Towards a Practical, Theoretically Sound Algorithm for Random Generation in Finite Groups

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Abstract

This work presents a new, simple \(O(\log^2 |G|)\) algorithm, the Fibonacci cube algorithm, for producing random group elements in black box groups. After the initial \(O(\log^2 |G|)\) group operations, \(\varepsilon\)-uniform random elements are produced using \(O((\log 1/\varepsilon) \log |G|)\) operations each. This is the first major advance over the ten year old result of Babai [Bab91], which had required \(O(\log^5 |G|)\) group operations. Preliminary experimental results show the Fibonacci cube algorithm to be competitive with the product replacement algorithm.

The new result leads to an amusing reversal of the state of affairs for permutation group algorithms. In the past, the fastest random generation for permutation groups was achieved as an application of permutation group membership algorithms and used deep knowledge about permutation representations. The new black box random generation algorithm is also valid for permutation groups, while using no knowledge that is specific to permutation representations. As an application, we demonstrate a new algorithm for permutation group membership that is asymptotically faster than all previously known algorithms.

1 Introduction

Quickly finding an element of a black box group is a problem of critical importance for many randomized algorithms for mathematical groups. (Black box groups are defined later.) Random group elements are especially important for computations with finite matrix groups, where few efficient deterministic algorithms are known.

Researchers requiring generation of such random elements tended to have a split personality. On the one hand, one could choose a theoretically sound algorithm with a complexity that was far too high to be practical. The best previous theoretical algorithm required \(O(\log^5 |G|)\) group multiplications to produce a random element [Bab91]. On the other hand, one could choose a heuristic for random elements such as the product replacement algorithm [CLGM’95], which could be demonstrated to have a bias away from the uniform distribution [BP02], but was “good enough” in practice.

This paper presents a simple \(O(\log^2 |G|)\) algorithm, the Fibonacci cube algorithm, which is easy to program. After the initial \(O(\log^2 |G|)\) group operations, \(\varepsilon\)-uniform random elements are produced using \(O((\log 1/\varepsilon) \log |G|)\) operations each. The algorithm is in Section 6.1. The main theoretical result is Theorem 7.3. The theoretical analysis of this paper currently has an unacceptably high coefficient of complexity, although experimental results show it to be competitive with the product
replacement algorithm. The conclusion points out opportunities to lower the theoretical coefficient by refining the complexity analysis.

A black box group is a group with an associated oracle, in which group elements are encoded as binary strings of some uniform length \( L \). The oracle can multiply, find inverses, and compare an element with the identity. Note that this implies an upper bound of \( 2^L \) on the group order.

A common use of black box groups is to model finite matrix groups over finite fields. A matrix group, \( \text{GL}(d, q) \) (dimension \( d \) over \( \text{GF}(q) \)), is a black box group with an encoding of length \( L = d^2 \log_2 q \), and its order is a priori bounded by \( 2^L \). Almost every paper in the recent development of matrix group algorithms assumes the availability of a random generation algorithm. In particular, the matrix recognition project [LG01] is a project to recognize matrix groups in \( \text{GL}(d, q) \) for values of \( d \) up to approximately 100, and for moderate size values of \( q \). That project relies heavily on the ability to compute nearly random group elements.

Surprisingly, even in the regime of permutation groups, the new black box algorithm for random generation is faster than the best known permutation algorithm both for the case of large and small base. Let \( n \) be the permutation degree. For large base, \( \log |G| \leq n \log n \), and so we have random generation in \( O(n^3 \log^2 n) \). For small base, if we assume a base size of \( O(\log n) \), then \( \log |G| \leq \log^2 n \).

Let \( G = \langle S \rangle \) be a finite black box group. We use \( \Pr(\cdot) \) to notate probability and \( \mathbb{E}(\cdot) \) to notate expectation. Random variables are denoted by capital letters, while group elements are denoted by lower case letters. Let \( U \) be a random variable on \( G \) with uniform distribution. We use the notation \( A \subset G \) for a proper subset of \( G \) and \( A \subseteq G \) for a subset of \( G \). Similarly, \( H < G \) denotes a proper subgroup of \( G \) and \( H \leq G \) denotes a subgroup of \( G \).

1.1 Previous work

The first polynomial time algorithm for random group elements was demonstrated by Babai [Bab91]. It runs in time \( O(\log^5 |G|) \). Unfortunately, the high complexity means that this algorithm is not used in computations. As Babai wrote in the Handbook of Combinatorics [Bab95]:

Reducing the exponent 5 would be of great significance since many algorithms in computational group theory rely on “randomly chosen” elements from the group. [Bab95]

A second heuristic, product replacement, was then proposed by Celler et al. [CLGM+95] as a practical way to find random elements of \( G \).

Other researchers asked how fast a product replacement algorithm would approach a uniform distribution in the class of generating \( k \)-sets for \( G \). Note that such a random \( k \)-set is distinct from a random group element. Diaconis and Saloff-Coste showed the algorithm to produce nearly random \( k \)-sets in sub-exponential time [DSC98], and Pak then showed it to operate in polynomial time [Pak01]. Pak requires the use of a \( k \)-tuple in which \( k = \Omega(\log |G| \log \log |G|) \). When \( k = \theta(\log |G| \log \log |G|) \), he achieves his best time of \( O(\log^9 |G| (\log \log |G|)^5) \). Babai and Pak [BP02] presented an important obstacle, whereby for the limiting distribution of \( k \)-sets, individual group elements are shown to be biased away from the identity.

1.2 Outline of contents

The primary result is Theorem 7.3. Informally, it shows that one can construct an initial, nearly random element using \( O(\log^2 |G|) \) group operations with further elements produced in \( O(\log |G|) \) time. The algorithm is given in Section 6.1. For a high level overview of the approach to the proof, see Section 6.2 after reading this section.
We use the notation $XY$ for the product of two $G$-valued random variables in analogy with $gh$ for $g, h \in G$. To illustrate the notation, if $E$ is a $\{0, 1\}$-random variable, then $XY^E = h$ if $X = h$ and $E = 0$, while $XY^E = hg$ if $X = h, Y = g$ and $E = 1$.

Although the algorithm for generating random group elements is a simple one, its justification is not simple. We will develop a sequence of random variables $R_0, R_1, \ldots$ such that $R_0$ is fixed at the identity and $R_{i+1} = R_ig_i^{E_i}$ where $g_i \in G$ is chosen from a random distribution based on $R_i$ and $E_i$ is a uniform random variable on $\{0, 1\}$. The $E_1, E_2, \ldots$ are pairwise independent and independent of the other random variables. For some fixed $t = \Omega(\log |G|)$, $g_1^{E_1} \cdots g_t^{E_t}$ is a nearly uniform random variable on $G$, and computing an element from its distribution requires at least $t$ group operations.

Section 3 provides some easy, well-known lemmas which form the foundation for the rest of the paper. The $\ell^2$ norm, $||X||$, of a random variable $X$ is defined in Section 2.2, along with some easy lemmas about it. One easily shows that $||R_i||$ is monotonically non-increasing as a function of $i$. The $\ell^2$ norm had previously been used by Diaconis and Saloff-Coste to analyze random walks on groups [DS93].

The main goal of the proof is to show that $||R_{i+1}|| \leq c||R_i||$ with probability at least $\rho > 0$ for some positive $c < 1$. Section 6.2 outlines the ideas of that proof. Section 6.3 provides that proof and concludes with the formal statement in Lemma 6.2 showing that for $t = \Omega((1/c) \log |G|)$, $R_i$ is semi-uniform. ($\Pr(R_i = g)$ is bounded away from 0.) That lemma then yields the main theorem, Theorem 7.3.

The main result relies on some technical results from Sections 3 to 5. As a matter of notation, we reserve upper case letters $E$, $I$, $J$, $K$, $T$ through $Z$ and $R_i$ for random variables.

Section 3 makes various assumptions on $X$ and $W$. It then asks for what positive $\alpha < 1$ and $\rho > 0$ can one conclude that for $g$ drawn from the distribution of $W$, $||Xg^{E_i}|| < \alpha||X||$ with probability at least $\rho$. Here, $E$ is an independent random variable on $\{0, 1\}$.

Section 4 asks the following question. Let $X$ and $Y$ be random variables on $G$ and let $J$ be a random variable on $\{0, 1\}$. (Often, we will take $\Pr(J = 0) = 1/2$ and $\Pr(J = 1) = 1/2$.) Assume that $R_j$ has the same probability density function as $X^JY^{1-J}$. (Note that the notation $X^J$ means that $X^J = X$ when $J = 1$, and $X^J$ is the group identity element when $J = 0$.) If $||Xg_i^{E_i}|| \leq \alpha||X||$ for some $0 < \alpha \leq 1$, then for what $\beta$ is it true that $||R_i g_i^{E_i}|| \leq \beta||R_i||$? In order to state the results more generally, that section writes $Z$ for $R_i$ and $W$ for $g_i^{E_i}$. As will be seen, Section 5.

Section 5 worries about the unusual case of being “stuck” in a proper subgroup. The fixed \{\{g_1, \ldots, g_t\}\} constructed to define the series $R_1, \ldots, R_{i+1}$ can all be contained in a proper subgroup. In such a case, a random $g_{i+1}$ drawn from $R_{i+1}$ will also be in the proper subgroup. Further, if $R_{i+1}$ has only a small probability of lying outside a proper subgroup, then the same problem arises.

The solution is to use the generators of $G$ to construct a group element $g \notin A$, whereupon $Xg_i^{E_i}$ is smaller than $X$ in the $\ell^2$ norm. We use random subproducts (Definition 4) as an efficient way to construct a $g_i \notin A$.

Section 6 contains Theorem 7.3 which may have independent interest. Informally, it states that for a set $A \subseteq G$ with $A = A^{-1}$, either a random $(u, v) \in A \times A$ satisfies $uv \notin A$ with at least some positive probability, or else $A$ is close to a subgroup $A'$ with $A'A'$ a subgroup of $G$.

Section 5 demonstrates the Fibonacci Cube algorithm, which constructs the $g_i$ in the definition of $R_i$. This is the main algorithm. This is enough to show that $R_i^{-1}R_t$ is semi-uniform for sufficiently large $t$.

Section 7 shows how to construct $\varepsilon$-uniform random elements from $\varepsilon$-semi-uniform random elements. It then summarizes the previous results in the main theorem, Theorem 7.3. Theorem 7.1 of that section is of independent interest, since it shows how to efficiently construct a uniform random variable from a semi-uniform random variable.
Section 8 presents some initial experimental results applying the Fibonacci cube algorithm to conjugacy classes. After a precomputation of about 100 group operations, one produces independent pseudo-random elements costing 20 group operations per random element. Those elements satisfy the $\chi^2$ goodness of fit test as having a distribution over the conjugacy classes that is close to uniformly random.

Section 9 produces an $O(\log^2 |G|)$ random generation algorithm for a variation of the product replacement algorithm, and Section 10 describes how to use the new Fibonacci cube algorithm to produce what is currently asymptotically fastest group membership algorithm — both for the general (large base) case and the special case of small base groups.

2 Preliminaries

The following easy lemmas and theorems are included for completeness. Note that throughout this paper, random variables are always denoted by upper case letters $E$, $I$, $J$, $K$, $T$ through $Z$ and by $\mathcal{R}_i$.

2.1 Probability and $\varepsilon$-uniform random variables

The following lemma is well-known and has an easy proof.

Lemma 2.1 (Markov’s inequality) Let $\xi$ be a nonnegative random variable and $\lambda > 1$ a real number. Then

$$\Pr(\xi \geq \lambda E(\xi)) \leq \frac{1}{\lambda}.$$

Corollary 2.2 Let $\xi$ be a random variable on the interval $[0, 1]$ and $\lambda > 1$ a real number. Then

$$\Pr(\xi > 1 - \lambda E(1 - \xi)) \geq 1 - \frac{1}{\lambda}.$$

Proof: Let $\zeta = 1 - \xi$ and note that $\zeta$ is nonnegative. Then $\Pr(\xi > 1 - \lambda E(1 - \xi)) \geq 1 - \frac{1}{\lambda} \Leftrightarrow \Pr(1 - \zeta > 1 - \lambda E(\zeta)) \geq 1 - \frac{1}{\lambda} \Leftrightarrow \Pr(\zeta < \lambda E(\zeta)) \geq 1 - \frac{1}{\lambda} \Leftrightarrow \Pr(\zeta \geq \lambda E(\zeta)) \leq \frac{1}{\lambda}$, and the last inequality follows from Markov’s inequality. $\Box$

Theorem 2.3 (Chernoff’s Bound [Che52]) Let $S_t$ be a random variable equal to the number of successes in $t$ independent Bernoulli trials in which the probability of success is $p$ ($0 < p < 1$). Let $0 < \varepsilon < 1$. Then

$$\Pr(S_t \leq (1 - \varepsilon)pt) \leq e^{-\varepsilon^2pt/2}.$$

Definition 1 A random subproduct on an ordered set $S = \{g_1, \ldots, g_k\} \subseteq G$ is given by $g_1^{\epsilon_1} \cdots g_k^{\epsilon_k}$ for $\epsilon_i$ independent, uniform random variables on $\{0, 1\}$. ($\Pr(\epsilon_i = 0) = 1/2$ and $\Pr(\epsilon_i = 1) = 1/2.$)

The following is a generalization of Proposition 2.1 of Cooperman and Finkelstein [CF93].

Lemma 2.4 (random subproduct) Let $H$ be a proper subgroup of $G = \langle S \rangle$ and let $r$ be a random subproduct on $S$. Then with probability at least $1/2$, $|Hr \setminus H| \geq |H|/2.$
Proof: Let \( S = \{g_1, \ldots, g_k\} \) and let \( j \leq k \) be the largest integer such that \( g_j \notin H \). Decompose the random subproduct \( r = g_1^{i_1} \cdots g_k^{i_k} \) as \( r = u g_j^{\epsilon_j} v \). If \( |H u \setminus H| \geq |H|/2 \), then with probability \( 1/2 \), \( \epsilon_j = 0 \), which implies \( |H r \setminus H| = |H u \setminus H| \). If \( |H u \setminus H| < |H|/2 \), then with probability \( 1/2 \), \( \epsilon_j = 1 \), which implies \( |H r \setminus H| = |H u g_j \setminus H| \geq |H u \cap H| = |H| - |H u \setminus H| > |H|/2 \). \( \square \)

**Lemma 2.5** Let \( X \) and \( Y \) be independent random variables on \( G \). Then \( \min_{h \in G} \Pr(X = h) \leq \Pr(XY = g) \leq \max_{h \in G} \Pr(X = h) \) for all \( g \in G \). Similarly, \( \min_{h \in G} \Pr(Y = h) \leq \Pr(XY = g) \leq \max_{h \in G} \Pr(Y = h) \).

**Proof:** Note \( \Pr(XY = g) = \sum_{h \in G} \Pr(X = h) \Pr(XY = g) = \sum_{h \in G} \Pr(X = h) \Pr(Y = h^{-1}g) = \sum_{f \in G} \Pr(Y = f^{-1}g) = \min_{h \in G} \Pr(X = h) \times \sum_{f \in G} \Pr(Y = f^{-1}g) \leq \max_{h \in G} \Pr(X = h) \times \sum_{f \in G} \Pr(Y = f^{-1}g) = \max_{h \in G} \Pr(X = h) \). A similar argument holds for \( \min_{h \in G} \Pr(Y = h) \) and \( \max_{h \in G} \Pr(Y = h) \). \( \square \)

**Lemma 2.6 (Babai and Szemerédi [BS84])** The following holds: \( g \notin A^{-1}A \iff Ag \cap A = \emptyset \iff |Ag \setminus A| = 2|A| \).

The proof is clear.

**Definition 2** A random variable on a group \( G \) is an \( \varepsilon \)-uniform random variable if \( |\Pr(X = g) - 1/|G|| \leq \varepsilon/|G| \) for all \( g \in G \). Note that a 0-uniform random variable is just a uniform random variable.

**Lemma 2.7 (\( \varepsilon \)-uniform random variable)** Let \( U \) and \( V \) be independent random variables on a group \( G \) and let \( \varepsilon \geq 0 \). If \( U \) is an \( \varepsilon \)-uniform random variable, then \( UV \) and \( VU \) are also \( \varepsilon \)-uniform.

**Proof:** \( |\Pr(UV = g) - 1/|G|| = \sum_{h \in G} (|\Pr(U = h) - 1/|G|| \Pr(V = h^{-1}g) = (|\Pr(U = h) - 1/|G|| (\sum_{h \in G} \Pr(V = h^{-1}g)) \leq \varepsilon 1/|G| \). A similar argument follows for \( VU \). \( \square \)

The next lemma shows that once a random variable \( U \) is found to be uniform on \( A \) for \(|A| > |G|/2 \), \( U^{-1}VU \) is \( \varepsilon \)-uniform for arbitrary random variable \( V \).

**Lemma 2.8** \( \alpha \) be a constant satisfying \( 1/2 < \alpha \leq 1 \). Let \( A \) be a subset of a group \( G \) such that \(|A| \geq \alpha|G| \). Let \( U_1, U_2 \) and \( V \) be independent random variables on \( G \). Let \( U_1 \) and \( U_2 \) be uniform on \( A \) with \( \Pr(U_1 = g) = \Pr(U_2 = g) = 0 \) for \( g \notin A \). Then

\[
\forall g \in G, \quad \frac{1 - \alpha}{\alpha} \frac{1}{|G|} \geq \Pr(U_1^{-1}VU_2 = g) - \frac{1}{|G|} \geq - \left( \frac{1 - \alpha}{\alpha} \right)^2 \frac{1}{|G|}.
\]

Hence, \( U_1^{-1}VU_2 \) is a \((1 - \alpha)/\alpha\)-uniform random variable on \( G \).

**Proof:** Note that \(|A| \geq \alpha|G| \) implies \(|A \cap Ag| \geq (2\alpha - 1)|G| \). So, \( \Pr(VU_2 \in Ag) \geq (2\alpha - 1) \times |G|/|A| \geq (2\alpha - 1)/\alpha \). Since \( U_1 \) and \( U_2 \) are independent, \( \Pr(U_1^{-1}VU_2 = g) = \Pr(VU_2 = U_1g) = \Pr(VU_2 \in Ag)/|A| \leq ((2\alpha - 1)/\alpha^2)/|G| \). Also, \( \Pr(U_1^{-1}VU_2 = g) = \Pr(VU_2 \in Ag)/|A| \leq 1/|A| = (1/\alpha)/|G| \). Subtracting \( 1/|G| \) from the lower and upper bounds on \( \Pr(U_1^{-1}VU_2 = g) \) completes the proof. \( \square \)

In fact, Lemma 2.8 can easily be generalized to \( U_1 \) uniform on \( A_1 \) for \(|A_1| \geq \alpha_1|G| \) and \( U_2 \) uniform on \( A_2 \) for \(|A_2| \geq \alpha_2|G| \), but the existing form suffices for our purposes.
2.2 The $\ell^2$ norm

Let $R$ denote the real numbers. Recall that the $\ell^2$ norm on $v = (v_1, \ldots, v_k) \in R^k$ is $\|v\|_2 = \sqrt{\sum_{i=1}^{k} (v_i)^2}$. Let $\mathcal{X}$ be the set of $G$-valued random variables for $G$ a group. Define the function $\varphi$ as the natural function from $\mathcal{X}$ to $R^{|G|}$, the $|G|$-dimensional vector space over the reals. Hence, if $X \in \mathcal{X}$ and $G = \{g_1, g_2, \ldots, g_{|G|}\}$, then define:

$$\varphi(X) = \left( \Pr(X = g_1), \Pr(X = g_2), \ldots, \Pr(X = g_{|G|}) \right)$$

$$\|X\| = \|\varphi(X)\|_2 = \sqrt{\sum_{g \in G} (\Pr(X = g))^2}$$

Note that $\|XY\|$ is a norm under multiplication, since $\|XY\| = \|\varphi(XY)\|_2 \leq \|\varphi(X)\|_2 \|\varphi(Y)\|_2 = \|X\| \|Y\|$ by the Cauchy-Schwartz inequality.

Observe that for two $G$-valued random variables $X$ and $Y$,

$$\|XY\| = \| \sum_{g \in G} \varphi(Xg) \Pr(Y = g) \|_2 = \| \sum_{g \in G} \varphi(gY) \Pr(X = g) \|_2.$$

**Lemma 2.9** For $X$ a random variable on the group $G$ and $g \in G$, $\|X\| = \|X^{-1}\| = \|Xg\|$.

**Proof:** $\|X\|^2 = \sum_{h \in G} (\Pr(X = h))^2 = \sum_{h \in G} (\Pr(X^{-1} = h^{-1}))^2 = \|X^{-1}\|^2$. Similarly, $\|X\|^2 = \sum_{h \in G} (\Pr(Xg = h)) = \|Xg\|^2$. □

**Lemma 2.10** If $X$ and $Y$ are independent $G$-valued random variables for $G$ a group, then $\|XY\| \leq \min(\|X\|, \|Y\|)$.

**Proof:** By the triangle inequality, $\|XY\| = \| \sum_{g \in G} \varphi(Xg) \Pr(Y = g) \|_2 \leq \sum_{g \in G} \|\varphi(Xg)\|_2 \times \Pr(Y = g) = \|X\| \sum_{g \in G} \Pr(Y = g) = \|X\|$, and similarly $\|XY\| \leq \|Y\|$. □

**Lemma 2.11** Let $X$ be a random variable on $G$. If $\Pr(X = g) \leq m$ for all $g \in G$, then $\|X\| \leq \sqrt{m}$.

**Proof:** $\|X\|$ is maximized when $\Pr(X = g) = m$ or $\Pr(X = g) = 0$ for all $g \in G$ except at most one $g' \in G$ for which $0 < \Pr(X = g') < m$. To see this, let $Y$ be a random variable with $\Pr(Y = g) \leq m$ such that $\|Y\|$ is maximal. If $x_1 = \Pr(Y = g_1)$, $x_2 = \Pr(Y = g_2)$, $0 < x_1 < m$, $0 < x_2 < m$ and $0 < \delta \leq x_2$, then $(x_1 + \delta)^2 + (x_2 - \delta)^2 = x_1^2 + x_2^2 + 2x_1x_2 - 2 \delta x_1 + 2 \delta^2 > x_1^2 + x_2^2$ when $x_1 > x_2$. This violates maximality of $\|Y\|$. So there is at most one $g' \in G$ such that $0 < \Pr(Y = g') < m$. Let $\Pr(Y = g') = m' < m$. Then $\|Y\| = \sqrt{m'^2 + ((1 - m')/m)m^2} < \sqrt{m}$. □

**Definition 3** The support of a random variable $X$ on a group $G$ is the set

$$\text{supp}(X) = \{ g \in G : \Pr(X = g) > 0 \}.$$

**Lemma 2.12** Let $X$ be a random variable on $G$. Then $\|X\| \geq 1/\sqrt{|\text{supp}(X)|}$.

**Proof:** Let $U$ be the uniform random variable on $\text{supp}(X)$ and observe that $\Pr(U = g) = 1/|\text{supp}(X)|$ for $g \in \text{supp}(X)$, taking the inner product of $\phi(U)$ and $\phi(X)$, the result follows from $1/|\text{supp}(X)| = \phi(U) \cdot \phi(X) \leq ||U|| ||X|| = ||X||/\sqrt{|\text{supp}(X)|}$. The inequality $\phi(U) \cdot \phi(X) \leq ||U|| ||X||$ is the Cauchy-Schwartz inequality. □
3 Reduction of probability in the $\ell^2$ norm

In this section, we derive estimates of the form $||Xg^E|| \leq \alpha ||X||$ for $E$ a uniform $\{0,1\}$-random variable and for fixed $g$ drawn from the distribution of $W$, with probability at least $\rho > 0$. The positive parameters $\alpha < 1$ and $\rho$ depend on the choice of $X$ and $W$. In applications, we will find $X$, $Y$ and $J$ such that $R_i$ has the same distribution as $X^JY^{1-J}$. Having shown $||Xg^E|| \leq \alpha ||X||$ in this section, Section 4 will allow us to conclude $||\mathcal{R}_gE_i|| < \alpha ||\mathcal{R}_i||$ with probability at least $\rho > 0$.

**Lemma 3.1** Let $X$ and $W$ be independent random variables on a group $G$. Let $E$ be a $\{0,1\}$-random variable and let $X$ and $E$ be independent. The notation $E_{g \in W}(f(g))$ denotes $E(f(W))$ for the function $f: G \to R$ into the real numbers $R$. Hence, $E_{g \in W}(||Xg^E||^2) \overset{\text{def}}{=} E(f(W))$ for $f(g) = ||Xg^E||$ for $g \in G$. Then

$$E_{g \in W}(||Xg^E||^2) \overset{\text{def}}{=} \left( \left( \Pr(E = 0) \right)^2 + \Pr(E = 1) \right) ||X||^2 + \sum_{h \in G} 2\Pr(E = 0) \Pr(E = 1) \Pr(X = h) \Pr(XW = h)$$

**Proof:** Lemma 2.9 tells us that $||X|| = ||Xg^{-1}||$. Without loss of generality, we can take $X$ and $W$ as independent. If $X$ and $W$ were dependent, then we would take $X'$ as an independent random variable with identical distribution to $X$, and note that $E_{g \in W}(||Xg^E||^2) = E_{g \in W}(||Xg^{E'}||^2)$. For $X$ and $W$ independent, $\sum_{g \in G} \Pr(X = hg^{-1}) \Pr(W = g) = \Pr(XW = h)$. The following equality then holds.

$$E_{g \in W}(||Xg^E||^2) = \sum_{g \in G} \left( \Pr(W = g) \sum_{h \in G} \left( \Pr(Xg^E = h) \right)^2 \right)$$

$$= \sum_{g \in G} \sum_{h \in G} \Pr(W = g) \left( \left( \Pr(E = 0) \Pr(X = h) \right)^2 + \left( \Pr(E = 1) \Pr(X = hg^{-1}) \right)^2 + 2\Pr(E = 0) \Pr(E = 1) \Pr(X = h) \Pr(XW = h) \right)$$

$$= \sum_{g \in G} \Pr(W = g)(\Pr(E = 0))^2 ||X||^2 + \sum_{g \in G} \Pr(W = g)(\Pr(E = 1))^2 ||Xg^{-1}||^2 + \sum_{h \in G} 2\Pr(E = 0) \Pr(E = 1) \Pr(X = h) \Pr(XW = h)$$

$$= \left( (\Pr(E = 0))^2 + (\Pr(E = 1))^2 \right) ||X||^2 + \sum_{h \in G} 2\Pr(E = 0) \Pr(E = 1) \Pr(X = h) \Pr(XW = h)$$

\[ \Box \]

**Theorem 3.2** Let $X$, $W$ and $Z$ be random variables on a group $G$. Let $E$ be a uniform $\{0,1\}$-random variable and let $X$ and $E$ be independent. Let $\lambda > 1$ and let $g \in G$ be drawn from the distribution of $W$. Let $\phi = \Pr(XW \in \text{supp}(X))$. Let $Z$ have a density function such that $\Pr(Z = g) = \Pr(XW = g) / \phi$ for $g \in \text{supp}(X)$ and $\Pr(Z = g) = 0$ for $g \notin \text{supp}(X)$. (The random variable $Z$ can be thought of as $XW$ conditioned on the event $XW \in \text{supp}(X)$.) Let $||Z|| \leq c ||X||$. Then with probability at least $1 - 1/\lambda$,

$$||Xg^E|| < \sqrt{\frac{\lambda + c\phi}{2}} ||X||.$$
Proof: Note $\sum_{h \in G} \Pr(X = h) \Pr(XW = h) \leq \sum_{h \in G} \Pr(X = h) \Pr(Z = h) \phi \leq \phi ||X|| ||Z|| \leq c\phi ||X||^2$, where the first inequality holds due to the Cauchy-Schwartz inequality. From Lemma 3.1, $E_{g \in G}(||Xg^E||^2) \leq ||X||^2(1 + c\phi)/2$. Define the function $f(g) = ||Xg^E||^2$ from $G$ to the real numbers. By Markov’s inequality, $\Pr(f(W) \geq \lambda(||X||^2(1 + c\phi)/2)) \leq 1/\lambda$, from which the theorem follows. □

The estimate of the next corollary is used for Case 2 in Section 6.

Corollary 3.3 Assume the same hypotheses as Theorem 3.2, with the exception that $XW$ is replaced by $WX$ in the definition of $\phi$ and of $Z$. Then with probability at least $1 - 1/\lambda$,

$$||g^E X|| < \sqrt{\frac{1 + c\phi}{2}} ||X||.$$

Proof: Replace $X$ by $X^{-1}$, $W$ by $W^{-1}$, and $g$ by $g^{-1}$ in Theorem 3.2. Then $\phi = \Pr(WX \in \text{supp}(X))$ and $\Pr(Z^{-1} = g) = \Pr(WX = g)/\phi$ for $g \in \text{supp}(X)$ and $\Pr(Z^{-1} = g) = 0$ otherwise. So $||Z^{-1}|| = ||Z|| \leq c||X||$. Also $||X^{-1}(g^{-1})^E|| = ||g^E X||$, where the last follows from Lemma 2.9. So, the result follows from Theorem 3.2 by considering $Z^{-1}$ instead of $Z$. □

Lemma 3.4 Under the assumptions of Lemma 3.4, and assuming $m \geq \Pr(W = g)$ for all $g \in G$,

$$E_{g \in G}(||Xg^E||^2) \leq \left((\Pr(E = 0))^2 + (\Pr(E = 1))^2\right) ||X||^2 + 2m \Pr(E = 0) \Pr(E = 1)$$

Proof: The lemma follows from Lemma 3.1 and $\Pr(XW = h) = \sum_{g \in G} \Pr(W = g) \Pr(X = hg^{-1}) \leq m \sum_{g \in G} \Pr(X = hg^{-1}) = m$. □

The estimate of the next theorem is used for Case 1 in Section 6.

Theorem 3.5 Let $X$ and $W$ be random variables on a group $G$. Let $E$ be a uniform $\{0, 1\}$-random variable and let $X$ and $E$ be independent. Assume $\Pr(W \notin \text{supp}(X)) \geq \delta$. Assume further that $\Pr(W = g) = \max_{h \in G} \Pr(W = h)$ for all $g \in \text{supp}(X)$. Let $\lambda > 1$ and let $g \in G$ be drawn from the distribution of $W$. Then with probability at least $1 - 1/\lambda$,

$$||Xg^E|| < \sqrt{\frac{1 - \delta/2}{\lambda}} ||X||.$$

Proof: Let $m = \max_{g \in G} \Pr(W = g)$. Then $m \text{supp}(X)| + \delta \leq 1$. So $|\text{supp}(X)| \leq (1 - \delta)/m$. Next, $||X||^2 \geq 1/|\text{supp}(X)| \geq m/(1 - \delta)$ by Lemma 2.1. So, $m \leq (1 - \delta)||X||^2$. Combining this inequality with Lemma 3.4 and $\Pr(E = 0) = 1/2$ yields $E_{g \in G}(||Xg^E||^2) \leq (1/2)||X||^2 + m/2 \leq (1 - \delta/2)||X||^2$. By Markov’s inequality (Lemma 2.1), this implies for the function $f(g) = ||Xg^E||^2$ from $G$ to the real numbers, that $\Pr(f(W) \geq \lambda(1 - \delta/2)||X||^2) \leq 1/\lambda$. This is equivalent to $\Pr\left(f(W) < \sqrt{\frac{\lambda(1 - \delta/2)||X||}{\lambda}}\right) \geq 1 - \frac{1}{\lambda}$, from which the theorem follows. □

4 Decomposition of a random variable

One key to this paper is that given random variables $Z$ and $X$, we can decompose $Z$ into $X$ and a new random variable $Y$, subject to a certain “domination condition”. In this section, the variable $Z$ plays the role of $R_i$ in the main algorithm, and the variable $W$ plays the role of $g_i^E$ in the main algorithm. Hence in the application to the main algorithm, $W$ can have only two values, $g_i$ and the identity element. Further, $\Pr(W = g_i) = \Pr(E_i = 1)$. 

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Definition 4 For $X$ and $Y$ random variables on a group $G$, the statement $X \xrightarrow{\text{prob}} Y$ means $\forall g \in G, \Pr(X = g) = \Pr(Y = g)$ (i.e. $X$ and $Y$ are identically distributed).

Lemma 4.1 (decomposition) Let $Z$ and $X$ be random variables on a group $G$ and let $I$ be a \{0,1\}-random variable with $I$ independent of $X$. Assume that $\Pr(I = 1)\Pr(X = g) \leq \Pr(Z = g)$ for all $g \in G$. Then there is a decomposition of $Z$ such that $Z \xrightarrow{\text{prob}} X^I Y^{1-I}$ for all $g \in G$, where $Y$ is a random variable on $G$ independent of $I$ and is unique up to probability density.

Proof: Choose $Y$ independent of $X$ and $I$ to have a probability density function satisfying $\Pr(I = 0)\Pr(Y = g) = \Pr(Z = g) - \Pr(I = 1)\Pr(X = g)$. □

In Section 3, this lemma will be used repeatedly for such decompositions as $\mathcal{R}_i \xrightarrow{\text{prob}} X^I Y^{1-I}$. This allows us to draw a $g \in G$ from the distribution of $\mathcal{R}_i$ with the knowledge that with probability $\Pr(I = 1)$, it is as if the group element had been drawn from the distribution of $X$. Since we can choose $X$ arbitrarily subject to the domination condition $\Pr(I = 1)\Pr(X = g) \leq \Pr(\mathcal{R}_i = g)$, this gives us a lot of flexibility.

Once an $\varepsilon$-uniform random variable is available for some $\varepsilon < 1$, the next lemma shows how to iterate to improve the uniformity.

Lemma 4.2 Let $X$ and $Y$ be independent random variables on a group $G$. Let $X$ be $\delta$-uniform and $Y$ be $\varepsilon$-uniform. Then $XY$ is a $\delta\varepsilon$-uniform random variable.

Proof: Let $U$, $V$, $I$ and $J$ be independent random variables. Let $U$ and $V$ be uniform on $G$ and let $I$ and $J$ be on $\{0,1\}$, where $\Pr(I = 0) = \delta$ and $\Pr(J = 0) = \varepsilon$. Further, by Lemma 4.1, we can write $X \xrightarrow{\text{prob}} U^I A^{1-I}$ and $Y \xrightarrow{\text{prob}} V^J B^{1-J}$ for some random variables, $A$ and $B$. Note that $\Pr(I = 0)\Pr(A = g) \leq 2\delta/|G|$ and $\Pr(I = 1)\Pr(U = g) = (1 - \delta)/|G|$ for all $g \in G$ and similarly for $J$, $V$, $B$ and $\varepsilon$. By Lemma 2.7, $UV$, $UB$ and $AV$ are all uniform. So there is a $\{0,1\}$-uniform random variable $W$ and a $\{0,1\}$-random variable $K$ such that $XY \xrightarrow{\text{prob}} W^K(AB)^{1-K}$ with $\Pr(K = 0) = \Pr(I = 0)$ and $\Pr(K = 1) = \delta\varepsilon$. So $\Pr(XY = g) \geq \Pr(K = 1)\Pr(W = g) = (1 - \delta\varepsilon)/|G|$. Also $\Pr(K = 0)\Pr(AB = g) \leq 2\delta\varepsilon/|G|$. So $\Pr(XY = g) \leq \Pr(K = 1)\Pr(W = g) + \Pr(K = 0)\Pr(AB = g) = (1 + \delta\varepsilon)/|G|$. □

The next lemma from linear algebra is a standard calculation on vectors in the $\ell^2$ norm. It is needed to prove the succeeding Theorem 4.4. The $\ell^2$ vectors of the lemma will correspond to vectors of dimension $|G|$, where a $G$-valued random variable is considered as $(\Pr(X = g_1), \ldots, \Pr(X = g_{|G|}))$.

Lemma 4.3 Let $c$ and $\alpha$ be constants. Let $x$, $y$, $x'$ and $y'$ be vectors in the $\ell^2$ norm. Assume $||x'||_2 \leq \alpha||x||_2$, $||y'||_2 \leq ||y||_2$, and $||x||_2 \geq c||y||_2$ for $0 < \alpha \leq 1$ and $c > 0$. Then $||x' + y'||_2 \leq \left((1 + \alpha c)/\sqrt{1 + c^2}\right)||x + y||_2$.

Proof: Note that $||x||_2 + ||y||_2 \leq \left((1 + c)/\sqrt{1 + c^2}\right)\sqrt{||x||_2^2 + ||y||_2^2}$. Let $d = (c - \alpha c)/(1 + c)$. The proof follows from $||x' + y'||_2 \leq ||x'||_2 + ||y'||_2 \leq ||x||_2 + ||y||_2 \leq (1 + d)||x||_2 + (1 - d)||y||_2 = \left((1 + \alpha c)/(1 + c)\right)(||x||_2 + ||y||_2)$. Note that for fixed $||x||_2$ and $||y||_2$, $||x + y||_2$ is minimized when $x$ and $y$ are perpendicular. In this case, define $c'$ such that $||x||_2 = c'||y||_2$ and observe that $\left((1 + \alpha c)/(1 + c)\right)||x||_2^2 + ||y||_2^2 = \left((1 + \alpha c)/(1 + c)\right)(c' + 1)/\sqrt{c^2 + 1} \sqrt{||x||_2^2 + ||y||_2^2} \leq \left((1 + \alpha c)/(1 + c^2)\right)\sqrt{||x||_2^2 + ||y||_2^2} = \left((1 + \alpha c)/(\sqrt{1 + c^2})\right)||x + y||_2$. □

The estimate of the next theorem is used for Cases 2 and 3 in Section 4.
**Theorem 4.4** Let $X, Y, Z$ and $W$ be random variables on a group $G$ and let $J$ be a $\{0,1\}$ random variable. Let $X, Y, W$ and $J$ be independent, and let $Z \overset{\text{prob}}{=} X^JY^{1-J}$. Let $\|XW\| \leq \alpha \|X\|$ and $\Pr(J = 1)\|X\| \geq c \Pr(J = 0)\|Y\|$ for some $0 < \alpha \leq 1$ and $c > 0$. Then

$$\|ZW\| \leq \left(1 + \alpha c/\sqrt{1 + c^2}\right) \|Z\|.$$ 

**Proof:** By Lemma 2.10, $\|YW\| \leq \|Y\|$. By Lemma 3.3, $\|ZW\| = \|\Pr(J = 1)\varphi(XW) + \Pr(J = 0)\varphi(YW)\|_2 \leq \left(1 + \alpha c/\sqrt{1 + c^2}\right) \|\Pr(J = 1)\varphi(X) + \Pr(J = 0)\varphi(Y)\|_2 = \left(1 + \alpha c/\sqrt{1 + c^2}\right) \|Z\|$. □

The estimate of the next theorem is used for Case 1 in Section 6.

**Theorem 4.5** Let $X, Y, Z$ and $W$ be random variables on a group $G$ and let $J$ be a $\{0,1\}$ random variable. Let $X, Y, W$ and $J$ be independent, and let $Z \overset{\text{prob}}{=} X^JY^{1-J}$. Assume constants $m, c$ and $\alpha$ satisfying the following. Let $\Pr(J = 0)\Pr(Y = g) \leq m$ for all $g \in G$, and further let $\Pr(J = 0)\Pr(Y = g) = m$ when $g \in \text{supp}(X)$. Let $\|XW\| \leq \alpha \|X\|$ for some $0 < \alpha \leq 1$. Let $c = (1 - \alpha^2)(\Pr(J = 1))^2/(mA\Pr(Z \notin A) + 1)$ for $A = \text{supp}(X)$. Then

$$\|ZW\| \leq \sqrt{1 - c} \|Z\|.$$ 

**Proof:** Note that $\Pr(J = 0)\Pr(YW = g) = \Pr(J = 0)\sum_{h \in G}\Pr(Y = h)\Pr(W = h^{-1}g) \leq m\sum_{h \in G}\Pr(W = h^{-1}g) = m$. Since $\Pr(X = g) = 0$ for $g \notin \text{supp}(X)$ and $\Pr(J = 0)\Pr(Y = g) = m$ for $g \in \text{supp}(X)$, we have

$$\sum_{g \in G}2\Pr(J = 1)\Pr(J = 0)\Pr(XW = g)\Pr(YW = g)$$

$$\leq 2m\Pr(J = 1)\sum_{g \in G}\Pr(XW = g)$$

$$= 2m\Pr(J = 1)$$

$$= 2m\Pr(J = 1)\sum_{g \in \text{supp}(X)}\Pr(X = g)$$

$$= 2\Pr(J = 1)\Pr(J = 0)\sum_{g \in G}\Pr(X = g)\Pr(Y = g).$$

By Lemma 2.11, $\|YW\| \leq \|Y\|$. Hence,

$$\|ZW\|^2 = \|\Pr(J = 1)\varphi(XW) + \Pr(J = 0)\varphi(YW)\|^2$$

$$= \sum_{g \in G}(\Pr(J = 1)\Pr(XW = g) + \Pr(J = 0)\Pr(YW = g))^2$$

$$= (\Pr(J = 1)\|XW\|^2 + (\Pr(J = 0)\|YW\|^2$$

$$\sum_{g \in G}2\Pr(J = 1)\Pr(J = 0)\Pr(XW = g)\Pr(YW = g)$$

$$\leq \alpha^2(\Pr(J = 1)\|X\|^2 + (\Pr(J = 0)\|Y\|^2$$

$$\sum_{g \in G}2\Pr(J = 1)\Pr(J = 0)\Pr(X = g)\Pr(Y = g)$$

$$= \|Z\|^2 - (1 - \alpha^2)(\Pr(J = 1)\|X\|^2$$

$$\leq (1 - c)\|Z\|^2$$
We must demonstrate membership in three cases: \(ac, ca, cd\).

\[\varphi(a, b)\]

that

\[w\]

From this it follows that

\[\text{Proof:}\]

To see this, note that

\[v\]

providing \((1 - \alpha^2) (\Pr(J = 1)||X||^2 \geq c||Z||^2) \) for \(c > 0\).

We find such a \(c\). Let \(A = \supp(X)\). Note that \(m|A| + \Pr(J = 1) = \Pr(J = 0)\Pr(Y \in A) + \Pr(J = 1) \leq 1\). One can show that \(||X||^2/||Z||^2\) is minimized when \(X\) is uniform on \(A\). In this case, \(||X||^2 = 1/|A|\) and we have \(||Z||^2 \leq m^2(\Pr(Z \notin A)/m + |A|(m + \Pr(J = 1)/|A|)^2 \leq m\Pr(Z \notin A) + 1/|A|\). So, if \(c = (1 - \alpha^2)(\Pr(J = 1))^2/(m|A|\Pr(Z \notin A) + 1), \) then \((1 - \alpha^2) (\Pr(J = 1)||X||^2 \geq c||Z||^2) \) \(\square\)

Section 3 constructs an element \(g \in G\) such that \(||R_ig^{E_i}|< c||R_i||\) for some \(c < 1\). However, it fails if, for example, \(X, W\) and \(XW\) all have identical distribution. This is part of a larger class of examples. If \(R_i\) is the uniform distribution on a proper subgroup \(H < G\), then any construction of \(X\) and \(W\) from \(R_i\) will fail to produce a \(g \in G\) with \(||R_ig^{E_i}| < c||R_i||\), since \(||R_i||\) is already minimized among random variables on \(H < G\). Hence, when the methods of Section 3 fail, we must demonstrate that this implies that \(R_i\) is close to a uniform distribution on a proper subgroup \(H < G\).

The following surprising lemma is the key. It shows that if one cannot escape a set \(A = B^{-1}B\) with reasonable probability simply by multiplying two random elements of the set \(A\), then one must be “stuck” in a proper subgroup. Loosely speaking, either the product of two random elements of \(A\) “escapes” from the set \(A\), or else \(A\) must be a “fuzzy subgroup” of \(G\) in the sense that \(A\) is close in probability to some subgroup of \(G\). In the latter case, we use the generators of \(G\) to construct a \(g \notin H\) so that \(R_ig^{E_i}\) “escapes” the set \(H\).

The proof proceeds by constructing a multiplication table for products of elements of \(A\). If \(gh \notin A\) for \(g, h \in A\), then we think of \(gh\) as a “hole” in the multiplication table. We then augment the multiplication table to include \(gh\), and show that the number of holes in the multiplication table for \(A \cup gh\) has been reduced.

**Lemma 5.1** Let \(A \subseteq G\) satisfy \(A = A^{-1}\). Let \(\delta < 1/4\). Assume \(\forall g \in A, |Ag \setminus A| \leq \delta|A|\). Then

\[|AA \setminus A| \leq \frac{\delta}{1 - 2\delta}|A|\]

Furthermore, \(AA\) is a subgroup of \(G\).

**Proof:** Define \(\phi(g) = |\{a \in A: ag \notin A\}|\). The hypothesis can be re-phrased as \(\forall g \in A, \phi(g) \leq \delta|A|\). From this it follows that

\[\forall g, h \in A, \phi(gh) \leq \phi(g) + \phi(h) \leq 2\delta|A|\]

Similarly, for all \(a, b, c, d \in A\), \(\phi(abc) \leq 3\delta|A|\) and \(\phi(abcd) \leq 4\delta|A|\).

So, for \(g, h \in A\) such that \(gh \notin A\), there are at least \((1 - 2\delta)|A|\) pairs \((u, v)\) such that \(gh = uv\). To see this, note that \(v = u^{-1}gh\) and so we are counting the number of pairs \((u, v) \in A \times A\) such that \(u^{-1}gh \in A\). This number is \(|A| - \phi(gh) \geq (1 - 2\delta)|A|\). Hence, \(|AA \setminus A| \leq \delta/(1 - 2\delta)|A|\) as required by the lemma.

It remains to show that \(AA\) is a group. Since it is closed under inverses, we must show that it is closed under multiplication. For \(a, b \in A\), it is clear that \(ab \in AA\). Given \(a \in A\) and \(c, d \in AA \setminus A\), we must demonstrate membership in three cases: \(ac, ca, cd \in AA\).

We first show that \(cd \in AA\), where \(c = gh, d = uv, c, d \in AA \setminus A\) and \(g, h, u, v \in A\). Note that \(\phi(ghuv) \leq 4\delta|A| < |A|\). Therefore, \(\exists w \in A\) such that \(wghuv \overset{\text{def}}{=} x \in A\). So, \(cd = (gh)(uv) = w^{-1}x \in AA\).
A similar argument holds to show that \( ca \in AA \), where \( c = gh, \) \( c \in AA \setminus A \) and \( c, g, h \in A \). It follows from noting that \( \phi(gha) \leq 3\delta|A| < |A| \). Finally, \( ac = c^{-1}a^{-1} \), and so the case of \( ac \) reduces to the previously proved case of \( ca \). \( \square \)

**Remark 1** Examination of the proof shows that the hypothesis could be weakened to \( \delta < 1/3 \) or further, at the cost of showing that \( A^k \) is a group for some sufficiently large \( k \).

One interpretation of Lemma 5.1 is that for random \( (u, v) \in A \times A \), \( uv \notin A \) with some constant probability, or \( AA \) is close to a group and so \( ug \notin A \) with some constant probability for some group generator \( g \).

**Theorem 5.2** Let \( k > 1 \) and \( 0 < \varepsilon < 1 \) be arbitrary constants and let \( \delta = (2 + k^2\varepsilon)/(k - 2) \). Assume \( \delta \leq 1/4 \). Let \( A \subseteq G = \langle S \rangle \) satisfy \( A = A^{-1} \). Then one of the following is true.

1. Given a random \( (u, v) \in A \times A \) drawn from a uniform distribution, \( uv \notin A \) with probability at least \( \varepsilon \).

2. \( \exists A' \subseteq A \) with \( |A \setminus A'| < 2|A|/k \) such that \( A' A' \) is a subgroup of \( G \). Furthermore

\[
|A' A' \setminus A'| \leq \frac{\delta}{1 - 2\delta}|A'|
\]

**Proof:** Define \( B \subseteq A \times A \) such that \( B = \{(g, h): gh \in A\} \). If \( |A \times A \setminus B| > \varepsilon|A \times A| \), then a random \( (u, v) \in A \times A \) satisfies \( uv \notin A \) with probability at least \( \varepsilon \), and we are done.

Otherwise, \( |A \times A \setminus B| \leq \varepsilon|A \times A| \). Note that \( k\varepsilon < 1 \). Let

\[
A' = \{g \in A: |Ag \setminus A| \leq k\varepsilon|A| \text{ and } |gA \setminus A| \leq k\varepsilon|A|\}.
\]

Note that \( A' = A'^{-1} \). Also \( |A \setminus A'| < 2|A|/k \). To see the latter, note that \( |\{g: |Ag \setminus A| > k\varepsilon|A|\}| < |\{(u, g): ug \notin A\}|/(k\varepsilon|A|) = |A \times A \setminus B|/(k\varepsilon|A|) \leq |A|/k \).

Therefore \( |(A'g \cap A) \setminus A'| \leq |A \setminus A'| \leq (2/k)|A| \). Also \( |A'g \setminus A| = |g^{-1}A' \setminus A| \leq k\varepsilon|A| \) for all \( g \in A \). Hence \( |A'g \setminus A'| \leq (2/k)|A| + k\varepsilon|A| \leq ((2 + k^2\varepsilon)/k)|A| \). But \( |A| < |A'|/(1 - 2/k) \) follows from \( |A| - |A'| = |A \setminus A'| < 2|A|/k \). (The coefficient \( 1/(1 - 2/k) \) is positive since \( \delta \leq 1/4 \) implies that \( k \geq 10 \).) Hence \( |A'g \setminus A'| < ((2 + k^2\varepsilon)/(k - 2))|A'| = \delta|A| \) for all \( g \in A' \).

Since \( \delta \leq 1/4 \) and \( |A'g \setminus A'| \leq \delta|A'| \) for all \( g \in A' \), we can invoke Lemma 5.1 on \( A' \) and conclude that \( A' A' \) is a subgroup of \( G \). The bounds on \( |A' A' \setminus A'| \) follow from the same lemma. \( \square \)

**Corollary 5.3** Assume \( A, k \) and \( \varepsilon \) as in Theorem 5.2. Let \( p = (1/2) - (1/k) \) and let \( r \) be a random subproduct on \( S \). Assume \( (u, v) \in A \times A \) drawn from a uniform distribution. Let \( g = uv^r \) for \( I \) a \( \{0, 1\} \) random variable with \( \Pr(I = 1) = p/(p + \varepsilon) \). Then \( g \notin A \) with probability at least \( \varepsilon/(p + \varepsilon) > \varepsilon - 2k\varepsilon^2/(k - 2) \).

**Proof:** Theorem 5.2 tells us that \( uv \notin A \) with probability at least \( \varepsilon \) or \( A' A' \) is a subgroup of \( G \) with \( |A \setminus A'| < 2|A|/k \). In the latter case, \( r \notin A' A' \) with probability at least \( 1/2 \). Hence, with probability at least \( 1/2 \), for \( h \in A' \), \( hr \notin A' A' \supseteq A' \). For \( u \) drawn at random from \( A \), \( ur \notin A \) with probability at least \( (1/2)|A'|/|A| = (1/2) - (1/k) = p \).

Let \( g = uv^r \). Then \( \Pr(g \notin A) \geq \min(\varepsilon \Pr(I = 1), p \Pr(I = 0)) = \varepsilon/(p + \varepsilon) = \varepsilon(k - 2)/(2k\varepsilon + k - 2) > \varepsilon - 2k\varepsilon^2/(k - 2) \). \( \square \)

**Remark 2** Consider the equation \( \delta = (2 + k^2\varepsilon)/k \) of Theorem 5.2. The variable \( \varepsilon \) is maximized when \( k = 4/\delta + 4 \). Taking \( \delta = 1/4 \) implies \( k = 20 \) and \( \varepsilon = 1/160 \) when it is maximized. In this case, Corollary 5.3 produces a \( g \notin A \) with probability at least \( \varepsilon > 0.006 \).
6 Fibonacci Cube algorithm for semi-uniform random generation

We now have all of the algorithmic components outlined in Section 1.2. The goal of this section is only to construct $g_i$ for which $g_1^{E_1} \cdots g_t^{E_t}$ is semi-uniform.

**Definition 5** A random variable on a group $G$ is an $\varepsilon$-semi-uniform random variable if $\Pr(X = g) \geq 1/|G| - \varepsilon/|G|$ for all $g \in G$. The random variable is semi-uniform if it is $\varepsilon$-semi-uniform for some $\varepsilon > 0$.

### 6.1 Algorithm

Given a random variable $R_i$ on a group $G = \langle S \rangle$, we wish to construct $g_i \in G$ such that $||R_i g_i^{E_i}||/||R_i|| < c < 1$ for some constant $c$ and for $E_1, E_2, \ldots$ independent uniform $\{0,1\}$-random variables. By Lemma 2.10, $||R_i h^{E_i}|| \leq ||R_i||$ for all $h \in G$. Hence, we will construct $g_i$ that has only some constant probability of satisfying $||R_i g_i^{E_i}||/||R_i|| < c < 1$. We then set $R_{i+1} = R_i g_i^{E_i}$, knowing that $||R_{i+1}|| = ||R_i g_i^{E_i}||$, even if $g_i$ did not succeed. We can then try again by constructing $g_{i+1}$. In Case 2 below, we define $R_{i+1} = g_i^{E_i} R_i$ instead of $R_{i+1} = R_i g_i^{E_i}$, but this does not change the spirit of the algorithm.

We call the algorithm below the Fibonacci Cube algorithm by allusion to the Fibonacci series. Like the Fibonacci series, each group element is derived from the previous elements of the series. It is a cube algorithm since $R_i = h_1^{E_1} \cdots h_k^{E_k}$ for exponents that are independent uniform $\{0,1\}$-random variables. The pseudo-code for the algorithm is simple.

**Algorithm Fibonacci-Cube**

**INPUT:** Black box group $G = \langle S \rangle$

**OUTPUT:** $R_t^{-1} \bar{R}_t$ for $\bar{R}_t$ an independent copy of $R_t$;

[ For large enough $t$, $\Pr(R_t^{-1} \bar{R}_t = g) \geq (3/4)(1-\beta^2)/|G|$ for all $g \in G$ ]

**PARAMETERS:** positive constants $a$, $b$ and $c$; $\alpha$ and $\rho$ dependent on $a$, $b$ and $c$ such that $||R_{i+1}|| \leq \alpha ||R_i||$ with probability at least $\rho$ unless $R_t^{-1} \bar{R}_t$ already satisfies the conditions on $R_t^{-1} \bar{R}_t$

Let $R_1$ be the identity element with probability 1

Let $t = \log |G|/\log \alpha^{-2\rho}$

For $i = 1$ to $t - 1$

Let $d = 1/a + 1/b + 1/c$

Let $j \in 1, 2$ or 3 with

probability $1/(ad)$, $1/(bd)$ or $1/(cd)$, respectively

Goto Case $j$

Case 1:

Choose $g_i$ from distribution of $R$

Set $R_{i+1} = R_i g_i^{E_i}$

Case 2:

Choose $g_i$ from distribution of $R$

Set $R_{i+1} = g_i^{E_i} R_i$

Case 3:

Choose $g_i$ from distribution of random subproducts on $S$

Set $R_{i+1} = R_i g_i^{E_i}$

Return $R_t^{-1} \bar{R}_t$ for $\bar{R}_t$ an independent copy of $R_t$
Note that the output of the algorithm is in terms of a random variable $R_t = h_1^{E_1} \cdots h_t^{E_t}$, where $(h_1, \ldots, h_t)$ is a reordering of $(g_1, \ldots, g_t)$. So, an implementation of the algorithm would need only to record the elements $(h_1, \ldots, h_t)$. An element from the distribution $R_i R_t$ is then computed as $(h_1^{-1})^{E_1} \cdots (h_t^{-1})^{E_t} (h_1)^{E_1} \cdots (h_t)^{E_t}$ where each of $E_1, \ldots, E_t, E_1, \ldots, E_t$ is independently equal to zero or one with probability $1/2$.

The random variable produced by the Fibonacci cube algorithm is used to produce a $\gamma$-uniform random element. One can then use Lemma 6.2 to produce $\varepsilon$-uniform random elements for arbitrarily small $\varepsilon$.

6.2 Overview of proof

The immediate goal is to prove Lemma 6.2, that $||R_{i+1}|| \leq c||R_i||$ with probability at least $\rho > 0$ for some positive $c < 1$.

In Cases 1 and 2, $R_{i+1} = R_i g_i^{E_i}$ or $R_{i+1} = g_i^{E_i} R_i$, for $g$ drawn from $W = R_i$. In Case 3, $R_{i+1} = R_i g_i^{E_i}$ for $g$ a random subproduct. The proof proceeds by decomposing both $R_i$ and $W$ as follows into products of random variables that are easier to analyze.

$$R_i \overset{\text{prob}}{=} X^J Y^{1-J}$$

$$W = R_i \overset{\text{prob}}{=} W' K T^{1-K} \quad \text{(Cases 1 and 2 only)}$$

$$W = W'$$

is a random subproduct on the group generators (Case 3).

The general approach in each case is to define $X$, $J$, $W'$ and $K$ so that $||X g_i^{E_i}|| \leq a ||X||$ for some positive $a < 1$ and for $g$ drawn from the distribution of $W'$, with probability $\rho > 0$. The results of Sections 3 or 4 are used here (Theorem 3.2 for Case 1, Corollary 3.3 for Case 2, and Theorem 5.2 for Case 3).

Then a result from Section 1 (Theorem 4.5 or Theorem 4.4) is used to show that $||X g_i^{E_i}|| \leq a ||X||$ implies $||R_i g_i^{E_i}|| \leq \beta ||R_i||$ for some positive $\beta < 1$ and for $g$ drawn from the distribution of $W'$, with probability $\rho > 0$.

Of course, one wishes to draw $g$ from the distribution of $W$, rather than from the distribution of $W'$. Since $W = W' K T^{1-K}$, a group element $g$ drawn from the distribution of $W$ can be considered to have been drawn from the distribution of $W'$ with probability $\Pr(K = 1)$. Hence, one observes that the previous result implies that $||R_i g_i^{E_i}|| \leq \beta ||R_i||$ for some positive $\beta < 1$ and for $g$ drawn from the distribution of $W$, with probability $\rho \Pr(K = 1) > 0$.

At any step of the algorithm, one does not know which of the three cases are satisfied by the current $R_i$. However, this is not a problem. One chooses the recipe of one of the three cases at random in deciding how to construct $g_i$ and $R_{i+1}$. If an incorrect case is chosen, Theorem 2.10 guarantees that $||R_{i+1}|| = ||R_i g_i^{E_i}|| \leq ||R_i||$. So, as long as a correct case is chosen with at least some positive probability, the algorithm makes progress.

The pseudo-code allows one to choose positive parameters $a$, $b$ and $c$ to determine the ratio of the probabilities for choosing each of the three cases. However, the algorithm succeeds with the same asymptotic estimates regardless of the choice of $a$, $b$ and $c$.

6.3 Proof

The analysis of the pseudo-code will be in terms of four parameters, $\beta$, $\delta$ and $\lambda$, such that $1 > \beta > 2\delta > 0$ and $\lambda > 1$. The parameter values will be chosen based on the requirements of the proof.
The analysis of Cases 1, 2, and 3 of this section applies for $|G| > \max(1/\delta, 1/(\beta - \delta))$. The analysis finds asymptotic bounds on the time to produce an $\varepsilon$-uniform random variable on $G$. For groups with order $|G| \leq \max(1/\delta, 1/(\beta - \delta))$, one can easily show that the pseudo-code succeeds in some constant time.

**Definition 6** Define

$$
\overline{A}_x \overset{\text{def}}{=} \{ g \in G: \Pr(\mathcal{R}_i = g) > x \}
$$

$$
m \overset{\text{def}}{=} \min_x \{ x: \Pr(\mathcal{R}_i \not\in \overline{A}_x) > \delta \}
$$

Note that $m$ and $\overline{A}_m$ implicitly depend on $\mathcal{R}_i$, and hence on $i$. Define $A_m \supseteq \overline{A}_m$ so that

$$
\forall B \supset A_m, \Pr(\mathcal{R}_i \not\in B) < \delta \leq \Pr(\mathcal{R}_i \not\in A_m).
$$

This need not uniquely define $A_m$, but any instance satisfying the defining conditions will suffice. The condition implies that $A_m$ is maximal in the sense that $\Pr(\mathcal{R}_i \not\in B) < \delta$ for all $B \supset A_m$.

**Lemma 6.1** Assume $\max_{g \in G} \Pr(\mathcal{R}_i = g) \leq 1 - \delta$. The set $A_m \subseteq G$ satisfies

$$
\delta \leq \Pr(\mathcal{R}_i \not\in A_m) < \delta + m.
$$

Also,

$$
\Pr(\mathcal{R}_i = g) \geq m \quad \text{for } g \in A_m
$$

$$
\Pr(\mathcal{R}_i = g) \leq m \quad \text{for } g \not\in A_m
$$

Further, if $m < \delta$, then

$$
\delta \leq \Pr(\mathcal{R}_i \not\in A_m) < 2\delta
$$

**Proof:** The first inequality follows easily from the definition of $A_m$ and $\max_{g \in G} \Pr(\mathcal{R}_i = g) \leq 1 - \delta$. For the next two inequalities, note that the definition of $\overline{A}_m$ implies there is a $g \in G$ such that $\Pr(\mathcal{R}_i = g) = m$. If there were only one such $g$, one would have $A_m = \overline{A}_m$. If there are multiple such $g$, then $\Pr(\mathcal{R}_i = g) = m$ for all $g \in \overline{A}_m \setminus A_m$. The last inequality follows from the first one and $m < \delta$. □

![Figure 1: Probability density function for \(\mathcal{R}_i\), shaded part (outside \(\overline{A}_m\)) has area > \delta and outside of \(A_m\), the area is > \delta](image)

In the rest of this section, we will isolate a “Case 0” to consider $m \geq \delta$ or $\max_{g \in G} \Pr(\mathcal{R}_i = g) \leq 1 - \delta$. In all other cases, Lemma 6.1 applies with its conclusion that

$$
\delta \leq \Pr(\mathcal{R}_i \not\in A_m) < 2\delta.
$$
Definition 7 Define the random variable $U_B$ on $G$ for a set $B \subseteq G$ by

$$\Pr(U_B = g) = \begin{cases} 1/|B| & \text{for } g \in B \\ 0 & \text{for } g \notin B \end{cases}$$

Recall that $1 > \beta > 2\delta > 0$ and $\lambda > 1$ below. The parameters $\beta$, $\delta$ and $\lambda$ are fixed throughout. The parameter $m$ and the set $A_m$ depend on $\mathcal{R}_i$ and hence on $i$. Intuitively, one may think of $1 - \beta$ as a constant against which $m||A_m||$ is measured. Similarly, one may think of $\delta$ as a constant against which $\Pr(\mathcal{R}_i \notin A_m)$ is measured. One thinks of $\Pr(\mathcal{R}_i \in A_m) - m||A_m||$ as “large” if it is larger than $\beta - \delta$. In each of the three cases, we will construct $g_i \in G$ and conclude that there is a $c' < 1$ and $\rho' > 0$ such that $||\mathcal{R}_i g'_i|| < c'||\mathcal{R}_i||$ with probability at least $\rho'$.

All cases are described in the following context:

$\mathcal{R}_i = X^JY^{1-J}$

$g_i$ drawn from $W = W'^K1^{1-K}$

Certain of the cases will also require $V_1$ and $V_2$, defined as independent random variables distributed identically to $U_{A_m}$. The two random variables depend on $\mathcal{R}_i$, and hence on $i$.

Case 0: $(m \geq \delta$ or $\max_{g \in G} \Pr(\mathcal{R}_i = g) > 1 - \delta)$ Note that $m \geq \delta$ implies $\max_{g \in G} \Pr(\mathcal{R}_i = g) \geq m \geq \delta$. Hence, $\max_{g \in G} \Pr(\mathcal{R}_i = g) \geq \min(\delta, 1 - \delta)$ and this case represents the initial situation, when the probability distribution of $G$ still includes at least one group element whose probability of occurrence is high. Since $\delta$ is a constant, we need only show that we can make constant progress. Specifically, after a constant number of steps, we need to show that $\max_{g \in G} \Pr(\mathcal{R}_i = g) < \min(\delta, 1 - \delta)$. Lemma 2.3 shows that if this is true for some $i$, then it will be true for all $j \geq i$.

One can show for arbitrary constant $\delta$ that there large enough constants $i$ and $\phi$, such that $|G| \geq \phi$ implies $\max_{g \in G} \Pr(\mathcal{R}_i = g) < \min(\delta, 1 - \delta)$. We omit the details.

Figure 2: Case 1: Left shaded part is unnormalized probability density for $X$; right shaded part is unnormalized probability density for $W'$ (shaded parts have area less than 1)

Case 1: $(m < \delta$ and $\max_{g \in G} \Pr(\mathcal{R}_i = g) \leq 1 - \delta$ and $m||A_m|| < 1 - \beta)$ Intuitively, if $||X||^2$ is larger than $\max_{g \in G} \Pr(W' = g)$, then we will make progress to a more uniform distribution via Theorem 3.3. We require that $||X||$ and $||W'||$ be sufficiently large. We enforce this condition through $\Pr(\mathcal{R}_i \notin A_m) \geq \delta$ and through $\Pr(\mathcal{R}_i \in A_m) - m||A_m|| > (1 - \delta) - (1 - \beta) = \beta - \delta$. This allows us to choose $X$ and $W'$ as in Figure 3.

Let $f_1(g) = \max(0, \Pr(\mathcal{R}_i = g) - m)$. Let $\Pr(J = 1) = \sum_{g \in G} f_1(g) = \Pr(\mathcal{R}_i \in A_m) - m||A_m||$. Define $X$ so that $\Pr(X = g) = f_1(g)/\Pr(J = 1) = f_1(g)/f_1(g')$. Let $f_2(g) = \min(m, \Pr(\mathcal{R}_i = g))$. Let $\Pr(K = 1) = \sum_{g \in G} f_2(g) = \Pr(\mathcal{R}_i \notin A_m) + m||A_m||$. Define $W'$ so that $\Pr(W' = g) = f_2(g)/\Pr(K = 1) = f_2(g)/f_2(g')$ and $W'$ is independent of $X$. Note
that $\Pr(J = 1) = \sum_{g \in G} f_1(g) > 1 - 2\delta - (1 - \beta) = \beta - 2\delta$. Note that $\Pr(K = 1) = \sum_{g \in G} f_2(g) = \Pr(R_i \notin A_m) + m|A_m|$ and hence $\delta \leq \Pr(K = 1) < (1 - \beta) + 2\delta = 1 + 2\delta - \beta$.

We wish to apply Theorem 3.3. Let $X$ and $W$ of Theorem 3.3 correspond to $X$ and $W'$ in our context. Denote the $\delta$ of Theorem 3.3 by $\delta' = \delta / \Pr(K = 1) > \delta / (1 + 2\delta - \beta)$ for $\delta$ in our context. The conclusion of the theorem then yields that for a fixed $g$ drawn from the distribution of $W'$, with probability at least $1 - 1/\lambda$, $||Xg^{E_i}|| \leq \sqrt{\lambda(1 - \delta'/2)||X||} < a||X||$, where

$$a = \sqrt{\frac{2 + 3\delta - 2\beta}{2 + 4\delta - 2\beta}}.$$

We have $||Xg^{E_i}||/||X||$ bounded above, and we wish to invoke Theorem 4.5 by identifying $Z$ with $R_i$ and $A = \text{supp}(X)$ with $A_m$. The conditions $\Pr(J = 0) \Pr(Y = g) = m$ for $g \in \text{supp}(X)$ and $\Pr(J = 0) \Pr(Y = g) \leq m$ hold also in our context. We invoke the theorem with $a$ as above, and $\Pr(Z \notin A_m) = \Pr(R_i \notin A_m) < 2\delta$. Recall that $\Pr(J = 1) > \beta - 2\delta$. So, $c = (1 - a^2)(\Pr(J = 1))^2/(m|A_m| \Pr(Z \notin A_m)) + 1$ 

$$c > \left(1 - \frac{2 + 3\delta - 2\beta}{2 + 4\delta - 2\beta}\right) \frac{\beta - 2\delta}{1 + 2(1 - \beta)\delta} \text{ def } = \overline{c}$$

$$c < 1 - a^2$$

in Theorem 4.5. The random variable $W$ of Theorem 4.5 corresponds to $g^{E_i}$ in our current context and $Z$ corresponds to $R_i$. To employ Theorem 4.5, we also require that $a < 1$, from which $c < 1 - a^2$ implies $\sqrt{1 - c} < 1$. For $\lambda > 1$ sufficiently small, $a < 1$.

Hence, $0 < \sqrt{1 - c} < 1$ and $c$ is a constant determined by $\lambda$, $\delta$ and $\beta$. So, we have $||R_i g^{E_i}|| \leq \sqrt{1 - \overline{c}}||R_i||$ with probability at least $(1 - 1/\lambda)$ for $g$ drawn from the distribution of $W'$. Since $W = W'^{K}T^{1-K}$, one sees that $||R_i g^{E_i}|| \leq \sqrt{1 - \overline{c}}||R_i||$ for $g$ drawn from the distribution of $W$ with probability at least $(1 - 1/\lambda) \Pr(K = 1) \geq (1 - 1/\lambda)\delta$.

**Figure 3:** Case 2: Left shaded part is unnormalized probability density for $X$; right shaded part is unnormalized probability density for $W'$ (shaded part has area less than 1)

**Case 2:** ($m < \delta$ and $\max_{g \in G} \Pr(R_i = g) \leq 1 - \delta$ and $m|A_m| \geq 1 - \beta$ and $\Pr(V_1^{-1}V_2 \in A_m) \leq 0.997$). Intuitively, if $\Pr(W'X \in A_m)$ is small, then we will make progress toward a more uniform distribution via Corollary 3.3. We enforce this through $\Pr(V_1^{-1}V_2 \in A_m) \leq 0.997$. We choose an $X$ close to $V_2$ and choose $W' = V_1^{-1}$ as in Figure 3. One knows that $||X||$ and $||W'||$ are sufficiently large, since $m|A_m| \geq 1 - \beta$.

Let $X$ be a random variable such that $\Pr(X = g) = \Pr(R_i = g) / \Pr(R_i \in A_m)$ for $g \in A_m$ and $\Pr(X = g) = 0$ for $g \notin A_m$. Set $W' = V_1^{-1}$. Set $\Pr(J = 1) = \Pr(R_i \in A_m)$ and note that $\Pr(J = 1) \geq 1 - 2\delta$. Similarly, set $\Pr(K = 1) = m|A_m|$ and note that $\Pr(K = 1) \geq 1 - \beta$.

One wishes to apply Corollary 3.3 with $X$ and $W'$. One shows that $||W'X|| \leq \sqrt{0.997}||X||$. By Lemma 4.5, $\Pr(W'X = g) \leq \max_{h \in G} \Pr(W' = h) = 1/|A_m|$. So $||W'X||$
is maximized when \( \Pr(W'X = g) \) equals \( 1/|A_m| \) or equals 0 for all \( g \). Note that \( \text{supp}(X) = A_m \). Define \( Z \) and \( \phi = \Pr(W'X \in A_m) \) as in Corollary 3.3. Hence, \( Z = W'X \mid W'X \in A_m \) (\( Z \) is the random variable \( W'X \) conditioned on the event \( W'X \in A_m \)). Note that one can write \( X \overset{\text{prob}}{=} \mathcal{N}_2 Y^{1 - J} \) for \( \Pr(J' = 1) = m|A_m|/\Pr(R_i \in A_m) \geq (1 - \beta)/(1 - \delta) \). Since \( \Pr(J = 1)|X| \geq c \Pr(J = 0)|Y| \), one sees that

\[
\Pr(W'X \notin A_m) \geq \left( \frac{(1 - \beta)/(1 - \delta)}{0.003(1 - \beta)/(1 - \delta)} \right) \Pr(W'V_2 \notin A_m) \geq 0.003(1 - \beta)/(1 - \delta).
\]

So \( \|W'X\| \leq \|Z\| \leq (1/|A_m|)\sqrt{|A_m|(1 - 0.003(1 - \beta)/(1 - \delta))} \leq \sqrt{1 - 0.003(1 - \beta)/(1 - \delta)} \|U_{A_m}\| \leq \sqrt{1 - 0.003(1 - \beta)/(1 - \delta)} \|X\| \leq \sqrt{0.997} \|X\|.
\]

Apply Corollary 3.3 with \( X \) and \( W' \) as above, and with \( c = \sqrt{0.997} \). With probability \( \Pr(K = 1) = m|A_m| \geq 1 - \beta \), a random \( g \) drawn from the distribution of \( R_i \) is as if \( g^{-1} \) were drawn from the distribution of \( W' \). Note that \( \phi < 1 \). Applying the corollary now yields \( \|g^{E_i}X\| < \sqrt{\lambda(1 + c\phi)/2} \|X\| < \sqrt{(1 + \sqrt{0.997})/2} \|X\| \) with probability at least \( 1 - 1/\lambda \) for \( g \) drawn from the distribution of \( W' \). We require \( \lambda > 1 \) to satisfy \( (1 + \sqrt{0.997})/2 < 1 \).

We wish to apply Theorem 4.4. (In fact, a variation of Theorem 4.4 is invoked for \( WZ \) instead of for \( ZW \).) The random variable \( Y \) is defined by \( R_i \overset{\text{prob}}{=} X^{J} \mathbf{Y}^{1 - J} \). To apply the theorem, we need a positive constant \( c \) such that \( \Pr(J = 1)||X|| \geq c \Pr(J = 0)||Y|| \). Note that \( \Pr(J = 1)||X|| = \Pr(J = 1)||U_{A_m}|| \). Note that \( \Pr(J = 0)||Y|| \leq \sqrt{(2\delta/m)m^2} = \sqrt{2m\delta} \). Recall that \( ||U_{A_m}|| = 1/\sqrt{|A_m|} \) by Lemma 2.12. Hence, one can choose

\[
c = (1 - 2\delta)/\sqrt{2\delta},
\]

since \( \Pr(J = 1)||X|| \geq \Pr(J = 1)||U_{A_m}|| \geq (1 - 2\delta)||U_{A_m}|| = c\sqrt{2\delta}||U_{A_m}|| > c\sqrt{2\delta m|A_m|}||U_{A_m}|| = c\sqrt{2m\delta/||U_{A_m}||^2}||U_{A_m}|| = c\sqrt{2m\delta} > c\Pr(J = 0)||Y||.
\]

Theorem 4.4 is then invoked with the above \( c \) and with \( a = \sqrt{\lambda(1 + \sqrt{0.997})/2} \). The \( W \) and \( Z \) of Theorem 4.4 correspond to \( g^{E_i} \) and \( R_i \) in our context. So, \( ||g^{E_i}R_i|| \leq \left( (1 + ac)/\sqrt{1 + c^2} \right) ||R_i|| \) with probability at least \( (1 - 1/\lambda) \) for \( g \) drawn from the distribution of \( W' \). Since \( W = W^{K} \mathbf{Y}^{1 - K} \), one sees that \( ||g^{E_i}R_i|| \leq \left( (1 + ac)/\sqrt{1 + c^2} \right) ||R_i|| \) for \( g \) drawn from the distribution of \( W \) with probability at least \( (1 - 1/\lambda) \Pr(K = 1) \geq (1 - 1/\lambda)(1 - \beta) \).

For the inequality \( ||g^{E_i}R_i|| \leq \left( (1 + ac)/\sqrt{1 + c^2} \right) ||R_i|| \) to be useful, we require that \( (1 + ac)/\sqrt{1 + c^2} < 1 \). This is true if \( a < 1 \) and \( c \) is sufficiently large. For the former, we need only require that \( \lambda > 1 \) be sufficiently small so that \( a = \sqrt{\lambda(1 + \sqrt{0.997})}/2 < 1 \). For the latter, it suffices to make \( \delta \) sufficiently small. We omit the computation of the explicit requirements for \( \delta \).

**Figure 4:** Case 3: Shaded part is unnormalized probability density for \( X \) (shaded part has area less than 1)

**Case 3:** (\( m < \delta \) and \( \max_{g \in G} \Pr(R_i = g) \leq 1 - \delta \) and \( m|A_m| \geq 1 - \beta \) and \( \Pr(V_1^{-1}V_2 \in A_m) > 0.997 \)) Intuitively, one constructs an \( A' \) close to \( A_m \) with \( A'A' \) a subgroup of \( G \) (Theorem 5.2).
The argument then splits, based on whether $A' A'$ is proper in $G$. If $A' A'$ is proper in $G$, then we choose an $X$ close to $V_1$ as in Figure 4. The random variable $W = W'$ will be the distribution of random subproducts on the generators of $G$. Under the conditions of Case 3, one then shows that a random subproduct $g_i$ drawn from $W$ has probability at least 1/2 of satisfying $A' g_i \cap A' = \emptyset$. Hence, $X g_i$ escapes from the “fuzzy subgroup” $A_m$ with high probability (Theorem 5.2). So $X g_i^{A_m}$ makes progress toward a uniform distribution. If, on the other hand, $A' A'$ is proper, then one can show that $U^{A_m} U^{A'}$ is already closed to uniform.

We will first construct $A' \subseteq A_m$ such that $A' A'$ is a group. The random variable $X$ is then defined such that $\Pr(X = g) = \Pr(R_i = g) / \Pr(R_i \in A')$ for $g \in A'$ and $\Pr(X = g) = 0$ for $g \notin A'$. Let $W = W'$ be the distribution of random subproducts on the generators of $G$. Let $\Pr(J = 1) = \Pr(R_i \in A')$. Note that $\Pr(J = 1) \geq 1 - \delta$.

Since $\Pr(V_i^{-1} V_2 \in A_m) > 0.997$, $\Pr(V_i^{-1} V_2 \notin A_m) = \Pr(V_i^{-1} V_1 = (V_2^{-1} V_1)^{-1} \notin A_m^-) \leq 0.003$. Recall that $V_1$, $V_2$ and $U_{A_m}$ are identically distributed. Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be independent random variables with the same distribution as $U_{A_m \cap A_m}$. Hence, $\Pr(\mathcal{V}_1^{-1} \mathcal{V}_2 \notin A_m \cap A_m^-) < \Pr(\mathcal{V}_1^{-1} \mathcal{V}_2 \notin A_m) + \Pr(\mathcal{V}_1^{-1} \mathcal{V}_2 \notin A_m^-) \leq 0.006$.

We claim $|A_m \cap A_m^-| > 0.976 |A_m|$. Since $V_1$ and $V_2$ are independent, $E(|A_m^{-1} V_2| \cap A_m^-)/|A_m^-| = \{(u, v) \in A_m: u^{-1} v \in A_m\}/|A_m^-|^2 = \Pr(V_i^{-1} V_2 \in A_m^-) > 0.997$. Similarly, $E(|A_m^{-1} V_1| \cap A_m^-)/|A_m^-| = \{(v, v) \in A_m: v^{-1} v \in A_m\}/|A_m^-|^2 = \Pr(V_i^{-1} V_2 \in A_m^-) > 0.997$. Applying Lemma 2.2 with its parameter $\lambda = 4$ yields $\Pr(|A_m^{-1} V_2| \cap A_m^-) > 0.988$ and $\Pr(|A_m^{-1} V_1| \cap A_m^-) > 0.988$ which, at least half of the elements $h \in A_m$ satisfy both $|A_m^{-1} h| \cap A_m^-/|A_m^-| > 0.988$ and $|A_m^{-1} h| \cap A_m^-/|A_m^-| > 0.988$. Choosing one such $h$ yields $|A_m^{-1} \cap A_m^-|/|A_m^-| > 0.976$.

Combining $|A'| \geq (9/10) |A_m \cap A_m^-|$, we obtain $|A_m \cap A_m^-| > 0.976 |A_m|$. Recall that $A' \subseteq A_m$, $A' = A'$ is a group, and $A' A' \subseteq A'/2$. If $|A_m| < (2/3) |G|$, then $|A'| < (2/3) |G|$ and so $A' A'$ is proper in $G (A' A' \subseteq G)$.

Assume for the remainder of this case that $|A_m| < (2/3) |G|$, and hence $A' A' \subseteq G$. We show that $\|X g^{E_1}\| = \|X\|/\sqrt{2}$ for $g$ drawn from $W$, with probability at least 1/2. Let $W$ be a uniform random variable on the random subproducts on the generators of the group $G$. By Lemma 2.4, $|A' A'| < |G|$ implies $\Pr(W \notin A' A') \geq 1/2$. Composing $\Pr(W \notin A' A') \geq 1/2$ with Lemma 2.6 implies that $\Pr(W \cap A' = \emptyset) = 1/2$. Since $X = U_{A'}$, $\|X g^{E_1}\| = \|X\|/\sqrt{2}$ with probability at least 1/2 for $g$ drawn from $W$.

We wish to apply Theorem 4.4. The random variable $X$ is defined by $R_i = X \cup Y^{1-J}$. To apply the theorem, we need a positive constant $c$ such that $\Pr(J = 1) = \|X\| \geq c \Pr(J = 0) / |Y|$. Recall that $\|U_{A_m}\| = 1/\sqrt{|A_m|}$ by Lemma 1.12, and similarly $\|U_{A_m}\| = 1/\sqrt{|A'|}$. Note that $|A'| > 0.85 |A_m|$ and $\Pr(J = 1) \geq 1 - \beta$ implies $\Pr(J = 1) = \Pr(J = 1) = \|A'|/|A_m| \Pr(J = 1) = \|U_{A_m}\| > 0.85 (1 - \beta) |U_{A_m}|$. Note that $\Pr(J = 0) = \|Y\| < (2\delta/m) m^2 = \sqrt{2m\delta} < \sqrt{2m\delta}$. Hence, one can choose

$$c = 0.85 (1 - \beta) / \sqrt{2\delta},$$

since $\Pr(J = 1) = \|X\| > 0.85 (1 - \beta) |U_{A_m}| = c \sqrt{2\delta} |U_{A_m}| > c \sqrt{2\delta m |A_m|} |U_{A_m}| = c \sqrt{2m\delta} |U_{A_m}|^2 = \sqrt{2m\delta} / |U_{A_m}|.$
Theorem 4.4 is then invoked with the above $c$ and with $a = 1/\sqrt{2}$. The $W$ and $Z$ of Theorem 4.4 correspond to $g^{E_i}$ and $R_i$ in our context. So, $\|g^{E_i}R_i\| \leq \left( (1 + ac)/\sqrt{1 + c^2} \right) \|R_i\|$ with probability at least $1 - 1/\lambda$ for $g$ drawn from the distribution of $W$.

For the inequality $\|g^{E_i}R_i\| \leq \left( (1 + ac)/\sqrt{1 + c^2} \right) \|R_i\|$ to be useful, we require that $(1 + ac)/\sqrt{1 + c^2} < 1$. This is true if $c > \sqrt{2}$. For this, it suffices to make $1 - \beta > 2\sqrt{\delta}$.

The preceding analysis demonstrates the following lemma.

**Lemma 6.2** Let $R_i$ be independent and identically distributed to $R_i$. For any choice of positive parameters $\alpha$, $b$ and $c$ in the Fibonacci cube algorithm, there are constants $\alpha < 1$, $\rho > 0$, $\beta > 0$ and $\iota > 0$ such that for $i > \iota \log |G|$ one of the following holds:

(i) $\|R_{i+1}\| \leq \alpha \|R_i\|$ with probability at least $\rho$; or

(ii) $\Pr(\|R_{i+1}\| = g) \geq (3/4)(1 - \beta)^2/|G|$ for all $g \in G$.

Further, Let $\phi > 1$. For $i \geq \iota \log |G| + (\phi/\rho)(1 + (1/2)\log_1/\alpha |G|)$, case ii above occurs with probability at least $1 - \exp(-(\phi(1 - 1/\phi)^2/4) \log_1/\alpha |G|)$.

**Proof:** The proof follows from the analysis of the three cases just presented. As discussed in the analysis of Case 0, after a constant number of steps of the Fibonacci cube algorithm, Case 0 will never again be revisited, with high probability. Therefore, after $\iota \log |G|$ steps, for some constant $\iota$, the probability of ever revisiting Case 0 will be less than $\exp(-\log |G|)$. Hence, we can ignore Case 0 for purposes of the analysis.

We show that $|A_m| < (2/3)|G|$ implies case i and that $|A_m| \geq (2/3)|G|$ implies case ii. Assume first that $|A_m| < (2/3)|G|$. In each of the three cases, we concluded that $\|R_{i+1}\| = \|R_ig^{E_i}\| \leq \alpha \|R_i\|$ with probability at least $\rho$ for appropriate $\alpha < 1$ and $\rho > 0$. (In Case 3, this conclusion need not hold if $|A_m| \geq (2/3)|G|$.) The parameters $\alpha$ and $\rho$ are defined in terms of $\beta$, $\delta$, $\lambda$ and $G$ for each of the three cases.

In order to make the parameters $\alpha$ independent of the particular case, one chooses $\alpha$ to be the maximum of the three definitions for each of the three cases. In order to make $\rho$ independent of the particular case, define $\rho_1$, $\rho_2$ and $\rho_3$ to be the probabilities for the three cases. Then let $\rho = \min(\rho_1/(ad), \rho_2/(bd), \rho_3/(cd))$ for $d = 1/a + 1/b + 1/c$. In particular, $\rho$ can be maximized by choosing $a = \rho_1$, $b = \rho_2$ and $c = \rho_3$, whereupon $\rho = 1/(1/\rho_1 + 1/\rho_2 + 1/\rho_3)$.

It remains to verify that the constants $\beta$, $\delta$ and $\lambda$ can be simultaneously chosen to meet the requirements of the analysis in Cases 1, 2 and 3. Recall that $1 > \beta > 2\delta > 0$ and $\lambda > 1$. Collecting the bounds from Case 1, we require $\lambda > 1$ to be sufficiently small that $\alpha = \sqrt{\frac{2 + 3\delta - 2\beta}{2 + 1\delta - 2\beta}}$. Collecting the bounds from Case 2, we require that $\lambda > 1$ such that $\lambda(1 + 0.997)/2 < 1$. We further require that $\delta$ be sufficiently small to satisfy $(1 + ac)/\sqrt{1 + c^2} < 1$ for $c = (1 - 2\delta)/\sqrt{2\delta}$. The bounds from Case 3 require that $\lambda > 1$ and $1 - \beta > 2\sqrt{\delta}$.

There can be at most $\log_1/\alpha \sqrt{|G|} = O(\log |G|)$ distinct instances of $i$ such that $\|R_{i+1}\| > \alpha \|R_i\|$. To see this, note that $\|R_0\| = 1$ and $\|R_i\| \geq \|U_G\| = 1/\sqrt{|G|}$ for all $i$ by Lemma 2.12 and that $\|R_{i+1}\| \leq \|R_i\|$ by Lemma 2.10.

With the probability in the statement of the lemma, we must show we are in Case 3 and $|A_m| \geq (2/3)|G|$ with the stated probability after the stated number of steps. We will then show that this implies case ii. We define the $i$-th step to be a *success* if $\|R_{i+1}\| \leq \alpha \|R_i\|$. So, at most $\log_1/\alpha \sqrt{|G|}$ successes may occur for distinct $i$. We know that for a given $i$, a success will occur with probability at least $\rho$, or else $|A_m| \geq (2/3)|G|$.
Consider Chernoff’s bound (Theorem 2.3). Assume a success with probability at most \( p = \rho \), and assume \( t = (1 + \log_{1/\alpha} \sqrt{|G|})/(\rho(1 - \epsilon)) \) trials. Chernoff’s bound predicts at least \( [(1 - \epsilon)pt] \geq \log_{1/\alpha} \sqrt{|G|} \) successes over \( t \) trials with probability at least \( 1 - \exp(-\epsilon^2 pt/2) \). We have seen that more than \( \log_{1/\alpha} \sqrt{|G|} \) successes are impossible. So, with probability at least \( 1 - \exp(-\epsilon^2 pt/2) \), we are in Case 3 and \( |A_m| \geq (2/3)|G| \) for some step \( j \) among the first \( t \) steps. Let \( \epsilon = 1 - 1/\phi \) for \( \phi > 1 \). This yields the probability of the lemma.

Hence, there is a \( j \) such that \( \mathcal{R}_j \) is in Case 3 and \( |A_m| \geq (2/3)|G| \). Combining the condition \( m|A_m| \geq 1 - \beta \) of Case 3 with \( |A_m| \geq (2/3)|G| \) implies that \( m \geq (1 - \beta)/((2/3)|G|) \). Define a \( \{0, 1\} \)-random variable \( J \) such that \( \Pr(J = 1) = m|A_m| \). Note \( \Pr(J = 1) \geq 1 - \beta \). Let \( \mathcal{R}_j \overset{\text{prob}}{=} X^J Y^{1-J} \) for \( X = U_{A_m} \). Let \( J \) be independent and distributed identically to \( J \). Similarly, let \( X \) be independent and distributed identically to \( X \). Then for \( V \) an arbitrary \( G \)-valued random variable, \( \mathcal{R}_j^{-1} V \mathcal{R}_j \overset{\text{prob}}{=} (X^{-1}VX)^{JY^{1-J}} \) for some \( G \)-valued independent random variable \( Y' \).

We show that for an arbitrary \( G \)-valued random variable \( V \), \( \Pr(\mathcal{R}_j^{-1} V \mathcal{R}_j = g) \geq (3/4)(1 - \beta)^2/|G| \) for all \( g \in G \) when \( |A_m| \geq (2/3)|G| \). With probability at least \( (1 - \beta)^2 \), \( J = J = 1 \). Hence, with probability at least \( (1 - \beta)^2 \), we can take \( \mathcal{R}_j^{-1} V \mathcal{R}_j = X^{-1}VX \). Applying Lemma 2.2 with \( A = A_m \) and \( \alpha = 2/3 \), one sees that \( X^{-1}VX \) is 1/2-uniform and that \( \Pr(X^{-1}VX = g) \geq (|G|/3)/|A_m|^2 \geq (3/4)/|G| \).

We have seen \( \Pr(\mathcal{R}_j^{-1} V \mathcal{R}_j = g) \geq (3/4)(1 - \beta)^2/|G| \). We show that \( \Pr(\mathcal{R}_i^{-1} \mathcal{R}_j = g) \geq (3/4)(1 - \beta)^2/|G| \) for all \( i \geq j \). To see this, define \( \mathcal{X} = \mathcal{R}_i^{-1} \mathcal{R}_j = V_i \mathcal{R}_j^{-1} V_j \mathcal{R}_j \mathcal{V}_j \). For \( U \) uniform on \( G \), we can write \( \mathcal{R}_j^{-1} V \mathcal{R}_j \overset{\text{prob}}{=} U^J Y^{1-J} \) for \( J \) an independent \( \{0, 1\} \)-random variable with \( \Pr(J = 1) = (3/4)(1 - \beta)^2 \). So \( \mathcal{X} \overset{\text{prob}}{=} (V_i UV_3)^J (V_i Y V_3)^{1-J} \). Lemma 2.2 shows that \( V_i UV_3 \) is uniform. So \( \Pr(\mathcal{X} = g) \geq \Pr(J = 1)/|G| = (3/4)(1 - \beta)^2/|G| \) for all \( g \in G \).

For some applications, Lemma 5.2 may suffice, since it promises to produce each group element with a minimum probability \( (3/4)(1 - \beta)^2/|G| \). For an \( \varepsilon \)-uniform random distribution, one must do a little more. The next section is concerned with producing an \( \varepsilon \)-uniform distribution.

### 7 Constructing \( \varepsilon \)-uniform from \( \varepsilon \)-semi-uniform

Lemma 5.2 shows that for \( i \) sufficiently large, Algorithm Fibonacci Cube constructs an \( \alpha \)-semi-uniform random variable, \( \mathcal{R}_i^{-1} \mathcal{R}_i \), with the stated probability for \( \alpha = (3/4)(1 - \beta)^2 \). This section shows that constructing a \( \varepsilon \)-semi-uniform random distribution is tantamount to constructing a \( \varepsilon \)-uniform random distribution. This is shown in the next theorem uses \( W = \mathcal{R}_i^{-1} \mathcal{R}_i \) in order to efficiently construct a \( \beta \)-uniform random variable.

**Theorem 7.1** Let \( G \) be a group. Let \( W \) be an \( \alpha \)-semi-uniform random variable on \( G \). Let \( P_0 \) be an arbitrary \( G \)-valued random variable. Let \( E_i \) be independent, uniform random variables on \( \{0, 1\} \). For all \( i > 0 \), define \( P_{i+1} = P_i g_i^{E_i} \) for \( g_i \) drawn from the distribution of \( W \). Let \( \gamma = 14/(11 + 3\alpha) \). Then, \( \Pr(P_i = g) \leq 7/8 \) for \( \gamma \geq 2 \log_2 \gamma + \log_2(64\gamma) \), with probability at least \( 1 - 1/\lambda \). Hence \( WP_i \) is a max\( (\alpha, 7/8) \)-uniform random variable with probability at least \( 1 - 1/\lambda \).

**Proof:** Define the set \( A_i = \{g: \Pr(P_i = g) \geq (7/4)/|G|\} \). Note that \( |G \setminus A_i| \geq (3/7)|G| \), since otherwise \( |A_i| > (4/7)|G| \), which implies \( \Pr(P_i \in A_i) = \sum_{g \in A_i} \Pr(P_i = g) > |A_i|(7/4)/|G| \).

Define \( T_i = \sum_{h \in A_i} (\Pr(P_i = h) - (7/4)/|G|)^2 \) for \( i \geq 0 \). We will find an upper bound on \( E(T_{i+1}) \) as compared to \( T_i \). Define \( x_h = \Pr(P_i = h) - (7/4)/|G| \). Hence \( x_{h(g^{-1}E_i)} = x_h/2 + x_{hg^{-1}}/2 \) since...
$E_i$ and $P_i$ are independent. Note that for $i \geq 0$,

$$T_i = \sum_{h \in A_i} (\Pr(P_i = h) - (7/4)/|G|)^2 = \sum_{h \in G} (\max(0, x_h))^2.$$  

We show that $T_{i+1} \leq T_i$ for any value of $g_i$. Recall that $T_{i+1} = P_i g_i^{E_i}$.

$$T_{i+1} = \sum_{h \in A_{i+1}} \left( \Pr(P_i g_i^{E_i} = h) - (7/4)/|G| \right)^2$$

$$= \sum_{h \in G} \left( \max(0, x_{h(g_i^{-1})E_i}) \right)^2$$

$$= \sum_{h \in G} \left( \max(0, x_h/2 + x_{h_{g_i^{-1}}}/2) \right)^2$$

$$\leq \frac{1}{4} \sum_{h \in G} \left( \max(0, x_h) \right)^2 + \frac{1}{4} \sum_{h \in G} \left( \max(0, x_{h_{g_i^{-1}}}) \right)^2 + \frac{1}{2} \sum_{h \in G} \max(0, x_h) \max(0, x_{h_{g_i^{-1}}})$$

$$\leq T_i/4 + T_i/4 + T_i/2$$

$$= T_i$$

where the Cauchy-Schwartz inequality was invoked to show

$$\sum_{h \in G} \max(0, x_h) \max(0, x_{h_{g_i^{-1}}}) \leq \sqrt{\sum_{h \in G} (\max(0, x_h))^2} \sqrt{\sum_{h \in G} (\max(0, x_{h_{g_i^{-1}}}))^2} = T_i.$$

Since $W$ is $\alpha$-semi-uniform, by Lemma 4.1 we can write $W = U^J V^{1-J}$ for $G$-valued random variables $U$ and $V$, with $U$, $V$ and $J$ independent, $U$ uniform, and $Pr(J = 1) = 1 - \alpha$. Note that $2(\frac{x_h}{2} + \frac{x_g}{2})^2 \leq (x_h^2 + x_g^2)$ follows from elementary algebra. Note that $x_g \leq 0$ for $g \notin A_i$. The notation $E_{g_i \in U} E(f(A_i))$ denotes $E(f(U))$ for a function $f(\cdot)$ from $G$ to the real numbers. Since $U$ and $P_i$ are independent, if one conditions on $J = 1$ (implying that $g_i$ is drawn from $U$), then the following is true.

$$E_{g_i \in U} (T_{i+1} \mid J = 1)$$

$$= E_{g_i \in U} \left( \sum_{h \in A_{i+1}} \left( \Pr(P_i g_i^{E_i} = h) - (7/4)/|G| \right)^2 \right)$$

$$= E_{g_i \in U} \left( \sum_{h \in G} \left( \max \left(0, x_{h(g_i^{-1})E_i} \right) \right)^2 \right)$$

$$= \frac{1}{|G|} \sum_{h \in G} \sum_{g \in G} \left( \max(0, \frac{x_h}{2} + \frac{x_{h_{g_i^{-1}}}}{2}) \right)^2$$

$$= \frac{1}{|G|} \sum_{h \in G} \sum_{g \in G} \left( \max(0, \frac{x_h}{2} + \frac{x_g}{2}) \right)^2$$

$$= \frac{1}{|G|} \sum_{h \in A_i} \left( \max(0, \frac{x_h}{2} + \frac{x_g}{2}) \right)^2 + \frac{2}{|G|} \sum_{h \in A_i} \left( \max(0, \frac{x_h}{2} + \frac{x_g}{2}) \right)^2$$

$$+ \frac{1}{|G|} \sum_{h \notin A_i} \left( \max(0, \frac{x_h}{2} + \frac{x_g}{2}) \right)^2$$

$$= T_i.$$
\[
\begin{align*}
&\leq \frac{|A_i|}{|G|} \sum_{h \in A_i} x_h^2 + \frac{2|G \setminus A_i|}{|G|} \sum_{h \in A_i} x_h^2/4 \\
&= \frac{|A_i|}{|G|} T_i + \frac{|G \setminus A_i|}{2|G|} T_i
\end{align*}
\]

Recalling that \(|G \setminus A_i| \geq (3/7)|G|\), one sees

\[
E_{g_i \in U}(T_i + 1) \leq \frac{|A_i|}{|G|} T_i + \frac{|G \setminus A_i|}{2|G|} T_i \leq \frac{11}{14} T_i.
\]

Let \(\gamma = 14/(1 + 3\alpha)\). Then \(E(T_{i+1}) \leq E(T_i)/\beta\). To see this, note \(E(T_{i+1}) = \Pr(J = 1) E(T_{i+1} | J = 1) + \Pr(J = 0) E(T_{i+1} | J = 0) \leq (1 - \alpha)(11/14) E(T_i) + \alpha E(T_i) = (11/14 + 3\alpha/14) E(T_i)\). An easy argument implies \(E(T_{i+k}) \leq E(T_i)/\beta^k\).

Let \(\lambda > 1\) and let \(t \geq 2 \log_\gamma(8\sqrt{\lambda}|G|) = \log_{1/\gamma}(1/(8\sqrt{\lambda}|G|)^2)\). Since \(T_0 \leq 1\), \(E(T_i) \leq 1/(8\sqrt{\lambda}|G|)^2\). For \(\lambda > 1\) in Markov’s inequality (Lemma 2.1), one has \(\Pr(T_i < 1/(8|G|)^2) = \Pr(T_i < \lambda/(8\sqrt{\lambda}|G|)^2) \geq \Pr(T_i < \lambda E(T_i)) \geq 1 - 1/\lambda\). So,

\[
\Pr(T_i < 1/(8|G|)^2) \geq 1 - 1/\lambda \text{ for } t \geq 2 \log_\gamma(8\sqrt{\lambda}|G|).
\]

Note that \(T_i = \sum_{h \in A_i} (\Pr(P_t = h) - (7/4)/|G|)^2 \leq 1/(8|G|)^2\) implies that \(\max_{h \in G} \Pr(P_t = h) = (7/4)/|G| \leq 1/(8|G|)^2\). So \(\max_{h \in G} \Pr(P_t = h) \leq 15/(8|G|)\). If \(\Pr(P_t = g) \leq (15/8)/|G|\) for all \(g \in G\), then by Lemma 2.3, \((1 - \alpha)/|G| \leq \min_{g \in G} \Pr(W = g) \leq \max_{g \in G} \Pr(WP_t = g) \leq \max_{g \in G} \Pr(P_t = g) < (15/8)|G|\). Hence \(P_t\) is max(\(\alpha, 7/8\))-uniform with the given probability. \(\square\)

**Corollary 7.2** Assume a random variable \(X\) on \(G\) is \(\alpha\)-semi-uniform. Assume it costs \(c\) group operations to compute a group element drawn from the distribution of \(X\). There is a fixed constant \(\gamma\) such that one can construct a \(\gamma\)-uniform random variable \(Y\) for which one can draw a group element from the distribution of \(Y\) using \(O(c + \log |G|/(1 - \alpha))\) group operations. The cost of constructing \(Y\) is \(O(c \log |G|/(1 - \alpha))\) group operations.

The proof of the corollary is clear.

**Theorem 7.3** Let \(G = \langle S \rangle\) be a black box group with \(|G| < L\). One can construct a \(\varepsilon\)-uniform \(X\) such that the cost of computing a group element from the distribution of \(X\) is \(O((\log(1/\varepsilon)) \log |G|)\) group operations. The cost of constructing \(X\) is \(O(\log^2 |G| + |S| \log |G|)\). Where \(|G|\) is not known a priori, one can replace \(|G|\) by \(L\) in the asymptotic estimates.

**Proof:** The timing of the Fibonacci Cube algorithm is immediate since \(O(\log |G| + |S|)\) group operations are required to compute each \(g_i\), and \(t = O((\log 1/\varepsilon) \log |G|)\). So, the timing of the pseudo-code is \(O(\log^2 |G| + |S| \log |G|)\). To compute an element from the distribution of \(\mathcal{R}_{t-1}\mathcal{R}_t\), then requires \(t = O((\log 1/\varepsilon) \log |G|)\) group multiplications, where each factor \(g_i^{E_i}\) of \(\mathcal{R}_t\) contributes at most one to the number of multiplications.

Lemma 6.2 shows that one can construct an \(\alpha\)-semiuniform random variable \(X_1 = \mathcal{R}_{t-1}\mathcal{R}_t\) for \(\alpha = (3/4)(1 - \beta)^2\) in \(O(b \log |G|)\) steps for \(b = (\phi/\rho)/(\log(1/\alpha))\). Hence, \(O(b^2 \log^2 |G| + |S| \log |G|)\) group operations are required to construct \(X_1\). Computing an element from the distribution of \(X_1\) costs \(O(b \log |G|)\) group operations.

Corollary 7.2 shows that one can construct a \(\gamma\)-uniform random variable \(X_2\) using \(O(b^2 \log^2 |G|/(1 - \alpha) + |S| \log |G|)\) group operations. One can compute a group element from the distribution of \(X_2\) using \(O((b + 1/(1 - \alpha)) \log |G|)\) group operations.
Since $\beta$, $\alpha$, $\phi$ and $\rho$ are all constants, this implies that one can construct $X_2$ using $O(\log^2 |G| + |S| \log |G|)$ group elements and one can compute an element from $X_2$ using $O(\log |G|)$ group elements.

It remains to construct an $\epsilon$-uniform random element from the give $\gamma$-uniform random element for arbitrary $\epsilon > 0$. We take the product of $\lceil \log_2(\epsilon/\log_2 \gamma) \rceil$ many $\gamma$-uniform elements drawn from the distribution of $R_t^{-1}R_t$. By Lemma 7.2, this suffices to produce an $\epsilon$-uniform random element.

To compute an $\epsilon$-uniform random element requires $O(\log 1/\epsilon) \gamma$-uniform random elements. So, the number of operations to produce an $\epsilon$-uniform random element is $O((\log 1/\epsilon) \log |G|)$. □

**Remark 3** Chernoff’s bound shows that the probability of error can be further reduced by a power of $n$ at the cost of multiplying $t$ by the factor $n$.

Theorem 7.3 states a complexity of $O(\log^2 |G| + |S| \log |G|)$ group operations. In the unusual case that $|S| > O(\log |G|)$, there is a black box algorithm to quickly produce a smaller generating set $[BCF+91] \subset [CF93]$. We quote that theorem here.

**Theorem 7.4 (from [BCF+91], Theorem 2.3)** Let $G = \langle S \rangle$ be a finite group. Let $L$ be a known upper bound on the length of all subgroup chains in $G$. Then for any fixed parameter $p$ such that $0 < p < 1$, with probability at least $p$ one can find a generating set $S'$ with $|S'| = O(L \log (1/(1-p)))$, using $O(|S| \log L \log (1/(1-p)))$ group operations.

# 8 Experimental Results

The current results are highly preliminary. For the Fibonacci cube algorithm, we initialize the first elements of the cube to be the group generators. We take the parameters $a = b = c = 1$ as a simple heuristic choice. We compute only $R_t^{-1}R_t$, which, in principle, is $\epsilon$-semi-uniform, but not necessarily $\epsilon$-uniform. We take $t = 20, 25, 30$. After the precomputation of the $g_1, \ldots, g_{20}$ that determine $R_t^{-1}R_t$, we draw 10,000 elements from the distribution of $R_t^{-1}R_t$.

The table shows the results of tests on the distribution of $R_t^{-1}R_t$ according to a partition into conjugacy classes. (The conjugacy class of $g \in G$ is $\{g^h : h \in G\}$.) The groups tested on are all simple groups. Later experiments will consider other parameters than $a = b = c = 1$. They will incorporate the ideas of Section 7. They will also look at distributions over other group partitions than that of conjugacy classes.

The $\chi^2$ distribution was applied with a critical value of 0.05. The $\chi^2$ test accepts the hypothesis of uniform randomness when the observed $\chi^2$ statistic satisfies $\chi^2 < \chi^2_{0.05}$.

The number of degrees of freedom in the $\chi^2$ test is one less than the number of conjugacy classes. However, in most tests, the smaller conjugacy classes showed fewer than five observations. Hence, the smallest conjugacy classes have been merged so that the smallest set in the partition has just enough conjugacy classes to have at least five observations. The number of degrees of freedom is then adjusted accordingly, as one less than the number of final categories.

These experimental results are intended only to demonstrate the quality of the random elements in computer experiments. In principle, the distribution $g_{20}^{-1}E_{20}, \ldots, g_1^{-1}E_0, g_1E_1, \ldots, g_{20}E_0$ can produce at most $2^{40} \approx 10^{12}$ group elements. This is not too much larger than the order of the groups being tested. Hence, the experimental distribution of the individual group elements is most likely not close to uniform. However, the $\chi^2$ test shows that an empirical computation will not be able to distinguish the distribution of group elements according to conjugacy class from a distribution based on uniformly random group elements.
The $\chi^2$ test accepts the hypothesis at the 0.05 significance level for all groups, except the McLaughlin group (McL). The McLaughlin group is accepted at the 0.01 significance level. By using the ideas of Section 7, we are able to pass the $\chi^2$ test for McL at the 0.05 significance level. We achieve $\chi = 15.5$ for 20 degrees of freedom using only 15 group operations per random element.

For the McLaughlin group, we use the product replacement algorithm to generate random elements. The algorithm requires $O(\log |G|)$ steps to produce a random element, where $|G|$ is the order of the group. We can modify the algorithm to produce an $\varepsilon$-uniform distribution of random elements by restricting the use of group operations to elements of a specified set $K$. The modified algorithm requires $O(\log^2 |G|)$ steps to produce a random element.

Detailed distributions are provided in the context of the McLaughlin group in the appendix.

### 9 Product Replacement

The Fibonacci cube algorithm can emulate a variation of the product replacement algorithm, which produces an $\varepsilon$-uniform random element in $O(\log^2 |G|)$ steps. This should be compared with the work of Pak [Pak00] to produce nearly random $k$-sets (not elements) in the limiting distribution in $O(\log^9 |G|)$ with $k = O(\log |G| \log \log |G|)$.

To see this, choose $k = O(\log |G|)$ and modify the product replacement algorithm so that at each step, a randomly chosen element, $g_i$, of the $k$-set is chosen and all other elements of the $k$-set are multiplied by $g_i$. Further, after the $i$-th element has been chosen, it should not be chosen again. After $O(\log |G|)$ steps, an element of the $k$-set that has not yet been chosen will have an $\varepsilon$-uniform distribution. The proof is modelled on the proof for the Fibonacci cube algorithm. Further details will be provided in a different paper.

### 10 Permutation Group Membership

Precomputation of a group membership data structure for permutation groups allows one to compute group orders, find random elements, test an arbitrary permutation for group membership, etc. There are at least four such group membership data structures: Sims’s Schreier vectors (or Schreier trees) [Sim71], Knuth’s data structure [Knu91], Jerrum’s labelled branchings [Jer86], and the deep sift data structure of Cooperman and Finkelstein [CF93].

If $n$ is the permutation degree and $b < n$ is the size of a base, then $b \leq \log |G| \leq b \log n$. Schreier vectors require $O(bn)$ group operations in the worst case, but $O(\log |G|)$ operations typically, to produce a random element. Knuth’s data structure and deep sift require $O(\log |G|)$ operations to produce a random element. Jerrum’s data structure requires $O(b)$ operations to produce a random element. While the first three data structures require space proportional to the time to produce a random element, Jerrum’s data structure has the disadvantage of requiring space for $\Theta(n)$ group elements.

Cooperman and Finkelstein [CF94, Theorem A] had previously demonstrated a random base change algorithm requiring $O(\log |G|)$ random group elements as input. The base change algorithm produces a group membership data structure, thus solving permutation group membership. The
original paper assumed that the random group elements came from a Schreier vector, but the result of this paper provides an alternative source of such random elements. Combining this paper with the random base change algorithm and any of the four group membership data structures yields a Monte Carlo group membership algorithm operating in $O(\log^2 |G|)$ group operations.

Prior to this, the fastest general algorithm was the deep sift algorithm of Cooperman and Finkelstein, requiring $O(n^2 \log^4 n)$ group operations, and the fastest small base algorithm Babai, Cooperman, Finkelstein and Seress \cite{BCFS91}, required $O(\log^3 |G| + b^2 \log^2 |G| \log(b + \log n)) b^2 (\log b) (\log^2 |G|) (\log n)/n$ group operations. In both cases, the $O(\log^2 |G|)$ group operations using the new Fibonacci cube algorithm represents a significant improvement.

Any of these Monte Carlo algorithms can be upgraded to Las Vegas by applying a strong generating test afterwards. The danger with Monte Carlo algorithms is that they may not produce enough group elements to form a full strong generating set. Cooperman and Finkelstein \cite{CF91} demonstrate a $O(\log^3 |G|)$ algorithm for testing if a set of group elements forms a strong generating set. (In fact, the algorithm is $O(n^4)$ for a permutation group acting on $n$ points.)

11 Conclusion

The Fibonacci cube algorithm has been demonstrated to produce a $\gamma$-uniform random variable in $O(\log^2 |G|)$ group operations. From that distribution $\varepsilon$-uniform elements with $O((\log 1/\varepsilon) \log |G|)$ group operations can be computed. The algorithm is asymptotically faster than previous theoretical algorithms and also empirically faster than the product replacement heuristic for many groups. The faster random generation algorithm also yields a faster permutation group membership algorithm.

The coefficient of complexity of the Fibonacci cube algorithm analyzed in this paper is still unacceptably high. This may not be an issue for computations that have an independent check for correctness, such as Las Vegas algorithms, since the experimental results are competitive. An expanded version of this paper will refine the analysis to produce a smaller coefficient of complexity.

The large coefficient arises due to the constant 0.997 arising from Section 5. In Theorem 5.2 we are too greedy in demanding that $A'A' = A' = A'^{-1}$ (and therefore $A'g \setminus A' = \emptyset$). If we prove only that $A'A'$ does not differ greatly from $A'$, we can still prove that a random subproduct has reasonable probability of allowing us to escape the set $A'$.

12 Acknowledgement

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Appendix: Computational Experiment

This appendix is a quick note on a computation suggested by Persi Diaconis. It is about a quick computational experiment. It is not intended to be a polished document.

I test McLaughlin’s group (McL). I compare the true distribution of elements according to the conjugacy classes, with the distribution according to conjugacy classes produced by the random generator of the paper.

I take the constants \( a = b = c = 1 \) in the Fibonacci Cube algorithm. I use only the Fibonacci Cube algorithm, which in principle produces only a semi-uniform random variable. That is, in principle, this distribution will satisfy only

\[
\forall g \in G, \quad \Pr(X = g) > \alpha/|G|
\]

The paper has an additional step for producing nearly uniform random variables. I will test the full algorithm at a later date. I suspect the full algorithm will represent an improvement. But for now even the semi-uniform random variables seem to be close enough to uniform.

The code was written using GAP 4.2. The test here is for McL (McLaughlin group), of order 898,128,000, with 2 generators, based on a permutation representation on 275 points. The representation is provided by Walter Kim, U. Chicago, Feb., 2000.

For McLaughlin’s group, there are 24 conjugacy classes. For each conjugacy class, \( C_i \), I compute an integer, \( \lceil C_i/10^6 \rceil \). This is for convenience, since GAP doesn’t handle floating point. Since \( \sum \lceil C_i/10^6 \rceil = 886 \), I test the random generator by generating exactly 886 elements, and test their distribution into conjugacy classes.

In each case, the first row is the distribution of elements produced by the random generator (the number of elements in each of the 24 conjugacy classes). The second row corresponds to the true distribution, normalized to the form \( \lfloor C_i/10^6 \rfloor \). The notation 30 terms means that \( R_{30} \) was computed in the notation of the paper. The \( O(\log^2 |G|) \) precomputation refers to the computation of \( g_1, \ldots, g_{30} \) for \( R_{30} = g_1^{E_1} \cdots g_{30}^{E_{30}} \). The 886 random elements are then each drawn from \( R_{30}^{-1} R_{30} \). This is the \( O(\log |G|) \) computation. On average, the \( O(\log |G|) \) computation of a random element from \( R_{30}^{-1} R_{30} \) costs 30 group operations per random element (29 multiplications and one inverse).

Note that for less than 20 terms, there are too many pseudo-random elements in the first, third, tenth and eleventh conjugacy classes. This experimental observation reflects the theoretical model, which states only that \( R_i^{-1} R_i \) is semi-uniform. A future experiment will also test the theory of Section 7 for converting semi-uniform to \( \varepsilon \)-uniform. This should be more efficient in producing \( \varepsilon \)-uniform random elements.

Experiment 1:

30 terms, 174 group operations for \( O(\log^2 |G|) \) precomputation. 30 ops/rand elt.
[ 0, 38, 0, 31, 20, 32, 22, 0, 43, 0, 2, 12, 82, 75, 55, 68, 62, 72, 78, 102, 2, 31, 25, 34 ]
[ 0, 35, 1, 29, 29, 29, 29, 0, 29, 0, 2, 9, 74, 64, 64, 64, 64, 81, 81, 112, 0, 33, 33, 24 ]

Experiment 2:

30 terms, 144 group operations for \( O(\log^2 |G|) \) precomputation. 30 ops/rand elt.
[ 0, 31, 0, 33, 34, 27, 19, 0, 24, 0, 1, 13, 76, 79, 72, 64, 59, 94, 75, 105, 3, 29, 28, 20 ]
[ 0, 35, 1, 29, 29, 29, 29, 0, 29, 0, 2, 9, 74, 64, 64, 64, 64, 81, 81, 112, 0, 33, 33, 24 ]

Experiment 3:

20 terms, 74 group operations for \( O(\log^2 |G|) \) precomputation. 20 ops/rand elt.
[ 0, 46, 3, 25, 23, 27, 35, 0, 19, 0, 2, 7, 66, 69, 49, 50, 71, 94, 72, 137, 1, 41, 29, 20 ]
[ 0, 35, 1, 29, 29, 29, 29, 0, 29, 0, 2, 9, 74, 64, 64, 64, 64, 81, 81, 112, 0, 33, 33, 24 ]
Experiment 4:
20 terms, 88 group operations for $O(\log^2 |G|)$ precomputation. 20 ops/rand elt.
[ 0, 32, 1, 32, 31, 31, 18, 0, 22, 1, 3, 11, 78, 66, 49, 67, 76, 81, 74, 120, 0, 39, 23, 31 ]
[ 0, 35, 1, 29, 29, 29, 29, 0, 29, 0, 2, 9, 74, 64, 64, 64, 64, 81, 81, 112, 0, 33, 33, 24 ]

Experiment 5:
15 terms, 48 group operations for $O(\log^2 |G|)$ precomputation. 15 ops/rand elt.
[ 1, 44, 23, 38, 32, 27, 33, 0, 25, 9, 8, 11, 72, 50, 59, 50, 56, 79, 72, 104, 4, 31, 33, 25 ]
[ 0, 35, 1, 29, 29, 29, 29, 0, 29, 0, 2, 9, 74, 64, 64, 64, 64, 81, 81, 112, 0, 33, 33, 24 ]

Experiment 6:
15 terms, 44 group operations for $O(\log^2 |G|)$ precomputation. 15 ops/rand elt.
[ 8, 41, 54, 44, 36, 12, 14, 0, 10, 17, 11, 6, 94, 34, 46, 50, 69, 82, 71, 54, 1, 38, 52, 42 ]
[ 0, 35, 1, 29, 29, 29, 29, 0, 29, 0, 2, 9, 74, 64, 64, 64, 64, 81, 81, 112, 0, 33, 33, 24 ]

Experiment 7:
10 terms, 21 group operations for $O(\log^2 |G|)$ precomputation. 10 ops/rand elt.
[ 0, 47, 54, 55, 53, 11, 14, 1, 6, 30, 8, 11, 96, 47, 66, 44, 42, 80, 74, 39, 4, 45, 37, 22 ]
[ 0, 35, 1, 29, 29, 29, 29, 0, 29, 0, 2, 9, 74, 64, 64, 64, 64, 81, 81, 112, 0, 33, 33, 24 ]

Experiment 8:
10 terms, 19 group operations for $O(\log^2 |G|)$ precomputation. 10 ops/rand elt.
[ 17, 49, 82, 27, 28, 10, 12, 0, 15, 30, 7, 7, 101, 36, 26, 40, 54, 110, 107, 12, 0, 46, 51, 19 ]
[ 0, 35, 1, 29, 29, 29, 29, 0, 29, 0, 2, 9, 74, 64, 64, 64, 64, 81, 81, 112, 0, 33, 33, 24 ]