On the absence of Volterra correct restrictions and extensions of the Laplace operator

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Abstract

At the beginning of the last century J. Hadamard constructed the well-known example illustrating the incorrectness of the Cauchy problem for elliptic-type equations. If the Cauchy problem for some differential equation is correct, then it is usually a Volterra problem, i.e., the inverse operator is a Volterra operator. At present, not a single Volterra correct restriction or extension for elliptic-type equations is known. In the present paper, we prove the absence of Volterra correct restrictions of the maximal operator \( \hat{L} \) and Volterra correct extensions of the minimal operator \( L_0 \) generated by the Laplace operator in \( L_2(\Omega) \), where \( \Omega \) is the unit disk.

1 Introduction

Let us present some definitions, notation, and terminology.

In a Hilbert space \( H \), we consider a linear operator \( L \) with domain \( D(L) \) and range \( R(L) \).

By the kernel of the operator \( L \) we mean the set

\[
\text{Ker } L = \{ f \in D(L) : Lf = 0 \}.
\]

Definition 1.1. An operator \( L \) is called a restriction of an operator \( L_1 \), and \( L_1 \) is called an extension of \( L \), briefly \( L \subset L_1 \), if:

1) \( D(L) \subset D(L_1) \),
2) \( Lf = L_1f \) for all \( f \) from \( D(L) \).

Definition 1.2. A linear closed operator \( L_0 \) in a Hilbert space \( H \) is called minimal if \( R(L_0) \neq H \) and there exists a bounded inverse operator \( L_0^{-1} \) on \( R(L_0) \).

Definition 1.3. A linear closed operator \( \hat{L} \) in a Hilbert space \( H \) is called maximal if \( R(\hat{L}) = H \) and \( \text{Ker } \hat{L} \neq \{0\} \).

Definition 1.4. A linear closed operator \( L \) in a Hilbert space \( H \) is called correct if there exists a bounded inverse operator \( L^{-1} \) defined on all of \( H \).

Definition 1.5. We say that a correct operator \( L \) in a Hilbert space \( H \) is a correct extension of minimal operator \( L_0 \) (correct restriction of maximal operator \( \hat{L} \) if \( L_0 \subset L \) (\( L \subset \hat{L} \)).

Definition 1.6. We say that a correct operator \( L \) in a Hilbert space \( H \) is a boundary correct extension of a minimal operator \( L_0 \) with respect to a maximal operator \( \hat{L} \) if \( L \) is simultaneously a correct restriction of the maximal operator \( \hat{L} \) and a correct extension of the minimal operator \( L_0 \), that is, \( L_0 \subset L \subset \hat{L} \).
Let $\hat{L}$ be a maximal linear operator in a Hilbert space $H$, let $L$ be any known correct restriction of $\hat{L}$, and let $K$ be an arbitrary linear bounded (in $H$) operator satisfying the following condition:

$$R(K) \subset \text{Ker} \hat{L}. \quad (1.1)$$

Then the operator $L^{-1}_K$ defined by the formula (see [1])

$$L^{-1}_K f = L^{-1} f + K f, \quad (1.2)$$

describes the inverse operators to all possible correct restrictions $L_K$ of $\hat{L}$, i.e., $L_K \subset \hat{L}$.

Let $L_0$ be a minimal operator in a Hilbert space $H$, let $L$ be any known correct extension of $L_0$, and let $K$ be a linear bounded operator in $H$ satisfying the conditions

a) $R(L_0) \subset \text{Ker} K$,

b) $\text{Ker} (L^{-1} + K) = \{0\}$,

then the operator $L^{-1}_K$ defined by formula (1.2) describes the inverse operators to all possible correct extensions $L_K$ of $L_0$ (see [1]).

Let $L$ be any known boundary correct extension of $L_0$, i.e., $L_0 \subset L \subset \hat{L}$. The existence of at least one boundary correct extension $L$ was proved by Vishik in [2]. Let $K$ be a linear bounded (in $H$) operator satisfying the conditions

a) $R(L_0) \subset \text{Ker} K$,

b) $R(K) \subset \text{Ker} \hat{L}$,

then the operator $L^{-1}_K$ defined by formula (1.2) describes the inverse operators to all possible boundary correct extensions $L_K$ of $L_0$ (see [1]).

**Definition 1.7.** A bounded operator $A$ in a Hilbert space $H$ is called *quasinilpotent* if its spectral radius is zero, that is, the spectrum consists of the single point zero.

**Definition 1.8.** An operator $A$ in a Hilbert space $H$ is called a *Volterra operator* if $A$ is compact and quasinilpotent.

**Definition 1.9.** A correct restriction $L$ of a maximal operator $\hat{L}$ ($L \subset \hat{L}$), a correct extension $L$ of a minimal operator $L_0$ ($L_0 \subset L$) or a boundary correct extension $L$ of a minimal operator $L_0$ with respect to a maximal operator $\hat{L}$ ($L_0 \subset L \subset \hat{L}$), will be called *Volterra* if the inverse operator $L^{-1}$ is a Volterra operator.

We denote by $\mathcal{S}_\infty(H, H_1)$ the set of all linear compact operators acting from a Hilbert space $H$ to a Hilbert space $H_1$. If $T \in \mathcal{S}_\infty(H, H_1)$, then $T^* T$ is a non-negative self-adjoint operator in $\mathcal{S}_\infty(H, H)$ and, moreover, there is a non-negative unique self-adjoint root $|T| = (T^* T)^{1/2}$ in $\mathcal{S}_\infty(H)$. The eigenvalues $\lambda_n(|T|)$ numbered, taking into account their multiplicity, form a monotonically converging to zero sequence of non-negative numbers. These numbers are usually called *s-numbers* of the operator $T$ and denoted by $s_n(T)$, $n \in \mathbb{N}$. We denote by $\mathcal{S}_p(H, H_1)$ the set of all compact operators $T \in \mathcal{S}_\infty(H, H_1)$, for which

$$|T|^p_p = \sum_{j=1}^\infty s^p_j(T) < \infty, \quad 0 < p < \infty.$$  

Obviously, if rank $|T| = r < \infty$, then $s_n(T) = 0$, for $n = r + 1, r + 2, \ldots$. Operators of finite rank certainly belong to the classes $\mathcal{S}_p(H, H_1)$ for all $p > 0$.

In the Hilbert space $L_2(\Omega)$, where $\Omega$ is the unit disk in $\mathbb{R}^2$ with boundary $\partial \Omega$, let us consider the minimal $L_0$ and maximal $\hat{L}$ operators generated by the Laplace operator

$$- \Delta u = - \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \quad (1.3)$$
The closure $L_0$ in the space $L_2(\Omega)$ of the Laplace operator (1.3) with the domain $C_0^\infty(\Omega)$ is the minimal operator corresponding to the Laplace operator.

The operator $\hat{L}$, adjoint to the minimal operator $L_0$ corresponding to the Laplace operator is the maximal operator corresponding to the Laplace operator (see [3]). Note that

$$D(\hat{L}) = \{u \in L_2(\Omega) : \hat{L}u = -\Delta u \in L_2(\Omega)\}.$$  

Denote by $L_D$ the operator, corresponding to the Dirichlet problem with the domain

$$D(L_D) = \{u \in W^{2,2}_0(\Omega) : u|_{\partial\Omega} = 0\}.$$  

Then, by virtue of (1.2), the inverse operators $L^{-1}$ to all possible correct restrictions of the maximal operator $\hat{L}$ corresponding to the Laplace operator (1.3) have the following form:

$$u \equiv L^{-1}f = L^{-1}_D f + Kf,$$  

where, by virtue of (1.1), $K$ is an arbitrary linear operator bounded in $L^2(\Omega)$ with

$$R(K) \subset \ker \hat{L} = \{u \in L_2(\Omega) : -\Delta u = 0\}.$$  

Then the direct operator $L$ is determined from the following problem:

$$\hat{L}u \equiv -\Delta u = f, \ f \in L_2(\Omega),$$  

$$D(L) = \{u \in D(\hat{L}) : [(I - K\hat{L})u]|_{\partial\Omega} = 0\},$$  

where $I$ is the unit operator in $L^2(\Omega)$. There are no other linear correct restrictions of the operator $\hat{L}$ (see [4]).

The operators $(L^*)^{-1}$, corresponding to the adjoint operators $L^*$

$$v \equiv (L^*)^{-1}g = L^{-1}_D g + K^*g,$$

describe the inverse operators to all possible correct extensions of the minimal operator $L_0$ if and only if $K$ satisfies the condition (see [4]):

$$\ker (L^{-1}_D + K^*) = \{0\}.$$  

Note that the last condition is equivalent to the following: $D(L) = L_2(\Omega)$. If the operator $K$ from (1.4) satisfies one more additional condition

$$KR(L_0) = \{0\},$$

then the operator $L$ corresponding to problem (1.5), (1.6), will turn out to be a boundary correct extension.

Now we state the main result.

2 Main results

We pass to the polar coordinate system:

$$x = r \cos \varphi, \ y = r \sin \varphi.$$  

Then the operator

$$\hat{L}u \equiv -\Delta u = -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = -u_{rr} - \frac{1}{r}u_r - \frac{1}{r^2}u_{\varphi\varphi} = f(r, \varphi),$$  

(2.1)
\[
D(\hat{L}) = \{ u \in L_2(\Omega) : \Delta u \in L_2(\Omega) \},
\]
is the maximal operator (see [5]). Any correct restriction \( L \) acts as the maximal operator \( \hat{L} \) on the domain
\[
D(L) = \{ u \in D(\hat{L}) : [(I - K\hat{L})u]_{r=1} = 0 \},
\]
(2.2)
where \( K \) is any bounded linear operator in \( L_2(\Omega) \) that \( R(K) \subset \text{Ker} \hat{L} \). To be Volterra of \( L \) is necessary compactness of \( L^{-1} \). From (1.4) note that \( L^{-1} \) is compact if and only if \( K \) is a compact operator. Then for \( K \) the Schmidt expansion takes place (see [6, p. 47(28)])
\[
K = \sum_{j=1}^{\infty} s_j (\cdot, Q_j) F_j
\]
(2.3)
where \( \{Q_j\}_{1}^{\infty} \) is orthonormal system in \( L_2(\Omega) \), \( \{F_j\}_{1}^{\infty} \) is orthonormal system in \( \text{Ker} \hat{L} \) and \( \{s_j\}_{1}^{\infty} \) is a monotone sequence of non-negative numbers converging to zero. The series on the right side of (2.3) converges in the uniform operator norm. We now state the main result of this paper.

**Theorem 2.1.** Let \( \hat{L} \) be a maximal operator generated by the Laplace (1.3) in \( L_2(\Omega) \). Then any correct restriction \( L \) of the maximal operator \( \hat{L} \), i.e., the problem (2.1) and (2.2) cannot be Volterra.

**Proof.** Let us prove by contradiction. Suppose that there exists a Volterra correct restriction \( L \). This is equivalent to the existence of a such compact operator \( K \) that the operator \( L \) has no non-zero eigenvalue. The general solution of the equation

\[
\hat{L}u = -\Delta u = -u_{rr} - \frac{1}{r} u_{r} - \frac{1}{r^2} u_{\varphi\varphi} = \lambda^2 u,
\]
from the space \( L_2(\Omega) \) has the form (see [7])

\[
u(r, \varphi) = u_0(r, \varphi) - \int_0^r u_0(\rho, \varphi) \frac{\partial}{\partial \rho} J_0(\lambda \sqrt{r(\rho - \rho)}) d\rho,
\]
where \( \lambda \) is any complex number, \( u_0(r, \varphi) \) is the solution of the equation

\[
\hat{L}u_0 \equiv -\Delta u_0 = -u_{0rr} - \frac{1}{r} u_{0r} - \frac{1}{r^2} u_{0\varphi\varphi} = 0,
\]
which is a harmonic function from the space \( L_2(\Omega) \) and

\[
J_0(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left( \frac{z}{2} \right)^{2n}
\]
is the Bessel function. Then, by virtue of (2.2) we obtain the equation

\[
u_0(1, \varphi) = \int_0^1 u_0(\rho, \varphi) \frac{\partial}{\partial \rho} J_0(\lambda \sqrt{1 - \rho}) d\rho
\]
\[- \lambda^2 \sum_{j=1}^{\infty} s_j F_j(1, \varphi) \cdot \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 u_0(\rho, \theta) \cdot Q_j(\rho, \theta) \rho d\rho d\theta\]
\[+ \lambda^2 \sum_{j=1}^{\infty} s_j F_j(1, \varphi) \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 \frac{Q_j(\rho, \theta)}{\rho} \cdot \int_0^\rho u_0(\tau, \theta) \frac{\partial}{\partial \tau} J_0(\lambda \sqrt{\rho(\rho - \tau)}) d\tau \rho d\rho d\theta = 0.
\]
(2.4)
The considered problem on the spectrum of the Laplace operator has no eigenvalues if and only if the equation (2.4) has no zeros as a function of \( \lambda \). The harmonic function \( u_0(\rho, \phi) \) does not depend on \( \lambda \). It is easy to notice that the left side of the equation is an entire function no higher than the first order. Then by virtue of Picard’s theorem (see [8, p. 264, 266]) this function have the form \( Ce^{d\lambda} \), where \( C(\varphi) \) and \( d(\varphi) \) are functions which are independent of \( \lambda \). If you notice that the left side of the equation (2.4) is even with respect to the sign of \( \lambda \), then \( d = 0 \). Equating these functions when \( \lambda = 0 \) we have \( C = u_0(1, \varphi) \). Then we get the following

\[
- \int_0^1 u_0(\rho, \varphi) \frac{\partial}{\partial \rho} J_0(\lambda \sqrt{1 - \rho}) d\rho \\
- \lambda^2 \sum_{j=1}^{\infty} s_j F_j(1, \varphi) \cdot \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 u_0(\rho, \theta) \cdot \overline{Q_j(\rho, \theta)} \rho d\rho d\theta \\
+ \lambda^2 \sum_{j=1}^{\infty} s_j F_j(1, \varphi) \cdot \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 \overline{Q_j(\rho, \theta)} \int_0^\rho u_0(\tau, \theta) \frac{\partial}{\partial \tau} J_0(\lambda \sqrt{\rho(\rho - \tau)}) d\tau d\rho d\theta = 0.
\]

(2.5)

Divide both sides of (2.5) by \( \lambda^2 \) and let \( \lambda \) tend to zero. Then

\[
\sum_{j=1}^{\infty} s_j F_j(1, \varphi) \cdot \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 u_0(\rho, \theta) \overline{Q_j(\rho, \theta)} \rho d\rho d\theta = -\frac{1}{4} \int_0^1 u_0(\rho, \varphi) d\rho.
\]

(2.6)

Under the condition that (2.6) is fulfilled we obtain

\[
- \int_0^1 u_0(\rho, \varphi) \left[ \frac{\partial}{\partial \rho} (\lambda \sqrt{1 - \rho}) + \frac{1}{4} \right] d\rho \\
+ \sum_{j=1}^{\infty} s_j F_j(1, \varphi) \cdot \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 \overline{Q_j(\rho, \theta)} \int_0^\rho u_0(\tau, \theta) \frac{\partial}{\partial \tau} J_0(\lambda \sqrt{\rho(\rho - \tau)}) d\tau d\rho d\theta = 0.
\]

(2.7)

On the left side of the equation (2.7) we make the change of variables: in the first summand \( t = \sqrt{1 - \rho} \), in the second summand \( t = \sqrt{\rho(\rho - \tau)} \). Then we have

\[
\int_0^1 u_0(1 - t^2, \varphi) \left[ \frac{J'_0(\lambda t)}{2\lambda t} + \frac{1}{4} \right] 2tdt \\
- \sum_{j=1}^{\infty} s_j F_j(1, \varphi) \cdot \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 J'_0(\lambda t) \int_0^t u_0(\rho^2 - t^2, \theta) \overline{Q_j(\rho, \theta)} \rho d\rho dt d\theta = 0.
\]

(2.8)

For the Bessel function has the following equalities

\[
\frac{J'_0(\lambda t)}{2\lambda t} + \frac{1}{4} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n+1)!^2} \left( \frac{\lambda t}{2} \right)^{2n} \cdot (n + 1),
\]

and

\[
\lambda J'_0(\lambda t) = \sum_{n=1}^{\infty} \frac{(-1)^n}{(n!)^2} \left( \frac{\lambda t}{2} \right)^{2n} \cdot n \cdot \frac{2}{t}.
\]
Substitute them into (2.8) and equate the coefficients of $\lambda^{2n}$ to zero

$$\int_0^1 u_0(1 - t^2, \varphi) \frac{-1}{4(n + 1)} \cdot t^{2n} \cdot 2dt$$

$$- \sum_{j=1}^{\infty} s_j F_j(1, \varphi) \cdot \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{n} t^{2n} \cdot \frac{2}{t} \int_t^1 u_0 \left( \frac{\rho^2 - t^2}{\rho}, \theta \right) Q_j(\rho, \theta) \rho d\rho dt \theta = 0.$$

We do the conversion of the following form

$$\frac{1}{n + 1} \cdot t^{2n} = \frac{2}{t^2} \int_0^1 \tau^{2n+1} d\tau, \quad n \cdot t^{2n} = \frac{t}{2} \cdot \frac{\partial}{\partial t} \left( t^{2n} \right).$$

Then

$$\int_0^1 t^{2n} \cdot \left\{ \int_t^1 u_0(1 - \tau^2, \varphi) \frac{d\tau}{\tau} \right. dt$$

$$- \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial}{\partial t} \sum_{j=1}^{\infty} s_j F_j(1, \varphi) \cdot \int_t^1 u_0 \left( \frac{\rho^2 - t^2}{\rho}, \theta \right) Q_j(\rho, \theta) \rho d\rho d\theta \right\} dt = 0.$$

In view of the completeness of the system of functions \( \{t^{2n}\}_{n=1}^{\infty} \) in \( L_2(0, 1) \) we obtain (see [9, p. 107])

$$t \int_t^1 u_0(1 - \tau^2, \varphi) \frac{d\tau}{\tau} - \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial}{\partial t} \sum_{j=1}^{\infty} s_j F_j(1, \varphi) \cdot \int_t^1 u_0 \left( \frac{\rho^2 - t^2}{\rho}, \theta \right) Q_j(\rho, \theta) \rho d\rho d\theta = 0.$$

Integrating this equation from \( t \) to 1, we get

$$\int_t^1 u_0(1 - \tau^2, \varphi) \frac{\tau^2 - t^2}{2\tau} d\tau$$

$$+ \frac{1}{2\pi} \int_0^{2\pi} \int_t^1 u_0 \left( \frac{\rho^2 - t^2}{\rho}, \theta \right) \sum_{j=1}^{\infty} s_j F_j(1, \varphi) Q_j(\rho, \theta) \rho d\rho d\theta = 0.$$

where \( 0 \leq t \leq 1, 0 \leq \varphi < 2\pi \). Note that the condition (2.9) contains the condition (2.6) as a particular case when \( t = 0 \). Condition (2.9) will turn out to be the Volterra criterion of the correct restriction \( L \), if it holds for any harmonic function \( u_0(r, \varphi) \) from \( L_2(\Omega) \).

By Poisson’s formula

$$u_0(r, \varphi) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r \cos(\varphi - \gamma) + r^2} u_0(1, \gamma) d\gamma,$$

the equality (2.9) is transformed to

$$\frac{1}{2\pi} \int_0^{2\pi} u_0(1, \gamma) \left\{ \int_t^1 \frac{1 - (1 - \tau^2)^2}{1 - 2(1 - \tau^2) \cos(\varphi - \gamma) + (1 - \tau^2)^2} \cdot \frac{\tau^2 - t^2}{2\tau} d\tau$$

$$+ \frac{1}{2\pi} \int_0^{2\pi} \int_t^1 \frac{1 - (\frac{\rho^2 - t^2}{\rho})^2}{1 - 2(\rho^2 - t^2) \cos(\theta - \gamma) + (\rho^2 - t^2)^2} \sum_{j=1}^{\infty} s_j F_j(1, \varphi) Q_j(\rho, \theta) \rho d\rho d\theta \right\} d\gamma = 0.$$
Considering the density of the set of functions \( u_0(1, \varphi) \) in \( L_2(0, 2\pi) \) for almost all values of \( t \) \((0 \leq t \leq 1)\), \( \varphi \) \((0 \leq \varphi < 2\pi) \) we obtain the equality

\[
\int_t^1 \frac{1 - (1 - \tau^2)^2}{1 - 2(1 - \tau^2) \cos(\varphi - \gamma) + (1 - \tau^2)^2} \cdot \frac{\tau^2 - t^2}{2\tau} \, d\tau
\]

\[
+ \frac{1}{2\pi} \int_0^{2\pi} \int_t^1 \frac{1 - \left( \frac{\rho^2 - t^2}{\rho} \right)^2}{1 - 2 \left( \frac{\rho^2 - t^2}{\rho} \right) \cos(\theta - \gamma) + \left( \frac{\rho^2 - t^2}{\rho} \right)^2} \cdot \sum_{j=1}^{\infty} s_j F_j(1, \varphi) Q_j(\rho, \theta) \rho \, d\rho \, d\theta = 0.
\]

Now the equation (2.10) is the Volterra criterion of the correct restriction \( L \) of the maximal operator \( \tilde{L} \), generated by the Laplace operator (1.3) in \( L_2(\Omega) \), where \( \Omega \) is the unit disk.

Further, we apply to the equation (2.10) the Poisson operator of the variables \( r \) and \( \varphi \). The first summand we transform with the formula of the superposition of two Poisson integrals (see [10, p. 140]), and in the second summand the harmonic function \( \hat{\text{f}} \) is generated by the Laplace operator (1.3) in \( \Omega \), \( \Omega \) is the unit disk.

From this equality using the orthonormality of the system \( \{ F_j(r, \varphi) \}_{j=1}^{\infty} \) we obtain the relation between the orthonormal systems \( \{ F_j \}_{j=1}^{\infty} \) and \( \{ Q_j \}_{j=1}^{\infty} \) of the following form

\[
\int_t^1 \left\{ \frac{1}{2\pi} \int_0^{2\pi} \int_t^1 \frac{1 - r^2(1 - \tau^2)^2}{1 - 2r(1 - \tau^2) \cos(\varphi - \gamma) + r^2(1 - \tau^2)^2} \cdot \frac{\tau^2 - t^2}{2\tau} \, d\tau \right\} \cdot \frac{\tau^2 - t^2}{2\tau} \, d\tau
\]

\[
= - \frac{1}{2\pi} \int_0^{2\pi} \int_t^1 \frac{1 - \left( \frac{\rho^2 - t^2}{\rho} \right)^2}{1 - 2 \left( \frac{\rho^2 - t^2}{\rho} \right) \cos(\theta - \gamma) + \left( \frac{\rho^2 - t^2}{\rho} \right)^2} \cdot \sum_{j=1}^{\infty} s_j Q_j(\rho, \theta) \rho \, d\rho \, d\theta, ~ j = 1, 2, \ldots
\]

In both parts of the equality (2.11) we use the expansion of the Poisson kernel

\[
\frac{1 - r^2}{1 - 2r \cos \varphi + r^2} = 1 + 2 \sum_{n=1}^{\infty} r^n \cos n\varphi.
\]

We obtain the equality of the two Fourier series in the orthogonal system \( \{ 1/2, \cos \gamma, \sin \gamma, \ldots, \cos n\gamma, \sin n\gamma, \ldots \} \) in \( L_2(0, 2\pi) \). Equating the coefficients, we get the following system of equations

\[
\begin{cases}
\frac{1}{2\pi} \int_0^{2\pi} \int_0^1 F_j(r, \varphi) \cdot r \, dr \, d\varphi \cdot \int_t^1 \frac{\tau^2 - t^2}{2\tau} \, d\tau = \frac{1}{2\pi} \int_0^{2\pi} \int_t^1 s_j Q_j(\rho, \theta) \rho \, d\rho \, d\theta,

\frac{1}{\pi} \int_0^{2\pi} \int_t^1 F_j(r, \varphi) \cdot r^{n+1} \cos n\varphi \, dr \, d\varphi \int_t^1 (1 - \tau^2)^n \frac{\tau^2 - t^2}{2\tau} \, d\tau

= - \int_t^1 \frac{1}{\pi} \int_0^{2\pi} s_j Q_j(\rho, \theta) \cdot \cos n\varphi \, \rho \, d\varphi \left( \frac{\rho^2 - t^2}{\rho} \right)^n \, d\rho,

\frac{1}{\pi} \int_0^{2\pi} \int_t^1 F_j(r, \varphi) \cdot r^{n+1} \cdot \sin n\varphi \, dr \, d\varphi \int_t^1 (1 - \tau^2)^n \frac{\tau^2 - t^2}{2\tau} \, d\tau

= - \int_t^1 \frac{1}{\pi} \int_0^{2\pi} s_j Q_j(\rho, \theta) \cdot \sin n\varphi \, \rho \, d\varphi \left( \frac{\rho^2 - t^2}{\rho} \right)^n \, d\rho, ~ j = 1, 2, \ldots, ~ n = 1, 2, \ldots
\end{cases}
\]
We denote

\[ A_{nj} = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 F_j(r, \varphi) \cos n\varphi \cdot r^n \cdot r dr d\varphi, \quad n = 0, 1, 2, \ldots \]

\[ B_{nj} = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 F_j(r, \varphi) \sin n\varphi \cdot r^n \cdot r dr d\varphi, \quad n = 1, 2, \ldots \]

From the first equation of the system (2.12) it is easy to find that

\[ \frac{1}{2\pi} \int_0^{2\pi} s_j Q_j(t, \theta) d\theta = \frac{1}{2} A_{0j} \cdot \ln t. \]

The second equation reduces to

\[ \int_t^1 (1 - \tau^2)^{n-2} - \frac{t^2}{2\tau} d\tau = - \int_t^1 (\rho^2 - t^2)^{n-2} w_n(\rho) \rho d\rho, \quad n = 1, 2, \ldots \]  (2.13)

if we denote by

\[ \omega_n(\rho) = \frac{1}{\pi} \int_0^{2\pi} s_j Q_j(\rho, \theta) \cos n\theta d\theta \frac{1}{A_{nj} \rho^n}. \]

The third equation is transformed into the same equation (2.13), if we denote by

\[ \omega_n(\rho) = \frac{1}{\pi} \int_0^{2\pi} s_j Q_j(\rho, \theta) \sin n\theta d\theta \frac{1}{B_{nj} \rho^n}. \]

We solve the equation (2.13) with respect to \( \omega_n(\rho) \). Note that

\[ \omega_1(t) = -\frac{1 - t^2}{2t^2}, \quad \omega_2(t) = -\frac{1 - t^4}{4t^4}; \]

Further, we get the recurrence relation

\[ (1 - t^2)^{n-k} = - \int_t^1 (\rho^2 - t^2)^{n-k-2} \cdot (n - k)(n - k - 1) \cdot 4t^2 \omega_n(\rho) \rho d\rho \]

\[ + k \int_t^1 (\rho^2 - t^2)^{n-k-1} \cdot (n - k) \cdot 4 \omega_n(\rho) \rho d\rho, \quad n = 2, 3, 4, \ldots, \quad k = 0, 1, 2, \ldots, n - 2. \]

This relation is equivalent to the Cauchy problem

\[ \omega_n'(t) + \frac{2n}{t} \omega_n(t) = \frac{1}{t}, \quad \omega_n(1) = 0. \]

Solving, we get

\[ \omega_n(t) = \frac{1 - t^{-2n}}{2n}. \]

Now we have the following relations between the orthonormal systems \( \{Q_j\}_1^{\infty} \) and \( \{F_j\}_1^{\infty} \):

\[
\begin{align*}
\frac{1}{2\pi} \int_0^{2\pi} s_j Q_j(t, \theta) d\theta &= \frac{1}{2} A_{0j} \cdot \ln t, \\
\frac{1}{\pi} \int_0^{2\pi} s_j Q_j(t, \theta) \cos n\theta d\theta &= -A_{nj} \cdot \frac{t^n - t^{-n}}{2n}, \\
\frac{1}{\pi} \int_0^{2\pi} s_j Q_j(t, \theta) \sin n\theta d\theta &= -B_{nj} \cdot \frac{t^n - t^{-n}}{2n}, \quad n = 1, 2, \ldots
\end{align*}
\] (2.14)
Satisfying the Volterra criterion (2.10), we obtained the relation (2.14). By assumption $Q_j(t, \theta)$ from $L_2(\Omega)$. Then the integral with respect to $t$ on the left-hand sides of the system of equations (2.14) exists. However, for an arbitrary orthonormal system \( \{F_j\}_1^\infty \), for $n = 1, 2, \ldots$, the integral on the right-hand sides of the system of equations (2.14) with respect to $t$ from 0 to 1 does not exist. This means that there are no orthonormal systems \( \{F_j\}_1^\infty \) and \( \{Q_j\}_1^\infty \) satisfying the equality (2.10). This contradicts our assumption that there exists a Volterra correct restriction $L$. Thus, Theorem 2.1 is proved.

**Corollary 2.2.** There does not exist a Volterra correct extension $L$ of the minimal operator $L_0$ generated by the Laplace operator (1.3) in a Hilbert space $L_2(\Omega)$, where $\Omega$ is the unit disk.

**Proof.** Suppose that there exists a Volterra correct extension $L$ of the minimal operator $L_0$. From $L_0 \subset L$ it follows that $L^* \subset L_0^* = \hat{L}$. The adjoint of a Volterra operator is a Volterra operator. Then we get a contradiction to Theorem 2.1. This completes the proof of Corollary 2.2.

In the author’s work (see [5]), it was proved that there are no Volterra correct extensions or restrictions for the $m$-dimensional Laplace operator in $L_2(\Omega)$, where $\Omega$ is a bounded domain in $\mathbb{R}^m$ with a sufficiently smooth boundary, if the operator $K$ from representation (1.4) that it belongs to the Schatten class $\mathfrak{S}_p(L_2(\Omega))$ for $0 < p \leq m/2$, where $m \geq 2$.

It was noticed that in the case $m = 1$ there exists many Volterra correct restrictions and extensions.

**Remark 2.3.** If in (2.11) the function $u(r, \varphi)$ does not depend on the angle $\varphi$, then we get one-dimensional equation

$$\hat{L}u \equiv -\Delta u = -u_{rr} - \frac{1}{r} u_r = f(r),$$

in the weighted space $L_2(r; 0, 1)$ with weight $r$. Then the Volterra criterion (2.10) and the equation (2.14) determine the operator $K$ of the following form

$$Kf = \int_0^1 f(t) \ln t \cdot tdt.$$

To it corresponds to the correct restriction $L$ with domain $D(L) = \{u \in W_2^1(r; 0, 1) : u(0) = 0\}$. Then the correct restriction $L$ is a Volterra, because its inverse operator

$$u(r) = L^{-1}f = \int_0^r \frac{t}{r} f(t)tdt,$$

is a Volterra in space $L_2(r; 0, 1)$.

**Remark 2.4.** Theorem 2.1 is true for every bounded simply connected domain in the plane, for which the Dirichlet problem is correct and there exists a conformal mapping onto the unit disk.

**Remark 2.5.** The generalization of Theorem 2.1 to the $m$-dimensional ball (where $m \geq 3$) does not cause problems but it is cumbersome to write down.

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