Lie symmetry reductions and conservation laws for fractional order coupled KdV system

Hossein Jafari¹, Hong Guang Sun² and Marzieh Azadi³,⁴*

Abstract

Lie symmetry analysis is achieved on a new system of coupled KdV equations with fractional order, which arise in the analysis of several problems in theoretical physics and numerous scientific phenomena. We determine the reduced fractional ODE system corresponding to the governing fractional PDE system.

In addition, we develop the conservation laws for the system of fractional order coupled KdV equations.

MSC: 35R11; 76M60

Keywords: Reduced fractional system; Coupled KdV system; Riemann–Liouville derivative; Lie symmetry method; Conservation laws

1 Introduction

Fractional partial differential equations (FPDEs) have a significant role to play in various fields such as chemistry, physics, fluid dynamics and biology, therefore obtaining solutions of such FPDEs is unavoidable [1, 2]. There are many numerical and theoretical methods for solving fractional order differential equations [1–4].

The Lie symmetry technique is one of the most useful techniques to conclude to solutions of nonlinear FPDEs, generally, Lie symmetries might be used to reduce the order of the original equation (system of equations) as well as the number of independent variables [5–11].

Lie symmetry analysis and conservation law have been applied to different type of fractional PDEs such as the time-fractional Caudrey–Dodd–Gibbon–Sawada–Kotera equation, time-fractional third-order evolution equation, the space-time-fractional nonlinear evolution equations, the time-fractional modified Zakharov–Kuznetsov equation, the time-fractional generalized Burgers–Huxley equation and the time-fractional dispersive long-wave equation [12–17]. In [8] the new coupled KdV system

\[
\begin{align*}
    u_t + u_{xxx} + 3uu_x + 3ww_x &= 0, \\
    v_t + v_{xxx} + 3vv_x + 3ww_x &= 0, \\
    w_t + w_{xxx} + \frac{3}{2}(uw)_x + \frac{3}{2}(vw)_x &= 0,
\end{align*}
\]

(1)

was derived and examined by the Lie symmetry method.
Unlike the case of ordinary partial differential equations (PDEs) symmetries of FPDEs have not considered comprehensively. The study of FPDEs through symmetries is remarkable and interesting [18–22].

In this paper, we consider the new coupled KdV system (1) of fractional order given by

\[
\begin{align*}
D_t^\alpha u + u_{xxx} + 3uu_x + 3ww_x &= 0, \\
D_t^\alpha v + v_{xxx} + 3vv_x + 3ww_x &= 0, \\
D_t^\alpha w + w_{xxx} + \frac{3}{2}(uw)_x + \frac{3}{2}(vw)_x &= 0,
\end{align*}
\]

where \(\alpha \in (0, 2)\).

The article is organized as follows. In Sect. 2, some definitions and properties of Lie group scheme to analysis of (2) are given. In Sect. 3, we find Lie point symmetries of system (2) and reduced system of this system. The conservation laws of (2) are obtained in Sect. 4. Discussion and conclusions are summarized in Sect. 5.

2 The symmetry group analysis of (2)

In this section, we briefly review some key definitions and properties of the fractional Lie group scheme to obtain infinitesimal function of the FPDE system.

Definition 1 The Riemann–Liouville fractional derivative of order \(\alpha\) [1, 2] is defined by

\[
D_t^\alpha u(x, t) = \frac{\partial^\alpha u(x, t)}{\partial t^\alpha} = \begin{cases} 
\frac{\partial^n u(x, t)}{\partial t^n}, & \alpha = n \in \mathbb{N}, \\
\frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial \tau^{n-\alpha}} \int_0^\tau u(x, \tau) (t-\tau)^{-\alpha} d\tau, & n-1 < \alpha < n.
\end{cases}
\]

For a fractional PDE system like (2) with two independent variables we have

\[
\begin{align*}
\frac{\partial^\alpha u(x, t)}{\partial t^\alpha} &= F(x, t, u, v, w, u_{(1)}, v_{(1)}, w_{(1)}, \ldots), \\
\frac{\partial^\alpha v(x, t)}{\partial t^\alpha} &= G(x, t, u, v, w, u_{(1)}, v_{(1)}, w_{(1)}, \ldots), \\
\frac{\partial^\alpha w(x, t)}{\partial t^\alpha} &= H(x, t, u, v, w, u_{(1)}, v_{(1)}, w_{(1)}, \ldots),
\end{align*}
\]

(3)

Throughout the article we use \((\bar{x}, \bar{u})\) instead of \((x, t, u, v, w)\).

Assume that (3) is invariant under the one parameter Lie group of infinitesimal transformations,

\[
\begin{align*}
t^* &= t + \epsilon \tau(\bar{x}, \bar{u}) + O(\epsilon^2), \\
x^* &= x + \epsilon \bar{x}(\bar{x}, \bar{u}) + O(\epsilon^2), \\
u^* &= u + \epsilon \eta^u(\bar{x}, \bar{u}) + O(\epsilon^2), \\
v^* &= v + \epsilon \eta^v(\bar{x}, \bar{u}) + O(\epsilon^2), \\
w^* &= w + \epsilon \eta^w(\bar{x}, \bar{u}) + O(\epsilon^2),
\end{align*}
\]

\[
\begin{align*}
D_t^\alpha u^* &= D_t^\alpha u + \epsilon \eta^{(\alpha)u}(\bar{x}, \bar{u}) + O(\epsilon^2), \\
D_t^\alpha v^* &= D_t^\alpha v + \epsilon \eta^{(\alpha)v}(\bar{x}, \bar{u}) + O(\epsilon^2), \\
D_t^\alpha w^* &= D_t^\alpha w + \epsilon \eta^{(\alpha)w}(\bar{x}, \bar{u}) + O(\epsilon^2),
\end{align*}
\]
\[
\frac{\partial^{j}u}{\partial x^{j}} = \frac{\partial^{j}u}{\partial x^{j}} + \epsilon \eta^{(j)\alpha}(\tilde{x}, \tilde{u}) + O(\epsilon^2), \\
\frac{\partial^{j}v}{\partial x^{j}} = \frac{\partial^{j}v}{\partial x^{j}} + \epsilon \eta^{(j)\beta}(\tilde{x}, \tilde{u}) + O(\epsilon^2), \\
\frac{\partial^{j}w}{\partial x^{j}} = \frac{\partial^{j}w}{\partial x^{j}} + \epsilon \eta^{(j)\gamma}(\tilde{x}, \tilde{u}) + O(\epsilon^2), \quad j = 1, 2, \ldots
\] (4)

where \( \xi, \tau, \eta^{\alpha}, \eta^{\beta}, \eta^{\gamma} \) are infinitesimals and \( \eta^{(a)u}, \eta^{(a)v}, \eta^{(a)w}, \eta^{(a)\alpha}, \eta^{(a)\beta}, \eta^{(a)\gamma} \) are extended infinitesimals. \( \epsilon \) is the group parameter.

According to Lie's algorithm, the infinitesimal generator of (2) is given by

\[
X = \xi(\tilde{x}, \tilde{u}) \frac{\partial}{\partial x} + \tau(\tilde{x}, \tilde{u}) \frac{\partial}{\partial t} + \eta^{\alpha}(\tilde{x}, \tilde{u}) \frac{\partial}{\partial u} + \eta^{\beta}(\tilde{x}, \tilde{u}) \frac{\partial}{\partial v} + \eta^{\gamma}(\tilde{x}, \tilde{u}) \frac{\partial}{\partial w} + \epsilon \sum_{j=1}^{\infty} \left[ \eta^{(j)\alpha}(\tilde{x}, \tilde{u}) \frac{\partial}{\partial u} + \eta^{(j)\beta}(\tilde{x}, \tilde{u}) \frac{\partial}{\partial v} + \eta^{(j)\gamma}(\tilde{x}, \tilde{u}) \frac{\partial}{\partial w} \right] + \ldots
\] (5)

The coupled KdV system of fractional order has at most \( \alpha \)-th order derivatives, therefore, the \( \alpha \)-prolongation of the generator should be considered in the form

\[
X^{(\alpha)} = \xi(\tilde{x}, \tilde{u}) \frac{\partial}{\partial x} + \tau(\tilde{x}, \tilde{u}) \frac{\partial}{\partial t} + \eta^{\alpha}(\tilde{x}, \tilde{u}) \frac{\partial}{\partial u} + \eta^{\beta}(\tilde{x}, \tilde{u}) \frac{\partial}{\partial v} + \eta^{\gamma}(\tilde{x}, \tilde{u}) \frac{\partial}{\partial w} + \epsilon \sum_{j=1}^{\infty} \left[ \eta^{(j)\alpha}(\tilde{x}, \tilde{u}) \frac{\partial}{\partial u} + \eta^{(j)\beta}(\tilde{x}, \tilde{u}) \frac{\partial}{\partial v} + \eta^{(j)\gamma}(\tilde{x}, \tilde{u}) \frac{\partial}{\partial w} \right] + \ldots
\] (6)

where

\[
\eta^{(a)u} = D_{1t}^{(\alpha)}(\eta^{u}) + \xi D_{1t}^{(\alpha)}(\eta^{u}) + \tau D_{1t}^{(\alpha)}(\eta^{u}) - \tau D_{1t}^{(\alpha+1)}(\eta^{u}) + \tau D_{1t}^{(\alpha+1)}(\eta^{u}), \\
\eta^{(a)v} = D_{2t}^{(\alpha)}(\eta^{v}) + \xi D_{2t}^{(\alpha)}(\eta^{v}) - \tau D_{2t}^{(\alpha+1)}(\eta^{v}) + \tau D_{2t}^{(\alpha+1)}(\eta^{v}), \\
\eta^{(a)w} = D_{3t}^{(\alpha)}(\eta^{w}) + \xi D_{3t}^{(\alpha)}(\eta^{w}) - \tau D_{3t}^{(\alpha+1)}(\eta^{w}) + \tau D_{3t}^{(\alpha+1)}(\eta^{w}).
\]

\( D_{1t}, D_{2t} \) and \( D_{3t} \) are the total derivative operators defined as

\[
D_{1t} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial u} + u_{1t} \frac{\partial}{\partial u_{1}} + u_{2t} \frac{\partial}{\partial u_{2}} + \ldots,
\]
\[ D_{2t} = \frac{\partial}{\partial t} + v_t \frac{\partial}{\partial v} + v_xt \frac{\partial}{\partial v_x} + v_xtt \frac{\partial}{\partial v_{xt}} + \cdots, \]
\[ D_{3t} = \frac{\partial}{\partial t} + w_t \frac{\partial}{\partial w} + w_xt \frac{\partial}{\partial w_x} + w_xtt \frac{\partial}{\partial w_{xt}} + \cdots. \]

**Definition 2** A vector \( X \) given by (5) is said to be a Lie point symmetry vector field for system (2), if

\[ X^{(a)} \left[ D_t^t u + u_{xxx} + 3u u_x + 3w w_x \right] = 0, \]
\[ X^{(a)} \left[ D_t^t v + v_{xxx} + 3v v_x + 3w w_x \right] = 0, \]
\[ X^{(a)} \left[ D_t^t w + w_{xxx} + \frac{3}{2} (uw)_x + \frac{3}{2} (vw)_x \right] = 0. \]

3 Lie symmetries and similarity reductions for (2)

We apply the \( \alpha \)-prolongation of \( X^{(a)} \) to Eq. (2). It gives the following claim.

**Theorem 1** Lie symmetry group of (2) is spanned by the following vector fields:

\[ X_1 = \frac{\partial}{\partial x}, \quad X_2 = \alpha x \frac{\partial}{\partial x} + 3t \frac{\partial}{\partial t} - 2\alpha u \frac{\partial}{\partial u} - 2\alpha v \frac{\partial}{\partial v} - 2\alpha w \frac{\partial}{\partial w}. \]  

**Proof** Let us consider the one parameter Lie group of infinitesimal transformation in \( x, t, u, v, w \) given by

\[ t \rightarrow t + \epsilon \xi'(\tilde{x}, \tilde{u}), \]
\[ x \rightarrow x + \epsilon \xi'(\tilde{x}, \tilde{u}), \]
\[ v \rightarrow v + \epsilon \eta'(\tilde{x}, \tilde{u}), \]
\[ u \rightarrow u + \epsilon \eta'(\tilde{x}, \tilde{u}), \]
\[ w \rightarrow w + \epsilon \eta'(\tilde{x}, \tilde{u}), \]

now we find the coefficient functions \( \xi, \tau, \eta^u, \eta^v, \eta^w \).

By applying the \( X^{(a)} \) to both sides of (2), we have

\[ X^{(a)} \left[ D_t^t u + u_{xxx} + 3u u_x + 3w w_x \right] = 0, \]
\[ X^{(a)} \left[ D_t^t v + v_{xxx} + 3v v_x + 3w w_x \right] = 0, \]
\[ X^{(a)} \left[ D_t^t w + w_{xxx} + \frac{3}{2} (uw)_x + \frac{3}{2} (vw)_x \right] = 0. \]  

We obtain the Lie point symmetries by expanding (8), and solving the resulting system using Maple as follows:

\[ \xi'(\tilde{x}, \tilde{u}) = c_1 + c_2 \alpha x, \quad \tau'(\tilde{x}, \tilde{u}) = 3c_2 t, \]
\[ \eta^u'(\tilde{x}, \tilde{u}) = -2c_2 \alpha u, \quad \eta^v'(\tilde{x}, \tilde{u}) = -2c_2 \alpha v, \quad \eta^w'(\tilde{x}, \tilde{u}) = -2c_2 \alpha w, \]
here $c_1$ and $c_2$ are arbitrary constants. Thus, the corresponding vector fields are

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = ax \frac{\partial}{\partial x} + 3t \frac{\partial}{\partial t} - 2axu \frac{\partial}{\partial u} - 2av \frac{\partial}{\partial v} - 2aw \frac{\partial}{\partial w}.$$  

Here we want to obtain symmetry reductions of (2), then the system (2) transforms into a system of fractional ODE.

In order to solve the following associated Lagrange equations:

$$\frac{dx}{\xi(x, u)} = \frac{dt}{\tau(x, u)} = \frac{du}{\eta^u(x, u)} = \frac{dv}{\eta^v(x, u)} = \frac{dw}{\eta^w(x, u)}.$$  

We consider the following cases.

- **Case 1**: $X_1 = \frac{\partial}{\partial x}$.

  In this case the symmetry $X_1$ gives rise to the group-invariant solution:

  $$r = t, \quad u = F(r), \quad v = G(r), \quad w = H(r),$$

  (9)

  substituting (9) into (2) results in the fact that $F(r)$, $G(r)$ and $H(r)$ fulfill the following differential equations:

  $$\frac{d^\alpha F(t)}{dt^\alpha} = 0, \quad \frac{d^\alpha G(t)}{dt^\alpha} = 0, \quad \frac{d^\alpha H(t)}{dt^\alpha} = 0.$$  

  By using a Laplace transformation we get

  $$F(t) = \frac{k_1}{\Gamma(\alpha)} t^{\alpha-1}, \quad G(t) = \frac{k_2}{\Gamma(\alpha)} t^{\alpha-1}, \quad H(t) = \frac{k_3}{\Gamma(\alpha)} t^{\alpha-1},$$

  where $k_1$, $k_2$ and $k_3$ are constant; therefore

  $$u(x, t) = \frac{k}{\Gamma(\alpha)} t^{\alpha-1}, \quad v(x, t) = \frac{k}{\Gamma(\alpha)} t^{\alpha-1}, \quad w(x, t) = \frac{k}{\Gamma(\alpha)} t^{\alpha-1}.$$

- **Case 2**: $X_1 = ax \frac{\partial}{\partial x} + 3t \frac{\partial}{\partial t} - 2axu \frac{\partial}{\partial u} - 2av \frac{\partial}{\partial v} - 2aw \frac{\partial}{\partial w}.$

  In this case, the group-invariant solution is

  $$r = tx^{\frac{3}{\alpha}}, \quad u = F(r)x^{-2}, \quad v = G(r)x^{-2}, \quad w = H(r)x^{-2},$$

  (10)

  substituting (10) into (2) leads to the following fractional ODE system:

  $$\frac{D^\alpha r}{dt^\alpha} F + k_1 F + k_2 r F^2 + k_3 r^2 F''(r) + k_4 r^3 F^{(3)}(r) + k_5 F^2(r) + k_6 r F(r)F'(r) + k_7 H^2(r) + k_8 r H(r)H'(r) = 0,$$

  $$\frac{D^\alpha r}{dt^\alpha} G + k'_1 G + k'_2 r G^2(r) + k'_3 r^2 G''(r) + k'_4 r^3 G^{(3)}(r) + k'_5 G^2(r) + k'_6 r G(r)G'(r) + k'_7 H^2(r) + k'_8 r H(r)H'(r) = 0,$$

  $$\frac{D^\alpha r}{dt^\alpha} H + k''_1 H(r) + k''_2 r H^2(r) + k''_3 r^2 H''(r) + k''_4 r^3 H^{(3)}(r) + k''_5 F(r)H(r) + k''_6 r F'(r)H'(r) + k''_7 r F(r)H''(r) + k''_8 r G(r)H'(r) + k''_9 G(r)H''(r) + k''_{10} r G'(r)H''(r) = 0,$$

  where $k_i = h_i(\alpha)$, $k'_i = g_i(\alpha)$, ($i = 1, 2, \ldots, 8$) and $k''_i = m_i(\alpha)$, ($j = 1, 2, \ldots, 10$) are constants.

  Note. For $\alpha = 1$, the Lie point symmetries provide similar results to those obtained by Adem and Khalique in [8].
4 Conservation laws

Now, we construct conservation laws for system (2) by using the Lie point symmetry (7).

The vectors $C_i = (C_i^1, C_i^2, C_i^3)$, ($i = 1, 2, 3$) are called conserved vectors for system (2), if they satisfy the conservation equations,

$$
D_t(C_i^1) + D_x(C_i^2)D_{xx} + u_{xxx} + 3u_{wx} + 3ww_x = 0, \\
D_t(C_i^2) + D_x(C_i^2)D_{xx} + v_{xxx} + 3v_{wx} + 3ww_x = 0, \\
D_t(C_i^3) + D_x(C_i^3)D_{xx} + w_{xxx} + 3(wu)_x + 3(vw)_x = 0.
$$

For system (2), a formal Lagrangian can be introduced as

$$
L = \Lambda^1(x, t)[D_t^1u + u_{xxx} + 3u_{wx} + 3ww_x] + \Lambda^2(x, t)[D_t^2v + v_{xxx} + 3v_{wx} + 3ww_x] + \Lambda^3(x, t)[D_t^3w + w_{xxx} + 3(uw)_x + 3(vw)_x] = 0,
$$

where $\Lambda^i(x, t)$, ($i = 1, 2, 3$), are new dependent variables.

The Euler–Lagrange operators are defined by

$$
\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + (D_t^1)^* \frac{\partial}{\partial D_t^1 u} - D_x \frac{\partial}{\partial u_x} + D_{xx} \frac{\partial}{\partial u_{xx}} - D_{xxx} \frac{\partial}{\partial u_{xxx}}, \\
\frac{\delta}{\delta v} = \frac{\partial}{\partial v} + (D_t^2)^* \frac{\partial}{\partial D_t^2 v} - D_x \frac{\partial}{\partial v_x} + D_{xx} \frac{\partial}{\partial v_{xx}} - D_{xxx} \frac{\partial}{\partial v_{xxx}}, \\
\frac{\delta}{\delta w} = \frac{\partial}{\partial w} + (D_t^3)^* \frac{\partial}{\partial D_t^3 w} - D_x \frac{\partial}{\partial w_x} + D_{xx} \frac{\partial}{\partial w_{xx}} - D_{xxx} \frac{\partial}{\partial w_{xxx}},
$$

here $(D_t^i)^*$ is the adjoint operator of $D_t^i$.

For the RL-fractional operators

$$(D_t^i)^* = (-1)^nI_t^{\alpha-n}(D_t^i) \equiv C_i D_t^i,$$

where

$$I_t^{\alpha-n}f(t, x) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f(\tau, x)}{(t-\tau)^{1+n-\alpha}} d\tau, \quad n = [\alpha] + 1.$$

The adjoint equations to the system (2) are written as

$$
F_1^* = \frac{\delta L}{\delta u} = 0, \quad F_2^* = \frac{\delta L}{\delta v} = 0, \quad F_3^* = \frac{\delta L}{\delta w} = 0.
$$

Replacing the formal Lagrangian (11) into (12), we have

$$
F_1^* = (D_t^1)^* \Lambda^1 - 3u\Lambda^1_x - \Lambda^1_{xxx} + 3\frac{1}{2}w\Lambda^3 - \frac{3}{2}w\Lambda^3_x = 0, \\
F_2^* = (D_t^2)^* \Lambda^2 - 3v\Lambda^2_x - \Lambda^2_{xxx} + 3\frac{1}{2}w\Lambda^3 - \frac{3}{2}w\Lambda^3_x = 0, \\
F_3^* = (D_t^3)^* \Lambda^3 - 3w\Lambda^1_x - 3w\Lambda^2_x + \frac{3}{2}(w_u)_x + \Lambda^3 - \frac{3}{2}(u + v)\Lambda^3_x - \Lambda^3_{xxx} = 0.
$$

(13)
Since in the system (2), there are no fractional derivatives involved w.r.t. \( x \), we have

\[
X^{(a)} + D_{1\tau}(\tau)L + D_{1x}(\xi)L = w \frac{\partial}{\partial u} + D_{1\tau}N_1^u + D_{1x}N_2^u,
\]

\[
X^{(a)} + D_{2\tau}(\tau)L + D_{2x}(\xi)L = w \frac{\partial}{\partial v} + D_{2\tau}N_1^v + D_{2x}N_2^v,
\]

\[
X^{(a)} + D_{3\tau}(\tau)L + D_{3x}(\xi)L = w \frac{\partial}{\partial w} + D_{3\tau}N_1^w + D_{3x}N_2^w,
\]

where

\[
W_i = (\eta^u + \eta^v + \eta^w) - \xi_i(u_x + v_x + w_x) - \tau_i(u_x + v_x + w_x), \quad i = 1, 2,
\]

are the Lie characteristic functions corresponding to the Lie symmetries \( X_1 \) and \( X_2 \).

If we have the RL-time-fractional derivative in the system (2) then the operators \( N^\alpha \) are given by

\[
N_1^u = \sum_{k=0}^{n-1} (-1)^k D_{1\tau}^{-\eta -1-k}(w)D_{1\tau}^{k} \left( \frac{\partial}{\partial D_{1\tau}^\alpha u} \right) \left( \frac{\partial}{\partial D_{1\tau}^\alpha u} \right),
\]

\[
N_2^v = \sum_{k=0}^{n-1} (-1)^k D_{2\tau}^{-\eta -1-k}(w)D_{2\tau}^{k} \left( \frac{\partial}{\partial D_{2\tau}^\alpha v} \right) \left( \frac{\partial}{\partial D_{2\tau}^\alpha v} \right),
\]

\[
N_3^w = \sum_{k=0}^{n-1} (-1)^k D_{3\tau}^{-\eta -1-k}(w)D_{3\tau}^{k} \left( \frac{\partial}{\partial D_{3\tau}^\alpha w} \right) \left( \frac{\partial}{\partial D_{3\tau}^\alpha w} \right),
\]

where \( J \) is the integral

\[
J(f, g) = \frac{1}{\Gamma(n - \alpha)} \int_{0}^{t} \int_{\mu}^{t} f(\tau, x)g(\mu, x) \frac{d\mu}{(\mu - \tau)^{n+1}} \, d\tau;
\]

the operators \( N^\alpha \) are defined by

\[
N_1^u = w \frac{\partial L}{\partial u_x} + D_{1x}(W_i) \frac{\partial L}{\partial u_{xx}} + D_{1\tau}^{2}(W_i) \frac{\partial L}{\partial u_{xxx}},
\]

\[
N_2^v = w \frac{\partial L}{\partial v_x} + D_{2x}(W_i) \frac{\partial L}{\partial v_{xx}} + D_{2\tau}^{2}(W_i) \frac{\partial L}{\partial v_{xxx}},
\]

\[
N_3^w = w \frac{\partial L}{\partial w_x} + D_{3x}(W_i) \frac{\partial L}{\partial w_{xx}} + D_{3\tau}^{2}(W_i) \frac{\partial L}{\partial w_{xxx}}.
\]

For any generator \( X \) of system (2), we have

\[
(X^{(a)}L + D_{1\tau}(\tau)L + D_{1x}(\xi)L) |_{D_{1\tau}^{\eta}u_{xxx} + 3\mu u_{xx} + 3\mu u_{x} = 0} = 0,
\]

\[
(X^{(a)}L + D_{2\tau}(\tau)L + D_{2x}(\xi)L) |_{D_{2\tau}^{\eta}v_{xxx} + 3\mu v_{xx} + 3\mu v_{x} = 0} = 0,
\]

\[
(X^{(a)}L + D_{3\tau}(\tau)L + D_{3x}(\xi)L) |_{D_{3\tau}^{\eta}w_{xxx} + 3\mu w_{xx} + 3\mu w_{x} = 0} = 0.
\]

These equalities yield the conservation laws

\[
D_{1\tau}(N_1^u L) + D_{1x}(N_1^u L) = 0,
\]
\[ D_{2x}(N_i^2 L) + D_{2x}(N_i^2 L) = 0, \]
\[ D_{3x}(N_i^2 L) + D_{3x}(N_i^2 L) = 0. \]

For the case, when \( \alpha \in (0, 1) \), using \( N_i^1 \) and \( N_i^2 \) \( (i = 1, 2, 3) \), one can get the components of the conserved vectors

\[
C_i^1 = (-1)^0 D_{tt}^{-1}(W_i) D_t^0 \frac{\partial L}{\partial D_t^u} - (-1)^1 J\left( W_i, D_t^1 \frac{\partial L}{\partial D_t^u} \right) = \Lambda_1^1 D_{tt}^{-1}(W_i) + J(W_i, \Lambda_1^1), \\
C_i^2 = (-1)^0 D_{tt}^{-1}(W_i) D_t^0 \frac{\partial L}{\partial D_t^v} - (-1)^1 J\left( W_i, D_t^1 \frac{\partial L}{\partial D_t^v} \right) = \Lambda_2^2 D_{tt}^{-1}(W_i) + J(W_i, \Lambda_2^2), \\
C_i^3 = (-1)^0 D_{tt}^{-1}(W_i) D_t^0 \frac{\partial L}{\partial D_t^w} - (-1)^1 J\left( W_i, D_t^1 \frac{\partial L}{\partial D_t^w} \right) = \Lambda_3^3 D_{tt}^{-1}(W_i) + J(W_i, \Lambda_3^3), \\
\]

and

\[
C_i^a = W_i \left( \frac{\partial L}{\partial u_x} - D_{1x} \frac{\partial L}{\partial u_{xx}} + D_{2x}^2 \frac{\partial L}{\partial u_{xxx}} \right) + D_{1x}(W_i) \left( \frac{\partial L}{\partial u_{xx}} - D_{1x} \frac{\partial L}{\partial u_{xxx}} \right) \\
+ D_{2x}^2(W_i) \frac{\partial}{\partial u_{xxx}} \\
= W_i \left( 3u \Lambda^1 + \frac{3}{2} w \Lambda^3 + \Lambda_{xx}^1 \right) - D_{1x}(W_i) \Lambda_x^1 + D_{2x}^2(W_i) \Lambda^1, \\
(14) \\
C_i^a = W_i \left( \frac{\partial L}{\partial v_x} - D_{2x} \frac{\partial L}{\partial v_{xx}} + D_{2x}^2 \frac{\partial L}{\partial v_{xxx}} \right) + D_{2x}(W_i) \left( \frac{\partial L}{\partial v_{xx}} - D_{2x} \frac{\partial L}{\partial v_{xxx}} \right) \\
+ D_{2x}^2(W_i) \frac{\partial}{\partial v_{xxx}} \\
= W_i \left( 3v \Lambda^2 + \frac{3}{2} w \Lambda^3 + \Lambda_{xx}^2 \right) - D_{2x}(W_i) \Lambda_x^2 + D_{2x}^2(W_i) \Lambda^2, \\
(15) \\
C_i^a = W_i \left( \frac{\partial L}{\partial w_x} - D_{3x} \frac{\partial L}{\partial w_{xx}} + D_{3x}^2 \frac{\partial L}{\partial w_{xxx}} \right) + D_{3x}(W_i) \left( \frac{\partial L}{\partial w_{xx}} - D_{3x} \frac{\partial L}{\partial w_{xxx}} \right) \\
+ D_{3x}^2(W_i) \frac{\partial}{\partial w_{xxx}} \\
= W_i \left( 3w \Lambda^1 + \frac{3}{2} h \Lambda^3 + \frac{3}{2} u \Lambda^3 + \Lambda_{xx}^3 \right) - D_{3x}(W_i) \Lambda_x^3 + D_{3x}^2(W_i) \Lambda^3, \\
(16) \\
\]

where \( i = 1, 2 \) and the functions \( W_i \) are

\[
W_1 = -(u_x + v_x + w_x), \\
W_2 = -2\alpha u - 2\alpha v - 2\alpha w - \alpha(x(u_x + v_x + w_x) - 3t(u_t + v_t + w_t)). \\
(17) \\
\]

Also, when \( \alpha \in (1, 2) \), we get the components of the conserved vectors

\[
C_i^1 = \Lambda_1^1 D_{tt}^{-1}(W_i) + J(W_i, \Lambda_1^1) - \Lambda_1^1 D_{tt}^{-2}(W_i) - J(W_i, \Lambda_1^1), \\
C_i^2 = \Lambda_2^2 D_{tt}^{-1}(W_i) + J(W_i, \Lambda_2^2) - \Lambda_2^2 D_{tt}^{-2}(W_i) - J(W_i, \Lambda_2^2), \\
C_i^3 = \Lambda_3^3 D_{tt}^{-1}(W_i) + J(W_i, \Lambda_3^3) - \Lambda_3^3 D_{tt}^{-2}(W_i) - J(W_i, \Lambda_3^3), \\
\]
where \( i = 1, 2 \) and the functions \( W_i \) in the form (17); also the conserved vectors \( C^1_x, C^2_x, C^3_x \) coincide with (14), (15) and (16).

5 Conclusions
In this paper, Lie symmetries and conservation laws have been studied for fractional order coupled KdV system (2). First, we obtained the fractional Lie point symmetries to the KdV system (2) with Riemann–Liouville derivative and we have shown that system (2) can be reduced to a nonlinear system of FDEs. Finally, conservation laws are constructed for system (2), the calculated conserved vectors, might be used for creating the particular solutions for the KdV system by the given method in [23, 24].

Acknowledgements
We would like thank the editor and reviewers for their time spent on reviewing this manuscript and their comments, helping us to improve the article.

Funding
Not applicable.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors’ contributions
All authors have read and approved the final manuscript.

Author details
1 Applied Analysis Research Group, Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam. 2 State Key Laboratory of Hydrology-Water Resources and Hydraulic Engineering, International Center for Simulation Software in Engineering and Sciences, College of Mechanics and Materials, Hohai University, Nanjing, 211100, China. 3 Department of Mathematics, University of Mazandaran, Babolsar, Iran. 4 Department of Mathematical Sciences, University of South Africa, Pretoria, South Africa.

Publisher’s Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 19 May 2020 Accepted: 29 November 2020 Published online: 11 December 2020

References
1. Baleanu, D., Diethelm, K., Scalas, E., Trujillo, J.J.: Fractional Calculus Models and Numerical Methods. Series on Complexity, Nonlinearity and Chaos. World Scientific, Singapore (2012)
2. Podlubny, I.: Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations to Methods of Their Solution and Some of Their Application. Mathematics in Science and Engineering, vol. 198. Academic Press, San Diego (1999)
3. Veeresha, P., Prakasha, D.G., Kumar, D., Baleanu, D., Singh, J.: An efficient computational technique for fractional model of generalized Hirota–Satsuma coupled KdV and coupled mKdV equations. J. Comput. Nonlinear Dyn. 15, 071003 (2020)
4. Singh, J., Kumar, D., Baleanu, D.: A new analysis of fractional fish farm model associated with Mittag-Leffler type kernel. Int. J. Biomath. 13(2), 2050010 (2020)
5. Olver, P.J.: Application of Lie Group to Differential Equation. Springer, New York (1986)
6. Bluman, G.W., Kumei, S.: Symmetries and Differential Equations. Springer, New York (1989)
7. Ibragimov, N.H.: CRC Handbook of Lie Group Analysis of Differential Equations, vol. 1–3. CRC Press, Boca Raton (1994)
8. Adem, A.R., Khalique, C.M.: Symmetry reductions exact solutions and conservation laws of a new coupled KdV system. Commun. Nonlinear Sci. Numer. Simul. 17, 3465–3475 (2012)
9. Molati, M., Khalique, C.M.: Lie symmetry analysis of the time-variable coefficient B-BBM equation. Adv. Differ. Equ. 2012, 212 (2012). https://doi.org/10.1186/1687-1847-2012-212
10. Goitiemang, T., Mothibi, D.M., Mutajtejeja, B., Motsumi, T.G.: Symmetry analysis and conservation laws of a further modified 3D Zakharov–Kuznetsov equation. Results Phys. 19, 103401 (2020)
11. Liu, Y., Teng, Q., Tai, W., et al.: Symmetry reductions of the (3 + 1)-dimensional modified Zakharov–Kuznetsov equation. Adv. Differ. Equ. 2019, 77 (2019). https://doi.org/10.1186/s13662-019-2017-4
12. Baleanu, D., Inc, M., Abdullahi, Y., Alyu, A.I.: Lie symmetry analysis, exact solutions and conservation laws for the time fractional Caudrey–Dodd–Gibbon–Sawada–Kotera equation. Commun. Nonlinear Sci. Numer. Simul. 59, 222–234 (2018)
13. Baleanu, D., Inc, M., Yusuf, A., Aliyu, A.I.: Time fractional third-order evolution equation: symmetry analysis, explicit solutions, and conservation laws. J. Comput. Nonlinear Dyn. 13(2), 021011 (2018)
14. Inc, M., Yusuf, A., Aliyu, A.I., Baleanu, D.: Lie symmetry analysis, explicit solutions and conservation laws for the spacetime fractional nonlinear evolution equations. Phys. A, Stat. Mech. Appl. 496, 371–383 (2018)
15. Baleanu, D., Inc, M., Yusuf, A., Aliyu, A.I.: Lie symmetry analysis, exact solutions and conservation laws for the time fractional modified Zakharov–Kuznetsov equation. Nonlinear Anal., Model. Control 22(6), 861–876 (2017)
16. Inc, M., Yusuf, A., Aliyu, A.I., et al.: Lie symmetry analysis and explicit solutions for the time fractional generalized Burgers–Huxley equation. Opt. Quantum Electron. 50, 94 (2018)
17. Muatjetjeja, B., Mogorosi, T.E.: Lie reductions and conservation laws of a coupled Jaulent–Miodek system. J. Appl. Nonlinear Dyn. 9(1), 109–114 (2020)
18. Liu, H.Z.: Complete group classifications and symmetry reductions of the fractional fifth-order KdV types of equations. Stud. Appl. Math. 131, 317–330 (2013)
19. Wang, G.W., Xu, T.Z.: Invariant analysis and exact solutions of nonlinear time fractional Sharma–Tasso–Olver equation by Lie group analysis. Nonlinear Dyn. 76, 571–580 (2014)
20. Huang, Q., Zhdanov, R.: Symmetries and exact solutions of the time fractional Harry–Dym equation with Riemann–Liouville derivative. Physica A 409, 110–118 (2014)
21. Hashemi, M.S.: Group analysis and exact solutions of the time fractional Fokker–Planck equation. Physica A 417, 141–149 (2015)
22. Lukashchuk, S.Y.: Conservation laws for time-fractional subdiffusion and diffusion-wave equations. Nonlinear Dyn. 80, 791–802 (2015)
23. Ibragimov, N., Avdonina, E.D.: Nonlinear self-adjointness, conservation laws, and the construction of solutions of partial differential equations using conservation laws. Russ. Math. Surv. 68(5), 889–921 (2013)
24. Avdonina, E.D., Ibragimov, N.H., Khamitova, R.: Exact solutions of gas dynamic equations obtained by the method of conservation laws. Commun. Nonlinear Sci. Numer. Simul. 18(9), 2359–2366 (2013)