Two New Piggybacking Designs with Lower Repair Bandwidth

Zhengyi Jiang, Hanxu Hou, Yungsiang S. Han, Patrick P. C. Lee, Bo Bai, and Zhongyi Huang

Abstract

Piggybacking codes are a special class of MDS array codes that can achieve small repair bandwidth with small sub-packetization by first creating some instances of an \((n, k)\) MDS code, such as a Reed-Solomon (RS) code, and then designing the piggyback function. In this paper, we propose a new piggybacking coding design which designs the piggyback function over some instances of both \((n, k)\) MDS code and \((n, k')\) MDS code, when \(k \geq k'\). We show that our new piggybacking design can significantly reduce the repair bandwidth for single-node failures. When \(k = k'\), we design a piggybacking code that is MDS code and we show that the designed code has lower repair bandwidth for single-node failures than all existing piggybacking codes when the number of parity node \(r = n - k \geq 8\) and the sub-packetization \(\alpha < r\).

Moreover, we propose another piggybacking codes by designing \(n\) piggyback functions of some instances of \((n, k)\) MDS code and adding the \(n\) piggyback functions into the \(n\) newly created empty entries with no data symbols. We show that our code can significantly reduce repair bandwidth for single-node failures at a cost of slightly more storage overhead. In addition, we show that our code can recover any \(r + 1\) node failures for some parameters. We also show that our code has lower repair bandwidth than locally repairable codes (LRCs) under the same fault-tolerance and redundancy for some parameters.

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Index Terms

Piggybacking, MDS array code, repair bandwidth, storage overhead, sub-packetization, fault tolerance

I. INTRODUCTION

Maximum distance separable (MDS) array codes are widely employed in distributed storage systems that can provide the maximum data reliability for a given amount of storage overhead. An \((n, k, \alpha)\) MDS array code encodes a data file of \(k\alpha\) data symbols to obtain \(n\alpha\) coded symbols with each of the \(n\) nodes storing \(\alpha\) symbols such that any \(k\) out of \(n\) nodes can retrieve all \(k\alpha\) data symbols, where \(k < n\) and \(\alpha \geq 1\). The number of symbols stored in each node, i.e., the size of \(\alpha\), is called sub-packetization level. We usually employ systematic code in practical storage systems such that the \(k\alpha\) data symbols are directly stored in the system and can be retrieve without performing any decoding operation. Note that Reed-Solomon (RS) codes are typical MDS codes with \(\alpha = 1\).

In modern distributed storage systems, node failures are common and single-node failures occur more frequently than multi-node failures. When a single-node fails, it is important to repair the failed node with the repair bandwidth (i.e., the total amount of symbols downloaded from other surviving nodes) as small as possible. It is shown in \cite{3} that we need to download at least \(\frac{\alpha}{n-k}\) symbols from each of the \(n-1\) surviving nodes in repairing one single-node failure. MDS array codes with minimum repair bandwidth for any single-node failure are called minimum storage regenerating (MSR) codes. There are many constructions of MSR codes to achieve minimum repair bandwidth in the literature \cite{4–11}. However, the sub-packetization level \(\alpha\) of high-code-rate (i.e., \(\frac{k}{n} > 0.5\)) MSR codes \cite{9} is exponential in parameters \(n\) and \(k\). A nature question is that can we design new MDS array codes with both sub-packetization and repair bandwidth as small as possible.

Piggybacking codes \cite{12, 13} are a special class of MDS array codes that have small sub-packetization and small repair bandwidth. The essential idea behind the piggybacking codes \cite{13} is as follows: by creating \(\alpha\) instances of \((n,k)\) RS codes and adding carefully well-designed linear combinations of some symbols as so-called piggyback functions from one instance to the others, we can reduce the repair bandwidth of single-node failure. Some further studies of piggybacking codes are in \cite{14–19}.
The existing piggybacking codes are designed based on some instances of an \((n, k)\) RS codes. The motivation of this paper is to significantly reduce the repair bandwidth by designing new piggybacking codes. In this paper, we propose new piggybacking codes by first creating some instances of both \((n, k)\) MDS code and \((n, k')\) MDS code, and then designing the piggyback functions that can significantly reduce repair bandwidth for single-node failures, when \(k \geq k'\).

A. Contributions

Our main contributions are as follows.

- First, we propose a new type of piggybacking coding design which is designed by both \((n, k)\) MDS code and \((n, k')\) MDS code, where \(k \geq k'\). We give an efficient repair method for any single-node failure for our piggybacking coding design and present an upper bound on repair bandwidth. When \(k > k'\), our codes are non-MDS codes and we show that our codes have much less repair bandwidth than that of existing piggybacking codes at a cost of slightly more storage overhead. The essential reason of repair bandwidth reduction of our codes is that we have more design space than that of existing piggybacking codes.

- Second, when \(k = k'\), we design new piggybacking codes that are MDS codes based on the proposed design. We show that the proposed piggybacking codes with \(k = k'\) have lower repair bandwidth than that of the existing piggybacking codes when \(r = n - k \geq 8\) and the sub-packetization is less than \(r\).

- Third, we design another piggybacking codes by designing and adding the \(n\) piggyback functions into the \(n\) newly created empty entries with no data symbols. We show that our piggybacking codes can tolerate any \(r + 1\) node failures under some conditions. We also show that our codes have lower repair bandwidth than that of both Azure-LRC \([20]\) and optimal-LRC \([21]\) under the same fault-tolerance and the same storage overhead for some parameters.

B. Related Works

Many works are designed to reduce the repair bandwidth of erasure codes which we discuss as follows.

1) Piggybacking Codes: Rashmi et al. present the seminal work of piggybacking codes \([12]\), \([13]\) that can reduce the repair bandwidth for any single-data-node with small sub-packetization. Another piggybacking codes called REPB are proposed \([15]\) to achieve lower repair bandwidth
for any single-data-node than that of the codes in [13]. Note that the piggybacking codes in [13], [15] only have small repair bandwidth for any single-data-node failure, while not for parity nodes. Some follow-up works [16], [18], [19] design new piggybacking codes to obtain small repair bandwidth for both data nodes and parity nodes. Specifically, when $r = n - k \leq 10$ and sub-packetization is $r - 1 + \sqrt{r - 1}$, OOP codes [16] have the lowest repair bandwidth for any single-node failure among the existing piggybacking codes; when $r \geq 10$ and sub-packetization is $r$, the codes in [19] have the lowest repair bandwidth for any single-node failure among the existing piggybacking codes.

Note that all the existing piggybacking codes are designed over some instances of an $(n, k)$ MDS code. In this paper, we design new piggybacking codes that are non-MDS codes over some instances of both $(n, k)$ MDS code and $(n, k')$ MDS codes with $k > k'$ that have much lower repair bandwidth for any single-node failures at a cost of slightly larger storage overhead.

2) MDS Array Codes: Minimum storage regenerating (MSR) codes [3] are a class of MDS array codes with minimum repair bandwidth for a single-node failure. Some exact-repair constructions of MSR codes are investigated in [4–6], [8], [10], [11], [22], [23]. The sub-packetization of high-code-rate MSR codes [5], [8], [10], [11], [23] is exponentially increasing with the increasing of parameters $n$ and $k$. Some MDS array codes have been proposed [24–29] to achieve small repair bandwidth under the condition of small sub-packetization; however, they either only have small repair bandwidth for data nodes [24–27], [30] or require large field sizes [28], [29].

3) Locally Repairable Codes: Locally repairable codes (LRCs) [20], [31] are non-MDS codes that can achieve small repair bandwidth for any single-node failure with sub-packetization $\alpha = 1$ by adding some local parity symbols. Consider the $(n, k, g)$ Azure-LRC [20] that is employed in Windows Azure storage systems, we first create $n - k - g$ global parity symbols by encoding all $k$ data symbols, divide the $k$ data symbols into $g$ groups and then create one local parity symbol for each group, where $k$ is a multiple of $g$. In the $(n, k, g)$ Azure-LRC, we can repair any one symbol except $n - k - g$ global parity symbols by locally downloading the other $k/g$ symbols in the group. Optimal-LRC [21], [32–34] is another family of LRC that can locally repair any one symbol (including the global parity symbols). One drawback of optimal-LRC is that existing constructions [21], [32–34] can not support all the parameters and the underlying field size should be large enough. In this paper, we propose new piggybacking codes by designing and adding the $n$ piggyback functions into the $n$ newly created empty entries with no data symbols that are also non-MDS codes and we show that our piggybacking codes have lower.
repair bandwidth when compared with Azure-LRC [20] and optimal-LRC under the same storage overhead and fault-tolerance, for some parameters.

The remainder of this paper is organized as follows. Section II presents two piggybacking coding designs. Section III shows new piggybacking codes with \( k = k' \) based on the first design. Section IV shows another new piggybacking codes based on the second design. Section V evaluates the repair bandwidth for our piggybacking codes and the related codes. Section VI concludes the paper.

II. TWO PIGGYBACKING DESIGNS

In this section, we first present two piggybacking designs and then consider the repair bandwidth of any single-node failure for the proposed piggybacking codes.

A. Two Piggybacking Designs

Our two piggybacking designs can be represented by an \( n \times (s+1) \) array, where \( s \) is a positive integer, the \( s+1 \) symbols in each row are stored in a node, and \( s+1 \leq n \). We label the index of the \( n \) rows from 1 to \( n \) and the index of the \( s+1 \) columns from 1 to \( s+1 \). Note that the symbols in each row are stored at the corresponding node.

In the following, we present our first piggybacking design. In the piggybacking design, we first create \( s \) instances of \( (n,k) \) MDS codes plus one instance of \( (n,k') \) MDS codes and then design the piggyback functions, where \( k \geq k' > 0 \). We describe the detailed structure of the design as follows.

1) First, we create \( s+1 \) instances of MDS codes over finite field \( \mathbb{F}_q \), the first \( s \) columns are the codewords of \( (n,k) \) MDS codes and the last column is a codeword of \( (n,k') \) MDS codes, where \( k' = k - h \), \( h \in \{0,1,\ldots,k-1\} \) and \( s + n + k + 2 \leq h \). Let \( \{a_i = (a_{i,1},a_{i,2},\ldots,a_{i,k})^T\}_{i=1}^s \) be the \( sk \) data symbols in the first \( s \) columns and \( (a_{i,1},a_{i,2},\ldots,a_{i,k},P_1^Ta_i,\ldots,P_s^Ta_i)^T \) be codeword \( i \) of the \( (n,k) \) MDS codes, where \( i = 1,2,\ldots,s \) and \( P_j^T = (\eta^{j-1},\eta^2(j-1),\ldots,\eta^{k(j-1)}) \) with \( j = 1,2,\ldots,r \), \( r = n - k \) and \( \eta \) is a primitive element of \( \mathbb{F}_q \). Let \( \{b = (b_1,b_2,\ldots,b_{k'})^T\} \) be the \( k' = k - h \) data symbols in the last column and \( (b_1,b_2,\ldots,b_{k'},Q_1^Tb,\ldots,Q_{h+r}^Tb)^T \) be a codeword of an \( (n,k') \) MDS code, where \( Q_j^T = (\eta^{j-1},\eta^2(j-1),\ldots,\eta^{k'(j-1)}) \) with \( j = 1,2,\ldots,h + r \). Note that the total number of data symbols in this code is \( sk + k' \).
2) Second, we add the *piggyback functions* of the symbols in the first $s$ columns to the parity symbols in the last column, in order to reduce the repair bandwidth. We divide the piggyback functions into two types: (i) piggyback functions of the symbols in the first $k' + 1$ rows in the first $s$ columns; (ii) piggyback functions of the symbols in the last $r + h - 1$ rows in the first $s$ columns. Fig. [I] shows the structure of two piggyback functions. For the first type of the piggyback functions, we add symbol $a_{i,j}$ (the symbol in row $j$ and column $i$) to the parity symbol $Q^T_{2+((j-1)s+i-1) \mod (h+r-1)} b$ (the symbol in row $k-h+2+((j-1)s+i-1) \mod (h+r-1)$ in the last column), where $i \in \{1, 2, \ldots, s\}$ and $j \in \{1, 2, \ldots, k-h+1\}$. For the second type of the piggyback functions, we add the symbol in row $j$ and column $i$ with $i \in \{1, 2, \ldots, s\}$ and $j \in \{k-h+2, \ldots, k+r\}$ to the parity symbol $Q^T_{t_{i,j}} b$ (the symbol in row $k-h+t_{i,j}$ in the last column), where

$$t_{i,j} = \begin{cases} 
  i + j - k + h, & \text{if } i + j \leq n \\
  i + j - n + 1, & \text{if } i + j > n
\end{cases} \quad \text{(1)}$$

The first piggybacking design described above is denoted by $C(n, k, s, k')$. When $h = 0$, we have $k = k'$ and the created $s+1$ instances are codewords of $(n, k)$ MDS codes. We will show the repair bandwidth in Section [III].

We present the second piggybacking design as follows. We create $s$ instances (in the first $s$ columns) of $(n, k)$ MDS codes over finite field $F_q$ and one additional empty column of length $n$, i.e., there is no data symbol in the last column, all the $n = k + r$ entries in the last columns are piggyback functions. We design the $k + r$ piggyback functions in the last column as follows. For $i \in \{1, 2, \ldots, s\}$ and $j \in \{1, 2, \ldots, k+r\}$, we add the symbol in row $j$ and column $i$ to the symbol in row $\hat{t}_{i,j}$ in the last column, where

$$\hat{t}_{i,j} = \begin{cases} 
  i + j, & \text{if } i + j \leq n \\
  i + j - n, & \text{if } i + j > n
\end{cases} \quad \text{(2)}$$

We denote the second piggybacking design by $C(n, k, s, k' = 0)$, the last parameter $k' = 0$ denotes that there is not data symbol in the last column. We will discuss the repair bandwidth in Section [IV].

Recall that in our first piggybacking design, the number of symbols to be added with piggyback functions in the last column is $h-1+r$ and $h \geq s - r + 2$ such that we can see that any two
Fig. 1. The structure of the first piggybacking design \( C(n, k, s, k') \), where \( k' > 0 \).

symbols used in computing both types of piggyback functions are from different nodes. Since

\[
k - h + t_{i,j} = \begin{cases} 
  k - h + i + j - k + h = i + j > j, & \text{when } i + j \leq n \\
  k - h + i + j - n + 1 < j, & \text{when } i + j > n
\end{cases}
\]

the symbol in row \( j \) with \( j \in \{k - h + 2, k - h + 3, \ldots, k + r\} \) and column \( i \) with \( i \in \{1, 2, \ldots, s\} \) is not added to the symbol in row \( j \) and column \( s + 1 \) in computing the second type of piggyback functions. In our second piggybacking design, since

\[
\hat{t}_{i,j} = \begin{cases} 
  i + j > j, & \text{when } i + j \leq n \\
  i + j - n < j, & \text{when } i + j > n
\end{cases}
\]

the symbol in row \( j \) with \( j \in \{1, 2, \ldots, k + r\} \) and column \( i \) with \( i \in \{1, 2, \ldots, s\} \) is not added to the symbol in row \( j \) and column \( s + 1 \) in computing the piggyback functions.

It is easy to see the MDS property of the first piggybacking design \( C(n, k, s, k') \). We can retrieve all the other symbols in the first \( s \) columns from any \( k \) nodes (rows). By computing all the piggyback functions and subtracting all the piggyback functions from the corresponding parity symbols, we can retrieve all the symbols in the last column. Fig. 2 shows an example of \( C(8, 6, 1, 3) \).
Note that the piggyback function of the second piggybacking design is different from that of the first piggybacking design. In the following of the section, we present the repair method for the first piggybacking design. We will show the repair method for the second piggybacking design in Section IV.

For $i \in \{2, 3, \ldots, h + r\}$, let $p_{i-1}$ be the piggyback function added on the parity symbol $Q_i^T b$ and $n_{i-1}$ be the number of symbols in the sum in computing piggyback function $p_{i-1}$. According to the design of piggyback functions, we have two set of symbols that are used in computing the $h + r - 1$ piggyback functions. The first set contains $s(k - h + 1)$ symbols (in the first $k - h + 1$ rows and in the first $s$ columns) and the second set contains $s(h + r - 1)$ symbols (in the last $h + r - 1$ rows and in the first $s$ columns). We have that the total number of symbols used in computing the $h + r - 1$ piggyback functions is $s(k + r)$, i.e.,

$$
\sum_{i=1}^{h+r-1} n_i = s(k + r).
$$

In our first piggybacking design, the number of symbols used in computing each piggyback function is given in the next lemma.

**Lemma 1.** In the first piggybacking design $C(n, k, s, k')$ with $k' > 0$, the number of symbols

| Node 1 | $a_{1,1}$ | $b_1$ |
|--------|-----------|-------|
| Node 2 | $a_{1,2}$ | $b_2$ |
| Node 3 | $a_{1,3}$ | $b_3$ |
| Node 4 | $a_{1,4}$ | $Q_1^T b$ |
| Node 5 | $a_{1,5}$ | $Q_2^T b + a_{1,1} + P_2^T a_1$ |
| Node 6 | $a_{1,6}$ | $Q_3^T b + a_{1,2} + a_{1,5}$ |
| Node 7 | $P_1^T a_1$ | $Q_4^T b + a_{1,3} + a_{1,6}$ |
| Node 8 | $P_2^T a_1$ | $Q_5^T b + a_{1,4} + P_1^T a_1$ |

Fig. 2. An example of $C(n, k, s, k')$, where $(n, k, s, k') = (8, 6, 1, 3)$. **Note**
used in computing the piggyback function \( p_r \) is

\[
\begin{align*}
n_r &= s + \left\lfloor \frac{s(k-r+1)}{h+r-1} \right\rfloor, \forall 1 \leq \tau \leq (k-h+1)s - \left\lceil \frac{(k-h+1)s}{h+r-1} \right\rceil(h+r-1) \\
n_r &= s + \left\lfloor \frac{s(k-h+1)}{h+r-1} \right\rfloor, \forall (k-h+1)s - \left\lceil \frac{(k-h+1)s}{h+r-1} \right\rceil(h+r-1) < \tau < h+r. (4)
\end{align*}
\]

Proof. In the design of the piggyback function, we add the symbol in row \( j \) and column \( i \) for \( i \in \{1, 2, \ldots, s\} \) and \( j \in \{1, 2, \ldots, k-h+1\} \) \((k-h+1)s \) symbol in the first set) to the symbol in row \( k-h+2 + (((j-1)s+i-1) \mod (h+r-1)) \) \((\text{piggyback function } p_1+((j-1)s+i-1) \mod (h+r-1)))\) in the last column. Therefore, we can see that the symbols in row \( j \) and column \( i \) are added to \( p_1 \) for all \( i \in \{1, 2, \ldots, s\} \), \( j \in \{1, 2, \ldots, k-h+1\} \) and \((j-1)s+i-1\) is a multiple of \( h+r-1 \). Note that

\[
\{(j-1)s+i-1| i=1,2,\ldots, s, \ j=1,2,\ldots, k-h+1\} = \{0,1,\ldots, (k-h+1)s-1\},
\]

we need to choose the symbol in row \( j \) and column \( i \) for \( i \in \{1, 2, \ldots, s\} \), \( j \in \{1, 2, \ldots, k-h+1\} \) such that \( \eta \) is a multiple of \( h+r-1 \) for all \( \eta \in \{0, 1, \ldots, (k-h+1)\} \). The number of symbols in the first set used in computing \( p_1 \) is \([\frac{(k-h+1)s}{h+r-1}]\). Given integer \( \tau \) with \( 1 \leq \tau \leq h+r-1 \), we add the symbol in row \( j \) and column \( i \) for \( i \in \{1, 2, \ldots, s\} \), \( j \in \{1, 2, \ldots, k-h+1\} \) such that \( \eta-\tau+1 \) is a multiple of \( h+r-1 \) for all \( \eta \in \{0, 1, \ldots, (k-h+1)s-1\} \) to \( p_r \). The number of symbols in the first set used in computing \( p_r \) is \([\frac{(k-h+1)s}{h+r-1}]\) if \((k-h+1)s+1 \leq \tau \leq h+r-1\) and \([\frac{(k-h+1)s}{h+r-1}]\) if \((k-h+1)s+1 > \tau \leq (k-h+1)s + \left\lceil \frac{(k-h+1)s}{h+r-1} \right\rceil(h+r-1)\). Therefore, the number of symbols in the first set used in computing \( p_r \) is \([\frac{(k-h+1)s}{h+r-1}]\) if \( 1 \leq \tau \leq (k-h+1)s + \left\lceil \frac{(k-h+1)s}{h+r-1} \right\rceil(h+r-1)\) and \([\frac{(k-h+1)s}{h+r-1}]\) if \( h+r-1 \geq \tau \). For the \((h+r-1)s\) symbols in the second set, we add the symbol in row \( j \) and column \( i \) with \( i \in \{1, 2, \ldots, s\} \) and \( j \in \{k-h+2, \ldots, k+r\} \) to the symbol in row \( k-h+t_{i,j} \) \((\text{piggyback function } p_{t_{i,j}})\) in the last column, where \( t_{i,j} \) is given in Eq. (1). Consider the piggyback function \( p_1 \), i.e., \( t_{i,j} = 2 \). When \( i = 1 \), according to Eq. (1), only when \( j = k+r \) for \( j \in \{k-h+2, \ldots, k+r\} \), we can obtain \( t_{i,j} = 2 \). When \( i = 2 \), according to Eq. (1), only when \( j = k+r-1 \) for \( j \in \{k-h+2, \ldots, k+r\} \), we can obtain \( t_{i,j} = 2 \). Similarly, for any \( i \) with \( i \in \{1, 2, \ldots, s\} \), only when \( j = k+r+1-i \) for \( j \in \{k-h+2, \ldots, k+r\} \), we can obtain \( t_{i,j} = 2 \). Since \( h \geq s-r+2 \), we have \( j = k+r+1-i \geq k+r+1-s > k-h+2 \), which is within \( \{k-h+2, \ldots, k+r\} \). In other words, for any \( i \) with \( i \in \{1, 2, \ldots, s\} \), we can find one and only one \( j \) with \( j \in \{k-h+2, \ldots, k+r\} \) such that \( t_{i,j} = 2 \). The number of symbols in the second set used in computing \( p_1 \) is \( s \). Similarly, we can show that the number of symbols in the second set used in computing \( p_r \) is \( s \) for
all $\tau = 1, 2, \ldots, h + r - 1$. Therefore, the total number of symbols used in computing $p_r$ is $n_r = s + \left\lceil \frac{(k-h+1)s}{h+r-1} \right\rceil$ for $\tau = 1, 2, \ldots, (k-h+1)s-\left\lceil \frac{(k-h+1)s}{h+r-1} \right\rceil(h+r-1)$ and $n_r = s + \left\lceil \frac{(k-h+1)s}{h+r-1} \right\rceil$ for $\tau = (k-h+1)s-\left\lceil \frac{(k-h+1)s}{h+r-1} \right\rceil(h+r-1)+1, (k-h+1)s-\left\lceil \frac{(k-h+1)s}{h+r-1} \right\rceil(h+r-1)+2, \ldots, h+r-1$. \hfill \Box

Next lemma shows that any two symbols in one row in the first $s$ columns are used in computing two different piggyback functions.

**Lemma 2.** In the first piggybacking design, if $s + 2 \leq h + r$, then the symbol in row $j$ in column $i_1$ and the symbol in row $j$ in column $i_2$ are used in computing two different piggyback functions, for any $j \in \{1, 2, \ldots, k + r\}$ and $i_1 \neq i_2 \in \{1, 2, \ldots, s\}$.

**Proof.** When $j \in \{1, 2, \ldots, k-h+1\}$, we add the symbol in row $j$ and column $i_1$ to the symbol in row $k-h+2+(((j-1)s+i_1-1) \mod (h+r-1))$ in the last column. Similarly, the symbol in row $j$ and column $i_2$ is added to the symbol in row $k-h+2+(((j-1)s+i_2-1) \mod (h+r-1))$ in the last column. Suppose that the two symbols in row $j$ and columns $i_1, i_2$ are added to the same piggyback function, we obtain that $(((j-1)s+i_1-1) \mod (h+r-1)) = (((j-1)s+i_2-1) \mod (h+r-1))$, i.e., $i_1 = i_2 \mod (h+r-1)$, which contradicts to $i_1 \neq i_2 \in \{1, 2, \ldots, s\}$ and $s + 2 \leq h + r$.

When $j \in \{k-h+2, k-h+2, \ldots, k+r\}$, we add two symbols in row $j$ column $i_1$ and row $j$ column $i_2$ to the symbol in the last column in row $k-h+t_{i_1,j}$ and row $k-h+t_{i_2,j}$, respectively, where $i_1 \neq i_2 \in \{1, 2, \ldots, s\}$ and

$$t_{i,j} = \begin{cases} 
    i + j - k + h, & \text{if } i + j \leq n \\
    i + j - n + 1, & \text{if } i + j > n.
\end{cases}$$

Suppose that the two symbols in row $j$ and columns $i_1, i_2$ are added to the same piggyback function, we obtain that $t_{i_1,j} = t_{i_2,j}$. If $i_1 + j \leq n$ and $i_2 + j \leq n$, we have that $i_1 = i_2$ which contradicts to $i_1 \neq i_2$. If $i_1 + j \leq n$ and $i_2 + j > n$, we have that $i_1 + r + h - 1 = i_2$ which contradicts to $i_1 \neq i_2 \in \{1, 2, \ldots, s\}$ and $s + 2 \leq h + r$. Similarly, we can obtain a contradiction if $i_1 + j > n$ and $i_2 + j \leq n$. If $i_1 + j > n$ and $i_2 + j > n$, we have that $i_1 = i_2$ which contradicts to $i_1 \neq i_2$. Therefore, in our first piggybacking design, any two symbols in the same row are not used in computing the same piggyback function. \hfill \Box

**B. Repair Process**

In the first piggybacking design, suppose that node $f$ fails, we present the repair procedure of node $f$ as follows, where $f \in \{1, 2, \ldots, k + r\}$.
We first consider that \( f \in \{1, 2, \ldots, k-h+1\} \), each of the first \( s \) symbols \( \{a_{1,f}, a_{2,f}, \ldots, a_{s,f}\} \) stored in node \( f \) is used in computing one piggyback function and we denote the corresponding piggyback function associated with symbol \( a_{i,f} \) by \( p_{i,f} \), where \( i = 1, 2, \ldots, s \) and \( t_{i,f} \in \{1, 2, \ldots, h+r-1\} \). We download \( k-h \) symbols in the last column from nodes \( \{1, 2, \ldots, k-h+1\} \setminus \{f\} \) to recover \( s+1 \) symbols \( b_f, Q_{t_{i,f}+1}^Tb, Q_{t_{i+1,f}+1}^Tb, \ldots, Q_{t_{s,f}+1}^Tb \) when \( f \in \{1, 2, \ldots, k-h\} \), or \( Q_i^Tb, Q_{t_{i,f}+1}^Tb, Q_{t_{i+1,f}+1}^Tb, \ldots, Q_{t_{s,f}+1}^Tb \) when \( f = k-h+1 \), according to the MDS property of the last instance. By Lemma 2, any two symbols in one row are used in computing two different piggyback functions. The piggyback function \( p_{i,f} \) is computed by \( n_{t_{i,f}} \) symbols, where one symbol is \( a_{i,f} \) and the other \( n_{t_{i,f}} - 1 \) symbols are not in node \( f \) (row \( f \)). Therefore, we can repair the symbol \( a_{i,f} \) in node \( f \) by downloading the parity symbol \( Q_{t_{i,f}+1}^Tb + p_{i,f} \) and \( n_{t_{i,f}} - 1 \) symbols which are used to compute \( p_{i,f} \) except \( a_{i,f} \), where \( i = 1, 2, \ldots, s \). The repair bandwidth is \( k-h+\sum_{i=1}^{s} n_{t_{i,f}} \) symbols.

When \( f \in \{k-h+2, k-h+3, \ldots, n\} \), each of the first \( s \) symbols stored in node \( f \) is used in computing one piggyback function and we denote the corresponding piggyback function associated with the symbol in row \( f \) and column \( i \) by \( p_{i,f} \), where \( i \in \{1, 2, \ldots, s\} \) and \( t_{i,f} \in \{1, 2, \ldots, h+r-1\} \). We download \( k-h \) symbols in the last column from nodes \( \{1, 2, \ldots, k-h\} \) to recover \( s+1 \) symbols \( Q_{f-k+h}^Tb, Q_{t_{i+1,f}+1}^Tb, Q_{t_{i+2,f}+1}^Tb, \ldots, Q_{t_{s,f}+1}^Tb \), according to the MDS property of the last instance. Recall that any symbol in row \( f \) in the first \( s \) columns is not used in computing the piggyback function in row \( f \). We can recover the last symbol \( Q_{f-k+h}^Tb + p_{f-k+h-1} \) stored in node \( f \) by downloading \( n_{f-k+h-1} \) symbols which are used to compute the piggyback function \( p_{f-k+h-1} \). Recall that any two symbols in one row are used in computing two different piggyback functions by Lemma 2. The piggyback function \( p_{i,f} \) is computed by \( n_{t_{i,f}} \) symbols, where one symbol is in row \( f \) column \( i \) and the other \( n_{t_{i,f}} - 1 \) symbols are not in node \( f \) (row \( f \)). We can repair the symbol in row \( f \) and column \( i \), for \( i \in \{1, 2, \ldots, s\} \), by downloading symbol \( Q_{t_{i,f}+1}^Tb + p_{i,f} \) and \( n_{t_{i,f}} - 1 \) symbols which are used to compute \( p_{i,f} \) except the symbol in row \( f \) and column \( i \). The repair bandwidth is \( k-h+n_{f-k+h-1}+\sum_{i=1}^{s} n_{t_{i,f}} \) symbols.

Consider the repair method of the code \( C(8,6,1,3) \) in Fig. 2. Suppose that node 1 fails, we can first download 3 symbols \( b_2, b_3, Q_1^Tb \) to obtain the two symbols \( b_1, Q_2^Tb \), according to the MDS property. Then, we download the following 2 symbols

\[
Q_2^Tb + a_{1,1} + P_2^Ta_1, P_2^Ta_1
\]
to recover $a_{1,1}$. The repair bandwidth of node 1 is 5 symbols. Similarly, we can show that the repair bandwidth of any single-node failure among nodes 2 to 4 is 5 symbols.

Suppose that node 5 fails, we can download the 3 symbols $b_1, b_2, b_3$ to obtain $Q^T_2 b, Q^T_3 b$, according to the MDS poverty. Then, we download the 2 symbols $a_{1,1}, P^T_2 a_1$ to recover $Q^T_2 b + p_1$. Finally, we download the 2 symbols $Q^T_3 b + p_2, a_{1,2}$ to recover $a_{1,5}$. The repair bandwidth of node 5 is 7 symbols. Similarly, we can show that the repair bandwidth of any single-node failure among nodes 6 to 8 is 7 symbols.

C. Average Repair Bandwidth Ratio of Code $C(n, k, s, k')$, $k' > 0$

Define the average repair bandwidth of data nodes (or parity nodes or all nodes) as the ratio of the summation of repair bandwidth for each of $k$ data nodes (or $r$ parity nodes or all $n$ nodes) to the number of data nodes $k$ (or the number of parity nodes $r$ or the number of all nodes $n$). Define the average repair bandwidth ratio of data nodes (or parity nodes or all nodes) as the ratio of the average repair bandwidth of $k$ data nodes (or $r$ parity nodes or all $n$ nodes) to the number of data symbols.

In the following, we present an upper bound of the average repair bandwidth ratio of all $n$ nodes, denoted by $\gamma^{\text{all}}$, for the proposed codes $C(n, k, s, k')$ when $k' > 0$.

**Theorem 3.** When $k' > 0$, the average repair bandwidth ratio of all $n$ nodes, $\gamma^{\text{all}}$, of codes $C(n, k, s, k')$, is upper bounded by

$$\gamma^{\text{all}} \leq \frac{(u + s)^2(h + r - 1)}{(k + r)(sk + k - h)} + \frac{k - h + s}{sk + k - h},$$

where $u = \left\lceil \frac{s(k-h+1)}{h+r-1} \right\rceil$.

**Proof.** Suppose that node $f$ fails, where $f \in \{1, 2, \ldots, n\}$, we will count the repair bandwidth of node $f$ as follows. Recall that the symbol in row $f$ and column $i$ is used to compute the piggyback function $p_{t_i,f}$, where $f \in \{1, 2, \ldots, n\}$ and $i \in \{1, 2, \ldots, s\}$. Recall also that the number of symbols in the sum in computing piggyback function $p_{t_i,f}$ is $n_{t_i,f}$. When $f \in \{1, 2, \ldots, k - h + 1\}$, according to the repair method in Section II-B, the repair bandwidth of node $f$ is $(k-h+\sum_{i=1}^{s} n_{t_i,f})$ symbols. When $f \in \{k - h + 2, \ldots, n\}$, according to the repair method in Section II-B, the
repair bandwidth of node \( f \) is \((k - h + n_{f-k+h-1} + \sum_{i=1}^{s} n_{t_i,f})\) symbols. The summation of the repair bandwidth for each of the \( n \) nodes is

\[
\sum_{f=1}^{k-h+1} (k - h + \sum_{i=1}^{s} n_{t_i,f}) + \sum_{f=k-h+2}^{k+r} (k - h + n_{f-k+h-1} + \sum_{i=1}^{s} n_{t_i,f})
\]

\[= (k + r)(k - h) + \sum_{f=1}^{k+r} (\sum_{i=1}^{s} n_{t_i,f}) + \sum_{f=k-h+2}^{k+r} n_{f-k+h-1}. \tag{5}\]

Next, we show that

\[
\sum_{f=1}^{k+r} (\sum_{i=1}^{s} n_{t_i,f}) = \sum_{i=1}^{h+r-1} n_i^2. \tag{6}\]

Note that \(\sum_{i=1}^{k+r} (\sum_{i=1}^{s} n_{t_i,f})\) is the summation of the repair bandwidth for each of the \((k + r)s\) symbols in the first \( s \) columns. The \((k + r)s\) symbols are used to compute the \( h + r - 1 \) piggyback functions and each symbol is used for only one piggyback function. For \( i = 1, 2, \ldots, h + r - 1 \), the piggyback function \( p_i \) is the summation of the \( n_i \) symbols in the first \( s \) columns and can recover any one of the \( n_i \) symbols (used in computing \( p_i \)) with repair bandwidth \( n_i \) symbols. Therefore, the summation of the repair bandwidth for each of the \( n_i \) symbols (used in computing \( p_i \)) is \( n_i^2 \). In other words, the summation of the repair bandwidth for each of the \((k + r)s\) symbols in the first \( s \) columns is the summation of the repair bandwidth for each of all the \((k + r)s\) symbols used for computing all \( h + r - 1 \) piggyback functions, i.e., Eq. (6) holds.

By Eq. (5), we have \(\sum_{f=k-h+2}^{n} n_{f-k+h-1} = \sum_{i=1}^{h+r-1} n_i = s(k + r)\). By Eq. (4), we have \(n_i \leq u + s, \forall i \in \{1, 2, \ldots, h + r\}\), where \(u = \lceil s(k-h+1) \rceil \). According to Eq. (5) and Eq. (6), we have

\[
\gamma^{all} = \frac{(k + r)(k - h) + \sum_{i=1}^{h+r-1} n_i^2}{(k + r)(sk + k - h)} + \sum_{f=k-h+2}^{n} n_{f-k+h-1} \leq \frac{k - h + s}{sk + k - h} + \frac{(u + s)^2(h + r - 1)}{(k + r)(sk + k - h)}. \]

Define "storage overhead" to be the ratio of total number of symbols stored in the \( n \) nodes to the total number of data symbols. We have that the storage overhead \( s^* \) of codes \( C(n, k, s, k') \) satisfies that

\[
\frac{k + r}{k} \leq s^* = \frac{(s + 1)(k + r)}{sk + k - h} \leq \frac{(s + 1)(k + r)}{sk} = \left(\frac{s + 1}{s}\right) \cdot \frac{k + r}{k}. \]
III. Piggybacking Codes $\mathcal{C}(n, k, s, k' = k)$

In this section, we consider the special case of codes $\mathcal{C}(n, k, s, k')$ with $k' = k$. When $k' = k$, we have $s \leq r - 2$ and the created $s + 1$ instances are codewords of $(n, k)$ MDS codes and the codes $\mathcal{C}(n, k, s, k' = k)$ are MDS codes. The structure of $\mathcal{C}(n, k, s, k' = k)$ is shown in Fig. 3.

![Fig. 3. The design of code $\mathcal{C}(n, k, s, k' = k)$, $s \leq r - 2$.](image)

In $\mathcal{C}(n, k, s, k' = k)$, we have $r - 1$ piggyback functions $\{p_i\}_{i=1}^{r-1}$, and each piggyback function $p_i$ is a linear combination of $n_i$ symbols that are located in the first $s$ columns of the $n \times (s + 1)$ array, where $i \in \{1, 2, \ldots, r - 1\}$. According to Eq. (3), we have

$$\sum_{i=1}^{r-1} n_i = s(k + r). \quad (7)$$

The average repair bandwidth ratio of all nodes of $\mathcal{C}(n, k, s, k' = k)$ is given in the next theorem.

**Theorem 4.** The lower bound and the upper bound of the average repair bandwidth ratio of all nodes $\gamma_{0, \text{all}}$ of $\mathcal{C}(n, k, s, k' = k)$ is

$$\gamma_{0, \text{all}} = \frac{k + s}{(s + 1)k} + \frac{s^2(k + r)}{(r - 1)(s + 1)k} \quad \text{and}$$

$$\gamma_{0, \text{all}} = \gamma_{0, \text{all}} + \frac{r - 1}{4k(k + r)(s + 1)}, \quad (8)$$

respectively.
Proof. By Eq. (5), the summation of the repair bandwidth for each of the $n$ nodes is

$$\gamma^0 = (k + r)k + \sum_{i=1}^{r-1} n^2_i + \sum_{i=1}^{r-1} n_i.$$  

By Eq. (7), we have

$$\gamma^0 = \frac{(k + r)k + \sum_{i=1}^{r-1} n^2_i + \sum_{i=1}^{r-1} n_i}{(k + r)(s + 1)k} = \frac{(k + r)(k + s) + \sum_{i=1}^{r-1} n^2_i}{(k + r)(s + 1)k} = \frac{(k + r)(k + s) + \sum_{i=1}^{r-1} n^2_i + \sum_{i=1}^{r-1} n_i^2 + \sum_{i=1}^{r-1} (n_i - n)^2}{(k + r)(s + 1)k}.$$  

Note that $\sum_{i\neq j} (n_i - n_j)^2 = t(r - 1 - t)$ by Eq. (4), where $t = s(k + 1) - \left\lfloor \frac{s(k+1)}{r-1} \right\rfloor (r - 1).$ According to Eq. (7), we have

$$\gamma^0 = \frac{(k + r)(k + s) + \sum_{i=1}^{r-1} n^2_i + \sum_{i=1}^{r-1} n_i^2 + \sum_{i=1}^{r-1} (n_i - n)^2}{(k + r)(s + 1)k}.$$  

By the mean inequality, we have $0 \leq t(r - 1 - t) \leq \frac{(r-1)^2}{4}$ and we can further obtain that

$$\frac{k + s}{(s + 1)k} + \frac{s^2(k + r)}{(r - 1)(s + 1)k} \leq \gamma^0 \leq \frac{k + s}{(s + 1)k} + \frac{s^2(k + r)}{(r - 1)(s + 1)k} + \frac{r - 1}{4k(k + r)(s + 1)}.$$  

According to Theorem 4, the difference between the lower bound and the upper bound of the average repair bandwidth ratio satisfies that

$$|\gamma^0 - \gamma^0_{\min}| \leq |\gamma^0_{\min} - \gamma^0_{\max}| = \frac{r - 1}{4k(k + r)(s + 1)} \leq \frac{r - 1}{8k(k + r)}.$$  

(10)

When $r \ll k$, the difference between $\gamma^0$ and $\gamma^0_{\min}$ can be ignored. When $r \ll k$, we present the repair bandwidth for $C(n, k, s, k') = k$ as follows.

**Corollary 5.** Let $r \ll k$ and $k \to +\infty$, then the minimum value of the average repair bandwidth ratio $\gamma^0$ of $C(n, k, s, k') = k$ is achieved when $s = \sqrt{r} - 1.$

**Proof.** When $r \ll k$ and $k \to +\infty$, we have that $\lim_{k \to +\infty} \gamma^0 = \lim_{k \to +\infty} \gamma^0_{\min}$ by Eq. (10) and by Eq. (8), we can further obtain that

$$\lim_{k \to +\infty} \gamma^0 = \lim_{k \to +\infty} \gamma^0_{\min} = \frac{s^2}{(r - 1)(s + 1)} + \frac{1}{s + 1}.$$  

(11)
We can compute that
\[ \frac{\partial \lim_{k \to +\infty} \gamma_{0}^{\text{all}}}{\partial s} = \frac{(s + 1)^2 - r}{(s + 1)^2(r - 1)}. \]
If \( s > \sqrt{r} - 1 \), then \( \frac{\partial \gamma(r,s)}{\partial s} > 0 \); if \( s < \sqrt{r} - 1 \), then \( \frac{\partial \gamma(r,s)}{\partial s} < 0 \); if \( s = \sqrt{r} - 1 \), then \( \frac{\partial \gamma(r,s)}{\partial s} = 0 \). Therefore, when \( s = \sqrt{r} - 1 \), \( \lim_{k \to +\infty} \gamma_{0}^{\text{all}} \) achieves the minimum value.

Note that \( s \) should be a positive integer, we can let \( s = \lfloor \sqrt{r} - 1 \rfloor \) or \( s = \lceil \sqrt{r} - 1 \rceil \), and then choose the minimum value of the average repair bandwidth ratio \( \gamma_{0}^{\text{all}} \) for \( C(n,k,s,k' = k) \).

Consider a specific example of code \( C(n = 20, k = 14, s = 1, k' = 14) \) in Fig. 4. Suppose that node 1 fails, we can first download 14 symbols \( a_{2,2}, a_{2,3}, \ldots, a_{2,14}, P_{1}^{T}a_{1} \) to obtain the two symbols \( a_{1,1}, P_{2}^{T}a_{2} \), according to the MDS property. Then, we download the following 4 symbols
\[ P_{2}^{T}a_{2} + (a_{1,1} + a_{1,6} + a_{1,11} + P_{6}^{T}a_{1}), a_{1,6}, a_{1,11}, P_{6}^{T}a_{1} \]
to recover \( a_{1,1} \). The repair bandwidth of node 1 is 18 symbols. Similarly, we can show that the repair bandwidth of any single-node failure among nodes 2 to 15 is 18 symbols.

Suppose that node 16 fails, we can download the 14 symbols \( a_{2,1}, a_{2,2}, \ldots, a_{2,14} \) to obtain \( P_{2}^{T}a_{2}, P_{3}^{T}a_{2} \), according to the MDS property. Then, we download the 4 symbols \( P_{3}^{T}a_{2} + (a_{1,2} + a_{1,7} + a_{1,12} + P_{2}^{T}a_{1}), a_{1,2}, a_{1,7}, a_{1,12} \) to recover \( P_{2}^{T}a_{1} \). Finally, we download the 4 symbols
\[ a_{1,1}, a_{1,6}, a_{1,11}, P_{6}^{T}a_{1} \]
to recover \( P_{2}^{T}a_{2} + p_{1} \). The repair bandwidth of node 16 is 22 symbols. Similarly, we can show that the repair bandwidth of any single-node failure among nodes 17 to

| Node 1 | \( a_{1,1} \) | \( a_{2,1} \) |
| --- | --- | --- |
| ... | ... | ... |
| Node 14 | \( a_{1,14} \) | \( a_{2,14} \) |
| Node 15 | \( P_{1}^{T}a_{1} \) | \( P_{1}^{T}a_{2} \) |
| Node 16 | \( P_{2}^{T}a_{1} \) | \( P_{2}^{T}a_{2} + (a_{1,1} + a_{1,6} + a_{1,11} + P_{6}^{T}a_{1}) \) |
| Node 17 | \( P_{3}^{T}a_{1} \) | \( P_{3}^{T}a_{2} + (a_{1,2} + a_{1,7} + a_{1,12} + P_{2}^{T}a_{1}) \) |
| Node 18 | \( P_{4}^{T}a_{1} \) | \( P_{4}^{T}a_{2} + (a_{1,3} + a_{1,8} + a_{1,13} + P_{6}^{T}a_{1}) \) |
| Node 19 | \( P_{5}^{T}a_{1} \) | \( P_{5}^{T}a_{2} + (a_{1,4} + a_{1,9} + a_{1,14} + P_{2}^{T}a_{1}) \) |
| Node 20 | \( P_{6}^{T}a_{1} \) | \( P_{6}^{T}a_{2} + (a_{1,5} + a_{1,10} + P_{1}^{T}a_{1} + P_{5}^{T}a_{2}) \) |

Fig. 4. An example of code \( C(n, k, s, k' = k) \), where \( (n, k, s) = (20, 14, 1) \).
20 is 22 symbols. Therefore, we know that in this example, the average repair bandwidth ratio of all nodes is \( \frac{15 \times 18 + 5 \times 22}{20 \times 28} \approx 0.68 \).

IV. PIGGYBACKING CODES \( C(n, k, s, k' = 0) \)

In this section, we consider the special case of codes \( C(n, k, s, k' = 0) \) with \( n \geq s + 1 \) based on the second piggybacking design. Recall that there is no data symbol in the last column and we add the \( n \) piggyback functions in the last column. Here, for \( i \in \{1, 2, \ldots, n\} \), we use \( p_i \) to represent the piggyback function in the last column in row (node) \( i \). Fig. 5 shows the structure of codes \( C(n, k, s, k' = 0) \).

For notational convenience, we denote the parity symbol \( P^T_ja_i \) by \( a_{i,k+j} \) in the following, where \( 1 \leq j \leq r, 1 \leq i \leq s \). Given an integer \( x \) with \( -s + 1 \leq x \leq k + r + s \), we define \( \overline{x} \) by

\[
\overline{x} = \begin{cases} 
  x + k + r, & \text{if } -s + 1 \leq x \leq 0 \\
  x, & \text{if } 1 \leq x \leq k + r \\
  x - k - r, & \text{if } k + r + 1 \leq x \leq k + r + s 
\end{cases}
\]

According to the design of piggyback functions in Section II-A, the symbol \( a_{i,j} \) is used to compute the piggyback function \( p_{i+j} \) for \( i \in \{1, 2, \ldots, s\} \) and \( j \in \{1, 2, \ldots, n\} \). Therefore, we can obtain that the piggyback function \( p_j = \sum_{i=1}^{s} a_{i,j-i} \) for \( 1 \leq j \leq k + r \). For any \( j \in \{1, 2, \ldots, k + r\} \) and \( i_1 \neq i_2 \in \{1, 2, \ldots, s\} \), the two symbols in row \( j \), columns \( i_1 \) and \( i_2 \) are added to two different piggyback functions \( p_{i_1+j} \) and \( p_{i_2+j} \), respectively, since \( \overline{i_1+j} \neq \overline{i_2+j} \) for \( n \geq s + 1 \).

The next theorem shows the repair bandwidth of codes \( C(n, k, s, k' = 0) \).
Theorem 6. The repair bandwidth of codes $C(n, k, s, k' = 0)$ is $s + s^2$.

Proof. Suppose that node $f$ fails, where $f \in \{1, 2, \ldots, n\}$. Similar to the repair method in Section II-B, we can first repair the piggyback function $p_f$ (the symbol in row $f$ and column $s + 1$) by downloading $s$ symbols $\{a_{j,f} \}_{j=1}^{s}$ to repair the symbol $p_f$. Note that the symbol $a_{j,f}$ is used to compute the piggyback function $p_{j+f}$ for $j \in \{1, 2, \ldots, s\}$, we can recover the symbol $a_{j,f}$ by downloading $p_{j+f}$ and the other $s - 1$ symbols used in computing $p_{j+f}$ except $a_{j,f}$. Therefore, the repair bandwidth of node $f$ is $s + s^2$ symbols.

By Theorem 6, the average repair bandwidth ratio of $C(n, k, s, k' = 0)$ is $\frac{s + 1}{k}$. The storage overhead of $C(n, k, s, k' = 0)$ is

$$\frac{(s + 1)(k + r)}{sk} = (1 + \frac{1}{s}) \cdot \frac{k + r}{k}.$$ 

We show in the next theorem that our codes $C(n, k, s, k' = 0)$ can recover any $r + 1$ failures under some condition.

Theorem 7. If $k > (s - 1)(r + 1) + 1$, then the codes $C(n, k, s, k' = 0)$ can recover any $r + 1$ failures.

Proof. Suppose that $r + 1$ nodes $f_1, f_2, \ldots, f_{r+1}$ fail, where $1 \leq f_1 < f_2 < \cdots < f_{r+1} \leq k + r$. In the following, we present a repair method to recover the failed $r + 1$ nodes.

For $i \in \{1, 2, \ldots, r\}$, let $t_i$ be the number of surviving nodes between two failed nodes $f_i$ and $f_{i+1}$, and $t_{r+1}$ be the number of surviving nodes between nodes $f_{r+1}$ and $k + r$ plus the number of surviving nodes between nodes 1 and $f_1$, i.e., $t_i = f_{i+1} - f_i - 1$ and $t_{r+1} = k + r - f_{r+1} + f_1 - 1$. It is easy to see that

$$\sum_{i=1}^{r+1} t_i = k - 1.$$ 

Let $t_{\max} = \max\{t_1, t_2, \ldots, t_{r+1}\}$ and without loss of generality, we assume that $t_{\max} = t_j$ with $j \in \{1, 2, \ldots, r + 1\}$. We have $t_j = t_{\max} \geq t_i$ for $i = 1, 2, \ldots, r + 1$ and

$$(r + 1)t_j \geq \sum_{i=1}^{r+1} t_i = k - 1 > (s - 1)(r + 1),$$

where the last inequality comes from that $k > (s - 1)(r + 1) + 1$. Therefore, we obtain that $t_j \geq s$. 

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We are now ready to describe the repair method for the failed $r+1$ nodes. We can first repair one symbol stored in the failed node $f_j$, some symbols that are used to repair the symbol in node $f_j$ are shown in Fig. 6. Recall that the symbol $p_{f_j+s}$ is located column $s+1$ in node $f_j+s$. We claim that node $f_j+s$ is not a failed node, since $t_j \geq s$. Recall also that $p_{f_j+s} = a_{s,f_j} + \sum_{i=1}^{s-1} a_{i,f_j+s-i}$.

We claim that node $f_j+s-i$ is not a failed node, for all $1 \leq i \leq s-1$, since $t_j \geq s$. Therefore, we can download the $s$ symbols $p_{f_j+s}, \{a_{i,f_j+s-i}\}_{i=1}^{s-1}$ to recover the symbol $a_{s,f_j}$. Once the symbol $a_{s,f_j}$ is recovered, we can recover the other failed $r$ symbols $\{a_{s,f_i}\}_{i=1,2,j-1,j+1,...,r+1}$ in column $s$ in nodes $f_1, \ldots, f_{j-1}, f_{j+1}, \ldots, f_{r+1}$, according to the MDS property of the MDS codes.

\[ \begin{array}{cccccc}
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\text{Node } f_j & a_1, f_j & \ldots & a_{s-2}, f_j & a_{s-1}, f_j & a_{s,f_j} \\
\text{Node } f_j+1 & a_{s-2}, f_j & \vdots & a_{s-1}, f_j & a_{s,f_j} & p_{f_j+1} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\text{Node } f_j+s & a_{1,f_j+s-2} & \ldots & a_{s-1,f_j+s-1} & a_{s,f_j+s-1} & p_{f_j+s} \\
\text{Node } f_j+s-1 & a_{1,f_j+s-1} & \ldots & & & \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\end{array} \]

Fig. 6. Piggyback functions of the codes $C(n, k, s, k' = 0)$, where one piggyback function and the corresponding symbols that are used to compute the piggyback function are with the same color.

Next, we present a repair method to recover the symbol $a_{s-1,f_j}$. First, recall that

\[ p_{f_j+s-1} = a_{s-1,f_j} + a_{s,f_j-1} + \sum_{i=1}^{s-2} a_{i,f_j+s-1-i} \]

node $f_j+s-1-i$ is not a failed node for all $1 \leq i \leq s-2$ and the symbol $a_{s,f_j-1}$ has already recovered. Therefore, we can recover $a_{s-1,f_j}$ by downloading the following $s$ symbols

\[ p_{f_j+s-1}, a_{s,f_j-1}, \{a_{i,f_j+s-1-i}\}_{i=1}^{s-2} \]

Once the symbol $a_{s-1,f_j}$ is recovered, we can recover the other failed $r$ symbols $\{a_{s-1,f_i}\}_{i=1,2,j-1,j+1,...,r+1}$ in column $s-1$ in nodes $f_1, \ldots, f_{j-1}, f_{j+1}, \ldots, f_{r+1}$, according to the MDS property of the MDS codes.

Similarly, we can recover $a_{s-\ell,f_j}$ by downloading the following $s$ symbols

\[ p_{f_j+s-\ell}, \{a_{s-\ell+i,f_j-i}\}_{i=1}^{\ell}, \{a_{s-\ell-i,f_j+i}\}_{i=1}^{s-\ell-1} \]
and further recover the other failed $r$ symbols $\{a_{s-\ell,f_i}\}_{i=1,2,\ldots,j-1,j+1,\ldots,r+1}$ in column $s-\ell$ in nodes $f_1, \ldots, f_{j-1}, f_{j+1}, \ldots, f_{r+1}$, according to the MDS property of the MDS codes, where $\ell = 1, 2, \ldots, s-1$.

We have recovered $(r+1)s$ symbols stored in the $r+1$ failed nodes and can recover all the other $r+1$ symbols in the $r+1$ failed nodes in the last column.

Recall that an $(n,k,g)$ Azure-LRC code [20] first computes $n-k-g$ global parity symbols by encoding all the $k$ data symbols, then divides the $k$ data symbols into $g$ groups each with $\frac{k}{g}$ symbols (suppose that $k$ is a multiple of $g$) and compute one local parity symbol for each group. Each of the obtained $n$ symbols is stored in one node and an $(n,k,g)$ Azure-LRC code can tolerate any $n-k-g+1$-node failures. In the following theorem, we will show that the proposed codes $\mathcal{C}(n-g,k,s,k' = 0)$ have lower repair bandwidth than the $(n,k,g)$ Azure-LRC code [20] under the condition that both the fault-tolerance and the storage overhead of two codes are the same.

**Theorem 8.** If $2g > n-k+1$ and $n^2 - k^2 < kg \cdot (n-k-g+1)$, then the proposed codes $\mathcal{C}(n-g,k,\frac{n-g}{g},k' = 0)$ (suppose that $\frac{n-g}{g}$ is an integer) have strictly less repair bandwidth than that of $(n,k,g)$ Azure-LRC code, under the condition that both the fault-tolerance and the storage overhead of two codes are the same.

**Proof.** Recall that the storage overhead of $(n,k,g)$ Azure-LRC code is $\frac{n}{k}$ and the fault-tolerance of $(n,k,g)$ Azure-LRC code is $n-k-g+1$. We should determine the parameter $s$ for $\mathcal{C}(n-g,k,s,k' = 0)$ such that the storage overhead is $\frac{n}{k}$ and the fault-tolerance is $n-k-g+1$.

When $s = \frac{n-g}{g}$, we have that the storage overhead of code $\mathcal{C}(n-g,k,s,k' = 0)$ is

$$\frac{(s+1) \cdot (k+r)}{sk} = \frac{\left(\frac{n-g}{g} + 1\right) \cdot (n-g)}{\left(\frac{n-g}{g}\right) \cdot k} = \frac{n}{k},$$

which is equal to the storage overhead of $(n,k,g)$ Azure-LRC code. By assumption, we have that $2g > n-k+1$, then we can obtain that

$$(s-1)(r+1) + 1 = \left(\frac{n-g}{g} - 1\right)(n-k-g+1) + 1 < \left(\frac{n-g}{g} - 1\right)g + 1 = n-2g+1 < k.$$

Therefore, the condition in Theorem[7] is satisfied, the fault-tolerance of our code $\mathcal{C}(n-g,k,\frac{n-g}{g},k' = 0)$ is $r+1 = n-k-g+1$ and the code length of $\mathcal{C}(n-g,k,\frac{n-g}{g},k' = 0)$ is $n-g$. In the following, we show that the average repair bandwidth ratio of all nodes of our $\mathcal{C}(n-g,k,\frac{n-g}{g},k' = 0)$ is strictly less than that of $(n,k,g)$ Azure-LRC code.
Let the average repair bandwidth ratio of all nodes of \((n, k, g)\) Azure-LRC code and \(C(n - g, k, \frac{n-g}{g}, k' = 0)\) be \(\gamma_1\) and \(\gamma_2\), respectively. In \((n, k, g)\) Azure-LRC code, we can repair any symbol in a group by downloading the other \(\frac{k}{g}\) symbols in the group and repair any global parity symbol by downloading the \(k\) data symbols. The average repair bandwidth ratio of all nodes of \((n, k, g)\) Azure-LRC code is

\[
\gamma_1 = \frac{(k + g) \cdot \frac{k}{g} + (n - k - g) \cdot k}{nk} = \frac{(n - k - g + 1) \cdot g + k}{ng}.
\]

According to Theorem 6, the average repair bandwidth ratio of all nodes of \(C(n - g, k, \frac{n-g}{g}, k' = 0)\) is \(\frac{s+1}{k}\). We have that

\[
\gamma_2 < \gamma_1 \iff \frac{s+1}{k} < \frac{(n - k - g + 1) \cdot g + k}{ng}
\]

\[
\iff \frac{n-g+1}{k} < \frac{(n - k - g + 1) \cdot g + k}{ng}
\]

\[
\iff n^2 - k^2 < kg \cdot (n - k - g + 1).
\]

By the assumption, we have that \(\gamma_2 < \gamma_1\). 

By Theorem 8, the storage overhead of \(C(n - g, k, \frac{n-g}{g}, k' = 0)\) and \((n, k, g)\) Azure-LRC code are the same and \(C(n - g, k, \frac{n-g}{g}, k' = 0)\) have strictly less repair bandwidth than that of \((n, k, g)\) Azure-LRC code, when the condition is satisfied. Since the storage overhead and the repair bandwidth of \(C(n, k + g, \frac{n}{g}, k' = 0)\) are strictly less than that of \(C(n - g, k, \frac{n-g}{g}, k' = 0)\). We thus obtain that the proposed codes \(C(n, k + g, \frac{n}{g}, k' = 0)\) have better performance than that of \((n, k, g)\) Azure-LRC code in terms of both storage overhead and repair bandwidth, when \(2g > n - k + 1\) and \(n^2 - k^2 < kg \cdot (n - k - g + 1)\).

Note that, with \(C(n, k + g, \frac{n}{g}, k' = 0)\), one can repair any single-node failure by accessing \(\frac{2n}{g}\) helper nodes; however, one only need to access \(\frac{k}{g}\) helper nodes in repairing any symbol in a group and access \(k\) helper nodes in repairing any global parity symbol with \((n, k, g)\) Azure-LRC code.

Recall that an \((n, k, g)\) optimal-LRC first encodes \(k\) data symbols to obtain \(n - k - g\) global parity symbols, then divides the \(n - g\) symbols (including \(k\) data symbols and \(n - k - g\) global parity symbols) into \(g\) groups each with \((n-g)/g\) symbols and encodes one local parity symbol for each group, where \(n - g\) is a multiple of \(g\). Each of the obtained \(n\) symbols is stored in one node and an \((n, k, g)\) optimal-LRC code can tolerate any \(n - k - g + 1\) node failures. The repair bandwidth of any single-node failure of optimal-LRC is \((n-g)/g\) symbols. Next theorem shows
that our codes $C(n, k + g, \frac{n-g}{g}, k' = 0)$ have better performance than $(n, k, g)$ optimal-LRC code, in terms of both storage overhead and repair bandwidth.

**Theorem 9.** If $2g > n - k + 1$, then the proposed codes $C(n, k + g, \frac{n-g}{g}, k' = 0)$ (suppose that $\frac{n-g}{g}$ is an integer) have strictly less storage overhead and less repair bandwidth than that of $(n, k, g)$ optimal-LRC code, under the same fault-tolerant capability.

**Proof.** When $s = \frac{n-g}{g}$, the storage overhead of $C(n, k + g, \frac{n-g}{g}, k' = 0)$ is

$$\frac{(s + 1) \cdot n}{s \cdot (k + g)} = \frac{(\frac{n-g}{g} + 1) \cdot n}{(\frac{n-g}{g}) \cdot (k + g)} = \frac{n^2}{(n - g) \cdot (k + g)},$$

while the storage overhead of $(n, k, g)$ optimal-LRC is $\frac{n}{k}$. We have that

$$\frac{n^2}{(n - g) \cdot (k + g)} < \frac{n}{k} \iff nk < (n - g)(k + g) \iff k + g < n.$$

The last inequality is obviously true. Therefore, $C(n, k + g, \frac{n-g}{g}, k' = 0)$ have strictly less storage overhead than that of $(n, k, g)$ optimal-LRC.

By assumption, we have that $2g > n - k + 1$, then we can obtain that

$$(s - 1)(r + 1) + 1 = (\frac{n-g}{g} - 1)(n - k - g + 1) + 1 < (\frac{n-g}{g} - 1)g + 1 = n - 2g + 1 < k + g.$$

The condition in Theorem 7 is satisfied, and the fault-tolerant of $C(n, k + g, \frac{n-g}{g}, k' = 0)$ is $r + 1 = n - k - g + 1$, which is equal to the fault-tolerant of $(n, k, g)$ optimal-LRC code.

Recall that the average repair bandwidth ratio of all nodes of $(n, k, g)$ optimal-LRC and $C(n, k + g, \frac{n-g}{g}, k' = 0)$ is $\frac{n-g}{k}g$ and $\frac{n-g}{k}g + \frac{s+1}{k+g}$, respectively. We have that

$$\frac{s + 1}{k + g} < \frac{n - g}{k \cdot g} \iff \frac{(\frac{n-g}{g} + 1)}{k + g} < \frac{n - g}{k \cdot g} \iff nk < (n - g)(k + g) \iff k + g < n.$$

The last inequality is obviously true. Therefore, $C(n, k + g, \frac{n-g}{g}, k' = 0)$ have strictly less repair bandwidth than that of $(n, k, g)$ optimal-LRC.

Consider the code $C(n = 7, k = 5, s = 2, k' = 0)$ which is shown in Fig. 7. Suppose that node 1 fails, we can first download 2 symbols $P_2^T a_1, P_1^T a_2$ to obtain the piggyback function $p_1 = P_2^T a_1 + P_1^T a_2$, then download 2 symbols $a_{1,1} + P_2^T a_2, P_2^T a_2$ to recover $a_{1,1}$ and finally download 2 symbols $a_{1,2} + a_{2,1}, a_{1,2}$ to recover $a_{2,1}$. The repair bandwidth of node 1 is 6 symbols. Similarly, the repair bandwidth of any single-node failure is 6 symbols.

We claim that the code $C(n = 7, k = 5, s = 2, k' = 0)$ can tolerate any $r + 1 = 3$ node failures. Suppose that nodes 2, 4 and 6 fail, we have that the number of surviving nodes between
two failed nodes are \( t_1 = 1, t_2 = 1 \) and \( t_3 = 2 \). We can first repair the symbol \( \mathbf{P}_2^T \mathbf{a}_2 \) in node 6 by downloading symbols \( \mathbf{P}_2^T \mathbf{a}_1 + \mathbf{P}_1^T \mathbf{a}_2, \mathbf{P}_2^T \mathbf{a}_1 \). Since the symbol \( \mathbf{P}_2^T \mathbf{a}_2 \) has been recovered, according to the MDS property of the second column, we can recover \( a_{2,2}, a_{2,4} \). Then, we can download symbols \( a_{2,5}, \mathbf{P}_1^T \mathbf{a}_1 + a_{2,5} \) to recover \( \mathbf{P}_1^T \mathbf{a}_1 \). Finally, since \( \mathbf{P}_1^T \mathbf{a}_1 \) has been repaired, we can recover \( a_{1,2}, a_{1,4} \) by the MDS property of the first column. Up to now, we have recovered the first two symbols in the three failed nodes. Since the third symbol in each of the three failed nodes is a piggyback function, we can recover the symbol by reading some data symbols. The repair method of any three failed nodes is similar as the above repair method.

V. Comparison

In this section, we evaluate the repair bandwidth for our piggybacking codes \( \mathcal{C}(n, k, s, k') \) and other related codes, such as existing piggybacking codes \([13], [18]\) and \((n, k, g)\) Azure-LRC code \([20]\) under the same fault-tolerance and the storage overhead.

A. Codes \( \mathcal{C}(n, k, s, k' = k) \) VS Piggybacking Codes

Recall that OOP codes \([16]\) have the lowest repair bandwidth for any single-node failure among the existing piggybacking codes when \( r \leq 10 \) and sub-packetization is \( r - 1 + \lfloor \sqrt{r - 1} \rfloor \) or \( r - 1 + \lceil \sqrt{r - 1} \rceil \), the codes in \([19]\) have the lowest repair bandwidth for any single-node failure among the existing piggybacking codes when \( r \geq 10 \) and sub-packetization is \( r \). REPB codes
[15] have small repair bandwidth for any single-data-node with sub-packetization usually less than \( r \). Moreover, the codes in [18] have lower repair bandwidth than the existing piggybacking codes when \( r \geq 10 \) and sub-packetization is less than \( r \). There are two constructions in [18], the first construction in [18] has larger repair bandwidth than the second construction. We choose codes in [15], [19] and the second construction in [18] as the main comparison.

Let \( C_1 \) be the second piggybacking codes in [18] and \( C_2 \) be the codes in [19]. Fig. 8 shows the average repair bandwidth ratio of all nodes for codes \( C_1, C_2, \) REPB codes [15] and the proposed codes \( \mathcal{C}(n, k, s, k' = k) \), where \( r = 8, 9 \) and \( k = 10, 11, \ldots, 100 \). Note that the sub-packetization of REPB codes and \( C_1 \) is inexplicit and usually less than \( r \). In Fig. 8 we choose the lower bound of the repair bandwidth of REPB codes and \( C_1 \). The sub-packetization of \( C_2 \) is \( r \).

The results in Fig. 8 demonstrate that the proposed codes \( \mathcal{C}(n, k, s, k' = k) \) have the lowest average repair bandwidth ratio of all nodes compared to the existing piggybacking codes, when the sub-packetization level is less than \( r \) and \( k \geq 30 \). Note that the sub-packetization of our codes \( \mathcal{C}(n, k, s, k' = k) \) is \( \sqrt{r} \), which is lower than that of \( C_2 \).

### B. Codes \( \mathcal{C}(n, k, s, k' = k - sr - 1) \) VS Piggybacking Codes

Recall that OOP codes [16] have the lowest repair bandwidth for any single-node failure among the existing piggybacking codes when \( r \leq 10 \), where the sub-packetization of OOP codes...
is \( r - 1 + \lfloor \sqrt{r-1} \rfloor \) or \( r - 1 + \lceil \sqrt{r-1} \rceil \), and the minimum value of average repair bandwidth ratio of all nodes \( \gamma^{all}_{OOP} \) is

\[
\gamma^{all}_{OOP} = \frac{1}{k + r} \cdot \left( k \cdot \frac{2\sqrt{r-1} + 1}{2\sqrt{r-1} + r} + \frac{r}{r} \cdot \left( \frac{\sqrt{r-1}}{r} + \frac{1}{r} + \frac{(r-1)^2 - \sqrt{(r-1)^3}}{kr} \right) \right).
\]  \hspace{1cm} (12)

In the following, we show that our codes \( C(n, k, s, k' = k - sr - 1) \) have strictly less repair bandwidth than OOP codes and thus have less repair bandwidth than all the existing piggybacking codes when \( r \leq 10 \).

**Lemma 10.** Let \( \gamma^{all}_{1} \) be the average repair bandwidth ratio of all nodes of \( C(n, k, s, k' = k - sr - 1) \).

When \( 2 \leq r \ll k, 2 + \sqrt{r-1} \leq s \), we have

\[
\lim_{k \to +\infty} \gamma^{all}_{1} < \lim_{k \to +\infty} \gamma^{all}_{OOP}.
\]

**Proof.** According to Theorem 3, since \( k - k' = h = sr + 1 \), we have

\[
\lim_{k \to +\infty} \gamma^{all}_{1} \leq \frac{s^2}{(sr + r)(s + 1)} + \frac{1}{s + 1}
\]
\[
= \frac{s^2}{(s + 1)^2} \cdot \frac{1}{r} + \frac{1}{s + 1} < \frac{1}{r} + \frac{1}{s + 1}.
\]

From Eq. (12), when \( r \geq 2 \), we have

\[
\left( \lim_{k \to +\infty} \gamma^{all}_{OOP} \right) - \frac{1}{r} = \frac{2\sqrt{r-1} + 1}{2\sqrt{r-1} + r} - \frac{1}{r}
\]
\[
= \frac{2\sqrt{r-1}}{2\sqrt{r-1} + r} \cdot \frac{r - 1}{r}
\]
\[
\geq \frac{2\sqrt{r-1}}{2\sqrt{r-1} + r} \cdot \frac{1}{2}
\]
\[
= \frac{\sqrt{r-1}}{2\sqrt{r-1} + r}.
\]  \hspace{1cm} (13)

When \( s \geq \sqrt{r-1} + 2 \) and \( r \geq 2 \), we have

\[
s \geq \sqrt{r-1} + 2
\]
\[
= (\sqrt{r-1} + 1) + 1
\]
\[
= (\frac{r - 1}{\sqrt{r-1}} + 1) + 1
\]
\[
\geq (\frac{r - 1}{\sqrt{r-1}} + 1) + \frac{1}{\sqrt{r-1}}
\]
\[
= \frac{r + \sqrt{r-1}}{\sqrt{r-1}}.
\]  \hspace{1cm} (14)
From Eq. (14) and Eq. (13), we have

\[
\frac{1}{s+1} \leq \left( \frac{r + \sqrt{r-1}}{\sqrt{r-1}} + 1 \right)^{-1} = \frac{\sqrt{r-1}}{2\sqrt{r-1} + r} \leq (\lim_{k \to +\infty} \gamma_{OOP}^{all}) - \frac{1}{r}.
\]

Therefore, we have

\[
\lim_{k \to +\infty} \gamma_{1}^{all} < \frac{1}{r} + \frac{1}{s+1} \leq \lim_{k \to +\infty} \gamma_{OOP}^{all}.
\]

It is easy to see that the storage overhead of \( C(n, k, s, k' = k - sr - 1) \) ranges from \( \frac{k+r}{k-r} \cdot \frac{k+r}{k} \). According to Lemma 10, our codes \( C(n, k, s, k' = k - sr + 1) \) have strictly less repair bandwidth than all the existing piggybacking codes when \( r \ll k \), at a cost of slightly more storage overhead.

C. The proposed Codes VS LRC

Next, we evaluate the repair bandwidth of our codes \( C(n, k + g, \frac{n-g}{g}, k' = 0) \), codes \( C(n, k + g, \frac{n-g}{g}, k' = 0) \), Azure-LRC [20] and optimal-LRC. According to Theorem 8, the repair bandwidth of our \( C(n, k + g, \frac{n-g}{g}, k' = 0) \) is strictly less than that of \( (n, k, g) \) Azure-LRC [20]. Fig. 9 shows the average repair bandwidth ratio of all nodes for \( C(n, k + g, \frac{n-g}{g}, k' = 0) \) and \( (n, k, g) \) Azure-LRC when the fault-tolerance is 8 and \( n = 100 \). The results demonstrate that \( C(n, k + g, \frac{n-g}{g}, k' = 0) \) have strictly less repair bandwidth than \( (n, k, g) \) Azure-LRC. Moreover, our \( C(n, k + g, \frac{n-g}{g}, k' = 0) \) have less storage overhead than \( (n, k, g) \) Azure-LRC. For example, \( C(n, k + g, \frac{n-g}{g}, k' = 0) \) have 44.92% less repair bandwidth and 6.16% less storage overhead than \( (n, k, g) \) Azure-LRC when \( (n, k, g) = (100, 73, 20) \).

According to Theorem 9, the repair bandwidth of our \( C(n, k + g, \frac{n-g}{g}, k' = 0) \) is strictly less than that of \( (n, k, g) \) optimal-LRC. Fig. 10 shows the average repair bandwidth ratio of all nodes for \( C(n, k + g, \frac{n-g}{g}, k' = 0) \) and \( (n, k, g) \) optimal-LRC under the same fault-tolerance and \( n = 100 \). The results demonstrate that \( C(n, k + g, \frac{n-g}{g}, k' = 0) \) have strictly less repair bandwidth than \( (n, k, g) \) optimal-LRC.
Fig. 9. Average repair bandwidth ratio of all nodes for codes \( C(n, k + g, \frac{n-g}{g}', k' = 0) \) and \((n, k, g)\) Azure-LRC, where \(10 \leq g \leq 20\), fault-tolerance is 8, and \(n = 100\).

Fig. 10. Average repair bandwidth ratio of all nodes for codes \( C(n, k + g, \frac{n-g}{g}', k' = 0) \) and \((n, k, g)\) optimal-LRC under the same fault-tolerance and code length \( n = 100\), where \(10 \leq g \leq 20\).

VI. CONCLUSION

In this paper, we propose two new piggybacking coding designs. We propose one class of piggybacking codes based on the first design that are MDS codes, have lower repair bandwidth than the existing piggybacking codes when \( r \geq 8 \) and the sub-packetization is \( \alpha < r \). We also propose another piggybacking codes based on the second design that are non-MDS codes and have better tradeoff between storage overhead and repair bandwidth, compared with Azure-LRC and optimal-LRC for some parameters. One future work is to generalize the piggybacking coding design over codewords of more than two different MDS codes. Another future work is to obtain the condition of \( C(n, k, s, k' = 0) \) that can recover any \( r + 2 \) failures.
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