The local structure of the free boundary in the fractional obstacle problem

Abstract: Building upon the recent results in [M. Focardi and E. Spadaro, On the measure and the structure of the free boundary of the lower-dimensional obstacle problem, Arch. Ration. Mech. Anal. 230 (2018), no. 1, 125–184] we provide a thorough description of the free boundary for the solutions to the fractional obstacle problem in $\mathbb{R}^{n+1}$ with obstacle function $\varphi$ (suitably smooth and decaying fast at infinity) up to sets of null $\mathcal{H}^{n-1}$ measure. In particular, if $\varphi$ is analytic, the problem reduces to the zero obstacle case dealt with in [M. Focardi and E. Spadaro, On the measure and the structure of the free boundary of the lower-dimensional obstacle problem, Arch. Ration. Mech. Anal. 230 (2018), no. 1, 125–184] and therefore we retrieve the same results:

(i) local finiteness of the $(n - 1)$-dimensional Minkowski content of the free boundary (and thus of its Hausdorff measure),
(ii) $\mathcal{H}^{n-1}$-rectifiability of the free boundary,
(iii) classification of the frequencies and of the blowups up to a set of Hausdorff dimension at most $(n - 2)$ in the free boundary.

Instead, if $\varphi \in C^{k+1}(\mathbb{R}^n)$, $k \geq 2$, similar results hold only for distinguished subsets of points in the free boundary where the order of contact of the solution with the obstacle function $\varphi$ is less than $k + 1$.

Keywords: Fractional obstacle problem, free boundary, rectifiability

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1 Introduction

Quasi-geostrophic flow models [10], anomalous diffusion in disordered media [4] and American options with jump processes [11] are some instances of constrained variational problems involving free boundaries for thin obstacle problems. In this paper we analyze the fractional obstacle problem with exponent $s \in (0, 1)$, a problem that can be stated in several ways, each motivated by different applications and suited to be studied with different techniques. We follow here the variational approach: given $\varphi : \mathbb{R}^n \to \mathbb{R}$ smooth and decaying sufficiently fast at infinity, one seeks for minimizers of the $H^s$-seminorm

$$[v]_{H^s}^2 := \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|v(x') - v(y')|^2}{|x' - y'|^{n+2s}} \, dx' \, dy', \quad s \in (0, 1),$$

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on the cone
\[ \mathcal{A} := \{ v \in \dot{H}^s(\mathbb{R}^n) : v(x') \geq \varphi(x') \}, \]
where \( \dot{H}^s(\mathbb{R}^n) \) is the homogeneous space defined as the closure in the \( H^s \) seminorm of \( C_0^\infty(\mathbb{R}^n) \) functions. Existence and uniqueness of a minimizer \( w \) follow for all \( s \in (0, 1) \) if \( n \geq 2 \) (the case \( n = 1 \) requires some care, see [30] and [3]). In addition, defining the fractional laplacian as
\[ (-\Delta)^s v(x') := c_{n,s} \text{PV.} \int_{\mathbb{R}^n} \frac{v(x') - v(y')}{|x' - y'|^{n+2s}} \, dy' \]
for \( v \in \dot{H}^s(\mathbb{R}^n) \), with \( c_{n,s} \) a suitable constant, the Euler–Lagrange conditions characterize \( w \) as a distributional solution to the system of inequalities
\[
\begin{aligned}
& w(x') \geq \varphi(x') \quad \text{for } x' \in \mathbb{R}^n, \\
& (-\Delta)^s w(x') = 0 \quad \text{for } x' \in \mathbb{R}^n \text{ such that } w(x') > \varphi(x'), \\
& (-\Delta)^s w \geq 0 \quad \text{in } \mathbb{R}^n.
\end{aligned}
\](1.1)

The most challenging regularity issues are then that of \( w \) itself and that of its free boundary
\[ \Gamma_\varphi(w) := \partial \{ x' \in \mathbb{R}^n : w(x') = \varphi(x') \}. \]

To investigate the fine properties of the solution \( w \) of (1.1) the groundbreaking paper by Caffarelli and Silvestre [5] introduces an equivalent local counterpart for the fractional obstacle problem in terms of the so-called \( a \)-harmonic extension argument. Indeed, it is inspired by the case \( s = \frac{1}{2} \), in which it is nothing but the harmonic extension problem. More precisely, setting \( a = 1 - 2s \) for \( s \in (0, 1) \) and \( m = |x_{n+1}|^2 L^{n+1} \), it turns out that any function \( w \) satisfying (1.1) is the trace of a function \( u \in \dot{H}^1(\mathbb{R}^{n+1}, dm) \) solving for \( x = (x', x_{n+1}) \in \mathbb{R}^{n+1} \),
\[
\begin{aligned}
& u(x', 0) \geq \varphi \quad \text{for } (x', 0) \in \mathbb{R}^n \times \{0\}, \\
& u(x', x_{n+1}) = u(x', -x_{n+1}) \quad \text{for all } x \in \mathbb{R}^{n+1}, \\
& \text{div}(|x_{n+1}|^a \nabla u(x)) = 0 \quad \text{for } x \in \mathbb{R}^{n+1} \setminus \{(x', 0) : u(x', 0) = \varphi(x')\}, \\
& \text{div}(|x_{n+1}|^a \nabla u(x)) \leq 0 \quad \text{in } \mathcal{A}(\mathbb{R}^{n+1}).
\end{aligned}
\](1.2)

In particular, note that \( u \) is the unique minimizer of the Dirichlet energy
\[ \int_{B_R} |\nabla \bar{u}|^2 \, dm \]
on the class \( \mathcal{A} := \{ \bar{v} \in \dot{H}^1(\mathbb{R}^{n+1}, dm) : \bar{v}(x', 0) \geq \varphi(x'), \bar{v}_{|\partial B_R} = u|_{\partial B_R} \} \) for every \( R > 0 \). Viceversa, the trace \( u(x', 0) \) on the hyperplane \( \{x_{n+1} = 0\} \) of the solution \( u \) to (1.2) is the solution to (1.1), as for all \( x' \in \mathbb{R}^n \) (cf. [5])
\[ \lim_{x_{n+1} \to 0^+} |x_{n+1}|^a \partial_{n+1} u(x', x_{n+1}) = -(-\Delta)^s u(x', 0). \]

One then is interested into the regularity of the free boundary \( \Gamma_\varphi(u) \) (with a slight abuse of notation we use the same symbol as for the analogous set for \( w \)) the topological boundary, in the relative topology of \( \mathbb{R}^n \), of the coincidence set of a solution \( u \)
\[ \Lambda_\varphi(u) := \{(x', 0) \in \mathbb{R}^{n+1} : u(x', 0) = \varphi(x')\}. \]
The locality of the operator
\[ L_\alpha(v) := \text{div}(|x_{n+1}|^a \nabla v(x)) \]
in (3.1) is the main advantage of the new formulation to perform the analysis of \( \Gamma_\varphi(u) \). Indeed, being \( \Gamma_\varphi(u) = \Gamma_\varphi(w) \) it permits the use of (almost) monotonicity-type formulas analogous to those introduced by Weiss and Monneau for the classical obstacle problem (cf. [6, 7, 27, 31]).

Optimal interior regularity for \( u \) has been established Caffarelli, Salsa and Silvestre in [9, Theorem 6.7 and Corollary 6.8] for any \( s \in (0, 1) \) (see also [8]). The particular case \( s = \frac{1}{2} \) had previously been addressed by Athanasopoulos, Caffarelli and Salsa in [1]. Instead, despite all the mentioned progresses, the current
picture for free boundary regularity theory is still incomplete. In this paper we go further on in this direction and deal with the nonzero obstacle case following the recent achievements in [17–19]. Drawing a parallel with the theory in the zero-obstacle case, the free boundary \( \Gamma_\varphi(u) \) can be split as a pairwise disjoint union of sets

\[
\Gamma_\varphi(u) = \text{Reg}(u) \cup \text{Sing}(u) \cup \text{Other}(u),
\]

(1.4)

termed in the existing literature as the subset of regular, singular and nonregular/nonsingular points, respectively. These sets are defined via the infinitesimal behavior of appropriate rescalings of the solution itself. More precisely, for \( x_0 \in \Gamma_\varphi(u) \) a function \( \varphi_{x_0} \) related to \( \varphi \) can be conveniently defined (cf. (3.46) and (3.48)) in a way that if

\[
u_{x_0,r}(y) := \frac{r^\alpha}{\| \varphi \|_{L^1(\mathbb{B}_r)}} \left( u(x_0 + ry) - \varphi_{x_0}(x_0 + ry) \right)
\]

then the family of functions \( \{ u_{x_0,r} \}_{r>0} \) is pre-compact in \( H^{1,0}_0(\mathbb{B}_r, \mathbb{B}_r) \) (see [9, Section 6]). The limits are called blowups of \( u \) at \( x_0 \), they are homogeneous solutions of a fractional problem with zero obstacle. The set of all such functions is denoted by \( \overline{B}(x_0) \). Their homogeneity \( \lambda(x_0) \) depends only on the base point \( x_0 \) and not on the extracted subsequence, and it is called infinitesimal homogeneity or frequency of \( u \) at \( x_0 \). It is indeed the limit value, as the radius vanishes, of an Almgren’s-type frequency function related to \( u \) which turns out to be non-decreasing in the radius. Given this, one defines

\[
\text{Reg}(u) := \{ x \in \Gamma_\varphi(u) : \lambda(x_0) = 1 + s \},
\]

\[
\text{Sing}(u) := \{ x \in \Gamma_\varphi(u) : \lambda(x_0) = 2m, m \in \mathbb{N} \},
\]

\[
\text{Other}(u) := \Gamma_\varphi(u) \setminus (\text{Reg}(u) \cup \text{Sing}(u)).
\]

According to the regularity of \( \varphi \) different results are known in literature:

(i) Regular points: in [9] (see also [16, 22] for alternative proofs) for \( \varphi \in C^{2,1}(\mathbb{R}^n) \) optimal one-sided \( C^{1,s} \)-regularity of solutions is established. Moreover, \( \text{Reg}(u) \) is shown to be locally a \( C^{1,n} \)-submanifold of codimension 2 in \( \mathbb{R}^{n+1} \).

(ii) Singular points: for \( \varphi \) analytic and \( a = 0 \) it is proved in [20] that \( \text{Sing}(u) \) is \((n-1)\)-rectifiable. The latter result has been very recently extended to the full range \( a \in (-1, 1) \) and to \( \varphi \in C^{k,1}(\mathbb{R}^n) \), \( k \geq 2 \), in [21]. Furthermore, fine properties of the singular set have been studied very recently by Fernández-Real and Jhaveri [14].

It is also worth mentioning the paper by Barrios, Figalli and Ros-Oton [3], in which the authors study the fractional obstacle problem (1.1) with nonzero obstacle \( \varphi \) having compact support and satisfying suitable concavity assumptions. Under these assumptions, they are able to fully characterize the free boundary, showing that \( \text{Other}(u) = \emptyset \) and that at every point of \( \text{Sing}(u) \) the blowup is quadratic, i.e. the only admissible value of \( m \) is 1. In addition, they are able to show that the singular set \( \text{Sing}(u) \) is locally contained in a single \( C^{1}\)-regular submanifold (see also [8] for the case of less regular obstacles, [13, 24–26] for higher regularity results on \( \text{Reg}(u) \) and [19] for the nonlinear case of the area functional).

For ease of expositions we start with the simpler case in which the obstacle function \( \varphi \) is analytic, actually the slightly milder assumption (1.5) below suffices (see Section 3 for related results in the case \( \varphi \in C^{k,1}(\mathbb{R}^n) \)). Indeed, after a suitable transformation (see Section 2.1) such a framework reduces to the zero obstacle. Thus, in view of [18, Theorems 1.1–1.3] we may deduce the following result.

**Theorem 1.1.** Let \( u \) be a solution to the fractional obstacle problem (1.2) with obstacle function \( \varphi: \mathbb{R}^n \rightarrow \mathbb{R} \) such that

\[
\{ \varphi > 0 \} \subset \mathbb{R}^n, \quad \varphi \text{ is real analytic on } \{ \varphi > 0 \}.
\]

(1.5)

Then:

(i) The free boundary \( \Gamma_\varphi(u) \) has finite \((n-1)\)-dimensional Minkowski content: more precisely, there exists a constant \( C > 0 \) such that

\[
\mathcal{L}^{n+1}(\mathcal{J}_r(\Gamma_\varphi(u))) \leq Cr^2 \quad \text{for all } r \in (0, 1),
\]

(1.6)

where \( \mathcal{J}_r(\Gamma_\varphi(u)) := \{ x \in \mathbb{R}^{n+1} : \text{dist}(x, \Gamma_\varphi(u)) < r \} \).
Moreover, there exists a subset $\Sigma(u) \subset \Gamma_\varphi(u)$ with Hausdorff dimension at most $n-2$ such that for every $x_0 \in \Gamma_\varphi(u) \setminus \Sigma(u)$ the infinitesimal homogeneity $\lambda(x_0)$ of $u$ at $x_0$ belongs to $\{2m, 2m - 1 + s, 2m + 2s\}_{m \in \mathbb{N} \setminus \{0\}}$.

The analysis is more involved in case $\varphi$ is not analytic, since one cannot in principle avoid contact points of infinite order between the solution and the obstacle, and the free boundary can be locally an arbitrary compact set $K \subset \mathbb{R}^n$ (explicit examples are provided in [15]). In view of this, we follow the existing literature and we consider obstacles $\varphi \in C^{k+1}(\mathbb{R}^n)$ and only those points in the free boundary in which $u$ has order of contact with $\varphi$ less than $k + 1$: given $u$ a solution to the fractional obstacle problem (1.2) and given a constant $\theta \in (0, 1)$ we set

$$
\Gamma_{\varphi, \theta}(u) := \left\{ x_0 \in \Gamma_\varphi(u) : \liminf_{r \to 0} r^{-((n+\alpha+2)(k+1-\theta))} H_{u_{x_0}}(r) > 0 \right\},
$$

where $H_{u_{x_0}}$ is defined in the sequel and it is related to the $L^2(\partial B_r, d\mathcal{H}^n)$ norm of $u_{x_0, r}$ (cf. Section 3 for more details). For this subset of points of the free boundary we can still prove some of the results stated in Theorem 1.1.

**Theorem 1.2.** Let $u$ be a solution to the fractional obstacle problem (1.2) with obstacle function $\varphi \in C^{k+1}(\mathbb{R}^n)$, $k \geq 2$, and let $\theta \in (0, 1)$. Then $\Gamma_{\varphi, \theta}(u)$ is $(n-1)$-rectifiable. Moreover, there exists a subset $\Sigma_\theta(u) \subset \Gamma_{\varphi, \theta}(u)$ with Hausdorff dimension at most $n-2$ such that for every $x_0 \in \Gamma_{\varphi, \theta}(u) \setminus \Sigma_\theta(u)$ the infinitesimal homogeneity $\lambda(x_0)$ of $u$ at $x_0$ belongs to $\{2m, 2m - 1 + s, 2m + 2s\}_{m \in \mathbb{N} \setminus \{0\}}$.

This note extends the results of [18] to the case of nonconstant obstacles. It is clear by the examples of arbitrary compact sets as contact sets of suitable solutions of the problem, that in general the free boundary for nonconstant smooth obstacles does not possess any structure and that the key ingredient for the analysis of the free boundary is the **analyticity** of the obstacles as shown in Theorem 1.1. Nevertheless, for a distinguished of the free boundary, characterized as those points of finite order of contact, e.g. the points $\Gamma_{\varphi, \theta}(u)$ already considered in the literature (see [21]), a partial regularity result still holds even in the framework of non-analytic obstacles, as proven in Theorem 1.2. The main novelty of this paper with respect to [18] consists in the analysis of the spatial dependence of the frequency for nonconstant obstacles: indeed, in this case the frequency is defined differently from point to point, by taking into account the geometry of the obstacle itself. It is not at all evident to which extent the oscillation of the frequency can be controlled. The results of Section 4 show that this kind of estimates are not completely rigid and extend to nonflat obstacles. Hence, this paper contributes to the program of broadening the results initially proven for the Signorini problem with zero obstacles to the case of the obstacle problem for the fractional Laplacian (see e.g. [1, 21, 22]), providing a generalization of the known results on the structure of the free boundary firstly proven in [18].

## 2 Analytic obstacles

In this section we deal with analytic obstacles. We report first on some results related to the Caffarelli–Silvestre $a$-harmonic extension argument that will be instrumental to reduce the analytic-type fractional obstacle problem to the lower-dimensional obstacle problem. We provide then the proof of Theorem 1.1.

### 2.1 Extension results

We start off with a lemma in which it is proved that there exists a canonical $a$-harmonic extension of a polynomial in the class of polynomials (see [21, Lemma 5.2]). We denote by $\mathcal{P}(\mathbb{R}^n)$ the finite-dimensional vector space of homogeneous polynomials of degree $l \in \mathbb{N}$ in $\mathbb{R}^n$. 
In this section we deal with the more demanding case of \( C \) quoted [18, Theorems 1.1–1.3] and note that it solves a zero lower-dimensional obstacle problem (1.2). Hence, we can conclude by the analysis of (1.2) to that of the following localized problem:

\[
\begin{aligned}
&\text{In each ball } B_{r} \text{ of a compact set } \Gamma \text{ all } \mathbb{R}^n, \\
&\text{for every } x \in \mathbb{R}^n, \\
&\text{for all } (x', x_{n+1}) \in \mathbb{R}^{n+1}. \\
\end{aligned}
\]

**Proof.** Let \( p \in \mathcal{P}(\mathbb{R}^n) \) and set

\[
\delta_{i}[p](x', x_{n+1}) := \sum_{j=0}^i p_{2j}(x')x_{n+1}^j,
\]

with \( p_{2j}(x') := -\frac{1}{(2j)!} \Delta p_{2j-2}(x') \) if \( j \in \{1, \ldots, \lfloor \frac{i}{2} \rfloor \} \) and \( p_0 := p \). It is then easy to verify that \( \delta_{i} \) satisfies all the stated properties. \( \square \)

**Remark 2.2.** In particular, \( \delta_{i} \) is a continuous operator. We will use in what follows that there exists a constant \( C = C(n, L) > 0 \) such that for every \( p \in \mathcal{P}(\mathbb{R}^n) \) and for every \( r > 0 \),

\[
\| \delta_{i}[p] \|_{L^\infty(B_r)} \leq C \| p \|_{L^\infty(B_r)}.
\]

We provide next the main result that reduces locally the analytic case to the zero obstacle case (cf. [21, Lemma 5.1]).

**Lemma 2.3.** Let \( \varphi : \Omega \to \mathbb{R} \) be analytic, \( \Omega \subset \mathbb{R}^n \) open. Then for all \( K \subset \subset \Omega \times (0) \) there exists \( r > 0 \) such that, for every \( x_0 \in K \), there exists a function \( \delta_{x_0}^s \varphi : B_r(x_0) \to \mathbb{R} \) even symmetric with respect to \( x_{n+1} \) such that

\[
\begin{aligned}
&\text{(i) } -\operatorname{div}(x_{n+1}^{a} \nabla \delta_{x_0}^s \varphi) = 0 \text{ in } \mathcal{D}'(\mathbb{R}^{n+1}), \\
&\text{(ii) } \delta_{x_0}^s \varphi(x', 0) = \varphi(x') \text{ for all } (x', 0) \in B_r(x_0), \\
&\text{(iii) } \delta_{x_0}^s \varphi \text{ is analytic in } B_r(x_0).
\end{aligned}
\]

**Proof.** For every \( x_0 \) as in the statement, we can locally expand \( \varphi \) in power series as \( \varphi(x') = \sum_{a} c_a(x' - x_0)^a \). Then we set \( \delta_{x_0}^s \varphi(x) := \sum_{a} c_a \delta_{x_0}[p_a](x - x_0) \) where \( p_a(x') := (x')^a \). From the explicit formulas in the proof of Lemma 2.1 it is easily verified that the power series defining \( \delta_{x_0}^s \varphi \) is converging in \( B_r(x_0) \) and gives an analytic \( a \)-harmonic extension even symmetric with respect to \( x_{n+1} \) in \( B_r(x_0) \) with \( r > 0 \) uniform on compact sets. \( \square \)

### 2.2 Proof of Theorem 1.1

Theorem 1.1 follows straightforwardly from [18, Theorems 1.1–1.3]. As explained in the introduction \( w(x') = u(x', 0) \) solves the fractional obstacle problem (1.1). By the maximum principle \( u(x', 0) > 0 \) for all \( x' \in \mathbb{R}^n \). Therefore, \( \Gamma_u \subset \{ \varphi > 0 \} \subset \mathbb{R}^n \). Let \( r > 0 \) be the radius in Lemma 2.3 corresponding to the compact set \( \Gamma_u \). By compactness we cover \( \Gamma_u \) with a finite number of balls \( B_r(x_i) \), with \( x_i \in \Gamma_u \). In each ball \( B_r(x_i) \) we consider the corresponding function \( u - \delta_{x_i}^s \varphi \), with \( \delta_{x_i}^s \varphi \) provided by Lemma 2.3, and note that it solves a zero lower-dimensional obstacle problem (1.2). Hence, we can conclude by the quoted [18, Theorems 1.1–1.3].

### 3 \( C^{k+1} \) obstacles

In this section we deal with the more demanding case of \( C^{k+1} \) obstacles, \( k \geq 2 \). It is convenient to reduce the analysis of (1.2) to that of the following localized problem:

\[
\begin{aligned}
&u(x', 0) \geq \varphi(x') \quad \text{for } (x', 0) \in B'_1, \\
u(x', x_{n+1}) = u(x', -x_{n+1}) \quad \text{for } x = (x', x_{n+1}) \in B_1, \\
&\operatorname{div}(x_{n+1}^{a} \nabla u(x)) = 0 \quad \text{for } x \in B_1 \setminus \{(x', 0) \in B'_1 : u(x', 0) = \varphi(x')\}, \\
&\operatorname{div}(x_{n+1}^{a} \nabla u(x)) \leq 0 \quad \text{in } \mathcal{D}'(B_1),
\end{aligned}
\]

(3.1)
for $\varphi \in C^{k+1}(B'_1)$. In what follows, we shall assume that $\|\varphi\|_{C^{k+1}(B'_1)} \leq 1$. This assumption can easily be matched by a simple scaling argument (cf. the proof of Theorem 1.2).

For any $x_0 \in B'_1$ we denote by $T_{k,x_0}[\varphi]$ the Taylor polynomial of $\varphi$ of order $k$ at $x_0$:

$$T_{k,x_0}[\varphi](x') := \sum_{|\alpha| \leq k} \frac{D^\alpha \varphi(x_0)}{\alpha!} p_\alpha(x' - x_0),$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $D^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$, $p_\alpha(x') := (x')^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $|\alpha| := \alpha_1 + \cdots + \alpha_n$, $\alpha! := \alpha_1! \cdots \alpha_n!$. We will repeatedly use that (recall that $\|\varphi\|_{C^{k+1}(B'_1)} \leq 1$)

$$|T_{k,x_0}[\varphi](x') - \varphi(x')| \leq \frac{1}{(k+1)!} |x'| - x_0|^{k+1}$$

(3.2)

and that

$$|T_{k,x_0}[\varphi](x') - \varphi(x') - \partial_e \varphi(x')| \leq 2|x'| - x_0|^k$$

(3.3)

for all unit vectors $e \in \mathbb{R}^{n+1}$ such that $e \cdot e_{n+1} = 0$.

Let then $\partial\{T_{k,x_0}[\varphi]\}$ be the $a$-harmonic extension of $T_{k,x_0}[\varphi]$, namely,

$$\partial\{T_{k,x_0}[\varphi]\}(x) = \sum_{|\alpha| \leq k} \frac{D^\alpha \varphi(x_0)}{\alpha!} \partial_{|\alpha|}[p_\alpha(-x_0)](x),$$

where $\partial_{|\alpha|}$ are the extension operators in Lemma 2.1. By the translation invariance of the operator, we point out that

$$\partial_{|\alpha|}[p_\alpha(-x_0)](x) = \partial_{|\alpha|}[p_\alpha](x-x_0).$$

(3.4)

Set

$$\varphi_{x_0}(x) := \varphi(x') - T_{k,x_0}[\varphi](x') + \partial\{T_{k,x_0}[\varphi]\}(x)$$

(3.5)

and

$$u_{x_0}(x) := u(x) - \varphi_{x_0}(x).$$

(3.6)

Recalling that $\partial\{T_{k,x_0}[\varphi]\}(x', 0) = T_{k,x_0}[\varphi](x')$, we have $\Lambda_{\varphi}(u) = \{(x', 0) \in B'_1 : u_{x_0}(x', 0) = 0\}$, and thus in particular $\Gamma_{\varphi}(u) = \partial_{B'_1}(x', 0) \in B'_1 : u_{x_0}(x', 0) = 0$, where $\partial_{B'_1}$ is the relative boundary in the hyperplane $\{x_{n+1} = 0\}$. We note that $u_{x_0}$ is not a solution of a fractional obstacle problem as in (3.1) with null obstacle, but rather of a related obstacle problem with drift as discussed in the sequel (cf. (3.14)).

First, from the regularity assumption on $\varphi$, from Lemma 2.1 and from estimate (3.2) we infer that $L_a(\varphi_{x_0})$ is a function in $L^1(B_1)$ (recall the definition of the operator $L_a$ given in (1.3)). Moreover, estimate (3.2) gives for all $x \in B_1 \setminus B'_1$,

$$|L_a(\varphi_{x_0})(x)| = |\text{div}(|x_{n+1}|^{a} \nabla(\varphi - T_{k,x_0}[\varphi])(x'))|$$

$$= |x_{n+1}|^a |\Delta(\varphi(x') - T_{k,x_0}[\varphi](x'))| \leq |x_{n+1}|^a |x' - x_0|^{k-1}. $$

(3.7)

In turn, this yields that the distribution $L_a(u_{x_0})$ is given by the sum of a function in $L^1(B_1)$ and of a nonpositive singular measure supported on $B'_1$, namely,

$$L_a(u_{x_0})(x) = \text{div}(|x_{n+1}|^{a} \nabla u(x)) - L_a(\varphi_{x_0}(x)) \mathcal{L}^n \setminus B_1.$$

(3.8)

The following result resumes the regularity theory developed in [9, Proposition 4.3].

**Theorem 3.1.** Let $u$ be a solution to the fractional obstacle problem (3.1) in $B_1$, $x_0 \in B'_1$, $r \in (0, 1)$, then $u_{x_0} \in C^{0,\min\{2a,1\}}(B_{1-r}(x_0))$, $\partial_{|a|} u_{x_0} \in C^{0,\alpha}(B_{1-r}(x_0))$ for $i = 1, \ldots, n$, and $|x_{n+1}|^a \partial_{x_{n+1}} u_{x_0} \in C^{0,\alpha}(B_{1-r}(x_0))$ for all $\alpha \in (0, 1-s)$. Moreover, there exists a constant $C_{3.1} = C_{3.1}(n, a, \alpha, r) > 0$ such that

$$\|u_{x_0}\|_{C^{0,\min\{2a,1\}}(B_{1-r}(x_0))} + \|\nabla^r u_{x_0}\|_{C^{0,\alpha}(B_{1-r}(x_0))} + \|\operatorname{sign}(x_{n+1})|x_{n+1}|^a \partial_{x_{n+1}} u_{x_0}\|_{C^{0,\alpha}(B_{1-r}(x_0))} \leq C_{3.1} \|u_{x_0}\|_{L^1(B_{1-r}(x_0), \mathcal{L}^n)},$$

(3.9)

where $\nabla^r u_{x_0} = (\partial_{x_1} u_{x_0}, \ldots, \partial_{x_n} u_{x_0})$ is the horizontal gradient.
In particular, the function \( u \) is analytic in \( \{ x_{n+1} > 0 \} \) (see, e.g., [23]) and the following boundary conditions holds:

\[
\lim_{x_{n+1} \to 0^+} x_{n+1}^a \partial_{n+1} u(x', x_{n+1}) = 0 \quad \text{for } x' \in B'_1 \setminus \Lambda_{\varphi}(u), \quad (3.10)
\]

\[
\lim_{x_{n+1} \to 0^+} x_{n+1}^a \partial_{n+1} u(x', x_{n+1}) \leq 0 \quad \text{for } x' \in B'_1. \quad (3.11)
\]

In particular,

\[
(u'(x), 0) - \varphi(x')) \quad \lim_{x_{n+1} \to 0^+} x_{n+1}^a \partial_{n+1} u(x', x_{n+1}) = 0 \quad \text{for } x' \in B'_1. \quad (3.12)
\]

Furthermore, for \( B_r(x_0) \subset B_1 \) and \( x_0 \in B'_1 \), an integration by parts implies that

\[
\int_{B_r(x_0)} |\nabla u|^2 |x|^{n+1-a} \, dx - \int_{B_r(x_0)} |\nabla \varphi(x)|^2 |x|^{n+1-a} \, dx \\
= \int_{B_r(x_0)} |\nabla u|^2 |x|^{n+1-a} \, dx + 2 \int_{B_r(x_0)} u \varphi_x \cdot \nabla \varphi |x|^{n+1-a} \, dx \\
= \int_{B_r(x_0)} |\nabla u|^2 |x|^{n+1-a} \, dx - 2 \int_{B_r(x_0)} u \varphi_x L_0(\varphi_{x_0}) \, dx + 2 \int_{\partial B_r(x_0)} u \varphi_x |x|^{n+1-a} \, dx,
\]

(3.13)

where in the second equality we have used that \( \partial [T_{k,x_0}(\varphi)] \) is even with respect to the hyperplane \( \{ x_{n+1} = 0 \} \) to deduce that

\[
\lim_{x_{n+1} \to 0^+} \partial_{n+1} \varphi(x)|x|^{n+1-a} = 0.
\]

In particular, since the last addend in (3.13) only depends on the boundary values of \( u_{x_0} \), it follows that \( u_{x_0} \) is a minimizer of the functional

\[
\int_{B_r(x_0)} |\nabla \varphi|^2 |x|^{n+1-a} \, dx - 2 \int_{B_r(x_0)} \varphi L_0(\varphi_{x_0}) \, dx
\]

(3.14)

among all functions \( \varphi \in u_{x_0} + H_0^1(B_r(x_0), dm) \) and satisfying \( \varphi(x', 0) \geq 0 \) on \( B'_1(x_0) \). Equivalently, we will say that \( u_{x_0} \) is a local minimizer of the functional in (3.14) subject to null obstacle conditions.

**Remark 3.2.** We record here some bounds that shall be employed extensively in what follows. By using the linearity and continuity of the extension operator \( \delta_k \) (cf. Remark 2.2), together with estimate (3.2) we get for all \( z \in B_r' \),

\[
|u_{x_0}(z) - u_{x_1}(z)| = |\varphi_{x_0}(z) - \varphi_{x_1}(z)| \\
\leq |T_{k,x_0}(\varphi)(z') - T_{k,x_1}(\varphi)(z')| + |\delta_k[T_{k,x_0}(\varphi)](z) - \delta_k[T_{k,x_1}(\varphi)](z)| \\
\leq \| T_{k,x_0}(\varphi) - T_{k,x_1}(\varphi) \|_{L^\infty(B'_1)} + \| \delta_k[T_{k,x_0}(\varphi)] - \delta_k[T_{k,x_1}(\varphi)] \|_{L^\infty(B'_1)} \\
\leq C(\max_{z \in B'_1} |z - x_0|^{k+1} + \max_{z \in B'_1} |z - x_1|^{k+1})
\]

(3.15)

for some constant \( C = C(n, a, k) > 0 \). Since \( \nabla(T_{k,x_i}(\varphi)) = T_{k-1,x_i}[\nabla \varphi], i \in \{0, 1\} \), arguing as above, using (3.3) rather than (3.2), we conclude that

\[
|\nabla(u_{x_0}(z) - u_{x_1}(z))| = |\nabla(\varphi_{x_0}(z) - \varphi_{x_1}(z))| \leq C(\max_{z \in B'_1} |z - x_0|^k + \max_{z \in B'_1} |z - x_1|^k)
\]

(3.16)

for some constant \( C = C(n, a, k) > 0 \).
3.1 A frequency-type function

Building upon the approach developed in [18], we consider a quantity strictly related to Almgren’s frequency function (see [12]) and instrumental for developing the free boundary analysis in the subsequent sections. Let \( \phi : [0, +\infty) \rightarrow [0, +\infty) \) be defined by

\[
\phi(t) := \begin{cases} 
1 & \text{for } 0 \leq t \leq \frac{1}{2}, \\
2(1-t) & \text{for } \frac{1}{2} < t \leq 1, \\
0 & \text{for } 1 < t,
\end{cases}
\]

then given the solution \( u \) to (3.1), a point \( x_0 \in B'_1 \) and the corresponding function \( u_{x_0} \) in (3.6), we define for all \( 0 < r < 1 - |x_0| \),

\[
I_{u_{x_0}}(r) := \frac{rG_{u_{x_0}}(r)}{H_{u_{x_0}}(r)},
\]

where

\[
G_{u_{x_0}}(r) := - \frac{1}{r} \int \phi \left( \frac{|x-x_0|}{r} \right) u_{x_0}(x)|\nabla u_{x_0}(x)| \left( \frac{x-x_0}{|x-x_0|} \right) |x_{n+1}|^d \, dx
\]

and

\[
H_{u_{x_0}}(r) := - \int \phi \left( \frac{|x-x_0|}{r} \right) u_{x_0}^2(x) |x-x_0| |x_{n+1}|^d \, dx. \tag{3.17}
\]

Here \( \dot{\phi} \) indicates the derivative of \( \phi \). Clearly, \( I_{u_{x_0}}(r) \) is well-defined as long as \( H_{u_{x_0}}(r) > 0 \), therefore when writing \( I_{u_{x_0}}(r) \) we shall tacitly assume that the latter condition is satisfied.

For later convenience, we introduce also the notation

\[
D_{u_{x_0}}(r) := \int \phi \left( \frac{|x-x_0|}{r} \right) |\nabla u_{x_0}(x)|^2 |x_{n+1}|^d \, dx
\]

and

\[
E_{u_{x_0}}(r) := \int \dot{\phi} \left( \frac{|x-x_0|}{r} \right) \left( \frac{|x-x_0|}{r} \right)^2 \left( |\nabla u_{x_0}(x)| \cdot \frac{x-x_0}{|x-x_0|} \right)^2 |x_{n+1}|^d \, dx.
\]

In particular, note that for all \( r > 0 \),

\[
H_{u_{x_0}}(r) E_{u_{x_0}}(r) - G_{u_{x_0}}^2(r) \geq 0 \tag{3.18}
\]

by Cauchy–Schwarz inequality.

**Remark 3.3.** In case \( \varphi = 0 \), then \( u_{x_0} = u \) for all \( x_0 \in B'_1 \) and \( G_u = D_u \). Thus, \( I_{u_{x_0}} \) boils down to the variant of Almgren’s frequency function used in [18].

**Remark 3.4.** If \( u \) is a solution to the fractional obstacle problem (3.1), then for every \( c > 0, x_0 \in B'_1 \) and \( r > 0 \) such that \( B_r(x_0) \subset B_1 \), the function \( \bar{u}(y) := cu(x_0 + ry) \) solves (3.1) on \( B_1 \) with obstacle function \( \bar{\phi}(y) := c\varphi(x_0 + ry) \). Therefore, if \( x_1 = x_0 + ry_1 \in B'_1 \) we have

\[
T_{k,y_1}[\bar{\phi}](y') = cT_{k,x_1}[\varphi](x_0 + ry')
\]

and

\[
\bar{u}_{y_1}(y) = cu_{x_1}(x_0 + ry).
\]

Thus, \( I_{\bar{u}_{y_1}}(\rho) = I_{u_{x_1}}(\rho r) \) for every \( \rho \in (0, 1) \).

In particular, this shows that the frequency function is scaling invariant, in the sequel we will use this property repeatedly.

3.2 Almost monotonicity of \( I_{u_{x_0}} \) at distinguished points

In this subsection we prove the quasi-monotonicity of \( I_{u_{x_0}} \) for a suitable subset of points of the free boundary. We show first some useful identities in a generic point \( x_0 \) of \( B'_1 \).
Lemma 3.5. Let $u$ be a solution to the fractional obstacle problem (3.1) in $B_1$. Then, for all $x_0 \in B_1'$ and $t \in (0, 1 - |x_0|)$, it holds

$$
Du_{x_0}(t) = Gu_{x_0}(t) - \int \phi\left(\frac{|x - x_0|}{t}\right)u_{x_0}(x)L_a(u_{x_0}(x))\,dx,
$$

(3.19)

$$
\frac{d}{dt}(H_{u_{x_0}}(t)) = \frac{n + a}{t}H_{u_{x_0}}(t) + 2G_{u_{x_0}}(t),
$$

(3.20)

$$
\frac{d}{dt}(D_{u_{x_0}(t)}) = \frac{n + a - 1}{t}D_{u_{x_0}}(t) + 2E_{u_{x_0}}(t) - \frac{2}{t} \int \phi\left(\frac{|x - x_0|}{t}\right)\nabla u_{x_0}(x) \cdot (x - x_0)L_a(u_{x_0}(x))\,dx.
$$

(3.21)

Remark 3.6. With an abuse of notation, the integration in the last addends in (3.19) and (3.21) is meant with respect to the reference measure $L_a(u_{x_0})$. Actually, we use this notation because from the proofs of (3.19) and (3.21) it turns out that one can consider equivalently its absolutely continuous part $-L_a(\varphi_{x_0})$, see (3.8).

Proof. To show (3.19), (3.20) and (3.21), we assume without loss of generality that $x_0 = 0$.

For (3.19) we consider the vector field $V(x) := \phi\left(\frac{|x|}{t}\right)u_0(x)|\nabla u_0(x)|x_{n+1}|^a$. Clearly, $V$ has compact support and $V \in C^{\infty}(B_1 \setminus B_1', \mathbb{R}^{n+1})$. Moreover, for $x_{n+1} \neq 0$,

$$
V(x) \cdot e_{n+1} = \phi\left(\frac{|x|}{t}\right)u_0(x)\partial_{n+1}u_0(x)|x_{n+1}|^a,
$$

so that $\lim_{|x' - x_0| \to 0} V(y) \cdot e_{n+1} = 0$. Indeed, recalling that $\mathcal{C}[T_{x_0}(\varphi)]$ is even with respect to the hyperplane $\{x_{n+1} = 0\}$ (cf. Lemma 2.1): if $(x', 0) \in \Lambda_\varphi(u)$, we exploit the regularity of $u$ resumed in Theorem 3.1 to conclude; instead, if $(x', 0) \notin \Lambda_\varphi(u)$, it suffices to use (3.10). Thus, the distributional divergence of $V$ is the $L^1(B_1)$ function given by

$$
\text{div } V(x) = \phi\left(\frac{|x|}{t}\right)u_0(x)\nabla u_0(x) \cdot \frac{x}{|x|} |x_{n+1}|^a + \phi\left(\frac{|x|}{t}\right)|\nabla u_0(x)|^2 |x_{n+1}|^a + \phi\left(\frac{|x|}{t}\right)u_0(x)L_a(u_0(x)).
$$

Therefore, (3.19) follows from the divergence theorem by taking into account that $V$ is compactly supported.

Next, (3.20) is a consequence of the direct computation

$$
\frac{d}{dt}(H_{u_{x_0}}(t)) = \frac{n + a}{t}H_{u_{x_0}}(t) - 2t^{a+1} \int \phi(|y|)u_0(y)\nabla u_0(y) \cdot \frac{y}{|y|} |y_{n+1}|^a\,dy
$$

and

$$
\frac{d}{dt}(D_{u_{x_0}}(t)) = \frac{n + a - 1}{t}D_{u_{x_0}}(t) + 2E_{u_{x_0}}(t) - \frac{2}{t} \int \phi\left(\frac{|x - x_0|}{t}\right)\nabla u_{x_0}(x) \cdot (x - x_0)L_a(u_{x_0}(x))\,dx.
$$

Finally, to prove (3.21) we consider the compactly supported vector field $W \in C^{\infty}(B_1 \setminus B_1', \mathbb{R}^{n+1})$ defined by

$$
W(x) = \phi\left(\frac{|x|}{t}\right)\left(\frac{|\nabla u_0(x)|^2}{2}x - (\nabla u_0(x) \cdot x)\nabla u_0(x)\right)|x_{n+1}|^a.
$$

Moreover, conditions (3.10)-(3.12) and Lemma 2.1 imply that $\lim_{x_{n+1} \to 0} W(y) \cdot e_{n+1} = 0$. Thus, $\text{div } W$ has no singular part in $B_1'$, and we can compute pointwise the distributional divergence as follows: for $x_{n+1} \neq 0$,

$$
\text{div } W(x) = \phi\left(\frac{|x|}{t}\right)\left(\frac{|\nabla u_0(x)|^2}{2}x - (\nabla u_0(x) \cdot x)\nabla u_0(x)\right)|x_{n+1}|^a + \frac{n + a - 1}{2} |\nabla u_0(x)|^2 |x_{n+1}|^a - \phi\left(\frac{|x|}{t}\right)(\nabla u_0(x) \cdot x)L_a(u_0(x)).
$$

Therefore, we infer that

$$
0 = \int \text{div } W(x) \,dx
$$

and

$$
= \int \phi\left(\frac{|x|}{t}\right)\frac{|\nabla u_0(x)|^2}{2} |x_{n+1}|^a \,dx + tE_{u_0}(t) + \frac{n + a - 1}{2} D_{u_0}(t) - \int \phi\left(\frac{|x|}{t}\right)(\nabla u_0(x) \cdot x)L_a(u_0(x))\,dx,
$$

and we conclude (3.21) by direct differentiation since

$$
\frac{d}{dt}(D_{u_{x_0}}(t)) = -\int \phi\left(\frac{|x|}{t}\right)\frac{|\nabla u_0(x)|^2}{2} |x_{n+1}|^a \,dx.
$$
As a consequence we derive a first monotonicity formula for \( H_{u_{x_0}} \) in \( B_1 \).

**Corollary 3.7.** Let \( u \) be a solution to the fractional obstacle problem (3.1). Then, for all \( x_0 \in B'_1 \) and \( r_0, r_1 \) with \( 0 < r_0 < r_1 < 1 - |x_0| \) such that \( H_{u_{x_0}} (t) > 0 \) for all \( t \in (r_0, r_1) \), we have

\[
H_{u_{x_0}} (r_1) \left( \frac{r_1}{r_0} \right)^{\alpha} = H_{u_{x_0}} (r_0) \left( \frac{r_0}{r_1} \right)^{\alpha} + \frac{r_0}{r_1} \int_{r_0}^{r_1} \frac{H_{u_{x_0}} (s)}{s^2} \, ds.
\]

(3.22)

In particular, if \( A_1 \leq H_{u_{x_0}} (t) \leq A_2 \) for every \( t \in (r_0, r_1) \), then

\[
(r_0, r_1) \ni r \mapsto \frac{H_{u_{x_0}} (r)}{r^{n+\alpha+2A_2}}
\]

is monotone decreasing,

(3.23)

\[
(r_0, r_1) \ni r \mapsto \frac{H_{u_{x_0}} (r)}{r^{n+\alpha+2A_1}}
\]

is monotone increasing.

(3.24)

Moreover, for all \( x_0 \in B'_1 \) and \( 0 < r < 1 - |x_0| \),

\[
\int_{B_r (x_0)} |u_{x_0}|^2 |x|^{n+1} \, dx \leq \frac{1}{1-r^{n+\alpha}} |x_0|^{n+1}.
\]

(3.25)

**Proof.** The proof of (3.22) (and hence of (3.23) and (3.24)) follows from the differential equation in (3.20). The proof of (3.25) is a simple consequence of a dyadic integration argument:

\[
\int_{B_r (x_0)} |u_{x_0}|^2 |x|^{n+1} \, dx = \sum_{j \in \mathbb{N}} \int_{B_{2^{-j} r} (x_0)} |u_{x_0}|^2 |x|^{n+1} \, dx \leq \sum_{j \in \mathbb{N}} \frac{r}{2^j} H_{u_{x_0}} \left( \frac{r}{2^j} \right) \leq r H_{u_{x_0}} (r),
\]

where in the last inequality we used that \( H_{u_{x_0}} (s) \leq H_{u_{x_0}} (r) \) for \( s \leq r \) by (3.24) (with \( A_1 = 0 \)).

We establish next an auxiliary lemma containing useful bounds for some quantities related to the \( L^2 \)-norm of \( u_{x_0} \), for points \( x_0 \) in the contact set.

**Lemma 3.8.** Let \( u \) be a solution to the fractional obstacle problem (3.1) in \( B_1 \). Then there is a positive constant \( C_{3,8} = C_{3,8} (n, \alpha) > 0 \) such that for every point \( x_0 \in \Lambda_\varphi (u) \) we have for all \( r \in (0, 1 - |x_0|) \),

\[
H_{u_{x_0}} (r) \leq C_{3,8} (r D_{u_{x_0}} (r) + r^{n+\alpha+2(k+1)})
\]

(3.26)

\[
\int_{B_r (x_0) \setminus B_{r/2} (x_0)} |u_{x_0} (x)|^2 |x|^{n+1} \, dx \leq C_{3,8} (r^2 D_{u_{x_0}} (r) + r^{n+\alpha+1+2(k+1)})
\]

(3.27)

\[
\int_{B_r (x_0) \setminus B_{r/2} (x_0)} |u_{x_0} (x)|^2 |x|^{n+1} \, dx \leq C_{3,8} (r^2 D_{u_{x_0}} (r) + r^{n+\alpha+1+2(k+1)})
\]

(3.28)

**Proof.** By the co-area formula for Lipschitz functions we check that

\[
H_{u_{x_0}} (r) = \frac{r}{2} \int_{\partial B_{\frac{r}{2}} (x_0)} |u_{x_0} (x)|^2 |x|^{n+1} \, d\mathcal{H}^n (x)
\]

(3.29)

and

\[
D_{u_{x_0}} (r) = \int_{B_{\frac{r}{2}} (x_0)} |\nabla u_{x_0} (x)|^2 |x|^{n+1} \, dx + \frac{2}{r} \int_{\frac{r}{2}}^{r} dr \int_{\partial B_r (x_0)} (r-t) |\nabla u_{x_0} (x)|^2 |x|^{n+1} \, d\mathcal{H}^n (x).
\]

Therefore, an integration by parts gives

\[
D_{u_{x_0}} (r) = \frac{2}{r} \int_{\frac{r}{2}}^{r} dr \int_{B_r (x_0)} |\nabla u_{x_0} (x)|^2 |x|^{n+1} \, dx.
\]

(3.30)

By (3.8), as \( x_0 \in \Lambda_\varphi (u) \), [9, Lemma 2.13] and [21, Lemma 6.3] yield the Poincaré inequality

\[
\frac{1}{r} \int_{\partial B_r (x_0)} |u_{x_0} (x)|^2 |x|^{n+1} \, d\mathcal{H}^n (x) \leq C \int_{B_r (x_0)} |\nabla u_{x_0} (x)|^2 |x|^{n+1} \, dx + Cr^{n+\alpha-1+2(k+1)}
\]

(3.31)
with $C = C(n, a) > 0$. Integrating the latter inequality on $(r, r)$, we find (3.26) in view of (3.30). Instead, by first multiplying formula (3.31) by $t$ and then integrating over $(0, r)$, we infer

$$
\int_{B_r(x_0)} |u_{x_i}(x)|^2 |x_{n+1}|^a \, dx \leq Cr^2 \int_{B_r(x_0)} |\nabla u_{x_i}(x)|^2 |x_{n+1}|^a \, dx + Cr^{n+a+1+2(k+1)}.
$$

In conclusion, (3.27) and (3.28) follow directly.

Next we show an explicit expression for the radial derivative of $I_{u_{x_i}}$ at all points $x_0 \in B'_1$. We follow here [18, Proposition 2.7].

**Proposition 3.9.** Let $u$ be a solution to the fractional obstacle problem (3.1) in $B_1$. Then, if $x_0 \in B'_1$ is such that $H_{u_{x_i}}(t) > 0$ for all $t \in [0, r_1]$, we have

$$
I_{u_{x_i}}(r_1) - I_{u_{x_i}}(r_0) = \int_{r_0}^{r_1} \left( \frac{2t}{H_{u_{x_i}}(t)} (H_{u_{x_i}}(t)E_{u_{x_i}}(t) - G_{u_{x_i}}^2(t)) + R_{u_{x_i}}(t) \right) \, dt
$$

(3.32)

for $0 < r_0 < r_1 < 1 - |x_0|$, with

$$
|R_{u_{x_i}}(t)| \leq C_{3.9} \frac{t^{n+a+2k+1}}{H_{u_{x_i}}(t)} \left( \left( \frac{D_{u_{x_i}}(t)}{t^{n+a+2k+1}} \right)^{\frac{1}{2}} + 1 \right)
$$

(3.33)

and $C_{3.9} = C_{3.9}(n, a) > 0$.

**Proof.** It is not restrictive to assume $x_0 = 0$. We use the identities in (3.19), (3.20) and (3.21) to compute (the lengthy details are left to the reader)

$$
\frac{d}{dt}(I_{u_{x_i}}(t)) = I_{u_{x_i}}(t) \left( \frac{1}{t} + \frac{d}{dt} \left( \frac{G_{u_{x_i}}(t)}{H_{u_{x_i}}(t)} \right) \right)
$$

$$
= 2I_{u_{x_i}}(t) \left( \frac{G_{u_{x_i}}(t)}{H_{u_{x_i}}(t)} - \frac{D_{u_{x_i}}(t)}{H_{u_{x_i}}(t)} \right) + R_{u_{x_i}}(t),
$$

where

$$
R_{u_{x_i}}(t) \coloneqq - \frac{1}{H_{u_{x_i}}(t)} \int \phi \left( \frac{|x|}{t} \right) \left( (n + a - 1)u_0(x) + 2(\nabla u_0(x) \cdot x) \right) L_a(u_0(x)) \, dx
$$

$$
+ \frac{1}{t} \int \phi \left( \frac{|x|}{t} \right) |x| u(x) L_a(u_0(x)) \, dx.
$$

From this we conclude (3.32) straightforwardly.

For (3.33), we estimate separately each term appearing in the integral defining $R_{u_{x_i}}(t)$. We start with

$$
\left| \int \phi \left( \frac{|x|}{t} \right) u_0(x) L_a(u_0(x)) \, dx \right| \leq t^{k-1} \int \phi \left( \frac{|x|}{t} \right) u_0(x) |x|^k |x_{n+1}|^a \, dx
$$

$$
\leq Ct^{n+1+k-1} \left( \int \phi \left( \frac{|x|}{t} \right) u_0(x) |x|^k |x_{n+1}|^a \, dx \right)^{\frac{1}{2}}
$$

(3.27)

$$
\leq Ct^{n+1+k+1} \left( \int \phi \left( \frac{|x|}{t} \right) u_0(x) |x|^k |x_{n+1}|^a \, dx \right)^{\frac{1}{2}}
$$

(3.28)

(3.34)

with $C_{3.34} = C_{3.34}(n, a) > 0$. Arguing similarly we infer

$$
\left| \int \phi \left( \frac{|x|}{t} \right) (\nabla u_0(x) \cdot x) L_a(u_0(x)) \, dx \right| \leq t \int \phi \left( \frac{|x|}{t} \right) \nabla u_0(x) |L_a(u_0(x))| \, dx
$$

$$
\leq t^{k-1} \int \phi \left( \frac{|x|}{t} \right) \nabla u_0(x) |x_{n+1}|^a \, dx \leq Ct^{n+1+k+1} D_{u_{x_i}}^k(t),
$$

(3.35)
and
\[
\frac{1}{t} \int \phi \left( \frac{|x|}{t} \right) |x| u_{\varphi}(x) L_a(u_{\varphi}(x)) \, dx \leq t^{-k-1} \int_{B \setminus B_2} |u_0(x)||x|^{n+1} \, dx
\]

\[
\leq Ct^{n+1+k+1} \left( \frac{1}{u_0} + t^{n+1+k+1} \right),
\]
with $C = C(n, a) > 0$. Therefore, (3.33) follows at once from (3.34)–(3.36).

Estimate (3.33) turns out to be useful to analyze the subsets of points $\varphi_{\varphi, \theta}(u)$ of $\varphi_{\varphi}(u)$, for every $\theta \in (0, 1)$ (cf. (1.8)). With fixed $\theta \in (0, 1)$, we then look at points of the free boundary in the subset

\[
\varphi_{\varphi, \theta, \delta}(u) := \{ x_0 \in \varphi_{\varphi}(u) \cap B \setminus B_2 : H_{u_0}(r) \geq \delta r^{n+2(k+1)} \text{ for all } r \in (0, \frac{1}{2}) \},
\]
(3.37)
with $\delta > 0$.

**Remark 3.10.** Note that $\varphi_{\varphi, \theta, \delta}(u) \subseteq \varphi_{\varphi, \theta', \delta}(u)$ if $\theta' \leq \delta$. Hence, in what follows it is enough to consider the values of $\delta$ small enough.

**Proposition 3.11.** For every $\delta > 0$, $\theta \in (0, 1)$, there exist $C_{3.11}, \varrho_{3.11} > 0$ such that for every $x_0 \in \varphi_{\varphi, \theta, \delta}(u)$, the function $0, \varrho_{3.11} \ni r \mapsto e^{C_{3.11}r^\theta} I_{u_0}(r)$ is non-decreasing. In particular, the ensuing limits exist finite and are equal

\[
\lim_{r \downarrow 0} \frac{r D_{u_0}(r)}{H_{u_0}(r)} = \lim_{r \downarrow 0} I_{u_0}(r) =: I_{u_0}(0^+).
\]

(3.38)

**Proof.** Since $x_0 \in \varphi_{\varphi, \theta, \delta}(u)$, formula (3.26) yields for $r \in (0, \frac{1}{2})$,

\[
C_{3.8} D_{u_0}(r) \geq \delta r^{n+2(k+1)} - C_{3.8} r^{n+2(k+1)}
\]
therefore, for $\varrho_{3.11}$ sufficiently small, we have for all $r \in (0, \varrho_{3.11})$, 

\[
D_{u_0}(r) \geq Cr^{n+2(k+1)}.
\]
(3.39)

In addition, from (3.19) and (3.34) we get for all $r \in (0, \varrho_{3.11})$, if $\varrho_{3.11}$ small enough,

\[
|G_{u_0}(r) - D_{u_0}(r)| \leq C_{3.34} D_{u_0}(r) \left( \frac{x_0}{r} \right)^{n+2k+1} \left( \frac{D_{u_0}(r)}{D_{u_0}(r)} + \frac{r^{n+2k+1}}{D_{u_0}(r)} \right)
\]

\[
\leq CD_{u_0}(r) \left( 2Cr^2 + 2C^2r^2 \right) \leq Cr^\theta D_{u_0}(r).
\]
(3.40)

Therefore, from (3.33), if $\varrho_{3.11}$ is sufficiently small, we get for all $r \in (0, \varrho_{3.11})$,

\[
|R_{u_0}(r)| \leq C_{3.5} \frac{I_{u_0}(r)}{r^{n+2k}} \left( \frac{D_{u_0}(r)}{D_{u_0}(r)} + 1 \right)
\]

\[
\leq C \frac{I_{u_0}(r)}{D_{u_0}(r)} \left( \frac{x_0}{r} \right)^{n+2k+1} \left( \frac{x_0}{r} \right)^{n+2k+1} \frac{r^{n+2k+1}}{D_{u_0}(r)} \leq Cr^\theta I_{u_0}(r).
\]
(3.41)

Hence, from (3.18), (3.32) and (3.41) we find

\[
\frac{d}{dr} (I_{u_0}(r)) \geq -Cr^{\theta-1} I_{u_0}(r)
\]

(3.42)

and the monotonicity of $(0, \varrho_{3.11}) \ni r \mapsto e^{C_{3.11}r^\theta} I_{u_0}(r)$ follows by direct integration. In addition, we also infer (3.38), because from (3.40) for all $r \in (0, \varrho_{3.11})$ we have

\[
(1 - Cr^\theta) \frac{r D_{u_0}(r)}{H_{u_0}(r)} \leq I_{u_0}(r) \leq (1 + Cr^\theta) \frac{r D_{u_0}(r)}{H_{u_0}(r)}
\]

(3.43)

The proof is complete.
Remark 3.12. The monotonicity for the truncated Almgren’s frequency function
\[ r(1 + C r^\theta) \frac{d}{dr} \log \max \{ H_{u_{20}}(r), r^{n+2(k+1-\theta)} \} \]
proved in [9] and [21] is essentially equivalent to Proposition 3.11.

We derive next an additive quasi-monotonicity formula for the frequency.

Corollary 3.13. For every \( A, \delta > 0, \theta \in (0, 1) \), there exist \( C_{3.13}, q_{3.13} > 0 \) with this property: if \( x_0 \in \mathcal{Z}_{A, \theta, \delta}(u) \) and \( I_{u_{20}}(q_{3.13}) \leq A \), then for all \( \Lambda \geq A C_{3.13} \) the function
\[ (0, q_{3.13}) \ni r \mapsto I_{u_{20}}(r) + \Lambda r^\theta \]
is non-decreasing.

Proof. Under the standing assumptions, the quasi-monotonicity of \( I_{u_{20}} \) and (3.42) yield that
\[ \frac{d}{dr}(I_{u_{20}}(r)) \geq -C e^{C_{3.8} A r^{\theta-1}} \]
for \( r \) sufficiently small. Hence, we conclude (3.44) at once by integration. \( \square \)

3.3 Lower bound on the frequency and compactness

We first show that the frequency of a solution \( u \) to (3.1) at points in \( \mathcal{Z}_{A, \theta, \delta}(u) \) is bounded from below by a universal constant.

Lemma 3.14. For every \( \delta > 0, \theta \in (0, 1) \), there exists a constant \( q_{3.14} > 0 \) such that, for all \( x_0 \in \mathcal{Z}_{A, \theta, \delta}(u) \) and \( r \in (0, q_{3.14}] \),
\[ I_{u_{20}}(r) \geq \frac{1}{2C_{3.8}}. \]

Proof. In view of (3.26) and since \( x_0 \in \mathcal{Z}_{A, \theta, \delta}(u) \), we have for all \( r \) sufficiently small,
\[ \frac{1}{C_{3.8}} \leq \frac{r D_{u_{20}}(r)}{H_{u_{20}}(r)} + \frac{r^{2\theta}}{\delta}. \]

Inequality (3.45) is a straightforward consequence of estimate (3.43) and the latter estimate provided that \( q_{3.14} \) is sufficiently small. \( \square \)

For the free boundary analysis developed in [18] it is mandatory to consider the critical set of a solution. In the current framework, the natural substitute for the critical set is given by
\[ \mathcal{N}_A(u) := \{(x', 0) \in B_1^1 : u(x', 0) - \varphi(x') = |\nabla'(u(x', 0) - \varphi(x'))| = \lim_{y \downarrow 0} y^a \partial_{n+1} u(x', y) = 0\}. \]

Notice that \( \Gamma_{\varphi}(u) \subseteq \mathcal{N}_A(u) \subseteq \Lambda_{\varphi}(u) \) (the first inclusion is a consequence of (3.10)).

We can then give the following compactness result. For \( u : B_1 \to \mathbb{R} \) solution of (3.1) and \( x_0 \in B_1' \), we introduce the rescalings
\[ u_{x_0, r}(y) := \frac{r^{n+2} u_{x_0}(x_0 + ry)}{H_{u_{20}}(r)} \quad \text{for all } r \in (0, 1 - |x_0|) \] \text{and all } y \in B_{1 - |x_0|}. \]

Note that \( u_{x_0, r} \) is a minimizer of the functional
\[ \int_{B_1} |\nabla v|^2 \chi_{x_{n+1}}^{2a} \ dx - 2 \int_{B_1} v L_A(\varphi_{x_{0, r}}) \ dx \]
with obstacle function
\[ \varphi_{x_{0, r}}(y) := \frac{r^{n+2} \varphi_{x_0}(x_0 + ry)}{H_{u_{20}}(r)}, \]
among all functions \( v \in u_{x_0, r} + H_{u_{20}}(B_1, dm) \) satisfying \( v(x', 0) \geq 0 \) on \( B_1' \).
Corollary 3.15. Let $\delta > 0$ and $\theta \in (0, 1)$ be given. Let $(u_i)_{i \in \mathbb{N}}$ be a sequence of solutions to the fractional obstacle problem (3.1) in $B_1$ with obstacle functions $\varphi_i$ equi-bounded in $C^{k+1}(B_1)$, and let $x_i \in \mathcal{K}_{\varphi_i, \theta, \delta}(u_i)$ be such that $\sup_{B} I_{(x_i)}(q_l) < +\infty$, for some $q_l \downarrow 0$. Then there exist a subsequence $l_j \uparrow \infty$ and a solution $v_{\infty}$ to the fractional obstacle problem (3.1) in $B_1$ with null obstacle function, such that on setting $v_j := (u_i)_{x_i, q_l}$ we have

$$
\begin{align*}
v_j & \to v_{\infty} \quad \text{in } H^1(B_1, dm), \\
v_j & \to v_{\infty} \quad \text{in } C^{0, \alpha}_{\text{loc}}(B_1), \text{ for all } \alpha < \min\{1, 2s\}, \\
|\nabla^j v_j | & \to |\nabla v_{\infty}| \quad \text{in } C^{0, \alpha}_{\text{loc}}(B_1), \text{ for all } \alpha < s, \\
\text{sign}(x) & \to |\nabla v_{\infty}| \quad \text{in } C^{0, \alpha}_{\text{loc}}(B_1), \text{ for all } \alpha < 1 - s.
\end{align*}
$$

(3.49) (3.50) (3.51) (3.52)

Proof. By taking into account inequality (3.43) in Proposition 3.11, we get for $l$ large,

$$
\frac{\varphi_l D_{(x_i, \theta, \delta)}(q_l)}{H_{(x_i, \theta, \delta)}(q_l)} \leq (1 + C\|\varphi_l\|_{C^{0, 1}(B_1)} q_l^{\theta}_{\|y_{x+1}\|^1}) I_{(x_i, \theta, \delta)}(q_l).
$$

In particular, we infer that $\sup_{B} D_{(x_i, \theta, \delta)}(q_l) < +\infty$. Thus, a subsequence $v_j := (u_i)_{x_i, q_l}$ converges weakly $H^1(B_1, dm)$ to some function $v_{\infty}$. Moreover, $v_j$ is a local minimizer of

$$
F_l(v) := \int_{B_1} |\nabla v|^2 |y_{x+1}|^1 a \, dy - 2 \int_{B_1} v L_a((\varphi_j)_{x_i, q_l}) \, dy
$$

among all functions $v \in v_j + H^1_0(B_1, dm)$ satisfying $v(x', 0) \geq 0$ on $B_1'$ (cf. (3.46)-(3.47)).

By taking into account that $x_i \in \mathcal{K}_{\varphi_i, \theta, \delta}(u_i)$, inequality (3.7) implies that for all $y \in B_1 \setminus B_1'$,

$$
|L_a((\varphi_j)_{x_i, q_l})(y)| \leq \frac{1}{\delta^2} \|\varphi_l\|_{C^{0, 1}(B_1)} q_l^{\theta}_{\|y_{x+1}\|^1} a.
$$

(3.53)

Therefore, one can easily show that the sequence $(F_j)$ $\Gamma(L^2(B_1, dm))$-converges to the functional

$$
F_{\infty} : L^2(B_1, dm) \to [0, +\infty]
$$

defined by

$$
F_{\infty}(v) := \int_{B_1} |\nabla v|^2 |y_{x+1}|^1 a \, dy
$$

if $v \in v_{\infty} + H^1_0(B_1, dm)$ with $v(x', 0) \geq 0$ on $B_1'$, and $+\infty$ otherwise on $L^2(B_1, dm)$. In addition, being the functionals $F_j$ equicoercive in $L^2(B_1, dm)$, $F_j(v_j) \to F_{\infty}(v_{\infty})$, so that by (3.53) the convergence of $(v_j)$ to $v_{\infty}$ is actually strong $H^1(B_1, dm)$.

Items (3.50)–(3.52) are then a straightforward consequence of Theorem 3.1 and (3.53) (cf. the arguments in [9, Lemma 6.2]).

A sharp lower bound on the frequency then follows.

Corollary 3.16. Let $\delta > 0$, $\theta \in (0, 1)$. If $x_0 \in \mathcal{K}_{\varphi, \theta, \delta}(u)$, then

$$
I_{(x_0)}(0^+) \geq 1 + s.
$$

(3.54)

Proof. Note that $I_{(x_0)}(0^+) = \lim_{r \downarrow 0} I_{(x_0)}(r) = \lim_{r \downarrow 0} I_{(x_0)}(1) = I_{(0^+)}(1)$ for some $v_{\infty}$ homogeneous solution to the fractional obstacle problem (3.1) with null obstacle function provided by Corollary 3.15. Thus, we conclude (3.54) by [9, Proposition 5.1] (see also [18, Corollary 2.12]).

4 Main estimates on the frequency

In this section we prove the principal estimates on the frequency that we are going to exploit in the sequel. We start with an elementary lemma. Recall that all obstacles functions $\varphi$ are assumed to satisfy the normalization condition $\|\varphi\|_{C^{0,1}(B_1')} \leq 1$. 

Lemma 4.1. Let $A, \delta > 0, \theta \in (0, 1)$. Then, there exist $C_{A,1}, \varrho_{A,1} > 0$ such that, if $u$ is a solution of the fractional obstacle problem (3.1) in $B_1$, with $0 \not\in \mathcal{Z}_{\varrho,\theta,\delta}(u)$ and $I_{u_2}(2\varrho) \leq A, \varrho \leq \varrho_{A,1}$, then for every $x \in B_{\delta}^c$,

$$\frac{1}{C_{A,1}} \leq \frac{H_{u_2}(q)}{H_{u_2}(\varrho)} \leq C_{A,1} \quad \text{and} \quad \frac{1}{C_{A,1}} \leq \frac{D_{u_2}(q)}{D_{u_2}(\varrho)} \leq C_{A,1},$$

(4.1)

and

$$|I_{u_2}(\varrho) - I_{u_2}(q)| \leq C_{A,1}.$$  

(4.2)

Remark 4.2. Note that as a byproduct of the first estimate in (4.1) in Lemma 4.1 frequencies at the scale $\varrho$ are well-defined at every point $x \in B_{\delta}^c$, recalling that $0 \not\in \mathcal{Z}_{\varrho,\theta,\delta}(u)$.

Proof. In order to prove (4.1), we argue by contradiction: we can assume that there exist $A, \delta > 0$ and solutions $u_t$ to the fractional obstacle problem with obstacles $\varphi_{j,t}, \|\varphi_t\|_{C^{1,1}(B_2^c)} \leq 1$, with $0 \not\in \mathcal{Z}_{\varrho,\theta,\delta}(u_t)$, such that $I_{u_2}(\varrho_j) \leq A$, for some $\varrho_j \downarrow 0$, and there exist points $x_j \in B_{\delta}^c$, contradicting one of the sets of inequalities in (4.1).

In particular, by almost monotonicity of the frequency function (cf. Proposition 3.11) and the lower bound on the frequency (cf. Corollary 3.16) we infer that $1 + s \leq I_{u_2}(\varrho_j) \leq A e^{C_1(2\varrho_j)\delta} \leq A e^{C_1(\theta)} := A' \quad \text{for all} \ t \in (0, 2\varrho_j]$. By Corollary 3.15, up to a subsequence, $v_j := (u_{j}\varrho_j)_{\varrho_j}$ converges strongly in $H^1(B_1, dm)$ to a function $v_{\infty}$ solution of the fractional obstacle problem in $B_2$ with zero obstacle function. We assume in addition that $\varrho_j^{-1}x_j \to x_{\infty} \in B_{\delta}^c$.

To prove the first set of inequalities in (4.1), we compute

$$\frac{H_{(u_{j})_2}(\varrho)}{H_{(u_{j})_2}(\varrho_j)} = \frac{2\varrho^{n+a}}{H_{(u_{j})_2}(\varrho)} \int_{B_1 \setminus B_{\frac{1}{2}}} (u_{j})_2^2(x_j + \varrho_j x) \frac{|x + 1|}{|x|} \, dx$$

$$= \frac{2\varrho^{n+a}}{H_{(u_{j})_2}(\varrho)} \int_{B_1 \setminus B_{\frac{1}{2}}} [u_j(x_j + \varrho_j x) - (\varphi_j)_2(x_j + \varrho_j x')]^2 \frac{|x + 1|}{|x|} \, dx$$

$$= 2 \int_{B_1 \setminus B_{\frac{1}{2}}} \left[ v_j(\varrho_j^{-1}x_j + x) + \frac{\varrho_j^{n+a}}{H_{(u_{j})_2}(\varrho)} ((\varphi_j)_2(x_j + \varrho_j x') - (\varphi_j)_2(x_j + \varrho_j x'))^2 \frac{|x + 1|}{|x|} \right] \, dx.$$  

(4.3)

Moreover, by estimate (3.15) and since $\varrho_j^{-1}x_j \to x_{\infty}$, we get for all $x' \in B_{\frac{1}{2}}$

$$|((\varphi_j)_2(x_j + \varrho_j x') - (\varphi_j)_2(x_j + \varrho_j x'))| \leq C \varrho_j.$$  

(4.4)

Therefore, recalling that $0 \not\in \mathcal{Z}_{\varrho,\theta,\delta}(u_j)$, from (4.4) we infer

$$\frac{\varrho_j^{n+a}}{H_{(u_{j})_2}(\varrho)} \int_{B_1 \setminus B_{\frac{1}{2}}} ((\varphi_j)_2(x_j + \varrho_j x') - (\varphi_j)_2(x_j + \varrho_j x'))^2 \frac{|x + 1|}{|x|} \, dx \leq C \varrho_j^{2\theta}.$$  

(4.5)

Since $\varrho_j \downarrow 0$, by contradiction

$$\lim_{j} \frac{H_{(u_{j})_2}(\varrho_j)}{H_{(u_{j})_2}(\varrho)} \in (0, \infty).$$

Moreover, by (4.3) and (4.5), by the strong $L^2(B_1, dm)$ and local uniform convergence of $v_j \to v_{\infty}$ we conclude that

$$2 \int_{B_1(x_{\infty}) \setminus B_{\frac{1}{2}}(x_{\infty})} \nu_{\infty}^2(y) \frac{|y + 1|}{|y - x_{\infty}|} \, dy = 2 \lim_{j} \int_{B_{\frac{1}{2}}(\varrho_j^{-1}x_j) \setminus B_{\frac{1}{2}}(\varrho_j^{-1}x_j)} \nu_j^2(y) \frac{|y + 1|}{|y - \varrho_j^{-1}x_j|} \, dy$$

$$= 2 \lim_{j} \int_{B_1 \setminus B_{\frac{1}{2}}} \nu_j^2(\varrho_j^{-1}x_j + x) \frac{|x + 1|}{|x|} \, dx$$

$$= \lim_{j} \frac{H_{(u_j)_2}(\varrho_j)}{H_{(u_j)_2}(\varrho_j)}.$$
Being the left-hand side finite, necessarily

\[ 2 \int_{B_1(x_{\omega}) \cap B_1} \psi_{\omega}^2(y) \frac{|y_{n+1}|^2}{|y-x_{\omega}|} \, dy = \lim_{j} \frac{H(u_{\omega,j})(q)}{H(u_{\omega,j})(q)} = 0. \]

Hence, \( \psi_{\omega} = 0 \) on \( B_1(x_{\omega}) \setminus B_1 \), and thus \( \psi_{\omega} = 0 \) on the whole of \( B_1 \) by analyticity. A contradiction to \( H_{\psi_{\omega}}(1) = 1 \) that follows from strong \( L^2(B_1, dm) \) convergence and the equality \( H_{\psi_{\omega}}(1) = 1 \) for all \( j \).

The second set of inequalities in (4.1) is proven by the same argument. Indeed, assuming that

\[ \lim_{j} \frac{D(u_{\omega,j})(q)}{D(u_{\omega,j})(q)} \in (0, \infty), \]

we have

\[ \frac{\phi_j^{n+a+1}}{D(u_{\omega,j})(q)} \int_{B_1} \phi(|x|)|\nabla((\phi_{\omega})_j(x+q|x|) - (\phi_{\omega})_j(x+q|x|))| |x_{n+1}|^a \, dx, \]

and since by (3.16) in Remark 3.2 and by (3.39),

\[ \frac{\phi_j^{n+a+1}}{D(u_{\omega,j})(q)} \int_{B_1} \phi(|x|)|\nabla((\phi_j^{-1}x_j + x))| |x_{n+1}|^a \, dx \leq C \phi_j^{n+a+1+2k}, \]

we get (recall \( \phi_j \downarrow 0 \))

\[ \lim_{j} \frac{D(u_{\omega,j})(q)}{D(u_{\omega,j})(q)} = \lim_{j} \frac{1}{4I_{u_{\omega,j}}(q)} \int_{B_1} \phi(|x|)|\nabla((\phi_j^{-1}x_j + x))| |x_{n+1}|^a \, dx. \]

By the strong convergence of \( v_j \) to \( v_{\omega} \) in \( H^2(B_1, dm) \), we infer that the left-hand side is finite and then actually 0, so that

\[ \int_{B_1} \phi(|x|)|\nabla v_{\omega}(x_{\omega} + x)| |x_{n+1}|^a \, dx = 0. \]

Thus, by analyticity \( v_{\omega} \) is constant on \( B_1 \), and we may conclude that

\[ \int_{B_1} \phi(|x|)|\nabla v_{\omega}(x)|^2 |x_{n+1}|^a \, dx = 0. \]

The latter equality contradicts

\[ \int_{B_1} \phi(|x|)|\nabla v_{\omega}(x)|^2 |x_{n+1}|^a \, dx \in [1 + s, 2A'], \]

that follows from strong \( H^1(B_1, dm) \) convergence and recalling that \( H_{v_j}(1) = 1 \) and \( 1 + s \leq I_{v_j}(1) \leq 2D_{v_j}(1) \leq A' \) for \( j \) big enough (cf. (3.43)).

Finally, (4.2) follows straightforwardly from (4.1) for \( \phi_{\omega,1} \) sufficiently small by taking into account (3.43):

\[ |I_{u_{\omega}}(r) - I_{u_{\omega}}(r)| = \frac{rI_{u_{\omega}}(r)}{I_{u_{\omega}}(r)} \leq C. \]

We introduce the following notation for the radial variation of (modified) frequency at a point \( x \in Z_{\phi,\theta,\delta}(u) \) of a solution \( u \) in \( B_1 \): given \( 0 < r_0 < r_1 < 1 - |x| \), we set

\[ \Delta_{\omega}^j(x) := I_{u_{\omega}}(r_1) + \Lambda r_1^a - (I_{u_{\omega}}(r_0) + \Lambda r_0^a). \]

Note that \( \Delta_{\omega}^j(x) \geq 0 \) if \( x \in Z_{\phi,\theta,\delta}(u) \), if \( r_1 \) is sufficiently small and if \( \Lambda \geq AC_{3.13} \) (cf. Corollary 3.13). We do not indicate the dependence of \( \Delta_{\omega}^j \) on \( \Lambda \) since such a parameter will be fixed according to the restriction above.
Lemma 4.3. Let \( A, \delta > 0, \theta \in (0, 1) \). Then there exist \( C_{A, \delta} \) and \( \varrho_{A, \delta} > 0 \) such that, if \( x_0 \in \mathcal{Z}_{\delta, \theta}(u) \), and \( I_{u_{x_0}}(r_1) \leq A \), with \( 2r_1 \leq \varrho_{A, \delta} \), then for every \( r_0 \in (r_1/2, r_1) \) we have

\[
\int_{B_{r_0}^+(x_0)} \left( \nabla u_{x_0}(z) \cdot (z - x_0) - I_{u_{x_0}} \left( \frac{r_0}{2} \right) u_{x_0}(z) \right)^2 \frac{|z_{n+1}|^\alpha}{|z|} \, dz \leq C_{A, \delta} H_{u_{x_0}}(r_1) \Delta_{\frac{1}{2}}^\theta \left( x_0 \right).
\]  

(4.6)

Proof. Without loss of generality we prove the result for \( x_0 = 0 \). We start off with the following computation:

\[
2 \int_{B_{r_1}^+ \setminus B_{r_2}^+} \left( \nabla u_{x_0}(z) \cdot z - I_{u_{x_0}}(t) u_{x_0}(z) \right)^2 \frac{|z_{n+1}|^\alpha}{|z|} \, dz = \int -\frac{1}{t} \left( \frac{|z|}{t} \right) \left[ (\nabla u_{x_0}(z) \cdot z - I_{u_{x_0}}(t) u_{x_0}(z))^2 \frac{|z_{n+1}|^\alpha}{|z|} \, dz \right]
\]

\[
= t^2 E_{u_{x_0}}(t) - 2t H_{u_{x_0}}(t) G_{u_{x_0}}(t) + I_{u_{x_0}}^2(t) H_{u_{x_0}}(t)
\]

\[
= \frac{t^2}{H_{u_{x_0}}(t)} (E_{u_{x_0}}(t) H_{u_{x_0}}(t) - G_{u_{x_0}}^2(t))
\]

\[
= \frac{t}{H_{u_{x_0}}(t)} \left( \frac{d}{dt}(I_{u_{x_0}}(t)) - R_{u_{x_0}}(t) \right).
\]  

(4.7)

We now use the integral estimate (whose elementary proof is left to the readers)

\[
\int_{B_{r_0}^+ \setminus B_{r_0}} f(z) \, dz \leq \rho_0^{-1} \int_{\rho_0}^{2\rho_0} \int_{B_\rho^+} f(z) \, dz \, dt \quad \text{for all } 0 < \rho_0 \leq \rho_1,
\]  

(4.8)

\( f \geq 0 \) a measurable function, in order to deduce

\[
\int_{B_{r_0}^+ \setminus B_{r_0}} \left( \nabla u_{x_0}(z) \cdot z - I_{u_{x_0}} \left( \frac{r_0}{2} \right) u_{x_0}(z) \right)^2 \frac{|z_{n+1}|^\alpha}{|z|} \, dz \leq \rho_0^{-1} \int_{\rho_0}^{2\rho_0} \int_{B_\rho^+} \left( \nabla u_{x_0}(z) \cdot z - I_{u_{x_0}}(t) u_{x_0}(z) \right)^2 \frac{|z_{n+1}|^\alpha}{|z|} \, dz \, dt
\]

\[
\leq \frac{4}{r_0} \int_{\rho_0}^{2\rho_0} \int_{B_\rho^+} \left[ \left( \nabla u_{x_0}(z) \cdot z - I_{u_{x_0}}(t) u_{x_0}(z) \right)^2 + \left( I_{u_{x_0}}(t) - I_{u_{x_0}} \left( \frac{r_0}{2} \right) \right)^2 u_{x_0}^2(z) \right] \frac{|z_{n+1}|^\alpha}{|z|} \, dz \, dt
\]

\[
\leq \frac{4}{r_0} \int_{\rho_0}^{2\rho_0} \int_{B_\rho^+} \left[ \frac{d}{dt}(I_{u_{x_0}}(t)) - R_{u_{x_0}}(t) \right] dt
\]

\[
+ \frac{16}{r_0} \left( I_{u_{x_0}}(r_1) - I_{u_{x_0}} \left( \frac{r_0}{2} \right) \right)^2 + (AC_{3.13})^2 \left( r_1^\theta - \left( \frac{r_0}{2} \right)^\theta \right)^2 \int_{\rho_0}^{2\rho_0} H_{u_{x_0}}(t) \, dt
\]

\[
\leq \frac{r_1}{r_0} H_{u_{x_0}}(r_1) \int_{\rho_0}^{2\rho_0} \left( \frac{d}{dt}(I_{u_{x_0}}(t)) - R_{u_{x_0}}(t) \right) dt
\]

\[
+ \frac{16 r_1}{r_0} H_{u_{x_0}}(r_1) \left( I_{u_{x_0}}(r_1) - I_{u_{x_0}} \left( \frac{r_0}{2} \right) \right)^2 + (AC_{3.13})^2 \left( r_1^\theta - \left( \frac{r_0}{2} \right)^\theta \right)^2 \left. \right|_{\rho_0}^{2\rho_0}.
\]  

(4.9)

In the last inequality we have used that, if \( \varrho_{A, \delta} \) is sufficiently small, then we have \( H_{u_{x_0}}(t) \leq H_{u_{x_0}}(r_1) \) for all \( t \leq r_1 \) by (3.24), and that \( \frac{d}{dt}(I_{u_{x_0}}(t)) - R_{u_{x_0}}(t) \geq 0 \) thanks to (4.7). Moreover, estimate (3.41) in Proposition 3.11, \( I_{u_{x_0}}(r_1) \leq A \), the quasi-monotonicity of the frequency function and the choice \( 2r_1 \leq \varrho_{A, \delta} \) imply

\[
\int_{\rho_0}^{2\rho_0} |R_{u_{x_0}}(t)| \, dt \leq AC_{3.11} e^{C_{1.11} r_1^\theta \left( \frac{r_0}{2} \right)^\theta}.
\]
Hence, from (4.9) we conclude that
\[
\int_{\mathbb{R}^2} \left( \nabla u_0(z) \cdot z - Iu_0 \left( \frac{r_0}{2} \right) u_0(z) \right)^2 \frac{|z|^{n+1}}{|z|} \, dz \leq C H_{1n}(r_1) \left( Iu_0(r_1) + \Lambda r_1 - Iu_0 \left( \frac{r_0}{2} \right) - \Lambda \left( \frac{r_0}{2} \right) \right),
\]
where we used that \( \frac{r_0}{r_1} \leq 8 \), and \( C > 0 \).

\[
\square
\]

### 4.1 Oscillation estimate of the frequency

The following lemma shows how the spatial oscillation of the frequency in two nearby points at a given scale is in turn controlled by the radial variations at comparable scales.

**Proposition 4.4.** Let \( A, \delta > 0, \theta \in (0, 1) \). Then there exist \( C_{A, \delta} > 0 \) such that, if \( \Phi \in \mathcal{X}_{\delta, \delta}(u) \), \( \tau \in (0, \frac{\delta}{144}) \) with \( Iu_0(72 \tau) \leq A \), then
\[
|Iu_{x_1}(10\tau) - Iu_{x_2}(10\tau)| \leq C_{A, \delta} \left( (\Lambda_{2\delta}^2(x_1))^\frac{1}{2} + (\Lambda_{2\delta}^2(x_2))^\frac{1}{2} \right) + C_{A, \delta} \tau^\theta
\]
for every \( x_1, x_2 \in B'_{\tau} \cap \mathcal{X}_{\delta, \delta}(u) \).

**Proof.** We start off noting that by Remark 4.2 and the choice \( 144 \tau < \delta_{2, \delta} \), if the constant \( \delta_{2, \delta} \) is suitably chosen, a simple scaling argument yields that \( Iu_0(10\tau) \) is well-defined for every \( x \in B'_{\tau} \).

To ease the readability of the proof we divide it in several substeps.

**Step 1.** With fixed \( x_1, x_2 \in B'_{\tau} \cap \mathcal{X}_{\delta, \delta}(u) \), let
\[
x_t := tx_1 + (1-t)x_2, \quad t \in [0, 1],
\]
and consider the map \( t \mapsto Iu_{x_t}(10\tau) \). The differentiability of the functions \( x \mapsto H_{x_t}(10\tau) \) and \( x \mapsto D_{x_t}(10\tau) \) yields that
\[
Iu_{x_{1t}}(10\tau) - Iu_{x_{2t}}(10\tau) = \int_0^1 \frac{d}{dt}(Iu_{x_t}(10\tau)) \, dt.
\]

Set \( e := x_1 - x_2 \); then \( e \cdot e_{n+1} = 0 \); and set for all \( y \in \mathbb{R}^{n+1} \),
\[
\delta_t(y) := \frac{d}{dt}(u_{x_t}(x_t + y)).
\]

Recalling the very definition of \( u_{x_t} \) in (3.6), it turns out that
\[
\delta_t(y) = \partial_x u(x_t + y) - \partial_x \varphi(x_t + y) + T_{k,x_t}[\partial_x \varphi](x_t + y) - \epsilon_0[T_{k,x_t}[\partial_x \varphi]](x_t + y),
\]
and
\[
\frac{d}{dt}(T_{k,x_t}[\varphi])(x_t + y) = T_{k,x_t}[\partial_x \varphi](x_t + y)
\]
by linearity (the details of the elementary computations are left to the readers)
\[
\frac{d}{dt}(\epsilon_0[T_{k,x_t}[\varphi]])(x_t + y) = \epsilon_0[T_{k,x_t}[\partial_x \varphi]](x_t + y).
\]

Moreover, from the very definition of \( u_{x_t} \) in (3.6) it is straightforward to prove that
\[
\partial_x u_{x_t}(x_t + y) = \partial_x u(x_t + y) - \partial_x \varphi(x_t + y) + T_{k-1,x_t}[\partial_x \varphi](x_t + y) - \epsilon_0[T_{k-1,x_t}[\partial_x \varphi]](x_t + y).
\]

Thus, from (4.11) and the latter equality, by direct calculation it follows that
\[
\delta_t(y) - \partial_x u_{x_t}(x_t + y) = \sum_{|a|=k} D^a(\partial_x \varphi)(x_t) \left( \frac{y!}{a!} \right) - \epsilon_0 \left( \sum_{|a|=k} D^a(\partial_x \varphi)(x_t) \frac{P_a(-x_t)}{a!} \right)(y),
\]
and thus we may conclude that
\[
|\delta_t(y) - \partial_x u_{x_t}(x_t + y)| \leq C |x_1 - x_2| |y|^k.
\]

Moreover, note also that
\[
\nabla \delta_t(y) = \frac{d}{dt}(\nabla u_{x_t}(x_t + y)).
\]
Step 2. Thanks to the previous formulas, for all $\lambda \in \mathbb{R}$ we infer
\[
\frac{d}{dt}(H_{u_\lambda}(10\tau)) = -2 \int \phi \left( \frac{|y|}{10\tau} \right) u_{\lambda}(x_\tau + y) \delta_t(y) \frac{|y_n + 1|^q}{|y|} dy
= -2 \int \phi \left( \frac{|y|}{10\tau} \right) \left( \delta_t(y) - \lambda u_{\lambda}(x_\tau + y) \right) u_{\lambda}(x_\tau + y) \frac{|y_n + 1|^q}{|y|} dy + 2 \lambda H_{u_\lambda}(10\tau). \tag{4.15}
\]
In addition, integrating by parts gives
\[
\frac{d}{dt}(D_{u_\lambda}(10\tau)) = 2 \int \phi \left( \frac{|y|}{10\tau} \right) \nabla \delta_t(y) \cdot \nabla u_{\lambda}(x_\tau + y) |y_n + 1|^q \frac{dy}{|y|}
= -\frac{1}{5\tau} \int \phi \left( \frac{|y|}{10\tau} \right) \delta_t(y) \nabla u_{\lambda}(x_\tau + y) \cdot y |y_n + 1|^q \frac{dy}{|y|} - 2 \int \phi \left( \frac{|y|}{10\tau} \right) \delta_t(y) L_\alpha(u_{\lambda}(x_\tau + y)) \frac{dy}{|y|}
+ 2 \lambda G_{u_\lambda}(10\tau) - 2 \int \phi \left( \frac{|y|}{10\tau} \right) \delta_t(y) L_\alpha(u_{\lambda}(x_\tau + y)) \frac{dy}{|y|}. \tag{4.16}
\]
Then, by formula (3.19) together with (4.15) and (4.16), we have that
\[
\frac{d}{dt}(I_{u_\lambda}(10\tau)) = I_{u_\lambda}(10\tau) \frac{d}{dt}(\frac{\frac{d}{dt}(G_{u_\lambda}(10\tau))}{H_{u_\lambda}(10\tau)} - \frac{d}{dt}(H_{u_\lambda}(10\tau)))
= -\frac{2}{H_{u_\lambda}(10\tau)} \int \phi \left( \frac{|y|}{10\tau} \right) \delta_t(y) - \lambda u_{\lambda}(x_\tau + y) \left( \nabla u_{\lambda}(x_\tau + y) \cdot y - I_{u_\lambda}(10\tau) u_{\lambda}(x_\tau + y) \right) \frac{dm(y)}{|y|}
+ \frac{10\tau}{H_{u_\lambda}(10\tau)} \int \phi \left( \frac{|y|}{10\tau} \right) \left( u_{\lambda}(x_\tau + y) \frac{d}{dt}(L_\alpha(u_{\lambda}(x_\tau + y))) - \delta_t(y) L_\alpha(u_{\lambda}(x_\tau + y)) \right) \frac{dy}{|y|}
=: f^{(1)}_t + f^{(2)}_t.
\]
In what follows we estimate separately the two terms $f^{(i)}_t$.

Step 3. We start off with $f^{(1)}_t$. With this aim, first note that $I_{u_2}(10\tau) \leq Ae^{72\tau C_{1,1}}$ by Proposition 3.11 since $144\tau < q_{4,4}$, provided the latter is small enough. In turn, as $x_\tau \in B_r'$, by (4.2) in Lemma 4.1 we infer that
\[
I_{u_\lambda}(10\tau) \leq C_{4,1} + Ae^{72\tau C_{1,1}}.
\]
We estimate separately the factors of the integrand defining $f^{(1)}_t$ (setting $x_t + y = z$). We start off with the first one as follows
\[
|\delta_t(z - x_t) - \lambda u_{\lambda}(z)| \leq |\delta_t(z - x_t) - \lambda u_{\lambda}(z)| + |\delta_t(z - x_t) - \partial_x u_{\lambda}(z)|
\leq |\partial_x u_{\lambda}(z)| + C|x_1 - x_2||z - x_1|^k,
\]
with $C = C(n, k) > 0$. Moreover, by choosing $\lambda := I_{u_2}(10\tau) - I_{u_\lambda}(10\tau)$, we infer
\[
|\partial_x u_{\lambda}(z)| - \lambda u_{\lambda}(z)| = |\nabla u_{\lambda}(z) \cdot e - \lambda u_{\lambda}(z)|
\leq |\nabla u_{\lambda}(z) \cdot (z - x_1) - I_{u_2}(10\tau) u_{\lambda}(z) + |\nabla u_{\lambda}(z) \cdot (z - x_2) - I_{u_\lambda}(10\tau) u_{\lambda}(z)|
\leq |\nabla u_{\lambda}(z) \cdot (z - x_1) - I_{u_2}(10\tau) u_{\lambda}(z) + |\nabla u_{\lambda}(z) \cdot (z - x_2) - I_{u_\lambda}(10\tau) u_{\lambda}(z)|
+ |\nabla u_{\lambda}(z) - u_{\lambda}(z) \cdot (z - x_1) - I_{u_2}(10\tau) u_{\lambda}(z) - u_{\lambda}(z)\|
+ |\nabla u_{\lambda}(z) - u_{\lambda}(z) \cdot (z - x_2) - I_{u_\lambda}(10\tau) u_{\lambda}(z) - u_{\lambda}(z)|.
\]
Using inequalities (3.15)–(3.16) in Remark 3.2, we estimate the last two addends as follows:
\[
|\nabla u_{\lambda}(z) - u_{\lambda}(z) \cdot (z - x_i) - I_{u_\lambda}(10\tau) u_{\lambda}(z) - u_{\lambda}(z)| \leq Cr^{k+1}
\]
for some constant $C > 0$, for $i = 1, 2$. In the last inequality we have used that $|z - x_i| \leq 12\tau$, being $z \in B_{10\tau}(x_t)$. Therefore, we have
\[
|\delta_t(z - x_t) - \lambda u_{\lambda}(z)| \leq |\nabla u_{\lambda}(z) \cdot (z - x_1) - I_{u_2}(10\tau) u_{\lambda}(z) + |\nabla u_{\lambda}(z) \cdot (z - x_2) - I_{u_\lambda}(10\tau) u_{\lambda}(z)| + Cr^{k+1}
=: \psi(z). \tag{4.17}
\]
For the second factor, we note that for $i = 1, 2$,
\[
|\nabla u_{ik}(z) \cdot (z - x_i) - I_{u_{ik}}(10r)u_{ik}(z)| \leq |\nabla u_{ik}(z) \cdot (z - x_i) - I_{u_{ik}}(10r)u_{ik}(z)| + |(u_{ik}(z) - u_{ik}(z)) \cdot (z - x_i)|
\]
\[
+ |I_{u_{ik}}(10r)||u_{ik}(z) - u_{ik}(z)| + |I_{u_{ik}}(10r) - I_{u_{ik}}(10r)||u_{ik}(z)|
\]
\[
\leq r^{k+1} + C|u_{ik}(z)|.
\]

To estimate the last three addends, we have used the very definition of $u_{ik}$ in (3.6), formula (4.2) and inequalities (3.15)–(3.16) in Remark 3.2, taking into account that $|z - x_i| \leq 12r$ being $z \in B_{10r}(x_i)$. Therefore, we get
\[
|\nabla u_{ik}(z) \cdot (z - x_i) - I_{u_{ik}}(10r)u_{ik}(z)| \leq |u_{ik}(z)| + C(|u_{ik}(z)| + |u_{ik}(z)|).
\]
(4.18)

By collecting (4.17) and (4.18), using Hölder inequality we conclude that there exists $C > 0$ such that
\[

t^{(1)} \leq \frac{C}{H_{u_{ik}}(10r)} \int -\phi(\frac{|z - x_i|}{10r}) \psi(z)(\psi(z) + |u_{ik}(z)| + |u_{ik}(z)|) \frac{|z_{n+1}|}{|z - x_i|} dz
\]
\[
\leq \frac{C}{H_{u_{ik}}(10r)} \left( \int -\phi(\frac{|z - x_i|}{10r}) \psi(z) \frac{|z_{n+1}|}{|z - x_i|} dz \right)^{\frac{1}{2}}
\]
\[
\cdot \left( \int -\phi(\frac{|z - x_i|}{10r}) (\psi(z) + |u_{ik}(z)|)^2 + |u_{ik}(z)|^2 \frac{|z_{n+1}|}{|z - x_i|} dz \right)^{\frac{1}{2}}.
\]
(4.19)

Clearly, we have that
\[
\int -\phi(\frac{|z - x_i|}{10r}) \psi(z) \frac{|z_{n+1}|}{|z - x_i|} dz
\]
\[
\leq C \int_{B_{10r}(x_i) \setminus B_{5r}(x_i)} |\nabla u_{ik}(z) \cdot (z - x_1) - I_{u_{ik}}(10r)u_{ik}(z)|^2 \frac{|z_{n+1}|}{|z - x_i|} dz
\]
\[
+ C \int_{B_{10r}(x_i) \setminus B_{5r}(x_i)} |\nabla u_{ik}(z) \cdot (z - x_2) - I_{u_{ik}}(10r)u_{ik}(z)|^2 \frac{|z_{n+1}|}{|z - x_i|} dz + C r^{n+a+2(k+1)}
\]
\[
\leq C \int_{B_{12r}(x_i) \setminus B_{5r}(x_i)} |\nabla u_{ik}(z) \cdot (z - x_1) - I_{u_{ik}}(10r)u_{ik}(z)|^2 \frac{|z_{n+1}|}{|z - x_i|} dz
\]
\[
+ C \int_{B_{12r}(x_i) \setminus B_{5r}(x_i)} |\nabla u_{ik}(z) \cdot (z - x_2) - I_{u_{ik}}(10r)u_{ik}(z)|^2 \frac{|z_{n+1}|}{|z - x_i|} dz + C r^{n+a+2(k+1)}
\]
\[
\leq C H_{u_{ik}}(24r)\Delta^{2k}_{\frac{a}{2}}(x_1) + C H_{u_{ik}}(24r)\Delta^{2k}_{\frac{a}{2}}(x_2) + C r^{n+a+2(k+1)}.
\]

In the second inequality, we have used that $B_{10r}(x_1) \setminus B_{5r}(x_1) \subset B_{12r}(x_1) \setminus B_{5r}(x_1)$ for $t \in [0, 1]$, and that $|z - x_i| \leq 2|z - x_i|$ as $z \in B_{10r}(x_1) \setminus B_{5r}(x_1)$, $i = 1, 2$. Moreover, in the third inequality we have applied estimate (4.6) in Lemma 4.3 to $x_1, x_2 \in B^* \cap \mathcal{P}_{\varphi, \theta, \delta}(u)$, with $r_1 = 24r$ and $r_0 = 3r$. Furthermore, thanks to Corollary 3.7, we conclude
\[
\int -\phi(\frac{|z - x_i|}{10r}) \psi(z) \frac{|z_{n+1}|}{|z - x_i|} dz \leq C H_{u_{ik}}(10r)\Delta^{2k}_{\frac{a}{2}}(x_1) + C H_{u_{ik}}(10r)\Delta^{2k}_{\frac{a}{2}}(x_2) + C r^{n+a+2(k+1)}.
\]
(4.20)

In addition, thanks to (3.25) and $|z - x_i| \geq 5r$ we get
\[
\int -\phi(\frac{|z - x_i|}{10r}) (|u_{ik}(z)|^2 + |u_{ik}(z)|^2) \frac{|z_{n+1}|}{|z - x_i|} dz \leq \frac{2}{5r} \left( \|u_{ik}\|^2_{L^2(B_{5r}(x_i), dm)} + \|u_{ik}\|^2_{L^2(B_{10r}(x_i), dm)} \right)
\]
\[
\leq 4H_{u_{ik}}(10r) + 4H_{u_{ik}}(10r).
\]
(4.21)

By collecting (4.19)–(4.21), we conclude that for some $C > 0$,
\[

t^{(1)} \leq \frac{C}{H_{u_{ik}}(10r)} (H_{u_{ik}}(10r)\Delta^{2k}_{\frac{a}{2}}(x_1) + H_{u_{ik}}(10r)\Delta^{2k}_{\frac{a}{2}}(x_2) + C r^{n+a+2(k+1)})
\]
\[
+ \frac{C}{H_{u_{ik}}(10r)} (H_{u_{ik}}(10r) + H_{u_{ik}}(10r))^\frac{1}{2} (H_{u_{ik}}(10r)\Delta^{2k}_{\frac{a}{2}}(x_1) + H_{u_{ik}}(10r)\Delta^{2k}_{\frac{a}{2}}(x_2) + C r^{n+a+2(k+1)})^\frac{1}{2}
\]
\[
\leq C(\Delta^{2k}_{\frac{a}{2}}(x_1) + (\Delta^{2k}_{\frac{a}{2}}(x_2))^\frac{1}{2}) + C(\Delta^{2k}_{\frac{a}{2}}(x_1) + (\Delta^{2k}_{\frac{a}{2}}(x_2))^\frac{1}{2}) + C \frac{r^2}{\delta^2} + C \frac{r^2}{\delta^2},
\]
where, in the last inequality, we have used Lemma 4.1 and that $x_1, x_2 \in B^* \cap \mathcal{P}_{\varphi, \theta, \delta}(u)$. 

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Finally, in view of the very definition of the spatial oscillation of the frequency and Corollary 3.13, we deduce for some constant depending on $A$ that

$$J_1^{(1)} \leq C((\Delta^{2\alpha\tau}(x_1))^\frac{1}{2} + (\Delta^{2\alpha\tau}(x_2))^\frac{1}{2}) + C\rho^q. \quad (4.22)$$

Step 4. We estimate next $J_1^{(2)}$. We start off noting that for all $y \in B_1 \setminus B'_1$ (cf. (3.3))

$$\frac{d}{dt}(L_d(u_{x_i}(x_t + y))) = |y_{n+1}|^2 \Delta \left( \frac{d}{dt}(T_{k,x_i}[\varphi](x_t + y') - \varphi(x_t + y')) \right)$$

$$= C_{y_{n+1}} \Delta(T_{k,x_i}[\partial_e \varphi](x_t + y') - \partial_e \varphi(x_t + y')) \leq C|x_t - x_i||y_{n+1}|^2 |y'| |y'|^{k-2}. \quad (4.12)$$

Then, arguing as in (4.4), thanks to estimate (3.15) in Remark 3.2, we get as $k \geq 2$

$$\left| \int \phi \left( \frac{|y|}{10\tau} \right) u_{x_i}(x_t + y) \frac{d}{dt}(L_d(u_{x_i}(x_t + y))) \, dy \right| \leq C\tau^{k-1} \left| \int \phi \left( \frac{|y|}{10\tau} \right) u_{x_i}(x_t + y)|y_{n+1}|^2 \, dy \right|$$

$$= C\tau^{k-1} \left| \int \phi \left( \frac{|z - x_i|}{10\tau} \right) u_{x_i}(x_t + y)|z_{n+1}|^2 \, dz \right|$$

$$\leq C\tau^{n+2k+1} + C\tau^{k-1} \int_{B_{10\tau}(x_i)} \phi \left( \frac{|z - x_i|}{40\tau} \right) u_{x_i}(z)|z_{n+1}|^2 \, dz \leq C\tau^{n+2k+1} + C\tau^{\frac{n+2}{10\tau} + k} \frac{1}{10\tau}. \quad (4.23)$$

In addition, (3.8) and (4.13) yield

$$\left| \int \phi \left( \frac{|y|}{10\tau} \right) \delta_t(y)L_d(u_{x_i}(x_t + y)) \, dy \right| \leq C\tau^{n+2k+1} + \left| \int \phi \left( \frac{|y|}{10\tau} \right) \delta_t u_{x_i}(x_t + y)L_d(u_{x_i}(x_t + y)) \, dy \right|$$

$$\leq C\tau^{n+2k+1} + C\tau^{k-1} \left| \int \phi \left( \frac{|y|}{10\tau} \right) |\nabla u_{x_i}(x_t + y)||y_{n+1}|^2 \, dy \right|$$

$$\leq C\tau^{n+2k+1} + C\tau^{\frac{n+1}{10\tau} + k} \frac{1}{10\tau}. \quad (4.24)$$

Therefore, by applying repeatedly Lemma 4.1, by taking into account (3.43) and by choosing $\theta > 0$ sufficiently small, we infer that

$$J_2^{(1)} \leq \frac{C}{H_{u_{x_i}}(10\tau)} \left( \tau^{n+2k+1} + \tau^{\frac{n+1}{10\tau} + k} \frac{1}{10\tau} \right) \left( \tau^{\frac{2\theta}{\delta} + \frac{\theta}{\delta^2} \left( \frac{\tau D_{u_{x_i}}(40\tau)}{H_{u_{x_i}}(10\tau)} \right)^\frac{1}{2} + \frac{\tau^{\theta} I_{u_{x_i}}(10\tau)}{\delta^2} \right)^\frac{1}{2}$$

$$\leq C\tau^{\frac{2\theta}{\delta} + \frac{\theta}{\delta^2} \left( \frac{\tau D_{u_{x_i}}(40\tau)}{H_{u_{x_i}}(10\tau)} \right)^\frac{1}{2} + \frac{\tau^{\theta} I_{u_{x_i}}(10\tau)}{\delta^2}} = C\tau^q, \quad (4.24)$$

since $\tau^{14A\rho} < \rho_{A,\rho_0}$.

The conclusion in (4.10) follows at once from estimates (4.22) and (4.24).

5 Proof of the main result

5.1 Mean-flatness

Here we show a control of the Jones’ $\beta$-number by the oscillation of the frequency. Given a Radon measure $\mu$ in $\mathbb{R}^{n+1}$, for every $x_0 \in \mathbb{R}^n$ and for every $r > 0$, we set

$$\beta_\mu(x_0, r) := \inf_{\mathcal{L}} \left( r^{n-1} \int_{B_r(x_0)} \text{dist}^2(y, \mathcal{L}) \, d\mu(y) \right)^\frac{1}{2}, \quad (5.1)$$

where the infimum is taken among all affine $(n-1)$-dimensional planes $\mathcal{L} \subset \mathbb{R}^{n+1}$.
If \( x_0 \in \mathbb{R}^{n+1} \) and \( r > 0 \) is such that \( \mu(B_r(x_0)) > 0 \), set \( \bar{x}_{x_0,r} \) the barycenter of \( \mu \) in \( B_r(x_0) \), i.e.

\[
x_{x_0,r} := \frac{1}{\mu(B_r(x_0))} \int_{B_r(x_0)} x \, d\mu(x)
\]

and

\[
B_{x_0}(v, w) := \int_{B_r(x_0)} ((x - \bar{x}_{x_0,r}) \cdot v) ((x - \bar{x}_{x_0,r}) \cdot w) \, d\mu(x) \quad \text{for all } v, w \in \mathbb{R}^{n+1}.
\]

Then

\[
\beta_\mu(x_0, r) = (r^{n-1}(\lambda_n + \lambda_{n+1}))^{\frac{1}{2}},
\]

where \( 0 \leq \lambda_{n+1} \leq \lambda_n \leq \cdots \leq \lambda_1 \) are the eigenvalues of the positive semidefinite bilinear form \( B_{x_0} \).

**Proposition 5.1.** Let \( A, \delta > 0, \theta \in (0, 1) \). Then there exist constants \( C_{5.1}, q_{5.1} > 0 \) with this property. Let \( 122r \leq q_{5.1}, q \in \mathbb{Z}_{q_{5.1}}, 0 \leq q_{5.1} > A \). Let \( \mu \) be a finite Borel measure with \( \text{sp}(\mu) \subset \mathbb{R}^{n+1} \cdot u \cdot (n \int \mathbb{R}^{n+1} \delta_{x_0}) \).

Then

\[
\beta_\mu^2(p, r) \leq \frac{C_{5.1}}{p^n} \left( \int_{B_r(p)} \Delta^{2r}(x) \, d\mu(x) + r^{2q} \mu(B_r(p)) \right).
\]

**Proof.** The proof is a variant of the [18, Proposition 4.2], which in turn follows closely the original arguments by Naber and Valtorta in [28, 29], therefore we only highlight the main differences.

Without loss of generality assume that \( p \in B_r(p) \cap T_{\mu}(u) \cap \mathbb{R}^{n+1} \cdot u \cdot (n \int \mathbb{R}^{n+1} \delta_{x_0}) \) is such that \( \mu(B_r(p)) > 0 \) (otherwise, there is nothing to prove). Let \( \{v_1, \ldots, v_{n+1}\} \) be any diagonalizing basis for the bilinear form \( B_p \) introduced in Section 5.1, with corresponding eigenvalues \( 0 \leq \lambda_{n+1} \leq \lambda_n \leq \cdots \leq \lambda_1 \).

Since \( \text{sp}(\mu) \cap \Gamma_{\mu}(u) \subset \mathbb{R}^n \), we may assume that \( v_{n+1} = e_{n+1}, \lambda_{n+1} = 1 \), so that \( \beta_\mu(p, r) = (r^{n-1} \lambda_n)^{\frac{1}{2}} \) by (5.2). Clearly, we may also assume that \( \lambda_n > 0 \).

From the very definitions of \( B_p \) and of its barycenter we deduce

\[
r^{n+1} \beta_\mu^2(p, r) \int_{B_{12r}(p) \setminus B_{10r}(p)} |\nabla u_p(z)|^2 \, dm(z)
\]

\[
\leq n \int_{B_r(p) \cap B_{12r}(x) \setminus B_{10r}(x)} ((z - x) \cdot \nabla u_p(z) - \alpha u_p(z))^2 \, dm(z) \, d\mu(x),
\]

where

\[
\alpha := \frac{1}{\mu(B_r)} \int_{B_r(p)} I_{u_p}(9r) \, d\mu(x).
\]

Next we estimate the two sides of (5.4).

For estimating the left-hand side of (5.4), we can show by compactness that

\[
D_{up}(12r) \leq C \int_{B_{12r}(p) \setminus B_{10r}(p)} |\nabla u_p(z)|^2 \, dm(z).
\]

Here we use the same contradiction argument in [18, Proposition 4.2] using the compactness given by Corollary 3.15.

For what concerns the right-hand side of (5.4) we proceed as follows. By the triangular inequality we have

\[
\text{r.h.s. of (5.4)} \leq 4n \int_{B_r(p) \cap B_{12r}(x) \setminus B_{10r}(x)} ((z - x) \cdot \nabla u_p(z) - I_{u_p}(9r)u_p(z))^2 \, dm(z) \, d\mu(x)
\]

\[
+ 4n \int_{B_r(p) \cap B_{12r}(x) \setminus B_{10r}(x)} ((z - x) \cdot \nabla (u_p - u_p)(z) - \alpha (u_p - u_p)(z))^2 \, dm(z) \, d\mu(x)
\]

\[
+ 4n \int_{B_r(p) \cap B_{12r}(x) \setminus B_{10r}(x)} (I_{u_p}(9r) - \alpha)^2 u_p^2(z) \, dm(z) \, d\mu(x) =: f_1 + f_2 + f_3.
\]
The addends $J_1$ and $J_3$ can be treated as in [18, Proposition 4.2]. Indeed, for $J_1$ we use Lemma 4.1 and Lemma 4.3 for a suitable choice of the constants to get

$$J_1 \leq Cr \int_{B_r(p)} H_{u_R}(2\lambda r) \Delta^{2r}_R(x) \, d\mu(x) \leq CrH_{u_R}(12r) \int_{B_r(p)} \Delta^{2r}_R(x) \, d\mu(x).$$  

(5.7)

For $J_3$ we use Jensen’s inequality, Proposition 4.4, Fubini’s Theorem, inequality (3.25) and (4.1) in Lemma 4.1 to get

$$J_3 \leq CrH_{u_R}(12r) \left( \int_{B_r(p)} \Delta^{2r}_R(x) \, d\mu(x) + r^\theta \mu(B_r(p)) \right).$$  

(5.8)

Note that the extra term with respect to [18, Proposition 4.2] arises as a consequence of the additional error term in Proposition 4.4.

To estimate $J_2$ in (5.6), we first note that

$$\nabla(T_{k,x}[\varphi]) = T_{k-1,x}[\nabla \varphi].$$

Then we use estimates (3.15) and (3.16) in Remark 3.2 to deduce that for all $x \in B_r(p)$ and $z \in B_{12r}(x) \setminus B_9(x)$ we have

$$((z - x) \cdot \nabla(u_x - u_p)(z) - a(u_x - u_p)(z))^2 \leq C(r^2|\nabla(T_{k,x}[\varphi])(z) - T_{k,p}[\varphi](z)|^2 + a^2|T_{k,x}[\varphi](z) - T_{k,p}[\varphi]|^2(z))$$

$$\leq Cr^{2(k+1)}.$$ 

Therefore, integrating the last estimate we conclude that

$$J_2 \leq C^{n+2k+3} \mu(B_r(p)).$$  

(5.9)

We can now collect estimates (5.5)–(5.9) and use Corollary 3.13 to get

$$r^{n+1}B^2_{\mu}(p, r)D_{u_R}(12r) \leq CrH_{u_R}(12r) \int_{B_r(p)} \left( \Delta^{2r}_R(x) + \Delta^{2r}_R(x) \right) \, d\mu(x)$$

$$+ Cr^{1+2\theta} \mu(B_r(p))H_{u_R}(12r) + Cr^{n+2k+3} \mu(B_r(p))$$

$$\leq CrH_{u_R}(12r) \left( \int_{B_r(p)} \Delta^{2r}_R(x) \, d\mu(x) + \left( r^\theta \mu(B_r(p)) \right) \right).$$

(5.10)

Finally, by assumption $p \in \mathcal{Z}_{\mu, \theta, \delta}$, then $e^{C_{1,1}:1} \varphi(12r) I_{u_R}(12r) \geq 1 + s$ (cf. Proposition 3.11, Corollary 3.16 and the choice $122r \leq 1$), so that the upper inequality in (3.43) yields (5.3).}

### 5.2 Rigidity of homogeneous solutions

In this subsection we extend the results on the rigidity of almost homogeneous solutions established in [18].

We denote by $\mathcal{H}_\lambda$ the space of all nonzero $\lambda$-homogeneous solutions to the thin obstacle problem (3.1) with zero obstacle,

$$\mathcal{H}_\lambda := \left\{ u \in H^1_{\text{loc}}(\mathbb{R}^{n+1}, \text{dm}) \setminus \{0\} : u(x) = |x|^{-1} u \left( \frac{X}{|X|} \right), \ u|_{B_1} \text{ solves (3.1) with } \varphi \equiv 0 \right\},$$

and set $\mathcal{H} := \bigcup_{\lambda \geq 1+\varepsilon} \mathcal{H}_\lambda$. The spine $S(u)$ of $u \in \mathcal{H}$ is the maximal subspace of invariance of $u$,

$$S(u) := \{ y \in \mathbb{R}^n \times \{0\} : u(x + y) = u(x) \text{ for all } x \in \mathbb{R}^{n+1} \}.$$ 

As observed in [18], the maximal dimension of the spine of a function in $\mathcal{H}$ is at most $n - 1$ and we set $u \in \mathcal{H}^{\text{top}}$ if $u \in \mathcal{H}$ and $\dim S(u) = n - 1$, and $\mathcal{H}^{\text{low}} := \mathcal{H} \setminus \mathcal{H}^{\text{top}}$. All functions in $\mathcal{H}^{\text{top}}$ are classified in [18, Lemma 5.3]. Note also that by Caffarelli, Salsa and Silvestre [9]

$$\mathcal{H}_{1+\varepsilon} \subseteq \mathcal{H}^{\text{top}}.$$  

(5.10)

We next introduce the notion of almost homogeneous solutions. Given $\delta > 0$ and $x_0 \in \mathcal{Z}_{\mu, \theta, \delta}$, we set

$$J_{u_{x_0}}(t) := e^{C_{1,1}:1: \theta} I_{u_{x_0}}(t) \text{ for all } t \in (0, \varphi_{3,11}].$$
Definition 5.2. Let $\eta > 0$ and let $u : B_l \to \mathbb{R}$ be a solution to thin obstacle problem (3.1) with obstacle $\varphi$ (as usual $\|\varphi\|_{C^{1,1}(B_l^*)} \leq 1$). Assume that $0 \in \mathcal{Z}_{\varphi, \theta, \delta}$ and $q \leq q_{3.11}$, $u$ is called $\eta$-almost homogeneous in $B_l$ if

$$J_{u_{\lambda}} \left( \frac{\theta}{2} \right) - J_{u_{\lambda}} \left( \frac{\theta}{4} \right) \leq \eta. $$

The following lemma justifies this terminology and it is the analog of [18, Lemma 5.5].

**Lemma 5.3.** Let $\varepsilon, A > 0$, $\theta \in (0, 1)$. There exists $\eta_{5,3} > 0$ with the following property: for every $\delta > 0$ there exists $q_{5,3}$ such that, if $u$ is an $\eta_{5,3}$-almost homogeneous solution of (3.1) in $B_\delta$ with $q \leq q_{5,3}$ and obstacle $\varphi$, $0 \in \mathcal{Z}_{\varphi, \theta, \delta}(u)$ and $I_{u_\lambda}(q_{5,3}) \leq A$, then

$$\|u_{\lambda\varphi} - w\|_{H^1(B_{\frac{1}{4}}, \text{dm})} \leq \varepsilon \tag{5.11}$$

for some homogeneous solution $w \in \mathcal{H}$. 

**Proof.** The proof follows by a contradiction argument similar to [18, Lemma 5.5]. Assume that for some $\varepsilon, A > 0$ we could find infinitesimal sequences of numbers $\delta_l, q_l$ and of $\frac{1}{4}$-almost homogeneous solutions $u_l$ of (3.1) in $B_{q_l}$ such that $0 \in \mathcal{Z}_{\varphi, \theta, \delta}(u_l)$ and

$$\inf_l \inf_{w \in \mathcal{H}} \|u_l|_{\varphi, q_l} - w\|_{H^1(B_{\frac{1}{4}}, \text{dm})} \geq \varepsilon, \tag{5.12}$$

and satisfying the bounds $I_{(u_l)_{\varphi}(q_l)} \leq A$.

Consider $v_l := (u_l)_{\varphi, q_l}$; then by Corollary 3.15 applied to $v_l$ there would be a subsequence, not relabeled, converging in $H^1(B_1, \text{dm})$ to a solution $v_{\infty}$ of the thin obstacle problem with zero obstacle. By Proposition 3.11 there is some $\Lambda'$ independent of $l$ such that $I_{(u_l)_{\varphi}(t)} \leq \Lambda'$ for all $t \in (0, q_l)$, then from (3.23) in Corollary 3.7 we would infer that

$$\begin{aligned}
- \int \varphi \left( \frac{2|x|}{t} \right) |v_l|^2 \frac{1}{|x|} |x_{n+1}| \, dx &= -\lim_l \int \varphi \left( \frac{2|x|}{t} \right) |v_l|^2 \frac{1}{|x|} |x_{n+1}| \, dx \leq \lim_l \frac{H_{(u_l)_{\varphi}(q_l)}}{H_{(u_l)_{\varphi}(q_l)}(t)} \geq 2^{-n+2\Lambda'},
\end{aligned}$$

in turn implying that $v_{\infty}$ is not zero. On the other hand, we would also get

$$I_{(v_{\infty})_{\varphi}(\frac{1}{2})} - I_{(v_{\infty})_{\varphi}(\frac{1}{4})} = \lim_l \left( I_{(u_l)_{\varphi}(\frac{1}{2})} - I_{(u_l)_{\varphi}(\frac{1}{4})} \right) = \lim_l \left( I_{(u_l)_{\varphi}(\frac{q_l}{2})} - I_{(u_l)_{\varphi}(\frac{q_l}{4})} \right) = 0,$$

and thus we would conclude that $v_{\infty} \in \mathcal{H}$ being a solution to the lower-dimensional obstacle problem with constant frequency (see for instance [18, Proposition 2.7]). We have thus contradicted (5.12). 

A rigidity property of the type shown in [18, Proposition 5.6] holds for the nonzero obstacle problem.

**Proposition 5.4.** Let $A, \tau > 0$, $\theta \in (0, 1)$. There exists $\eta_{5,4} > 0$ with this property. For every $\delta > 0$ there exists $q_{5,4}$ such that, if $u$ is an $\eta_{5,4}$-almost homogeneous solution of (3.1) in $B_\delta$ with $q \leq q_{5,4}$ and obstacle $\varphi$, $0 \in \mathcal{Z}_{\varphi, \theta, \delta}(u)$ and $I_{u_\lambda}(q_{5,4}) \leq A$, then the following dichotomy holds:

(i) Either for every point $x \in B_{\frac{1}{8}} \cap \mathcal{Z}_{\varphi, \theta, \delta}(u)$ we have

$$\left| J_{u_\lambda} \left( \frac{l}{2} \right) - J_{u_\lambda} \left( \frac{l}{16} \right) \right| \leq \tau, \tag{5.13}$$

(ii) or there exists a linear subspace $V \subset \mathbb{R}^n \times \{0\}$ of dimension $n - 2$ such that

$$\begin{aligned}
y \in B_{\frac{1}{8}} \cap \mathcal{Z}_{\varphi, \theta, \delta}(u),
y \mapsto \text{dist}(y, V) < \tau \varepsilon.
\end{aligned} \tag{5.14}$$

**Proof.** The proof proceeds by contradicion and follows the strategy developed in [18, Proposition 5.6]. Let $A, \tau > 0$ be given constants and assume that there exist infinitesimal sequences $\delta_l, q_l$ and a sequence $(u_l)_{l \in \mathbb{N}}$ of $\frac{1}{4}$-almost homogeneous solutions in $B_{q_l}$ such that $0 \in \mathcal{Z}_{\varphi, \theta, \delta}(u_l)$, $I_{u_{\lambda\varphi}}(q_l) \leq A$ and such that:

(i) there exists $x_l \in B_{\frac{1}{8}} \cap \mathcal{Z}_{\varphi, \theta, \delta}(u_l)$ for which

$$\left| J_{u_{\lambda\varphi}} \left( \frac{q_l}{2} \right) - J_{u_{\lambda\varphi}} \left( \frac{q_l}{16} \right) \right| \leq \eta_{5,4},$$

(ii) or there exists $x_l \in B_{\frac{1}{8}} \cap \mathcal{Z}_{\varphi, \theta, \delta}(u_l)$ such that

$$\left| J_{u_{\lambda\varphi}} \left( \frac{q_l}{2} \right) - J_{u_{\lambda\varphi}} \left( \frac{q_l}{16} \right) \right| \leq \eta_{5,4}.$$
We consider the rescaled functions \( v_t := (u_t)_{2\varrho t} : B_2 \to \mathbb{R} \). By the compactness result in Corollary 3.15 we deduce that, up to passing to a subsequence (not relabeled), there exists a nonzero function \( v_{co} \) solution to the thin obstacle problem (3.1) in \( B_1 \) with null obstacle such that \( v_t \to v_{co} \) in \( H^1(B_1, \mathbb{R}) \). Moreover, \( v_{co} \in \mathcal{F} \) thanks to Lemma 5.3.

If \( v_{co} \in \mathcal{F}^{\text{top}} \), then (5.15) is contradicted. Indeed, up to choosing a further subsequence, we can assume that \( z_t := \varrho_t^{-1} x_t \to z_{co} \in B_{\frac{1}{2}} \). Note that the points \( z_t \in \mathcal{N}(v_t) \), as \( x_t \in \Gamma_{\varphi_i}(u_t) \), so that

\[
v_t(z_t) = \frac{\varrho_t^{2\varrho t}}{H_{(u_t)^2}(\varrho_t)}(u_t)(x_t) = 0.
\]

In addition, by (3.10) and being \( \epsilon_\varphi \epsilon_\Gamma \) even with respect to \( \{x_{n+1} = 0\} \) (cf. Lemma 2.1), for all \( l \) we infer that

\[
\lim_{t \to 0} \epsilon^a_{n+1} v_t(z_t', t) = \frac{\varrho_t^{2\varrho t}}{H_{(u_t)^2}(\varrho_t)} \lim_{t \to 0} \epsilon^a_{n+1} (u_t(x_t', t) - \epsilon_\varphi \epsilon_\Gamma (x_t', t)) = 0.
\]

Hence, we conclude that \( z_{co} \in \mathcal{N}(v_{co}) \) in view of (3.52). Moreover, by taking into account the very definition of \( v_t \) and Remark 3.4 we get by scaling

\[
\left| I_{(v_{co})_{n+1}} \left( \frac{1}{2} \right) - I_{(v_{co})_{n+2}} \left( \frac{1}{2} \right) \right| = \left| \frac{1}{2} \int \frac{\partial}{\partial x} \left| \nabla v_{co} \right|^2 |x_{n+1}|^a \, dx - \frac{1}{2} \int \frac{\partial}{\partial x} \left| \nabla v_{co} \right|^2 |x_{n+2}|^a \, dx \right|
\]

which is a contradiction to the constancy of the frequency at critical points of the homogeneous solution \( v_{co} \in \mathcal{F}^{\text{top}} \) (see [18, Lemma 5.3]). The fourth equality is justified by taking into account that \( x_t \in \mathcal{Z}_{\varphi_i, \vartheta, \delta_i(u)} \) (cf. (3.37) and (3.39)), and in view of estimates (3.15) and (3.16) in Remark 3.2, in turn implying for all \( z \in B_{\frac{1}{2}}(x_t) \) (recall that \( \varrho_t^{-1} x_t \to z_{co} \))

\[
|\varrho_t(z_t) - \varrho_t^n(z) - \varrho_t(z_t)| \leq C \varrho_t^{k+1}, \quad |\nabla(\varrho_t(z_t) - \varrho_t^n(z_t))| \leq C \varrho_t^k.
\]

Moreover, (3.43) can be employed in the last two equalities as \( x_t \in B_{\frac{1}{2}}(x_t) \cap \mathcal{Z}_{\varphi_i, \vartheta, \delta_i(u)} \).

Instead, if \( v_{co} \in \mathcal{F}^{\text{low}} \), we show a contradiction to (5.16) with \( V \) any \((n-2)\)-dimensional subspace containing \( S(v_{co}) \). Indeed, let \( y_t \) be as in (5.16) for such a choice of \( V \). By compactness, up to passing to a subsequence (not relabeled), \( z_t := \varrho_t^{-1} y_t \to z_{co} \) for some \( z_{co} \in B_{\frac{1}{2}} \) with \( \text{dist}(z_{co}, V) \geq \tau \). In addition, arguing as before

\[
\left| I_{(v_{co})_{n+1}} \left( \frac{1}{2} \right) - I_{(v_{co})_{n+2}} \left( \frac{1}{2} \right) \right| = \lim_{t \to \infty} \left| \frac{\varrho_t^2 D(u_{n+1})(\varrho_t)}{H_{(u_{n+1})^2}(\varrho_t)} - \frac{\varrho_t^2 D(u_{n+2})(\varrho_t)}{H_{(u_{n+2})^2}(\varrho_t)} \right| \geq \tau.
\]

Again, note that (3.43) can be employed since \( y_t \in B_{\frac{1}{2}}(x_t) \cap \mathcal{Z}_{\varphi_i, \vartheta, \delta_i(u)} \). By [18, Proposition 2.7, Lemma 5.2] it follows that \( z_{co} \in S(v_{co}) \), thus contradicting \( S(v_{co}) \subseteq \overline{V} \) and \( \text{dist}(z_{co}, V) \geq \tau \). \( \square \)
5.3 Proof of Theorem 1.2

We start off noting that it suffices to prove that $\Gamma_{\varphi,\theta}(u) \cap B_1(x_0)$ satisfies all the conclusions for all $x_0 \in \Gamma_{\varphi,\theta}(u)$. For all $R \in (0, 1)$, we can find a finite number of balls $B_{1/2}^i(x_i)$, $x_i \in \Gamma_{\varphi,\theta}(u)$ for $i \in \{1, \ldots, M\}$, whose union covers $\Gamma_{\varphi,\theta}(u) \cap B_1(x_0)$. We shall choose appropriately $R$ in what follows. Moreover, with fixed $i \in \{1, \ldots, M\}$, by horizontal translation we may reduce to $x_i = 0 \in \Gamma_{\varphi,\theta}(u)$ without loss of generality.

Then, recalling the definition of $\Gamma_{\varphi,\theta}(u)$ in (1.8) we have that

$$\Gamma_{\varphi,\theta}(u) \cap B'_1 = \bigcup_{j \in \mathbb{N}} \mathcal{X}^R_{\varphi,\theta,1}(u),$$

where

$$\mathcal{X}^R_{\varphi,\theta,1}(u) := \left\{ x_0 \in \Gamma_{\varphi}(u) \cap B'_1 : H_{\varphi,\theta}(r) \geq \frac{r^{n+\alpha+2k(1-\theta)}}{j} \text{ for all } r \in (0, \frac{R}{2}) \right\}.$$

Hence, we may establish the result for $\mathcal{X}^R_{\varphi,\theta,1}(u)$ with $j \in \mathbb{N}$ fixed.

Note that as $\partial \Gamma_{\varphi}(u)$, the function

$$\tilde{u}(y) := u(Ry) - u(0)$$

solves the fractional obstacle problem (3.1) in $B_1$ with obstacle function $\tilde{\varphi}(\cdot) := \varphi(R \cdot) - \varphi(0)$. Moreover,

$$\Gamma_{\varphi,\theta}(\tilde{u}) \cap B'_1 = \frac{1}{R}(\Gamma_{\varphi,\theta}(u) \cap B'_1),$$

with $\tilde{u}_2(\cdot) = u_2(R \cdot)$ if $z \in \Gamma_{\varphi,\theta}(u) \cap B'_1$, being $T_{k,1}(\tilde{\varphi})(\cdot) = T_{k,2}(\varphi)(R \cdot)$. Thus, we get that $z \in \mathcal{X}^R_{\varphi,\theta,1}(u)$ if and only if $\tilde{z} \in B'_1 \cap \mathcal{X}_{\varphi,\theta,1}(\varphi)$. In addition, it is easy to check that

$$\|\tilde{\varphi}\|_{C^{1,1}(B'_1)} \leq R \|\varphi\|_{C^1(B'_1, \mathbb{R}^n)}.$$

We choose $R > 0$ sufficiently small so that $\|\tilde{\varphi}\|_{C^{1,1}(B'_1)} \leq 1$ and the smallness conditions on the radii in all the statements of Sections 3–5 are satisfied.

In such a case the proof, of the main results can be obtained by following verbatim [18, Sections 6–8]. Indeed, [18, Proposition 6.1], that leads both to the local finiteness of the Minkowski content of $\mathcal{X}_{\varphi,\theta,1}(\varphi)$ and to its $(n - 1)$-rectifiability, is based on a covering argument that exploits the lower bound on the frequency in Corollary 3.16, the control of the mean oscillation via the frequency in Proposition 5.1, the rigidity of almost homogeneous solutions in Proposition 5.4, the discrete Reifenberg theorem by Naber and Valtorta [28, Theorem 3.4, Remark 3.9], and the rectifiability criterion either by Azzam and Tolsa [2] or by Naber and Valtorta [28, 29]. Therefore, the only extra-care needed in the current setting is to start the covering argument from a scale which is small enough to validate the conclusions of the lemmas and propositions of the previous sections.

Finally, the classification of blowup limits is exactly that stated in [18, Theorem 1.3], and proved in [18, Section 8], in view of Lemma 3.14 and Corollary 3.15.

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References

[1] I. Athanasopoulos, L. A. Caffarelli and S. Salsa, The structure of the free boundary for lower dimensional obstacle problems, Amer. J. Math. 130 (2008), no. 2, 485–498.
[2] J. Azzam and X. Tolsa, Characterization of $n$-rectifiability in terms of Jones’ square function: Part II, Geom. Funct. Anal. 25 (2015), no. 5, 1371–1412.
[3] B. N. Barrios, A. Figalli and X. Ros-Oton, Global regularity for the free boundary in the obstacle problem for the fractional Laplacian, Amer. J. Math. 140 (2018), no. 2, 415–447.
[4] J.-P. Bouchaud and A. Georges, Anomalous diffusion in disordered media: Statistical mechanisms, models and physical applications, *Phys. Rep.* **195** (1990), no. 4–5, 127–293.

[5] L. Caffarelli and L. Silvestre, An extension problem related to the fractional Laplacian, *Comm. Partial Differential Equations* **32** (2007), no. 7–9, 1245–1260.

[6] L. A. Caffarelli, The regularity of free boundaries in higher dimensions, *Acta Math.* **139** (1977), no. 3–4, 155–184.

[7] L. A. Caffarelli, The obstacle problem revisited, *J. Fourier Anal. Appl.* **4** (1998), no. 4–5, 383–402.

[8] L. A. Caffarelli, D. De Silva and O. Savin, The two membranes problem for different operators, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **34** (2017), no. 4, 899–932.

[9] L. A. Caffarelli, S. Salsa and L. Silvestre, Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian, *Invent. Math.* **171** (2018), no. 2, 425–461.

[10] L. A. Caffarelli and A. Vasseur, Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation, *Ann. of Math. (2)* **171** (2010), no. 3, 1903–1930.

[11] R. Cont and P. Tankov, *Financial Modelling with Jump Processes*, Chapman & Hall/CRC Financ. Math. Ser., Chapman & Hall/CRC, Boca Raton, 2004.

[12] C. De Lellis and E. Spadaro, Regularity of area minimizing currents III: Blow-up, *Ann. of Math. (2)* **183** (2016), no. 2, 577–617.

[13] D. De Silva and O. Savin, $C^{m,\alpha}$ regularity of certain thin free boundaries, *Indiana Univ. Math. J.* **64** (2015), no. 5, 1575–1608.

[14] X. Fernández-Real and Y. Jhaveri, On the singular set in the thin obstacle problem: Higher order blow-ups and the very thin obstacle problem, preprint (2020), https://arxiv.org/abs/1812.01515.

[15] X. Fernández-Real and X. Ros-Oton, Free boundary regularity for almost every solution to the Signorini problem, in preparation.

[16] M. Focardi and E. Spadaro, An epiperimetric inequality for the thin obstacle problem, *Adv. Differential Equations* **21** (2016), no. 1–2, 153–200.

[17] M. Focardi and E. Spadaro, Correction to: On the measure and the structure of the free boundary of the lower dimensional obstacle problem [MR3840912], *Arch. Ration. Mech. Anal.* **230** (2018), no. 2, 783–784.

[18] M. Focardi and E. Spadaro, On the measure and the structure of the free boundary of the lower dimensional obstacle problem, *Arch. Ration. Mech. Anal.* **230** (2018), no. 1, 125–184.

[19] M. Focardi and E. Spadaro, How a minimal surface leaves a thin obstacle, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **37** (2020), no. 4, 1017–1046.

[20] N. Garofalo and A. Petrosyan, Some new monotonicity formulas and the singular set in the lower dimensional obstacle problem, *Invent. Math.* **177** (2009), no. 2, 415–461.

[21] N. Garofalo and X. Ros-Oton, Structure and regularity of the singular set in the obstacle problem for the fractional Laplacian, *Rev. Mat. Iberoam.* **35** (2019), no. 5, 1309–1365.

[22] F. Geraci, An epiperimetric inequality for the lower dimensional obstacle problem, *ESAIM Control Optim. Calc. Var.* **25** (2019), Paper No. 39.

[23] E. Hopf, Über den funktionalen, insbesondere den analytischen Charakter der Lösungen elliptischer Differentialgleichungen zweiter Ordnung, *Math. Z.* **34** (1932), no. 1, 194–233.

[24] Y. Jhaveri and R. Neumayer, Higher regularity of the free boundary in the obstacle problem for the fractional Laplacian, *Adv. Math.* **311** (2017), 748–795.

[25] H. Koch, A. Petrosyan and W. Shi, Higher regularity of the free boundary in the elliptic Signorini problem, *Nonlinear Anal.* **126** (2015), 3–44.

[26] H. Koch, A. Rüland and W. Shi, The variable coefficient thin obstacle problem: Higher regularity, *Adv. Differential Equations* **22** (2017), no. 11–12, 793–866.

[27] R. Monneau, On the number of singularities for the obstacle problem in two dimensions, *J. Geom. Anal.* **13** (2003), no. 2, 359–389.

[28] A. Naber and D. Valtorta, Rectifiable-Reifenberg and the regularity of stationary and minimizing harmonic maps, *Ann. of Math. (2)* **185** (2017), no. 1, 131–227.

[29] A. Naber and D. Valtorta, The singular structure and regularity of stationary varifolds, *J. Eur. Math. Soc. (JEMS)* **22** (2020), no. 10, 3305–3382.

[30] L. Silvestre, Regularity of the obstacle problem for a fractional power of the Laplace operator, *Comm. Pure Appl. Math.* **60** (2007), no. 1, 67–112.

[31] G. S. Weiss, A homogeneity improvement approach to the obstacle problem, *Invent. Math.* **138** (1999), no. 1, 23–50.