Local existence and uniqueness for the non-resistive MHD equations in homogeneous Besov spaces

*Jinlu Li*\(^1,2\) \quad Wenke Tan\(^3\) \quad and \quad Zhaoyang Yin\(^1,4\)

\(^1\)Department of Mathematics, Sun Yat-sen University, Guangzhou, 510275, China

\(^2\)School of Mathematics and Computer Sciences, Gannan Normal University, Ganzhou 341000, China

\(^3\)Department of Mathematics and Computer Science, Hunan Normal University, Changsha 410006, China

\(^4\)Faculty of Information Technology, Macau University of Science and Technology, Macau, China

Abstract

In the paper, we consider the Cauchy problem of the non-resistive MHD equations in homogeneous Besov spaces. We prove the local existence and uniqueness of the solution to the non-resistive MHD equations by using the iterative scheme and compactness arguments. Our obtained result improves considerably the recent results in \([2, 11]\).

Key Words: The non-resistive MHD equations; local existence and uniqueness; iterative scheme; compactness arguments.

Mathematics Subject Classification (2010): 35Q35, 35A01, 35A02, 76W05

1 Introduction

In this paper, we consider the following non-resistive MHD equations \((d \geq 2)\):

\[
\begin{aligned}
\partial_t u + u \cdot \nabla u - \Delta u + \nabla P &= B \cdot \nabla B, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^d, \\
\partial_t B + u \cdot \nabla B &= B \cdot \nabla u, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^d, \\
\text{div} u &= \text{div} B = 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^d, \\
(u, B)|_{t=0} &= (u_0, B_0), \quad x \in \mathbb{R}^d,
\end{aligned}
\]

\[(1.1)\]

*Email Address: lijl29@mail2.sysu.edu.cn (J. Li), tanwenkeybfq@163.com (W. Tan), mcsyzy@mail.sysu.edu.cn (Z. Yin, Corresponding author).*

1
where the unknowns are the vector fields \( u = (u^1, u^2, \cdots, u^d) \), \( B = (B^1, B^2, \cdots, B^d) \) and the scalar function \( P \). Here, \( u \) and \( B \) are the velocity and magnetic, respectively, while \( P \) denotes the pressure.

Fefferman et al. showed local-in-time existence of strong solutions to (1.1) in \( \mathbb{R}^d \), \( d = 2, 3 \) with the initial data \( (u_0, B_0) \in H^s(\mathbb{R}^d) \times H^s(\mathbb{R}^d) \), \( s > \frac{d}{2} \) in [7] and \( (u_0, B_0) \in H^{s-1+\varepsilon}(\mathbb{R}^d) \times H^s(\mathbb{R}^d) \), \( s > \frac{d}{2}, 0 < \varepsilon < 1 \) in [8]. Chemin et al. in [2] proved the local existence of solutions to (1.1) in \( \mathbb{R}^d \), \( d = 2, 3 \) with the initial data \( (u_0, B_0) \in \mathfrak{B}_{2,1}^{\frac{d}{2}-1}(\mathbb{R}^d) \times \mathfrak{B}_{2,1}^{\frac{d}{2}}(\mathbb{R}^d) \) and also proved the corresponding solution is unique in 3D case. Recently, Wan in [11] obtained the uniqueness of the solution in the 2D case by using mixed space-time Besov spaces.

However, the local existence and uniqueness of the solution for the Cauchy problem of the non-resistive MHD equations in homogeneous Besov spaces has not been studied yet. Whether or not the solution for the Cauchy problem of the non-resistive MHD equations exists locally in time and is unique in homogeneous Besov spaces is an open problem which was proposed by Chemin et al. in [2]. In the paper, our aim is to solve this open problem by establishing the local existence and uniqueness of the solution for the Cauchy problem (1.1) in homogeneous Besov spaces.

For convenience, we transform the system (1.1) into an equivalent form of compressible type. By using \( \text{div} u = \text{div} B = 0 \), we have

\[
\begin{align*}
  u \cdot \nabla u &= \text{div}(u \otimes u), \quad B \cdot \nabla B = \text{div}(B \otimes B), \quad B \cdot \nabla u = \text{div}(u \otimes B).
\end{align*}
\]

Therefore, the system (1.1) is formally equivalent to the following equations

\[
\begin{align*}
  \partial_t u - \Delta u &= \mathbb{P} \text{div}(B \otimes B - u \otimes u), \\
  \partial_t B + u \cdot \nabla B &= \text{div}(u \otimes B), \\
  (u, B)|_{t=0} &= (u_0, B_0),
\end{align*}
\]

where \( \mathbb{P} = I + \nabla(-\Delta)^{-1} \text{div} \) and \( \text{div} u_0 = \text{div} B_0 = 0 \).

To solve (1.2), the main difficulty is that the system is only partially parabolic, owing to the magnetic equation which is of hyperbolic type. This precludes any attempt to use the Banach fixed point theorem in a suitable space. Therefore, we would like to present a general functional framework to deal with the local existence and uniqueness of the solution for the compressible fluids of (1.2) in the homogeneous Besov spaces.

Our main result can be stated as follows:

**Theorem 1.1.** Let \( d \geq 2 \). Assume that the initial data \( u_0 \in \mathfrak{B}_{p,1}^{\frac{d}{2}-1}(\mathbb{R}^d) \), and \( B_0 \in \mathfrak{B}_{p,1}^{\frac{d}{2}}(\mathbb{R}^d) \). Then there exists a positive time \( T \) such that

(a) Local existence: If \( p \in [1,2d] \), then the system (1.1) has a solution \( (u, B) \in E_T^p \) with

\[
E_T^p \triangleq \left( C([0,T]; \mathfrak{B}_{p,1}^{\frac{d}{2}-1}(\mathbb{R}^d)) \cap L^1_T(\mathfrak{B}_{p,1}^{\frac{d}{2}}(\mathbb{R}^d)) \right)^d \times \left( C([0,T]; \mathfrak{B}_{p,1}^{\frac{d}{2}}(\mathbb{R}^d)) \right)^d.
\]
(b) Uniqueness: If \( p \in [1, 2d] \), then the uniqueness of the solution holds in \( E^p_T \).

**Remark 1.2.** Note that when \( d = 3 \), one has \( \mathfrak{B}^{\frac{1}{2}}_{2,1}(\mathbb{R}^3) \subset \mathfrak{B}^{\frac{1}{4}}_{2,1}(\mathbb{R}^3) \) and \( \mathfrak{B}^{\frac{3}{2}}_{2,1}(\mathbb{R}^3) \subset \mathfrak{B}^{\frac{3}{4}}_{2,1}(\mathbb{R}^3) \). This shows that our obtained result in Theorem 1.1 improves considerably the corresponding result in [2].

**Remark 1.3.** For \( d = 2 \), we also obtain a new result compared with the recent result in [2]. Since \( \mathfrak{B}^{0}_{2,1}(\mathbb{R}^2) \subset \mathfrak{B}^{0}_{2,1}(\mathbb{R}^2) \) and \( \mathfrak{B}^{1}_{2,1}(\mathbb{R}^2) \subset \mathfrak{B}^{1}_{2,1}(\mathbb{R}^2) \), then our obtained result in Theorem 1.1 and the corresponding result in [2] don’t contain each other.

**Remark 1.4.** When \( d = 2 \), scrutinizing our proof of the uniqueness (in Theorem 1.1) for (1.1) in homogeneous Besov spaces, we can also gain the uniqueness for (1.1) in nonhomogeneous Besov spaces. This implies that our obtained uniqueness result in Theorem 1.1 covers the recent result in [11], and our proof is more brief than that in [11].

The paper is organized as follows. In Section 2, we recall the Littlewood-Paley theory and give some properties of homogeneous Besov spaces. In Section 3, we prove the local existence and the uniqueness of the solution to the system (1.2).

**Notations.** In the following, we denote by \( \langle \cdot, \cdot \rangle \) the action between \( S'(\mathbb{R}^d) \) and \( S(\mathbb{R}^d) \). Given a Banach space \( X \), we denote its norm by \( \| \cdot \|_X \). Since all spaces of functions are over \( \mathbb{R}^d \), for simplicity, we drop \( \mathbb{R}^d \) in our notations of function spaces if there is no ambiguity.

## 2 Littlewood-Paley Analysis

In this section, we first recall the Littlewood-Paley theory, the definition of homogeneous Besov spaces and some useful properties. Then, we state some applications in the linear transport equation and the heat conductive equation.

First, let us introduce the Littlewood-Paley decomposition. Choose a radial function \( \varphi \in S(\mathbb{R}^d) \) supported in \( \widetilde{C} = \{ \xi \in \mathbb{R}^d, \frac{4}{3} \leq \xi \leq \frac{5}{3} \} \) such that

\[
\sum_{j \in \mathbb{Z}} \varphi(2^{-j} \xi) = 1 \quad \text{for all } \xi \neq 0.
\]

The frequency localization operator \( \hat{\Delta}_j \) and \( \hat{S}_j \) are defined by

\[
\hat{\Delta}_j f = \varphi(2^{-j} D) f = \mathcal{F}^{-1}(\varphi(2^{-j} \cdot) \mathcal{F} f), \quad \hat{S}_j f = \sum_{k \leq j - 1} \hat{\Delta}_k f \quad \text{for } j \in \mathbb{Z}.
\]

With a suitable choice of \( \varphi \), one can easily verify that

\[
\hat{\Delta}_j \hat{\Delta}_k f = 0 \quad \text{if } |j - k| \geq 2, \quad \hat{\Delta}_j (\hat{S}_{k-1} f \hat{\Delta}_k f) = 0 \quad \text{if } |j - k| \geq 5.
\]
Next we recall Bony’s decomposition from [1]:

\[ uv = \dot{T}_u v + \dot{T}_v u + \dot{R}(u, v), \]

with

\[ \dot{T}_u v = \sum_{j \in \mathbb{Z}} \dot{S}_j \dot{\Delta}_j v, \quad \dot{R}(u, v) = \sum_{j \in \mathbb{Z}} \dot{\Delta}_j u \Delta_j v, \quad \Delta_j v = \sum_{|j' - j| \leq 1} \dot{\Delta}_j' v. \]

The following Bernstein lemma will be stated as follows (see [1]):

**Lemma 2.1.** Let \( 1 \leq p \leq q \leq \infty \) and \( B \) be a ball and \( C \) a ring of \( \mathbb{R}^d \). Assume that \( f \in L^p \), then for any \( \alpha \in \mathbb{N}^d \), there exists a constant \( C \) independent of \( f, j \) such that

\[
\text{Supp } \hat{f} \subset \lambda B \Rightarrow \sup_{|\alpha| = k} \| \partial^\alpha f \|_{L^q} \leq C^{k+1} \lambda^{k+d(\frac{1}{p} - \frac{1}{q})} \| f \|_{L^p},
\]

\[
\text{Supp } \hat{f} \subset \lambda C \Rightarrow C^{-k-1} \lambda^k \| f \|_{L^p} \leq \sup_{|\alpha| = k} \| \partial^\alpha f \|_{L^p} \leq C^{k+1} \lambda^k \| f \|_{L^p}.
\]

Now, we will introduce the definition of the homogeneous Besov space. We denote the space \( \mathcal{Z}'(\mathbb{R}^d) \) by the dual space of \( \mathcal{Z}(\mathbb{R}^d) = \{ f \in \mathcal{S}(\mathbb{R}^d); D^\alpha \hat{f}(0) = 0; \forall \alpha \in \mathbb{N}^d \} \), which can be identified by the quotient space of \( \mathcal{S}'(\mathbb{R}^d)/\mathcal{P} \) with the polynomials space \( \mathcal{P} \). The formal equality \( f = \sum_{j \in \mathbb{Z}} \hat{\Delta}_j f \) holds true for \( f \in \mathcal{Z}'(\mathbb{R}^d) \) and is called the homogenous Littlewood-Paley decomposition.

The operators \( \hat{\Delta}_j \) help us recall the definition of the homogenous Besov space (see [1][10])

**Definition 2.2.** Let \( s \in \mathbb{R}, 1 \leq p, r \leq \infty \). The homogeneous Besov space \( \dot{\mathcal{B}}^s_{p, r} \) is defined by

\[ \dot{\mathcal{B}}^s_{p, r} = \{ f \in \mathcal{Z}'(\mathbb{R}^d); \| f \|_{\dot{\mathcal{B}}^s_{p, r}} < +\infty \}, \]

where

\[ \| f \|_{\dot{\mathcal{B}}^s_{p, r}} \triangleq \left\| \left( 2^{ks} \| \hat{\Delta}_k f \|_{L^p} \right)_k \right\|_r. \]

It is easy to check that \( \| Pf \|_{\dot{\mathcal{B}}^s_{p, r}} \leq C \| f \|_{\dot{\mathcal{B}}^s_{p, r}} \) for some positive constant \( C \). We also need to use Chemin-Lerner type Besov spaces introduced in [1],

**Definition 2.3.** Let \( s \in \mathbb{R}, 1 \leq p, q, r \leq \infty \) and \( T \in (0, \infty] \). The functional space \( \dot{L}^q_T(\dot{\mathcal{B}}^s_{p, r}) \) is defined as the set of all the distributions \( f(t) \) satisfying

\[ \| f \|_{\dot{L}^q_T(\dot{\mathcal{B}}^s_{p, r})} \triangleq \left\| \left( 2^{ks} \| \hat{\Delta}_k f(t) \|_{L^q_T(L^p)} \right)_k \right\|_r < +\infty. \]
By Minkowski’s inequality, it is easy to find that
\[ \|f\|_{L^r_p(\mathfrak{A}^s_{p,r})} \leq \|f\|_{L^r_T(\mathfrak{A}^s_{p,r})} \quad \text{if} \quad q \leq r, \quad \|f\|_{L^r_T(\mathfrak{A}^s_{p,r})} \geq \|f\|_{L^r_p(\mathfrak{A}^s_{p,r})} \quad \text{if} \quad q \geq r. \]

We state the following logarithmic interpolation inequality which will be useful in the sequel (see \[5\]).

**Lemma 2.4.** (see Proposition 2.8 in \[4\]) Let \( s \in \mathbb{R} \). Then, for any \( 1 \leq p, q \leq \infty \) and \( 0 < \varepsilon \leq 1 \), we have
\[
\|f\|_{L^q_T(\mathfrak{B}^s_{p,q})} \leq C \left( \varepsilon \right) \|f\|_{L^q_T(\mathfrak{B}^s_{p,\infty})} \log(e + \frac{\|f\|_{L^q_T(\mathfrak{B}^s_{p,\infty})}}{\|f\|_{L^q_T(\mathfrak{B}^s_{p,\infty})}}).
\]

Next, we give the important product acts on homogenous Besov spaces or Chemin-Lerner type Besov spaces by collecting some useful lemmas from \[3, 4\].

**Lemma 2.5.** (see Proposition 1.8 in \[4\]) Let \( 1 \leq p, q, q_1, q_2 \leq \infty \) with \( \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q} \). Then there hold
(a) if \( s_2 \leq \frac{d}{p} \), we have
\[
\|T_g f\|_{\mathfrak{B}^{s_1+2 s_2 - \frac{d}{p}}_{p,1}} \leq C \|f\|_{\mathfrak{B}^{s_1}_{p,1}} \|g\|_{\mathfrak{B}^{s_2}_{p,1}},
\]
\[
\|\hat{T}_g f\|_{L^q_T(\mathfrak{B}^{s_1+2 s_2 - \frac{d}{p}}_{p,1})} \leq C \|f\|_{L^q_T(\mathfrak{B}^{s_1}_{p,1})} \|g\|_{L^q_T(\mathfrak{B}^{s_2}_{p,1})}.
\]
(b) if \( s_1 + s_2 > d \max\{0, \frac{2}{p} - 1\} \), we have
\[
\|\hat{R}(f, g)\|_{\mathfrak{B}^{s_1+2 s_2 - \frac{d}{p}}_{p,1}} \leq C \|f\|_{\mathfrak{B}^{s_1}_{p,1}} \|g\|_{\mathfrak{B}^{s_2}_{p,1}},
\]
\[
\|\hat{R}(f, g)\|_{L^q_T(\mathfrak{B}^{s_1+2 s_2 - \frac{d}{p}}_{p,1})} \leq C \|f\|_{L^q_T(\mathfrak{B}^{s_1}_{p,1})} \|g\|_{L^q_T(\mathfrak{B}^{s_2}_{p,1})}.
\]

**Lemma 2.6.** (see Lemma 2.6 in \[3\]) Let \( s_1, s_2 \leq \frac{d}{p} \), \( s_1 + s_2 > d \max\{0, \frac{2}{p} - 1\} \) and \( 1 \leq p, q, q_1, q_2 \leq \infty \) with \( \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q} \). Then there hold
\[
\|f g\|_{\mathfrak{B}^{s_1+2 s_2 - \frac{d}{p}}_{p,1}} \leq C \|f\|_{\mathfrak{B}^{s_1}_{p,1}} \|g\|_{\mathfrak{B}^{s_2}_{p,1}},
\]
\[
\|f g\|_{L^q_T(\mathfrak{B}^{s_1+2 s_2 - \frac{d}{p}}_{p,1})} \leq C \|f\|_{L^q_T(\mathfrak{B}^{s_1}_{p,1})} \|g\|_{L^q_T(\mathfrak{B}^{s_2}_{p,1})}.
\]

**Lemma 2.7.** (see Lemma 2.7 in \[3\]) Let \( s_1 \leq \frac{d}{p} \), \( s_2 < \frac{d}{p} \), \( s_1 + s_2 \geq d \max\{0, \frac{2}{p} - 1\} \) and \( 1 \leq p, q, q_1, q_2 \leq \infty \) with \( \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q} \). Then there hold
\[
\|f g\|_{\mathfrak{B}^{s_1+2 s_2 - \frac{d}{p}}_{p,\infty}} \leq C \|f\|_{\mathfrak{B}^{s_1}_{p,\infty}} \|g\|_{\mathfrak{B}^{s_2}_{p,\infty}},
\]
\[
\|f g\|_{L^q_T(\mathfrak{B}^{s_1+2 s_2 - \frac{d}{p}}_{p,\infty})} \leq C \|f\|_{L^q_T(\mathfrak{B}^{s_1}_{p,\infty})} \|g\|_{L^q_T(\mathfrak{B}^{s_2}_{p,\infty})}.
\]
Finally, we will present the priori estimates of the linear transport equation

\[ \partial_t f + v \cdot \nabla f = g, \quad f(0, x) = f_0, \]  

and the heat conductive equation

\[ \partial_t u - \Delta u = G, \quad u(0, x) = u_0, \]  

in homogenous Besov spaces. The following estimates will be frequently used in the sequel (see [6]).

**Lemma 2.8.** (see Proposition 1.8 in [6]) Let \( s \in (-d \min\{\frac{1}{p}, 1 - \frac{1}{p}\} - 1, 1 + \frac{4}{p}) \), \( 1 \leq p, r \leq \infty \) and \( s = 1 + \frac{d}{p} \) if \( r = 1 \). Let \( v \) be a vector field such that \( \nabla v \in L^1_T(\dot{B}^{\frac{d}{p}}_{p,r} \cap L^\infty) \) and \( \text{div} v = 0 \). Assume that \( f_0 \in \dot{B}^s_{p,r}, g \in L^1_T(\dot{B}^s_{p,r}) \) and \( f \) is the solution of (2.1). Then there holds for \( t \in [0, T] \),

\[ ||f||_{L^\infty_t(\dot{B}^s_{p,r})} \leq e^{CV(t)}(||f_0||_{\dot{B}^s_{p,r}} + \int_0^t e^{-CV(\tau)}||g(\tau)||_{\dot{B}^s_{p,r}} d\tau), \]

or

\[ ||f||_{L^\infty_t(\dot{B}^s_{p,r})} \leq e^{CV(t)}(||f_0||_{\dot{B}^s_{p,r}} + ||g||_{L^1_T(\dot{B}^s_{p,r})}), \]

where \( V(t) = \int_0^t ||\nabla v||_{\dot{B}^{\frac{d}{p}}_{p,r} \cap L^\infty} d\tau \).

**Lemma 2.9.** (see Proposition 1.9 in [6]) Let \( s \in \mathbb{R} \) and \( 1 \leq q, q_1, p, r \leq \infty \) with \( q_1 \leq q \). Assume that \( u_0 \in \dot{B}^s_{p,r} \) and \( G \in L^{q_1}_T(\dot{B}^{s - \frac{2+\frac{4}{q}}{q}}_{p,r}) \). Then (2.2) has a unique solution \( u \in L^{q}_T(\dot{B}^{s + \frac{4}{q}}_{p,r}) \) satisfying

\[ ||u||_{L^{q}_T(\dot{B}^{s + \frac{4}{q}}_{p,r})} \leq C(||u_0||_{\dot{B}^s_{p,r}} + ||G||_{L^{q_1}_T(\dot{B}^{s - \frac{2+\frac{4}{q}}{q_1}}_{p,r})}). \]

### 3 Local existence and uniqueness in homogeneous Besov spaces

In this section, we will show the local existence and uniqueness of the solution to the system (1.2) with the initial data in homogeneous Besov spaces. However, because the whole system is not fully parabolic, the strong convergence of the sequence is shown for a weaker norm corresponding to a loss of one derivative. For that reason, the uniqueness issue is tractable by taking advantage of a logarithmic interpolation inequality together with Osgood’s lemma.
In the following, we divide the proof of Theorem 1.1 into four steps to prove the local existence and the uniqueness of the solution to the system (1.2).

**Step 1: An iterative scheme.** We will use an iterative scheme to obtain the approximating sequence to the system (1.2) by combining the linear transport equation and the heat conductive equation. Set \( u_0^n \triangleq \hat{S}_n u_0 \) and \( B_0^n = \hat{S}_n B_0 \), and define the first term \((u^0, B^0)\) of the approximating sequence to be

\[
 u^0 \triangleq e^{t \Delta} u_0^0, \quad B^0 \triangleq e^{t \Delta} B_0^0.
\]

Starting from the above term \((u^0, B^0)\), we define by induction a sequence \((u^n, B^n)_{n \in \mathbb{N}}\) of smooth functions by solving the following linear transport and heat conductive equations:

\[
\begin{cases}
\partial_t u^{n+1} - \Delta u^{n+1} = \mathbb{P} \text{div}(-u^n \otimes u^n + B^n \otimes B^n), \\
\partial_t B^{n+1} - u^n \cdot \nabla B^{n+1} = \text{div}(u^n \otimes B^n), \\
(u_0^{n+1}, B_0^{n+1}) = \hat{S}_{n+1}(u_0, B_0).
\end{cases}
\]

(3.1)

**Step 2: Uniform estimates.** Taking advantage of Lemmas 2.8 - 2.9, we shall bound the approximating sequence in the expected solution space on some fixed time interval. That is, for all \( T > 0 \) and \( n \in \mathbb{N} \), we have \((u^n, B^n) \in E_T^p\). Now, we claim that there exists some \( T > 0 \) independent of \( n \) such that the solution \((u^n, B^n)_{n \in \mathbb{N}}\) satisfies the following inequalities for some positive constants \( C_0 > 1 \) and \( \eta < 1 \) (to be determined later):

\[
(H_1) : \quad ||u^n||_{L_T^\infty(\mathbb{B}_{p,1}^{\frac{d}{p}-1})} + ||B^n||_{L_T^\infty(\mathbb{B}_{p,1}^\frac{d}{p})} \leq C_0 E_0,
\]

\[
(H_2) : \quad ||u^n||_{L_T^p(\mathbb{B}_{p,1}^{\frac{d}{p}+1})} + ||u^n||_{L_T^p(\mathbb{B}_{p,1}^\frac{d}{p})} \leq \eta,
\]

where \( E_0 = ||u_0||_{\mathbb{B}_{p,1}^{\frac{d}{p}-1}} + ||B_0||_{\mathbb{B}_{p,1}^\frac{d}{p}} \). Now, we suppose that \( T \) satisfies the following inequality:

\[
||e^{t \Delta} u_0||_{L_T^p(\mathbb{B}_{p,1}^{\frac{d}{p}+1})} + ||e^{t \Delta} u_0||_{L_T^p(\mathbb{B}_{p,1}^\frac{d}{p})} \leq \eta^2. \tag{3.2}
\]

It is easy to check the conditions \((H_1) - (H_2)\) hold true for \( n = 0 \) by \((3.2)\). In what follows, we will show that if the conditions \((H_1) - (H_2)\) hold true for \( n \), then they hold true for \( n + 1 \). First of all, we get by Lemma 2.6

\[
||\text{div}(-u^n \otimes u^n + B^n \otimes B^n)||_{L_T^1(\mathbb{B}_{p,1}^{\frac{d}{p}-1})} \leq C||-u^n \otimes u^n + B^n \otimes B^n||_{L_T^1(\mathbb{B}_{p,1}^\frac{d}{p})}
\]

\[
\leq C||u^n||_{L_T^2(\mathbb{B}_{p,1}^{\frac{d}{p}})} ||u^n||_{L_T^2(\mathbb{B}_{p,1}^\frac{d}{p})} + C||B^n||_{L_T^\infty(\mathbb{B}_{p,1}^{\frac{d}{p}})} ||B^n||_{L_T^1(\mathbb{B}_{p,1}^\frac{d}{p})}. \tag{3.3}
\]
It follows from Lemma 2.9 and (3.3) that
\[ ||u^{n+1}||_{L_T^\infty(\mathcal{B}_{p,1}^{\frac{d}{p}-1})} \leq C||u_0^{n+1}||_{\mathcal{B}_{p,1}^{\frac{d}{p}-1}} + C||\text{div}(-u^n \otimes u^n + B^n \otimes B^n)||_{L_T^1(\mathcal{B}_{p,1}^{\frac{d}{p}})} \]
\[ \leq C||u_0||_{\mathcal{B}_{p,1}^{\frac{d}{p}-1}} + C||u^n||^2_{L_T^2(\mathcal{B}_{p,1}^{\frac{d}{p}})} + CT||B^n||^2_{L_T^\infty(\mathcal{B}_{p,1}^{\frac{d}{p}})}. \quad (3.4) \]

By Lemma 2.6 and Lemma 2.8, we have
\[ ||B^{n+1}||_{L_T^\infty(\mathcal{B}_{p,1}^{\frac{d}{p}})} \leq Ce^{CU_n(T)}(||B_0^n||_{\mathcal{B}_{p,1}^{\frac{d}{p}}} + ||B^n||_{L_T^1(\mathcal{B}_{p,1}^{\frac{d}{p}})}) \]
\[ \quad \leq Ce^{CU_n(T)}(||B_0||_{\mathcal{B}_{p,1}^{\frac{d}{p}}} + ||B^n||_{L_T^\infty(\mathcal{B}_{p,1}^{\frac{d}{p}})}||u^n||_{L_T^1(\mathcal{B}_{p,1}^{\frac{d}{p}+1})}), \quad (3.5) \]
where \( U(t) = \int_0^t ||u^n||_{\mathcal{B}_{p,1}^{\frac{d}{p}+1}} \, d\tau. \) Then, combining (3.4)-(3.5) and the conditions \((H_1)-(H_2),\) we obtain
\[ ||u^{n+1}||_{L_T^\infty(\mathcal{B}_{p,1}^{\frac{d}{p}-1})} + ||B^{n+1}||_{L_T^\infty(\mathcal{B}_{p,1}^{\frac{d}{p}})} \leq Ce^{C_0E_0} + CC_0^2E_0^2T + C_0\eta e^{C_0C_0E_0} + C_0\eta^2. \quad (3.6) \]

Next, we also get by Lemma 2.9 and (3.2) that
\[ ||u^{n+1}||_{L_T^1(\mathcal{B}_{p,1}^{\frac{d}{p}+1})} + ||u^n||_{L_T^2(\mathcal{B}_{p,1}^{\frac{d}{p}})} \leq C(||e^{t\Delta}u_0^{n+1}||_{L_T^1(\mathcal{B}_{p,1}^{\frac{d}{p}+1})} + ||e^{t\Delta}u_0^n||_{L_T^2(\mathcal{B}_{p,1}^{\frac{d}{p}})}) + C||\text{div}(-u^n \otimes u^n + B^n \otimes B^n)||_{L_T^1(\mathcal{B}_{p,1}^{\frac{d}{p}})} \]
\[ \leq C(||e^{t\Delta}u_0||_{L_T^1(\mathcal{B}_{p,1}^{\frac{d}{p}+1})} + ||e^{t\Delta}u_0||_{L_T^2(\mathcal{B}_{p,1}^{\frac{d}{p}})}) + C||u^n||^2_{L_T^2(\mathcal{B}_{p,1}^{\frac{d}{p}})} + CT||B^n||^2_{L_T^\infty(\mathcal{B}_{p,1}^{\frac{d}{p}})} \]
\[ \leq C\eta^2 + CC_0^2E_0^2T. \quad (3.7) \]

Therefore, if choosing \( \eta < \min\{\frac{1}{4C}, E_0\}, C_0 > 8(C + 1) \) and \( T < \frac{\eta^2}{C_0E_0+1}, \) then we infer from (3.6)-(3.7) that
\[ ||u^{n+1}||_{L_T^\infty(\mathcal{B}_{p,1}^{\frac{d}{p}-1})} + ||B^{n+1}||_{L_T^\infty(\mathcal{B}_{p,1}^{\frac{d}{p}})} \leq C_0E_0, \]
\[ ||u^{n+1}||_{L_T^1(\mathcal{B}_{p,1}^{\frac{d}{p}+1})} + ||u^n||_{L_T^2(\mathcal{B}_{p,1}^{\frac{d}{p}})} \leq \eta. \]

This implies the conditions \((H_1)-(H_2)\) hold true for \( n + 1. \) Therefore, we deduce that the approximate sequence \( (u^n, B^n)_{n\in\mathbb{N}} \) is uniformly bounded in \( E_T^p \) independent of \( n. \)

**Step 3: Existence of a solution.** Now, we will use the compactness argument in Besov spaces for the approximating sequence \( (u^n, B^n)_{n\in\mathbb{N}} \) to get some solution.
(u, B) which also satisfies the system (1.2) in the sense of distributions. Since \( u^n \) is uniformly bounded in \( L_T^\infty(\mathfrak{B}_{p,1}^{d-1}) \), the interpolation inequality yields that \( u^n \) is uniformly bounded in \( L_T^q(\mathfrak{B}_{p,1}^{d-1+\frac{d}{q}}) \) for \( 1 \leq q \leq \infty \). Then, we get by Lemma 2.6 and \( \text{div} u^n = 0 \),
\[
||u^n \cdot \nabla B^{n+1} - \text{div}(u^n \otimes B^n)||_{L_T^2(\mathfrak{B}_{p,1}^d)} = ||\text{div}(B^{n+1} \otimes u^n - u^n \otimes B^n)||_{L_T^2(\mathfrak{B}_{p,1}^d)} \\
\leq C||u^n||_{L_T^\infty(\mathfrak{B}_{p,1}^d)}||B^{n+1}||_{L_T^\infty(\mathfrak{B}_{p,1}^d)} + C||u^n||_{L_T^\infty(\mathfrak{B}_{p,1}^d)}||B^n||_{L_T^\infty(\mathfrak{B}_{p,1}^d)}.
\]
We infer that \( \partial_t B^{n+1} \) is uniformly bounded in \( L_T^2(\mathfrak{B}_{p,1}^d) \). On the other hand, by Lemma 2.5 we have
\[
||\text{div}(u^n \otimes u^n)||_{\mathfrak{B}_{p,1}^{d-\frac{3}{2}}} \leq C||u^n||_{\mathfrak{B}_{p,1}^{d-1}}||u^n||_{\mathfrak{B}_{p,1}^{d+\frac{1}{2}}},
\]
\[
||\text{div}(B^n \otimes B^n)||_{\mathfrak{B}_{p,1}^d} \leq C||B^n||_{\mathfrak{B}_{p,1}^d}||B^n||_{\mathfrak{B}_{p,1}^d}.
\]
Then, from the second equation of the system (3.1) and the above inequalities, we infer that \((\partial_t u^n)_{n \in \mathbb{N}}\) is uniformly bounded in \( L_T^2(\mathfrak{B}_{p,1}^{d-\frac{3}{2}} + \mathfrak{B}_{p,1}^d) \).

Let \((\chi_j)_{j \in \mathbb{N}}\) be a sequence of smooth functions with value in \([0, 1]\) supported in the ball \( B(0, j + 1) \) and equal to 1 on \( B(0, j) \). The above argument ensures that \((B^n)_{n \in \mathbb{N}}\) is uniformly bounded in \( C_T^\frac{1}{2}([0, T]; \mathfrak{B}_{p,1}^{d-\frac{3}{2}}) \) and \((u^n)_{n \in \mathbb{N}}\) is uniformly bounded in \( C_T^\frac{1}{2}([0, T]; \mathfrak{B}_{p,1}^{d-\frac{3}{2}} + \mathfrak{B}_{p,1}^d) \). Then, By Proposition 2.93 in [4] and the embedding relation \( \mathfrak{B}_{p,1}^{d-\frac{3}{2}} \hookrightarrow \mathfrak{B}_{p,1}^{d-\frac{1}{2}} \), we get that for any \( j \in \mathbb{N} \), \((\chi_j B^n)_{n \in \mathbb{N}}\) is uniformly bounded in \( C_T^\frac{1}{2}([0, T]; \mathfrak{B}_{p,1}^{d-\frac{3}{2}} + \mathfrak{B}_{p,1}^d) \) and \((\chi_j u^n)_{n \in \mathbb{N}}\) is uniformly bounded in \( C_T^\frac{1}{2}([0, T]; \mathfrak{B}_{p,1}^{d-\frac{3}{2}}) \). Now, according to Theorem 2.94 in [4], the map \( z \rightarrow \chi_j z \) is compact from \( \mathfrak{B}_{p,1}^{d-\frac{3}{2}} \) to \( \mathfrak{B}_{p,1}^{d-\frac{1}{2}} \) and \( \mathfrak{B}_{p,1}^{d-\frac{1}{2}} \) to \( \mathfrak{B}_{p,1}^{d-\frac{3}{2}} \). Thus, by applying Ascoli’s theorem (see Theorem 2.1 in [4]) and Cantor’s diagonal process, there exists some function \((u_j, B_j)\) such that for any \( j \in \mathbb{N} \), \( \chi_j u \) tends to \( u_j \) and \( \chi_j B \) tends to \( B_j \). As \( \chi_j \chi_j+1 = \chi_j \), we have, in addition, \( u_j = \chi_j u_{j+1} \) and \( B_j = \chi_j B_{j+1} \). From that, we can easily deduce that there exists \((u, B)\) such that for all \( \chi \in \mathcal{D}(\mathbb{R}^d)\),
\[
\begin{cases}
\chi B^n \rightarrow \chi B & \text{in } C([0, T]; \mathfrak{B}_{p,1}^{d-\frac{3}{2}}), \\
\chi u^n \rightarrow \chi u & \text{in } C([0, T]; \mathfrak{B}_{p,1}^{d-\frac{1}{2}}),
\end{cases}
\]

(3.8)
as \( n \) tends to \( \infty \) (up to a subsequence). By the interpolation, we also have
\[
\begin{cases}
\chi B^n \rightarrow \chi B & \text{in } C([0, T]; \mathfrak{B}_{p,1}^{d-s}), \text{ for all } 0 < s \leq 1, \\
\chi u^n \rightarrow \chi u & \text{in } L^1([0, T]; \mathfrak{B}_{p,1}^{d+s}), \text{ for all } -\frac{3}{2} \leq s < 1.
\end{cases}
\]

(3.9)
Combining the uniform bounds which we have proved in Step 2 and the Fatou property for Besov spaces, we readily get

\[
(u, B) \in (L_T^{\infty}(\mathcal{B}^{\frac{p}{p+1}}_{p,1}) \cap L_T^{2}(\mathcal{B}^{\frac{p}{p+1}}_{p,1}))^d \times (L_T^{\infty}(\mathcal{B}^{\frac{d}{p}}_{p,1}))^d.
\] (3.10)

With (3.8)-(3.9), it is a routine process to verify that \((u, B)\) satisfies the system (1.2) in the sense of distributions. For simplicity, we only check the nonlinear term \(\text{div}(u^n \otimes B^n)\). For any \(\phi \in (\mathcal{D}(\mathbb{R}^d))^d\), there exists some \(\chi \in \mathcal{D}(\mathbb{R}^d)\) such that \(\phi = \phi \chi\). Therefore, letting \(p'\) be the Hölder conjugate of \(p\), i.e. \(\frac{1}{p} + \frac{1}{p'} = 1\), we infer from Lemma 2.7 that

\[
\left| \int_0^T \langle \text{div}(u^n \otimes B^n - u \otimes B), \phi \rangle dt \right| \leq \int_0^T \langle \text{div}((\chi u^n) \otimes (\chi B^n) - (\chi u) \otimes (\chi B)), \phi \rangle dt
\]
\[
\leq C \|\chi u^n - \chi u\|_{L^1_t(\mathcal{B}^{\frac{d}{q}}_{p,1})} \|\chi B^n\|_{L^\infty_t(\mathcal{B}^{\frac{d}{q}}_{p,1})} \|\nabla \phi\|_{\mathcal{B}^{\frac{-q}{q}}_{p',1}}
\]
\[
+ C \|\chi B^n - \chi B\|_{L^1_t(\mathcal{B}^{\frac{d}{q}}_{p,1})} \|\chi u\|_{L^2_t(\mathcal{B}^{\frac{d}{q}}_{p,1})} \|\nabla \phi\|_{\mathcal{B}^{\frac{d}{q}}_{p',1}} \quad \text{as} \quad n \to \infty,
\]

which implies \(\text{div}(u^n \otimes B^n)\) tends to \(\text{div}(u \otimes B)\) in the sense of distributions. Furthermore, we also deduce from Lemma 2.7 and (3.10) that the right-hand terms of the first equation and second equation of (1.2) belong to \(L^1_T(\mathcal{B}^{\frac{d}{p-1}}_{p,1})\). Then, by Lemma 2.9 and (3.11), we have

\[
(u, B) \in (L_T^{\infty}(\mathcal{B}^{\frac{d}{p-1}}_{p,1}) \cap L_T^{1}(\mathcal{B}^{\frac{d}{p-1}}_{p,1}))^d \times (L_T^{\infty}(\mathcal{B}^{\frac{d}{p}}_{p,1}))^d.
\]

Finally, following the argument of Theorem 3.19 in [1], we can show that \((u, B) \in E_T^p\).

**Step 4: Uniqueness of the solution.** Assume that \((u^1, B^1)\) and \((u^2, B^2)\) are two solutions of the system with the same initial data. Set \(\delta u = u^1 - u^2\) and \(\delta B = B^1 - B^2\). Then, \((\delta u, \delta B)\) satisfies

\[
\begin{aligned}
\partial_t \delta u - \Delta \delta u &= \text{Pdiv}(-u^1 \otimes \delta u - \delta u \otimes u^2 + B^1 \otimes \delta B + \delta B \otimes B^2), \\
\partial_t \delta B + u^1 \cdot \nabla \delta B &= \text{div}(u^1 \otimes \delta B + \delta u \otimes B^2) - \delta u \cdot \nabla B^2.
\end{aligned}
\] (3.11)

We first claim that \(\delta B \in \left(L_T^{\infty}(\mathcal{B}^{\frac{d}{p}}_{p,\infty})\right)^d\). This claim relies on the following inequality
which can be deduced from Lemma 2.6 and \( \text{div}(\delta u) = \text{div}(\delta B) = 0: \)

\[
\| \delta B \|_{L^\infty(\mathbb{R}^d, T)} \leq C e^{CU^1(t)} \left\| \text{div}(u^1 \otimes \delta B + \delta u \otimes B^2) - \delta u \cdot \nabla B^2 \right\|_{L^1_{t}((\mathbb{R}^d)_{p,1})} \\
\leq C e^{CU^1(t)} \| u^1 \otimes \delta B + \delta u \otimes B^2 - B^2 \otimes \delta u \|_{L^1_{t}((\mathbb{R}^d)_{p,1})} \\
\leq C e^{CU^1(t)} \left( \| \delta u \|_{L^1_{t}((\mathbb{R}^d)_{p,1})} \| B^2 \|_{L^\infty((\mathbb{R}^d)_{p,1})} + \| \delta B \|_{L^\infty((\mathbb{R}^d)_{p,1})} \| u^1 \|_{L^1_{t}((\mathbb{R}^d)_{p,1})} \right),
\]

(3.12)

where \( U^1(t) = \int_0^t \| u^1 \|_{H_{p,1}}^2 \, dt. \) Using Lemmas 2.6–2.7 and the fact \( \text{div}(\delta u) = \text{div}(\delta B) = 0, \) we have

\[
\| \text{div}(u^1 \otimes \delta B + \delta u \otimes B^2) - \delta u \cdot \nabla B^2 \|_{L^1_{t}((\mathbb{R}^d)_{p,1})} \\
= \| \delta B \cdot \nabla u^1 + \text{div}(\delta u \otimes B^2 - B^2 \otimes \delta u) \|_{L^1_{t}((\mathbb{R}^d)_{p,1})} \\
\leq C \| \delta u \|_{L^1_{t}((\mathbb{R}^d)_{p,1})} \| B^2 \|_{L^\infty((\mathbb{R}^d)_{p,1})} + C \| \delta B \|_{L^\infty((\mathbb{R}^d)_{p,1})} \| u^1 \|_{L^1_{t}((\mathbb{R}^d)_{p,1})}.
\]

(3.13)

We apply Lemma 2.8 and (3.13) to get for all \( t \in [0, T], \)

\[
\| \delta B \|_{L^\infty(\mathbb{R}^d, T)} \leq C e^{CU^1(t)} \left( \| \delta u \|_{L^1_{t}((\mathbb{R}^d)_{p,1})} \| B^2 \|_{L^\infty((\mathbb{R}^d)_{p,1})} + \| \delta B \|_{L^\infty((\mathbb{R}^d)_{p,1})} \| u^1 \|_{L^1_{t}((\mathbb{R}^d)_{p,1})} \right).
\]

(3.14)

It follows from Lemma 2.7 that

\[
\| \text{div}(-u^1 \otimes \delta u - \delta u \otimes u^2 + B^1 \otimes \delta B + \delta B \otimes B^2) \|_{L^1_{t}((\mathbb{R}^d)_{p,1})} \\
\leq C \| -u^1 \otimes \delta u - \delta u \otimes u^2 + B^1 \otimes \delta B + \delta B \otimes B^2 \|_{L^1_{t}((\mathbb{R}^d)_{p,1})} \\
\leq C (\| u^1 \|^2_{L^2((\mathbb{R}^d)_{p,1})} + \| u^2 \|^2_{L^2((\mathbb{R}^d)_{p,1})}) \| \delta u \|^2_{L^2((\mathbb{R}^d)_{p,1})} \\
+ C (\| B^1 \|^2_{L^2((\mathbb{R}^d)_{p,1})} + \| B^2 \|^2_{L^2((\mathbb{R}^d)_{p,1})}) \| \delta B \|^2_{L^2((\mathbb{R}^d)_{p,1})}.
\]

(3.15)

Then, we infer from Lemma 2.9 and (3.15) that for any \( t \in [0, T], \)

\[
\| \delta u \|_{L^1_{t}((\mathbb{R}^d)_{p,1})} + \| \delta u \|_{L^2((\mathbb{R}^d)_{p,1})} \\
\leq C \| (u^1, u^2) \|_{L^2((\mathbb{R}^d)_{p,1})} \| \delta u \|_{L^2((\mathbb{R}^d)_{p,1})} + C (\| B^1 \|_{L^2((\mathbb{R}^d)_{p,1})} + \| B^2 \|_{L^2((\mathbb{R}^d)_{p,1})}) \| \delta B \|_{L^2((\mathbb{R}^d)_{p,1})}.
\]

(3.16)
We take $\bar{T}$ small enough such that $||| (u^1, u^2) |||_{L_T^2 (B^{\frac{d}{p}, \infty}) \cap L_T^1 (B^{\frac{d}{p}, 1})} \ll 1/C$. Thus, combining (3.14) and (3.16), we infer that for any $t \in [0, \bar{T}]$,

$$||| \delta u |||_{L_t^1 (B^{\frac{d}{p}, \infty})} \leq C \int_0^t ||| \delta B |||_{B^{\frac{d}{p}, \infty}} (||| B^1 |||_{B^{\frac{d}{p}, 1}} + ||| B^2 |||_{B^{\frac{d}{p}, 1}}) \, dt, \quad (3.17)$$

and

$$||| \delta B |||_{L_t^\infty (B^{\frac{d}{p}, \infty})} \leq C e^{C \nu_1 (t)} ||| \delta u |||_{L_t^1 (B^{\frac{d}{p}, \infty})} ||| B^2 |||_{L_t^\infty (B^{\frac{d}{p}, 1})}. \quad (3.18)$$

From Lemma 2.4, it follows that

$$||| \delta u |||_{L_t^1 (B^{\frac{d}{p}, 1})} \leq C ||| \delta u |||_{L_t^1 (B^{\frac{d}{p}, \infty})} \log \left( e + \frac{||| \delta u |||_{L_t^1 (B^{\frac{d}{p}, \infty})} + ||| \delta u |||_{L_t^1 (B^{\frac{d}{p}, 1})}}{||| \delta u |||_{L_t^1 (B^{\frac{d}{p}, \infty})}} \right),$$

which together with (3.17) and (3.18) yields that for any $t \in [0, \bar{T}]$,

$$||| \delta u |||_{L_t^1 (B^{\frac{d}{p}, 1})} \leq A_T \int_0^t ||| \delta u |||_{L_t^1 (B^{\frac{d}{p}, \infty})} \log (e + \frac{C_T}{||| \delta u |||_{L_t^1 (B^{\frac{d}{p}, \infty})}}) \, dt,$$

where

$$A_T = C \exp \{ C ||| u^1 |||_{L_T^1 (B^{\frac{d}{p}, 1})} ||| B^2 |||_{L_T^\infty (B^{\frac{d}{p}, 1})} (||| B^1 |||_{L_T^\infty (B^{\frac{d}{p}, 1})} + ||| B^2 |||_{L_T^\infty (B^{\frac{d}{p}, 1})}) \},$$

and

$$C_T = ||| \delta u |||_{L_T^1 (B^{\frac{d}{p}, \infty})} + ||| \delta u |||_{L_T^1 (B^{\frac{d}{p}, 1})}.$$

Note that $A_T$ is integrable on $[0, T]$, and

$$\int_0^1 \frac{1}{r \log (e + C_T r^{-1})} \, dr = +\infty.$$

An application of Osgood’s lemma yields that $(\delta u, \delta B) = (0, 0)$ on $[0, \bar{T}]$, and a continuity argument ensures that $(u^1, B^1) = (u^2, B^2)$ on $[0, T]$. Therefore, combining the above four steps, we complete the proof of Theorem 1.1.

**Acknowledgements.** Li and Yin were partially supported by NNSFC (No.11671407 and No.11271382), FDCT (No. 098/2013/A3), Guangdong Special Support Program (No. 8-2015), and the key project of NSF of Guangdong province (No. 2016A030311004). Tan was partially supported by NNSFC (No. 11301174). The authors thank the referee for valuable comments and suggestions.
References

[1] H. Bahouri, J. Chemin and R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations*, Grundlehren der Mathematischen Wissenschaften, 343, Springer, Heidelberg, 2011.

[2] J. Chemin, D. McCormick, J. Robinson and J. Rodrigo, *Local existence for the non-resistive MHD equations in Besov spaces*, Adv. Math., **286** (2016), 1-31.

[3] Q. Chen, C. Miao and Z. Zhang, *Well-posedness in critical spaces for the compressible Navier-Stokes equations with density dependent viscosities*, Rev. Mat. Iberoam., **26** (2010), 915-946.

[4] R. Danchin, *Local theory in critical spaces for compressible viscous and heat-conductive gases*, Comm. Partial Differential Equations, **26** (2001), 1183-1233.

[5] R. Danchin, *Density-dependent incompressible viscous fluids in critical spaces*, Proc. Roy. Soc. Edinburgh Sect. A, **133** (2003), 1311-1334.

[6] R. Danchin, *On the uniqueness in critical spaces for compressible Navier-Stokes equations*, Nonlinear Differential Equations Appl., **12** (2005), 111-128.

[7] C. Fefferman, D. McCormick, J. Robinson and J. Rodrigo, *Higher order commutator estimates and local existence for the non-resistive MHD equations and related models*, J. Funct. Anal., **267** (2014), 1035-1056.

[8] C. Fefferman, D. McCormick, J. Robinson and J. Rodrigo, *Local existence for the non-resistive MHD equations in nearly optimal Sobolev spaces*, Arch. Ration. Mech. Anal., **223** (2017), 677-691.

[9] E. Feireisl, *Dynamics of Viscous Compressible Fluids*, Oxford University Press, Oxford, 2004.

[10] H. Triebel, *Theory of Function Spaces*, Monographs in Mathematics, Birkhäuser Verlag, Basel, 1983.

[11] R. Wan, *On the uniqueness for the 2D MHD equations without magnetic diffusion*, Nonlinear Anal. Real World Appl., **30** (2016), 32-40.