RECENT RESULTS ON THERMAL CASIMIR FORCE BETWEEN DIELECTRICS AND RELATED PROBLEMS

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We review recent results obtained in the physics of the thermal Casimir force acting between two dielectrics, dielectric and metal, and between metal and semiconductor. The detailed derivation for the low-temperature behavior of the Casimir free energy, pressure and entropy in the configuration of two real dielectric plates is presented. For dielectrics with finite static dielectric permittivity it is shown that the Nernst heat theorem is satisfied. Hence, the Lifshitz theory of the van der Waals and Casimir forces is demonstrated to be consistent with thermodynamics. The nonzero dc conductivity of dielectric plates is proved to lead to a violation of the Nernst heat theorem and, thus, is not related to the phenomenon of dispersion forces. The low-temperature asymptotics of the Casimir free energy, pressure and entropy are derived also in the configuration of one metal and one dielectric plate. The results are shown to be consistent with thermodynamics if the dielectric plate possesses a finite static dielectric permittivity. If the dc conductivity of a dielectric plate is taken into account this results in the violation of the Nernst heat theorem. We discuss both the experimental and theoretical results related to the Casimir interaction between metal and semiconductor with different charge carrier density. Discussions in the literature on the possible influence of spatial dispersion on the thermal Casimir force are analyzed. In conclusion, the conventional Lifshitz theory taking into account only the frequency dispersion remains the reliable foundation for the interpretation of all present experiments.

Keywords: Casimir force; Lifshitz theory; thermal corrections.

1. INTRODUCTION

The Casimir effect is the force and also the specific polarization of the vacuum arising in restricted quantization volumes and originating from the zero-point oscillations of quantized fields. This force acts between two closely spaced macrobodies, between an atom or a molecule and macrobody or between two atoms or molecules. During more than fifty years, passed after the discovery of the Casimir effect, it has attracted much theoretical attention because of numerous applications in quantum field theory, atomic physics, condensed matter physics, gravitation and cosmology,

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mathematical physics, and in nanotechnology (see monographs\textsuperscript{2–4} and reviews\textsuperscript{5–7}). In multidimensional Kaluza-Klein supergravity the Casimir effect was used\textsuperscript{8} as a mechanism for spontaneous compactification of extra spatial dimensions and for constraining the Yukawa-type corrections to Newtonian gravity.\textsuperscript{6,10,11,12,13} In quantum chromodynamics the Casimir energy plays an important role in the bag model of hadrons.\textsuperscript{3} In cavity quantum electrodynamics the Casimir interaction between an isolated atom and a cavity wall leads to the level shifts of atomic electrons depending on the position of the atom near the wall.\textsuperscript{2} Both the van der Waals and Casimir forces are used\textsuperscript{14,15} for the theoretical interpretation of recent experiments on quantum reflection and Bose-Einstein condensation of ultracold atoms on or near the cavity wall of different nature. In condensed matter physics the Casimir effect turned out to be important for interaction of thin films, in wetting processes, and in the theory of colloids and lattice defects.\textsuperscript{16} The Casimir force was used to actuate nanoelectromechanical devices\textsuperscript{17} and to study the absorption of hydrogen atoms by carbon nanotubes.\textsuperscript{18} Theoretical work on the calculation of the Casimir energies and forces stimulated important achievements in mathematical physics and in the theory of renormalizations connected with the method of generalized zeta function and heat kernel expansion.\textsuperscript{6,19} All this made the Casimir effect the subject of general interdisciplinary interest and attracted permanently much attention in the scientific literature.

The last ten years were marked by the intensive experimental investigation of the Casimir force between metallic test bodies (see Refs.\textsuperscript{20–33}). During this time the agreement between experiment and theory on the level of 1-2% of the measured force was achieved. This has become possible due to the use of modern laboratory techniques, in particular, of atomic force microscopes and micromechanical torsional oscillators. Metallic test bodies provide advantage in comparison with dielectrics because their surfaces avoid charging. In Refs.\textsuperscript{34,35} where the importance of the Casimir effect for nanotechnology was pioneered, it was demonstrated that at separations below 100 nm the Casimir force becomes larger than the typical electrostatic forces acting between the elements of microelectromechanical systems. Bearing in mind that the miniaturization is the main tendency in modern technology, it becomes clear that the creation of new generation of nanotechnological devices with further decreased elements and separations between them would become impossible without careful account and calculation of the Casimir force.

Successful developments of nanotechnologies based on the Casimir effect calls for more sophisticated calculation methods of the Casimir forces. Most of theoretical output produced during the first decades after Casimir’s discovery did not take into account experimental conditions and real material properties of the boundary bodies, such as surface roughness, finite conductivity and nonzero temperature. The basic theory giving the unified description of both the van der Waals and Casimir forces was elaborated by Lifshitz\textsuperscript{36,37,38} shortly after the publication of Casimir’s paper. It describes the boundary bodies in terms of the frequency dependent dielec-
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The static dielectric permittivity $\epsilon(\omega)$ at nonzero temperature $T$. In the applications of the Lifshitz theory to dielectrics it was supposed that the static dielectric permittivity (i.e., the dielectric permittivity at zero frequency) is finite. The case of ideal metals was obtained from the Lifshitz theory by using the so-called Schwinger’s prescription, i.e., that the limit $|\epsilon(\omega)| \to \infty$ should be taken first and the static limit $\omega \to 0$ second. For ideal metals the same result, as follows from the Lifshitz theory combined with the Schwinger’s prescription, was obtained independently in the framework of thermal quantum field theory in Matsubara formulation. However, the cases of real dielectrics and metals (which possess some nonzero dc conductivity at $T > 0$ and finite dielectric permittivity at nonzero frequencies, respectively) remained practically unexplored for a long time. The case of semiconductor boundary bodies was also unexplored despite of the crucial role of semiconductor materials in nanotechnology.

Starting in 2000, several theoretical groups in different countries attempted to describe the Casimir interaction between real metals at nonzero temperature in the framework of Lifshitz theory. They have used different models of the metal conductivity and arrived to controversial conclusions. In Ref. using the dielectric function of the Drude model, quite different results than for ideal metals were obtained. According to Ref. 41, at short separations (low temperatures) the thermal correction to the Casimir force acting between real metals is several hundred times larger than between ideal ones. In addition, at large separations of a few micrometers (high temperatures) a two times smaller magnitude of the thermal Casimir force was found than between ideal metals (the latter is known as “the classical limit”). In Refs. 44, 45 the dielectric permittivity of the plasma model was used to describe real metals and quite different results were obtained. At short separations the thermal correction appeared to be small in qualitative agreement with the case of ideal metals. At large separations for real metals the familiar classical limit was reproduced. Later the approach of Ref. 41 was supported in Refs. 46, 47. The plasma model approach can be used at such separations $a$ that the characteristic frequency $c/(2a)$ belongs to the region of infrared optics. Later a more general framework, namely the impedance approach was suggested, which is applicable at any separation larger than the plasma wavelength. It was supported in Refs. 48, 49. In the region of the infrared optics, the impedance approach leads to practically the same results as the plasma model approach. As was shown in Refs. 50, 51 the Drude model approach leads to the violation of the Nernst heat theorem when applied to perfect metal crystal lattices with no impurities. This approach was also excluded by experiment at 99% confidence in the separation region from 300 to 500 nm and at 95% confidence in the wider separation region from 170 to 700 nm. On the contrary, the plasma model and impedance approaches were shown to be in agreement with thermodynamics and consistent with experiment. The polemic between different theoretical approaches to the description of the thermal Casimir force in the case of real metals can be found in Refs. 47, 54–56.
These findings on the application of the Lifshitz theory to real metals have inspired a renewed interest in the Casimir force between dielectrics. As was mentioned above, at nonzero temperature dielectrics possess an although small but not equal to zero dc conductivity. In Ref. 57, 58 the van der Waals force arising from the dc conductivity of a dielectric plate was shown to lead to large effect in noncontact atomic friction, a phenomenon having so far no satisfactory theoretical explanation. This brings up the question: Is it necessary or possible to take into account the dc conductivity of dielectrics in the Lifshitz theory? Recall that in the case of a positive answer the static dielectric permittivity of a dielectric material would be infinitely large. It is amply clear that the resolution of the above issue should be in accordance with the fundamentals of thermodynamics. For this reason, it is desirable to investigate the low-temperature behavior of the Casimir free energy and entropy for two dielectric plates both with neglected and included effects of the dc conductivity. 

A major breakthrough in the investigation of this problem was achieved in the year 2005. In Ref. 60 a new variant of perturbation theory was developed in a small parameter proportional to the product of the separation distance between the plates and the temperature. As a result, the behavior of the Casimir free energy, entropy and pressure at low temperatures was found analytically. If the static dielectric permittivity is finite, the thermal correction was demonstrated to be in accordance with thermodynamics. This solves positively the fundamental problem about the agreement between the Lifshitz theory and thermodynamics for the case of two dielectric plates. In Ref. 60 it was shown that, on the contrary, the formal inclusion of a small conductivity of dielectric plates at low frequencies into the model of their dielectric response leads to a violation of the Nernst heat theorem. This result gives an important guidance on how to extrapolate the tabulated optical data for the complex refractive index to low frequencies in numerous applications of the van der Waals and Casimir forces. All these problems and related ones arising for semiconductor materials are discussed in this review.

In Sec. 2 we derive the analytical behavior of the Casimir free energy, entropy and pressure in the configuration of two parallel dielectric plates at both low and high temperature. It is demonstrated that if the static dielectric permittivity is finite the Lifshitz theory is in agreement with thermodynamics. Sec. 3 contains the derivation of the low-temperature behavior for the Casimir free energy and entropy between two dielectric plates with included dc conductivity. In this case the Lifshitz theory is found to be in contradiction with the Nernst heat theorem. The conclusion is made that the conductivity properties of a dielectric material at a constant current are unrelated to the van der Waals and Casimir forces and must not be included into the model of dielectric response. In Sec. 4 we consider the thermal Casimir force between metal and dielectric. This problem was first investigated in Ref. 61. It was found that for dielectric plate with finite static dielectric permittivity the Nernst heat theorem is satisfied but the Casimir entropy may take negative values. Here we not only reproduce an analytical proof of the Nernst heat theorem but also
find the next perturbation orders in the expansion of the Casimir free energy and entropy in powers of a small parameter. The results obtained in high-temperature limit are also provided. Sec. 5 is devoted to the Casimir interaction between metal and dielectric plates with included dc conductivity of the dielectric material. We demonstrate that in this case the Nernst heat theorem is violated. Sec. 6 contains the discussion of semiconductors which present a wide variety of electric properties varying from metallic to dielectric. We consider the Casimir interaction between metal and semiconductor test bodies and formulate the criterion when it is appropriate to include the dc conductivity of a semiconductor into the model of dielectric response. In doing so, the results of recent experiments on the measurement of the Casimir force between metal and semiconductor test bodies are taken into account. To this point the assumption has been made that metal, dielectric and semiconductor materials of the Casimir plates possess only temporal dispersion, i.e., can be described by the dielectric permittivity depending only on frequency. In Sec. 7 we discuss recent controversial results by different authors (see, e.g., Refs. [64] – [68]) attempting to take into account also spatial dispersion. As is shown in this section, the way of inclusion of spatial dispersion into the Lifshitz theory, used in Refs. [64] – [68] is unjustified. We argue that the account of spatial dispersion cannot influence theoretical results obtained with the help of usual, spatially local, Lifshitz theory within presently used ranges of experimental separations. Sec. 8 contains our conclusions and discussion.

2. NEW ANALYTICAL RESULTS FOR THE THERMAL CASIMIR FORCE BETWEEN DIELECTRICS

We consider two thick dielectric plates (semispaces) described by the frequency-dependent dielectric permittivity \( \varepsilon(\omega) \) and restricted by the parallel planes \( z = \pm a/2 \) with a separation \( a \) between them, in thermal equilibrium at temperature \( T \). The Lifshitz formula for the free energy of the van der Waals and Casimir interaction between the plates is given by

\[
F(a, T) = \frac{k_B T}{2\pi} \sum_{l=0}^{\infty} \left( 1 - \frac{\delta l_0}{2} \right) \int_0^{\infty} k_\perp dk_\perp \times \left\{ \ln \left[ 1 - r_\parallel(\xi_l, k_\perp)e^{-2aq_l} \right] + \ln \left[ 1 - r_\perp(\xi_l, k_\perp)e^{-2aq_l} \right] \right\},
\]

where the reflection coefficients for two independent polarizations of electromagnetic field are defined as

\[
r_\parallel(\xi_l, k_\perp) = \frac{\varepsilon_l q_l - k_l}{\varepsilon_l q_l + k_l}, \quad r_\perp(\xi_l, k_\perp) = \frac{k_l - q_l}{k_l + q_l}.
\]

Here \( k_\perp \) is the magnitude of the wave vector in the plane of plates, \( \xi_l = 2\pi k_B T l / h \) are the Matsubara frequencies, \( k_B \) is the Boltzmann constant, \( \varepsilon_l = \varepsilon(i\xi_l) \), and

\[
q_l = \sqrt{\frac{\xi_l^2}{c^2} + k_\perp^2}, \quad k_l = \sqrt{\varepsilon(i\xi_l) \frac{\xi_l^2}{c^2} + k_\perp^2}.
\]
The problems in the application of the Lifshitz theory to real materials discussed in the Introduction are closely connected with the values of the reflection coefficients at zero Matsubara frequency. For later use we discuss it for the various cases.

- For ideal metals it holds
  \[ r_{\parallel}(0, k_\perp) = r_{\perp}(0, k_\perp) = 1. \tag{4} \]

- For real metals described by the dielectric function of the Drude model,
  \[ \varepsilon(i\xi_l) = 1 + \frac{\omega_p^2}{\xi_l[\xi_l + \nu(T)]}, \tag{5} \]
  where \( \omega_p \) is the plasma frequency and \( \nu(T) \) is the relaxation parameter, it holds
  \[ r_{\parallel}(0, k_\perp) = 1, \quad r_{\perp}(0, k_\perp) = 0. \tag{6} \]

  Eq. (6) results in the discontinuity between the cases of ideal and real metals and leads to the violation of the Nernst heat theorem for metallic plates having perfect crystal lattices.

- For real metals described by the dielectric function of the plasma model,
  \[ \varepsilon(i\xi_l) = 1 + \frac{\omega_p^2}{\xi_l}, \tag{7} \]
  from Eq. (2) it follows
  \[ r_{\parallel}(0, k_\perp) = 1, \quad r_{\perp}(0, k_\perp) = \frac{\sqrt{c^2k_\perp^2 + \omega_p^2} - ck_\perp}{\sqrt{c^2k_\perp^2 + \omega_p^2} + ck_\perp}. \tag{8} \]

  Here, in the limit of ideal metals \( (\omega_p \to \infty) \) the continuity is preserved because \( r_{\perp}(0, k_\perp) \) in Eq. (5) goes to unity. The free energy calculated with the permittivity (5) is also consistent with thermodynamics.

- For dielectrics and semiconductors the dielectric permittivities at the imaginary Matsubara frequencies are given by the Ninham-Parsegian representation
  \[ \varepsilon(i\xi_l) = 1 + \sum_j \frac{C_j}{1 + \xi_l^2 / \omega_j^2}, \tag{9} \]
  where the parameters \( C_j \) are the absorption strengths satisfying the condition
  \[ \sum_j C_j = \varepsilon_0 - 1 \tag{10} \]
  and \( \omega_j \) are the characteristic absorption frequencies. Here, the static dielectric permittivity \( \varepsilon_0 \equiv \varepsilon(0) \) is supposed to be finite. Although Eq. (10) is an
approximate one, it gives a very accurate description for many materials.\cite{71}

By the substitution of Eq. (9) in Eq. (2) one arrives at

\[ r_{\parallel}(0, k_{\perp}) \equiv r_0 = \frac{\varepsilon_0 - 1}{\varepsilon_0 + 1}, \quad r_{\perp}(0, k_{\perp}) = 0. \] (11)

Note that the vanishing of the transverse reflection coefficient for dielectrics at zero frequency in Eq. (11) has another meaning than for the Drude metals in Eq. (6). For Drude metal the parallel reflection coefficient is equal to the physical value for real photons at normal incidence, i.e., to unity, and the transverse one vanishes instead of taking unity, its physical value. This results in the violation of the Nernst heat theorem for perfect crystal lattices. In the case of dielectrics both reflection coefficients at zero frequency in Eq. (11) depart from the physical value for real photons which is equal to \((\sqrt{\varepsilon_0 - 1})/(\sqrt{\varepsilon_0 + 1})\). In this case, however, one of them is larger and the other one is smaller than the physical value. As we will see below, this leads to the preservation of Nernst’s heat theorem confirming that Eq. (9), despite being approximate, describes the material properties of dielectric and semiconductor plates in a thermodynamic consistent way.

Now we derive the analytic representation for the Casimir free energy in Eq. (1) at low temperatures. For convenience in calculations, we introduce the dimensionless variables

\[ \zeta_l = \frac{\xi_l}{\xi_c} = \frac{2a \xi_l}{c} = \tau l, \quad y = 2aq_l, \] (12)

where \(\xi_c = c/(2a)\) is the characteristic frequency, \(\tau = 4\pi k_B aT/(\hbar c)\), and \(q_l\) was defined in Eq. (3). Then the Lifshitz formula (1) takes the form

\[ F(a, T) = \frac{\hbar c\tau}{32\pi^3 a^3} \sum_{l=0}^{\infty} \left(1 - \frac{\delta_{l0}}{2}\right) \int_0^{\infty} dy f(\zeta_l, y), \] (13)

where

\[ f(\zeta, y) = f_{\parallel}(\zeta, y) + f_\parallel(\zeta, y), \] (14)

\[ f_{\parallel}(\zeta, y) = y \ln \left[1 - r_{\parallel}^\perp(\zeta, y)e^{-y}\right], \] (15)

and reflection coefficients (2), in terms of variables (12), being given by

\[ r_{\parallel}(\zeta_l, y) = \frac{\varepsilon_l y - \sqrt{y^2 + \zeta_l^2(\varepsilon_l - 1)}}{\varepsilon_l y + \sqrt{y^2 + \zeta_l^2(\varepsilon_l - 1)}}, \quad r_{\perp}(\zeta_l, y) = \frac{\sqrt{y^2 + \zeta_l^2(\varepsilon_l - 1)} - y}{\sqrt{y^2 + \zeta_l^2(\varepsilon_l - 1)} + y}. \] (16)

To separate the temperature independent contribution and thermal correction in Eq. (13) we apply the Abel-Plana formula,\cite{36}

\[ \sum_{l=0}^{\infty} \left(1 - \frac{\delta_{l0}}{2}\right) F(l) = \int_0^{\infty} F(t) \, dt + i \int_0^{\infty} dt \frac{F(it) - F(-it)}{e^{2\pi t} - 1}, \] (17)

where \(F(z)\) is an analytic function in the right half-plane. Here, taking it as

\[ F(x) = \int_x^{\infty} dy f(x, y) \] (18)
and using Eq. (17), we can identically rearrange Eq. (13) to the form
\[ F(a, T) = E(a) + \Delta F(a, T), \]
(19)
where \( E(a) \) is the energy of the van der Waals or Casimir interaction at zero temperature,
\[ E(a) = \frac{\hbar c}{32\pi^2a^3} \int_0^\infty d\zeta \int_{\zeta}^\infty dy f(\zeta, y), \]
(20)
and \( \Delta F(a, T) \) is the thermal correction to this energy,
\[ \Delta F(a, T) = \frac{i\hbar c\tau}{32\pi^2a^3} \int_0^\infty dt \frac{F(i\tau t) - F(-i\tau t)}{e^{2\pi t} - 1}. \]
(21)
Note that, in fact, Eq. (21) describes the dependence of the free energy on the temperature arising from the dependence on temperature of the Matsubara frequencies. Thus, \( \Delta F(a, T) \) in (21) coincides with the thermal correction to the energy, defined as \( F(a, T) - F(a, 0) \), only for plate materials with temperature independent properties.

The asymptotic expressions for the energy \( E(a) \) at both short and large separations are well known. Below we find the asymptotic expressions for the thermal correction (21) under the conditions \( \tau \ll 1 \) and \( \tau \gg 1 \). Taking into account the definition of \( \tau \) in Eq. (12), the asymptotic expressions at \( \tau \ll 1 \) are applicable both at small and large separations if the temperature is sufficiently low.

We begin with condition \( \tau \ll 1 \). Let us substitute Eq. (9) in Eqs. (14) – (16), expand the function \( f(x, y) \) in powers of \( x = \tau t \), and then integrate the obtained expansion with respect to \( y \) from \( x \) to infinity in order to find \( F(x) \) in Eq. (18) and \( F(ix) - F(-ix) \) in Eq. (21).

It is easy to check that \( f(\zeta, y) \) does not contribute to the leading, second, order in the expansion of \( F(ix) - F(-ix) \) in powers of \( x \). Thus, we can restrict ourselves by the consideration of the expansion
\[ f(\parallel, y) = y \ln(1 - r_0^2e^{-y}) + \frac{2\varepsilon_0 r_0^2}{\varepsilon_0 + 1} \frac{x^2}{y(e^y - r_0^2e^{-y})} + \frac{4\xi^2 r_0^2}{(\varepsilon_0 + 1)\omega_1^2} \frac{y x^2}{e^y - r_0^2} + O(x^3), \]
(22)
where \( r_0 \) was defined in Eq. (11). Note that for simplicity we consider here only one oscillator in Eq. (9) and put \( \omega_j = \omega_1 \). The case of several oscillator modes can be considered in an analogous way.

As a next step, we integrate Eq. (22) term by term according to Eq. (18), expand the partial results in powers of \( x \) and sum up the obtained series. Thereby we obtain the following expressions:
\[ Z_1(x) = \int_x^\infty y \, dy \ln(1 - r_0^2e^{-y}) = -\sum_{n=1}^\infty \frac{(1 + nx)e^{-nx}}{n^3} r_0^{2n} \]
\[ = -\text{Li}_3(r_0^2) - \frac{x^2}{2} \ln(1 - r_0^2) + O(x^3), \]
(23)
Again, applying the Abel-Plana formula (17), we represent the pressure as follows,

\[ Z_2(x) = \frac{2\varepsilon_0 r_0^2 x^2}{\varepsilon_0 + 1} \int_r^\infty \frac{dy}{y} \left( e^{y} - r_0^2 \right) = \frac{2\varepsilon_0 x^2}{\varepsilon_0 + 1} \sum_{n=1}^{\infty} r_0^{2n} \text{Ei}(-nx), \quad (24) \]

\[ Z_3(x) = \frac{4\varepsilon_0^2 \xi_0^2 x^2}{(\varepsilon_0 + 1)\omega_0^2} \int_r^\infty \frac{ydy}{e^{y} - r_0^2} = \frac{4\varepsilon_0^2 x^2}{(\varepsilon_0 + 1)\omega_0^2} \sum_{n=1}^{\infty} r_0^{2n} \frac{(1 + nx)e^{-nx}}{n^2}, \]

where \( \text{Li}_n(z) \) is the polylogarithm function and \( \text{Ei}(z) \) is the exponential integral function.

From these equations it follows

\[ Z_1(ix) - Z_1(-ix) = \text{O}(x^3), \quad Z_3(ix) - Z_3(-ix) = \text{O}(x^5), \quad (26) \]

\[ Z_2(ix) - Z_2(-ix) = 2i\pi \frac{\varepsilon_0}{\varepsilon_0 + 1} \frac{r_0^2}{1 - r_0^2} x^2 + O(x^3), \quad (27) \]

and, thus, \( Z_1 \) and \( Z_3 \) do not contribute to the leading order in the expansion of \( F(ix) - F(-ix) \). The latter is determined by \( Z_2 \) only. As a result, we arrive at

\[ F(ix) - F(-ix) = i\pi \frac{(\varepsilon_0 - 1)^2}{2(\varepsilon_0 + 1)} x^2 - i\alpha x^3 + O(x^4), \quad (28) \]

where \( r_0 \) was substituted from Eq. (11) and \( \alpha \) was introduced for the still unknown real coefficient of the next to leading order resulting from \( Z_1 \) and \( Z_2 \) as well as, possibly, from \( f_L(\zeta, y) \). At this stage it is difficult to determine the value of this coefficient because all powers in the expansion of \( f(x, y) \) contribute to it. Remarkably, the two leading orders depend only on the static dielectric permittivity \( \varepsilon_0 \) and are not influenced by the dependence of the dielectric permittivity on the frequency contained in \( Z_3 \).

Substituting Eq. (28) in Eq. (21) and using Eq. (19), we obtain

\[ \mathcal{F}(a, T) = E(a) - \frac{hc}{32\pi^2 a^4} \left( \frac{\zeta(3)}{8\pi^2} \frac{(\varepsilon_0 - 1)^2}{\varepsilon_0 + 1} x^3 - C_4 r^4 + O(r^5) \right), \quad (29) \]

where \( C_4 \equiv \alpha/240 \) and \( \zeta(z) \) is the Riemann zeta function.

So far we have considered the free energy. The thermal pressure is obtained as

\[ P(a, T) = -\frac{\partial \mathcal{F}(a, T)}{\partial a} = P_0(a) - \frac{hc}{32\pi^2 a^4} \left[ C_4 r^4 + O(r^5) \right], \quad (30) \]

where \( P_0(a) = -\partial E(a)/\partial a \) is the Casimir pressure at zero temperature.

In order to determine the value of the coefficient \( C_4 \) of the leading term, we express the pressure directly through the Lifshitz formula

\[ P(a, T) = -\frac{hc\tau}{32\pi^2 a^4} \sum_{l=0}^{\infty} \left( 1 - \frac{\Delta_0}{2} \right) \int_{\zeta_l}^{\infty} y^2 dy \left[ \frac{r_{\parallel}^2(\zeta_l, y)}{e^y - r_{\parallel}^2(\zeta_l, y)} + \frac{r_{\perp}^2(\zeta_l, y)}{e^y - r_{\perp}^2(\zeta_l, y)} \right]. \quad (31) \]

Again, applying the Abel-Plana formula (17), we represent the pressure as follows,

\[ P(a, T) = P_0(a) + \Delta P(a, T), \quad (32) \]
where the thermal correction to $P_0(a)$, the pressure at zero temperature, is
\[
\Delta P(a, T) = -\frac{i\hbar c}{32\pi^2 a^4} \int_0^\infty dt \frac{\Phi(i\tau) - \Phi(-i\tau)}{e^{2\pi\tau} - 1}
\]
and the function $\Phi(x)$ is given by
\[
\Phi(x) \equiv \Phi_\parallel(x) + \Phi_\perp(x), \quad \Phi_\parallel,\perp(x) = \int_x^\infty \frac{y^2 r^2_\parallel,\perp(x, y)}{e^{y} - r^2_\parallel,\perp(x, y)}. \tag{34}
\]
First, we determine the leading term of the expansion of $\Phi_\perp(x)$ in powers of $x$. For this purpose, let us introduce the new variable $v = y/x$ and note that the reflection coefficient $r_\perp(x, v)$ depends on $x$ only through the frequency dependence of $\varepsilon$ given by Eq. (9). Thus, we can rewrite and expand Eq. (34) as follows:
\[
\Phi_\perp(x) = x^3 \int_1^\infty dv \frac{v^2 r^2_\perp(x, v)}{e^{v x} - r^2_\perp(x, v)} = x^3 \int_1^\infty dv \frac{v^2 r^2_\perp(v)}{1 - r^2_\perp(v)} + O(x^4), \tag{35}
\]
where, according to Eq. (10),
\[
r_\perp(v) \equiv r_\perp(0, v) = \frac{\sqrt{v^2 + \varepsilon_0 - 1} - v}{\sqrt{v^2 + \varepsilon_0 - 1} + v}. \tag{36}
\]
Integration in Eq. (35) with account of Eq. (36) results in
\[
\Phi_\perp(x) = \left[1 - \frac{\varepsilon_0 (3 - \varepsilon_0)}{2\varepsilon_0}\right] \frac{x^3}{6} + O(x^5), \tag{37}
\]
from which it follows:
\[
\Phi_\perp(ix) - \Phi_\perp(-ix) = -i \left[1 - \frac{1}{2} \sqrt{\varepsilon_0 (3 - \varepsilon_0)}\right] \frac{x^3}{3} + O(x^5). \tag{38}
\]
The expansion of $\Phi_\parallel(x)$ from Eq. (34) in powers of $x$ is somewhat more cumbersome. It can be performed in the following way. As is seen from the second equality in Eq. (28), the dependence of the dielectric permittivity on frequency contributes to $F(ix) - F(-ix)$ starting from only the 5th power in $x$. Bearing in mind the connection between free energy and pressure, we can conclude that the dependence on the frequency contributes to $\Phi_\parallel(ix) - \Phi_\parallel(-ix)$ starting from the 4th order. We are looking for the lowest (third) order expansion term of $\Phi_\parallel(ix) - \Phi_\parallel(-ix)$. Because of this, it is permissible to disregard the frequency dependence of $\varepsilon$ and describe the dielectric by its static dielectric permittivity.

To begin with, we identically rearrange $\Phi_\parallel(x)$ in Eq. (34) by subtracting and adding the two first expansion terms of the function under the integral in powers of $x$,
\[
\Phi_\parallel(x) = \int_x^\infty dy \left[y^2 \frac{r^2_\parallel(x, y)}{e^{y} - r^2_\parallel(x, y)} - y^2 \frac{r^2_\perp}{e^{y} - r^2_\perp(x, y)} + x^2 \frac{2\varepsilon_0}{\varepsilon_0 + 1} \frac{r^2_\parallel e^{-y}}{(1 - r^2_\parallel e^{-y})^2}\right] + \int_x^\infty y^2 dy \frac{r^2_\perp}{e^{y} - r^2_\perp(x, y)} - x^2 \frac{2\varepsilon_0}{\varepsilon_0 + 1} \int_x^\infty dy \frac{r^2_\parallel e^{-y}}{(1 - r^2_\parallel e^{-y})^2}. \tag{39}
\]
and consider these three integrals separately. The first integral in terms of the new variable \( v = y/x \) reads

\[
Q_1(x) \equiv x^3 \int_1^\infty dv \left[ v^2 \frac{r_0^2(v)}{e^{v x} - r_0^2(v)} - v^2 \frac{r_0^2}{e^{v x} - r_0^2} + \frac{2 \varepsilon_0}{\varepsilon_0 + 1} \frac{r_0^2 e^{-v x}}{1 - r_0^2 e^{-v x}} \right].
\]  

(40)

where, in accordance with Eq. (10),

\[
r_\parallel(v) \equiv r_\parallel(0, v) = \frac{\varepsilon_0 v - \sqrt{\varepsilon_0^2 + 1 - 1}}{\varepsilon_0 v + \sqrt{\varepsilon_0^2 + 1 - 1}}.
\]

Expanding \( Q_1(x) \) in powers of \( x \) and explicitly calculating the remaining integrals for the lowest, third, power of \( x \) results in

\[
Q_1(x) = x^3 \int_1^\infty dv \left[ v^2 \frac{r_0^2(v)}{1 - r_\parallel(v)} - v^2 \frac{r_0^2}{1 - r_0^2} + \frac{2 \varepsilon_0}{\varepsilon_0 + 1} \frac{r_0^2}{1 - r_0^2} \right] + O(x^4)
\]

(42)

\[
= \frac{x^3}{6} \left[ 1 - \frac{\sqrt{\varepsilon_0}}{2} \left( 1 + 3 \varepsilon_0 - 2 \varepsilon_0^2 \right) + \frac{2 \varepsilon_0}{\varepsilon_0 + 1} \frac{r_0^2}{1 - r_0^2} \right] + O(x^4).
\]

The second and third integrals on the right-hand side of Eq. (39) are simply determined with the following result:

\[
Q_2(x) \equiv \int_x^\infty dy \frac{y^2}{e^{y x} - y_0^2} = \sum_{n=1}^\infty \frac{2 + 2 n x + n^2 x^2}{n^3} e^{-n x}
\]

(43)

\[
= 3 \text{Li}_3(r_0^2) - \frac{x^3}{3} \frac{r_0^2}{1 - r_0^2} + O(x^4),
\]

\[
Q_3(x) \equiv -\frac{2 \varepsilon_0 x^2}{\varepsilon_0 + 1} \int_x^\infty dy \frac{r_0^2 e^{-y}}{(1 - r_0^2 e^{-y})^2} = -\frac{2 \varepsilon_0 x^2}{\varepsilon_0 + 1} \frac{r_0^2 e^{-x}}{1 - r_0^2 e^{-x}}
\]

(44)

\[
= -\frac{2 \varepsilon_0}{\varepsilon_0 + 1} \left[ \frac{x^2}{1 - r_0^2} - x^3 \frac{r_0^2}{1 - r_0^2} \right] + O(x^4).
\]

Substituting Eqs. (42), (43) and (44) into \( \Phi_\parallel(x) = Q_1(x) + Q_2(x) + Q_3(x) \), we arrive at

\[
\Phi_\parallel(ix) - \Phi_\parallel(-ix) = -i \left[ 1 + \frac{\sqrt{\varepsilon_0}}{2} \left( 2 \varepsilon_0^3 - 3 \varepsilon_0 - 1 \right) \right] \frac{x^3}{3} + O(x^4).
\]

(45)

Then, by summing Eqs. (38) and (45), the result is obtained

\[
\Phi(ix) - \Phi(-ix) = -i \left( 2 + \varepsilon_0^{5/2} - \varepsilon_0^{3/2} - 2 \sqrt{\varepsilon_0} \right) \frac{x^3}{3} + O(x^4).
\]

(46)

Now we substitute Eq. (40) in Eq. (33) and perform integration. Finally, from Eq. (32) the desired expression for the Casimir pressure is derived

\[
P(a, T) = P_0(a) - \frac{\hbar c}{32\pi^2 a^4} \left[ \left( \sqrt{\varepsilon_0 - 1} \left( \varepsilon_0^3 - \varepsilon_0^{3/2} - 2 \right) \right) \frac{r_0^2}{720} + O(x^5) \right].
\]

(47)

By comparison with Eq. (30) the explicit form of the coefficient \( C_4 \) is found as

\[
C_4 = \left( \sqrt{\varepsilon_0 - 1} \left( \varepsilon_0^3 - \varepsilon_0^{3/2} - 2 \right) \right)/720
\]

(48)
and, thus, both two first perturbation orders in the expansion for the free energy \( F(a, T) \) are determined.

Equations (29), (47) and (48) solve the fundamental problem of the thermodynamic consistency of the Lifshitz theory in the case of two dielectric plates. From Eqs. (29) and (48) the entropy of the van der Waals and Casimir interaction between plates takes the form

\[
S(a, T) = -\frac{\partial F(a, T)}{\partial T} = \frac{3k_B \zeta(3)}{64\pi^3 a^2 (\varepsilon_0 - 1)} \tau^2 \times \left[ 1 - \frac{2\pi^2 (\varepsilon_0 + 1)(\varepsilon_0^{3/2} + 2\varepsilon_0 + 2\sqrt{\varepsilon_0} + 2)}{135\zeta(3)(\sqrt{\varepsilon_0} + 1)^2} \tau + O(\tau^2) \right].
\]

As is seen from Eq. (49), in the limit \( \tau \to 0 \) (\( T \to 0 \)) the lower order contributions to the entropy are of the second and the third powers in the small parameter \( \tau \). Thus, the entropy vanishes when the temperature goes to zero as it must be in accordance with the third law of thermodynamics (the Nernst heat theorem).

A similar behavior was obtained for ideal metals \(^{10,72,73}\) and for real metals described by the plasma model \(^{15,52}\). For example, in the case of plates made of ideal metal the entropy at low temperatures is given by

\[
S(a, T) = \frac{3k_B \zeta(3)}{32\pi^3 a^2} \tau^2 \left[ 1 - \frac{2\pi^2 (\varepsilon_0 + 1)(\varepsilon_0^{3/2} + 2\varepsilon_0 + 2\sqrt{\varepsilon_0} + 2)}{135\zeta(3)(\sqrt{\varepsilon_0} + 1)^2} \tau + O(\tau^2) \right].
\]

Note, however, that the expansion coefficients in Eq. (50) cannot be obtained as a straightforward limit \(|\varepsilon_0| \to \infty\) in Eq. (49) and the above equations for the free energy and pressure. The mathematical reason is that it is impermissible to interchange the limiting transitions \( \tau \to 0 \) and \(|\varepsilon_0| \to \infty\) in the power expansions of functions depending on \( \varepsilon_0 \) as a parameter.

Remarkably, the low-temperature behavior of the free energy, pressure and entropy of nonpolar dielectrics in Eqs. (29), (47) and (49) is universal, i.e., is determined only by the static dielectric permittivity. The absorption bands included in Eq. (9) do not influence the low-temperature behavior. A more simple derivation of the results (29), (47)–(49) for dielectrics with constant \( \varepsilon \) is contained in Ref. \(^{74}\). As was demonstrated above, all these results remain unchanged if the dependence of dielectric permittivity on frequency is taken into account.

In Ref. \(^{60}\) more general results were obtained related to two dissimilar dielectric plates with dielectric permittivities \( \varepsilon^{(1)}(\omega) \) and \( \varepsilon^{(2)}(\omega) \). For brevity here we present only the final expressions for the low-temperature behavior of the Casimir free energy, pressure and entropy between dissimilar plates. They are as follows \(^{60}\)

\[
\mathcal{F}(a, T) = E(a) - \frac{\hbar c}{32\pi^2 a^2} \tau^3 \times \left[ \frac{\zeta(3)}{8\pi^2} \left( \frac{\varepsilon_0^{(1)}}{\varepsilon_0^{(1)} + 1} + \frac{\varepsilon_0^{(2)}}{\varepsilon_0^{(2)} + 1} \right) \left( \frac{\varepsilon_0^{(1)} - 1}{\varepsilon_0^{(1)} + \varepsilon_0^{(2)}} \right) \left( \varepsilon_0^{(2)} \right) - C_4 \tau + O(\tau^2) \right].
\]
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It is easily seen that in the limit ε → 1 the Lifshitz formula. Dielectric properties of the plates can be described by the equations (29), (47)–(49) having obtained above. Note that in the application region of low-temperature asymptotic expressions the entropy of the Casimir interaction between dielectric plates is nonnegative.

\[ P(a, T) = P_0(a) - \frac{\hbar c}{32\pi^2 a^4} \left[ C_4 \tau^4 + O(\tau^5) \right], \]  

\[ S(a, T) = \frac{k_B}{2\pi a^2} \tau^2 \]

\[ \times \left[ \frac{3\zeta(3)}{32\pi^2} \left( \varepsilon_0^{(1)} + \varepsilon_0^{(2)} \right) \left( \varepsilon_0^{(1)} + 1 \right) \left( \varepsilon_0^{(2)} + 1 \right) \left( \varepsilon_0^{(1)} - 1 \right) \left( \varepsilon_0^{(2)} - 1 \right) - C_4 \tau + O(\tau^2) \right]. \]

Here \( \varepsilon_0^{(1,2)} \equiv \varepsilon^{(1,2)}(0) \) and the coefficient \( C_4 \) is given by

\[ C_4 = \frac{1}{720} \left\{ \frac{1}{\left( \sqrt{\varepsilon_0^{(1)}} + \sqrt{\varepsilon_0^{(2)}} \right)^2} \right\} \]

\[ \times \left[ - \left( \varepsilon_0^{(1)} + \varepsilon_0^{(2)} \right)^2 \left( 2\varepsilon_0^{(1)} + 2\varepsilon_0^{(2)} + \sqrt{\varepsilon_0^{(1)} \varepsilon_0^{(2)}} - \varepsilon_0^{(1)} \varepsilon_0^{(2)} \right) \right.

\[ + \varepsilon_0^{(1)} \varepsilon_0^{(2)} \sqrt{\varepsilon_0^{(1)} \varepsilon_0^{(2)}} \left( 5\varepsilon_0^{(1)} \varepsilon_0^{(2)} - 3\varepsilon_0^{(1)} - 3\varepsilon_0^{(2)} + 1 \right) \]

\[ + \sqrt{\varepsilon_0^{(1)} \varepsilon_0^{(2)}} \left( \sqrt{\varepsilon_0^{(1)}} - \sqrt{\varepsilon_0^{(2)}} \right)^2 \left( \varepsilon_0^{(1)} \varepsilon_0^{(2)} - \varepsilon_0^{(1)} \varepsilon_0^{(2)} \right) \}

\[ - 3 \left( \varepsilon_0^{(1)} \varepsilon_0^{(2)} \right)^2 \left( \varepsilon_0^{(1)} - 1 \right) \left( \varepsilon_0^{(2)} - 1 \right) \left( \sqrt{\varepsilon_0^{(1)}} - \sqrt{\varepsilon_0^{(2)}} \right) \}

\[ \left\{ \right. \left( \sqrt{\varepsilon_0^{(1)}} - \sqrt{\varepsilon_0^{(2)}} \right) \left( \sqrt{\varepsilon_0^{(1)}} + \sqrt{\varepsilon_0^{(2)}} \right) \}

\[ \text{Artanh} \left( \sqrt{\varepsilon_0^{(1)}} - \sqrt{\varepsilon_0^{(2)}} \right) \left( \sqrt{\varepsilon_0^{(1)}} + \sqrt{\varepsilon_0^{(2)}} \right) \}

\[ \left. \left( \sqrt{\varepsilon_0^{(1)}} - \sqrt{\varepsilon_0^{(2)}} \right) \left( \sqrt{\varepsilon_0^{(1)}} + \sqrt{\varepsilon_0^{(2)}} \right) \right\} \]. \]

It is easily seen that in the limit \( \varepsilon_0^{(1)} = \varepsilon_0^{(2)} = \varepsilon_0 \) equations (51), (52) coincide with equations (29), (47)–(49) having obtained above. Note that in the application region of low-temperature asymptotic expressions the entropy of the Casimir interaction between dielectric plates is nonnegative.

The obtained analytic behavior of the free energy, pressure and entropy at low temperatures can be compared with the results of numerical computations using the Lifshitz formula. Dielectric properties of the plates can be described by the static dielectric permittivity or more precisely using the optical tabulated data for the complex index of refraction. As an example, in Fig. 1 we present the thermal corrections to the Casimir energy (a) and pressure (b) at a separation \( a = 400 \text{ nm} \) as functions of temperature in the configuration of two dissimilar plates made of high-resistivity Si and SiO\(_2\). The dielectric permittivities of both materials along the imaginary frequency axis were computed in Ref. 75 using the optical data of Ref. 76. The precise thermal corrections computed by taking into account these permittivities are shown by the solid lines and corrections computed by our analytical asymptotic expressions are shown by the long-dashed lines. Short-dashed lines indicate the results computed by the Lifshitz formula with constant dielectric permittivities of Si and SiO\(_2\) equal to \( \varepsilon_0^{(1)} = 11.67 \) and \( \varepsilon_0^{(2)} = 3.84 \), respectively. As is seen in Fig. 1a,b at \( T < 60 \text{ K} \) the results obtained using the analytical asymptotic
Fig. 1. Magnitudes of the thermal corrections to the energy (a) and pressure (b) in configuration of two plates, one made of Si and another one of SiO$_2$, at a separation $a = 400$ nm as a function of temperature calculated by the use of different approaches: by the Lifshitz formula and tabulated optical data (solid lines), by the Lifshitz formula and static dielectric permittivities (short-dashed lines), by the asymptotic expressions in Eqs. (51) and (52) (long-dashed lines).

expressions practically coincide with the solid lines computed using the tabulated optical data for the materials of the plates.

Now we return to the case of two similar dielectric plates and consider the asymptotic expressions under the condition $\tau \gg 1$, i.e., at high temperatures (large separations). It is well known that in this case the approximation of static
dielectric permittivity works good and the main contribution is given by the zero-frequency term of the Lifshitz formula \(^{13}\)

\[ F(a, T) = \frac{\hbar c \tau}{16\pi^2 a^3} \int_{0}^{\infty} y dy \ln (1 - r_0^2 e^{-y}) \]  

(53)

(the other terms being exponentially small). Performing the integration in Eq. (53) we obtain

\[ F(a, T) = -\frac{k_B T}{16\pi a^2} \text{Li}_3 (r_0^2). \]  

(54)

In a similar manner for the Casimir pressure and entropy at \(\tau \gg 1\) it follows

\[ P(a, T) = -\frac{k_B T}{8\pi a^3} \text{Li}_3 (r_0^2), \quad S(a, T) = \frac{k_B}{16\pi a^2} \text{Li}_3 (r_0^2). \]  

(55)

Equations (54) and (55) are simply generalized\(^{10}\) for the case of two dissimilar dielectric plates by performing the replacement \(r_0^2 \rightarrow r_0^{(1)}_0 \), \(r_0^{(2)}_0\), where \(r_0^{(1,2)}_0\) are defined by Eq. (11) with the static dielectric permittivities of dissimilar plates \(\varepsilon_0^{(1,2)}\).

### 3. IS THE DC CONDUCTIVITY RELATED TO THE CASIMIR INTERACTION BETWEEN DIELECTRICS?

As was discussed in the previous section, the zero-frequency term in formula \(^{11}\), i.e., the contribution with \(l = 0\), is of prime importance and determines many of the basic properties of the Casimir interaction. In the above consideration we have described dielectric materials by Eq. (9) with finite static dielectric permittivity \(\varepsilon_0\). This resulted in Eq. (11) where one reflection coefficient at zero frequency is larger and the other one is smaller than the physical value for real photons at normal incidence. However, in the Lifshitz theory, the departure of both coefficients from their physical values is coordinated in such a way that the Nernst heat theorem remains valid.

It is common knowledge that at nonzero temperatures dielectric materials possess a negligibly small but not equal to zero dc conductivity. From physical intuition it is reasonable to expect that the influence of this conductivity on the van der Waals and Casimir forces should be also negligible. In Ref. \(^{57}\) it was shown that, on the contrary, the inclusion of small dielectric dc conductivity in the model of dielectric response leads to a large effect in dispersion forces. This raises the question if dc conductivity is related to dispersion forces or if, on the contrary, the zero-frequency contribution should be understood not literally but as an analytic continuation from the region of high frequencies determining the physical phenomenon of dispersion forces.

To illustrate this problem in more details, we consider the asymptotic behavior of the free energy and entropy at low temperature with included dc conductivity. What this means is that, instead of the dielectric permittivity \(\varepsilon(i\xi_l)\) given in Eq. (2), one uses\(^{57,58}\)

\[ \tilde{\varepsilon}(i\xi_l) = \varepsilon(i\xi_l) + \frac{4\pi\sigma_0}{\xi_l} = \varepsilon_l(i\xi) + \frac{\beta(T)}{l}, \]  

(56)
where $\sigma_0$ is the dc conductivity of the plate material and $\beta(T) = 2\hbar\sigma_0/(k_B T)$. The conductivity of dielectrics depends on temperature as $\sigma_0 \sim \exp(-b/T)$ where $b$ is determined by the energy gap $\Delta$ which differs for different materials. The smallness of the dc conductivity of dielectrics can be illustrated by the example of SiO$_2$ where at $T = 300$ K it holds $\beta \sim 10^{-12}$. Thus, the role of the dc conductivity is really negligible for all $l \geq 1$. In addition, $\beta(T)$ quickly decreases with decrease of $T$ and, as a consequence, remains negligible at any $T$. In spite of this, the substitution of Eq. (56) into the reflection coefficients (10) leads to different result than in Eq. (11):

$$\tilde{r}_\parallel(0, y) = 1, \quad \tilde{r}_\parallel(0, y) = 0.$$  

Equation (57) is in some analogy to Eq. (6), obtained for metals described by the Drude model, which leads to the violation of the Nernst heat theorem in the case of perfect crystal lattices.

Now we substitute the dielectric permittivity $\tilde{\varepsilon}_l \equiv \tilde{\varepsilon}(\xi_l)$ in Eqs. (13) - (16) instead of $\varepsilon_l$ and find the Casimir free energy $\tilde{F}(a, T)$ with included dc conductivity. For convenience, we separate the zero-frequency term, subtract and add the usual zero-frequency contribution for dielectric without dc conductivity. The result is

$$\tilde{F}(a, T) = \frac{k_B T}{16\pi a^2} \int_0^\infty dy \left[ \ln (1 - e^{-y}) - \ln (1 - r_0^2 e^{-y}) \right]$$

$$+ \frac{k_B T}{16\pi a^2} \int_0^\infty dy \ln \left(1 - r_0^2 e^{-y}\right)$$

$$+ \frac{k_B T}{8\pi a^2} \sum_{l=1}^\infty \int_{\xi_l}^\infty dy \left\{ \ln \left[1 - \tilde{r}_\parallel^2(\xi_l, y)e^{-y}\right] + \ln \left[1 - \tilde{r}_\perp^2(\xi_l, y)e^{-y}\right] \right\},$$

where $r_0$ was defined in Eq. (11). Let us expand the last, third integral on the right-hand side of Eq. (58) in powers of the small parameters $\beta/l$. Then, we combine the zero-order contribution in this expansion with the second integral on the right-hand side of Eq. (58) and obtain the Casimir free energy $F(a, T)$ calculated with the dielectric permittivities $\varepsilon_l$. The first integral on the right-hand side of Eq. (59) is calculated explicitly. Then, Eq. (58) can be rewritten as

$$\tilde{F}(a, T) = F(a, T) - \frac{k_B T}{16\pi a^2} \left[ \zeta(3) - \text{Li}_3 \left(r_0^2\right) \right] + R(a, T).$$

Here, $R(a, T)$ is of order $O(\beta/l)$. It represents the first and higher-order contributions in the expansion of the third integral on the right-hand side of Eq. (58) in powers of $\beta/l$. Restricting its explicit form to the first order contribution we get

$$R(a, T) = R_1(a, T) + O\left[ (\beta/l)^2 \right],$$

$$R_1(a, T) = \frac{k_B T}{4\pi a^2} \sum_{l=1}^\infty \frac{\beta}{\sqrt{\xi_l^2 + \varepsilon_l y}} \int_0^\infty dy \frac{y^2 e^{-y}}{\sqrt{y^2 + \varepsilon_l}} \left\{ \frac{2 - \varepsilon_l}{\left[\sqrt{y^2 + \varepsilon_l} - (\xi_l - 1) + \varepsilon_l y \right]^2} \right\}.$$
Calculating the entropy by the first equality in Eq. (49), we arrive at
\[
\tilde{S}(a, T) = S(a, T) + \frac{k_B}{16\pi a^2} \left[ \zeta(3) - \text{Li}_3 \left( \frac{r_0^2}{\epsilon_0^2} \right) \right] - \frac{\partial R(a, T)}{\partial T},
\]
where \( S(a, T) \) is the entropy for the plates with the dielectric permittivity \( \epsilon_1 \) given by Eq. (49).

Let us now show that the quantity \( R(a, T) \) exponentially goes to zero with the decrease of \( T \). First we consider only the integral in Eq. (61), expand the integrated function in powers of \( \tau \) (we recall that \( \zeta_l = \tau l \)), restrict ourselves to the main contribution, resulting for \( \tau = 0 \), and rearrange it appropriately:
\[
-2 \int_{\zeta_l}^{\infty} dy \frac{ye^{-y}}{(\epsilon_0^2 - 1)^2} = -\frac{2}{(\epsilon_0^2 - 1)} \int_{\zeta_l}^{\infty} dy \frac{r_0}{1 - r_0^2 e^{-y}} = -\frac{2}{(\epsilon_0^2 - 1)} \sum_{n=1}^{\infty} r_0^n e^{-ny} = -\frac{2}{(\epsilon_0^2 - 1)} \sum_{n=1}^{\infty} \frac{r_0^n}{n^2} e^{-n\zeta_l}.
\]
Substituting this in Eq. (61), we find
\[
R_1(a, T) = -\frac{k_B T}{4\pi a^2 (\epsilon_0^2 - 1)} \sum_{n=1}^{\infty} r_0^n \left( \sum_{l=1}^{\infty} \frac{e^{-n\tau l}}{l} + n\tau \sum_{l=1}^{\infty} e^{-n\tau l} \right) = -\frac{k_B T}{4\pi a^2 (\epsilon_0^2 - 1)} \sum_{n=1}^{\infty} \frac{r_0^n}{n^2} \left[ -\ln \left( 1 - e^{-n\tau} \right) + \frac{n\tau}{e^{n\tau} - 1} \right] = \frac{k_B \text{Li}_2 \left( \frac{r_0^2}{\epsilon_0^2} \right)}{2\pi a^2 (\epsilon_0^2 - 1)} T \beta \ln \tau + T \beta O(\tau^0).
\]
Here, the last line is obtained by using the equality
\[
-\ln \left( 1 - e^{-n\tau} \right) + \frac{n\tau}{e^{n\tau} - 1} = -\ln \tau + 1 - \ln n + O(\tau^2),
\]
substituting only its leading term and observing the definition of the integral logarithm, \( \text{Li}_2 (z) = (1/2) \sum_{n=1}^{\infty} z^n / n^2 \). Taking into account that \( \beta \sim (1/T) \exp(-b/T) \), we get the conclusion that the temperature dependence of \( R_1(a, T) \) is given by
\[
R_1(a, T) \sim e^{-b/T} \ln T.
\]
Thus, both \( R_1(a, T) \) and its derivative with respect to \( T \) in Eqs. (65) and (62) go to zero. The terms of the second and higher powers in \( \beta \) in Eq. (60) go to zero even faster than \( R_1 \) when \( T \to 0 \).

Finally, in the limit \( T \to 0 \) from Eq. (62) it follows
\[
\tilde{S}(a, 0) = \frac{k_B}{16\pi a^2} \left[ \zeta(3) - \text{Li}_3 \left( \frac{r_0^2}{\epsilon_0^2} \right) \right] > 0.
\]

The right-hand side of this equation depends on the parameter of the system under consideration (the separation distance \( a \)) and implies a violation of the Nernst heat theorem. An analogous result was obtained in the case of two dissimilar dielectrics.
The violation of the Nernst heat theorem in the Casimir interaction for dielectrics originates from the inclusion of the dc conductivity in the model of dielectric response. This violation is, however, of different nature than the one discussed above in the case of Drude metals. In the case of dielectrics the entropy at zero temperature is positive but in the case of Drude metals it is negative. In the case of metals the violation is caused by the vanishing contribution from the transverse electric mode at zero frequency whereas the other reflection coefficient takes the physical value 1 [see Eq. (6)]. For dielectrics the situation is quite opposite. In this case the transverse reflection coefficient at zero frequency is always equal to zero [compare Eqs. (11) and (57) in the absence and in the presence of the contribution from dc conductivity]. Here the violation occurs due to the unity value of the parallel reflection coefficient in Eqs. (57) which departs from the value $r_0 = (\varepsilon_0 - 1)/(\varepsilon_0 + 1)$ coordinated with the zero value of the transverse coefficient in Eq. (11).

One can conclude that the dc conductivity of a dielectric is not related to the nature of the van der Waals and Casimir forces and must not be included in the model of dielectric response. Ignoring this rule results in a violation of thermodynamics. Physically it is amply clear that there is no fluctuating field of zero frequency and that for such high-frequency phenomena as the van der Waals and Casimir forces the low-frequency behavior should be obtained by analytic continuation from the region of high frequencies. This permits to conclude that the correct procedure consists in the substitution of the finite static dielectric permittivities into the zero-frequency term of the Lifshitz formula, as Lifshitz and his collaborators really did[36,37,38].

4. THERMAL CASIMIR FORCE BETWEEN DIELECTRIC AND METAL PLATES

The Casimir interaction between metal and dielectric plates suggests the interesting opportunity to verify the thermodynamic consistency of the Lifshitz theory with different models of the dielectric response. This configuration was first investigated in Ref. 61 where it was proved that the Casimir entropy is in accordance with the demands of the Nernst heat theorem if the static permittivity of the dielectric plate is finite. In Ref. 61 however, only the first leading terms in the low-temperature asymptotic expressions for the free energy and entropy were obtained and the Casimir pressure was derived only in the dilute approximation. Here we derive the more precise low-temperature behavior for the Casimir free energy, pressure and entropy in the configuration of one plate made of ideal metal and another plate made of dielectric with any finite static dielectric permittivity.

For the configuration of metal and dielectric plates the Lifshitz formula takes the form analogous to Eq. (63)\

$$\mathcal{F}(a, T) = \frac{\hbar c T}{32\pi^2 a^3} \sum_{l=0}^{\infty} \left(1 - \frac{\delta_{0l}}{2}\right) \int_{\zeta_l}^{\infty} ydy$$ (68)
\[ \times \left\{ \ln \left[ 1 - r_{\parallel}^M(\zeta_i, y) r_{\parallel}^D(\zeta_i, y) e^{-y} \right] + \ln \left[ 1 - r_{\perp}^M(\zeta_i, y) r_{\perp}^D(\zeta_i, y) e^{-y} \right] \right\}. \]

Here the reflection coefficients \( r_{\parallel,\perp}^{M,D} \) for metal and dielectric, respectively, are given by Eqs. \((66)\) where \( \varepsilon_i \) should be changed for \( \varepsilon_i^{M,D} = \varepsilon_i^{M,D}(i\xi_i) \).

For an ideal metal \( r_{\parallel,\perp}^M(\zeta, y) = 1 \) and Eq. \((68)\) takes the more simple form

\[ \mathcal{F}(a, T) = \frac{\hbar c \tau}{32\pi^2 a} \sum_{l=0}^{\infty} \left( 1 - \frac{\delta_{0l}}{2} \right) \int_0^\infty y dy \]

\[ \times \left\{ \ln \left[ 1 - r_{\parallel}(\zeta, y) e^{-y} \right] + \ln \left[ 1 - r_{\perp}(\zeta, y) e^{-y} \right] \right\} \]

(here and below we omit the index \( D \) near the reflection coefficient and permittivity of a dielectric plate). We admit that the dielectric permittivity calculated at Matsubara frequencies \( \varepsilon_i \equiv \varepsilon_0, \) i.e., is equal to its static value and find the asymptotic behavior of Eq. \((69)\) at small \( \tau. \) [In analogy with Sec. 2 it is possible to prove that the deviations of \( \varepsilon(i\xi_i) \) from \( \varepsilon_0 \) at high frequencies do not influence the low-temperature behavior of the Casimir free energy, pressure and entropy. It can be shown also that the results of this section are valid not only for ideal metal plate but for plate made of real metal as well.] The free energy \((69)\) can be represented by Eqs. \((68) - (71)\), with the function \( f(\zeta, y) \) replaced by

\[ \hat{f}(\zeta, y) = y \ln \left[ 1 - r_{\parallel}(\zeta, y) e^{-y} \right] + y \ln \left[ 1 - r_{\perp}(\zeta, y) e^{-y} \right] \]

\[ = \hat{f}_\parallel(\zeta, y) + \hat{f}_\perp(\zeta, y). \] (70)

In the case of one dielectric and one metal plate both \( \hat{f}_\parallel \) and \( \hat{f}_\perp \) contribute to \( F(ix) - F(-ix) \). The expansion of \( \hat{f}(x, y) \) in powers of \( x \) takes the form

\[ \hat{f}(x, y) = y \ln(1 - r_0 e^{-y}) - \left( \frac{\varepsilon_0}{4y} e^{-y} - \frac{\varepsilon_0}{\varepsilon_0 + 1} \sum_{n=1}^{\infty} r_0^n e^{-ny} \right) x^2 + O(x^3). \] (71)

Now we integrate Eq. \((71)\) in accordance with Eq. \((68)\) to find the function \( F(x) \). The integral of the first term on the right-hand side of Eq. \((71)\) is evaluated using the new variable \( v = y - x \):

\[ \int_x^\infty y dy \ln(1 - r_0 e^{-y}) = \int_0^\infty v dv \ln(1 - r_0 e^{-v}) + O(x^2), \] (72)

where the coefficient near the first-order contribution in \( x \) vanishes. As a result, this term could contribute to \( F(ix) - F(-ix) \) only starting from the third expansion order. The integrals of the second-order terms on the right-hand side of Eq. \((71)\) are simply calculated using the formulas

\[ \int_x^\infty dy \frac{e^{-y}}{y} = -Ei(-x), \quad \int_x^\infty dy \frac{e^{-ny}}{y} = -Ei(-nx). \] (73)

Finally, we obtain

\[ F(ix) - F(-ix) = i\pi \frac{(\varepsilon_0 - 1)^2}{4(\varepsilon_0 + 1)} x^2 - i\gamma x^3 + O(x^4), \] (74)
where the unknown third order expansion coefficient is designated as \( \gamma \).

Substituting Eq. (74) in Eq. (21) and using Eq. (19), we find the free energy in the system metal-dielectric in the form

\[
\mathcal{F}(a,T) = E(a) - \frac{\hbar c}{32\pi^2 a^3} \left[ \frac{1}{16\pi^2} \frac{(\varepsilon_0 - 1)^2}{\varepsilon_0 + 1} + K_4 \tau^4 + O(\tau^5) \right],
\]

where \( K_4 \equiv \gamma/240 \).

The Casimir pressure in the configuration of metal and dielectric plates obtained from Eq. (75) is equal to

\[
P(a,T) = P_0(a) - \frac{\hbar c}{32\pi^2 a^4} [K_4 \tau^4 + O(\tau^5)].
\]

The direct application of the Lifshitz formula gives the expression for the pressure analogical to Eq. (31),

\[
P(a,T) = -\frac{\hbar c}{32\pi^2 a^4} \sum_{n=0}^{\infty} \left( 1 - \frac{\delta_0}{2} \right) \int_{\xi}^{\infty} y^2 dy \left[ \frac{r_{\parallel}(\xi,y)}{e^y - r_{\parallel}(\xi,y)} + \frac{r_{\perp}(\xi,y)}{e^y - r_{\perp}(\xi,y)} \right].
\]

Using the Abel-Plana formula (17), Eq. (77) can be represented in the form of Eqs. (32), (33) where

\[
\Phi_{\parallel,\perp}(x) = \int_{x}^{\infty} dy \frac{y^2 r_{\parallel,\perp}(x,y)}{e^y - r_{\parallel,\perp}(x,y)}.
\]

Again, we deal first with \( \Phi_{\perp}(x) \). By adding and subtracting the asymptotic behavior of the integrated function at small \( x \),

\[
\frac{y^2 r_{\perp}(x,y)}{e^y - r_{\perp}(x,y)} = \frac{1}{4}(\varepsilon_0 - 1)x^2 e^{-y} + O(x^3),
\]

and introducing the new variable \( v = y/x \), the function \( \Phi_{\perp}(x) \) can be identically rearranged and expanded in powers of \( x \) as follows:

\[
\Phi_{\perp}(x) = \frac{1}{4}(\varepsilon_0 - 1)x^2 e^{-x} + x^3 \int_{1}^{\infty} dv \left[ v^2 \sum_{n=1}^{\infty} r_{\perp}(v) e^{-nx} - \frac{1}{4}(\varepsilon_0 - 1)e^{-vx} \right]
\]

\[
= \frac{1}{4}(\varepsilon_0 - 1)x^2 (1-x) + x^3 \int_{1}^{\infty} dv \left[ \frac{v^2 r_{\perp}(v)}{1 - r_{\perp}(v)} - \frac{\varepsilon_0 - 1}{4} \right] + O(x^4).
\]

The integral on the right-hand side of Eq. (80) is converging and can be simply calculated with the result

\[
\Phi_{\perp}(x) = \frac{\varepsilon_0 - 1}{4} x^2 - \frac{1}{6}(\varepsilon_0\sqrt{\varepsilon_0 - 1}) x^3 + O(x^4).
\]

To deal with \( \Phi_{\parallel}(x) \) we add and subtract in Eq. (78) the two first expansion terms of the integrated function in powers of \( x \),

\[
\Phi_{\parallel}(x) = \int_{x}^{\infty} y^2 dy \left[ \frac{r_0}{e^y - r_0} - \frac{\varepsilon_0 r_0 e^{-y} x^2}{y^2(\varepsilon_0 + 1)(1 - r_0 e^{-y})^2} \right]
\]

\[
+ \int_{x}^{\infty} y^2 dy \left[ \frac{r_{\parallel}(x,y)}{e^y - r_{\parallel}(x,y)} - \frac{r_0}{y^2(\varepsilon_0 + 1)(1 - r_0 e^{-y})^2} + \frac{\varepsilon_0 r_0 e^{-y} x^2}{y^2(\varepsilon_0 + 1)(1 - r_0 e^{-y})} \right].
\]
The asymptotic expansion of the first integral on the right-hand side of Eq. (82) is given by
\[
2\text{Li}_3(r_0) - \frac{\varepsilon_0(\varepsilon_0 - 1)}{2(\varepsilon_0 + 1)} x^2 + \frac{1}{12} (\varepsilon_0 - 1)(3\varepsilon_0 - 2)x^3 + O(x^4),
\] (83)
and of the second one by
\[
\left[ -\frac{1}{4} \varepsilon_0(\varepsilon_0 - 1) - \frac{1}{6} \varepsilon_0\sqrt{\varepsilon_0 - 1} + \frac{1}{2} \varepsilon_0(\varepsilon_0 - 1)\sqrt{\varepsilon_0} \right] x^3 + O(x^4). \tag{84}
\]
By summing Eqs. (83) and (84) we find
\[
\Phi_{\parallel}(x) = 2\text{Li}_3(r_0) - \frac{\varepsilon_0(\varepsilon_0 - 1)}{2(\varepsilon_0 + 1)} x^2 - \frac{1}{6} \left[ (\varepsilon_0 - 1) + (\varepsilon_0\sqrt{\varepsilon_0 - 1} - 3\varepsilon_0(\varepsilon_0 - 1)\sqrt{\varepsilon_0}) \right] x^3 + O(x^4).
\] (85)
Finally we add Eq. (85) to Eq. (81) and arrive at
\[
\Phi(ix) - \Phi(-ix) = -i\frac{2}{3} \left( 1 - 2\varepsilon_0\sqrt{\varepsilon_0} + \varepsilon_0^2\sqrt{\varepsilon_0} \right) x^3 + O(x^4). \tag{86}
\]
Substituting this in Eq. (83) and using Eq. (32), the asymptotic expression for the Casimir pressure is obtained
\[
P(a, T) = P_0(a) - \frac{\hbar c}{32\pi^2 a^4} \left[ \frac{1}{360} \left( 1 - 2\varepsilon_0\sqrt{\varepsilon_0} + \varepsilon_0^2\sqrt{\varepsilon_0} \right) \tau^4 + O(\tau^5) \right]. \tag{87}
\]
Comparing Eqs. (87) and (76), the explicit expression for the coefficient $K_4$ reads
\[
K_4 = \frac{1}{360} \left( 1 - 2\varepsilon_0\sqrt{\varepsilon_0} + \varepsilon_0^2\sqrt{\varepsilon_0} \right). \tag{88}
\]
Now we are in a position to find the asymptotic behavior of the entropy in the limit of low temperatures in the configuration of two parallel plates one of which is metallic and the other one dielectric. By calculating the negative derivative of Eq. (75) with respect to temperature, one arrives at
\[
S(a, T) = \frac{3k_B\zeta(3)(\varepsilon_0 - 1)^2}{128\pi^3 a^2(\varepsilon_0 + 1)} \tau^2 \times \left[ 1 - \frac{8\pi^2(\varepsilon_0 + 1)(1 - 2\varepsilon_0\sqrt{\varepsilon_0} + \varepsilon_0^2\sqrt{\varepsilon_0})}{135\zeta(3)(\varepsilon_0 - 1)^2} \tau + O(\tau^2) \right]. \tag{89}
\]
This equation is in analogy to Eq. (49) obtained for the case of two dielectrics. As is seen from Eq. (89), the entropy of the Casimir and van der Waals interactions between metal and dielectric plates vanishes when the temperature goes to zero, i.e., the Nernst heat theorem is satisfied. Note that the first term of order $\tau^2$ on the right-hand side of Eq. (89) was already obtained in Ref. 61. It is notable also that at low temperatures the entropy goes to zero remaining positive. At the same time, as was shown in Ref. 61, at larger temperatures entropy is nonmonotonous and may take negative values. This interesting property distinguishes the configuration of metal and dielectric plates from two dielectric plates. In the latter configuration the
negative entropy appears only for nonphysical dielectrics with anomalously large and frequency independent dielectric permittivities.\textsuperscript{53}

Now let us consider the case $\tau \gg 1$, i.e., high temperatures (large separations). In the same way as for two dielectric plates, here the main contribution to the free energy is given by the term with $l = 0$ in Eq. (69),

$$F(a, T) = \frac{\hbar c}{64\pi^2 a^2} \int_0^\infty y dy \ln \left(1 - r_0 e^{-y}\right). \quad (90)$$

Performing the integration in Eq. (90), we obtain

$$F(a, T) = -\frac{k_B T}{16\pi a^2} \text{Li}_3 (r_0). \quad (91)$$

From this equation, for the Casimir pressure and entropy at $\tau \gg 1$ it follows

$$P(a, T) = -\frac{k_B T}{8\pi a^4} \text{Li}_3 (r_0), \quad S(a, T) = -\frac{k_B}{16\pi a^2} \text{Li}_3 (r_0). \quad (92)$$

The results (91) and (92) are analogous to (54) and (55) for two dielectric plates.

5. THE PROBLEM ORIGINATING FROM THE ACCOUNT OF DIELECTRIC DC CONDUCTIVITY

In the previous section it was supposed that the static dielectric permittivity $\varepsilon_0$ of the dielectric plate is finite. Now we will deal with the configuration of metal and dielectric plates with included dc conductivity of the dielectric material. In doing so the permittivity of the dielectric plate is given by

$$\hat{\varepsilon}(i\xi_l) = \varepsilon_0 + \frac{\beta(T)}{l}, \quad (93)$$

where all notations were introduced in and below Eq. (56). Thus, the reflection coefficients at zero frequency satisfy Eq. (4) for a plate made of ideal metal and Eq. (57) for a plate made of dielectric with included dc conductivity. Let us find the low-temperature behavior of the Casimir entropy and verify the consistency of the Lifshitz theory with thermodynamics in this nonstandard situation.

For this purpose we substitute the dielectric permittivity (93) in Eq. (99) instead of $\varepsilon_0$ and find the Casimir energy $\tilde{F}(a, T)$ with included dc conductivity of a dielectric plate. In the same way as in Sec. 3, it is convenient to separate the zero-frequency term of $\tilde{F}(a, T)$ and subtract and add the usual zero-frequency contribution for metal-dielectric plates computed with the dielectric permittivity $\varepsilon_0$,

$$\tilde{F}(a, T) = \frac{k_B T}{16\pi a^2} \int_0^\infty y dy \left[\ln \left(1 - e^{-y}\right) - \ln \left(1 - r_0 e^{-y}\right)\right]$$

$$+ \frac{k_B T}{16\pi a^2} \int_0^\infty y dy \ln \left(1 - r_0 e^{-y}\right)$$

$$+ \frac{k_B T}{8\pi a^2} \sum_{l=1}^{\infty} \int_{\gamma_l}^\infty y dy \left\{\ln \left[1 - \tilde{r}_l (\gamma_l, y) e^{-y}\right] + \ln \left[1 - \tilde{r}_l (\gamma_l, y) e^{-y}\right]\right\}. \quad (94)$$
Here the reflection coefficients $\tilde{r}_{\parallel,\perp}$ are calculated with the permittivity (93). We expand the third integral on the right-hand side of Eq. (94) in powers of the small parameter $\beta/l$. The zero-order contribution in this expansion together with the second integral of Eq. (94) form the Casimir free energy $\tilde{F}(a, T)$ calculated with dielectric permittivity $\varepsilon_0$. The first integral on the right-hand side of Eq. (94) is calculated explicitly. As a result, Eq. (94) is rearranged to

$$\tilde{F}(a, T) = F(a, T) - \frac{k_B T}{16 \pi a^2} \left[ \zeta(3) - \text{Li}_3(r_0) \right] + Q(a, T),$$

(95)

where $Q(a, T)$ contains the first and higher-order contributions in the expansion of the third integral on the right-hand side of Eq. (94) in powers of $\beta/l$. The explicit form of the main first-order term in $Q(a, T)$ is the following:

$$Q_1(a, T) = \frac{k_B T}{8 \pi a^2} \sum_{l=1}^{\infty} \frac{\beta}{l} \int_{\zeta_l}^{\infty} \frac{dy \, y^2 e^{-y}}{\sqrt{y^2 + \zeta_l^2 (\varepsilon_0 - 1)}} \left\{ \frac{(2 - \varepsilon_0) \zeta_l^2 - 2y^2}{\left[\sqrt{y^2 + \zeta_l^2 (\varepsilon_0 - 1)} + \varepsilon_0 y\right]^2} \right\} \left\{ \frac{1}{1 - r_\parallel(\zeta_l, y)e^{-y}} - \frac{\zeta_l^2}{\left[\sqrt{y^2 + \zeta_l^2 (\varepsilon_0 - 1)} + y\right]^2} \right\}.$$  

(96)

In the same way as in Sec. 3, we expand the integrated function in Eq. (96) in powers of $\tau$ (bearing in mind that $\zeta_l = \tau l$) and preserve only the main contribution at $\tau = 0$:

$$Q_1(a, T) = -\frac{k_B T}{4 \pi a^2} \sum_{l=1}^{\infty} \frac{\beta}{l} \int_{\zeta_l}^{\infty} \frac{dy \, y e^{-y}}{(\varepsilon_0^2 - 1) (1 - r_0 e^{-y})} + n \tau \sum_{l=1}^{\infty} \frac{e^{-n \tau l}}{l}.$$  

(97)

Dealing with this expression in the same way as with Eq. (64), we arrive at

$$Q_1(a, T) \sim e^{-b/T} \ln T.$$  

(98)

The Casimir entropy in the configuration of metal and dielectric plates with included dc conductivity of the dielectric plate is obtained as minus derivative of Eq. (95) with respect to temperature,

$$\tilde{S}(a, T) = S(a, T) + \frac{k_B}{16 \pi a^2} \left[ \zeta(3) - \text{Li}_3(r_0) \right] - \frac{\partial Q(a, T)}{\partial T}.$$  

(99)

Using Eq. (98), the calculation of the limiting value at $T \to 0$ is straightforward:

$$\tilde{S}(a, 0) = \frac{k_B}{16 \pi a^2} \left[ \zeta(3) - \text{Li}_3(r_0) \right] > 0.$$  

(100)

From this equation it follows that the inclusion of the dc conductivity of dielectric plate in the configuration metal-dielectric results in a violation of the Nernst heat theorem. In the above this result was obtained for a metallic plate made of ideal metal. It can be shown that it remains valid for a metallic plate made of real metal with finite conductivity.
Thus, both configurations (two dielectric plates or one metal plate and one dielectric) lead to the same conclusion that when the dc conductivity is included in the model of dielectric response of the dielectric plate, the Lifshitz theory loses its consistency with thermodynamics. This confirms the conclusion made in Sec. 3 that the actual properties of dielectric materials at very low frequencies are in fact not related to the van der Waals and Casimir forces.

6. QUALITATIVE DISCUSSION OF CASIMIR INTERACTION BETWEEN METAL AND SEMICONDUCTOR

As was mentioned in the Introduction, semiconductors possess a wide variety of electric and optical properties ranging from metallic to dielectric. This opens the possibility to modulate the van der Waals and Casimir forces by changing the charge carrier density. Bearing in mind the discussed above problems on the consistency of the Lifshitz theory with thermodynamics, semiconductors can provide us with a test for the validity of different approaches. Thus, if for good dielectric the dc conductivity does not play any real role in the van der Waals and Casimir forces, the question arises on how much it should be increased in order to become a relevant factor in the description of dispersion forces.

In Ref. 81 the Casimir force acting between two Si plates was calculated using the simple analytic expression for Si dielectric permittivity as a function of frequency. The complete tabulated optical data of Si were used in Ref. 75 to calculate the van der Waals interaction of different atoms with a Si wall. The first attempt to measure the van der Waals force between a glass lens and a Si plate and to modify it by light due to the change of carrier density was undertaken in Ref. 82. However, glass is a dielectric and therefore the electric forces due to localized point charges could not be controlled. This might explain that no force change occurred on illumination at small separations where the effect should be most pronounced.

The first measurements of the Casimir force between a gold coated sphere and a single crystal Si plate were performed in Refs. 62, 63 by means of the atomic force microscope. The experiments used a p-type B doped Si plate of resistivity $\rho = 0.0035 \, \Omega \, \text{cm}$. The chosen resistivity of the plate is in some sense intermediate between the resistivity of metals (which is usually two or three orders of magnitude lower) and the resistivity of dielectrics (it can be by about a factor of $10^5$ larger; for instance, high-resistivity “dielectric” Si has the resistivity $\rho_0 = 1000 \, \Omega \, \text{cm}$). Thus, the used Si plate had a relatively large absorption typical for semiconductors but it was also enough conductive to avoid the accumulation of charges.

In Fig. 2, taken from Ref. 63, the differences of the theoretical and mean experimental Casimir forces acting between Au sphere and Si plate are presented as functions of separation. In Fig. 2a the theoretical force $F_{\text{theor}}$ is computed using the Lifshitz formula and the dielectric permittivity of a Si plate with the relatively low resistivity $\rho$ used in experiment. This dielectric permittivity goes to infinity as $\xi^{-1}$ with decreasing frequency (see the solid line in Fig. 3). In Fig. 2b the theo-
Fig. 2. Differences of the theoretical and mean experimental Casimir forces versus separation. Theoretical forces are computed (a) for the Si plate used in experiment and (b) for dielectric Si. Solid and dashed lines indicate 95 and 70% confidence intervals, respectively.

Theoretical Casimir force $\tilde{F}\text{theor}$ is computed using the dielectric permittivity of the Si plate made of “dielectric” Si with high resistivity $\rho_0$. The dielectric permittivity of high-resistivity Si is shown by the dashed line in Fig. 3. It is characterized by the finite static value $\varepsilon_{\text{Si}}(0) = 11.67$. The solid and dashed lines in Fig. 2a,b indicate the 95% and 70% confidence intervals, respectively. As is seen from Fig. 2, the theoretical approach using the dielectric permittivity of high-resistivity “dielectric” Si is excluded by experiment within the separation range from 60 to 110 nm at 70% confidence. At the same time, the theory using the dielectric permittivity of Si with a low resistivity $\rho$ is consistent with experiment.
The above results suggest an approach on how to correctly determine the possible role of the low-frequency conductivity properties in dispersion forces. As is seen from Fig. 3, the dielectric permittivity of low-resistivity Si (solid line) significantly departs from the dielectric permittivity of “dielectric” Si in the region around the important dimensional parameter of the problem, the characteristic frequency \( c/(2a) \sim 10^{14} - 10^{15} \text{rad/s} \). That is why the low-resistivity sample cannot be described at low frequencies by the static dielectric permittivity of Si equal to \( \varepsilon_{Si}(0) = 11.67 \). To describe it, the term \( \beta(T)/l \), like in Eq. (93), should be added to \( \varepsilon_{Si}(0) \). Note that in this case \( \beta(T)/l > 1 \) at the first Matsubara frequencies with \( l = 1, 2, 3, \ldots \) and, thus, this quantity cannot be considered as a small parameter. At the same time, for a high-resistivity sample the inclusion of the dc conductivity would lead to deviations from the dashed line in Fig. 3 only at frequencies \( \xi < 10^8 \text{rad/s} \), which are much less than the characteristic frequency. This comparison permits to make a conclusion in what experimental situations the conductivity properties of semiconductors at low frequencies should be taken into account and when they should be omitted as being not related to dispersion forces.

The future experiments on the modification of semiconductor charge carrier density by laser light\(^{83}\) will bring a more clear understanding of this problem on the connection between the low-frequency material properties and dispersion forces.
7. DOES SPATIAL DISPERSION LEAD TO AN IMPORTANT IMPACT ON THERMAL CASIMIR FORCE?

The presented above new analytic results on the low-temperature behavior of the Casimir free energy, pressure and entropy between dielectrics or between dielectric and metal are based on the conventional Lifshitz theory which describes dielectric materials by means of the frequency dependent dielectric permittivity. In fact, the assumption that the material of the plates possesses only the frequency dispersion means that the components of electric displacement are connected with the components of electric field by the relation

\[ D_k(r, \omega) = \varepsilon_{kl}(r, \omega) E_l(r, \omega). \]  

This equation is central in all different derivations of the Lifshitz formula (see, e.g., Refs. 2, 6, 55, 56, 70, 84). The effects of spatial nonlocality (spatial dispersion) are in fact essential only at shortest separations between the plates comparable with atomic dimensions and also for metals at sufficiently large separations (typically of about 2–3 \( \mu \text{m} \)) in the frequency region of the anomalous skin effect. The Casimir force in the latter region was described by the Lifshitz theory reformulated in terms of the Leontovich impedance. 85

Recently the spatial dispersion came to the attention in connection with the thermal Casimir force. 64, 65, 66, 67, 68 In particular, it was claimed 64, 65 that for real metals at any separation the account of spatial dispersion leads to practically the same result (6) for the reflection coefficients at zero frequency as was obtained earlier using the Drude model dielectric function (5). This conclusion, if it is correct, not only returns us to the contradiction with experiment 30, 32, 33 but also casts doubts on all results obtained by means of the conventional Lifshitz theory accounting for only the frequency dispersion. It is natural when the spatial dispersion contributes a small fraction of a percent as it is generally believed in numerous applications of the Lifshitz theory. It is, however, quite another matter when the account of the spatial dispersion results in some “dramatic effects” 64, i.e., in several hundred times larger thermal correction than is obtained in the local case. Below we demonstrate that the conclusions of Refs. 64–68 are in fact not reliable because they use the Lifshitz theory of dispersion forces outside of its application range. 86

To find the electromagnetic modes associated with an empty gap between the plates, Refs. 64–68 use the standard continuity boundary conditions,

\[ E_{1t} = E_{2t}, \quad B_{1n} = B_{2n}, \quad D_{1n} = D_{2n}, \quad B_{1t} = B_{2t}, \]  

which are commonly applied in the derivation of the Lifshitz formula for spatially local nonmagnetic materials. Here \( \mathbf{B} \) is the magnetic induction, \( \mathbf{n} \) is the normal to the boundary directed inside the medium, the subscripts \( n, t \) refer to the normal and tangential components, respectively, the subscript 1 refers to the vacuum and subscript 2 to the plate material. In Refs. 64–68 the spatial dispersion is described by the longitudinal and transverse dielectric permittivities depending on the wave vector and frequency: \( \varepsilon_{kl} = \varepsilon_{kl}(\mathbf{q}, \omega) \). However, as is shown below, in the theory
of the Casimir effect both the boundary conditions \((102)\) and permittivities \(\varepsilon_{kl} = \varepsilon_{kl}(q, \omega)\) are inapplicable.

We start from the boundary conditions and recall the set of Maxwell equations in a metal describing the Casimir effect,

\[
\begin{align*}
\text{rot}E + \frac{1}{c} \frac{\partial B}{\partial t} &= 0, \\
\text{div}D &= 0, \\
\text{rot}B - \frac{1}{c} \frac{\partial D}{\partial t} &= 0 \\
\text{div}B &= 0.
\end{align*}
\]

Equations \((103)\) do not contain any external, i.e., independent on \(E, D\) and \(B\), current or charge densities. The definition of the electric displacement is

\[
\frac{\partial D}{\partial t} = \frac{\partial E}{\partial t} + 4\pi i,
\]

where the volume current \(i\) is induced by \(E\) and \(B\) and takes into account the conduction electrons.

In electrodynamics with spatial dispersion the electric field and magnetic induction are finite at the boundary surfaces, whereas the electric displacement can tend to infinity.\(^{87}\) Then, integrating Eqs. \((103)\) over the thickness of the boundary layer as is done in Ref. \(^{88}\) we reproduce the first two conditions in Eq. \((102)\) and arrive at the modified third and fourth conditions,\(^{87,89}\)

\[
\begin{align*}
E_1 &= E_2, \\
B_1 &= B_2, \\
D_2 - D_1 &= 4\pi \sigma, \\
\left[ n \times (B_2 - B_1) \right] &= \frac{4\pi}{c} j,
\end{align*}
\]

where the induced surface charge and current densities are given by

\[
\sigma = \frac{1}{4\pi} \int_1^2 \text{div}[n \times [D \times n]] dl, \\
j = \frac{1}{4\pi} \int_1^2 \frac{\partial D}{\partial t} dl.
\]

Note that the boundary conditions \((105)\) are obtained from the macroscopic Maxwell equations for physical fields. They should not be mixed with the boundary conditions arising in perturbative theories and for the fictitious fields (see below).

In linear electrodynamics for a medium with time-independent properties without spatial dispersion the material equation connecting the electric displacement and electric field takes the form

\[
D_k(r, t) = \int_{-\infty}^{t} \varepsilon_{kl}(r, t - t') E_l(r, t') dt'.
\]

According to this equation, the electric displacement at a point \(r\) and moment \(t\) is determined by the electric field at the same point \(r\) at different moments \(t' \leq t\) (the spatial dispersion is absent but the temporal may be present). It is easily seen that the substitution of Eq. \((107)\) in Eq. \((106)\) leads to \(\sigma = 0, j = 0\) and, as a result, the boundary conditions \((105)\) coincide with the standard continuity conditions \((102)\). It is unjustified, however, to use conditions \((102)\) in the presence of spatial dispersion. In Refs. \(^{87,90}\) a few examples are presented illustrating that in this case neither \(\sigma\) nor \(j\) is equal to zero.

We now turn to a discussion of the use of dielectric permittivity \(\varepsilon_{kl}(q, \omega)\) in the theory of the Casimir effect with account of spatial dispersion. In the presence of
only frequency dispersion, it is possible to perform the Fourier transformation of the fields

\[ E(r, t) = \int_{-\infty}^{\infty} E(r, \omega) e^{-i\omega t} d\omega, \quad D(r, t) = \int_{-\infty}^{\infty} D(r, \omega) e^{-i\omega t} d\omega, \quad (108) \]

in Eq. (107) and arrive at Eq. (101) where

\[ \epsilon_{kl}(r, \omega) = \int_{0}^{\infty} \hat{\epsilon}_{kl}(r, \tau)e^{i\omega \tau} d\tau \quad (109) \]

is the frequency-dependent dielectric permittivity and \( \tau \equiv t - t' \). In fact Eqs. (101), (108) and (109) are used in parallel with the boundary conditions (102) in all derivations of the Lifshitz formula. 2, 6, 36, 37, 38, 69, 70, 84

If the material of the plates is characterized not only by temporal but also spatial dispersion, Eq. (107) is generalized to

\[ D_k(r, t) = \int_{-\infty}^{t} dt' \int d\tau \hat{\epsilon}_{kl}(r, r', t - t') E_l(r', t'). \quad (110) \]

If the material medium were uniform in space (i.e., all points were equivalent), the kernel \( \hat{\epsilon} \) would not depend on \( r \) and \( r' \) separately, as in Eq. (110), but on the difference \( R \equiv r - r' \). In this case, by performing the Fourier transformation,

\[ E(r, t) = \int_{-\infty}^{\infty} d\omega \int dq E(q, \omega) e^{i(qr - \omega t)}, \]
\[ D(r, t) = \int_{-\infty}^{\infty} d\omega \int dq D(q, \omega) e^{i(qr - \omega t)}, \quad (111) \]

and substituting it in Eq. (110), one could introduce the dielectric permittivities

\[ \epsilon_{kl}(q, \omega) = \int_{0}^{\infty} d\tau \int dR \hat{\epsilon}_{kl}(R, \tau)e^{-i(qR - \omega \tau)}, \quad (112) \]

as Refs. 64–68 do, and rearrange Eq. (110) to the form

\[ D_k(q, \omega) = \epsilon_{kl}(q, \omega) E_l(q, \omega). \quad (113) \]

In the Casimir effect, however, the material medium is not uniform due to the presence of a macroscopic gap between the two plates (half spaces). Because of this, the assumption that the kernel \( \hat{\epsilon} \) depends only on \( R \) and \( \tau \) is wrong. As a result, it is not possible to introduce the dielectric permittivity \( \epsilon_{kl}(q, \omega) \) depending on the wave vector and frequency. In fact, for systems with spatial dispersion in the presence of boundaries the kernel \( \hat{\epsilon} \) depends not only on \( R \) and \( \tau \) but also on the distance from the boundary. In this complicated situation the following approximate phenomenological approach is sometimes applicable. For electromagnetic waves with a wavelength \( \lambda \) the kernel \( \hat{\epsilon}(r, r', \tau) \) in Eq. (110) differs essentially from zero only in a certain vicinity of the point \( r \) with characteristic dimensions \( l \ll \lambda \) (for nonmetallic condensed media \( l \) is of the order of the lattice constant). Then it is reasonable to assume that \( \hat{\epsilon} \) is a function of \( R=r-r' \), except for a layer of thickness \( l \) adjacent to the boundary surface. If one is mostly interested in bulk phenomena
and neglects the role and influence of a subsurface layer, the quantity $\varepsilon_{kl}(q, \omega)$ may be employed as a reasonable approximation.

This approximate phenomenological approach is widely applied in the theoretical investigation of the anomalous skin effect (see, e.g., Ref. 91). Note that in Ref. 91 some kind of fictitious infinite system was introduced and electromagnetic fields in this system are discontinuous on the boundary surface. This discontinuity should not be confused with the discontinuity of physical fields of a real system in the presence of spatial dispersion described by Eqs. (105) and (106). (There is also another approach to the description of the anomalous skin effect in polycrystals using the generalizations of the local Leontovich impedance which takes into account the shape of Fermi surface 92). The frequency and wave vector dependent dielectric permittivity in the presence of boundaries is also approximately applied in the theory of radiative heat transfer 93 or in the study of electromagnetic interaction of molecules with metal surfaces. 94 In all these applications the boundary effects are usually taken into account by the boundary conditions (105) supplemented by so called “additional boundary conditions”.

It is unlikely, however, that the approximate phenomenological approach using such quantity as $\varepsilon_{kl}(q, \omega)$ in the presence of boundaries would be applicable in the theory of the Casimir force where the boundary effects on the zero-point electromagnetic oscillations are of prime importance. It is notable also that this approach faces serious theoretical difficulties including the violation of the law of conservation of energy 95. It is then not surprising that the application of this approach in Refs. 64, 65 results in the contribution to the Casimir free energy from the transverse electric mode which is in contradiction with experiment. 30, 32, 33

One more shortcoming of Refs. 64–68 is that they substitute the dielectric permittivity $\varepsilon_{kl}(q, \omega)$, depending on both wave vector and frequency, into the conventional Lifshitz formula derived in the presence of only temporal dispersion. In the famous review paper 96 it has been noticed, however, that with the inclusion of spatial dispersion the free energy of a fluctuating field takes the form $F = F_L + \Delta F$, where $F_L$ is given by the conventional Lifshitz expression derived in a spatially local case and written in terms of the Fresnel reflection coefficients, and $\Delta F$ is an additional term which can be expressed in terms of the thermal Green’s function of the electromagnetic field and polarization operator. Review 96 calls as not reliable the results of, e.g., Ref. 97 obtained by the substitution of dielectric permittivity $\varepsilon_{kl}(q, \omega)$, taking account of spatial dispersion, into the conventional Lifshitz formula. It can be true that the Lifshitz formula written in terms of general reflection coefficients is applicable in both spatially local and nonlocal situations. However, as far as the exact reflection coefficients in a spatially nonlocal case are unknown, the use of some approximate phenomenological models, elaborated in literature for applications different than the Casimir effect, may lead to incorrect results for $\Delta F$ and create inconsistencies with experiment.
To conclude this section, the results of Refs. [64–68] on the influence of spatial nonlocality on the Casimir interaction are shown to be not reliable. Although at present there is no fundamental theory of the thermal Casimir force incorporating spatial dispersion, there is no reason to expect that it can play any significant role in the frequency region of infrared optics (experimental separations) or normal skin effect (i.e., at separations between plates greater than 4–5 $\mu$m).

8. CONCLUSIONS AND DISCUSSION

In the foregoing we have presented the derivation of analytic asymptotic expressions for the free energy, pressure and entropy of the Casimir interaction between two dielectric plates and between metal and dielectric plates at low and high temperatures. It was shown that the low-temperature behavior of the Casimir interaction between dielectrics and between dielectric and metal is determined by the static dielectric permittivities of nonpolar dielectrics. The obtained results were shown to be in agreement with thermodynamics when the static dielectric permittivities of dielectrics are finite. In particular, the entropy of the Casimir interaction goes to zero when the temperature vanishes, i.e., the Nernst heat theorem is satisfied. This demonstrates the consistency of the original Lifshitz’s approach to the van der Waals forces between dielectrics which disregards the small conductivity of dielectrics at constant current.

The second important result shown above is that the inclusion of the dc conductivity of dielectrics into the model of dielectric response leads not to some small corrections to the characteristics of the Casimir interaction, as one could expect, but makes the Lifshitz theory inconsistent with thermodynamics leading to the violation of the Nernst heat theorem. This reveals that real material properties at very low, quasistatic frequencies are in fact not related to the phenomenon of the van der Waals and Casimir forces which is actually determined by sufficiently high frequencies. In this case the zero-frequency contribution to the Casimir force should be understood not literally but as analytic continuation to zero of the material physical behavior in the region around the characteristic frequency.

The presented results provide a basis for the calculation of the van der Waals and Casimir forces between real materials. Such calculations are much needed for numerous applications of the Casimir force discussed in the Introduction, in particular for the applications in nanotechnology and for constraining predictions of fundamental physical theories beyond the Standard Model. Bearing in mind that semiconductors are the main constituent materials in nanotechnological devices, it is a subject of high priority to understand the Casimir effect with semiconductor boundaries. In this connection we have discussed new experimental results and theoretical ideas on the Casimir interaction between a metal sphere and semiconductor plate. It was stressed that by changing the charge carrier density in the semiconductor it is possible to bring it in different intermediate states between metallic and dielectric. In this case the problem arises when the conductivity properties of
semiconductor are not related to dispersion forces and when they are becoming relevant. A criterion for the resolution of this problem was formulated based on the relation between the typical frequency at which the dc conductivity properties come into play and the characteristic frequency of the Casimir effect. In fact the thermal Casimir interaction between semiconductors remains an open question and much work should be done both in experiment and theory to gain a better insight into this subject.

The last major problem discussed in this review is whether or not the spatial dispersion influences essentially the thermal Casimir force between real materials. In the present literature there is no agreement on this subject. We adduced arguments in favor of the statement that in the region of experimental separations the influence of the spatial dispersion on the Casimir force is negligible small. The statements on the opposite, contained in the literature, were shown to be not reliable because they are obtained by the application of the Lifshitz theory outside of its application range. At the same time it was ascertained that at the moment there is no consistent fundamental theory of the van der Waals and Casimir forces taking the spatial dispersion into account. This is the problem to solve in the foreseeable future.

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