Smarr’s formula in 11-dimensional supergravity

Patrick A Haas

Department of Physics and Astronomy, University of Southern California, Los Angeles, CA 90089, United States of America

E-mail: pa.haas@gmx.de

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Abstract
We examine the Smarr formula in 11-dimensional spacetime compactified on a general six-dimensional, Ricci-flat manifold. We show that non-zero mass for smooth and horizonless solutions can only be provided by cohomology. Furthermore, we confirm the result that there are no solitons without topology and prove the fact that Chern–Simons terms in the mass formula only appear in order to generate a purely topological integral.

Keywords: Smarr formula, 11-dimensional supergravity, Komar mass, black holes, cohomological fluxes, topology, supergravity solitons

1. Introduction

In quantum mechanics the main issues associated with the classical description of a black hole are the singularity and the ‘information paradox’ following from the problem of accounting for the entropy stored inside and associated with the horizon. One way physicists have addressed this is by searching for a consistent quantum description of a black hole.

In 2002, Samir Mathur proposed a concept within the framework of string theory: The ‘fuzzball’ program [1]. In particular, the idea of centrally compressed microstates enclosed by an event horizon with nothing but vacuum in between gets replaced by a ‘fuzzy’ structure comprised of strings and branes at the horizon scale; and since this structure is not a classical fluid, it does not disagree with Buchdahl’s theorem [2].

The right value of the entropy would be precisely summed up to by properly counting the degenerate energy states of this string-brane constellation. Consequently, both the issue of the singularity and the information paradox might be resolved in this way. Also, in this picture there are no horizons anymore either. Consistency is still preserved though, since the extension of this fuzzball exactly equals the classical radius of the event horizon. On top of all that, Mathur’s solution limits to the known results for spacetime outside a black hole.
In particular, the spectrum of possible state configurations within the system gives rise to the microstate geometries, and their counting is reflected by the degree of degeneracy of the energy states.

If one considers the fuzzball solution in the supergravity limit, the microstate geometries lead to smooth, horizonless, and asymptotically flat solutions representing solitons if time-independent.

Earlier solutions of supergravity, like the ‘t Hooft–Polyakov monopole [3, 4] and other monopole and instaton solutions [5–8], have indeed been solitonic, but their characteristics are based on the topology of the given space of (gauge) field configurations rather than spacetime’s topology.

Supergravity solitons, or spacetime solitons, smooth massive and horizonless configurations in space, have long been doubted to exist though, especially since some theorems appear to rule them out [9–13]. In particular, those works showed that smoothness and absence of non-trivial topology necessarily mean zero mass according to Smarr’s formula in 3+1 dimensions; non-trivial 1-cycles could contribute to the mass formula, but the solutions would not be simply connected.

The predominant view was that there are ‘no solitons without horizons’; in other words, for a globally smooth four-dimensional spacetime, free of singularities and inner boundaries, they were clearly ruled out by those ‘no-go theorems’, especially since a Smarr formula would under such circumstances never give a non-zero result for the solitonic mass.

Gibbons and Warner showed that these ‘no-go theorems’ can be circumvented through non-trivial topology in form of 2-cycles on the solution’s spatial hypersurfaces [14]. In order to arrange non-trivial topology, in particular, homological cycles, on space-like hypersurfaces without the need for singularities and inner boundaries, one needs at least four spatial dimensions. It is the very cohomological fluxes on those cycles which give rise to a non-zero mass in virtue of a generalized Smarr formula within the framework of the Komar integral formalism [15]. This will be elaborated in more detail in this work.

An additional important finding by Gibbons and Warner was the general contribution of Chern–Simons interactions to solitonic solutions.

Since then, further work has been done upon the abovestated results, exploring the implications for the mass in various physical situations, like [16].

In this paper, the idea is studied in 11-dimensional supergravity, the classical limit of M-Theory. In particular, we generalize and examine these matters in an 11-dimensional spacetime, six dimensions of which are compactified on a Ricci-flat manifold. We also show that the Chern–Simons terms, conjectured in earlier works as necessary supplement to pure topology, in fact only play a secondary role in combining all the fluxes in the integral of the mass formula to a closed differential form and thus ensure the soliton’s mass to be purely topological.

Hence, for the existence of solitons in supergravity one still has ‘No solitons without topology’, and the Chern–Simons interactions entirely support this circumstance without adding any extra support mechanisms.

In section 2, we write out the bosonic part of the 11-dimensional supergravity action [17], allow an arbitrary constant in front of the Chern–Simons term as a ‘gauge bar’ for these interactions to explicitly see their manifestation and role in the final mass formula, and reexamine the work in [14] under these more general circumstances.

In section 3, we set up the 11-dimensional Komar integral, determine its normalization based on the known result in five-dimensions [14], and derive the 11-dimensional version of Smarr’s formula reflecting the mass of the solitonic solutions purely by topology.

In section 4, we recap the five-dimensional calculations from [14], allow in addition for non-trivial 1-forms, and again apply an arbitrary Chern–Simons coefficient, in order to show
both the purely topological nature of the mass and the process of getting there from the more general 11-dimensional case by special choices of fields and compact geometry\(^1\).

2. Preliminaries

2.1. The 11-dimensional supergravity action and equations of motion

11-dimensional supergravity has in its bosonic sector the graviton, \(g_{MN}\), and the 3-form potential, \(C_{[MNK]}\).\(^2\) The latter gives rise to a field strength 4-form, \(F = dC\).

The bosonic action \([17]\) is

\[
S_{11} = \int d^{11}x \sqrt{-g} \left( R - \frac{1}{2} |F|^2 \right) - \frac{\alpha}{6} \int C \wedge F \wedge F,
\]

(1)

where we have introduced an arbitrary constant coefficient, \(\alpha\), for the Chern–Simons term to examine the impact of the Chern–Simons interactions on the soliton mass in an explicit sense.

The Einstein equations resulting from (1) are

\[
R_{MN} - \frac{1}{2} g_{MN} R = \frac{1}{12} F_{MRSK} F^N_{RSK} - \frac{1}{96} g_{MN} F_{RSKL} F^{RSKL},
\]

(2)

which may be written,

\[
R_{MN} = \frac{1}{12} F_{MRSK} F^N_{RSK} - \frac{1}{144} g_{MN} F_{RSKL} F^{RSKL}.
\]

(3)

The Maxwell equations resulting from (1) are

\[
\nabla_N F^N_{RSK} = J_{CS, \alpha}^{RSK},
\]

(4)

with the 11-dimensional Chern–Simons 3-form current\(^3\),

\[
J_{CS, \alpha}^{RSK} = \frac{\alpha}{1152} \epsilon_{RSM_1...M_6} F^{M_7...M_{11}} F^{M_1...M_6}.
\]

(5)

Define a dual 7-form,

\[
G = *_{11} F \Leftrightarrow G_{M_1...M_7} = \frac{1}{24} \epsilon_{M_1...M_7} F^{M_8...M_{11}}.
\]

(6)

The equation of motion for \(G\) is simply the Bianchi identity for \(F\) and vice versa,

\[
\nabla_R G^{M_1...M_7}_{M_1...M_7} = \frac{1}{24} \epsilon^{M_1...M_7} F_{M_8...M_{11}} \nabla_R F^{M_8...M_{11}} = 0,
\]

(7)

and, with (4),

\[
\nabla_{[M_1} G_{M_2...M_7]} = \frac{35 \alpha}{8} F_{[M_2...M_7} F_{M_8...M_{11}]} \Leftrightarrow dG = \frac{\alpha}{2} F \wedge F.
\]

(8)

Note that

\[
G_{M_1...M_7} G^{N_1...N_7} = -150 \delta^{[N}_{M_1} F_{K_1...K_7]} F_{K_1...K_7]} = 120 F_{M_1K_1K_2} F^{K_3K_4K_5} - 30 \delta^{[N}_{M_1} F_{K_1...K_7]} F^{K_1...K_7]},
\]

(9)

\(^1\) For detailed elaboration on flux compactification, see [18–20].

\(^2\) We indicate the whole 11-dimensional spacetime with a capital latin, the non-compact five-dimensional spacetime with a greek, and the compact six-dimensional space with a small latin index.

\(^3\) The Levi-Civita tensor for curved spacetimes is related to the Levi-Civita symbol of Minkowski spacetime like

\[\epsilon_{\mu_1...\mu_7} = (-g)^{-\frac{1}{2}} \epsilon_{\mu_1...\mu_7} \Leftrightarrow \epsilon_{\mu_1...\mu_7} = (-g)^{\frac{1}{2}} \epsilon_{\mu_1...\mu_7}.\]
which allows us to rewrite (3) as

\[ R_{MN} = \frac{1}{18} F_{MRSK} F_N^{RSK} + \frac{1}{4320} G_{MS_1...S_6} G_N^{S_1...S_6}. \]  
(10)

2.2. Invariances

We assume that the matter fields have the symmetries of the metric, that is, they are invariant under a Killing vector, \( K \),

\[ L_K F = 0 = L_K G, \]  
(11)

where \( L_K \) is the corresponding Lie derivative. Cartan’s magic formula,

\[ L_K \omega = d (i_K \omega) + i_K (d \omega), \]  
(12)

applied to the 4-form \( F \), yields

\[ 0 = d (i_K F) \leftrightarrow K^M F_{MRNS} = 3 \nabla_{[N \lambda_{RS}] K} H^{(3)}_{MRN}, \]  
(13)

where \( \lambda \) are the magnetostatic 2-form potentials of \( G \) and the electrostatic 2-form potentials of \( F \), respectively, and \( H^{(3)} \) is a closed but not exact 3-form, that is, \( H^{(3)} \in H^3 (\mathcal{M}_{11}) \).

For \( G \) we find

\[ 0 = d (i_K G) + i_K (d G) \leftrightarrow d (i_K G) = -\alpha \left( d \lambda + H^{(3)} \right) \wedge F = -\alpha d \left( \lambda \wedge F - H^{(3)} \wedge C \right), \]  
\[ \leftrightarrow d \left( i_K G + \alpha \lambda \wedge F - \alpha H^{(3)} \wedge C \right) = 0, \]  
(14)

where we used (8) and (13) in the second step, and so

\[ K^M G_{M_1...M_6} = 6 \nabla_{[M_1 \lambda_{M_2...M_6]} - 15 \alpha \lambda_{[M_1M_2} F_{M_3...M_6]} + 20 \alpha H^{(3)}_{[M_1M_2M_3M_4M_5]} + H^{(6)}_{M_1...M_6}, \]  
(15)

where \( \lambda \) is a generic 5-form and \( H^{(6)} \in H^6 (\mathcal{M}_{11}) \) a closed but not exact 6-form.

From (13) and (15) follows with (4), (5) and (8),

\[ K^M F_{MRSK} F_N^{RSK} = -3 \nabla_K \left( \lambda^{SK} F_{SKN}^R \right) + \frac{\alpha}{384} \lambda^{SK} F_{SKNM_1...M_6} F^{N_1...N_6}, \]  
(16)

and hence, equation (10) can be rewritten as

\[ K^M R_{MN} = -\frac{1}{720} \nabla_K \left( 120 \lambda^{SK} F_{SKN}^R - \lambda_{S_1...S_6} G^{S_1...S_6}_{N}, \right) + \frac{1}{18} H^{(3)}_{RSK} F_N^{RSK} \]  
\[ + \frac{1}{4320} \left( 20 \alpha H^{(3)}_{S_1S_2S_3} G_{S_1S_2S_3} + H^{(6)}_{S_1...S_6} \right) G_N^{S_1...S_6}. \]  
(18)

As in [14] the \( \lambda (\ast F \wedge \lambda) \) terms cancel out, and it is important to note that this happens independently of the choice of the parameter, \( \alpha \). However, explicit Chern–Simons terms indeed go along with the inclusion of \( H^{(3)} \). As we will describe below, the analogue of this in the
analysis of [14] was omitted for the assumption of simple-connectedness of the four-dimensional slices, \( \Sigma \).

3. Komar integrals in 11-dimensional supergravity

If \( K \) is a Killing vector, then the Komar integral [14, 21–24] is

\[
\int_{\partial \Sigma} \ast dK = \int_{\partial \Sigma} (\partial_M K_N - \partial_N K_M) d\Sigma^{MN} = -2 \int_{\Sigma} R_{MN} K^M d\Sigma^N. \tag{19}
\]

If \( K \) is timelike at infinity, we can use a coordinate with \( K \approx \partial / \partial t \), so near infinity the 1-form is then

\[
K \approx g^0_0 \partial / \partial t. \tag{20}
\]

We know how to get the conserved mass, \( M \), from the five-dimensional Komar integral in [14], assuming

\[
M = \int_{\Sigma_0} T^{(D)}_{00} d\Sigma^{(D-1)} = A_D \int_{\partial S^{D-2}} \ast dK,
\tag{21}
\]

where \( A_D \) is a normalization, and in particular,

\[
A_5 = -\frac{3}{32 \pi G_5}. \tag{22}
\]

We can set up the formula for eleven dimensions,

\[
M = A_{11} \int_{M_{10}} d \ast_{11} dK = A_{11} \int_{M_6 \times M_4} d \ast_5 dK \wedge d\text{vol}_6 |_{r = \infty},
\]

\[
= A_{11} \text{vol}_6 \int_{M_6} d \ast_5 dK = A_{11} \text{vol}_6 \int_{S^9} \ast_5 dK, \tag{23}
\]

and conclude the relation of the normalization factors directly,

\[
A_{11} = \frac{A_5}{\text{vol}_6} = -\frac{3}{32 \pi G_5 \text{vol}_6}, \tag{24}
\]

where \( \text{vol}_6 \) is the volume of the \( M_6 \) at space’s infinity.

Hence, the complete 11-dimensional Komar integral is:

\[
M = -\frac{3}{32 \pi G_5 \text{vol}_6} \int_{S^9} \ast_{11} dK = \frac{3}{32 \pi G_5 \text{vol}_6} \left[ 2 \int_{\Sigma_{10}} R_{MN} K^M d\Sigma^N + \int_{\partial \Sigma_{10}} (\partial_M K_N - \partial_N K_M) d\Sigma^{MN} \right], \tag{25}
\]

also allowing for interior boundaries in addition to the sphere at infinity, \( S^9 \), in case horizons are considered as well.

Using (18) in (25) and assuming that the terms in the divergence part fall off sufficiently fast at infinity (only interior boundary terms survive after applying Stoke’s theorem), the conserved mass is given by

\[
M = \frac{3}{16 \pi G_5 \text{vol}_6} \left\{ \int_{\Sigma_{10}} \left[ \frac{1}{18} H^{(3)}_{\text{cK}} F_{sK} R_{sK} + \frac{1}{4320} \left( 20 \alpha_0 H^{(3)}_{\text{cK}, sK} G_{sK, sK} + H^{(6)}_{\text{cK}, sK} \right) G_{sK, sK} \right] d\Sigma^N \right. \\
+ \left. \int_{\partial \Sigma_{10}} \frac{1}{720} \left( 120 \lambda_{sK} F_{sKMN} + \Lambda_{sK, sK} G_{sK, sK} + \frac{1}{2} (\partial_M K_N - \partial_N K_M) \right) d\Sigma^{MN} \right\}. \tag{26}
\]
Rewritten in terms of differential forms, (26) becomes

\[
M = \frac{1}{32\pi G \text{vol}_6} \left\{ \int_{\Sigma^0} \left[ H^{(3)} \wedge (2G - \alpha C \wedge F) - H^{(6)} \wedge F \right] + \int_{\partial\Sigma^m} (2\Lambda \wedge G - \Lambda \wedge F + 3*_{11} dK) \right\}.
\]

(27)

For solitonic mass sourced by non-trivial topology within smooth, horizonless solutions, the contributions from interior boundaries are dropped:

\[
M = \frac{1}{32\pi G \text{vol}_6} \int_{\Sigma^0} \left[ H^{(3)} \wedge (2G - \alpha C \wedge F) - H^{(6)} \wedge F \right].
\]

(28)

Note that the differential form,

\[2G - \alpha C \wedge F,
\]

is closed by virtue of (8). Since \(F, H^{(3)}\) and \(H^{(6)}\) are closed by definition, the integral is purely topological for all values of \(\alpha\). This means that the contribution of explicit Chern–Simons terms in the Komar mass formula is precisely to turn the latter into an integral over cohomology—and does not add any further terms aside from pure topology.

Furthermore, the cohomological nature of the integrand ensures a non-zero value for the mass in case of non-trivial (non-exact) flux forms, like \(H^{(3)}\) and \(H^{(6)}\), without the need of singularities or horizons (inner boundaries).

4. Recap of the five-dimensional case

We are going to repeat the calculations done in [14], in addition allowing for non-trivial 1-forms in the cohomology. Like above, an arbitrary constant coefficient of the Chern–Simons term is introduced to show the interaction’s pure support of topology. Finally, we will compare the achieved result to the 11-dimensional case.

The action \([25, 26]\) is

\[
S = \int \left( \star_5 R - Q_{IJ} dX^I \wedge \star_5 dX^J - Q_{IJ} F^I \wedge F^J - \frac{\beta}{6} C_{IJK} F^I \wedge F^J \wedge A^K \right),
\]

where \(C_{IJK} = |\epsilon_{IJK}|\) and \(X^I, I = 1, 2, 3\), are scalar fields arising from reducing the 11-dimensional metric,

\[
dx_1^2 = dx_1^2 + \left( \frac{Z_2 Z_3}{Z_1^2} \right)^\frac{1}{4} (dx_2^2 + dx_3^2) + \left( \frac{Z_1 Z_3}{Z_2^2} \right)^\frac{1}{4} (dx_4^2 + dx_5^2) + \left( \frac{Z_1 Z_2}{Z_3^2} \right)^\frac{1}{4} (dx_6^2 + dx_7^2),
\]

(30)

with the reparametrization,

\[
X^1 = \left( \frac{Z_2 Z_3}{Z_1^2} \right)^\frac{1}{4}, X^2 = \left( \frac{Z_1 Z_3}{Z_2^2} \right)^\frac{1}{4}, X^3 = \left( \frac{Z_1 Z_2}{Z_3^2} \right)^\frac{1}{4},
\]

(31)

to fulfill the constraint \(X^1 X^2 X^3 = 1\).

Moreover, there is a metric for the kinetic terms,

\[
Q_{IJ} = \frac{1}{2} \text{diag} \left( \left( \frac{1}{X^1} \right)^2, \left( \frac{1}{X^2} \right)^2, \left( \frac{1}{X^3} \right)^2 \right),
\]

(32)
necessary also for the dualization of the field strength\(^4\), \(\mathcal{F}^I = d A^I\),

\[
\mathcal{G}_I = Q^I (\ast_5 \mathcal{F}^I). \tag{33}
\]

The analysis done for equation (5.8) in [14] leads with an arbitrary Chern–Simons coefficient to

\[
d \mathcal{G}_I = \frac{\beta}{4} C_{JKL} \mathcal{F}^J \wedge \mathcal{F}^K. \tag{34}
\]

4.1. Incorporating 1-forms

In [14], non-trivial 1-forms have not been considered, since their contribution was assumed to not provide new interesting physics, but here we incorporate them for completeness of the further below stated dictionary of the fields. In particular, they correspond to the 11-dimensional 3-form.

Equation (5.13) of [14] can be extended to

\[
K^\rho \mathcal{F}^I_{\rho \mu} = \partial_\mu \lambda^I + H^{(1)I}_\mu. \tag{35}
\]

As a consequence we get

\[
d (i_k \mathcal{G}_I) = -i_k d \mathcal{G}_I = -\frac{\beta}{4} C_{JKL} i_k \left( \mathcal{F}^J \wedge \mathcal{F}^M \right) = -\frac{\beta}{2} C_{JKL} \left( d \lambda^J + H^{(1)J}_\mu \right) \wedge \mathcal{F}^M,
\]

so

\[
K^\rho \mathcal{G}^I_{\rho \mu \nu} = 2 \partial_\mu \Lambda_{\nu I} - \frac{\beta}{2} C_{JKL} \left( \lambda^J \mathcal{F}^M_{\mu \nu} - 2 H^{(1)L}_\mu A^M_{\nu} \right) + H^{(2)}_{\mu \nu}. \tag{37}
\]

Note. The arbitrary constant coefficient, \(\beta\), of the Chern–Simons term appears throughout the calculation, analogously to the 11-dimensional case.

It follows

\[
K^\mu (Q^I \mathcal{F}^J_{\mu \nu} \mathcal{F}^J_{\nu \rho}) = \nabla_\rho \left( Q^I \lambda^J \mathcal{F}^J_{\nu \rho} \right) + \frac{\beta}{16} C_{JKL} \epsilon_{\alpha \beta \gamma \delta} \lambda^J \mathcal{F}^{(1)J} \wedge \mathcal{F}^{K \gamma \delta} + Q^I H^{(1)J} \mathcal{F}^J_{\nu \rho} \tag{38}
\]

\[
K^\mu \left( Q^I \mathcal{G}^J_{\mu \rho \sigma} \mathcal{G}^J_{\nu \rho \sigma} \right) = 2 \nabla_\rho \left( Q^I \Lambda_{\nu \rho} \mathcal{G}^J_{\nu \rho \sigma} \right) - \frac{\beta}{4} C_{JKL} \epsilon_{\alpha \beta \gamma \delta} \lambda^J \mathcal{F}^{L \nu \rho \sigma} + Q^I \left( \beta C_{JKL} H^{(1)J}_\rho A^M_{\sigma} + H^{(2)}_{\rho \sigma} \right) \mathcal{G}^J_{\nu \rho \sigma}. \tag{39}
\]

and hence

\[
K^\mu R_{\mu \nu} = \frac{1}{3} \nabla_\rho \left( 2 Q^I \lambda^J \mathcal{F}^J_{\nu \rho} + Q^I \Lambda_{\nu \rho} \mathcal{G}^J_{\nu \rho} \right) + \frac{2}{3} Q^I H^{(1)J}_\rho \mathcal{F}^J_{\nu \rho} + \frac{1}{6} Q^I \left( \beta C_{JKL} H^{(1)J}_\rho A^M_{\sigma} + H^{(2)}_{\rho \sigma} \right) \mathcal{G}^J_{\nu \rho \sigma}. \tag{40}
\]

\(^4\)We use calligraphic script at some places to avoid confusion between the five- and 11-dimensional objects.
Excluding inner boundaries, the Komar mass formula becomes
\[
\frac{16\pi G}{3} M = \int_{\Sigma_t} R_{\mu\nu} F^\mu F^\nu d\Sigma^t = \int_{\Sigma_t} \left[ \frac{2}{3} Q_{i\ell} H^{(1)\ell\nu} F_{\nu\rho} + \frac{1}{6} Q_{i\ell} \left( \beta C_{H^H} H^{(1)\ell\nu} A^{\mu\nu} + H^{(2)\mu\nu} \right) \right] d\Sigma^t. \tag{41}
\]

The generalized version of Smarr's formula given in [14] is now
\[
M = \frac{1}{32\pi G_5} \int_{\Sigma_4} \left[ H^{(1)}_1 \wedge (4G^I - \beta C^{IJK} A_J \wedge F_K) - 2H^{(2)}_1 \wedge F^I \right]. \tag{42}
\]

Also here note that the differential form,
\[
4G^I - \beta C^{IJK} A_J \wedge F_K,
\]

is closed in virtue of (34). As in (28) the explicit Chern–Simons contributions do only support the purely topological form of the integrand for all values of $\beta$.

### 4.2. Dimensional reduction

The five-dimensional mass formula in [14] and the one above can obviously be reproduced by dimensional reduction of the 11-dimensional expression (28).

The five-dimensional fields embed into the 11-dimensional ones via
\[
C = A^1 \wedge dx^5 \wedge dx^6 + A^2 \wedge dx^7 \wedge dx^8 + A^3 \wedge dx^9 \wedge dx^{10} \tag{43}
\]
\[
F = F^1 \wedge dx^5 \wedge dx^6 + F^2 \wedge dx^7 \wedge dx^8 + F^3 \wedge dx^9 \wedge dx^{10}. \tag{44}
\]

In order to express $G = \star_1 F$ in terms of the $\mathcal{G}_I = Q_{ij} \star_5 F^j$, we go to a representation in frames:
\[
e^0 = Z^{-1} (dt + k) \quad e^i = \sqrt{\gamma_i} dx^i \quad e^5 = \left( \frac{2\pi}{\sqrt{Z} \gamma Z} \right)^{\frac{1}{2}} dx^5 \quad e^6 = \left( \frac{2\pi}{\sqrt{Z} \gamma Z} \right)^{\frac{1}{2}} dx^6
\]
\[
e^7 = \left( \frac{2\pi}{\sqrt{Z} \gamma Z} \right)^{\frac{1}{2}} dx^7 \quad e^8 = \left( \frac{2\pi}{\sqrt{Z} \gamma Z} \right)^{\frac{1}{2}} dx^8 \quad e^9 = \left( \frac{2\pi}{\sqrt{Z} \gamma Z} \right)^{\frac{1}{2}} dx^9 \quad e^{10} = \left( \frac{2\pi}{\sqrt{Z} \gamma Z} \right)^{\frac{1}{2}} dx^{10}. \tag{45}
\]

We compute explicitly the first term of (44) and then find the other two by analogy. It holds:
\[
\star_1 \left( e^0 \wedge e^i \wedge e^5 \wedge e^6 \right) = \star_5 \left( e^0 \wedge e^i \wedge e^7 \wedge \ldots \wedge e^{10} \right), \tag{46}
\]

which can be rewritten with (45) to
\[
\left( \frac{Z_i Z_j}{Z_5} \right)^{\frac{1}{2}} \star_1 \left( e^0 \wedge e^i \wedge dx^5 \wedge dx^6 \right) = \left( \frac{Z_i Z_j}{Z_5} \right)^{\frac{1}{2}} \left( \frac{Z_i Z_j}{Z_5} \right)^{\frac{1}{2}} \star_5 \left( e^0 \wedge e^i \wedge dx^7 \wedge \ldots \wedge dx^{10} \right) \tag{47}
\]

and thus becomes with (31)–(33)
\[
\star_1 \left( F^1 \wedge dx^5 \wedge dx^6 \right) = \frac{1}{X_5^2} \star_5 F^1 \wedge dx^7 \wedge \ldots \wedge dx^{10} = 2Q_{11} \star_5 F^1 \wedge dx^7 \wedge \ldots \wedge dx^{10} = 2G_{11} \wedge dx^7 \wedge \ldots \wedge dx^{10}. \tag{48}
\]

Analogously proceeded for $G_2$ and $G_3$, we finally achieve
\[
G = 2 \left( G_1 \wedge dx^7 \wedge dx^8 \wedge dx^{10} + G_2 \wedge dx^5 \wedge dx^6 \wedge dx^8 \wedge dx^{10} + G_3 \wedge dx^5 \wedge dx^6 \wedge dx^7 \wedge dx^{10} \right). \tag{49}
\]

With (13), (15), (35), (37), (43), (44) and (49) we find the relations between the non-trivial forms,
\[
H^{(1)} = H^{(1)1} \wedge dx^5 \wedge dx^6 + H^{(1)2} \wedge dx^7 \wedge dx^8 + H^{(1)3} \wedge dx^9 \wedge dx^{10} \tag{50}
\]
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\[ H^{(6)} = 2H^{(2)}_1 \wedge dx^7 \wedge dx^8 \wedge dx^9 \wedge dx^{10} + 2H^{(2)}_2 \wedge dx\wedge dx^9 \wedge dx^{10} + 2H^{(2)}_3 \wedge dx^5 \wedge dx^6 \wedge dx^7 \wedge dx^8, \]  

(51)

and the remaining forms,

\[ \lambda^{(2)} = \lambda^1 dx^5 \wedge dx^6 + \lambda^2 dx^7 \wedge dx^8 + \lambda^3 dx^9 \wedge dx^{10} \]  

(52)

\[ \Lambda^{(5)} = 2\Lambda^{(1)}_1 \wedge dx^7 \wedge dx^8 \wedge dx^9 \wedge dx^{10} + 2\Lambda^{(1)}_2 \wedge dx^5 \wedge dx^6 \wedge dx^9 \wedge dx^{10} + 2\Lambda^{(1)}_3 \wedge dx^5 \wedge dx^6 \wedge dx^7 \wedge dx^8. \]  

(53)

If we assume our compact space to be a 6-torus, that is, \( M_6 = T^6 = T^2 \times T^2 \times T^2 \), with \( d\text{vol}_6 = dx^5 \wedge ... \wedge dx^{10} \), \( \text{vol}_6 = (2\pi)^6 \prod_{m=1}^6 r_m \), the \( r_m \) being the radii of the tori, and use the above stated dictionary, the 11-dimensional integral amounts to

\[
\begin{align*}
\int_{\Sigma_{11}} \left[ H^{(3)}_1 \wedge (2G - \alpha C \wedge F) - H^{(6)} \wedge F \right] \\
= \int_{\Sigma_4 \times T^6} \left[ H^{(1)}_i \wedge (4G^I - \alpha C^{JK}A_J \wedge F_K) - 2H^{(2)}_j \wedge F^j \right] \wedge d\text{vol}_6 \\
= \text{vol}_6 \int_{\Sigma_4} \left[ H^{(1)}_1 \wedge (4G^I - \alpha C^{JK}A_J \wedge F_K) - 2H^{(2)}_1 \wedge F^j \right].
\end{align*}
\]  

(54)

From this result we see with (28) and (42) that if \( \alpha = \beta \) the Komar masses of both dimensional cases are related like

\[ M_{(11)} = M_{(5)}. \]  

(55)

5. Conclusion

We have derived the equations of motion following from 11-dimensional supergravity and from that inferred a mass formula of solitonic solutions via compactifying on a simply connected and Ricci-flat 6-manifold. Furthermore, we gave the Chern–Simons term an arbitrary constant coefficient in the action to see in how far the results are influenced by this parameter. We finally obtained a generalized version of Smarr’s formula and also showed how to arrive back at the five-dimensional theory by performing the 11-dimensional Komar integral over \( T^6 \).

The most intriguing aspect of the calculations done both in [14] and here is the proof of the possibility of constructing massive soliton solutions without the need of horizons and so showing that the techniques used in microstate geometries are the only methods that can support solitons. Moreover, we could show that making the Chern–Simons term arbitrary does not change this fact; the incorporation of Chern–Simons interactions does not yield extra pieces in the mass formula in addition to the topological terms, but is only significant for the purely topological nature of the soliton mass.

The question, whether the 11-dimensional generalization does indeed contain new physics in its spectrum of different topological mass terms, is yet to be investigated.
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ORCID iDs

Patrick A Haas  
https://orcid.org/0000-0002-8742-7407

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