Application of Noether’s theorem to extended particles

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Abstract. We consider an approach similar to field theory for the application of Noether theorem to extended particles. We obtain Euler-Lagrange equations for the extended particles as well as an equation binding the internal and external currents. The concrete case of spin 1/2 is considered in detail.

1. Introduction
Previously, we have constructed a model of extended particles based on a fiber bundle structure ([1] and references therein). The extended particle is represented by a bilocal field $\psi(x, \xi)$ where $x$ belongs to the base manifold and $\xi$ to a fiber. The base has been interpreted as the usual (external) space-time which may be Riemannian while the fiber corresponds to an internal space-time accounting for the extension of the particles and having a definite symmetry (Galilei, Minkowski, or de Sitter symmetry). The bilocal field $\psi(x, \xi)$ describes an external quantum mode located at $x$ and an internal one located at $\xi$. Both modes compose the particle as in the two-body problem ($x$ being the center of mass and $\xi$ the relative motion). The basic foundations of the model have been drawn from a functional theory [2], a semi-classical geometric model [3], and a geometro-stochastic theory [4, 5].

This led us to the conclusion that it may be worth constructing a Lagrangian variant of our model. The present work constitutes one aspect of this construction program. We consider flat external and internal space-times and a vanishing connection and carry out a simple Lagrangian calculation [6]. In section 2, we determine the equations of motion for a set of fields $u_i(x, \xi)$ which are no more than the usual Euler-Lagrange equations with an extra term corresponding to the internal motion. In section 3, Noether’s theorem is used to determine the currents of the theory. The latter will be split into external and internal components (whose expressions are analogous to the well known ones [6]) and to additional components representing a mixing between the external and internal spaces. In section 4, we apply the formalism to a specific Lagrangian of bispinor fields before we conclude the work.

2. Euler Lagrange equations
We consider a set of bilocal fields $u_i(x, \xi)$ where $x = \left(x^k\right)$ corresponds to the external Minkowski space-time and $\xi = (\xi^\alpha)$ corresponds to the internal Minkowski space-time so that both superscripts $k$ and $\alpha$ run from 0 to 3. The Lagrangian density is chosen as usual with a
distinction between the external and internal derivatives

\[
L = L(u_i(x,\xi), u_{i;k}(x,\xi), u_{i;\alpha}(x,\xi))
\]

(1)

\[
u_{i;k} = \partial_k u_i = \partial u_i / \partial x^k ; \quad u_{i;\alpha} = \partial_\alpha u_i = \partial u_i / \partial \xi^\alpha
\]

Euler-Lagrange equations correspond to an extremum of the action

\[
A = \int L \, dx d\xi
\]

(2)

\[
\delta A = \int \bar{\delta} L \, dx d\xi = 0
\]

(3)

with respect to variations of the fields at fixed space-time points

\[
u_i'(x,\xi) = v_i(x,\xi) + \delta v_i
\]

(4)

where \(\delta v_i\) is the variation of the form of the function \(v_i\). The Lagrangian density variation is

\[
\bar{\delta} L = \bar{\delta} v_i \partial L / \partial v_i + \bar{\delta} v_{i;k} \partial L / \partial v_{i;k} + \bar{\delta} v_{i;\alpha} \partial L / \partial v_{i;\alpha}
\]

(5)

Using the expression of \(\bar{\delta} L\) in \(\delta A = 0\) and the well known integration by parts with vanishing variations of the fields at the boundaries, the Euler-Lagrange Equations are derived

\[
\partial L / \partial v_i - \partial_k [\partial L / \partial v_{i;k}] - \partial_\alpha [\partial L / \partial v_{i;\alpha}] = 0
\]

(6)

Up to this point, nothing new is obtained for these obvious equations of motion which are ordinary Euler-Lagrange equations with an additional term \(\partial_\alpha [\partial L / \partial v_{i;\alpha}]\) due to the internal degrees of freedom. However, when the Lagrangian density is suitably chosen, these equations will contain influences between the external and internal motions. This will be shown after calculating the currents in a general form.

3. Noether’s theorem

Noether’s theorem states that the invariance of the action with respect to a \(n\) parameter group leads to the existence of \(n\) conserved quantities provided the equations of motion hold true.

Hence, we consider a transformation of the external and internal space-time variables

\[
x' = x + \delta x
\]

(7)

\[
\xi' = \xi + \delta \xi
\]

(8)

The fields undergo the usual transformations

\[
u_i'(x',\xi') = v_i(x,\xi) + \delta v_i
\]

(9)

\[
\delta v_i = \bar{\delta} v_i + \delta x^k v_{i;k} + \delta \xi^\alpha v_{i;\alpha}
\]

(10)

so that the lagrangian density variation becomes

\[
\delta L = \bar{\delta} L + \delta x^k \partial L / \partial x^k + \delta \xi^\alpha \partial L / \partial \xi^\alpha
\]

(11)

Invariance of the action means that

\[
\delta A = \int \delta L \, dx d\xi + \int L \, (dx d\xi) = 0
\]

(12)
where the variation of the integration measure
\[ \delta (dx d\xi) = (1 - |J|) dx d\xi \] (13)
is given by the Jacobian determinant
\[ |J| \approx 1 + \frac{\partial \delta x^k}{\partial x^k} + \frac{\partial \delta \xi^\alpha}{\partial \xi^\alpha} \] (14)
Replacing all the above values in the invariance condition without using the equations of motion, one finds the following equation
\[ \frac{\delta L}{\delta u_i} \delta u_i = \partial_k (\delta \omega^n \Theta^k_n) + \partial_\alpha (\delta \omega^n \Theta^\alpha_n) \] (15)
where \( \delta \omega^n \) are the parameters of the symmetry. The Lagrangian derivatives are given by
\[ \frac{\delta L}{\delta u_i} = \partial_L/\partial u_i - \partial_k [\partial_L/\partial u_{i;k}] - \partial_\alpha [\partial_L/\partial u_{i;\alpha}] \] (16)
and the currents are
\[ \Theta^k_n = \frac{-1}{\delta \omega^n} \left[ \delta u_i \partial L/\partial u_{i;k} + \delta x^k L \right] \] (17)
\[ \Theta^\alpha_n = \frac{-1}{\delta \omega^n} \left[ \delta u_i \partial L/\partial u_{i;\alpha} + \delta \xi^\alpha L \right] \] (18)
It is at this stage that the symmetry of the extended particle comes into play. In the general case, we have
\[ \delta x^k = \delta \omega^n \left( [I_n]^{i;k} x^l + [I_n]^{i;\alpha} \xi^\alpha \right) \] (19)
\[ \delta \xi^\alpha = \delta \omega^n \left( [I_n]^{i;\alpha} x^l + [I_n]^{\alpha;\beta} \xi^\beta \right) \] (20)
The generators \( I \) of the symmetry are denoted \( T \) in the fields representation
\[ \delta u_i = \delta \omega^n [T_n] u_i \] (21)
\[ \delta u_i = \delta \omega^n \left( [T_n] u_i - u_{i;k} \left( [I_n]^{i;k} x^l + [I_n]^{i;\alpha} \xi^\alpha \right) - u_{i;\alpha} \left( [I_n]^{\alpha;\beta} x^l + [I_n]^{\alpha;\beta} \xi^\beta \right) \right) \] (22)
The currents can be decomposed
\[ \Theta^k_n = \theta^k_n + \sigma^k_n \] (23)
\[ \Theta^\alpha_n = \theta^\alpha_n + \sigma^\alpha_n \] (24)
The components
\[ \theta^k_n = -\partial L/\partial u_{i;k} \left( [T_n] u_i - u_{i;j} [I_n]^{j;l} x^l \right) - L [I_n]^{i;k} x^l \] (25)
\[ \theta^\alpha_n = -\partial L/\partial u_{i;\alpha} \left( [T_n] u_i - u_{i;\lambda} [I_n]^{\lambda;\beta} \xi^\beta \right) - L [I_n]^{\alpha;\beta} \xi^\beta \] (26)
have expressions which are identical to those corresponding to point-like external or internal particles. Depending on the subscript \( n \), they may correspond to external or internal components.
We chose the following expression for the Lagrangian:

\[
\sigma_n^k = -\partial L/\partial u_{i;k} \left( -u_{i;j} [I_n]_j^i \xi^\alpha - u_{i;\alpha} \left( [I_n]_\alpha^j x^j + [I_n]^{ij}_\alpha \xi^j \right) \right) - L [I_n]_\alpha^k \xi^\alpha \quad (27)
\]

\[
\sigma_n^\alpha = -\partial L/\partial u_{i;\alpha} \left( -u_{i;k} \left( [I_n]^k_i \xi^j - u_{i;\beta} [I_n]^{ij}_\alpha \xi^j \right) \right) - L [I_n]^\alpha_i x^j \quad (28)
\]

contain explicit mixing between external and internal space-time components.

If we consider the case where the external and internal symmetries act separately \(([I_n]_\alpha^k = [I_n]_\beta^i = 0)\), then the latter expressions simplify extremely:

\[
\sigma_n^k = \partial L/\partial u_{i;k} \left( u_{i;\alpha} [I_n]^{\alpha}_\beta \xi^\beta \right) \quad (29)
\]

\[
\sigma_n^\alpha = \partial L/\partial u_{i;\beta} \left( u_{i;k} [I_n]^k_i \xi^\beta \right) \quad (30)
\]

4. Bispinor fields

Let us construct a Lagrangian density assuming that the fields \(u(x, \xi)\) are bispinors \(u(x, \xi) = u^A(x, \xi)\) where the first index \(A\) transforms by an external Lorentz group representation and the second component \((A')\) transforms with an internal Lorentz group. The conjugate fields \(\bar{u} = u^* (\gamma^0 \otimes \Gamma^0)\) are expressed with external and internal Dirac matrices \(\gamma\) and \(\Gamma\), respectively. We chose the following expression for the Lagrangian:

\[
L = \frac{i}{2} \left[ \bar{\gamma}^k \partial_k u - \partial_k \bar{\gamma}^k u \right] + \frac{i}{2} \left[ \bar{\xi}^\alpha \partial_\alpha u - \partial_\alpha \bar{\xi}^\alpha u \right] - \frac{1}{2} \left[ \partial_k \bar{\xi}^k \Gamma^\alpha \partial_\alpha u + \partial_\alpha \bar{\xi}^k \Gamma^\alpha \partial_k u \right] - m \bar{u} u \quad (31)
\]

Obvious simplifying notations have been used such that \(\gamma^k = \gamma^k \otimes 1, \Gamma^\alpha = 1 \otimes \Gamma^\alpha\), and \(\gamma^k \Gamma^\alpha = \gamma^k \otimes \Gamma^\alpha\). The equations of motion

\[
\left[ i \left( \gamma^k \partial_k + \Gamma^\alpha \partial_\alpha \right) - m + \left( \gamma^k \partial_k \otimes \Gamma^\alpha \partial_\alpha \right) \right] u = 0 \quad (32)
\]

\[
\bar{u} \left[ -i \left( \partial_k \gamma^k + \partial_\alpha \Gamma^\alpha \right) - m + \left( \partial_k \gamma^k \otimes \partial_\alpha \Gamma^\alpha \right) \right] = 0 \quad (33)
\]

contain, in addition to the external and internal kinetic terms, a mixing term \(\gamma^k \partial_k \otimes \Gamma^\alpha \partial_\alpha\) due to the chosen form of the Lagrangian.

As to the currents, let us begin with translations in external and internal spaces. In this case, \(n\) is either \(l\) or \(\beta\). The generators are such that \([I_n]^k_i x^j\) and \([I_n]^{ij}_\alpha \xi^j\) should be replaced by \(\delta^k_l\) and \(\delta^\alpha_\beta\), and \(T_n = 0\). The external and internal components of the energy-momentum tensor retain their usual expressions:

\[
\theta^k_l = \partial L/\partial u_{i;k} (u_{i;l}) - L \delta^k_l \quad (34)
\]

\[
\theta^\alpha_\beta = \partial L/\partial u_{i;\alpha} (u_{i;\beta}) - L \delta^\alpha_\beta \quad (35)
\]

The mixing components read:

\[
\theta^k_\beta = \theta^\alpha_k = 0 \quad (36)
\]

\[
\sigma^k_\beta = \frac{i}{2} \left[ \bar{\gamma}^k \gamma^k + \bar{\xi}^\alpha \partial_\alpha \left( \gamma^k \otimes \Gamma^\alpha \right) \right] \partial_\beta \xi \quad (37)
\]

\[
\sigma^\alpha_k = \frac{i}{2} \left[ \bar{\xi}^\alpha \gamma^k + \bar{\xi}^k \partial_k \left( \gamma^k \otimes \Gamma^\alpha \right) \right] \partial_\alpha \xi \quad (38)
\]
For Lorentz rotation in the external plane \((m, n)\) and the internal plane \((\mu, \nu)\), the generators act in the following form

\[
I_{(n,m)}^k \delta^k_n - x_n \delta^k_m = x_m \delta^k_n - x_n \delta^k_m \tag{39}
\]

\[
I_{(\nu,\mu)}^{\alpha \beta} \xi^{\beta} = \xi_\mu \delta^{\alpha}_\nu - \xi_\nu \delta^{\alpha}_\mu \tag{40}
\]

The mixing components are

\[
\theta^k_{(\nu,\mu)} = \frac{i}{2} \left[ i \gamma^k + i \partial_\alpha \bar{u} \left( \gamma^k \otimes \Gamma^\alpha \right) \right] \left[ 1 \otimes (\Gamma_\nu \Gamma_\mu / 2) \right] u \tag{41}
\]

\[
\theta^\alpha_{(n,m)} = \frac{i}{2} \left[ i \Gamma^\alpha + i \partial_\kappa \bar{u} \left( \gamma^k \otimes \Gamma^\alpha \right) \right] \left[ (\gamma_n \gamma_m / 2) \otimes 1 \right] u \tag{42}
\]

\[
\sigma^k_{(\nu,\mu)} = \frac{i}{2} \left[ i \gamma^k + i \partial_\alpha \bar{u} \left( \gamma^k \otimes \Gamma^\alpha \right) \right] (\xi_\mu \partial_\nu - \xi_\nu \partial_\mu) u \tag{43}
\]

\[
\sigma^\alpha_{(n,m)} = \frac{i}{2} \left[ i \Gamma^\alpha + i \partial_\kappa \bar{u} \left( \gamma^k \otimes \Gamma^\alpha \right) \right] (x_m \partial_\alpha - x_\alpha \partial_m) u \tag{44}
\]

5. Conclusion

We considered a direct product symmetry for a lagrangian of bilocal fields representing extended particles. Very simple calculations lead to complicated currents with external, internal and mixed components. The chosen Lagrangian has thrown some light on these expressions but the results are not conclusive. We hope that forthcoming works will sharpen the physical interpretation of each component.

References

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