Lambda}(\{0\}), \text{ a.e.}(x).$$

We also have that $\sum_{\lambda \in \Lambda} |f(x - \lambda)|$ is finite a.e. (see Remark 1 following the definition of tiling), hence, for almost every $x \in \mathbb{R}^d$

$$\sum_{\lambda \in \Lambda} |f(x - \lambda) - f_{\epsilon}(x - \lambda)| = \sum_{\lambda \in \Lambda} |f(x - \lambda)| \cdot |1 - \psi_{\epsilon}(x - \lambda)|,$$

which tends to 0 as $\epsilon \to 0$. This proves

$$\sum_{\lambda \in \Lambda} f(x - \lambda) = \hat{f}(0) \cdot \hat{\delta}_{\Lambda}(\{0\}), \text{ a.e.}(x).$$

§2. Proof of the main result

We now make some remarks that relate the property of $E_\Lambda$ being a basis for $L^2(\Omega)$ to a certain function tiling $\mathbb{R}^d$ with $\Lambda$.

Assume that $\Omega$ is a bounded open set of measure 1. Notice first that

$$\langle e_\lambda, e_x \rangle_\Omega = \hat{1}_\Omega(x - \lambda).$$

The set $E_\Lambda$ is an orthonormal basis for $L^2(\Omega)$ if and only if for each $f \in L^2(\Omega)$

$$\|f\|_\Omega^2 = \sum_{\lambda \in \Lambda} |\langle e_\lambda, f \rangle_\Omega|^2,$$

and, by the completeness of the exponentials in $L^2$ of a large cube containing $\Omega$, it is necessary and sufficient that

$$\sum_{\lambda \in \Lambda} \left| \hat{1}_\Omega(x - \lambda) \right|^2 = 1,$$

(4)

for each $x \in \mathbb{R}^d$. In other words a necessary and sufficient condition for $(\Omega, \Lambda)$ to be a spectral pair is that $\left| \hat{1}_\Omega \right|^2 + \Lambda$ is a tiling at level 1. Notice also that $\left| \hat{1}_\Omega \right|^2$ is the Fourier Transform of $1_\Omega \ast \hat{1}_\Omega$ which has support equal to the set $\overline{\Omega - \Omega}$. We use the notation $\hat{f}(x) = f(-x)$.
Proof of Theorem 1: Write $K = \Omega - \Omega$, which is a symmetric, open convex set. Assume that $(\Omega, \Lambda)$ is a spectral pair. We can clearly assume that $0 \in \Lambda$. It follows that $|\mathbf{1}_\Omega|^2 + \Lambda$ is a tiling and hence that $\Lambda$ has uniformly bounded density, has density equal to 1 and $\hat{\delta}_\Lambda(\{0\}) = 1$.

By Theorem 2 (with $f = |\mathbf{1}_\Omega|^2$, $\hat{f} = \mathbf{1}_\Omega * \mathbf{1}_\Omega(-x)$) it follows that

$$\text{supp} \hat{\delta}_\Lambda \subseteq \{0\} \cup K^c.$$ 

Let $H = K/2$ and write

$$f(x) = \mathbf{1}_H * \mathbf{1}_H(x) = \int_{\mathbb{R}^d} \mathbf{1}_H(y) \mathbf{1}_H(y - x) \, dy.$$ 

The function $f$ is supported in $K$ and has nonnegative Fourier Transform

$$\hat{f} = |\mathbf{1}_H|^2.$$ 

We have

$$\int_{\mathbb{R}^d} \hat{f} = f(0) = \text{vol} H$$
and

$$\hat{f}(0) = \int_{\mathbb{R}^d} f = (\text{vol} H)^2.$$ 

By the Brunn-Minkowski inequality (see for example [G94, Ch. 3]), for any convex body $\Omega$,

$$\text{vol} \frac{1}{2} (\Omega - \Omega) \geq \text{vol} \Omega,$$

with equality only in the case of symmetric $\Omega$. Since $\Omega$ has been assumed to be non-symmetric it follows that

$$\text{vol} H > 1.$$ 

For

$$1 > \rho > \left( \frac{1}{\text{vol} H} \right) ^ {1/d}$$
consider

$$g(x) = f(x/\rho)$$
which is supported properly inside $K$, and has

$$g(0) = f(0) = \text{vol} H, \quad \int_{\mathbb{R}^d} g = \rho^d \int_{\mathbb{R}^d} f = \rho^d (\text{vol} H)^2.$$ 

Since supp $g$ is properly contained in $K$ Theorem 3 implies that $\hat{g} + \Lambda$ is a tiling at level $\int \hat{g} \cdot \text{dens} \Lambda = \int \hat{g} = g(0) = \text{vol} H$. However, the value of $\hat{g}$ at 0 is $\int g = \rho^d (\text{vol} H)^2 > \text{vol} H$, and, since $\hat{g} \geq 0$ and $\hat{g}$ is continuous, this is a contradiction.

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