MIXED LOCAL-NONLOCAL OPERATORS: MAXIMUM PRINCIPLES, EIGENVALUE PROBLEMS AND THEIR APPLICATIONS

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Abstract. In this article we consider a class of non-degenerate elliptic operators obtained by superpositioning the Laplacian and a general nonlocal operator. We study the existence-uniqueness results for Dirichlet boundary value problems, maximum principles and generalized eigenvalue problems. As applications to these results, we obtain Faber-Krahn inequality and a one-dimensional symmetry result related to the Gibbons’ conjecture. The latter results substantially extend the recent results of Biagi et. al. [7, 9] who consider the operators of the form $-\Delta + (-\Delta)^s$ with $s \in (0, 1)$.

1. Introduction

In this article we consider an operator in $\mathbb{R}^d$ which is a combination of a local and a nonlocal operator. In particular, we consider operators of the form

$$Lu = \Delta u + \int_{\mathbb{R}^d} (u(x + y) - u(x) - 1_{\{|y| \leq 1\}}y \cdot \nabla u(x))j(y)dy,$$

where $j : \mathbb{R}^d \setminus \{0\} \to [0, \infty)$ is a jump kernel satisfying

$$\int_{\mathbb{R}^d} (1 \wedge |y|^2)j(y)dy < \infty. \quad (1.2)$$

Operators of the form (1.1) appears naturally in the study of Lévy process. More precisely, the generator of a $d$-dimensional Lévy process is given by the following general structure

$$Au = \text{trace}(aD^2u) + b \cdot \nabla u + \int_{\mathbb{R}^d} (u(x + y) - u(x) - 1_{\{|y| \leq 1\}}y \cdot \nabla u(x))\nu(dy),$$

where $a$ is a non-negative definite matrix, $b \in \mathbb{R}^d$ and $\nu$ is a Lévy measure satisfying

$$\int_{\mathbb{R}^d} (1 \wedge |y|^2) \nu(dy) < \infty.$$

The local elliptic operator corresponds to $\nu = 0$. For $\nu(dy) = |y|^{-d-2s}dy$, the nonlocal part corresponds to the well-studied fractional Laplacian. In this article we set $a = I, b = 0$ and $\nu(dy) = j(y)dy$ where $j$ satisfies (1.2). Let us denote by

$$\psi(z) = \int_{\mathbb{R}^d} (1 - e^{iz \cdot \xi} + 1_{\{|z| \leq 1\}}iz \cdot \xi) j(\xi)d\xi. \quad (1.3)$$

Let $Y$ be a pure-jump Lévy process with Lévy-Khinchine exponent given by $\psi$ and $B$ be a Brownian motion, independent of $Y$, running twice as fast as the standard $d$-dimensional Brownian motion. Let $X = B + Y$. We also assume that all the processes are defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. It is well-known that $X$ is a strong Markov process and the semigroup generated by $X$ is
determined by the generator (1.1). Furthermore, the Lévy-Khinchine representation of $X$ is given by
\[ \mathbb{E}[e^{iz \cdot X_t}] = e^{-t(|z|^2 + \psi(z))} \quad \text{for all } z \in \mathbb{R}^d \text{ and } t > 0, \]
where $\mathbb{E}[]$ denotes the expectation with respect to the measure $\mathbb{P}$. For more details on this topic we refer to the book of Sato [46]. We impose the following assumption on $\psi$.

(A1) For some constant $C > 0$ we have $|\text{Im}(\psi(p))| \leq C(|p|^2 + \text{Re}(\psi(p)))$ for all $p$ and for all $r > 0$ we have $\sup_{|p| \leq r}(|p|^2 + \text{Re}(\psi(p))) > 0$.

It is easy to see that for $j$ symmetric (that is, $j(y) = j(-y)$) we have
\[ \psi(z) = \int_{\mathbb{R}^d} (1 - \cos(z \cdot \xi))j(\xi)d\xi \geq 0, \]
and thus, (A1) holds.

In this article we are concerned with equations of the form
\[ Lu = f(u, x) \quad \text{in } D, \quad \text{and } u = g \quad \text{in } D^c. \tag{1.4} \]
Integro-differential operators such as (1.4) became quite popular very recently. There is a large body of works dealing with elliptic operators with both local and nonlocal parts. But most of the works restricted the nonlocal term to be the fractional Laplacian [1, 3, 4, 7, 8, 9, 17, 23, 26, 44]. However, there are many practical situations; for instance, in biology [24, 41], mathematical finance [14, 42], where the Lévy measure need not be of fractional Laplacian type. This gives us a motivation to consider an integro-differential equation with a general Lévy measure. By a solution we shall always mean a viscosity solution in the sense of Caffarelli and Silvestre [22].

**Definition 1.1** (Viscosity solution). A bounded function $u : \mathbb{R}^d \to \mathbb{R}$, upper (lower) semi-continuous in a domain $\bar{D}$, is said to be a viscosity subsolution (supersolution) to
\[ Lu = f \quad \text{in } D, \]
written as $Lu \geq f$ ($Lu \leq f$), if for any point $x \in D$ and a neighbourhood $N_x$ of $x$ in $D$, there exists a function $\varphi \in C^2(N_x)$ so that $\varphi - u$ attains minimum (maximum) 0 in $N_x$ at the point $x$, then letting
\[ v(y) := \begin{cases} 
\varphi(y) & \text{for } y \in N_x, \\
u(y) & \text{otherwise,} \end{cases} \]
we have $Lv(x) \geq f(x)$ ($Lv(x) \leq f$, respectively). We say $u$ is a viscosity solution if it is both sub and supersolution.

We can also restrict ourselves to the test functions attaining strict minimum or maximum in the definition above. Our first result establishes the existence and uniqueness of solution.

**Theorem 1.1.** Assume (A1). Let $D$ be an open, bounded Lipschitz domain in $\mathbb{R}^d$. Also, assume that $f \in C(\bar{D})$ and $g \in C_b(\mathbb{R}^d)$. Then there exists a unique viscosity solution $u \in C_b(\mathbb{R}^d)$ to
\[ Lu = -f \quad \text{in } D, \quad \text{and } u = g \quad \text{in } D^c. \tag{1.5} \]
Furthermore, the unique solution can be written as
\[ u(x) = \mathbb{E}_x \left[ \int_0^{\tau} f(X_t) \, dt \right] + \mathbb{E}_x [g(X_{\tau})], \quad x \in D, \tag{1.6} \]
where $\tau = \tau_D$ denotes the first exit time of $X$ from $D$, that is,
\[ \tau_D = \inf \{ t > 0 : X_t \notin D \}. \]
\[ \mathbb{E}_x[\cdot] \] denotes the conditional expectation operator conditioned on \( X_0 = x \), that is, \( \mathbb{E}_x[\cdot] \) is the expectation operator with respect to the law of the Lévy process \( X + x \) where \( X_0 = 0 \). When \( f \in C^\alpha(D) \) and \( g \in C^{2+\alpha}(D^c) \) for some \( \alpha > 0 \), the existence of a unique classical solution to (1.5) is known from the work of Garroni and Menaldi [32]. In [4] Barles, Chasseigne and Imbert establish the existence of a viscosity solutions for a large class of nonlinear integro-differential operators. Unlike ours, [4] (see also [5]) requires the operators to be strictly monotone in the zeroth order term. For the proof of Theorem 1.1 we follow the approach of Cabré-Caffarelli [21] and Caffarelli-Silvestre [22]. The following result plays a key role in the proof of Theorem 1.1 and many other proofs in this article (compare it with [22, Theorem 5.9]).

**Theorem 1.2.** Let \( D \) be an open bounded set, \( u \) and \( v \) be two bounded functions such that \( u \) is upper-semicontinuous and \( v \) is lower-semicontinuous in \( \overline{D} \). Also, assume that \( Lu \geq f \) and \( Lv \leq g \) in the viscosity sense in \( D \), for two continuous functions \( f \) and \( g \). Then \( L(u - v) \geq f - g \) in \( D \) in the viscosity sense.

Furthermore, for a bounded function \( u \) which is upper-semicontinuous in \( \overline{D} \) and satisfies \( Lu \geq 0 \) in \( D \), we have \( \sup_D u \leq \sup_{D^c} u \).

Proof of Theorem 1.1 can be found in Section 5 whereas the proof of Theorem 1.2 follows from Theorems 5.1 and 5.2. The stochastic representation (1.6) of \( u \) plays a key role in this article. In particular, using this representation of the solution we can establish an Alexandrov-Bakelman-Pucci (ABP) maximum principle.

**Theorem 1.3** (ABP-maximum principle). Assume (A1) and let \( D \) be an open, bounded set. Let \( f : D \to \mathbb{R} \) be continuous and \( u \in C_b(\mathbb{R}^d) \) be a viscosity subsolution to

\[
Lu = -f \quad \text{in } \{u > 0\} \cap D, \quad \text{and} \quad u \leq 0 \quad \text{in } D^c.
\]

Then for every \( p > \frac{d}{2} \), there exists a constant \( C = C(d, p, \text{diam}(D)) \), satisfying

\[
\sup_D u \leq C \|f^+\|_{L^p(D)}.
\]

In [40, Theorem 3.2] Mou and Święch consider the Pucci extremal operators and establish the ABP estimate for strong solutions. Similar estimate for viscosity solutions can be found in Mou [39]. It should be observed that the ABP estimates in [39, 40] holds for \( p > p_0 \) where \( p_0 \) is some number in \([d/2, d)\). Theorem 1.3 shows that we can choose \( p_0 = d/2 \) for \( L \). Also, compare this result with [20, Theorem 1.9]. For a proof of Theorem 1.3, see Section 2. Recently, Sobolev regularity and maximum principles for the operator \(-\Delta + (-\Delta)^s\), \( s \in (0, 1) \), are studied by Biagi et. al. in [8]. We also mention the work of Alibaud et. al. [2] where the authors provide a complete characterization of the translation-invariant integro-differential operators that satisfy the Liouville property in the whole space.

In view of Theorems 1.1 and 1.3 we can define a generalized principal eigenvalue for \( L \) in the spirit of Berestycki, Nirenberg and Varadhan [11]. By \( C_+(D) \) \( (C_{b,+}(\mathbb{R}^d)) \) we denote the set of all positive (bounded and non-negative) continuous functions in \( D \) (in \( \mathbb{R}^d \), respectively). Given any bounded domain \( D \), the (Dirichlet) generalized principal eigenvalue of \( L \) in \( D \) is defined to be

\[
\lambda_D = \sup\{\lambda : \exists v \in C_+(D) \cap C_{b,+}(\mathbb{R}^d) \text{ satisfying } Lv + cv + \lambda v \leq 0 \text{ in } D\},
\]

where \( c \in C_b(D) \). We prove the existence of a unique eigenfunction.

**Theorem 1.4.** Grant (A1). Let \( D \) be a bounded Lipschitz domain satisfying a uniform exterior sphere condition. Let \( c \in C(\overline{D}) \). There exists a unique \( \psi_D \in C_b(\mathbb{R}^d) \) satisfying

\[
L\psi_D + c\psi_D = -\lambda_D\psi_D \quad \text{in } D,
\]

\[
\psi_D = 0 \quad \text{in } D^c,
\]

\[
\psi_D > 0 \quad \text{in } D, \quad \psi_D(0) = 1.
\]
Moreover, if \( u \in C_{b+}(\mathbb{R}^d) \) is positive in \( D \) and satisfies
\[
Lu + cu \leq -\lambda u \quad \text{in} \quad D,
\]
for some \( \lambda \in \mathbb{R} \) then \( \lambda \leq \lambda_D \). Furthermore, if \( \lambda = \lambda_D \) and \( u = 0 \) in \( D^c \), then we have \( u = k\psi_D \) for some \( k > 0 \). Furthermore, \( \lambda_D \) is the only Dirichlet eigenvalue with a positive eigenfunction.

For some recent works dealing with generalized eigenvalue problems of integro-differential operator we refer [12, 16, 45]. The proof of Theorem 1.4 is quite standard which uses Krein-Rutman theorem and a narrow domain maximum principle (Corollary 2.1). For a proof see Theorem 3.1.

We next concentrate on the Faber-Krahn inequality for the operator \( L \). In particular, we prove

**Theorem 1.5** (Faber-Krahn inequality). Assume (A1). Also, assume that \( j(y) = j(|y|) \) and \( j \) is radially decreasing. Then for any bounded, open set \( D \) with \( |\partial D| = 0 \) we have
\[
\lambda_D \geq \lambda_B, \quad \text{(1.8)}
\]
where \( B \) is ball around 0 satisfying \( |B| = |D| \).

As is well-known Faber-Krahn inequality was first proved independently by Faber [27] and Krahn [37] for the Laplacian. See also [35, Chapter 2]. Very recently, Biagi et. al. [7] establish Faber-Krahn inequality for the operator \(-\Delta + (\Delta)^s\) for \( s \in (0, 1) \). Their method uses Schwarz symmetrization combined with the Polya-Szegö inequality and [31, Theorem A.1]. Since the inequality in [31, Theorem A.1] holds for a more general class of kernel \( j \), it might be possible to mimic the proof of [7] in an appropriate variational set-up and Sobolev space to establish (1.8). However, our viscosity solution approach does not impose any additional regularity on the solution. Using Theorem 1.4, we find a probabilistic representation of the principal eigenvalue which together with the Brascamp-Lieb-Luttinger inequality gives us Theorem 1.5. Also, note that our condition on \( j \) is very general and our proof works in dimension one. Proof of Theorem 1.5 can be found in Section 3.

As another application of ABP maximum principle and Theorem 1.2 we also study symmetry properties of the positive solutions of semilinear equations. Thanks to Theorem 1.2, symmetry of the positive solutions can be established using the standard method of moving plane [16, 30, 34]. See Section 4 for more detail. Another interesting application of Theorem 1.2 is the one-dimensional symmetry result related to the Gibbons’ conjecture. More precisely, we prove the following

**Theorem 1.6.** Assume (A1). Let \( u \in C_b(\mathbb{R}^d) \) solve
\[
Lu(x) = f(u(x)) \quad \text{for} \quad x \in \mathbb{R}^d,
\]
\[
\lim_{x_n \to \pm \infty} u(x',x_n) = \pm 1 \quad \text{uniformly for} \quad x' \in \mathbb{R}^{d-1},
\]
where \( f \in C^1(\mathbb{R}) \) satisfying
\[
\inf_{|r| \geq 1} f'(r) > 0.
\]
Then there exists a strictly increasing function \( u_0 : \mathbb{R} \to \mathbb{R} \) satisfying
\[
u(y, t) = u_0(t) \quad \text{for all} \quad y \in \mathbb{R}^{d-1}, t \in \mathbb{R}.
\]

The above problem is inspired by a conjecture of G. W. Gibbons [33] which was formulated for the classical Laplacian operator. The classical Gibbons’ conjecture was proved by several researchers using different approaches; see for instance, [6, 10, 28]. In [29] Farina and Valdinoci prescribed a unified approach to this problem which also works for several other classes of operators. Using the approach of [29], a similar problem is treated in [9] for the operator \(-\Delta + (\Delta)^s\) with \( s \in (0, 1) \). For the proof of Theorem 1.6 we also broadly follow the approach of [29] but, thanks to Theorem 1.2, we do not impose any additional regularity assumption on \( u \). Proof of Theorem 1.6 can be found in Section 4.
The rest of the paper is organized as follows. In Section 2 we prove the ABP maximum principle and the Hopf’s lemma. The existence of principal eigenfunction and Faber-Krahn inequality are discussed in Section 3. Section 4 contains the proof of Gibbons’ conjecture whereas Section 5 provides the proofs of the existence and uniqueness results for Dirichlet boundary value problems.

2. Maximum principles

In this section we prove a Alexandrov-Bakelman-Pucci (ABP) maximum principle (Theorem 2.1), a Hopf’s lemma (Theorem 2.2) and a strong maximum principle (Theorem 2.3). We begin with the following estimate on the exit time.

Lemma 2.1. Assume (A1). Let $D \subset \mathbb{R}^d$ be a bounded domain. For every $k \in \mathbb{N}$ we have

$$\sup_{x \in D} \mathbb{E}_x [\tau^k] \leq k! \left( \sup_{x \in D} \mathbb{E}_x [\tau] \right)^k.$$ 

Moreover, there exists a constant $\theta = \Theta(d, \text{diam}(D))$, monotonically increasing with respect to $\text{diam}(D)$, such that

$$\sup_{x \in D} \mathbb{E}_x [\tau^k] \leq k! \theta^k.$$ 

Proof. Proof follows from [15, Lemma 3.1], (A1) and [48, Remark 4.8].

Using Lemma 2.1 we find an ABP type estimate for the semigroup subsolutions. This is the content of our next lemma.

Lemma 2.2. Let $D$ be any bounded domain and $u : \mathbb{R}^d \to \mathbb{R}$ be a bounded function satisfying

$$u(x) \leq \mathbb{E}_x [u(X_\tau)] + \mathbb{E}_x \left[ \int_0^\tau f(X_s) \, ds \right] \text{ for all } x \in D,$$

with $f \in L^p(D)$, for some $p > \frac{d}{2}$, where $\tau$ denotes the first exit time from $D$. Then there exists a constant $C_1 = C_1(p,d,\text{diam}(D))$ such that

$$\sup_{x \in D} u^+ \leq \sup_{D^c} u^+ + C_1 \|f\|_{L^p(D)}.$$ 

Proof. For simplicity of notation we extend $f$ by zero outside of $D$. From the given condition it is easily seen that

$$u(x) \leq \sup_{D^c} u^+ + \mathbb{E}_x \left[ \int_0^\tau |f(X_s)| \, ds \right], \quad x \in D.$$ 

Thus, we only need to estimate the rightmost term in above expression. Recall that $X = B + Y$ where $B$ is a $d$-dimensional Brownian motion, running twice as fast as standard Brownian motion, and is independent of $Y$. Let $\nu_t$ be the transition probability of $Y_t$, starting from 0, that is,

$$\nu_t(A) = \mathbb{P}(Y_t \in A), \quad A \in \mathcal{B}(\mathbb{R}^d).$$

Let

$$p_t^B(y) = \frac{(4\pi t)^{-d/2}}{\sqrt{\pi}} e^{-\frac{|y|^2}{4t}},$$

be the transition density of $B_t$ starting from 0. Then the transition density of $X_t$, starting from $x$, is given by

$$p_t(x,y) = \int_{\mathbb{R}^d} p_t^B(z-x-y) \nu_t(dz), \quad t > 0.$$ 

In particular, for any $t \in (0,\infty)$ and $x,y \in \mathbb{R}^d$, we have

$$p_t(x,y) = \int_{\mathbb{R}^d} \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x-y|^2}{4t}} \nu_t(dz).$$
we obtain a narrow domain maximum principle.

Next we write
\[
\mathbb{E}_x \left[ \int_0^\tau |f(X_s)| ds \right] = \mathbb{E}_x \left[ \int_0^\infty \mathbbm{1}_{\{\tau > s\}} |f(X_s)| ds \right] \\
\leq \mathbb{E}_x \left[ \int_0^1 |f(X_s)| ds \right] + \mathbb{E}_x \left[ \int_1^\infty \mathbbm{1}_{\{\tau > s\}} |f(X_s)| ds \right].
\]
(2.2)

We estimate the first term on the rhs as
\[
\mathbb{E}_x \left[ \int_0^1 |f(X_s)| ds \right] = \int_0^1 \int_{\mathbb{R}^d} |f(y)| p_s(x,y) dy ds \\
\leq \|f\|_{L^p(D)} \int_0^1 \left[ \int_{\mathbb{R}^d} (p_s(x,y))^{p'} dy \right]^\frac{1}{p'} ds \\
\leq \frac{1}{(4\pi)^{d/2p}} \|f\|_{L^p(D)} \int_0^1 s^{-\frac{d}{p'}} ds = \frac{1}{(4\pi)^{d/2p}} \frac{2p}{2p - d} \|f\|_{L^p(D)},
\]
where in the third line we use (2.1) and \( p, p' \) are Hölder conjugates. To deal with the rightmost term in (2.2) we choose \( k \in \mathbb{N} \) with \( k > p' \). Using Lemma 2.1 we then calculate
\[
\mathbb{E}_x \left[ \int_1^\infty \mathbbm{1}_{\{\tau > s\}} |f(X_s)| ds \right] \leq \int_1^\infty (\mathbb{P}_x(\tau > s))^{\frac{1}{p'}} \mathbb{E}_x [ |f(X_s)|^p ]^{\frac{1}{p'}} ds \\
\leq \frac{1}{(4\pi)^{d/2p}} \|f\|_{L^p(D)} \int_1^\infty (\mathbb{P}_x(\tau > s))^{\frac{1}{p'}} ds \\
\leq \frac{1}{(4\pi)^{d/2p}} \|f\|_{L^p(D)} \int_1^\infty s^{-\frac{k}{p'}} \mathbb{E}_x \left[ \tau^k \right]^{\frac{1}{p'}} ds \\
\leq \frac{1}{(4\pi)^{d/2p}} \|f\|_{L^p(D)} \frac{p'}{k - p'} (k! \theta^k)^{\frac{k}{p'}}.
\]
Combining these estimates in (2.2) completes the proof.

Using Lemma 2.1 and the arguments of [16, Theorem 3.1] then gives us the ABP maximum principle.

Theorem 2.1 (ABP-maximum principle). Assume (A1). Let \( f : D \to \mathbb{R} \) be continuous and \( u \in \mathcal{C}_b(\mathbb{R}^d) \) be a viscosity subsolution to
\[ Lu = -f \quad \text{in } \{u > 0\} \cap D, \quad \text{and } \quad u \leq 0 \quad \text{in } D^c. \]
Then for every \( p > \frac{d}{2} \), there exists a constant \( C = C(d, p, \text{diam}(D)) \) satisfying
\[ \sup_D u \leq C \|f\|_{L^p(D)}. \]

As an application of Theorem 2.1 we obtain a narrow domain maximum principle.

Corollary 2.1 (Maximum principle for narrow domains). Assume (A1). Let \( u \in \mathcal{C}_b(\mathbb{R}^d) \) be a viscosity subsolution to
\[ Lu + c u = 0 \quad \text{in } D, \quad \text{and } \quad u \leq 0 \quad \text{in } D^c, \]
for some continuous function \( c : D \to \mathbb{R} \). Then there exists a constant \( \varepsilon = \varepsilon(d, \|c\|_{L^\infty(D)}, \text{diam}(D)) \) such that \( u \leq 0 \) in \( \mathbb{R}^d \), whenever \( |D| < \varepsilon \).
Theorem 2.1

Proof. The result follows from Theorem 2.1 by choosing \( f = \|c\|_{L^\infty(D)}u^+ \) in \( \{ u > 0 \} \).

\[ \square \]

Corollary 2.2. Assume (A1) and let \( D \) be a bounded open set. Let \( u, v \in C_b(\mathbb{R}^d) \) satisfy

\[
Lu + cu \geq f, \quad Lv + cv \leq g \quad \text{in} \quad D,
\]

for some \( c, f, g \in C(D) \). Also, assume that \( c \leq 0 \) and \( f \geq g \) in \( D \). Then, if \( u \leq v \) in \( D^c \), we have \( u \leq v \) in \( \mathbb{R}^d \).

Proof. Using Theorem 1.2 we get that \((L + c)w \geq 0 \) in \( D \) with \( w \leq 0 \) in \( D^c \) where \( w = u - v \). Note that, moving \( cw \) on the rhs, we can take \( f = 0 \) on \( \{ w > 0 \} \) in Theorem 2.1. Then applying Theorem 2.1, we obtain \( w \leq 0 \) in \( \mathbb{R}^d \). Hence the proof.

Next we prove the Hopf’s lemma. At this point we mention a recent interesting work of Klimsiak and Komorowski [36] where an abstract Hopf’s type lemma is obtained for the semigroup solutions of a general integro-differential operator.

Theorem 2.2 (Hopf’s lemma). Let \( Lu + cu \leq 0 \) in \( D \) where \( u \in C_b(\mathbb{R}^d) \) and \( c \in C_b(\overline{D}) \). Suppose that \( u > 0 \) in \( D \) and non-negative in \( \mathbb{R}^d \). Then there exists \( \eta > 0 \) such that for any \( x_0 \in \partial D \) with \( u(x_0) = 0 \) we have

\[
\frac{u(x)}{(r - |x - z|)} \geq \eta,
\]

for all \( x \in B_r(z) \cap B_\overline{\mathbb{R}^d}(x_0) \), where \( B_r(z) \subset D \) is a ball that touches \( \partial D \) at \( x_0 \).

Proof. Since \( u > 0 \) in \( D \), without any loss of generality, we may assume that \( c \leq 0 \). Let \( K = B_r(z) \cap B_\overline{\mathbb{R}^d}(x_0) \) and define

\[
v(x) = e^{-\alpha q(x)} - e^{-\alpha r^2},
\]

where \( q(x) = |z - x|^2 \land 9r^2 \). Clearly \( v > 0 \) in \( B_r(z) \), \( v(x) = 0 \) on \( \partial B_r(z) \), and \( v \leq 0 \) in \( \mathbb{R}^d \setminus B_r(z) \).

For \( x \in B_{2r}(z) \) we have

\[
\Delta v = \alpha e^{-\alpha |x - z|^2} \left( 4\alpha |x - z|^2 - 2d \right).
\]

Fix any \( x \in B_{2r}(z) \). Using the convexity of \( x \mapsto e^x \) we first note that, for \( |y| \leq 1 \),

\[
v(x + y) - v(x) - \mathbf{1}_{\{ |y| \leq 1 \}} y \cdot \nabla v(x)
= e^{-\alpha (x+y-z)^2} - e^{-\alpha |x-z|^2} + 2\alpha \mathbf{1}_{\{ |y| \leq 1 \}} y \cdot (x - z) e^{-\alpha |x-z|^2}
\geq -e^{-\alpha |x-z|^2} \left( |x+y-z|^2 - |x-z|^2 - 2\mathbf{1}_{\{ |y| \leq 1 \}} y \cdot (x - z) \right) = -\alpha e^{-\alpha |x-z|^2} |y|^2.
\]

Therefore, for \( x \in B_{2r}(z) \), we have

\[
\int_{\mathbb{R}^d} (v(x + y) - v(x) - \mathbf{1}_{\{ |y| \leq 1 \}} y \cdot \nabla v(x)) j(y) dy
= \int_{|y| \leq 1} (v(x + y) - v(x) - \mathbf{1}_{\{ |y| \leq 1 \}} y \cdot \nabla v(x)) j(y) dy + \int_{|y| > 1} (v(x + y) - v(x)) j(y) dy
\geq -\alpha e^{-\alpha |x-z|^2} \int_{|y| \leq 1} |y|^2 j(y) dy + \int_{|y| > 1} (v(x + y) - v(x)) j(y) dy
\geq -\alpha e^{-\alpha |x-z|^2} \int_{|y| \leq 1} |y|^2 j(y) dy + \alpha \int_{|y| > 1} (e^{-9\alpha r^2} - 1) j(y) dy,
\]

where in the last line we used \( |x - z + y|^2 \land 9r^2 \leq |x - z|^2 + 9r^2 \). Thus, using (1.2) and \( x \in B_{2r}(z) \), we obtain

\[
Lu(x) + c(x)v(x)
\geq \alpha e^{-\alpha |x-z|^2} \left[ 4\alpha |x - z|^2 - 2d - \int_{|y| \leq 1} |y|^2 j(y) dy + \alpha^{-1} \int_{|y| > 1} (e^{-9\alpha r^2} - 1) j(y) dy \right].
\]
- \|c\|_{L^\infty(D)} \alpha^{-1} \left(1 - e^{-\alpha(r^2 - |x-z|^2)}\right).

For $|x-z| \geq r/2$, we can choose $\alpha$ large enough so that

$$Lv + cv > 0 \quad \text{for} \quad \frac{r}{2} \leq |x-z| < 2r.$$  \tag{2.3}

Let $m = \min_{D_2^+} u$ where $D_2^+ = \{ y \in D : \text{dist}(y, \partial D) \geq \frac{r}{2}\}$. Defining $w = mv$, we have $L(w - u) + c(w - u) \geq 0$ in $B_r(z) \setminus B_{r/2}(z)$, by Theorem 1.2, and $w - u \leq 0$ in $(B_r(z) \setminus B_{r/2}(z))^c$. Thus as a consequence of Corollary 2.2, we obtain

$$u \geq mv = me^{-\alpha r^2}(e^{\alpha r^2 - |x-z|^2} - 1) \geq me^{-\alpha r^2} \alpha(r^2 - |x-z|^2).$$

This completes the proof. \qed

As a by-product of the proof of Hopf’s lemma above we obtain a strong maximum principle (compare with Ciomaga [25]).

**Theorem 2.3** (Strong maximum principle). Assume (A1) and let $c \in C_b(D)$. Let $Lu + cu \leq 0$ in $D$ and $u \in C_b(\mathbb{R}^d)$ be non-negative in $\mathbb{R}^d$. Then either $u > 0$ in $D$ or it is identically 0 in $D$.

**Proof.** Since $u$ is non-negative, without any loss of generality, we may assume that $c \leq 0$. If we take $K = \{u = 0\} \cap D$, then we want to show that $K \cap D$ is either empty or $D$. Suppose, to the contrary, $K \cap D$ is non-empty and is not equal to the set $D$. This means $D \setminus K$ is also non-empty. Hence we can find a point $z \in D \setminus K$ and $r$ small, such that $x_0 \in \partial B_r(z)$ for some $x_0 \in K \cap D$, $B_{2r}(z) \subset D$ and $u > 0$ in $B_r(z)$.

Now we consider the function $v$ that we constructed in Theorem 2.2, that is

$$v(x) = e^{-\alpha q(x)} - e^{-\alpha r^2},$$

where $q(x) = |z - x|^2 \wedge 9r^2$. Again we have $v > 0$ in $B_r(z)$, $v(x) = 0$ on $\partial B_r(z)$, and $v \leq 0$ in $\mathbb{R}^d \setminus B_r(z)$. Also by (2.3) we have

$$Lv + cv > 0 \quad \text{in} \quad B_{2r}(z) \setminus B_{r/2}(z).$$  \tag{2.4}

Let $m = \min_{B_{2r}(z)} u$ and define $w = mv$. Then following similar argument as in Theorem 2.2 we have $u \geq w$ in $\mathbb{R}^d$. Since $w \in C^2(B_{2r}(z))$ and $w(x_0) = u(x_0) = 0$, we use $w$ as a test function and define

$$\phi(y) := \begin{cases} w(y) & \text{for } y \in B_r(x_0), \\ u(y) & \text{otherwise.} \end{cases}$$

Then by the definition of viscosity supersolution we have

$$L\phi(x_0) + c(x_0)\phi(x_0) \leq 0.$$

Clearly, $w \leq \phi$ in $\mathbb{R}^d$, $\nabla\phi(x_0) = \nabla w(x_0)$ and $\Delta\phi(x_0) = \Delta w(x_0)$. Thus $Lw(x_0) + c(x_0)w(x_0) \leq 0$ which contradicts the fact that $Lw(x_0) + c(x_0)w(x_0) > 0$ (see (2.4)). Hence $K \cap D$ is either empty or $D$. This completes the proof. \qed
3. Generalized principal eigenvalue and Faber-Krahn inequality

In this section we study the generalized eigenvalue problem of $L$ (Theorem 3.1) and then we establish a Faber-Krahn inequality (Theorem 3.2). We begin with the following boundary estimate which will be useful.

**Lemma 3.1.** Assume (A1). Let $D$ be a bounded domain satisfying an uniform exterior sphere condition with radius $r > 0$. Let $u \in C_c(\mathbb{R}^d)$ be a viscosity solution to
\[ Lu = f \quad \text{in} \quad D, \quad u = 0 \quad \text{in} \quad D^c, \]
for some $f \in L^\infty(D)$. Then there exists a constant $C$, dependent on $r, d, \text{diam}(D)$, satisfying
\[ |u(x)| \leq C \| f \|_{L^\infty(D)} \| v \|_{L^\infty(D)} \text{dist}(x, \partial D) \quad \text{for} \quad x \in D. \]

**Proof.** Let $B$ be ball containing $D$. Then $v(x) = \mathbb{E}_x[\tau_B]$ solves (see Theorem 1.1)
\[ Lu = -1 \quad \text{in} \quad B, \quad u = 0 \quad \text{in} \quad B^c. \]

Applying comparison principle, Theorem 1.2, it then follows that
\[ |u(x)| \leq \| f \|_{L^\infty(D)} v(x), \quad x \in \mathbb{R}^d. \tag{3.1} \]

Without any loss of generality we may assume $r \in (0, 1)$. By [39, Lemma 5.4] there exists a bounded, Lipschitz continuous function $\varphi$, with Lipschitz constant $r^{-1}$, satisfying
\[ \begin{cases} 
\varphi = 0, & \text{in } B_r, \\
\varphi > 0, & \text{in } B^c_r, \\
\varphi \geq \varepsilon, & \text{in } B^c_{(1+\delta)r}, \\
L\varphi \leq -1, & \text{in } B^c_{(1+\delta)r}, 
\end{cases} \]

for some constant $\varepsilon, \delta$, where $B_r$ denotes the ball of radius $r$ around $0$. Now for any point $y \in \partial D$ we can find another point $z \in D^c$ such that $\overline{B_r(z)}$ touches $\partial D$ at $y$. Defining $w(x) = \varepsilon^{-1} \| f \|_{L^\infty(D)} \| v \|_{L^\infty(D)} \varphi(x - z)$ and using (3.1) it follows that $|u(x)| \leq w(x)$ in $B_{(1+\delta)r}(z) \cap D$, by comparison principle. This relation holds for any $y \in \partial D$. Now for any point $x \in D$ with $\text{dist}(x, \partial D) < \delta r$ we can find $y \in \partial D$ satisfying $\text{dist}(x, \partial D) = |x - y| < \delta r$. By previous estimate we obtain
\[ |u(x)| \leq \varepsilon^{-1} \| f \|_{L^\infty(D)} \| v \|_{L^\infty(D)} \varphi(x - z) \leq \varepsilon^{-1} \| f \|_{L^\infty(D)} \| v \|_{L^\infty(D)} (\varphi(x - z) - \varphi(y - z)) \leq \varepsilon^{-1} \| f \|_{L^\infty(D)} \| v \|_{L^\infty(D)} r^{-1} \text{dist}(x, \partial D). \]

Now the proof follows from (3.1). \qed

Now fix a Lipschitz domain $D$ satisfying a uniform exterior sphere condition. In view of Theorem 1.1 we can define a map $\mathcal{T} : C(\overline{D}) \rightarrow C_0(\overline{D})$ (the space of continuous function in $\overline{D}$ vanishing on the boundary) as follows: $\mathcal{T}[f] = u$ where $u$ is the unique viscosity solution to
\[ Lu = -f \quad \text{in} \quad D, \quad u = 0 \quad \text{in} \quad D^c. \]

**Lemma 3.2.** Under (A1), the map $\mathcal{T}$ is a bounded linear, compact operator.

**Proof.** It is evident that $\mathcal{T}$ is a bounded linear operator. So we only need to show that $\mathcal{T}$ is compact. Let $\mathcal{K}$ be a bounded subset of $C(\overline{D})$, that is, there exists a constant $\kappa$ such that $\| f \|_{\infty} \leq \kappa$ for all $f \in \mathcal{K}$. Define $\mathcal{G} = \{ u : u = \mathcal{T}[f] \text{ for some } f \in \mathcal{K} \}$. By Lemma 3.1, $\mathcal{G}$ is bounded in $C_0(\overline{D})$. Let us now show that $\mathcal{G}$ is also equicontinuous. Consider $\varepsilon > 0$. For $\delta_1 > 0$ let us define
\[ D^{\delta_1}_r = \{ x \in D : \text{dist}(x, \partial D) > \delta_1 \}. \]

Using Lemma 3.1, we can then choose $\delta_1 > 0$ small enough to satisfy
\[ \sup_{D \setminus D^{\delta_1}_r} |u(x)| < \varepsilon/2 \quad \text{for all} \quad u \in \mathcal{G}. \tag{3.2} \]
Again, by [39, Theorem 4.2], there exists \( \alpha > 0 \) satisfying
\[
\sup_{x \neq y, x, y \in D_1} \frac{|u(x) - u(y)|}{|x - y|^\alpha} \leq \kappa_2 \quad \text{for all } u \in G,
\] (3.3)
for some constant \( \kappa_2 \). Choose \( \delta \in (0, \delta_1) \) satisfying \( \kappa_2 \delta^\alpha < \varepsilon \). Then, from (3.2) and (3.3), we obtain
\[
|u(x) - u(y)| < \varepsilon \quad \text{for all } u \in G, \ x, y \in D, \ |x - y| \leq \delta.
\]
Therefore \( G \) is equicontinuous. Hence by Arzelà-Ascoli theorem we have \( G \) compact, completing the proof. \( \square \)

From Lemma 3.2 and Corollary 2.2 we get the following existence result.

**Lemma 3.3.** Grant (A1). Let \( D \) be a bounded Lipschitz domain satisfying a uniform exterior sphere condition. Suppose \( c \in C(\bar{D}) \) with \( c \leq 0 \). Then for any \( f \in C(\bar{D}) \) there exists a unique solution \( u \) to
\[
Lu + cu = -f \quad \text{in } D, \quad \text{and} \quad u = 0 \quad \text{in } D^c.
\] (3.4)

**Proof.** Fixing \( f \in C(\bar{D}) \), we define a map \( F : C_0(D) \rightarrow C_0(D) \) as
\[
u = F(v) = T[f + cv],
\]
where \( T \) is same as in Lemma 3.2. From Lemma 3.2 it follows that \( F \) is continuous and compact. Consider a set
\[
E = \{ v \in C_0(D) : v = \lambda F(v) \quad \text{for some } \lambda \in [0, 1] \}.
\]

Claim: \( E \) is bounded in \( C_0(D) \).

Suppose, to the contrary, that \( E \) is not bounded. Then there exist \( v_n \in E \), for \( n \in \mathbb{N} \), such that \( \|v_n\|_{L^\infty(D)} \rightarrow \infty \) as \( n \rightarrow \infty \). So we have couples \( (v_n, \lambda_n) \) satisfying \( v_n = \lambda_n F(v_n) \) which gives us
\[
Lv_n = \lambda_n (-f - cv_n) \quad \text{in } D.
\]

Letting \( w_n = \|v_n\|_{L^\infty(D)}^{-1} v_n \) we get from above that
\[
Lw_n = \lambda_n \frac{-f}{\|v_n\|_{L^\infty(D)}} - c\lambda_n w_n.
\]
Since \( \|w_n\|_{L^\infty(D)} = 1 \) for all \( n \) we see that
\[
\|\lambda_n \frac{f}{\|v_n\|_{L^\infty(D)}} + c\lambda_n w_n\|_{L^\infty(D)} \leq \kappa_2,
\]
for some \( \kappa_2 > 0 \). Therefore, using Lemma 3.2, \( \{w_n : n \geq 1\} \) is equicontinuous and hence, up to a subsequence, \( w_n \rightarrow w \) in \( C_0(D) \). By the stability property of the viscosity solutions we obtain
\[
Lw = 0 - c\lambda w \quad \text{in } D \quad \text{and} \quad w = 0 \quad \text{in } D^c,
\]
for some \( \lambda \in [0, 1] \). Hence \( w \) solves the equation
\[
Lw + c\lambda w = 0 \quad \text{in } D,
\]
\[
w = 0 \quad \text{in } D^c.
\]
From Corollary 2.2 we see that \( w = 0 \) in \( \mathbb{R}^d \). But this contradicts the fact that \( \|w\|_{L^\infty(D)} = 1 \), and this proves our claim.

Applying Leray-Schauder theorem we must have a fixed point of \( F \). This gives the existence of solution for (3.4). Uniqueness again follows from the above arguments. \( \square \)

Now we are ready to prove the existence of principal eigenfunction.
Theorem 3.1. Assume (A1). Let $D$ be a bounded domain satisfying a uniform exterior sphere condition. Let $c \in C(D)$. There exists a unique $\psi_D \in C_b(\mathbb{R}^d)$, satisfying

$$L\psi_D + c\psi_D = -\lambda_D \psi_D \quad \text{in } D,$$

$$\psi_D = 0 \quad \text{in } D^c,$$

$$\psi_D > 0 \quad \text{in } D, \quad \psi_D(0) = 1.$$ Moreover, if $u \in C_b(\mathbb{R}^d)$ is positive in $D$ and satisfies

$$Lu + cu \leq -\lambda u \quad \text{in } D,$$

for some $\lambda \in \mathbb{R}$ then $\lambda \leq \lambda_D$. Furthermore, if $\lambda = \lambda_D$ and $u = 0$ in $D^c$, then we have $u = k\psi_D$ for some $k > 0$. Furthermore, $\lambda_D$ is the only Dirichlet eigenvalue with a positive eigenfunction.

Proof. The proof technique is quite standard and follows by combining Krein-Rutman theorem with Lemmas 3.2 and 3.3. Replacing $c$ by $c - ||c||_{L^\infty(D)}$ we can assume that $c \leq 0$. Using Lemma 3.3 we define a map $T_1 : C_0(D) \to C_0(D)$ as follows: $T_1[u] = v$ if and only if

$$Lu + cv = -u \quad \text{in } D, \quad \text{and } v = 0 \quad \text{in } D^c.$$ Since, by comparison principle (see Corollary 2.2), $||v|| \leq ||u|| \max\{||w_+||, ||w_-||\}$ where

$$Lw_\pm + cw_\pm = \pm 1 \quad \text{in } D, \quad \text{and } w_\pm = 0 \quad \text{in } D^c,$$

it follows from Lemma 3.2 that $T_1$ is a compact, bounded linear map. Again, if $u_1 \leq u_2$ in $C_0(D)$, by comparison principle (see Corollary 2.2) we get $T_1[u_1] \leq T_1[u_2]$. Furthermore, if $u_1 \leq u_2$, then $T_1[u_1] < T_1[u_2]$ in $D$ by Theorem 3.3. Let $f \geq 0$ be nonzero compactly supported function in $D$. Then, for $T_1[f] = v$, we have $v > 0$ in $D$ and therefore, we can find $M > 0$ satisfying $MT_1[f] > f$ in $D$. Denote by $\mathcal{P}$ the cone of non-negative functions in $C_0(D)$. From comparison principle Corollary 2.2, it is easily seen that $T_1(\mathcal{P}) \subset \mathcal{P}$. Therefore, Krein-Rutman applies to $T_1$ and we find $\lambda_D > 0$ and $\psi_D \in C_0(D)$ with $\psi_D > 0$ in $D$ satisfying

$$L\psi_D + c\psi_D + \lambda_D \psi_D = 0 \quad \text{in } D,$$

$$\psi_D = 0 \quad \text{in } D^c. \quad (3.5)$$

Now we focus on the second part of the theorem. Consider a non-negative function $u \in C_b(\mathbb{R}^d) \cap C_+(D)$ satisfies

$$Lu + cu + \lambda u \leq 0 \quad \text{in } D,$$

for some $\lambda \in \mathbb{R}$. Suppose, to the contrary, that $\lambda > \lambda_D$. Then using Corollary 2.1, Theorem 2.3 and the proof of [16, Theorem 3.2] we find $\gamma > 0$ satisfying $u = \gamma \psi_D$ in $D$. Since minimum of two viscosity supersolutions is also a supersolution, we have $\gamma \psi_D = \min\{u, \gamma \psi_D\}$ and

$$L(\gamma \psi_D) + (c + \lambda)(\gamma \psi_D) \leq 0 \quad \text{in } D.$$ Applying Theorem 1.2, we see that $(\lambda - \lambda_D)\psi_D \leq 0$ in $D$ which is a contradiction. Hence $\lambda \leq \lambda_D$. Rest of the proof follows from [16, Theorem 3.2]. \ \Box

Our next aim is to prove Faber-Krahn inequality and to do so we need certain continuity property of the principal eigenvalue with respect to the domains. To do so we need the following condition.

(A4) The domain $D$ is Lipschitz and bounded with uniform exterior sphere condition of radius $r$. Furthermore, there exists a collection of bounded, Lipschitz decreasing domains $\{D_n\}$ such that $\cap_n D_n = \bar{D}$ and each $D_n$ satisfies uniform exterior sphere condition of radius $r$.

It can be easily seen that convex domains, $C^{1,1}$ domains satisfy the above condition. In the next lemma we prove result on continuity of $\lambda_D$.

Lemma 3.4. Assume (A1) and (A4). Denote by $\lambda_n = \lambda_{D_n}$. Then $\lambda_n \to \lambda_D$ as $n \to \infty$. 
Proof. From Theorem 3.1 we notice that $\lambda_n \leq \lambda_{n+1}$. Let $\lim_{n \to \infty} \lambda_n = \lambda$. Evidently, $\lambda \leq \lambda_D$. Using condition (A4), Lemma 3.1 and the proof of Lemma 3.2 it follows that $\{\psi_n\}$ is equicontinuous in $\mathbb{R}^d$ where $\psi_n$ is the principal eigenfunction corresponding to $\lambda_n$. We also normalize $\psi_n$ to satisfy $\|\psi_n\|_{L_\infty(B^d)} = 1$. Applying Arzelá-Ascoli we can extract a subsequence of $\psi_n$ converging to $\psi$ and by the stability property of the viscosity solution we obtain

$$L\psi + (c + \lambda)\psi = 0 \quad \text{in } D, \quad \text{and} \quad \psi = 0 \quad \text{in } D^c.$$ 

Since $\psi \geq 0$, by strong maximum principle Theorem 2.3, we must have $\psi > 0$ in $D$. Then, by Theorem 3.1, we must have $\lambda = \lambda_D$. Hence the proof. \qed

Next we find a representation of the principal eigenvalue which is crucial for the proof of Faber-Krahn inequality.

Lemma 3.5. Consider the setting of Lemma 3.4 and let $c = 0$. Let $\lambda_D$ be the corresponding principal eigenvalue. Then

$$\lambda_D = -\lim_{t \to \infty} \frac{1}{t} \log \mathbb{P}_x(\tau > t) \quad \text{for all } x \in D. \quad (3.6)$$

Proof. The principal eigenpair $(\psi_D, \lambda_D)$ satisfies

$$L\psi_D + \lambda_D\psi_D = 0 \quad \text{in } D, \quad \text{and} \quad \psi_D = 0 \quad \text{in } D^c.$$ 

From the arguments of [13, Lemma 3.1] we then have

$$\psi_D(x) = \mathbb{E}_x \left[ e^{\lambda_D t} \psi_D(X_t) \mathbbm{1}_{\{\tau > t\}} \right], \quad x \in D. \quad (3.7)$$ 

Using (3.7), Lemma 3.4 and the proof of [15, Corollary 4.1] we get (3.6). This completes the proof. \qed

Now we are ready to prove the Faber-Krahn inequality.

Theorem 3.2 (Faber-Krahn inequality). Let $z \mapsto j(z)$ be isotropic and radially decreasing. Let $D$ be any bounded domain satisfying $|\partial D| = 0$. Then

$$\lambda_D \geq \lambda_B,$$

where $B$ is ball around 0 satisfying $|B| = |D|$.

Proof. By the assumption on $j$, (A1) holds. We note that $p_t(x, y) = p_t(y - x)$ where

$$\int_{\mathbb{R}^d} e^{i\xi \cdot x} p_t(x) dx = e^{-t(|\xi|^2 + \psi(\xi))}.$$ 

From [50] we know that $p_t$ is isotropic unimodal, that is, $p_t$ is radially decreasing. We first assume that $D$ is a smooth domain. Without any loss of generality we may also assume $0 \in D$. Then by Markov property

$$\mathbb{P}_0(\tau > t) = \lim_{m \to \infty} \mathbb{P}_0(X_{\frac{m}{m}} \in D, X_{\frac{2m}{m}} \in D, \ldots, X_{\frac{m^d}{m}} \in D)$$

$$= \lim_{m \to \infty} \int_D \cdots \int_D p_{\frac{m}{m}}(x_1)p_{\frac{m}{m}}(x_2 - x_1) \cdots p_{\frac{m}{m}}(x_m - x_{m-1}) dx_1 dx_2 \cdots dx_m$$

$$\leq \lim_{m \to \infty} \int_B \cdots \int_B p_{\frac{m}{m}}(x_1)p_{\frac{m}{m}}(x_2 - x_1) \cdots p_{\frac{m}{m}}(x_m - x_{m-1}) dx_1 dx_2 \cdots dx_m$$

$$= \mathbb{P}_0(\tau_B > t),$$

where in the third line we used Brascamp-Lieb-Luttinger inequality [19, Theorem 3.4]. Therefore,

$$- \lim_{t \to \infty} \frac{1}{t} \log \mathbb{P}_0(\tau > t) \geq - \lim_{t \to \infty} \frac{1}{t} \log \mathbb{P}_0(\tau_B > t).$$
Applying Lemma 3.5 we then have
\[
\lambda_D \geq \lambda_B.
\] (3.8)

Now given a bounded domain \( D \) with \( |\partial D| = 0 \) we consider a decreasing sequence of smooth domains \( D_n \) such that \( \cap_{n \geq 1} D_n = \bar{D} \) and \( |D_n| \to |D| \) as \( n \to \infty \). Let \( B_n \) be a ball centered at \( 0 \) and \( |B_n| = |D_n| \). It is also easily seen that \( B \) and \( \{B_n\} \) satisfies condition (A4). Using (3.8) and monotonicity of \( \lambda_D \) with respect to domains, we get that
\[
\lambda_D \geq \lambda_{D_n} \geq \lambda_{B_n}.
\]

Now let \( n \to \infty \) and apply Lemma 3.4 to conclude
\[
\lambda_D \geq \lambda_B.
\]

Hence the proof. \( \square \)

4. Symmetry of positive solutions and Gibbons’ problem

In this section we prove radial symmetry of positive solutions and a one-dimensional symmetry result related to the Gibbons’ conjecture.

**Theorem 4.1.** Assume (A1). Also, assume that \( j \) is radially decreasing in \( \mathbb{R}^d \setminus \{0\} \) and strictly decreasing in a neighbourhood of 0. Suppose that \( D \) is convex in the direction of the \( x_1 \) axis, and symmetric about the plane \( \{x_1 = 0\} \). Also, let \( f : [0, \infty) \to \mathbb{R} \) be locally Lipschitz continuous. Consider any solution \( u \in \mathcal{C}_b(\mathbb{R}^d) \) of
\[
Lu = f(u) \quad \text{in } D,
\]
\[
u = 0 \quad \text{in } D^c,
\]
\[
u > 0 \quad \text{in } D.
\]

Then \( u \) is symmetric with respect to \( x_1 = 0 \) and strictly decreasing in the \( x_1 \) direction.

**Proof.** We use the method of moving plane appeared in the seminal work Gidas, Ni and Nirenberg [34] which was motivated by a work of Serrin [47]. The proof can be easily completed following the arguments of [30]. We provide a proof for the sake of completeness. Define
\[
\Sigma_\lambda = \left\{ x = (x_1, x') \in D : x_1 > \lambda \right\} \quad \text{and} \quad T_\lambda = \left\{ x = (x_1, x') \in \mathbb{R}^d : x_1 = \lambda \right\},
\]
\[
u_\lambda(x) = u(x_\lambda) \quad \text{and} \quad w_\lambda(x) = u_\lambda(x) - u(x),
\]
where \( x_\lambda = (2\lambda - x_1, x') \). For a set \( A \) we denote by \( \mathcal{R}_A A \) the reflection of \( A \) with respect to the plane \( T_\lambda \). Also, define
\[
\lambda_{\text{max}} = \sup \{ \lambda > 0 : \Sigma_\lambda \neq \emptyset \}.
\]

We note that for any \( \lambda \in (0, \lambda_{\text{max}}) \), \( u_\lambda \) is a viscosity solution of
\[
Lu_\lambda = f(u_\lambda) \quad \text{in } \Sigma_\lambda,
\]

and therefore, from Theorem 1.2 we obtain
\[
Lu_\lambda = f(u_\lambda) - f(u) \quad \text{in } \Sigma_\lambda.
\]

Define \( \Sigma_\lambda^- = \{ x \in \Sigma_\lambda : w_\lambda < 0 \} \). Since \( w_\lambda \geq 0 \) on \( \partial \Sigma_\lambda \), it follows that \( w_\lambda = 0 \) on \( \partial \Sigma_\lambda^- \). Hence the function
\[
v_\lambda = \begin{cases} w_\lambda & \text{in } \Sigma_\lambda^- \\ 0 & \text{elsewhere,} \end{cases}
\]
is in \( \mathcal{C}_b(\mathbb{R}^d) \). We claim that for every \( \lambda \in (0, \lambda_{\text{max}}) \)
\[
Lv_\lambda \leq f(u_\lambda) - f(u) \quad \text{in } \Sigma_\lambda^-,
\] (4.1)
in the viscosity sense. To see this, let \( \varphi \) be a test function that touches \( v_\lambda \) from below at a point \( x \in \Sigma^-_\lambda \). Then \( \varphi + (w_\lambda - v_\lambda) \in \mathcal{C}_b(x) \) and touches \( w_\lambda \) at \( x \) from below. Denote \( \zeta_\lambda(x) = w_\lambda - v_\lambda \). It then follows that

\[
L(\phi + \zeta_\lambda)(x) \leq f(u_\lambda(x)) - f(u(x)).
\]

Since \( \zeta_\lambda = 0 \) in a small neighbourhood of \( x \) in \( \Sigma^-_\lambda \) we have \( \Delta \zeta_\lambda(x) = 0 \). Again, since \( j \) is radial, to show (4.1) it is enough to show that

\[
\int_{\mathbb{R}^d} (\zeta_\lambda(x + z) - \zeta_\lambda(x)) j(z) dz \geq 0.
\]

This can be done by following the argument of [30, p. 8] and the fact \( j \) is radially decreasing.

If \( \lambda < \lambda_{\text{max}} \) is sufficiently close to \( \lambda_{\text{max}} \), then \( w_\lambda > 0 \) in \( \Sigma_\lambda \). Indeed, note that if \( \Sigma^-_\lambda \neq \emptyset \), then \( v_\lambda \) satisfies (4.1). Denoting

\[
c(x) = \frac{f(u_\lambda(x)) - f(u(x))}{u_\lambda(x) - u(x)},
\]

it then follows that

\[
Lv_\lambda - c(x)v_\lambda \leq 0 \quad \text{in} \quad \Sigma^-_\lambda.
\]

Thus choosing \( \lambda \) sufficiently close to \( \lambda_{\text{max}} \) it follows from Corollary 2.1 that \( v_\lambda \geq 0 \) in \( \mathbb{R}^d \). Hence \( \Sigma^-_\lambda = \emptyset \) and we have a contradiction. To show that \( w_\lambda > 0 \) in \( \Sigma_\lambda \), we assume to the contrary that \( w_\lambda(x_0) = 0 \) for some \( x_0 \in \Sigma_\lambda \). Consider a non-negative test function \( \varphi \in \mathcal{C}_b(x_0) \) crossing \( w_\lambda \) from below with the property that \( \varphi = 0 \) in \( B_j(x_0) \subset \Sigma_\lambda \) and \( \varphi = w_\lambda \) in \( B_{2r}(x_0) \). Furthermore, choose \( r \) small enough such that \( B_{2r}(x_0) \subset \Sigma_\lambda \) and \( \varphi \geq 0 \) in \( \Sigma_\lambda \). Then we obtain

\[
L\varphi(x_0) \leq 0. \tag{4.2}
\]

Since \( \Delta \varphi(x_0) = 0 \) we get

\[
I[\varphi](x_0) := \frac{1}{2} \int_{\mathbb{R}^d} (\varphi(x_0 + z) + \varphi(x_0 - z) - 2\varphi(x_0)) j(z) dz \leq 0.
\]

Next we compute \( I[\varphi](x_0) \). Note that \( \varphi \geq 0 \) in \( R_\lambda := \{ x \in \mathbb{R}^d : x_1 \geq \lambda \} \). We have

\[
I[\varphi](x_0) = \int_{\mathbb{R}^d} \varphi(z) j(|x_0 - z|) dz = \int_{R_\lambda} \varphi(z) j(|x_0 - z|) dz + \int_{R_\lambda \setminus B_{2r}(x_0)} \varphi(z) j(|x_0 - z|) dz
\]

\[
= \int_{R_\lambda} \varphi(z) j(|x_0 - z|) dz + \int_{R_\lambda} w_\lambda(z) j(|x_0 - z|) dz
\]

\[
= \int_{R_\lambda} \varphi(z) j(|x_0 - z|) dz - \int_{R_\lambda} w_\lambda(z) j(|x_0 - z|) dz
\]

\[
\geq \int_{R_\lambda \setminus B_{2r}(x_0)} w_\lambda(z) (j(|x_0 - z|) - j(|x_0 - z|)) dz - \int_{B_{2r}(x_0)} w_\lambda(z) j(|x_0 - z|) dz.
\]

Since \( |z_\lambda - x_0| > |z - x_0| \) and \( j(|z_\lambda - x_0|) \geq j(|z - x_0|) \) thus the first term in the above expression is non-negative. In fact, since

\[
\lim_{r \to 0} \int_{B_{2r}(x_0)} w_\lambda(z) j(|x_0 - z_\lambda|) dz = 0,
\]

using (4.2), we obtain

\[
\lim_{r \to 0} \int_{R_\lambda \setminus B_{2r}(x_0)} w_\lambda(z) j(|x_0 - z_\lambda|) dz \leq 0. \tag{4.3}
\]

Since \( w_\lambda \) is continuous and \( j \) is strictly decreasing in a neighbourhood of 0, from (4.3) we get \( w_\lambda = 0 \) in \( B_\delta(x_0) \) for some \( \delta > 0 \). Thus \( \{ w_\lambda = 0 \} \cap \Sigma_\lambda \) is an open set. Hence \( \{ w_\lambda = 0 \} \) forms a connected
component of \( \Sigma_\lambda \) which in turn, implies that \( \{ w_\lambda = 0 \} \cap \partial \Sigma_\lambda \cap \partial D \neq \emptyset \). This is a contradiction. Hence we must have \( w_\lambda > 0 \) in \( \Sigma_\lambda \).

Now from the above argument and Step 2 in [30, p. 10] we can show that \( \inf \{ \lambda > 0 : w_\lambda > 0 \text{ in } \Sigma_\lambda \} = 0 \). Also, strict monotonicity of \( u \) in the \( x_1 \) direction can be obtained by following the calculations in Step 3 of [30]. \( \square \)

As a consequence of Theorem 4.1 we obtain.

**Corollary 4.1.** Grant the setting of Theorem 4.1. Then every solution to

\[
Lu = f(u) \quad \text{in } B_1(0), \quad u = 0 \quad \text{in } B_1^c(0), \quad \text{and } \quad u > 0 \quad \text{in } B_1(0),
\]

is radial and strictly decreasing in \( |x| \).

Remaining part of this section is devoted to the Gibbons’ problem. Let \( u : \mathbb{R}^d \to \mathbb{R} \) be a solution to the problem

\[
\begin{cases}
Lu(x) = f(u(x)), & \text{for } x \in \mathbb{R}^d, \\
\lim_{x_n \to \pm \infty} u(x', x_n) = \pm 1, & \text{uniformly for } x' \in \mathbb{R}^{d-1}.
\end{cases}
\]  

We also suppose that \( f \in C^1(\mathbb{R}) \) satisfying

\[
\inf_{|r| \geq 1} f'(r) > 0. \tag{4.5}
\]

We show that \( u \) is one-dimensional.

**Theorem 4.2.** Assume (A1). Let \( u \in C_0(\mathbb{R}^d) \) solve (4.4) where \( f \) satisfies (4.5). Then there exists a strictly increasing function \( u_0 : \mathbb{R} \to \mathbb{R} \) satisfying

\[
u(y, t) = u_0(t) \quad \text{for all } y \in \mathbb{R}^d, \ t \in \mathbb{R}.
\]

We need the following lemma to prove Theorem 4.2.

**Lemma 4.1.** Let \( w \in C_b(\mathbb{R}^d) \) satisfy

\[
Lw - c(x)w = 0 \quad \text{in } \mathbb{R}^d,
\]

with

\[
w(x) \geq 0 \quad \text{in } \mathbb{R}^d \setminus U \text{ and } c(x) \geq \kappa \text{ in } U
\]

for some open set \( U \subseteq \mathbb{R}^d \) and some constant \( \kappa > 0 \). Also, assume that \( c \in C_0(\mathbb{R}^d) \). Then

\[
w(x) \geq 0 \quad \text{for all } x \in \mathbb{R}^d.
\]

**Proof.** Suppose, to the contrary, that \( m = \inf_{\mathbb{R}^d} w < 0 \). If \( m \) is attend then the proof can be completed from maximum principle. In general, without loss of generality, we may choose a sequence \( x_k \in \mathbb{R}^d \) satisfying

\[
\lim_{k \to \infty} w(x_k) = m \quad \text{and} \quad w(x_k) \leq \frac{m}{2} < 0 \quad \forall \ k \in \mathbb{N}. \tag{4.6}
\]

By given condition, for every \( k \in \mathbb{N} \), we have

\[
x_k \in U \quad \text{and} \quad c(x_k) \geq \kappa > 0.
\]

By [39, Theorem 4.1] there exist \( \hat{\kappa}, \alpha > 0 \), dependent on \( j, d, \|c\|_{L^\infty(\mathbb{R}^d)} \), such that

\[
\sup_{x \neq y; x, y \in B_{1/2}(0)} \frac{|w(x) - w(y)|}{|x - y|^{\alpha}} \leq \hat{\kappa} \|w\|_{L^\infty(\mathbb{R}^d)}. \tag{4.7}
\]

Translating the center of the ball it is evident from (4.7) that \( w \in C^\alpha_b(\mathbb{R}^d) \). We note that for some \( \delta > 0 \) we have \( \text{dist}(x_k, U^c) > \delta \). Otherwise, along some subsequence, we must have \( |x_k - z_k| \to 0 \) as \( k \to \infty \), for some \( z_k \in U^c \). Since \( w \in C^\alpha_b(\mathbb{R}^d) \) and \( w \geq 0 \) in \( U^c \), we get

\[
w(x_k) \leq w(z_k) - \hat{\kappa}|x_k - z_k|^\alpha \to 0, \quad \text{as } k \to \infty.
\]
This is a contradiction to (4.6). Thus, we must find $\delta > 0$ so that $B_\delta(x_k) \subseteq U.$

Let us now define $v_k(x) = w(x_k + x).$ Using (4.6) and (4.7), we restrict $\delta$ small enough so that

$$w(y) < \frac{m}{4} \text{ in } B_\delta(x_k), \quad \text{for all } k.$$ 

Thus, by the given condition on $c$, it follows that

$$Lv_k \leq \kappa \frac{m}{4} \text{ in } B_\delta(0). \quad (4.8)$$

Since $\{v_k\}$ forms a equicontinuous family, using Arzelà-Ascoli theorem, we can find a $v \in C_b(\mathbb{R}^d)$ satisfying $v_k \to v$ along some subsequence, uniformly over compacts. Using the stability property of viscosity supersolutions, it follows from (4.8) that

$$Lv \leq \kappa \frac{m}{4} \text{ in } B_\delta(0). \quad (4.9)$$

On the other hand

$$v(0) = \lim_{k \to \infty} v_k(0) = \lim_{k \to \infty} w(x_k) = m \leq \lim_{k \to \infty} w(x + x_k) = v(x),$$

for all $x \in \mathbb{R}^d.$ Thus $x = 0$ is a minimum point for $v$ in $\mathbb{R}^d.$ Then $\varphi \equiv m$ is a bonafide test function at $x = 0.$ From (4.9) we then obtain

$$0 > \frac{m}{4} \kappa \geq L\varphi(0) \geq 0,$$

which is a contradiction. Therefore, we must have $m \geq 0$ which completes the proof. \hfill \Box

Now we can complete the proof of Theorem 4.2.

**Proof of Theorem 4.2.** We broadly follow the idea of [9, 29] without imposing any stronger regularity assumption on $u.$ Fix a unit vector $\nu = (\nu_1, \ldots, \nu_d)$ such that $\nu_d > 0$ and we write $\nu = (\nu', \nu_d).$ We also define

$$\Gamma_h[u](x) = u(x + h\nu) - u(x).$$

We first show that $\Gamma_h[u](x) > 0$ for all $x \in \mathbb{R}^d$ and for all $h > 0.$ Observe that

$$f(u(x + h\nu)) - f(u(x)) = c_h(x)\Gamma_h[u](x),$$

where

$$c_h(x) = \int_0^1 f'((1 - t)u(x) + tu(x + h\nu)) \, dt.$$ 

Using (4.5) we can choose $\delta \in (0, \frac{1}{2})$ such that $f' \geq \kappa_1$ in $(-\infty, -1 + \delta] \cup [1 - \delta, \infty)$ for some $\kappa_1 > 0.$

Again, by (4.4), we may take $M > 0$ satisfying

$$u(x) \geq 1 - \delta \quad \text{for } x_d \geq M, \quad \text{and } u(x) \leq -1 + \delta \quad \text{for } x_d \leq -M. \quad (4.10)$$

**Claim:** If $x \in \{\Gamma_h[u] < 0\} \cap \{x_d \geq M\},$ then $c_h(x) \geq \kappa_1.$

If $x \in \{\Gamma_h[u] < 0\} \cap \{x_d \leq -M\},$ then $u(x + h\nu) < u(x) \leq -1 + \delta,$ by (4.10), which implies

$$(1 - t)u(x) + tu(x + h\nu) \leq -1 + \delta,$$

for all $t \in [0, 1].$ Similarly, if $x \in \{\Gamma_h(u) < 0\} \cap \{x_d \geq M\},$ then $u(x) > u(x + h\nu) \geq 1 - \delta,$ since $x_d + h\nu_d > x_d \geq M.$ Thus

$$(1 - t)u(x) + tu(x + h\nu) \geq 1 - \delta,$$

for all $t \in [0, 1].$ Hence we have $c_h(x) \geq \kappa_1$ for $x \in \{\Gamma_h[u] < 0\} \cap \{x_d \geq M\}.$

Next we claim that if $h \geq \frac{2M}{\nu_d},$ then $\Gamma_h[u](x) > 0$ for any $x \in \mathbb{R}^d.$ Fix any $h \geq \frac{2M}{\nu_d}$ and let $U = \{\Gamma_h(u) < 0\}.$ For $x_d = -M$ we have $x_d + h\nu_d \geq M$ and therefore,

$$\Gamma_h[u](x) \geq \inf_{x_d \geq M} u(x) - \sup_{x_d \leq -M} u(x) \geq 1 - \delta - (1 + \delta) = 2(1 - \delta) > 0.$$
Thus $U \subset \{x_d = -M\}^c$. We write $U = U^- \cup U^+$ where $U^- = U \cap \{x_d < -M\}$ and $U^+ = U \cap \{x_d > -M\}$. By above claim, $c_h(x) \geq \kappa_1$ for $x \in U^-$. Again, for $x \in U^+$ we have $x_d + h \nu_d > M$ by our choice of $h$ and hence, we have $u(x) > u(x + h \nu) \geq 1 - \delta$. Thus

$$(1 - t)u(x) + tu(x + h \nu) \geq 1 - \delta,$$

for all $t \in [0, 1]$. Hence we have $c_h(x) \geq \kappa_1$ for all $x \in U$ and $\Gamma_h[u] \geq 0$ in $U^c$. Applying Theorem 1.2 we also have

$$L \Gamma_h[u] = c_h \Gamma_h[u] \quad \text{in } \mathbb{R}^d.$$

By Lemma 4.1 we then obtain $\Gamma_h[u] \geq 0$ in $\mathbb{R}^d$. We can apply strong maximum principle Theorem 2.3 to get $\Gamma_h(u)(x) > 0$ in $\mathbb{R}^d$, since $\Gamma_h[u] > 0$ on $\{x_d = -M\}$. This proves the claim that $\Gamma_h[u] > 0$ in $\mathbb{R}^d$ and for $h \geq \frac{2M}{\nu_d}$. Define

$$h_o = \inf \{h > 0 : \Gamma_s[u](x) > 0 \text{ for all } x \in \mathbb{R}^d \text{ with } |x_d| \leq M, \text{ for all } s \geq h\}. \quad (4.11)$$

From the above argument we have $h_o \in [0, \frac{2M}{\nu_d}]$. We show that $h_o = 0$. Suppose, to the contrary, that $h_o > 0$. Then for any $\varepsilon \in (0, h_o)$

$$\Gamma_{h_o + \varepsilon}[u](x) = u(x + (h_o + \varepsilon) \nu) - u(x) > 0 \quad \text{for all } x \in \{ |x_d| \leq M \},$$

and

$$\Gamma_{h_o - \varepsilon}[u](x_k) = u(x_k + (h_o - \varepsilon) \nu) - u(x_k) \leq 0 \quad \text{for some } x_k \in \{ |x_d| \leq M \},$$

for any sequence $\varepsilon_k \to 0$ as $k \to \infty$. Since $u$ is continuous, we obtain

$$\Gamma_{h_o}[u](x) = \lim_{\varepsilon \to 0} u(x + (h_o + \varepsilon) \nu) - u(x) \geq 0, \quad (4.12)$$

for any $x \in \{ |x_d| \leq M \}$. Repeating an argument similar to above would give $\Gamma_{h_0}[u] \geq 0$ in $\mathbb{R}^d$. Now we define $w_k = u(x + x_k)$. Since $u \in C^0(\mathbb{R}^d)$ by (4.7), $\{w_k\}$ is equicontinuous and therefore, $w_k \to w_\infty$ along some subsequence, uniformly on compacts. As a consequence,

$$c_k(x) := c_{h_o - \varepsilon_k}(x + x_k) = \int_0^1 f'((1 - t)w_k(x) + tw_k(x + (h_o - \varepsilon_k) \nu)) \, dt$$

$$\to \int_0^1 f'((1 - t)w_\infty(x) + tw_\infty(x + h_o \nu)) \, dt := c_\infty(x)$$

uniformly over compacts. From the stability property of viscosity solution we then obtain

$$L \Gamma_{h_o}[w_\infty](x) = f(w_\infty(x + h_o \nu)(x)) - f(w_\infty(x)) - c_\infty(x) \Gamma_{h_o}[w_\infty](x) \quad \text{in } \mathbb{R}^d.$$

Again, by (4.12)

$$\Gamma_{h_o}[w_\infty](x) = \lim_{k \to \infty} \Gamma_{h_o}[u](x + x_k) \geq 0.$$ 

Also, from (4.7) we have

$$\Gamma_{h_o}[w_\infty](0) = \lim_{k \to \infty} u(x_k + h_o \nu) - u(x_k)$$

$$\leq \lim_{k \to \infty} u(x_k + (h_o - \varepsilon_k) \nu) - u(x_k) + (\varepsilon_k)^\alpha ||u||_{C^0}$$

$$\leq 0.$$

Hence $\Gamma_{h_o}w_\infty(0) = 0$. By strong maximum principle (Theorem 2.3) we must have $\Gamma_{h_o}w_\infty(x) = 0$ for all $x \in \mathbb{R}^d$. By a simple iteration this also gives

$$w_\infty(x + jh_o \nu) = w_\infty(x)$$

for any $x$ and $j \in \mathbb{Z}$. Choosing $j \in \mathbb{N}\cap\{\frac{2M}{\nu_d} \wedge \infty\}$ we see that $j h_o \nu_d + (x_d)_k \geq M$ and $-j h_o \nu_d + (x_d)_k \leq -M$ (since $x_k \in \{ |x_d| \leq M \}$) and therefore, by (4.10) we obtain $u(x_k + jh_o \nu) \geq 1 - \delta$ and $u(x_k - jh_o \nu) \leq -1 + \delta$ for all $k$. Hence

$$2(1 - \delta) \leq \lim_{k \to \infty} u(x_k + jh_o \nu) - u(x_k - jh_o \nu)$$
\[\frac{\partial}{\partial \tau}(x) = w_\infty(jh_\tau) - w_\infty(-jh_\tau) = 0,\]

which is a contradiction. Thus \(h_\circ \) in (4.11) must be 0. In other words, for any \(h > 0\), \(\Gamma_h[u] > 0\) in \(\{|x_d| \leq M\}\). Since \(c_\circ(x) > k_1\) for any \(x \in \{\Gamma_h[u] < 0\} \cap \{|x_d| \geq M\}\), by the same argument as above we have \(\Gamma_h[u] > 0\) in \(\mathbb{R}^d\).

Thus we have proved that \(\Gamma_h^\mu[u](x) := u(x + h\nu) - u(x) \geq 0\) for all \(\nu \in S^{d-1}\) with \(\nu_d > 0\) and all \(h \geq 0\). Taking \(\mu = -\nu\) we obtain for all \(h \geq 0\) that

\[\Gamma_h^\mu[u] \leq 0 \quad \text{for all } x \in \mathbb{R}^d, \quad \text{and for all } \mu \in S^{d-1} \text{ with } \mu_d < 0,\]

as

\[
\Gamma_h^\mu[u](x) = u(x + h\mu) - u(x) = u(\tilde{x}) - u(\tilde{x} + h(-\mu))
\]

\[= -(u(\tilde{x} + h\nu) - u(\tilde{x})) = -\Gamma_h^\mu[u](\tilde{x}) \leq 0,
\]

where \(\tilde{x} = x + h\mu\). Now letting \(\mu_d \nrightarrow 0\) and \(\nu_d \searrow 0\) it follows from above that

\[\Gamma_h^\mu[u] = 0 \quad \text{for all } x \in \mathbb{R}^d, \quad \text{and for all } \omega \in S^{d-1} \text{ with } \omega_d = 0.
\]

In particular, this gives \(\partial_x u = 0\) for \(i = 1, 2, \ldots, d - 1\). Again, \(u_0\) is strictly increasing follows from (4.11) and the fact that \(h_\circ = 0\). This completes the proof of the theorem.

\[\square\]

## 5. Existence and uniqueness results

Our goal in this section is to prove the existence of a unique viscosity solution to

\[Lu = -f \quad \text{in } D, \quad \text{and } u = g \quad \text{in } D^c,\]

where \(f \in C(\overline{D})\) and \(g \in C(D^c)\). Denote by \(C_0^2(x)\) the space of all bounded functions in \(\mathbb{R}^d\) that are twice continuously differentiable in some neighbourhood around \(x\). Also, recall the definition of viscosity solutions from Definition 1.1.

First we prove Theorem 1.2. To do so, we need the notion inf and sup convolution. Given a bounded, upper-semicontinuous function \(u\), the sup-convolution approximation \(u^\varepsilon\) is given by

\[u^\varepsilon(x) = \sup_{y \in \mathbb{R}^d} u(x + y) - \frac{|y|^2}{\varepsilon} = \sup_{y \in \mathbb{R}^d} u(y) - \frac{|x - y|^2}{\varepsilon} = u(x^\ast) - \frac{|x - x^\ast|^2}{\varepsilon}.
\]

Likewise, for a bounded and lower-semicontinuous function \(u\), the inf-convolution \(u_\varepsilon\) is given by

\[u_\varepsilon = \inf_{y \in \mathbb{R}^d} u(x + y) + \frac{|y|^2}{\varepsilon} = \inf_{y \in \mathbb{R}^d} u(y) + \frac{|x - y|^2}{\varepsilon}.
\]

**Lemma 5.1.** Let \(D\) be an open bounded set and \(f\) is continuous function in \(D\). If \(u\) is a bounded, upper-semicontinuous function such that \(Lu \geq f\) in \(D\), then \(Lu^\varepsilon \geq f - d_\varepsilon\) in \(D_1 \subset D\) where \(d_\varepsilon \to 0\) in \(D_1\), as \(\varepsilon \to 0\), and depends on the modulus of continuity of \(f\).

An analogous statement also holds for supersolutions.

**Proof.** Fix \(x_0 \in D_1\), and let \(\varphi\) be a test function that touches \(u^\varepsilon\) from above at \(x_0\) in some neighbourhood \(B_\rho(x_0) \subset D_1\) and \(\varphi = u^\varepsilon\) in \(B_\rho^c(x_0)\). We define

\[Q(x) = \varphi(x + x_0 - x_0^\ast) + \frac{1}{\varepsilon}|x_0 - x_0^\ast|^2.
\]

We observe from the definition of \(u^\varepsilon\) that \(|x_0 - x_0^\ast| \leq M\) where \(M = (2\|u\|_{L^\infty})^{1/2}\). Hence we can pick \(\varepsilon_1\) such that for all \(\varepsilon \leq \varepsilon_1\) and \(x_0 \in D_1\) we have \(x_0^\ast \in D\). It then follows from the definition that \(u(x) \leq u^\varepsilon(x + x_0 - x_0^\ast) + \frac{1}{\varepsilon}|x_0 - x_0^\ast|^2\). Thus, for \(|x - x_0^\ast| < r\) we then get

\[u(x) \leq \varphi(x + x_0 - x_0^\ast) + \frac{1}{\varepsilon}|x_0 - x_0^\ast|^2 = Q(x)
\]
and $u(x_0^*) = Q(x_0^*)$. Hence $Q$ touches $u$ by above at $x_0^*$ in $B_r(x_0^*)$. Define

$$w(x) = \begin{cases} Q(x) & x \in B_r(x_0^*), \\ u(x) & x \in \mathbb{R}^d \setminus B_r(x_0^*). \end{cases}$$

Then by the definition of viscosity subsolution we have $Lw(x_0^*) \geq f(x_0^*)$, that is,

$$\Delta Q(x_0^*) + \int_{\mathbb{R}^d} (w(x_0^* + y) - Q(x_0^*) - 1_{(|y| \leq 1)} y \cdot \nabla Q(x_0^*)) j(y) dy \geq f(x_0^*).$$

Now we observe that $\Delta Q(x_0^*) = \Delta \varphi(x_0)$, $\nabla Q(x_0) = \nabla \varphi(x_0)$ and $Q(x_0^* + y) - Q(x_0^*) = \varphi(x_0 + y) - \varphi(x_0)$. Since $\varphi \geq u^\varepsilon \geq u$, we obtain

$$L \varphi(x_0) \geq f(x_0) - |f(x_0^*) - f(x_0)|.$$ 

Since $|x_0 - x_0^*| \leq M \varepsilon^\frac{1}{4}$, choosing $d_\varepsilon(x) = \sup_{y \in B_M \varphi(x_0)} |f(x_0^*) - f(x_0)|$ gives us the desired result. \(\square\)

Next we show that difference of sub and supersolution gives us a subsolution.

**Lemma 5.2.** Let $D$ is open bounded set, $u$ and $v$ be two bounded functions such that $u$ is upper-semicontinuous and $v$ is lower-semicontinuous in $\mathbb{R}^d$, $Lu \geq f$ and $Lv \leq g$ in $D$ for two continuous function $f$ and $g$ then $L(u - v) \geq f - g$ in $D$.

**Proof.** Fix $D_1 \subseteq D$ and $\varepsilon > 0$. Let $P \in C^2(\mathbb{R}^d)$ be such that $u^\varepsilon - v_\varepsilon \leq P$ in $B_r(x_0) \subset D_1$ and $u^\varepsilon(x_0) - v_\varepsilon(x_0) = P(x_0)$. Without loss of generality we may also assume that $P$ is a paraboloid and $B_{2r}(x_0) \subset D$. Take $\delta > 0$ and define

$$w(x) = v_\varepsilon(x) - u^\varepsilon(x) + \phi(x) + \delta (|x - x_0| \wedge r)^2 - \delta r^2,$$

where $0 < r_1 < \delta \wedge \frac{r}{2}$ and

$$\phi(x) = \begin{cases} P(x) & x \in B_r(x_0) \\ u^\varepsilon(x) - v_\varepsilon(x) & x \in \mathbb{R}^d \setminus B_r(x_0). \end{cases}$$

We see that $w \geq 0$ on $\partial B_{r_1}(x_0)$, $w(x_0) < 0$ and $w > \frac{\delta r^2}{2}$ on $\mathbb{R}^d \setminus B_r(x_0)$. For any $x \in B_{r_1}(x_0)$ there exists a convex paraboloid $P^x$ on opening $K$ such that it touches $w$ from above at $x$ in $B_{r_1}(x)$, where $K(= 4/\varepsilon)$ is a constant independent of $x$. Using [21, Lemma 3.5] and $w(x_0) < 0$ we obtain

$$0 < \int_{A \cap \{w = \Gamma_w\}} \det D^2 \Gamma_w,$$

where $\Gamma_w$ is the convex envelope of $w$ in $B_{2r}(x_0)$ and $A \subset B_{r_1}(x_0)$ satisfies $|B_{r_1}(x_0) \setminus A| = 0$, $\Gamma_w$ is second order differentiable in $A$ and $\Gamma_w \in C^{1,1}(B_r^*)$. Furthermore, $u^\varepsilon$, $v^\varepsilon$ (and hence $w$) are punctually second order differentiable in $A$ [21, Theorem 5.1]. Thus $Lu^\varepsilon(x)$, $Lv^\varepsilon(x)$ are defined in the classical sense for $x \in A$ and from **Lemma 5.1** we have

$$Lu^\varepsilon(x) \geq f(x) - d_\varepsilon \quad \text{and} \quad Lv^\varepsilon(x) \leq g(x) + d_\varepsilon.$$

We note that since the contact set $\{w = \Gamma_w\}$ are the points of minimum for $w - \Gamma_w$ and $w$ is differentiable at the points of $A$ (as it is punctually twice differentiable), we have $\nabla w = \nabla \Gamma_w$ on $A \cap \{w = \Gamma_w\}$. Therefore, since $\Gamma_w$ is convex and $\Gamma_w \leq w$, we have

$$\Delta w(x) \geq 0 \quad \text{and} \quad \int_{x+y \in B_r(x)} (w(x+y) - w(x) - 1_{(|y| \leq 1)} y \cdot \nabla w(x)) j(y) dy \geq 0 \quad \text{for} \ x \in A \cap \{w = \Gamma_w\},$$

using the fact that for $x \in A \cap \{w = \Gamma_w\}$ we have

$$w(x + y) - w(x) - 1_{(|y| \leq 1)} y \cdot \nabla w(x) \geq \Gamma_w(x + y) - \Gamma_w(x) - 1_{(|y| \leq 1)} y \cdot \nabla w(x) \geq 0,$$

for all $x + y \in B_r(x)$. 


Now from (5.2) it follows that
\[ |\{w = \Gamma_w \cap A\} > 0, \]
and therefore, there is one point \(x_1^0 \in \{w = \Gamma_w \cap A\} \) where \(Lu^\varepsilon(x)\), \(Lv^\varepsilon(x)\) can be computed classically. At this point we thus have
\[
f(x_1^0) - d_\varepsilon \leq Lu^\varepsilon(x_1^0) = Lv_\varepsilon(x_1^0) - Lw(x_1^0) + L\phi(x_1^0) + \delta L(|\bullet - x_0| \wedge r)^2(x_1^0)
\leq g(x_1^0) + d_\varepsilon + L\phi(x_1^0) - \int_{|y| \geq r} (w(x_1^0 + y) - w(x_1^0))j(y)dy \]
\[ + \int_{|y| \leq r} y \cdot \nabla \Gamma_w(x_1^0)j(y)dy + \delta L(|\bullet - x_0| \wedge r)^2(x_1^0). \]

Letting \(r_1 \to 0\), we see that \(x_1^0 \to x_0\) and \(\nabla \Gamma_w(x_1^0) \to \nabla \Gamma(x_0)\). Since \(\Gamma_w\) attains its minimum at \(x_0\) we have \(\nabla \Gamma_w(x_0) = 0\). Also, \(w(x_1^0 + y) - w(x_1^0) \to w(x_0 + y) - w(x_0)\). Hence, by the dominated convergence theorem, we have
\[
f(x_0) - d_\varepsilon \leq g(x_0) + d_\varepsilon + L\phi(x_0) - \int_{|y| \geq r} (w(x_0 + y) - w(x_0))j(y)dy + \delta L(|\bullet - x_0| \wedge r)^2(x_0).
\]
Since \(w(x_0 + y) - w(x_0) \geq 0\) for all \(|y| \geq r\), we have
\[
f(x_0) - d_\varepsilon \leq g(x_0) + d_\varepsilon + L\phi(x_0) + \delta L(|\bullet - x_0| \wedge r)^2(x_0).
\]
Now, we let \(\delta \to 0\) to find that
\[
f(x_0) - g(x_0) - 2d_\varepsilon \leq L\phi(x_0).
\]
This gives us \(L(u^\varepsilon - v_\varepsilon) \geq f - g - 2d_\varepsilon\) in \(D\), in the viscosity sense. At the end, we let \(\varepsilon \to 0\) and use the stability property of viscosity solution to obtain our desired result. \(\Box\)

Now applying a standard approximation argument together with Lemma 5.2 we obtain (cf. [22, Theorem 5.9])

**Theorem 5.1.** Let \(D\) be an open bounded set, \(u\) and \(v\) be two bounded functions such that \(u\) is upper-semicontinuous and \(v\) is lower-semicontinuous in \(\overline{D}\) also \(Lu \geq f\) and \(Lv \leq g\) in the viscosity sense in \(D\), for two continuous functions \(f\) and \(g\), then \(L(u - v) \geq f - g\) in \(D\) in the viscosity sense.

Now we can state our comparison result.

**Theorem 5.2.** Let \(u\) be a bounded function in \(\mathbb{R}^d\) which is upper-semicontinuous in \(\overline{D}\) and satisfies \(Lu \geq 0\) in \(D\). Then \(\sup_D u \leq \sup_{D^c} u\).

**Proof.** From [39, Lemma 5.5] we can find a non-negative function \(\chi \in C^2(\overline{D}) \cap C_b(\mathbb{R}^d)\) satisfying
\[
L\chi \leq -1 \quad \text{in} \quad D.
\]
The above equation holds in the classical sense. For \(\varepsilon > 0\), we let \(\phi_M\) to be
\[
\phi_M(x) = M + \varepsilon \chi.
\]
Then \(L\phi_M \leq -\varepsilon\) in \(D\).

Let \(M_0\) be the smallest value of \(M\) for which \(\phi_M \geq u\) in \(\mathbb{R}^d\). We show that \(M_0 \leq \sup_{D^c} u\). Suppose, to the contrary, that \(M_0 > \sup_{D^c} u\). Then there must be a point \(x_0 \in D\) for which \(u(x_0) = \phi_{M_0}(x_0)\). But in that case \(\phi_{M_0}\) would touch \(u\) from above at \(x_0\) and thus, by the definition of the viscosity subsolution, we would have that \(L\phi_{M_0}(x_0) \geq 0\). This is a contradiction. Therefore, \(M_0 \leq \sup_{D^c} u\) which implies that for every \(x \in \mathbb{R}^d\)
\[
u \leq \phi_{M_0} \leq M_0 + \varepsilon \sup_{D^c} \chi \leq \sup u + \varepsilon \sup \chi.
\]
The result follows by taking \(\varepsilon \to 0\). \(\Box\)
Now we prove the existence of viscosity solution to (5.1).

**Lemma 5.3.** Assume (A1) and let $D$ be a bounded Lipschitz domain. Define

$$ u(x) = \mathbb{E}_x \left[ \int_0^\tau f(X_t) \, dt \right] + \mathbb{E}_x[g(X_\tau)], \quad x \in D, $$

where $\tau$ denotes the first exit time of $X$ from $D$. Then $u \in \mathcal{C}_b(\mathbb{R}^d)$ and solves (5.1) in the viscosity sense.

**Proof.** Since $u(x) = g(x)$ in $D^c$, we only need to show that $u \in \mathcal{C}(\bar{D})$. Let $x_n \in D$ and $x_n \to z \in \bar{D}$. Define

$$ \tau_n = \inf \{ t > 0 : X^t_n = x_n + X_t \notin D \}. $$

Here $\tau_n$ is the first exit time of a process starting from $x_n$. In a similar manner one can define the first exit time $\tau_z$ of a process starting from $z$ as

$$ \tau_z = \inf \{ t > 0 : X^t_z = z + X_t \notin D \}. $$

First suppose that $z \in \partial D$. Since $D$ is Lipschitz, it satisfies the exterior cone condition and hence regular with respect to $X$ [38, 49]. This means $P_z(\tau_D = 0) = 1$. Therefore, for every $\delta > 0$, $X^z$ intersects $(\bar{D})^c$ before time $\delta$, almost surely. Since

$$ \sup_{t \in [0,M]} |x_n + X_t - (z + X_t)| \leq |x_n - z| \to 0, \quad \text{as} \quad n \to \infty, $$

for every fixed $M$, it implies that $\tau_n \to 0$ and $X^D_n \to z$, almost surely. Therefore, using Lemma 2.1 and dominated convergence theorem, it follows that $u(x_n) \to g(z)$ as $n \to \infty$.

For the remaining part we assume that $z \in D$. For a fixed $M > 0$, next we show that $\tau_n \wedge M \to \tau_z \wedge M$ almost surely. Denote by $\Omega_M$ the event in (5.3). Then $\mathbb{P}(\Omega_M) = 1$. Since $D$ is regular we have $P(\tau_z = \tau_z) = 1$, where $\tau_z$ is the first exit time of a process from $\bar{D}$ starting at $z$. Denote by $\tilde{\Omega} = \{ \tau_z = \tau_z \}$. Let $\varepsilon > 0$. We claim that, on $\Omega_M \cap \tilde{\Omega}$, $\tau_n \wedge M \leq \tau_z \wedge M + \varepsilon$ for all large $n$. Also, we only need to show it on $\{ \tau_z < M \}$. For $\omega \in \Omega_M \cap \tilde{\Omega} \cap \{ \tau_z < M \}$, there exists $s \in [\tau_z(\omega), \tau_z(\omega) + \frac{\varepsilon}{2}]$ such that $X^z_s(\omega) \in \overline{D^c}$ which implies

$$ \text{dist}(X^z_s(\omega), \overline{D}) > 0. $$

By (5.3), it then implies that

$$ \tau_n(\omega) \leq s \leq \tau_z(\omega) + \frac{\varepsilon}{2}, $$

for all large $n$. This proves the claim.

Next we show that, on $\tilde{\Omega} \cap \Omega_M$, we have $\tau_z \wedge M - \varepsilon \leq \tau_n \wedge M$ for all large $n$. Now since $\tau_z(\omega) \wedge M - \varepsilon < \tau_z(\omega)$, for all $s \in [0, \tau_z(\omega) \wedge M - \varepsilon]$ we have $X^z_s(\omega) \in D^c$. Applying (5.3) we get $X^D_n \in D^c$ for all $s \in [0, \tau_z(\omega) \wedge M - \varepsilon]$ and for all large $n$. Thus $\tau_z \wedge M - \varepsilon \leq \tau_n \wedge M$ for all large $n$. Thus for every $\varepsilon > 0$ and $\omega \in \Omega_M \cap \tilde{\Omega}$, we have $N(\omega)$ satisfying

$$ \tau_z(\omega) \wedge M - \varepsilon \leq \tau_n(\omega) \wedge M \leq \tau_z(\omega) \wedge M + \varepsilon $$

for all $n \geq N(\omega)$. Hence we proved $M \wedge \tau_n \to M \wedge \tau_z$ pointwise in $\omega \in \Omega_M \cap \tilde{\Omega}$, as $n \to \infty$. Since $M$ is arbitrary, this also implies $\tau_n \to \tau_z$ almost surely.

Next we want to show that exit location converges, that is, $X^D_n \to X^z$ as $n \to \infty$, almost surely. We recall the Lévy system formula (cf. [18, p. 65]),

$$ \mathbb{E}_x \left[ \sum_{0 \leq s \leq t} f(X_{s^-}, X_s)1_{\{X_{s^-} \neq X_s\}} \right] = \mathbb{E}_x \left[ \int_0^t \int_{\mathbb{R}^d} f(X_s, y)j(y - X_s) \, dy \, ds \right] $$

(5.4)
which holds for all non-negative \( f \in \mathcal{B}_b(\mathbb{R}^d \times \mathbb{R}^d) \). Now we put \( f_\varepsilon(x, y) = \mathbb{1}_{\{x \in D\}} \mathbb{1}_{\{|x-y| > \varepsilon\}} \mathbb{1}_{\{y \in \partial D\}} \), in the above formula and using the fact \( |\partial D| = 0 \) we obtain

\[
\mathbb{E}_x \left[ \sum_{0 < s \leq t} f_\varepsilon(X_{s-}, X_s) \mathbb{1}_{\{X_{s-} \neq X_s\}} \right] = 0, \quad \text{for all } \varepsilon > 0.
\]

Since \( \varepsilon \) and \( t \) are arbitrary, this gives us

\[
\mathbb{P}_x(\{X_{\tau_z} \in \partial D, X_{\tau_z-} \neq X_{\tau_z}\}) = 0. \tag{5.5}
\]

Again, choosing

\[
\hat{f}_\varepsilon(x, y) = \mathbb{1}_{\{x \in \partial D\}} \mathbb{1}_{\{y \in D^+_\varepsilon\}}, \quad \text{for } D^+_\varepsilon = \{y : \text{dist}(y, D) > \varepsilon\}
\]

in (5.4) and since \( X_s \) has transition density (see Lemma 2.2), it follows that

\[
\mathbb{E}_x \left[ \sum_{0 < s \leq t} \hat{f}_\varepsilon(X_{s-}, X_s) \mathbb{1}_{\{X_{s-} \neq X_s\}} \right] = 0.
\]

Since \( \varepsilon, t \) are arbitrary, we get

\[
\mathbb{P}_x(\{X_{\tau_z} \in \partial D, X_{\tau_z} \in (\bar{D})^c\}) = 0. \tag{5.6}
\]

We claim that for any \( M > 0 \) we have

\[
X_{\tau_z \wedge M}^n \to X_{\tau_z \wedge M}^z \quad \text{as } n \to \infty, \text{ almost surely.}
\]

We will be interested in the case where \( \tau_z < M \) since, given \( t = M \), function \( t \mapsto X_t \) is almost surely continuous at \( t = M \). Now fix \( \omega \) so that it is in the complement of the events in (5.5) and (5.6). Since process \( X_t \) is càdlàg (right continuous with a finite left limit) and \( \tau_n \wedge M \to \tau_z \wedge M \) almost surely, we only need to consider a situation where \( \tau_n \wedge M \not\nearrow \tau_z \wedge M \). On set \( \{\tau_z < M\} \), we have \( \tau_n \not\nearrow \tau_z \).

If \( t \to X^n_t(\omega) \) is continuous at \( \tau_z(\omega) = t \), then we have the claim. So we let \( X_{\tau_z-}^z(\omega) \neq X_{\tau_z}^z(\omega) \). Since \( \omega \) is in the complement of the events in (5.5) and (5.6), we have \( X_{\tau_z-}^z(\omega) \in D^c \) and \( X_{\tau_z}^z(\omega) \in \overline{D}^c \). But \( x_n \to z \) and \( X^z \restriction_{[0, \tau_z]}(\omega) \) is in \( D^c \), then \( X^n \restriction_{[0, \tau_n]}(\omega) \) is in \( D^c \) for large \( n \), contradicting the fact that \( X^n_{\tau_n}(\omega) \in D^c \). So this case can not happen and we get that \( X^n_{\tau_n \wedge M} \to X_{\tau_z \wedge M}^z \) as \( n \to \infty \), almost surely. Since \( M \) is arbitrary and \( \tau_n \to \tau_z \), it gives us

\[
X_{\tau_n} \to X_{\tau_z}^z \quad \text{as } n \to \infty, \text{ almost surely.} \tag{5.7}
\]

Now we are ready to show that \( u(x_n) \to u(z) \), Applying dominated convergence theorem and using (5.7) we get

\[
\mathbb{E}_{x_n}[g(X_T^n)] \to \mathbb{E}_x[g(X_T)]
\]

Since \( \tau_n \wedge M \to \tau_z \wedge M \) almost surely and for all \( \omega, \sup_{t \in [0, M]} |X^n_t - X^n_T| \to 0 \) (see (5.3)), we have

\[
\int_0^{\tau_n \wedge M} f(X^n_s)ds \to \int_0^{\tau_z \wedge M} f(X^z_s)ds,
\]

almost surely. Hence by dominated convergence we have

\[
\mathbb{E} \left[ \int_0^{\tau_n \wedge M} f(X^n_s)ds \right] \to \mathbb{E} \left[ \int_0^{\tau_z \wedge M} f(X^z_s)ds \right].
\]

Now since \( M \) is arbitrary, using Lemma 2.1, it follows that

\[
\mathbb{E} \left[ \int_0^{\tau_n} f(X^n_s)ds \right] \to \mathbb{E} \left[ \int_0^{\tau_z} f(X^z_s)ds \right].
\]

This completes the proof.

It is standard to show that \( u \) is a viscosity solution (cf. [15, Remark 3.2] [43, Theorem 2.2]).
Now we can complete the proof of Theorem 1.1.

Proof of Theorem 1.1. Proof follows from Theorems 5.1 and 5.2 and Lemma 5.3.

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