The centre of generic algebras of small PI algebras

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Abstract

Verbally prime algebras are important in PI theory. They are well known over a field \( K \) of characteristic zero: 0 and \( K\langle T \rangle \) (the trivial ones), \( M_n(K), M_n(E), M_{ab}(E) \). Here \( K\langle T \rangle \) is the free associative algebra with free generators \( T \), \( E \) is the infinite dimensional Grassmann algebra over \( K \), \( M_n(K) \) and \( M_n(E) \) are the \( n \times n \) matrices over \( K \) and over \( E \), respectively. Moreover \( M_{ab}(E) \) are certain subalgebras of \( M_{a+b}(E) \), defined below. The generic algebras of these algebras have been studied extensively. Procesi gave a very tight description of the generic algebra of \( M_n(K) \). The situation is rather unclear for the remaining nontrivial verbally prime algebras.

In this paper we study the centre of the generic algebra of \( M_{11}(E) \) in two generators. We prove that this centre is a direct sum of the field and a nilpotent ideal (of the generic algebra). We describe the centre of this algebra. As a corollary we obtain that this centre contains nonscalar elements thus we answer a question posed by Berele.

The verbally prime algebras (also called T-prime) play a crucial role in the theory of the ideals of identities (also called T-ideals) of associative algebras. A T-ideal is called T-prime if it is prime in the class of all T-ideals. Let \( K \) be a field and denote by \( K\langle T \rangle \) the free associative algebra freely generated by the set \( T \) over \( K \). If \( \text{char } K = 0 \) then the nontrivial T-prime T-ideals are those of the polynomial identities of the following algebras: \( M_n(K), M_n(E), M_{ab}(E) \). We denote here by \( E \) the infinite dimensional Grassmann (or exterior) algebra over \( K \). The algebra \( M_{ab}(E) \) is a subalgebra of \( M_{a+b}(E) \). It consists of the block matrices having blocks \( a \times a \) and \( b \times b \) on the main diagonal with entries from \( E_0 \), and all remaining entries from \( E_1 \). Here \( E_0 \) is the centre of \( E \) and \( E_1 \) is the anticommuting part of \( E \). In order to be more precise, assume \( V \) is

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a vector space with a basis $e_1, e_2, \ldots$, and let $E$ be the Grassmann algebra of $V$. Then $E$ has a basis consisting of all elements of the type $e_{i_1} \ldots e_{i_k}$ where $i_1 < \cdots < i_k$, $k \geq 0$, and multiplication induced by $e_i e_j = -e_j e_i$. Hence $E_0$ is the span of all of the above elements with even $k$ while $E_1$ is the span of those with odd $k$.

The above classification of the T-prime algebras was obtained by Kemer, as a part of the theory that led him to the positive solution of the Specht problem, see [13] for an account of Kemer’s theory.

Although polynomial identities in T-prime algebras have been extensively studied the concrete information is quite scarce. Thus the polynomial identities for $M_n(K)$ are known only for $n \leq 2$, see [19] [20] [9] when $K$ is of characteristic 0, and [14] [8], when $|K| = \infty$, char $K = p > 2$. The identities satisfied by the Grassmann algebra $E$ are well known, see [15] when char $K = 0$, and the references of [12] for the remaining cases for $K$. The identities of $M_{11}(E)$ were described in characteristic 0 by Popov, [16]. Recall that the paper [16] gives a basis of the identities satisfied by $E \otimes E$ but it is well known (see for example [15]) that the latter algebra satisfies the same identities as $M_{11}(E)$ when char $K = 0$. Our knowledge about the identities even of $M_3(K)$ and of $M_2(E)$ is quite limited; it should be noted that no working methods are available in order to describe them.

Let $A$ be a PI algebra and suppose $I = T(A)$ is its T-ideal in $K(T)$. The quotient $K(T)/I$ is the relatively free algebra, also called the generic algebra of $A$. Thus one may want to study the generic algebras of the T-prime algebras. It is worth mentioning that these generic algebras admit quite natural models as matrices over certain algebras. The generic algebra for $M_n(K)$ is called the generic matrix algebra. It is a fundamental object in Invariant theory, and enjoys very many good properties; one associates the study of the generic matrix algebras with Procesi, see for example [17] [18] and also [11]. Concrete models for the generic algebras for $M_{ab}(E)$ and for $M_n(E)$ were described by Berele [5]. Moreover the study of these generic algebras led to descriptions of their trace rings and to many interesting results in Invariant theory, see for example [3] [4] [6] [7]. The detailed knowledge of the generic algebra for $M_n(K)$ led Procesi [18] to the description of the trace identities of this algebra, a result obtained independently by Razmyslov as well, see [20]. The Razmyslov and Procesi’s theorem states that the trace identities for $M_n(K)$ all follow from the Cayley–Hamilton characteristic polynomial. Razmyslov proved an analogue of this assertion for the algebras $M_{ab}(E)$ as well.

In [5, Corollary 21] it was proved that the centre of the generic algebra of $M_{ab}(E)$ is a direct sum of the base field and a nilpotent ideal of the centre. Moreover the author of [5] asked whether the centre of that generic algebra contains any non-scalar elements.

In this paper we describe completely the centre of the generic algebra in two generators of $M_{11}(E)$. It follows from our description that it is a direct sum of the field and a nilpotent ideal (of the generic algebra). Moreover we obtain a detailed information about that nilpotent ideal. As a corollary we show that there are very many non-scalar elements in the centre.
By using this description of the centre we were able to obtain, in characteristic 0, a basis of the polynomial identities satisfied by the generic algebra of $M_{11}(E)$ in two generators. Clearly these differ significantly from the identities of $M_{11}(E)$. This last result requires quite a lot of work, and will be published in a forthcoming paper.

1 Preliminaries

We fix an infinite field $K$ of characteristic different from 2. All algebras and vector spaces we consider will be over $K$. We denote by $K(T)$ the free (unitary) associative algebra freely generated over $K$ by the infinite countable set $T = \{t_1, t_2, \ldots \}$. One may conveniently view $K(T)$ as the algebra of polynomials in the non-commuting variables $T$. If $T_k$ is a finite set with $k$ elements, say $T_k = \{t_1, \ldots, t_k\}$ then the free algebra in $k$ generators is denoted by $K(T_k)$. The polynomial $f(t_1, \ldots, t_n) \in K(T)$ is a polynomial identity for the algebra $A$ if $f(a_1, \ldots, a_n) = 0$ for all $a_i \in A$. The set of all polynomial identities satisfied by $A$ is denoted by $T(A)$, it is its T-ideal. Here we suppose $T(A) \subseteq K(T)$. Set $T_k(A) = T(A) \cap K(T_k)$. Furthermore we denote $U(A) = K(T)/T(A)$ and $U_k(A) = K(T_k)/T_k(A)$ the relatively free algebras of $A$ of infinite rank and of rank $k$, respectively. With some abuse of notation we shall use the same letters $t_i$ for the free generators of $K(T)$ and for their images under the canonical projection on $U(A)$; analogously for the rank $k$ case.

The algebra $A$ is 2-graded if $A = A_0 \oplus A_1$, a direct sum of vector subspaces such that $A_iA_j \subseteq A_{i+j}$ where the latter sum is taken modulo 2. Such algebras are often called superalgebras. A typical example is the Grassmann algebra $E = E_0 \oplus E_1$ as above. We call the elements from $A_0 \cup A_1$ homogeneous. When $a \in A_i$ we denote its homogeneous degree $\deg a = i$, $i = 0, 1$. If $A$ is 2-graded and moreover $ab - (-1)^{\deg a \deg b}ba = 0$ for all homogeneous $a$ and $b$ then $A$ is called a supercommutative algebra. Clearly the Grassmann algebra is supercommutative. Next we recall the construction of the free supercommutative algebra, see for example [5] Lemma 1]. Let $X$ and $Y$ be two sets and form the free associative algebra $K(X \cup Y)$. It is 2-graded assuming the elements of $X$ of degree 0 and those of $Y$ of degree 1. Denote by $I$ the ideal generated by all $ab - (-1)^{\deg a \deg b}ba$ where $a$, $b$ are homogeneous, and put $K[X; Y] = K(X \cup Y)/I$. It is immediate to see that $K[X; Y] \cong K[X] \otimes_K E(Y)$. Here $K[X]$ is the polynomial algebra in $X$ and $E(Y)$ is the Grassmann algebra of the vector space with basis $Y$. Thus if $Y = \{y_1, y_2, \ldots \}$ then $E(Y)$ will have a basis consisting of the products $y_iy_j \cdots y_k$, $i_1 < \cdots < i_k$, and multiplication induced by $y_iy_jy_k = -y_jy_i$. Below we also recall the construction of the generic algebras for the T-prime algebras.

Suppose $X = \{x^r_{ij}\}$, $Y = \{y^r_{ij}\}$ where $1 \leq i, j \leq n$, $r = 1, 2, \ldots$; observe that we use $r$ as an upper index, not as an exponent. Define the matrices $A_r = (x^r_{ij})$, $B_r = (x^r_{ij} + y^r_{ij})$, $C_r = (x^r_{ij})$ where $z = x$ whenever $1 \leq i, j \leq a$ or $a + 1 \leq i, j \leq a + b$, and $z = y$ for all remaining possibilities for $i$ and $j$. Suppose $a + b = n$, and consider the following subalgebras of $M_n(K[X; Y])$. The first
is generated by the generic matrices $A_r, K[A_r \mid r \geq 1]$. It is well known it is isomorphic to the relatively free (or universal) algebra $U(M_n(K))$ of $M_n(K)$. In [3, Theorem 2] it was shown that $U(M_n(E)) \cong K[B_r \mid r \geq 1]$, and that $U(M_{ab}(E)) \cong K[C_r \mid r \geq 1]$. Moreover the relatively free algebras of finite rank $k$, denoted by $U_k$, can be obtained by letting $r = 1, \ldots, k$, that is by taking the first $k$ matrices.

We recall another fact from [3] that we shall exploit. It was shown in [3, Theorem 20] that if $f$ is a central polynomial for $M_{ab}(E)$, without constant term, then for some $m$ the polynomial $f^m$ is an identity for $M_{ab}(E)$. It follows that the centre of $U_k(M_{ab}(E))$ must be a direct sum of $K$ and a nilpotent ideal of the centre, see [3, Corollary 21].

We shall need information about the polynomial identities of $M_{11}(E)$. These were described by Popov in characteristic 0, see the main theorem of [10]. As we mentioned above, in [10] it was proved that the $T$-ideal of $E \otimes E$ is generated by the two polynomials

$$[[t_1, t_2]^2, t_1], \quad [[t_1, t_2], [t_3, t_4], t_5]$$

where $[a, b] = ab - ba$ is the usual commutator. We consider the commutators left normed that is $[a, b, c] = [[a, b], c]$, and so on in higher degree.

The algebra $K(T)$ is multigraded by the degree of its monomials in each variable. We work with the infinite field $K$ therefore every $T$-ideal is generated by its multihomogeneous elements, see for example [10, Section 4.2]. Thus from now on we shall work with multihomogeneous polynomials only.

The algebra $E \otimes E$ is PI equivalent to $M_{11}(E)$ in characteristic 0, so the polynomials (1) generate the $T$-ideal of $M_{11}(E)$ as well. This is a result due to Kemer, see [10]. Kemer proved that the tensor product of two $T$-prime algebras (in characteristic 0) is PI equivalent to a $T$-prime algebra, and described precisely these PI equivalences. Note that if $\text{char}K = p > 2$ then the algebras $M_{11}(E)$ and $E \otimes E$ are not PI equivalent, see for example [2], or [1]. While the former paper proved directly the non-equivalence the latter proved it by computing the GK dimensions of the corresponding relatively free algebras (these turn out to be different).

2 The free supercommutative algebra

In this section we denote by $F = U_2(M_{11}(E))$ the relatively free algebra of rank 2 for $M_{11}(E)$. As mentioned before we have that $F = K[C_1, C_2]$. From now on we consider the free supercommutative algebra $K[X, Y]$ where $X = \{x_1, x_2, x'_1, x'_2\}$, $Y = \{y_1, y_2, y'_1, y'_2\}$, and set $C_1 = \begin{pmatrix} x_1 & y_1 \\ y'_1 & x'_1 \end{pmatrix}$, $C_2 = \begin{pmatrix} x_2 & y_2 \\ y'_2 & x'_2 \end{pmatrix}$. Clearly the algebra $F$ satisfies all identities of $M_{11}(E)$.

The algebra $K[X, Y]$ is graded by the integers, taking into account the degree with respect to the variables in $Y$ only: $K[X, Y] = \oplus_{n \in \mathbb{Z}} K[X, Y]^{(n)}$. Here $K[X, Y]^{(n)}$ is the span of all monomials of degree $n$ in the variables from $Y$. It is immediate that the $n$-th homogeneous component is zero unless $0 \leq n \leq 4$. 

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The canonical 2-grading on \( K[X; Y] \) and the \( \mathbb{Z} \)-grading just defined are related as follows:

\[
K[X; Y]_0 = K[X; Y]^{(0)} + K[X; Y]^{(2)} + K[X; Y]^{(4)}; \\
K[X; Y]_1 = K[X; Y]^{(1)} + K[X; Y]^{(3)}.
\]

The next facts are quite obvious; we collect them in a lemma for further reference.

**Lemma 1** Consider the polynomial algebra \( K[X] \subseteq K[X; Y] \).

1. For every \( n \), \( 0 \leq n \leq 4 \), \( K[X; Y]^{(n)} \) is a free module over \( K[X] \), with a basis \( B_n \), where \( B_0 = \{1\} \), and

\[
B_1 = \{y_1, y_2, y'_1, y'_2\}, \quad B_2 = \{y_1y_2, y_1y'_1, y_1y'_2, y_2y'_1, y_2y'_2, y'_1y'_2\} \\
B_3 = \{y_1y_2y'_1, y_1y_2y'_2, y_1y'_1y'_2, y_2y'_1y'_2\}, \quad B_4 = \{y_1y_2y'_1y'_2\}.
\]

2. The free supercommutative algebra \( K[X; Y] \) is a free module over \( K[X] \) with a basis \( B = B_0 \cup B_1 \cup B_2 \cup B_3 \cup B_4 \).

3. Every ideal of \( K[X; Y] \) is a \( K[X] \)-submodule of \( K[X; Y] \).

In effect one may extend the scalars as follows. Let \( K(X) \) be the field of fractions of \( K[X] \), and consider the Grassmann algebra \( E(Y) \) on \( Y \) over \( K(X) \), that is \( E(Y) = E(Y) \otimes_K K(X) \).

**Lemma 2** The matrices \( C_1 \) and \( C_2 \) are not zero divisors in \( F \).

**Proof.** Suppose \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in F \) is such that \( C_1 A = 0 \). Then one obtains

\[
x_1a + y_1c = 0, \quad y'_1a + x'_1c = 0, \quad x_1b + y_1d = 0, \quad y'_1b + x'_1d = 0.
\]

It follows from the second equation that \( x'_1c = -y'_1a \). Now multiply the first equation by \( x'_1 \) and substitute \( x'_1c \) by its equal and get \( a(x_1x'_1 - y_1y'_1) = 0 \).

Extending the scalars as above, and noting that the set \( B \) from Lemma 1 is a basis of the vector space \( E(Y) \) over \( K(X) \), we immediately obtain \( a = 0 \) and consequently \( c = 0 \). In the same manner but using the last two equations we obtain \( b = d = 0 \). In this way \( C_1 \) is not a left zero divisor. Analogously one shows it is not a right zero divisor, and the same for \( C_2 \). \( \diamond \)

We define an automorphism \( ' \) on \( K[X; Y] \) by setting \( x_i \mapsto x'_i, \ x'_i \mapsto x_i, \ y_i \mapsto y'_i \) and \( y'_i \mapsto y_i \). Thus the automorphism \( ' \) is of order two.

We also define the following polynomials in \( K[X] \):

\[
q_n(x_1, x'_1) = \sum_{i=0}^n x_1^i x'^{-i}_1; \quad Q_n(x_2, x'_2) = q_n(x_2, x'_2);
\]

\[
r_n(x_1, x'_1) = \sum_{i=0}^{n-1} (n-i)x_1^{n-1-i}x'^i_1; \quad R_n(x_2, x'_2) = r_n(x_2, x'_2);
\]

\[
s_n(x_1, x'_1) = r_n(x'_1, x_1); \quad S_n(x_2, x'_2) = s_n(x_2, x'_2).
\]
Lemma 3 The following relations hold among the above polynomials.

\[
\begin{align*}
    r_n &= q_{n-1} + x_1 r_{n-1}; \\
    s_n &= q_{n-1} + x_1' s_{n-1}; \\
    s_n + r_n &= (n + 1) q_{n-1}; \\
    q_n &= x_1^n + x_1' q_{n-1} = x_1^n + x_1 q_{n-1}; \\
    (x_1' - x_1) q_{n-1} &= x_1^n - x_1^n; \\
    x_1^n x_1'^m - x_1^m x_1'^n &= (x_1' - x_1)(q_n q_{m-1} - q_m q_{n-1}).
\end{align*}
\]

Proof. The proof consists of an easy induction. 

It is immediate to check, once again by induction, that for every \( m \) and \( n \),

\[
\begin{align*}
    C_1^m &= x_1^n + y_1 y'_1 r_{n-1} \\
    y_1 q_{n-1} - y_1 y'_1 s_{n-1} \\
    C_2^m &= x_2^n + y_2 y'_2 r_{m-1} \\
    y_2 q_{m-1} - y_2 y'_2 s_{m-1}.
\end{align*}
\]

Note that the \( y_i \) and the \( y'_i \) anticommute and this produces the minus signs at the \( (2, 2) \)-entries of the above matrices.

Therefore for the product \( C_1^m C_2^m \) we have

\[
C_1^m C_2^m = \begin{pmatrix}
    x_1^n x_2'^m + a + d & y_1 x_2'^m q_{m-1} + y_2 x_2'^n q_{m-1} + c \\
    y_1 x_2'^m q_{m-1} + y_2 x_2'^n q_{m-1} + c' & x_1^n x_2'^m + d' + d''
\end{pmatrix}
\]

where \( a, a' \in K[X; Y]^{(2)}, d, d' \in K[X; Y]^{(4)}, \) and \( c, c' \in K[X; Y]^{(3)} \).

As the elements of the algebra \( F \) are linear combinations of products of the above type we obtain immediately the proof of the following lemma.

Lemma 4 Let \( A = (a_{ij}) \in F \), then \( a_{22} = a'_{11} \) and \( a_{21} = a'_{12} \).

We shall need the following elements in order to describe the centre of \( F \).

\[
\begin{align*}
    h_1 &= y_1 y_2 y'_1 y'_2; \\
    h_2 &= y_1 y_2 (y'_1 (x'_2 - x_2) - y'_2 (x'_1 - x_1)); \\
    h_3 &= y'_1 y'_2 (y_1 (x'_2 - x_2) - y_2 (x'_1 - x_1)); \\
    h_4 &= (y'_1 (x'_2 - x_2) - y'_2 (x'_1 - x_1)) (y_1 (x'_2 - x_2) - y_2 (x'_1 - x_1)).
\end{align*}
\]

Once again using the fact that \( y_1, y'_1, y_2, y'_2 \) anticommute we obtain that the elements \( h_1, h_2, h_3, h_4 \) satisfy the following relations in \( K[X; Y] \).

\[
\begin{align*}
    h_1 y_1 &= h_1 y'_1 = h_1 y_2 = h_1 y'_2 = 0; \\
    h_2 y_1 &= h_2 y_2 = h_3 y'_1 = h_3 y'_2 = 0; \\
    h_2 y'_1 &= h_3 y_1 = (x'_1 - x_1) h_1; \\
    h_2 y'_2 &= h_3 y_2 = (x'_2 - x_2) h_1; \\
    h_4 y_1 &= (x'_1 - x_1) h_2; \\
    h_4 y_2 &= (x'_2 - x_2) h_2; \\
    h_4 y'_1 &= -(x'_1 - x_1) h_3; \\
    h_4 y'_2 &= -(x'_2 - x_2) h_3.
\end{align*}
\]
3 The centre of $F_2(M_{11}(E))$

Take a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in F \cong K[C_1, C_2]$. Assume it is central in $F$, that is $[A, C_1] = [A, C_2] = 0$. Then

\[
by_1' + cy_1 = 0; \quad by_2' + cy_2 = 0; \\
(a - d)y_1 + (x_1' - x_1)b = 0; \quad (a - d)y_2 + (x_2' - x_2)b = 0; \\
(a - d)y_1' + (x_1' - x_1)c = 0; \quad (a - d)y_2' + (x_2' - x_2)c = 0.
\]

We multiply the third equation by $(x_2' - x_2)$, the fourth by $(x_1' - x_1)$ and subtract, obtaining $(d - a)((x_2' - x_2)y_1 - (x_1' - x_1)y_2) = 0$. Similarly from the last two equations we get $(d - a)((x_2' - x_2)y_1' - (x_1' - x_1)y_2') = 0$. Therefore

\[
d - a \in Ann((x_2' - x_2)y_1 - (x_1' - x_1)y_2) \cap Ann((x_2' - x_2)y_1' - (x_1' - x_1)y_2').
\]

Denote by $J$ the intersection of the two annihilators in the right hand side above. Then $J$ is an ideal of $K[X; Y]$ hence $J$ is a $K[X]$-submodule as well.

**Proposition 5** The $K[X]$-module $J$ is spanned by $\{h_1, h_2, h_3, h_4\}$.

**Proof.** It is immediate that $J$ contains $h_1, h_2, h_3, h_4$. We shall prove that $J$ is contained in the $K[X]$-module spanned by $\{h_1, h_2, h_3, h_4\}$. Let $f \in J$ and write $f = f_0 + f_1 + f_2 + f_3 + f_4$ where $f_i \in K[X; Y]^{(i)}$. First note that both $((x_2' - x_2)y_1 - (x_1' - x_1)y_2)$ and $((x_2' - x_2)y_1' - (x_1' - x_1)y_2')$ lie in $K[X; Y]^{(1)}$ in the $\mathbb{Z}$-grading defined above. Thus $f$ annihilates the latter two polynomials if and only if every $f_i$ does. Therefore we may and shall assume $f$ is homogeneous in the $\mathbb{Z}$-grading.

Suppose first $f \in K[X; Y]^{(0)} \cong K[X]$. As $f \in J$ then it follows easily that $f = 0$.

Let $f \in K[X; Y]^{(1)} \cap J$, then $f = \alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_1' + \alpha_4 y_2'$ for some $\alpha_i \in K[X]$. But $f(y_1(x_2' - x_2) - y_2(x_1' - x_1)) = 0$, hence

\[
(x_1' - x_1)(-\alpha_1 y_1 y_2 + \alpha_3 y_1 y_2' + \alpha_4 y_2 y_2') - (x_2' - x_2)(-\alpha_2 y_1 y_2 - \alpha_3 y_1 y_2' - \alpha_4 y_1 y_2') = 0.
\]

As $B_2$ is a $K[X]$-basis for the free module $K[X; Y]^{(2)}$ it follows $\alpha_3 = \alpha_4 = 0$. In the same way, using the fact that $f$ annihilates $y_1(x_2' - x_2) - y_2(x_1' - x_1)$, we get $\alpha_1 = \alpha_2 = 0$. This implies $J \cap K[X; Y]^{(1)} = 0$.

Let $f = \alpha_1 y_1 y_2 + \alpha_2 y_1 y_2' + \alpha_3 y_2 y_2' + \alpha_4 y_2 y_2'$, then $f \in K[X; Y]^{(2)} \cap J$. As above, from $f(y_1(x_2' - x_2) - y_2(x_1' - x_1)) = 0$ we obtain, in $K[X; Y]^{(3)}$, a linear combination of $y_1 y_2 y_1', y_1 y_2 y_2', y_1 y_1 y_2', y_2 y_1 y_2'$, with coefficients respectively

\[
\alpha_2(x_1' - x_1) + \alpha_4(x_2' - x_2); \quad \alpha_3(x_1' - x_1) + \alpha_5(x_2' - x_2); \quad \alpha_6(x_2' - x_2); \quad -\alpha_6(x_1' - x_1).
\]

Thus $\alpha_6 = 0$, $\alpha_2(x_1' - x_1) + \alpha_4(x_2' - x_2) = 0$, and $\alpha_3(x_1' - x_1) + \alpha_5(x_2' - x_2) = 0$.

Analogously, from $f(y_1(x_2' - x_2) - y_2(x_1' - x_1)) = 0$ we obtain that $\alpha_1 = 0$, $\alpha_2(x_1' - x_1) + \alpha_3(x_2' - x_2) = 0$, and $\alpha_4(x_1' - x_1) + \alpha_5(x_2' - x_2) = 0$.  

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Now the $\alpha_i$ are polynomials in $K[X]$. Consider the field of fractions $K(X)$ of this polynomial ring, and resolve the corresponding linear system of four equations in $K(X)$. One obtains that the solution depends on one parameter $\beta \in K(X)$:

$$
\alpha_2 = \beta; \quad \alpha_3 = -\frac{x'_1 - x_1}{x'_2 - x_2}\beta; \quad \alpha_4 = -\frac{x'_1 - x_1}{x'_2 - x_2}\beta; \quad \alpha_5 = \frac{(x'_1 - x_1)^2}{(x'_2 - x_2)^2}\beta,
$$

and of course $\alpha_1 = \alpha_6 = 0$. Since we are looking for a solution of the system in $K[X]$ then one must have $\beta = (x'_2 - x_2)^2\alpha$ for some $\alpha \in K[X]$, and the solution in $K[X]$ will be

$$
\alpha_2 = (x'_2 - x_2)^2\alpha; \quad \alpha_3 = -(x'_1 - x_1)(x'_2 - x_2)\alpha;
\alpha_4 = -(x'_1 - x_1)(x'_2 - x_2)\alpha; \quad \alpha_5 = (x'_1 - x_1)^2\alpha.
$$

When we substitute these in the expression of $f$ we get $f = \alpha h_4$.

Let $f = \alpha_1 y_1 y'_1 + \alpha_2 y_1 y'_2 + \alpha_3 y_1 y'_3 + \alpha_4 y_2 y'_2 \in K[X; Y]^{(3)} \cap J$ where $\alpha_i \in K[X]$. Proceeding as above we get the equalities

$$
(\alpha_4(x'_2 - x_2) + \alpha_3(x'_1 - x_1))y_1 y'_1 y_2 y'_2 = 0
$$

$$
(\alpha_2(x'_2 - x_2) + \alpha_1(x'_1 - x_1))y_1 y'_1 y_2 y'_2 = 0
$$

therefore $\alpha_2(x'_2 - x_2) + \alpha_1(x'_1 - x_1) = 0$ and $\alpha_4(x'_2 - x_2) + \alpha_3(x'_1 - x_1) = 0$.

As above we work first in $K(X)$ and then go back to $K[X]$. We find that the solution in $K[X]$ is

$$
\alpha_1 = -(x'_2 - x_2)\alpha; \quad \alpha_2 = (x'_1 - x_1)\alpha; \quad \alpha_3 = -(x'_2 - x_2)\beta; \quad \alpha_4 = (x'_1 - x_1)\beta
$$

where $\alpha, \beta \in K[X]$. Substituting these values in $f$ we obtain $f = \alpha h_2 + \beta h_3$, and this case is dealt with.

Finally it is immediate to see that $K[X; Y]^{(4)} = K[X] \cdot h_1$, and thus we conclude the proof. \hfill \diamond

**Corollary 6** Let the matrix $A$ be as at the beginning of this section. Then $A$ commutes with $C_1$ and $C_2$ if and only if $b = f_4 h_2$, $c = -f_4 h_3$, $d = a + f_1 h_1 + f_4 h_4$ for some $f_1, f_4 \in K[X]$. \hfill \diamond

**Proof.** We already saw that $d-a \in J$. Thus $d-a = f_1 h_1 + f_2 h_2 + f_3 h_3 + f_4 h_4$, $f_i \in K[X]$. By means of homogeneity we get $d-a \in K[X; Y]_0$, it follows that $d-a = f_1 h_1 + f_4 h_4$. Now substituting $d-a$ in the system (2) we have $b = f_4 h_2$ and $c = -f_4 h_3$. On the other hand if $a, b, c, d$ satisfy the conditions of the statement it is immediate that $[A, C_1] = [A, C_2] = 0$. \hfill \diamond

**Remark 1.** We just proved that $[A, C_1] = [A, C_2] = 0$ if and only if

$$
A = aI + f_1 \begin{pmatrix} 0 & 0 \\ 0 & h_1 \end{pmatrix} + f_4 \begin{pmatrix} 0 & h_2 \\ -h_3 & h_4 \end{pmatrix}.
$$
2. Therefore if $A$ is central in $F$ then $a_{12}, a_{21} \in K[X;Y]^{(3)}$.

An element $a \in F$ will be called strongly central if it is central, and moreover, for every $b \in F$ the element $ab$ is central in $F$ (thus $ba = ab$ will be strongly central as well).

Let us fix the following matrices in $F$:

$$A_0 = \begin{pmatrix} h_1 & 0 \\ 0 & h_1 \end{pmatrix}; \quad A_1 = \begin{pmatrix} 0 & 0 \\ 0 & h_1 \end{pmatrix}; \quad A_2 = \begin{pmatrix} 0 & h_2 \\ -h_3 & h_4 \end{pmatrix}; \quad A_3 = \begin{pmatrix} h_4 & 0 \\ 0 & h_4 \end{pmatrix}.$$  

**Lemma 7** Let $a = a_0A_0 + a_1A_1 + a_2A_2 + a_3A_3$, for some $a_i \in K[X]$. If $a \in F$ then $a$ is strongly central.

**Proof.** The matrices $A_0$ and $A_1$ are clearly strongly central. Also $A_2$ and $A_3$ are central. One computes

$$A_2C_i = (x'_i - x_i) \begin{pmatrix} h_1 & 0 \\ 0 & -h_1 \end{pmatrix} + x'_i \begin{pmatrix} 0 & h_2 \\ -h_3 & h_4 \end{pmatrix}.$$  

Hence $A_2C_i$ is a linear combination (over $K[X]$) of $A_0$, $A_1$ and $A_2$. Iterating we will have $A_2C_i, C_{i_2} \ldots C_{i_r}$ is central for $i_j = 1, 2$ and $r = 1, 2, \ldots$, and $A_2$ is strongly central. One checks in a similar manner that

$$A_3C_i = x_i \begin{pmatrix} h_4 & 0 \\ 0 & h_4 \end{pmatrix} + (x'_i - x_i) \begin{pmatrix} 0 & h_2 \\ -h_3 & h_4 \end{pmatrix}.$$  

That is $A_3C_i$ is a combination of $A_2$ and $A_3$ and iterating as above we show that $A_3$ is strongly central.  

**Lemma 8** Let $f(t_1, t_2) = [t_1, t_2, t_3, \ldots, t_m]$ be a left normed commutator, $i_j = 1, 2$. Suppose that $\deg t_1 = n$, $\deg t_2 = m$, $n + m = k$. Then for every $n$ and $m$ one has $f(C_1, C_2) = (x'_1 - x_1)^{n-1}(x'_2 - x_2)^{m-1}A(k)$ where

$$A(k) = \begin{pmatrix} F(k) \\ (-1)^k(y_2(x'_2 - x_1) - y_1'(x'_2 - x_2)) \end{pmatrix} \frac{y_1(x'_2 - x_2) - y_2(x'_1 - x_1)}{F(k)},$$  

$$F(k) = \frac{y_1(x'_2 - x_2) - y_2(x'_1 - x_1)y'_1 + (-1)^k y_1(y'_2(x'_1 - x_1) - y'_1(x'_2 - x_2))}{x'_i - x_i}.$$  

In the last expression we use the shorthand $i$ for $i_k$.

**Proof.** The proof consists of an induction on $k$. The base of the induction is $k = 2$; then $F(2) = y_1y'_2 + y'_1y_2$. If $f = f_k$ is the commutator of the statement then one computes $[f_k, C_i]$ directly by induction.  

**Remark** It follows from the above lemma that if $u$ is a left normed commutator in $t_1$ and $t_2$ then $u(C_1, C_2)$ does not depend on the order of the variables starting with the third and up to the last but one. In other words any permutation of the variables in $u$ that preserves the first two and the last one, leaves $u$ invariant.
Lemma 9 Let $u_1(t_1, t_2)$ and $u_2(t_1, t_2)$ be two left-normed commutators of degrees at least two in $F$, and denote by $u = u_1u_2$ their product. Then $u$ is strongly central in $F$.

Proof. Suppose deg$_{t_1} u_j = n_j$ and deg$_{t_2} u_j = m_j$, $j = 1, 2$. Using the notation of Lemma 8, we have

$$u_1(C_1, C_2)u_2(C_1, C_2) = (x'_1 - x_1)^{n_1 + n_2 - 2}(x'_2 - x_2)^{m_1 + m_2 - 2}A(k_1)A(k_2)$$

where deg$_t u_j = k_j$. But it is immediate to see that $F(k_1)F(k_2) = \alpha h_1$, and also

$$F(k_1)(y_1(x'_2 - x_2) - y_2(x'_1 - x_1)) = (-1)^{k_1}h_2,$$

$$F(k_1)(y'_2(x'_1 - x_1) - y'_1(x'_2 - x_2)) = h_3$$

where $\alpha \in K[X]$. Then

$$A(k_1)A(k_2) = F(k_1)F(k_2)I + \begin{pmatrix} (-1)^{k_2}h_4 & -((-1)^{k_1} + (-1)^{k_2})h_2 \\ ((-1)^{k_1} + (-1)^{k_2})h_3 & (-1)^{k_1}h_4 \end{pmatrix}.$$ 

Therefore $u_1(C_1, C_2)u_2(C_1, C_2) = \alpha A_0 + (-1)^{k_2}A_3 - ((-1)^{k_1} + (-1)^{k_2})A_2$ is strongly central in $F$.

Remark Let $u_1$ and $u_2$ be two commutators, deg $u_1 \equiv$ deg $u_2$ (mod 2). Then $u = u_1u_2$ is central element in $F$ but it is not a scalar multiple of $I$. In particular

$$[C_1, C_2]^2 = \begin{pmatrix} -2h_1 + h_4 & -2h_2 \\ 2h_3 & -2h_1 - h_4 \end{pmatrix}$$

is central in $F$ but is not a scalar. This answers Berele’s question from [5] for the case of $M_{11}(E)$ and two generators. In this same case we shall give below the precise answer to Berele’s question.

We consider unitary algebras. Let $L(T)$ be the free Lie algebra freely generated by $T$; suppose further $L(T) \subset K(T)$. That is we consider the vector space $K(T)$ with the commutator operation $[a, b] = ab - ba$, and take $L(T)$ as the Lie subalgebra generated by $T$. Choose an ordered basis of $L(T)$ such that the variables from $T$ precede the longer commutators. As $K(T)$ is the universal enveloping algebra of $L(T)$ one has that a basis of $K(T)$ consists of 1 and all products $t_{i_1}^{m_1} \cdots t_{i_k}^{m_k} u_{j_1} \cdots u_{j_m}$ where $i_1 < \cdots < i_k$, and the $u_{j_i}$ are commutators of degree at least two. Clearly all this holds for $K(t_1, t_2)$ and for its homomorphic image $F = K[C_1, C_2]$. Therefore every element of $F$ is a linear combination of products of the type $C_1^n C_2^m u_1 \cdots u_r$ where the $u_i$ are commutators. Moreover we can assume all commutators left normed, and of the type $[C_1, C_2, \ldots]$. 

Proposition 10 Let $f(C_1, C_2) = C_1^n C_2^m u_1^{k_1} \cdots u_r^{k_r} \in F$ where the $u_i$ are left normed commutators, deg $u_i \geq 2$. The element $f$ is central in $F$ if and only if $k_1 + \cdots + k_r \geq 2$, or else $m = n = k_1 = \cdots = k_r = 0$. 

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Proof. It follows from Lemma 9 that the product of two commutators is strongly central. Thus if \( k_1 + \cdots + k_r \geq 2 \) then \( f(C_1, C_2) \) is central. It remains to prove that \( C_1^n C_2^m \) and \( C_1^m C_2^n \) are not central where \( u \) is a left normed commutator. (In the former we assume \( n + m > 0 \).) Take first \( C_1^n C_2^m \). By the form of the product (computed just before Lemma 8) it follows that the (1,2) entry of \( C_1^n C_2^m \) is \( \alpha y_1 + \beta y_2 + \gamma \) where \( (\alpha, \beta) \neq (0, 0) \) and \( \gamma \in \mathcal{K}[X;Y]^{(3)} \). Now by the remark following Corollary 6 we have that \( C_1^n C_2^m \) cannot be central as long as \( n + m > 0 \).

One proceeds in a similar manner when \( f = C_1^n C_2^m u \) where \( u \) is a left normed commutator. The (1,2) entry of \( f \) will be \( x_1^n x_2^m (y_1 (x'_2 - x_2) - y_2 (x'_1 - x_1)) + b \) for some \( b \in \mathcal{K}[X;Y]^{(3)} \). Once again the remark mentioned above yields that \( f \) cannot be central.

\[ \diamond \]

Proposition 11 Let \( u_1, \ldots, u_r \) be left normed commutators, \( \deg u_j = n \), \( \deg u_j = m \) for all \( j \). Suppose \( u_j = [t_1, t_2, t_3, \ldots, t_{k_j}], j = 1, \ldots, r \). Put \( f(C_1, C_2) = \sum \alpha_j u_j(C_1, C_2) \in F \) where \( \alpha_j \in \mathcal{K}[X] \). Then the following three conditions are equivalent.

1. The element \( f(C_1, C_2) \) is strongly central in \( F \).
2. The element \( f(C_1, C_2) \) is central in \( F \).
3. The sum \( \alpha_1 + \cdots + \alpha_r = 0 \) in \( \mathcal{K}[X;Y] \).

Proof. Clearly (1) implies (2). We prove now that (2) implies (3). Suppose \( f(C_1, C_2) \) is central in \( F \). By Lemma 8 the (1,2) entry of every commutator \( u_i \) equals \( (x'_1 - x_1)^{n-1}(x'_2 - x_2)^{m-1}(y_1(x'_2 - x_2) - y_2(x'_1 - x_1)) \). Hence the (1,2) entry of \( f(C_1, C_2) \) equals

\[
\sum \beta_i(y_1(x'_2 - x_2) - y_2(x'_1 - x_1)) = (y_1(x'_2 - x_2) - y_2(x'_1 - x_1)) \sum \beta_i
\]

where \( \beta_i = (x'_1 - x_1)^{n-1}(x'_2 - x_2)^{m-1} \alpha_i \). Thus if \( \sum \alpha_i \neq 0 \) then the (1,2) entry of \( f(C_1, C_2) \) is a non-zero multiple of \( y_1(x'_2 - x_2) - y_2(x'_1 - x_1) \) by some element of \( \mathcal{K}[X;Y]_0 \) and cannot belong to \( \mathcal{K}[X;Y]^{(3)} \). By the remark following Corollary 6 \( f(C_1, C_2) \) cannot be central.

In order to complete the proof we have to prove that (3) implies (1). Suppose \( \sum \alpha_i = 0 \). It was observed in the remark preceding Lemma 8 that if \( u_i \) and \( u_j \) have the same rightmost variable then \( u_i(C_1, C_2) = u_j(C_1, C_2) \). Thus we divide the commutators \( u_j \) into two types according to their rightmost variable. Clearly if all of them end with \( t_1 \) then \( \sum \alpha_j u_j(C_1, C_2) = u_1(C_1, C_2) \sum \alpha_j = 0 \). Hence suppose \( u_1 \) ends with \( t_1 \) while \( u_2 \) ends with \( t_2 \). Write \( \sum \alpha_j u_j = \beta_1 u_1 + \beta_2 u_2 \) where \( \beta_q \) is the sum of all \( \alpha_j \) such that \( u_j \) ends with \( t_q \), \( q = 1, 2 \). Then \( \beta_1 + \beta_2 = \sum \alpha_j = 0 \) and it suffices to prove \( u_1 - u_2 \) is strongly central. But \( u_1 - u_2 = (x'_1 - x_1)^{n-1}(x'_2 - x_2)^{m-1}(F_1(k) - F_2(k))I \) where we denote by \( F_j(k) \) the expression \( F(k) \) from Lemma 8 obtained by \( u_j, j = 1, 2 \). Clearly \( F_1(k) - F_2(k) = 0 \) if \( k \) is even, and \( F_1(k) - F_2(k) = -2(x'_1 - x_1)^{-1}(x'_2 - x_2)^{-1}h_4 \)
if $k$ is odd. In this way either $u_1 - u_2 = 0$ or $u_1 - u_2$ is a multiple of $A_3$. In both cases it is strongly central in $F$. 

**Remark** We observe that in the previous proposition if we suppose $\alpha_j \in K[X; Y]_0$, the statement of the proposition remains valid replacing the condition (3) by the condition

(3’) The sum $\alpha_1 + \cdots + \alpha_r \in K[X; Y]^{[4]}$

Let $f(C_1, C_2) \in F = K[C_1, C_2]$. Then $f$ can be written as

$$f(C_1, C_2) = \sum_{n,m \geq 0} \alpha_{nm} C_1^n C_2^m + \sum_{n_j, m_j} C_1^{n_j} C_2^{m_j} \beta_{ij} u_{ij} + g(C_1, C_2).$$

Here $\alpha_{nm}, \beta_{ij} \in K$, $u_{ij}$ are left normed commutators as in Proposition\[11\] and moreover $g(C_1, C_2) = \sum u \gamma u C_1^n C_2^m \cdot \cdots \cdot u_k$ where $u_i$ are left normed commutators. Define $I_{ij}$ as the set of all indices $p$ such that $\deg t_i, u_{pj} = r_{ij}, \deg t_2 u_{pj} = s_{ij}$ for some integers $r_{ij}$ and $s_{ij}$.

**Theorem 12** Using the notation above, $f(C_1, C_2)$ is central in $F$ if and only if $\alpha_{nm} = 0$ for all $n$ and $m$, $n + m \geq 1$, and moreover, for every $i$ and $j$ the equalities $\sum_p \beta_{ij} p_{ij} = 0$ hold where $p \in I_{ij}$. Furthermore $f(C_1, C_2)$ is strongly central if and only if it is central and $\alpha_{00} = 0$.

**Proof.** We already proved that $f(C_1, C_2)$ is central provided that $\alpha_{nm} = 0$ when $n + m \geq 1$ and all sums $\sum_{p \in I_{ij}} \beta_{ij} p_{ij} = 0$. Such an element is strongly central if and only if $\alpha_{00} = 0$. We shall prove the converse. Clearly $g(C_1, C_2)$ is strongly central and $\alpha_{00} I$ is central.

So suppose $\sum_{n+m \geq 1} \alpha_{nm} C_1^n C_2^m + \sum_{n_j, m_j} C_1^{n_j} C_2^{m_j} \sum_i \beta_{ij} u_{ij}$ is central. The computation of $C_1^{n_j} C_2^{m_j}$ done just before Lemma\[4\] yields that the (1, 2) entry of $\sum_{n+m \geq 1} \alpha_{nm} C_1^n C_2^m$ will be equal to

$$\sum_{n+m \geq 1} \alpha_{nm}(q - 1)x_{1}^{m} y_{1} + Q_{m-1} x_{1}^{m} y_{2} + \mu, \quad \mu \in K[X; Y]^{(3)}.$$

Analogously the (1, 2) entry of $\sum_{n_j, m_j} C_1^{n_j} C_2^{m_j} \sum_i \beta_{ij} u_{ij}$ is

$$\sum_{n_j, m_j, i} \beta_{ij} x_{1}^{n_i} x_{2}^{m_j} (x_1' - x_1)^{s_{ij} - 1}(x_2' - x_2)^{s_{ij} - 1}(y_1(x_2' - x_2) - y_2(x_1' - x_1)) + \rho.$$

Here $r_{ij} = \deg t_i, u_{ij}, s_{ij} = \deg t_2 u_{ij}$, and $\rho \in K[X; Y]^{(3)}$. Since our element is central its (1, 2) entry lies in $K[X; Y]^{(3)}$. Thus we obtain that the sum

$$\sum_{n+m \geq 1} \alpha_{nm}(q - 1)x_{1}^{m} y_{1} + Q_{m-1} x_{1}^{m} y_{2} + \rho$$

$$\sum_{n_j, m_j, i} \beta_{ij} x_{1}^{n_i} x_{2}^{m_j} (x_1' - x_1)^{s_{ij} - 1}(x_2' - x_2)^{s_{ij} - 1}(y_1(x_2' - x_2) - y_2(x_1' - x_1))$$

...
must vanish. But the set $B_1$ is a basis of $K[X;Y]^{(1)}$ therefore

$$\sum_{n+m \geq 1} \alpha_{nm} q_{n-1}x_2^m + \sum_{n, m, j} \beta_{ij}x_1^n x_2^m (x_1' - x_1)^{r_{ij}} (x_2' - x_2)^{s_{ij}} = 0$$

$$\sum_{n+m \geq 1} \alpha_{nm} q_{m-1}x_1^n - \sum_{n, m, j} \beta_{ij}x_1^n x_2^m (x_1' - x_1)^{r_{ij}} (x_2' - x_2)^{s_{ij}-1} = 0.$$  

Multiplying the first equation by $(x_1' - x_1)$, the second by $(x_2' - x_2)$ and summing up we will obtain that

$$0 = \sum_{n+m \geq 1} \alpha_{nm} (q_{n-1}x_2^m(x_1' - x_1) + q_{m-1}x_1^n(x_2' - x_2))$$

$$= \sum_{n+m \geq 1} \alpha_{nm} (x_2'^m(x_1^n - x_1^n) + x_1^m(x_2'^m - x_2^m))$$

$$= \sum_{n+m \geq 1} \alpha_{nm} (x_2'^m x_1^n - x_1^n x_2^m).$$

Therefore $\alpha_{nm} = 0$ whenever $n + m \geq 1$. So we are left with the sum

$$\sum_{n, m, j} \beta_{ij}x_1^n x_2^m (x_1' - x_1)^{r_{ij}} (x_2' - x_2)^{s_{ij}} (y_1(x_2' - x_2) - y_2(x_1' - x_1)) = 0.$$  

Thus we have that $\sum_{n, m, j} \beta_{ij}x_1^n x_2^m (x_1' - x_1)^{r_{ij}} (x_2' - x_2)^{s_{ij}} = 0$. Similarly $\sum_{n, m, j} \beta_{ij}x_1^n x_2^m (x_1' - x_1)^{r_{ij}} (x_2' - x_2)^{s_{ij}-1} = 0$. By homogeneity we deduce that for each $j$ it holds $\sum_{i} \beta_{ij}(x_1' - x_1)^{r_{ij}} (x_2' - x_2)^{s_{ij}} = 0$. Recalling the definition of the sets $I_{ij}$ we have $\sum_{p \in I_{ij}} \beta_{pj} = 0$ and we are done. \hfill \bigcirc \triangleleft

**Corollary 13** For the centre $Z(F)$ we have $Z(F) = K \oplus I$ where $I$ is a nilpotent ideal of $F$ (and not only of $Z(F)$).

We observe that the last Corollary, together with Theorem[12] gives a precise answer to the question of Berele, and that $I$ is a nilpotent ideal actually of $F$, not only of the centre.

A further remark is relevant. It is interesting to note that when one deals with 3 generators, say the generic matrices $C_1$, $C_2$, $C_3$, then the element $[C_1, C_2, [C_1, C_3]]$ is central in the generic algebra of three generators. But $C_2[C_1, C_2, [C_1, C_3]]$ is not. Therefore the analogue of the above nilpotent ideal is an ideal of the centre only.

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