Cube-magic labelings of grids

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Abstract

We show that the vertices and edges of a $d$-dimensional grid graph $G = (V, E)$ ($d \geq 2$) can be labeled with the integers from $\{1, \ldots, |V|\}$ and $\{1, \ldots, |E|\}$, respectively, in such a way that for every subgraph $H$ isomorphic to a $d$-cube the sum of all the labels of $H$ is the same. As a consequence, for every $d \geq 2$, every $d$-dimensional grid graph is $Q_d$-supermagic where $Q_d$ is the $d$-cube.

1 Introduction

The graphs considered in this paper are finite, undirected and simple. For a graph $G$, we denote its vertex set by $V(G)$ and its edge set by $E(G)$. A graph labeling, as introduced in [8], is an assignment of integers to the vertices or edges, or both, subject to certain conditions. Over the years, a large variety of different types of graph labelings have been studied, see [2] for an extensive survey.

For a graph $H$, we say that a graph $G$ admits an $H$-covering if every edge of $G$ belongs to at least one subgraph of $G$ which is isomorphic to $H$. A graph $G = (V, E)$ which admits an $H$-covering is called $H$-magic if there exists a bijection $F : V \cup E \to \{1, 2, \ldots, |V| + |E|\}$ and a constant $c = c(F)$, which we call the $H$-magic sum of $F$, such that

$$\sum_{v \in V(H')} F(v) + \sum_{e \in E(H')} F(e) = c$$

for every subgraph $H' \subseteq G$ with $H' \cong H$. If in addition $F(V) = \{1, \ldots, |V|\}$ then we say that the graph $G$ is $H$-supermagic. The case where $H$ is a single edge was studied in [1], and the general concept for arbitrary graphs $H$ was introduced in [3]. Since then $H$-magic and $H$-supermagic labelings have been studied for a variety of graphs $H$ ([4, 5, 6, 7, 9]).

In this paper we show that for an integer $d \geq 2$, a $d$-dimensional grid graph $G$ is $Q_d$-supermagic where $Q_d$ denotes the $d$-cube. For $d = 2$, the 2-cube is the same as a 4-cycle $C_4$ and our result is a consequence of Theorem 1 in [4] which gives sufficient conditions for the cartesian product of a graph and a path to be $C_4$-supermagic.

The structure of the paper is as follows. In Section 2 we fix some notation and state our main result. Section 3 contains the proof which is by induction on the dimension $d$, where the base case $d = 2$ is contained in Section 3.1 and the induction step in Section 3.2.
2 Notation and main result

For integers $k \leq \ell$ we denote the sets $\{1, \ldots, k\}$ and $\{k, k+1, \ldots, \ell\}$ by $[k]$ and $[k, \ell]$, respectively. For integers $d \geq 2$ and $n_1 \geq n_2 \geq \cdots \geq n_d \geq 2$, let $\text{GRID}(d_1, \ldots, d_d)$ denote the $n_1 \times \cdots \times n_d$-grid graph, i.e., the cartesian product of $d$ paths of lengths $n_1, \ldots, n_d$. In other words, the vertex set of $G(d_1, \ldots, n_d)$ is $V = [n_1] \times \cdots \times [n_d]$ and edge set

$$E = \left\{ \{x, y\} : x, y \in V \text{ and } \sum_{i=1}^{d} |x_i - y_i| = 1 \right\}.$$  

The graph $\text{GRID}(2, 2, \ldots, 2)$ is called the $d$-cube and will be denoted by $Q_d$.

To simplify the presentation of our proof we will label the vertices and the edges separately and then combine the labelings to obtain the $Q_d$-supermagic labeling. A vertex labeling $f : V \rightarrow \{1, 2, \ldots, |V|\}$ for a graph $G = (V, E)$ is called $H$-magic if there exists a constant $c = c(f)$, called the H-magic sum of $f$ such that

$$\sum_{v \in V(H')} f(v) = c$$

for every subgraph $H' \subseteq G$ with $H' \cong H$. Similarly, an edge labeling $g : E \rightarrow \{1, 2, \ldots, |E|\}$ for a graph $G = (V, E)$ is called $H$-magic if there exists a constant $c' = c'(g)$, called the $H$-magic sum of $f$ such that

$$\sum_{e \in E(H')} f(e) = c'$$

for every subgraph $H' \subseteq G$ with $H' \cong H$. An $H$-magic vertex labeling $f$ and an $H$-magic edge labeling $g$ with $H$-magic sums $c = c(f)$ and $c' = c'(g)$ can be combined to obtain an $H$-supermagic labeling $F$ with $H$-supermagic sum $c + |E(H)||V(G)|$ by setting $F(v) = f(v)$ for all $v \in V(G)$ and $F(e) = g(e) + |V(G)|$ for all $e \in E(G)$.

**Theorem 1.** Let $d \geq 2$ and $n_1 \geq n_2 \geq \cdots \geq n_d \geq 2$ be positive integers, and let $G = \text{GRID}(n_1, \ldots, n_d)$. Then $G$ admits a $Q_d$-magic vertex labeling $f$ and a $Q_d$-magic edge labeling $g$.

Based on the observation about combining $H$-magic vertex and edge labelings we obtain the following corollary.

**Corollary 1.** Let $d \geq 2$ and $n_1 \geq n_2 \geq \cdots \geq n_d \geq 2$ be positive integers. Then $\text{GRID}(n_1, \ldots, n_d)$ is $Q_d$-supermagic.

3 Proof of the main result

We proceed by induction on $d$. In Section 3.1 we treat the base case $d = 2$, and present explicit vertex and edge labelings for grid graphs $\text{GRID}(n_1, n_2)$. In Section 3.2 we assume $d \geq 3$, and we describe how labelings $f$ and $g$ for $\text{GRID}(n_1, \ldots, n_d)$ can be constructed from the labelings $\hat{f}$ and $\hat{g}$ for $\text{GRID}(n_1, \ldots, n_{d-1})$.

3.1 The base case $d = 2$

In order to describe the labeling in a compact way we use $[P]$ to denote the indicator function for a statement $P$, i.e.,

$$[P] = \begin{cases} 1 & \text{if } P \text{ is true}, \\ 0 & \text{if } P \text{ is false}. \end{cases}$$
We define a vertex labeling \( f : [n_1] \times [n_2] \rightarrow [n_1 n_2] \) by

\[
f(i, j) = \begin{cases} 
(i-1)n_2 + j & \text{if } i \text{ is odd and } j \text{ is odd}, \\
(i-1)n_2 + (n_2 + 1 - j) & \text{if } i \text{ is even and } j \text{ is even}, \\
(n_1 - i)n_2 + j + [2 \mid n_1 \wedge 2 \nmid n_2] & \text{if } i \text{ is odd and } j \text{ is even}, \\
(n_1 - i)n_2 + (n_2 + 1 - j + [2 \mid n_1 \wedge 2 \nmid n_2]) & \text{if } i \text{ is even and } j \text{ is odd},
\end{cases}
\]

and an edge labeling \( g : E \rightarrow [2n_1 n_2 - n_1 - n_2] \) by

\[
g((i, j), (i, j + 1)) = (i-1)(2n_2 - 1) + j \quad \text{for } (i, j) \in [n_1] \times [n_2 - 1], \tag{2}
\]

\[
g((i, j), (i + 1, j)) = (n_1 - i)(2n_2 - 1) + 1 - j \quad \text{for } (i, j) \in [n_1 - 1] \times [n_2]. \tag{3}
\]

**Example 1.** The construction is illustrated in Figure 1 for the graph Grid(5,3).

![Figure 1: A C4-supermagic labeling for Grid(5,3).](image)

Figure 1: A \( C_4 \)-supermagic labeling for Grid(5,3). The vertex labels \( f(i, j) \) are given by (1), and the edge labels are \( g(e) + 15 \) where \( g \) is defined by (2) and (3).

**Lemma 1.** The function \( f \) defined by (1) is a \( Q_2 \)-magic vertex labeling for Grid\((n_1, n_2)\) with \( Q_2 \)-magic sum

\[
c(f) = \begin{cases} 
2(n_1 n_2 + 1) & \text{if } n_1 \text{ is odd or } n_2 \text{ is even}, \\
2(n_1 n_2 + 2) & \text{if } n_1 \text{ is even and } n_2 \text{ is odd}.
\end{cases}
\]

**Proof.** For every \((i, j) \in [n_1 - 1] \times [n_2 - 1]\) we have

\[
f(i, j) + f(i, j + 1) + f(i + 1, j) + f(i + 1, j + 1)
\]

\[
= (i-1)n_2 + j + (n_1 - i)n_2 + (j + 1) + [2 \mid n_1 \wedge 2 \nmid n_2] + (n_1 - (i + 1))n_2
\]

\[
+ n_2 + 1 - (j + 1) + [2 \mid n_1 \wedge 2 \nmid n_2] + ((i + 1) - 1)n_2 + n_2 + 1 - (j + 1)
\]

\[
= (2n_1 - 2)n_2 + 2n + 2 + 2[2 \mid n_1 \wedge 2 \nmid n_2] + 2(n_1 n_2 + 1 + [2 \mid n_1 \wedge 2 \nmid n_2]) \quad \Box
\]

**Lemma 2.** The function \( g \) defined by (2) and (3) is a \( Q_2 \)-magic edge labeling for Grid\((n_1, n_2)\) with \( Q_2 \)-magic sum \( c(g) = (2n_1 - 1)(2n_2 - 1) + 1 \).

**Proof.** For every \((i, j) \in [n_1 - 1] \times [n_2 - 1]\), we have

\[
g((i, j), (i, j + 1)) + g((i, j), (i + 1, j)) + g((i + 1, j), (i + 1, j + 1)) + g((i, j + 1), (i + 1, j + 1))
\]

\[
= (i-1)(2n_2 - 1) + j + (n_1 - i)(2n_2 - 1) + 1 - j + ((i + 1) - 1)(2n_2 - 1) + j
\]

\[
+ (n_1 - i)(2n_2 - 1) + 1 - (j + 1)
\]

\[
= (2n_1 - 1)(2n_2 - 1) + 1 \quad \Box
\]
Combining these two labelings as described in Section 2 we obtain a $Q_2$-supermagic labeling $F$ with $Q_2$-supermagic sum

$$c(F) = 10n_1n_2 - 2n_1 - 2n_2 + \begin{cases} 4 & \text{if } n_1 \text{ is odd or } n_2 \text{ is even}, \\ 6 & \text{if } n_1 \text{ is even and } n_2 \text{ is odd}. \end{cases}$$

### 3.2 The induction step

We now assume $d \geq 3$. By induction, there exist $Q_{d-1}$-magic labelings $\tilde{f}$ and $\tilde{g}$ with $Q_{d-1}$-magic sums $S = c(f)$ and $S' = c'(g)$ for $\text{Grid}(n_1, \ldots, n_{d-1})$. We define $f : [n_1] \times \cdots \times [n_d] \to \{1, \ldots, \prod_{i=1}^d n_i\}$ by

$$f(x_1, \ldots, x_d) = \begin{cases} \tilde{f}(x_1, \ldots, x_{d-1}) + (x_d - 1) \prod_{i=1}^{d-1} n_i & \text{if } x_1 + \cdots + x_{d-1} \text{ is even}, \\ \tilde{f}(x_1, \ldots, x_{d-1}) + (n_d - x_d) \prod_{i=1}^{d-1} n_i & \text{if } x_1 + \cdots + x_{d-1} \text{ is odd}. \end{cases} \quad (4)$$

**Example 2.** The vertex labeling (4) is illustrated in Figure 2 for the graph $\text{Grid}(5, 3, 3)$ where $\tilde{f}$ is the vertex labeling for $\text{Grid}(5, 3)$ presented in Example 1.

![Figure 2: A $Q_3$-magic vertex labeling with magic sum $c(f) = 184$ for $\text{Grid}(5, 3, 3)$.](image)

**Lemma 3.** The function $f$ defined by (4) a $Q_d$-magic vertex labeling with $Q_d$-magic sum

$$c(f) = 2S + 2^{d-1}(n_d - 1) \prod_{i=1}^{d-1} n_i.$$
Figure 3: A vertex-labeled $Q_3$-subgraph of $\text{GRID}(5, 3, 3)$. The triples next to the vertices indicate their position in the grid.

by (4) the vertex labels for the top and the bottom square of the cube in Figure 3 are

\[ 27 = 12 + 15, \quad 22 = 7 + 15, \quad 20 = 5 + 15 \]
\[ 42 = 12 + 2 \times 15, \quad 7 = 7, \quad 38 = 8 + 2 \times 15, \quad 5 = 5 \]

and therefore the sum of the vertex labels is $2(12 + 7 + 8 + 5 + 4 + 15) = 2 \times 15$, where $S$ is the magic sum of the vertex labeling in Figure 3.

Proof of Lemma 3. Fix $i \in [n_d]$ and $(x_1, \ldots, x_{d-1}) \in [n_1 - 1] \times \cdots \times [n_{d-1} - 1]$. Let $H \cong Q_{d-1}$ be the subgraph of $\text{GRID}(n_1, \ldots, n_d)$ induced by

\[ V(H) = \{(x_1 + \varepsilon_1, \ldots, x_{d-1} + \varepsilon_{d-1}, i) : (\varepsilon_1, \ldots, \varepsilon_{d-1}) \in \{0, 1\}^{d-1}\}. \]

Using the fact that exactly half of the $2^{d-1}$ vertices of $H$ have even coordinate sum, we obtain from (4):

\[
\sum_{v \in V(H)} f(v) = \sum_{\varepsilon \in \{0, 1\}^{d-1}} f(x_1 + \varepsilon_1, \ldots, x_{d-1} + \varepsilon_{d-1}) + 2^{d-2}(i - 1) \prod_{i=1}^{d-1} n_i + 2^{d-2}(n_d - i) \prod_{i=1}^{d-1} n_i \\
= S + 2^{d-2} \prod_{i=1}^{d-1} n_i(i - 1 + n_d - i) = S + 2^{d-2}(n_d - 1) \prod_{i=1}^{d-1} n_i. \tag{5}
\]

A subgraph $H$ of $\text{GRID}(n_1, \ldots, n_d)$ is isomorphic to $Q_d$ if and only if its vertex set is

\[ V(H) = \{(x_1 + \varepsilon_1, \ldots, x_{d-1} + \varepsilon_{d-1}, x_d + \varepsilon_d) : (\varepsilon_1, \ldots, \varepsilon_{d-1}, \varepsilon_d) \in \{0, 1\}^d\} \]

for some $(x_1, \ldots, x_d) \in [n_1 - 1] \times \cdots \times [n_d - 1]$. Using (5), this implies

\[
\sum_{v \in V(H)} f(v) = 2 \left( S + 2^{d-2}(n_d - 1) \prod_{i=1}^{d-1} n_i \right). \tag{6}
\]

We define an edge labeling $g : E \rightarrow \{1, \ldots, |E|\}$ for $G = \text{GRID}(n_1, \ldots, n_d)$ as follows. Let $N = |V(\text{GRID}(n_1, \ldots, n_{d-1}))|$ and $M = |E(\text{GRID}(n_1, \ldots, n_{d-1}))|$. For an edge $e = \{x, y\}$ with $y_d = x_d + 1$, we set

\[
g(e) = \tilde{f}(x_1, \ldots, x_{d-1}) + n_d M + \begin{cases} 
(x_d - 1)N & \text{if } x_1 + \cdots + x_{d-1} \text{ is odd,} \\
(n_d - 1 - x_d)N & \text{if } x_1 + \cdots + x_{d-1} \text{ is even.}
\end{cases} \tag{6}
\]

For the remaining edges we distinguish two cases.
Case 1. $d$ is odd. For an edge $e = \{x, y\}$ with $y_i = x_i + 1$, $i \leq d - 1$, we set

$$
g(e) = \tilde{g}(\{(x_1, \ldots, x_{d-1}), (y_1, \ldots, y_{d-1})\}) + \begin{cases} 
(x_d - 1)M & \text{if } i \text{ is odd}, \\
(n_d - x_d)M & \text{if } i \text{ is even}.
\end{cases} \quad (7)
$$

Case 2. $d$ is even. For an edge $e = \{x, y\}$ with $y_i = x_i + 1$, $i \leq d - 2$, we set

$$
g(e) = \tilde{g}(\{(x_1, \ldots, x_{d-1}), (y_1, \ldots, y_{d-1})\}) + \begin{cases} 
(x_d - 1)M & \text{if } i \text{ is odd}, \\
(n_d - x_d)M & \text{if } i \text{ is even}.
\end{cases} \quad (8)
$$

For an edge $e = \{x, y\}$ with $y_{d-1} = x_{d-1} + 1$, we set

$$
g(e) = \tilde{g}(\{(x_1, \ldots, x_{d-1}), (y_1, \ldots, y_{d-1})\}) + \begin{cases} 
(x_d - 1)M & \text{if } \sum_{i=1}^{d-2} x_i \text{ is odd}, \\
(n_d - x_d)M & \text{if } \sum_{i=1}^{d-2} x_i \text{ is even}.
\end{cases} \quad (9)
$$

Example 3. The edge labeling given by (6) to (9) is illustrated in Figure 4 for the graph $\text{Grid}(5,3,3)$. The underlying labelings for $\tilde{f}$ and $\tilde{g}$ for $\text{Grid}(5,3)$ are the labelings from Example 1.

Lemma 4. The function $g$ defined by (6) to (9) is a $Q_d$-magic edge labeling with $Q_d$-magic sum

$$
c'(g) = S + 2S' + 2^{d-2}(n_d - 2)N + 2^{d-2}(2n_d + (d - 1)(n_d - 1)) M.
$$

Again, we illustrate the basic idea before going into the formal proof. Figure 5 shows the edge labels for the same subgraph of $\text{Grid}(5,3,3)$ as in the illustration for Lemma 3 (see Figure 3). The top and bottom squares of the cube in Figure 5 correspond to the square labeled 12–7–8–5 in Figure 6. In this example we have $N = 15$ and $M = 22$, and according to (6), the labels of the vertical edges are

$$
78 = 12 + 3 \times 22, \quad 88 = 7 + 3 \times 22 + 15, \quad 74 = 8 + 3 \times 22, \quad 86 = 5 + 3 \times 22 + 15,
$$

Figure 4: A $Q_3$-magic edge labeling with magic sum 594 for $\text{Grid}(5,3,3)$. 
so the sum of the labels of the vertical edges is $S + 12 \times 22 + 2 \times 15$, where $S = 12 + 7 + 8 + 5$ is the vertex-magic sum in Figure 6. According to (7), the labels of the edges in the top and the bottom square are

$$15 = 15, \quad 55 = 11 + 2 \times 22, \quad 14 = 14, \quad 50 = 6 + 4 \times 22,$$

$$37 = 15 + 22, \quad 33 = 11 + 22, \quad 36 = 14 + 22, \quad 28 = 6 + 22,$$

and therefore the sum of these edge labels is $2(15 + 11 + 14 + 6 + 4 \times 22) = 2S + 8 \times 22$, where $S' = 15 + 11 + 14 + 6$ is the edge-magic sum in Figure 6. Adding all edge labels we obtain $S + 2S' + 2 \times 15 + 20 \times 22$ as claimed in the lemma.

**Proof of Lemma 4** Fix a subgraph $H \subseteq \text{GRID}(n_1, \ldots, n_d)$ with $H \cong Q_d$. Its vertex set is

$$V(H) = \{(x_1 + \varepsilon_1, \ldots, x_{d-1} + \varepsilon_{d-1}, x_d + \varepsilon_d) : (\varepsilon_1, \ldots, \varepsilon_d) \in \{0,1\}^d\}$$

for some $(x_1, \ldots, x_d) \in [n_1 - 1] \times \cdots \times [n_d - 1]$. We partition the vertex set as $V(H) = V_0(H) \cup V_1(H)$ and the edge set as $E(H) = E_0(H) \cup E_1(H) \cup E_2(H)$ where

$$V_\varepsilon(H) = \{y \in V(H) : y_d = x_d + \varepsilon\} \quad \text{for } \varepsilon \in \{0,1\},$$

$$E_\varepsilon(H) = \{(y, z) \in E(H) : y_d = z_d = x_d + \varepsilon\} \quad \text{for } \varepsilon \in \{0,1\},$$

$$E_2(H) = \{(y, z) \in E(H) : y_d = x_d, z_d = x_d + 1\}.$$

Note that $|V_0(H)| = |V_1(H)| = |E_0(H)| = |E_2(H)| = 2^{d-1}$ and $|E_0(H)| = |E_1(H)| = |E(Q_{d-1})| = (d-1)2^{d-2}$. Using the fact that $y_1 + \cdots + y_{d-1}$ is even for exactly half of the edges $(y, z) \in E_2(H)$ and that the subgraph of $G$ induced by $V_0(H)$ is isomorphic to $Q_{d-1}$, we obtain from (6),

$$\sum_{e \in E_2(H)} g(e) = \sum_{y \in V_0(H)} \tilde{f}(y_1, \ldots, y_{d-1}) + 2^{d-1}n_dM + 2^{d-2}(x_d - 1)N + 2^{d-2}(n_d - 1 - x_d)N$$

$$= S + 2^{d-1}n_dM + 2^{d-2}(n_d - 2)N. \quad (10)$$

For the edges in $E_0(H) \cup E_1(H)$ we use the fact that each of the sets $V_\varepsilon(H)$ induces a subgraph isomorphic to $Q_{d-1}$. In addition, if $d$ is even then exactly half of the indices $i \in \{1, \ldots, d-1\}$ are even, and if $d$ is even then
• exactly half of the indices \( i \in \{1, \ldots, d - 2\} \) are even, and

• for exactly half of the vertices \((y_1, \ldots, y_{d-2}, x_{d-1}, x_d + \varepsilon)\) the sum \( y_1 + \cdots + y_{d-2} \) is even.

In both cases we conclude that for exactly half of the edges in \( E_{\varepsilon}(H) \) the term added to \( \tilde{g}(\{(x_1, \ldots, x_{d-1}), (y_1, \ldots, y_{d-1})\}) \) in (6) to (9) is \( (x_d - 1)M \), and for the other half it is \( (n_d - x_d)M \). This implies

\[
\sum_{\{x,y\} \in E_{\varepsilon}(H)} g(\{x,y\}) = \sum_{\{x,y\} \in E_{\varepsilon}(H)} \tilde{g}(\{(x_1, \ldots, x_{d-1}), (y_1, \ldots, y_{d-1})\}) \\
+ (d - 1)2^{d-3}(x_d - 1)M + (d - 1)2^{d-3}(n_d - x_d)M \\
= S' + (d - 1)2^{d-3}(n_d - 1)M. \quad (11)
\]

Combining (10) and (11), the function \( g \) is a \( Q_d \)-magic labeling with \( Q_d \)-magic sum

\[
c'(g) = S + 2^{d-1}n_dM + 2 \left( S' + (d - 1)2^{d-3}(n_d - 1)M \right) \\
= S + 2S' + 2^{d-2}(n_d - 2)N + 2^{d-2} (2n_d + (d - 1)(n_d - 1))M. \quad \square
\]

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