Independent Sets in Vertex-Arrival Streams

Graham Cormode\textsuperscript{1}, Jacques Dark\textsuperscript{1}, and Christian Konrad\textsuperscript{2}

\textsuperscript{1}Department of Computer Science, University of Warwick, Coventry, UK, 
\{g.cormode, j.dark\}@warwick.ac.uk
\textsuperscript{2}Department of Computer Science, University of Bristol, Bristol, UK, 
christian.konrad@bristol.ac.uk

We consider the classic maximal and maximum independent set problems in three models of graph streams:

In the edge-arrival model we see a stream of edges which collectively define a graph; this model has been well-studied for a variety of problems. We first show that the space complexity for a one-pass streaming algorithm to find a maximal independent set is quadratic (i.e. we must store all edges). We further show that the problem does not become much easier if we only require approximate maximality. This contrasts strongly with the other two vertex-based models, where one can greedily find an exact solution using only the space needed to store the independent set.

In the “explicit” vertex stream model, the input stream is a sequence of vertices making up the graph, where every vertex arrives along with its incident edges that connect to previously arrived vertices. Various graph problems require substantially less space to solve in this setting than for edge-arrival streams. We show that every one-pass \(c\)-approximation streaming algorithm for maximum independent set (MIS) on explicit vertex streams requires space \(\Omega(n^{2/c^7})\), where \(n\) is the number of vertices of the input graph, and it is already known that space \(\tilde{\Theta}(n^{2/c^2})\) is necessary and sufficient in the edge arrival model (Halldórrsson et al. 2012). The MIS problem is thus not significantly easier to solve under the explicit vertex arrival order assumption. Our result is proved via a reduction to a new multi-party communication problem closely related to pointer jumping.

In the “implicit” vertex stream model, the input stream consists of a sequence of objects, one per vertex. The algorithm is equipped with a function that can map a pair of objects to the presence or absence of an edge, thus defining the graph. This model captures, for example, geometric intersection graphs such as unit disc graphs. Our final set of results consists of several improved upper and lower bounds for ball intersection graphs, in both explicit and implicit streams.
1 Introduction

The streaming model supposes that, rather than being loaded into memory all at once, the input is received piece-by-piece over a period of time. Only a sublinear amount of memory (in the input size) is made available, preventing any algorithm from “seeing” even a constant fraction of the whole input at once.

In graph streams (see [19] for an excellent survey), we distinguish between the “edge-arrival” model, where the stream consists of individual edges arriving in any order, and the “vertex-arrival” model, where the stream consists of batches of edges incident to a particular vertex—as each vertex “arrives” we are given all the edges from the new vertex to previously arrived vertices. We will shorten the names to edge streams and vertex streams, respectively. Problems on edge streams are always at least as hard as on vertex streams (as any vertex stream is also a valid edge stream).

There is a further variant which we will call “implicit” vertex streams (as opposed to the normal explicit representation). In this model, the stream consists of a series of small (polylog(n)-sized) identifiers, one per vertex. We are additionally provided with some function or oracle which maps a pair of identifiers to a Boolean output indicating whether the two vertices are connected or not. This implicitly defines a graph over the list of identifiers received. Geometric intersection graphs, received as a stream of geometric objects, are the most natural member of this class. For example, a unit interval intersection graph might be received as a series of points in \( \mathbb{R} \). Then a pair of vertices \( x, y \) are adjacent if and only if \( |x - y| \leq 1 \).

Explicit and implicit vertex streams are closely related but distinct, with neither being strictly “harder” than the other. For example: it is easy to count exactly the number of edges in \( O(1) \) space (of words) for an explicit vertex stream, however, doing so for an implicit stream requires linear space, otherwise we cannot hope to know how many edges are incident to the final vertex. On the other hand: implicit vertex streams can be stored entirely in \( \tilde{O}(n) \) space\(^1\), whereas explicit vertex streams require \( \Omega(n^2) \) space to store the full structure.

Maximum independent set is an important problem on graphs. The task is to find a largest subset of vertices which have no edges between them. Unfortunately, the offline problem is NP-hard to find a maximum set in a general graph [16], and even hard to approximate within a factor of \( n^{1-\epsilon} \), for any \( \epsilon > 0 \) [20]. It is also known to be hard (requiring \( \tilde{O}(n^{3/2}) \) space to \( c \)-approximate on an \( n \)-vertex graph) in the edge streaming model, despite being allowed unlimited computation [14]. However, we can do much better for graphs of bounded independence, given as vertex streams [8].

In this paper, we study the hardness of approximate maximum independent set in the explicit and implicit vertex streaming model. First, we motivate the study of vertex streams by showing a large space complexity gap between the edge and (both) vertex models for the related problem of finding a maximal independent set. The gap remains non-trivial even when we only ask for a set which is “nearly” maximal. Then, we propose a new communication problem (closely related to index and pointer jumping) and use it to show that MIS for general graphs cannot be much easier in the explicit vertex streaming model than in the edge streaming model. Last, we show various improved upper and lower bounds for certain geometric intersection graphs in both vertex streaming models.

1.1 Our Contributions

For a graph \( G \), let \( \alpha(G) \) denote the size of a maximum independent set. Further, let \( \chi(G) \) denote the chromatic number, i.e., the number of colors needed in any (legal) coloring of the input graph. We arrange our contributions according to the three models of streaming graphs that we consider (defined more formally below):

- **Edge streams.** We show that solving maximal independent set in an edge stream requires \( \Omega(n^2) \)

\(^1\)We use \( \tilde{O} \), \( \tilde{\Theta} \), and \( \tilde{\Omega} \) to mean \( O \), \( \Theta \), and \( \Omega \) (respectively) with log factors suppressed.
space, significantly more than the $O(\alpha(G))$ sufficient to solve it on an explicit or implicit vertex stream. We further show that even finding any independent set which covers all but an $n^{-\epsilon}$ fraction of the vertices for $\epsilon > 0$ (a nearly maximal independent set) in an edge stream requires $n^{2-o(1)}$ space, while covering just a constant fraction of the vertices requires $n^{1+\Omega(\frac{1}{\log \log n})}$ space. The techniques rely on reductions from problems in communication complexity — initially, a relatively simple reduction from INDEX to show the hardness of MIS, but a more involved reduction to a graph-based variant of INDEX based on Ruzsa-Szemerédi graphs for the approximate relaxation.

- **Explicit vertex streams.** For general explicit vertex streams, we show that $c$-approximating $\alpha(G)$ requires $\Omega(\frac{\alpha(G)}{c})$ space. We further observe that the construction leads to the same space bound for approximating $\chi(G)$, the chromatic number. We rely on the communication complexity of a novel multiparty generalization of the well-known INDEX problem, which is closely related to pointer-jumping problems but allows us to more directly show hardness. The hardness of the MIS problem itself then relies on a careful construction based on erasure codes to bound the size of cliques in our hard instance graphs.

- **Implicit vertex streams.** Next, we show several results for geometric intersection graphs. We can 3-approximate MIS for a stream of unit squares in the plane using $O(\alpha(G))$ space, and achieving better than a $\frac{5}{2}$-approximation to $\alpha(G)$ requires $\Omega(n)$ space. Unit interval intersection graphs given as an explicit vertex stream require $\Omega(n)$ space to get a better than $\frac{5}{4}$ approximation to $\alpha(G)$, making it harder than the implicit stream equivalent. Figures 2 and 1 illustrate these results and put them in context to previously known bounds (see also Section 1.3). The two-dimensional upper bounds can be viewed as a generalization of the one-dimensional bounds, with more work to cover the increased number of cases that occur in 2D. However, the lower bounds involve an intricate packing argument, to show that the MIS size can be used to recover encoded information, which is used in conjunction with our multiparty INDEX variant problem to demonstrate approximation hardness.
1.2 Problem Definitions

Our problems are defined with respect to a graph \( G = (V, E) \) with \( n \) vertices (\(|V| = n\)).

**Definition 1** (Independent Sets). An **independent set** of \( G \) is a subset of the vertices \( U \subseteq V \) that contains no edges ((\( U \times U \)) \( \cap E = \emptyset \)). We use \( \alpha(G) \) to refer to the maximum size of any independent set of \( G \). An MIS (maximum independent set) of \( G \) is any independent set of size \( \alpha(G) \). A maximal independent set is an independent set that no other independent sets contain as a proper subset.

We are also interested in allowing approximations:

**Definition 2** (Approximate MIS). A \( c \)-approximate MIS of \( G \) is an independent set of size at least \( \frac{\alpha(G)}{c} \). An algorithm is said to return a \( c \)-approximation to \( \alpha(G) \) if it returns a value \( \gamma \) satisfying \( \frac{1}{c} \alpha(G) \leq \gamma \leq \alpha(G) \).

The one-sided nature of the \( c \)-approximation requirement for \( \alpha(G) \) follows by analogy with the problem of finding a \( c \)-approximate MIS, where of course no overestimate is possible. A two-sided approximation can of course be made one-sided by rescaling by \( \sqrt{c} \).

**Definition 3** (Graph Streams). We define the three different input arrival models as follows:

- **An edge stream** consists of a sequence of \( m \) edges \( S = \langle e_1, e_2, \cdots, e_m \rangle \) over a pre-determined vertex set \( V \), arriving one-by-one in arbitrary order. Then the edge set is given by \( E = \{ e_i : i \in [m] \} \).

- **An explicit vertex stream** is a sequence of \( n \) vertices \( S = \langle v_1, v_2, \cdots, v_n \rangle \) arriving one-by-one in arbitrary order. Each vertex \( v_i \) arrives with a list of all the edges between \( v_i \) and previous vertices \( \{ v_j : j < i \} \). Thus, each edge \( \{ v_i, v_j \} \) is presented exactly once in the stream, with the arrival of vertex \( v_{\max\{i,j\}} \).

- **An implicit vertex stream** consists of a sequence of \( n \) identifiers \( S = \langle I_1, I_2, \cdots, I_n \rangle \). Each identifier represents a vertex of the graph and is taken from some universe \( \mathcal{U} \) with \( \mathcal{U} \in O(2^{\text{polylog}(n)}) \). Thus, the identifiers have a succinct representation of \( \text{polylog}(n) \) bits.

We are equipped with oracle access to a symmetric function \( \sigma : \mathcal{U} \times \mathcal{U} \rightarrow \{0,1\} \) which determines the presence or absence of an edge between a particular pair of vertices based on their identifiers. So the edge set of the streamed graph is \( E_\sigma = \{ \langle i, j \rangle : \sigma(I_i, I_j) = 1 \} \).

Implicit vertex streams can be understood as being defined over a large pre-defined graph on \( O(2^{\text{polylog}(n)}) \) vertices, so that the stream describes the induced sub-graph arising by selecting \( n \) of these vertices.

We also mention several special classes of graphs which have natural implicit representations or are easier to approximate MIS on.

**Definition 4** (Bounded Independence Graphs).

- **A graph** \( G \) **has** \( f \)-**bounded independence** for some bounding function \( f : \mathbb{N} \rightarrow \mathbb{N} \) if every \( r \)-neighborhood (the set of vertices within distance \( r \)) of every vertex of \( G \) contains no independent set larger than \( f(r) \).

- **A family of graphs** \( \mathcal{F} \) **has polynomially bounded independence** if there is a polynomial \( P \) such that every graph \( G \in \mathcal{F} \) has \( P \)-bounded independence.

Bounded-independence graphs admit greedy approximation algorithms. Consider the greedy algorithm that maintains an independent set by arbitrarily ordering the vertices, and adding each subsequent vertex so long as the set stays independent. For an \( f \)-bounded independence graph this greedy procedure will give an \( f(1) \)-approximation to MIS. This algorithm can be implemented in any vertex stream (implicit or explicit) by following the vertex arrival order and using \( O(\alpha(G)) \) space.
**Definition 5 (Intersection Graphs).** The geometric intersection graph \( G = (V, E) \) of objects drawn from some geometric space \( \{X_1, X_2, \ldots, X_n\} \) has one vertex associated with each object, and an edge between a pair of vertices if and only if the corresponding pair of objects intersect: \( V = [n], E = \{\{i, j\} : X_i \cap X_j \neq \emptyset\} \).

For example, an \( l_p^d \)-ball graph is the geometric intersection graph of \( n \) closed \( l_p \) balls in \( d \) dimensions, each of the form \( B^p(x, r) = \{y \in \mathbb{R}^d : \|x - y\|_p \leq r\} \subset \mathbb{R}^d \). Of particular interest is the unit \( l_p^d \)-ball graph, where all the balls have radius 1. By specifying \( p \) and \( d \), we can define intersection graphs on disks, spheres, squares, rectangles, hyper-cubes, and generalize to any combination of these and other geometric objects. Many of these classes have polynomially bounded independence (as long as the ratio between the sizes of the smallest ball which can cover any of the objects and the largest ball which can fit inside any object is constant). These geometric intersection graphs also form natural implicit vertex streams, assuming that the geometry is discretised to allow \( \text{polylog}(n) \) sized representation of the shapes.

### 1.3 Prior Work

**Edge Streams**  
Halldórsson et al. [14] showed that for general graphs in the edge-arrival model \( \Omega \left( \frac{n^2}{c^2 \log^c n} \right) \) space is required to obtain a \( c \)-approximation to the maximum independent set size (or maximum clique size) if \( c = \Omega(\log n) \), and \( \Omega \left( \frac{n^2}{c^2} \right) \) is required if \( c = o(\log n) \) [14]. A corresponding \( \tilde{O} \left( \frac{n^2}{c^2} \right) \) space random sampling algorithm shows this is tight up to logarithmic factors. Braverman et al. [5] showed that space \( \Omega \left( \frac{n^2}{c^2} \right) \) is needed, even if \( c = o(\log n) \), where \( m \) is the number of edges of the input graph. This bound though only holds for small values of \( m \).

**Explicit Vertex Streams**  
The work of Halldórsson et al. [12] gives an \( O(n \log n) \) space streaming algorithm which can find an independent set of expected size at least \( \beta(G) = \sum_{v \in V} \frac{1}{\deg(v)+1} \). On general graphs, this only gives a \( \Theta(n) \)-approximation, but for polynomially bounded independence graphs, this gives a \( \text{polylog}(n) \)-approximation [13].

In our prior work, it was shown how to return an estimate \( \gamma \in \Omega \left( \frac{\beta(G)}{\log n} \right) \) with \( \gamma \leq \alpha(G) \) from an explicit vertex arrival stream using only \( O(\log^3 n) \) space [8]. This result for example gives a \( O \left( \frac{\log^2 n}{\log \log n} \right) \)-approximation on unit interval graphs (see Figure 1). However, the technique samples vertices based on their degree and so does not extend to implicit vertex streams.

Braverman et al. [5] showed that in a vertex arrival model, where every vertex arrives together with all its incident edges (as opposed to the explicit vertex stream model considered here where every vertex arrives together with its incident edges connecting to vertices that have previously arrived), space \( \Omega \left( \frac{n^2}{c^2} \right) \) is required for computing a \( c \)-approximate MIS. In their construction the input graph has \( \Theta(nc) \) edges, which thus yields a lower bound of \( \Omega \left( \frac{n^2}{c^2} \right) \). Observe that our lower bound for explicit vertex streams is \( \Omega \left( \frac{n^2}{c^2} \right) \), which is a quadratic improvement for constant \( c \).

**Implicit Vertex Streams**  
In [10], it was shown that it is possible to \( \frac{3}{2} \)-approximate MIS for the intersection graph of a unit interval stream using \( \tilde{O}(\alpha(G)) \) space. In the same space, a 2-approximation is possible for arbitrary interval streams. Both these results are shown to be tight: any \( (\frac{3}{2} - \epsilon) \)-approximation for unit intervals—or \( (2 - \epsilon) \) for general intervals—requires \( \Omega(n) \) space. By clever use of sampling, the result can be adapted to provide an approximation of \( \alpha(G) \) of \( \frac{3}{2} + \epsilon \) for unit intervals and \( 2 + \epsilon \) for general intervals with space only \( \text{polylog}(n, \epsilon^{-1}) \) [6].
2 Bounds for Maximal Independent Set

In this section, we consider streaming algorithms for the maximal independent set problem. Vertex-arrival streams (both explicit and implicit) are well-suited to the maximal independent set problem, since they allow the implementation of the Greedy algorithm for independent sets, which greedily add every incoming vertex \( v \) to an initially empty independent set \( I \) if this is possible, i.e., if \( I \cup \{ v \} \) is an independent set. This yields the following result:

**Fact 1.** The Greedy algorithm for independent sets is a one-pass \( O(n \log n) \) space maximal independent set algorithm in the vertex-arrival order streaming model (for both implicit and explicit vertex streams).

Since the only space required by the algorithm is to store a valid independent set, the storage is in fact bounded by \( O(\alpha(G)) \), the space needed to store an MIS, which is bounded by \( O(n \log n) \).

This fact raises the question how well we can solve the maximal independent set problem in edge-arrival streams, which we address in the remainder of this section. We will first show that computing a maximal independent set in one pass in the edge-arrival model is not possible using sublinear space, i.e., space \( \Omega(n^2) \) is required. We then ask whether we can compute an independent set that is approximately maximal in a single pass:

**Definition 6** (Approximate Maximality). Let \( G = (V, E) \) be an \( n \)-vertex graph, and let \( I \subseteq V \) be an independent set. Then \( I \) is \( \delta \)-maximal, if \( |I \cup \Gamma_G[I]| \geq \delta n \).

A \( \delta \)-maximal independent set \( I \) covers a \( \delta \)-fraction of the vertices, or, in other words, when removing \( I \) and its neighbors \( \Gamma_G[I] \) from the graph, then \( (1 - \delta)n \) vertices are remaining. We will next show that establishing approximate maximality in edge-arrival streams requires strictly more space that computing a maximal independent set in vertex-arrival streams (i.e., \( \omega(n) \) space), even if \( \delta = \frac{24}{25} \). Regarding stronger approximate maximality, our lower bound yields that computing a \( (1 - \frac{1}{n^\epsilon}) \)-maximal independent set requires space \( \Omega(n^2 - o(1)) \), for every \( \epsilon > 0 \).

Interestingly, if we allow an algorithm to perform multiple passes, then sublinear space algorithms for maximal independent set can be obtained. Such algorithms are in fact immediately implied by the correlation clustering algorithms given in Ahn et al. \[^1\]. Their result yields the following theorem:

**Theorem 1.** There is a \( O(\log \log n) \)-pass streaming algorithm for maximal independent set that uses space \( \tilde{O}(n) \).

2.1 Lower Bound for Maximal Independent Set in Edge-arrival Streams

We give a reduction to the well-known two-party communication complexity problem INDEX:

**Definition 7.** In the two-party communication problem INDEX, Alice holds an \( N \)-bit string \( X \in \{0, 1\}^N \) and Bob holds an index \( \sigma \in [N] \). Alice sends a single message to Bob, who, upon receipt, outputs \( X_\sigma \).

It is well-known that Alice essentially needs to send all \( N \) bits to Bob:

**Theorem 2** ([18]). The randomized constant error communication complexity of INDEX is \( \Omega(N) \).

**Theorem 3.** Every randomized constant error one-pass streaming algorithm in the edge arrival model that computes a maximal independent set requires \( \Omega(n^2) \) space.
Hence, if required by the streaming algorithm is therefore at least the communication complexity of $I_G$ of the input graph $B$ as the $n$ vertex set $X$ that if $I = \Gamma(y) \cap \Gamma(x) = \emptyset$ for every $y \in X$ and $x \in Y$, then there must be an edge between these vertex sets, there must be an edge between $X$ and $Y$. This cannot cause a conflict, because no edges were previously present between $G_1$ and $G_2$. Alice simulates a streaming algorithm for maximal independent set on the stream of edges of $G_1$, and sends the memory state to Bob.

Let $I$ be the output maximal independent set computed by the algorithm. We argue that Bob can decide whether edge $(a_1, b_1)$ is in the input graph and hence determine the value of $X_\sigma$. First, suppose that $X_\sigma = 1$. Then, for every $j \in \{1, 2\}$, there exist $i_1, i_2 \in I$, i.e., $a_j$ and $b_j$ are not both included in $I$. We argue now that if $X_\sigma = 0$, then $\{a_1, b_1\} \subseteq I$ or $\{a_2, b_2\} \subseteq I$. Suppose that this is not the case, i.e., there are vertices $x_1 \in \{a_1, b_1\}$ and $x_2 \in \{a_2, b_2\}$ which are not included in $I$. Since $I$ is maximal, there exists a vertex $y_1 \in I \cap \Gamma_G(x_1)$ and $y_2 \in I \cap \Gamma_G(x_2)$. Furthermore, since $a_1, b_1, a_2, b_2$ can only have edges to $B_1^*, A_1^*, B_2^*$, $A_2^*$ respectively, we have $y_1 \in A_1^* \cup B_1^*$ and $y_2 \in A_2^* \cup B_2^*$. However, since Bob added all edges between these vertex sets, there must be an edge between $y_1$ and $y_2$, which contradicts $I$ being an IS. Hence, if $X_\sigma = 0$, then $\{a_1, b_1\} \in I$ or $\{a_2, b_2\} \in I$ and we can thus determine the value of $X_\sigma$. The space required by the streaming algorithm is therefore at least the communication complexity of INDEX, which is $\Omega(N) = \Omega(n^2)$, by Theorem 2.

### 2.2 Lower Bound for Approximate Maximality in Edge-arrival Streams

We now extend the lower bound given in the previous subsection to approximate maximality. Central to our construction are Ruzsa-Szemerédi graphs, which have previously been used for the construction of space lower bounds for streaming algorithms for the maximum matching problem [11, 17, 3]:

**Definition 8 (Ruzsa-Szemerédi graph).** A bipartite graph $G$ is an $(r, s)$-Ruzsa-Szemerédi graph if its edge
Lemma 1. Let a randomized constant error one-pass streaming algorithm in the edge arrival model that computes a random bit, while in RS-I N D EX M or each induced matching C outputs at least

Theorem 4 (11). In a lower bound is given for the task of computing a maximum matching. Their hardness stems from the fact that it is hard to learn many edges of M under the distribution described in the definition of RS-INDEX.

Figure 4: Sketch proof for theorem 3 that at least one pair a, b is in any maximal IS when Xσ = 0.

set can be partitioned into r induced matchings each of size s.

Recall that a matching M ⊆ E in a graph G = (V, E) is induced, if the edge set of the vertex-induced subgraph G[V(M)] equals M, i.e., there are no other edges interconnecting V(M) different from M.

Our lower bound for approximate maximality is obtained by a reduction to the two-party communication problem RS-INDEX, defined as follows:

Definition 9 (RS-INDEX). Let H be an (r, s)-Ruzsa-Szemeredi graph with induced matchings M1, M2, . . . , Mr. For each induced matching Mi, let Mi′ ⊆ Mi be a uniform random subset of size s/2 (we assume that s is even). The RS-INDEX problem is a one-way two-party communication problem, where H, and, in particular, M1, M2, . . . , Mr are known by both parties. In addition, Alice holds the graph G = H[∪iMi′], and Bob holds a uniform random index i ∈ {1, 2, . . . , r}. Alice sends a single message to Bob, who, upon receipt, outputs at least C · s edges of Mi′, for an arbitrary small constant C.

Observe that this problem is similar in spirit to INDEX: In INDEX, Bob needs to learn one uniform random bit, while in RS-INDEX, Bob needs to learn the presence of many edges of Mi′. A lower bound on the communication complexity of RS-INDEX is implicit in 11.3.

Theorem 4 (11). The randomized constant error communication complexity of RS-INDEX is Ω(r · s).

Equipped with the RS-INDEX problem, we now give a reduction from approximate maximality to RS-INDEX, which yields our lower bound for streaming algorithms:

Lemma 1. Let r, s, n be integers such that there is an n-vertex (r, s)-Ruzsa-Szemeredi graph. Then, every randomized constant error one-pass streaming algorithm in the edge arrival model that computes a (1 − 1/n)-maximal independent set requires Ω(r · s) space.

Proof. Let H be an n-vertex (r, s)-Ruzsa-Szemeredi graph, and let G be Alice’s input graph for the RS-INDEX problem derived from H. Let M1, M2, . . . , Mr denote the induced matchings in H, let Vi = V(Mi), and let Mi′ ⊆ Mi denote the subset of edges of matching Mi that is included in G. Let i be Bob’s input. Furthermore, let A be a constant error randomized one-pass streaming algorithm for the edge-arrival model that computes a (1 − 1/n)-maximal independent set on a graph on N vertices. We now show how A can be used to solve RS-INDEX:

In 11 a lower bound is given for the task of computing a maximum matching. Their hardness stems from the fact that it is hard to learn many edges of M under the distribution described in the definition of RS-INDEX.
Given $G$, let $\tilde{G}$ be the graph obtained from $G$, where every induced matching $M'_1$ in $G$ is replaced by edges $M'_2 := M_1 \setminus M'_1$ (observe that $E(G) \cup E(\tilde{G}) = E(H)$). Alice now constructs two disjoint copies $G_1$ and $G_2$ of $\tilde{G}$, runs algorithm $A$ on $G_1 \cup G_2$ (on an arbitrary ordering of their edges), and sends the message state to Bob. Bob constructs the edge set $F$ that connects every vertex $v_1 \in V(G_1) \setminus V_i$ with every vertex $v_2 \in V(G_2) \setminus V_i$, where $V_i$ and $V_2$ are the copies of the vertices $V_i$ in graphs $G_1$ and $G_2$, respectively, and continues the execution of $A$ on $F$. Let $I$ be the independent set produced by algorithm $A$.

Observe that the graph processed by algorithm $A$ contains $N = 2n$ vertices. Since $I$ is $(1 - \frac{s}{6n})$-maximal, we have $|V \setminus \Gamma[I]| \leq N - (1 - \frac{s}{6n})N = s/6 - 6$. This allows us to identify $\Omega(s)$ edges of $M'_2$ as follows:

Let $a, b$ be the incident vertices to an arbitrary edge of $M'_2$, let $a_1, b_1$ be the copies of $a, b$ in $G_1$, and let $a_2, b_2$ be the copies of $a, b$ in $G_2$. Observe that $a_1$ and $b_1$ are not connected in $G_1$, and $a_2$ and $b_2$ are not connected in $G_2$. We now claim that if all vertices $a_1, b_1, a_2, b_2$ are covered by $I$, i.e., $\{a_1, b_1, a_2, b_2\} \subseteq \Gamma[I]$, then either $\{a_1, b_1\} \subseteq I$ or $\{a_2, b_2\} \subseteq I$ (or both). Indeed, suppose that this is not the case. Then there are vertices $x_1 \in \{a_1, b_1\}$ and $x_2 \in \{a_2, b_2\}$ with $x_1, x_2 \notin I$. Let $y_1 \in I$ be a vertex incident to $x_1$, and let $y_2 \in I$ be a vertex incident to $x_2$. By the construction of the input graph, $y_1 \in V(G_1) \setminus V_i$, and $y_2 \in V(G_2) \setminus V_i$. Observe however that the edge $y_1y_2$ was included by Bob, which implies that $y_1, y_2$ are not independent, a contradiction. Hence, either $\{a_1, b_1\} \subseteq I$ or $\{a_2, b_2\} \subseteq I$ (or both) holds. Observe that this implies that the algorithm identified that there is no edge between $a_1, b_1$, which in turn implies that we learned one edge of $M'_2$. Hence, for every pair of vertices $a, b$ of $M'_2$, either at least one vertex among $\{a_1, b_1, a_2, b_2\}$ is not covered by $I$, or we learn one edge of $M'_2$. Since there are $s/2$ edges in $M'_2$, and at most $s/6$ vertices of the input graph are not covered by $I$, we learn at least $s/2 - s/6 = \Omega(s)$ edges of $M'_2$, which thus solves RS-INDEX. By Theorem 4, algorithm $A$ therefore requires space $\Omega(r \cdot s)$.

In [11] it is shown that there are $n$-vertex $(n^{\Theta(\frac{1}{\log \log n})}, (\frac{1}{4} - \epsilon)n)$ Ruzsa-Szemerédi graphs, for every $\epsilon > 0$, and in [2], it is shown that there are $n$-vertex Ruzsa-Szemerédi graphs with $\Theta(n^{2-o(1)})$ edges such that each matching is of size $n^{1-o(1)}$. Combined with Lemma 1 we obtain our main theorem:

**Theorem 5.** Every randomized constant error one-pass streaming algorithm that computes a $\frac{24}{25}$-maximal independent set requires space $n^{1+\Omega(\frac{1}{\log \log n})}$, and every such algorithm computing a $(1 - \frac{1}{n^2})$-maximal independent set requires space $\Omega(n^{2-o(1)})$, for every $\epsilon > 0$.

### 3 Maximum Independent Set in Explicit Vertex Streams

We first introduce and show the hardness of a “chained index” problem, which we then use to show the hardness of approximating the size $\alpha(G)$ (and hence also for finding an MIS).

#### 3.1 Chained Index Communication Problem

We define a multi-party communication problem, which allows us to prove new lower bounds on several streaming problems. The problem is closely related to pointer jumping and generalizes the classic 2-party index communication problem to more parties by “chaining” together multiple instances which have the same answer but are otherwise independent. In our setting, each party (other than the last) holds a vector that contains (somewhere) a bit which is “the answer” to the instance. Each party (other than the first) knows where the answer bit is located in the previous party’s vector. Communication is one-way and private, with each player receiving a message from the previous player and then sending a message to the next player. This rules out any trivial solution where a party can simply look up the bit announced by a later party. Formally,
Communication proceeds as follows: $P_1$ sends a single message to $P_2$, then $P_2$ communicates to $P_3$, and so on, with each party sending exactly one message to its immediate successor. After all messages are sent, $P_k$ must correctly output $z$, succeeding with probability at least $2/3$. If the promise condition is violated, any output is considered correct.

There is a trivial communication upper bound of $O(n)$ bits: for instance, simply have the penultimate party send $X^{(k-1)}$ to the final party who can then return $X^{(k-1)}_{\sigma_{k-1}}$. We now show lower bounds by a reduction from a different multi-party communication problem.

Definition 11 ([9]). The Boolean conservative one-way $k$-party pointer jumping problem $\text{JUMP}_k$ consists of a constant $\alpha \in [n]$ and $k-1$ functions $\{f_i\}_{i=2}^k$. The first $k-2$ are of the form $f_i : [n] \rightarrow [n]$, and the final one is of the form $f_{k-1} : [n] \rightarrow \{0,1\}$. We use $f_{i:j}$ to refer to the composition $f_{i:(j-1)} \circ f_j$ of functions $f_i \circ f_{i+1} \circ \cdots \circ f_{j-1} \circ f_j$. The input is divided as follows:

- The first party $P_1$ knows all the functions $\{f_i\}_{i=2}^k$
- The second party $P_2$ knows $\alpha$ and every $f_j$ for $j \geq 3$
- Each other party $P_i$ knows $f_{2;i-1}(\alpha)$ and every $f_j$ for $j \geq i + 1$

Each party sends exactly one message in ascending order to their immediate successor, i.e. $P_1$ sends to $P_2$, then $P_2$ sends to $P_3$, and so on. After all messages are sent, $P_k$ must correctly output $f_{2:k}(\alpha)$ with probability at least $2/3$. The conservative version of one-way $k$-party pointer jumping problem was introduced and studied in [9], showing $\Omega\left(\frac{n}{k^2}\right)$ hardness for $k \in o\left(\frac{n}{\log n}\right)$ for a version with a non-Boolean final layer. Later, [7] extended this to all $k$ and to the Boolean version.

Theorem 6 (Theorem 2 in [7]). Any communication scheme $\mathcal{A}$ which solves $\text{JUMP}_k$ must communicate at least $\Omega\left(\frac{n}{k^2}\right)$ bits.

Proof. We prove the claim by showing that any instance of $\text{JUMP}_k$ can be reduced to an instance of $\text{CHAIN}_k$ without any communication. Hence, any algorithm which solves $\text{CHAIN}_k$ can solve $\text{JUMP}_k$ with no change in the communication cost. Combining this with the lower bound of Theorem 6 gives the result.

Fix an instance of $\text{JUMP}_k$. For each $i$ let $X^{(i)}$ be the binary vector whose $j^{th}$ entry is $f_{i+1:k}(j)$. For each $i$ let $\sigma_i = f_{2:i}(\alpha)$. Now we observe three facts:

- Every $\{X^{(i)}_{\sigma_i}\}_{i=1}^{k-1}$ is equal to $f_{2:k}(\alpha)$

4Using the convention that $(g \circ h)(x) = h(g(x)).
• Each party $P_i$ for $i < k$ knows all the information required to compute $X^{(i)}$
• Each party $P_i$ for $i > 1$ knows all the information required to compute $\sigma_{i-1}$

So we have constructed (with no communication) a $k$-party chained index problem which, if solved, will tell us exactly $f_{2:k}(\alpha)$. It therefore follows that the communication cost for any solution to CHAIN$_k$ is at least that for JUMP$_k$.

In particular, for constant $k$, we have a tight bound on the communication complexity of the $k$-party chained index problem of $\Theta(n)$. We conjecture that a dependence on $k$ is not necessary.

**Conjecture 1.** Any communication scheme which solves CHAIN$_k$ requires $\Omega(n)$ communication.

If proven, this would give a hardness of $\Omega(n^2)$ for theorem X. Although the two problems of JUMP$_k$ and CHAIN$_k$ may look fairly similar, we find that the form of CHAIN$_k$ is much more convenient to show lower bounds for independent set and related problems, as we show in the subsequent sections.

### 3.2 MIS hardness in explicit vertex streams

We show a new lower bound for the vertex streaming space complexity of approximate maximum independent set.

**Theorem 8.** Any algorithm for the explicit vertex stream model which finds a $c$-approximation to $\alpha(G)$ with probability at least $2/3$ requires $\Omega\left(\frac{n^2}{c^3}\right)$ space.

For ease of argument, we will actually prove an equivalent result for the problem of clique number approximation, and then note that the complement of the constructed graph can be used with the same arguments to prove Theorem X. To see this equivalence, note that an MIS of a graph is a maximum clique in its complement, and in a vertex streaming model an algorithm can simulate operation on the complement by taking the complement of each vertex as it arrives. Importantly, in the edge-arrival model this reduction is not possible, since the model does not allow to ‘subtract’ the observed edges from the complete graph.

**Theorem 9.** Any algorithm for the explicit vertex stream model which finds a $c$-approximation to the size of the largest clique $\omega(G)$ with probability at least $2/3$ requires $\Omega\left(\frac{n^2}{c^2}\right)$ space.

The heart of our construction is to use an erasure code to encode a length $O\left(\frac{n^2}{c^3}\right)$ binary vector on $O\left(\frac{n}{c}\right)$ vertices, with each bit corresponding to the presence or absences of a clique of size $2c$. The use of the erasure code is to ensure that no pair of these cliques can share an edge. We can then chain together $2c$ such gadgets to encode an instance of CHAIN$_{2c}$ such that if the correct answer is 1, the resulting graph has an independent set of size $4c^2$, while if the correct answer is 0 the graph has no independent sets of size larger than $4c - 1$. Any (one-sided) $c$-approximation algorithm could distinguish these two cases, and so the result is proved.

First we define our clique gadget.

**Lemma 2.** For any positive integers $n$ and $c^2 < \frac{n}{8}$, there exists a graph on $n$ vertices containing $\frac{n^2}{16c^2}$ edge-disjoint cliques of size $2c$ and no cliques of size larger than $2c$. 
Proof. We construct the sets from an erasure code with block size $2c$ and message size 2. Choose a prime $p$ such that $\frac{4c}{p^2} < p \leq \frac{n}{2c}$ (which is guaranteed to exist). Now take $2c < p$ groups of vertices, each of size $p$. Label the groups $V_i$ (for $i \in [2c]$) and label the items in each group $V_i$ as $v^i_j$ (for $j \in [p]$). Leftover vertices are added to the final graph as isolated vertices.

For each polynomial $P \in \mathbb{GF}(p^2)$ we define $K_P$ to be the clique over vertices $\{v^i_P(i) | i \in [2c]\}$. This can be viewed as taking each of the $p^2$ possible edges between $V_1$ and $V_2$ and extending them “linearly” to the other layers (see Figure 5). Clearly $\mathcal{K} = \{K_P | P \in \mathbb{GF}(p^2)\}$ consists of $p^2 > \frac{n^2}{16c^2}$ cliques, each of size $2c$.

We next show that they are pairwise edge-disjoint and that their union contains no larger cliques.

Each clique contains exactly one vertex from each group $V_i$, so for two cliques to share an edge there must be distinct polynomials $P, Q \in \mathbb{GF}(p^2)$ that have the same value at two different points: $P(i) = Q(i)$ and $P(j) = Q(j)$ for $i \neq j$ — a contradiction. Finally, because no clique contains a pair of vertices from a single $V_i$, their union can contain no internal edges on any $V_i$. So any clique can contain at most 1 vertex from each $V_i$, giving a maximum size of $2c$. Hence, $\bigcup_{P \in \mathbb{GF}(p^2)} K_P$ is a graph with the required properties.

Proof of Theorem. Suppose we have an algorithm $\mathcal{C}$ for explicit vertex streams which can, with probability at least $\frac{2}{3}$, produce a $c$-approximation to $\omega(G)$, the size of the largest clique. We will show that such an algorithm can be used to solve $\text{CHAIN}_{2c}$, by communicating its state $2c - 1$ times.
Fix an instance of CHAIN$_{2c}$ with vectors of length $b = \frac{n^2}{2c}$. Our lower bound implies any algorithm that can solve this must send at least one message of size $\Omega \left( \frac{n}{b} \right) = \Omega \left( \frac{n^2}{c} \right)$. Take $n$ vertices and partition the nodes into $2c$ groups of size $\frac{n}{2c}$. Each group will be added to the stream by one of the parties.

**Intra-party edges.** First, consider the group of nodes associated with party $P_i$. We will encode the bits of $X^{(i)}$ onto the internal edges of this group using the construction from Lemma 2. The size $\frac{n}{c}$ sub-graph can fit $b$ cliques of size $2c$. We include the edges of clique $j$ if and only if $X^{(i)}_j = 1$. This is well-defined as the cliques are edge-disjoint. Label the clique in party $P_i$ corresponding to bit $j$ of $X^{(i)}$ as $K^{j}_{i}$. The final party $P_{2c}$ has no associated vector. Instead, it constructs a single clique of size $2c$ and leaves the other vertices isolated.

**Inter-party edges.** We also need edges between the sub-graphs associated with different parties. Each party $P_i$ will connect all its vertices to some of the vertices belonging to previous parties ($P_j$ for $j < i$). These edges are considered to belong to party $P_i$, as they will be added by this party in the vertex streaming model. For each $j < i$ the party $P_i$ connects every one of its vertices to all of $K^{j}_{\sigma_j}$ (the clique corresponding to index $\sigma_j$). For this to happen, $P_i$ must know all $\sigma_j$ for $j < i$. This information is not known initially, but can be appended to the communications between players with only $O(c)$ overhead.

Now that we have our construction, we need to show bounds on $\omega(G)$ for the two cases. First, consider when every $X^{(i)}_{\sigma_i} = 1$. In this case we have each of the cliques $K^{i}_{\sigma_i}$ present and connected together, forming a clique of size $4c^2$. Now consider the case when every $X^{(i)}_{\sigma_i} = 0$. Consider a clique $K$ in the graph. If $K$ contains multiple vertices belonging to one party $P_i$, then it can contain none from any subsequent party $P_j$ ($j > i$), and at most one from each preceding party $P_l$ ($l < i$). Hence the size of any clique is bounded by $4c - 1$. To see why this holds, observe that for any $i < 2c$, our clique can contain only one vertex from $K^{i}_{\sigma_i}$, as none of its edges are included in the graph. So to contain multiple vertices from party $P_i$, the clique $K$ must contain a vertex $v$ from some $K^{i}_{j}$ with $j \neq \sigma_i$. But then all subsequent parties $P_j$ ($j > i$) will have no vertices adjacent to $v$, so cannot contribute anything to $K$. So the best we can do is include one vertex from each $K^{i}_{\sigma_i}$, and then $2c$ from party $P_{2c}$ giving a clique of size $4c - 1$.

To complete the proof, observe that this gap in clique sizes can be distinguished by a $c$-approximation algorithm, and any streaming algorithm gives a communication protocol by having each party update the algorithm state with their information and then passing it to the next player. 

![Figure 7: Example lower bound instance with 4 players for theorem](image)

Cliques corresponding to $\sigma_1$, $\sigma_2$, and $\sigma_3$ are shown in bold red—other cliques are omitted.
Interestingly, the same construction gives us hardness for approximating the chromatic number of a graph. This is notably not possible in the 2-party edge stream construction in \([14]\), as the random graphs used as gadgets have large chromatic number with high probability (see \([4]\)).

**Corollary 1.** Any explicit vertex streaming algorithm to find a \(c\)-approximation to \(\chi(G)\) (the chromatic number), succeeding with probability at least \(2/3\) requires \(\Omega\left(\frac{n^2}{c^7}\right)\) space.

**Proof.** Consider the construction in the proof of Theorem 8. In the case of all \(X_{\sigma_i}^{(i)} = 1\), the graph contains a clique of size \(4c^2\), so it requires at least as many colours.

Conversely, in the case of every \(X_{\sigma_i}^{(i)} = 0\), we can construct a \(4c\)-colouring of the graph. First colour each of the nodes in each \(K_{\sigma_i}^i\) with the \(i\)th colour (this is allowed, as they have no internal edges). Now, the remaining vertices in each party are not adjacent to any uncoloured vertices from other parties, so we simply need to be able to complete the colouring of each party in isolation with \(2c\) new colours and we are finished. This is easily done, as each party’s sub-graph is \(2c\)-partite by construction. \(\square\)

### 4 Maximum Independent Set in Geometric Intersection Graphs

In this section we present a collection of results around geometric intersection graphs given as explicit or implicit vertex streams. This represents a first study of how the difficulty of these problems differs between the models, and with other factors such as dimension.

Recall that a geometric intersection graph is a graph where nodes correspond to geometric objects, and edges indicate whether or not a particular pair of objects intersect. These graphs can be described implicitly, by just giving the collection of geometric objects, or explicitly as a collection of vertices and edges under the promise that some geometric representation exists. For sufficiently complex geometry, every graph will have a geometric representation (simply take hyper-rectangles of a high enough dimension), so to define meaningful classes, we must limit the universe of objects.

We will consider the \(l^p\) closed balls in \(\mathbb{R}^d\), particularly for \(p = 1, 2,\) and \(\infty\). A \(d\)-dimensional \(l^p\) ball is uniquely defined by its center and radius. The ball with center \(c\) and radius \(r\) is exactly the region \(\{x \in \mathbb{R}^d : \|x - c\|_p \leq r\}\). For the implicit representation, we discretize the space of possible centers as \([M]^d\) and the space of possible radii as \([M]\). Then, the input stream is defined as follows:

**Definition 12.** A \(d\)-dimensional \(l^p\) ball stream consists of sequence of \(n\) pairs in \([M]^d \times [M]\) indicating the center and radius of each ball respectively.

This defines an implicit vertex stream, with the intersection function:

\[
\sigma\left((p_1, w_1), (p_2, w_2)\right) = \begin{cases} 
1, & \text{if } \|p_1 - p_2\|_p \leq w_1 + w_2 \\
0, & \text{otherwise.}
\end{cases}
\]

Typically \(M\) will be some polynomial in \(n\), so that the balls can be represented in polylog\((n)\) space. We also define a dilation parameter, which will capture some of the difficulty of the problem.

**Definition 13.** Let the dilation, \(\Delta = \frac{r_{\text{max}}}{r_{\text{min}}}\), be the ratio between the largest and smallest radii in the stream.

We refer to the interesting special case of \(\Delta = 1\), when all the balls are uniform in size, as a unit ball stream.
### 4.1 Interval Graphs: \( d = 1 \)

In \( d = 1 \), the choice of \( p \) is irrelevant, as any \( l^p \) ball is simply an interval. As discussed in Section 1.3 given an interval (ball) stream we can compute a \( \frac{3}{2} \)-approximation to MIS in \( \tilde{O}(\alpha(G)) \) space, and any better approximation requires \( \Omega(n) \) space. A natural question is how this compares with the space complexity for an interval intersection graph given as an explicit vertex stream.

**Theorem 10.** Any algorithm with constant error probability that returns a \((\frac{5}{3} - \varepsilon)\)-approximation of \( \alpha(G) \) for a unit interval intersection graph given as an explicit vertex stream requires \( \Omega(n) \) space.

**Proof.** We will show this bound by a reduction from the 2-party INDEX communication problem for an \( n \)-bit vector.

Consider an instance of INDEX with bit vector \( X \in \{0, 1\}^n \) and index to be queried \( \sigma \in [n] \). We will construct a \( 2n + 3 \) vertex graph as an explicit vertex stream.

Label the vertices \( x, y, z \) and \( a_i, b_i \) for \( i \in [n] \). Split the \( a_i \)'s into two sets based on the bit vector \( X \): \( A_1 = \{a_i\} \cdot X_i = 1 \) and \( A_0 = \{a_i\} \cdot X_i = 0 \). Similarly let \( B_1 = \{b_i\} \cdot X_i = 1 \) and \( B_0 = \{b_i\} \cdot X_i = 0 \). Now the first party creates the following subgraph in the stream: a clique consisting of all the vertices in \( A_1 \), a second clique made from \( B_1 \), and a third clique containing \( A_0 \cup B_0 \cup \{x\} \).

So far this represents a valid interval graph, which can be interpreted as three adjacent “stacks” of intervals. Now, the second player adds \( y \) with edges to every \( a_i \) except \( a_\sigma \) and then adds \( z \) with edges to every \( b_i \) except \( b_\sigma \). This can still be viewed as a valid interval graph, but we now require some intervals from each stack to be “shifted” to overlap with the two new intervals.

In the case of \( X_\sigma = 0 \), the resulting graph has \( \alpha(G) = 3 \). Otherwise, \( \alpha(G) = 5 \). Hence, any algorithm giving a better than \( \frac{5}{3} \) (one-sided) approximation factor could distinguish them and solve INDEX.

This shows that MIS for interval graphs is strictly more difficult in explicit vertex streams than implicit ones.
4.2 Square Graphs: $d = 2$

Observe that in $d = 2$, the geometry defined by $l^1$ and $l^\infty$ balls is equivalent, after fixed scaling and rotation. Thus any instance in one geometry can be transformed into an equivalent (intersection-wise) instance of the other, after ensuring that the discretization is not too coarse.

Our first result for 2 dimensions is a 3-approximation algorithm for MIS on a unit square stream. This is a generalization the algorithm of [6] for unit interval streams — we perform a decomposition of the plane into 2-by-3 strips, similar to their decomposition of the line into length 3 segments.

**Theorem 11.** For $d = 2$ and $\Delta = 1$ we can 3-approximate MIS for an $l_1$ or $l_\infty$ ball stream using $\tilde{O}(\alpha(G))$ space.

**Proof.** Let $r$ be the radius of the balls, and let $w = 2r$. Without loss of generality, we consider the $l_\infty$ version of the problem.

First we look at the problem restricted to squares in the half open strip $[0, 3w) \times [0, 2w)$, referring to the first axis as “left-right”, and the second as “up-down”. At most 2 non-overlapping closed unit squares can fit fully within the open region, and for them not to overlap one must be left of the other. Hence, by storing the leftmost and rightmost squares seen within the strip, we can determine exactly an MIS in the region.

Now, consider the whole of $[M]^2$. We partition this up into $3w$-by-$2w$ half-open strips and consider only the squares which fall exactly within one of the strips. By solving MIS within each strip as before and taking the union, we can solve exactly MIS on this substream. This requires us to store at most twice as many squares as the solution.

Finally, we consider 6 different copies of the partitioning shifted by 0, $w$, or $2w$ horizontally and 0 or $w$ vertically. Any square from the stream must be fully contained in a strip in at least 2 of the 6 partitionings. In particular, this holds for every square in a fixed MIS of the full stream. Using $G_{x,y}$ for the graph of the substream of squares that fit exactly in the partitioning shifted by $(x, y)$, we know the following:

$$\sum_{x=0,w,2w} \left( \sum_{y=0,w} (\alpha(G_{x,y})) \right) \geq 2\alpha(G).$$

That is, the sum of the sizes of the substream MIS’s for the 6 partitionings is at least 2 times the size of the true MIS. Therefore

$$\max_{x=0,w,2w} \left( \max_{y=0,w} (\alpha(G_{x,y})) \right) \geq \frac{1}{3} \alpha(G).$$

Therefore, we simply take the largest of the 6 substream MIS’s and this must give a 3-approximation. In total, we must store at most 12 squares per MIS square.

As in [6] for unit intervals, this immediately leads to a sublinear space algorithm for estimating $\alpha(G)$ with only a $(1 + \epsilon)$ factor loss in approximation factor, through a combination of distinct elements and sampling.

**Corollary 2.** For $d = 2$ and $\Delta = 1$ we can $(3 + \epsilon)$-approximate $\alpha(G)$ with constant probability for an $l_1$ or $l_\infty$ ball stream using $O(\epsilon^{-2} \log \epsilon^{-1} + \log n)$ space.

**Proof.** Observe that if we can get a $(1 + \epsilon)$-approximation to each $\alpha(G_{x,y})$ from the proof of theorem 11 then we are done.

Each strip can have 0, 1, or 2 disjoint squares in it. To approximate $\alpha(G_{x,y})$ we estimate $\gamma$, the number of strips of a given partitioning which are non-empty, and $\delta$, the average number of disjoint squares in the non-empty strips. Then $\alpha(G_{x,y}) \approx \gamma \delta$.

Observe that approximating $\gamma$ is a distinct elements problem, which we can $(1 + \epsilon)$-approximate with constant probability in $O(\epsilon^{-2} + \log n)$ space (see [13]).

Then $\delta$ can be approximated by using nearly-uniform permutations to keep a nearly-uniform sample of the non-empty strips. For full details see [6, lemma 16].
One might speculate whether this decomposition approach could afford a better approximation factor based on some different partitioning of the place. We give evidence for the negative, since any larger strips result in the fixed-size sub-problems not being solvable exactly, as the following result shows.

**Theorem 12.** Given a stream of $w$-by-$w$ squares (2 dimensional $l_\infty$ balls) contained in a $(2 + \delta)w$-by-$(2 + \delta)w$ region, achieving a $(\frac{2}{3} - \epsilon)$-approximation to $\alpha(G)$, with constant probability of success, for any $\epsilon, \delta > 0$ requires $\Omega(n)$ space.

**Proof.** We show this by a reduction from 2-party INDEX. Fix an instance with bit vector $X \in \{0, 1\}^n$ and query index $\sigma \in [n]$. For the lower bound, we use squares of width $w = \frac{4n}{\delta}$, which meets the requirements for an implicit vertex stream as long as $\delta$ is constant.

Party one constructs the following collection of squares arranged along a diagonal line: for each $i \in [n]$ with $X_i = 1$ include the square centered on $(\frac{2n}{\delta} + 2i, \frac{2n}{\delta} + 2n + 2 - 2i)$. Now, observe that the parties could use a $(\frac{2}{3} - \epsilon)$-approximation streaming algorithm to allow party two to determine $X_\sigma$. Simply have party one run the algorithm on its collection, then pass the state to party two and append squares centered on $(\frac{2n}{\delta} + 2n + 2 - 2\sigma)$ and $(\frac{2n}{\delta} + 2\sigma, \frac{2n}{\delta} + 2n + 3 - 2\sigma)$. If $X_\sigma = 1$, there exists a square in the original collection sandwiched between the two new squares giving $\alpha(G) = 3$. Otherwise, $\alpha(G) = 2$. □

Our next result for two dimensions is a stronger lower bound for approximating $\alpha(G)$ of a stream of unit squares in an unrestricted region, based on a reduction from the chained index communication problem used in our main result in Section 3.

**Theorem 13.** Achieving a $(\frac{2}{3} - \epsilon)$-approximation of $\alpha(G)$, with constant probability of success, on an $l_1$ or $l_\infty$ ball stream with $d = 2$ and $\Delta = 1$ requires $\Omega(n)$ space for any $\epsilon > 0$.

**Proof.** We show the lower bound for $l_\infty$, which then implies the result for $l_1$. This proof works by reducing from the 3-party chained index problem. As with the general result in Section 3, more than 2 parties are necessary in order to give a bound for approximation factors greater than 2.

Suppose we have an instance of CHAIN$_3$ with $n$-bit vectors. We will describe a way for each party to construct a collection of unit $l_1$ balls based on the part of the input they hold, such that their union has small or large $\alpha(G)$ depending on the solution to the communication problem. Thus, a streaming algorithm for approximating $\alpha(G)$ can be used to solve the communication problem, giving a space bound.

For the lower construction we use domain size $M = 10n^3$ and ball radius $r = 2n^2$, which are small enough to allow succinct polylog $n$ sized descriptions of the balls (as required for an implicit vertex stream), but large enough to create the gadgets we require. Fix integer $k \in [n]$. The first party has the bit vector $X^{(1)}$. For each entry with $X^{(1)}_i = 1$, add balls centered at $(i(4n + 3) + (j + 1)(4n^2 + 3n), 4n^2)$ and $(i(4n + 3) + (j + 1)(4n^2 + 3n), 8n^2)$ for each $j \in [k]$. Essentially, this makes two horizontal lines of balls stacked on top of each other. There are potentially $nk$ ball locations along the line with centers $4n + 3$ apart. The first $n$ locations are associated with the $n$ entries of $X^{(1)}$; we place a ball if $X^{(1)}_i = 1$, and omit it otherwise; then this is repeated $k$ times in succession. The two lines produce a collection of balls $G_1$ of size at most $2nk$, with $\alpha(G_1) = 2k$. Importantly, the collection of balls associated with any index $X^{(1)}_i = 1$ forms an MIS.

Party two will obliviously add their own set of balls, such that $\alpha(G)$ will increase exactly when the answer bit $X^{(1)}_{\sigma_1} = 1$. The second party has its bit vector $X^{(2)}$ and the index $\sigma_1$ of the answer bit in the first party’s bit vector. For each entry with $X^{(2)}_i = 1$ add a ball centered at $(\sigma_1(4n + 3) + (j + \frac{3}{2})(4n^2 + 3n), 6n^2 - n + 2i)$ for each $j \in [k]$. Essentially, this produces $k$ columns of $n$ balls. The columns are spaced with centers $n^2 + 3n$ apart, lined up to fit in the gaps between the balls corresponding to bit $X^{(1)}_{\sigma_1}$ of the first party (if those balls are present). Each column is constructed as follows: place $n$ balls spaced with centers 1 apart, each associated with one of the entries of $X^{(2)}$, but exclude each ball whose $X^{(2)}_i = 0$. The result is a
collection of balls $G_2$ of size $nk$ with $\alpha(G_2) = k$. Again, the collection of balls associated with any index $X_i^{(1)} = 1$ forms an MIS.

Now, party three will add the final collection of balls $G_3$. Party three knows $\sigma_1$ (this can be appended to any message from player two) and $\sigma_2$, the index of the answer bit in party two’s bit vector. This party adds balls centered at \((\sigma_1(4n + 3) + (j + \frac{3}{2})(4n^2 + 3n), 10n^2 - n + 2\sigma_2 + 1)\) and \((\sigma_1(4n + 3) + (j + \frac{3}{2})(4n^2 + 3n), 2n^2 - n + 2\sigma_2 - 1)\) for each $j \in [k]$. These balls sit at the top and bottom of each column from party 2 sandwiching the ball corresponding to $X_{\sigma_2}^{(2)}$ (if present). This final collection has an independent set of size $2k$.

At this point we wish to determine $\alpha(G)$ for the union $G = G_1 \cup G_2 \cup G_3$. In the case that $X_{\sigma_1}^{(1)} = X_{\sigma_2}^{(2)} = 1$, we can take the MIS of $G_1$ associated with $X_{\sigma_1}^{(1)}$ and the MIS of $G_2$ associated with $X_{\sigma_2}^{(2)}$ along with all the balls in $G_3$ to form an MIS of $G$ of size $5k$. However, if $X_{\sigma_1}^{(1)} = X_{\sigma_2}^{(2)} = 0$, then choosing any ball from $G_1$ (other than the left-most $\sigma_1 - 1$ balls in each row) excludes every ball in a column of $G_2$ and a ball from $G_3$. Similar exclusions occur between the other pairs of collections. The result is that the best we can do is to choose an MIS from $G_1$ corresponding to an index smaller than $\sigma_1$ along with 1 ball from each of $G_2$ and $G_3$ in the rightmost column, giving a total of $\alpha(G) = 2k + 2$.

This shows that a streaming algorithm achieving an approximation factor better than $\frac{5k}{2k+2}$ must use $\Omega(n)$ space. This holds for any constant $k$ (just take $n$ large enough to allow that $k$), giving the result.

![Figure 9](image)

(a) Balls are excluded or included based on corresponding $X_i^{(1)}$.
(b) First party adds 2 rows each made of $k$ consecutive copies of figure 9a.
(c) Similarly, choose balls based on $X^{(2)}$.
(d) Second party adds copies of figure 9c to fit in between balls for $X_{\sigma_1}^{(1)}$ (other balls from the first party omitted).
(e) Final party adds $2k$ balls to fit in between balls for $X_{\sigma_1}^{(1)}$ and $X_{\sigma_2}^{(2)}$ (other balls from previous parties omitted).

Figure 9: Example for theorem 13 with $X^{(1)} = (1, 1, 0, 1)$, $X^{(2)} = (1, 0, 0, 1)$, $\sigma_1 = 2$, $\sigma_2 = 4$, and $k = 2$.

If we are allowed a combination of large and small balls, we can slightly improve the lower bound up to the maximum possible for a 3-party construction.

**Theorem 14.** Achieving a $(3 - \epsilon)$-approximation of $\alpha(G)$, with constant probability of success, on an $l_1$ or $l_\infty$ ball stream with $d = 2$ and arbitrary $\Delta$ requires $\Omega(n)$ space for any $\epsilon > 0$. 17
Proof. We adapt the construction from Theorem 13 as follows: the first party inserts $k$ rows stacked on top of each other, rather than 2, the second party inserts copies of its columns between every consecutive pair of rows from the first party, and the third party places smaller balls in between the columns of party 2 such that they are independent of the squares corresponding to the answer bit but overlap the squares either side.

This results in a gap of $k^2 + k + 1$ to $3k^2$ for the two cases, giving the result. \qed

5 Conclusion

We have addressed the complexity of Maximal and Maximum Independent Set (and various relaxations and related problems) under three natural models of graph streams: edge-arrival, explicit vertex arrival, and implicit vertex arrival. The central problems of maximal and maximum independent set separate the models: maximal independent set has high space complexity in the edge-arrival model but an easy greedy algorithm in the vertex models; however, MIS has high space complexity for explicit vertex arrival (and hence also the harder edge arrival model), while there are small-space constant factor approximations for some cases of implicit vertex arrival.

A natural extension is to consider weighted versions of the MIS problem, where each vertex is assigned a weight and the aim is to maximize the weight of the independent set. Our upper bounds generalize naturally to this case. Consider, for example, the two-dimensional geometric case ($l_1$ or $l_{\infty}$). A naive first approach is to round all weights to the nearest power of epsilon, and solve each weight class separately, before combining the results. An improved result follows by observing that for each “strip” in the decomposition of Theorem 13 we only need to track at most $\tilde{O}(\epsilon^{-1})$ weight classes: if a given strip contains a ball of weight $w$ then we can ignore any intervals of weight less than $\epsilon w$ without affecting the quality of the solution by more than a $(1+\epsilon)$ factor. Thus we obtain a $(3+\epsilon)$-approximate weighted MIS for the unit square case using only $\tilde{O}(\log W + \alpha(G))$. It remains to consider the complexity of other weighted variants of MIS problems.

There are a number of other natural open questions that follow from our study:

- Is there a multi-pass lower bound for maximal independent set in edge streams?

- Are there $o(\alpha(G))$ space algorithms for achieving constant factor approximations to $\alpha(G)$ for classes of geometric intersection graphs given as explicit vertex streams?

- Can we close the gap between the upper and lower bounds for approximating MIS in a unit square stream?

- Is there a constant factor approximation algorithm for MIS on streams of arbitrary sized squares?

References

[1] Kook Jin Ahn, Graham Cormode, Sudipto Guha, Andrew McGregor, and Anthony Wirth. “Correlation Clustering in Data Streams”. In: Proceedings of the 32Nd International Conference on International Conference on Machine Learning - Volume 37. ICML’15. Lille, France: JMLR.org, 2015, pp. 2237–2246. URL: http://dl.acm.org/citation.cfm?id=3045118.3045356

[2] Noga Alon, Ankur Moitra, and Benny Sudakov. “Nearly Complete Graphs Decomposable into Large Induced Matchings and Their Applications”. In: Proceedings of the Forty-fourth Annual ACM Symposium on Theory of Computing. STOC ’12. New York, New York, USA: ACM, 2012, pp. 1079–1090. ISBN: 978-1-4503-1245-5. DOI: 10.1145/2213977.2214074 URL: http://doi.acm.org/10.1145/2213977.2214074
[3] Sepehr Assadi, Sanjeev Khanna, Yang Li, and Grigory Yaroslavtsev. “Maximum Matchings in Dynamic Graph Streams and the Simultaneous Communication Model”. In: Proceedings of the Twenty-seventh Annual ACM-SIAM Symposium on Discrete Algorithms. SODA ’16. Arlington, Virginia: Society for Industrial and Applied Mathematics, 2016, pp. 1345–1364. ISBN: 978-1-61997-33-1. URL: http://dl.acm.org/citation.cfm?id=2884435.2884528

[4] Béla Bollobás. “The chromatic number of random graphs”. In: Combinatorica 8.1 (1988), pp. 49–55.

[5] Vladimir Braverman, Zaoxing Liu, Tejasvam Singh, N. V. Vinodchandran, and Lin F. Yang. “New Bounds for the CLIQUE-GAP Problem Using Graph Decomposition Theory”. In: Algorithmica 80.2 (2018), pp. 652–667. ISSN: 1432-0541. DOI: 10.1007/s00453-017-0277-5 URL: https://doi.org/10.

[6] Sergio Cabello and Pablo Pérez-Lantero. “Interval selection in the streaming model”. In: Theoretical Computer Science 702 (2017), pp. 77–96.

[7] Amit Chakrabarti. “Lower bounds for multi-player pointer jumping”. In: Computational Complexity, 2007. CCC’07. Twenty-Second Annual IEEE Conference on. IEEE, 2007, pp. 33–45.

[8] Graham Cormode, Jacques Dark, and Christian Konrad. “Approximating the Caro-Wei Bound for Independent Sets in Graph Streams”. In: Combinatorial Optimization. Ed. by Jon Lee, Giovanni Rinaldi, and A. Ridha Mahjoub. Cham: Springer International Publishing, 2018, pp. 101–114. ISBN: 978-3-319-96151-4.

[9] Carsten Damm, Stasys Jukna, and Jiří Sgall. “Some bounds on multiparty communication complexity of pointer jumping”. In: computational complexity 7.2 (1998), pp. 109–127.

[10] Yuval Emek, Magnús M Halldórsson, and Adi Rosén. “Space-constrained interval selection”. In: ACM Transactions on Algorithms (TALG) 12.4 (2016), p. 51.

[11] Ashish Goel, Michael Kapralov, and Sanjeev Khanna. “On the communication and streaming complexity of maximum bipartite matching”. In: Proceedings of the Twenty-Third Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2012, Kyoto, Japan, January 17-19, 2012. 2012, pp. 468–485.

[12] Bjarni V Halldórsson, Magnús M Halldórsson, Elena Losievskaja, and Mario Szegedy. “Streaming algorithms for independent sets”. In: International Colloquium on Automata, Languages, and Programming. Springer. 2010, pp. 641–652.

[13] Magnús M. Halldórsson and Christian Konrad. “Computing Large Independent Sets in a Single Round”. In: Distrib. Comput. 31.1 (Feb. 2018), pp. 69–82. ISSN: 0178-2770. DOI: 10.1007/s00446-017-0298-y URL: https://doi.org/10.1007/s00446-017-0298-y

[14] Magnús M Halldórsson, Xiaoming Sun, Mario Szegedy, and Cheng Wang. “Streaming and communication complexity of clique approximation”. In: International Colloquium on Automata, Languages, and Programming. Springer. 2012, pp. 449–460.

[15] Daniel M Kane, Jelani Nelson, and David P Woodruff. “An optimal algorithm for the distinct elements problem”. In: Proceedings of the twenty-ninth ACM SIGMOD-SIGACT-SIGART symposium on Principles of database systems. ACM. 2010, pp. 41–52.

[16] R. M. Karp. “Reducibility among combinatorial problems”. In: Complexity of Computer Computations. Ed. by R. E. Miller and J. W. Thatcher. Plenum Press, 1972, pp. 85–103.

[17] Christian Konrad. “Maximum Matching in Turnstile Streams”. In: Algorithms - ESA 2015. Ed. by Nikhil Bansal and Irene Finocchi. Berlin, Heidelberg: Springer Berlin Heidelberg, 2015, pp. 840–852. ISBN: 978-3-662-48350-3.
[18] I. Kremer, N. Nisan, and D. Ron. “On Randomized One-round Communication Complexity”. In: *computational complexity* 8.1 (1999), pp. 21–49. ISSN: 1420-8954. DOI: 10.1007/s000370050018 URL: http://dx.doi.org/10.1007/s000370050018

[19] Andrew McGregor. “Graph Stream Algorithms: A Survey”. In: *SIGMOD Rec.* 43.1 (May 2014), pp. 9–20. ISSN: 0163-5808. DOI: 10.1145/2627692.2627694 URL: http://doi.acm.org/10.1145/2627692.2627694

[20] David Zuckerman. “Linear Degree Extractors and the Inapproximability of Max Clique and Chromatic Number”. In: *Theory of Computing* 3.1 (2007), pp. 103–128.