Stochastic and Deterministic Framework for Modeling the Effect of Antennas Location Error on DOA Estimation

EMMANUEL UFITEYEZU AND YUN LI, (Member, IEEE)
Chongqing Key Laboratory of Mobile Communications Technology, Chongqing University of Posts and Telecommunications, Chongqing 400065, China
Corresponding author: Emmanuel Ufiteyezu (ufiteyezue@yahoo.fr)

This work was supported by the Chongqing Key Lab of Mobile Communications Technology, Chongqing University of Posts and Telecommunications.

ABSTRACT This article focuses on the estimation of the direction of Arrival based on the interpolation approaches of the sparse array. Such as the minimum redundancy array, co-prime array, and the nested array have attracted much attention, especially for a non-linear array antenna. We analyze the effect of antennas locations errors on overall estimation performance for deterministic and stochastic error models and to improve the estimator performance in the case of uncorrelated sources of signals in spatially-smoothed MUSIC scenarios. In two scenarios: we determine the Cramér-Rao bound in the case of the antennas locations are perfect, and in other cases, we take into account antennas locations errors. For the first scenario, the Cramér-Rao bound converges as the SNR increase. For the second scenario, the Signal-to-noise ratio increases to infinity, to achieve the same Cramér-Rao bound as in the first scenarios. There is a significant gap between the Cramér-Rao bound values in a situation where antennas locations errors are considered, and when other locations errors are not considered. This disparity shows that unknown antenna locations errors have a significant impact on the estimated achievable performance of the direction of Arriva of scattered linear matrices.

INDEX TERMS Allocation errors, DoA estimation, difference co-array antenna, sparse array error allocation.

I. INTRODUCTION

In recent decades, the problem of estimating the direction of the arrival (DOA) of signals using antenna array plays a vital role in the domain of array processing that usually focuses on the uniform linear array (ULA) system model and no uniform linear array (NULA) configuration. It is known that a traditional high-resolution DOA estimation algorithm like MUltiple SIgnal Classification (MUSIC) or Estimation of Signal Parameter via Rotational Invariance Techniques (ESPRIT), can resolve \( N \) sources of signals with a uniform linear array of \( N - 1 \) antennas; hence it has \( M \) degree of freedoms. It requires a significant number of antennas to resolve more resources of signals which can be very expensive in terms of investment, reducing array costs by having fewer antennas are extremely desirable. In this growing evolution or 5G, one fundamental problem, we could solve in array processing techniques is to make an estimation of unknown, spatial, and temporal parameters of a signal by exploiting the available information from antenna arrays. Sparse arrays identify \( O(M^2) \) no correlated sources of signals by using \( M \) physical antennas. This technique is based on the different positions of physical antennas. A sparse array such as nested arrays [1], co-prime arrays [2], and Minimum redundancy array (MRA) [3].

However, many situations imply a system model in which the numbers of physical antennas are less than the number of unknown sources. To address these challenges, numerous techniques are based on creating the same effect with more virtual antenna elements than those actual physical antennas, and their different techniques of signal processing were presented in [1]–[4]. MRA has no standard expression. Also, MRA estimates the covariance matrix conforming to a wide range array of antennas deserves a challenging and tedious iterative operation. To overcome this trouble, a method known as nested arrays was proposed [2], within the nested array, we have no uniform arrays designed via integrating multiple ULAs in which each sub-array has a
distinct inter-antennas spacing, the nested linear array (NLA) and the co-prime linear array (CLA) proposed by Piya and P.P in [3], [5].

This paper focuses on linear array geometries, in which all antennas are positioned along a line. The spacing between the antennas may or may not be uniform. Fig.1a and Fig.1b illustrate the Uniform Linear Array (ULA), a non-uniform linear array, and sub-arrays. In order to respond, the enhanced Matrix method from the minimum redundancy array is extended to the nested array and co-prime array for direction-of-arrival estimation. Numerical examples of ways these new structures of no uniform are constructed.

The underlying array assumption is accurately calibrated, and the difference co-array model is developed. Our setting provides a performance baseline for various arrays. We will show that even mutual interference, nested arrays, can still estimate DOA in a reasonable amount of error.

The effect of different parameters on the Bit Error Rate (BER) of the reconfigurable MIMO scheme, including the DOA estimation error, angular propagation, and antenna beamwidth, was examined [6]. The results of the simulation showed that because of the DOA estimate error, the BER of the MIMO scheme rises [24], [26].

Moreover, it has been shown that the reduced angular propagation scheme works better than the bigger angular propagation scheme with a tiny DOA assessment error [8]. However, this connection is overturned for the significant DOA estimation error.

While in DOA estimation error, the BER of a system with a smaller beam width has better efficiency. For reconfigurable antenna communication schemes, these findings can be used to design a suitable angle-of-arrival estimation algorithm. This may lead to a degradation of the DAO estimate results compared to the instance where more virtual components were mapped to the real matrix of the virtual ULA. With reduced interpolation tolerance, the lateral lobes become more prominent due to the trade-off between the interpolation error location and the beam formation performance. Indeed, a sufficient precision of the transmission of the interpolation must be guaranteed in the sector of interest. When interpolation errors are smaller in the area of interest, the interpolation accuracy of the transmission networks is higher, but there are fewer degrees of freedom for sidelobe control.

Different works have focused on the development of analyzing the sensitivity of DOA estimators and the achievement of boundaries limits in the case of the antennas’ positions errors.

In [6], a hybrid Cramér Rao bound (CRB) linked to a calibrated sparse antenna array and sources of the signal location has been derived for two dimensions.

In [6], [25], they carried out subspace-based DOA and derived performance analysis of estimator in the case of antennas error locations. In [7], they analyzed the probability of resolution of the MUSIC approach while considering the error location model.

The authors recently assessed the performance of non – uniform linear array and uniform linear array (ULA) samplers in [8], in the case of a grid-based on the error model in the location of a radiated element based on Mean Square Error (MSE) and the CRB. These results could be used to develop algorithms for the sparse array-like co-prime arrays, nested arrays, and MRA (minimum redundancy arrays) model. However, one dimension perturbation along the antenna array is assumed, and their analysis concluded that the DOA lay on a predefined grid.

Larsson and Stoica [9], for instance, suggested a maximum probability estimator based on the parameterization of Cholesky and evaluated its asymptotic performance. Their model, however, is based on standardized linear arrays (ULA) and needs a system of sequential antennas error location.

In practice, zero error antenna location does not exist, so the sequential hypothesis may not be accurate. Recent advances in completing the Matrix [10], [11], [29] and
The main contributions of this research are listed below.

1. We built the analytical basis for the proposed signal processing model;
2. We have introduced a new set of spatial smoothing rules to maximize the enhanced levels of freedom;
3. We carried out the detection of the source number and the estimation of the DOA on the basis of the decomposition of the vector;
4. Based on the signal model constructed for nested vector antenna networks, we showed that the detected signals were increased to $O(M^2)$ with the only $M$ antennas.

We predicted benefit errors by exploiting a partial Toeplitz shape of the covariance matrix for non-uniform linear arrays with the antenna location perturbations, which include nested arrays and co-prime arrays. Then, rather than estimating phase errors, we directly estimated the DOA by solving a sparse total least-square problem.

We have statistically analyzed the performance of DOA estimators based on co-array. More specifically, we have considered two commonly co-array used Multiple Signal Classification (MUSIC) Estimators, namely MUSIC based on a Direct Augmentation (DA-MUSIC) and MUSIC based on a Spatial Smoothing (SS-MUSIC). We first deduced the asymptotic first-order statistics (i.e., there are many snapshots) of the DOA estimation error for the two MUSIC estimators. Based on their first-order statistics, we analytically derived an asymptotic simplified expression of Mean Square Error (MSE) for the two estimators by exploiting the co-array properties. We have shown that both estimators have the same asymptotic MSE. This observation implies that, with enough snapshots, we can replace SS-MUSIC by DA-MUSIC in order to reduce computational complexity without sacrificing accuracy. Besides, we analyzed the asymptotic behavior of the derived MSE in high SNR regions.

In section II, we introduce a signal model of differential co-arrays based on this signal model of a sparse antenna array error locations. In Section III, after theoretical and numerical analysis of deterministic and stochastic error models, we derive an asymptotic closed-form expression of SS-MUSIC MSE, and we analyze the performance of the proposed stochastic and we present simulation results. In Section IV, we present our conclusion and synthesize final remarks.

II. THE SIGNAL AND SYSTEM MODEL

In the estimation of direction of arrival (DOA) of narrow-band signals using the array of antennas, the two different types of the data model are currently used:

The deterministic model assumes that the signals are a nonrandom, and stochastic model, assuming that the signals are random.

These two models, a deterministic model, and a stochastic model lead to different maximum likelihood and different CRB on DOA accuracy [15], [16].

The DOA estimation approach could also be predicted to have different statistical properties under the two models.

The deterministic likelihood is statistically less efficient than stochastic probability [15], [16].

The method of direction estimation is asymptotically equivalent to stochastic likelihood.

The stochastic probability and direction estimation method allow the unconditional CRB, and the accuracy of the stochastic DOA model estimate is a lower limit of the
asymptotic statistical accuracy of any DOA estimate based on the covariance matrix of the data sample. It can not be reached [15], [16]. DOA stochastic accuracy and DOA deterministic accuracy decrease monotonically as the number of antennas or snapshots increases, and they increase monotonically as the number of sources increases [15], [16].

Two influential papers in the DOA context are the papers by Stoica [15] and Stoica and Nehorai, [16]. The specific CRB expressions are given in Eqs. (2.11) and (3.1) of [15] are valid only when $N < M$ (fewer sources than antennas). The analytical expressions are based on the inverse of the matrix $A^H A$ [15]. Nevertheless, for our work, the assumption $N < M$ is not fundamental to the existence of CRB of the DOA parameters because even when $N \geq M$, with prior information, the FIM can remain invertible under some conditions, as we shall prove it in this section II of signal and system model. So it is possible to get more useful expressions which do not involve $(A^H A)^{-1}$.

Analysis of the CRB presented in [31], where they considered only uniform linear array (ULA), our results showed that given a fixed number of antennas, nested arrays and co-prime significantly outperform ULAs. This finding theoretically confirmed the advantage of using co-prime and nested arrays when the number of antennas is a limiting factor. We showed that when the aperture is fixed, we demonstrated the trading of the number of spatial samples and the number of temporal samples. Our results will help in choosing between sparse linear arrays and ULAs in practical problems.

In [32], the authors derived a hybrid Cramér-Rao bound on calibration and source localization for general two-dimensional arrays in the presence of an antennas error location. The authors showed the condition when the CRB goes to zero as the SNR approaches infinity. Nevertheless, Our results when we consider one dimension array and the presence of antenna location error or not. There is a significant gap between the Cramér-Rao bound values in a situation where antennas error locations are considered, and when other locations errors are not considered.

In [8], [33], the authors conducted a thorough performance analysis of subspace-based DOA estimators in the presence of a model of antenna location errors. In [24], they analyzed the MUSIC algorithm resolution probability. The above analysis, however, is based on the physical array and the number of the source model, which is usually less than the number of antennas. Our analysis is based on a stochastic and deterministic error model, and it can be applied when the sources are less or more than the number of antennas. In [9], [11]–[13], the authors derived the CRB for arbitrary arrays in the one-source case, and numerically analyzed this CRB for several sparse linear arrays. Still, our paper considers more than one source of the signal.

It provides an illustrative example of the relationship between the physical array and the corresponding virtual ULA.

### TABLE 1. Notations.

| Notations | Definitions |
|-----------|-------------|
| $(\cdot)^T$ | The transpose of matrix, |
| $(\cdot)^*$ | Conjugate matrix, |
| $(\cdot)^H$ | Complex conjugate transpose |
| $(\cdot)^r$ | Moore-Penrose inversion |
| $\lceil \cdot \rceil$ | Integer ceiling |
| $E[\cdot]$ | Expectation (mean) |
| $\odot$ | Hadamard product |
| $\otimes$ | Kronecker product |
| $\odot$ | Khatri-Rao product |
| $\text{vec}()$ | vectorization |
| $\text{diag}()$ | Diagonal elements of a matrix |
| $a_{i,j}$ | $(i,j)$-th element of matrix $A$ |
| $a_i$ | The $i$-th column of matrix $A$ |
| $|\cdot|_{H}$ | The cardinality of matrix $A$ |
| $tr(A)$ | The trace of matrix $A$ |
| $\|A\|_F$ | The nuclear norm |
| $\text{Re}(A)$ | The Real part of matrix $A$ |
| $\text{Im}(A)$ | The Imaginary part of matrix $A$ |
| $\Pi = AA^*$ | A Projection into the range space of matrix $A$ |
| $\Pi_a = I - AA^*$ | projection matrix onto the null space of $A$ |
| $e_N^{i\theta}$ | The $i$-th natural base vector in $\mathbb{R}^N$ |
| $x(t)$ | the incident signal vector |
| $n(t)$ | the incident noise vector |
| $L$ | the snapshot number |
| $M$ | the number of physical antennas |
| $N$ | the number of signal sources |
| $M_v$ | the number of virtual antennas in co-array |
| $D$ | the antennas location “different co-array,” |
| $\hat{d}_{\theta,\delta}$ | The perturbed steering matrix |
| $A' \odot A$ | The steering matrix of the virtual array |
| $\hat{R}$ | The real covariance matrix |
| $\hat{\hat{R}}$ | The maximum likelihood of a covariance matrix. |

We assume the obvious statistical assumptions of the properties of the sources of signals and noise:

**Assumption 1**: The sources of signals are temporarily and based on the unconditional spatial uncorrelated signal model [17].

**Assumption 2**: The angles of the direction of the arrival (DOA) are distinct, means ($\theta_k \neq \theta_l \forall k \neq l$), and belongs to $(-\frac{\pi}{2}, \frac{\pi}{2})$.

**Assumption 3**: The white additive noise is temporally, spatially uncorrelated, and circularly symmetrical, while Gaussian random signals are uncorrelated from $N$ sources of signals.
According to assumptions 1-3, the auto-correlation of the matrix \( y(t) \) given by:

\[
R_{yy} = E[y(t)y^H(t) + \sigma_n^2 I] \tag{1}
\]

\( A(\theta) = [a(\theta_1), a(\theta_2), \ldots, a(\theta_N)] \) is the array steering matrix and \( P = diag(p_1, p_2, \ldots, p_N) \): the sources of the signals correlation vector. Vectorization \( R_{yy} \) from (1), we get a \((M^2 \times 1)\) vector:

\[
r = vec(R_{yy}) = (A^* \odot A)p + \sigma_n^2 vec(I) \tag{2}
\]

\( p = [p_1, p_2, \ldots, p_N]^T = [\sigma_1^2, \sigma_2^2, \ldots, \sigma_N^2] \tag{3}
\]

\[
Z_m = [e_1^T e_2^T \ldots e_M^T]^T \tag{4}
\]

The matrix \( A^* \odot A \) from an equation (2) is the manifold matrix of the co-array model corresponding to the different sources of signals at difference co-array, \( A \) which is the steering matrix. The \( p \) vectors are the power of the sources of signals, and the vector \( \sigma_n^2 vec(I) \) is an Additive white Gaussian noise (AWGN).

With \( e_i \) is a sparse vector with a majority are zeros elements except \( q_i \) at \( i \)-th position. Vectorizing \( R_{yy} \) in (3) behaves like a new longer received source signals with a new steering matrix \( A^* \odot A \).

We determine the vector \( r \) of which single measurement steering matrix embedded in the difference co-array of sparse antenna positions are determined by:

\[
D_{co} = \{(d_h - d_m, 0), d_h, d_m \in D\}. \tag{5}
\]

For sparse linear arrays, \( D_{co} \) accommodates a uniform linear array of \( 2M_{co} - 1 \) antennas, \( M_{co} \) which is a number of antennas for any sub-array, as in Fig. 1b; antennas with \( M_{co} > M \) [18], antennas centered at the origin.

By using spatial averaging techniques, we determine a new augmented vector \( z = Fr \) that has the same vector property as the ULA vector, where \( F \in \mathbb{R}^{2M_{co} \times M_{co}} \) is the co-array selection matrix, they can find a precise \( F \) definition in [18]. We apply a spatial smoothing technique, and we could formulate an augmented matrix covariance of \( R_{yy} \), its order is \( M_{co} \times M_{co} \), such that:

\[
R_{yy} = \frac{1}{M_{co}} \sum_{l=1}^{M_{co}} \Gamma_l z z^H \Gamma_l^H \tag{6}
\]

where \( \Gamma_l = [0_{M_{co} \times (l-1)} I_{M_{co} \times M_{co}} 0_{M_{co} \times (M_{co}-l-1)}] \) represent the selection matrix for the \( M_{co} \) sub-array of a co-array model. We use vector \( u = [u_1, u_2, \ldots, u_M]^T \) along X-axis to indicate antenna locations errors and vector \( v = [v_1, v_2, \ldots, v_M]^T \) along Y-axis, and the antennas with error locations are then determined by:

\[
\tilde{D} = \{(d_1 + u_1, v_1), (d_2 + u_2, v_2), \ldots, (d_M + u_M, v_M)\}
\]

If the antennas error location is large and not negligible, the basic structure of linear array will be wrecked, and this results in an incredibly difficult to characterize a large DOA estimation error. Our performance analyzes will, therefore, focus on cases where small antenna position errors occur. Additional hypotheses to 1-3 are added as follows:

**Assumption 4:** Let \( \delta = [u^T v^T]^T \) is a collection of antenna location errors parameter. Compared with the distance between two antennas, \( d \), antennas error locations are small. Under assumption 1-4 and the number of Snapshots \( L \), the output of the disturbing array antenna for the deterministic model may be expressed in as follow:

\[
\tilde{y}(t) = \tilde{A}(\theta, \delta(t))x(t) + n(t) \quad t = 1, 2, \ldots, L \tag{7}
\]

We examine in detail how these degradations relate to antenna errors location in the following sections. To avoid complications, we have analyzed antenna errors’ positions within the following sections; however, by the usage of the stochastic error version and its effects on the DOA performance estimation of SS-MUSIC.

We suggest the following additional presumption to avoid complications and to get an idea of the effect of the locations of stochastic antenna errors location:

**Assumption 5:** Antennas elements error locations \( \delta(t) \) are Independent and identically distributed random variables.

The source of signals \( x(t) \) and the additive white Gaussian noise \( n(t) \) are uncorrelated.

Since \( \tilde{A}(\delta(t)) \) is a nonlinear function of a random variable \( \delta(t) \), \( \tilde{y}(t) \) as a deterministic output function error model, and the complex circular symmetric Gaussian distribution no longer follows.

The application of the MUSIC spatial smoothing algorithm to \( R_{yy} \) be known as SS-MUSIC [19], more sources than the number of antennas, can be determined.

**A. THE STOCHASTIC MODELING OF ANTENNA ERRORS LOCATIONS**

We are considering the nested array with \( M \) antennas, including two concatenated Uniform Linear arrays (ULAs).

Assuming the inner ULA has \( M_1 \) antennas with \( d \)-spacing between successive antennas and the outer ULA has \( M_2 \) antennas with spacing \( (M + 1)d \). We consider that \( N \) narrowband signals are impinging on the array from directions \( \{\theta_1, \theta_2, \ldots, \theta_N\} \). Errors location vary in each snapshot, at \( l \)-snapshot is determined by:

\[
y(l) = A(\theta)x(l) + n(l) \nonumber \]

\[
A(\theta) = [a(\theta_1), a(\theta_2), \ldots, a(\theta_N)] = e^{2\pi j \lambda d \sin \theta} \nonumber \]

At which \( l \)-snapshot, the \( y(l) \) is the output signal, \( x(l) \) is the input signal source, \( n(l) \) is noise, \( A(\theta) \) is steering vector matrix, \( \tilde{a}(\theta) \) is steering vector according to antenna position, \( \lambda \) is a wavelength, \( (d, d) \) is \( i \)-th antenna position in the physical array.

\[
R_{yy} = E[y(y(l)H(l))] = A(\theta)R_{xx}A^H(\theta) + \sigma_n^2 I \tag{9}
\]

\[
R_{xx} = diag(\sigma_1^2, \ldots, \sigma_N^2) \tag{10}
\]

\[
R_{nn} = diag(q_1 \delta_1^2, \ldots, q_M \delta_M^2) \tag{11}
\]
where $R_{yy}$, $R_{xx}$, $R_{nu}$ are the covariance matrix for the output signal, the input signal, and the noise signal, respectively, $q_m\delta$ is the noise power of the $m$-th antenna.

From the matrix $A^* \odot A$, the repeated rows are eliminated and sorting the remaining rows, which are equivalent to remove the corresponding rows of $r$ and sorting the rest of them to get a new vector: $r_1 = A_1 p + \delta^2 z_1$.

Where $z_1$ is a vector, all elements are zero except an $q_n \in \{q_1, \ldots, q_M\}$ at the central position.

Reducing the computational burden, we determine a full-rank Hermitian Toeplitz matrix $T_1$ from $r_1$ to make a covariance matrix [17], [27]. If $M$ is even, the Toeplitz matrix is defined as:

$$T_1 = \begin{bmatrix} [r_1]_{\frac{M^2}{2}+\frac{M}{2}} & [r_1]_{\frac{M^2}{2}+\frac{M}{2}-1} & \cdots & [r_1]_1 \\ [r_1]_{\frac{M^2}{2}+\frac{M}{2}+1} & [r_1]_{\frac{M^2}{2}+\frac{M}{2}} & \cdots & [r_1]_2 \\ \vdots & \vdots & \ddots & \vdots \\ [r_1]_{\frac{M^2}{2}+\frac{M}{2}-1} & [r_1]_{\frac{M^2}{2}+\frac{M}{2}-2} & \cdots & [r_1]_{\frac{M^2}{2}+\frac{M}{2}} \end{bmatrix}$$

$$= \sigma^2 A(\theta_1) a^H(\theta_1) + \cdots + \sigma^2 A(\theta_M) a^H(\theta_M) + q_m\delta^2 I$$

$$= \Lambda R_{xx} A^H + q_m\delta^2 I$$

$$A(\theta) = [a(\theta_1), a(\theta_2), \ldots, a(\theta_M)]$$

$$a(\theta) = \left[e^{j2\pi \sin \theta d} \quad \cdots \quad e^{j2\pi \sin \theta M\frac{N}{2} + (N-1)ld}\right]^T$$

Since all sources of signal directions are totally different. $A$, $I$ are a full column rank Vandermonde matrix and identity matrix, respectively. From (12), $T_1$ can play the same role as the received sources covariance matrix of the \(\left(\begin{array}{c} N^2 \\ N + 1 \end{array}\right)\) elements ULA.

**B. PROPOSED ALGORITHM**

We estimate from a deterministic number of L snapshots as:

$$R_{yy} = \frac{1}{L} \sum_{l=1}^{L} y(l)y^H(l)$$

where $L$ is the number of snapshots. The Toeplitz matrix is $T_1 = \Lambda R_{xx} A^H + E$. Here $R_{xx}$ is the source of the signals component, which is not zero when we consider the small size of the sample. $E$ could not be diagonal in the case of sample size, as it has a noise component that is not zero. In this paper, we utilize the technique of full information in the case of a small sample size. We consider all values of different covariance to form some Toeplitz matrices.

The range of specific factors in the distinction co-array formed by using the $i$-th antenna in the nested array is denoted via the integer value function. When $M$ is even, the distinctive co-array is particular most effective if the element belongs to the set of:

$$D_1 = \{d_i - d_j, M_1 + 1 \leq i \leq M_2, 1 \leq j \leq M_1\}.$$  

**Proposition 1:** The following properties hold for the positive integer elements of $\eta(i)$:

1. $\eta(i) = 1$ for $i = M_1 + 1$
2. $\eta(i) = M_1$ for $M_1 + 1 < i < M_1 + M_2$
3. $\eta(i) = M_1 + 1$ for $i = M_1 + M_2$
4. $\eta(i) = 0$ for $1 \leq i \leq M_1$

$\eta(i)$ is a function that represents the number of unique elements formed by the $i$-th antennas in the nested array.

**Property 1:** Can be easily verified as follows: $i = M_1 + 1$ represents the first antenna position in the outer ULA of the two-level nested array. The value of $\eta(M_1 + 1)$ is one at $d_{M_1+1} - d_1$.

**Property 2:** Can be verified as follows: $M_1 + 1 < i < M_1 + M_2$ represents the positions from the second to the penultimate antennas in the outer ULA of the two-level nested array. Let $d_{M_1+i}$ be an antenna position in this interval, then $M_1$ corresponding unique elements are $\{d_{M_1+i} - d_j, 1 \leq j \leq M_1\}$.

**Property 3:** Can be verified as follows: $i = M_1 + M_2$ represents the last antenna position in the outer ULA of the two-level nested array. Its corresponding $M_1 + 1$ unique elements in the difference co-array are $\{d_{M_1+i} - d_j, 1 \leq j \leq M_1 + 1\}$.

**Property 4:** Means that for every antenna position in the inner ULA of the two-level nested array, there will be no unique elements in the difference co-array.

The corresponding negative integer elements set, -$D_1$, has the same results. In conclusion, for a two-level nested array, the number of unique elements in the difference co-array is $M_1 - M + 4$. Then the number of elements that appear more than once is $2M - 5$.

We obtain the pseudo output power spectrum by using the result in [20], for nested arrays as:

$$P(\theta) = \frac{1}{\frac{M^2}{2} + \frac{M}{2}} \times \max \left| G^H(\theta)F^{-1}(\theta) \right|$$

$$F = \sum_{i=1}^{2M-5} T_i^H T_i \in \mathbb{C}^{(\frac{M^2}{2} + \frac{M}{2}) \times (\frac{M^2}{2} + \frac{M}{2})}$$

$$G = \left[T_1^H a(\theta) \ldots T_{2M-5}^H a(\theta)\right] \in \mathbb{C}^{(\frac{M^2}{2} + \frac{M}{2}) \times (2M-5)}$$

Therefore, on the presumption of a limited number of $L$ snapshots, the stochastic error model distribution is slightly hard to derive.

As an alternative, as deduced [18], the antenna error location impact prevails only when there is a sufficiently large number of snapshots. Therefore, we are able to analyze the effect of antenna location errors on overall estimation performance for the stochastic error model while an infinite number of snapshots are available.

Under the hypothesis assumption 1-4, the disturbed covariance matrix can be determined as follow:

$$\tilde{R}_{yy} = E \left[y(l)y^H(l)\right]$$

$$= E \left[\tilde{A}(\delta(l))x(l)\tilde{A}^H(\delta(l))\right] + E \left[\tilde{A}(\delta(l))x(l)n^H(l)\right]$$
As the sources of signals and AWGN has zero mean and are decorated when the terms crossed are gone.

The term $\delta$ is determined as:

$$S = \sum_{i=1}^{N} \sum_{l=1}^{N} E \left[ \hat{a}(\theta_i, \delta(l)) x_i(l) x_i^*(l) \hat{a}^H(\theta_i, \delta(l)) \right]$$

(18)

Whose element $(n, m)$-th is indicated by:

$$S_{nm} = \sum_{i=1}^{N} \sum_{l=1}^{N} E \left[ \hat{a}(\theta_i, \delta(l)) \hat{a}^H(\theta_i, \delta(l)) x_i(l) x_i^*(l) \right]$$

(19)

We can decouple the expectations assessments to $\delta(l)$ and $x(l)$ using assumption 5, and we determine the autocorrelation as follow:

$$E\{x_i(l) x_i^*(l)\} = \delta$$

If $i = l$ or otherwise is zero. We have to consider the terms only where $i = l$. We can write (19) as:

$$S_{nl} = \sum_{n=1}^{N} \sum_{l=1}^{N} p_i E \left[ \hat{a}_m(\theta_n, \delta(l)) \hat{a}_l(\theta_n, \delta(l)) \right]$$

$$= \sum_{n=1}^{N} \sum_{l=1}^{N} p_i a_m(\theta_n) a^*_l(\theta_n) E \{ e^{j(t_{n, l} - t_{n, m})} \}$$

$$= \sum_{n=1}^{N} \sum_{l=1}^{N} p_i a_m(\theta_n) a^*_l(\theta_n) \phi_\delta(t_{n, l} - t_{n, m})$$

(20)

where $\phi_\delta(t)$ is the characteristic function of $\delta(t)$ and

$$t_{n, l} = \frac{2\pi}{\lambda} \left[ e^l \sin \theta_n \right]$$

(21)

$e^l$ is a vector of $M$-dimension with only the $n$-th element being one and the other element being zero.

Let $y(l) = A(\theta) x(l) + n(l)$ be a $M \times M$ dimensioned matrix whose $(n, m)$-th element is given by $\phi_\delta(t_{n, l} - t_{n, m})$. We could then express $\mathbf{R}$ as:

$$\mathbf{R} = \sum_{k=1}^{N} p_n [a(\theta_n) a^H(\theta_n)] \phi_n + \sigma_n^2 \mathbf{I}$$

$$\delta_l \sim \mathbb{N}(0, \mathbf{C})$$

$$\mathbf{R} = \mathbf{A}(\theta, \delta) \mathbf{P}^H(\theta, \delta)$$

(22)

Here, the impact of the antennas error location is encoded in matrices $\mathbf{P}$. Since $t_{n, m}$ depending on the $n$-th DOA, the effect of antennas locations error depend on DOA and cannot be treated as Gaussian colored noise. Vectorizing (19) leads to

$$\mathbf{v} = [(A^* \otimes A) \phi] \mathbf{p} + \sigma_n^2 \text{vect} \mathbf{I}$$

$$\Phi = [\text{vect}(\phi_1), \text{vect}(\phi_2), \ldots, \text{vect}(\phi_N)]$$

(23)

Comparison between equations (23) and (2), we remark that the direction matrix $A^* \otimes A$ is modulated by $\Phi$ in case of a stochastic error model.

To have a good idea of (19) and (20), we consider the case when $\delta(l)$ are zero-mean Gaussian distribution and covariance matrix $C$. We decompose matrix $\mathbf{C}$ as:

$$\begin{bmatrix} c_{uu} & c_{uv} \\ c_{uv} & c_{vv} \end{bmatrix}$$

(61)

where $c_{uv}$ and $c_{uu}$ are the covariances of error locations alongside $x$-axis and $y$-axis, respectively, $c_{uv}$ and $c_{vv}$ are cross-covariance. Then the corresponding feature of the $\delta(l)$ is performed by $\phi_\delta(t) = \exp(-\frac{1}{2} t^2 \mathbf{C})$.

Substituting $t_{n, l}$ into $\phi_\delta(t)$ and we obtain this in the Gaussian case by extending the terms of the exponent.

$$\phi_\delta(m, l) = \exp \left\{ \frac{-2\pi^2}{\lambda^2} \left[ \mu_1(m, l) \sin^2(\theta_n) \right] + \mu_2(m, l) \cos^2(\theta_n) \right\}$$

where

$$\mu_1(m, l) = c_{ua}(l, l) + c_{uu}(m, m) - 2c_{ua}(l, m)$$

$$\mu_2(m, l) = c_{va}(l, l) + c_{vv}(m, m) - 2c_{va}(l, m)$$

$$\mu_3(m, l) = c_{uv}(l, l) + c_{uv}(m, m) - c_{uv}(l, m) - c_{uv}(l, m)$$

(24)

We also observe that $\Phi_\delta(m, l)$ it depends on the $n$-th DOA. Consequently, for a widespread covariance matrix, the impact of random antennas elements location errors nonetheless depends on DOA. Although, as shown below, the statement shows $\Phi_\delta(m, l)$ is independent of $n$ for some of these covariance matrices.

**Proposition 2:** Let $\delta_l \sim \mathbb{N}(0, \mathbf{C})$, additive white Gaussian with zero mean and covariance matrix $\mathbf{C}$, and then $\Phi_\delta(n = 1, 2, \ldots, N)$ arrays manifolds are not dependent on the DOAs if and only if $u_1(l, m) = u_2(l, m) = u_3(l, m) = 0$ holds for every $m, l = 1, 2, 3, \ldots, M$.

A notable case that meets the conditions specified in proposition 2 is when $\mathbf{C} = \sigma_n^2 \mathbf{I}$ this results in the following a corollary.

**Corollary 1:** Let $\delta_l \sim \mathbb{N}(0, \mathbf{C})$ then

$$\mathbf{R} = \mathbf{C}_1 \mathbf{A} \mathbf{P}^H \mathbf{A}^H + \frac{1}{\mathbf{C}_1} \left[ \sigma_n^2 + (1 - \mathbf{C}_1) \sum_{n=1}^{N} k_n \right]$$

(25)

where $\mathbf{C}_1 = \exp(-\frac{4\pi^2 \sigma_n^2}{\lambda^2})$.

The subspace signal does not change. Nevertheless, the effective SNR is reduced because of $0 < \mathbf{C}_1 < 1$. In this particular case, we can compare SS-MUSIC’s asymptotic MSE for $n$-th DOA, but replaced by the effective noise variance with the original noise variance $\sigma_n^2$:

$$\frac{1}{\mathbf{C}_1} \left[ \sigma_n^2 + (1 - \mathbf{C}_1) \sum_{k=1}^{N} k_n \right]$$
FIGURE 2. Spatially smoothing MUSIC process.

Spatial smoothing MUSIC (SS MUSIC) steps:

Step 1: Sample autocorrelation function on the difference co-array: \( \hat{X}_D \).

Step 2: ULA segment of \( \hat{X}_D \).

Step 3: Hermitian Toeplitz matrix \( \hat{R} \) (indefinite matrix).

Step 4: MUSIC on \( \hat{R} \) resolves uncorrelated sources.

Step 1 and 2 handling bits effects for normalized covariance matrix [33] and steps 3 and 4 s for spatially smoothing MUSIC process.

C. THE DETERMINISTIC MODELING OF ANTENNA ERROR LOCATION

In signal processing and system model for different techniques for DoA estimation, the co-array based MUSIC algorithm, like direct augmentation based MUSIC (DA-MUSIC) and SS-MUSIC, exist and express the same asymptotic MSE [18]. However, in this section, we compute a subspace estimate via SS-MUSIC and to improve estimator performance in the case of uncorrelated sources of signal scenarios, antennas locations errors, and the ability to employ SS-MUSIC to obtain DOA estimation. We will analyze, however, the effect of antenna location errors on SS-MUSIC’s DOA performance estimation.

We focus on the deterministic part of the model where we have a model that the data we want to fit is determined by the disturbed covariance matrix:

\[ \hat{R} = \hat{\Lambda}(\theta, \delta) \hat{P} \hat{A}^H(\theta, \delta). \]  (26)

where

\[ \hat{\Lambda} = \exp \left[ j \sum_{i=1}^{2\pi} (d_i \sin \theta_n + u_i \sin \theta_n + v_i \cos \theta_n) \right] \]

The difference co-array of model observation is determined by:

\[ \hat{r} = (\hat{\Lambda}^* \otimes \hat{\Lambda}) \mathbf{p} + \sigma^2 \mathbf{I} \]  (27)

where \((\hat{\Lambda}^* \otimes \hat{\Lambda})\) is a manifold matrix in a case of difference co-array errors, whose antennas error location are determined by:

\[ \hat{D}_{co} = \{ (d_m - d_n + u_m - u_n, v_m - v_n) \}, m, n = 1, 2, \ldots, M \].

The disturbed differences are no longer modified and divided a ULA into several overlapped sub-arrays of equivalent form.

As a result, we can apply the SS-MUSIC algorithm approach to the disrupted model of the co-array network without error compensation, and it can result in DOA estimates performance degradation. To maintain the relationship between the disturbance of co-array antennas and the direction of arrival estimation errors begins with a disturbing direction matrix \( \hat{\Lambda} \) since \( \hat{\Lambda} \) is analytical, and its value is close to \( \delta = 0 \). Through a first-order Taylor, we can linearize \( \hat{\Lambda} \), around \( \delta = 0 \) under an assumption 4:

\[ \hat{\Lambda} = \mathbf{A} + \mathbf{U} \hat{\Lambda}_u + \mathbf{V} \hat{\Lambda}_v + \mathbf{O}(\delta) \]  (28)

\[ \mathbf{U} = \text{diag}(u_1, u_2, \ldots, u_M) \]  (29)

\[ \hat{\Lambda}_u = j \frac{2\pi}{\lambda} \mathbf{A} \mathbf{D}_s \]  (30)

\[ \hat{\Lambda}_v = j \frac{2\pi}{\lambda} \mathbf{A} \mathbf{D}_c \]  (31)

\[ \mathbf{D}_s = \text{diag}(\sin \theta_1, \sin \theta_2, \ldots, \sin \theta_N) \]  (32)

\[ \mathbf{D}_c = \text{diag}(\cos \theta_1, \cos \theta_2, \ldots, \cos \theta_N) \]  (33)

With respect to \( \delta \), the higher-order term is denoted by \( \mathbf{O}(\delta) \), the disturbing matrix of covariance \( \hat{R}_{yy} \) can then be approximated as:

\[ \hat{R}_{yy} = \hat{R}_{yy} + \mathbf{U} \hat{\Lambda}_u \mathbf{P}_H + \mathbf{A} \hat{\Lambda}_v^H \mathbf{U} \]

\[ + \mathbf{V} \hat{\Lambda}_v \mathbf{P}_H + \mathbf{A} \hat{\Lambda}_v^H + \mathbf{O}(\delta) \]  (34)

The estimated \( \hat{R} \) with \( \hat{R} = \frac{1}{\ell} \sum_{i=1}^{\ell} \mathbf{y}(t) \mathbf{y}^H(t) \).

The difference between the variances \( \hat{R} \) and \( \mathbf{R} \), we break down this difference into two components:

\[ \Delta \mathbf{R} = \hat{\mathbf{R}} - \mathbf{R} = (\hat{\mathbf{R}} - \mathbf{R}) + (\mathbf{R} - \mathbf{R}) \]

W: indicates the estimation errors from deterministic snapshots. Z: Indicates the estimation resulting from antennas location errors.

In the presence of antennas error location, to obtain an asymptotic MSE of SS-MUSIC algorithm, we use the following Theorem [18]:

**Theorem 1:** \( \hat{\theta}^{SS} \) is the value estimation of the \( n \)-th DOA by SS-MUSIC, and \( \Delta \mathbf{R} = \mathbf{v}(\Delta \mathbf{R}) \).

In the central ULA, The virtual antennas in the co-array differences \( D_{co} \) with \( 2M_{co} - 1 \) antennas.

If \( \Delta \mathbf{R} \) is Hermitian and \( \Delta \mathbf{R} \) is very small and the errors of the DoA estimation of the SS-MUSIC are determined by:

\[ \Delta \theta_n := \hat{\theta}_n^{SS} - \theta_n = - (\gamma_n p_n)^{-1} \mathbf{y}_n^T \Delta \mathbf{R} + \mathbf{O}(\| \Delta \mathbf{R} \|) \]  (36)

where:

\[ \mathbf{y}_n = \mathbf{f}_n^T \mathbf{f}_n^T (\beta_n \otimes \alpha_n) \]

\[ \alpha_n = - e_n \mathbf{A}_{co}^T \]

\[ \beta_n = \prod_{\mathbf{A}_{co}} \hat{\mathbf{a}}_{co}(\theta_n) \]

\[ \gamma_n = \mathbf{A}_{co}(\theta_n)^T H \mathbf{A}_{co}(\theta_n) \]

\[ \hat{\mathbf{a}}_{co}(\theta_n) = \frac{\partial \mathbf{a}^2(\theta_n)}{\partial \theta_n} \]

**MCO:** Uniform linear array antennas have \( \mathbf{A}_{co} \) as a beam steering matrix of \( M_{co} \) antennas ULA, and antennas are located following these distances distributions given by \( \{0, d_0, \ldots, (M_{co} - 1)d_0\} \), where
\( \Gamma_i = [0_{M_i \times (i-1)} \, 1_{M_i \times M_0} \, 0_{M_i \times (M_0-i)}] \), and \( F \) have defined in [18], the co-array selection matrix and \( \Gamma_i \) is the selection matrix for the \( i \)-th sub-array. We combine (34) and theorem 1 to verify if the matrix \( \tilde{X} \) is Hermitian in case of antennas radiated element position errors and consider only the low order terms, we obtain:

\[
\Delta \theta_n \doteq -\langle \gamma_n \rho_n \rangle^{-1} \langle [\xi_n^T(z + w)] \rangle \quad (37)
\]

where \( \doteq \) denotes the first order of equality up, \( w = \text{vect}(W) \) and \( z = \text{vect}(Z) \). We now investigate the asymptotic MSE behavior in case of a large number of snapshots; the asymptotic MSE is given by:

\[
E[\Delta \theta_n] \doteq \frac{E\left[\frac{\langle [\xi_n^T(z + w)]^2 \rangle}{\rho_n^2} \right]}{\rho_n^2} \quad (38)
\]

By using this relation \( \langle (\mathbf{A}_n \mathbf{B}) \rangle = \langle \mathbf{A} \rangle \langle \mathbf{B} \rangle - \Delta \langle \mathbf{A} \rangle \langle \mathbf{B} \rangle \), we can develop a numerator equation from (38) as follows:

\[
\text{vect}(\mathbf{D}) \mathbf{X} = (\mathbf{X}^T \odot \mathbf{I}) \mathbf{d}
\]

\[
\text{vect}(\mathbf{AXB}) = (\mathbf{B}^T \odot \mathbf{A}) \text{diag}(\mathbf{X})
\]

\[
\mathbf{B}_u = \mathbf{I} \odot (\mathbf{A}^H \mathbf{A}_u^H) + (\mathbf{A}^H \mathbf{A}_u^H)^{-1} \odot \mathbf{I}, \quad \mathbf{B}_v = \mathbf{I} \odot (\mathbf{A}^H \mathbf{A}_v^H) + (\mathbf{A}^H \mathbf{A}_v^H)^{-1} \odot \mathbf{I}.
\]

\[
\{l_1, l_2, \ldots, l_{M_1}\} \subseteq \{1, 2, 3, \ldots, M\}
\]

\[
\{l_1, l_2, \ldots, l_{M_2}\} \subseteq \{1, 2, 3, \ldots, M\}
\]

(39)

Because \( E(\mathbf{w}) = 0 \)

\[
E \left[ \frac{\langle [\xi_n^T(z + w)]^2 \rangle}{\rho_n^2} \right] = \frac{\langle [\xi_n^H (\tilde{R} \odot \tilde{R}^T) \xi_n] \rangle}{L} + \langle [\xi_n^T(z + w)]^2 \rangle \quad (37)
\]

To determine the final expression of MSE, we expand \( z \) as a function of \( \delta \) as following:

Let a diagonal matrix \( \mathbf{D} = \text{vect}(\mathbf{d}) \) and:

\[
\text{vect}(\mathbf{DX}) = (\mathbf{X}^T \odot \mathbf{I}) \mathbf{d}
\]

\[
\text{vect}(\mathbf{XD}) = (\mathbf{I} \odot \mathbf{X}^T) \mathbf{d}
\]

For a proper matrix shape \( \mathbf{X} \). For the proof look in [22].

\[
\text{vect}(\mathbf{AXB}) = (\mathbf{B}^T \odot \mathbf{A}) \text{diag}(\mathbf{X})
\]

(40)

We can expand \( z \) as \( \mathbf{B} \delta \) + \( \mathbf{O}(\delta) \)

where \( \mathbf{B} = [\mathbf{B}_u \mathbf{B}_v] \) and

\[
\mathbf{B}_u = \mathbf{I} \odot (\mathbf{A}^H \mathbf{A}_u^H) + (\mathbf{A}^H \mathbf{A}_u^H)^{-1} \odot \mathbf{I}, \quad \mathbf{B}_v = \mathbf{I} \odot (\mathbf{A}^H \mathbf{A}_v^H) + (\mathbf{A}^H \mathbf{A}_v^H)^{-1} \odot \mathbf{I}.
\]

There are two terms in the asymptotic MSE (21). As a result, the covariance matrix’s estimation errors will disappear as \( L \) numbers of snapshots reach infinity for the first term.

We noted that the terms are also affected by the antennas error locations. After dividing by \( L \), the effect becomes negligible; however, increasing the number of \( L \) snapshots, as this number reaches infinity, the second term of an antenna error location will not disappear, resulting in a constant bias between estimates of DOA values.

**Corollary 1:** Provides the value of asymptotic MSE for specific errors in the locations of the radiated antennas, \( \delta \), we are also interested under different possible realizations of location errors of radiating antenna elements. We assume that the ideal hybrid CRB and location errors of radiating antenna elements \( \delta \) are Gaussian prior with mean equal zero and variance \( \mathbf{C} \) [1], and we assess an asymptotic MSE average. We assume the following Corollary 2.

**Corollary 2:** \( \delta \sim \mathbb{N}(\mathbf{0}, \mathbf{C}) \), and a small value \( ||\mathbf{C}|| \) is enough to make this high-order moment of \( \delta^2 \) to \( o(||\mathbf{C}||) \).

The average SS-MUSIC asymptotic MSE (AAMSE) in the presence of deterministic errors in the location of the antennas determined by:

\[
\frac{1}{p_n^2 ||\mathbf{C}||^2} \left( \frac{1}{L} \mathbb{E} \left[ \xi_n^H (\tilde{R} \odot \tilde{R}^T) \xi_n \right] + \mathbb{E}(\mathbf{B}^T \xi_n)^T \mathbb{E}[\mathbf{B}^T \xi_n] \right)
\]

(42)

For proof [18].

**Corollary 3:** We are assuming that power \( \mathbf{p} \) is shared by all sources. Let \( \varepsilon(\theta_n) \) indicates the average asymptotic MSE(AAMSE) of the \( n \)-th DOA. We assume in a Corollary \( \sigma_n^2 \) fixed we have:

\[
\lim_{n \to \infty} \varepsilon(\theta_n) = \frac{1}{2} \mathbb{E} \left[ \xi_n^H (\mathbf{A}^H \mathbf{A})^* \xi_n \right] + \mathbb{E}(\mathbf{B}^T \xi_n)^T \mathbb{E}[\mathbf{B}^T \xi_n]
\]

(43)

In an equation (42), we have two parts, the left part of the equal sign was a limiting expression of SS-MUSIC’s asymptotic MSE when antennas error locations are absent as the SNR tends to infinity that is non-zero in the presence of multiple sources.

The right part of the equation (42) represents the result of antennas error locations. Since the term is not dependent on source power, then we deduce that by increasing the SNR alone, the SS-MUSIC DOA estimation bias caused by the antennas error locations could not be attenuated.

**D. THE CRB METRIC BASED MUSIC**

Cramér-Rao Lower Bound (CRLB) gives the unbiased minimum variance objective estimator. The CRB provides a reduced limit on variances of parameter unbiased estimates (e.g., DOA). Closed-form expressions for the CRB provide insights into the array performance dependence to different parameters such as the number of antennas \( M \) in the array,
the array geometry, the number of sources \(N\), the number of snapshots \(L\), the signal-to-noise ratio (SNR), etc.

In this situation, we propose the Cramér–Rao lower bound for sparse uniform linear arrays in the deterministic error model, co-prime, and MRA. We take into consideration the antenna error locations as basic parameters that are unknown like DOA, additive noise power, and source power.

To achieve a general augmented expression of the Fisher Information Matrix (FIM), we suppose that the precise locations of the antennas are partially known.

The underlying assumption involves the case when the antennas error locations are unknown among all antennas.

Let \([i_1, i_2, \ldots, i_{M_l}] \subseteq [1, 2, 3, \ldots, M]\) and \([l_1, l_2, \ldots, l_{M_l}] \subseteq [1, 2, 3, \ldots, M]\) indicate different indices of antennas with unknown antennas’ error imperfections in the direction of the x-axis and the direction of the y-axis, respectively. The real vector collects the all unknown parameters \((2N + M_1 + M_2 + 1) \times 1:\)

\[
\eta = [\theta^T, p^T, u_{i_1}, \ldots, u_{i_{M_l}}, v_{l_1}, \ldots, v_{l_{M_l}}, \sigma_n^2] \tag{44}
\]

The Fisher Information Matrix of deterministic error model is derived by:

**Proposition 3**: Consider assumptions 1–5, FIM is determined by [7]:

\[
G = NM^H (\bar{R}^T \otimes \bar{R})^{-1} M \tag{45}
\]

Here

\[
\bar{r} = (\hat{\Lambda}^* \otimes \hat{\Lambda} + \Lambda^* \otimes \Lambda_0) \tag{46}
\]

where

\[
\frac{\partial \hat{r}}{\partial \theta} = (\hat{\Lambda}^\circ \otimes \hat{\Lambda} + \Lambda^* \otimes \Lambda_0) \tag{47}
\]

And

\[
L_1 = \left[ e_{M_l}^{(i_1)} e_{M_l}^{(i_2)} \ldots e_{M_l}^{(i_{M_l})} \right] \tag{48}
\]

\[
L_2 = \left[ e_{M}^{(l_1)} e_{M}^{(l_2)} \ldots e_{M}^{(l_{M_l})} \right] \tag{49}
\]

\[
\hat{\Lambda}_0 = \left[ \frac{\partial \hat{\Lambda}(\theta_1)}{\partial \theta_1} \frac{\partial \hat{\Lambda}(\theta_2)}{\partial \theta_2} \ldots \frac{\partial \hat{\Lambda}(\theta_n)}{\partial \theta_n} \right] \tag{50}
\]

If the Fisher Information Matrix is nonsingular and the CRB determination of DOAs could be quickly evaluated by the Fisher Information Matrix inversion. However, the CRB does not often exist because of the rank deficiency caused by antenna physical location errors. Even if we limit the disturbance only along the X-axis, the local ambiguity still happens because we can get the same direction by widening or reducing the full variety along the X-axis and adjusting the DOA accordingly.

Where there are such local ambiguities, the set of unknown parameters will be locally unidentifiable, leading to FIM’s singular information. We suppose that FIM is not unique in the next observation. Furthermore, using our bypass, we conclude that for sources up to \(O(M^2)\) FIM can stay non-unique. Because of FIM (34) in [16], it is easy to demonstrate that the respective CRB relies on SNR rather than the absolute values \(\sigma_n\).

In this situation, even if the SNR gets closer infinity, the CRB location of error-free remains positive. In the presence of antenna location errors, this egregious conduct still occurs.

### III. NUMERICAL RESULTS

In this section, we describe the numerical simulations results of deterministic and stochastic model showing how DOA performance estimation for sparse linear arrays is affected by antennas locations errors. We just consider the model of stochastic error. We use two sets of sparse linear arrays during the simulations for a complete and detailed comparison. The first set is made up of four different linear sparse arrays with the same number of antennas:

- Co-prime \(M_1 = 5; M_2 = 7\): \([0, 4, 5, 8, 10, 12, 15, 16, 20, 25, 30, 35]d\);
- MRA 12: \([0, 1, 4, 10, 16, 22, 28, 30, 33, 35, 38, 41]d\);
- Nested \(M_1 = 6, M_2 = 6\): \([1, 2, 3, 4, 5, 6, 7, 17, 21, 28, 35, 42]d\);
- MHA \(6, 6\): \([0, 1, 4, 10, 12, 17, 20, 22, 24, 27, 29, 32]d\).

The second set comprises of four distinctly different sparse linear arrays with the same aperture:

- Co-prime \(M_1 = 2; M_2 = 4\): \([0, 2, 3, 4, 6, 9] d\);
- MRA 6: \([0, 1, 2, 6, 9, 11, 13] d\);
- Nested \(M_1 = 3; M_2 = 3\): \([0, 1, 2, 3, 4, 8, 12] d\).
- Nested \(1, 5\): \([0, 1, 3, 5, 7, 9] d\).

Through all simulations, the SNR is defined as:

\[
SNR = 10 \log_{10} \min_{n=1,2,\ldots,N} \frac{P_n}{\sigma_n^2} \tag{49}
\]

We get the results after \(L\) trials, and we determine the empirical MSE with:

\[
MSE = E[x_i(l) \hat{x}_i^*(l)] = P_l \tag{50}
\]

where \(\theta_n^l\) is the estimate of \(\theta_n^l\).

In our results simulations, we consider two sets of sparse antenna arrays of 12 antennas, the first step, and we have 14 sources of signals for Fig.3a and Fig.3b, and the second set we have six sources of signals for Fig.4. We assume that the sources of signals are uniformly distributed between \((-\pi, \pi/2)\) with the same power, and the number of sources chosen for both simulations is greater than or equal to the
The method presented can resolve more sources than the array elements and distinguish uncorrelated and coherent narrowband sources that impinge simultaneously on the far-field of a uniform linear array (ULA) by using: a) SS-MUSIC, b) DA-MUSIC using c) MVDA.

The (CRB) \cite{17, 23} is also determined, and the RMSE is formulated as:

\[ RMSE = \sqrt{\frac{1}{N_s N} \sum_{i=1}^{N_s} \sum_{j=1}^{N_i} (\hat{\theta}_{i,j} - \theta_i) } \]  \hspace{1cm} (51)

We observe that although there is a small variety of snapshots, the empirical MSE starts from the analytical MSE. As the number of snapshots increases, the empirical MSE addresses the analytic approximation. Indeed, our analytic approximation is based entirely on the assumption of an unlimited number of snapshots.

Fig. 3.a., the assumed DOAs are detected effectively by using individual accurate peaks for every of assumed resources. The MUSIC spectrum depicted in Fig. 3.a. suggests that the DOA estimation algorithm has resolved the locations of the resources effectively in each array configurations. The similar level of peaks and valleys among them means the same resolution power for middle angles inside the spectrum. Similar research can be executed for the Minimum variance distortionless response (MVDR). The peaks generated within the MVDR spectrum, are not as sharp because the MUSIC spectrum, so the MVDR resolution may grow to be lower than MUSIC. Consequently, the angular area among angles that are resolved cannot be as close as MUSIC.

In fig. 3. b. This direct MVDR using the co-prime matrix has only twelve elements and may identify fourteen sources; the number of resources is higher than the number of antennas.

Also, in this section, with a numerical example, we show the prevalence of our proposed algorithm. We take into account the two nested degrees to be a spatially uncorrelated white Gaussian random method with a matrix of correlation:

\[ R_{nn} = \delta^2 \text{diag}(5, 1, 6, 0.3, 3, 0.1). \]

For our simulations, \( M_c \) is the number of Monte Carlo runs and is 600.

The sampling interval of zero 0.1° is uniform inside the spatial grid. Inside the first example, we consider \( N = 12 \) uncorrelated sources that impinge on the antennas array as follows: \( s = (-62°, -50°, -35°, -22°, -10°, -5°, 5°, 19°, 25°, 33°, 47°, 59°) \). The number of snapshots is set to be 5000, and SNR is 20dB.

Fig. 5 indicates that the mean square error for co-prime is higher than the other three arrays. This is because the co-prime array is the only one of the four arrays whose difference array does not correspond to a complete ULA.
As a result, the central ULA portion of the Matrix lease is the smallest of the four, resulting in a much higher MSE.

The central ULA part of the co-prime array is, therefore, the smallest of the four, resulting in a much better MSE.

In this simulation, we just consider the model of deterministic error. The closed-form expressions available in literature theory are not applicable in this context of antenna error locations; we propose and validate new expression (21) through simulations.

Also, we give numerical examples showing our proposed algorithm dominance. Consider a nesting array of two levels $M = \text{antennas}$ ($M_1 = 3$, $M_2 = 3$). Un correlated noise signals are spatially Gaussian random process with covariance matrix is $R_{nn} = \delta^2 \text{diag}\{5, 1, 1.6, 0.3, 2, 0.1\}$ and SNR is:

$$\text{SNR} = 10 \log_{10} \left( \frac{\sigma_1^2 + \sigma_2^2 + \ldots + \sigma_M^2}{tr(R_{nn})} \right) \quad (52)$$

The (CRB) [21] is also determined, and the RMSE is formulated as (51).

In our second simulation, where the number of Monte Carlo runs 650 trials, and the sampling interval of $0.1^\circ$ is uniform in the spatial grid. We consider the number of signals $N = 10$ to impinge on thees of signal from the direction of:

$s = (-62^\circ, -50^\circ, -36^\circ, -21^\circ, -9^\circ, 5^\circ, 18^\circ, 32^\circ, 46^\circ, 58^\circ)$

The snapshot number is $L = 500$, and SNR varies from $-20\text{dB}$ to $20\text{dB}$. The performance of our algorithm proposal for MUSIC versus SNR is compared in this simulation.

In fig. 6, we should observe that the MSE approaches the CRB in high SNR regions and that the stochastic CRB is tighter than the deterministic CRB.

With the additional assumption of uncorrelated sources, we expect a lower CRB. As SNR tends to infinity, all three CRBs converge to the same one.

Fig. 7 shows that the SS-MUSIC algorithm’s RMSE curve becomes relatively flat in high SNRs, as the base mismatch caused by the predefined spatial sampling grids limits the accuracy of the estimation. Similarly, for fixed searching interval values for the MUSIC algorithm for the spectrum search process for the interpolation-based co-prime virtual array algorithm and SS-MUSIC limit the accuracy of estimation, resulting in a flat RMSE curve when the SNR is greater than $10\text{dB}$.

The snapshot number is 550. Fig.8 displays RMSE vs. SNR performance. Our algorithm has the same performance as MUSIC in the event of no uniform noise. We find it interesting to be consistent with our theoretical results with the MSE simulation results. Spatial smoothing MUSIC MSE converges to stagnate as the SNR tends to be infinite in the presence of antenna error location, which is also consistent with our corollary analysis 3.

We remark that the MSE values in the presence of antennas locations errors; they do not decrease with SNR increase. In this simulation confirms that we cannot mitigate the problem of antennas errors’ locations by attempting to increase SNR alone. As illustrated in Fig.9 for different types of sparse antennas arrays have the same number of antennas. We determine RMSE vs. $\sigma_t^2 p$.

Among four sparse antennas’ arrays, considered in our experiments, the MRA attains the largest aperture, while the smallest aperture is in the co-prime array.

When all four arrays have similar calibrations, MRA is less sensitive than others to the antennas error locations. These results are in line with our analysis approach (21).
A compelling argument is that MRA 5, the nested array (1, 5), and nested array (4, 2) contribute almost the same central ULA parts in their differential arrays.

RMSE vs. Co-prime SNR (2, 3) at different levels of perturbation. About 1500 trials, the empirical results are averaged. At last, we show the covariance of antenna error locations, $\sigma^2$ affects considerably the SS-MUSIC MSE regions. We assume six sources placed between $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and set the snapshot number to 6000.

In fig.9 demonstrate how DOA estimate errors change in the presence of antennas error locations for different sparse arrays' antennas types.

IV. CONCLUSION

The sparse array antennas attracted considerable attention as they are able to resolve several identifiable sources of the signal exceeding the number of physical antennas. We studied the co-array signal model and derived the asymptotic MSE expression for a co-array-based MUSIC algorithm SS-MUSIC. In this situation, for the sparse array, the closed-form available in the literature does not apply. We consider the sparse arrays in the presence of antennas error locations and propose a new approach expression for SS-MUSIC’s CRB and asymptotic MSE to fill this gap, in this situation, the CRLB stagnates positively as the SNR tends to be infinite, and we observe that for sparse arrays and ULA with the same number of antennas. It is proved that the ULA antennas are more robust than sparse array antennas. We additionally derived the CRB for non-uniform linear arrays and analyzed the statistical performance of typical no uniform arrays. Our outcomes will gain future research on overall performance evaluation and the optimal design of non-uniform linear arrays.

REFERENCES

[1] A. Moffet, “Minimum-redundancy linear arrays,” IEEE Trans. Antennas Propag., vol. 16, no. 2, pp. 172–175, Mar. 1968.

[2] C.-L. Liu and P. P. Vaidyanathan, “High order super nested arrays,” in Proc. IEEE Sensor Array Multichannel Signal Process. Workshop, Jul. 2016, pp. 1–5.

[3] P. P. Vaidyanathan and P. Pal, “Sparse sensing with co-prime samplers and arrays,” IEEE Trans. Signal Process., vol. 59, no. 2, pp. 573–586, Feb. 2011.

[4] S. Qin, Y. D. Zhang, and M. G. Amin, “Generalized coprime array configurations for direction-of-arrival estimation,” IEEE Trans. Signal Process., vol. 63, no. 6, pp. 1377–1390, Mar. 2015.

[5] P. Pal and P. P. Vaidyanathan, “Nested arrays: A novel approach to array processing with enhanced degrees of freedom,” IEEE Trans. Signal Process., vol. 58, no. 8, pp. 4167–4181, Aug. 2010.

[6] A. L. Swindlehurst and T. Kailath, “A performance analysis of subspace-based methods in the presence of model errors. I. The MUSIC algorithm,” IEEE Trans. Signal Process., vol. 40, no. 7, pp. 1758–1774, Jul. 1992.

[7] M. Wang, Z. Zhang, and A. Nehorai, “Performance analysis of coarray-based MUSIC in the presence of sensor location errors,” IEEE Trans. Signal Process., vol. 66, no. 12, pp. 3074–3085, Jun. 2018.

[8] A. Koochakzadeh and P. Pal, “Performance of uniform and sparse non-uniform samplers in the presence of modeling errors: A Cramér-Rao bound based study,” IEEE Trans. Signal Process., vol. 65, no. 6, pp. 1607–1621, Mar. 2017.

[9] E. G. Larsson and P. Stoica, “High-resolution direction finding: The missing-data case,” IEEE Trans. Signal Process., vol. 49, no. 5, pp. 950–958, May 2001.

[10] E.J. Candès and B. Recht, “Exact low-rank matrix completion via convex optimization,” in Proc. 46th Ann. Allerton Conf. Commun., Control, Comput., Urbana-Champaign, IL, USA, Sep. 2008, pp. 806–812.

[11] E.J. Candès and Y. Plan, “Matrix completion with noise,” Proc. IEEE, vol. 98, no. 6, pp. 925–936, Jun. 2010.

[12] V. Chandrasekaran, B. Recht, P. A. Parrilo, and A. S. Willsky, “The convex geometry of linear inverse problems,” Found. Comput. Math., vol. 12, no. 6, pp. 805–849, 2012.

[13] G. Tang, B. N. Bhaskar, P. Shah, and B. Recht, “Compressed sensing off the grid,” IEEE Trans. Inf. Theory, vol. 59, no. 11, pp. 7465–7490, Nov. 2013.

[14] K. Han and A. Nehorai, “Wideband Gaussian source processing using a linear nested array,” IEEE Signal Process. Lett., vol. 20, no. 11, pp. 1110–1113, Nov. 2013.

[15] P. Stoica and A. Nehorai, “Performance study of conditional and unconditional direction-of-arrival estimation,” IEEE Trans. Acoust., Speech, Signal Process., vol. 38, no. 10, pp. 1783–1795, Oct. 1990.

[16] P. Stoica and A. Nehorai, “MUSIC, maximum likelihood, and Cramér-Rao bound,” in Proc. Int. Conf. Acoust., Speech, Signal Process. (ICASSP), New York, NY, USA, vol. 4, 1998, pp. 2296–2299.

[17] T.E. Bogale and L. B. Le, “Beamforming for multiuser massive MIMO systems: Digital versus hybrid analog-digital,” in Proc. IEEE Global Commun. Conf. (GLOBECOM), Dec. 2014, pp. 4066–4071.

[18] M. Wang and A. Nehorai, “Coarrays, MUSIC, and the Cramér-Rao bound,” IEEE Trans. Signal Process., vol. 65, no. 4, pp. 933–946, Nov. 2017.

[19] F. Ademaj, M. Taranetz, and M. Rupp, “Evaluating the spatial resolution of 2D antenna arrays for massive MIMO transmissions,” in Proc. Eur. Signal Process. Conf., Nov. 2016, pp. 1995–1999.

[20] D. Pearson, S. U. Pillai, and Y. Lee, “An algorithm for near-optimal placement of sensor elements,” IEEE Trans. Inf. Theory, vol. 36, no. 6, pp. 1280–1284, Nov. 1990.

[21] M. Wang, Z. Zhang, and A. Nehorai, “Direction finding using sparse linear arrays with missing data,” in Proc. IEEE Int. Conf. Acoust. Speech, Signal Process., Mar. 2017, pp. 3066–3069.

[22] G. A. F. Seber, A Matric Handbook for Statisticians. Hoboken, NJ, USA: Wiley, 2009.

[23] M. Ishiguro, “Minimum redundancy linear arrays for a large number of antennas,” Radio Sci., vol. 15, no. 6, pp. 1163–1170, 1980.

[24] X. Zhang, Y. Li, W. Wang, and W. Shen, “Ultra-wideband 8-port MIMO antenna array for 5G metal-frame smartphones,” IEEE Access, vol. 7, pp. 72273–72282, 2019.

[25] T. D. Zhao, and Z. Ye, “DOA estimation and self-calibration algorithm for a nonuniform linear array,” in Proc. Int. Symp. Intell. Signal Process. Commun. Syst., Chengdu, China, 2010, pp. 1–4.

[26] Z. Tan and A. Nehorai, “Sparse direction of arrival estimation using co-prime arrays with off-grid targets,” IEEE Signal Process Lett., vol. 21, no. 1, pp. 26–29, Jan. 2014.

[27] J. Ramirez, J. Odom, and J. Kroll, “Exploiting array motion for augmentation of co-prime arrays,” in Proc. IEEE 8th Int. Sensor Array Multichannel Signal Process., A Coruna, Spain, Jun. 2014, pp. 525–528.
EMMANUEL UFITEYEZU received the B.S. degree in electrical engineering from the University of Guelma, Guelma, Algeria, in 2009, and the M.S. degree in electrical engineering and systems of telecommunications from the University of Sciences and Technology, Houari Boumediene, Algiers, Algeria, in 2011. He is currently pursuing the Ph.D. degree in information and communication engineering with the Chongqing University of Posts and Telecommunications.

From 2012 to 2017, he was a Research Assistant with the College of Sciences and Technology, University of Rwanda, where he is currently an Assistant Lecturer with the Electrical and Electronics Department. His research interests include wireless networks, wireless sensors, millimeter waves, massive MIMO, mobile communications, and antennas.

YUN LI received the Ph.D. degree in communication engineering from the University of Electronic Science and Technology of China. He is currently a Professor of electrical engineering with the College of Communications, Chongqing University of Posts and Telecommunications, China. He has coauthored more than 150 journal/conference papers. His research interests include mobile cloud computing, cooperative/relay communications, green wireless communications, wireless ad hoc networks, sensor networks, and virtual wireless networks. He has also served as a TPC member of numerous conferences, including the IEEE GLOBECOM, the IEEE WCNC, WiCON, CNC2012, WOCC, IWCMC, and WiCOM. He served as the Co-Chair of the ChinaCom 2010 WCN Symposium and the IEEE RWS2011 DSPAW Symposium. He is on the Editorial Board of IEEE Access and Wiley Security and Communication Networks. He is the Executive Associate Editor of Digital Communications and Networks (DCN) (Elsevier/CQUPT).

* * *