Random Periodic Processes, Periodic Measures and 
Strong Law of Large Numbers

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Abstract

We first prove that a random periodic path of a random dynamical system on a Polish space gives a periodic measure of the skew product flow. In the case of a Markovian dynamical system, it gives a periodic measure of the Markov semigroup. Conversely, in general a periodic measure cannot give a random periodic path in the original probability space. But we can construct an enlarged probability space including the trajectories of the random dynamical systems on the lifted cylinder and noise paths, on which the pull back path of the random dynamical system is a random periodic process. The law of the random periodic process is the periodic measure. We can further prove the strong law of large numbers (SLLN) of the random periodic processes/periodic measures. This is a new class of random processes satisfying the SLLN, in addition to the classical results about a sequence of independent and identically distributed random variables (Kolmogorov) and stationary processes/invariant measures (Birkhoff). The results are true for the cocycles and time non-homogeneous periodic semi-flows. The periodic semi-flows can be lifted to a cocycle on a cylinder.

Keywords: random periodic processes, periodic measures, invariant measures, random dynamical systems, strong law of large numbers.

1 Introduction

The idea regarding stochastic differential equations as random dynamical systems went back to late 1970's with a number of seminal works ([3, 4, 15, 27, 28, 34] etc). Later this was developed to include stochastic partial differential equations in [20, 35, 22, 24]. With these foundational works in hands, similar to the case of deterministic dynamical systems, to study the long time behaviour lies in the centre of the theory of random dynamical systems.

Fixed points and periodic paths are two basic notions in the theory of dynamical systems capturing their long time behaviour and equilibrium. The concept of the stationary solution has been known for
some time and is the corresponding notion of fixed points in the stochastic counterpart. Their existence has been subject to intensive studies ([14], [21], [31], [33], [38], [40], [44]). Moreover, it is well known that a stationary solution gives rise to the existence of an invariant measure ([1]). The converse part is not true in general ([36]). But in an enlarged probability space, an invariant measure is a random Dirac measure, so there exists a pathwise stationary solution ([1]). The study of invariant measures has been one of the central problems in the areas of ergodic theory and stochastic partial differential equations in the last few decades ([5], [6], [13], [21], [25], [29] to name but a few). The notion of periodic solution is another key concept in the theory of dynamical systems, and has occupied a central space in the theory of dynamical systems since Poincaré’s pioneering work. A notion of random periodic solutions should play a similar role in the theory of random dynamical systems. Periodic phenomena exist in many real world problems e.g. biology, economics, chemical reactions, climate dynamics etc. But, by nature, many real world systems are very often subject to the influence of internal or external randomness. Periodicity and randomness may often mix together. For instance, the maximum daily temperature in any particular region is a random process, however, it certainly has periodic nature driven by the divine clock due to the revolution of the earth around the sun.

Physicists have attempted to study random perturbations to periodic solutions for some time by considering a first linear approximation or asymptotic expansions in small noise regime, e.g. see [41], [43]. One of the obstacles to make a systematic progress was the lack of a rigorous mathematical definition of random periodic solution and appropriate mathematical tools. For a random path with some periodic nature, it was not clear what a mathematically correct relation between the random position\[ Y(t, \omega) \] at time\( t \) and\[ Y(t + \tau, \omega) \] at time\( t + \tau \) after a period\( \tau \) should be. In the meantime, as\[ Y(t, \omega) \] is a true path or solution, so it is not true in general that\[ Y(t, \omega) = Y(t + \tau, \omega) \]. The approach in [43] was to seek\[ Y(t + \tau, \omega) \] in a neighbourhood of\[ Y(t, \omega) \] which was applicable to small noise perturbations.

New observation was that for each fixed\( t \in I^+ \), \{\[ Y(t + k\tau, \omega) \]\}_k\in\mathbb{N} should be a random stationary solution of the discretised random dynamical system\[ \Phi(k\tau, \omega) \] ([18], [19], [30], [45]). Here\[ \Phi : I^+ \times \Omega \times X \rightarrow X \] is a random dynamical system over a metric dynamical system \((\Omega, \mathcal{F}, P, (\theta_t)_{t \in I})\). This then led to the rigorous definition of pathwise random periodicity\[ Y(t + \tau, \omega) = Y(t, \theta(\tau)\omega) \] and existence of random periodic solutions for cocycles and stochastic semiflows defined e.g. by SDEs and SPDEs. The concept of pathwise random periodic paths (solutions) is the stochastic counterpart of periodic paths in the theory of dynamical systems. It gives rigour and a clearer understanding to physically interesting problems of certain random phenomena with a periodic nature. On the other hand, similar to the stationary solution, the random periodic path also represents a long time limit of the underlying random dynamical system, therefore is one of the basic objects in its geometric structure. The existence of random periodic solutions of various stochastic systems and stochastic (partial) differential equations was obtained in [18], [19], [30], [45]. More recently there have been more progresses on the study of random periodic solutions including [8] on random attractors of the stochastic TJ model in climate dynamics, [42] on bifurcations of stochastic reaction diffusion equations, [2] on stochastic lattice systems, [17], in the case of anticipating random periodic solutions, [16] on stochastic functional differential equations.

An alternative way to understand random periodic behaviours of SDEs is to study periodic measures which describe periodicity in the sense of distributions ([29]). There are a few works in the literature attempting to study statistical solutions of certain types of stochastic differential equations.
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with periodic forcings. This was motivated in the context of studying the climate change problem when the seasonal cycle is taken into considerations (22, 32), in the context of the Brussellator arising in chemical reactions (39), and Ornstein-Uhlenbeck processes (9).

In this paper, we will first discuss the relation between random periodic paths and periodic measures. In particular, we will prove that a random periodic path of a random dynamical system gives rise to a periodic measure of the skew product flow in the product measurable space and Markov semigroup in phase space respectively. For a cocycle random dynamical system, we will construct an invariant measure as the time average of the periodic measure in a time interval of one period. Conversely, in general a periodic measure cannot give a random periodic solution in the original probability space. We will construct a new probability space which is enlarged by adding trajectories of the random dynamical systems on the lifted phase space to be part of the new noise paths. We can then extend the random dynamical systems naturally over the enlarged probability space and proved that the pull-back of the random dynamical systems is a random periodic path. These results reveal the natural relation between pathwise random periodicity and periodicity in the sense of distributions. In fact, one can prove that the law of the random periodic path is the very periodic measure. For this reason, we call the random periodic path the random periodic process in the future. Both cocycles and semi-flows can possess random periodic processes (see 18, 19, 45). We will discuss these two cases separately in the following sections.

One importance of the relationship of the random periodic processes and periodic measures is to enable us to establish the strong law of large numbers (SLLN). The SLLN was known for sequences of independent and identically distributed random variables (Kolmogorov’s SLLN theorem) and stationary processes/invariant measures (Birkhoff’s ergodic theory). We will add in this investigation to prove that the SLLN holds for random periodic processes/periodic measures. We would like to mention that in a different direction, the SLLN was also investigated recently in [7] for independent and identically distributed random variables in the sense of non-additive probability (37). The SLLN result proved here implies that if \( Y(s, \omega), s \in R \) is a random periodic solution of a Markov dynamical system on a Polish space \( X \) over a metric dynamical system \( (\Omega, \mathcal{F}, P, (\theta_t)_{t \in R}) \), with its law \( \mathcal{L}(Y(s)) = \rho_s \), where there exists a constant \( \tau > 0 \) such that \( Y(s + \tau, \omega) = Y(s, \theta_{\tau} \omega) \) for all \( \omega \in \Omega \) and \( \rho_{s+\tau} = \rho_s \), then under an exponential convergence condition of the transition semigroup to the periodic measure (see Condition (EC) in section 2), we have as \( T \to \infty \),

\[
\frac{1}{T} \int_0^T f(Y(s, \omega)) ds \to \int_X f(x) \bar{\rho}(dx) \text{ a.s.,}
\]

for any \( f \in L^1(X, \mathcal{B}_X, \bar{\rho}(dx)) \). Here \( \bar{\rho} = \frac{1}{\tau} \int_0^\tau \rho_s ds \).

2 The cocycle case

Let \( X \) be a Polish space and \( \mathcal{B}_X \) be its Borel \( \sigma \)-algebra, a common notation \( I \) be either \( R \), two-sided continuous time or \( \{0, \pm 1, \pm 2, \cdots\} \), two-sided discrete time, two different scenarios of time. Denote by a common notation \( \nu \) the Lebesgue measure in the continuous time case or the counting measure in the
discrete time case on \( I \). We use \( I^+ (I^-) \) to denote one-sided time, i.e. \( I^+ = R^+ (I^- = R^-) \), one-sided continuous time; and \( I^+ = \{0, 1, 2, \cdots \} (I^- = \{0, -1, -2, \cdots \}) \), one-sided discrete time.

In this section, we consider a measurable cocycle random dynamical system \( \Phi \) on \((X, \mathcal{B}_X)\) over a metric dynamical systems \((\Omega, \mathcal{F}, P, (\theta(t))_{t \in I^+})\) with a one-sided time set \( I^+ \), \( \Phi : I^+ \times \Omega \times X \to X \). It is \((\mathcal{B}_{I^+} \otimes \mathcal{F} \otimes \mathcal{B}_X, \mathcal{B}_X)\)-measurable and satisfies the cocycle property:

\[
\Phi(0, \omega) = id_X \text{ and } \Phi(t + s, \omega) = \Phi(t, \theta(s)\omega)\Phi(s, \omega)
\]

for all \( s, t \in I^+ \) and \( \omega \in \Omega \). The map \( \theta(t) : \Omega \to \Omega \) is \( P\)-measure preserving and measurably invertible. Therefore it can be extended to \( I^- \) as well by setting \( \theta(t) = \theta(-t)^{-1} \) when \( t \in I^- \). There is no need to require the map \( \Phi(t, \omega) : X \to X \) to be invertible, which enables the work to be applicable to both SDEs and SPDEs. Therefore we can consider a cocycle measurable random dynamical system \( \Phi \) on \((X, \mathcal{B}_X)\) over a metric dynamical systems \((\Omega, \mathcal{F}, P, (\theta(t))_{t \in I^+})\) on a one-sided time set \( I^+ \).

First recall the definition of random periodic paths (solutions) given in [18], [19]. The definition of stationary solutions was well known. But we include it here to make a comparison between the concepts of random periodic paths and stationary solutions. The same remark also applies to invariant measures in Definitions 2.2 and 2.5.

**Definition 2.1** A random periodic path of period \( \tau \) of the random dynamical system \( \Phi : I^+ \times \Omega \times X \to X \) is an \( \mathcal{F} \)-measurable map \( Y : I \times \Omega \to X \) such that for any \( t \in I^+, s \in I \) and \( \omega \in \Omega \),

\[
\Phi(t, \theta(s)\omega)Y(s, \omega) = Y(t + s, \omega), \quad Y(s + \tau, \omega) = Y(s, \theta(\tau)\omega).
\]  
(2.2)

It is a stationary solution of \( \Phi \) if \( Y(s, \omega) = Y(0, \theta(s)\omega) =: Y_0(\omega) \) for all \( s \in I \), i.e. \( Y_0 : \Omega \to X \) is a stationary solution if for any \( t \in I^+ \) and \( \omega \in \Omega \),

\[
\Phi(t, \omega)Y_0(\omega) = Y_0(\theta(t)\omega).
\]  
(2.3)

The first part of the definition of the random periodic path suggests that a random periodic path \( Y(t, \omega) \) is indeed a pathwise trajectory of the random dynamical system. The second part of the definition says that it has some periodicity. But it is different from a periodic path in the deterministic case, \( Y(t + \tau, \omega) \) is not equal to \( Y(t, \omega) \), but \( Y(t, \theta(\tau)\omega) \). We call this random periodicity. Starting at \( Y(t, \omega) \), after a period \( \tau \), trajectory does not return to \( Y(t, \omega) \), but to \( Y(t, \cdot) \) with different realisation \( \theta(\tau)\omega \). So it is neither completely random, nor completely periodic, but a mix of randomness and periodicity.

Let \( \phi(s, \omega) = Y(s, \theta(-s)\omega), s \in I \). It is easy to see that \( \phi \) satisfies the definition given in [45]

\[
\Phi(t, \omega)\phi(s, \omega) = \phi(t + s, \theta(t)\omega), \quad \phi(s + \tau, \omega) = \phi(s, \omega), t \in I^+, s \in I.
\]  
(2.4)

Therefore \( \phi \) is a periodic function and the graph

\[
L^\omega = \{ (\phi(s, \omega) : s \in [0, \tau) \cap I) \}
\]  
(2.5)
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is a closed curve. It is easy to see from the first formula in (2.4) that \( L^\omega \) is an invariant set, i.e.
\[ \Phi(t, \omega)L^\omega = L^\theta(t, \omega), \quad t \in I^+. \]
But needless to say that random periodic solution gives more detailed information about the dynamics of the random dynamical system than a general invariant set.

Unlike the periodic solution of deterministic dynamical systems, the random dynamical system does not follow the closed curve, but move from one closed curved to another when time evolves. This is fundamentally different from the deterministic case, which makes it hard to study. However, this natural definition in random case makes it possible to gain new understanding of random phenomena with some periodic nature, where strict deterministic periodicity is not applicable.

First we prove that the random Dirac measure with the support on sections of the random periodic curve \( L^\omega \) is the periodic measure of the random dynamical system and its time average is an invariant measure. To make this clear, we consider a standard product measurable space \((\bar{\Omega}, \bar{F}) = (\Omega \times X, \mathcal{F} \otimes \mathcal{B}_X)\) and the skew-product of the metric dynamical system \((\Omega, \mathcal{F}, P, (\theta(t))_{t \in I})\) and the cocycle \(\Phi(t, \omega)\) on \(X\), \(\bar{\Theta}(t): \bar{\Omega} \rightarrow \bar{\Omega}\), \(\bar{\Theta}(t)(\bar{\omega}) = (\theta(t, \omega), \Phi(t, \omega)x)\), where \(\bar{\omega} = (\omega, x)\), \(t \in I^+\).

Recall
\[ P_P(\Omega \times X) := \{\mu: probability\ measure\ on\ (\Omega \times X, \mathcal{F} \otimes \mathcal{B}_X)\ with\ marginal\ P\ on\ (\Omega, \mathcal{F})\} \]
and
\[ P(X) := \{\rho: probability\ measure\ on\ (X, \mathcal{B}_X)\} \]

**Definition 2.2** A map \(\mu : I \rightarrow P_P(\Omega \times X)\) is called a periodic probability measure of period \(\tau\) on \((\Omega \times X, \mathcal{F} \otimes \mathcal{B}_X)\) for the random dynamical system \(\Phi\) if
\[ \mu_{\tau+s} = \mu_s and \bar{\Theta}(t)\mu_s = \mu_{t+s}, \quad for\ any\ t \in I^+, s \in I. \] (2.7)
It is an invariant measure if it also satisfies \(\mu_s = \mu_0\) for any \(s \in I\) i.e. \(\mu_0\) is an invariant measure of \(\Phi\) if \(\mu_0 \in P_P(\Omega \times X)\) and
\[ \bar{\Theta}(t)\mu_0 = \mu_0, \quad for\ any\ t \in I^+. \] (2.8)

**Theorem 2.3** If a random dynamical system \(\Phi : I^+ \times \Omega \times X \rightarrow X\) has a random periodic path \(Y : I \times \Omega \rightarrow X\), it has a periodic measure on \((\Omega \times X, \mathcal{F} \otimes \mathcal{B}_X)\) \(\mu : I \rightarrow P_P(\Omega \times X)\) given by
\[ \mu_s(A) = \int_{\Omega} \delta_{Y(s, \omega)}(A_{\theta(s)\omega})P(d\omega), \] (2.9)
where \(A_\omega\) is the \(\omega\)-section of \(A\). Moreover, the time average of the periodic measure defined by
\[ \bar{\mu} = \frac{1}{\nu([0, \tau) \cap I)} \int_{[0, \tau) \cap I} \mu_s d\nu(s), \] (2.10)
is an invariant measure of \(\Phi\) whose random factorisation has the support \(L^\omega\) defined by (2.5).
Proof: It is obvious that $P$ is the marginal measure of $\mu_s$ on $(\Omega, \mathcal{F})$, so $\mu_s \in \mathcal{P}_P(\Omega \times X)$. To check \((2.11)\), first note for $t \in I^+$, $\bar{\Theta}(t)^{-1}(A) = \{(\omega, x) : (\theta(t)\omega, \Phi(t, \omega)x) \in A\}$. Then it is easy to see that for $t \in I^+$

$$
(\bar{\Theta}_t^{-1}(A))_\omega = \{x : (\theta(t)\omega, \Phi(t, \omega)x) \in A\} = \{x : \Phi(t, \omega)x \in A_{\theta(t)\omega}\} = \Phi^{-1}(t, \omega)A_{\theta(t)\omega}.
$$

Thus for $t \in I^+$

$$
(\bar{\Theta}_t \mu_s)(A) = \mu_s(\bar{\Theta}_t^{-1}(A)) = \int_{\Omega} \delta_{Y(s,\omega)}((\bar{\Theta}_t^{-1}(A))_{\theta(s)\omega})P(d\omega) = \int_{\Omega} \delta_{Y(s,\omega)}(\Phi^{-1}(t, \theta(s)\omega)A_{\theta(t)\theta(s)\omega})P(d\omega) = \int_{\Omega} \delta_{\Phi(t,\theta(s)\omega)Y(s,\omega)}(A_{\theta(t+s)\omega})P(d\omega) = \int_{\Omega} \delta_{Y(t+s,\omega)}(A_{\theta(t+s)\omega})P(d\omega) = \mu_{t+s}(A). \tag{2.12}
$$

Second from the definition of random periodic path and the probability preserving property of $\theta$, we have

$$
\mu_{s+r}(A) = \int_{\Omega} \delta_{Y(s,\theta(r)\omega)}(A_{\theta(s)\theta(r)\omega})P(d\omega) = \int_{\Omega} \delta_{Y(s,\omega)}(A_{\theta(s)\omega})P(d\omega) = \mu_s(A).
$$

Thus $\mu_s, s \in I$ defined by \((2.7)\) is a periodic measure as claimed in the theorem. To see $\bar{\mu}$ defined by \((2.10)\) is an invariant measure, note for any $A \in \mathcal{F} \otimes \mathcal{B}_X$ and $t \in I^+ \cap [0, \tau)$, by what we have proved for $\mu_s$, \(\bar{\Theta}(t)\bar{\mu}(A)

$$
= \frac{1}{\nu([0, \tau) \cap I)} \int_{[0, \tau) \cap I} \bar{\Theta}(t) \mu_s(A) d\nu(s)
$$

$$
= \frac{1}{\nu([0, \tau) \cap I)} \int_{[0, \tau) \cap I} \mu_{t+s}(A) d\nu(s)
$$

$$
= \frac{1}{\nu([0, \tau) \cap I)} \int_{[t, \tau+t) \cap I} \mu_s(A) d\nu(s)
$$

$$
= \frac{1}{\nu([0, \tau) \cap I)} \int_{[t, \tau) \cap I} \mu_s(A) d\nu(s) + \frac{1}{\nu([0, \tau) \cap I)} \int_{[\tau, \tau+t) \cap I} \mu_s(A) d\nu(s)
$$

$$
= \frac{1}{\nu([0, \tau) \cap I)} \int_{[t, \tau) \cap I} \mu_s(A) d\nu(s) + \frac{1}{\nu([0, \tau) \cap I)} \int_{[0, \tau) \cap I} \mu_{t+s}(A) d\nu(s)
$$

$$
= \frac{1}{\nu([0, \tau) \cap I)} \int_{[t, \tau) \cap I} \mu_s(A) d\nu(s) + \frac{1}{\nu([0, \tau) \cap I)} \int_{[0, \tau) \cap I} \mu_s(A) d\nu(s)
$$

$$
= \bar{\mu}(A). \tag{2.13}
$$

A similar argument can be used to prove that any $t \in I^+$,

$$
\bar{\Theta}(t)\bar{\mu}(A) = \bar{\mu}(A). \tag{2.14}
$$
Thus $\bar{\mu}$ is an invariant measure. To see its support, by (2.10), (2.9) and Fubini’s Theorem, for any $A \in \bar{\mathcal{F}}$,

$$
\bar{\mu}(A) = \frac{1}{\nu([0, \tau) \cap I]} \int_{[0, \tau) \cap I} \mu_s(A) d\nu(s) = \frac{1}{\nu([0, \tau) \cap I]} \int_{[0, \tau) \cap I} \delta_{Y(s, \theta(-s)\omega)}(A_{\omega}) P(\omega) d\nu(s) = \int_{\Omega} \frac{1}{\nu([0, \tau) \cap I]} \int_{[0, \tau) \cap I} \delta_{Y(s, \theta(-s)\omega)}(A_{\omega}) d\nu(s) P(\omega).
$$

This leads to its factorisation given by

$$(\bar{\mu})_\omega = \frac{1}{\nu([0, \tau) \cap I]} \int_{[0, \tau) \cap I} \delta_{Y(s, \theta(-s)\omega)} d\nu(s) = \frac{1}{\nu([0, \tau) \cap I]} \int_{[0, \tau) \cap I} \delta_{\phi(s, \omega)} d\nu(s).$$

It follows easily that $L^\omega$ is the support of $(\bar{\mu})_\omega$.

**Remark 2.4** For a random periodic path $Y$, it is easy to see that the factorization of $\mu_s$ defined in Theorem 2.3 is

$$(\mu_s)_\omega = \delta_{Y(s, \theta(-s)\omega)},
$$

and satisfies

$$(\mu_{s+\tau})_\omega = (\mu_s)_\omega, \quad \Phi(t, \omega)(\mu_s)_\omega = (\mu_{t+s})_{\theta(t)\omega}.
$$

Now consider a Markovian cocycle random dynamical system $\Phi$ on a filtered dynamical system $(\Omega, \mathcal{F}, P, (\theta_t)_{t \in I}, (\mathcal{F}_s^t)_{s \leq t})$, i.e. for any $s, t, u \in I$, $s \leq t$, $\theta_u^{-1}\mathcal{F}_s^t = \mathcal{F}_{s+u}^{t+u}$ and for any $t \in I^+$, $\Phi(t, \cdot)$ is measurable with respect to $\mathcal{F}_t$. We also assume the random periodic solution $Y(s)$ is adapted, that is to say that for each $s \in I$, $Y(s, \cdot)$ is measurable with respect to $\mathcal{F}_{s+u}^{t+u}$.

Denote the transition probability of Markovian process $\Phi(t, \omega)x$ on the Polish space $X$ with Borel $\sigma$-field $\mathcal{B}_X$ by (c.f. Arnold [1], Da Prato and Zabczyk [13])

$$P(t, x, B) = P(\{\omega : \Phi(t, \omega)x \in B\}), \quad t \in I^+, \quad B \in \mathcal{B}_X,
$$

and for any probability measure $\rho$ on $(X, \mathcal{B}_X)$, define

$$(P^*(t)\rho)(B) = \int_X P(t, x, B)\rho(dx), \quad \text{for any } t \in I^+, \quad B \in \mathcal{B}_X.
$$

**Definition 2.5** A measure function $\rho : I \rightarrow \mathcal{P}(Y)$ is called a periodic measure of period $\tau$ on the phase space $(X, \mathcal{B}_X)$ for the Markovian random dynamical systems $\Phi$ if it satisfies

$$\rho_{s+\tau} = \rho_s \quad \text{and} \quad \rho_{t+s} = P^*(t)\rho_s, \quad s \in I, \quad t \in I^+.
$$

It is called an invariant measure if it also satisfies $\rho_s = \rho_0$ for all $s \in I$, i.e. $\rho_0$ is an invariant measure for the Markovian random dynamical system $\Phi$ if
\[ \rho_0 = P^*(t)\rho_0, \text{ for all } t \in I^+. \]  

(2.18)

**Theorem 2.6** Assume the Markovian cocycle \( \Phi : I^+ \times \Omega \times X \to X \) has an adapted random periodic path \( Y : I \times \Omega \to X \). Then the measure function \( \rho : I \to \mathcal{P}(X) \) defined by

\[ \rho_s := E(\mu_s). = E\delta_{Y(s, \theta(-s))} = E\delta_{Y(s,)}, \]  

(2.19)

which is the law of the random periodic path \( Y \), is a periodic measure of \( \Phi \) on \((X, \mathcal{B}_X)\). Its time average \( \bar{\rho} \) over a time interval of exactly one period defined by

\[ \bar{\rho} = \frac{1}{\nu([0, \tau) \cap I)} \int_{[0, \tau) \cap I} \rho_s d\nu(s), \]  

(2.20)

is an invariant measure and satisfies that for any \( B \in \mathcal{B}_X, t \in I \)

\[ \bar{\rho}(B) = E\left( \frac{1}{\nu([0, \tau) \cap I)} \int_{[0, \tau) \cap I} \rho_s d\nu(s) \right) \]  

(2.21)

**Proof:** Firstly it is easy to see from the definition of random periodic path that for any \( B \in \mathcal{B}_X, \)

\[ \rho_{s+\tau}(B) = P(\{\omega : Y(s + \tau, \omega) \in B\}) \]  

\[ = P(\{\omega : Y(s, \theta^s \omega) \in B\}) \]  

\[ = P(\{\omega : Y(s, \omega) \in B\}) = \rho_s(B). \]

Secondly, from (2.15) we have \((\mu_{t+s})_\omega = \delta_{Y(t+s, \theta(-t-s)\omega)} = \delta_{\Phi(t, \theta(-t)\omega)Y(s, \theta(-t-s)\omega)}\). Therefore for any \( B \in \mathcal{B}_X, t \in I^+, \) by measure preserving property of \( \theta, \) independency of \( \Phi(t, \theta(s)\omega) \) and \( \mathcal{F}_{-\infty}^\infty, \)

\[ \rho_{t+s}(B) = E\delta_{\Phi(t, \theta(s)\omega)Y(s, \omega)}(B) \]  

\[ = P(\{\omega : \Phi(t, \theta(s)\omega)Y(s, \omega) \in B\}) \]  

\[ = \int_X P(t, x, B)P(\omega : Y(s, \omega) \in dx) \]  

\[ = \int_X P(t, x, B)\rho_s(dx) \]  

\[ = P^*(t)\rho_s(B). \]  

(2.22)

Therefore \( \rho \) satisfies Definition 2.5, so is a periodic measure on \((X, \mathcal{B}_X)\). To prove the second part of the theorem, similar to the computation in (2.13), we have for any \( t \in [0, \tau) \cap I^+, \)

\[ \int_{[0, \tau) \cap I} \rho_{t+s}d\nu(s) = \int_{[0, \tau) \cap I} \rho_s d\nu(s), \]

and by using Fubini Theorem,

\[ \int_{[0, \tau) \cap I} P^*(t)\rho_s d\nu(s) = P^*(t)(\int_{[0, \tau) \cap I} \rho_s d\nu(s)). \]
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It then follows easily that $\bar{\rho}$ is an invariant measure of $\Phi$ satisfying (2.18). To prove the last part of the theorem, from (2.20), (2.19), and using Fubini’s Theorem, we know for any $B \in \mathcal{B}_X$,

$$\bar{\rho}(B) = \frac{1}{\nu([0, \tau] \cap I)} \int_{[0, \tau] \cap I} P(Y(s, \omega) \in B) d\nu(s)$$

$$= \frac{1}{\nu([0, \tau] \cap I)} \int_{[0, \tau] \cap I} (E\delta_{Y(s, \omega)}(B)) d\nu(s)$$

$$= E(\frac{1}{\nu([0, \tau] \cap I)} \int_{[0, \tau] \cap I} \delta_{Y(s, \omega)}(B) d\nu(s))$$

$$= E(\frac{1}{\nu([0, \tau] \cap I)} \int_{[0, \tau] \cap I} \delta_{Y(s, \omega)}(B) d\nu(s))$$

$$= E(\frac{1}{\nu([0, \tau] \cap I)} m\{s \in [0, \tau) : Y(s, \omega) \in B\})).$$

However, since $\bar{\rho}$ is an invariant measure, so from (2.22) we know that for any $t \in I^+$

$$\bar{\rho}(B) = P^*(t) \bar{\rho}(B)$$

$$= \frac{1}{\nu([0, \tau] \cap I)} \int_{[0, \tau] \cap I} P^*(t) \rho_s(B) d\nu(s)$$

$$= \frac{1}{\nu([0, \tau] \cap I)} \int_{[0, \tau] \cap I} \rho_{t+s}(B) d\nu(s)$$

$$= E(\frac{1}{\nu([0, \tau] \cap I)} \int_{[0, \tau] \cap I} \delta_{Y(t+s, \omega)}(B) d\nu(s))$$

$$= E(\frac{1}{\nu([0, \tau] \cap I)} \int_{[t, t+\tau] \cap I} \delta_{Y(s, \omega)}(B) d\nu(s))$$

$$= E(\frac{1}{\nu([0, \tau] \cap I)} m\{s \in [t, t+\tau) : Y(s, \omega) \in B\})).$$

For $t \in I^-$, it is easy to verify $P^*(-t) \rho_{t+s} = \rho_s$ and therefore

$$\bar{\rho}(B) = P^*(-t) \bar{\rho}(B)$$

$$= \frac{1}{\nu([0, \tau] \cap I)} \int_{[t, t+\tau] \cap I} P^*(-t) \rho_{s+t}(B) d\nu(s)$$

$$= \frac{1}{\nu([0, \tau] \cap I)} \int_{[t, t+\tau] \cap I} \rho_s(B) d\nu(s)$$

$$= \frac{1}{\nu([0, \tau] \cap I)} \int_{[t, t+\tau] \cap I} \delta_{Y(s, \omega)}(B) d\nu(s))$$

$$= \frac{1}{\nu([0, \tau] \cap I)} \int_{[t, t+\tau] \cap I} \delta_{Y(s, \omega)}(B) d\nu(s))$$

$$= E(\frac{1}{\nu([0, \tau] \cap I)} m\{s \in [t, t+\tau) : Y(s, \omega) \in B\}).$$

So we can see that (2.21) is true for any $t \in I$.

Identity (2.21) says that the expected time spent inside a Borel set by the random periodic path over a time interval of exactly one period starting at any time is invariant, i.e. independent of the starting time. This shows that the random periodicity of a random periodic path by means of invariant measures. In the following we will push the above observation further to study the strong law of large
numbers which says on long run, the average time that the random periodic path spends on on a Borel set \(B\) over one period is equal to \(\bar{\rho}(B)\) a.s.

In order to prove this result, we use a weak law of large numbers (WLLN) for sequences of dependent random variables with long range decay covariance. This result may not be new and the proof is quite easy. But we can’t find any reference. So we include it here for convenience without claiming its originality.

**Lemma 2.7 (WLLN)** Let \(X_n\) be a sequence of identical distributed variables and \(S_n := \sum_{i=1}^{n} X_i\) denote the \(n\)-th partial sum of \(X_n\). Assume that \(|\text{cov}(X_i, X_j)| \leq M |i-j|^\alpha\), where \(M > 0\), \(0 < \alpha < 1\). Then for any \(\epsilon > 0\),

\[
\lim_{n \to \infty} P\left(\left|\frac{S_n}{n} - E X_1\right| > \epsilon\right) = 0.
\]

**Proof:** For any \(\epsilon > 0\) and \(n \in N\), by Chebyshev’s inequality, we have

\[
P\left(\left|\frac{S_n}{n} - E X_1\right| > \epsilon\right) \leq \frac{1}{\epsilon^2} \text{Var}\left(\frac{S_n}{n}\right) = \frac{1}{\epsilon^2 n^2} \text{Var}(S_n) \leq \frac{M}{\epsilon^2 n^2} \sum_{i,j=1}^{n} \alpha| i-j |.
\]

We can explicitly compute \(\sum_{i,j=1}^{n} \alpha| i-j |\) as follows:

\[
\sum_{i,j=1}^{n} \alpha| i-j | = n + 2 \sum_{i=1}^{n-1} \sum_{j=1}^{i-1} \alpha| i-j | = n + 2 \sum_{i=1}^{n} \frac{\alpha^i - \alpha}{\alpha - 1} = \frac{2\alpha(\alpha^n - 1)}{(\alpha - 1)^2} + \frac{\alpha + 1}{1 - \alpha n}.
\]

Therefore,

\[
\lim_{n \to \infty} P\left(\left|\frac{S_n}{n} - E X_1\right| > \epsilon\right) \leq \lim_{n \to \infty} \frac{M}{\epsilon^2 n^2} \left(\frac{2\alpha(\alpha^n - 1)}{(\alpha - 1)^2} + \frac{\alpha + 1}{1 - \alpha n}\right) = 0.
\]

‡‡

Instead of considering the whole interval \([0, T]\), we can consider a ”window” in the first period \([0, \tau]\), and identical copies of the window in the subsequent periods. We will prove the SLLN in those windows. The case of continuous internal \([0, T]\) \((T \to \infty)\) is a special case of the results that we will prove here. For this, let \(F_0 \subset [0, \tau) \cap I\) be a given Borel set on \(R^1\) and assume \(\nu(F_0) > 0\). Define for \(k = 0, 1, 2, \cdots\),

\[
F_k := \{t = k\tau + t_0 : t_0 \in F_0\}, \quad G_N := \bigcup_{k=0}^{N-1} F_k \text{ and } G_{\infty} := \bigcup_{k=0}^{\infty} F_k.
\]

**Lemma 2.8** Assume \(Y : I \times \Omega \to X\) is an adapted random periodic path of the Markovian random dynamical system \(\Phi\) and for any \(y \in X\), \(\lim_{k \to \infty} \Phi(s + k\tau, \theta(-k\tau)\omega)y = Y(s, \omega)\) a.s.. Then for any \(B \in B_X\),

\[
P\left(\left|\frac{S_n}{n} - E X_1\right| > \epsilon\right) \leq \frac{M}{\epsilon^2 n^2} \left(\frac{2\alpha(\alpha^n - 1)}{(\alpha - 1)^2} + \frac{\alpha + 1}{1 - \alpha n}\right) = 0.
\]
As \( k \to \infty \).

**Proof:** Note for any \( B \in B_X \),

\[
P(s, y, B) = EI_B(\Phi(s, \theta(-k\tau)\omega)y),
\]

and by the definition of \( F_k \) and the fact that \( Y(s, \omega) = Y(s + k\tau, \theta(-k\tau)\omega) \), we have

\[
\frac{1}{\nu(F_0)} \int_{F_k} \rho_s(B) d\nu(s) = \frac{1}{\nu(F_0)} \int_{F_0} EI_B(Y(s, \omega)) d\nu(s) = \frac{1}{\nu(F_0)} \int_{F_k} EI_B(Y(s, \theta(-k\tau)\omega)) d\nu(s).
\]

Thus, by using Lebesgue’s dominated convergence theorem, we have that

\[
\frac{1}{\nu(F_0)} \int_{F_k} P(s, y, B) d\nu(s) - \frac{1}{\nu(F_0)} \int_{F_k} EI_B(Y(s, \theta(-k\tau)\omega)) d\nu(s)
\]

\[
= E \frac{1}{\nu(F_0)} \int_{F_k} (I_B(\Phi(s - k\tau + k\tau, \theta(-k\tau)\omega)y) - I_B(Y(s - k\tau, \omega))) d\nu(s)
\]

\[
= E \frac{1}{\nu(F_0)} \int_{F_0} (I_B(\Phi(s + k\tau, \theta(-k\tau)\omega)y) - I_B(Y(s, \omega))) d\nu(s)
\]

\[
\to 0,
\]

as \( k \to \infty \). Then (2.24) follows from Lebesgue's dominated convergence theorem again. \( \dagger \dagger \)

We now strengthen the above assumption in the following condition to require that the convergence in Lemma 2.9 is exponentially fast, i.e.

**Condition (EC):** There exist constants \( \delta_1, \delta_2 > 0 \) such that for any \( k \geq 1 \) and any \( B \in B_X \),

\[
\int_X \left| \frac{1}{\nu(F_0)} \int_{F_k} (P(s, y, B) - \rho_s(B)) d\nu(s) \right| d\mu(dy) \leq \delta_1 e^{-\delta_2 k}.
\]

Under this condition, we call the transition probability \( P(t, x, \cdot) \), \( t \in I^+ \) and the periodic measure \( \rho_s \in \mathcal{P}(X) \), \( s \in I \) satisfies Condition (EC). Note no other objects are involved in this condition.

Define

\[
J_k := E \left[ \left( \frac{1}{\nu(F_0)} \int_{F_k} (I_B(Y(s, \omega)) - \rho_s(B)) d\nu(s) \right) \cdot \left( \frac{1}{\nu(F_0)} \int_{F_0} (I_B(Y(s, \omega)) - \rho_s(B)) d\nu(s) \right) \right].
\]

**Lemma 2.9** Under the condition (EC), we have for any \( k \geq 1 \),

\[
|J_k| \leq \delta_1 e^{-\delta_2 (k-1)}.
\]
Proof: When \( k \geq 1 \), note when \( k \tau \leq s < (k + 1)\tau \),
\[
Y(s, \omega) = \Phi(s - \tau, \theta(\tau)\omega)Y(\tau, \omega).
\]

Then taking the conditional expectation we have
\[
J_k = E\left[ E\left[ \frac{1}{\nu(F_0)} \int_{F_k} (I_B(\Phi(s - \tau, \theta(\tau)\omega)Y(\tau, \omega)) - \rho_s(B))d\nu(s)|\mathcal{F}_{-\infty} \right] \cdot \frac{1}{\nu(F_0)} \int_{F_0} (I_B(Y(s, \omega)) - \rho_s(B))d\nu(s) \right]
= E\left( \frac{1}{\nu(F_0)} \int_{F_k} (P(s - \tau, Y(\tau, \omega), B) - \rho_s(B))d\nu(s) \cdot \frac{1}{\nu(F_0)} \int_{F_0} (I_B(Y(s, \omega)) - \rho_s(B))d\nu(s) \right).
\]

So by condition (EC),
\[
|J_k| \leq E\left| \frac{1}{\nu(F_0)} \int_{F_k} (P(s - \tau, Y(\tau, \omega), B) - \rho_s(B))d\nu(s) \right|
= \int_X \left| \frac{1}{\nu(F_0)} \int_{F_k} (P(s - \tau, y, B) - \rho_s(B))d\nu(s) \right| \rho_0(dy)
\leq \delta_1 e^{-\delta_2(k-1)}.
\]

We first prove a weak law of large numbers.

Lemma 2.10 (WLLN) Assume \( Y : I \times \Omega \rightarrow X \) is an adapted random periodic path of the Markovian random dynamical system \( \Phi \), and its law and the transition probability of \( \Phi \) satisfy condition (EC). Then as \( N \rightarrow \infty \),
\[
\frac{1}{\nu(G_N)} \int_{G_N} I_B(Y(s, \omega))d\nu(s) \rightarrow \frac{1}{\nu(F_0)} \int_{F_0} \rho_s(B)d\nu(s) \text{ in probability.}
\]

Proof: Define
\[
\xi_k(\omega) := \frac{1}{\nu(F_0)} \int_{F_k} I_B(Y(s, \omega))d\nu(s).
\]

Note first that we can prove \( E\xi_k(\omega) = \frac{1}{\nu(F_0)} \int_{F_0} \rho_s(B)d\nu(s) \) by using a similar method as in the proof of Lemma 2.11. Now for any \( m > n \), from the measure preserving property of \( \theta \) and the random periodicity of \( Y \), we have
\[
J_{mn} := cov(\xi_m, \xi_n)
= E\left[ \left( \frac{1}{\nu(F_0)} \int_{F_m} I_B(Y(s, \omega))d\nu(s) - \frac{1}{\nu(F_0)} \int_{F_0} \rho_s(B)d\nu(s) \right) \cdot \left( \frac{1}{\nu(F_0)} \int_{F_n} I_B(Y(s, \omega))d\nu(s) - \frac{1}{\nu(F_0)} \int_{F_0} \rho_s(B)d\nu(s) \right) \right]
\]
Then by Lemma 2.7, we have that

\[ \nu \int_{F_m} I_B(Y(s, \theta(-n\tau)\omega)) d\nu(s) \to \frac{1}{\nu(F_0)} \int_{F_0} \rho_s(B) d\nu(s) \]

On the other hand,

\[ \nu \int_{F_m} I_B(Y(s-n\tau, \omega)) d\nu(s) \to \frac{1}{\nu(F_0)} \int_{F_0} \rho_s(B) d\nu(s) \]

So by Lemma 2.9,

\[ \nu \int_{F_m} I_B(Y(s, \omega)) d\nu(s) \to \frac{1}{\nu(F_0)} \int_{F_0} \rho_s(B) d\nu(s) \]

So by Lemma 2.8, \( |J_{mn}| \leq \delta_1 e^{\delta_2 e^{-\delta_2 (m-n)}} \), when \( m > n \). It is easy to know that when \( m = n \), \(|J_{nn}| \leq 1 \).

On the other hand,

\[ \frac{1}{\nu(G_N)} \int_{G_N} I_B(Y(s, \omega)) d\nu(s) = \frac{1}{N} \frac{1}{\nu(F_0)} \sum_{m=0}^{N-1} \int_{F_m} I_B(Y(s, \omega)) d\nu(s) \]

\[ = \frac{1}{N} \sum_{m=0}^{N-1} \xi_m(\omega). \]

Then by Lemma 2.8, we have that

\[ \frac{1}{\nu(G_N)} \int_{G_N} I_B(Y(s, \omega)) d\nu(s) \to E\xi_m(\omega) = \frac{1}{\nu(F_0)} \int_{F_0} \rho_s(B) d\nu(s), \]

in probability as \( N \to \infty \).

Now we can prove the SLLN from the WLLN with the help of Birkhoff’s ergodic theorem.

**Theorem 2.11 (SLLN)** Assume the same conditions as in Lemma 2.10. Then as \( R \ni T \to \infty \),

\[ \frac{1}{\nu([0, T) \cap G_\infty)} \int_{[0, T) \cap G_\infty} I_B(Y(s, \omega)) d\nu(s) \to \frac{1}{\nu(F_0)} \int_{F_0} \rho_s(B) d\nu(s) \text{ a.s.} \]

In particular, if Condition (EC) holds for \( F_0 = [0, \tau) \), then as \( T \to \infty \),

\[ \frac{1}{\nu([0, T) \cap \mathbb{T})} \int_{[0, T) \cap \mathbb{T}} I_B(Y(s, \omega)) d\nu(s) \to \bar{\rho}(B) \text{ a.s.} \]

**Proof:** Note that

\[ \xi_m(\omega) = \frac{1}{\nu(F_0)} \int_{F_m} I_B(Y(s, \omega)) d\nu(s) \]

\[ = \frac{1}{\nu(F_0)} \int_{F_m} I_B(Y(s-m\tau, \theta(m\tau)\omega)) d\nu(s) \]

\[ = \frac{1}{\nu(F_0)} \int_{F_0} I_B(Y(s, \theta(m\tau)\omega)) d\nu(s) \]
\[ = \xi_0(\theta(m\tau)\omega). \]

Then by Birkhoff’s ergodic theorem, there exists \( \xi^*(\omega) \) such that
\[
\frac{1}{\nu(G_N)} \int_{G_N} I_B(Y(s, \omega))d\nu(s) = \frac{1}{N} \sum_{m=0}^{N-1} \xi_m(\omega) = \frac{1}{N} \sum_{m=0}^{N-1} \xi_0(\theta(m\tau)\omega) \rightarrow \xi^*(\omega) \text{ a.s.,} \tag{2.26}
\]
as \( N \rightarrow \infty \). On the other hand, by Theorem 2.10, the left hand side of (2.26) converges to \( \frac{1}{\nu(F_0)} \int_{F_0} \rho_s(B)d\nu(s) \) in probability. So there exist a subsequence \( N_k \rightarrow \infty \), as \( k \rightarrow \infty \), such that as \( k \rightarrow \infty \),
\[
\frac{1}{\nu(G_{N_k})} \int_{G_{N_k}} I_B(Y(s, \omega))d\nu(s) \rightarrow \frac{1}{\nu(F_0)} \int_{F_0} \rho_s(B)d\nu(s) \text{ a.s.}
\]
Thus
\[
\frac{1}{\nu(F_0)} \int_{F_0} \rho_s(B)d\nu(s) = \xi^*(\omega) \text{ a.s.}
\]
That is to say as integer sequence \( N \rightarrow \infty \),
\[
\frac{1}{\nu(G_N)} \int_{G_N} I_B(Y(s, \omega))d\nu(s) \rightarrow \frac{1}{\nu(F_0)} \int_{F_0} \rho_s(B)d\nu(s) \text{ a.s.}
\]
By a standard argument, we can get the result for \( T \rightarrow +\infty \).

Conversely, we now assume a Markovian random dynamical system has a periodic measure \( \rho_s \in \mathcal{P}(X) \). In general, with the original probability space, similar to the case that an invariant measure does not give a stationary process, neither a periodic measure gives a random periodic path. In the following, we will construct an enlarged probability space and an extended random dynamical system, on which the pull-back flow is a random periodic solution. This construction is much more demanding than constructing the periodic measure from a random periodic path.

Now we consider a Markovian random dynamical system. If it has a periodic measure on \((X, \mathcal{B}_X)\), then we can construct a periodic measure on the product measurable space \((\Omega \times X, \mathcal{F} \otimes \mathcal{B}_X)\). Here we use Crauel’s construction of invariant measures on the product space from invariant measures of transition semigroup on phase space.

**Theorem 2.12** Assume the Markovian random dynamical system \( \Phi \) has a periodic measure \( \rho : I \rightarrow \mathcal{P}(X) \) on \((X, \mathcal{B}_X)\). Then for any \( s \in I \)
\[
(\mu_s)_\omega := \lim_{n \to \infty} \Phi(n\tau + s, \theta(-n\tau - s)\omega)\rho_0, \tag{2.27}
\]
exists. Let
\[
\mu_s(dx, d\omega) = (\mu_s)_\omega(dx) \times P(d\omega).
\]
Then \( \mu_s \) is a periodic measure on the product measurable space \((\Omega \times X, \mathcal{F} \otimes \mathcal{B}_X)\) for \( \Phi \) and \( E(\mu_s) = \rho_s \), \( s \in I \).
Moreover, it is easy to see that
\[ P^r(nτ)ρ_0 = ρ_{nτ} = ρ_0. \]

This means that ρ_0 is a forward invariant measure under \( P^r(nτ) \), \( n ∈ Z^+ \). By Crauel [10, 11], we know that the following limit exists
\[
(μ_0)_ω := \lim_{n→∞} Φ(nτ, θ(−nτ)ω)ρ_0.
\]

By cocycle property of Φ, we have that for any \( B ∈ B_X \) for any \( s ∈ I^+ \),
\[
\lim_{n→∞} Φ(nτ + s, θ(−nτ − s)ω)ρ_0(B) = \lim_{n→∞} (Φ(s, θ(−s)ω) ◦ Φ(nτ, θ(−nτ)θ(−s)ω)ρ_0)(B)
= \lim_{n→∞} (Φ(nτ, θ(−nτ)θ(−s)ω)ρ_0)(Φ(s, θ(−s)ω)^{-1}B)
= (μ_0)_ω(Φ(s, θ(−s)ω)^{-1}B)
= Φ(s, θ(−s)ω)(μ_0)_ω(B). \tag{2.28}
\]

When \( s ∈ I^− \), we can also obtain that the above limit still exists by decomposing \( s = −mτ + s_0 \), \( s_0 ∈ [0, τ) \), and considering
\[
\lim_{n→∞} Φ(nτ + s, θ(−nτ − s)ω)ρ_0(B)
= \lim_{n→∞} (Φ(s + mτ, θ(−(s + mτ))ω) ◦ Φ((n − m)τ, θ(−(n − m)τ)θ(−(s + mτ))ω)ρ_0)(B).
\]

Now, from the cocycle property and (2.27) and the argument of taking limits in (2.28), we know that for \( t ∈ I^+ \),
\[
Φ(t, ω)(μ_s)_ω = \lim_{n→∞} Φ(t, ω) ◦ Φ(nτ + s, θ(−nτ − s)ω)ρ_0
= \lim_{n→∞} Φ(nτ + t + s, θ(−nτ − t − s)θ(t)ω)ρ_0
= (μ_{t+s})_ω. \tag{2.29}
\]

It follows that for any \( A ∈ F ⊗ B_X \), by (2.11) and (2.29), for \( t ∈ I^+ \)
\[
(\hat{O}(t)μ_s)(A) = \int_Ω (μ_s)_ω ((\hat{O}^{-1}_t(A))_ω) P(dω)
= \int_Ω (μ_s)_ω (Φ^{-1}(t, ω)A_{θ(t)ω}) P(dω)
= \int_Ω (Φ(t, ω)(μ_s)_ω)(A_{θ(t)ω}) P(dω)
= \int_Ω (μ_{t+s})_ω(A_{θ(t)ω}) P(dω)
= \int_Ω (μ_{t+s})_ω(A_ω) P(dω) = μ_{t+s}(A).
\]

Moreover, It is easy to see that
\[
(μ_{s+t})_ω = \lim_{n→∞} Φ((n + 1)τ + s, θ(−(n + 1)τ − s)ω)ρ_0 = (μ_s)_ω,
\]
\begin{align*}
\mu_{s+\tau} &= \mu_s.
\end{align*}

Then \( \mu_s \) is a periodic measure on the product measurable space \((\Omega \times X, \mathcal{F} \otimes \mathcal{B}_X)\) for \( \Phi \).

Next let’s prove for any \( B \in \mathcal{B}_X, \ s \in I, \ E(\mu_s)_\omega(B) = \rho_s(B) \). First, we will show that for any \( B \in \mathcal{B}_X, \ E(\mu_0)_\omega(B) = \rho_0(B) \). In fact, by the Lebesgue’s dominated convergence theorem, the Fubini theorem and measure preserving property of \( \theta \),

\begin{align*}
E(\mu_0)_\omega(B) &= \int_{\Omega} \lim_{n \to \infty} \Phi(n\tau, \theta(-n\tau)\omega)\rho_0(B)P(d\omega) \\
&= \lim_{n \to \infty} \int_{\Omega} \rho_0(\Phi(n\tau, \theta(-n\tau)\omega)^{-1}(B))P(d\omega) \\
&= \lim_{n \to \infty} \int_{\Omega} \int_X 1_{\Phi(n\tau, \theta(-n\tau)\omega)^{-1}B}(x) d\rho_0(x)P(d\omega) \\
&= \lim_{n \to \infty} \int_X \int_{\Omega} 1_B(\Phi(n\tau, \theta(-n\tau)\omega)x) P(d\omega)d\rho_0(x) \\
&= \lim_{n \to \infty} \int_X P(n\tau, x, B)d\rho_0(x) \\
&= \lim_{n \to \infty} P^*(n\tau)\rho_0(B) \\
&= \rho_0(B).
\end{align*}

Similarly and also applying the above result, we have for \( s \in I^+ \),

\begin{align*}
E(\mu_s)_\omega(B) &= \int_{\Omega} \lim_{n \to \infty} \Phi(s + n\tau, \theta(-s - n\tau)\omega)\rho_0(B)P(d\omega) \\
&= \lim_{n \to \infty} \int_X \int_{\Omega} 1_B(\Phi(s + n\tau, \theta(-s - n\tau)\omega)x) P(d\omega)d\rho_0(x) \\
&= \lim_{n \to \infty} \int_X P(s + n\tau, x, B)d\rho_0(x) \\
&= \lim_{n \to \infty} \int_X \int_X P(s, y, B)P(n\tau, x, y)dyd\rho_0(x) \\
&= \lim_{n \to \infty} \int_X P(s, y, B) \int_X P(n\tau, x, y)d\rho_0(x)dy \\
&= \int_X P(s, y, B)d\rho_0(y) \\
&= P^*(s)\rho_0(B) \\
&= \rho_s(B).
\end{align*}

If \( s \in I^- \), there exists \( m \in \mathbb{Z}^+, \ s_0 \in [0, \tau) \) such that \( s = -m\tau + s_0 \). So

\begin{align*}
E(\mu_s)_\omega(B) &= \int_{\Omega} \lim_{n \to \infty} \Phi(s + n\tau, \theta(-s - n\tau)\omega)\rho_0(B)P(d\omega) \\
&= \int_{\Omega} \lim_{n \to \infty} \Phi(s_0 + (n\tau - m\tau), \theta(-s_0 - (n\tau - m\tau)\omega)\rho_0(B)P(d\omega)
\end{align*}
Theorem 2.13 Assume that a random dynamical system $\Phi$ generates a periodic probability measure $\mu$ on the product measurable space $(\bar{\Omega}, \bar{\mathcal{F}}) = (\Omega \times X, \mathcal{F} \otimes \mathcal{B})$. Then a measure $\hat{\mu}$ on the measurable space $(\hat{\Omega}, \mathcal{F})$ defined by

$$
\hat{\mu}(I_r \times A) = \frac{1}{\nu(I_r)} \int_{I_r} \mu_s(A) d\nu(s), \quad \hat{\mu}(\emptyset \times A) = 0, \text{ for any } A \in \mathcal{F} \otimes \mathcal{B}_X.
$$

(2.31)

is a probability measure and $\hat{\Theta}(t) : \hat{\Omega} \to \hat{\Omega}$ defined by (2.30) is measure $\hat{\mu}$-preserving, and

$$
\hat{\Theta}(t_1) \hat{\Theta}(t_2) = \hat{\Theta}(t_1 + t_2), \text{ for any } t_1, t_2 \in I^+.
$$

(2.32)

If we extend $\Phi$ to a map over the metric dynamical system $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mu}, (\bar{\Theta}(t))_{t \in I^+})$ by

$$
\bar{\Phi}(t, \bar{\omega}) = \Phi(t, \omega), \quad t \in I^+,
$$

(2.33)

then $\bar{\Phi}$ is a RDS on $X$ over $\bar{\Theta}$ and has a random periodic path $\bar{Y} : I^+ \times \bar{\Omega} \to X$ constructed as follows: for any $\bar{\omega}^* = (s, \omega^*, x^*(\omega^*)) \in \bar{\Omega},$

$$
\bar{Y}(t, \bar{\omega}^*) := \Phi(t + s, \theta(-s)\omega^*) x^*(\theta(-s)\omega^*), \quad t \in I^+.
$$

(2.34)
Proof: It is easy to see that the proof of (2.32) is a matter of straightforward computations and $\hat{\mu}$ is a probability measure. To verify $\hat{\Theta}(t)\hat{\mu} = \hat{\mu}$, for any $t \in I^+$, first using (2.7) and a similar argument as (2.14), we have that for any $t \in [0, \tau) \cap I^+$,

$$
\hat{\Theta}(t)\hat{\mu}(I_\tau \times A) = \frac{1}{\nu(I_\tau)} \int_{I_\tau} \mu_s(\hat{\Theta}^{-1}(t)A) d\nu(s)
$$

$$
= \frac{1}{\nu(I_\tau)} \int_{I_\tau} \mu_{s+t}(A) d\nu(s)
$$

$$
= \frac{1}{\nu(I_\tau)} \int_{I_\tau} \mu_s(A) d\nu(s)
$$

$$
= \hat{\mu}(I_\tau \times A).
$$

It is trivial to note that $\hat{\Theta}(t)\hat{\mu}(\emptyset \times A) = \hat{\mu}(\emptyset \times A)$. So $\hat{\Theta}(t)$ is $\hat{\mu}$-preserving for $t \in [0, \tau) \cap I^+$. This can be easily generalised to any $t \in I^+$ using the group property of $\hat{\Theta}$. Moreover, it is trivial to see that $\hat{\Phi}$ is a cocycle on $X$ over $\hat{\Theta}$. Again, the construction of $\hat{Y}$ given by (2.33) is key to the proof, from which the actual proof itself is quite straightforward. In fact, for $\hat{\omega} = (s, \omega, x)$, we have $\hat{Y}(t, \hat{\omega}) = \Phi(t+s, \theta(-s)\omega)x$.

Moreover, for any $r, t \in I^+$, we have by the cocycle property that

$$
\hat{\Phi}(r, \hat{\Theta}(t)\hat{\omega})\hat{Y}(t, \hat{\omega}) = \Phi(r, \theta(t)\omega)\Phi(t+s, \theta(-s)\omega)x
$$

$$
= \Phi(r + t + s, \theta(-s)\omega)x = \hat{Y}(r + t, \hat{\omega}). \tag{2.35}
$$

Note that $\hat{\Theta}(\tau)\hat{\omega} = (s, \theta(\tau)\omega, \Phi(\tau, \omega)x)$, so we have by the cocycle property

$$
\hat{Y}(t, \hat{\Theta}(\tau)\hat{\omega}) = \Phi(t+s, \theta(\tau-s)\omega)\Phi(\tau, \theta(-s)\omega)x
$$

$$
= \Phi(t+s+\tau, \theta(-s)\omega)x = \hat{Y}(\tau + t, \hat{\omega}). \tag{2.36}
$$

The proof is completed.

Remark 2.14 It is not clear how to extend the definition of $Y$ to $I^-$ in general. However, if the cocycle $\Phi(t, \omega) : X \to X$ is invertible for any $t \in I^+$ and $\omega \in \Omega$, for instance in the case of SDEs in a finite dimensional space with some suitable conditions, it is obvious to extend $Y$ to $I^-$. 

One implication of Theorems 2.12 and 2.13 is that starting from a periodic measure $\rho_s \in \mathcal{P}(X)$, one can construct a (enlarged) probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mu})$ and extended random dynamical system, with which the pull-back of the random dynamical system is a random periodic path. In the following we will prove that the transition probability of $\hat{\Phi}(t, \hat{\omega})x$ is actually the same as $P(t, x, \cdot)$ and the law of the random periodic solution $\hat{Y}$ is $\rho_s$, i.e.

$$
\hat{L}(\hat{Y}(s, \cdot)) = \rho_s, \text{ for any } s \in I,
$$

and $\rho_{s+\tau} = \rho_s$. We call $\hat{Y}$ a random periodic process as its law is periodic. Moreover, we can prove the SLLN of the random periodic process $\hat{Y}$ and its associated periodic measure $\rho_s$. This kind of result was previously known for invariant measure and stationary processes as Birkhoff’s ergodic theorem (see e.g. Da Prato and Zabczyk [13]). But random periodic processes are more general than stationary processes. Though there are many stationary processes, many processes in the real world are only
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random periodic, not stationary, e.g. the daily maximum temperature. We believe that the SLLN in
this paper opens a new scope of investigating of random periodic processes and their applications in
many real world problems. In the following, by \( \hat{\mathcal{E}} \) we denote the expectation on \((\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mu})\)

**Lemma 2.15** Assume \( \rho_s \) is a periodic measure with respect to the semigroup transition probability of
a Markovian random dynamical system \( \Phi \). Let the metric dynamical system \((\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mu}, \hat{\Theta}(t))_{t \in \mathbb{Z}^+})\), the
extended random dynamical system \( \hat{\Phi} \) and the random periodic process \( \hat{Y} \) be defined in Theorem 2.13.
Then for any \( B \in \mathcal{B}_X \)

\[
\hat{\mu}\{\hat{\omega} : \hat{Y}(t, \hat{\omega}) \in B\} = \rho_t(B),
\]
and

\[
\hat{P}(t, y, B) = \hat{\mu}\{\hat{\omega} : \hat{\Phi}(t, \hat{\omega})y \in B\} = P(t, y, B).
\]

Thus \( \rho \) is a periodic measure with respect to \( \hat{\mu} \) as well.

**Proof:** Note in (2.31), for any \( A \in \mathcal{F} \otimes \mathcal{B}_X \), by the periodicity of \( \mu_s \) and measure preserving property
of \( \theta \),

\[
\hat{\mu}(I_\tau \times A) = \frac{1}{\nu(I_\tau)} \int_{I_\tau} \mu_s(A) d\nu(s) \\
= \frac{1}{\nu(I_\tau)} \int_{I_\tau} \mu_{-s}(A) d\nu(s) \\
= \frac{1}{\nu(I_\tau)} \int_{I_\tau} \int_\Omega (\mu_{-s})_\omega(A_\omega) P(\omega) d\nu(s) \\
= \frac{1}{\nu(I_\tau)} \int_{I_\tau} \int_\Omega (\mu_{-s})_{\theta(-s)\omega}(A_{\theta(-s)\omega}) P(\omega) d\nu(s).
\]

From the proof of Theorem 2.13 we know that, for any \( \hat{\omega} = (s, \omega, x) \), \( \hat{Y}(t, \hat{\omega}) = \Phi(t + s, \theta(-s)\omega)x \),
is a random periodic process on the probability space \((\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mu}, (\hat{\Theta}(t))_{t \in \mathbb{Z}^+})\). Then for any \( t \in \mathbb{Z}^+ \) and
\( B \in \mathcal{B}_X \), by (2.29) and definition of \( \hat{Y} \),

\[
\hat{\mu}(\hat{\omega} : \hat{Y}(t, \hat{\omega}) \in B) = \int_\Omega \int_{I_\tau} \int_X I_B(\hat{\Phi}(t, \hat{\omega})) P(\omega) d\nu(s)
\]

\[
= \frac{1}{\nu(I_\tau)} \int_{I_\tau} \int_X \int\mathcal{F} \hat{\mathcal{F}}(\hat{\Phi}(t, \hat{\omega}))(\mu_{-s})_{\theta(-s)\omega}(dx) P(\omega) d\nu(s)
\]

\[
= \frac{1}{\nu(I_\tau)} \int_{I_\tau} \int_X \int\mathcal{F} \hat{\mathcal{F}}(\hat{\Phi}(t + s, \theta(-s)\omega)x)(\mu_{-s})_{\theta(-s)\omega}(dx) P(\omega) d\nu(s)
\]

\[
= \frac{1}{\nu(I_\tau)} \int_{I_\tau} \int_X \int\mathcal{F} (\Phi(t + s, \theta(-s)\omega))(\mu_{-s})_{\theta(-s)\omega}(dy) P(\omega) d\nu(s)
\]

\[
= \frac{1}{\nu(I_\tau)} \int_{I_\tau} \int_X \int\mathcal{F} (\mu_t)(B) = E[(\mu_t)(B)] = \rho_t(B).
\]

Now we consider \( \hat{\Phi}(t, \hat{\omega}) = \Phi(t, \omega) \), the extended random dynamical system on \( X \) over the probability
space \((\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mu})\). For any \( y \in X \), note again \( \hat{\omega} = (s, \omega, x) \),
The last claim follows easily from the above two results already proved.

\[ \hat{P}(t, y, B) = \hat{\mu}(\hat{\omega} : \hat{\Phi}(t, \hat{\omega})y \in B) \]
\[ = \int_{\Omega} I_{B}(\hat{\Phi}(t, \hat{\omega})y)|\hat{\mu}(d\hat{\omega}) \]
\[ = \frac{1}{\nu(I_{r})} \int_{I_{r}} \int_{\Omega} \int_{X} I_{B}(\hat{\Phi}(t, \hat{\omega})y)(\mu_{s})_{\omega}(dx)P(d\omega)d\nu(s) \]
\[ = \frac{1}{\nu(I_{r})} \int_{I_{r}} \int_{\Omega} \int_{X} I_{B}(\Phi(t, \omega)y)(\mu_{s})_{\omega}(dx)P(d\omega)d\nu(s) \]
\[ = \int_{\Omega} I_{B}(\Phi(t, \omega)y)P(d\omega) \]
\[ = P(t, y, B). \]

The last claim follows easily from the above two results already proved.

Now define
\[ \hat{J}_{k} := \hat{E}\left[ \left( \frac{1}{\nu(F_{0})} \int_{F_{k}} (I_{B}(\hat{\Phi}(t, \hat{\omega}) - \rho_{s}(B))d\nu(s) \right) \left( \frac{1}{\nu(F_{0})} \int_{F_{0}} (I_{B}(\hat{\Phi}(t, \hat{\omega}) - \rho_{s}(B))d\nu(s) \right) \right] \]

We also have:

**Lemma 2.16** Assume the semigroup transition probability \( P(t, x, \cdot) \) and periodic measure \( \rho \) satisfy Condition (EC). Then the random periodic process \( \hat{Y} \) which is given in Theorem 2.13 has exponentially decay correction in different periods, i.e. for any \( k \geq 1 \),
\[ |\hat{J}_{k}| \leq \delta_{1}e^{\delta_{2}k}. \]

**Proof:** When \( k \geq 1 \), note when \( k\tau \leq t < (k+1)\tau \),
\[ \hat{Y}(t, \hat{\omega}) = \hat{\Phi}(t - \tau, \hat{\Theta}(\tau)\hat{\omega})\hat{Y}(\tau, \hat{\omega}). \]

Then taking the conditional expectation and using Lemma 2.15 for \( k \geq 1 \),
\[ \hat{J}_{k} = \hat{E}\left[ \hat{E}\left[ \frac{1}{\nu(F_{0})} \int_{F_{k}} (I_{B}(\hat{\Phi}(t - \tau, \hat{\Theta}(\tau)\hat{\omega})\hat{Y}(\tau, \hat{\omega}) - \rho_{s}(B))d\nu(t) \right] \right] \]
\[ = \hat{E}\left( \frac{1}{\nu(F_{0})} \int_{F_{k}} (P(t - \tau, \hat{Y}(\tau, \hat{\omega}), B) - \rho_{s}(B))d\nu(t) \right) \]
\[ \cdot \left( \frac{1}{\nu(F_{0})} \int_{F_{0}} (I_{B}(\hat{\Phi}(t, \hat{\omega}) - \rho_{s}(B))d\nu(t) \right) \]

So by condition (EC),
\[ |\hat{J}_{k}| \leq \hat{E}\left( \frac{1}{\nu(F_{0})} \int_{F_{k}} (P(t - \tau, Y(\tau, \omega), B) - \rho_{s}(B))d\nu(t) \right] \]
\[ = \int_{X} \mu_{s}(d\nu) \int_{F_{k}} (P(t - \tau, y, B) - \rho_{s}(B))d\nu(t) \]
\[ = \hat{\mu}(\hat{\omega} : \hat{Y}(\tau, \hat{\omega}) \in dy) \]
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\[
\begin{align*}
= \int_X \left( \frac{1}{\nu(F_0)} \int_{F_k} (P(t-\tau,y,B) - \rho_t(B))d\nu(t) \right) \rho_0(dy) \\
\leq \delta_1 e^{\delta_2} e^{-\delta_2 k}.
\end{align*}
\]

Then we can prove the weak law of large numbers.

**Lemma 2.17 (WLLN)** Assume the same condition as in Theorem 2.16. Then the random periodic process \( \hat{Y} \) and its law \( \rho \) satisfy WLLN, i.e. as \( N \to \infty \),

\[
\frac{1}{\nu(G_N)} \int_{G_N} I_B(\hat{Y}(s,\hat{\omega}))d\nu(s) \to \frac{1}{\nu(F_0)} \int_{F_0} \rho_t(B)d\nu(t) \text{ in probability.}
\]

**Proof:** Define

\[
\hat{\xi}_k(\hat{\omega}) := \frac{1}{\nu(F_0)} \int_{F_k} I_B(\hat{Y}(t,\hat{\omega}))d\nu(t).
\]

Then for any \( m > n \), from the measure preserving property of \( \theta \) and the random periodicity of \( \hat{Y} \), similar to (2.25), we have

\[
J_{mn} := \text{cov}(\hat{\xi}_m, \hat{\xi}_n)
\]

\[
= \hat{E} \left[ \left( \frac{1}{\nu(F_0)} \int_{F_0} \rho_t(B)d\nu(t) \right) \left( \frac{1}{\nu(F_0)} \int_{F_0} \rho_t(B)d\nu(t) \right) \right]
\]

\[
= \hat{E} \left[ \left( \frac{1}{\nu(F_0)} \int_{F_0} I_B(\hat{Y}(t,\hat{\omega}))d\nu(t) - \frac{1}{\nu(F_0)} \int_{F_0} \rho_t(B)d\nu(t) \right) \cdot \left( \frac{1}{\nu(F_0)} \int_{F_0} I_B(\hat{Y}(t,\hat{\omega}))d\nu(t) - \frac{1}{\nu(F_0)} \int_{F_0} \rho_t(B)d\nu(t) \right) \right].
\]

So by Lemma 2.16, \( |J_{mn}| \leq \delta_1 e^{\delta_2} e^{-\delta_2 (m-n)} \), when \( m > n \). It is easy to know that when \( m = n \), \( |J_{nn}| \leq 1 \). On the other hand,

\[
\frac{1}{\nu(G_N)} \int_{G_N} I_B(\hat{Y}(s,\hat{\omega}))d\nu(s) = \frac{1}{N \nu(F_0)} \sum_{m=0}^{N-1} \int_{F_m} I_B(\hat{Y}(s,\hat{\omega}))d\nu(s)
\]

\[
= \frac{1}{N} \sum_{m=0}^{N-1} \hat{\xi}_m(\hat{\omega}).
\]

Thus by Lemma 2.7, we have

\[
\frac{1}{\nu(G_N)} \int_{G_N} I_B(\hat{Y}(t,\hat{\omega}))d\nu(t) \to \hat{E} \hat{\xi}_m(\hat{\omega}) = \frac{1}{\nu(F_0)} \int_{F_0} \rho_t(B)d\nu(t),
\]

in probability as \( N \to \infty \). \( \Box \)

Now we can prove the SLLN theorem for random periodic processes/periodic measure.

**Theorem 2.18 (SLLN)** Assume the same conditions as in Lemma 2.16. Then as \( R \ni T \to \infty \),
\[
\frac{1}{\nu([0,T] \cap G_\infty)} \int_{[0,T] \cap G_\infty} I_B(\hat{Y}(t, \hat{\omega}))d\nu(t) \rightarrow \frac{1}{\nu(F_0)} \int_{F_0} \rho_t(B)d\nu(t) \quad \text{a.s.} \tag{2.37}
\]

In particular, if Condition (EC) holds for \(F_0 = [0, \tau]\), then as \(T \rightarrow \infty\),

\[
\frac{1}{\nu([0,T] \cap I)} \int_{[0,T] \cap I} I_B(\hat{Y}(t, \hat{\omega}))d\nu(t) \rightarrow \bar{\rho}(B) \quad \text{a.s.} \tag{2.38}
\]

\textbf{Proof:} It suffices to note from (2.36) that

\[
\hat{\xi}_m(\hat{\omega}) = \frac{1}{\nu(F_0)} \int_{F_0} I_B(\hat{Y}(t - m\tau, \hat{\Theta}(m\tau)\hat{\omega}))d\nu(t)
\]

\[
= \frac{1}{\nu(F_0)} \int_{F_0} I_B(\hat{Y}(t, \hat{\Theta}(m\tau)\hat{\omega}))d\nu(t)
\]

\[
= \hat{\xi}_0(\hat{\Theta}(m\tau)\hat{\omega}).
\]

Then the proof of the theorem can be completed by a similar augment as the proof of Theorem 2.37. \^\^\†

In the following we can prove that (2.37) (2.38 as well) can be represented in a different way using general test functions.

\textbf{Theorem 2.19 (SLLN)} Assume the same conditions as in Lemma 2.16. Then for any \(f \in L^1(X, \mathcal{B}_X, \bar{\rho}_{F_0}(dx))\), as \(R \ni T \rightarrow \infty\),

\[
\frac{1}{\nu([0,T] \cap G_\infty)} \int_{[0,T] \cap G_\infty} f(\hat{Y}(t, \hat{\omega}))d\nu(t) \rightarrow \frac{1}{\nu(F_0)} \int_{F_0} f(x)\rho_t(dx)d\nu(t) \quad \text{a.s.} \tag{2.39}
\]

In particular, if Condition (EC) holds for \(F_0 = [0, \tau]\), then as \(T \rightarrow \infty\),

\[
\frac{1}{\nu([0,T] \cap I)} \int_{[0,T] \cap I} f(\hat{Y}(t, \hat{\omega}))d\nu(t) \rightarrow \int_{X} f(x)\bar{\rho}(dx) \quad \text{a.s.} \tag{2.40}
\]

\textbf{Proof:} Define for any \(B \in \mathcal{B}_X\)

\[
\bar{\rho}_{F_0,T}(B) = \frac{1}{\nu([0,T] \cap G_\infty)} \int_{[0,T] \cap G_\infty} I_B(\hat{Y}(t, \hat{\omega}))d\nu(t), \quad \bar{\rho}_{F_0}(B) = \frac{1}{\nu(F_0)} \int_{F_0} \rho_t(B)d\nu(t).
\]

Note \(\bar{\rho}_{F_0,T}(B)\) is a random probability measure on \((X, \mathcal{B}_X)\) and satisfies

\[
\bar{\rho}_{F_0,T}(B) \rightarrow \bar{\rho}_{F_0}(B), \text{ for any } B \in \mathcal{B}_X, \text{ as } T \rightarrow \infty.
\]

For any \(f \in L^1(X, \mathcal{B}_X, \bar{\rho}_{F_0}(dx))\), there is a sequence of simple functions \(f_n\) such that

\[
\int_X |f_n(x) - f(x)|\bar{\rho}_{F_0}(dx) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{2.41}
\]

Therefore for any \(\epsilon > 0\), there exists an \(m^* > 0\) such that when \(m \geq m^*\),

\[
\int_X |f_m(x) - f(x)|\bar{\rho}_{F_0}(dx) < \frac{\epsilon}{4}. \tag{2.42}
\]

It follows from (2.37) easily that as \(T \rightarrow \infty\),
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\[ \int_X f_m^r(x)\hat{\rho}_{F_0,T}(dx) - \int_X f_m^r(x)\hat{\rho}_{F_0}(dx) \to 0 \text{ a.s..} \]

Now we can apply Lebesgue’s dominated convergence theorem to deduce that as \( T \to \infty \),

\[ \hat{E}\| \int_X f_m^r(x)\hat{\rho}_{F_0,T}(dx) - \int_X f_m^r(x)\hat{\rho}_{F_0}(dx) \| \to 0. \]

Thus, there exists \( T^* \) such that when \( T > T^* \)

\[ \hat{E}\| \int_X f_m^r(x)\hat{\rho}_{F_0,T}(dx) - \int_X f_m^r(x)\hat{\rho}_{F_0}(dx) \| < \frac{\epsilon}{2}. \quad (2.43) \]

Note for any integer \( N \) such that \( N\tau > T^* \)

\[ \hat{E}\| \int_X f(x)\hat{\rho}_{F_0,N\tau}(dx) - \int_X f(x)\hat{\rho}_{F_0}(dx) \| 
\]

\[ = \hat{E}\| \int_X f_m(x)\hat{\rho}_{F_0,N\tau}(dx) - \int_X f_m(x)\hat{\rho}_{F_0}(dx) \| 
\]

\[ + \hat{E}\| \int_X |f(x) - f_m(x)||\hat{\rho}_{F_0,N\tau}(dx) + \int_X (f(x) - f_m(x))\hat{\rho}_{F_0}(dx) \| 
\]

\[ < \frac{\epsilon}{2} + \int_X |f(x) - f_m(x)||\hat{\rho}_{F_0}(dx) + \frac{\epsilon}{4} < \epsilon. \]

Here we used the fact that \( \hat{E}\rho_{F_0,N\tau} = \hat{\rho}_{F_0}. \) Thus \( \hat{E}\| \int_X f(x)\hat{\rho}_{F_0,N\tau}(dx) - \int_X f(x)\hat{\rho}_{F_0}(dx) \| \to 0 \) as \( N \to \infty \). By Chebyshev’s inequality we know that \( \int_X f(x)\hat{\rho}_{F_0,N\tau}(dx) \to \int_X f(x)\hat{\rho}_{F_0}(dx) \) in probability as \( N \to \infty \). Therefore there exists a subsequence \( N_k \) with \( N_k \to \infty \) as \( k \to \infty \) such that \( \int_X f(x)\hat{\rho}_{F_0,N_k\tau}(dx) \to \int_X f(x)\hat{\rho}_{F_0}(dx) \) a.s. as \( k \to \infty \). On the other hand, using a similar argument of Birkhoff’s ergodic theorem as before, we know that

\[ \int_X f(x)\hat{\rho}_{F_0,N\tau}(dx) = \sum_{i=0}^{N-1} \frac{1}{N\nu(F_0)} \int_{F_0} \int_X f(x)\delta_{Y(s,\hat{\omega}(i\tau)\omega)}(dx)d\nu(s) \]

converges almost surely. Thus we have as \( N \to \infty \),

\[ \int_X f(x)\hat{\rho}_{F_0,N\tau}(dx) \to \int_X f(x)\hat{\rho}_{F_0}(dx) \text{ a.s..} \quad (2.44) \]

Now by a standard argument to pass the limit as \( R \ni T \to \infty \), we completed the proof of the following theorem.

\[ \hat{\rho} \]

**Remark 2.20** Note it is obvious that \( (2.39) \) \( (2.40) \) implies \( (2.37) \) \( (2.38) \) by taking \( f = I_R \). Therefore \( (2.39) \) \( (2.40) \) and \( (2.37) \) \( (2.38) \) are equivalent. In other cases considered in this paper, the SLLN with test functions can also be given similarly. We omit them as they become obvious with this remark.
3 The semi-flow case

As in the last section, denote by \((\Omega, \mathcal{F}, P, (\theta(s))_{s \in I})\) a metric dynamical system and \(\theta(s) : \Omega \rightarrow \Omega\) is assumed to be measurably invertible for all \(s \in I\). Denote \(\Delta := \{(t, s) \in I^2, s \leq t\}\). Consider a stochastic semi-flow \(u : \Delta \times \Omega \times X \rightarrow X\), which satisfies the following standard condition

\[
u(t, r, \omega) = u(t, s, \omega) \circ u(s, r, \omega), \quad \text{for all } r \leq s \leq t, \ r, s, t \in I. \tag{3.1}\]

As in the cocycle case in the last section, we do not assume the map \(u(t, s, \omega) : X \rightarrow X\) to be invertible for \((t, s) \in \Delta, \omega \in \Omega\). We call \(u\) is a \(\tau\)-periodic stochastic semi-flow if it satisfies an additional periodicity property: there exists a constant \(\tau > 0\) such that

\[
u(t + \tau, s + \tau, \omega) = u(t, s, \theta(\tau)\omega). \tag{3.2}\]

Remark 3.1 (i). The periodicity assumption (3.2) is very natural. It can be verified for SDEs or SPDEs with time periodic coefficients in the same manner as verifying the cocycle property for autonomous stochastic systems. In the cocycle case, (3.2) holds for all \(\tau \geq 0\) i.e.

\[
u(t, s, \omega) = u(t - s, 0, \theta(s)\omega) \quad \text{for all } s \leq t, \ s, t \in I.\]

(ii) The periodicity assumption (3.2) plays a crucial role to enable us to lift the semiflow \(u\) to a cocycle on the cylinder \((I \cap [0, \tau)) \times X\).

The definition of random periodic paths (solutions) for stochastic semi-flow was given in [18], [19]:

Definition 3.2 A random periodic path of period \(\tau\) of the semi-flow \(u : \Delta \times \Omega \times X \rightarrow X\) is an \(\mathcal{F}\)-measurable map \(Y : I \times \Omega \rightarrow X\) such that for any \((t, s) \in \Delta\) and \(\omega \in \Omega\),

\[
u(t, s, \omega)Y(s, \omega) = Y(t + s, \omega), \quad Y(s + \tau, \omega) = Y(s, \theta(\tau)\omega). \tag{3.3}\]

The following lemma tells how to lift a periodic stochastic semi-flow to a cocycle.

Lemma 3.3 We lift the \(\tau\)-periodic stochastic semi-flow \(u : \Delta \times \Omega \times X \rightarrow X\) to a random dynamical system on a cylinder \(\tilde{X} := (I \cap [0, \tau)) \times X\) by the following:

\[
\tilde{\Phi}(t, \omega)(s, x) = (t + s \mod \tau, u(t + s, s, \theta(-s)\omega)x), \quad \text{for any } (s, x) \in \tilde{X}, \ t \in I^+. \tag{3.4}
\]

Then \(\tilde{\Phi} : I^+ \times \Omega \times \tilde{X} \rightarrow \tilde{X}\) is a cocycle on \(\tilde{X}\) over the metric dynamical system \((\Omega, \mathcal{F}, P, (\theta(s))_{s \in I})\). Moreover, assume \(Y : I \times \Omega \rightarrow X\) is a random periodic solution of the semi-flow \(u\) with period \(\tau > 0\). Then \(\tilde{Y} : I \times \Omega \rightarrow \tilde{X}\) defined by

\[
\tilde{Y}(s, \omega) := (s \mod \tau, Y(s, \omega)), \tag{3.5}
\]

is a random periodic solution of the cocycle \(\tilde{\Phi}\) on \(\tilde{X}\) over the metric dynamical system \((\Omega, \mathcal{F}, P, (\theta(s))_{s \in I})\).

Proof: From (3.1), (3.2), we have for any \((s, x) \in \tilde{X}, t_1, t_2 \in I^+\)
\[
\tilde{\Phi}(t_2, \theta(t_1)\omega) \circ \tilde{\Phi}(t_1, \omega)(s, x) \\
= (t_2 + t_1 + s \mod \tau, u(t_2 + (t_1 + s \mod \tau), t_1 + s \mod \tau, \theta(-(t_1 + s \mod \tau))\theta(t_1)\omega) \\
\circ u(t_1 + s, s, \theta(-s)\omega)x) \\
= (t_2 + t_1 + s \mod \tau, u(t_2 + (t_1 + s - k\tau), t_1 + s - k\tau, \theta(-(t_1 + s - k\tau))\theta(t_1)\omega) \\
\circ u(t_1 + s, s, \theta(-s)\omega)x) \\
= (t_2 + t_1 + s \mod \tau, u(t_2 + (t_1 + s), t_1 + s, \theta(-(t_1 + s))\theta(t_1)\omega) \circ u(t_1 + s, s, \theta(-s)\omega)x) \\
= \tilde{\Phi}(t_1 + t_2, \omega)(s, x), \\
\]

where we take \(k = \lceil \frac{t}{\tau} \rceil\), \(\lceil \cdot \rceil\) represents the integer part. This means that \(\tilde{\Phi}\) is a cocycle.

Now assume \(Y : I \times \Omega \rightarrow X\) is a random periodic solution of the semi-flow \(u\), and set \(\tilde{Y} : I \times \Omega \rightarrow \tilde{X}\) defined by (3.3). Then one can easily verify that for any \(s \in I, t \in I^+\),

\[
\tilde{\Phi}(t, \theta(s)\omega)\tilde{Y}(s, \omega) = \tilde{Y}(t + s, \omega),
\]

and

\[
\tilde{Y}(s + \tau, \omega) = \tilde{Y}(s, \theta(\tau)\omega).
\]

That is to say that \(\tilde{Y}\) is a random periodic solution of the cocycle \(\tilde{\Phi}\) on \(\tilde{X}\). \(\ddagger\ddagger\)

Now we define the skew product \(\tilde{\Theta} : \Delta \times \tilde{\Omega} \rightarrow \tilde{\Omega}\) of the metric dynamical system \((\Omega, \mathcal{F}, P, (\theta(s))_{s \in I})\) and the semi-flow \(u\) by

\[
\tilde{\Theta}(t + s, s)(\omega, x) = (\theta(t)\omega, u(t + s, s, \theta(-s)\omega)x), \ t \in I^+, \ s \in I.
\]

(3.6)

Here \(\tilde{\Omega} = \Omega \times X\) as in the last section. One can easily verify that for any \(t_1, t_2 \in I^+, s \in I\),

\[
\tilde{\Theta}(t_2 + t_1 + s, t_1 + s) \circ \tilde{\Theta}(t_1 + s, s) = \tilde{\Theta}(t_2 + t_1 + s, s).
\]

(3.7)

**Theorem 3.4** Assume the \(\tau\)-periodic stochastic semi-flow \(u : \Delta \times \Omega \times X \rightarrow X\) has a random periodic solution \(Y : I \times \Omega \rightarrow X\). Define

\[
(\mu_s)\omega(B) = \delta_{Y(s, \theta(-s)\omega)}(B).
\]

(3.8)

Then

\[
\mu_s(dx, d\omega) = (\mu_s)\omega(dx) \times P(d\omega)
\]

(3.9)

is a periodic measure of the skew product \(\tilde{\Theta}\) on the product measurable space \((\Omega \times X, \mathcal{F} \otimes \mathcal{B}_X)\), i.e.

\[
\tilde{\Theta}(t + s, s)\mu_s = \mu_{t+s}, \ \mu_{s+t} = \mu_s, \ \text{for all} \ t \in I^+, \ s \in I,
\]

(3.10)

which is equivalent to
\[ u(t + s, s, \theta(-s)\omega)(\mu_s)_\omega = (\mu_{t+s})_{\theta(t)\omega} \text{ and } (\mu_{t+s})_{\omega} = (\mu_s)_{\omega}, \quad (3.11) \]

for all \( t \in I^+, s \in I, \omega \in \Omega. \)

**Proof:** We use the lifting-up and mapping-down procedure. First, from Lemma 3.3, we know that \( \tilde{Y}(s, \omega) \) defined in (3.5) is a random periodic solution of cocycle \( \tilde{\Phi} \) on \( \tilde{X} \). Following the result of Theorem 2.3 about the relation of random periodic solutions and periodic measures for cocycle, there is a periodic measure \( \tilde{\mu}_s \) on the product measurable space \((\Omega \times \tilde{X}, \mathcal{F} \otimes \mathcal{B}_{\tilde{X}})\) defined as

\[ \tilde{\mu}_s(\tilde{A}) = \int_{\Omega} \delta_{\tilde{Y}(s,\omega)}(\tilde{A}_{\theta(s)\omega})P(d\omega) \]

for any set \( \tilde{A} \in \mathcal{F} \otimes \mathcal{B}_{\tilde{X}} \). Then for any \( t \in I^+, s \in I, \)

\[ (\tilde{\mu}_{t+s})_{\omega} = (\tilde{\mu}_s)_{\omega}, \text{ and } \tilde{\Phi}(t, \omega)(\tilde{\mu}_s)_{\omega} = (\tilde{\mu}_{t+s})_{\theta(t)\omega}, \quad (3.12) \]

where \( (\tilde{\mu}_s)_{\omega} \) is the factorisation of \( \tilde{\mu}_s(d\tilde{x}, d\omega) = (\tilde{\mu}_s)_{\omega}(d\tilde{x}) \times P(d\omega). \) In fact,

\[ (\tilde{\mu}_s)_{\omega} = \delta_{\tilde{Y}(s,\theta(-s)\omega)}. \]

So for any \( C \in \mathcal{B}_{I \cap [0, \tau)}, B \in \mathcal{B}_X, \)

\[ (\tilde{\mu}_s)_{\omega}(C \times B) = \delta_{(s \mod \tau, Y(s, \theta(-s)\omega))}(C \times B) \]
\[ = \delta_{(s \mod \tau)}(C)\delta_{Y(s, \theta(-s)\omega)}(B) \]
\[ = \delta_{(s \mod \tau)}(C)(\mu_s)_{\omega}(B), \quad (3.13) \]

where \( (\mu_s)_{\omega} \) is defined in (3.8). Then (3.11) follows from (3.12) and (3.13) and the definition of \( \tilde{\Phi} \) in (3.4). Now from (3.9), for any \( A \in \mathcal{F} \otimes \mathcal{B}_X, s \in I \)

\[ \mu_s(A) = \int_{\Omega} (\mu_s)_{\omega}(A_{\omega})P(d\omega). \]

Note for any \( t \in I^+ \)

\[ \tilde{\Theta}(t + s, s)^{-1}A = \{(\omega, x) : (\theta(t)\omega, u(t + s, s, \theta(-s)\omega)x) \in A\} \]
\[ = \{(\omega, x) : \omega \in \Omega \text{ and } u(t + s, s, \theta(-s)\omega)x \in A_{\theta(t)\omega}\} \]
\[ = \{(\omega, x) : \omega \in \Omega \text{ and } x \in u(t + s, s, \theta(-s)\omega)^{-1}A_{\theta(t)\omega}\}, \]

thus

\[ (\tilde{\Theta}(t + s, s)\mu_s)(A) = \mu_s(\tilde{\Theta}(t + s, s)^{-1}A) \]
\[ = \int_{\Omega} (\mu_s)_{\omega}(u(t + s, s, \theta(-s)\omega)^{-1}A_{\theta(t)\omega})P(d\omega) \]
\[ = \int_{\Omega} u(t + s, s, \theta(-s)\omega)(\mu_s)_{\omega}(A_{\theta(t)\omega})P(d\omega) \]
\[ = \int_{\Omega} (\mu_{t+s})_{\theta(t)\omega}(A_{\theta(t)\omega})P(d\omega). \]
say that for any $s$  

if (3.10) is true, then (3.11) can be also verified using the factorisation of $\mu$ as $\mu(dx,d\omega) = (\mu_s)(dx) \times P(d\omega)$, the definition of $\bar{\Theta}$ and (3.14).

Moreover, for any $t$, $\mu_{t+s} = \mu_s$ follows from $(\mu_{t+s})_\omega = (\mu_s)_\omega$ easily. Therefore (3.10) is verified. Conversely, if (3.10) is true, then (3.11) can be also verified using the factorisation of $\mu_s$ as $\mu_s(dx,d\omega) = (\mu_s)_\omega(dx) \times P(d\omega)$, the definition of $\bar{\Theta}$ and (3.14).

Now consider the case when $u(t+s,s,\cdot)$ is a Markovian semi-flow on a filtered dynamical system $(\Omega, \mathcal{F}, P, (\theta_t)_{t \in I}, (\mathcal{F}_t^I)_{s \leq t})$, i.e. for any $s,t,u \in I, s \leq t$, we have $\theta_u^{-1}\mathcal{F}_s^I = \mathcal{F}_{s+u}^I$ and $u(t+s,s,\cdot)$ is independent with $\mathcal{F}^s_{-\infty}$. We also assume the random periodic solution $Y(s,\cdot)$ is adapted, that is to say that for any $s \in I$, $Y(s,\cdot)$ is measurable with respect to $\mathcal{F}^s_{-\infty} := \bigvee_{r \leq s} \mathcal{F}^s_r$.

Denote the transition semigroup of $u$ by

$$P(t+s,s,x,B) = P(\{\omega : u(t+s,s,\omega)x \in B\}), \text{ for any } B \in \mathcal{B}_X, \ t \in I^+, \ s \in I.$$  

Note that $u$ is not a homogenous Markov process, so its transition probability depends on the starting time $s$. For any probability measure $\rho$ on $(X, \mathcal{B}_X)$, define

$$(P^s(t+s,s)\rho)(B) = \int_X P(t+s,s,x,B)\rho(dx), \text{ for any } B \in \mathcal{B}_X, \ s \in I, \ t \in I^+.$$  

**Theorem 3.5** Assume that the $\tau$-periodic Markovian semi-flow $u : \Delta \times \Omega \times X \to X$ has an adapted random periodic solution $Y : I \times \Omega \to X$. Then it has a periodic measure $\rho_s$ on $(X, \mathcal{B}_X)$ defined by $\rho_s(B) = E\delta_{Y(s,\omega)}(B) = P(\{\omega : Y(s,\omega) \in B\}), \ B \in \mathcal{B}_X, \ s \in I.$  

satisfying for any $s \in I, \ t \in I^+$,  

$$P^s(t+s,s)\rho_s = \rho_{t+s}, \ \rho_{s+\tau} = \rho_s.$$  

Moreover, for any $t \in I$,  

$$E(m\{s \in [0,\tau] : Y(s,\cdot) \in B\}) = E(m\{s \in [t,t+\tau] : Y(s,\cdot) \in B\}).$$  

**Proof:** Consider the lifted cocycle $\tilde{\Phi}$. Denote its transition probability by  

$$\tilde{P}(t, (s,x), \tilde{B}) = P(\{\omega : \tilde{\Phi}(t,\omega)(s,x) \in \tilde{B}\}), \text{ for any } \tilde{B} \in \mathcal{B}_{\tilde{X}}, \ t \in I^+.$$  

One can easily see that  

$$\tilde{P}(t, (s,x), C \times B) = \delta_{t+s \mod \tau}(C)P(t+s,s,x,B),$$  

for any $t \in I^+, \ (s,x) \in \tilde{X}, \ C \in \mathcal{B}_{[0,\tau] \cap I}, \ B \in \mathcal{B}_X$. For any probability measure $\tilde{\rho}$ on $(\tilde{X}, \mathcal{B}_{\tilde{X}})$, define  

$$(\tilde{P}^s(t)\tilde{\rho})(\tilde{B}) = \int_{\tilde{X}} \tilde{P}(t, \tilde{x}, \tilde{B})\tilde{\rho}(d\tilde{x}), \text{ for any } \tilde{B} \in \mathcal{B}_{\tilde{X}}, \ t \in I^+.$$
Following the result for the cocycle case, Theorem 2.6, define
\[ \tilde{\rho}_s(B) = E(\tilde{\mu}_s)(\tilde{B}) = P(\{\omega : \tilde{Y}(s, \omega) \in \tilde{B}\}), \text{ for any } \tilde{B} \in \mathcal{B}_{\tilde{X}}. \]

Then
\[ \tilde{P}^*(t)\tilde{\rho}_s = \tilde{\rho}_{t+s}, \text{ and } \tilde{\rho}_{s+\tau} = \tilde{\rho}_s, \ s \in I, \ t \in I^+. \]  
(3.20)

Define
\[ \rho_s(B) = E\delta_{Y(s,\omega)}(B) = P(\{\omega : Y(s,\omega) \in B\}), \ B \in \mathcal{B}_X. \]  
(3.21)

Then it is easy to see that for any \( C \in \mathcal{B}_{[0, \tau) \cap I} \) and \( B \in \mathcal{B}_X \),
\[ \tilde{\rho}_s(C \times B) = \delta_{s \mod \tau}(C)\rho_s(B), \]  
(3.22)

and
\[ (\tilde{P}^*(t)\tilde{\rho}_s)(C \times B) = \delta_{(t+s \mod \tau)}(C)(P^*(t+s, s)\rho_s(B)). \]  
(3.23)

From (3.20), (3.21), (3.22), (3.23), we know immediately (3.16) by taking \( C = [0, \tau) \cap I \). On the other hand, if we define \( \hat{\rho}_s \) as in (3.22), it is easy to derive (3.23) from (3.18), (3.19). Then we can get (3.20) if (3.16) holds.

We cannot construct an invariant measure on the space \((X, \mathcal{B}_X)\). But following the construction of the invariant measure in the case of cocycle from a random periodic solution, we can do this for the lifted cocycle on the cylinder \( \tilde{X} \). For this, define
\[ \bar{\rho} = \frac{1}{\nu([0, \tau) \cap I)} \int_{[0, \tau) \cap I} \tilde{\rho}_s d\nu(s). \]

Then
\[ \tilde{P}^*(t)\bar{\rho} = \bar{\rho}. \]

Then following Theorem 2.6, we can obtain for any \( C \in \mathcal{B}_{[0, \tau) \cap I}, \ B \in \mathcal{B}_X, \ t \in I \)
\[ \bar{\rho}(C \times B) = E(\frac{1}{\nu([0, \tau) \cap I)}) \{s \in [0, \tau) : (s, Y(s, \cdot)) \in C \times B\} \]
\[ = E(\frac{1}{\nu([0, \tau) \cap I)}) \{s \in [t, t+\tau) \cap I : (s \mod \tau, Y(s, \cdot)) \in C \times B\}. \]  
(3.24)

We can take \( C = [0, \tau) \cap I \) to obtain (3.17).

Similar to the cocycle case, we can also prove a SLLN theorem for the periodic Markovian semi-flow and random periodic paths. We only state the results without including their proofs as they are similar to the cocycle case. First similar with Lemma 2.8 we have:

**Lemma 3.6** Assume that \( Y : I \times \Omega \rightarrow X \) is an adapted random periodic path of a periodic Markovian semi-flow \( u \) and for any \( x \in X, \lim_{k \rightarrow \infty} u(s, -k\tau, \omega)x = Y(s, \omega) \ a.s. \). Then for any \( B \in \mathcal{B}_X \),
random periodic processes, periodic measures and law of large numbers

\[
\int_X \left| \frac{1}{\nu(F_0)} \int_{F_0} P(s, \tau, x, B) d\nu(s) - \frac{1}{\nu(F_0)} \int_{F_0} \rho_{s}(B) d\nu(s) \right| \rho_0(dx) \to 0,
\]

\text{as } k \to \infty.

**Condition (ECS):** Assume there exist constants \(\delta_1, \delta_2 > 0\) such that for any \(k \geq 1,\)

\[
\int_X \left| \frac{1}{\nu(F_0)} \int_{F_0} (P(s, 0, B) - \rho_{s}(B)) d\nu(s) \right| \rho_0(dx) \leq \delta_1 e^{-\delta_2 k}.
\]

**Theorem 3.7 (SLLN)** Assume that \(Y : I \times \Omega \to X\) is an adapted random periodic path of a periodic Markovian semi-flow \(u\) and the law \(\rho_{s} \in \mathcal{P}(X)\) of \(Y(s)\) satisfies Condition (ECS). Then as \(R \ni T \to \infty,\)

\[
\frac{1}{\nu([0,T] \cap G_{\infty})} \int_{[0,T] \cap G_{\infty}} I_B(Y(s, \omega)) d\nu(s) \to \frac{1}{\nu(F_0)} \int_{F_0} \rho_{s}(B) d\nu(s) \text{ a.s.}
\]

In particular, if Condition (ECS) holds for \(F_0 = [0, \tau)\), then as \(T \to \infty,\)

\[
\frac{1}{\nu([0,T] \cap I)} \int_{[0,T] \cap I} I_B(Y(s, \omega)) d\nu(s) \to \bar{\rho}(B) := \frac{1}{\nu([0, \tau) \cap I)} \int_{[0, \tau) \cap I} \rho_{s}(B) d\nu(s) \text{ a.s.}
\]

In the following, we will give a construction of the random periodic solution from a periodic measure for stochastic semi-flow.

Similar as in the last section, set \(I_\tau = [0, \tau) \cap I\) of the additive modulo \(\tau, B_{I_\tau} = \{\emptyset, I_\tau\}, \hat{\Omega} = I_\tau \times \Omega \times X, \hat{\mathcal{F}} = B_{I_\tau} \otimes \mathcal{F} \otimes \mathcal{B}_X\) For \(\hat{\omega} = (s, \omega, x) \in \hat{\Omega},\) define for \(t \in I^+\), the skew triple product of the metric dynamical system \((\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}, (\theta(s))_{s \in I})\), the semiflow \(u\) and the rotation on the circle \(I_\tau,\)

\[
\hat{\Theta}(t) \hat{\omega} = (s + t \mod \tau, \theta(t)\omega, u(t + s, s, \theta(-s)\omega)x).
\]

**Theorem 3.8** Assume that the \(\tau\)-periodic Markovian semi-flow \(u\) has a periodic measure \(\mu_s(dx, d\omega) = (\mu_s)_\omega(dx) \times P(d\omega)\) on the product measurable space \((\Omega \times X, \mathcal{F} \otimes \mathcal{B}_X)\) satisfying (3.11) (or (3.10)). Then a measure \(\hat{\mu}\) on the measurable space \((\hat{\Omega}, \hat{\mathcal{F}})\) defined by

\[
\hat{\mu}(I_\tau \times A) = \frac{1}{\nu(I_\tau)} \int_{I_\tau} \mu_s(A) d\nu(s), \quad \hat{\mu}(\emptyset \times A) = 0, \quad \text{for any } A \in \mathcal{F} \otimes \mathcal{B}_X,
\]

is a probability measure and the skew product \(\hat{\Theta}(t) : \hat{\Omega} \to \hat{\Omega}, t \in I^+,\) defined by (3.20) is measure \(\hat{\mu}\)-preserving, and satisfies the semi-group property: \(\hat{\Theta}(t_1) \hat{\Theta}(t_2) = \hat{\Theta}(t_1 + t_2)\) for any \(t_1, t_2 \in I^+.\) Moreover, for \(\hat{\omega} = (s, \omega, x)\)

\[
\hat{\Theta}(t, \hat{\omega}) = u(t + s, s, \theta(-s)\omega),
\]

is a cocycle random dynamical system over the metric dynamical system \((\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mu}, \hat{\Theta}(t))_{t \in I^+}\). The cocycle \(\hat{\Phi}\) has a random periodic path \(\hat{Y}\) given by the following: for \(\hat{\omega}^* = (s, \omega^*, u^*(t_2, t_1, \theta(-t_2)\omega^*)x) \in \hat{\Omega}, \hat{Y} : I^+ \times \hat{\Omega} \to X\) is defined by

\[
\hat{Y}(t, \hat{\omega}^*) := u(t + s, 0, \theta(-s)\omega^*)u^*(t_2 - s, t_1 - s, \theta(-t_2)\omega^*).
\]
Proof: We first show the semigroup property of \(\hat{\Theta} \). For this, from (3.26), the definition of \(\hat{\Theta} \), and (3.1), (3.2), we have that

\[
\hat{\Theta}(t_2)\hat{\Theta}(t_1)\hat{\omega} = \left(s + t_1 + t_2 \mod \tau, \theta(t_1 + t_2)\omega, u(t_2 + (s + t_1 \mod \tau), s + t_1 \mod \tau, \theta(-s + t_1 \mod \tau)\theta(t_1)\omega) \circ u(t_1 + s, s, \theta(-s)\omega)x\right)
\]

\[
= \left(s + t_1 + t_2 \mod \tau, \theta(t_1 + t_2)\omega, u(t_2 + t_1 + s - k\tau, s + t_1 - k\tau, \theta(-s - t_1 + k\tau)\theta(t_1)\omega) \circ u(t_1 + s, s, \theta(-s)\omega)x\right)
\]

\[
= \left(s + t_1 + t_2 \mod \tau, \theta(t_1 + t_2)\omega, u(t_2 + t_1 + s, s + t_1, \theta(-s)\omega) \circ u(t_1 + s, s, \theta(-s)\omega)x\right)
\]

\[
= \left(s + t_1 + t_2 \mod \tau, \theta(t_1 + t_2)\omega, u(t_2 + t_1 + s, s, \theta(-s)\omega)\right)
\]

\[
\hat{\Theta}(t_1 + t_2)\hat{\omega},
\]

where we take \(k = \left\lfloor \frac{t_1}{\tau} \right\rfloor \), \([ \lfloor \cdot \rfloor \) represents the integer part.

Now assume the stochastic semi-flow has periodic measure \((\mu_s)_\omega\) satisfying (3.11). It is obvious that \(\hat{\mu} \) defined by (3.27) is a probability measure on the measurable space \((\hat{\Theta}, \mathcal{F})\). We now need to prove that \(\hat{\Theta}(t)\hat{\mu} = \hat{\mu} \) for all \(t \in I^+\). Note that

\[
\hat{\Theta}(t)(s, \omega, x) = (t \mod \tau, \hat{\Theta}(t + s)(\omega, x)).
\]

So for any \(A \in \mathcal{F} \otimes \mathcal{B}_{\mathbb{R}}, t \in [0, \tau) \cap I^+\),

\[
\hat{\Theta}(t)\hat{\mu}(I_r \times A)
\]

\[
= \hat{\mu}(\hat{\Theta}(t)^{-1}(I_r \times A))
\]

\[
= \hat{\mu}\left((s, \omega, x) : (s + t \mod \tau, \theta(t)\omega, u(t + s, s, \theta(-s)\omega)x) \in I_r \times A\right)
\]

\[
= \hat{\mu}\left(\{(s, \omega, x) : (s \in [0, \tau - t), (\theta(t)\omega, u(t + s, s, \theta(-s)\omega)x) \in A\}
\]

\[
\cup\{(s, \omega, x) : (s \in [\tau - t, \tau), (\theta(t)\omega, u(t + s, s, \theta(-s)\omega)x) \in A\}\}
\]

\[
= \frac{1}{\nu(I_r)} \int_{[0,\tau-t) \cap \Omega} \mu_s(\hat{\Theta}(t + s, s)^{-1}A)dv(s)
\]

\[
+ \frac{1}{\nu(I_r)} \int_{[\tau-t,\tau) \cap \Omega} \mu_s(\hat{\Theta}(t + s, s)^{-1}A)dv(s)
\]

\[
= \frac{1}{\nu(I_r)} \int_{[0,\tau-t) \cap \Omega} \mu_{t+s}(A)dv(s)
\]

\[
+ \frac{1}{\nu(I_r)} \int_{[\tau-t,\tau) \cap \Omega} \mu_{t+s}(A)dv(s)
\]

\[
= \frac{1}{\nu(I_r)} \int_{[0,\tau) \cap \Omega} \mu_s(A)dv(s)
\]

\[
+ \frac{1}{\nu(I_r)} \int_{[\tau,\tau+t) \cap \Omega} \mu_s(A)dv(s)
\]

\[
= \frac{1}{\nu(I_r)} \int_{I_r} \mu_s(A)dv(s)
\]

\[
= \hat{\mu}(I_r \times A).
\]
This can be generalised to any \( t \in I^+ \) using the group property of \( \hat{\Theta} \). Thus \((\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mu}, (\hat{\Theta}_t)_{t \in I^+}) \) is a metric dynamical system. We can prove \( \hat{\Phi} \) defined by (3.28) is a cocycle on \( X \) over the \((\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mu}, (\hat{\Theta}_t)_{t \in I^+}) \). In fact, from the definitions of \( \hat{\Phi} \), \( \hat{\Theta} \), and (3.1), (3.2), we know for any \( r, t \in I^+ \), \( \hat{\omega} = (s, \omega, x) \in \hat{\Omega} \),

\[
\hat{\Phi}(r, \hat{\Theta}(t)\hat{\omega}) \circ \hat{\Phi}(t, \hat{\omega}) = u(r + (t + s \mod \tau), t + s \mod \tau, \theta(-(t + s \mod \tau))\theta(t)\omega) \circ u(t + s, s, \theta(-s)\omega) = u(r + t + s, t + s, \theta(-s)\omega) \circ u(t + s, s, \theta(-s)\omega) = u(r + t + s, s, \theta(-s)\omega) = \hat{\Phi}(r + t, \hat{\omega}).
\]

The claim is asserted.

Now we prove that \( \hat{Y} \) defined by (3.29) is a random periodic solution of \( \hat{\Phi} \). First note if \( \hat{\omega} = (s, \omega, x) \in \hat{\Omega} \),

\[
\hat{Y}(t, \hat{\omega}) = u(t + s, 0, \theta(-s)\omega)x \text{ for any } t \in I^+.
\]

Thus from (3.1), (3.2) and definition of \( \hat{\Phi} \) and \( \hat{Y} \), we have for any \( r, t \in I^+ \)

\[
\hat{\Phi}(r, \hat{\Theta}(t)\hat{\omega})\hat{Y}(t, \hat{\omega}) = u(r + (s + t \mod \tau), s + t \mod \tau, \theta(-(s + t \mod \tau))\theta(t)\omega) \circ u(t + s, 0, \theta(-s)\omega)x = u(r + s + t, t + s, \theta(-s)\omega) \circ u(t + s, 0, \theta(-s)\omega)x = u(r + s + t, 0, \theta(-s)\omega)x = \hat{Y}(r + t, \hat{\omega}).
\]

Moreover,

\[
\hat{Y}(t + \tau, \hat{\omega}) = u(t + \tau + s, 0, \theta(-s)\omega)x,
\]

and note

\[
\hat{\Theta}(\tau)\hat{\omega} = (s, \theta(\tau)\omega, u(\tau + s, s, \theta(-(s + \tau))\omega) \circ \theta(\tau)\omega)x,
\]

so from the definition of \( \hat{Y} \),

\[
\hat{Y}(t, \hat{\Theta}(\tau)\hat{\omega}) = u(t + s, 0, \theta(-s)\theta(\tau)\omega) \circ u(\tau, 0, \theta(-s)\omega)x = u(t + s + \tau, \tau, \theta(-s)\omega) \circ u(\tau, 0, \theta(-s)\omega)x = u(t + s + \tau, 0, \theta(-s)\omega)x.
\]

Thus

\[
\hat{Y}(t + \tau, \hat{\omega}) = \hat{Y}(t, \hat{\Theta}(\tau)\hat{\omega}).
\]

That is to say \( \hat{Y} \) is a random periodic solution as claimed. \( \dagger \dagger \)

In the last part of this section, we consider a periodic Markovian semi-flow \( u \). Starting from a periodic measure on \((X, \mathcal{B}_X)\) satisfying (3.10), we give a construction of the periodic measure on the product measurable space \((\Omega \times X, \mathcal{F} \otimes \mathcal{B}_X)\) and a therefore random periodic path of the Markovian semiflow \( u \) over the enlarged metric dynamical system \((\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mu}, (\hat{\Theta}_t)_{t \in I^+})\). Note we ignore the order of
\[ (\Theta(t)\omega) = (\theta(t)\omega, \Phi(t, \omega)(s, x)). \] (3.30)

In this sense, we can regard \( \hat{\Theta} \) as the skew product of the metric dynamical system \((\Omega, \mathcal{F}, P, (\theta(s))_{s \in I})\) and the lifted cocycle \( \hat{\Phi} : I^+ \times \tilde{X} \to \tilde{X} \). The idea is to lift the semi-flow to the cocycle on the cylinder \( \tilde{X} \) and then apply Theorem 2.12 to construct a periodic measure \( \tilde{\mu}_s \) on the product space \((\Omega \times \tilde{X}, \mathcal{F} \otimes B_{\tilde{X}})\). However, instead of applying Theorem 2.13, which only gives a random periodic path of the lifted cocycle \( \tilde{\Phi} \) on the cylinder \( \tilde{X} \), here we go further to project the periodic measure to a periodic measure of the semi-flow \( u \) on the space \((\Omega \times X, \mathcal{F} \otimes B_X)\). Then we are ready to construct a random periodic path of the semiflow \( u \) on \( X \) over the enlarged probability space \((\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mu}, (\hat{\Theta}(t))_{t \in I^+})\) by applying Theorem 3.3.

**Theorem 3.9** Assume a periodic Markovian semi-flow \( u : \Delta \times \Omega \times X \to X \) has a periodic measure \((\rho_s)_{s \in I}\) on \((X, \mathcal{B}_X)\) in the sense of (3.10). Then the semi-flow \( u \) has a periodic measure \( \mu_s \), \( s \in I \) on \((\Omega \times X, \mathcal{F} \otimes B_X)\) in the sense of (3.11) (or (3.10)) and \( E(\mu_s) = \rho_s \).

**Proof:** As \( u \) has a periodic measure \( \rho_s \) in the sense of (3.10), from the proof of Theorem 3.6, we know that the lifted cocycle \( \hat{\Phi} \) on \( \tilde{X} \) over the probability space \((\Omega, \mathcal{F}, P)\) has a periodic measure \( \tilde{\mu}_s \) defined by (3.22) satisfying (3.24). Then by using Theorem 2.12, there exists a periodic measure \( \tilde{\mu}_s, s \in I \), of \( \tilde{\Phi} \) on the product measure space \((\Omega \times \tilde{X}, \mathcal{F} \otimes B_{\tilde{X}})\) over the metric dynamical system \((\Omega, \mathcal{F}, P, (\theta_s)_{s \in I})\), i.e. for any \( s \in I, t \in I^+ \),

\[ (\tilde{\mu}_s)(\{s \mod \tau\} \times B) = (\tilde{\mu}_{s+\tau})(\{s \mod \tau\} \times B). \] (3.31)

Define for any \( B \in \mathcal{B}_X \),

\[ (\mu_s)(B) = (\tilde{\mu}_s)(\{s \mod \tau\} \times B). \] (3.32)

Then

\[ (\mu_{s+\tau})(B) = (\tilde{\mu}_{s+\tau})(\{s + \tau \mod \tau\} \times B) = (\tilde{\mu}_s)(\{s \mod \tau\} \times B) = (\mu_s)(B). \] (3.33)

Moreover, for any \( C \in \mathcal{B}_{[0, \tau]} \) and \( B \in \mathcal{B}_X \), from the definition of \( \tilde{\Phi} \), for \( t \in I^+ \),

\[ (\tilde{\Phi}(t, \omega)(\tilde{\mu}_s)(\{r \mod \tau\} \times X : (t + r \mod \tau, u(t + r, r, \theta(-r)\omega)x) \in C \times B)). \]

Now we consider the case \( s \in [0, \tau) \). Take \( C = \{s \mod \tau\} \), then by Condition (3.2) again, we have

\[ (\tilde{\Phi}(t, \omega)(\tilde{\mu}_s)(\{s \mod \tau\} \times B) = (\tilde{\mu}_s)(\{s \mod \tau\} \times B) = (\mu_s)(u(t + s, \theta(-s)\omega)^{-1}B) = (\mu_s)(u(t + s, \theta(-s)\omega)^{-1}B) \]
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\[ = u(t + s, s, \theta(-s)\omega)(\mu_s)_{\omega}(B). \]

On the other hand,

\[ (\hat{\mu}_{t+s})_{\theta(t)\omega}((s + t \mod \tau) \times B) = (\mu_{t+s})_{\theta(t)\omega}(B). \]

Then it follows from (3.29) that for any \( s \in [0, \tau), t \in I^+ \),

\[ u(t + s, s, \theta(-s)\omega)(\mu_s)_{\omega} = (\mu_{t+s})_{\theta(t)\omega}. \tag{3.34} \]

For general \( s \in I \), there is a unique \( m \in \mathbb{Z}, s_0 \in [0, \tau) \) such that \( s = m\tau + s_0 \). From (3.34) we have

\[ u(t + s - m\tau, s - m\tau, \theta(-s + m\tau)\omega)(\mu_{s-m\tau})_{\omega} = (\mu_{t+s-m\tau})_{\theta(t)\omega}. \]

It then follows from (3.2) and (3.33) that

\[ u(t + s - m\tau, s - m\tau, \theta(-s + m\tau)\omega)(\mu_{s-m\tau})_{\omega} = u(t + s, s, \theta(-s)\omega)(\mu_s)_{\omega} \]

and

\[ (\mu_{t+s-m\tau})_{\theta(t)\omega} = (\mu_{t+s})_{\theta(t)\omega}. \]

Thus we proved (3.34) holds for all \( s \in I \). So \((\mu_s)_{s \in I}\) is a periodic measure on \((\Omega \times X, \mathcal{F} \times \mathcal{B}_X)\) of the semi-flow \( u \) as claimed. Moreover, From (3.32) and Theorem 2.12 we have for any \( B \in \mathcal{B}_X \),

\[ E(\mu_s)_{\omega}(B) = E(\hat{\mu}_s)_{\omega}([s \mod \tau] \times B) = \hat{\rho}_s([s \mod \tau] \times B) = \delta_{s \mod \tau}([s \mod \tau])\rho_s(B) = \rho_s(B). \]

We can combine Theorem 3.8 and 3.9 to construct a random periodic solution \( \hat{Y} \) on the enlarged probability space \((\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mu}, (\hat{\theta}(t))_{t \in I^+})\), from the periodic measure \( \rho_s \) for the semi-flow as well. However, unlike in the case of cocycles considered in the last section, the law of \( \hat{Y} \) under the probability measure \( \hat{\mu} \) is, though still periodic, but not \( \rho_s \) any more. Note in the proof \( \hat{\mathcal{L}}(\hat{Y}(s)) = \rho_s \) in the cocycle case, the shift invariant plays a key role. This shift invariance can be explained as follows: Let \( \Phi \) be a cocycle considered in Section 2 and define

\[ u(t, s, \omega) = \Phi(t - s, \theta(s)\omega), \quad t \geq s. \]

Then for any \( r \geq t \geq s \),

\[ u(r, t, \omega) \circ u(t, s, \omega) = \Phi(r - t, \theta(t)\omega) \circ \Phi(t - s, \theta(s)\omega) = \Phi(r - s, \theta(s)\omega) = u(r, s, \omega). \]

But for any \( r \in R, t \geq s \),

\[ u(t + r, s + r, \omega) = \Phi((t + r) - (s + r), \theta(s + r)\omega) = \Phi(t - s, \theta(s)\theta(r)\omega). \]
Thus,

$$u(t + r, s + r, \omega) = u(t, s, \theta(r)\omega). \quad (3.35)$$

The last identity \((3.35)\) is the shift invariance for cocycles. However, \((3.35)\) is not true for general semi-flows. It is noted that for periodic semi-flow satisfying \((3.1)\) and \((3.2)\), \((3.35)\) is true for \(r = \tau\). But this is not enough to prove the result that the law of \(\hat{Y}\) is \(\rho_s\), which is a key step to prove the law of large numbers for periodic measures. Similarly as in the cocycle case, we have the following SLLN for lifted semi-flows:

**Theorem 3.10** Assume \(\rho_s\) is a periodic measure with respect to the Markovian semigroup \(P(t + s, s, \cdot)\) of the periodic Markovian semi-flow \(u(t + s, s, \omega)\) on the space \(X\) over a metric dynamical system \((\Omega, \mathcal{F}, P, (\theta(t))_{t \in I})\) in the sense of \((3.10)\). Then \(P(t + s, s, \cdot)\) can be lifted to a Markovian semigroup \(\tilde{P}(t, \tilde{x}, \cdot)\) of the Markovian cocycle \(\hat{\Phi}(t, \cdot)\) on the cylinder \(\tilde{X} = I_\tau \times X\). \(\rho_s\) can be lifted to a \(\tau\)-periodic measure \(\tilde{\rho}_s\) of the semigroup \(\tilde{P}(t, \tilde{x}, \cdot)\). Furthermore, we can construct an enlarged probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mu})\) and measure preserving map \(\tilde{\Theta}(t) : \tilde{\Omega} \to \tilde{\Omega}\) and extended random dynamical system \(\tilde{\Phi}\) in the same way as in Theorem 2.13, but \(X\) replaced by \(\tilde{X}\). Then \(\tilde{\Phi}\) has a random periodic solution given by: for any \(\tilde{\omega}^* = (s, \omega^*, \tilde{x}^* (\omega^*)) \in \tilde{\Omega}, \)

$$\tilde{Y}(t, \tilde{\omega}^*) := \Phi(t + s, \theta(-s)\omega^*) \tilde{x}^* (\theta(-s)\omega^*), \quad t \in I^+.$$ 

Assume further that the transition semigroup \(P(t + s, s, \cdot)\) and the periodic measure \(\rho_s\) satisfy the Condition (ECS) uniformly w.r.t. subsets of \(F_0\) with positive \(\nu\) measure. Then we have the SLLN, i.e. for any \(\tilde{B} \in \mathcal{B}(\tilde{X}), \)

$$\frac{1}{\nu([0,T] \cap G_\infty)} \int_{[0,T] \cap G_\infty} I_{\tilde{B}}(\tilde{Y}(s, \tilde{\omega}))d\nu(s) \to \frac{1}{\nu(F_0)} \int_{F_0} \tilde{\rho}_s(\tilde{B})d\nu(s) \quad \text{a.s.}$$

In particular, if Condition (ECS) holds for \(F_0 = [0, \tau]\), then as \(T \to \infty, \)

$$\frac{1}{\nu([0,T] \cap I)} \int_{[0,T] \cap I} I_{\tilde{B}}(\tilde{Y}(s, \tilde{\omega}))d\nu(s) \to \tilde{\rho}(\tilde{B}) \quad \text{a.s.}$$

**Proof:** In order to prove a law of large number theorem for a periodic measure, we consider lifted cylinder space \(\tilde{X}\). Consider the lifted measure and transition semigroup defined by

$$\tilde{\rho}_s(C \times B) = \delta_{(s \mod \tau)}(C)\rho_s(B),$$

and

$$\tilde{P}(t, (s, x), C \times B) = \delta_{(t+s \mod \tau)}(C)P(t + s, s, x, B),$$

for any \(C \in \mathcal{B}_I\) and \(B \in \mathcal{B}_X\). It is easy to see that they satisfy

$$\tilde{P}^s(t)\tilde{\rho}_s = \tilde{\rho}_{s+t}, \quad \text{and} \quad \tilde{\rho}_{s+\tau} = \tilde{\rho}_s, \quad s \in I, \ t \in I^+. $$

Moreover, for any \(k \geq 1, \ \tilde{y} = (s, x), \) if \(\nu(C \cap F_0) > 0, \)
Now we can use the idea of Theorem 2.12 to consider an enlarged probability space (construct a periodic measure and by (3.4), i.e. Moreover, for any \( s \in I, t \in I^+ \),

\[
\int_X \left| \frac{1}{\nu(F_0)} \int_{F_k} \left( \hat{P}(t, \tilde{y}, C \times B) - \hat{\rho}_t(C \times B) \right) d\nu(t) \right| \hat{\rho}_0(dy) \leq \delta_1 e^{-\delta_2 k},
\]

where \( C_k \) is an identical copy of \( C \) in \([k\tau, (k+1)\tau)\) defined by the same way as \( F_k \). When \( \nu(C \cap F_0) = 0 \), the above is automatically true. Thus we have

\[
\int_X \left| \frac{1}{\nu(F_0)} \int_{F_k} \left( \hat{P}(t, \tilde{y}, C \times B) - \hat{\rho}_t(C \times B) \right) d\nu(t) \right| \hat{\rho}_0(dy) \leq \delta_1 e^{-\delta_2 k}.
\] (3.36)

That is to say the lifted semigroup \( \tilde{P} \) and periodic measure \( \hat{\rho} \) satisfy Condition (EC). Let \( \tilde{\Phi} \) defined by (3.4), i.e.

\[
\tilde{\Phi}(t, \omega)(s, x) = (t + s \mod \tau, u(t + s, s, \theta(-s)\omega)x), \text{ for any } (s, x) \in \tilde{X}, \ t \in I^+.
\]

Then \( \tilde{P} \) is the transition semigroup of \( \tilde{\Phi} \). Now we follow the procedure given in Theorem 2.13 to construct a periodic measure \( \mu_s \) on the product measurable space \((\Omega \times \tilde{X}, F \otimes B_{\tilde{X}})\) with the following decomposition

\[
\tilde{\mu}_s(\tilde{d}x, d\omega) = (\hat{\mu}_s)_\omega(\tilde{d}x) \times \tilde{P}(d\omega).
\]

and

\[
E(\hat{\mu}_s) = \tilde{\rho}_s, \ s \in I.
\]

Moreover, for any \( s \in I, t \in I^+ \),

\[
\tilde{\Phi}(t, \omega)(\tilde{\mu}_s)_\omega = (\tilde{\mu}_{t+s}\theta(t)\omega), \ (\tilde{\mu}_{s+\tau})_\omega = (\tilde{\mu}_s)_\omega.
\] (3.37)

Now we can use the idea of Theorem 2.12 to consider an enlarged probability space \((\hat{\Omega}, \hat{F}, \hat{\mu})\), where \( \hat{\Omega} = I_r \times \Omega \times \tilde{X}, \hat{F} = B_{I_r} \otimes F \times B_{\tilde{X}}, \) and

\[
\hat{\mu}(I_r \times \tilde{A}) = \frac{1}{\nu(I_r)} \int_{I_r} \tilde{\mu}_s(\tilde{A}) d\nu(s), \ \hat{\mu}(\emptyset \times \tilde{A}) = 0, \text{ for any } \tilde{A} \in F \otimes B_{\tilde{X}}.
\]

Define the skew product \( \hat{\Theta} : I^+ \times \hat{\Omega} \to \hat{\Omega} \) as follows: for any \( \hat{\omega} = (s, \omega, \tilde{x}) \in \hat{\Omega}.

\[
\hat{\Theta}(t)\hat{\omega} = (s + t \mod \tau, \theta(t)\omega, \tilde{\Phi}(t, \omega)\tilde{x}), \ t \in I^+.
\]

Now define \( \hat{\Phi} : I^+ \times \hat{\Omega} \times \tilde{X} \to \tilde{X} \) by: for any \( (r, x) \in \tilde{X},

\[
\hat{\Phi}(t, \hat{\omega})(r, x) = \hat{\Phi}(t, \omega)(r, x) = (t + r \mod \tau, u(t + r, r, \theta(-r)\omega)x).
\]
Finally, we need to check the law of \( \hat{\omega}^* = (s, \omega^*, \bar{x}^*(\omega^*)) \in \hat{\Omega} \),

\[
\hat{Y}(t, \hat{\omega}^*) := \hat{\Phi}(t + s, \theta(-s)\omega^*)\bar{x}^*(\theta(-s)\omega^*), \quad t \in I^+.
\]

So first for \( \hat{\omega}^* = (s, \omega, \bar{x}) \), \( \bar{x} = (r, x) \),

\[
\hat{Y}(t, \hat{\omega}) = \hat{\Phi}(t + s, \theta(-s)\omega)(r, x) = (t + s + r \mod \tau, u(t + s + r, r, \theta(-(s + r))\omega)x).
\]

We can check this is a random periodic solution of the random dynamical system \( \hat{\Phi} \) over the metric dynamical system \( (\tilde{\Omega}, \tilde{F}, \tilde{\mu}, (\tilde{\Theta}(t))_{t \in I^+}) \). To see this, for any \( t_1, t \in I^+ \),

\[
\begin{align*}
\hat{\Phi}(t_1, \tilde{\Theta}(t)\hat{\omega})\hat{Y}(t, \hat{\omega}) &= \hat{\Phi}(t_1, \theta(t)\omega)(t + s + r \mod \tau, u(t + s + r, r, \theta(-(s + r))\omega)x) \\
&= (t_1 + t + s + r \mod \tau, u(t_1 + t + s + r, t + s + r, \theta(-(s + r))\omega) \\
&\quad \circ u(t + s + r, r, \theta(-(s + r))\omega)x) \\
&= (t_1 + t + s + r \mod \tau, u(t_1 + t + s + r, r, \theta(-(s + r))\omega)x) \\
&= \hat{Y}(t_1 + t, \hat{\omega}).
\end{align*}
\]

Moreover,

\[
\begin{align*}
\hat{Y}(t + \tau, \hat{\omega}) &= (\tau + t + s + r \mod \tau, u(\tau + t + s + r, r, \theta(-(s + r))\omega)x) \\
&= (t + s + r \mod \tau, u(\tau + t + s + r, r, \theta(-(s + r))\omega)x).
\end{align*}
\]

Note also that

\[
\begin{align*}
\hat{\Theta}(\tau)\hat{\omega} &= (s \mod \tau, \theta(\tau)\omega, \hat{\Phi}(\tau, \omega)(r, x)) \\
&= (s \mod \tau, \theta(\tau)\omega, (r \mod \tau, u(\tau + r, r, \theta(-r)\omega)x)).
\end{align*}
\]

So

\[
\begin{align*}
\hat{Y}(t, \hat{\Theta}(\tau)\hat{\omega}) &= \left(t + s + r \mod \tau, u(t + (s + r \mod \tau), r \mod \tau, \theta(-(s + r \mod \tau))\theta(\tau)\omega) \\
&\quad \circ u(\tau + r, r, \theta(-(r + s))\omega)x\right) \\
&= \left(t + s + r \mod \tau, u(t + s + r, r, \theta(-(s + r))\theta(\tau)\omega) \\
&\quad \circ u(\tau + r, r, \theta(-(r + s))\omega)x\right) \\
&= \left(t + s + r \mod \tau, u(\tau + t + s + r, \tau + r, \theta(-(s + r))\omega) \\
&\quad \circ u(\tau + r, r, \theta(-(r + s))\omega)x\right) \\
&= \left(t + s + r \mod \tau, u(\tau + t + s + r, r, \theta(-(s + r))\omega)x\right) \\
&= \hat{Y}(t + \tau, \hat{\omega}).
\end{align*}
\]

Finally, we need to check the law of \( \hat{Y} \). For any \( C \times B \in \mathcal{B}_X \),
\[ \hat{\mu}(\{\hat{\omega}: \hat{Y}(t, \hat{\omega}) \in C \times B\}) \]
\[ = \hat{E}[IC_{XB}(\hat{Y}(t, \hat{\omega}))] \]
\[ = \frac{1}{\nu(I_{\tau})} \int_{I_{\tau}} \int_{\Omega} \int_{X} IC_{XB}(t + s + r \mod \tau, u(t + s + r, r, \theta(-(s + r))\omega)x) \]
\[ (\hat{\mu}_{s})_{\omega}(dr, dx)P(d\omega)ds \]
\[ = \frac{1}{\nu(I_{\tau})} \int_{I_{\tau}} \int_{\Omega} \int_{X} IC_{XB}(r, x)\hat{\Phi}(t + s, \theta(-s)\omega)(\hat{\mu}_{s})_{\theta(-s)\omega}(dr, dx)P(d\omega)ds \]
\[ = \frac{1}{\nu(I_{\tau})} \int_{I_{\tau}} \int_{\Omega} \int_{X} IC_{XB}(r, x)(\hat{\mu}_{t})_{\theta(t+s)\omega}(dr, dx)P(d\omega)ds \]
\[ = \int_{I_{\tau}} \int_{\Omega} \int_{X} IC_{XB}(r, x)(\hat{\mu}_{t})_{\omega}(dr, dx)P(d\omega) \]
\[ = \hat{\rho}(C \times B). \]

Finally, as \( \hat{P}(t, x, \cdot) \) and \( \hat{\rho} \) satisfy condition (EC), we have the SLLN by Theorem 2.18. 

\[ \]
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