Free vibration of a taut cable with a parallel-connected viscous mass damper and a grounded cross-tie

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Abstract
Recent studies have shown that the combination of external dampers and cross-ties can overcome their respective deficiencies while retaining their respective merits. Inspired by the advantages of a viscous damper (VD) and a cross-tie on a single stay cable, the coupled damping effect of a parallel-connected viscous mass damper (PVMD) and a grounded cross-tie on a single stay cable is investigated in this paper. The complex wave number equations of the cable–PVMD–cross-tie system are first formulated through the complex modal analysis. Subsequently, the asymptotic and iterative complex wave numbers are compared to evaluate the applicability of the asymptotic and iterative solutions. Furthermore, parametric studies are carried out to investigate the effects of the stiffness coefficients and the installation positions of the cross-tie on the first supplemental modal damping ratio and frequency of the cable. Finally, the installation positions of the cross-tie are optimized to achieve the optimal vibration control of the cable with the PVMD and the cross-tie.

Keywords
Stay cable, vibration control, viscous mass damper, cross-tie, modal damping ratio

Introduction
In recent years, structural vibration control has received extensive attentions, especially the vibration control of stay cables in cable-stayed bridges.¹-³ Stay cables, as the critical load-bearing component of cable-stayed bridges, are highly vulnerable to environmental excitations such as wind, wind-rain, and parametric excitations.⁴-⁶ Frequent vibrations may shorten the cable life and impair public confidence in the safety of cable-stayed bridges. Consequently, several countermeasures have been proposed to suppress the excessive vibrations of stay cables, including changing the surface characteristics of the cable to improve its aerodynamic characteristics,⁷,⁸ connecting cables via the cross-tie,⁹ and attaching external dampers on cables.¹⁰-¹² Among these countermeasures, attaching external dampers on cables is the most common and effective measure. However, it is difficult to effectively mitigate vibrations for super-long cables in large-span cable-bridges, such as the longest cable in the Stonecutters Bridge (536 m) and the longest cable in the Sutong Bridge (577 m). Hence, it is necessary to combine available countermeasures to efficiently mitigate these cable vibrations.¹³-¹⁵

To improve the control performance of passive dampers, a semi-active control strategy using magneto-rheological (MR) damper is proposed.¹⁶ MR dampers, as one of the most representative semi-active control devices, will provide a semi-active control method for stay cables to balance effectiveness and practicality. It is found that the negative stiffness characteristic of the semi-active MR dampers helps to improve the supplemental modal damping ratio of stay cables.¹⁷,¹⁸ Inspired by the advantages of the negative stiffness characteristic of the semi-active MR dampers, the passive negative stiffness dampers (NSDs), including pre-spring NSDs¹⁹,²⁰ and magnetic NSDs,²¹,²² have received more attention in the field of cable vibration control. Moreover, a series of theoretical and experimental investigations have proved the

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effectiveness of NSDs in mitigating cable vibrations.\textsuperscript{23,25} Shi et al.\textsuperscript{23,24} demonstrated that the negative stiffness characteristic of the NSDs was conducive to providing more sufficient supplemental damping than that of passive viscous dampers. Javanbakht et al.\textsuperscript{25} verified that the NSDs presented superior performance over the conventional positive stiffness dampers (PSDs) in cable vibration control. However, excessive passive negative stiffness may induce the instability problem to the cable.\textsuperscript{22} Recent studies have shown that passive VD and inertial mass elements are connected in parallel to consist of a parallel-connected viscous mass damper (PVMD), which can improve the vibration control performance of structures.\textsuperscript{26-29} Compared with conventional VD, PVMD can provide more supplemental damping for stay cables, which has been verified by both theoretical analyses\textsuperscript{30-33} and experimental investigations.

A combination strategy of using both cross-ties and external dampers is also proposed to suppress the cable vibrations. Connecting cables via cross-ties can effectively enhance the in-plane stiffness and frequencies of stay cables,\textsuperscript{36-38} while it is difficult to effectively improve the supplemental damping of stay cables.\textsuperscript{39} It has been found that the hybrid techniques of using cross-ties and dampers can overcome their respective deficiencies while retaining their respective merits.\textsuperscript{40-43} Among them, Zhou et al.\textsuperscript{43} simplified the cross-ties as a discrete spring to investigate the damping and frequency of a taut cable with a damper and a spring. Di et al.\textsuperscript{44} proposed a simplified analytical method to investigate the effects of cross-ties on the dynamics of a hybrid cable network system. Furthermore, recent studies have shown that when two dampers are installed at opposite cable ends, the combined damping effect is approximately the sum of the contributions from each damper.\textsuperscript{45-47} In addition, Yang et al.\textsuperscript{48} explored a novel optimal design of two viscous dampers near to a cable anchorage, which can achieve a potential improvement in damping ratios of cables for lower and higher vibration modes. Thus, the combination strategy of using two dampers, either or a damper and a cross-tie, can achieve superior cable vibration control performance over that of using a single damper.

Inspired by the advantages of a VD and a grounded cross-tie on a single stay cable, the coupled damping effects of a single stay cable with a parallel-connected viscous mass damper (PVMD) and a grounded cross-tie are further investigated in this paper. The paper is organized as follows. Firstly, the complex wave number equations of the cable–PVMD–cross-tie system are formulated by using the complex modal analysis. Then, the asymptotic and iterative solutions for the PVMD and the grounded cross-tie, either at the same cable end or at opposite cable ends, are obtained, respectively. Furthermore, the applicability of the iterative solution is evaluated, and the coupled damping effect of the cable with a PVMD and a cross-tie is further discussed when the cross-tie is connected at different positions of the cable. Finally, the maximum first supplemental modal damping ratio of a cable with a PVMD and a grounded cross-tie is highlighted when the cross-tie is installed at the cable mid-span.

**Formulation of the cable–PVMD–cross-tie system**

A taut cable with a PVMD and a grounded cross-tie is shown in Figure 1. The length, the tension, and the mass per unit length of the cable are denoted as \( l, T, \) and \( m \), respectively. The coordinate system defines that the \( x \)-axis and the \( u \)-axis are along the cable chord and the transverse direction. The PVMD and the cross-tie are respectively fixed at distances \( x_1 \) and \( x_2 \) from the left end of the cable \((x_2 \geq x_1)\). The distance between the cross-tie and the right end of the cable is denoted as \( x'_2 = l - x_2 \). The stiffness coefficient of the cross-tie is denoted as \( k_b \). The damping coefficient and inertial mass of the PVMD are denoted as \( c_d \) and \( b_d \), respectively. The free vibration of the cable–damper–cross-tie system in the transverse direction can be described by the following equation\textsuperscript{45}

\[
T \frac{\partial^2 u(x,t)}{\partial x^2} - m \frac{\partial^2 u(x,t)}{\partial t^2} = \sum_{j=1}^{2} f_j(t) \delta(x - x_j)
\]  

(1)

where \( u(x,t) \) is the cable transverse displacement; \( \delta(\cdot) \) is the Dirac delta function; \( f_j(t) \) is the damping force of the PVMD located at \( x = x_j \); and \( f_2(t) \) is the concentrated force of the cross-tie located at \( x = x'_2 \).

Equation (1) satisfies the boundary conditions at the cable ends, and there is

![Figure 1. A taut cable with a PVMD and a grounded cross-tie.](image-url)
\[ u(0,t) = u(l,t) = 0 \] (2)

At the damper and the cross-tie location, \( u(x) \) needs to satisfy the following boundary conditions

\[ u(x_1^-, t) = u(x_1^+, t) = u(x_1,t) \]
\[ u(x_2^-, t) = u(x_2^+, t) = u(x_2,t) \]
\[ T \left( \frac{\partial u(x,t)}{\partial x} \bigg|_{x_1^+} - \frac{\partial u(x,t)}{\partial x} \bigg|_{x_2^-} \right) = f_j(t), \ j = 1, 2 \] (3)

Substituting equation (4) into equation (1), \( U(x) \) in each span needs to satisfy a homogeneous equation\(^{45,46}\)

\[ \frac{d^2 U(x)}{dx^2} + \beta_n^2 U(x) = 0 \]

\[ 0 \leq x \leq x_1 \]
\[ x_1 \leq x \leq x_2 \]
\[ 0 \leq x^* \leq x_2^* \]

where \( \beta_n = \omega_n \sqrt{m/T} \) refers to the wave number.

The general solution of equation (6) can be written in the form

\[ U(x) = \begin{cases} 
U(x_1) \frac{\sin \beta_n x}{\sin \beta_n x_1} & 0 \leq x \leq x_1 \\
U(x_1) \frac{\sin \beta_n (x_2 - x)}{\sin \beta_n (x_2 - x_1)} + U(x_2) \frac{\sin \beta_n (x - x_1)}{\sin \beta_n (x_2 - x_1)} & x_1 \leq x \leq x_2 \\
U(x_2) \frac{\sin \beta_n x^*}{\sin \beta_n x_2^*} & 0 \leq x^* \leq x_2^* 
\end{cases} \] (7)

Substituting equation (7) into equation (3) yields

\[ \left\{ \begin{array}{l}
\cot \beta_n x_1 + \cot \beta_n (x_2 - x_1) - \frac{U_2}{U_1} \frac{1}{\sin \beta_n (x_2 - x_1)} = \frac{F_1/T}{\beta_n U_1} \\
- \frac{U_1}{U_2} \frac{1}{\sin \beta_n (x_2 - x_1)} + \cot \beta_n (x_2 - x_1) + \cot \beta_n x_2^* = \frac{F_2/T}{\beta_n U_2} 
\end{array} \right. \] (8)

Substituting \( U_1/U_2 \) from the second equation of equation (8) into the first equation of equation (8) and rearranging, the characteristic equation of the wave number \( \beta_n \) can be derived as

\[ \left( \cot \beta_n x_1 + \frac{F_1/T}{\beta_n U_1} \right) \left( \cot \beta_n x_2^* + \frac{F_2/T}{\beta_n U_2} \right) + \left( \cot \beta_n x_1 + \cot \beta_n x_2^* + \frac{F_2/T}{\beta_n U_2} \right) \cot \beta_n (x_2 - x_1) = 1 \] (9)
The forces of the PVMD and the cross-tie can be expressed as

\[ f_1(t) = c_\theta \frac{\partial u(x_1, t)}{\partial t} + b_\theta \frac{\partial^2 u(x_1, t)}{\partial t^2}, \quad f_2(t) = k_\theta u(x_2, t) \]  

(10)

When the PVMD and the grounded cross-tie are installed at the same cable end, that is, \( x_1 \leq x_2 < l/2 \), substituting equation (10) into equation (9), equation (9) can be rearranged to the form relating to \( x_1 \) and \( x_2 \) as

\[ \tan \beta_n l = \frac{A + iB}{C + iD} \]

(11)

\[ A = -\chi \sin^2 \beta_n x_1 + \kappa \sin^2 \beta_n x_2 - \chi \kappa \sin \beta_n x_1 \sin \beta_n x_2 \sin \beta_n (x_2 - x_1) \]  

(11a)

\[ B = \eta \sin^2 \beta_n x_1 + \kappa \eta \sin \beta_n x_1 \sin \beta_n x_2 \sin \beta_n (x_2 - x_1) \]  

(11b)

\[ C = 1 - \chi \sin \beta_n x_1 \cos \beta_n x_2 \sin \beta_n x_2 - \kappa \chi \sin \beta_n x_1 \cos \beta_n x_2 \sin \beta_n (x_2 - x_1) \]  

(11c)

\[ D = \eta \sin \beta_n x_1 \cos \beta_n x_2 + \kappa \eta \sin \beta_n x_1 \cos \beta_n x_2 \sin \beta_n (x_2 - x_1) \]  

(11d)

where \( \kappa = k_\theta / \beta_n T \) denotes the dimensionless stiffness coefficient of the cross-tie and \( \eta = c_\theta / \sqrt{mT} \) and \( \chi = b_\theta \omega_0 / \sqrt{mT} \) denote the dimensionless damping coefficient and the dimensionless inertial mass of the PVMD, respectively.

When the PVMD and the grounded cross-tie are installed at opposite cable ends, that is, \( 0 < x_2 < l/2 \), equation (9) can be rearranged to the form relating to \( x_1 \) and \( x_2^* \) as

\[ \tan \beta l = \frac{E + iF}{G + iH} \]

(12)

\[ E = -\chi \sin^2 \beta_n x_1 + \kappa \sin^2 \beta_n x_2^* - \chi \kappa \sin \beta_n x_1 \sin \beta_n x_2^* \sin \beta_n (x_2^* + x_1) \]  

(12a)

\[ F = \eta \sin^2 \beta_n x_1 + \kappa \eta \sin \beta_n x_1 \sin \beta_n x_2^* \sin \beta_n (x_2^* + x_1) \]  

(12b)

\[ G = 1 - \chi \sin \beta_n x_1 \cos \beta_n x_2^* \cos \beta_n (x_2^* + x_1) \]  

(12c)

\[ H = \eta \sin \beta_n x_1 \cos \beta_n x_2^* + \kappa \eta \sin \beta_n x_1 \cos \beta_n x_2^* \cos \beta_n (x_2^* + x_1) \]  

(12d)

**The PVMD and the grounded cross-tie at the same cable end**

**Asymptotic solution and iterative solution.** The iterative solution of the complex wave number \( \beta_n \) is obtained by the fixed-point iteration method.\(^{39}\) The initial value of the iterative equation is the undamped complex wave number \( \beta_n^0 \). Substituting \( \beta_n^0 \) into the right side of equation (11), the \( n^{th} \) modal resulting value of \( \beta_n^0 \) is obtained. Similarly, with current estimate \( \beta_n^j \) \((j=1, 2, 3, \ldots)\), a new estimate \( \beta_n^{j+1} \) will be derived. The iterative equation can be written in the form

\[ \beta_n^{j+1} = \frac{A(\beta_n^j) + iB(\beta_n^j)}{C(\beta_n^j) + iD(\beta_n^j)} \]

(13)

\[ \beta_n^0 = \frac{n\pi}{T}, \quad n = 1, 2, 3, \ldots \]

The \( n^{th} \) supplemental modal damping ratio of the cable can be calculated by

\[ \xi_n = \frac{\text{Im}(\omega_n)}{|\omega_n|} = \frac{\text{Im}(\beta_n)}{|\beta_n|} = \frac{\text{Im}(\Delta \beta_n)}{|\beta_n|} \]

(14)
Finally, the supplemental modal damping ratio of the cable with a PVMD and a grounded cross-tie can be calculated by equation (14) after solving iteratively the wave number.

The following assumptions are introduced: (1) the PVMD and the cross-tie are installed near the left cable end, that is, \(x_1 = l, x_2 = l, x_1 < x_2\) and (2) the complex wave number \(\beta_n\) of the cable has a small perturbation \(\Delta \beta_n = \beta_n - \beta^0_n = \beta^0_n\) from the undamped value \(\beta^0_n = n\pi / l\). These assumptions yield the following approximations:

\[
\tan \beta^0_n l = \beta^0_n x_1, \quad \sin \beta^0_n x_1, \quad \sin \beta^0_n x_2, \quad \cos(\beta^0_n x_1) = \cos(\beta^0_n x_2) \approx 1
\]  

The asymptotic solution for the complex wave number \(\beta_n\) can be expressed as

\[
\frac{\Delta \beta_n}{\beta^0_n} = \frac{\kappa x_1}{1 + \kappa x_1 - \kappa \beta^0_n \left( \frac{x_2}{x_1} - 1 \right)} + i\varepsilon \left[ 1 + \frac{\kappa x_1}{1 + \kappa x_1 - \kappa \beta^0_n \left( \frac{x_2}{x_1} - 1 \right)} \right]^x_1
\]  

where \(\kappa = k \beta x_1 = k_0 x_1 / T, \varepsilon = \eta \beta x_1 = c_0 x_1 \omega^0_n / T,\) and \(\beta^0_n = \beta x_1 = b_0 x_1 (\omega^0_n)^2 / T\).

Substituting equation (16) into equation (14), the asymptotic solution of the \(n^{th}\) supplemental modal damping ratio of the cable can be derived as

\[
\zeta_n = \frac{\varepsilon}{\left( 1 - \kappa x_1 + \kappa \beta^0_n \left( \frac{x_2}{x_1} - 1 \right) \right)^2 + \varepsilon^2 \left[ 1 + \kappa \beta^0_n \left( \frac{x_2}{x_1} - 1 \right) \right]^\frac{x_1}{T}}
\]  

**Comparison of asymptotic and iterative solutions**

The PVMD and the grounded cross-tie are installed at the positions of \(x_1=1\%l\) and \(x_2=2\%l\) from the left cable end, respectively. Figure 2 compares the asymptotic and iterative complex wave numbers of a cable with a PVMD and a cross-tie.
when the dimensionless stiffness coefficient of the cross-tie $\kappa = 0.01/(n\pi x_1/l)$. When the dimensionless inertial mass remains constant and the dimensionless damping coefficient of PVMD increases from zero to infinity, the complex wave number traces a semicircle with a positive imaginary part. The semicircle trace starts from an undamped frequency and eventually terminates at a clamped frequency on the real axis. The top point of the semicircle trace suggests the maximum supplemental modal damping ratio of the cable. It can be seen from Figure 2 that the maximum supplemental modal damping ratio of the cable gradually increases with the increase of the inertial mass of the PVMD. Moreover, the asymptotic complex wave number is in good agreement with the iterative complex wave number when the dimensionless inertial mass $\chi = 0/(n\pi x_1/l)$, $\chi = 0.3/(n\pi x_1/l)$, and $\chi = 0.6/(n\pi x_1/l)$. However, the asymptotic and iterative complex wave numbers significantly deviate from each other for the large inertial mass $\chi = 0.9/(n\pi x_1/l)$, and the basic assumption of the asymptotic solution is no longer valid under this circumstance. Hence, the iterative solutions are used to accurately predict the maximum supplemental modal damping ratio of the cable as well as the corresponding parameters of the PVMD and the cross-tie in the following discussions.

**Parametric studies**

Figure 3 shows the variations of the first supplemental modal damping ratio of a cable with a VD (PVMD) and a grounded cross-tie with the damping coefficient of the VD (PVMD) under the various stiffness coefficients of the cross-tie. It can be observed from Figure 3 that the maximum supplemental damping ratio of a cable with a PVMD and a grounded cross-tie is significantly increased compared with that of a cable with a VD and a grounded cross-tie, which indicates the coupled damping effect of a PVMD and a grounded cross-tie on a stay cable is superior to that of a VD and a grounded cross-tie on a stay cable. However, increasing the dimensionless stiffness coefficient of the cross-tie has negative effects on improving the maximum supplemental damping ratio of the cable.

Figure 4 shows the variations of the first supplemental modal damping ratio of a cable with a VD (PVMD) and a grounded cross-tie with the damping coefficient of the VD (PVMD) under various installation positions of the cross-tie. It can be seen from Figure 4 that the maximum supplemental modal damping ratio of the cable decreases with the cross-tie moving away from the VD or the PVMD. Thus, when the PVMD and the grounded cross-tie are installed at the same cable end, the increase of installation length of the cross-tie is unfavorable to the damping improvement of the cable.

**The PVMD and the grounded cross-tie at opposite cable ends**

Asymptotic solution and iterative solution. Similar to the case of a PVMD and a ground cross-tie at the same cable end, when the PVMD and the grounded cross-tie are installed at opposite cable ends, the iterative solution of the complex wave number $\beta_n$ is also obtained by the fixed-point iteration method, in which the iterative equation is given by

![Figure 3](attachment:image.png)

*Figure 3.* Variations of the first modal damping ratio of the cable with the damping coefficient under various stiffness coefficients of the cross-tie ($x_1=1\%, x_2=2\%$).
\[ \beta_n^{j+1} l = n \pi + \arctan \left( \frac{E(\beta_n^j)}{G(\beta_n^j) + iH(\beta_n^j)} \right) \]

\[ \beta_n^0 = \frac{n \pi}{l} n = 1, 2, 3, \ldots \]  

(18)

After solving the wave number, the supplemental modal damping ratio of a cable with a PVMD and a cross-tie at opposite cable ends can also be calculated by equation (14). It is assumed that the PVMD is installed near the left cable end, while the cross-tie is installed near the right cable end, that is, \( x_1 = l \), \( x_2^* = l \). The approximate solution for the complex wave number \( \beta_n \) can be expressed as

\[ \Delta \beta_l = \frac{\kappa}{C_{18}} \frac{x_1}{C_{19}} + \frac{1}{2} \frac{C_{18}}{C_{19}} \frac{x_1}{C_{21}} \]  

(19)

Substituting equation (19) into equation (14), the asymptotic solution of the \( n^{th} \) supplemental modal damping ratio of the cable can be expressed as

\[ \xi_n = \frac{\tau \left( 1 + \frac{\tau^2}{x_1} \right)^2}{\left( \frac{\tau}{C_{18}} - \frac{C_{18}}{x_1} \right)^2 + \left( \frac{\tau}{C_{19}} - \frac{C_{19}}{x_1} \right)^2} \frac{x_1}{l} \]  

(20)

**Comparison of asymptotic and iterative solutions**

The PVMD and the grounded cross-tie at opposite cable ends are installed at the positions of \( x_1 = 1\%l \) from the left cable end and \( x_2^* = 1\%l \) from the right cable end. Figure 5 compares the asymptotic and iterative complex wave numbers of a cable with a PVMD and a grounded cross-tie when the dimensionless stiffness coefficients of the cross-tie \( \kappa = 0.01/(n \pi x_1/l) \). It can be seen from Figure 5 that the maximum supplemental modal damping ratio of the cable gradually increases with the increase of the inertial mass of the PVMD. When the dimensionless inertia mass \( \chi = 0/(n \pi x_1/l), \chi = 0.3/(n \pi x_1/l), \) and \( \chi = 0.6/(n \pi x_1/l) \), the asymptotic complex wave number coincides well with the iterative complex wave number, while the asymptotic and iterative complex wave numbers significantly deviate from each other for the large inertial mass \( \chi = 0.9/(n \pi x_1/l) \). Similarly, the basic assumption of the asymptotic solution is no longer valid when the PVMD has a large inertial mass, which further verified that the applicability of the iterative solution is superior to the asymptotic solution. In addition, compared with installing the PVMD and the cross-tie at the same cable end, the asymptotic and iterative complex wave numbers of the cable with a PVMD and a cross-tie at opposite cable ends are more consistent.
Parametric studies

Figure 5. Comparisons of the asymptotic and iterative complex wave numbers of a cable with a PVMD and a grounded cross-tie ($x = x_1 = x^*_2 = 1/\ell, \kappa=0.01/(n\pi x/l)$).

Figure 6. Variations of the first supplemental modal damping ratio of the cable with the damping coefficient under different stiffness coefficients of the cross-tie ($x = x_1 = x^*_2 = 1/\ell$).

Figure 6 shows the variations of the first supplemental modal damping ratio of the cable with a VD (PVMD) and a grounded cross-tie with the damping coefficient of the VD (PVMD) under various stiffness coefficients of the cross-tie. As illustrated in Figure 6, the maximum supplemental damping ratio of a cable with a PVMD and a grounded cross-tie is significantly increased compared with that of a cable with a VD and a grounded cross-tie, which indicates the coupled damping effect of a PVMD and a grounded cross-tie on a stay cable is superior to that of a VD and a grounded cross-tie on a stay cable. Moreover, increasing the dimensionless stiffness coefficient of the cross-tie is conducive to improving the maximum supplemental modal damping ratio of the cable.

Figure 7 shows the variations of the first supplemental modal damping ratio of the cable with a VD (PVMD) and a grounded cross-tie with the damping coefficient of the VD (PVMD) under various installation positions of the cross-tie. It can be seen from Figure 7 that the maximum supplemental modal damping ratio of the cable slightly increases with the cross-tie moving away from the right cable end. Consequently, when the PVMD and the grounded cross-tie are installed at
The increase of installation length of the cross-tie helps to achieve the damping improvement of the cable.

Optimization for the installation positions of the cross-tie

To analyze the effects of the installation positions of the cross-tie on the damping of a cable with a PVMD and a grounded cross-tie, the optimization for the installation positions of the cross-tie is carried out in this section. Figure 8 shows the variations of the first supplemental modal damping ratio of the cable with the installation positions of the cross-tie and the damping coefficients of the PVMD. It can be observed from Figure 8 that the first modal damping ratio of the cable reaches its maximum value when the installation position of the cross-tie is $x_2/l=0.51$ and the damping coefficient of the PVMD is $\eta=0.26$. The variations of the first supplemental modal damping ratio of the cable with the installation positions and the stiffness coefficients of the cross-tie are further illustrated in Figure 9. It can be seen from Figure 9 that the first supplemental modal damping ratio of the cable reaches the maximum when the cross-tie is installed at the cable mid-span, in which the first supplemental modal damping ratio of the cable increases significantly with the installation position of the cross-tie.

**Optimal installation positions of the cross-tie**

To analyze the effects of the installation positions of the cross-tie on the damping of a cable with a PVMD and a grounded cross-tie, the optimization for the installation positions of the cross-tie is carried out in this section. Figure 8 shows the variations of the first supplemental modal damping ratio of the cable with the installation positions of the cross-tie and the damping coefficients of the PVMD. It can be observed from Figure 8 that the first modal damping ratio of the cable reaches its maximum value when the installation position of the cross-tie is $x_2/l=0.51$ and the damping coefficient of the PVMD is $\eta=0.26$. The variations of the first supplemental modal damping ratio of the cable with the installation positions and the stiffness coefficients of the cross-tie are further illustrated in Figure 9. It can be seen from Figure 9 that the first supplemental modal damping ratio of the cable reaches the maximum when the cross-tie is installed at the cable mid-span, in which the first supplemental modal damping ratio of the cable increases significantly with the installation position of the cross-tie.
increase of the stiffness coefficient of the cross-tie. Thus, the cable mid-span is the optimal installation position for the cross-tie to improve the vibration control performance of the cable for the first mode.

**The cross-tie at the cable mid-span**

To further analyze the effects of the stiffness coefficients of the cross-tie on the damping and frequency of a cable with a PVMD and a grounded cross-tie, the PVMD and the cross-tie are installed at the positions of $x_1=2\%l$ and $x_2=50\%l$ from the left cable end. **Figure 10** shows the first modal complex wave numbers of a cable with a PVMD and a grounded cross-tie under various stiffness coefficients of the cross-tie. It is noteworthy that the frequency and the damping ratio of the cable increase significantly with the increase of the stiffness coefficient of the cross-tie.
Figure 11. Variations of the first supplemental modal damping ratio of the cable with the damping coefficients of PVMD and stiffness coefficients of the cross-tie ($x_1=2\%l$, $x_2=50\%l$, $\chi=0.9/(n\pi x_1/l)$).

Figure 12. The maximum first supplemental modal damping ratio of the cable ($x_1=1\%l$, $x_2=50\%$).

Figure 11 shows the influence of the damping coefficients of PVMD and the stiffness coefficients of the cross-tie on the first supplemental modal damping ratio of the cable. It can be seen from Figure 11 that when the stiffness coefficient of the cross-tie is determined, the first supplemental modal damping ratio of the cable increases with the increase of the damping coefficient of the PVMD, reaching the maximum value at the optimum damping coefficient of the PVMD, and then decreases with the further increase of the optimum damping coefficient of PVMD. Furthermore, the optimum damping coefficient of the PVMD decreases with the increase of the stiffness coefficient of the cross-tie, which helps to reduce the damping cost to achieve optimal vibration control performance of the cable.

Figure 12 compares the maximum first supplemental modal damping ratio of a cable with a PVMD and a cross-tie, a single PVMD, as well as a VD and a cross-tie. It is found in Figure 12 that when the cross-tie is installed at the cable mid-span, the maximum first supplemental modal damping ratio of a cable with PVMD and cross-tie is greater than that of a cable with VD and cross-tie or single PVMD. In addition, the maximum first supplemental modal damping ratio of the cable increases with the increase of the stiffness coefficient of the cross-tie.

Conclusions

In this paper, the coupled damping effect of a cable with a parallel-connected viscous mass damper (PVMD) and a grounded cross-tie is investigated. The coupled damping effect of a PVMD and a grounded cross-tie on a stay cable is compared with that of a viscous damper (VD) and a grounded cross-tie on a stay cable. Moreover, the effects of the stiffness coefficients and
installation positions of the cross-tie on the damping of the cable are emphatically analyzed. The main conclusions are summarized as follows:

1. The supplemental modal damping ratio of a cable with a PVMD and a grounded cross-tie is significantly improved compared with that of a cable with a VD and a grounded cross-tie, which indicates the coupled damping effect of a PVMD and a grounded cross-tie on a stay cable is superior to that of a VD and a grounded cross-tie on a stay cable.

2. When a PVMD and a ground cross-tie are installed at the same cable end, increasing the dimensionless stiffness coefficient of the cross-tie has negative effects on improving the maximum supplemental damping ratio of the cable, and the maximum supplemental modal damping ratio of the cable decreases with the cross-tie moving away from the PVMD.

3. When a PVMD and a ground cross-tie are installed at opposite cable ends, increasing the dimensionless stiffness coefficient of the cross-tie is beneficial to enhance the maximum supplemental damping ratio of the cable, and the maximum supplemental modal damping ratio of the cable increases with the cross-tie moving close to the cable mid-span.

4. The cable mid-span is determined as the optimal installation position of the cross-tie to improve the vibration control performance of the cable for the first mode. When the cross-tie is installed at the cable mid-span, the first supplemental modal damping ratio of the cable reaches the maximum.

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