REDUCTION OF THE DIMENSION OF NUCLEAR C*-ALGEBRAS

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Abstract. We show that for a large class of C*-algebras $A$, containing arbitrary direct limits of separable type I C*-algebras, the following statement holds: If $A \in A$ and $B$ is a simple projectionless C*-algebra with trivial K-groups that can be written as a direct limit of a system of (nonunital) recursive subhomogeneous algebras with no dimension growth then the stable rank of $A \otimes B$ is one. As a consequence we show that if $A \in A$ and $W$ is the C*-algebra constructed in [12] then the stable rank of $A \otimes W$ is one. We also prove the following stronger result: If $A$ is separable C*-algebra that can be written as a direct limit of C*-algebras of the form $C_0(X) \otimes M_n$, where $X$ is locally compact and Hausdorff, then $A \otimes W$ can be written as a direct limit of a sequence of 1-dimensional noncommutative CW-complexes.

1. Introduction

The notion of covering dimension of a topological space has led to different dimension theories for C*-algebras; for instance, the stable rank, real rank, decomposition rank, and nuclear dimension. Each of these dimension theories have had important applications to the theory of C*-algebras. It was shown in [31] and [32] that simple separable nonelementary unital C*-algebras with finite decomposition rank or more generally with finite nuclear dimension absorb the Jiang-Su algebra tensorially. As a consequence, new classification results have been obtained for simple C*-algebras. For C*-algebras that can be written as direct limits of subhomogeneous algebras one can also associate a dimension, namely the infimum over all such direct limit decompositions of the supremum of the covering dimension of the spectrum of all the C*-algebras appearing on the given direct limit. For instance, it was shown in [10] that for simple separable unital AH-algebras with very slow dimension growth this dimension is at most three. This result was later used in [6] to classify this class of C*-algebras.

In this paper we study different notions of dimension for certain C*-algebras of the form $A \otimes B$, where $B$ is simple, nuclear, either projectionless or unital with no nonzero projections but its unit, and with a specified direct limit decomposition. We are particularly interested in two cases: the first case is when $B$ is the C*-algebra $W$ constructed in [12] and the second is when $B$ is the Jiang-Su algebra $Z$. It was shown in [27] that if $A$ is a
simple finite C*-algebra then the stable rank of $A \otimes \mathcal{Z}$ is one. However, it is not known in general what the exact value of the stable rank is when $A$ is not simple. In Proposition 3.9 we partially answer this question. We compute the stable rank in the case that $A$ is a commutative C*-algebra whose spectrum is a CW-complex. It is believed that the decomposition rank (in the stably finite case) and the nuclear dimension of C*-algebras of the form $A \otimes \mathcal{Z}$ is at most two. This has recently been confirmed in [29] when $A$ is a commutative C*-algebra.

The C*-algebra $W$ is a simple separable nuclear C*-algebra that is stably finite, stably projectionless, has a unique tracial state, and has trivial K-groups. This algebra should be considered as a stably finite analog of the Cuntz algebra $O_2$. It should play central role in the classification of projectionless C*-algebras. Let $A$ be a C*-algebra and let $T(A)$ denote the cone of lower semicontinuous traces on $A$ with values in $[0, \infty]$ (note that the traces are not required to be densely finite). It has been shown in [7] that $T(A)$ belongs to the category of compact Hausdorff non-cancellative cones with jointly continuous addition and jointly continuous scalar multiplication. Our main motivation for studying C*-algebras of the form $A \otimes W$ is the following conjecture of Leonel Robert:

**Conjecture 1.1.** If $A$ and $B$ are separable nuclear C*-algebras then

$$T(A) \cong T(B) \iff A \otimes W \otimes \mathcal{K} \cong B \otimes W \otimes \mathcal{K},$$

where the isomorphism between $T(A)$ and $T(B)$ is assumed to be a linear homeomorphism.

This conjecture has been shown to be true for AF-algebras and for $O_2$-absorbing algebras. In fact, in the $O_2$-absorbing case this conjecture is nothing more than Kirchberg Classification Theorem of $O_2$-absorbing algebras. As a consequence of Theorem 1.4 below we obtained the following result: if $A$ is a separable direct limit of homogeneous C*-algebras, then the tensor product $A \otimes W$ is a direct limit of a sequence of 1-dimensional noncommutative CW-complexes (these are subhomogeneous algebras of 1-dimensional spectrum). This result reduces the proof of Robert’s Conjecture for direct limits of homogeneous algebras to prove a classification result for direct limits of 1-dimensional noncommutative CW-complexes (or shortly NCCW-complexes). This result also implies that the decomposition rank and the nuclear dimension of $A \otimes W$ is one. Direct limits of sequences of 1-dimensional NCCW-complexes has been classified in [26] in the case that the building blocks have trivial $K_1$-groups. Unfortunately, this classification result can not be applied directly in our case since all the building blocks obtained in the direct limit decomposition of $A \otimes W$ have non-trivial $K_1$-groups. Another consequence of Theorem 1.4 is that if $A$ is a C*-algebra in the class $\mathcal{A}$ defined below then the stable rank of $A \otimes W$ is one. In particular, by Theorem 3.3 the stable rank of the tensor product of $W$ with a direct limit of separable type I C*-algebras is one.
Definition 1.2. Let $\mathcal{A}$ be a class of $C^*$-algebras. We say that a $C^*$-algebra $B$ is locally contained in $\mathcal{A}$ if for every $\epsilon > 0$ and every finite subset $F$ of $B$ there exists a $C^*$-algebra $A \in \mathcal{A}$ and a *-homomorphism $\phi: A \to B$ such that the distance from $x$ to $\phi(A)$ is less than $\epsilon$ for every $x \in F$.

Definition 1.3. Let us denote by $\mathcal{A}$ the smallest class of $C^*$-algebras that satisfies the following properties:

(i) $C_0(X) \in \mathcal{A}$ for every locally compact space $X$.
(ii) If $A \in \mathcal{A}$ then $A \otimes M_n(\mathbb{C}) \in \mathcal{A}$ for every $n \in \mathbb{N}$.
(iii) If $A \in \mathcal{A}$ then every hereditary sub-$C^*$-algebra of $A$ belongs to $\mathcal{A}$. In particular, every closed two-sided ideal of $A$ belongs to $\mathcal{A}$.
(iv) If $A \in \mathcal{A}$ then every quotient of $A$ belongs to $\mathcal{A}$.
(v) If $A,C \in \mathcal{A}$ and if
\[ 0 \to A \to B \to C \to 0 \]
is an exact sequence of $C^*$-algebras then $B \in \mathcal{A}$.
(vi) If $A$ is locally contained in $\mathcal{A}$ then $A \in \mathcal{A}$.

The following theorem is our main result (see later in Sections 2 and 3 for the definition of a RSH$_0$-algebra and of a 1-dimensional NCCW-complex):

Theorem 1.4. Let $B$ be a direct limit of a system of RSH$_0$-algebras with no dimension growth. The following statements hold:

(1) If $B$ has a finite number of ideals then $\text{sr}(B) = 1$, if and only if, for every ideal $I$ of $B$ the index map $\delta: K_1(A/I) \to K_0(I)$ is trivial.
(2) If $B$ is simple, projectionless, and $K_0(B) = K_1(B) = 0$ then $\text{sr}(A \otimes B) = 1$ for every $C^*$-algebra $A \in \mathcal{A}$.
(3) If $B$ is simple, $K_1(B) = 0$, and $B$ is either projectionless or it is unital and its only non-zero projection is its unit then $\text{sr}(A \otimes B) = 1$ for every $C^*$-algebra $A$ that is approximately contained in the class of RSH$_0$-algebras with 1-dimensional spectrum.

Moreover, if $B$ is a simple direct limit of a sequence of 1-dimensional NCCW-complexes with $K_0(B) = K_1(B) = 0$, and $A$ is approximately contained in the class of $C^*$-algebras that are stably isomorphic to a commutative $C^*$-algebra, then $A \otimes B$ is approximately contained in the class of $C^*$-algebras that are stably isomorphic to 1-dimensional NCCW-complexes. In particular the decomposition rank and the nuclear dimension of $A \otimes B$ is one. If in addition $A$ is separable then $A \otimes B$ can be written as an inductive limit of $C^*$-algebras that are stably isomorphic to 1-dimensional NCCW-complexes.

Recursive subhomogeneous algebras (shortly RSH-algebras) were introduced and studied in [20] and [19]. This class of $C^*$-algebras arise naturally in the study of the crossed product $C^*$-algebras obtained from a minimal homeomorphism of a compact metric space. In [19] it is shown that if $B$ is a unital simple direct limit of a system of RSH-algebras with no dimension growth then the stable rank of $B$ is one. This result was generalized in [15] for certain simple inductive limits of some stable version of RSH-algebras.
In the first part of Theorem 1.4 we prove a similar result for C*-algebras with a finite number of ideals that can be written as the direct limit of some non-unital version of RSH-algebras. The proof of this result follows closely the proof of the unital simple case. The second and last part of Theorem 1.4 can be applied when $\mathcal{B}$ belongs to the class of C*-algebras classified in [23]; in particular, for the C*-algebra $\mathcal{W}$. These results are quite surprising since they imply that the tensor product of $\mathcal{W}$ with an arbitrary commutative C*-algebra can be decomposed as an inductive limit of 1-dimensional NCCW-complexes; and thus, it is of stable rank one. In other words, this result state that the stable rank and the dimension of the spectrum of a C*-algebra are independent when there are no K-theoretical obstructions. The third part of Theorem 1.4 can be applied to the Jiang-Su algebra. This result provide some evidence for the following statement: the operation of taking tensor products with the Jiang-Su algebra does not increase the stable rank.

The paper consists of three sections. In Section 2 we study pullbacks of C*-algebras; specifically, the structure of the ideals, quotients, and unitizations of pullbacks. In this section we also introduce and study the class of RSH$_{0}$-algebras. In Section 3 we prove the results stated in Theorem 1.4.

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2. Preliminary definitions and results

2.1. Continuous lattices. Let us briefly recall the definition of a continuous lattice ([9, Definition I-1.6.]). Let $L$ be a lattice and let $a$ and $b$ be elements of $L$, we say that $a$ is compactly contained in $b$, denoted by $a \ll b$, if for every increasing net $(b_i)_{i \in I}$ with $b \leq \sup_i b_i$, there exists $j$ such that $a \leq b_j$. We say that $L$ is a continuous lattice if $L$ is complete (i.e., every subset of $L$ has a least upper bound or supremum), and if $a = \sup_a \ll a'$ for every $a \in L$. An element $a \in A$ is said to be compact if $a \ll a$.

Let $A$ be a C*-algebra and let Lat$(A)$ denote the lattice of closed two-sided ideals of $A$. It was shown in [9, Proposition I-1.21.2] that Lat$(A)$ is a continuous lattice. Given a subset $S$ of $A$ we denote by Ideal$(S)$ the closed two-sided ideal generated by $S$. If $\phi: A \to B$ is a *-homomorphism of C*-algebras then the map $\hat{\phi}: \text{Lat}(A) \to \text{Lat}(B)$ defined by $\hat{\phi}(I) = \text{Ideal}(\phi(I))$ is a morphism in the category of continuous lattice. That is, a lattice map that preserves the compact containment relation and suprema of arbitrary subsets.

Throughout this paper an ideal of a C*-algebra will mean a closed two-sided ideal except when we refer to the Pedersen ideal (i.e., the smallest dense two sided-ideal of a C*-algebra ([21, Theorem 5.6.1])). The Pedersen ideal of a C*-algebra $A$ will be denoted by Ped$(A)$. 


The following proposition characterize the relation of compact containment of ideals. This is a refinement of [9, Proposition I-1.21.1].

**Proposition 2.1.** Let $A$ be a $C^*$-algebra and let $I$ and $J$ be closed two-sided ideals of $A$. Then $I \ll J$ if and only if there exist $a \in J_{+}$ and $\epsilon > 0$ such that $I \subseteq \text{Ideal}((a - \epsilon)_+)$. 

**Proof.** It was shown in [9, Proposition I-1.21.1] that $I \subseteq \text{Ideal}((a - \epsilon)_+)$ implies $I \ll J$. Let us prove the opposite implication. By [9, Proposition I-1.21.1] it is sufficient to show that for every $a_1, a_2, \ldots, a_n \in \text{Ped}(A)_{+}$ there exist $a \in A$ and $\epsilon > 0$ such that 

$$\text{Ideal}(a_1, a_2, \ldots, a_n) \subseteq \text{Ideal}((a - \epsilon)_+).$$

Suppose that $a_i = (b_i - \delta)_+$, where $b_1, b_2, \cdots, b_n \in A_{+}$ and $\delta > 0$. Set $a = \sum_{j=1}^{n} b_j$. Take an approximate unit $(u_i)_{i=1}^{\infty}$ of the hereditary algebra $\text{Her}(a)$ such that $u_i a u_i \leq (a - 1/i)_+$ (e.g., $u_i = h_i(a)$ with $h_i(t) = \frac{1}{t}(t - 1/i)_+$). Then $u_i b_j u_i \to b_j$ since $b_1, b_2, \cdots, b_n \in \text{Her}(a)$. By [13, Lemma 2.2] there exist $i \geq 1$ and $d_1, d_2, \cdots, d_n \in A$ such that 

$$(b_j - \delta)_+ = d_j^* u_i b_j u_i d_j,$$

for all $1 \leq j \leq n$. It follows that 

$$\text{Ideal}((b_1 - \delta)_+, (b_2 - \delta)_+, \cdots, (b_n - \delta)_+) \subseteq \text{Ideal}(u_i b_1 u_i, u_i b_2 u_i, \cdots, u_i b_n u_i) \subseteq \text{Ideal}(\sum_{j=1}^{n} u_i b_j u_i) \subseteq \text{Ideal}((a - 1/i)_+).$$

Now let us prove the general case. By the construction of the Pedersen ideal of a $C^*$-algebra (see the proof of [22, Theorem 5.6.1]) there are $c_1, c_2, \cdots, c_n \in A_{+}$ and continuous functions $f_1, f_2, \cdots, f_n : (0, \infty) \to [0, \infty)$ with compact support such that 

$$a_1 + a_2 + \cdots + a_n \leq f_1(c_1) + f_2(c_2) + \cdots + f_n(c_n).$$

Choose positive elements $(b_i)_{i=1}^{n} \in A$ and a real number $\delta > 0$ such that $f_i(c_i) = (b_i - \delta)_+$. It follows that 

$$\text{Ideal}(a_1, a_2, \cdots, a_n) \subseteq \text{Ideal}(a_1 + a_2 + \cdots + a_n) \subseteq \text{Ideal}(f_1(c_1), f_2(c_2), \cdots, f_n(c_n)) \subseteq \text{Ideal}((b_1 - \delta)_+, (b_2 - \delta)_+, \cdots, (b_n - \delta)_+).$$

Set $a = \sum_{j=1}^{n} b_j$. Then by the previous argument there exists $\epsilon > 0$ such that 

$$\text{Ideal}((b_1 - \delta)_+, (b_2 - \delta)_+, \cdots, (b_n - \delta)_+) \subseteq \text{Ideal}((a - \epsilon)_+).$$

$\square$

**Corollary 2.2.** Let $A$ be a $C^*$-algebra. The following are equivalent: 

(1) $I$ is a compact ideal of $A$ (i.e., $I \ll I$);
There is $a \in I_+ \cap \epsilon > 0$ such that $I = \text{Ideal}((a-\epsilon)_+)$.

(3) The spectrum of $I$ is compact.

Proof. The equivalence of (i) and (ii) follows by the previous proposition. The equivalence of (i) and (iii) follows using the standard identification of the closed two-sided ideals of a $C^*$-algebra with the open subsets of its spectrum.

Let $\phi: A \to B$ be a $*$-homomorphism of $C^*$-algebras. We denote by $\tilde{\phi}: \text{Lat}(A) \to \text{Lat}(B)$ the lattice map defined by $\tilde{\phi}(I) = \text{Ideal}(\phi(I))$. The following proposition was proved in [25, Proposition 3.1.9 (ii)]:

Proposition 2.3. Let $A = \lim_{\to} (A_n, \phi_{n,m})$ be an inductive limit of $C^*$-algebras. Let $I, J, K$ be ideals of $A_k$ such that $I \ll J$ and $\tilde{\phi}_{k,\infty}(J) \leq \tilde{\phi}_{k,\infty}(K)$.

Then there is $m \geq k$ such that $\tilde{\phi}_{k,m}(I) \ll \tilde{\phi}_{k,m}(K)$.

Lemma 2.4. Let $A = \lim_{\to} (A_n, \phi_{n,m})$ be an inductive limit of $C^*$-algebras. Suppose that $A$ has compact spectrum. Then there exist $k \geq 1$ and ideals $I_n \subseteq A_n$, $n \geq k$, with compact spectrum such that $\tilde{\phi}_{n,n+1}(I_n) = I_{n+1}$ and $A = \lim_{\to} (I_n, \phi_{n,m}|_{I_n})$.

Proof. Let $A$ be as in the statement of the lemma. Then $A \ll A$ by Corollary 2.2. Also, we have

$$A = \bigcup_n \text{Ideal}(\phi_{n,\infty}(A_n)) = \sup_n \tilde{\phi}_{n,\infty}(A_n).$$

By the definition of the compact containment relation there is $m$ such that $\tilde{\phi}_{m,\infty}(A_m) = A$. Write $A_m = \sup_{I \ll A_m} I$. Then

$$A = \tilde{\phi}_{m,\infty}(A_m) = \sup_{I \ll A_m} \tilde{\phi}_{m,\infty}(I).$$

Since $A \ll A$ there are ideals $J, K \subseteq A_m$ with $J \ll K \ll A_m$ such that $\tilde{\phi}_{m,\infty}(J) = \tilde{\phi}_{m,\infty}(K) = A$.

This implies, by Proposition 2.3 applied to $I' = K$, $J' = A_n$, and $K' = J$, that $\tilde{\phi}_{m,k}(K) \ll \tilde{\phi}_{m,k}(J)$ for some $k \geq m$. Hence,

$$\tilde{\phi}_{m,k}(J) \ll \tilde{\phi}_{m,k}(K) \ll \tilde{\phi}_{m,k}(J).$$

In other words, the spectrum of $\tilde{\phi}_{m,k}(J)$ is compact. The statement of the lemma now follows by taking $I_n = \tilde{\phi}_{n,m}(J)$.

Lemma 2.5. Let $A = \lim_{\to} (A_n, \phi_{n,m})$ be an inductive limit of $C^*$-algebras such that $\tilde{\phi}_{n,n+1}(A_n) = A_{n+1}$ for all $n$. Let $a \in A_m$, for some $m$, be such that $\text{Ideal}(\phi_{m,\infty}(a)) = A$ and suppose that $A_m$ has compact spectrum. Then there exists $l \geq m$ such that $\text{Ideal}(\phi_{m,k}(a)) = A_k$ for all $k \geq l$. 

□
Proof. Let \( a \in A_m \) be as in the statement of the lemma. Since \( A_m \) has compact spectrum, \( A_m \ll A_m \) by Corollary 2.2. Therefore,
\[
A = \hat{\phi}_{m,\infty}(A_m) \ll \hat{\phi}_{m,\infty}(A_m) = A,
\]
\[
A_n = \hat{\phi}_{m,n}(A_m) \ll \hat{\phi}_{m,n}(A_m) = A_n,
\]
for all \( n \geq m \). In other words, \( A \) and \( A_n \), with \( n \geq m \), have compact spectrum. Using the definition of the compact containment relation and that \( A \ll A = \sup_n \hat{\phi}_{n,\infty}(A_n) \) we get
\[
\hat{\phi}_{n,\infty}(A_n) = A = \text{Ideal}(\phi_{m,\infty}(a)),
\]
for all \( n \). This implies, by Proposition 2.3 applied to \( I = A_m, J = A_m \), and \( K = \text{Ideal}(a) \), that there exists \( l \geq m \) such that
\[
\hat{\phi}_{m,k}(A_m) \subseteq \text{Ideal}(\phi_{m,k}(a)),
\]
for all \( k \geq l \). Since, we also have
\[
\text{Ideal}(\phi_{m,k}(a)) \subseteq \hat{\phi}_{m,k}(A_m), \quad \hat{\phi}_{m,k}(A_m) = A_k,
\]
we get \( \text{Ideal}(\phi_{m,k}(a)) = A_k \) for all \( k \geq l \).
\]

2.2. Ideals, quotients, and unitizations of pullbacks. Recall that if \( A, B, \) and \( C \) are C*-algebras and \( \phi: A \to C \) and \( \psi: B \to C \) are *-homomorphisms then the pullback \( A \oplus_C B \) is given by
\[
A \oplus_C B = \{(a,b) \in A \oplus B : \phi(a) = \psi(b)\}.
\]
We denote by \( \pi_1: A \oplus_C B \to A \) and \( \pi_2: A \oplus_C B \to B \) the projections maps associated to this pullback. That is, the maps defined by \( \pi_1(a,b) = a \) and \( \pi_2(a,b) = b \) for \( (a,b) \in A \oplus_C B \).

The following is Proposition 3.1. of [21].

**Proposition 2.6.** A commutative diagram of C*-algebras

\[
\begin{array}{ccc}
X & \xrightarrow{\pi_2} & B \\
\pi_1 \downarrow & & \downarrow \psi \\
A & \xrightarrow{\phi} & C
\end{array}
\]
is a pullback if and only if the following conditions hold:

(i) \( \text{Ker}(\pi_1) \cap \text{Ker}(\pi_2) = \{0\} \),

(ii) \( \psi^{-1}(\phi(A)) = \pi_2(X) \),

(iii) \( \pi_1(\text{Ker}(\pi_2)) = \text{Ker}(\phi) \).

2.2.1. Ideals of Pullbacks. The following theorem describes the lattice of ideals of a pullback of C*-algebras.

**Theorem 2.7.** Let \( A, B, \) and \( C \) be C*-algebras and let \( \phi: A \to C \) and \( \psi: B \to C \) be *-homomorphisms with \( \phi \) is surjective. Let \( \mathcal{L}_{A,B} \) be the subset of \( \text{Lat}(A) \times \text{Lat}(B) \) consisting of pairs of ideals \((I,J)\) such that:

(1) \( \phi(I) = \psi(J) \),
(2) $\text{Ideal}(I \cap \phi^{-1}(\psi(J))) = I$.

If $L_{A,B}$ is endowed with the order

$$(I_1, J_1) \leq (I_2, J_2) \quad \text{if} \quad I_1 \leq I_2, J_1 \leq J_2,$$

then:

(i) $L_{A,B}$ is a continuous lattice and the map $\alpha: \text{Lat}(A \oplus_C B) \to L_{A,B}$

defined by

$$\alpha(M) = (\bar{\pi}_1(M), \bar{\pi}_2(M)),$$

is an isomorphism of lattices. Moreover, the inverse of $\alpha$ is given by

the map $\beta: L_{A,B} \to \text{Lat}(A \oplus_C B)$ defined by

$$\beta(I, J) = I \oplus_{\phi(I)} J,$$

where $I \oplus_{\phi(I)} J$ denotes the pullback taken with respect to the restriction maps $\phi|_I: I \to \phi(I)$ and $\psi|_J: J \to \phi(I)$ (this pullback may be identified with the subset of $A \oplus_C B$ given by $\{(a, b) \in A \oplus_C B \mid a \in I, b \in J\}$);

(ii) For every subset $T$ of $L_{A,B}$ the supremum of $T$ in $L_{A,B}$ is equal to

the supremum of $T$ in $\text{Lat}(A) \times \text{Lat}(B)$;

(iii) $(I_1, J_1) \ll (I_2, J_2)$ in $L_{A,B}$ if and only if $(I_1, J_1) \ll (I_2, J_2)$ in

$\text{Lat}(A) \times \text{Lat}(B)$.

Proof. (i) Let $M$ be an ideal of $A \oplus_C B$. Then since $\phi$ is surjective $\pi_2$ is

surjective by (ii) of Proposition 2.6. Hence, $\pi_2(M)$ is an ideal of $B$ and

$\bar{\pi}_2(M) = \pi_2(M)$. Let $I = \bar{\pi}_1(M)$ and $J = \pi_2(M)$. Then it is clear that

$$\phi(I) = \text{Ideal}(\psi(J)) = \hat{\psi}(J).$$

Also, since $\pi_1(M) \subseteq I$ and $\pi_1(M) \subseteq \phi^{-1}(\psi(J))$ we have $I = \text{Ideal}(I \cap \phi^{-1}(\psi(J)))$. It follows that $(I, J) \in L_{A,B}$ and that the map $\alpha$ is well defined.

It is straightforward to check using Proposition 2.6 that the diagram

$$\begin{array}{ccc}
M & \xrightarrow{\pi_2|_M} & J \\
\downarrow{\pi_1|_M} & & \downarrow{\psi|_J} \\
I & \xrightarrow{\phi|_I} & \phi(I)
\end{array}$$

is a pullback. Therefore $M = I \oplus_{\phi(I)} J$ or what is the same $(\beta \circ \alpha)(M) = M$.

Since $M$ is an arbitrary ideal of $A \oplus_C B$ we conclude that $\beta \circ \alpha = \text{id}_{\text{Lat}(A \oplus_C B)}$.

Now let us show that $\alpha \circ \beta = \text{id}_{L_{A,B}}$. Let $(I, J) \in L_{A,B}$. Then $\pi_2(I \oplus_{K} J) = J$ by (ii) of Proposition 2.6. Let $a$ be an element of $I \cap \phi^{-1}(\psi(J))$. Choose $b \in J$ such that $\phi(a) = \psi(b)$. Then $(a, b) \in I \oplus_{\phi(I)} J$ and $a \in \pi_1(I \oplus_{\phi(I)} J)$.

Since $a$ is arbitrary this implies that

$$I \cap \phi^{-1}(\psi(J)) \subseteq \pi_1(I \oplus_{\phi(I)} J) \subseteq I.$$

Therefore $\bar{\pi}_1(I \oplus_{\phi(I)} J) = I$ by (2) of the definition of $L_{A,B}$. We now have

$$(\alpha \circ \beta)(I, J) = \alpha(I \oplus_{\phi(I)} J) = (\bar{\pi}_1(I \oplus_{\phi(I)} J), \bar{\pi}_2(I \oplus_{\phi(I)} J)) = (I, J).$$
Therefore, \( \alpha \circ \beta = \text{id}_{L_{A,B}} \).

The maps \( \alpha \) and \( \beta \) are clearly order-preserving. Hence, they are isomorphisms of lattices that are inverse of each other. Since \( \text{Lat}(A \oplus_C B) \) is a continuous lattice it follows that \( L_{A,B} \) is a continuous lattice.

(ii) Let \( T \) be a subset of \( L_{A,B} \) and let
\[
\tilde{I} = \text{Ideal}\{I \mid (I, J) \in T\}, \quad \tilde{J} = \text{Ideal}\{J \mid (I, J) \in T\}.
\]
Then
\[
(\tilde{I}, \tilde{J}) = \text{Ideal}\{(I, J) \mid (I, J) \in T\} = \sup\{(I, J) \mid (I, J) \in T\},
\]
where the supremum is taken in \( \text{Lat}(A) \times \text{Lat}(B) \). Let us see that \( (\tilde{I}, \tilde{J}) \in L_{A,B} \). We have
\[
\phi(\tilde{I}) = \text{Ideal}\{(\phi(I) \mid (I, J) \in T)\}
= \text{Ideal}\{(\tilde{\psi}(J) \mid (I, J) \in T)\}
= \text{Ideal}\{\text{Ideal}(\psi(J)) \mid (I, J) \in T\}
= \text{Ideal}\{\psi(J) \mid (I, J) \in T\}
= \tilde{\psi}(\tilde{J}).
\]
Hence, \( (\tilde{I}, \tilde{J}) \) satisfies condition (1) of the definition of \( L_{A,B} \).

Let \( (I, J) \in T \) and let \( G = I \cap \phi^{-1}(\psi(J)) \). Then \( \text{Ideal}(G) = I \), \( G \subseteq \tilde{I} \), and
\[
G \subseteq \phi^{-1}(\psi(J)) \subseteq \phi^{-1}(\psi(\tilde{J})).
\]
Hence, \( G \subseteq I \cap \phi^{-1}(\psi(\tilde{J})) \). Using that \( \text{Ideal}(G) = I \) we get
\[
I \subseteq \text{Ideal}(I \cap \phi^{-1}(\psi(\tilde{J}))).
\]
Therefore,
\[
\tilde{I} = \text{Ideal}\{I \mid (I, J) \in T\} \subseteq \text{Ideal}(I \cap \phi^{-1}(\psi(\tilde{J}))).
\]
Since \( \text{Ideal}(I \cap \phi^{-1}(\psi(\tilde{J}))) \) \( \subseteq \tilde{I} \) we conclude that \( \tilde{I} = \text{Ideal}(I \cap \phi^{-1}(\psi(\tilde{J}))) \). This shows that \( (\tilde{I}, \tilde{J}) \) satisfies condition (2) of the definition of \( L_{A,B} \). Therefore, \( (\tilde{I}, \tilde{J}) \in L_{A,B} \). Since the order in \( L_{A,B} \) is induced by the order in \( \text{Lat}(A) \times \text{Lat}(B) \), \( (\tilde{I}, \tilde{J}) \) is the supremum of the set \( T \) in \( L_{A,B} \).

(iii) Suppose that \( (I_1, J_1) \ll (I_2, J_2) \) in \( L_{A,B} \). Then
\[
I_1 \oplus_{\phi(I_1)} J_1 = \beta(I_1, J_1) \ll \beta(I_2, J_2) = I_2 \oplus_{\phi(I_2)} J_2.
\]
By Proposition 2.1 there are \( (a, b) \in I_2 \oplus_{\phi(I_2)} J_2 \) and \( \epsilon > 0 \) such that
\[
I_1 \oplus_{\phi(I_1)} J_1 \subseteq \text{Ideal}((a, b) - \epsilon)_{+} = \text{Ideal}((a - \epsilon)_{+}, (b - \epsilon)_{+}).
\]
This implies that
\[
(I_1, J_1) = \alpha(I_1 \oplus_{\phi(I_1)} J_1) \subseteq (\text{Ideal}((a - \epsilon)_{+}), \text{Ideal}((b - \epsilon)_{+})).
\]
Therefore, \( I_1 \ll I_2 \) and \( J_1 \ll J_2 \) by Proposition 2.1. In other words, \( (I_1, J_1) \ll (I_2, J_2) \) in \( \text{Lat}(A) \times \text{Lat}(B) \).
Let \((I_1, J_1), (I_2, J_2) \in \text{Lat}(A) \times \text{Lat}(B)\) be such that \((I_1, J_1) \ll (I_2, J_2)\) in \(\text{Lat}(A) \times \text{Lat}(B)\). Let \(((I_\lambda, J_\lambda))_{\lambda \in \Lambda} \subseteq \text{L}_{A,B}\) be an increasing net such that \((I_2, J_2) = \sup_{\lambda \in \Lambda} (I_\lambda, J_\lambda)\), where the supremum is taken in \(\text{L}_{A,B}\). Then by the second part of the theorem the supremum of \(((I_\lambda, J_\lambda))_{\lambda \in \Lambda}\) in \(\text{Lat}(A) \times \text{Lat}(B)\) is equal to \((I_2, J_2)\). Since \((I_1, J_1) \ll (I_2, J_2)\) in \(\text{Lat}(A) \times \text{Lat}(B)\) there exists \(\lambda \in \Lambda\) such that \((I_1, J_1) \subseteq (I_\lambda, J_\lambda)\) in \(\text{Lat}(A) \times \text{Lat}(B)\). This implies that \((I_1, J_1) \leq (I_\lambda, J_\lambda)\) in \(\text{L}_{A,B}\). Since the increasing net \(((I_\lambda, J_\lambda))_{\lambda \in \Lambda}\) is arbitrary, we conclude by the definition of the compact containment relation that \((I_1, J_1) \ll (I_2, J_2)\) in \(\text{L}_{A,B}\).

\[\square\]

**Corollary 2.8.** Let \(A, B, \) and \(C\) be \(C^\ast\)-algebras and let \(\phi: A \to C\) and \(\psi: B \to C\) be \(*\)-homomorphisms with \(\phi\) is surjective. If \(M\) be an ideal of \(A \oplus_B C\) then there are ideals \(I \subseteq A, J \subseteq B, \) and \(K \subseteq C\) such that

\[M = I \oplus_K J, \quad \phi(I) = K, \quad \text{Ideal}(\psi(J)) = K.\]

Moreover, if \(M\) has compact spectrum then the ideals \(I, J, \) and \(K\) may be taken with compact spectrum.

**2.2.2. Quotients of pullbacks.** Let \(I \subseteq A, J \subseteq B, \) and \(K \subseteq C\) be ideals such that \(\phi(I) = K\) and \(\psi(J) \subseteq K\). Then the maps \(\phi\) and \(\psi\) induce \(*\)-homomorphisms \(\overline{\phi}: A/I \to C/K\) and \(\overline{\psi}: B/J \to C/K\) between the quotient \(C^\ast\)-algebras. Therefore, we can form the pullback \((A/I) \oplus_C (B/J)\). In the following proposition we show that this pullback is natural isomorphic to the quotient of \(A \oplus_B C\) by the ideal \(I \oplus_K J\).

**Proposition 2.9.** Let \(A, B, \) and \(C\) be \(C^\ast\)-algebras and let \(\phi: A \to C\) and \(\psi: B \to C\) be \(*\)-homomorphisms with \(\phi\) is surjective. If \(I \subseteq A, J \subseteq B, \) and \(K \subseteq C\) are ideals such that \(\phi(I) = K\) and \(\psi(J) \subseteq K\) then the map

\[\gamma: (A \oplus_B C)/(I \oplus_K J) \to (A/I) \oplus_C (B/J),\]

given by

\[\gamma((a, b) + I \oplus_K J) = (a + I, b + J),\]

is a \(*\)-isomorphism.

**Proof.** Let \(\rho: A \oplus_B C \to (A/I) \oplus_C (B/J)\) be the map defined by \(\rho(a, b) = (a + I, b + J)\) for every \((a, b) \in A \oplus_B C\). Note that \(\rho\) is well defined and that the kernel of \(\rho\) is \(I \oplus_K J\).

Let us show that \(\rho\) is surjective. Let \((a + I, b + J) \in (A/I) \oplus_C (B/J)\).

Then \(\overline{\phi}(a + I) = \overline{\psi}(b + J)\) and so \(\phi(a) - \psi(b) \in K\). Since \(\phi\) is surjective there exists \(a' \in A\) such that \(\phi(a') = \overline{\psi}(b)\). It follows that \(\phi(a - a') \in K\). Since by assumption \(\phi(I) = K\) there exists \(a'' \in I\) such that \(\phi(a'') = \phi(a - a')\).

We now have \(\phi(a - a'') = \phi(a') = \phi(b)\). Hence, \((a - a'', b) \in A \oplus_B C\) and \(\rho(a - a'', b) = (a + I, b + J)\). In other words, \(\rho\) is surjective.

Since \(\rho\) is surjective and the kernel of \(\rho\) is \(I \oplus_K J\), \(\rho\) induces an isomorphism \(\overline{\gamma}: (A \oplus_B C)/(I \oplus_K J) \to (A/I) \oplus_C (B/J)\). It is trivial to see that \(\overline{\gamma} = \gamma\).

\[\square\]
2.2.3. Unitization of Pullbacks. Let $A$ be a $C^*$-algebra. Let us denote by $\tilde{A}$ the $C^*$-algebra obtained by adjoining a unit to $A$ (note that if $A$ is unital $\tilde{A} \cong A \oplus \mathbb{C}$). If $\phi: A \to B$ is a *-homomorphism then it extends to a unique *-homomorphism $\tilde{\phi}: \tilde{A} \to \tilde{B}$ such that $\tilde{\phi}(1\_A) = 1\_B$. If $B$ is unital then we denote by $\phi^\dagger: A \to \tilde{B}$ the *-homomorphism that agrees with $\phi$ on $A$ and that satisfies $\phi^\dagger(1\_A) = 1\_B$.

The following proposition can be easily verified using Proposition 2.6.

**Proposition 2.10.** Let $A$, $B$ and $C$ be $C^*$-algebras and let $\phi: A \to C$ and $\psi: B \to C$ be *-homomorphisms with $\phi$ surjective. The following statements hold:

1. $(A \oplus_C B)^\sim \cong \tilde{A} \oplus_C \tilde{B}$. Specifically, the diagram

$$
\begin{array}{ccc}
(A \oplus_C B)^\sim & \xrightarrow{\tilde{\pi}_2} & \tilde{B} \\
\downarrow{\tilde{\pi}_1} & & \downarrow{\tilde{\psi}} \\
A & \xrightarrow{\tilde{\phi}} & C
\end{array}
$$

is a pullback;

2. If $A$ is unital then $(A \oplus_C B)^\sim \cong A \oplus_C \tilde{B}$. Specifically, the diagram

$$
\begin{array}{ccc}
(A \oplus_C B)^\sim & \xrightarrow{\pi^\dagger_2} & \tilde{B} \\
\downarrow{\pi^\dagger_1} & & \downarrow{\psi^\dagger} \\
A & \xrightarrow{\phi} & C
\end{array}
$$

is a pullback.

2.3. **Recursive subhomogeneous algebras.** In this subsection we introduce a nonunital version of recursive subhomogeneous algebras (cf., [20]).

**Definition 2.11.** Let $A$ be a $C^*$-algebra. We say that $A$ is a recursive subhomogeneous algebra vanishing at infinity, shortly $RSH_0$-algebra, if $A$ is isomorphic to the $k$-th term, for some $k$, of a sequence of iterated pullbacks of the form:

$$
P_i = \begin{cases} 
C_0(X_0, M_{n_0}), & \text{if } i = 0, \\
C_0(X_i, M_{n_i}) \oplus C_0(X^{(0)}_i, M_{n_i}) & \text{for } 1 \leq i \leq k,
\end{cases}
$$

with $X_i$ locally compact and Hausdorff, $X^{(0)}_i$ a closed subspace of $X_i$, $n_i$ a positive integer, and where the maps $C_0(X_i, M_{n_i}) \to C_0(X^{(0)}_i, M_{n_i})$ are always restriction maps. We call $(P_i, X_{i+1}, X^{(0)}_{i+1}, n_{i+1}, \rho_i)_{i=0}^{k-1}$ a decomposition of $A$. Here $\rho_i: P_i \to C_0(X^{(0)}_{i+1}, M_{n_{i+1}})$ denotes the *-homomorphism appearing in the $i$-th pullback. Associated to this decomposition are:

1. its length $k$;
(2) its base spaces $X_0, X_1, \ldots, X_k$ and total space $X = \prod_{i=0}^k X_i$;
(3) its topological dimension $\text{dim} X$ (i.e., the covering dimension of $X$), and topological dimension function $d: X \to \mathbb{N} \cup \{0\}$, defined by $d(x) = \text{dim}(X_k)$ when $x \in X_k$;
(4) its matrix sizes $n_0, n_1, \ldots, n_k$, and matrix size function $m: X \to \mathbb{N} \cup \{0\}$, defined by $m(x) = n_k$ when $x \in X_k$;
(5) its standard representation $\sigma: A \to \bigoplus_{i=0}^k C_0(X_i, M_{n_i})$, defined by using the natural inclusion of a pullback on the direct sum of its projection $C^*$-algebras;
(6) the evaluation maps $\text{ev}_x: A \to M_{n_i}$ for $x \in X_i$, defined as the composition of $\sigma$ with the evaluation map at $x$.

Note that by definition (see [20, Definition 1.2]) a $C^*$-algebra $A$ is a recursive subhomogeneous algebra, shortly a RSH-algebra, if it admits a RSH-decomposition in which all the algebras and all the *-homomorphisms are unital.

**Proposition 2.12.** Let $A$ be a RSH$_0$-algebra. The following statements hold:

1. If $I$ is an ideal of $A$ then $I$ and $A/I$ are RSH$_0$-algebras;
2. $A$ has compact spectrum if and only if $A$ has a decomposition
   \[
   (P_i, X_{i+1}, X_{i+1}^{(0)}, n_i+1, \rho_i)_{i=0}^{k-1}, \quad \rho_i: P_i \to C_0(X_{i+1}^{(0)}, M_{n_i+1}),
   \]
   such that
   1. $X_i$ and $X_i^{(0)}$ are compact for all $i$;
   2. $\rho_i: P_i \to C(X_{i+1}^{(0)}, M_{n_i+1})$ satisfies
      \[
      \text{Ideal}(\rho_i(P_i)) = C_0(X_{i+1}^{(0)}, M_{n_i+1}),
      \]
      for all $i$;
   3. there is $l$, with $0 \leq l \leq k$, such that $P_l$ is unital, $P_{l+1}$ is non-unital (if $l < k$), and $(P_i, X_{i+1}, X_{i+1}^{(0)}, n_i+1, \rho_i)_{i=0}^{l-1}$ is a RSH-decomposition of $P_l$ (i.e., all the maps and all the algebras in the decomposition are unital);
3. If $A$ has compact spectrum and
   \[
   (P_i, X_{i+1}, X_{i+1}^{(0)}, n_i+1, \rho_i)_{i=0}^{k-1}, \quad \rho_i: P_i \to C_0(X_{i+1}^{(0)}, M_{n_i+1}),
   \]
   is a decomposition of $A$ satisfying (a), (b), and (c) of (ii) then $\tilde{A}$ is RSH-algebra with decomposition $(Q_i, Y_{i+1}, Y_{i+1}^{(0)}, m_i+1, \mu_i)_{i=0}^k$ given by
   \[
   (Q_i, Y_{i+1}, Y_{i+1}^{(0)}, m_i+1, \mu_i) = \begin{cases}
   (P_i, X_{i+1}, X_{i+1}^{(0)}, n_i+1, \rho_i) & \text{if } 0 \leq i \leq l-1, \\
   (P_l, \{y_i\}, \varnothing, 1, \mu_l: P_l \to 0) & \text{if } i = l, \\
   (P_{l+1}, X_i, X_i^{(0)}, n_i+1, \rho_i) & \text{if } l+1 \leq i \leq k.
   \end{cases}
   \]

(In particular, $Q_{l+1} \cong Q_l \oplus \mathbb{C}$.)
Let \( X \) be a locally compact Hausdorff space then \( C_0(X) \otimes A \) is a RSH\(_0\)-algebra. Moreover, if \( X \) is compact and \( A \) has compact spectrum then \( C(X) \otimes A \) has compact spectrum;

If \( A \) has compact spectrum and \( p \in A \) is a projection then \( pAp \) is a RSH-algebra.

In addition, if a RSH\(_0\)-decomposition of \( A \) is given then the decompositions of \( I \) and \( A/I \) in (i), \( A \) in (ii), and \( pAp \) in (v) can be taken with topological dimension no greater than the topological dimension of the given decomposition of \( A \).

**Proof.** The first part of the proposition is a consequence of Corollary 2.8 and Proposition 2.9. The second part follows using Corollary 2.8 and that every ideal with compact spectrum of a C*-algebra of the form \( M_n(C(X)) \), with \( X \) locally compact and Hausdorff, has the form \( M_n(C(Y)) \) for some compact open subset \( Y \) of \( X \). The third part follows by Proposition 2.10; that is, using that the following diagrams are pullbacks:

\[
\begin{array}{ccc}
\tilde{P}_l & \rightarrow & P_l \\
\downarrow & & \downarrow \\
\mathbb{C} (\cong C(\{y_l\})) & \rightarrow & 0
\end{array}
\quad \begin{array}{ccc}
\tilde{P}_{i+1} & \rightarrow & \tilde{P}_i \\
\downarrow (\pi_2^{i+1}) & & \downarrow \rho_i \\
C(X_{i+1}, M_{n_{i+1}}) & \rightarrow & C(X_{i+1}^{(0)}, M_{n_{i+1}})
\end{array}
\]

for \( l \leq i \leq k - 1 \), where \( \pi_1^{i+1} : P_{i+1} \rightarrow C(X_{i+1}, M_{n_{i+1}}) \) and \( \pi_2^{i+1} : P_{i+1} \rightarrow P_i \) denote the projection maps. The fourth part of the proposition is a consequence of [21, Theorem 3.9]. Let us prove the last part of the proposition. By (i) and (ii) the C*-algebra \( \tilde{A} \) is a RSH-algebra. Hence, \( pAp = p\tilde{A}p \) is a RSH-algebra by [20, Corollary 1.11].

The following theorem is a corollary of the previous proposition and [17, Corollary 2.1]:

**Proposition 2.13.** Let \( A \) be a separable C*-algebra that can be written as the inductive limit of subhomogeneous algebras. Then \( A \) can be written as the inductive limit of RSH\(_0\)-algebras of finite topological dimension.

**Proof.** Let \( A \) be as in the statement of the theorem. Since the class of subhomogeneous C*-algebras is closed under the operation of adjoining a unit the C*-algebra \( \tilde{A} \) is the inductive limit of unital subhomogeneous C*-algebras. It follows now by [17, Corollary 2.1] that \( \tilde{A} \) is the inductive limit of RSH-algebras. Since \( A \) is an ideal of \( \tilde{A} \) it can be written as an inductive limit of ideals of the finite stage C*-algebras. By (i) of Proposition 2.12 these ideals are RSH\(_0\)-algebras that can be taken to have topological dimension no greater than the topological dimension of the given algebras. In particular, they can be taken with finite topological dimension. \( \Box \)
**Definition 2.14.** Let $A$ be a $RSH_0$-algebra with compact spectrum. We say that a decomposition of $A$ is a standard decomposition if it satisfies (a), (b), and (c) of (ii) of Proposition 2.12. The integer $l$ associated to this decomposition as in (c) will be called the unital degree of the decomposition.

Let $A$ be a C*-algebra. We denote by $U(A)$ and $U_0(A)$ the unitary group of the unitization $\tilde{A}$ of $A$ and the subgroup of $U(A)$ consisting of the unitaries that are connected to the identity. If $a \in A$ we denote by $|a|$ the element $(a^*a)^{\frac{1}{2}}$.

The following proposition is a minor modification of [19, Proposition 3.4] (the assumption on the dimension of the subspaces $E_x$ is only assumed to hold on the total space of $A$ instead of the total space of $A$).

**Proposition 2.15.** Let $A$ be a $RSH_0$-algebra with compact spectrum. Suppose that a standard decomposition of $A$ is given and let $X$, $m$, and $d$ be its total space, matrix size function, and topological dimension function (as in Definition 2.11). Let $\alpha, \epsilon > 0$. Let $a \in \tilde{A}$, and suppose that for every $x \in X$ there is a subspace $E_x \in \mathbb{C}^m(x)$ with $\dim(E_x) \geq \frac{1}{2}d(x)$ and such that $\|ev_x(a)\xi\| < \alpha\|\xi\|$ for $\xi \in E_x \setminus \{0\}$. Then there is a unitary $u \in U_0(A)$ such that

$$\|u|a| - a\| < 2\alpha + \epsilon. \tag{2.1}$$

**Proof.** Let $k$, $l$, and $X$ be the length, unital degree, and total space of the given standard decomposition of $A$. By (iii) of Proposition 2.11 the total space of $\tilde{A}$ is the disjoint union of $X$ and a singleton space $\{x_0\}$. Let us prove the following by induction in the number $k - l$: Let $\epsilon > 0$. Let $a \in \tilde{A}$, with $\|a\| \leq 1$, be as in the statement of the proposition and let $p(x) = \chi_{(-\infty, \alpha)}(ev_x(|a|))$ for $x \in X \cup \{x_0\}$. Then there is a unitary $u \in U_0(A)$ such that

$$\|ev_x(u|a| - a)(1 - p(x))\| < \epsilon,$$

for all $x \in X \cup \{x_0\}$. As in the proof of [19, Proposition 3.4] the previous inequality and the inequality $\|ev_x(a)p(x)\| \leq \alpha$, for $x \in X \cup \{x_0\}$, imply the inequality (2.1) (the former clearly holds by the definition of $p$).

If $k - l = 0$ then $A$ is unital and so it is a RSH-algebra. Also, $\tilde{A} \cong A \oplus \mathbb{C}$, where $\mathbb{C}$ is identified with $\mathbb{C}(\{x_0\})$. Let us write $a$ as pair $(b, \lambda) \in A \oplus \mathbb{C}$. Then $b$ satisfies the conditions of [19, Proposition 3.4]. Therefore, by the result stated in proof of the same proposition (see the second paragraph of the proof) there is a unitary $v \in A$ such that

$$\|ev_x(v|b| - b)(1 - p(x))\| < \epsilon,$$

for all $x \in X$. The case $k - l = 0$ now follows by taking $u = (v, \lambda')$, where $\lambda' = 1$ if $\lambda = 0$ and $\lambda' = \frac{\lambda}{|\lambda|}$ otherwise.

Now suppose that the result holds for all standard decompositions with $k - l = N > 0$ and suppose that for the given standard decomposition
of $A$ one has $k - l = N + 1$. Write $A = B \oplus_{C(X^{(0)}, M_n)} C(X, M_n)$, where $B$
 denotes the $(k - 1)$-pullback in the given decomposition. Then the difference
between the length of $B$ and its unital degree is $N$. Also, if $Y$ denote
the total space of $B$ then the total spaces of $\tilde{B}$, $A$, and $\hat{A}$ are $Y \sqcup\{x_0\}$, $X \sqcup Y$, and $X \sqcup Y \sqcup\{x_0\}$, respectively. Let $b$ be the image of $a$ in $\tilde{B}$ and let $a_0$ be the
image of $a$ in $C(X, M_n)$ under the projections maps given by the pullback.
Then $\chi_{(-\infty, \alpha)}(ev_y(|b|)) = p(y)$ for $y \in Y \sqcup\{x_0\}$. For the given values of $\alpha$, $\epsilon$, and $n$, let $\delta$ be as in
[19] Lemma 3.3. Since $b \in \tilde{B}$ satisfies the conditions of
the proposition and $\tilde{B}$ satisfies the induction assumptions there is a unitary
$v \in U_0(B)$ such that

$$
||ev_y(v|b| - b)(1 - p(y))|| < \delta
$$

for all $y \in Y$. The rest of proof now follows line by line the proof of
[19] Proposition 3.4 starting in the fourth paragraph. \hfill \Box

**Definition 2.16.** Let $(A_i, \phi_{i,j})_{i,j}$ be a system of $RSH_0$-algebras. For each $i$ let $X_i$ denote the total space of $A_i$, (ii) of Definition [2.11]. We say that the system $(A_i, \phi_{i,j})_{i,j}$ has no dimension growth if $\sup_i (\dim X_i) < \infty$.

**Lemma 2.17.** Let $A = \lim_{\leftarrow}(A_i, \phi_{i,j})$ be an inductive limit of a system of
$RSH_0$-algebras with no dimension growth. Suppose that for each $i \geq 1$ the
$C^*$-algebra $A_i$ has compact spectrum and that the given decomposition of $A_i$
is standard. Let $X_i$ be the total space of $A_i$. Let $a \in \tilde{A}_k$, for some $k$, be a
noninvertible element. Let $\epsilon > 0$. Suppose that for every $N \in \mathbb{N}$ there exists
$j \geq k$ such that for every $x \in X_j$,

$$
\text{rank} \left( \text{ev}_x(\phi_{k,j}(f_{\epsilon}(|a|))) \right) \geq N,
$$

for some continuous function $f_{\epsilon}: (0, \infty) \to [0, 1]$ satisfying $f_{\epsilon}(x) > 0$ for
$x \in (0, \epsilon)$ and $f_{\epsilon}(x) = 0$ for $x \in \mathbb{R}_+$. Then there exists a unitary $u \in \tilde{A}$
such that

$$
||\phi_{k,\infty}(a) - u|\phi_{k,\infty}(a)|| < 5\epsilon.
$$

**Proof.** Let $m_i: X_i \to \mathbb{N}$ be the matrix size function of $A_i$. Let $N$ be a
number such that $N > \sup_i \dim X_i$ ($N$ is finite since by assumption the
system $(A_i, \phi_{i,j})_{i,j}$ has no dimension growth). Let $\epsilon > 0$ and let $a$ be as in
the statement of the lemma. Then there is $j \geq k$ such that

$$
\text{rank} \left( \text{ev}_x(\phi_{k,j}(f_{\epsilon}(|a|))) \right) \geq N,
$$

for all $x \in X_j$. Given $x \in X_j$ let $E_x$ denote the subspace of $\mathbb{C}^{m_j(x)}$
defined by $E_x = \text{ev}_x(\phi_{k,j}(f_{\epsilon}(|a|)))\mathbb{C}^{m_j(x)}$. Then by the previous inequality

$$
\dim E_x \geq N > \frac{\dim X_j}{2}.
$$
Let \( g_\epsilon : [0, \infty) \to [0, 1] \) be a continuous function such that \( g_\epsilon(x) = 1 \) if \( x \in [0, \epsilon] \) and \( g_\epsilon(x) = 0 \) if \( x \in [2\epsilon, \infty) \). Then

\[
\|a|g_\epsilon(|a|)\| < 2\epsilon, \quad \text{ev}_x(\tilde{\phi}_{k,j})(g_\epsilon(|a|))\xi = \xi
\]

for all \( \xi \in E_x \). It follows that

\[
\|\text{ev}_x(\tilde{\phi}_{k,j}(a))\xi\| = \|\text{ev}_x(\tilde{\phi}_{k,j}(|a|))\xi\| = \|\text{ev}_x(\tilde{\phi}_{k,j}(|a| \cdot g_\epsilon(|a|)))\xi\| < 2\epsilon\|\xi\|
\]

for all \( \xi \in E_x \setminus \{0\} \). This shows that \( \tilde{\phi}_{k,j}(a) \in \tilde{A}_j \) satisfies the conditions of Proposition \[2.16\]. Therefore, there is a unitary \( v \in U(\tilde{A}_i) \) such that

\[
\|\tilde{\phi}_{k,j}(a) - v|\tilde{\phi}_{k,j}(a)\| < 5\epsilon.
\]

Using that \(*\)-homomorphisms are contractive we conclude that

\[
\|\tilde{\phi}_{k,\infty}(a) - u|\tilde{\phi}_{k,\infty}(a)\| < 5\epsilon,
\]

where \( u = \tilde{\phi}_{j,\infty}(u) \in U(\tilde{A}) \).

Let \( A \) and \( B \) be C*-algebras a let \( \phi : A \to B \) be a \(*\)-homomorphism. Recall that \( \hat{\phi} : \text{Lat}(A) \to \text{Lat}(B) \) denotes the lattice map defined by \( \hat{\phi}(I) = \text{Ideal}(I) \).

**Proposition 2.18.** Let \( A = \varinjlim(A_n, \phi_{n,m}) \) be an inductive limit of RSH\(_0\)-algebras. Suppose that \( A \) has compact spectrum. Then \( A \) can be written as inductive limit \( A = \varinjlim(B_n, \psi_{n,m}) \), where \( B_n \) are RSH\(_0\)-algebras with compact spectrum and the maps \( \psi_{n,m} \) are injective and such that \( \hat{\psi}_{n,m+1}(B_n) = B_n \) (i.e., \( \text{Ideal}(\psi_{n,m+1}(B_n)) = B_n \)). Moreover, if the system \((A_n, \phi_{n,m})\) has no dimension growth the new system \((B_n, \psi_{n,m})\) can be taken to have no dimension growth.

**Proof.** Let \( \phi_{n,\infty} : A_n \to A \) denote the \(*\)-homomorphisms given by the inductive limit decomposition of \( A \). These maps satisfy that \( \phi_{n+1,\infty} \circ \phi_n = \phi_{n,\infty} \) for all \( n \). For each \( n \geq 1 \) let \( I_n \) denote the kernel of the map \( \phi_n \). Then the kernel of \( \phi_n \) is contained in \( I_n \) and \( \phi_n(I_n) \subseteq I_{n+1} \). It follows that the map \( \phi_n : A_n \to A_{n+1} \) induces an injective map \( \hat{\phi}_n : A_n/I_n \to A_{n+1}/I_{n+1} \). Also, it is not difficult to show that

\[
A_1/I_1 \xrightarrow{\phi_{1,1}} A_2/I_2 \xrightarrow{\phi_{2,2}} A_3/I_3 \xrightarrow{\phi_{3,3}} \cdots \to A,
\]

is an inductive limit decomposition of \( A \). By (i) of Proposition \[2.12\] the C*-algebras \( A_n/I_n, \) \( n = 1, 2, \cdots \), are RSH\(_0\)-algebras. Furthermore, the dimension of the total space of \( A_n/I_n \) is at most the dimension of the total space of \( A_n \). Therefore, we may assume that the maps in the inductive limit decomposition of \( A \) are injective.

Now let us use Lemma \[2.4\] to construct a sequence \( \{k_n\}_{n \in \mathbb{N}} \) and C*-algebras \( B_n \in A_{k_n} \) such that \( B_n = \hat{\phi}_{n,k_n}(A_n) \), \( B_n \) has compact spectrum,
and $\tilde{\phi}_{k_n,k_n+1}(B_n) = B_{n+1}$. Note that if such sequence of $C^*$-algebras exists then

$$B_1 \xrightarrow{\phi_{k_1,k_2}} B_2 \xrightarrow{\phi_{k_2,k_3}} B_3 \xrightarrow{\phi_{k_3,k_4}} \cdots \xrightarrow{\phi_{k_n,k_{n+1}}} A,$$

is an inductive limit decomposition of $A$ satisfying the conclusions of the proposition. To see this note that the $C^*$-algebras $B_n$ satisfy

$$\bigcup_{n=1}^{\infty} \phi_{k_n,\infty}(B_n) = \bigcup_{n=1}^{\infty} \phi_{k_n,\infty}(\tilde{\phi}_{n,k_n}(A_n)) = \bigcup_{n=1}^{\infty} \phi_{n,\infty}(A_n) = A.$$

In addition, the algebras $B_n$ are RSH$_0$-algebras by (iii) of Proposition 2.12 and the dimension of the total space of $B_n$ is at most the dimension of the total space of $A_{k_n}$. Therefore, if the system $(A_n, \phi_n)_{n \in \mathbb{N}}$ has no dimension growth then the system $(B_n, \tilde{\phi}_{k_n,k_n+1})_{n \in \mathbb{N}}$ has no dimension growth.

Let us proceed by induction to construct the sequences $(k_n)_{n \in \mathbb{N}}$ and $(B_n)_{n \in \mathbb{N}}$. By Lemma 2.4 applied to the case $n = 1$ there is a positive integer $k_1 > 2$ such that the ideal $\tilde{\phi}_{1,k_1}(A_1)$ has compact spectrum, and $\phi_{2,k_2}(A_2) = \tilde{\phi}_{1,k_1}(A_1)$ for all $k \geq k_1$. Set $B_1 = \tilde{\phi}_{1,k_1}(A_1)$. Suppose that we have constructed the integer $k_n$ and the algebra $B_n$ and let us construct $k_{n+1}$ and $B_{n+1}$. By Lemma 2.4 applied to the number $n + 1$ there is $k_{n+1} > k_n$ such that the ideal $\tilde{\phi}_{n+1,k_{n+1}}(A_{n+1})$ has compact spectrum and $\phi_{n+1,k}(A_{n+1}) = \tilde{\phi}_{n,k}(A_n)$ for all $k \geq k_{n+1}$. Set $B_{n+1} = \tilde{\phi}_{n+1,k_{n+1}}(A_{n+1})$. Then,

$$B_{n+1} = \tilde{\phi}_{n+1,k_{n+1}}(A_{n+1}) = \tilde{\phi}_{n,k_{n+1}}(A_n) = \tilde{\phi}_{k_n,k_{n+1}}(B_n).$$

We have found a $C^*$-algebra $B_{n+1}$ with compact spectrum satisfying $B_{n+1} = \tilde{\phi}_{k_n,k_{n+1}}(B_n) = \text{Ideal}(\tilde{\phi}_{k_n,k_{n+1}}(B_n))$. This concludes the proof of the proposition. \hfill $\square$

3. Main results

3.1. Reduction of the stable rank. In this subsection we prove the stable rank results stated in Theorem 1.4.

**Lemma 3.1.** Let $A$ be a $C^*$-algebra such that every quotient of $A$ is projectionless and every quotient of $\bar{A}$ is finite. Let $a \in A$ be such that $a - 1 \in A$. Let $0 < \delta < \min(1, \|a\|)$ and let $f \in C_0(0, \infty)$ be such that $f(x) > 0$ for $x \in (0, \delta)$ and $f(x) = 0$ for $x \in [\delta, \infty)$. If $I$ denote the closed two-sided ideal generated by $f(|a|)$, with $|a| = (a^*a)^{\frac{1}{2}}$, then the image of $a$ in $\bar{A}/I$ is invertible.

**Proof.** Since $a - 1 \in A$ we have $|a| - 1 \in A$. Therefore,

$$f(|a|) \in C^*([|a|] - 1) \subset A.$$

It follows that $I = \text{Ideal}(f(|a|)) \subset A$.

Suppose that the image $[a]_I$ of $a$ in the quotient $\bar{A}/I$ is not invertible. Then $|[a]|_I$ is not invertible by the finiteness of $\bar{A}/I$. Since $I = \text{Ideal}(f(|a|))$
we have \( f(||a||) = [f(||a||)]_I = 0 \). Hence, the element \( ||a|| \) has a gap in its spectrum (here we are using that \( \delta < ||a|| \)). Now using functional calculus on the element \( a \) we can find a nonzero projection \( p \in \tilde{A}/I \). Note that 
\( p < 1/\tilde{A}/I \) since \( 0 \in \text{sp}([||a||]) \). Also, since \( p \in \tilde{A}/I \) there are \( q \in A/I \) and \( \lambda \in \mathbb{C} \) such that \( p = \lambda 1_{\tilde{A}/I} - q \). This implies using that \( p \) is a projection that \( \lambda = 1 \) and \( q \) is a projection in \( A/I \). But this is impossible since \( A/I \) is projectionless and \( p < 1 \). \( \square \)

**Theorem 3.2.** Let \( A \) be the inductive limit of a system of RSH\(_0\)-algebras with no dimension growth. Assume that \( A \) has a finite number of ideals. Then the stable rank of \( A \) is one if and only if for every ideal \( I \) of \( A \) the index map \( \delta: K_1(A/I) \to K_0(I) \) is zero.

**Proof.** First, let us assume that \( A \) is an infinite dimensional simple C\(^*\)-algebra of the form \( A = \lim(A_n, \phi_{n,m}) \), where \((A_n, \phi_{n,m})_{n,m}\) is a system of RSH\(_0\)-algebras with no dimension growth. Since \( A \) is simple it has compact spectrum; thus, by Proposition 2.18 we may assume that the C\(^*\)-algebras \( A_n \) have compact spectrum, that the maps \( \phi_{n,m} : A_n \to A_m \) are injective, and that \( \text{Ideal}(\phi_{n,m}(A_n)) = A_m \) for all \( m \geq n \).

Suppose that there is a non-zero projection \( p \in A \). Since every projection in the inductive limit C\(^*\)-algebra is the image of a projection from a finite stage algebra there are \( N > 0 \) and projections \( p_n \in A_n \), with \( n > N \), such that \( \phi_{n,m}(p_n) = p_m \) and \( \phi_{n,\infty}(p_n) = p \). The C\(^*\)-algebras \( p_n A_n p_n \) are RSH-algebras and \( p A p = \lim p_n A_n p_n \) (the first statement follows by (iv) of Proposition 2.12). Hence, by [19, Theorem 3.6] the stable rank of \( p A p \) is one. Since \( A \) is simple it is stable isomorphic to \( p A p \), thus \( \text{tsr}(A) = 1 \) by [21, Theorem 3.6].

Now suppose that \( A \) is projectionless. Then the C\(^*\)-algebras \( A_n \) are projectionless since the maps \( \phi_{n,m} \) are injective. Let us consider the unitization \( \tilde{A} \) of \( A \). Then \( \tilde{A} = \lim(\tilde{A}_n, \tilde{\phi}_n) \). In order to show that \( \text{tsr}(\tilde{A}) = 1 \) it is enough to prove that for every \( n \geq 1 \) and every element \( a \in \tilde{A}_n \), with \( a - 1 \in A_n \), the element \( \tilde{\phi}_{n,\infty}(a) \) can be approximated by invertibles in \( \tilde{A} \). This is clear since the elements of the form \( \lambda \tilde{\phi}_{n,\infty}(a) \), with \( a \) as above and \( 0 \neq \lambda \in \mathbb{C} \), are dense in \( \tilde{A} \).

Let \( a \in A_n \), for some \( n \), be such that \( a - 1 \in A_n \). Set \( b = \tilde{\phi}_{n,\infty}(a) \). If \( b \) is invertible then the claim holds, so let us assume that \( b \) is not invertible.

Let \( 0 < \epsilon < \min(1, ||b||) \) and let \( f \in C_0(0, \infty) \) be such that \( f(x) > 0 \) for \( x \in (0, \epsilon) \) and \( f(x) = 0 \) for \( x \in (\epsilon, \infty) \). Consider the element \( f(||b||) \). By functional calculus \( f(||b||) \in \text{C}^*(||b|| - 1) \subset A \). In addition, the element \( f(||b||) \) is non-zero (if \( f(||b||) = 0 \) then by Lemma 3.1 applied to \( I = \{0\} \) we would get that \( b \) is invertible). Therefore, \( \text{Ideal}(f(||b||)) = A \).

Let \( N \in \mathbb{N} \). Choose \( m > n \) such that there are mutually orthogonal nonzero positive elements \( a_1, a_2, \ldots, a_N \in \text{Her}(\phi_{n,m}(f(||a||))) \). (If this is not
possible, then [19 Lemma 1.6] implies that
\[ \dim(\text{Her}(\phi_{n,m}(f(|a|)))) \leq (N - 1)^2, \]
for all \( m \geq n \). So, \( \dim(\text{Her}(f(|b|))) \leq (N - 1)^2 \). This implies that \( \text{Her}(f(|b|)) \)
has a projection, which is not possible since \( A \) is projectionless.) Since \( a_1, a_2, \cdots, a_N \) are nonzero and the maps \( \phi_{n,m} \) are injective \( \text{Ideal}(\phi_{n,\infty}(a_i)) = A \) for all \( i \). Hence, by Lemma 2.5 there is \( k \geq m \) such that \( \text{Ideal}(\phi_{m,k}(a_i)) = A_k \) for all \( i \). In particular, this implies that \( \phi_{m,k}(a_i)(x) \neq 0 \) for every \( x \in X_k \) and every \( i \), where \( X_k \) denotes the total space of \( A_k \). Therefore, since the elements \( a_1, a_2, \cdots, a_N \) are mutually orthogonal and \( \phi_{m,k}(a_i) \in \text{Her}(\phi_{n,k}(f(|a|))) \) for all \( i \) we have
\[ \text{rank}(ev_x(\phi_{n,l}(f(|a|)))) \geq N, \]
for all \( x \in X_l \) and all \( l \geq k \). Since \( N \in \mathbb{N} \) is arbitrary \( \phi_{n,\infty}(a) \) can be approximated by invertibles in \( \tilde{A} \) by Lemma 2.17.

This ends the proof in the case that \( A \) is a simple C*-algebra. The proof of the general case follows by applying inductively [15 Lemma 3].

Let \( \mathcal{A} \) be the class of C*-algebras defined in Definition 1.3. Then \( \mathcal{A} \) is closed under direct limits by (vi). Also, it is closed under the operation of taking pullbacks of C*-algebras (in fact, by [21 Proposition 3.4] this is true for any class of C*-algebras that satisfies (v), (iv), and the second part of (iii)). In particular, this implies that \( \mathcal{A} \) contains the class of RSH\(_0\)-algebras since \( M_n(C_0(X)) \in \mathcal{A} \) for every \( n \) and every locally compact Hausdorff space \( X \), by (i) and (ii). The class \( \mathcal{A} \) also contains C*-algebras with infinite projections (e.g., the Toeplitz algebra), simple C*-algebras with arbitrary stable rank (e.g., Villadsen’s algebras), and even simple C*-algebras that can not be classified using the Elliott invariant (e.g., Tom’s algebras).

**Proposition 3.3.** Every separable type I C*-algebra belongs to \( \mathcal{A} \).

**Proof.** By the previous remark \( \mathcal{A} \) contains the class of RSH\(_0\)-algebras. Hence, by Theorem 2.13 it also contains the class of separable subhomogeneous C*-algebras. Now, let us show that \( \mathcal{A} \) contains the class of separable continuous trace C*-algebras. Let \( A \) be a separable continuous trace C*-algebra then the set of elements of \( a \in A \) with uniformly bounded trace (i.e., there exists \( n \in \mathbb{N} \) such that \( \text{rank}(\pi(a)) \leq n \) for all irreducible representation \( \pi \)) form a dense self-adjointed ideal of \( A \) (not necessarily closed). Note that for such elements the hereditary algebra \( a^* a \) is a subhomogeneous algebra. Also, since this ideal is dense in \( A \) and \( A \) is separable, \( A \) can be written as a direct limit of such hereditary algebras. Therefore, \( A \in \mathcal{A} \).

Suppose now that \( A \) is an arbitrary separable type I C*-algebra. By [22 Theorem 6.2.11] and by the separability of \( A \), \( A \) has a composition series \( (I_n)_{n=0}^\infty \) such that the quotients \( I_{n+1}/I_n \) are continuous trace C*-algebras. Hence, by (v) of the above definition \( I_n \in \mathcal{A} \) for all \( n \). Since \( A = \bigcup_n I_n \) we deduce that \( A \in \mathcal{A} \) since \( \mathcal{A} \) is closed under direct limits. \( \square \)
Lemma 3.4. For every $\epsilon > 0$ there is $\delta > 0$ such that: if $A$ is a C*-algebra satisfying $U(B) = U_0(B)$ for every hereditary sub-C*-algebra $B$ of $A$, and if $a \in \widetilde{A}$ is a positive element with $a - 1 \in A$ and $\|a\| \leq M$, and $u \in \widetilde{A}$ is a unitary with $u - 1 \in A$ satisfying
\[ \|ua - a\| < \delta, \]
then there exists a path of unitaries $(u_t)_{t \in [0,1]}$ in $\widetilde{A}$ with $u_t - 1 \in A$, $u_0 = u$, and $u_1 = 1$, such that
\[ \|u_t a - a\| < \epsilon, \]
for all $t \in [0,1]$.

Proof. The universal C*-algebra generated by elements $a$ and $u$ satisfying the relations:
\[ u^* u = uu^* = 1, \quad 0 \leq a \leq M \cdot 1, \quad ua = a, \]
is isomorphic to the algebra of complex-value continuous functions over the graph consisting of a circle with an interval attached to it at a point. Since this C*-algebra is weakly semiprojective the relations mention above are weakly stable.

Let $0 < \epsilon < 1$ and let $\epsilon' = \min(\frac{\epsilon}{2}, \frac{\epsilon}{3M+1})$. Then by the weak stability of the relations \textcolor[rgb]{0.00,0.00,0.00}{(3.1)} there exists $0 < \delta < \epsilon'$ such that: if $a \in \widetilde{A}$ is a positive element and $u \in U_0(A)$ satisfy $\|ua - a\| < \delta$ then there are $a' \in \widetilde{A}$, with $0 \leq a' \leq M \cdot 1$, and $u' \in U_0(A)$ such that
\[ u'a' = a', \quad \|u - u'\| < \epsilon', \quad \|a - a'\| < \epsilon'. \]
Let $\pi : \widetilde{A} \to \mathbb{C}$ be the quotient map. Then we have
\[ \pi(u')\pi(a') - \pi(a') = 0, \quad \|\pi(a') - 1\| < \epsilon' < \frac{1}{2}, \quad \|\pi(u') - 1\| < \epsilon' < \frac{1}{2}. \]
It follows that $\pi(u') = 1$. In other words $u' - 1 \in A$ and
\[ \left\| a - \frac{a'}{\pi(a')} \right\| \leq \|a - a'\| + \left\| \frac{a'}{\pi(a')} \pi(a') - 1 \right\| < \epsilon' + 2M \epsilon' = (2M + 1)\epsilon'. \]
If we replace $a'$ by $\frac{a'}{\pi(a')}$ then we get a positive element $a' \in \widetilde{A}$ and a unitary $u' \in \widetilde{A}$ in the connected component of the identity such that
\[ a' - 1, u' - 1 \in A, \quad u'a' = a', \]
\[ \|u - u'\| < \epsilon', \quad \|a - a'\| < (2M + 1)\epsilon'. \]
Write $u' = x + 1$, with $x \in A$. Then $x a' = 0$. Also, since $x$ is a normal element $x \in x^* Ax$. In particular, $u'$ is unitary of the sub-C*-algebra generated by $x^* Ax$ and the unit of $A$. Note that this C*-algebra can be identified with $(x^* Ax)$. Hence, by hypothesis there is a path of unitaries $(v_t)_{t \in [0,1]}$ in $(x^* Ax)$ such that $v_0 = 1$ and $v_1 = u'$. Moreover, this path of unitaries
may be taken such that \( v_t - 1 \in \overline{x^*Ax} \) for all \( t \). Note that with this choice \((v_t)_{t \in [0,1]}\) satisfies \( v_1a' - a' = 0 \). Now it follows that
\[
(3.3) \quad \|v_t a - a\| \leq \|v_t(a - a')\| + \|a' - a\| < 2\epsilon' \leq \epsilon.
\]
Let \( z_t = tu + (1 - t)a' \), with \( t \in [0,1] \). Then by (3.2)
\[
\|u - z_t\| = \|(1 - t)(u - a')\| < \epsilon' < \frac{1}{2}.
\]
Hence \( z_t \) is invertible for all \( t \) and by [3, Lemma 1] we have
\[
\|1 - z_t\| \leq \|1 - z_t^*z_t\|^\frac{1}{2} \leq \left(\|u - z_t\| \cdot \|u\|\right)^\frac{1}{2} < (\epsilon' + \epsilon')^\frac{1}{2} < 2\epsilon'.
\]
Since \( z_t \) is invertible it has a polar decomposition \( z_t = w_t^*z_t \) with \( w_t \) a unitary in \( \tilde{A} \). Note that the path of unitaries \((w_t)_{t \in [0,1]}\) satisfies \( w_0 = u' \), \( w_1 = u \), and \( w_t - 1 \in A \). Also, by the previous inequalities we have
\[
\|u - w_t\| \leq \|u - w_t\| + \|w_t(z_t - 1)\| < \epsilon' + 2\epsilon' = 3\epsilon'.
\]
This implies that
\[
(3.4) \quad \|w_t a - a\| = \|(v_t - u)a\| + \|ua - a\| < 3M\epsilon' + \delta \leq (3M + 1)\epsilon' \leq \epsilon.
\]
The proof of the lemma now follows by taking
\[
u_t = \begin{cases}v_{2t}, & \text{if } t \in [0, \frac{1}{2}], \\w_{2t - 1}, & \text{if } t \in [\frac{1}{2}, 1].\end{cases}
\]
Indeed, \( u_0 = 1, u_1 = u, u_t - 1 \in A \), and by (3.3) \( \|u_t a - a\| < \epsilon \) for all \( t \in [0,1] \). \( \square \)

Let \( X \) be a compact Hausdorff space an let \( V \) be an open subset of \( X \). We denote by \( \partial V, V \), and \( X \setminus V \) the boundary, the closure, and the complement in \( X \) of the set \( V \). Let \( B \) be a C*-algebra. In the following we use the natural identifications:
\[
C(X, B) \cong C(X) \otimes B,
\]
\[
C(X, B)/C_0(V, B) \cong C(X \setminus V, B),
\]
\[
C(X, B)^\sim/C_0(V, B) \cong C(X \setminus V, B)^\sim.
\]
Also, if \( B \) is non-unital we identify \( C(X, B)^\sim \) with the subalgebra of \( C(X, \tilde{B}) \) generated by \( C(X, B) \) and the unit of \( C(X, \tilde{B}) \). In this way, the image of an element \( a \in C(X, B)^\sim \) in the quotient by the ideal \( C(V, B) \) may be identified with the restriction \( a_{X \setminus V} \) of \( a \) to the closed set \( X \setminus V \).

**Lemma 3.5.** Let \( B \) be a C*-algebra and let \( X = [0,1]^k \), for some positive integer \( k \). Let \( a \) be an element of \( C(X, B)^\sim \) such that \( a - 1 \in C(X, B) \). Suppose that for every \( \epsilon > 0 \) there are open subsets \( U, V, W \subseteq X \), with \( \overline{U} \subset V \subset \overline{V} \subset W \), such that the stable rank of \( C(\partial V, B) \) is one, every ideal of \( C(\partial V, B) \) has trivial \( K_1 \)-group, \( a_{X \setminus U} \) is in the closure of the invertible elements of \( C(X \setminus U, B)^\sim \), and there is a unitary \( u \in C(\overline{W}, B)^\sim \) such that
\[
\|a_{\overline{W}} - u\| < \epsilon.
\]
Then \( a \) is in the closure of the invertible elements of \( C(X, B)^\sim \).

**Proof.** Let \( a \in C(X, B)^\sim \) with \( a - 1 \in C(X, B) \) and let \( \epsilon > 0 \). Let us show that there is a unitary \( w \in C(X, B)^\sim \) such that \( \| a - w \| < \epsilon \). (This clearly implies that \( a \) is the closure of the invertible elements of \( C(X, B)^\sim \) since \( \epsilon \) is arbitrary and \( |a| \) can be approximated by invertible elements.)

Choose \( 0 < \delta < \epsilon \) satisfying the statement of Lemma 3.4 for \( \frac{\epsilon}{2} \). Choose open subsets \( U, V, W \subseteq X \) satisfying the assumptions of the lemma for the number \( \frac{\delta}{2} \). Then there are unitaries \( u_0 \in C(W, B)^\sim \) and \( u_1 \in C(X \setminus U, B)^\sim \) such that

\[
\| a_W - u_0 |a_W| \| < \frac{\delta}{2}, \quad \| a_{X \setminus U} - u_1 |a_{X \setminus U}| \| < \frac{\delta}{2}.
\]  

Moreover, by a simple perturbation argument we may assume that \( u_0 - 1 \in C(W, B) \) and \( u_1 - 1 \in C(X \setminus U, B) \).

Now let us restrict the inequalities above to the boundary \( V \). Then, it follows that

\[
\| (u_0)_{\partial V} |a_{\partial V}| - (u_1)_{\partial V} |a_{\partial V}| \| < \delta.
\]

By choice of \( V \), the stable rank of \( C(\partial V, B) \) is one, and the \( K_1 \)-group of every ideal of \( C(\partial V, B) \) is trivial. Hence, every hereditary sub-C*-algebra of \( C(\partial V, B) \) has stable rank one and trivial \( K_1 \)-group. This implies, by [21, Theorem 2.10], the unitary group of every hereditary sub-C*-algebra of \( C(\partial V, B) \) is connected. It follows now by the preceding inequality and Lemma 3.4 applied to the element \( |a_{\partial V}| \) and to the unitary \( (u_0)_{\partial V} (u_1)_{\partial V} \), that there is a path of unitaries \( u(t) : [0, 1] \to C(\partial V, B)^\sim \), with \( u(t) - 1 \in C(\partial V, B) \), such that \( u(0) = (u_0)_{\partial V}, u(1) = (u_1)_{\partial V} \) and

\[
\| u(t) |a_{\partial V}| - u_1 |a_{\partial V}| \| < \frac{\epsilon}{2},
\]

for all \( t \in [0, 1] \). Using this inequality and the second inequality of (3.5) restricted to the boundary of \( V \) we get

\[
\| a_{\partial V} - u(t) |a_{\partial V}| \| < \epsilon,
\]

for all \( t \in [0, 1] \). Let \( d : X \to [0, \infty) \) be a metric in \( X \) and let

\[
F_n = \{ x \in X \mid d(x, \partial V) \leq 1/n \}.
\]

Then \( \bigcup_n F_n = \partial V \) and \( \lim C(F_n, B)^\sim = C(\partial V, B)^\sim \), where the connecting *-homomorphisms are the obvious quotient maps. Since any path of unitaries in an inductive limit C*-algebra can be lifted approximately to finite stage C*-algebra there are \( F_n \subseteq \overline{V} \cap (X \setminus U) \) and a path of unitaries \( v(t) : [0, 1] \to C(F_n, B)^\sim \) such that \( v(0) = (u_0)_{F_n}, v(1) = (u_1)_{F_n} \), and

\[
\| a_{F_n} - v(t) |a_{F_n}| \| < \epsilon,
\]  

(3.6)
for all $t \in [0,1]$. Moreover, since $u(t) - 1 \in C(\partial V, B)$ we may choose $v(t)$ such that $v(t) - 1 \in C(F_n, B)$. Let $w$ be the element defined by

$$
w(x) = \begin{cases} 
    u_0(x), & \text{if } x \in V \setminus F_n, \\
    u_1(x), & \text{if } x \in X \setminus (V \cup F_n), \\
    v(g(x))(x), & \text{if } x \in F_n,
\end{cases}
$$

where $g: F_n \to [0,1]$ is given by

$$
g(x) = \frac{d(x, V \setminus F_n)}{\max\{d(x, X \setminus (V \cup F_n)), d(x, V \setminus F_n)\}.
$$

Note that $g(x)$ is continuous and $g(x) = 0$ if $x \in F_n \cap V \setminus F_n$, and $g(x) = 1$ if $x \in F_n \cap X \setminus (V \cup F_n)$. Hence, $w$ is a well defined and it is a unitary of $C(X, B)$ since $w(x) - 1 \in B$ for all $x \in X$. It follows now by equations (3.5) and (3.6) that

$$
\|a - w|a|| < \epsilon.
$$

\[\square\]

Recall that a C*-algebra $B$ is locally contained in a class of C*-algebras $A$ if for every $\epsilon > 0$ and every finite subset $F$ of $B$ there exists a C*-algebra $A \in A$ and a *-homomorphism $\phi: A \to B$ such that the distance from $x$ to $\phi(A)$ is less than $\epsilon$ for every $x \in F$.

**Theorem 3.6.** Let $A$ be the class of C*-algebras defined in Definition 1.3. Let $B$ be a simple inductive limit of a system of $RSH_0$-algebras with no dimension growth. The following statements hold:

1. If $B$ is projectionless and $K_0(B) = K_1(B) = 0$ then $sr(A \otimes B) = 1$ for every C*-algebra $A \in A$.
2. If $B$ is either projectionless or it is unital with no nonzero projection but its unit and $K_1(B) = 0$ then $sr(A \otimes B) = 1$ for every C*-algebra $A$ that is approximately contained in the class of $RSH_0$-algebras with 1-dimensional spectrum.

**Proof.** (i) Let $\mathcal{D}$ be the subclass of $A$ consisting of the C*-algebras $A \in A$ such that $sr(A \otimes B) = 1$. Then by the results of [24] (i.e., Theorems 6.1, 4.4, 6.4, 4.3, and the proof of Theorem 5.1) and by [18 Lemma 3] it follows that (ii), (iii), (iv), (v), and (vi) of Definition 1.3 holds if we replace $A$ by $\mathcal{D}$. Hence, $\mathcal{D}$ agrees with $A$ if and only if $C_0(X) \in \mathcal{D}$, for every locally compact Hausdorff space $X$.

Let $X$ be a locally compact Hausdorff space. Let us show that the stable rank of $C_0(X, B)$ is one. First we consider the case of compact metric spaces $X$ of finite covering dimension. The proof in this case will proceed by induction on the covering dimension of $X$. If the covering dimension of $X$ is zero then $C(X)$ is an AF-algebra. Hence, $sr(C(X, B)) = 1$ since by Theorem 3.2 the stable rank of $B$ is one. Now let us assume that $sr(C(X, B)) = 1$ for every compact metric space $X$ of covering dimension at most $n - 1$. 


Let \( B = \lim (B_i, \phi_{i,j}) \) be as in the statement of the theorem. By Proposition 2.18 we may assume that the C*-algebras \( B_i \) have compact spectrum, that the *-homomorphisms \( \phi_{i,j} \) are injective, and that \( \text{Ideal}(\phi_{i,i+1}(B_i)) = B_{i+1} \) for all \( i \). Also, we have

\[
C(X, B)^\sim = \lim_{\rightarrow} (C(X, B_i)^\sim, (\text{id}_X \otimes \phi_{i,j})^\sim),
\]

where \( \text{id}_X : C(X) \to C(X) \) denotes the identity map. In particular, this implies that any element of \( C(X, B)^\sim \) can be approximated by the images in \( C(X, B_i)^\sim \) of elements of the C*-algebras \( C(X, B_i)^\sim \). In addition, any element of \( C(X, B_i)^\sim \) can be approximated by elements of the form \( \lambda(b + 1) \), with \( \lambda \in \mathbb{C} \setminus \{0\} \) and \( b \in C(X, B_i) \). Therefore, \( \text{sr}(C(X, B)) = 1 \) if for every \( i \geq 1 \) and for every element \( a \in C(X, A_i)^\sim \), with \( a - 1 \in C(X, A_i) \), the element \( (\text{id}_X \otimes \phi_{i,\infty})^\sim(a) \) can be approximated by invertible elements of \( C(X, B)^\sim \).

Let \( X = [0, 1]^n \). Let \( a \in C(X, B_i)^\sim \), for some \( i \), be such that \( a - 1 \in C(X, B) \) and such that \( (\text{id}_X \otimes \phi_{i,\infty})^\sim(a) \) is not invertible. Let \( \epsilon > 0 \). Consider the elements

\[
(3.7) \quad a' = (\text{id}_X \otimes \phi_{i,\infty})^\sim(a), \quad b = f(|a|), \quad b' = (\text{id}_X \otimes \phi_{i,\infty})^\sim(b),
\]

where \( |a| \) denotes the element \( (a^*a)^{\frac{1}{2}} \), \( 0 < \epsilon < \min(1, \|a\|) \), and \( f \in C_0(0, \infty) \) is such that \( f(x) > 0 \) for \( x \in (0, \epsilon) \) and \( f(x) = 0 \) for \( x \in (\epsilon, \infty) \). By functional calculus \( b \in C(X, B_i) \) and as a result \( b' \in C(X, B) \). As a consequence, the ideal generated by \( b' \) has the form \( C_0(Y, B) \) for some open subset \( Y \) of \( X \) (to deduce this we are also using the simplicity of \( B \)). This implies by Lemma 3.1 applied to the element \( a' \) and to the ideal \( C_0(Y, B) \), that \( a'_{X \setminus Y} \in C(X \setminus Y) \) is invertible. Choose an increasing sequence of open subsets \( (Y_j)_{j=1}^\infty \) such that \( \overline{Y_j} \subset Y_{j+1} \), \( Y = \bigcup_j Y_j \), and the covering dimension of the boundary \( \partial Y_j \) of \( Y_j \) is at most \( n - 1 \) for all \( j \). It follows that,

\[
C(X \setminus Y, B)^\sim = \lim_{\rightarrow} C(X \setminus Y_j, B)^\sim.
\]

Hence, since \( a'_{X \setminus Y} \) is invertible in \( C(X \setminus Y, B)^\sim \) there is \( j \) such that \( a'_{X \setminus Y_k} \) is invertible in \( C(X \setminus Y_k, B)^\sim \) for all \( k \geq j \). Set

\[
(3.8) \quad U = Y_j, \quad V = Y_{j+1}, \quad W = Y_{j+2}.
\]

Then \( a'_{X \setminus U} \) is invertible and \( \overline{U} \subset V \subset \overline{V} \subset W \subset \overline{W} \subset U \).

The element \( b' \) generates the ideal \( C_0(U, B) \). Therefore, \( b'_W \) generates \( C(\overline{W}, B) \) as an ideal. This implies using that \( \overline{W} \) is compact that

\[
K = \min_{x \in \overline{W}} \|b'_W(x)\| \neq 0.
\]

For each \( 1 \leq j \leq N \) let \( g_j \) be positive continuous function with support on the interval \( \left( \frac{j-1}{N}K, \frac{j}{N}K \right) \). Set \( b'_j = g_j(b'_W) \). Then \( b'_j(x) \neq 0 \) for every \( x \in \overline{W} \) (Note that \( b'_j(x) = 0 \) for some \( x \in \overline{W} \) implies that \( b_j(x) \in B \) has a gap in its spectrum. In turn, this implies that \( B \) contains a nonzero projection which contradics the assumption on \( B \).) Also, the elements \( (b'_j)_i^N \) are mutually orthogonal.
The closed subset $W \subset [0,1]^n$ has covering dimension at most $n$. Hence, the C*-algebra $C(W,B)$ can be written as and the inductive limit of a system of RSH$_0$-algebras with no dimension growth. Specifically,

$$C(W,B) = \lim_{\to} (C(W,B_j), \text{id}_W \otimes \phi_{j,k}).$$

(3.9)

Note that the C*-algebras $C(W,B_j)$ have compact spectrum, the homomorphisms $\text{id}_W \otimes \phi_{j,k}$ are injective, and the image of $\text{id}_W \otimes \phi_{j,j+1}$ generates $C(W,B_{j+1})$ as an ideal for all $j$.

Let $b_1, b_2, \cdots, b_N \in C(W,B_i)$ be the elements defined by $b_j = g_j(b_W)$. Then by the definition of $b'$ (see (3.7)) we have

$$b'_j = (\text{id}_W \otimes \phi_{i,l})(b_j).$$

By Lemma 2.5 applied to the inductive limit (3.9) and to the elements $(b_j)_{j=1}^N$, there exists $k > i$ such that

$$\text{Ideal}((\text{id}_W \otimes \phi_{i,l})(b_j)) = C(W,B_l),$$

for all $j$ and all $l \geq k$. This implies that

$$\text{ev}_{x,y}((\text{id}_W \otimes \phi_{i,l})(b_j)) \neq 0,$$

for all $1 \leq j \leq N$, $l \geq k$, $x \in W$, and $y$ in the total space of $B_l$. As a consequence, since $(b_i)_{i=1}^N$ are pairwise orthogonal elements of the C*-algebra generated by $b'_W$ we get

$$\text{rank} \left( \text{ev}_{x,y}((\text{id}_W \otimes \phi_{i,l})(b_j)) \right) \geq N,$$

for all $l \geq k$, $x \in W$, and $y$ in the total space of $B_l$. This implies by Lemma 2.17 that there exists a unitary $u \in C(W,B)$ such that

$$\|a'_W - u|a'_W|\| < 5\epsilon.$$

We have shown that given $\epsilon > 0$ there are open subsets $U, V, W \subset X$ such that $U \subset V \subset \overline{V} \subset W$, the covering dimension of $\partial V$ is at most $n - 1$, $a'_{X \setminus U}$ is invertible, and $a'_W$ satisfies the inequality above. Also, note that every ideal of $C(\partial V, B)$ has trivial $K_1$-group and by the induction hypothesis $\text{sr}(C(\partial V, B)) = 1$. Therefore, by Lemma 3.5 the element $a'$ is invertible. This shows that $\text{sr}(C([0,1]^n, B)) = 1$.

Let $X$ be a CW-complex of dimension at most $n$. Then $C(X)$ admits a recursive subhomogeneous decomposition with base spaces homeomorphic to cubes of dimension at most $n$. This implies by [21 Theorem 3.9] and [2 Proposition 4.1 (i)] that $\text{sr}(C(X,B)) = 1$. Now, suppose that $X$ is an arbitrary compact metric space of covering dimension at most $n$. Then, by [8 Theorem 1.13.2], $X$ is homeomorphic to an inverse limit of CW-complexes of dimension at most $n$. This implies that $C(X,B)$ can be written as a direct limit of a sequence of C*-algebras of the form $C(Y_i,B)$, where each space $Y_i$ is a CW-complex of dimension at most $n$. It follows now by [24 Theorem 5.1] that the stable rank of $C(X,B)$ is one. This concludes the proof by induction.
By [8, Theorem 3.3.5] every compact Hausdorff space \( X \) is homeomorphic to an inverse limit of compact metric spaces of finite covering dimension. Hence, as above, we conclude that \( \text{sr}(C(X, B)) = 1 \) by [24, Theorem 5.1] (this theorem is stated for direct limits of sequences of C*-algebras but it is easy to see that it holds for arbitrary direct limits). Now, let \( X \) be an arbitrary locally compact Hausdorff space. Then \( C_0(X, B) \) is an ideal of \( C(\tilde{X}, B) \), where \( \tilde{X} \) denotes the one-point compactification of \( X \). Since \( \tilde{X} \) is compact and Hausdorff \( \text{sr}(C(\tilde{X}, B)) = 1 \). Therefore, by [24, Theorem 4.4], \( \text{sr}(C_0(X, B)) = 1 \).

(ii) Let \( B \) be a unital C*-algebra containing no nontrivial projection but its unit. Let \( a \in B \) be a non-invertible positive element of \( B \). Then the hereditary sub-C*-algebra \( aBa \) is projectionless. In addition, \( B \) and \( aBa \) are stable isomorphic since \( B \) is simple. Therefore, \( K_1(aBa) = 0 \) and by [24, Theorem 6.4] the stable rank of the tensor product of \( B \) with a C*-algebra \( A \) is one if and only if the stable rank of the tensor product of \( aBa \) with \( A \) is one. This shows that it is sufficient to prove (ii) for projectionless C*-algebras.

Let \( B \) be a projectionless C*-algebra as in (ii). Then by repeating the first step of the induction in the proof of (i) we conclude that the stable rank of \( C([0, 1], B) \) is one. (Note that in the proof of (i) the assumption on the K-groups of \( B \) was only used to assure that every ideal of \( C(\partial V, B) \) has trivial K-group. This clearly holds if the K-group of \( B \) is trivial and \( V \) is an open subset of \([0, 1]\).) Also, as in the proof of (i) this implies that the stable rank of \( C_0(X) \otimes B \) is one for every locally compact 1-dimensional space \( X \). Now, using [24, Theorem 6.1], we get that the stable rank of \( C([0, 1], M_n) \otimes B \) is one for every \( n \). Therefore, by [24 Proposition 4.1 (i)] the stable rank of \( A \otimes B \) is one for any RSH\(_0\)-algebra with 1-dimensional spectrum. The case that \( A \) is locally contained in the class of RSH\(_0\)-algebra with 1-dimensional spectrum follows using the ideas of the proof of [24, Theorem 5.1] and that the C*-algebra \( A \otimes B \) is approximately contained in the class of C*-algebras that are tensor products of \( B \) with RSH\(_0\)-algebras with one dimensional spectrum.

\[ \square \]

**Corollary 3.7.** If \( A \) is the inductive limit of separable type I C*-algebras and \( B \) is as in Theorem 3.6 then \( \text{sr}(A \otimes B) = 1 \).

An important example of a simple stably projectionless C*-algebra with trivial K-groups is the C*-algebra \( W \) constructed in [12]. This C*-algebra is an inductive limit of RSH\(_0\)-algebra; thus, the first part of Theorem 3.6 applies to \( W \). In other words, the following statement hold:

**Corollary 3.8.** Let \( A \) be a C*-algebra in \( A \). Then \( \text{sr}(A \otimes W) = 1 \).

Another important example of a simple C*-algebra for which Theorem 3.6 applies is the Jiang-Su algebra \( Z \). This C*-algebra is constructed as an inductive limit of RSH-algebras and is such that \( K_1(Z) = 0 \) and \( K_0(Z) = \mathbb{Z} \). For this C*-algebra we can proof the following:
Proposition 3.9. The following statements hold:

1. If $A$ is a C*-algebra that is locally contained in the class of RSH$_0$-algebras with 1-dimensional spectrum then $sr(A \otimes \mathbb{Z}) = 1$.

2. If $0 \to A \to B \to C \to 0$ is an exact sequence of C*-algebras with nonzero index map $\delta: K_1(A/I) \to K_0(I)$ then $sr(B \otimes \mathbb{Z}) \neq 1$.

3. If $X$ is CW-complex then
   
   $$sr(C(X) \otimes \mathbb{Z}) = \begin{cases} 
   1 & \text{if } \dim(X) \leq 1, \\
   2 & \text{if } \dim(X) > 1.
   \end{cases}$$

Proof. The first part of proposition is a corollary of the second part of Theorem 3.6. Let us now prove (ii) and (iii).

(ii) Let us proceed by contradiction. Suppose that $sr(B \otimes \mathbb{Z}) = 1$. Then $sr(A \otimes \mathbb{Z}) = sr(C \otimes \mathbb{Z}) = 1$. Using the Kunneth formula and that $K_0(\mathbb{Z}) = \mathbb{Z}$ and $K_1(\mathbb{Z}) = 0$ we get the following commutative diagram involving the respective index maps:

$$
\begin{array}{ccc}
K_1(C \otimes \mathbb{Z}) & \xrightarrow{\delta'} & K_0(A \otimes \mathbb{Z}) \\
\downarrow \cong & & \downarrow \cong \\
K_1(C) & \xrightarrow{\delta} & K_0(A)
\end{array}
$$

By assumption $\delta$ is nonzero thus $\delta'$ is nonzero. But this contradicts the fact that $sr(B \otimes \mathbb{Z}) = 1$ by [18, Lemma 3].

(iii) By [28, Proposition 1.5] the stable rank of $C(X) \otimes \mathbb{Z}$ is at most two for any CW-complex $X$. If $\dim(X) \leq 1$ then $sr(C(X) \otimes \mathbb{Z}) = 1$ by (i). Now let us consider the case $\dim(X) > 1$. We will proceed by contradiction. Suppose that $sr(C(X) \otimes \mathbb{Z}) = 1$. Since the dimension of $X$ is greater than one $X$ contains a closed subspace homeomorphic to the unit disc $\mathbb{D}$. It follows that both $C(\overline{\mathbb{D}}) \otimes \mathbb{Z}$ and $C(\mathbb{T}) \otimes \mathbb{Z}$ are quotients of $C(X) \otimes \mathbb{Z}$ and that $C(\mathbb{D}) \otimes \mathbb{Z}$ is an ideal of $C(X) \otimes \mathbb{Z}$. Hence,

$$sr(C(\overline{\mathbb{D}}) \otimes \mathbb{Z}) = sr(C_0(\mathbb{D}) \otimes \mathbb{Z}) = sr(C(\mathbb{T}) \otimes \mathbb{Z}) = 1.$$  

Consider the three term exact sequence:

$$0 \to C_0(\mathbb{D}) \to C(\overline{\mathbb{D}}) \to C(\mathbb{T}) \to 0.$$  

Then the index map $\delta: K_1(C(\mathbb{T})) \to K_0(C(\mathbb{D}))$ is non-zero. This implies that the respective index map corresponding to the exact sequence

$$0 \to C_0(\mathbb{D}) \otimes \mathbb{Z} \to C(\overline{\mathbb{D}}) \otimes \mathbb{Z} \to C(\mathbb{T}) \otimes \mathbb{Z} \to 0,$$

is nonzero. But this contradicts the fact that $sr(C(\overline{\mathbb{D}}) \otimes \mathbb{Z}) = 1$ by [18, Lemma 3].

□

Proposition 3.10. Let $A$ and $B$ be C*-algebras with $B$ as in Theorem 3.6. Suppose that a quotient of $A \otimes K$ contains a properly infinite projection. Then $sr(A \otimes B) \neq 1$.  


Proof. Suppose that $I$ is an ideal of $A$ such that the quotient C*-algebra $(A/I) \otimes K$ contains a properly infinite projection. In other words, there is an embedding of $O_{\infty}$ into $(A/I) \otimes K$. By the classification theorem of Kirchberg and Phillips $O_{\infty} \otimes B \cong O_2 \otimes K$; thus, $O_2$ embeds into $(A/I) \otimes B \otimes K$. It follows that, sr$(A/I) \otimes B \otimes K) \leq$ sr$(A \otimes B \otimes K)$ and that sr$(A \otimes B \otimes K) = 1$ if and only if sr$(A \otimes B) = 1$ we conclude that sr$(A \otimes B) \neq 1$. □

3.2. Reduction of the topological dimension. Let $B$ denote the class of C*-algebras that are stably isomorphic to commutative C*-algebras. Let $C$ denote the class of C*-algebras that are stably isomorphic to 1-dimensional NCCW-complexes. Recall that a C*-algebra $A$ is a 1-dimensional noncommutative CW-complex (or shortly a NCCW-complex) if it can be expressed as a pullback diagram of the form:

$$
\begin{array}{ccc}
A & \rightarrow & E \\
\downarrow & & \downarrow \phi \\
C_0([0,1], F) & \rightarrow & F \oplus F
\end{array}
$$

(3.10)

where $E$ and $F$ are finite dimensional C*-algebras. (We do not assume that the map from $E$ to $F \oplus F$ is unital)

Lemma 3.11. If $A$ is a 1-dimensional NCCW-complex and $I$ is an ideal of $A$ then $I$ can be written as an inductive limit of a sequence of 1-dimensional NCCW-complexes.

Proof. Let $A$ be a 1-dimensional NCCW-complex as in the pullback diagram (3.10) and let $F = \bigoplus_{i=1}^k M_{n_i}$. Let $I$ be an ideal of $A$. Then by Theorem 2.7 there are open subsets $(U_i)_{i=1}^k$ of $[0,1]$ and an ideal $E'$ of $E$ such that

$$
\begin{array}{ccc}
I & \rightarrow & E' \\
\downarrow & & \downarrow \phi|_{E'} \\
\bigoplus_{i=1}^k C_0(U_i, M_{n_i}) & \rightarrow & F \oplus F
\end{array}
$$

is a pullback diagram. Decompose each $U_i$ as the union of disjoint open subintervals of $[0,1]$. Denote by $V_i$ and $W_i$ the union of the open subintervals in the decomposition of $U_i$ that contain either the point 0 or the point 1, and the union of the open subintervals in this decomposition that do not contain either point. Let $B$ be the C*-algebra obtained by the pullback diagram

$$
\begin{array}{ccc}
B & \rightarrow & E' \\
\downarrow & & \downarrow \phi|_{E'} \\
\bigoplus_{i=1}^k C_0(V_i, M_{n_i}) & \rightarrow & F \oplus F
\end{array}
$$

Then $B$ is a 1-dimensional NCCW-complex and $A \cong B \oplus \bigoplus_{i=1}^k C_0(W_i, M_{n_i})$. For each $i$ write $W_i$ as the union of an increasing sequence of open subsets
$W^{(j)}_i$, $j = 1, 2, \ldots$, where each $W^{(j)}_i$ is the disjoint union of a finite number of open intervals. It follows that

$$B \oplus \bigoplus_{i=1}^k C_0(W_i, M_{n_i}) = \lim_{\rightarrow} B \oplus \bigoplus_{i=1}^k C_0(W^{(j)}_i, M_{n_i}).$$

Now, each algebra $C_0(W^{(j)}_i, M_{n_i})$ is a 1-dimensional NCCW-complex. Therefore, $A$ can be written as an inductive limit of 1-dimensional NCCW-complexes.

**Theorem 3.12.** Let $A$ be a $C^*$-algebra that is locally contained in $B$ and let $B$ be a simple inductive limit of 1-dimensional NCCW-complexes such that $K_0(B) = K_1(B) = 0$. Then $A \otimes B$ is locally contained in $C$. Moreover, if $A$ is separable then $A \otimes B$ can be written as an inductive limit of a sequence of $C^*$-algebras in $C$.

**Proof.** Since $B$ is an inductive limit of 1-dimensional NCCW-complexes $C([0,1]) \otimes B$ is an inductive limit of 2-dimensional NCCW-complexes. By Theorem 5.2 the stable rank of $C([0,1]) \otimes B$ is one. Also, $K_0(C([0,1]) \otimes B) = 0$. Hence, by Theorem 5.3 Theorem 4.2 (iv) the $C^*$-algebra $C([0,1]) \otimes B$ is an inductive limit of 1-dimensional NCCW-complexes. By induction the same is true for the $C^*$-algebras $C([0,1]^n) \otimes B$, $n = 1, 2, \ldots$. Since $C([0,1]^n) = \lim_{\rightarrow} C([0,1]^n)$ and 1-dimensional NCCW-complexes are semiprojective it follows that $C([0,1]^n) \otimes B$ is an inductive limit of 1-dimensional NCCW-complexes. Using semiprojectivity again we get that any quotient of $C([0,1]^n) \otimes B$ is the inductive limit of a sequence of 1-dimensional NCCW-complexes. If $X$ is a compact metric space then it can realized as a close subspace of $[0,1]^n$. Hence, the $C^*$-algebra $C(X) \otimes B$ is isomorphic to a quotient of $C([0,1]^n) \otimes B$. This implies that $C(X) \otimes B$ can be written as the inductive limit of a sequence of 1-dimensional NCCW-complexes. In particular, $C(X) \otimes B \in C$. By [3] Theorem 3.3.5 every compact Hausdorff space is homeomorphic to an inverse limit of compact metric spaces. In particular, this implies that any abelian unital $C^*$-algebra is locally contained in the class of abelian $C^*$-algebras whose spectrum are compact metric spaces. It follows that $C(X) \otimes B \in C$ for every compact Hausdorff space $X$. Since the class $C$ is closed under the operation of taking tensor products with matrix algebras and inductive limits we get $C(X, K) \otimes B = \lim_{\rightarrow} C(X, M_{n_i}) \otimes B \in C$. If $A$ is a hereditary subalgebra of $C(X, K)$ then clearly $A \otimes B \in C$. This concludes the proof of the first part of the theorem. (Note that if $X$ is locally compact and $\tilde{X}$ denotes the one point compactification of $X$ then $C_0(X, K)$ is a hereditary subalgebra of $C(\tilde{X}, K)$, thus $C_0(X, K) \in C$.)

Now assume that $A$ is separable. We have shown that $A \otimes B \in C$. Note that this implies that $A \otimes K \otimes B$ is locally contained in the class of $C^*$-algebras that are ideals of 1-dimensional NCCW-complexes. Hence $A \otimes K \otimes B$ is locally contained in the class of 1-dimensional NCCW-complexes by Lemma 3.11. Now using the semiprojectivity of 1-dimensional NCCW-complexes
and the separability of $A \otimes K \otimes B$ we can write $A \otimes K \otimes B$ as an inductive limit of a sequence of 1-dimensional NCCW-complexes. The second part of the theorem follows using the following result proved in \cite[Corollary 4]{4}: If $A = \lim A_n$ and the stable rank of $A$ is one then every hereditary subalgebra of $A$ can be written as inductive limit of hereditary subalgebras of the C*-algebras $A_n$.

**Corollary 3.13.** Let $A$ and $B$ be as in the theorem above. Then $\text{dr}(A \otimes B) = 1$.

**Proof.** The decomposition rank of a 1-dimensional NCCW-complex is at most 1 by \cite[Theorem 1.6]{30}. Hence by \cite[Corollary 3.11 and Proposition 3.10]{14} the decomposition rank of a C*-algebra that is stable isomorphic to a 1-dimensional NCCW-complex is at most 1. Therefore, the decomposition rank of any C*-algebra in $\mathcal{C}$ is at most one. By the definition of decomposition rank and Averson’s Extension Theorem the same holds for C*-algebras that are locally contained in $\mathcal{C}$. In particular, $\text{dr}(A \otimes B) \leq 1$. The decomposition rank of $A \otimes B$ can not be zero since this would imply that $A \otimes B$ is an AF-algebra (see \cite[Example 6.1 (i)]{14}) contradicting the fact that $A \otimes B$ is projectionless.

**Corollary 3.14.** If $A$ is a C*-algebra that is locally contained in $\mathcal{B}$. Then $A \otimes \mathcal{W}$ is locally contained in $\mathcal{C}$. In particular, $\text{dr}(A \otimes \mathcal{W}) = 1$.

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