NONCOMMUTATIVE SMOOTH PROJECTIVE CURVES:
LOCAL AND GLOBAL SKEWNESS

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Abstract. Let \( H \) be a noncommutative smooth projective curve over a perfect field \( k \). The square root \( s(H) \) of the dimension of the function (skew-) field over its centre we call the skewness of \( H \), since it is a global measure of noncommutativity. For each point \( x \) we consider a triple of natural numbers, each of which measures skewness locally. Using local properties of the Auslander-Reiten translation, we prove a local-global principle: for each point \( x \) the product of these three numbers coincides with \( s(H) \). We show links to the Hasse principle. We apply our results to Klein surfaces with an even number of segmentation points on its ovals. These can naturally be regarded as real noncommutative smooth projective curves, which we call Witt curves. Many examples of genus zero and one are discussed. In particular we present an elliptic Witt curve which is a noncommutative Fourier-Mukai partner of the Klein bottle.

Contents

1. Introduction 1
2. Basic concepts 4
3. Homogeneous coordinate rings and localizations 6
4. Serre duality and the bimodule of a homogeneous tube 10
5. Tubes and their complete local rings 11
6. Local-global principle of skewness 15
7. Maximal orders and ramifications 16
8. The genus and the Euler characteristic 21
9. Local-global principle for global function fields 23
10. Examples I: genus zero 24
11. The real case: Witt curves 26
12. Examples II: Real elliptic curves 31
13. Examples III: insertion of weights; some tubular cases 33
References 34

1. Introduction

In this article we study categories \( \mathcal{H} \) which have the same formal properties as categories \( \text{coh}(X) \) of coherent sheaves over a smooth projective curve over a perfect field \( k \). We call \( \mathcal{H} \) a noncommutative smooth projective curve. They are in particular noncommutative projective schemes in the sense of Artin and Zhang [7]. In this setting the function field \( k(\mathcal{H}) \) is a skew field, finite dimensional over its

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centre \( K \). This centre is of the form \( K = k(X) \) for a unique smooth projective curve \( X \). We will show that \( \mathcal{H} \) is uniquely determined by its function field \( k(\mathcal{H}) \), and that there is moreover a bijective correspondence between noncommutative smooth projective curves over \( k \) and algebraic function skew fields in one variable over \( k \).

The natural number \( s(\mathcal{H}) \), whose square is the dimension of \( k(\mathcal{H}) \) over \( k(X) \), we call the skewness of \( \mathcal{H} \), since it is a global measure for the degree of noncommutativity of \( \mathcal{H} \). The case \( s(\mathcal{H}) = 1 \) is just the classical (commutative) case \( \mathcal{H} = \text{coh}(X) \). For each (closed) point \( x \) we will define a triple of natural numbers \( e(x), e^*(x) \) and \( e_\tau(x) \), each of which locally measures a different aspect of skewness at \( x \). Whereas the multiplicity \( e(x) \) was introduced and studied by Ringel already in [62] and the number \( e^*(x) \) was introduced in [38], the third local number \( e_\tau(x) \), which we call the \( \tau \)-multiplicity, appears for the first time in the present article. Here, \( \tau: \mathcal{H} \to \mathcal{H} \) is the Auslander-Reiten translation (also known as Serre functor) which appears in Serre duality \( \text{Ext}^1(E, F) = D \text{Hom}(F, \tau E) \). The number \( e_\tau(x) \) reflects a local property of \( \tau \) at \( x \), namely it is defined as the order of this functor restricted to the subcategory of skyscraper sheaves concentrated in \( x \).

The major result of this paper is a local-global principle of skewness, which says that for each point \( x \) the product of these three local numbers equals the global skewness \( s(\mathcal{H}) \). Our proof is based on the study of local properties of the Auslander-Reiten translation. This establishes also a new approach to the arithmetic number theory of function fields. For instance, in case \( k \) is a finite field, we will show that the Hasse local-global principle for central simple algebras over global function fields can be expressed as saying that \( s(\mathcal{H}) = 1 \) is equivalent to \( e_\tau(x) = 1 \) everywhere. We will extend this principle by showing that this is also equivalent to \( e(x) = 1 \) everywhere.

Another major aim of this article is to develop the foundations of the theory of noncommutative smooth projective curves, where our focus differs from that of [45]: homogeneous coordinate rings, noncommutative localizations (both, in the Ore-Asano and in the Serre-Grothendieck-Gabriel sense), equivalence to the concept of algebraic function skew fields in one variable, ramifications, genus, Euler characteristic, Hurwitz equation etc. A fundamental result, inspired by work of Reiten and van den Bergh [60], is the characterization of our categories \( \mathcal{H} \) as categories \( \text{coh}(\mathcal{A}) \) of coherent \( \mathcal{A} \)-modules, where \( \mathcal{A} \) is a maximal \( \mathcal{O}_X \)-order in a central simple \( k(X) \)-algebra, where \( X \) is a (commutative) smooth projective curve over \( k \).

The theory lives on interesting examples. Best understood is the case of genus zero, which is of importance in the representation theory of finite dimensional algebras. As another class of examples we extensively study the real case. It is well-known that (commutative) real smooth projective curves correspond to the concept of Klein surfaces [2]. Based on work of Witt [74] on central skew field extensions of real algebraic function fields in one variable we will extend this correspondence to the noncommutative case by defining Witt curves, so that we have a complete picture of the real noncommutative smooth projective curves. As a particularly interesting case we will treat the real elliptic curves. We will show that the Klein bottle (a commutative elliptic curve) has as Fourier-Mukai partner an elliptic Witt curve whose function field is not commutative. This shows, that the function field is no derived invariant.

We will now explain the notion of noncommutative regular projective curve (which is slightly more general than “smooth”). The axioms are essentially taken from work of Lenzing and Reiten [45]. Let \( k \) be a field and \( \mathcal{H} \) a small connected \( k \)-category satisfying the following properties (NC 1) to (NC 6'), generalizing those of
categories coh($X$) of coherent sheaves over a regular projective curve $X$ (condition (NC 7) will be a consequence of the others):

(NC 1) $\mathcal{H}$ is abelian and each object in $\mathcal{H}$ is noetherian.
(NC 2) All morphism and extension spaces in $\mathcal{H}$ are of finite $k$-dimension.
(NC 3) There is an autoequivalence $\tau$ on $\mathcal{H}$ (called the Auslander-Reiten translation) such that Serre duality $\text{Ext}^1_k(X, Y) = \text{DHom}_\mathcal{H}(Y, \tau X)$ holds, where $D = \text{Hom}_k(-, k)$.
(NC 4) $\mathcal{H}$ contains an object of infinite length.

It follows from Serre duality that $\mathcal{H}$ is a hereditary category, that is, $\text{Ext}^n_k$ vanishes for all $n \geq 2$. Let $\mathcal{H}_0$ be the Serre subcategory of $\mathcal{H}$ formed by the objects of finite length, and let $\mathcal{H}_+^+$ be the full subcategory of objects not containing a simple object. Then each indecomposable object of $\mathcal{H}$ lies in $\mathcal{H}_+$ or in $\mathcal{H}_0$. Moreover, $\mathcal{H}_0 = \bigoplus_{x \in \mathcal{X}} \mathcal{U}_x$ (for some index set $\mathcal{X}$) where $\mathcal{U}_x$ are connected uniserial categories, called tubes. The objects in $\mathcal{U}_x$ are called (skyscraper sheaves) concentrated in $x$. We also write $\mathcal{H} = \text{coh}(\mathcal{X})$. In order to avoid some degenerated cases, we further assume:

(NC 5) $\mathcal{X}$ consists of infinitely many points.

In this case each tube $\mathcal{U}_x$ has only finitely many simple objects. Since we are mainly interested in arithmetic effects, we assume for simplicity also that

(NC 6') $\text{Ext}^1(S, S) \neq 0$ (equivalently: $\tau S \simeq S$) for each simple object $S$.

That is, we restrict to non-weighted curves, which are also called homogeneous. For each point $x$ we denote by $S_x$ the unique simple sheaf concentrated in $x$. We call $\mathcal{X}$ (or also $\mathcal{H} = \text{coh}(\mathcal{X})$) a noncommutative regular projective curve over $k$; in case $k$ is a perfect field (which will be the standard assumption for most of our main results), we call $\mathcal{H}$ a noncommutative smooth projective curve; we refer to the discussion in 7.1. It follows that the quotient category $\tilde{\mathcal{H}} = \mathcal{H}/\mathcal{H}_0$ is semisimple with one simple object, thus $\tilde{\mathcal{H}} \simeq \text{mod}(k(\mathcal{H}))$ for a skew field $k(\mathcal{H})$, which we call the function field. It follows from [40] that:

(NC 7) The function field $k(\mathcal{H})$ is of finite dimension over its centre $Z(k(\mathcal{H}))$, which is an algebraic function field in one variable over $k$.

We will later show (Theorem 7.11) that, as in the commutative case, $\mathcal{H}$ is uniquely determined by its function field $k(\mathcal{H})$. We call the natural number

$$s(\mathcal{H}) = [k(\mathcal{H}) : Z(k(\mathcal{H}))]^{1/2}$$

the (global) skewness of $\mathcal{H}$. A main result will be the following local-global principle, which we already mentioned:

**Theorem 1.1** (Local-global principle of skewness). Let $\mathcal{X}$ be a noncommutative smooth projective curve over a perfect field $k$ with $\mathcal{H} = \text{coh}(\mathcal{X})$ and structure sheaf $L$. For each point $x \in \mathcal{X}$ we have

$$s(\mathcal{H}) = e(x) \cdot e^*(x) \cdot c_\tau(x),$$

where $e(x) = [\text{Ext}^1(S_x, L) : \text{End}(S_x)]$, $e^*(x) = [\text{End}(S_x) : Z(\text{End}(S_x))]^{1/2}$ and $c_\tau(x)$ is the order of the Auslander-Reiten translation $\tau : \mathcal{H} \to \mathcal{H}$ on the homogeneous tube $\mathcal{U}_x$. The condition $c_\tau(x) > 1$ holds for at most finitely many $x$.

The main tool for the proof of this equation, and also of several other (partly well-known) identities, is the following result on the structure of the complete local rings.

**Theorem 1.2.** The assumptions are the same as in Theorem 1.1. For each point $x \in \mathcal{X}$ the full subcategory $\mathcal{U}_x$ of skyscraper sheaves concentrated in $x$ is equivalent to the category of finite length modules over the skew power series ring...
End(S_x)[[T, τ^-]]. Here the twist τ^- is given by the restriction of the inverse of the Auslander-Reiten translation to the simple object S_x concentrated in x.

Actually, both theorems hold for arbitrary fields k and for those x ∈ X such that \text{End}(S_x)/k is separable, which is automatic if k is perfect. We will give an inseparable example where the equation does not hold (Example 10.6).

We remark that the present setting is part of the larger context of \textit{weighted}, or \textit{orbifold}, regular/smooth projective curves [45]. We think that the arithmetic aspects mainly studied in this paper are best described in its pure form in the non-weighted cases. Since many problems for weighted curves rely on and can be reduced to the underlying non-weighted curves, the present investigation will also improve the background for the weighted cases. We will consider the more general orbifold context here only at two places, in Theorem 7.10 and in the final section. We will treat different topics of the orbifold situation in a forthcoming paper.

The author thanks Helmut Lenzing for drawing his attention to the paper [74] of E. Witt, and also for giving him access to the slides [43]. The real case, mainly on the basis of Witt’s paper, forms a substantial part of this article, illustrating the theory in a very graphic way. The author also thanks Dieter Vossieck for various useful critical comments on a former version.

2. Basic concepts

Let \( \mathcal{H} \) be a noncommutative regular projective curve over the field \( k \).

\textbf{2.1 (Rank function).} Let \( \mathcal{H} \to \tilde{\mathcal{H}} = \mathcal{H}/\mathcal{H}_0 = \text{mod}(k(\mathcal{H})) \), \( X \mapsto \tilde{X} \) be the quotient functor. The \( k(\mathcal{H}) \)-dimension on \( \mathcal{H}/\mathcal{H}_0 \) induces the \textit{rank function} \( \text{rk}: \mathcal{H}/\mathcal{H}_0 \to \mathbb{Z} \) of \( \mathcal{H} \). For an indecomposable object \( E \in \mathcal{H} \) we have \( \text{rk}(E) = 0 \) if \( E \in \mathcal{H}_0 \) and \( \text{rk}(E) > 0 \) if \( E \in \mathcal{H}_+. \) An indecomposable object \( L \) with \( \text{rk}(L) = 1 \) is called a \textit{line bundle}. The function field \( k(\mathcal{H}) \) is isomorphic to the endomorphism ring of \( \tilde{L} \) in \( \tilde{\mathcal{H}} \).

We fix a line bundle \( L \) in \( \mathcal{H} \) and consider it as a \textit{structure sheaf} of \( \mathcal{H} \). The parametrizing set of all the (homogeneous) tubes we denote by \( X \), and regard it as the set of closed points. We write \( \mathcal{H} = \text{coh}(X) \), and also call \( X \) (instead of \( \mathcal{H} \)) a noncommutative regular projective curve. Thus, actually we always consider the pair \( (\mathcal{H}, L) \) together with the underlying curve (point set) \( X \).

\textbf{2.2 (Almost split sequences).} We recall that a short exact sequence
\[
\mu: \quad 0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0
\]
in \( \mathcal{H} \) is called \textit{almost split}, [10], if it does not split, if \( A \) and \( C \) are indecomposable, and if every morphism \( X \to C \), which is not a split epimorphism, factors through \( \beta \). Then \( \alpha \) is, up to isomorphism, uniquely determined by \( C \), and conversely. For every indecomposable \( C \) (resp. \( A \)) there is an almost split sequence (2.1) ending (starting) in \( C \) (in \( A \)); then \( \tau C = A \) and \( \tau^- A = C \), which define mutually quasi-inverse autoequivalences \( \tau, \tau^-: \mathcal{H} \to \mathcal{H} \), which appear in the Serre duality. For categories of coherent sheaves \( \tau \) is also known as Serre functor; we will reserve this term for the derived category of \( \mathcal{H} \).

The almost split sequences are fundamental in the definition of the \textit{Auslander-Reiten quiver} of \( \mathcal{H} \): its vertices are the isomorphism classes of indecomposable objects in \( \mathcal{H} \), and the arrows between classes of indecomposables are given by the so-called \textit{irreducible} morphisms, which are the components of the maps which occur in the corresponding almost split sequences.

\textbf{2.3 (Tubes).} Let \( x \in X \) be a (closed) point and \( \mathcal{U} = \mathcal{U}_x \) the corresponding connected uniserial category in \( \mathcal{H}_0 \). Up to isomorphism there is precisely one simple object \( S = S_x \) in \( \mathcal{U} \). Such categories are also called \textit{homogeneous tubes}. For each \( n \geq 1 \) we
denote by $S[n]$ the (up to isomorphism) unique indecomposable object in $U$ of length $n$. Thus we have $U = \text{add}(\{S[n] \mid n \geq 1\})$. We have injections $\iota_n : S[n] \to S[n + 1]$ and surjections $\pi_n : S[n + 1] \to S[n]$. The Auslander-Reiten translation satisfies $\tau S[n] \cong S[n]$. We will usually identify them, $\tau S[n] = S[n]$. We then have almost split sequences $\mu_n : 0 \to S \overset{\iota_n}{\longrightarrow} S[1] \overset{\pi_n}{\longrightarrow} S \to 0$, and for $n \geq 2$:

$$\mu_n : 0 \to S[n] \overset{(\tau \pi_n^{-1}, \iota_n, 1)^{n-1}}{\longrightarrow} S[n - 1] \oplus S[n + 1] \overset{(1, \tau \pi_n^{-1})^{n-1}}{\longrightarrow} S[n] \to 0.$$ 

The $\iota_n$ and $\pi_n$ are the irreducible maps in $U$.

2.4 ($\tau$-multiplicity). Let $U_x$ be a (homogeneous) tube with simple object $S_x$. The Auslander-Reiten translation $\tau$ restricts to an autoequivalence of $U_x$. Up to isomorphism it fixes all indecomposable objects $S_x[n]$. If we consider a skeleton of $\text{ind}(U_x)$, we can assume that equality $\tau S_x[n] = S_x[n]$ holds for all $n \geq 1$. The action on morphisms induces, in particular, an automorphism of $D_x = \text{End}(S_x)$, that is, an element $\tau$ in $\text{Aut}(D_x/k)$. We define $\text{Gal}(D_x/k) = \text{Aut}(D_x/k)/\text{Inn}(D_x/k)$, the factor group modulo inner automorphisms. We call

$$(2.2) \quad e_\tau(x) = \text{ord}_{\text{Gal}(D_x/k)}(\tau),$$

the order of (the class of) $\tau$ in $\text{Gal}(D_x/k)$, the $\tau$-multiplicity of $x$.

2.5 (Tubular shifts). Let $x \in X$ be a point and $U = U_x$ be the corresponding homogeneous tube in $H$. The indecomposable objects in $U$ form the Auslander-Reiten component containing the simple object $S = S_x$ with support $\{x\}$. Then $\text{End}(S)$ is a division algebra over $k$, and $\text{Ext}^1(S, S)$ is non-zero, one-dimensional as $\text{End}(S)$-vector space. Thus $S$ is, in the terminology of [64], a spherical object. (In [64] only the case $\text{End}(S) = k$ is considered.) For every object $E$ in $H$, which has no indecomposable summand in $U$, one has the $S$-universal extension $0 \to E \to E(x) \to E_x \to 0$ of $E$ with $E_x = \text{Ext}^1(S, E) \otimes_{\text{End}(S)} S$. The assignment $E \mapsto E(x)$ induces an autoequivalence $\sigma_x : H \to H$, called the tubular shift associated with $x$. We refer to [44], [38]. This is also known as Seidel-Thomas twist, [64]. For $n \in \mathbb{Z}$ we write $E(nx) = \sigma_x^n(E)$.

The assignment $E \mapsto E_x$ is also functorial. If $f \in \text{Hom}(E, E')$, then we call $f_x \in \text{Hom}(E_x, E'_x)$ the corresponding fibre map.

2.6 (Multiplicity and Comultiplicity). Let $(H, L)$ be a noncommutative regular projective curve. The dimensions

$$(2.3) \quad e(x) = [\text{Ext}^1(S_x, L) : \text{End}(S_x)]$$

are called multiplicities, [62], [44], [38]. In particular, we have the $S$-universal extension

$$(2.4) \quad 0 \to L \overset{\pi_x}{\longrightarrow} L(x) \to S_x^e(x) \to 0$$

of $L$. The number

$$(2.5) \quad e^*(x) = [\text{End}(S_x) : Z(\text{End}(S_x))]^{1/2}$$

we called comultiplicities in [38], since (in case of genus zero) for almost all $x \in X$ the product of $e(x)$ and $e^*(x)$ coincides with the skewness $s(H)$, [38, Cor. 2.3.5]. It was left open in [38] whether $e(x) \cdot e^*(x)$ is always a divisor of $s(H)$, not to speak about what the description of the cofactor could be. To answer this question, and without being restricted to the case of genus zero, was the main motivation of this article.

We remark that the comultiplicity, like the skewness, can be expressed in terms of polynomial identity (PI) degree.
2.7 (Orbit algebras). In (Noncommutative) Algebraic Geometry orbit algebras are important tools for constructing homogeneous coordinate rings. We refer to the survey [37]. If $E$ is an object in $\mathcal{H}$ and $\sigma : \mathcal{H} \to \mathcal{H}$ an endofunctor, then we denote by $\Pi(E,\sigma)$ the positively $\mathbb{Z}$-graded orbit algebra $\bigoplus_{n \geq 0}\text{Hom}(E,\sigma^nE)$. The multiplication is defined on homogeneous elements $f : E \to \sigma^mE$, $g : E \to \sigma^nE$ by the rule $g \ast f = \sigma^m(g) \circ f : E \to \sigma^{m+n}E$.

The special cases we are interested in are $\Pi(L,\sigma_x)$ with $L$ the structure sheaf and $\sigma_x : \mathcal{H} \to \mathcal{H}$ a tubular shift. The homogeneous element $\pi_x$ from (2.4) is central, [38, Lem. 1.7.1]. We denote the (homogeneous) ideal of $\Pi(L,\sigma_x)$ generated by $\pi_x$ by $P_x$. We will later see that $P_x$ is a homogeneous prime ideal. Whereas in [38], [37] we fixed one autoequivalence $\sigma$ (with additional good properties) and one coordinate algebra $\Pi(L,\sigma)$ for $\mathcal{H}$, we will in this paper for every point $x$ make use of its “own” orbit algebra $\Pi(L,\sigma_x)$ for the investigation of the numbers $e(x)$, $e^*(x)$ and $e_*(x)$.

2.8 (PI-degree). We will make use of some ring-theoretic tools like the polynomial identity (PI) degree. If $R$ is a ring (always assumed to be associative and with identity), we denote by $Z(R)$ its centre. If $D$ is a skew field which is of finite dimension over its centre, then the PI-degree of $D$ equals the square root of this dimension, [63, Thm. 1.5.23]. If $R$ is a PI-ring without zero-divisors, then Posner’s theorem [3, Thm. 7] tells us that the PI-degree of $R$ equals the PI-degree of its quotient division ring.

By $\text{mod}(R)$ we denote the category of finitely presented (right) $R$-modules, by $\text{mod}_0(R)$ the full subcategory of the modules of finite length. Usually, finite length is equivalent to finite dimensional over the base-field $k$.

3. Homogeneous coordinate rings and localizations

We continue to assume that $(\mathcal{H},L,X)$ is a noncommutative regular projective curve over the field $k$. In this section we show that, via the Serre construction, $\mathcal{H}$ is a noncommutative noetherian projective scheme in the sense of Artin-Zhang [7], and accordingly $X$ a projective spectrum. Moreover, via localization we study rings locally at a point $x \in X$.

Lemma 3.1. Each vector bundle has a line bundle filtration.

Proof. We refer to [45, Prop. 1.6]. \hfill \Box

Lemma 3.2. Let $0 \to L \xrightarrow{\pi} L(x) \to S^e \to 0$ be the $S$-universal sequence of $L$ with $S = S_x$ and $e = e(x)$. For $n \geq 1$ we have the exact sequence

$$0 \to L \xrightarrow{\pi^n} L(nx) \to S[n]^e \to 0. $$

Proof. By induction on $n$. For $n = 1$ the assertion is trivial. Let $n > 1$. Write $\mu : 0 \to L \xrightarrow{\pi^n} L(nx) \to E \to 0$. By induction hypothesis, from the snake lemma we obtain that $E$ appears as the middle term of a short exact sequence

$$0 \to S^e \to E \to S[n-1]^e \to 0. $$

Write $E = E_1 \oplus \ldots \oplus E_m$ with $E_i = S[t_i]$ indecomposable. By uniseriality we have $\text{Soc}(E_i) = S$. This yields $S^e = \text{Soc}(S^e) \subseteq \text{Soc}(E) = S^m$, and thus $m \geq e$. On the other hand, assume that $m > e$. Let $u_i : S \to S[t_i] = E_i \xrightarrow{\beta_i} E$ a monomorphism. By the definition of $e = e(x)$, there are $f_1,\ldots, f_m$ in $\text{End}(S)$, not all of them zero, such that $0 = \sum_{i=1}^m \mu \circ u_i f_i = \mu \circ (\sum_{i=1}^m u_i f_i)$. Denoting $\sum_{i=1}^m u_i f_i$ by $0 \neq h : S \to E$, the short exact sequence $\mu \circ h$ splits, and we obtain, that $S$ embeds into $L(nx)$, which gives a contradiction. We conclude $m = e$. Let $R = \text{End}(S[\infty])$ be the complete
local ring with maximal ideal \( m \) such that \( \mathcal{U} = \text{mod}_0(R) \). Since \( S^c \) is annihilated by \( m \) and \( S[n-1]^c \) by \( m^{n-1} \), we deduce from sequence (3.1) that \( E \) is annihilated by \( m^n \), and thus all \( \ell_i \leq n \). Since the length of \( E \) is \( n \cdot c \), we get \( \ell_i = n \) for all \( i \). This completes the proof of the lemma. \( \square \)

**Lemma 3.3.** Let \( E \) be an indecomposable vector bundle and \( S \) be a simple sheaf. Then \( \text{Hom}(E, S) \neq 0 \).

**Proof.** Using connectedness of \( \mathcal{H} \) this is shown like in [44, (S11)] or [60, Cor. IV.1.8]. \( \square \)

**Lemma 3.4.** Let \( L \) and \( L' \) be line bundles, and let \( x \in \mathbb{K} \) be a point. Then \( \text{Hom}(L(-nx), L') \neq 0 \) for \( n \gg 0 \).

**Proof.** By the preceding lemma we have an exact sequence \( 0 \to L(-nx) \to L \to S[n]^c \to 0 \) for each \( n \geq 0 \). Applying \( \text{Hom}(\_, L') \) gives \( 0 \to \text{Hom}(L, L') \to \text{Hom}(L(-nx), L') \to \text{Ext}^1(S[n]^c, L') \to \text{Ext}^1(L, L') \). By Lemma 3.3 we have \( d := \dim \text{Hom}(L', S) > 0 \), and thus \( \dim \text{Ext}^1(S[n]^c, L') = d \cdot n \gg 0 \) for \( n \gg 0 \). From this follows the claim. \( \square \)

**Lemma 3.5.** For each \( x \in \mathbb{K} \) the pair \((L, \sigma_x)\) is ample in the sense of [7]. Accordingly,

\[
(3.2) \quad \mathcal{H} \cong \frac{\text{mod}^\Sigma_x(\Pi(L, \sigma_x))}{\text{mod}^0_0(\Pi(L, \sigma_x))}.
\]

In particular, a noncommutative regular projective curve \( \mathcal{H} \) is a noncommutative projective scheme in the sense of Artin-Zhang [7].

**Proof.** (Compare the proof of [60, Lem. IV.4.1]) We have the inverse system \( \ldots \to L(\alpha) \to L(\beta) \to L \) of subobjects with zero intersection. By [60, Lem. IV.1.3] there is a line bundle \( L' \subseteq L \) such that \( \text{Ext}^1(U, L') = 0 \) for all subobjects (line bundles) \( U \subseteq L' \). Moreover, for \( n \gg 0 \) we have \( L(-nx) \subseteq L' \), and we conclude \( \text{Ext}^1(L(-nx), L) = 0 \).

Let \( E \in \mathcal{H} \). Let \( F \subseteq E \) be the largest subobject such there is an epimorphism \( G := \oplus_{i=1}^n L(-\alpha_i x) \to F \), and let \( C = E/F \). We assume that \( C \neq 0 \), and will show that this yields a contradiction. If \( C \) is of finite length, then it follows from Lemma 3.2 that a finite direct sum of copies of \( L \) maps onto \( C \). Thus we can assume that \( C \) is a vector bundle, and it suffices to assume that \( C \) is a line bundle. We have an exact sequence \( 0 \to K \to G \to F \to 0 \) with \( G \) a finite direct sum of \( \sigma \)-shifts of \( L \). By the preceding paragraph there is \( n_0 \) such that \( \text{Ext}^1(L(-nx), G) = 0 \), and then \( \text{Ext}^1(L(-nx), F) = 0 \) for all \( n \geq n_0 \).

By Lemma 3.4 we have a non-trivial morphism \( L(-nx) \to C \) for some \( m \geq n_0 \). Since \( \text{Ext}^1(L(-nx), F) = 0 \), this lifts to a non-trivial morphism \( L(-nx) \to E \), giving a contradiction. \( \square \)

**Lemma 3.6.** The homogeneous ideal \( P_x \) in \( \Pi(L, \sigma_x) \) generated by \( \pi_x \) is prime.

**Proof.** This follows like in [38, Thm. 1.2.3]. We only need to show that for \( n \gg 0 \) sufficiently large we have \( \text{Hom}(L, \tau L(-nx)) = 0 \), as in [38, Lem. 1.2.2]. To this end, by Lemma 3.4 for \( n \gg 0 \) there is a non-zero morphism \( g: \tau L(-nx) \to L \). We assume that there is a non-zero morphism \( f: L \to \tau L(-nx) \). Both, \( f \) and \( g \), are monomorphisms, and \( g \circ f: L \to L \) is an isomorphism, thus \( g \) is an isomorphism. Enlarging \( n \) further, we see that there is \( m > 0 \) such that \( L \) and \( L(mx) \) are isomorphic. But then, repeating the argument just given, also \( (\pi_x)^m \) would be an isomorphism. But this is not true by Lemma 3.2, giving a contradiction. Thus \( \text{Hom}(L, \tau L(-nx)) = 0 \). \( \square \)
Lemma 3.7. For each $x \in \mathbb{X}$ the ring $\Pi(L, \sigma_x)$ is a graded noetherian domain which has a central prime element $\pi_x$ of degree one, and the quotient division ring of degree-zero fractions $s^{-1}r$ (with $r$, $s$ homogeneous of the same degree, $s \neq 0$) is the function field $k(\mathcal{H})$.

Proof. Noetherianness follows from the proof of [38, Prop. 1.4.4] also in this more general setting (right-noetherianness is also shown in [7, Thm. 4.5]). Since non-zero morphisms between line bundles are monomorphisms, the orbit algebra $\Pi(L, \sigma_x)$ is a graded domain. By the preceding lemma the homogeneous element $\pi_x$ is central and prime. The assertion about the function field follows like in [60, Lem. IV.4 Step 4]. □

Lemma 3.8. Let $x \in \mathbb{X}$ be of multiplicity $e(x)$ and with simple sheaf $S_x$.

(1) For a non-zero homogeneous element $s \in \Pi(L, \sigma_x)$ the following conditions are equivalent:

- $s \in C(P_x)$, that is, $s$ is regular modulo $P_x$.
- The cokernel of $s$ lies in $\coprod_{y \neq x} \mathcal{U}_y$.
- The fibre map $s_x \in \text{End}(S_x^{(x)})$ is an isomorphism.

(2) The set $C(P_x)$ is a denominator set.

(3) For the graded localization $R_x^{\gr} = \Pi(L, \sigma_x)_{C(P_x)}$ the graded Jacobson radical is generated by the central element $\pi_x 1^{-1}$ and is the only non-zero graded prime ideal.

(4) As graded rings, $R_x^{\gr}/\text{rad}^{\gr}(R_x^{\gr}) \simeq M_{e(x)}(\text{END}(S_x))$, where $\text{END}(S_x)$ is a graded skew field.

Proof. (1) The equivalence of the three conditions is shown in [38, Lem. 2.2.1].

(2) Since $\Pi(L, \sigma_x)$ is graded noetherian (left and right) and $\pi_x$ is central, this follows from a graded version of [66, Thm. 4.3].

(3), (4) (For analogous ungraded statements we refer to [51, Thm. 4.3.18] and [28, Lem. 14.18].) Localizing the universal exact sequence $0 \to L \to \pi_x L(x) \to S_x^{(x)} \to 0$ we get, like in [38, Prop. 2.2.8], a short exact sequence $0 \to R_x^{\gr} \to R_x^{\gr}(x) \to S_x^{(x)} \to 0$ of graded $R_x^{\gr}$-modules, where $S_x$ is simple. As graded rings thus $R_x^{\gr}/(\pi_x 1^{-1}) \simeq M_{e(x)}(\text{END}(S_x))$. The graded Jacobson radical $\text{rad}^{\gr}(R_x^{\gr})$ is the principal ideal generated by $\pi_x 1^{-1}$; clearly, $1 - \pi_x r \in C(P_x)$ for each $r$, so that $\pi_x 1^{-1}$ lies in the radical. The canonical surjective ring homomorphism $R_x^{\gr}/(\pi_x 1^{-1}) \to R_x^{\gr}/\text{rad}^{\gr}(R_x^{\gr})$ is an isomorphism, by simplicity of the graded ring on the left hand side. □

We denote by $R_x$ the degree-zero component of the localization $\Pi(L, \sigma_x)_{C(P_x)}$.

Proposition 3.9. Let $\mathbb{X}$ be a noncommutative regular projective curve over the field $k$. Let $x \in \mathbb{X}$ be a point. There is an isomorphism

$$R_x/\text{rad}(R_x) \simeq M_{e(x)}(\text{End}(S_x))$$

of rings.

Proof. Since $\text{rad}(R_x)$ is the degree zero part of $\text{rad}^{\gr}(R_x^{\gr})$, the assertion follows from the preceding lemma. □

For $x \in \mathbb{X}$ we denote by $\mathcal{H}_x = \mathcal{H}/(\coprod_{y \neq x} \mathcal{U}_y)$ the quotient category, modulo a Serre subcategory, where all tubes except $\mathcal{U}_x$ are “removed”, and by $p_x: \mathcal{H} \to \mathcal{H}_x$ the quotient functor.

Lemma 3.10. The object $L_x = p_x(L)$ is an indecomposable projective generator of $\mathcal{H}_x$. Accordingly, for the ring $V_x = \text{End}_{\mathcal{H}_x}(L_x)$ we have $\mathcal{H}_x \simeq \text{mod}(V_x) = \cdots$
mod⁺(V_x)\cup\text{mod}_0(V_x),\text{ with }\text{mod}_0(V_x)\simeq U_x \text{ the finite length modules and mod}_+(V_x) \text{ the finitely generated torsionfree modules.}

\textbf{Proof.} Let y \in X be another point, y \neq x. Then \pi_y \text{ induces an isomorphism } L_x(-y) \simeq L_x. \text{ Using ampieness of the pair } (L,\sigma_y) \text{ we see that } L_x \text{ is a generator for } \mathcal{H}_x. \text{ It is easy to see that for each exact sequence } \eta: 0 \to A \to B \to L \to 0 \text{ in } \mathcal{H}\text{ the exact sequence } p_x(\eta) \text{ in } \mathcal{H}_x \text{ splits, showing that } L_x \text{ is a projective object in } \mathcal{H}_x. \text{ From this it follows that } \text{Hom}_{\mathcal{H}}(L_x,-): \mathcal{H}_x \to \text{mod}(V_x) \text{ is an equivalence. It is easy to see that } p_x \text{ induces an injective homomorphism } V_x \to k(\mathcal{H}) \text{ of rings, and thus } V_x \text{ is a domain. We infer that } L_x \text{ is indecomposable.} \text{ } \square

\textbf{Proposition 3.11.} \text{ There is an isomorphism of rings } R_x \simeq V_x.

\textbf{Proof.} By using the definition of morphisms in the quotient category we see easily that \begin{align*}
R_x \subseteq V_x. \text{ Let } 0 \to L' \xrightarrow{s} L \to C \to 0 \text{ be an exact sequence in } \mathcal{H} \text{ with } L' \text{ a line bundle and } C \in \prod_{y \neq x} U_y. \text{ By ampieness of } (L,\sigma_x) \text{ there is an epimorphism } f = (f_1,\ldots,f_n): \bigoplus_{i=1}^n L(-\alpha_i x) \to L' \text{ (with } \alpha_i \geq 1). \text{ If we assume that each } C_i = \text{Coker}(f_i) \text{ has a non-zero summand in } U_x, \text{ then there is an epimorphism } C_i \to S_x, \text{ and thus we can write } f_i = \pi_x \circ f'_i. \text{ But then } f = \pi_x \circ f' \text{ is not surjective, giving a contradiction. Thus there is } i \text{ such that } C_i \in \prod_{y \neq x} U_y. \text{ We conclude that there is } f: L(-\alpha x) \to L' \text{ such that } s \circ f: L(-\alpha x) \to L \text{ is a non-zero homogeneous element in } \Pi(L,\sigma_x) \text{ with } \text{Coker}(sf) \in \prod_{y \neq x} U_y, \text{ that is, } sf \in C(P_x). \text{ Thus we can write } rs^{-1} = (rf)(sf)^{-1}, \text{ from which we infer the converse inclusion.} \text{ } \square

\textbf{Corollary 3.12.} \text{ For each } x \in X \text{ we have } U_x \simeq \text{mod}_0(R_x).

\textbf{Corollary 3.13.} \text{ Each ring } R_x \text{ is a noncommutative Dedekind domain with unique non-zero prime ideal given by } \text{rad}(R_x).

\textbf{Proof.} \text{ } R_x \text{ is right hereditary since } \text{mod}(R_x) \simeq \mathcal{H}_x \text{ is hereditary. Since } R_x \text{ is noetherian, by [65, Cor. 3] it is also left hereditary. It follows from Proposition 3.9 that the radical } J = \text{rad}(R_x) \text{ is the only (two-sided) maximal ideal. As in [51, 4.3.20] one shows } \bigcap_{n \geq 0} J^n = 0. \text{ Thus, if } r \in R_x, r \neq 0, \text{ then there is } v(r) = n \text{ with } r \in J^n \text{ but } r \not\in J^{n+1}. \text{ Let } I \text{ be a non-zero idempotent ideal in } R_x. \text{ Let } 0 \neq r \in I \text{ with } v(r) \text{ minimal. From the condition } I = I^2 \text{ we get } v(r) = 0, \text{ and then } I = R_x \text{ since } J \text{ is maximal. Thus } R_x \text{ is Dedekind by [51, 5.6.3]. By [51, 5.2.9] each non-zero ideal is of the form } J^n, \text{ and it follows that } J \text{ is the only non-zero prime ideal.} \text{ } \square

We also consider the category \( \widehat{\mathcal{H}} = \text{Qcoh X} = \frac{\text{Mod}^b(\Pi(L,\sigma_x))}{\text{Mod}_0(\Pi(L,\sigma_x))} \) of quasicoherent sheaves, where \text{Mod}_0 \text{ denotes the localizing Serre subcategory of torsion (that is, locally finite dimensional) graded modules. This is a hereditary, locally noetherian Grothendieck category. In this we can consider the Prüfer sheaf } S_x[\infty] \text{ for } x \in X, \text{ which is the union } \bigcup_{n \geq 1} S_x[n], \text{ that is, the direct limit of the direct system } (S_x[n],\epsilon_{n}), \text{ and thus is a quasicoherent torsion sheaf. We now have the main result of this section.}

\textbf{Proposition 3.14.} \text{ For the } \text{rad}(R_x)\text{-adic completion of } R_x \text{ we have }
\hat{R}_x \simeq M_{e(x)}(\text{End}(S_x[\infty])).

\textbf{Proof.} \text{ By [39, Thm. 21.31] the completion } \hat{R}_x \text{ is semiperfect, it satisfies }\begin{align*}
R_x / \text{rad}(R_x) &\simeq R_x / \text{rad}(R_x) \simeq M_{e(x)}(D_x), \quad (3.3)
\end{align*}
\text{ and from [39, Thm. 23.10] it follows that } \hat{R}_x \simeq M_{e(x)}(\widehat{E}_x) \text{ for a complete local ring } \widehat{E}_x. \text{ Moreover, for the categories of finite dimensional modules we have } \text{mod}_0(\widehat{E}_x) \simeq \text{mod}_0(R_x) \simeq \text{mod}_0(R_x) \simeq U_x. \text{ The result follows now, since the complete local ring}
End(S_x[∞]) is uniquely determined such that mod_0(End(S_x[∞])) \simeq \mathcal{U}_x, by [25, IV. Prop. 13].

4. Serre Duality and the Bimodule of a Homogeneous Tube

In [26] P. Gabriel defined the species of a uniserial category \( \mathcal{U} \). In the most basic situation, when there is (up to isomorphism) only one simple object \( S \) in \( \mathcal{U} \), like in the case of a homogeneous tube, then this species is just the bimodule \( \text{End}(S) \text{End}(S) \). In order to describe this bimodule more precisely, we derive from [46] some general facts about Serre duality.

Let \( \mathcal{H} \) be a noncommutative regular projective curve over the field \( k \). We call a \( k \)-bilinear map \( \langle \cdot , \cdot \rangle : \mathcal{V} \times \mathcal{W} \to k \) a perfect pairing, if for each non-zero \( x \in \mathcal{V} \) there exists \( y \in \mathcal{W} \) with \( \langle x|y \rangle \neq 0 \), and if for each non-zero \( y \in \mathcal{W} \) there is \( x \in \mathcal{V} \) with \( \langle x|y \rangle \neq 0 \). For each indecomposable object \( X \in \mathcal{H} \) we fix an almost split sequence \( \mu_X : 0 \to \tau X \to E \to X \to 0 \) and a \( k \)-linear map \( \kappa_X : \text{Ext}^1(X, \tau X) \to k \) with \( \kappa_X(\mu_X) = 0 \). Similarly, for \( Y \in \mathcal{H} \) indecomposable and an almost split sequence \( \mu_Y : 0 \to Y \to F \to \tau Y \to 0 \) we fix \( \kappa_{\tau Y} : \text{Ext}^1(\tau Y, Y) \to k \) with \( \kappa_{\tau Y}(\mu_{\tau Y}) \neq 0 \). Then

\[
\langle \cdot , \cdot \rangle : \text{Ext}^1(X, Y) \times \text{Hom}(\tau Y, X) \to k, (\eta, f) \mapsto \kappa_{\tau Y}(\eta \cdot f)
\]
is a perfect pairing, and similarly so is

\[
\langle \cdot , \cdot \rangle : \text{Hom}(Y, \tau X) \times \text{Ext}^1(X, Y) \to k, (g, \eta) \mapsto \kappa_X(g \cdot \eta).
\]

From these perfect pairings we obtain Serre duality

\[
\text{Hom}(Y, \tau X) \xrightarrow{\psi_{XY}} \text{D} \text{Ext}^1(X, Y) \xrightarrow{\phi_{XY}} \text{Hom}(\tau Y, X),
\]
where \( \psi_{XY} : f \mapsto \langle f|\cdot \rangle \) and \( \phi_{XY} : g \mapsto \langle \cdot|g \rangle \) are isomorphisms, natural in \( X \) and \( Y \).

**Proposition 4.1.** Let \( X \in \mathcal{H} \) be indecomposable such that End(\( X \)) is a skew field. Denote by \( \mu : 0 \to \tau X \xrightarrow{\mu} E \xrightarrow{\nu} X \to 0 \) the almost split sequence ending in \( X \). For all \( f \in \text{End}(X) \) we have

\[
\tau(f) \cdot \mu = \mu \cdot f.
\]

**Proof.** The isomorphism \( \psi_{XY} \) from (4.1) is natural in \( X \) and \( Y \) and thus, in particular, an isomorphism of End(\( X \) − End(\( Y \))-bimodules. Then we have the following rules:

\[
\langle f|g \eta \rangle = \langle f|g \eta \rangle, \quad \langle f|g \eta \rangle = \langle f|g \eta \rangle, \quad \langle \eta|fg \rangle = \langle \eta|fg \rangle, \quad \langle f|g \eta \rangle = \langle \tau(g)f|\eta \rangle
\]

(compare [46, (3.2)]). The last equality is just End(\( X \))-linearity. Moreover, by definition of the End(\( X \)) − End(\( Y \))-bimodule structure on D Ext^1(\( X \), \( Y \)) we have \( \langle g|\eta f \rangle = f \cdot \langle g|\eta \rangle \) for all \( f \in \text{End}(X) \). Let now \( Y = \tau X \) and \( \mu \in \text{Ext}^1(X, \tau X) \) be the almost split sequence. Since \( D = \text{End}(X) \cong \text{End}(\tau X) \) is a skew field, \( \mu \) is an onedimensional \( D - D \)-bimodule, in particular \( \text{D} \mu = \mu \text{D} \). Thus for each \( f \in D = \text{End}(X) \) there is a unique \( f' \in D = \text{End}(\tau X) \) such that \( f' \mu = \mu f \). We have to show that \( f' = \tau(f) \). Let \( \eta \in \text{Ext}^1(X, \tau X) \). Then there is an \( h \in \text{End}(X) \) such that \( \eta = \mu \cdot h \). First we have

\[
\langle f'|\mu \rangle = \langle 1|f'|\mu \rangle = \langle 1|\mu f \rangle = \langle \mu f \rangle = \langle \tau(f)|\mu \rangle,
\]
and then

\[
\langle f'|\eta \rangle = \langle f'|\mu \cdot h \rangle = h \cdot \langle f'|\mu \rangle = h \cdot \langle \tau(f)|\mu \rangle = \langle \tau(f)|\mu \cdot h \rangle = \langle \tau(f)|\eta \rangle.
\]

Since \( \langle \cdot , \cdot \rangle \) is a perfect pairing we conclude \( f' = \tau(f) \), finishing the proof. \( \Box \)
Corollary 4.2. Let \( \text{End}(X) \) be a skew field. Let \( f \in \text{End}(X) \) such that there is a commutative diagram

\[
\begin{array}{c}
0 \rightarrow \tau X \xrightarrow{u} E \xrightarrow{v} X \rightarrow 0 \\
\vert \quad \vert \quad \vert \quad \vert \\
\mu : 0 \rightarrow \tau X \xrightarrow{u} E \xrightarrow{v} X \rightarrow 0.
\end{array}
\]

Then \( f' = \tau(f) \) holds.

Proof. We show that from the assumptions \( f' \cdot \mu = \mu \cdot f \) follows. By the preceding proposition then \( f' \cdot \mu = \tau(f) \cdot \mu \), thus \( (f' - \tau(f)) \cdot \mu = 0 \); since \( \mu \neq 0 \) and \( \text{End}(\tau X) \) is a skew field, this yields \( f' = \tau(f) \).

If \( f \neq 0 \), then \( f \) is an isomorphism, and since \( \mu \) does not split, \( f' \neq 0 \) follows easily. Dually, if \( f = 0 \), then also \( f' = 0 \) holds. Thus we may assume that \( f \), and then also \( f' \) and \( g \) are isomorphisms. One computes that

\[
f' \cdot \mu : 0 \rightarrow \tau X \xrightarrow{u} E \xrightarrow{f' \cdot v} X \rightarrow 0
\]

and

\[
\mu \cdot f : 0 \rightarrow \tau X \xrightarrow{\mu \cdot v} E \xrightarrow{v} X \rightarrow 0.
\]

Then we have the commutative exact diagram:

\[
\begin{array}{c}
0 \rightarrow \tau X \xrightarrow{u} E \xrightarrow{f' \cdot v} X \rightarrow 0 \\
\vert \quad \vert \quad \vert \quad \vert \\
\mu : 0 \rightarrow \tau X \xrightarrow{u} E \xrightarrow{v} X \rightarrow 0,
\end{array}
\]

thus \( f' \cdot \mu = \mu \cdot f \) as claimed. \( \square \)

As a special case we get the following description of the bimodule of a homogeneous tube.

Corollary 4.3. Let \( U \) be a homogeneous tube in \( \mathcal{H} \) with simple object \( S = \tau S \), almost split sequence \( \mu : 0 \rightarrow S \rightarrow S[2] \rightarrow S \rightarrow 0 \) and division algebra \( D = \text{End}(S) \). Then the bimodule of \( U \), that is, the \( D \)-bimodule \( E = \text{Ext}^1(S, S) \), is given by \( E = D \cdot \mu = \mu \cdot D \) with relations \( \mu \cdot d = \tau(d) \cdot \mu \) (for all \( d \in D \)), where \( \tau \in \text{Aut}(D/k) \) is induced by the Auslander-Reiten translation. \( \square \)

5. Tubes and their complete local rings

Let \( k \) be a field. If \( D \) is a division algebra over \( k \) and \( \sigma \in \text{Aut}(D/k) \), then we denote by \( D[[T, \sigma]] \) the ring of formal power series \( \sum_{n \geq 0} a_n T^n \) over \( D \), subject to the relation \( T a = \sigma(a) T \) for all \( a \in D \). We denote by \( \text{Gal}(L/k) \) the factor group \( \text{Aut}(D/k)/\text{Inn}(D/k) \). By the theorem of Noether-Skolem this is a subgroup of \( \text{Gal}(Z(D)/k) \).

Let \( (R, m) \) be a (not necessarily commutative) local ring with Jacobson radical \( m \). We write \( \text{gr}(R) = \bigoplus_{n \geq 0} m^n/m^{n+1} \). This is a graded local ring, with graded Jacobson radical given by \( \text{gr}_+(R) = \bigoplus_{n \geq 1} m^n/m^{n+1} \). Its \( \text{gr}_+ \)-adic completion is given by \( \hat{\text{gr}}(R) = \prod_{n \geq 0} m^n/m^{n+1} \), with multiplication given by the Cauchy product.

Proposition 5.1. Let \( k \) be a field. Let \( U \) be a (homogeneous) tube over a non-commutative regular projective curve over \( k \) with simple object \( S = \tau S \), \( D = \text{End}(S) \) the endomorphism skew field. Let \( \tau \in \text{Gal}(D/k) \) be the automorphism (modulo inner) induced by the Auslander-Reiten translation \( \tau \). Let \( S[\infty] \) be the corresponding Prüfer sheaf and \( R = \text{End}(S[\infty]) \) its endomorphism ring. Then \( R \) is a complete
local ring with maximal ideal \( m = R\pi = \pi R \), where \( \pi \) is a surjective endomorphism of \( S[\infty] \) having kernel \( S \). Each one-sided ideal is two-sided, and if non-zero then of the form \( m^n = R\pi^n = \pi^n R \). Moreover, \( U \cong \text{mod}_0(R) \), and there are isomorphisms
\[
gr(R) \cong D[T, \tau^{-1}] \quad \text{(of graded rings)} \quad \text{and} \quad \hat{gr}(R) \cong D[[T, \tau^{-1}]].
\]

Proof. (1) With the notations from 2.3, the Prüfer object \( S[\infty] \) is the direct limit of the direct system \( (S[n], \iota_n) \). The direct limit closure \( \overline{U} \) of \( U \) in \( \mathcal{H} \) is a hereditary locally finite Grothendieck category in which \( S[\infty] \) is an indecomposable injective cogenerator. Its endomorphism ring is the inverse limit of the inverse system of rings \( \text{End}(S[n], p_n) \), where the \( p_n \) are the surjective restriction maps.

It is well-known (we refer to \([61], [62], [26], \) and \([70, \text{Prop. 4.10}]\) that \( R \) is a complete local domain with \( U \cong \text{mod}_0(R) \), having the properties stated in the proposition. The two isomorphisms of rings remain to show. Since \( U \) is hereditary, \( \hat{gr}(R) \) is, by \([26, 8.5]\), isomorphic to the complete tensor algebra \((\text{[26, 7.5]}\)) \( \Omega \) of the species of \( U \), which is given by the \( D-D \)-bimodule \( E = \text{Ext}^1(S, S) \).

(2) We now determine the complete tensor algebra of the bimodule \( E \). Let \( \mu \in \mathbb{E} \) denote the almost split sequence \( 0 \rightarrow S \rightarrow S[2] \rightarrow S \rightarrow 0 \). We have \( E = D\mu = \mu D \), and from Proposition 4.1 we get \( \mu \cdot d = \tau(d) \cdot \mu \) for each \( d \in D \). For each natural number \( n \) there is a canonical isomorphism \( D^\beta_D \otimes_D E_D \cong (\mu D \otimes_D E_D)^n \cong D^n E_D \), where \( E^n \) as left \( D \)-module is isomorphic to \( D^n \), and the right \( D \)-module structure on \( E^n \) is given by \( \tau \)-twist: \((x_1, \ldots, x_n) \cdot d = (x_1 \tau(d), \ldots, x_n \tau(d)) \).

We denote by \( U' \) the category of small representations \((\text{[26]}\) of the species given by the division ring \( D \) and one loop labelled by the \( D-D \)-bimodule \( E \). The indecomposable of length \( n \) is given by \( S'[n] \), which is the representation \( D^n \xrightarrow{\beta} E^n \), with \( g \) nilpotent right \( D \)-linear and given by the indecomposable Jordan matrix
\[
J_n = J_n(0) = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}
\]
to the eigenvalue \( 0 \). That is, we have
\[
g(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}) = J_n \cdot \begin{pmatrix} \tau(x_1) \\ \vdots \\ \tau(x_n) \end{pmatrix}.
\]

If
\[
\begin{array}{ccc}
D^n & \xrightarrow{g} & E^n \\
\downarrow{f} & \quad & \downarrow{f'} \\
D^n & \xrightarrow{g} & E^n
\end{array}
\]
is an endomorphism of \( S'[n] \), where \( f \) and \( f' \) are right \( D \)-linear maps given by the same \( n \times n \)-matrix \( A = (a_{ij}) \) with entries in \( D \), then for the matrices \( A \cdot J_n = J_n \cdot A^\tau \) holds, where \( A^\tau = (\tau(a_{ij})) \). Similar relations hold for morphisms of representations. This yields that we can \( A \) write as
\[
(5.1) \quad A = a_0 \cdot J_n + a_1 \cdot J_n + a_2 \cdot (J_n)^2 + \cdots + a_{n-1} \cdot (J_n)^{n-1}
\]
with unique \( a_0, \ldots, a_{n-1} \in D \), and where \( a \cdot (J_n)^\ell \) is given by replacing the side-diagonal given by \( 1, 1, \ldots, 1 \) by the elements \( a, \tau^{-1}(a), \ldots, \tau^{-(n-\ell-1)}(a) \). Clearly \( (a \cdot J_n) \cdot (b \cdot I_n) = (ab) \cdot I_n \) holds, and \( D \cdot I_n \) forms a subalgebra of \( \text{End}(S'[n]) \) isomorphic to \( D \). Moreover,
\[
(5.2) \quad (J_n)^\ell \cdot (a \cdot I_n) = \tau^{-\ell}(a) \cdot (J_n)^\ell
\]
holds, and the Jacobson radical of $\text{End}(S'[n])$ is generated, as a left ideal and as a right ideal, by the map with matrix $J_n$.

We denote by $\iota'_n: S'[n] \to S'[n+1]$ and $\pi'_n: S'[n+1] \to S'[n]$ the morphisms given by the matrices

$$
\begin{pmatrix}
I_n \\
0
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
0 \\
I_n
\end{pmatrix},
$$

respectively. This yields that $\iota'_n \circ \pi'_n$ is given by multiplication with $J_{n+1}$, and $\pi'_n \circ \iota'_n$ by $J_n$. If $f'$ is the restriction of the endomorphism $f$ to $S'[n-1]$, that is, $\iota'_{n-1} \circ f' = f \circ \iota'_n$ holds, and if $A'$ denotes the corresponding $(n-1) \times (n-1)$-matrix, then equation (5.1) yields

$$
A' = a_0 \cdot I_{n-1} + a_1 \cdot J_{n-1} + a_2 \cdot (J_{n-1})^2 + \cdots + a_{n-2} \cdot (J_{n-1})^{n-2}.
$$

Thus, sending $f$ to its restriction $f'$, gives a, clearly surjective, homomorphism $\mu'_n: \text{End}(S'[n]) \to \text{End}(S'[n-1])$ of $k$-algebras.

Starting with the almost split sequence $\mu'_1$ with end term $S' = S'[1]$ we have the following direct system of short exact sequences

$$
\begin{array}{cccc}
\mu'_1: & 0 & \to & S'[1] \\
& & \downarrow & \iota'_1 \\
& & S'[2] & \to & S'[1] \\
& & \downarrow & \pi'_1 \\
& & 0 & \to & 0
\end{array}
$$

$$
\begin{array}{cccc}
\mu'_2: & 0 & \to & S'[1] \\
& & \downarrow & \iota'_2 \\
& & S'[3] & \to & S'[2] \\
& & \downarrow & \pi'_2 \\
& & 0 & \to & 0
\end{array}
$$

$$
\begin{array}{cccc}
\mu'_3: & 0 & \to & S'[1] \\
& & \downarrow & \iota'_3 \\
& & S'[4] & \to & S'[3] \\
& & \downarrow & \pi'_3 \\
& & 0 & \to & 0
\end{array}
$$

and its direct limit

$$
\mu'_\infty: \quad 0 \to S' \to S'[\infty] \xrightarrow{\pi'} S'[\infty] \to 0.
$$

The ring $R' = \text{End}(S'[\infty])$ is isomorphic to the inverse limit of the $\text{End}(S'[n])$ (with respect to the inverse system given by the $\mu'_n$). As in (1), $R'$ is a complete local ring with maximal ideal $m' = R' \pi' = \pi' R'$, and with $\text{mod}_0(R') \simeq U'$. Thus there is an automorphism $\sigma \in \text{Aut}(R'/k)$ with $\pi' f = \sigma(f) \pi'$ for all $f \in R'$.

Each $f \in \text{End}(S'[\infty])$ has a unique expression $f = (f_1, f_2, f_3, \ldots)$ with $f_n \in \text{End}(S'[n])$ and $f_n = f(S[n])$ the restriction of $f$ to $S'[n]$ for each $n$. The restriction of $\pi'$ to $S'[n]$ is given by $\iota'_{n-1} \circ \pi'_n$, hence by the matrix $J_n$. We conclude that $f$ has a unique expression as formal power series

$$
f = \sum_{n=0}^{\infty} a_n \pi'^n.
$$

From (5.2) we deduce

$$
\pi' a = \tau^-(a) \pi'
$$

for all $a \in D$. Thus $R' = D[[\pi', \tau^-]] \simeq D[[T, \tau^-]]$. On the other hand, by [26, 7.5] the complete tensor algebra $\Omega$ of $E$ is a complete local ring also satisfying $U' \simeq \text{mod}_0(\Omega)$. It follows, by [25, IV. Prop. 13], that $R' \simeq \Omega \simeq \text{gr}(R)$. Finally this yields $\text{gr}(R) \simeq D[T, \tau^-]$ as graded rings.

In the separable (perfect) case, we can apply the Wedderburn-Malcev theorem [57, Thm. 11.6] in order to get the following.
Proposition 5.2. With the notations of the preceding proposition, assume that $D/k$ is separable (that is, $Z(D)/k$ is a separable field extension). Then $R \simeq \hat{\text{gr}}(R)$.

Proof. The proof is based on [26, 8.5]. For $n \geq 1$ let $B_n$ be the finite dimensional $k$-algebra $\text{End}(S[n]) \simeq R/m^n$, where $R$ is the endomorphism ring of $S[\infty]$ with maximal ideal $m$. The Wedderburn-Malcev theorem implies that the projection $B_n \to B_n/\text{rad}(B_n) \simeq D$ splits. Thus $B_n = D_n \oplus \text{rad}(B_n)$, with a subalgebra $D_n$ of $B_n$ isomorphic to $D$. Then $B_n$ becomes a $D$-$D$-bimodule, and $\text{rad}(B_n)$ contains a submodule which is isomorphic to $V_n = \text{rad}(B_n)/\text{rad}^2(B_n)$. Thus there is a surjective homomorphism from the tensor algebra of $V_n$, and then also from the complete tensor algebra $\Omega$ of the species of $U$, onto $B_n$. We get an isomorphism $\Omega/\text{rad}^n(\Omega) \simeq B_n$. A more detailed analysis shows that this can be done inductively in such a way that we obtain an isomorphism of inverse systems of rings. Taking inverse limits we get $\Omega \simeq R$. This finishes the proof. 

If $U = U_x$ satisfies this separability condition, we call $x$ (resp. $U$) a separable point (tube). If $k$ is a perfect field, then all points are separable.

Theorem 5.3. Let $k$ be a field. Let $U$ be a separable tube over a noncommutative regular projective curve over $k$ with simple object $S$ and $D = \text{End}(S)$ the endomorphism skew field. Let $\tau \in \text{Gal}(D/k)$ be the automorphism induced by the Auslander-Reiten translation $\tau$. Let $S[\infty]$ be the corresponding Prüfer sheaf. Then

$$
\text{End}(S[\infty]) \simeq D[[T, \tau^{-1}]].
$$

In particular, $U \simeq \text{mod}_0(D[[T, \tau^{-1}]])$. 

We set

$$
\text{Aut}_\tau(D/k) = \{\sigma \in \text{Aut}(D/k) \mid \sigma \tau = \tau \sigma\},
$$

and finally $\text{Gal}_\tau(D/k) = \text{Aut}_\tau(D/k)/\text{Inn}_\tau(D/k)$. Clearly $\text{Inn}_\tau(D/k) = \text{Inn}(D/k) \cap \text{Aut}(D/k)$, so that $\text{Gal}_\tau(D/k)$ can be regarded as a subgroup of $\text{Gal}(D/k)$. Trivially $\tau \in \text{Aut}_\tau(D/k)$ holds, so that the order of $\tau$ in $\text{Gal}(D/k)$ is the same as the order of $\tau$ in $\text{Gal}_\tau(D/k)$.

Corollary 5.4. Let $x$ be a separable point, $U = U_x$ and $D = \text{End}(S_x)$. Then $\text{Aut}(U/k) \simeq \text{Gal}_\tau(D/k)$.

Proof. We have $\text{gr}(R) \simeq D[T, \tau^{-1}]$ and $U \simeq \text{mod}_0^s(\text{gr}(R))/s^2$, the orbit category with respect to the degree shift $s$; in different words, this is the category of finite dimensional $\text{gr}(R)$-modules which are annihilated by some power of $T$. A graded automorphism of $\text{gr}(R)$ is uniquely determined by its action on degrees zero and one, and is thus of the form

$$
\sum a_i T^i \mapsto \sum f(a_i) N_i(b) T^i,
$$

with $f \in \text{Aut}(D/k)$ and $b \in D^\times$, satisfying $f \tau^{-1}(a) \cdot b = b \cdot \tau^{-1}(f(a))$ for all $a \in D$. Here $N_i(b)$ is defined as $b \cdot \tau^{-1}(b) \cdots \tau^{-i}(b)$. We define the group of graded inner automorphisms of $\text{gr}(R)$, denoted by $\text{Inn}(\text{gr}(R))$, generated by automorphisms of the form $u \mapsto u^{-1} T^\tau u$ ($u \in D^\times$), and by automorphisms induced by $T^\tau \mapsto N_i(b) T^\tau$ (with $b \in Z(D)^\times$). Each graded automorphism $\sigma = (f, b)$ of $\text{gr}(R)$ induces an autoequivalence $F^\sigma$ on $U$, and $F^\sigma \simeq 1_U$ if and only if $\sigma$ is a graded inner automorphism. We refer to [38, Prop. 3.2.3] for a similar statement. On the other hand, each automorphism of $U$ is uniquely determined by its action on the bimodule $\text{Ext}^1(S, S)$, and thus on $R/m = D$ and $m/m^2$, and thus induces a graded automorphism of $\text{gr}(R) = D(m/m^2)$. 


Considering the skeleton of $\mathcal{U}$ and requiring that automorphisms are the identity on objects (e.g., equality $\tau S = S$), the automorphism $F^\sigma$ on $S$ commutes with $\tau$ on $S$, which follows from the diagram in Corollary 4.2. We thus can assume that $f \in \text{Aut}_\tau(D/k)$. Then also $b \in \mathbb{Z}(D)^\times$. We write $\text{Aut}_\tau(\text{gr}(R))$ for the subgroup of the automorphisms with these properties, and $\text{Im}_\tau(\text{gr}(R)) = \text{Im}(\text{gr}(R)) \cap \text{Aut}_\tau(\text{gr}(R))$. We conclude $\text{Aut}(\mathcal{U}/k) \simeq \text{Aut}_\tau(\text{gr}(R))/\text{Im}_\tau(\text{gr}(R)) \simeq \text{Gal}_\tau(D/k)$, finishing the proof. □

6. LOCAL AND GLOBAL SKEWNESS

For a point $x$ of a noncommutative regular projective curve over an arbitrary field we write $e^\ast(x)$ for the PI-degree of $\text{End}(S_x[\infty])$.

**Theorem 6.1** (General skewness principle). Let $\mathcal{H}$ be a noncommutative regular projective curve over an arbitrary field $k$. For all points $x \in \mathcal{X}$ the following hold:

1. $e(x) \cdot e^\ast(x) = s(\mathcal{H})$.
2. $e^\ast(x)$ divides $e^\ast(x)$.

**Proof.** (1) By Proposition 3.14 the PI-degree of $R_x$ is $e(x) \cdot e^\ast(x)$. Since by [14, Thm.13] the ring $R_x$ and its completion $\hat{R}_x$ have the same PI-degree. The PI-degree of $R_x$ coincides with the PI-degree of its quotient division ring $k(\mathcal{H})$, which is $s(\mathcal{H})$. Thus we get the equation.

(2) By a theorem of Bergman-Small (see [63, Thm. 1.10.70]), applied to the surjective ring homomorphism $R_x \to R_x/\text{rad}(R_x) \simeq M_{e(x)}(D_x)$, the PI-degree of the factor, which is $e(x) \cdot e^\ast(x)$, divides the PI-degree of $R_x$, which is $s(\mathcal{H})$. Together with (1) we get that $e^\ast(x)$ divides $e^\ast(x)$. □

**Lemma 6.2.** Let $x$ be a point. Denote by

$$\hat{D}_x = D_x((T, \tau^-))$$

the skew Laurent power series ring over $D_x$ in the variable $T$. It is a skew field of dimension $e^\ast(x)^2 \cdot e_x(x)^2$ over its centre. Moreover, it is $v_x$-complete, where the valuation $v_x$ is given by $v_x(\sum_m a_i T^i) = (1/2)^i$, with $\ell$ the infimum of indices $i$ with $a_i \neq 0$.

**Proof.** Let $r = e_x(x)$ and $\sigma = (d) = u^{-1}du$ for some $u \in \text{Fix}(\tau)^\times$. By [57, 19.7], the centre of $D_x((T, \tau^-))$ is given by

$$\hat{K}_x = K_x((uT^\alpha)) \quad \text{with} \quad K_x = Z(D_x) \cap \text{Fix}(\tau^-).$$

From this the assertion about the centre follows. Completeness is shown in [57, 19.7]. □

**Proposition 6.3.** Let $x$ be a separable point.

1. We have $e^\ast(x) = e^\ast(x) \cdot e_x(x)$.
2. $e_x(x)$ coincides with the order of $\tau \in \text{Aut}(\mathcal{U}_x/k)$, the group of (isomorphism classes of) autoequivalences on the tube $\mathcal{U}_x$.

**Proof.** (1) We have $\text{End}(S_x[\infty]) \simeq D_x[[T, \tau^-]]$. By Posner’s theorem (see [3, Thm. 7]) the PI-degree of $D_x[[T, \tau^-]]$ coincides with the PI-degree of its quotient division ring, which is $D_x((T, \tau^-))$. The assertion follows from the preceding lemma.

(2) This follows from Corollary 5.4. □

The following local-global principle is the main result on skewness.
Theorem 6.4 (Local-global principle of skewness). Let \( \mathcal{H} \) be a noncommutative regular projective curve over a field. Then for each separable point \( x \in \mathcal{X} \) the formula
\[
e(x) \cdot e^*(x) \cdot e_\tau(x) = s(\mathcal{H})
\]
holds.

For a perfect field, we obtain the main part of Theorem 1.1.

Proof. Follows from Theorem 6.1 (1) and Proposition 6.3 (1).

Definition 6.5. Let \( \mathcal{X} \) be a noncommutative smooth projective curve over a perfect field. We call a point \( x \in \mathcal{X} \) a separation point, if \( e_\tau(x) > 1 \).

7. Maximal orders and ramifications

Following [59, 60], we will use in this section an alternative description of noncommutative curves in terms of hereditary and maximal orders. Here our main result is that the \( \tau \)-multiplicities \( e_\tau(x) \) are the ramification indices of the underlying maximal order \( \mathcal{A} \). We will temporarily, in Theorem 7.10, also permit weighted curves. This will allow to characterize the non-weighted situation in terms of orders. Namely, the weights \( p(x) \) correspond to the local types of the, in general, hereditary order \( \mathcal{A} \), which measure the deviation of \( \mathcal{A}_x \) from being maximal. For excellent expositions on orders and valuation theory we refer to [9], [15, 16], [31], [58], [72], and the unpublished [5].

7.1. By a (commutative) curve we mean a one-dimensional scheme over \( k \), which we always assume to be integral, separated, noetherian and of finite type over \( k \). A curve \( \mathcal{X} \) is regular (or non-singular) if all local rings \( \mathcal{O}_{\mathcal{X},x} \) are regular, equivalently, discrete valuation domains; in particular they are hereditary. We recall that in particular, \( \mathcal{X} \) is even an isomorphism. In particular, \( \mathcal{X} \) itself is projective over \( k \). We call \( \mathcal{X} \) the centre curve of \( \mathcal{H} \) (or \( \mathcal{X} \)). If \( \mathcal{O} = \mathcal{O}_X \) is the structure sheaf of \( X \), we denote by \( (\mathcal{O}_x, \mathfrak{m}_x) \) the local rings \( (x \in \mathcal{X}) \) and by \( k(x) = \mathcal{O}_x/\mathfrak{m}_x \) the residue class fields. We call \( \mathcal{H} \) (or \( \mathcal{X} \)) smooth, if its centre curve \( \mathcal{X} \) is smooth.

Example 7.3. Let \( k = \mathbb{R} \) be the field of real numbers and \( R = \mathbb{C}[X;Y;\sigma] \) the twisted polynomial algebra, graded by total degree, where \( X \) is central and \( Yz = \sigma(z)Y \) for each \( z \in \mathbb{C} \), with \( \sigma(z) = \bar{z} \) the complex conjugation. Then \( \mathcal{H} = \text{mod}^\mathbb{C}(R)/\text{mod}_0^\mathbb{C}(R) \) is a noncommutative smooth projective curve (we refer to [38] for more details). The function field is \( \mathbb{C}(T, \sigma) \), its centre given by \( \mathbb{R}(T^2) \).

The centre of \( R \) is \( S = \mathbb{R}[X,Y^2] \), and \( \text{Proj}(S) \) is the centre curve of \( \mathcal{H} \). It is isomorphic to the projective spectrum of \( S' = \mathbb{R}[X,Y] \), graded by total degree, having function field \( \mathbb{R}(T) \), which as \( \mathbb{R} \)-algebra is isomorphic to \( \mathbb{R}(T^2) \). Thus the centre curve of \( \mathcal{H} \) is isomorphic to the projective line \( \mathbb{P}_1(\mathbb{R}) \).
Proposition 7.4. Let \((\mathcal{H}, L)\) be a noncommutative regular projective curve.

1. For each \(x \in X\) the graded ring \(\Pi(L, \sigma_x)\) is finitely generated as module over its centre.
2. If \(s(\mathcal{H}) = 1\), then \(\Pi(L, \sigma_x)\) is commutative.

Proof. Like in [38, Prop. 4.3.3] we have a graded inclusion \(\Pi(L, \sigma_x) \subseteq k(\mathcal{H})[T]\), where \(T\) is a central variable. From this, (2) follows immediately; and (1) follows with [40] and [6, Thm. 0.1(ii)]. □

Corollary 7.5. Let \((\mathcal{H}, L)\) be a noncommutative regular projective curve over \(k\) with \(s(\mathcal{H}) = 1\). Then there is a (commutative) regular projective curve \(X\) over \(k\) such that \(\mathcal{H} \simeq \text{coh}(X)\), and the points of \(X\) are in bijective correspondence with the (closed) points of \(X\).

Proof. Let \(S\) be the commutative graded ring \(\Pi(L, \sigma_x)\) for some \(x \in X\) and \(X = \text{Proj}(S)\).

□

Corollary 7.6. Let \(k\) be an algebraically closed field. Then \(\mathcal{H}\) is a noncommutative smooth projective curve over \(k\) if and only if \(\mathcal{H}\) is equivalent to the category \(\text{coh}(X)\) of coherent sheaves over a (commutative) smooth projective curve \(X\).

Proof. By Tsen’s theorem [69] we have \(s(\mathcal{H}) = 1\) over an algebraically closed field. Since \(k\) is perfect, by [21, I.5.3.2] regular is the same as smooth. □

7.7 (Weighted noncommutative projective curves). In order to avoid pathological cases, we will restrict to a subclass of the weighted curves considered in [45]: we define a small, connected \(k\)-category \(\mathcal{H}\) to be a weighted noncommutative regular projective curve over \(k\), if there exists a non-weighted noncommutative regular projective curve \(\mathcal{H}'\) over \(k\) such that \(\mathcal{H}\) is obtained from \(\mathcal{H}'\) by insertion of weights into a finite number of points of \(\mathcal{H}'\) in the sense of the \(p\)-cycle construction from [42] (we refer also to [38, Sec. 6.1]). This is equivalent to say that \(\mathcal{H}\) satisfies conditions (NC 1) to (NC 5), and

- (NC 6) For all points \(x \in X\) there are (up to isomorphism) precisely \(p(x) < \infty\) simple objects in \(\mathcal{U}_x\), and for almost all \(x\) we have \(p(x) = 1\).

Thus there is a finite number of so-called exceptional simple sheaves \(S_i\), that is, with \(\text{Ext}^1(S, S) = 0\). Since \(\mathcal{H}/\mathcal{H}_0 \simeq \mathcal{H}'/\mathcal{H}_0\), we have \(k(\mathcal{H}) \simeq k(\mathcal{H}')\), and thus (NC 7) also holds in the weighted cases. We call \(\mathcal{H}\) smooth, if \(\mathcal{H}'\) is smooth.

Let \(p(x) \geq 1\) be the number of isoclasses of simple objects concentrated in the point \(x \in X\). Then always \(p(x) < \infty\) holds, and for all but finitely many \(x \in X\) we have \(p(x) = 1\). We call a line bundle \(L \in \mathcal{H}\) special, if for each \(x \in X\) there is (up to isomorphism) precisely one simple sheaf \(S_i\) concentrated in \(x\) with \(\text{Hom}(L, S_i) \neq 0\). We consider a pair \((\mathcal{H}, L)\) with \(L\) a special line bundle, which we consider as the structure sheaf of \(\mathcal{H}\). Like in [60, Lem. IV.4.1] it is shown, that \(\mathcal{H}\) is of the form \(\text{mod}^\tau(R)/\text{mod}_0^\tau(R)\) for some positively \(\mathbb{Z}\)-graded domain \(R\), finitely generated over its centre.

The following proposition is almost tautological.

Proposition 7.8 (Reduction to the non-weighted case). Let \(\mathcal{H}\) be a weighted noncommutative regular projective curve with the exceptional points given by \(x_1, \ldots, x_t\), of weights \(p_i = p(x_i) > 1\). Choose for every \(i = 1, \ldots, t\) one simple sheaf \(S_i\) concentrated in \(x_i\). Let \(\mathcal{I} \subseteq \mathcal{H}\) be the system \(\{r^jS_i | i = 1, \ldots, t; j = 1, \ldots, p_i - 1\}\).

1. The right perpendicular category \(\mathcal{H}' = \mathcal{I}^\perp \subseteq \mathcal{H}\) is a full, exact subcategory of \(\mathcal{H}\), and is a (non-weighted) noncommutative regular projective curve.
2. There is a special line bundle \(L\) in \(\mathcal{H}\).
Proof. This is similar to the proof of [41, Prop. 1]. For (2) we remark that the full exact embedding $\mathcal{H}' \subseteq \mathcal{H}$ preserves the rank. So any line bundle in $\mathcal{H}'$ gives rise to a special line bundle in $\mathcal{H}$. □

7.9 (Orders over the centre curve). Let $X$ be the centre curve with function field $K = k(X)$. Let $A$ be a finite dimensional central simple $K$-algebra. As in [5] we call a torsionfree, coherent $O_X$-algebra $A$ an $O_X$-order in $A$, if the generic fibre of $A$ is isomorphic to $A$, or equivalently, if $A \otimes_{O_X} K \simeq A$. An order $A$ is called maximal if it is not contained properly in another order. Then all stalks $A_x = A \otimes O_x$ are maximal $O_x$-orders in $A$. An order $A$ is called hereditary, if all stalks $A_x$ are hereditary $O_x$-orders in $A$. Each maximal order is hereditary. The $O_X$-order $A$ is called an Azumaya algebra of degree $n$, if $A$ is locally-free of rank $n^2$, and if for each $x \in X$ the geometric fibre $A(x) = A_x \otimes_{O_x} k(x) = A_x / \text{rad}(A_x)$ is a full matrix algebra with centre $k(x)$. Equivalently (by [5, Prop. 1.9.2]): For each $x$ we have $[A(x) : k(x)] = n^2$. Azumaya algebras over $O_X$ are maximal orders (by [5, Prop. 1.8.2]).

We now have the following fundamental description of noncommutative regular projective curves.

Theorem 7.10. Let $k$ be a field.

(1) For a $k$-category $\mathcal{H}$ the following two conditions are equivalent:
   (a) $\mathcal{H}$ is a, possibly weighted, noncommutative regular projective curve over $k$.
   (b) There is a (commutative) regular projective curve $X$ over $k$, a (finite dimensional) central simple $k(X)$-algebra $A$ and a torsionfree coherent sheaf $\mathcal{A}$ of hereditary $O = O_X$-orders in $A$ such that $\mathcal{H} \simeq \text{coh}(\mathcal{A})$, the category of coherent $\mathcal{A}$-modules.

(2) If the equivalent conditions in (1) hold, then $X$ is the centre curve. Accordingly, the points of $X$ correspond bijectively to the points of $X$, and for each $x \in X$ its weight $p(x)$ is the local type (in the sense of [58, p. 369]) of the hereditary $O$-order $A$ at $x$. Accordingly, $p(x) > 1$ if and only if $A_x$ is not maximal (there is only a finite number of such points $x$).

(3) In (1) we have that $\mathcal{H}$ is non-weighted if and only if $A$ is a maximal $O$-order in $A$.

Proof. (1) This is shown like in [60, Prop. III.2.3]. For the fact that the centre of a hereditary order is a Dedekind domain, we refer to [31, Thm. 2.6]. By [71] the category $\text{coh}(A)$ has Serre duality. We recall the construction of $A$ and $\mathcal{A}$ if $\mathcal{H}$ is given. Let $X$ be the underlying centre curve. Let $R$ be a positively $\mathbb{Z}$-graded coordinate algebra of $\mathcal{H}$ and $S$ its centre. Let $x_1, \ldots, x_t$ be a set of homogeneous generators of $S$ over the field $S_0$. Let $n$ be the least common multiple of their degrees. Then

$$T = \begin{pmatrix} R & R(1) & \ldots & R(n-1) \\ R(-1) & R & \ldots & R(n-2) \\ \vdots & \vdots & \ddots & \vdots \\ R(-n+1) & R(-n+2) & \ldots & R \end{pmatrix}$$

is graded Morita-equivalent to $R$ and strongly $\mathbb{Z}$-graded; thus $\text{mod}^\mathbb{Z}(T) \simeq \text{mod}(T_0)$. Let $\mathcal{T}$ be the corresponding sheaf of graded rings. We set $\mathcal{A} = M_n(k(\mathcal{H}))$, which is of finite dimension over its centre $k(X)$, and $\mathcal{A} = \mathcal{T}_0 \subseteq A$, equipped canonically with the structure of an $O_X$-module.
(2) The assertion is clear from the structure of hereditary orders [15, 16], we refer also to [58, Ch. 9], and the Auslander-Goldman criterion [9, Thm. 2.3] for maximality.

(3) This follows from (2), since by [58, (40.8)] the order $\mathcal{A}$ is maximal if and only if all $\mathcal{A}_x$ are maximal. □

We switch back to the non-weighted case. The next theorem extends [29, Prop. (7.4.18)] to this noncommutative setting, and it gives a positive answer to [38, Probl. 4.3.9], even in this much more general context.

**Theorem 7.11.** Two noncommutative regular projective curves $\mathcal{H}$ and $\mathcal{H'}$ over a field $k$ are isomorphic (that is, they are equivalent as $k$-categories) if and only if their function fields $k(\mathcal{H})$ and $k(\mathcal{H'})$ are isomorphic.

**Proof.** If $\mathcal{H} \simeq \mathcal{H'}$, then $H_0 \simeq H'_0$ and $H/H_0 \simeq H'/H'_0$, and consequently $k(H)$ and $k(H')$ are isomorphic. Assume conversely, that the function fields $k(H)$ and $k(H')$ are isomorphic and have the common centre $K = k(X)$. By parts (1) and (3) of the preceding theorem, there are maximal orders $\mathcal{A}$ and $\mathcal{A}'$ in Morita-equivalent central simple $K$-algebras $A$ and $A'$, respectively, such that $\mathcal{H} \simeq \text{coh}(\mathcal{A})$ and $\mathcal{H'} \simeq \text{coh}(\mathcal{A'})$. Since $X$ is, in particular, a normal curve, by [5, Prop. 1.9.1 (ii)] (for an affine version we refer to [58, Cor. (21.7)]; for a similar result on hereditary orders over a smooth curve we refer to [19, Thm. 7.6]) it follows, that $\mathcal{A}$ and $\mathcal{A}'$ are Morita-equivalent, that is (by definition), $\text{Qcoh}(\mathcal{A}) \simeq \text{Qcoh}(\mathcal{A'})$. Then clearly $\text{coh}(\mathcal{A}) \simeq \text{coh}(\mathcal{A'})$, and thus $\mathcal{H} \simeq \mathcal{H'}$ follows. □

Since maximal $O_X$-orders over a regular projective curve $X$ in a central simple $k(X)$-algebra always exist, by [5, Prop. 1.8], we have even more:

**Corollary 7.12.** The assignments

\[ \mathcal{H} \mapsto k(\mathcal{H}) \quad \text{and} \quad A \mapsto \text{coh}(A), \]

where $A$ is a maximal order in $A$ (whose centre is of the form $k(X)$), induce mutually inverse bijections between the sets of

- **noncommutative regular projective curves over $k$, up to equivalence of categories; and**
- **algebraic function skew fields of one variable over $k$, up to isomorphism.** □

Let $\mathcal{H} = \text{coh}(\mathcal{A})$ be a noncommutative smooth projective curve over a perfect field $k$, with a maximal order $A$ in $A = M_n(k(H))$ as above. Let $K = k(X)$ be the centre, as above. Let $x \in X$. We denote by $\hat{E}_x = \text{End}(S_n[\infty])$ the endomorphism ring of the corresponding Prüfer sheaf, which is a complete local domain, the maximal ideal generated by $\pi_x$. By Proposition 3.14 we have $\hat{R}_x \simeq M_n(\hat{E}_x)$. If $\hat{K}_x$ denotes the quotient field of the $m_x$-adic completion $\hat{O}_x$ of $O_x$, then

\[
A \otimes_K \hat{K}_x \simeq M_n(k(\mathcal{H}) \otimes_K \hat{K}_x) \simeq M_{n \cdot e(x)}(\hat{D}_x),
\]

with $\hat{D}_x$ a skew field (unique up to isomorphism) with centre $\hat{K}_x$; compare Proposition 7.15 below. Analogously to the global situation we make the following local definition.

**Definition 7.13.** We call the number

\[
(7.2) \quad s(x) = [\hat{D}_x : \hat{K}_x]^{1/2}
\]

the local skewness at $x$.

By (7.1) we get the following relationship between global and local skewness

\[
(7.3) \quad s(\mathcal{H}) = e(x) \cdot s(x),
\]
and with the skewness principle we have

\[(7.4)\quad s(x) = e^*(x) \cdot e_\tau(x).\]

The following results make the situation quite explicit.

**Lemma 7.14.**

1. \(\mathcal{O}_x\) is the centre of \(R_x\).
2. Let \(\mathcal{F}_x\) be the multiplicative set \(\mathcal{O}_x \setminus \{0\}\). The central localization \(\mathcal{F}_x^{-1}R_x\) is equal to the function field \(k(\mathcal{H})\).

**Proof.** (1) We have \(\text{mod}(A_x) \simeq H_x \simeq \text{mod}(R_x)\). Since \(L_x\) is an indecomposable projective generator of \(H_x\), there is \(n(x) \geq 1\) such that as module \(A_x \simeq (L_x)^{n(x)}\), thus as ring \(A_x \simeq M_{n(x)}(R_x)\). So it is sufficient to show that \(\mathcal{O}_x\) is the centre of \(A_x = A \otimes_\mathcal{O} \mathcal{O}_x\). Let \(U \subseteq X\) be an affine open subset with \(x \in U\). Then \(\mathcal{O}(U)\) is a Dedekind domain with quotient field \(K\), and \(\mathcal{A}(U)\) is a maximal \(\mathcal{O}(U)-\)order in \(A\), whose centre \(\mathcal{A}(U) \cap K = \mathcal{O}(U)\), since \(\mathcal{O}(U)\) is integrally closed. Now, localization at \(x\) is compatible with the centres \([63, \text{Prop. 1.7.4}]\) and yields the result.

(2) \(k(\mathcal{H})\) is the quotient division ring of \(R_x\). By \([63, \text{Thm. 1.7.9}]\) the ring \(\mathcal{F}_x^{-1}R_x\) is a skew field with centre \(K\). From \(R_x \subseteq \mathcal{F}_x^{-1}R_x \subseteq k(\mathcal{H})\) the result follows. \qed

**Proposition 7.15.** Let \(x\) be separable. The skew field \(\hat{D}_x\) and its centre \(\hat{K}_x\) from \((7.1)\) agree with the (skew) Laurent power series rings in \((6.1)\) and \((6.2)\), respectively. Moreover, \(k(x) = K_x := Z(D_x) \cap \text{Fix}(\tau)\).

**Proof.** Write \(r = e_\tau(x)\). Using the preceding lemma, we apply \(\mathcal{F}_x^{-1}\) (that is, central localization) to the isomorphism \(R_x \otimes_\mathcal{O} \hat{\mathcal{O}}_x \simeq M_{e_\tau(x)}(\hat{E}_x)\). By Hom-tensor properties of the localization \([13, \text{II.}\S 2.7]\), we obtain the isomorphism \(Q(R_x) \otimes_K \mathcal{F}_x^{-1}\hat{\mathcal{O}}_x \simeq M_{e_\tau(x)}(\mathcal{F}_x^{-1}\hat{E}_x)\), where \(Q(-)\) stands for quotient division ring. Moreover, since \(\hat{\mathcal{O}}_x\) and \(\hat{E}_x\) are (skew) power series rings in one variable by Theorem 5.3, clearly \(\mathcal{F}_x^{-1}\hat{\mathcal{O}}_x \simeq Q(\hat{\mathcal{O}}_x)\) and \(\mathcal{F}_x^{-1}\hat{E}_x \simeq Q(\hat{E}_x)\) hold, since in each case the uniformizer becomes invertible, so that we get the corresponding (skew) Laurent series rings. We conclude \(\hat{D}_x \simeq Q(\hat{E}_x) \simeq D_x((T, \tau^{-})).\) Moreover, for the centre we deduce \(k(x)((T)) \simeq \hat{K}_x \simeq K_x((uT^\tau))\), from which also \(K_x = k(x)\) follows. \qed

We can now derive well-known identities for well-studied local invariants, and also their relationship with the \(\tau\)-multiplicity. As usual, define the

\[(7.5)\quad \text{inertial degree} \quad f_{in}(x) = [\hat{D}_x / \text{rad}(\hat{D}_x) : k(x)] = [D_x : k(x)]\]

and the

\[(7.6)\quad \text{ramification index} \quad e_{ra}(x) = [\Gamma_{\hat{D}_x} : \Gamma_{\hat{K}_x}]\]

(the index of the corresponding discrete value groups) of (the skew field part of the completion of) \(A\) in \(x\). If \(e_{ra}(x) > 1\), then \(x\) is called a \textit{ramification point} of \(A\).

By Proposition 7.15 it is easy to see that the ramification index coincides with the number \(e\) such that \(m_{e}\hat{E}_x = \text{rad}(\hat{E}_x)^e\), and also with the number

\[(7.7)\quad e'(x) = [Z(D_x) : k(x)].\]

(Compare also \([5, \text{Sec. 1.3}]\).) With this number we clearly have

\[(7.8)\quad f_{in}(x) = e^*(x)^2 \cdot e'(x)\]

and moreover

\[(7.9)\quad e'(x) = e_{ra}(x) = e_\tau(x).\]

Indeed, the right equation follows from Proposition 7.15, since the Galois group \(\text{Gal}(Z(D_x)/k(x))\) is cyclic, generated by the restriction of \(\tau\). Together with the
skewness principle we deduce further the following well-known fundamental equation
\[(7.10) \quad e_{ra}(x) \cdot f_{in}(x) = s(x)^2\]
(compare [52, Thm. 3] and [53, p. 359]). We now have the main result of this section.

**Theorem 7.16.** Let \( \mathcal{H} \) be a noncommutative smooth projective curve over the perfect field \( k \). The maximal \( \mathcal{O}_X \)-order \( \mathcal{A} \) in \( A \) from Theorem 7.10 with \( \mathcal{H} = \text{coh}(\mathcal{A}) \) has the following properties:

1. On the (Zariski-open) unramified locus \( U \subseteq X \) the order \( \mathcal{A} \) is an Azumaya-algebra.
2. For \( x \in X \) we have that \( e_{\mathcal{A}}(x) \) coincides with the ramification index of \( \mathcal{A} \) in \( x \). In particular, the separation points are just the ramification points of \( \mathcal{A} \), and there are only finitely many of them.

**Proof.**

1. By [58, p. 372] there are only finitely many ramification points of \( \mathcal{A} \). Thus the unramified locus \( U \subseteq X \) is open. The restriction \( \mathcal{A}|_U \) is an Azumaya algebra by [5, Cor. 1.9.6].
2. This follows from the preceding discussion. \( \square \)

8. The genus and the Euler characteristic

Let \((\mathcal{H},L)\) be a noncommutative regular projective curve over \( k \). We assume (without loss of generality) that \( k \) is the centre of \( \mathcal{H} \). We set \( \kappa = \dim_k \text{End}(L) \).

The Euler form is defined by
\[
\langle E, F \rangle = \dim_k \text{Hom}(E, F) - \dim_k \text{Ext}^1(E, F)
\]
for objects \( E, F \in \mathcal{H} \). We call
\[
g(\mathcal{H}) = \dim_{\text{End}(L)} \text{Ext}^1(L, L)
\]
the genus and the number
\[
\chi(\mathcal{H}) = \langle L, L \rangle = \kappa \cdot (1 - g(\mathcal{H}))
\]
the Euler characteristic of \( \mathcal{H} \). (Here, \( \text{End}(L) \) plays the role of the field of constants. The normalization factor fits to the case \( k = \mathbb{R} \). Over the complex numbers the value should be doubled.) With this we have
\[
g(\mathcal{H}) = 0 \iff \chi(\mathcal{H}) > 0 \quad \text{and} \quad g(\mathcal{H}) = 1 \iff \chi(\mathcal{H}) = 0.
\]

For \( F \in \mathcal{H} \) we define
\[
\deg(F) = \frac{1}{\kappa} \cdot \langle L, F \rangle - \frac{1}{\kappa} \cdot \langle L, L \rangle \cdot \text{rk}(F).
\]
We obtain

**Proposition 8.1** (Riemann-Roch formula).
\[
\frac{1}{\kappa} \cdot \langle E, F \rangle = (1 - g(\mathcal{H})) \cdot \text{rk}(E) \cdot \text{rk}(F) + \left\lfloor \frac{\text{rk}(E)}{\deg(E)} \cdot \frac{\text{rk}(F)}{\deg(F)} \right\rfloor.
\]

**Proof.** First we remark that \( \deg \) is additive on short exact sequences, and that \( \langle -, - \rangle \) and the right hand side of the formula are bilinear. If \( E = L \), then the formula is just the definition of the degree. If \( E, F \in \mathcal{H}_0 \), then both sides are zero. If one of \( E \) or \( F \) belongs to \( \mathcal{H}_0 \), then it is easy to see that the formula holds, by definition of \( \deg \) and \( \text{rk} \). If \( \sigma \in \text{Aut}(\mathcal{H}) \) is point-fixing, then both sides remain equal, if replacing the pair \( (E, F) \) by \( (\sigma E, \sigma F) \) (for all \( E, F \in \mathcal{H} \)). If \( L' \) is a line bundle, by Lemma 3.4 we have an exact sequence \( 0 \to L(-nx) \to L' \to C \to 0 \) (any point, \( n \gg 0, C \in \mathcal{U}_x \)). From this the formula holds for \( E = L' \) and \( F \in \mathcal{H} \).
Using line bundle filtrations, the formula holds for every $E \in \mathcal{H}_+$ and $F \in \mathcal{H}$, and then generally.

In particular, if $\omega = \omega_{\mathcal{H}} = \tau L$ denotes the dualizing sheaf in $\mathcal{H}$, then

\begin{equation}
\deg(\omega) = -\frac{2}{\kappa} (L, L) = 2 \cdot (g(\mathcal{H}) - 1).
\end{equation}

Let $\mathcal{H} = \text{coh}(\mathcal{A})$ be a noncommutative regular projective curve with centre curve $X$ and with $\mathcal{A}$ a maximal $\mathcal{O}_X$-order in a central simple $k(X)$-algebra. Following [5] we call $e = (e_1, \ldots, e_n)$ the ramification vector of $\mathcal{A}$ if $e_1, \ldots, e_n$ are all ramification indices $> 1$, and moreover, for each ramification point $x$ of $\mathcal{A}$ its ramification index $e_x(x)$ appears precisely $[k(x) : k]$ times in $e$. If for all ramifications points $x_1, \ldots, x_t$ (pairwise different) the numbers $f_i = [k(x_i) : k]$ are given, then we will also write more precisely $e = (e_1^{f_1}, \ldots, e_t^{f_t})$.

**Proposition 8.2** ([5, (4.1.6)]). Let $\mathcal{H} = \text{coh}(\mathcal{A})$ be a noncommutative smooth projective curve of skewness $s(\mathcal{H})$, which we assume to be prime to the characteristic of $k$, and with centre curve $X$. Let $\mathcal{A}$ be a maximal $\mathcal{O}_X$-order in $k(\mathcal{H})$ and $e = (e_1, \ldots, e_n)$ its ramification vector. Then we have for the Euler characteristics

\begin{equation}
\chi(\mathcal{H}) = s(\mathcal{H})^2 \cdot \left( \chi(X) - \frac{1}{2} \sum_{i=1}^n \left( 1 - \frac{1}{e_i} \right) \right).
\end{equation}

**Proof.** Using the formulae $\text{Ext}^1_X(\mathcal{A}, \omega_X) \simeq \text{Ext}^1_{\mathcal{A}}(\mathcal{A}, \omega_{\mathcal{A}})$ (for $i = 0, 1$) and Serre duality, see [71, (2)+(4)], we have $\chi(\mathcal{H}) = \langle L, L \rangle = \langle \mathcal{A}, \mathcal{A} \rangle = \chi(\mathcal{A})$. Now the assertion follows from [5, Lem. 4.1.5].

We will treat the genus zero case in the next section.

**Definition 8.3.** We call a (non-weighted) noncommutative regular projective curve of genus one a (noncommutative) elliptic curve.

For elliptic curves there is the following analogue of Atiyah’s classification of vector bundles over an elliptic curve over an algebraically closed field [8]:

**Proposition 8.4.** Let $\mathcal{H} = \text{coh}(\mathcal{X})$ be a noncommutative elliptic curve. Then the following holds:

1. Each indecomposable object $E$ in $\mathcal{H}$ is semistable of slope $\mu(E) = \frac{\deg(E)}{\text{rk}(E)}$ in $\mathbb{Q} \cup \{ \infty \}$ and satisfies $\text{Ext}^1(E, E) \neq 0$.

2. For each $q$ subcategory of semistable objects of slope $q$ is non-trivial and forms a tubular family, again parametrized by a noncommutative elliptic curve $\mathcal{X}'$, so that $\mathcal{H}' = \text{coh}(\mathcal{X}')$ is derived-equivalent to $\mathcal{H}$.

We remark that one major difference to Atiyah’s result is, that in general $\mathcal{H}'$ may be not isomorphic to $\mathcal{H}$. In this situation, $\mathcal{H}'$ is called a Fourier-Mukai partner of $\mathcal{H}$. We will give an example later.

**Proof.** (1) This follows like in [27, Prop. 5.5].

(2) The proof of [38, Prop. 8.1.6] works also in this situation. (We refer also to [45, Sec. 5].)

**Examples 8.5.**

1. Each (commutative) smooth projective curve of genus one is elliptic in our sense. In particular, the Klein bottle is an elliptic curve without $\mathbb{R}$-rational (= real) points (it is a Klein surface without boundary).

2. [5, Prop. 4.2.1] Let $X$ be a smooth projective curve over a field $k$, $K$ its function field and $D$ a finite dimensional central division $K$-algebra, whose degree is prime to the characteristic of $k$. Let $\mathcal{A}$ be a maximal $\mathcal{O}_X$-order in $D$ and $e$ its ramification vector.
(a) We have \( \chi(A) > 0 \) if and only if \( X \) has genus zero and \( e = (e), (e_1, e_2), (2, 2, e), (2, 3, 3), (2, 3, 4) \) or \( (2, 3, 5) \).

(b) \( A \) is elliptic (that is, the noncommutative curve \( \text{coh}(A) \) is elliptic), if either \( X \) has genus one and \( A \) is Azumaya, or \( X \) has genus zero and \( A \) has ramification vector \( e = (2, 3, 6), (2, 4, 6), (3, 3, 3) \) or \( (2, 2, 2, 2) \). This follows directly from (8.4).

(c) In case of the field of real numbers we will discuss (a) and (b) in more detail when treating the real elliptic curves. We remark that in that case ramification points \( x \) always belong to the boundary, and so the numbers \( |k(x) : k| \) equal 1, so that the ramification indices of \( x \) are not duplicated in \( e \).

**Proposition 8.6.** Let \( \mathcal{H} \) be a noncommutative regular projective curve. If \( g(\mathcal{H}) > 1 \), then all Auslander-Reiten components of \( \mathcal{H}_+ = \text{vect}(X) \) have as underlying directed graph \( \mathbb{Z} \mathcal{A}_\infty \), and the category \( \mathcal{H} \) is wild.

**Proof.** This follows from [45, Prop. 4.7]. \( \square \)

9. **Local-global principle for global function fields**

We call a noncommutative curve **multiplicity free**, if \( e(x) = 1 \) for all \( x \in \mathbb{X} \), and **unramified**, if \( e_+(x) = 1 \) for all \( x \in \mathbb{X} \). We recall that \( x \) is called **inert**, if \( f_{\text{in}}(x) = s(\mathcal{H})^2 \); equivalently, if \( x \) satisfies \( e(x) = 1 \) and \( e_+(x) = 1 \).

From the three numbers \( e(x) \), \( e^*(x) \) and \( e_-(x) \) only the first, \( e(x) \), is not a purely local datum, since it involves the relationship between the tube \( \mathcal{U}_x \) and the structure sheaf \( \mathcal{L} \). The equation (7.3), \( s(\mathcal{H}) = e(x) \cdot s(x) \), comparing global skewness with the local skewness at \( x \), might suggest, that the multiplicity \( e(x) \) additionally contributes to the non-commutativity of \( \mathcal{H} \) independently of \( s(x) \). But actually it happens in many cases that \( s(x) = 1 \) everywhere is equivalent to \( e(x) = 1 \) everywhere. In this section we show that this is the case when \( k \) is a finite field. We can regarded this as an extension of the Hasse principle. This well-known local-global principle for central simple algebras, here in the version over global function fields (that is, algebraic function fields in one variable over a finite field), [55, Thm. 8.1.17], [23, VII.10], is also known as the Albert-Brauer-Hasse-Noether theorem. In terms of global and local skewness it can be stated as follows.

**Theorem 9.1** (Hasse principle). Let \( k \) be a finite field and \( \mathcal{H} \) a noncommutative smooth projective curve over \( k \). The following are equivalent:

1. \( s(\mathcal{H}) = 1 \).
2. \( s(x) = 1 \) for all \( x \in \mathbb{X} \). \( \square \)

This is indeed just another formulation of the Hasse principle: a central simple \( k(X) \)-algebra \( A \) is of the form \( A = M_m(k(\mathcal{H})) \). Then \( A \simeq M_m(k(X)) \) is equivalent to \( s(\mathcal{H}) = 1 \). For \( x \in \mathbb{X} \) the condition \( A \otimes_K \mathbb{K}_x \simeq M_n(\mathbb{K}_x) \) is equivalent to \( s(x) = 1 \).

We now have the following extension of this local-global principal.

**Theorem 9.2.** Let \( k \) be a finite field and \( \mathcal{H} \) a noncommutative smooth projective curve over \( k \).

1. For almost all \( x \in \mathbb{X} \) we have \( e(x) = s(\mathcal{H}) \).
2. If \( s(\mathcal{H}) > 1 \), then there are no inert points.
3. The following are equivalent:
   a. The skewness is \( s(\mathcal{H}) = 1 \).
   b. The curve \( \mathbb{X} \) is multiplicity free.
   c. The curve \( \mathbb{X} \) is unramified.

**Proof.** By Wedderburn’s theorem on finite skew fields we have \( e^*(x) = 1 \) for all \( \mathbb{X} \). Moreover, from the skewness principle we infer \( s(x) = e_-(x) \). Since \( \mathbb{X} \) has...
infinitely many points, (1), (2) and the equivalences of (a) and (b) in (3) follow then from (7.4). The equivalence of (a) and (c) is just the Hasse principle. □

The preceding theorem is a generalization of [38, Cor. 2.3.8+2.3.9].

**Lemma 9.3.** Let \( (\mathcal{H}, L) \) be a noncommutative smooth projective curve over a finite field \( k \). For each \( x \in X \) we have

\[
\deg(S_x) = \frac{s(\mathcal{H})}{k} \cdot [k(x) : k].
\]

**Proof.** We have \([\text{Hom}(L,S_x) : k] = [\text{Hom}(L,S_x) : \text{End}(S_x)] \cdot [\text{End}(S_x) : k] = e(x) \cdot e^*(x)^2 \cdot e_T(x) \cdot [k(x) : k] = s(\mathcal{H}) \cdot [k(x) : k] \), by the skewness principle, since \( e^*(x) = 1 \). The assertion now follows from the definition of the degree. □

10. **Examples I: Genus Zero**

Noncommutative projective curves of genus zero are important in the representation theory of finite dimensional algebras, in particular (but not only) for the tame algebras. For details we refer to [38] and [37]. Recall that genus zero means that \( \text{Ext}^1(L,L) = 0 \). The noncommutative regular genus zero curves \( \mathcal{H} \) over a field \( k \) correspond to the so-called tame bimodules \( FMG \), where \( F \) and \( G \) are finite dimensional division algebras over \( k \), and \( M \) is an \( F\)-\( G \) bimodule on which \( k \) is acting centrally and \( \dim_F M \cdot \dim MG = 4 \). The corresponding bimodule algebra

\[
\Lambda = \begin{pmatrix}
G & 0 \\
M & \mathcal{H}
\end{pmatrix}
\]

is a finite dimensional tame hereditary \( k \)-algebra, whose category \( \operatorname{mod}(\Lambda) \) of finite dimensional right modules is derived equivalent to \( \mathcal{H} \). We refer to [24, 20, 11] as references for the representation theory of tame bimodules and tame hereditary algebras. We define \( \varepsilon = 1 \), if \( (\dim_F M, \dim MG) = (2, 2) \), and \( \varepsilon = 2 \), if this dimension pair is given by \( (1, 4) \) or \( (4, 1) \). We further set

\[
(10.1) \quad f(x) = \frac{1}{\varepsilon} \cdot [\text{Ext}^1(S_x, L) : \text{End}(L)] = \frac{1}{\varepsilon} \cdot \deg(S_x).
\]

A point \( x \in X \) is called rational, if \( f(x) = 1 \). In the genus zero case, rational points always exist, [44, Prop. 4.1].

The proof of the following theorem, which improves the finiteness property from Theorem 7.16 in case genus zero, is independent from the results in Sections 7 and 8.

**Theorem 10.1.** Let \( \mathcal{H} \) be a noncommutative smooth projective curve of genus zero over a perfect field. There are at most three separation points.

**Proof.** Using Theorem 7.16 this follows from Examples 8.5. We give another, independent (not using ramifications) argument rather focusing on the nature of the \( \tau \)-multiplicities. Let \( \Lambda \) be the corresponding tame hereditary bimodule algebra with centre \( k \). There is a finite field extension \( K/k \) such that \( \Lambda^K = \Lambda \otimes_k K \) is a tame hereditary algebra with centre \( K \), and such that the underlying noncommutative curve arises by insertion of weights from the projective line \( \mathbb{P}_1(K) \); since \( \Lambda^K \) is tame hereditary (by [20, Thm. 7.1]), there are at most three non-trivial weights. Moreover, there is the formula \( \tau_{\Lambda}(X)^K \simeq \tau_{\Lambda^K}(X^K) \) (by [20, p. 505]; in general \( X^K \) is not indecomposable if \( X \) is). If \( \mathcal{U}_x \) is a (homogeneous) tube over \( \Lambda \), then \( (\mathcal{U}_x)^K \) lies in a finite union of tubes over \( \Lambda^K \), and for at most three points \( x \) exceptional tubes are involved. (This follows from [33, Lem. 4.1].) Since \( s(\mathbb{P}_1(K)) = 1 \), we have that \( \tau_{\Lambda^K} \) is the identity functor on all homogeneous tubes. By the formula \( \text{Hom}_{\Lambda^K}(X^K, Y^K) \simeq \text{Hom}_{\Lambda}(X,Y)^{[K,k]} \), we see, that only for these at most three points \( \tau_{\mathcal{H}} \) might act non-trivially on \( \mathcal{U}_x \). □

For genus zero there is the following analogue of Theorem 9.2.
Theorem 10.2 ([38, Thm. 4.3.1]). Let $\mathcal{H}$ be a noncommutative regular projective curve of genus zero. The following are equivalent:

1. The curve is commutative, that is, $s(\mathcal{H}) = 1$.
2. For all points $x \in \mathcal{X}$ we have $e(x) = 1$.
3. For all rational points $x \in \mathcal{X}$ we have $e(x) = 1$. 

Example 10.3 (Finite fields). Let $k$ be a finite field and $\mathcal{H}$ over $k$ of genus zero. Then there is an efficient tubular shift $\sigma = \sigma_x$. (For the definition of an efficient automorphism of $\mathcal{H}$ we refer to [38, Def. 1.1.3]. It is such that the pair $(L, \sigma)$ is ample and the orbit algebra $\Pi(L, \sigma)$ has “good” properties.) Then $\tau \sigma^{\alpha/\epsilon}$ is a ghost, that is, an autoequivalence of $\mathcal{H}$ which acts trivially on objects (but not necessarily isomorphic to the identity functor). Moreover, for all points $p$ we have $[k(p) : k] = \frac{\text{ord}_p}{\text{ord}_p f(p)}$, by Lemma 9.3. Without loss of generality we can assume that $k$ is the centre of $\mathcal{H}$. There are two possible cases, which describe all genus zero cases over $k$:

1. $\varepsilon = 1$. [37, Prop. 4.1]. Then $M = M(K, \alpha) = K \oplus K$ where $K$ acts canonically from the left and by $(x, y) : z = (xz, y\alpha(z))$ from the right, and $\alpha : K \to K$ is a $k$-automorphism. Since $k$ is the centre of $M$, it is the fixed field of $\alpha$, so that $\text{Gal}(K/k)$ is cyclic, generated by $\alpha$, and $k$ is the centre of $M$. Let $n = [K : k]$. Then $s(\mathcal{H}) = n$. The case $n = 1$ is the usual Kronecker over $k$. So assume $n \geq 2$. We have that $R = \Pi(L, \sigma_x)$ is isomorphic to $K[X; Y, \alpha]$. The two points $x$ and $y$ (corresponding to $X$ and $Y$) are the only rational points of multiplicity 1, and the only rational points $p$ such that $e_x(p) > 1$; moreover, $e_x(p) = s(\mathcal{H})$; compare [38, Cor. 5.4.2]. Since also $\tau = \sigma_x^{-1} \sigma_y^{-1}$, these are the only separation points. The ramification vector is $e = (n^1, n^1)$.

2. $\varepsilon = 2$. Then $M = K \oplus K$, where $[K : k] = 4$. Indeed, a priori we have $M = F \cdot K_K$ with $F/k$ a finite field extension and $M_K = K_K$, with $[K : F] = 4$, and $\alpha \in \text{Gal}(K/k)$; the left $F$-structure is given by $f \cdot x = \alpha(f)x$. It is easy to see that for all $\alpha$ we obtain isomorphic bimodules (in the sense that the induced hereditary bimodule $k$-algebras are isomorphic, compare [38, 5.1.3]). Thus we can assume that $M = F \cdot K_K$ is equipped with the canonical structure induced by the subring $F \subseteq K$. Then $F$ is the centre (in the sense of [38, 0.5.5]) of the bimodule $M$; thus, we can assume $F = k$.

There is a unique intermediate field $F$ of degree two over $k$, which is of the form $F = k(\alpha)$. This defines, by [37, Lem. 2.6], the simple regular representation

$$S_x = (k^2 \otimes K \overset{1, \alpha}{\longrightarrow} K)$$

with $f(x) = 1$ and $\text{End}(S_x) = F$, hence $e(x) = 1$. By uniqueness of $F$, we have that $x$ is the only rational point $p$ with $e(p) = 1$. Hence $e_x(x) = 2$ and $e_x(p) = 1$ for all other rational points. Since $[k(p) : k] = f(p)$ for all $p$ and $\text{End}(L) = k$ is the field of constants, we deduce then from (8.4) (since $g(\mathcal{H}) = 0$) that besides $x$ there is precisely one additional ramification point $p$, and this must satisfy $f(p) = 2$ and $e(p) = 1$. Thus the ramification vector is $e = (2^1, 2^2)$. Assume additionally $\text{char}(k) \neq 2$. Then, by [37, Thm. 5.1],

$$\Pi(L, \sigma_x) \simeq k(X, Y, Z)/\left(\begin{array}{c}
XY - YX,
XZ - ZX,
YZ +ZY + a_1X^2,
Z^2 + a_0Y^2 - a_0X^2
\end{array}\right)$$

for certain $a_0, a_1, c_0 \in k$. The point $x$ corresponds to the prime element given by the class of $X$ in $R$. The second ramification point $p$ corresponds to a prime element $\pi_p$ in $R$ of degree 2, and which is also irreducible, that is, not a product of two elements of degree 1. It follows that (up to multiplication with a non-zero element from $k$) there is precisely one such prime.
Example 10.4 (Three separation points). Let \( M = \mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}(\sqrt{2}, \sqrt{3})^2 \cdot \mathbb{Q}(\sqrt{2}, \sqrt{3}) \).

\[
\Pi(L, \sigma_x) \simeq k(X, Y, Z)/\left( \begin{array}{l} XY - YX, \ ZX - ZX, \\ YZ + ZY, \ Z^2 + 2Y^2 - 3X^2 \end{array} \right).
\]

Here, the three rational points \( x, y, z \) (corresponding to the variables \( X, Y, Z \), respectively) satisfy \( e(p) = 1 \), and thus \( e_\tau(p) = 2 = s(\mathcal{H}) \). Moreover, by [37, Prop. 7.1] we have

\[ \tau = \sigma_x \sigma_y \sigma_z^{-1}, \]

by which it follows again, that \( x, y, z \) are the only separation points. The ramification vector is \( \mathbf{e} = (2^1, 2^1, 2^1) \).

Example 10.5. The following example shows, that if \( x \) is a separation point, then in general \( e(x) \cdot e^*(x) = 1 \) does not hold. We consider the simple bimodule from [38, Ex. 5.7.3] with skewness \( s(\mathcal{H}) = 4 \): Let \( k = \mathbb{Q} \) and \( F = (\mathbb{Q}(-1/\sqrt{3}, 0)) \) be the skew field of quaternions over \( \mathbb{Q} \) on generators \( i, j \) with relations \( i^2 = -1 = j^2, \ ij = -ji, \ K = \mathbb{Q}(\sqrt{3}, \sqrt{2}) \) and \( M \) be the bimodule \( S(k \oplus K)_F \) with the canonical \( K \)-action, and where the \( F \)-action on \( M \) is defined by

\[
(x, y) \cdot i = \frac{1}{\sqrt{-3}}(\sqrt{2}x + y, x - \sqrt{2}y), \quad (x, y) \cdot j = (y, -x)
\]

for all \( x, y \in K \). By [38, Prop. 5.7.5], each rational point \( x \) satisfies \( e(p) = 2 \) or \( e(x) = 4 \) (and both cases occur), and moreover \( e^*(x) = 1 \); those with \( e(x) = 2 \) are separation with \( e(x) \cdot e^*(x) = 2 \).

Example 10.6 (An inseparable (counter-) example). In [37, Ex. 6.3(2)] we exhibited a purely inseparable example where \( \text{Aut}(\mathbb{X}) = 1 \) is trivial, with \( s(\mathcal{H}) = 2 \) and \( \tau^* = \sigma_x \), where \( x \) is a rational point with \( e(x) = 1 \), moreover not separable, with \( K = \text{End}(S_x) \) commutative, that is \( e^*(x) = 1 \), and moreover, \( \text{Gal}(K/k) = 1 \) trivial. Thus, \( \tau \) acts like the identity on every tube \( U_y \) for \( y \neq x \) and trivially also on \( \text{End}(S_x) \). Thus \( e(x) \cdot e^*(x) \cdot e_\tau(x) = 1 \neq 2 = s(\mathcal{H}) \), so that the special skewness principle in Theorem 6.4 does not hold in this example. We conclude further that \( e^*(x) = 2 \), so that \( s(x) \neq e^*(x) \).

We recall the construction of the example: let \( F \) be any field of characteristic two and \( K = F(u) \) be the field of rational functions in one variable \( u \) over \( F \). We then consider the field extension \( K/k \) of degree four, where \( k = F(u^2) \), and the tame bimodule \( kK_k \) over \( k \). The intermediate field \( k(u^2) \) yields a rational point \( x \) with \( e(x) = 1 \) and \( \text{End}(S_x) \simeq k(u^2) \). (In particular, \( x \) is not a separable point.) Then \( \text{Aut}(\mathbb{X}) \simeq \text{Gal}(K/k) = 1 \) and \( \text{Gal}(\text{End}(S_x)/k) = 1 \). We have

\[
\Pi(L, \sigma_x) \simeq k(X, Y, Z)/\left( \begin{array}{l} XY - YX, \ ZX - ZX, \\ YZ + ZY, \ Z^2 + 2Y^2 - 3X^2 \end{array} \right).
\]

11. The real case: Witt curves

If \( k \) is algebraically closed, then by Corollary 7.6 each noncommutative smooth projective curve over \( k \) is actually commutative. For \( k = \mathbb{C} \) the field of complex numbers it is well-known that the three concepts smooth projective curves \( X \) over \( \mathbb{C} \), algebraic function fields \( K \) in one variable over \( \mathbb{C} \), and compact Riemann surfaces \( S \) are equivalent/dual; here \( K \) is the field of meromorphic functions on \( S \) (which are the holomorphic functions \( \alpha : S \to \mathbb{C}^* \) to the Riemann sphere) and also the function field \( k(X) \). Over the field \( k = \mathbb{R} \) of real numbers there are similar correspondences, where the Riemann surfaces are replaced by the Klein surfaces \( K \). [2, 54]. The real points on \( K \) (if any) form the boundary \( \partial K \). By Harnack’s theorem \( \partial K \) has at most \( g(K) + 1 \) components, called ovals, since they are homeomorphic to \( S^1 \). Each Klein surface \( K \) is of the form \( S/\sigma \), where \( S \) is a compact Riemann surface and \( \sigma : S \to S \)
an antiholomorphic involution, [54, Thm. 1.1]. The ovals are given by the set \( S^\sigma \) of fixed points of \( \sigma \). By a theorem of Weichold [54, p. 56], every \( K \) is, topologically, uniquely determined by a triple \((g, t, s)\), where \( g = g(K) = g(S) \) is the genus of the Riemann surface \( S \), \( t \) is the number of ovals, and \( s = 0 \) if \( S \setminus S' \) is connected, and \( s = 1 \) otherwise. Moreover, precisely the triples \((g, t, s)\) with \( s = 0 \) and \( t \leq g \), or \( s = 1, t \equiv g + 1 \pmod{2} \) and \( 1 \leq t \leq g + 1 \) occur.

Finite dimensional central skew fields over the function field \( K = k(K) \) of a Klein surface \( K \) are, if non-trivial, quaternion skew fields of the form \((\mathbb{Q}_K, \alpha)\), where \( \alpha: K \to \mathbb{S}^2 \) is a skew field if and only if \( \alpha \) is an antiholomorphic (holomorphic) function. This follows easily from Tsen’s theorem, [74]. We recall that such an algebra is of dimension four over \( K \) on generators \( i, j \) and relation \( i^2 = -1, j^2 = \alpha \) and \( ji = -ij \). We will show that these skew fields correspond to the real noncommutative smooth projective curves with \( s(\mathcal{H}) > 1 \).

Witt gave in [74] a global and local description of these quaternion skew fields. On a Klein surface \( K \) we choose an even number of marked points (which we call segmentation points) on each of its ovals. Each oval is then a union of an even number (maybe zero) of closed intervals whose endpoints are the given segmentation points. Additionally we attach to each segment (or complete oval) a sign + or − with the only rule that signs change at segmentation points. For such a configuration there is a meromorphic function \( \alpha: K \to \mathbb{S}^2 \) in the real algebraic function field \( K = k(K) \) which on \( \partial K \) is real-valued and changing the signs (that is, has a zero or a pole of odd order) precisely in the segmentation points and is non-negative on curves with a + sign (we write \( \alpha(x) > 0 \), locally) and non-positive on curves with a − sign (\( \alpha(x) < 0 \); we refer to [74, III.], [2, Thm. 2.4.5]). Moreover, if \( \beta \) is another such function, to the same configuration of segments, then \((\frac{\alpha - 1}{\alpha - 1}) \simeq (\frac{\beta - 1}{\beta - 1})\), [74, p. 10]. It is clear, that conversely any function \( \alpha \in k(K) \) yields a configuration as described. We recall that \( \alpha \) is called positive definite if it never becomes negative on an oval, and definite, if there is no sign-change on an oval. So, \((\frac{\alpha - 1}{\alpha - 1}) \) does not split (it is a skew field) if and only if \( \alpha \) is not positive definite.

11.1 (Local data). Witt [74, p. 10] described (function-theoretically) the local data. We assume that \( \alpha \) is not positive definite. For convenience we already use the notions \( S_x \) and \( e(x) \) in each concluding statement (“Thus…”), which will get a proper meaning only below when we define the notion of a Witt curve.

- If \( x \) is inner then \((\frac{\alpha - 1}{\alpha - 1}) \) splits. Thus \( \operatorname{End}(S_x) = \mathbb{C} \) and \( e(x) = 2 \).
- If \( x \) is a boundary point (\( x = x' \)), then
  - If \( \alpha(x) > 0 \) (that is, \( \alpha \) does not change its positive sign in a neighbourhood of \( x \)) then \((\frac{\alpha - 1}{\alpha - 1}) \) splits. Thus \( \operatorname{End}(S_x) = \mathbb{R} \) and \( e(x) = 2 \).
  - If \( \alpha(x) < 0 \) (that is, \( \alpha \) does not change its negative sign) then \((\frac{\alpha - 1}{\alpha - 1}) \) does not split, and \( x \) is inert in \((\frac{\alpha - 1}{\alpha - 1}) \). Thus \( \operatorname{End}(S_x) = \mathbb{H} \) and \( e(x) = 1 \).
  - Otherwise (if \( \alpha \) changes the sign in \( x \), that is, \( x \) is segmentation) \((\frac{\alpha - 1}{\alpha - 1}) \) does not split. Thus \( \operatorname{End}(S_x) = \mathbb{C} \) and \( e(x) = 1 \).

Thus the interior of a connected closed segment of an oval, whose endpoints are segmentation points, is “coloured” real, in case \( \alpha \) is nonnegative on this segment, and quaternion, in case \( \alpha \) is negative on this segment. If \( \alpha \) is not positive definite then we call \( \alpha: K \to \mathbb{S}^2 \) a Witt function (and \( K \) with the induced configuration, formally, a Witt surface); we will give \((K, \alpha)\) a canonical structure of a noncommutative smooth projective curve below. Some of the following considerations are reformulations of results of Section 7.

Let \( K \) be a Klein surface with function field \( k(K) \) and \( \alpha \) a Witt function. Let \( A = (\frac{\alpha - 1}{\alpha - 1}) \) the corresponding quaternion skew field. Let \( A(x) = A_x \otimes_{O_x} k(x) = \).
$A_x/m_x A_x$ with $k(x) = \mathcal{O}_x/m_x$ be the geometric fibre. If $\alpha > 0$, then $A(x) \simeq M_2(\mathbb{R})$ is split on an oval $O$, if $\alpha < 0$, then $A(x) \simeq \mathbb{H}$ on $O$. The segmentation points are just the ramification points of $A$. There is the injective homomorphism
\begin{equation}
\beta : \text{Br}(K) \to \text{Br}(K)
\end{equation}
of Brauer groups. Here, $\text{Br}(K)$ consists of (classes of) Azumaya algebras $A$. The homomorphism $\beta$ sends the class of $A$ to the class of $A_\xi$, where $\xi$ is the generic point. In the image of $\beta$ are precisely those $A = \left(\frac{\alpha - 1}{K}\right)$, which are unramified on $K$. (We refer to [22].) In other words, if $A$ is unramified on $K$, then it can be equipped with the structure of a unique Azumaya algebra $A$. In [17, Thm. 1.3.7] (also [18]) it is shown that the category $\mathcal{H} = \text{coh}(A)$ of coherent $A$-modules is equivalent to the category $\text{coh}(K, \alpha)$ of $\alpha$-twisted coherent sheaves on $K$.

**Proposition 11.2.** Assume the Witt function $\alpha$ is definite. Let $A$ be the corresponding Azumaya algebra. The category $\mathcal{H} = \text{coh}(A)$ of coherent $A$-modules is a noncommutative smooth projective curve with $s(\mathcal{H}) = 2$.

**Proof.** Since $\alpha$ is not positive definite, by Witt’s theorem [74] the quaternion algebra $A = \left(\frac{\alpha - 1}{K}\right)$ does not split. Since $\alpha$ is definite, $A$ is unramified. The assertion follows from Theorem 7.10. \hfill \Box

**Remark 11.3.** It is even shown [56, Thm. 4.9] that the set of tubes has a structure of a scheme, and the above bijection is an isomorphism $X \to K$. (We also refer to [4].)

We now treat the general (ramified) situation. Denote by $U = K \setminus \{x_1, \ldots, x_n\}$ the Zariski-open unramified locus. The given quaternion algebra $A = \left(\frac{\alpha - 1}{K}\right)$ defines an Azumaya algebra $A_U$ in $\text{Br}(U)$. We remark that the general problem of extending Azumaya algebras is object of active research. We refer to [47].

**Theorem 11.4.** Each Witt function $\alpha : K \to \mathbb{S}^2$ gives rise to a noncommutative smooth projective curve $\mathcal{H}$ of skewness $s(\mathcal{H}) = 2$, and with the following properties:

1. The centre curve is $K$.
2. The function field is $k(\mathcal{H}) = \left(\frac{\alpha - 1}{K}\right)$.
3. Up to an equivalence of categories, $\mathcal{H}$ is uniquely determined by $K$ and the coloured segments of the ovals.

**Proof.** Let $K$ and $\alpha$ be given. Denote by $\mathcal{O} = \mathcal{O}_K$ the structure sheaf of $K$. With $K = k(\mathcal{O})$ let $A$ be the quaternion skew field $\left(\frac{\alpha - 1}{K}\right)$ over $K$. Let $U \subseteq K$ be the unramified locus and $j : U \to K$ the inclusion. The function field of $U$ is $K$. By [5, Cor. 1.9.6] there exists an Azumaya $\mathcal{O}_U$-algebra $A'$ in $A$. By [5, Prop. 1.8.1] there is an $\mathcal{O}$-order $\mathcal{B}$ in $A$ with $j^*\mathcal{B} = A'$. By [5, Prop. 1.8.2] there is a maximal $\mathcal{O}$-order $A$ in $A$ containing $\mathcal{B}$. Then, by Theorem 7.10, $\mathcal{H} = \text{coh}(A)$ is a noncommutative smooth projective curve over $K$. Moreover, conditions (1) and (2) are clearly satisfied.

(3) Let $\alpha' : K \to \mathbb{S}^2$ be another Witt function. It is a part of Witt’s theorem [74, p. 10] that if $\alpha$ and $\alpha'$ yield the same segments with colourings, then the function fields $\left(\frac{\alpha - 1}{K}\right)$ and $\left(\frac{\alpha' - 1}{K}\right)$ are isomorphic. Thus the assertion follows from Theorem 7.11. \hfill \Box

**Definition 11.5.** Let $(K, \alpha)$ be a Klein surface with a Witt function $\alpha : K \to \mathbb{S}^2$. (Recall that this means that $\alpha$ is not positive definite.) We call the noncommutative smooth projective curve constructed in the preceding theorem Witt curve (associated with $(K, \alpha)$).

**Theorem 11.6.** Each noncommutative smooth projective curve $\mathcal{H}$ over $k = \mathbb{R}$ with $s(\mathcal{H}) > 1$ is a Witt curve.
Let $11.10.$ be a Witt curve. We recall that for each $x$ is non-zero and thus injective. There are the natural embeddings $R \hookrightarrow \mathbb{H}$. By the uniqueness part of Theorem 11.4 then $H$ is the Witt curve associated with $(K, \alpha)$.

**Remark 11.7.** By the preceding theorems we have a complete picture of the non-commutative smooth projective curves over the real numbers: those corresponding to the compact (Riemann and) Klein surfaces $(s(H) = 1)$, and the Witt curves $(s(H) = 2)$.

We now apply the skewness principle to Witt curves.

**Theorem 11.8.** Let $H$ be a Witt curve. Then $e_r(x) = 2$ if and only if $x$ is a segmentation point.

In other words: precisely for the segmentation points $\tau$ is acting non-trivially – by complex conjugation – on the corresponding tubes.

**Proof.** This follows from Theorem 7.16, but we give a more direct argument using the local description 11.1 of Witt and the skewness principle. We have $s(H) = 2$. Since $e_r(x)$ is the order of $\tau$ in $\text{Gal}(\text{End}(S_x)/\mathbb{R})$, the case $e_r(x) > 1$ is only possible when $\text{End}(S_x) = \mathbb{C}$. When $\text{End}(S_x) = \mathbb{C}$, then $e^*(x) = 1$, and from the formula $2 = s(H) = e(x) \cdot e^*(x) \cdot e_r(x) = e(x) \cdot e_r(x)$ we get $e_r(x) > 1$ if and only if $e(x) = 1$, that is (by 11.1), $x$ is a segmentation point.

In particular we conclude that over $k = \mathbb{R}$ any even natural number is possible for the number of separation points.

**Corollary 11.9.** Let $H$ be a real noncommutative smooth projective curve. The following are equivalent:

1. The curve is commutative, that is, $s(H) = 1$.
2. The curve $X$ is multiplicity free.
3. Each local skewness $s(x)$ is trivial.
4. All boundary points $x \in X$ (if any) are real.

The equivalence of (1) and (3) is again the Hasse principle, in this case due to Witt [74, I.]. Indeed, if $s(H) = 2$, then there is always a quaternion point $x$, and thus $s(x) = e^*(x) = 2$. These points are the inert points. In contrast to Theorem 9.2, unramified real curves need not be commutative.

11.10. Let $H$ be a Witt curve. The following table summarizes some local data

| point $x$ | $e(x)$ | $e^*(x)$ | $e_r(x)$ | $k(x)$ | $Z(D_x)$ | $D_x$ | $\hat{D}_x$ |
|-----------|--------|----------|----------|--------|----------|-------|-------------|
| inner     | 2      | 1        | 1        | $\mathbb{C}$ | $\mathbb{C}$ | $\mathbb{C}$ | $\mathbb{C}((T))$ |
| real      | 2      | 1        | 1        | $\mathbb{R}$ | $\mathbb{R}$ | $\mathbb{R}$ | $\mathbb{R}(T)$ |
| quaternion| 1      | 2        | 1        | $\mathbb{R}$ | $\mathbb{R}$ | $\mathbb{H}$ | $\mathbb{H}((T))$ |
| segmentation | 1     | 1      | 2        | $\mathbb{R}$ | $\mathbb{C}$ | $\mathbb{C}$ | $\mathbb{C}((T, \sigma))$ |

**Table 1.** Local data of a Witt curve

A Klein surface has the constant field $\mathbb{R}$. We will now determine the constant field of a Witt curve. We recall that for each $x \in X$ we have $R_x/\text{rad}(R_x) \simeq M_{e(x)}(D_x)$, with $D_x = \text{End}(S_x)$. The composition $\text{End}(L) = \mathbb{H}(L, \sigma_x) \to R_x \to R_x/\text{rad}(R_x)$ is non-zero and thus injective. There are the natural embeddings $\mathbb{R} \subseteq M_2(\mathbb{R}) \subseteq M_2(\mathbb{C})$, $\mathbb{C} \subseteq M_2(\mathbb{R}) \subseteq M_2(\mathbb{C})$ and $\mathbb{H} \subseteq M_2(\mathbb{C})$. Via these inclusions we can form the intersection of all the $R_x/\text{rad}(R_x)$ in $M_2(\mathbb{C})$. 
Lemma 11.11 (Field of constants). Let $(\mathcal{H}, L)$ be a Witt curve.

1. $\text{End}(L) = \bigcap_{x \in X} R_x / \text{rad}(R_x)$.
2. Let $r$ and $q$ be the numbers of completely real and quaternion coloured ovals, respectively, and $n = 2m$ the total number of segmentation points. Then

$$\text{End}(L) \cong \begin{cases} C & \text{if } m > 0 \text{ or } r > 0, \\ H & \text{if } m = 0 \text{ and } r = 0. \end{cases}$$

Part (1) also holds for a Klein surface if and only if it has a non-empty boundary.

Proof. If $m = 0$, then, since $s(\mathcal{H}) = 2$, we have $q > 0$. Each real point $x$ satisfies $e(x) = 2$, by the skewness principle, and thus $R_x / \text{rad}(R_x) \cong M_2(\mathbb{R})$. Since $\mathbb{H} \cap M_2(\mathbb{R}) = \mathbb{C}$, the intersection $\bigcap_{x \in X} R_x / \text{rad}(R_x)$ equals $\mathbb{C}$ if $m > 0$ or $r > 0$, and $\mathbb{H}$ in the other cases. In both situations, we can choose a boundary point $x$ with $e(x) = 1$ such that $L_x = R_x / \text{rad}(R_x)$ is $\mathbb{C}$ or $\mathbb{H}$, respectively. It is easy to see that each $h \in \text{End}(L_x)$ can be lifted to $L$, so that there is $f \in \text{End}(L)$ with fibre map $f_x = h$. (Indeed, we can assume $h \neq 0$, hence it is an isomorphism. Forming the pullback $\eta \cdot h$, where $\eta$ is the $S$-universal exact sequence starting in $L$, we use that the resulting sequence is again $S$-universal.) For the additional statement we remark that for a Klein surface without boundary (like the Klein bottle) $\text{End}(L)$ is $\mathbb{R}$, but the intersection is $\mathbb{C}$. □

Proposition 11.12 (Hurwitz formula). Let $\mathcal{H}$ be a Witt curve with $n = 2m$ segmentation points, and let $r$ be the number of completely real coloured ovals. Then

$$g(\mathcal{H}) = \begin{cases} 2g(K) - 1 + m & \text{if } m > 0 \text{ or } r > 0, \\ g(K) & \text{if } m = 0 \text{ and } r = 0. \end{cases}$$

Proof. Let $\mathcal{A}$ be the maximal order such that $\mathcal{H} \cong \text{coh}(\mathcal{A})$. Let $L \in \mathcal{H}$ be the structure sheaf. We have to compute $g(\mathcal{H}) = |\text{Ext}^1_K(L, L) : \text{End}_K(L)|$. By formula (8.4) we have for the Euler characteristics

$$\chi(\mathcal{A}) = 4\chi(K) - n.$$

With $\kappa = \dim_k \text{End}(L)$ we obtain $\kappa(1 - g(\mathcal{H})) = \chi(\mathcal{A}) = 4\chi(K) - n = 4(1 - g(K)) - n$, thus

$$g(\mathcal{H}) = \frac{4}{\kappa} g(K) - \frac{4}{\kappa} + 1 + \frac{2m}{\kappa}.$$  

Now the assertion follows with Lemma 11.11. □

Remark 11.13. (1) Our definition of the genus differs from the definition of the genus of function skew fields in [73], since we always have $g(\mathcal{H}) \geq 0$. The analogues formula obtained in [49] is $4g(K) - 3 + n$, which we would get for $\kappa = 1$. For example, the function fields $\mathbb{H}(T)$ and $\mathbb{C}(T, \sigma)$ have genus $0$ by our definition, but genus $-3$ by the other definitions. But the conditions $g \leq 0$ and $g = 1$ are equivalent for both definitions.

(2) We form a double cover $\pi : K' \to K$ of Klein surfaces as in the proof of Witt’s theorem in [2, Thm. 2.4.5]: $K'$ is obtained by gluing two copies of $K$ together at the closures of the segments, or complete ovals, where $\alpha(x) < 0$. The boundary of $K'$ is then given by two copies of the segments (or ovals), where $\alpha(x) > 0$, and each pair of these segments (if not an entire oval) is bounded by two segmentation points of $K$. The connected components are then ovals, each of which contains precisely zero or two of the segmentation points from $K$. Setting $\alpha' = \pi^*(\alpha) = \alpha \circ \pi$, then $\alpha'$ is positive on the segments. Thus the former segmentation points are not longer segmentation points on $K'$. By construction $k(K') = k(K)(\sqrt{\alpha})$. Since $K'$ is a
Riemann surface only in case \( m = 0 \) and \( r = 0 \), formula (11.2) (for \( H \)) coincides with the Hurwitz equation for the covering \( \pi: K' \to K \) in \([50, \text{Thm. 2}]\).

**Corollary 11.14.** Let \( \mathcal{H} = (K, \alpha) \) be a Witt curve with \( n > 0 \) segmentation points. Then:

- \( g(\mathcal{H}) = 0 \) if and only if \( g(K) = 0 \) (hence \( K \) is the compact disc) and \( n = 2 \).
- \( g(\mathcal{H}) = 1 \) if and only if \( g(K) = 0 \) and \( n = 4 \). \( \square \)

**Example 11.15** (Genus zero). There are two Witt curves of genus zero; we describe the corresponding Witt surfaces, Figure 1.

1. The closed disc \( D_H \), the boundary coloured quaternion. This is the projective spectrum of the graded polynomial ring \( \mathbb{H}[X,Y] \) with central variables \( X, Y \) of degree 1. The function field is given by \( \mathbb{H}(T) \).

2. The closed disc \( D_2 \), with two segmentation points on its boundary. This is the projective spectrum of the graded skew-polynomial algebra \( \mathbb{C}[X; Y, \sigma] \). Accordingly, the function field is \( \mathbb{C}(T, \sigma) \).

**Figure 1.** The Witt curves with \( g(\mathcal{H}) = 0 \): \( D_H \) and \( D_2 \).

**12. Examples II: Real elliptic curves**

In the following examples we treat all real elliptic curves, that is, the Klein and Witt surfaces with \( g(\mathcal{H}) = 1 \). Here, we classify them only topologically, not up to isomorphism, where one has to add a real parameter to each topological case.

**Example 12.1** (Commutative real elliptic curves). There are (up to parameters) three real elliptic curves with \( s(\mathcal{H}) = 1 \), with corresponding Klein surfaces given by the annulus \( A \), the Klein bottle \( K \) and the Möbius band \( M \). We refer to the book \([1]\).

**Example 12.2** (Elliptic Witt curves). As only real elliptic curves with \( s(\mathcal{H}) > 1 \) we have (up to parameters) the corresponding Witt surfaces: the annuli \( A_{R,H} \) and \( A_{H,R} \), where one and both ovals, respectively, are coloured quaternion, the Möbius band \( M_{H} \), with quaternion coloured boundary, and the closed disc \( D_{2,2} \) with four segmentation points on the boundary, Figure 2; there is a moduli parameter \( \lambda > 0 \) involved, so the general case is not as symmetric as in the figure.

**Lemma 12.3.** For the real elliptic curves \( (\mathcal{H}, L) \) Table 2 describes the endomorphism ring of the structure sheaf \( L \), the endomorphism ring of a simple sheaf \( S = S_\varepsilon \) of lowest degree with \( e(x) = 1 \), its degree \( \deg(S) \), and the number of orbits in \( \hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\} \) of the action of \( \text{Aut}(D^0(\mathcal{H})) \) on the slopes. In case of two orbits, the orbits are given by those fractions \( \frac{a}{b} \) (with \( a \in \mathbb{Z}, b \in \mathbb{N} \) coprime) with a even or odd, respectively.
Figure 2. Elliptic Witt curves: $D_{2,2,2}$, $A_{R,H}$, $A_{H,H}$, $M_H$

| End($L$) | $A$ | $M$ | $K$ | $A_{R,H}$ | $A_{H,H}$ | $M_H$ | $D_{2,2,2}$ |
|---------|-----|-----|-----|-----------|-----------|-------|------------|
| End($S$) | $R$ | $R$ | $R$ | $C$       | $H$       | $H$   | $C$        |
| deg($S$) | 1   | 1   | 2   | 1         | 1         | 1     | 1          |
| #orbits | 1   | 1   | 2   | 1         | 1         | 1     | 1          |

Table 2. Data of real elliptic curves

Proof. We get End($L$) from Lemma 11.11. The data for $S$ are easy to see. The object $L$ and $S$ define tubular mutations $\sigma_L$, $\sigma_S$: $D^b(H) \to D^b(H)$, respectively, which on $K_0(H)$ act as follows (with $a := [L]$, $s := [S]$ and $\kappa = \dim_k \text{End}(L)$):

$$(12.1) \quad \sigma_L(y) = y + \frac{\langle a, y \rangle}{\kappa} a, \quad \sigma_S(y) = y + \frac{\langle s, y \rangle}{|\text{End}(S)|} s.$$ 

With $\deg(y) = \frac{1}{\kappa} \langle a, y \rangle$ and $\text{rk}(y) = \frac{1}{\kappa \deg(y)} \langle y, s \rangle$ one obtains like in [36, Lem. 6.1] the numbers of orbits as claimed. (In case $D_{2,2,2}$ we take a simple object $S = S_x$ with $e(x) = 1$, $\deg(S) = 1$ and $\text{End}(S) = \mathbb{C}$.)

12.4 (Fourier-Mukai partners). Let $H$ and $H'$ be noncommutative regular projective curves over $k$. We call $H'$ a Fourier-Mukai partner of $H$, if there is an exact equivalence $D^b(H) \to D^b(H')$.

We recall that, since $H$ is hereditary, $D^b(H)$ is the repetitive category of $H$, that is, $D^b(H) = \bigvee_{n \in \mathbb{Z}} H[n]$, and for $E, F \in H$ we have $\text{Hom}_{D^b(H)}(E[n], F[n]) = \text{Ext}^n_H(E, F)$. In the elliptic case, $H = \bigvee_{q \in \hat{Q}} t_q$, where $t_q$ for each $q \in \hat{Q}$ is the tubular family given by the semistable objects of slope $q$. If $\alpha \in \hat{Q}$, then we define the interval category $H(\alpha)$ as the full subcategory in $D^b(H)$ given as the
additive closure of $\bigcup_{q>\alpha} t_q[-1] \cup \bigcup_{q\leq \alpha} t_q$. Then $H(\alpha) \simeq \text{coh}(X_0)$, where $X_0$ is the elliptic curve parametrizing the tubular family $t_\alpha$, which can be proved like [38, Prop. 8.1.6]. Obviously $D^b(H) = D^b(H(\alpha))$.

From this it follows easily, that if $H$ is elliptic and $D^b(H) \to D^b(H')$ an exact equivalence, then $H'$ is equivalent to $H(\alpha)$ for some $\alpha$.

It was already observed in [43] that the Klein bottle must have a Fourier-Mukai partner different from a Klein bottle. The first part of the following statement is a non-weighted analogue of [35].

**Theorem 12.5.**

(1) The Klein bottle $\mathbb{K}$ (with any parameter) has as a Fourier-Mukai partner a Witt curve given by the annulus $A_{R,\mathbb{K}}$ with two differently coloured ovals (with a suitable parameter).

(2) If $H$ is a noncommutative real elliptic curve, which is neither a Klein bottle, nor an annulus $A_{R,\mathbb{K}}$, then each Fourier-Mukai partner of $H$ is isomorphic to $H$ itself.

**Proof.** (1) A Klein bottle $\mathbb{K}$ has no boundary. Thus all simple sheaves have endomorphism ring $\mathbb{C}$, the complex numbers. On the other hand, the structure sheaf $\mathcal{O}_\mathbb{K}$ (which is stable and of slope 0) has endomorphism ring $\mathbb{R}$. Thus by Proposition 8.4 the subcategory of semistable bundles of slope 0 is parametrized by a noncommutative projective curve $H$ of $g(H) = 1$ with $H \not\simeq \mathbb{K}$, and $H$ is derived-equivalent to $\mathbb{K}$. There must be a simple sheaf $S$ in $H$ with $\text{End}(S) \simeq \mathbb{R}$. By Table 2 the only possibility is then $H = A_{R,\mathbb{K}}$ (with a suitable parameter).

(2) This follows from Table 2. □

The theorem describes a situation where the conclusion of a theorem of Bondal-Orlov [12] does not hold. It also shows that the recent result [48] does not extend to the non-algebraically closed base-fields.

**Corollary 12.6.** The skewness, and thus the function field, of a noncommutative smooth projective curve is no derived invariant. □

**Remark 12.7** (Calabi-Yau). Let $\mathcal{T}$ be a triangulated $k$-category with finite dimensional Hom-spaces and with Serre duality $\text{Hom}_\mathcal{T}(X,Y) = D \text{Hom}_\mathcal{T}(Y, SX)$, where $S$ is an exact autoequivalence of $\mathcal{T}$, the (triangulated) Serre functor. If $S^n \simeq [n]$, the $n$-th suspension functor (with $n \geq 1$ minimal), then $\mathcal{T}$ is called triangulated *Calabi-Yau of (fractional) dimension $\frac{m}{n}$* (we refer to [34]). If $\mathcal{T} = D^b(H)$, with $H$ a noncommutative regular projective curve with Auslander-Reiten translation $\tau$, then $S = \tau \circ [1]$ is the Serre functor of $\mathcal{T}$. If $g(H) = 1$, then the functor $\tau$ is of finite order $p$, and then $S^p \simeq [p]$, that is, $\mathcal{T}$ is Calabi-Yau of dimension $\frac{p}{2}$. The preceding discussion shows that the derived category of the elliptic Witt curve $\mathbb{D}_{2,2,2,2}$ has Calabi-Yau dimension $\frac{2}{3}$; all the others have dimension $\frac{1}{4}$.

13. **Examples III: insertion of weights; some tubular cases**

We conclude the article by discussing very briefly what happens with the $\tau$-multiplicities $e_\tau(x)$ when weights are involved.

**Lemma 13.1.** Let $\mathcal{H}$ be a noncommutative regular projective curve over a field $k$ and $x$ a separable point. Assume that on the tube $\mathcal{U}_x$ the inverse Auslander-Reiten translation acts (as functor) like $\sigma_x$. Let $p > 1$ be a weight and let $\mathcal{H}$ be a weighted noncommutative curve arising from $H$ by insertion of the weight $p$ into $x$. Let $\mathcal{U}_x$ be the corresponding tube of rank $p$. Then the order of $\tau_{\mathcal{U}_x}$ in $\text{Aut}(\mathcal{U}_x/k)$ is $e_\tau(x)^{-1} \cdot p$.

**Proof.** Working with $p$-cycles in $x$ (compare [42]) one sees easily that $(\tau_{\mathcal{U}_x})^p$ acts on $\text{End}(S_x)$ like $\sigma_x^{-1}$ and hence like $\tau_{\mathcal{H}}$. □
Example 13.2. We exhibit two tubular examples over the field $k = \mathbb{R}$ of real numbers. In both cases the underlying non-weighted curve $\mathcal{H}$ is the Witt curve $D_2 \times D_2$ in Figure 1. The weight-points $z$ are drawn in blue, the weight $p(z)$ is given besides. In both cases, one of the weight-points $z$ is also a segmentation point, the other weighted point is not.

We recall that the homogeneous coordinate algebra of $\mathcal{H}$ is given by $\mathbb{C}[X;Y,\sigma]$. The points $x$ and $y$ associated with the prime elements $X$ and $Y$, respectively, are the two segmentation points. Moreover, we know that $\tau^{-} = \sigma_x \circ \sigma_y$. So $\tau^{-}$ is acting on the tube $\mathcal{U}_x$ like $\sigma_x$, so the preceding lemma can be applied.

(a) Weight sequence $(3,3)$. (See Figure 3.) The least common multiple is $\bar{p} = 3$. The second weight-point is real. (The case when the second weight-point is quaternion is similar.) On the tube associated with the weighted segmentation point $x$, the Auslander-Reiten translation $\tau_{\mathcal{H}(3,3)}$ has order $p(x) \cdot e_{\tau}(x) = 3 \cdot 2 = 6$. It follows, that the Calabi-Yau dimension is $\frac{6}{2} = 3$ (and not $\frac{6}{3} = 2$).

(b) We consider the same situation, but with weight sequence $(2,4)$. (See Figure 3.) The least common multiple is $\bar{p} = 4$. The weighted segmentation point $x$ has weight $p(x) = 2$. Hence the order of $\tau_{\mathcal{H}(2,4)}$ on $\mathcal{U}_x$ is $p(x) \cdot e_{\tau}(x) = 2 \cdot 2 = 4$, thus the same order as on the other exceptional tube. Therefore the Calabi-Yau dimension is $\frac{4}{2} = 2$.

For further, similar examples we refer to [38, Table A.5].

Remark 13.3 (The real tubular zoo). Let $k = \mathbb{R}$ be the field of real numbers. Allowing weightings, there are (up to parameters) 39 real noncommutative smooth projective curves of (orbifold) genus one:

- Non-weighted. 8 elliptic curves. With centre $\mathbb{R}$: the Klein bottle $\mathbb{K}$, the Möbius band $\mathbb{M}$ (the oval coloured real or quaternion), the annulus $\mathbb{A}$ (there are three possibilities to colour the two ovals). The disc $D_{2,2,2,2}$ with four segmentation points. With centre $\mathbb{C}$: The torus $T$.
- Weighted. 31 tubular curves. Those 27 with centre $\mathbb{R}$ are shown in the tables [38, Appendix A]; with centre $\mathbb{C}$ there are the 4 types $(2,4,4), (2,3,6), (3,3,3)$ and $(2,2,2,2)$. 17 of these have $s(\mathcal{H}) = 1$ (these are the parabolic (or flat, that is, of curvature zero) 2-orbifolds shown in [68, Thm. 13.3.6]), and 22 have $s(\mathcal{H}) = 2$.

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