Optimal architecture for a non-deterministic noiseless linear amplifier

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Non-deterministic quantum noiseless linear amplifiers are a new technology with interest in both fundamental understanding and new applications. With a noiseless linear amplifier it is possible to perform tasks such as improving the performance of quantum key distribution and purifying lossy channels. Previous designs for noiseless linear amplifiers involving linear optics and photon counting are non-optimal because they have a probability of success lower than the theoretical bound given by the theory of generalised quantum measurement. This paper develops a theoretical model which reaches this limit. We calculate the fidelity and probability of success of this new model for coherent states and Einstein-Podolsky-Rosen (EPR) entangled states.

I. NOISELESS AMPLIFICATION AS A GENERAL MEASUREMENT

An ideal noiseless amplifier performs the operation $g^0\hat{a}$ [2], that is it takes an input state $|\psi\rangle$ to $g^0\hat{a} |\psi\rangle$. This operator takes the coherent state $|\alpha\rangle$ to the coherent state $|g\alpha\rangle$ and is inherently not unitary. This suggests that a measurement process with post-selection on the measurement outcomes is required to implement it. The case we are most interested in here is where $g > 1$. In this situation the operator is unbounded and can only be implemented perfectly over the entire Hilbert space via a measurement process with probability zero. In many experimental situations the action of this operator on states with high occupation number are not important as they have negligible amplitude. Therefore this operator is generally chosen to be truncated at some occupation number $N$, which will be chosen depending on the desired performance of an experimental apparatus. This truncation allows for non-zero probabilities of successfully implementing the desired amplification transformation. Lower values of $N$ will generally result in higher probabilities of success at the cost of a lower fidelity of operation when compared to the ideal operation. In current experiments with low energy inputs $N = 1$ is sufficient to achieve high fidelity, and this very simple case has non-trivial implications.

When constructing a measurement which implements the amplification, it suffices to consider the case where there is only two outcomes, a success outcome and a failure outcome. When a success outcome is achieved the state is transformed in the required way. Measurement outcomes, which we will label $i$ are represented by $S$ for success and $F$ for failure. The action on the input state due to each measurement result can be represented by the generally non-unitary operator $\hat{M}_i$ called the measurement operator. The probability of success for this measurement outcome when the measurement is applied to the state $|\psi\rangle$ is given by

$$P_i = \langle \psi | \hat{M}_i^\dagger \hat{M}_i |\psi\rangle$$

and the resultant output state having achieved the result
is

\[ |\psi_i'\rangle = \frac{\hat{M}_i |\psi\rangle}{\sqrt{P_i}}. \] (2)

To ensure that these operators define a probability measure the condition

\[ \hat{M}_S \hat{M}_S^\dagger + \hat{M}_F \hat{M}_F^\dagger = \hat{I} \] (3)

must be satisfied [17].

To implement the amplification we require \( \hat{M}_S \propto g^a \). To ensure (3) holds over the entire Hilbert space it would be necessary for \( \hat{M}_S = 0g^a \) as the eigenvalues of \( g^5 \) are unbounded for \( g > 1 \) and \( \hat{M}_F \hat{M}_F^\dagger \) must be a positive operator. Now we can make the truncation of this operator to achieve a non-zero probability. We do this by requiring the action on the first \( N \) Fock states to be proportional to the those same elements for the perfect amplification operator and leaving the action on higher occupation number states arbitrary. In this case the success measurement operator can be written as

\[ \hat{M}_S = N \sum_{n=0}^N g^n |n\rangle \langle n| + \sum_{n=N+1}^\infty S_n |n\rangle \langle n|, \] (4)

where \( S_n \) is a sequence of complex numbers with norm between zero and one. This will then allow the operation to satisfy (3) with \( N \) playing the role of the proportionality constant and will in general be non-zero. The probability of success for an arbitrary input state \( |\psi\rangle \) is prepared in the \( |F\rangle \) state before the interaction. The interaction is given by the unitary operator

\[ \hat{U} = \hat{M}_S \otimes |S\rangle \langle F| + \hat{M}_F \otimes |F\rangle \langle F| + \hat{B}_1 \otimes |F\rangle \langle S| + \hat{B}_2 \otimes |S\rangle \langle S| \] (8)

where \( \hat{M}_S \) is the operator which will be applied to the system input state when a success result is measured and \( \hat{M}_F \) is the operator applied to the system on measuring the failure result. The particular form of the operators \( \hat{B}_{1,2} \) are not of concern as they are dependent on the apparatus being initialised in the \( |S\rangle \) state. They are included to include enough freedom to ensure that \( \hat{U} \) remains unitary. Using the Kronecker product representation of the tensor product the unitarity requirement can be written as

\[ \left( \begin{array}{c} \hat{M}_F^\dagger \\ \hat{B}_1^\dagger \\ \hat{B}_2^\dagger \end{array} \right) \left( \begin{array}{c} \hat{M}_S^\dagger \\ \hat{B}_1 \\ \hat{B}_2 \end{array} \right) = \left( \begin{array}{cc} \hat{I} & 0 \\ 0 & \hat{I} \end{array} \right) \] (9)

which can be rewritten as

\[ \hat{M}_F^\dagger \hat{M}_F + \hat{B}_1^\dagger \hat{B}_1 + \hat{B}_2^\dagger \hat{B}_2 = \hat{I} \] (10)

\[ \hat{B}_1^\dagger \hat{B}_1 + \hat{B}_2^\dagger \hat{B}_2 = 0 \] (11)

\[ \hat{B}_1 \hat{B}_1 + \hat{B}_2 \hat{B}_2 = \hat{I} \] (12)
FIG. 1. A bosonic system (labeled “System”) interacts with a two-level apparatus (labeled “Apparatus”). The apparatus is prepared into a $Z$ axis spin eigenstate. The interaction applies a conditional unitary rotation where the conditioning depends on the number of bosons in the input. The apparatus is measured and if the spin has flipped then a success is heralded.

Provided $\hat{M}_S$ and $\hat{M}_F$ define a set of measurement operators (in particular the requirement in equation (6)) then the first and last equations are always satisfied if $\hat{B}_1 = \pm \hat{M}_S$ and $\hat{B}_2 = \pm \hat{M}_F$. The second equation could never be satisfied had we swapped the success and failure operators in this assignment. If $\hat{M}_S$ and $\hat{M}_F$ are Hermitian and commute, as is the case we are considering here, then we can always satisfy the second equation by choosing $\hat{B}_1 = -\hat{M}_S$ and $\hat{B}_2 = \hat{M}_F$.

Now we can substitute our success operator from equation (13) into this interaction unitary. This unitary can then be rearranged to be written as

$$\hat{U} = \sum_{n=0}^{\infty} |n\rangle \langle n| \otimes \hat{R}_n$$

where $\hat{R}_n$ is defined as

$$\hat{R}_n = \left( \begin{array}{cc} \sqrt{1 - G_n^2} & G_n \\ -G_n & \sqrt{1 - G_n^2} \end{array} \right),$$

$$G_n = \min(1, g^{n-N}).$$

The operator $\hat{R}_n$ is a Pauli Y-rotation of $\theta = 2\arcsin(\min(1, g^{n-N}))$ on the heralding qubit which depends on the number of bosons in the input mode. This unitary can be generated by the Hamiltonian

$$\hat{H} = \frac{\hbar}{\tau} \left[ \sum_{n=0}^{\infty} \arcsin(\min(1, g^{n-N})) |\hat{Y}| \otimes |n\rangle \langle n| \right]$$

$$= \frac{\hbar}{\tau} \arcsin(\min(1, g^{a^\dagger a-N})) \otimes \hat{Y},$$

where $\tau$ is the interaction time which is chosen to ensure the appropriate that the rotation parameter $\theta$ is implemented.

A. Low photon number limit

In the limit of low amplitude inputs we can implement the amplifier with $N = 1$. The system can then be considered a qubit and the gate between the system and the apparatus is locally equivalent to a standard controlled rotation. To see this, we take the unitary from equation (13)

$$\hat{U}_{N=1} = |0\rangle \langle 0| \otimes \left( \begin{array}{cc} \sqrt{1-1/g^2} & 1/g \sqrt{1-1/g^2} \\ -1/g & \sqrt{1-1/g^2} \end{array} \right) + |1\rangle \langle 1| \otimes \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)$$

(17)

and then decompose it into

$$\hat{U}_{N=1} = -(X \otimes X)(I \otimes Z)C(R_y(\theta))(X \otimes I)$$

(18)

where $X$ and $Z$ are the standard Pauli matrices and $C(R_y(\theta))$ is a controlled Pauli Y rotation by $\theta$ and $\theta$ is as defined above with $n = 0$ and $N = 1$. Applying this unitary to states of the form $|0\rangle + \alpha |1\rangle$ where $\alpha$ is small results in the probability of success for the noiseless amplification of $\frac{1}{\sqrt{r}}$.

III. COHERENT STATE INPUTS

We can now calculate the performance of this model for particular situations. First we will calculate the action on coherent states. Coherent states are an ideal test of the particular situations. First we will calculate the action on coherent states. Coherent states are an ideal test of the

$$g^{a^\dagger a} |\alpha\rangle = e^{(g^2-1)|\alpha|^2/2} |\alpha\rangle.$$  

(19)

This can then be used to calculate the probability of success and the fidelity of our model amplifier for coherent state inputs denoted by $P_c$ and $F_c$ respectively,

$$P_c = \langle \alpha | \hat{M}_S^\dagger \hat{M}_S | \alpha \rangle = e^{-|\alpha|^2} \left[ g^{-2N} \sum_{n=0}^{N} g^{2n} \frac{|\alpha|^{2n}}{n!} + \sum_{n=N+1}^{\infty} \frac{|\alpha|^{2n}}{n!} \right],$$

(20)

$$F_c = P_c^{-1} |\langle g\alpha | M_s | \alpha \rangle|^2 = P_c^{-1} e^{-(1+g^2)|\alpha|^2} \left[ g^{-N} \sum_{n=0}^{N} g^{2n} \frac{|\alpha|^{2n}}{n!} + \sum_{n=N+1}^{\infty} g^n \frac{|\alpha|^{2n}}{n!} \right]^2.$$  

(21)
These expressions can be written in terms of incomplete gamma functions

\[ P_c = P(N + 1, |\alpha|^2) + g^{-2N}e^{(g^2 - 1)|\alpha|^2}Q(N + 1, |g\alpha|^2), \]  

\[ F_c = P_c^{-1}e^{-(1+g^2)|\alpha|^2} \left| g^{-N}e^{g|\alpha|^2}P(N + 1, |g\alpha|^2) + e^{g|\alpha|^2}Q(N + 1, g|\alpha|^2) \right|^2 \]  

where \( Q(N, \lambda) \) is the regularised incomplete gamma function defined as

\[ Q(N, \lambda) = \Gamma(N, \lambda)/\Gamma(N) \]  

where \( \Gamma(N, \lambda) \) is the incomplete gamma function, \( \Gamma(N) \) is the complete gamma function and \( P(N, \lambda) = 1 - Q(N, \lambda) \). The appearance of the incomplete gamma functions here is expected as this function is the cumulative distribution function for the Poissonian distribution which is the distribution that would result when measuring a coherent state in the Fock basis. In this form these equations can be rapidly computed numerically for particular values of \( g, \alpha \) and \( N \). Figure 2 shows \( P_c \) and \( F_c \) for \( \alpha = 0.8 \) and \( N = 1, 2, 3, 4 \). The probability drops away from 1 for small gains and the rate at which this occurs increases as \( N \) increases. The fidelity initially stays close to 1 for small amplitudes but eventually drops and the gain at which this occurs increases as \( N \) increases. Whilst these properties are evident in the figure, they are general features given that \( \alpha \) is fixed.

Low fidelity operation is not of great interest for building a device which performs linear amplification. Therefore we will set a bound on performance that is deemed acceptable. Quantitatively we will require a minimum fidelity \( F \geq 0.99 \). The fidelity will increase towards 1 as \( N \) increases hence in any particular situation we can choose an \( N \) to achieve this fidelity requirement. Figure 3 shows the effect of enforcing this minimum acceptable fidelity. The most notable effect that can be seen is the discontinuous jumps in the probability of success. A jump occurs when the cut-off \( N \) is incremented to enforce the minimum fidelity. This means that the probability of success is made up of pieces from the probabilities like what is shown in figure 2 for \( \alpha = 0.8 \). Also of note, is that for low amplitude inputs (here \( \alpha = 0.1 \)) then choosing \( N = 1 \) provides an acceptable reproduction of linear amplification over a wide range of gain (here \( 1 \leq g \leq 3 \)).

**IV. EPR STATE INPUTS**

An important application of this type of amplification is distilling continuous variable entanglement. The action of the amplifier is easiest to calculate for an ideal Einstein-Podolsky-Rosen (EPR) state

\[ |EPR\rangle = \sqrt{1 - \chi^2} \sum_{n=0}^{\infty} \chi^n |n, n\rangle, \]  

where the parameter \( 0 \leq \chi < 1 \) is representative of the strength of the continuous variable entanglement. The
Two separate equations can be obtained from this, and transmission \( \eta \) state in the limit as the output state does not converge towards a particular \( N \) coefficients in the summation diverge. What this means gives the relations

\[
\eta' = \frac{g^2}{f^2} \eta,
\]

which can be inverted to give

\[
\chi' = f \chi,
\]

and the gain of the amplification \( g \) (equations 31, 32 and 33).

The possibility of non-convergence of the output state, just as seen for pure EPR inputs, is present here as well. Convergence will be achieved provided \( \chi' < 1 \).

We will consider \( \eta \) to be a fixed value and choose \( \chi' \) to be a fixed in the sense that some target squeezing strength is desired. In this way we can avoid choosing gains for which the output is not convergent.

We will focus here on the ability of the state to demonstrate the EPR paradox [19,20]. This is achieved by EPR criterion \( \varepsilon_{EPR} < 1 \) where

\[
\varepsilon_{EPR} = V_{B|A}^+ V_{B|A}^- (34)
\]

and \( V_{B|A}^\pm \) is the conditional variance of the \( B \) mode on \( A \) and the superscript represents the quadrature in which the variance is calculated. The conditional variance is defined as

\[
V_{B|A}^\pm = \min_{0 \leq g \leq 1} \langle (X_B^\pm + gX_A^\pm)^2 \rangle, (35)
\]

and for the EPR state with one sided loss the optimisation gives [23]

\[
V_{B|A}^+ = V_{B|A}^- = 1 - \frac{2 \chi^2 \eta}{1 + \chi^2}, (36)
\]

and hence the EPR criterion in this case is

\[
\varepsilon_{EPR} = \left( 1 - \frac{2 \chi^2 \eta}{1 + \chi^2} \right)^2. (37)
\]

When the amplifier succeeds, both the effective squeezing and transmission are greater than their initial counterparts. The amplifier has a purifying action on this state. This means that it is possible to reach a lower EPR criterion than would be otherwise possible.

Figure 4 shows the EPR criterion for an output squeezing of \( \chi = 0.5 \) with a channel transmission of \( \eta = 0.25 \). The lowest EPR condition possible without amplification given the channel loss (i.e. \( \chi \to 1 \)) is achieved by amplifying the lossy EPR state when \( g \approx 2.5 \).

The state conditional on achieving success is
The EPR criterion as a function of gain with a target output squeezing of \( \chi' = 0.5 \) and an initial transmission of \( \eta = 0.25 \). The red line indicates the initial EPR criterion (no amplification), while the green line represents the maximum EPR criterion that could be obtained by using a squeezed state with \( \chi \to 1 \) without making use of a noiseless amplifier.

\[
M_s|EPR_t \rangle = \sum_{t=0}^{\infty} \sum_{n=t}^{\infty} \min(g^{-N}, 1) \chi^n \sqrt{\left( \frac{n}{t} \right)^n \eta^t (1 - \eta)^{n-t} |n, t, n-t \rangle}.
\]

The probability of success for our model amplifier on this type of input state can be simply computed as

\[
P_{EPR} = \langle EPR | M_s^\dagger M_s | EPR \rangle = (1 - \chi^2)^2 \left( \frac{g^{-2N}}{1 - \chi^2(1 + (g^2 - 1)\eta)} + \sum_{n=N+1}^{\infty} \chi^{2n} \sum_{t=N+1}^{n} \left( \frac{n}{t} \right) (1 - g^{2(t-N)}) \eta^t (1 - \eta)^{n-t} \right).
\]

A sum can be removed from this equation by using the relationship

\[
\sum_{t=N+1}^{n} \left( \frac{n}{t} \right) a^t b^{n-t} = (a + b)^n I_{\frac{a+b}{a+b}}(N+1, n-N),
\]

where \( I_x(a, b) \) is the regularised incomplete beta function \( [22] \), giving

\[
P_{EPR} = (1 - \chi^2)^2 \left( \frac{g^{-2N}}{1 - \chi^2} + \sum_{n=N+1}^{\infty} \left( \chi^{2n} I_{\frac{1}{\chi^2}}(N+1, n-N) - g^{-2N} \chi^{2n} I_{\frac{1}{\chi^2}}(N+1, n-N) \right) \right).
\]

To compute fidelity is more difficult because when the loss mode is traced out the resulting state is mixed. We can calculate a lower bound on the fidelity by computing the fidelity of the amplified state compared to the purified lossy EPR state with squeezing \( \chi' \) and loss \( \eta' \), i.e.

\[
F_{EPR} = P^{-1}_{EPR} \left| \langle EPR | M_s | EPR \rangle \right|^2
\]

\[
\frac{\sqrt{F_{EPR}P_{EPR}}}{\sqrt{(1 - \chi^2)(1 - \eta^2)}} = \frac{g^{-N}}{1 - (g\sqrt{\eta\eta'} + \sqrt{(1 - \eta)(1 - \eta')})\chi'} + \sum_{n=N+1}^{\infty} (\chi\chi')^n \sum_{t=N+1}^{n} \left( \frac{n}{t} \right) (1 - g^{t-N}) \sqrt{\eta\eta'} \sqrt{(1 - \eta)(1 - \eta')}^{n-t}
\]

\[
\eta_1 = \sqrt{\eta\eta'} + \sqrt{(1 - \eta)(1 - \eta')} = \frac{1 - \eta + g\eta}{f}
\]

\[
\eta_2 = g\sqrt{\eta\eta'} + \sqrt{(1 - \eta)(1 - \eta')} = f
\]
where \( f \) is defined in Equation 33.

The probability and fidelity for \( N = 1 \) to 5 with \( \chi' = 0.5 \) and \( \eta = 0.3 \) are shown in figure 6. The probabilities drop exponentially with gain, but the fidelity drops slowly. This is because as the gain increases a lower \( \chi \) is used to ensure that \( \chi' \) stays fixed. The asymptotic behaviour of these functions as \( g \to \infty \) is

\[
P_{EPR} = g^{-2N} \left( \frac{1 - \chi'^{2N+2}}{1 - \chi^2} \right) + O(g^{-2N-1}), \quad (46)
\]

\[
F_{EPR} = g^{-2N} \left( \frac{(1 - \chi'^{2N+2})^2}{1 - \chi^2} \right) + O(g^{-2N-1}). \quad (47)
\]

Hence we find that the fidelity asymptotically approaches a constant value

\[
F_{EPR} = 1 - \chi'^{2N+2} + O(g^{-1}). \quad (48)
\]

The fidelity will always be 1 at \( g = 1 \) and for larger \( g \) then approaches this constant value from above. Therefore this number constitutes a lower bound on the fidelity.

As was indicated before in the analysis for coherent state inputs, the low fidelity operation is not usually of interest. When designing an experiment there is usually some minimum fidelity and probability of success that is deemed acceptable. The order of magnitude for these is dependant on the on the experimental conditions. We will now consider these factors to further analyse the action of this model amplifier.

We can use this expression for the limiting case of fidelity to explicitly compute a maximum \( N \) under restrictions in the fidelity and entanglement. A fidelity minimum is chosen \( F_{min} < 1 \) and at all times the performance of amplification must always be higher than this number. Also, if there is a maximum \( \chi' < 1 \) for which amplifications cannot exceed after successful amplification, then it must be true that

\[
N \leq \frac{\log (1 - F_{min})}{2 \log \chi'} - 1 \quad (49)
\]

Note that this requirement is independent of the probability of success.

To consider both a probability and fidelity bound we consider a numerical optimisation of the EPR criterion for an amplified EPR state which results a particular output squeezing \( \chi' \) which has undergone one sided loss \( 1 - \eta \).

The optimisation we will consider here enforces a fidelity greater than 0.99 and the probability of success greater than either 0.1, 0.01 and 0.001. Because of the monotonic nature of the fidelity and probability under such conditions, we find that this optimisation always occurs on the boundary of either the probability constraint or the fidelity constraint. Figure 7 shows the results of this optimisation when \( \chi' = 0.5 \) and 0.8 as a function of loss.

The results of this optimisation are best understood by starting at the case where \( \eta = 1 \). For this case we want to find if we are at the boundary of the fidelity or probability constraints whilst ensuring that both constraints are satisfied. Also, the largest possible gain which achieves the fidelity constraint will occur at the lowest value for \( N \). Therefore we seek the gain and lowest \( N \) such that our fidelity and probability constraints are satisfied. As the loss is increased, less signal is amplified and the fidelity and probability increase. Therefore a larger gain can be chosen which still satisfies the constraints. This continues until such point as the input signal is weak enough so that the next lowest \( N \) satisfies the constraints. This results in a discontinuous jump in the output. Also, if the probability was the saturated constraint, when \( N \) is decremented this may change to the fidelity constraint being the one that is saturated. As loss is increased further there will be a point where the saturation of these constraints will swap. This results in sharp corners appearing in the maximised curves for the gain and EPR criterion.

The figures also show a comparison of this best EPR criterion to particular situations not involving any amplification process. The amplification process always produces a lower EPR criterion when compared with doing no amplification. However, it is probably of more interest to compare the situation to that of assuming the entanglement source could in principle produce a maximally entangled EPR state (i.e. \( \chi = 1 \)). Because of the loss, the EPR criterion for this limiting case is not zero. Our amplification model can succeed in producing a lower EPR criterion than that of the maximally entangled source. As shown in Figure 7 this improvement occurs in high loss situations. The parameters for which this improvement occurs will depend on the value of \( \chi' \) chosen. But as shown in figure 7 the range of losses for which this occurs can cover a significant range.

V. CONCLUSION

This paper has demonstrated a new model which could be used as a noiseless phase insensitive linear amplifier. We have presented a unitary for the non-conditional evolution of a coupled harmonic oscillator system and a heralding qubit. This evolution can then be used as a probabilistic amplifier by measuring the heralding qubit after the unitary evolution. The evolution is not that of a linear optical transformation, but does achieve the
highest theoretically possible probability of success. The action of our noiseless amplification model on a coherent state and an EPR state was computed. For an EPR state undergoing one sided loss, we found that for sufficiently high loss it is possible for the amplifier to achieve an EPR criterion lower than that possible using an unamplified infinite squeezed source passing through the same loss. By choosing our parameters such that we target a particular level of two-mode squeezing when the amplification succeeds, we have shown that, for the case of single sided loss, the fidelity of the amplification has a lower bound. This model and the results we have computed here may be used as a guide to future experiments which wish to operate near the optimal probability of success.

VI. ACKNOWLEDGEMENTS

This research was conducted by the Australian Research Council Centre of Excellence for Quantum Computation and Communication Technology (Project number CE110001027).

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FIG. 7. Plots of the lowest possible EPR criterion, $N$, gain $g$, lower bound of fidelity and probability of success which achieve the lowest possible EPR criterion, $\varepsilon_{EPR}$ such that $\mathcal{F}_{\min} > 0.99$, probability of success $P > 0.1$ (solid line), 0.01 (dashed line) and 0.001 (dotted line) and output squeezing of $\chi' = 0.5$ shown in (a) and $\chi' = 0.8$ shown in (b). The x-axis of each plot is the channel throughput $\eta$. The plots showing the optimised EPR criterion have curves showing the EPR criterion when no amplification is performed (*, dash-dot line) and the EPR criterion with no amplification with an infinitely squeezed source with the same loss (**, dash-dot-dot line).
function is then \( I_z(a, b) = B_z(a, b)/B_1(a, b) \).

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