THE GRAVITATIONAL POTENTIAL RECONSTRUCTION FROM PECULIAR VELOCITY AND WEAK LENSING MEASUREMENTS

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ABSTRACT

We present an analytic method for rapidly forecasting the accuracy of the gravitational potential reconstruction from measurements of radial peculiar velocities of every galaxy cluster with $M > M_{\text{th}}$ in solid angle $\theta^2$ and over a redshift range $z_{\text{min}} < z < z_{\text{max}}$. These radial velocities can be determined from measurements of the kinetic and thermal Sunyaev-Zeldovich effects. For a shallow survey with $0.2 < z < 0.4$, one mode of the potential (on length scales $\approx 60$ Mpc) can be reconstructed for every approximately eight cluster velocity determinations. Deeper surveys require measurements of more clusters per a signal-to-noise ratio greater than 1. Accuracy is limited by the “undersampling noise” because of the nonobservation of a large fraction of mass that is not in galaxy clusters. Determining the potential will allow us to study in detail the relationship between galaxies and their surrounding large-scale density fields over a wide range of redshifts and to test the gravitational instability paradigm on very large scales. Observation of weak lensing by large-scale structure is complementary since lensing is sensitive to the tangential modes that do not affect the velocity.

Subject headings: cosmology: observations — cosmology: theory — galaxies: evolution — galaxies: formation

1. INTRODUCTION

Reconstruction of the density field from galaxy peculiar velocities was pioneered by Dekel, Bertschinger, & Faber (1990). The radial component of the peculiar velocities of galaxies can be determined by inferring the distance and then subtracting off the Hubble flow contribution to the redshift.

Galaxy peculiar velocity determinations, because they rely on distance determinations, are only useful at $z \lesssim 0.1$. Galaxy cluster velocities determined from observations of the Sunyaev-Zeldovich (SZ) effects (Sunyaev & Zeldovich 1980) do not rely on a distance determination. The velocity signal arises from the cluster’s radial motion with respect to the cosmic microwave background (CMB). High-resolution ($\approx 1''$), multifrequency observations of clusters can be used to determine radial velocities with a nearly distance-independent accuracy. Errors of $\sim 100$ km s$^{-1}$ may be achievable (Holder 2002; Nagai, Kravtsov, & Kosowsky 2003).

Here we present a method for rapidly forecasting the accuracy of density field reconstruction from peculiar velocity measurements and apply it to surveys with various redshift ranges. We also show how weak lensing observations can provide complementary information (Mellier 1999; Bartelmann & Schneider 2001).

Our analytic method for forecasting results assumes a uniform and continuous velocity field map. Even if this map were derived from noiseless peculiar velocity measurements, there would still be an effective noise contribution due to the fact that most of the mass in the universe is not in galaxy clusters. Fortunately, as we quantify below, this contribution is negligible for relevant scales, so this noise from undersampling is not overwhelmingly large.

We consider three different survey types labeled “SDSS,” “DEEP/VIRMOS,” and “SZ” with redshift ranges $0.2 < z < 0.4$, $0.7 < z < 1.4$, and $0.2 < z < 2$, respectively. Cluster redshifts are required to convert SZ measurements into gravitational potential maps. Thus, the redshift ranges of two survey types are subsets of those of optical surveys SDSS (Sloan Digital Sky Survey), DEEP II (Deep Extragalactic Evolutionary Probe; Davis et al. 2001), and VIRMOS (Visible-Infrared Multi-Object Spectrographs; Le Fèvre et al. 2001). The peculiar velocity surveys that we imagine could be realized by a targeted multifrequency SZ follow-up on galaxy clusters identified in those optical surveys. The ambitious, deep “SZ” survey is modeled after proposed SZ surveys that can locate galaxy clusters over a large $z$ range. Again, the radial velocities can be determined from a multifrequency follow-up (if necessary), although the redshift determinations are an unsolved challenge.

Radial peculiar velocities are directly sensitive to radial gradients in the gravitational potential. Gravitational lensing is caused by tangential gradients in the gravitational potential; thus, in principle, lensing provides complementary information for potential reconstruction. However, we show below that lensing is actually unlikely to add much to the density field reconstruction. The reason is simple: lensing observations are two-dimensional, whereas velocity observations are three-dimensional. The conclusion may be different if one considers the lensing of many source populations with differing redshift distributions (Hu & Keeton 2002).

2. FORECASTING ERRORS

To begin, we assume that we have a continuous and uniform map of the velocity field in a box of volume $V$. The map has white noise with finite weight per comoving volume $w$; i.e., the variance of the error on the average velocity in some subvolume $V_i$ is $\sigma^2_{\text{v},i} = 1/(w V_i)$. For specificity, we set $h = 1 - \Omega_m = \Omega_\Lambda = 0.7$ and $\sigma_8 = 1$.

For the moment, we ignore light cone effects and assume that the map is of the velocity field at the present time. This idealization maintains homogeneity and allows a completely analytic analysis of the reconstruction errors in Fourier space. With the assumption of a potential flow (Peebles 1993),

$$v_k = i \frac{k}{H_0} \Phi_k(t_0) x(t),$$

where $x(t) \equiv \frac{2}{3} \frac{dD}{da} (t) \frac{E(t) a^2(t)}{\Omega_m}$,
and \( E^2(t) = H^2(t)/H_0^2 = \Omega_m a^3(t) + \Omega_x \). The noise and signal contributions to the variance of \( \Phi_\xi \) are diagonal, with diagonal entries given by

\[
N(k) = \left( \frac{H_0}{k x_0} \right)^2, \\
S(k) = P_k(k) = \frac{9}{4} \left( \frac{H_0}{k} \right)^4 \Omega_m^{2/3} \frac{2\pi^2}{k^3} \Delta^2(k),
\]

respectively, where \( \Delta^2(k) = k^3 P(k)/(2\pi^2) \) and \( P_k(k) \) is the matter power spectrum.

The square of the signal-to-noise ratio (S/N) for the gravitational potential map is thus

\[
\frac{S(k)}{N(k)} = \frac{\Delta^2(k) \text{ w}^2}{w_y^2} \left( \frac{x_0}{1.1} \right)^2 \left( \frac{\Omega_m h^2}{0.21} \right)^2 \left( \frac{k}{0.1 \text{ Mpc}^{-1}} \right)^{-5},
\]

where \( w_y^2 = (100 \text{ km s}^{-1})^2 (64 \text{ Mpc})^3 \) and \( x_0 \equiv x(t_0) = 1.1 \) and \( \Delta^2(0.1 \text{ Mpc}^{-1}) = 0.8 \) for our fiducial model. We therefore expect measurements with \( S/N > 1 \) for every mode with \( 2\pi/L \geq \Delta \approx 0.1 \text{ Mpc}^{-1} \), where \( L^3 = \mathcal{V} \). Furthermore, there are many of these modes for a box of size \( L \); the number in Fourier-space volume \( (\Delta k)^3 \) is \( 4 \times 10^3 [(k/0.1 \text{ Mpc}^{-1}) \times (L/1000 \text{ Mpc})^2] \).

We now address our value of \( w_y \) above. Evaluating the total gravitational potential, using only the fraction of the mass that is contained in clusters, entails an inherent undersampling of the gravitational potential. The resulting uncertainty we refer to as undersampling noise and evaluate it as follows. The comoving number density of galaxy clusters with a lower mass threshold of \( 10^{14} h^{-1} M_\odot \) at \( z = 1 \) is about \( 1 \) (64 Mpc)\(^{-3} \) volume (Holder, Haiman, & Mohr 2001), which roughly corresponds to the volume of a 40 Mpc radius sphere. However, a sphere containing this limiting mass in a uniform background only has a comoving radius of about \( R = 9 \) Mpc. Estimating the velocity field averaged over \( R = 40 \) Mpc from just one sample of the velocity field measured over 9 Mpc introduces an undersampling error with a variance (at \( z = 1 \)) of \( \langle (\nu_{v_0} - \nu_y)^2 \rangle = (117 \text{ km s}^{-1})^2 \). Thus, we cannot do better than \( w^{-1} = (117 \text{ km s}^{-1})^2 (64 \text{ Mpc})^3 \) at \( z = 1 \). We calculate the variance of the undersampling error at any redshift as \( w^{-1} = \langle (\nu_{v_0} - \nu_y)^2 \rangle / 4 \pi R^3(z) \), where \( 4/3 \pi R^3(z) = \bar{n}^{-1}(z) \); \( \bar{n}(z) \) is the number density of clusters more massive than our threshold mass.

Our calculation of the undersampling noise is highly idealized. We assume that all clusters have a mass at the threshold of \( 10^{14} h^{-1} M_\odot \). Some of the clusters are more massive; including this effect would reduce our estimate of the undersampling noise. We also used linear perturbation theory to calculate \( \langle (\nu_{v_0} - \nu_y)^2 \rangle \), although we expect this to be quite accurate due to the small contribution from modes with wavelengths less than 9 Mpc. The undersampling noise is highly sensitive to the number density of clusters; it will be higher if the actual number density of clusters is sparser than in our fiducial cosmology.

The dashed line in Figure 1 shows the redshift dependence of this undersampling noise, which is due to the evolution in \( \bar{n}(z) \). According to Holder (2002) and Nagai et al. (2003), there is also a fundamental limit of about 100 km s\(^{-1} \) on how well one can infer the peculiar velocity of a galaxy cluster from the (highly nonuniform) velocities of the gas. The result of adding these errors in quadrature is represented by the solid line in Figure 1. Wiener filtering can reduce the error.

The angular resolution, sensitivity, and frequency coverage required to achieve a velocity measurement with a given precision have not yet been worked out in detail. From prior work (Haeblein & Tegmark 1996; Holzapfel et al. 1997; Kashlinsky & Atrio-Barandela 2000; Aghanim, Górski, & Puget 2001), which ignores the velocity substructure of the gas, we find that measurements with errors of \( 8 \mu \text{K} \) translate into \( \sigma_v = 100 \text{ km s}^{-1} \), which is well matched to the Holder/Nagai limit. A multimode detector on a large telescope with a sensitivity of \( 100 \mu \text{K} \) could achieve \( 8 \mu \text{K} \) on 20 clusters in 1 hr, or 16,000 in a month.

We now consider radial velocities on our past light cone. The isotropic, \( z \)-dependent geometry suggests the mode decomposition used by, e.g., Stebbins (1996):

\[
\Phi(\hat{\gamma}, r) = \sum_{lm} \int_0^{\bar{n}} dk \frac{k^3 \tilde{\Phi}_{lm}(k)}{H_0} j_l(kr) Y_{lm}(\hat{\gamma}).
\]

The radial velocity generated by this potential is

\[
v_r(\hat{\gamma}, r) = \frac{ix(r)}{H_0} \sum_{lm} \int_0^{\bar{n}} dk \frac{k^3 \tilde{\Phi}_{lm}(k)}{H_0} j_l(kr) Y_{lm}(\hat{\gamma}),
\]

where we have changed our independent variable from \( t \) to a conformal distance \( r \) and \( j_l(y) \equiv d^l y j_l(y) \). Note that all the time dependence on the right-hand side is carried by \( x(r) \); thus, we are discussing the reconstruction of the gravitational potential on our past light cone extrapolated by linear theory to what it would be today. Note that this extrapolation assumes a precise knowledge of the cosmological parameters, which we assume will come from, e.g., CMB and supernova observations.

Our discretized model of the data is

\[
v_i = \sum_{k, m} A_i(i; b, l, m) \tilde{\Phi}_{km} + n_i,
\]

where \( n \) is the error (whose statistical properties were just dis-
cussed above). \( A_x \) is implicitly defined by equation (4), and \( b \) enumerates the \( k \) bins:

\[
\Phi_{blm} = \frac{1}{\Delta k_b} \int_{-\Delta k_b/2}^{+\Delta k_b/2} dk \Phi_{blm}(k).
\]  (6)

The minimum-variance estimate of \( \Phi_{blm} \) is

\[
\hat{\Phi}_{blm} = \sum_{b'l'm'} W_{sl}^{\dagger} (blm, b'Tm') A_i (i; b', l', m') N_{ij}^{-1} v_{ij},
\]

where \( N_{ij} = \langle n_n \rangle \), \( W_{sl}^{\dagger} (blm, b'Tm') = \langle \delta \Phi_{slm} \delta \Phi_{pcm} \rangle \) is the noise covariance matrix of \( \Phi_{blm} \) given by

\[
W_{sl}(blm, b'Tm') = \sum_{i'} A_i (i; b, l, m) N_{ij}^{-1} A_i^*(i'; b', l', m'),
\]

and \( \delta \Phi_{pcm} \equiv \Phi_{pcm} - \hat{\Phi}_{pcm} \).

With the uniform sampling assumption, \( W_{sl} \) for reconstructing \( \Phi_{blm} \) is given by

\[
W_{sl}(blm, b'Tm') = \frac{\Delta k_b}{H_0} \frac{\Delta k_{l'}}{H_0} \delta_{m'} \delta_{m''} k_b k_{l'}
\times \int dr r^2 \delta(r) w(r) y_j(k_r r) y_j^*(k_r r).
\]  (7)

The matrix is block-diagonal with zero entries for any elements with \( l \neq l' \) or \( m \neq m' \). As one expects from statistical isotropy, \( W_{sl} \) does not depend on \( m \).

The signal matrix is diagonal, although the effect of binning the uncorrelated \( k \)-modes is to reduce the variance by \( \Delta k: \langle \Phi_{blm} \Phi_{bl} \rangle = P_F(k)/\Delta k \). In § 3, we compare signal and noise by plotting \( k^2 \Delta k P_F(k) \), where the \( k^2 \) factor makes a dimensionless quantity and the \( \Delta k \) makes the corresponding signal quantity, \( k^2 P_F(k) = (4\pi)^2 k^2 P_F(k) \), independent of \( \Delta k \).

The weight matrix for potential reconstruction from a weak lensing convergence map with uniform weight per solid angle \( w \) is

\[
W_{sl}(blm, b'Tm') = \delta_{m'} \delta_{m''} w l^2 (l + 1)^2 \Delta k_b \Delta k_{l'} k_b k_{l'} I_{l}(k_r r, \theta),
\]

where

\[
I_{l}(k_r r, \theta) = \int_0^\infty dr r^2 \delta(r) W(k_r \theta)
\]

\[(\text{Stebbins} 1996), \text{and} \ W(k_r \theta) = 2I_{l}(k_r \theta) \text{/(kr)}, (\text{Jain} \& \text{Seljak} \text{1997}) \text{incorporates the fact that we smooth the signal over radius} \ \theta = 2.5. \text{Again, this is block-diagonal in} \ l \text{and} \ m. \text{However, the structure in the radial wavenumber} \ k \text{is highly degenerate. For each} \ l, \ m \text{-submatrix, the} \ k, \ k' \text{-dependence can be written as an outer product of a single vector. Such matrices have only one nonzero eigenvalue; i.e., only one particular linear combination of} \ k \text{-modes can be determined. This degeneracy is expected since lensing measurements are inherently two-dimensional, unlike the peculiar velocity measurements.}

3. RESULTS

In Figure 2, we compare signal and noise for \( \Delta k = k \) for several values of \( l \) assuming \( w(z) \) as shown in Figure 1. One can see that \( \text{S/N} > 1 \) measurements are possible on large spatial and angular scales. The sharp low-\( k \) cutoffs are an effect of the finite size of the surveys: \( k_{min} = \pi / L \), where \( L \) is the radial depth of the survey because \( j(kr) \approx 0 \) for \( k < l r \). The velocity results can be understood quantitatively by multiplying the \( \text{S/N} \) from equation (3) by the number of modes in a bin of width \( \Delta k: \Delta k L/(2\pi) \).

Now we turn to the weak lensing panel of Figure 2. We assume a noise level of \( w = 9 \times 10^5 \), which can be achieved with a galaxy number density of \( n = 30 \text{galaxy arcmin}^{-2} \) and an intrinsic galaxy ellipticity of \( \sigma_\epsilon = 0.2 \) (Kaiser 1998; van Waerbeke, Bernardeau, & Mellier 1999). We further assume a negligible width to the distribution of these galaxies centered at \( z_s = 1 \). The results can be understood analytically from equation (8) and the approximation (Stebbins 1996) \( I_l = [\pi/(2l)]^{1/2} [1 - 1/(kr)] \) for \( kr > l \) and \( I_l = 0 \) otherwise. Unlike with velocity, the higher \( l \)-values are reconstructed with less noise because of the sensitivity of the convergence to tangential gravitational potential variations.

The variance expected on a measurement of \( k_b^2 \Phi_{pcm} \) is actually \( k_b^2 W_{sl}^{\dagger} (blm, blm) \), not the \( k_b^2 W_{sl}^{\dagger} (blm, blm) \) plotted in Figure 2. The inverse is not well defined because some linear combinations of the \( \Phi_{pcm} \) remain unconstrained by the measurements. We have determined the modes that are well constrained by use of a \( \text{S/N} \) eigenmode analysis (e.g., Bond 1995).

The \( \text{S/N} \) eigenmode transformation (also called the KL transformation after its original discoverers; Karhunen 1947) is a transformation to a basis in which both the signal and the noise are diagonal and the noise has unit variance. The elements of the diagonal signal matrix are given by the eigenvalues of \( W_{sl}^{\dagger} P_F / \Delta k W_{sl}^{\dagger} \), shown in Figure 3. We see that for SDSS for each \( l, m \) with \( l = 25 \), there are about 20 modes with \( \text{S/N} > 1 \). As one would expect from Figure 2, these modes are linear combinations of \( \Phi_{pcm} \) with \( k_b \) between 0.02 and 0.20.

The total weight from a combined weak lensing and peculiar velocity survey is the sum of the individual weights. The resulting eigenvalue spectrum is identical to the peculiar velocity survey eigenvalue spectrum, except for the increased value of one eigenvalue per \( l, m \)-mode.

4. DISCUSSION

The largest \( l \)-value of modes with \( \text{S/N} > 1 \) is \( l_{max} \), which entails a minimum survey size of \( \theta_{min} = 5^\circ(80/h_{max}) \). From Figure 3,
but also increases spectral resolution [i.e., decreases $\delta l = 2\pi/\theta = 12(30/\theta^2)$] and increases the number of angular modes with $S/N > 1$:

$$N_{\theta} = \frac{\theta^2}{4\pi} \sum_{l_{\min}}^{l_{\max}} (2l + 1) = 27\left(\frac{\theta}{30^\circ}\right)^2 \left(\frac{l_{\max}}{35}\right)^2.$$ (9)

Thus, for a $\theta = 30^\circ$ survey with $0.2 < z < 0.4$, the total number of modes with $S/N \approx 1$ is roughly $N_{\theta} \times 15 \approx 360$. Covering $\pi$ steradians would require measuring 36,000 clusters and would deliver $\sim 4500$ modes.

5. CONCLUSIONS

Galaxy cluster peculiar velocities can be used to make high $S/N$ maps of the gravitational potential, and therefore matter density, on very large scales. A limiting source of uncertainty in these maps is what we have called the undersampling noise owing to the fact that only a small fraction of mass is in clusters. The undersampling noise can be reduced by lowering the mass threshold, but the lower the mass of the cluster, the harder the measurement of its peculiar velocity. Much more work is needed to understand the demand on experimental resources and to optimize observing strategies. We have made several arguments that favor shallower surveys over deeper ones.

The $\Phi$ and density maps will have numerous applications. Cross-correlations with various tracers of the density field (red galaxies, blue galaxies, infrared galaxies, lensing of galaxies, lensing of the CMB, the integrated Sachs-Wolfe effect on CMB, etc.) will be of great interest. The greatest drawback to the maps is their inability to probe scales below the typical separation between clusters.

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REFERENCES

Aghanim, N., Górski, K. M., & Puget, J.-L. 2001, A&A, 374, 1
Bartelmann, M., & Schneider, P. 2001, Phys. Rep., 340, 291
Bond, J. R. 1995, Phys. Rev. Lett., 74, 4369
Davis, M., Newman, J. A., Faber, S. M., & Phillips, A. C. 2001, in Deep Fields, ed. S. Cristiani, A. Renzini, & R. E. Williams (Berlin: Springer), 241
Dekel, A., Bertschinger, E., & Faber, S. M. 1990, ApJ, 364, 349
Haehnelt, M. G., & Tegmark, M. 1996, MNRAS, 279, 545
Holder, G. P. 2002, preprint (astro-ph/0207600)
Holder, G., Haiman, Z., & Mohr, J. J. 2001, ApJ, 560, L111
Holzapfel, W. L., Ade, P. A. R., Church, S. E., Mauskopf, P. D., Rephael, Y., Wilbanks, T. M., & Lange, A. E. 1997, ApJ, 481, 35
Hu, W., & Keeton, C. R. 2002, Phys. Rev. D, 66, 63506
Jain, B., & Seljak, U. 1997, ApJ, 484, 560
Kaiser, N. 1998, ApJ, 498, 26
Karhunen, K. 1947, Über lineare Methoden in der Wahrscheinlichkeitsrechnung (Helsinki: Kirjapaino oy. sana)
Kashlinsky, A., & Atrio-Barandela, F. 2000, ApJ, 536, L67
Le Fevre, O., et al. 2001, in Deep Fields, ed. S. Cristiani, A. Renzini, & R. E. Williams (Berlin: Springer), 236
Melier, Y. 1999, ARA&A, 37, 127
Nagai, D., Kravtsov, A. V., & Kosowsky, A. 2003, ApJ, in press (astro-ph/ 0208308)
Peebles, P. J. E. 1993, Principles of Physical Cosmology (Princeton: Princeton Univ. Press)
Stebbins, A. 1996, preprint (astro-ph/9609149)
Suniaev, R. A., & Zeldovich, Ya. B. 1980, MNRAS, 190, 413
van Waerbeke, L., Bernardeau, F., & Mellier, Y. 1999, A&A, 342, 15