On classification of Mori contractions:
Elliptic curve case

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Abstract. We study Mori’s three-dimensional contractions $f: X \to Z$. It is proved that on the “good” model $(X, S)$ there are no elliptic components of $\text{Diff}_S$ with coefficients $\geq 6/7$.

1. Introduction

1.1. Let $f: X \to Z \ni o$ be an extremal log terminal contraction over $\mathbb{C}$, that is:

- $X$ is a normal algebraic $\mathbb{Q}$-factorial threefold with at worst log terminal singularities,
- $f$ is a projective morphism such that $f_* \mathcal{O}_X = \mathcal{O}_Z$,
- $\rho(X/Z) = 1$ and $-K_X$ is $f$-ample.

We assume that $\dim(Z) \geq 1$ and regard $(Z \ni o)$ as a sufficiently small Zariski neighborhood. Such contractions naturally appear in the Minimal Model Program [KMM]. By $\text{Exc}(f) \subset X$ denote the exceptional locus of $f$.

According to the general principle introduced by Shokurov [Sh] all such contractions can be divided into two classes: exceptional and nonexceptional. A contraction $f: X \to Z \ni o$ such as in [Sh] is said to be exceptional if for any complement $K_X + D$ near $f^{-1}(o)$ there is at most one divisor $E$ of the function field $\mathcal{K}(X)$ with discrepancy $a(E, D) = -1$. The following is a particular case of the theorem proved in [Sh1] and [P2] (see also [PSh]).

Theorem 1.2. Notation as above. Assume that $f: X \to Z \ni o$ is nonexceptional. Then for some $n \in \{1, 2, 3, 4, 6\}$ there is a member $F \in |-nK_X|$ such that the pair $(X, \frac{1}{n}F)$ is log canonical near $f^{-1}(o)$.

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Thus nonexceptional contractions have a “good” member in $|−nK_X|$, $n \in \{1, 2, 3, 4, 6\}$. The most important case is the case of Mori contractions, i.e. when $X$ has only terminal singularities:

**Conjecture 1.3.** Notation as in [1.4]. Assume that $X$ has at worst terminal singularities. Then $f : X \to Z \ni o$ is nonexceptional.

Similar to the classification of three-dimensional terminal singularities, this fact should be the key point in the classification Mori contractions. For example it is very helpful in the study of three-dimensional flips [K], [M], [Sh].

Methods of [Sh1], [P2], [PS] use inductive procedure of constructing divisors in $|−nK|$. This procedure works on so-called good model of $X$ over $Z$. Roughly speaking, a good model is a birational model $\overline{Y}$ equipped with a prime divisor $\overline{S}$ such that the pair $(\overline{Y}, \overline{S})$ is plt and $-(K_{\overline{Y}} + \overline{S})$ is nef and big over $Z$.

If $f$ is exceptional, then $\overline{S}$ is a projective surface. Adjunction Formula 2.1 gives us that $(\overline{S}, \text{Diff}_{\overline{S}})$ is a klt log del Pezzo surface. Moreover, exceptionality of $f$ implies that the projective log pair $(\overline{S}, \text{Diff}_{\overline{S}})$ is exceptional, by definition this means that any complement $K_{\overline{S}} + \text{Diff}_{\overline{S}}$ is klt [P2, Prop. 2.4]. Thus our construction gives the following correspondence:

| Exceptional contractions $f : X \to Z$ as in [1.1] | Exceptional log del Pezzos $(\overline{S}, \Delta = \text{Diff}_{\overline{S}})$ |

**1.4.** For exceptional log del Pezzos $(\overline{S}, \Delta)$ Shokurov introduced the following invariant:

$$\delta = \delta(\overline{S}, \Delta) = \text{number of divisors } E \text{ of } K(\overline{S})$$

with discrepancy $a(E, \Delta) \leq -6/7$.

He proved that $\delta \leq 2$, classified log surfaces with $\delta = 2$ and showed that in the case $\delta = 1$ the (unique) divisor $E$ with $a(E, \Delta) \leq -6/7$ is represented by a curve of arithmetical genus $\leq 1$ (see [Sh1], [P3]).

The aim of this short note is to exclude the case of Mori contractions with $\delta = 1$ and elliptic curve $E$:

**Theorem 1.5.** Notation as in [1.4]. Assume that $\delta(\overline{S}, \text{Diff}_{\overline{S}}) = 1$. Write $\text{Diff}_{\overline{S}} = \sum \delta_i \overline{\Delta}_i$, where $\overline{\Delta}_i$ are irreducible curves. If $\delta_{i_0} \geq 6/7$ for some $i_0$, then $p_a(\overline{\Delta}_{i_0}) = 0$.

The following example shows that Theorem 1.5 cannot be generalized to the klt case.
**Example 1.6 (IP).** Let \((Z \ni o)\) be the hypersurface canonical singularity \(x_1^2 + x_3^3 + x_4^{11} + x_4^{12} = 0\) and let \(f: X \to Z\) be the weighted blowup with weights \((66, 44, 12, 11)\). Then \(f\) satisfies conditions of 1.1 and we have case 3.2. It was computed in [IP] that \(S\) is the weighted projective plane \(\mathbb{P}(3, 2, 1)\) and \(\text{Diff}_S = \frac{10}{11}C + \frac{1}{2}L\), where \(C\) is an elliptic curve.

Log del Pezzo surfaces of elliptic type (like \((S, \text{Diff}_S)\) in the above theorem) were classified by T. Abe [Ab]. Our proof uses different, very easy arguments.

2. Preliminary results

In this paper we use terminologies of Minimal Model Program [KMM], [Ut]. For the definition of \textit{complements} and their properties we refer to [Sh, Sect. 5], [Ut, Ch. 19] and [P3]

**Definition 2.1 ([Sh, Sect. 3], [Ut, Ch. 16]).** Let \(X\) be a normal variety and let \(S \neq \emptyset\) be an effective reduced divisor on \(X\). Let \(B\) be a \(\mathbb{Q}\)-divisor on \(X\) such that \(S\) and \(B\) have no common components. Assume that \(K_X + S\) is lc in codimension two. Then the \textit{different} of \(B\) on \(S\) is defined by 

\[
K_S + \text{Diff}_S(B) \equiv (K_X + S + B)|_S.
\]

Usually we will write simply \(\text{Diff}_S\) instead of \(\text{Diff}_S(0)\).

**Theorem 2.2 (Inversion of Adjunction [Ut, 17.6]).** Let \(X\) be a normal variety and let \(D\) be a boundary on \(X\). Write \(D = S + B\), where \(S = \lfloor D \rfloor\). Assume that \(K_X + S + B\) is \(\mathbb{Q}\)-Cartier. Then \((X, S + B)\) is plt near \(S\) iff \(S\) is normal and \((S, \text{Diff}_S(B))\) is klt.

**Definition 2.3 ([P4]).** Let \(X\) be a normal variety and let \(g: Y \to X\) be a birational contraction such that the exceptional locus of \(g\) contains exactly one irreducible divisor, say \(S\). Assume that \(K_Y + S\) is plt and \(-(K_Y + S)\) is \(f\)-ample. Then \(g: (Y \supset S) \to X\) is called a \textit{plt blowup} of \(X\).

The key point in the proof of Theorem [L5] is the following proposition.

**Proposition 2.4.** Let \((X \ni P)\) be a three-dimensional terminal singularity and let \(g: (Y, S) \to X\) be a plt blowup with \(f(S) = P\). Write \(\text{Diff}_S = \sum \delta_i \Delta_i\), where \(\Delta_i\) are irreducible curves, and assume that \(\delta_0 \geq 6/7\) for some \(i_0\). Further, assume that \(S\) is smooth at singular points of \(\Delta_{i_0}\). Then \(p_a(\Delta_{i_0}) = 0\).
Lemma 2.5 (cf. [P1, Cor. 5]). Let \((X \ni P)\) be a three-dimensional terminal singularity and let \(g : (Y, S) \to X\) be a plt blowup with \(f(S) = P\). Then there is a boundary \(\Upsilon \geq \text{Diff}_S\) on \(S\) such that

(i) \(\lfloor \Upsilon \rfloor \neq 0\);

(ii) \(-(K_S + \Upsilon)\) is ample.

Moreover, \(K_S + \text{Diff}_S\) has a non-klt 1, 2, 3, 4, or 6-complement.

Proof. Regard \((X \ni P)\) as an analytic germ. It was shown in [YPG, Sect. 6.4] that the general element \(F \in |-K_X|\) has a normal Du Val singularity at \(P\). By Inversion of Adjunction 2.2, \(K_X + F\) is plt. Consider the crepant pull-back \(K_Y + aS + F_Y = f^*(K_X + F)\), where \(F_Y\) is the proper transform of \(F\) and \(a < 1\). Since both \(K_Y + S\) and \(g^*K_X\) are \(\mathbb{Q}\)-Cartier, so are \(S\) and \(F_Y\). Clearly, \(-(K_Y + S + F_Y)\) is \(f\)-ample. Let \(\Upsilon' := \text{Diff}_S(F_Y)\). Then \(\lfloor \Upsilon' \rfloor \neq 0\) and \(-(K_S + \Upsilon')\) is ample. Therefore \(\Upsilon\) can be found in the form \(\Upsilon = \text{Diff}_S(F_Y) + t(\Upsilon' - \text{Diff}_S(F_Y))\) for suitable \(0 < t \leq 1\).

Take \(\Delta = \text{Diff}_S(F_Y) + \lambda(\Upsilon - \text{Diff}_S(F_Y))\) for \(0 < \lambda \leq 1\) so that \(K_S + \Delta\) is lc but not klt (and \(-(K_S + \Delta)\) is ample). By [Sh1, Sect. 2] (see also [P3, 5.4.1]) there exists either an 1, 2, 3, 4, or 6-complement of \(K_S + \Delta\) which is not klt.

A very important problem is to prove the last lemma without using [YPG, Sect. 6.4], i.e. the classification of terminal singularities. This can be a way in higher-dimensional generalizations.

Proof of Proposition 2.4. Put \(C := \Delta_i\delta_0\) and let \(\delta_0 = 1 - 1/m, m \geq 7\). Assume that \(p_a(C) \geq 1\). Let \(\Upsilon\) be such as in Lemma 2.3 and let \(K_S + \Theta\) be a non-klt 1, 2, 3, 4, or 6-complement of \(K_S + \text{Diff}_S\). Using that the coefficients of \(\text{Diff}_S\) are standard [Sh, Prop. 3.9] it is easy to see that \(\Theta \geq \text{Diff}_S\) and \(\Theta \geq C\) [P3, Sect. 4.7]. In particular, \(K_S + C\) is lc.

Further, \(C \not\subset \lfloor \Upsilon \rfloor\). Indeed, otherwise by Adjunction we have

\[-\deg K_C \geq -\deg(K_C + \text{Diff}_C(\Upsilon - C)) = -(K_S + \Upsilon) \cdot C > 0\]

This implies \(p_a(C) = 0\), a contradiction.

By Lemma 2.6 below, \(\Theta = C\), \(p_a(C) = 1\), \(S\) is smooth along \(C\) and has only Du Val singularities outside. Therefore, \(\text{Diff}_S = (1 - 1/m)C\) and \(-K_S \equiv C \equiv -m(K_S + (1 - 1/m)C)\) is ample (see 2.3). Thus \(S\) is a del Pezzo surface with at worst Du Val singularities. Since \(C \not\subset \lfloor \Upsilon \rfloor\), we can write \(\Upsilon = \alpha C + L + \Upsilon^o\), where \(L\) is an irreducible
curve, $1 > \alpha \geq 1 - 1/m \geq 6/7$, $C \not\subset \text{Supp}(\Upsilon^\circ)$ and $\Upsilon^\circ \geq 0$. Further,
\[
0 < K_S \cdot (K_S + \Upsilon) = K_S \cdot (K_S + \alpha C + L + \Upsilon^\circ) \\
\leq K_S \cdot ((1 - \alpha)K_S + L) \leq \frac{1}{7}K_S^2 + K_S \cdot L. \tag{1}
\]
Thus $K_S^2 > -7K_S \cdot L \geq 7$.

Let $S_{\text{min}} \to S$ be the minimal resolution. By Noether's formula, $K_S^2 + \rho(S_{\text{min}}) = K_S^2_{\text{min}} + \rho(S_{\text{min}}) = 10$. Thus, $8 \leq K_S^2 \leq 9$ and $\rho(S_{\text{min}}) \leq 2$. In particular, $S$ either is smooth or has exactly one singular point which is of type $A_1$. By (1), $-K_S \cdot L = 1$. Similar to (1) we have
\[
0 < -(K_S + \Upsilon) \cdot L \leq -(1 - \alpha)K_S \cdot L - L^2.
\]
Hence $L^2 < 1 - \alpha \leq 1/7$, so $L^2 \leq 0$. This means that the curve $L$ generates an extremal ray on $S$ and $\rho(S) = 2$. Therefore, $S$ is smooth and $K_S^2 = 8$. In this case, $S$ is a rational ruled surface ($\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_n$).

Proof. By Adjunction, $K_C + \text{Diff}_C(\Xi) = 0$. Hence $\text{Diff}_C(\Xi) = 0$. This shows that $C \cap \text{Supp}(\Xi) = \emptyset$ and $S$ has no singularities at points of $C \setminus \text{Sing}(C)$ (see [U4], Prop. 16.6, Cor. 16.7]). Let $\mu: \tilde{S} \to S$ be the minimal resolution and let $\tilde{C}$ be the proper transform of $C$ on $\tilde{S}$. Define $\tilde{\Xi}$ as the crepant pull-back: $K_{\tilde{S}} + \tilde{C} + \tilde{\Xi} = \mu^*(K_S + C + \Xi)$. It is sufficient to show that $\tilde{\Xi} = 0$. Assume the converse. Replace $(S, C + \Xi)$ with $(\tilde{S}, \tilde{C} + \tilde{\Xi})$. It is easy to see that all the assumptions of the lemma holds for this new $(S, C + \Xi)$. Contraction of $(-1)$-curves again preserve the assumptions. Since $C$ and $\text{Supp}(\Xi)$ are disjoint, whole $\Xi$ cannot be contracted. Thus we get $S \simeq \mathbb{P}^2$ or $S \simeq \mathbb{F}_n$ (a rational ruled surface). In both cases simple computations gives us $\Xi = 0$. \qed
3. Construction of a good model

Notation as in 1.1. We recall briefly the construction of \([\mathbf{P}2]\) (see also \([\mathbf{PSH}]\)). Assume that \(f: X \to Z \ni o\) is exceptional. Let \(K_X + F\) be a complement which is not klt. There is a divisor \(S\) of \(\mathcal{K}(X)\) such that \(a(S, F) = -1\). Since \(f\) is exceptional, this divisor is unique.

3.1. First we assume that the center of \(S\) on \(X\) is a curve or a point. Then \([F] = 0\). Let \(g: Y \to X\) be a minimal log terminal modification of \((X, F)\) \([\mathbf{Ut}, \text{17.10}]\), i.e. \(g\) is a birational projective morphism such that \(Y\) is \(\mathbb{Q}\)-factorial and \(K_Y + S + A = (K_X + F)^g\) is dlt, where \(A\) is the proper transform of \(F\). In our situation, \(K_Y + S + A\) is plt. By \([\mathbf{Ut}, \text{Prop. 2.17}]\), \(K_Y + S + (1 + \varepsilon)A\) is also plt for sufficiently small positive \(\varepsilon\).

3.1.1. Run \((K + S + (1 + \varepsilon)A)\)-Minimal Model Program over \(Z\):

\[
\begin{array}{ccc}
Y & \xrightarrow{g} & Y' \\
\downarrow & \downarrow & \downarrow \\
X & \xrightarrow{f} & Z
\end{array}
\] (2)

Note that \(K_Y + S + (1 + \varepsilon)A \equiv \varepsilon A \equiv -\varepsilon(K_Y + S)\). At the end we get so-called good model, i.e. a log pair \((Y, S + A)\) such that one of the following holds:

(3.1.A) \(\rho(Y/Z) = 2\) and \(-\rho(K_Y + S)\) is nef and big over \(Z\);

(3.1.B) \(\rho(Y/Z) = 1\) and \(-\rho(K_Y + S)\) is ample over \(Z\).

3.2. Now assume that the center of \(S\) on \(X\) is of codimension one. Then \(S = [F]\). In this case, we put \(g = \text{id}\), \(Y = X\) and \(A = F - S\). If \(-\rho(K_X + S)\) is nef, then we also put \(Y = X\), \(\bar{S} = S\). Assume \(-\rho(K_X + S)\) \(\equiv A\) is not nef. Since \(A\) is effective, \(f\) is birational. If \(f\) is divisorial, then it must contract a component of \(\text{Supp}(A)\). Thus \(S\) is not a compact surface, a contradiction (see \([\mathbf{P}2, \text{Prop. 2.2}]\)). Therefore \(f\) is a flipping contraction. In this case, in diagram (2) the map \(X = Y \to Y\) is the corresponding flip.

Since \(f\) is exceptional, in both cases 3.1 and 3.2 we have \(f(g(S)) = g(S) = o\) (see \([\mathbf{P}2, \text{Prop. 2.2}]\)). Adjunction Formula \([2.1]\) gives us that \((\bar{S}, \text{Diff}_{\bar{S}})\) is a klt log del Pezzo surface, i.e. \((\bar{S}, \text{Diff}_{\bar{S}})\) is klt and \(-\rho(K_{\bar{S}} + \text{Diff}_{\bar{S}})\) is nef and big (see \([\mathbf{P}2, \text{Lemma 2.4}]\)). Moreover, exceptionality of \(f\) implies that the pair \((\bar{S}, \text{Diff}_{\bar{S}})\) is exceptional, i.e. any complement \(K_{\bar{S}} + \text{Diff}_{\bar{S}}\) is klt.

**Proposition 3.3.** Let \(f: X \to Z \ni o\) be as in 1.1. Assume that \(f\) is exceptional. Furthermore,

(i) if \(f\) is divisorial, we assume that the point \((Z \ni o)\) is terminal;

(ii) if \(f\) is birational, then \(\rho(Y/Z) = 2\) and \(-\rho(K_Y + S)\) is nef and big over \(Z\).

(iii) if \(f\) is birational and \(\rho(Y/Z) = 1\), then \(-\rho(K_Y + S)\) is ample over \(Z\).

(iv) if \(f\) is birational and \(\rho(Y/Z) \geq 2\), then \(-\rho(K_Y + S)\) is nef and big over \(Z\).
(ii) in the case \( \dim(Z) = 1 \), we assume that singularities of \( X \setminus f^{-1}(o) \) are canonical.

Then case (3.1.B) does not occur.

**Proof.** Assume the converse. Then \( \rho(Y/Z) = 1 \) and \( q: \varnothing \to Z \) is also an exceptional contraction as in [1.1]. First, we consider the case when \( f \) is divisorial. Then \( q \) is a plt blowup of a terminal point \((Z \ni o)\) and \( q(\varnothing) = o \) (see [P2] Prop. 2.2). By Lemma 2.3 \((\varnothing, \text{Diff}_{\varnothing})\) has a non-klt complement. This contradicts [P2] Prop. 2.4. Clearly, \( f \) cannot be a flipping contraction (because, in this case, the map \( Y \to \varnothing \) must be an isomorphism in codimension one). If \( \dim(Z) = 2 \), then \( q \) is not equidimensional, a contradiction.

Finally, we consider the case \( \dim(Z) = 1 \) (and \( S \) is the central fiber). Let \( F \) be a general fiber of \( q \) (a del Pezzo surface with at worst Du Val singularities). Consider the exact sequence

\[
0 \to \mathcal{O}_Y(-K_Y - F) \to \mathcal{O}_Y(-K_Y) \to \mathcal{O}_F(-K_F) \to 0
\]

By Kawamata-Viehweg Vanishing [KMM Th. 1-2-5], \( R^1q_*\mathcal{O}_Y(-K_Y-F) = 0 \). Hence there is the surjection

\[
H^0(Y, \mathcal{O}_Y(-K_Y)) \to H^0(F, \mathcal{O}_F(-K_F)) \to 0.
\]

Here \( H^0(F, \mathcal{O}_F(-K_F)) \neq 0 \) (because \(-K_F\) is Cartier and ample). Therefore, \( H^0(Y, \mathcal{O}_Y(-K_Y)) \neq 0 \). Let \( \overline{G} \in | - K_Y | \) be any member. Take (positive) \( c \in \mathbb{Q} \) so that \( K_Y + \varnothing + c\overline{G} \) is lc and not plt. Clearly \( c \leq 1 \), so \(- (K_Y + \varnothing + c\overline{G}) \equiv -(1 - c)K_Y \) is \( q \)-nef. By Base Point Free Theorem [KMM Th. 3-1-1] there is a complement \( K_Y + \varnothing + c\overline{G} + L \), where \( nL \in | - n(K_Y + \varnothing + c\overline{G}) | \) for sufficiently big and divisible \( n \) and this complement is not plt, a contradiction with exceptionality (see [P2] Prop. 2.4).

**4. Proof of Theorem 1.5**

In this section we use notation and assumptions of [1.1] and Theorem 1.5.

If \( g = \text{id} \), then \( Y = X \) and \( \varnothing \) have only terminal singularities (see 3.2 and [KMM 5-1-11]). Then \( \text{Diff}_{\varnothing} = 0 \), a contradiction. From now on we assume that \( g \neq \text{id} \). Denote \( \overline{C} := \Delta_{\omega_0} \) and let \( \delta_{\omega_0} = 1 - 1/m \). Since \( \delta(\varnothing, \text{Diff}_{\varnothing}) = 1 \), there are no divisors \( E \neq \overline{C} \) of \( \mathcal{K}(\varnothing) \) with \( a(E, \text{Diff}_{\varnothing}) \leq -6/7 \). This gives us that \( \varnothing \) is smooth at \( \text{Sing}(\overline{C}) \) whenever \( \text{Sing}(\overline{C}) \neq \varnothing \) (see [P3] Lemma 9.1.8). By our assumptions, \( \varnothing \) is singular along \( \overline{C} \). Moreover, at the general point of \( \overline{C} \) we have an analytic isomorphism
Lemma 4.1. Notation as above. Assume that \( p_a(C) \geq 1 \). Then the map \( Y \to C \) is an isomorphism at the general point of \( C \). Moreover, if \( P \in C \) is a singular point, then \( Y \to C \) is an isomorphism at \( P \). In particular, the proper transform \( C \) of \( C \) is a curve with \( p_a(C) \geq 1 \).

4.2. First we show that Lemma 4.1 implies Theorem 1.5. Assume \( p_a(C) \geq 1 \). Clearly, \( C \subset S \). By Lemma 4.1, \( p_a(C) \geq 1 \) and
\[
\# \text{Sing}(C) \geq \# \text{Sing}(C) \tag{4}
\]
From (3), we have
\[
\text{Diff}_S = (1 - 1/m)C + \text{(other terms)}.
\]

4.3. Consider the case when \( g(S) \) is a point. By Lemma 2.3, as in the proof of Proposition 2.4, one can show that there exists 1, 2, 3, 4, or 6-complement of the form \( K_S + C + \text{(other terms)} \) on \( S \). By Adjunction, \( p_a(C) = 1 \). Therefore, \( p_a(C) = 1 \) and we have equality in (4). Thus \( S \) is smooth at \( \text{Sing}(C) \) (whenever \( \text{Sing}(C) \neq \emptyset \)). We have a contradiction by Proposition 2.4.

4.4. Consider the case when \( g(S) \) is a curve. Note that \( S \) is rational (because \( (S, \text{Diff}_S) \) is a klt log del Pezzo, see e. g. [P3, Sect. 5.5]) and so is \( g(S) \). Consider the restriction \( g_S : S \to g(S) \). Since \( p_a(C) \geq 1 \), \( C \) is not a section of \( g_S \). Let \( \ell \) be the general fiber of \( g_S \). Then \( \ell \simeq \mathbb{P}^1 \) and
\[
2 = -K_S \cdot \ell > \text{Diff}_S \cdot \ell \geq (1 - 1/m)C \cdot \ell \geq \frac{6}{7}C \cdot \ell.
\]
Thus \( C \cdot \ell = 2 \) and \( C \) is a 2-section of \( g_S \). Moreover,
\[
(\text{Diff}_S -(1 - 1/m)C) \cdot \ell < 2 - 2(1 - 1/m) = 2/m < 1/2.
\]
Hence \( \text{Diff}_S \) has no horizontal components other than \( C \). Let \( P := g(\ell) \), let \( X' \) be a germ of a general hyperplane section through \( P \) and let \( Y' := g^{-1}(X') \). Consider the induced (birational) contraction \( g' : Y' \to X' \). Since singularities of \( X \) are isolated, \( X' \) is smooth. By Bertini Theorem, \( K_{X'} + \ell \) is plt. Further, \( Y' \) has exactly two singular points \( C \cap \ell \) and these points are analytically isomorphic to \( \mathbb{C}^2/Z_m(1, q) \) (see [3]). This contradicts the following lemma.

Lemma 4.5 (see [P3, Sect. 6]). Let \( \phi : Y' \to X' \ni o' \) be a birational contraction of surfaces and let \( \ell := \phi^{-1}(o')_{\text{red}} \). Assume that \( K_{Y'} + \ell \) is plt and \( X' \ni o' \) is smooth. Then \( Y' \) has on \( \ell \) at most two singular points. Moreover, if \( Y' \) has on \( \ell \) exactly two singular points,
then these are of types $\frac{1}{m_1}(1,q_1)$ and $\frac{1}{m_2}(1,q_2)$, where $\gcd(m_i,q_i) = 1$ and $\gcd(m_1,m_2) = 1$.

**Proof.** We need only the second part of the lemma. So we omit the proof of the first part. Let $\text{Sing}(Y') = \{P_1, P_2\}$. We use topological arguments. Regard $Y'$ as a analytic germ along $\ell$. Since $(X' \ni o')$ is smooth, $\pi_1(Y' \setminus \ell) \simeq \pi_1(X' \setminus \{o'\}) \simeq \pi_1(X') = \{1\}$. On the other hand, for a sufficiently small neighborhood $Y' \supset U \ni P_i$ the map $\pi_1(U_i \cap \ell \setminus P_i) \to \pi_1(U_i \setminus P_i)$ is surjective (see [K, Proof of Th. 9.6]). Using Van Kampen’s Theorem as in [Mo, 0.4.13.3] one can show that

$$\pi_1(Y' \setminus \{P_1, P_2\}) \simeq \langle \tau_1, \tau_2 \rangle / \langle \tau_1^{m_1} = \tau_2^{m_2} = 1, \tau_1 \tau_2 = 1 \rangle.$$

This group is nontrivial if $\gcd(m_1, m_2) \neq 1$, a contradiction. 

**Proof of Lemma 4.1.** The map $Y \dashrightarrow \overline{Y}$ is a composition of log flips:

$$Y = Y_0 \dashrightarrow Y_1 \dashrightarrow \cdots \dashrightarrow Y_N = \overline{Y}$$

$$\downarrow g \quad \searrow W_1 \quad \swarrow W_2 \quad \cdots \quad \searrow W_N \quad \swarrow$$

where every contraction $\searrow$ is $(K + S + (1+\varepsilon)A)$-negative and every $\swarrow$ is $(K + S + (1-\varepsilon)A)$-negative. Kawamata-Viehweg Vanishing [KMM, Th. 1-2-5] implies that exceptional loci of these contractions are trees of smooth rational curves [Mo, Cor. 1.3]. Thus Lemma 4.1 is obvious if the curve $\overline{U}$ is nonrational. From now on we assume that $\overline{U}$ is a (singular) rational curve.

**Lemma 4.6.** Notation as above. Let $S_i$ be the proper transform of $S$ on $Y_i$. If $f$ is not a flipping contraction, then $-S_i$ is ample over $W_i$ for $i = 1,\ldots,N$. In particular, all nontrivial fibers of $Y_i \to W_i$ are contained in $S_i$.

**Proof.** We claim that $-S_i$ is not nef over $Z$ for $i = 1,\ldots,N-1$. Indeed, assume $\text{Exc}(f) \neq f^{-1}(o)$. Take $o' \in f(\text{Exc}(f))$, $o' \neq o$ and let $\ell \subset g^{-1}(f^{-1}(o'))$ be any compact irreducible curve. Clearly, $Y \dashrightarrow Y_i$ is an isomorphism along $\ell$. Let $\ell_i$ be the proper transform of $\ell$ on $Y_i$. Since $S \cdot \ell = 0$, we have $S_i \cdot \ell_i = 0$. The curve $\ell_i$ cannot generate an extremal ray (because extremal contractions on $Y_1,\ldots,Y_{N-1}$ are flipping). If $-S_i$ is nef over $Z$, then taking into account that $\rho(Y_i/Z) = 2$ we obtain $S_i \equiv 0$, a contradiction. Thus we may assume that $\text{Exc}(f) = f^{-1}(o)$ is a (prime) divisor. Then the exceptional locus of $Y_i \to Z$ is compact. If $-S_i$ is nef, this implies $S_i = \text{Exc}(Y_i \to Z)$. Again we have a contradiction because the proper transform of $\text{Exc}(f)$ does not coincide with $S_i$. 

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We prove our lemma by induction on $i$. It is easy to see that $-S$ is ample over $W_0 := X$. The Mori cone $NE(Y_i)$ is generated by two extremal rays. Denote them by $R_i$ and $Q_i$, where $R_i$ (resp. $Q_i$) corresponds to the contraction $Y_i \rightarrow W_i$ (resp. $Y_i \rightarrow W_{i+1}$). Suppose that our assertion holds on $Y_{i-1}$, i.e. $S_{i-1} \cdot R_{i-1} < 0$. By our claim above, $S_{i-1} \cdot Q_{i-1} > 0$ and after the flip $Y_{i-1} \rightarrow Y_i$ we have $S_i \cdot R_i < 0$. This completes the proof of the lemma.

4.7. Let us consider the case when $f$ is not a flipping contraction. Let $C(i)$ be the proper transform of $C$ on $Y_i$. If $p_a(C(i)) \geq 1$, then $Y_i \rightarrow W_i$ cannot contract $C(i)$. Thus $C(i+1)$ is well defined. Now we need to show only that on each step of (5) no flipping curves $\text{Exc}(Y_i \rightarrow W_i)$ pass through singular points of $C(i)$. (Then $Y_i \rightarrow Y_{i+1}$ is an isomorphism near singular points of $C(i)$ and we are done). By Lemma 4.6 all flipping curves $\text{Exc}(Y_i \rightarrow W_i)$ are contained in $S_i$. Therefore we can reduce problem in dimension two. The last claim $\text{Exc}(Y_i \rightarrow W_i) \cap \text{Sing}(C(i)) = \emptyset$ easily follows by the lemma below.

**Lemma 4.8.** Let $\varphi: S \rightarrow \tilde{S} \ni \hat{0}$ be a birational contraction of surfaces and let $\Delta = \sum \delta_i \Delta_i$ be a boundary on $S$ such that $K_S + \Delta$ is klt and $-(K_S + \Delta)$ is $\varphi$-ample. Put $\Theta := \sum_{\delta_i \geq 6/7} \Delta_i$. Assume that $\varphi$ does not contract components of $\Theta$. Then $\Theta$ is smooth on $\varphi^{-1}(\hat{0}) \setminus \text{Sing}(S)$.

**Proof.** Assume the converse and let $P \in \text{Sing}(\Theta) \cap (\varphi^{-1}(\hat{0}) \setminus \text{Sing}(S))$. Let $\Gamma$ be a component of $\varphi^{-1}(\hat{0})$ passing through $P$. Then $\Gamma \simeq \mathbb{P}^1$. There is an $n$-complement $\Delta^+ = \sum \delta_i^+ \Delta_i$ of $K_S + \Delta$ near $\varphi^{-1}(\hat{0})$ for $n \in \{1, 2, 3, 4, 6\}$ (see [Sh, Th. 5.6], [Ut, Cor. 19.10], or [P3, Sect. 6]). By the definition of complements, $\delta_i^+ \geq \min \{1, \lfloor (n+1) \delta_i \rfloor / n \}$ for all $i$. In particular, $\delta_i^+ = 1$ whenever $\delta_i \geq 6/7$, i.e. $\Theta \leq \Delta^+$. This means that $K_S + \Theta$ is lc. Since $P \in \text{Sing}(\Theta)$, $K_S + \Theta$ is not plt at $P$. Therefore $\Theta = \Delta^+$ near $P$ and $\Gamma$ is not a component of $\Delta^+$. We claim that $K_S + \Gamma$ is lc. Indeed, $\Gamma$ is lc at $P$ (because both $\Gamma$ and $S$ are smooth at $P$). Assume that $K_S + \Gamma$ is not lc at $Q \neq P$. Then $K_S + (1-\varepsilon)\Gamma + \Delta^+$ is not lc at $P$ and $Q$ for $0 < \varepsilon \ll 1$. This contradicts Connectedness Lemma [Sh, 5.7], [Ut, 17.4]. Thus $K_S + \Gamma$ is lc and we can apply Adjunction:

$$(K_S + \Delta^+ + \Gamma)|_{\Gamma} \geq (K_S + \Theta + \Gamma)|_{\Gamma} = \text{Diff}_{\Gamma}(\Theta).$$

Since $K_S + \Delta^+ \equiv 0$ over $\tilde{S}$ and $\Gamma \simeq \mathbb{P}^1$, we have $\deg \text{Diff}_{\Gamma}(\Theta) < 2$. On the other hand, the coefficient of $\text{Diff}_{\Gamma}(\Theta)$ at $P$ is equal to $(\Theta \cdot \Gamma)_{P} \geq 2$, a contradiction. \qed
4.9. Finally let us consider the case when $f$ is a flipping contraction. If $-S_i$ is ample over $W_i$ for $i = 1, \ldots, N$, then we can use arguments of [4,7]. From now on we assume that $S_I$ is nef over $W_I$ for some $1 \leq I \leq N$.

Let $L$ be an effective divisor on $Z$ passing through $o$. Take $c \in \mathbb{Q}$ so that $K_X + cf^*L$ is lc but not klt. By Base Point Free Theorem [KMM, 3-1-1], there is a member $M \in \left| -n(K_X + cf^*L) \right|$ for some $n \in \mathbb{N}$ such that $K_X + cf^*L + \frac{1}{n}M$ is lc (but not klt). Thus, we may assume $F = cf^*L + \frac{1}{n}M$. Let $K_Y + S + B' = g^*(K_X + cf^*L)$ be the crepant pull-back. Write $B = B' + B''$, where $B', B'' \geq 0$. Then $-(K_Y + S + B')$ is nef over $Z$ and trivial on fibers of $g$. Run $(K_Y + S + B')$-MMP over $Z$. Since $K_Y + S + B' \equiv -B'' \neq 0$, this $\mathbb{Q}$-divisor cannot be nef until $S$ is not contracted. Therefore after a number of flips we get a divisorial contraction:

$$Y \twoheadrightarrow Y_1 \twoheadrightarrow \cdots \twoheadrightarrow Y_N = \overline{Y} \twoheadrightarrow \cdots Y_{N'} \twoheadrightarrow X'.$$

Since $\rho(Y_i/Z) = 2$ the cone $\nabla E(Y_i/Z)$ has exactly two extremal rays. Hence the sequence (5) is contained in (6).

**Claim 4.10.** $S_j$ is nef over $W_j$ and $-S_j$ is ample over $W_{j+1}$ for $I \leq j \leq N'$.

**Proof.** Clearly, $-S_I$ is ample over $W_{I+1}$ (because $S_I$ cannot be nef over $Z$). After the flip $Y_I \twoheadrightarrow Y_{I+1}$ we have that $S_{I+1}$ is ample over $W_{I+1}$. Continuing the process we get our claim. \hfill $\Box$

Further, $X'$ has only terminal singularities. Indeed, $X'$ is $\mathbb{Q}$-factorial, $\rho(X'/Z) = 1$ and $X' \to Z$ is an isomorphism in codimension one. Therefore, one of the following holds:

(i) $-K_{X'}$ is ample over $Z$, then $X' \simeq X$;
(ii) $K_{X'}$ is numerically trivial over $Z$, then so is $K_X$, a contradiction;
(iii) $K_{X'}$ is ample over $Z$, then $X \twoheadrightarrow X'$ is a flip and $X'$ has only terminal singularities [KMM, 5-1-11].

This shows also that $Y_{N'} \to X'$ is a plt blowup. Then we can replace $X$ with $X'$ and apply arguments of [4,7]. This finishes the proof of Theorem [1,3]. \hfill $\Box$

**Concluding Remark.** Shokurov’s classification of exceptional log del Pezzos with $\delta \geq 1$ uses reduction to the case $\rho = 1$. More precisely, this method uses the following modifications: $\overline{S} \leftarrow S^\bullet \to S^\circ$, where $S^\bullet \to \overline{S}$ is the blow up of all divisors with discrepancy $a(E, \text{Diff}_{\overline{S}} \geq 6/7$, $S^\bullet \to S^\circ$ is a sequence of some extremal contractions and $\rho(S^\circ) = 1$. Then all divisors with discrepancy $\geq 6/7$ are nonexceptional on $S^\circ$. In our case, a smooth elliptic curve with coefficient $\geq 6/7$ on $S^\circ$
cannot be contracted to a point on $S$ (because the singularities of $S$ are rational). By Theorem 1.5 this case does not occur. The situation in the case of a singular rational curve with coefficient $\geq 6/7$ on $S^o$ which is contracted to a point on $S$ is more complicated. This case will be discussed elsewhere.

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