Petal-shape probability areas: complete quantum state discrimination

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We find the allowed complex numbers associated with the inner product of \( N \) equally separated pure quantum states. The allowed areas on the \textit{unitary complex plane} have the form of petals. A point inside the petal-shape represents a set of \( N \) linearly independent (LI) pure states, and a point on the edge of that area represents a set of \( N \) linearly dependent (LD) pure states. For each one of those LI sets we study the complete discrimination of its \( N \) equi-separated states combining sequentially the two known strategies: first the unambiguous identification protocol for LI states, followed, if necessary, by the error-minimizing measurement scheme for LD states. We find that the probabilities of success for both unambiguous and ambiguous discrimination procedures depend on both the module and the phase of the involved \textit{inner product complex number}. We show that, with respect to the phase-parameter, the maximal probability of discriminating unambiguously the \( N \) non-orthogonal pure states holds just when there no longer be probability of obtaining ambiguously information about the prepared state by applying the second protocol if the first one was not successful.

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I. INTRODUCTION

The discrimination among different nonorthogonal pure quantum states becomes a fundamental issue in Quantum Information and Computation Theory \cite{1,2}. There are two schemes in order to achieve the task of recognizing nonorthogonal states. The first scheme is the \textit{unambiguous quantum states discrimination} which has been written many interesting works: some generic \cite{3,4,5,6,7,8} and others concerned with its applications \cite{9,10,11,12}. That scheme requires that, whenever an outcome is projected after the measurement process, one can, with nonzero probability, infer the prepared state without error, this is, unambiguously. This can be performed at the expense of allowing for a nonzero probability of inconclusive outcomes. An \textit{optimum unambiguous states discrimination} protocol is performed when the probability of success is maximum. Thus, nowadays we know that, for discriminating with certainty any one amongst \( N \) nonorthogonal and linearly independent (LI) pure states, it will be required to map or to represent those states onto two orthogonal subspaces which are called the \textit{conclusive-subspace} and the \textit{inconclusive-subspace}. In the first \( N \)-dimensional conclusive-subspace, each possible state to be discriminated has no null component on only one state belonging to an orthonormal basis whereas, in the inconclusive-subspace each state has no null component on one state of a set of \( N \) linearly dependent (LD) states. In the second scheme, called the \textit{error-minimizing measurement} protocol, the recognition of non-orthogonal states accepts errors in the outcome results at the expense of that inconclusive outcomes are not allowed. The advantage of this scheme with respect to the former is that it can be applied to ambiguously discriminate LD states \cite{13,14,15}. In the second scheme an optimum ambiguous measurement process minimizes the probability of making a wrong guess about the prepared state.

In this article we study the \textit{complete quantum states discrimination} scheme among \( N \) equally separated pure quantum states. This \textit{complete scheme} consists in the consecutive application of both protocols described above. To be specific, first is applied the \textit{unambiguous quantum states discrimination} protocol and, if it is not successful, then the system is mapped onto the \textit{inconclusive-subspace} and then the \textit{error-minimizing measurement} protocol is applied. This \textit{complete scheme} allows getting all the obtainable information about the prepared state. Before addressing that problem we consider the characterization of the allowed complex numbers associated with the inner product of the \( N \) equally-separated pure quantum states. We find that the allowed areas on the \textit{unitary complex plane} have the form of petals and that a point inside a petal-shape represents a LI set of \( N \) pure states, and a point on the edge of a petal represents a LD set of \( N \) pure states.

II. \( N \) EQUALLY SEPARATED STATES

Let us assume that the separation between two normalized states is their inner product \cite{16}, with its module going from 0 to 1 and its phase being between 0 and \( 2\pi \). The 0 value of the module corresponds to two orthogonal states and the 1 value refers to equal or parallel states. Two different states are always LI and this property can be characterized completely by the fact that the separation-module is different from 1. However, for more
than two states the fact that the separation-modules among them are different from 1 guarantees neither the LI nor the LD property. Thus an interesting question arises in this regard: given a set $A_N(\alpha)$ which has $N$ normalized and equi-separated states, i.e.

$$A_N(\alpha) \ni \{ |\alpha_1 \rangle, |\alpha_2 \rangle, \ldots, |\alpha_N \rangle \}: \langle \alpha_k | \alpha_{k'} \rangle = \alpha, \ \forall \ k \neq k',$$

what are the allowed values of $\alpha$? For which of those values the $A_N(\alpha)$ is a set of LI or LD states?

We know that $A_N(\alpha)$ is a LI set when the identity

$$\sum_{k=1}^{N} A_k |\alpha_k \rangle = \textnormal{null}$$

is satisfied if and only if the $N$ coefficients $A_k$'s are all zero, otherwise it is a LD set. Projecting Eq. (1) onto each $|\alpha_k \rangle$ state we obtain an $N$ times $N$ homogeneous linear equations system, being the $A_k$'s the $N$ unknown quantities. From that homogeneous linear equations system one finds that the $A_N(\alpha)$ is a LI set when the following $N$ times $N$ matrix has determinant different from zero:

$$D = \det \begin{pmatrix} 1 & \alpha & \alpha & \cdots & \alpha \\
\alpha^* & 1 & \alpha & \cdots & \alpha \\
\alpha^* & \alpha^* & 1 & \cdots & \alpha \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\alpha^* & \alpha^* & \alpha^* & \cdots & 1 \end{pmatrix}_{N \times N}$$

$$= \begin{array}{c}
1 + \alpha^* \sum_{k=0}^{N-2} \left( \frac{1 - \alpha^*}{1 - \alpha} \right)^k (\alpha - 1)^{N-1}, \\
\alpha (\alpha^* - 1)^N - \alpha^* (\alpha - 1)^N
\end{array}$$

The $\alpha$-roots of $D$ correspond to sets of LD states. Solutions such as $|\alpha| < 0$, $|\alpha| > 1$, and $|\alpha| = \text{complex number}$ are not allowed.

First of all, we consider $\alpha = x$ real. It is easy to show from Eq. (2b) that, in this case, the $N$ states $\{|\alpha_k \rangle\}$, are LI if and only if

$$-\frac{1}{N - 1} < x < 1.$$  

For $x = 1$ the $N$ states are LD and all equals whereas, for $x = -1/(N - 1)$ the $N$ states are LD and form a symmetric structure in a $(N - 1)$-dimensional subspace. The range $-1 \leq x < -1/(N - 1)$ is forbidden for $N$ equally-separated normalized states. For instance, a structure of three equi-separated states becomes LD just for $x = -1/2$ which means that for this value of $x$ they can only lie on a 2-dimensional plane on which they are separated by an angle of $2\pi/3$. That family of three states is the well known trine set $\{14, 12\}$. Four equi-separated states become LD just for $x = -1/3$, i.e., they collapse symmetrically at a 3-dimensional subspace to form a tetrahedron.

Secondly, we consider the case of purely imaginary, i.e. $\alpha = iy$. From Eq. (2c) we find that $A_N(iy)$ is a LI set for

$$-\sin \frac{\pi}{2N} < y < \sin \frac{\pi}{2N}.$$  

The extreme values, $\pm \sin(\pi/2N)/\cos(\pi/2N)$, correspond to $N$ equi-separated LD states whereas the range $\sin(\pi/2N)/\cos(\pi/2N) < |y| \leq 1$ is not allowed for normalized states.

In the general case we find that $N$ equally-separated and normalized LI states can only have inner product $\alpha = |\alpha|e^{i\theta}$ which satisfies the constraint

$$0 \leq |\alpha| < |\langle \alpha_k^{LD}(\theta) | \alpha_{k'}^{LD}(\theta) \rangle|, \quad 0 \leq \theta < 2\pi,$$  

with

$$|\langle \alpha_k^{LD}(\theta) | \alpha_{k'}^{LD}(\theta) \rangle| = \sin \frac{\pi - \theta}{N} = \sin \left( \theta + \frac{\pi - \theta}{N} \right).$$

whereas, at the contour defined by $|\alpha|$ the $N$ equally-separated states are LD. Therefore, the allowed values of the inner product, $\alpha$, for $N$ equi-separated states are inside the region defined by the Eqs. (3) and (4). Thus, each point in that zone represents a LI set $A_N(\alpha)$ and each point on the outline (4) represents a LD set of $N$ equi-separated states. In other words, for a given phase, $\theta$, there are infinite LI sets $A_N(|\alpha|e^{i\theta})$ and only one LD set, $\{|\alpha_k^{LD}(\theta)\}, k = 1, 2, \ldots, N\}$ and all those families (LI's and the LD with $\theta$ fixed) preserve the phase of their inner product changing only the module. Figure 1 shows those allowed region-values of $\alpha$ with grey-degradation representing the probability petal-shapes of success for: a) $N = 3$, b) $N = 7$, c) $N = 11$ and d) $N = 31$. The clearest grey at $(0,0)$ means 1, black contour means 0, and white means the forbidden values. Thus, it is worth emphasizing that along a radius ($\theta$ fixed) the sets $A_N(|\alpha|e^{i\theta})$ preserve the phase, changing only the module of the involved inner product and just in the maximum allowed value of $|\alpha|$ the $N$ states become LD lying symmetrically on a $(N - 1)$-dimensional subspace. We can notice that the allowed surface in the complex unitary-circle decreases as $N$ increases, going from a disk for $N = 2$, passing thru forms like a petal for $N > 2$ up to tends to be much closer to the positive real axis for $N \gg 2$. Because of that reason we call those areas $(N > 2)$ probability petal-shapes.

### III. COMPLETE QUANTUM PURE STATES DISCRIMINATION

In the previous section we have characterized the sets of $N$ equi-separated states. In this section we shall describe the complete quantum pure states discrimination scheme for $N$ equi-separated and LI pure states.

Let us begin by supposing that a quantum system of interest is prepared with probability $p_k$ in the state $|\alpha_k\rangle$
which belongs to the LI set \( A_N(\alpha) \). The complete quantum pure states discrimination scheme is performed as follows: by means of a joint unitary operation, \( U \), an ancillary system, initially in a known normalized state \( \mid \alpha_0 \rangle \), is coupled to the system of interest in such a way that:

\[
\hat{U} \mid \alpha \rangle \mid \alpha_0 \rangle = \sqrt{1-|s|^2} \mid \perp \rangle_a + s|\alpha_{LD}(\theta)\rangle \mid + \rangle_a,
\]

where the two ancillary normalized states, \( \mid \perp \rangle_a \) and \( \mid + \rangle_a \), are orthogonal. The set \{ \mid k \rangle, k = \{1, 2, \ldots, N\} \} \mid k' \rangle = \delta_{k,k'} \} defines an orthonormal basis in the unambiguous-subspace of the system of interest. The \{ \mid \alpha_{LD} \rangle \} are \( N \) LD states lying on the ambiguous-subspace of the system of interest. That LD set allows preserving the phase of the inner products and it is unique as we saw in Section II. The \( s \) probability amplitude helps to preserve the module of the inner product and, due to the symmetry, it does not depend on \( k \).

Hence, by performing a measurement process on the auxiliary system on the \{ \mid \perp \rangle_a, \mid + \rangle_a \} basis, the system of interest is mapped with probability \( P_s = 1 - |s|^2 \) onto the unambiguous-subspace and with probability \( P_l = |s|^2 \) onto the ambiguous-subspace. Therefore, after that measurement procedure we have the possibility of obtaining unambiguously the information about the prepared state or of obtaining inconclusive information about the prepared state by implementing the so called LD states recognition with minimal-error protocol on the \{ \mid \alpha_{LD} \rangle \} set [13, 14, 15].

### A. Unambiguous discrimination probability

Since the inner product is preserved under a unitary transformation, from Eq. (5) we get:

\[
\langle \alpha_k | \alpha_{k'} \rangle = |s|^2 \langle \alpha_{LD}^{\prime} | \alpha_{LD} \rangle \langle \alpha_{k'}^{LD} | \alpha_{k}^{LD} \rangle, \\
|\alpha\rangle |e^{i\theta} = |s|^2 \langle \alpha_{LD}^{\prime} | \alpha_{LD} \rangle |e^{i\theta}.
\]

From where we obtain

\[
|s|^2 = \frac{|\alpha|^2}{|\alpha_{k}^{LD}(\theta) |\alpha_{k'}^{LD}(\theta)|}.
\]

Therefore, the probability of discriminating unambiguously the \( |\alpha_k\rangle \) state is

\[
P_k = p_k(1 - |s|^2) = p_k \left( 1 - \frac{|\alpha|^2}{|\alpha_{k}^{LD}(\theta) |\alpha_{k'}^{LD}(\theta)|} \right),
\]

and the probability of discriminating unambiguously whichever be the prepared state, becomes

\[
P_s = \sum_k P_k,
\]

\[
= 1 - \frac{|\alpha|^2}{|\alpha_{k}^{LD}(\theta) |\alpha_{k'}^{LD}(\theta)|},
\]

\[
= 1 - |\alpha| \frac{\sin \pi \theta}{\sin \left( \theta + \frac{\pi}{N} \right)}. \quad (7)
\]

First of all we emphasize two limits: i) for \( N = 2 \) and independently of the \( \theta \) value the (7) probability of success, \( P_s \), becomes the Peres’s formula [3], \( 1 - |\alpha| \), which also holds for all \( N \geq 2 \) when \( \theta = 0 \), ii) the (7) probability of success is zero on the outline (4) which is in agreement with the fact that a set of LD states can not be unambiguously discriminated. We can also notice that the (7) probability of success depends on both the phase and the module of the involved \( \alpha \). Specifically \( P_s \) is linear with respect to \( |\alpha| \) having its maximum value, 1, at \( |\alpha| = 0 \) and its minimum value, 0, at the \( |\alpha| = |\langle \alpha_{k}^{LD}(\theta) |\alpha_{k'}^{LD}(\theta)| \rangle| \) cut-contour (see Fig. II). Figure IIa shows that linear behavior considering \( N = 7 \) for different phases: \( \theta = 0 \) (solid), \( \theta = \pi/11 \) (dashes), \( \theta = \pi/5 \) (dots), and \( \theta = \pi \) (dash-dots). On the other hand, we can notice that for a given \( N > 2 \); for \( 0 < |\alpha| \leq 1/(N-1) \), all the range \( 0 \leq \theta < 2\pi \) is allowed, whereas for \( 1/(N-1) < |\alpha| < |\langle \alpha_{k'}^{LD}(\theta) \rangle |\alpha_{k'}^{LD}(\theta)| \) the range of the phase is restricted having a wide window of values non-allowed. Figure IIb shows the (7) probability as a function of the phase considering \( N = 7 \) for different modules of \( \alpha \): \( |\alpha| = 1/17 \) (solid), \( |\alpha| = 1/8 \) (dashes), \( |\alpha| = 1/5 \) (dots), and \( |\alpha| = 1/3 \) (dash-dots).

We emphasize that for a given and allowed module \( |\alpha| \) the maximal probability of discriminating conclusively \( N(> 2) \) equally-separated states holds for the phase \( \theta = 0 \) and curiously, in this case, the ambiguous-subspace is 1-dimensional since the \( N \) equispaced LD states, \{ \mid \alpha_{k}^{LD}(\theta) \rangle \}, are all equals. In other words, in the optimal case all the possible information to be gotten about the prepared states, is acquired unambiguously. On the other hand, for \( \theta \neq 0 \) each set, at the contour of the petal-shape, has \( N \) equispaced LD states symmetrically distributed in a \((N-1)\)-dimensional Hilbert subspace. In those cases \( \theta \neq 0 \), at the ambiguous-subspace there can even be information about the prepared state.
FIG. 2: Probability of success, $P_s$: a) as a function of $|\alpha|$ for the different values of the phase $\theta = 0$ (solid), $\theta = \pi/11$ (dashes), $\theta = \pi/5$ (dots), and $\theta = \pi$ (dash-dots); b) as a function of $\theta$ for the different modules $|\alpha| = 1/17$ (solid), $|\alpha| = 1/8$ (dashes), $|\alpha| = 1/5$ (dots), and $|\alpha| = 1/3$ (dash-dots). For these graphics we have considered $N = 7$.

and, even though this can no longer be unambiguously obtained, it can be inconclusively obtained by means of the so called LD states recognition with minimal-error protocol [12, 13, 14]. Therefore, we can say that the non-maximal cases ($\theta \neq 0$) of obtaining certainly the precedence of the state is at the expense of that part of the information that can even be got with minimal-error.

B. Ambiguous discrimination probability for equi-separated LD states

When the unambiguous discrimination states process fails, the state of the system of interest is projected, with probability $p_k$, onto the state $|\alpha_k^{LD}(\theta)\rangle$, see Eq. (5). Those states belong to a LD set of $N$ equi-separated states symmetrically distributed at a $(N - 1)$--dimensional Hilbert subspace. We would like to emphasize that for this analysis the $N$ equi-separated LD states must be different. In other words, our analysis is valid for $\theta \neq 0$ since the $\theta = 0$ case is a singularity because the $N$ states collapse to only one when $\theta$ goes to $0$ (since $|\alpha_k^{LD}(0)|^2$ and $|\alpha_k^{LD}(0)|^2 = 1$). In addition, an obvious restriction to our analysis is $N > 2$.

Therefore, in this minimal-error discrimination scheme the lowest error probability will be achieved by means of a measurement process of the observable $\hat{\Omega}$ whose eigenstates $\{|\omega_k\rangle, k = 1, 2, \ldots, N, \omega_k|\omega_k\rangle = \delta_{k,k'}\}$ are in a one-to-one correspondence with each possible state of the LD set, $\{|\alpha_k^{LD}(\theta)\rangle, k = 1, 2, \ldots, N\}$, of the system of interest. Let us consider that each eigenstate $|\omega_k\rangle$ has two components: one of them is lengthways parallel to its corresponding $|\alpha_k\rangle$ and the other one has a direction $|\Delta_k\rangle$ which is orthogonal to the ambiguous-subspace, this is,

$$|\omega_k\rangle = \sqrt{1-|\alpha|^2}|\Delta_k\rangle + a|\alpha_k^{LD}(\theta)\rangle.$$  

(8)

We can notice that $|\alpha_k^{LD}(\theta)\rangle$ has component $a$ on its associated $|\omega_k\rangle$ and has component $a(|\alpha_k^{LD}(\theta)\rangle|\alpha_k^{LD}(\theta)\rangle)$ over any $|\omega_k\rangle$ with $k > k'$. From the identity

$$\sum_{m=1}^{N} \langle\alpha_k^{LD}(\theta)|\omega_m\rangle\langle\omega_m|\alpha_k^{LD}(\theta)\rangle = 1$$

$$|a|^2(N-1)|\langle\alpha_k^{LD}(\theta)|\alpha_k^{LD}(\theta)\rangle|^2 + |a|^2 = 1,$$

we obtain the square module of the $a$ probability amplitude:

$$|a|^2 = \frac{1}{1 + (N-1)|\langle\alpha_k^{LD}(\theta)|\alpha_k^{LD}(\theta)\rangle|^2}.$$  

(9)

From Eqs. (5), (8), and (4) we can infer that the probability, $P_{ci}$, that the state will be correctly identified is:

$$P_{ci} = \sum_{k=1}^{N} p_k|a|^2$$

$$= \frac{1}{1 + (N-1)\sin^2(\theta/2+\omega/2N)}.$$  

(10)

We notice that the minimum $P_{ci}$ probability holds for $\theta \rightarrow 0$ whereas the maximum value is reached at $\theta = \pi$. In other words, $P_{ci}$ increases in the range $0 < \theta \leq \pi$ going from $1/N$ to $1-1/N$. On the other hand, in the range $0 < \theta \leq \pi$ and for $N \gg 2$ the $P_{ci}$ probability has the behavior

$$P_{ci}^{(N \gg 2)} \approx 1 - \frac{(\theta - \pi)^2}{N \sin^2 \theta} + O\left(\frac{1}{N^2}\right).$$

From here we can see that the probability of correctly identifying the $N$ equi-separated LD states comes closer to $1$ for $N \gg 2$; therefore, in this limit, the probability of erroneously inferring the prepared state comes closer to $0$.

According to the described complete quantum pure states discrimination scheme, see Eq. (5), the total probability, $P$, of obtaining information about the prepared state of the system of interest becomes:

$$P = P_s + (1-P_s)P_{ci}$$

$$= 1 - \frac{|\alpha|(N-1)\sin(\omega/2N)}{1 + (N-1)\sin^2(\theta/2+\omega/2N)}.$$  

(11)

We can define two more probabilities which give us knowledge about the complete quantum pure states discrimination scheme: The $P_{err} = (1-P_s)(1-P_{ci})$ which is the probability of inferring the prepared state with error and $P_{no-\ell} = 1 - P$ which corresponds to the probability of not obtaining information about the prepared state of the system of interest. We notice that the $P$ probability is linear and $P_{err}$, which is equal to $P_{no-\ell}$, is quadratic with respect to the allowed $|\alpha|$. In the Subsection IIIA we find that for a given and allowed $|\alpha|$ the maximum
value of $P_\alpha$ is reached at $\theta = 0$ and that probability decreases as $\theta$ goes from 0 to $\pi$. On the other hand, in Subsection III B we see that $P_\alpha$ increases as $\theta$ goes from 0 to $\pi$. However, since $P$ is a composition of $P_\alpha$ and $P_s$, it is not easy to estimate its behavior with respect to the $\theta$ parameter for a fixed $|\alpha|$. Figure 3 shows the $P$ (black) and $P_s$ (grey) probabilities as functions of $\theta$ for different values of $|\alpha|$: $|\alpha| = 1/17$ (solid), $|\alpha| = 1/8$ (dashed), $|\alpha| = 1/5$ (dotted), and $|\alpha| = 1/3$ (dash-dotted). In order to compare with results of Fig. 2b we have considered again $N = 7$ for each curve of Fig. 3.

![Figure 3: The $P$ (black) and $P_s$ (grey) probabilities as functions of $\theta$ for different values of $|\alpha|$: $|\alpha| = 1/17$ (solid), $|\alpha| = 1/8$ (dashed), $|\alpha| = 1/5$ (dotted), and $|\alpha| = 1/3$ (dash-dotted). Here we have considered $N = 7$.](image)

We have characterized the families of $N$ equally separated states finding the allowed values of the involved inner product. We find that the allowed surface in the complex unitary-circle decreases as $N$ increases and it looks like petal-shapes. We studied the unambiguous discrimination of those $N$ nonorthogonal pure quantum states, finding the probability of success. That probability depends on both the module and the phase of the inner product and curiously its maximum value, with respect to the phase, arises just when the respective LD states become parallel in such a way that ambiguous information can not be obtained. In all the other cases the protocol can be complemented with a minimal-error discrimination scheme. In this form we have proposed the complete quantum pure states discrimination scheme of $N$ equally-separated LI pure states which allows obtaining all the possible information about the prepared state of the system of interest.

IV. CONCLUSIONS

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