Bifurcation of heteroclinic orbits via an index theory

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Abstract

Heteroclinic orbits for one-parameter families of nonautonomous vectorfields appear in a very natural way in many physical applications. Inspired by a recent bifurcation result for homoclinic trajectories of nonautonomous vectorfield proved by author in [Pej08], we define a new $\mathbb{Z}_2$-index and we construct a index theory for heteroclinic orbits of nonautonomous vectorfield. We prove an index theorem, by showing that, under some standard transversality assumptions, the $\mathbb{Z}_2$-index is equal to the parity, a homotopy invariant for paths of Fredholm operators of index 0. As a direct consequence of the index theory developed in this paper, we get a new bifurcation result for heteroclinic orbits.

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Introduction

Index theory reveals is central role in many problems of Classical Mechanics like in the study of linear and spectral stability of periodic solutions of differential systems, in the investigations on the existence and multiplicity of elliptic closed characteristics on energy hypersurfaces, in bifurcation theory etc. One-parameter families of vectorfields appear very naturally by linearizing a 1D partial differential evolution equation along a special solution like, for instance, a travelling wave or a steady solitary wave solution, etc. In all of these cases, in fact, the parameter actually is represented by the spectral parameter.

In spite of the fact that in the Hamiltonian world many index theorems are available in the literature (cf. [HP17] and references therein), no results at all are known in the non-Hamiltonian case. The most striking difference between the Hamiltonian and non-Hamiltonian case is that in the former there is a natural homotopy invariant which encodes the topology of the solutions space; it is defined in terms

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of the fundamental solution of the Hamiltonian system known in literature as Maslov-type index (cf., for instance, [CLM94, RS93, LZ00] references therein) whilst in the latter case essentially no homotopy invariant were detected so far. Nevertheless, some decades ago a \( \mathbb{Z}_2 \)-homotopy invariant for paths of Fredholm operators of index 0 termed *parity* was defined in the non-Hamiltonian realm. (We refer the interested reader to [FP91a, FP91b, PR98]). In this respect, we have to mention that recently, authors in [LZ00] defined an integer-valued homotopy invariant for path of essentially hyperbolic operators by generalizing to this class of Fredholm operators the classical notion of spectral flow very well-known in the self-adjoint case. However, for this class of operators, no finite dimensional counterpart (like the Maslov-type index) as been discovered.

Inspired by the definition of the Evans function, one of the main purpose of this paper, is to construct a \( \mathbb{Z}_2 \)-homotopy invariant, in terms of the determinant of a path of matrices naturally associated to an ordered pair of paths of linear subspaces parametrized by a bounded interval. To this pair of paths we naturally associate some bundles (by pulling back the tautological bundle on the Grassmannian) and out of these trivial bundles we construct a new bundle on \( S^1 \) through the classical clutching procedure.

Our first main result Theorem 1 claims that, under suitable assumptions on the one-parameter family of nonautonomous vectorfields, the parity of the path of operators arising by linearizing the system along a solution, coincides with the \( \mathbb{Z}_2 \)-index constructed in terms of the invariant unstable and stable subspaces.

It is well-known that the parity plays a central in order to detect the bifurcation from the trivial branch (cf. [FP91a, FP91b, PR98, Pej08]). A similar role is played by the spectral flows in the Hamiltonian case. (For further details, we refer the interested reader to [FPR99, PPT04, PP05, MPP07, PW13, PW14b] and references therein). As direct consequence of the \( \mathbb{Z}_2 \) Index Theory we prove a sufficient condition for detecting the bifurcation along a trivial branch of heteroclinic orbits, in terms of the \( \mathbb{Z}_2 \)-index. We conclude by observing that the bifurcation result proved in Theorem 2 is completely different in the essence from the main result, recently proved by author in [Pej08] in which the bifurcation was related to a non-trivial twist of the asymptotic stable and unstable bundles at infinity.

## 1 Description of the problem and main results

Let \( g \) be the one-parameter nonautonomous vectorfield defined by

\[
g : [0, 1] \times \mathbb{R} \times \mathbb{R}^n \ni (\lambda, t, z) \mapsto g(\lambda, t, z) \in \mathbb{R}^n
\]

and we assume that \( g \) as well as \( D_\lambda g \) are bounded. Let \( z_-, z_+ \in \mathbb{R}^n \) be two zeroes of \( g \), meaning that \( g(\lambda, t, z_-) = g(\lambda, t, z_+) = 0 \) for every \( \lambda, t \in [0, 1] \times \mathbb{R} \) and let us consider the first order differential system

\[
\begin{align*}
z'(t) - g(\lambda, t, z(t)) &= 0, & t \in \mathbb{R} \\
\lim_{t \to -\infty} z(t) &= z_-, & \lim_{t \to +\infty} z(t) &= z_+.
\end{align*}
\]

We assume that \( z_\lambda \) is a solution of the system given in Equation (1.1). By linearising the vectorfield along \( z_\lambda \), we get the following linear one-parameter family of first order systems

\[
\begin{align*}
u'(t) - S_\lambda(t) u(t) &= 0, & t \in \mathbb{R} \\
\lim_{t \to +\infty} u(t) &= 0
\end{align*}
\]

where we set \( S_\lambda(t) := D_\lambda g(\lambda, t, z_\lambda(t)) \). We introduce the following assumptions.

(A1) The smooth family of matrices \( S : [0, 1] \times \mathbb{R} \to \text{Mat}(n, \mathbb{R}) \) such that \( S_\lambda := S(\lambda, \cdot) : \mathbb{R} \to \text{Mat}(n, \mathbb{R}) \) converges uniformly w.r.t. \( \lambda \) to families

\[
S_\lambda^+ = \lim_{t \to +\infty} S_\lambda(t), \quad S_\lambda^- = \lim_{t \to -\infty} S_\lambda(t), \quad \lambda \in [0, 1].
\]

We assume that both \( S_\lambda^+ \) and \( S_\lambda^- \) are hyperbolic, i.e. the spectrum does not lie to the imaginary axis; namely

\[
\text{sp}(S_\lambda^+) \cap i\mathbb{R} = \text{sp}(S_\lambda^-) \cap i\mathbb{R} = \emptyset.
\]
Remark 1.1. It is worth noticing that assumption (A1) implies that the families \( \lambda \mapsto S^\pm_\lambda \) and \( \lambda \mapsto S^-_\lambda \) are continuous.

Let \( \gamma(\lambda,\tau) : \mathbb{R} \to \text{Mat}(n, \mathbb{R}) \) be the linear flow of the system given in Equation (1.2), i.e. the (fundamental) matrix-valued solution of the linear initial value problem

\[
\begin{align*}
\gamma'_\lambda(t) &= S_\lambda(t) \gamma(\lambda,\tau)(t), \\
\gamma(\lambda,\tau)(\tau) &= I.
\end{align*}
\]

We recall that the stable and unstable subspaces of the linear system given in Equation (1.2) are

\[
E^s_\lambda(\tau) := \left\{ v \in \mathbb{R}^{2n} \mid \lim_{t \to +\infty} \gamma(\lambda,\tau)(t) v = 0 \right\} \quad \text{and} \quad E^u_\lambda(\tau) := \left\{ v \in \mathbb{R}^{2n} \mid \lim_{t \to -\infty} \gamma(\lambda,\tau)(t) v = 0 \right\}.
\]

By invoking [AM03, Proposition 1.2], Assumption (A1), implies the following uniform convergence result on the invariant manifolds

\[
\lim_{\tau \to +\infty} E^s_\lambda(\tau) = V^- (S^+_\lambda) \quad \text{and} \quad \lim_{\tau \to -\infty} E^u_\lambda(\tau) = V^+ (S^-_\lambda),
\]

where the convergence is meant in the (gap-metric) topology of the Grassmannian manifold; furthermore for any fixed \( \tau_0 \), \( \lambda \mapsto E^s_\lambda(\tau_0) \) and \( \lambda \mapsto E^u_\lambda(\tau_0) \) are continuous.

Notation 1.2. We denote by \( \cap \) the transversality between two linear subspaces meaning that their sum is the whole space. In symbols, if \( V, W \subset \mathbb{R}^n \), \( V \cap W \) means that \( V + W = \mathbb{R}^n \).

Under the non-degeneracy assumption given in (A2), we get that

\[
E^s_1(0) \cap E^u_1(0) \cap E^s_0(0).
\]

In particular the \( \mathbb{Z}_2 \)-index \( \iota(E^s_1(0), E^u_1(0); \lambda \in [0, 1]) \) (cf. Definition 2.1), is well-defined.

We set \( \mathcal{H} := W^{1,2}(\mathbb{R}, \mathbb{R}^n) \) and \( \mathcal{H} := L^2(\mathbb{R}, \mathbb{R}^n) \). Under the assumptions (A1) and (A3) it is well-known (cf. [Pej08, Proposition 3.1]) that for each \( \lambda \in [0, 1] \), the operator

\[
A_\lambda = \frac{d}{dt} - S_\lambda : \mathcal{H} \mapsto \mathcal{H}
\]

is Fredholm of index 0. Moreover, by assumption (A2), the operators \( A_0, A_1 \) are both invertible. Thus it remains well-defined a homotopy invariant known in literature as parity, namely \( \sigma(A_\lambda, \lambda \in [0, 1]) \in \tilde{KO}([0, 1], \partial([0, 1])) \cong \mathbb{Z}_2 \), where \( \tilde{KO} \) denotes the reduced Grothendieck group. We refer to [FP91a, FP91b] and references therein, for further details. The first main result of the present paper is a sort of (mod 2) spectral flow formula.

**Theorem 1. (A \( \mathbb{Z}_2 \)-index theorem)** Under the assumptions (A1)-(A2) and (A3), the following equality holds

\[
\sigma(A_\lambda, \lambda \in [0, 1]) = \iota(E^s_1(0), E^u_1(0); \lambda \in [0, 1]).
\]

\(^1\) We recall that \( T \in \text{Mat}(n, \mathbb{R}) \) is termed hyperbolic if it has no eigenvalues on the imaginary axis. Thus in this case the spectrum of a hyperbolic operator \( T \) consists of two isolated closed components (one of which may be empty)

\[
\sigma(T) \cap \{ z \in \mathbb{C} \mid \Re(z) < 0 \} \quad \text{and} \quad \sigma(T) \cap \{ z \in \mathbb{C} \mid \Re(z) > 0 \}.
\]
A direct application of Theorem 1 is in bifurcation theory since its non-triviality is sufficient in order to detect bifurcation. It is well-known, in fact, that (cf. [PR98, Theorem 6.1]), that the nontrivial parity implies bifurcation from the trivial branch. More precisely, a point $\lambda_+ \in [0, 1]$ is a bifurcation point for heteroclinic solutions of the system given in Equation (1.1) from the trivial branch $\lambda \mapsto z_\lambda$, if there exists a sequence $(\lambda_k, z_k)_{k \in \mathbb{N}} \subset [0, 1] \times \mathbb{W}$, where $z_k$ are solutions of Equation (1.1), $z_k \neq z_\lambda$ and $(\lambda_k, z_k) \mapsto (\lambda_+, z_\lambda)$.

**Theorem 2.** Let $z_\lambda$ be a family of (heteroclinic) solutions of the system given in Equation (1.1) where the restpoints $z_-$ and $z_+$ are hyperbolic. We assume that $\iota(E^s_\lambda(0), E^u_\lambda(0); \lambda \in [0, 1]) \neq 0$. Then, for all $\epsilon > 0$ sufficiently small there exists a (nontrivial) solution $(\lambda, z)$ of the system given in Equation (1.1) such that $\|z - z_\lambda\|_{W^{1,2}} = \epsilon$.

For $\lambda = 0, 1$ the systems are termed **boundary non degenerate**, if the following transversality condition holds

$$V^-(S^+_0) \pitchfork V^+(S^-_0) \text{ and } V^-(S^+_1) \pitchfork V^+(S^-_1).$$

In this case it is possible to associate to the heteroclinic orbit $z_0$ (resp. $z_1$) the $\mathbb{Z}_2$-index, termed **geometrical parity**, $t_{\text{geo}}(z_0)$ that counts mod 2 the number of nontrivial intersections between the path of stable and unstable subspaces parametrized by $\tau \in \mathbb{R}$. (We refer the reader to Definition 2.12 and Definition 2.14 for the rigorous statements). As direct consequence of the homotopy invariance of this index, we immediately get

$$\iota(E^s_\lambda(0), E^u_\lambda(0); \lambda \in [0, 1]) = t_{\text{geo}}(z_0) + t_{\text{geo}}(z_1) + \iota(V^-(S^+_\lambda), V^+(S^-_\lambda); \lambda \in [0, 1]) \pmod{2}$$

In the special case in which both paths $\lambda \mapsto V^-(S^+_\lambda)$ and $\lambda \mapsto V^+(S^-_\lambda)$ are independent on $\lambda$, then $\iota(V^-(S^+_\lambda), V^+(S^-_\lambda); \lambda \in [0, 1]) = 0$. Thus in this case the bifurcation is detected by the following condition

$$t_{\text{geo}}(z_0) \neq t_{\text{geo}}(z_1) \pmod{2}.$$

### 2 A new index for heteroclinic orbits and the geometrical parity

The aim of this Section is to define the $\mathbb{Z}_2$-index, the **geometrical parity** of a non-degenerate heteroclinic orbit as well as to listen their basic properties.

We start by briefly recalling some basic facts about the Grassmannian and to fix our notations. (For all of this we refer the interested reader to the beautiful book [MS78, Chapter 5]). We denote by $Gr_k(n, \mathbb{R})$ the set of all $k$-dimensional linear subspaces of $\mathbb{R}^n$. As homogeneous space, it is well-known that

$$Gr_k(n, \mathbb{R}) = O(n)/\{O(k) \times O(n-k)\}$$

where $O$ denotes the orthogonal group. In particular, $Gr_k(n, \mathbb{R})$ is a $k(n-k)$-dimensional compact smooth manifold (in general it has the structure of a smooth algebraic variety) whose topology is induced by the gap-metric

$$d(V, W) := \|P_V - P_W\|$$

where $P_V$ and $P_W$ denote the orthogonal projections in $\mathbb{R}^n$ onto the subspaces $V$ and $W$, respectively.

A $k$-frame in $\mathbb{R}^n$ is a $k$-tuple of linearly independent vectors of $\mathbb{R}^n$ and the collections of all $k$-frames form an open subset of the $k$-fold Cartesian product of $\mathbb{R}^n$ called the **Stiefel manifold** and denoted by $V_k(n, \mathbb{R})$. There is a canonical function $\gamma : V_k(n, \mathbb{R}) \rightarrow Gr_k(n, \mathbb{R})$ which maps each $k$-frame to the $k$-dimensional linear subspace it spans. By [MS78, Lemma 5.1, pag.31-33] we also get that the correspondence $X \mapsto X^{-\perp}$ which assigns to each $k$-dimensional linear subspace the $(n-k)$-dimensional linear orthogonal subspace, defines a homeomorphism between the $Gr_k(n, \mathbb{R})$ and $Gr_{n-k}(n, \mathbb{R})$. We denote by $\gamma^k(n, \mathbb{R})$ the **tautological line bundle (or universal bundle)** over the Grassmannian manifold $Gr_k(n, \mathbb{R})$. Let

\(^{4}\text{It is worth noticing that in the case of homoclinic orbit this condition is equivalent to the hyperbolicity of the equilibrium point.}\)
$V \in \mathcal{C}^0([a, b], \text{Gr}_k(n, \mathbb{R}))$ and $W \in \mathcal{C}^0([a, b], \text{Gr}_{n-k}(n, \mathbb{R}))$, and we assume the following transversality condition at the endpoints

(2.1) $V(a) \cap W(a)$ and $V(b) \cap W(b)$.

For every $t \in [a, b]$, let $E_V(t) := \{v_1(t), \ldots, v_k(t)\}$ and $E_W(t) := \{w_1(t), \ldots, w_{n-k}(t)\}$ be two frames generating $V(t)$ and $W(t)$ respectively. We consider $M \in \mathcal{C}^0([a, b], \text{Mat}(n, \mathbb{R}))$ whose columns are given by $v_j$ and $w_i$; i.e.,

$$M(t) := \begin{bmatrix} v_1(t) & \cdots & v_k(t) \\ \vdots & \ddots & \vdots \\ v_1(t) & \cdots & v_k(t) \end{bmatrix}.$$ 

By the transversality assumption given in Equation (2.1), it readily follows that the endpoints of the path $M$, namely $M(a), M(b)$ are nondegenerate matrices (in the sense that the determinant of $M(a)$ and $M(b)$ do not vanish). Thus we are entitled to introduce the definition of the $Z_2$-index.

**Definition 2.1.** We term $Z_2$-index of the pair $V$ and $W$, the integer

$$\iota(V(t), W(t); t \in [a, b]) := \begin{cases} 0 & \text{if } \det (M(a) \cdot M(b)) > 0 \\ 1 & \text{if } \det (M(a) \cdot M(b)) < 0 \end{cases}.$$ 

**Lemma 2.2.** The $Z_2$-index given in Definition 2.1 is well-posed.

**Proof.** By a straightforward calculation, it readily follows that this definition is independent on the choice of the frames. Let $E_V$ and $E_W$ be two continuous frames for $V$ and $W$ respectively, pointwise given by $E_V(t) := \{\hat{v}_1(t), \ldots, \hat{v}_k(t)\}$ and $E_W(t) := \{\hat{w}_1(t), \ldots, \hat{w}_{n-k}(t)\}$ and let us define the continuous path of matrices

$$\hat{M}(t) := \begin{bmatrix} \hat{v}_1(t) & \cdots & \hat{v}_k(t) \\ \vdots & \ddots & \vdots \\ \hat{v}_1(t) & \cdots & \hat{v}_k(t) \end{bmatrix}.$$ 

Thus, for every $t \in [a, b]$, there exists $G_1(t) \in \text{GL}(k)$ and $G_2(t) \in \text{GL}(n-k)$ such that $\hat{M}(t) = M(t)G(t)$ for $G(t) := \text{diag}(G_1(t), G_2(t))$. In particular, $G(t)$ is nondegenerate for every $t \in [a, b]$ and $\text{sgn} \big( \det(G(t)) \big)$ is independent on $t$. Thus

$$\text{sgn} \big( \det (\hat{M}(a) \cdot \hat{M}(b)) \big) = \text{sgn} \big( \det (M(a) \cdot M(b)) \big) \text{sgn} \big( \det (G(a) \cdot G(b)) \big) = \text{sgn} \big( \det (M(a) \cdot M(b)) \big).$$ 

This conclude the proof. □

We now list some properties (omitting the proofs) of the $Z_2$-index which are straightforward consequences of Definition 2.1.

**Properties of the $Z_2$-index**

**Property I. (Reparametrisation Invariance)** Let $\psi : [c, d] \to [a, b]$ be a continuous function such that $\psi(c) = a$ and $\psi(d) = b$, or $\psi(c) = b$ and $\psi(d) = a$. Then

$$\iota(V(t), W(t); t \in [a, b]) = \iota((V \circ \psi)(t), (W \circ \psi)(t); t \in [a, b]).$$

**Property II. (Homotopy invariance Relative to the Ends)** Let

$$[0, 1] \times [a, b] \ni (s, t) \mapsto (V_s(t), W_s(t)) \in \text{Gr}_k(n, \mathbb{R}) \oplus \text{Gr}_{n-k}(n, \mathbb{R})$$

be a continuous two-parameter family subspaces such that $V_s(a) \cap W_s(a)$ and $V_s(b) \cap W_s(b)$. Then

$$\iota(V_0(t), W_0(t); t \in [a, b]) = \iota(V_1(t), W_1(t); t \in [a, b]).$$


\textbf{Property III. (Path Additivity)} If \( c \in (a, b) \) such that \( V(c) \cap W(c) \), then
\[
\iota(V(t), W(t); t \in [a, b]) \equiv \iota(V(t), W(t); t \in [a, c]) + \iota(V(t), W(t); t \in [c, b]) \pmod{2}
\]

\textbf{Property IV. (Symmetry property)}
\[
\iota(V(t), W(t); t \in [a, b]) = \iota(W(t), V(t); t \in [a, b]).
\]

\textbf{Property V. (Sum Additivity)} For \( i = 1, 2 \), let \( (V_i, W_i) \in \text{Gr}_{k_i}(n_i, \mathbb{R}) \oplus \text{Gr}_{n_i - k_i}(n_i, \mathbb{R}) \). Then
\[
\iota((V_1 \oplus V_2)(t), (W_1 \oplus W_2)(t); t \in [a, b]) \equiv \iota(V_1(t), W_1(t); t \in [a, b])
\]
\[
+ \iota(V_2(t), W_2(t); t \in [a, b]) \pmod{2}
\]

\textbf{Remark 2.3.} We remark that the \( \mathbb{Z}_2 \)-index defined above actually depends on the whole path and not just on its endpoints.

\textbf{Notation 2.4.} We denote by \( \mathcal{P}([a, b], k, n) \) (resp. by \( \mathcal{P}^*([a, b], k, n) \)) the space of all ordered pairs of continuous paths of subspaces (resp. with transversal ends)
\[
\mathcal{P}([a, b], k, n) = \{ Z \in \mathcal{B}^o([a, b], \text{Gr}_k(n, \mathbb{R}) \times \text{Gr}_{n-k}(n, \mathbb{R})) \mid Z(t) = (V(t), W(t)) \}
\]
and we let
\[
\mathcal{P}^*([a, b], k, n) = \{ Z \in \mathcal{P}([a, b], k, n) \mid V(a) \cap W(a) \text{ and } V(b) \cap W(b) \}.
\]

\subsection{2.1 A Vector Bundle over the circle}

The aim of this subsection is to construct a vector bundle over \( S^1 \) whose triviality is determined by the vanishing of the \( \mathbb{Z}_2 \)-index. Given the ordered pair of subspace \( (V, W) \in \mathcal{P}^*([0, 1], k, n) \), we define the path \( \mathcal{W} : [0, 2] \to \text{Gr}_k(n, \mathbb{R}) \) as follows
\[
\mathcal{W}(t) = \begin{cases} W(t) & \text{for } t \in [0, 1], \\ W(2 - t) & \text{for } t \in [1, 2]. \end{cases}
\]

\textbf{Remark 2.5.} Actually the path \( \mathcal{W} \) on the interval \([1, 2]\) geometrically coincides with \( W \) on the interval \([0, 1]\) but travelled in the opposite direction.

\textbf{Lemma 2.6.} There exists a continuous path \( \tilde{V} : [0, 2] \to \text{Gr}_k(n, \mathbb{R}) \) such that
\begin{enumerate}
\item \( \tilde{V}|_{[0, 1]} = V \).
\item \( \tilde{V} \) is closed, namely \( \tilde{V}(0) = \tilde{V}(2) \).
\item For every \( t \in [0, 1] \) the following transversality condition holds
\[
\tilde{V}(t) \cap \mathcal{W}(t) \quad t \in [1, 2].
\]
\end{enumerate}

\textbf{Proof.} We start to define the path \( \tilde{V} : [0, 2] \to \text{Gr}_k(n, \mathbb{R}) 
\]
\[
\tilde{V}(t) = \begin{cases} V(t) & \text{for } t \in [0, 1], \\ \tilde{V}(t) & \text{for } t \in [1, 2]. \end{cases}
\]

where \( \tilde{V} \) is such that \( \tilde{V}(1) = V(1) \) and \( \tilde{V}(2) = V(0) \). Clearly the path \( \tilde{V} \) given in Equation (2.2) is closed, being \( \tilde{V}(0) = V(0) \) and \( \tilde{V}(2) = V(2) = V(0) \). From the definition, it holds also that \( \tilde{V}|_{[0, 1]} = V \); furthermore it is easy to check that \( \tilde{V}(1) \cap \mathcal{W}(1) \) and \( \tilde{V}(2) \cap \mathcal{W}(2) \). These last two facts readily follows from the definitions of \( \tilde{V} \) and \( \mathcal{W} \) and from the fact that the ends of the two paths \( V \) and \( W \) are transversal.
Now, if \( \tilde{V}(t) \cap W(t) \) for all \( t \in [1, 2] \) the result follows. If not, it just enough to observe that the path \( \tilde{W}^⊥ : [1, 2] \to \text{Gr}_k(n, R) \) pointwise given by the orthogonal complement to \( W \) clearly satisfies the following transversality condition
\[
\tilde{W}^⊥(t) \cap W(t) \quad t \in [1, 2].
\]
Thus, if \( \tilde{V}(1) = \tilde{W}^⊥(1) \) and \( \tilde{V}(2) = \tilde{W}^⊥(2) \), it is just enough to define \( \tilde{V}(t) = \tilde{W}^⊥(2 - t) \) for all \( t \in [1, 2] \). Otherwise, we reduce to the previous situation as follows. For \( \epsilon \in (0, 1) \), let us consider the continuous path
\[
\tilde{V}(t) := \begin{cases} 
\tilde{W}^⊥(1 + \epsilon) \quad \text{for all } t \in [1, 1 + \epsilon] \\
\tilde{W}^⊥(2 - t) \quad \text{for all } t \in [1 + \epsilon, 2 - \epsilon] \\
\tilde{W}^⊥(2 - \epsilon) \quad \text{for all } t \in [2 - \epsilon, 2]. 
\end{cases}
\]
By choosing \( \epsilon > 0 \) sufficiently small and observing that the transversality is an open condition, the result readily follows. This conclude the proof. \( \square \)

By passing to the quotient of \([0, 2]\) with respect to its boundary, the function \( \tilde{V} : [0, 2] \to \text{Gr}_k(n, R) \) induces a map, that with a slight abuse of notation, we still will denote by the same symbol, \( \tilde{V} : S \to \text{Gr}_k(n, R) \) for \( S := R/(2Z) \). Let \( \pi : E(\gamma^k(n, R)) \to \text{Gr}_k(n, R) \) denote the (standard) tautological bundle projection onto its first factor. In shorthand notation we set \( E(\gamma^k) := E(\gamma^k(n, R)) \). We now consider the pull-back bundle of \( E(\gamma^k) \) through \( \tilde{V} \); thus we have the following commutative diagram
\[
\begin{array}{ccc}
\tilde{V}^* E(\gamma^k) & \xrightarrow{\mu} & E(\gamma^k) \\
\tilde{V}^* (\pi) & \xrightarrow{\pi} & \text{Gr}_k(n, R) \\
S & \xrightarrow{\tilde{V}} & \text{Gr}_k(n, R)
\end{array}
\]
where as usually, \( \tilde{V}^* (\pi) \) denotes the pull-back projection \( \pi \) through \( \tilde{V} \). The next result gives a necessary and sufficient condition on the triviality of the pull-back of the tautological bundle induced by \( \tilde{V} \) in terms of the triviality of the \( \mathbb{Z}_2 \)-index.

**Lemma 2.7.** The vector bundle constructed by pulling back the tautological bundle through \( \tilde{V} \) is trivial if only if
\[
\iota (V, W; t \in [0, 1]) = 0.
\]

**Proof.** We start to observe that as direct consequence of third property stated in Lemma 2.6 as well as by the homotopy invariance of the \( \mathbb{Z}_2 \)-index with respect to its ends, we get that
\[
\iota (V, W; t \in [0, 1]) = \iota \left( \tilde{V}, \tilde{W}; t \in [0, 2] \right). \tag{2.3}
\]
We define the constant path \( \tilde{W} : [0, 2] \to \text{Gr}_k(n, R) \) as follows \( \tilde{W}(t) \equiv W(0) \) and again as consequence of the homotopy invariance property of the \( \mathbb{Z}_2 \)-index, we have
\[
\iota \left( \tilde{V}, \tilde{W}; t \in [0, 2] \right) = \iota \left( \tilde{V}, \tilde{W}; t \in [0, 2] \right). \tag{2.4}
\]
We consider the \( k \)-frame \( \mathcal{F}_V(t) = \{e_1(t), \ldots, e_k(t)\} \) for the subspace \( V(t) \) and the (constant) \( (n - k) \)-frame \( \mathcal{F}_W(0) = \{e_{k+1}, \ldots, e_n\} \) for \( W(0) \). As before, we define the \( n \times n \) matrix
\[
M(t) := \begin{bmatrix} e_1(t) & \cdots & e_k(t) & e_{k+1} & \cdots & e_n \end{bmatrix}.
\]
It is immediate to observe that the vector bundle over \( S \), namely \( \tilde{V}^* (\pi) : \tilde{V}^* E(\gamma^k) \to S \) is trivial if and only if \( \det (M(0) \cdot M(2)) > 0 \), which is equivalent to state that \( \iota \left( \tilde{V}, \tilde{W}; t \in [0, 2] \right) = 0 \). Now, the conclusion readily follows by invoking Equations (2.3)-(2.4). \( \square \)
2.2 A new index for heteroclinic orbits of nonautonomous vector fields

This subsection is to define a $\mathbb{Z}_2$-index in the case of heteroclinic orbits of a one-parameter family of nonautonomous systems. We start by setting $n = 2k$ and to consider the symplectic real vector space $(\mathbb{R}^{2k}, \omega)$ where $\omega$ is the standard symplectic form. We denote by $\Lambda(k)$ the Lagrangian Grassmannian manifold, namely the set of all Lagrangian subspaces of $(\mathbb{R}^{2k}, \omega)$. It is well-known that it is a real compact and connected analytic $\frac{1}{2}k(k + 1)$-dimensional submanifold of the Grassmannian manifold $\text{Gr}_k(2k, \mathbb{R})$.

**Notation 2.8.** We denote by $\mathcal{LP}([a, b], k)$ the space of all ordered pairs of continuous paths of Lagrangian subspaces

$$\mathcal{LP}([a, b], k) := \{ Z \in \mathcal{C}^0([a, b], \Lambda(k) \times \Lambda(k)) | Z(t) = (V(t), W(t)) \}$$

and we let

$$\mathcal{LP}^\ast([a, b], k) := \{ Z \in \mathcal{LP}([a, b], k) | V(a) \cap W(a) \text{ and } V(b) \cap W(b) \}.$$

To each pair ordered path of Lagrangian subspaces

$$(V, W) : [a, b] \ni t \mapsto (V(t), W(t)) \in \Lambda(k),$$

we associate a homotopy invariant known in literature as Maslov index, that will be denoted by

$$\iota^{\text{CLM}}(V(t), W(t); t \in [a, b]).$$

(We refer the interested reader to the beautiful papers [CLM94, RS93].)

**Proposition 2.9.** Let $(V, W) \in \mathcal{LP}^\ast([a, b], k)$. Then we have

$$\iota^\ast(V(t), W(t); t \in [a, b]) \equiv \iota^{\text{CLM}}(V(t), W(t); t \in [a, b]) \pmod{2}.$$

**Proof.** Following authors in [CLM94, Section 4], up to a slight perturbation, we can assume that the crossing instants (intersections) between the paths $V$ and $W$ are simply, meaning that they are $1$-dimensional and transversal. By codimensional arguments, this is generically true, and by the invariance property of the index $\iota^{\text{CLM}}$ (with free endpoints in the case of transversal ends), is actually independent on this choice. We assume that $t_0 \in (a, b)$ is a crossing instant such that $\dim (V(t_0) \cap W(t_0)) = 1$. There exists $\varepsilon > 0$ such that for $t \in [t_0 - \varepsilon, t_0 + \varepsilon)$, $V(t) = \text{Gr}(A(t))$ and $W(t) = \text{Gr}(B(t))$ where $A$ and $B$ are smooth path of symmetric matrices (generating the Lagrangian subspaces). In this case, the local contribution to the $\iota^{\text{CLM}}$ as well as to $\iota$ coincide. By a direct computation it follows that the crossing form at the crossing instant $t_0$ is given by

$$\Gamma(W, V, t_0) : \ker (B(t_0) - A(t_0)) \longrightarrow \mathbb{R} : u \mapsto \Gamma(W, V, t_0)[u] = ([\hat{B}(t_0) - \hat{A}(t_0)]u, u).$$

By invoking Kato’s selection theorem, $B(t) \cong \text{diag} (\lambda_1(t), \ldots, \lambda_k(t))$ where, for $j = 1, \ldots, k$, $\lambda_j(t)$ represent the repeated eigenvalues of $B(t) - A(t)$ according to its own multiplicity.

Since $\dim \ker (B(t) - A(t)) = 1$, there exists only one changing-sign eigenvalue at $t_0$; let’s say $\lambda_1$. Thus, we get that

$$\sgn \Gamma(W, V, t_0) = \begin{cases} 1 & \text{if } \hat{\lambda}_1(t_0) > 0, \\ -1 & \text{if } \hat{\lambda}_1(t_0) < 0. \end{cases}$$

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To conclude the proof we now define the matrix

\[ M(t) := \begin{bmatrix} I & I \\ A(t) & B(t) \end{bmatrix} \]

having nullity (i.e. dimension of the kernel) precisely 1 and let \( C \) be the block upper triangular matrix defined by \( C := \begin{bmatrix} I & -I \\ 0 & I \end{bmatrix} \). We observe

\[ M(t) \cdot C := \begin{bmatrix} I & 0 \\ A(t) & B(t) - A(t) \end{bmatrix} ; \quad \text{thus we get} \]

\[ \det (M(t)C) = \det (M(t)) = \det (B(t) - A(t)). \]

In particular, \( \ker M(t_0) = \ker (B(t_0) - A(t_0)) \) (which is 1-dimensional). By this arguments and by taking into account Definition 2.1, we get that

\[ \iota (V(t), W(t); t \in [t_0 - \varepsilon, t_0 + \varepsilon]) = 1 \]

This conclude the proof.

From now on, we assume that \([a, b]\) is an unbounded interval (either \(a = -\infty\) or \(b = +\infty\)). Thus we have the following three kind of unbounded intervals, namely \((-\infty, b],[a, +\infty)\) and finally \((-\infty, +\infty)\). We assume that there exists \(T > 0\) such that \(V(t) \cap W(t)\) for every \(t \leq -T, t \geq T\) and finally \(|t| \geq T\), in the first, second and finally in the third case respectively.

**Definition 2.10.** Under the previous notation, we define the \(Z_2\)-index as follows:

\[ \iota (V(t), W(t); t \in (-\infty, b]) := \iota (V(t), W(t); t \in [-T, b]) \]

\[ \iota (V(t), W(t); t \in [a, +\infty)) := \iota (V(t), W(t); t \in [a, T]) \]

\[ \iota (V(t), W(t); t \in (-\infty, +\infty)) := \iota (V(t), W(t); t \in [-T, T]). \]

**Remark 2.11.** Directly by the definition and by the path additivity property of the \(Z_2\)-index, it readily follows that the it is well-defined in the sense that it is independent on \(T\).

Let \(S : \mathbb{R} \to \text{Mat}(n, \mathbb{R})\) be a a continuous path of matrices and we assume that there exist \(S^\pm\) which are hyperbolic and such that

\[ \lim_{t \to \pm\infty} S(t) = S^\pm \]

For \(\tau \in \mathbb{R}\), we let \(\gamma_\tau : \mathbb{R} \to \text{Mat}(n, \mathbb{R})\) be the associated matrix-valued solution such that \(\gamma_\tau (\tau) = I\), and we denote by \(E^s(\tau)\) and \(E^u(\tau)\) respectively the stable and unstable subspace. By invoking [AM03, Proposition 2.1], we immediately get the following convergence result

\[ \lim_{\tau \to +\infty} E^s(\tau) = V^-(S^+) \quad \text{and} \quad \lim_{\tau \to -\infty} E^u(\tau) = V^+(S^-). \]

**Definition 2.12.** Under the previous notation, we assume the following transversality condition is fulfilled

\[ V^+(S^-) \cap V^-(S^+) \quad \text{and} \quad E^s(0) \cap E^u(0). \]

We define the \(Z_2\)-index of the path \(S\), as follows

\[ \iota(S) := \iota (E^s(t), E^u(-t); t \in [0, +\infty)). \]

**Remark 2.13.** We observe that by taking into account the convergence stated in Equation (2.5) as well as Definition 2.10, the index given in Definition 2.12 is well-defined.

Thus we are entitle to introduce the following definition.

**Definition 2.14.** Let \(z^-\) and \(z^+\) two hyperbolic restpoints. We term geometrical parity of the heteroclinic orbit \(x\) connecting them, the \(Z_2\)-index of the linear path \(S\) arising by linearizing the nonautonomous vectorfield along \(x\); thus in symbol

\[ \iota_{\text{geo}} (x) := \iota(S) \]

where \(\iota(S)\) is given in Definition 2.12.
In the special case in which the system is Hamiltonian, as direct consequence of Proposition 2.9 as well as Definition 2.10, Definition 2.12 and finally Definition 2.14, we get the following result.

**Corollary 2.15.** Let \( x \) be heteroclinic solution of the nonautonomous Hamiltonian vectorfield between the hyperbolic restpoints \( z_- \) and \( z_+ \). Thus, we have

\[
\iota(x) = \iota^{\text{CLM}}(x) \pmod{2}.
\]

By the homotopy invariance of the \( \mathbb{Z}_2 \)-index, we get the following result. (Cf. [HP17], for further details).

**Proposition 2.16.** Let us consider the system given in Equation (1.2) and we assume conditions (A1)-(A2)-(A3). If

\[
V^-(S_0^+) \triangleright V^+(S_0^-) \text{ and } V^-(S_1^+) \triangleright V^+(S_1^-)
\]

then, we have

\[
\iota(E_x^*(0), E_y^*(0); \lambda \in [0, 1]) \equiv \iota_{\text{geo}}(S_0) + \iota_{\text{geo}}(S_1) + \iota\left(V^-(S_x^-); V^+(S_y^-)\right) \lambda \in [0, 1] \pmod{2}.
\]

### 3 Parity for path of Fredholm operator and the Index theorem

In this section we introduce the other last main ingredient of the \( \mathbb{Z}_2 \)-index prove Theorem 1 and Theorem 2.

Let \( X, Y \) be two real and separable Hilbert spaces and \( \mathcal{F}_0(X, Y) \) be denote the set of all Fredholm operators of index 0. Given a continuous path \( T : [0, 1] \to \mathcal{F}_0(X, Y) \) having invertible endpoints, there is a homotopy invariant of \( T \) termed parity of \( T \) and denoted by \( \sigma(T(\lambda), \lambda \in [0, 1]) \) which is an element of the (reduced real Grothendieck group), \( \widetilde{K_0} \); in symbols

\[
\sigma(T(\lambda), \lambda \in [0, 1]) \in \widetilde{K_0}([0, 1], \partial([0, 1])) \cong \mathbb{Z}_2.
\]

In a geometrical fashion, the parity can be generically seen as a mod 2 intersection index between a continuous path in \( \mathcal{F}_0(X, Y) \) and the one-codimensional submanifold of all degenerate operators. (For further details, we refer the interested reader to [FP91b, Section 3]). As was proved by authors in [FP91a, Section 2], if \( T : P \to \mathcal{F}_0(X, Y) \) is a continuous family of linear Fredholm operators of index 0 parametrised by \( P \), the parity is equivalent to the nonorientability of the index bundle (actually an equivalence class of vector bundles) of the path \( T \), namely \( \text{Ind}(T) \in K_0(P) \) and in particular it is measured by \( w_1(\text{Ind}(T)) \in H^1(P, \mathbb{Z}_2) \) the first Stiefel-Whitney class of \( \text{Ind}(T) \).

For the sake of the reader, we explain what the parity of \( T \) means, in the special case of finite-dimensional vector space and for continuous families parametrised by \( S^1 \). We recall that vector bundles over the spheres, could be constructed by means of the trivial bundle on disks (homeomorphic to the upper lower hemisphere), through the clutching functions. More precisely, let \( E_2, E_1 \) be two real vector bundles over \([0, 1]\) such that \( \dim E_2 = \dim E_1 = k \) and let \( T : E_2 \to E_1 \) be a bundle morphism such that \( T|_{\partial([0, 1])} \) is invertible. We set \( I_j := [0, 1] \times \{j\} \) for \( j = 0, 1 \) and we consider the disjoint union

\[
I_0 \coprod I_1
\]

obtained by identifying the four points (two by two) in \( \partial([0, 1]) \times \{0\} \) and \( \partial([0, 1]) \times \{1\} \) as follows

\[
(0, 0) \cong (0, 1) \text{ and } (1, 0) \cong (1, 1).
\]

Under this identification, \( \partial([0, 1]) \times \{0\} \cup \partial([0, 1]) \times \{1\} \) is homeomorphic to \( S^1 \). Since by assumption \( T|_{\partial([0, 1])} \) is a bundle isomorphism, we get (for any such a bundle map \( T \)), a well-defined bundle \( E_T \) over \( S^1 \). We have

\[
\sigma(T(\lambda), \lambda \in [0, 1]) := w_1(E_T) = \begin{cases} 0 & \text{if } E_T \text{ is orientable} \\ 1 & \text{if } E_T \text{ is unorientable} \end{cases}.
\]
Furthermore, it is obvious that the constructed bundle $E$ is orientable if and only if $det T(0) \cdot det T(1) > 0$. This complete the proof.

We are now in position to discuss the infinite dimensional case. Since $[0, 1]$ is compact, there exists a subspace $V \subset Y$, such that

$$\text{Ran} T(s) + V = Y, \quad s \in [0, 1].$$

We let $E_s := T^{-1}(s)(V)$ and we observe that dim $E_s = \text{dim} V$. From Equation (3.1) it easily follows that the set $E := \{(s, x)|T(s)x \in V\}$ is the total space of a (trivial) vector bundle over $[0, 1]$.

**Definition 3.2.** Let $X, Y$ be two real and separable Hilbert spaces and let

$$T : ([0, 1], \partial([0, 1])) \to (\mathfrak{F}_0(X, Y), \text{GL}(X, Y))$$

be a continuous path of pairs. We define the *parity* of $T$, $\sigma (T(s), s \in [0, 1]) \in \widetilde{KO}([0, 1], \partial([0, 1])) \cong \mathbb{Z}_2$ as the first Stiefel-Whitney class of the bundle $E_T \to S^1$

$$\sigma (T(s), s \in [0, 1]) = w_1(E_T).$$

**Remark 3.3.** It is well-known that this definition is well-posed in the sense that it is independent on the choice of $V$. (Cf. [FP91a, FP91b] and references therein).

Let us consider the (smoooth) path of first order differential operators, pointwise defined by

$$A_\lambda := \frac{d}{dt} - s_\lambda(t), \quad t \in \mathbb{R}$$

arising by the system given in Equation (1.2). and we observe (cf. [Pej08, Proposition 3.1]) that for each $\lambda \in [0, 1]$, the operator $A_\lambda$ is a bounded Fredholm operator of index 0 from $W$ into $\mathcal{C}$. We recall that under the assumption (A2), both the operators $A_0$ and $A_1$ are invertible. For $\tau > 0$, we let

$$E_\lambda(\tau) := \{ x \in W^{1, 2}([\tau, \tau], \mathbb{R}^n) \mid x(-\tau) \in E^n(-\tau), x(\tau) \in E^s(\tau) \}$$

and we define $A_{\lambda, \tau}$ to be the restriction of $A_\lambda$ to $E_\lambda(\tau)$, namely

$$A_{\lambda, \tau} := A_\lambda|_{E_\lambda(\tau)}.$$
We recall that the adjoint $A^*_\lambda$ of $A_\lambda$ is the (closed) unbounded operator on $\mathcal{H}$ densely defined on $\mathcal{W}$ given by
\begin{equation}
A^*_\lambda := -\frac{d}{dt} - S^T_\lambda(t).
\end{equation}
As before, for $\tau > 0$, we define the subspace
\[ F_\lambda(\tau) := \left\{ x \in W^{1,2}([-\tau, \tau], \mathbb{R}^n) \mid x(-\tau) \in [E^n(\tau)]^\perp, x(\tau) \in [E^n(\tau)]^\perp \right\} \]
and we denote by $A^*_{\lambda,\tau}$ to be the restriction of $A^*_\lambda$ to $F_\lambda(\tau)$, namely
\[ A^*_{\lambda,\tau} := A^*_\lambda|_{F_\lambda(\tau)}. \]
Given $x \in E_\lambda(\tau)$ (resp. $x \in F_\lambda(\tau)$) we extend $x$ on the whole of $\mathbb{R}$ as follows; we let
\begin{equation}
\epsilon_{\lambda,\tau}(x)(t) := \begin{cases}
\gamma_{\lambda,\tau}(t)x(t) & \text{if } t \geq \tau \\
x(t) & \text{if } t \in [-\tau, \tau] \\
\gamma_{\lambda,-\tau}(t)x(-\tau) & \text{if } t \leq -\tau.
\end{cases}
\end{equation}
where $\gamma_{\lambda,\tau}$ is the matrix-valued solution of the system defined in Equation (1.2) (resp. in Equation (3.2)). It such that $\gamma_{\lambda,\tau}(\tau) = I$. It is immediate to check that $x \in \ker A^*_{\lambda,\tau}$ (resp. $x \in \ker A^*_\lambda$) if and only if $\epsilon_{\lambda,\tau}(x) \in \ker A_\lambda$ (resp. $x \in \ker A_\lambda$) and by this claim the next result readily follows.

**Lemma 3.4.** The following equality holds
\[ \dim \ker A_\lambda = \dim \ker A^*_{\lambda,\tau} = \dim \ker A^*_\lambda = \dim \ker A^*_{\lambda,\tau}. \]

The path $\lambda \mapsto A^*_{\lambda,\tau}$ plays a central role in the next Proposition which represent the main ingredient for proving Theorem 1.

**Proposition 3.5.** For $\tau > 0$ sufficiently large, the following equality holds:
\[ \sigma(A^*_{\lambda,\tau}, \lambda \in [0, 1]) = \sigma(A^*_\lambda, \lambda \in [0, 1]). \]

**Remark 3.6.** The main idea behind the proof of Proposition 3.5 is that, if $\tau > 0$ is sufficiently large, the arising vector bundles constructed through $A_\lambda$ and $A^*_{\lambda,\tau}$ are (bundle) isomorphic; thus the first Stiefel-Whitney classes coincide.

**Notation 3.7.** In what follows, we set
\[ \mathcal{W}_\tau := W^{1,2}([-\tau, \tau], \mathbb{R}^n) \quad \mathcal{H}_\tau = L^2([-\tau, \tau], \mathbb{R}^n) \]
We define the following *restriction* and *prolongation operator* respectively denoted by $\chi$ and $E_2$ and defined as follows
\[ \chi_\tau : \mathcal{H} \longrightarrow \mathcal{H}_\tau : x(t) \longmapsto x(t)|_{[-\tau, \tau]} \quad \text{and} \quad E_2 : \mathcal{H}_\tau \rightarrow \mathcal{H} \text{ defined by setting} \]
\[ E_2(x)(t) := \begin{cases}
0 & \text{if } |t| \geq \tau \\
x(t) & \text{if } t \in [-\tau, \tau].
\end{cases} \]

**Lemma 3.8.** For each $\lambda \in [0, 1]$, there exists a finite dimensional subspace $V_\lambda$ of $\mathcal{W}$, such that
\[ V_\lambda + \text{Ran } A_\lambda = \mathcal{H}. \]
Proof: We let $\lambda_0 \in (0, 1)$. For $\epsilon > 0$ sufficiently small, we consider the interval $[\lambda_0 - \epsilon, \lambda_0 + \epsilon] \subset [0, 1]$. If $\epsilon > 0$ is small enough, then there exists a finite dimensional subspace $V_\lambda$ of $\mathcal{H}$ such that $\ker A_\lambda^* = V_\lambda$. To see this, we let $\tilde{A}_\lambda := A_\lambda A_\lambda^*$ and we observe that $\lambda \mapsto \tilde{A}_\lambda$ is a continuous path of selfadjoint Fredholm operators such that

$$\ker \tilde{A}_\lambda = \ker A_\lambda^*.$$  

Let $\Gamma$ be a small circle around the origin chosen in such a way $\Gamma \cap \sigma(\tilde{A}_\lambda^*) = \emptyset$ for each $\lambda \in [\lambda_0 - \epsilon, \lambda_0 + \epsilon]$ and let us consider the projector operator

$$P_\lambda := \frac{1}{2\pi i} \int_{\Gamma} [z - (\tilde{A}_\lambda^*)]^{-1} dz,$$

and $V_\lambda := \text{Ran} P_\lambda \supset \ker \tilde{A}_\lambda = \ker A_\lambda^*$. Since $\lambda \mapsto P_\lambda$ is continuous on $[\lambda_0 - \epsilon, \lambda_0 + \epsilon]$, it follows that the path $\lambda \mapsto V_\lambda$ is. Moreover by the Fredholmness of $A_\lambda^*$, we get that $\dim V_\lambda < +\infty$. \hfill \Box

Lemma 3.9. We let $V_{\lambda, \tau} := \chi_{\tau} V_\lambda$ where $V_\lambda \in \mathcal{H}$ has been defined in Lemma 3.8. Thus, we have

$$V_{\lambda, \tau} + \text{Ran} A_{\lambda, \tau} = \mathcal{H}_\tau.$$

Proof: The proof of this result follows directly by arguing as in the proof of Lemma 3.8 once observed that

$$V_{\lambda, \tau} \supset \ker A_{\lambda, \tau}^*.$$

In fact, if $u \in \ker A_{\lambda, \tau}^*$, then $e_{\lambda, \tau}(u) \in \ker A_{\lambda}^* \subset V_\lambda$. In order to conclude, it is enough to observe that $u = \chi_\tau(e_{\lambda, \tau} u) \in V_{\lambda, \tau}$. \hfill \Box

Lemma 3.10. There exists $\tau_0 > 0$ such that, if $\tau \geq \tau_0$, then we have

$$\|\chi_\tau(y)\|_2 \geq \frac{1}{2} \|y\|_2, \quad \forall y \in V_\lambda.$$

Thus the linear map $\chi_\tau : V_\lambda \to V_{\lambda, \tau}$ is injective (and hence an isomorphism).

Proof: For each $\lambda \in [0, 1]$, let $P_\lambda : \mathcal{H} \to \mathcal{H}$ be the projector operator onto (the finite-dimensional vector space) $V_\lambda$. Let $\{e_1, \ldots, e_n\}$ be a unitary $n$-frame for $\text{Ran} P_\lambda$ and, for each $i \in \{1, \ldots, n\}$, we let

$$e_i(\lambda) := P_\lambda e_i.$$

Thus, there exists $\epsilon_1 > 0$ sufficiently small such that $\{e_1(\lambda), \ldots, e_n(\lambda)\}$ is a $n$-dimensional frame which depends continuously on $\lambda \in [0, \epsilon_1]$. We denote by $V$ the total space of the (trivial) vector bundle over $[0, 1]$ and by $S_{[0, \epsilon_1]}(V)$ the total space of the sphere bundle over $[0, \epsilon_1]$. We now consider the continuous function

$$f_\tau : S_{[0, \epsilon_1]}(V) \to \mathbb{R} \text{ defined by } f_\tau(y) := \|\chi_\tau(y)\|_2.$$  

By compactness of $S_{[0, \epsilon_1]}(V)$, the function $f_\tau$ is actually uniformly continuous and by the very definition of $\chi_\tau$, we infer that

$$\lim_{\tau \to +\infty} f_\tau(y) = 1.$$  

By uniformly continuity of $f_\tau$, we get that there exists $\tau_1 > 0$ sufficiently large such that

$$f_\tau(y) \geq \frac{1}{2} \quad \text{for every } y \in S_{[0, \epsilon_1]}(V) \text{ and for every } \tau \geq \tau_1.$$  

By compactness of $[0, 1]$ and by choosing an $\epsilon_0 > 0$ (maybe smaller than $\epsilon_1$), there exists $\tau_0 > 0$ such that

$$f_\tau(y) \geq \frac{1}{2} \quad \text{for every } y \in S_{[0, 1]}(V) \text{ and } \tau \geq \tau_0$$  

and by this the thesis readily follows. This conclude the proof. \hfill \Box

Lemma 3.11. We let $K_{\lambda, \tau} := E_2(\chi, \ker A_\lambda^*)$ and let $\tau_0$ as in Lemma 3.10. Then, for every $\tau \geq \tau_0$, we get
(i) \( \dim \ker A_\lambda^* = \dim K_{\lambda, \tau} \)

(ii) \( K_{\lambda, \tau} \cap \text{Ran} A_\lambda = \{0\} \).

Proof. In order to prove the first item, it is enough to observe that by taking into account Lemma 3.10, the map \( \chi_\tau : V_\lambda \to V_{\lambda, \tau} \) is injective and hence also its restriction on \( \ker A_\lambda^* \).

In order to prove the second statement, we argue by contradiction as follows. If not, there exists \( \neq y = A_\lambda x \) such that \( y \in K_{\lambda, \tau} \). In particular, by definition of \( K_{\lambda, \tau} \), there exists \( \bar{y} \in \ker A_\lambda^* \) such that \( y = E_2\chi_\tau(\bar{y}) \). We now observe that

\[
\langle y, \bar{y} \rangle_2 = \langle A_\lambda x, \bar{y} \rangle_2 = \langle x, A_\lambda^* \bar{y} \rangle_2 = 0
\]

where the last equality follows by the fact that \( \bar{y} \in \ker A_\lambda^* \). Furthermore

\[
\langle y, \bar{y} \rangle_2 = \langle E_2\chi_\tau(\bar{y}), \bar{y} \rangle_2 = \|\chi_\tau(\bar{y})\|^2 \neq 0.
\]

Summing up Equation (3.4) and Equation (3.5) we get a contradiction. This conclude the proof. \( \square \)

**Lemma 3.12.** For \( \tau > 0 \) sufficiently large there exists \( \nabla_\lambda \subset \mathcal{H} \) finite-dimensional subspace such that \( \nabla_\lambda + \text{Ran} A_\lambda = \mathcal{H} \).

Proof. By taking into account Lemma 3.4 we infer also that \( \chi_\tau(\ker A_\lambda^*) = \ker A_{\lambda, \tau} \subset V_{\lambda, \tau} \) and by invoking Lemma 3.11 we infer that for every \( \tau \geq \tau_0 \) we get

\[
E_2(V_{\lambda, \tau}) + \text{Ran} A_\lambda = \mathcal{H}.
\]

In order to conclude the proof, it is just enough to define \( \nabla_\lambda := E_2(V_{\lambda, \tau_0}) \). \( \square \)

**Proof of Proposition 3.5** By the previous Lemmas, we deduce that there exists \( V_{\lambda, \tau} \) and \( \nabla_\lambda \) finite dimensional subspaces respectively of \( \mathcal{H}_\tau \) and \( \mathcal{H} \) such that

\[
V_{\lambda, \tau} + \text{Ran} A_{\lambda, \tau} = \mathcal{H}_\tau \quad \text{and} \quad \nabla_\lambda + \text{Ran} A_\lambda = \mathcal{H}.
\]

We now set \( E_\lambda = A_{\lambda, \tau}^{-1} V_{\lambda, \tau} \) and \( \nabla_\lambda = A_\lambda^{-1} \nabla_\lambda \). Let \( y \in V_{\lambda, \tau} \) and \( A_{\lambda, \tau}(x) = y \) in particular \( x \in \mathcal{W}_\tau \) is such that

\[
\begin{cases}
  \frac{dx(t)}{dt} = S_\lambda(t)x, & t \in [-\tau, \tau] \\
  x(-\tau) \in E_\lambda^*(\bar{\tau}), & x(\bar{\tau}) \in E_\lambda(\bar{\tau}).
\end{cases}
\]

We now set \( \bar{y} := E_2 y \) and we let \( A_{\lambda, \tau}^{-1}(\bar{x}) = \bar{y} \). By this, readily follows that \( \bar{x} = e_{\lambda, \tau} x \) where \( e_{\lambda, \tau} \) has been defined in Equation (3.3). By this argument it follows that \( e_{\lambda, \tau} : E_\lambda \to \nabla_\lambda \) is an isomorphism with inverse \( \chi_\tau : \nabla_\lambda \to E_\lambda \). We also observe that \( \chi_\tau : \nabla_\lambda \to V_{\lambda, \tau} \) is an isomorphism with inverse \( E_2 : V_{\lambda, \tau} \to \nabla_\lambda \). By the commutativity of the following diagram

\[
\begin{array}{ccc}
  e_{\lambda, \tau} & \chi_\tau & E_2 \\
  \nabla_\lambda & \nabla_\lambda & \mathcal{H} \\
  \downarrow & \downarrow & \downarrow \\
  E_{\lambda, \tau} & V_{\lambda, \tau} & \mathcal{H}
\end{array}
\]

we get that \( w_1(\nabla_\lambda) = w_1(E_{\lambda, \tau}) \) and this conclude the proof. \( \square \)

Let us consider the following change of variables obtained by setting \( s = \frac{t + \tau}{2\tau} \in [0, 1] \). Then, the operator \( A_{\lambda, \tau} \) can be rewritten as follows

\[
A_{\lambda, \tau} = \frac{1}{2\tau} \frac{d}{ds} S_\lambda((2s - 1)\tau) : \mathcal{W}_\lambda(\tau) \subset L^2([0, 1]; \mathbb{R}^n) \to L^2([0, 1]; \mathbb{R}^n),
\]
where

\[ W_\lambda(\tau) := \{ u \in W^{1,2}([0, 1], \mathbb{R}^n) | u(0) \in E_\lambda^u(-\tau) \text{ and } u(1) \in E_\lambda^u(\tau) \}. \]

For each \( \lambda \in [0, 1] \), we define the following operator

\[ \bar{A}_{\lambda, \tau} = \frac{d}{ds} - 2\tau S_\lambda((2s - 1)\tau) \]

and we observe that, in contrast with \( A_{\lambda, \tau} \), \( \bar{A}_{\lambda, \tau} \) is well-defined also for \( \tau = 0 \); in fact \( \bar{A}_{\lambda, 0} = \frac{d}{dt} \) on the domain \( W_\lambda(0) \) and by taking into account the homotopy invariant of the parity, we get

\[ \sigma (\bar{A}_{\lambda, \tau}, \lambda \in [0, 1]) = \sigma (A_{\lambda, \tau}, \lambda \in [0, 1]). \]

**Lemma 3.13.** The following equality holds:

\[ \sigma (\bar{A}_{\lambda, 0}, \lambda \in [0, 1]) = \iota (E_\lambda^u(0), E_\lambda^s(0); \lambda \in [0, 1]). \]

**Proof.** We start to observe that since \( \bar{A}_{\lambda, 0}(0) = -\frac{d}{dt} \), its kernel consists of all constant functions. Let us denote by \( V \) be the space of all constant functions on \( L^2([0, 1], \mathbb{R}^n) \) and we observe that it is isomorphic to \( \mathbb{R}^n \). By a direct computation, we get that the space \( E_\lambda = \bar{A}_{\lambda, 0}^{-1} V \) can be characterised as follows

\[ E_\lambda = \{ z \in W^{1,2}([0, 1], \mathbb{R}^n) | z(t) = x + t(y - x), x \in E_\lambda^u(0), y \in E_\lambda^u(0) \}, \]

and \( \bar{A}_{\lambda, 0} z = y - x \). Clearly we have the following isomorphism \( E_\lambda \approx E_\lambda^u(0) \oplus E_\lambda^s(0) \), and

\[ \bar{A}_{\lambda, 0} : E_\lambda^u(0) \oplus E_\lambda^s(0) \rightarrow \mathbb{R}^n, (x, y) \rightarrow y - x. \]

Let \( \{ e_1(\lambda), \cdots, e_k(\lambda) \} \) be a frame of \( E_\lambda^u(0) \) and \( \{ e_{k+1}(\lambda), \cdots, e_n(\lambda) \} \) be a frame of \( E_\lambda^s(0) \). We set

\[ I(\lambda) := [e_1(\lambda) | \cdots | e_k(\lambda) | e_{k+1}(\lambda) | \cdots | e_n(\lambda)]. \]

Then, we have

\[ \iota (E_\lambda^u(0), E_\lambda^s(0); \lambda \in [0, 1]) = 0 \iff \det (T(0) \cdot T(1)) > 0. \]

We observe that every bundle on the interval is trivial and hence also the vector bundle \( E \) over \([0, 1] \) having fibre \( E_\lambda \) is. In particular, the trivialisation map is given by

\[ [0, 1] \times \mathbb{R}^n \ni (\lambda, e_1, \ldots, e_n) \mapsto (e_1(\lambda), \ldots, e_n(\lambda)). \]

Thus we have the isomorphism of \( E_\lambda^u(0) \oplus E_\lambda^s(0) \equiv \mathbb{R}^n \) and, under this trivialisation, we get

\[ \bar{A}_{\lambda, 0} = \begin{bmatrix} -e_1(\lambda) & \cdots & -e_k(\lambda) & e_{k+1}(\lambda) & \cdots & e_n(\lambda) \end{bmatrix} \]

and hence

\[ \text{sgn} ( \det \bar{A}_{0,0} \cdot \bar{A}_{1,0}) = \text{sgn} ( \det T(0) \cdot T(1)) \]

which conclude the proof. \( \square \)

**Proof of Theorem 1.** It readily follows by summing up the conclusion proved in Proposition 3.5 and Lemma 3.13. This conclude the proof. \( \square \)

**Proof of Theorem 2** It follows by taking into account Theorem 1 and by invoking Theorem 1 and [PR98, Theorem 6.1]. This conclude the proof. \( \square \)
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