The \(m\)-Core-EP Inverse in Minkowski Space

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Received: 5 January 2021 / Revised: 6 July 2021 / Accepted: 10 July 2021 / Published online: 25 November 2021
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Abstract
In this paper, we introduce the \(m\)-core-EP inverse in Minkowski space, consider its properties, and get several sufficient and necessary conditions for the existence of the \(m\)-core-EP inverse. We give the \(m\)-core-EP decomposition in Minkowski space, and note that not every square matrix has the decomposition. Furthermore, by applying the \(m\)-core-EP inverse and the \(m\)-core-EP decomposition, we introduce the \(m\)-core-EP order and give some characterizations of it.

Keywords Minkowski space · \(m\)-Core-EP inverse · \(m\)-Core-EP decomposition · \(m\)-Core-EP order

Mathematics Subject Classification 15A09 · 15A57 · 15A24

1 Introduction

In this paper, we use the following notations. The symbol \(\mathbb{C}_{n,n}\) is the set of \(n \times n\) matrices with complex entries. \(A^*\), \(\mathcal{R}(A)\) and \(\text{rk}(A)\) represent the conjugate transpose, range space (or column space) and rank of \(A \in \mathbb{C}_{n,n}\), respectively. The smallest positive integer \(k\), which satisfies \(\text{rk}(A^{k+1}) = \text{rk}(A^k)\), is called the index of \(A\) and is denoted

\[\text{rk}(A^{k+1}) = \text{rk}(A^k)\]
by \( \text{Ind}(A) \). In particular,

\[
\mathbb{C}_{n}^{\text{CM}} = \left\{ A \mid A \in \mathbb{C}_{n,n}, \ \text{rk}(A^2) = \text{rk}(A) \right\}.
\]

In 1907, Hermann Minkowski proposed the Minkowski space (shorted as \( \mathcal{M} \)). In 2000, Meenakshi [8] introduced generalized inverse into \( \mathcal{M} \), and got its existence conditions. The Minkowski metric matrix can be written as

\[
G = \begin{bmatrix} 1 & 0 \\ 0 & -I_{n-1} \end{bmatrix}, \quad G = G^* \quad \text{and} \quad G^2 = I_n.
\] (1.1)

The Minkowski adjoint of a matrix \( A \in \mathbb{C}_{n,n} \) is defined as \( A^\sim = GA^*G \). Let \( A, B \in \mathbb{C}_{n,n} \). It is easy to check that \( (AB)^\sim = B^\sim A^\sim \) and \( (A^\sim)^\sim = A \).

The Minkowski inverse of a matrix \( A \in \mathbb{C}_{n,n} \) in \( \mathcal{M} \) is defined as the matrix \( X \in \mathbb{C}_{n,n} \) satisfying the following four conditions [8]:

\[
\begin{align*}
(1) \quad & AXA = A, \\
(2) \quad & XAX = X, \\
(3^m) \quad & (AX)^\sim = AX, \\
(4^m) \quad & (XA)^\sim =XA,
\end{align*}
\] (1.2)

where \( X \) is denoted by \( A^m \). It is worthy to note that the Minkowski inverse \( A^m \) exists if and only if

\[
\text{rk} \left( A^\sim A \right) = \text{rk} \left( AA^\sim \right) = \text{rk}(A), [8].
\] (1.3)

If \( A^m \) exists, then it is unique [8,12,14].

In [18], Wang et al. defined the \( m \)-core inverse in \( \mathcal{M} \): Let \( A \in \mathbb{C}_{n}^{\text{CM}} \), if there exists \( X \in \mathbb{C}_{n,n} \) satisfying the following three equations:

\[
\begin{align*}
(1) \quad & AXA = A, \\
(2^l) \quad & AX^2 = X, \\
(3^m) \quad & (AX)^\sim = AX,
\end{align*}
\] (1.4)

then \( X \) is called the \( m \)-core inverse of \( A \), and is denoted by \( A^{\oplus} \). If \( A^{\ominus} \) exists, then it is unique. For \( A \in \mathbb{C}_{n}^{\text{CM}} \), \( A \) is \( m \)-core invertible if and only if

\[
\text{rk}(A^\sim A) = \text{rk}(A). \] (1.5)

In [17], Wang introduced the core-EP decomposition: Let \( A \in \mathbb{C}_{n,n} \) with \( \text{rk}(A^k) = r \) and \( \text{Ind}(A) = k \). Then

\[
A = A_1 + A_2, \] (1.6)

where \( A_1 \in \mathbb{C}_{n}^{\text{CM}}, \ A_2^k = 0 \) and \( A_1^*A_2 = A_2A_1 = 0 \). Here one or both of \( A_1 \) and \( A_2 \) can be null.

Furthermore, there exists an \( n \times n \) unitary matrix \( U \) such that

\[
A_1 = U \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix} U^* \quad \text{and} \quad A_2 = U \begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix} U^*.
\] (1.7)
where \( S \in \mathbb{C}_{r,n-r} \), \( T \in \mathbb{C}_{r,r} \) is invertible, \( N \in \mathbb{C}_{n-r,n-r} \) is nilpotent, and \( N^k = 0 \).

When \( A \in \mathbb{C}_{n}^{CM} \), it is obvious that \( N = 0 \), that is, \( A = A_1 \). Then the core inverse \( A^{\boxtimes} \) of \( A \) is

\[
A^{\boxtimes} = U \begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*,
\]

(1.8)

where \( A^{\boxtimes} \) denotes the core inverse of \( A \), and the core inverse \( A^{\boxtimes} \) of \( A \) is the unique solution of \( AXA = A \), \( AX^2 = X \), and \( (AX)^* = AX \).

Denote

\[
U^*GU = \begin{bmatrix} \hat{G}_1 & \hat{G}_2 \\ \hat{G}_3 & \hat{G}_4 \end{bmatrix},
\]

(1.9)

in which \( \hat{G}_1 \in \mathbb{C}_{r,r} \).

When \( A \in \mathbb{C}_{n}^{CM} \) with \( \text{rk}(A^{\sim}A) = \text{rk}(A) \), by applying (1.6), (1.7) and (1.9), we have

\[
\text{rk}(A^{\sim}A) = \text{rk}(GA^*GA) = \text{rk}
\begin{bmatrix}
G
\begin{bmatrix}
T^* & 0 \\
S^* & 0
\end{bmatrix}
U^*GU
\begin{bmatrix}
T & S \\
0 & 0
\end{bmatrix}
U^*
\end{bmatrix}
\]

\[
= \text{rk}
\begin{bmatrix}
T^* & 0 \\
S^* & 0
\end{bmatrix}
\begin{bmatrix}
\hat{G}_1 & \hat{G}_2 \\
\hat{G}_3 & \hat{G}_4
\end{bmatrix}
\begin{bmatrix}
T & S \\
0 & 0
\end{bmatrix}
\]

\[
= \text{rk}
\begin{bmatrix}
T^* & S^*
\end{bmatrix}
\begin{bmatrix}
\hat{G}_1 & \hat{G}_2
\end{bmatrix}
\begin{bmatrix}
T & S \\
0 & 0
\end{bmatrix}
= \text{rk}
\begin{bmatrix}
T^* & S^*
\end{bmatrix}
\hat{G}_1
\begin{bmatrix}
T & S
\end{bmatrix}
\]

\[
\leq \text{rk}(\hat{G}_1) \leq r.
\]

When \( \text{rk}(A^{\sim}A) = r \), it follows that \( \text{rk}(\hat{G}_1) = r \), that is, \( \hat{G}_1 \) is invertible. On the contrary, suppose that \( \hat{G}_1 \) is invertible, by applying \( r = \text{rk}(A) \geq \text{rk}(A^{\sim}A) \geq \text{rk}(T^*\hat{G}_1T) = r \), we have \( \text{rk}(A^{\sim}A) = r \). Therefore, we conclude that \( \text{rk}(A) = \text{rk}(A^{\sim}A) \) if and only if \( \hat{G}_1 \) is invertible. It follows using (1.5) that \( A \) is m-core invertible if and only if \( \hat{G}_1 \) is invertible.

Furthermore, the m-core inverse of \( A \) can be expressed as form

\[
X = U \begin{bmatrix} T^{-1}\hat{G}_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*G.
\]

(1.10)

We give the outline of the proof as follows.

Since \( G^* = G \), we have

\[
\hat{G}_1^* = \hat{G}_1.
\]

(1.11)
By (1.6), (1.7), (1.9), (1.10) and (1.11), we have

\[ A \mathcal{X} A = U \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix} U^* U \begin{bmatrix} T^{-1} \hat{G}_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* G U \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix} U^* = U \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix} U^* = A, \quad (1.12) \]

\[ A \mathcal{X}^2 = U \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix} U^* U \begin{bmatrix} T^{-1} \hat{G}_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* G U \begin{bmatrix} T^{-1} \hat{G}_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* G \]

\[ = U \begin{bmatrix} T^{-1} \hat{G}_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* G = \mathcal{X}, \]

\[ (A \mathcal{X})^\sim = \left( U \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix} U^* U \begin{bmatrix} T^{-1} \hat{G}_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* G \right)^\sim = \left( U \begin{bmatrix} \hat{G}_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* G \right)^\sim \]

\[ = U \begin{bmatrix} (\hat{G}_1^{-1})^* & 0 \\ 0 & 0 \end{bmatrix} U^* G = U \begin{bmatrix} \hat{G}_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* G = A \mathcal{X}. \quad (1.14) \]

Therefore, by applying (1.12), (1.13), (1.14) and (1.4), we conclude that \( \mathcal{X} \) is the m-core inverse of \( A \), and

\[ A^{\text{m-core}} = U \begin{bmatrix} T^{-1} \hat{G}_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* G. \]

In recent years, generalized core inverse has been extensively studied by many researchers, particularly to the core-EP inverse. The core-EP inverse of \( A \) is the unique matrix [13], which satisfies

\[ (1) X A^{k+1} = A^k, \quad (2) X A X = X, \quad (3) (A X)^* = A X, \quad (4) \mathcal{R}(X) \subseteq \mathcal{R}(A^k), \]

and is denoted by \( A^{\text{core}} \), where \( \text{Ind}(A) = k \).

In [7], Ma and Stanimirović studied the characterizations, approximation and perturbations of the core-EP inverse, and applied SMS algorithm for computing the core-EP inverse. In [2], Ferreyra et al. proposed generalized the core-EP inverse to rectangular matrices and gave properties of the weighted core-EP inverse. In [19], Zhou et al. proposed three limit representations of the core-EP inverse. In [9], Mosić studied the weighted core-EP inverse of an operator between Hilbert space. In [5], Ji and Wei applied the core-EP, weighted core-EP inverse of matrices to study constrained systems of linear equations. More details of the core-EP inverse and its applications can be seen in [3,4,7,10,11,15].

In this paper, we introduce a generalization of the m-core inverse for square matrices of an arbitrary index. We also give some of its characterizations, properties and applications.
2 The m-Core-EP Inverse in Minkowski Space

Let $A \in \mathbb{C}_{n, n}$ with $\text{Ind}(A) = k$, $A = A_1 + A_2$ be of the core-EP decomposition of $A$, and $A_1$ and $A_2$ be as in (1.7). Then $T \in \mathbb{C}_{r, r}$ is invertible and

$$A^k = U \begin{bmatrix} T^k & \hat{T} \\ 0 & 0 \end{bmatrix} U^*,$$

and

$$A^{k+1} = U \begin{bmatrix} T^{k+1} & \hat{T} \\ 0 & 0 \end{bmatrix} U^*,$$

where $\hat{T} = T^{k-1}S + T^{k-2}SN + \cdots + TSN^{k-2} + SN^{k-1}$, and $\overline{T} = T^k S + T^{k-1}SN + \cdots + TSN^{k-1} + SN^k = T^k S + T^{k-1}SN + \cdots + TSN^{k-1}$. It is obvious that $T^{-1}\overline{T} = \hat{T}$.

When $A \in \mathbb{C}_{n}^{\mathbb{C}M}$, it is easy to check that

$$A^* = U \begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix} U^*.$$

**Lemma 2.1** Let $A \in \mathbb{C}_{n, n}$ with $\text{Ind}(A) = k$, then $\text{rk} \left((A^k) \sim A^k\right)$ if and only if $G_1$ is invertible, in which $G_1 \in \mathbb{C}_{\text{rk}(A^k), \text{rk}(A^k)}$ is given as in

$$U^*G_U = \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix},$$

where $U$ is defined as in (1.7).

**Proof** Let $A \in \mathbb{C}_{n, n}$ be as in (1.6) and (1.7), $A^k$ be as in (2.1), and $U^*G_U$ is given as in (2.4). Then

$$\text{rk} \left((A^k) \sim A^k\right) = \text{rk} \left(G(A^k)^*G A^k\right) = \text{rk} \left(GU \begin{bmatrix} (T^k)^* & 0 \\ \hat{T}^* & 0 \end{bmatrix} U^*GU \begin{bmatrix} T^k & \hat{T} \\ 0 & 0 \end{bmatrix} U^*\right)$$

$$= \text{rk} \left( \begin{bmatrix} (T^k)^* & 0 \\ \hat{T}^* & 0 \end{bmatrix} \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix} \begin{bmatrix} T^k & \hat{T} \\ 0 & 0 \end{bmatrix} \right)$$

$$= \text{rk} \left( \begin{bmatrix} (T^k)^* & 0 \\ \hat{T}^* & 0 \end{bmatrix} \begin{bmatrix} G_1 & G_2 \\ 0 & 0 \end{bmatrix} \right) = \text{rk} \left( \begin{bmatrix} (T^k)^* & G_1 \end{bmatrix} \begin{bmatrix} T^k & \hat{T} \end{bmatrix} \right)$$

$$\leq \text{rk}(G_1) \leq r.$$

When $\text{rk} \left((A^k) \sim A^k\right) = r$, it follows that $\text{rk}(G_1) = r$, that is, $G_1$ is invertible. On the contrary, suppose that $G_1$ is invertible, by applying $r = \text{rk} \left(A^k\right) \geq \text{rk} \left((A^k) \sim A^k\right) \geq \text{rk} \left((T^k)^*G_1 T^k\right) = r$, we have $\text{rk} \left((A^k) \sim A^k\right) = r$. Therefore, we conclude that $\text{rk}(A^k) = \text{rk} \left((A^k) \sim A^k\right)$ if and only if $G_1$ is invertible. \qed
Remark 2.2 Let $A \in \mathbb{C}_{n,n}$ with $\text{Ind}(A) = k$ and $\text{rk}(A^k) = \text{rk}((A^k) \sim A^k)$. According to the $(U^*GU)^* = U^*GU$, we obtain

$$G_1^* = G_1 \quad \text{and} \quad G_3^* = G_2.$$ 

Let $A \in \mathbb{C}_{n,n}$ with $\text{Ind}(A) = k$, and consider

\begin{align*}
(1) \quad XAX &= X, \\
(2^k) \quad XA^{k+1} &= A^k, \\
(3^m) \quad (AX) \sim &= AX, \\
(4^r) \quad \mathcal{R}(X) &\subseteq \mathcal{R}(A^k).
\end{align*}

(2.5)

Theorem 2.3 Let $A \in \mathbb{C}_{n,n}$ with $\text{Ind}(A) = k$. If there exists a solution of the four matrix equations (1), $(2^k)$, $(3^m)$ and $(4^r)$, then the solution is unique.

Proof Let $A \in \mathbb{C}_{n,n}$ be as given in (1.6) and (1.7). We suppose that both $X_1$ and $Y_1$ satisfy (2.5). Let $U$ be defined as in (1.7). Denote

$$X_1 = U \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} U^* \quad \text{and} \quad Y_1 = U \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} U^*.$$ 

(2.6)

Next, we prove $X_1 = Y_1$.

From $X_1A^{k+1} = A^k$ and $Y_1A^{k+1} = A^k$, we have

$$X_1A^{k+1} = Y_1A^{k+1}. \quad (2.7)$$

By applying (2.6) and (2.7), we have

$$U \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} U^* \begin{bmatrix} T^{k+1} & 0 \\ 0 & 0 \end{bmatrix} U^* = U \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} U^* \begin{bmatrix} T^{k+1} & 0 \\ 0 & 0 \end{bmatrix} U^*,$$

that is,

$$\begin{bmatrix} X_{11} & X_{21} \\ X_{21} & X_{22} \end{bmatrix}\begin{bmatrix} T^{k+1} & 0 \\ 0 & T \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{21} \\ Y_{21} & T^{k+1} \end{bmatrix} \begin{bmatrix} T^{k+1} & 0 \\ 0 & T \end{bmatrix}.$$

Since $T$ is invertible, we get

$$X_{11} = Y_{11} \quad \text{and} \quad X_{21} = Y_{21}. \quad (2.8)$$

Applying $\mathcal{R}(X_1) \subseteq \mathcal{R}(A^k)$ gives that there exists a matrix $Z$ such that

$$X_1 = A^k Z. \quad (2.9)$$

Write

$$Z = U \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} U^*.$$

(2.10)
Substituting (2.6) and (2.10) into (2.9), we have
\[
\begin{bmatrix}
X_{11} & X_{12} \\
X_{21} & X_{22}
\end{bmatrix} = \begin{bmatrix}
T^k Z_{11} + \hat{T} Z_{21} & T^k Z_{12} + \hat{T} Z_{22} \\
0 & 0
\end{bmatrix},
\]
so \(X_{21} = 0\) and \(X_{22} = 0\). Similarly, we get \(Y_{21} = 0\) and \(Y_{22} = 0\).

Since \((AX_1)^\sim = AX_1\),
\[
GX_1^* A^* G = AX_1. \quad (2.11)
\]
By applying (1.7), (2.4), (2.6), (2.11), \(X_{21} = 0\) and \(X_{22} = 0\), we have
\[
0 = GX_1^* A^* G - AX_1
= GU \begin{bmatrix} X_{11}^* & 0 \end{bmatrix} \begin{bmatrix} T^* & 0 \\ S^* & N^* \end{bmatrix} U^* - U \begin{bmatrix} T & S \\ 0 & N \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} \\
0 & 0 \end{bmatrix} U^*
= GU \left( X_{12}^* T^* G_1 \begin{bmatrix} X_{11}^* T^* G_1 & X_{12}^* T^* G_2 \\
X_{12}^* T^* G_1 & X_{12}^* T^* G_2 \end{bmatrix} - \begin{bmatrix} G_1 T X_{11} & G_1 T X_{12} \\
G_3 T X_{11} & G_3 T X_{12} \end{bmatrix} \right) U^*.
\]
Then \(X_{11}^* T^* G_2 = G_1 T X_{12}\), that is, \(X_{12} = T^{-1} G_1^{-1} X_{11}^* T^* G_2\). Similarly, we get \(Y_{12} = T^{-1} G_1^{-1} Y_{11}^* T^* G_2\). Applying \(X_{11} = Y_{11}\), we have \(X_{12} = Y_{12}\). Therefore, we get \(X_1 = Y_1\). \(\square\)

**Definition 2.4** Let \(A \in \mathbb{C}_{n,n}\) with \(\text{Ind}(A) = k\). If there exists \(X \in \mathbb{C}_{n,n}\) satisfying the system of equations (2.5), then \(X\) is called the m-core-EP inverse of \(A\) in \(M\), and is denoted by \(A^\oplus\).

In the following theorems of this section, using matrix decomposition, Drazin inverse, group inverse, m-core inverse and Minkowski inverse, we derive several characterizations of the m-core-EP inverse.

**Theorem 2.5** Let \(A \in \mathbb{C}_{n,n}\) with \(\text{Ind}(A) = k\). Then \(A\) is m-core-EP invertible if and only if
\[
\text{rk}(A^k) = \text{rk}((A^k)^\sim A^k). \quad (2.12)
\]
Furthermore, let the decomposition of \(A\) be as in (1.6), then \(A^\oplus\) has the decomposition in the form
\[
A^\oplus = U \begin{bmatrix} T^{-1} G_1^{-1} & 0 \\
0 & 0 \end{bmatrix} U^* G, \quad (2.13)
\]
where \(G_1\) is of the form (2.4).

**Proof** “\(\Rightarrow\)” From the equation \((2^k)\), we have \(\mathcal{R}(A^k) \subseteq \mathcal{R}(X)\). It follows from the equation \((4^r)\) that \(\mathcal{R}(A^k) = \mathcal{R}(X)\). Therefore, there exists a nonsingular \(P\) such that \(X = A^k P\).
From $XAX = XAA^k P = X (A^k P) P^{-1} A P = X^2 P^{-1} A P = X$, we get $\text{rk}(X) = \text{rk}(X^2 P^{-1} A P) \leq \text{rk}(X^2) \leq \text{rk}(X)$, that is,

$$\text{rk}(X) = \text{rk}(X^2) = \text{rk}(A^k). \quad (2.14)$$

Applying $XAX = X$ and $(AX)^\sim = AX$, we get $X^2 = XAXA^k P = XP^\sim A^\sim (A^k)^\sim A^k P$, then $\text{rk}(X^2) \leq \text{rk}((A^k)^\sim A^k) \leq \text{rk}(A^k)$.

It follows from (2.14) that we get (2.12).

“$\Leftarrow$” Let $\text{rk}(A^k) = \text{rk}((A^k)^\sim A^k)$. Applying Lemma 2.1, we get that $G_1$ is invertible. Write

$$X = U \begin{bmatrix} T^{-1} G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* G, \quad (2.15)$$

where $T \in \mathbb{C}_{r,r}$ is given in (1.7).

By applying (2.4) and (2.15), we have

$$XAX = U \begin{bmatrix} T^{-1} G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* G \begin{bmatrix} T & S \\ 0 & N \end{bmatrix} U^* \begin{bmatrix} T^{-1} G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* G = U \begin{bmatrix} T^{-1} G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix} \begin{bmatrix} T & S \\ 0 & N \end{bmatrix} \begin{bmatrix} T^{-1} G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* G = U \begin{bmatrix} T^{-1} G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* G = X.$$ 

By applying (2.2) and (2.15), then

$$X A^k X^{-1} = U \begin{bmatrix} T^{-1} G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* G \begin{bmatrix} T^{k+1} T^\ast \end{bmatrix} U^* \begin{bmatrix} T^{-1} G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* G \begin{bmatrix} T^{k+1} T^\ast \end{bmatrix} U^* = U \begin{bmatrix} T^{-1} G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix} \begin{bmatrix} T^{k+1} T^\ast \end{bmatrix} U^* = U \begin{bmatrix} T^{k+1} T^\ast \end{bmatrix} U^* = A^k.$$ 

By applying (2.4) and (2.15), we have

$$AX = U \begin{bmatrix} T & S \\ 0 & N \end{bmatrix} \begin{bmatrix} T^{-1} G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* G = U \begin{bmatrix} G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* G.$$ 

Since

$$ (AX)^\sim = G(AX)^* G = GG^* \begin{bmatrix} (G_1^{-1})^* & 0 \\ 0 & 0 \end{bmatrix} U^* G = U \begin{bmatrix} (G_1^{-1})^* & 0 \\ 0 & 0 \end{bmatrix} U^* G, \quad \text{and} \quad (Ax)^\sim = Ax^\sim A, \quad \text{where} \quad (A^\sim)^\sim = A.$$
applying Remark 2.2, we get \((AX)\sim = AX\).

Let
\[
\bar{X} = U \begin{bmatrix}
(T^{k+1})^{-1} & (T^{k+1})^{-1}G_1^{-1}G_2 \\
0 & 0
\end{bmatrix} U^*.
\]  
(2.16)

It follows from applying (2.1), (2.15), (2.16) and
\[
A^k\bar{X} = A^k U \begin{bmatrix}
(T^{k+1})^{-1} & (T^{k+1})^{-1}G_1^{-1}G_2 \\
0 & 0
\end{bmatrix} U^* = U \begin{bmatrix}
T^{-1}G_1^{-1} & 0 \\
0 & 0
\end{bmatrix} U^* G = X,
\]

that \(R(X) \subseteq R(A^k)\).

Hence, \(A\) is \(m\)-core-EP invertible, and \(A \circ \) is of the form (2.13). \(\square\)

It is well known that every complex matrix is core-EP invertible. But, from (2.12) we see that it is untrue to say that all complex matrices are \(m\)-core-EP invertible. Even when one matrix \(A\) is core-EP invertible and is \(m\)-core-EP invertible, it is generally true that the two generalized inverses are different.

**Example 2.6** Let
\[
A = \begin{bmatrix}
3-\sqrt{6}+\sqrt{3}-\sqrt{2} & \sqrt{6}+2\sqrt{3}-2\sqrt{2} & 3+\sqrt{6}-\sqrt{3}+\sqrt{2} \\
\frac{9}{6} & \frac{9}{6} & \frac{9}{6} \\
\frac{3-\sqrt{6}+\sqrt{3}+\sqrt{2}}{6} & \frac{3-\sqrt{6}+\sqrt{3}+\sqrt{2}}{6} & \frac{3-\sqrt{6}+\sqrt{3}+\sqrt{2}}{6}
\end{bmatrix}
\]

with \(\text{Ind}(A) = 2\) and \(\text{rk}(A^2) = 1\). There exists a unitary matrix
\[
U = \begin{bmatrix}
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}}
\end{bmatrix},
\]
such that
\[
A = U \begin{bmatrix}
1 & 1 & 1 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix} U^*.
\]

By applying (2.4), we have
\[
U^* G U = \begin{bmatrix}
G_1 & G_2 \\
G_3 & G_4
\end{bmatrix} = \begin{bmatrix}
0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\
\frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{3}} \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\
\frac{2}{\sqrt{2}} & -\frac{2}{3\sqrt{2}} & -\frac{2}{3\sqrt{2}}
\end{bmatrix},
\]
where \(G_1 \in \mathbb{C}_{1,1}\). Since \(G_1 = 0\) is singular, by applying Lemma 2.1 and Theorem 2.5, then \(A \circ \) does not exist.
Example 2.7 Let \( A = \begin{bmatrix} 0 & \frac{4}{3} & -\frac{1}{3} \\ -\frac{1}{3} & 1 & -\frac{1}{3} \\ -\frac{2}{3} & -\frac{2}{3} & 0 \end{bmatrix} \) with \( \text{Ind}(A) = 2 \) and \( \text{rk}(A^2) = 1 \). There exists a unitary matrix

\[
U = \begin{bmatrix}
\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{2}{3} \\
-\frac{2}{3} & -\frac{2}{3} & -\frac{1}{3}
\end{bmatrix},
\]

such that

\[
A = U \begin{bmatrix}
1 & 1 & 1 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix} U^*,
\]

\( T = 1 \). In [17, Theorem 3.2], we know that

\[
A^\dagger = U \begin{bmatrix}
T^{-1} & 0 \\
0 & 0
\end{bmatrix} U^*.
\]

Then

\[
A^\dagger = U \begin{bmatrix}
T^{-1} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} U^* = \begin{bmatrix}
\frac{4}{9} & \frac{2}{9} & -\frac{4}{9} \\
\frac{2}{9} & \frac{4}{9} & -\frac{2}{9} \\
-\frac{4}{9} & -\frac{2}{9} & \frac{4}{9}
\end{bmatrix}.
\]

By applying (2.4), we have

\[
U^* G U = \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix} = \begin{bmatrix}
-\frac{1}{9} & \frac{4}{9} & -\frac{8}{9} \\
\frac{4}{9} & \frac{7}{9} & -\frac{4}{9} \\
-\frac{8}{9} & -\frac{4}{9} & \frac{1}{9}
\end{bmatrix},
\]

where \( G_1 \in \mathbb{C}_{1,1} \). Since \( G_1 = -\frac{1}{9} \) is nonsingular, by applying Lemma 2.1 and Theorem 2.5, then \( A \) is m-core-EP invertible. Therefore,

\[
A^\ominus = U \begin{bmatrix}
T^{-1} & G_1^{-1} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} U^* G = \begin{bmatrix}
-4 & 2 & -4 \\
-2 & 1 & -2 \\
4 & -2 & 4
\end{bmatrix}.
\]

From above, we know that \( A^\dagger \neq A^\ominus \).

Lemma 2.8 [1,16] Let \( A \in \mathbb{C}_{n,n} \) with \( \text{Ind}(A) = k \), and the Drazin inverse of \( A \) is defined as the unique matrix \( X \in \mathbb{C}_{n,n} \) satisfying the equations:

\[
(1^k)A^k X A = A^k, \quad (2)X A X = X, \quad (3)A X = X A, \quad (2.17)
\]
and \( X \) is denoted by \( A^D \). In particular, when \( A \in \mathbb{C}_n^{CM} \), \( X \) is called the group inverse of \( A \), and we denote \( X = A^\sharp \).

**Lemma 2.9** [1,16] Let \( A \in \mathbb{C}_{n,n} \) with \( \text{Ind}(A) = k \), then

\[
A^D = A^k(A^{k+1})^\sharp. \tag{2.18}
\]

Let \( A = A_1 + A_2 \) be of the core-EP decomposition of \( A \), and \( A_1 \) and \( A_2 \) be as in (1.7), by applying (2.2) and (2.3), we have

\[
(A^{k+1})^\sharp = U \begin{bmatrix} (T^{k+1})^{-1} & (T^{k+1})^{-2}T \\ 0 & 0 \end{bmatrix} U^* \tag{2.19}
\]

By applying (1.14), (2.1) and (2.19), we can check that

\[
A^D = U \begin{bmatrix} T^{-1} & T^{-k-2}T \\ 0 & 0 \end{bmatrix} U^* \quad \text{and} \quad (A^k)^\boxplus = U \begin{bmatrix} (T^k)^{-1} G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* G, \tag{2.20}
\]

in which \( G_1 \) is given as in (2.4). Furthermore, by applying (2.1) and (2.20), we can obtain

\[
A^k A^D (A^k)^\boxplus = U \begin{bmatrix} T^k & \hat{T} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T^{-1} & T^{-k-2}T \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (T^k)^{-1} G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* G = U \begin{bmatrix} T^{-1} G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* G = A^\boxplus.
\]

Therefore, we get one characterization of the \( m \)-core-EP inverse.

**Theorem 2.10** Let \( A \in \mathbb{C}_{n,n} \) with \( \text{Ind}(A) = k \). If \( A \) is \( m \)-core-EP invertible, then

\[
A^{\boxplus} = A^D (A^k)^\boxplus. \tag{2.21}
\]

**Theorem 2.11** Let \( A \in \mathbb{C}_{n,n} \) with \( \text{Ind}(A) = k \). If the \( m \)-core-EP inverse exists, then we have

\[
A^{\boxplus} = A_1^{\boxplus}, \tag{2.22}
\]

where \( A_1 \) is given as in (1.6).

**Proof** Let \( A_1 \) be as in (1.6), then applying (1.7), we get

\[
A_1^{\boxplus} = U \begin{bmatrix} T^{-1} G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* G,
\]

where \( G_1 \) is given as in (2.4). By applying (2.13), we have (2.22). \( \square \)
Theorem 2.12 Let $A \in \mathbb{C}_{n,n}$ with $\text{Ind}(A) = k$ and $\text{rk}(A^k) = \text{rk}((A^k)^{\sim}A^k) = \text{rk}(A^k(A^k)^{\sim}) = r$.

Then

$$A^{\oplus} = A_1^m A_1^m. \quad (2.23)$$

**Proof** Let the core-EP decomposition of $A$ be as in (1.6), and $A_1$ and $A_2$ be as in (1.7). Then $A_1$ is group invertible, and $(A_1 + A_2)^k = A_1^k + A_1^{k-1}A_2 + A_1^{k-2}A_2^2 + \cdots + A_1 A_2^{k-1} + A_2^k$, in which $k = \text{Ind}(A)$.

Applying $\text{rk}(A^k) = \text{rk}((A^k)^{\sim}A^k) = \text{rk}(A^k(A^k)^{\sim}) = r$, we get $\text{rk}(A_1) = \text{rk}(A_1 A_1^\sim) = \text{rk}(A_1 A_2)$. It follows from (1.3) that $A_1$ is Minkowski invertible.

Write

$$\mathcal{X} = A_1^m A_1^m.$$

Applying $A_2 A_1 = 0$, $A_1^m A_1 = A_1 A_1^m$, $A_2^k = 0$, $A_1^m A_1 = A_1$ and $A_1 A_1^m A_1 = A_1$, we have

$$\mathcal{X} A^{k+1} = A_1^m A_1^m (A_1 + A_2)^{k+1}$$

$$= A_1^m A_1^m (A_1^{k+1} + A_1^{k-1} A_2 + A_1^{k-2} A_2^2 + \cdots + A_2^{k+1})$$

$$= A_1^m A_1^m (A_1^{k+1} + A_1^{k-1} A_2 + A_1^{k-2} A_2^2 + \cdots + A_2^{k-1})$$

$$= A_1^m + A_1^{k-1} A_2 + A_1^{k-2} A_2^2 + \cdots + A_1 A_2^{k-1} + A_2^k$$

$$= (A_1 + A_2)^k = A^k, \quad (2.24)$$

$$(A \mathcal{X})^{\sim} = ((A_1 + A_2) A_1^m A_1^m)^{\sim} = (A_1 A_1^m A_1^m + A_2 A_1^m A_1^m)^{\sim}$$

$$= (A_1 A_1^m A_1^m + A_2 A_1^m A_1^m)^{\sim} = (A_1^m)^{\sim} = A_1^m$$

$$= A_1 A_1^m = (A_1 + A_2) A_1^m A_1^m = A \mathcal{X} \quad (2.25)$$

and

$$\mathcal{X} A \mathcal{X} = A_1^m A_1^m (A_1 + A_2) A_1^m A_1^m = (A_1^m A_1^m A_1^m A_1^m + A_1^m A_1^m A_1^m A_2) A_1^m A_1^m$$

$$= A_1^m A_1^m A_1^m A_1^m + A_1^m A_1^m A_1^m A_2 A_1 A_1^m = \mathcal{X}. \quad (2.26)$$

By (1.6), (1.7) and (2.1), we have

$$A_1^\sim = U \begin{bmatrix} T^{-1} & T^{-2} S \\ 0 & 0 \end{bmatrix} U^*. \quad (2.27)$$

It is easy to check that

$$\mathcal{R}(\mathcal{X}) \subseteq \mathcal{R}(A^k). \quad (2.28)$$
Therefore, by applying (2.24), (2.25), (2.26), (2.28) and Definition 2.4, we get (2.23).

\[ \square \]

3 The $m$-Core-EP Decomposition

In this section, we introduce a decomposition (called $m$-core-EP decomposition) in Minkowski space, prove that the decomposition is unique, derive several characterizations of it, and apply it to study the $m$-core-EP inverse.

Let $A \in \mathbb{C}_{n \times n}$ with $\text{Ind}(A) = k$ be as in (1.6), and $\text{rk}(A^k) = \text{rk}((A^k)^{\sim} A^k)$.

Write

\[ \hat{A}_1 = U \begin{bmatrix} T & S + G_1^{-1}G_2N \\ 0 & 0 \end{bmatrix} U^* \quad \text{and} \quad \hat{A}_2 = U \begin{bmatrix} 0 & -G_1^{-1}G_2N \\ 0 & N \end{bmatrix} U^*, \]

in which $U, T, N$ and $S$ are as given in (1.7), and $G_1$ and $G_2$ are as given in (2.4). By applying (1.4) and (3.1), we obtain

\[ \hat{A}_1 \hat{A}_1 = G \hat{A}_1^* G \hat{A}_1 \]

\[ = GU \begin{bmatrix} T^* \\ S^* + N^* G_2^{-1}(G_1^{-1})^* \\ 0 \end{bmatrix} U^* \begin{bmatrix} T & S + G_1^{-1}G_2N \\ 0 & 0 \end{bmatrix} U^* \]

\[ = GU \begin{bmatrix} T^* \\ S^* + N^* G_2^{-1}(G_1^{-1})^* \\ 0 \end{bmatrix} \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix} \begin{bmatrix} T & S + G_1^{-1}G_2N \\ 0 & 0 \end{bmatrix} U^* \]

\[ = GU \begin{bmatrix} T^*G_1T \\ S^*G_1T + N^*G_2^2T \\ S^*G_1S + S^*G_2N + N^*G_2^2S + N^*G_2^2G_1^{-1}G_2N \end{bmatrix} U^*. \]

Then $\text{rk}(\hat{A}_1) = \text{rk}(T^*G_1T) \leq \text{rk}(\hat{A}_1^{\sim} \hat{A}_1) \leq \text{rk}(\hat{A}_1)$, that is,

\[ \text{rk}(\hat{A}_1^{\sim} \hat{A}_1) = \text{rk}(\hat{A}_1). \] (3.2)

Applying $N^k = 0$, we have

\[ \hat{A}_2^k = U \begin{bmatrix} 0 & -G_1^{-1}G_2N^k \\ 0 & N \end{bmatrix} U^* = U \begin{bmatrix} 0 & -G_1^{-1}G_2N^k \\ 0 & N \end{bmatrix} U^* = 0, \]

\[ \hat{A}_1 \hat{A}_2 = G \hat{A}_1^* G \hat{A}_2 \]

\[ = GU \begin{bmatrix} T^* \\ S^* + N^* G_2^{-1}(G_1^{-1})^* \\ 0 \end{bmatrix} \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix} \begin{bmatrix} 0 & -G_1^{-1}G_2N \\ 0 & N \end{bmatrix} U^* \]

\[ = GU \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} U^* = 0 \] (3.4)

and

\[ \hat{A}_2 \hat{A}_1 = U \begin{bmatrix} 0 & -G_1^{-1}G_2N \\ 0 & N \end{bmatrix} \begin{bmatrix} T & S + G_1^{-1}G_2N \\ 0 & 0 \end{bmatrix} U^* = 0. \] (3.5)
By applying (3.2)–(3.5), we get the following Theorem 3.1.

**Theorem 3.1** (The $m$-core-EP decomposition) Let $A \in \mathbb{C}_{n,n}$ with $\text{Ind}(A) = k$ and $\text{rk}(A^k) = \text{rk}((A^k)^\sim A^k) = r$. Then $A$ can be written as the sum of matrices $\widehat{A}_1$ and $\widehat{A}_2$, i.e., $A = \widehat{A}_1 + \widehat{A}_2$, where

(i) $\widehat{A}_1 \in \mathbb{C}^{CM}_{n} \text{ with } \text{rk}(\widehat{A}_1) = \text{rk}(\widehat{A}_1^\sim \widehat{A}_1)$;
(ii) $\widehat{A}_2^k = 0$;
(iii) $\widehat{A}_1^\sim \widehat{A}_2 = \widehat{A}_2 \widehat{A}_1 = 0$.

Furthermore, $\widehat{A}_1$ and $\widehat{A}_2$ have the form (3.1). Here one or both of $\widehat{A}_1$ and $\widehat{A}_2$ can be null.

**Theorem 3.2** Let $A \in \mathbb{C}_{n,n}$ with $\text{Ind}(A) = k$, and let the $m$-core-EP decomposition of $A$ be as in Theorem 3.1. Then

$$A^\oplus = \widehat{A}_1^{\oplus}.$$  \hspace{1cm} (3.6)

**Proof** Let $\widehat{A}_1$ and $\widehat{A}_2$ be as in Theorem 3.1. Applying (1.4), we have

$$\widehat{A}_2 \widehat{A}_1^{\oplus} = \widehat{A}_2 \widehat{A}_1 \left(\widehat{A}_1^{\oplus}\right)^2 = 0.$$  \hspace{1cm} (3.7)

Therefore,

$$\widehat{A}_1^{\oplus} \widehat{A}_1^{\oplus} = \widehat{A}_1^{\oplus} \widehat{A}_1 \widehat{A}_1^{\oplus} + \widehat{A}_1^{\oplus} \widehat{A}_2 \widehat{A}_1^{\oplus} = \widehat{A}_1^{\oplus}$$  \hspace{1cm} (3.8)

and

$$\left(A \widehat{A}_1^{\oplus}\right)^\sim = \left(\widehat{A}_1\widehat{A}_1^{\oplus}\right)^\sim = A \widehat{A}_1^{\oplus}.$$  \hspace{1cm} (3.9)

From $\text{Ind}(A) = k$, $\widehat{A}_2^k = 0$ and $\widehat{A}_1^{\oplus} \widehat{A}_1 = \widehat{A}_1^\# \widehat{A}_1$, we have

$$\widehat{A}_1^{\oplus} A^{k+1} = \widehat{A}_1^{\oplus} \left(\widehat{A}_1^{k+1} + \widehat{A}_1^k \widehat{A}_2 + \cdots + \widehat{A}_1 \widehat{A}_2^k + \widehat{A}_2^{k+1}\right)$$

$$= \widehat{A}_1^\# \widehat{A}_1 \left(\widehat{A}_1^{k-1} + \widehat{A}_1^{k-2} \widehat{A}_2 + \cdots + \widehat{A}_2^{k-1}\right)$$

$$= \widehat{A}_1^k + \widehat{A}_1^{k-1} \widehat{A}_2 + \cdots + \widehat{A}_1 \widehat{A}_2^{k-1} + \widehat{A}_2^k$$

$$= (\widehat{A}_1 + \widehat{A}_2)^k = A^k.$$

$$\blacksquare$$
Since \( \widehat{A}_1 \left( \widehat{A}_1^m \right)^2 = \widehat{A}_1^m \), then \( A \left( \widehat{A}_1^m \right)^2 = (\widehat{A}_1 + \widehat{A}_2) \left( \widehat{A}_1^m \right)^2 = \widehat{A}_1^m \), and
\[
A^k \left( \widehat{A}_1^m \right)^{k+1} = A^{k-1} A \left( \widehat{A}_1^m \right)^2 \left( \widehat{A}_1^m \right)^{k-1} = A^{k-2} A \left( \widehat{A}_1^m \right)^2 \left( \widehat{A}_1^m \right)^{k-2} = \cdots = A A \left( \widehat{A}_1^m \right)^2 \widehat{A}_1^m = A A \widehat{A}_1^m A \widehat{A}_1^m = \widehat{A}_1^m.
\]

Therefore,
\[
\mathcal{R} \left( \widehat{A}_1^m \right) \subseteq \mathcal{R} \left( A^k \right).
\]

Therefore, applying (3.8)–(3.11), we get (3.6).

**Remark 3.3** Let \( A \in \mathbb{C}_{n,n} \) with \( \text{Ind}(A) = k \) and \( \text{rk} \left( A^k \right) = \text{rk} \left( (A^k)^- A^k \right) \). And let \( A_1 \) and \( \widetilde{A}_1 \) be as given in Theorems 2.11 and 3.1, respectively. It is interesting to see that \( A_1^\oplus = A_1^m = A_1^\ominus \).

It can be observed from Example 2.6 where \( G_1 = 0 \) is singular that, after applying Lemma 2.1 and Theorem 2.5, a matrix has a core-EP decomposition, but it not necessary has \( m \)-core-EP inverse or \( m \)-core-EP decomposition. The matrix has a \( m \)-core-EP decomposition, if and only if the \( m \)-core-EP inverse exists. Therefore, \( m \)-core-EP decomposition is different from core-EP decomposition.

**Example 3.4** Let \( A = \begin{bmatrix} 16+4\sqrt{5} & 2+8\sqrt{5} & 10-8\sqrt{5} \\ -8+3\sqrt{5} & -1+6\sqrt{5} & -5-6\sqrt{5} \\ \sqrt{5} & 2\sqrt{5} & -2\sqrt{5} \end{bmatrix} \) with \( \text{Ind}(A) = 2 \) and \( \text{rk}(A^2) = 1 \). There exists a unitary matrix
\[
U = \begin{bmatrix} 2 & 2 & 1 \\ \sqrt{5} & 3\sqrt{5} & 3 \sqrt{5} \\ 0 & 3 & -2 \end{bmatrix},
\]
such that
\[
A = U \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} U^*.
\]

By applying (2.4), we have
\[
U^* GU = \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix} = \begin{bmatrix} 3 & \frac{8}{3} & \frac{4}{3} \\ \frac{5}{3} & \frac{15}{4} & \frac{15}{4} \\ \frac{15}{4} & \frac{45}{9} & \frac{9}{5} \end{bmatrix},
\]
where $G_1 \in \mathbb{C}_{1,1}$. Since $\frac{3}{2}$ is nonsingular, by applying Lemma 2.1 and Theorem 2.5, we observe that $A$ is $m$-core-EP invertible. Then

$$
A^\oplus = \begin{bmatrix}
\frac{2}{3} & \frac{2}{\sqrt{3}} & \frac{1}{3} \\
-\frac{1}{3} & \frac{2}{3} & \frac{3}{2} \\
0 & \frac{2}{\sqrt{3}} & -\frac{2}{3}
\end{bmatrix}
\begin{bmatrix}
5 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\frac{1}{3} & -\frac{1}{3} & 0 \\
\frac{1}{3} & \frac{2}{3} & \frac{5}{2} \\
0 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{bmatrix}
= \begin{bmatrix}
\frac{4}{3} & \frac{2}{3} & 0 \\
-\frac{2}{3} & -\frac{1}{3} & 0 \\
0 & 0 & 0
\end{bmatrix}.
$$

$$
A_1 = \begin{bmatrix}
\frac{2}{3} & \frac{2}{\sqrt{3}} & \frac{1}{3} \\
-\frac{1}{3} & \frac{2}{3} & \frac{3}{2} \\
0 & \frac{2}{\sqrt{3}} & -\frac{2}{3}
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\frac{1}{3} & -\frac{1}{3} & 0 \\
\frac{1}{3} & \frac{2}{3} & \frac{5}{2} \\
0 & 0 & -1
\end{bmatrix}
= \begin{bmatrix}
16+2\sqrt{5} & 2+4\sqrt{5} & 10-4\sqrt{5} \\
-8-\sqrt{5} & -1-2\sqrt{5} & -5+2\sqrt{5} \\
0 & 0 & 0
\end{bmatrix}.
$$

$$
A_1^{\oplus} = \begin{bmatrix}
\frac{2}{3} & \frac{2}{\sqrt{3}} & \frac{1}{3} \\
-\frac{1}{3} & \frac{2}{3} & \frac{3}{2} \\
0 & \frac{2}{\sqrt{3}} & -\frac{2}{3}
\end{bmatrix}
\begin{bmatrix}
5 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\frac{1}{3} & -\frac{1}{3} & 0 \\
\frac{1}{3} & \frac{2}{3} & \frac{5}{2} \\
0 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{bmatrix}
= \begin{bmatrix}
\frac{4}{3} & \frac{2}{3} & 0 \\
-\frac{2}{3} & -\frac{1}{3} & 0 \\
0 & 0 & 0
\end{bmatrix}.
$$

$$
\bar{A}_1 = \begin{bmatrix}
\frac{2}{3} & \frac{2}{\sqrt{3}} & \frac{1}{3} \\
-\frac{1}{3} & \frac{2}{3} & \frac{3}{2} \\
0 & \frac{2}{\sqrt{3}} & -\frac{2}{3}
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\frac{1}{3} & -\frac{1}{3} & 0 \\
\frac{1}{3} & \frac{2}{3} & \frac{5}{2} \\
0 & 0 & -1
\end{bmatrix}
= \begin{bmatrix}
\frac{48+22\sqrt{5}}{45} & \frac{6+11\sqrt{5}}{45} & \frac{30-44\sqrt{5}}{45} \\
\frac{24-11\sqrt{5}}{45} & -\frac{3-22\sqrt{5}}{45} & -\frac{15+22\sqrt{5}}{45} \\
0 & 0 & 0
\end{bmatrix}.
$$

$$
\bar{A}_1^{\oplus} = \begin{bmatrix}
\frac{2}{3} & \frac{2}{\sqrt{3}} & \frac{1}{3} \\
-\frac{1}{3} & \frac{2}{3} & \frac{3}{2} \\
0 & \frac{2}{\sqrt{3}} & -\frac{2}{3}
\end{bmatrix}
\begin{bmatrix}
5 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\frac{1}{3} & -\frac{1}{3} & 0 \\
\frac{1}{3} & \frac{2}{3} & \frac{5}{2} \\
0 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{bmatrix}
= \begin{bmatrix}
\frac{4}{3} & \frac{2}{3} & 0 \\
-\frac{2}{3} & -\frac{1}{3} & 0 \\
0 & 0 & 0
\end{bmatrix}.
$$

It can be observed from above that $\tilde{A}_1 \neq A_1$. However, $A^{\oplus} = \tilde{A}_1^{\oplus} = A_1^{\oplus}$.

**Theorem 3.5** Let $A \in \mathbb{C}_{n,n}$ with $\text{Ind}(A) = k$ and $\text{rk}(A^k) = \text{rk}((A^k)^\sim A^k) = r$. The $m$-core-EP decomposition of $A$ is unique.

**Proof** Suppose that $A = \tilde{A}_1 + \tilde{A}_2$ is the $m$-core-EP decomposition of $A$. Let $A = \tilde{B}_1 + \tilde{B}_2$ be another $m$-core-EP decomposition of $A$. By Theorem 3.2, we know that

$$
\tilde{A}_1^{\oplus} = \tilde{B}_1^{\oplus} = A^{\oplus}. \quad (3.12)
$$

Premultiplying both sides of (3.12) with $A$, then

$$
\tilde{A}_1 \tilde{A}_1^{\oplus} + \tilde{A}_2 \tilde{A}_2^{\oplus} = \tilde{B}_1 \tilde{B}_1^{\oplus} + \tilde{B}_2 \tilde{B}_2^{\oplus}. \quad (3.13)
$$

Since $\tilde{A}_2 \tilde{A}_2^{\oplus} = 0$ and $\tilde{B}_2 \tilde{B}_2^{\oplus} = 0$, we get

$$
\tilde{A}_1 \tilde{A}_1^{\oplus} = \tilde{B}_1 \tilde{B}_1^{\oplus}. \quad (3.14)
$$

Postmultiplying both sides of (3.14) with $A$, then

$$
\tilde{A}_1 \tilde{A}_1^{\oplus} \tilde{A}_1 + \tilde{A}_1 \tilde{A}_1^{\oplus} \tilde{A}_2 = \tilde{B}_1 \tilde{B}_1^{\oplus} \tilde{B}_1 + \tilde{B}_1 \tilde{B}_1^{\oplus} \tilde{B}_2. \quad (3.15)
$$

Because the $m$-core inverse $\tilde{A}_1^{\oplus}$ satisfies $\tilde{A}_1^{\oplus} \tilde{A}_1^{\oplus} \tilde{A}_1 = \tilde{A}_1^{\oplus}$, applying $\tilde{A}_1 \tilde{A}_1^{\oplus} = (\tilde{A}_1 \tilde{A}_1^{\oplus})^\sim$, we get $\tilde{A}_1^{\oplus} \tilde{A}_2 = 0$. In the same way, we have $\tilde{B}_1^{\oplus} \tilde{B}_2 = 0$. 

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It follows that \( \hat{A}_1 = \hat{B}_1 \), that is, the \( m \)-core-EP decomposition of a given matrix is unique.

\[ \hat{A}_1 = A^k (A^k)^{\oplus} A \quad \text{and} \quad \hat{A}_2 = A - A^k (A^k)^{\oplus} A. \]  

(3.16)

**Proof** By applying (1.6), (2.1) and (2.4), we have

\[
A^k(A^k)^{\oplus} A = U \begin{bmatrix} T^k & \hat{T} \\ 0 & 0 \end{bmatrix} U^* U \begin{bmatrix} T^{-k}G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* GU \begin{bmatrix} T & S \\ 0 & N \end{bmatrix} U^*
\]

\[
= U \begin{bmatrix} T^k & \hat{T} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T^{-k}G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix} \begin{bmatrix} T & S \\ 0 & N \end{bmatrix} U^*
\]

\[
= U \begin{bmatrix} T & S + G_1^{-1}G_2N \\ 0 & 0 \end{bmatrix} U^* = \hat{A}_1,
\]

where \( G_i (i = 1, 2, 3, 4) \) are given as in (2.4). Since \( A = \hat{A}_1 + \hat{A}_2, \hat{A}_2 = A - \hat{A}_1 = A - A^k (A^k)^{\oplus} A \). Therefore, we get (3.16).

\[ \hat{A}_1 = AA^{\oplus} A \quad \text{and} \quad \hat{A}_2 = A - AA^{\oplus} A. \]  

(3.17)

**Proof** By applying (1.6) and (2.13), we have

\[
AA^{\oplus} A = U \begin{bmatrix} T & S \\ 0 & N \end{bmatrix} U^* U \begin{bmatrix} T^{-1}G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* GU \begin{bmatrix} T & S \\ 0 & N \end{bmatrix} U^*
\]

\[
= U \begin{bmatrix} T & S \\ 0 & N \end{bmatrix} \begin{bmatrix} T^{-1}G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix} \begin{bmatrix} T & S \\ 0 & N \end{bmatrix} U^*
\]

\[
= U \begin{bmatrix} T & S + G_1^{-1}G_2N \\ 0 & 0 \end{bmatrix} U^* = \hat{A}_1,
\]

where \( G_i (i = 1, 2, 3, 4) \) are given as in (2.4). Since \( A = \hat{A}_1 + \hat{A}_2, \hat{A}_2 = A - \hat{A}_1 = A - AA^{\oplus} A \). Therefore, we get (3.17).

**4 The \( m \)-Core-EP Order**

In [18], Wang et al. considered the \( m \)-core partial order by applying the \( m \)-core inverse in \( M \), which is characterized by

\[
A \overset{\oplus}{\leq} B \Leftrightarrow A^{\ominus} A = A^{\ominus} B \quad \text{and} \quad AA^{\ominus} = BA^{\ominus}.
\]  

(4.1)
Furthermore, it has following property.

**Lemma 4.1** [18] Let \( A, B \in \mathbb{C}_{n}^{m} \) with \( \text{rk} (A \sim A) = \text{rk} (A) = r > 0 \) and \( \text{rk} (B \sim B) = \text{rk} (B) = s \geq r \). If \( A \preceq B \), then

\[
A \circledast B \circledast m = A \circledast m \quad \text{and} \quad B \circledast m B A \circledast m = A \circledast m. \tag{4.2}
\]

In this section, we introduce a new order (called the \( m \)-core-EP order), consider its properties and get some characterizations of it. It is true that the \( m \)-core-EP order is a generalization of the \( m \)-core partial order, but it is a pre-order not a partial order. Let \( A, B \in \mathbb{C}_{n,n} \), with \( \text{rk}(A^{k}) = \text{rk}((A^{k}) \sim A^{k}) = r \). We define the \( m \)-core-EP order, which is characterized by

\[
A \preceq B \iff A \circledast B = A \circledast B \quad \text{and} \quad A \circledast B = B \circledast A \circledast B. \tag{4.3}
\]

**Theorem 4.2** Let \( A, B \in \mathbb{C}_{n,n} \), \( \text{Ind}(A) = k \), \( \text{Ind}(B) = t \), \( \text{rk}(A^{k}) = \text{rk}((A^{k}) \sim A^{k}) = r \) and \( \text{rk}(B^{t}) = \text{rk}((B^{t}) \sim B^{t}) = s \geq r \). If \( A \preceq B \), then there exists a unitary matrix \( \hat{U} \) such that

\[
A = \hat{U} \begin{bmatrix} T & S_{1} \\ S_{1} & N_{11} \end{bmatrix} \begin{bmatrix} S_{2} & S_{2} \end{bmatrix} \hat{U}^{*},
\]

\[
B = \hat{U} \begin{bmatrix} T & S_{1} + G_{1}^{-1} \begin{bmatrix} G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} N_{11} - \hat{T} \\ \hat{N}_{13} \end{bmatrix} S_{2} + G_{1}^{-1} \begin{bmatrix} G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} \hat{N}_{12} - \hat{S} \\ \hat{N}_{14} - \hat{N} \end{bmatrix} \end{bmatrix} \hat{U}^{*},
\]

where \( \begin{bmatrix} \hat{N}_{11} & \hat{N}_{12} \\ \hat{N}_{13} & \hat{N}_{14} \end{bmatrix} \in \mathbb{C}_{n-r,n-r} \) and \( \hat{N} \in \mathbb{C}_{n-s,n-s} \) are nilpotent, and \( T \in \mathbb{C}_{r,r} \), \( \hat{T} \in \mathbb{C}_{s-r,s-r} \), \( G_{1} \in \mathbb{C}_{r,r} \), and \( \hat{G}_{1} = \begin{bmatrix} G_{1} & G_{21} \\ G_{31} & G_{41} \end{bmatrix} \in \mathbb{C}_{s,s} \) are invertible, and

\[
\hat{U}^{*} \hat{G} \hat{U} = \begin{bmatrix} G_{1} & G_{21} & G_{22} \\ G_{31} & G_{41} & G_{42} \\ G_{32} & G_{43} & G_{44} \end{bmatrix}.
\]

**Proof** Let \( A \) be of the form (1.6). Applying (4.3) and Theorem 2.5, we observe that

\[
AA \circledast = U \begin{bmatrix} G_{1}^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^{*} G \quad \text{and} \quad A \circledast A = U \begin{bmatrix} I_{r} & T^{-1} S + T^{-1} G_{1}^{-1} G_{2} N \\ 0 & 0 \end{bmatrix} U^{*}.
\]

Suppose that \( U^{*} B U \in \mathbb{C}_{n,n} \) is partitioned as

\[
U^{*} B U = \begin{bmatrix} Q & Z \\ P & M \end{bmatrix}.
\]
where $Q \in \mathbb{C}_{r,r}$. Applying (2.4), we give

$$BA^\oplus = U \begin{bmatrix} Q(G_1T)^{-1} & 0 \\ M(G_1T)^{-1} & 0 \end{bmatrix} U^*G,$$

$$A^\oplus B = U \begin{bmatrix} T^{-1}Q + (G_1T)^{-1}G_2M & T^{-1}Z + (G_1T)^{-1}G_2P \\ 0 & 0 \end{bmatrix}.$$ 

Since $AA^\oplus = BA^\oplus$ and $A^\oplus A = A^\oplus B$, we derive $Q = T$, $M = 0$ and $Z = S - G_1^{-1}G_2P + G_1^{-1}G_2N$. Therefore,

$$B = U \begin{bmatrix} T & S - G_1^{-1}G_2P + G_1^{-1}G_2N \\ 0 & P \end{bmatrix} U^*.$$  \hfill (4.7)

Let

$$P = U_1 \begin{bmatrix} \tilde{T} & \tilde{S} \\ 0 & \tilde{N} \end{bmatrix} U_1^*,$$  \hfill (4.8)

where $\tilde{T}$ is invertible, $\tilde{N}$ is nilpotent, and $U_1$ is unitary. Denote

$$\hat{U} = U \begin{bmatrix} I_r & 0 \\ 0 & U_1 \end{bmatrix}.$$ 

Applying (1.6) and (1.7), we have

$$A = \hat{U} \begin{bmatrix} T & SU_1 \\ 0 & U_1^*NU_1 \end{bmatrix} \hat{U}^* \text{ and } N = U_1 \begin{bmatrix} \tilde{N}_{11} & \tilde{N}_{12} \\ \tilde{N}_{13} & \tilde{N}_{14} \end{bmatrix} U_1^*.$$ \hfill (4.9)

Applying (4.9) and (2.4), we have

$$\hat{U}^*G\hat{U} = \begin{bmatrix} I_r & 0 \\ 0 & U_1^* \end{bmatrix} U^*GU \begin{bmatrix} I_r & 0 \\ 0 & U_1 \end{bmatrix} = \begin{bmatrix} G_1 & G_2U_1 \\ U_1^*G_3 & U_1^*G_4U_1 \end{bmatrix}.$$ \hfill (4.10)

Furthermore, denote

$$SU_1 = \begin{bmatrix} S_1 & S_2 \end{bmatrix}, \quad G_2U_1 = \begin{bmatrix} G_{21} & G_{22} \end{bmatrix},$$

$$U_1^*G_3 = \begin{bmatrix} G_{31} \\ G_{32} \end{bmatrix}, \quad \text{ and } \quad U_1^*G_4U_1 = \begin{bmatrix} G_{41} & G_{42} \\ G_{43} & G_{44} \end{bmatrix},$$ \hfill (4.11)

where $G_{41} \in \mathbb{C}_{s-r,s-r}$. Substituting (4.11) into (4.10), we obtain (4.6).

By applying Lemma 2.1 to $\text{rk} \left( (A^k) \sim A^k \right) = \text{rk} \left( A^k \right)$ and $\text{rk} \left( (B^t) \sim B^t \right) = \text{rk} \left( B^t \right)$, we conclude that $G_1 \in \mathbb{C}_{r,r}$, $\begin{bmatrix} G_1 & G_{21} \\ G_{31} & G_{41} \end{bmatrix} \in \mathbb{C}_{s,s}$ and $\tilde{T} \in \mathbb{C}_{s-r,s-r}$ are invertible.
Substituting (4.11) and (4.8) into (4.7), and \(G_1\) is given as in (2.4), we obtain

\[
B = U \begin{bmatrix}
T & S - G_1^{-1} G_2 U_1 \begin{bmatrix}
\tilde{T} \\
0
\end{bmatrix} \begin{bmatrix} S \\ N \end{bmatrix} \end{bmatrix}
\begin{bmatrix} U_1^* \\ 0 \end{bmatrix} + G_1^{-1} G_2 \begin{bmatrix} U_1^* \\ 0 \end{bmatrix} \\
0 & \begin{bmatrix}
\tilde{T} \\
0
\end{bmatrix} \begin{bmatrix} S \\ N \end{bmatrix} \end{bmatrix}
= \hat{U} \begin{bmatrix}
T \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} - G_1^{-1} \begin{bmatrix} G_{21} \\ G_{22} \end{bmatrix} \begin{bmatrix}
\tilde{T} \\
0
\end{bmatrix} \begin{bmatrix} S \\ N \end{bmatrix} + G_1^{-1} (G_2 U_1) (U_1^* N U_1) \\
0 & \begin{bmatrix}
\tilde{T} \\
0
\end{bmatrix} \begin{bmatrix} S \\ N \end{bmatrix} \end{bmatrix}
\end{bmatrix} \hat{U}^*,
\]

that is, (4.5).

**Theorem 4.3** Let \(A, B \in \mathbb{C}_{n,n}\), \(\text{Ind}(A) = k\), \(\text{Ind}(B) = t\), \(\text{rk}(A^k) = \text{rk}((A^k)\sim A^k) = r\) and \(\text{rk}(B^t) = \text{rk}((B^t)\sim B^t) = s \geq r\). Then

\[
A \preceq B \iff A^{k+1} = B A^k \text{ and } A^{-} A^k = B^{-} A^k.
\]

**Proof** \(\leftarrow\leftarrow\) Let \(A\) be of the form (1.6). Applying (2.13), we have

\[
(A^{k+1}) \oplus = U \begin{bmatrix}
(T^{k+1})^{-1} G_1^{-1} & 0 \\
0 & 0
\end{bmatrix} U^* G, \quad A \oplus = U \begin{bmatrix}
T^{-1} G_1^{-1} & 0 \\
0 & 0
\end{bmatrix} U^* G
\]

and

\[
A^k (A^{k+1}) \ominus = U \begin{bmatrix}
T^k & \hat{T} \\
0 & 0
\end{bmatrix} U^* U \begin{bmatrix}
(T^{k+1})^{-1} G_1^{-1} & 0 \\
0 & 0
\end{bmatrix} U^* G = A \ominus.
\]

Since \(A^{k+1} (A^{k+1}) \ominus = A A^k (A^{k+1}) \ominus = B A^k (A^{k+1}) \ominus\), we get \(AA \ominus = BA \ominus\). Since \(A^{-} A^k = B^{-} A^k\) and \(A^{-} A^k (A^{k+1}) \ominus = B^{-} A^k (A^{k+1}) \ominus\), we get \(A^{-} A \ominus = B^{-} A \ominus\). Taking the Minkowski transpose of both sides, we have \((A \ominus) A = (A \ominus) B\). Premultiplying both sides by \(A \ominus A\), then \(A \ominus A (A \ominus) A = A \ominus A (A \ominus) A\). From

\[
A^{-} = G A \ast^* G = G U \begin{bmatrix}
T^* & 0 \\
S^* & N^*
\end{bmatrix} U^* G,
\]

\[
(A \ominus)^{-} = G G^* U \begin{bmatrix}
(T^{-1} G_1^{-1})^* & 0 \\
0 & 0
\end{bmatrix} U^* G = U \begin{bmatrix}
(T^{-1} G_1^{-1})^* & 0 \\
0 & 0
\end{bmatrix} U^* G.
\]
and

\[ A \bowtie A \sim (A \bowtie)^{\sim} = U \begin{bmatrix} T^{-1}G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* G U \begin{bmatrix} T^* & 0 \\ S^* & N^* \end{bmatrix} \]

\[ U^* G \begin{bmatrix} (T^{-1}G_1^{-1})^* & 0 \\ 0 & 0 \end{bmatrix} U^* G \]

\[ = U \begin{bmatrix} T^{-1}G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T^* & 0 \\ S^* & N^* \end{bmatrix} \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix} \begin{bmatrix} (T^{-1}G_1^{-1})^* & 0 \\ 0 & 0 \end{bmatrix} U^* G \]

\[ = U \begin{bmatrix} T^{-1}G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* G = A \bowtie, \]

we get \( A \bowtie A = A \bowtie B. \)

\( \Rightarrow \) Let \( A = A_1 + A_2 \) be the core-EP decomposition of \( A \), and \( A_1 \) and \( A_2 \) be as in (1.7). Applying (4.3), we have \( B \) of the form (4.7). Then

\[ B A^k = U \begin{bmatrix} T & S - G_1^{-1}G_2P + G_1^{-1}G_2N \\ 0 & P \end{bmatrix} U^* U \begin{bmatrix} T^k & 0 \\ 0 & 0 \end{bmatrix} U^* \]

\[ = U \begin{bmatrix} T^{k+1} & 0 \\ 0 & 0 \end{bmatrix} U^* = U \begin{bmatrix} T^{k+1} & 0 \\ 0 & 0 \end{bmatrix} U^* = A^{k+1}, \]

\[ A^{-1} A^k = GU \begin{bmatrix} T^* & 0 \\ S^* & N^* \end{bmatrix} U^* GU \begin{bmatrix} T^* & 0 \\ S^* & N^* \end{bmatrix} U = GU \begin{bmatrix} T^* & 0 \\ S^* & N^* \end{bmatrix} \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix} \begin{bmatrix} T^* & 0 \\ 0 & 0 \end{bmatrix} U^* \]

\[ = GU \begin{bmatrix} T^* G_1^T & T^* G_1^T \\ S^* G_1^T + N^* G_3^T & S^* G_1^T + N^* G_3^T \end{bmatrix} U^*, \]

and

\[ B^{-1} A^k = GU \begin{bmatrix} T^* & 0 \\ (S - G_1^{-1}G_2P + G_1^{-1}G_2N)^* & P^* \end{bmatrix} U^* GU \begin{bmatrix} T^k & 0 \\ 0 & 0 \end{bmatrix} U^* \]

\[ = GU \begin{bmatrix} T^* & 0 \\ (S - G_1^{-1}G_2P + G_1^{-1}G_2N)^* & P^* \end{bmatrix} \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix} \begin{bmatrix} T^k & 0 \\ 0 & 0 \end{bmatrix} U^* \]

\[ = GU \begin{bmatrix} T^* G_1^T & T^* G_1^T \\ S^* G_1^T + N^* G_2^T & S^* G_1^T + N^* G_2^T \end{bmatrix} U^*. \]

By applying Remark 2.2, we have \( A \sim A^k = B \sim A^k. \)

\( \square \)

**Theorem 4.4** Let \( A, B \in \mathbb{C}_{n,n} \), Ind(\( A \)) = \( k \), Ind(\( B \)) = \( t \), \( \text{rk} \ (A^k) = \text{rk} \ ((A \sim)^{\sim} A^k) = r \) and \( \text{rk} \ (B^t) = \text{rk} \ ((B \sim)^{\sim} B^t) = s \geq r. \) Then

\[ A \bowtie B \Leftrightarrow \hat{A}_1 \bowtie \hat{B}_1, \]

where \( A = \hat{A}_1 + \hat{A}_2 \) and \( B = \hat{B}_1 + \hat{B}_2 \) are the m-core-EP decompositions of \( A \) and \( B \), respectively.
Proof Let $A$ be as in (1.6). Applying (4.6) and (2.13), then

$$\widehat{A}_1 = AA^\oplus A$$

$$= \hat{U} \begin{bmatrix} T & S_1 & S_2 \\ 0 & \widehat{N}_{11} & \widehat{N}_{12} \\ 0 & \widehat{N}_{13} & \widehat{N}_{14} \end{bmatrix} \begin{bmatrix} T^{-1}G_1^{-1} & 0 & 0 \\ 0 & G_1 & G_2 \\ 0 & G_3 & G_4 \end{bmatrix} \begin{bmatrix} T & S_1 & S_2 \\ 0 & \widehat{N}_{11} & \widehat{N}_{12} \\ 0 & \widehat{N}_{13} & \widehat{N}_{14} \end{bmatrix} \hat{U}^*$$

$$= \hat{U} \begin{bmatrix} T & \beta \hat{S} + G_1^{-1} \hat{G}_2 \hat{N} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \hat{U}^*.$$ 

and

$$\widehat{A}_2 = A - \widehat{A}_1$$

$$= \hat{U} \begin{bmatrix} 0 & -G_1^{-1}(G_2 \widehat{N}_{11} + G_2 \widehat{N}_{13}) & -G_1^{-1}(G_2 \widehat{N}_{12} + G_2 \widehat{N}_{14}) \\ 0 & 0 & 0 \\ 0 & 0 \end{bmatrix} \hat{U}^*.$$ 

Write

$$\alpha = S_1 + G_1^{-1}(G_2 \widehat{N}_{11} - G_2 \widehat{T} + G_2 \widehat{N}_{13}),$$

$$\beta = S_2 + G_1^{-1}(G_2 \widehat{N}_{12} - G_2 \widehat{S} + G_2 \widehat{N}_{14} - G_2 \widehat{N}).$$

Let $B$ be as in (4.5). Applying (2.13), we have

$$B^\oplus = \hat{U} \begin{bmatrix} T^{-1} - T^{-1} \alpha \hat{T}^{-1} & 0 \\ 0 & 0 \end{bmatrix} \hat{G}_1^{-1} \hat{G}_2 \hat{N} \hat{U}^* G.$$ 

(4.12)

By applying (4.5) and (4.12), we obtain

$$\widehat{B}_1 = BB^\oplus B$$

$$= \hat{U} \begin{bmatrix} T & \alpha & \beta \\ 0 & \hat{T} & \hat{S} \\ 0 & \hat{N} \end{bmatrix} \hat{U}^* \begin{bmatrix} T^{-1} - T^{-1} \alpha \hat{T}^{-1} & 0 \\ 0 & 0 \end{bmatrix} \hat{G}_1^{-1} \hat{G}_2 \hat{N} \hat{U}^*$$

$$= \hat{U} \begin{bmatrix} T & \alpha \hat{T} & \beta \\ 0 & 0 & \hat{S} \hat{N} \end{bmatrix} \hat{U}^*$$

and

$$\widehat{B}_2 = B - \widehat{B}_1 = \hat{U} \begin{bmatrix} 0 & -\hat{G}_1^{-1} \hat{G}_2 \hat{N} \\ 0 & \hat{N} \end{bmatrix} \hat{U}^*.$$ 

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Applying

\[ \hat{A}_1 \Theta = \hat{U} \begin{bmatrix} T^{-1}G_1^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \hat{U}^* G, \]

we get

\[ \hat{A}_1 \Theta \hat{B}_2 = \hat{U} \begin{bmatrix} T^{-1}G_1^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \hat{U}^* G \begin{bmatrix} 0 & -\hat{G}_1^{-1} \hat{G}_2 \hat{N} \\ \hat{G}_1 \hat{G}_2 \hat{G}_3 \hat{G}_4 & 0 & -\hat{G}_1^{-1} \hat{G}_2 \hat{N} \end{bmatrix} \hat{U}^* = 0, \]

\[ \hat{B}_2 \hat{A}_1 \Theta = \hat{U} \begin{bmatrix} 0 & -\hat{G}_1^{-1} \hat{G}_2 \hat{N} \\ 0 & 0 & 0 \end{bmatrix} \hat{U}^* \begin{bmatrix} T^{-1}G_1^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \hat{U}^* G = 0, \]

\[ \hat{A}_1 \Theta \hat{A}_2 = \hat{U} \begin{bmatrix} T^{-1}G_1^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \hat{U}^* G \begin{bmatrix} 0 & -\hat{G}_1^{-1}(G_2 \hat{N}_{11} + G_2 \hat{N}_{13}) - \hat{G}_1^{-1}(G_2 \hat{N}_{12} + G_2 \hat{N}_{14}) \\ \hat{G}_1 \hat{G}_2 \hat{G}_3 \hat{G}_4 & 0 & -\hat{G}_1^{-1}(G_2 \hat{N}_{11} + G_2 \hat{N}_{13}) - \hat{G}_1^{-1}(G_2 \hat{N}_{12} + G_2 \hat{N}_{14}) \end{bmatrix} \hat{U}^* = 0, \]

and

\[ \hat{A}_1 \Theta \hat{A}_3 \Theta = \hat{U} \begin{bmatrix} 0 & -\hat{G}_1^{-1}(G_2 \hat{N}_{11} + G_2 \hat{N}_{13}) - \hat{G}_1^{-1}(G_2 \hat{N}_{12} + G_2 \hat{N}_{14}) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \hat{U}^* \begin{bmatrix} T^{-1}G_1^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \hat{U}^* G = 0. \]
Since $A \leq B$ and $A^E = A^1$, then $\hat{A}^1 = \hat{A}^1B$ and $A\hat{A}^1 = B\hat{A}^1$. For $A = \hat{A}_1 + \hat{A}_2$ and $B = \hat{B}_1 + \hat{B}_2$, we have

\[
\begin{align*}
\hat{A}^1 & = \hat{A}_1 + \hat{A}_2 = \hat{A}_1^m \hat{B}_1 + \hat{A}_2^m \hat{B}_2 \\
\hat{A}^1 & = \hat{A}_1 + \hat{A}_2 = \hat{B}_1^m \hat{A}_1 + \hat{B}_2^m \hat{A}_2.
\end{align*}
\]

It follows that $\hat{A}_1^m \hat{A}_1 = \hat{A}_1^m \hat{B}_1$ and $\hat{A}^1 \hat{A}_1^m = \hat{B}_1 \hat{A}_1^m$, that is, $\hat{A}_1 \leq \hat{B}_1$.

‘$\leftarrow$’ Since $\hat{A}_1 \leq \hat{B}_1$, by applying (4.1), we have

\[
\hat{A}_1^m \hat{A}_1 = \hat{A}_1^m \hat{B}_1 \quad \text{and} \quad \hat{A}^1 \hat{A}_1^m = \hat{B}_1 \hat{A}_1^m.
\]

By applying (3.7), (4.2) and Theorem 3.1, we have $\hat{A}_2 \hat{A}_1^m = 0$, $\hat{A}_1^m \hat{A}_2 = \hat{A}_1^m \hat{A}_1^m \hat{A}_2 = \hat{A}_1^m (\hat{A}_1^m \hat{A}_1^m) \hat{A}_2 = \hat{A}_1^m (\hat{A}_1^m \hat{A}_1^m) \hat{A}_2 = 0$, $\hat{A}_1^m \hat{B}_2 = \hat{A}_1^m \hat{B}_1 \hat{B}_2 = \hat{A}_1^m (\hat{B}_1 \hat{B}_1^m) \hat{B}_2 = \hat{A}_1^m (\hat{B}_1 \hat{B}_1^m) \hat{B}_2 = 0$ and $\hat{B}_2 \hat{A}_1^m = \hat{B}_2 \hat{B}_1 \hat{A}_1^m = 0$. Furthermore, by applying (3.6), we have $A^E \hat{A}_1 = A^E B_1$, $\hat{A}_1 A^E = \hat{B}_1 A^E$, $A^E A^E = \hat{A}_1 A^E A^E = 0$, $A^E B_2 = \hat{A}_1 \hat{B}_2 = 0$, $\hat{A}_2 A^E = \hat{A}_2 A^E = 0$ and $\hat{B}_2 A^E = \hat{B}_2 A^E = 0$. Therefore, it follows that $A^E A^E = A^E \hat{A}_1 + A^E \hat{A}_2 = A^E \hat{A}_1 = A^E \hat{B}_1 = A^E \hat{B}_2 = A^E B_2 = A^E B$ and $A^E A^E = \hat{A}_1 A^E + \hat{A}_2 A^E = \hat{A}_1 A^E = \hat{B}_1 A^E = \hat{B}_2 A^E = A B$, that is, $A \leq B$.

\[\Box\]

Example 4.5 Let

\[
A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

Then $A \leq B$ and $B \leq A$. However, $A \neq B$. Therefore, the m-core-EP order is not antisymmetric.

Theorem 4.6 The m-core-EP order is not a partial order but merely a pre-order.

Proof Reflexivity of the relation is obvious. Suppose $A \leq B$ and $B \leq C$, in which $A = \hat{A}_1 + \hat{A}_2$, $B = \hat{B}_1 + \hat{B}_2$ and $C = \hat{C}_1 + \hat{C}_2$ are the m-core-EP decomposition of $A$, $B$ and $C$, respectively. Then $\hat{A}_1 \leq \hat{B}_1$ and $\hat{B}_1 \leq \hat{C}_1$. Therefore, $\hat{A}_1 \leq \hat{C}_1$. By applying Theorem 4.4, we have $A \leq C$.

\[\Box\]

Acknowledgements The authors wish to extend their sincere gratitude to the referees for their precious comments and suggestions.

Funding H. Wang was supported partially by the Research Fund Project of Guangxi University for Nationalities (no. 2019KJQD03), Guangxi Natural Science Foundation (no. 2018GXNSFAA138181) and the Special Fund for Bagui Scholars of Guangxi (no. 2016A17). H. Wu was supported partially by the
Declarations

Disclosure statement No potential conflict of interest was reported by the authors.

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