The Deconfinement Phase Transition of
$Sp(2)$ and $Sp(3)$ Yang-Mills Theories
in 2 + 1 and 3 + 1 Dimensions

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Abstract

Some time ago, Svetitsky and Yaffe have argued that — if the deconfinement phase transition of a $(d+1)$-dimensional Yang-Mills theory with gauge group $G$ is second order — it should be in the universality class of a $d$-dimensional spin model symmetric under the center of $G$. For $d = 3$ these arguments have been confirmed numerically only in the $SU(2)$ case with center $\mathbb{Z}(2)$, simply because all $SU(N)$ Yang-Mills theories with $N \geq 3$ seem to have non-universal first order phase transitions. The symplectic groups $Sp(N)$ also have the center $\mathbb{Z}(2)$ and provide another extension of $SU(2) = Sp(1)$ to general $N$. Using lattice simulations, we find that the deconfinement phase transition of $Sp(2)$ Yang-Mills theory is first order in $3+1$ dimensions, while in $2+1$ dimensions stronger fluctuations induce a second order transition. In agreement with the Svetitsky-Yaffe conjecture, for $(2+1)$-d $Sp(2)$ Yang-Mills theory we find the universal critical behavior of the 2-d Ising model. For $Sp(3)$ Yang-Mills theory the transition is first order both in $2+1$ and in $3+1$ dimensions. This suggests that the size of the gauge group — and not the center symmetry — determines the order of the deconfinement phase transition.

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1 Introduction

The SU(N) Yang-Mills theory has a Z(N) center symmetry [1, 2] that is spontaneously broken at high temperatures. The corresponding order parameter is the Polyakov loop [3, 4] whose expectation value \( \langle \Phi \rangle = \exp(-\beta F) \) determines the free energy \( F \) of a static quark as a function of the inverse temperature \( \beta = 1/T \). In the low-temperature confined phase the center symmetry is unbroken, i.e. \( \langle \Phi \rangle = 0 \), and hence the free energy of a single static quark is infinite. In the high-temperature deconfined phase, on the other hand, the center symmetry is spontaneously broken, i.e. \( \langle \Phi \rangle \neq 0 \), and the free energy of a quark is finite. If the deconfinement phase transition is second order, its long-range physics is dominated by fluctuations in the Polyakov loop order parameter. In this case the details of the underlying dynamics become irrelevant and only the center symmetry and the dimensionality of space determine the universality class. As was first pointed out by Svetitsky and Yaffe, for an SU(N) Yang-Mills theory in \( d + 1 \) space-time dimensions the effective theory describing the fluctuations of the order parameter is a \( d \)-dimensional Z(N)-symmetric scalar field theory for the Polyakov loop [5]. For recent reviews on this subject we refer the reader to refs. [6, 7].

For \((d + 1)\)-dimensional SU(2) Yang-Mills theory the center symmetry is Z(2) and hence the effective theory is a \( d \)-dimensional Z(2)-symmetric scalar field theory for the real-valued Polyakov loop. The simplest theory in this class is a \( \Phi^4 \) theory with the Euclidean action

\[
S[\Phi] = \int d^d x \left[ \frac{1}{2} \partial_i \Phi \partial^i \Phi + V(\Phi) \right].
\]  

(1.1)

The scalar potential is given by

\[
V(\Phi) = a\Phi^2 + b\Phi^4,
\]

(1.2)

where \( b > 0 \) for stability reasons. Indeed, for \( a = 0 \) this theory has a second order phase transition in the universality class of the \( d \)-dimensional Ising model. However, this does not guarantee that the deconfinement phase transition in SU(2) Yang-Mills theory is also second order. In particular, one could imagine that the effective potential

\[
V(\Phi) = a\Phi^2 + b\Phi^4 + c\Phi^6,
\]

(1.3)

also involves a \( \Phi^6 \) term which is marginally relevant in three dimensions. Then the coefficient \( c \) has to be positive in order to ensure that the potential is bounded from below, but the coefficient \( b \) of the \( \Phi^4 \) term can now become negative. Then the phase transition becomes first order. For \( a = b = 0 \) there is a tricritical point at which the order of the phase transition changes. Still, this does not happen in \((3 + 1)\)-d SU(2) Yang-Mills theory, where — using lattice simulations — it has indeed been shown that the deconfinement phase transition is second order [8–13] and has the same critical exponents as the 3-d Ising model [14, 15]. Similarly, \((2 + 1)\)-d SU(2)
Yang-Mills theory has a second order deconfinement phase transition [16] in the universality class of the 2-d Ising model [17].

For $N \geq 3$, the effective theory for a $(d + 1)$-dimensional $SU(N)$ Yang-Mills theory with the center $\mathbb{Z}(N)$ should be a $d$-dimensional $\mathbb{Z}(N)$-symmetric scalar field theory for the complex-valued Polyakov loop $\Phi = \Phi_1 + i\Phi_2$ [18]. A simple representative of this class of theories is defined by the action

$$ S[\Phi] = \int d^dx \left[ \frac{1}{2} \partial_i \Phi^* \partial_i \Phi + V(\Phi) \right], $$

(1.4)

with

$$ V(\Phi) = a|\Phi|^2 + b|\Phi|^4 + c|\Phi|^6 + d \text{Re}(\Phi^N). $$

(1.5)

Note that $\text{Im}(\Phi^N)$ is also $\mathbb{Z}(N)$ invariant, but not invariant under charge conjugation. Hence, this term cannot appear in the effective action.

For $N = 3$ the cubic term in the action

$$ \text{Re}(\Phi^3) = \Phi_1 (\Phi_1^2 - 3\Phi_2^2), $$

(1.6)

breaks the $U(1)$ symmetry of the quadratic and quartic terms down to $\mathbb{Z}(3)$. For $d = 3$, the presence of this term renders the phase transition first order [18]. Also the 3-d 3-state Potts model [19–24] has a first order phase transition. The absence of universal behavior in 3-d $\mathbb{Z}(3)$-symmetric models suggests that the deconfinement phase transition in $SU(3)$ Yang-Mills theory should also be first order. Indeed this has been confirmed in great detail in lattice simulations [25–30]. In (2 + 1)-d $SU(3)$ Yang-Mills theory stronger fluctuations lead to a second order phase transition in the universality class of the 2-d 3-state Potts model [31].

Next, we consider the $N = 4$ case. For $c > 0$ the $\mathbb{Z}(4)$-symmetric scalar potential leads to a second order phase transition in the universality class of the 3-d $\mathbb{Z}(4)$-symmetric chiral clock model which corresponds to two decoupled Ising models. However, the deconfinement phase transition of $(3 + 1)$-d $SU(4)$ Yang-Mills theory does not seem to fall into that universality class. It is inconvenient to study the deconfinement phase transition in lattice simulations of $SU(4)$ Yang-Mills theory due to a first order bulk phase transition at zero temperature. The existing lattice data show that the deconfinement transition is first order [32–36]. Again, in (2 + 1)-d $SU(4)$ Yang-Mills theory stronger fluctuations seem to induce a second order phase transition [37, 38].

For $N \geq 5$, depending on the values of the various parameters in the potential $V(\Phi)$, the phase transition can again be first or second order. In case of a second order phase transition, the corresponding scalar field theory should be in the universality class of the $\mathbb{Z}(N)$-symmetric chiral clock model which, for $d = 3$, happens to be the one of the $U(1)$-symmetric XY-model [39]. Thus, for $N \geq 5$, the discrete $\mathbb{Z}(N)$ symmetry is not visible at the critical point and is, in fact, dynamically
enhanced to a continuous $U(1)$ symmetry. This should not be too surprising. In particular, for $N \geq 7$ the term $\text{Re}(\Phi^N)$ which breaks the $U(1)$ symmetry of the other terms down to $\mathbb{Z}(N)$ is irrelevant in three dimensions. However, again numerical simulations — in this case of $SU(6)$ and $SU(8)$ Yang-Mills theory [36] — indicate a first order deconfinement phase transition. This suggests that all $(3+1)$-d $SU(N)$ Yang-Mills theories with $N \geq 3$ have a first order deconfinement phase transition without universal behavior. In particular, the universality arguments of [5] then apply only to the $SU(2)$ case, not to $N \geq 3$ or to the $N = \infty$ limit. It should be mentioned that other arguments may suggest a second order phase transition at large $N$ [40].

It is interesting to ask if Svetitsky and Yaffe’s universality arguments can be applied beyond $SU(2)$ Yang-Mills theory, along another direction in the space of Lie groups. In particular, since $SU(2) \cong SO(3)$, one can ask if $SO(N)$ Yang-Mills theories show universal behavior at their deconfinement phase transitions. Numerical studies of $(3+1)$-d $SO(3)$ gauge theory are complicated by a bulk phase transition in which the lattice theory sheds off its $\mathbb{Z}(2)$ center monopole lattice artifacts [41–45]. Beyond this phase transition, in the continuum limit, one would expect $SO(3)$ and $SU(2)$ Yang-Mills theories to be equivalent. Lattice studies of the deconfinement phase transition of $SO(3)$ Yang-Mills theory [46–49] are consistent with this expectation.

In order to avoid complications due to lattice artifacts, it is best to work with the universal covering group of $SO(N)$ which is $Spin(N)$. For example, $Spin(3) = SU(2)$. The center of $Spin(N)$ is $\mathbb{Z}(2)$ for odd $N$, $\mathbb{Z}(2) \otimes \mathbb{Z}(2)$ for $N = 4k$, and $\mathbb{Z}(4)$ for $N = 4k + 2$. Let us first discuss the family of $Spin(N)$ with odd $N$ and center $\mathbb{Z}(2)$. The simplest case is $Spin(3) = SU(2)$ which we already discussed. Since $Spin(5) = Sp(2)$, this is a case that we will concentrate on later in this paper. In contrast to $Spin(3)$, for $d = 3$, we find a first order deconfinement phase transition. We are unaware of numerical lattice studies of $Spin(N)$ Yang-Mills theories with $N \geq 7$. Next, we consider $Spin(N)$ with $N = 4k$ and center $\mathbb{Z}(2) \otimes \mathbb{Z}(2)$. Now the simplest case is $Spin(4) = SU(2) \otimes SU(2)$. A $Spin(4)$ lattice Yang-Mills theory with the standard Wilson action factorizes into two $SU(2)$ Yang-Mills theories. Hence, its deconfinement phase transition is in the universality class of two decoupled Ising models. The next case in this family is $Spin(8)$ which has not been studied numerically. The last family is $Spin(N)$ with $N = 4k + 2$ with the center $\mathbb{Z}(4)$. Then the simplest case is $Spin(6) = SU(4)$. As we already discussed, lattice simulations have shown that in $(3+1)$-d $SU(4)$ Yang-Mills theory the deconfinement phase transition is first order. Although this need not necessarily be the case for larger $N$, due to the increasing number of gauge degrees of freedom we expect $(3+1)$-d $Spin(N)$ gauge theories to have first order transitions for all $N \geq 5$.

There is a last possible direction in the space of Lie groups which we explore in this paper. This is the sequence of symplectic Lie groups $Sp(N)$ which are simply connected and hence are their own universal covering groups. This sequence has
the center $\mathbb{Z}(2)$ for all $N$ and includes $SU(2) = Sp(1)$. The groups $Sp(N)$ are pseudo-real and thus provide a natural extension of $SU(2)$ to larger $N$. In contrast to $SU(N)$, the study of $Sp(N)$ Yang-Mills theories allows us to change the size of the group without changing the center. Hence, we can investigate the order of the deconfinement phase transition as a function of the size of the gauge group, keeping the available Ising universality class fixed. In fact, we will argue that the order of the phase transition is controlled by the size of the gauge group and not by the center symmetry. Our studies of $(3 + 1)$-d $Sp(2)$ and $Sp(3)$ Yang-Mills theories show that a first order phase transition arises although the 3-d Ising universality class is available. The order of the phase transition is a dynamical issue which does not simply follow from the center symmetry. The Lie groups larger than $SU(2)$ have many generators and thus give rise to a large number of deconfined gluons. The number of confined glueball states, on the other hand, is essentially independent of the gauge group. Hence, there is a drastic change in the number of relevant degrees of freedom on the two sides of the deconfinement phase transition. This can induce the abrupt changes in thermodynamical quantities that are characteristic for a first order phase transition. This suggests that the deconfinement phase transition of $(3 + 1)$-d $Sp(N)$ Yang-Mills theory is first order for all $N \geq 2$. In $2 + 1$ dimensions we find that stronger fluctuations drive the phase transition of $Sp(2)$ Yang-Mills theory second order. Due to the larger number of gauge degrees of freedom, $(2 + 1)$-d $Sp(3)$ Yang-Mills theory still has a first order phase transition. We expect this to be the case for all $N \geq 3$.

For completeness, let us also discuss the exceptional Lie groups $G(2)$, $F(4)$, $E(6)$, $E(7)$, and $E(8)$ that do not fall in the main sequences $SU(N)$, $Spin(N)$, or $Sp(N)$. The groups $G(2)$, $F(4)$, and $E(8)$ have a trivial center and thus need not have a deconfinement phase transition at all. In fact, recently we have argued that $G(2)$ Yang-Mills theory should only have a crossover between its low- and high-temperature regimes [50]. The group $E(6)$ has the center $\mathbb{Z}(3)$. Just as in the $SU(3)$ case, a $(3 + 1)$-d $E(6)$ Yang-Mills theory is expected to have a first order deconfinement phase transition because no universality class with $\mathbb{Z}(3)$ symmetry seems to exist in three dimensions. However, even for $(2 + 1)$-d $E(6)$ Yang-Mills theory, where the 2-d 3-state Potts model universality class is available, we expect a first order phase transition due to the large size of $E(6)$ (which has rank 6 and 78 generators). Finally, $E(7)$ has the center $\mathbb{Z}(2)$. If the deconfinement phase transition of $E(7)$ Yang-Mills theory is second order, it should hence have Ising critical exponents. However, for a group as large as $E(7)$ (with rank 7 and 133 generators) the large number of gauge degrees of freedom again suggests a first order deconfinement phase transition.

Also taking into account the numerical results for the various small Lie groups, the arguments from above lead us to conjecture that, in $3 + 1$ dimensions, only $SU(2)$ and its trivial extension $Spin(4) = SU(2) \otimes SU(2)$ Yang-Mills theories have a second order deconfinement phase transition. All other Yang-Mills theories are
expected to have a first order phase transition; possible exceptions are Yang-Mills theories with gauge group $G(2)$, $F(4)$, and $E(8)$ which have a trivial center and may hence just have a crossover. In that case, in 3 + 1 dimensions Svetitsky and Yaffe's universality arguments apply only to $SU(2)$.

In 2 + 1 dimensions stronger fluctuations arise and may result in a second order phase transition. Indeed, for (2+1)-d $Sp(2)$ Yang-Mills theory we find a second order phase transition. In agreement with Svetitsky and Yaffe’s arguments, a finite-size scaling analysis shows critical behavior in the 2-d Ising universality class. For the first time, this confirms the Svetitsky-Yaffe conjecture beyond $SU(N)$. For (2 + 1)-d $Sp(3)$ Yang-Mills theory, on the other hand, the transition is first order. This is again consistent with the increasing number of gauge degrees of freedom. Hence, in 2 + 1 dimensions we expect the transition to be first order for all $N \geq 3$.

The rest of this paper is organized as follows. In section 2 basic properties of $Sp(N)$ groups are reviewed. Section 3 introduces $Sp(N)$ Yang-Mills theory on the lattice, including heatbath and overrelaxation algorithms for its numerical simulation. Evidence for a first order deconfinement phase transition is reported for both $Sp(2)$ and $Sp(3)$ Yang-Mills theory in 3 + 1 dimensions and for $Sp(3)$ in 2 + 1 dimensions. A finite-size scaling analysis of the deconfinement phase transition in (2 + 1)-d $Sp(2)$ Yang-Mills theory shows that it is second order and in the universality class of the 2-d Ising model. We also study the static quark potential in (3 + 1)-d $Sp(2)$ Yang-Mills theory using the multi-level algorithm introduced by Lüscher and Weisz [51]. The resulting string tension is used to express the critical temperature of the deconfinement phase transition in physical units. Finally, section 4 contains our conclusions. Summaries of this work have already appeared in [52, 53].

2 The Symplectic Group $Sp(N)$

The group $Sp(N)$ is a subgroup of $SU(2N)$ which leaves the skew-symmetric matrix

$$J = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} = i\sigma_2 \otimes \mathbb{1},$$

(2.1)

invariant. Here $\mathbb{1}$ is the $N \times N$ unit-matrix and $\sigma_2$ is the imaginary Pauli matrix. The elements $U \in SU(2N)$ that belong to the subgroup $Sp(N)$ satisfy the constraint

$$U^* = JUJ^\dagger.$$  

(2.2)

Consequently, $U$ and $U^*$ are related by the unitary transformation $J$. Hence the 2N-dimensional fundamental representation of $Sp(N)$ is pseudo-real. The matrix $J$ itself also belongs to $Sp(N)$. This implies that in $Sp(N)$ Yang-Mills theory charge conjugation is just a global gauge transformation. This property is familiar from $SU(2) = Sp(1)$ Yang-Mills theory.
Indeed, matrices that obey the constraint eq.(2.2) form a group because for $U, V \in Sp(N)$ we have
\[(UV)^* = U^*V^* = JUJ^\dagger JVJ^\dagger = J(UV)J^\dagger. \tag{2.3}\]
The inverse $U^\dagger$ also obeys the constraint because
\[(U^\dagger)^* = (U^*)^\dagger = (JUJ^\dagger)^\dagger = JU^\dagger J^\dagger, \tag{2.4}\]
and obviously the unit-matrix also belongs to $Sp(N)$. The constraint eq.(2.2) implies the following form of a generic $Sp(N)$ matrix
\[U = \begin{pmatrix} W & X \\ -X^* & W^* \end{pmatrix}, \tag{2.5}\]
where $W$ and $X$ are complex $N \times N$ matrices. Since $U$ must still be an element of $SU(2N)$, these matrices must satisfy $WW^\dagger + XX^\dagger = 1$ and $WX^T = XW^T$. Note that the eigenvalues of $U$ come in complex conjugate pairs. Since center elements are multiples of the unit-matrix, in this case eq.(2.5) immediately implies $W = W^\ast$. Hence, the center of $Sp(N)$ is $Z(2)$.

Writing $U = \exp(iH)$, where $H$ is a Hermitean traceless matrix, eq.(2.2) implies that the generators $H$ of $Sp(N)$ satisfy the constraint
\[H^* = -JHJ^\dagger = JHJ. \tag{2.6}\]
This relation leads to the following generic form,
\[H = \begin{pmatrix} A & B \\ B^* & -A^* \end{pmatrix}, \tag{2.7}\]
where $A$ and $B$ are $N \times N$ matrices. The Hermiticity condition $H = H^\dagger$ implies $A = A^\dagger$ and $B = B^T$. Note that, since $A$ is Hermitean, $H$ is automatically traceless. The Hermitean $N \times N$ matrix $A$ has $N^2$ degrees of freedom and the complex symmetric $N \times N$ matrix $B$ has $(N+1)N$ degrees of freedom. Hence the dimension of the group $Sp(N)$ is $N^2 + (N+1)N = (2N+1)N$. There are $N$ independent diagonal generators of the maximal Abelian Cartan subgroup. Hence the rank of $Sp(N)$ is $N$. The $N = 1$ case is equivalent to $iSU(2)$, while the $N = 2$ case is equivalent to $SO(5)$, or more precisely to its universal covering group $Spin(5)$. Since $Sp(2)$ has rank 2, the weight diagrams of its representations can be drawn in a 2-d plane. The weight diagrams of the fundamental representation $\{4\}$, the $SO(5)$ vector representation $\{5\}$, and the adjoint representation $\{10\}$ are depicted in figures 1, 2, and 3, respectively.

3 $Sp(N)$ Yang-Mills Theory on the Lattice
In this section we consider $Sp(N)$ Yang-Mills theory on the lattice. First, we discuss the action, the measure, and important observables. Then we describe the simulation techniques and present results of numerical computations.
3.1 Action, Measure, and Observables

The construction of $Sp(N)$ Yang-Mills theory on the lattice is straightforward. The link parallel transporter matrices $U_{x,\mu} \in Sp(N)$ are group elements in the fundamental $\{2N\}$ representation. We consider the standard Wilson plaquette action

$$S[U] = -\frac{2}{g^2} \sum \text{Tr} U = -\frac{2}{g^2} \sum_{x,\mu<\nu} \text{Tr} (U_{x,\mu} U_{x+\hat{\mu},\nu} U_{x+\hat{\nu},\mu} U_{x,\nu}^\dagger),$$

where $g$ is the bare gauge coupling. The partition function then takes the form

$$Z = \int \mathcal{D}U \exp(-S[U]),$$

where the path integral measure

$$\int \mathcal{D}U = \prod_{x,\mu} \int_{Sp(N)} dU_{x,\mu},$$

is a product of local Haar measures of the group $Sp(N)$ for each link. Both the action and the measure are invariant under gauge transformations

$$U'_{x,\mu} = \Omega_x U_{x,\mu} \Omega_{x+\hat{\mu}}^\dagger,$$

with $\Omega_x \in Sp(N)$. The Polyakov loop

$$\Phi_x = \text{Tr}(P \prod_{t=1}^{N_t} U_{x,t,d+1})$$

is the trace of a path ordered product of link variables along a loop wrapping around the periodic Euclidean time direction. Here $N_t = 1/T$ is the extent of the lattice in
Figure 2: The weight diagram for the \{5\} representation of \(Sp(2)\) (the vector representation of \(SO(5)\)).

Euclidean time, which determines the temperature \(T\) in lattice units. The lattice action is invariant under global \(Z(2)\) center symmetry transformations

\[ U'_{\vec{x},N_t,d+1} = -U_{\vec{x},N_t,d+1}, \tag{3.6} \]

while the Polyakov loop changes sign, i.e. \(\Phi'_{\vec{x}} = -\Phi_{\vec{x}}\). The expectation value of the Polyakov loop is given by

\[ \langle \Phi \rangle = \frac{1}{Z} \int \mathcal{D}U \frac{1}{L^d} \sum_{\vec{x}} \Phi_{\vec{x}} \exp(-S[U]). \tag{3.7} \]

Here \(L^d\) is the spatial lattice volume, again in lattice units. As a consequence of the \(Z(2)\) center symmetry, with periodic boundary conditions in the spatial directions the expectation value of the Polyakov loop always vanishes, i.e. \(\langle \Phi \rangle = 0\), even in the deconfined phase. This is simply because spontaneous symmetry breaking — in the sense of a non-vanishing order parameter — does not occur in a finite volume. Alternatively, one may say that the expectation value of the Polyakov loop vanishes because the presence of a single static quark is incompatible with the Gauss law on a torus [54]. Since it always vanishes, on a finite torus the expectation value of the Polyakov loop does not contain any useful information about confinement or deconfinement. In the finite-size scaling analysis presented below we therefore consider the expectation value of the magnitude of the Polyakov loop \(\langle |\Phi| \rangle\). This quantity is always non-zero in a finite volume, but vanishes in the confined phase in the infinite volume limit. Furthermore, using the Polyakov loop, one can define its probability distribution

\[ p(\Phi) = \frac{1}{Z} \int \mathcal{D}U \delta \left( \Phi - \frac{1}{L^d} \sum_{\vec{x}} \text{Tr}(P \prod_{t=1}^{N_t} U_{\vec{x},t,d+1}) \right) \exp(-S[U]), \tag{3.8} \]
which does indeed allow one to distinguish confined from deconfined phases. In the confined phase $p(\Phi)$ has a single maximum at $\Phi = 0$, while in the deconfined phase it has two degenerate maxima at $\Phi \neq 0$. If the deconfinement phase transition is first order, the confined and the two deconfined phases coexist and one can simultaneously observe three maxima close to the phase transition. At a second order phase transition, on the other hand, the high- and low-temperature phases become indistinguishable. The two maxima of the deconfined phase then merge and smoothly turn into the single maximum of the confined phase. Three coexisting maxima then do not occur in a large volume.

In a pure glue theory another quantity of physical interest is the static quark potential $V_{QQ}(\vec{R})$. Note that, since the fundamental representation of $Sp(N)$ is pseudo-real, in $Sp(N)$ gauge theories quarks and anti-quarks are indistinguishable. At any temperature, $V_{QQ}(\vec{R})$ can be derived from the 2-point correlation function of the Polyakov loop

$$\langle \Phi_x \Phi_y \rangle = \exp(-N_t V_{QQ}(\vec{x} - \vec{y})).$$

In the zero-temperature limit, $N_t \to \infty$, at large distances $R = |\vec{R}| = |\vec{x} - \vec{y}|$ the static quark potential is linearly rising,

$$V_{QQ}(\vec{R}) \sim V_0 + \frac{c}{R} + \sigma R,$$

with the slope given by the string tension $\sigma$. At finite temperature $V(\vec{R})$ measures the free energy of a static quark pair at the distance vector $\vec{R}$. The string tension is a temperature-dependent function which decreases with increasing temperature. In the deconfined phase it vanishes and the potential levels off.

As physical quantities that are useful in the finite-size scaling analysis presented...
below we also introduce the Polyakov loop susceptibility
\[ \chi = \sum_{x} \langle \Phi_0 \Phi_x \rangle = L^d \langle \Phi^2 \rangle, \tag{3.11} \]
as well as the Binder cumulant [55]
\[ g_R = \frac{\langle \Phi^4 \rangle}{\langle \Phi^2 \rangle^2} - 3. \tag{3.12} \]
The susceptibility measures the strength of fluctuations in the order parameter while the Binder cumulant measures the deviation from a Gaussian distribution of those fluctuations. We also consider the specific heat which takes the form
\[ C_V = \frac{1}{L^d N_t} (\langle S^2 \rangle - \langle S \rangle^2). \tag{3.13} \]
In a finite volume \( C_V \) has a maximum close to the critical coupling \( 4N/g_c^2 \) of the infinite volume theory. We denote the value of the specific heat at the maximum by \( C_V^{\text{max}} \). Another interesting observable is the latent heat
\[ L_H = \frac{1}{L^d N_t} (\langle S \rangle_c - \langle S \rangle_d), \tag{3.14} \]
which measures the difference of the expectation values of the action in the confined and the deconfined phase. Note that this quantity is defined only at the phase transition in the infinite volume limit. It vanishes for a second order phase transition and is non-zero for a first order transition. Even in a finite (but sufficiently large) volume, \( L_H \) can be evaluated in a Monte Carlo simulation because — up to very rare tunneling events — every configuration can be unambiguously associated with the confined or the deconfined phase. In the large volume limit, the latent heat and the maximum of the specific heat are related by [56]
\[ C_V^{\text{max}} = L^d N_t \frac{L_H^2}{4}. \tag{3.15} \]

3.2 Simulation Techniques and Basic Results

The \( Sp(N) \) lattice Yang-Mills theory with the standard Wilson action can be simulated with heat-bath [57] and microcanonical overrelaxation [58–60] algorithms similar to the ones for \( SU(N) \). The main idea, originally due to Cabibbo and Marinari [61], is to work sequentially in various \( SU(2) = Sp(1) \) subgroups. In the \( Sp(2) \) case we use four different \( SU(2) \) subgroups: two of them operating along the two axes in the weight diagram in figure 1, and two of them operating along the two diagonals. Under the first two \( SU(2) \) subgroups the four states of the fundamental \( Sp(2) \) representation decompose into one \( SU(2) \) doublet and
two $SU(2)$ singlets $\{4\} = \{2\}_{SU(2)} \oplus \{1\}_{SU(2)} \oplus \{1\}_{SU(2)}$. Under the other two $SU(2)$ subgroups the fundamental representation decomposes into two $SU(2)$ doublets $\{4\} = \{2\}_{SU(2)} \oplus \{2\}_{SU(2)}$. To ensure ergodicity, two $SU(2)$ subgroups (one from each pair) are sufficient. For the general $Sp(N)$ case a minimal set of $N$ appropriately chosen $SU(2)$ subgroups is sufficient.

We have implemented the heat-bath and microcanonical overrelaxation algorithms for $Sp(2)$ and $Sp(3)$. First, we have looked for a potential bulk phase transition at zero temperature separating a strong from a weak coupling confined phase. Fortunately, in contrast to $SU(N)$ with $N \geq 4$, both in $Sp(2)$ and in $Sp(3)$ no bulk phase transition (which could obscure the finite temperature transition) has been found. In particular, different hot and cold starts did not show metastability. The absence of a bulk phase transition makes it easier to take the continuum limit than, for example, in simulations of $SU(4)$ Yang-Mills theory. Our Monte Carlo data for the expectation value of the plaquette are compared with analytic weak and strong coupling expansions in figures 4 and 5. At leading order of the weak coupling expansion, for the plaquette expectation value one finds

$$\frac{1}{2N} \langle \text{Tr} U_0 \rangle = 1 - \frac{(2N + 1)g^2}{4(d + 1)} + \mathcal{O}(g^4),$$  \hspace{1cm} (3.16)

(where $d + 1$ is the dimension of space-time), while at strong coupling

$$\frac{1}{2N} \langle \text{Tr} U_0 \rangle = \frac{1}{Ng^2} + \mathcal{O}\left(\frac{1}{g^{10}}\right).$$ \hspace{1cm} (3.17)

For comparison with potential future studies, some of our Monte Carlo data are listed in table 1.

### 3.3 Order of the Deconfinement Phase Transitions in $Sp(2)$ and $Sp(3)$ Yang-Mills Theories

The $SU(2) = Sp(1)$ Yang-Mills theory has a second order deconfinement phase transition both in $2 + 1$ and in $3 + 1$ dimensions. Since all $Sp(N)$ groups have the same center $\mathbb{Z}(2)$, one might have expected all $Sp(N)$ Yang-Mills theories to have second order deconfinement phase transitions. However, it should be noted that Svetitsky and Yaffe made no statement about the order of the phase transition. Their conjecture just states that, if the transition is second order, it should be in the Ising universality class.

In order to investigate the order of the deconfinement phase transition, we have simulated $(2 + 1)$-d $Sp(2)$ Yang-Mills theory at finite temperature for $N_t = 2, 4, \text{ and } 6$, at various spatial sizes ranging from $L = 10$ to 100. The probability distribution of the Polyakov loop, depicted in figure 6, indeed shows the characteristic features
Table 1: Plaquette expectation values $\langle Tr U_\square \rangle/2N$ for 3-d and 4-d $Sp(2)$ and $Sp(3)$ Yang-Mills theories on $8^3$ and $8^4$ lattices.
of a second order phase transition. In particular, approaching the phase transition from the confined side, fluctuations broaden the distribution, which then evolves into a two-peak structure on the deconfined side. The finite-size scaling analysis presented in the next subsection indeed confirms that the transition is second order and in the 2-d Ising universality class.

Interestingly, a corresponding study in (3+1)-d $Sp(2)$ Yang-Mills theory shows a first order transition. In this case, we have performed numerical simulations for $N_t = 2, 3, 4, 5, \text{ and } 6$ with $L = 8, 10, \ldots, 20$. The probability distribution of the Polyakov loop, displayed in figure 7, clearly shows three maxima, indicating coexistence of the two deconfined and the confined phase. This signal becomes more pronounced on larger volumes. However, since tunneling events are then suppressed, due to limited statistics the deconfined peaks are sampled unevenly. Figure 8 shows the Polyakov loop susceptibility $\chi$ as a function of $8/g^2$ for different spatial sizes $L$, keeping $N_t$
fixed. At $8/g_c^2 = 6.4643(3)$ the resulting behavior $\chi \sim L^3$ quantitatively confirms the first order nature of the transition. Figure 9 shows the maximum of the specific heat per volume, $C_{V_{\text{max}}}/L^3$, as a function of the inverse volume $1/L^3$. The linear behavior is characteristic of a first order phase transition. A linear extrapolation of $C_{V_{\text{max}}}/L^3$ to the infinite volume limit (see eq.(3.15)) is consistent with a direct measurement of the latent heat $L_H$, which again supports the first order nature of the transition. Hence, despite the fact that the 3-d Ising universality class is available, $(3+1)$-d $Sp(2)$ Yang-Mills theory does not fall into it and instead displays non-universal first order behavior. As the spatial dimension is increased from two to three, the coefficient $b$ of the $\Phi^4$ term in the effective potential $V(\Phi)$ of eq.(1.3) changes sign, thus driving the transition first order. This effect does not happen in the Ising model or in $SU(2) = Sp(1)$ Yang-Mills theory, but it does happen in $Sp(2)$ Yang-Mills theory. One may wonder if this effect is a lattice artifact. As we will see later, this seems not to be the case. Our data indicate that the $(3+1)$-d
Figure 6: Polyakov loop probability distributions for (2+1)-d $Sp(2)$ Yang-Mills theory on a $40^2 \times 2$ lattice at three different couplings $8/g^2 = 10.3$, 10.4, and 10.5 close to the phase transition.

$Sp(2)$ Yang-Mills theory has a first order deconfinement phase transition even in the continuum limit.

We have also investigated the deconfinement phase transition in $Sp(3)$ Yang-Mills theory, both in 2+1 and in 3+1 dimensions. Figure 10 shows the Polyakov loop distribution on a $40^2 \times 2$ lattice at $12/g^2 = 21.375$. Three distinct peaks are clearly visible, indicating a first order phase transition. Interestingly, compared to the (2+1)-d $Sp(2)$ case, the phase transition has changed from second to first order. We attribute this to the larger size of the group $Sp(3)$. The larger number of $Sp(3)$ gluons in the deconfined phase increases the difference between the relevant degrees of freedom on the two sides of the phase transition, thus driving the transition first order. Similar behavior has been observed in 3+1 dimensions. Figure 11 displays the Monte Carlo time history of the Polyakov loop on a $6^3 \times 2$ lattice at
Figure 7: Polyakov loop probability distributions for (3+1)-d Sp(2) Yang-Mills theory on a $10^3 \times 2$ lattice at three different couplings $8/g^2 = 6.464$, 6.465, and 6.466 close to the phase transition.

$12/g^2 = 13.83$ with multiple tunneling events between the coexisting confined and deconfined phases. We have not attempted to extrapolate our Sp(3) results to the continuum limit. Since the phase transition is very strongly first order at finite lattice spacing, we expect that it remains first order in the continuum limit.
Figure 8: Polyakov loop susceptibility per spatial volume, $\chi/L^3$, in $(3+1)$-d $Sp(2)$ Yang-Mills theory for $L = 8, 10, \ldots, 18$ and $N_t = 2$ as a function of the coupling $8/g^2$.

3.4 Finite-Size Scaling Analysis of the $(2+1)$-d $Sp(2)$ Deconfinement Phase Transition

A second order phase transition is characterized by a correlation length $\xi$ which diverges as the critical point is approached

$$\xi \sim x^{-\nu} = \left( \frac{g_c^2}{g^2} - 1 \right)^{-\nu}. \quad (3.18)$$

Here $x$ is a measure of the distance from criticality and the exponent $\nu$ is particular to the universality class of the phase transition. Note that the critical coupling $g_c$ depends on $N_t$, which is kept fixed in the finite-size scaling analysis. The universality class is determined by the dimensionality of space and by the underlying symmetries. As well as the correlation length, the order parameter and the susceptibility also
Figure 9: The maximum of the specific heat per volume, $C_V^{\text{max}} / L^3$, as a function of the inverse volume $1/L^3$ for $(3 + 1)$-d $Sp(2)$ Yang-Mills theory. The linear extrapolation to the infinite volume limit is in agreement with the measured latent heat $L_H$.

have particular power-like behavior close to the critical point

$$\langle |\Phi| \rangle \sim x^\beta, \quad \chi \sim x^{-\gamma}. \quad (3.19)$$

The exponents $\nu, \beta,$ and $\gamma$ characterize the critical behavior and are specific to a given universality class.

A finite system cannot have a phase transition in the sense of non-analytic behavior. In particular, the correlation length stays finite because it is limited by the system size. The correlation length sets a natural distance scale and the ratio $L/\xi$ specifies the spatial size $L$ of the system in these units. Close to criticality one can define the scaled variable $y = xL^{1/\nu} \sim (L/\xi)^{1/\nu}$. Then, as the spatial volume increases, physical quantities approach criticality in a specific way. Finite-size scaling is a method that relates the finite volume scaling behavior to the universal prop-
properties of the critical system in the thermodynamic limit. For instance, the order parameter behaves as
\[ \langle |\Phi| \rangle \sim L^{-\beta/\nu} F(xL^{1/\nu}), \tag{3.20} \]
where \( F(y) \) is a universal finite-size scaling function. By measuring the order parameter \( \langle |\Phi| \rangle \) for a variety of spatial volumes and couplings \( 8/g^2 \) close to the critical point, one can determine the critical exponents of the universality class. In order to check if our system is in the Ising universality class, we first assume the 2-d Ising critical exponents, \( \nu = 1, \beta = 1/8, \) and \( \gamma = 7/4, \) and only vary the critical coupling \( 8/g_c^2. \) If corrections to scaling are small, then all the data should fall onto a universal curve for the correct value of \( 8/g_c^2. \) Figure 12 is the finite-size scaling plot for \( \langle |\Phi| \rangle L^{\beta/\nu} \) obtained with fixed temporal extent \( N_t = 2 \) and spatial volumes ranging from \( L = 26 \) to 100. The data fall beautifully onto a universal curve for \( 8/g_c^2 = 10.45(1), \) indicating that (2 + 1)-d \( Sp(2) \) Yang-Mills theory has a second or-

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Figure 10: *Polyakov loop probability distributions for (2 + 1)-d Sp(3) Yang-Mills theory on a 40\(^2\) × 2 lattice at 12/g\(^2\) = 21.375.*
Figure 11: Monte Carlo time history of the Polyakov loop in (3 + 1)-d Sp(3) Yang-Mills theory on a $6^3 \times 2$ lattice at $12/g^2 = 13.83$.

der deconfinement transition in the universality class of the 2-d Ising model. Figure 12 also contains data from the $SU(2)$ Yang-Mills theory, which is known to fall in the 2-d Ising universality class. Indeed the data from the two theories fall on the same universal curve.

The value of the Binder cumulant $g_R$ at the critical point $x = 0$ is independent of the spatial size $L$, provided that corrections to scaling are negligible (which is true for sufficiently large volumes). Just like the critical exponents, the value $g_R(x = 0)$ is another characteristic of the universality class. For the 2-d Ising model, its value is $g_R(x = 0) = -1.837(8)$ [62]. From Figure 13, we see that this is in good agreement with the value measured in (2 + 1)-d $Sp(2)$ Yang-Mills theory, further evidence that the phase transition belongs to the expected universality class. As $g_R(xL^{1/\nu})$ changes rapidly over a small range around the critical point, this can be used to determine quite accurately the critical coupling $8/g_c^2$ as a function of the temporal
Figure 12: Finite-size scaling plot for $\langle |\Phi| \rangle L^{\beta/\nu}$ in $(2 + 1)$-d $Sp(2)$ Yang-Mills theory at $N_t = 2$. Some $SU(2)$ data are also included.

extent $N_t$ without having to assume the values of the universal exponents. At the critical coupling, the measured values of $g_R$ are independent of $L$ and in very good agreement with the 2-d Ising value quoted above. Figure 14 shows the finite-size scaling function of the susceptibility $\chi L^{-\gamma/\nu}$. Again all data fall on a universal curve consistent with the critical exponents of the 2-d Ising model.

3.5 Continuum Limit of $(3 + 1)$-d $Sp(2)$ Yang-Mills Theory

We have presented numerical results indicating a first order deconfinement phase transition for $(3 + 1)$-d $Sp(2)$ and for $(2 + 1)$-d and $(3 + 1)$-d $Sp(3)$ Yang-Mills theory. In the $Sp(2)$ case the Euclidean time extents $N_t = 2, 3, 4, 5$, and 6 have been investigated. First order signals were obtained in all cases and the critical couplings
Figure 13: Finite-size scaling plot for the Binder cumulant $g_R$ in (2 + 1)-d Sp(2) Yang-Mills theory at $N_t = 2$. Some SU(2) data are also included. The intersection of the dashed lines indicates the Ising value $g_R(x = 0) = -1.837(8)$.

$8/g_c^2$ have been determined for each of these $N_t$ values and are listed in table 2.

In this subsection we ask if the transition remains first order in the continuum limit. To test for scaling, we have measured the string tension $\sigma$ in the zero temperature limit close to the critical couplings $8/g_c^2$. In the scaling regime the dimensionless ratio $T_c/\sqrt{\sigma}$ — where $T_c = 1/N_t$ is the critical temperature in lattice units — should become independent of $N_t$. We have determined the string tension $\sigma$ (as well as the parameters $V_0$ and $c$ of eq.(3.10)) from the static quark potential $V_{QQ}(\vec{R})$ exploiting the Lüscher-Weisz multi-level algorithm [51]. An example of a typical Polyakov loop correlator together with a fit based on eq.(3.10) is depicted in figure 15. The relevant numerical data are summarized in table 3. For $N_t = 2, 3, 4,$ and $5$ the critical coupling $8/g_c^2$ has been determined quite accurately. In particular, the statistical error of the string tension is comparable with the systematic uncertainty resulting from
Figure 14: Finite-size scaling plot for $\chi L^{-\gamma/\nu}$ in $(2 + 1)$-d $Sp(2)$ Yang-Mills theory at $N_t = 2$.

the error in $8/g_c^2$. This is not the case for $N_t = 6$ where the error in $8/g_c^2$ leads to a larger systematic uncertainty in the string tension evaluated at the critical coupling. Not surprisingly, the result for $N_t = 2$ is not in the scaling regime. Figure 16 shows the extrapolation of $T_c/\sqrt{\sigma}$ for $N_t = 3, 4,$ and 5 to the continuum limit. As expected, the cut-off effects are consistent with proportionality to the lattice spacing squared. Indeed, our data seem to be in the scaling region, which indicates that the deconfinement phase transition of $(3 + 1)$-d $Sp(2)$ Yang-Mills theory remains first order in the continuum limit. Extrapolating to that limit, we find $T_c/\sqrt{\sigma} = 0.6875(18)$. This value is very close to the known result $T_c/\sqrt{\sigma} = 0.7091(36)$ for $SU(2) = Sp(1)$ [36], but significantly larger than the $SU(3)$ result $T_c/\sqrt{\sigma} = 0.6462(30)$ [36]. This may suggest that $SU(2) = Sp(1)$ Yang-Mills theory is closer to the large $N$ limit of $Sp(N)$ than to the one of $SU(N)$. It would be interesting to study this question by investigating $T_c/\sqrt{\sigma}$ in $Sp(N)$ Yang-Mills theory with $N \geq 3$. 24
It should be pointed out that we observe scaling of dimensionless ratios of physical quantities but no asymptotic scaling of individual quantities with the perturbative $\beta$-function of the bare coupling constant. For example, a change of scale by a factor of 2 from $N_t = 3$ to $N_t = 6$ requires a shift in the critical bare coupling $8/g_c^2$ by $7.611(14) - 7.1228(4) = 0.488(15)$. On the other hand, asymptotic scaling with the 1-loop $\beta$-function of $Sp(N)$ Yang-Mills theory corresponds to a shift in $4N/g^2$ by

$$\frac{11N(N + 1)}{6\pi^2} \ln 2 \approx 0.773 \quad \text{for } N = 2.$$  

Similar violations of asymptotic scaling are familiar from numerical simulations of $SU(N)$ Yang-Mills theory [63]. In particular, asymptotic scaling is expected to set in only for bare couplings much closer to the continuum limit.

| $1/T_c$ | $2$ | $3$ | $4$ | $5$ | $6$ |
|---------|-----|-----|-----|-----|-----|
| $8/g_c^2$ | 6.4643(3) | 7.1228(4) | 7.339(1) | 7.486(4) | 7.611(14) |

Table 2: Critical couplings $8/g_c^2$ for $N_t = 2, 3, 4, 5, 6$ for $(3 + 1)$-d $Sp(2)$ Yang-Mills theory.

Table 3: The string tension $\sigma$ for $(3 + 1)$-d $Sp(2)$ Yang-Mills theory at various couplings $8/g^2$.

| $8/g^2$ | $L^4 \times N_t$ | $\sigma$ |
|---------|-----------------|---------|
| 6.4643  | $12^4$          | 0.71(1) |
| 7.123   | $12^4$          | 0.2902(16) |
| 7.340   | $12^4$          | 0.1484(12) |
| 7.490   | $12^4$          | 0.0904(11) |
|         | $16^4$          | 0.0911(3) |
| 7.620   | $16^4$          | 0.0617(3) |
| 7.635   | $12^4$          | 0.0584(7) |
|         | $16^4$          | 0.0595(3) |
|         | $16^3 \times 20$| 0.0597(2) |

4 Conclusions

It is interesting to systematically investigate the deconfinement phase transition of Yang-Mills theories. The only case with rank 1 is $SU(2) = Spin(3) = Sp(1)$ Yang-Mills theory which has a second order deconfinement phase transition with
Figure 15: Polyakov loop correlation function in (3 + 1)-d Sp(2) Yang-Mills theory at $8/g^2 = 7.635$ for $L = 16$ and $N_t = 20$. The line is a fit to $A[\exp(-N_tV_{QQ}(R)) + \exp(-N_tV_{QQ}(L-R))]$ with the static quark potential $V_{QQ}(R)$ given by eq.(3.10). The resulting string tension $\sigma = 0.0597(2)$ is quite accurately measured.

Ising critical exponents. The rank 2 groups include $SU(3)$, $SO(4) \simeq Spin(4) = SU(2) \otimes SU(2)$, $SO(5) \simeq Spin(5) = Sp(2)$, and $G(2)$. While (3 + 1)-d $SU(3)$ Yang-Mills theory has a first order deconfinement phase transition, in the $SU(2) \otimes SU(2)$ case the transition is second order and in the universality class of two decoupled 3-d Ising models. The present study shows that in (3 + 1)-d $Sp(2)$ Yang-Mills theory the transition is again first order. This leaves $G(2)$ as the only unexplored case of rank 2. We have recently conjectured that, due to the triviality of its center, $G(2)$ Yang-Mills theory has no deconfinement phase transition — just a crossover between a low- and a high-temperature regime [50]. It would be interesting to investigate this in numerical simulations. In particular, this would complete the numerical study of gauge groups of rank one and two. A systematic investigation could then proceed to the rank 3 groups $SO(6) \simeq Spin(6) = SU(4)$, $Sp(3)$, and $SO(7) \simeq Spin(7)$. As we now know, not only $SU(4)$ but also $Sp(3)$ Yang-Mills theory has a first order
Figure 16: Continuum extrapolation of the dimensionless ratio $T_c/\sqrt{\sigma}$.

deconfinement phase transition. Like $Sp(3)$, the group $Spin(7)$ also has the center $\mathbb{Z}(2)$. It would be interesting to see if $(3 + 1)$-d $Spin(7)$ Yang-Mills theory has a second order deconfinement phase transition in the 3-d Ising universality class. Since — just like $Sp(3)$ — $Spin(7)$ is a large group (with 21 generators), we expect that it has a first order transition.

Let us summarize the results for the order of the deconfinement phase transition in $(3 + 1)$-d Yang-Mills theories. Based on lattice calculations with the gauge groups $SU(3)$, $SU(4)$, $SU(6)$, and $SU(8)$, it seems natural to assume that all $SU(N)$ Yang-Mills theories with $N \geq 3$ have a first order transition. For $SU(2) = Sp(1) = Spin(3)$ the transition is second order. The same is true for $SU(2) \otimes SU(2) = Spin(4)$. Based on our $Sp(2)$ and $Sp(3)$ results, we expect that all $Sp(N)$ Yang-Mills theories with $N \geq 2$ have a first order deconfinement phase transition. Since for $SU(4) = Spin(6)$ the transition is again first order, we expect the same for all $Spin(N)$ with $N \geq 5$. For $E(6)$ with the center $\mathbb{Z}(3)$ the transition should be first order, for the same reason as for $SU(3)$: no universality class with $\mathbb{Z}(3)$
symmetry seems to exist in three dimensions. In addition, due to the large size of $E(6)$, we would expect a first order transition even if a corresponding universality class was available. Again due to its large number of generators, and despite its center $\mathbb{Z}(2)$, we expect $E(7)$ Yang-Mills theory to have a first order transition. The remaining exceptional groups $G(2)$, $F(4)$, and $E(8)$ have a trivial center and thus need not have a deconfinement phase transition at all. For $G(2)$ we expect just a crossover, but we cannot rule out a first order phase transition. Consequently, only for $SU(2) = Sp(1) = Spin(3)$ and for its trivial extension $SU(2) \otimes SU(2) = Spin(4)$ the transition is second order and in the 3-d Ising universality class. Svetitsky and Yaffe’s universality arguments do not apply to the other Yang-Mills theories in $3 + 1$ dimensions.

In $(2+1)$ dimensions a few Yang-Mills theories have a second order deconfinement phase transition: they include those with $Sp(2)$, $Spin(4)$, $SU(2)$ [16,17], $SU(3)$ [17,31], and, perhaps, $SU(4)$ [37,38] gauge groups. It remains to be seen if other $(2 + 1)$-d Yang-Mills theories with gauge group $Spin(N)$ with $N \geq 7$ or $SU(N)$ with $N \geq 5$ belong on this list. Due to the large size of these groups we find this unlikely. For example, $Sp(3)$ and $Spin(7)$ have 21 and $SU(5)$ has 24 generators. Since we find $(2 + 1)$-d $Sp(3)$ Yang-Mills theory to have a first order deconfinement phase transition, we expect the same for $Spin(N)$ with $N \geq 7$ and for $SU(N)$ with $N \geq 5$.

Our results suggest that the size of the gauge group — and not the center symmetry — determines the order of the deconfinement phase transition. This should not be too surprising. The larger Lie groups have many generators and thus give rise to a large number of deconfined gluons. The number of confined glueball states, on the other hand, is essentially independent of the gauge group. For a large gauge group the drastic change in the number of relevant degrees of freedom on the two sides of the deconfinement phase transition may easily drive it first order.

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