The effective potential of $N$-vector models: a field-theoretic study to $O(\epsilon^3)$.

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Abstract

We study the effective potential of three-dimensional $O(N)$ models. In statistical physics, the effective potential represents the free-energy density as a function of the order parameter (Helmholtz free energy), and, therefore, it is related to the equation of state. In particular, we consider its small-field expansion in the symmetric (high-temperature) phase, whose coefficients are related to the zero-momentum $2j$-point renormalized coupling constants $g_{2j}$. For generic values of $N$, we calculate $g_{2j}$ to three loops in the field-theoretic approach based on the $\epsilon$-expansion. The estimates of $g_{2j}$, or equivalently of $r_{2j} \equiv g_{2j}/g_4^{-1}$, are obtained by a constrained analysis of the series that takes into account the exact results in one and zero dimensions.

Keywords: Field theory, Critical phenomena, $O(N)$ models, Effective potential, Equation of state. $n$-point renormalized coupling constants, $\epsilon$-expansion.

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I. INTRODUCTION

According to the universality hypothesis, most features of continuous phase transitions do not depend on the microscopic details of the systems, but only on their global properties such as the dimensionality and the symmetry of the order parameter (see e.g. Ref. [1]). The $O(N)$-symmetric universality classes describe many three-dimensional systems characterized by short-range interactions and an $N$-component order parameter. We mention the liquid-vapour transition in classical fluids ($N = 1$), the $\lambda$-transition in superfluid helium ($N = 2$), the critical properties of isotropic ferromagnetic materials ($N = 3$) and of long polymers ($N \to 0$). The case $N = 4$ is relevant for high-energy physics: it should describe the critical behavior of finite-temperature QCD with two flavours at the chiral-symmetry restoring phase transition [2]. Universality implies that critical exponents, as well as other universal quantities, are the same for all models belonging to the same $O(N)$-symmetric class. Thus, the universal results obtained for a representative of a given class, such as the $O(N)$-symmetric $\phi^4$ Hamiltonian, can be used to predict the critical behavior of all systems in the same class.

The effective potential (Helmholtz free energy) is related to the (Gibbs) free energy of the model. Indeed, if $M_a \equiv \langle \phi_a \rangle$ is the magnetization and $H$ the magnetic field, one defines

$$F(M) = MH - \frac{1}{V} \log Z(H),$$

where $Z(H)$ is the partition function (the dependence on the temperature is understood). The global minimum of the effective potential determines the value of the order parameter which characterizes the phase of the model. In the high-temperature or symmetric phase the minimum is unique with $M = 0$, while in the low-temperature or broken phase $F(M)$ presents a flat region around the origin [3], i.e. $F(M)$ is constant for $|M| \leq M_0$ where $M_0$ is the magnetization at the coexistence curve.

In the high-temperature phase the effective potential admits an expansion around $M = 0$:

$$\Delta F \equiv F(M) - F(0) = \sum_{j=1}^{\infty} \frac{1}{(2j)!} a_{2j} M^{2j}. \quad (2)$$

The coefficients $a_{2j}$ can be expressed in terms of renormalization-group invariant quantities. We introduce a renormalized magnetization

$$\phi^2 = \frac{\xi(t, H = 0)^2 M(t, H)^2}{\chi(t, H = 0)}, \quad (3)$$

where $t$ is the reduced temperature, $\chi$ and $\xi$ are respectively the magnetic susceptibility and the second-moment correlation length obtained from the two-point function of the order parameter $\phi$, i.e.

$$\langle \phi_a(0) \phi_b(x) \rangle \equiv \delta_{ab} G(x), \quad (4)$$

$$\chi = \int dx \ G(x), \quad \xi^2 = \frac{1}{2d} \int dx \ x^2 G(x).$$
Then one may write

\[ \Delta F = \frac{1}{2} m^2 \varphi^2 + \sum_{j=2} m^{d-j(d-2)} \frac{1}{(2j)!} g_{2j} \varphi^{2j}. \]  

(5)

Here \( m = 1/\xi \), \( g_{2j} \) are functions of \( t \) only, and \( d \) is the space dimension. In field theory \( \varphi \) is the expectation value of the zero-momentum renormalized field. For \( t \to 0 \) the quantities \( g_{2j} \) approach universal constants (which we indicate with the same symbol) that represent the zero-momentum \( 2j \)-point renormalized coupling constants. A simpler parametrization of the small-field expansion of the effective potential can be obtained by performing a further rescaling

\[ \varphi = \frac{m^{(d-2)/2}}{\sqrt{g_4}} z, \]  

(6)

which allows us to write the free energy as

\[ \Delta F = \frac{m^d}{g_4} A(z), \]  

(7)

where

\[ A(z) = \frac{1}{2} z^2 + \frac{1}{4!} z^4 + \sum_{j=3}^1 \frac{1}{(2j)!} r_{2j} z^{2j}, \]  

(8)

and

\[ r_{2j} = \frac{g_{2j}}{g_4^{-1}} \quad j \geq 3. \]  

(9)

The function \( A(z) \) is related to the equation of state. Indeed one can show that \( z \propto t^{-\beta} M \), and that the equation of state can be written in the form

\[ H \propto t^{3d} \frac{\partial A(z)}{\partial z}. \]  

(10)

The small-field expansion of the effective potential provides the starting point for the determination of approximate representations of the equation of state that are valid in the whole critical region. This requires an analytic continuation in the complex \( t \)-plane in order to reach the coexistence curve from the symmetric phase [1,4]. This can be achieved by using parametric representations [3,7], which implement in a rather simple way the known analytic properties of the equation of state (Griffith’s analyticity). This idea was successfully applied to the Ising model, for which one can construct a systematic approximation scheme based on polynomial parametric representations [4] and on a global stationarity condition [3], leading to an accurate determination of the critical equation of state and of the universal ratios of amplitudes that can be extracted from it [1,3,5]. In view of an application of such an approach to \( O(N) \) models [6] with \( N > 1 \), we decided to improve the

\[^{1}\text{For } N > 1, \text{ this approach is more difficult because of the presence of the Goldstone singularity at the coexistence curve, which must be somehow taken into account by the approximate parametric representations considered.}\]
estimates of the coefficients $r_{2j}$ appearing in Eq. (8). It is worth mentioning that a better
determination of the equation of state, and therefore of the universal ratios of amplitudes
such as the ratio of the specific heat amplitudes $A^+/A^−$, is particularly important in the
case $N = 2$, which describes the $\lambda$-transition in $^4$He. A recent Space Shuttle experiment [10]
made a very precise measurement of the heat capacity of liquid helium to within 2 nK from
the $\lambda$-transition, obtaining extremely accurate estimates of the exponent $\alpha$ and of the ratio
$A^+/A^−$. These results represent a challenge for theorists, who, until now, have not been
able to compute universal quantities at the same level of accuracy (see e.g. Refs. [8,11–13]
for recent theoretical estimates of $\alpha$ and $A^+/A^−$).

Several approaches can be employed to investigate the $O(N)$-vector models, such as
field-theoretical methods starting from the $\phi^4$ formulation of the theory

$$
\mathcal{H} = \int d^d x \left[ \frac{1}{2} \partial_\mu \phi(x) \partial_\mu \phi(x) + \frac{1}{2} r \phi^2 + \frac{1}{4!} g_0 (\phi^2)^2 \right], \quad (11)
$$
lattice techniques performing high- and low-temperature expansions, Monte Carlo simu-
lations, etc... In order to study the small-field expansion of the effective potential, we consider
the field-theoretic approach based on the $\epsilon \equiv 4 − d$ expansion [14]. We extend the series of
$r_{2j}$ to $O(\epsilon^3)$ (corresponding to a three-loop calculation) for generic values of $N$, and obtain
new estimates from their analysis. Earlier estimates of $r_{2j}$ based on the $\epsilon$-expansion [15]
were obtained using $O(\epsilon^2)$ series for generic values of $N$ and $O(\epsilon^3)$ series for the Ising model,
which were derived from the scaling equation of state known to $O(\epsilon^2)$ for generic values of
$N$ [16] and to $O(\epsilon^3)$ for the Ising model [17,18]. Since the $\epsilon$-expansion is asymptotic, one needs to perform a resummation of the series
in order to obtain reliable estimates. This can be efficiently done by exploiting its Borel
summability and the knowledge of the large-order behavior [19]. Moreover, one may exploit
exact results for low-dimensional models and perform constrained analyses of the $\epsilon$-series.
The basic assumption is that the zero-momentum 2$j$-point renormalized couplings $g_{2j}$, and
therefore the ratios $r_{2j}$, are analytic and quite smooth in the domain $4 > d > 0$ (thus
$0 < \epsilon < 4$). This can be verified in the large-$N$ limit [20,14]. One may then perform a polyno-
mial interpolation among the values of $d$ where the constants $r_{2j}$ are known, and then
analyze the series of the difference. As we shall see, the analysis of the $O(\epsilon^3)$ series of $r_{2j}$
leads to a substantial improvement of the estimates of the first few $r_{2j}$ with respect to earlier
results obtained from their $O(\epsilon^2)$ series [15]. As a by-product of our analysis we also obtain
new estimates for the first few $r_{2j}$ in the two-dimensional $O(N)$ models.

The zero-momentum four-point coupling $g \equiv g_4$ plays an important role in the field-
theoretic perturbative expansion at fixed dimension [21], which provides an accurate de-
scription of the critical region in the symmetric phase. In this approach, any universal
quantity is obtained from a series in powers of $g$ ($g$-expansion), which is then resummed and
evaluated at the fixed-point value of $g$, $g^*$ (see e.g. Refs. [4]). Accurate estimates of $g^*$ have
been obtained by calculating the zero of the Callan-Symanzik $\beta$-function associated to $g$ (see
e.g. Refs. [3,22,23,13,24,1]). These results have been substantially confirmed by computa-
tions using different approaches, such as $\epsilon$-expansion [20], high-temperature expansion (see,
e.g., Refs. [4,23] and references therein), Monte Carlo simulations [20,28], etc... In this paper
we reconsider the determination of $g^*$ from its $\epsilon$-expansion. In Ref. [20] we calculated it to
O(\varepsilon^3), but unfortunately the series published there contains a numerical mistake\(^2\). For this reason, we report here the correct series and the results of the new analysis. We anticipate that the changes with respect to the estimates reported in Ref. [20] are very small.

The field-theoretic method based on the $g$-expansion at fixed dimension has been recently considered in the calculation of the zero-momentum couplings $g_{2j}$ with $j > 2$: for the Ising model five-loop series [34] are available, while for generic values of $N$, $g_6$ and $g_8$ have been determined to four and three loops respectively [35,36]. The effective potential has also been studied by approximately solving the exact renormalization-group equations, providing some estimates of the coefficients $g_{2j}$ [55]. For the Ising model accurate estimates have been obtained from the analysis of lattice high-temperature expansions, see e.g. Refs. [37,38,39]. For $N > 1$, which is the case considered in this paper, only the high-temperature of $g_6$ has been computed [37]: however, the results of its analysis are rather imprecise.

The paper is organized as follows. In Sec. II we present the $O(\varepsilon^3)$ series of $r_{2j}$ that we have calculated. In Sec. III we give the results of the analyses of the series of $r_{2j}$, which are then compared with the available estimates obtained in other approaches. The appendix is dedicated to the calculation of the three-loop integrals involved in the computation of the zero-momentum $n$-point irreducible functions.

## II. EXPANSION OF $r_{2j}$ TO $O(\varepsilon^3)$. 

In the framework of the $\varepsilon$-expansion we have calculated, to three loops, the one-particle irreducible correlation functions at zero momentum

$$\Gamma_{2j} \equiv \Gamma^{(2j)}_{\alpha_1\alpha_2...\alpha_j\alpha_j}(0, ..., 0).$$

(12)

The number of diagrams one has to evaluate to compute $\Gamma_{2j}$ increases with increasing $j$. For example, the three-loop one-vertex irreducible diagrams necessary to compute $\Gamma_6$, $\Gamma_8$, $\Gamma_{10}$ are 16, 36, 64 respectively. In our calculation we employed a symbolic manipulation package developed in MATHEMATICA. It generates the diagrams using the algorithm described in Ref. [38], performs the necessary index contractions to determine the $N$ dependence of each diagram, and compute the corresponding integral according to the procedure described in App. A.

The coefficients $r_{2j}$ of the expansion of $A(z)$ in powers of $z$ can be written in terms of $\Gamma_{2j}$ as

$$r_{2j} \equiv \frac{g_{2j}}{g^{j-1}} = \frac{(2j)!}{2^j3^{j-1}j!} \frac{(N + 2)^{j-2}}{\prod_{i=2}^{j-1}(N + 2i)} \frac{\Gamma_{2j}\Gamma_j^{j-2}}{\Gamma_2^j\Gamma_2^{j-1}}.$$  

(13)

From the three-loop expansion of $\Gamma_{2j}$, one can derive the series of $r_{2j}$ to $O(\varepsilon^3)$. In the following we report the series of the first few $r_{2j}$, i.e. $r_6$, $r_8$ and $r_{10}$, that we will analyze in the following section. Writing

\(^2\)The numerical expressions for the Feynman graphs appearing in Ref. [20] have been checked in Ref. [21]. All numerical estimates are in agreement, except that of the constant $H$. The correct value is given in the Appendix.
\[ r_{2j} = \sum_{i=1}^{r} r_{2j,i} \epsilon^i, \]  

we find

\[ r_{6,1} = \frac{5(26 + N)}{6(8 + N)} \]  

\[ r_{6,2} = \frac{98 + 33 N + 4 N^2}{(8 + N)^3} + \frac{40 \lambda (-8 + 7 N + N^2)}{3 (8 + N)^3} \]  

\[ r_{6,3} = -\frac{5}{6} \left(17264 + 9968 N + 2574 N^2 + 319 N^3 + 7 N^4\right) \]  

\[ + \frac{6}{3 (8 + N)^5} \lambda (-2176 - 172 N + 152 N^2 + 9 N^3) + \frac{20 (Q_1 + \gamma_E \lambda) (N - 1)}{3 (8 + N)^2} \]  

\[ + \frac{640}{(8 + N)^3} \left(N - 1\right) Q_2 + \frac{20 (682 + 49 N - 2 N^2) \zeta(3)}{(8 + N)^4} \]

\[ r_{8,1} = \frac{-35(80 + N)}{18(8 + N)} \]  

\[ r_{8,2} = \frac{35 (31904 + 7610 N + 578 N^2 + 3 N^3)}{54 (8 + N)^3} - \frac{5600 \lambda (-8 + 7 N + N^2)}{27 (8 + N)^3} \]  

\[ r_{8,3} = \frac{35 (-259712 - 112232 N - 16204 N^2 - 422 N^3 + 13 N^4)}{18 (8 + N)^5} \]  

\[ + \frac{35 \lambda (105472 + 72528 N - 384 N^2 - 469 N^3)}{81 (8 + N)^4} + \frac{2800 (Q_1 + \gamma_E \lambda) (1 - N)}{27 (8 + N)^2} \]  

\[ + \frac{85120 (1 - N) Q_2}{9 (8 + N)^3} + \frac{70 (-29824 - 3010 N + 29 N^2) \zeta(3)}{9 (8 + N)^4} \]

\[ r_{10,1} = \frac{35(242 + N)}{3(8 + N)} \]  

\[ r_{10,2} = -\frac{35 (2083280 + 453428 N + 28580 N^2 + 63 N^3)}{108 (8 + N)^3} + \frac{162400 \lambda (-8 + 7 N + N^2)}{27 (8 + N)^3} \]  

\[ r_{10,3} = \frac{35 (157284800 + 62464976 N + 8716080 N^2 + 388468 N^3 - 110 N^4 + 27 N^5)}{108 (8 + N)^5} \]  

\[ + \frac{140 \lambda (-739808 - 822816 N - 35058 N^2 + 3359 N^3)}{81 (8 + N)^4} \]  

\[ + \frac{81200 (Q_1 + \gamma_E \lambda) (N - 1)}{27 (8 + N)^2} + \frac{1433600 (N - 1) Q_2}{9 (8 + N)^3} \]  

\[ + \frac{140 (463924 + 65932 N + 1585 N^2) \zeta(3)}{9 (8 + N)^4} \]

where
\[ \lambda = 1.171953619344729445... \]  
\[ Q_1 = -2.695258053506736953... \]  
\[ Q_2 = 0.400685634386531428... \]

For \( N = 1 \) the above expressions reproduce the \( O(\epsilon^3) \) series that can be derived from the \( O(\epsilon^3) \) equation of state calculated in Refs. [17,18]. We could also have computed the \( O(\epsilon^3) \) series of \( r_{2j} \) for some additional \( j > 5 \). Since, as we shall see, with increasing \( j \), longer and longer series are necessary to obtain acceptable three-dimensional estimates from their analysis, we decided not to go beyond \( r_{10} \).

We now report the series of the fixed-point values of the zero-momentum four-point coupling \( g \equiv g_4 \), which corrects that given in Ref. [20], which was plagued by a numerical mistake in one of the three-loop integrals. In the framework of the \( \epsilon \)-expansion, we found convenient to consider the rescaled coupling \( \bar{g} \), defined by

\[ \bar{g} = \frac{1}{2(4\pi)^{d/2}} \frac{(N+8)}{3} \Gamma \left( 2 - \frac{d}{2} \right) g, \]  

(19)

and expand it in powers of \( \epsilon \). Due to the \( O(\epsilon^{-1}) \) factor multiplying \( g \) in the definition of \( \bar{g} \), the \( O(\epsilon^4) \) of \( g^* \) corresponds to the \( O(\epsilon^3) \) of \( \bar{g}^* \). The \( \epsilon \)-expansion of \( \bar{g}^* \) to \( O(\epsilon^3) \) is given by

\[ \bar{g}^*(\epsilon) = \sum_{k=0} g_k \epsilon^k, \]  

(20)

with

\[ \bar{g}_0 = 1 \]  
\[ \bar{g}_1 = \frac{3(14 + 3N)}{(8 + N)^2} \]  
\[ \bar{g}_2 = \frac{1224 + 520N + 58N^2 - 2N^3}{(8 + N)^4} - \frac{12(22 + 5N)\zeta(3)}{(8 + N)^3} - \frac{\lambda(62 + 13N)}{3(8 + N)^2} \]  
\[ \bar{g}_3 = \frac{341312 + 225312N + 57572N^2 + 5404N^3 - 99N^4 + 4N^5}{8(8 + N)^6} - \frac{(22 + 5N)\pi^4}{15(8 + N)^3} \]  
\[ + \frac{2 (-3880 - 772N + 431N^2 + 90N^3) \zeta(3)}{(8 + N)^5} + \frac{40 (186 + 55N + 2N^2) \zeta(5)}{(8 + N)^4} \]  
\[ - \frac{\lambda(6500 + 2700N + 327N^2 + 4N^3)}{2(8 + N)^4} - \frac{(Q_1 + \gamma_E \lambda)(62 + 13N)}{6(8 + N)^2} \]  
\[ - \frac{8 (62 + 19N) Q_2}{(8 + N)^3} - \frac{8 H (22 + 5N)}{(8 + N)^3}, \]

where

\[ H = -2.155952487340794361... \]  

(22)

Finally we report the exact results in \( d = 1 \) and \( d = 0 \) that we will use in our constrained analyses of the \( \epsilon \)-series. In \( d = 1 \) we have [15].
\[ r_6 = 5 - \frac{5N(N - 1)^2(8N + 7)}{(N + 1)(N + 4)(4N - 1)^2}, \]  
\[ r_8 = \frac{175}{3} - \frac{35N(N - 1)^2(256N^3 + 3037N^2 + 1705N - 588)}{3(N + 1)(N + 4)(N + 6)(4N - 1)^3}. \]

\[ r_{10} = 1225 - 175N(N - 1)^2(N + 1)^{-2}(N + 3)^{-1}(N + 4)^{-2}(N + 6)^{-1}(N + 8)^{-1}(4N - 1)^{-4} \times 
(149184 - 886968N - 690826N^2 + 4219985N^3 + 6283975N^4 + 2913758N^5 + 552223N^6 + 44405N^7 + 1664N^8), \]

for \( N \geq 1 \), and

\[
\begin{align*}
\left. \begin{array}{l}
r_6 = 5, \\
r_8 = \frac{175}{3}, \\
r_{10} = 1225,
\end{array} \right\} \quad \text{for } N \leq 1.
\]

In \( d = 0 \) we have

\[ r_6 = \frac{10(N + 8)}{3(N + 4)}, \]
\[ r_8 = \frac{70(N^2 + 14N + 120)}{3(N + 4)(N + 6)}, \]
\[ r_{10} = \frac{280(10752 + 3136N + 256N^2 + 30N^3 + N^4)}{(N + 4)^2(N + 6)(N + 8)}. \]

for \( N \geq 1 \). It is not clear how to determine the value of \( r_{2j} \) for \( N = 0 \). Similarly to the \( d = 1 \) case, one may conjecture that their values are independent of \( N \) for \( N \leq 1 \), and therefore equal to those for \( N = 1 \).

### III. ANALYSES OF THE SERIES.

Since the \( \epsilon \)-expansion is asymptotic, it requires a resummation to get estimates for \( d = 3 \), i.e. \( \epsilon = 1 \), which is usually performed assuming its Borel summability. The analysis of the series in powers of \( \epsilon \) can be performed by using the method proposed in Ref. [19], which is based on the knowledge of the large-order behaviour of the series. It is indeed known that the \( n \)-th coefficient of the series behaves as \( \sim (-a)^n \Gamma(n + b_0 + 1) \) for large \( n \). The constant \( a \), which characterizes the singularity of the Borel transform, does not depend on the specific quantity; it is given by \( \left[38,40\right] \ a = 3/(N + 8) \). The coefficient \( b_0 \) depends instead on the quantity at hand.

Consider a generic quantity \( R \) whose \( \epsilon \)-expansion is

\[ R(\epsilon) = \sum_{k=0}^{\infty} R_k \epsilon^k. \]

According to Ref. [19], one generates new series \( R_p(\alpha, b; \epsilon) \) according to
\[ R_p(\alpha, b; \epsilon) = \sum_{k=0}^{p} B_k(\alpha, b) \int_0^\infty dt \, t^b \, e^{-t} \frac{u(ct)^k}{[1 - u(ct)]^\alpha}, \]  \tag{27}

where

\[ u(x) = \frac{\sqrt{1 + ax} - 1}{\sqrt{1 + ax} + 1}. \]  \tag{28}

The coefficients \( B_k(\alpha, b) \) are determined by the requirement that the expansion in \( \epsilon \) of \( R_p(\alpha, b; \epsilon) \) coincides with the original series. For each \( \alpha, b \) and \( p \) an estimate of \( R \) is simply given by \( R_p(\alpha, b; \epsilon = 1) \).

We follow Ref. [20] to derive the estimates and their uncertainty. We determine an integer value of \( b \), \( b_{\text{opt}} \), such that

\[ R_K(\alpha, b_{\text{opt}}; \epsilon = 1) \approx R_{K-1}(\alpha, b_{\text{opt}}; \epsilon = 1), \]  \tag{29}

for \( \alpha < 1 \), where \( K \) is the highest order of the known terms in the series. In a somewhat arbitrary way we consider as our final estimate the average of \( R_K(\alpha, b; \epsilon = 1) \) with \(-1 < \alpha \leq 1/2 \) and \(-2 + b_{\text{opt}} \leq b \leq 2 + b_{\text{opt}} \). The error we report is the variance of the values of \( R_K(\alpha, b; \epsilon = 1) \) with \(-1 < \alpha \leq 1/2 \) and \([b_{\text{opt}}/3 - 1] \leq b \leq [4b_{\text{opt}}/3 + 1] \). This procedure was already discussed and tested in Refs. [20,15]. It seems to provide reasonable estimates and error bars, at least when a sufficiently large number of terms is known. Nonetheless, the method is ad hoc, and its reliability may depend on the quantity at hand. It is therefore important to check the accuracy of the error estimates for each quantity considered, for example by comparing, when it is possible, results obtained from different series representing the same quantity.

It has been noted that, if one assumes a sufficiently large analytic domain in \( d \) for the quantity at hand, the estimates from its \( \epsilon \)-expansion can be improved using its known value in lower dimensions \[11,13,20,15,14\]. One can check explicitly in the large-\( N \) limit that the zero-momentum couplings \( g_{2j} \) are analytic for \( 0 < d < 4 \). One may then conjecture that the analytic properties of \( g_{2j} \) emerged in the large-\( N \) limit are also valid for finite fixed values of \( N \). This point was already discussed in Refs. [20,15].

Suppose that the values of the generic quantity \( R \) are known for a set of dimensions \( \epsilon_1, \ldots, \epsilon_k \). In this case one may use as zeroth order approximation the value for \( \epsilon = 1 \) of the polynomial interpolation through \( \epsilon = 0, \epsilon_1, \ldots, \epsilon_k \) and then use the series in \( \epsilon \) to compute the deviations. More precisely, let us suppose that exact values \( R_{\text{ex}}(\epsilon_1), \ldots, R_{\text{ex}}(\epsilon_k) \) are known for the set of dimensions \( \epsilon_1, \ldots, \epsilon_k, k \geq 2 \). Then define

\[ Q(\epsilon) = \sum_{i=1}^{k} \left[ \frac{R_{\text{ex}}(\epsilon_i)}{(\epsilon - \epsilon_i)} \prod_{j=1,j\neq i}^{k} (\epsilon_i - \epsilon_j)^{-1} \right], \]  \tag{30}

and

\[ S(\epsilon) = \frac{R(\epsilon)}{\prod_{i=1}^{k}(\epsilon - \epsilon_i)} - Q(\epsilon), \]  \tag{31}

and finally

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\begin{equation}
R_{\text{imp}}(\epsilon) = [Q(\epsilon) + S(\epsilon)] \prod_{i=1}^{k}(\epsilon - \epsilon_i).
\end{equation}

The resummation procedure is applied to \(S(\epsilon)\) and the final estimate is obtained by computing \(R_{\text{imp}}(\epsilon = 1)\). If the polynomial interpolation in \(d\) is a good approximation, one should find that the \(\epsilon\)-series which gives the deviations has smaller coefficients than the original one. Consequently one expects that also the errors in the resummation are reduced. In the cases considered, we find that, as expected, the coefficients of the series \(S(\epsilon)\) decrease in size with \(k\), the number of exact values that are used to constrain the series.

To begin with, we present the results of the analyses of the series for four-point coupling \(\bar{g}^*\), cf. Eq. (21). Table I shows the three-dimensional results for various constrained analyses, using the exact results in \(d = 1\) and \(d = 0\) and the accurate estimates for two-dimensional models [20]. Note that, for each \(N\), the results of the various constrained analyses are consistent with each other. The two-dimensional estimates used in the constrained analyses are those already considered in Ref. [20], except for \(N = 3\). In this case we used the recent two-dimensional estimate \(\bar{g}^* = 1.7778(45)\) obtained from a form-factor bootstrap approach [15], which should be more reliable than the high-temperature result \(\bar{g}^* = 1.724(9)\) considered in Ref. [20]. For comparison, we note that for the two-dimensional \(O(3)\) model our new estimate from the \(\epsilon\)-expansion is \(\bar{g}^* = 1.75(3)\). In Table I we present the two-dimensional results obtained by analyses constrained in \(d = 1\) and \(d = 0\).

Tables II and III supersede the corresponding ones, i.e. Tables 1 and 2, of Ref. [20]. The changes are small, so that the discussion presented there remains valid. In Table IV we report the additional recent estimates which appeared after Ref. [20]. These results have been obtained from: (i) a reanalysis of the six-loop \(\beta\)-function in the framework of the fixed-dimension \(g\)-expansion [3]; (ii) the analyses of the high-temperature expansions (HT) of \(O(N)\) \(\sigma\)-models on the cubic and bcc lattices [28]; (iii) high-temperature expansions of improved Hamiltonians (IHT) for which the leading scaling corrections are suppressed [4,13,46]. The agreement among the various estimates is globally good. Let us only note that for \(N = 3\) the error we obtained from our analysis of the \(\epsilon\)-series seems to be underestimated. A more complete list of references presenting estimates of \(\bar{g}^*\) can be found in Refs. [20,8,9].

Since the series of \(r_{2j}\) begins with a term of order \(\epsilon\), we analyzed the \(O(\epsilon^2)\) series of the quantity \(r_{2j}/\epsilon\). The constrained analyses were performed using the exact results of \(r_{2j}\) for \(d = 1\) and \(d = 0\). We mention that in the case of the Ising model (\(N = 1\)) one may also use the precise two-dimensional estimates obtained from the analyses of the available high-temperature series for the Ising model on the square and triangular lattice [17,18]. These results were exploited in the analysis of the \(O(\epsilon^3)\) series of the Ising model presented in Ref. [15], allowing us to further improve the estimates of \(r_{2j}\). A discussion of the large-\(N\) limit of \(r_{2j}\) can be found in Ref. [15].

Tables IV and V show respectively our three- and two-dimensional results for \(r_6\). There we report the estimates obtained from an unconstrained analysis and from analyses constrained in various dimensions. Like the case of \(g^*\), the results of the various analyses are consistent with each other and the error decreases when additional lower dimensional values are used to constrain the analysis, supporting the estimate and the error obtained by the \(d = 0,1\) constrained analyses.

Tables VI and VII present respectively our three- and two-dimensional results for \(r_8\). In
this case, the results of the different analyses are not in complete agreement, indicating a possible underestimate of the uncertainty. Therefore, we believe that the final estimates of \( r_8 \) obtained from the \( d = 0,1 \) constrained analyses have an error larger than what is given by our algorithmic procedure. In Table VIII where we report our final estimates, we multiply the errors by a factor of two: this quoted uncertainty should be more realistic.

In the case of \( r_{10} \), the analyses, even those constrained, of its \( O(\epsilon^3) \) series do not provide satisfactory estimates, but give only an order of magnitude. This may be explained by looking at the coefficients of the series of \( r_{2j} \): they increase rapidly with \( j \), and, for \( r_{10} \), they are already much larger than its final three-dimensional estimate. We just mention that the \( d = 0,1 \) constrained analyses give \( r_{10} = 29(34) \) for \( N = 2 \), \( r_{10} = 16(24) \) for \( N = 3 \), \( r_{10} = 9(17) \) for \( N = 4 \), where the reported errors are those obtained from our algorithmic procedure.

Let us compare our results with the available estimates from other approaches. For \( N \neq 1 \) there are not many published results: we are only aware of the estimates of \( g_6 \) and \( g_8 \) presented in Refs. [31,37,34] (from which we can derive estimates of \( r_6 \) and \( r_8 \) using their results for \( g_4 \)). Table VIII presents a summary of the available estimates of \( r_6 \), \( r_8 \) and \( r_{10} \) for several values of \( N \). For the sake of completeness, we report also the results for \( N = 1 \) although the three-loop series were already known and no new results we have obtained in this work (a more complete list of results for the Ising model can be found in Ref. [9]).

In Ref. [31] \( g_6 \) and \( g_8 \) were estimated from a Padé-Borel resummation of their \( d = 3 \) \( g \)-expansion, calculated to four and three loops respectively. The authors of Ref. [31] argue that the uncertainty on their estimates of \( g_6 \) is approximately 0.3%, while they consider their values for \( g_8 \) much less accurate. These results are in good agreement with ours, especially those for \( r_6 \). We have also redone the analysis of the four-loop series of \( r_6 \) calculated in Ref. [31] using the same method employed here for the \( \epsilon \)-expansion (the results are reported in Table VIII). In Table VIII we also show some results for the \( N = 2 \) model obtained from the analysis of the 20th-order high-temperature expansion of an improved lattice Hamiltonian [46]: \( r_6 \) is in good agreement with the \( \epsilon \)- and \( g \)-expansion, while for \( r_8 \) IHT seems to give the most precise estimate. We also mention that Ref. [34] uses a renormalization-group approach (ERG) in which the exact RG equation is approximately solved (no estimates of the errors are presented there). Concerning \( r_{10} \), the result for \( N = 2 \) can be compared with the estimate coming from IHT: \( r_{10} = -13(7) \) [46].

As already mentioned in the introduction, the estimates of the first few \( r_{2j} \) that we have presented in this work may be very useful for the determination of the whole critical equation of state in \( O(N) \) models with \( N > 1 \). Work is indeed in progress [46]. This approach was already successfully applied to the Ising model. In the cases \( N > 1 \) the presence of the Goldstone singularity requires a more careful treatment.

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APPENDIX A: THREE-LOOP DIAGRAMS.

In the three-loop calculations of the zero-momentum irreducible 2\(j\)-point functions one
deals with four families of integrals:

\[
B_2(n_1, n_2, n_3) \equiv \int \frac{d^dk_1}{(2\pi)^d} \frac{d^dk_2}{(2\pi)^d} \Delta(k_1)^{n_1} \Delta(k_2)^{n_2} \Delta(k_1 + k_2)^{n_3}, \tag{A1}
\]

\[
B_3(n_1, n_2, n_3, n_4) \equiv \int \frac{d^dk_1}{(2\pi)^d} \frac{d^dk_2}{(2\pi)^d} \frac{d^dk_3}{(2\pi)^d} \Delta(k_1)^{n_1} \Delta(k_2)^{n_2} \Delta(k_3)^{n_3} \Delta(k_1 + k_2 + k_3)^{n_4}, \tag{A2}
\]

\[
G(n_1; n_2, n_3; n_4, n_5) \equiv \int \frac{d^dk_1}{(2\pi)^d} \frac{d^dk_2}{(2\pi)^d} \frac{d^dk_3}{(2\pi)^d} \Delta(k_1)^{n_1} \Delta(k_2)^{n_2} \Delta(k_3)^{n_4} \times \Delta(k_1 + k_2)^{n_3} \Delta(k_1 + k_3)^{n_5}, \tag{A3}
\]

\[
M(n_1, n_2, n_3; n_4, n_5, n_6) \equiv \int \frac{d^dk_1}{(2\pi)^d} \frac{d^dk_2}{(2\pi)^d} \frac{d^dk_3}{(2\pi)^d} \Delta(k_1)^{n_1} \Delta(k_1 + k_2)^{n_2} \Delta(k_1 + k_3)^{n_3} \times \Delta(k_2 - k_3)^{n_4} \Delta(k_3)^{n_5} \Delta(k_2)^{n_6}, \tag{A4}
\]

where \(\Delta(k)\) is the massive propagator,

\[
\Delta(k) \equiv \frac{1}{k^2 + 1}. \tag{A5}
\]

In order to compute these integrals, we have used an algorithm that expresses each quantity
in terms of four basic integrals: \(B_2(1, 1, 1), B_3(1, 1, 1, 1), G(1, 1, 1, 1, 1)\) and \(M(1, 1, 1, 1, 1, 1)\). The algorithm is based on the integration-by-parts technique \[49–51\]. The reduction of the
integrals of type \(B_3\) is accomplished using the method presented in Ref. \[52\], noting that
these integrals are related to \(B_N(0, 0, n_1, n_2, n_3, n_4)\), as defined in Ref. \[52\]. The integral
\(B_3(1, 1, 1, 1)\) can be expressed in terms of the constant \(B_4\) defined in \[52\] since

\[
B_4 = -\frac{4}{3\epsilon^3}(1 - 3\epsilon) - \frac{14}{3\epsilon^4} \frac{\Gamma(1 - \epsilon/2)\Gamma(1 + \epsilon)^2\Gamma(1 + 3\epsilon/2)}{\Gamma(1 + \epsilon/2)^2\Gamma(1 + 2\epsilon)} \nonumber
\]

\[
+ \frac{(1 - \epsilon)(4 - 3\epsilon)(2 - 3\epsilon)}{\epsilon^2(2 - \epsilon)^2} N_{d-3}^{-3} B_3(1, 1, 1, 1) \Gamma(1 - \epsilon/2)^{-3} \Gamma(1 + \epsilon/2)^{-3}, \tag{A6}
\]

where \(d = 4 - \epsilon\) and

\[
N_{d} \equiv \frac{2}{(4\pi)^{d/2}\Gamma(d/2)}. \tag{A7}
\]

Using the expansion of \(B_4\) in powers of \(\epsilon\) reported in \[52\], we obtain

\[
B_3(1, 1, 1, 1) = N_d^3 \left\{ \frac{2}{\epsilon^3} + \frac{5}{6\epsilon^2} + \frac{1}{8\epsilon}(1 + 2\pi^2) - \frac{103}{96} + \frac{5\pi^2}{48} + O(\epsilon) \right\}. \tag{A8}
\]

Let us now describe the algorithm for the other three cases. Let us begin with \(B_2(n_1, n_2, n_3)\). The basic relations are
where $1^\pm B_2(n_1, n_2, n_3) = B_2(n_1 \pm 1, n_2, n_3)$ and so on. Using (A9) and the relation obtained interchanging 2 and 3 we can reduce each $B_2(n_1, n_2, n_3)$ to integrals of the form $B_2(m_1, 1, 1)$ and to $B_2(m_1, m_2, 0)$. The latter terms are the product of two one-loop integrals. To deal with the former ones, sum to (A9) the relation obtained interchanging 2 and 3 in (A9). Then, use Eq. (A10) to eliminate the terms $(n_2 2^+ + n_3 3^+)B_2$ and $(n_2 2^+ + n_3 3^+)1^-B_2$. The new relation can be used to express each $B_2(m_1, 1, 1)$ in terms of $B_2(1, 1, 1)$ and of the product of one-loop integrals. The integral $B_2(1, 1, 1)$ has been computed exactly in any dimension, obtaining [53, 54]:

\[
B_2(1, 1, 1) = -\frac{2N_d^2}{\epsilon^2(1-\epsilon)(2-\epsilon)} \Gamma(1+\epsilon/2)^2 \Gamma(2-\epsilon/2)^2 \\
\times \left\{ 3^{1/2-\epsilon/2} \frac{\pi \Gamma(\epsilon)}{\Gamma(\epsilon/2)^2} + \frac{3}{2}(1-\epsilon) \right\} F_1(1, \epsilon/2; 3/2; 1/4). \tag{A11}
\]

Expanding in powers of $\epsilon$ we obtain

\[
B_2(1, 1, 1) = N_d^2 \left\{ -\frac{3}{2\epsilon^2} - \frac{3}{4\epsilon} - \frac{3\lambda}{4} - \frac{\pi^2}{8} + \epsilon \left[ -\frac{3}{4} - \frac{\pi^2}{16} + \frac{3}{8}(\lambda + \gamma_E \lambda + Q_1) \right] + O(\epsilon^2) \right\}. \tag{A12}
\]

where

\[
\lambda \equiv \frac{3\sqrt{\pi}}{4} F_1 + \sqrt{3\pi} (\gamma_E + \log 3) = \frac{2}{\sqrt{3}} \text{Cl}_2 \left( \frac{\pi}{3} \right) = \frac{1}{3} \psi' \left( \frac{1}{3} \right) - \frac{2\pi^2}{9}, \tag{A13}
\]

\[
Q_1 \equiv \frac{3\sqrt{\pi}}{4} F_2 - \sqrt{3\pi} (\gamma_E + \log 3)^2 - \frac{\sqrt{3\pi^3}}{24}, \tag{A14}
\]

$\psi(x)$ is the logarithmic derivative of the $\Gamma$-function and $\text{Cl}_2(x)$ is Clausen’s polylogarithm [55]. Here $F_1$ and $F_2$ are the following sums:

\[
F_1 \equiv \frac{1}{2} \sum_{n=0}^{\infty} \frac{\Gamma(n+1)\psi(n+1)}{\Gamma(n+3/2)} 4^{-n}, \tag{A15}
\]

\[
F_2 \equiv \frac{1}{4} \sum_{n=0}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+3/2)} \left( \psi'(n+1) + \psi(n+1)^2 \right) 4^{-n}. \tag{A16}
\]

Let us now discuss the integrals $G(n_1; n_2, n_3; n_4, n_5)$. First of all, note that if one of the indices is zero, $G$ can be written as a $B_3$ integral or as a product of a $B_2$ and of a one-loop integral. We will now show that any $G(n_1; n_2, n_3; n_4, n_5)$ can be expressed in terms of $G(1; 1, 1; 1, 1)$ and of $G$-integrals in which one of the indices is zero. Using the integration-by-parts technique, we obtain the following relations:
Using these expressions, we obtain finally for $\epsilon$

$G$

and $is the massless propagator $1$

The integral $\text{[57,58]}$. We obtain:

First we consider Eq. (A17) and sum to it the three relations obtained interchanging $(2 \leftrightarrow 3)$, $(23 \leftrightarrow 45)$, and $(23 \leftrightarrow 54)$. Then, we use Eq. (A19) eliminating $(n_2 2^+ + n_3 3^+ + n_4 4^+ + n_5 5^+)G$ and $(n_2 2^+ + n_3 3^+ + n_4 4^+ + n_5 5^+)1^-G$. Replacing $n_1$ by $n_1 - 1$, we obtain a relation which (for $n_1 \geq 2$) expresses $G(n_1; n_2, n_3, n_4, n_5)$ in terms of $G(m_1; m_2, m_3, m_4, m_5)$ with $m_1 < n_1$. Therefore, in a finite number of steps we express the original integrals in terms of $G(1; n_2, n_3, n_4, n_5)$ and of integrals in which one of the indices is zero. Then, we consider Eq. (A17) and use the relation obtained previously to eliminate $1^+G$. The new relation can be used to reduce $n_3$ to 1. Interchanging 2 and 3 we can similarly reduce $n_2$ to 1. We end up with $G(1; 1, 1; n_4, n_5)$ and with terms in which one index is zero. To further reduce the integrals we consider Eq. (A18). Replacing $n_4$ by $n_4 - 1$ we obtain a relation which reduces each integral to the form $G(1; 1, 1, 1, n_5)$. Finally, we consider (A20) and use Eq. (A18) to eliminate the terms $4^+G$ and $4^+5^-G$, obtaining a relation that reduces $n_5$ to 1. We should now compute $G(1, 1, 1, 1, 1)$. Using the integration-by-parts technique we first obtain the exact relation

$$G(1, 1, 1, 1, 2) = G_0(1, 1, 1, 1, 2) - \frac{\epsilon}{2} M_0(1, 1, 1, 1, 1).$$

The integral $G_0(n_1; n_2, n_3; n_4, n_5)$ is given by Eq. (A3) replacing $\Delta(k_2)^{n_2}$ with $\Delta_0(k_2)^{n_2}$ and $M_0(n_1, n_2, n_3; n_4, n_5, n_6)$ by Eq. (A4) replacing $\Delta(k_2 - k_3)^{n_4}$ with $\Delta_0(k_2 - k_3)^{n_4}; \Delta_0(k)$ is the massless propagator $1/k^2$. For our purposes we only need the divergent part of $M_0(1, 1, 1, 1, 1, 1)$ (the next contribution is reported in [57]):

$$M_0(1, 1, 1, 1, 1, 1) = N_4^3 \frac{\z(3)}{2\epsilon} [1 + O(\epsilon)].$$

The integral $G_0(1, 1, 1, 1, 2)$ can be computed exactly using the Mellin-Barnes technique [57,58]. We obtain:

$$G_0(1, 1, 1, 1, 2) = \frac{\pi^3 N_4^3}{32 \sin^3(\pi\epsilon/2)} \frac{(2 - \epsilon)^2}{(1 - \epsilon)} \left\{ 3F_2(\epsilon/2, \epsilon, 1; 1 - \epsilon/2, 1/2 + \epsilon/2; 1/4) 

+ \frac{1}{2}(1 - \epsilon) 2F_1(1, \epsilon/2; 3/2; 1/4) + \sqrt{\pi} \frac{\Gamma \left( \frac{1 + \epsilon}{2} \right)}{\Gamma \left( \frac{\epsilon}{2} \right)} 3^{-1/2 - \epsilon/2} 2^{\epsilon - 1} - 1 

- 2^{-\epsilon} \frac{\Gamma \left( \frac{3}{2} \right) \Gamma \left( \frac{1 + \epsilon}{2} \right) \Gamma \left( 1 - \frac{\epsilon}{2} \right)}{\Gamma \left( \frac{1 + \epsilon}{2} \right) \left( \frac{1}{2} + \epsilon \right)} 2F_1(\epsilon, 3\epsilon/2; 1/2 + \epsilon; 1/4) \right\}. \tag{A23}$$

Using these expressions, we obtain finally for $\epsilon \to 0,$
\[
G(1;1,1;1,1) = N_d^3 \left\{ -\frac{1}{\epsilon^3} - \frac{4}{3\epsilon^2} + \frac{1}{\epsilon} \left( -\frac{25}{12} + \frac{3\lambda}{2} - \frac{\pi^2}{8} \right) 
- \frac{19}{6} - \frac{\pi^2}{6} + \frac{3}{4} (Q_2 + \gamma_E \lambda + 2\lambda) - 3Q_2 + \frac{7}{4}\zeta(3) \right\} + O(\epsilon),
\]  
(A24)

where

\[
Q_2 \equiv \frac{\sqrt{\pi}}{4} \sum_{n=1}^{\infty} \frac{\Gamma(n)^2}{\Gamma(n+1/2)} \frac{4^{-n}}{n!} \left( \psi(n+1/2) - \psi(n+1) + 2\log 2 + \frac{2}{n} \right).
\]  
(A25)

Finally we consider \(M(n_1, n_2, n_3; n_4, n_5, n_6)\). Again we will assume all indices to be positive: if one is zero, the integral can be rewritten as an integral of the \(G\) family. The basic relations we need are

\[
(d - 2n_6 - n_1 - n_5)M = 
\left[ -(2n_6 6^+ + n_1 1^+ + n_5 5^+) + n_5 5^+(6^- - 4^-) + n_1 1^+(6^- - 2^-) \right] M,
\]  
(A26)

\[
\begin{align*}
&[n_5 - n_3 - n_5 5^+ + n_3 3^+ \\
&+ n_5 5^+(1^- - 3^-) - n_3 3^+(1^- - 5^-) + n_4 4^+(2^- + 5^- - 6^- - 3^-)] M = 0.
\end{align*}
\]  
(A27)

To compute this family of integrals we rewrite Eq. \(A27\) as \(n_3 3^+ M = (n_5 5^+ + \ldots)M\). Then, replacing \(n_3\) by \(n_3 - 1\), we obtain a relation that reduces each integral to \(M(n_1, n_2, 1; n_4, n_5, n_6)\), or to \(M\)'s in which one index is zero, and that, therefore, can be rewritten as \(G\)-integrals. Using the relations that are obtained replacing \((123456) \rightarrow (312645)\) and \((123456 \rightarrow 321654)\) in Eq. \(A27\), we can similarly reduce \(n_4\) and \(n_5\) to 1. To further simplify the integrals, we consider Eq. \(A27\) replacing \((123456 \rightarrow 231564)\) and rewrite it as \(M = 1^-(n_6 6^+ + \ldots)M/(n_1 - 1)\). If we apply this relation to \(M(n_1, n_2, 1; 1, 1, n_6)\) we obtain three types of terms: (a) \(M(m_1, m_2, 1; 1, 2, m_6)\) with \(m_1 + m_2 + m_6 + 1 < n_1 + n_2 + n_6\); (b) \(M(m_1, m_2, 1; 1, 1, m_6)\) with \(m_1 + m_2 + m_6 < n_1 + n_2 + m_6\); (c) \(M(m_1, m_2, 1; 1, 1, m_6)\) with \(m_1 + m_2 + m_6 = n_1 + n_2 + m_6\) and \(m_1 < n_1\). Terms of type (a) can be eliminated applying repeatedly the relation we used to reduce \(n_5\), generating terms of type (b). Repeating the procedure, we end up with integrals in which either one index is zero or \(n_1 = 1\). An analogous procedure can be used to reduce \(n_2\) to 1. At the end of this reduction we should only consider \(M(1, 1, 1; 1, 1, n_6)\). We now consider Eq. \(A26\) and use the relations we considered in the reduction of \(n_5\) and \(n_1\) to eliminate \(5^+ M\) and \(1^+ M\). Then, replacing \(n_6\) by \(n_6 - 1\), we obtain \(M = 6^- (\ldots)M/(n_6 - 1)\). This relation generates integrals of type \(G\) and \(1^+ 6^- 6^- M\), \(5^+ 6^- 6^- M\), and \(6^- M\). The first two terms can be eliminated by applying repeatedly previous relations. At the end of the procedure any integral \(M\) is expressed in terms of \(M(1, 1, 1; 1, 1, 1)\) and of integrals of type \(G\). The former integral has been computed in \(56\) obtaining in \(d = 4 - \epsilon\),

\[
M(1, 1, 1; 1, 1, 1) = N_d^3 \left( \frac{1}{2\epsilon} \zeta(3) + H + O(\epsilon) \right),
\]  
(A28)

where

\[
H = -\frac{1}{2} C_2^2 \left( \frac{\pi}{3} \right) + 2 \text{Li}_4 \left( \frac{1}{2} \right) - \frac{17\pi^4}{720} - \frac{\pi^2}{12} \log^2 2 + \frac{1}{12} \log^4 2,
\]  
(A29)
and Cl$_2$(x) and Li$_4$(x) are polylogarithms \[55\]. A numerical determination of $H$ was given in Ref. \[20\], rewriting

$$H = \frac{\pi^4}{80} + \frac{3}{4} \zeta(3) \log \frac{3}{2} + 3 \hat{H},$$

(A30)

where $\hat{H}$ is the integral appearing in Eq. (B.19) of Ref. \[20\]. Numerically $\hat{H} \approx -0.4346277$, so that $H \approx -2.155953$, in agreement with the numerical results of \[56\]. Unfortunately, in Ref. \[20\], we used $\hat{H} \approx -0.04346277$ — a factor of ten smaller — obtaining an incorrect estimate of $H$. 
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TABLES

TABLE I. Three-dimensional estimates of $\bar{g}^*$ from an unconstrained analysis, “unc”, and constrained analyses in various dimensions. For the analyses which use the estimates in $d = 2$ we report two errors: the first one gives the uncertainty of the resummation of the series, the second one expresses the change in the estimate when the two-dimensional result varies within one error bar.

| $N$ | $d = 1$ | $d = 0, 1$ | $d = 2$ | $d = 1, 2$ | $d = 0, 1, 2$ |
|-----|---------|------------|---------|------------|-------------|
| 0   | 1.38(7) | 1.42(2)    | 1.407(18+2) | 1.396(16+4) |
| 1   | 1.40(8) | 1.44(3)    | 1.41(3)  | 1.424(22+0) | 1.408(19+1) | 1.408(12+1) |
| 2   | 1.39(7) | 1.42(3)    | 1.43(2)  | 1.426(19+6) | 1.427(10+11) | 1.425(8+16) |
| 3   | 1.39(7) | 1.40(3)    | 1.41(2)  | 1.410(19+1) | 1.420(7+2)  | 1.426(7+2)  |
| 4   | 1.37(7) | 1.38(3)    | 1.38(2)  | 1.384(21+5) | 1.389(10+11) | 1.393(5+16) |
| 8   | 1.31(5) | 1.30(3)    | 1.30(2)  | 1.301(19+1) | 1.304(9+1)  | 1.307(4+2)  |
| 16  | 1.210(26) | 1.203(17) | 1.200(12) | 1.202(12+0) | 1.201(7+0)  | 1.202(4+0)  |
| 24  | 1.160(17) | 1.155(12) | 1.151(9)  | 1.152(9+0)  | 1.150(5+0)  | 1.150(4+0)  |
| 32  | 1.129(14) | 1.125(9)  | 1.122(7)  | 1.122(7+0)  | 1.120(4+0)  | 1.119(3+0)  |
| 48  | 1.091(10) | 1.089(6)  | 1.087(4)  | 1.087(4+0)  | 1.085(3+0)  | 1.085(2+0)  |

TABLE II. Two-dimensional estimates of $\bar{g}^*$ obtained from analyses constrained at $d = 1$ and at $d = 0, 1$.

| $N$ | $d = 1$ | $d = 0, 1$ |
|-----|---------|------------|
| 0   | 1.72(4) |           |
| 1   | 1.84(6) | 1.76(5)   |
| 2   | 1.80(7) | 1.82(3)   |
| 3   | 1.73(8) | 1.75(3)   |
| 4   | 1.66(9) | 1.67(4)   |
| 8   | 1.46(6) | 1.46(3)   |
| 16  | 1.29(4) | 1.28(2)   |
| 24  | 1.21(3) | 1.20(2)   |
| 32  | 1.17(3) | 1.16(1)   |
| 48  | 1.12(2) | 1.11(1)   |
TABLE III. Three-dimensional estimates of $g^* \equiv g^*(N + 8)/(48\pi)$. The two results of Ref. [25] are relative to the cubic and bcc lattice respectively.

| $N$ | $\epsilon$-exp. | $d = 3$ g-exp. | HT [25] | IHT |
|-----|------------------|----------------|---------|-----|
| 0   | 1.396(20)        | 1.413(6)       | 1.388(5), 1.387(5) |
| 1   | 1.408(13)        | 1.411(4)       | 1.408(7), 1.407(6) | 1.402(2) [9] |
| 2   | 1.425(24)        | 1.403(3)       | 1.411(8), 1.406(8) | 1.396(4) [46] |
| 3   | 1.426(9)         | 1.391(4)       | 1.409(10), 1.406(8) |
| 4   | 1.393(21)        | 1.377(5)       | 1.392(10), 1.394(10) |

TABLE IV. Three-dimensional estimates of $r_6$ for various values of $N$ from an unconstrained analysis of the $\epsilon$-expansion and constrained analyses in $d = 1$ and $d = 0, 1$. The result in brackets for $N = 0$ has been obtained using our conjectured value for $r_6$ at $d = 0$, see Eq. [25].

| $N$ | unc | $d = 1$ | $d = 0, 1$ |
|-----|-----|---------|------------|
| 0   | 2.180(80) | 2.148(22) | 2.146(15) |
| 1   | 2.077(69) | 2.057(31) | 2.065(18) |
| 2   | 1.980(65) | 1.955(28) | 1.969(12) |
| 3   | 1.889(63) | 1.859(21) | 1.867(9)  |
| 4   | 1.812(66) | 1.778(23) | 1.780(8)  |
| 8   | 1.580(78) | 1.546(25) | 1.537(15) |
| 16  | 1.333(38) | 1.310(17) | 1.300(18) |
| 32  | 1.125(13) | 1.117(4)  | 1.110(9)  |
| 48  | 1.036(10) | 1.033(2)  | 1.029(4)  |

TABLE V. Two-dimensional estimates of $r_6$ for various values of $N$ from constrained analyses in $d = 1$ and $d = 0, 1$. The result in brackets for $N = 0$ has been obtained using our conjectured value at $d = 0$.

| $N$ | $d = 1$ | $d = 0, 1$ |
|-----|---------|------------|
| 0   | 3.745(47) | [3.740(23)] |
| 1   | 3.671(68) | 3.691(28)  |
| 2   | 3.494(58) | 3.530(18)  |
| 3   | 3.308(41) | 3.328(12)  |
| 4   | 3.155(44) | 3.159(12)  |
| 8   | 2.747(45) | 2.721(19)  |
| 16  | 2.368(34) | 2.335(24)  |
| 32  | 2.074(10) | 2.052(13)  |
| 48  | 1.950(4)  | 1.937(6)   |
TABLE VI. Three-dimensional estimates of $r_8$ for various values of $N$ from an unconstrained analysis of the $\epsilon$-expansion and constrained analyses in $d = 1$ and $d = 0, 1$. The result in brackets for $N = 0$ has been obtained using our conjectured value for $r_8$ at $d = 0$.

| $N$ | unc  | $d = 1$          | $d = 0, 1$          |
|-----|------|------------------|---------------------|
| 0   | 0.1(2.3) | 2.19(1.16) | [3.13(53)]          |
| 1   | −0.4(1.9) | 1.76(80) | 2.75(39)          |
| 2   | −0.8(1.7) | 0.75(75) | 2.08(45)          |
| 3   | −1.2(1.5) | 0.01(48) | 0.97(28)          |
| 4   | −1.4(1.1) | −0.50(31) | 0.19(23)          |
| 8   | −1.8(4) | −1.39(18) | −1.18(9)          |
| 16  | −1.7(4) | −1.57(7) | −1.54(4)          |
| 32  | −1.3(1) | −1.23(3) | −1.24(2)          |
| 48  | −0.97(5) | −0.96(1) | −0.969(4)          |

TABLE VII. Two-dimensional estimates of $r_8$ for various values of $N$ from constrained analyses in $d = 1$ and $d = 0, 1$. The result in brackets for $N = 0$ has been obtained using our conjectured value for $r_8$ at $d = 0$.

| $N$ | $d = 1$          | $d = 0, 1$          |
|-----|------------------|---------------------|
| 0   | 24.6(2.3) | [27.0(9)] |
| 1   | 23.8(1.3) | 26.5(5) |
| 2   | 19.7(1.4) | 23.2(6) |
| 3   | 16.1(1.0) | 18.8(4) |
| 4   | 13.5(8) | 15.4(3) |
| 8   | 8.1(4) | 8.7(2) |
| 16  | 5.0(2) | 5.1(1) |
| 32  | 3.87(5) | 3.82(2) |
| 48  | 3.71(2) | 3.64(1) |
TABLE VIII. Three-dimensional estimates of $r_6$ and $r_8$. When the original reference reports only estimates of $g_{2j}$ the errors we quote for $r_{2j}$ have been calculated by considering the estimates of $g_{2j}$ as uncorrelated.

| $N$  | $r_6$         | $d = 3$  | $g$-exp. | HT  | ERG  |
|------|---------------|----------|----------|-----|------|
| 0    | 2.148(22) [this work] | 2.11(9) [this work] | 2.048(5) | 1.92 | 34   |
|      | 2.1(3) [15]   |          |          |     |      |
| 1    | 2.058(11) [15] | 2.053(8) [34] | 2.048(5) | 1.92 | 34   |
|      | 2.12(12) [34] | 2.060 [31] |          |     |      |
|      | 2.157(18) [36] |          |          |     |      |
| 8    | 2.48(28) [15] | 2.47(25) [34] | 2.28(8)  | 2.18 | 34   |
|      | 2.42(30) [34] | 2.496 [31] |          |     |      |
|      | 2.7(4) [35]   |          |          |     |      |
| 10   | -20(15) [34]  | -25(18) [34] | -13(4)  |     |      |
|      | -12.0(1.1) [34] |          | -10(2)  |     |      |
|      |               |          | -4(2)   |     |      |
| 2    | 1.969(12) [this work] | 1.967 [31] | 1.951(14) | 1.83 | 34   |
|      | 1.94(11) [31] |          |          |     |      |
|      | 1.970(40) [this work] |          |          |     |      |
| 8    | 2.1(0.9) [this work] | 1.641 [31] | 1.36(9)  | 1.4  | 34   |
|      | 3.5(1.3) [13] |          |          |     |      |
| 3    | 1.867(9) [this work] | 1.880 [31] | 2.1(6)   | 1.74 | 34   |
|      | 1.84(9) [13]   |          |          |     |      |
|      | 1.884(32) [this work] |          |          |     |      |
| 8    | 1.0(0.6) [this work] | 0.975 [31] |          |     |      |
|      | 2.1(1.0) [13]  |          |          |     |      |
| 4    | 1.780(8) [this work] | 1.803 [31] | 1.9(6)   | 1.65 | 34   |
|      | 1.75(7) [13]   |          |          |     |      |
|      | 1.809(27) [this work] |          |          |     |      |
| 8    | 0.2(0.4) [this work] | 0.456 [31] |          |     |      |
|      | 1.2(1.0) [13]  |          |          |     |      |