Response Analysis of Asymmetric Monostable Harvesters Driven by Color Noise and Band-Limited Noise

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Abstract: In this paper, we investigate the response of asymmetric potential monostable energy harvesters (MEHs) excited by color noise and band-limited noise. The motivation for this study is that environmental vibrations always have the characteristic of randomness, and it is difficult to modulate a perfectly symmetric MEH. For the excitation of exponentially correlated color noise, the moment differential equation was applied to evaluate the output performance of the asymmetric potential MEHs. Numerical and theoretical analyses were carried out to investigate the influence of noise intensity and internal system parameters on the output power of the system. Our results demonstrate that the output performance of the asymmetric MEH decreases with the increase in the correlation time, which determines the character of the color noise. On the contrary, the increase in the asymmetric degree enhances the output power of the asymmetric MEH subjected to color noise. For the band-limited noise excitation, numerical simulation is undertaken to consider the response of the asymmetric MEHs, and outcomes indicate that the frequency bandwidth and center frequency have a significant influence on the output performance. Regarding the asymmetric potential, its appearance leads the MEHs to generate higher output power at lower frequencies and this phenomenon is more obvious with the increase in the degree of asymmetry. Finally, we observed that the characteristics of the response bandwidth of asymmetric MEHs subjected to band-limited noise excitation are similar to the response under harmonic excitation.

Keywords: monostable harvester; asymmetric potential; color noise; band-limited noise

1. Introduction

Vibrational energy harvesting is a green energy technology that can transduce the vibrational energy existing in the surroundings into available electrical energy. This technology has great potential application value as it has a promising ability to supply electricity for low-power consumption wireless sensors and portable electronics. There are three transmission mechanisms applied to convert vibrational energy into electricity, namely, the piezoelectric [1], electromagnetic [2], and electrostatic [3] energy harvesters. Due to the merits of high energy density and easy fabrication in miniaturization, piezoelectric energy harvesters have garnered a great deal of attention. Initially, linear energy harvesters were designed as different structures and then optimized for enhancing energy harvesting efficiency. However, linear energy harvesters only perform well near their resonance frequencies, and a slight change in the excitation frequency will decrease their performance.

To address these problems, a large number of studies have focused on broadening the frequency band of response by applying the mechanical and magnetic coupling approaches [4]. Particularly, the investigations of monostable [5], bistable [6], and tristable [7] energy harvesters have attracted a lot of attention. As an example, Stanton et al. [8] proposed a nonlinear monostable energy harvester (MEH) by applying a permanent magnet end mass that interacted with the field of oppositely poled stationary magnets. By adjusting the nonlinear magnetic forces between the magnets, both hardening and softening
responses can be obtained for enhancing the energy harvesting performance. Fan et al. [9] presented an MEH by introducing symmetric magnetic attraction to a piezoelectric cantilever beam and a pair of stoppers to confine the maximum deflection of the beam. Under excitation with an amplitude of 3 m/s², a 54% increase in the operating bandwidth and a 253% increase in the magnitude of output power were obtained compared to its linear counterpart. Due to the snap-through characteristics, bistable oscillators were widely applied in energy harvesting for the reason that they could generate considerably more energy in a wider frequency range. By utilizing the device proposed by Moon and Holmes [10], Erturk et al. [6,11] proposed a bistable energy harvester (BEH) and established the model by introducing piezoelectric coupling and Kirchhoff laws. Numerical, theoretical, and experimental data all indicated that the BEH can achieve chaotic and large-amplitude interwell oscillation, thus resulting in an 800% increase in the power amplitude. By introducing two rotatable external magnets to the energy harvesting systems, Zhou et al. [12] proposed a BEH and pointed out that the system could cover a broad low-frequency range of 4–22 Hz by changing the magnet orientation. Zou et al. [13] proposed a new magnetically coupled bistable piezoelectric energy harvesting approach for underwater applications. The bistable harvesters were also designed and optimized by applying the post-buckled beam [14], flexure hinge mechanism [15], inner stress of composite materials [16], as well as multiple degrees of freedom (DOF) systems [17]. To further improve the energy harvesting performance, the tristable [18], quad-stable [19], and quin-stable [20] piezoelectric energy harvesters have also been investigated extensively.

In the environment, the vibration has the characteristic of randomness, and the energy always distributes in a wide frequency range. Therefore, the investigation of piezoelectric energy harvesters subjected to random excitations has great significance. Numerical, experimental, and theoretical methods have all been applied to evaluate the output of the nonlinear harvesters. For example, Litak et al. [21] numerically studied the response of a BEH excited by stationary Gaussian white noise. Simulated data showed that the BEH exhibited the phenomenon of stochastic resonance which can be applied to optimize the output power when knowing the variance of the excitation. Lin et al. [22] experimentally studied the output of a magnetically coupled piezoelectric cantilever under the excitation of a 1/f vibration spectrum, and indicated that an increase of 50% in output voltage was observed when compared with its linear counterpart. Theoretically, Jiang et al. [23] approximately determined the output performance of a nonlinear piezoelectric energy harvester excited by Gaussian white noise excitations by an equivalent linearization method and numerical simulation demonstrated the effectiveness of the results. Xu et al. [24] considered the response of a piezoelectric MEH by utilizing the stochastic averaging method. Agreements between the analytical results and Monte Carlo simulations verified the effectiveness of the proposed technique. Applying the Fokker–Plank–Kolmogorov (FPK) equation, Daqaq [25] derived the exact joint probability density function of the response of a uni-modal electromagnetic harvester under Gaussian white noise excitation and showed that the nonlinearities in the stiffness do not provide any enhancement over the linear harvesters.

Much effort has been devoted to evaluating the output performance of multistable energy harvesters with perfectly symmetric potential energy functions. However, it is difficult or even impossible to achieve a nonlinear energy harvester with perfectly symmetric potentials. Halvorsen [26] considered an asymmetric quartic potential in a BEH and indicated that the asymmetric potential has a great influence on the output. He et al. [27] studied the asymmetric potential characterized by Halvorsen [28] and showed that the asymmetric potential has a positive effect on the MEH and a negative effect on the BEH. More recently, Zhou et al. [29] investigated the nonlinear dynamics of an asymmetric tristable harvester and demonstrated that the power depended on the asymmetry. Wang et al. [30] studied the influence of the asymmetric potential on BEHs and indicated that the performance can be enhanced by inclining the system with a certain bias angle. Furthermore, Wang et al. [31,32] investigated the response of monostable, bistable, and
tristable piezoelectric energy harvesters under Gaussian white noise. Although the asymmetry in energy harvesters has been investigated extensively, the influence of asymmetric potentials on the output performance of MEHs under color noise and band-limited noise is still an open issue.

Therefore, this paper focuses on the influence of asymmetric potential energy functions on the response of monostable energy harvesters (MEHs) subjected to color noise and band-limited noise. The method of moment differential equation was applied to determine the response of MEHs under color noise. The influence of system parameters on the output performance of the asymmetric MEHs was studied and numerical results agree well with the theoretical outcomes. For the band-limited noise, numerical investigations are undertaken and the influence of center frequency, frequency bandwidth, and the degree of asymmetry on the response was considered. The paper is organized as follows: in Section 2, the dynamic model of the asymmetric MEH is presented. Theoretical and numerical investigations of the influence of asymmetric potentials on MEH subjected to color noise are presented in Section 3 and the response of asymmetric MEH under band-limited noise excitation is considered in Section 4. In the final Section 5, the conclusion is provided.

2. Model of the Asymmetric Monostable Energy Harvester (MEH)

Figure 1 depicts the schematic plot of the piezoelectric monostable energy harvester (MEH) composed of a stainless steel layer, two piezoelectric layers at the roots, a tip magnet, and two external rotational magnets. By adjusting the magnet polarity and system parameters \( d, h, \) and \( \vartheta \), a monostable energy harvesting system could be achieved. In the experiments, an MEH with a perfectly symmetric potential energy function is difficult or even impossible to obtain due to the asymmetry in materials and structures. Therefore, the MEH obtained is always with an asymmetric potential energy function. Based on the Hamilton principle and Lagrange equation, the dynamic model of the asymmetric potential MEH can be expressed as follows:

\[
\begin{align*}
    m\ddot{z}^{\prime}(t) + c\dot{z}^{\prime}(t) + k_1z + k_2z^2 + k_3z^3 - \theta \ddot{\vartheta}(t) & = F(t) \\
    C_p\ddot{\vartheta}(t) + \frac{\tau(t)}{\varrho} + \theta \ddot{\vartheta}(t) & = 0 
\end{align*}
\]

(1)

in which \( m \) is the equivalent mass, \( c \) is the equivalent damping, \( \theta \) is the equivalent electromechanical coupling coefficient, \( C_p \) is the equivalent capacitance, and \( R \) is the load resistance. \( k_1, k_2, \) and \( k_3 \) are the coefficients of linear, quadratic, and cubic forces, respectively. \( z, \varpi, \) and \( \vartheta \) are the tip displacement of the cantilever, the output voltage across the load resistance, and the time, respectively. In addition, (\( \cdot \)'\) refers the derivate with respect to time \( t \).

Figure 1. Schematic plot of the asymmetric monostable energy harvester (MEH).
In order to simplify the analysis process, we introduce the variables $x = \frac{z}{l_0}, t = \frac{C_p t_0}{\pi}$ to obtain the following non-dimensional model.

$$\begin{align*}
\dot{x}(t) + 2\xi \dot{x}(t) + x(t) + \beta x^2(t) + \delta x^3(t) - \kappa^2 v(t) &= \eta(t) \\
\dot{v}(t) + \alpha v(t) + \ddot{x}(t) &= 0
\end{align*}$$

where $l_0$ and $\omega_0$ are the length scale and resonance frequency. In Equation (2), $x, v,$ and $\eta(t)$ refer to the dimensionless displacement, voltage, and ambient random excitations, respectively, which can be modeled by color noise and band-limited noise. Additionally, $\xi$ is the damping ratio, $\alpha$ is the time constant, and $\kappa^2$ is the dimensionless electromechanical coupling coefficient. $\beta$ and $\delta$ are the coefficients of dimensionless quadratic and cubic nonlinear forces, respectively. Furthermore, the upper dot is the derivate of dimensionless time $t$. For the non-dimensional model, the average output power in a certain time range could be calculated as:

$$P = \alpha E\left[v^2\right]$$

3. Response of the Asymmetric MEHs under Color Noise

In this section, the response of asymmetric MEHs under color noise was considered by the method of moment differential equation. For the color noise, it can be modeled with zero mean and an exponential correlation function expressed as follows:

$$\langle \eta(t)\eta(s) \rangle = \frac{D}{\tau} \exp\left[-\frac{|t-s|}{\tau}\right]$$

where $D$ is the noise intensity and $\tau$ is the correlation time. In this case, $\eta(t)$ can be obtained by applying a first-order low-pass filter passing Gaussian white noise as follows:

$$\dot{\eta}(t) = -\frac{1}{\tau} \eta(t) + \frac{1}{\tau} n(t)$$

where $n(t)$ is the Gaussian white noise with a mean value of zero and noise intensity of $D$.

3.1. Moment Differential Equation Analysis

For the convenience of calculation, we define $x_1 = x, x_2 = \dot{x}, x_3 = v, x_4 = \eta(t)$ and then the model of the MEH excited by color noise can be expressed as follows:

$$\begin{align*}
\frac{dx_1}{dt} &= x_2 \\
\frac{dx_2}{dt} &= -2\xi x_2 - x_1 - \beta x_1^2 - \delta x_1^3 + \kappa^2 x_3 + x_4 \\
\frac{dx_3}{dt} &= -\alpha x_3 - x_2 \\
\frac{dx_4}{dt} &= -\frac{1}{\tau} x_4 + \frac{1}{\tau} n(t)
\end{align*}$$

Based on the theory of moment differential equation [33], the first-order statistical moments can be obtained as follows:

$$\frac{dE(X_1)}{dt} = E(X_2)$$

$$\frac{dE(X_2)}{dt} = -2\xi E(X_2) - E(X_1) - \beta E\left(X_1^2\right) - \delta E\left(X_1^3\right) + \kappa^2 E(X_3) + E(X_4)$$

$$\frac{dE(X_3)}{dt} = -\alpha E(X_3) - E(X_2)$$

$$\frac{dE(X_4)}{dt} = -\frac{1}{\tau} E(X_4)$$
In the meantime, the second-order statistical moments are expressed as:

\[
\frac{dE(X_1^2)}{dt} = 2E(X_1)E(X_2) \tag{11}
\]

\[
\frac{dE(X_1X_2)}{dt} = E(X_1^2) - 2\xi E(X_1X_2) - E(X_2) - \beta E(X_1^3) - \delta E(X_2) + \kappa^2 E(X_3) + E(X_1X_4) \tag{12}
\]

\[
\frac{dE(X_2^2)}{dt} = -4\xi E(X_2^2) - 2E(X_1X_2) - 2\beta E(X_1^2X_2) + 2\delta E(X_1X_2X_2) + 2\kappa^2 E(X_2X_3) + 2E(X_2X_4) \tag{13}
\]

\[
\frac{dE(X_1^3)}{dt} = -2\alpha E(X_3^2) - 2E(X_2X_3) \tag{14}
\]

\[
\frac{dE(X_1X_3)}{dt} = -\alpha E(X_1X_3) - E(X_1X_2) + E(X_2X_3) \tag{15}
\]

\[
\frac{dE(X_1X_4)}{dt} = E(X_2X_4) - \frac{1}{\tau} E(X_1X_4) \tag{16}
\]

\[
\frac{dE(X_2X_4)}{dt} = -\frac{1}{\tau} E(X_2X_4) - 2\xi E(X_2X_4) - E(X_1X_4) - \beta E(X_1^2X_4) - \delta E(X_1X_4) \tag{17}
\]

\[
+\kappa^2 E(X_3X_4) + E(X_4^2) \quad \frac{dE(X_3X_4)}{dt} = -\alpha E(X_3X_4) - E(X_2X_4) - \frac{1}{\tau} E(X_3X_4) \tag{18}
\]

\[
\frac{dE(X_2^2)}{dt} = -2\frac{1}{\tau} E(X_2X_4) + \frac{1}{2}D \tag{19}
\]

By setting the first and second-order statistical moments from Equation (7) to Equation (19) to be zero, we can obtain a series of algebraic equations. Based on the theory of the Gaussian closure technique [33], the third and fourth statistical moments could be expressed by the moments of the first two orders. Therefore, by using the expressions and denoting $\chi = E(X_1^2), z = E(X_2^2), l = E(X_1), y = E(X_2^2), m = E(X_1X_4)$, we can finally obtain the following algebraic equations characterizing the output performance:

\[
l + \beta \chi + 2\delta l \chi = 0 \tag{20}
\]

\[
y - \chi - 2\beta l \chi - 3\delta \chi^2 - \kappa^2 z + m = 0 \tag{21}
\]

\[
-y - 2\xi y - 2\alpha \kappa^2 z + \frac{1}{\tau} m = 0 \tag{22}
\]

\[
\kappa^2 z - y + 2\xi \alpha z + z + 2\beta \lambda z + 3\delta \lambda z + \kappa^2 z - \frac{1}{\tau \alpha + 1} m = 0 \tag{23}
\]

\[
-\frac{1}{\tau} m - 2\xi \frac{m}{\tau} - m - 2\beta ml + 3\delta m \chi - \frac{\kappa^2}{\tau \alpha + 1} m + \frac{D}{\tau} = 0 \tag{24}
\]

To consider the influence of system parameters on the output performance of the asymmetric potential MEH, the parameters are set to be $\xi = 0.015$, $\beta = 1$, $\delta = 1$, $\kappa^2 = 0.003$, $\alpha = 0.5$, $\tau = 2$, and $D = 0.05$, unless other values are specially mentioned.

### 3.2. Theoretical and Numerical Analysis

In the numerical simulations, the data length of the noise is set to be 16,000 with a sampling frequency of 5 Hz. Then, the target moments are obtained by averaging five independent simulations. Figure 2a numerically and theoretically illustrates the influence of noise intensity on the mean value of response displacement, mean square voltage, and output power of the asymmetric MEH. It is seen that the mean value of response displacement decreases with an increase in noise intensity. Due to the type of asymmetric potential induced by the quadratic coefficient, it is noted that the mean value of response displacement shows a negative value. For a positive quadratic coefficient, the mean value
of response displacement will increase with the increase in noise intensity. Thus, the absolute value of mean displacement increases with the increase in the absolute value of the quadratic coefficient. The mean square voltage and output power exhibit the same increasing trend with an increase in the noise intensity. When it comes to the damping ratio, Figure 2b demonstrates that its increase results in the decrease in the absolute value of mean displacement, thus reducing the mean square voltage and average power output of the asymmetric MEH. Furthermore, it should be noticed the simulated data agree well with the theoretical outcomes.

![Figure 2](image_url)

**Figure 2.** Influence of (a) noise intensity and (b) damping ratio on the mean displacement, mean square voltage, and power output.

The influences of the time constant ratio and electromechanical coupling coefficient on the response of asymmetric MEH are depicted in Figure 3a,b, respectively. It is observed that the variation of the time constant ratio does not affect the response of displacement. The mean square value of output voltage decreases with the increase in the time constant ratio, while the average output power first increases and then decreases with an increase in the value of the time constant ratio. Therefore, there is an optimum time constant ratio for the asymmetric MEH to produce maximum output power. From Figure 3b, one can notice that that the increase in the electromechanical coupling coefficient has a negative effect on the output performance of the asymmetric MEH, decreasing the absolute mean value of response displacement, mean square voltage, and average power output.

Regarding the influence of quadratic and cubic coefficients on the mean displacement, mean square voltage, and power output of the asymmetric MEH, the numerical and theoretical results are illustrated in Figure 4a,b. It can be seen that the quadratic and cubic coefficients have a reverse effect on the output performance. For the quadratic coefficient, its increase results in a rise in the absolute value of mean displacement; thus, the mean square voltage output and average output power are all increasing with the increasing trend in the quadratic coefficient. Therefore, the appearance of asymmetric potential in MEH enhances the output performance, and the increase in the degree of asymmetry makes the average power much higher. The reason for this phenomenon is that the increase in the degree of asymmetry makes the nonlinear restoring force have a smaller value on the left of the equilibrium position. On the contrary, the increase in the cubic coefficient leads to higher values of the nonlinear restoring forces on both sides of the equilibrium position, thus decreasing the mean displacement, mean square voltage, and power output.
Influence of (a) time constant ratio and (b) electromechanical coupling coefficient on the mean displacement, mean square voltage, and power output.

Furthermore, Figure 5 explains the influence of correlation time on the response of asymmetric potential MEH. It is seen that the mean displacement increases with an increase in the correlation time, while the mean square voltage and average power output decreases. In other words, the increase in correlation time has a negative effect on the output performance of the asymmetric MEH. Of note, in Figures 2–5 some outliers arise, which may be due to the short data length applied and the low simulation number for averaging. Increasing the data length and the simulation number for averaging could improve the fit of the simulation results with the theoretical results.
Figure 4. Influence of (a) quadratic coefficient and (b) cubic coefficient on the mean displacement, mean square voltage, and power output.

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Figure 5. Influence of correlation time on the mean displacement, mean square voltage, and power output.

In order to investigate the response of asymmetric MEH subjected to color noise more specifically, Figure 6 illustrates the typical nonlinear restoring forces and potential energy functions of the asymmetric MEHs with different values of the quadratic coefficient: 0, 1.0, 1.5, and 1.9. The results showed that the increase in the quadratic coefficient results causes the potential energy functions to have smaller values on the left side of the equilibrium position. Under color noise with an intensity of 0.05, the time history of voltage responses of four asymmetric MEHs with different values of the quadratic coefficient are shown in Figure 7a–d. With the increase in the quadratic coefficient, the output voltage increases, and the corresponding average output power values are 0.0720, 0.0839, 0.1081, and 0.1511. Figure 8 shows the probability density function (PDF) of the dimensionless displacement response of the four asymmetric MEHs. For the symmetric MEH with a quadratic coefficient equal to 0, symmetric unimodal distribution is obtained, showing that the oscillation of the MEH concentrates around the equilibrium position symmetrically. For the asymmetric potential MEHs, the system exhibits a smaller probability density of oscillation around the equilibrium position and the right side, with larger nonlinear restoring forces. Meanwhile, for the left side, where nonlinear restoring forces show smaller values, a larger probability of oscillation is achieved.

Figure 6. (a) Nonlinear restoring forces and (b) potential functions of the asymmetric MEHs for different values of quadratic coefficients.
shown in Figure 7a–d. With the increase in the quadratic coefficient, the output voltage increases, and the corresponding average output power values are 0.0720, 0.0839, 0.1081, and 0.1511. Figure 8 shows the probability density function (PDF) of the dimensionless displacement response of the four asymmetric MEHs. For the symmetric MEH with a quadratic coefficient equal to 0, symmetric unimodal distribution is obtained, showing that the oscillation of the MEH concentrates around the equilibrium position symmetrically. For the asymmetric potential MEHs, the system exhibits a smaller probability density of oscillation around the equilibrium position and the right side, with larger nonlinear restoring forces. Meanwhile, for the left side, where nonlinear restoring forces show smaller values, a larger probability of oscillation is achieved.

Figure 6. (a) Nonlinear restoring forces and (b) potential functions of the asymmetric MEHs for different values of quadratic coefficients.

Figure 7. Voltage responses of asymmetric MEHs with different values of quadratic coefficients.

Figure 8. PDF Probability density function of dimensionless displacement for different values of quadratic coefficient.

4. Response of the Asymmetric MEHs under Band-limited Noise

In this section, we focus on considering the response characteristics of asymmetric potential MEHs subjected to band-limited noise excitation. Before discussing the response under band-limited noise excitation, the response under harmonic excitation is emphasized. Under an up-sweep frequency excitation with a level of 0.2, the voltage responses of four MEHs with different values of the quadratic coefficient are shown in Figure 9a. The MEHs could generate a larger voltage in a wider frequency range. With the increase in asymmetric degree, the response bandwidth was slightly enlarged around the cut-off frequency (COF-1) while the response amplitude was slightly decreased. For a down-sweep frequency excitation, the response voltage of the four MEHs was depicted in Figure 9b. The results show that the response frequency band and cut-off frequency (COF-2) shifts to the lower frequencies with an increase in the asymmetry. This analysis demonstrates that the response of asymmetric MEHs exhibits the property of multiple solutions between COF-1 and COF-2. Under band-limited noise excitation, the property of multiple solutions of MEHs cannot be achieved, and consequently, the response frequency is mainly distributed in lower frequencies.
In the numerical simulations, the band-limited noise used for excitation was obtained by passing the Gaussian white noise with certain noise intensity through a band-pass filter. For a certain frequency bandwidth of 0.05, Figure 10 illustrates the influence of center frequency on the average output power of the asymmetric MEH ($\beta = 1$) excited by band-limited noise with different noise intensities. Given a certain center frequency, the asymmetric MEH generates a larger average output power when excited by band-limited noise with larger noise intensity. With the center frequency shifting to high frequency, the average output power of the asymmetric MEH first increases and then decreases, and achieves its maximum value around the COF-2. On the whole, it is observed that the asymmetric MEH outputs larger power in a wide frequency range when subjected to band-limited noise with larger noise intensity, and the characteristic is similar to the response of MEH under harmonic excitation. For a certain center frequency of 0.18, which is near the COF-2, the effect of frequency bandwidth on the output performance of the asymmetric MEH excited by band-limited noise with different noise intensity is diagramed in Figure 11. For a lower excitation intensity of 0.01, the average output power quickly increases to a certain value and then basically remains unchanged with the center frequency shifting to high frequency. With the increase in noise intensity, the trend of output power increasing with the increase in frequency bandwidth becomes slow. Of course, the MEH generates larger output power for larger noise intensity.

**Figure 9.** Influence of quadratic coefficient on the response frequency band of the asymmetric MEH under harmonic excitation with a level of 0.2: (a) up-sweep frequency; (b) down-sweep frequency.

**Figure 10.** Influence of center frequency on the output performance of the asymmetric MEH excited by band-limited noise with different noise intensity ($\Delta = 0.05$).
In the numerical simulations, the band-limited noise used for excitation was obtained by passing the Gaussian white noise with certain noise intensity through a band-pass filter. The simulated results are illustrated in Figure 12. For a given frequency bandwidth, the average output power increases first and then decreases with the center frequency moving to high frequencies, and the average output achieved maximum values near the COF-2. When the frequency bandwidth is 0.025, the average output power is obtained when the center frequency equals 0.175. When with the frequency bandwidth increases to 0.05, 0.1, and 0.15, maximum power is achieved for the center frequency of 0.19, 0.216, and 0.23, respectively. Therefore, the increase in frequency bandwidth makes the system generate considerable power in a wider frequency range and leads to an increase in the center frequency at which the maximum output is reached. Furthermore, Figure 13 illustrates the effect of frequency bandwidth on the output performance of the asymmetric MEH excited by band-limited noise with different center frequencies. When the center frequency is small and far from the COF-2, the output power shows an increasing trend with the increase in frequency bandwidth. For a center frequency of 0.2 near the COF-2, the power output increases quickly to a certain value and then maintains a slower growth trend with the increase in frequency bandwidth. For a center frequency of 0.4, much larger than COF-2, the generated power first remains small and then quickly increases when the frequency bandwidth is about 0.25. The reason for this phenomenon is that the increase in frequency bandwidth leads to the frequency range around COF-2 being contained in the excitation frequency. Furthermore, it was observed that the power remains unchanged when the frequency bandwidth further increases.

Figure 11. Influence of frequency bandwidth on the output performance of the asymmetric MEH excited by band-limited noise with different noise intensity (CF = 0.18).

For a noise intensity of 0.05, we investigated the influence of center frequency on the output of the asymmetric MEH excited by band-limited noise with different frequency bandwidths, and the simulated results are illustrated in Figure 12. For a given frequency bandwidth, the average output power increases first and then decreases with the center frequency moving to high frequencies, and the average output achieved maximum values near the COF-2. When the frequency bandwidth is 0.025, the average output power is obtained when the center frequency equals 0.175. When with the frequency bandwidth increases to 0.05, 0.1, and 0.15, maximum power is achieved for the center frequency of 0.19, 0.216, and 0.23, respectively. Therefore, the increase in frequency bandwidth makes the system generate considerable power in a wider frequency range and leads to an increase in the center frequency at which the maximum output is reached. Furthermore, Figure 13 illustrates the effect of frequency bandwidth on the output performance of the asymmetric MEH excited by band-limited noise with different center frequencies. When the center frequency is small and far from the COF-2, the output power shows an increasing trend with the increase in frequency bandwidth. For a center frequency of 0.2 near the COF-2, the power output increases quickly to a certain value and then maintains a slower growth trend with the increase in frequency bandwidth. For a center frequency of 0.4, much larger than COF-2, the generated power first remains small and then quickly increases when the frequency bandwidth is about 0.25. The reason for this phenomenon is that the increase in frequency bandwidth leads to the frequency range around COF-2 being contained in the excitation frequency. Furthermore, it was observed that the power remains unchanged when the frequency bandwidth further increases.

Figure 12. Influence of center frequency on the output performance of the asymmetric MEH excited by band-limited noise with different frequency bandwidth.
In addition, we consider the effect of asymmetric degree on the output performance of the MEH, and Figure 14 explains the influence of center frequency on the output power of the MEH excited by band-limited noise with different values of the quadratic coefficient. For all MEHs, the output performance is first enhanced and then degraded with the increase in center frequency and the phenomenon was similar to the response under harmonic excitation. For a symmetric MEH with $\beta$ equaling 0, the average power reaches a maximum value at center frequency 0.22. When the asymmetric degree $i$ for $\beta$ increases to 1, 1.5, and 1.9, the maximum power is achieved at a frequency of 0.211, 0.191, and 0.166, respectively. In other words, the increase in asymmetric degree leads the MEH to generate a larger output power at lower frequencies and this condition is similar to the response under down-sweep frequency harmonic excitation.

5. Conclusions

This paper presents a detailed analysis of the response of asymmetric potential monostable energy harvesters (MEHs) excited by exponentially correlated color noise and band-limited noise excitations. For the excitation of exponentially correlated color noise, the mean square voltage and output power of the asymmetric potential MEHs are determined theoretically by the method of moment differential equation. Numerical and theoretical results are in good agreement and the results show that the increase in noise intensity and degree of asymmetry can enhance the power output. However, the increase in parameters such as damping ratio, electromechanical coupling coefficient, cubic coefficient, and corre-
lation time results in less power generated. Regarding the time constant ratio, there is an optimum value to maximize the output performance of the asymmetric MEHs.

Numerical simulations were carried out to consider the response of asymmetric MEHs under band-limited noise excitation. Our simulated data illustrate that the degree of asymmetry influences the output power significantly. With the increase in the degree of asymmetry, the MEH tends to generate a larger output power at lower frequency ranges, and this phenomenon is similar to the response of asymmetric MEHs under down-sweep frequency harmonic excitation. Regarding the center frequency, the asymmetric MEHs generate considerable average output power near the resonance frequency, which is determined by the nonlinear restoring forces. For the frequency bandwidth, its appropriate increase could enhance the performance and its influence will be small for a large frequency bandwidth. This work provides insight into strategies to design MEHs accounting for the response characteristics of random excitations.

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